A NEW $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-INVECTOR OF DESSINS D’ENFANTS

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Abstract. We study the action of $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ on the category of Belyi functions (finite, étale covers of $\mathbb{P}^1_{\mathbb{Q}} \setminus \{0, 1, \infty\}$). We describe a new combinatorial $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-invariant for a certain class of Belyi functions. As a corollary, we obtain that for all $k < 2\sqrt{3}$ and all positive integers $N$, there is an $n \leq N$ such that the set of degree $n$ Belyi functions of a particular rational Nielsen class must split into at least $\Omega \left( k\sqrt{N} \right)$ Galois orbits. In addition, we define a new version of the Grothendieck-Teichmüller group $\hat{GT}$ into which $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ embeds.

1. Introduction

In his Esquisse d’un Programme [7], Grothendieck described a research program to understand the structure of $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$. One idea is that there is a faithful, outer action of $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ on the Teichmüller tower of profinite mapping class groups (the étale fundamental groups of the moduli spaces $M_{g,n}$ of curves of genus $g$ with $n$ ordered marked points over $\overline{\mathbb{Q}}$). Grothendieck conjectured that the group of outer automorphisms of the Teichmüller tower is in fact isomorphic to $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$, and that the action is “generated” on the dimension 1 moduli spaces with “relations” in dimension 2. The moduli space $M_{0,4}$ is of dimension 1, and is isomorphic to $\mathbb{P}^1_{\overline{\mathbb{Q}}} \setminus \{0, 1, \infty\}$, and therefore as part of the program, one wishes to study the action of $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ on the category of étale covers of $\mathbb{P}^1_{\mathbb{Q}} \setminus \{0, 1, \infty\}$. Grothendieck’s dessins d’enfants encode the covers combinatorially, and one can try to understand the faithful action of $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ on them. A first step is to determine a set of invariants, perhaps algebraic, arithmetic, geometric, or topological in nature, that can distinguish distinct $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-orbits of dessins. In this paper, we construct a new invariant for a certain class of dessins and describe it combinatorially. The Grothendieck-Teichmüller group $\hat{GT}$ is the group of inertia-preserving automorphisms of the tower of genus 0 profinite mapping class groups, and $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ embeds into $\hat{GT}$. It is unknown whether the embedding $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \hookrightarrow \hat{GT}$ is an isomorphism. We also construct a new version of $\hat{GT}$ into which $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ still embeds.

The key idea is to consider commutative squares of the form:

\[
\begin{array}{ccc}
Y & \xleftarrow{t} & X \\
\downarrow & & \downarrow \\
\mathbb{P}^1 & \xleftarrow{\ell \cdot \frac{t}{(t+1)^2}} & \mathbb{P}^1
\end{array}
\]
with \(X\) the normalization of the fibered product \(Y \times_{\mathbb{P}^1} \mathbb{P}^1\). In certain cases, \(\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})\)-invariants of the left morphism extend to \(\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})\)-invariants of the right morphism. In particular, by considering the cycle types of the monodromy generators of the left morphism as a \(\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})\)-invariant, we partition the set of possible right morphisms into \(\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})\)-invariant subsets. We describe this new invariant combinatorially as the \textit{square-root cycle type class}. It can help distinguish \(\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})\)-orbits of Belyi functions that have the same monodromy cycle type over 0 and \(\infty\). In Theorems 3.4 and 3.5 we prove that our invariant is substantially finer than the rational Nielsen class (and therefore substantially finer than the monodromy group and the monodromy cycle type). In particular, we prove that for all \(k < 2\sqrt{2}\) and all positive integers \(N\), there is an \(n \leq N\) such that the set of degree \(n\) Belyi functions of a particular rational Nielsen class must split into at least \(\Omega\left(k\sqrt{N}\right)\) Galois orbits.

By viewing the commutative squares as pulling back étale covers of a genus 0 smooth Deligne-Mumford curve to \(\mathbb{P}^1 \setminus \{0, 1, \infty\}\), we obtain a constraint satisfied by the image of \(\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})\) in \(\hat{\Gamma}\) in Theorem 3.7. The constraint is stronger than the \(\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})\)-invariance of the square-root cycle type class because it is equivalent to the \(\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})\)-equivariance of a certain push-forward morphism \(t_*\) étale fundamental groups, which implies the \(\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})\)-invariance of the square-root cycle type class. The fact that the square-root cycle type class invariant is sufficiently fine to distinguish many \(\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})\)-orbits of Belyi functions that lie within the same rational Nielsen class demonstrates the strength of our constraint on the image of \(\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})\) in \(\hat{\Gamma}\). In Theorem 3.8 we consider a different morphism from \(\mathbb{P}^1 \setminus \{0, 1, \infty\}\) to a genus 0 smooth Deligne-Mumford curve to obtain a second new version of \(\hat{\Gamma}\) into which \(\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})\left(\sqrt{-3}\right)\) embeds.

The structure of this paper is as follows. In Section 2 we recall the basic definitions and discuss previous work. In Section 3 we state our main results, and in Section 4 we prove the basic properties of our new invariant. In Section 5 we prove our main theorems and we prove that our invariant is stronger than the rational Nielsen class invariant in certain cases. In Section 6 we prove that \(\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})\) and \(\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})\left(\sqrt{-3}\right)\) embed into our first and second new versions of \(\hat{\Gamma}\), respectively, and in Section 7 we give concluding remarks and state an open problem. Elementary computations are deferred to Appendix A.

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### 2. Previous Work

Unless otherwise specified, a curve will mean a smooth, irreducible, projective, algebraic curve over \(\mathbb{C}\), or equivalently a compact Riemann surface. We will denote by \(\mathbb{P}^1\) the complex projective line \(\mathbb{P}^1_\mathbb{C}\). Fix an embedding \(\overline{\mathbb{Q}} \hookrightarrow \mathbb{C}\). Let \(\chi: \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \hat{\mathbb{Z}}^\times\) denote the cyclotomic character.
Figure 1. Generators for $\pi_1\left(\mathbb{P}^1 \setminus \{0, 1, \infty\}, \overrightarrow{01}\right)$. The base-point is the tangent vector $\overrightarrow{01}$ at 0. The homotopy class $x_0$ is given by moving from 0 toward 1, in the counterclockwise around 0, and back to 0 along the segment between 0 and 1. The homotopy class $x_1$ is defined similarly. The homotopy class $x_\infty$ is defined by moving from 0 to $P$ along the segment, traversing the large circle clockwise, returning to $P$, and then returning to 0 along the segment. It is evident that $x_0 x_1 x_\infty = 1$.

Fundamental groups are topological unless otherwise specified. We fix a generating set $x_0, x_1, x_\infty$ of $\pi_1\left(\mathbb{P}^1 \setminus \{0, 1, \infty\}, \overrightarrow{01}\right)$ such that $x_0 x_1 x_\infty = 1$ and in $\mathbb{C} \setminus \{0, 1\}$ in Figure 1 the loops have winding numbers of 1, 0, $-1$ about 0 and 0, 1, $-1$ about 1, respectively. Sending the generators $x, y$ of $F_2$, the free group on two letters, to $x_0, x_1$, respectively, yields an isomorphism $F_2 \cong \pi_1\left(\mathbb{P}^1 \setminus \{0, 1, \infty\}, \overrightarrow{01}\right)$.

By a weak action of a group $G$ on a category $\mathcal{C}$, we mean a group homomorphism from $G$ to the group of equivalences from $\mathcal{C}$ to $\mathcal{C}$, modulo natural isomorphism. Let $\hat{G}$ denote the profinite completion, $G^{ab}$ the abelianization, and $G'$ the derived subgroup of a group $G$. Given a profinite group $G$, and elements $g_1, g_2 \in G$ and $f \in \hat{F}_2$, let $f(g_1, g_2)$ denote the image of $f$ under the continuous homomorphism from $\hat{F}_2$ to $G$ that sends the generators $x, y$ to $g_1, g_2$, respectively. When group-theoretic constructions are applied to a profinite group, we take the construction
to be in the category of profinite groups (e.g. the derived subgroup of a profinite group is the closed subgroup topologically generated by the commutator elements).

For a positive integer \( n \), let \([n] = \{1, 2, \ldots, n\}\). Given a partition \( \lambda \vdash n \), let the ramification number of \( \lambda \), which we denote by \( \text{ram}(\lambda) \), equal \( n - k \), where \( k \) is the number of non-empty parts of \( \lambda \). We can extend the definition of ram to permutations \( \sigma \in S_n \) by defining the ramification number of \( \sigma \) to be the ramification number of the cycle type partition of \( \sigma \).

2.1. Tangential base-points. Deligne \[3\] §15 gives a full treatment of tangential base-points. Let \( X \) be a smooth, proper curve over an algebraically closed field \( k \) of characteristic 0, let \( S \) be finite set of closed points of \( X \), and let \( X = X \setminus S \).

Given a closed point \( p \in S \) and a local coordinate \( z \in \mathcal{O}_{X,p} \), we have a morphism \( f : \text{Spec} \, k[[z]] \to \overline{X} \) that sends the closed point of \( \text{Spec} \, k[[z]] \) to \( p \). Localizing at \( z \) yields a morphism \( f_z : \text{Spec} \, k((z)) \to X \). We can pass to the field \( K \) of formal Puiseux series over \( k \), which is an algebraic closure of \( k((z)) \), to obtain a morphism \( f_z : \text{Spec} \, K \to X \). This yields a fiber functor from the category \( \text{Ét}(X) \) of finite étale covers of \( X \) to the category of finite sets, and hence an étale fundamental group \( \pi^\text{ét}_1(X, (p, z)) \). By \[3\] Lemme 15.25, if \( z_1 \in \mathcal{O}_{X,p} \) is another local coordinate such that \( z - z_1 \) vanishes to order at least 2 at \( p \), then \( \pi^\text{ét}_1(X, (p, z_1)) \) are canonically isomorphic. In the case in which \( X = \mathbb{P}^1_k \), then the tangential base-point \( \overline{0}1 \) is given by choosing \( p \) to be 0 and \( z \) to be the usual coordinate on \( \mathbb{P}^1 \).

Suppose that a fixed embedding of \( k \) into \( \mathbb{C} \) is given. Then, the local coordinate \( z \) yields a tangent vector \( \frac{\partial}{\partial z} \in T_p(\mathbb{C}^n) \), where \( \mathbb{C}^n \) denotes base-change to \( \mathbb{C} \) followed by analytification. It follows from \[3\] Construction 15.17 that there is a canonical isomorphism between \( \pi_1(\mathbb{C}^n, \frac{\partial}{\partial z}) \) and \( \pi^\text{ét}_1(X, (p, z)) \). If \( c \in \mathbb{R}_{>0} \), then there is a canonical isomorphism between \( \pi_1(\mathbb{C}^n, \frac{\partial}{\partial z}) \) and \( \pi_1(\mathbb{C}^n, c \frac{\partial}{\partial z}) \). It follows that if \( z_1 \in \mathcal{O}_{X,p} \) is a local coordinate such that there exists a constant \( c \in \mathbb{R}_{>0} \cap k \) such that \( z_1 - c z \) vanishes to order at least 2 at \( p \), then there are canonical isomorphisms

\[
\pi^\text{ét}_1(X, (p, z)) \cong \pi_1\left(\mathbb{C}^n, \frac{\partial}{\partial z_1}\right) \cong \pi_1\left(\mathbb{C}^n, \frac{\partial}{\partial z_1}\right) \cong \pi^\text{ét}_1(X, (p, z_1)).
\]

Similar results hold even if \( X \) is a smooth, one-dimensional Deligne-Mumford stack over \( k \).

2.2. The action of \( \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \) on profinite fundamental groups. Let \( \mathfrak{p} \) be a geometric (potentially tangential) point of \( \mathbb{P}^1_{\overline{\mathbb{Q}}} \setminus \{0, 1, \infty\} \), let \( p \) be the corresponding geometric point of \( \mathbb{P}^1_{\overline{\mathbb{Q}}} \), and let \( p_c \) be the base-change of \( \mathfrak{p} \) to \( \mathbb{C} \). There is an isomorphism between étale fundamental groups and profinite completions of topological fundamental groups \[8\] Exposé X, Corollaire 1.8:

\[
\pi^\text{ét}_1\left(\mathbb{P}^1_{\overline{\mathbb{Q}}} \setminus \{0, 1, \infty\}, \mathfrak{p}\right) \cong \pi^\text{et}_1\left(\mathbb{P}^1_{\mathbb{C}} \setminus \{0, 1, \infty\}, p_c\right) \cong \pi_1(\mathbb{P}^1_{\mathbb{C}} \setminus \{0, 1, \infty\}, p_c) \cong \hat{F}_2,
\]

where the first two isomorphisms are canonical and the last given by our choice of generators for \( \pi_1(\mathbb{P}^1_{\mathbb{C}}, p_c) \). Furthermore, there is a homotopy exact sequence of étale fundamental groups \[8\] Exposé IX, Théorème 6.1:

\[
1 \to \pi^\text{et}_1\left(\mathbb{P}^1_{\overline{\mathbb{Q}}} \setminus \{0, 1, \infty\}\right) \to \pi^\text{et}_1\left(\mathbb{P}^1_{\overline{\mathbb{Q}}} \setminus \{0, 1, \infty\}\right) \to \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to 1.
\]
This induces an outer action
\[
\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \text{Out}(\hat{F}_2).
\]
Choosing the tangential base-point $\overrightarrow{01}$ gives a splitting of the homotopy exact sequence of Equation 11 and hence a lifting of the outer action of Equation 2 to an action
\[
\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \text{Aut}(\hat{F}_2).
\]
The scheme $\mathbb{P}^1_{\mathbb{Q}} \setminus \{0, 1, \infty\}$ can be replaced by any quasi-compact, geometrically connected scheme $X$ over $\mathbb{Q}$ and $\mathbb{P}^1_{\mathbb{Q}} \setminus \{0, 1, \infty\}$ (resp. $\mathbb{P}^1_{\mathbb{C}} \setminus \{0, 1, \infty\}$) by the base-change of $X$ to $\overline{\mathbb{Q}}$ (resp. $\mathbb{C}$), but the choice of $\mathbb{P}^1_{\mathbb{Q}} \setminus \{0, 1, \infty\}$ has special properties, such as Theorem 2.1, to be outlined in the next subsection.

2.3. Belyi functions and dessins d’enfants. A Belyi function is a finite, étale, connected cover of $\mathbb{P}^1_{\mathbb{Q}} \setminus \{0, 1, \infty\}$. Due to [8, Exposé X, Corollaire 1.8], we can equivalently view a Belyi function as a finite, étale, connected cover of $\mathbb{P}^1_{\mathbb{C}} \setminus \{0, 1, \infty\}$, which is a meromorphic function on a curve $X$ that is unbranched outside $\{0, 1, \infty\}$.

A dessin d’enfant is a bipartite, connected graph $G$ with parts $V_0, V_1$ together with an embedding $G \to X$ where $X$ is a compact, oriented, topological 2-manifold, whose image is the 1-skeleton of a CW-complex structure on $X$.

The following data are then equivalent [11]:

1. an isomorphism class of Belyi functions of degree $n$;
2. an isomorphism class of dessin d’enfants with $n$ edges; and
3. a conjugacy class of transitive representations $(F_2 \cong \pi_1(\mathbb{P}^1 \setminus \{0, 1, \infty\}, \overrightarrow{01})) \to S_n$.

To a Belyi function $f$, we associate the dessin $f^{-1}(\{0, 1\})$ with $V_0 = f^{-1}(0)$ and $V_1 = f^{-1}(1)$, and the monodromy representation of $h : F_2 \cong \pi_1(\mathbb{P}^1 \setminus \{0, 1, \infty\}, \overrightarrow{01}) \to S_n$. It follows from the Riemann Existence Theorem that one can associate a Belyi function to any dessin or transitive permutation representation $F_2 \to S_n$.

There is a natural action of $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ on the category of Belyi functions: viewing the category of Belyi functions as the category of étale covers of $\mathbb{P}^1_{\mathbb{Q}} \setminus \{0, 1, \infty\}$ and given an automorphism $\sigma \in \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$, we can base-change by $\text{Spec} \sigma$. There is an action of $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ on the category of representations of $\hat{F}_2$ on finite sets where $\sigma \in \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ acts by sending $h$ to $h \circ \alpha(\sigma)$; the image of $h$ is defined only up to isomorphism because $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ acts canonically only by outer automorphisms.

The category of Belyi functions is equivalent to the category of representations of $F_2$ on finite sets, (where $F_2$ is identified with $\pi_1(\mathbb{P}^1 \setminus \{0, 1, \infty\}, \overrightarrow{01})$) which is in turn equivalent to the category of representations of $\hat{F}_2$ on finite sets and therefore Equation 2 yields a weak action of $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ on the category of Belyi functions. The fact that the two actions are equivalent follows from the definition of the exact sequence in Equation 2 and the fact that the group of isomorphism classes of self-equivalences of the category of representations of $\hat{F}_2$ on finite sets is canonically isomorphic to $\text{Out}(\hat{F}_2)$.

A key result regarding the action of $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ follows from following theorem of Belyi.
Theorem 2.1 ([1], Theorem 4). A curve admits a Belyi function if it is defined over $\mathbb{Q}$.

By considering the action of $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ on the $j$-invariants of smooth genus $1$ curves over $\mathbb{Q}$, it follows the actions of $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ on $\hat{F}_2$, the category of Belyi functions, and the set of isomorphism classes of dessins are faithful [1].

2.4. The embedding of $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ into $\hat{G}T$. The outer action of $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ on $\hat{F}_2$ yields an injection of $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ into the (profinite) Grothendieck-Teichmüller group $\hat{G}T$.

Definition 2.2. Let $\hat{G}T_u$ be the set of pairs $(\lambda, f)$ with $\lambda \in \hat{\mathbb{Z}}^\times$ and $f \in \hat{F}_2'$ such that the endomorphism of $\hat{F}_2$ defined by $x \mapsto x^\lambda$ and $y \mapsto f^{-1}y^\lambda f$ is an automorphism of $\hat{F}_2$. The composite of two elements $(\lambda, f)$ and $(\lambda, f')$ of $\hat{G}T_u$ is defined as

$$(\lambda, f) \cdot (\lambda, f') = (\lambda \lambda', f' \cdot (x^\lambda, f^{-1}y^\lambda f)).$$

The unit of $\hat{G}T_u$ is $(1, 1)$.

The natural monoid homomorphism $\hat{G}T_u \to \text{Out} (\hat{F}_2)$ is injective, and the image is the subgroup of outer automorphisms that preserve the conjugacy classes of the inertia subgroups $\langle x \rangle$, $\langle y \rangle$, and $\langle xy \rangle$. It follows that $\hat{G}T_u$ is a group, and has the natural structure of a subgroup of $\text{Out} (\hat{F}_2)$ that lifts to a subgroup of $\text{Aut} (\hat{F}_2)$.

An important property of $\hat{G}T_u$ is the following result.

Proposition 2.3 ([1], Proposition 2). Suppose that $\theta \in \text{Out} (\hat{F}_2)$ preserves the conjugacy classes of the inertia subgroups $\langle x \rangle$, $\langle y \rangle$, $\langle (xy)^{-1} \rangle$. Then, $\theta$ has a unique lift $\tilde{\theta}$ to $\hat{G}T_u$.

Definition 2.4 ([13], §1.1). Let $\hat{G}T$ be the set of pairs $(\lambda, f) \in \hat{G}T_u$ such that

(i) $f(x, y)f(y, x) = 1$;
(ii) $f(x, y)x^m f(z, x)z^m f(y, z)z^m = 1$, where $z = (xy)^{-1}$ and $m = \frac{\lambda - 1}{2}$;
(iii) $f(x_{34}, x_{45})f(x_{51}, x_{12})f(x_{23}, x_{34})f(x_{45}, x_{31})f(x_{12}, x_{23}) = 1$ in the profinite mapping class group $\hat{\Gamma}_{0, 5}$, where $x_{ij}$ are the standard generators of $\hat{\Gamma}_{0, 5}$.

Let $\hat{G}T_0$ be the set of elements of $\hat{G}T_u$ satisfying conditions (i) and (ii).

The crucial property of $\hat{G}T$ is the following theorem.

Theorem 2.5 ([13], §1.2 and §3.2). $\hat{G}T$ (resp. $\hat{G}T_0$) is a group, hence a subgroup of $\text{Out} (\hat{F}_2)$ that lifts to a subgroup of $\text{Aut} (\hat{F}_2)$. Furthermore, the image of the inclusion of $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ into $\text{Out} (\hat{F}_2)$ given by Equation 2 lies inside $\hat{G}T$. The composite inclusion $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \hookrightarrow \hat{G}T \hookrightarrow \text{Aut} (\hat{F}_2)$ corresponds to splitting the homotopy exact sequence $\mathbb{Q}$ at the tangential basepoint $\overline{0}$, as in Equation 3.

Definition 2.6. Let $g : \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \hookrightarrow \hat{G}T$ be the inclusion defined by Theorem 2.5.

For $\sigma \in \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$, let $f_\sigma \in \hat{F}_2'$ be such that $g(\sigma) = (\lambda, f_\sigma)$ for some $\lambda$.

Remark 2.7 ([6], Proposition 3.2). If $g(\sigma) = (\lambda, f_\sigma)$, then $\lambda = \chi(\sigma)$. 


2.5. Gal($\overline{\mathbb{Q}}/\mathbb{Q}$)-

-Invartants. Theorem 2.5 yields Gal($\overline{\mathbb{Q}}/\mathbb{Q}$)-invariants of Belyi functions and dessins d’enfants. Fix a Belyi function $f : X \to \mathbb{P}^1$ of degree $n$. We obtain an associated dessin $\Gamma \subseteq X$ and a monodromy representation $h : \pi_1(\mathbb{P}^1 \setminus \{0, 1, \infty\}, \overline{0}) \to S_n$. Let $\lambda_i + n$ denote the cycle type of $\sigma_i = h(x_i)$ for $i \in \{0, 1, \infty\}$. One can easily verify (from Theorem 2.5 or otherwise) that the cycle type of the monodromy $(\lambda_0, \lambda_1, \lambda_\infty)$ is Gal($\overline{\mathbb{Q}}/\mathbb{Q}$)-invariant. In fact, $\lambda_0$ is the degree multiset of $V_0$, $\lambda_1$ is the degree multiset of $V_1$, and $\lambda_\infty$ is the multiset of half the number of edges bounding each face of $\Gamma$ [12, p.4].

Another Gal($\overline{\mathbb{Q}}/\mathbb{Q}$)-invariant is the monodromy group, defined as the image of the monodromy representation $h$, which is Gal($\overline{\mathbb{Q}}/\mathbb{Q}$)-invariant by definition of the action of Gal($\overline{\mathbb{Q}}/\mathbb{Q}$) on the category of Belyi functions. A third invariant is the rational Nielsen class, which is the set of triples

$$\left\{ ([\sigma_0], [\sigma_1], [\sigma_\infty]) \mid \lambda \in \mathbb{Z}^\times \right\},$$

where $[u]$ denotes the conjugacy class of $u$ in the monodromy group of $f$; the Gal($\overline{\mathbb{Q}}/\mathbb{Q}$)-invariance of the rational Nielsen class follows immediately from Remark 2.7. Let $N(n)$ denote the family of pairs of a group $G$ acting transitively on $[n]$ and a rational Nielsen class in $G$.

There are other combinatorial invariants, such as the Ellenberg’s braid group invariant [6], Wood’s Belyi-extending map invariant [18], and Serre’s lifting invariant. Zapponi [19] defined an invariant for plane trees (equivalently, Belyi polynomials) that is merely a sign $\pm 1$, but that is particularly interesting in that it is not combinatorial.

## 3. Statements of the main results

### 3.1. A new Gal($\overline{\mathbb{Q}}/\mathbb{Q}$)-invariant for Belyi function with monodromy of cycle type $(\lambda, \mu, \lambda)$.

**Definition 3.1.** Let $n$ be a positive integer, let $\lambda, \mu \vdash n$, and let $f$ a Belyi function with monodromy of cycle type $(\lambda, \mu, \lambda)$. Suppose that $f$ has monodromy generators $\sigma_0, \sigma_1, \sigma_\infty$, over $0, 1, \infty$, respectively. The square-root class of $f$, denoted by Sqrt$(f)$, is defined as

$$\text{Sqrt}(f) = \left\{ (\sigma_0, \tau_1, \tau_1^{-1}\sigma_0^{-1}) \in S_n^3 \mid \tau_1^2 = \sigma_1 \text{ and } \sigma_\infty = \tau_1^{-1}\sigma_0\tau_1 \right\}.$$

Because $\sigma_0, \sigma_1, \sigma_\infty$ are only defined up to simultaneous conjugation in $S_n$, each element of Sqrt$(f)$ is only defined up to such conjugation.

**Definition 3.2.** Let the square-root cycle type class of $f$, denoted by SqCt$(f)$, be the multiset of triples $(\lambda_0, \lambda_1, \lambda_\infty)$ where $\lambda_i$ is the cycle type of $\tau_i$, for $(\tau_0, \tau_1, \tau_\infty) \in \text{Sqrt}(f)$.

For each positive integer $n$, the action of Gal($\overline{\mathbb{Q}}/\mathbb{Q}$) on the set of conjugacy classes of representations of $F_2$ in $S_n$ induces an action of Gal($\overline{\mathbb{Q}}/\mathbb{Q}$) on the power set of the set of such representations. Hence, for all $\sigma \in \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ and all Belyi functions $f$, one can define $\sigma(\text{Sqrt}(f))$. A key property of the square-root class is its Gal($\overline{\mathbb{Q}}/\mathbb{Q}$)-equivariance. This yields a key property of the square-root cycle type class, which is that it is Gal($\overline{\mathbb{Q}}/\mathbb{Q}$)-invariant, and in certain cases it can distinguish Gal($\overline{\mathbb{Q}}/\mathbb{Q}$)-orbits of dessins that are indistinguishable by the monodromy group and the rational Nielsen class. The square-root cycle type class is a purely combinatorial
invariant, albeit difficult to compute explicitly. In order to state the final properties of the square-root cycle type class, we define the genus of an element of $\text{SqCt}(f)$; for all $\lambda = (\lambda_0, \lambda_1, \lambda_\infty)$ with $\lambda_i \vdash n$, let

$$g(\lambda) = \frac{\sum_{i \in \{0,1,\infty\}} \text{ram}(\lambda_i)}{2} - n + 1.$$  

We can naturally extend $g$ to take arguments that are elements of $S_n$ instead. If $\sigma_i$ is a permutation of cycle type $\lambda_i$ for $i \in \{0,1,\infty\}$ such that $\sigma_0\sigma_1\sigma_\infty = 1$ and the $\sigma_i$ generate a transitive subgroup of $S_n$, the Riemann-Hurwitz formula implies that this is simply the genus of the curve $X$ that admits a Belyi function with monodromy of cycle type $(\sigma_0, \sigma_1, \sigma_\infty)$.

Now, we are prepared to state the key facts regarding the square root class and the square-root cycle type class.

**Theorem 3.3** (Properties of $Sqrt$ and $\text{SqCt}$). The function $Sqrt$ is $\text{Gal}(\overline{Q}/Q)$-equivariant and thus the function $\text{SqCt}$ is $\text{Gal}(\overline{Q}/Q)$-invariant. Let $f: X \to \mathbb{P}^1$ be a Belyi function and suppose that $X$ has genus $g$. Then,

(a) $|\text{SqCt}(f)|$ is at most the number of non-trivial involutions on $X$, and in particular, if $g > 1$, then $|\text{SqCt}(f)| \leq 84(g - 1) - 1$;

(b) if there exist odd positive integers $k, c$ and a triple $(\mu_0, \mu_1, \mu_\infty) \in \text{SqCt}(f)$ such that $\mu_1$ has $c$ parts of size $k$ and no parts of size $2k$, then $|\text{SqCt}(f)| = 1$;

(c) if $g > 1$, then there exists at most one triple $\lambda = (\lambda_0, \lambda_1, \lambda_\infty) \in \text{SqCt}(f)$ such that $g(\lambda) = 0$.

3.2. The monodromy cycle type and the rational Nielsen class are imprecise invariants. For all positive integers $n$, let

$$\text{Cl}(N) = \max_{n \leq N} \max_{\lambda_1, \lambda_2, \lambda_3 \vdash n} \left( \text{number of Gal}(\overline{Q}/Q)\text{-orbits of Belyi functions with monodromy of cycle type } (\lambda_1, \lambda_2, \lambda_3) \right).$$

Using the tools of Section 3.4, we derive the following optimized lower bound.

**Theorem 3.4.** For all positive integers $N$, we have

$$\text{Cl}(N) \geq \frac{1}{16} 2^{\sqrt{2N}}.$$

For a positive integer $N$, let

$$\text{Cl}'(N) = \max_{n \leq N} \max_{c \in N(n)} \left( \text{number of Gal}(\overline{Q}/Q)\text{-orbits of Belyi functions with rational Nielsen class } c \right).$$

We also prove the following theorem.

**Theorem 3.5.** For all $k < 2^{\sqrt{2}}$, we have

$$\text{Cl}'(N) = \Omega(k^{\sqrt{N}}).$$

The monodromy groups of the rational Nielsen classes achieving the given asymptotic inequality can be chosen to be $A_n$.

**Remark 3.6.** Theorem 3.4 is not special case of Theorem 3.5 because it provides an explicit constant as well as a base of $2^{\sqrt{2}}$ instead a base of arbitrarily close to $2^{\sqrt{2}}$. 
3.3. New versions of $\widehat{GT}$. By reinterpreting our work as an explicit computation of the push-forward map on étale fundamental groups induced by a morphism from $\mathbb{P}^1 \setminus \{0, 1, \infty\}$ to a particular Deligne-Mumford curve, we derive a constraint satisfied by the image of $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ in $\widehat{GT}$. In particular, we prove the following theorem.

**Theorem 3.7.** Let $N$ be the closed normal subgroup of $\widehat{F}_2$ topologically generated by $(xy)^2$. Consider the set $\widehat{GT}_2$ of elements $(\lambda, f) \in \widehat{GT}_0$ such that $f(x, y^2)f(x, y)^{-1} \in N$. The set $\widehat{GT}_2$ is a subgroup of $\widehat{GT}_0$, and if $\sigma \in \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$, then $g(\sigma) \in \widehat{GT}_2$.

We consider a different morphism from $\mathbb{P}^1 \setminus \{0, 1, \infty\}$ to a Deligne-Mumford stack to obtain the following theorem.

**Theorem 3.8.** Let $N_2$ be the closed normal subgroup of $\widehat{F}_2$ topologically generated by $y^3$ and $(xy)^3$. Consider the set $\widehat{GT}_3$ of elements $(\lambda, f) \in \widehat{GT}_0$ such that there exist $m, n \in \mathbb{Z}$ with $m + n \equiv 0 \pmod{3}$ and

$$y^{-1}f(x, yxy^{-1})x^mf(x, y)^{-1}y^3f(x, y)x^n \in N_2.$$

The set $\widehat{GT}_3$ is a subgroup of $\widehat{GT}_0$, and if $\sigma \in \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}(\sqrt{-3}))$, then $g(\sigma) \in \widehat{GT}_3$.

**Remark 3.9.** It follows that we have a chain of groups

$$\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}(\sqrt{-3})) \ni g(\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})) \subseteq \widehat{GT}_2 \cap \widehat{GT} \subseteq \widehat{GT},$$

and as usual we have not determined if either inclusion is proper.

**Remark 3.10.** We also have a chain of groups

$$\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}(\sqrt{-3})) \ni g(\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}(\sqrt{-3}))) \subseteq \widehat{GT}_3 \cap \widehat{GT} \subseteq \widehat{GT}.$$

We justify why the second inclusion is proper. Let $\sigma$ denote complex conjugation, and consider $g(\sigma) = (1, -1) \in \widehat{GT}$. We have $y^{-1}x^my^{-1}x^n \notin N_2$ for all $m, n \in \mathbb{Z}$ with $m + n \equiv 0 \pmod{3}$ because $y^{-1}x^my^{-1}x^n$ does not even vanish on descent to $\left(\widehat{F}_2/N_2\right)^{ab}$. It follows that $g(\sigma) \notin \widehat{GT}_3$, but clearly $g(\sigma) \in \widehat{GT}$.

**Theorem 3.8** demonstrates that acting non-trivially on third roots of unity is the only obstruction for an element of $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ to define an automorphism of $\widehat{F}_2$ that lies in $\widehat{GT}_3$.

3.4. Tools to prove the lower bounds. In this section, we state the specific consequences of Theorem 3.8 that we use to prove the lower bounds stated in Section 1.2. First, we describe a coarse analogue of $\text{SqCt}$. Let $n$ be a positive integer, and let $\lambda, \mu \vdash n$. We define a set $M(\lambda, \mu)$, of which $\text{SqCt}(f)$ will be a subset for all Belyi functions $f$ of monodromy cycle type $(\lambda, \mu, \lambda)$. First, we define an auxiliary set $M'(\lambda, \mu)$. Suppose that $\mu$ has $\ell_i$ parts of size $i$ for all $i$, and let $\lambda_0 = n$.

$$M'(\lambda, \mu) = \left\{ (u_0, u_1, \ldots, u_n) \mid \frac{\ell_i}{\ell_c} \leq u_i \leq \ell_i \text{ for } i \text{ and } i = 0, u_i = \frac{\ell_i}{\ell_c} \text{ for non-zero even } i, r + u_0 + u_1 + \cdots + u_n - n \right\},$$

where $r$ is the number of parts of $\lambda$. Given a $(n + 1)$-tuple $u = (u_0, u_1, \ldots, u_n) \in M'(\lambda, \mu)$, we associate partitions $\alpha(u), \beta(u) \vdash n$. The partition $\alpha(u)$ is defined by having $2u_0 - \ell_0$ parts of size $1$ and $\ell_0 - u_0$ parts of size $2$, and $\beta(u)$ is defined by having $2u_k - \ell_k$ parts of size $k$ for $k$ odd, and $\ell_k - u_k + 2u_k - \ell_k$ parts of size $k$ for $k$ even.
It is clear that $\alpha(u), \beta(u) \not| n$. Let $M(\lambda, \mu) = \{(\lambda, \beta(u), \alpha(u)) \mid u \in M'(\lambda, \mu)\}$. The constraints on $M'(\lambda, \mu)$ are chosen so that elements of $M(\lambda, \mu)$ are consistent in that the existence of a Belyi function with monodromy cycle types given by any element of $M(\lambda, \mu)$ would not violate the Riemann-Hurwitz formula.

One specific application of part (b) of Theorem 3.3 is the following theorem.

**Theorem 3.11 (Orbit-Splitting Theorem).** Let $n$ be a positive integer and let $\lambda, \mu \not| n$. Then, there are at least $|M(\lambda, \mu) \cap B| \text{Gal}(\overline{Q}/Q)$-orbits of Belyi functions with monodromy of cycle type $(\lambda, \mu, \lambda)$.

An existence result for Belyi functions, due to Edmonds-Kulkarni-Stong [5], yields the following corollary.

**Corollary 3.12 (n-cycle Orbit-Splitting Theorem).** If $\lambda = n \not| n$, then $M(\lambda, \mu) \subseteq B$. Hence, if $\mu \not| n$, then there are at least $|M(\lambda, \mu)| \text{Gal}(\overline{Q}/Q)$-orbits of Belyi functions with monodromy of cycle type $(n, \mu, n)$.

In certain cases, the constraint that $\mu_c = \ell_c$ is odd for some odd $c$ in the definition of $M'(\lambda, \mu)$ is restrictive, in that there are $\lambda, \mu$ for which the Orbit-Splitting Theorem gives weak bounds on the number of $\text{Gal}(\overline{Q}/Q)$-orbits of Belyi functions with monodromy of cycle type $(\lambda, \mu, \lambda)$. We prove an alternate form that applies even in those cases, but is weaker in other cases. For example, consider $n = 11$, $\lambda = 11$ and $\mu = 2222111$. The $n$-cycle Orbit-Splitting Theorem implies that there are at least 0 $\text{Gal}(\overline{Q}/Q)$-orbits of Belyi functions with monodromy of cycle type $(\lambda, \mu, \lambda)$; the alternate form will imply that there are at least 2 $\text{Gal}(\overline{Q}/Q)$-orbits.

Once again, let $n$ be a positive integer, and let $\lambda, \mu \not| n$. Suppose that $\mu$ has $\ell$ parts of size $i$ for all $i$, and let $\lambda_0 = n$. Let

$$M_0'(\lambda, \mu) = \left\{ (u_0, u_1, \ldots, u_n) \mid \ell_i \leq u_i \leq \ell_i \text{ for odd } i \text{ and } i = 0, \right.$$ non-zero even $i$, and $r + u_0 + u_1 + \cdots + u_n = n + 2 \right\},$$

where $r$ is the number of parts of $\lambda$. Define $M_0(\lambda, \mu) = \{(\lambda, \beta(u), \alpha(u)) \mid u \in M_0'(\lambda, \mu)\}$. We prove the following analogue of the Orbit-Splitting Theorem, which follows from Theorem 3.3(c).

**Theorem 3.13 (Orbit-Splitting Theorem, Alternate Form).** Let $n$ be a positive integer and let $\lambda, \mu \not| n$. Suppose that $\lambda$ has $r$ parts and $\mu$ has $s$ parts, and $2r + s < n$.

Then, there are at least $|M_0(\lambda, \mu) \cap B| \text{ and } \text{Gal}(\overline{Q}/Q)$-orbits of Belyi functions with monodromy of cycle type $(\lambda, \mu, \lambda)$.

Similar to the $n$-cycle Orbit-Splitting Theorem, we obtain the following corollary.

**Corollary 3.14 (n-cycle Orbit-Splitting Theorem, Alternate Form).** If $\lambda = n \not| n$, then $M_i(\lambda, \mu) \subseteq B$ for $i = 0, 1$. Hence, if $\mu \not| n$ has less than $n - 2$ parts, then there are at least $|M_0(\lambda, \mu)| \text{Gal}(\overline{Q}/Q)$-orbits of Belyi functions with monodromy of cycle type $(n, \mu, n)$.

4. Proof of Theorem 3.3

Let $f$ and $t$ be affine coordinates centered at 0 on $\mathbb{P}^1_1$ and $\mathbb{P}^1_2$, respectively. Define the morphism $t = \frac{4f}{t(1)} : \mathbb{P}^1_1 \rightarrow \mathbb{P}^1_2$. Notice that $t'(0) = 4$, and therefore $t$ preserves the topological tangential base-point $\mathbb{1}$. 

4.1. Computation of the morphism $t_*$ of topological fundamental groups.

The morphism $t$ defines a morphism from $\mathbb{P}^1 \setminus \{-1, 0, 1, \infty\}$ to $\mathbb{P}^1 \setminus \{0, 1, \infty\}$. We choose generators for $\pi_1 \left( \mathbb{P}^1 \setminus \{-1, 0, 1, \infty\}, 0 \right)$ as in Figure 2. It is clear that

\[ t_* y_0 = x_0 \]
\[ t_* y_1 = x_1^2 \]
\[ t_* y_{-1} = x_\infty^2 \]

where the equalities are up to base-point fixing homotopy. Because $y_{-1}y_0y_1y_\infty = 1$ and $t_*$ is a homomorphism, we have

\[ t_* y_\infty = t_* \left( y_1^{-1} y_0^{-1} y_{-1} \right) = x_1^{-2} x_0^{-1} x_\infty^{-2} = x_1^{-2} x_0^{-1} x_0 x_1 x_0 x_1 = x_1^{-1} x_0 x_1. \]
4.2. **Proof that $\text{Sqrt}$ is $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-equivariant.** Let $n$ be a positive integer, and let $g : X \to \mathbb{P}^1_t$ be a Belyi function of degree $n$ defined on an algebraic curve $X$. Let $X'$ be the normalization of $X \times_{\mathbb{P}^1_t} \mathbb{P}^1$, and let $f : X' \to \mathbb{P}^1_f$ be the projection. The curve $X'$ may not be irreducible.

**Definition 4.1.** We write $f = \Sigma(g)$, so that $\Sigma$ defines a function from the set of isomorphism classes of Belyi functions to the set of isomorphism classes of morphisms of curves $X' \to \mathbb{P}^1$, where $X'$ is not necessarily irreducible.

\[
\begin{array}{ccc}
X & \xrightarrow{\alpha} & X' \\
g \downarrow & & \downarrow f \\
\mathbb{P}^1_t & \xleftarrow{t : \mathbb{P}^1_f} & \mathbb{P}^1_f
\end{array}
\]

The projection $\alpha : X' \to X$ induces a bijection between the fibers $g^{-1}(\overline{01})$ and $(g')^{-1}(\overline{01})$. We order the fiber $g^{-1}(\overline{01})$, which gives an order on $(g')^{-1}(\overline{01})$ via the restriction of $\alpha$. Using these orders, we can define the monodromy of $f$ and $g$ as fixed representations (not isomorphism classes of representations) of $\pi_1(\mathbb{P}^1_f \setminus \{-1, 0, 1, \infty\}, \overline{01})$ and $\pi_1(\mathbb{P}^1_t \setminus \{0, 1, \infty\}, \overline{01})$ on $[n]$. Let $p_k \in S_n$ be the image of $x_k$ under the representation of $\pi_1(\mathbb{P}^1_t \setminus \{0, 1, \infty\}, \overline{01})$ for $k \in \{0, 1, \infty\}$, and let $\sigma_k$ be the image of $y_k$ under the monodromy representation of $\pi_1(\mathbb{P}^1_f \setminus \{-1, 0, 1, \infty\}, \overline{01})$ for $k \in \{-1, 0, 1, \infty\}$.

For all Belyi functions $g$, the fact that $\Sigma(g)$ is étale outside $\{-1, 0, 1, \infty\}$ follows from the fact that étaleness is preserved under base-change. The following proposition is immediate by lifting loops.

**Proposition 4.2.** Let $g : X \to \mathbb{P}^1_t$ be a Belyi function, with monodromy generators $\tau_0, \tau_1, \tau_\infty$. Then, $\Sigma(g)$ is unbranched outside $\{-1, 0, 1, \infty\}$. Let $\sigma_{-1}, \sigma_0, \sigma_1, \sigma_\infty$ be the monodromy of the function $\Sigma(g)$ over $-1, 0, 1, \infty$, respectively (the permutations are defined up to simultaneous conjugation in $S_n$ because we fixed loops of winding number $1$ about each branch point in both $\mathbb{P}^1_t$ and $\mathbb{P}^1_f$). Then, we have $\sigma_0 = \tau_0$, $\sigma_1 = \tau_1^2$, $\sigma_{-1} = \tau_\infty^2$, and $\sigma_\infty = \tau_1^{-1} \tau_0 \tau_1$.

We are now ready to link the constructions of this subsection to the square-root class.

**Definition 4.3.** Let $f : X \to \mathbb{P}^1_t$ be a Belyi function. Define

\[
\text{Sqrt}'(f) = \{g \mid \Sigma(g) \cong f\},
\]

and call $\text{Sqrt}'(f)$ the fibered product square-root class of $f$.

**Theorem 4.4.** (a) The function $\text{Sqrt}'$ is $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-equivariant.

(b) Let $n$ be a positive integer, and let $f : X \to \mathbb{P}^1_t$ be a Belyi function of degree $n$.

Then, $\text{Sqrt}(f)$ is the set of monodromy triples of elements of $\text{Sqrt}'(f)$.

In particular, the function $\text{Sqrt}$ is $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-equivariant.

**Proof.** We begin by proving part (a). We treat Belyi functions as finite étale covers of $\mathbb{P}^1_{\mathbb{Q}} \setminus \{0, 1, \infty\}$. The fact that $\Sigma$ is $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-equivariant then follows from the fact that base-changing by $\text{Spec} \sigma$ preserves fibered products and normalizations for all $\sigma \in \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$.

Part (b) follows immediately from Proposition 4.2. \qed
4.3. Proof of Part (a). Let \( f : X \to \mathbb{P}^1 \) be a Belyi function. The key to the proof of this part is to construct an injection from \( \text{Sqrt}'(f) \) to the set of involutions on \( X \). The remainder of the statement follows from Hurwitz’s Automorphism Theorem.

Proof of Theorem 3.3(a). Let \( \text{Inv}(X) \) denote the set of non-trivial involutions on \( X \). We construct an injection \( i : \text{Sqrt}'(f) \to \text{Inv}(X) \). Let \( g \in \text{Sqrt}'(f) \), so that we have a diagram

\[
\begin{array}{ccc}
X & \xrightarrow{\alpha} & X' \\
\downarrow{g} & & \downarrow{f} \\
\mathbb{P}^1 & \xrightarrow{\iota} & \mathbb{P}^1
\end{array}
\]

with \( X \) the normalization of the fibered product \( Y \times_{\mathbb{P}^1} \mathbb{P}^1 \). The bottom morphism is of degree 2, and the vertical morphisms are of degree \( n \), which implies that the top morphism \( \alpha \) is also of degree 2. There is an involution \( \iota : X \to X \), which is the unique deck transformation for the restriction of \( \alpha \) to its unramified locus. Let \( i(g) = \iota \). Note that \( \alpha : X \to Y \) is the quotient of \( X \) by \( \iota \), so that \( \iota \) determines \( \alpha \) up to composition by an automorphism of \( Y \).

To prove that \( i \) is injective, it suffices to prove that \( \alpha, f, \) and the bottom morphism uniquely determine \( g \). This is obvious, because \( \alpha \) is surjective and Diagram 2 is required to commute. Therefore, \( |\text{Sqrt}'(f)| \leq |\text{Inv}(X)| \), and the fact that \( |\text{Sqrt}(f)| \leq |\text{Inv}(X)| \) follows by Theorem 4.4(b). \( \square \)

4.4. Proof of Part (b). We transfer to representations of \( F_2 \) to analyze the fibered product square-root class. Fix a generating set \( F_2 = \langle x, y \rangle \) and a positive integer \( n \). For all positive integers \( k \), let \([k]\) denote the set \( \{1, 2, \ldots, k\} \). Let \( T_r \) be the set of conjugacy classes of transitive representations \( m : F_2 \to S_n \) such that there exists an odd positive integer \( c \) such that \( m(y) \) contains an odd number of cycles of length \( c \) and no cycle of length \( 2c \). Let \( \xi \) denote the representation of \( F_2 \) on \( S_2 \) with \( \xi(x) = (1)(2) \) and \( \xi(y) = (12) \).

Proposition 4.5. Let \( n \) be a positive integer. Let \( m \in T_r \) be a permutation representation \( m : F_2 \to S_n \), and let \( m' \) be a permutation representation \( m' : F_2 \to S_n \).

(a) The product representation \( m \times \xi \) is transitive.
(b) If \( m \times \xi \cong m' \times \xi \), then \( m \cong m' \).

Proof. Suppose that \( m, m' \) satisfy the conditions of the proposition. Let \( m_\xi = m \times \xi \) and let \( m'_\xi = m \times \xi \).

First, we prove part (a). Let \( (a, b), (a', b') \in [n] \times [2] \), and we will prove that there exists a word \( w \in F_2 \) such that \( m_\xi(w)(a, b) = (a', b') \). By definition, the permutation \( m(y) \in S_n \) must have an odd cycle in its cycle decomposition. Suppose that \( (p_1 p_2 \cdots p_k) \) is a cycle in \( m(y) \) with \( k \) odd. Let \( w_0, w_1 \in F_2 \) be such that \( m(w_0)(a) = p_1 \) and \( m(w_1)(p_1) = a' \) because \( m \) is transitive, such \( w_0, w_1 \) exist. If \( \xi(w_1 w_0)(b) = b' \), then we can take \( w = w_1 w_0 \) because \( m(w_1 w_0)(a) = a' \). Hence, we can assume that \( \xi(w_1 w_0) \neq b' \). Let \( w = w_1 y^k w_0 \). Because \( k \) is odd, \( \xi(w)(b) = b' \), and it is easy to see that \( m(w)(a) = a' \). It follows that \( m'_\xi \) is transitive.

We now prove part (b). Part (a) implies that \( m' \) is also transitive. There is an automorphism \( \alpha \) of \([n] \times [2] \) such that \( \alpha \circ (m \times \xi) = m' \times \xi \). Let \( G \) be the kernel of \( \xi \); it is a normal subgroup of index 2 in \( F_2 \). Note that the \( m \)-action (resp.
Furthermore, \(k\) contain 2 representations are \([c]\). Proposition 4.6. Following proposition.

Proof. Let \(C\) be the category of \(\acute{\text{e}}\)tale covers of \(\mathbb{P}^1 \setminus \{0, 1, \infty\}\). The function \(K\) is the object function of a contravariant functor from \(C\) to \(\text{FinSet}^F\), the category
of finite permutation representations of $F_2$. It is well-known that $K$ is in fact an equivalence of categories. In particular, $K$ preserves products. But, $\Sigma(g) = g \times t$ (in $C$), and the conclusion follows.

Proof of Theorem 3.3(b). Let $f : X \to \mathbb{P}^1$ be a Belyi function of odd degree $n$, let $(\tau_0, \tau_1, \tau_\infty) \in \text{Sqrt}(f)$, and let $\mu \vdash n$ be the cycle type of $\tau_1$. Suppose that $k, c$ are odd positive integers such that $\mu$ has $c$ parts of size $k$ and no parts of size $2k$. Let $m$ be the representation of $F_2$ on $S_n$ that sends $x$ to $\tau_0$ and $y$ to $\tau_1$. By Proposition 4.5 if a representation $m' : F_2 \to S_n$ satisfies $m \times \xi \sim m' \times \xi$, then in fact $m \sim m'$.

Suppose that $\tau' = (\tau'_0, \tau'_1, \tau'_\infty) \in \text{Sqrt}(f)$ and $m' : F_2 \to S_n$ is the corresponding representation. It follows from Theorem 4.4(b) and Proposition 4.6 that $m' \times \xi \sim K(f \circ t) \equiv m \times \xi$, which implies that $m \sim m'$. Therefore, $(\tau'_0, \tau'_1, \tau'_\infty)$ is conjugate to $(\tau_0, \tau_1, \tau_\infty)$. Since the choice of $\tau'$ was arbitrary, we have $|\text{Sqrt}(f)| = 1$ and the result follows.

4.5. Proof of Part (c). Let $n$ be an odd positive integer, and let $\lambda, \mu \vdash n$. We use the fact that a hyperelliptic curve admits a unique involution with a genus 0 quotient in the proof of Theorem 3.3(a).

Proof of Theorem 3.3(c). Let $T_0$ denote the set of isomorphism classes of Belyi functions whose domains are $\mathbb{P}^1$. Note that $g(\lambda_0, \lambda_1, \lambda_\infty)$ is the genus of a curve that admits a Belyi function with monodromy cycle type $(\lambda_0, \lambda_1, \lambda_\infty)$, if such a curve exists. Therefore, by Theorem 4.4(b), it suffices to prove that the restriction of $\Sigma$ to $T_0$ is injective.

Consider two commutative squares

$$
\begin{array}{ccc}
X & \xleftarrow{\alpha} & X' \\
\downarrow g & & \downarrow f \\
\mathbb{P}^1_t & \xleftarrow{(f+1)t} & \mathbb{P}^1_f \\
\end{array}
$$

and

$$
\begin{array}{ccc}
X & \xleftarrow{\alpha'} & X' \\
\downarrow g' & & \downarrow f \\
\mathbb{P}^1_t & \xleftarrow{(f+1)t} & \mathbb{P}^1_f \\
\end{array}
$$

where in both diagrams $X$ is the normalization of the fibered product $\mathbb{P}^1_t \times_{\mathbb{P}^1_f} \mathbb{P}^1_f$, and the left morphisms are Belyi functions of degree $n$. Because a hyperelliptic curve of genus at least 2 admits a unique degree 2 function to $\mathbb{P}^1$, there must be an automorphism $\beta$ of the top left copy of $\mathbb{P}^1_f$ such that $\alpha' = \beta \circ \alpha$. Hence, we have $g \circ \alpha = t \circ f$ and $(g' \circ \beta) \circ \alpha = t \circ f$. Because $\alpha$ is surjective, it follows that $g = g' \circ \beta$.

5. Proofs of the Orbit-Splitting Theorems and the lower bounds on $\text{Cl}(n)$ and $\text{Cl}'(n)$

In Section 5.1 we review a result that guarantees the existence of Belyi functions with particular prescribed monodromy cycle types. In Section 5.2 we prove the Orbit-Splitting Theorems using Theorem 3.3. In Section 5.3 we review some group-theoretic preliminaries that we use in the proofs of Theorems 3.4 and 3.5 which we give in Section 5.4.
5.1. Hurwitz existence problem. We investigate Belyi functions with monodromy of fixed cycle type. Let $\mathcal{B}$ be the set of monodromy cycle types of Belyi functions. Determining $\mathcal{B}$ is an unsolved case of the Hurwitz existence problem, which deals with the possible sequences of monodromy cycle types of étale covers of arbitrary curves over $\mathbb{C}$ with removed points, but is a purely group-theoretic question regarding finite permutation representations of the fundamental groups of Riemann surfaces with points removed.

In the case of $\mathbb{P}^1 \setminus \{0, 1, \infty\}$, the question is: given a finite group $G$ and conjugacy classes $c_0, c_1, c_\infty$, how many triples $(\sigma_0, \sigma_1, \sigma_\infty)$ are there of elements $\sigma_i \in c_i$ such that $\sigma_0 \sigma_1 \sigma_\infty = 1$? There is a formula for the number of solutions in terms of the characters of $G$ (see, for example, Serre [16, Theorem 7.2.1]), but this is not simple to evaluate in general. Edmonds-Kulkarni-Stong [5] construct a family of elements of $\mathcal{B}$.

Theorem 5.1 ([5], Proposition 5.2). Let $n$ be a positive integer, and let $\alpha, \beta \vdash n$. Let $P$ be the total number of parts of $\alpha, \beta$. A Belyi function with monodromy of cycle type $(\alpha, \beta, n)$ exists if and only if $P \equiv n + 1 \pmod{2}$ and $P \leq n + 1$.

Necessity follows immediately from the Riemann-Hurwitz formula, and sufficiency is proven constructively. If one of the partitions is not $n$, the Riemann-Hurwitz condition on the total number of parts of the three partitions is not in general sufficient.

5.2. Proofs of the Orbit-Splitting Theorems. Fix an integer $n$ and partitions $\lambda, \mu \vdash n$. For $\alpha, \beta \vdash n$, let $S_{\alpha, \beta}$ be the set of isomorphism classes of Belyi functions with monodromy of cycle type $(\alpha, \beta, \lambda)$. Let

$$S = \bigcup_{(\lambda, \beta, \alpha) \in M(\lambda, \mu) \cap \mathcal{B}} S_{\alpha, \beta} \quad \text{and} \quad S_0 = \bigcup_{(\lambda, \beta, \alpha) \in M_0(\lambda, \mu) \cap \mathcal{B}} S_{\alpha, \beta}.$$ 

Let $f \in \Sigma(S) \cup \Sigma(S_0)$. Proposition 4.2 implies that $f$ is unbranched outside $\{0, 1, \infty\}$ and has monodromy of cycle type $(\lambda, \mu, \lambda)$. By Propositions 4.5(a) and 4.6, the monodromy of $f$ acts transitively on the fiber above the base-point, and it follows that the domain of $f$ is irreducible and that $f$ is a Belyi function. The Orbit-Splitting Theorem, in its ordinary and alternate forms, follow from Theorem 3.3 parts (b) and (c), respectively.

Proof of the Orbit-Splitting Theorem. By Theorem 3.3(b), we have $|\Sigma f| = 1$ for all $f \in \Sigma(S)$. By construction, $\Sigma f$ can take any value in $M(\lambda, \mu) \cap \mathcal{B}$ as $f$ ranges over $S$. Because $\Sigma f$ is $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-invariant, the theorem follows.

Proof of the Orbit-Splitting Theorem, Alternate Form. By Theorem 3.3(b) and the construction of $S_0$, $\Sigma f$ contains exactly one element $(\lambda_0, \lambda_1, \lambda_\infty)$ such that $g(\lambda_0, \lambda_1, \lambda_\infty) = 0$ for all $f \in \Sigma(S_0)$. Denote this element by $R(f)$. Because $\Sigma f$ is $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-invariant, so is $R(f)$. By construction of $S_0$, $R(f)$ can take all values in $M'(\lambda, \mu) \cap \mathcal{B}$ as $f$ ranges over $S_0$, and the theorem follows.

By construction, the assertion that $M(\lambda, \mu) \subseteq \mathcal{B}$ would not violate the Riemann-Hurwitz formula. The fact that $M(\lambda, \mu) \subseteq \mathcal{B}$ when $\lambda = n \vdash n$ is immediate by Theorem 5.1 and the $n$-cycle Orbit-Splitting Theorems follow.
5.3. **Primitive subgroups of \( S_n \).** In order to control the monodromy groups of the Belyi functions that we consider, we need a result on primitive subgroups of \( S_n \), from Dixon-Mortimer [4] but due to Jordan. We also need a result describing permutation groups that contain short length cycles.

**Theorem 5.2** ([4], Example 3.3.1). Let \( n \geq 9 \), let \( G \) be a subgroup of \( S_n \), and suppose that there exists a nonidentity \( \sigma \in G \) with at least \( n - 4 \) fixed points. If \( G \) does not contain a transposition or a 3-cycle, then \( G \) is not primitive.

**Theorem 5.3** ([4], Theorem 3.3E). Let \( q \) be a prime, and let \( n > q + 2 \). If a primitive subgroup \( G \) of \( S_n \) contains a \( q \)-cycle, then \( G \) contains \( A_n \).

The form that we will need is the following proposition, which is immediate from Theorems 5.2 and 5.3.

**Proposition 5.4.** Let \( p > 7 \) be a prime, and let \( G \) be a subgroup of \( S_p \) that contains a \( p \)-cycle and a double transposition. Then \( G \) contains \( A_p \).

**Proof.** A subgroup of \( S_p \) that contains a \( p \)-cycle is primitive, and a double transposition in \( S_p \) has \( p - 4 \) fixed points. By Theorem 5.2, \( G \) contains a 2-cycle or a 3-cycle. In both cases, Theorem 5.3 implies that \( G \) contains \( A_p \), as claimed. \( \square \)

**Remark 5.5** (Noam Elkies, private communication). The proposition is false for \( p = 5, 7 \). For \( p = 5 \), one can take \( G = D_{10} \), and for \( p = 7 \), one can take \( G = PGL_3(\mathbb{Z}/2\mathbb{Z}) \).

5.4. **Proofs of Theorems 3.4 and 3.5.** We derive Theorems 3.4 and 3.5 from the Orbit-Splitting Theorem and the results quoted in the preceding section. First, we begin with a few computational lemmata, whose proofs are deferred to Appendix A.

For positive integers \( t \) and \( k \) with \( k \leq t \), let

\[
f_t(k) = \left\lfloor \frac{4t + 2}{2k - 1} \right\rfloor.
\]

For a positive integer \( t \), let

\[
n_0(t) = 2t + 1 + \sum_{i=1}^{t} 2(2k - 1)(f_i(k) - 1).
\]

**Lemma 5.6.** For all positive integers \( t \), we have

\[4t^2 + 12t + 1 < n_0(t) < 6(t + 1)^2 - 4.\]

**Lemma 5.7.** Let \( t \) be a positive integer. Then, we have

\[
\sum_{k=1}^{t} (f_i(k) - 1) \leq \frac{n_0(t)}{4}.
\]

**Lemma 5.8.** Let \( t \) be a positive integer. Then, we have

\[
\prod_{k=1}^{t} f_i(k) > 2^{2t}.
\]

**Proof of Theorem 3.4.** Fix a positive integer \( t \), and let \( n = n_0(t) \). We prove a lower bound on \( \text{Cl}(n) \) that will imply the theorem. Let \( \lambda = n \vdash n \). Define the partition \( \mu \vdash n \) to have \( 2f(k) - 2 \) parts of size \( 2k - 1 \) for \( 1 \leq k \leq t \) and 1 part of size \( 2t + 1 \).
We claim that

\[ |M'(\lambda, \mu)| \geq \prod_{k=1}^{t} f_t(k). \]

Let \( S \) be the set of tuples \((v_0, v_1, \ldots, v_n)\) such that \( f_t(k) - 1 \leq v_{2k-1} \leq 2f_t(k) - 2 \) for all \( 1 \leq k \leq t \), \( v_{2t+1} = 1 \), \( v_i = 0 \) for all \( i > 2t + 1 \) and \( i = 2, 4, \ldots, 2t \), and \( v_0 = n - r(v) \), where

\[ r(v) = \sum_{k=1}^{t} (2f_t(k) - 2 - v_k). \]

Notice that \( v_{2t+1} = 1 \), and \( \mu \) has 1 part of size \( 2t + 1 \) and no parts of size \( 4t + 2 \). Hence, to prove Equation 5, it suffices to prove that \( S \subseteq M'(\lambda, \mu) \). It suffices to prove that \( r(v) \leq \frac{n}{2} \). Indeed, we have

\[ \frac{r(v)}{2} \leq \sum_{k=1}^{t} (f_t(k) - 1). \]

Lemma 5.7 implies that \( r(v) \leq \frac{n}{2} \) for all \( t, v \).

The \( n \)-cycle Orbit-Splitting Theorem 3.12 implies that \( \text{Cl}(n) \geq |M(\lambda, \mu)| \geq \prod_{k=1}^{t} f(k) \). By Lemma 5.8 it follows that \( \text{Cl}(n) \geq 2^{2t} \), and Lemma 5.6 yields that \( \text{Cl}(6(t+1)^2) \geq 2^{2t} \).

We now let \( t \) vary. Let \( N \geq 24 \) be a positive integers. If \( 6(t+1)^2 \leq N < 6(t+2)^2 \), then we have

\[ \log_2 \text{Cl}(N) \geq 2t > 2 \left( \sqrt{\frac{N}{6}} - 2 \right) = \sqrt{\frac{2N}{3}} - 4. \]

It follows that

\[ \text{Cl}(N) \geq \frac{1}{16} 2^{2\sqrt{2N}}. \]

The bound is trivial for \( N < 24 \), and thus we have established the result for all \( N \).

\[ \square \]

\textbf{Remark 5.9.} A simpler construction can establish that \( \text{Cl}(N) = \Omega \left( 2^{\sqrt{2N}} \right) \).

\textbf{Proof of Theorem 3.5.} As in the previous proof, let \( t \) be a positive integer. Let

\[ n_1(t) = 4 + 2t + 1 + \sum_{i=1}^{t} 2(2k-1) \left\lfloor \frac{4t+2}{2k-1} - 1 \right\rfloor. \]

Let \( n(t) \) be the smallest prime number that is at least \( n_1(t) \). Let \( \epsilon(t) = \frac{n(t)}{n_0(t)} - 1. \)

Fix \( t \), and let \( n = n(t) \). It is clear that \( n_1(t) > 2 \), which implies that \( n \equiv n_1(t) \) (mod 2). Let \( 2\alpha + 1 = 2t + 1 + n - n_0 \). Let \( \lambda = (n) + n \), and let \( \mu + n \) be the partition of \( n \) with \( f(k) \) parts of size \( 2k - 1 \) for \( 1 \leq k \leq t \), two parts of size 2, and one part of size \( n - n_0(t) \). By Lemma 5.6 we have \( n_1(t) \leq n_0(t) + 4 < 6(t+1)^2 \), which implies that \( n < 6(t+1)^2(1 + \epsilon(t)) \).

We claim that

\[ |M(\lambda, \mu) \cap \mathcal{B}| \geq \prod_{k=1}^{t} f(k). \]
Let $S$ be the set of tuples $(v_0, v_1, \ldots, v_n)$ such that $f(k) - 1 \leq v_{2k+1} \leq 2f_i(k) - 2$ for all $1 \leq k \leq t$, $v_{n-n_0(t)} = 1$, $v_0 = n - r(v)$ where
\[
r(v) = 1 + \sum_{k=1}^{t} (2f_i(k) - 2 - v_k),
\]
and $v_i = 0$ for all other $i$. It follows from Lemma 5.7 that $r(v) \leq \frac{n}{2}$ for all $v, t$, which implies that $S \subseteq M'(\lambda, \mu)$. Notice that $v_{n-n_0(t)} = 1$, and $\mu$ has 1 part of size $n - n_0(t)$ and no parts of size $2n - 2n_0(t)$. Equation 5 follows.

Let $f$ be a Belyi function with monodromy of cycle type $(\lambda, \mu, \lambda)$ and monodromy generators $\sigma_0, \sigma_1, \sigma_\infty$ over $0, 1, \infty$, respectively. By definition, the permutation $\sigma_1^{(2t-1)!}$ is a double transposition. Because
\[
n \geq n_1(t) = n_0(t) + 4 \geq n_0(1) + 4 = 9,
\]
Proposition 5.4 implies that the monodromy group $G$, which is generated by $\sigma_0$ and $\sigma_1$, contains $A_n$. The fact that $\sigma_0$ and $\sigma_1$ are even implies that $G = A_n$. There are two conjugacy classes of $n$-cycles in $A_n$, so that $\sigma_0$ and $\sigma_\infty$ can lie in the same conjugacy class or in different conjugacy classes. Because $\sigma_0$ and $\sigma_1$ are only defined up to conjugation in $S_n$, the case of both monodromy generators being in one conjugacy class lies in the same rational Nielsen class as the case of both monodromy generators being in the other rational Nielsen class. Furthermore, the $S_n$-conjugacy class of permutations of cycle type $\lambda$ forms a single $A_n$-conjugacy class. Thus, there are at most two possible rational Nielsen classes of Belyi functions with monodromy of cycle type $(\lambda, \mu, \lambda)$.

By the $n$-cycle Orbit-Splitting Theorem 6.12 there are at least $|M(\lambda, \mu)| \geq \prod_{k=1}^{t} f(k)$ Belyi functions with monodromy of cycle type $(\lambda, \mu, \lambda)$. The previous paragraph and Lemma 5.8 then yield that
\[
\text{Cl}'(6(t + 1)^2(1 + \epsilon(t))) \geq \frac{1}{2} \prod_{k=1}^{t} f(k) > 2^{2t-1}
\]
for all positive integers $t$.

We now let $t$ vary. It follows from Lemma 5.6 that $\lim_{t \to \infty} \frac{n_1(t)}{n_0(t)} = 1$. Because
\[
\lim_{t \to \infty} n_0(t) = \infty,
\]
the Prime Number Theorem implies that
\[
\lim_{t \to \infty} (1 + \epsilon(t)) = \lim_{t \to \infty} \frac{n(t)}{n_0(t)} = \lim_{t \to \infty} \frac{n(t)}{n_0(t)} 1.
\]
Fix a constant $k < 2\sqrt{T}$. Let $T$ be a positive integer such that
\[
1 + \epsilon(t) < \frac{2}{3 (\log_2 k)^2}
\]
for all $t > T$; such a $T$ exists because $\lim_{t \to \infty} \epsilon(t) = 0$. Let $P = n_0(T)(1 + \epsilon(T))$, and let $N \geq P$. There exist an integer $t \geq T$ such that
\[
n_0(t + 1)(1 + \epsilon(t + 1)) \leq N < n_0(t + 2)(1 + \epsilon(t + 2)).
\]
Then, by Lemma 5.7, we have that $N < 6(t + 2)^2(1 + \epsilon(t + 2))$. It follows that
\[
t > \sqrt{\frac{N}{6(1 + \epsilon(t + 2))}} - 2.
\]
The fact that $C_l'$ is non-decreasing implies that
\[ \log_2 C_l'(N) \geq 2t - 1 > \sqrt{\frac{2N}{3(1 + \epsilon(t + 2))}} - 5 > \sqrt{N} \log_2 k - 5. \]

The theorem follows.

6. Proof of Theorems 3.7 and 3.8

To prove Theorem 3.7, we exploit the fact that the morphism $t = \frac{4f}{f + 1}$ is defined over $\mathbb{Q}$ through the $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-equivariance of the induced homomorphism $t_*$ on étale fundamental groups. A crucial fact is that $t'(0) = 4 \in \mathbb{R}_{>0}$, and therefore $t$ preserves the topological tangential base-point $0\overline{1}$. To prove Theorem 3.8, we replace $t$ by a morphism $v$ that is only defined over $\mathbb{Q}(\sqrt{-3})$. Because $v$ does not preserve the tangential base-point $0\overline{1}$, we have to use only the outer action of $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ on fundamental groups.

We need to deal with the étale fundamental groups of Deligne-Mumford curves, which we obtain by adding stabilizers to points on a curve. We do so using Cadman’s $r$th root construction [2, Definition 2.2.1], which given a scheme $X$, an effective Cartier divisor $D$, and a positive integer $r$ returns an Artin stack that is scheme-like away from $D$ and that has a $\mu_r$ stabilizer over $D$. More precisely, given a scheme $X$, an ordered $n$-tuple $D = (D_1, D_2, \ldots, D_n)$ of effective Cartier divisors on $X$, and an ordered $n$-tuple $r = (r_1, r_2, \ldots, r_n)$ of positive integers, we let $X_{D,r}$ be the Artin stack defined by Cadman [2, Definition 2.2.4], which we call the result of taking $r_i$th roots of $D_i$. We work over fields of characteristic 0, in which case $r$ is invertible on $X$ and therefore $X_{D,r}$ is actually a Deligne-Mumford stack by [2, Corollary 2.3.4]. The construction is left unchanged on simultaneous permutation of the divisors $D$ and the integers $r$.

6.1. Proof of Theorem 3.7

Let $N$ be the closed subgroup of $\hat{F}_2$ defined in the statement of Theorem 3.7. Define the homomorphism $h : \hat{F}_2 \to \hat{F}_2/N$ by $x \mapsto x$ and $y \mapsto y^2$. The key is that $h$ is $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-equivariant. Lemma 6.1 allows us to descend from $\hat{F}_2$ to $\hat{F}_2/N$, while Lemma 6.2 gives a criterion for an element of $\hat{GT}_0$ to be an element of $\hat{GT}_2$. Lemma 6.3 proves that elements of $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ satisfy the criterion of Lemma 6.2, from which Theorem 3.7 is immediate.

Lemma 6.1. Let $\alpha \in \hat{GT}_0$. Then, $\alpha$ descends to an automorphism of $\hat{F}_2/N$, which we denote by $\bar{\alpha}$. Furthermore, $\bar{\alpha}$ defines a homomorphism from $\hat{GT}_0$ to $\text{Aut} \left( \hat{F}_2/N \right)$.

Proof. By [9] Theorem 2, the automorphism $\alpha$ preserves the conjugacy class of the inertia subgroup $\langle (xy)^{-1} \rangle$, and thus preserves $N$. The lemma follows.

Lemma 6.2. Let $\alpha \in \hat{GT}_0$. Then, we have $\alpha \in \hat{GT}_2$ if and only if the following diagram commutes:

\[
\begin{array}{cccc}
\hat{F}_2 & \xrightarrow{h} & \hat{F}_2/N \\
\alpha \downarrow & & \downarrow \bar{\alpha} \\
\hat{F}_2 & \xrightarrow{h} & \hat{F}_2/N
\end{array}
\]
Proof. First, we claim that the centralizer $C_y$ of $y$ in $\hat{F}_2/N$ is the set of (profinite) powers of $y$. We can exhibit $\hat{F}_2/N$ as the coproduct of $\hat{Z}$ and $\hat{Z}/2\hat{Z}$ in the category of profinite groups, where the generator 1 of the first factor corresponds to $y$ and the generator of the second factor corresponds to $xy$. By \cite{H} Theorem B, $C_y$ is the set of (profinite) powers of $y$ in $\hat{F}_2/N$. Actually, it follows from \cite{H} Theorem B] that $C_y$ is the centralizer of any non-zero power of $y$ in $\hat{F}_2/N$.

Fix $\alpha = (\lambda, f) \in \hat{G}T_0$. The image of $x$ (resp. $y$) under the composite of the right and top homomorphisms in Diagram \[ is $x^\lambda$ (resp. $f(x, y)^{-1}y^{2\lambda}f(x, y)$), while $x$ (resp. $y$) is sent to $x^\lambda$ (resp. $f(x, y^2)^{-1}y^{2\lambda}f(x, y^2)$) under the composite of the bottom and left homomorphisms. Hence, Diagram \[ commutes if and only if

$$f(x, y)^{-1}y^{2\lambda}f(x, y) = f(x, y^2)^{-1}y^{2\lambda}f(x, y^2).$$

This is equivalent to the $f(x, y^2)f(x, y)^{-1}$ being an element of $C_y$.

$\Rightarrow$: Suppose that $f(x, y^2)f(x, y)^{-1} \in N$. Then clearly we have $f(x, y^2)f(x, y)^{-1} \in C_y$, and therefore Diagram \[ commutes.

$\Leftarrow$: Suppose that Diagram \[ commutes. Because

$$f(x, y^2)f(x, y)^{-1} \in C_y,$$

after lifting to $\hat{F}_2$, we have

$$f(x, y^2)f(x, y)^{-1}y^n \in N$$

for some $n \in \hat{Z}$. Passing to $\left(\hat{F}_2/N\right)^{ab} \cong \hat{Z} \oplus \hat{Z}/2$, where $y$ is identified with the generator 1 of $\hat{Z}$ and $xy$ is identified with the generator of $\hat{Z}/2$, we have $y^n = 1$. It follows that $n = 0$, and therefore $\alpha \in \hat{G}T_2$. \[ \square \]

Lemma 6.3. Let $\sigma \in \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$. Then, the following diagram commutes:

$$\begin{array}{ccc}
\hat{F}_2 & \xrightarrow{h} & \hat{F}_2/N \\
g(\sigma) \downarrow & & \downarrow g(\sigma) \\
\hat{F}_2 & \xrightarrow{h} & \hat{F}_2/N.
\end{array}$$

The key object we introduce is the Deligne-Mumford stack $S$ obtained from $\mathbb{P}^1_\overline{\mathbb{Q}}$ by removing 0, 1 and adding a $\mu_2$-stabilizer at $\infty$. More precisely, $S$ is obtained from $\mathbb{P}^1 \setminus \{0, 1\}$ by applying Cadman’s $r$th root construction to $X$ to take a 2nd root of the effective divisor ($\infty$).

We can choose generators for the topological fundamental group of the analytification $S^{\text{an}}_\mathbb{C}$ of the base-change of $S$ to $\mathbb{C}$ as in Figure \[. Therefore, we fix an isomorphism of $\pi_1(S^{\text{an}}, \overline{0})$ with $F_2/\hat{N}$, where $\hat{N}$ is the normal subgroup of $F_2$ generated by $(xy)^2$. This yields a fixed isomorphism between $\pi_1^{\text{et}}(S, \overline{0})$ and $\hat{F}_2/N$.

Proof. Note that $t$ defines a morphism of stacks from $\mathbb{P}^1_\overline{\mathbb{Q}} \setminus \{0, 1, \infty\}$ to $S$. Furthermore, $t$ exhibits $\mathbb{P}^1 \setminus \{0, 1, \infty\}$ as an étale cover of $S$, which demonstrates that $S$ is a Deligne-Mumford stack over $\overline{\mathbb{Q}}$. 


Consider the diagram of Deligne-Mumford stacks over \( \overline{\mathbb{Q}} \):
\[
\begin{array}{ccc}
P^1_{\overline{\mathbb{Q}}} \setminus \{0, 1, \infty\} & \to & S \\
& \downarrow & \\
P^1_{\overline{\mathbb{Q}}} \setminus \{0, 1, \infty\} & \xrightarrow{t} & S. 
\end{array}
\]

We identify the étale fundamental group of \( P^1_{\overline{\mathbb{Q}}} \setminus \{0, 1, \infty\} \) with \( \hat{F}_2 \) by identifying the topological fundamental group with \( F_2 \) as in Section 4 (see Figure 1). Hence, we have a diagram:
\[
\begin{array}{ccc}
\hat{F}_2 & \xrightarrow{g} & \hat{F}_2/\text{N} \\
& \downarrow & \downarrow \\
\hat{F}_2/\text{t} \quad & \xrightarrow{\text{q}} & \hat{F}_2/\text{N},
\end{array}
\]
where \( q \) is the quotient map. Because \( q \) is equivariant with respect to the action of \( \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \) on the fundamental groups defined by lifting the canonical outer action at the tangential base-point \( \overline{01} \), the homomorphism \( \sigma \mapsto \bar{g}(\sigma) \) defines the lifting of the \( \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \) action on \( \pi^\text{ét}_{1}(S, \overline{01}) \).

We claim that \( t \) preserves the action of \( \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \) on étale fundamental groups based at the tangential base-point \( \overline{01} \). We recall the algebraic definition of the tangential base-point \( \overline{01} \). Let \( z \) be an affine coordinate on \( P^1 \) such that \( z(0) = 0 \) and \( z(1) = 1 \), and so that \( z \) has a pole at \( \infty \). There is a natural morphism \( f : \text{Spec } \overline{\mathbb{Q}}[z] \to P^1_{\overline{\mathbb{Q}}} \setminus \{1, \infty\} \) that sends the closed point of \( \text{Spec } \overline{\mathbb{Q}}[z] \) to 0. Localizing at \( z \) yields a morphism \( f_z : \text{Spec } \overline{\mathbb{Q}}((z)) \to P^1_{\mathbb{Q}} \setminus \{0, 1, \infty\} \). Let \( \theta : \overline{\mathbb{Q}}((z)) \to K \) be an algebraic closure for \( \overline{\mathbb{Q}}((z)) \). Let \( \phi : K \to K \) be the natural continuous \( \overline{\mathbb{Q}} \)-algebra automorphism that sends \( z \) to \( \frac{4z}{(z+1)^2} \). We have a commuting square:
\[
\begin{array}{ccc}
\text{Spec } K & \xrightarrow{g} & P^1_{\overline{\mathbb{Q}}} \setminus \{0, 1, \infty\} \\
\downarrow{\text{Spec } \phi} & & \downarrow{t} \\
\text{Spec } K & \xrightarrow{\text{t}} & S.
\end{array}
\]

where \( i : P^1_{\mathbb{Q}} \setminus \{0, 1, \infty\} \to S \) is the natural open immersion. The discussion of Section 2.1 shows that definition of \( t_* \) is compatible with the identifications of the étale fundamental group with the profinite completions of the appropriate topological fundamental groups, because \( t'(0) = 4 \in \mathbb{R}_{>0} \).

It follows from Proposition 4.2 that \( t_*(x) = x \) and \( t_*(y) = y^2 \), and thus \( t_* = h \). Because \( \text{Spec } \phi \) and \( t \) are defined over \( \mathbb{Q} \), the homomorphism \( t_* = h \) is equivariant with respect to the action of \( \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \) on the étale fundamental groups of \( P^1_{\mathbb{Q}} \) and \( S \) based at \( \overline{01} \) (as defined in Equation 3). The lemma follows.

Theorem 3.7 is then immediate from Lemmata 6.2 and 6.3.

6.2. Proof of Theorem 3.8

Let \( \omega = e^{2\pi \sqrt{-3} / 3} \). Consider the function \( v = -\frac{3\sqrt{-3}(z-1)}{(z+\omega^2)^3} \) from \( P^1 \setminus \{0, 1, \infty\} \) to \( P^1 \setminus \{0\} \). Because \( v'(0) = 3\sqrt{-3} \), the morphism \( v \) does not preserve the topological tangential base-point \( \overline{01} \). Hence, we must deal with
Gal(\overline{\mathbb{Q}}/\mathbb{Q})\)-actions on \(\hat{F}_2\) for isomorphisms of \(\hat{F}_2\) with étale fundamental groups based at different base-points. To this end, we require the following proposition.

Proposition 6.4. Let \(b_1, b_2\) be base-points (potentially tangential) for \(\mathbb{P}^1 \setminus \{0, 1, \infty\}\), let \(p\) be a path from \(b_1\) to \(b_2\) in \(\mathbb{P}^1 \setminus \{0, 1, \infty\}\) and let \(p^*\) denote the induced isomorphism from \(\pi_1(\mathbb{P}^1 \setminus \{0, 1, \infty\}, b_2) \to \pi_1(\mathbb{P}^1 \setminus \{0, 1, \infty\}, b_1)\). Let

\[ i_1, i_2 : F_2 \to \pi_1(\mathbb{P}^1 \setminus \{0, 1, \infty\}, b_1), \pi_1(\mathbb{P}^1 \setminus \{0, 1, \infty\}, b_2) \]

be isomorphisms such that \(i_1(x)\) and \(i_2(x)\) (resp. \(i_1(y)\) and \(i_2(y)\)) have winding numbers 1 and 0 (resp. 0 and 1) about 0 and 1, respectively. Then, \(i_1^{-1} \circ p^* \circ i_2\) is an inner automorphism of \(F_2\).

The key tool we use is the fact that the natural map

\[ \text{Out}(F_2) \to \text{Aut}(F_2^{ab}) \cong \text{GL}_2(\mathbb{Z}) \]

is an isomorphism (see [17, §0.1]).

Proof. Because \(\mathbb{P}^1 \setminus \{0, 1, \infty\}\) is path-connected, it suffices to prove that if \(i_1, i_2\) are isomorphisms from \(F_2\) to \(\pi_1(\mathbb{P}^1 \setminus \{0, 1, \infty\}, b)\) such that \(i_1(x)\) and \(i_2(x)\) (resp. \(i_1(y)\) and \(i_2(y)\)) have winding numbers 1 and 0 (resp. 0 and 1) about 0 and 1, respectively, then \(i_1^{-1} \circ i_2\) is an inner automorphism of \(F_2\). By the constraint on winding numbers, the automorphism \(i_1^{-1} \circ i_2\) descends to the identity on \(F_2^{ab} \cong \mathbb{Z}^2\), and the conclusion follows by Equation (8) \(\square\)

In particular, the outer action of \(\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})\) on \(\hat{F}_2\) in Equation (8) is independent of the choice of (geometric, potentially tangential) base-point \(b\) and isomorphism of \(F_2\) with \(\pi_1(\mathbb{P}^1 \setminus \{0, 1, \infty\}, b)\). For the remainder of this section, we often suppress base-points and identifications of \(F_2\) or \(\hat{F}_2\) with the appropriate fundamental groups when discussing monodromy representations and the action of \(\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})\).

We follow the proof of Theorem 3.7. Let \(N_2\) be the closed subgroup of \(\hat{F}_2\) defined in the statement of Theorem 3.8. Define the homomorphism \(h_2 : \hat{F}_2 \to \hat{F}_2/N_2\) by \(x \mapsto x\) and \(y \mapsto xy^{-1}\). The key is that \(h_2\) is \(\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}(\sqrt{-3}))\)-equivariant. Similar to the proof of Theorem 3.7, Lemma 6.3 allows us to descend from \(\hat{F}_2\) to \(\hat{F}_2/N_2\), while Lemma 6.6 gives a criterion for an element of \(\hat{GT}_0\) to be an element of \(\hat{GT}_3\). Lemma 6.7 proves that elements of \(\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}(\sqrt{-3}))\) satisfy the criterion of Lemma 6.6 from which Theorem 3.8 is immediate.

Lemma 6.5. Let \(\alpha \in \hat{GT}_0\). Then, \(\alpha\) descends to an automorphism of \(\hat{F}_2/N_2\), which we denote by \(\bar{\alpha}\). Furthermore, \(\bar{\cdot}\) defines a homomorphism from \(\hat{GT}_0\) to \(\text{Aut}(\hat{F}_2/N_2)\).

Proof. By [9, Theorem 2], the automorphism \(\alpha\) preserves the conjugacy class of the inertia subgroups \(\langle y \rangle\) and \(\langle xy^{-1} \rangle\), and thus preserves \(N_2\). The proof follows. \(\square\)

Lemma 6.6. Let \(\alpha \in \hat{GT}_0\). Then, we have \(\alpha \in \hat{GT}_3\) if and only if the following diagram commutes up to an inner automorphism of \(\hat{F}_2/N_2\):

\[ \begin{array}{ccc}
\hat{F}_2 & \xrightarrow{\alpha} & \hat{F}_2 / N_2 \\
\downarrow h_2 & & \downarrow \bar{\alpha} \\
\hat{F}_2 & \xrightarrow{h_2} & \hat{F}_2 / N_2 \\
\end{array} \]

(9)
Proof. First, we claim that the centralizer $C_x$ of $x$ in $\mathring{F}_2/N_2$ is the set of (profinite) powers of $x$. We can exhibit $\mathring{F}_2/N_2$ as the coproduct of $\mathring{Z}$ and $\mathbb{Z}/3\mathbb{Z}$ in the category of profinite groups, where the generator $1$ of the first factor corresponds to $x$ and the generator of the second factor corresponds to $y$. By [10, Theorem B], $C_x$ is the set of (profinite) powers of $x$ in $\mathring{F}_2/N_2$. Actually, it follows from [10, Theorem B] that $C_x$ is the centralizer of any non-zero power of $x$ in $\mathring{F}_2/N_2$.

Fix $\alpha = (\lambda, f) \in \mathring{GT}_3$. The image of $x$ (resp. $y$) under the composite of the right and top homomorphisms in Diagram [9] is $x^\lambda$ (resp. $y$),

\[
f(x, y)^{-1} y^\lambda f(x, y)x^\lambda f(x, y)^{-1} y^{-\lambda} f(x, y),
\]

while $x$ (resp. $y$) is sent to $x^\lambda$ (resp.

\[
f(x, xy^{-1})^{-1} yx^\lambda y^{-1} f(x, yxy^{-1})
\]

under the composite of the bottom and left homomorphisms. Hence, Diagram [9] commutes up to an inner automorphism of $\mathring{F}_2/N_2$ if and only if there exists $g \in \mathring{F}_2/N_2$ such that $x^\lambda = g^{-1}x^\lambda g$ and

\[
f(x, y)^{-1} y^\lambda f(x, y)x^\lambda f(x, y)^{-1} y^{-\lambda} f(x, y) = g^{-1} f(x, yxy^{-1})^{-1} yx^\lambda y^{-1} f(x, yxy^{-1}) g.
\]

This is equivalent to the existence of an element $g \in C_x$ such that

\[
y^{-1} f(x, y^{-1}xy)g f(x, y)^{-1} y^\lambda f(x, y) \in C_x.
\]

$\Rightarrow$. Suppose that

\[
y^{-1} f(x, y^{-1}xy)x^m f(x, y)^{-1} y^\lambda f(x, y)x^n \in N_2.
\]

Let $g = x^m$. Then, clearly Equation [10] is satisfied and therefore Diagram [10] commutes.

$\Leftarrow$. Suppose that Diagram [7] commutes up to conjugation by $g$. Because

\[
y^{-1} f(x, y^{-1}xy)g f(x, y)^{-1} y^\lambda f(x, y) \in C_x,
\]

after lifting to $\mathring{F}_2$, we have

\[
y^{-1} f(x, y^{-1}xy)g f(x, y)^{-1} y^\lambda f(x, y)x^n \in N
\]

for some $n \in \mathring{Z}$. We have $g \in C_x$, which implies that $g = x^m$ for some $m \in \mathring{Z}$. Passing to $\left(\mathring{F}_2/N_2\right)^{ab} \cong \mathbb{Z}/3 \oplus \mathbb{Z}/3$, we have $m + n \equiv 0 \pmod{3}$. It follows that $\alpha \in \mathring{GT}_3$. $\Box$

Lemma 6.7. Let $\sigma \in \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}(\sqrt{-3}))$. Then, the following diagram commutes up to an inner automorphism of $\mathring{F}_2/N_2$:

\[
\begin{array}{c}
\mathring{F}_2 \xrightarrow{h_2} \mathring{F}_2/N_2 \\
\downarrow{g(\sigma)} \quad \quad \downarrow{g(\sigma)}
\end{array}
\]

\[
\xrightarrow{h_2} \mathring{F}_2/N_2.
\]

The key object we introduce is the Deligne-Mumford stack $S_2$ obtained from $\mathbb{P}^1_{\overline{\mathbb{Q}}}$ by removing 0 and adding $\mu_3$-stabilizers at 1 and $\infty$. The stack $S_2$ is the result of applying Cadman’s $r$th root construction to take 3rd roots of the effective divisors (1) and ($\infty$). We can choose generators for the topological fundamental group of the analytification $S_2^{an}$ of the base-change of $S_2$ to $\mathbb{C}$ as in Figure [1]. Therefore, we fix
an isomorphism of $\pi_1(S_{2,0}^\eta, \overrightarrow{01})$ with $F_2/\hat{N}_2$, where $\hat{N}_2$ is the normal subgroup of $F_2$ generated by $y^3$ and $(xy)^3$. This yields a fixed isomorphism between $\pi^\text{\acute e t}_1(S_2, \overrightarrow{01})$ and $\hat{F}_2/N_2$.

For a local coordinate $z_1 \in \mathcal{O}_{P_1,0}$ such that $3\sqrt{3}z_1 - z$ vanishes to order at least 2 at 0, let $\frac{1}{3\sqrt{3}}\overrightarrow{01}$ denote the natural tangential base-point corresponding to $z_1$. This corresponds to the topological tangent vector $\frac{1}{3\sqrt{3}}\overrightarrow{01}$.

Proof. The only points of ramification of $v$ are $-\omega$ and $-\omega^2$, where $v$ has ramifications of order 2. The corresponding branch points are $v(-\omega) = 1$ and $v(-\omega^2) = \infty$, respectively. Therefore, $v$ defines a morphism of stacks from $\mathbb{P}^1_\mathbb{Q} \setminus \{0, 1, \infty\}$ to $S$. Furthermore, $v$ exhibits $\mathbb{P}^1 \setminus \{0, 1, \infty\}$ as an étale cover of $S_2$, which demonstrates that $S_2$ is a Deligne-Mumford stack over $\overline{\mathbb{Q}}$.

Consider the diagram of Deligne-Mumford stacks over $\overline{\mathbb{Q}}$:

$$
\begin{array}{ccc}
\mathbb{P}^1_\mathbb{Q} \setminus \{0, 1, \infty\} & \xrightarrow{v} & S_2.
\end{array}
$$

We identify the étale fundamental group $\pi^\text{\acute e t}_1(\mathbb{P}^1 \setminus \{0, 1, \infty\}, \frac{1}{3\sqrt{3}}\overrightarrow{01})$ of the left copy of $\mathbb{P}^1 \setminus \{0, 1, \infty\}$ with $\hat{F}_2$ by identifying the topological fundamental group with $F_2$ as in Figure 3 using the generators $x \mapsto w_0$ and $y \mapsto w_1$. For the étale fundamental group $\pi^\text{\acute e t}_1(\mathbb{P}^1 \setminus \{0, 1, \infty\}, \overrightarrow{01})$ of the top copy of $\mathbb{P}^1 \setminus \{0, 1, \infty\}$, we use the generating set of Figure 11 with $x \mapsto x_0$ and $y \mapsto x_1$. Hence, we have a diagram:

$$
\begin{array}{ccc}
\hat{F}_2 & \xrightarrow{q} & \hat{F}_2/N_2,
\end{array}
$$

where $q$ is the quotient map. Because $q$ is equivariant with respect to the outer action of $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ on the fundamental groups, the homomorphism $\sigma \mapsto g(\sigma)$ defines a lifting of the outer $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ action on the étale fundamental group of $S$. Because the algebraic tangential base-point $\frac{1}{3\sqrt{3}}\overrightarrow{01}$ is defined over $\mathbb{Q}$ and the base-point $\overrightarrow{01}$ is defined over $\mathbb{Q}$, the étale fundamental group all have canonical outer actions of $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}(\sqrt{-3}))$. It follows from Proposition 6.4 that the two outer actions of $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}(\sqrt{-3}))$ on $\hat{F}_2$, which are à priori different (because they are induced by different identifications of $\hat{F}_2$ with the étale fundamental group of $\mathbb{P}^1 \setminus \{0, 1, \infty\}$) are in fact the same. Because $v$ is defined over $\mathbb{Q}$, $v_*$ is equivariant with respect to the outer action of $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}(\sqrt{-3}))$ on $\hat{F}_2$ and $\hat{F}_2/N_2$, in the sense that the diagram

$$
\begin{array}{ccc}
\hat{F}_2 & \xrightarrow{v_*} & \hat{F}_2/N_2
\end{array}
$$

is a pullback.
Figure 3. Generators for $\pi_1 \left( \mathbb{P}^1 \setminus \{-\omega^2, -\omega, 0, 1, \infty\}, \frac{1}{\sqrt[3]{-3}} \overrightarrow{01} \right)$.

The homotopy classes $w_0, w_1, w_\infty$ are defined similarly to Figure 1.

The homotopy class $w_{-\omega}$ (resp. $w_{-\omega^2}$) is defined by going from 0 to $Q$ (resp. $R$) along marked arc, moving along the marked path to the circle around $-\omega$ (resp. $-\omega^2$), traversing the circle around $-\omega$ (resp. $-\omega^2$) counterclockwise, returning to $Q$ (resp. $R$) around the marked path, and then returning to 0 along the marked arc. It is evident that $w_{-\omega} w_0 w_{-\omega^2} w_1 w_\infty = 1$.

commutes up to an inner automorphism of $\hat{F}_2 / N_2$.

Therefore, it suffices to prove that $v_* = h_2$. Notice that $v$ sends the tangential base-point $\frac{1}{\sqrt[3]{-3}} \overrightarrow{01}$ to $\overrightarrow{01}$. We can treat $v$ as an unramified map $v^*$ from the complex analytic space $\mathbb{P}^1 \setminus \{-\omega^2, \omega^2, 0, 1, \infty\}$ to $\mathbb{P}^1 \setminus \{0, 1, \infty\}$. We compute the morphism

$$v_*^* : \pi_1 \left( \mathbb{P}^1 \setminus \{-\omega^2, \omega^2, 0, 1, \infty\}, \frac{1}{\sqrt[3]{-3}} \overrightarrow{01} \right) \to \pi_1 \left( \mathbb{P}^1 \setminus \{0, 1, \infty\}, \overrightarrow{01} \right)$$

in order to compute the morphism $v_*$. We fix generators $w_{\omega^2}, w_{-\omega^2}, w_0, w_1, w_\infty$ for

$$\pi_1 \left( \mathbb{P}^1 \setminus \{-\omega^2, \omega^2, 0, 1, \infty\}, \frac{1}{\sqrt[3]{-3}} \overrightarrow{01} \right)$$
in Figure 3. It is evident that
\[
\begin{align*}
v_*^\# w - \omega &= x_1^3 \\
v_*^\# w_0 &= x_0 \\
v_*^\# w - \omega^2 &= x_1 x_\infty x_1^{-1} \\
v_*^\# w_1 &= x_1 x_0 x_1^{-1}
\end{align*}
\]

It follows that \( v_* = h_2 \) by reintroducing the points \(-\omega\) and \(-\omega^2\) to the source line, because in \( \pi_1 \left( \mathcal{S}_{2\ell, \ell}, [0,1] \right) \), we have \( x_1^3 = x_\infty = 1 \). □

Theorem 3.8 is immediate from Lemmata 6.6 and 6.7.

7. Concluding remarks and open problems

7.1. Generalizing the square-root class. Let \( t : \mathbb{P}_1 \to \mathbb{P}_1 \) be a morphism of curves satisfying \( t(\{0,1,\infty\}) \subseteq \{0,1,\infty\} \). Given a Belyi function \( f : X \to \mathbb{P}_1 \), we can form the generalized square-root class of \( f \), defined by

\[
\text{Sqrt}_t(f) = \{ \text{Belyi functions } g : X' \to \mathbb{P}_1 \mid g \times_{\mathbb{P}_1} t \cong f \}.
\]

It is clear that if \( t \) is defined over a number field \( K \), then the function \( \text{Sqrt}_t(f) \) is \( \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \)-equivariant. We recover the ordinary square-root class for the choice of \( t = 4f \).

However, if \( t \) is of degree greater than 1, then \( \text{Sqrt}_t(f) \) will be empty for most Belyi functions \( f \), and therefore we do not recover a very general invariant. In our case, where \( t = \frac{4f}{(f+1)^2} \), the monodromy cycle types of \( f \) above 0 and \( \infty \) must be the same in order for \( \text{Sqrt}(f) \) to be nonempty. We give an example that suggests that one may be able to reformulate the invariant in a manner that is applicable more generally.

7.2. Example: Belyi functions with monodromy of cycle type \((n,(2g+1)11\cdots 1,n)\). We apply the Orbit-Splitting Theorem to the case of Belyi functions with monodromy of cycle type \((n,(2g+1)11\cdots 1,n)\). An explicit count of \( M(n,(2g+1)11\cdots) \) and an application of the \( n \)-cycle Orbit-Splitting Theorem 3.11 yield the following result.

**Proposition 7.1.** Let \( g \) be a positive integer and let \( n \geq 4g+1 \) be an odd positive integer. Then, there are at least \( \left\lfloor \left( \frac{4}{g} + 1 \right)^2 \right\rfloor \) \( \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \)-orbits classes of Belyi maps with monodromy of cycle type \((n,(2g+1)11\cdots 1,n)\).

In the case of \( g = 1 \) and \( n = 5,7,9 \), we constructed the Belyi functions and explicitly verified the following conjecture, which suggests that the square-root cycle type class can be adapted to an invariant that describes the combinatorial action of \( \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \) on the groups of divisors or principal divisors.

**Conjecture 7.2.** Let \( n \) be an odd positive integer, \( X \) an algebraic curve, and \( f : X \to \mathbb{P}_1 \) a Belyi function with monodromy of cycle type \((n,311\cdots 1,n)\). Let \( P \) and \( O \) be the locations of the ramifications of order \( n-1 \) on \( X \), and let \( T \) be the
location of the ramification of order 2. Then, \( \text{SqCt}(f) = \{(22 \cdots 2111, 322 \cdots 2, n)\} \) if and only if

\[
(T) \sim \frac{n + 1}{2} (P) - \frac{n - 1}{2} (O)
\]

as divisors on \( X \).

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Appendix A. Proofs of Lemmata 5.6, 5.7, and 5.8

Proof of Lemma 5.6. We have

\[ n \leq 2t + 1 + \sum_{k=1}^{t} 2(2k - 1) \left( \frac{4t + 2}{2k - 1} - 1 \right) = 2t + 1 + 2 \sum_{k=1}^{t} (4t + 3 - 2k) \]

\[ = 2t + 1 + t(6t + 6) = 6t^2 + 12t + 1 < 6(t + 1)^2 \]

and

\[ n > 2t + 1 + \sum_{k=1}^{t} 2(2k - 1) \left( \frac{4t + 2}{2k - 1} - 2 \right) = 6t^2 + 12t + 1 - \sum_{k=1}^{t} 2(2k - 1) \]

\[ = 4t^2 + 12t + 1. \]

□

Proof of Lemma 5.7. We have

\[ \sum_{k=1}^{t} (f(k) - 1) \leq \sum_{k=1}^{t} \left( \frac{4t + 1}{2k - 1} - 1 \right) = -t + (4t + 1) \sum_{k=1}^{t} \frac{1}{2k - 1}. \]

Applying the bound

\[ \log(m + 1) \leq \sum_{k=1}^{m} \frac{1}{k} \leq \log m + 1, \]

which holds for all positive integers \( m \), we have

\[ \sum_{k=1}^{t} (f(k) - 1) \leq -t + (4t + 1) \left( \log(2t - 1) + 1 - \frac{\log(t)}{2} \right). \]

Therefore, we have

\[ 2 \sum_{k=1}^{t} (f(k) - 1) \leq 6t + 2 + (4t + 1) \log(4t). \]

It follows that \( 2 \sum_{k=1}^{t} (f(k) - 1) \leq 2t^2 + 6t + \frac{1}{2} \leq \frac{n(t)}{2} \) for \( t \geq 8 \), where the second inequality is by Lemma 5.6. We can easily verify the lemma for \( t \leq 7 \), and the lemma follows. □

Proof of Lemma 5.8. Fix \( t \), and let \( M \) denote the left-hand side. We have

\[ M > \prod_{k=1}^{t} \left( \frac{4t + 2}{2k - 1} - 1 \right) = \prod_{k=1}^{t} \frac{(4t + 3 - 2k)}{(2k - 1)}. \]

Recall that

\[ (2m - 1)!! = \prod_{k=1}^{m} (2k - 1) = \frac{(2m)!}{2^m (m!)}. \]

Returning to \( M \), we have

\[ M > \frac{(4t + 1)!!}{(2t + 1)!!(2t - 1)!!} = \frac{(4t + 2)!2^{t+1}2^t}{(2t + 2)!((2t)!)^2 2^{2t+1}} = \frac{(4t + 2)!2^{t+1}(t + 1)!2^t!}{(2t + 1)!(2t + 2)!(2t)!} = \frac{(4t + 2)!}{(2t + 1)!(2t + 2)!} \cdot \frac{(2^t)!}{2^t} \]
We now apply Stirling’s formula with error bounds, which is the well-known inequality
\[ e^{\frac{1}{12m+1}} < \frac{m!}{\sqrt{2\pi m} \left( \frac{m}{e} \right)^m} < e^{\frac{1}{12m}}. \]

It follows that
\[ e^{\frac{1}{12m+1} - \frac{1}{6}} < \frac{(2m)\sqrt{\pi m}}{2^m} < e^{\frac{1}{12m} - \frac{2}{12m+1}}. \]

In particular, we have
\[ -\frac{1}{6m} < \log \left( \frac{(2m)\sqrt{\pi m}}{2^m} \right) < 0. \]

Applying this bound to \( M \), we have
\[ M > 2^{2t} \sqrt{2e^{\frac{1}{12m+1}}} > 2^{2t}. \]

\[ \square \]