A note on the fuzzy sphere area spectrum, black-hole luminosity and the quantum nature of spacetime

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Abstract – Noncommutative corrections to the classical expression for the fuzzy sphere area are found out through the asymptotic expansion for its heat kernel trace. As an important consequence, some quantum gravity deviations in the luminosity of black holes must appear. We calculate these deviations for a static, spherically symmetric, black hole with a horizon modeled by a fuzzy sphere. The results obtained could be verified through the radiation of black holes formed in the Large Hadron Collider (LHC).

Introduction. – The search for the understanding of the quantum nature of spacetime is one of the most difficult challenges physics has faced. Several theories have been applied for reach this mission. Even though we do not have yet a complete theory of quantum gravity, some results obtained by approaches like string theory and loop quantum gravity can give us some clues about the spacetime behavior in the Planck regime. Some results favor a quantum nature for the spacetime, related with a noncommutative behavior of the geometry in the Planck scale \cite{1,2}

In fact, the use of noncommutative spaces in order to investigate the quantum behavior of physical systems is not new in physics, being the quantum-mechanical phase space a prime example. Actually, Heisenberg was the first to suggest extending noncommutativity to the coordinates of the physical space as a possible way of removing the infinite quantities appearing in field theories in his letters to Ehrenfest \cite{3} and Peierls \cite{4}. At that time Heisenberg could not formulate this idea mathematically. The first papers on the subject were published in 1947 by Snyder \cite{5} and Yang \cite{6}, regaining attention over the last decades \cite{7–10} due to its appearance in some prominent quantum gravity frameworks, like string theory \cite{1,11–14} and loop quantum gravity \cite{15–17}.

The motivation behind such spaces is that they have a potential for replacing the classical geometric description of spacetime by incorporating quantum fluctuations as follows: a modified commutation relation for the space-time coordinates, $[x^\mu, x^\nu] = i\theta^{\mu\nu}$, incorporates a spacetime uncertainty relation \cite{18},

$$\Delta x^\mu \cdot \Delta x^\nu \geq \frac{1}{2} |\theta^{\mu\nu}|,$$

which spoils the intuitive notion of point at distances shorter than $\sqrt{\theta}$. This is suspected to happen if we probe distances of the order of Planck’s length $l_P \sim 10^{-35}$ m, when gravitational quantum effects are not negligible. Besides, the semiclassical limit of quantum gravity theories is expected to be described by a field theory on some noncommutative spacetime \cite{19}.

Among the realizations of noncommutative geometry there are the so-called fuzzy spaces. A fuzzy space can be roughly defined as a quantum representation of a manifold. Its construction is based on the fact that coadjoint orbits of Lie groups are symplectic manifolds, and hence can be
quantized (under certain conditions). The quantization procedure introduces a parameter analogue to Planck’s constant, being the classical manifold limit recovered when this parameter goes to zero. In the case where the Lie group is compact and semi-simple, its quantum version is compact and identified as a finite-dimensional matrix algebra on which the group acts. Moreover, it preserves the symmetries of its classical version. Then a fuzzy space, realized as a matrix algebra, provides a concrete method to model the spacetime noncommutativity. We refer to ref. [20] for a broad introduction to the subject.

A possible way to verify the noncommutative properties of the spacetime is to investigate quantum gravity corrections nearby a black hole. Actually, the use of noncommutative geometry to address black-hole physics is mainly motivated by the hope that black holes might play a major role in our attempts to shed some light on the quantum nature of gravity such as the role played by atoms in the early development of quantum mechanics. In this way, noncommutativity has been used to investigate possible quantum gravity effects in black-hole physics, too [21]. It includes some fuzzy spaces approaches, where fuzzy manifolds have been successfully used as a way to quantize the event horizon of a black hole. In these approaches the authors argued that a solution for the information loss paradox can be obtained using the Hopf algebra structure of the noncommutative manifold to model a topology change in the black hole [22–24].

Due to the possibility of the formation of micro black holes in the Large Hadron Collider (LHC), it is interesting to investigate how quantum gravity effects could appear in the radiation emitted by these black holes. Following the idea that noncommutative geometry can be used to model these effects, in this work, we shall address how fuzzy spaces can be used to investigate quantum gravity corrections in the black-hole luminosity, modeling the black-hole event horizon as a fuzzy sphere.

A crucial point is to know the quantum geometrical features of the manifold which have been used to model the black-hole event horizon. In this way, several approaches in order to investigate the correct spectrum of the fuzzy sphere geometrical quantities have been proposed [23–27]. To address how the quantum gravity fluctuations in the black-hole horizon will modify the black-hole luminosity, the method we will use in the present work will consist in investigating the fuzzy sphere area spectrum through the asymptotic expansion for the heat kernel trace.

The major motivation to use this method is related to the fact that, since a fuzzy space is defined in terms of the spectral triple, its geometrical properties must be recovered algebraically [28]. Actually, following the standard procedure for the classical sphere, its geometry can be determined by the knowledge of the Laplacian spectrum. In this way, our first goal here is to extrapolate such approach to a fuzzy space, in order to investigate its effects on the classical area formula. As we will see in the following sections, such extrapolation leads to an exponential correction to the sphere area, which vanishes either in the commutative limit or in the low-energy limit.

This paper is organized as follows. The next section is devoted to organize the main concepts used in our discussions, mainly the relation between the heat kernel trace and the geometrical area. The third section contains the actual area spectrum derivation and, in the fourth section, we employ the derived area spectrum to discuss modifications in the luminosity of a Schwarzschild black hole. After this, we make the final remarks.

For the physical quantities, we employ dimensionless units such that $\hbar = c = G = k_B = 1$. With these definitions all quantities are written in terms of the Planck mass $m_P = 2.176 \times 10^{-8}$ kg, length $l_P = 1.616 \times 10^{-35}$ m, time $t_P = 5.381 \times 10^{-44}$ s, energy $E_P = 1.956 \times 10^{9}$ J, etc. As an example, the luminosity of the Sun can be written as $L_\odot = 1.1 \times 10^{-26} t_P/E_P$.

**Preliminaries.** In the present section we review some concepts of the classical geometry of Riemannian manifolds which will help us to calculate the noncommutative corrected area of a fuzzy sphere. Throughout this section, let $\mathcal{M}$ denote a smooth (orientable) Riemannian manifold with metric $g$ of dimension $n = \dim(\mathcal{M})$.

**Heat kernel.** By definition, the heat equation on $\mathcal{M}$ is defined as

$$\frac{\partial u}{\partial t} + \Delta_2 u = 0, \quad u : [0, \infty) \times \mathcal{M} \rightarrow \mathbb{R}, \quad (2)$$

where $\Delta_2$ is the Laplace-Beltrami operator with respect to the second variable (the spatial variable over the manifold $\mathcal{M}$). The function $u(t, x)$ can represent the temperature at the point $x \in \mathcal{M}$ at time $t$.

Given an initial distribution $u(t = 0, x) = f(x)$ for $x \in \mathcal{M}$ and a Dirichlet condition at the boundary $\partial \mathcal{M}$ of $\mathcal{M}$, there is a fundamental solution for (2), called the heat kernel, which is a function $K : (0, \infty) \times \mathcal{M} \times \mathcal{M} \rightarrow \mathcal{M}$ satisfying the following conditions:

1) $K(t, x, y) = C^1$ in $t$ and $C^2$ in $x$ and $y$;
2) $K$ solves the heat equation (2),

$$\frac{\partial K}{\partial t} + \Delta_y K = 0, \quad (3)$$

for $t \in (0, \mathbb{R})$;
3) $K$ satisfy the condition $K(t, x, y) = 0 \iff x \in \partial \mathcal{M}$;
4) the equality

$$\lim_{t \to 0^+} \int_{\mathcal{M}} dV_y(\mathcal{M}) K(t, x, y)f(y) = f(x), \quad (4)$$

where the subscript in the integration measure $dV_y$ denotes the variable of integration, holds uniformly for every function $f$ continuous on $\mathcal{M}$ and vanishing on $\partial \mathcal{M}$.
Heat trace. Once we have defined properly the Laplace operator $\Delta$ on $\mathcal{M}$, its spectrum $\{\lambda_n\}_{n \in \mathbb{N}}$ for an arbitrary index set $I$ together with its eigenfunctions $f_n$, $\Delta f_n = \lambda_n f_n$, determines the unique heat kernel as follows:

$$K(t, x, y) = \sum_{n \in I} e^{-\lambda_n t} f_n(x) f_n(y),$$

with absolute and uniform convergence for $t > 0$.

The heat trace of the heat kernel is defined by

$$\theta(t) = \int_{\mathcal{M}} dV_x K(t, x, x),$$

which can be written in a simpler form using decomposition (5):

$$\theta(t) = \sum_{n \in I} e^{-\lambda_n t},$$

where the summation is made over all eigenvalues, taking into account its multiplicity.

Asymptotic expansion and geometrical quantities. The heat trace (7) has an asymptotic expansion in the region $t \to 0^+$, as shown in [29];

$$\theta(t) = (4\pi t)^{-n/2} \sum_{i=0}^{N} c_i t^{i/2} + \mathcal{O}(t^{(N+1)/2}),$$

where the coefficients $c_i$ are called Seeley-de Witt coefficients, and it depends on the geometry of the manifold $\mathcal{M}$. If $\mathcal{M}$ is compact and has a compact $(n - 1)$-dimensional boundary $\mathcal{B} = \partial \mathcal{M}$, the first few coefficients of (8) are explicitly shown to encode the following geometric information [30]:

$$c_0 = \text{vol}(\mathcal{M}), \quad c_1 = -\frac{\sqrt{\pi}}{2} \text{vol}(\mathcal{B}), \quad c_2 = \frac{1}{3} \int_{\mathcal{M}} S - \frac{1}{6} \int_{\mathcal{B}} J,$$

where $S$ is the scalar curvature of $\mathcal{M}$ and $J$ is the mean curvature at its boundary $\partial \mathcal{M}$.

The fuzzy sphere. The triple $(\mathcal{H}, A, \Delta)$, where $\mathcal{H} = L^2(\mathcal{M})$ is the Hilbert space of square-integrable functions on $\mathcal{M}$, $\Delta$ is the Laplace-Beltrami operator and $A = \mathcal{C}^\infty(\mathcal{M})$ is the algebra of smooth (bounded) functions on $\mathcal{M}$, was shown to contain all geometrical information about the manifold; that is, the geometry of $\mathcal{M}$ can be formulated in terms of $A$ [31].

In a similar way, the quantum (fuzzy) version $\mathcal{M}_F$ of $\mathcal{M}$ can be defined by a sequence of triples $\mathcal{M}_F := \{(\mathcal{H}_N, A_N, \Delta_N)\}_{N \in \mathbb{N}}$, parametrized by a natural number $N$, where $\mathcal{H}_N = \mathcal{M}_N(\mathbb{C})$ is the algebra of $N \times N$ matrices with complex entries, $\mathcal{H}_N = \mathbb{C}^{N^2}$ is the Hilbert space over which $A_N$ acts and $\Delta_N$ is a representation of the Laplace operator on the matrix space. The Hilbert space inner product is

$$\langle \Phi, \Psi \rangle = \frac{1}{N} \text{Tr}(\Phi^\dagger \Psi).$$

The fuzzy sphere $S_F^2$ [32] is a very simple example of a fuzzy space, which appears as vacuum solutions in Euclidean gravity [33,34]. A simple recipe to obtain $A_N$ is to replace the global coordinates $x_1, x_2, x_3$ of the sphere $S^2$ of radius $R$, which satisfy $x_1^2 + x_2^2 + x_3^2 = R^2$, by a set of operators $X_1, X_2, X_3$ satisfying

$$\sum_{a=1}^{3} (X_a)^2 = R^2 \text{Id},$$

where $\text{Id}$ is the identity operator, whose representation is proportional to the unitary irreducible representation $j \in \mathbb{N}/2$ of the rotation generators $J_a$ of the $SU(2)$ algebra,

$$[J_a, J_b] = i \epsilon_{abc} J_c, \quad \sum_{a=1}^{3} (J_a)^2 = j(j + 1) \text{Id}.$$

From this we can see that $X_a$ are represented by $(2j + 1) \times (2j + 1)$ Hermitian matrices satisfying

$$X_a = \lambda J_a, \quad [X_a, X_b] = i \lambda \epsilon_{abc} X_c,$$

with

$$\lambda = \frac{R}{\sqrt{j(j+1)}},$$

and that its components can be decomposed into $2j + 1$ irreducible representations of $SU(2)$ algebra with angular momentum $\ell = 0, 1, \ldots, 2j$. Thus, defining the index in the quantum (fuzzy) version $\mathcal{M}_F$ of $\mathcal{M}$ (defined above eq. (10)) as $N = 2j + 1$, eq. (14) is rewritten as

$$\lambda = \frac{R}{\sqrt{N^2 - 1}}.$$ 

The parameter $\lambda$ has dimension of length, and plays a role analogue to Planck’s constant in quantum mechanics, as a quantization parameter. From (13) we can see that in the limit $\lambda \to 0$ ($N \to \infty$) the matrices $X_a$ become commutative, and we recover the standard commutative sphere.

Laplace operator. There are several choices for the Laplace operator over a fuzzy space, once one is given a covariant derivative or a metric. However, as pointed out in ref. [32], all the choices agree in the commutative limit.

For the fuzzy sphere we follow the choice used in refs. [35–38], which is based on its spherical symmetry. In this case the derivatives are replaced by the adjoint action of the symmetry generators, and we define the Laplace operator $\Delta_N$ as

$$\Delta_N \Psi = \frac{1}{R^2} \sum_{a=1}^{3} \left[ J_a(N), \left[ J_a(N), \Psi \right] \right];$$

then, solving the eigenvalue equation for (16) we can find the same spectrum of the usual commutative Laplacian, except by a cutoff at $\ell = N - 1$; that is,

$$\lambda_\ell = \frac{\ell(\ell + 1)}{R^2}, \quad \ell = 0, 1, \ldots, N - 1.$$ 

This spectrum coincides up to order $N - 1$ with the usual continuum counterpart, when the Laplace operator acts on the space of functions on a sphere.
Fuzzy sphere area spectrum. – In this section, we show how the heat kernel trace can be used to evaluate the area of a fuzzy sphere, extrapolating the classical results in the quantum regime. Such extrapolation will be used as a physical artifact to find signatures of the noncommutativity in the classical realm, revealing some features of the spacetime in the quantum regime which must be described by a full quantum gravity theory.

To begin with, we have that, since the Laplace operator spectrum is given by (17), the heat trace (7) for the fuzzy sphere can be written as the finite sum

$$\theta(t) = \sum_{0 \leq \ell < N} (2\ell + 1)e^{-\lambda_\ell t},$$

(18)

where the factor $(2\ell + 1)$ takes into account the multiplicity of each eigenvalue $\lambda_\ell$.

Since $\ell$ is a pure number, the eigenvalues (17) have dimension of $[\text{length}]^{-2}$, and hence the expansion parameter $t$ has dimension of $[\text{energy}]^{-2}$ in natural units; defining

$$\Lambda = \frac{1}{\sqrt{t}}$$

(19)

we have that $\Lambda$ has dimension of energy. We interpret such parameter as the energy scale where we probe the fuzziness of the geometry.

With such interpretation, comparing the heat trace with the spectral action of Connes, noncommutative geometry leads to a reasonable cutoff for $\Lambda$ of the order of the Planck scale: $\Lambda \sim MP$. Thus, conversely to the commutative case, the asymptotic expansion (8) must be taken not at $t \to 0$ but at a finite value $t \to t_\ast = MP^{-2}$. Hence, if we define

$$\alpha(s) = 4\pi t \theta(t) = 4\pi t \sum_{0 \leq k < N} (2k + 1)e^{-\lambda_k t}.$$  

(20)

The extrapolation of the classical behavior of the heat trace shown in eqs. (8) and (9) relates the area $A$ to the coefficient of the power $(t - t_\ast)^{i\nu}$ of the power series expansion of (20).

In order to evaluate such coefficient, we employ the Euler-Maclaurin summation formula

$$\sum_{a \leq k < b} f(k) = \int_a^b dx f(x) + \sum_{1 \leq k \leq M} \frac{b_k}{k!} f^{(k-1)}(x)\bigg|_a^b + R_M,$$

(21)

where $a$ and $b$ are integers such that $a \leq b$, $b_k$ denote the Bernoulli numbers $b_0 = 1$, $b_1 = -1/2$, $b_2 = 1/6$, ..., and $R_M$ denotes the remainder when we truncate the series at order $M \geq 1$:

$$R_M = \frac{(-1)^{M+1}}{M!} \int_a^b dx B_M(\{x\}) f^{(M)}(x),$$

(22)

where $B_M(\{x\})$ is the Bernoulli polynomial and $\{x\}$ is a shorthand for the fractional part of $x$. Then, taking

$$f(x) = 4\pi t (2x + 1) \exp\left(-\frac{x(x+1)}{R^2}\right).$$

(23)

we can verify that due to the factor $4\pi t$, all derivatives $f^{(i\nu)}(x)$ will contribute to terms involving positive powers of $t$, and hence the only term which contributes to the power $t^{\nu}$ of the Taylor expansion is the integral on the right-hand side of (21); thus, the area spectrum $A_N^F$ for the fuzzy sphere is simply the first term of (21)

$$A_N^F = 4\pi R^2 \left(1 - e^{-N(N+1)/M^2R^2}\right).$$

(24)

To give us a better insight of this formula, we define a normalized area spectrum, by scaling (24) with the classical area:

$$\tilde{A}_N = \frac{A_N^F}{4\pi R^2} = 1 - e^{-N(N+1)/\epsilon^2}, \quad \epsilon = \Lambda R.$$  

(25)

The behavior for the spectrum (25) is shown in fig. 1. From eq. (24) we found out that the exponential correction quickly vanishes in the commutative limit $N \to \infty$, and in the limit $\Lambda \to 0$, where the fundamental scale goes to zero, agreeing with the classical expansion $t \to 0$ for the heat trace.

The magnitude order of the noncommutativity parameter $N$ must be emphasized; being $M_P \sim 1.22 \times 10^{19}$ GeV, in order to observe a correction for the area of order of $1\%$ we should have

$$\frac{N(N+1)}{R^2} \sim 6.87 \times 10^{56} \text{ eV}^2,$$

(26)

such that, using the estimated Schwarzschild radius $R \sim 10^{20} \text{ eV}^{-1}$ for the M87 black hole [39], we conclude that $N = 10^{56}$ still corresponds to a strong quantum regime. This partly agrees with the interpretation for $N$ as the number of quanta of area $N \sim A/\ell_P$ in entropic gravity [40].

Effects on the black-hole thermodynamics and luminosity. – The quantum nature of spacetime can be revealed by a black hole. In this way, some authors have argued that the black-hole event horizon can
be modeled by a noncommutative manifold like a fuzzy sphere [21–24]. If this is true, some modifications in the description of the black-hole evaporation process must appear as a consequence of the fuzzy sphere spectrum (24).

In fact, after a quick calculation, one can find that the temperature of a Schwarzschild black hole whose event horizon area is given by (24) possesses a temperature

\[ T^F_N = \frac{T_{BH}}{\sqrt{1 - e^{-\frac{N(N+1)}{\epsilon^2}}}} \]  

and an entropy given by

\[ S^F_N = S_{BH} \left(1 - e^{-\frac{N(N+1)}{\epsilon^2}}\right), \]

where \( T_{BH} \) and \( S_{BH} \) are the classical Bekenstein-Hawking temperature and entropy, respectively.

It would be very welcoming, given the possible formation of micro black holes in the LHC, to investigate the consequences that the area spectrum (24) can bring to the black-hole luminosity. We have that Page’s semiclassical results points to a 1/M^2-dependence for black-hole luminosity [41]. These results are also found in the equally spaced spectrum proposed by Bekenstein [42]. However we can see in eq. (24) that quantum effects can change such behavior.

In order to investigate black-hole luminosity, we will use the same method due to Mäkelä [43]. In this way, we will assume that transitions where \( \delta \omega \) is small are preferred. It is due to the fact that black-hole radiation involves the creation of virtual particle-antiparticle pairs near the black-hole horizon, and “swallowing” one member of some of the pairs. The greater the energy \( \delta \omega \) of a particle of any virtual pair, the smaller the probability that the pair will live long enough so that the black hole will be able to swallow one of the virtual particles.

In this way, in terms of the minimum energy \( \delta \omega_0^F \), black-hole luminosity will be given by

\[ L^F \sim \frac{\delta \omega_0^F}{\tau_F} = (\delta \omega_0^F)^2, \]

where we have used the uncertainty relation \( \tau_F \sim 1/\delta \omega_0^F \) and the superscript “F” indicates that the quantities are fuzzy, or quantum corrected.

To find out the value of the energy \( \delta \omega_0^F \) of a particle emitted by a Schwarzschild black hole, we have \( A_{BH} = 16\pi(M^F)^2 \), in a way that \( \delta M^F = \delta A^F / 32\pi M^F \) and

\[ \delta \omega_0^F = \frac{(A_N - A_{N-1})}{8\sqrt{\pi A_N}}. \]

Hence, black-hole luminosity will be given by

\[ L^F \sim (\delta \omega_0^F)^2 = \frac{(A_N - A_{N-1})^2}{64\pi A_N}, \]

and we define a normalized luminosity in an analogous way as the area: \( \tilde{L} = L/4\pi R^2 = (A_N - A_{N-1})^2/64\pi A_N \).

It is interesting to express the black-hole luminosity in terms of black-hole classical mass \( M \)

\[ L \sim \delta \omega_0^2 = \frac{1}{1024\pi^2 M^2} \gamma, \]

where \( \gamma = (4\pi R^2)^\frac{1}{2}(A_N - A_{N-1})^2. \) As we can see, as the black hole shrinks, the factor \( \gamma \) deviates increasingly from a constant value, in a way that the black-hole luminosity also deviates increasingly from Page’s \( M^{-2} \) semiclassical result. Thus, the factor \( \gamma \) becomes important as the black hole becomes smaller in order to confirm the quantum nature of the spacetime.

Black-hole normalized luminosity is shown in fig. 2. Since the luminosity increases as the fundamental energy scale \( \epsilon \) increases, noncommutative corrections can also be observed if the fundamental energy scale could be probed, like in large extra-dimensional models where particle collision at LHC can produce micro black holes [44,45].

**Conclusions.** In this paper we have performed an investigation on the area spectrum of a 2-sphere, with a correction due to quantum fluctuations of space through the asymptotic expansion for the heat kernel trace. This investigation can shed some light on the nature of space-time in the Planck scale which can be reflected in quantum gravity corrections on the luminosity of the black holes, and verified in the LHC.

We can verify that, in our approach, in order to observe a correction for the area of order of 1% we should have \( \frac{N(N+1)}{\epsilon^2} \sim 6.87 \times 10^{46} \text{eV}^2 \). In this way, using the estimated Schwarzschild radius \( R \sim 10^{20} \text{eV}^{-1} \) for the M87 black hole [39] we conclude that \( N = 10^{50} \) still corresponds to a strong quantum regime. It indicates that \( N \) can be interpreted as the number of quanta area \( N \sim A/\ell_P \) as occurs in entropic gravity approach [40].

The found out quantum gravity corrections become stronger as we get close to the fundamental cutoff scale, and since such corrections decrease the area, we should expect that our approach breaks close to the Planck scale.
where a full theory of quantum gravity will be necessary. In spite of this, the results obtained in this paper can shed some light on the quantum nature of the spacetime in the Planck scale. Other ingredients, like extra dimensions, can be added later in order to provide more precise experimental signatures of noncommutativity.

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