Nearly unstable integer-valued ARCH process and unit root testing

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Abstract
This paper introduces a Nearly Unstable INteger-valued AutoRegressive Conditional Heteroscedastic (NU-INARCH) process for dealing with count time series data. It is proved that a proper normalization of the NU-INARCH process weakly converges to a Cox–Ingersoll–Ross diffusion in the Skorohod topology. The asymptotic distribution of the conditional least squares estimator of the correlation parameter is established as a functional of certain stochastic integrals. Numerical experiments based on Monte Carlo simulations are provided to verify the behavior of the asymptotic distribution under finite samples. These simulations reveal that the nearly unstable approach provides satisfactory and better results than those based on the stationarity assumption even when the true process is not that close to nonstationarity. A unit root test is proposed and its Type-I error and power are examined via Monte Carlo simulations. As an illustration, the proposed methodology is applied to the daily number of deaths due to COVID-19 in the United Kingdom.

Keywords
count time series, Cox–Ingersoll–Ross diffusion process, inference, limit theorems, stochastic integral
1 | INTRODUCTION

First-order nearly unstable continuous autoregressive processes have been well explored in the literature, see for example, Chan and Wei (1987), Phillips (1987), Chan et al. (2019), and the references therein. In these works, it is assumed that the model approaches the nonstationarity region as the sample size increases. More specifically, a nearly unstable continuous process \( \{Y_t^{(n)}\}_{t \in \mathbb{N}} \) is defined by

\[
Y_t^{(n)} = \rho_n Y_{t-1}^{(n)} + \eta_t, \quad t \in \mathbb{N},
\]

for \( n \in \mathbb{N} \), where \( \{\eta_t\}_{t \in \mathbb{N}} \) is a white noise, \( \rho_n = 1 - b/n \) for \( b > 0 \), and \( Y_0^{(n)} = 0 \) for all \( n \).

In the past few years, nearly unstable discrete processes have emerged based on the Integer-valued AutoRegressive (INAR) approach (Al-Osh & Alzaid, 1987; McKenzie, 1985). The first attempt on this subject was due to Ispány et al. (2003). More specifically, a nearly unstable INAR(1) process \( \{X_t^{(n)}\}_{t \in \mathbb{N}} \) is defined by

\[
X_t^{(n)} = \alpha_n \circ X_{t-1}^{(n)} + \epsilon_t^{(n)}, \quad t \in \mathbb{N},
\]

where \( \circ \) is the thinning operator proposed by Steutel and van Harn (1990), given by

\[
\alpha_n \circ X_{t-1}^{(n)} = \sum_{j=0}^{X_{t-1}^{(n)}} B_j^{(n)} \text{ with } \{B_j^{(n)}\}_{j \in \mathbb{N}} \overset{\text{iid}}{\sim} \text{Bernoulli}(\alpha_n), \text{ for } \alpha_n \in (0, 1), \text{ and } \{\epsilon_t^{(n)}\}_{t \in \mathbb{N}} \text{ is a sequence of independent and identically distributed (iid) random variables with } \epsilon_t^{(n)} \text{ being independent of the counting series } \{B_j^{(n)}\}_{j \in \mathbb{N}} \text{ for all } k \leq t, \text{ for } t \in \mathbb{N}. \]

These authors assumed that \( \alpha_n \) approaches 1 (nonstationarity) when \( n \to \infty \) as given in Chan and Wei (1987) in the continuous context. By assuming \( \mu_c = E(\epsilon_t) \) is known, the conditional least squares (CLS) estimator of \( \alpha_n \) was explored by Ispány et al. (2003). They showed that, under nearly nonstationarity and assuming finite second moment for \( \epsilon_t \), the CLS estimator weakly converges to a normal distribution at the rate \( n^{3/2} \). Other related works dealing with nearly unstable INAR (Galton–Watson/branching) processes are due to Wei and Winnicki (1990), Winnicki (1991), Ispány et al. (2005, 2014), Rahimov (2007, 2008, 2009), Drost et al. (2009), Barczy et al. (2011), Barczy et al. (2014, 2016), and Guo and Zhang (2014). Practical situations demonstrating evidence of a nearly unstable INAR model are discussed for instance by Hellström (2001).

Another popular way for dealing with count time series data is the Integer-valued Generalized AutoRegressive Conditional Heterokedastic (INGARCH) models by Ferland et al. (2006), Fokianos et al. (2009), Fokianos and Fried (2010), Zhu (2011, 2012), Fokianos and Tjøstheim (2011), Christou and Fokianos (2015), Gonçalves et al. (2015), Davis and Liu (2016), Silva and Barreto-Souza (2019), Weiß et al. (2022), which constitute in some sense an integer-valued counterpart of the classical GARCH models by Bollerslev (1986). The INARCH methodology is the focus of this paper. Like the existing literature on nearly unstable continuous and INAR processes that assumes first-order autoregressive dependence, in this paper we consider the first-order autoregressive version of the INARCH approach, which is known as INARCH(1) (Integer-valued AutoRegressive Conditional Heteroscedastic).

The main goal in this paper is to introduce a Nearly Unstable INARCH (denoted by NU-INARCH) process for dealing with count time series data. To the best of our knowledge, this is the first time that a nearly unstable count time series model is being proposed based on the INARCH approach; all existing nearly unstable discrete processes in the literature consider the INAR approach. We establish the weak convergence of the NU-INARCH process (when
properly normalized) in the Skorohod space. With this result at hand, we derive the asymptotic distribution of the conditional least squares estimator of the correlation parameter as a functional of certain stochastic integrals. An equally important contribution of this paper is to develop a unit root test (URT) for the INARCH model, where the asymptotic distribution of the statistics under the null hypothesis is provided. Note that although URTs are well explored in the continuous case, only sporadic results are available for the discrete case. A few works dealing with this relevant problem, based on the INAR approach, are due to Hellström (2001) and Drost et al. (2009).

The paper is organized as follows. In Section 2, the NU-INARCH model is introduced and a fluctuation theorem is established, which involves the Cox–Ingersoll–Ross diffusion process. The asymptotic distribution of the CLS estimator for the correlation parameter is derived in Section 3 under the nearly unstable and stationarity assumptions. Section 4 provides simulated results about the asymptotic distribution of the CLS estimator under both nearly unstable and stationary approaches and also compares them in terms of confidence interval coverages. A unit root test for the INARCH process is proposed in Section 5 and its performance is evaluated via Monte Carlo simulations. An empirical application about the daily number of deaths due to COVID-19 in the United Kingdom, which exhibits a nearly unstable/nonstationary behavior, is provided in Section 6. Concluding remarks and future research are addressed in Section 7.

2 | MODEL AND THE FLUCTUATION THEOREM

In this section, we define the nearly unstable INARCH process and obtain its weak convergence (under a proper normalization) in the space of the nonnegative càdlàg functions endowed with the Skorokhod topology.

**Definition 1.** We say that a sequence \( \{X_t^{(n)}\}_{t \in \mathbb{N}} \) is a first-order nearly unstable integer-valued ARCH process (in short NU-INARCH) if

\[
X_t^{(n)} | F_{t-1}^{(n)} \sim \text{Poisson}(\lambda_t^{(n)}),
\]

\[
\lambda_t^{(n)} \equiv E(X_t^{(n)} | F_{t-1}^{(n)}) = \beta + \alpha_n X_{t-1}^{(n)}, \quad t \geq 1,
\]

for \( n \in \mathbb{N} \), where \( F_{t-1}^{(n)} = \sigma(X_{t-1}^{(n)}, \ldots, X_0^{(n)}) \), \( \beta > 0 \), and \( \alpha_n = 1 - \frac{\gamma_n}{n} \), with \( \lim_{n \to \infty} \gamma_n = \gamma \in \mathbb{R} \), and \( X_0^{(n)} = 0 \) for all \( n \).

**Remark 1.** The parameterization of \( \alpha_n \) in (2) was first proposed by Chan and Wei (1987) and subsequently used in Ispány et al. (2003). This parameterization will be used to establish theoretical results for the unstable INARCH process, including a unit root test. For the stable INARCH case, we consider \( \alpha_n = \alpha \) being a constant (without the underscript \( n \)). For the simulated and real data illustrations in this paper, we consider the stable case and show how our methodology can be applied when \( \alpha \) is close to 1.

In the next proposition, we provide the mean, variance, and autocovariance function of the NU-INARCH process. These results will be important to establish the proper normalization in order to obtain a nontrivial limit for the counting process.
Proposition 1. Let \( \{X_t^{(n)}\}_{t \in \mathbb{N}} \) be a nearly unstable INARCH process. Then, its marginal mean and variance, and autocorrelation function are given respectively by

\[
E(X_t^{(n)}) = \beta \frac{1 - \alpha_n^t}{1 - \alpha_n},
\]

\[
\text{Var}(X_t^{(n)}) = \frac{\beta}{1 - \alpha_n} \left\{ \frac{1 - \alpha_n^2}{1 - \alpha_n} - \alpha_n \frac{1 - \alpha_n^t}{1 - \alpha_n} \right\},
\]

\[
\text{cov}(X_{t+k}^{(n)}, X_t^{(n)}) = \alpha_n^k \text{Var}(X_t^{(n)}), \quad t, k \in \mathbb{N}_0 \equiv \{0, 1, 2, \ldots\}.
\]

Proof. We have that \( E(X_t^{(n)}) = E\left( E(X_t^{(n)} \mid F_{t-1}^{(n)}) \right) = \beta + \alpha_n E(X_{t-1}^{(n)}) \). By using recursion \( t \) times, we obtain the result for the marginal mean. For the variance, it follows that

\[
\text{Var}(X_t^{(n)}) = E\left( \text{Var}(X_t^{(n)} \mid F_{t-1}^{(n)}) \right) + \text{Var}\left( E(X_t^{(n)} \mid F_{t-1}^{(n)}) \right) = \beta + \alpha_n E(X_{t-1}^{(n)}) + \alpha_n^2 \text{Var}(X_{t-1}^{(n)}) = \beta \frac{1 - \alpha_n^t}{1 - \alpha_n} + \alpha_n^2 \text{Var}(X_{t-1}^{(n)}) = \beta \frac{1 - \alpha_n^t}{1 - \alpha_n} + \alpha_n \left\{ \sum_{k=0}^{t-1} \alpha_n^k - \alpha_n \sum_{k=0}^{t-1} \alpha_n^k \right\} = \beta \frac{1 - \alpha_n^t}{1 - \alpha_n} + \alpha_n \left\{ \frac{1 - \alpha_n^{2t}}{1 - \alpha_n} - \alpha_n \frac{1 - \alpha_n^t}{1 - \alpha_n} \right\},
\]

where we have used in the second equality that \( \text{Var}(X_{t}^{(n)} \mid F_{t-1}^{(n)}) = E(X_t^{(n)} \mid F_{t-1}^{(n)}) \).

Finally, for \( k, t \in \mathbb{N}_0 \), the autocorrelation function becomes

\[
\text{cov}(X_{t+k}^{(n)}, X_t^{(n)}) = E\left( \text{cov}(X_{t+k}^{(n)}, X_t^{(n)} \mid F_t^{(n)}) \right) + \text{cov}\left( E(X_{t+k}^{(n)} \mid F_t^{(n)}), E(X_t^{(n)} \mid F_t^{(n)}) \right) = \text{cov}\left( E(X_{t+k}^{(n)} \mid F_t^{(n)}), X_t^{(n)} \right) = \alpha_n \text{cov}(X_{t+k-1}^{(n)}, X_t^{(n)}) = \alpha_n^k \text{Var}(X_t^{(n)}),
\]

where we have used in the second equality that \( \text{cov}(X_{t+k}^{(n)}, X_t^{(n)} \mid F_t^{(n)}) = 0 \) almost surely and in the third equality that \( E(X_{t+k}^{(n)} \mid F_t^{(n)}) = E\left( E(X_{t+k}^{(n)} \mid F_{t+k-1}^{(n)}) \mid F_t^{(n)} \right) = \beta + \alpha_n E(X_{t+k-1}^{(n)} \mid F_t^{(n)}) \) since \( F_t^{(n)} \subseteq F_{t+k-1}^{(n)} \) for \( k \geq 1 \).

From Proposition 1, for \( s > 0 \) (nonnegative real-valued), we have that \( E(X_{(n)}^{(s)} \mid n) = \beta \gamma^{-1} n (1 - e^{-\gamma s}) = \mathcal{O}(n) \) and \( \text{Var}(X_{(n)}^{(s)} \mid n) \approx \beta \gamma^{-2} n^2 (1 - e^{-\gamma s})^2 / 2 = \mathcal{O}(n^2) \). Then define the normalized process \( \mathcal{X}^{(s)}(n) \equiv X_{(n)}^{(s)} / n \) and obtain that \( \mathcal{X}^{(s)}(n) \equiv \mathcal{O}(1) \), for \( t \geq 0 \). In the following theorem, we establish the weak convergence of the process \( \{ \mathcal{X}^{(s)}(n); \ t \geq 0 \} \) as \( n \to \infty \). We introduce some notation before presenting such a result. Denote by \( D^+[0, \infty) \) the space of the nonnegative càdlàg (right continuous with left limits) functions on \([0, \infty)\) and \( C_0^\infty[0, \infty) \) the space of infinitely differentiable functions on \([0, \infty)\) having compact supports.

Theorem 1. The stochastic process \( \{ \mathcal{X}^{(s)}(n); \ t \geq 0 \} \) weakly converges in \( D^+[0, \infty) \) endowed with the Skorokhod topology to a diffusion process \( \{ \mathcal{X}(s); \ t \geq 0 \} \) given by the solution of the stochastic differential equation

\[
ds \mathcal{X}(s) = (\beta - \gamma \mathcal{X}(s)) ds + \sqrt{\mathcal{X}(s)} dB(s), \quad s > 0,
\]

and \( \mathcal{X}(0) = 0 \), as \( n \to \infty \), where \( \{ B(s); \ t \geq 0 \} \) is a standard Brownian motion.
Remark 2. The process \( \{\mathcal{X}(s); \ s \geq 0\} \) appearing in Theorem 1, Equation (3), is known in the literature as the Cox–Ingersoll–Ross (CIR) process (Cox et al., 1985).

Proof. We have that \( X_t^{(n)}|X_{t-1}^{(n)} = nx \sim \text{Poisson}(\beta + \alpha nx) \), with \( x \in E_n = \{j/n : j = 0, 1, 2, \ldots\} \); we here denote \( Z_x^{(n)} \sim \text{Poisson}(\beta + \alpha nx) \) and \( Z_x^{(n)} \equiv Z_x^{(n)} / n \). By definition, \( X_0^{(n)} = 0 \) for all \( n \). Note that \( Z_x^{(n)} \) is a Markov chain assuming values in \( E_n \). For \( h \in C^\infty_c[0, \infty) \), define \( T_n h(x) \equiv E\left[h(Z_x^{(n)})\right] \). From theorem 6.5 in chapter 1 and corollary 8.9 in chapter 4 of Ethier and Kurtz (1986), to obtain the desired result, it is enough to show that

\[
\lim_{n \to \infty} \sup_{x \in E_n} |\epsilon_n(x)| = 0, \quad h \in C^\infty_c[0, \infty),
\]

with \( \epsilon_n(x) = n(T_n h(x) - h(x)) - (\beta - \gamma x)h'(x) - \frac{1}{2}xh''(x) \), where \( h'(\cdot) \) and \( h''(\cdot) \) denote the first and second derivatives of \( h(\cdot) \), respectively.

For \( Z_x^{(n)} \neq x \), we have that

\[
\int_0^1 h''(x + v(Z_x^{(n)} - x)) dv = \frac{h'(Z_x^{(n)} - h'(x)}{Z_x^{(n)} - x},
\]

and

\[
\int_0^1 vh''(x + v(Z_x^{(n)} - x)) dv = \frac{h'(Z_x^{(n)} - h'(x)}{Z_x^{(n)} - x} - \frac{h(Z_x^{(n)} - h(x)}{(Z_x^{(n)} - x)^2}.
\]

By combining (5) and (6), we obtain that

\[
n\left(h(Z_x^{(n)} - h(x)\right) = \int_0^1 n(Z_x^{(n)} - x)^2(1 - v)h''(x + v(Z_x^{(n)} - x)) dv + n(Z_x^{(n)} - x)h'(x).
\]

Note that Equation (7) also holds for \( Z_x^{(n)} = x \). Further, we can write

\[
-\frac{1}{2}E\left(n(Z_x^{(n)} - x)^2\right)h''(x) = E\left(-\int_0^1 n(Z_x^{(n)} - x)^2(1 - v)h''(x) dv\right).
\]

We now use the Equations (7) and (8) to express \( \epsilon_n(x) \) as follows:

\[
\epsilon_n(x) = E\left(\int_0^1 n(Z_x^{(n)} - x)^2(1 - v)\left(h''(x + v(Z_x^{(n)} - x)) - h''(x)\right) dv\right)
\]

\[
+ h'(x)\left(E\left(n(Z_x^{(n)} - x)\right) - (\beta + nx - \gamma x)\right) + \frac{1}{2}h''(x)\left(E\left(n(Z_x^{(n)} - x)^2\right) - x\right)
\]

\[
:= \epsilon_n^{(1)}(x) + \epsilon_n^{(2)}(x) + \epsilon_n^{(3)}(x).
\]

We will show that \( \lim_{n \to \infty} \sup_{x \in E_n} |\epsilon_n^{(j)}(x)| = 0 \), for \( j = 1, 2, 3 \). This result, Equation (9), and the triangular inequality imply that (4) holds and therefore conclude the proof of the theorem.
To show the case \( j = 1 \), we argue as in the proof of theorem 3.1 in chapter 9 of Ethier and Kurtz (1986). Then, the result follows by showing that \( \lim_{n \to \infty} |\varepsilon_n^{(1)}(x_n)| = 0 \) for any convergent sequence \( \{x_n\}_{n \in \mathbb{N}} \), where \( x_n \to \infty \) is allowed. Without loss of generality, suppose that the support of \( h(\cdot) \) is contained in the interval \([0, c]\), for constant \( c > 0 \). For \( \nu \in (0, 1) \) and \( x \in E^n \equiv E_n - \{0\} \), it follows that \( x + \nu(\tilde{Z}_{x_n}^{(n)} - x) > x(1 - \nu) \) and therefore the integral involved in \( \varepsilon_n^{(1)}(x) \) equals 0 under the region \( x(1 - \nu) > c \) (\( h''(z) = 0 \) for \( z > c \)), that is \( \nu < 1 - c/x \). Define \( \omega^*(x) = \max\{0, 1 - c/x\} \) for \( x > 0 \), \( \omega^*(0) = 1 \), \( \omega_*(x) = \min\{1, c/x\} \) for \( x > 0 \), and \( \omega_*(0) = 1 \). Hence, it follows that

\[
|\varepsilon_n^{(1)}(x_n)| = \left| E \left( \int_{\omega^*(x_n)}^1 n(\tilde{Z}_{x_n}^{(n)} - x_n)^2(1 - \nu)(h''(x_n + \nu(\tilde{Z}_{x_n}^{(n)} - x_n)) - h''(x_n)) d\nu \right) \right|
\leq nE \left( \int_{\omega^*(x_n)}^1 n(\tilde{Z}_{x_n}^{(n)} - x_n)^2(1 - \nu)2\|h''\| d\nu \right) = nE \left( \int_{\omega^*(x_n)}^1 (\tilde{Z}_{x_n}^{(n)} - x_n)^2 \|h''\| \omega_*(x_n)^2 \right), \tag{10}
\]

where \( \{x_n\}_{n \in \mathbb{N}} \) is some nonnegative real sequence.

Further, we have that \( E \left( \tilde{Z}_{x_n}^{(n)} - x_n \right)^2 \equiv n^{-2}(\beta + \gamma_n x_n^2) + n^{-1} a_n x_n \). Consider \( x_n \to 0 \), then \( nE \left( \tilde{Z}_{x_n}^{(n)} - x_n \right)^2 \to 0 \) and \( \omega_*(x_n) \to 1 \). These results give us that the right-hand side of (10) goes to 0 as \( n \to \infty \). We obtain the same conclusion when \( x_n \to \infty \) since \( nE \left( \tilde{Z}_{x_n}^{(n)} - x_n \right)^2 = \Theta(n^{-1} x_n^2) \) and \( \omega_*(x_n)^2 = \Theta(x_n^{-2}) \), and hence \( \lim_{n \to \infty} nE \left( \tilde{Z}_{x_n}^{(n)} - x_n \right)^2 \|h''\| \omega_*(x_n)^2 = \lim_{n \to \infty} \Theta(n^{-1} x_n^{-1}) = 0 \). Suppose now that \( x_n \to x \in (0, \infty) \). We can establish the weak convergence of \( \sqrt{n(\tilde{Z}_{x_n}^{(n)} - x_n)} \) via its characteristic function as follows:

\[
E \left( \exp \{i t \sqrt{n(\tilde{Z}_{x_n}^{(n)} - x_n)} \right) = \exp \left\{ -it \sqrt{n} x_n + (\beta + \gamma_n x_n)(\frac{e^{it} - 1}{\sqrt{n}}) \right\} = \exp \left\{ it n^{-1/2}(\beta - \gamma_n x_n) - (n^{-1} \beta + a_n x_n) t^2/2 + \Theta(n^{-3/2}) \right\} \to \exp \left\{ -xt^2/2 \right\}, \quad t \in \mathbb{R},
\]

as \( n \to \infty \). Therefore, \( \sqrt{n(\tilde{Z}_{x_n}^{(n)} - x_n)} \overset{d}{\to} N(0, x) \). Now, by using the Mean Value Theorem, we have that \( |h''(x + \nu) - h''(x)| \leq |\nu| \|h''''\| \) and then

\[
|\varepsilon_n^{(1)}(x_n)| \leq n^{-1/2}E \left( |n^{1/2}(\tilde{Z}_{x_n}^{(n)} - x_n)\right)^3 \|h''''\| \int_0^1 \nu(1 - \nu) d\nu \to 0,
\]

as \( n \to \infty \), where we have used that \( \lim_{n \to \infty} E \left( |n^{1/2}(\tilde{Z}_{x_n}^{(n)} - x_n)|^3 \right) = 2^{3/2} \pi^{-1/2} \chi^3 \) by the above normal weak convergence and by the Dominated Convergence Theorem. For the case \( j = 2 \), it follows that

\[
\sup_{x \in E_n} |\varepsilon_n^{(2)}(x)| = \sup_{x \in E_n} x|h'(x)||\gamma_n - \gamma| \leq \sup_{x \in E_n} x|h'(x)||I(0 \leq x \leq c)||\gamma_n - \gamma| \leq c\|h'\||\gamma_n - \gamma| \to 0.
\]

as \( n \to \infty \). In a similar fashion, for \( j = 3 \), it can be shown that \( \lim_{n \to \infty} \sup_{x \in E_n} |\varepsilon_n^{(3)}(x)| = 0 \), which concludes the proof.
Remark 3. As mentioned by the referees, our NU-INARCH process can be interpreted as a branching process with immigration (or INAR process) where both offspring and immigration are Poisson distributed; for instance, see Weiss (2015). Our Theorem 1 could be obtained as a special case of theorem 3.1 by Sriram (1994), which holds under general assumptions; for similar results involving a branching process with immigration under a pure critical case, see Wei and Winnicki (1989). On the other hand, it is necessary to show that some conditions holds, like a Lindeberg-type condition of the offspring distributions. Anyway, we would like to highlight that the proof of our Theorem 1 can be straightforwardly adapted beyond the Poisson distribution. In that case, the NU-INARCH interpretation as a branching process with immigration can be lost so theorem 3.1 by Sriram (1994) is not longer applicable.

3 | CONDITIONAL LEAST SQUARES

In this section, we provide the asymptotic distribution of the conditional least squares estimator of \( a_n \) for the nearly unstable INARCH process. The parameter \( \beta \) is assumed to be known. This can be seen as a nuisance parameter since our main interest relies on the parameter \( a_n \) that controls the dependence in the model. In the empirical illustration, we discuss how to deal with the unknown \( \beta \) case.

The CLS estimator of \( a \) is obtained by minimizing the \( Q \)-function given by

\[
Q(a) = \sum_{t=2}^{n}(X_t - E(X_t|X_{t-1}))^2 = \sum_{t=2}^{n}(X_t - \beta - aX_{t-1})^2.
\]

Hence, we obtain explicitly the CLS estimator of \( a \), say \( \hat{a}_n \), which is given by

\[
\hat{a}_n = \frac{\sum_{t=2}^{n}X_{t-1}(X_t - \beta)}{\sum_{t=2}^{n}X_{t-1}^2}.
\]  

Remark 4. A joint estimation of \( a \) and \( \beta \) using the \( Q \)-function produces the estimators \( \tilde{\beta} = \bar{X} - \hat{a}_n\bar{X}_s \approx \bar{X}(1 - \hat{a}_n) \) and \( \tilde{a}_n = \frac{\sum_{t=2}^{n}(X_{t-1} - \bar{X})(X_{t} - \bar{X})}{\sum_{t=2}^{n}(X_{t-1} - \bar{X})^2} \) where \( \bar{X} = \frac{1}{n-1}\sum_{t=2}^{n}X_t \) and \( \bar{X}_s = \frac{1}{n-1}\sum_{t=2}^{n}X_{t-1} \). Since the estimator \( \tilde{a}_n \) can produce values greater than 1, \( \tilde{\beta} \) can be negative, which is an inadmissible estimate since our model requires \( \beta > 0 \). This fact motivates us to leave out \( \beta \) from the CLS estimation. To overcome this issue, in our empirical illustration, we replace \( \beta \) by its conditional maximum likelihood (CML) estimate, which ensures its positiveness.

Remark 5. A referee queried about the issue of estimating \( a \) and \( \beta \) jointly using conditional maximum likelihood estimation (CMLE). The reason for not considering the CMLEs of \( a \) and \( \beta \) is that we do not have an explicit form for the estimators, which is a crucial point to derive unit root tests (URTs). This is why most, if not all, of the URTs are based on CLS instead of CML. Our URT is constructed by assuming that \( \beta \) is known and the idea of replacing it with its CML estimate can be seen as a heuristic solution; we have provided a small Monte Carlo simulation in our application showing that such a solution works well. At the same time, we think that the best solution is to estimate both parameters using the same method. Given the aforementioned challenges, we believe that one alternative could be to consider a two-step CLS estimation by combining the usual \( Q \) function (which involves the first conditional moment \( E(X_t|X_{t-1}) \) and another similar estimating equation based on a higher-order moment,
say \( E(X_i^*|X_{i-1}) \), where \( s \geq 2 \) is such that \( \hat{\beta} \) is an admissible estimator. We believe that this idea deserves investigation and hope to explore it in a future paper.

We begin by deriving the asymptotic distribution of \( \hat{\alpha}_n \) under the stationary assumption, where we denote the count time series by \( \{X_t\}_{t \in \mathbb{N}} \) (no need for the superscript \((n)\)). This case will be contrasted to the nearly unstable INARCH process through simulation in the following section.

**Theorem 2.** Assume that \( X_1, \ldots, X_n \) is a trajectory from a stationary Poisson INARCH(1) model, that is \( \alpha_n = \alpha < 1 \). Then, the CLS estimator \( \hat{\alpha}_n \) given in (11) satisfies

\[
\sqrt{n}(\hat{\alpha}_n - \alpha) \xrightarrow{d} N(0, \tilde{\sigma}^2),
\]
as \( n \to \infty \), where

\[
\tilde{\sigma}^2 = \frac{(1-\alpha)(1-\alpha^2)}{(1+\beta(1+\alpha))^2} \left\{ 1 + \beta(1-\alpha) + \frac{\alpha(2 + \beta^{-1})}{1-\alpha} - \frac{\alpha^2(1-\alpha)\beta^{-1}}{1-\alpha^3} + \frac{1 + \beta(1+\alpha)}{1-\alpha} \right\}.
\]

**Proof.** From Fokianos et al. (2009), we have that \( \{X_t\} \) is strictly stationary and ergodic since \( \alpha < 1 \). Hence, we can use theorem 3.2 from Tjøstheim (1986) to establish the asymptotic normality of the CLS estimator \( \hat{\alpha}_n \). The other conditions necessary to obtain this weak convergence can be straightforwardly checked in our case and therefore are omitted. Applying this theorem, we get that the asymptotic variance, say \( \tilde{\sigma}^2 \), assumes the form \( \tilde{\sigma}^2 = R/U^2 \), with \( U = E\left( \frac{\partial E(X_t|F_{t-1})}{\partial \alpha} \right)^2 \) and \( R = E\left( \frac{\partial E(X_t|F_{t-1})}{\partial \alpha} \right)^2 \) Var\( (X_t|F_{t-1}) \) = \( \beta E(X_{t-1}^2) + \alpha E(X_{t-1}^3) \). Explicit expression for the marginal moments of a Poisson INARCH(1) model are given in Weiß (2010). Using these results and the notation considered there with \( f_k \equiv \frac{\beta}{\prod_{i=0}^{k-1}(1-\alpha^i)} \), for \( k \in \mathbb{N} \), we obtain \( U = f_2(1 + \beta(1+\alpha)) \) and \( R = \frac{df_2(1+\beta)}{1-\alpha} + \alpha f_2 - \alpha^2(1-\alpha)f_3 + \alpha f_2(1 + \beta(1+\alpha)) + \beta f_2(1 + \beta(1+\alpha)) \). Direct algebraic manipulations conclude the proof. \( \blacksquare \)

From now on assume that \( \{X_t^{(n)}\}_{t \in \mathbb{N}} \) is a nearly unstable INARCH process as given in Definition 1. Define \( W_t^{(n)} = X_t^{(n)} - E(X_t^{(n)}|F_{t-1}) \), \( X_t^{(n)} = X_t^{(n)} \), and \( W_t^{(n)} = \sum_{k=1}^{[\alpha n]} W_k^{(n)} \), for \( t \in \mathbb{N}_0 \) and \( s \geq 0 \), where \([x]\) denotes the integer-part of \( x \in \mathbb{R} \). Like in the nearly unstable INAR process by Ispány et al. (2003), we can express \( \hat{\alpha}_n - \alpha_n \) as

\[
\hat{\alpha}_n - \alpha_n = \frac{\sum_{t=2}^{n} \frac{X_t^{(n)} - 1}{X_{t-1}^{(n)}} W_t^{(n)}}{\sum_{t=2}^{n} (X_{t-1}^{(n)})^2} \approx \int_0^1 \frac{1}{X_t^{(n)}} W^{(n)}(s) dW^{(n)}(s) / n \int_0^1 (X_t^{(n)})^2 ds. \tag{12}
\]

In the following lemma, we provide the asymptotic behavior of the autocovariance function of the process \( \{W_t^{(n)}(s); s \geq 0\} \); note that \( E(W_t^{(n)}(s)) = 0 \). This will be important to identify the proper normalization of \( \hat{\alpha}_n - \alpha_n \) in (12) yielding a nontrivial weak limit.

**Lemma 1.** For \( s, v \geq 0 \), we have that \( \text{cov}(W_t^{(n)}(s), W_t^{(n)}(v)) \approx n^2 C_W(s \wedge v) \), where \( C_W(u) = \beta \gamma^{-2}(\gamma u + e^{-\gamma u} - 1) \) for \( u \geq 0 \), \( s \wedge v = \min(s, v) \), and \( a_n \approx b_n \) denoting that \( \lim_{n \to \infty} a_n/b_n = 1 \) for real sequences \( \{a_n\} \) and \( \{b_n\} \).
Proof. It is straightforward that \( E(W^{(n)}(s)) = 0 \) and \( \text{cov}(W^{(n)}_k, W^{(n)}_l) = 0 \) for \( k \neq l \). Further, \( \text{Var}(W^{(n)}_k) = \text{Var}(X^{(n)}_k) + \alpha_n^2 \text{Var}(X^{(n)}_{k-1}) - 2\alpha_n \text{cov}(X^{(n)}_k, X^{(n)}_{k-1}) = \text{Var}(X^{(n)}_k) - \alpha_n^2 \text{Var}(X^{(n)}_{k-1}) \), where the last equality follows from the expression of the covariance given in Proposition 1. After using the expression of the variance given in that lemma, we obtain that \( \text{Var}(W^{(n)}_k) = \beta \frac{1-\alpha_n^k}{1-\alpha_n} \). 

From the above results and Proposition 1, we obtain that

\[
\text{cov}(W^{(n)}(s), W^{(n)}(v)) = \sum_{k=1}^{\lfloor ns \rfloor} \text{Var}(W^{(n)}_k) = \frac{\beta}{1 - \alpha_n} \left\{ \lfloor ns \rfloor \wedge \lfloor nv \rfloor - \alpha_n \frac{\lfloor ns \rfloor \wedge \lfloor nv \rfloor}{1 - \alpha_n} \right\}
\]

\[
\approx n^2 \beta \gamma^{-2}(\gamma u + e^{-\gamma u} - 1) = n^2 C_W(s \wedge v).
\]

Lemma 1 and Theorem 1 give us that \( \hat{\alpha}_n - \alpha_n = O_p(n^{-1}) \). We now are able to establish the asymptotic distribution of the CLS estimator \( \hat{\alpha}_n \) under the nearly unstable INARCH process as follows.

**Theorem 3.** Let \( \{\mathcal{X}(s); s \geq 0\} \) be the diffusion process given in (3). Then, the CLS estimator \( \hat{\alpha}_n \) satisfy the following weak convergence

\[
n(\hat{\alpha}_n - \alpha_n) \overset{d}{\rightarrow} \frac{\int_0^1 \mathcal{X}(s)d\mathcal{W}(s)}{\int_0^1 \mathcal{X}(s)^2ds} = \frac{\int_0^1 \mathcal{X}(s)^{3/2}dB(s)}{\int_0^1 \mathcal{X}(s)^2ds}, \tag{13}
\]

as \( n \to \infty \), where \( d\mathcal{W}(s) = \sqrt{\mathcal{X}(s)}dB(s) \), for \( s > 0 \), with \( \mathcal{W}(0) = 0 \).

Proof. Define \( W^{(n)}(s) = W^{(n)}(s)/n \), for \( s > 0 \). We have that

\[
n(\hat{\alpha}_n - \alpha_n) = \frac{\int_0^1 \mathcal{X}^{(n)}(s)dW^{(n)}(s)}{\int_0^1 \mathcal{X}^{(n)}(s)^2ds} = \frac{\int_0^1 \mathcal{X}^{(n)}(s)dW^{(n)}(s)}{\int_0^1 \mathcal{X}^{(n)}(s)^2ds},
\]

where both numerator and denominator have the same order of magnitude \( O_p(n^2) \).

For \( s > 0 \), it follows that

\[
W^{(n)}(s) = \sum_{k=1}^{\lfloor ns \rfloor} (X^{(n)}_k - \beta - \alpha_n X^{(n)}_{k-1}) = X^{(n)}_{\lfloor ns \rfloor} + \frac{\gamma}{n} \sum_{k=1}^{\lfloor ns \rfloor} X^{(n)}_{k-1} - \beta \lfloor ns \rfloor,
\]

and then \( W^{(n)}(s) \) can be expressed by

\[
W^{(n)}(s) = \mathcal{X}^{(n)} \left( \frac{\lfloor ns \rfloor}{n} \right) + \gamma \int_0^{\lfloor ns \rfloor/n} \mathcal{X}^{(n)}(u)du - \beta \frac{\lfloor ns \rfloor}{n}.
\]

Define the functions \( \Phi_n \) (\( n = 1, 2, \ldots \)) and \( \Phi \) mapping \( D^+[0, \infty) \) into \( D(\mathbb{R}_+, \mathbb{R}^2) \) as \( \Phi_n(x)(s) = \left( x(s), x \left( \frac{\lfloor ns \rfloor}{n} \right) + \gamma \int_0^{\lfloor ns \rfloor/n} x(u)du - \beta \frac{\lfloor ns \rfloor}{n} \right) \) and \( \Phi(x)(s) = (x(s), x(s) + \gamma \int_0^s x(u)du - \beta s) \). Hence, it follows that \( (\mathcal{X}^{(n)}(s), W^{(n)}(s)) = \Phi_n(\mathcal{X}^{(n)})(s) \). Using the fact that the CIR process has almost sure continuous trajectories and
similar arguments given in the proof of proposition 4.1 of Ispány et al. (2003), we obtain that \( \Phi_n(\lambda^{(n)}) \) weakly converges to \( \Phi(\lambda) \) as \( n \to \infty \).

In particular, we have that \( \mathcal{W}^{(n)}(s) \) weakly converges to \( \mathcal{W}(s) = \lambda(s) + \gamma \int_0^s \lambda(u)du - \beta s \). From the definition of \( \lambda \), we have that \( \lambda(s) = \beta s - \gamma \int_0^s \lambda(u)du + \int_0^s \sqrt{\lambda(u)}dB(u) \) and, therefore, \( \mathcal{W}(s) = \int_0^s \sqrt{\lambda(u)}dB(u) \). In other words, \( d\mathcal{W}(t) = \sqrt{\lambda(t)}dB(t) \). The above results and the continuous mapping theorem give us that
\[
\int_0^1 \lambda^{(n)}(s)d\mathcal{W}^{(n)}(s) \xrightarrow{d} \int_0^1 \lambda(s)d\mathcal{W}(s) = \int_0^1 \lambda(s)^{3/2}dB(s).
\]

The above arguments are straightforwardly extended to establish the joint weak convergence
\[
\left( \lambda^{(n)}(s), \mathcal{W}^{(n)}(s), \int_0^1 (\lambda^{(n)}(u))^2du \right) \Rightarrow \left( \lambda(s), \mathcal{W}(s), \int_0^1 (\lambda(u))^2du \right),
\]
in \( D(\mathbb{R}_+, \mathbb{R}^2) \) as \( n \to \infty \). Then, the desired result given in (13) is obtained by applying the continuous mapping theorem.

4 | SIMULATED EXPERIMENTS

In this section, we present simulated results illustrating the behavior of the asymptotic distributions of the normalized CLS estimator under the nearly unstable and stable cases. All the numerical results of this paper were obtained by using the statistical software R (R Development Core Team, 2021). We conduct Monte Carlo simulations with 10,000 replications, where we generate Poisson INARCH(1) trajectories with \( \beta = 1, \alpha = 0.98, 0.99, 0.999 \), and initially a sample size of \( n = 500 \). Note that the chosen values for \( \alpha \) here indicate nearly unstable count processes. For each replication, we compute the CLS estimate of \( \alpha \) using (11) and then its standardized estimate as \( n(\hat{\alpha}_n - \alpha) \) and \( \sqrt{n}(\hat{\alpha}_n - \alpha) \) according to the nearly unstable (Theorem 3) and stable/stationary (Theorem 2) cases, respectively.

A generator from the asymptotic distribution given on the right-hand side of (13) was implemented. To compute the stochastic integrals involved there, we used Riemann-sum approximations as described, for instance, by Rogers and Williams (2000) (section 47 of chapter IV). For the generation of the CIR process, we considered the Euler approximation; please see Deelstra and Delbaen (1998) for more details. Hence, for instance, we can obtain its quantiles and also plot the associated density function by generating samples and then applying a nonparametric density estimator (here the Gaussian kernel is considered), which are important for what follows. We present the histograms and qq-plots of the standardized CLS estimates along with their associated asymptotic density/quantiles under the stable and nearly unstable cases in Figures 1 and 2, respectively. From Figure 1, it is evident that the normal approximation is not adequate and it is worsening when \( \alpha \) gets closer to 1, which is expected since these results are based on stationarity. On the other hand, the histograms and qq-plots regarding the nearly unstable approximation given in Figure 2 show an excellent agreement between the empirical standardized estimates and the theoretical asymptotic distribution for all scenarios.

A natural question is what happens when \( \alpha \) is not close to 1. To address this point, we run additional simulations with \( \alpha = 0.7, 0.8, 0.9 \), and the remaining settings as before. Figures 3 and 4 exhibit histograms and qq-plots of the standardized CLS estimates of \( \alpha \) obtained from a Monte
FIGURE 1 Histograms and qq-plots of the standardized estimates $\sqrt{n}(\hat{\alpha}_n - \alpha)$ for $\alpha = 0.98, \alpha = 0.99$, and $\alpha = 0.999$, along with their associated limiting normal density/quantiles given in Theorem 2 (under stationarity). The sample size is $n = 500$.

FIGURE 2 Histograms and qq-plots of the standardized estimates $n(\hat{\alpha}_n - \alpha)$ for $\alpha = 0.98, \alpha = 0.99$, and $\alpha = 0.999$, along with their associated limiting density/quantiles given in Theorem 3 (under nearly nonstationarity). The sample size is $n = 500$. 
FIGURE 3  Histograms and qq-plots of the standardized estimates $\sqrt{n}(\hat{\alpha}_n - \alpha)$ for $\alpha = 0.7$, $\alpha = 0.8$, and $\alpha = 0.9$, along with their associated limiting normal density/quantiles given in Theorem 2 (under stationarity). The sample size is $n = 500$.

FIGURE 4  Histograms and qq-plots of the standardized estimates $n(\hat{\alpha}_n - \alpha)$ for $\alpha = 0.7$, $\alpha = 0.8$, and $\alpha = 0.9$, along with their associated limiting density/quantiles given in Theorem 3 (under nearly nonstationarity). The sample size is $n = 500$. 
\( \bar{\alpha} \) and \( \alpha \)

\[ \sqrt{n} (\hat{\alpha}_n - \alpha) \] for \( \alpha = 0.98 \), \( \alpha = 0.99 \), and \( \alpha = 0.999 \), along with their associated limiting normal density/quantiles given in Theorem 2 (under stationarity).

The sample size is \( n = 1000 \).

Carlosimulation for the stationary and nearly nonstationary Poisson INARCH processes. From Figure 3, we observe some deviation from the normality even for the case \( \alpha = 0.7 \). This is well evidenced by the qq-plots. Surprisingly, the results based on the nearly unstable methodology work quite satisfactorily even for \( \alpha = 0.7 \). These conclusions can be drawn again in Figure 4, where we note a good agreement between the empirical standardized CLS estimates and the theoretical asymptotic distribution derived in Theorem 3.

All the configurations considered here are repeatedly again with a sample size \( n = 1000 \). Figures 5 and 6 give us the histograms and qq-plots of the standardized CLS estimates under the stable and nearly unstable Poisson INARCH processes, respectively, under the settings \( \alpha = 0.98, 0.99, 0.999 \). The plots regarding the settings \( \alpha = 0.7, 0.8, 0.9 \) for the stable and nearly unstable cases are reported in Figures 7 and 8, respectively.

The conclusions are quite similar to the case \( n = 500 \) for the configurations nearly to nonstationarity \( \alpha = 0.98, 0.99, 0.999 \). Regarding the configurations where \( \alpha = 0.7, 0.8, 0.9 \), although there is an improvement in the results based on the stationary case (compared to \( n = 500 \)), deviations from the normality can still be observed. In contrast, the nearly unstable approach again works very well and provides the best outcomes. As a short conclusion, we recommend using the nearly unstable-based approach even when the fitted model may in practice not be too close to the nonstationarity region because the proposed methodology works well and perform better than the stationary-based approach.

As requested by one of the referees, we now provide some simulated results to investigate what happens when we have enough information on stationarity, for instance \( n = 10,000 \) and...
FIGURE 6  Histograms and qq-plots of the standardized estimates $n(\hat{\alpha}_n - \alpha)$ for $\alpha = 0.98$, $\alpha = 0.99$, and $\alpha = 0.999$, along with their associated limiting density/quantiles given in Theorem 3 (under nearly nonstationarity). The sample size is $n = 1000$.

FIGURE 7  Histograms and qq-plots of the standardized estimates $\sqrt{n}(\hat{\alpha}_n - \alpha)$ for $\alpha = 0.7$, $\alpha = 0.8$, and $\alpha = 0.9$, along with their associated limiting normal density/quantiles given in Theorem 2 (under stationarity). The sample size is $n = 1000$. 
FIGURE 8  Histograms and qq-plots of the standardized estimates $n(\hat{\alpha}_n - \alpha)$ for $\alpha = 0.7$, $\alpha = 0.8$, and $\alpha = 0.9$, along with their associated limiting density/quantiles given in Theorem 3 (under nearly nonstationarity). The sample size is $n = 1000$.

$\alpha = 0.7$, and when the model is clearly stationary, with $\alpha = 0.3$ and $n = 1000$. Figure 9 shows the histograms and qq-plots of the standardized estimates $n(\hat{\alpha}_n - \alpha)$ for $(\alpha, n) = (0.7, 10,000)$ and $(\alpha, n) = (0.3, 1000)$ obtained from a Monte Carlo simulation with 1000 replications. The associated limiting density/quantiles in the plots are related to Theorem 3, the nearly unstable case. As seen, the approximation in these cases is not working well as before. This is expected because we have enough information about the stationarity. The plots regarding the stationary case are provided in Figure 10, where a better approximation can be observed.

Our interest now is to evaluate the coverages of the confidence intervals based on the asymptotic results under the nearly unstable and stable assumptions. To construct the confidence intervals under the nearly unstable case, we obtain the quantiles of the asymptotic distribution of $n(\hat{\alpha}_n - \alpha_n)$ by using empirical quantiles of generated samples, where the generation of this asymptotic distribution is described in the second paragraph of this section. In Table 1, we provide the empirical coverages of confidence intervals, from a Monte Carlo simulation with 10,000 replications, for $\alpha$ with significance level at 10%, 5%, and 1% based on Theorem 3 (under nearly nonstationarity). The sample size is $n = 500$ and we consider $\alpha = 0.999, 0.99, 0.98, 0.9, 0.8, 0.7$. These results show that inference on the correlation parameter using our methodology is satisfactory since the coverages are close to the nominal levels for all cases considered, even when $\alpha$ is not close to the nonstationarity region.
FIGURE 9 histograms and q-q plots of the standardized estimates $n(\hat{\alpha}_n - \alpha)$ for $(\alpha, n) = (0.7, 10,000)$ and $(\alpha, n) = (0.3, 1000)$, along with their associated limiting density/quantiles given in Theorem 3 (under nearly nonstationarity).

5 | UNIT ROOT TEST

In this section, we propose a statistical procedure for testing unit root in a Poisson INARCH(1) model with correlation parameter $\alpha$. The null and alternative hypotheses are respectively $H_0 : \alpha = 1$ and $H_1 : \alpha < 1$. To this end, we consider the nearly unstable approach and the statistic $n(\hat{\alpha}_n - 1)$, which is inspired by the traditional unit root test for the continuous AR(1) model of Dickey and Fuller (1979). Under the conditions of Theorem 3, we have that

$$n(\hat{\alpha}_n - 1) = n(\hat{\alpha}_n - \alpha_n) - \gamma_n \overset{d}{\rightarrow} D_{\gamma, \bar{\beta}} - \gamma,$$  

(14)
FIGURE 10  Histograms and qq-plots of the standardized estimates $\sqrt{n}(\hat{\alpha}_n - \alpha)$ for $(\alpha, n) = (0.7, 10,000)$ and $(\alpha, n) = (0.3, 1000)$, along with their associated limiting normal density/quantiles given in Theorem 2 (under stationarity).

TABLE 1  Empirical coverages of the 90%, 95%, and 99% confidence intervals for $\alpha$ based on the nearly unstable approach. Sample size $n = 500$.  

| $\alpha \rightarrow$ | 0.999 | 0.99 | 0.98 | 0.9 | 0.8 | 0.7 |
|----------------------|--------|------|------|-----|-----|-----|
| 90%                  | 0.934  | 0.917| 0.897| 0.892| 0.920| 0.915|
| 95%                  | 0.967  | 0.952| 0.939| 0.945| 0.968| 0.966|
| 99%                  | 0.989  | 0.984| 0.982| 0.986| 0.993| 0.990|
TABLE 2  The 10%, 5%, and 1% quantiles of the distribution of \( D_{0, \beta} \) for \( \beta = 0.25, 0.5, 1, 1.25, 1.5 \).

| \( \beta \) | 0.25 | 0.5 | 1 | 1.25 | 1.5 |
|---|---|---|---|---|---|
| \( q_{0.10} \) | -14.291 | -7.687 | -4.125 | -3.405 | -2.929 |
| \( q_{0.05} \) | -19.118 | -10.314 | -5.444 | -4.472 | -3.823 |
| \( q_{0.01} \) | -31.015 | -16.574 | -8.587 | -6.993 | -5.949 |

TABLE 3  Empirical significance levels obtained from a Monte Carlo study to evaluate the proposed unit root test under some sample sizes and nominal significance levels at 10%, 5%, and 1%.

| \( n \) | 50 | 80 | 100 | 200 | 300 | 400 | 500 | 1000 | 2000 | 5000 |
|---|---|---|---|---|---|---|---|---|---|---|
| 10% | 0.116 | 0.104 | 0.109 | 0.101 | 0.105 | 0.104 | 0.103 | 0.101 | 0.103 | 0.104 |
| 5% | 0.061 | 0.056 | 0.057 | 0.054 | 0.055 | 0.054 | 0.049 | 0.054 | 0.054 | 0.049 |
| 1% | 0.014 | 0.013 | 0.014 | 0.012 | 0.010 | 0.010 | 0.010 | 0.011 | 0.010 | 0.010 |

as \( n \to \infty \), where \( D_{\gamma, \beta} \) is a random variable (depending on \( \gamma \) and \( \beta \)) following the asymptotic distribution given in the right-hand side of (13). We can approach the null hypothesis of interest through our methodology by taking \( \gamma \to 0 \). In this case, the distribution of the right-hand side of (14) approaches that of \( D_{0, \beta} \), which has the associated \( \mathcal{X} \) process satisfying the stochastic differential equation \( d\mathcal{X}(s) = \beta ds + \sqrt{\mathcal{X}(s)} dB(s), s \geq 0 \). Denote by \( q_\zeta \) the \( \zeta \)-quantile of the distribution of \( D_{0, \beta} \), for \( \zeta \in (0, 1) \), that is \( P(D_{0, \beta} \leq q_\zeta) = \zeta \). These quantiles can be obtained from Monte Carlo simulation as done in Section 4. Table 2 gives us quantiles of the distribution of \( D_{0, \beta} \) for some values of \( \beta \).

Based on the above discussion, we propose the following decision rule for testing \( H_0 : \alpha = 1 \) against \( H_1 : \alpha < 1 \) with significance level at \( \zeta \times 100\% : \)

- Reject \( H_0 \) in favor of \( H_1 \) if \( n(\hat{\alpha}_n - 1) < q_\zeta \).

To evaluate the finite-sample performance of the proposed unit root test (URT), we run a Monte Carlo simulation with 10000 replications. We set \( \beta = 1 \) and sample sizes \( n = 50, 80, 100, 200, 300, 400, 500, 1000, 2000, 5000 \). In Table 3, we provide the empirical significance levels with nominal levels at 10%, 5%, and 1%. We observe that the URT is yielding the desired Type-I error even for small sample sizes (for instance, \( n = 50, 80 \)).

Aiming at the investigation of the test power, another Monte Carlo simulation is considered under the same setup as before and with a significance level at 5%. We consider \( \alpha = 0.999, 0.99, 0.98, 0.95, 0.9, 0.8, 0.7 \) and compute the proportion of rejections of the null hypothesis in each scenario. The results are presented in Table 4. As expected, the power increases when either we are going away from the null hypothesis or the sample size increases. The proportions of rejections given in that table indicate that the proposed URT is working satisfactorily and offers a promising device in dealing with count time series based on the INGARCH approach.

6  |  REAL DATA APPLICATION

We here apply the proposed methodology to the daily number of deaths due to COVID-19 in the United Kingdom from January 30, 2020, to June 4, 2021, so yielding \( n = 492 \) observations.
Empirical power obtained from a Monte Carlo study to evaluate the proposed unit root test under some sample sizes and values of $\alpha$.

| $n$ ↓ | $\alpha$ → | 0.999 | 0.99 | 0.98 | 0.95 | 0.9 | 0.8 | 0.7 |
|-------|-------------|-------|------|------|------|-----|-----|-----|
| 50    |             | 0.057 | 0.085| 0.115| 0.258| 0.623| 0.983| 1.000|
| 80    |             | 0.114 | 0.173| 0.264| 0.618| 0.973| 1.000| 1.000|
| 100   |             | 0.166 | 0.270| 0.413| 0.865| 1.000| 1.000| 1.000|
| 200   |             | 0.216 | 0.425| 0.695| 0.998| 1.000| 1.000| 1.000|
| 300   |             | 0.265 | 0.614| 0.930| 1.000| 1.000| 1.000| 1.000|
| 400   |             | 0.314 | 0.798| 0.995| 1.000| 1.000| 1.000| 1.000|
| 500   |             | 0.367 | 0.927| 1.000| 1.000| 1.000| 1.000| 1.000|
| 1000  |             | 0.434 | 0.998| 1.000| 1.000| 1.000| 1.000| 1.000|
| 2000  |             | 0.551 | 1.000| 1.000| 1.000| 1.000| 1.000| 1.000|
| 5000  |             | 0.839 | 1.000| 1.000| 1.000| 1.000| 1.000| 1.000|

Note: Significance level at 5%.

This dataset is publicly available at the site https://coronavirus.data.gov.uk. The plot of the daily number of deaths and its associated autocorrelation function are provided in Figure 11, which reveals a nearly unstable/nonstationary behavior.

We assume that the time series comes from an NU-INARCH(1) process. The aim of this application is to illustrate that the theoretical results found in this paper can reveal the unit root behavior for a real dataset. We first need to deal with $\beta$, which is unknown and can be seen as a nuisance parameter; our primary interest in this paper relies on the correlation parameter $a_n$. One strategy is to estimate $\beta$ through the conditional maximum likelihood method, which consists in maximizing $\ell \propto \sum_{t=2}^{n}(\alpha_t \log \lambda_t - \lambda_1)$, and then assume it known in what follows. This procedure gives $\beta = 0.269$. At the end of this application, we will evaluate such an approach by performing a small Monte Carlo simulation study.

Using (11), we obtain the estimate for the correlation parameter equal to $\hat{\alpha}_n = 0.997$, which is very close to 1. We obtain the standard error of the $a_n$ estimate (s.e.$(\hat{\alpha}_n)$) using the asymptotic distribution stated in Theorem 2, which gives the s.e.$(\hat{\alpha}_n) \approx 0.014$. We perform the URT proposed in
Section 5 for testing the hypothesis $H_0 : \alpha = 1$ against $H_1 : \alpha < 1$. We obtain $n(\hat{\alpha}_n - 1) = -1.257 > -17.952 = q_{0.05}$ and therefore we do not reject the null hypothesis on the unit root with significance level at $5\%$. The quantile $q_{0.05}$ was obtained by generating a random sample of size 100,000 from the asymptotic distribution given in the right side of (13) (with $\gamma = 0$) and then getting its $5\%$ empirical quantile (as described in Section 4). The density function of $D_{0.0, 0.269}$ based on Gaussian kernel and 100,000 Monte Carlo replications is provided in Figure 12 along with vertical lines denoting the statistic test and the $0.05$-quantile (of the $D_{0.0, 0.269}$ distribution). The associated $p$-value is $0.704$, which shows that we obtain the same indication by using any usual significance level.

In Figure 13, we present the count time series data and the predicted means based on the fitted NU INARCH model, which reveals a good agreement between the observed time series and the model.

Remark 6. A referee suggested checking for a possible sensitivity of $\alpha$ estimate when considering different $\beta$ estimates. To pursue that, we jointly estimated $\alpha$ and $\beta$ by CLS to investigate such a possible sensitivity, and found $\hat{\alpha} = 0.996$, which is very close to the estimate obtained by combining CLS and CML (given in the paper), that is, $\hat{\alpha} = 0.997$. Therefore, we conclude that the $\alpha$ estimate was not sensitive when considering different $\beta$ estimates, at least for the dataset considered in the paper.

We conclude this application by evaluating our strategy by estimating $\beta$ and assuming known. To do this, we run a small Monte Carlo simulation with 1000 replications. In each loop, we generate an NU-INARCH model with $\beta = 0.269$, $\alpha = 0.997$, and $n = 492$ (specifications of the application), construct confidence intervals for $\alpha$ based on both approaches with fixed and non-fixed (estimated as done in this section and then assumed known) $\beta$, and check if they contain the “true” value. The empirical coverages of the $90\%$, $95\%$, and $99\%$ confidence intervals under both

![Figure 12](image-url)
FIGURE 13  Left: Number of deaths (points) and the predicted mean $\hat{E}(Y_t|Y_{t-1}) = \beta + \hat{\alpha}Y_{t-1}$ (solid line). Right: Number of deaths against predicted means.

TABLE 5  Empirical coverages of 90%, 95%, and 99% confidence intervals for the correlation parameter $\alpha$ based on a Monte Carlo simulation under the NU-INGARCH model with the settings $\beta = 0.269$, $\alpha = 0.997$, and $n = 492$.

|               | Fixed $\beta$ | Nonfixed $\beta$ |
|---------------|---------------|------------------|
| 90%           | 0.927         | 0.911            |
| 95%           | 0.960         | 0.948            |
| 99%           | 0.986         | 0.979            |

Note: Both approaches with fixed and nonfixed (estimated) $\beta$ are reported.

approaches are reported in Table 5. As can be seen from this table, the proposed solution given here in the application provides the expected nominal coverages and works even better than the fixed $\beta$ case for the 90% and 95% coverages; the 99% coverages are very close to each other.

7  DISCUSSION AND FUTURE RESEARCH

A nearly unstable INARCH(1) process was introduced and weak convergence of a normalized version was established. The asymptotic distribution of the CLS estimator of the correlation parameter was derived under both nearly unstable and stable cases, which have been explored via Monte Carlo simulations. We also proposed a unit root test and checked its performance in terms of yielding the desired Type-I error and power through simulation. The nearly unstable INARCH approach was applied to the daily number of deaths due to the COVID-19 in the UK, which exhibits a nonstationary behavior. The proposed URT has provided evidence for the existence of a unit root, which is in agreement with the descriptive analysis.

We have assumed that the conditional distribution in (1) is Poisson, but the methods presented in this paper can be easily adapted for other distributional assumptions such as negative binomial or more generally mixed Poisson distributions, among others. More specifically, the same strategy given in Proposition 1 and Lemma 1 can be employed to find the proper normalizations
for the processes $\{X^{(n)}(nt), \ t \geq 0\}$ and $\{W^{(n)}(t), \ t \geq 0\}$ in these cases. After obtaining these results, the asymptotic distributions of the normalized count process and CLS estimator are established following the same steps as given in Theorems 1 and 3, respectively. We also believe that extending the results for higher-order INGARCH models and the development of goodness-of-fit tools for the NU-INARCH process deserve future investigation.

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