A recurrence relation for the Li/Keiper constants in terms of the Stieltjes constants

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Abstract

A recurrence relation for the Li/Keiper constants in terms of the Stieltjes constants is derived in this paper.

In addition, we also report a formula for the Stieltjes constants in terms of the higher derivatives, \( \zeta^{(n)}(0) \), of the Riemann zeta function evaluated at \( s = 0 \). A formula for the Stieltjes constants in terms of the (exponential) complete Bell polynomials containing the eta constants \( \eta_n \) as the arguments is also derived.

CONTENTS

1. Introduction 1

2. A brief survey of the Stieltjes constants 3

3. A recurrence relation for the Li/Keiper constants 6

4. The eta constants 27

5. A formula for the Stieltjes constants in terms of the higher derivatives of the Riemann zeta function \( \zeta^{(n)}(0) \) 34

6. A formula for the Stieltjes constants in terms of the (exponential) complete Bell polynomials containing the eta constants \( \eta_n \) as the arguments 39

Appendix A: A brief survey of the (exponential) complete Bell polynomials 42

1. Introduction

The Riemann xi function \( \xi(s) \) is defined as

\[
(1.1) \quad \xi(s) = \frac{1}{2} s(s - 1) \pi^{-s/2} \Gamma(s/2) \zeta(s)
\]
and we see from the functional equation for the Riemann zeta function $\zeta(s)$

\[(1.2) \quad \zeta(1-s) = 2(2\pi)^{-s} \Gamma(s) \cos(\pi s/2) \zeta(s)\]

that $\xi(s)$ satisfies the functional equation

\[(1.3) \quad \xi(s) = \xi(1-s)\]

In 1996, Li [24] defined the sequence of numbers $\lambda_n$ by

\[(1.4) \quad \lambda_n = \frac{1}{(n-1)!} \frac{d^n}{ds^n} \left[ s^{n-1} \log \xi(s) \right]_{s=1}\]

and proved that a necessary and sufficient condition for the non-trivial zeros $\rho$ of the Riemann zeta function to lie on the critical line $s = \frac{1}{2} + it$ is that $\lambda_n$ is non-negative for every positive integer $n$. Earlier in 1991, Keiper [21] showed that if the Riemann hypothesis is true, then $\lambda_n > 0$ for all $n \geq 1$.

Li also showed that

\[(1.5) \quad \lambda_n = \sum_{\rho} \left[ 1 - \left(1 - \frac{1}{\rho} \right)^n \right]\]

Taking logarithms of (1.1) and noting that $\frac{s}{2} \Gamma\left(\frac{s}{2}\right) = \Gamma\left(1 + \frac{s}{2}\right)$ we see that

\[(1.6) \quad \log \xi(s) = \log \Gamma\left(1 + \frac{s}{2}\right) - \frac{s}{2} \log \pi + \log[(s-1)\xi(s)]\]

We also have the Maclaurin expansion about $s = 1$

\[(1.7) \quad \log \xi(s) = -\log 2 - \sum_{k=1}^{\infty} (-1)^k \frac{\sigma_k}{k} (s-1)^k\]

where the constant term in (1.7) arises because

$$\lim_{s \to 1} \xi(s) = \frac{1}{2} \pi^{-1/2} \Gamma(1/2) \lim_{s \to 1} [(s-1)\xi(s)] = \frac{1}{2}$$

We note that the coefficients $\sigma_k$ are defined by
\[ \sigma_k = \frac{(-1)^{k+1}}{(k-1)!} \frac{d^k}{ds^k} \log \zeta(s) \bigg|_{s=1} \]

and we see that
\[
\frac{d}{ds} \log \zeta(s) = \frac{\xi'(s)}{\xi(s)} = -\sum_{k=1}^{\infty} (-1)^{k} \sigma_k (s-1)^{k-1}
\]

and comparing this with (1.4) we immediately see that \( \lambda_1 = \sigma_1 \).

Differentiating (1.6) results in
\[
(1.8) \quad \frac{d}{ds} \log \zeta(s) = \frac{1}{2} \psi \left( 1 + \frac{s}{2} \right) - \frac{1}{2} \log \pi + \frac{d}{ds} \left[ (s-1) \zeta(s) \right] \]

which, referring to (1.4), is clearly relevant in computing \( \lambda_1 \).

Before proceeding any further we need to recall some details regarding the Stieltjes constants.

2. A brief survey of the Stieltjes constants

The Stieltjes constants \( \gamma_n(u) \) are the coefficients in the Laurent expansion of the Hurwitz zeta function \( \zeta(s,u) \) about \( s = 1 \)

\[
(2.1) \quad \zeta(s,u) = \sum_{n=0}^{\infty} \frac{1}{(n+u)^s} = \frac{1}{s-1} + \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \gamma_n(u)(s-1)^n
\]

and \( \gamma_0(u) = -\psi(u) \), where \( \psi(u) \) is the digamma function which is the logarithmic derivative of the gamma function \( \psi(u) = \frac{d}{du} \log \Gamma(u) \). It is easily seen from the definition of the Hurwitz zeta function that \( \zeta(s,1) = \zeta(s) \) and accordingly that \( \gamma_n(1) = \gamma_n \). Further details of the Stieltjes constants are contained in [5] to [13] inclusive.

The generalised Euler-Mascheroni constants \( \gamma_n \) (or Stieltjes constants) are the coefficients of the Laurent expansion of the Riemann zeta function \( \zeta(s) \) about \( s = 1 \)

\[
(2.2) \quad \zeta(s) = \frac{1}{s-1} + \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \gamma_n(s-1)^n
\]
Since \( \lim_{s \to 1} \left[ \zeta(s) - \frac{1}{s-1} \right] = \gamma \) it is clear that \( \gamma_0 = \gamma \). It may be shown, as in [20, p.4], that

\[
\gamma_n = \lim_{N \to \infty} \left[ \sum_{k=1}^{N} \frac{\log^n k}{k} \right] = \lim_{N \to \infty} \left[ \sum_{k=1}^{N} \frac{\log^n k}{k} - \int_{1}^{N} \frac{\log^n t}{t} \, dt \right]
\]

where, throughout this paper, we define \( \log^0 1 = 1 \).

We now refer to the Hasse identity [19] for the Hurwitz zeta function \( \zeta(s, u) \) which is valid for all \( s \in \mathbb{C} \) provided \( \text{Re}(s) \neq 1 \)

\[
\zeta(s, u) = \frac{1}{s-1} \sum_{i=0}^{\infty} \frac{1}{i+1} \sum_{j=0}^{i} \left( \frac{i}{j} \right) (-1)^j (u+j)^{s-1}
\]

and we have the well-known limit

\[
\lim_{s \to 1} (s-1) \zeta(s, u) = \sum_{i=0}^{\infty} \frac{1}{i+1} \sum_{j=0}^{i} \left( \frac{i}{j} \right) (-1)^j = \sum_{i=0}^{\infty} \frac{1}{i+1} \delta_{i,0} = 1
\]

where \( \delta_{i,j} \) is the Kronecker delta symbol.

We see from (2.4) that

\[
d_{s=1}^{n+1}[(s-1)\zeta(s, u)] = (-1)^{n+1} \sum_{i=0}^{\infty} \frac{1}{i+1} \sum_{j=0}^{i} \left( \frac{i}{j} \right) (-1)^j \log^{n+1}(u+j)
\]

and thus

\[
d_{s=1}^{n+1}[(s-1)\zeta(s, u)] = (-1)^{n+1} \sum_{i=0}^{\infty} \frac{1}{i+1} \sum_{j=0}^{i} \left( \frac{i}{j} \right) (-1)^j \log^{n+1}(u+j)
\]

We previously showed in [15] that

\[
\gamma_n(u) = -\frac{1}{n+1} \sum_{i=0}^{\infty} \frac{1}{i+1} \sum_{j=0}^{i} \left( \frac{i}{j} \right) (-1)^j \log^{n+1}(u+j)
\]

and comparing this with (2.7) allows us to conclude for \( n \geq 0 \) that

\[
d_{s=1}^{n+1}[(s-1)\zeta(s, u)] = (-1)^n (n+1) \gamma_n(u)
\]
This may also be more directly obtained by differentiating (2.1) \( p + 1 \) times where we see that

\[
\frac{d^{p+1}}{ds^{p+1}}[(s-1)\zeta(s,u)] = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \gamma_n(u)(n+1)n(n+1)\cdots(n+1-p)(s-1)^{n-p}
\]

With \( u = 1 \) in (2.9) we have

\[
(2.10) \quad \frac{d^{n+1}}{ds^{n+1}}[(s-1)\zeta(s)] \bigg|_{s=1} = (-1)^n(n+1)\gamma_n
\]

We now evaluate (1.8) at \( s = 1 \) and obtain

\[
\lambda_1 = \left. \frac{d}{ds} \left[ \log \xi(s) \right] \right|_{s=1} = \frac{1}{2} \psi\left( \frac{3}{2} \right) - \frac{1}{2} \log \pi + \left. \frac{d}{ds} [(s-1)\zeta(s)] \right|_{s=1}
\]

Using [30, p.20]

\[
\psi\left( n + \frac{1}{2} \right) = -\gamma - 2\log 2 + 2 \sum_{k=0}^{n-1} \frac{1}{2k+1}
\]

we see that

\[
(2.11) \quad \psi\left( \frac{3}{2} \right) = -\gamma - 2\log 2 + 2
\]

From (2.10) we have

\[
(2.12) \quad \left. \frac{d}{ds} [(s-1)\zeta(s)] \right|_{s=1} = \gamma_0 = \gamma
\]

and we therefore easily compute the first Li/Keiper constant

\[
(2.13) \quad \lambda_1 = \left. \frac{d}{ds} \left[ \log \xi(s) \right] \right|_{s=1} = \frac{1}{2} \psi\left( \frac{3}{2} \right) + \gamma - \frac{1}{2} \log \pi = -\frac{1}{2} \log \pi + \frac{1}{2} \gamma + 1 - \log 2
\]

We note from (2.11) that \( \psi\left( \frac{3}{2} \right) \) is positive; in addition we see that \( \gamma - \frac{1}{2} \log \pi > 0 \) and therefore \( \lambda_1 \) is positive. Its approximate value is [5]

\[
(2.14) \quad \sigma_1 = \lambda_1 = 0.023...
\]
However, in order to determine \( \lambda_n \) we would need to calculate \( \frac{d^n}{ds^n}[s^{-1} \log \xi(s)] \) and reference to (1.6) shows that this in turn would require us to compute

\[
\frac{d^n}{ds^n}[s^{-1} \log((s-1)\xi(s))] = \frac{d^{n-1}}{ds^{n-1}} \left[ (n-1)s^{n-2} \log((s-1)\xi(s)) + s^{n-1} \frac{d}{ds}[(s-1)\xi(s)] \right]
\]

It is remarkable that determining the veracity, or otherwise, of the most famous hypothesis in mathematics is so dependent upon obtaining a “manageable” formula for the higher derivatives of the quotient of two functions.

Formulae for the Li/Keiper constants have been given by Bombieri and Lagarias [4], Maślanka [26] and Coffey [13], the latter being based on an identity determined by Matsuoka [27]. However, it is not easy to discern particular values of \( \lambda_n \) from these representations.

It is because of this inherent difficulty that we now proceed to go for second best and derive a recurrence relationship for the \( \lambda_n \) constants.

3. A recurrence relation for the Li/Keiper constants

We recall (1.7) above

\[
\log \xi(s) = -\log 2 - \sum_{k=1}^{\infty} (-1)^i \frac{\sigma_k}{k} (s-1)^k
\]

and since \( \log \xi(s) = \log \xi(1-s) \) we see that

\[
\log \xi(s) = -\log 2 - \sum_{k=1}^{\infty} \frac{\sigma_k}{k} s^k
\]

We therefore have from (1.6)

\[
(3.1) \quad \log \Gamma \left(1+\frac{s}{2}\right) + \log 2 - \frac{s}{2} \log \pi + \log((s-1)\xi(s)) = -\sum_{k=1}^{\infty} \frac{\sigma_k}{k} s^k
\]

and differentiation gives us

\[
(3.2) \quad \frac{1}{2} \psi \left(1+\frac{s}{2}\right) - \frac{1}{2} \log \pi + \frac{d}{ds} \left[\frac{(s-1)\xi(s)}{(s-1)\xi(s)}\right] = -\sum_{k=1}^{\infty} \frac{\sigma_k}{k} ks^{k-1}
\]
We now multiply (3.2) across by \((s - 1)\zeta(s)\) to obtain

(3.3)

\[
\frac{1}{2}(s - 1)\zeta(s)\psi\left(1 + \frac{s}{2}\right) - \frac{1}{2}\log \pi (s - 1)\zeta(s) + \frac{d}{ds}[(s - 1)\zeta(s)] = -\sum_{k=1}^{\infty} \frac{\sigma_k}{k} s^{k-1}[(s - 1)\zeta(s)]
\]

and with \(s = 1\) we have

\[
\frac{1}{2}\psi\left(\frac{3}{2}\right) - \frac{1}{2}\log \pi + \gamma = -\sum_{k=1}^{\infty} \frac{\sigma_k}{k}
\]

Using

\[
\log \xi(s) = -\log 2 - \sum_{k=1}^{\infty} \frac{\sigma_k}{k} s^k
\]

we see that

\[
\frac{d^n}{ds^n}[s^{n-1}\log \xi(s)] = -\sum_{k=1}^{\infty} \frac{\sigma_k}{k} (k + n - 1) \cdots k s^{k+n-1}
\]

and hence referring to (1.4) we have

\[
\lambda_n = \frac{1}{(n-1)!} \left. \frac{d^n}{ds^n}[s^{n-1}\log \xi(s)] \right|_{s=1} = -\frac{1}{(n-1)!} \sum_{k=1}^{\infty} \frac{\sigma_k}{k} (k + n - 1) \cdots k
\]

We then see that for \(n \geq 1\)

(3.4) \((n-1)! \lambda_n = -\sum_{k=1}^{\infty} \frac{\sigma_k}{k} k \cdots (k + n - 1)\)

and for example we have

\[
\lambda_1 = -\frac{1}{0!} \sum_{k=1}^{\infty} \frac{\sigma_k}{k} k = -\sum_{k=1}^{\infty} \sigma_k = \sigma_1
\]

\[
\lambda_2 = -\frac{1}{1!} \sum_{k=1}^{\infty} \frac{\sigma_k}{k} k(k + 1)
\]

\[
\lambda_3 = -\frac{1}{2!} \sum_{k=1}^{\infty} \frac{\sigma_k}{k} k(k + 1)(k + 2)
\]
We then see that
\[
\frac{1}{2} \psi \left( \frac{3}{2} \right) + \gamma - \frac{1}{2} \log \pi = \lambda_1
\]
which is the same as (2.11).

Differentiating equation (3.3) gives us
\[
(s-1) \zeta(s) \frac{1}{2} \psi^{(1)} \left( 1 + \frac{s}{2} \right) + \frac{1}{2} \psi \left( 1 + \frac{s}{2} \right) \frac{d}{ds} [(s-1) \zeta(s)] - \frac{1}{2} \log \pi \frac{d}{ds} [(s-1) \zeta(s)] + \frac{d^2}{ds^2} [(s-1) \zeta(s)]
\]
\[
= - \sum_{k=1}^{\infty} \frac{\sigma_k}{k} s^{k-1} \frac{d}{ds} [(s-1) \zeta(s)] - \sum_{k=1}^{\infty} \frac{\sigma_k}{k} k(k-1) s^{k-2} [(s-1) \zeta(s)]
\]
and with \( s = 1 \) we have
\[
\frac{1}{4} \psi^{(1)} \left( \frac{3}{2} \right) + \frac{1}{2} \psi \left( \frac{3}{2} \right) \gamma - \frac{1}{2} \gamma \log \pi - 2 \gamma_1 = -\gamma \sum_{k=1}^{\infty} \sigma_k - \sum_{k=1}^{\infty} \frac{\sigma_k}{k} k(k-1)
\]
We have \( k(k-1) = k(k+1) - 2k \) and hence
\[
-\gamma \sum_{k=1}^{\infty} \sigma_k - \sum_{k=1}^{\infty} \frac{\sigma_k}{k} k(k-1) = -\gamma \sum_{k=1}^{\infty} \sigma_k - \sum_{k=1}^{\infty} \frac{\sigma_k}{k} k(k+1) + 2 \sum_{k=1}^{\infty} \frac{\sigma_k}{k} k
\]
and, therefore, using (3.4), we have
\[
\frac{1}{4} \psi^{(1)} \left( \frac{3}{2} \right) + \frac{1}{2} \psi \left( \frac{3}{2} \right) \gamma - \frac{1}{2} \gamma \log \pi - 2 \gamma_1 = (\gamma - 2) \lambda_1 + \lambda_2
\]
Since \( \psi (1 + x) = \psi (x) + \frac{1}{x} \) we see that
\[
\psi^{(n)} (1 + x) = \psi^{(n)} (x) + (-1)^n n! x^{-n-1}
\]
and with \( x = 1/2 \) we have
\[
\psi^{(n)} \left( \frac{3}{2} \right) = \psi^{(n)} \left( \frac{1}{2} \right) + (-1)^n n! 2^{n+1}
\]
We have the well-known result from [30]
\[
\psi^{(n)} \left( \frac{1}{2} \right) = (-1)^{n+1} n! [2^{n+1} - 1] \zeta(n+1)
\]
and hence we have

\[(3.5.1) \quad \psi^{(n)}(3/2) = (-1)^{n+1} n! \left( \left[ 2^{n+1} - 1 \right] \zeta(n+1) - 2^{n+1} \right) \]

In fact we have from [30, p.22]

\[(3.5.2) \quad \psi^{(n)}(x) = (-1)^n n! \zeta(n+1, x) \]

and hence we deduce that for \( n \geq 1 \)

\[(3.5.3) \quad \psi^{(n)}(3/2) = (-1)^{n+1} c_n \quad \text{where} \quad c_n > 0 \]

We thus obtain

\[\psi^{(1)}\left(\frac{3}{2}\right) = 3\zeta(2) - 4\]

We have from (2.11)

\[\psi\left(\frac{3}{2}\right) = -\gamma - 2\log 2 + 2\]

and from (2.13)

\[\lambda_1 = -\frac{1}{2} \log \pi + \frac{1}{2} \gamma + 1 - \log 2\]

Therefore we have the second Li/Keiper constant

\[(3.6) \quad \lambda_2 = \frac{3}{4} \zeta(2) + 1 + \gamma - \gamma^2 - 2 \log 2 - \log \pi - 2\gamma_1\]

Differentiating equation (3.3) \( n + 1 \) times using the Leibniz rule gives us

\[\sum_{m=0}^{n+1} \binom{n+1}{m} \frac{d^m}{ds^m} [(s-1)\zeta(s)] - \frac{1}{2^{n+2-m}} \psi^{(n+1-m)}\left(1 + \frac{s}{2}\right) \frac{1}{2} \log \pi \frac{d^{n+1}}{ds^{n+1}} [(s-1)\zeta(s)]\]

\[(3.7) \quad + \frac{d^{n+2}}{ds^{n+2}} [(s-1)\zeta(s)] = -\sum_{k=1}^{\infty} \sum_{m=0}^{n+1} \binom{n+1}{m} \frac{d^m}{ds^m} \left[ks^{k-1}\right] \frac{d^{n+1-m}}{ds^{n+1-m}} [(s-1)\zeta(s)]\]
First of all, isolating the term for $m = 0$ and evaluating at $s = 1$ we have since 

$$\lim_{s \to 1} [(s-1) \zeta(s)] = 1$$

$$S_1 = \sum_{m=0}^{n+1} \binom{n+1}{m} \frac{d^m}{ds^m} [(s-1) \zeta(s)] \frac{1}{2^{n+2-m}} \psi^{(n+1-m)} \left(1 + \frac{s}{2}\right)_{k=1}$$

$$= \frac{1}{2^{n+2}} \psi^{(n+1)} \left(1 + \frac{s}{2}\right) + \sum_{m=1}^{n+1} \binom{n+1}{m} \frac{d^m}{ds^m} [(s-1) \zeta(s)] \frac{1}{2^{n+2-m}} \psi^{(n+1-m)} \left(1 + \frac{s}{2}\right)_{k=1}$$

and then rebasing the index to $m = p + 1$ results in 

$$S_1 = \frac{1}{2^{n+2}} \psi^{(n+1)} \left(1 + \frac{s}{2}\right) + \sum_{p=0}^{n+1} \binom{n+1}{p+1} \frac{d^{p+1}}{ds^{p+1}} [(s-1) \zeta(s)] \frac{1}{2^{n+1-p}} \psi^{(n-p)} \left(1 + \frac{s}{2}\right)_{k=1}$$

Noting from (2.9) that 

$$\frac{d^{p+1}}{ds^{p+1}} [(s-1) \zeta(s)]_{k=1} = (-1)^p (p+1) \gamma_p$$

we obtain 

$$S_1 = \frac{1}{2^{n+2}} \psi^{(n+1)} \left(\frac{3}{2}\right) + \sum_{p=0}^{n} \binom{n+1}{p+1} (-1)^p (p+1) \gamma_p \frac{1}{2^{n+1-p}} \psi^{(n-p)} \left(\frac{3}{2}\right)$$

Carrying out similar operations for the summation on the right-hand side of (3.7), where this time we isolate the term for $m = n+1$, we get 

$$S_2 = -\sum_{k=1}^{\infty} \sigma_k \sum_{m=0}^{n+1} \binom{n+1}{m} \frac{d^m}{ds^m} [k s^{k-1}] \frac{d^{n+1-m}}{ds^{n+1-m}} [(s-1) \zeta(s)]_{k=1}$$

$$= -\sum_{k=1}^{\infty} \sigma_k \frac{d^{n+1}}{ds^{n+1}} [k s^{k-1}] - \sum_{k=1}^{\infty} \sigma_k \sum_{m=0}^{n} \binom{n+1}{m} \frac{d^m}{ds^m} [k s^{k-1}] \frac{d^{n+1-m}}{ds^{n+1-m}} [(s-1) \zeta(s)]_{k=1}$$

$$= -\sum_{k=1}^{\infty} \frac{\sigma_k}{k} k(k-1)...(k-n-1) - \sum_{k=1}^{\infty} \frac{\sigma_k}{k} \sum_{m=0}^{n} \binom{n+1}{m} k(k-1)...(k-m) \frac{d^{n+1-m}}{ds^{n+1-m}} [(s-1) \zeta(s)]_{k=1}$$

$$= -\sum_{k=1}^{\infty} \frac{\sigma_k}{k} k(k-1)...(k-n-1) - \sum_{k=1}^{\infty} \frac{\sigma_k}{k} \sum_{m=0}^{n} \binom{n+1}{m} k(k-1)...(k-m)(-1)^{n-m} (n-m+1) \gamma_{n-m}$$
Therefore, the evaluation of (3.7) at \( s = 1 \) results in

\[
\frac{1}{2^{n+2}} \psi^{(n+1)} \left( \frac{3}{2} \right) + \sum_{m=0}^{n} \binom{n+1}{m+1} (-1)^m (m+1) \psi^m \frac{1}{2^{n+1-m}} \psi^{(n-m)} \left( \frac{3}{2} \right)
\]

\[-\frac{1}{2} (-1)^n (n+1) \gamma_n \log \pi + (-1)^{n+1} (n+2) \gamma_{n+1}
\]

\[
= -\sum_{k=1}^{\infty} \frac{\sigma_k}{k} k(k-1)\ldots(k-n-1) - \sum_{k=1}^{\infty} \sum_{m=0}^{n} \binom{n+1}{m} k(k-1)\ldots(k-m)(-1)^{n-m} (n-m+1) \gamma_{n-m}
\]

which may be written as

\[
(3.8) \quad \frac{1}{2^{n+2}} \psi^{(n+1)} \left( \frac{3}{2} \right) + \sum_{m=0}^{n} \binom{n+1}{m+1} (-1)^m (m+1) \psi^m \frac{1}{2^{n+1-m}} \psi^{(n-m)} \left( \frac{3}{2} \right)
\]

\[
+ \frac{1}{2} (-1)^{n+1} (n+1) \gamma_n \log \pi + (-1)^{n+1} (n+2) \gamma_{n+1}
\]

\[
= -\sum_{k=1}^{\infty} \frac{\sigma_k}{k} k(k-1)\ldots(k-n-1) - \sum_{m=0}^{n} \binom{n+1}{m} (-1)^{n-m} (n-m+1) \gamma_{n-m} \sum_{k=1}^{\infty} \frac{\sigma_k}{k} k(k-1)\ldots(k-m)
\]

The right-hand side of this equation clearly requires a modicum of simplification which we now endeavour to provide.

The rising factorial \([x]^p\) is defined by

\([x]^p = x(x+1)\ldots(x+p-1)\)

and it is shown in [2, p.36] that

\([x]^p = \sum_{j=1}^{p} \frac{p!}{j!(p-1)!} [x]_j\)

where the falling factorial \([x]_j\) is defined by

\([x]_j = x(x-1)\ldots(x-j+1) = \sum_{i=0}^{j} s(j,i)x^i\)

and \(s(j,i)\) are known as the Stirling numbers of the first kind. We therefore have
Letting $x \to -x$ we see that

$$x(x-1)...(x-p+1) = (-1)^p \sum_{j=1}^{p} \frac{p!}{j!} (-1)^j x(x+1)...(x+j-1)$$

and with $x = k$ we have

(3.9) $\quad k(k-1)...(k-p+1) = (-1)^p \sum_{j=1}^{p} \frac{p!}{j!} (-1)^j k(k+1)...(k+j-1)$

We then have for the first term in the right-hand side of (3.8)

$$S_3 = -\sum_{k=1}^{n} \sigma_k\frac{k}{k} k(k-1)...(k-n-1) = (-1)^n \sum_{j=1}^{n+1} \frac{(n+2)!}{j!} (-1)^j \sum_{k=1}^{n} \sigma_k k(k+1)...(k+j-1)$$

Using (3.4) for $n \geq 1$

$$(n-1)!\lambda_n = -\sum_{k=1}^{n} \sigma_k k \cdots (k+n-1)$$

we may then express $S_3$ in terms of the Li/Keiper constants as

(3.10) $\quad S_3 = (-1)^n \sum_{j=1}^{n+1} \frac{(n+2)!}{j!} (-1)^j (j-1)!\lambda_j = (-1)^n \sum_{j=1}^{n+1} \frac{(n+2)!}{j!} (-1)^j \lambda_j$

We now consider the second term in the right-hand side of (3.8)

$$S_4 = -\sum_{m=0}^{n} \binom{n+1}{m} (-1)^{n-m} (n-m+1)\gamma_{n-m} \sum_{k=1}^{n} \sigma_k k(k-1)...(k-m)$$

which on using (3.9) becomes

$$= -\sum_{m=0}^{n} \binom{n+1}{m} (-1)^{n-m} (n-m+1)\gamma_{n-m} (-1)^{m+1} \sum_{j=1}^{m+1} \frac{(m+1)!}{j!} (-1)^j \sum_{k=1}^{n} \sigma_k k(k+1)...(k+j-1)$$

$$= \sum_{m=0}^{n} \binom{n+1}{m} (-1)^{n-m} (n-m+1)\gamma_{n-m} (-1)^{m+1} \sum_{j=1}^{m+1} \frac{(m+1)!}{j!} (-1)^j (j-1)!\lambda_j$$

12
(3.11)

\[
\sum_{m=0}^{n} \binom{n+1}{m} (-1)^{n-m}(n-m+1) \gamma_{n-m} (-1)^{m+1} \sum_{j=1}^{m+1} \frac{(m+1)!}{j!} \left(-1\right)^{j} \lambda_{j}
\]

Using (3.10) and (3.11) we may then write (3.8) as

\[
(3.12) \quad \frac{1}{2^{n+2}} \psi^{(n+1)} \left( \frac{3}{2} \right) + \sum_{m=0}^{n} \binom{n+1}{m} (-1)^{m}(m+1) \gamma_{m} \frac{1}{2^{n+1-m}} \psi^{(n-m)} \left( \frac{3}{2} \right)
\]

\[+ \frac{1}{2} (-1)^{n+1} (n+1) \gamma_{n} \log \pi + (-1)^{n+1} (n+2) \gamma_{n+1}
\]

\[= (-1)^{n} \sum_{j=1}^{n+2} \frac{(n+2)!}{j!} \left(-1\right)^{j} \lambda_{j}
\]

\[+ \sum_{m=0}^{n} \binom{n+1}{m} (-1)^{n-m}(n-m+1) \gamma_{n-m} (-1)^{m+1} \sum_{j=1}^{m+1} \frac{(m+1)!}{j!} \left(-1\right)^{j} \lambda_{j}
\]

After some algebra and using the following elementary binomial identities

\[
\binom{n+1}{m+1} (m+1) = (n+1) \binom{n}{m} \quad (n+1) \binom{n}{m} (n-m+1) = (m+1) \binom{n}{m+1}
\]

\[\frac{1}{j!} \left(-1\right)^{j} \lambda_{j} = \frac{1}{n+2} \left(-1\right)^{j} \lambda_{j}
\]

\[\frac{1}{m+1} \left(-1\right)^{j} \lambda_{j} = \frac{1}{m+1} \left(-1\right)^{j} \lambda_{j}
\]

we have the required recurrence relation

\[
(3.13) \quad \frac{1}{2^{n+2}} \psi^{(n+1)} \left( \frac{3}{2} \right) + \frac{(n+1)}{2^{n+1}} \sum_{m=0}^{n} \binom{n}{m} (-1)^{m} \gamma_{m} \sum_{m=0}^{n+1} \frac{(m+1)!}{j!} \left(-1\right)^{j} \lambda_{j}
\]

\[+ \frac{1}{2} (-1)^{n+1} (n+1) \gamma_{n} \log \pi + (-1)^{n+1} (n+2) \gamma_{n+1}
\]

\[= (-1)^{n} (n+1) \sum_{j=1}^{n+2} \frac{(n+2)!}{j!} \left(-1\right)^{j} \lambda_{j} + (-1)^{n+1} \sum_{m=0}^{n+1} \binom{n}{m+1} (m+1) \gamma_{n-m} \sum_{j=1}^{m+1} \frac{(m+1)!}{j!} \left(-1\right)^{j} \lambda_{j}
\]

For example with \( n = 0 \) we get
(3.14) \[ \frac{1}{4} \psi'(\tfrac{3}{2}) + \frac{1}{2} \gamma \psi\left(\tfrac{3}{2}\right) - \frac{1}{2} \gamma \log \pi - 2\gamma_1 = -2\lambda_1 + \lambda_2 + \gamma \lambda_1 \]

which was previously derived in (3.5). Using (2.13) this may be expressed as

\[ \frac{1}{4} \psi'(\tfrac{3}{2}) - \gamma^2 - 2\gamma_1 + 2\lambda_1 = \lambda_2 \]

and referring to (4.4) this becomes

\[ \frac{1}{4} \psi'(\tfrac{3}{2}) - \eta_1 + 2\lambda_1 = \lambda_2 \]

and we note that \( \psi'(\tfrac{3}{2}) \) and \( \eta_1 \) are both positive (however at this stage we still need to substitute numerical values of the various constants to determine that \( \lambda_2 \) is non-negative).

\[ \square \]

In what follows we show how it is possible to determine the signs of the first few \( \lambda_n \) with a modicum of numerical calculations.

Using the Jacobi theta function, Coffey [5] showed in 2003 that even derivatives of \( \xi(s) \) are positive for all real values of \( s \), while odd derivatives are positive for \( s \geq \frac{1}{2} \) and negative for \( s < \frac{1}{2} \). In particular, \( \xi^{(n)}(1) \) is positive for \( n \geq 1 \). This was also proved by Freitas [18] in a different manner in 2005.

We note from (1.7) that

\[ \xi'(s) = -\xi(s) \sum_{k=1}^{\infty} (-1)^k \frac{\sigma_k}{k} k(s-1)^{k-1} = -\xi(s) \sum_{k=1}^{\infty} \frac{\sigma_k}{k} k s^{k-1} \]

Since \( \xi(1) = \xi(0) = \frac{1}{2} \) this results in

(3.15) \[ \xi'(1) = \frac{1}{2} \sigma_1 = \frac{1}{2} \lambda_1 \]

and we immediately deduce that both \( \lambda_1 \) and \( \sigma_1 \) are positive.
The (odd) derivative \( \xi'(0) = \frac{1}{2} \sum_{k=1}^{\infty} \sigma_k = -\frac{1}{2} \lambda_1 \) which again shows that \( \lambda_1 \) is positive and that \( \sum_{k=1}^{\infty} \sigma_k \) is negative.

We have the second derivative

\[
\xi^{(2)}(s) = -\xi(s) \sum_{k=1}^{\infty} \frac{\sigma_k}{k} k(k-1)s^{k-2} - \xi^{(1)}(s) \sum_{k=1}^{\infty} \frac{\sigma_k}{k} s^{k-1}
\]

and thus

\[
\xi^{(2)}(1) = -\xi(1) \sum_{k=1}^{\infty} \frac{\sigma_k}{k} k(k-1) - \xi^{(1)}(1) \sum_{k=1}^{\infty} \frac{\sigma_k}{k} \]

Since \( k(k-1) = k(k+1) - 2k \) we have

\[
\xi^{(2)}(1) = -\xi(1) \sum_{k=1}^{\infty} \frac{\sigma_k}{k} k(k+1) + 2\xi(1) \sum_{k=1}^{\infty} \frac{\sigma_k}{k} k - \xi^{(1)}(1) \sum_{k=1}^{\infty} \frac{\sigma_k}{k} k
\]

and hence, using (3.4) and (3.15), we obtain

(3.16) \[ \xi^{(2)}(1) = \frac{1}{2} \lambda_2 - \lambda_1 + \frac{1}{2} \lambda_1^2 \]

Since \( \xi^{(2)}(1) \) is positive, we determine that

(3.17) \[ \lambda_2 > \lambda_1(2 - \lambda_1) \]

and from (2.14) we see that \( 2 - \lambda_1 > 0 \). We may therefore conclude that \( \lambda_2 \) is also positive.

Since \( 1 > \lambda_1 \) we have \( \lambda_1 > \lambda_1^2 \) and from (3.17) we have

\[ \lambda_2 > 2\lambda_1 - \lambda_1^2 > 2\lambda_1 - \lambda_1 \]

and therefore we also have

(3.18) \[ \lambda_2 > \lambda_1 \]

Continuing as above we have
\[ \xi^{(3)}(s) = -\xi(s) \sum_{k=1}^{\infty} \frac{\sigma_k}{k} k(k-1)(k-2) s^{k-3} - 2\xi^{(1)}(s) \sum_{k=1}^{\infty} \frac{\sigma_k}{k} k(k-1) s^{k-2} - \xi^{(2)}(s) \sum_{k=1}^{\infty} \frac{\sigma_k}{k} k s^{k-1} \]

and using the identity \( k(k-1)(k-2) = k(k+1)(k+2) - 6k(k+1) + 6k \) we obtain

\[ \xi^{(3)}(s) = -\xi(s) \sum_{k=1}^{\infty} \frac{\sigma_k}{k} k(k+1)(k+2) s^{k-3} + 6\xi(s) \sum_{k=1}^{\infty} \frac{\sigma_k}{k} k(k+1) s^{k-3} - 6\xi(s) \sum_{k=1}^{\infty} \frac{\sigma_k}{k} k s^{k-3} \]

\[-2\xi^{(1)}(s) \sum_{k=1}^{\infty} \frac{\sigma_k}{k} k(k+1) s^{k-2} + 4\xi^{(1)}(s) \sum_{k=1}^{\infty} \frac{\sigma_k}{k} k s^{k-2} - \xi^{(2)}(s) \sum_{k=1}^{\infty} \frac{\sigma_k}{k} k s^{k-1} \]

In particular we have

\[ \xi^{(3)}(1) = -\xi(1) \sum_{k=1}^{\infty} \frac{\sigma_k}{k} k(k+1)(k+2) + 6\xi(1) \sum_{k=1}^{\infty} \frac{\sigma_k}{k} k(k+1) - 6\xi(1) \sum_{k=1}^{\infty} \frac{\sigma_k}{k} k \]

\[-2\xi^{(1)}(1) \sum_{k=1}^{\infty} \frac{\sigma_k}{k} k(k+1) + 4\xi^{(1)}(1) \sum_{k=1}^{\infty} \frac{\sigma_k}{k} k - \xi^{(2)}(1) \sum_{k=1}^{\infty} \frac{\sigma_k}{k} k \]

Employing (3.16) this becomes

\[ \xi^{(3)}(1) = -\frac{1}{2} \sum_{k=1}^{\infty} \frac{\sigma_k}{k} k(k+1)(k+2) + 3\sum_{k=1}^{\infty} \frac{\sigma_k}{k} k(k+1) - 3\sum_{k=1}^{\infty} \frac{\sigma_k}{k} k \]

\[ -\lambda_1 \sum_{k=1}^{\infty} \frac{\sigma_k}{k} k(k+1) + 2\lambda_1 \sum_{k=1}^{\infty} \frac{\sigma_k}{k} k - \frac{1}{2} (\lambda_2 - 2\lambda_1 + \lambda_1^2) \sum_{k=1}^{\infty} \frac{\sigma_k}{k} k \]

and we obtain

\[ \xi^{(3)}(1) = \lambda_3 - 3\lambda_2 + 3\lambda_1 + \lambda_1 \lambda_2 - 2\lambda_1^2 + \frac{1}{2} \lambda_1 (\lambda_2 - 2\lambda_1 + \lambda_1^2) \]

Since \( \xi^{(3)}(1) \) is positive we have

\[ \lambda_3 > 3(\lambda_2 - \lambda_1) - 3\lambda_1 \left( \frac{1}{2} \lambda_2 - \lambda_1 \right) - \frac{1}{2} \lambda_1^3 \]

\[ = 3(\lambda_2 - \lambda_1) - 3\lambda_1 (\lambda_2 - \lambda_1) + \frac{3}{2} \lambda_1 \lambda_2 - \frac{1}{2} \lambda_1^3 \]
and we therefore deduce that $\lambda_3$ is positive. It remains to be determined whether or not the above inequality also implies that $\lambda_3 > \lambda_2$.

Using the other version of (1.7) gives us

\[
(3.19) \quad \xi^{(2)}(s) = -\xi(s) \sum_{k=1}^{\infty} (-1)^k (k-1)\sigma_k (s-1)^{k-2} - \xi^{(1)}(s) \sum_{k=1}^{\infty} (-1)^k \sigma_k (s-1)^{k-1}
\]

and for $s = 1$ we get

\[
\xi^{(2)}(1) = -\xi(1)\sigma_2 + \xi^{(1)}(1)\sigma_1 = \frac{1}{2} [\sigma_1^2 - \sigma_2^2]
\]

Hence we see that

\[
(3.20) \quad \sigma_1^2 > \sigma_2
\]

Letting $s = 0$ in (3.19) results in

\[
\xi^{(2)}(0) = -\frac{1}{2} \sum_{k=1}^{\infty} \frac{\sigma_k}{k} k(k-1) - \xi^{(1)}(s) \sum_{k=1}^{\infty} \frac{\sigma_k}{k} k = \frac{1}{2} \lambda_2 - \lambda_1 + \frac{1}{2} \lambda_1^2
\]

which is the same as (3.16) since

\[
\xi^{(n)}(s) = (-1)^n \xi^{(n)}(1-s)
\]

and therefore

\[
\xi^{(n)}(0) = (-1)^n \xi^{(n)}(1)
\]

A further differentiation gives us
\[ \xi^{(3)}(s) = -\xi(s) \sum_{k=1}^{\infty} (-1)^k \frac{\sigma_k}{k} k(k+1)(k+2)(s-1)^{k-3} + 6\xi(s) \sum_{k=1}^{\infty} (-1)^k \frac{\sigma_k}{k} k(k+1)(s-1)^{k-3} \]

\[ -6\xi(s) \sum_{k=1}^{\infty} (-1)^k \frac{\sigma_k}{k} k (s-1)^{k-3} - 2\xi^{(1)}(s) \sum_{k=1}^{\infty} (-1)^k \frac{\sigma_k}{k} k(k+1)(s-1)^{k-2} \]

\[ + 4\xi^{(1)}(s) \sum_{k=1}^{\infty} (-1)^k \frac{\sigma_k}{k} k (s-1)^{k-2} - \xi^{(2)}(s) \sum_{k=1}^{\infty} (-1)^k \frac{\sigma_k}{k} k(s-1)^{k-1} \]

It will be seen in (6.2) below that

\[ \xi^{(n+1)}(1) = \frac{1}{2} (-1)^n n! \sigma_{n+1} + \sum_{k=1}^{n} \binom{n}{k} (-1)^{n-k+1} \xi^{(k)}(1)(n-k+1)! \sigma_{n-k} \]

We now attempt to generalise the above specific results which we obtained for \( \lambda_1, \lambda_2 \) and \( \lambda_3 \).

It is shown in [22] that

\[ \frac{d^m}{dx^n} e^{f(x)} = e^{f(x)} Y_m \left( f^{(1)}(x), f^{(2)}(x), ..., f^{(m)}(x) \right) \]

where the (exponential) complete Bell polynomials \( Y_n(x_1, ..., x_n) \) are defined by \( Y_0 = 1 \) and for \( n \geq 1 \)

\[ Y_n(x_1, ..., x_n) = \sum_{\pi(n)} \frac{n!}{k_1! k_2! ... k_n!} \left( \frac{x_1}{1!} \right)^{k_1} \left( \frac{x_2}{2!} \right)^{k_2} ... \left( \frac{x_n}{n!} \right)^{k_n} \]

where the sum is taken over all partitions \( \pi(n) \) of \( n \), i.e. over all sets of integers \( k_j \) such that

\[ k_1 + 2k_2 + 3k_3 + ... + nk_n = n \]

The complete Bell polynomials have integer coefficients and the first five are set out below (Comtet [14, p.307])

\[ Y_1(x_i) = x_i \]
\[ Y_2(x_1, x_2) = x_1^2 + x_2 \]
\[ Y_3(x_1, x_2, x_3) = x_1^3 + 3x_1x_2 + x_3 \]
\[ Y_4(x_1, x_2, x_3, x_4) = x_1^4 + 6x_1^2x_2 + 4x_1x_2^2 + 3x_2^2 + x_4 \]
\[ Y_5(x_1, x_2, x_3, x_4, x_5) = x_1^5 + 10x_1^3x_2 + 10x_1^2x_3 + 15x_1x_2^2 + 5x_1x_4 + 10x_2x_3 + x_5 \]

Suppose that \( h'(x) = h(x)g(x) \) and let \( f(x) = \log h(x) \). We see that

\[ f'(x) = \frac{h'(x)}{h(x)} = g(x) \]

and then using (3.22) above we have

\[ \frac{d^m}{dx^m} h(x) = \frac{d^m}{dx^m} e^{\log h(x)} = h(x)Y_m\left(g(x), g^{(1)}(x), \ldots, g^{(m-1)}(x)\right) \]

We now write (1.7) as

\[ \xi'(s) = \xi(s)g(s) \]

where \( g(s) = -\sum_{k=1}^{\infty} \frac{\sigma_k}{k} ks^{k-1} \) and \( g(1) = \lambda_1 \). We then have

\[ \xi^{(m)}(s) = \xi(s)Y_m\left(g(s), g^{(1)}(s), \ldots, g^{(m-1)}(s)\right) \]

where

\[ g^{(r)}(s) = -\sum_{k=1}^{\infty} \frac{\sigma_k}{k} k(k-1) \cdots (k-r) s^{k-1-r} \]

and

\[ g^{(r)}(1) = -\sum_{k=1}^{\infty} \frac{\sigma_k}{k} k(k-1) \cdots (k-r) \]

Using (3.9)

\[ k(k-1) \cdots (k-r) = (-1)^r r! \sum_{j=1}^{r-1} \binom{r}{j} (-1)^j k(k+1) \cdots (k+j-1) \]

we have
\[ g^{(r)}(1) = -(-1)^{r+1} \sum_{j=1}^{r+1} \frac{(r+1)!}{j!} \binom{r}{j-1} (-1)^j (j-1)! \lambda_j \]

and using (3.4) this becomes

\[ = (-1)^{r+1} \sum_{j=1}^{r+1} \frac{(r+1)!}{j!} \binom{r}{j-1} (-1)^j (j-1)! \lambda_j \]

\[ = (-1)^{r+1} (r+1)! \sum_{j=1}^{r+1} \frac{r}{j} (-1)^j \frac{\lambda_j}{j} \]

We then have

(3.26) \[ g^{(r)}(1) = (-1)^{r+1} r! \sum_{j=1}^{r+1} \binom{r+1}{j} (-1)^j \lambda_j \]

and applying the binomial inversion formula

(3.27) \[ a_n = \sum_{k=0}^{n} \binom{n}{k} (-1)^k b_k \quad \iff \quad b_n = \sum_{k=0}^{n} \binom{n}{k} (-1)^k a_k \]

to the following expression (where we define \( \lambda_0 = g^{(-1)}(1) = 0 \))

\[ \frac{(-1)^r g^{(r-1)}(1)}{(r-1)!} = \sum_{j=0}^{r} \binom{r}{j} (-1)^j \lambda_j \]

we obtain

(3.28) \[ \lambda_j = \sum_{j=0}^{r} \binom{r}{j} \frac{g^{(j-1)}(1)}{(j-1)!} \]

We have

\[ g^{(j-1)}(s) = -\sum_{k=1}^{\infty} (-1)^j \frac{\sigma_k}{k} k(k-1) \cdots (k - j +1)(s-1)^{k-j} \]

which results in

\[ g^{(j-1)}(1) = -(j-1)!(-1)^j \sigma_j \]
and therefore from (3.28) we obtain the well-known result [6]

\[ \lambda_j = \sum_{j=1}^{r} (-1)^j \binom{r}{j} \sigma_j \]  

(3.29)

Particular values of (3.26) are

\[ g(1) = \lambda_1 \]

\[ g^{(1)}(1) = \lambda_2 - 2\lambda_1 \]

\[ g^{(2)}(1) = 6\lambda_1 - 6\lambda_2 + 2\lambda_3 \]

Using (3.25) we see that

\[ \xi^{(m)}(1) = \frac{1}{2} Y_m \left( g(1), g^{(1)}(1), \ldots, g^{(m-1)}(1) \right) \]

\[ \xi^{(1)}(1) = \frac{1}{2} Y_1(g(1)) = \frac{1}{2} \lambda_1 \]

\[ \xi^{(2)}(1) = \frac{1}{2} Y_2(g(1), g^{(1)}(1)) \]

\[ = \frac{1}{2} \left[ g^2(1) + g^{(1)}(1) \right] \]

\[ = \frac{1}{2} \left[ \lambda_1^2 + \lambda_2 - 2\lambda_1 \right] \]

\[ \xi^{(3)}(1) = \frac{1}{2} Y_3(g(1), g^{(1)}(1), g^{(2)}(1)) \]

and, since \( Y_3(x_1, x_2, x_3) = x_1^3 + 3x_1x_2 + x_3 \), this becomes

\[ = \frac{1}{2} \left[ g^3(1) + 3g(1)g^{(1)}(1) + g^{(2)}(1) \right] \]

\[ \xi^{(3)}(1) = \frac{1}{2} \left[ \lambda_1^3 + 3\lambda_1(\lambda_2 - 2\lambda_1) + 6\lambda_1 - 6\lambda_2 + 2\lambda_3 \right] \]
Riordan [28] reports that

\[(3.30) \quad Y_{m+1}(x_1, \ldots, x_{m+1}) = \sum_{k=0}^{m} \binom{m}{k} Y_{m-k}(x_1, \ldots, x_{m-k})x_{k+1} = \sum_{k=0}^{m} \binom{m}{k} Y_k(x_1, \ldots, x_k)x_{m-k+1}\]

and for example we have

\[Y_2(x_1, x_2) = Y_f(x_1)x_1 + Y_0x_2 = x_1^2 + x_2\]

From (3.25) we see that

\[\xi^{(m+1)}(s) = \xi(s) Y_{m+1}\left(g(s), g^{(1)}(s), \ldots, g^{(m)}(s)\right)\]

\[= \sum_{k=0}^{m} \binom{m}{k} \xi(s) Y_k \left(g(s), g^{(1)}(s), \ldots, g^{(k-1)}(s)\right)g^{(m-k)}(s)\]

We therefore have

\[(3.31) \quad \xi^{(m+1)}(s) = \sum_{k=0}^{m} \binom{m}{k} \xi^{(k)}(s)g^{(m-k)}(s)\]

which simply corresponds with the Leibniz differentiation rule using \(\xi'(s) = \xi(s)g(s)\).

This gives us

\[\xi^{(2)}(s) = \xi(s)g^{(1)}(s) + \xi^{(1)}(s)g(s)\]

\[\xi^{(2)}(1) = \frac{1}{2}[\lambda_2 - 2\lambda_1] + \frac{1}{2}\lambda_1^2\]

We therefore have

\[\frac{1}{2}[\lambda_2 - 2\lambda_1] + \frac{1}{2}\lambda_1^2 > 0\]

which corresponds with (3.16). \(\square\)

We note from (1.6) that
\[ \log \xi(s) = \log \Gamma\left(1 + \frac{s}{2}\right) - \frac{s}{2} \log \pi + \log[(s-1)\xi(s)] \]

and therefore
\[ g(s) = \frac{\xi'(s)}{\xi(s)} = \psi\left(1 + \frac{s}{2}\right) - \frac{1}{2} \log \pi - \frac{1}{s-1} + \frac{\xi'(s)}{\xi(s)} \]

We will note from (4.1) that
\[ \frac{\xi'(s)}{\xi(s)} + \frac{1}{s-1} = -\sum_{k=0}^{\infty} \eta_k (s-1)^k \]

and we therefore obtain
\[ g(s) = \frac{\xi'(s)}{\xi(s)} = \frac{1}{2} \psi\left(1 + \frac{s}{2}\right) - \frac{1}{2} \log \pi - \sum_{k=0}^{\infty} \eta_k (s-1)^k \]

and we see that
\[ g(1) = g^{(0)}(1) = \frac{1}{2} \psi\left(\frac{3}{2}\right) - \frac{1}{2} \log \pi - \eta_0 = \lambda_1 \]

For \( r \geq 1 \)
\[ g^{(r)}(s) = \frac{1}{2^{r+1}} \psi^{(r)}\left(1 + \frac{s}{2}\right) - \sum_{k=0}^{\infty} \eta_k (k-1) \cdots (k-r+1)(s-1)^{k-r} \]

\[ g^{(r)}(1) = \frac{1}{2^{r+1}} \psi^{(r)}\left(\frac{3}{2}\right) - r! \eta_r \]

\[ = \frac{1}{2^{r+1}} \psi^{(r)}\left(\frac{3}{2}\right) - r! \eta_r - \delta_{r,0} \frac{1}{2} \log \pi \]

where we have introduced the Kronecker delta \( \delta_{r,0} \) to ensure that this equation is also valid for \( r = 0 \)
\[ g^{(r)}(1) = (-1)^{r+1} r! \sum_{j=0}^{r+1} \binom{r+1}{j} (-1)^j \lambda_j \]

This may be written as

23
\begin{align*}
(-1)^{r+1} r! \sum_{j=0}^{r+1} \binom{r+1}{j} (-1)^j \lambda_j &= \frac{1}{2^{r+1}} \psi^{(r)} \left( \frac{3}{2} \right) - r! \eta_r - \delta_r, 0 \frac{1}{2} \log \pi \\
\text{where we start the summation at } j = 0 \text{ by defining } \lambda_0 &= 0 .
\end{align*}

We now let \( r \to r - 1 \) and define \( \eta_{-1} = 0 \) and \( \psi^{(-1)} \left( \frac{3}{2} \right) = 0 \)

\begin{align*}
(-1)^r (r-1)! \sum_{j=0}^{r} \binom{r}{j} (-1)^j \lambda_j &= \frac{1}{2^r} \psi^{(r-1)} \left( \frac{3}{2} \right) - (r-1)! \eta_{r-1} - \delta_{r-1, 0} \frac{1}{2} \log \pi \\
\sum_{j=0}^{r} \binom{r}{j} (-1)^j \lambda_j &= (-1)^r \left[ \frac{r}{2^r r!} \psi^{(r-1)} \left( \frac{3}{2} \right) - \eta_{r-1} - \delta_{r-1, 0} \frac{r}{2^r r!} \log \pi \right]
\end{align*}

and applying the binomial inversion formula (3.27) we obtain

\begin{align*}
\lambda_r &= \sum_{j=0}^{r} \binom{r}{j} \left[ \frac{j}{2^j j!} \psi^{(j-1)} \left( \frac{3}{2} \right) - \eta_{j-1} - \delta_{j-1, 0} \frac{j}{2^j j!} \log \pi \right] \\
\lambda_r &= \sum_{j=1}^{r} \binom{r}{j} \left[ \frac{j}{2^j j!} \psi^{(j-1)} \left( \frac{3}{2} \right) - \eta_{j-1} \right] - \frac{1}{2} \binom{r}{1} \log \pi \\
(3.33) \quad \lambda_r &= \sum_{j=2}^{r} \binom{r}{j} \frac{j}{2^j j!} \psi^{(j-1)} \left( \frac{3}{2} \right) + \frac{1}{2} \binom{r}{j} \psi \left( \frac{3}{2} \right) - \sum_{j=1}^{r} \binom{r}{j} \eta_{j-1} - \frac{1}{2} \binom{r}{1} \log \pi \\
\text{From (3.5.1) we have } \psi^{(j-1)} \left( \frac{3}{2} \right) &= (-1)^j (j-1)! \left[ \left( \frac{2j-1}{2j} \right) \varsigma(j) - 2^j \right] \\
\text{and we obtain } \\
\lambda_r &= \sum_{j=2}^{r} \binom{r}{j} (-1)^j \left[ 1 - \frac{1}{2^j} \right] \varsigma(j) - \sum_{j=2}^{r} \binom{r}{j} (-1)^j + \frac{1}{2} \binom{r}{1} \psi \left( \frac{3}{2} \right) - \sum_{j=1}^{r} \binom{r}{j} \eta_{j-1} - \frac{1}{2} \binom{r}{1} \log \pi \\
\text{Using (2.11) } \\
\psi \left( \frac{3}{2} \right) &= - \gamma - 2 \log 2 + 2 \\
\text{and noting that }
\end{align*}
\[
\sum_{j=2}^{r} \binom{r}{j} (-1)^j = \sum_{j=0}^{r} \binom{r}{j} (-1)^j + r - 1 = r - 1
\]

we simply obtain another derivation of Coffey’s result [C1] for \( r \geq 2 \)

\[(3.34) \quad \lambda_r = \sum_{j=2}^{r} \binom{r}{j} (-1)^j \left[ 1 - \frac{1}{2^j} \right] \zeta(j) + 1 - \sum_{j=1}^{r} \binom{r}{j} \eta_{j-1} + 1 - \frac{1}{2} r (\gamma + 2 \log 2 + \log \pi) \]

(and we note that the term +1 has been inadvertently omitted in equation (3) in [7] and in equation (12) in [13]).

A much simpler derivation of (3.34) is shown below.

We see that (3.33) may be written as

\[
\lambda_r = \sum_{j=2}^{r} \binom{r}{j} \frac{1}{(j-1)!} \left[ \frac{1}{2^j} \psi^{(j-1)} \left( \frac{3}{2} \right) - (j-1)! \eta_{j-1} \right] + r \gamma + \frac{1}{2} r \psi \left( \frac{3}{2} \right) - \frac{1}{2} r \log \pi
\]

\[
= \sum_{j=2}^{r} \binom{r}{j} \frac{1}{(j-1)!} \left[ \frac{1}{2^j} \psi^{(j-1)} \left( \frac{3}{2} \right) - (j-1)! \eta_{j-1} \right] + \frac{1}{2} r (\gamma - 2 \log 2 + 2 - \log \pi)
\]

and we note from (3.32) that

\[
\frac{d}{ds} \log \zeta(s) + \frac{1}{2} \log \pi = \frac{1}{2} \psi \left( 1 + \frac{s}{2} \right) - \sum_{k=0}^{\infty} \eta_k (s-1)^k
\]

Differentiating this \( j-1 \) times gives us

\[
\frac{d^{j-1}}{ds^{j-1}} \log \zeta(s) + \delta_{j,1} \frac{1}{2} \log \pi = \frac{1}{2} \psi^{(j-1)} \left( 1 + \frac{s}{2} \right) - \sum_{k=0}^{\infty} \eta_k k(k-1) \cdots (k-j) (s-1)^{k-j+1}
\]

and evaluating this at \( s = 1 \) results in

\[
\left. \frac{d^{j-1}}{ds^{j-1}} \log \zeta(s) \right|_{s=1} + \delta_{j,1} \frac{1}{2} \log \pi = \frac{1}{2} \psi^{(j-1)} \left( \frac{3}{2} \right) - (j-1)! \eta_{j-1}
\]

We then have

\[
\sum_{j=2}^{r} \binom{r}{j} \frac{1}{(j-1)!} \left[ \frac{1}{2^j} \psi^{(j-1)} \left( \frac{3}{2} \right) - (j-1)! \eta_{j-1} \right] + r \gamma + \frac{1}{2} r \psi \left( \frac{3}{2} \right) - \frac{1}{2} r \log \pi
\]
\[
\lambda_r = \sum_{j=2}^{r} \binom{r}{j} \frac{1}{j!} \left. \frac{d^j}{ds^j} \log \xi(s) \right|_{s=1} \frac{1}{2} r(\gamma + 2 \log 2 + \log \pi)
\]

Applying the Leibniz differentiation rule to (1.4) we see that

\[
\lambda_r = \frac{1}{(r-1)!} \left. \frac{d^r}{ds^r} \left[ s^{r-1} \log \xi(s) \right] \right|_{s=1} \\
= \frac{1}{(r-1)!} \sum_{j=0}^{r} \binom{r}{j} \frac{1}{j!} \left. \frac{d^j}{ds^j} \log \xi(s) \frac{d^{r-j}}{ds^{r-j}} s^{r-1} \right|_{s=1} \\
= \sum_{j=0}^{r} \binom{r}{j} \frac{1}{j!} \left. \frac{d^j}{ds^j} \log \xi(s) \right|_{s=1} + r \left. \frac{d}{ds} \log \xi(s) \right|_{s=1} \\
= \sum_{j=0}^{r} \binom{r}{j} \frac{1}{j!} \left[ \frac{1}{2j} \psi^{(j-i)} \left( \frac{3}{2} \right) - (j-1)! \eta_{j-1} \right] + \frac{1}{2} r(\gamma - 2 \log 2 + 2 - \log \pi)
\]

which corresponds with (3.33) above.

We have

\[
\lambda_r = \sum_{j=2}^{r} \binom{r}{j} \frac{1}{(j-1)!} \left[ \frac{1}{2j} \psi^{(j-i)} \left( \frac{3}{2} \right) - (j-1)! \eta_{j-1} \right] + \frac{1}{2} r(\gamma - 2 \log 2 + 2 - \log \pi)
\]

Using (3.5.2)

\[
\psi^{(j-i)} \left( \frac{3}{2} \right) = (-1)^i (j-1)! \xi(j, 3/2)
\]

we may write this as

\[
(3.35) \quad \lambda_r = \sum_{j=2}^{r} \binom{r}{j} (-1)^j \left[ \frac{\xi(j, 3/2)}{2j} - (-1)^j \eta_{j-1} \right] + \frac{1}{2} r(\gamma - 2 \log 2 + 2 - \log \pi)
\]

and it may be possible to simplify the first summation by using the following Hermite integral representation of the Hurwitz zeta function (see for example [31, p.120])

\[
\xi(s,u) = \frac{u^{-s}}{2} + \frac{u^{1-s}}{s-1} + \int_0^\infty \frac{(u+ix)^{-s} - (u-ix)^{-s}}{e^{2\pi x} - 1} \, dx
\]
It may be noted that
\[ \gamma - 2 \log 2 + 2 - \log \pi > 0 \]

4. The \( \eta \) constants

The \( \eta \) constants are defined by reference to the logarithmic derivative of the Riemann zeta function

\[
\frac{d}{ds}[\log x(s)] = \frac{\zeta'(s)}{\zeta(s)} = -\frac{1}{s-1} - \sum_{k=0}^{\infty} \eta_k (s-1)^k \quad |s-1| < 3
\]

and we may also note that this is equivalent to

\[
\frac{d}{ds} \log[(s-1)x(s)] = \frac{\zeta'(s)}{\zeta(s)} + \frac{1}{s-1} = -\sum_{k=0}^{\infty} \eta_k (s-1)^k
\]

We then see that

\[
\frac{d^{n+1}}{ds^{n+1}} \log[(s-1)x(s)] = -\sum_{k=0}^{\infty} \eta_k k(k-1)...(k-n+1)(s-1)^{k-n}
\]

and hence we get

\[
\lim_{s \to 1} \frac{d^{n+1}}{ds^{n+1}} \log[(s-1)x(s)] = -n! \eta_n
\]

We see from (4.2) that

\[
\frac{d}{ds}[(s-1)x(s)] = -\sum_{k=0}^{\infty} \eta_k (s-1)^k [(s-1)x(s)]
\]

and thus

\[
\frac{d^{n+1}}{ds^{n+1}} [(s-1)x(s)] = -\frac{d^n}{ds^n} \sum_{k=0}^{\infty} \eta_k (s-1)^k [(s-1)x(s)]
\]

We consider the value at \( s = 1 \)

\[
\left. \frac{d^{n+1}}{ds^{n+1}} [(s-1)x(s)] \right|_{s=1} = -\lim_{s \to 1} \frac{d^n}{ds^n} \sum_{k=0}^{\infty} \eta_k (s-1)^k [(s-1)x(s)]
\]

and using (2.10)
\[
\frac{d^{n+1}}{ds^{n+1}}[(s-1)\zeta(s)]_{s=1} = (-1)^n (n+1)\gamma_n
\]

we obtain via the Leibniz differentiation formula

\[
= -n!\eta_n + \sum_{j=1}^{n} \binom{n}{j} (n-j)!\eta_{n-j}(-1)^j j\gamma_{j-1}
\]

where we have isolated the first term.

\[
= -n!\eta_n + n!\sum_{j=1}^{n} (-1)^j j\eta_{n-j}\gamma_{j-1}
\]

Hence we obtain the recurrence relation

\[
(4.4) \quad (-1)^n (n+1)\gamma_n = -n!\eta_n + n!\sum_{j=1}^{n} (-1)^j j\eta_{n-j}\gamma_{j-1}
\]

where for \( n = 0,1,2 \) we have respectively:

\[
\gamma = -\eta_0
\]

\[
2\gamma_1 = \eta_1 + \eta_0\gamma = \eta_1 - \gamma^2
\]

\[
\frac{3}{2}\gamma_2 = -\eta_2 - 3\gamma_1\gamma - \gamma^3
\]

Equation (4.4) is equivalent to Coffey’s recurrence relation (4.5) below.

Letting \( u = 1 \) in (2.1) gives us

\[
(s-1)\zeta(s) = 1 + \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \gamma_n (s-1)^{n+1}
\]

and defining

\[
g(s) = (s-1)\zeta(s)
\]

\[
L(s) = \log[(s-1)\zeta(s)]
\]

we have

\[
\frac{d}{ds}(s-1)\zeta(s) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \gamma_n (n+1)(s-1)^n
\]

28
and thus
\[ g^{(1)}(1) = \gamma_0 = \gamma \]
\[ g^{(k+1)}(1) = (-1)^k (k + 1) \gamma_k \]

We see that
\[ \frac{d}{ds} \log[(s-1)\zeta(s)] = \frac{(s-1)\zeta'(s) + \zeta(s)}{(s-1)\zeta(s)} \]

which may be written as
\[ (s-1)\zeta(s)L^{(1)}(s) = (s-1)\zeta'(s) + \zeta(s) = \frac{d}{ds}[(s-1)\zeta(s)] \]
or
\[ g(s)L^{(1)}(s) = g^{(1)}(s) \]

We then have
\[ \frac{d^{(n)}}{ds^{(n)}} \left[g(s)L^{(1)}(s)\right] = g^{(n+1)}(s) \]

and applying the Leibniz rule gives us
\[ g^{(n+1)}(s) = \sum_{k=0}^{n} \binom{n}{k} g^{(k)}(s)L^{(n+1-k)}(s) \]

Letting \( s = 1 \) gives us
\[ (-1)^n (n+1) \gamma_n = g^{(0)}(1)L^{(n+1)}(1) - \sum_{k=1}^{n} \binom{n}{k} (-1)^{k-1} k \gamma_{k-1} (n-k)! \eta_{n-k} \]
\[ = -n! \eta_n - \sum_{k=1}^{n} \binom{n}{k} (-1)^k k \gamma_{k-1} (n-k)! \eta_{n-k} \]
\[ = -n! \eta_n - n! \sum_{k=1}^{n} \frac{(-1)^k}{(k-1)!} \gamma_{k-1} \eta_{n-k} \]

Hence we have
\[ n! \eta_n = (-1)^{n+1} (n+1) \gamma_n - n! \sum_{k=1}^{n} \frac{(-1)^k}{(k-1)!} \gamma_{k-1} \eta_{n-k} \]

and reindexing the sum gives us
\begin{equation}
(4.5) \quad n!\eta_n = (-1)^{n+1}(n+1)\gamma_n + (-1)^{n+1}n!\sum_{k=0}^{n-1} \frac{(-1)^{k+1}}{(n-k-1)!} \gamma_{n-k-1}\eta_k
\end{equation}

as originally reported by Coffey [5].

A formula for the Stieltjes constants in terms of the (exponential) complete Bell polynomials containing the eta constants $\eta_n$ as the arguments is shown below in equation (6.1).

Eliminating $\log \xi(s)$ from (1.6) and (1.7) results in

$$
\log \Gamma\left(1 + \frac{s}{2}\right) - \frac{s}{2} \log \pi + \log[(s-1)\xi(s)] = -\log 2 - \sum_{k=1}^{\infty} (-1)^k \frac{\sigma_k}{k} (s-1)^k
$$

and, since $\xi(s) = \xi(1-s)$, we have the equivalent representation

$$
\log \Gamma\left(1 + \frac{s}{2}\right) - \frac{s}{2} \log \pi + \log[(s-1)\xi(s)] = -\log 2 - \sum_{k=1}^{\infty} \frac{\sigma_k}{k} s^k
$$

For convenience, we define $L(s)$ by $L(s) = \log[(s-1)\xi(s)]$

Upon differentiating the above two equations we obtain

$$
\frac{1}{2} \psi\left(1 + \frac{s}{2}\right) - \frac{1}{2} \log \pi + L^{(1)}(s) = -\sum_{k=1}^{\infty} (-1)^k \frac{\sigma_k}{k} (s-1)^{k-1}
$$

$$
\frac{1}{2} \psi\left(1 + \frac{s}{2}\right) - \frac{1}{2} \log \pi + L^{(1)}(s) = -\sum_{k=1}^{\infty} \frac{\sigma_k}{k} s^{k-1}
$$

For $n \geq 1$ we have the higher derivatives

$$
\frac{1}{2^{n+1}} \psi^{(n)}\left(1 + \frac{s}{2}\right) + L^{(n+1)}(s) = -\sum_{k=1}^{\infty} (-1)^k \frac{\sigma_k}{(k-1)(k-2)...(k-n)(s-1)^{k-1-n}}
$$

$$
\frac{1}{2^{n+1}} \psi^{(n)}\left(1 + \frac{s}{2}\right) + L^{(n+1)}(s) = -\sum_{k=1}^{\infty} \frac{\sigma_k}{(k-1)(k-2)...(k-n)} s^{k-1-n}
$$

and therefore with $s = 1$ we obtain
\[
\frac{1}{2^{n+1}} \psi^{(n)}(3/2) + L^{(n+1)}(1) = (-1)^{n+1} n! \sigma_{n+1}
\]

We have
\[
\psi^{(n)}(3/2) = (-1)^{n+1} n! [2^{n+1} - 1] \zeta(n + 1) + (-1)^n n! 2^{n+1}
\]

and hence we get
\[
(-1)^{n+1} n! \left[ 1 - \frac{1}{2^{n+1}} \right] \zeta(n + 1) + (-1)^n n! + L^{(n+1)}(1) = (-1)^{n+1} n! \sigma_{n+1}
\]

Using \( L^{(n+1)}(1) = -n! \eta_n \) results in
\[
\sigma_{n+1} = (-1)^{n+1} \eta_n - \left[ 1 - \frac{1}{2^{n+1}} \right] \zeta(n + 1) + 1
\]

Letting \( s = 0 \) gives us
\[
\frac{1}{2^{n+1}} \psi^{(n)}(1) + L^{(n+1)}(0) = -n! \sigma_{n+1}
\]

and therefore we get
\[
L^{(n+1)}(0) = -n! \sigma_{n+1} - \frac{1}{2^{n+1}} (-1)^{n+1} n! \zeta(n + 1)
\]

and substituting for \( \sigma_{n+1} \) gives us
\[
= -n! \left( (-1)^{n+1} \eta_n - \left[ 1 - \frac{1}{2^{n+1}} \right] \zeta(n + 1) + 1 \right) - \frac{1}{2^{n+1}} (-1)^{n+1} n! \zeta(n + 1)
\]

Hence we get for \( n \geq 1 \)
\[
L^{(n+1)}(0) = (-1)^n n! \eta_n + \left( 1 - \frac{1}{2^{n+1}} \left[ 1 - (-1)^n \right] \right) n! \zeta(n + 1) - n!
\]

and with \( n = 1 \) we have
\[
L^{(2)}(0) = \frac{1}{2} \zeta(2) - \eta_1 - 1
\]

We will also see from (4.6) below that
\[ L^{(2)}(0) = -2\zeta''(0) - \log^2(2\pi) - 1 \]

and we therefore obtain
\[ \zeta''(0) = \frac{1}{2} \eta_1 - \frac{1}{4} \zeta(2) - \frac{1}{2} \log^2(2\pi) \]

Substituting (4.4) \( \eta_1 = 2\gamma + \gamma^2 \) we then have an expression for \( \zeta''(0) \)
\[ \zeta''(0) = \gamma + \frac{1}{2} \gamma^2 - \frac{1}{4} \zeta(2) - \frac{1}{2} \log^2(2\pi) \]

as previously derived by Ramanujan [3] and Apostol [1].

We have
\[ L^{(1)}(s) = -\sum_{k=0}^{\infty} \eta_k (s-1)^k \]

and hence
\[ L^{(1)}(1) = -\eta_0 = \gamma \]

\[ L^{(n+1)}(s) = -\sum_{k=0}^{\infty} \eta_k k(k-1)...(k-n+1)(s-1)^{k-n} \]

Coffey [13] has shown that the sequence \( (\eta_n) \) has strict sign alteration
\[ \eta_n = (-1)^{n+1} \varepsilon_n \]

where \( \varepsilon_n \) are positive constants and therefore we have
\[ L^{(n+1)}(1) = -n!\eta_n = (-1)^n n!\varepsilon_n \]

We therefore note that \( L^{(1)}(1) \) is positive and \( L^{(2)}(1) \) is negative and the signs strictly alternate thereafter.

We see that
\[ L^{(1)}(0) = -\sum_{k=0}^{\infty} (-1)^k \eta_k = \sum_{k=0}^{\infty} \varepsilon_k \text{ is positive} \]

In fact we have
\[ L^{(1)}(0) = -\sum_{k=0}^{\infty} (-1)^k \eta_k = \log(2\pi) - 1 \]

\[ L^{(n+1)}(0) = -\sum_{k=0}^{\infty} \eta_k (k-1)...(k-n+1)(-1)^{k-n} = (-1)^n \sum_{k=0}^{\infty} \varepsilon_k (k-1)...(k-n+1) \]

and therefore we note that the signs of \( L^{(n+1)}(0) \) also strictly alternate.

We see that

\[ L^{(2)}(s) = \frac{(s-1)\zeta^n(s) + 2\zeta'(s)}{(s-1)\zeta(s)} - \left[ L^{(1)}(s) \right]^2 \]

Referring to (2.9) we then obtain

\[ L^{(2)}(1) = -2\gamma_1 - \left[ L^{(1)}(1) \right]^2 = -2\gamma_1 - \gamma^2 \]

and we therefore obtain (as in 4.4))

\[ \eta_1 = 2\gamma_1 + \gamma^2 \]

Since \( \eta_1 \geq 0 \) we deduce that

\[ 2\gamma_1 + \gamma^2 \geq 0 \]

Since \( \eta_2 \leq 0 \) we see that

\[ \frac{3}{2} \gamma_2 - \gamma^3 - 3\gamma_1 \leq 0 \]

We have

\[ L^{(2)}(0) = -2\zeta^{(2)}(0) + 4\zeta^{(1)}(0) - \left[ L^{(1)}(0) \right]^2 \]

\[ = -2\zeta^{(2)}(0) + 4\zeta^{(1)}(0) - \left[ \log(2\pi) - 1 \right]^2 \]

\[ = -2\zeta^{(2)}(0) - 2\log(2\pi) - \left[ \log(2\pi) - 1 \right]^2 \]

and therefore
We recall Hasse’s formula (2.4) with \( u = 1 \)

\[
(s - 1)\zeta(s) = (s - 1)\zeta(s, 1) = \sum_{n=0}^{\infty} \frac{1}{n+1} \sum_{k=0}^{n} \left( \frac{n}{k} \right) \frac{(-1)^k}{(1+k)^s}
\]

and with \( s \to s - 1 \) we have

\[
s\zeta(1-s) = -\sum_{n=0}^{\infty} \frac{1}{n+1} \sum_{k=0}^{n} \left( \frac{n}{k} \right) (-1)^k (1+k)^s
\]

We refer to the functional equation for the Riemann zeta function (1.2)

\[
2(2\pi)^{-s} \Gamma(s) \cos(\pi s/2) \zeta(s) = \zeta(1-s)
\]

and, multiplying by \( s \), we see that

\[
f(s) = s\zeta(1-s) = 2(2\pi)^{-s} \Gamma(s+1) \cos(\pi s/2) \zeta(s) = -\sum_{n=0}^{\infty} \frac{1}{n+1} \sum_{k=0}^{n} \left( \frac{n}{k} \right) (-1)^k (1+k)^s
\]

Differentiation results in

\[
f^{(p)}(s) = -\sum_{n=0}^{\infty} \frac{1}{n+1} \sum_{k=0}^{n} \left( \frac{n}{k} \right) (-1)^k (1+k)^s \log^p (1+k)
\]

and we have the particular value at \( s = 0 \)

\[
f^{(p)}(0) = -\sum_{n=0}^{\infty} \frac{1}{n+1} \sum_{k=0}^{n} \left( \frac{n}{k} \right) (-1)^k \log^p (1+k)
\]

We recall the following expression for the Stieltjes constants (2.8)

\[
\gamma_p(u) = -\frac{1}{p+1} \sum_{n=0}^{\infty} \frac{1}{n+1} \sum_{k=0}^{n} \left( \frac{n}{k} \right) (-1)^k \log^{p+1} (u+k)
\]

which gives us

(4.6) \( L^{(2)}(0) = -2\psi^{(2)}(0) - \log^2 (2\pi) - 1 \)
\[
\gamma_p = \gamma_p(1) = -\frac{1}{p+1} \sum_{n=0}^{\infty} \frac{1}{n+1} \sum_{k=0}^{n}(\frac{n}{k})(-1)^k \log^{p+1}(1+k)
\]

and we therefore obtain

(5.1) \[f^{(p+1)}(0) = (p+1)\gamma_p\]

We have

\[
\log f(s) = \log 2 - s \log(2\pi) + \log \Gamma(s+1) + \log \cos(\pi s / 2) + \log \zeta(s)
\]

and differentiation results in

\[
\frac{f'(s)}{f(s)} = -\log(2\pi) + \psi(s+1) - \frac{\pi}{2} \tan(\pi s / 2) + \frac{\zeta'(s)}{\zeta(s)}
\]

We note that

\[f(0) = 2\zeta(0) = -1\]

\[f'(0) = f(0) \left[ -\log(2\pi) + \psi(1) + \frac{\zeta'(0)}{\zeta(0)} \right] = \log(2\pi) + \gamma + 2\zeta'(0)\]

and, referring to (5.1), we see that

\[
\log(2\pi) + \gamma + 2\zeta'(0) = \gamma_0 = \gamma
\]

We then deduce the well-known result

(5.2) \[\zeta'(0) = -\frac{1}{2} \log(2\pi)\]

We may write

\[f''(s) = g(s)f(s)\]

where

\[
g(s) = \frac{d}{ds} \left[ \log(2\pi)^{\gamma} + \log \Gamma(s+1) + \log \cos(\pi s / 2) + \log \zeta(s) \right]
\]

\[= -\log(2\pi) + \psi(s+1) - \frac{\pi}{2} \tan(\pi s / 2) + \frac{\zeta'(s)}{\zeta(s)}\]
We note that
\[ g(0) = -\gamma \text{ and } f'(0) = \gamma \]

\[ f''(s) = \left[ -\log(2\pi) + \psi(s + 1) - \frac{\pi}{2} \tan(\pi s / 2) + \frac{\zeta'(s)}{\zeta(s)} \right] f'(s) \]

\[ f'''(s) = \left[ -\log(2\pi) + \psi(s + 1) - \frac{\pi}{2} \tan(\pi s / 2) + \frac{\zeta'(s)}{\zeta(s)} \right] f'(s) \]

\[ + \left[ \psi'(s + 1) - \left( \frac{\pi}{2} \right)^2 \sec^2(\pi s / 2) + \frac{\zeta(s)\zeta''(s) - [\zeta'(s)]^2}{\zeta^2(s)} \right] f(s) \]

We then see that
\[ f'''(0) = -\gamma^2 - \left[ \psi'(1) - \left( \frac{\pi}{2} \right)^2 + \frac{\zeta(0)\zeta''(0) - [\zeta'(0)]^2}{\zeta^2(0)} \right] \]

We know from (3.5.2) that
\[ \psi'(1) = \zeta(2) = \frac{\pi^2}{6} \]

and we then have
\[ f'''(0) = -\gamma^2 + \frac{\pi^2}{12} + 2\zeta''(0) + \log^2(2\pi) \]

We also have from (5.1)
\[ f'''(0) = 2\gamma_i \]

and we thus obtain
\[ \zeta''(0) = \gamma_i + \frac{1}{2} \gamma^2 - \frac{1}{24} \pi^2 - \frac{1}{2} \log^2(2\pi) \]

This concurs with the result previously obtained by Ramanujan [3] and Apostol [1].

As noted by Coffey [9] for differentiable functions \( f(s) \) and \( g(s) \) such that
then we have in terms of the (exponential) complete Bell polynomials

\[ f^{(n)}(s) = f(s) Y_n\left(g(s), g'(s), \ldots, g^{(n-1)}(s)\right) \]

where, in the case under review, we have

\[ g(s) = \frac{d}{ds} \left[ \log(2\pi)^{-s} + \log \Gamma(s+1) + \log \cos(\pi s / 2) + \log \zeta(s) \right] = \frac{d}{ds} \sum_{p=1}^{4} \log g_p(s) \]

\[ f^{(n)}(s) = f(s) Y_n\left(\frac{d}{ds} \sum_{p=1}^{4} \log g_p(s), \frac{d^2}{ds^2} \sum_{p=1}^{4} \log g_p(s), \ldots, \frac{d^{n-1}}{ds^{n-1}} \sum_{p=1}^{4} \log g_p(s)\right) \]

We have [28]

\[ Y_n(x_1 + y_1, \ldots, x_n + y_n) = \sum_{k=0}^{n} \binom{n}{k} Y_{n-k}(x_1, \ldots, x_{n-k}) Y_k(y_1, \ldots, y_k) \]

and this may be generalised to Bell polynomials with additional arguments as follows

\[ Y_n(a_1 + b_1 + c_1 + d_1, \ldots, a_n + b_n + c_n + d_n) = \sum_{k=0}^{n} \binom{n}{k} Y_{n-k}(a_1, \ldots, a_{n-k}) Y_k(b_1 + c_1 + d_1, \ldots, b_k + c_k + d_k) \]

\[ Y_k(b_1 + c_1 + d_1, \ldots, b_k + c_k + d_k) = \sum_{j=0}^{k} \binom{k}{j} Y_{k-j}(b_1, \ldots, b_{k-j}) Y_j(c_1 + d_1, \ldots, c_j + d_j) \]

\[ Y_j(c_1 + d_1, \ldots, c_j + d_j) = \sum_{l=0}^{j} \binom{j}{l} Y_{j-l}(c_1, \ldots, c_{j-l}) Y_l(d_1, \ldots, d_l) \]

and we end up with

\[ Y_n(a_1 + b_1 + c_1 + d_1, \ldots, a_n + b_n + c_n + d_n) = \sum_{k=0}^{n} \binom{n}{k} Y_{n-k}(a_1, \ldots, a_{n-k}) \sum_{j=0}^{k} \binom{k}{j} Y_{k-j}(b_1, \ldots, b_{k-j}) \sum_{l=0}^{j} \binom{j}{l} Y_{j-l}(c_1, \ldots, c_{j-l}) Y_l(d_1, \ldots, d_l) \]

We note from (A.5) in Appendix A that
\[
\frac{d^m}{dx^m} e^{h(x)} = e^{h(x)} Y_m \left( h^{(1)}(x), h^{(2)}(x), \ldots, h^{(m)}(x) \right)
\]

and letting \( h(x) \to \log h(x) \) we have

\[
\frac{d^m}{dx^m} h(x) = \frac{d^m}{dx^m} e^{\log h(x)} = e^{\log h(x)} Y_m \left( \frac{d}{dx} \log h(x), \frac{d^2}{dx^2} \log h(x), \ldots, \frac{d^m}{dx^m} \log h(x) \right)
\]

and we see that

\[
\frac{1}{h(x)} \frac{d^m}{dx^m} h(x) = Y_m \left( \frac{d}{dx} \log h(x), \frac{d^2}{dx^2} \log h(x), \ldots, \frac{d^m}{dx^m} \log h(x) \right)
\]

We therefore have

\[
f^{(n)}(s) = f(s) Y_n \left( \sum_{p=1}^{4} \log g_p(s), \sum_{p=1}^{4} \frac{d^2}{ds^2} \log g_p(s), \ldots, \sum_{p=1}^{4} \frac{d^{n-1}}{ds^{n-1}} \log g_p(s) \right)
\]

\[
= \frac{1}{g_1(s)g_2(s)g_3(s)g_4(s)} \sum_{k=0}^{n} \binom{n}{k} \frac{d^{n-k}}{ds^{n-k}} g_1(0) \sum_{j=0}^{k} \binom{k}{j} \frac{d^{k-j}}{ds^{k-j}} g_2(0) \sum_{l=0}^{j} \binom{j}{l} \frac{d^{j-l}}{ds^{j-l}} g_3(0) \frac{d^l}{ds^l} g_4(0)
\]

and with \( s = 0 \) this becomes using (5.1)

\[
\gamma_{n-1} = 2 \sum_{k=0}^{n} \binom{n}{k} \frac{d^{n-k}}{ds^{n-k}} g_1(0) \sum_{j=0}^{k} \binom{k}{j} \frac{d^{k-j}}{ds^{k-j}} g_2(0) \sum_{l=0}^{j} \binom{j}{l} \frac{d^{j-l}}{ds^{j-l}} g_3(0) \frac{d^l}{ds^l} g_4(0)
\]

where

\[
g(s) = \frac{d}{ds} \left[ \log(2\pi)^{-s} + \log \Gamma(s+1) + \log \cos(\pi s / 2) + \log \zeta(s) \right] = \frac{d}{ds} \sum_{p=1}^{4} \log g_p(s)
\]

Having regard to the four components \( g_p(s) \), the following derivatives are easily computed

\[
\frac{d^m}{ds^m} (2\pi)^{-s} = (-1)^m (2\pi)^{-s} \log^m (2\pi)
\]

\[
\frac{d^m}{ds^m} (2\pi)^{-s} \bigg|_{s=0} = (-1)^m \log^m (2\pi)
\]
\[
\frac{d^m}{ds^m} \cos \left( \frac{\pi s}{2} \right) = \left( \frac{\pi}{2} \right)^m \cos \left( \frac{\pi s + m\pi}{2} \right)
\]

\[
\frac{d^m}{ds^m} \cos \left( \frac{\pi s}{2} \right) \bigg|_{s=0} = \left( \frac{\pi}{2} \right)^m \cos \left( \frac{m\pi}{2} \right)
\]

and we obtain

(5.5)

\[
\gamma_{n-1} = 2\sum_{k=0}^{n} \binom{n}{k} \Gamma^{(n-k)}(1) \sum_{j=0}^{k} \binom{k}{j} (-1)^{k-j} \log^{k-j} (2\pi) \sum_{l=0}^{j} \binom{j}{l} \left( \frac{\pi}{2} \right)^{j-l} \cos \left( \frac{j-l\pi}{2} \right) \zeta^{(l)}(0)
\]

This extends the formula previously obtained by Apostol [1]. We note from (A.7) in Appendix A that the derivatives of the gamma function may be expressed in terms of the (exponential) complete Bell polynomials

\[
\Gamma^{(m)}(1) = Y_m(-\gamma, x_1, \ldots, x_{m-1})
\]

where \( x_p = (-1)^{p+1} p! \zeta(p+1) \).

\[\square\]

Using (3.3) we see that

\[
\frac{d}{ds} [(s-1)\zeta(s)] = -(s-1)\zeta(s) \frac{d}{ds} \left[ \sum_{k=1}^{\infty} (-1)^k \frac{\sigma_k}{k} (s-1)^k + \log \Gamma \left( 1 + \frac{s}{2} \right) + \log \pi^{s/2} \right]
\]

As above we see that

\[
Y_n(a_1 + b_1 + c_1, \ldots, a_n + b_n + c_n) = \sum_{k=0}^{n} \binom{n}{k} Y_{n-k}(a_1, \ldots, a_{n-k}) \sum_{j=0}^{k} \binom{k}{j} Y_{k-j}(b_1, \ldots, b_{k-j}) Y_j(c_1, \ldots, c_j)
\]

but it is not clear whether the above methodology will result in an easily manipulated formula.

6. A formula for the Stieltjes constants in terms of the (exponential) complete Bell polynomials containing the eta constants \( \eta_n \) as the arguments

We refer to (4.1)
\[ \frac{\zeta'(s)}{\zeta(s)} = -\frac{1}{s-1} - \sum_{k=0}^{\infty} \eta_k (s-1)^k \]

and it is easily seen that this may equivalently be written as

\[ \frac{d}{ds}[(s-1)\zeta(s)] = -(s-1)\zeta(s)\sum_{k=0}^{\infty} \eta_k (s-1)^k \]

As noted by Coffey [9] for differentiable functions \( f(s) \) and \( g(s) \) such that

\[ f'(s) = g(s)f(s) \]

then we have in terms of the (exponential) complete Bell polynomials

\[ f^{(n+1)}(s) = f(s)Y_{n+1} \left( g(s), g'(s), ..., g^{(n)}(s) \right) \]

In this particular case we have

\[ f(s) = (s-1)\zeta(s) \quad f(1) = 1 \quad \text{and} \quad g(s) = -\sum_{k=0}^{\infty} \eta_k (s-1)^k \quad g(1) = -\eta_0 = \gamma \]

\[ g^{(j)}(s) = -\sum_{k=0}^{\infty} \eta_k k(k-1) \cdots (k-j+1)(s-1)^{k-j} \]

\[ g^{(j)}(1) = -j!\eta_j \]

We recall from (2.10) that

\[ \frac{d^{n+1}}{ds^{n+1}}[(s-1)\zeta(s)] \bigg|_{s=1} = (-1)^n(n+1)\gamma_n \]

and we then obtain the relationship for \( n \geq 0 \)

\[ (6.1) \quad (-1)^n(n+1)\gamma_n = Y_{n+1}(\gamma, -1!\eta_1, ..., -n!\eta_n) \]

For example with \( n = 0 \) and using (3.24) \( Y_1(x) = x \) we recover \( \gamma_0 = Y_1(\gamma) = \gamma \). Similarly with \( n = 1 \) we have using \( Y_2(x_1, x_2) = x_1^2 + x_2 \)

\[ -2\gamma_1 = Y_2(\gamma, -1!\eta_1) = \gamma^2 - \eta_1 \]

which gives us
\[ \eta_1 = \gamma^2 + 2\gamma_1 \]

As a third example, using \( Y_3(x_1, x_2, x_3) = x_1^3 + 3x_1x_2 + x_3 \) we obtain

\[ 3\gamma_2 = Y_3(\gamma, -1!\eta_1, -2!\eta_2) = \gamma^3 - 3\gamma\eta_1 - 2\eta_2 \]
\[ 2\eta_2 = \gamma^3 - 3\gamma\eta_1 - 3\gamma_2 = \gamma^3 - 3\gamma(\gamma^2 + 2\gamma_1) - 3\gamma_2 \]

and therefore we have

\[ 3\gamma_2 = -2\gamma^3 - 6\gamma\gamma_1 - 2\eta_2 \]

We may write (6.1) as

\[ (-1)^n(n + 1)\gamma_n = Y_{n+1}(-0!\eta_0, -1!\eta_1, ..., -n!\eta_n) \]

Using the recurrence relation [28]

\[
Y_{n+1}(x_1, ..., x_{n+1}) = \sum_{k=0}^{n} \binom{n}{k} Y_{n-k}(x_1, ..., x_{n-k}) x_{k+1} = \sum_{k=0}^{n} \binom{n}{k} Y_k(x_1, ..., x_k) x_{n-k+1}
\]

\[= x_{n+1} + \sum_{k=1}^{n} \binom{n}{k} Y_k(x_1, ..., x_k) x_{n-k+1} \]

where \( x_r = -(r-1)!\eta_{r-1} \) We see that

\[ (-1)^n(n + 1)\gamma_n = -n!\eta_n + \sum_{k=1}^{n} \binom{n}{k} (-1)^k k\gamma_{k-1}(n-k)!\eta_{n-k} \]

and hence we recover (4.4).

From (1.7) we have

\[ \xi''(s) = -\xi(s) \sum_{k=1}^{\infty} \frac{\sigma_k}{k} k(s-1)^{k-1} \]

and in this case we designate \( g(s) = -\sum_{k=1}^{\infty} (-1)^k \frac{\sigma_k}{k} k(s-1)^{k-1} \)
\[ g^{(j)}(s) = -\sum_{k=0}^{\infty} (-1)^k \frac{\sigma}{k} k(k-1)\cdots(k-j)(s-1)^{k-j} \]

\[ g^{(j)}(1) = (-1)^j j! \sigma_{j+1} \]

\[ \xi^{(n)}(s) = \xi(s)Y_n\left(g(s), g'(s), \ldots, g^{(n-1)}(s)\right) \]

\[ \xi^{(n)}(1) = \frac{1}{2} Y_n\left(\sigma_1, -1! \sigma_2, \ldots, (-1)^{n-1}(n-1)! \sigma_n\right) \]

We have

\[ \xi^{(n+1)}(1) = \frac{1}{2} Y_n\left(\sigma_1, -1! \sigma_2, \ldots, (-1)^n n! \sigma_{n+1}\right) \]

and using

\[ Y_{n+1}(x_1, \ldots, x_{n+1}) = x_{n+1} + \sum_{k=1}^{n} \binom{n}{k} Y_k(x_1, \ldots, x_k) x_{n-k+1} \]

we obtain

\[ (6.2) \quad \xi^{(n+1)}(1) = \frac{1}{2} (-1)^n n! \sigma_{n+1} + \sum_{k=1}^{n} \binom{n}{k} (-1)^{n-k+1} \xi^{(k)}(1)(n-k+1)! \sigma_{n-k} \]

**Appendix A**

**A brief survey of the (exponential) complete Bell polynomials**

The (exponential) complete Bell polynomials may be defined by \( Y_0 = 1 \) and for \( n \geq 1 \)

\[ Y_n(x_1, \ldots, x_n) = \sum_{\pi(n)} \frac{n!}{k_1! k_2! \ldots k_n!} \left( \frac{x_1}{1!} \right)^{k_1} \left( \frac{x_2}{2!} \right)^{k_2} \cdots \left( \frac{x_n}{n!} \right)^{k_n} \]

where the sum is taken over all partitions \( \pi(n) \) of \( n \), i.e. over all sets of integers \( k_j \) such that

\[ k_1 + 2k_2 + 3k_3 + \ldots + nk_n = n \]

The complete Bell polynomials have integer coefficients and, by way of illustration, the first six are set out below [14, p.307]
\[(A.3)\]

\[Y_1(x_1) = x_1\]

\[Y_2(x_1, x_2) = x_1^2 + x_2\]

\[Y_3(x_1, x_2, x_3) = x_1^3 + 3x_1x_2 + x_3\]

\[Y_4(x_1, x_2, x_3, x_4) = x_1^4 + 6x_1^2x_2 + 4x_1x_2^2 + 3x_2^2 + x_4\]

\[Y_5(x_1, x_2, x_3, x_4, x_5) = x_1^5 + 10x_1^3x_2 + 10x_1^2x_3 + 15x_1x_2^2 + 5x_1x_2x_3 + 10x_2x_3 + x_5\]

\[Y_6(x_1, x_2, x_3, x_4, x_5, x_6) = x_1^6 + 6x_1x_5 + 15x_1x_2x_4 + 10x_2^3 + 15x_1^2x_4 + 15x_3 + 60x_1x_2x_3 + 20x_1^3x_3 + 45x_1^2x_2^2 + 15x_1^4x_1 + x_6\]

The complete Bell polynomials are also given by the exponential generating function in Comtet’s book \([14, p.134]\)

\[(A.4)\]

\[\exp\left(\sum_{j=1}^{\infty} x_j \frac{t^j}{j!}\right) = 1 + \sum_{n=1}^{\infty} Y_n(x_1, \ldots, x_n) \frac{t^n}{n!} = \sum_{n=0}^{\infty} Y_n(x_1, \ldots, x_n) \frac{t^n}{n!}\]

Let us now consider a function \(f(x)\) which has a Taylor series expansion around \(x\): we have

\[e^{f(x+t)} = \exp\left(\sum_{j=0}^{\infty} f^{(j)}(x) \frac{t^j}{j!}\right) = e^{f(x)} \exp\left(\sum_{j=1}^{\infty} f^{(j)}(x) \frac{t^j}{j!}\right)\]

\[= e^{f(x)} \left[1 + \sum_{n=1}^{\infty} Y_n\left(f^{(1)}(x), f^{(2)}(x), \ldots, f^{(n)}(x)\right) \frac{t^n}{n!}\right]\]

We see that

\[\frac{d^m}{dx^m} e^{f(x)} = \frac{\partial^m}{\partial x^m} e^{f(x+t)} \bigg|_{t=0} = \frac{\partial^m}{\partial t^m} e^{f(x+t)} \bigg|_{t=0}\]

and we therefore obtain (as noted by Kölblig \([22]\) and Coffey \([9]\))

\[(A.5)\]

\[\frac{d^m}{dx^m} e^{f(x)} = e^{f(x)} Y_m\left(f^{(1)}(x), f^{(2)}(x), \ldots, f^{(m)}(x)\right)\]

Differentiating \((A.5)\) we see that
\[ \frac{d}{dx} Y_m \left( f^{(1)}(x), f^{(2)}(x), \ldots, f^{(m)}(x) \right) = Y_{m+1} \left( f^{(1)}(x), f^{(2)}(x), \ldots, f^{(m+1)}(x) \right) - f^{(1)}(x) Y_m \left( f^{(1)}(x), f^{(2)}(x), \ldots, f^{(m)}(x) \right) \]

As an example of (A.5), letting \( f(x) = \log \Gamma(x) \) we obtain

\[ (A.6) \quad \frac{d^m}{dx^m} e^{\log \Gamma(x)} = \Gamma^{(m)}(x) = \Gamma(x) Y_m \left( \psi(x), \psi^{(1)}(x), \ldots, \psi^{(m-1)}(x) \right) \]

\[ = \int_0^\infty t^{x-1} e^{-t} \log^m t \, dt \]

and since [30, p.22]

\[ \psi^{(p)}(x) = (-1)^{p+1} p! \zeta(p+1, x) \]

we may express \( \Gamma^{(m)}(x) \) in terms of \( \psi(x) \) and the Hurwitz zeta functions. In particular, Coffey [9] notes that

\[ (A.7) \quad \Gamma^{(m)}(1) = Y_m(-\gamma, x_1, \ldots, x_{m-1}) \]

where \( x_p = (-1)^{p+1} p! \zeta(p+1) \).

Values of \( \Gamma^{(m)}(1) \) are reported in [30, p.265] for \( m \leq 10 \) and the first three are

\[ \Gamma^{(1)}(1) = -\gamma \]

\[ \Gamma^{(2)}(1) = \zeta(2) + \gamma^2 \]

\[ \Gamma^{(3)}(1) = -[2\zeta(3) + 3\gamma \zeta(2) + \gamma^3] \]

The general form is

\[ \Gamma^{(m)}(1) = (-1)^m \sum_{j=1}^m \varepsilon_{mj} \]

where \( \varepsilon_{mj} \) are positive constants. □
The following is extracted from an interesting series of papers written by Snowden [29, p.68].

Let us consider the function \( f(x) \) with the following Maclaurin expansion

\[
\log f(x) = b_0 + \sum_{n=1}^{\infty} \frac{b_n}{n} x^n
\]

and we wish to determine the coefficients \( a_n \) such that

\[
f(x) = \sum_{n=0}^{\infty} a_n x^n
\]

By differentiating (A.8) and multiplying the two power series, we get

\[
n a_n = \sum_{k=1}^{n} b_k a_{n-k}
\]

Upon examination of this recurrence relation Snowden reports it is easy to see that

\[
n! a_n = a_0 [b_1, -b_2, b_3, \ldots, (-1)^{n+1} b_n]
\]

where the symbol \([c_1, c_2, c_3, \ldots, c_n]\) is defined as the \(n \times n\) determinant

\[
\begin{vmatrix}
c_1 & c_2 & c_3 & c_4 & \cdots & c_n \\
(n-1) & c_1 & c_2 & c_3 & \cdots & c_{n-1} \\
0 & (n-2) & c_1 & c_2 & \cdots & c_{n-2} \\
0 & 0 & (n-3) & c_1 & \cdots & c_{n-3} \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & 0 & 0 & 1 \end{vmatrix}
\]

Since \( \log f(0) = \log a_0 = b_0 \) we have

\[
f(x) = e^{b_0} \left[ 1 + \sum_{n=1}^{\infty} [b_1, -b_2, b_3, \ldots, (-1)^{n+1} b_n] \frac{x^n}{n!} \right]
\]

Multiplying (A.9) by \( \alpha \) it is easily seen that (after correcting the misprint in Snowden’s paper)
\[(A.13)\]
\[f^\alpha(x) = e^{ab_0} \left[ 1 + \sum_{n=1}^{\infty} \left[ a b_1, -a b_2, a b_3, \ldots, (-1)^{n+1} a b_n \frac{x^n}{n!} \right] \right]\]

and, in particular, with \( \alpha = -1 \) we obtain

\[(A.14)\]
\[
\frac{1}{f(x)} = e^{-b_0} \left[ 1 + \sum_{n=1}^{\infty} \left[ -b_1, b_2, -b_3, \ldots, (-1)^n b_n \frac{x^n}{n!} \right] \right]
\]

Differentiating (A.13) with respect to \( \alpha \) would give us an expression for \( f^\alpha(x) \log f(x) \).

Differentiating (A.8) gives us

\[
f'(x) = f(x) \sum_{k=1}^{n} b_k x^{k-1} = f(x)g(x)
\]

and hence

\[
f^{(n)}(x) = f(x)Y_n\left(g(x), \ldots, g^{(n-1)}(x)\right)
\]

Alternatively, differentiating (A.9) results in

\[
f^{(n)}(x) = \sum_{k=0}^{n} a_k k(k-1) \cdots (k-n+1)x^{k-n}
\]

and letting \( x = 0 \) gives us

\[(A.15)\]
\[n!a_n = e^{b_0}Y_n\left(b_1, 1!b_2, \ldots, (n-1)!b_n\right)\]

From (A.11) we then see that

\[(A.16)\]
\[Y_n\left(b_1, 1!b_2, \ldots, (n-1)!b_n\right) = [b_1, -b_2, b_3, \ldots, (-1)^{n+1} b_n]\]

and more generally

\[(A.17)\]
\[Y_n\left(x_1, \ldots, x_n\right) = \begin{bmatrix} \frac{x_1}{0!}, \frac{x_2}{1!}, \ldots, \frac{x_n}{(n-1)!} \end{bmatrix} (-1)^{n+1} \frac{x_n}{(n-1)!}
\]

We now write (A.13) as

\[
f^\alpha(x) = e^{ab_0} \left[ 1 + \sum_{n=1}^{\infty} D_n(\alpha) \frac{x^n}{n!} \right]
\]
where for convenience we have designated \( D_n(\alpha) \) by

\[
D_n(\alpha) = [\alpha b_1, -\alpha b_2, \alpha b_3, \ldots, (-1)^{n+1} \alpha b_n]
\]

Letting \( \alpha = 0 \) we see that

\[
1 = 1 + \sum_{n=1}^{\infty} D_n(0) \frac{x^n}{n!}
\]

and therefore we have

\[
D_n(0) = 0
\]

which is only to be expected since all of the elements of the top row of the defining determinant are equal to zero.

Differentiation with respect to \( \alpha \) results in

\[
f^{\alpha}(x) \log f(x) = e^{\alpha b_0} \sum_{n=1}^{\infty} D'_n(\alpha) \frac{x^n}{n!} + b_0 e^{\alpha b_0} \left[ 1 + \sum_{n=1}^{\infty} D_n(\alpha) \frac{x^n}{n!} \right]
\]

Letting \( \alpha = 0 \) gives us

\[
\log f(x) = \sum_{n=1}^{\infty} D'_n(0) \frac{x^n}{n!} + b_0 \left[ 1 + \sum_{n=1}^{\infty} D_n(0) \frac{x^n}{n!} \right]
\]

\[
= b_0 + \sum_{n=1}^{\infty} D'_n(0) \frac{x^n}{n!}
\]

We started out with (A.8)

\[
\log f(x) = b_0 + \sum_{n=1}^{\infty} \frac{b_n}{n} x^n
\]

and, hence equating coefficients in the two power series, we obtain

\[
D'_n(0) = (n-1)! b_n
\]

We have
\[
\frac{d^m}{dx^m} f^\alpha(x) = e^{\alpha h} \sum_{n=1}^{\infty} D_n(\alpha) \frac{n(n-1) \cdots (n-m+1)x^{n-m}}{n!}
\]

\[
\frac{d}{d\alpha} \frac{d^m}{dx^m} f^\alpha(x) = e^{\alpha h} \sum_{n=1}^{\infty} D'_n(\alpha) \frac{n(n-1) \cdots (n-m+1)x^{n-m}}{n!}
\]

\[
+ b_0 e^{\alpha h} \sum_{n=1}^{\infty} D_n(\alpha) \frac{n(n-1) \cdots (n-m+1)x^{n-m}}{n!}
\]

and hence

\[
\frac{d}{d\alpha} \frac{d^m}{dx^m} f^\alpha(x) \bigg|_{x=1, \alpha=0} = \sum_{n=1}^{\infty} D'_n(0) \frac{n(n-1) \cdots (n-m+1)}{n!}
\]

We then have

\[
\frac{d}{d\alpha} \frac{d^m}{dx^m} f^\alpha(x) \bigg|_{x=1, \alpha=0} = \sum_{n=1}^{\infty} \frac{b_n}{n} n(n-1) \cdots (n-m+1)
\]

which is equivalent to

\[
\frac{d^m}{ds^m} \log f(x) \bigg|_{x=1} = \sum_{n=1}^{\infty} \frac{b_n}{n} n(n-1) \cdots (n-m+1)
\]

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