ON THE EXPONENT OF THE SCHUR MULTIPLIER OF A PAIR OF FINITE $p$-GROUPS

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In this paper, we find an upper bound for the exponent of the Schur multiplier of a pair $(G, N)$ of finite $p$-groups, when $N$ admits a complement in $G$. As a consequence, we show that the exponent of the Schur multiplier of a pair $(G, N)$ divides $\exp(N)$ if $(G, N)$ is a pair of finite $p$-groups of class at most $p-1$. We also prove that if $N$ is powerfully embedded in $G$, then the exponent of the Schur multiplier of a pair $(G, N)$ divides $\exp(N)$.

Keywords: Pair of groups; Schur multiplier of a pair; finite $p$-groups.

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1. Introduction and Motivation

In 1998, Ellis [2] extended the theory of the Schur multiplier for a pair of groups. By a pair of groups $(G, N)$ we mean a group $G$ with a normal subgroup $N$ of $G$. The Schur multiplier of a pair $(G, N)$ of groups is a functorial abelian group $M(G, N)$ whose principal feature is a natural exact sequence

$$H_3(G) \to H_3(G/N) \to M(G, N) \to M(G) \to M(G/N)$$

$$\to N/[N, G] \to (G)^{ab} \to (G/N)^{ab} \to 0$$
in which $H_3(G)$ is the third homology group of $G$ with integer coefficients. In particular, if $N = G$, then $M(G, G)$ is the usual Schur multiplier $M(G)$.

It has been a considerable question that when $\exp(M(G))$ divides $\exp(G)$, in which $\exp(G)$ denotes the exponent of $G$. Macdonald and Wamsley (see [1]) constructed an example of a group of exponent 4, whereas its Schur multiplier has exponent 8, hence the conjecture is not true in general. In 1973, Jones [4] proved that the exponent of the Schur multiplier of a finite $p$-group of exponent $8$, hence the conjecture is not true in general. In 1973, Jones [4] proved that the exponent of the Schur multiplier of a finite $p$-group of class $c \geq 2$ and exponent $p^e$ is at most $p^{c(c-1)}$ and hence $\exp(M(G))$ divides $\exp(G)$ when $G$ is a $p$-group of class 2. In 1987, Lubotzky and Mann [6] proved that $\exp(M(G))$ divides $\exp(G)$ when $G$ is a powerful $p$-group. A result of Ellis [3] shows that if $G$ is a $p$-group of class $k \geq 2$ and exponent $p^e$, then $\exp(M(G)) \leq p^{\lceil k/2 \rceil}$, where $\lceil k/2 \rceil$ denotes the smallest integer $n$ such that $n \geq k/2$. Moravec [8] showed that $\lceil k/2 \rceil$ can be replaced by $2 \log_2 k$ which is an improvement if $k \geq 11$. He [8] also proved that if $G$ is a metabelian group of exponent $p$, then $\exp(M(G))$ divides $p$. Kayvanfar and Sanati [5] proved that if $G$ is a $p$-group, then $\exp(M(G))$ divides $\exp(G)$ when $G$ is a finite $p$-group of class 3, 4 or 5 with some conditions. The authors [7] extended the result and proved that $\exp(M(G))$ divides $\exp(G)$ when $G$ is a finite $p$-group of class at most $p - 1$.

On the other hand, Ellis [2] proved that $\exp(M(G, N))$ divides $|N|$ for any pair $(G, N)$ of finite groups, in which $|N|$ denotes the order of $N$. Now a question that can naturally arise, is whether $\exp(M(G, N))$ divides $\exp(N)$ when $N$ is a proper normal subgroup of $G$. In this paper, first we present an example to give a negative answer to the question. Second, we give some conditions under which the exponent of $M(G, N)$ divides the exponent of $N$.

In Sec. 2, we give an upper bound for $\exp(M(G, N))$ in terms of $\exp(N)$, when $(G, N)$ is a pair of finite $p$-groups such that $N$ admits a complement in $G$, and apply it to prove that if $(G, N)$ is a pair of finite $p$-groups of class at most $p - 1$ (i.e. $|N, p-1G| = 1$), then $\exp(M(G, N))$ divides $\exp(N)$. Finally in Sec. 3, we show that if $(G, N)$ is a pair of finite $p$-groups and $N$ is powerfully embedded in $G$, then $\exp(M(G, N))$ divides $\exp(N)$.

2. Nilpotent Pairs of $p$-Groups

Macdonald and Wamsley [1] gave an example which shows that $\exp(M(G, G))$ does not divide $\exp(G)$, in general. The following example shows that $\exp(M(G, N))$ does not divide $\exp(N)$ when $N$ is a proper normal subgroup of $G$.

Example 1. Let $D = A >\triangleleft \langle x_1 \rangle$, where $A = \langle x_2 \rangle \times \langle x_3 \rangle \times \langle x_4 \rangle \times \langle x_5 \rangle \cong \mathbb{Z}_4 \times \mathbb{Z}_4 \times \mathbb{Z}_4 \times \mathbb{Z}_2$ and $x_1$ is an automorphism of order 2 of $A$ acting in the following way:

$[x_2, x_1] = x_2^2, \quad [x_3, x_1] = x_3^2, \quad [x_4, x_1] = x_4^2, \quad [x_5, x_1] = 1$.

There exists an automorphism $a$ of $D$ of order 4 acting on $D$ as follows:

$[x_1, a] = x_3, \quad [x_2, a] = x_2^2x_3^2x_4^2, \quad [x_3, a] = x_5, \quad [x_4, a] = x_2^2, \quad [x_5, a] = x_3^2$. 

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Form $N = D > \langle a \rangle$ and put $G = N > \langle b \rangle$, where $b^2 = 1$ and $[x_1, b] = x_2$, $[x_2, b] = x_2^2x_4x_5$, $[x_3, b] = x_4$, $[x_4, b] = x_2^2x_4^2$, $[x_5, b] = x_2^3x_4^2$, $[a, b] = x_1$. Moravec [8] showed that the group $G$ is a nilpotent group of class 6 and exponent 4 and $M(G) \cong \mathbb{Z}_2 \times \mathbb{Z}_4 \times \mathbb{Z}_2 \times \mathbb{Z}_2$. Ellis [2] proved that if $G = K > \langle q \rangle$, then $M(G) \cong M(G, K) \oplus M(Q)$. This implies that $M(G, N) \cong M(G)$ which does not divide $\exp(M(G, N)) = 8$. Therefore $\exp(M(G, N)) = 4$.

Here we first give an upper bound for the exponent of $M(G, N)$ in terms of the exponent of $N$, when $(G, N)$ is a pair of finite $p$-groups such that $N$ admits a complement in $G$. Since our proof relies on commutator calculations, we need to state the following lemmas.

**Lemma 2.1 ([9])**. Let $x_1, x_2, \ldots, x_r$ be any elements of a group and $\alpha$ be a non-negative integer. Then

$$\left( x_1x_2 \ldots x_r \right)^\alpha = x_1^\alpha x_2^\alpha \cdots x_r^\alpha f_1(\alpha)^{f_2(\alpha)} \cdots,$$

where $\{i_1, i_2, \ldots, i_r\} = \{1, 2, \ldots, r\}$ and $v_1, v_2, \ldots$ are commutators of weight at least two in the letters $x_i$’s in ascending order and

$$f_i(\alpha) = a_1 \left( \frac{\alpha}{1} \right) + a_2 \left( \frac{\alpha}{2} \right) + \cdots + a_w \left( \frac{\alpha}{w} \right), \quad (2.1)$$

with $a_1, \ldots, a_w \in \mathbb{Z}$ and $w_i$ is the weight of $v_i$ in elements $x_1, \ldots, x_r$.

**Lemma 2.2 ([9])**. Let $\alpha$ be a fixed integer and $G$ be a nilpotent group of class at most $k$. If $b_1, \ldots, b_r \in G$ and $r < k$, then

$$[b_1, \ldots, b_{i-1}, b_i^\alpha, b_{i+1}, \ldots, b_r] = [b_1, \ldots, b_r]^\alpha v_1^{f_1(\alpha)} v_2^{f_2(\alpha)} \cdots,$$

where $v_1, v_2, \ldots$ are commutators in $b_1, \ldots, b_r$ of weight strictly greater than $r$, and every $b_j$, $1 \leq j \leq r$, appears in each commutator $v_i$. The $f_i(\alpha)$ are of the form (2.1), with $a_1, \ldots, a_w \in \mathbb{Z}$ and $w_i$ is the weight of $v_i$ (in $b_1, \ldots, b_r$) minus $(r - 1)$.

It is noted by Struik [9] that the above lemma can be proved similarly if $[b_1, \ldots, b_{i-1}, b_i^\alpha, b_{i+1}, \ldots, b_r]$ and $[b_1, \ldots, b_r]$ are replaced by arbitrary commutators (that is monomial commutators with parentheses arranged arbitrarily).

To prove the main results we require the following notions.

**Definition 2.3.** A relative central extension of a pair $(G, N)$ of groups consists of a group homomorphism $\sigma : M \to G$ together with an action of $G$ on $M$ such that

(i) $\sigma(M) = N$;

(ii) $\sigma(m^g) = g^{-1}\sigma(m)g$, for all $m \in M, g \in G$;

(iii) $m^{\sigma(m)} = m_1^{-1}mm_1$, for all $m \in M, g \in G$;

(iv) $G$ acts trivially on ker $\sigma$.

Let $(G, N)$ be a pair of groups and $\sigma : M \to G$ be a relative central extension of $(G, N)$. The $G$-commutator subgroup of $M$ is defined the subgroup $[M, G]$ generated
by all the $G$-commutators $[m, g] = m^{-1}m^g$, where $m^g$ is the action of $g$ on $m$, for all $g \in G$, $m \in M$. Also for all positive integer $n$, we define

$$Z_n(M, G) = \{m \in M \mid [m, g_1, g_2, \ldots, g_n] = 1, \text{ for all } g_1, g_2, \ldots, g_n \in G\},$$

in which $[m, g_1, g_2, \ldots, g_n]$ denotes $\cdots[[m, g_1], g_2], \ldots, g_n]$. It is easy to see that $Z_n(M, G) \subseteq Z_{n+1}(M, G)$.

Let $(G, N)$ be a pair of groups and $k$ be a positive integer. We define $\gamma_k(N, G) = [N, kG]$ in which $[N, kG] = \cdots[[N, G], G], \ldots, G]$. A pair $(G, N)$ of groups is called nilpotent of class $k$ if $\gamma_k(N, G) = 1$ and $\gamma_k(N, G) \neq 1$. It is clear that any pair of finite $p$-groups is nilpotent.

**Definition 2.4.** A relative central extension $\sigma : N^* \rightarrow G$ of a pair $(G, N)$ is called a covering pair if there exists a subgroup $A$ of $N^*$ such that

(i) $A \leq Z(N^*, G) \cap [N^*, G]$;

(ii) $A \cong M(G, N)$;

(iii) $N \cong N^*/A$.

Ellis proved that any pair of finite groups has at least one covering pair [2, Theorem 5.4].

Hereafter in this section, we suppose that $(G, N)$ is a pair of finite groups and $K$ is the complement of $N$ in $G$. Also, suppose that $\sigma : N^* \rightarrow G$ is a covering pair of $(G, N)$ with a subgroup of $A$ of $N^*$ such that $A \leq Z(N^*, G) \cap [N^*, G]$, $A \cong M(G, N)$ and $N \equiv N^*/A$. Then for all $k \in K$, the homomorphism $\psi_k : N^* \rightarrow N^*$ defined by $n^k \rightarrow n_k^k$ is an automorphism of $N^*$ in which $n_k^k$ is induced by the action of $G$ on $N^*$. Considering the homomorphism $\psi : K \rightarrow \text{Aut}(N^*)$ given by $\psi(k) = \psi_k$ for all $k \in K$, we form the semidirect product of $N^*$ by $K$ and denote it by $G^* = N^*K$. Then it is easy to check that the subgroups $[N^*, G]$ and $Z(N^*, G)$ are contained in $[N^*, G^*]$ and $Z(N^*, G^*)$, respectively. If $\delta : G^* \rightarrow G$ is the map given by $\delta(n^k) = \sigma(n^k)$, for all $n^k \in N^*$ and $k \in K$, then $\delta$ is an epimorphism with $\ker\delta = \ker\sigma$.

**Lemma 2.5.** By the above notation, let $(G, N)$ be a nilpotent pair of finite groups of class $k$ and $\exp(N) = p^e$. Then every commutator of weight $w$ ($w \geq 2$) in $[N^*, w\cdot 1_G^*]$ has an order dividing $p^{e+m(k+1-w)}$, where $m = \lfloor\log_p k\rfloor$.

**Proof.** We use reverse induction on $w$ to prove the lemma. Since $(G, N)$ is nilpotent of class $k$ and $N \cong N^*/A$, $G \cong G^*/A$ and $A \leq Z(N^*, G^*)$, we have $[N^*, k+1G^*] = 1$. On the other hand, $\exp(N) = p^e$ implies that $[N^*, p^eG^*] = 1$. Hence the result follows for $w \geq k + 1$ by Lemma 2.2. Now assume that $l < k + 1$ and the result is true for all $w > l$. We will prove the result for $l$. Put $\alpha = p^{e+m(k+1-l)}$ with...
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$m = \lfloor \log_p k \rfloor$ and let $u = [n, x_2, \ldots, x_l]$ be a commutator of weight $l$, where $n \in N^*$ and $x_2, \ldots, x_l \in G^*$. Then by Lemma 2.2, we have

$$[n^n, x_2, \ldots, x_l] = [n, x_2, \ldots, x_l]^n v_1^{f_1(\alpha)} v_2^{f_2(\alpha)} \cdots,$$

where $v_i$ is a commutator on $n, x_2, \ldots, x_l$ of weight $w_i$ such that $l < w_i \leq k + 1$, and $f_i(\alpha) = a_1(i_1) + a_2(i_2) + \cdots + a_k(i_k)$, where $k_i = w_i - l + 1 \leq k$, for all $i \geq 1$.

One can easily check that $p^l$ divides $(p^{e_m})$ with $m = \lfloor \log_p k \rfloor$, for any prime $p$ and any positive integers $t, s$ with $s \leq k$. This implies that $p^{e+m(k-l)}$ divides the $f_i(\alpha)$'s and so by induction hypothesis $v_i^{f_i(\alpha)} = 1$, for all $i \geq 1$. On the other hand, it is clear that $[n^n, x_2, \ldots, x_l] = 1$. Therefore $w^\alpha = 1$ and this completes the proof. $\square$

**Theorem 2.6.** If $(G, N)$ is a nilpotent pair of finite groups of class $k$ and $N$ is a $p$-group of exponent $p^e$, then exp($[N^*, G^*]$) divides $p^{e+m(k-1)}$, where $m = \lfloor \log_p k \rfloor$.

**Proof.** Every element $g \in [N^*, G^*]$ can be expressed as $g = y_1 y_2 \cdots y_n$, where $y_i = [n_i, g_i]$ for $n_i \in N^*, g_i \in G^*$. Put $\alpha = p^{e+m(k-1)}$. By Lemma 2.1, we have

$$g^\alpha = y_1^{\alpha_1} y_2^{\alpha_2} \cdots y_n^{\alpha_n} v_1^{f_1(\alpha)} v_2^{f_2(\alpha)} \cdots,$$

where $\{i_1, i_2, \ldots, i_n\} = \{1, 2, \ldots, n\}$ and $v_i$ is a basic commutator of weight $w_i$ in $y_1, y_2, \ldots, y_n$, with $2 \leq w_i \leq k$, for all $i \geq 1$, and also $f_i(\alpha)$ is of the form (2.1). Hence by an argument similar to the proof of Lemma 2.5 $p^{e+m(k-2)}$ divides $f_i(\alpha)$. Then applying Lemma 2.5, we have $v_i^{f_i(\alpha)} = 1$, for all $i \geq 1$, and $y_j^{\alpha} = 1$, for all $j$, $1 \leq j \leq n$. We therefore have $g^\alpha = 1$ and the desired result follows. $\square$

An upper bound for the exponent of the Schur multiplier of some pairs of finite groups is given in the following theorem.

**Theorem 2.7.** Let $(G, N)$ be a nilpotent pair of finite groups of class $k$ such that exp($N$) = $p^e$. Then exp($M(G, N)$) is a divisor of $p^{e+m(k-1)}$, where $m = \lfloor \log_p k \rfloor$.

**Proof.** The result follows by Theorem 2.6 and the fact that $M(G, N) \cong A \leq [N^*, G] \leq [N^*, G^*]$. $\square$

The following corollary gives a condition under which the exponent of the Schur multiplier of a pair $(G, N)$ divides the exponent of $N$.

**Corollary 2.8.** Let $(G, N)$ be a pair of finite $p$-groups of class at most $p - 1$. Then exp($M(G, N)$) divides exp($N$).

**Remark 2.9.** Let $G$ be a finite $p$-group of class $k$ with exp($G$) = $p^e$. Since $M(G, G) = M(G)$, Theorem 2.7 implies that exp($M(G)$) divides $p^{e+\lfloor \log_p k \rfloor(k-1)}$. It is easy to see that this bound improves the bound $p^{(2e+\lfloor \log_p k \rfloor))}$ given by Moravcev [8]. For example for any $p$-group $G$ of class $k$, $2 \leq k \leq p - 1$ with exp($G$) = $p^e$, we have $p^{e+\lfloor \log_p k \rfloor(k-1)} \leq p^{(2e+\lfloor \log_p k \rfloor)}$. 1350053-5
Remark 2.10. Let \((G, N)\) be a pair of finite nilpotent groups of class at most \(k\). Let \(S_1, S_2, \ldots, S_n\) be all the Sylow subgroups of \(G\). By [2, Corollary 1.2], we have
\[
M(G, N) = M(S_1, S_1 \cap N) \times \cdots \times M(S_n, S_n \cap N).
\]
Put \(m_i = \lfloor \log_p k \rfloor\), for all \(i\), \(1 \leq i \leq n\). Then by Theorem 2.7, we have
\[
\exp(M(G, N)) \bigg| \prod_{i=1}^{n} p_i^{e_i + m_i(k-1)},
\]
where \(p_i^{e_i} = \exp(S_i)\).

3. Pairs of Powerful \(p\)-Groups

In 1987, Lubotzky and Mann [6] defined powerful \(p\)-groups which are used for studying \(p\)-groups. They gave some bounds for the order, the exponent and the number of generators of the Schur multiplier of a powerful \(p\)-group. Also, they showed that \(\exp(M(G))\) divides \(\exp(G)\) when \(G\) is a powerful \(p\)-group. The purpose of this section is to show that if \((G, N)\) is a pair of finite \(p\)-groups and \(N\) is powerfully embedded in \(G\), then the exponent of \(M(G, N)\) divides the exponent of \(N\). Throughout this section \(\Omega_i(G)\) denotes the subgroup of \(G\) generated by all \(p^i\)th powers of elements of \(G\). It is easy to see that \(\Omega_i+j(G) \subseteq \Omega_i(\Omega_j(G))\), for all positive integers \(i, j\).

Definition 3.1. (i) A \(p\)-group \(G\) is called powerful if \(p\) is odd and \(G' \leq \Omega_1(G)\), or \(p = 2\) and \(G' \leq \Omega_2(G)\).

(ii) Let \(G\) is a \(p\)-group and \(N \leq G\). Then \(N\) is powerfully embedded in \(G\) if \(p\) is odd and \([N, G] \leq \Omega_1(N)\), or \(p = 2\) and \([N, G] \leq \Omega_2(N)\).

Any powerfully embedded subgroup is itself a powerful \(p\)-group and must be normal in the whole group. Also a \(p\)-group is powerful exactly when it is powerfully embedded in itself. While it is obvious that factor groups and direct products of powerful \(p\)-groups are powerful, this property is not subgroup-inherited [6]. The following lemma gives some properties of powerful \(p\)-groups.

Lemma 3.2 ([6]). The following statements hold for a powerful \(p\)-group \(G\).

(i) \(\gamma_i(G), G', \Omega_i(G), \Phi(G)\) are powerfully embedded in \(G\).

(ii) \(\Omega_i(\Omega_j(G)) = \Omega_{i+j}(G)\).

(iii) Each element of \(\Omega_i(G)\) can be written as \(a^{p^i}\), for some \(a \in G\) and hence \(\Omega_i(G) = \{g^{p^i} : g \in G\}\).

(iv) If \(G = \langle a_1, a_2, \ldots, a_d \rangle\), then \(\Omega_i(G) = \langle a_1^{p^i}, a_2^{p^i}, \ldots, a_d^{p^i} \rangle\).

Lemma 3.3 ([6]). Let \(N\) be powerfully embedded in \(G\). Then \(\Omega_i(N)\) is powerfully embedded in \(G\).

The proof of the following lemma is straightforward.
Lemma 3.4. Let $M$ and $G$ be two groups with an action of $G$ on $M$. Then for all $m, n \in M$, $g, h \in G$, and any integer $k$ we have the following equalities:

(i) $[mn, g] = [m, g]^n[n, g]$;
(ii) $[m, gh] = [m, h][m, g]^h$;
(iii) $[m^{-1}, g]^{-1} = [m, g]^{-1}$;
(iv) $[m, g^{-1}]^{-1} = [m, g]^{-1}$;
(v) $[m^{-1}, h]g[m, [g, h^{-1}]]h^{-1}[m^{-1}, h^{-1}, g]m = 1$;
(vi) $[m^{k}, g] = [m, g]^k[m, g, m]^{k(k-1)/2} \pmod{[M, 3G]}$.

Lemma 3.5. Let $(G, N)$ be a pair of finite $p$-groups and $\sigma : N^* \to G$ be a relative central extension of $(G, N)$. Suppose that $M$ and $K$ are two normal subgroups of $N^*$. Then $M \leq K$ if $M \leq K[M, G]$.

Proof. Applying Lemma 3.4 we have

$$M \leq K[M, G] \leq K[K[M, G], G] \leq K[K, G][M, G, G] \leq \cdots \leq K[M, G]_i,$$

for all $i \geq 1$. On the other hand, since $G$ is a finite $p$-group, there exists an integer $l$ such that $[N, lG] = 1$. Hence $[N^*, lG] = 1$ and the result follows.

Lemma 3.6. Let $(G, N)$ be a pair of finite $p$-groups and $\sigma : N^* \to G$ be a relative central extension of $(G, N)$. Let $M$ be a normal subgroup of $H$. Then the following statements hold.

(i) If $p > 2$, then $\mathcal{U}_1(M, G) \subseteq \mathcal{U}_1([M, G])[M, 3G]$.
(ii) If $p = 2$, then $\mathcal{U}_2(M, G) \subseteq \mathcal{U}_2([M, G])\mathcal{U}_1([M, 2G])[M, 3G]$.

Proof. (i) It is enough to show that $[m^p, g] \in \mathcal{U}_1([M, G])[M, 3G]$, for all $m \in M, g \in G$. By Lemma 3.4 $[m^p, g] = [m, g]^p[m, g, m]^{p(p-1)/2} \pmod{[M, 3G]}$.

Since $p$ is odd and $p \equiv 1 \pmod{2}$ we have $[m, g]^p[m, g, m]^{p(p-1)/2} \in \mathcal{U}_1([M, G])$.

Now the result holds.

(ii) The proof is similar to (i).

Lemma 3.7. Let $(G, N)$ be a pair of finite $p$-groups and $\sigma : N^* \to G$ be a relative central extension of $(G, N)$. Suppose that $K \leq N^*$. Then the following statements hold.

(i) If $p > 2$, then $[K, G] \leq \mathcal{U}_1(K)$ if and only if $[K/[K, 2G], G] \leq \mathcal{U}_1(K/[K, 2G])$.
(ii) If $p = 2$, then $[K, G] \leq \mathcal{U}_2(K)$ if and only if $[K/[K, 2G], G] \leq \mathcal{U}_2(K/[K, 2G])$.
(iii) If $p = 2$, then $[K, G] \leq \mathcal{U}_2(K)$ if and only if $[K/\mathcal{U}_1([K, G]), G] \leq \mathcal{U}_2(K/\mathcal{U}_1([K, G]))$.

Proof. (i) Let $[K, G] \leq \mathcal{U}_1(K)$ and put $H = [K, 2G]$. Then

$$\left[ \frac{K}{H}, G \right] \leq \mathcal{U}_1(K)H \leq \mathcal{U}_1(K) = \mathcal{U}_1 \left( \frac{K}{H} \right),$$

as desired. Sufficiency follows by Lemma 3.5.
(ii) The proof is similar to (i).

(iii) Necessity follows as for (i). Let \([K/\mathcal{U}_1([K, G]), G] \leq \mathcal{U}_2(K/\mathcal{U}_1([K, G])). Then \([K, G] \leq \mathcal{U}_2(K/\mathcal{U}_1([K, G]))\). On the other hand, \(\mathcal{U}_1([K/\mathcal{U}_1([K, G]), G])\). Thus \([K/\mathcal{U}_1([K, G]), G]\) is abelian and so \(\Phi([K/\mathcal{U}_1([K, G]), G]) = 1\). This implies that \(\Phi([K, G]) = \mathcal{U}_1([K, G]).\) Therefore \([K, G] \leq \mathcal{U}_2(K)\). □

The following useful remark is a consequence of Lemma 3.7.

**Remark 3.8.** Let \((G, N)\) be a pair of finite \(p\)-groups and \(\sigma : N^* \to G\) be a relative central extension of \((G, N)\). Let \(K \leq N^*\). Then to prove that \([K, G] \leq \mathcal{U}_1(K)\) \(([K, G] \leq \mathcal{U}_2(K)\) for \(p = 2\)) we can assume that

(i) \([K, 2G] = 1\);

(ii) \(\mathcal{U}_1(K) = 1 \) \((\mathcal{U}_2(K) = 1 \) for \(p = 2\)) and try to show that \([K, G] = 1\);

(iii) \(\mathcal{U}_1([K, G]) = 1\) whenever \(p = 2\).

**Lemma 3.9.** Let \((G, N)\) be a pair of finite \(p\)-groups and \(\sigma : N^* \to G\) be a covering pair of \((G, N)\). Let \(N\) be powerfully embedded in \(G\).

(i) If \(p > 2\), then \([\mathcal{U}_n([N^*, G]), G] \leq \mathcal{U}_1(\mathcal{U}_n([N^*, G]))\).

(ii) If \(p = 2\), then \([\mathcal{U}_n([N^*, G]), G] \leq \mathcal{U}_2(\mathcal{U}_n([N^*, G]))\).

**Proof.** \(N^*\) has a subgroup \(A\) such that \(A \leq Z(N^*, G) \cap [N^*, G], A \cong M(G, N)\) and \(N \cong N^*/A\).

(i) Let \(p > 2\). We use induction on \(n\). If \(n = 0\), then by Remark 3.8 we may assume that \([N^*, G], G, G] = 1, \mathcal{U}_1([N^*, G]) = 1\) and we should show that \([N^*, G], G] = 1\). Since \(N\) is powerfully embedded in \(G\), we have \([N, G] \leq \mathcal{U}_1(N)\), and therefore \([N^*, G] \leq \mathcal{U}_1(N^*)A\). Now we claim that \(\mathcal{U}_1(N^*) \leq Z(N^*, G)\). To prove the claim, let \(a \in N^*\) and \(b \in G\). Since \(\gamma_3((a, [N^*, G])) = 1\), we have \(c((a, a^p)) \leq c((a, [N^*, G])) \leq 2 \) \((c(H)\) denotes the nilpotency class of \(H)\). On the other hand, Lemma 3.4 implies that

\[(a^p)^b = a^p[a^p, b] \equiv a^p[a, b]^p[a, b, a]^{p(p-1)/2} \mod ([a, b]G].\]

Therefore \((a^p)^b = a^p\) since \([N^*, G], G, G] = 1\) and \(\mathcal{U}_1([N^*, G]) = 1\). Hence \(\mathcal{U}_1(N^*) \leq Z(N^*, G)\) as desired. Thus \([N^*, G] \leq \mathcal{U}_1(N^*)A \leq Z(N^*, G)\) and the result follows for \(n = 0\).

Now suppose that the induction hypothesis is true for \(n = k\). The first step of induction implies that \([N^*, G]\) is powerful. Using Lemmas 3.5 and 3.6, one can see that if \(H\) is a subgroup of \(N^*\) and \([H, G] \leq \mathcal{U}_1(H)\), then \([\mathcal{U}_1(H), G] \leq \mathcal{U}_1(\mathcal{U}_1(H))\). Hence by Lemma 3.2 and induction hypothesis we have

\([\mathcal{U}_{k+1}([N^*, G]), G] = [\mathcal{U}_1(\mathcal{U}_k([N^*, G])), G] \leq \mathcal{U}_1(\mathcal{U}_1(\mathcal{U}_k([N^*, G])))\]

which completes the proof.
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(ii) Let $p = 2$. The proof is similar to (i), but we need to prove that if $H$ is a subgroup of $N^*$ and $[H,G] \leq \mathcal{U}_2(H)$, then $[\mathcal{U}_1(H),G] \leq \mathcal{U}_2(\mathcal{U}_1(H))$. By Remark 3.8, for $a \in H, b \in G$ we have $[a^4, b] = [a^2, b]^2 = 1$. So $a^4 \in Z(H,G)$ and $\mathcal{U}_2(H) \leq Z(H,G)$. Then $[H,G] \leq \mathcal{U}_2(H) \leq Z(H,G)$. Therefore $[a^2, b] = [a, b]^2$ and

$$[\mathcal{U}_1(H),G] = \mathcal{U}_1([H,G]).$$

(3.1)

On the other hand, since $\mathcal{U}_2(H) \leq Z(H,G)$, we have

$$\mathcal{U}_1(\mathcal{U}_2(H)) = \langle (a_1^4 \cdots a_k^4) | a_i \in H \rangle = \langle a_1^8 \cdots a_k^8 \rangle = \mathcal{U}_3(H) \leq \mathcal{U}_2(\mathcal{U}_1(H)).$$

Hence (3.1) implies that $[\mathcal{U}_1(H),G] \leq \mathcal{U}_2(\mathcal{U}_1(H))$ which completes the proof of the above claim. \hfill \Box

Lemma 3.10. Let $H$ and $G$ be two arbitrary groups with an action of $G$ on $N$. If $x \in H$ and $g \in G$, then

$$[x^n,g] = [x,g]^n c,$$

where $M = \langle x, [x,g] \rangle$ and $c \in \gamma_2(M)$.

Proof. Applying Lemma 2.1, we have

$$[x^n,g] = (x^n)^{-1} (x^n)^g = (x)^{-n} (x^g)^n = (x)^{-n} (x[g]^n c),$$

where $M = \langle x, [x,g] \rangle$, $c \in \gamma_2(M)$. \hfill \Box

Now we can state the main result of this section.

Theorem 3.11. Let $(G,N)$ be a pair of finite p-groups in which $N$ is powerfully embedded in $G$. Then $\exp(M(G,N))$ divides $\exp(N)$.

Proof. Let $p > 2$ and $\sigma : N^* \to G$ be a covering pair of $(G,N)$ with a subgroup $A$ such that $A \leq Z(N^*,G) \cap [N^*,G]$, $A \cong M(G,N)$ and $N \cong N^*/A$. It is enough to show that $\exp([N^*,G]) = \exp(N^*/Z(N^*,G))$. For this we use induction on $k$ and show that

$$\mathcal{U}_k([N^*,G]) = \mathcal{U}_k(N^*,G).$$

(3.2)

If $k = 0$, then (3.2) holds. Now assume that (3.2) holds, for $k = n$. Working in powerful $p$-group $N^*/A$ we get $\mathcal{U}_{n+1}(N^*/A) = \mathcal{U}_1(\mathcal{U}_n(N^*/A))$ by Lemma 3.2. Hence

$$\frac{\mathcal{U}_{n+1}(N^*) A}{A} = \frac{\mathcal{U}_1(\mathcal{U}_n(N^*) A)}{A} = \frac{\mathcal{U}_1(\mathcal{U}_n(N^*) A) A}{A}.$$
Then Lemmas 3.6 and 3.9 and induction hypothesis imply that

$$[U_{n+1}(N^*), G] = [U_1(U_n(N^*)A), G] \leq U_1([U_n(N^*)A], G)$$

$$\leq U_1([U_n(N^*)A], 2G)$$

$$\leq U_1([U_n([N^*, G]), G]) = U_{n+1}([N^*, G]).$$

For the reverse inclusion, we show that

$$U_{n+1}([N^*, G]) \equiv 1 \pmod{U_{n+1}(N^*, G)}.$$

Since by (3.4), $$[U_{n+1}(N^*), G] = U_1(U_n(N^*)A, G) = U_1(U_n(N^*)A, G),$$ it follows that $$U_1(U_n(N^*)A) \leq Z(N^*, G) \pmod{U_{n+1}(N^*, G)}.$$

On the other hand, since $$N$$ is powerfully embedded in $$G$$, we have

$$[U_n(N^*), G] = [U_n(N^*)A, G] \leq U_1(U_n(N^*)A)$$

$$\leq Z(N^*, G) \pmod{U_{n+1}(N^*, G)}.$$

Therefore $$[U_n(N^*), G] \equiv 1 \pmod{U_{n+1}(N^*, G)}.$$

Moreover, by Lemma 3.10 we have

$$[U_1(U_n(N^*)A), G][U_n(N^*), G, G] = U_1([U_n(N^*), G][U_n(N^*), G, G]).$$

It follows that $$U_1([U_n(N^*), G]) \equiv 1 \pmod{U_{n+1}(N^*, G)}.$$ Then by induction hypothesis, we have

$$U_{n+1}([N^*, G]) = U_1(U_n[N^*, G]) = U_1([U_n(N^*), G]) \equiv 1 \pmod{U_{n+1}(N^*, G)}.$$

This completes the proof for odd primes $$p$$. The proof for the case $$p = 2$$ is similar.

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