GENERALIZED COMPLEX MONGE-AMPERE TYPE EQUATIONS
ON CLOSED HERMITIAN MANIFOLDS

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Abstract. We study generalized complex Monge-Ampère type equations on closed Hermitian manifolds. We derive a priori estimates and then prove the existence of admissible solutions.

1. Introduction

Let \((M, \omega)\) be a compact Hermitian manifold of complex dimension \(n \geq 2\) and \(\chi\) another Hermitian metric on \(M\). In local coordinate charts, we shall write \(\omega = \sqrt{-1} \sum_{i,j} g_{ij} dz^i \wedge d\bar{z}^j\) and \(\chi = \sqrt{-1} \sum_{i,j} \chi_{ij} dz^i \wedge d\bar{z}^j\). For a real \(C^2(M)\) function \(u\), we will use the notation \(\chi_u = \chi + \sqrt{-1} \partial \bar{\partial} u\).

For a smooth positive real function \(\psi\) on \(M\), we are concerned with the generalized complex Monge-Ampère type equation

\[
\begin{cases}
\chi^n u = \psi \sum_{\alpha=1}^n c_\alpha \chi_u^{n-\alpha} \wedge \omega^\alpha,
\chi_u > 0,
\sup_M u = 0,
\end{cases}
\]

where \(c_\alpha\)'s are nonnegative real constants, and \(\sum_{\alpha=1}^n c_\alpha > 0\). Following [14, 7, 10], we assume that there is a Hermitian metric \(\chi' \in [\chi]\) satisfying

\[
n \chi' \wedge \chi' = \psi \sum_{\alpha=1}^{n-1} c_\alpha (n-\alpha) \chi_u^{n-\alpha-1} \wedge \omega^\alpha.
\]

By an appropriate approximation as in [15], a smooth \(\chi'\) is available. Without loss of generality, we can assume \(\chi = \chi'\) throughout this paper. If \(\chi\) and \(\omega\) are both Kähler and \(\psi\) is a constant, \(\psi\) is uniquely determined by

\[
\psi = c := \frac{\int_M \chi^n}{\sum_{\alpha=1}^n \int_M \chi_u^{n-\alpha} \wedge \omega^\alpha}.
\]

The generalized complex Monge-Ampère type equation is an extension of Donaldson’s equation [5]. In fact, Donaldson’s equation corresponds to the case that \(c_2 = \cdots = c_n = 0\) and \(\psi\) is constant,

\[
\chi_u = \psi \chi_u^{n-1} \wedge \omega.
\]

Chen also found the equation when he studied the lower bound of Mabuchi energy. The equation is studied by Chen [1, 2], Weinkove [23, 24], Song and Weinkove [14] by \(J\)-flow. Their results were extended to the complex Monge-Ampère type equations by Fang, Lai and Ma [7] also using parabolic flow method. More general cases were treated by Guan and the author [10], the author [15, 17] on...
Hermitian manifolds using continuity method. Later, the author [16] reproduced the results in [15] by parabolic flow method. In these works, the cone condition analogous to condition (1.2) is sufficient and necessary.

Fang, Lai and Ma [7] stated a conjecture for the solvability of (1.1) under condition (1.2). Admitting the famous work of Yau [25], Collins and Székelyhidi [4] proved the conjecture by continuity method starting from the complex Monge-Ampère equation. We adopt piecewise continuity method due to the author [15] to prove general results without the solvability results by Yau [25], Cherrier [3], Tosatti and Weinkove [19, 20].

For general Hermitian manifolds, we have the following result.

**Theorem 1.1.** Let \((M,\omega)\) be a closed Hermitian manifold of complex dimension \(n\) and \(\chi\) also a Hermitian metric. Suppose that

\[
n\chi^{n-1} > \psi \sum_{\alpha=1}^{n-1} c_{\alpha} (n-\alpha) \chi^{n-\alpha-1} \wedge \omega^\alpha.
\]

If there is a \(C^2\) function \(v\) satisfying \(\chi_v > 0\) and

\[
\chi_v^n \leq \psi \sum_{\alpha=1}^{n} c_{\alpha} \chi_v^{n-\alpha} \wedge \omega^\alpha,
\]

then there exist a unique admissible solution \(u\) and a unique constant \(b\) such that

\[
\left\{
\begin{align*}
\chi_u^n &= e^{b\psi} \sum_{\alpha=1}^{n} c_{\alpha} \chi_u^{n-\alpha} \wedge \omega^\alpha, \\
\chi_u &> 0, \quad \sup_M u = 0.
\end{align*}
\right.
\]

A corollary immediately follows from Theorem 1.1 and the fact that the inequality (1.2) does not contain the \(\omega^n\) term.

**Corollary 1.2.** Let \((M,\omega)\) be a closed Hermitian manifold of complex dimension \(n\) and \(\chi\) also a Hermitian metric. Suppose that

\[
n\chi^{n-1} > \psi \sum_{\alpha=1}^{n-1} c_{\alpha} (n-\alpha) \chi^{n-\alpha-1} \wedge \omega^\alpha.
\]

Then there is a constant \(N \geq 0\) such that if \(c_n \geq N\) there exist a unique admissible solution \(u\) and a unique constant satisfying the equation (1.5).

**Remark 1.3.** Corollary 1.2 indeed includes the case of complex Monge-Ampère type equation, which has a stronger cone condition than all others.

**Theorem 1.4.** Let \((M,\omega)\) be a closed Kähler manifold of complex dimension \(n\) and \(\chi\) also Kähler. Suppose that

\[
n\chi^{n-1} > \psi \sum_{\alpha=1}^{n-1} c_{\alpha} (n-\alpha) \chi^{n-\alpha-1} \wedge \omega^\alpha.
\]

If \(\psi \geq c\) for all \(x \in M\), then there exist a unique admissible solution \(u\) and a unique constant \(b\) satisfying the equation (1.5).
In order to prove the solvability of generalized complex Monge-Ampère type equations, we shall obtain a priori, which are the fundamental of the study.

**Theorem 1.5.** Let \((M, \omega)\) be a closed Hermitian manifold of complex dimension \(n\) and \(u\) be a smooth admissible solution to the equation (1.1). Suppose that

\[
 n\chi^{n-1} > \psi \sum_{\alpha=1}^{n-1} c_{\alpha}(n-\alpha)\chi^{n-\alpha-1} \wedge \omega^{\alpha}.
\]

Then there are uniform \(C^\infty\) a priori estimates of \(u\).

In this paper, we prove the uniform estimate and partial second order estimates. The gradient estimate can be obtained as in [10]. Higher order estimates can be achieved by Evans-Krylov theorem [6, 12] and Schauder estimate.

**Remark 1.6.** Following the argument in [4], the condition (1.6) can be defined in the viscosity sense.

### 2. Preliminary

We denote by \(\nabla\) the Chern connection of \(g\). As in [10, 15], we express

\[
 X := \chi u,
\]

and thus

\[
 X_{ij} = \chi_{ij} + \partial_i \bar{\partial} j u.
\]

Also, we denote the coefficients of \(X^{-1}\) by \(X^{ij}\). Assume that at the point \(p, g_{ij} = \delta_{ij}\) and \(X_{ij}\) is diagonal in a specific chart. Such local coordinates are called normal coordinate charts around \(p\).

We recall the formula

\[
 X_{ij} - X_{jji} = R_{jji}X_{ii} - R_{ijj}X_{jj} + 2\Re \left\{ \sum_p T_{ij}^p \chi_{ipj} \right\} - \sum_p T_{ij}^p T_{ij}^p \chi_{p} - G_{ijj},
\]

where

\[
 G_{ijj} = \chi_{jj} - \chi_{ij} + \sum_p R_{ij} \chi_{pi} - \sum_p R_{ij} \chi_{pj} + 2\Re \left\{ \sum_p T_{ij}^p \chi_{ipj} \right\} - \sum_p T_{ij}^p T_{ij}^p \chi_{p}.
\]

Let \(S_{\alpha}(\lambda)\) denote the \(\alpha\)-th elementary symmetric polynomial of \(\lambda \in \mathbb{R}^n\),

\[
 S_{\alpha}(\lambda) = \sum_{1 \leq i_1 < \cdots < i_\alpha \leq n} \lambda_{i_1} \cdots \lambda_{i_\alpha}.
\]

For a square matrix \(A\), define \(S_{\alpha}(A) = S_{\alpha}(\lambda(A))\) where \(\lambda(A)\) denote the eigenvalues of \(A\). Further, write \(S_{\alpha}(X) = S_{\alpha}(\lambda(X))\) and \(S_{\alpha}(X^{-1}) = S_{\alpha}(\lambda^*(X^{-1}))\) where \(\lambda(A)\) and \(\lambda^*(A)\) denote the eigenvalues of a Hermitian matrix \(A\) with respect to \(\{g_{ij}\}\) and to \(\{g^{ij}\}\), respectively. Unless otherwise indicated we shall use \(S_{\alpha}\) to denote \(S_{\alpha}(X^{-1})\) when no possible confusion would occur. In local coordinates, equation (1.1) can be written in the form

\[
 \sum_{\alpha=1}^n \frac{c_{\alpha}}{C_{\alpha}} S_{\alpha}(\chi_{u}^{-1}) = \frac{1}{\psi}.
\]
As in [7, 10, 15], differentiating $S_\alpha$ twice and applying the strong concavity of $S_\alpha$ [11], we have

\[(2.7) \partial_l(S_\alpha) = -\sum_i S_{\alpha-1;i}(X^{i\bar{i}})^2 X_i\bar{l}\]

and

\[(2.8) \bar{\partial}_l \partial_l(S_\alpha) \geq \sum_{i,j} S_{\alpha-1;i}(X^{i\bar{i}})^2 X_j\bar{l}X_{\bar{j}i} - \sum_i S_{\alpha-1;i}(X^{i\bar{i}})^2 X_i\bar{l},\]

where for $\{i_1, \ldots, i_s\} \subseteq \{1, \ldots, n\}$,

\[(2.9) S_{k;i_1,\ldots,i_s}(\lambda) = S_k(\lambda|_{\lambda_{i_1}=\ldots=\lambda_{i_s}=0}).\]

By convention, $S_{0;k} = 1$. Then inequality (1.2) is equivalent to

\[(2.10) \frac{1}{\psi} > \sum_{\alpha=1}^n \frac{c_{\alpha}}{C_{\alpha} n} S_{\alpha;k}(\chi^{-1})\]

for all $k$.

Differentiating equation (2.6) twice and applying (2.7), (2.8),

\[(2.11) \partial_l(\psi^{-1}) = -\sum_{\alpha=1}^n \frac{c_{\alpha}}{C_{\alpha} n} S_{\alpha-1;i}(X^{i\bar{i}})^2 X_i\bar{l}\]

and

\[(2.12) \bar{\partial}_l \partial_l(\psi^{-1}) \geq \sum_{\alpha=1}^n \frac{c_{\alpha}}{C_{\alpha} n} \left[ \sum_{i,j} S_{\alpha-1;i}(X^{i\bar{i}})^2 X_j\bar{l}X_{\bar{j}i} - \sum_i S_{\alpha-1;i}(X^{i\bar{i}})^2 X_i\bar{l}\right].\]

We may assume

\[(2.13) \epsilon \omega \leq \chi \leq \epsilon^{-1} \omega\]

for some $\epsilon > 0$.

3. THE UNIFORM ESTIMATE

In this section, we derive the uniform estimate directly. However, the $C^2$ estimate in Section 4 also implies a uniform estimate as shown in [19].

Following the work of Tosatti and Weinkove [19, 20], it suffices to show that there is a constant $C$ such that

\[(3.1) \int_M |\partial e^{-\frac{\psi}{\omega}u}|^2 \omega^n \leq C p \int_M e^{-pu} \omega^n\]

for $p$ large enough. We refer the readers to [23, 19, 20, 22] for more details.

Applying the technique in [17, 18], we obtain the following lemma on closed Hermitian manifolds.

**Lemma 3.1.** Let $u$ be a smooth admissible solution to equation (1.1) with

\[(3.2) n\chi^{n-1} > \psi \sum_{\alpha=1}^{n-1} c_\alpha (n-\alpha) \chi^{n-\alpha-1} \wedge \omega^\alpha.\]

Then there are uniform constants $C, p_0$ such that for all $p \geq p_0$, inequality (3.1) holds true.
Proof. Consider the integral
\begin{equation}
I = \int_M e^{-pu} \left( (\chi_u^n - \chi^n) - \psi \sum_{\alpha=1}^{n-1} c_\alpha (\chi_u^{n-\alpha} \wedge \omega^\alpha - \chi^{n-\alpha} \wedge \omega^\alpha) \right) \, dt.
\end{equation}
(3.3)

It is easy to see that for some constant $C$,
\begin{equation}
I \leq C \int_M e^{-pu} \chi^n.
\end{equation}
(3.4)

On the other hand, by integration by parts,
\begin{align*}
I &= p \int_M \int_0^1 e^{-pu} \sqrt{1} \partial \bar{u} \wedge \partial \chi \wedge \chi_{tu}^{n-1} dt \\
&\quad + n(n-1) \int_M \int_0^1 e^{-pu} \sqrt{1} \partial \bar{u} \wedge \partial \chi \wedge \chi_{tu}^{-2} dt \\
&\quad - \sum_{\alpha=1}^{n-1} c_\alpha(n-\alpha) \int_M \int_0^1 e^{-pu} \sqrt{1} \partial \bar{u} \wedge \partial \psi \wedge \chi_{tu}^{n-\alpha} \wedge \omega^\alpha dt \\
&\quad - \sum_{\alpha=1}^{n-2} c_\alpha(n-\alpha)(n-\alpha-1) \int_M \int_0^1 e^{-pu} \psi \sqrt{1} \partial \bar{u} \wedge \partial \chi \wedge \chi_{tu}^{n-\alpha-2} \wedge \omega^\alpha dt \\
&\quad - \sum_{\alpha=1}^{n-1} c_\alpha(n-\alpha) \alpha \int_M \int_0^1 e^{-pu} \psi \sqrt{1} \partial \bar{u} \wedge \partial \omega \wedge \chi_{tu}^{n-\alpha-1} \wedge \omega^\alpha dt.
\end{align*}
(3.5)

Since $-\sum_{\alpha=1}^{n-1} c_\alpha S_\alpha (A^{-1})$ and $S_{n-1}^2 (A)$ are concave with respect to positive definite $(n-1) \times (n-1)$ Hermitian matrix $A$, the first term in (3.5) is greater than
\begin{equation}
2a_1 p \left( \int_M e^{-pu} \sqrt{1} \partial \bar{u} \wedge \partial \chi \wedge \chi_{tu}^{n-1} + \int_0^2 \int_M e^{-pu} \sqrt{1} \partial \bar{u} \wedge \partial \chi \wedge \chi_{tu}^{-1} dt \right)
\end{equation}
(3.6)

for some small $a_1 > 0$. Applying Schwarz’s inequality pointwise and a key inequality in [17], we can find a lower bound for the last four terms in (3.5), for any $\epsilon_1 > 0$,
\begin{equation}
- \epsilon_1 p \int_0^1 \int_M e^{-pu} \sqrt{1} \partial \bar{u} \wedge \partial \chi \wedge \chi_{tu}^{n-1} dt - \frac{C_1}{\epsilon_1 p} \int_0^1 \int_M e^{-pu} \chi_{tu}^{n} dt.
\end{equation}
(3.7)

As shown in [17] [18], if $p$ is large enough,
\begin{equation}
\frac{1}{p} \int_0^1 \int_M e^{-pu} \chi_{tu}^{n} dt \leq C_2 \int_0^1 \int_M e^{-pu} \sqrt{1} \partial \bar{u} \wedge \partial \chi \wedge \chi_{tu}^{n-1} dt + \frac{2}{p} \int_M e^{-pu} \chi^n.
\end{equation}
(3.8)

By the observation that
\begin{equation}
\int_0^1 \int_M e^{-pu} \sqrt{1} \partial \bar{u} \wedge \partial \chi \wedge \chi_{tu}^{n-1} dt \leq 2^n \int_0^1 \int_M e^{-pu} \sqrt{1} \partial \bar{u} \wedge \partial \chi \wedge \chi_{tu}^{n-1} dt,
\end{equation}
we choose $2^n \epsilon_1 = a_1$ and then let $p$ sufficiently large. Therefore, we obtain
\begin{equation}
I \geq 2a_1 p \int_M e^{-pu} \sqrt{1} \partial \bar{u} \wedge \partial \chi \wedge \chi_{tu}^{n-1} - \frac{C_3}{p} \int_M e^{-pu} \chi^n.
\end{equation}
(3.10)
When more geometric information is available, we are able to extend the result to more general cases. For example, we can prove the following lemma on closed Kähler manifolds.

**Lemma 3.2.** Let $u$ be a smooth admissible solution to the equation

\[
\begin{aligned}
\chi_u^m \wedge \omega^{n-m} &= \psi \sum_{\alpha=1}^{m} c_\alpha \chi_u^{m-\alpha} \wedge \omega^{n-m+\alpha}, \\
\chi_u \in \Gamma^m \cap [\chi], \quad \sup_M u = 0,
\end{aligned}
\]

where $1 \leq m \leq n$, $\chi \in \Gamma^m$ is a smooth real $(1, 1)$ form, $c_\alpha$'s are nonnegative real constants and $\sum_{\alpha=1}^{m} c_\alpha > 0$. If

\[
m\chi^{m-1} \wedge \omega^{n-m} > \psi \sum_{\alpha=1}^{m-1} c_\alpha (m - \alpha) \chi^{m-\alpha-1} \wedge \omega^{n-m+\alpha},
\]

inequality \((3.1)\) holds true for some uniform constants $C, p_0$ such that for all $p \geq p_0$.

**Proof.** Consider the integral

\[
I = \int_M e^{-pu} \left[ (\chi_u^m - \chi^m) \wedge \omega^{n-m} - \psi \sum_{\alpha=1}^{m-1} c_\alpha (\chi_u^{m-\alpha} - \chi^{m-\alpha}) \wedge \omega^{n-m+\alpha} \right] dt.
\]

By integration by parts and $d\omega = 0$,

\[
I = p \int_0^1 \int_M e^{-pu} \sqrt{-1} \partial \bar{\partial} u \wedge \left[ \frac{m \chi^{m-1} \wedge \omega^{n-m} - \psi \sum_{\alpha=1}^{m-1} c_\alpha (m - \alpha) \chi^{m-\alpha-1} \wedge \omega^{n-m+\alpha}}{2} \right] dt
\]

\[
+ m(m-1) \int_0^1 \int_M e^{-pu} \sqrt{-1} \partial u \wedge \partial \chi \wedge \chi^{m-2} \wedge \omega^{n-m}
\]

\[
- \sum_{\alpha=1}^{m-2} c_\alpha (m - \alpha)(m - \alpha - 1) \int_0^1 \int_M e^{-pu} \sqrt{-1} \partial u \wedge \partial \chi \wedge \chi^{m-\alpha-2} \wedge \omega^{n-m+\alpha} dt
\]

\[
- \sum_{\alpha=1}^{m-1} c_\alpha (m - \alpha) \int_0^1 \int_M e^{-pu} \sqrt{-1} \partial u \wedge \partial \psi \wedge \chi^{m-\alpha-1} \wedge \omega^{n-m+\alpha} dt.
\]

The remaining argument is the same as above. \qed
4. The second order estimate

In this section, we prove the partial second order estimates. Note that the sharp form of estimates also implies the uniform estimate as shown in [19].

In order to obtain the second order estimate, we need the following lemma. There are more general statements by Guan [9], Collins and Székelyhidi [4], Székelyhidi [8] respectively. However, for completeness we include a proof following [7, 4].

**Lemma 4.1.** There are constants $N$, $\theta > 0$ such that when $w \geq N$ at a point $p$,

$$\sum_{\alpha=1}^{n} \frac{c_{\alpha}}{C_{n}} \sum_{i} S_{\alpha-1;i}(X^{i})^{2} u_{i}^{\alpha} \leq -\theta \left(\frac{c_{\alpha}}{C_{n}} \sum_{i} S_{\alpha-1;i}(X^{i})^{2} + 1\right),$$

under normal coordinates around $p$.

**Proof.** Without loss of generality, we may assume that $X_{11} \geq \cdots \geq X_{n\bar{n}}$. By direct calculation,

$$\sum_{i} S_{\alpha-1;i}(X^{i})^{2} u_{i}^{\alpha} \leq c_{\alpha} - \epsilon \sum_{i} S_{\alpha-1;i}(X^{i})^{2}$$

(4.2)

$$\leq c_{\alpha} - \epsilon \left(\sum_{i} S_{\alpha} - \frac{n-\alpha}{n} S_{\alpha} S_{1}\right)$$

$$= \left(1 - \frac{\epsilon}{n} S_{1}\right) c_{\alpha}$$

which means that if the inequality (4.1) does not hold for any $\theta > 0$,

$$\sum_{i} X^{i} \leq \frac{2n}{\epsilon}$$

(4.3)

Then we have

$$X^{i}_{11} \geq \cdots \geq X_{n\bar{n}} \geq \frac{\epsilon}{2n}.$$  

(4.4)

Now we follow the argument in [31]

$$\sum_{\alpha=1}^{n} \frac{c_{\alpha}}{C_{n}} \sum_{i} S_{\alpha-1;i}(X^{i})^{2} u_{i}^{\alpha} \leq \sum_{\alpha=1}^{n} \frac{c_{\alpha}}{C_{n}} \sum_{i} S_{\alpha-1;i}(X^{i})^{2} - \sum_{\alpha=1}^{n} \frac{c_{\alpha}}{C_{n}} S_{\alpha-1;i}(X^{i})^{2}$$

(4.5)

$$\leq -\sum_{\alpha=1}^{n} \frac{c_{\alpha}}{C_{n}} (S_{\alpha:1} - S_{\alpha} ((\chi|1)^{-1})) - \sum_{\alpha=1}^{n} \frac{\alpha c_{\alpha}}{C_{n}} (S_{\alpha:1} - S_{\alpha}),$$

where $(\chi|1)$ is the $(1,1)$ minor matrix of $\chi$ in local charts. Then by the condition (1.6) there is a constant $\sigma > 0$ such that

$$\sum_{\alpha=1}^{n} \frac{c_{\alpha}}{C_{n}} \sum_{i} S_{\alpha-1;i}(X^{i})^{2} u_{i}^{\alpha} \leq -\frac{\sigma}{\psi} - \sum_{\alpha=1}^{n} \frac{(\alpha + 1) c_{\alpha}}{C_{n}} (S_{\alpha:1} - S_{\alpha})$$

$$= -\frac{\sigma}{\psi} + \sum_{\alpha=1}^{n} \frac{(\alpha + 1) c_{\alpha}}{C_{n}} X^{1\bar{1}} S_{\alpha-1;1}$$

(4.6)

$$\leq -\frac{\sigma}{\psi} + \sum_{\alpha=1}^{n} \frac{(\alpha + 1)\alpha}{n} \left(\frac{2n}{\epsilon}\right)^{\alpha-1} X^{1\bar{i}}.$$  

So if $X_{11}$ is large enough, we achieve the inequality (4.1).
Proposition 4.2. Let \( u \in C^4(M) \) be a solution to equation (1.1) and \( w = \Delta u + \text{tr} \chi \). Then there are uniform positive constants \( C \) and \( A \) such that
\[
\sup_M w \leq C e^{A(u - \inf_M u)},
\]
where \( C \) depends on the given geometric quantities.

Proof. Let us consider the function \( e^\phi w \) where \( \phi \) is to be specified later. We may assume that \( w \gg 1 \).

Suppose that \( e^\phi w \) attains its maximal value at some point \( p \in M \). Choose a local chart near \( p \) such that \( g_{ij} = \delta_{ij} \) and \( X_{ij} \) is diagonal at \( p \). Therefore, at the point \( p \), we have
\[
(4.8) \quad \frac{\partial_l w}{w} + \partial_l \phi = 0,
\]
(4.9) \quad \frac{\bar{\partial}_l w}{w} + \bar{\partial}_l \phi = 0,

and
\[
(4.10) \quad \frac{\bar{\partial}_l \partial_l w}{w} - \frac{\bar{\partial}_l w \partial_l w}{w^2} + \bar{\partial}_l \partial_l \phi \leq 0.
\]

It is easy to see that
\[
(4.11) \quad \partial_l w = \sum_i X_{i\bar{l}l},
\]
and
\[
(4.12) \quad \bar{\partial}_l \partial_l w = \sum_i X_{i\bar{i}\bar{l}}.
\]

As in [3, 21, 15], direct calculation shows that
\[
\sum_{i,j,l} S_{a-1;i}(X^{\bar{i}j})^2 X_{j\bar{i}l} X_{i\bar{l}l} - \sum_{i,j,l} S_{a-1;i}(X^{\bar{i}j})^2 \bar{X}_{i\bar{l}l}
\geq - \sum_i S_{a-1;i}(X^{\bar{i}j})^2 \bar{\partial}_i \partial_i w - \frac{2}{w} \sum_{i,j} S_{a-1;i}(X^{\bar{i}j})^2 2 \text{Re} \{ \sum_k \bar{T}_{i\bar{j}k} \chi_{k\bar{j}} \bar{\partial}_l w \}
\]
(4.13) \quad + \sum_i S_{a-1;i}(X^{\bar{i}j})^2 \frac{|\partial_i w|^2}{w} - \alpha C_1 S_a - C_2 w \sum_i S_{a-1;i}(X^{\bar{i}j})^2
\geq \alpha C_1 S_a - C_2 w \sum_i S_{a-1;i}(X^{\bar{i}j})^2 2 \text{Re} \{ \sum_k \bar{T}_{i\bar{j}k} \chi_{k\bar{j}} \bar{\partial}_l w \}
\]
\[
- \alpha C_1 S_a - C_2 w \sum_i S_{a-1;i}(X^{\bar{i}j})^2
\]
where \( \bar{T} \) denotes the torsion with respect to the Hermitian metric \( \chi \).
Substituting (11.8) and (4.13) into the sum of (2.12),

\[
\sum_i \partial_i \partial_i (\psi^{-1}) + nC_1 \psi^{-1} \geq \sum_{\alpha=1}^{n} \frac{c_\alpha}{C_n} \left[ w \sum_i S_{\alpha-1;i} (X^{\bar{i}}) \partial_i \partial_i \phi \right. \\
+ 2 \sum_{i,j} S_{\alpha-1;i} (X^{\bar{i}}) \partial_i \partial_j \phi \left( \sum_k \tilde{T}_{ij} \chi_{kj} \partial_k \phi \right) - C_2 w \sum_i S_{\alpha-1;i} (X^{\bar{i}})^2 \right].
\]

(4.14)

By applying a trick due to Phong and Sturm [13], we use the function

\[
\phi := -Au + \frac{1}{u - \inf_M u + 1} = -Au + E_1.
\]

(4.15)

Without loss of generality, we assume \( C, A \gg 1 \) throughout this section. It is easy to see that

\[
\partial_i \phi = -(A + E_1^2) u_i
\]

(4.16)

and

\[
\partial_i \partial_i \phi = -(A + E_1^2) u_{\bar{i}} + 2E_1^2 |u_i|^2.
\]

(4.17)

Then, we have

\[
w \sum_i S_{\alpha-1;i} (X^{\bar{i}})^2 \partial_i \partial_i \phi = 2E_1^2 w \sum_i S_{\alpha-1;i} (X^{\bar{i}})^2 |u_i|^2 - (A + E_1^2) \sum_i S_{\alpha-1;i} (X^{\bar{i}})^2 u_{\bar{i}},
\]

and

\[
2 \sum_{i,j} S_{\alpha-1;i} (X^{\bar{i}})^2 \partial_i \partial_j \phi \left( \sum_k \tilde{T}_{ij} \chi_{kj} \partial_k \phi \right)
\]

\[
\geq -wE_1^2 \sum_i S_{\alpha-1;i} (X^{\bar{i}})^2 |u_i|^2 - \frac{C_3 A^2}{wE_1} \sum_i S_{\alpha-1;i} (X^{\bar{i}})^2.
\]

(4.18)

Therefore,

\[
C_2 w \sum_{\alpha=1}^{n} \frac{c_\alpha}{C_n} \sum_i S_{\alpha-1;i} (X^{\bar{i}})^2 + C_4
\]

\[
\geq -(A + E_1^2) \sum_{\alpha=1}^{n} \frac{c_\alpha}{C_n} \sum_i S_{\alpha-1;i} (X^{\bar{i}})^2 u_{\bar{i}} - \frac{C_3 A^2}{wE_1} \sum_{\alpha=1}^{n} \frac{c_\alpha}{C_n} \sum_i S_{\alpha-1;i} (X^{\bar{i}})^2.
\]

(4.19)

For \( A \gg 1 \) which is to be determined later, there are two cases in consideration: (1) \( w > AE_1^{-\frac{2}{\beta}} \geq A > N \), where \( N \) is the crucial constant in Lemma 4.1 (2) \( w \leq AE_1^{-\frac{2}{\beta}} \).

In the first case, by Lemma 4.1

\[
C_2 w \sum_{\alpha=1}^{n} \frac{c_\alpha}{C_n} \sum_i S_{\alpha-1;i} (X^{\bar{i}})^2 + C_4 \geq Au \left( \sum_{\alpha=1}^{n} \frac{c_\alpha}{C_n} \sum_i S_{\alpha-1;i} (X^{\bar{i}})^2 + 1 \right).
\]

(4.21)

This gives a bound \( w \leq 1 \) at \( p \) if we choose \( A \theta > \max\{C_2, C_4\} \). It contradicts the assumption \( w \gg 1 \).

In the second case,

\[
we^\phi \leq we^\phi p \leq AE_1^{-\frac{2}{\beta}} e^{-Au + 1} \leq Ae^2 e^{-A \inf_M u}
\]

(4.22)
and hence

\[ w \leq Ae^2 e^{A u - E_{\inf M}} u \leq Ae^2 e^{A u - A_{\inf M}} u \leq C e^{A (u - \inf M)}. \]

5. Solving the equations

In this section, we give proofs of the solvability results stated in introduction section.

Proof of Theorem 1.1. Define \( \varphi \) by

\[ \chi^n = \varphi \sum_{\alpha=1}^n c_{\alpha} \chi^{n-\alpha} \land \omega^\alpha. \]

We use the continuity method and consider the family of equations

\[ \chi^n_{u_t} = \psi^t \varphi 1 - t e^{b_t} \sum_{\alpha=1}^n c_{\alpha} \chi^{n-\alpha}_{u_t} \land \omega^\alpha, \quad \text{for } t \in [0, 1], \]

where \( \chi_{u_t} > 0 \), where \( b_t \) is a constant for each \( t \). We consider the set

\[ \mathcal{T} := \{ t' \in [0, 1] | \exists u_t \in C^2(M) \text{ and } b_t \text{ solving 5.1 for } t \in [0, t'] \}. \]

As shown in [15], the continuity method works if we can guarantee: (1) \( 0 \in \mathcal{T} \); (2) we have uniform \( C^\infty \) estimates for all \( u_t \). The first requirement is naturally met; for the second requirement, we only need to show that \( \psi^t \varphi 1 - t e^{b_t} \leq \psi \) for all \( t \in [0, 1] \). At the maximum point \( u_t - v \), we have \( \psi^t \varphi 1 - t e^{b_t} \leq \psi \), and thus \( b_t \leq 0 \). Therefore \( \psi^t \varphi 1 - t e^{b_t} \leq \psi \).

Proof of Theorem 1.4. Define \( \varphi \) by

\[ \chi^n = \varphi \sum_{\alpha=1}^n c_{\alpha} \chi^{n-\alpha} \land \omega^\alpha. \]

Definitely,

\[ n \chi^{n-1} > \varphi \sum_{\alpha=1}^{n-1} c_{\alpha} (n - \alpha) \chi^{n-\alpha-1} \land \omega^\alpha. \]

Therefore, we can find a \( C^2 \) function \( h \) satisfying \( h(x) \geq \max \{ \varphi(x), \psi(x) \} \) for all \( x \in M \) and

\[ n \chi^{n-1} > h \sum_{\alpha=1}^{n-1} c_{\alpha} (n - \alpha) \chi^{n-\alpha-1} \land \omega^\alpha. \]

Since \( h \geq \varphi \), by Theorem 1.1 there exists and admissible solution \( u_0 \) and a constant \( b_0 \leq 0 \) such that

\[ \chi^n_{u_0} = e^{b_0} h \sum_{\alpha=1}^n c_{\alpha} \chi^{n-\alpha}_{u_0} \land \omega^\alpha. \]
Now we apply continuity method again from $\chi_{u_0}$ and consider the family of equations

\begin{equation}
\chi^n = \psi^t h^{1-t} e^{bi} \sum_{\alpha=1}^n \chi_{u_0}^{n-\alpha} \wedge \omega^\alpha, \quad \text{for } t \in [0, 1].
\end{equation}

According to the argument in [15], we only need to show that $\psi^t h^{1-t} e^{bi} \leq h$ for all $t \in [0, 1]$. Note that, we use the new condition (5.8) to obtain $C^\infty$ estimates.

Integrating equation flow (5.8),

\begin{equation}
\int_M \chi^n = e^{bi} \sum_{\alpha=1}^n \int_M \psi^t h^{1-t} \chi_{u_0}^{n-\alpha} \wedge \omega^\alpha \\
\geq e^{bi} \sum_{\alpha=1}^n \int M \chi^{n-\alpha} \wedge \omega^\alpha,
\end{equation}

which implies $b_1 \leq 0$. Therefore,

\begin{equation}
\psi^t h^{1-t} e^{bi} \leq \psi^t h^{1-t} \leq h.
\end{equation}

\[\square\]

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