1. Introduction and Results

Let $\Omega$ be a smooth ($C^\infty$) bounded pseudoconvex domain in $\mathbb{C}^n$. Let $U$ be a small enough neighborhood of a point $z \in b\Omega$ so that there exist smooth vector fields $L_1, L_2, \ldots, L_n$ in $U$, of type $(1,0)$, pointwise orthonormal, so that $L_1, \ldots, L_{n-1}$ are tangential to $b\Omega$ at the boundary, and $L_n$ is the (complex) normal. We use the customary notation $\omega_1, \omega_2, \ldots, \omega_n$ to denote the $(1,0)$-forms dual to $L_1, \ldots, L_n$. For a boundary point $z \in b\Omega \cap U$, $L_1(z), \ldots, L_{n-1}(z)$ form an orthonormal basis for $T_z^{1,0}(b\Omega)$. $(U, L_1, \ldots, L_n, \omega_1, \ldots, \omega_n)$ are usually referred to as a special boundary chart and/or frame. The Levi form of the boundary is defined via $[L, \overline{L}] = L(L, \overline{L})T \mod T^{1,0} \oplus T^{0,1}$, where $T$ is the familiar ‘bad’ direction inside the tangent space, purely imaginary, normalized and chosen so that $L$ is positive semi-definite.

Denote by $\lambda_j(z)$, $1 \leq j \leq n-1$, the eigenvalues of the Levi form of $b\Omega$ at the point $z \in b\Omega$, ordered increasingly. Strictly speaking, we mean the eigenvalues of the matrix that represents the Levi form with respect to a basis $L_1, \ldots, L_{n-1}$ as above; as long as we insist on orthonormal bases, these eigenvalues do not depend on the basis chosen. We say that the Levi form of $b\Omega$ satisfies a comparable eigenvalues condition at level $q$ in $U \cap b\Omega$, if there exists a constant $C > 0$ such that $C(\lambda_1(z) + \cdots + \lambda_{n-1}(z)) \leq \sum_{j=1}^{q} \lambda_{j}(z) \leq \lambda_1(z) + \cdots + \lambda_{n-1}(z)$ for any $q$-tuple $(j_1, \ldots, j_q)$ and
$z \in b\Omega$. That is, the sum of any $q$ eigenvalues is comparable to the trace. Note that the second inequality is trivially satisfied because $\Omega$ is pseudoconvex. This condition is easily seen to be equivalent to sums of $q$ eigenvalues being comparable. We say that the Levi form of $b\Omega$ satisfies a comparable eigenvalues condition at level $q$ if every point $z \in b\Omega$ has a neighborhood $U$ so that the condition is satisfied in $U \cap b\Omega$. Because $b\Omega$ is compact, we may take the constant $C$ to be independent of $z \in b\Omega$.

The comparable eigenvalues condition is important because it is equivalent to an $L^2$ estimate in the $\overline{\partial}$-Neumann problem that is better than the usual estimate on a pseudoconvex domain. Namely, if $b\Omega$ satisfies a comparable eigenvalues condition at level $q$, then we have the estimate

$$\sum_{j=1}^{n} \|\overline{\partial} f_j\|^2 + \sum_{j=1}^{n-1} \|\partial f_j\|^2 \leq C(\|\overline{\partial} f\|^2 + \|\overline{\partial}^* f\|^2 + \|f\|^2)$$

for any $f \in \text{Dom}(\overline{\partial}) \cap \text{Dom}(\overline{\partial}^*) \cap C^\infty_{(0,q)}(\overline{\Omega})$ that is supported in a special boundary chart. For proofs, see [8], Théorème 3.1 for $q = 1$, and [1], Théorème 3.7 for $q > 1$. (Since we work on pseudoconvex domains, the term $\|f\|^2$ on the right hand side is dominated by the others, and so will not be needed.) The first term in (1) is always dominated by the right hand side, in view of the Morrey-Kohn-Hörmander inequality, however, the second term is not in general. So the point of (1) is that all, not just the barred, complex tangential derivatives of $f$ are controlled by $\|\overline{\partial} f\| + \|\overline{\partial}^* f\|$. Such estimates are referred to as maximal estimates. We refer the reader to the introduction of [12] for an account of the genesis of this terminology, and the important role such estimates play in the theory of the $\overline{\partial}$-Neumann problem.

Next we define the Hankel operators on $(0,q)$-forms for $0 \leq q \leq n$ as follows. Let $K^2_{(0,q)}(\Omega)$ denote the set of square integrable $\overline{\partial}$-closed $(0,q)$-forms on $\Omega$ and $P_q : L^2_{(0,q)}(\Omega) \to K^2_{(0,q)}(\Omega)$ be the Bergman projection. The Hankel operator with symbol $\phi \in L^\infty(\Omega)$ is the operator $H^q_{\phi} : K^2_{(0,q)}(\Omega) \to L^2_{(0,q)}(\Omega)$,

$$H^q_{\phi} f = \phi f - P_q(\phi f)$$

for $f \in K^2_{(0,q)}(\Omega)$. When $\phi \in C^1(\overline{\Omega})$, Kohn’s formula, $P_q = I - \overline{\partial}^* N_{q+1} \overline{\partial}$, implies that

$$H^q_{\phi} f = \overline{\partial}^* N_{q+1} (\overline{\partial} \phi \wedge f)$$

for $f \in K^2_{(0,q)}(\Omega)$. Here, $N_{q+1}$ denotes the $\overline{\partial}$-Neumann operator on $(0,q+1)$-forms. In the following theorem, $A^2_{(0,q)}(\Omega) \subset K^2_{(0,q+1)}(\Omega)$ denotes the space of $(0,q)$-forms with square integrable holomorphic coefficients, and $\mathbb{D}^q$ is the unit polydisc, i.e. the $q$-fold product of the unit disc in $\mathbb{C}^q$.

\(^2\text{Note that } \lambda_1 + \cdots + \lambda_{n-1} = \left(\frac{n-1}{q-1}\right)^{-1} \sum_{|J|=q} (\lambda_{j_1} + \cdots + \lambda_{j_q}), \text{ where the summation is over strictly increasing multi-indices } J. \text{ Thus if the } q\text{-sums compare, the trace also compares to any } q\text{-sum. A similar observation for } (\lambda_1 + \cdots + \lambda_{q+1}) \text{ shows that if the comparable eigenvalues condition holds at level } q, \text{ it also holds at level } (q+1).\)
Our first result gives a necessary condition for compactness of Hankel operators; the case \( q = 1 \) and \( n = 2 \) is in [6] (for symbols in \( C(\overline{\Omega}) \)), the case \( q = 1 \) but general \( n \) is in [3] (for symbols in \( C^\infty(\overline{\Omega}) \)).

**Theorem 1.** Let \( \Omega \) be a bounded convex domain in \( \mathbb{C}^n \) for \( n \geq 2 \). Assume that \( \phi \in C^1(\overline{\Omega}) \) and there exists a holomorphic embedding \( \psi : D^q \to b\Omega \) for some \( 1 \leq q \leq n - 1 \) such that \( \phi \circ \psi \) is not holomorphic. Then \( H^{q-1}_\phi \) is not compact on \( A^2_{(0,q-1)}(\Omega) \) (and a fortiori not on \( K^2_{(0,q-1)}(\Omega) \)).

That is, for \( H^{q-1}_\phi \) to be compact (even on \( A^2_{(0,q-1)}(\Omega) \)) it is necessary that the symbol \( \phi \) is holomorphic along (the regular part of) \( q \)-dimensional, and thus higher dimensional, varieties in the boundary. Since \( \Omega \) is convex, such varieties are necessarily contained in affine varieties, see [10], Theorem 1.1 and section 2, and [6], Lemma 2. The proof of Theorem 1 given in section 2, combines ideas from [10] and [3] in a fairly straightforward way.

In view of the ‘if and only if’ nature of the results in [10], one might expect that the converse of Theorem 1 also holds. This is known for (convex) domains in \( C^2 \) ([6], Theorem 3), and we can verify it in many cases, but the general case remains open.

Suppose we have an estimate whose right hand side depends only on \( \|\overline{\partial} f\| \) and \( \|\overline{\partial}^* f\| \) (possibly modulo ‘weak’ terms), as in (1), at the level of \( (q+1) \)-forms. In addition to (1), examples include compactness estimates ([4] below) and subelliptic estimates ([5] below). In order to derive an analogous estimate for \( q \)-forms, it is natural to take a \( q \)-from \( f \) and produce \( (q+1) \)-forms \( f^k := f \wedge \overline{\omega}_k \), \( k = 1, \ldots , (n-1) \) (say \( f \) is supported in a local frame), control the relevant norm of \( f \) by those of the \( f^k \), apply the known estimate to the \( (q+1) \)-forms \( f^k \), and finally control \( \|\overline{\partial} (f^k)\| \) and \( \|\overline{\partial}^* (f^k)\| \) by \( \|\overline{\partial} f\| \) and \( \|\overline{\partial}^* f\| \). This is no problem for \( \overline{\partial} (f^k) = \overline{\partial} f \wedge \overline{\omega}_k + (-1)^q f \wedge \overline{\partial} (\overline{\omega}_k) \); it is controlled by \( \|\overline{\partial} f\| + \|f\| \), hence by \( \|\overline{\partial} f\| + \|\overline{\partial}^* f\| \). The form \( \overline{\partial}^* (f^k) \) takes more care. First, if \( f \) is smooth enough (say in \( C^\infty_{(0,q)}(\overline{\Omega}) \) for simplicity), then \( f^k = f \wedge \overline{\omega}_k \) is in \( \text{dom}(\overline{\partial}^*) \) if \( f \) is. Indeed, since the normal components of both \( f \) and \( \overline{\omega}_k \) vanish on the boundary, so does that of \( f \wedge \overline{\omega}_k \). Computation of \( \overline{\partial}^* (f^k) \) then reveals that to control \( \|\overline{\partial}^* (f^k)\| \), one needs not only \( \|\overline{\partial}^* f\| \) and \( \|f\| \), but also \( \|L_k f\| \) (see (40) in section 5 below). So in order for the above scheme to work, the latter term needs to be controlled by \( \|\overline{\partial} f\| + \|\overline{\partial}^* f\| \). This, however, is precisely what the condition of maximal estimates ensures. Theorems 2 and 3 below take advantage of this observation.

**Theorem 2.** Let \( \Omega \) be a smooth bounded convex domain in \( \mathbb{C}^n \), \( n \geq 2 \) and \( 1 \leq q \leq n - 1 \). Assume that the Levi form of \( b\Omega \) satisfies a comparable eigenvalues condition at level \( q \). Let \( \phi \in C^1(\overline{\Omega}) \) such that \( \phi \circ \psi \) is holomorphic for every holomorphic embedding \( \psi : D^{n-1} \to b\Omega \). Then the Hankel operator \( H^{q-1}_\phi : K^2_{(0,q-1)}(\Omega) \to L^2_{(0,q-1)}(\Omega) \) is compact.

Note that the symbol \( \phi \) is assumed holomorphic only on \( (n-1) \)-dimensional varieties, while the Hankel operator is on \( (0,q-1) \)-forms. Combined with Theorem 1, this ‘discrepancy’ leads to the following corollary. Its gist is that on convex domains, varieties in the boundary, apart from
the ones in top dimension, are obstructions to maximal estimates (equivalently, to comparable eigenvalues conditions).

**Corollary 1.** Let $\Omega$ be a smooth bounded convex domain in $\mathbb{C}^n$, $n \geq 2$ and $1 \leq q \leq n - 1$. Denote by $A$ the union of the (non-trivial) $(n - 1)$-dimensional varieties in $b\Omega$. Assume that $\Omega$ satisfies maximal estimates for $(0,q)$-forms. Then $b\Omega \setminus \overline{A}$ contains no $q$-dimensional analytic varieties.

Note that $(n - 1)$-dimensional varieties in $b\Omega$ are contained in an affine $(n - 1)$-dimensional variety in the boundary, see [6], Lemma 2, [10], section 2.

Convex domains in $\mathbb{C}^2$ (where maximal estimates hold trivially on $(0,1)$-forms) show that the requirement in the corollary that the varieties be outside $\overline{A}$ cannot be dropped.

A portion of the technique in the proof of Theorem 2 leads to an interesting percolation phenomenon for compactness and subellipticity in the $\overline{\partial}$-Neumann problem on domains with maximal estimates. We first recall these notions.

The $\overline{\partial}$-Neumann problem is said to satisfy compactness estimates for $(0,q)$-forms if the following holds: for every $\varepsilon > 0$, there exists a constant $C_\varepsilon$ such that

$$
\|f\|_2^2 \leq \varepsilon \left( \|\overline{\partial} f\|_2^2 + \|\partial^* f\|_2^2 \right) + C_\varepsilon \|f\|_{-1}^2, \quad f \in \text{dom}(\overline{\partial}) \cap \text{dom}(\partial^*) \subset L^2_{(0,q)}(\Omega).
$$

Here, $\|f\|_{-1}$ denotes the Sobolev-($-1$) norm. The $\overline{\partial}$-Neumann problem is said to be subelliptic for $(0,q)$-forms if there exists $\varepsilon > 0$ and a constant $C$ such that

$$
\|f\|_{\varepsilon}^2 \leq C \left( \|\overline{\partial} f\|_2^2 + \|\partial^* f\|_2^2 \right), \quad f \in \text{dom}(\overline{\partial}) \cap \text{dom}(\partial^*) \subset L^2_{(0,q)}(\Omega).
$$

Again, the subscript $\varepsilon$ denotes the Sobolev-$\varepsilon$ norm. We say that the $\overline{\partial}$-Neumann problem is subelliptic of order $\varepsilon$. The relevance of estimates (4) and (5) stems from their equivalence to compactness and subellipticity, respectively, of the $\overline{\partial}$-Neumann operator $N_q$ ([16], [13], [7]).

Compactness and subellipticity in the $\overline{\partial}$-Neumann problem are known to percolate up the $\partial$ complex ([16], Proposition 4.5 and reference there, [14]); the point of Theorem 3 is that they percolate down to level $q$ when the Levi from of $b\Omega$ satisfies a comparable eigenvalues condition at level $q$ (equivalently: when there are maximal estimates for $(0,q)$-forms).

**Theorem 3.** Let $\Omega$ be a smooth bounded pseudoconvex domain in $\mathbb{C}^n$ for $n \geq 2$. Assume that the Levi form of $b\Omega$ satisfies a comparable eigenvalues condition at level $q$ for some $q$, $1 \leq q \leq n - 1$. Then

(i) The $\overline{\partial}$-Neumann problem satisfies compactness estimates for $(0,q)$-forms if and only if it satisfies such estimates for $(0, n - 1)$-forms.

(ii) The $\overline{\partial}$-Neumann problem is subelliptic of order $\varepsilon$ for $(0,q)$-forms if and only if it is subelliptic of order $\varepsilon$ for $(0, n - 1)$-forms, $0 < \varepsilon \leq 1/2$.

The following corollary for domains with comparable eigenvalues of the Levi form is immediate, but we formulate it for emphasis: if compactness or subelliptic estimates hold at some form level, corresponding estimates hold at all levels.
Corollary 2. Let $\Omega$ be a smooth bounded pseudoconvex domain in $\mathbb{C}^n$, $n \geq 2$. Assume that the eigenvalues of the Levi form of $b\Omega$ are comparable (equivalently, $\Omega$ admits maximal estimates). Then

i) The $\overline{\partial}$-Neumann problem satisfies compactness estimates for $(0,q_0)$-forms for some $q_0$, $1 \leq q_0 \leq (n-1)$, if and only if it satisfies compactness estimates for $(0,q)$-forms for all $q$, $1 \leq q \leq (n-1)$.

ii) The $\overline{\partial}$-Neumann problem is subelliptic on $(0,q_0)$-forms for some $q_0$, $1 \leq q_0 \leq (n-1)$, if and only if it is subelliptic on $(0,q)$-forms for all $q$, $1 \leq q \leq (n-1)$.

We remark that the $\overline{\partial}$-Neumann problem is always subelliptic (and hence also compact) on $(0,n)$-forms.

The rest of the paper is organized as follows. In section 2 we prove Theorem 1. Theorem 2 and Corollary 1 are shown in section 3. Section 4 contains the proof of Theorem 3. In the appendix, section 5, we compute $\overline{\partial}^+(f \wedge \overline{\omega})$ for $f \in \operatorname{dom}(\overline{\partial}^+)$.\

2. PROOF OF THEOREM 1

Proof of Theorem 1. The proof combines ideas from [10] and [6, 3]. (In turn, these ideas can be traced back at least to [2, 9].) In particular, we follow the geometric setup in the proof of the implication $(1) \Rightarrow (2)$ in Theorem 1.1 in [10]. If $b\Omega$ contains a complex variety of dimension $q$ as in Theorem 1 its convex hull is an affine variety in $b\Omega$ ([6], Lemma 2, see also [10], section 2) of dimension at least $q$. $\phi$ is not holomorphic on this variety; consequently, there is a $q$-dimensional affine variety in $b\Omega$ on which $\phi$ is not holomorphic. After a suitable affine change of coordinates, we may assume that $(2\mathbb{D})^q \times \{0\} = \{(z',0) \in \mathbb{C}^n; z' \in (2\mathbb{D})^q\} \subset b\Omega$, where $\mathbb{D}$ is the unit disc in $\mathbb{C}$ and $z' = (z_1, \ldots, z_q)$, and that $\partial \phi / \partial \overline{z}_1(z) \neq 0$ when $|z_1| < 1$. Let $z'' = (z_{q+1}, \ldots, z_n)$. We set $\Omega_1 := \{z'' \in \mathbb{C}^{n-q}; (0,z'') \in \Omega\}$, and $\Omega_2 := \{z'' \in \mathbb{C}^{n-q}; 2z'' \notin \Omega_1\}$. Convexity of $\Omega$ implies that $\mathbb{D}^q \times \Omega_2 \subset \Omega$ ([10], page 636): every point in this set is the midpoint of a line segment joining a point in $\mathbb{D}^q \times \{0\}$ to a point in $\{0\} \times \Omega_1$.

The crucial analytic fact from [10] is the following. There exists a bounded sequence $\{F_j\}_{j=1}^\infty \subset A^2(\Omega)$ such that the sequence $\{f_j\}_{j=1}^\infty$ of restrictions to $\Omega_2$, given by $f_j(z'') := F_j(0,z'')$, belongs to $A^2(\Omega_2)$, but does not admit a convergent subsequence.\

For the rest of the argument, we follow [6, 3], with appropriate modifications. Choose a radially symmetric non-negative function $\chi \in C_0^\infty(\mathbb{D})$ such that $\chi(\xi) = 1$ for $|\xi| \leq 1/2$ and $\chi(\xi) = 0$ for $|\xi| \geq 3/4$. We denote $\int_{\{\xi| \leq 3/4\}} \chi(\xi) dV(\xi) = c_\chi > 0$. Because $\partial \phi / \partial \overline{z}_1 \neq 0$ when $|z_1| < 1$, we can define $\gamma \in C(\overline{\Omega})$ via the formula

\begin{equation}
\gamma(z',z'') \frac{\partial \phi(z',z'')}{\partial z_1} = \chi(z_1) \cdots \chi(z_q).
\end{equation}

The crux of the matter is that the restriction operator from $A^2(\Omega_1)$ to $A^2(\Omega_2)$ is not compact; the proof involves estimates on the Bergman kernel of $\Omega_1$. The Ohsawa-Takegoshi extension theorem then allows to pass from a sequence on $\Omega_1$ with the required property to a suitable sequence on $\Omega$.
Note that for \( z'' \) fixed, \( \gamma(\cdot, z'') \) is compactly supported in \( \mathbb{D}^q \), uniformly in \( z'' \). We will eventually have to approximate \( \gamma \) by a smooth function, so let \( \gamma_1 \in C^\infty(\overline{\Omega}) \) such that for \( z'' \in \Omega_2, \gamma_1(\cdot, z'') \) is compactly supported in \( \mathbb{D}^q \). Let \( F \in A^2(\Omega) \) and \( \alpha = F d\bar{z}_2 \wedge \cdots \wedge d\bar{z}_q \in A^2_{(0,q-1)}(\Omega) \). Denote by \( \langle \cdot, \cdot \rangle \) the standard pointwise inner product on forms in \( \mathbb{C}^q \). Then, for \( z'' \in \Omega_2 \), the mean value property for holomorphic functions gives

\[
\begin{align*}
(7) \quad (c_\chi)^q F(0, z'') &= \int_{\mathbb{D}^q} \chi(z_1) \cdots \chi(z_q) F(z', z'') dV(z') \\
&= \int_{\mathbb{D}^q} \gamma(z', z'') \frac{\partial \phi(z', z'')}{\partial z_1} F(z', z'') dV(z') \\
&= \int_{\mathbb{D}^q} \langle \partial \phi \wedge \alpha, \overline{\gamma} d\bar{z}_1 \wedge \cdots \wedge d\bar{z}_q \rangle dV(z') + \int_{\mathbb{D}^q} \langle \partial \phi \wedge \alpha, (\overline{\gamma} - \overline{\gamma}_1) d\bar{z}_1 \wedge \cdots \wedge d\bar{z}_q \rangle dV(z').
\end{align*}
\]

Because \( F \) is holomorphic, \( \overline{\partial} \alpha = 0 \) implies

\[
\begin{align*}
(8) \quad \overline{\partial} \phi \wedge \alpha &= \overline{\partial}(\phi \alpha) = \overline{\partial}(\phi \alpha - P_{q-1}(\phi \alpha)) = \overline{\partial} H^{q-1}_\phi \alpha.
\end{align*}
\]

Denote by \( \overline{\partial}_z \) the formal adjoint of the \( \overline{\partial} \)-operator in the \( z' \) variables. Inserting (8) into the first term in the third line of (7) shows that

\[
\begin{align*}
(9) \quad \int_{\mathbb{D}^q} \langle \partial \phi \wedge \alpha, \overline{\gamma}_1 d\bar{z}_1 \wedge \cdots \wedge d\bar{z}_q \rangle dV(z') &= \int_{\mathbb{D}^q} \langle \overline{\partial} H^{q-1}_\phi \alpha, \overline{\gamma}_1 d\bar{z}_1 \wedge \cdots \wedge d\bar{z}_q \rangle dV(z') \\
&= \int_{\mathbb{D}^q} \langle H^{q-1}_\phi \alpha, \overline{\partial}_z (\overline{\gamma}_1 d\bar{z}_1 \wedge \cdots \wedge d\bar{z}_q) \rangle dV(z').
\end{align*}
\]

We have used here that terms in \( \overline{\partial} H^{q-1}_\phi \alpha \) and \( H^{q-1}_\phi \alpha \) that contain differentials \( d\bar{z}_s \) with \( s \geq (q+1) \) drop out upon taking inner products with \( \overline{\gamma}_1 d\bar{z}_1 \wedge \cdots \wedge d\bar{z}_q \) or \( \overline{\partial}_z (\overline{\gamma}_1 d\bar{z}_1 \wedge \cdots \wedge d\bar{z}_q) \), respectively, and that for \( z'' \in \Omega_2 \) fixed, \( \gamma \) is compactly supported in \( \mathbb{D}^q \).

Now let \( \{ F_j \}_{j=1}^\infty \subset A^2(\Omega) \) be a bounded sequence whose sequence of restrictions \( \{ f_j \}_{j=1}^\infty \subset A^2(\Omega_2) \) does not admit a convergent subsequence, and set \( \alpha_j = F_j d\bar{z}_2 \wedge \cdots \wedge d\bar{z}_q \in A^2_{(0,q-1)}(\Omega) \) (with the convention from above when \( q = 1 \)). Then we have from (7) and (9)

\[
(10) \quad (c_\chi)^q (f_j(z'') - f_k(z'')) = \int_{\mathbb{D}^q} \langle H^{q-1}_\phi (\alpha_j - \alpha_k), \overline{\partial}_z (\overline{\gamma}_1 d\bar{z}_1 \wedge \cdots \wedge d\bar{z}_q) \rangle dV(z') \\
+ \int_{\mathbb{D}^q} \langle \partial \phi \wedge (\alpha_j - \alpha_k), (\overline{\gamma} - \overline{\gamma}_1) d\bar{z}_1 \wedge \cdots \wedge d\bar{z}_q \rangle dV(z')
\]

for any \( j, k \), and thus

\[
(11) \quad |f_j(z'') - f_k(z'')|^2 \leq C_{\gamma_1} \int_{\mathbb{D}^q} |H^{q-1}_\phi (\alpha_j - \alpha_k)(z', z'')|^2 dV(z') \\
+ \int_{\mathbb{D}^q} |(\alpha_j - \alpha_k)(z', z'')| \langle (\overline{\gamma} - \overline{\gamma}_1)(z', z'') \rangle dV(z'),
\]

\footnote{When \( q = 1 \), this definition is to be interpreted as \( \alpha = F \).}
where $C_{\gamma_1}$ is a constant that depends on $\gamma_1$. Integrating both sides of (11) with respect to $z'' \in \Omega_2$ gives
\[
\|f_j - f_k\|_{A^2(\Omega_2)}^2 \lesssim C_{\gamma_1} \|H^{q-1}_\phi(\alpha_j - \alpha_k)\|_{L^2_{(0,q-1)}(\Omega)}^2 + \left( \sup_{D^q \setminus \bar{\Omega}_2} |\gamma - \gamma_1| \right) \|\alpha_j - \alpha_k\|_{L^2_{(0,q-1)}(\Omega)}^2. \tag{12}
\]
Assume now that $H^{q-1}_\phi$ is compact on $A^2_{(0,q-1)}(\Omega)$. Then the sequence $\{H^{q-1}_\phi \alpha_j\}_{j=1}^\infty \subset L^2_{(0,q-1)}(\Omega)$ has a subsequence, say $\{H^{q-1}_\phi \alpha_{j_s}\}_{s=1}^\infty$, that is convergent, and so is a Cauchy sequence. Because $\{\alpha_j\}_{j=1}^\infty$ is bounded in $L^2_{(0,q-1)}(\Omega)$, and we can make $(\sup_{D^q \setminus \bar{\Omega}_2} |\gamma - \gamma_1|)$ as small as we wish, (12) implies that the sequence $\{f_{j_s}\}_{s=1}^\infty \subset A^2(\Omega_2)$ is Cauchy as well, and therefore is convergent. This is a contradiction. Therefore, $H^{q-1}_\phi$ cannot be compact on $A^2_{(0,q-1)}(\Omega)$. \qed

It is worth noting that in the last part of this proof (from (7) on), the two key steps are the exploitation of nonanalyticity of $\phi$ via the introduction of $\overline{\partial} \phi$ into the mean value equation (7), and the observation (8). The extra complications in the formulas arise from approximating $\gamma$ by a smooth function. This step is needed because we can only assert that $\gamma \in C(\bar{\Omega})$ from $\phi \in C^1(\bar{\Omega})$, yet in (9), $\gamma$ (resp. $\gamma_1$) is differentiated (via $\partial_x^\epsilon \phi$). These complications could be avoided by assuming $\phi \in C^2(\bar{\Omega})$.

3. Proofs of Theorem 2 and Corollary 11

We start with the proof of Theorem 2.

Proof of Theorem 2. It suffices to show that for every $\epsilon > 0$, there exists $C_\epsilon$ so that we have the family of estimates
\[
\|H^{q-1}_\phi f\|^2 \leq \epsilon \|f\|^2 + C_\epsilon \|f\|_{K^2_{(0,q-1)}(\Omega)}^2, \quad f \in K^2_{(0,q-1)}(\Omega). \tag{13}
\]
Because $K^2_{(0,q-1)}(\Omega)$ embeds compactly into $W^{-1}_{(0,q-1)}(\Omega)$, this family of estimates will imply that $H^{q-1}_\phi : K^2_{(0,q-1)}(\Omega) \to L^2_{(0,q-1)}(\Omega)$ is compact (16, Lemma 4.3; in fact, compactness of $H^{q-1}_\phi$ is equivalent to this family of estimates). Note that the left hand side of (13) equals
\[
\langle H^{q-1}_\phi f, H^{q-1}_\phi f \rangle = \langle \overline{\partial} N_q(\overline{\partial} \phi \wedge f), \overline{\partial} N_q(\overline{\partial} \phi \wedge f) \rangle = \langle N_q(\overline{\partial} \phi \wedge f), \overline{\partial} \phi \wedge f \rangle. \tag{14}
\]
We will estimate the right hand side of (14).

Denote by $A$ the union of all the $(n - 1)$-dimensional analytic (then actually affine, by convexity, 10, Lemma 2) varieties in the boundary. Near the boundary, the split of forms into their normal and tangential components is well defined. A detailed discussion may be found in 16, section 2.9. The tangential component $(\overline{\partial} \phi)_{\text{Tan}}$ of $\overline{\partial} \phi$ vanishes at points of $A$. For $\epsilon > 0$, denote by $U_\epsilon$ a neighborhood of $A$ in $\mathbb{C}^n$ such that $|((\overline{\partial} \phi)_{\text{Tan}}| < \epsilon$ on $U_\epsilon \cap \bar{\Omega}$, and choose a cutoff function $\chi_1 \in C^\infty_0(U_\epsilon)$ with $\chi_1 \equiv 1$ near $\overline{A}$. Then
\[
\left| \langle \chi_1 N_q(\overline{\partial} \phi \wedge f), (\overline{\partial} \phi)_{\text{Tan}} \wedge f \rangle \right| \leq \|N_q(\overline{\partial} \phi \wedge f)\| \|\chi_1((\overline{\partial} \phi)_{\text{Tan}} \wedge f)\| \lesssim \epsilon \|f\|^2. \tag{15}
\]
To estimate the contribution from the normal component $(\overline{\partial} \phi)_{\text{Norm}}$ of $\overline{\partial} \phi$, notice that only the normal component $(\chi_1 N_q (\overline{\partial} \phi \wedge f))_{\text{Norm}}$ will be involved (as $(\overline{\partial} \phi)_{\text{Norm}} \wedge f$ has vanishing tangential component). Estimate 2.91 in [16] provides the estimate

$$\| (\chi_1 N_q (\overline{\partial} \phi \wedge f))_{\text{Norm}} \|_1 \lesssim \| \overline{\partial} (\chi_1 N_q (\overline{\partial} \phi \wedge f)) \| + \| \overline{\partial}^* (\chi_1 N_q (\overline{\partial} \phi \wedge f)) \| \lesssim \| N_q (\overline{\partial} \phi \wedge f) \| + \| \overline{\partial} \phi \wedge f \| \lesssim \| f \|.$$  

Because $W^1(\Omega)$ imbeds compactly into $L^2(\Omega)$, the map $f \to \chi_1 N_q (\overline{\partial} \phi \wedge f)_{\text{Norm}}$ is compact from $K^2_{(0,q-1)}(\Omega)$ to $L^2_{(0,q)}(\Omega)$. Therefore ([16], Lemma 4.3)

$$\| \chi_1 N_q (\overline{\partial} \phi \wedge f)_{\text{Norm}} \| \leq \varepsilon \| f \| + C_\varepsilon \| f \|^{-1}.$$  

(i.e. for all $\varepsilon > 0$, there exists $C_\varepsilon$ such that (17) holds). This gives

$$\left| \langle \chi_1 N_q (\overline{\partial} \phi \wedge f), (\overline{\partial} \phi)_{\text{Norm}} \wedge f \rangle \right| \lesssim (\varepsilon \| f \| + C_\varepsilon \| f \|^{-1}) \| f \| \leq 2 \varepsilon \| f \|^2 + C_\varepsilon \| f \|^{-2}.$$  

Here, we have used the usual small constant–large constant estimate on $\| f \|^{-1} \| f \|$, and we have allowed $C_\varepsilon$ to change its value. It remains to estimate $\langle (1 - \chi_1) N_q (\overline{\partial} \phi \wedge f), \overline{\partial} \phi \wedge f \rangle$.

In estimating this latter contribution, we use two observations. The first is that functions in $C(\Omega)$ that vanish on $\overline{A}$ are compactness multipliers for $(0,n-1)$-forms ([14], Proposition 1 and Theorem 3). The second observation is that, more or less, norms of $(0,q)$-forms can be estimated by norms of certain associated $(0,n-1)$-forms (for which we can then apply the compactness estimates).

We elaborate on the second observation. Suppose $u \in L^2_{(0,q)}(\Omega)$ is supported in a special boundary chart, with vanishing normal component, say $u = \sum_{|I|=q, n \notin I} u_I \overline{\omega}_I$. Fix a multi-index $I$ of length $(n-1-q)$, with $n \notin I$. Then

$$u \wedge \overline{\omega}_I = \sum_{|J|=q, n \notin J} u_J \overline{\omega}_J \wedge \overline{\omega}_I = \epsilon^I_{(1,\ldots,n-1)} u_{I^c} \overline{\omega}_1 \wedge \cdots \wedge \overline{\omega}_{n-1},$$  

where $I^c$ is the increasingly ordered multi-index of length $q$ which as a set is the complement of $I$ in $\{1,\ldots,n-1\}$, and $\epsilon^I_{(1,\ldots,n-1)}$ denotes the usual Kronecker symbol. (19) shows that taking the wedge product with $\overline{\omega}_I$ singles out precisely one coefficient of $u$ (namely $u_{I^c}$). If we now let $I$ vary over all multi-indices of length $(n-1-q)$ (and not containing $n$), $I^c$ will vary over all indices of length $q$. Therefore, to estimate $\| u \|$, it suffices to estimate $\| u \wedge \overline{\omega}_I \|$, for all such $I$. The point of not having $n$ in $I$ is that we will want to use compactness estimates on $u \wedge \overline{\omega}_I$, which requires the form to be in the domain of $\overline{\partial}^*$. In order to apply this scheme to the form $N_q (\overline{\partial} \phi \wedge f)$, which appears on the right hand side of (14), we need to localize and also take care of normal components. To this end, choose cutoff functions $\chi_2,\ldots,\chi_m$ so that together with $\chi_1$ from above, they form a partition of unity near $b\Omega$, and so that for $2 \leq s \leq m$, $\chi_s$ is supported in a special boundary chart. Moreover, $\chi_2,\ldots,\chi_m$ can be chosen so that the supports stay close enough to $b\Omega$ that splitting forms into
their tangential and normal components is well defined. Also set \( \chi_0 := 1 - (\chi_1 + \cdots + \chi_m) \) on \( \Omega \); then \( \chi_0 \in C_0^\infty(\Omega) \).

Fix an \( s \) with \( 2 \leq s \leq m \), and consider \( \chi_s N_q(\overline{\partial} \phi \wedge f) \). Note that multiplication by \( \chi_s \) preserves the domain of \( \overline{\partial}^* \). The normal component of a form in \( \text{dom}(\overline{\partial}) \cap \text{dom}(\overline{\partial}^*) \) is in \( W_q^1(\Omega) \), and so is also in \( \text{dom}(\overline{\partial}) \cap \text{dom}(\overline{\partial}^*) \) ([16], section 2.9). Therefore, so is the tangential component. Moreover, we have as in (16) and (17)

\[
(\chi_s N_q(\overline{\partial} \phi \wedge f))_{\text{Norm}} \lesssim \|\overline{\partial} \phi \wedge f\| \lesssim \|f\|, \tag{20}
\]

and

\[
\|\chi_s N_q(\overline{\partial} \phi \wedge f)\|_{\text{Norm}} \leq \epsilon\|f\| + C_\epsilon\|f\|_{-1}. \tag{21}
\]

For economy of notation, let us denote the tangential component of \( \chi_s N_q(\overline{\partial} \phi \wedge f) \) by \( u_s \). Then \( u_s \in \text{dom}(\overline{\partial}) \cap \text{dom}(\overline{\partial}^*) \), and the discussion above applies: we only have to estimate the forms \( u_s \wedge \overline{\omega_I} \), where \( I \) varies over all multi-indices of length \( (n - 1 - q) \) that do not contain \( n \). Note that for such \( I \), \( u_s \wedge \overline{\omega_I} \in \text{dom}(\overline{\partial}) \cap \text{dom}(\overline{\partial}^*) \) (see Lemma 2 in the appendix). If \( \chi_s \) is a cutoff function supported in \( \overline{\Omega \setminus \overline{A}} \), it is a compactness multiplier for \( (0, n - 1) \)-forms ([4], Proposition 1 and Theorem 3). If in addition \( \chi \equiv 1 \) on the support of \( \chi_s \), then \( u_s \wedge \overline{\omega_I} = \chi(u_s \wedge \overline{\omega_I}) \). Therefore, for all \( \epsilon > 0 \), there exists \( C_\epsilon \) such that

\[
\|u_s \wedge \overline{\omega_I}\| \leq \|\chi(u_s \wedge \overline{\omega_I})\| \leq \epsilon \left(\|\overline{\partial}(u_s \wedge \overline{\omega_I})\| + \|\overline{\partial}^*(u_s \wedge \overline{\omega_I})\|\right) + C_\epsilon\|u_s \wedge \overline{\omega_I}\|_{-1}. \tag{22}
\]

Estimate (33) in Lemma 2 gives

\[
\|\overline{\partial}(u_s \wedge \overline{\omega_I})\| + \|\overline{\partial}^*(u_s \wedge \overline{\omega_I})\| \lesssim \|\overline{\partial} u_s\| + \|\overline{\partial}^* u_s\| \lesssim \|\overline{\partial}(\chi_s N_q(\overline{\partial} \phi \wedge f))\| + \|\overline{\partial}^*(\chi_s N_q(\overline{\partial} \phi \wedge f))\| + \|f\| \lesssim \|\chi_s N_q(\overline{\partial} \phi \wedge f)\| + \|\overline{\partial}^* N_q(\overline{\partial} \phi \wedge f)\| + \|f\| \lesssim \|f\|. \tag{23}
\]

In the second estimate, we have used (20), which implies that \( \|\overline{\partial} u_s\| \lesssim \|\overline{\partial}(\chi_s N_q(\overline{\partial} \phi \wedge f))\| + \|f\| \), as well as the analogous estimate for \( \|\overline{\partial}^* u_s\| \) (since \( u_s = \chi_s N_q(\overline{\partial} \phi \wedge f) - (\chi_s N_q(\overline{\partial} \phi \wedge f))_{\text{Norm}} \)). We point out that estimating the term \( \|\overline{\partial}(u_s \wedge \overline{\omega_I})\| \) is straightforward; it is only in estimating \( \|\overline{\partial}^*(u_s \wedge \overline{\omega_I})\| \) that the assumption on maximal estimates is needed. For the last term in (22), we observe that because the forms \( \omega_I \) are smooth up to the boundary

\[
\|u_s \wedge \overline{\omega_I}\|_{-1} \lesssim \|u_s\|_{-1} \leq \epsilon\|f\| + C_\epsilon\|f\|_{-1}, \tag{24}
\]

for a suitable \( C_\epsilon \). The second inequality follows again with [16], Lemma 4.3, because the map \( f \mapsto u_s \) is continuous into \( L^2_{(0,q)}(\Omega) \), hence compact into \( W_{(0,q)}(\Omega) \).
Combining (21) through (24), we find
\[
\| \chi_s N_q (\bar{\partial} \phi \wedge f) \| \leq \varepsilon \| f \| + C_\varepsilon \| f \|_{-1},
\]
again for $C_\varepsilon$ suitably big. Therefore (as in (17), (18)),
\[
\langle \chi_s N_q (\bar{\partial} \phi \wedge f), \bar{\partial} \phi \wedge f \rangle \lesssim \varepsilon \| f \|^2 + C_\varepsilon \| f \|^2_{-1}.
\]

It remains to estimate the contribution from the factor $\chi_0$ to the right hand side of (14). This is a consequence of interior elliptic regularity. A short argument is as follows. Because $\chi_0$ vanishes on the boundary, it is a compactness multiplier, so that [4], Proposition 1 and Remark 2 give the same estimate as (25), but with $\chi_0$ in place of $\chi_s$. In turn, we obtain, as in (26),
\[
\langle \chi_0 N_q (\bar{\partial} \phi \wedge f), \bar{\partial} \phi \wedge f \rangle \lesssim 2\varepsilon \| f \|^2 + C_\varepsilon \| f \|^2_{-1}.
\]
We have used that in (32) in [4], it is immaterial whether the estimate is stated with $\| \cdot \|$ or with $\| \cdot \|^2$.

(14) together with (15), (18), (26), and (27) establish the family of estimates in (13). This completes the proof of Theorem 2. □

We complete this section by proving Corollary 1.

Proof of Corollary 1 We argue indirectly. Let $V$ a $q$-dimensional analytic variety in $b\Omega \setminus \overline{A}$. We may assume that $V \cap A = \emptyset$, and furthermore, that $V$ is smooth (otherwise, choose a small enough subset of $V$ near a regular point of $V$ in $b\Omega \setminus \overline{A}$). Choose a symbol $\phi \in C^\infty (\overline{\Omega})$ that vanishes identically on $\overline{A}$ and is not holomorphic on $V$. Because we have maximal estimates for $(0, q)$-forms (i.e. the comparable eigenvalues condition at level $q$), and $\phi$ is (trivially) holomorphic on every $(n-1)$-dimensional variety in the boundary, Theorem 2 implies that $H^q_{\phi} : K^2_{(0,q-1)}(\Omega) \rightarrow L^2_{(0,q-1)}(\Omega)$ is compact. This contradicts Theorem 1. □

4. PROOF OF THEOREM 3

Proof of Theorem 3 First note that both compactness and subellipticity of the $\bar{\partial}$-Neumann problem are known to percolate up (see for example [16], Proposition 4.4 and the remark following its proof). Therefore, we only have to show the downward percolation in both (i) and (ii) under the assumptions in Theorem 3. We do this for (ii) first.

To prove the downward percolation in (ii), let $1 \leq q \leq (n-1)$. We need to prove the estimate
\[
\| f \|_\varepsilon^2 \lesssim \| \bar{\partial} f \|^2 + \| \bar{\partial}^* f \|^2, \quad f \in \text{dom}(\bar{\partial}) \cap \text{dom}(\bar{\partial}^*) \subset L^2_{(0,q)}(\Omega),
\]
provided such an estimate (with the same $\varepsilon$) holds for $(0, n-1)$-forms. The argument follows section 3 closely.
Via a partition of unity, it again suffices to check this estimate for forms supported in a special boundary chart. Thus,

\[(29) \quad \| f \|_\varepsilon \lesssim \| f_{\text{Norm}} \|_\varepsilon + \| f_{\text{Tan}} \|_\varepsilon ,\]

where \( f_{\text{Norm}} \) and \( f_{\text{Tan}} \) denote the normal and tangential components of \( f \), respectively (see again [16], section 2.9). The Sobolev-1 estimate for \( f_{\text{Norm}} \) ([16], Lemma 2.12) says that

\[(30) \quad \| f_{\text{Norm}} \|_\varepsilon \lesssim \| f_{\text{Norm}} \|_1 \lesssim \| \partial f \| + \| \partial^* f \|.\]

As in section 3, equation (19), one can see that in order to estimate the second term on the right hand side of (29), it suffices to estimate \( \| f_{\text{Tan}} \wedge \omega_I \|_\varepsilon \) for all (increasing) multi-indices \( I \) of length \( n - 1 - q \) with \( n \not\in I \). The form \( (f_{\text{Tan}} \wedge \omega_I) \) is a \((0, n-1)\)-form, and we can use the subelliptic estimate that is assumed at this form level. The result is

\[(31) \quad \| f_{\text{Tan}} \wedge \omega_I \|_\varepsilon \lesssim \| \partial (f_{\text{Tan}} \wedge \omega_I) \| + \| \partial^* (f_{\text{Tan}} \wedge \omega_I) \|.\]

Lemma 2 in section 5 says that the right hand side in (31) is dominated by \( \| \partial f_{\text{Tan}} \| + \| \partial^* f_{\text{Tan}} \| \). In turn, this sum is dominated by \( \| \partial f \| + \| \partial^* f \| \) (in view of the second inequality in (30) and since \( f_{\text{Tan}} = f - f_{\text{Norm}} \), as in (23). With this, and (30) and (31), the estimate (28) is established.

The proof for the downward induction in (i) is analogous.

5. Appendix

The comparable eigenvalues conditions in Theorems 2 and 3 are used only to see that \( (u \wedge \omega_I) \in \text{dom}(\partial^*) \) if \( u \in \text{dom}(\partial) \cap \text{dom}(\partial^*) \), and to control \( \| \partial^* (u \wedge \omega_I) \| \), where \( u \) is supported in an appropriate boundary chart (and \( n \not\in I \)). This requires a computation which we present it in this appendix; no originality is claimed. Throughout this section \( \Omega \) denotes a smooth bounded pseudoconvex domain.

**Lemma 1.** Assume that the Levi form of \( b\Omega \) satisfies a comparable eigenvalues condition at level \( q \) for some \( 1 \leq q \leq (n - 1) \). Let \( u \in \text{dom}(\partial) \cap \text{dom}(\partial^*) \subset L^2_{(0,q)}(\Omega) \) be supported in a special boundary chart. Then \( L_k u \) (computed in the sense of distributions) is actually in \( L^2_{(0,q)}(\Omega) \), and

\[(32) \quad \| L_k u \| \lesssim \| \partial u \| + \| \partial^* u \| , \quad 1 \leq k \leq (n-1) .\]

**Proof.** The comparable eigenvalues condition entails maximal estimates for \((0,q)\)-forms, as in (1). A little care is needed because (1) is an *a priori* estimate: it is assumed that the form is smooth up to the boundary. However, forms smooth up to the boundary are dense in the graph norm of \( \partial \oplus \partial^* \) ([16], Proposition 2.3), and the proof shows that the approximation can be done with forms supported in the same boundary chart. If the approximating sequence is \( \{ u_j \}_{j=1}^\infty \), then (1) shows that \( \{ L_k u_j \}_{j=1}^\infty \) is Cauchy, hence converges, in \( L^2_{(0,q)}(\Omega) \). But it also converges to \( L_k u \) in the sense of distributions. Therefore, \( L_k u \in L^2_{(0,q)}(\Omega) \), and (32) holds.
What is needed in the proofs of Theorems 2 and 3 is contained in the next Lemma.

**Lemma 2.** Assume that the Levi form of \( b \Omega \) satisfies a comparable eigenvalues condition at level \( q \) for some \( q \) with \( 1 \leq q \leq (n - 1) \). Let \( u \in \text{dom}(\overline{\partial}) \cap \text{dom}(\overline{\partial}^* ) \subset L^2_{(0,q)}(\Omega) \) be supported in a special boundary chart. Then \( u \wedge \overline{\omega}^I \in \text{dom}(\overline{\partial}) \cap \text{dom}(\overline{\partial}^* ) \), and

\[
\| \overline{\partial}^+(u \wedge \overline{\omega}^I)\| + \| \overline{\partial}(u \wedge \overline{\omega}^I)\| \lesssim \| du \| + \| \overline{\partial}^* u\| ,
\]

for any multi-index \( I \) of length \((n - 1 - q)\) that does not contain \( n \).

Note that the issue is only with the first term on the left hand side of (33), the estimate for \( \| \overline{\partial}(u \wedge \overline{\omega}^I)\| \) is trivial, since \( \| u \| \lesssim \| du \| + \|\overline{\partial}^* u\| \).

**Proof of Lemma 2.** We only need to prove the case \(|I| = 1\), i.e. \( \overline{\omega}^I = \overline{\omega}^k \) for some \( k \) with \( 1 \leq k \leq (n - 1) \); the general case then follows inductively.

First recall that the adjoint of wedging with \( \overline{\omega}^k \) is essentially an interior product with \( \overline{L}_k \). Indeed, if \( v = \sum'_{|J|=q+1} v_J \overline{\omega}^J \in L^2_{(0,q+1)}(\Omega) \), then

\[
\langle u \wedge \overline{\omega}^k, v \rangle = \langle \sum'_{|K|=q} u_K(\overline{\omega}_K \wedge \overline{\omega}^k), \sum'_{|J|=q+1} v_J \overline{\omega}^J \rangle = \sum'_{|K|=q} \langle u_K, v_{KK} \rangle = \langle u, v_k \rangle ,
\]

where \( v_k \) is the \((0,q)\)-form \( v_k = \sum'_{|K|=q} v_{KK} \overline{\omega}_K \). We have slightly abused notation: the various appearances of \( \langle \cdot, \cdot \rangle \) denote the inner product between \((0,q+1)\)-forms, functions, and \((0,q)\)-forms, respectively. For \( v = \overline{\partial}g \), (34) gives

\[
\langle u \wedge \overline{\omega}^k, \overline{\partial}g \rangle = \langle u, (\overline{\partial}g)_k \rangle .
\]

Next, we compare \((\overline{\partial}g)_k \) to \( \overline{\partial}(g_k) \) \footnote{This amounts to a \( \overline{\partial} \) version of the Cartan formula \( i(X)d + di(X) = \text{Lie}_X \) (see for example [15], Theorem 2.11). The difference in sign in (38) below (i.e. \( \overline{\partial}(g_k) \) instead of \( -\overline{\partial}(g_k) \)) results from the definition of \( (\cdot)_k \), which corresponds to inserting \( \overline{L}_k \) into the last slot rather than the first. This affects \( \overline{\partial}(g)_k \) by a factor \((-1)^q\), and \( \overline{\partial}(g_k) \) by a factor \((-1)^{q-1}\).}

Assume for the moment that \( g \in \text{dom}(\overline{\partial}) \) is smooth up to the boundary, say \( g = \sum'_{|J|=q} g_J \overline{\omega}^J \) in the local boundary frame (we do not assume that \( g \) is supported in that chart). Then \( g_k = \sum'_{|K|=q-1} g_{KK} \overline{\omega}_K \). Also

\[
(\overline{\partial}g)_k = \sum'_{|J|=q} \sum^n_{m=1} \left( (\overline{L}_m g_J)\overline{\omega}_m \wedge \overline{\omega}^k \right)_k + O(||g||)
\]

\[
= \sum'_{|K|=q-1} \sum^n_{m=1} (\overline{L}_m g_{KK}) (\overline{\omega}_m \wedge \overline{\omega}_K)_k + (-1)^q \sum'_{|J|=q, k \notin J} (\overline{L}_k g_J) \overline{\omega}_J + O(||g||) .
\]

The first sum in the second line results form those \( j \) with \( k \in J \). Note that \((\overline{\omega}_m \wedge \overline{\omega}_K)_k = \overline{\omega}_m \wedge \overline{\omega}_K \) when \( m \neq k \). When \( m = k \), the term vanishes. Therefore, modulo terms that are \( O(||g||) \), this

\[
ed^S_{(m,k_1,\cdots,k_{q-1})} \overline{\omega}_S = \overline{\omega}_m \wedge \overline{\omega}_K .
\]
Finally, as noted above, we also have

\begin{equation}
\|\overline{\partial}(u \wedge \omega_k)\| \lesssim \|\overline{\partial}u\| + \|\overline{\partial}^* u\|.
\end{equation}

Estimates (41) and (42) give (33) when \(|I| = 1\). The general case now follows inductively. \(\Box\)

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