Quantum mechanics in terms of realism

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Abstract. We expound an alternative to the Copenhagen interpretation of the formalism of nonrelativistic quantum mechanics. The basic difference is that the new interpretation is formulated in the language of epistemological realism. The $\psi$ function is no longer interpreted as a probability amplitude of the observed behaviour of elementary particles but as an objective physical field representing the particles themselves. The particles are thus extended objects whose extension varies in time according to the variation of $\psi$. They are considered as fundamental regions of space with some kind of nonlocality. There is no wave-particle duality. The point-particle-like behaviour is explained by spatial contraction in a deterministic reduction process.

Key words: foundations of quantum mechanics, interpretation, realism, wavepackets, measurement, reduction, collapse, entanglement, nonlocality, Einstein-Podolsky-Rosen problem, Bell inequality

*The present version is another thorough revision of the 1996 version, taking experimental and theoretical progress into account.*
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1. REALISM AND WAVE ASPECT

1.1 Difficulties in Present-Day Quantum Theory

There is no doubt that quantum theory is one of the most successful physical theories. Yet there is also no doubt that it contains serious difficulties. These difficulties are nowadays felt more and more strongly by those concerned with the unification of quantum theory and relativity and the future basis of physics. The difficulties may be divided into two kinds: conceptual and mathematical.

The conceptual difficulties are related to the so-called Copenhagen interpretation. Any physical theory consists of a mathematical formalism, that is, a set of mathematical symbols and the rules for connecting these among themselves, and a set of interpretation rules, connecting the symbols of the mathematical formalism with the concepts of our sensory experience. The Copenhagen interpretation represents that set of interpretation rules that is presented more or less explicitly in current textbooks on quantum mechanics. Actually, it is difficult to say who exactly constitutes the “Copenhagen school”, supporting the Copenhagen interpretation; certainly Bohr and Heisenberg, but also Dirac, Pauli and von Neumann [1], [2]. Also, many versions of “the Copenhagen interpretation”, from orthodox to liberal, can be found when different authors or textbooks are consulted.

The difficulties of the Copenhagen interpretation may be characterized in the following way:

1) The wave function \( \psi(x, t) \) is not taken as an objective physical field like the electromagnetic field, but as a probability amplitude. And the probabilities to which it refers are not the probabilities that something is true or something will happen, whether it is observed or not (as in statistical mechanics) but the probabilities of specified outcomes of measurements or observations. Most importantly, the observer is not just another physical object but the linguistic ego, something which appears nowhere as a mathematical symbol in the formalism. An electron does not have an exact location as long as we do not observe it, but it does have one when we observe it. “The ‘trajectory’ arises only by our observing it” [3]. In this way the Copenhagen interpretation speaks of position and of momentum, angular momentum components and others only as “observables”, not as real properties which objects have regardless of whether we observe them or not. The observer and the measurement are therefore indispensable elements in defining the theory. In all other physical theories the observer’s only function is to test and apply the theory, not to define it. The Copenhagen interpretation thus rejects the language of epistemological realism. In my opinion, this is the most serious difficulty with that interpretation.

2) According to the Copenhagen interpretation it is impossible, in principle, to explain the probabilistic behaviour in quantum physics as the result of some
underlying deterministic processes. This means that the Copenhagen interpretation rejects determinism. This point will be examined further in Chap. 2.

3) In the Copenhagen view concepts that refer to pointlike particles (with single, sharp position) are applied to micro-objects, whereas the $\psi$ function is the solution of a field equation, namely a partial differential equation, like Schrödinger's, and may show up wave-like behaviour. This has been called the “wave-particle duality”, and it has been asserted that a unifying picture of micro-objects cannot exist.

As is well known Einstein, von Laue, Schrödinger, Planck, and de Broglie never accepted the Copenhagen interpretation. In fact, any really thoughtful scholar finds it difficult to digest. Inquiring students are usually silenced by authoritarian statements such as that they do not yet know enough and will understand later, or that their questions are not relevant to physics.

In particular, the defenders of the Copenhagen interpretation assert that a unifying picture is really unnecessary. They say that the formalism plus some working rules for its application give us the correct prescriptions for calculating the probabilities of the outcomes of any experiment, and that that is all we want. But as if some tectonic tensions were felt, discussions concerning the foundations of quantum mechanics continue. Moreover, it is noticeable that in the course of historical development as well as in the mind of any particular scientist the final mathematical formalism describing a set of physical phenomena emerges from a more or less pictorial view, conception or model. A good picture is very helpful since it has the same logical structure as the region of reality which it aims to represent, and it leads to a correct mathematical formulation of this reality. An example is Faraday’s intuitive picture of lines of force and their subsequent mathematical formulation by Maxwell. “It is mainly with the hope of making these [Faraday’s] ideas the basis of a mathematical method that I have undertaken this treatise”, Maxwell writes [5]. A bad picture leads to no or to an only partially correct formalism. In this latter case it may happen that the emerging formalism describes the known phenomena correctly in its initial stage, but when it is developed further to include more and more experimental facts it sooner or later comes off the track. This is what I think has happened to quantum theory. I think that the lack of a good picture is responsible for the mathematical difficulties.

The present-day mathematical difficulties arise with the attempt to extend non-relativistic quantum mechanics into the relativistic domain, that is, into quantum electrodynamics and relativistic quantum field theory. Here, marvelous successes in nuclear and particle physics and relativistic quantum field theory have been obtained. Nevertheless, divergent integrals have shown up in the perturbation expansions as solutions of the basic equations. Even if these integrals are made finite by means of renormalization procedures or are avoided by means of Epstein-Glaser methods [6], [7] nobody knows whether the expansions converge, and nobody has found an exact solution of the equations including interactions in the real world of 3+1 dimensions, although enormous efforts have been undertaken [8], [9, Sec. 11.1]. Thus Dirac [10], [11] writes:
I feel pretty sure that the changes which will be needed to get over the present difficulties facing quantum theory and appearing as a resistance between the quantum theory and relativity will be very drastic just as drastic as the change from Bohr orbits to the quantum mechanics of Heisenberg and Schrödinger and therefore one should not become too much attached to the present quantum mechanics. One shouldn’t build up ones whole philosophy as though this present quantum mechanics were the last word. If one does that, one is on very uncertain ground and one will in some future time have to change one’s standpoint entirely.

In the present paper we are, however, only concerned with nonrelativistic quantum mechanics.

1.2 Realism

About the nature of the expected changes Einstein writes [12] - [15, p. 667] - [17]. In particular in [16, p. 6]:

But in any case my conception starts from a thesis which is strongly rejected by most present-day theoreticians: There is something like the “real state” of a physical system, which independent of any observation or measurement exists objectively and which can in principle be described by means of physical terms [Which adequate terms or basic concepts have to be employed for this is in my opinion unknown at the present moment (material points? field? concepts that have still to be invented?)]. Because of its “metaphysical” nature, this thesis of reality does not have the purpose of providing a statement of fact: it has really only a programmatic character. However, everybody, including the quantum theoreticians, sticks consistently to this thesis of reality so long as he does not discuss the foundations of quantum theory. Nobody doubts, for example, that there has been at a certain time a certain position of the moon’s center of gravity even if no real or potential observer existed.

This thesis of Einstein’s is what we mean by epistemological realism. We do not attempt to give a fool-proof definition of realism. We emphasize, however, that the type of realism adopted here does not insist that physical objects with their properties exist independently of whether or not we observe them (this would be naive realism); it only means that the laws of nature can be formulated as if that were the case.

Realism is not a matter to be proved or disproved, it is a way of speaking, a language. Compare the language of realism with English, and the Copenhagen language with German. Nobody will deny that everything that can be expressed in German can also be expressed in English, although some things can be expressed in a much shorter and simpler way in the one language than in the other. With realism quantum mechanics is formulated in the same language as classical mechanics and every other physical theory, and the power that lies in a realist language is available for quantum mechanics, too. To quote Wittgenstein [18]:
For this is what disputes between Idealists, Solipsists and Realists look like. The one party attack the normal form of expression as if they were attacking a statement, the others defend it, as if they were stating facts recognized by every reasonable human being.

If we were to distinguish our type of realism from naive realism we would call it epistemological or linguistic realism.

Von Laue [19], Schrödinger [20], [21], [22], and Planck [23], [24] have always shared a realist attitude with Einstein. And in the course of the years the number of physicists who openly advocate realism in quantum theory has continually increased. Jammer’s book [25] already quotes Bohm, Bunge, de Broglie, Jaynes, Ludwig, Popper and Renninger. And we want to add the papers by Janossy [24], de Broglie [26], Bunge and Kahay [27] - [29], Bell [30] - [32], Rayski [33], Lévy-Leblond [34], Stapp [35], [36], Roberts [37], Maxwell [38], Burgos [39], Popper [40], [41], Pearle [42], Bohm, Hiley and Kaloyerou [43], Rohrlich [44], Dorling [45], and Dieks [46]. Actually, it is difficult to do justice to everybody because there are several types of realism, because statements in favour of realism range from very outspoken to rather casual. Scientists I found particularly outspoken in favor of realism are Popper, Bunge, and Bell. Bell [31, p. 40] in particular postulated “beables” to replace the “observables” (Sec. 2.1), and his work will concern us in Chap. 4.

Thus in the present work we show how one can overcome the conceptual difficulties of quantum mechanics by interpreting the formalism in terms of realism. The formalism is: (1) that presented in the usual textbooks [47], [48], plus: (2) a mathematical description of the reduction (collapse) process, introduced in Sec. 2.1 [49]. Actually, since 1996 a number of other “ontological” or realist interpretations have appeared, but according to them they have nothing to do with my work.

The essential point is that the wave function \( \psi(x, t) \) is taken as an objective physical field, comparable in this respect to the function \( F_{\mu \nu}(x, t) \) as the source-free electromagnetic field. This implies that \( \psi \) is not merely a device for calculating the probabilities of specified outcomes of observations; it does not merely describe “knowledge” [50].

It is mainly for convenience of presentation that the nonrelativistic formalism is chosen, with the familiar Schrödinger equation and wave function as the basis of the reinterpretation. We think that the concepts developed will prove fruitful in the relativistic domain as well; at least we do not know of any argument that would point to the contrary. Everything that can be described by the Schrödinger equation can also be described by the Klein-Gordon, Dirac etc. equation. Thus we might as well have used a Lorentz scalar, spinor, vector, etc. instead of the Schrödinger scalar \( \psi(x, t) \). Moreover, we include photons in our considerations, that is, we treat classical electromagnetic radiation pulses on the same footing as pulses of Schrödinger or similar waves.

The realist conception of the wave function already resolves the problems of Wigner’s friend and Schrödinger’s cat. Usually the whole measurement apparatus
consists of a long chain of sub-apparatuses (amplifier, channel analyzer, transmitter ...). In the most orthodox version of the Copenhagen interpretation it is my becoming conscious of the result that completes the measurement. Now, a friend of mine may form a sub-apparatus in that chain in that he, for example, reads off the pointer position on a display and then telephones it to me. The difficulty arises as soon as I credit my friend with the same capabilities as I have because this implies that the result has already obtained in the apparatus due to his being conscious of it. This is essentially the conflict between the Copenhagen description where the observer is the linguistic ego and any realist description, where the observer is just another physical object and the result appears in a physical process that occurs whether it is noticed or not.

Another difficulty with the Copenhagen view is described in the example of Schrödinger’s cat [20, p. 812]. Consider a closed box containing a cat, a certain amount of radioactive nuclei, a Geiger counter and a cat-killing device, all protected against the cat. Circumstances are arranged so that the probability that the Geiger counter discharges at the decay of at least one nucleus within one hour is just 1/2. If the counter discharges it triggers the cat-killing device, which consists of a hammer and a flask of prussic acid. The flask is smashed, the acid is released, and the cat is poisoned. The probability that after one hour the cat is dead is 1/2. Since the box is closed we cannot know after an hour whether the cat is dead or alive, unless we cautiously open the box and look into it.

In orthodox quantum mechanics, where the wave function represents our knowledge, there is one wave function \( \psi_L \) that represents our knowledge that there is a living cat in the box and another function \( \psi_D \) that there is a dead cat, but the situation is not described by the sum of the probabilities but by the superposition of the probability amplitudes

\[
(1.1) \quad c_L \psi_L + c_D \psi_D
\]

in the two-dimensional Hilbert space of dead and living cats. (1.1) is then interpreted as the wave function of neither a dead nor a living cat but a superposition of both. Only when we look into the box a reduction occurs and the cat’s wave function becomes either \( \psi_L \) or \( \psi_D \).

The cat is a macro-object, and in the realm of macro-objects the language of realism is spoken: the cat is either alive or dead even if we do not observe it. The radioactive nucleus is a micro-object, and in the realm of micro-objects the language of realism is forbidden by the verdict of Copenhagen. In the example there is a chain of reactions beginning in the microworld with the decay of the unstable nucleus and ending in the macroworld with the death of the cat. If both the micro- and the macro-object are described by a \( \psi \) function, the character of the \( \psi \) function must change when the chain of reactions crosses the borderline between the two realms. This is another difficulty.

In our interpretation the language of realism is spoken in the microworld as well as in the macroworld, and the character of the \( \psi \) function is always that of a real
physical field representing real physical objects, micro- or macroscopic. The point is however that these objects, when their wavepackets are superposed, must exist at the same time \[^{[51]}\]. Probabilities may refer to different times, but the wavepackets may not. The wavepackets $\psi_L$ and $\psi_D$, on the contrary, exist at different times, $\psi_L$ before and $\psi_D$ after the decay of the radioactive nucleus. No such superposition is met elsewhere in the standard formalism of quantum mechanics, not even for micro_objects.

In fact I do think that Schrödinger considered the cat example in order to point out to what incredible features the superposition (1.1) would lead.

Thus, although the particular superposition (1.1) for the cat is not allowed in realism, this does not mean that there is no superposition at all of wavefunctions representing macro_objects. The restriction is that these wavefunctions must represent something that really exists \textit{at the same time}. Examples are the recent experiments with large molecules and clusters \[^{[52]}\], \[^{[53]}\], if one accepts these to be already macro_objects.

1.3 Classical Wavepackets

The identification of an elementary particle with a field means that quantum mechanics becomes a field theory, albeit a special one. The Schrödinger equation, or any of the quantum equations of motion, in any case is a field equation, that is, a partial differential equation, with the solution $\psi$ depending on the four independent variables $x, y, z$ and $t$. The equations of motion of the point particles of classical mechanics, on the contrary, are ordinary differential equations for the three functions $x(t), y(t)$ and $z(t)$. On this point Einstein \[^{[54]}\] writes:

\begin{quote}
The most difficult point for such a field theory at present is how to include the atomic structure of matter and energy. For the theory in its basic principles is not an atomic one in so far as it operates exclusively with continuous functions of space, in contrast to classical mechanics whose most important feature, the material point, squares with the atomistic structure of matter ... And yet a theory may perfectly well exist, which is in a genuine sense an atomistic one (and not merely on the basis of a particular interpretation), in which there is no localizing of the particles in a mathematical model. For example, in order to include the atomistic character of electricity, the field equations only need to involve that a three-dimensional volume of space on whose boundary the electrical density vanishes everywhere, contains a total electrical charge of an integral amount. Thus in a continuum theory, the atomistic character could be satisfactorily expressed by integral propositions without localizing the particles which constitute the atomistic system. Only if this sort of representation of the atomistic structure be obtained could I regard the quantum problem within the framework of a continuum theory as solved.
\end{quote}

The idea that elementary particles are not pointlike but are extended objects has
repeatedly appeared in the literature. However, the size has always been considered to be fixed, for example equal to the Compton length of the electron $\lambda_C = \frac{\hbar}{mc}$, the classical electron radius $r_{cl} = \frac{e^2}{mc^2}$, or the Planck length $l_P = \left(\frac{\hbar G}{c^3}\right)^{1/2}$. In the interpretation presented here the size of any individual particle is variable, namely equal to the size of the $\psi$ function traditionally associated with it. Mathematically, the $\psi$ function need not have a sharp boundary but for our purposes it may be considered to have the extension given by the usual standard deviation $\Delta x := \langle(x - \langle x \rangle)^2\rangle^{1/2}$, which varies in time according to the variation of $\psi(x, t)$.

![Figure 1: The double-slit experiment.](image)

The most convincing experiments showing the wave aspect of the elementary particles is the double-slit experiment. Consider a beam of electrons with average momentum $p$ and little spread about this value. The beam is directed towards a wall, as shown in Fig. 1. The wall contains two parallel slits, which can be opened and closed. Behind the wall there is a detecting screen which registers the intensity $I(x)$ of the beam (number of electron counts per second) as a function of the distance $x$ from the center O. If only slit 1 is open the intensity function $I_1(x)$ will look like $I_1$; if only slit 2 is open it will look like $I_2$. If both slits are open the intensity function is not, however, the sum $I_1 + I_2$, but will look like $I_{12}$. The shape of $I_{12}$ is obtained simply by regarding the beam of electrons as a plane wave with a wavelength $\lambda = \frac{\hbar}{p}$ and calculating the interference of waves originating from slit 1 and slit 2. These interference effects constitute the difference between $I_{12}$ and $I_1 + I_2$.

The important point is that no matter how low the intensity of the incoming beam, the intensity function on the screen, when both slits are open, is always given by $I_{12}$, provided we compensate for the lower incoming intensity by a longer exposure time in order to have the same total amount of energy (or total number of electrons) deposited on the screen. We may adjust the incident intensity until it is so low that it corresponds to one incoming electron per hour. Thus, even a single electron must correspond to a number of wave trains capable of interference with one another.

The electron covers both slits, so that in any single passage both slits are involved and determine the final interference pattern. Only if the wave function has a transverse width (normal to the direction of its motion) that is smaller than
the distance between the slits will no double-slit interference effects be observed.

In order to emphasize that we consider the elementary particles not as pointlike but as extended objects we call them wavepackets (one word). The function $\psi(x, t)$ then is the mathematical representation of a wavepacket. Sometimes we shall neglect the difference between the wavepacket and its mathematical representation and just call $\psi$ a wavepacket. The term ‘wavepacket’ does not, however, in any way mean a restriction to a linear superposition of plane waves, and even when it is mathematically expressed as such it does not mean that plane waves are physical constituents of the particles. It just means the region(s) of non-vanishing $\psi$. Compare it in this respect with a water drop or a soap bubble. A particle may be regarded as a “matter pulse”, on an equal footing with an electromagnetic radiation pulse. Indeed, in our interpretation the radiation pulse, under certain conditions, is also a particle, namely a photon. For each kind of particle the “matter” field is specified by additional parameters like mass or charge.

1.4 The Heisenberg Relations

The Heisenberg relations

\begin{equation}
\Delta x \Delta p_x \geq \hbar/2
\end{equation}

play a central role in the continuing discussions on the physical meaning of quantum theory. Many different interpretations have been advanced [25], and it is not our intention to review them here. Rather, we shall pick out a few points that serve to clarify the present interpretation.

The derivations of the Heisenberg relations presented in most textbooks consider operators for position and momentum and their Hermiticity and noncommutativity, and the Schwarz inequality for the state vectors. This might suggest that the Heisenberg relations are basic structural features of the quantum mechanical formalism. We do not think so. The relation can also be derived by using only classical wavepackets, as is amply demonstrated by Rapp [55].

This equation as it stands is still a classical field equation [56]. The appearance of the typical quantum constant $\hbar$ in it only indicates the kind of fields to which the equation refers, namely to Schrödinger, de Broglie or matter fields whose characteristic wave quantities $\omega$ and $\lambda$ are related to the particle quantities $E$ and $p$ by the relations $E = \hbar \omega$ and $p = \hbar/\lambda$. Even the special boundary conditions introduced in order to get “quantized” solutions, which can be distinguished from one another by parameters that take on discrete (eigen)values, are not enough to provide a genuine quantum character. The energy eigenvalues in the hydrogen atom, for example, follow from the normalizability postulate. Eigenvalues already appear in classical macro-physics, and it is significant that the title of Schrödinger’s famous papers was “Quantization as a problem of eigenvalues”.

In 1928 Schrödinger said [57]:
One may believe either (1) that matter has really a wave structure. Then the uncertainty principle is an immediate consequence. Or (2) one may think that the uncertainty principle is the more fundamental.

Of course, ours is the first option. Moreover, Dirac in 1972 said [58]:

So if one asks what is the main feature of quantum mechanics, I feel inclined now to say that it is not noncommutative algebra. It is the existence of probability amplitudes which underly all atomic processes.

The importance of the Heisenberg relations stems from the importance of the wavepacket nature of the elementary particles. In the macroworld only the first moment of the function $|\psi(x, t)|^2$, that is, the centre value $\langle x \rangle$ plays a role (Ehrenfest theorem), whereas in the microworld the higher moments come into play. Already the first step, the inclusion of only the second moments $\Delta x$, marks the essential differences from the macroworld. This is why the wavepacket concept captures more characteristics of the elementary particle than does the point particle concept.

The function $\psi(x, t)$ represents all of the wavepacket (particle), whether we know it or not. It need not be an eigenfunction of some operator. Only when the wavepacket passes a specific physical situation (e.g. a magnetic field) it assumes the form of a superposition of eigenfunctions of the associated observable. And in a subsequent collapse (Sec.2.1) it assumes the form of a single eigenfunction. In the case of a Heisenberg relation, different physical situations are necessary for the wavepacket to assume eigenfunctions of different observables. This concords with Bohr’s statement. [15, p. 233]:

As repeatedly stressed, the principal point is here that such measurements demand mutually exclusive experimental arrangements.

Once we are given the shape of the wavepacket (which includes its ranges), no uncertainty is left concerning this packet. We therefore do not speak of uncertainty relations; rather we would speak of complementarity, or just Heisenberg or Born-Heisenberg relations. As $\Delta p_x$ is the width of the wavepacket $\psi(p_x)$ in momentum space, where $\psi(p_x)$ is the Fourier transform of $\psi(x)$ (cf. Appendix A), ‘Fourier reciprocity relation’ would appear the most fitting denomination.

A classical analogy may help: Consider a certain amount of dough of any form. Then shape a tower (“experimental arrangement”). Let its height (“$\Delta x$”) be large. Then its base (“$\Delta p$”) is small. Alternatively shape a pancake. Its height is small, but its base is large. And you cannot shape a tower and a pancake with the same procedure.

The relations (1.2) may be generalized to any two quantities which are represented by noncommuting Hermitean operators $A$ and $B$ [59], [60]

$$\Delta A \Delta B \geq \frac{1}{2} |\langle [A, B] \rangle| > 0,$$

and our realist interpretation is essentially the same as that of the original Heisenberg relations. This interpretation is independent of whether the operators have discrete
or continuous eigenvalues. For example, the relation between the components of angular momentum

$$\Delta l_x \Delta l_y \geq \frac{1}{2}\hbar |\langle l_z \rangle| > 0,$$

means that the particle which is an eigenpacket of $l_z(\neq 0)$ cannot at the same time be an eigenpacket of $l_x$ and $l_y$, rather it has the finite ranges $\Delta l_x$ and $\Delta l_y$. When the same packet in an appropriate physical situation changes into an eigenpacket of $l_x$, say, then its range $\Delta x$ shrinks to zero while it acquires the finite ranges $\Delta l_y$ and $\Delta l_z$. 
2 ELIMINATION OF THE PARTICLE ASPECT

2.1 Reduction and Measurement

So far we have only considered the wave aspect, which does not present any genuine quantum feature. The quantum features come in with the particle aspect. This aspect will now be discussed – and eliminated.

To see why the Copenhagen interpretation holds the concept of classical particles let us return to Fig. 1 in Sec. 1.3, and let us observe the detecting screen behind the wall with the two slits in it. What is observed is that the interference picture $I_{12}$ is built up gradually as time proceeds. It is made of a distribution of pointlike spots whose spatial density finally exhibits the interference picture. This is impressively shown in Fig. 2, from an experiment corresponding to the two-slit experiment, done with electrons [61]. Moreover, from other experiments (some cited in [62], [63]) one knows that a one-particle wavepacket never produces more than one spot at a time.

These observations are indeed suggestive of being produced by point-particles. Now, since the double-slit interferences make it impossible for the particle to have a (sharp) position at all times, that position is ascribed to them only at the moment of measurement. Thus, “position” is no longer a permanent objective property of a particle but an “observable”, which comes into existence only in the act of observation. If there is no observation, a position must not be ascribed to the particle. As with position, other quantities like the components of momentum and angular momentum must not be ascribed to the particle, except at the moment of the respective measurement. All these quantities are mere “observables”. Thus, the Copenhagen interpretation has both a wave and a particle inside the wave. This is the pernicious wave-particle duality.

We do away with that duality. There are no classical point-particles in the microworld. Here we replace the concept of a classical point-particle by the concept of a quantum wavepacket. A quantum wavepacket suffers a reduction or collapse, and thereby it can contract to a pointlike extension in space (more on this below). Those effects that formerly were ascribed to the action of a classical point-particle, now are ascribed to the collapse of the quantum wavepacket. And when we speak of a particle in our theory we always mean a one-quantum wavepacket. Notice that

![Figure 2: Experiment on the buildup of an electron interference pattern [61]](image-url)
the spots in the detection screen are not points but have some extension. A spot is the result of a cascade of processes which is initiated by the ionization of an atom in the detecting screen by the incoming wavepacket. The localization is thus limited in practice by the extension of the black spot and in principle by the extension of the initiating atom or of the smallest wavepacket in the screen with which the incoming packet interacts.

The concept of reduction in quantum mechanics was first mentioned shortly by Heisenberg [3, p. 186] and was worked out very clearly by von Neumann [14]. It is therefore often called von Neumann’s axiom. It assumes that immediately after the measurement the wavepacket is a superposition of eigenfunctions of the respective operator which belong to the interval of eigenvalues specified by the measurement apparatus [47, p. 298], [48, p. 221]. This covers the case of continuous as well as discrete eigenvalues. In the special case of a discrete spectrum, a non-degenerate eigenvalue and sufficient measurement accuracy, the wavepacket immediately after the measurement will be a completely specified eigenfunction of the chosen observable.

In our conception, reduction is different. It is conceived to be a real physical process quite independent of any measurement, though measurement needs it (see below). It occurs when a certain physical criterion is satisfied.

Actually, a quantitative physical criterion for the occurrence of the reduction process has been formulated in detail in [49]. The reduction formulas (4.1), (2.1) below are on an equal footing with the Schrödinger equation in describing the temporal evolution of the wavepackets. We here only want to sketch those features which are used in the present paper. Reduction, in our picture, occurs when two wavepackets, \( \psi_1 \) and \( \psi_2 \), in ordinary space for the first time overlap and satisfy a certain criterion. Both then contract suddenly to the place of overlap. The criterion depends on the parameters of both wavepackets in a symmetric way:

\[
|\alpha_1 - \alpha_2| \leq \frac{1}{2} \alpha_s
\]

\[
\left( \int_{\mathbb{R}^3} |\psi_1(\mathbf{r}, t)| |\psi_2(\mathbf{r}, t)| d^3r \right)^2 \geq \frac{\alpha}{2\pi}.
\]

\( \alpha_s \) is Sommerfeld’s fine-structure constant. \( \alpha_1 \) and \( \alpha_2 \) are the absolute phase constants of wavepacket 1 and 2, respectively. They are nonlocal ‘hidden’ variables. and in our reduction they do play an important role. They are physical because there are situations where they can be determined. In the superposition the absolute phases become relative and determine the position of the interference fringes. The phase constant \( \alpha \) in (2.2) is the smaller one of \( \alpha_1 \) and \( \alpha_2 \). In an ensemble of wavepackets, the phase constants are taken to be pseudorandom numbers. That is, they only seem to be random, but in reality they are determined by certain initial conditions. These phase constants determine when and where a reduction occurs. We call this micro-determinism.

Now comes quantum measurement. By this we mean one where the wavepacket nature of the elementary particles cannot be neglected, that is, where a location
is measured with an error interval that is smaller than, or of the order of, the corresponding spatial width of the wavepacket.

As every physicist can verify, in any quantum mechanical measurement the measurement apparatus fans out the incoming wavepacket in ordinary space into spatially separated eigenpackets of the chosen observable (self-adjoint operator). When one of these eigenpackets and some wavepacket in the sensitive region of the apparatus satisfy the criterion, the reduction associates the place of contraction with an eigenvalue of the observable. There must be special wavepackets in the sensitive region which under contraction initiate an avalanche of effects that result in an observable spot. These wavepackets in the screen are very small, so the region of overlap of such a wavepacket with the incoming wavepacket is also very small, and the whole of the incoming wavepacket has contracted to this very small region. This mimics the action of a pointlike particle.

In order to predict the outcome of a particular measurement, the observer would have to know the shape of the incoming wavepacket and the phase constant of every special grain in the screen of the measuring apparatus. This is virtually impossible. Moreover, the observer cannot know effects coming from events outside his past light-cone [64].

We thus have determinism in the microworld (micro-determinism), but unpredictability (macro-indeterminism) in the macroworld, just as in throwing dice.

2.2 The Stern-Gerlach Experiment

The Stern-Gerlach experiment is well suited for a demonstration of the new concept of measurement outlined above. In quantum mechanics it has traditionally been regarded as the prototype of a measurement.

First let us briefly recall the facts. We consider hydrogen atoms in the ground state which move in the $y$ direction with velocity $\nu$ through an inhomogeneous magnetic field $B$ produced by a Stern-Gerlach magnet [65] - [68], [47, Sec. III.10]. The magnet is positioned so that along the path of the atoms both $B$ and $\nabla B_z$ point in the $z$ direction. This is then the “spin-reference axis”, or simply the “axis” of the apparatus. The hydrogen atom has a permanent magnetic moment $\vec{\mu}$ which comes from the magnetic dipole moment of the electron, the contribution of the proton being negligible. Therefore it is mainly the electron that interacts with the magnetic field, and it is the spin of the electron $(2.3)$

$$ s = -(m/e)\vec{\mu} $$

that determines the precession in the magnetic field. Accordingly, in (2.3) $m$ is the electron mass. We might thus just speak of electrons moving through the Stern-Gerlach magnet. We note, however, that the Stern-Gerlach magnet does not work for free electrons. This is due to the Lorentz force and to the spreading of the electron wavepacket [69] - [71, p. 214]. If we want to perform the Stern-Gerlach experiment for free electrons we may scatter electrons by atoms [71, Chap. IX], [72]. We call such devices Stern-Gerlach-type apparatuses.
The experimental arrangement is chosen so that the laws of classical mechanics and electrodynamics predict that the atom when it has spent the time $\Delta t$ in the Stern-Gerlach magnet will be deflected along the $z$ direction by the angle

$$\alpha_z = \frac{p_z}{p_y} = \mu_z (\partial B_z / \partial z) \Delta t / p_y,$$

where $p_y$ and $p_z$ are the momentum components of the atom, and $\mu_z$ is the $z$ component of $\vec{\mu}$. When a beam of atoms goes through the Stern-Gerlach magnet with the spins of the atomic electrons initially oriented at random, $\mu_z$ can take on all values between $+\mu$ and $-\mu$, and the deflection angles can take on all values between the corresponding extreme values $\pm \mu (\partial B_z / \partial z)(\Delta t / p_y)$. On the screen behind the magnet, the sensitive region, one would therefore observe one single spot elongated along the $z$ direction. What is actually observed, however, is two separate spots corresponding to the above two extreme values of $\alpha_z$, with

$$\mu = \frac{e\hbar}{2m} = \frac{ec}{4\pi\chi},$$

corresponding to the electron spin value $s = \hbar/2$ in formula (2.3). The upper spot on the screen thus corresponds to spin-up electrons and the lower spot to spin-down electrons with respect to the axis of the apparatus.

So far our considerations of the Stern-Gerlach experiment have been independent of our interpretation since they have been formulated with beams consisting of many atomic electrons. Difficulties arise when the behaviour of the individual electron wavepackets of the beam is considered. Any single wavepacket is fanned out into two coherent parts, one corresponding to spin up and the other corresponding to spin down, and it covers both the upper and the lower path in portions that can be calculated by the standard formulas (cf. formula (2.5) below). According to the orthodox version of the Copenhagen interpretation it is only in a subsequent measurement, for example, when a black spot at the proper “up” position on the screen is observed, that the packet contracts and is reduced to a pure spin-up eigenpacket. When we choose not to look at the measurement device, no reduction can occur [73].

In the Copenhagen interpretation the Stern-Gerlach experiment is called a measurement of the initial $z$ component of the spin of the atomic electron [74], [75, p. 593] In the present interpretation this is different. Consider an atomic electron which is initially described by a wave function that is a product of a spatial and a spin function. Such a wavepacket never has a definite pointlike position, but it always has a definite spin component, in the sense that the wave function can always be written as a spin-up eigenfunction of the spin-component operator $s_z$, with some axis $z'$, which of course need not coincide with the axis $z$ of the Stern-Gerlach apparatus. This is connected with the fact that the group SU(2) is locally isomorphic to O(3). We may call the direction of the axis $z'$ the spin direction of the electron before it entered the apparatus.

Now the inhomogeneous magnetic field accomplishes that the eigenfunctions of the spin component of the incoming electron become spatially separated: the spin-up component eigenfunction goes upward (say) and the spin-down eigenfunction
downward. Both functions then enter the sensitive screen in which contraction can occur. It is only when the function contracts at a cluster in the upper region of the screen, say, that the electron has become a pure spin-up electron with respect to the apparatus axis $z$, whatever the electron’s initial spin direction $z'$. Because of total angular momentum conservation, the angular momentum of the apparatus is thereby also changed. This has been verified experimentally in the case of photon polarization apparatuses, which in principle function like Stern-Gerlach apparatuses.

In our interpretation, the operation of the Stern-Gerlach apparatus on an individual incoming electron and the observation of its respective final position on the screen is not a measurement of the electron’s initial spin component. We may indeed use the Stern-Gerlach apparatus for such a measurement, but not without further steps: a large number of equal electrons must be given. Let the electrons enter the magnet one after the other. Then the direction of the axis of the apparatus must be varied, and from the abundance ratios of the up and down spots for the various chosen directions the original spin direction can be derived, using the standard formulas of quantum mechanics. For example, let the spin of the incoming electrons be restricted to lie in the $x$-$z$ plane perpendicular to the direction of motion ($y$ axis). The probability of inducing a spot in the “up” or “down” position, respectively, is then given by

$$P_{\text{up/down}} = \frac{1}{2}(1 \pm \cos \vartheta)$$

where $\vartheta$ denotes the angle between the spin axis of the incoming electron and the axis of the apparatus. The ratio of the corresponding abundances will give $(P_{\text{up}})/(P_{\text{down}}) = \tan^2(\vartheta/2)$, hence $\vartheta$ and the spin direction of the incoming electrons becomes known. Alternatively, we may rotate the apparatus round the $y$ axis until a position is obtained where only spots at the “up” position are observed. This signifies $\vartheta = 0$ in (2.5), and the apparatus axis coincides with the spin direction of the incoming electrons.

If the electron were a classical gyroscope, the measurement of the place where it hits the screen behind the magnet could be considered as a measurement of its initial spin component (formula (2.4)). However, the electron is not a classical gyroscope, and to describe the quantum situation in the same way as the classical situation is misleading. In our interpretation, expressing a wave function as a superposition of certain eigenfunctions in general is no more than a mathematical procedure. Only in special physical situations like the one within the Stern-Gerlach magnet are the spatially separated eigenfunctions physical parts of a wavepacket.

### 2.3 Micro-Determinism and Macro-Indeterminism

As explained in Sec. 2.1 we have micro-determinism but macro-indeterminism (unpredictability). Of course macro-determinism is only possible with micro-determinism.

Both determinism and indeterminism are compatible with realism. Determinism means a programme. It means the expectation that as science advances we will be
able to make more and more phenomena predictable, by means of laws of nature, and that this process is infinite.

It is well known that Einstein favoured determinism: “At any rate I am convinced that he is not playing dice.” And this has often been considered as Einstein’s main criticism of quantum theory. However, as indicated by Einstein and emphasized by Pauli in a letter to Born:

Einstein (as he explicitly repeated to me) does not consider the concept of “determinism” to be as fundamental as it is frequently held to be . . . Einstein’s point of departure is “realistic” rather than “deterministic.”

What Einstein had in mind is obviously micro- as well as macro-determinism.

We comment on this topic here because the Copenhagen interpretation takes the stand of strict indeterminism, micro and thus also macroscopic, and apodictically decrees a definite limit to the described process of the deterministic programme. It maintains that the probabilities in quantum mechanics are unlike those in statistical mechanics and can never be explained by underlying determining processes that would specify the physical situation in more detail. The probability statements in quantum mechanics, according to this interpretation, are the last word. Even if we knew all the laws and all the wavepackets in the world, we would not be able, in principle, to calculate the exact future result of an individual measurement. Only in some degenerate cases can we obtain probabilities that reach the value 1 and thus give certainty. In general, identical initial conditions do not lead to identical results.

Why does the Copenhagen interpretation assume such an extreme position? Admittedly, so long as no theory is available that can specify the hypothetical underlying processes postulated in the deterministic attitude, it might seem reasonable, from the viewpoint of economy of concepts, to eliminate the concept of these processes altogether. This would give the additional bonus that the indeterminacy no longer points at an inability of the quantum theorists to build a complete theory, but is a property of nature.

It seems that the attitude of the Copenhagen school received additional support from von Neumann’s demonstration that some basic features of quantum mechanical states are incompatible with the introduction of additional hidden variables besides $\psi$ in order to further specify the physical situations and to restore micro-determinism. This statement seems to have been taken to mean that no deterministic theory at all is possible. When Bell examined the case he found that those basic features of the quantum mechanical states which von Neumann postulated also for the states in a hidden-variable theory, are actually more than can reasonably be postulated in such a theory. Thus, von Neumann’s proof, although mathematically correct, leaves the real question untouched and does not exclude deterministic hidden-variable theories. The same conclusion had been reached by Grete Hermann in 1935.

On the other hand, Bell’s investigations revealed that any hidden-variable theory which after averaging over the hidden variables reproduces the formulas of quantum mechanics must have a grossly nonlocal structure. A more detailed account of this
specific aspect will be given in Chap. 4. Here, the essential lesson is that von Neumann’s proof does not exclude micro-deterministic theories, and the apodictic exclusion of micro-determinism in the Copenhagen interpretation is unjustified.

Let us, therefore, consider what a deterministic programme might look like in a realist interpretation. In fact, such a programme is presented in [19]. The point is that in an ensemble of quantum wavepackets their overall phase constants $\alpha$ in the reduction criterion (2.1), (2.2) are taken as pseudorandom numbers, determined by certain initial conditions, in the spirit of the theory of deterministic chaos. The criterion (2.1), (2.2) then reproduces the Born probability rules in measurements.
3 NONLOCALITY

3.1 One-Particle Nonlocality

In addition to the property of reduction ascribed to the quantum wavepackets in the preceding sections in order to account for the experimental results, there is the property of nonlocality, which will be described in this and the subsequent sections.

These quantum properties may appear rather strange. This is the price one has to pay. It is not to be expected that the difficulties that have beset quantum theory for more than 90 years can be overcome by some cheap trick. On the other hand, the experimental observations now will fit into a coherent picture.

We will now consider more closely how the black spot in an emulsion screen is brought about. Consider a beam of electromagnetic radiation falling onto a screen. Let us first treat the beam classically as a continuous field with the time-averaged energy flux density \( \bar{S} = \epsilon_o c E^2 \) in W/m\(^2\), say. The registering screen, on the other hand, is conceived to consist of atoms in the sense of quantum mechanics. The incoming radiation will then cause black spots at certain places on the screen, and their density classically is proportional to the energy flux density of the radiation at these places. The resulting pattern will exhibit a granular structure but this in itself does not demonstrate that the incoming radiation has particle or quantum properties; it only shows that the screen has, which we have presupposed anyway. The wind is not made of particles only because it causes an integral number of trees to fall in the forest, as Marshall and Santos put it [86].

To see what may be called the quantum or particle aspect of the incoming radiation recall that according to Sec. 2.1 the incoming radiation in the limit of low intensity (not energy) does not excite more than one atom at a time, even if the energy \( h\nu \) would have sufficed for this. The surplus energy goes into the kinetic energy of the atomic electron [87]. The black spot is the result of a cascade of processes which is initiated by the ionization of an atom. This atom, in order to become ionized, needs some threshold energy \( E_{\text{thr}} \). Of course, knowing quantum mechanics we assume that \( h\nu \geq E_{\text{thr}} \), but this is not sufficient for ionization in the classical picture. Imagine that the atom gets this energy by absorbing and accumulating all the energy of the incoming classical radiation which arrives on its area \( \sigma \), like a dust collector that is put in a stream of polluted air. With the energy flux density of the radiation \( \bar{S} \) the energy accumulated by the atom during the time \( t \) is \( E = \bar{S}\sigma t \). When the atom has accumulated the energy \( E_{\text{thr}} \) it becomes ionized and initiates the formation of the black spot. The accumulation time needed for this is

\[
 t_{\text{acc}} = E_{\text{thr}} / (\bar{S}\sigma).
\]

Now, the value for \( t_{\text{acc}} \) according to this formula turns out to be of the order of hours or years in situations where effects are actually observed immediately after the arrival of the radiation.
As an example consider the interference experiments of Reynolds et al. [88], [89]. Light of wavelength $\lambda = 4.358 \times 10^{-7}$ m and $\Delta \lambda = 10^{-12}$ m passing a Fabry-Perot interferometer produced an interference pattern on the multi alkali cathode of an image intensifier tube. The entire interference pattern had an area of $1.5 \times 3.1 \text{ mm}^2 = 47 \times 10^{-6} \text{ m}^2$. From the size of the coherence length of the light (cf. Appendix A, (A25) f.) and from the density of the excited atoms in the source it could be concluded that there was only one photon in the apparatus at a time. The wavelength of $4.358 \times 10^{-7}$ m means photons of energy $2.85 \text{ eV}$. So, a light energy of $15 \times 2.85 \text{ eV} = 43 \text{ eV/s} (\bar{S} = 1.5 \times 10^{-13} \text{ W/m}^2)$ passed over the area of the pattern.

Each photon wavepacket covers the whole interference pattern. With a minimum linear size of the photon-absorbing molecule of $8 \times 10^{-10}$ m, the energy passing per second over the area of a molecule located in an interference maximum ($\approx 2 \times$ average energy) may be approximately $2 \times 15 \times (64 \times 10^{-20}/47 \times 10^{-6}) \times 2.85 \text{ eV/s}$. With a threshold energy of $1.36 \text{ eV} (\lambda = 0.9 \times 10^{-6} \text{ m})$ necessary for the emission of an electron [90], the accumulation time is $1.36/(11.6 \times 10^{-13}) \text{ s} = 1.17 \times 10^{12} \text{ s} = 37500$ years!

The interference pattern would thus only appear after centuries, but then it would appear fully in one flash. Actually the interference patterns were obtained in 15 seconds, and in various runs with reduced exposure time the authors verified that the pattern is built up gradually as time proceeds. The first black spots are induced immediately after the arrival of the radiation. In fact, upper limits of the time lags between the arrival of the radiation and the ionization of the molecule as short as $3 \times 10^{-9}$ s [91] and $10^{-10}$ s [92] have been reported. Of course the total energy absorbed by all atoms during the whole accumulation time is the same in both cases. The difference is that classical theory would have it absorbed all in the last moment (at $t_{\text{acc}}$), whereas experiment shows that it is absorbed in many small portions distributed over the accumulation time.

Even with somewhat different assumptions one arrives at the same conclusion, as already shown by Campbell [93], Planck [94], Mandel [95], Paul [96], and others [97]. It is thus safe to conclude that the energy for ionizing the atom is not the energy contained in the cylinder that the atom has cut out of the field up to the moment of its ionization.

Now, the field quanta are spatially extended wavepackets of mean energy $\langle E \rangle = h\langle \nu \rangle$, and in the act of measurement they contract to an effectively pointlike region (Sec. 2.1). And what we have here concluded for the photon wavepackets we conceive to hold for any wavepackets.

This contraction is different from the shrinking (or spreading) of the wavepacket governed by the Schrödinger equation (Appendix A). In order to account for the situations described above, the contraction must occur with superluminal velocity in the reference system of the measurement apparatus:

With the dimension of the interference pattern ($\lesssim$ lateral dimension of the wavepacket) in [88], [89] of 3.1 cm and a time lag between the arrival of the radiation and the ionization of the atom of $10^{-10}$ s [92] the contraction velocity would be $3.1 \times 10^8 \text{ m/s}$. This is only slightly larger than light velocity, but wavepackets with
larger dimensions are easily met:

In fact, the one-particle wavepacket may consist of several non-overlapping spatially well separated parts. In [98], for example, each photon of a low intensity radiation beam was split by a beam splitter into two separate parts, and either part was directed into a different detector. No coincidence counts between the two detectors could be observed. That is, if the photon is detected in detector 1 it immediately contracts to a small spot in that detector, so that there is no longer any part of the wavepacket at detector 2. The detectors were separated by 20 m. With a time lag between arrival and detection of $10^{-10}$ s [99] ($0.074 \times 10^{-10}$ s [100]) the contraction had to occur at a velocity of 20 $c$ (270 $c$).

In other words, the count in one detector effects that there is no count in the other detector. In view of the multi-particle nonlocality to be considered in Sec. 3.2 we regard this as one-particle nonlocality [103]. And we ascribe this one-particle nonlocality also to massive wavepackets, for example to the atoms in the Stern-Gerlach apparatus (Sec. 2.2) and the neutrons in the single-crystal neutron interferometer [104].

There are no parts of an electron wavepacket, say, which could dynamically interact with each other. We want to call this internal structurelessness of the wavepackets. In fact, we may regard the success of the Schrödinger equation as a confirmation of the absence of dynamic self-interactions (i.e. those that are explicitly represented by interaction terms in the Schrödinger equation) [105]. Consider the Schrödinger equation for an electron in an electrical potential $V(x, t)$

$$ i \hbar \frac{\partial \psi(x, t)}{\partial t} = -\frac{\hbar^2}{2m} \Delta \psi(x, t) - eV(x, t) \psi(x, t). $$

Let us for the moment regard the quantity $\rho(x, t) := -e|\psi(x, t)|^2$ as the charge density of the electron, and let us write the potential $V(x, t)$ as the sum of two terms

$$ V(x, t) = V_o(x) + V_e(x, t), $$

where $V_o(x)$ is the potential produced by the atomic nucleus (proton) plus some outside charges, and $V_e(x, t)$ is the potential produced by the charge distribution of the electron itself. This can be written as

$$ V_e(x, t) = \frac{1}{4\pi \epsilon_o} \int \frac{\rho(x', t)}{|x - x'|} d^3x', $$

$V_e(x, t)$ represents some action of the electron on itself. Inserting (3.2) and (3.3) into (3.1) leads us to the nonlinear integro-differential equation

$$ i \hbar \frac{\partial \psi(x, t)}{\partial t} = -\frac{\hbar^2}{2m} \Delta \psi(x, t) - eV_o(x) \psi(x, t) $$

$$ + \psi(x, t) \frac{e^2}{4\pi \epsilon_o} \int \frac{|\psi(x', t)|^2}{|x - x'|} d^3x', $$

(3.4)
which differs from the familiar Schrödinger equation by the last, self-interaction term. On the other hand, it is the familiar Schrödinger equation (3.1), and not Eq. (3.4), that gives the correct results, for example for the hydrogen eigenfunctions.

Of course, the absence of dynamic interactions between spatial parts of a wavepacket does not exclude the existence of recoil effects of the emitted radiation on the emitting wavepacket as a whole, as it is considered by Barut and collaborators in their approach to quantum electrodynamics without canonical quantization [106].

In this context we may quote Lorentz [107]:

In speculating on the structure of these minute particles we must not forget that there may be many possibilities not dreamt of at present; it may very well be that other internal forces serve to ensure the stability of the system, and perhaps, after all, we are wholly on the wrong track when we apply to the parts of an electron our ordinary notion of force.

and also Dirac [109]:

it is possible for a signal to be transmitted faster than light through the interior of an electron. The finite size of the electron now reappears in a new sense, the interior of the electron being a region of failure, not of the field equations of electromagnetic theory, but of some of the elementary properties of space-time.

The contraction in any case shows that the quantum wavepacket must be a special object of its own kind. We have already stated that a one-quantum wavepackets can never induce more than one effect at a time. This is just another aspect of the here described contraction to one single place. Does the contraction occur even with infinite velocity? In which reference system? [110]. In any case a velocity inside the wavepacket has no direct physical meaning: there cannot be observers sitting inside the wavepacket at determined positions and reading off synchronized clocks. Might the failure of the elementary properties of spacetime mentioned by Dirac go so far that there is no space at all (i.e. no distance to travel) inside the wavepacket, as speculated in the 1996 version of this article? Note also Bell’s remark in [111]:

Behind the apparent Lorentz invariance of the phenomena, there is a deeper level which is not Lorentz invariant.

3.2 Entanglement and Multi-Particle Nonlocality

In the preceding sections single elementary particles or single field quanta were considered. Now we shall extend our considerations to include systems of several particles.

Multi-particle systems in quantum mechanics are described by a configuration space wave function

$$\Psi(x_1, x_2, \ldots, x_N, t).$$
This does not prevent us from maintaining a realist interpretation. Essentially, what the configuration-space formalism effects is to introduce correlations between wavepackets, as we shall see below (and in Chap. 4).

It is of particular importance to distinguish between multi-particle wave functions that can be written (perhaps after some transformation to a different system of eigenfunctions) as a product of one-particle functions and those that cannot.

In those that can, the particles are independent of each other. In those that cannot, the particles, viz. the wavepackets representing them, in a sense are dependent on each other and are called entangled, a term coined by Schrödinger [112], [113], and they form a system of entangled wavepackets.

We will consider these entangled wavepackets more closely. A two-particle entangled wavepacket, for example, may be written in the form

\[ \Psi = a_1 \varphi_1(x_1; u_1, m_1) \varphi_2(x_2; u_2, m_1) + a_2 \varphi_1(x_2; u_2, m_2) \varphi_2(x_1; u_1, m_2). \]

The function \( \varphi_1 \) (in both parts) represents one particle, and \( \varphi_2 \) the other. The parameters \( u_1, u_2 \) determine the spatial shapes of the wavepackets, which includes their centre position and their width. The parameters \( m_1, m_2 \) are additional properties, for example spin or polarization components [114] or energy and arrival time [115]. The time variable \( t \) is the same for all, and is omitted. For simplicity only one spatial dimension \( x \) is considered.

It may happen that the wavepackets \( \varphi_1 \) and \( \varphi_2 \) develop so as to occupy disconnected regions \( R, L \) of space, that is, the distance between their centres being large compared with their widths. Let us write this as

\[ \Psi = a_1 \varphi_1(R; m_1) \varphi_2(L; m_2) + a_2 \varphi_1(R; m_2) \varphi_2(L; m_1). \]

The two wavepackets are still entangled because both components \( m_1 \) and \( m_2 \) appear in each packet. In the realist interpretation Eq. (3.6) means that particle 1 neither has the property \( m_1 \) nor the property \( m_2 \), and the same holds for particle 2. Considering the experiments where such processes occur [116] - [118] it seems that a necessary condition for independent particles getting entangled is that the wavepackets must overlap to some degree

\[ \varphi_1(x, t) \varphi_2(x, t) \neq 0 \]

at some instant of time. It does not seem that dynamic interactions are sufficient for establishing entanglement. An electron and a proton, for example, in principle interact with each other via the Coulomb force even if the proton is in the Andromeda galaxy and the electron on Earth. I do not think that there is any physicist who would assume that the two are entangled, with the properties of entanglement described below, if they never satisfied condition (3.7). An exact mathematical specification of the entangling condition is not the concern of the present article. Its concern is only to emphasize that it is a real physical process, occurring with real physical wavepackets, not just a loss of ‘information’ or ‘knowledge’. 
How do entangled wavepackets become disentangled? This also happens in the reduction [49], sketched in Sec. 2.1. We here extend that sketch and mention that when one of the overlapping wavepackets on his part is entangled with another one, these two become disentangled.

We now come to the most remarkable feature of entangled wavepackets. As early as 1932 [119] Ehrenfest had emphasized that the mere use of a non-product configuration space wave function implies some kind of sinister action at a distance. Indeed, the conditional probability that particle 1 acts (induces an effect) in $d^3x_1$ about $x_1$, if particle 2 acts in $d^3x_2$ about $x_2$, is

$$P(x_1|x_2, t)d^3x_1 = \frac{|\Psi(x_1, x_2, t)|^2d^3x_1d^3x_2}{d^3x_2 \int |\Psi(x_1, x_2, t)|^2d^3x_1},$$

and this depends on $x_2$, that is, on the result of a simultaneous action of the second particle. On the other hand, the distance between the particles in ordinary space $|x_1 - x_2|$ may be many kilometers long, and the correlations are independent of whether or not there are dynamic interactions between the two particles. With a product form of $\Psi(1, 2)$ the probability (3.8) would be independent of $x_2$, and there would be no correlations.

Indeed, the two events may occur at spacelike intervals of spacetime, that is, if a signal from one event to the other were to connect the two, this signal would have to proceed with superluminal speed. There are many experimental confirmations of this. In [108] a lower bound of that speed was found to be four orders of magnitude larger than the speed of light. As in the case of one-particle nonlocality an interesting question is: can there be a reference system where the speed is infinite [110]?

What has been said here concerning the case of two wavepackets holds also for $N$ entangled one-particle wavepackets [121].

A macro-body, which in the classical Newton-Euler description is a system of point particles, in the present description is a system of wavepackets, where in addition to the dynamic interactions between them there are entanglement and disentangling contractions between them.

Why are the correlations between spacelike separated events (spacelike correlations, for short) such a remarkable feature? Could it not be that the spacelike distance is caused by so far unknown common causes in the overlap of the past light cones of the events, like the consecutive illumination of a series of places on the Moon by searchlight pulses emitted from a place on the earth? In Chap. 4 we shall return to this question and show that the observed spacelike correlations predicted by quantum mechanics can lead to to a violation of the Bell inequality [122], which excludes common causes in the past.

We will call such special spacelike correlation, ascribable to entanglement, multiparticle nonlocality. Therefore the range of the nonlocality in our theory is limited to the extension of the system of entangled wavepackets. And one-particle and multiparticle nonlocality are conceived to be basically of the same nature, and we will just speak of nonlocality in both cases.
As we also shall see in Sec. 4.2 this nonlocality does not lead to a superluminal connection between cause and effect and does not allow superluminal signaling.
3.3 Similar (Identical) and Condensed Wavepackets

The case of identical particles deserves special consideration. Following Dirac’s book [123] we call the particles ‘similar’ rather than ‘identical’. One reason for this is the ambiguity in the meaning of the word identical. We may say “Lord Kelvin and William Thompson are identical”, which means that the two are one and the same person; but we may also speak of identical twins, which means two different persons.

In quantum mechanics the effects of similarity go beyond those met in classical mechanics [124]. In classical mechanics similarity or indistinguishability always means essentially the indistinguishability of equal billiard balls. Imagine one billiard ball in your right hand and the other in your left hand. You are blindfolded and somebody else takes the balls out of your hands, then puts them back. If you look at them again you cannot tell whether or not they have been interchanged. However, if you were not blindfolded, you could follow their paths and decide which ball was initially in your right hand and which was in your left hand. Or, imagine a situation where the balls are in rapid movement around each other so that you can see nothing but a fuzzy cloud of whirling balls. Nevertheless, when you are allowed to use more refined methods of observation, you will always be able to follow the paths of the balls individually.

Clearly, in classical physics with its mass points representing the centres of mass of impenetrable bodies, there is no indistinguishability that could not be resolved in principle. In quantum physics this is no longer true. There are situations, associated with wave function overlap, where the observer is unable, in principle, to distinguish the particles, in the sense that he is unable to follow the path of a given particle unmistakably through all processes. In other words, for him the particles lose their individuality. In the realist programme this is conceived not as any incapability on the part of the observer but as an objective physical fact.

The wave function representing a system of similar particles must be symmetric or antisymmetric under the exchange of function parameters [124]. Consider the product function

\[ \Psi = \varphi_1(R \uparrow) \varphi_2(L \downarrow) \]

representing one particle with spin up (\( \uparrow \)) in the spatial region \( R \) and another particle with spin down (\( \downarrow \)) in the separate region \( L \). Considering the case where there has been the possibility of spin flip in a previous overlap of the two wavepackets so that they can no longer be identified by their spin components, function (3.9) can be superposed with the exchange function

\[ \Psi = \varphi_1(L \downarrow) \varphi_2(R \uparrow) \]

and we obtain the (anti)symmetric function

\[ \Psi = \varphi_1(R \uparrow) \varphi_2(L \downarrow) \pm \varphi_1(L \downarrow) \varphi_2(R \uparrow) \]

However, as emphasized by Ghirardi et al. [125], though this is no product state, it does not mean entanglement, with the property of nonlocality. The single particles
all have definite properties of their own. True, in function (3.11) neither particle 1 nor particle 2 have definite spin values; nevertheless there is one particle in \( R \), whichever of the two it is, with definite spin up and another particle in \( L \) with definite spin down. This does not suffice to violate the Bell inequality (Appendix C, Eq. (C12) and following).

An entangled wavepacket is

\[
\Psi = \varphi_1(R \uparrow) \varphi_2(L \downarrow) \pm \varphi_1(L \uparrow) \varphi_2(R \downarrow).
\]

This function indeed means that the particle in region \( R \), whichever of the two it is, has neither the property \( \uparrow \) nor the property \( \downarrow \), and the same holds for the particle in region \( L \). Eq. (3.12) is the type of function which usually is the base of the discussions of the experiments designed to verify a violation of the Bell inequality (e.g. \([115], [126], [127]\)). In fact, Eq. (3.12) represents 2 of the 4 “Bell states”, which can lead to maximal violation (cf. Sec. 4.3). The 2 others are

\[
\Psi = \varphi_1(R \uparrow) \varphi_2(L \uparrow) \pm \varphi_1(R \downarrow) \varphi_2(L \downarrow).
\]

Thus, (anti)symmetrizing a product of two similar one-quantum wavepackets by itself does not guarantee entanglement with its nonlocality. This is an exception of the statements made in Sec. 3.2.

Another interesting special case are the condensed wavepackets, which are formed when the similar wavepackets, such as \( \varphi_1 \) and \( \varphi_2 \) in Eqs. (3.12) or (3.13), are equal in all respects. Any configuration space function, we recall, can be expanded in terms of a complete set of one-particle functions in ordinary space \( \varphi_r(x, t) \) \([47, Sec. VII.6]\).

\[
\Psi_{SA}(x_1, \ldots, x_N, t) = \sum_{r_1, \ldots, r_N} c(r_1, \ldots, r_N, t) \varphi_{r_1}(x_1, t) \cdots \varphi_{r_N}(x_N, t).
\]

The \( x_i \) may include the spin components. The functions \( \varphi_{r_i} \) may be taken as functions of time too, as in the Dirac or interaction picture. \( c(r_1, \ldots, r_N, t) \) is then the transformed wave function. In the case of similar wavepackets it is completely determined if we specify the number of times each of the arguments \( r_1, r_2, r_3, \ldots \) occurs in it. These are the occupation numbers \( n_{r_i} \), and the set \( |n_1, n_2, n_3, \ldots, t \rangle \) is another representation of the wave function \( \Psi_{SA}(x_1, \ldots, x_N, t) \). The set \( |n_1, n_2, n_3, \ldots, t \rangle \) is the state (vector, wave function) in the occupation-number, \( N \), or Fock representation \([128] - [130]\). The change in occupation numbers in the course of time, due to interactions, is described by the creation and annihilation operators \( a^\dagger \) and \( a \). This works for bosons \( (n_{r_i} = 0, \ldots, \infty) \) and, with minor additions, for fermions \( (n_{r_i} = 0, 1) \).

The occupation-number representation is well suited for our interpretation because numbering of quanta within one wavepacket \( \varphi_{r_i}(x_i, t) \) is not even mentioned in it. The term “occupation number” is, however, likely to mislead one to think that the quanta and the wavepackets filled with them are two different things. In our
interpretation there are only wavepackets and therefore we choose a different formulation: the one-particle basis functions are wavepackets, and occupation numbers of 2 or more mean that two or more wavepackets have condensed to form one single wavepacket, even though this is still normalized to 1 in the current formalism. Thus boson wavepackets can condense, but fermi packets are excluded from doing so. This is our formulation of the Pauli exclusion principle.

The inverse process we call decondensation. The change in occupation numbers will then be described as condensation and decondensation of wavepackets. Condensation and decondensation occur only between Bose but not between Fermi packets. In Chap. 5 we shall show how these processes lead to a new derivation of the quantum-statistical Bose and Fermi distributions and to the corresponding fluctuations in the wavepacket picture.

Another manifestation of the multi-quantum condensed wavepackets is the ‘photon bunching’ in thermal radiation; that is, the observation that the photon coincidence rate in small temporal coincidence windows is higher than can be explained by random coincidences [131], [132], [133]. By now, multi-quantum wavepackets have also been isolated experimentally: in the Bose-Einstein condensates of atoms [134], [135] and of photons [136].

In Sec. 5.2 we shall also see that there are many other objects, conceived long ago in radiation theory, that are very similar to the condensed packets and may be taken to be just other aspects of them.
4 NONLOCALITY AND SUPERLUMINAL SIGNALING

In this chapter we return to the nonlocality described in Secs. 3.1 and 3.2 and proof that the observed spacelike correlations cannot be ascribed to common causes in the past, and that they do not allow superluminal signaling.

4.1 The EPR Problem and Nonlocality

As an introduction we consider the problem of “simultaneous elements of reality” and of action at a distance formulated by Einstein, Podolsky and Rosen (EPR) in 1935 [137]. This paper in its time had raised new interest in the nonlocality problem. The title of the EPR paper is

“Can Quantum-Mechanical Description of Physical Reality Be Considered Complete?”

The authors wanted to demonstrate that the answer is “no”. For a physical theory to be complete it is necessary that

“every element of the physical reality must have a counterpart in the physical theory”,

and reality is characterized by the following sufficient criterion:

“If, without in any way disturbing a system, we can predict with certainty (i.e., with probability equal to unity) the value of a physical quantity, then there exists an element of physical reality corresponding to this physical quantity.”

EPR consider two systems, 1 and 2 (imagine two protons) that have interacted from time $t = 0$ to $t = T$, after which time they are well separated, so, EPR assume, there is no longer any interaction between them. Let the (exactly calculable) wave function of the combined system $1+2$ after $T$ be $\Psi(1,2)$. The number 1 stands for all variables used to describe the first system and 2 for those of the second system. In general the function $\Psi(1,2)$ cannot be written as a product of one function $\varphi_1(1)$ of the variables 1 and one function $\varphi_2(2)$ of the variables 2, and hence we cannot describe the state in which either one of the two systems is left after the interaction. This state, according to the Copenhagen interpretation, can only be known by a subsequent measurement: Let $m_1, m_2, m_3, \ldots$ be the eigenvalues of some physical quantity $M$ pertaining to system 1 and $u_1(1), u_2(1), u_3(1), \ldots$ the corresponding orthonormal eigenfunctions; then $\Psi(1,2)$ can be expanded into a series of these eigenfunctions with coefficients that are functions of the variables 2

\begin{equation}
\Psi(1,2) = \sum_{r=1}^{\infty} \zeta_r(2) u_r(1).
\end{equation}

Although not necessary for the argument, we assume for simplicity of presentation, that the eigenvalues are discrete. The functions $\zeta_r(2)$ are not normalized and in general are not orthogonal to each other, but this is not relevant here. Suppose that the quantity $M$ is measured on system 1 and that the value $m_7$ is found. According to reduction, the first system after the measurement is left in the state $u_7(1)$ [i.e. the first wavepacket assumes the form $u_7(1)$]. Hence the sum (4.1) is reduced to the
single term $\zeta_7(2)u_7(1)$, and due to the product form of this term the second system is left in the state $\zeta_7(2)$, apart from normalization.

The set of functions $u_n(1)$ is determined by the choice of the physical quantity $M$. If, instead of $M$, we had chosen a different quantity $N$, with eigenvalues $n_1, n_2, n_3, \ldots$ and orthonormal eigenfunctions $v_1(1), v_2(1), v_3(1), \ldots$ we would have obtained a different expansion

$$\Psi(1, 2) = \sum_{s=1}^{\infty} \eta_s(2) v_s(1),$$

where the $\eta_s(2)$ are the new coefficient functions. If the quantity $N$ is now measured and the value $n_5$ is found, then system 1 is left in the function $v_5(1)$ and system 2 in the function $\eta_5(2)$.

Therefore, as a consequence of two different measurements performed on the first system, the second system may be left in states with two essentially different wave functions. On the other hand, at the time of measurement the two systems, according to EPR, no longer interact, that is, no real change can take place in the second system as a result of anything that may be done to the first system. Thus it is possible to assign two different types of wave functions, $\zeta$ and $\eta$, to the same physical reality, namely to system 2 after the interaction.

It is even possible to choose noncommuting operators $M$ and $N$, operating in system 1, in such a way that the two sets of wave functions $\zeta_r, \eta_s$ of system 2 are discrete eigenfunctions of two noncommuting operators, for example of the two operators $s_x$ and $s_z$ of the spin component of a proton in $x$ direction and in $z$ direction respectively. Such a case was first considered by Bohm \[75, p. 614\]. Let the two protons interact at O (Fig. 3) and let the scattering proceed through an intermediate state of zero total spin (singlet spin state). The general expression (4.1) in this particular case becomes \[47, p. 562\]

$$\Psi(1, 2) = \frac{1}{\sqrt{2}} \left( |\uparrow\rangle^{(1)} |\downarrow\rangle^{(2)} - |\downarrow\rangle^{(1)} |\uparrow\rangle^{(2)} \right) A(1, 2),$$

where the spin projections up $|\uparrow\rangle$ and down $|\downarrow\rangle$ refer to an arbitrary axis. $A(1,2)$ is the spatial part and the bracket is the spin part of the wave function. Notice that the
spin part follows solely from spin algebra (Clebsch-Gordon coefficients) and happens to be antisymmetric in the particle labels, regardless of whether the particles are similar. In the case of similar particles the spatial part may take care of the correct symmetry.

After the interaction the protons propagate with opposite momentum \(|\mathbf{p}|\) towards the observers A and B respectively. Each observer is equipped with a Stern-Gerlach-type apparatus (e.g. a scattering device with counters; a Stern-Gerlach magnet with registering screen would not do in this case, see Sec. 2.2). The (spin-reference) axes of the apparatuses can be oriented in any direction. Observer A may thus put the axis of his apparatus either in the \(x\) direction or in the \(z\) direction, thereby obtaining either the \(x\) or the \(z\) spin component of the first proton. He is then in a position to predict with certainty, and without in any way disturbing the second proton, either the value of the \(x\) or the value of the \(z\) component of the spin of the second proton. According to the EPR criterion of reality both components must be elements of physical reality. Therefore, the values of both must enter into the complete description of reality. On the other hand, in the formalism of quantum mechanics no wave function can contain both an eigenvalue of some operator \(M\) and an eigenvalue of an operator \(N\) that does not commute with \(M\). Therefore EPR conclude that the quantum-mechanical description of reality by the wave function is not complete.

Einstein seems to consider the elementary particles like dice, where all of their numbers are permanent properties even if we can see only one number at a time. Bohr, on the other hand, considers the particular properties as coming into existence in interaction with the environment. Our conception, specified in Sec. 2.1, in this point is in line with Bohr’s view (cf. the analogy with dough in Sec. 1.4).

In the disputes following the publication of the EPR paper the adherents of orthodox quantum mechanics pointed out that the conclusions of EPR were only valid provided the two systems after the interaction are truly independent of each other in every respect. Present quantum mechanics, however, conceives the two particles to be entangled, that is, inseparably incorporated into the single wave function (4.1), so that we cannot operate on the one particle “without in any way disturbing” the other, and only a reduction of the sum to one of its terms by means of a reduction can achieve a separation of the two particles.

This is independent of whether \(\zeta, u, \eta, \nu\) and \(v\) in expressions (4.1) and (4.2) represent Schrödinger scalar wave functions or relativistic Dirac spinors or other tensors. We may consider the functions \(\zeta, u\) in (4.1) or \(\eta, \nu\) in (4.2) as functions not only of the space coordinates but of time as well, with the same time variable \(t\) in all functions (cf. Sec. 3.2). Thus, the moment of the observation at the one place achieves the simultaneous reduction at the other place. Since the time of this subsequent observation is at the observer’s disposal, he or she may perform it with an arbitrarily long delay after the interaction, so that the wave functions of the two systems can be taken to be separated from one another by an arbitrarily large distance.

On the other hand, things may be arranged so that observer A operates on system 1 such a short time before B operates on system 2 that no light signals could connect these two events. What happens at B then depends on what A has done in
a region that is separated from B by a spacelike distance in spacetime.

Thus in regarding the two systems as independent, EPR are not in accordance with quantum mechanics; hence they cannot maintain that it provides only an incomplete description of physical reality. But then the original question “Can quantum-mechanical description of physical reality be considered complete?” is replaced by another question: “Are these spacelike correlations a feature of physical reality?” Einstein agreed that the EPR conclusion rests on the assumption of complete independence of the two systems after the interaction, but the assumption of non-independence in the form suggested by orthodox quantum mechanics appeared to him an unacceptable “spooky action at a distance” [138], [12], [13], [15, p. 84, 682, 683].

The question is so important for our conception of nature that, in spite of the fact that the spacelike correlations mentioned are predicted by quantum mechanics and that quantum mechanics has been confirmed in innumerable situations, one would wish this particular prediction to be tested in specific experiments. We shall come to these experiments in Sec. 4.4.

4.2 Superluminal Signaling

Here we want to consider the question whether the spacelike correlations permit the transmission of signals or messages with superluminal speed from one person to another. Such a transmission would mean a drastic violation of relativistic causality because we may consider A’s sending a message “the cause” and B’s receiving it “the effect”, and with superluminal transmission these cause and effect could appear in reverse order of time in a different Lorentz system.

Let us try to construct an early-warning system. Consider Fig. 3 of Sec. 4.1. Imagine B to be the Earth and O and A two space stations. Invaders (the Borg) from a distant star are expected to approach the Earth from the direction of A. The task of A is to inform the Earth as soon as the invaders have been seen (emergency case). For this purpose the auxiliary space station O continually emits pairs of scattered protons, say at a rate of 1 pair per second, and the protons are to pass through Stern-Gerlach-type apparatuses on station A and on the Earth respectively. The distance between O and the Earth is made only a little larger than the distance between O and A, so that A receives its proton such a short time (which still may amount to some hours) before the Earth receives its proton that no light signal could have informed the Earth of A’s operation. In routine cases, the apparatuses at both A and B have their axes in +z direction, and in the emergency case A turns the axis of its apparatus into the +x direction. One might think that this changes the probability of an up or down result in apparatus B on Earth, so from the changes in the ratio of up and down results the physicists on Earth would soon learn (before any light signal could be sent from A to B) that the invaders had been seen. Now, the joined probability that proton 1 in the Stern-Gerlach-type apparatus A becomes an $r_A$-proton (= up proton if $r_A = +1$, down proton if $r_A = -1$) and that proton 2 in apparatus B becomes an $r_B$-proton, according to the approved formulas of quantum
mechanics, is (Appendix C)

\[ P(r_A, r_B | a, b) = \frac{1}{2} (1 - r_A r_B \cos \vartheta), \]

where the unit vector \( a \) specifies the axis of apparatus A, \( b \) that of apparatus B, and \( \vartheta \) (\( 0 \leq \vartheta \leq \pi \)) is the angle between \( a \) and \( b \). Hence the probability that B observes the result \( r_B \), whatever the result \( r_A \), is just

\[ P(r_B | a, b) = \sum_{r_A} P(r_A, r_B | a, b) = \frac{1}{2}, \]

regardless of the axis \( a \) (as well as of \( b \)), and in this way the early-warning system will not work. In fact, the above arrangement cannot transmit any message, superluminal or subluminal; the superluminal case is only the most interesting aspect of this general incapability.

We may try to exploit the fact that the change of A’s axis \( a \), if it does not change B’s probabilities, will at least change the correlations between A’s and B’s results \( r_A \) and \( r_B \) respectively. However, the physicists on Earth do not know this. They only know the results of their own apparatus, that is, the lower line of Fig. 4, but not the upper line. Thus, they do not know the correlation of their results with those of A, still less can they realize that there had been any change in those correlations. Either of the two lines of Fig. 4 is just a random series; the probability of an up result is equal to that of a down result, before and after the emergency case. So, the early-warning system does not work this way either.

| Event no. | Before | Emergency Case | After |
|-----------|--------|----------------|-------|
| \( r_A \) | 1 2 3 4 5 6 | 7 8 9 10 11 12 |
| Event no. | 1 2 3 4 5 6 | 7 8 9 10 11 12 |
| \( r_B \) | - - + - + - | + - + - + + |

Figure 4: Records of results of observers A and B before and after the emergency case.

One may try more general apparatuses than just Stern-Gerlach-type ones. These also will not work. It can be shown quite generally that no faster-than-light warning system can be built with devices obeying the formulas of quantum mechanics. The proof is shown in Appendix D.

On the other hand we note that the formulas of quantum mechanics only give probabilities for the various possible results. If the physicist at A could arrange
with certainty that his proton always goes into the up state (say) with respect
to his axis, that is, if there were macro-determinism (predictability) superluminal
messages would be possible. For then, with A’s and B’s axes parallel, the physicist
at B would register only down protons. In the emergency case let A turn his
apparatus upside down. From then on B would obtain only up protons, and the
first of these would tell B that A has seen the invaders. For the construction of an
early-warning system one might, therefore, try situations in the grey zone between
quantum and classical physics, hoping that here the probability features of quantum
mechanics have already sufficiently approached classical deterministic behaviour
while the superluminal features persist. An attempt has been made in that direction
by Herbert [139], who used the amplification of a weak beam of light. But it was
soon shown that the proposal would not work because the amplification of arbitrary
states by one and the same apparatus is impossible because it is at variance with the
linearity of the quantum mechanical operators [140] - [143]. Thus, the very theory,
quantum mechanics, that predicts superluminal features also predicts that these
features cannot be used for transmitting superluminal messages from one person to
another.

What, then, is the remarkable feature of the quantum mechanical formula (4.4)
expressing superluminal features, i.e. spacelike correlations? To see this we have to
consider Bell’s inequality.

4.3 Bell’s Inequality

A priori, spacelike correlations, i.e. correlations between spacelike separated events,
may be thought of as caused by arrangements in the past, for example by the
searchlight spots on the Moon mentioned at the end of Section 3.2, or the letters
running over the lights of a billboard. Such effects are brought about by common
causes in the past, i.e. by events in the overlap of the past light-cones of the correlated
events. The remarkable feature of formula (4.4) is that the correlations described by
it cannot be accomplished with the above mentioned arrangements. This is shown
by means of the Bell inequality. Spacelike correlations with no common causes in
the past exhibit what we have called multiparticle nonlocality (Section 3.2) or just
nonlocality.

The Bell test of whether spacelike correlations exhibit nonlocality is that the
joint-probability formula leads to expectation values of the product $r_A r_B$ of the
dichotomic variables $r_A$ and $r_B$

$$E(a, b) := P(+, + | a, b) + P(−, − | a, b) − P(+, − | a, b) − P(−, + | a, b)$$

that violate Bell’s inequality

(4.5) $$K := |E(a, b) + E(a, b') + E(a', b) − E(a', b')| ≤ 2$$

for some choice of the parameters $a, b, a', b'$. (For convenience we write $a$ and $b$
instead of $a$ and $b$). Actually, there are many versions of the Bell inequality; the
particular version (4.5) was first written down in [144]. It is easy to see that formula (4.4) leads to $E(a, b) = -\cos \vartheta$, and this may violate (4.5). For example, choose vectors $a$ and $b$ that lie in planes normal to the direction of propagation of the protons, and let $a$ form the angle $0^\circ$, $b = 45^\circ$, $a' = 90^\circ$, and $b' = -45^\circ$ relative to some standard direction. This choice results in $K = 2\sqrt{2} = 2.83 > 2$.

Thus the quantum mechanical formula (4.4) for spacelike correlations implies nonlocality. In order to see why Bell’s inequality accomplishes this we must look at the assumptions that are made in its derivation. In Appendix B it is shown that Bell’s inequality is satisfied if the joint probability can be written in the form

$$P(r_A, r_B | a, b) = \int P_1(r_A | a, \lambda) P_2(r_B | b, \lambda) f(\lambda) \, d\lambda.$$  

(4.6)

This form means that we first go to a more detailed level of description by introducing the additional variable $\lambda$ into the joint probability in the integrand, and that on this level we write the joint probability in the special form [145]

$$P(r_A, r_B | a, b, \lambda) = P_1(r_A | a, \lambda) P_2(r_B | b, \lambda).$$  

(4.7)

The variable $\lambda$ fluctuates with the probability density

$$f(\lambda) \geq 0, \quad \int f(\lambda) \, d\lambda = 1,$$

(4.8)

and $f(\lambda)$ and the range of $\lambda$ do not depend on $a, b, r_A$, and $r_B$. Actually, $\lambda$ stands for any set of variables that might be relevant in determining the probabilities. The product form (4.7) is more than just the separability $P(r_A, r_B | a, b, \lambda) = P_1(r_A | a, b, \lambda) \times P_2(r_B | a, b, \lambda)$ since in (4.7) the first factor does not depend on $b$ nor the second on $a$. Thus, not only are the events $r_A$ and $r_B$ statistically independent for given $\lambda$, but the probability that $A$ obtains the result $r_A$ [i.e., $\sum_{r_B} P(r_A, r_B | a, b, \lambda) = P_1(r_A | a, \lambda)$] is also independent of $B$’s parameter $b$; and similarly $P_2$ is independent of $a$. There is however still a link between $A$ and $B$, namely the common variable $\lambda$ and after the integration the probability (4.6) need no longer have the product form $P(r_A, r_B | a, b) = P_1(r_A | a) P_2(r_B | b)$, and correlations between $r_A$ and $r_B$ may arise that depend on $a$ and $b$.

Nevertheless, in order to calculate the probabilities $P_1(r_A | a, \lambda)$ at $A$ it is sufficient to take into account simply the common parameter $\lambda$ and the local parameter $a$, but not the remote parameter $b$; and analogously for $B$. This is why the correlations based on the probability formula (4.6), satisfying the Bell inequality, are called locally explicable [146].

Compare formula (4.4) with the formula

$$P(r_A, r_B | a, b) = \frac{1}{4} (1 - \frac{1}{2} r_A r_B \cos \vartheta).$$

(4.9)

All considerations in Sec. 4.2 would remain unaltered if instead of formula (4.4) we used formula (4.9). Formula (4.9) results if we assume, in a “semiclassical” model, that the two protons in Fig. 4 were completely independent after their interaction at
O (cf. [147], [148], [149]), one having the spin $\vec{\sigma}$ and the other $-\vec{\sigma}$ (total spin zero) already at O, and that the probability formulas for the single protons were still those of standard quantum mechanics ((C.9) of Appendix C)

$$P(r_A | a, \vec{\sigma}) = \frac{1}{2} (1 + r_A \vec{\sigma} a) = \frac{1}{2} (1 + r_A \cos \alpha).$$

Then proton 1 would arrive at A with $\vec{\sigma}$, proton 2 at B with $-\vec{\sigma}$, and the conditional joint probability would be

$$P_{SC}(r_A, r_B | a, b, \vec{\sigma}) = \frac{1}{2} (1 + r_A \vec{\sigma} a) \frac{1}{2} (1 - r_B \vec{\sigma} b).$$

The integration over an isotropic distribution of $\vec{\sigma}$ would then result in

$$P_{SC}(r_A, r_B | a, b) = \frac{1}{4|4\pi|} \int d\varphi \sin \alpha \alpha \sin \alpha \frac{1}{2} (1 + r_A \vec{\sigma} a) \frac{1}{2} (1 - r_B \vec{\sigma} b)$$

$$= \frac{1}{4\pi} \int d\varphi \sin \alpha \alpha \sin \alpha \frac{1}{2} (1 + r_A \cos \alpha) (1 - r_B [\sin \vartheta \cos \varphi \sin \alpha + \cos \vartheta \cos \alpha])$$

$$= \frac{1}{4} (1 - \frac{1}{3} r_A r_B \cos \vartheta).$$

Expression (4.10) differs from the quantum-mechanical formula (4.4) only by the factor $1/3$ inside the last bracket. This has however the consequence that the expectation is now

$$E(a, b) = -\frac{1}{3} \cos \vartheta$$

and the Bell inequality (4.5)

$$|E(a, b) + E(a, b') + E(a', b) - E(a', b')| \leq 4/3 < 2$$

is always satisfied. The variable $\vec{\sigma}$ in the semiclassical model corresponds to the $\lambda$ introduced in formula (4.6) because it is independent of $a$ and $b$. In the quantum-mechanical case it is not so, because there the value of $\vec{\sigma}$ for proton 2 before it enters B's apparatus is influenced by A's variable $\vec{\sigma}$ being either $+a$ or $-a$.

Thus correlations described by formulas such as (4.4), which do not satisfy the Bell inequality, mean local inexplicability, i.e. nonlocality [146].

It is always assumed that the choice of the parameters $a$ and $b$ can be made at any time at will by the experimenters. If we accept a strictly deterministic view, where free will is an illusion, it is possible to assume that all correlations arise from common causes in the past. Nevertheless, it is also possible to assume that there are particular correlations which do not arise in this way. Some support of this view is seen in the superluminal contraction of the one-particle wavepackets discussed in Sec. 3.1 and in the arguments given in [49] (cf. also [110]).

Finally, we note that in the above considerations about the Bell inequality the question of determinism is not involved because the parameters $a, b$ and $\lambda$ only determine the probability of an outcome, not the outcome itself. Whether this probability is reducible to some underlying constellations of additional variables
is left open. Neither is the question of realism involved because it is left open whether the outcomes come into existence by our observation or arise independently of the observer. The Bell inequality is just about nonlocality.

4.4 Experiments

Are the formulas of quantum mechanics that lead to a violation of the Bell inequality confirmed in specific experiments? Many experiments have been performed [117], [150], [127], [151], [152] and the result is that they generally confirm quantum mechanics. Most experiments were concerned with the Bell inequality in its different but essentially equivalent forms. There are also other experiments confirming the nonlocal features [153], [154], but the experiments related to the Bell inequality seem to be the most stringent ones and have been subjected to the closest scrutiny. In all of them, except in two early cases which are now considered unreliable, a violation of the respective variant of the Bell inequality has been found. Moreover, the particular type of violation was exactly that predicted by the formulas of quantum mechanics. The probability $P(r_A, r_B|a, b)$ in formula (4.6) or the average $E(a, b)$ are measured by means of normalized coincidence rates and appropriate average values. Not all experiments were absolutely conclusive because simplifying though very plausible assumptions had to be made. These assumptions were necessary because the filters and detectors employed were not ideal, because the two photons in the atomic-cascade experiments are not strictly antiparallel, and because of other reasons. Clauser and Horne [145], for example, introduced the “no-enhancement assumption”, which means that the photon detection probabilities $P_1$ and $P_2$ in (4.7), for every value of the variable $\lambda$, can at most be reduced but not enhanced by a polarization filter placed in front of the detector (“detection loophole”). In subsequent experiments most of the simplifying assumptions have been gradually eliminated or reduced in their influence.

For example, in all pre-1982 experiments the spin-(polarization)-reference axes were fixed before the individual measurements were done, hence the measurements were not separated by spacelike intervals (“locality loophole”). Therefore the result of A could, in principle, have been transmitted to B with light (or even under-light) velocity before the measurement by B had taken place and so could have influenced B’s result. Of course, in the actual experiments any mechanism that might, according to current knowledge, have permitted this was excluded; still the possibility was only excluded technically, not in principle. In 1982 Aspect et al. [154] performed an experiment in which this was excluded in principle. They used variable polarizers that jumped between two orientations in a time that was short compared with the photon transit time. In this experiment, too, a violation of Bell’s inequality and a confirmation of the quantum mechanical formulas was found (see also [155], [157], [158], [159], [160]). Thus, the experiments provide overwhelming evidence that nonlocality is indeed a feature of physical reality.

Finally, let us have a look at the spatial separations of the wavepackets between which spacelike correlations have been observed in the experiments.
(1) In the proton-proton scattering experiment of Lamehi-Rachti and Mittig [72] the distance $OA$ in Fig. 4 was about 5 cm. After scattering at $O$ the protons had a kinetic energy of 6 MeV, and the length of the proton packets was calculated from the lifetime of the intermediate singlet $s$-wave state to be $4 \times 10^{-15}$ m. A proton packet of the above energy, for which $4 \times 10^{-15}$ m is the minimum width, spreads out to an extension of 2.3 cm while its centre traverses the distance of 5 cm [Appendix A, Eq. (A23)]. Thus, the separation between the two proton packets is about 4 times their width as measured by the standard deviation.

(2) Some experiments for testing Bell’s inequality employ photons from a cascade decay of excited atoms are employed. The length of the photon packets is estimated from the mean lives of the decaying levels, which gives values of the order of 1.5 – 3 m. This is comparable to or even larger than the dimensions $(OA)$ of the apparatuses used up to 1980. In the subsequent experiments [161], [156] the apparatuses $A$ and $B$ are separated by about 13 m. This is 8 times the estimated length of the photon packets.

(3) In correlation experiments with photon pairs from $e^+e^-$ annihilation [162] - [164] the individual photon packets are usually assumed to have lengths of the order of $7\,–\,15$ cm while the distance between $O$ and $A$ was up to 2.5 m. This is 16 to 35 times the packet length.

Admittedly, the lengths ascribed to the individual wavepackets may be larger than assumed, in particular they may be larger than the usually adopted standard deviation $\Delta y$. The value of $\Delta y$ is often calculated from $\Delta p_y$ by means of the Heisenberg relation with the equality sign $\Delta y \Delta p_y = \hbar/2$. However, the equality sign can only hold for a Gaussian form of the wavepacket, and even for a Gaussian form it holds only at one instant of time; at other times the length may have spread out to values considerably larger than the minimum value. Also, the length $\Delta y$ of a wavepacket is often taken to be the coherence length of the beam in which it takes part [72], [162], [163]. Actually, the coherence length of the beam is of the order of a lower bound for the length of the constituent wavepackets (Appendix A).

Brendel et al. [165] used pairs of parametrically down-converted photons and measured the correlations in coincidence counts over distances of 55 cm. At the same time they measured the length of the individual photon wavepackets and obtained values of less than 10 cm. There is thus very little overlap between the photon wavepackets. However, at the same time they also obtained the high value of 87% for visibility in the coincidence rate as a function of wavepacket separation. This cannot be explained as interference of wavepackets in ordinary space. And there are by now many other photon-correlation experiments that point in the same direction [114], [166], [167]. In fact, spacialike correlations between photons (length of the order of $30\,\mu$m) entangled over more than 10 km [168], [169], 16 km [108], and 143 km [170] have been observed.
5 QUANTUM STATISTICS WITH WAVEPACKETS

5.1 Field Quantization

The Schrödinger equation in ordinary (3+1)-dimensional space is a classical field equation, and the discrete eigenfunctions following from imposing the usual normalizability and uniqueness conditions are no more than the standing waves of classical physics. This is just a consequence of de Broglie’s idea of matter waves. We would thus not call that appearance of discrete eigenvalues and eigenfunctions real quantum effects but would reserve this denomination to effects that cannot be explained in the mentioned way. At first, Schrödinger seems to have believed that there are no such other effects, but he was opposed by Heisenberg [171, 172] who pointed, among other things, to the photoelectric effect and to the Planck radiation law.

Schrödinger’s equation in its general form is not an equation in ordinary space but in (3N + 1)-dimensional configuration space, and this goes beyond de Broglie’s conception. The Schrödinger or de Broglie function \( \psi(x, t) \) in ordinary space in itself does not tell us whether it refers to one or to more particles or to particles at all. It is just a field and expresses only the wave aspect. With the introduction of the configuration-space function \( \Psi(x_1, x_2, \ldots, x_N, t) \), however, the number \( N \) of particles or quanta is explicitly introduced.

In 1927 in his fundamental paper on the quantum theory of the emission and absorption of radiation Dirac [173] derived, among other things, the Einstein \( A \) and \( B \) coefficients and hence Planck’s law by means of a new procedure, which had first been introduced by Born and Jordan [174] and in the famous three-man work on matrix mechanics by Born, Heisenberg and Jordan [175]. The procedure consisted in turning some canonically conjugate variables of the Hamilton formalism into operators satisfying the canonical commutation relation, such as between position and momentum. Here the now familiar creation and annihilation operators \( a^\dagger \) and \( a \) (in present notation) showed up for the first time. In the same paper Dirac, and later Tomonaga [176], showed that this procedure was equivalent with Schrödinger’s configuration space treatment with symmetrical wave functions. The equivalence was subsequently elaborated and extended by Jordan, Klein and Wigner [177] - [181] to include antisymmetric wave functions, i.e. fermions, in which case anticommutators were to replace the commutators. In 1932 Fock [128] gave a lucid summary of these developments, and he showed that the restriction to a configuration space of fixed dimension can easily be overcome. This he made particularly clear by casting Schrödinger’s configuration-space formalism into the form of the Fock, occupation-number, or \( N \) representation [128, 182]. In this representation the total number of particles no longer appears explicitly, and this makes it possible to apply the formalism to systems in which the total number of particles is not conserved, as for example to the photons in a cavity.

The method of commutation relations was then further developed into a comprehensive scheme by Heisenberg and Pauli [183, 184]. They no longer derived the commutation relations from configuration space but set out from ordinary-space
fields $\psi(x,t)$ and introduced the commutation relations by way of postulate. In this way the quanta of the fields (i.e. the particles) arise from interpreting the operator $a^\dagger a$, which has only non-negative integer eigenvalues, as a particle-number operator. Moreover, and most important, they extended the formalism to include Lorentz invariant interactions and hence retardation between the similar particles. Retardation effects cannot be taken into account in Schrödinger’s configuration space, so the two schemes are no longer equivalent. Heisenberg’s and Pauli’s scheme is canonical quantization. In it the well known difficulties with the diverging integrals, irreparable by simple normal ordering, began. Thus, in my opinion, this is where something went wrong with the relativistic formulation of quantum theory, and I suspect that this is related to the general negative attitude towards nonlocality at that time, as reflected, for example, by Pauli’s classification [185] of Landau’s and Peierls’ nonlocal density as “unnatural”. In fact, some features, which had been thought to be explicable only in the formalism of canonical quantization, could by now be shown to be explicable within the formalism of quantum mechanics [62], [186].

Thus here we stop, and we conclude our treatise with the re-interpretation of the nonrelativistic field-quantization formalism as far as it is equivalent with Schrödinger’s configuration-space formalism, including symmetrization and particle non-conservation. Our treatment still includes quantization of the electromagnetic radiation field, though not in the way of canonical quantization, but in the spirit of Einstein. There are no retardation effects between the photons, because there are no direct interactions between them.

In the following sections we shall derive the Bose, Planck and Fermi distributions as well as the corresponding fluctuations, by speaking of alteration, condensation and decondensation, introduced in Sec. 3.3, of the realist quantum wavepackets, rather than of distributing particles over phase-space cells.

5.2 The Many Aspects of the Condensed Wavepackets

We consider similar particles of mass $m$ in a cavity of volume $V$ at temperature $T$, and we write the general quantum statistical distribution function in the well-known form

$$N(p,T)dp = \frac{4\pi V p^2 dp}{k^3} \times \left\{ \exp \left[ \left( \sqrt{p^2 c^2 + m^2 c^4} - \mu \right) \frac{1}{kT} \right] \pm 1 \right\}^{-1}$$

where $N(p,T)dp$ means, in the usual interpretation, the time averaged number of particles in $V$ whose absolute value of momentum lies in the interval $dp$ about $p$. The plus sign refers to fermions and the minus sign to bosons. $\mu$ is the chemical potential [fugacity $z = \exp(\mu/kT)$]. In the special case of photons we have $m = 0$, $p = h\nu/c$, $\mu = 0$, and formula (5.1) with the minus sign reduces to the Planck distribution for polarised radiation.

We have written the distribution (5.1) as the product of two factors. The first
factor is

\begin{equation}
(5.2) \quad g_p = \frac{4\pi V}{h^3} p^2 dp = \frac{4\pi V \epsilon \sqrt{\epsilon^2 - (mc^2)^2}}{h^3 c^3} d\epsilon
\end{equation}

where \( \epsilon = \sqrt{p^2 c^2 + (mc^2)^2} \) is the total energy of a particle. In this section we will consider only this factor; the second factor will be considered in the next section. In the special case of photons (5.2) becomes

\begin{equation}
(5.3) \quad g_\nu = 4\pi V \nu^2 d\nu / c^3,
\end{equation}

and in this case it has a long history:

In 1899 it was calculated by Planck [187] as the proportionality factor between the mean energy of electromagnetic radiation in \( V \) and \( d\nu \) and the mean energy of a charged oscillator with radiation damping. In 1900 and 1905 Rayleigh [188] and Jeans [189] considered the factor as the number of degrees of freedom of the ether inside the cavity, this number in turn being considered equal to the easily calculable number of eigenvibrations (modes of vibration) within \( d\nu \) of the ether. In 1914 von Laue [190] decomposed the cavity radiation into mutually independent radiation bundles, each converging to its focal region and then diverging. To these bundles he attributed degrees of freedom and obtained (5.3) as the sum of the degrees of all these bundles (see below). Bose, in his famous paper of 1924 [191], considered the factor (5.3) as the number of phase-space cells of size \( h^3 \). Such cells had already been considered by Planck in 1906 [192] in the special case of harmonic oscillators. In 1925 Landé [193] called those of von Laue’s radiation bundles that had just one degree of freedom elementary light-quantum bundles or just quantum bundles, and he proposed to identify these with Bose’s quantum phase-space cells.

In present-day quantum mechanics (5.2) is the number of eigenvalues of the Hamilton operator for a free particle in \( V \) that fall into the energy interval \( d\epsilon \) which corresponds to \( dp \). Each eigenvalue is multiply counted according to its order of degeneracy. In other words, (5.2) is the number of eigenstates in \( V \) and \( dp \). For photons we are thus effectively back at Rayleigh’s and Jeans’ determination. In quantum mechanics (5.2) holds, however, for any kind of particle, not just photons, because de Broglie waves are associated with each kind of particle. Finally, in canonically quantized radiation theory (5.3) is the number of oscillators. But in contrast to Planck’s oscillators, which represent atoms (“resonators”) interacting with the radiation field, these oscillators are to represent the field itself, a point of view that had already been indicated by Ehrenfest in 1906 [194].

Now we want to show that (5.2) or (5.3) can also be taken as the number of (condensed) wavepackets in the cavity covering the momentum interval \( \Delta p \). For this purpose we employ the fact that (5.3) is the total number of degrees of freedom of von Laue’s radiation bundles and that a bundle of \( F \) degrees of freedom may be taken to consist of \( F \) wavepackets. Von Laue defines the number \( F \) of degrees of freedom of a bundle of length \( l \) (from wall to wall of the cavity), spectral range \( d\nu \) (equal to the spectral range of the radiation considered), convergence half angle \( \alpha \)
and focal cross section $A$ with the help of the theory of optical resolving power and
the counting of Fourier coefficients. He arrives at the expression

$$(5.4) \quad F = \frac{Ald\nu}{ac}$$

where $a$ is the minimum focal area that is possible for a bundle of convergence
angle $\alpha$ (cf. Appendix A formula (A16), with $\alpha = \nu_{\text{arc}} / c$). Now imagine that the
bundle of $F$ degrees of freedom consists of a stream of wavepackets, all moving
parallel to the axis of the bundle and going side by side through its focal area. The
convergence of the bundle to the focal plane and its subsequent divergence comes
about by the contraction and subsequent spreading in the transverse direction of each
one of these wavepackets, assuming that all packets have their minimum transverse
extension in the focal plane. In front of and behind the focal plane the wavepackets
may overlap in the lateral direction. We then write von Laue’s degrees of freedom
$F$ as the product of three factors: $F = N_1N_2N_3$ where $N_1 = d\nu / \Delta \nu$, $N_2 = A/a$, and
$N_3 = l/(\Delta y)$. Each factor is the ratio of some quantity relating to the bundle divided
by the corresponding quantity relating to the packets. $\Delta \nu$ is the frequency range of
a wavepacket of total length $\Delta y$. $\Delta \nu$ is related to $\Delta y$ by $\Delta \nu = rc/(4\pi \Delta y)$, $(r \geq 1)$,
which follows from the Fourier reciprocity (Heisenberg) relation $\Delta y \Delta p_y = \hbar/2$ with
$\Delta p_y = (h/c) \Delta \nu$. With this relation we obtain $N_1N_2N_3 \leq (4\pi/r)F$. The number $r$
may be set equal to $4\pi$ because the packets in the cavity do not all have their
minimum phase-space extension $(r = 1)$. Moreover, there is always some degree of
arbitrariness in the exact definition of the widths $\Delta y, \Delta p_y$ etc., resulting in some
arbitrariness in $r$. Thus we take $r = 4\pi$ to mean the average extension of the
wavepackets in the cavity with an appropriate definition of the widths, and with
this we obtain

$$(5.5) \quad N_1N_2N_3 = F.$$ 

Now, by the definitions given above $N_1$ is the number of spectral types of wavepackets
in the bundle, as defined by their individual frequency ranges $\Delta \nu$ (colors), $N_2$ is the
number of wavepackets of a particular spectral type that go side by side and $N_3$
those that go one after the other through the focal area of the bundle. It follows
that the product $N_1N_2N_3$ is equal to the total number of wavepackets that make up
the bundle, and the relation (5.5) means that this number is equal to the number
of degrees of freedom of the bundle. Thus the total number of wavepackets can
be identified with the total number (5.3) of degrees of freedom, or of von Laue’s
bundles, if we imagine the radiation to consist only of elementary bundles. This not
only holds for photon packets but for packets of any kind. The considerations by
von Laue can easily be extended to matter waves. One just has to replace $\nu/c$ by
$1/\lambda$ in (5.4) and the ensuing text, and then to replace $1/\lambda$ by $p/h$, and with $r = 4\pi$
the same result obtains.

Now, the discrete energy values $n\hbar \nu$ may be attributed to each of Planck’s
oscillators or Jeans’ degrees of freedom [195], and $n$ quanta may occupy each of
Bose’s cells. These are then our condensed wavepackets representing \( n \) quanta. Empty wavepackets, without a quantum, are also included in formula (5.2). This is a convenient means of indicating that there is space left for more wavepackets to show up in \( V \) and \( dp \) (cf. Bose [191] and Schrödinger [196]). Equivalently, one may say that (5.2) is the maximal possible number of existing (non-empty) wavepackets.

The condensed wavepackets also resemble the degenerate light pulses of Mandel [197]. Mandel introduced the degeneracy parameter \( \delta \), meaning the average number of photons in a light beam that are to be found in the same cell of phase space. He expressed the phase space volume with the help of a certain coherence volume defined in the theory of optical coherence. We obtain the same formulas when we take the phase-space volume to be the product of the ranges of the wavepacket at the moment of its minimum extension (Appendix A).

Finally we want to point out to the “light molecules”, “quantum multiples”, “\( n \)-quantum rays” and “radiation bundles” first mentioned parenthetically by Joffé (1911) [198] and considered more closely by Ishiwaara (1912) [199], Wolfke (1921) [200], de Broglie (1922) [201], Bothe (1923,1924) [202] and especially by Schrödinger (1924) [196] and Bothe (1927) [203]. These authors noticed that the Planck distribution can be written in the form of a sum and that the \( n \)-th term can be interpreted as a contribution from objects that are composed of \( n \) light quanta (see formula (5.16) below). It seems that these ideas retreated under the blow of Dirac’s quantization of radiation in 1927 [173], but it is seen that they also strongly resemble the condensed wavepackets.

The wavepackets, not the single quanta (if we were to take these for a moment as entities of their own) are the statistically independent objects, and condensation of wavepackets is our means of expressing the “mutual influence of the molecules [i.e. quanta] which for the time being is of a quite mysterious nature” mentioned by Einstein in 1925 [204].

5.3 The Balance Relation

Now we turn to the second factor in (5.1)

\[
\left\{ \exp \left[ \left( \sqrt{p^2c^2 + m^2c^4} - \mu \right) \frac{1}{kT} \right] \pm 1 \right\}^{-1}.
\]

We want to derive this factor by means of Einstein’s method of balance relations between transition rates [205], formulated in terms of wavepackets. Einstein’s original treatment of 1917 did not explicitly take into account that photons are bosons and not fermions; it would give the same distribution function in both cases. Of course, the Fermi distribution was published only in 1926 [206]. The fact that photons are bosons can only be taken into account when phase-space regions or energy intervals are subdivided into those fundamental units that are counted by formula (5.2). It is not enough to consider the number of photons in a given energy interval, as in Einstein’s procedure of 1917, but one must further specify how the photons are distributed over the various fundamental units within this interval. This was done
by Einstein later in 1924 in Bose’s quite different phase-space cell approach. The fact that Einstein did obtain the Planck distribution by his 1917 method, in spite of not accounting for the subdivision into fundamental units, is due to the special way he formulated the balancing equations. We shall return to this point below.

With the subdivision into fundamental units we shall be able to derive both the Bose and the Fermi distribution on an equal footing by means of the method of balancing equations between wavepackets. The Fermi distribution has already been obtained in the framework of this method by several authors, though in a different way [129], [207] - [211]. Our procedure is inspired by the comprehensive treatment by Oster [209] and the remarkable paper by Bothe [203]. The general mechanism in all these works is exchange of quanta between fundamental units. These units have sometimes been taken to be the discrete energy states of atoms or oscillators which emit and absorb photons. We emphasize, however, that the fundamental units are not restricted to discrete energy states. That they may well be small but finite energy intervals, centred about any energy values, had already been pointed out by Pauli [212] and by Einstein and Ehrenfest [213] when considering photons scattered by free electrons in the Compton effect.

In our interpretation the fundamental units are the wavepackets and we are going to consider processes taking place between these. As we have discussed in Sec. 3.3 this is our interpretation of the change in “occupation numbers”, which in the occupation-number representation is described by means of the creation and annihilation operators. The one-particle basis functions in the expansion (3.14) now are energy eigenfunctions or narrow superpositions of these. Specifically, we consider two types of elementary processes. Either of them is decomposed in alteration, condensation and decondensation. Alteration changes the energy of a wavepacket, whereas condensation and decondensation change the number of quanta it represents. Thus the first type is:

(1) An $s$-quantum wavepacket of kind 1 which represents $s$ quanta in the energy interval $\epsilon_1^1 \ldots \epsilon_1^1 + d\epsilon_1^1$ (an $s$-packet in $d\epsilon_1^1$, for short) decondenses into an $(s - n)$-packet and an $n$-packet in $d\epsilon_1^1$. The $n$-packet exchanges energy and momentum with an $n'$-packet of kind 2 in $d\epsilon_2^1$ whereby it is altered and goes into the energy interval $d\epsilon_1^1$ and then condenses with an $r$-packet in $d\epsilon_1^1$ to form an $(r + n)$-packet in $d\epsilon_1^1$. Simultaneously an $s'$-packet of kind 2 in $d\epsilon_2^1$ decondenses into an $(s' - n')$-packet plus an $n'$-packet in $d\epsilon_2^1$. The $n'$-packet is altered in an interaction with the $n$-packet in $d\epsilon_1^1$ whereby it goes into the interval $d\epsilon_2^1$ and then condenses with an $r'$-packet in $d\epsilon_1^1$ to form an $(r' + n')$-packet in $d\epsilon_1^1$. A graphical scheme is presented in Fig. 5.

Conservation of energy requires

$$ n \ (\epsilon_1^1 - \epsilon_2^1) + n' \ (\epsilon_2^2 - \epsilon_2^1). $$

The energy intervals $d\epsilon$ are chosen so that they correspond to the respective intervals $dp$ in absolute value of momentum used in formula (5.1), i.e. $d\epsilon = (d\epsilon/dp)dp$. Effectively, if we may use here the picture of quanta as standard portions of water, an $s$-packet in $d\epsilon_1^1$ gives $n$ quanta to an $r$-packet in $d\epsilon_1^1$, and an $s'$-packet in $d\epsilon_2^1$...
Figure 5: Scheme of the considered processes (1) and (2) between the wavepackets of kind 1 and kind 2.

gives $n'$ quanta to an $r'$-packet in $d\epsilon_2$. Thus, process (1) leads from the state $|a\rangle$ characterized by the 4 packets which represent $s, r, s'$ and $r'$ quanta respectively, to some state $|b\rangle$ characterized by the 4 packets that represent $s - n, r + n, s' - n'$ and $r' + n'$ quanta respectively. The probability of such a transition is denoted by $W_1$. The process comprises most particular physical situations that lead to Fermi or Bose distributions as special cases.

There is no interaction between photons, so here we need the second kind of wavepacket (atoms, electrons etc.). Electrons interact with each other, and kind 1 and kind 2 may be the same. In the Boltzmann case there is no change in the number of quanta but only an alteration in the energies of the wavepackets. This can be described by putting $r = r' = 0, n = s', n' = s'$ in in the scheme of Fig. 5 and dropping $p(0, \epsilon)$ and $q(0, \epsilon)$ from the Eqs. (5.7) and (5.8) below, which then lead to the Boltzmann distribution for the wavepackets (of energy $\epsilon s$).

(2) The second type of processes is this: an $(r + n)$-packet of kind 1 in $d\epsilon_1$ (which need not be the same packet as that at the end of process (1)) decondenses into an $r$-packet and an $n$-packet. The $n$-packet exchanges energy and momentum with an $n'$-packet of kind 2 whereby it goes into the energy interval $d\epsilon_1$ and then condenses...
with an \((s - n)\)-packet of that interval. Simultaneously an \((r' + n')\)-packet of kind 2 in \(d\epsilon_2\) decondenses into an \(r'\)-packet and an \(n'\)-packet. The \(n'\)-packet is altered in an interaction with the \(n\)-packet of kind 1 whereby it goes into \(d\epsilon_2\) and then condenses with an \((s' - n')\)-packet. Conservation of energy is again guaranteed by Eq. (5.6).

The initial (final) \(s, r, s'\) and \(r'\)-packets of process (2) have the same momenta etc. as the final (initial) \(s, r, s'\) and \(r'\)-packets of process (1) and differ from those only by spatial translations. Thus, process (2) goes back from state \(|b\rangle\) to state \(|a\rangle\). The probability of this transition is denoted by \(W_2\).

Process (2) is not the time reversed ("converse") process to process (1), but may be called the "reverse" process, after Dirac [214]. Only the reverse process can lead to statistical equilibrium [213], but only the converse process is suggested to exist and to occur at the same rate as the original process on account of the general principle of time-reversal invariance of basic processes. Now, in an isotropic medium the reverse process can be obtained from the converse one by successive reflections in three mutually perpendicular mirrors at rest relative to the system as a whole, and their frequencies of occurrence must be equal [214]. Thus, in statistical equilibrium the two processes (1) and (2) also occur at equal rates. We now consider these rates. Rate 1 is the mean number of processes (1) that occur per second in the volume \(V\). According to the above-given description it should be equal to

\[
p(s, \epsilon_1) d\epsilon_1 p(r, \epsilon_1) d\epsilon_1 q(s', \epsilon_2) d\epsilon_2 q(r', \epsilon_2) d\epsilon_2 W_1
\]

where \(p(s, \epsilon_1) d\epsilon_1\) is the mean (time averaged) number of \(s\)-packets of kind 1 in \(V\) that represent quanta in \(d\epsilon_1\), \(q(s', \epsilon_2) d\epsilon_2\) is the mean number of \(s'\)-packets of kind 2 in \(V\) and \(d\epsilon_2\), and so on. Analogously, for process (2) the rate is

\[
p(s - n, \epsilon_1) d\epsilon_1 p(r + n, \epsilon_1) d\epsilon_1 q(s' - n', \epsilon_2) d\epsilon_2 q(r' + n', \epsilon_2) d\epsilon_2 W_2,
\]

and the two rates (5.7) and (5.8) have to be equated.

Now, according to the preceding section our wavepackets and hence the states \(|a\rangle\) and \(|b\rangle\) mean pure states of quantum mechanics, and since the probability of a transition in quantum mechanics is the same for a process that goes from \(|a\rangle\) to \(|b\rangle\) as for a process that goes from \(|b\rangle\) to \(|a\rangle\) (Hermitean operators), the probabilities \(W_1\) and \(W_2\) are equal and disappear from the balancing equation. We shall thus obtain the statistical distribution functions without using any special property of the transition probabilities. The differentials \(d\epsilon\) also cancel, and our balance relation acquires the simple and symmetric form

\[
p(s, \epsilon_1) p(r, \epsilon_1) q(s', \epsilon_2) q(r', \epsilon_2)
\]

\[
= p(s - n, \epsilon_1) p(r + n, \epsilon_1) q(s' - n', \epsilon_2) q(r' + n', \epsilon_2).
\]

The relation is reminiscent of the relation for chemical equilibrium between several kinds of molecules. It still comprises the Bose and Fermi cases. There is no spontaneous emission term, i.e. one that would be independent of the number of wavepackets. A general solution is

\[
p(s, \epsilon) = a(\epsilon) \exp\left[-(b\epsilon - c)s\right]
\]
\[ q(s, \epsilon) = a'(\epsilon) \exp[-(b\epsilon - c')s]. \]

[Insert and use (5.6)]. Notice that only the parameter \( b \) (which shortly will be identified with \( 1/kT \)) is the same for the two kinds of packets. In any other respect the distribution function for kind-1 packets is independent of the distribution function for kind-2 packets.

The parameters \( c \) and \( b \) are obtained via the thermodynamic relations \( \frac{\partial S}{\partial E_{\text{tot}}} = \frac{1}{T} \) and \( \frac{\partial S}{\partial N_{\text{tot}}} = -\frac{\mu}{T} \) where

\[
S = k \ln \prod_{\{de_i\}} \frac{g_p!}{[p(0, \epsilon_i)de_i]! [p(1, \epsilon_i)de_i]! \cdots}
\]

is Bose’s or Natanson’s formula (in our notation) for the entropy of the total system, with \( N_{\text{tot}} = \sum_{\{de_i\}} Ndp \) and \( E_{\text{tot}} = \sum_{\{de_i\}} \epsilon_i Ndp \). The total energy here is thought to be subdivided into a set of intervals \( \{de_i\} \). The entropy (5.11) does not depend explicitly on the numbers \( s \). Formula (5.11) also implies that the wavepackets (as units representing \( s \) quanta) are thermodynamically independent [200], [87]. The number of all wavepackets in \( V \) and \( de \), including the empty ones, is given by (5.2), so we have

\[
\sum_{\{s\}} p(s, \epsilon)de = g_p.
\]

The total number of quanta in \( V \) and \( de \) is \( Ndp \), so

\[
\sum_{\{s\}} s \ p(s, \epsilon)de = Ndp.
\]

With (5.10), (5.13) and Stirling’s approximation \( \ln p! \approx p \ln p \) the entropy in thermal equilibrium may be written as

\[
S = k \left( bE_{\text{tot}} - cN_{\text{tot}} - \sum_{\{de_i\}} g_p \ln [a(\epsilon_i)de_i/g_p] \right)
\]

hence the above-mentioned thermodynamic relations lead to

\[
b = 1/kT, \quad c = \mu/kT.
\]

So far all mathematical operations could be carried out even if the numbers \( s, s', r, r' \) were not integers.

### 5.4 The Bose and Fermi Distributions

Now we go to special cases. First we take the kind-1 packets to be boson packets. In this case the numbers \( s, r \) and \( n \) are non-negative integers and (5.12) becomes

\[
\sum_{s=0}^{\infty} p(s, \epsilon)de = g_p = \frac{ade}{1 - \exp[-(\epsilon - \mu)/kT]}.
\]

From this we obtain

\[
ade = g_p (1 - \exp[-(\epsilon - \mu)/kT]) \quad \text{and} \quad p_B(s, \epsilon)de = g_p \left( 1 - \exp[-(\epsilon - \mu)/kT] \right) \exp[-s(\epsilon - \mu)/kT].
\]
This formula coincides with Bose’s expression for the number of phase space cells occupied with $s$ quanta. The total number of quanta (5.13) becomes

\[
N dp = \sum_{s=0}^{\infty} s \ p(s, \epsilon) \ d\epsilon = \frac{g_p}{\exp[(\epsilon - \mu)/kT] - 1}
\]

and we identify these quanta, not the wavepackets, with the particles in the usual interpretation of formula (5.1). Thus we have arrived at the desired Bose-Einstein distribution function, Eq. (5.1) with the minus sign.

Let us further consider photons, as a special kind of bosons, and let us consider the processes where photons are absorbed and emitted by atoms. In this case we take the packets of kind 1 to be the photons and the packets of kind 2 to be the atoms. In one respect the situation goes beyond the scheme of Fig. 5, in that the number of photons is no longer conserved. In its interaction with the atom the photon is absorbed and exists no longer. Thus, in Fig. 5 the arrow that points from the $n$-packet in $d\epsilon_1^t$ (second line) to the $(r + n)$-packet in $d\epsilon_1^f$ (third line) no longer exists, and the $(r + n)$-packet remains an $r$-packet. Equivalently, one may say that the $n$-packet turns into an empty packet ($n = 0$).

Thus, in the balance relation (5.9) the function $p(r + n, \epsilon_1^f)$ on the right-hand side becomes equal to the function $p(r, \epsilon_1^f)$ on the left-hand side and disappears from the equation. In the energy-conservation relation (5.6) we have to put $\epsilon_1^f = 0$. What the atom does beyond satisfying the energy conservation in absorbing and emitting a photon is irrelevant. Likewise, in the reverse process the arrow that points from the $(r + n)$-packet in $d\epsilon_1^f$ (fourth line) to the $n$-packet in $d\epsilon_1^f$ (fifth line) no longer exists, or equivalently, refers to an empty packet. This does not, however, affect the balance equation, which thus is

\[
p(s, \epsilon_1^t) \ q(s', \epsilon_2^s) = p(s - 1, \epsilon_1^t) \ q(s' - n', \epsilon_1^t) \ q(r' + n', \epsilon_2^s).
\]

The solution is again given by (5.10), but only if $c = 0$ in $p(s, \epsilon)$ there, so we have obtained Planck’s law.

There is no spontaneous emission in our treatment. Let us compare this with Einstein’s treatment. Einstein’s balancing equation (Sec. 3 in his 1917 paper [205]) is

\[
\exp(-\epsilon_n/kT) \ \rho = \exp(-\epsilon_m/kT) \ (\rho + A_m^n/B_m^n)
\]

where $\rho = h\nu N dp/(V d\nu)$ is $h\nu$ times the (time averaged) number of photons per unit volume and per unit frequency interval in the cavity. This equation, unlike our Eq. (5.9) or (5.17), is concerned with the number of quanta, not with the number of wavepackets. The second term in the bracket, $A_m^n/B_m^n = 4\pi h\nu^3/c^3$ (polarised radiation), is independent of $\rho$ and is the spontaneous emission term. One may obtain Einstein’s Eq. (5.18) from our Eq. (5.17) if one puts back the transition probabilities $W_1$ and $W_2$ into this equation and uses a special property of them. The left-hand side of (5.17) effectively means a process where an atom absorbs a photon from an $s$-photon packet, and the right-hand side a process where an atom
emits a photon into an \((s - 1)\)-packet. In order to compare with Einstein’s 1917
treatment, which disregards the wavepacket structure of the radiation, one has to
sum Eq. (5.17), with \(W_1\) and \(W_2\) restituted, over all photon packets, i.e. over all
values of \(s\) (with \(n=1\)):

\[
q(s', \epsilon^1_2) q(r', \epsilon^1_2) \sum_{s=0}^{\infty} p(s, \epsilon^1_1) W_1(s, \alpha) =
\]

\[
= q(s' - n', \epsilon^1_2) q(r' + n', \epsilon^1_2) \sum_{s=0}^{\infty} p(s - 1, \epsilon^1_1) W_2(s - 1, \beta).
\]

\(\alpha\) and \(\beta\) are the other arguments in \(W_1\) and \(W_2\), which do not depend on \(s\). Now we
use the special property

\[
W_1(s, \alpha) = W_2(s - 1, \beta) = f \cdot s
\]

where \(f\) may depend on anything but \(s\). This fits with Dirac’s statement that the
probability of a transition in which a boson is absorbed from (emitted into) state \(x\)
is proportional to the number of bosons originally in state \(x\) (in state \(x\), plus one)
\[^{[173]}\]

With Eq. (5.20) one may write (5.19) in the form

\[
\frac{q(s', \epsilon^1_2)}{q(s' - n', \epsilon^1_2)} \frac{d\epsilon_1}{C_1} \sum_{s=1}^{\infty} s p(s, \epsilon^1_1) = \frac{q(r' + n', \epsilon^1_2)}{q(r', \epsilon^1_2)} \frac{d\epsilon_1}{C_2} \sum_{s=1}^{\infty} s p(s - 1, \epsilon^1_1)
\]

\[
= C_2 \left[ d\epsilon_1 \sum_{s=1}^{\infty} (s - 1) p(s - 1, \epsilon^1_1) + d\epsilon_1 \sum_{s=1}^{\infty} p(s - 1, \epsilon^1_1) \right],
\]

and with (5.14) and (5.16) one obtains \(C_1 N dp = C_2 (N dp + g_p)\). If one multiplies this
by \(h\nu/(\nu d\nu)\) and observes that \(C_1/C_2 = \exp[n'(\epsilon^1_2 - \epsilon^1_2)/kT]\) and \(n' = 1\) one obtains
Einstein’s balancing equation (5.18). We note that it is only the special form (5.18)
of the balancing equation that requires the special property (5.20) of the transition
probabilities in order to arrive at the Planck distribution.

Second, we take the kind-1 packets to be fermion packets. In this case we have
only zero- and one-quantum packets, and the sums in (5.14) and (5.16) range only
from 0 to 1. Thus

\[
\sum_{s=0}^{1} p(s, \epsilon) d\epsilon = a d\epsilon (1 + \exp[-(\epsilon - \mu)/kT]) = g_p.
\]

Hence

\[
p_F(s, \epsilon) d\epsilon = g_p (1 + \exp[-(\epsilon - \mu)/kT])^{-1} \exp[-s(\epsilon - \mu)/kT]
\]
and the total number of quanta is

\[ Ndp = \sum_{s=0}^{1} s p(s, \epsilon) d\epsilon = \frac{g_p}{\exp[(\epsilon - \mu)/kT] + 1}, \]

which is the desired Fermi-Dirac distribution, i.e. formula (5.1) with the plus sign.

The balance relation (5.9) in the Fermi case may be specified to read (s = n = 1, r = 0)

\[ p(1, \epsilon_1^i) p(0, \epsilon_1^f) q(s', \epsilon_2^i) q(r', \epsilon_2^f) = p(0, \epsilon_1^i) p(1, \epsilon_1^f) q(s' - n', \epsilon_2^i) q(r' + n', \epsilon_2^f) \]

and may also be given a specific interpretation: a fermion 1-packet in \( d\epsilon_1 \) is altered in an interaction with a wavepacket of kind 2 whereby it goes into \( d\epsilon_1' \), conservation of energy requiring \( \epsilon_1 - \epsilon_1' = n'(\epsilon_2' - \epsilon_2^f) \). Then it goes into a region of phase space within \( d\epsilon_1' \) that is not yet occupied by a (non-empty) wavepacket. The transition rate is proportional to the size of this region, expressed in fundamental units, that is, to the number of empty packets in \( d\epsilon_1' \), \( p(0, \epsilon_1^f) d\epsilon_1' \).

### 5.5 Quantum Count Fluctuations

Finally we extend our considerations on wavepackets in a cavity to include fluctuations. To be definite, we consider a small subvolume \( v \) of the total cavity volume \( V \). We imagine that the subvolume is homogeneously filled with detectors (groups of sensitive atoms). The detectors are sensitive only within the interval \( p \ldots p + dp \) of the absolute value of momentum or the corresponding energy interval \( \epsilon \ldots \epsilon + d\epsilon \), and we assume that within this interval the sensitivity is constant. These detectors are switched on during the interval \( \Delta t \), and the number of counts is registered. This procedure is repeated a great many times, where the time intervals between the repetitions are large compared with \( \Delta t \). We then ask for the mean square deviation, or variance, \( (\Delta m)^2 \) of the number of counts. Again we shall treat both boson and fermion wavepackets on an equal footing.

The subvolume \( v \) together with the interval \( dp \) define a certain volume of phase space and with this a certain number \( g_v \) of (empty plus non-empty) wavepackets, given by formula (5.2) with \( V \) replaced by \( v \). The number of wavepackets with which the counter can interact during \( \Delta t \) is larger than \( g_v \) because (1) the switch-on time \( \Delta t \) may be so long that many sets of wavepackets, each set filling the counter volume at one time, may pass through the counter during \( \Delta t \), and (2) the counter can also interact with wavepackets that only partially extend into it. The (integer) number of wavepackets that partially and/or totally cover the phase-space region of the counter during \( \Delta t \) is denoted by \( g \), where \( g \geq g_v \) and \( g \geq 1 \). A count is always an interaction of the counter with a wavepacket, not with a quantum. The number of quanta represented by the \( g \) packets fluctuates because between two measurements some few-quantum packets may have replaced some many-quantum packets and vice versa. This is the only source of fluctuations. Fluctuations that arise from a non-empty wavepacket leaving the region without another non-empty packet entering
it are already accounted for because our number of wavepackets includes empty packets, so that a non-empty packet leaving the region is equivalent with an empty packet entering it.

First we want to consider a special property of a condensed wavepacket which we shall use below. As the condensed packet arises from the process where some of the one-particle functions in the expansion of ΨS (3.14) become equal we may describe it by means of the product

\[ \Psi_S = \varphi(x_1, s_1, t) \varphi(x_2, s_2, t) \cdots \varphi(x_N, s_N, t) \]

where the \( s_i \) signify the spin variables, and the same \( \varphi \) is used in all factors. \( \varphi(x, s, t) \) here does not necessarily mean the lowest-energy one-particle state, as it does in Bose-Einstein condensation proper \[210\]. We may mention that in the treatment of laser coherence \[217, 218\] the wave function of a stationary \( N \)-photon state can also be written as a product of the type (5.22). The condensed packet is thus effectively described by the one function \( \varphi(x, s, t) \) in ordinary space. Consider the expression

\[ P_1 = \int_{x_1 \in D^3} \int_{x_i \in R^3 (i \geq 2)} \cdots \int_{x_N \in D^3} |\Psi_S(x_1, x_2, \ldots, x_N, t)|^2 d^3 x_1 d^3 x_2 \cdots d^3 x_N \]

\[ = \int_{x \in D^3} |\varphi(x, t)|^2 d^3 x. \]

\( x \in D^3 \) means \( x_a \leq x \leq x_b, y_a \leq y \leq y_b, z_a \leq z \leq z_b. \) \( R^3 \) means total space. If necessary \( x, D^3 \) and \( R^3 \) are to include the spin variables. For \( D^3 = d^3 x \) the expression reduces to \( P_1 = |\varphi(x, t)|^2 d^3 x. \)

In the Copenhagen interpretation expression (5.23) means the probability that particle 1 of a system of \( N \) similar particles is found in the spatial region \( D^3 \), irrespective of where the other \( N - 1 \) particles are found. In our interpretation it is the probability that wavepacket 1 acts in \( D^3 \), irrespective of where the other \( N - 1 \) wavepackets act. Actually, the wavepackets are equal, so what we really want is an expression for the probability that any one packet acts in \( D^3 \), irrespective of where the other \( N - 1 \) packets act, which is \( N \cdot P_1 \). In the same way the standard formalism gives the probability \( P_2 \) that any \( m \) wavepackets of the \( N \) act in \( D^3 \) while the other \( N - m \) do not act in \( D^3 \)

\[ P_2 = \binom{N}{m} \int_{x_1 \in D^3} \cdots \int_{x_m \in D^3} \int_{x_{m+1} \in R^3} \cdots \int_{x_N \in R^3} |\Psi_S(x_1, x_2, \ldots, x_N, t)|^2 \]

\[ \times d^3 x_1 d^3 x_2 \cdots d^3 x_N \]

where a prime at the integral sign means that in the integration over the variables \( x_i \), the region \( D^3 \) has to be excluded. The combinatorial factor \( \binom{N}{m} \) is just the number of ways \( m \) billiard balls (or particle labels) can be chosen from \( N \). In the case of the condensed packet of the product form (5.22) \( P_2 \) reduces to

\[ P_2 = \binom{N}{m} \eta^m (1 - \eta)^{N-m} \]
with
\( \eta = \int_{x \in D^3} |\varphi(x, t)|^2 d^3x. \)

This is the binomial distribution. It is just the probability that \( N \) independent trials with probabilities \( \eta \) for success and \( 1 - \eta \) for failure result in \( m \) successes and \( N - m \) failures. We may thus say that \( P_2 \) is the probability that the condensed packet acts with \( m \) of its \( N \) quanta in \( D^3 \).

Now, of all the quanta of a wavepacket only some fraction will be counted. The probability of \( m \) counts from an \( n \)-quantum wavepacket is formula (5.25) with \( n \) instead of \( N \):
\[
(5.27) \quad b(m; n, \eta) = \binom{n}{m} \eta^m (1 - \eta)^{n-m}
\]
where \( \eta = \bar{m}/n \) (0 \( \leq \eta \leq 1 \), \( (\Delta m)^2 = \bar{m}(1 - \eta) \)) now is the average fraction of quanta of the packet that are counted during the interval \( \Delta t \). In (5.26) we would have \( \eta = 1 \) if \( D^3 = R^3 \). Now, the detector volume \( \nu \) may happen to be much larger than a wavepacket, and in this case \( \nu \) is effectively equivalent to \( R^3 \), leading to \( \eta = 1 \) in Eq. (5.26). Eq. (5.27) would then always give zero for \( m \neq n \) and be useless. It is, nevertheless, possible to maintain formula (5.27) even in this situation if we take into account that the efficiency \( \eta \) is not only limited by the finite counter volume, as accounted for in Eq. (5.26), but also by the intrinsic counter efficiency and short interval \( \Delta t \), so that even when \( D^3 \to R^3 \) the efficiency \( \eta \) may be less than unity, and we can maintain formulas (5.25) and (5.27), independently of whether the counter volume covers the wavepacket totally or partially. The difference between these two cases is absorbed in the numerical value of \( \eta \), that value suffering an additional decrease when we go from the case of total to that of partial spatial covering.

What, then, is the probability \( W(m; g) \) of counting \( m \) quanta from \( g \) wavepackets? To answer this question we first evaluate the probability \( w(n; g) \) that the \( g \) packets represent \( n \) quanta and then the probability \( B(m; n) \) that of these \( n \) quanta \( m \) are counted, and we write \( W(m; g) = \sum_{n=m}^{\infty} w(n; g) B(m; n) \).

We first consider boson packets. The probability that a randomly chosen boson packet is an \( s \)-quantum packet is given by (5.15). From this we obtain the average number \( \bar{s} \) of quanta represented by one packet averaged over all packets in \( V \) and \( dp \) (or the corresponding \( d\epsilon \))
\[
(5.28) \quad \bar{s} = \sum_{s=0}^{\infty} s p(s) d\epsilon/g_p \exp[-(\epsilon - \mu)/kT](1 - \exp[-(\epsilon - \mu)/kT])^{-1}
\]
so that \( \exp[-(\epsilon - \mu)/kT] = \bar{s}/(1 + \bar{s}) \) and
\[
(5.29) \quad p(s) d\epsilon/g_p = \frac{1}{(1 + \bar{s}) (1 + 1/\bar{s})^s}.
\]
The probability that of the \( g \) packets in \( \nu \) the first one represents \( s_1 \), the second \( s_2 \),... and the \( g \)-th \( s_g \) quanta, with \( \sum_{i=1}^{g} s_i = n \), is the product
\[
(5.30) \quad \prod_{i=1}^{g} p(s_i) d\epsilon/g_p = \frac{1}{(1 + \bar{s})^g (1 + 1/\bar{s})^n}.
\]
We are, however, not interested in the particular distribution (which quanta are represented by which packets), so we have to form a sum of expressions (6.30), one for each distribution. Since (6.30) is the same for any distribution we need only to multiply (6.30) by the number of possible distributions, given by the well-known combinatorial expression

\[
\binom{g + n - 1}{n} = \frac{(g + n - 1)!}{(g - 1)! n!}.
\]

Thus the probability that the \(g\) packets represent \(n\) quanta is

\[
w(n; g) = \binom{g + n - 1}{n} \frac{1}{(1 + s)^g} \frac{1}{(1 + 1/s)^n}.
\]

This formula was already obtained by Mandel [219] in a related context.

Next we evaluate the probability \(B(m; n)\) that if \(n\) quanta are present \(m\) are counted. Here we take advantage of the fact that this probability is independent of how these quanta are represented by the different wavepackets. This is seen in the following way: assume that all the \(n\) quanta are from one and the same wavepacket. Then the probability of \(m\) counts is given by the binominal distribution (5.27). Next assume that the quanta are from two wavepackets, one with \(n_1\) and one with \(n_2 = n - n_1\) quanta. The probability of \(m\) counts would then be

\[
\sum_{m_1} \sum_{m_2} \binom{n_1}{m_1} \eta^{m_1}(1 - \eta)^{n_1 - m_1} \binom{n_2}{m_2} \eta^{m_2}(1 - \eta)^{n_2 - m_2}.
\]

But due to a special folding property of the binominal distribution [220, p. 173, 268], this is equal to \(b(m; n, \eta)\) of (5.27). And this would remain so if the quanta were from any number of packets. We may thus assume that all \(n\) quanta are from one and the same wavepacket whence \(B(m; n) = b(m; n, \eta)\) of (5.27). In deriving this result we have used the same value of \(\eta\) for all wavepackets. This requires a justification because when the phase-space volume of a wavepacket \(\Delta^3 x \Delta^3 p\) and that of the counter \(4\pi \mu^2 dp\) with which it interacts are comparable, their overlap and with this the value of \(\eta\) may vary appreciably from one wavepacket to the next, even if, as we assume, all wavepackets in the cavity have very nearly the same size in phase space. In this case the above folding theorem in fact no longer holds generally, although still in special cases \((\eta \ll 1, \eta \approx 1, \text{Poisson approximation, normal approximation [220, Chap. 6])}. When, however, the phase-space volume of the counter is large (in each direction) compared with that of a wavepacket the counter covers almost each wavepacket completely, and since we consider a counter with constant sensitivity over its whole phase-space volume, \(\eta\) is still the same for all packets. When, in the opposite limit, the counter is small compared with the wavepacket, it is true that \(\eta\) may vary considerably because the counter may cover regions of the wavepacket with varying \(|\psi|^2\) (in \(x\) space or in \(p\) space) since we have not assumed that \(|\psi|^2\) is constant over the wavepacket. But now, whichever region of a wavepacket is covered
by the counter, $\eta$ will always be small, and under this condition the folding theorem still holds in the form

$$\sum_{m_1} \sum_{m_2} b(m_1; n_1, \eta_1) b(m_2; n_2, \eta_2) - b\left(m; n_1 + n_2, \frac{n_1 \eta_1 + n_2 \eta_2}{n_1 + n_2}\right)$$

$$\propto (\eta_1 - \eta_2)^2 + \text{terms of higher order in } \eta_1 \text{ and } \eta_2,$$

so that $B(m; n) = b(m; n, \eta)$ still holds with $\eta$ representing some average over different regions of the wavepacket. Therefore we consider $B(m; n) = b(m; n, \eta)$ as an acceptable approximation.

With this the probability $W(m; g)$ of counting $m$ quanta when $g$ packets are in $4\pi v p^2 dp$ is

$$W(m; g) = \sum_{n=m}^{\infty} w(n; g) b(n; m, \eta) = \sum_{n=m}^{\infty} w(n; g) \binom{n}{m} \eta^m (1 - \eta)^{n-m}$$

$$= \sum_{n=m}^{\infty} \binom{g+n-1}{n} \frac{1}{(1+s)^g} \frac{1}{(1+1/s)^n} \binom{n}{m} \eta^m (1 - \eta)^{n-m}.$$  \hfill (5.33)

Formula (5.33) is the same as that obtained via the standard quantization formalism \[221\], as it should be. Substituting $l = n - m$ and using the binomial identity $(l+g-1+m) = (-1)^l (-g^m)$ and Newton’s binomial formula $\sum_{l=0}^{\infty} (-g^m) \left(-\frac{1}{1+1/s}\right)^l = \left(1 - \frac{1-\eta}{1+1/s}\right)^{-g-m}$ we obtain

$$W(m; g) = \binom{g+m-1}{m} \frac{1}{(1+\eta s)^g} \frac{1}{(1+1/(\eta s))^m}.$$ \hfill (5.34)

From this we can calculate the variance $(\Delta m)^2 := \sum_{m=0}^{\infty} (m - \bar{m})^2 W(m; g) = \eta s g (1 + \eta s)$ [use the generating function \[220\] p. 266 of $W(m; g)$]. Observing that $\eta s g = \bar{m}$ is the average number of counted quanta from $g$ packets we may finally write the variance of the number $m$ of counts as

$$(\Delta m)^2 = \bar{m} \left(1 + \frac{\bar{m}}{g}\right).$$ \hfill (5.35)

Comparing (5.34) with (5.32) we see that the distribution, and hence variance, of the counted quanta has the same form as that of the existing quanta.

Our procedure is easily carried over to fermion packets. We only have to observe that there are only zero- and one-quantum packets. Of course (5.22) no longer holds, but (5.27) still does since for $m \leq n \leq 1$ it reduces to three trivial expressions. Further, instead of (5.29) we have to take (5.21) which we write in the form (cf. (5.28))

$$p(s) d\epsilon/gp = (1 - \bar{s}) \frac{1}{(1/s - 1)^s}.$$ \hfill (5.36)
and instead of (5.31) we have to take

\[
\binom{g}{n} = \frac{g!}{(g-n)!n!}.
\]

(5.34) is then replaced by

\[
(5.37) \quad W(m; g) = \binom{g}{m}(1-\eta\bar{s})^g \frac{1}{(1/(\eta\bar{s}) - 1)^m}
\]

and (5.35) by

\[
(\Delta m)^2 = \bar{m}(1 - \frac{\bar{m}}{g}).
\]

For \( g = 1 \) Formula (5.34) becomes (\( \bar{m} = \eta \bar{s}g \))

\[
(5.38) \quad W_B(m; 1) = \frac{\bar{m}^m}{(1 + \bar{m})^{m+1}} \quad \text{(Bose)}
\]

and (5.37)

\[
W_F(m; 1) = \frac{\bar{m}^m}{(1 - \bar{m})^{m-1}} \quad \text{(Fermi)}.
\]

For \( g \to \infty \) both (5.34) and (5.37) approach the Poisson distribution

\[
(5.39) \quad W_{BM}(m; \infty) = \frac{\bar{m}^m}{m!}e^{-\bar{m}} \quad \text{(Boltzmann)}.
\]

It seems that our results not only hold for cavities but also for beams, at least in some situations. This we infer from the observation that the distributions (5.38) and (5.39) are also obtained for stationary beams of chaotic light in standard quantized radiation theory \[222\] and are confirmed by experiment \[223\]. Formula (5.38) obtains when the counting time \( \Delta t \) is short compared with the coherence time \( \tau_c \), \( \Delta t \ll \tau_c \), and (5.39) obtains in the opposite limit, \( \Delta t \gg \tau_c \). These limits can be compared with the limits \( g = 1 \) and \( g \gg 1 \) of our treatment because the wavepacket structure of the radiation field reflects its coherence properties, in that the spatial size of a photon wavepacket is a measure of the size of the coherence region with length \( L_c = c\tau_c \) (Appendix A). The limit \( \Delta t \gg \tau_c \) means that many wavepackets can interact with the counter during \( \Delta t \), either consecutively, if the packets are large, or simultaneously, if they are small compared with the counter volume \( v \). Thus \( g \gg 1 \) in this limit. In the opposite limit, \( \Delta t \ll \tau_c \), the wavepackets can interact with the counter only during such a short time interval that their movement is negligible. Whether we have \( g \gg 1 \) or \( g = 1 \) now depends on further specifications. When the packets are large and pass over the counter one after the other we have \( g = 1 \). This was in fact implicitly assumed in the (plane-wave) calculations referred to by Loudon \[222\] and was explicitly stated in the experimental verification \[223\] of Formula (5.38). So, here \( \Delta t \gg \tau_c \) means \( g \gg 1 \), and \( \Delta t \ll \tau_c \) means \( g = 1 \), and the situations are those covered by the above formulas.
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APPENDIX A: Wavepacket Spreading Formulas

The wavepackets considered here are the usual free packets of Schrödinger or de Broglie waves. Since the formulas are rather spread out in the literature and are often restricted to special cases we have collected here some more general results for easy reference.

The general wavepacket is written in the form

\[ \psi(x, t) = (2\pi)^{-3/2} \int_{-\infty}^{+\infty} \tilde{\psi}(k) \exp[i(kx - \omega(k)t)] d^3k \]

where \( \tilde{\psi}(k) \), and hence \( \psi(x, t) \), is normalized

\[ \int_{-\infty}^{+\infty} |\tilde{\psi}(k)|^2 d^3k = 1. \]

The (three-dimensional) Fourier transform of \( \psi(x, t) \) is

\[ \tilde{\psi}(k, t) = \tilde{\psi}(k) \exp[-i\omega(k)t]. \]

This is the wavepacket in \( k \) (momentum) space. Mathematically the wavepacket need not have a sharp boundary in \( x \) space (or in \( k \) space), but for practical purposes it may be considered to have a finite extension given, for example, by the standard deviation

\[ \Delta x := ((x - \langle x \rangle)^2)^{1/2} \]

with

\[ \langle x \rangle(t) = \int_{-\infty}^{+\infty} \psi^*(x, t) x \psi(x, t) d^3x, \]

and with analogous expressions for the widths \( \Delta y \) and \( \Delta z \). In the special case of the Gaussian form (and \( t = 0 \))

\[ |\psi(x)|^2 = (2\pi\sigma^2)^{-1/2} \exp[-(x - \langle x \rangle)^2/(2\sigma^2)] \]

\( \Delta x \) of (A2) is equal to the parameter \( \sigma \) and is the distance between the position of the maximum and the point where the distribution has fallen off to \( \exp(-1/2) = 0.61 \) of the maximum value. Sometimes the form of \( \psi(x) \) or of \( \tilde{\psi}(k) \) is such that the integrals in (A2) etc. diverge, for example for \( \tilde{\psi}(k) \propto \sin(ak)/k \) and for \( \tilde{\psi}(k) \propto (1 + a^2k^2)^{-1/2} \). In such cases one uses other definitions of the width, for instance the distance between the maximum and the first zero, the half width, the equivalent width [224] or the overall width [225].
The dispersion law \( \omega(k) \) is determined by the relation between the energy \( E \) and the momentum \( p \) of the object the wavepacket is to represent provided we make use of the Einstein-Planck relation
\[
\nu = E/h \text{ or } \omega = E/\hbar
\]
and the de Broglie relation
\[
p = \hbar k
\]
where
\[
k \equiv |k| = 2\pi/\lambda.
\]
Thus, the relativistic relation for a free particle
\[
E = E_{\text{tot}} = \pm (p^2 c^2 + m^2 c^4)^{1/2}
\]
leads to
\[
(A3) \quad \omega(k) = \pm c(k^2 + \kappa^2)^{1/2}
\]
where
\[
\lambda_C = 1/\kappa = \hbar/(mc)
\]
is the Compton length belonging to the rest-mass parameter \( m \).

The most general solution of a relativistic wave equation would be a function of the type (A1) in which \( \omega(k) \) from (A3) has the positive sign plus a function (A1) in which \( \omega(k) \) has the negative sign. That is, the most general form would be a superposition of waves with positive as well as those with negative frequencies viz. energies. Although we shall consider the general relativistic formula (A3) we will restrict ourselves to positive-energy wave functions in order to have a simple connection with the results of nonrelativistic quantum mechanics.

Let us consider the time dependence of the wavepacket’s “centre” \( \langle x \rangle \) and “width” \( \sigma(t) = (|\Delta x(t)|^2 + |\Delta y(t)|^2 + |\Delta z(t)|^2)^{1/2} \)
where
\[
|\Delta x(t)|^2 := \int (x - \langle x \rangle)^2 |\psi(x, t)|^2 d^3x
\]
and analogously for the \( y \) and \( z \) components. A very general calculation of these quantities has been given by Bradford [226]. His treatment works in three dimensions and does not require a special form either for \( \tilde{\psi}(k) \) or for \( \omega(k) \), except for the usual convergence requirements of the integrals that appear in the averaging procedures. In particular, the treatment is valid in the nonrelativistic as well as in the relativistic domain. The result for the centre is
\[
(A4) \quad \langle x \rangle(t) = \langle x \rangle(0) + \langle \bar{v}_g \rangle t
\]
with
\[ \langle x \rangle(0) = -\int \text{Im} \left\{ \tilde{\psi}^* \nabla_k \tilde{\psi} \right\} d^3k \]
(A5)
\[ \langle \vec{v}_g \rangle = \int |\tilde{\psi}|^2 \nabla_k \omega \ d^3k \]
\[ \tilde{\psi} = \tilde{\psi}(k). \]

According to (A4) and (A5) the centre moves at constant velocity \( \langle \vec{v}_g \rangle \), which is the mean group velocity.

The result for the width is
\[ \sigma^2(t) = \sigma^2(t_0) + [\Delta v_g]^2 (t - t_0)^2 \]
where
\[ \sigma^2(t_0) = \sigma^2(0) - [\Delta v_g]^2 t_0^2 \]
(A.7)
\[ \sigma^2(0) = -\int \text{Re} \left\{ \tilde{\psi}^* \nabla_k \tilde{\psi} \right\} d^3k - |\langle x \rangle(0)|^2 \]
\[ t_0 = \left( \int \text{Im} \left\{ \tilde{\psi}^* \nabla_k \tilde{\psi} \right\} \nabla_k \omega \ d^3k + \langle x \rangle(0) \langle \vec{v}_g \rangle \right) \frac{1}{[\Delta v_g]^2} \]
and
\[ [\Delta v_g]^2 := \int |\tilde{\psi}|^2 (\nabla_k \omega - \langle \vec{v}_g \rangle)^2 \ d^3k. \]

Formula (A6) shows that the width varies hyperbolically with time. This type of variation is even independent of the form of the dispersion law \( \omega(k) \), except when \( \Delta v_g \) is zero: there is either a hyperbolic dependence or none. The minimum extension of the wavepacket occurs at \( t = t_0 \); before \( t_0 \) the wavepacket shrinks, after \( t_0 \) it spreads out.

When we write \( \tilde{\psi}(k) \) in the polar form
\[ \tilde{\psi}(k) = \rho(k) \exp[i\alpha(k)] \]
we are led to
\[ \langle x \rangle(0) = -\langle \nabla_k \alpha \rangle \]
(A8)
\[ t_0 = \left( \langle \nabla_k \alpha \cdot \nabla_k \omega \rangle - \langle \nabla_k \alpha \rangle \langle \nabla_k \omega \rangle \right) \frac{1}{[\Delta v_g]^2}. \]
(A9)

We thus can fix the time and the place of the minimum extension by an appropriate choice of the phase \( \alpha(k) \), that is, by \( \langle \nabla_k \alpha \rangle \) and \( \langle \nabla_k \alpha \cdot \nabla_k \omega \rangle \). Resolving the two Eqs. (A8) and (A9) for these two quantities we obtain
\[ \langle \nabla_k \alpha \rangle = \langle \nabla_k \omega \rangle t_0 - \langle x \rangle(t_0) \]
(A10)
\[ \langle \nabla_k \alpha \nabla_k \omega \rangle = \langle [\nabla_k \omega]^2 \rangle t_0 - \langle \nabla_k \omega \rangle \langle x \rangle (t_0). \]

For example, the especially simple form with three parameters \( \xi_0, \tau_0 \) and \( \eta_0 \)
\[ \alpha(k) = -k \xi_0 + \omega(k) \tau_0 + \eta_0 \]
leads to
\[ \langle \nabla_k \alpha \rangle = -\xi_0 + \tau_0 \langle \nabla_k \omega \rangle \]
which, by comparison with (A10), shows that the parameters \( \xi_0 \) and \( \tau_0 \) coincide with the initial values
\[ \xi_0 = \langle x \rangle (t_0), \quad \tau_0 = t_0, \]
and \( \eta_0 \) is an arbitrary constant.

In what follows we will always assume \( \langle \nabla_k \alpha \rangle = \langle \nabla_k \alpha \nabla_k \omega \rangle = 0 \), so that the minimum extension occurs at \( t = t_0 = 0 \), and the centre of the packet at that time is \( \langle x \rangle (0) = 0 \). Formulas (A6) and (A7) then simplify to
\[ \sigma^2(t) = \sigma^2(0) + [\Delta v_g]^2 t^2 \]
and
\[ \sigma^2(0) = -\int \text{Re} \left\{ \tilde{\psi}^* \nabla_k^2 \tilde{\psi} \right\} d^3k. \]

Let us now consider the spreading velocity
\[ (A11) \quad v_s := \frac{\partial \sigma(t)}{\partial t} = [\Delta v_g]^2 t/\sigma(t) \]
which for large \( t \) tends to the asymptotic spreading velocity
\[ v_{s\infty} := \lim_{t \to \infty} v_s = \Delta v_g. \]
The packet spreads out to double its initial \( (t = 0) \) extension in a time \( \tau_2 = \sqrt{3} \sigma(0)/\Delta v_g. \) After this time the spreading velocity is \( v_s = (\sqrt{3}/2) \Delta v_g = 0.87 v_{s\infty}. \)
That is, 87\% of the asymptotic spreading velocity is already reached when the packet has doubled its initial extension. The asymptotic value may thus be used in all practical estimates. At the time \( t_1 \) when the asymptotic spreading velocity has (practically) been reached, the time for the wavepacket to further double its extension is \( \sigma(t_1)/\Delta v_g \) which is somewhat smaller than \( \tau_2 \) provided we identify \( \sigma(t_1) \) with \( \sigma(0) \) in this comparison.

To proceed further we must make specific assumptions about \( \tilde{\psi}(k) \). We shall assume a nearly unidirectional and quasimonochromatic packet, that is, a narrow packet in \( k \) space; that is, \( \tilde{\psi}(k) \) is assumed to be appreciably different from zero only in a narrow region concentrated about the point \( k_0 = (0, k_0, 0) \) so that
\[ (A12) \quad \Delta k_x, \Delta k_y, \Delta k_z \ll |k_0| \equiv k_0 = 2\pi/\lambda_0. \]
With the help of the Fourier reciprocity (Heisenberg) relations for the considered wavepackets

\begin{equation}
\Delta x(0) \Delta k_x = \frac{1}{2} r, \text{ etc. with } r \geq 1.
\end{equation}

where \( r \) depends on the form of the wavepacket, and \( r=1 \) can only occur for a Gaussian form for \( \tilde{\psi}(k) \), the condition (A12) can be written as a condition in ordinary space

\begin{equation}
\Delta x(0), \Delta y(0), \Delta z(0) \gg \frac{r}{(2k_0)}.
\end{equation}

It is then possible to express \( \Delta v_g \) as a function of \( \Delta k_x \) and \( \Delta k_y \). In the case where \( k_0 \neq 0 \) it is reasonable to consider separately the longitudinal spreading, along the direction \( y \) of the centre, and the transverse spreading, normal to that direction, say in the \( x \) direction. We then expand \( \nabla_x \omega = c^2 k / \omega(k) \) in a three-dimensional Taylor series about \( k_0 \) and break the series off after the quadratic terms. After a straightforward but tedious calculation one arrives at

\begin{equation}
u_{sx\infty} := \lim_{t \to \infty} \partial \Delta x(t) / \partial t
= \Delta v_{gx} := \left( \int |\tilde{\psi}|^2 (\partial \omega / \partial k_x) - \langle \tilde{v}_{gx} \rangle \right) d^3 k \right)^{1/2}
\end{equation}

\begin{equation}
\frac{c^2}{\omega_0} \Delta k_x = \frac{c}{(k_0^2 + \kappa^2)^{1/2}} \Delta k_x
\end{equation}

\begin{equation}
= \frac{r c}{2(k_0^2 + \kappa^2)^{1/2} \Delta x(0)},
\end{equation}

where (A13) has been used for obtaining (A16). Likewise we obtain

\( \tilde{v}_0 \equiv \langle \tilde{v}_g \rangle = (0, k_0 c^2 / \omega_0, 0) \)

\( v_0 = k_0 c^2 / \omega_0, \quad \omega_0 = c(k_0^2 + \kappa^2)^{1/2}, \)

and with this we may write (A15) as

\begin{equation}
u_{sx\infty} = \frac{c}{\kappa} \Delta k_x \left( 1 - (v_0/c)^2 \right)^{1/2}
\end{equation}

\begin{equation}
\geq \frac{c (1 - (v_0/c)^2)^{1/2}}{2 \kappa \Delta x(0)}.
\end{equation}

In the same way we arrive at the longitudinal asymptotic spreading velocity

\begin{equation}
u_{sy\infty} = \left( \frac{c \kappa}{\omega_0} \right)^2 \frac{c^2}{\omega_0} \Delta k_y = \frac{c}{\kappa} \Delta k_y \left( 1 - (v_0/c)^2 \right)^{3/2}
\end{equation}
\[(A20) \quad \geq \frac{c (1 - (v_0/c)^2)^{3/2}}{2 \kappa \Delta y(0)} \]

and

\[(A21) \quad \frac{v_{y\infty}}{v_0} = \frac{\Delta k_y}{k_0} \times \left(1 + \left(\frac{k_0}{\kappa}\right)^2\right)^{-1} = \frac{\Delta k_y}{k_0} \times \left(1 - \left(\frac{v_0}{c}\right)^2\right). \]

We thus have the interesting result that the spreading velocities of a narrow packet in \(k\) space do not depend on the detailed form of the wavepacket over and above its second central moments \(\Delta k_x, \Delta k_y\). In particular, formulas (A15) to (A21) hold in the nonrelativistic as well as in the relativistic domain.

Figure 6: Relation between minimum extension and spreading angle.

Fig. 6 pictures \(\sigma(t)\) according to (A6), and according to (A16) shows that the smaller the minimum extension \(\Delta x(0)\), at the waist of the packet, the larger is the spreading angle \(\alpha\):

\[(A22) \quad \tan \alpha := \frac{v_{x\infty}}{v_0} = \frac{r \frac{c}{2(k_0^2 + \kappa^2)^{1/2}}}{\Delta x(0)}v_0, \]

where \(r \geq 1\) (Heisenberg relation) depending on the form of the wavepacket \((r = 1\) for Gaussian form).

In the nonrelativistic domain it is \(k_0^2 \ll \kappa^2\) and (A22) becomes

\[(A23) \quad \tan \alpha = r \frac{\lambda_{\text{deBr}}}{4\pi \Delta x(0)}, \quad \lambda_{\text{deBr}} = \frac{\hbar}{mv_0}. \]

For photons it is \(\kappa = m = 0, v_0 = c\), and (A22) becomes

\[(A24) \quad \tan \alpha = r \frac{\lambda_0}{4\pi \Delta x(0)}, \]

where \(\lambda_0 = \frac{\hbar}{mv_0}\).
where $\lambda_0$ is the centre wavelength of the photon packet. Relation (A24) coincides, except for some numerical factors $\approx 1$, with Verdet’s condition for the cone of coherence [227], [228], and also with the angular distance between the maximum and the first zero in diffraction at a slit of width $2\Delta x(0)$, though the above formulas were obtained without any use of holes, slits, or microscopes.

From (A20) it might appear that $v_{sx\infty} \to \infty$ if $\Delta x(0) \to 0$. This is not true because by (A13) $\Delta x(0) \to 0$ would imply $\Delta k_x \to \infty$, and our assumption of a narrow wavepacket would no longer hold. In fact, closer inspection shows that the spreading velocity (A11) is always limited when $\Delta x(0) \to 0$ [229]. For equal widths $\Delta k_x = \Delta k_y$ the transverse spreading is always larger than the longitudinal one since by (A17) and (A19) it is then

$$\frac{v_{sx\infty}}{v_{sy\infty}} = \frac{1}{1 - (v_0/c)^2} \geq 1.$$  

In the nonrelativistic domain we have

$$k_0^2 \ll \kappa^2,$$

and condition (A12) now takes the form

$$\Delta k_x, \Delta k_y, \Delta k_z \ll \kappa.$$  

The integrals involved in the averaging procedures may then be restricted to regions where the dispersion law (A3) can be approximated by $\omega(k) = mc^2/\hbar + \hbar k^2/(2m)$ from which one obtains the nonrelativistic relation $\Delta \vec{u}_g = \hbar \Delta k/m = \Delta p/m$. With (A13) the last condition in ordinary space reads

$$\Delta x(0), \Delta y(0), \Delta z(0) \gg 1/(2\kappa) = \frac{1}{2} \lambda_C.$$  

In the relativistic domain the spreading becomes slower as the velocity $v_0$ of the packet (centre) approaches $c$. In the zero-mass limit, $\kappa \to 0$, $\omega_0 \to c k_0$, formulas (A15) and (A21) lead to

$$v_{sx\infty} \to \frac{c}{k_0} \Delta k_x$$

$$v_{sy\infty} \to 0.$$  

Thus for photons there is a finite transverse spreading but no longitudinal spreading, in accordance with the well known absence of spreading of one-dimensional electromagnetic pulses composed of unidirectional waves.

Let us finally consider another length, the coherence length $\Delta c y$ of a wavepacket (in the $y$ direction). We want to show that this length is equal to the coherence length of the beam in which the wavepackets take part and that $\Delta c y$ does not spread out in time.

The coherence length of a wavepacket is defined by means of the autocorrelation function

$$\gamma(b) = \int \psi^*(x, y, z, t) \psi(x + b, z, t) \, d^3 x.$$  

(A25)
The function $|\gamma(b)|$ is maximal (=1) at $b = 0$, and $\Delta_y$ is defined as that value of $b$ where $|\gamma(b)|$ has decayed for the first time to $\exp(-1/2)$. $\Delta_y$ is closely related to the “mean peak width” of Hilgevoord and Uffink [225], [230], [231].

The coherence length $L_c$ of a beam is defined by means of the “contrast” or “visibility”

$$V(b) = (I_{\text{max}} - I_{\text{min}})/(I_{\text{max}} + I_{\text{min}})$$

where $I_{\text{max}}$ ($I_{\text{min}}$) is the maximal (minimal) intensity in the interference pattern obtained by dividing the beam into two sub-beams, delaying the one sub-beam by the distance $b$, and then reuniting the two. The length $L_c$ is defined as that value of $b$ where $V(b)$ has decayed to $\exp(-1/2)$. It is a quantity that is easy to measure.

In order to see that $\Delta_y$ equals $L_c$ suppose that the beam is a stream of equal wavepackets and that the subdivision of the beam means the subdivision of each single wavepacket. The reunited beam then means reunited wavepackets and for each wavepacket we have $\psi_f = \psi(x, y + b, z, t) + \psi(x, y, z, t)$. The probability that it will cause a count in the final counter, at any moment of time and at any place within the large counter (assuming 100% detection efficiency) is proportional to

$$W = \int_{-\infty}^{+\infty} |\psi_f|^2 d^3x$$

$$= \int |\psi(x, y, z, t)|^2 d^3x + \int |\psi(x, y + b, z, t)|^2 d^3x$$

$$+ 2\text{Re} \int \psi^*(x, y, z, t) \psi(x, y + b, z, t) d^3x.$$ 

The function $\psi_f$ need not be normalized because the original beam may be divided into more than the two sub-beams considered. The integrals in the second line of (A26) are equal and we denote them by $A$. The third line may then be written as $2A\text{Re}\gamma(b)$ when we use (A25) and observe that $\psi$ in (A25) is normalized but in (A26) perhaps not. Writing $\text{Re}\gamma = |\gamma| \cos \alpha$ we have

$$W(b) = 2A + 2A|\gamma(b)| \cos(\alpha(b)).$$

Now, the detection probability $W(b)$ of the wavepacket is proportional to the registered time averaged intensity $I(b)$ of the final beam, and when one assumes that $\cos(\alpha(b))$ in typical cases varies much more rapidly than does $|\gamma(b)|$ one obtains $W(b) = |\gamma(b)|$ and with this $\Delta_y = L_c$.

$\gamma(b)$ and with it $\Delta_y$ is independent of time because $\gamma(b)$ is just the mean value of the $y$-translation operator in the state $\psi(x, y, z, t)$, and this operator commutes with the Hamilton operator for free packets [232]. Thus, whereas the length $\Delta_y(t)$ of the packet may spread out with time, the coherence length $\Delta_y$ does not and is usually proportional to the minimum length $\Delta y_{\text{min}} = \Delta y(0)$ (assumed to occur at $t = 0$). For a Gaussian packet, for example, one obtains $\Delta_y = 2\Delta y_{\text{min}}$ [233].
There is no strict relation between the coherence length $\Delta_c y$ and the length $\Delta y(t)$ of the wavepacket. Any beam, with long or with short coherence length, may be considered to be one very long wavepacket, simply by superposing the shorter wavepackets that originally were conceived as its constituents. All that can be said is that the coherence length is of the order of a lower bound for the length of the constituent wavepackets. The difference between $\Delta y(t)$ and $\Delta_c y$ may always be small in the case of photon wavepackets which propagate in $y$ direction, since photon packets do not spread out in longitudinal but only in transverse directions. In fact, it is often found that for photon packets the coherence length as measured by means of the interference pattern extends over almost the whole length of the packet as measured by means of the distance the light travels during the mean life of the decaying state. In these cases the packet length is always close to its lower bound. In the case of wavepackets describing massive particles, however, there is considerable spreading even in the direction of propagation. In the neutron-interference experiment of Kaiser et al. [234], for example, the length $\Delta y(t)$ of the neutron packet at the time of its registration may be larger than the measured coherence length $\Delta_c y = 20 \, \text{Å}$ by more than a factor of $10^5$.

APPENDIX  B: Derivation of the Bell Inequality

In this appendix we derive the inequality (4.5)

\begin{equation}
    K := |E(a, b) + E(a, b') + E(a', b) - E(a', b')| \leq 2
\end{equation}

of Sec. 4.3 following Bell [235]. $E(a, b)$ is the expectation of the product $r_A r_B$ of two dichotomic variables, $r_A$ and $r_B$, either of which can take on the values $+1$ and $-1$ only:

\begin{equation}
    E(a, b) := P(+, + | a, b) + P(-, - | a, b) - P(+, - | a, b) - P(-, + | a, b)
\end{equation}

and

\begin{equation}
    P(r_A, r_B | a, b, \lambda) = \int P(r_A, r_B | a, b, \lambda) f(\lambda) d\lambda
\end{equation}

\begin{equation}
    r_A, r_B \in \{-1, +1\}
\end{equation}

\begin{equation}
    P(r_A, r_B | a, b, \lambda) = P_1(r_A | a, \lambda) P_2(r_B | b, \lambda)
\end{equation}

\begin{equation}
    f(\lambda) \geq 0, \quad \int f(\lambda) d\lambda = 1.
\end{equation}

Since $f(\lambda)$ does not depend on $r_A$ and $r_B$ it is possible to write (B2) as

\begin{equation}
    E(a, b) = \int d\lambda f(\lambda) \times \{P_1(+ | a, \lambda) P_2(+ | b, \lambda) + P_1(- | a, \lambda) P_2(- | b, \lambda) \}
\end{equation}
\[- P_1(+|a, \lambda) P_2(-|b, \lambda) - P_1(-|a, \lambda) P_2(+|b, \lambda),\]

and since the first factor on the right-hand side of condition (B3) does not depend on \( r_B \), nor the second on \( r_A \) (outcome independence), we may factorize the integrand

\[
E(a, b) = \int d\lambda f(\lambda) \{ P_1(+|a, \lambda) - P_1(-|a, \lambda) \} \{ P_2(+|b, \lambda) - P_2(-|b, \lambda) \}
\]

where

\[
\bar{A}(a, \lambda) = P_1(+|a, \lambda) - P_1(-|a, \lambda)
\]
\[
\bar{B}(b, \lambda) = P_2(+|b, \lambda) - P_2(-|b, \lambda).
\]

The probability nature of \( P_1 \) and \( P_2 \) means that

\[
0 \leq P_1 \leq 1, \quad 0 \leq P_2 \leq 1,
\]

hence

\[
|\bar{A}(a, \lambda)| \leq 1, \quad |\bar{B}(b, \lambda)| \leq 1.
\]

Using (B5) and the fact that \( \bar{A}(a, \lambda) \) does not depend on \( b \) nor \( \bar{B}(b, \lambda) \) on \( a \) (parameter independence) we have

\[
E(a, b) \pm E(a, b') = \int d\lambda f(\lambda) \bar{A}(a, \lambda) [\bar{B}(b, \lambda) \pm \bar{B}(b', \lambda)],
\]

and using (B6) for \( \bar{A}(a, \lambda) \), (B4) for \( f(\lambda) \) and the fact that \( f(\lambda) \) does not depend on \( a \) and \( b \) we obtain

\[
|E(a, b) \pm E(a, b')| \leq \int d\lambda f(\lambda) |\bar{B}(b, \lambda) \pm \bar{B}(b', \lambda)|.
\]

Likewise

\[
|E(a', b) \mp E(a', b')| \leq \int d\lambda f(\lambda) |\bar{B}(b, \lambda) \mp \bar{B}(b', \lambda)|.
\]

Now, we have

\[
|\bar{B}(b, \lambda) \pm \bar{B}(b', \lambda)| + |\bar{B}(b, \lambda) \mp \bar{B}(b', \lambda)| = 2 \max(|\bar{B}(b, \lambda)|, |\bar{B}(b', \lambda)|).
\]

This can be seen by observing that

\[
|x \pm y| + |x \mp y| = 2 \max(|x|, |y|),
\]

which in turn may be obtained by considering separately the various possible cases of positive and negative \( x \) and \( y \). For example, for \( x \) positive and \( y \) negative it is \( x = |x|, y = -|y| \) and \( x + y = |x| - |y| = \max(|x| - |y|, |y| - |x|) \), and \( x - y = |x| + |y| = |x| + |y| \), so that \( x + y + |x - y| = \max(|x| - |y| + |x| + |y|, |y| - |x| + |x| - |y|). \)
\[ |x| + |y| = \max(2|x|, 2|y|) \]. Then (B6) means \(|x| \leq 1, |y| \leq 1\) so that (B9) with (B6) can be written as

\[ |\bar{B}(b, \lambda) \pm \bar{B}(b', \lambda)| + |\bar{B}(b, \lambda) \mp \bar{B}(b', \lambda)| \leq 2. \]

With the normalization (B4) in (B7) and (B8) we arrive at

\[ |E(a, b) \pm E(a, b')| + |E(a', b) \mp E(a', b')| \leq 2 \]

and this includes the desired Bell inequality (B1), (4.5).

**APPENDIX C: EPR Joint Probability Formulas**

Here we derive formula (4.4) of Sec. 4.2

(4.4) \quad \begin{align*}
P(r_A, r_B|a, b) &= \frac{1}{4}(1 - r_A r_B \cos \vartheta) \\
&= \frac{1}{4} \left| \left( |+\rangle_1 |-\rangle_2 - |-\rangle_1 |+\rangle_2 \right) A(1, 2) \right|^2 \\
&\quad + \frac{1}{4} \left| \left( |+\rangle_1 |+\rangle_2 - |-\rangle_1 |+\rangle_2 \right) A(1, 2) \right|^2,
\end{align*}

from the rules of quantum mechanics. We have two spin-\(\frac{1}{2}\) particles (similar or not) in a state of zero total spin (spin singlet state). The particles move in opposite directions and each enters a Stern-Gerlach-type apparatus where it is deflected upwards or downwards with respect to the axis of its respective apparatus (cf. Fig. 4 in Sec. 4.1). Particle 1 enters apparatus A, which has its axis in the direction of the unit vector \(a\), and particle 2 enters apparatus B with axis \(b\). We first calculate the joint probability \(P(+, -|a, b)\) that A obtains an up deflection and B a down deflection.

The wave function of the two-particle system in the spin singlet state is (4.5)

\[ \Psi(1, 2) = \frac{1}{\sqrt{2}} \left[ |+\rangle_1 |-\rangle_2 - |-\rangle_1 |+\rangle_2 \right] A(1, 2), \]

where the up \(|+\rangle\) and down \(|-\rangle\) eigenfunctions of the spin component refer to a fixed but arbitrary axis. This is an entangled wave function. Only the spin part needs to be considered here. It happens to be antisymmetric, independent of whether it refers to similar or dissimilar particles.

Let the two eigenfunctions of the one-particle spin component operator of apparatus A be \(|a+\rangle\) and \(|a-\rangle\), and those of apparatus B \(|b+\rangle\) and \(|b-\rangle\). The operation of a Stern-Gerlach-type apparatus on either of the two particles leads to either \(|a+\rangle^{(1)} |b-\rangle^{(2)}\) or \(|b-\rangle^{(1)} |a+\rangle^{(2)}\) with equal probability. Hence

\[ P(+, -|a, b) = \frac{1}{2} \left| \left( |a+\rangle^{(1)} |b-\rangle^{(2)}, \Psi(1, 2) \right) \right|^2 \]

whether the particles are similar or not. In the case of dissimilar particles the factors 1/2 reflect the fact that we are not interested in distinguishing the particles but only in the average result. In the case of similar particles the probability expressions should be invariant under particle-label permutation (Sec. 3.3), and this is now the reason for the factors 1/2.
To be able to evaluate expression (C2) we need the eigenfunctions \( |a+\rangle, |b-\rangle \) etc. in terms of the eigenfunctions \(|+\rangle, |-\rangle\), which refer to the fixed but arbitrary axis employed in (C1). We let this axis coincide with A’s axis \( \mathbf{a} \), so that

(C3) \[ |a+\rangle = |+\rangle, \quad |a-\rangle = |-\rangle. \]

Let the axis of B form an angle \( \vartheta \) with the axis of A We then have to express the eigenfunctions \(|b+\rangle, |b-\rangle\) in the rotated system B in terms of the eigenfunctions of system A [236] [47, p. 1073]

\[ |b+\rangle = \cos \frac{\vartheta}{2} \exp[i(\beta + \gamma)/2] |+\rangle + i \sin \frac{\vartheta}{2} \exp[-i(\beta - \gamma)/2] |-\rangle \]

(C4) \[ |b-\rangle = i \sin \frac{\vartheta}{2} \exp[i(\beta - \gamma)/2] |+\rangle + \cos \frac{\vartheta}{2} \exp[-i(\beta + \gamma)/2] |-\rangle. \]

The angles \( \beta \) and \( \gamma \) define possible rotations of the other axes but will not appear in the final formulas. Inserting (C1), (C3) and (C4) into (C2) and observing the orthonormality of the functions \(|+\rangle, |-\rangle\) for the respective particles we obtain

(C5) \[ P(\+,-|a,b) = \frac{1}{2} \left( \cos \frac{\vartheta}{2} \right)^2 = \frac{1}{4} (1 + \cos \vartheta). \]

Proceeding in the same way in the other cases (A up, B up; A down, B up; A down, B down) we obtain

(C6) \[ P(-,+|a,b) = P(\+,-|a,b), \]

(C7) \[ P(\+,+|a,b) = P(-,-|a,b) = \frac{1}{2} \left( \sin \frac{\vartheta}{2} \right)^2 = \frac{1}{4} (1 - \cos \vartheta). \]

(C5), (C6) and (C7) may be summarized in the form

(C8) \[ P(r_A, r_B|a,b) = \frac{1}{4} (1 - r_A r_B \cos \vartheta) \]

where \( r_A, r_B \in \{-1, +1\} \). This is formula (4.4) of Sec. 4.2.

It is amusing to notice that we can also obtain formula (4.4) by proceeding as if the following situation were to hold: after any single interaction either of the two protons of Fig. 4 in Sec. 4.1 has a definite direction of spin (i.e. is a spin-up eigenfunction of some \( s_{z'} \), cf. Sec. 2.2), say \( \vec{\sigma} \) and \( -\vec{\sigma} \), respectively, where \( \vec{\sigma} \) is a unit vector, and the total spin is zero. The direction of \( \vec{\sigma} \) varies from one interaction to the other in such a way that there is spherical symmetry on the average. The Stern-Gerlach-type apparatus which obtains its proton first, say A, turns the spin of its proton into either up or down direction with respect to its axis, say into direction \( + \mathbf{a} \), and at the same time turns the spin of the other proton into the opposite direction \( - \mathbf{a} \). Thus here angular momentum is conserved within the system of the two protons, and the apparatuses are not involved in angular-momentum conservation. Then B’s apparatus turns the spin of its proton from direction \( - \mathbf{a} \) into either up or down
direction with respect to B’s axis \( \mathbf{b} \), without, however, influencing the spin of proton 1 any more. Here, angular-momentum conservation involves proton 2 and apparatus B, as mentioned in Sec. 2.2 on the Stern-Gerlach experiment.

The spin direction \( \mathbf{\sigma} \) plays the role of the parameter \( \lambda \) in the general consideration of Sec. 4.3. From the point of view of quantum mechanics the parameter \( \mathbf{\sigma} \), interpreted in the above way, is hidden. Of course, in the one-particle states |\( + \rangle \) etc. \( \mathbf{\sigma} \) is not hidden but explicitly specifies the spin direction, but the two-particle state \( \Psi(1, 2) \) of (C1), which is the only state existing after the interaction at O, has spherical symmetry, and there can be no parameter specifying any direction in such a state in quantum mechanics. This is why we used the proviso as if.

In order to prove our above assertion regarding the as-if derivation of (4.4) we observe that the conditional probability of obtaining the result \( r_A \) (i.e. either + or −) with respect to the axis \( \mathbf{a} \), given that the spin of the proton before it enters the apparatus points in the direction \( \mathbf{b} \), is, by (C3) and (C4),

\[
P(r_A|\mathbf{a}, \mathbf{b}) = \frac{1}{2} (1 + r_A \mathbf{b})
\]

and if we replace the axis \( \mathbf{b} \) by the axis \( \mathbf{\sigma} \) and the angle \( \vartheta \) between \( \mathbf{a} \) and \( \mathbf{b} \) by the angle \( \alpha \) between \( \mathbf{a} \) and \( \mathbf{\sigma} \) we get the probability that A obtains the result \( r_A \) given that the spin of the proton before it entered the apparatus pointed in the direction \( \mathbf{\sigma} \)

\[
(C9) \quad P(r_A|\mathbf{a}, \mathbf{\sigma}) = \frac{1}{2} (1 + r_A \mathbf{\sigma}) = \frac{1}{2} (1 + r_A \cos \alpha).
\]

However, the probability of B obtaining the result \( r_B \) is not the analogous formula \((1 + r_B(-\mathbf{\sigma})\mathbf{b})/2\), because A has turned not only the spin of proton 1 into the direction \( r_A \mathbf{a} \) but also the spin of proton 2 into direction \( -r_A \mathbf{a} \) and then separated the two protons. Thus here we have to replace \(-\mathbf{\sigma}\) by \(-r_A \mathbf{a}\), and B’s probability is

\[
(C10) \quad P(r_B|\mathbf{b}, -r_A \mathbf{a}) = \frac{1}{2} (1 + r_B(-r_A \mathbf{a})\mathbf{b}).
\]

The conditional joint probability of A obtaining \( r_A \) and B obtaining \( r_B \) is given by the product of (C9) with (C10)

\[
P(r_A, r_B|\mathbf{a}, \mathbf{b}, \mathbf{\sigma}) = \frac{1}{2} (1 + r_A \mathbf{\sigma}) \frac{1}{2} (1 - r_A r_B \cos \vartheta).
\]

If we integrate over all directions of \( \mathbf{\sigma} \), assuming an isotropic distribution, we obtain

\[
(C11) \quad P(r_A, r_B|\mathbf{a}, \mathbf{b}) = \frac{1}{4} \pi \int_{-\pi}^{\pi} d\varphi \int_{0}^{\pi} \sin \alpha \, d\alpha \frac{1}{4} (1 + r_A \cos \alpha) (1 - r_A r_B \cos \vartheta)
\]

where the system of coordinates \((x, y, z)\) for the integration is chosen such that the \( z \) axis is in the direction \( \mathbf{a} \), and \( \mathbf{b} \) lies in the \( x-z \) plane:

\[
\mathbf{a} = (0, 0, 1), \quad \mathbf{b} = (\sin \vartheta, 0, \cos \vartheta),
\]
so that
\[ \vec{\sigma} = (\sin \alpha \cos \varphi, \sin \alpha \sin \varphi, \cos \alpha), \]
\[ \mathbf{ab} = \cos \vartheta, \quad \vec{\sigma} \mathbf{a} = \cos \alpha, \quad \vec{\sigma} \mathbf{b} = \sin \vartheta \cos \varphi \sin \alpha + \cos \vartheta \cos \alpha. \]

It is not difficult to verify that (C11) leads to formula (C8) or (4.4). Such an as-if procedure is in fact possible in any EPR situation, not only in that of two spin-\( \frac{1}{2} \) particles in the singlet state.

**APPENDIX D: EPR Probabilities in Different Systems of Eigenfunctions**

We shall show here that not only Stern-Gerlach-type devices (as in Sec. 4.2), but any devices that obey the formulas of quantum mechanics are incapable of building faster-than-light warning systems [237] - [241]. The proof uses the fact that in quantum mechanics the apparatuses are represented by operators. Different apparatuses mean different operators, and different operators in general mean different systems of eigenfunctions, and these can be transformed into one another. The normalized quantum-mechanical wave function for a system of two similar particles can be written in the form

\[ \Psi_{SA}(1, 2) = C \sum_{k=1}^{\infty} [\zeta_k(1) u_k(2) \pm \zeta_k(2) u_k(1)], \]

which is formula (4.1) from Sec. 4.1 with a properly symmetrized function. The \( u_k(x) \) form a complete set of orthonormal eigenfunctions of some operator representing the apparatus of experimenter B. \( C \) is a real overall normalization constant, which need not be equal to \( 1/\sqrt{2} \) because the functions \( \zeta_k(x) \), which describe the particle at A, are not presupposed to be orthonormal. With the expansion

\[ \zeta_k(x) = \sum_l a_{lk} w_l(x), \]

where the \( w_l(x) \) form a complete set of orthonormal eigenfunctions of some operator representing the apparatus of experimenter A, (D1) can be written as

\[ \Psi_{SA}(1, 2) = C \sum_{lk} a_{lk} [w_l(1)u_k(2) \pm w_l(2)u_k(1)]. \]

We first calculate the probability \( P_1(u_n, w_m) \) of a transition where the state (D3) changes into either the state \( w_m(1)u_n(2) \) or the state \( w_m(2)u_n(1) \) with equal probability. This is the probability that in B’s apparatus there will be a particle (“whichever of the two it is”) with state \( u_n \) and in A’s apparatus a particle with state \( w_m \). It is [see the remark on (C2) in Appendix C]

\[ P_1(u_n, w_m) = \frac{1}{2} \left| \left( C \sum_{kl} a_{lk} (w_l(1)u_k(2) \pm w_l(2)u_k(1)), w_m(1)u_n(2)) \right) \right|^2 \]
\[ + \frac{1}{2} \left( C \sum_{kl} a_{lk} (u_l(1) u_k(2) \pm w_l(2) u_k(1)) , w_m(2) u_n(1) \right)^2 \]

\[ = \frac{C^2}{2} \sum_{kl} a_{lk}^* \left( (u_l(1) u_k(2) , w_m(1) u_n(2)) \pm (w_l(2) u_k(1) , w_m(1) u_n(2)) \right)^2 \]

\[ + \frac{C^2}{2} \sum_{kl} a_{lk}^* \left( (w_l(1) u_k(1) , w_m(2) u_n(1)) \pm (w_l(1) u_k(2) , w_m(2) u_n(1)) \right)^2 . \]

With \((u_l(1) u_k(2) , w_m(1) u_n(2)) = (w_l(2) u_k(1) , w_m(2) u_n(1)) = \delta_{lm} \delta_{kn}\) this becomes

\[ P_1(u_n, w_m) = \frac{C^2}{2} \left| a_{mn}^* \pm \sum_{kl} a_{lk}^* \left( w_l(2) , u_k(1) \right) \left( u_k(1) , w_m(1) \right) \right|^2 \]

\[ + \frac{C^2}{2} \left| a_{mn}^* \pm \sum_{kl} a_{lk}^* \left( w_l(1) , u_k(1) \right) \left( u_k(2) , w_m(2) \right) \right|^2 . \]

As the scalar products in (D4) are zero the formula reduces to

\[ P_1(u_n, w_m) = C^2 |a_{mn}|^2 . \]

The scalar products are zero because the final wave function \(u_l\) of the particle in apparatus B and the final wave function \(w_l\) of the particle in apparatus A are well separated from each other and do not overlap. The interaction of a wavepacket from the entangled system (D1) with that apparatus that operated first, had led to reduction and to disentanglement of the system.

Second we consider the probability of B observing that his particle assumes the state \(u_n\) irrespective of the state of A’s particle. This is obtained by summing the probability \(P_1(u_n, w_m)\) over all states of A’s particle

\[ P_2(u_n) = C^2 \sum_m |a_{mn}|^2 , \]

and we want to show that this probability is unchanged when A uses a different apparatus. Let the eigenfunctions of the new operator corresponding to the new apparatus be \(w'_m(x)\). They are related to the eigenfunctions \(w_m(x)\) of the original operator by

\[ w'_m(x) = \sum_k U_{mk} w'_k(x) . \]

\(U_{mk}\) is a unitary matrix \((\sum_{m} U_{jm}^* U_{km} = \delta_{jk})\), and the index \(k\) may even be continuous and the sum an integral. Actually, the transformation (D6) need not even be unitary \cite{241, 242}, but we will not pursue this here. By inserting (D6) into (D3) we can write \(\Psi_{SA}(1, 2)\) in the form

\[ \Psi_{SA}(1, 2) = C \sum_{mn} a_{mn} \left[ u_n(2) \sum_k U_{mk} w'_k(1) \pm u_n(1) \sum_k U_{mk} w'_k(2) \right] \]
\[ = C \sum_{l} \sum_{k} a_{jk} U_{jl} \left[ w'_{l}(1)u_{k}(2) \pm w'_{l}(2)u_{k}(1) \right]. \]

The probability that this changes into either \( w'_{m}(1)u_{n}(2) \) or \( w'_{m}(2)u_{n}(1) \) is \( C^2|b_{mn}|^2 \), by analogy with (D3) and (D5). Hence

\[
P_2(u_n) = C^2 \sum_{m} |b_{mn}|^2 = C^2 \sum_{m} b_{mn}^{*}b_{mn} = C^2 \sum_{m} a_{jn}^{*}U_{jm}^{*}a_{kn}U_{km}
\]

\[
= C^2 \sum_{m} a_{jn}^{*}a_{kn} \sum_{m} U_{jm}^{*}U_{km} = C^2 \sum_{k} |a_{kn}|^2,
\]

and this coincides with (D5), concluding the proof of our assertion.

Finally we want to show that even if A chooses not to use his apparatus and to do nothing this will make no difference. In this case, when B’s particle assumes the state \( u_n \), A’s particle will assume some correlated state \( \zeta_n \). The probability of B’s particle assuming the state \( u_n \) is the probability of the transition where the state \( \Psi_{SA}(1, 2) \) changes either into the state \( \zeta_n(1)u_n(2) \) or into the state \( \zeta_n(2)u_n(1) \).

When \( \zeta_n \) is expressed in terms of the \( w_m \), according to formula (D2), the two states become

\[
\sum_{m} a_{mn}w_{m}(1)u_{n}(2) \quad \text{and} \quad \sum_{m} a_{mn}w_{m}(2)u_{n}(1),
\]

respectively, and the transition probability becomes

\[
(D7) \quad P'_{2}(u_n) = \frac{1}{2} \left| \left( \Psi_{SA}(1, 2), \sum_{m} a_{mn}w_{m}(1)u_{n}(2) \right) \right|^{2} \times \left| \sum_{m} a_{mn}w_{m}(1)u_{n}(2) \right|^{-2}
\]

\[
+ \frac{1}{2} \left| \left( \Psi_{SA}(1, 2), \sum_{m} a_{mn}w_{m}(2)u_{n}(1) \right) \right|^{2} \times \left| \sum_{m} a_{mn}w_{m}(2)u_{n}(1) \right|^{-2}.
\]

The denominators are different from 1 because the \( a_{mn} \) come in via the \( \zeta_n(x) \) in formula (D2), and the \( \zeta \)'s are not normalized. With (D3) the first term of expression (D7) becomes

\[
T_1 = \left| \sum_{klm} a_{lk}a_{mn} \left[ (w_l(1), w_m(1))(u_k(2), u_n(2)) + (w_l(2), u_n(2))(u_k(1), w_m(1)) \right] \right|^{2}
\]

\[
\times \left| \sum_{mk} a_{mn}^{*}a_{kn} \left( w_m(1)u_n(2), w_k(1)u_n(2) \right) \right|^{-1} \times \frac{C^2}{2}
\]

\[
= \frac{C^2}{2} \sum_{m} |a_{mn}a_{mn}|^2 \times \left| \sum_{m} a_{mn}^{*}a_{mn} \right|^{-1} = \frac{C^2}{2} \sum_{m} |a_{mn}|^2.
\]

The second term of (D7) leads to the same expression, so

\[
P'_{2}(u_n) = C^2 \sum_{m} |a_{mn}|^2 = P_2(u_n),
\]

which is what we wanted to show.
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