ON A CLASS OF DEGENERATE AND SINGULAR MONGE-AMPÈRE EQUATIONS

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Abstract: In this paper we shall prove the existence, uniqueness and global Hölder continuity for the Dirichlet problem of a class of Monge-Ampère type equations which may be degenerate and singular on the boundary of convex domains. We will establish a relation of the Hölder exponent for the solutions with the convexity for the domains.

Key Words: existence, uniqueness, global regularity, degenerate, singular, Monge-Ampère equation

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1. Introduction

In this paper we study the Monge-Ampère type equation
\[ \det D^2 u = F(x, u) \text{ in } \Omega, \]
\[ u = 0 \text{ on } \partial \Omega, \]
where \( \Omega \) is a bounded convex domain in \( \mathbb{R}^n \) \( (n \geq 2) \), and \( F \) satisfies the following (1.2)-(1.3):
\[ F(x, t) \in C(\Omega \times (-\infty, 0)) \text{ is non-decreasing in } t \text{ for any } x \in \Omega; \]
\[ \text{there are constants } A > 0, \alpha \geq 0, \beta \geq n + 1 \text{ such that} \]
\[ 0 < F(x, t) \leq Ad_x^{\beta-n-1}|t|^{-\alpha} \quad \forall (x, t) \in \Omega \times (-\infty, 0), \]
where \( d_x = \text{dist}(x, \partial \Omega) \). Obviously, this problem is singular and degenerate at the boundary of the domain.

The particular case of problem (1.1) includes a few geometric problems. When \( F = |t|^{-(n+2)} \) and \( u \) is a solution to problem (1.1), then the Legendre transform of \( u \) is a complete affine hyperbolic sphere \([3, 4, 6, 11, 13]\), and \((-u)^{-1} \sum u_{x_ix_j}dx_idx_j \) gives the Hilbert metric (Poincaré metric) in the convex domain \( \Omega \) \([18]\). When \( F = f(x)|t|^{-p} \), problem (1.1) may be obtained from \( L^p \)-Minkowski problem \([19]\) and the Minkowski problem in centro-affine geometry \([7, 12]\). Also see p.440-441 in \([14]\). Generally, problem (1.1) can be applied to construct non-homogeneous complete Einstein-Kähler metrics on a tubular domain \([4, 5]\).

Cheng and Yau in \([4]\) proved that if \( \Omega \) is a strictly convex \( C^2 \)-domain and \( F \in C^k \) \( (k \geq 3) \) satisfies (1.2)-(1.3), then problem (1.1) admits an unique convex generalized solution \( u \in C(\bar{\Omega}) \). Moreover, \( u \in C^{k+1,\varepsilon}(\Omega) \cap C^\gamma(\bar{\Omega}) \) for any \( \varepsilon \in (0, 1) \) and some \( \gamma = C(\beta, \alpha, A, n, \partial \Omega) \in (0, 1) \). We should emphasize that their methods need the strict convexity and the smoothness of \( \Omega \), and the differentiability of \( F \).

In this paper we find that the global Hölder regularity for problem (1.1) is independent of the smoothness of \( \Omega \) and \( F \), and the Hölder exponent depends only on the convexity of the domain. As a result, we can remove the smoothness of \( \Omega \) as well as the differentiability of \( F \) in \([4]\). Moreover, using the concept of \((a, \eta)\) type introduced in \([11]\) to describe the convexity of the domain, we obtain a relation of the Hölder exponent for \( u \) with the convexity for \( \Omega \).
We have noticed that there are many papers on global regularity for equations of Monge-Ampère type. See, for example, [2 8 10 17 21 22 24] and the references therein. But, generally speaking, those results require that the domain $\Omega$ should be strictly convex and $\partial \Omega \in C^{1,1}$.

Our first result is stated as the following

**Theorem 1.1.** Supposed that $\Omega$ is a bounded convex domain in $\mathbb{R}^n$ and $F(x,t)$ satisfies (1.2)-(1.3). Let

\begin{equation}
\gamma_1 := \begin{cases} 
\frac{\beta-n+1}{n+\alpha}, & \text{if } \beta < \alpha + 2n - 1, \\
\text{any number in } (0,1), & \text{if } \beta \geq \alpha + 2n - 1.
\end{cases}
\end{equation}

Then problem (1.1) admits an unique convex generalized solution $u \in C^{\gamma_1}(\Omega)$. Furthermore, $u \in C^{2,\gamma_1}(\Omega)$ if $F(x,t) \in C^{0,1}(\Omega \times (-\infty, 0))$.

Here a generalized solution means the well-known Alexandrov solution. See, for example, [8 9 23] for the details.

To improve the regularity for the solution obtained in Theorem 1.1, we use the $(a, \eta)$ type in [11] to describe the convexity of $\Omega$. From now on, we denote

$$ x = (x_1, x_2, \ldots, x_n) = (x', x_n), \quad x' = (x_1, \ldots, x_{n-1}) $$

and

$$ |x'| = \sqrt{x_1^2 + \ldots + x_{n-1}^2}. $$

**Definition 1.1.** Supposed that $\Omega$ is a bounded convex domain in $\mathbb{R}^n$, and $x_0 \in \partial \Omega$. $x_0$ is called to be $(a, \eta)$ type if there are numbers $a \in [1, +\infty)$ and $\eta > 0$, after translation and rotation transforms, we have

$$ x_0 = 0 \quad \text{and} \quad \Omega \subseteq \{ x \in \mathbb{R}^n | x_n \geq \eta |x'|^a \}. $$

$\Omega$ is called $(a, \eta)$ type domain if every point of $\partial \Omega$ is $(a, \eta)$ type.

**Remark 1.1.** The convexity requires that the number $a$ should be no less than 1. The less is $a$, the more convex is the domain. There is no $(a, \eta)$ type domain for $a \in [1, 2)$, although part of $\partial \Omega$ may be $(a, \eta)$ type point for $a \in [1, 2)$.

**Definition 1.2.** We say that a domain $\Omega$ in $\mathbb{R}^n$ satisfies exterior (or interior) sphere condition with radius $R$ if for each $x_0 \in \partial \Omega$, there is a $B_R(y_0) \supseteq \Omega$ (or $B_R(y_0) \subseteq \Omega$, respectively) such that $\partial B_R(y_0) \cap \partial \Omega \ni x_0$.

In [11], we have proved that $(2, \eta)$ type domain is equivalent to the domain satisfies exterior sphere condition.

The following two theorems show the relation of the Hölder exponent for $u$ on $\bar{\Omega}$ with the convexity for $\Omega$. 

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Theorem 1.2. Suppose that $\Omega$ is $(a, \eta)$ type domain in $\mathbb{R}^n$ with $a \in (2, +\infty)$, and $F$ satisfies (1.2)-(1.3). Let Let

\begin{equation}
\gamma_2 := \begin{cases}
\frac{\beta-n+1}{n+\alpha} + \frac{2n-2}{a(n+\alpha)}, & \text{if } \beta < \alpha + 2n - 1 - \frac{2n-2}{a}, \\
\text{any number in}(0,1), & \text{if } \beta \geq \alpha + 2n - 1 - \frac{2n-2}{a}.
\end{cases}
\end{equation}

Then the convex generalized solution to problem (1.1)

\begin{equation}
\gamma_2(\Omega).
\end{equation}

Furthermore $u \in C^{2,\gamma_2}(\Omega)$ if $F(x, t) \in C^{0,1}(\Omega \times (-\infty, 0))$.

Theorem 1.3. Let $\Omega$ be a bounded convex domain in $\mathbb{R}^n$ and $u$ be a convex generalized solution to problem (1.1).

(i) Suppose that $\Omega$ satisfies exterior sphere condition and $F$ satisfies (1.2)-(1.3). Let

\begin{equation}
\gamma_3 := \begin{cases}
\frac{\beta}{n+\alpha}, & \text{if } \beta < \alpha + n, \\
\text{any number in}(0,1), & \text{if } \alpha + n \leq \beta < \alpha + n + 1, \\
1, & \text{if } \beta \geq \alpha + n + 1.
\end{cases}
\end{equation}

Then

\begin{equation}
u \in C^{\gamma_3}(\overline{\Omega}).
\end{equation}

Furthermore $u \in C^{2,\gamma_3}(\Omega)$ if $F(x, t) \in C^{0,1}(\Omega \times (-\infty, 0))$.

(ii) If $\Omega$ satisfies interior sphere condition with radius $R$ and $F$ satisfies (1.2) and

\begin{equation}
Ad_x^{\beta-n-1}|t|^{\alpha} \leq F(x, t), \forall (x, t) \in \Omega \times (-\infty, 0)
\end{equation}

for some constants $A > 0$, then

\begin{equation}
|u(y)| \geq C(d_y)^{\gamma_4}, \forall y \in \Omega
\end{equation}

for some constant $C = C(\beta, \alpha, A, n, R) > 0$, where

\begin{equation}
\gamma_4 := \frac{\beta}{n+\alpha} \in (0,1),
\end{equation}

Remark 1.2. The Hölder regularity result of Theorem 1.1 can be viewed as the limit case of Theorem 1.2 as $a \to \infty$. Theorem 1.3 (i) shows that Theorem 1.2 is true for $a = 2$, since a $(2, \eta)$ type domain is equivalent to that the domain satisfies exterior sphere condition.

In the following Sections 2, 3, and 4, we will prove Theorems 1.1, 1.2, and 1.3, respectively.
2. Proof of Theorem 1.1

We start at a primary result which is useful to proving that a convex function in \( \Omega \) is Hölder continuous in \( \overline{\Omega} \).

**Lemma 2.1.** Let \( \Omega \) be a bounded convex domain and \( u \in C(\overline{\Omega}) \) be a convex function in \( \Omega \) with \( u|_{\partial \Omega} = 0 \). If there are \( \gamma \in (0, 1] \) and \( M > 0 \) such that

\[
\tag{2.1} |u(x)| \leq Md_x^{-\gamma}, \quad \forall x \in \Omega,
\]

then \( u \in C^\gamma(\overline{\Omega}) \) and

\[
|u|_{C^\gamma(\overline{\Omega})} \leq M\{1 + [\text{diam}(\Omega)]^{-\gamma}\}.
\]

**Proof.** This was proved in [11]. Here we copy the arguments for the convenience.

For any two point \( x_1, x_2 \in \Omega \), consider the line determined by \( x_1 \) and \( x_2 \). The line will intersect \( \partial \Omega \) at two points \( y_1 \) and \( y_2 \). Without loss generality we assume the four points are \( y_1, x_1, x_2, y_2 \) in order. By restricted onto the line, \( u \) is one dimension convex function. By the monotonic proposition of convex functions, we have

\[
|u(x_2) - u(x_1)| \leq \max\{|u(y_1 + (x_2 - x_1)) - u(y_1)|, |u(y_2) - u(y_2 - (x_2 - x_1))|\}.
\]

Moreover, since \( y_1 \in \partial \Omega \), by the assumption (2.1) we have

\[
|u(y_1 + (x_2 - x_1)) - u(y_1)| = |u(y_1 + (x_2 - x_1))| \\
\leq M\{\text{dist}(y_1 + x_2 - x_1, \partial \Omega)\}^{\gamma} \\
\leq M|x_2 - x_1|^{\gamma}.
\]

Similarly,

\[
|u(y_2) - u(y_2 - (x_2 - x_1))| \leq M|x_2 - x_1|^{\gamma}.
\]

The above three inequalities, together with (2.1), implies the desired result. \( \square \)

To prove Theorem 1.1, we need an a priori estimate result as follows, which holds without strictly convexity of \( \Omega \) or any smoothness of \( \Omega \) and of \( F \).

**Lemma 2.2.** Supposed that \( \Omega \) is a bounded convex domain in \( \mathbb{R}^n \) and \( F(x, t) \) satisfies (1.2) and (1.3). If \( u \) is a convex generalized solution to problem (1.1), then \( u \in C^{\gamma_1}(\overline{\Omega}) \) and

\[
\tag{2.2} |u|_{C^{\gamma_1}(\overline{\Omega})} \leq C(\alpha, \beta, A, \text{diam}(\Omega), n),
\]

where \( \gamma_1 \) is given by (1.4).

**Proof.** First, we may assume

\[
\tag{2.3} \beta < \alpha + 2n - 1.
\]
Since for the case $\beta \geq \alpha + 2n - 1$, we take a $\hat{\beta} < \alpha + 2n - 1$ such that $\frac{\hat{\beta} - n + 1}{n + \alpha}$ can be any number in $(0, 1)$. (Note $n \geq 2$). Obviously, (1.3) still holds with $\beta$ replaced by $\hat{\beta}$. Hence, this case is reduced to the case (2.3).

Next, we assume for the time being that

$$0 \in \mathbb{R}^n.$$ (2.4)

Then we are going to construct a sub-solution to problem (1.1).

For brevity, write $l = \text{diam} (\Omega)$. Set

$$W = -Mx_n^\gamma \cdot \sqrt{N^2l^2 - r^2}$$

where $r = \sqrt{x_1^2 + \ldots + x_{n-1}^2}$. We will choose positive constants $\gamma, M, N$ such that $W$ is a sub-solution to problem (1.1) under the assumptions (2.3) and (2.4).

For $i, j \in \{1, 2, \ldots, n-1\}$, write $W_i = \frac{\partial W}{\partial x_i}, W_{ij} = \frac{\partial^2 W}{\partial x_i \partial x_j}$. Then we have

$$W_i = Mx_n^\gamma \cdot \frac{x_i}{\sqrt{N^2l^2 - r^2}},$$

$$W_{ij} = Mx_n^\gamma \cdot \frac{1}{\sqrt{N^2l^2 - r^2}} (\delta_{ij} + \frac{x_ix_j}{\sqrt{N^2l^2 - r^2}}),$$

$$W_n = -M\gamma x_n^{\gamma - 1} \cdot \sqrt{N^2l^2 - r^2},$$

$$W_{in} = M\gamma x_n^{\gamma - 1} \cdot \frac{x_i}{\sqrt{N^2l^2 - r^2}},$$

$$W_{nn} = M\gamma (1 - \gamma)x_n^{\gamma - 2} \cdot \sqrt{N^2l^2 - r^2}.$$

Denote

$$D^2W := \left( \begin{array}{cc} G & \xi \\ \xi^T & W_{nn} \end{array} \right)$$

where $\xi^T = (W_{n1}, \ldots, W_{n(n-1)})$, and $G$ is the $(n - 1)$-order matrix. Then

$$\text{det}D^2W = \text{det}G \cdot (W_{nn} - \xi^T G^{-1} \xi).$$

Since all the eigenvalues of $G$ are

$$Mx_n^\gamma \frac{1}{\sqrt{N^2l^2 - r^2}}, \ldots, Mx_n^\gamma \frac{1}{\sqrt{N^2l^2 - r^2}}, Mx_n^\gamma \frac{1}{(N^2l^2 - r^2)\sqrt{N^2l^2 - r^2}},$$

$$\text{det}G = Mn^{-1}N^2l^2x_n^{(n-1)\gamma} \cdot \left( \frac{1}{\sqrt{N^2l^2 - r^2}} \right)^{n+1}.$$

It is direct to verify that

$$G\xi = \frac{N^2l^2 Mx_n^\gamma}{(N^2l^2 - r^2)^{\frac{n+1}{2}}} \xi.$$
It follows that
\[
\xi^T G^{-1} \xi = \left( \frac{N^2l^2 - r^2}{N^2l^2 M x_n} \right)^\frac{1}{2} |\xi|^2 \\
= \frac{M \gamma^2}{N^2l^2} x_n^{\gamma - 2} r^2 \sqrt{N^2l^2 - r^2}.
\]

Hence, we obtain that
\[
det D^2 W = det G (W_{nn} - \xi^T G^{-1} \xi) \\
= M^{n-1} N^2l^2 x_n^{(n-1)\gamma} (\frac{1}{\sqrt{N^2l^2 - r^2}})^n+1 M \gamma x_n^{\gamma - 2} \sqrt{N^2l^2 - r^2} \\
\cdot \left[ 1 - (1 + \frac{r^2}{N^2l^2}) \gamma \right] \\
= M^n N^2l^2 \gamma x_n^{n+\gamma - 2} (\frac{1}{\sqrt{N^2l^2 - r^2}})^n [1 - (1 + \frac{r^2}{N^2l^2}) \gamma].
\]

(2.5)

We want to prove
\[
det D^2 W \geq F(x, W) \text{ in } \Omega.
\]

Since (1.3) and (2.4) implies that
\[
F(x, W) \leq Ad^\beta - n - 1 |W|^{-\alpha} \leq Ax_m^{\beta - n - 1} |W|^{-\alpha},
\]
we see that (2.6) can be deduced from
\[
(2.7) \quad det D^2 W \geq Ax_m^{\beta - n - 1} |W|^{-\alpha} \text{ in } \Omega,
\]
which is equivalent to
\[
(2.8) \quad det D^2 W \cdot \frac{1}{A} x_n^{n+1 - \beta} |W|^{\alpha} \geq 1 \text{ in } \Omega.
\]

By (2.5), (2.8) is nothing but
\[
(2.9) \quad \frac{1}{A} M^{n+\alpha} N^2l^2 \gamma x_n^{(n+\alpha)\gamma - (\beta - n + 1)} [1 - (1 + \frac{r^2}{N^2l^2}) \gamma] \cdot (\sqrt{N^2l^2 - r^2})^{\alpha - n} \geq 1 \text{ in } \Omega.
\]

Now we choose \( \gamma = \frac{\beta - n + 1}{n+\alpha} \) such that
\[
(n + \alpha) \gamma - (\beta - n + 1) = 0.
\]

Since \( \gamma \in (0, 1) \) by (2.3) and \( r = |x'| \leq diam(\Omega) = l \) in \( \Omega \), we first take \( N = C(\gamma) \) large enough such that
\[
1 - (1 + \frac{r^2}{N^2l^2}) \gamma > 0.
\]
Noting \( N^2l^2 - r^2 \in [(N^2 - 1)l^2, N^2l^2] \), we then take \( M = C(A, \alpha, \gamma, N, n, l) \) large enough such that

\[
\frac{1}{A} M^{n+\sigma} N^2l^2 x_n^{(n+\alpha)\gamma - (\beta-n+1)} [1 - (1 + \frac{r^2}{N^2l^2})\gamma] \cdot (\sqrt{N^2l^2 - r^2})^{\alpha-n} \geq 1,
\]

we obtain (2.9) and thus have proved (2.6).

Finally, for any point \( y \in \Omega \), letting \( z \in \partial \Omega \) be the nearest boundary point to \( y \), by some translations and rotations, we assume \( z = 0 \), \( \Omega \subseteq \mathbb{R}^n \) and the line \( yz \) is the \( x_n \)-axis. This is to say that (2.4) is satisfied. Therefore we have (2.6). Obviously, \( W \leq 0 \) on \( \Omega \). Hence, \( W \) is a sub-solution to problem (1.1). By comparison principle for generalized solutions (see [8, 9, 23] for example), we have

\[
|u(y)| \leq |W(y)| \leq MNl^{\beta-n+1} = MNld^{\beta-n+1},
\]

which, together with Lemma 2.1, implies the desired result (2.2).

Note that we have used the fact that problem (1.1) is invariant under translation and rotation transforms, since \( detD^2u \) is invariant and \( F(x, u) \) is transformed to the one satisfying the same condition as \( F \). This fact will be again used a few times in the following.

\[\square\]

**Proof of Theorem 1.1.** We prove the theorem by three steps.

**Step 1.** Suppose that \( \Omega \) is bounded convex but \( F(x, t) \in C^k(\Omega \times (-\infty, 0)) \) \((k \geq 3)\) satisfies (1.2) and (1.3).

We choose a sequence of bounded and strictly convex domains \( \{\Omega_i\} \) such that

\[
\Omega_i \in C^2 \quad \text{and} \quad \Omega_i \subseteq \Omega_{i+1}, i = 1, 2, \ldots, \bigcup_{i=1}^{\infty} \Omega_i = \Omega.
\]

Then by Theorem 5 in [4], there exists a convex generalized solution \( u_i \) to problem (1.1) in the domain \( \Omega_i \) for each \( i \). We assume \( u_i(x) = 0 \) for all \( x \in \mathbb{R}^n \setminus \Omega_i \). By Lemma 2.2, We have the uniform estimations

\[
|u_i|_{C^{\beta-n+1}(\Omega_i)} = |u_i|_{C^{\beta-n+1}(\Omega_i)} \leq C(\alpha, \beta, A, diam(\Omega), n),
\]

which implies that there is a subsequence, still denoted by itself, convergent to a \( u \) in the space \( C(\Omega) \). Moreover, by (2.11) again, we have

\[
|u|_{C^{\beta-n+1}(\Omega)} \leq C(\alpha, \beta, A, diam(\Omega), n).
\]

By the well-known convergence result for convex generalized solutions (see Lemma 1.6.1 in [9] for example), we see that \( u \) is a convex generalized solution to problem (1.1).
Step 2. Drop the restriction on the smoothness for $F$.

Suppose $F_j \in C^k (\Omega \times (-\infty,0)) \ (k \geq 3)$ satisfy the same assumption as $F$ in the Step 1 and $F_j$ locally uniform convergence to $F$ in as $j \to \infty$. (For example we can take $F_j = F \ast \eta_{\epsilon_j}$, $\epsilon_j$ convergence to $0$ as $j$ tend to $+\infty$.) Then by the result of Step 1, for each $j$, there exists a convex generalized solution $u_j \in C^{\frac{\beta-n+1}{n+\alpha}}(\Omega)$ to problem (1.1) with $F$ replaced by $F_j$. Moreover, we have

$$|u_j|_{C^{\frac{\beta-n+1}{n+\alpha}}(\Omega)} \leq C(\alpha, \beta, A, \text{diam}(\Omega), n)$$

(2.12)

for all $j$. Using this estimate, Lemma 1.6.1 in [9], and the same argument as in Step 1, we obtain a solution $u$ to problem (1.1), which is the limit of a subsequence of $u_j$ in the space space $C(\Omega)$. Furthermore, we have $u \in C^{\frac{\beta-n+1}{n+\alpha}}(\Omega)$ by (2.12). The uniqueness for (1.1) is directly from the comparison principle (see [8, 9, 23] for example).

Step 3. We are going to prove $u \in C^{\frac{\beta-n+1}{n+\alpha}}(\Omega)$ if $F(x,t) \in C^{0,1}(\Omega \times (-\infty,0))$.

It is enough to prove

$$u \in C^{\frac{\beta-n+1}{n+\alpha}}(\Omega_1)$$

(2.13)

for any convex $\Omega_1 \subset \subset \Omega$.

Taking a convex $\Omega'$ such that $\Omega_1 \subset \subset \Omega' \subset \subset \Omega$, if there exists $z \in \text{cl} \Omega' \subset \subset \Omega$ such that $u(z) = 0$, then $u \equiv 0$ in $\Omega$ by convexity and the boundary condition $u|_{\partial \Omega} = 0$. Hence we obtain (2.13). Otherwise, $u(x) < 0$ for all $x \in \text{cl} \Omega'$. Then $F(x,u(x)) \in C^{\frac{\beta-n+1}{n+\alpha}}(\Omega')$ and is positive on $\text{cl} \Omega'$. By the Caffarelli’s local $C^{2,\alpha}$ regularity in [11] (also see [15] for another proof), we obtain (2.13), too.

### 3. Proof of Theorem 1.2

In this section we establish the relation between the H"older exponent and the convexity of the domain $\Omega$ and thus prove Theorem 1.2.

Assume that $\Omega$ is a $(a, \eta)$ type domain with $a \in (2, \infty)$, $F$ satisfies (1.2)-(1.3), and $u$ is the unique solution to problem (1.1) as in Theorem 1.1. To prove Theorem 1.2, it is sufficient to prove (1.6). See the Step 3 in the proof of Theorem 1.1.

As (2.3) we may assume

$$\beta < \alpha + 2n - 1 - \frac{2n-2}{a}.$$  

(3.1)

Hence, in the following we have

$$\gamma_2 = \frac{\beta-n+1}{n+\alpha} + \frac{2n-2}{a(n+\alpha)} \in (0,1).$$
By Lemma 2.1, (1.6) can be deduced from
\[(3.2) \quad |u(y)| \leq C d_y^{\gamma_2}, \quad \forall y \in \Omega\]
for some positive constant \(C = C(a, n, \alpha, \eta, A, \text{diam}\Omega)\).

We are going to prove (3.2). For any \(y \in \Omega\), we can find \(z \in \partial \Omega\), such that \(|y - z| = d_y\). Since the domain \(\Omega\) is \((a, \eta)\) type and the problem (1.1) is invariant under translation and rotation transforms, we may assume \(z = 0\), and take the line determined by \(z\) and \(y\) as the \(x_n - axis\) such that
\[\Omega \subseteq \{ x \in \mathbb{R}^n | x_n \geq \eta |x'|^a \} \].

We will prove (3.2) by three steps.

**Step 1.** Let
\[W(x_1, ..., x_n) = W(r, x_n) = -\left(\frac{x_n}{r} \right)^2 - x_1^2 - ... - x_{n-1}^2 \right) \frac{1}{b}, \]
where \(r = |x'| = \sqrt{x_1^2 + \ldots + x_{n-1}^2}\), \(b\) and \(\varepsilon\) are positive constants to be determined. We want to find a sufficient condition for which \(W\) is a sub-solution to problem (1.1).

For \(i, j \in \{1, 2, ..., n-1\}\), by direct computation we have
\[W_i = W_r \frac{x_i}{r}, \]
\[W_{ij} = W_r \delta_{ij} + (W_r - W_{rr}) \frac{x_i x_j}{r^2}, \]
\[W_{nn} = W_r \frac{x_n}{r}. \]

Let
\[D^2W := \begin{pmatrix} G & \xi \\ \xi^T & W_{nn} \end{pmatrix} \]
where \(\xi = (W_{n1}, ..., W_{n(n-1)})\), and \(G\) is the matrix of \(n - 1\) order all of which eigenvalues are
\[W_r, ..., W_r, W_{rr}, \]
and one of which eigenvector with respect to the eigenvalue \(W_{rr}\) is \(\xi\). As obtaining (2.5), we have
\[\text{det}D^2W = (W_r)^{n-2} W_{rr} (W_{nn} - |W_{rn}|^2 W_{rr}). \]

Obviously, \(W \leq 0\) on \(\partial \Omega\). Therefore we conclude that \(W\) is a sub-solution to problem (1.1) if and only if
\[(3.4) \quad H[W] := (W_r)^{n-2} (W_{rr} W_{nn} - |W_{rn}|^2) [F(x, W)]^{-1} \geq 1 \text{ in } \Omega. \]
We use the expression of $W$ to compute

$$W_r = \frac{2}{b} ((\frac{x_n}{\varepsilon})^{\frac{2}{\alpha}} - r^2)^{\frac{1}{b} - 1} \cdot r,$$

$$W_n = -\frac{2}{ab} ((\frac{x_n}{\varepsilon})^{\frac{2}{\alpha}} - r^2)^{\frac{1}{b} - 1} \cdot (\frac{x_n}{\varepsilon})^{\frac{2}{\alpha} - 1} \cdot \frac{1}{\varepsilon},$$

$$W_{rr} = \frac{4}{b} (1 - \frac{1}{b}) ((\frac{x_n}{\varepsilon})^{\frac{2}{\alpha}} - r^2)^{\frac{1}{b} - 2} \cdot r^2 + \frac{2}{b} ((\frac{x_n}{\varepsilon})^{\frac{2}{\alpha}} - r^2)^{\frac{1}{b} - 1},$$

$$W_{nn} = \frac{4(b - 1)}{a^2b^2} ((\frac{x_n}{\varepsilon})^{\frac{2}{\alpha}} - r^2)^{\frac{1}{b} - 2} \cdot (\frac{x_n}{\varepsilon})^{\frac{2}{\alpha} - 2} \cdot (\frac{1}{\varepsilon})^2 \cdot r^2 + \frac{2(a - 2)}{a^2b} (\frac{x_n}{\varepsilon})^{\frac{2}{\alpha} - 2} \cdot (\frac{1}{\varepsilon})^2,$$

$$W_{rn} = \frac{4(1 - b)}{ab^2} ((\frac{x_n}{\varepsilon})^{\frac{2}{\alpha}} - r^2)^{\frac{1}{b} - 2} \cdot (\frac{x_n}{\varepsilon})^{\frac{2}{\alpha} - 1} \cdot r \cdot \frac{1}{\varepsilon}.$$

Using the expression of $W$ again we have

$$(3.5) \quad W_r = \frac{2}{b} |W|^{1-b} \cdot r,$$

$$W_n = -\frac{2}{ab} |W|^{1-b} \cdot (\frac{x_n}{\varepsilon})^{\frac{2}{\alpha} - 1} \cdot \frac{1}{\varepsilon},$$

$$W_{rr} = \frac{4(b - 1)}{b^2} |W|^{1-2b} \cdot r^2 + \frac{2}{b} |W|^{1-b},$$

$$W_{nn} = \frac{4(b - 1)}{a^2b^2} |W|^{1-2b} \cdot (\frac{x_n}{\varepsilon})^{\frac{2}{\alpha} - 2} \cdot \frac{1}{\varepsilon^2} + \frac{2(a - 2)}{a^2b} |W|^{1-b} \cdot (\frac{x_n}{\varepsilon})^{\frac{2}{\alpha} - 2} \cdot \frac{1}{\varepsilon^2},$$

$$W_{rn} = \frac{4(1 - b)}{ab^2} |W|^{1-2b} (\frac{x_n}{\varepsilon})^{\frac{2}{\alpha} - 1} \cdot r \cdot \frac{1}{\varepsilon}.$$

Hence,

$$W_{rr} \cdot W_{nn} - (W_{rn})^2 = \frac{8(a - 2)(b - 1)}{a^2b^3} |W|^{2-3b} \cdot (\frac{x_n}{\varepsilon})^{\frac{2}{\alpha} - 2} \cdot r^2 \cdot (\frac{1}{\varepsilon})^2$$

$$+ \frac{8(b - 1)}{a^2b^3} |W|^{2-3b} \cdot (\frac{x_n}{\varepsilon})^{\frac{2}{\alpha} - 2} \cdot \frac{1}{\varepsilon^2}$$

$$+ \frac{4(a - 2)}{a^2b^2} |W|^{2-2b} \cdot (\frac{x_n}{\varepsilon})^{\frac{2}{\alpha} - 2} \cdot (\frac{1}{\varepsilon})^2$$

$$:= I_1 + I_2 + I_3.$$

To estimate $I_1, I_2$ and $I_3$, we will choose a small $\delta = C(a, \alpha, \beta, n) > 0$. Now for this $\delta$, we choose a small $\varepsilon = C(\delta, a, \eta) > 0$ such that

$$(3.7) \quad \varepsilon (\frac{1}{\delta})^{\frac{2}{\alpha}} \leq \eta.$$

Then we have

$$(3.8) \quad \Omega \subseteq \{x \in \mathbb{R}^n | x_n \geq \eta | x' |^a \} \subseteq \{x \in \mathbb{R}^n | \delta (\frac{x_n}{\varepsilon})^{\frac{2}{\alpha}} \geq r^2 \}.$$
By (3.8) we have

$$|W|^b = (\frac{x_n}{\varepsilon})^\frac{\delta}{2} - r^2 \in [(1 - \delta)(\frac{x_n}{\varepsilon})^\frac{\delta}{2}, (\frac{x_n}{\varepsilon})^\frac{\delta}{2}]$$

Since $a > 2$, we have two cases: $a \geq \frac{2\alpha + 2}{\beta - n + 1}$ and $a < \frac{2\alpha + 2}{\beta - n + 1}$ if $\frac{2\alpha + 2}{\beta - n + 1} > 2$.

**Step 2.** Assume that $\frac{2\alpha + 2}{\beta - n + 1} > 2$ and $2 < a < \frac{2\alpha + 2}{\beta - n + 1}$. We want to find $b > 1$ and $\varepsilon > 0$ such that (3.4) is satisfied, by which we will prove (3.2).

Since $a > 2$ and $b > 1$, $I_1, I_2$ and $I_3$ in (3.6) are all positive.

$$W_{rr} \cdot W_{nn} - (W_{rn})^2 \geq I_2 = \frac{8(b - 1)}{a^2b^3} |W|^{2 - 3b} \cdot (\frac{x_n}{\varepsilon})^{\frac{\delta}{2} - 2} \cdot (\frac{1}{\varepsilon})^2.$$  

Observe that $d_x \leq x_n$ in $\Omega$. Hence, by (1.3), (3.4) and (3.5) we obtain

$$H[W] = (\frac{W}{r})^{n-2}(W_{rr}W_{nn} - |W_{rn}|^2)[F(x, W)]^{-1}$$

$$\geq (\frac{2}{b})^{n-2} \cdot |W|^{(1-b)(n-2)} \cdot \frac{8(b - 1)}{a^2b^3} |W|^{2 - 3b} \cdot (\frac{x_n}{\varepsilon})^{\frac{\delta}{2} - 2} \cdot (\frac{1}{\varepsilon})^2 \cdot \frac{1}{A} |x|^{n+1-\beta} |W|^\alpha$$

$$\geq (\frac{2}{b})^{n-2} \cdot |W|^{(1-b)(n-2)} \cdot \frac{8(b - 1)}{a^2b^3} |W|^{2 - 3b} \cdot (\frac{x_n}{\varepsilon})^{\frac{\delta}{2} - 2} \cdot (\frac{1}{\varepsilon})^2 \cdot \frac{1}{A} |x|^{n+1-\beta} |W|^\alpha.$$  

It follows from (3.9) that

$$x_n \leq \varepsilon(\frac{1}{1 - \delta})^{\frac{\delta}{2}} |W|^{\frac{ab}{2}} \text{ in } \Omega,$$

$$\frac{x_n}{\varepsilon}^{\frac{\delta}{2} - 2} \geq \left[(\frac{1}{1 - \delta})^{\frac{\delta}{2}} |W|^{\frac{ab}{2}} \right]^{\frac{\delta}{2} - 2},$$

$$|x|^n \geq |\varepsilon(\frac{1}{1 - \delta})^{\frac{\delta}{2}} |W|^{\frac{ab}{2}} |W|^n.$$  

Therefore, we arrive at

$$H[W] \geq (\frac{2}{b})^{n-2} \cdot |W|^{(1-b)(n-2)} \cdot \frac{8(b - 1)}{a^2b^3} |W|^{2 - 3b} \cdot (\frac{1}{1 - \delta})^{2-a} |W|^{\frac{ab}{2}(\frac{n}{2} - 2)}$$

$$\cdot (\frac{1}{\varepsilon})^2 \cdot \frac{1}{A} \cdot \varepsilon^{n+1-\beta} (\frac{1}{1 - \delta})^{\frac{\beta}{2}(n+1-\beta)} |W|^{\frac{ab}{2}(n+1-\beta)} |W|^\alpha$$

$$= (\frac{1}{\varepsilon})^{\beta-n+1} \cdot \frac{2}{A} \cdot \frac{1}{b} |W|^{n-2} \cdot \frac{8(b - 1)}{a^2b^3} \cdot (\frac{1}{1 - \delta})^{2-a+(\frac{\beta}{2}(n+1-\beta)+\alpha)}$$

$$\cdot |W|^{(1-b)(n-2) + 2 - 3b + \frac{ab}{2}(\frac{n}{2} - 2) + \frac{ab}{2}(n + 1 - \beta) + \alpha}.$$  

Now, we set

$$(1 - b)(n - 2) + 2 - 3b + \frac{ab}{2}(\frac{n}{2} - 2) + \frac{ab}{2}(n + 1 - \beta) + \alpha = 0$$
which is equivalent to
\[
 b = \frac{2(n + \alpha)}{a(\beta - n + 1) + 2n - 2}.
\]
Since \( a \in (2, \frac{2\alpha + 2}{\beta - n + 1}) \), we see that \( b > 1 \) by (3.1). Observing that \( \beta - n + 1 > 0 \), we can choose \( \varepsilon = C(a, \eta, A, \alpha, \beta, n) > 0 \) small enough again, such that \( H[W] \geq 1 \). This proves (3.4), which is to say that \( W \) is a sub-solution to problem (1.1). By comparison principle, we have
\[
|u(x)| \leq |W(x)|, \quad \forall x \in \Omega.
\]
Restricting this inequality onto the \( x_n \) axis, we obtain
\[
|u(y)| \leq \left( \frac{y_n}{\varepsilon} \right)^{\frac{2n}{a} - 2} = \left( \frac{d_y}{\varepsilon} \right)^{\frac{2n}{a} + \frac{2n - 2}{a(n + \alpha)}},
\]
which is (3.2) exactly.

**Step 3.** Assume that \( a \geq \frac{2\alpha + 2}{\beta - n + 1} \). Note that \( a > 2 \) by the assumption of the theorem. We will find \( b \in (0, 1) \) and \( \varepsilon > 0 \) such that the function \( W \) is a sub-solution to problem (1.1), and thus prove (3.2).

By (3.9) we have
\[
I_1 \geq \frac{8(a - 2)(b - 1)}{a^2b^3}|W|^{2 - 3b} \cdot \left( \frac{x_n}{\varepsilon} \right)^{\frac{a}{2} - 2} \cdot \delta \left( \frac{x_n}{\varepsilon} \right)^{\frac{a}{2}} \cdot \left( \frac{1}{\varepsilon} \right)^2 = \delta(a - 2)I_2.
\]
Since \( a > 2 \), \( b \in (0, 1) \) and (3.9) yields
\[
\left( \frac{x_n}{\varepsilon} \right)^{\frac{a}{2} - 2} \leq |W|^{b - a},
\]
we obtain
\[
I_1 + I_2 \geq (1 + \delta(a - 2))I_2
\]
\[
\geq (1 + \delta(a - 2)) \frac{8(b - 1)}{a^2b^3}|W|^{2 - 3b} \cdot |W|^{b - a} \cdot \left( \frac{1}{\varepsilon} \right)^2
\]
\[
= (1 + \delta(a - 2)) \frac{8(b - 1)}{a^2b^3}|W|^{2 - ab} \cdot \left( \frac{1}{\varepsilon} \right)^2.
\]
Again by (3.9), we have
\[
\left( \frac{x_n}{\varepsilon} \right)^{\frac{a}{2} - 2} \geq \left( \frac{1}{1 - \delta} \right)^{1-a}|W|^{b(1-a)}.
\]
Hence, we have
\[
I_3 \geq \frac{4(a - 2)}{a^2b^2}|W|^{2 - 2b} \cdot \left( \frac{1}{1 - \delta} \right)^{1-a}|W|^{b(1-a)} \cdot \left( \frac{1}{\varepsilon} \right)^2
\]
\[
= \frac{4(a - 2)}{a^2b^2} \left( \frac{1}{1 - \delta} \right)^{1-a} \cdot |W|^{2 - 2b} \cdot \left( \frac{1}{\varepsilon} \right)^2.
\]
Therefore, we obtain
\[ W_{rr} \cdot W_{nn} - (W_{rn})^2 = I_1 + I_2 + I_3 \]
\[ \geq [(1 + \delta(a - 2)) \frac{8(b - 1)}{a^2 b^3} + \frac{4(a - 2)}{a^2 b^2} \frac{1}{1 - \delta}] W^{2-b-ab} \cdot \frac{1}{\varepsilon}^2 \]
\[ := \sigma(a, b, \delta)|W|^{2-b-ab} \cdot \frac{1}{\varepsilon}^2, \]
where
\[ \sigma(a, b, \delta) = (1 + \delta(a - 2)) \frac{8(b - 1)}{a^2 b^3} + \frac{4(a - 2)}{a^2 b^2} \frac{1}{1 - \delta} \]

Using above estimates, together with (1.3) and (3.9) we have
\[ H[W] = \left( \frac{W_r}{W} \right)^{n-2}(W_{rr}W_{nn} - |W_{rn}|^2)(F(x, W))^{-1} \]
\[ \geq \left( \frac{2}{b} \right)^{n-2} \cdot |W|^{(1-b)(n-2)} \cdot \sigma(a, b, \delta)|W|^{2-b-ab} \cdot \frac{1}{\varepsilon} \cdot (F(x, W))^{-1} \]
\[ \geq \left( \frac{2}{b} \right)^{n-2} \cdot |W|^{(1-b)(n-2)} \cdot \sigma(a, b, \delta)|W|^{2-b-ab} \cdot \frac{1}{\varepsilon} \cdot \frac{1}{A} \cdot d_n^{n+1-\beta}|W|^{\alpha} \]
\[ \geq \left( \frac{2}{b} \right)^{n-2} \cdot |W|^{(1-b)(n-2)} \cdot \sigma(a, b, \delta)|W|^{2-b-ab} \cdot \frac{1}{\varepsilon} \cdot \frac{1}{A} x_n^{n+1-\beta}|W|^{\alpha} \]
\[ = \left( \frac{1}{\varepsilon} \right)^{\beta-n+1} \left( \frac{2}{b} \right)^{n-2} \frac{1}{A} \cdot \sigma(a, b, \delta)|W|^{2-b-ab} \cdot |W|^{(1-b)(n-2)} \cdot \left( \frac{|x_n|}{\varepsilon} \right)^{n+1-\beta}|W|^{\alpha} \]
\[ \geq \left( \frac{2}{b} \right)^{n-2} \frac{1}{A} \cdot \sigma(a, b, \delta)|W|^{2-b-ab} \cdot |W|^{(1-b)(n-2)} \cdot \left( \frac{1}{1-\delta} \right)^{\frac{a(n+1-\beta)}{2}} \cdot |W|^{\alpha} \]
\[ = \left( \frac{2}{b} \right)^{n-2} \frac{1}{A} \cdot \left( \frac{1}{1-\delta} \right)^{\frac{a(n+1-\beta)}{2}} \sigma(a, b, \delta)|W|^{2-b-ab+(1-b)(n-2) + \frac{ab(n+1-\beta)\alpha}{2} + \alpha}. \]

Now, we set
\[ 2 - b - ab + (1-b)(n-2) + \frac{ab(n+1-\beta)}{2} + \alpha = 0, \]
which is equivalent to
\[ b = \frac{2(n+\alpha)}{a(\beta - n + 1) + 2n - 2}. \]
Since \( a \geq \frac{2n+2}{\beta-n+1}, \) we see that \( b \in (0, 1]. \) Of course, we also need
\[ \sigma(a, b, \delta) = (1 + \delta(a - 2)) \frac{8(b - 1)}{a^2 b^3} + \frac{4(a - 2)}{a^2 b^2} \frac{1}{1 - \delta}(1-a) > 0, \]
which is equivalent to
\[ (a - 2)(1 - \delta)^{a-1} > (1 + \delta(a - 2))(\frac{2(1-b)}{b}). \]
Since \( \gamma_2 = \frac{\beta-n+1}{n+\alpha} + \frac{2n-2}{a(n+\alpha)} \in (0, 1) \) by (3.1), we see that
\[
a - 2 > \frac{a(\beta - n + 1) + 2n - 2}{n + \alpha} - 2 = \left( \frac{2(1 - b)}{b} \right).
\]
Using this and taking \( \delta = C(a, \alpha, \beta, n) > 0 \) small enough, we obtain (3.12) and thus (3.11).

Finally, choosing a positive \( \varepsilon = C(a, \eta, A, \alpha, \beta, n) \) smaller if necessary, by (3.10) and (3.11) we obtain that \( H[W] \geq 1 \) in \( \Omega \), which implies \( W \) is an sub-solution to problem (1.1) by (3.4). As in the end of Step 2, we have proved (3.2).

4. Proof of Theorem 1.3

As the proof of Theorem 1.2, the proof of (i) of Theorem 1.3 follows directly from
\[
|u(y)| \leq C d_y^{\gamma_n}, \quad \forall y \in \Omega
\]
for some positive constant \( C = C(a, n, \alpha, \eta, A, \text{diam} \Omega) \).

For any \( y \in \Omega \), we can find \( z \in \partial \Omega \), such that \( |y - z| = d_y \). Since the domain \( \Omega \) satisfies exterior sphere condition with radius \( R \) and the problem (1.1) is invariant under translation and rotation transforms, we may assume
\[
z = 0 \in \partial \Omega \cap \partial B_R(y_0), \quad \Omega \subseteq B_R(y_0).
\]
Since \( z = 0 \) satisfies \( |y - z| = d_y \), the tangent plane of \( \Omega \) at \( z = 0 \) is unique. And it is easy to check \( y \) is on the line determined by \( 0 \) and \( y_0 \). Hence \( d_y = |y| = |y_0| - |y_0 - y| = R - |y_0 - y| \).

Consider the function
\[
W(x) = -M(R^2 - |x - y_0|^2)^b = -M(R^2 - r^2)^b,
\]
where \( r = |x - y_0| \), \( M \) and \( b \) are positive constants to be determined later. As (3.3), we obtain that
\[
det D^2 W = \left( \frac{W_r}{r} \right)^{n-1} W_{rr}.
\]
But
\[
W_r = 2Mbr(R^2 - r^2)^{b-1},
\]
\[
W_{rr} = 2Mb(R^2 - r^2)^{b-2}[R^2 - (2b - 1)r^2].
\]
Hence
\begin{equation}
(4.4) \quad detD^2W = (2Mb)^n(R^2 - r^2)^{n(b-1)-1}[R^2 - (2b - 1)r^2].
\end{equation}

Observing that $W \leq 0$ on $\partial \Omega$, we see that $W$ is a sub-solution to problem (1.1) if and only if
\begin{equation}
(4.5) \quad H[W] := (2Mb)^n(R^2 - r^2)^{n(b-1)-1}[R^2 - (2b - 1)r^2][F(x, W)]^{-1} \geq 1
\end{equation}
for all $x \in \Omega$ and $r = |x - y_0|$.  

First, we consider the case
\begin{equation}
(4.6) \quad \beta < n + \alpha + 1.
\end{equation}
As (2.3), we need only to consider the case $\beta < n + \alpha$. We take
\begin{equation}
(4.7) \quad b = \frac{\beta}{n + \alpha} = \gamma_3.
\end{equation}
Then in this case $b \gamma_3 \in (0, 1)$ and $|2b - 1| < 1$. Hence,
\begin{equation}
(4.8) \quad R^2 - (2b - 1)r^2 \geq (1 - |2b - 1|)R^2.
\end{equation}
It follows from (4.2) that
\begin{equation}
(4.9) \quad d_x \leq R - |x - y_0| = R - r, \quad \forall x \in \Omega.
\end{equation}
Therefore, by (1.3), (4.5), (4.8) and (4.9) that
\begin{equation}
(4.10) \quad H[W] \geq (1 - |2b - 1|)R^2(2Mb)^n(R^2 - r^2)^{n(b-1)-1} \frac{1}{A}(d_x)^{n+1-\beta}|W|^\alpha
\end{equation}
\begin{equation}
\geq (1 - |2b - 1|)R^2 \frac{1}{A}(2Mb)^n(R^2 - r^2)^{n(b-1)-1}(R - r)^{n+1-\beta}|W|^\alpha
\end{equation}
\begin{equation}
= (1 - |2b - 1|)R^2 \frac{1}{A}M^\alpha(2Mb)^n(R^2 - r^2)^{n(b-1)+b\alpha-1}(R - r)^{n+1-\beta}
\end{equation}
\begin{equation}
= (1 - |2b - 1|)R^2 \frac{1}{A}M^{\alpha+n}(2b)^n(R + r)^{n(b-1)+b\alpha-1}(R - r)^{n(b-1)+b\alpha+n-\beta}.
\end{equation}
Note that
\begin{equation}
(4.11) \quad n(b - 1) + b\alpha + n - \beta = 0
\end{equation}
by (4.7). Hence, by (4.10) and (4.11) we can choose a large $M = C(A, b, R, \alpha, n, \beta)$ such that
\begin{equation}
(4.12) \quad H[W] \geq 1 \quad \text{in} \quad \Omega.
\end{equation}

Next, we consider the case
\begin{equation}
\beta \geq n + \alpha + 1.
\end{equation}
In this case, we take
\[ b = 1 = \gamma_3. \]

Then, by (1.3) and (4.4) we have
\[
H[W] = (2M)^n[F(x, W)]^{-1} \geq \frac{1}{A} 2^n M^{n+\alpha} (R + r)^{\alpha} (R - r)^{\alpha + n + 1 - \beta} = \frac{1}{A} 2^n M^{n+\alpha} (R + r)^{\alpha}.
\]

Therefore, (4.12) still holds true.

To sum up, we have obtained (4.5). By comparison principle, we see that
\[
(4.13) \quad W(x) \leq u(x) \leq 0.
\]

In particular, we obtain that
\[
|u(y)| \leq |W(y)| = M(R + |y - y_0|)^{\gamma_3} (R - |y - y_0|)^{\gamma_3} \leq M(2R)^{\gamma_3} (d_y)^{\gamma_3}.
\]

This is desired (4.1) and hence we have proved the (i) of Theorem 1.3.

To prove (ii) of Theorem 1.3, we notice that \( u \in C(\overline{\Omega}) \) and \( u < 0 \) in \( \Omega \) and \( u = 0 \) on \( \partial \Omega \). By comparing the graph of the convex function \( u \) with the cone whose vortex is \( (x_0, u(x_0)) \) and whose upper bottom is \( \overline{\Omega} \), where \( u(x_0) = \min_{\Omega} u \), we see easily that (1.10) is true for \( \gamma_4 \geq 1 \). Hence, we need only to consider that case \( \gamma_4 < 1 \) in the following, which implies that \( \beta < n + 1 \).

Since (1.10) holds naturally for all \( y \in \{ x \in \Omega : d_x \geq \frac{R}{2} \} \), where \( R \) is the radius of the interior sphere for the \( \Omega \). Hence, it is sufficient to prove
\[
(4.14) \quad |u(y)| \geq (d_y)^{\gamma_4}, \quad \forall y \in \{ x \in \Omega : d_x < \frac{R}{2} \}.
\]

Take such a \( y \). We can find \( z \in \partial \Omega \), such that \( |y - z| = d_y \), we may assume
\[
(4.15) \quad z = 0 \in \partial \Omega \bigcap \partial B_R(y_0), \quad B_R(y_0) \subseteq \Omega.
\]

Since the tangent plane of \( \Omega \) at \( z = 0 \) is unique. And it is easy to check \( y \) is on the line determined by \( 0 \) and \( y_0 \). Hence \( d_y = |y| = |y_0| - |y_0 - y| = R - |y_0 - y| \).

Observing that in this case, instead of (4.8) we have
\[
(4.16) \quad d_x \geq R - |x - y_0| = R - r, \quad \forall x \in B_R(y_0).
\]
First, we require \( b \in (0, 1) \), which implies \( 2b - 1 \in (-1, 1) \). Similarly to the arguments of (i), by (4.16) we find that the function \( W \), given by (4.3), satisfies

\[
H[W] \leq \frac{1}{A} (2Mb)^n [R^2 - r^2]^{n(b-1) - 1} [R^2 - (2b - 1)r^2] d_x^{m+1-\beta} |W|^\alpha
\]

(4.17)

\[
\leq \frac{1}{A} M^\alpha (2Mb)^n 2R^2 [R^2 - r^2]^{n(b-1) - 1 + b\alpha} (R - r)^{n+1-\beta}
\]

\[
\leq \frac{1}{A} M^{\alpha + n} (2b)^n 2R^2 (2R)^{n(b-1) - 1 + b\alpha} (R - r)^{n(b-1) + b\alpha + n - \beta}.
\]

Taking \( b = \frac{\beta}{n+\alpha} = \gamma_4 \in (0, 1) \) we have

(4.18)

\[
n(b - 1) + b\alpha + n - \beta = 0.
\]

Using (4.17)-(4.18), we see that \( W \) is a super-solution to problem (1.1) in the domain \( B_R(y_0) \) for sufficiently small \( M = C(A, b, R, \alpha, n, \beta) > 0 \). Since \( u \) is a solution on \( \Omega \) and \( u|_{\partial B_R(y_0)} \leq 0 \), thus \( u \) is a sub-solution on \( B_R(y_0) \). Therefore, we have

\[
|u(y)| \geq |W(y)|
\]

\[
= M(R + |y - y_0|)^{\gamma_4} (R - |y - y_0|)^{\gamma_4}
\]

\[
\geq MR^{\gamma_3} (d_y)^{\gamma_4},
\]

Which is the desired (4.14) exactly. In this way, the proof of Theorem 1.3 has been completed.
References

[1] Caffarelli, L.A., Interior $W^{2,p}$ estimates for solutions of Monge-Ampère equations, Ann. Math. 131 (1990), 135-150.

[2] Caffarelli, L.A., Nirenberg, L., Spruck, J., The Dirichlet problem for nonlinear second-order elliptic equations I, Monge-Ampère equation, Comm. Pure Appl. Math. 37 (1984), 369–402.

[3] Calabi, E., Complete affine hypersurfaces I, Symposia Mathematica 10 (1972), 19-38.

[4] Cheng, S.Y., Yau, S.T., On the regularity of the Monge-Ampère equation $\det \frac{\partial^2 u}{\partial x_i \partial x_j} = F(x, u)$, Comm. Pure Appl. Math. 30 (1977), 41–68.

[5] Cheng, S.Y., Yau, S.T., On the existence of a complete Kähler metric on noncompact complex manifolds and the regularity of Fefferman’s equation, Comm. Pure Appl. Math. 33 (1980), 507–544.

[6] Cheng, S.Y., Yau, S.T., Complete affine hypersurfaces I, The completeness of affine metrics, Comm. Pure Appl. Math. 39 (1986), 839-866.

[7] Chou, K.S., Wang, X.-J., The $L_p$-Minkowski problem and the Minkowski problem in centroaffine geometry, Adv. Math. 205 (2006), 33–83.

[8] Figalli, A., The Monge-Ampère equation and its Applications, European Math Soc Publ House, CH-8092 Zurich, Switzerland, 2017.

[9] Gutiérrez, C. E., The Monge-Ampère equation, Birkhauser, Boston, 2001.

[10] Gilbarg, D., Trudinger, N.S., Elliptic partial differential equations of second order, Springer-Verlag, New York, 1983.

[11] Jian, H.Y., Li, Y., Optimal boundary regularity for a singular Monge-Ampère equation, J. Differential Equations, 264 (2018), 6873-6890.

[12] Jian, H.Y., Lu, J., Zhang, G., Mirror symmetric solutions to the centro-affine Minkowski problem, Calc. Var. Partial Differential Equations, (2016), 55:41.

[13] Jian, H.Y., Lu, J., Wang, X.-J., Boundary expansion of solutions to nonlinear singular elliptic equations, preprint, May 2015.

[14] Jian, H.Y., Wang, X.-J., Bernstein theorem and regularity for a class of Monge-Ampère equation, J. Diff. Geom. 93 (2013),431-469.

[15] Jian, H.Y., Wang, X.-J., Continuity estimates for the Monge-Ampère equation, SIAM J. Math. Anal. 39 (2007), 608–626

[16] Jian, H.Y., Wang, X.-J., Zhao Y. W., Global smoothness for a singular Monge-Ampère equation, Journal of Differential Equations, 263(2017), 7250-7262.

[17] Le, N. O., Savin, O., Schauder estimates for degenerate Monge-Ampère equations and smoothness of the eigenfunctions, Invent. Math. 207 (2017), 389–423.

[18] Loewner, C., Nirenberg, L., Partial differential equations invariant under conformal or projective transformations, In Contributions to Analysis, pages 245-272, Academic Press, 1974.

[19] Lutwak, E., The Brunn-Minkowski-Firey theory I, Mixed volumes and the Minkowski problem, J. Diff. Geom. 38 (1993), 131-150.

[20] Pogorelov, A.V., The Minkowski multidimensional problem, J. Wiley, New York, 1978.

[21] Savin, O., pointwise $C^{2,\alpha}$ estimates at the boundary for the Monge-Ampère equation, J. Amer. Math. Soc., 26(1), 63-99 (2013)

[22] Trudinger, N.S., Wang, X.-J., Boundary regularity for the Monge-Ampère and affine maximal surface equations, Ann. Math. (2) 167 (2008), 993–1028.
[23] Trudinger, N.S., Wang, X.-J., The Monge-Ampère equation and its geometric Applications, Handbook of geometric analysis. No. 1, Adv. Lect. Math. (ALM), vol. 7, Int. Press, Somerville, MA, 2008, 467-524.

[24] Urbas, J.I.E., Global Hölder estimates for equations of Monge-Ampère type, Invent. Math. 91 (1988), 1–29.

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