Existence and uniqueness of invariant measures for SPDEs with two reflecting walls *

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Abstract

In this article, we study stochastic partial differential equations with two reflecting walls $h^1$ and $h^2$, driven by space-time white noise with non-constant diffusion coefficients under periodic boundary conditions. The existence and uniqueness of invariant measures is established under appropriate conditions. The strong Feller property is also obtained.

Key Words: stochastic partial differential equations with two reflecting walls; white noise; heat equation; invariant measures; coupling; strong Feller property.

MSC: Primary 60H15; Secondary 60J35

1 Introduction

Consider the following stochastic partial differential equations (SPDEs) with two reflecting walls

\[
\begin{cases}
\frac{\partial u(x,t)}{\partial t} = \frac{\partial^2 u(x,t)}{\partial x^2} + f(u(x,t)) + \sigma(u(x,t))\dot{W}(x,t) \\
\quad \quad \quad \quad \quad \quad \quad \quad \quad \quad + \eta(x,t) - \xi(x,t); \\
u(x,0) = u_0(x) \in C(S^1); \\
h^1(x) \leq u(x,t) \leq h^2(x), \text{ for } (x,t) \in Q.
\end{cases}
\]

(1.1)

$Q := S^1 \times \mathbb{R}_+, \ S^1 := \mathbb{R} (mod 2\pi)$, or $\{e^{i\theta}; \ \theta \in \mathbb{R}\}$ denotes a circular ring and the random field $W(x,t) := W(\{e^{i\theta}; \ 0 \leq \theta \leq x\} \times [0,t])$ is a regular Brownian sheet defined on a filtered probability space $(\Omega, P, \mathcal{F} \cap \mathcal{F}_t)$. The

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random measures $\xi$ and $\eta$ are added to equation (1.1) to prevent the solution from leaving the interval $[h^1, h^2]$.

We assume that the reflecting walls $h^1(x), h^2(x)$ are continuous functions satisfying

(H1) $h^1(x) < h^2(x)$ for $x \in S^1$;

(H2) $\frac{\partial^2 h^i}{\partial x^2} \in L^2(S^1)$, where $\frac{\partial^2}{\partial x^2}$ is interpreted in a distributional sense.

We also assume that the coefficients: $f, \sigma: \mathbb{R} \rightarrow \mathbb{R}$ satisfy

(F1) there exists $L > 0$ such that

$$|f(z_1) - f(z_2)| + |\sigma(z_1) - \sigma(z_2)| \leq L|z_1 - z_2|, \ z_1, z_2 \in \mathbb{R};$$

The following is the definition of a solution of a SPDE with two reflecting walls $h^1, h^2$.

**Definition 1.1.** A triplet $(u, \eta, \xi)$ is a solution to the SPDE (1.1) if

(i) $u = \{u(x,t); (x,t) \in Q\}$ is a continuous, adapted random field (i.e., $u(x,t)$ is $\mathcal{F}_t$-measurable $\forall t \geq 0, x \in S^1$) satisfying $h^1(x) \leq u(x,t) \leq h^2(x)$, a.s;

(ii) $\eta(dx,dt)$ and $\xi(dx,dt)$ are positive and adapted (i.e. $\eta(B)$ and $\xi(B)$ is $\mathcal{F}_t$-measurable if $B \subset S^1 \times [0,t]$) random measures on $Q$ satisfying

$$\eta(S^1 \times [0,T]) < \infty, \ \xi(S^1 \times [0,T]) < \infty$$

for $T > 0$;

(iii) for all $t \geq 0$ and $\phi \in C^\infty(S^1)$ we have

$$\int_Q (u(x,t) - h^1(x))\eta(dx,dt) = \int_Q (h^2(x) - u(x,t))\xi(dx,dt) = 0.$$

The existence and uniqueness of the solution of equation (1.1) is established in [13], see also [11] for SPDEs with one reflecting barrier. SPDEs with reflection were first studied by Nualart and Pardoux in [4]. Interesting properties were obtained in [12].
The aim of this paper is to establish the existence and uniqueness of invariant measures, as well as the strong Feller property of fully non-linear SPDEs with two reflecting walls (1.1).

For SPDEs without reflection, the existence and uniqueness of invariant measures has been studied by many people, see Sowers [9], Mueller [3], Peszat and Zabczyk [7], Da Prato and Zabczyk [2]. For SPDEs with reflection, when the diffusion coefficient $\sigma$ is a constant, existence and uniqueness of invariant measures was obtained by Otobe [5], [6]. The strong Feller property of SPDEs has been studied by several authors, see Peszat and Zabczyk [7], Da Prato and Zabczyk [2]. The strong Feller property of SPDEs with reflection at 0 was first proved in [14].

For the existence of invariant measures, our approach is to use Krylov-Bogolyubov theorem. To this end, the continuity of the solution with respect to the solutions of some random obstacle problems plays an important role. For the uniqueness, we adapted a coupling method used by Mueller [3]. Because of the reflection, we need to establish a kind of uniform coupling for approximating solutions. The strong Feller property of SPDEs with two reflecting walls will be obtained in a similar way as that in Zhang [14].

The rest of the paper is organized as follows. In Section 2, we give the proof of the existence and uniqueness of invariant measures. Section 3 establishes the strong Feller property.

2 Existence and Uniqueness of Invariant Measures

Denote by $\mathcal{B}(C(S^1))$ the $\sigma$-field of all Borel subsets of $C(S^1)$ and by $\mathcal{M}(C(S^1))$ the set of all probability measures defined on $(C(S^1), \mathcal{B}(C(S^1)))$. We denote by $u(x, t, u_0)$ the solution of equation (1.1) and by $P_t(u_0, \cdot)$ the corresponding transition function

$$P_t(u_0, \Gamma) = P(\mathcal{u}(\cdot, t, u_0) \in \Gamma), \Gamma \in \mathcal{B}(C(S^1)), \ t > 0,$$

where $u_0$ is the initial condition. For $\mu \in \mathcal{M}(C(S^1))$ we set

$$P_t^* \mu(\Gamma) = \int_{C(S^1)} P_t(x, \Gamma) \mu(dx),$$

where $t \geq 0, \ \Gamma \in \mathcal{B}(C(S^1))$.

**Definition 2.1.** A probability measure $\mu \in \mathcal{M}(C(S^1))$ is said to be invariant or stationary with respect to $P_t$, $t \geq 0$, if and only if $P_t^* \mu = \mu$ for each $t \geq 0$. 

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The initial condition \(u_0(x)\) satisfies (F2) \(u_0(x) \in C(S^1)\) satisfy \(h^1(x) \leq u_0(x) \leq h^2(x)\), for \(x \in S^1\).

**Theorem 2.1.** Suppose the hypotheses (H1)-(H2), (F1)-(F2) hold. Then there exists an invariant measure to equation (1.1) on \(C(S^1)\).

**Proof.** According to Krylov-Bogolyubov theorem (see [2]), if the family \(\{P_t(u_0, \cdot); t \geq 1\}\) is tight, then there exists an invariant measure for equation (1.1). So we need to show that for any \(\varepsilon > 0\) there is a compact set \(K \subset C(S^1)\) such that

\[
P(u(t) \in K) \geq 1 - \varepsilon, \quad \text{for any } t \geq 1.
\]

where \(u(t) = u(t, u_0) = u(\cdot, t, u_0)\). On the other hand, for any \(t \geq 1\), we have by the Markov property

\[
P(u(t) \in K) = \mathbb{E}(P_1(u(t - 1), K)). \tag{2.1}
\]

Thus it is enough to show \(P(u(1, u(t - 1)) \in K) \geq 1 - \varepsilon\), for any \(t \geq 1\). As \(h^1(\cdot) \leq u(t - 1)(\cdot) \leq h^2(\cdot)\), it suffices to find a compact subset \(K \subset C(S^1)\) such that

\[
P_1(g, K) \geq 1 - \varepsilon, \quad \text{for all } g \in C(S^1) \text{ with } h^1 \leq g \leq h^2. \tag{2.2}
\]

Put

\[
v(x, t, g) = \int_0^t \int_{S^1} G_{t-s}(x, y)f(u(y, s, g))dyds + \int_0^t \int_{S^1} G_{t-s}(x, y)\sigma(u(y, s, g))W(dy, ds), \tag{2.3}
\]

where \(G_{t}(x, y)\) is the Green's function of the heat equation on \(S^1\). Then \(u\) can be written in the form (see [4], [1] and [10])

\[
u(x, t, g) = \int_{S^1} G_t(x, y)g(y)dy = v(x, t, g) + \int_0^t \int_{S^1} G_{t-s}(x, y)\eta(g)(dx, dt) - \int_0^t \int_{S^1} G_{t-s}(x, y)\xi(g)(dx, dt),
\]

where \(\eta(g), \xi(g)\) indicates the dependence of the random measures on the initial condition \(g\). Put

\[
\bar{u}(x, t, g) = u(x, t, g) - \int_{S^1} G_t(x, y)g(y)dy
\]
Then \((\tilde{u}, \eta, \xi)\) solves a random obstacle problem. From the relationship between \(\tilde{u}\) and \(v\) proved in Theorem 4.1 in [13], we have the following inequality
\[
\|\tilde{u}(g) - \hat{u}(\hat{g})\|_1^1 \leq 2\|v(g) - v(\hat{g})\|_1^1,
\]
where \(\|\omega\|_1^1 := \sup_{x \in S^1, t \in [0,1]} |\omega(x, t)|\). So \(\tilde{u}\) is a continuous functional of \(v\) and denoted by \(u = \Phi(v)\), where \(\Phi(\cdot) : C(S^1 \times [0,1]) \rightarrow C(S^1 \times [0,1])\) is continuous. In particular, \(\tilde{u}(\cdot, 1, g)\) is also a continuous functional of \(v\), from \(C(S^1 \times [0,1])\) to \(C(S^1)\). We denote this functional by \(\Phi_1\), i.e. \(\tilde{u}(\cdot, 1, g) = \Phi_1(v(\cdot, g))\), where \(v(\cdot, g) = v(\cdot, g)\). If \(K''\) is a compact subset of \(C(S^1 \times [0,1])\), then \(K' = \Phi_1(K'')\) is a compact subset in \(C(S^1)\) and
\[
P(\tilde{u}(\cdot, 1, g) \in K') = P(\tilde{u}(\cdot, 1, g) \in \Phi_1(K'')) \
\geq P(v(\cdot, g) \in K''). \quad (2.4)
\]
Next, we want to find a compact set \(K''(\subset C(S^1 \times [0,1])\) such that
\[
P(v(\cdot, g) \in K'') \geq 1 - \varepsilon, \quad \text{for all } g \in C(S^1) \text{ with } h^1 \leq g \leq h^2. \quad (2.5)
\]
For \(0 < \alpha < \frac{1}{4}\) and \(\kappa > 0\), from Proposition A.1 in [8] and using a similar proof to that of Corollary 3.4 in [10], there exists a random variable \(Y(g)\) such that with probability one, for all \(x, y \in S^1\) and \(s, t \in (0, 1)\),
\[
|v(x, t, g) - v(y, s, g)| \leq Y(g)(d((x, t), (y, s)))^{\alpha - \kappa} \text{ and } \mathbb{E}(Y(g))^{\frac{1}{2}} \leq C_0, \quad (2.6)
\]
where \(d((x, t), (y, s)) := (r^2(x, y) + (t - s)^2)^{\frac{1}{4}}\) with \(r(x, y)\) the length of the shortest arc of \(S^1\) connecting \(x\) with \(y\) and \(C_0\) is independent of \(g\).

Define
\[
\|v\|_\alpha = \sup \left\{ \frac{|v(x, t) - v(y, s)|}{d^\alpha((x, t), (y, s))} : \right. \\
(x, t), (y, s) \in S^1 \times [0,1], (x, t) \neq (y, s) \left. \right\}, \quad \text{for } \alpha < \frac{1}{4}.
\]

By the Arzela-Ascoli theorem, for all \(r > 0\), \(K_r := \{v; \|v\|_\alpha \leq r\}\) is a compact subset of \(C(S^1 \times [0,1])\). In view of (2.6), we see that for given \(\varepsilon > 0\), there exists \(r_0\) such that
\[
P(v(\cdot, g) \in K_r^\varepsilon) \leq \varepsilon, \quad \text{for all } g \text{ with } h^1 \leq g \leq h^2.
\]
Choosing \(K'' = K_{r_0}\), we obtain (2.5). Hence \(P(\tilde{u}(\cdot, 1, g) \in K') \geq 1 - \varepsilon\) for all \(g \in C(S^1)\) with \(h^1 \leq g \leq h^2\). On the other hand, it is easy to see that there is a compact subset \(K_0 \subset C(S^1)\) such that
\[
\int_{S^1} G_1(x, y)g(y)dy; \quad h^1 \leq g \leq h^2 \subset K_0
\]
Define $K = K' + K_0$. We have

$$P_1(g, K) = P(u(\cdot, 1, g) \in K) \geq P(\tilde{u}(\cdot, 1, g) \in K') \geq 1 - \varepsilon,$$

for all $g \in C(S^1)$ with $h^1 \leq g \leq h^2$. This finishes the proof. \hfill \square

For the uniqueness of invariant measures, we need the following proposition. For simplicity, we put $u(x, t) = u(x, t, u_0)$.

**Proposition 2.1.** Under the assumption in Theorem 2.1, for any $p \geq 1$, $T > 0$, $\sup_{\varepsilon, \delta} \mathbb{E}(\|u^{\varepsilon, \delta}\|_\infty^p) < \infty$ and $u^{\varepsilon, \delta}$ converges uniformly on $S^1 \times [0, T]$ to $u$ as $\varepsilon, \delta \to 0$ a.s, where $u, u^{\varepsilon, \delta}$ are the solutions of equation (1.1) and the penalized SPDEs

$$\begin{cases}
\frac{\partial u^{\varepsilon, \delta}(x,t)}{\partial t} = \frac{\partial^2 u^{\varepsilon, \delta}(x,t)}{\partial x^2} + f(u^{\varepsilon, \delta}(x,t)) + \sigma(u^{\varepsilon, \delta}(x,t))\dot{W}(x,t) \\
u^{\varepsilon, \delta}(x,0) = u_0(x).
\end{cases}$$

**Proof.** Let $v^{\varepsilon, \delta}$ be the solution of equation

$$\begin{cases}
\frac{\partial v^{\varepsilon, \delta}(x,t)}{\partial t} = \frac{\partial^2 v^{\varepsilon, \delta}(x,t)}{\partial x^2} + f(u^{\varepsilon, \delta}(x,t)) + \sigma(u^{\varepsilon, \delta}(x,t))\dot{W}(x,t); \\
v^{\varepsilon, \delta}(x,0) = u_0(x).
\end{cases}$$

Set $\Phi^{\varepsilon, \delta}(t) = \sup_{s \leq t, y \in S^1} (v^{\varepsilon, \delta}(y,s) - h^2(y))^+$. Note that $\Phi^{\varepsilon, \delta}(t)$ is increasing w.r.t. $t$ and $v^{\varepsilon, \delta} - \Phi^{\varepsilon, \delta} \leq h^2$. $z^{\varepsilon, \delta}(x, t) := v^{\varepsilon, \delta}(x, t) - \Phi^{\varepsilon, \delta}(t) - u^{\varepsilon, \delta}(x, t)$ is a solution of equation

$$\begin{cases}
\frac{\partial z^{\varepsilon, \delta}}{\partial t} + \frac{\partial \Phi^{\varepsilon, \delta}}{\partial t} = \frac{\partial^2 z^{\varepsilon, \delta}}{\partial x^2} - \frac{1}{3}(u^{\varepsilon, \delta} - h^1)^- - \frac{1}{\varepsilon}(u^{\varepsilon, \delta} - h^2)^+; \\
z^{\varepsilon, \delta}(x, 0) = 0.
\end{cases}$$

Multiplying (2.8) by $(z^{\varepsilon, \delta})^+$ and using $((u^{\varepsilon, \delta} - h^2)^+, (z^{\varepsilon, \delta})^+) = 0$ we get $(z^{\varepsilon, \delta})^+ = 0$. Hence,

$$u^{\varepsilon, \delta} \geq v^{\varepsilon, \delta} - \Phi^{\varepsilon, \delta}.$$

Similarly, setting $\bar{z}^{\varepsilon, \delta}(x, t) = u^{\varepsilon, \delta}(x, t) - v^{\varepsilon, \delta}(x, t) - \sup_{s \leq t, y \in S^1} (v^{\varepsilon, \delta}(y, s) - h^1(s))^-$, we can show that

$$u^{\varepsilon, \delta} \leq v^{\varepsilon, \delta} + \sup_{s \leq t, y \in S^1} (v^{\varepsilon, \delta} - h^1)^-. $$
As \( \sup_{\epsilon,\delta} \mathbb{E}(\|u^{\epsilon,\delta}\|_T^p) < \infty \), the above two inequalities implies

\[
\sup_{\epsilon,\delta} \mathbb{E}(\|u^{\epsilon,\delta}\|_T^p) < \infty.
\]

Since \( u^{\epsilon,\delta} \) is increasing in \( \delta \) by the comparison theorem of SPDEs (see [1]), we can show \( u^\epsilon := \lim_{\delta \downarrow 0} u^{\epsilon,\delta} \) exists a.s. and \( u^\epsilon \) solves

\[
\begin{aligned}
\frac{\partial u^\epsilon(x,t)}{\partial t} &= \frac{\partial^2 u^\epsilon(x,t)}{\partial x^2} + f(u^\epsilon(x,t))W(x,t) \\
& \quad + \frac{1}{\epsilon}(u^\epsilon(x,t) - h^1(x))^+; \\
u^\epsilon(x,0) &\geq h^1(x); \\
u^\epsilon(x,0) &= u_0(x),
\end{aligned}
\]

where \( \eta^\epsilon(dx,dt) := \lim_{\delta \downarrow 0} \frac{u^{\epsilon,\delta}(x,t) - h^1(x))^+}{\delta} dx dt \). Also, by comparison, we know that \( u^\epsilon \) is decreasing as \( \epsilon \downarrow 0 \). Let \( v^\epsilon \) be the solution of equation (2.7) replacing \( u^{\epsilon,\delta} \) by \( u^\epsilon \). Setting \( \tilde{z}^\epsilon(x,t) = u^\epsilon(x,t) - v^\epsilon(x,t) - \sup_{s \leq t, y \in S^1} (v^\epsilon(y,s) - h^1(y))^- \), we can show

\[
u^\epsilon \leq v^\epsilon + \sup_{s \leq t, y \in S^1} (v^\epsilon - h^1)^-.
\]

In addition, by the definition of \( u^\epsilon \), \( u^\epsilon \geq h^1 \). Hence, \( u := \lim_{\epsilon \downarrow 0} u^\epsilon = \lim_{\delta \downarrow 0} \lim_{\epsilon \downarrow 0} u^{\epsilon,\delta} \) exists a.s.

The continuity of \( u \) can be proved similarly as in Theorem 4.1 in [1]. The uniform convergence of \( u^{\epsilon,\delta} \) w.r.t. \( (x,t) \) follows from Dini’s theorem.

\[
\square
\]

The following result is the uniqueness of invariant measures.

**Theorem 2.2.** Under the assumptions in Theorem 2.1 and that \( \sigma \geq L_0 \) for some constant \( L_0 > 0 \), there is a unique invariant measure for the equation (1.1).

**Proof.** We will adopt the coupling method used in Mueller [3] to SPDEs with reflection. Let \( u^1(x,0) \) and \( u^2(x,0) \) be two initial values having distributions given by two invariant probabilities \( \mu_1 \) and \( \mu_2 \). Then \( u^1(x,t) \) and \( u^2(x,t) \) also have these distributions for any \( t > 0 \). Thus

\[
\text{Var}(\mu_1 - \mu_2) \leq \mathbb{P}(\sup_{x \in S^1} |u^1(x,t) - u^2(x,t)| \neq 0).
\]
Thus, for given two initial functions \( u^1(x, 0) \) and \( u^2(x, 0) \), it is sufficient to construct two coupled processes \( u^1(x, t) \), \( u^2(x, t) \) satisfying equation (2.11), driven by different white noises on a probability space \((\Omega, \mathcal{F}, P)\), such that

\[
\lim_{t \to \infty} P\left( \sup_{x \in S^1} |u^1(x, t) - u^2(x, t)| \neq 0 \right) = 0. \tag{2.10}
\]

We first assume \( u^1(x, 0) \geq u^2(x, 0), \ x \in S^1 \). We want to construct two independent space-time white noises \( W_1(x, t) \), \( W_2(x, t) \) defined on a probability space \((\Omega, \mathcal{F}, P)\), and a solution \( u, v \) of the following SPDEs with two reflecting walls

\[
\frac{\partial u(x,t)}{\partial t} = \frac{\partial^2 u(x,t)}{\partial x^2} + f(u(x,t)) + \sigma(u(x,t)) \dot{W}_1(x,t) + \eta_1(x,t) - \xi_1(x,t),
\]

\[
\frac{\partial v(x,t)}{\partial t} = \frac{\partial^2 v(x,t)}{\partial x^2} + f(v(x,t)) + \eta_2(x,t) - \xi_2(x,t) + \sigma(v(x,t)) \left[ (1 - |u - v| \wedge 1) \frac{1}{2} \dot{W}_1(x,t) + (|u - v| \wedge 1) \frac{1}{2} \dot{W}_2(x,t) \right],
\]

\[
u(x,0) = u^1(x,0), \quad v(x,0) = u^2(x,0).
\tag{2.11}
\]

Note that the coefficients in the second equation in (2.11) is not Lipschitz. The existence of a solution of equation (2.11) is not automatic. In the following, using a similar method as that in the paper [3], we will give a construction of a solution on some probability space. The construction will also be used to prove the successful coupling

\[
\lim_{t \to \infty} P\left( \sup_{x \in S^1} |u(x,t) - v(x,t)| \neq 0 \right) = 0.
\]

For \( 0 \leq z \leq 1 \), set

\[
f_n(z) = \left( z + \frac{1}{n} \right) \frac{1}{2} - \left( \frac{1}{n} \right) \frac{1}{2},
\]

\[
g_n(x) = \left( 1 - f_n(z)^2 \right) \frac{1}{2}.
\]

We have \( f_n(z)^2 + g_n(z)^2 = 1 \) and that \( f_n(z) \to z^\frac{1}{2}, \ g_n(z) \to (1 - z)^\frac{1}{2} \) uniformly as \( n \to \infty \), for \( z \in S^1 \).

Let \( W_1(x, t), \ W_2(x, t) \) be two independent space-time white noises defined on a probability space \((\Omega, \mathcal{F}, P)\). Let \( \bar{u}, \bar{v} \) be the unique solution of
the following SPDEs with two reflecting walls

\[
\frac{\partial \pi(x, t)}{\partial t} = \frac{\partial^2 \pi(x, t)}{\partial x^2} + f(\pi(x, t)) + \sigma(\pi(x, t))\dot{W}_1(x, t) + \eta_1(x, t) - \xi_1(x, t),
\]

\[
\frac{\partial \pi^n(x, t)}{\partial t} = \frac{\partial^2 \pi^n(x, t)}{\partial x^2} + f(\pi^n(x, t)) + \eta_{2n}(x, t) - \xi_{2n}(x, t)
\]

\[
+ \sigma(\pi^n(x, t)) \left[ g_n(|\bar{u} - \pi^n| \wedge 1)\dot{W}_1(x, t) + f_n(|\bar{u} - \pi^n| \wedge 1)\dot{W}_2(x, t) \right],
\]

\[
\pi(x, 0) = u^1(x, 0), \quad \pi^n(x, 0) = u^2(x, 0).
\] (2.12)

The existence and uniqueness of \((\pi, \pi^n)\) is guaranteed because of the Lipschitz continuity of the coefficients. Put

\[
\dot{\bar{u}}(x, t) = \int_{S^1} G_1(x, y)u^1(y, 0)dy + \int_0^t \int_{S^1} G_{t-s}(x, y)f(\pi(y, s))dyds
\]

\[
+ \int_0^t \int_{S^1} G_{t-s}(x, y)\sigma(\pi(y, s))\dot{W}_1(dy, ds)
\] (2.13)

and

\[
\dot{\pi^n}(x, t) = \int_{S^1} G_1(x, y)u^2(y, 0)dy + \int_0^t \int_{S^1} G_{t-s}(x, y)f(\pi^n(y, s))dyds
\]

\[
+ \int_0^t \int_{S^1} G_{t-s}(x, y)\sigma(\pi^n(y, s))\dot{W}^n(dy, ds),
\] (2.14)

where

\[
\dot{W}^n(x, t) = \left[ g_n(|\bar{u} - \pi^n| \wedge 1)\dot{W}_1(x, t) + f_n(|\bar{u} - \pi^n| \wedge 1)\dot{W}_2(x, t) \right]
\]

is another space-time white noise on \((\Omega, \mathcal{F}, \mathbb{P})\). From the proof of Theorem 2.1, it is known that there exists a continuous functional \(\Phi\) from \(C(S^1 \times [0, T])\) into \(C(S^1 \times [0, T])\) (for ant \(T > 0\)) such that \(\bar{u} = \Phi(\dot{u})\) and \(\bar{v}^n = \Phi(\dot{v}^n)\). On the other hand, following the same proof of Lemma 3.1 in [3] it can be shown that the sequence \(\dot{u}, \dot{v}^n, n \geq 1\) is tight. As the images under the continuous map \(\Phi\), the vector \((\bar{u}, \bar{v}^n, \bar{W}_1, \bar{W}_2)\) is also tight. By Skorohod’s representation theorem, there exist random fields \((u, v^n, W_1, W_2)\), \(n \geq 1\) on some probability space \((\Omega, \mathcal{F}, \mathbb{P})\) such that \((u, v^n, W_1, W_2)\) has the same law as \((\bar{u}, \bar{v}^n, \bar{W}_1, \bar{W}_2)\) and that the following SPDEs with two reflecting walls
hold

\[
\frac{\partial u(x,t)}{\partial t} = \frac{\partial^2 u(x,t)}{\partial x^2} + f(u(x,t)) + \sigma(u(x,t))\dot{W}_1(x,t) \\
+ \eta_1(x,t) - \xi_1(x,t),
\]

\[
\frac{\partial v^n(x,t)}{\partial t} = \frac{\partial^2 v^n(x,t)}{\partial x^2} + f(v^n(x,t)) + \eta^n_2(x,t) - \xi^n_2(x,t) \\
+ \sigma(v^n(x,t))\left[g_n(|u - v^n| \wedge 1)\dot{W}_1(x,t) + f_n(|u - v^n| \wedge 1)\dot{W}_2(x,t)\right],
\]

\[
u(x,0) = u^1(x,0), \quad v^n(x,0) = u^2(x,0).
\] (2.15)

Furthermore, \(v^n \to v\) uniformly almost surely as \(n \to \infty\). By a similar proof as that of Theorem 4.1 in [13] we can prove that the limit \((u, v)\) satisfies the following SPDEs with two reflecting walls

\[
\frac{\partial u(x,t)}{\partial t} = \frac{\partial^2 u(x,t)}{\partial x^2} + f(u(x,t)) + \sigma(u(x,t))\dot{W}_1(x,t) \\
+ \eta_1(x,t) - \xi_1(x,t),
\]

\[
\frac{\partial v(x,t)}{\partial t} = \frac{\partial^2 v(x,t)}{\partial x^2} + f(v(x,t)) + \eta_2(x,t) - \xi_2(x,t) \\
+ \sigma(v(x,t))\left[(1 - |u - v| \wedge 1)\dot{W}_1(x,t) + (|u - v| \wedge 1)\dot{W}_2(x,t)\right],
\]

\[
u(x,0) = u^1(x,0), \quad v(x,0) = u^2(x,0).
\] (2.16)

The next step is to show that \(u, v\) admits a successful coupling. To this end, consider the following approximating SPDEs

\[
\begin{cases}
\frac{\partial u^{\varepsilon, \delta}}{\partial t} = \frac{\partial^2 u^{\varepsilon, \delta}}{\partial x^2} + f(u^{\varepsilon, \delta}) + \frac{1}{\delta}(u^{\varepsilon, \delta} - h^1)^+ \frac{1}{\varepsilon}(u^{\varepsilon, \delta} - h^2)^+ + \sigma(u^{\varepsilon, \delta})\dot{W}_1; \\
\frac{\partial v^{\varepsilon, \delta}}{\partial t} = \frac{\partial^2 v^{\varepsilon, \delta}}{\partial x^2} + f(v^{\varepsilon, \delta}) + \frac{1}{\delta}(v^{\varepsilon, \delta} - h^1)^+ \frac{1}{\varepsilon}(v^{\varepsilon, \delta} - h^2)^+ + \sigma(v^{\varepsilon, \delta})\left[g_n(|u^{\varepsilon, \delta} - v^{\varepsilon, \delta}| \wedge 1)\dot{W}_1(x,t) + f_n(|u^{\varepsilon, \delta} - v^{\varepsilon, \delta}| \wedge 1)\dot{W}_2(x,t)\right]; \\
 u^{\varepsilon, \delta}(x,0) = u^1(x,0), \quad v^{\varepsilon, \delta}(x,0) = u^2(x,0).
\end{cases}
\] (2.17)

We may and will assume that \(f(u)\) is non-increasing. Otherwise, we consider \(\tilde{u} := e^{-Lt}u, \quad \tilde{v} := e^{-Lt}v\), where \(L\) is the Lipschitz constant in (F1), which
satisfy

\[
\frac{\partial \tilde{u}(x,t)}{\partial t} = \frac{\partial^2 \tilde{u}(x,t)}{\partial x^2} + e^{-Lt} f(e^{Lt} \tilde{u}(x,t)) - L\tilde{u}(x,t) \\
+ e^{-Lt} \sigma(e^{Lt} \tilde{u}(x,t)) \tilde{W}_1(x,t) + \eta_3(x,t) - \xi_3(x,t),
\]

\[
\frac{\partial \tilde{v}^n(x,t)}{\partial t} = \frac{\partial^2 \tilde{v}^n(x,t)}{\partial x^2} + e^{-Lt} f(e^{Lt} \tilde{v}(x,t)) - L\tilde{v}(x,t) + \eta^n_3(x,t) - \xi^n_3(x,t) \\
+ e^{-Lt} \sigma(e^{Lt} \tilde{v}^n(x,t)) \left[ g_n(|e^{Lt} \tilde{u} - e^{Lt} \tilde{v}^n| \land 1) \tilde{W}_1(x,t) \right] \\
+ f_n(|e^{Lt} \tilde{u} - e^{Lt} \tilde{v}^n| \land 1) \tilde{W}_2(x,t) ,
\]

\[u(x,0) = u^1(x,0), \quad v^n(x,0) = u^2(x,0).\]

The new drift \(e^{-Lt} f(e^{Lt} x) - Lx\) is non-increasing. Also, if \(\tilde{u}, \tilde{v}\) satisfy a successful coupling, so does \(u, v\). Note that all the coefficients in (2.17) are Lipschitz continuous. We can apply Proposition 2.1 to conclude that \(u^{\varepsilon, \delta}(x,t) \to u(x,t), v^{n, \varepsilon, \delta}(x,t) \to v^n(x,t)\) uniformly on \(S^1 \times [0,T]\) (for any \(T > 0\)) as \(\varepsilon, \delta \to 0\). As \(u^1(x,0) \geq u^2(x,0)\), as lemma 3.1 in [3], we can show that \(u^{\varepsilon, \delta} \geq v^{n, \varepsilon, \delta}\). Let

\[U^{n, \varepsilon, \delta}(t) = \int_{S^1} (u^{\varepsilon, \delta}(x,t) - v^{n, \varepsilon, \delta}(x,t)) dx.\] (2.18)

It follows from the above equation that

\[U^{n, \varepsilon, \delta}(t) = \int_{S^1} (u_1(x,0) - u_2(x,0)) dx + \int_0^t C^{n, \varepsilon, \delta}(s) ds + M^{n, \varepsilon, \delta}(t),\] (2.19)

where

\[C^{n, \varepsilon, \delta}(t) = \int_{S^1} \left\{ f(u^{\varepsilon, \delta}) - f(v^{n, \varepsilon, \delta}) + \frac{1}{\delta} (u^{\varepsilon, \delta} - h^1)^-(x,t) - \frac{1}{\delta} (v^{n, \varepsilon, \delta} - h^1)^-(x,t) \right. \]

\[- \left( \frac{1}{\varepsilon} (u^{\varepsilon, \delta} - h^2)^+(x,t) - \frac{1}{\varepsilon} (v^{n, \varepsilon, \delta} - h^2)^+(x,t) \right) \} dx \leq 0,
\]

\[M^{n, \varepsilon, \delta}(t) = \int_0^t \int_{S^1} \sigma(u^{\varepsilon, \delta}(x,s)) W_1(dx,ds) \]

\[- \int_0^t \int_{S^1} \sigma(v^{n, \varepsilon, \delta}(x,s)) g_n(|u^{\varepsilon, \delta} - v^{n, \varepsilon, \delta}| \land 1) \tilde{W}_1(dx,ds) \]

\[- \int_0^t \int_{S^1} \sigma(v^{n, \varepsilon, \delta}(x,s)) f_n(|u^{\varepsilon, \delta} - v^{n, \varepsilon, \delta}| \land 1) \tilde{W}_2(dx,ds).\]
Observe that
\[
\lim_{\varepsilon, \delta \to 0} U^{n, \varepsilon, \delta}(t) = U^n(t) := \int_{S^1} (u(x, t) - v^n(x, t)) dx, \quad (2.20)
\]
and
\[
\lim_{\varepsilon, \delta \to 0} M^{n, \varepsilon, \delta}(t) = M^n(t) := \int_0^t \int_{S^1} \sigma(u(x, s)) W_1(dx, ds) \\
- \int_0^t \int_{S^1} \sigma(v^n(x, s)) g_n(|u - v^n| \wedge 1) \dot{W}_1(dx, ds) \\
- \int_0^t \int_{S^1} \sigma(v^n(x, s)) f_n(|u - v^n| \wedge 1) \dot{W}_2(dx, ds). \quad (2.21)
\]
Letting $\varepsilon, \delta \to 0$ in (2.19) we see that
\[
U^n(t) = \int_{S^1} (u_1(x, 0) - u_2(x, 0)) dx + A^n(t) + M^n(t), \quad (2.22)
\]
where $A^n(t) = \lim_{\varepsilon, \delta \to 0} \int_0^t C^{n, \varepsilon, \delta}(s) ds$ is a continuous, adapted non-increasing process. Now, sending $n$ to $\infty$ we obtain
\[
U(t) = \int_{S^1} (u_1(x, 0) - u_2(x, 0)) dx + A(t) + M(t), \quad (2.23)
\]
where
\[
U(t) = \int_{S^1} (u(x, t) - v(x, t)) dx,
\]
\[
M(t) = \int_0^t \int_{S^1} \sigma(u(x, s)) W_1(dx, ds) \\
- \int_0^t \int_{S^1} \sigma(v(x, s))(1 - |u - v| \wedge 1) \frac{1}{2} \dot{W}_1(dx, ds) \\
- \int_0^t \int_{S^1} \sigma(v(x, s))(|u - v| \wedge 1) \frac{1}{2} \dot{W}_2(dx, ds),
\]
and $A(t) = \lim_{n \to \infty} A^n(t)$ a continuous, adapted non-increasing process. The existence of the limits of $A^n$ follows from the existence of the limit of $U^n$ and
$M^n$. Now we can modify the proof in [3] to obtain the successful coupling of $u$ and $v$. In view of the assumption on $\sigma$ and the boundedness of the walls $h^1, h^2$, it is easy to verify that
\[
\frac{d <M > (t)}{dt} \geq C_0 U(t)
\] 
for some positive constant $C_0$. Thus, there exists a non-negative adapted process $V(t)$ such that
\[
\frac{d <M > (t)}{dt} = U(t)V(t), \quad V(t) \geq C_0.
\] 
Let
\[
\phi(t) = \int_0^t V(s)ds, \\
X(t) = U(\phi^{-1}(t)).
\] 
Then the time-changed process $X$ satisfies the following equation
\[
X(t) = U(0) + \tilde{A}(t) + \int_0^t X^\sharp(s)dB(s),
\] 
where $B$ is a Brownian motion and $\tilde{A}$ is an adapted non-increasing process. Let $Y(t) = 2X^\sharp(t)$. Applying Ito's formula (before $Y$ hits 0) we obtain
\[
Y(t) = Y(0) + 2 \int_0^t \frac{1}{Y(s)}d\tilde{A}(s) - \frac{1}{2} \int_0^t \frac{1}{Y(s)}ds + B(t).
\] 
As $\tilde{A}$ is non-increasing, it follows that
\[
0 \leq Y(t) \leq Y(0) + B(t).
\] 
The property of one dimensional Brownian motion implies that $Y$ hits 0 with probability 1. Hence
\[
\lim_{t \to \infty} P\left( \sup_{x \in S^1} |u(x, t) - v(x, t)| \neq 0 \right) = 0.
\] 
Next let us consider the general case, i.e. we do not assume $u^1(x, 0) \geq u^2(x, 0), x \in S^1$. Consider a solution $v, u^1, u^2$ of the following SPDEs with
two reflecting walls
\[
\frac{\partial v(x,t)}{\partial t} = \frac{\partial^2 v(x,t)}{\partial x^2} + f(v(x,t)) + \sigma(v(x,t))\dot{W}_1(x,t) + \eta_v(x,t) - \xi_v(x,t),
\]
\[
\frac{\partial u^i(x,t)}{\partial t} = \frac{\partial^2 u^i(x,t)}{\partial x^2} + f(u^i(x,t)) + \eta_{u^i}(x,t) - \xi_{u^i}(x,t)
\]
\[+ \sigma(u^i(x,t))[(1 - |v - u^i| \wedge 1)\frac{1}{2}\dot{W}_1(x,t) + (|v - u^i| \wedge 1)\frac{1}{2}\dot{W}_2(x,t)],
\]
\[v(x,0) = \max_{i=1,2}\{u^i(x,0)\}.
\]

By following the arguments in the first part, we have
\[
\lim_{t \to \infty} P\left(\sup_{x \in S^1} |v(x,t) - u^i(x,t)| \neq 0\right) = 0, \ i = 1, 2.
\]
The inequality
\[
0 \leq \sup_{x \in S^1} |u^1(x,t) - u^2(x,t)| \leq \sum_{i=1}^{2} \left( \sup_{x \in S^1} |v(x,t) - u^i(x,t)| \right)
\]
implies
\[
\lim_{t \to \infty} P\left(\sup_{x \in S^1} |u^1(x,t) - u^2(x,t)| \neq 0\right) = 0.
\]

\[\square\]

3 Strong Feller property

In this section, we consider the strong Feller property of the solution of equation (1.1). Let $H = L^2(S^1)$. If $\varphi \in B_b(H)$ (the Banach space of all real bounded Borel functions, endowed with the sup norm), we define, for $x \in S^1$, $0 \leq t \leq T$ and $g \in H$,
\[
P_t\varphi(g) = \mathbb{E}\varphi(u(x,t,g)).
\]

**Definition 3.1.** The family \{\(P_t\)\} is called strong Feller if for arbitrary \(\varphi \in B_b(H)\), the function \(P_t\varphi(\cdot)\) is continuous for all \(t > 0\).

**Theorem 3.1.** Under the hypotheses (H1)-(H2), (F1)-(F3) and that \(p_1 \leq |\sigma(\cdot)| \leq p_2\) for some constants \(p_1, p_2 > 0\), then for any \(T > 0\) there exists a constant \(C'_T\) such that for all \(\varphi \in B_b(H)\) and \(t \in (0, T]\),
\[
|P_t\varphi(u^1_0) - P_t\varphi(u^2_0)| \leq \frac{C'_T\|\varphi\|_{\infty}|u^1_0 - u^2_0|_H}{\sqrt{t}}, \quad (3.1)
\]
for $u_1^0, u_2^0 \in H$ with $h^1(x) \leq u_0^1(x) \leq h^2(x)$, where $\|\varphi\|_\infty = \sup_{u_0} |\varphi(u_0)|$.

In particular, $P_t, \ t > 0$, is strong Feller.

**Proof.** Choose a non-negative function $\phi \in C^\infty_0(R)$ with $\int_R \phi(x) = 1$ and denote

$$f_n(\zeta) = n \int_R \phi(n(\zeta - y)) f(y) dy,$$

$$\sigma_n(\zeta) = n \int_R \phi(n(\zeta - y)) \sigma(y) dy,$$

$$k_n(\zeta, x) = n \int_R \phi(n(\zeta - y))(y - h^1(x))^- dy,$$

$$l_n(\zeta, x) = n \int_R \phi(n(\zeta - y))(y - h^2(x))^+ dy.$$  

So $f_n, \sigma_n, k_n, l_n$ are smooth w.r.t. $\zeta$. Let

$$u^{\varepsilon, \delta}_n(x, t, u_0) = \int_{S^1} G_t(x, y) u_0(y) dy + \int_0^t \int_{S^1} G_{t-s}(x, y) f_n(u^{\varepsilon, \delta}_n(y, s, u_0)) dy ds$$

$$+ \int_0^t \int_{S^1} G_{t-s}(x, y) \sigma_n(u^{\varepsilon, \delta}_n(y, s, u_0)) W(dy, ds)$$

$$+ \frac{1}{\delta} \int_0^t \int_{S^1} G_{t-s}(x, y) k_n(u^{\varepsilon, \delta}_n(y, s, u_0), y) dy ds$$

$$- \frac{1}{\varepsilon} \int_0^t \int_{S^1} G_{t-s}(x, y) l_n(u^{\varepsilon, \delta}_n(y, s, u_0), y) dy ds.$$  

Since $f_n(\zeta) \to f(\zeta), \sigma_n(\zeta) \to \sigma(\zeta), k_n(\zeta, x) \to (\zeta - h^1(x))^-$ and $l_n(\zeta, x) \to (\zeta - h^2(x))^+$ as $n \to \infty$, we can show that for any fixed $\varepsilon, \delta$ and $p \geq 1$,

$$\lim_{n \to \infty} \sup_{t \in [0,T]} \mathbb{E} \left( |u^{\varepsilon, \delta}_n(t, \cdot, u_0) - u^{\varepsilon, \delta}(t, \cdot, u_0)|^p_H \right) = 0.$$  

By Lemma 7.1.5 in [2] and Proposition 2.1, it is enough to prove that there exists a constant $C'_T$, independent of $\varepsilon, \delta$ and $n$, such that

$$|P^{n,\varepsilon, \delta}_t \varphi(u_0^1) - P^{n,\varepsilon, \delta}_t \varphi(u_0^2)| \leq \frac{C'_T}{\sqrt{t}} \|\varphi\|_\infty |u_0^1 - u_0^2|_H,$$  

(3.2)

where $P^{n,\varepsilon, \delta}_t \varphi(u_0) := \mathbb{E}(\varphi(u^{\varepsilon, \delta}(\cdot, u_0)))$ and $u_0^1, u_0^2 \in H$.  

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From Theorem 5.4.1 in [2], \( u_{n}^{\varepsilon,\delta}(\cdot, u_{0}) \) is continuously differentiable w.r.t. \( u_{0} \). Denote by \( X_{n}^{\varepsilon,\delta}(x, t) := (Du_{n}^{\varepsilon,\delta}(\cdot, u_{0}))(x, t) \) the directional derivative of \( u_{n}^{\varepsilon,\delta}(\cdot, u_{0}) \) at \( u_{0} \) in the direction of \( \bar{u}_{0} \) and it satisfies the mild form of a SPDE

\[
X_{n}^{\varepsilon,\delta}(x, t) = \int_{S^1} G_t(x, y)\bar{u}_0(y)dy + \int_{0}^{t} \int_{S^1} G_{t-s}(x, y)f_{n}'(u_{n}^{\varepsilon,\delta}(y, s, u_{0}))X_{n}^{\varepsilon,\delta}(y, s)dyds
+ \int_{0}^{t} \int_{S^1} G_{t-s}(x, y)\sigma_{n}'(u_{n}^{\varepsilon,\delta}(y, s, u_{0}))X_{n}^{\varepsilon,\delta}(y, s)W(dy, ds)
+ \frac{1}{\delta} \int_{0}^{t} \int_{S^1} G_{t-s}(x, y)\frac{\partial}{\partial \zeta} k_{n}(u_{n}^{\varepsilon,\delta}(y, s, u_{0}), y)X_{n}^{\varepsilon,\delta}(y, s)dyds
- \frac{1}{\varepsilon} \int_{0}^{t} \int_{S^1} G_{t-s}(x, y)\frac{\partial}{\partial \zeta} l_{n}(u_{n}^{\varepsilon,\delta}(y, s, u_{0}), y)X_{n}^{\varepsilon,\delta}(y, s)dyds.
\]

Since \( \frac{\partial}{\partial \zeta} k_{n}(u_{n}^{\varepsilon,\delta}(y, s, u_{0}), y) \leq 0, \frac{\partial}{\partial \zeta} l_{n}(u_{n}^{\varepsilon,\delta}(y, s, u_{0}), y) \geq 0 \), we use the similar arguments as that in [14] and to get

\[
\sup_{\varepsilon,\delta \geq 0, t \in [0,T]} E\left( \int_{S^1} (X_{n}^{\varepsilon,\delta}(y, t))^2 dy \right) \leq C|\bar{u}_0|^2_H,
\]

where \( C \) is a constant. By Elworthy-Li formula (Lemma 7.1.3 in [2]), we obtain

\[
| \langle DP_{t}\varphi(u_0), \bar{u}_0 \rangle |^2 \leq \frac{C}{p_{1}(t)} \| \varphi \|_\infty^2 |\bar{u}_0|^2_H.
\]

This implies inequality (3.2) which completes the proof. \( \square \)

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