Bipartite mixed states as quantum teleportation channels studied under coherent and incoherent basis

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Abstract
Quantum coherence and quantum entanglement are two different manifestations of the superposition principle. In this article we show that the right choice of basis to be used to estimate coherence is the separable basis. The quantum coherence estimated using the Bell basis does not represent the coherence in the system, since there is a coherence in the system due to the choice of the basis state. We first compute the entanglement and quantum coherence in the two qubit mixed states prepared using the Bell states and one of the state from the computational basis. The quantum coherence is estimated using the \( \ell_1 \)-norm of coherence, the entanglement is measured using the concurrence and the mixedness is measured using the linear entropy. Then we estimate these quantities in the Bell basis and establish that coherence should be measured only in separable basis, whereas entanglement and mixedness can be measured in any basis. We then calculate the teleportation of these mixed states and find the regions where the states have a fidelity greater than the classical teleportation fidelity. We also examine the violation of the Bell-CHSH inequality to verify the quantum nonlocal correlations in the system. The estimation of the above mentioned quantum correlations, teleportation fidelity and the verification of Bell-CHSH inequality is also done for bipartite states obtained from the tripartite systems by

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the tracing out of one of their qubits. We find that for some of these states teleportation is possible even when the Bell-CHSH inequality is not violated, signifying that nonlocality is not a necessary condition for quantum teleportation.

Keywords: Coherence, Concurrence, Linear Entropy, Teleportation Fidelity, Mixedness, Bell-CHSH inequality

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1 Introduction:

In quantum mechanics, superposition gives rise to two interesting phenomena, one of which is coherence and the other is known as entanglement[1, 2]. Although the entanglement and coherence are two different manifestations of the superposition principle, they are different in their nature and properties. The quantum entanglement measures the non-local correlations in the systems and the quantum coherence is a measure of the total quantumness in the system. While quantum entanglement, when measured in any reference basis, retains its trait, quantum coherence is a basis dependent quantity [3, 4]. This has generated an upsurge of interest on coherence and, for the last several years the exploration of this feature has gained momentum and has been studied in many different areas, alongside the study of entanglement [5, 6, 7, 8, 9]. A considerable amount of research works on entanglement of bipartite and as well as of multipartite system have already been done[10, 11, 12, 13]. The focus of this article will be on bringing coherence and entanglement together in a bipartite two qubit mixed state scenario and to observe how these two features together contribute to the capacity of transferring information via such mixed states in a protocol such as teleportation. Such a capacity of states is, called teleportation fidelity.

Another interesting fact in the class of mixed states is that there exists a mixed entangled state, known as Werner state, which does not violate Bell’s inequality[14]. Moreover Werner state can be used as a quantum teleportation channel (average optimal teleportation fidelity exceeding $\frac{2}{3}$) even without violating the Bell-CHSH inequality[13]. We know Bell inequality marks the boundary between the classical and quantum natures. The relation between Bell violation and entanglement has been studied already for a class of two qubit maximally entangled mixed states [10, 11, 12, 13]. Hence it would also be interesting to examine how coherence and Bell violation are connected to one another for the class of states considered in this paper. It has been observed that in most of the cases the computational techniques of measuring entanglement, coherence, mixedness et. al involve computational basis. In this article, we shall make a comparative study of the quantification of these features with respect to both computational basis and Bell basis. Recently in an article [15] it has been shown that for any general two qubit density matrix $\rho$, $\ell_1$— norm of coherence, which is a measure of quantum coherence is always greater than or equal to concurrence of that density matrix. We will show by considering a class of mixed states that, the relation between coherence and concurrence is actually dependent on the choice of the basis under which they are measured and in a product basis where the multipartite basis is a product of the individual bases, then the $\ell_1$— norm of coherence is greater than or equal to that of the entanglement as measured by concurrence. When we use an entangled basis like Bell basis this inequality does not hold. This fact motivates us to explore how teleportation fidelities of the class of mixed states behave in accordance with coherence, entanglement and mixedness along with the Bell violation for these class of mixed states.

The paper is organized as follows. In section 2 we discuss the quantifiers of coherence, entangle-
ment and mixedness. It is to be pointed out that, although relative entropy of coherence is defined in the next section, in most of the calculations we have considered the $l_1$– norm of coherence. The relative entropy and $\ell_1$– norm, however, satisfies all the characteristics of good coherence measure. The mixedness of the class of states will also be studied and their relationship with both coherence and concurrence is investigated. The basic objective of studying any quantum mechanical state (pure or mixed) is to check its utility in information processing. Also we describe the two different classes of mixed states which we use in our investigations. In section 3, we compute the different measures for the two class of states introduced in the previous section in both the computational basis and in the Bell basis. The teleportation fidelity of these two classes of states is computed in the section 4 and the Bell-CHSH violation for these states is investigated in section 5. A comprehensive analysis of two qubit reduced density matrices derived from tripartite states is given in section 6. For all these states, we find quantum coherence, concurrence, mixedness, teleportation and the Bell-CHSH inequality. Finally we present our conclusions in section 7.

2 Quantumness measures and Quantum States

2.1 Measures of coherence, entanglement and mixedness

2.1.1 Quantum coherence:

Quantum coherence originates from the principle of superposition and hence it manifests itself in the off-diagonal elements of the density matrix corresponding to the state. A widely used quantum coherence quantifier is the $\ell_1$-norm measure and in our work we use the $C_{\ell_1}$ to represent this. Since the quantum coherence is a basis dependent quantity, we fix the reference basis $|i\rangle$ for a given quantum state. The $l_1$– norm of coherence is defined as

$$C_{\ell_1}(\rho) = \sum_{i,j;i\neq j} |\rho_{ij}|,$$

where $\rho_{ij} = \langle i| \rho |j\rangle$ is the matrix elements corresponding to the $i^{th}$ row and $j^{th}$ column. Hence if we consider a bipartite two qubit system in the standard computational basis, the reference basis is fixed at $\{|00\rangle, |01\rangle, |10\rangle, |11\rangle\}$. An alternative method of quantifying coherence is the relative entropy of coherence, denoted by $C_r$ and which is based on the relative entropy. Thus for a state $\rho$, the relative entropy of coherence is defined as

$$C_r(\rho) = \min_{\sigma \in I} S(\rho \parallel \sigma),$$

where $S$ is the von-Neumann entropy. The minimum is taken over the set of the incoherent states $I$ which are states without quantum coherence. In Ref. [4] it was proved that the expression for the relative entropy also reduces to the form

$$C_r(\rho) = S(\rho_d) - S(\rho),$$

where $\rho_d$ is the dephased state in the reference basis $\{|i\rangle\}$ i.e. the state obtained from $\rho$ by deleting all off-diagonal entries. Any good measure of coherence must satisfy the properties for quantum coherence proposed in [4] and both the $l_1$– norm of coherence and relative entropy of coherence are the most general coherence monotones satisfying these properties.
2.1.2 Concurrence:

Entanglement is an important quantum resource which arises due to nonlocal correlations between quantum systems. There are several measures for quantifying entanglement, like concurrence, relative entropy of entanglement, negativity and log negativity. Of these measures concurrence is the widely accepted measure for two qubit system and it works equally for pure and mixed states. The concurrence of a quantum state $\rho$ is

$$C(\rho) = \max \{0, \sqrt{\lambda_1} - \sqrt{\lambda_2} - \sqrt{\lambda_3} - \sqrt{\lambda_4}\}, \quad (4)$$

where $\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \lambda_4$ are the eigenvalues of the matrix $\rho \tilde{\rho}$. The spin-flipped density matrix $\tilde{\rho}$ is

$$\tilde{\rho} = (\sigma_y \otimes \sigma_y) \rho^*(\sigma_y \otimes \sigma_y), \quad (5)$$

where $\sigma_y$ is the Pauli spin matrix in the $y$-basis and $\tilde{\rho}$ is in the same basis as $\rho$, and $\rho^*$ is the complex conjugate of the density matrix $\rho$.

2.1.3 Mixedness:

A quantum state is more often studied as an isolated system. This is not exactly true and quantum states are in contact with an external environment which influences these states externally. The environment causes a degradation of the quantumness of these states and causes them to decohere and become classical. The quantumness of a state is maximal when it is pure and once it interacts with the environment, it usually loses its quantumness and consequently it appears mixed. A completely classical state is the maximal mixed state. When the state changes from being pure to becoming mixed, there is an entropy introduced which is quantified using the linear entropy. For an arbitrary $d$-dimensional quantum mixed state $\rho$, the mixedness is defined using the normalized linear entropy $L(\rho)$ and is defined as[16]

$$L(\rho) = \frac{d}{d-1} (1 - \text{Tr}(\rho^2)). \quad (6)$$

Here the quantity $\text{Tr}(\rho^2)$ describes the purity of the quantum system. For a two-qubit system, the value of $L$ ranges from 0 to 1. The entropy $L(\rho) = 0$ for any pure state, and the maximum value $L(\rho) = 1$ is attained for the maximally mixed state $I_4$.  

2.2 Quantum states

The Bell states are a set of maximally entangled two qubit pure states and they are as follows:

$$|\phi^\pm\rangle = \frac{1}{\sqrt{2}} \{ |00\rangle_{AB} \pm |11\rangle_{AB} \}; \quad |\varphi^\pm\rangle = \frac{1}{\sqrt{2}} \{ |01\rangle_{AB} \pm |10\rangle_{AB} \}. \quad (7)$$

Here $A$ and $B$ denote the two particles Alice and Bob holding one qubit each. These four states are orthogonal to each other and forms a basis which is in general referred to as the Bell basis. The elements of the computational basis corresponding to the two qubit system can be represented in
terms of the Bell basis elements as

\[
|00\rangle = \frac{1}{\sqrt{2}}\{|\phi^+\rangle_{AB} + |\phi^-\rangle_{AB}\}, \\
|01\rangle = \frac{1}{\sqrt{2}}\{|\varphi^+\rangle_{AB} + |\varphi^-\rangle_{AB}\}, \\
|10\rangle = \frac{1}{\sqrt{2}}\{|\varphi^+\rangle_{AB} - |\varphi^-\rangle_{AB}\}, \\
|11\rangle = \frac{1}{\sqrt{2}}\{|\phi^+\rangle_{AB} - |\phi^-\rangle_{AB}\}.
\]

(8)

While we note that these two different bases can equally describe the Hilbert space of the two qubit systems, the computational basis is inherently separable whereas the Bell basis encodes nonlocal correlations in the basis vectors. Following ref. [20, 21] we can construct a class of mixed states as given below:

\[
\rho^1 = p_1|\phi^\pm\rangle_{AB}\langle\phi^\pm| + (1 - p_1)|00\rangle_{AB}\langle00| \\
\rho^2 = p_1|\phi^\pm\rangle_{AB}\langle\phi^\pm| + (1 - p_1)|11\rangle_{AB}\langle11| \\
\rho^3 = p_1|\varphi^\pm\rangle_{AB}\langle\varphi^\pm| + (1 - p_1)|01\rangle_{AB}\langle01| \\
\rho^4 = p_1|\varphi^\pm\rangle_{AB}\langle\varphi^\pm| + (1 - p_1)|10\rangle_{AB}\langle10|,
\]

(9)

and

\[
\varrho^1 = p_2|\phi^\pm\rangle_{AB}\langle\phi^\pm| + (1 - p_2)|01\rangle_{AB}\langle01| \\
\varrho^2 = p_2|\phi^\pm\rangle_{AB}\langle\phi^\pm| + (1 - p_2)|10\rangle_{AB}\langle10| \\
\varrho^3 = p_2|\varphi^\pm\rangle_{AB}\langle\varphi^\pm| + (1 - p_2)|00\rangle_{AB}\langle00| \\
\varrho^4 = p_2|\varphi^\pm\rangle_{AB}\langle\varphi^\pm| + (1 - p_2)|11\rangle_{AB}\langle11|.
\]

(10)

where \(0 \leq p_1 \leq 1\) and \(0 \leq p_2 \leq 1\) are the respective mixing parameters. In future discussions in this manuscript, we refer to the states described in Eq. (9) as the class-1 mixed states. Similarly, the states given in Eq. (10) are referred to as the class-2 mixed states. Each one of these equations Eq. (9) and Eq. (10) contain 16 different mixed states and hence in total we have thirty two different mixed states. A quantum state in \(C^2 \otimes C^2\) or \(C^2 \otimes C^3\) is separable if and only if it obeys the PPT criteria. Any violation of this criteria in the \(C^2 \otimes C^2\) or \(C^2 \otimes C^3\) implies that the state is entangled. Using this criteria we can show that the classes shown in Eq. (9) and (10) are mixed entangled states under the conditions \(0 < p_1 < 1\) and \(0 < p_2 < 1\). When \(p_1\) and \(p_2\) are zero, they are separable pure states and when \(p_1\) and \(p_2\) are both one, they are pure entangled states. In the following sections we investigate the coherence, concurrence and mixedness of these two class of states. For the sake of simplicity we denote the computational basis with the notation \(B_1 = \{|00\rangle, |01\rangle, |10\rangle, |11\rangle\}\) and the Bell basis \(B_2 = \{|\phi^+\rangle, |\varphi^+\rangle, |\varphi^-\rangle, |\phi^-\rangle\}\).

3 Choice of measurement basis

3.1 Measurement in the computational basis

The quantum coherence, concurrence and the mixedness of the two class of states viz class-1 states given through Eq. (9) and the class-2 states given in Eq. (10) are computed in the computational
Quantum Correlations

\[ C_\ell(\rho) \]

\[ C(\rho) \]

\[ L(\rho) \]

\[ C_\ell(\varrho) \]

\[ C(\varrho) \]

\[ L(\varrho) \]

Figure 1: The quantum coherence measured using the \( \ell_1 \) norm of coherence \( C_\ell \), the entanglement measured using the concurrence and the mixedness estimated using the linear entropy \( L \) are given for (a) class-1 type of states as a function of the probability \( p_1 \) and (b) class-2 type of states as a function of the probability \( p_2 \). In this plot we use the computational basis for estimating the above mentioned quantifiers.

basis. For the class-1 states the results obtained are

\[
C_\ell(\rho^i) = p_1, \\
C(\rho^i) = p_1, \\
L(\rho^i) = \frac{4}{3} p_1 (1 - p_1), \forall i \in \{1, 2, 3, 4\},
\]

(11)

The results corresponding to the class-2 states are

\[
C_\ell(\varrho^i) = p_2, \\
C(\varrho^i) = p_2, \\
L(\varrho^i) = \frac{8}{3} p_2 (1 - p_2), \forall i \in \{1, 2, 3, 4\}.
\]

(12)

From the analytic expressions we find that the coherence and concurrence are equal to each other. This happens due to the following reason: These two class of states belong to the wider classification of states called the \( X \)-states in which the density matrix contains elements only along the main diagonal and the anti-diagonal. A generic \( X \)-state has the following form as given below:

\[
\rho^m = \begin{pmatrix}
  a & 0 & 0 & g \\
  0 & b & f & 0 \\
  0 & f^* & c & 0 \\
  g^* & 0 & 0 & d
\end{pmatrix}.
\]

(13)

For this state one can find the analytic expression of concurrence to be:

\[
C(\rho^m) = 2 \max \left\{ 0, |f| - \sqrt{ad}, |g| - \sqrt{bc} \right\}.
\]

(14)

For the class-1 set of states in Eq. (9) either the set \( \{a, g, g^*, d\} \) or the set \( \{b, c, f, f^*\} \) which will give us the result to be either \( C(\rho) = 2g \) or \( C(\rho) = 2f \). The class-2 set of states in Eq. (10) has
five elements which again gives us the result of either \( C(\rho) = 2g \) or \( C(\rho) = 2f \) depending on which element is present. The variation of quantum coherence, concurrence and mixedness is shown in Fig. (1) as a function of the mixing parameters. For the class-1 set of states the result are shown in Fig. 1(a). In Fig. 1(b), the results corresponding to class-2 set of states are given. From both these plots we find that the coherence and the concurrence increases linearly with \( p_1 \) and \( p_2 \) as expected from the analytic result. The mixedness is maximum at \( p_1 = 0.5 \) and \( p_2 = 0.5 \) when the states are equally present. In the extreme cases when \( p_1 = p_2 = 0 \) and \( p_1 = p_2 = 1 \) the states are pure and consequently the mixedness vanishes.

### 3.2 Measurement in the Bell basis

In this part we calculate the concurrence, quantum coherence and mixedness of the two classes of states namely the class-1 and the class-2 in the Bell basis. It is well known that the Bell states are the set of maximally entangled states in the two qubit space. They are linearly independent, hence forms a basis and hence any two-qubit state whether entangled or not can be expressed in the Bell basis. In this subsection we look at the implications of computing the quantifiers like concurrence, coherence and mixedness in the Bell basis. We begin by expressing the density matrix in the Bell basis \( \rho^{(b)} \). To compute the concurrence we find \( \tilde{\rho}^{(b)} \) as well in the same Bell basis using Eq. (5). From the eigenvalues of the density matrix we can find the concurrence of the system. For computing the quantum coherence, we use the density matrix in the Bell basis \( \rho^{(b)} \) and find the distance to the closest diagonal state \( \rho_d^{(b)} \) using the \( \ell_1 \)-norm coherence measure. The mixedness of the state \( \rho^{(b)} \) is found using the expression in Eq. (6). Below we summarize the results of our calculations. For the class-1 type mixed states we have

\[
\begin{align*}
C_{\ell_1}(\rho^i) &= 1 - p_1, \\
C(\rho^i) &= p_1, \\
L(\rho^i) &= \frac{4}{3}p_1(1 - p_1), \forall i \in \{1, 2, 3, 4\}, \\
\end{align*}
\]

and for the class-2 type mixed states we have

\[
\begin{align*}
C_{\ell_1}(\varrho^i) &= 1 - p_2, \\
C(\varrho^i) &= p_2, \\
L(\varrho^i) &= \frac{8}{3}p_2(1 - p_2), \forall i \in \{1, 2, 3, 4\}.
\end{align*}
\]

The variation of the quantum coherence, concurrence and mixedness with the mixing probabilities for the class-1 and class-2 type of states are shown through the plots in Fig. 2(a) and Fig. 2(b) respectively. From the plots we can see that the coherence and the concurrence are inversely proportional to each other. When \( p_1 (p_2) \) is zero, the coherence \( C_{\ell_1}(\rho) \) \( (C_{\ell_1}(\varrho)) \) is maximum and goes to zero when \( p_1 (p_2) \) is unity. In the case of concurrence we find that \( C(\rho)(C(\varrho)) \) is minimum when the probabilities are zero and maximum when the probabilities are set to one. This behavior is in contrast to the results obtained when these quantifiers are measured in the computational basis where we observed that the coherence and concurrence varied in the same manner. Now we explain the reason behind the contrasting results when we use these two different measurement bases. The computational basis is a separable basis in the sense that the two qubit basis states can be written as a tensor product of the single qubit basis states. Hence these basis states do not contain any type of quantum correlations or coherence within them. Meanwhile if we look at the Bell basis the two qubit basis states are the entangled states and they cannot be written as tensor product of the
individual basis states. Hence these basis states inherently contain entanglement and coherence within them. When we use the concurrence measure Eq. (4) to calculate the entanglement, we find the eigenvalues of the product of the density matrix and its spin-flipped version, with both these matrices expressed in the same Bell basis. This computation gives us a basis independent result and hence the concurrence is the same when we use either the computational basis or the Bell basis. In fact the concurrence measure is an invariant quantity with respect to the choice of the basis. From the results on quantum coherence we find that the two different calculations arising from the computational basis and the Bell basis give opposite results. This is because when we use the Bell basis for calculating quantum coherence, the density matrix is transformed to the Bell basis and then we use the same procedure as described through Eq. (1). It is expected that this coherence need not be quantitatively same as the coherence measured using the computational basis, since quantum coherence as measured using Eq. (1) is a basis dependent quantity. But what we are observing is a qualitative difference in behaviour where the relation between entanglement and coherence changes from being directly proportional to inversely proportional. This difference is because in the Bell basis there is coherence hidden within the basis which is not accounted for. Hence in the calculation of coherence we should always choose a basis in which the multiqubit basis states are tensor product of the single qubit basis states to ensure no coherence is left unaccounted for.

The mixedness of these quantum states can be computed from the linear entropy of the system which is defined in Eq. (6). In general the mixedness can take values from 0 to 1. From the plots of the two class of states we find that the maximal value of mixedness of the class-1 states is lower than that of the class-2 states. This is because the class-1 states have only two diagonal elements whereas the class-2 states have three diagonal elements and mixedness is the distance to the maximally mixed state $I/4$ where $I$ is the Identity matrix which has four diagonal elements. When the mixing parameter is 0.5, the class-1 states and the class-2 states attain their maximal mixedness which are $L(\rho) = 1/3$ and $L(\varrho) = 2/3$ respectively. The qualitative and quantitative nature of mixedness is the same when it is computed either in the computational basis or the Bell basis.
4 Teleportation fidelity of the class of mixed states:

Teleportation provides us a different notion of quantum inseparability. In this method, a given quantum state is separated into classical and quantum parts. Later these quantum and classical parts can be combined to reconstruct the state with perfect fidelity. For this process, we use an entangled pair of states as a quantum channel. In general a singlet state $|\psi^-\rangle = \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle)$ is used and the state is shared between Alice (sender) and Bob (receiver). Now Alice decides to send an unknown reference state of the form $\alpha|0\rangle + \beta|1\rangle$ to Bob. To achieve this Alice performs Bell measurements jointly on her qubit and the reference qubit. Her measurement results are then conveyed to Bob by classical communication. Finally Bob applies some unitary operators on his qubit depending upon Alice’s measurement results. In this process the unknown quantum state is destroyed at Alice’s end and is recreated at Bob’s end.

The teleportation protocol as introduced by Bennett et al. [22] was based on using pure states as quantum channel. Later Popescu in Ref. [23] found that the pairs of mixed states can also be used for teleportation a feature known as probabilistic teleportation [24] or imperfect teleportation. However there is a limitation to kind of quantum state which can be used in teleportation. This limitation is captured by the measure called teleportation fidelity. A quantum channel is useful for teleportation only if its fidelity exceeds $\frac{2}{3}$ which is the maximum possible fidelity achievable by means of Local operations and classical communication (LOCC) and is known as the classical fidelity [23, 26]. A well known two qubit mixed state is the Werner state and has the form

$$\rho_w = \frac{1 - F_w}{3} I_4 + \frac{4F_w - 1}{3} |\psi^-\rangle\langle\psi^-|$$

where $|\psi^-\rangle = (|01\rangle - |10\rangle)/\sqrt{2}$ is the maximally entangled singlet and $F_w$ is the maximal singlet fraction in the Werner state. When the maximal singlet fraction $F_w$ lies in the range $\frac{1}{2} \leq F_w \leq \frac{3+\sqrt{2}}{4\sqrt{2}}$ the Werner state satisfies the Bell-CHSH inequality, but the state is still entangled in this region. This is because a quantum state which violates Bell-CHSH inequality is said to exhibit nonlocality a form of quantum correlation which is much stronger than entanglement and in the hierarchy of quantum correlations is more rarer than entanglement [25]. Hence Werner states can be used as quantum teleportation channel with average optimal fidelity exceeding $\frac{2}{3}$ even though it does not violate the Bell-CHSH inequality in the given region [13].

The utility of the class of maximally and non-maximally entangled states in quantum teleportation has been studied in Ref. [13]. Below we examine the teleportation fidelity of the class of states defined in the previous section. For an arbitrary density matrix $\rho$, the optimal teleportation fidelity is

$$f_T(\rho) = \frac{1}{2} \left[ 1 + \frac{N(\rho)}{3} \right]$$

where $N(\rho) = \sum_{i=1}^{3} \sqrt{u_i}$. Here, $u_i$’s are the eigenvalues of the matrix $T^\dagger T$. The elements of $T$, denoted by $t_{st}$ are

$$t_{st} = T^\dagger (\rho \sigma_s \otimes \sigma_t)$$

and $\sigma_j$’s denote the Pauli spin matrices. In terms of the quantity $N(\rho)$, one can say that any mixed state $\rho$ is useful for standard teleportation if and only if [27] $N(\rho) > 1$. For the class-1 states defined in Eq. (9), $\forall i, N(\rho^i) > 1$ with $0 < p_1 \leq 1$ and the optimal teleportation fidelity defined in Eq. (18) for these states is

$$f_T(\rho^i) = \frac{2 + p_1}{3}.$$
Now for the class-2 states defined in Eq. (10), we have \( \forall j, N(\rho^j) > 1 \) when \( 0.5 \leq p_2 \leq 1 \). The optimal teleportation fidelity of these states is

\[
f_T(\rho^j) = \frac{1 + 2p_2}{3}. \tag{21}
\]

These results hold both in the computational basis and Bell basis. It is easy to observe analytically that the teleportation fidelity of the class-1 states always exceeds the classical teleportation value of 2/3. Hence these states can be used as teleportation channels for all values of mixing parameter. As for class-2 states it is observed that in the range \( 0 \leq p_2 < 1/2 \) the teleportation fidelity of the states is less than the classical teleportation fidelity and is not suitable for quantum teleportation. For the range \( 1/2 \leq p_2 \leq 1 \) however, the teleportation fidelity of the states is greater than the classical teleportation fidelity of 2/3 and hence it is suitable for quantum teleportation. Hence we find the class-1 states are better candidates for quantum teleportation in comparison to class-2.

5 Bell-CHSH violation for the class of mixed states

The paradox which arose due to the Einstein-Podolsky-Rosen thought experiment also known as EPR paradox, where they concluded that quantum mechanical description of nature is incomplete since it goes against our common sense perceptions of locality and reality. To verify the existence of non-local correlations, Bell proposed an inequality and a simpler form of this inequality is the CHSH inequality. The verification of the Bell-CHSH inequality for the class-1 and class-2 mixed states is defined in Eq. (9) and in Eq. (10) is carried out in this section.

Any quantum state described by the density operator \( \rho \) violates the Bell-CHSH inequality if and only if the following condition is satisfied:

\[
M(\rho) = \max_{i > j} (u_i + u_j) > 1. \tag{22}
\]

Here \( u_i \)'s are eigenvalues of the matrix \( T^\dagger T \) [29, 30]. The elements of the matrix \( T \) are defined in the Eq. (19) and \( \sigma_3 \)'s are the Pauli matrices.

Bell violation of the class-1 states in the computational basis: For the class-1 states defined in Eq. (9), the eigenvalues of the matrix \( T^\dagger T \) are \( u_1 = p_1^2 + 1, u_2 = p_2^2 + 1 \) and \( u_3 = 2p_1^2 \). The state \( M(\rho^j) \) exceeds 1 for \( 0 < p_1 \leq 1, \forall i \). Hence we can conclude that the class-1 type of mixed states violates the Bell-CHSH inequality for all values of the mixing parameter.

Bell violation of the class-2 states in the computational basis: The eigenvalues of the \( T^\dagger T \) matrix are \( v_1 = p_2^2, v_2 = p_2^2 \) and \( v_3 = (2p_2 - 1)^2 \). Using this we compute the three possible different summations namely \( e_1 = v_2 + v_1 = 2p_2^2, e_2 = v_3 + v_1 = 5p_2^2 - 4p_2 + 1 \) and \( e_3 = v_3 + v_2 = 5p_2^2 - 4p_2 + 1 \). In the range \( \frac{1}{3} < p_2 < 1 \), \( M(\rho^j) = 2p_2^2 \) and for the region \( 0 < p_2 < \frac{1}{3} \), \( M(\rho^j) = 5p_2^2 - 4p_2 + 1 \) based on the maximum value. We find that the quantity \( M(\rho^j) = 5p_2^2 - 4p_2 + 1 \) for \( 0.8 < p_2 < 1 \). Similarly, the quantity \( M(\rho^j) = 2p_2^2 > 1 \) when \( 0.7071 < p_2 < 1 \). Hence we see that the class 2 mixed states \( \rho^j \) violate Bell-CHSH inequality \( 0.7071 < p_2 < 1, \forall j \). While these quantum states are correlated for any value of \( p_2 \), the states have non-locality only in the region \( 0.7071 < p_2 < 1 \).

From these observations we find that the class-1 type mixed states violate the Bell-CHSH inequality for all values of the mixing parameter \( p_1 \). In the case of the class-2 type mixed states, the state violates Bell inequality when \( p_2 > 0.7071 \), while for the region \( 0.5 < p_2 \leq 0.7071 \) the states satisfies the Bell-CHSH inequality and in both these cases these type of mixed states have a teleportation fidelity which is higher than the classical teleportation fidelity of 2/3. If we use the Bell basis the results obtained are consistent with those of the computational basis which is shown in this section.
6 Mixed bipartite states derived from tripartite states

The bipartite mixed states in Eq.(9) and Eq.(10) are probabilistic mixtures of two qubit pure states. To achieve this experimentally one might have to generate pure states and it is well known that perfect generation of pure states is an experimental challenge. An alternative is to generate tripartite quantum states and obtain the two qubit mixed states by tracing out one of the qubit. We use the bipartite mixed states obtained from the reduced states of some well known pure as well as three qubit mixed states. In the first subsection we look at the reduced two qubit density matrices obtained from pure three qubit systems. The second section presents results where the two qubit mixed states are obtained from three qubit mixed states.

6.1 Bipartite mixed states derived from pure tripartite states

In this part we consider three qubit pure entangled states and consider the reduced density matrices arising out of them by tracing out any one of the qubit. The three qubit pure states can be entangled in two different ways namely the three way entangled GHZ state and the two way entangled $W$ state. In the $GHZ$ state all the three qubits are entangled in such a way that loss of any one of the qubits destroys all the entanglement in the system. When a qubit is traced out in a three qubit $W$ state the remaining bipartite system still has residual entanglement in the system. These two states are classified into two distinct SLOCC (Stochastic Local Operations and Classical Communication) classes since a state in a given class cannot be transformed to a state in another class using only SLOCC operations.

$SLOCC$ class of states: The GHZ state is tripartite maximally entangled state in which the loss of even a single qubit makes the quantum state disentangled. The GHZ state has the form $|GHZ\rangle = (|000\rangle + |111\rangle)/\sqrt{2}$. and for this state the two qubit reduced density matrix is

$$\rho_{AB}^g = \rho_{BC}^g = \rho_{AC}^g = \begin{pmatrix} \frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 \end{pmatrix}, \quad (23)$$

where $\rho_{AB}^g = \text{Tr}_C \rho_{ABC}^g$ is the reduced two qubit density matrix and $\rho_{ABC}^g = |GHZ\rangle\langle GHZ|$. Computing the various quantum correlations for this state (23) we find the concurrence $C = 0$, coherence $C_{\ell_1} = 0$ and mixedness $L = 2/3$. The teleportation fidelity of the quantum state $f_T(\rho_{AB}^g) = f_T(\rho_{BC}^g) = f_T(\rho_{CA}^g) = 0$. Since there is no entanglement in the two qubit reduced density matrix of a $GHZ$ state, the state (23) is not useful for any kind of teleportation.

A $W$ state is a three qubit entangled state, in which entanglement is shared only between the pairs of qubits and there is no genuine entanglement as in a $GHZ$ state. The three qubit $W$ state reads: $|W\rangle = (|001\rangle + |010\rangle + |100\rangle)/\sqrt{3}$. The reduced density of the $W$ state is

$$\rho_{AB}^W = \rho_{BC}^W = \rho_{AC}^W = \begin{pmatrix} \frac{1}{3} & 0 & 0 & 0 \\ 0 & \frac{1}{3} & \frac{1}{3} & 0 \\ 0 & \frac{1}{3} & \frac{1}{3} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \quad (24)$$

Representing the two qubit reduced density matrix $\rho^m = \rho_{AB}^W = \rho_{BC}^W = \rho_{AC}^W$, we give the coherence,
concurrency, mixedness and teleportation fidelity below:

\[
\begin{align*}
C_{\ell_1}(\rho^m) &= \frac{2}{3}, \\
C(\rho^m) &= \frac{2}{3}, \\
L(\rho^m) &= \frac{16}{27}, \\
f_T(\rho^m) &= \frac{7}{9}.
\end{align*}
\] (25)

The bipartite reduced density matrices obtained from the $W$-states have a teleportation fidelity higher than $2/3$ and so they can be used in Quantum teleportation.

**$W\bar{W}$ state:** The three qubit $W\bar{W}$ state is a pure state which is a linear superposition of the $W$ state and the $\bar{W}$ state (which in fact is a spin flipped version of the $W$ state). The expression for this tripartite state reads:

\[
|W\bar{W}\rangle = \frac{1}{\sqrt{2}}(|W\rangle + |\bar{W}\rangle).
\] (27)

where

\[
|W\rangle = \frac{1}{\sqrt{3}}(|001\rangle + |010\rangle + |100\rangle),
\]

\[
|\bar{W}\rangle = \frac{1}{\sqrt{3}}(|110\rangle + |101\rangle + |011\rangle).
\] (28)

The $W\bar{W}$ state has its coherence distributed in both three way and two way manner, apart from this single qubit coherences are also present in the system. For this system, we find the tangle $\tau = 1/3$ and hence we know that the three qubit entanglement is also present in the system. The two qubit reduced density matrix also has entanglement suggesting that the bipartite entanglement is also present in the system. In the computational basis the two qubit reduced density matrix of this state is

\[
\rho_{AB} = \rho_{BC} = \rho_{AC} = \begin{pmatrix}
\frac{1}{6} & \frac{1}{6} & \frac{1}{6} & 0 \\
\frac{1}{6} & \frac{1}{6} & \frac{1}{6} & 0 \\
\frac{1}{6} & \frac{1}{6} & \frac{1}{6} & 0 \\
0 & 0 & 0 & \frac{1}{6}
\end{pmatrix},
\] (29)

This two qubit mixed state has coherence value of $C_{\ell_1} = 2$, concurrence $C = 1/3$, mixedness $L = 10/27$ and teleportation fidelity $f_T = 7/9$. To examine the Bell-CHSH inequality violation we compute $M(\rho_{W\bar{W}})$ (where $\rho_{W\bar{W}} = \rho_{WAB} = \rho_{WBC} = \rho_{WAC}$) which comes out to be $8/9$. As $M(\rho_{W\bar{W}}) < 1$, the state satisfies Bell-CHSH inequality.

**Star state:** The three qubit quantum states considered above viz $GHZ$, $W$ and the $W\bar{W}$ states are all symmetric states in that the reduced density matrices are all identical irrespective of which qubit is traced over. As an example of asymmetric states we consider the three qubit star states. In these states there is a central qubit which is entangled to two peripheral qubits. These peripheral qubits are not entangled between themselves and the form of a three qubit star state is:

\[
|S\rangle_{ABC} = \frac{1}{2}[|000\rangle + |100\rangle + |101\rangle + |111\rangle].
\] (30)
When we trace out the central qubit, the reduced density matrix of the two qubit system is

\[
\rho_{AB}^s = \begin{pmatrix}
\frac{1}{4} & 0 & \frac{1}{4} & 0 \\
0 & 0 & 0 & 0 \\
\frac{1}{4} & 0 & \frac{1}{4} & \frac{1}{4} \\
0 & 0 & \frac{1}{4} & \frac{1}{4}
\end{pmatrix}.
\]

(31)

Although the state has zero concurrence, it has finite amount of coherence \(C_{\ell_1} = 1\). This bipartite partition is separable and hence is not of use in computing quantum teleportation fidelity. If we trace out any one of the peripheral qubit we get the following reduced density matrix:

\[
\rho_{BC}^s = \rho_{AC}^s = \begin{pmatrix}
\frac{1}{4} & \frac{1}{4} & 0 & \frac{1}{4} \\
\frac{1}{4} & \frac{1}{4} & 0 & \frac{1}{4} \\
0 & 0 & 0 & 0 \\
\frac{1}{4} & \frac{1}{4} & 0 & \frac{1}{4}
\end{pmatrix}.
\]

(32)

The state has a concurrence \(C = 1/2\), coherence \(C_{\ell_1} = 3/2\) and mixedness \(L = 1/3\). The teleportation fidelity of this state is \((7 + 2\sqrt{2})/2\) which is higher than the classical teleportation fidelity of \(2/3\) and hence can be used for quantum teleportation. These reduced density matrices satisfy the Bell-CHSH inequality.

### 6.2 Bipartite mixed states derived from mixed tripartite states

In this subsection, we consider bipartite mixed states obtained by tracing out one qubit in a three qubit mixed state which is a convex combination of three qubit pure states. We investigate various features like quantum coherence, entanglement, mixedness and teleportation fidelity for these states.

**Mixture of GHZ and W states:** We consider the genuinely entangled three qubit GHZ state and the bipartite entangled W state and consider a tripartite state which is a convex mixture of both these states. The form of this tripartite mixed state is

\[
\rho_{ABC}^{gw} = p|GHZ\rangle_{ABC}\langle GHZ| + (1 - p)|W\rangle_{ABC}\langle W|, \quad 0 \leq p \leq 1,
\]

(33)

where \(p\) is the mixing parameter. Now if we trace out any one of the qubits (A, B or C) from the tripartite state Eq. (33), the reduced density matrix \(\rho^m\) in the computational basis is

\[
\rho^m = \begin{pmatrix}
\frac{p+2}{6} & 0 & 0 & 0 \\
0 & \frac{1-p}{3} & \frac{1-p}{3} & 0 \\
0 & \frac{1-p}{3} & \frac{1-p}{3} & 0 \\
0 & 0 & 0 & \frac{p}{2}
\end{pmatrix}.
\]

(34)

The coherence, concurrence and mixedness along with teleportation fidelity of the state (34) in computational basis are

\[
C_{\ell_1}(\rho^m) = \frac{2(1 - p)}{3},
\]

\[
C(\rho^m) = \frac{2(1 - p)}{3} - 2\sqrt{\frac{p(p + 2)}{6}}, \quad 0 \leq p < 0.292
\]

\[
L(\rho^m) = \frac{2}{27} (8 - 13p^2 + 14p),
\]

\[
f_T(\rho^m) = \frac{7 - 4p}{9}, \quad 0 \leq p < \frac{1}{4}.
\]

(35)
The application of this state as a teleportation channel has been investigated in [13] where it has been shown that the state is useful for quantum teleportation when $p$ lies between 0 and 0.25. It is also to be noted that for $0.25 < p < 0.292$ though the state is entangled yet it cannot be used as a teleportation channel. The results of these investigations are shown in the plots in Fig. (3).

![Figure 3: The quantum coherence measured using the $\ell_1$ norm of coherence $C_{\ell_1}$, the entanglement measured using the concurrence and the teleportation fidelity $f_T$ are given for (a) mixture of GHZ and $W$ states and (b) a mixture of $W$ state and $\bar{W}$ states both as a function of the probability $p$.](image)

**Mixture of $W$ and $\bar{W}$ states:** Next we consider a convex mixture of $|W\rangle$ and $|\bar{W}\rangle$ states and investigate the teleportation capability of two qubit reduced states obtained from these states. The expression for the three qubit mixed states read:

$$
\rho_{W\bar{W}}^{ABC} = p|W\rangle_{ABC}\langle W| + (1-p)|\bar{W}\rangle_{ABC}\langle \bar{W}| \quad 0 \leq p \leq 1.
$$

(36)

Tracing out any one of the three qubits i.e., $A$ or $B$ or $C$, the three qubit state $\rho_{W\bar{W}}^{ABC}$ reduces to a two qubit density matrix $\rho^m$ which is

$$
\rho^m = \begin{pmatrix}
\frac{p}{3} & 0 & 0 & 0 \\
0 & \frac{1}{3} & \frac{1}{3} & 0 \\
0 & \frac{1}{3} & \frac{1}{3} & 0 \\
0 & 0 & 0 & \frac{1-p}{3}
\end{pmatrix},
$$

(37)

where $\rho^{WW} = \rho_{AB}^{WW} = \rho_{BC}^{WW} = \rho_{AC}^{WW}$. For this state, the coherence, concurrence, mixedness and teleportation fidelity are

$$
C_{\ell_1}(\rho^m) = \frac{2}{3},
$$

$$
C(\rho^m) = \frac{2}{3} - \frac{2}{3} \sqrt{p(1-p)},
$$

$$
L(\rho^m) = \frac{8}{27} (2 - p^2 + p),
$$

$$
f_T(\rho^m) = \frac{7}{9}.
$$

(38)

Thus we find the teleportation fidelity to be greater than the classical teleportation fidelity of $2/3$ and hence the states can be useful in quantum teleportation.
Table 1: Summary of the quantum correlations, teleportation fidelity and Bell-CHSH inequality

| Quantum states                | Quantum Coherence $C_{\ell_1}$ | Concurrence $C$ | Mixedness $L$ | Teleportation Fidelity $f_T$ |
|------------------------------|---------------------------------|-----------------|---------------|-----------------------------|
| class-1 states with mixing parameter $p_1$ | $p_1$ | $p_1$ | $4(p_1(1-p_1))/3$ | $(2+p_1)/3$ |
| class-2 states with mixing parameter $p_2$ | $p_2$ | $p_2$ | $8(p_2(1-p_2))/3$ | $(1+2p_2)/3$ |
| GHZ state 2-qubit reduced form | 0 | 0 | $2/3$ | 0 |
| W state 2-qubit reduced form | $2/3$ | $2/3$ | $16/27$ | $7/9$ |
| $|WW\rangle$ 2-qubit reduced form | 2 | $1/3$ | $10/27$ | $7/9$ |
| Star state 2-qubit reduced form | 2 | $3/2$ | $1/3$ | $(7+2\sqrt{2})/2$ |
| $p\rho_{GHZ} + (1-p)\rho_W$ 2-qubit reduced form | $\frac{2}{3}(1-p)$ | $2\left(\frac{1-p}{3} - \sqrt{\frac{p(p+2)}{6}}\right)$ | $\frac{2}{27}(8 - 13p^2 + 14p)$ | $(7 - 4p)/9$ |
| $p\rho_{W} + (1-p)\rho_W$ 2-qubit reduced form | $2/3$ | $2/3 - 2(\sqrt{p(1-p)})/3$ | $\frac{8}{27}(2 - p^2 + p)$ | $7/9$ |

7 Conclusion:

In this work, we study the quantum coherence and correlations of several bipartite mixed states. Particularly, we find the quantum coherence, entanglement and mixedness of these quantum systems. To estimate quantum coherence we use the well-known $\ell_1$ norm of coherence, in which we sum over the off-diagonal elements of the two qubit density matrix. The entanglement is measured using the concurrence measure which is suitable for both pure and mixed states. Using the linear entropy we find the amount of mixedness in the states. For the quantum states we consider a wider class of states considering all possibilities. Initially we consider the two qubit states created using a mixture of the maximally entangled Bell state with any one of the states from the computational basis. Here we find that we can create two classes of states viz class-1 type and the class-2 type of states. In the class-1 type of states the basis vector being mixed with is also a constituent of the Bell state, whereas in the class-2 type of states it is not a constituent of the Bell state with which it is being mixed. Apart from this we also consider bipartite mixed states derived from either a tripartite pure or mixed state by tracing out one of their qubits. A complete list of the states considered in this study are given in Table 1, where we have summarized the properties of each individual state.

The quantum coherence, entanglement and mixedness of both the class-1 and class-2 type of states are computed both in the computational as well as in the Bell basis. In the computational basis, the quantum coherence measured using the $\ell_1$-norm of coherence is equal to the entanglement estimated using the concurrence. This is because we are using a specific class of $X$-states in which only two diagonal elements are present. The mixedness of the class-1 (class-2) states varies such that they are zero when either $p_1$ ($p_2$) is zero or when $p_1$ ($p_2$) is unity and the maximal value is attained midway when the probabilities are half. When we estimate the coherence and concurrence
in the Bell basis we find them to have opposite behavior where the coherence is maximal when the concurrence is zero and the concurrence is maximal for vanishing coherence. This is because in the Bell basis there is a coherence within the basis elements which is not accounted for when we compute the $\ell_1$-norm of coherence. But concurrence measurements are basis independent and will give the same result irrespective of the basis chosen for the computation. From this observation we learn that one should always use a incoherent basis (separable basis) for the computation of quantum coherence in the system. The use of either a coherent basis or an entangled basis might always lead to wrong results. The mixedness in the bell basis is identical to the one obtained in the computational basis because the linear entropy is a basis independent evaluation. Since the issue of the right kind of basis is settled, we computed the teleportation fidelity of both the class-1 and class-2 type of states and examined their suitability for quantum teleportation. The teleportation fidelity of the class-1 type of states is greater than the classical teleportation fidelity of 2/3 for all values of the mixing parameter. Hence we conclude that these type of states can always be used for quantum teleportation. In the case of the class-2 type of states, the teleportation fidelity is higher than the classical value only when $0.5 \leq p \leq 1$ and so these states can be used in quantum teleportation only in this range. Next we verify the Bell-CHSH inequality for both these class of states. We find that the class-1 type of states violates the Bell-CHSH inequality for all values of the mixing parameter. For the class-2 type of states we observe that the Bell-CHSH inequality is violated for $p > 0.7071$. In the region $0.5 \leq p \leq 0.7071$, the states satisfy the Bell-CHSH inequality but they can still be used for quantum teleportation. Finally we also investigate reduced two qubit states obtained from the three qubit pure and mixed states. Here we consider a comprehensive set of pure states viz $GHZ$ state, $W$ state, $\bar{W}$ state, star state. For the mixed states we consider the mixture of (i) $GHZ$ and $W$ state and (ii) $W$ and $\bar{W}$ state. The computed correlations as well as their teleportation fidelity is listed in Table 1. These results gives us an idea of the suitability of various states for quantum teleportation. An interesting future work will be the examination of the teleportation fidelity of these different states under classical noise like telegraphic noise as well as dissipative and dephasing noise. We are currently working on this problems and the results will form the discussions of our future work.

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Data Availability Statement

The authors confirm that the data supporting the findings of this study are available within the article

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