Unitary matrix integrals, symmetric polynomials, and long-range random walk models

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Abstract

Unitary matrix integrals over symmetric polynomials play an important role in a wide variety of applications, including random matrix theory, gauge theory, number theory, and enumerative combinatorics. This paper derives novel results on such integrals and applies these and other identities to correlation functions of long-range random walk (LRRW) models. We demonstrate a duality between distinct LRRW models, which we refer to as quasi-local particle-hole duality. We then generalize an identity due to Diaconis and Shahshahani which computes unitary matrix integrals over products of power sum symmetric polynomials, and derive two novel expressions for unitary matrix integrals over Schur polynomials which can be directly applied to LRRW correlation functions. In passing, we demonstrate that going from any (particle-hole) configuration A to another configuration B by consecutively moving particles by n steps involves hopping over either an even or an odd number of particles, depending only on the choice of A and B. In principle, all results demonstrated in this work can be checked in experimental setups such as trapped ion systems, where LRRW models appear as an effective description. We further suggest specific correlation functions which may be applied to the benchmarking of such experimental setups.

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1 Introduction

1.1 Background

This work presents a study of unitary matrix integrals over symmetric polynomials and their application to correlation function of long-range random walk (LRRW) models. We derive novel results on weighted unitary integrals over symmetric polynomials. These have a wide variety of mathematical and physical applications, including to Random Matrix Models (RMM) and quantum chaos, gauge theories, number theory, and enumerative combinatorics. Connections between RMM’s and random walk models have been explored extensively, see e.g. [1], [2] for an early review on RMM’s and [3], [4] for a more recent one. Random walkers are called vicious when their paths are not allowed to intersect, i.e. when there exists the restriction that no two random walkers may occupy the same site. To be precise, vicious random walker models belong to the class of random-turns models when, at each time step, we move a single random walker. On the other hand, there are the lock-step models, where, at each time step, all random walkers are moved. For an early treatment, see [5], see also e.g. [6], [7], [8], [9], [10], [11], [12]. The relation between RMM’s and vicious random walkers can be seen to arise from the fact
that the joint eigenvalue distribution of unitary RMM’s is proportional to the squared Vandermonde determinant. This leads to a vanishing probability that any two eigenvalues coincide, which is known as level repulsion in random matrix theory. In physical terms, these non-intersecting random walkers can be interpreted as fermionic or hard-core bosonic particles. This leads to an interpretation of the eigenvalue probability density of models related to classical Lie groups in terms of the ground state density of non-interacting and non-intersecting particles which are subject to confining potential and certain boundary conditions. This language naturally leads to a so-called $\tau$-functions of integrable hierarchies of differential equations with many deep results obtained over the years \cite{13}, \cite{14}, \cite{15}, \cite{16}, \cite{17}, \cite{18}. Further, various correlation functions and important quantities in string theory are expressed in terms of these objects, see e.g. \cite{19} for recent developments. We also mention here applications in algebraic geometry and topology, see \cite{4} for a review.

It was shown by Bogoliubov \cite{20} that the time-dependent correlation functions of the XX0-model are the generating functions of nearest neighbour vicious random walkers, i.e. those which can take only a single step to the left or right. Here, the XX0-model refers to the XX-model at zero magnetic field. It was found that these correlation functions can be expressed as certain weighted unitary matrix integrals over Schur polynomials \cite{20}, \cite{21}. In general, spin configurations which start with an infinite sequence of particles and end with an infinite sequence of holes correspond uniquely to certain Young diagrams. Time-dependent correlation functions of the XX0-model then take the form of unitary matrix integrals over a product of Schur polynomials associated to the corresponding Young diagrams. In the remainder of this work, we will refer to down spins as particles and up spins as holes. In the nearest neighbour case, the corresponding RMM is the Gross-Witten-Wadia (GWW) model, where the inverse square of the coupling constant of the GWW model is proportional to the time parameter of the correlation function of the XX0-model. This relation between random walk correlation functions and matrix integrals over Schur polynomials was generalized to the case of LRRW models \cite{22}, where the particles can not only take a single step to the left or right at each time step but can generally move over greater distances. The particles in question behave as hard-core bosons rather than fermions, as the wave function does not acquire a minus sign when a particle hops over another one. In the long-range case, the weight function of the corresponding matrix model encodes the particular choice of LRRW model. In particular, the $n^{th}$ hopping parameter $a_n$, which allows particles to hop a distance of $n$ steps, equals the $n^{th}$ Fourier coefficient of the weight function. The hopping parameters are required to decay as $a_n \sim n^{-1-\epsilon}$ for $\epsilon > 0$ as $n \to \infty$, corresponding to weight functions which satisfy the strong Szegö limit theorem.

The generalization to LRRW models allows for the application to long-range one-dimensional systems, such as those characterized by dipole-dipole interactions. For more recent examples of such systems, see e.g. \cite{23}. Another physically motivated context where long-range systems are of interest is that of Anderson localization in long-range low-dimensional hopping models \cite{24}, \cite{25}, \cite{26}. These systems can be simulated experimentally with trapped ions, which have been demonstrated to exhibit great tunability in terms of the range of the the relevant hopping amplitude (see e.g. \cite{27} for a review). Further applications include the dynamics of the Loschmidt echo in 1D systems, already considered
The Loschmidt echo, a measure of chaoticity in quantum systems \cite{29}, can be regarded as a particular realization of the $\tau$-function mentioned above. The Loschmidt echo is also related to other important probes of many-body system dynamics, such as out-of-time-correlation functions. Importantly, the Loschmidt echo is an experimentally measurable quantity, see e.g. \cite{31}, \cite{32}, \cite{33}. In the context of combinatorics, we mention also a relation to the plane partitions, see e.g. \cite{29} for a recent account thereof.

In this paper, we consider physically relevant quantities, such as Loschmidt echo and various correlation functions in LRRW models, for which we employ identities on Young diagrams and the closely related theory of symmetric functions. We summarise the outline of the paper, including these results, in the remainder of the introduction.

1.2 Structure of the paper and main results

This work is structured as follows. In section 2 we review the basics of symmetric function theory and state several expressions that will be useful to us later. Then, in section 3 we review the relation between correlation functions of LRRW models and weighted unitary integrals over Schur polynomials. We consider a special class of correlation functions between configurations which begin with an infinite string of particles and end with an infinite string of holes with a finite region of mixed particles and holes in between. The Young diagrams of the aforementioned Schur polynomials characterize the particle-hole configurations that occupy a finite interval in between the infinite strings of particles and holes. To be precise, we consider the following type of correlation function

$$F_{\lambda;\mu}(\tau) = \left\langle \varnothing \left| \sigma_{j_1}^+ \cdots \sigma_{j_N}^+ e^{-\tau H} \sigma_{l_1}^- \cdots \sigma_{l_N}^- \right| \varnothing \right\rangle, \quad \lambda_r = j_r + r, \quad \mu_s = l_s + s,$$

(1)

where $|\varnothing\rangle$ is a state with holes at all lattice sites, where we remind the reader that we refer to up spins as holes and down spins as particles. Defining the matrix model average $\langle \ldots \rangle$ for some weight function $f$ as in \cite{104}, it was found \cite{20}, \cite{21}, \cite{22} that this can be written (in our notation) as

$$\frac{F_{\lambda;\mu}(\tau)}{F_0(\tau)} = \left\langle s_{\lambda}(U)s_{\mu}(U^{-1}) \right\rangle,$$

(2)

where the dependence on the particular choice of LRRW model and the generalized time parameter $\tau$ are captured by the weight function $f$. Section 4 treats the evaluation of the objects appearing in (2) using the strong Szegö limit theorem and Gessel-type identities.

Sections 5 and 6 present our results, starting in 5.1 with the generalization of the result due to Diaconis and Shahshahani in \cite{34}. Writing $p_{\mu} = p_{\mu_1}p_{\mu_2} \ldots$, our result in \cite{157} reads

$$\left\langle p_{\rho}(U)p_{\mu}(U^{-1}) \right\rangle = \sum_{n=0}^{\hat{n}} C_n.$$

(3)

where $C_n$ and $\hat{n}$ are the contribution arising from performing $n$ contractions in $\left\langle p_{\rho}p_{\mu} \right\rangle$ and the maximum number of such contractions defined in \cite{155} and \cite{156}, respectively. Essentially, this result arises from
the application of Wick’s theorem and the fact that $\langle p_n(U) p_k(U^{-1}) \rangle - \langle p_n(U) \rangle \langle p_k(U^{-1}) \rangle = n \delta_{n,k}$, which we derived a previous work [35]. In sections 5.2.1 and 5.2.2 we derive expansions of general correlation functions ($\langle s_{\lambda s_{\nu}} \rangle$) in terms of Schur or power sum polynomials, respectively. Specifically, section 5.2.1 establishes a relation between correlation functions $\langle s_{\lambda s_{\nu}} \rangle$ and $\langle s_{\lambda \setminus \{\alpha\}} \rangle$, where $\lambda \setminus \{\alpha\}$ is a diagram obtained from $\lambda$ by the removal of border strips of sizes $\alpha_1, \alpha_2, \ldots$. In section 5.2.2 we express $\langle s_{\lambda s_{\nu}} \rangle$ in terms of $\langle p_{\mu} \rangle$, where the expansion coefficients are determined by removing border strips from $\lambda$ and $\nu$ such that the resulting diagram is the same for both $\lambda$ and $\nu$. These results have a direct interpretation in terms of particle-hole configurations which is treated in section 6.2, for which reason we state these results with the particle-hole interpretation below.

We first consider the results derived in section 6.1, where we apply identities involving elementary and complete homogeneous symmetric polynomials to LRRW correlation functions. The most striking result that arises in this way involves the well-known involution between elementary and homogeneous polynomials corresponding to the transposition of Young diagrams, which exchange rows and columns. Consider any two LRRW models, referred to as model A and B, whose hopping parameters are related by $a_k \rightarrow (-1)^{k+1} a_k$, and write their correlation functions as $F_{\lambda,\mu}^{(1)}(\tau)$ and $F_{\lambda,\mu}^{(1)}(\tau)$, respectively. Our result is given by equation (216), which reads

$$F_{\lambda,\mu}^{(1)}(\tau) = F_{\lambda,\mu}^{(2)}(\tau),$$

where $^t$ refers to transposition of diagrams, which, combined with a parity transformation, implements a particle-hole transformation. Therefore, equation (4) states that the correlation functions of models A and B are equal after performing local particle-hole and parity transformations, which only act non-trivially on the interval in between the infinite strings of particles and holes. For this reason, we refer to this result as Quasi-Local Particle-Hole duality (QLPH). In this way, a basic property of symmetric polynomials leads to a surprising result on LRRW models. Further such examples, involving e.g. the Pieri formula, are given in equations (200), (201), (206), (207), (208).

Section 6.2 treats the application of identities involving power sum polynomials and border strips, including results derived in section 5.2. We find that the multiplication of power sum polynomials of degree $n$ and the corresponding addition of border strips is related to fermionic particles hopping $n$ steps to the right, whereas removal of such a border strip corresponds to hopping $n$ steps to the left. This allows us to interpret various results in terms of an auxiliary fermionic (rather than hard-core bosonic) system. This reasoning can be applied to the calculation of $\chi_{\lambda \setminus \alpha}$, which are the irreducible characters of the symmetric group, as we explain just above section 6.2.1. Further, combined with identities we found in [35], it leads immediately to results such as (235) and (228). In section 6.2.1 we use the relation between border strips and particle hopping to characterize the action of the hamiltonian in terms of Young diagrams.

In section 6.2.2 we use the fact that $\chi_{\lambda \setminus \alpha}$, for $\alpha = (n^k)$, is cancellation-free [36] to derive a result that appears a priori unrelated to the rest of the work. In particular, start with a particle-hole configuration, which we call $\lambda$, and consider all ways to move (not necessarily distinct) particles by $n$ steps, where $n$
is any finite positive integer. By taking a particle and moving it by \( n \) steps (from site \( j \) to \( j + n \)), it may in the process ‘jump over’ \( \leq n - 1 \) other particles (occupying sites \( j + 1 \) to \( j + n - 1 \)). Our result then states that all ways to go from \( \lambda \) to any other configuration \( \nu \) involves jumping over either an even or an odd number of particles. Although this is a rather simple statement that could easily be understood by a layperson, it is, to the best of the authors’ knowledge, a novel result. When applied to a fermionic model where fermions can hop only by \( \pm n \) sites for a single choice of \( n \), this implies that there is no destructive interference between various ways to arrive at the same fermionic state, allowing for rapid spread through Hilbert space upon time evolution. The same reasoning can be applied to certain LRRW correlation functions, leading to (244) and (246).

In sections 6.2.3 and 6.2.4, we apply our results from section 5.2 to LRRW correlation functions. The expansion in 5.2.1, including its particle-hole interpretation in 6.2.3, given in (148) and (252), reads

\[
\langle s_\lambda s_\nu \rangle_c = \sum_{\alpha} \frac{1}{z_\alpha} \left( \sum_{\alpha_1, \ldots, \alpha_k} (-1)^{ht(T_n)} \langle s_{\lambda,1}\{\alpha_1\} \rangle \right) \times (\lambda \rightarrow \nu) = \sum_{\alpha} \frac{1}{z_\alpha} (-1)^{ht(T_n)} \left\{ \text{Distinct ways to take } \ell(\alpha) \text{ particles in } \lambda \right. \\
\left. \text{and move them } \alpha_1, \alpha_2, \ldots \text{ steps to the left, thereby hopping over } T_\alpha \text{ particles} \right\} \times (\lambda \rightarrow \nu). \quad (5)
\]

The expansion in terms of power sum polynomials derived in section 5.2.2, including its particle-hole interpretation in 6.2.4, is given by equations (178) and (255) as

\[
\langle s_\lambda s_\nu (U^{-1}) \rangle = \sum_{\omega, \gamma} p_\omega(y)p_\gamma(x) \sum_{\{\eta\}, \{\xi\}} (-1)^{ht(T_\eta) + ht(T_\xi)} \delta_{\lambda \setminus \{\eta\}, \nu \setminus \{\xi\}}
\]

\[
= \sum_{\omega, \gamma} p_\omega(y)p_\gamma(x) \frac{1}{z_\omega z_\gamma} (-1)^P \left\{ \text{Distinct ways to move particles in } \lambda \text{ and } \nu \right. \\
\left. \text{to the left by } \gamma_1, \gamma_2, \ldots \text{ and } \omega_1, \omega_2, \ldots \text{ steps, respectively, so hopping over } P \text{ other particles} \right. \\
\left. \text{and ending up in the same configuration.} \right\} \quad (6)
\]

The expansion in (6) is particularly convenient in applications where one has access to the power sum polynomials, such as for LRRW models where they are given by \( p_k = k\tau a_{\pm k} \), in terms of the hopping parameters \( a_k \) and the generalized time parameter \( \tau \). We therefore believe that the expression in equation (6) is the most useful one derived in this work. Then, in section 6.2.5 we suggest applications of some of our results to experimental benchmarking. We finish with our conclusions, as well as possible generalizations and applications of our results, which include both theoretical and experimental suggestions.

### 2 Symmetric polynomials and Young diagrams

We review here some aspects of symmetric polynomials and Young diagrams that will be useful to us later, starting with some simple examples. Take a set of variables \( x = (x_1, x_2, \ldots) \). The elementary
symmetric polynomials are then defined as

$$e_k(x) = \sum_{i_1 < \cdots < i_k} x_{i_1} \ldots x_{i_k} .$$  \hspace{1cm} (7)

Some examples include

$$e_0 = 1 ,$$
$$e_1(x_1) = x_1 ,$$
$$e_1(x_1, x_2) = x_1 + x_2 ,$$
$$e_2(x_1, x_2) = x_1x_2 .$$  \hspace{1cm} (8)

Closely related are the complete homogeneous symmetric polynomials, defined as

$$h_k(x) = \sum_{i_1 \leq \cdots \leq i_k} x_{i_1} \ldots x_{i_k} ,$$  \hspace{1cm} (9)

which contain all monomials of degree $k$. Note the difference in the summation bounds between (7) and (9). Some examples of $h_k$ include

$$h_0 = 1 ,$$
$$h_1(x_1) = x_1 ,$$
$$h_1(x_1, x_2) = x_1 + x_2 ,$$
$$h_2(x_1, x_2) = x_1^2 + x_2^2 + x_1 x_2 .$$  \hspace{1cm} (10)

Another type of symmetric polynomial is the power-sum polynomial,

$$p_k(x) = \sum_j x_j^k = x_1^k + x_2^k + \ldots$$  \hspace{1cm} (11)

Writing $z = e^d$, the generating functions of $e_k$ and $h_k$ and their relation with power sums are as follows [see e.g \[37\]],

$$E(x; z) = \sum_{k=0}^{\infty} e_k(x) z^k = \prod_{k=1}^{\infty} \left( 1 + x_k z \right) = \exp \left[ \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} p_k(x) z^k \right] ,$$
$$H(x; z) = \sum_{k=0}^{\infty} h_k(x) z^k = \prod_{k=1}^{\infty} \frac{1}{1 - x_k z} = \exp \left[ \sum_{k=1}^{\infty} \frac{1}{k} p_k(x) z^k \right] .$$  \hspace{1cm} (12)

From the above expressions, it is clear that $H(x; z)E(x; -z) = 1$. Checking every order of $z$ then gives, for all $n \geq 1$ and any choice of $x$ [e.g. (2.6') from \[37\]],

$$\sum_{r=0}^{n} (-1)^r h_{n-r}(x) e_r(x) = 0 .$$  \hspace{1cm} (13)
Partitions play an important role in the study of symmetric polynomials. A partition of \( n \in \mathbb{Z}^+ \) is a sequence of non-negative integers \( \lambda = (\lambda_1, \lambda_2, \ldots, \lambda_{\ell(\lambda)}) \), which we will order for convenience as \( \lambda_1 \geq \lambda_2 \geq \ldots \), satisfying \( \sum_j \lambda_j = n \). The size (or weight) of a partition is given by the sum of its terms \( |\lambda| = \sum_j \lambda_j \) and its length \( \ell(\lambda) \) is the largest value of \( j \) such that \( \lambda_j \neq 0 \). Closely related to partitions of \( n \) are compositions of \( n \), consisting also of a sequence of non-negative integers which sum to \( n \), but where a different ordering in these integers defines a different composition. A weak composition of \( n \) is a composition which may include zeroes as its entries, that is, an ordered set of non-negative integers which sum up to \( n \). A partition of \( n \) corresponds to a Young (or Ferrers) diagram containing \( n \) cells, or ‘boxes’. We will use these terms interchangeably. As an example, the diagram corresponding to a partition of 12 given by \( \lambda = (6, 4, 2, 1) \) is given below, where \( \lambda_j \) equals the number of cells in the \( j \)th row.

![Diagram](image)

We will denote a diagram consisting of \( b \) rows of \( a \) cells by \((a^b)\). For a partition \( \lambda \), we will write

\[
e_\lambda = \prod_{j>1} e_{\lambda_j} , \quad h_\lambda = \prod_{j>1} h_{\lambda_j} , \quad p_\lambda = \prod_{j>1} p_{\lambda_j} .
\]

Further, we write

\[
z_\lambda = \prod_{j>1} j^{m_j} m_j! , \quad m_j(\lambda) = \text{Card}\{ k : \lambda_k = j \},
\]

i.e. \( m_j(\lambda) \) is the number of rows in \( \lambda \) of length \( j \). We also write \( \varepsilon_\lambda = (-1)^{|\lambda| - \ell(\lambda)} \). Newton’s identities then read

\[
h_n = \sum_{|\lambda|=n} \varepsilon_\lambda^{-1} p_\lambda , \quad e_n = \sum_{|\lambda|=n} \varepsilon_\lambda \varepsilon_\lambda^{-1} p_\lambda .
\]

In terms of the complete exponential Bell polynomial \( B_n \), we have

\[
h_n = \frac{1}{n!} B_n \left( p_1, p_2, 2! p_3, \ldots, (n-1)! p_n \right) , \quad e_n = \frac{(-1)^n}{n!} B_n \left( -p_1, -p_2, -2! p_3, \ldots, -(n-1)! p_n \right) .
\]

\(^1\)Complete Bell polynomials \( B_n(x_1 \ldots x_n) \) can be defined by their generating function, \( \sum_{n=0}^{\infty} B_n(x_1 \ldots x_n) t^n/n! = \exp(\sum_{j=1}^{\infty} x_j t^j/j! ) \).
Another type of symmetric polynomial is the Schur polynomial. Schur polynomials play an important role as characters of irreducible representations, often referred to as irreps, of general linear groups and subgroups thereof. Schur polynomials are associated to a partition $\lambda$ and a set of variables $x = (x_1, x_2, \ldots)$ in the following way. For a choice of $\lambda$, a semistandard Young tableau (SSYT) is given by positive integers $T_{i,j}$ satisfying $1 \leq i \leq \ell(\lambda)$ and $1 \leq j \leq \lambda_i$. These integers are required to increase weakly along every row and increase strictly along every column, i.e. $T_{i,j} \geq T_{i,j+1}$ and $T_{i,j} > T_{i+1,j}$ for all $i, j$. Label by $\alpha_i$ the number of times that the number $i$ appears in the SSYT. We then define

$$ x^T = x_1^{\alpha_1} x_2^{\alpha_2} \ldots. $$

(19)

The Schur polynomial $s_{\lambda}(x)$ is given by [38].

$$ s_{\lambda}(x) = \sum_T x^T, $$

(20)

where the sum runs over all SSYT’s corresponding to $\lambda$ i.e. all possible ways to inscribe the diagram corresponding to $\lambda$ with positive integers that increase weakly along rows and strictly along columns. If $\lambda_j = 0$ for all $j$, then $\lambda$ is the empty partition, which we denote by $\lambda = \emptyset$. The Schur polynomial of the empty partition is set to unity, i.e.

$$ s_{\emptyset}(x) = 1, $$

(21)

which is independent of the choice of variables $x$. We give an example of an SSYT corresponding to a Young diagram $\lambda = (3,2)$ and with non-zero variables $x_1, x_2, x_3$.

(22)

From [31] one can see that the contribution of this SSYT is given by $x_1^2 x_2 x_3^2$. Summing over all monomials corresponding to all SSYT’s then gives the Schur polynomial $s_{(3,2)}(x_1, x_2, x_3)$. We emphasize that this is a symmetric polynomial, as this may not be obvious from the definition. Consider the Schur polynomials corresponding to a row or a column of $n$ cells, shown below for $n = 4$.

(23)

One can see that

$$ s_{(1^n)} = e_n, \quad s_{(n)} = h_n, $$

(24)
for any choice of \( x \). That is, the Schur polynomial of a column or row of \( n \) cells is given by the degree \( n \) elementary or complete homogeneous symmetric polynomial, respectively. Equation (24) simply follow from the requirement for SSYT’s that integers increase weakly along rows and strongly along columns, compare with (7) and (9). It follows that we can exchange between \( e_n \) and \( h_n \) by transposing diagrams, that is, by reflecting across the main diagonal of the diagram, as this exchanges rows and columns. For a diagram \( \lambda \), its transpose is denoted as \( \lambda^t \). Since transposition is a reflection, it is an involution, i.e. \( (\lambda^t)^t = \lambda \). It is clear that \( (n^t)^t = (1^n) \) i.e. this involution maps rows to columns and vice versa. As a less trivial example, take \( \lambda = (5, 4, 2, 1) \), shown below on the left, and \( \lambda^t = (4, 3, 2^2, 1) \), shown on the right.

![Diagram](image)

The power sum polynomials can also be expressed in terms of Schur polynomials, in this case in the form of a sum,

\[
p_n = \sum_{r=0}^{n-1} (-1)^r s_{(n-r, 1^r)}.
\]

Here, \((a, 1^b)\) is a hook-shaped diagram consisting of a row with \( a \) cells followed by \( b \) rows with a single cell. For example, \( \lambda = (5, 1^3) \) is given by the following diagram.

![Diagram](image)

Although equation (26) may not be immediately obvious, it can easily be seen to arise in simple examples. If \( x = (x_1, x_2) \) and \( n = 1 \), we have the following SSYT’s.

\[
\begin{align*}
1 & + 2 \\
\end{align*}
\]

Which gives \( p_1(x) = s_{(1)}(x_1, x_2) = x_1 + x_2 \). For \( n = 2 \), the following SSYT’s contribute,

\[
\begin{align*}
1 & 1 + 1 & 2 + 2 & 2 & - 1 & 2 \\
\end{align*}
\]

Note that the rightmost SSYT corresponding to \((1^2)\) contributes with a minus sign, so we get \( p_2(x) = s_{(2)}(x_1, x_2) - s_{(1^2)}(x_1, x_2) = x_1^2 + x_2^2 \). The reader may convince oneself that this generalizes to higher
Schur polynomials have a natural generalization to so-called **skew Schur polynomials**, which are associated to skew diagrams. Skew diagrams are constructed from two (non-skew) diagrams $\lambda$ and $\mu$ such that $\mu \subseteq \lambda$, which means that $\mu_i \leq \lambda_i$, $\forall \ i$. The skew diagram denoted by $\lambda/\mu$ is then the complement of $\mu$ in the diagram corresponding to $\lambda$. For $\lambda = (4,3,2)$ and $\mu = (3,1)$, the skew diagram $\lambda/\mu$ is given by the following, where we indicate in black those cells which are removed from $\lambda$.

![Skew Diagram](image)

The skew diagram on the right hand side is a **border strip**, which is a connected skew diagram not containing a 2 by 2 subdiagram. This is an important class of skew diagrams which we will encounter again later. For a general skew diagram, define a skew semistandard Young tableau corresponding to $\lambda/\mu$ as above, namely, as an array of positive integers $T_{ij}$ satisfying $1 \leq i \leq \ell(\lambda)$ and $\mu_i \leq j < \lambda_i$ which increase weakly along rows and strictly along columns. We then define the **skew Schur polynomial** corresponding to $\lambda/\mu$ as

$$s_{\lambda/\mu} = \sum_T x^T,$$

where the sum again runs over all SSYT’s corresponding to $\lambda/\mu$. Note that if $\mu = \emptyset$, we have $s_{\lambda/\mu} = s_\lambda$, and if $\lambda = \mu$, $s_{\mu/\mu} = s_\emptyset = 1$. Let us consider $\lambda = (3,2)$ and $\mu = (1)$. Below, we give a skew SSYT corresponding to the skew partition $\lambda/\mu$, which would contribute $x_1^2x_2x_3$ to the skew Schur polynomial.

![Skew SSYT](image)

From the strong increase of integers along the rows of a (skew) SSYT, it follows that,

$$s_{\lambda/\mu}(x_1, \ldots, x_n) = 0 \text{ unless } 0 \leq \lambda_i - \mu_i \leq n \text{ for all } i \geq 1.$$  

Note that an example of (33) is given by the fact that $e_k(x_1, \ldots, x_N) = s_{(1^k)}(x_1, \ldots, x_N) = 0$ for $k > N$.

We have the following expressions for skew Schur polynomials,

$$s_{\lambda/\mu} = \sum_\nu C_{\lambda\mu}^\nu s_\nu,$$

and products of non-skew Schur polynomials,

$$s_\lambda s_\mu = \sum_\nu C_{\lambda\mu}^\nu s_\nu.$$
The Littlewood-Richardson coefficients $c^\lambda_{\mu\nu}$ are given by the number of Littlewood-Richardson tableaux of shape $\nu/\lambda$ and weight $\mu$. A Littlewood-Richardson tableau is an SSYT such that, when we read its entries from right to left and top to bottom, any positive integer $j$ appears at least as many times as $j + 1$. Note from (35) that $c^\lambda_{\mu\nu} = c^\lambda_{\nu\mu}$. For example, of the SSYT’s pictured below, the one on the left is a Littlewood-Richardson tableau while the one on the right is not.

\[
\begin{array}{ccc}
1 & 1 & \\
2 & 2 & \\
1 & 3 & \\
\end{array} & \begin{array}{ccc}
1 & 2 & \\
2 & 2 & \\
1 & 3 & \\
\end{array} 
\]

A special case of the Littlewood-Richardson rule is the Pieri formula [37]

\[s_\lambda h_n = \sum_\nu s_\nu ,\]  

where the sum is over all $\nu$ such that $\nu/\lambda$ is a horizontal strip, i.e. a skew diagram with at most one cell in each column. Applying the involution which transposes all diagrams, and therefore exchanges $h_n = s_{(n)}$ with $e_n = s_{(1^n)}$, we have

\[s_\lambda e_n = \sum_\nu s_\nu ,\]  

where the sum is now over all $\nu$ such that $\nu/\lambda$ is a vertical strip, i.e. a skew diagram with at most one cell in each row. The Pieri formula states that $c^\nu_{\lambda\mu}$ for $\mu = (n)$ is equal to 1 when $\nu/\lambda$ is a horizontal strip, and zero otherwise. Applying this to $s_\lambda/(n)$, we have

\[s_\lambda/(n) = \sum_\nu s_\nu ,\]  

where the sum is now over all $\nu$ such that $\lambda/\nu$ is a horizontal strip. From [38], we also have

\[s_\lambda/(1^n) = \sum_\nu s_\nu ,\]  

where $\lambda/\nu$ is a vertical strip. The Pieri formula $s_\lambda h_n = \sum_\nu s_\nu$ is illustrated below for $\lambda = (2,1)$ and $n = 2$, with the cells that are added onto $\lambda$ are indicated in gray. It is clear that $\nu/\lambda$ is a horizontal strip for all diagrams $\nu$ on the right hand side, as there are no two gray cells in any column. For equation (38), one should simply transpose the diagrams.

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We can use the Pieri rule to demonstrate $\sum_{r=0}^n (-1)^r h_{n-r} e_r = 0$, equation (13). In particular, for
$a, b \geq 1$, we have

\[ h_a e_b = s_{(a,1^b)} + s_{(a+1,1^{b-1})} . \]  \hfill (42)

The example for $a = 4$ and $b = 3$ gives the following.

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\begin{array{c}a, b \geq 1$, we have
\[ h_a e_b = s_{(a,1^b)} + s_{(a+1,1^{b-1})} . \]  \hfill (42)

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\begin{array{c}a, b \geq 1$, we have
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Similar to the Pieri formula, equation (46) can be inverted to given the following expression for skew Schur polynomials,

\[
\sum_{r=0}^{n-1} (-1)^r s_{\lambda/(n-r,1^r)} = \sum_{\nu} (-1)^{ht(\lambda/\nu)} s_{\nu},
\]

(48)

where the sum is now over all \( \nu \) such that \( \lambda/\nu \) is a border strip of size \( n \). We will also denote these border strips by (e.g.) \( \eta \) with \( |\eta| = n \) and write \( \lambda \backslash \eta \) for the representation obtained from \( \lambda \) after removing the border strip \( \eta \).

Schur polynomials can be expressed in determinantal form. First of all, the (antisymmetric) Vandermonde determinant can be expressed as

\[
\det(x_j^{(N-k)} \mid_{j,k=1}^N) = \prod_{1 \leq j < k \leq N} (x_j - x_k).
\]

(49)

We then have

\[
s_{\lambda}(x_j) = \frac{\det(x_j^{N-k+\lambda_k} \mid_{j,k=1}^N)}{\det(x_j^{N-k} \mid_{j,k=1}^N)}.
\]

(50)

(Skew) Schur polynomials can be expressed in terms of elementary symmetric polynomials or complete homogeneous symmetric polynomials via the following determinantal expressions, known as the Jacobi-Trudi identities,

\[
s_{(\mu/\lambda)} = \det(h_{\mu_j - \lambda_k - j + k}^{(\mu)} \mid_{j,k=1}^{\mu_1}) = \det(e_{\mu_j - \lambda_k - j + k}^{(\mu)} \mid_{j,k=1}^{\mu_1}),
\]

\[
s_{(\mu/\lambda)^t} = \det(e_{\mu_j - \lambda_k - j + k}^{(\mu)} \mid_{j,k=1}^{\mu_1}) = \det(h_{\mu_j - \lambda_k - j + k}^{(\mu)} \mid_{j,k=1}^{\mu_1}).
\]

(51)

Again, we see that the expressions in terms of \( h_j \) and \( e_j \) are related by transposition of the skew diagram, \( (\mu/\lambda) \rightarrow (\mu/\lambda)^t \).

Schur polynomials can also be expanded in terms of power sum polynomials,

\[
s_{\lambda} = \sum_{\alpha} \frac{\chi_{\lambda}^{\alpha}}{z_{\alpha}} p_{\alpha},
\]

(52)

where the sum is over all partitions \( \alpha \), \( z_{\alpha} \) is defined in (16), and where \( \chi_{\lambda}^{\alpha} \) is the character of the symmetric group \( S_n \) with \( n = |\lambda| \) of an irrep \( \lambda \) associated to a permutation of cycle type \( \alpha \), see e.g. [37], [38]. In fact, \( \alpha \) can generally be a weak composition, and it can be shown that \( \chi_{\lambda}^{\alpha} \) does not depend on the order of the entries of \( \alpha \). However, in (52), we only sum over a single \( \alpha \) corresponding to each cycle type, which is equivalent to summing over partitions. Equation (52) generalizes to the case of a skew partition \( \lambda/\mu \) instead of \( \lambda \), but we will not be needing this expression. The inverse of
\( p_{\alpha} = \sum_{\lambda} \chi_{\lambda}^\alpha s_{\lambda} \). \hspace{1cm} (53)

To describe the characters \( \chi_{\lambda}^\alpha \) appearing in these expansions, we first define a *border-strip tableau* (BST) of shape \( \lambda \) and type \( \alpha \) as follows. We take a diagram \( \lambda \) and inscribe it with positive integers such that

1. The integers are weakly increasing along both rows and columns
2. The cells of \( \lambda \) that are inscribed by \( j \) form a border strip of length \( \alpha_j \)

The resulting object is a border strip tableau, which we denote as \( T \in BST(\lambda, \alpha) \). We show an example below for \( \lambda = (7, 4^2, 3, 1) \) and \( \alpha = (4^2, 5, 3) \) (remember that \( \alpha \) is a composition and its entries are not generally in non-decreasing order) where cells belonging to a single border strip share the same color.

\[
\begin{array}{cccccc}
1 & 1 & 2 & 2 & 2 & 2 \\
1 & 3 & 3 & 5 & 5 & 5 \\
3 & 3 & 5 & 5 & 5 & 5 \\
4 & 4 & 5 & 5 & 5 & 5 \\
4 & 4 & 5 & 5 & 5 & 5 \\
\end{array}
\]

(54)

Denoting the border strips of length \( \alpha_j \) that appear in \( T \) as \( B_j \), the height of \( T \) is defined as

\[
ht(T) = \sum_{j=1}^{\ell(\alpha)} ht(B_j) .
\]

(55)

For example, for the above BST for \( \lambda = (6, 4^2, 3, 1) \) and \( \alpha = (4, 3, 5, 3) \), we have

\[
ht(T) = 1 + 0 + 1 + 1 + 2 = 5 .
\]

(56)

We then have

\[
\chi_{\lambda}^\alpha = \sum_{T \in BST(\lambda, \alpha)} (-1)^{ht(T)} .
\]

(57)

This is known as the Murnaghan-Nakayama rule, see e.g. [37] or [38]. It can be shown that \( \chi_{\lambda}^\alpha \) does not depend on the order of the entries in \( \alpha \), i.e. on the order in which we add border strips of lengths \( \alpha_j \). The Murnaghan-Nakayama rule also generalizes to skew diagrams \( \lambda/\mu \), but we will not be needing this. From \( (53) \), we have

\[
p_n = \sum_{\lambda} \chi_{\lambda}^\alpha s_{\lambda} = \sum_{r=0}^{n-1} (-1)^r s_{(n-r, 1^r)} .
\]

(58)
This arises simply from the fact that any Young diagram consisting of a single border strip is a hook shape, as this is the only type of non-skew shape that has no two by two subdiagram. This gives a sum over all hook shapes containing \( n \) cells, \( (n - r, 1^r) \), where the sign appears from the fact that \( \text{ht}((n - r, 1^r)) = r \). We thus see how equation (26) arises as a special case of (53). To calculate \( \chi^\lambda_{\alpha} \), one can use the following recursive formula [e.g. (2.4.4) [39]]

\[
\chi^\lambda_{\mu} = \sum_{\rho} (-1)^{\text{ht}(\rho)} \chi^{\lambda\setminus\rho}_{\mu - \mu_1},
\]

(59)

where the sum runs over all border strips \( \rho \) of \( \lambda \) containing \( \mu_1 \) cells, \( \lambda\setminus\rho \) is the results of removing \( \rho \) from \( \lambda \), and \( \mu - \mu_j = (\mu_1, \mu_2, \ldots, \mu_{j-1}, \mu_{j+1}, \ldots) \).

Let us consider the object \( p^k_n \). Using (53), we have

\[
p^k_n = \sum_{\lambda} \chi^\lambda_{(n^k)} S^\lambda.
\]

(60)

From (57) (or (40) with \( \lambda = \emptyset \)), the \( \chi^\lambda_{(n^k)} \) appearing in (60) are of the following form

\[
\chi^\lambda_{(n^k)} = \sum_{T \in \text{BST}(\lambda, (n^k))} (-1)^{\text{ht}(T)}
\]

(61)

where the sum is over all border strip tableaux of shape \( \lambda \) and type \( \alpha = (n^k) \). For \( \alpha = (n^k) \), it has been shown that the expansion in (61) is cancellation-free [corollary 10, [36]]. That is, for any fixed choice of \( \lambda \), all BST’s appear with the same sign, so that

\[
\chi^\lambda_{(n^k)} = \pm \sum_{T \in \text{BST}(\lambda, (n^k))} 1.
\]

(62)

We illustrate this with an example, with \( \lambda = (6, 5, 2^2, 1) \) and \( \alpha = (n^k) = (4^4) \). This gives the following border strip tableaux, where border strips again share the same color.

\[
\text{(63)}
\]

We see that the heights of the tableaux are given, from left to right, by 4, 6, 4, 6, so that \((-1)^{\text{ht}(T)} = 1 \). Note that the two leftmost BST’s appear with multiplicity two, as one can remove the green and blue border strips in either order. This means that \( \chi^{(6,5,2^2,1)}_{(4^4)} = 6 \). We see, then, that all BST’s contribute with the same sign as they are all of even height. Equation (62) states that the BST’s always appear with the same sign for any choice of \( \lambda \) and \( (n^k) \).
3 Correlation functions of long-range random walk models

We will summarise here the derivation which relates correlation functions of one-dimensional LRRW models to weighted integrals over $U(N)$ with insertion of Schur polynomials. The derivation summarized here follows [20], [21], and [22]. We first consider the XX0-model, that is, the XX-model with zero magnetic field,

$$\hat{H} = \sum_{m,n} \Delta_{n,m} \sigma_n^+ \sigma_m^- \, , \quad \Delta_{m,n} = \delta_{n+1,m} + \delta_{n-1,m} \, .$$  \hfill (64)

We start with state with holes at all sites

$$|\uparrow\rangle = \otimes_n |\uparrow_n\rangle = \otimes_n \left( \begin{array}{c} 1 \\ 0 \end{array} \right)_n \, ,$$  \hfill (65)

which satisfies

$$\hat{H} |\uparrow\rangle = 0 \, .$$  \hfill (66)

Define the correlation function

$$F_{j;l}(\tau) = \left\langle \uparrow| \sigma_j^+ e^{-\tau \hat{H}} \sigma_l^- |\uparrow\rangle \right\rangle \, .$$  \hfill (67)

For now, we consider $\tau$ to be a general complex number, but we will mostly be interested in the case where $\tau = it$ where $t$ is a real-valued time parameter. Using

$$[\sigma_n^+, \sigma_m^-] = \sigma_n^z \delta_{m,n} \, , \quad [\sigma_n^z, \sigma_m^\pm] = \pm 2 \sigma_n^\pm \delta_{m,n} \, ,$$  \hfill (68)

we have

$$\left[ \hat{H} , \sigma_k^- \right] = \sum_m \Delta_{m,k} \sigma_m^- \sigma_k^- \, ,$$  \hfill (69)

which we apply to find

$$\frac{d}{d\tau} F_{j;l}(\tau) = - \left\langle \uparrow| \sigma_j^+ e^{-\tau \hat{H}} \sigma_l^- |\uparrow\rangle \right\rangle = \sum_m \Delta_{l,m} F_{j;m}(\tau) \, .$$  \hfill (70)

In particular, for the XX0-model, where $\Delta_{m,n} = \delta_{n+1,m} + \delta_{n-1,m}$,

$$\frac{d}{d\tau} F_{j;l}(\tau) = F_{j;l+1}(\tau) + F_{j;l-1}(\tau) \, .$$  \hfill (71)

We will generally consider the case where $\Delta_{m,n} = \Delta_{m-n} = a_{m-n}$, i.e. the hopping amplitude depends only on the (positive or negative) distance between lattice sites $m$ and $n$. In this case, $\Delta_{m,n}$ is a Toeplitz matrix, i.e. it is constant along its diagonals. Taking $L + 1$ lattice sites, as in [20], the
The Hamiltonian is generally of the form,

\[ \hat{H} = -\sum_{m=0}^{L} \frac{(L-1)/2}{n=1} \left( a_n \sigma^+_m \sigma^+_{m+n} + a_n \sigma^-_m \sigma^-_{m+n} \right) + \frac{\hbar}{2} \sum_{m=0}^{L} \left( \sigma^z_m - 1 \right), \]  

(72)

where we demand that \( a_{-k} = a_k^* \), the complex conjugate of \( a_k \). We then have

\[ \left[ \hat{H}, \sigma_k \right] = -\sum_n \left( a_n \sigma^-_{k-n} \sigma^z_n + a_n \sigma^-_{k+n} \sigma^z_k \right) - \hbar \sigma_k^-, \]  

(73)

so that

\[ \frac{d}{d\tau} F_{j,l}(\tau) = \sum_n \left( a_n F_{j-n,l} + a_n F_{j+n,l} \right) + \hbar F_{j,l} + \hbar F_{j,l}. \]  

(74)

It is clear that \( F_{j,l} \) only depends on \( |j - l| \). The generating function \( f(z;\tau) = \sum_{j\in\mathbb{Z}} F_{j,l} z^{j-l} \) is given by \[20, 22\]

\[ f(z;\tau) = f_0(z) \exp \left( \tau \sum_{k\in\mathbb{Z}} a_k z^k \right), \]  

(75)

for \( f_0(z) = f(z;\tau = 0) \). Consider now the multi-particle correlation function, with \( N \leq L \),

\[ F_{j_1,\ldots,j_N;l_1,\ldots,l_N}(\tau) = \left\langle \left\langle \sigma^+_{j_1} \ldots \sigma^+_{j_N} e^{-\tau \hat{H}} \sigma^-_{l_1} \ldots \sigma^-_{l_N} \right\rangle \right\rangle. \]  

(76)

This satisfies,

\[ \frac{d}{d\tau} F_{j_1,\ldots,j_N;l_1,\ldots,l_N}(\tau) = \sum_{k,m} \left( a_k F_{j_1,\ldots,j_N;l_1,\ldots,l_{m+k}\ldots,l_N}(\tau) + a_k F_{j_1,\ldots,j_N;l_1,\ldots,l_{m-k}\ldots,l_N}(\tau) \right) + \]  

\[ + N\hbar F_{j_1,\ldots,j_N;l_1,\ldots,l_N}(\tau). \]  

(77)

Remember that the summations over \( k \) and \( m \) are over different ranges. The solution to equation (77) is of determinantal form \[20, 21, 22\],

\[ F_{j_1,\ldots,j_N;l_1,\ldots,l_N}(\tau) = \det \left( F_{j_r,l_s}(\tau) \right)_{r,s=1}^N. \]  

(78)

From the initial condition \( F_{j,l}(0) = \delta_{j,l} \), it follows that,

\[ F_{j,l}(\tau) = \frac{1}{L+1} \sum_{s=0}^{L} \left( \sum_{k} \left( a_k e^{ik\theta_s} + a_{-k} e^{-ik\theta_s} \right) \right) e^{i(j-l)\theta_s}, \quad \theta_s = \frac{2\pi}{L+1} (s - L/2). \]  

(79)
Using the determinantal expression for Schur functions in equation (50), we have

$$F_{j_1, \ldots, j_N; l_1, \ldots, l_N}(\tau) = \frac{1}{(L-1)!^N} \sum_{\{s\}} \exp \left[ \tau \sum_{\tau=1}^N \left( \sum_{k} \left( a_k e^{i\theta_k s} + a_{-k} e^{-i\theta_k s} \right) \right) \right] \prod_{1 \leq r < s \leq N} |e^{i\theta_r} - e^{i\theta_s}|^2$$

with $\lambda_r = j_r - N + r$ and $\mu_s = l_s - N + s$. Since we will be taking $L, N \to \infty$, we shift $j_r$ and $l_r$ by $N$, such that

$$\lambda_r = j_r + r \quad \mu_s = l_s + s \quad (81)$$

This is merely a convenient relabelling of our lattice sites. We now take $L \to \infty$ and $N \to \infty$ such that $L - N = \infty$. This means simply that those configurations start with an infinite sequence of particles and end with an infinite sequence of holes. Demanding that the hopping parameters $a_k$ decay at least as $a_k \sim k^{-1-\epsilon}$ for some $\epsilon > 0$, we then have $\mathcal{N}_{20, 21, 22}$. We include in equation (82) a normalization factor $\mathcal{N}$ which is determined by demanding

$$F_{j_1, \ldots, j_N; l_1, \ldots, l_N}(0) = \prod_{k=1}^M \delta_{j_k, l_k} \quad (83)$$

Note that (82) is the expression for an integral over $U(N)$ weighted by some weight function $f$ with insertion of Schur polynomials $s\lambda$ and $s\mu$, as mentioned at the start of this section. In particular, writing $s\lambda(U) = s\lambda(e^{i\theta_k})$, were $e^{i\theta_k}$ are the eigenvalues of $U$, we have

$$F_{j_1, \ldots, j_N; l_1, \ldots, l_N}(\tau) = \int_{U(N)} dU \det(f(U)) s\lambda(U)s\mu(U^{-1}) \quad (84)$$

When we take the limit $\tau \to 0$, we have $f = 1$, which recovers the circular unitary ensemble (CUE) i.e. the integral over the Haar measure on $U(N)$. In this case $s_{\lambda/\mu}(x) = 0$ for any (generally) skew diagram $\lambda/\mu \neq \emptyset$, which greatly simplifies many calculations.

If $\lambda = \emptyset = \mu$, we have $j_r = -r = l_r$, and we are simply considering the return probability for $N$ adjacent particles $[40, 41]$, see also $[42]$. In general, we will write

$$F_{\lambda; \mu}(\tau) = F_{j_1, \ldots, j_N; l_1, \ldots, l_N}(\tau) \quad (85)$$

where equation (81) expresses the relation between $\lambda$, $\mu$ and $\{j_r\}, \{l_r\}$, respectively. Taking $\lambda = \emptyset = \mu$
then corresponds to the following autocorrelation function,

\[ F_{\emptyset;\emptyset}(\tau) = \langle \ldots, \uparrow, \downarrow, \ldots, \uparrow, \ldots | e^{-\tau H} | \ldots, \downarrow, \ldots, \uparrow, \ldots \rangle. \]  

(86)

We will write the state corresponding to an empty diagram as \( |\emptyset\rangle \). This can be illustrated as follows, where a particle is indicated by a black dot and hole by a white dot and the vertical line separates lattice sites 0 and 1.

\[ |\emptyset\rangle = \ldots \bullet \bullet \bullet \bullet \bullet \bullet \bullet \bullet \bullet \bullet \bullet \ldots \]  

(87)

For non-empty \( \lambda, \mu \), \( F_{\lambda;\mu} \) corresponds to a correlation function where the particles are shifted by a finite number of steps. The well-known association between Young diagrams and 1D configurations of spins (or fermions, or any other binary variable) is as follows. We number the external edges (i.e. those on the lower right) of the diagram corresponding to \( \lambda \) such that the main diagonal passes between sites 0 and 1. We then associate a particle to all vertical edges and a hole to all horizontal edges. This association is illustrated below for \( \lambda = (5,3,1^2) \), where we add a dotted diagonal line which separates lattice sites 0 and 1.

(88)

The configuration corresponding to the above diagram is illustrated below, with again a dotted line separating sites 0 and 1.

\[ |\lambda\rangle = \ldots \bullet \bullet \bullet \bullet \bullet \bullet \bullet \bullet \bullet \bullet \bullet \ldots \]  

(89)

It is clear from the above association between diagrams and spin configurations that \( \lambda \) affects particles from position \( j = -\ell(\lambda) + 1 \) up to \( j = \lambda_1 \), which means that an interval containing \( \ell(\lambda) \) particles and \( \lambda_1 \) holes is affected. In particular, the state \( |\lambda\rangle \) has an hole at \( j = -\ell(\lambda) + 1 \) and a particle at \( j = \lambda_1 \), and the remaining \( \ell(\lambda) - 1 \) particles and \( \lambda_1 - 1 \) holes are are distributed over sites \( j = \{-\ell(\lambda) + 2, -\ell(\lambda) + 3, \ldots, \lambda_1 - 1\} \), which can be seen explicitly for \( \lambda = (5,3,1^2) \) above.

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\(^2\)We made grateful use of the illustrations in the excellent review \[43\] by Zinn-Justin, which we adapt here for our purposes.
4 Evaluating unitary matrix integrals

We briefly review the evaluation of weighted unitary group integrals over Schur polynomials. We start from an absolutely integrable function on the unit circle in $\mathbb{C}$,

$$f(e^{i\theta}) = \sum_{k \in \mathbb{Z}} d_k e^{ik\theta}. \quad (90)$$

We require that $f(e^{i\theta})$ satisfies the assumptions of Szegö’s theorem. That is, we write $f(e^{i\theta})$ as

$$f(e^{i\theta}) = \exp \left( \sum_{k \in \mathbb{Z}} c_k e^{ik\theta} \right), \quad (91)$$

and demand that

$$\sum_{k \in \mathbb{Z}} |c_k| < \infty, \quad \sum_{k \in \mathbb{Z}} |k||c_k|^2 < \infty. \quad (92)$$

Writing,

$$D_N(f) = \det(T_N(f)) = \det(d_{j-k})_{j,k=1}^N, \quad (93)$$

the strong Szegö limit theorem then states that $[44],

$$\lim_{N \to \infty} \frac{D_N(f)}{e^{-N\epsilon_0}} = \exp \left( \sum_{k=1}^\infty kc_k e^{-k} \right). \quad (94)$$

The above determinant and various generalization thereof can be related to integrals over the group of $N$ by $N$ unitary matrices with some weight function $f$ with the insertion of Schur polynomials. Take the following matrix,

$$T_{N}^{\lambda,\mu}(f) = (d_{j-\lambda_j-k+\mu_k})_{j,k=1}^N. \quad (95)$$

Then $[45], [46],

$$D_N^{\lambda,\mu}(f) := \det(T_N^{\lambda,\mu}(f)) = \int_{U(N)} s_{\lambda}(U^{-1})s_{\mu}(U) \det f(U) dU$$

$$= \frac{1}{N!(2\pi)^N} \int_0^{2\pi} s_{\lambda}(e^{-i\theta_1}, \ldots, e^{-i\theta_M})s_{\mu}(e^{i\theta_1}, \ldots, e^{i\theta_M}) \prod_{j=1}^M f(e^{i\theta_j}) \prod_{1 \leq j < k \leq N} |e^{i\theta_j} - e^{i\theta_k}|^2 d\theta_j$$

$$= \det(d_{j-\lambda_j-k+\mu_k})_{j,k=1}^N. \quad (96)$$

From the above expression, one can see that $D_N^{\lambda,\mu}(f)$ can be expressed as a minor of $T_{N+k}(f)$, where $k = \max\{\lambda_1, \mu_1\}$. We remind the reader that a minor of some matrix $M$ is the determinant of a matrix obtained from $M$ by removing some of its rows and columns. In our case, the rows and columns which
are removed are specified by \( \lambda \) and \( \mu \).

For two functions of the form
\[
a(e^{i\theta}) = \sum_{k \leq 0} a_k e^{ik\theta}, \quad b(e^{i\theta}) = \sum_{k \geq 0} b_k e^{ik\theta},
\]
the associated Toeplitz matrix satisfies
\[
T(ab) = T(a)T(b).
\]

Let us therefore write \( f(e^{i\theta}) \) as follows
\[
f(e^{i\theta}) = H(p e^{i\theta})H(y; e^{-i\theta}),
\]
where \( H(x; z) \) is the generating function of the homogeneous symmetric polynomials \( h_k \) given in (12) and where we assume that \( h_k(x) \) and \( h_k(y) \) are square-summable, i.e.
\[
\sum_k h_k < \infty.
\]

We can also define our weight function as
\[
f(e^{i\theta}) = E(p e^{i\theta})E(y; e^{-i\theta}),
\]
at the cost replacing \( h_j \) by \( e_j \) everywhere, which is equivalent to transposing all diagrams. We repeat that the CUE corresponds to \( f = 1 \), which corresponds to \( x_j = 0 \) for all \( j \) for both \( f(z) = H(x; z)H(y; z^{-1}) \) and \( f(z) = E(x; z)E(y; z^{-1}) \). We will consider the case where \( f(z) = H(x; z)H(y; z^{-1}) \) unless stated otherwise.

It was shown by Gessel [47] that \( D_N(f) \) can be expressed in terms of Schur polynomials as
\[
D_N(f) = \sum_{\ell(\nu) \leq N} s_\nu(x)s_\nu(y).
\]
Equation (100) can be similarly expressed in terms of (skew) Schur polynomials [45], [48], [49].
\[
\int s_\lambda(U)s_\mu(U^{-1}) \det f(U) dU = \sum_{\ell(\rho) \leq N} s_{\rho/\lambda}(x)s_{\rho/\mu}(y).
\]

We will define the usual expectation value \( \langle \ldots \rangle \) with respect to the matrix model with weight function \( f \). That is,
\[
\langle s_\lambda(U)s_\mu(U^{-1}) \rangle := \int s_\lambda(U)s_\mu(U^{-1}) \det f(U) dU / \sqrt{\det f(U)}.
\]

Since these expectation values are invariant under exchange of any two eigenvalues of \( U \) and the Schur
polynomials form a basis of all symmetric polynomials, we can expand general expectation values as (sums over) \( \langle s_\lambda s_\mu \rangle \). We now take the limit \( N \to \infty \). From (103) and the fact that [Chapter I.5, example 26 in [37]],

\[
\sum_\rho s_{\rho/\lambda}(x)s_{\rho/\mu}(y) = \sum_\nu s_{\nu/\lambda}(x)s_{\nu/\mu}(y) \sum_\eta s_\eta(x)s_\eta(y) ,
\]

where the sums run over all partitions, we have [43, 49]

\[
\langle s_\lambda(U) s_\mu(U^{-1}) \rangle = \sum_\nu s_{\nu/\lambda}(y)s_{\nu/\mu}(x) ,
\]

where the sum runs over all \( \nu \) such that \( \lambda, \mu \subseteq \nu \). As noted at equation (101), we can also define \( f(z) = E(x; z)E(y; z^{-1}) \) at the cost of transposing all diagrams. In this case, one has

\[
\langle s_\lambda(U) s_\mu(U^{-1}) \rangle = \sum_\nu s_{(\lambda/\mu)^*}(y)s_{(\mu/\nu)^*}(x) ,
\]

If we take \( \mu = \emptyset \) in (106), the only choice for \( \nu \) that contributes to the above sum is \( \nu = \emptyset \) as well, and we have

\[
\langle s_\lambda(U) \rangle = s_\lambda(y) , \quad \langle s_\mu(U^{-1}) \rangle = s_\mu(x) .
\]

We also define the connected expectation value ,

\[
\langle s_\lambda(U) s_\mu(U^{-1}) \rangle_c := \langle s_\lambda(U) s_\mu(U^{-1}) \rangle - \langle s_\lambda(U) \rangle \langle s_\mu(U^{-1}) \rangle ,
\]

which corresponds to subtracting the contribution corresponding to \( \nu = \emptyset \) in (106).

5 Unitary matrix integrals and Wick’s theorem

In [35] we found that, for \( k, n \in \mathbb{Z}\setminus\{0\} \), and \( f(z) = H(x; z)H(y; z^{-1}) \) satisfying the strong Szegő limit theorem,

\[
\langle \text{tr} U^n \text{tr} U^{-k} \rangle = n\delta_{n,k} + p_n(y)p_k(x) .
\]

For the CUE, \( x_j = 0 = y_j \) for all \( j \), so that \( p_n(x) = 0 \) for all \( n \). Therefore \( \langle \text{tr} U^n \text{tr} U^{-k} \rangle_{\text{CUE}} = n\delta_{n,k} \) [50]. We briefly summarise derivation of our more general result in (110) as we will be using similar arguments throughout the remainder of this paper. We repeat that, for \( n \in \mathbb{Z}^+ \),

\[
\text{tr} U^n = p_n(e^{i\theta}) = \sum_{r=0}^{n-1} (-1)^r s_{(n-r,1^r)}(U) .
\]
It is clear that this also holds when we replace $U$ with $U^{-1}$. Using (108), we have

$$\langle \text{tr} U^n \rangle = p_n(y) , \quad \langle \text{tr} U^{-n} \rangle = p_n(x) .$$

(112)

Then,

$$\langle \text{tr} U^n \text{tr} U^{-k} \rangle = \sum_{\nu} \sum_{s=0}^{k-1} (-1)^s s_{(k-s,1^r)/\nu}(x) \sum_{r=0}^{n-1} (-1)^r s_{(n-r,1^s)/\nu}(y) .$$

(113)

The first sum on the right hand side runs over all representations $\nu$ satisfying $\nu \subseteq (n-r,1^r)$ as well as $\nu \subseteq (k-s,1^s)$. We list below the contributions arising for various choices of $\nu$. We will often omit the variables $x$ and $y$ henceforth.

1. If $\nu$ is the empty partition $\nu = \emptyset$, $s_{\lambda/\nu} = s_{\lambda}$ i.e. the skew Schur polynomials reduces to the usual (non-skew) Schur polynomial. This contributes

$$\sum_{s=0}^{k-1} (-1)^s s_{(k-s,1^r)} \sum_{r=0}^{n-1} (-1)^r s_{(n-r,1^s)} = p_n p_k .$$

(114)

2. If $\nu = \lambda$, the skew Schur polynomial $s_{\lambda/\nu} = s_{\lambda/\lambda} = 1$. For $\lambda = (n-r,1^r)$ and $\mu = (k-s,1^s)$, one can only have $\lambda = \nu = \mu$ if $k = n$ and $r = s$. For $k = n$ and $\nu = (n-r,1^r)$ and we sum over $r$, we get a contribution of the following form

$$\sum_{r=0}^{n-1} (s_{\emptyset})^2 = n .$$

(115)

This is where the term $n \delta_{n,k}$ in (110) originates.

3. Take $k \leq n$ without loss of generality. The only remaining choice for $\nu$ is $\nu \neq \emptyset$ and $\nu \neq (n-r,1^r)$. For $\lambda = (a,1^b)$ and $\nu = (c,1^d)$ such that $\nu \subseteq \lambda$, we have $\lambda/\nu = (a-c) \times (1^{b-d})$, e.g. for $\lambda = (4,1^2)$ and $\nu = (2,1)$, we have the following.

```
  /  /
|  /|
| / |
|/ | 
```

Fixing a single such $\nu = (a,1^b)$ and considering only the sum over $r$ in (113), we have

$$\sum_{r=0}^{n-1} (-1)^r s_{(n-r,1^s)/(a,1^b)} = \sum_{r=5}^{n-a} (-1)^r h_{n-r-a} e_{r-b} = 0 ,$$

(117)

where we applied (123). That is, we get zero contribution for all $\nu \neq \emptyset, (n-r,1^r)$.
Combining the above arguments leads to equation (110). Of course, the same reasoning can be applied to other expectation values involving $\text{tr}U^n$, such as

$$
\langle \text{tr}U^{-n}\text{tr}\lambda U \rangle_c = \sum_{\nu \neq \emptyset} \sum_{r=0}^{n-1} (-1)^r s_{(n-r,1^r)/\nu} s_{\lambda/\nu}
$$

(118)

Again, the sum is over all $\nu \neq \emptyset$ such that $\nu \subseteq \lambda, (n-r,1^r)$. Fixing any $\nu \subseteq (n-r,1^r)$ in (118) with $\nu \neq (n-r,1^r)$ gives zero upon summing over $r$ upon application of (117). Therefore, we only get a nonzero answer for terms for which $\nu = (n-1,1^r) \subseteq \lambda$. That is,

$$
\langle \text{tr}U^{-n}\text{tr}\lambda U \rangle_c = \sum_{\nu} (-1)^r s_{\lambda/(n-r,1^r)}
$$

(119)

where the sum is over $\min(0,n-\lambda_1) \leq r \leq \min(n-1,\lambda_1+1)$. Using (118), we can express this as

$$
\langle \text{tr}U^{-n}\text{tr}\lambda U \rangle_c = \sum_{\nu} (-1)^{ht(\lambda/\nu)} s_{\nu}
$$

(120)

where the sum is over all $\nu$ such that $\lambda/\nu$ is a border strip of size $n$. For example, take $\lambda = (6,4,3,3,2)$ and $n = 4$. We show the diagrams $\nu$ appearing in (120) below, where the cells that are removed are again given in black.

As mentioned, the above results were already derived by the authors in [35]. By applying Wick’s theorem to (110), we will now proceed to compute more complicated objects. Using the connected version of (110) (which equals the CUE result),

$$
\langle \text{tr}U^n\text{tr}U^{-k} \rangle_c = n\delta_{n,k}
$$

(122)

we will consider how Wick’s theorem arises for some relatively simple cases. For example, Wick’s theorem tells us that

$$
\langle (\text{tr}U^2)^2 \text{tr}U^{-1} \rangle_c = 2 \langle \text{tr}U \text{tr}U^{-2} \rangle_c \sqrt{\text{tr}U^{-2}} = 0
$$

(123)

Using (119), the diagrams corresponding to $\text{tr}U^2$ are as follows.

(124)
We take the square of the above expression and apply the Pieri formulas (37) and (38) to find the diagrams contributing to $(\text{tr} U^2)^2$. These are as follows.

\[
\begin{array}{c}
\text{Diagram 1} - \text{Diagram 2} = 2 \\
\text{Diagram 3} - \text{Diagram 4} + \text{Diagram 5} \\
\end{array}
\] (125)

We will denote the above sum over diagrams as $\sum_{\lambda} b_\lambda s_\lambda$, i.e. $b_{(4)} = 1$, $b_{(3,1)} = -1$, $b_{(2,2)} = 2$, $b_{(2,1,1)} = -1$, $b_{(1,1)} = 1$. Applying (106), we have

\[
\langle (\text{tr} U^2)^2 \text{tr} U^{-1} \rangle_c = \sum_{\lambda} b_\lambda s_\lambda \square
\] (126)

That is, we take $\sum_{\lambda} b_\lambda s_\lambda$ and find the skew diagrams $\lambda/\square$, found by removing a single cell from $\lambda$ which has no cells to its right or below it. From these constraints, it follows that the resulting object after removing an cell is still a valid, non-skew diagram. This gives the following sum over diagrams, which evidently mutually cancel.

\[
\begin{array}{c}
\text{Diagram 6} - \text{Diagram 7} - \text{Diagram 8} + 2 \text{Diagram 9} - \text{Diagram 10} - \text{Diagram 11} + \text{Diagram 12} = 0
\end{array}
\]

We thus explicitly confirm that Wick’s theorem is satisfied for the case of equation (123). Consider now $\langle (\text{tr} U^2)^2 \text{tr} U^{-2} \rangle_c$. Using (110), this should give

\[
\langle (\text{tr} U^2)^2 \text{tr} U^{-2} \rangle_c = 2 \langle \text{tr} U^2 \text{tr} U^{-2} \rangle_c \langle \text{tr} U^2 \rangle = 4 \langle \text{tr} U^2 \rangle
\] (127)

We will check this explicitly as well. Applying (120) to $\sum_{\lambda} b_\lambda s_\lambda$ gives

\[
\langle (\text{tr} U^2)^2 \text{tr} U^{-2} \rangle_c = \sum_{\lambda} b_\lambda \left( s_{\lambda/(2)} - s_{\lambda/(1^2)} \right) = \sum_{\lambda} b_\lambda \sum_{\nu} (-1)^{ht(\lambda/\nu)} s_\nu ,
\] (128)

where the sum is over all $\nu$ such that $\lambda/\nu$ is a border strip of size 2, i.e. $\lambda/\nu = \square$ or $\lambda/\nu = \square$. In terms of diagrams, this is given below. The first three diagrams corresponds to $\lambda/\nu = \square$ whereas the latter three correspond to $\lambda/\nu = \square$, which appear with a minus sign due to the factor $(-1)^{ht(\lambda/\nu)}$.

\[
\begin{array}{c}
\text{Diagram 13} - \text{Diagram 14} + 2 \text{Diagram 15} - 2 \text{Diagram 16} + \text{Diagram 17} - \text{Diagram 18} = 4 \text{Diagram 19} - 4 \text{Diagram 20}
\end{array}
\]
We see that \( \langle (\text{tr}U^2)^2 \text{tr}U^{-2} \rangle_e = 2 \langle \text{tr}U^2 \text{tr}U^{-2} \rangle_e \langle \text{tr}U^2 \rangle = 4 \square - 4 \square = 4 \langle \text{tr}U^2 \rangle \), thus confirming (127). Lastly, we will briefly check

\[
\langle (\text{tr}U^2)^2 (\text{tr}U^{-1})^2 \rangle_e = 0 .
\]

We have

\[
\square \times \square = \square + \square.
\]

Note that \( \square \) appears with a positive sign, instead of with a minus sign as it does for \( \text{tr}U^2 \). Then,

\[
\langle (\text{tr}U^2)^2 (\text{tr}U^{-1})^2 \rangle_e = s_\square \sum_\lambda b_\lambda s_\square + \sum_\lambda b_\lambda \left[ s_{\lambda/2} + s_{\lambda/1^2} \right] ,
\]

where we applied the fact that \( \sum_\lambda b_\lambda s_\square = 0 \), see the diagrams below equation (126). Note that (131) gives an equation very similar to (128), but now \( s_{\lambda/1^2} \) carries a positive sign. This gives the same six diagrams as below equation (128), except that the last three diagrams are multiplied by \( -1 \).

\[
\square \square \square \square \square \square - \square \square \square \square \square \square + 2 \square \square \square \square \square \square + 2 \square \square \square \square \square \square - \square \square \square \square \square \square + \square \square \square \square \square \square = 0
\]

This confirms equation (129). From these relatively simple examples, one can explicitly see how Wick’s theorem arises from equation (106) and the multiplication rules for Young diagrams given in section 2. We proceed to apply Wick’s theorem to the computation of more general objects in the remainder of this section.

5.1 Generalization of an identity due to Diaconis and Shahshahani

We use Wick’s theorem and equation (110) to generalize an identity due to Diaconis and Shahshahani [34], see also [50]. Writing again \( p_\rho = p_{\rho_1} p_{\rho_2} \ldots p_{\rho_\ell} \) and \( m_j(\rho) = \text{Card}\{k : \rho_k = j\} \), we wish to calculate

\[
\langle p_\rho(U) p_\mu(U^{-1}) \rangle = \langle (\text{tr}U^{j_1})^{m_1(\rho)} (\text{tr}U^{j_2})^{m_2(\rho)} \ldots (\text{tr}U^{-k_1})^{m_1(n)} (\text{tr}U^{-k_2})^{m_2(n)} \ldots \rangle .
\]

We start with a simpler object that we can easily apply Wick’s theorem to. Using \( \langle \text{tr}U^j \rangle = p_j \), it follows that performing \( n \) contractions on \( \langle (\text{tr}U^j)^a (\text{tr}U^{-j})^b \rangle \) leads to the following expression

\[
C_{n,j} := \frac{a! b! j^n}{(a - n)! (b - n)! n!} p_j(x)^{b - n} p_j(y)^{a - n} , \quad n \leq \text{Min}(a, b) .
\]
Further, we have $C_{n,j} = 0$ for $n \geq \text{Min}(a,b) + 1$. Equation (134) arises as follows. There are \( \binom{a+b}{(a-n)(b-n)n} \) ways to perform \( n \) contractions between \( (\text{tr}U^n) \) and \( (\text{tr}U^{-n}) \), and the contracted terms contribute \( \langle \text{tr}U^n \text{tr}U^{-n} \rangle^a = j^n \). The \( a+b-2n \) uncontracted traces contribute \( \langle \text{tr}U^n \rangle^{a-n} \langle \text{tr}U^{-n} \rangle^{a-n} = p_j(x)^{b-n} p_j(y)^{a-n} \). We now consider all possible ways to perform \( n \) contractions between \( p_\rho(U) \) and \( p_\mu(U^{-1}) \). This leads to a sum over \( \alpha = (\alpha_1, \alpha_2, \ldots) \) which are partitions of \( n \), which specify the contractions that are performed. In particular, \( m_j(\alpha) \) gives the number of \( \text{tr}U^j \) and \( \text{tr}U^{-j} \) which are contracted. The contribution coming from \( n \) contractions in \( \langle p_\rho(U)p_\mu(U^{-1}) \rangle \) can then be written as

$$C_n = \sum_\alpha \prod_j C_{m_j(\alpha),j},$$

where the sum is over \( \alpha \) that are partitions of \( n \), i.e. which satisfy \( |\alpha| = n \). We denote by \( \tilde{n} \) is the maximal number of contractions one can perform, which is given by

$$\tilde{n} = \text{Max}(n) = \sum_{j \geq 1} \text{Min}(m_j(\rho), m_j(\mu)).$$

Summing over all possible contractions and applying (134) and (135), we arrive at our result

$$\langle p_\rho(U)p_\mu(U^{-1}) \rangle = \sum_{n=0}^{\tilde{n}} C_n.$$

As mentioned before, this is a generalization of a result in [24], which considered the CUE, where \( f = 1 \) so that \( p_j(x) = 0 \) for all \( j \neq 0 \). Therefore, one only gets a nonzero result when \( \rho = \mu \). In our notation, their result reads

$$\langle p_\rho(U)p_\mu(U^{-1}) \rangle_{\text{CUE}} = z_\rho \delta_{\rho,\mu},$$

where \( \delta_{\rho,\mu} = 1 \) for \( \rho = \mu \) and zero otherwise. Note that (135) is only the last term in the full expansion in (137), corresponding to \( n = \tilde{n} = \sum_j m_j(\rho) \), so that all power sums in \( p_\rho(U) \) and \( p_\mu(U^{-1}) \) are contracted.

### 5.2 Border strips and their application to unitary matrix integrals over Schur polynomials

We will derive two expressions for general \( \langle s_\lambda(U)s_\nu(U^{-1}) \rangle \) which rely on removing border strips from \( \lambda \) and \( \nu \). The first of these, equation (142), relates \( \langle s_\lambda(U)s_\nu(U^{-1}) \rangle \) to sums over \( \langle s_\mu(U) \rangle \langle s_\rho(U^{-1}) \rangle \), where \( \mu \) and \( \rho \) are related to \( \lambda \) and \( \nu \) by the removal of border strips, respectively. The second expression, in equation (178), provides an expansion of \( \langle s_\lambda(U)s_\nu(U^{-1}) \rangle \) in terms of the power sums \( p_k(x) \) and \( p_k(y) \). The latter expression appears to be particularly useful, as power sums are simpler objects than general Schur polynomials. In the context of LRRW models, \( p_k(x) \) and \( p_k(y) \) are given by \( \pm \tau k a_{\pm k} \), where \( a_{\pm k} \) are the hopping parameters in (12). Equation (178) therefore provides an expansion in \( \tau \), where the expansion coefficients depend on \( a_k \) which can be read off from the hamiltonian. We will
treat these applications in section 6.2.

5.2.1 Expansion in Schur polynomials

From (48), (53), and (52), we have

$$\langle s_\lambda p_n \rangle_c = \sum_{\eta} (-1)^{\text{ht}(\eta)} \langle s_{\lambda \setminus \eta} \rangle = \sum_{\mu} \frac{\chi_\lambda^\mu}{z_\mu} \langle p_{\mu - (n)} \rangle n m_n(\mu) , \quad (139)$$

where the sum is over all $\mu$ containing a row of size $n$ and $\mu - (n)$ is the remainder of $\mu$ after removing a row of size $n$. We remind the reader that $\lambda \setminus \eta$ is the diagram that results from $\lambda$ after removing border strip $\eta$ with $|\eta| = n$. Inserting two identical power sums gives

$$\langle s_\lambda p_n^2 \rangle_c = \sum_{\eta, \zeta} (-1)^{\text{ht}(\eta) + \text{ht}(\zeta)} \langle s_{(\lambda \setminus \eta) \setminus \zeta} \rangle + 2 \langle p_n \rangle \langle s_\lambda p_n \rangle_c$$

$$= \sum_{\mu} \frac{\chi_\lambda^\mu}{z_\mu} \langle p_{\mu - (n^2)} \rangle 2 n^2 m_n(\mu)(m_n(\mu) - 1) + 2 \langle p_n \rangle \langle s_\lambda p_n \rangle_c , \quad (140)$$

where we consecutively remove border strips $\eta$ and $\zeta$ satisfying $|\eta| = n = |\zeta|$, resulting in the partition $(\lambda \setminus \eta) \setminus \zeta$. The sum on the right hand side is over all $\mu$ containing (at least) two rows of length $n$, and $\mu - (n^2)$ is the remainder of $\mu$ after removing two such rows. The term $2 \langle p_n \rangle \langle s_\lambda p_n \rangle_c$ arises from a single contraction between $p_n$ and $s_\lambda$. Continuing this reasoning, considering $\langle s_\lambda (p_n)^k \rangle$ and performing all $k$ contractions gives

$$C(n, k; \lambda) = \sum_{\mu} \frac{\chi_\lambda^\mu}{z_\mu} \langle p_{\mu - (nk)} \rangle k n^k \frac{m_n(\mu)!}{(m_n(\mu) - k)!} = \sum_{\eta_1, \ldots, \eta_k} (-1)^{\text{ht}(T_\eta)} \langle s_{\lambda \setminus \{\eta_j\}} \rangle . \quad (141)$$

The sum on the left is over all $\mu$ containing at least $k$ rows of length $n$. The sum on the right is over $k$ border strips satisfying $|\eta_j| = n$, $\lambda \setminus \{\eta\}$ is the diagram obtained after removing all $\eta_j$ from $\lambda$, and $T_\eta$ is the BST consisting of the union of $\eta_1, \ldots, \eta_k$. It follows that the term in (141) gives zero if it is not possible to construct a subdiagram of $\lambda$ with $k$ border strips of size $n$, e.g., simply if $|\lambda| < nk$. Note that $C(n, 1; \lambda) = \langle s_\lambda p_n \rangle_c$ and $C(n, 2; \lambda) = \langle s_\lambda p_n^2 \rangle_c - 2 \langle p_n \rangle \langle s_\lambda p_n \rangle_c$.

We now consider

$$\langle s_\lambda \sigma(U) s_{\nu}(U^{-1}) \rangle_c = \sum_{\mu, \nu} \frac{\chi_\lambda^\mu}{z_\mu} \frac{\chi_\nu^\nu}{z_{\nu}} \langle p_{\mu}(U) p_{\nu}(U^{-1}) \rangle . \quad (142)$$

When we consider those $\mu$ and $\rho$ that contain a row of size $n$ and we contract a single copy of $p_n$ between $p_{\mu}(U)$ and $p_{\rho}(U^{-1})$, we get the following,

$$A(n, 1; \lambda, \nu) = \sum_{\mu, \nu} \frac{\chi_\lambda^\mu}{z_\mu} \frac{\chi_\nu^\nu}{z_{\nu}} n m_n(\mu) m_n(\nu) \langle p_{\mu - (n)}(U) \rangle \langle p_{\nu - (n)}(U^{-1}) \rangle = \frac{1}{n} \langle s_\lambda p_n \rangle_c \langle s_\nu p_n \rangle_c . \quad (143)$$
Using \(139\) gives

\[
A(n, 1; \lambda, \nu) = \frac{1}{n} C(n, 1; \lambda) C(n, 1; \nu) = \frac{1}{n} \left( \sum_{\eta} (-1)^{ht(\eta)} \left\langle s_{\lambda|\eta} \right\rangle \right) \left( \sum_{\zeta} (-1)^{ht(\zeta)} \left\langle s_{\nu|\zeta} \right\rangle \right),
\]

where the sums are again over border strips satisfying \(|\eta| = n = |\zeta|\). Consider now \(\mu\) and \(\rho\) that contain at least two rows of length \(n\) contract two copies of \(p_n\) between \(p_{\mu}(U)\) and \(p_{\rho}(U^{-1})\),

\[
A(n, 2; \lambda, \nu) = \sum_{\mu, \rho} \frac{\chi^\lambda_{\mu} \chi^\nu_{\rho} n^2}{z_\mu z_\rho} m_n(\mu)(m_n(\mu) - 1) m_n(\rho)(m_n(\rho) - 1) \left\langle p_{\mu-(n)}(U) \right\rangle \left\langle p_{\rho-(n)}(U^{-1}) \right\rangle
\]

\[
= \frac{1}{2n^2} C(n, 2; \lambda) C(n, 2; \nu).
\]

More generally, performing \(k\) contractions between \((\text{tr} U^n)^k\) and \((\text{tr} U^{-1})^k\) results in

\[
A(n, k; \lambda, \nu) = \frac{1}{k! n^k} C(n, k; \lambda) C(n, k; \nu).
\]

Consider a partition \(\alpha\) and, as above, contract over \(\alpha_1\) copies of \(p_1(U)\) and \(p_1(U^{-1})\), \(\alpha_2\) copies of \(p_2(U)\) and \(p_2(U^{-1})\), etc. This gives the following.

\[
A(n, \alpha; \lambda, \nu) = \frac{1}{z_\alpha} \prod_{j \geq 1} C(n, \alpha_j; \lambda) C(n, \alpha_j; \nu).
\]

We can apply the above expression and \(141\) to find

\[
\left\langle s_{\lambda s_{\nu}} \right\rangle_c = \sum_{\alpha} \frac{1}{z_\alpha} \prod_{j \geq 1} C(n, \alpha_j; \lambda) C(n, \alpha_j; \nu)
\]

\[
= \sum_{\alpha} \frac{1}{z_\alpha} \left( \sum_{\alpha_1, \ldots, \alpha_k} (-1)^{ht(T_{\alpha})} \left\langle s_{\lambda|\{\alpha\}} \right\rangle \right) \times (\lambda \to \nu).
\]

The above equation can be interpreted as follows. For any \(\alpha\), consider all ways to remove \(\alpha_1\) border strips of unit size (single cells) from \(\lambda\) and \(\nu\), then remove \(\alpha_2\) border strips of size 2, then \(\alpha_3\) of size 3, and so on. Remember that a different order of removal results in the same diagrams (including multiplicities). The resulting diagrams are given by \(\lambda\setminus\{\alpha\}\) and idem for \(\lambda \to \nu\). Equation \(148\) expresses general expectation values \(\left\langle s_{\lambda s_{\nu}} \right\rangle_c\) in terms of the non-skew Schur polynomials corresponding to \(\lambda\setminus\{\alpha\}\) and \(\nu\setminus\{\alpha\}\). On the other hand, equation \(106\), which we started with, gives an expansion in terms of skew Schur polynomials. There are various applications where an expression in terms of skew Schur polynomials is desirable, including the spin correlation functions we will be considering in the next section.

We consider now the case where one can form BST’s of shapes \(\lambda\) and \(\nu\) from \(\alpha_j\) border strips of size \(j\), summed over \(j\), such that we can fully contract between \(p_{\mu}(U)\) and \(p_{\rho}(U^{-1})\). That is, we consider the case where we can completely tile both \(\lambda\) and \(\nu\) with \(\alpha_1\) single cells, \(\alpha_2\) border strips of size 2 (dominoes), and so on. This clearly requires \(|\lambda| = |\nu| = \sum_{j \geq 1} j \alpha_j\), which is a necessary (but not
sufficient) condition for $\chi_\lambda^{s_\alpha}, \chi_\nu^{s_\alpha} \neq 0$. Consider the CUE, where $s_{\lambda/\mu} = 0$ for any $\lambda/\mu \neq \emptyset$. Applying the Murnaghan-Nakayama formula,

$$\sum_{\alpha_1, \ldots, \alpha_k} (-1)^{ht(T_\alpha)} = \chi_\lambda^{s_\alpha},$$

we then find that

$$\langle s_\lambda s_\nu \rangle_{\text{CUE}} = \sum_{\alpha} z_\alpha^{-1} \chi_\lambda^{s_\alpha} \chi_\nu^{s_\alpha} = \delta_{\lambda, \nu}.$$  

(150)

This is just the orthonormality property for symmetric group characters, e.g. [Proposition 7.17.6b, [38]]. Another way to arrive at the same expression is to directly plug (see equation (138))

$$\langle p_\mu p_\rho \rangle_{\text{CUE}} = z_\mu \delta_{\mu, \rho},$$

(151)

into equation (152) to find the orthonormality relation in (150).

We work out the explicit example for $\lambda = (3, 2)$, which is a sufficiently small partition that we can still apply (108) for comparison. Indeed, applying (108) to $\langle s_{(3, 2)} s_{(3, 2)} \rangle_c = \sum_{\nu} s_{(3, 2)}^{s_{(3, 2)}} s_{(3, 2)}^{s_{(3, 2)}}$ gives the following diagrams $(3, 2)/\nu$.

\[
\begin{align*}
\begin{array}{cccc}
\hline
& & & \\
& & & \\
\end{array} & \begin{array}{cccc}
\hline
& & & \\
& & & \\
\end{array} & \begin{array}{cccc}
\hline
& & & \\
& & & \\
\end{array} & \begin{array}{cccc}
\hline
& & & \\
& & & \\
\end{array} & \begin{array}{cccc}
\hline
& & & \\
& & & \\
\end{array} & \begin{array}{cccc}
\hline
& & & \\
& & & \\
\end{array} & \begin{array}{cccc}
\hline
& & & \\
& & & \\
\end{array} & \begin{array}{cccc}
\hline
& & & \\
& & & \\
\end{array}
\end{align*}
\]

Note that \[ \begin{array}{cccc}
\hline
& & & \\
& & & \\
\end{array} \] appears with multiplicity 2 as it arises from $\nu = (3, 1)$ and $\nu = (2^2)$. By applying (48) for $n = 1$ (or, equivalently, (39) or (40)) to the leftmost diagram, we find the following.

\[
\begin{align*}
\begin{array}{cccc}
\hline
& & & \\
& & & \\
\end{array} & \begin{array}{cccc}
\hline
& & & \\
& & & \\
\end{array} & \begin{array}{cccc}
\hline
& & & \\
& & & \\
\end{array} & \begin{array}{cccc}
\hline
& & & \\
& & & \\
\end{array} & \begin{array}{cccc}
\hline
& & & \\
& & & \\
\end{array} & \begin{array}{cccc}
\hline
& & & \\
& & & \\
\end{array} & \begin{array}{cccc}
\hline
& & & \\
& & & \\
\end{array}
\end{align*} = \begin{array}{cccc}
\hline
& & & \\
& & & \\
\end{array} + \begin{array}{cccc}
\hline
& & & \\
& & & \\
\end{array}
\]

(153)

This leads to

$$\langle s_{(3, 2)} s_{(3, 2)} \rangle_c = (s_{(2, 2)} + s_{(3, 1)})^2 + (s_{(3)} + s_{(2, 1)})^2 + s_{(1, 2)}^2 + (s_{(2)} + s_{(1, 2)})^2 + s_{(1, 2)}^2 + 2s_{(1)}^2 + 1.$$  

(154)

Note that the above equation is strictly speaking only correct when $x = y$. For $x \neq y$, we have e.g. $\langle s_{(2, 2)}(x) + s_{(3, 1)}(x) \rangle (s_{(2, 2)}(y) + s_{(3, 1)}(y))$ instead of $(s_{(2, 2)} + s_{(3, 1)})^2$, but we write it in this way to avoid clutter.

We now apply (148) to compute $\langle s_{(3, 2)} s_{(3, 2)} \rangle_c$. The diagrams on the right hand side of (148) are given below, where we indicate the multiplicity and indicate the choice of $\alpha$ for which diagrams arise. For example, the two leftmost diagrams arise for $\alpha = \square$, i.e. they arise from contracting $p_1$. 


Further, we get $\varnothing$ for all $\alpha = (3, 2)$, $(4, 1)$, $(3, 1^2)$, $(2, 1^3)$, $(1^5)$, where $\varnothing$ appear with multiplicity 5 for $\alpha = (1^5)$. Equation (148) then gives the following, where the Schur polynomials appear in the same order as the diagrams above.

$$
\langle s_{(3, 2)} s_{(3, 2)} \rangle_c = \left( s_{(2, 2)} + s_{(3, 1)} \right)^2 + \frac{1}{2} \left( 2 s_{(2, 1)} + s_{(3)} \right)^2 + \frac{1}{3!} \left( 3 s_{(2)} + 2 s_{(1^2)} \right)^2 + \frac{1}{3} s_{(1^2)}^2 + \frac{5}{2} s_{(3)}^2 \\
+ \frac{1}{2} s_{(2)}^2 + \left( \frac{1}{3} + \frac{1}{8} + \frac{1}{4} + \frac{1}{4} + \frac{5^2}{4!} \right) s_{(1)} + 1 .
$$

The number 1 on the right hand side arises from those $\alpha$ which lead to $\varnothing$, in particular,

$$
\frac{1}{s_{(3, 2)}} + \frac{1}{s_{(4, 1)}} + \frac{1}{s_{(3, 1^2)}} + \frac{1}{s_{(2, 1^2)}} + \frac{1}{s_{(2^2, 1)}} + \frac{1}{s_{(1^3, 1)}} = \frac{1}{6} + \frac{1}{4} + \frac{1}{6} + \frac{1}{12} + \frac{1}{8} + \frac{5}{4!} = 1 .
$$

The reader may check that expression (156) and (157) are identical. The above example may not appear to give a very convincing argument in favor of equation (148) over (106), as the application of (148) appears to be more complicated for $\lambda = (3, 2)$. Indeed, for partitions containing few cells, such as $\lambda = (3, 2)$, it is convenient to use (106), as the skew partitions $\lambda/\nu$ can easily be related to non-skew partitions, as in (154). However, for larger partitions, this is no longer the case, as (148), (106), and (39) can no longer be applied. When considering larger partitions, then, equation (148) still can be used to express general objects $\langle s_{\lambda} s_{\nu} \rangle_c$ in terms of non-skew Schur polynomials. This will allow us to express complicated correlation functions in terms of simpler ones in the following section, but can be used more generally in situations where an expression for $\langle s_{\lambda} s_{\nu} \rangle$ in terms of non-skew Schur polynomials is desirable.

### 5.2.2 Expansion in power sum polynomials

For certain applications, such as the long-range spin-$1/2$ models we will be considering in the next section, an expansion of $\langle s_{\lambda}(U) s_{\nu}(U^{-1}) \rangle$ may be particularly useful. One way to do so is to use the expansion of Schur polynomials in terms of power sum polynomials in (22). We will in the following section, below equation (235), that a convenient way to calculate the symmetric group characters $\chi^{\lambda}_{\alpha}$ appearing in (22) may be to use the relation with particles hopping between lattice sites with fermionic statistics. One may then apply Wick’s to the resulting expression to find an expansion of $G_{\lambda, \mu}$ in terms of $p_k(x)$ and $p_k(y)$. Although the aforementioned relation to fermionic particles may prove a convenient method for calculating $\chi^{\lambda}_{\alpha}$, it would be more effective to use their recursive structure in (259) to find an expression that does not require the computation of all $\chi^{\lambda}_{\alpha}$. We will do so here, with the
resulting expression in (178). However, we will first apply the former, more complicated, method to \( \langle s_\lambda(U)s_\lambda(U^{-1}) \rangle \) for \( \lambda = (3, 2) \), before applying (178) for comparison. This provides both an explicit check of equation (178) as well as a demonstration of its effectiveness for computing \( \langle s_\lambda(U)s_\lambda(U^{-1}) \rangle \).

More general and complicated examples of \( \langle s_\lambda(U)s_\nu(U^{-1}) \rangle \) are treated in section 6.2.4, where we consider the application to long-range spin-\( \frac{1}{2} \) models.

Consider \( \lambda = (3, 2) \). By applying (59), we find that the non-zero characters \( \chi^\lambda_\alpha \) are

\[
\chi^{(3,2)}_{(1^\nu)} = 5 , \quad \chi^{(3,2)}_{(2,1^\nu)} = 1 , \quad \chi^{(3,2)}_{(2^2,1)} = 1 , \\
\chi^{(3,1^2)}_{(1^\nu)} = 1 , \quad \chi^{(3,2)}_{(3,2)} = -1 , \quad \chi^{(3,2)}_{(4,1)} = 1 .
\]

(158)

Applying (52) gives

\[
s^{(3,2)} = \frac{p_1^5}{24} + \frac{p_1^3p_2}{12} + \frac{p_1^2p_3}{6} + \frac{p_2^3p_1}{8} - \frac{p_2p_1^4}{6} + \frac{p_1p_4}{4} \\
= \frac{p_1^{(1^\nu)}}{24} + \frac{p_1^{(2,1^\nu)}}{12} + \frac{p_1^{(3,1^2)}}{6} + \frac{p_1^{(2^2,1)}}{8} - \frac{p_1^{(3,2)}}{6} + \frac{p_1^{(4,1)}}{4} .
\]

(159)

We then apply Wick’s theorem to \( \langle s^{(3,2)}(U)s^{(3,2)}(U^{-1}) \rangle \), where we indicate with \( p_\alpha \) the copies of \( p_\alpha \) we contract. That is, \( p_4 \) indicates we contract a single copy of \( p_4 \), while \( p_2^2 \) indicates we contract two copies of \( p_2 \), and so on.

\[
\langle s^{(3,2)}(U)s^{(3,2)}(U^{-1}) \rangle = \frac{p_1(x)p_1(y)}{p_4} + \frac{3}{62} \left( \frac{p_2(x)p_2(y)}{p_4} - \frac{p_2(y)p_1(x)^2}{p_3} - \frac{p_2(x)p_1(y)^2}{p_3} + \frac{p_1(x)^2p_1(y)^2}{p_3} \right) \\
+ \frac{p_1(x)p_1(y)}{p_2^2} + \ldots
\]

(160)

We will apply the above method to find the prefactors of \( p_\omega(y)p_\gamma(x) \) for some examples of \( \omega, \gamma \). Consider first \( \gamma = \omega = (4) \), leading to a term proportional to \( p_4(x)p_4(y) \). This term is rather simple to find as we only get a contribution from the rightmost term in (159). In particular, we have

\[
\frac{1}{4} \langle p_{(4,1)}(U)p_{(4,1)}(U^{-1}) \rangle = \frac{p_4(x)p_4(y)}{4^2} + \frac{p_1(x)p_1(y)}{4} ,
\]

(161)

so that \( p_4(x)p_4(y) \) appears with a prefactor \( \frac{1}{16} \) in the expansion of \( \langle s^{(3,2)}(U)s^{(3,2)}(U^{-1}) \rangle \). We consider now those terms proportional to \( p_3(x)p_3(y) \) arising from (159), where the dots below refer to terms not proportional to \( p_3(x)p_3(y) \) and where we omit omit writing \( (U^{\pm 1}) \) explicitly henceforth.

\[
\frac{1}{6} \langle p_{(3,1^2)}(U)p_{(3,1^2)} \rangle + \frac{1}{6} \langle p_{(3,2)}(U)p_{(3,2)} \rangle = \frac{p_4(x)p_3(y)}{36} (2 + 2) + \ldots \\
= \frac{p_3(x)p_3(y)}{9} + \ldots
\]

(162)
For $p_2(x)p_2(y)$, we find

$$\frac{\langle p_{(2,1^3)}p_{(2,1^3)} \rangle}{(12)^2} + \frac{\langle p_{(2^2,1)}p_{(2^2,1)} \rangle}{8^2} + \frac{\langle p_{(3,2)}p_{(3,2)} \rangle}{6^2} = p_2(x)p_2(y) \left( \frac{3}{12} + \frac{8}{8^2} + \frac{3}{6^2} \right) + \ldots$$

$$= \frac{p_2(x)p_2(y)}{4} + \ldots$$

(163)

We see a pattern emerge as we get $\frac{p_j(x)p_j(y)}{j^2}$ for all $j$ considered thus far. However, we will see that this pattern does not continue down to $j = 1$. In particular, one can see that

$$\frac{\langle p_{(1^5)}p_{(1^5)} \rangle}{(24)^2} + \frac{\langle p_{(2,1^3)}p_{(2,1^3)} \rangle}{(12)^2} + \frac{\langle p_{(3,1^2)}p_{(3,1^2)} \rangle}{6^2} = \frac{p_1(x)p_1(y)}{4} + \ldots$$

(164)

is given by

$$p_1(x)p_1(y) \left( \frac{5!}{(4!^3)2!} \frac{6^2}{12^2} + \frac{12}{8^2} + \frac{8}{6^2} + \frac{4}{4^2} \right) + \ldots = 2p_1(x)p_1(y) + \ldots$$

(165)

We will see below that the fact that we get $2\frac{p_{(2)}p_{(2)}}{2^2}$ for $j = 1$ is related that we can remove two border strips of size one, i.e. single cells, from $\lambda = (3, 2)$. Consider the term proportional to $p_1(x)^2p_1(y)^2$, which is receives contributions from

$$\frac{\langle p_{(1^5)}p_{(1^5)} \rangle}{(24)^2} + \frac{\langle p_{(2,1^3)}p_{(2,1^3)} \rangle}{(12)^2} + \frac{\langle p_{(3,1^2)}p_{(3,1^2)} \rangle}{6^2}.$$  

(166)

In particular, this gives

$$p_1(x)^2p_1(y)^2 \left( \frac{5!}{(4!^3)2!} \frac{6^2}{12^2} + \frac{12}{8^2} + \frac{3}{6^2} \right) = \frac{5p_1(x)^2p_1(y)^2}{4}.$$  

(167)

Lastly, we consider a mixed term, namely, the term proportional to $p_2(x)p_1(y)^2 + p_1(x)^2p_2(y)$. This is given by

$$\frac{\langle p_{(1^5)}p_{(2,1^3)} \rangle}{12 \cdot 24} + \frac{\langle p_{(3,1^2)}p_{(3,2)} \rangle}{6^2} + \frac{\langle p_{(2,1^3)}p_{(2^2,1)} \rangle}{12 \cdot 8} = p_2(x)p_1(y)^2 \left( \frac{5}{24} - \frac{1}{12} + \frac{1}{8} \right) = \frac{p_2(x)p_1(y)^2}{4}.$$  

(168)

Of course, inverting the order of $p_\mu(U)$ and $p_\nu(U^{-1})$ in $\langle p_\mu(U)p_\nu(U^{-1}) \rangle$ above gives the same contribution with $x \leftrightarrow y$. The end result is therefore $\frac{p_2(x)p_1(y)^2 + p_1(x)^2p_2(y)}{4}$.

The method above can be used to express general averages $\langle s_{\gamma}(U)s_{\nu}(U^{-1}) \rangle$ in terms of $p_\gamma(x)$ and $p_\nu(y)$ for some partitions $\gamma, \omega$. However, it is evident that this method is quite laborious, and quickly becomes more complicated for larger $\lambda, \nu$. We will proceed to derive an expression which deals precisely with these issues, as it provides an iterative method for expanding in $p_\gamma(x)$ and $p_\nu(y)$ specified by the successive removal of border strips from $\lambda, \nu$. In particular, we will revert the order of expansion in section 5.1.2.3 and start from the term with all $p_\mu$ contracted, see equation (150), then consider the term where a single $p_j$ is not contracted, then two uncontracted power sums, and so on. For simplicity, we
start by considering autocorrelation function up to the subleading term, which gives

\[
\langle s_\lambda(U) s_\lambda(U^{-1}) \rangle = \sum_{\mu,\rho} \frac{\chi_\lambda^\mu \chi_\lambda^\rho}{z_\mu z_\rho} \langle p_\mu(U)p_\rho(U^{-1}) \rangle = 1 + \sum_{j \geq 1} \sum_{\mu} \frac{(\chi_\lambda^\mu)^2 m_j(\mu)}{z_\mu} j p_j(x) p_j(y) + \ldots \tag{169}
\]

where sum is over all \(\mu\) containing a row of size \(j\). We have \(\mu = \rho\) since this is the only way to contract all of \(p_\mu\) with \(p_\rho\) except for two copies of \(p_j\) (one from \(p_\mu\) and the other from \(p_\rho\)). From (16), it is clear that

\[
z_\mu = z_{\mu - j} j m_j(\mu) . \tag{170}
\]

Using the recursive formula for \(\chi_\mu^\lambda\) in (159) and the orthogonality of symmetric group characters in (150), the rightmost term in equation (169) then gives

\[
\sum_j \frac{1}{j^2} \sum_{\mu} \left( \frac{\sum_{\eta} (-1)^{h(\eta)} \chi_\mu^\eta}{z_{\mu - j}} \right) \left( \frac{\sum_{\zeta} (-1)^{h(\zeta)} \chi_\mu^\zeta}{z_{\mu - j}} \right) p_j(x) p_j(y) = \sum_{j \geq 1} \frac{p_j(x) p_j(y)}{j^2} \sum_\eta 1 , \tag{171}
\]

where \(\eta\) and \(\zeta\) are border strips of size \(j\). Equation (171) then tells us that the coefficient \(p_j(x) p_j(y)\) in the power sum expansion of \(\langle s_\lambda(U) s_\lambda(U^{-1}) \rangle\) is given by \(\frac{1}{j^2}\) times the number of ways to remove a border strip of size \(j\) from \(\lambda\). Consider another term in this expansion, where \(\mu = (j, \alpha), \rho = (k^2, \alpha)\) with \(j = 2k\), and we contract over \(p_\alpha\), leading to the term proportional to \(p_j(x) p_k(y)^2\). Va the same reasoning as above, this is given by

\[
\frac{p_j(x) p_k(y)^2}{2jk^2} \sum_{\eta, \zeta, \xi} (-1)^{h(\eta) + h(\zeta) + h(\xi)} \delta_{\lambda \eta \zeta} \delta_{\eta \zeta} \xi , \tag{172}
\]

where the sums are over all \(\eta\), which are border strips of size \(j\), and \(\zeta\) and \(\xi\), which are border strips of size \(k\). We see that this is the coefficient in the power sum expansion of \(\langle s_\lambda(U) s_\lambda(U^{-1}) \rangle\) are found by removing border strips \(\{\eta\}\) and \(\{\zeta\}\) from \(\lambda\) such that \(\lambda \backslash \{\eta\} = \lambda \backslash \{\zeta\}\). Continuing this procedure gives the following. Consider not contracting \(p_\omega(U)\) and \(p_\gamma(U^{-1})\) for some \(\omega\) and \(\gamma\), leading to the term proportional to \(p_\omega(y) p_\gamma(x)\). In general this leads to a sum over \(\mu\) and \(\rho\) (in the middle expression in (171)) of the form

\[
\mu = (\omega, \alpha) , \quad \rho = (\gamma, \alpha) , \tag{173}
\]

so that \(m_j(\mu) - m_j(\omega) = m_j(\alpha) = m_j(\rho) - m_j(\gamma)\) for all \(j\), and \(|\omega| = |\eta|\). Since we sum over \(\mu\) and \(\rho\) for a fixed choice of \(\omega\) and \(\gamma\), this effectively leads to a sum over \(\alpha\). Since there are \(\frac{m_j(\mu)!}{m_j(\omega)! m_j(\gamma)!}\) ways to perform \(m_j(\alpha)\) contractions of \(p_j\) appearing in \(p_\mu\) and \(p_\rho\), we have

\[
\langle p_\mu p_\rho \rangle = \cdots + p_\omega(y) p_\gamma(x) \prod_{j \geq 1} \frac{1}{m_j(\alpha)!} \frac{m_j(\mu)!}{(m_j(\mu) - m_j(\alpha))!} \frac{m_j(\rho)!}{(m_j(\rho) - m_j(\alpha))!} j^{m_j(\alpha)} + \ldots \tag{174}
\]
where we show only the term proportional to $p_\omega p_\gamma(x)$. Using the fact that,

$$z_\mu = z_\alpha \prod_{j=1}^d j^{m_j(\mu)-m_j(\alpha)} \frac{m_j(\mu)!}{m_j(\alpha)!},$$

and again applying the orthogonality of symmetric group characters, we find that the term proportional to $p_\omega p_\gamma(x)$ in the expansion of $\langle s_\lambda s_\nu \rangle$ is given by

$$\frac{p_\omega(y)p_\gamma(x)}{z_\omega z_\gamma} \sum_{\{\eta\},\{\xi\}} (-1)^{\text{ht}(T_\eta)+\text{ht}(T_\xi)} \delta_{\lambda\setminus\{\eta\},\lambda\setminus\{\xi\}},$$

where the sum is over border strips $\eta_1, \eta_2, \ldots$ satisfying $|\eta_j| = \omega_j$, as well as border strips $\xi_j$ satisfying $|\xi_j| = \gamma_j$. Further, $T_\eta$ and $T_\xi$ are the (generally disconnected) skew diagrams consisting of the unions of the border strips in $\eta$ and $\xi$, respectively, and $\text{ht}(T)$ is defined in (55). It is clear that the examples in (171) and (172) arise as special cases of (176). The autocorrelation function can then be expanded as a sum $\omega$ and $\gamma$, each contributing a term of the form appearing in (176). This gives

$$\langle s_\lambda(U)s_\lambda(U^{-1}) \rangle = \sum_{\omega,\gamma} \frac{p_\omega(y)p_\gamma(x)}{z_\omega z_\gamma} \sum_{\{\eta\},\{\xi\}} (-1)^{\text{ht}(T_\eta)+\text{ht}(T_\xi)} \delta_{\lambda\setminus\{\eta\},\lambda\setminus\{\xi\}},$$

where the sum is again over border strips $\eta_j$ satisfying $|\eta_j| = \omega_j$, as well as border strips $\xi_j$ satisfying $|\xi_j| = \gamma_j$, for each choice of $\omega, \gamma$. We can consider more general correlation functions, of the form $\langle s_\lambda(U)s_\nu(U^{-1}) \rangle$. This gives the following

$$\langle s_\lambda(U)s_\nu(U^{-1}) \rangle = \sum_{\omega,\gamma} \frac{p_\omega(y)p_\gamma(x)}{z_\omega z_\gamma} \sum_{\{\eta\},\{\xi\}} (-1)^{\text{ht}(T_\eta)+\text{ht}(T_\xi)} \delta_{\lambda\setminus\{\eta\},\nu\setminus\{\xi\}}.$$

The above expression is particularly useful for our purposes, as it provides an expression for $\langle s_\lambda(U)s_\lambda(U^{-1}) \rangle$ in terms of $p_h(x)$ and $p_k(y)$. Its application is straightforward, as it involves considering the different ways to remove border strips from $\lambda$ and $\nu$ such that the resulting diagrams are identical, as expressed by the presence of $\delta_{\lambda\setminus\{\eta\},\nu\setminus\{\xi\}}$.

We now apply (178) to $\langle s_\lambda(U)s_\lambda(U^{-1}) \rangle$ and compare with the method applied above, resulting in equations (161)–(168). In particular, consider all distinct ways to remove a border strip of size one, i.e. a single cell, from $\lambda = (3, 2)$. This results in the following diagrams.

\[
\text{(179)}
\]

Removing a single cell twice results in the following diagrams.

\[
\text{(180)}
\]
We get multiplicity two for \(p_2, q_1\) on the right as one can remove the single cells indicated in black in either order. Removing border strips of sizes 2, 3, 4 results in the diagrams below, read from left to right.

Note that the height \(ht\) of the border strips of sizes 3 and 4 is given by 1, so that \((-1)^{ht} = -1\). It is easy to see that no border strips of size \(\geq 5\) can be removed from \(\lambda = (3,2)\), since \(\lambda_1 + \ell(\lambda) - 1 = 4\). From the fact that there are two distinct ways to remove a single cell from \(\lambda\), and only a single way to remove border strips of sizes 2, 3, 4, we see that the corresponding contributions are given by

\[
2p_1(x)p_1(y) + \frac{p_2(x)p_2(y)}{4} + \frac{p_3(x)p_3(y)}{9} + \frac{p_4(x)p_4(y)}{16},
\]

thus recovering equations (161)–(165) via a significantly simpler method. Consider now the term proportional to \(p_1(x)^2p_1(y)^2\). The two diagrams which arise from consecutively removing two single cells are \((2,1)\) and \((3)\), where \((2,1)\) appears with multiplicity two, as demonstrated above. We thus see that there are four ways to take two copies of \((3,2)\), consecutively remove two single cells, and end up with \((2,1)\). Conversely, there is only a single way to end up with \((3)\) via the aforementioned procedure. Combined with \((z_{(1^2)})^2 = 4\), this demonstrates that we have

\[
\frac{5p_1(x)^2p_1(y)^2}{4}
\]

appearing in the expansion of \(\langle s_{(3,2)}s_{(3,2)}\rangle\), which confirms (166). Lastly, we consider \(p_2(x)p_1(y)^2 + p_1(x)^2p_2(y)\). One may remove either a border strip of size two or two single cells and end up in \((3)\), a single row of 3 cells, see the two lefmost diagrams above (182). Combined with \(z_{(1^2)} = 2 = z_{(2)}\), this results in

\[
\frac{p_2(x)p_1(y)^2 + p_1(x)^2p_2(y)}{4},
\]

thus confirming (168). Although the effectiveness of (178) is already clear for \(\lambda = (3,2)\), this is still a rather small partition. For larger \(\lambda\), it becomes increasingly harder to compute \(\chi^\lambda_{\alpha}\), increasing the advantage of (178). We will work out more complicated examples in the following section, where we apply the above results to correlation functions of long-range spin-\(\frac{1}{2}\) models.

### 6 Applying symmetric polynomial theory to long-range random walk models

We will now apply some of the results above to spin correlation functions by using their relation to weighted \(U(N)\) integrals over Schur polynomials in equation (84) and identities from symmetric polynomial theory, as well as results we derived in section 5. We first consider identities relating
to elementary and complete homogeneous symmetric polynomials, before moving on to power sum symmetric polynomials and border strips.

6.1 Elementary and complete homogeneous symmetric polynomials

We remind the reader that the weight function in (75) is given by

$$f(e^{i\theta}; \tau) = \exp \left( \tau \sum_{k \in \mathbb{Z}} a_k e^{i\theta} \right),$$

where $a_k$ are the hopping parameters of the hamiltonian in (72). As noted before, the hamiltonian is a Toeplitz matrix, and $a_k$ is the number on its $k^{th}$ diagonal. By comparing with (12), we see that we can write the weight function as

$$f(z) = H(x; z) H(y; z^{-1}),$$

with the following identification, for $k \geq 1$,

$$\tau a_k = \frac{p_k(x)}{k},$$

$$\tau a_{-k} = \frac{p_k(y)}{k}. \quad (187)$$

Alternatively, we can write

$$f(z) = E(x; z) E(y; z^{-1}),$$

by identifying, for $k \geq 1$,

$$\tau a_k = \frac{(-1)^{k+1} p_k(x)}{k},$$

$$\tau a_{-k} = \frac{(-1)^{k+1} p_k(y)}{k}, \quad (189)$$

and by transposing the diagrams as in (107). When $\tau \to 0$, i.e. the CUE limit, we have

$$F_{\lambda;\mu}(0) = \left< s_\lambda(U) s_\mu(U^{-1}) \right>_{\text{CUE}} = \delta_{\lambda,\mu}, \quad (190)$$

which is again simply the orthonormality of Schur polynomials as the irreducible characters of $U(N)$. By using the strong Szegő limit theorem, we can compute $F_{\varnothing;\varnothing}$. Assuming $a_0 = 0$, i.e. zero on-site energy, we have

$$F_{\varnothing;\varnothing}(\tau) = \exp \left( \tau^2 \sum_{k=1}^\infty ka_k^2 \right). \quad (191)$$
If we have $a_0 \neq 0$, we get an additional multiplicative term $e^{-N\tau a_0}$ on the right, where one should remember that we take $N \to \infty$. Considering $a_1 = -1 = a_1$ and $a_k = 0$ otherwise, i.e. the XX0-model, and choosing $\tau = it$, we recover the result of [41] and [40], see also [42].

$$F_{\emptyset;\emptyset}(it) = e^{-t^2}. \quad (192)$$

Remember that equations (103a) and (100) state that

$$F_{\lambda;\mu}(\tau) = F_{\emptyset;\emptyset}(\tau) \langle s_\lambda(s_\mu(U^{-1})) \rangle_\tau, \quad (193)$$

where $\langle \ldots \rangle_\tau$ is given in equation (104) with weight function given by $f(z; \tau)$ in (75). We therefore define

$$G_{\lambda;\mu}(\tau) := \frac{F_{\lambda;\mu}(\tau)}{F_{\emptyset;\emptyset}(\tau)} = \langle s_\lambda(U) s_\mu(U^{-1}) \rangle_\tau, \quad (194)$$

i.e. we express correlations in terms of their proportionality to $F_{\emptyset;\emptyset}(\tau)$. We will also write this as

$$G_{\lambda;\mu}(\tau) = \langle \lambda|\mu \rangle_\tau. \quad (195)$$

If $\mu = \emptyset$ (or $\lambda = \emptyset$), we will simply write

$$G_{\lambda;\emptyset}(\tau) = \langle \lambda|\emptyset \rangle_\tau = \langle \lambda \rangle_\tau. \quad (196)$$

We also define the following - connected - correlation function,

$$G^c_{\lambda;\mu}(\tau) := \langle \lambda|\mu \rangle_\tau - \langle \lambda \rangle_\tau \langle \mu \rangle^*_\tau. \quad (197)$$

We consider now some explicit examples. Using (18), we can express

$$\langle e_k \rangle_\tau = e_k(y) = G_{(1^k);\emptyset}, \quad \langle h_k \rangle_\tau = h_k(y) = G_{(k);\emptyset}, \quad (198)$$

in terms of $p_n(y)$ with $n \leq k$. For $h_k$ and $e_k$ with $k = 4$, the diagrams and corresponding spin configurations are given below on the left and right, respectively.

\[\text{Diagram 1}\]
\[\text{Diagram 2}\]
That is, $h_n$ correspond to taking only the single rightmost particle and moving it $n$ steps to the right, whereas $e_n$ corresponds to taking the $n$ rightmost particles and moving them all a single step to the right. We noted above that with $x = (x_1, \ldots, x_K)$, we have $e_j(x) = 0$ for $j > K$ (and likewise for $y$). This means that, for any $n, m > K$,

$$F_{(1^m);(1^n)} = \sum_{j=0}^{K} e_j e_j$$

That is, when we move $n > K$ adjacent particles by a single step, the effect is the same as to move $K$ adjacent particles a single step.

Not only can we read off $F_{\emptyset;(n)}$ and $F_{\emptyset;(1^n)}$. From the corollary of the Pieri formula in (39), we have that, for any $\lambda$,

$$G_{\lambda;(n)}^c = \langle s_{\lambda}(U) h_n(U^{-1}) \rangle_c = \sum_{j=1}^{n} h_{n-j} s_{\lambda/(j)} = \sum_{j=1}^{n} h_{n-j} \sum_{\nu} s_{\nu}$$

where the rightmost sum is over all $\nu$ such that $\lambda/\nu$ is a horizontal strip of length $j$. Take, for example, $\lambda = (3, 2)$,

which corresponds to

$$|(3, 2)\rangle = \cdots \bullet \circ \circ \bullet \circ \cdots$$

We remind the reader that the dots on the left (right) refer to an infinite string of particles (holes). We take $\lambda = (3, 2)$ and $n = 2$. For this choice of $\lambda$, the diagrams contributing to the sum over $\nu$ in (201) are given below equation (178).

We assign particles and holes to these diagrams and remind ourselves that

$$G_{(1);\emptyset}(\tau) = \langle \cdots \bullet \circ \circ \bullet \circ \cdots \rangle_{\tau}$$
We then see that equation (201) is given by

\[ G_{(3,2);(2)}(\tau) = \left\langle \cdots \od\od\od\od\cdots | \cdots \od\od\od\od\cdots \right\rangle \tau \]

\[ = \left( \left\langle \cdots \od\od\od\od\cdots \right\rangle \tau + \left\langle \cdots \od\od\od\od\cdots \right\rangle \tau \right) \times G_{(1);(2)}(\tau) \]

\[ + \left\langle \cdots \od\od\od\od\cdots \right\rangle \tau + \left\langle \cdots \od\od\od\od\cdots \right\rangle \tau \]  

(205)

The second line above corresponds to \( \nu^1 \) in (201), whereas the bottom line corresponds to \( \nu^2 \). By comparing with equation (202), we see that the sum over \( \nu^1 \) contains all diagrams related to (202) by moving a single particle a single step to the left. The sum over \( \nu^2 \) consists of all ways to move one or two particles by a total of two steps, with the restriction that we cannot move two adjacent particles. This is the interpretation of the corollary of the Pieri formula in (39), and it generalizes to \( h_n \) for any \( n \in \mathbb{Z}^+ \). In particular, we have the following.

\[ s_{\lambda/(n)} = \sum \left\{ \text{Distinct ways to move } \leq n \text{ particles } n \text{ steps (in total) to the left without moving any two adjacent particles.} \right\} \]  

(206)

We can apply the same reasoning to \( \left\langle s_{\lambda(U)} e_n(U^{-1}) \right\rangle \) by using (40). Similar to (201), we have

\[ G_{\lambda/(1^n)} = \sum_{j=1}^{n} c_{n-j} s_{\lambda/(1^j)} = \sum_{j=1}^{n} c_{n-j} \sum_{\nu^j} s_{\nu} \]  

(207)

where \( \sum_{\nu^j} \) is now a sum over all \( \nu \) such that \( \lambda/\nu \) is a vertical strip of length \( j \). This can be interpreted as follows.

\[ s_{\lambda/(1^n)} = \sum \left\{ \text{Distinct ways to move } n \text{ particles a single step to the left.} \right\} \]  

(208)

Consider the case where \( x = y \) (e.g. \( \tau = \beta \in \mathbb{R} \) and \( a_k = a_{-k} \)). We can then apply the Pieri formula once again, in this case to the products \( h_{n-j} s_{\lambda/(j)} \) in (201), as

\[ h_{n-j} s_{\lambda/(j)} = \sum_{\nu} s_{\nu} \]  

(209)

where the sum is over all \( \nu \) obtained from \( \lambda \) by removing a horizontal strip of size \( j \) and then adding to the resulting diagram a horizontal strip of size \( n - j \). We consider again \( G_{(3,2);(2)} = s_{(3,2)/(1)} h_1 + s_{(3,2)/(2)} \), where \( s_{(3,2)/(2)} \) is given by the diagrams above equation (204). Applying the Pieri formula to \( s_{(3,2)/(1)} h_1 = s_{(3,2)/(1)} s_{(1)} \) gives the following diagrams, where the cell that is added again indicated in gray. It is clear that \( \lambda = (3, 2) \) appears twice, as there are two ways to remove (and then add again) a single cell from \( \lambda \), given below (203).
We now briefly consider the single particle correlation functions $F_{j;l}$. The generating function $f(z) = \sum_j F_{j;l} z^{j-l}$ for one-particle correlations is written in (70). Writing again $f(z) = H(x;z)H(y;z^{-1})$ for some $x$, we have

$$f(z) = \prod_j (1 - x_j z)^{-1}(1 - y_j z^{-1})^{-1} = \sum_{j,k=0}^{\infty} h_j(x)h_k(y)z^{j-k} =: \sum_{m=0}^{\infty} d_m(z^m + z^{-m}) ,$$

(211)

where $h_j$ are complete homogeneous symmetric polynomials. Then,

$$d_m = \sum_{j=m}^{\infty} h_j(y)h_{m-j}(x) = \langle h_M(U)h_{M-m}(U^{-1}) \rangle = G_{j;l}(\tau) , \quad j-l = m .$$

(212)

where $M$ is the largest number $k$ such that $h_k \neq 0$, which is generally infinite. Remember that the $G_{j;l}(\tau)$ are the single-particle wavefunctions at site $j$ and time $\tau$ (where $x = (x_1, x_2, \ldots$ depends on $\tau$) for a particle released from site $l$. The last two equalities in (212) relate $F_{j;l}$ to single-particle wavefunctions at site $M$ and time $\tau$ for a particle released from site $M - m$. Since $M$ is infinite for most choices of hopping parameters $a_k \sim p_k$, this correspond to releasing a particle at infinity and finding its wavefunction $m$ lattice sites away from where it was released at some Euclidean time $\tau$.

The fact that $h_{M-m}(U^{-1}) = G_{j;l}(\tau)$ for $M \rightarrow \infty$ agrees with the intuition that a single particle at infinity does not feel the presence of the $N-1$ remaining particles that are infinitely far away from it.

Lastly, we note that one can apply the Jacobi-Trudi identity (51) to $h_n$ and $e_n$ to find any skew Schur polynomials on the right hand side of (106). This, in effect, provides a way to compute any correlation function. Although the resulting expressions generally grow quickly as one considers large partitions, the application of this method itself is rather simple as it only requires the computation of a determinant. It can therefore be easily implemented and applied to any $\langle s_\lambda s_\mu \rangle$. In 6.2 we will outline various other, closely interrelated, methods to calculate any $\langle s_\lambda s_\mu \rangle$, based on power sum polynomials rather than elementary and complete homogeneous symmetric polynomials.

### 6.1.1 The involution between $e_n$ and $h_n$ and quasi-local particle-hole duality

As mentioned before, we can replace the generating function $f_1(z) = H(x;z)H(y;z^{-1})$ by $f_2(z) = E(x;z)E(y;z^{-1})$ at the cost of transposing the partitions appearing in the correlation functions, see (106) and (107). From (12), it follows that switching between $f_1(z)$ and $f_2(z)$ corresponds to taking

$$p_k(x) \rightarrow (-1)^{k+1}p_k(x) .$$

(213)
From equations (187) and (189), it is clear that replacing as $p_k(x) \rightarrow (-1)^{k+1} p_k(x)$ corresponds to $a_k \rightarrow (-1)^{k+1} a_k$. In particular, define two hamiltonians for the same choice of $a_n$,

$$\hat{H}_1 = - \sum_{m=0}^{\infty} \sum_{n} a_n \left( \sigma_m^+ \sigma_{m+n}^- + \sigma_m^- \sigma_{m+n}^+ \right) ,$$

$$\hat{H}_2 = - \sum_{m=0}^{\infty} \sum_{n} (-1)^{n+1} a_n \left( \sigma_m^+ \sigma_{m+n}^- + \sigma_m^- \sigma_{m+n}^+ \right) . \quad (214)$$

Correlation functions for $\hat{H}_1$ correspond to weight function $f_1(z)$, whereas those for $\hat{H}_2$ correspond to $f_2(z)$. Let us compare correlation functions for $\hat{H}_1$ and $\hat{H}_2$, which we will write as $F_{\lambda;\mu}^{(1)}(\tau)$ and $F_{\lambda';\mu'}^{(2)}(\tau)$, respectively. From Szegö’s theorem, we know that $F_{\emptyset;\emptyset}$ only depends on $p_k(x)p_k(y)$, so taking $p_k \rightarrow (-1)^{k+1} p_k$, has no effect on $F_{\emptyset;\emptyset}$. Therefore,

$$F_{\emptyset;\emptyset}^{(1)}(\tau) = F_{\emptyset;\emptyset}^{(2)}(\tau) , \quad (215)$$

that is, the return probability for $N$ adjacent particles for $\hat{H}_1$ is identical to that of $\hat{H}_2$ for any $\tau$. Moreover, by comparing (106) and (107), we see that, for general $\lambda$ and $\mu$,

$$F_{\lambda;\mu}^{(1)}(\tau) = F_{\lambda';\mu'}^{(2)}(\tau) . \quad (216)$$

This establishes the following duality between correlation functions of $\hat{H}_1$ and $\hat{H}_2$. It is well known that transposition of diagrams combined with a parity transformation (exchanging between left and right) implements the exchange between particles and holes, or particles and holes. This follows from the fact that transposition maps vertical edges to horizontal ones and vice versa. However, we are not implementing particle-hole symmetry in (216) but instead we establish an equality between different correlation functions corresponding to different models. Let us consider the spin configuration associated to $\lambda = (5, 4, 2, 1^3)$ and $\lambda' = (6, 3, 2^2, 1)$, given on the left and right below, respectively.

\[ \text{(217)} \]
These correspond to the following states,

\[ |\lambda\rangle = \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \ , \]

\[ |\lambda^t\rangle = \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \ . \]  

(218)

where the vertical lines on the left and right correspond to the diagonal lines at the lower left and upper right corners of \( \lambda \) and \( \lambda^t \) in (217), respectively. We see that the regions in between the vertical lines for \( |\lambda\rangle \) and \( |\lambda^t\rangle \) are related by performing a parity transformation and exchanging between particles and holes. Only a finite interval \( \ell \leq r \leq \lambda_1 \) is affected by the above relation and we have \( j_r^L = j_r \) otherwise. We thus establish a bijection between the correlation functions corresponding to \( \hat{H}_1 \) and \( \hat{H}_2 \), given by the involution of diagrams, which we refer to as quasi-local particle-hole duality. It follows trivially from the above treatment that if \( a_{2k} = 0 \), \( \forall k \in \mathbb{Z}^+ \), then \( H(x; z) = E(x; z) \), as can be seen immediately from (12). It follows that when all even hopping parameters \( a_{2k} \) are zero, we have \( F^{(1)}_{\lambda;\mu} = F^{(2)}_{\lambda;\mu} \), so we can suppress the superscript. In this case, it follow from (216) that

\[ F_{\lambda;\mu}(\tau) = F_{\lambda^t;\mu^t}(\tau) . \]

(219)

That is, any system with \( a_{2k} = 0 \) satisfies quasi-local particle-hole duality with itself.

### 6.2 Power sum symmetric polynomials and border strips

We consider now the spin configurations corresponding to power sum polynomials, which are a very natural basis for this application due to their proportionality to \( \tau a_k \). Consider specifically the case where \( \tau = it \). Then, from (187), we have

\[ p_k(y) = -it a_k^* = (-p_k(x))^* = e^{i\phi} p_k(x) , \quad \phi := \pi - 2 \arg(p_k(x)) . \]

(220)

Since \( p_k(x) \) only differs from \( p_k(y) \) by a phase, we can apply the usual multiplication rules to objects of the form \( s_\lambda(x)s_\mu(y) \), which is an advantage over the usage of \( h_j, e_j \).

Remember that power sum polynomials are expanded as an alternating sum over hook-shaped Schur polynomials, see equation (26). Note that

\[ \langle \text{tr} U^n \rangle_\tau = p_n = \sum_{r=0}^{n-1} (-1)^r G_{(n-r,1^r);\emptyset} , \]

(221)

can be read off from the hamiltonian by the identification in either (187) or (189). As an example, the spin configuration corresponding to a hook shape \( \lambda = (4,1^2) \) is as follows.
That is, a hook-shaped diagram corresponds to taking the rightmost particle and moving it a steps to the right, then taking the next b particles and moving them a single step to the right. More generally, consider

\[
\frac{1}{\sqrt{n}} \langle p_n | s_{\lambda} \rangle = \frac{1}{\sqrt{n}} \sum_{r=0}^{n-1} (-1)^r G_{\lambda; (n-r, 1^r)} = \frac{1}{\sqrt{n}} \sum_{r=0}^{n-1} (-1)^r \langle (n-1, 1^r) | \lambda \rangle .
\]  

(223)

Using equation (45), this is given by

\[
\frac{1}{\sqrt{n}} \sum_{r=0}^{n-1} (-1)^r s_{\lambda; (n-r, 1^r)} = \frac{1}{\sqrt{n}} \sum_{\nu} (-1)^{|\nu|} s_{\nu} ,
\]  

(224)

where the sum \(\sum_{\nu}\) is over \(\nu\) such that \(\nu/\lambda\) is a border strip of size n, as is illustrated below equation (123). Note that there is only a single Schur polynomial \(s_{\nu}\) appearing on the right hand side, as opposed to (201) and (207), which contain factors of \(e_{n-j}\) and \(h_{n-j}\), respectively. We consider a few specific examples of (224), already noted in [35], before treating it in generality. In particular, consider \(\langle s_{\lambda} p_n \rangle_c\) with \((n-r, 1^r) \not\subseteq \lambda \forall r \in \{0, \ldots, n-1\}\), such that \(\lambda_1 + \lambda_1' - 1 < n\). This gives

\[
\langle \text{tr} U^{-n} \text{tr} \lambda U \rangle_c = 0 .
\]  

(225)

Writing \(\lambda\) in Frobenius notation as \(\lambda = (a_1, \ldots, a_k | b_1, \ldots, b_k)\) with \(a_j\) satisfying \(a_1 > \cdots > a_k\) and similar for \(b_j\). In this case, \(a_1 + b_1 + 1\) equals the hook-length of the top left cell of \(\lambda\), that is, the number cells to it right and below it, plus one. Equation (225) then states that \(\langle \text{tr} U^{-n} \text{tr} \lambda U \rangle_c = 0\) if \(a_1 + b_1 + 1 < n\). In terms of spin configurations, (225) states that \(\sum_{r=0}^{n-1} (-1)^r G_{\lambda; (n-r, 1^r)} = 0\) if \(n\) is greater than the distance between the leftmost hole and rightmost particle in the configuration corresponding to \(\lambda\) (that is, one plus the number of lattice sites in between the leftmost hole and rightmost particle).

Further, write \(m = a_1 + b_1 + 1 - n\) and consider \(\lambda = (a|b) = (a_1, \ldots, a_k | b_1, \ldots, b_k)\) such that \(m \leq a_1 - a_2 - 1\) and \(m \leq b_1 - b_2 - 1\), respectively. Take \(\mu = (a_2, \ldots, a_k | b_2, \ldots, b_k)\), obtained by removing the first row and column from \(\lambda\). Any \((n-r, 1^r) \subseteq \lambda\) then satisfies

\[
\lambda/(n-r, 1^r) = (a_1 + 1, 1^{b_1})/(n-r, 1^r) \times \mu = (a_1 + 1 - n + r) \times (1^{b_1-r}) \times \mu .
\]  

(226)

Consider the example of \((6, 4, 3, 1^2)/(5, 1^3)\), which is given by the diagram below.
We then apply (13) to find

\[
\langle s_\lambda(U)\text{tr}U^{-n}\rangle_c = \sum_{r=n-a_1-1}^{b_1} (-1)^r s_\lambda((n-r,1^r)) = \pm s_\mu \sum_{k=0}^{m} (-1)^k h_{m-k} \epsilon_k = 0. 
\] (228)

In terms of particles, this can be interpreted by assigning configurations to e.g. \( \lambda = (6,4,3,1^2) \) above. We see that the correlation function corresponding to \( \langle s_\lambda(U)\text{tr}U^{-n}\rangle_c \) vanishes when the distance between the leftmost hole and rightmost particle, as well as the distance between the rightmost particle and the second-to-rightmost particle (and vice versa for holes), are sufficiently small compared to \( n \).

We now consider (224) more generally. Here, \( \nu \) is related to \( \lambda \) by the removal of a border strip, i.e. a connected skew diagram not containing a subdiagram that is a 2 by 2 block. In terms of spin configurations, a 2 by 2 block corresponds to moving two adjacent particles by two steps to the right, see below.

In equation (224), fact that a border strip has no 2 by 2 subdiagram therefore states that (the configuration corresponding to) \( \nu \) is related to \( \lambda \) by moving particles left by \( n \) steps without moving two or more adjacent particles by two or more steps. The number of rows that \( \lambda/\nu \) occupies (which is \( \text{ht}(\lambda/\nu) + 1 \)) equals the number of particles that are moved, which follows immediately from the fact that vertical edges on the boundary of a diagram correspond to particles. From the fact that the border strip is connected, it follows that it occupies only consecutive rows. For example, the skew diagram on the left is a border strip, whereas the one on the right is not.

This means that only consecutive (but not necessarily adjacent) particles are affected. However, connectedness is a stronger condition, and for border strips this leads to the following observation.
For and skew diagram $\lambda/\nu$, we call the *outer rim* the (horizontal and vertical) edges on the bottom right of the diagram of $\lambda/\nu$, as in e.g. \cite{36}. Conversely, the *inner rim* consists of the edges on the top left of $\lambda/\nu$. For a border strip containing $n$ cells, we number the edges of the inner and outer rims by $j = \{1, \ldots, n+1\}$. We will refer to the $j^{th}$ edge on the outer rim as the $j^{th}$ outer edge, and likewise for the inner rim. We consider an explicit example, where $\lambda = (8, 6^2, 4, 1)$ and $\nu = (5^2, 3, 2, 1)$, leading to a border strip of size $n = 9$ below. On the right hand side, we number the vertical edges of the inner and outer rims.

One can see in the above example that the first edge on the inner rim and the ledge edge on the outer rim are both vertical, which is clearly true for any (skew) diagram. Further, we see that all other vertical edges occupy the same positions on the inner and outer rims, namely $\{3, 6, 7\}$. If e.g. the third edge on the inner rim were horizontal, the resulting skew diagram would be disconnected and therefore not a valid border strip, shown below.

It is clear that this holds generally for border strips. In particular, the inner and outer edges of any border strip are identical for $j = 2, 3, \ldots, n$, and the outer (inner) edge for $j = 1$ is horizontal (vertical), whereas the outer (inner) edge for $j = n + 1$ is vertical (horizontal).

The configurations corresponding to $\lambda$ and $\nu$ are related by taking the configuration on the outer rim of $\lambda/\nu$ (as a subset of the outer rim of $\lambda$) and replacing it by the configuration corresponding to the inner rim of $\lambda/\nu$. What we effectively get is the following. We take a particle at some site $k$ and move it to site $k - n$, and we get a factor $(-1)^{ht(\lambda/\nu)}$. Here, $ht(\lambda/\nu)$ equals the number of particles the affected particle jumps over, that is, the number of particles occupying sites $\{k - n + 1, \ldots, k - 1\}$, which is simply fermionic statistics. Below, we show the border strip $\lambda/\nu$ with empty cells as a subset of $\lambda$, where we indicate particles and holes. The inner edge $j$ and outer edge $j$ for $j = 2, \ldots, n$ are connected by diagonal lines, it is clear that they are identical. Further, the outer (inner) rim starts with a horizontal (vertical) edge, and ends with a vertical (horizontal) edge.
The states corresponding to $\lambda$ and $\nu$ are given by

$$|\lambda\rangle = \cdots \bullet \bullet \bullet \cdots ,$$

$$|\nu\rangle = \cdots \bullet \bullet \bullet \cdots .$$

(234)

It is clear that they are identical except for a single particle which has moved $n = 9$ steps to the left, whereby it jumps over 3 other particles leading to a factor $(-1)^3$. Summarizing the above, we have

$$\frac{1}{\sqrt{n}} \sum_{r=0}^{n-1} (-1)^r G_{\lambda,(n-r,1)} = \sum (-1)^P \text{Distinct ways to move a single particle by } n \text{ steps, thereby jumping over } P \leq n - 1 \text{ other particles.}$$

(235)

We can apply the reasoning presented above to the calculation of $\chi^\lambda_\alpha$, by considering $p_\alpha = \sum_{\lambda} \chi^\lambda_\alpha s_\lambda$ in terms of fermionic particles hopping on a 1D lattice. In particular, starting from the empty partition and fixing some $\alpha$, we consider all way to consecutively take a particle and move it $\alpha_j$ steps to the right and then sum over $j \geq 1$. Further, we add a multiplicative factor $-1$ for each other particle which is being hopped over in this fashion. We then add the resulting numbers $(\pm 1)$ for all cases where the end result of this process is the configuration $\lambda$. The outcome of this computation is precisely $\chi^\lambda_\alpha$. This might provide a convenient method for computing $\chi^\lambda_\alpha$, as it involves moving particles around on a line instead of a border strip tiling problem. Although the problems evidently identical, the former appear simper to implement.

### 6.2.1 The action of the hamiltonian in terms of Young diagrams

When we consider the action of the hamiltonian in (72) in terms of diagrams, the following picture arises. We first consider the XX0-model, in which case the action of $H^n$ on some state with $k$ particles can be described in terms of $k$ non-intersecting (vicious) random walkers that are allowed to take a single step to the left or right at each time step, for $n$ time steps [20]. For the XX0-model, considering $H |\lambda\rangle$ in terms of diagrams then corresponds to summing over all ways to add one cell to and to remove one cell from $\lambda$, as this gives vicious random walkers that can take a single step to the left or right. For a general long-range hamiltonian such as in (72), the action of $H$ is given by vicious random walkers which can take $n$ steps left of right, weighted by $a_n$ and summed over $n$. We saw above that for $\nu$ related to $\lambda$ by removing a border strip of size $n$, this corresponds to taking a single particle and
moving it by \( n \) steps, so it appears that

\[
H |\lambda\rangle = \sum_n a_n \left( \sum_{\eta} |\lambda\eta\rangle + \sum_{\nu} |\nu\rangle \right),
\]

(236)

where \( \eta \) is a border strip of size \( n \) and \( \nu \) is related to \( \lambda \) by the addition of a border strip of size \( n \), and we sum over all such \( \eta \) and \( \nu \). Note that we do not get the minus sign that we get when considering \( \langle p_n(U) s_{\lambda}(U^{-1}) \rangle_c \) as in (224). For \( n = 1 \), when we consider all ways to start with the empty partition and look at all ways to successively add single cells, we simply get Young’s lattice. The action of the XX0-model on some state \( |\lambda\rangle \) then corresponds to moving from \( \lambda \) along all edges in the Young’s lattice.

Starting from the empty partition, we get

\[
H^n |\varnothing\rangle = \sum_{m=0}^{n} d_{m:n} Y_m, \quad d_{m:n} = \frac{n!}{(\frac{n-m}{2})! (\frac{n+m}{2})!},
\]

(237)

where \( Y_m \) is the subset of the Young’s lattice containing \( n \) cells and \( d_{m:n} \) is the number of paths traversing a distance \( m \) in \( n \) unit (left and right) steps. For a general hamiltonian as in (22), the picture is clearly more involved. Take, for example, the case where \( a_4 \neq 0 \) and \( a_n = 0 \) for \( n \neq 4 \). If we have \( \lambda = (5, 3, 1) \), given below,

\[
|\lambda\rangle = \cdots \circ \circ \circ \circ \cdots
\]

(238)

which corresponds to

\[
|\lambda\rangle = \cdots \circ \circ \circ \circ \cdots
\]

(239)

It is easy to see that there are two ways to move a single particle four steps to the left, and there are six ways to move a particle four steps to the right. Correspondingly, the action of the hamiltonian \( H |\lambda\rangle \) produces the following eight diagrams.
Consider a generalization of the Young’s lattice in the form of a graph where each edge connects two diagrams that are related by addition or removal of some size $n$ border strip, for all $n$. The action $H^n |\lambda\rangle$ of a general long-range hamiltonian as in (72) corresponds to an $n$-step random walk on this graph, where edges corresponding to a border strip of size $n$ carry weight $a_n$. This, then, provides a general description of the action of a hamiltonian in (72) in terms of (addition and removal of) border strips.

### 6.2.2 Equal size border strip tableaux and cancellation-free fermionic models

We saw that adding or removing a border strip $\eta$ and multiplying with $(-1)^{ht(\eta)}$ implements fermionic statistics, and (in (62)) that $\chi^{(\nu)}_{(n)}$ is cancellation-free. This leads to the following observation. Take a long-range fermionic hamiltonian $\hat{H}_f$, where fermions are only allowed to hop $n$ steps. The action of $\hat{H}_f$ on some fermionic state associated to a diagram $\lambda$ as above is given by

$$s_{\lambda} p_n + \sum_{r=0}^{n-1} (-1)^r s_{\lambda\langle n-r,\lambda_r \rangle} = \sum_{\nu=\lambda\langle n-r,\lambda_r \rangle} (-1)^{ht(\eta)} s_{\lambda\eta} + \sum_{\nu} (-1)^{ht(\nu/\lambda)} s_{\nu},$$

where, again, $\eta$ is a border strip of size $n$ and $\nu$ is related to $\lambda$ by the addition of a border strip of size $n$. Compare with equation (236), especially the factors $(-1)^{ht}$. We can keep iterating this step to find $s_{\lambda}(p_n)^k$. From (62), it then follows that all diagrams appearing in the expansion of $\left(\hat{H}_f\right)^k |\lambda\rangle$ have the same sign. We saw that adding or removing border strips $\lambda/\nu$ and multiplying by $(-1)^{ht(\lambda/\nu)}$ corresponds to letting fermions hop over a distance of $|\lambda/\nu|$ lattice sites. It follows that, for a fermionic model where particles can only hop $n$ sites, all distinct ways to go from some configuration $\lambda$ to another configuration $\mu$ appear with the same sign. In other words, all different ways to go from any configuration $\lambda$ to any other configuration $\mu$ involves fermions hopping over either an even or an odd number of other fermions. Physically, this means that there is no interference between various realizations of the same fermion state. This entails that fermionic states will spread through Hilbert space very rapidly upon time evolution, as its spread is not restricted by destructive interference.
The same reasoning can be applied to various expectation values in the long-range spin system \(^{(24)}\). In particular, we will consider \(\langle (\text{tr}U^n)^k s_\lambda(U^{-1}) \rangle_c\). Repeatedly applying \((48)\), which we have used at various points above, we get

\[
\langle (\text{tr}U^n)^k s_\lambda(U^{-1}) \rangle_c = \sum_{j=1}^k \sum_{\nu_{n,j}} (-1)^{ht(T_j)} s_{\nu_{n,j}}(x)p_{\nu}(y)^{k-j}.
\]

(242)

where \(\nu_{n,j}\) is related to \(\lambda\) by the consecutive removal of \(j\) border strips of size \(n\). For example, consider again \(\lambda = (6, 5, 2^2, 1)\) and \((n^k) = (4^4)\). Then, the diagrams \(\nu_{4,2}\) are given by removing the green and blue regions from the diagrams below equation \((62)\), for \(\nu_{4,3}\) the orange regions are removed, and after removing the red regions we end up with \(\nu_{4,4} = \emptyset\). Note that \(\chi_{(n^k)}^\lambda = 0\) for most \(\lambda\), one sufficient (but far from necessary) condition for this being that \(nk \neq |\lambda|\). In general, consecutively removing border strips of some size \(n\) leads eventually to what is referred to as the \(n\)-core of \(\lambda\) \((38)\). In general, this \(n\)-core may be non-empty even when \(nk = |\lambda|\). From \((220)\), we have

\[
s_{\nu_{n,j}}(x)p_{\nu}^{-j}(x) = \sum_{\mu} d_{n,j,k}^{\mu} s_\mu(x),
\]

(243)

for some coefficients \(d_{n,j,k}^{\mu}\). From \((46)\), we know that \(s_\lambda p_n^k\) is expanded as a sum over all diagrams related to \(\lambda\) by subsequently adding a \(k\) border strips of size \(n\). On the other hand, \(\nu_{n,j}\) is related to \(\lambda\) by removal of \(j\) border strips of size \(n\). Therefore,

\[
\langle (\text{tr}U^n)^k s_\lambda(U^{-1}) \rangle_c = \sum_{j=1}^k \sum_{\nu_{n,j}} e^{i\phi(k-j)} (-1)^{ht(T_j)+ht(T_2)} s_{\nu_{n,j}}^{-j}(x).
\]

(244)

In the above expression, the sum is over all \(\nu_{n,j}^{-j}\) constructed by first removing \(j\) border strips of size \(n\) from \(\lambda\) (which results in \(\nu_{n,j}\)) and then adding \(k-j\) border strips of size \(n\). Further, \(T_1\) is given by the BST constructed from of the union of the \(j\) border strips that are removed from \(\lambda\), and \(T_2\) is the tableau that is the union of the \(k-j\) border strips that are added to \(\nu_{n,j}\) to construct \(\nu_{n,j}^{-j}\). For \(\lambda = (6, 5, 2^2, 1)\), we consider

\[
\langle (\text{tr}U^4)^2 s_\lambda(U^{-1}) \rangle_c.
\]

(245)

As mentioned above, contracting both copies of \(\text{tr}U^4\) with \(s_\lambda\) gives rise to the diagrams below \((62)\) after removing the green and blue border strips. The \(\tilde{\nu}_{1,1}^k\) are given by the ways to remove from and then add to \(\lambda\) a border strip of size four. We see that \(\lambda\) appears in equation \((245)\) with a multiplicity two, as there are two distinct border strips of size four that one can remove from \(\lambda\), namely, the green and blue border strips on the leftmost diagram below \((62)\). For the full expression expectation value \(\langle (\text{tr}U^n)^k s_\lambda(U^{-1}) \rangle_c\), we have

\[
\langle (\text{tr}U^n)^k s_\lambda(U^{-1}) \rangle_c = \pm \sum_{j=1}^k e^{i\phi(k-j)} \left\{ \begin{array}{l}
\text{Distinct ways to consecutively move } j \text{ particles } n \text{ sites to the left and then move } k-j \text{ particles } n \text{ steps to the right.}
\end{array} \right\}
\]

(246)
Note that the particles that are moved by \(n\) steps are not necessarily distinct. Further, the right hand side appears with a positive or negative sign depending only on the final configuration. This again follows from the fact that \(\chi^\lambda\) is cancellation-free. As such, the correlation function corresponding to \(\langle (\text{tr} U^n)^k s_\lambda(U^{-1}) \rangle_c\) in fact counts the number of ways to go from some configuration \(\lambda\) to any other by the procedure on the right hand side of (246).

### 6.2.3 Schur polynomial expansions of correlation functions

We consider the expansions for general correlation functions derived in section 5.2. We first consider the application of (148) to \(\langle s_\lambda(U)s_\nu(U^{-1}) \rangle\). Equation (148) expresses \(\langle s_\lambda(U)s_\nu(U^{-1}) \rangle\) as a sum over diagrams obtained form \(\lambda\) and \(\nu\) by removing border strips of size \(\alpha_1, \alpha_2, \ldots\), for some partition \(\alpha\). In particular, it establishes a relation between \(x_{s_\lambda s_\nu} \) and the correlation functions \(\langle s_\lambda(U) s_{\nu(U^{-1})} \rangle\) involving only a single non-trivial spin configuration \(\lambda\setminus\{\alpha\}\) (and likewise for \(\nu\setminus\{\alpha\}\)). The sum is over all ways to start from \(\lambda\) and \(\nu\) and move \(\alpha_j\) particles \(j\) steps to the left with fermionic statistics (in the form of \((-1)^P\)). Consider the autocorrelation for \(\nu = \lambda = (8, 6^2, 4, 1)\), which we show below.

\[
(247)
\]

Equation (148) gives an expression in terms of diagrams obtained by removing border strips from \(\lambda\). Below, we show from left to right all border strips of size 1, 2, 3 that can be removed from \(\lambda\).

\[
(248)
\]

In the middle diagram above, the green cell is shared between both a horizontal and a vertical strip of size two. We can thus remove four border strips of both size 1 and 2, and two of size 3. It is easy to see that there is a single border strip of size 4 that can be removed from \(\lambda\), three border strips for both size 5 and 6, and so on, up to a single border strip of size \(\lambda_1 + \ell(\lambda) - 1 = 12\). Applying (148) then gives

\[
\langle s_\lambda(U) s_\lambda(U^{-1}) \rangle_c = \left(s_{(7,6^2,4,1)} + s_{(8,6,5,4,1)} + s_{(8,6^2,3,1)} + s_{(8,6^2,4)}\right)^2 + \\
+ \frac{1}{2} \left(s_{(6^3,4,1)} - s_{(8,5^2,4,1)} + s_{(8,6,4^2,1)}\right)^2 + \frac{1}{3} \left(-s_{(8,5,4,1^2)} + s_{(8,6^2,1^2)}\right)^2 + \ldots
\]

(249)
In the above expression, the first, second, and third term correspond to the removal of border strips of size one, two, and three, respectively. Further, we again write \((s_\mu + s_\rho + \ldots)^2\) instead of \((s_\mu(x) + s_\rho(x) + \ldots)\ (s_\mu(y) + s_\rho(y) + \ldots)\).

Consider now \(\langle s_\lambda(U)s_\nu(U^{-1})\rangle\) with \(\lambda\) as above and \(\nu = (8^3, 4, 1)\), the latter of which is shown below. We have here \(\lambda \subseteq \nu\), and we indicate \(\nu/\lambda\) in gray.

\[
\begin{array}{ccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccc}
\end{array}
\]

(250)

Consider again all ways to remove border strips from \(\nu\), and apply (148). This then leads to

\[
\langle s_\lambda(U)s_\nu(U^{-1})\rangle = (s_{(7,6^2,4,1)} + s_{(8,6^2,5,4,1)} + s_{(8,6^2,3,1)} + s_{(8,6^2,4)}) (s_{(8^2,7,4,1)} + s_{(8^2,3,1)} + s_{(8^2,4)}) + \frac{1}{2} (s_{(6^3,4,1)} - s_{(8,5^2,4,1)} + s_{(8,6^2,4,1)}) (s_{(8,7^2,4,1)} + s_{(8^2,6,4,1)} + s_{(8^2,3,2,1)}) + \ldots
\]

(251)

From the relation between removal or addition of border strips as in (148) and fermionic particles hopping on a line, we can interpret (148) in the following manner.

\[
\langle s_\lambda s_\nu \rangle = \sum_{\alpha} \frac{1}{z_\alpha} (-1)^{ht(T_\alpha)} \left\{ \text{Distinct ways to take } \ell(\alpha) \text{ particles in } \lambda \text{ and move them } \alpha_1, \alpha_2, \ldots \text{ steps to the left, thereby hopping over } T_\alpha \text{ particles} \right\} \times (\lambda \rightarrow \nu) \quad (252)
\]

Note that the \(\ell(\alpha)\) particles mentioned above are not required to be distinct.

### 6.2.4 Power sum expansions of correlation functions

From the fact that, for \(\tau = \text{it}\), \(p_k(x)\) and \(p_k(y)\) are related by a phase, it is useful to express the resulting terms as an expansion in terms of \(p_k(x)\) and \(p_k(y)\) to reveal these phases. To do so, one may use the relation between Schur and power sum polynomials in equation (62) and apply the Murnaghan-Nakayama rule, as done for the relatively simple example of \(\langle s_\lambda(U)s_\lambda(U^{-1})\rangle\) for \(\lambda = (3, 2)\) in section 5.2.2. However, as noted there, even this relatively simple example is already quite complicated, as it requires the computation of \(\chi^\lambda_{\alpha}\) for all \(\alpha\). For these purposes, it is more convenient to employ (178), which provides an expansion of \(\langle s_\lambda(U)s_\lambda(U^{-1})\rangle\) in terms of \(p_k(x)\) and \(p_k(y)\). The prefactors appearing in this expansion depend on the number of ways to remove border strips from \(\lambda\) and \(\nu\) in such a way that the resulting diagram is the same for both \(\lambda\) and \(\nu\). Taking again \(\nu = \lambda = (8, 6^2, 4, 1)\), for which the diagrams resulting from removal of border strips were treated in section 6.2.3 above. Applying
\[ \langle s_\lambda(U)s_\lambda(U^{-1}) \rangle_c = 4p_1(x)p_1(y) + p_2(x)p_2(y) + \frac{2}{9}p_3(x)p_3(y) + \frac{1}{16}p_4(x)p_4(y) + \frac{1}{2}(p_1^2(x)p_2(y) + p_2(x)p_1^2(y)) + \frac{1}{6}(p_3(x)p_1(y)p_2(y) + p_4(y)p_1(x)p_2(x)) + \ldots \] (253)

In the above expression, the first line on the right hand side is given by (171), which is a special case of (178). As hinted at previously, the terms proportional to \( \frac{p_1(x)p_1(y)}{j^2} \) in the first line of (253) arise as follows. The denominator is given by the inverse of \( z_{(j)}^2 \) and the numerator equals the number of distinct ways to remove a border strip of size \( j \) from \( \lambda = (3, 2) \). The second line gives mixed power sums obtained from not contracting a single \( p_j(U^\pm) \) and two copies of \( p_k(U^\pm) \), \( p_m(U^\pm) \). One can see that there are four ways to remove two border strips of unit size and arrive at a diagram that can alternatively be obtained by removing a single border strip of size 2. This follows from the fact that there are four ways to remove a border strip of size two, which can alternatively be achieved by removing the two cells of such a border strip successively. However, due to the factors \((-1)^{ht(T)}\) in (178), one of those four contributes with a negative sign, leading to a prefactor of \( \frac{(2/2)}{z_{(j)}z_{(j)}} = \frac{1}{2} \) multiplying \( p_2^2p_2 \) in the second line of (253). There is a similar cancellation occurring form the term proportional to \( p_3p_1p_2 \). This expansion can then be continued by removing further border strips. Consider again \( \langle s_\lambda(U)s_\nu(U^{-1}) \rangle \) with \( \lambda = (8, 6^2, 4, 1) \) and \( \nu = (8^3, 4, 1) \), as at the end of section 3.2.3 applying (178) leads to

\[ \langle s_\lambda(U)s_\nu(U^{-1}) \rangle = -\frac{2}{3}p_1(x)p_3(x) + \frac{1}{4}p_2^2(x) + \frac{1}{12}p_4(x)^4 + \frac{1}{2}p_2^2(x)p_1(x)p_1(y) + \ldots \] (254)

This can be seen by considering diagram corresponding to \( \nu \) above (251), and covering the gray 2 by 2 square with border strips of sizes from 1 to 3. We get no contribution proportional to \( p_2(x)p_1(x)^2 \) since the two ways to cover a 2 by 2 diagram with a single border strip of size 2 and 2 border strips of size 1 appear with opposite sign. The above examples show that (178) is rather powerful when applied to \( \langle s_\lambda(U)s_\nu(U^{-1}) \rangle \) for large \( \lambda, \nu \). Indeed, as hinted at above, it is particularly useful for the following three reasons.

1. Equation (178) provides a controlled expansion of general correlation functions \( \langle s_\lambda(U)s_\nu(U^{-1}) \rangle \) in terms of power sums. These power sums can be directly read off from the hamiltonian, see (187) and (189).

2. This expansion can be straightforwardly applied, including to correlation functions involving large diagrams \( \lambda, \nu \). Using the comments below (229), the removal of border strips is related to fermionic particles hopping on a line, which could further simplify the application of this method.

3. The power sums \( p_k(x) \) and \( p_k(y) \) are proportional to \( \tau \), so that (178) provides an expansion in terms of powers of \( \tau \). Depending on the application and the range of \( \tau \) one would like to consider, this expansion can be truncated at any desirable order that provides sufficient precision.
From the results in this section, it follows that equation (178) has the following interpretation in terms of particles and holes.

\[
\langle s_\lambda(U)s_\nu(U^{-1}) \rangle = \sum_{\omega,\gamma} \frac{p_\omega(y)p_\gamma(x)}{z_\omega z_\gamma} (-1)^p \left\{ \begin{array}{l}
\text{Distinct ways to move particles in } \lambda \text{ and } \nu \\
\text{to the left by } \gamma_1, \gamma_2, \ldots \text{ and } \omega_1, \omega_2, \ldots \text{ steps,}
\end{array} \right. \\
\text{respectively, so hopping over } P \text{ other particles}
\text{and ending up in the same configuration.}
\]

(255)

6.2.5 Correlations for power sums and applications to experimental benchmarking

We give here some suggestions for the benchmarking of experimental setups using correlation functions involving power sum polynomials. In particular, in experimental contexts such as trapped ions, one could measure the correlation functions corresponding to \(\langle p_n(U^{\pm 1}) \rangle = \langle \text{tr}U^{\pm n} \rangle = p_n \) experimentally. Remember that the \(p_n\) can be read off from the hamiltonian as in (187) or (189) due to their direct proportionality to \(a_k\). Therefore, one could measure \(\langle p_n(U^{\pm 1}) \rangle\) and compare them with the intended values of \(a_k\) to benchmark experimental setups. Further, one can use equation (110),

\[
\langle |\text{tr}U^n|^2 \rangle = n + p_n(x)p_n(y),
\]

(256)

and consider \(\text{tr}U^n\) as a superposition of states (when properly normalized), as before. Then, the correlation function corresponding \(\langle |\text{tr}U^n|^2 \rangle\) is proportional to \(F_{\Theta}^3\Theta\) with proportionality given by \(n + p_n(x)p_n(y)\), up to normalization. When we have certain hopping parameters \(a_k \neq 0\), the only way to get independence from time is to have no dependence on the power sums \(p_k \sim \tau k a_k\), as the power sums contain a factor of \(\tau\). The \(\tau\)-independence of the connected part of the correlation function corresponding to (110) therefore seems quite unusual, and might offer an effective way to benchmark experimental setups. Remember that \(p_\lambda\) form a basis for all symmetric polynomials. In case \(\ell(\lambda) \geq 2\) so that \(p_\lambda = p_{\lambda_1}p_{\lambda_2} \ldots\), Wick’s theorem tells us that \(\langle p_\lambda p_\mu \rangle_c\) will contain where not all \(p_{\lambda_j}\) and \(p_{\mu_k}\) are contracted. These contribute terms containing \(p_k(x) \sim \tau\) (and/or \(p_j(y) \sim \tau\)). Therefore, \(\frac{1}{\tau^n} \langle |\text{tr}U^n|^2 \rangle\) is the only non-zero connected correlation function that does not depend on \(\tau\). Lastly, we mention that one could use \(\langle \text{tr}U^n\text{tr}U^{-k} \rangle_c = n\delta_{n,k}\) (after proper normalization) to see if the system is truly translationally invariant, as its derivation is predicated on the assumption of translational invariance.

7 Conclusions

This work focused on weighted \(U(N)\) integrals over Schur polynomials, written as \(\langle s_\lambda(U)s_\nu(U^{-1}) \rangle\), and their relation with correlation functions of LRRW models. The weighting is given by a weight function \(f(z)\), which is required to satisfy Szegő’s strong limit theorem, corresponding to hopping parameters \(a_k\) which decay faster than \(k^{-1}\) for \(k \rightarrow \infty\). Writing the weight function in terms of generating functions of elementary or complete homogeneous symmetric polynomials as \(f(z) = E(x;z)E(y;z^{-1})\) or \(f(z) = H(x;z)H(y;z^{-1})\), respectively, general correlation functions can be expressed in terms of Schur polynomials with variables \(x\) and \(y\). In particular, the power sum polynomials are related to the
hopping parameters as $p_k(x) = \pm \tau a_k$ and $p_k(y) = \pm \tau a_k$ for $k \geq 1$. By using our earlier result that $\langle \text{tr} U^n \text{tr} U^{-k} \rangle_c = n \delta_{n,k}$ and applying Wick’s theorem, we derive various identities. In particular, we compute $\langle p_{\alpha}(U)p_{\beta}(U^{-1}) \rangle$, generalizing a result due to Diaconis and Shahshahani, who computed said object in the CUE, corresponding to $f = 1$. Further, we derive two expressions for correlation functions $\langle s_{\lambda}(U)s_{\nu}(U^{-1}) \rangle$ for general $\lambda$ and $\nu$, both of which are obtained by removing border strips from $\lambda$ and $\nu$. In particular, the first of these expressions is an expansion in terms of diagrams related to $\lambda$ and $\nu$ by the removal of border strips. The second is an expansion in terms of $p_k(x)$ and $p_k(y)$, where the expansion coefficients are determined by the number of ways to remove border strips from $\lambda$ and $\nu$ and arrive at the same diagram, up to a sign determined by the height of the (generally disjoint) border strip tableau that is removed. Due to the proportionality of the power sum polynomials and the hopping parameters, the latter expansion is particularly useful for our purposes.

We applied these results to correlation functions of LRRW models, where we consider configurations starting with an infinite number of particles (down spins) and ending with and infinite number of holes (up spins). Before applying our own results, we consider various standard relations in the theory of symmetric polynomials, such as the Pieri formula, which leads to a simple calculation and interpretation of various correlation functions. Most interestingly, the involution between elementary and complete homogeneous symmetric polynomials, corresponding to the transposition of diagrams, leads to a duality between models related by switching the sign of the even hopping parameters, $a_k \rightarrow (-1)^{k+1}a_k$. In particular, we find that the correlation functions of these two models equal each other when we perform a particle-hole and parity transformation to the finite interval in between the infinite strings of particles and holes. We therefore refer to this duality as quasi-local particle-hole duality. Further, we note that the Jacobi-Trudi identity allows one to calculate any $\langle s_{\lambda}(U)s_{\nu}(U^{-1}) \rangle$ directly. Although the number of terms appearing in this expansion grows quickly with the size of $\lambda$, $\nu$, it is simple to implement as it only involves the calculation of a determinant of a matrix with known entries.

We then considered the basis of power sum polynomials $p_k(x)$ and $p_k(y)$ and the closely related border strips, which is particularly useful in the application to LRRW models due to their aforementioned relation to hopping parameters $a_k$ and generalized time $\tau$. The addition or removal of a border strip $T$ of size $n$ and height $\text{ht}(T)$ corresponds to moving a particle a distance $n$ to the right or left, respectively, thereby hopping over $\text{ht}(T)$ other particles. This provides an interpretation of various expansions of correlation functions in terms of particles hopping with fermionic exchange statistics, even though the LRRW model consists of hard-core bosons rather than fermions. For example, the second expansion of $\langle s_{\lambda}s_{\nu} \rangle$ we derived involving removal of border strips can be interpreted as replacing the (hard-core boson) particles in $\lambda$ and $\nu$ by fermions and considering all distinct ways to move them any number of steps to the left such that the final configurations are identical. Due to the above relation between power sums and hopping parameters, this expression provides an expansion in powers of $\tau$ where the numbers appearing in the expansion coefficients can be read off from the Hamiltonian.

The mathematical results derived in section 5 may be generally applied to the case where the strong Szegö limit theorem applies. The expansion of general correlation functions in terms of power sum
polynomials is particularly useful in such applications where these objects can be accessed directly, such as for LRRW models, where they are provided as input. In section 6, we saw that the addition or removal of border strips with sign \((-1)^{ht}\) can be interpreted as fermionic particles hopping to the right or left, respectively. One may apply this ‘physical’ interpretation of this process to the power sum expansion of \(\langle s_\lambda s_\nu \rangle\), by considering all ways to move fermionic particles in \(\lambda\) and \(\nu\) to the left to arrive at the same configuration. Further, one may apply the same fermionic interpretation to the Murnaghan-Nakayama rule for the calculation of general \(\chi^\lambda_\mu\). That is, \(\chi^\lambda_\mu\) equals the sum over distinct ways to move particles \(\mu_j\) steps to the right so that the result is \(\lambda\), up to a plus or minus sign for an even or odd numbers of particles hopped over in this way, respectively. The fact that \(\chi^\lambda_\mu\) is cancellation-free leads to the conclusion that all ways to go from any configuration to any other configuration by taking only \(n\) steps involves either an even or an odd number of particles being hopped over. This appears to be a novel result as well (at least it was not found elsewhere by the authors) which may have independent mathematical applications and/or generalizations.

As mentioned in the introduction, LRRW models such as we consider here have seen increasing activity from both experimental and theoretical physicists due to the experimental accessibility of such systems and the novel and surprising phenomena they exhibit. We believe our work can be applied along some of these lines of research, including the consideration of localization by addition of (diagonal) disorder \([24],[25]\). Adding such disorder will generally break translational invariance, and the expressions in this work will no longer apply. One may also consider translationally invariant disorder such as random hopping parameters, as in \([51],[26]\), in which case the results derived here would still apply as long as the hopping parameters satisfy the aforementioned asymptotic fall-off conditions. All results in section 6 can, in principle, be checked experimentally, where the expressions in this work would be expected to hold with reasonable accuracy up to the time that finite size effects start to occur. Besides checking our results in experimental setups, correlation functions involving superpositions of states corresponding to power sum polynomials may be used for experimental benchmarking.

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