Non-axisymmetric Magnetorotational Instabilities in Cylindrical Taylor-Couette Flow

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(Dated: December 29, 2009)

We study the stability of cylindrical Taylor-Couette flow in the presence of azimuthal magnetic fields, and show that one obtains non-axisymmetric magnetorotational instabilities, having azimuthal wavenumber \(m = 1\). For \(\Omega_o/\Omega_i\) only slightly greater than the Rayleigh value \((r_i/r_o)^2\), the critical Reynolds and Hartmann numbers are \(Re_c \sim 10^1\) and \(Ha_c \sim 10^2\), independent of the magnetic Prandtl number \(Pm\). These values are sufficiently small that it should be possible to obtain these instabilities in the PROMISE experimental facility.

PACS numbers: 47.20.-k, 47.65.+a, 95.30.Qd

The magnetorotational instability (MRI) was discovered in 1959 by Velikhov \(^1\), who considered cylindrical Taylor-Couette flow in the presence of an axial magnetic field, and obtained instabilities in otherwise hydrodynamically stable flows. Several decades later, it was recognized that much the same instability plays a crucial role in the dynamics of astrophysical accretion disks \(^2\). This prompted renewed interest in the MRI in Taylor-Couette context as well, in particular the possibility of studying it in laboratory experiments. Following Velikhov, it was originally suggested \(^3\) to impose an axial magnetic field. However, this ‘standard’ MRI (SMRI) has one very considerable disadvantage, namely that the rotation rates required to achieve it are enormous.

The relevant parameter turns out to be not the hydrodynamic Reynolds number \(Re = \Omega r_o^2/\nu\), but rather the magnetic Reynolds number \(Rm = \Omega r_o^2/\eta\), where \(\nu\) is the viscosity and \(\eta\) the magnetic diffusivity. The SMRI sets in when \(Re \sim 10\). Re is then given by \(Rm/Pm\), where \(Pm = \nu/\eta\) is the magnetic Prandtl number, a material property of the fluid. Typical values are \(\sim 10^{-5}\) for liquid sodium, and \(\sim 10^{-6}\) for gallium. \(Re\) must therefore exceed \(10^6\) or even \(10^7\), which unfortunately leads to increasingly strong end-effects \(^13\). These can perhaps be overcome \(^14\), but the SMRI has not been obtained yet.

An alternative approach was suggested by \(^15\), who showed that in a combined axial and azimuthal magnetic field, the relevant parameter is \(Rm\) rather than \(Re\) – that is, the scaling with \(Pm\) is altered – and that the resulting ‘helical’ MRI (HMRI) occurs when \(Re \sim 10^2\), several orders of magnitude less than what would be required for the SMRI. This new design was quickly implemented in the PROMISE facility \(^6\), and does indeed yield modes in good agreement with the theoretical predictions. Note though that end-effects inevitably play an important role in this set-up as well, particularly due to the traveling wave nature of the HMRI. The implications for the PROMISE results continue to be debated \(^16\), \(^17\).

In this work we start with a purely azimuthal field. Velikhov \(^1\) had already considered this as well, and showed that it does not yield any axisymmetric instabilities like the SMRI or the (continuously connected) HMRI. It can, however, yield non-axisymmetric instabilities, as \(^18\) first demonstrated in an astrophysical context (where the magnetic fields in accretion disks may indeed be predominantly azimuthal rather than axial). The
possibility of obtaining non-axisymmetric instabilities is also particularly exciting, as it would help to circumvent Cowling’s theorem, disallowing purely axisymmetric dynamics action.

In the Taylor-Couette problem considered here, this ‘azimuthal’ MRI (AMRI) was briefly noted by [19], but only in a parameter regime that is not experimentally accessible. We show here that for rotation ratios Ω_o/Ω_i only slightly greater than the Rayleigh limit (r_i/r_o)^2, the relevant parameters are sufficiently small that it should be achievable in the PROMISE facility.

Given the basic state consisting of an azimuthal magnetic field B_0 = B_0(r_i/r)̇e_φ, imposed by running a current down the central axis, as well as an angular velocity profile Ω(r), imposed by differentially rotating the inner and outer cylinders, we begin by linearizing the governing equations about it. The perturbation flow u and field b may be expressed as

\[ u = \nabla \times (e_φ) + \nabla \times \nabla \times (f e_φ), \]
\[ b = \nabla \times (g e_φ) + \nabla \times \nabla \times (h e_φ). \]

Taking the (φ, z, t) dependence to be exp(Imφ + ikr + γt), the perturbation equations become

\[
\begin{align*}
\text{Re} & \gamma(C_2e + C_3f) + C_4e + C_5f = \text{Re}E_1 + \text{Re}F_1 + H^2G_1 + H^2H_1, \\
\text{Re} & \gamma(C_3e + C_4f) + C_5e + C_6f = \text{Re}E_2 + \text{Re}F_2 + H^2G_2 + H^2H_2, \\
\text{Rm} & \gamma(C_1g + C_2h) + C_3g + C_4h = E_1 + F_3 + \text{Rm}G_3 + \text{Rm}H_3, \\
\text{Rm} & \gamma(C_2g + C_3h) + C_4g + C_5h = E_4 + F_4 + \text{Rm}G_4 + \text{Rm}H_4.
\end{align*}
\]

The operators C_n are defined by C_nρ = ̇e_φ \cdot (∇ \times)^n(p ̇e_φ), and work out to be

\[
\begin{align*}
C_1 &= 0, & C_2 &= \Delta, & C_3 &= -2mk^2r^2, \\
C_4 &= -\Delta \partial_r^2 + (m^2r^{-2} - k^2)(r^{-1} \partial_r - r^{-2}) + \Delta^2, \\
C_5 &= 4mk(r^{-2} \partial_r^2 - r^{-3} \partial_r + (1 - m^2) r^{-6} - k^2 r^{-2}), \\
C_6 &= \Delta \partial_r^4 - 2(m^2r^{-2} - k^2) r^{-1} \partial_r^2 + (5m^2r^{-4} - 3k^2r^{-2} - 2\Delta^2) \partial_r^2 + (3m^2(2m^2 - 3)r^{-4} + (4m^2 + 3)k^2r^{-2} - 2k^4) r^{-1} \partial_r + m^2(9 - 10m^2)r^{-6} - 3k^2r^{-4} + 2k^4r^{-2} + \Delta^3,
\end{align*}
\]

where \( \partial_r = \partial/\partial r \), and \( \Delta = m^2r^{-2} + k^2 \). The other quantities are

\[
E_1 = -im\DeltaΩe, \quad E_2 = ik\ΔΩe,
\]
\[
E_3 = 0, \quad E_4 = imr^{-2}Δe,
\]
\[
F_1 = ik(\ΔΩ + \Delta rΩ')f,
\]
\[
F_2 = -imΩ(C_4 + 4k^2r^{-2})f - im(\Delta^2 + 3r^{-1}Ω')f,
\]
\[
F_3 = imr^{-2}Δf, \quad F_4 = -ikr^{-2}Δf,
\]
\[
G_1 = imr^{-2}Δg, \quad G_2 = -ikr^{-2}Δg,
\]
\[
G_3 = 0, \quad G_4 = -imΩg,
\]
\[
H_1 = -2imkr^{-4}h, \quad H_2 = imr^{-2}C_4h + 4imk^2r^{-4}h,
\]
\[
H_3 = -imΔΩh, \quad H_4 = ik(2m^2r^{-2}Ω - ΔrΩ')h,
\]

where primes denote d/dr, and \( \Delta = 4m^2r^{-2} + 2k^2 \). All of these terms are easily derivable using MAPLE, or some other symbolic algebra package.

Length has been scaled by r_i, time by Ω_i^−1, Ω by Ω_i, u by Ω_i r_i, B_0 by B_0, and by RmB_0. The two Reynolds numbers Re and Rm are as above; the Hartmann number \( Ha = B_0 r_i / √ρμν \), where \( ρ \) is the fluid’s density and \( μ \) the magnetic permeability. Another parameter that appears implicitly is the rotation ratio \( µ = Ω_o/Ω_i \), which enters into the details of Ω(r) = c1 + c2/r^2. The radius ratio is fixed at r_i/r_o = 1/2, as in the PROMISE experiment.

The radial structure of e, f, g and h was expanded in terms of Chebyshev polynomials, typically up to \( N = 30 – 60 \). These equations and associated boundary conditions (no slip for u, insulating for b) then reduce to a large (4N × 4N) matrix eigenvalue problem, with the eigenvalue being the growth or decay rate γ of the given mode. This numerical implementation is very different from that of [19], in which the individual components of u and b were used, and discretized in r by finite differencing. Both codes yielded identical results though in every instance where we benchmarked one against the other.

Figure 1 shows the results for \( m = 1 \), the most unstable wavenumber. At each point in the Ha-Re-plane, we repeatedly solve the basic eigenvalue problem to find the axial wavenumber k that yields the largest Re(γ). We see that if \( µ \) is only slightly greater than the Rayleigh limit 0.25, values as small as Ha ~ 10^2 and Re ~ 10^3 are already sufficient to achieve instability. As \( µ \) is increased, increasingly large values are required. Note also that these results are independent of the Prandtl number; \( Pr_m = 10^{-5}, 10^{-6} \), or indeed even 0 all yield identical results (where we recall that Pr_m enters the equations via Rm = Pr_mRe).

The crucial question then is whether Ha ~ 10^2 and Re ~ 10^3 are achievable in the PROMISE facility. Re ~ 10^3 is certainly possible; this is precisely the range where the HMRI has already been obtained. Ha = 10^2 is somewhat more challenging, corresponding to a current of 13 kA along the central axis, roughly twice what was required for the HMRI. Once the latest upgrade is complete though, currents up to 20 kA will be achievable (F. Stefani, private communication).
FIG. 1: The grey-shaded regions show where \( \text{Re}(\gamma) > 0 \). The contour interval is 0.01, indicating that these instabilities grow on the basic rotational timescale \( \Omega^{-1} \), but with a somewhat smaller multiplicative factor than for the SMRI. (a) \( \hat{\mu} = 0.25 \), (b) \( \hat{\mu} = 0.26 \), (c) \( \hat{\mu} = 0.27 \), (d) \( \hat{\mu} = 0.28 \).

FIG. 2: \( \text{Re}_c \) as a function of \( \hat{\mu} \), optimized over \( k \) and \( H_a \).

Figure 2 quantifies how the critical Reynolds number increases with \( \hat{\mu} \). That is, we now optimize over \( H_a \) as well as \( k \), and compute the minimum value of \( \text{Re} \) that still allows instability. The behavior is remarkably similar to the transition from the HMRI to the SMRI, as shown in Fig. 1 of [15]. In both cases \( \text{Re}_c \) is \( \sim 10^3 \), and independent of \( \text{Pm} \) for \( \hat{\mu} \) only slightly greater than the Rayleigh value, but then increases dramatically, and scales as \( \text{Pm}^{-1} \) once \( \hat{\mu} \) is sufficiently large.

Having obtained this non-axisymmetric instability in a purely azimuthal field, and demonstrated that it should be achievable in the PROMISE experiment, it is of further interest to add an axial field again, and investigate at what point one switches back to the previous axisymmetric HMRI. We therefore modify the equations to impose a field of the form \( \mathbf{B}_0 = B_0 [ (r^2 / r) \hat{e}_\phi + \delta \hat{e}_z ] \), and explore what happens as \( \delta \) is increased from 0.

At this point we must also consider the handedness of both the basic state and the resulting instabilities. For a purely azimuthal field, the basic state has no handedness, that is, it is invariant to reversing the sign of \( z \). As a result, instabilities that spiral either to the left (for which \( mk > 0 \)) or to the right (for which \( mk < 0 \)), necessarily have exactly the same critical Reynolds and Hartmann numbers. For a combined azimuthal Reynolds and axial field though, the basic state itself has a handedness [15, 20], so left and right spiraling instabilities must be considered separately.

Figure 3 shows the results for \( \hat{\mu} = 0.26 \) and \( \text{Pm} = 0 \). An axial field as weak as \( \delta = 0.02 \) is already enough to induce a clear asymmetry between the left and right spirals, but both are otherwise still similar to the \( \delta = 0 \) results from Fig. 1(b), included here as the dotted line. For \( \delta = 0.03 \) another new feature emerges, the curve labeled 0. This is precisely the previous \( m = 0 \), axisymmetric HMRI. At this value of \( \delta \) the non-axisymmetric modes are still preferred though. Further increasing \( \delta \), the asymmetry between left and right spirals gradually becomes greater, and both curves shift upward slightly, indicating that these modes are suppressed by the addition of an axial field. In contrast, the HMRI is strongly excited, so much so that by \( \delta = 0.05 \) it is already the preferred mode.

We can at least begin to understand why \( m = 0 \) and 1 behave so differently by noting that if \( m = 0 \) and \( \delta = 0 \), the instability equation for the field component \( h \) reduces to just free decay, \( \text{Rm} \gamma C_g h + C_l h = 0 \). However, in the absence of this part of the field \( \nabla \times \nabla \times (h \hat{e}_z) \), there is no radial component to provide the coupling between different radii that ultimately drives the MRI, since the other part of the field \( \nabla \times (g \hat{e}_r) \) has no radial component.

This is essentially Velikhov's original proof that a purely azimuthal field does not yield any axisymmetric instabilities. See also [21], who extend Velikhov's analysis from ideal to diffusive fluids. To obtain an axisymmetric instability, we therefore require \( \delta \neq 0 \). This couples \( h \) to the other components again, thereby allowing the HMRI.
FIG. 4: The left and right spiral modes, at the two dots indicated in Fig. 3(c). On the left Re = 1910, Ha = 110, k = 4.2, Im(γ) = −0.24; on the right Re = 1510, Ha = 130, k = −3.0, Im(γ) = −0.27. Arrows denote the meridional flow (ur, uz), normalized such that the maximum (uφ2 + uz2)1/2 is 1. Contours show uφ, with a contour interval of 0.2, grey positive and white negative.

Future work will consider the nonlinear interactions among these different modes. Exactly symmetric left and right spirals in non-magnetic Taylor-Couette flow already allow a rich variety of possibilities, including both traveling and standing waves. It remains to be seen which of these occurs here for δ = 0. By judiciously adjusting δ, Ha and Re, it should also be possible in this problem to preferentially select either the left or right modes, or indeed the axisymmetric HMRI. The regime δ ∼ 0.04, where all three modes have comparable critical Hartmann and Reynolds numbers, is likely to yield particularly rich dynamics. Taylor-Couette flows in predominantly azimuthal magnetic fields of this type clearly deserve further attention, both experimental and theoretical.

This work was supported by the Science and Technology Facilities Council under Grant No. PP/E001092/1.

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