Generating and Revealing a Quantum Superposition of Electromagnetic Field
Binomial States in a Cavity

R. Lo Franco, G. Compagno, A. Messina, and A. Napoli
Dipartimento di Scienze Fisiche ed Astronomiche,
Università di Palermo, via Archirafi 36, 90123 Palermo, Italy
(Dated: April 1, 2022)

We introduce the $N$-photon quantum superposition of two orthogonal generalized binomial states of electromagnetic field. We then propose, using resonant atom-cavity interactions, non-conditional schemes aimed at generating “optimized” quantum superpositions for the two-photon case in a single-mode high-$Q$ cavity. We finally discuss the implementation of the proposed schemes.

PACS numbers: 42.50.Dv, 03.65.-w, 32.80.-t

Since the birth of the Schrödinger cat phenomenon \[1\], the possibility of generating and detecting macroscopic quantum superpositions has been holding much interest in several frameworks \[2–4\]. A macroscopic quantum superposition of electromagnetic field states is usually meant as a superposition of two coherent states with classically different phases \[5, 6\]. In the context of cavity quantum electrodynamics (CQED), such a state has been generated by dispersive coupling between a circular Rydberg atom and a small coherent cavity field, the quantum decoherence of the superposition being there observed by probe atoms \[7\]. Other schemes have been proposed to generate or detect quantum superpositions of this kind, for example in a dispersive medium \[8\], in a nanomechanical resonator \[9\] and in a cavity \[10–13\] and a free-propagating light pulse was also recently prepared in such a state \[14\]. Nevertheless, two different coherent states can never be made exactly orthogonal and therefore the coherent states of a quantum superposition are not completely distinguishable. In the CQED experiment of Ref. \[7\], it is for example necessary to adjust the detuning between the atomic transition and the cavity frequency to partially distinguish the two components of the superposition. Thus, to propose schemes aimed at generating “optimized” quantum superpositions, defined as quantum superpositions of two orthogonal, distinguishable electromagnetic field states with nonzero mean fields, appears to be an attractive challenge.

It is well known that the binomial states of electromagnetic field are characterized by a finite maximum number of photons $N$, interpolate between coherent state and number state and also exhibit nonzero mean fields \[15–17\]. In addition, it is always possible to find orthogonal couples among the $N$-photon binomialized states \[18\]. Such features make the generalized binomial states promising candidates to construct optimized quantum superpositions and to study the general problem of classical-quantum border and quantum measurement. The point is then on how to generate and reveal such states. This paper addresses this issue, by exploiting standard resonant atom-cavity interactions in the CQED framework.

The dynamics of the resonant interaction between a two-level atom and a single-mode cavity field of frequency $\omega$ is described by the Jaynes-Cummings Hamiltonian $H_{JC} = \hbar \omega \sigma_z/2 + \hbar \omega a^\dagger a + i \hbar g (\sigma_+ a - \sigma_- a^\dagger)$, where $a$ and $a^\dagger$ are the field annihilation and creation operators, $\sigma_z = | ↑⟩⟨↑ | - | ↓⟩⟨↓ |$, $\sigma_+ = (\sigma_-)^\dagger = | ↑⟩⟨↓ |$ the pseudo-spin atomic operators, $| ↑⟩$ and $| ↓⟩$ being respectively the excited and ground state of the two-level atom, and $g$ is the atom-field coupling constant. The $H_{JC}$ based time evolution of the states $| ↑⟩| n⟩ ≡ | ↑⟩| n⟩$ and $| ↓⟩| n⟩ ≡ | ↓⟩| n⟩$, with $a^\dagger a^n = n| n⟩$, is \[19\]

\[
| ↑⟩| n⟩ \rightarrow cos(g\sqrt{n+1}t)| ↑⟩| n⟩ - sin(g\sqrt{n+1}t)| ↓⟩| n+1⟩, \\
| ↓⟩| n⟩ \rightarrow cos(g\sqrt{n}t)| ↓⟩| n⟩ + sin(g\sqrt{n}t)| ↑⟩| n-1⟩,
\]

where $t$ is the atom-cavity interaction time.

The normalized $N$-photon generalized binomial state is given by \[15\]

\[
|N, p, \phi⟩ = \sum_{n=0}^{N} \left[ \binom{N}{n} p^n (1-p)^{N-n} \right]^{1/2} e^{i n \phi} | n⟩,
\]

where $0 \leq p \leq 1$ is the probability of single photon occurrence and $\phi$ is the mean phase \[10\]. The orthogonality property $\langle N, p, \phi | N, 1-p, \pi + \phi \rangle = 0$ \[15\] allows us to define the $N$-photon quantum superposition of two orthogonal generalized binomial states (NQSB) as \[15\]

\[
|Ψ_{N}^{(N)}⟩ = \mathcal{N}[|N, p, \phi⟩ + \eta|N, 1-p, \pi + \phi⟩],
\]

where $\eta$ is a complex number and $\mathcal{N} = 1/\sqrt{1 + |\eta|^2}$. It should be noted that, for $p = 0.1$, the NQSB is reduced to a quantum superposition of the number states $|0⟩, |N⟩$. The NQSB effectively represents a macroscopic superposition of electromagnetic field states if $N \gg 1$. However, in order to remain in the grasp of the current experimental feasibility, we shall concentrate on both the generation and the revealing of the quantum superposition in the case $N = 2$. We shall show that the 2QSB $|Ψ_{2}^{(2)}⟩$ may be generated in a cavity by the experimental...
scheme sketched in Fig. 1 and its components and coherence may be revealed by the schemes sketched in Figs. 2 and 3.

**Generating the quantum superposition.**—In the generation scheme of Fig. 1 the cavity \( C \) is initially prepared in the vacuum state |0\>, and a couple of two-level atoms, namely 1 and 2, is prepared in the state \(|\psi\rangle = \mathcal{N}(|\uparrow_1\downarrow_2\rangle + \eta_0 |\downarrow_1\uparrow_2\rangle)\), with \( \eta_0 \) real. Entangled atomic states of this form have already been obtained using a cavity as atomic entanglement catalyst [21, 22], providing in addition that the two atoms enter the Ramsey zone, as well as the cavity, one at a time. Each atom first crosses a “preparing” Ramsey zone \( R_p \). The Ramsey zone interaction makes the \( j \)-th atom undergo the following transformations:

\[
|\uparrow_j\rangle \xrightarrow{R} \cos(\theta_j/2)|\uparrow_j\rangle - e^{i\varphi_j}\sin(\theta_j/2)|\downarrow_j\rangle, \\
|\downarrow_j\rangle \xrightarrow{R} e^{-i\varphi_j}\sin(\theta_j/2)|\uparrow_j\rangle + \cos(\theta_j/2)|\downarrow_j\rangle, \tag{4}
\]

where the parameters \( \theta_j \) (“Ramsey pulse”) and \( \varphi_j \) are fixed by adjusting the classical field amplitude and the atom-field interaction time. The \( j \)-th atom then resonantly interacts with \( C \) for a time \( T_j \) (\( j = 1, 2 \)). The atom-cavity interaction times \( T_j \) can be obtained by selecting either different velocities for each atom or the same velocity for the two atoms (“monokinetic atomic beam”) and applying a Stark shift inside the cavity for a time such as to have the desired resonant interaction time [21, 22]. The appropriate atomic velocity may be selected by laser induced atomic pumping [23]. We shall show that a 2QSB state can be efficiently generated by appropriately choosing the Ramsey zone settings and the atom-cavity interaction times.

In accordance with the scheme of Fig. 1 atom 1 crosses \( R_p \) set with a “pulse” \( \theta_1 \) such that \( \cos(\theta_1/2) = \sqrt{7}, \sin(\theta_1/2) = \sqrt{1-\sqrt{7}} \), and with \( \varphi_1 \) to be related to the mean phase \( \phi \) appearing in Eq. 3. After a free evolution time \( \tau_1 \) between \( R_p \) and \( C \), atom 1 interacts with the cavity \( C \) for a given time \( T_1 \). After it exits \( C \), atom 2 crosses the Ramsey zone \( R_p \), freely evolves for a time \( \tau_2 \) from \( R_p \) to \( C \) and then interacts with \( C \) for a time \( T_2 \). Let us indicate with \( T \) the time elapsed between the exit of the atom 1 from \( C \) and the entrance of the atom 2 in \( C \). After the passage of atom 1, the \( R_p \) parameters must be reset at \( \theta_2 = \theta_1 + \pi \), so that \( \cos(\theta_2/2) = -\sqrt{1-\sqrt{7}} \), and \( \sin(\theta_2/2) = \sqrt{7} \) in Eq. 1, and \( \varphi_2 = \varphi_1 + \omega(\tau_1 + T - \tau_2) \). Taking into account Eqs. 1 and 4 it is possible to demonstrate that, if \( T_1 = (4m+1)\pi/2g \) (\( m \) non-negative integer) and \( T_2 \) is such that the two equalities \( \sin(gT_2/2 + \pi/4) = 1 \), \( \sin(g\sqrt{2T_2}) = 1 \) are simultaneously satisfied, when the second atom leaves \( C \), the state of the total system (atom 1 + atom 2 + cavity) turns out to be factorized as \(|\Psi_S^{(2)}\rangle \downarrow_1\downarrow_2\rangle\). It is worth noting that, choosing \( T_2 = 41\pi/4g \), both equalities above are satisfied within the error due to the typical experimental interaction time uncertainties [24]. Thus, the cavity field after the passage of the two atoms coincides with the quantum superposition of a couple of orthogonal two-photon generalized binomial states (2QSB)

\[
|\Psi_S^{(2)}\rangle = \mathcal{N}|[2, p, \phi, \eta_0 e^{i\gamma}[2, 1-p, \pi + \phi]]\rangle, \tag{5}
\]

where \( \phi = -[\varphi_1 + \omega(\tau_1 + T)] \), and \( \gamma = \omega(t_{R_2} - t_{R_1} - T_1) \). \( t_{R_1}, t_{R_2} \) being respectively the interaction times of the atoms 1 and 2 with \( R_p \). It is of relevance that our procedure to generate a 2QSB in a cavity does not require a final atomic measurement and then it is a non-conditional scheme.

We shall now analyze the possibility to probe the generated \( |\Psi_S^{(2)}\rangle \) state. Generally speaking, to probe a quantum superposition requires a measurement procedure permitting both to resolve the two components and to reveal their relative quantum coherence. In the following, we present a procedure appropriate for the 2QSB \(|\Psi_S^{(2)}\rangle \) based on two-level probe atoms that “read” the cavity field. Our considerations will be developed for the maximal 2QSB of Eq. (5), corresponding to \( \eta_0 = \pm 1 \), that is

\[
|\Psi_S^{(2)}\rangle_\pm = ||[2, p, \phi] \pm e^{i\gamma}[2, 1-p, \pi + \phi]]||\sqrt{2}. \tag{6}
\]

**Revealing the two components.**—The experimental scheme we propose is illustrated in Fig. 2. It exploits two consecutive probe atoms both in their ground state interacting one at a time with the apparatus. The atom 1 resonantly interacts with \( C \) for an appropriate time \( T_{P_1} \), after a delay time \( t_1 \) from \( C \) to \( R_d \) it crosses the “decoding” Ramsey zone \( R_d \) and it is finally measured by field ionization detectors. After this measurement, atom 2 enters the cavity \( C \). Let us denote with \( T' \) the time interval between the exit of the atom 1 from \( C \) and the entrance of the atom 2 in \( C \). Atom 2 resonantly interacts...
FIG. 3: Experimental scheme for revealing the coherence of the 2QSB. $R_c$ is the “coherence decoding" Ramsey zone. The cavity $C$ is prepared in the 2QSB $|Ψ_{S}^{(2)}⟩$±.

with $C$ for a time $T_{P_1}$, takes a time $t_2$ to go from $C$ to $R_d$, crosses $R_d$ and its internal state is finally measured.

Let us suppose the cavity prepared in the state $|2, p, φ⟩(2, 1 - p, π + φ)$ and perform the experiment previously described fixing $T_{P_1} = 41π/4g$, $T_{P_2} = (4m + 1)π/2g$, the $R_d$ parameters $θ_{d_1} = θ_{d_2} = θ_d$ such that $\cos(θ_d/2) = \sqrt{p}$, $\sin(θ_d/2) = \sqrt{1 - p}$ and $ϕ_{d_1} = -ϕ + ωt_1$, $ϕ_{d_2} = -ϕ + ω(t_2 + t_1)$. Under these conditions, the probability of finding the two atoms in the states $|↑⟩|↑⟩$ at the end of the experiment is equal to one. This statement readily follows from the unitary evolutions $|Ψ_S^{(2)}⟩±$ evolves into

$$|0⟩\left\{\frac{1 ± e^{i(γϕ - 2ϕ_0)}}{2}\left[|↑⟩|↑⟩ + e^{iα}|↓⟩|↓⟩⟩\right]\right\} - e^{iϕ_0}\frac{1 ± e^{i(γϕ - 2ϕ_0)}}{2}\left[|↑⟩|↑⟩ + e^{iβ}|↓⟩|↓⟩⟩\right\}, \tag{9}$$

where $|0⟩$ is the cavity vacuum state, $ϕ_0 = 2ϕ - ω(t_2 - t_1 + Δϕ + Δϕ_0)$, $α = 2ϕ_0 + Δω$ and $β = Δω$, with $t$ being the sum of some characteristic times of the procedure. From Eq. (9) it is readily seen that, setting $ϕ_c = (γ - ϕ_0)/2$, the following unitary evolutions are obtained:

$$|Ψ_S^{(2)}⟩+ → |0⟩|↑⟩|↑⟩ + e^{iα}|↓⟩|↓⟩⟩\right\} / √2,$n

$$|Ψ_S^{(2)}⟩- → |0⟩|↑⟩|↑⟩ + e^{iβ}|↓⟩|↓⟩⟩\right\} / √2. \tag{10}$$

Note that all the free evolution times can be determined from the atomic velocities, the delay time $T_0$ between the two atoms and the geometrical parameters. Eq. (10) says that the unitary evolution of the probe atoms and the cavity field $|Ψ_S^{(2)}⟩+$ (|$Ψ^{(2)}⟩-$) generates a vanishing probability amplitude for the atomic states $|↑⟩|↑⟩$ and $|↓⟩|↓⟩$ (|$↑⟩|↑⟩$ and $|↓⟩|↓⟩$), equally distributing the probability between the other two possible outcomes $|↑⟩|↓⟩$ and $|↓⟩|↑⟩$. Therefore, after repeating this experiment many times, including the preparation of the cavity field, we are able to confirm the quantum coherence (“sign" and relative phase $γ$) of the initial cavity field state $|Ψ_S^{(2)}⟩+$ or $|Ψ^{(2)}⟩-$. In fact, if the outcomes of the repeated measurements always give “parallel" atoms then the cavity field is with certainty in the quantum superposition $|Ψ_S^{(2)}⟩+$; otherwise, if the outcomes always give “antiparallel" atoms then the cavity field is with certainty in the quantum superposition $|Ψ^{(2)}⟩-$. We now briefly analyze the experimental feasibility of the proposed schemes. They require precise atom-cavity interaction times. However, the experimental uncertainty of the selected velocity $Δv$ induces an error $ΔT$ on the interaction time such that $ΔT/T ≈ Δv/v$. In current laboratory experiments it is possible to select a given atomic velocity such that $Δv/v < 10^{-2}$. This error does not appear to sensibly affect our schemes.
For circular Rydberg atomic levels and microwave superconducting cavities with quality factors $Q \sim 10^8 - 10^{10}$ the required inequality on the mean lifetimes can indeed be satisfied, being $\tau_{at} \sim 10^{-5} - 10^{-2}$ s, $\tau_{cav} \sim 10^{-4} - 10^{-1}$ s and $T \sim 10^{-5} - 10^{-4}$ s \cite{12, 13}. Moreover, the typical mean lifetimes of circular Rydberg atomic levels $\tau_{at}$ are such that the atoms do not decay during the entire sequence of the schemes \cite{14, 15}. The delay time $\tau_0$ between the two atoms can be adjusted so that they cross the experimental apparatus one at a time, as required by our schemes. Recent laboratory developments open promising perspectives for a better and easy control of a well-defined atom numbers sequence \cite{16, 17} and for a high efficiency atomic detection in microwave CQED experiments \cite{18}.

Finally, because the binomial states interpolate between number and coherent states, an estimate of the time scale of 2QSB decoherence can be provided by the corresponding experiment on coherent states superpositions with small mean photon numbers \cite{7}. In this experiment the decoherence time comes out shorter than the photon decay time of the cavity thus it may be taken as a good indication of the mesoscopic nature of the superposition. We also wish to observe that our state check procedure gives an unambiguous signal when the state superposition is perfect. However if, because of decoherence or state preparation, the final superposition is not perfect, our procedure is yet able to measure the degree of coherence (or the state preparation fidelity) of the 2QSB. Although a detailed ab initio analysis is required for the general case, we give here a quantitative simple example of this aspect. In fact, if the initial state of the system leads to a final state of the form given in the first line of Eq. (10) plus the term $\delta \lvert \uparrow \downarrow \rangle$ and with a new global normalization factor $N = 1/\sqrt{2 + |\delta|^2}$, the probability of detecting the first atom in $\lvert \uparrow \rangle$ and the second atom in $\lvert \downarrow \rangle$ is $P(\lvert \uparrow \downarrow \rangle) = (N|\delta|^2)$. So, the state preparation fidelity is given by $F = (1 + |\delta|^2/2)^{-1} = 1 - P(\lvert \uparrow \downarrow \rangle)$ and it is therefore determined by the detection outcomes.

In this paper, we have defined the $N$-photon quantum superposition of two orthogonal generalized binomial states of electromagnetic field (NQSB). Our main result is the proposal of non-conditional schemes to generate and reveal such an “optimized” quantum superposition in a single-mode high-$Q$ cavity in the case $N = 2$. We wish to emphasize that the orthogonality of the two generalized binomial states forming this state plays a crucial role to reveal, by resonant probe atoms, the quantum nature of the superposition. The implementation of the proposed schemes has been also analyzed, showing how the unavoidable errors characterizing the current experiments do not seem to sensibly affect them. Because of the orthogonality property of generalized binomial states with any $N$, our generation and revealing procedure of their quantum superposition may be in principle extended to the cases with $N$ larger than two. This would lead, for $N \gg 1$, to a regime of macroscopic quantum superpositions, the highest value of $N$ being only limited by the experimental capabilities. The results of this paper can provide the basis for both new knowledge about the foundations of quantum theory (measurement process, quantum-classical border) and applications in quantum information processing, in analogy with the superpositions of coherent states \cite{19, 20}.

\begin{thebibliography}{20}
\bibitem{1} E. Schrödinger, Naturwissenschaften \textbf{23}, 807 (1935).
\bibitem{2} J. A. Wheeler and W. H. Zurek, \textit{Quantum Theory of Measurement} (Princeton Univ. Press, Princeton, NJ, 1983).
\bibitem{3} H. Jeong and T. C. Ralph, Phys. Rev. Lett. \textbf{97}, 100401 (2006).
\bibitem{4} E. G. Cavalcanti and M. D. Reid, Phys. Rev. Lett. \textbf{97}, 170405 (2006).
\bibitem{5} S. Haroche, in \textit{Les Houches Session LIII 1990, Course 13, Cavity Quantum Electrodynamics} (Elsevier Science Publishers B.V., New York, 1992).
\bibitem{6} S. Haroche and J. M. Raimond, \textit{Exploring the Quantum: Atoms, Cavities, and Photons} (Oxford University Press, USA, Oxford, New York, 2006).
\bibitem{7} M. Brune et al., Phys. Rev. Lett. \textbf{77}, 4887 (1996).
\bibitem{8} B. Yurke and D. Stoler, Phys. Rev. Lett. \textbf{57}, 13 (1986).
\bibitem{9} L. Tian, Phys. Rev. B \textbf{72}, 195411 (2005).
\bibitem{10} J. M. C. Malbouisson and B. Baseia, J. Mod. Opt. \textbf{46}, 2015 (1999).
\bibitem{11} Y. xi Liu, L. F. Wei, and F. Nori, Phys. Rev. A \textbf{71}, 063620 (2005).
\bibitem{12} P. P. Munhoz and A. Vidiella-Barranco, J. Mod. Opt. \textbf{52}, 1557 (2005).
\bibitem{13} F. Casagrande and A. Lulli, Eur. Phys. J. D \textbf{36}, 123 (2005).
\bibitem{14} A. Ourjoumtsev, R. Tualle-Brouri, J. Laurat, and P. Grangier, Science \textbf{312}, 83 (2006).
\bibitem{15} D. Stoler, B. E. A. Saleh, and M. C. Teich, Opt. Acta \textbf{32}, 345 (1985).
\bibitem{16} A. Vidiella-Barranco and J. A. Roversi, Phys. Rev. A \textbf{50}, 5233 (1994).
\bibitem{17} A. Vidiella-Barranco and J. A. Roversi, J. Mod. Optics \textbf{42}, 2475 (1995).
\bibitem{18} R. Lo Franco, G. Compagno, A. Messina, and A. Napoli, Phys. Rev. A \textbf{72}, 053806 (2005).
\bibitem{19} M. O. Scully and M. S. Zubairy, \textit{Quantum Optics} (Cambridge University Press, 1997).
\bibitem{20} J. M. Raimond, M. Brune, and S. Haroche, Rev. Mod. Phys. \textbf{73}, 565 (2001).
\bibitem{21} E. Hagley et al., Phys. Rev. Lett. \textbf{79}, 1 (1997).
\bibitem{22} L. Davidovich et al., Phys. Rev. A \textbf{50}, R895 (1994).
\bibitem{23} S. Haroche, Phys. Scripta \textbf{102}, 128 (2002).
\bibitem{24} R. Lo Franco, G. Compagno, A. Messina, and A. Napoli, Phys. Rev. A \textbf{74}, 045803 (2006).
\bibitem{25} P. Maioli et al., Phys. Rev. Lett. \textbf{94}, 113601 (2005).
\bibitem{26} A. Auffeves et al., Phys. Rev. Lett. \textbf{91}, 230405 (2003).
\bibitem{27} H. Jeong and T. C. Ralph, in \textit{Quantum Information with Continuous Variables of Atoms and Light}, edited by N. J. Cerf, G. Leuchs, and E. S. Polzik (Imperial College Press, 2007).
\end{thebibliography}