A GENERALIZED HILBERT OPERATOR ACTING ON CONFORMALLY INVARIANT SPACES

DANIEL GIRELA\textsuperscript{1*} and NOEL MERCHÁN\textsuperscript{1}

Abstract. If $\mu$ is a positive Borel measure on the interval $[0,1)$ we let $\mathcal{H}_\mu$ be the Hankel matrix $\mathcal{H}_\mu = (\mu_{n,k})_{n,k \geq 0}$ with entries $\mu_{n,k} = \mu_{n+k}$, where, for $n = 0, 1, 2, \ldots$, $\mu_n$ denotes the moment of order $n$ of $\mu$. This matrix induces formally the operator

$$\mathcal{H}_\mu(f)(z) = \sum_{n=0}^{\infty} \left( \sum_{k=0}^{\infty} \mu_{n,k} a_k \right) z^n$$

on the space of all analytic functions $f(z) = \sum_{k=0}^{\infty} a_k z^k$, in the unit disc $D$. This is a natural generalization of the classical Hilbert operator. The action of the operators $\mathcal{H}_\mu$ on Hardy spaces has been recently studied. This paper is devoted to study the operators $\mathcal{H}_\mu$ acting on certain conformally invariant spaces of analytic functions on the disc such as the Bloch space, $BMOA$, the analytic Besov spaces, and the $Q_s$ spaces.

1. Introduction

Let $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ denote the open unit disc in the complex plane $\mathbb{C}$ and let $\mathcal{H}ol(\mathbb{D})$ be the space of all analytic functions in $\mathbb{D}$ endowed with the topology of uniform convergence in compact subsets. We also let $H^p \ (0 < p \leq \infty)$ be the classical Hardy spaces. We refer to [18] for the notation and results regarding Hardy spaces.

If $\mu$ is a finite positive Borel measure on $[0,1)$ and $n = 0, 1, 2, \ldots$, we let $\mu_n$ denote the moment of order $n$ of $\mu$, that is, $\mu_n = \int_{[0,1)} t^n d\mu(t)$, and we let $\mathcal{H}_\mu$ be the Hankel matrix $(\mu_{n,k})_{n,k \geq 0}$ with entries $\mu_{n,k} = \mu_{n+k}$. The matrix $\mathcal{H}_\mu$ induces formally an operator, which will be also called $\mathcal{H}_\mu$, on spaces of analytic functions by its action on the Taylor coefficients: $a_n \mapsto \sum_{k=0}^{\infty} \mu_{n,k} a_k, \ n = 0, 1, 2, \ldots$. To be precise, if $f(z) = \sum_{k=0}^{\infty} a_k z^k \in \mathcal{H}ol(\mathbb{D})$ we define

$$\mathcal{H}_\mu(f)(z) = \sum_{n=0}^{\infty} \left( \sum_{k=0}^{\infty} \mu_{n,k} a_k \right) z^n,$$

whenever the right hand side makes sense and defines an analytic function in $\mathbb{D}$.

If $\mu$ is the Lebesgue measure on $[0,1)$ the matrix $\mathcal{H}_\mu$ reduces to the classical Hilbert matrix $\mathcal{H} = ((n+k+1)^{-1})_{n,k \geq 0}$, which induces the classical Hilbert operator $\mathcal{H}$ which has extensively studied recently (see [1, 13, 14, 16, 24]).

2010 Mathematics Subject Classification. Primary 47B35; Secondary 30H10.

Key words and phrases. Hilbert operators, conformally invariant spaces, Carleson measures.
Galanopoulos and Peláez [19] described the measures $\mu$ so that the generalized Hilbert operator $H_\mu$ becomes well defined and bounded on $H^1$. Chatzifountas, Girela and Peláez [12] extended this work describing those measures $\mu$ for which $H_\mu$ is a bounded operator from $H^p$ into $H^q$, $0 < p, q < \infty$.

Obtaining an integral representation of $H_\mu$ plays a basic role in these works. If $\mu$ is as above, we shall write throughout the paper

$$I_\mu(f)(z) = \int_{[0,1)} \frac{f(t)}{1-tz} \, d\mu(t),$$

whenever the right hand side makes sense and defines an analytic function in $D$. It turns out that the operators $H_\mu$ and $I_\mu$ are closely related. In fact, in [19] and [12] the measures $\mu$ for which the operator $I_\mu$ is well defined in $H^p$ ($0 < p < \infty$) are characterized and it is proved that for such measures we have $H_\mu(f) = I_\mu(f)$ for all $f \in H^p$. These measures are Carleson-type measures.

If $I \subset \partial D$ is an arc, $|I|$ will denote the length of $I$. The Carleson square $S(I)$ is defined as $S(I) = \{re^{it} : e^{it} \in I, \quad 1 - \frac{|I|}{2\pi} \leq r < 1\}$.

If $s > 0$ and $\mu$ is a positive Borel measure on $D$, we shall say that $\mu$ is an $s$-Carleson measure if there exists a positive constant $C$ such that

$$\mu(S(I)) \leq C|I|^s, \quad \text{for any interval } I \subset \partial D.$$

If $\mu$ satisfies $\lim_{|I| \to 0} \frac{\mu(S(I))}{|I|^s} = 0$, then we say that $\mu$ is a vanishing $s$-Carleson measure.

A 1-Carleson measure, respectively, a vanishing 1-Carleson measure, will be simply called a Carleson measure, respectively, a vanishing Carleson measure.

We recall that Carleson [11] proved that $H^p \subset L^p(d\mu)$ ($0 < p < \infty$), if and only if $\mu$ is a Carleson measure. This result was extended by Duren [17] (see also [18, Theorem 9.4]) who proved that for $0 < p \leq q < \infty$, $H^p \subset L^q(d\mu)$ if and only if $\mu$ is a $q/p$-Carleson measure.

Following [32], if $\mu$ is a positive Borel measure on $D$, $0 \leq \alpha < \infty$, and $0 < s < \infty$ we say that $\mu$ is an $\alpha$-logarithmic $s$-Carleson measure if there exists a positive constant $C$ such that

$$\mu(S(I)) \left(\log \frac{2\pi}{|I|}\right)^\alpha \leq C, \quad \text{for any interval } I \subset \partial D.$$

If $\mu(S(I)) \left(\log \frac{2\pi}{|I|}\right)^\alpha = o(|I|^s)$, as $|I| \to 0$, we say that $\mu$ is a vanishing $\alpha$-logarithmic $s$-Carleson measure.

A positive Borel measure $\mu$ on $[0,1)$ can be seen as a Borel measure on $D$ by identifying it with the measure $\tilde{\mu}$ defined by

$$\tilde{\mu}(A) = \mu(A \cap [0,1)), \quad \text{for any Borel subset } A \text{ of } D.$$

In this way a positive Borel measure $\mu$ on $[0,1)$ is an $s$-Carleson measure if and only if there exists a positive constant $C$ such that

$$\mu([t, 1)) \leq C(1-t)^s, \quad 0 \leq t < 1,$$
and we have similar statements for vanishing $s$-Carleson measures and for $\alpha$-logarithmic $s$-Carleson and vanishing $\alpha$-logarithmic $s$-Carleson measures.

Our main aim in this paper is studying the operators $\mathcal{H}_\mu$ acting on conformally invariant spaces.

It is a standard fact that the set of all disc automorphisms (i.e., of all one-to-one analytic maps $f$ of $\mathbb{D}$ onto itself), denoted $\text{Aut}(\mathbb{D})$, coincides with the set of all Möbius transformations of $\mathbb{D}$ onto itself:

$$\text{Aut}(\mathbb{D}) = \{ \lambda \varphi_a : |a| < 1, |\lambda| = 1 \},$$

where $\varphi_a(z) = (a - z)/(1 - \overline{a}z)$.

A space $X$ of analytic functions in $\mathbb{D}$, defined via a semi-norm $\rho$, is said to be conformally invariant or Möbius invariant if whenever $f \in X$, then also $f \circ \varphi \in X$ for any $\varphi \in \text{Aut}(\mathbb{D})$ and, moreover, $\rho(f \circ \varphi) \leq C \rho(f)$ for some positive constant $C$ and all $f \in X$. A great deal of information on conformally invariant spaces can be found in [5, 15, 30].

Let us start considering the Bloch space and $BMOA$. The Bloch space $\mathcal{B}$ consists of all analytic functions $f$ in $\mathbb{D}$ with bounded invariant derivative:

$$f \in \mathcal{B} \iff \|f\|_\mathcal{B} \overset{\text{def}}{=} |f(0)| + \sup_{z \in \mathbb{D}} (1 - |z|^2) |f'(z)| < \infty.$$  

The little Bloch space $\mathcal{B}_0$ is the closure of the polynomials in the above norm of $\mathcal{B}$ and consists of all functions $f$ analytic in $\mathbb{D}$ for which

$$\lim_{|z| \to 1} (1 - |z|^2)|f'(z)| = 0.$$  

A classical source for the Bloch space is [3]; see also [34]. Rubel and Timoney [30] proved that $\mathcal{B}$ is the biggest “natural” conformally invariant space.

The space $BMOA$ consists of those functions $f$ in $H^1$ whose boundary values have bounded mean oscillation on the unit circle $\partial \mathbb{D}$ as defined by F. John and L. Nirenberg. There are many characterizations of $BMOA$ functions. Let us mention the following:

*If $f$ is an analytic function in $\mathbb{D}$, then $f \in BMOA$ if and only if*

$$\|f\|_{BMOA} \overset{\text{def}}{=} |f(0)| + \|f\|_* < \infty,$$

*where*

$$\|f\|_* \overset{\text{def}}{=} \sup_{a \in \mathbb{D}} \|f \circ \varphi_a - f(a)\|_{H^2}.$$  

It is clear that the seminorm $\| \cdot \|_*$ is conformally invariant. If

$$\lim_{|a| \to 1} \|f \circ \varphi_a - f(a)\|_{H^2} = 0$$

we say that $f$ belongs to the space $VMOA$. We mention [9, 21] as general references for the spaces $BMOA$ and $VMOA$. Let us recall that

$$H^\infty \subsetneq BMOA \subsetneq \bigcap_{0<p<\infty} H^p \quad \text{and} \quad BMOA \subsetneq \mathcal{B}.$$
Other important Möbius invariant spaces are the analytic Besov spaces \( B^p \) \((1 < p < \infty)\) and the \( Q_s \)-spaces \((s > 0)\). These spaces will be considered in Section 3.

We close this section noticing that, as usual, we shall be using the convention that \( C = C(p, \alpha, q, \beta, \ldots) \) will denote a positive constant which depends only upon the displayed parameters \( p, \alpha, q, \beta \ldots \) (which sometimes will be omitted) but not necessarily the same at different occurrences. Moreover, for two real-valued functions \( E_1, E_2 \) we write \( E_1 \lesssim E_2 \), or \( E_1 \gtrsim E_2 \), if there exists a positive constant \( C \) independent of the arguments such that \( E_1 \leq CE_2 \), respectively \( E_1 \geq CE_2 \). If we have \( E_1 \lesssim E_2 \) and \( E_1 \gtrsim E_2 \) simultaneously then we say that \( E_1 \) and \( E_2 \) are equivalent and we write \( E_1 \asymp E_2 \).

2. The operator \( \mathcal{H}_\mu \) acting on \( BMOA \) and the Bloch space

We start characterizing those \( \mu \) for which the operator \( I_{\mu} \) is well defined in \( BMOA \) and in the Bloch space. It turns out that they coincide.

**Theorem 2.1.** Let \( \mu \) be a positive Borel measure on \([0, 1)\). Then the following conditions are equivalent:

(i) \( \int_{[0,1)} \log \frac{2}{1-t} \, d\mu(t) < \infty \).

(ii) For any given \( f \in \mathcal{B} \), the integral in (1.1) converges for all \( z \in \mathbb{D} \) and the resulting function \( I_{\mu}(f) \) is analytic in \( \mathbb{D} \).

(iii) For any given \( f \in \text{BMOA} \), the integral in (1.1) converges for all \( z \in \mathbb{D} \) and the resulting function \( I_{\mu}(f) \) is analytic in \( \mathbb{D} \).

**Proof.**

(i) \( \Rightarrow \) (ii). It is well known that there exists a positive constant \( C \) such that

\[
|f(z)| \leq C \|f\|_{\mathcal{B}} \log \frac{2}{1-|z|}, \quad (z \in \mathbb{D}), \text{ for every } f \in \mathcal{B}, \quad \text{(2.1)}
\]

(see [3, p. 13]). Assume (i) and set \( A = \int_{[0,1)} \log \frac{2}{1-t} \, d\mu(t) \). Using (2.1) we see that

\[
\int_{[0,1)} |f(t)| \, d\mu(t) \leq C \|f\|_{\mathcal{B}} \int_{[0,1)} \log \frac{2}{1-t} \, d\mu(t) = AC \|f\|_{\mathcal{B}}, \quad f \in \mathcal{B}. \quad \text{(2.2)}
\]

This implies that

\[
\int_{[0,1)} \frac{|f(t)|}{|1-tz|} \, d\mu(t) \leq \frac{AC \|f\|_{\mathcal{B}}}{1-|z|}, \quad (z \in \mathbb{D}), \quad f \in \mathcal{B}. \quad \text{(2.3)}
\]

Using (2.2), (2.3), and Fubini’s theorem we see that if \( f \in \mathcal{B} \) then:

- For every \( n \in \mathbb{N} \), the integral \( \int_{[0,1]} t^n f(t) \, d\mu(t) \) converges absolutely and

\[
\sup_{n \geq 0} \left| \int_{[0,1]} t^n f(t) \, d\mu(t) \right| < \infty.
\]
The integral\[ \int_{[0,1)} \frac{f(t)}{1-tz} \, d\mu(t) = \sum_{n=0}^{\infty} \left( \int_{[0,1)} t^n f(t) \, d\mu(t) \right) z^n, \quad z \in \mathbb{D}. \]

Thus, if $f \in B$ then $I_\mu(f)$ is a well defined analytic function in $\mathbb{D}$ and

$$I_\mu(f)(z) = \sum_{n=0}^{\infty} \left( \int_{[0,1)} t^n f(t) \, d\mu(t) \right) z^n, \quad z \in \mathbb{D}. $$

The implication (ii) \implies (iii) is clear because $BMOA \subset B$.

(iii) \implies (i). Suppose (iii). Since the function $F(z) = \log \frac{2}{1-z}$ belongs to $BMOA$, $I_\mu(F)(z)$ is well defined for every $z \in \mathbb{D}$. In particular

$$I_\mu(F)(0) = \int_{[0,1)} \log \frac{2}{1-t} \, d\mu(t)$$

is a complex number. Since $\mu$ is a positive measure and $\log \frac{2}{1-t} > 0$ for all $t \in [0,1)$, (i) follows. □

Our next aim is characterizing the measures $\mu$ so that $I_\mu$ is bounded in $BMOA$ or $B$ and seeing whether or not $I_\mu$ and $H_\mu$ coincide for such measures. We have the following results.

**Theorem 2.2.** Let $\mu$ be a positive Borel measure on $[0,1)$ with $\int_{[0,1)} \log \frac{2}{1-t} \, d\mu(t) < \infty$. Then the following three conditions are equivalent:

(i) The measure $\nu$ defined by $d\nu(t) = \log \frac{2}{1-t} \, d\mu(t)$ is a Carleson measure.

(ii) The operator $I_\mu$ is bounded from $B$ into $BMOA$.

(iii) The operator $I_\mu$ is bounded from $BMOA$ into itself.

**Theorem 2.3.** Let $\mu$ be a positive Borel measure on $[0,1)$ with $\int_{[0,1)} \log \frac{2}{1-t} \, d\mu(t) < \infty$. If the measure $\nu$ defined by $d\nu(t) = \log \frac{2}{1-t} \, d\mu(t)$ is a Carleson measure, then $H_\mu$ is well defined on the Bloch space and $H_\mu(f) = I_\mu(f)$, for all $f \in B$.

Theorem 2.2 and Theorem 2.3 together yield the following.

**Theorem 2.4.** Let $\mu$ be a positive Borel measure on $[0,1)$ such that the measure $\nu$ defined by $d\nu(t) = \log \frac{2}{1-t} \, d\mu(t)$ is a Carleson measure. Then the operator $H_\mu$ is bounded from $B$ into $BMOA$.

**Proof of Theorem 2.2.** Since $\int_{[0,1)} \log \frac{2}{1-t} \, d\mu(t) < \infty$, (2.1) implies that

$$\int_{[0,1)} |f(t)| \, d\mu(t) < \infty, \quad \text{for all } f \in B.$$
and this implies that
\[
\int_0^{2\pi} \left| \frac{f(t)g(e^{i\theta})}{1 - re^{i\theta}t} \right| d\mu(t)d\theta < \infty, \quad 0 \leq r < 1, \ f \in \mathcal{B}, \ g \in H^1.
\]

Using this, Fubini’s theorem and Cauchy’s integral representation of $H^1$-functions [18, Theorem 3.6], we deduce that whenever $f \in \mathcal{B}$ and $g \in H^1$ we have
\[
\int_0^{2\pi} I_\mu(f)(re^{i\theta})g(e^{i\theta})d\theta = \int_0^{2\pi} \left( \int_{[0,1]} \frac{f(t)d\mu(t)}{1 - re^{i\theta}t} \right) g(e^{i\theta})d\theta
\]
\[
= \int_{[0,1]} f(t) \left( \int_0^{2\pi} \frac{g(e^{i\theta})d\theta}{1 - re^{i\theta}t} \right) d\mu(t) = \int_{[0,1]} f(t) \overline{g(rt)} d\mu(t), \quad 0 \leq r < 1.
\]

(i) $\Rightarrow$ (ii). Assume that $\nu$ is a Carleson measure and take $f \in \mathcal{B}$ and $g \in H^1$. Using (2.4) and (2.1), we obtain
\[
\left| \int_0^{2\pi} I_\mu(f)(re^{i\theta})g(e^{i\theta})d\theta \right| = \left| \int_{[0,1]} f(t)\overline{g(rt)} d\mu(t) \right|
\]
\[
\lesssim \|f\|_B \int_{[0,1]} |g(rt)| \log \frac{2}{1 - t} d\mu(t) = \|f\|_B \int_{[0,1]} |g(rt)| d\nu(t).
\]

Since $\nu$ is a Carleson measure
\[
\int_{[0,1]} |g(rt)| d\nu(t) \lesssim \|g_r\|_{H^1} \leq \|g\|_{H^1}.
\]

Here, as usual, $g_r$ is the function defined by $g_r(z) = g(rz)$ ($z \in \mathbb{D}$).

Thus, we have proved that
\[
\left| \int_0^{2\pi} I_\mu(f)(re^{i\theta})g(e^{i\theta})d\theta \right| \lesssim \|f\|_B \|g\|_{H^1}, \quad f \in \mathcal{B}, \ g \in H^1.
\]

Using Fefferman’s duality Theorem (see [21, Theorem 7.1]) we deduce that if $f \in \mathcal{B}$ then $I_\mu(f) \in BMOA$ and
\[
\|I_\mu(f)\|_{BMOA} \lesssim \|f\|_B.
\]

The implication (ii) $\Rightarrow$ (iii) is trivial because $BMOA \subset \mathcal{B}$.

(iii) $\Rightarrow$ (i). Assume (iii). Then there exists a positive constant $A$ such that
\[
\|I_\mu(f)\|_{BMOA} \leq A\|f\|_{BMOA}, \text{ for all } f \in BMOA.
\]

Set
\[
F(z) = \log \frac{2}{1 - z}, \quad z \in \mathbb{D}.
\]

It is well known that $F \in BMOA$. Then $I_\mu(F) \in BMOA$ and
\[
\|I_\mu(F)\|_{BMOA} \leq A\|F\|_{BMOA}.
\]

Then using again Fefferman’s duality theorem we obtain that
\[
\left| \int_0^{2\pi} I_\mu(F)(re^{i\theta})g(e^{i\theta})d\theta \right| \lesssim \|g\|_{H^1}, \quad g \in H^1.
\]
Using (2.4) and the definition of $F$, this implies
\[
\left| \int_{[0,1]} g(rt) \log \frac{2}{1-t} \, d\mu(t) \right| \lesssim \|g\|_{H^1}, \quad g \in H^1. \tag{2.5}
\]

Take $g \in H^1$. Using Proposition 2 of [12] we know that there exists a function $G \in H^1$ with $\|G\|_{H^1} = \|g\|_{H^1}$ and such that
\[
|g(s)| \leq G(s), \quad \text{for all } s \in [0,1).
\]
Using these properties and (2.5) for $G$, we obtain
\[
\int_{[0,1]} \left| g(rt) \log \frac{2}{1-t} \right| d\mu(t) \lesssim \|G\|_{H^1} \leq C \|G\|_{H^1} = C \|g\|_{H^1}
\]
for a certain constant $C > 0$, independent of $g$. Letting $r$ tend to 1, it follows that
\[
\int_{[0,1]} \left| g(t) \log \frac{2}{1-t} \right| d\mu(t) \lesssim \|g\|_{H^1}, \quad g \in H^1.
\]
This is equivalent to saying that $\nu$ is a Carleson measure. \(\square\)

It is worth noticing that for $\mu$ and $\nu$ as in Theorem 2.1, $\nu$ being a Carleson measure is equivalent to $\mu$ being an 1-logarithmic 1-Carleson measure. Actually, we have the following more general result.

**Proposition 2.5.** Let $\mu$ be a positive Borel measure on $[0,1)$, $s > 0$, and $\alpha \geq 0$. Let $\nu$ be the Borel measure on $[0,1)$ defined by
\[
d\nu(t) = \left( \log \frac{2}{1-t} \right)^\alpha d\mu(t).
\]
Then, the following two conditions are equivalent.
(a) $\nu$ is an $s$-Carleson measure.
(b) $\mu$ is an $\alpha$-logarithmic $s$-Carleson measure.

**Proof.**
(a) $\Rightarrow$ (b). Assume (a). Then there exists a positive constant $C$ such that
\[
\int_{[t,1)} \left( \log \frac{2}{1-u} \right)^\alpha d\mu(u) \leq C(1-t)^s, \quad t \in [0,1).
\]
Using this and the fact that the function $u \mapsto \log \frac{2}{1-u}$ is increasing in $[0,1)$, we obtain
\[
\left( \log \frac{2}{1-t} \right)^\alpha \int_{[t,1)} d\mu(u) \leq \int_{[t,1)} \left( \log \frac{2}{1-u} \right)^\alpha d\mu(u) \leq C(1-t)^s, \quad t \in [0,1).
\]
This shows that $\mu$ is an $\alpha$-logarithmic $s$-Carleson measure.
(b) $\Rightarrow$ (a). Assume (b). Then there exists a positive constant $C$ such that
\[
\left( \log \frac{2}{1-t} \right)^\alpha \mu([t,1)) \leq C(1-t)^s, \quad 0 \leq t < 1. \tag{2.6}
\]
For $0 \leq u < 1$, set $F(u) = \mu([0, u)) - \mu([0, 1)) = -\mu([u, 1))$. Integrating by parts and using (2.6), we obtain

\[
\nu([t, 1)) = \int_{[t, 1)} \left( \log \frac{2}{1-u} \right)^{\alpha} d\mu(u)
\]

\[
= \left( \log \frac{2}{1-t} \right)^{\alpha} \mu([t, 1)) + \alpha \int_{[t, 1)} \mu([u, 1)) \left( \log \frac{2}{1-u} \right)^{\alpha-1} \frac{du}{1-u}
\]

\[
\leq C (1-t)^s + \alpha \int_{t}^{1} \frac{(1-u)^{s-1}}{\log \frac{2}{1-u}} du
\]

\[
\lesssim (1-t)^s, \quad 0 \leq t < 1.
\]

Thus, $\nu$ is an $s$-Carleson measure.

The following lemma will be needed in the proof of Theorem 2.3.

**Lemma 2.6.** Let $\mu$ be a positive Borel measure in $[0, 1)$ such that the measure $\nu$ defined by $d\nu(t) = \log \frac{1}{1-t} d\mu(t)$ is a Carleson measure. Then the sequence of moments $\{\mu_n\}$ satisfies

\[
\mu_n = O \left( \frac{1}{n \log n} \right), \quad \text{as } n \to \infty.
\]

Actually, we shall prove the following more general result.

**Lemma 2.7.** Suppose that $0 \leq \alpha \leq \beta$, $s \geq 1$, and let $\mu$ be a positive Borel measure on $[0, 1)$ which is a $\beta$-logarithmic $s$-Carleson measure. Then

\[
\int_{[0,1)} t^k \left( \log \frac{2}{1-t} \right)^{\alpha} d\mu(t) = O \left( \frac{(\log k)^{\alpha-\beta}}{k^s} \right), \quad \text{as } k \to \infty.
\]

Using Proposition 2.5, Lemma 2.6 follows taking $\alpha = 0$, $\beta = 1$, and $s = 1$ in Lemma 2.7.

**Proof of Lemma 2.7.** Arguing as in the proof of the implication (b) $\Rightarrow$ (a) of Proposition 2.5, integrating by parts and using the fact that $\mu$ is a $\beta$-logarithmic 1-Carleson measure, we obtain

\[
\int_{[0,1)} t^k \left( \log \frac{2}{1-t} \right)^{\alpha} d\mu(t)
\]

\[
= k \int_{0}^{1} \mu([t, 1)) t^{k-1} \left( \log \frac{2}{1-t} \right)^{\alpha} dt + \alpha \int_{0}^{1} \mu([t, 1)) t^k \left( \log \frac{2}{1-t} \right)^{\alpha-1} \frac{dt}{1-t}
\]

\[
\lesssim k \int_{0}^{1} (1-t)^{k-1} \left( \log \frac{2}{1-t} \right)^{\alpha-\beta} dt + \alpha \int_{0}^{1} (1-t)^{s-1} t^k \left( \log \frac{2}{1-t} \right)^{\alpha-\beta-1} dt.
\]
Now, we notice that the weight functions
\[ \omega_1(t) = (1 - t)^s \left( \log \frac{2}{1-t} \right)^{\alpha - \beta} \] and
\[ \omega_2(t) = (1 - t)^s - 1 \left( \log \frac{2}{1-t} \right)^{\alpha - \beta - 1} \]
are regular in the sense of [29] (see [29, p. 6] and [2, Example 2]). Then, using Lemma 1.3 of [29] and the fact that the \( \omega_j \)'s are also decreasing, we obtain
\[
\int_0^1 (1 - t)^s t^{k-1} \left( \log \frac{2}{1-t} \right)^{\alpha - \beta} dt \lesssim \int_{1 - \frac{1}{k}}^1 (1 - t)^s t^{k-1} \left( \log \frac{2}{1-t} \right)^{\alpha - \beta} dt \leq \frac{(\log k)^{\alpha - \beta}}{k^{s+1}}
\]
and
\[
\int_0^1 (1 - t)^s - 1 t^k \left( \log \frac{2}{1-t} \right)^{\alpha - \beta - 1} dt \lesssim \int_{1 - \frac{1}{k}}^1 (1 - t)^s - 1 t^k \left( \log \frac{2}{1-t} \right)^{\alpha - \beta - 1} dt \leq \frac{(\log k)^{\alpha - \beta - 1}}{k^s}.
\]
Using these two estimates in (2.7) yields
\[
\int_{[0,1)} t^k \left( \log \frac{2}{1-t} \right)^{\alpha} d\mu(t) \lesssim \frac{(\log k)^{\alpha - \beta}}{k^s}
\]
finishing the proof. □

We shall also use the characterization of the coefficient multipliers from \( B \) into \( \ell^1 \) obtained by Anderson and Shields in [4].

**Theorem A.** A sequence \( \{\lambda_n\}_{n=0}^{\infty} \) of complex numbers is a coefficient multiplier from \( B \) into \( \ell^1 \) if and only if

\[
\sum_{n=1}^{\infty} \left( \sum_{k=2^n+1}^{2^{n+1}} |\lambda_k|^2 \right)^{1/2} < \infty.
\]

Bearing in mind Definition 1 of [4], Theorem A reduces to the case \( p = 1 \) in Corollary 1 in p. 259 of [4].

We recall that if \( X \) is a space of analytic functions in \( \mathbb{D} \) and \( Y \) is a space of complex sequences, a sequence \( \{\lambda_n\}_{n=0}^{\infty} \subset \mathbb{C} \) is said to be a multiplier of \( X \) into \( Y \) if whenever \( f(z) = \sum_{n=0}^{\infty} a_n z^n \in X \) one has that the sequence \( \{\lambda_n a_n\}_{n=0}^{\infty} \) belongs to \( Y \). Thus:

By saying that \( \{\lambda_n\}_{n=0}^{\infty} \) is a coefficient multiplier from \( B \) into \( \ell^1 \) we mean that

If \( f(z) = \sum_{n=0}^{\infty} a_n z^n \in B \) then \( \sum_{n=0}^{\infty} |\lambda_n a_n| < \infty. \)

Actually, using the closed graph theorem, we can assert the following:

A complex sequence \( \{\lambda_n\}_{n=0}^{\infty} \) is a multiplier from \( B \) to \( \ell^1 \) if and only if there exists a positive constant \( C \) such that whenever \( f(z) = \sum_{n=0}^{\infty} a_n z^n \in B \), we have that

\[ \sum_{n=0}^{\infty} |\lambda_n a_n| \leq C \|f\|_B. \]
Proof of Theorem 2.3. Suppose that $\nu$ is a Carleson measure. Then, using Lemma 2.6, we see that there exists $C > 0$ such that
\[
|\mu_n| \leq \frac{C}{n \log n}, \quad n \geq 2.
\] (2.8)

It is clear that
\[
k^2 \log^2 k \geq 2^{2n} (\log 2)^2, \quad \text{if } 2^n + 1 \leq k \leq 2^{n+1} \text{ for all } n.
\]
Then it follows that
\[
\sum_{n=1}^{\infty} \left( \sum_{k=2^n+1}^{2^{n+1}} \frac{1}{k^2 \log^2 k} \right)^{1/2} \lesssim \sum_{n=1}^{\infty} \left( \frac{2^n}{n^2 2^{2n}} \right)^{1/2} = \sum_{n=1}^{\infty} \frac{1}{n^{2n/2}} < \infty.
\]

Using this, (2.8) and Theorem A, we obtain:

The sequence of moments $\{\mu_n\}_{n=0}^{\infty}$ is a multiplier from $B$ to $\ell^1$. (2.9)

Take now $f \in B$, $f(z) = \sum_{n=0}^{\infty} a_n z^n$ ($z \in \mathbb{D}$). Using the simple fact that the sequence $\{\mu_n\}_{n=0}^{\infty}$ is a decreasing sequence of positive numbers and (2.9), we see that there exists $C > 0$ such that
\[
\sum_{k=0}^{\infty} |\mu_{n+k} a_k| \leq \sum_{k=0}^{\infty} |\mu_k a_k| \leq C \|f\|_B, \quad n = 0, 1, 2, \ldots
\] (2.10)

This implies that $\mathcal{H}_\mu(f)(z)$ is well defined for all $z \in \mathbb{D}$ and that, in fact, $\mathcal{H}_\mu(f)$ is an analytic function in $\mathbb{D}$. Furthermore, since (2.10) also implies that we can interchange the order of summation in the expression defining $\mathcal{H}_\mu(f)(z)$, we have
\[
\mathcal{H}_\mu(f)(z) = \sum_{n=0}^{\infty} \left( \sum_{k=0}^{\infty} \mu_{n+k} a_k \right) z^n = \sum_{k=0}^{\infty} a_k \left( \sum_{n=0}^{\infty} \mu_{n+k} z^n \right)
\]
\[
= \sum_{k=0}^{\infty} a_k \left( \sum_{n=0}^{\infty} \frac{\int_{[0,1]} t^{n+k} z^n \, d\mu(t)}{1-tz} \right) = \sum_{k=0}^{\infty} \frac{\int_{[0,1]} f(t) t^k}{1-tz} \, d\mu(t)
\]
\[
= \int_{[0,1]} \frac{f(t)}{1-tz} \, d\mu(t) = I_\mu(f)(z), \quad z \in \mathbb{D}.
\]
\[
\square
\]

We have the following result regarding compactness.

Theorem 2.8. Let $\mu$ be a positive Borel measure on $[0, 1)$ with $\int_{[0,1]} \log \frac{2}{1-t} \, d\mu(t) < \infty$. If the measure $\nu$ defined by $d\nu(t) = \log \frac{2}{1-t} \, d\mu(t)$ is a vanishing Carleson measure then:

(i) The operator $I_\mu$ is a compact operator from $B$ into $BMOA$.
(ii) The operator $I_\mu$ is a compact operator from $BMOA$ into itself.
Before embarking on the proof of Theorem 2.8 it is convenient to recall some facts about Carleson measures and to fix some notation.

If $\mu$ is a Carleson measure on $\mathbb{D}$, we define the Carleson-norm of $\mu$, denoted $\mathcal{N}(\mu)$, as

$$
\mathcal{N}(\mu) = \sup_{I \text{ subarc of } \partial \mathbb{D}} \frac{\mu(S(I))}{|I|}.
$$

We let also $\mathcal{E}(\mu)$ denote the norm of the inclusion operator $i : H^1 \to L^1(d\mu)$. It turns out that these quantities are equivalent: There exist two positive constants $A_1, A_2$ such that

$$
A_1 \mathcal{N}(\mu) \leq \mathcal{E}(\mu) \leq A_2 \mathcal{N}(\mu), \quad \text{for every Carleson measure } \mu \text{ on } \mathbb{D}.
$$

For a Carleson measure $\mu$ on $\mathbb{D}$ and $0 < r < 1$, we let $\mu_r$ be the measure on $\mathbb{D}$ defined by

$$
d\mu_r(z) = \chi_{\{r < |z| < 1\}} d\mu(z).
$$

We have that $\mu$ is a vanishing Carleson measure if and only if

$$
\mathcal{N}(\mu_r) \to 0, \quad \text{as } r \to 1.
$$

Proof of Theorem 2.8. Since $BMOA$ is continuously contained in the Bloch spaces, it suffices to prove (i).

Suppose that $\nu$ is a vanishing Carleson measure. Let $\{f_n\}_{n=1}^\infty$ be a sequence of Bloch functions with $\sup_{n \geq 1} \|f_n\|_B < \infty$ and such that $\{f_n\} \to 0$, uniformly on compact subsets of $\mathbb{D}$. We have to prove that $I_\mu(f_n) \to 0$ in $BMOA$.

The condition $\sup_{n \geq 1} \|f_n\|_B < \infty$ implies that there exists a positive constant $M$ such that

$$
|f_n(z)| \leq M \log \frac{2}{1 - |z|}, \quad z \in \mathbb{D}, \quad n \geq 1. \quad (2.11)
$$

Recall that for $0 < r < 1$, $\nu_r$ is the measure defined by

$$
d\nu_r(t) = \chi_{\{r < t < 1\}} d\nu(t).
$$

Since $\nu$ is a vanishing Carleson measure, we have that $\mathcal{N}(\nu_r) \to 0$, as $r \to 1$, or, equivalently,

$$
\mathcal{E}(\nu_r) \to 0, \quad \text{as } t \to 1. \quad (2.12)
$$

Take $g \in H^1$ and $r \in [0, 1)$. Using (2.11) we have

$$
\int_{[0,1]} |f_n(t)||g(t)| d\mu(t) = \int_{[0,r]} |f_n(t)||g(t)| d\mu(t) + \int_{[r,1]} |f_n(t)||g(t)| d\mu(t)
\leq \int_{[0,r]} |f_n(t)||g(t)| d\mu(t) + M \int_{[r,1]} \log \frac{2}{1 - t} |g(t)| d\mu(t)
= \int_{[0,r]} |f_n(t)||g(t)| d\mu(t) + M \int_{[0,1]} |g(t)| d\nu_r(t)
\leq \int_{[0,r]} |f_n(t)||g(t)| d\mu(t) + M \mathcal{E}(\nu_r) \|g\|_{H^1}.
$$
Using (2.12) and the fact that \( \{f_n\} \to 0 \), uniformly on compact subsets of \( \mathbb{D} \), it follows that
\[
\lim_{n \to \infty} \int_{[0,1)} |f_n(t)||g(t)| \, d\mu(t) = 0, \quad \text{for all } g \in H^1.
\]
Bearing in mind (2.4), this yields
\[
\lim_{n \to \infty} \left( \lim_{r \to 1} \left| \int_{0}^{2\pi} I_\mu(f_n)(re^{i\theta})g(e^{i\theta}) \, d\theta \right| \right) = 0, \quad \text{for all } g \in H^1.
\]
By the duality relation \((H^1)^* = BMOA\), this is equivalent to saying that \( I_\mu(f_n) \to 0 \) in \( BMOA \). □

3. The operator \( \mathcal{H}_\mu \) acting on \( Q_s \) spaces and Besov spaces

If \( 0 \leq s < \infty \), we say that \( f \in Q_s \) if \( f \) is analytic in \( \mathbb{D} \) and \( \|f\|_{Q_s} \overset{\text{def}}{=} (|f(0)|^2 + \rho_{Q_s}(f)^2)^{1/2} < \infty \), where
\[
\rho_{Q_s}(f) \overset{\text{def}}{=} \left( \sup_{a \in \mathbb{D}} \int_{\mathbb{D}} |f'(z)|^2 g(z, a)^s \, dA(z) \right)^{1/2}.
\]
Here, \( g(z, a) \) is the Green’s function in \( \mathbb{D} \), given by \( g(z, a) = \log |1 - \frac{a}{z}| \), while \( dA = \frac{dz \, dq}{\pi} \) is the normalized area measure on \( \mathbb{D} \). All \( Q_s \) spaces \( (0 \leq s < \infty) \) are conformally invariant with respect to the semi-norm \( \rho_{Q_s} \) (see e.g., [31, p. @1] or [15, p. 47]).

These spaces were introduced by Aulaskari and Lappan in [6] while looking for new characterizations of Bloch functions. They proved that for \( s > 1 \), \( Q_s \) is the Bloch space. Using one of the many characterizations of the space \( BMOA \) (see, e.g., [9, Theorem 5] or [21, Theorem 6.2]) we see that \( Q_1 = BMOA \). In the limit case \( s = 0 \), \( Q_s \) is the classical Dirichlet space \( \mathcal{D} \) of those analytic functions \( f \) in \( \mathbb{D} \) satisfying \( \int_{\mathbb{D}} |f'(z)|^2 \, dA(z) < \infty \).

It is well known that \( \mathcal{D} \subset VMOA \). Aulaskari, Xiao and Zhao proved in [8] that
\[
\mathcal{D} \subsetneq Q_{s_1} \subsetneq Q_{s_2} \subsetneq BMOA, \quad 0 < s_1 < s_2 < 1.
\]
We mention the book [31] as an excellent reference for the theory of \( Q_s \)-spaces.

It is well known that the function \( F(z) = \log |\frac{2}{1-z}| \) belong to \( Q_s \), for all \( s > 0 \), (in fact, it is proved in [7] that the univalent functions in all \( Q_s \)-spaces \( (0 < s < \infty) \) are the same). Using this we easily see that Theorem 2.1 and Theorem 2.4 can be improved as follows.

**Theorem 3.1.** Let \( \mu \) be a positive Borel measure on \([0,1)\). Then the following conditions are equivalent:

(i) \( \int_{[0,1)} \log \frac{2}{1-t} \, d\mu(t) < \infty \).

(ii) For any given \( s \in (0, \infty) \) and any \( f \in Q_s \), the integral in (1.1) converges for all \( z \in \mathbb{D} \) and the resulting function \( I_\mu(f) \) is analytic in \( \mathbb{D} \).
We remark that condition (ii) with $s \geq 1$ includes the points (ii) and (iii) of Theorem 2.1.

**Theorem 3.2.** Let $\mu$ be a positive Borel measure on $[0, 1]$ with $\int_{[0,1]} \log \frac{2}{1-t} \, d\mu(t) < \infty$. Then the following two conditions are equivalent:

(i) The measure $\nu$ defined by $d\nu(t) = \log \frac{2}{1-t} \, d\mu(t)$ is a Carleson measure.

(ii) For any given $s \in (0, \infty)$, the operator $I_\mu$ is bounded from $Q_s$ into $BMOA$.

We remark that (ii) with $s > 1$ reduces to condition (ii) of Theorem 2.2, while (ii) with $s = 1$ reduces to condition (iii) of Theorem 2.2.

These results cannot be extended to the limit case $s = 0$. Indeed, the function $F(z) = \log \frac{2}{1-z}$ does not belong to the Dirichlet space $D$. The Dirichlet space is one among the analytic Besov spaces.

For $1 < p < \infty$, the analytic Besov space $B^p$ is defined as the set of all functions $f$ analytic in $D$ such that

$$\|f\|_{B^p} \overset{\text{def}}{=} (\|f(0)\|^p + \rho_p(f)^p)^{1/p} < \infty,$$

where

$$\rho_p(f) = \left( \int_D (1 - |z|^2)^{p-2} |f'(z)|^p \, dA(z) \right)^{1/p}.$$

All $B^p$ spaces ($1 < p < \infty$) are conformally invariant with respect to the seminorm $\rho_p$ (see [5, p. 112] or [15, p. 46]). We have that $D = B^2$. A lot of information on Besov spaces can be found in [5, 15, 23, 33, 34]. Let us recall that

$$B^p \subsetneq B^q \subsetneq VMOA, \quad 1 < p < q < \infty.$$

From now on, if $1 < p < \infty$ we let $p'$ denote the exponent conjugate to $p$, that is, $p'$ is defined by the relation $\frac{1}{p} + \frac{1}{p'} = 1$. If $f \in B^p$ ($1 < p < \infty$) then, see [23] or [33],

$$|f(z)| = o \left( \left( \log \frac{1}{1-|z|} \right)^{1/p'} \right), \quad \text{as } |z| \to 1, \quad (3.1)$$

and there exists a positive constant $C > 0$ such that

$$|f(z)| \leq C\|f\|_{B^p} \left( \log \frac{2}{1-|z|} \right)^{1/p'}, \quad z \in \mathbb{D}, \quad f \in B^p. \quad (3.2)$$

Clearly, (3.1) or (3.2) imply that the function $F(z) = \log \frac{2}{1-z}$ does not belong to $B^p$ ($1 < p < \infty$), a fact that we have already mentioned for $p = 2$. Our substitutes of Theorem 2.1 and Theorem 2.2 for Besov spaces are the following.

**Theorem 3.3.** Let $1 < p < \infty$ and let $\mu$ be a positive Borel measure on $[0, 1)$. We have:
(i) If \( \int_{[0,1)} \left( \log \frac{2}{1-t} \right)^{1/p'} \, d\mu(t) < \infty \), then for any given \( f \in B^p \), the integral in (1.1) converges for all \( z \in \mathbb{D} \) and the resulting function \( I_{\mu}(f) \) is analytic in \( \mathbb{D} \).

(ii) If for any given \( f \in B^p \), the integral in (1.1) converges for all \( z \in \mathbb{D} \) and the resulting function \( I_{\mu}(f) \) is analytic in \( \mathbb{D} \), then \( \int_{[0,1)} \left( \log \frac{2}{1-t} \right)^{\gamma} \, d\mu(t) < \infty \) for all \( \gamma < \frac{1}{p'} \).

**Theorem 3.4.** Suppose that \( 1 < p < \infty \) and let \( \mu \) be a positive Borel measure on \([0,1)\). Let \( \nu \) be the measure defined by

\[
d\nu(t) = \left( \log \frac{2}{1-t} \right)^{1/p'} \, d\mu(t).
\]

(i) If \( \nu \) is a Carleson measure, then the operator \( I_{\mu} \) is bounded from \( B^p \) into \( \text{BMOA} \).

(ii) If \( \nu \) is a vanishing Carleson measure then the operator \( I_{\mu} \) is compact from \( B^p \) into \( \text{BMOA} \).

These results follow using the growth condition (3.2), the fact that if \( \gamma < \frac{1}{p'} \) then the function \( f(z) = \left( \log \frac{2}{1-z} \right)^{\gamma} \) belongs to \( B^p \) (see [23, Theorem 1]), and with arguments similar to those used in the proofs of Theorem 2.1, Theorem 2.2, and Theorem 2.8. We omit the details.

Let us work next with the operator \( \mathcal{H}_{\mu} \) directly. In order to study its action on the Besov spaces we need some results on the Taylor coefficients of functions in \( B^p \). The following result was proved by Holland and Walsh in [23, Theorem 2].

**Theorem B.** (i) Suppose that \( 1 < p \leq 2 \). Then there exists a positive constant \( C_p \) such that if \( f \in B^p \) and \( f(z) = \sum_{k=0}^{\infty} a_k z^k \) (\( z \in \mathbb{D} \)) then

\[
\sum_{k=1}^{\infty} k^{p-1} |a_k|^p \leq C_p \rho_p(f)^p.
\]

(ii) If \( 2 \leq p < \infty \) then there exists \( C_p > 0 \) such that if \( f(z) = \sum_{k=0}^{\infty} a_k z^k \) (\( z \in \mathbb{D} \)) with \( \sum_{k=1}^{\infty} k^{p-1} |a_k|^p \) < \( \infty \) then \( f \in B^p \) and

\[
\rho_p(f)^p \leq C_p \sum_{k=1}^{\infty} k^{p-1} |a_k|^p.
\]

If \( p \neq 2 \) the converses to (i) and (ii) are false.

Theorem B is the analogue for Besov spaces of results of Hardy and Littlewood for Hardy spaces (Theorem 6.2 and Theorem 6.3 of [18]).

In spite of the fact that the converse to (ii) is not true, the membership of \( f \) in \( B^p \) (\( p > 2 \)) implies some summability conditions on the Taylor coefficients \( \{a_k\} \) of \( f \). Indeed, Pavlović has proved the following result in [28, Theorem 2.3].
Theorem C. Suppose that $2 < p < \infty$. Then there exists a positive constant $C_p$ such that if $f \in B^p$ and $f(z) = \sum_{k=0}^{\infty} a_k z^k$ ($z \in \mathbb{D}$) then

$$\sum_{k=1}^{\infty} k |a_k|^p \leq C_p \rho_p(f)^p.$$ 

These results allow us to obtain conditions on $\mu$ which are sufficient to ensure that $\mathcal{H}_\mu$ is well defined on the Besov spaces.

Theorem 3.5. Let $\mu$ be a finite positive Borel measure on $[0, 1]$.

(i) If $1 < p \leq 2$ and $\sum_{k=1}^{\infty} \frac{\mu_k^p}{k} < \infty$, then the operator $\mathcal{H}_\mu$ is well defined in $B^p$.

(ii) If $2 < p < \infty$ and $\sum_{k=1}^{\infty} \frac{\mu_k^p}{k^{p'/p}} < \infty$, then the operator $\mathcal{H}_\mu$ is well defined in $B^p$.

Proof. Suppose that $1 < p < \infty$ and $f \in B^p$, $f(z) = \sum_{k=0}^{\infty} a_k z^k$ ($z \in \mathbb{D}$). Since the sequence of moments $\{\mu_n\}_{n=0}^{\infty}$ is clearly decreasing we have

$$\sum_{k=1}^{\infty} |\mu_n + k a_k| \leq \sum_{k=1}^{\infty} |\mu_k| |a_k|, \quad \text{for all } n \geq 0.$$ 

Consequently, we have:

(i) If $1 < p \leq 2$ and $f \in B^p$, $f(z) = \sum_{k=0}^{\infty} a_k z^k$ ($z \in \mathbb{D}$), then

$$\sum_{k=1}^{\infty} |\mu_n + k a_k| \leq \sum_{k=1}^{\infty} |\mu_k||a_k| = \sum_{k=1}^{\infty} k^{1-\frac{1}{p}} |a_k| \frac{\mu_k}{k^{1/p}}, \quad n \geq 0.$$ 

Then using Hölder inequality and Theorem B (i), we obtain

$$\sum_{k=1}^{\infty} |\mu_n + k a_k| \leq \left( \sum_{k=1}^{\infty} k^{p-1} |a_k|^p \right)^{1/p} \left( \sum_{k=1}^{\infty} |\mu_k|^{p'} \frac{k^{1/p'}}{k} \right)^{1/p'} \leq C \rho_p(f) \left( \sum_{k=1}^{\infty} \frac{|\mu_k|^{p'}}{k} \right)^{1/p'}, \quad n \geq 0.$$ 

Then it is clear that the condition $\sum_{k=1}^{\infty} |\mu_k|^{p'/k} < \infty$ implies that the power series appearing in the definition of $\mathcal{H}_\mu(f)$ defines an analytic function in $\mathbb{D}$.

(ii) If $2 < p < \infty$ and $f \in B^p$, $f(z) = \sum_{k=0}^{\infty} a_k z^k$ ($z \in \mathbb{D}$), then

$$\sum_{k=1}^{\infty} |\mu_n + k a_k| \leq \sum_{k=1}^{\infty} |\mu_k||a_k| = \sum_{k=1}^{\infty} k^{1-\frac{1}{p}} |a_k| \frac{\mu_k}{k^{1/p}}, \quad n \geq 0.$$
Then using Hölder inequality and Theorem B (ii), we obtain
\[
\sum_{k=1}^{\infty} |\mu_{n+k} a_k| \leq \left( \sum_{k=1}^{\infty} k |a_k|^p \right)^{1/p} \left( \sum_{k=1}^{\infty} |\mu_k| |p'|^{kp'/p} \right)^{1/p'} \leq C \rho_p(f) \left( \sum_{k=1}^{\infty} |\mu_k| |p'|^{kp'/p} \right)^{1/p'}, \quad n \geq 0.
\]

Then we see that the condition \(\sum_{k=1}^{\infty} |\mu_k| |p'|^{kp'/p} < \infty\) implies that the power series appearing in the definition of \(\mathcal{H}_\mu(f)\) defines an analytic function in \(D\).

\(\square\)

Let us turn to study when is the operator \(\mathcal{H}_\mu\) bounded from \(B^p\) into itself. Let us mention that Bao and Wulan [10] considered an operator which is closely related to the operator \(\mathcal{H}_\mu\) acting on the Dirichlet spaces \(D_\alpha\) \((\alpha \in \mathbb{R})\) which are defined as follows:

For \(\alpha \in \mathbb{R}\), the space \(D_\alpha\) consists of those functions \(f(z) = \sum_{n=0}^{\infty} a_n z^n\) analytic in \(D\) for which
\[
\|f\|_{D_\alpha} \overset{\text{def}}{=} \left( \sum_{n=0}^{\infty} (n+1)^{1-\alpha} |a_n|^2 \right)^{1/2} < \infty.
\]

Let us remark that \(D_0\) is the Dirichlet spaces \(D = B^2\), while \(D_1 = H^2\).

Bao and Wulan proved that if \(\mu\) is a positive Borel measure on \([0,1)\) and \(0 < \alpha < 2\), then the operator \(\mathcal{H}_\mu\) is bounded from \(D_\alpha\) into itself if and only if \(\mu\) is a Carleson measure. Let us remark that this does not include the case \(\alpha = 0\). In fact, the following results are proved in [10].

**Theorem D.**

(i) There exists a positive Borel measure \(\mu\) on \([0,1)\) which is a Carleson measure but such that \(\mathcal{H}_\mu(B^2) \not\subset B^2\).

(ii) Let \(\mu\) be a positive Borel measure on \([0,1)\) such that the operator \(\mathcal{H}_\mu\) is a bounded operator from \(B^2\) into itself. Then \(\mu\) is a Carleson measure.

We can improve these results and, even more, we shall obtain extensions of these improvements to all \(B^p\) spaces \((1 < p < \infty)\). More precisely we are going to prove the following results.

**Theorem 3.6.** Suppose that \(1 < p < \infty\) and \(0 < \beta \leq \frac{1}{p}\). Then there exists a positive Borel measure \(\mu\) on \([0,1)\) which is a \(\beta\)-logarithmic \(1\)-Carleson measure but such that the operator \(\mathcal{H}_\mu\) does not apply \(B^p\) into itself.

Next we prove that \(\mu\) being a \(\beta\)-logarithmic \(1\)-Carleson measure for a certain \(\beta\) is a necessary condition for \(\mathcal{H}_\mu\) being a bounded operator from \(B^p\) into itself.

**Theorem 3.7.** Suppose that \(1 < p < \infty\) and let \(\mu\) be a positive Borel measure on \([0,1)\) such that the operator \(\mathcal{H}_\mu\) is bounded from \(B^p\) to itself. Then \(\mu\) is a \(\gamma\)-logarithmic \(1\)-Carleson measure for any \(\gamma < 1 - \frac{1}{p}\).
Finally, we obtain a sufficient condition for the boundedness of $\mathcal{H}_\mu$ from $B^p$ into itself.

**Theorem 3.8.** Suppose that $1 < p < \infty$, $\gamma > 1$, and let $\mu$ be a positive Borel measure on $[0,1]$ which is a $\gamma$-logarithmic 1-Carleson measure. Then the operator $\mathcal{H}_\mu$ is a bounded operator from $B^p$ into itself.

We shall need a number of results on Besov spaces, as well as some lemmas, to prove these three theorems. First of all we notice that the Besov spaces can be characterized in terms of “dyadic blocks”. In order to state this in a precise way we need to introduce some notation.

For a function $f(z) = \sum_{n=0}^\infty a_n z^n$ analytic in $D$, define the polynomials $\Delta_j f$ as follows:

$$\Delta_j f(z) = \sum_{k=2^j}^{2^{j+1}-1} a_k z^k, \quad \text{for } j \geq 1,$$

$$\Delta_0 f(z) = a_0 + a_1 z.$$

Mateljević and Pavlović proved in [25, Theorem 2.1] (see also [27, Theorem C]) the following result.

**Theorem E.** Let $1 < p < \infty$ and $\alpha > -1$. For a function $f$ analytic in $D$ we define

$$Q_1(f) \overset{\text{def}}{=} \int_D |f(z)|^p (1 - |z|)^\alpha \, dA(z), \quad Q_2(f) \overset{\text{def}}{=} \sum_{n=0}^\infty 2^{-n(p+1)} \|\Delta_n f\|_{H^p}^p.$$

Then, $Q_1(f) \asymp Q_2(f)$.

Theorem E readily implies the following result.

**Corollary 3.9.** Suppose that $1 < p < \infty$ and $f$ is an analytic function in $D$. Then

$$f \in B^p \iff \sum_{n=0}^\infty 2^{-n(p-1)} \|\Delta_n f\|_{H^p}^p < \infty.$$

Furthermore,

$$\rho_p(f)^p \asymp \sum_{n=0}^\infty 2^{-n(p-1)} \|\Delta_n f\|_{H^p}^p.$$

Using Corollary 3.9 we can prove that the converses of (i) and (ii) in Theorem B hold if the sequence of Taylor coefficients $\{a_n\}$ decreases to 0. This is the analogue for Besov spaces of the result proved in [22] by Hardy and Littlewood for Hardy spaces (see also [27, 7.5.9] and [35, Chapter XII, Lemma 6.6]).

**Theorem 3.10.** Suppose that $1 < p < \infty$ and let $\{a_n\}_{n=0}^\infty$ be a decreasing sequence of non-negative numbers with $\{a_n\} \to 0$, as $n \to \infty$. Let $f(z) = \sum_{n=0}^\infty a_n z^n$ ($z \in D$). Then

$$f \in B^p \iff \sum_{n=1}^\infty n^{p-1} a_n^p < \infty.$$

Furthermore, $\rho_p(f)^p \asymp \sum_{n=1}^\infty n^{p-1} a_n^p$. 
Proof. For every $n$, we have
\[
z (\Delta_n f') (z) = \sum_{k=2^n+1}^{2^{n+1}} k a_k z^k.
\]
Since the sequence $\lambda = \{k\}_{k=0}^\infty$ is an increasing sequence of non-negative numbers, using Lemma A of [27] we see that
\[
\|z (\Delta_n f')\|_{H^p} \asymp 2^{np} \|\Delta_n f\|_{H^p}. \tag{3.3}
\]
Now, set $h(z) = \sum_{n=0}^{\infty} z^n (z \in \mathbb{D})$. Since the sequence $\tilde{\lambda} = \{a_n\}_{n=0}^\infty$ is a decreasing sequence of non-negative numbers, using the second part of Lemma A of [27], we see that
\[
a^{p}_{2^n} \|\Delta_n h\|_{H^p} \lesssim \|\Delta_n f\|_{H^p} \lesssim a^{p}_{2^{n-1}} \|\Delta_n h\|_{H^p}. \tag{3.4}
\]
Notice that $h(z) = \frac{1}{1-z}$ ($z \in \mathbb{D}$). Then it is well known that $M_p(r, h) \asymp (1-r)^{1/p-1}$ (recall that $1 < p < \infty$). Following the notation of [25], this can be written as $h \in H \left(p, \infty, 1 - \frac{1}{p}\right)$. Then using Theorem 2.1 of [25] (see also [26, p. 120]), we deduce that $\|\Delta_n\|_{H^p} \asymp 2^{n(p-1)}$. Using this and (3.4), it follows that
\[
2^{n(p-1)} a^{p}_{2^n} \lesssim \|\Delta_n f\|_{H^p} \lesssim 2^{n(p-1)} a^{p}_{2^{n-1}}. \tag{3.5}
\]
Using Corollary 3.9, (3.3), and (3.5), we see that
\[
\rho_p(f)^p \asymp \sum_{n=0}^{\infty} 2^{-n(p-1)} \|z (\Delta_n f')\|_{H^p} \asymp \sum_{n=0}^{\infty} 2^{np} \|\Delta_n f\|_{H^p} \asymp \sum_{n=0}^{\infty} 2^{np} a^{p}_{2^n}.
\]
Now, the fact that $\{a_n\}$ is decreasing implies that $\sum_{n=0}^{\infty} 2^{np} a^{p}_{2^n} \asymp \sum_{n=1}^{\infty} n^{p-1} a^{p}_n$ and, then it follows that $\rho_p(f)^p \asymp \sum_{n=1}^{\infty} n^{p-1} a^{p}_n$. \(\square\)

Remark 3.11. If $f$ is an analytic function in $\mathbb{D}$, $f(z) = \sum_{n=0}^{\infty} a_n z^n$ ($z \in \mathbb{D}$), and $1 < p < \infty$ then any of the two conditions $f \in B^p$ and $\sum_{n=1}^{\infty} n^{p-1} |a_n|^p < \infty$ implies that $\{a_n\} \to 0$. Consequently, the condition $\{a_n\} \to 0$ can be omitted in the hypotheses of Theorem 3.10.

Suppose that $\beta \geq 0$, $s \geq 1$, $1 < p < \infty$, and $\mu$ is a positive Borel measure on $[0, 1)$ which is a $\beta$-logarithmic $s$-Carleson measure. Using Lemma 2.7 and Theorem 3.5, it follows that $\mathcal{H}_\mu$ is well defined on $B^p$. Also, it is easy to see that $\int_{[0,1)} (\log \frac{2}{1-t})^{1/p'} \, d\mu(t) < \infty$, a fact that, using Theorem 3.3 (i), shows that $I_\mu$ is also well defined in $B^p$. Using then standard arguments it follows that $I_\mu$ and $\mathcal{H}_\mu$ coincide in $B^p$. Let us state this as a lemma.

Lemma 3.12. Suppose that $\beta \geq 0$, $s \geq 1$, $1 < p < \infty$, and $\mu$ is a positive Borel measure on $[0, 1)$ which is a $\beta$-logarithmic $s$-Carleson measure. Then the operators $\mathcal{H}_\mu$ and $I_\mu$ are well defined in $B^p$ and $\mathcal{H}_\mu(f) = I_\mu(f)$, for all $f \in B^p$.

Proof of Theorem 3.6. Let $\mu$ be the Borel measure on $[0,1)$ defined by
\[
d\mu(t) = \left(\log \frac{2}{1-t}\right)^{-\beta} \, dt.
\]
Since the function \( x \mapsto \left( \log \frac{2}{1-x} \right)^{-\beta} \) is decreasing in \([0, 1)\), we have
\[
\mu ([t, 1)) = \int_t^1 \left( \log \frac{2}{1-x} \right)^{-\beta} \, dx \leq (1-t) \left( \log \frac{2}{1-t} \right)^{-\beta}, \quad 0 \leq t < 1.
\]
Hence, \( \mu \) is a \( \beta \)-logarithmic 1-Carleson measure. Then, taking \( \alpha = 0 \) in Lemma 2.7, we see that
\[
\mu_k = O \left( \frac{1}{k(\log k)^\beta} \right).
\]
On the other hand,
\[
\mu_k \geq \int_0^{1-\frac{1}{k}} t^k \left( \log \frac{2}{1-t} \right)^{-\beta} \, dt \geq \frac{1}{(\log k)^\beta} \int_0^{1-\frac{1}{k}} t^k \, dt \geq \frac{1}{k(\log k)^\beta}.
\]
Thus, we have seen that \( \mu \) is a \( \beta \)-logarithmic 1-Carleson measure which satisfies
\[
\mu_n \asymp \frac{1}{n(\log n)^\beta}.
\] (3.6)
Take \( p \in (1, \infty) \) and \( \alpha > \frac{1}{p} \) and set
\[
a_n = \frac{1}{(n+1)(\log(n+2))^\alpha}, \quad n = 0, 1, 2, \ldots,
\]
and
\[
g(z) = \sum_{n=0}^\infty a_n z^n, \quad z \in \mathbb{D}.
\]
Notice that \( \{a_n\} \downarrow 0 \) and that \( \sum_{n=0}^\infty n^{p-1} |a_n|^p < \infty \). Hence, \( g \in B^p \).

We are going to prove that \( \mathcal{H}_{\mu}(g) \notin B^p \). This implies that \( \mathcal{H}_{\mu}(B^p) \notin B^p \), proving the theorem.

We have \( \mathcal{H}_{\mu}(g)(z) = \sum_{n=0}^\infty (\sum_{k=0}^\infty \mu_{n+k} a_k) z^n \). Notice that \( a_k \geq 0 \) for all \( k \) and that the sequence of moments \( \{\mu_n\} \) is a decreasing sequence of non-negative numbers. Then it follows that the sequence \( \{\sum_{k=0}^\infty \mu_{n+k} a_k\} \) of the Taylor coefficients of \( \mathcal{H}_{\mu}(g) \) is decreasing. Consequently, we have that
\[
\mathcal{H}_{\mu}(g) \in B^p \iff \sum_{n=1}^\infty n^{p-1} \left| \sum_{k=0}^\infty \mu_{n+k} a_k \right|^p < \infty. \quad (3.7)
\]
Using the definition of the sequence \( \{a_k\} \), (3.6) and the simple inequalities \( \frac{k}{n+k} \geq \frac{1}{n+1} \) and \( \log(n+k) \leq (\log n)(\log k) \) which hold whenever \( k, n \geq 10 \), say, we obtain
\[
\sum_{n=1}^\infty n^{p-1} \left| \sum_{k=0}^\infty \mu_{n+k} a_k \right|^p \geq \sum_{n=1}^\infty n^{p-1} \left( \sum_{k=10}^\infty \mu_{n+k} a_k \right)^p \geq \sum_{n=10}^\infty \left( \sum_{k=10}^\infty \frac{1}{(n+k)(\log(n+k))^\beta} \right)^p \frac{1}{k(\log k)^\beta} \left( \sum_{k=10}^\infty \frac{1}{k^2 (\log k)^{\alpha+\beta}} \right)^p \]
\[
\geq \sum_{n=10}^\infty \frac{1}{n(\log n)^{p\beta}} \left( \sum_{k=10}^\infty \frac{1}{k^2 (\log k)^{\alpha+\beta}} \right)^p = \infty.
\]
Bearing in mind (3.7), this implies that $H_\mu(g) \notin B^p$ as desired. □

**Proof of Theorem 3.7.** Suppose that $1 < p < \infty$ and $\gamma < 1 - \frac{1}{p}$. Let $\mu$ be a positive Borel measure on $[0,1)$ such that the operator $H_\mu$ is a bounded operator from $B^p$ into itself. Set $\alpha = 1 - \gamma$, 

$$a_k = \frac{1}{k(\log k)^\alpha}, \quad k \geq 2,$$

and

$$f(z) = \sum_{k=2}^\infty a_k z^k, \quad z \in \mathbb{D}.$$ 

Since $\alpha > \frac{1}{p}$, using Theorem 3.10 we see that $f \in B^p$. By our assumption $H_\mu(f) \in B^p$, that is, $\|H_\mu(f)\|_{B^p} < \infty$. We have

$$H_\mu(f)(z) = \sum_{n=0}^\infty \left( \sum_{k=2}^\infty \mu_n a_k \right) z^n.$$

Since $a_k \geq 0$ for all $k$ and $\{\mu_n\}$ is a decreasing sequence of non-negative numbers, it follows that the sequence $\{\sum_{k=2}^\infty \mu_n a_k\}_{n=0}^\infty$ is a decreasing sequence of non-negative numbers. Then, using Theorem 3.10 we obtain

$$\|H_\mu(f)\|_{B^p}^p \gtrsim \sum_{n=1}^\infty n^{p-1} \left( \sum_{k=2}^\infty \mu_n a_k \right)^p$$

$$\gtrsim \sum_{n=1}^\infty n^{p-1} \left( \sum_{k=2}^\infty \frac{1}{k(\log k)^\alpha} \int_{[0,1)} x^{n+k} d\mu(x) \right)^p$$

$$\gtrsim \sum_{n=1}^\infty n^{p-1} \left( \sum_{k=2}^\infty \frac{1}{k(\log k)^\alpha} \int_{[t,1)} x^{n+k} d\mu(x) \right)^p$$

$$\gtrsim \sum_{n=1}^\infty n^{p-1} \left( \sum_{k=2}^\infty \frac{t^{n+k}}{k(\log k)^\alpha} \right) \mu(\{t, 1\})^p$$

$$= \sum_{n=1}^\infty n^{p-1} t^{np} \left( \sum_{k=2}^\infty \frac{t^k}{k(\log k)^\alpha} \right) \mu(\{t, 1\})^p, \quad \text{for all } t \in (0, 1).$$

Now, it is well known that $\sum_{k=2}^\infty \frac{t^k}{k(\log k)^\alpha} \asymp (\log \frac{2}{1-t})^{1-\gamma} = (\log \frac{2}{1-t})^\gamma$ (see [35, Vol. I, p. 192]). Then it follows that

$$\|H_\mu(f)\|_{B^p}^p \gtrsim \left( \log \frac{2}{1-t} \right)^\gamma \left( \sum_{n=1}^\infty n^{p-1} t^{np} \right) \mu(\{t, 1\})^p$$

$$\gtrsim \left( \log \frac{2}{1-t} \right)^\gamma \frac{1}{(1-t)^p} \mu(\{t, 1\})^p.$$

Since $\|H_\mu(f)\|_{B^p} < \infty$, this shows that $\mu$ is a $\gamma$-logarithmic 1-Carleson measure. □
The following lemma will be used to prove Theorem 3.8. It is an adaptation of [20, Lemma 7] to our setting. The proof is very similar to that of the latter but we include it for the sake of completeness.

**Lemma 3.13.** Let $p, \gamma,$ and $\mu$ be as in Theorem 3.8. Then, there exists a constant $C = C(p, \gamma, \mu) > 0$ such that if $f \in B^p$, $g(z) = \sum_{k=0}^{\infty} c_k z^k \in \text{Hol}(\mathbb{D})$, and we set

$$h(z) = \sum_{k=0}^{\infty} c_k \left( \int_0^1 t^{k+1} f(t) \, d\mu(t) \right) z^k,$$

then

$$\|\Delta_n h\|_{H^p} \leq C \left( \int_0^1 t^{2n-2+1} |f(t)| \, d\mu(t) \right) \|\Delta_n g\|_{H^p}, \quad n \geq 3.$$

**Proof.** For each $n = 1, 2, \ldots$, define

$$\Upsilon_n(s) = \int_0^1 t^{2n+1} f(t) \, d\mu(t), \quad s \geq 0.$$

Clearly, $\Upsilon_n$ is a $C^\infty(0, \infty)$-function and

$$|\Upsilon_n(s)| \leq \int_0^1 t^{2n-2+1} |f(t)| \, d\mu(t), \quad s \geq \frac{1}{2}. \quad (3.8)$$

Furthermore, since $\sup_{0<x<1} (\log \frac{1}{x})^2 x^{1/2} = C(2) < \infty$, we have

$$|\Upsilon_n'(s)| \leq \int_0^1 \left( \log \frac{1}{t^{2n}} \right)^2 t^{2n-1} |f(t)| \, d\mu(t)$$

$$\leq C(2) \int_0^1 t^{2n-1} |f(t)| \, d\mu(t) \leq C(2) \int_0^1 t^{2n-2+1} |f(t)| \, d\mu(t), \quad s \geq \frac{3}{4}. \quad (3.9)$$

Then, using (3.8) and (3.9), for each $n = 1, 2, \ldots$, we can take a function $\Phi_n \in C^\infty(\mathbb{R})$ with $\text{supp}(\Phi_n) \in \left(\frac{3}{4}, 4\right)$, and such that

$$\Phi_n(s) = \Upsilon_n(s), \quad s \in [1, 2],$$

and

$$A_{\Phi_n} = \max_{s \in \mathbb{R}} |\Phi_n(s)| + \max_{s \in \mathbb{R}} |\Phi_n'(s)| \leq C \int_0^1 t^{2n-2+1} |f(t)| \, d\mu(t).$$

Following the notation used in [20, p. 236], we can then write

$$\Delta_n h(z) = \sum_{k=2^n}^{2^{n+1}-1} c_k \left( \int_0^1 t^{k+1} f(t) \, d\mu(t) \right) z^k$$

$$= \sum_{k=2^n}^{2^{n+1}-1} c_k \Phi_n \left( \frac{k}{2^n} \right) z^k = W_{2^n} \Phi_n * \Delta_n g(z).$$
So by using part (iii) of Theorem B of [20], we have
\[
\| \Delta_n h \|_{H^p} = \| W^{\Phi_n}_{2^n} \ast \Delta_n g \|_{H^p} \leq C_p A_{\Phi_n} \| \Delta_n g \|_{H^p}
\leq C \left( \int_0^1 t^{2^{n-2}+1} |f(t)| \, d\mu(t) \right) \| \Delta_n g \|_{H^p}.
\]
\[
\square
\]

**Proof of Theorem 3.8.** By the closed graph theorem it suffices to show that \( \mathcal{H}_\mu(B^p) \subset B^p \).

Take \( f \in B^p \). Since \( \mu \) is a \( \gamma \)-logarithmic 1-Carleson measure, using Lemma 3.12 we see that
\[
\mathcal{H}_\mu(f)(z) = I_\mu(f)(z) = \sum_{n=0}^{\infty} \left( \int_{[0,1]} t^n f(t) \, d\mu(t) \right) z^n, \quad z \in \mathbb{D}.
\]
Also, using Corollary 3.9, we see that
\[
\mathcal{H}_\mu(f) \in B^p \iff \sum_{n=1}^{\infty} 2^{-n(p-1)} \| \Delta_n (\mathcal{H}_\mu(f))' \|_{H^p}^p < \infty. \tag{3.10}
\]
Now, we have
\[
\Delta_n (\mathcal{H}_\mu(f))'(z) = \sum_{k=2^n}^{2^{n+1}-1} (k+1) \left( \int_{[0,1]} t^{k+1} f(t) \, d\mu(t) \right) z^k.
\]
Using Lemma 3.13 we obtain that
\[
\| \Delta_n (\mathcal{H}_\mu(f))' \|_{H^p} \lesssim \left( \int_{[0,1]} t^{2^{n-2}+1} |f(t)| \, d\mu(t) \right) \| \Delta_n F \|_{H^p}
\]
with \( F(z) = \sum_{k=0}^{\infty} (k+1) z^k \) (\( z \in \mathbb{D} \)). Now, we have that \( M_p(r, F) = O \left( \frac{1}{(1-r)^{2-\frac{1}{p'}}} \right) \)
and then it follows that \( \| \Delta_n F \|_{H^p} = O \left( 2^{n(2-\frac{1}{p})} \right) \) (see, e.g., [25]). Using this and the estimate \( |f(t)| \lesssim \left( \log \frac{2}{1-t} \right)^{1/p'} \), we obtain
\[
\| \Delta_n (\mathcal{H}_\mu(f))' \|_{H^p} \lesssim 2^{n(2-\frac{1}{p})} \left( \int_{[0,1]} t^{2^{n-2}+1} \left( \log \frac{2}{1-t} \right)^{1/p'} \, d\mu(t) \right),
\]
which using the fact that \( \mu \) is a \( \gamma \)-logarithmic 1-Carleson measure and Lemma 2.7 implies
\[
\| \Delta_n (\mathcal{H}_\mu(f))' \|_{H^p} \lesssim 2^{n(2-\frac{1}{p})} 2^{-n \frac{1}{p'}} \gamma = 2^{n/p'} n^{(1-\gamma)} \gamma.
\]
This, together with the fact that \( \gamma > 1 \), implies that
\[
\sum_{n=1}^{\infty} 2^{-n(p-1)} \| \Delta_n (\mathcal{H}_\mu(f))' \|_{H^p}^p \lesssim \sum_{n=1}^{\infty} 2^{-n(p-1)} 2^{n/p'} n^{(1-\gamma)-1} = \sum_{n=1}^{\infty} n^{p(1-\gamma)-1} < \infty.
\]
Bearing in mind (3.10), this shows that \( \mathcal{H}_\mu(f) \in B^p \) and finishes the proof. \( \square \)
Acknowledgements.

The authors wish to express their gratitude to the referee who read the paper very carefully and made a good number of suggestions for improvement.

This research is supported by a grant from “El Ministerio de Economía y Competitividad”, Spain (MTM2014-52865-P) and by a grant from la Junta de Andalucía FQM-210. The second author is also supported by a grant from “El Ministerio de de Educación, Cultura y Deporte”, Spain (FPU2013/01478).

References

[1] A. Aleman, A. Montes-Rodríguez and A. Sarafoleanu, The eigenfunctions of the Hilbert matrix, Const. Approx. 36 n. 3, (2012), 353–374.

[2] A. Aleman and A. Siskakis, Integration operators on Bergman spaces, Indiana Univ. Math. J. 46 (1997), No. 2, 337-356.

[3] J. M. Anderson, J. Clunie and Ch. Pommerenke, On Bloch functions and normal functions, J. Reine Angew. Math. 270 (1974), 12–37.

[4] J. M. Anderson and A. L. Shields, Coefficient Multipliers of Bloch Functions, Trans. Amer. Math. Soc. 224 (1976), n. 2, 255–265.

[5] J. Arazy, S. D. Fisher and J. Peetre, Möbius invariant function spaces, J. Reine Angew. Math. 363 (1985), 110–145.

[6] R. Aulaskari and P. Lappan, Criteria for an analytic function to be Bloch and a harmonic or meromorphic function to be normal, Complex Analysis and its Applications (Harlow), Pitman Research Notes in Math, vol. 305, Longman Scientific and Technical, 1994, 136–146.

[7] R. Aulaskari, P. Lappan, J. Xiao and R. Zhao, On α-Bloch Spaces and Multipliers on Dirichlet Spaces, J. Math. Anal. Appl. 209 (1997), 103–121.

[8] R. Aulaskari, J. Xiao and R. Zhao, On subspaces and subsets of BMOA and UBC, Analysis 15 (1995), 101–121.

[9] A. Baernstein, Analytic functions of Bounded Mean Oscillation In: Aspects of Contemporary Complex Analysis, Editors: D. A. Brannan and J. G. Clunie., Academic Press, London, New York (1980), pp. 3–36.

[10] G. Bao and H. Wulan, Hankel matrices acting on Dirichlet spaces, J. Math. Anal. Appl. 409 (2014), no. 1, 228-235.

[11] L. Carleson, Interpolation by bounded analytic functions and the corona problem, Ann. of Math. 76 (1962), 547-559.

[12] Ch. Chatzifountas, D. Girela and J. Á. Peláez, A generalized Hilbert matrix acting on Hardy spaces, J. Math. Anal. Appl. 413 (2014), no. 1, 154-168.

[13] E. Diamantopoulos, Hilbert matrix on Bergman spaces, Illinois J. of Math. 48, n. 3, Fall (2004), 1067–1078.

[14] E. Diamantopoulos and A. G. Siskakis, Composition operators and the Hilbert matrix, Studia Math. 140 (2000), 191–198.

[15] J. J. Donaire, D. Girela and D. Vukotić, On univalent functions in some Möbius invariant spaces, J. Reine Angew. Math. 553 (2002), 43–72.

[16] M. Dostanić, M. Jevtić and D. Vukotić, Norm of the Hilbert matrix on Bergman and Hardy spaces and a theorem of Nehari type, J. Funct. Anal. 254 (2008), 2800–2815.

[17] P. L. Duren, Extension of a Theorem of Carleson, Bull. Amer. Math. Soc. 75 (1969), 143–146.

[18] P. L. Duren, Theory of $H^p$ Spaces, Academic Press, New York-London 1970. Reprint: Dover, Mineola, New York 2000.

[19] P. Galanopoulos and J. A. Peláez, A Hankel matrix acting on Hardy and Bergman spaces, Studia Math. 200, 3, (2010), 201–220.
24 D. GIRELA AND N. MERCHÁN

[20] P. Galanopoulos, D. Girela, J. A. Peláez and A. Siskakis, Generalized Hilbert operators, Ann. Acad. Sci. Fenn. Math. 39 (2014) 39 (2014), no. 1, 231-258.

[21] D. Girela, Analytic functions of bounded mean oscillation. In: Complex Function Spaces, Mekrijärvi 1999 Editor: R. Aulaskari. Univ. Joensuu Dept. Math. Rep. Ser. 4, Univ. Joensuu, Joensuu (2001) pp. 61–170.

[22] G. H. Hardy and J. E. Littlewood, Some new properties of Fourier coefficients, J. London. Math. Soc. 6. (1931), 3–9.

[23] F. Holland and D. Walsh, Growth estimates for functions in the Besov spaces A_p, Proc. Roy. Irish Acad. Sect. A 88 (1988), 1–18.

[24] B. Lanucha, M. Nowak and M. Pavlović, Hilbert matrix operator on spaces of analytic functions, Ann. Acad. Sci. Fenn. Math. 37 (2012), 161–174.

[25] M. Mateljević and M. Pavlović, $L^p$ behaviour of the integral means of analytic functions, Studia Math. 77 (1984), 219–237.

[26] M. Pavlović, Introduction to function spaces on the Disk, Posebna Izdanja [Special Editions], vol. 20, Matematički Institut SANU, Beograd, 2004.

[27] M. Pavlović, Analytic functions with decreasing coefficients and Hardy and Bloch spaces, Proc. Edinburgh Math. Soc. Ser. 2 56, 2, (2013), 623–635.

[28] M. Pavlović, Invariant Besov spaces. Taylor coefficients and applications, Technical Report, available at ResearchGate, 9 pp.

[29] J. A. Peláez and J. Rättyä, Weighted Bergman spaces induced by rapidly increasing weights, Mem. Amer. Math. Soc. 227 (2014), no. 1066, vi+124 pp.

[30] L. E. Rubel and R. M. Timoney, An extremal property of the Bloch space, Proc. Amer. Math. Soc. 75 (1979), no. 1, 45–49.

[31] J. Xiao, Holomorphic Q classes, Lecture Notes in Mathematics 1767, Springer-Verlag, 2001.

[32] R. Zhao, On logarithmic Carleson measures, Acta Sci. Math. (Szeged) 69 (2003), no. 3-4, 605–618.

[33] K. Zhu, Analytic Besov spaces, J. Math. Anal. Appl. 157 (1991), 318–336.

[34] K. Zhu, Operator Theory in Function Spaces, Second Edition, Math. Surveys and Monographs, 138 (2007).

[35] A. Zygmund, Trigonometric Series, Vol. I and Vol. II, Second edition, Camb. Univ. Press, Cambridge, 1959.

1 Análisis Matemático, Universidad de Málaga, Campus de Teatinos, 29071 Málaga, Spain.

E-mail address: girela@uma.es; noel@uma.es