OPTIMAL APPROXIMANTS AND ORTHOGONAL POLYNOMIALS IN SEVERAL VARIABLES II: FAMILIES OF POLYNOMIALS IN THE UNIT BALL

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Abstract. We obtain closed expressions for weighted orthogonal polynomials and optimal approximants associated with the function \( f(z) = 1 - \frac{1}{\sqrt{2}}(z_1 + z_2) \) and a scale of Hilbert function spaces in the unit 2-ball having reproducing kernel \((1 - \langle z, w \rangle)^{-\gamma}, \gamma > 0\). Our arguments are elementary but do not rely on reduction to the one-dimensional case.

1. Introduction

This note continues recent work in [12] concerning certain families of polynomials connected with approximation in spaces of analytic functions, and orthogonal polynomials in weighted spaces. In the paper [12], we discussed the notion of optimal approximants to \( 1/f \) for a holomorphic function \( f \) belonging to a Hilbert function space in \( \mathbb{C}^n \), and pointed out connections with orthogonal polynomials in certain weighted spaces, with weight determined by the same target function \( f \). We presented some elementary examples of optimal approximants and orthogonal polynomials in several variables, and to obtain concrete closed-form representations of these objects, we relied on one-variable results together with suitable transformations.

In this note, we present a further family of examples of weighted orthogonal polynomials and optimal approximants in several variables. We use a direct, elementary approach to go beyond cases that admit easy reduction to essentially one-variable problems. For simplicity, we focus on two variables, the target function \( f = 1 - \frac{1}{\sqrt{2}}(z_1 + z_2) \), and a scale of spaces of functions in the unit ball \( \mathbb{B}^2 = \{(z_1, z_2) \in \mathbb{C}^2: |z_1|^2 + |z_2|^2 < 1\} \), but some of our arguments potentially extend to higher dimensions, at the price of more cumbersome notation and more involved proofs.

We consider a scale of reproducing kernel Hilbert spaces that have recently featured in work of Richter and Sunkes [11]. For further background on this kind of spaces, see for instance [14] [7] [5] and the references therein. Fix \( \gamma > 0 \) and let \( H_\gamma \) denote the reproducing kernel Hilbert space in \( \mathbb{B}^d \) associated with the reproducing kernel

\[
k_\gamma(z; w) = \frac{1}{(1 - \langle z, w \rangle)^\gamma}, \quad z, w \in \mathbb{B}^d.
\]
The $\mathcal{H}_\gamma$ include well-known spaces such as the Drury-Arveson space ($H_d^2 = H_1^2$), the Hardy space of $\mathbb{B}^2$ ($H^2(\partial \mathbb{B}_d) = \mathcal{H}_d$), and the Bergman space of the 2-ball ($A^2(\mathbb{B}_d) = \mathcal{H}_{d+1}$). In two variables, the norm in $\mathcal{H}_\gamma$ of an analytic function $f = \sum_{m=0}^\infty \sum_{n=0}^\infty \hat{f}(m, n) z_1^m z_2^n$ can be expressed as

$$
\|f\|_\gamma^2 = \sum_{m=0}^\infty \sum_{n=0}^\infty a_{m,n} \hat{f}(m,n) \|z_1^m z_2^n\|^2,
$$

where

$$
a_{m,n} = \begin{cases} 1 & m = n = 0, \\ \frac{m!n!}{(\gamma+m+n-1)\cdot(\gamma+1)} & \text{otherwise.} \end{cases}
$$

We observe that polynomials are dense in all the $\mathcal{H}_\gamma$, monomials are orthogonal, and multiplication by the coordinate functions furnish bounded linear operators.

We now state the definition of optimal approximants; see [6, 13, 2, 11, 12] for more comprehensive discussions and references. Enumerating the monomials in two variables in some fixed way, we write $\chi_j$ for the $j$th monomial in this ordering, and set $P_n = \text{span}\{\chi_j : j = 0, \ldots, n\}$. In this note, we work with degree lexicographic order. Monomials are ordered by increasing total degree, and ties between two monomials of the same total degree are broken lexicographically. See [10, 9] and the references therein for background material. Explicitly, we have

$$1 < z_1 < z_2 < z_1^2 < z_1 z_2 < z_2^2 < z_1^3 < z_1^2 z_2 < \cdots,$$

so that $\chi_4 = z_1 z_2$, $\chi_5 = z_2^2$, and so on. For pairs of natural numbers $(j, k)$ and $(m, n)$, we will take $(j, k) \prec (m, n)$ to signify that $z_1^j z_2^k < z_1^m z_2^n$.

**Definition 1** (Optimal approximants). Let $f \in \mathcal{H}_\gamma$ be given. We define the $n$th order optimal approximant to $1/f$ in $\mathcal{H}_\gamma$, relative to $P_n$, as $p_n^* = \text{Proj}_{\gamma,f} P_n \|1/f\|_{\mathcal{H}_\gamma}$, where $\text{Proj}_{\gamma,f} : \mathcal{H}_\gamma \rightarrow f \cdot P_n$ is the orthogonal projection onto the closed subspace $f \cdot P_n \subset \mathcal{H}_\gamma$.

Given some $f \in \mathcal{H}_\gamma$, optimal approximants can be viewed as polynomial substitutes for the function $1/f$, the point being that $1/f$ may fall outside of $\mathcal{H}_\gamma$. Optimal approximants arise in several contexts, for instance cyclicity problems and filtering theory, see [13, 12]. The papers [8, 14] discuss some methods for computing optimal approximants, but closed formulas are only known in a few instances. Multi-variable examples have so far only been obtained as a consequence of one-variable results.

**Definition 2** (Weighted orthogonal polynomials). Let $f \in \mathcal{H}_\gamma$ be fixed. We say that a sequence $(\phi_j)_{j \in \mathbb{N}} \subset \mathbb{C}[z_1, z_2]$ consists of weighted orthogonal polynomials with respect to $f$ if $(\phi_j)$ is an orthogonal basis for the Hilbert space $\mathcal{H}_{\gamma,f}$ with inner product given by $\langle g, h \rangle_{\gamma,f} = \langle g f, h f \rangle_{\mathcal{H}_\gamma}$. There is an important connection between optimal approximants and orthogonal polynomials, as is explained in [3, 12]. Namely, if $(p_n^*)$ denote the optimal approximants to $1/f$, $f \in \mathcal{H}_\gamma$, and $(\phi_n)$ are orthogonal polynomials
in the weighted space $\mathcal{H}_{\gamma,f}$, respectively, then

\begin{equation}
(1.3) \quad p^*_n(z) = \sum_{k=0}^{n} (1, f\psi_k)_{\mathcal{H}_f} \psi_k(z),
\end{equation}

where $\psi_k = \phi_k / \|\phi_k\|_{\gamma,f}$. This means that if we determine $\{\phi_k\}_k$ explicitly for some given weight $f$, then we also obtain formulas for the optimal approximants to $1/f$. Implementing this strategy in practice in $\mathcal{H}_\gamma$ and for the function $f = 1 - \frac{1}{\sqrt{2}} (z_1 + z_2)$ is the goal of this note.

2. A FAMILY OF ORTHOGONAL POLYNOMIALS

We begin with an elementary lemma.

**Lemma 1.** Let $f(z_1, z_2) = 1 - a(z_1 + z_2)$ and let $\mathcal{H}$ be a reproducing kernel Hilbert space in which the monomials are orthogonal. Consider $\mathcal{H}_f$, the space weighted by $f$ with inner product $(g, h)_{\mathcal{H}_f} := (gf, hf)_{\mathcal{H}}$. For nonnegative integers $j, k, m, n$, we have

\[
\langle z_1^j z_2^k, z_1^m z_2^n \rangle_f =
\begin{cases}
\|z_1^j z_2^k\|^2 + a^2 \|z_1^{j+1} z_2^k\|^2 + a^2 \|z_1^j z_2^{k+1}\|^2 & \text{if } m = j, n = k, \\
-a \|z_1^j z_2^k\|^2 & \text{if } m = j - 1, n = k, \\
-a \|z_1^{j+1} z_2^k\|^2 & \text{if } m = j, n = k - 1, \\
-a \|z_1^j z_2^{k+1}\|^2 & \text{if } m = j + 1, n = k, \\
a^2 \|z_1^{j+1} z_2^k\|^2 & \text{if } m = j + 1, n = k - 1, \\
a^2 \|z_1^j z_2^{k+1}\|^2 & \text{if } m = j - 1, n = k + 1, \\
0 & \text{otherwise}.
\end{cases}
\]

**Proof.** This amounts to expanding the inner product and reading off terms. \hfill \Box

Recall the standard definition of the *Pochhammer symbol* for $\gamma$ real:

\[(\gamma)_n = \gamma \cdot (\gamma + 1) \cdots (\gamma + n - 1), \quad n \geq 0.\]
Throughout the rest of the paper we shall use the notation 
\[ z^j z^k = \sqrt{\frac{\gamma}{2}} (z^j + z^k), \]
for convenience. For any purely imaginary number \( \gamma > 0 \), set 
\[ J^k = \sqrt{\frac{\gamma}{2}} z^k, \quad K^k = \sqrt{\frac{\gamma}{2}} z^{\bar{k}}. \]

**Theorem 2.** In \( H_\gamma \), weighted by \( f(z_1, z_2) = 1 - \sqrt{\frac{\gamma}{2}} (z_1 + z_2) \), let \( \phi_{j,k} \) be the first orthogonal polynomial containing \( z_j^1 z_k^2 \) (with respect to degree lexicographic order). Then \( \phi_{j,k} \) has the form

\[ \phi_{j,k}(z_1, z_2) = \sum_{m=0}^{j} \sum_{n=0}^{k} \hat{\phi}_{j,k}(m, n) z_1^m z_2^n \]

where the coefficients \( \hat{\phi}_{j,k}(m, n) \) are given by

\[ \hat{\phi}_{j,k}(m, n) = \left( \sqrt{\frac{\gamma}{2}} \right)^{j+k-m-n} \frac{(\gamma)_{m+n+1}}{(\gamma)_{j+k+1}} \frac{j! k!}{m! n! (j-m)! (k-n)!}. \]

Moreover,

\[ \| \phi_{j,k} \|_f^2 = \frac{\gamma + j + k + 1}{\gamma + j + k} j! k! \gamma_j^{j+k} \]

**Proof.** We shall prove this using strong induction. First, \( \phi_{0,0}(z_1, z_2) = 1 \), and by Lemma 1

\[ \| \phi_{0,0} \|_f^2 = 1 = \| 1 \|_f^2 + \left( \sqrt{\frac{\gamma}{2}} \right)^2 \| z \|_f^2 + \left( \sqrt{\frac{\gamma}{2}} \right)^2 \| z \|_f^2 = 1 + \frac{1}{2\gamma} + \frac{1}{2\gamma} = \frac{\gamma + 1}{\gamma} \]

as needed. Now consider \( \phi_{j,k} \) and assume that for all \( (J, K) \prec (j,k) \), the polynomial \( \phi_{J,K} \) has the desired form, coefficients, and norm. Using the Gram-Schmidt algorithm,

\[ \phi_{j,k}(z_1, z_2) = z_j^1 z_k^2 - \sum_{(J,K) \prec (j,k)} \frac{\langle z_j^1 z_k^2, \phi_{J,K} \rangle_\gamma}{\| \phi_{J,K} \|_f^2} \phi_{J,K}. \]

Each \( \phi_{J,K} \) has the form \( (2.1) \), and by Lemma 1 we see that there are only three \( \phi_{J,K} \) with \( (J, K) \prec (j,k) \) that give a non-zero inner product: \( \phi_{j-1,k}, \phi_{j,k-1}, \) and \( \phi_{j+1,k-1} \). Noting that \( \hat{\phi}_{J,K}(J, K) = 1 \) and applying Lemma 1 gives that

\[ \langle z_j^1 z_k^2, \phi_{j,k-1} \rangle_\gamma = -\sqrt{\frac{\gamma}{2}} \frac{j! k!}{(\gamma + j + k - 1) \cdots (\gamma + 1) \gamma} \]

(2.5)

\[ \langle z_j^1 z_k^2, \phi_{j-1,k} \rangle_\gamma = -\sqrt{\frac{\gamma}{2}} \frac{j! k!}{(\gamma + j + k - 1) \cdots (\gamma + 1) \gamma} \]

(2.6)

\[ \langle z_j^1 z_k^2, \phi_{j+1,k-1} \rangle_\gamma = \langle z_j^1 z_k^2, z_1^{j+1} z_2^{k-1} \rangle_\gamma + \hat{\phi}_{j+1,k-1}(j, k-1) z_j^1 z_k^2 \]

(2.7)

\[ = \langle z_j^1 z_k^2, z_1^{j+1} z_2^{k-1} \rangle_\gamma + \hat{\phi}_{j+1,k-1}(j, k-1) \langle z_j^1 z_k^2, z_1^{j-1} z_2^{k-1} \rangle_\gamma. \]
The right hand side of (2.7) is equal to zero: by Lemma [1]

\[ (2.8) \quad \langle z_{j,k}^1, z_{j,k}^{1, j+1,k-1} \rangle_f = \frac{1}{2} \left[ \frac{(j + 1)k!}{(\gamma + j + 1 + k - 1) \cdots (\gamma + 1) \cdot \gamma} \right], \]

and by the inductive hypothesis about the norm of \( \phi_{j+1,k-1} \) and Lemma [1]

\[ \hat{\phi}_{j+1,k-1}(j,k - 1) \langle z_{j,k}^1, z_{j,k}^{1, j+1,k-1} \rangle_f = \frac{\sqrt{2}}{2} \frac{j + 1}{\gamma + j + k} \cdot \left( -\frac{\sqrt{2}}{2} \right) \frac{j!k!}{(\gamma + j + k - 1) \cdots (\gamma + 1) \cdot \gamma} \]

\[ = -\frac{1}{2} \left[ \frac{(j + 1)k!}{(\gamma + j + k) \cdots (\gamma + 1) \cdot \gamma} \right]. \]

Because of this cancellation, which is the key to getting the form the form (2.1), the only preceding orthogonal polynomials that contribute terms to \( \phi_{j,k} \) are \( \phi_{j,k-1} \) and \( \phi_{j-1,k} \), so we have

\[ \phi_{j,k}(z_1, z_2) = z_{j,k}^1 z_{j,k}^2 - \frac{\langle z_{j,k}^1, \phi_{j,k-1} \rangle_f \phi_{j,k-1} - \langle z_{j,k}^1 z_{j,k}^2, \phi_{j-1,k} \rangle_f}{\| \phi_{j,k-1}\|^2_f} \phi_{j-1,k} \]

\[ = z_{j,k}^1 z_{j,k}^2 + \frac{\sqrt{2}}{2} \frac{j!k!}{(\gamma + j + k) \cdots (\gamma + 1) \cdot \gamma} \left( \frac{1}{\| \phi_{j,k-1}\|^2_f} \phi_{j,k-1}^2 + \frac{1}{\| \phi_{j-1,k}\|^2_f} \phi_{j-1,k}^2 \right). \]

Using the inductive hypothesis about the norms and simplifying, we obtain

\[ (2.10) \quad \phi_{j,k}(z_1, z_2) = z_{j,k}^1 z_{j,k}^2 + \frac{\sqrt{2}}{2} \frac{1}{\gamma + j + k} (k \phi_{j,k-1} + j \phi_{j-1,k}). \]

This recursive formula can now be used to recover individual coefficients \( \hat{\phi}_{j,k}(m,n) \) using the coefficients \( \hat{\phi}_{j,k-1}(m,n) \) and \( \hat{\phi}_{j-1,k}(m,n) \). We know that \( \hat{\phi}_{j,k}(j,k) = 1 \), and in the case where \( m = j \) (or, similarly, where \( n = k \)) we have \( \hat{\phi}_{j-1,k}(j,n) = 0 \) (similarly, \( \hat{\phi}_{j,k-1}(m,k) = 0 \)). Let us first consider the case where \( m = j \) and \( n = 0, 1, \ldots, k - 1 \), noting that the case where \( n = k \) and \( m = 0, 1, \ldots, j - 1 \) proceeds similarly:

\[ \hat{\phi}_{j,k}(j,n) = \frac{\sqrt{2}}{2} \frac{1}{\gamma + j + k} \left( k \hat{\phi}_{j,k-1}(j,n) + j \hat{\phi}_{j-1,k}(j,n) \right) \]

\[ = \frac{\sqrt{2}}{2} \frac{1}{\gamma + j + k} \left( \frac{\sqrt{2}}{2} \right)^{j+k-1-j-n} \frac{(\gamma + j + n) \cdots (\gamma + 1) \cdot \gamma}{(\gamma + j + k - 1) \cdots (\gamma + 1) \cdot \gamma} \]

\[ \cdot \left( k \frac{j!k}{(j,n)!} \frac{1}{(j-1,n)!} \right) \]

\[ = \frac{\sqrt{2}}{2} \frac{\sqrt{2}^{k-n} (\gamma + j + n) \cdots (\gamma + 1) \cdot \gamma}{(\gamma + j + k) \cdots (\gamma + 1) \cdot \gamma} \frac{k!}{(k-1-n)!} \]

\[ = \frac{\sqrt{2}}{2} \frac{\sqrt{2}^{k-n} (\gamma + j + n) \cdots (\gamma + 1) \cdot \gamma}{(\gamma + j + k) \cdots (\gamma + 1) \cdot \gamma} \frac{k!}{(k-1-n)!} \frac{1}{n! (k-1-n)!} \]

and this is what is obtained from substituting \( m = j \) in (2.2).
Now we consider the case where \( n < k \) and \( m < j \):

\[
\hat{\phi}_{j,k}(m,n) = \frac{\sqrt{2}}{2} \frac{1}{\gamma + j + k} \left( k\hat{\phi}_{j,k-1}(m,n) + j\hat{\phi}_{j-1,k}(m,n) \right)
\]

\[
= \frac{\sqrt{2}}{2} \frac{1}{\gamma + j + k} \left( \sqrt{2} \right)^{j+k-1-m-n} \frac{(\gamma + m + n) \cdots (\gamma + 1) \cdot \gamma}{(\gamma + j + k - 1) \cdots (\gamma + 1) \cdot \gamma} \\
\quad \cdot \left( k \left( \frac{j! (k-1)! (j + k - 1 - m - n)!}{m! n! (k-1 - n)!} \right) \\
\quad \quad + j \left( \frac{(j-1)! k! (j - 1 + k - m - n)!}{m! n! (j - 1 - m)! (k - n)!} \right) \right)
\]

\[
= \frac{\sqrt{2}}{2} \left( \frac{j+k-m-n}{2} \right) \frac{(\gamma + m + n) \cdots (\gamma + 1) \cdot \gamma}{(\gamma + j + k) \cdots (\gamma + 1) \cdot \gamma} \cdot \frac{j! k! (j + k - m - n)!}{m! n! (j - m)! (k - n)!}
\]

as needed. All that remains is to establish \([2,3]\). We use the recursive form \([3]\) and expand the inner product:

\[
\langle \phi_{j,k}, \phi_{j,k} \rangle_f = \left\langle z_1^j z_2^k, z_1^j z_2^k \right\rangle_f + \frac{\sqrt{2}}{2} \frac{k}{\gamma + j + k} \left\langle z_1^j z_2^k, \phi_{j,k-1} \right\rangle_f
\]

\[
+ \frac{\sqrt{2}}{2} \frac{j}{\gamma + j + k} \left\langle z_1^j z_2^k, \phi_{j-1,k} \right\rangle_f + \frac{\sqrt{2}}{2} \frac{k}{\gamma + j + k} \left\langle \phi_{j-1,k}, z_1^j z_2^k \right\rangle_f
\]

\[
+ \frac{\sqrt{2}}{2} \frac{j}{\gamma + j + k} \left\langle \phi_{j,k-1}, z_1^j z_2^k \right\rangle_f + \frac{1}{2} \frac{k^2}{(\gamma + j + k)^2} \|\phi_{j,k-1}\|_f^2
\]

\[
+ \frac{1}{2} \frac{k}{(\gamma + j + k)^2} \left\langle \phi_{j,k-1}, \phi_{j-1,k} \right\rangle_f + \frac{1}{2} \frac{j^2}{(\gamma + j + k)^2} \|\phi_{j-1,k}\|_f^2
\]

\[
+ \frac{1}{2} \frac{k}{(\gamma + j + k)^2} \left\langle \phi_{j-1,k}, \phi_{j-1,k} \right\rangle_f + \frac{1}{2} \frac{j}{(\gamma + j + k)^2} \|\phi_{j-1,k}\|_f^2.
\]
Substituting the inductive values of the norms, (2.5), (2.6), and recalling that the \( \phi \) are orthogonal, we obtain
\[
\langle \phi_{j,k}, \phi_{j,k} \rangle_f = \frac{j!k!}{(\gamma + j + k - 1) \cdots (\gamma + 1) \cdot \gamma} + \frac{1}{2(\gamma + j + k) \cdots (\gamma + 1) \cdot \gamma} j!(k+1)! + \frac{\sqrt{2}}{2(\gamma + j + k)^2} \left( -\frac{\sqrt{2} j!k!}{(\gamma + j + k - 1) \cdots (\gamma + 1) \cdot \gamma} \right)
\]
and simplifying yields
\[
\langle \phi_{j,k}, \phi_{j,k} \rangle_f = \frac{j!k!}{(\gamma + j + k - 1) \cdots (\gamma + 1) \cdot \gamma} + \frac{j!k!}{(\gamma + j + k) \cdots (\gamma + 1) \cdot \gamma} \left( \frac{j + 1}{2} + \frac{k + 1}{2} \right) + \frac{j!k!}{(\gamma + j + k) \cdots (\gamma + 1) \cdot \gamma} \left( \frac{j + 1}{2} + \frac{k + 1}{2} \right) \cdot \left( \frac{\gamma + j + k}{\gamma + j + k + 1} \cdot \frac{\gamma}{\gamma + j + k} \right).
\]

\textbf{Corollary 3.} The orthogonal polynomials given in Theorem 2 can be written recursively as
\[
\phi_{j,k} = z^j w^k + \frac{\sqrt{2}}{2} \frac{1}{\gamma + j + k} (k \phi_{j,k-1} + j \phi_{j-1,k}).
\]

3. A FAMILY OF OPTIMAL APPROXIMANTS

Making use of the formula (1.3), we obtain information about optimal approximants to \( 1/(1 - 1/\sqrt{2} (z_1 + z_2)) \). We again set \( \psi_{j,k} = \phi_{j,k}/\|\phi_{j,k}\|_{\gamma,f} \).

\textbf{Lemma 4.} Let \( \gamma > 0 \) be fixed. Then for \( j, k \in \mathbb{N} \),
\[
\langle 1, f \psi_{j,k} \rangle_{\gamma} \psi_{j,k} = \frac{\hat{\phi}_{j,k}(0,0)}{\|\phi_{j,k}\|^2} \phi_{j,k} = \left( \frac{\sqrt{2}}{2} \right)^{j+k} \frac{j!k!}{j!k! \gamma + j + k + 1} \hat{\phi}_{j,k}.
\]

\textbf{Proof.} From the power series expression for the norm in \( \mathcal{H}_\gamma \), we have \( \langle 1, f \psi_{j,k} \rangle_{\gamma} = \langle f \psi_{j,k} \rangle(0) = \tilde{\psi}_{j,k}(0,0) = \hat{\psi}_{j,k}(0,0) \), and by definition, \( \hat{\psi}_{j,k}(0,0) = \hat{\phi}_{j,k}(0,0)/\|\phi_{j,k}\|_{\gamma,f} \) which is real by (2.2).
It remains to compute
\[ \phi_{j,k}(0, 0) = \left( \frac{\sqrt{2}}{2} \right)^{j+k} \frac{\gamma}{(\gamma)^{j+k+1}} (j+k)! \]
and, simplifying, we obtain
\[ \frac{\phi_{j,k}(0, 0)}{\|\phi_{j,k}\|_2^2} = \left( \frac{\sqrt{2}}{2} \right)^{j+k} \frac{(j+k)! \gamma}{j!k! \gamma + j + k + 1}. \]

\[ \square \]

Setting \( \Phi_{j,k} = \sum_{n=0}^{j} \sum_{m=0}^{k} \hat{\Phi}_{j,k}(m,n) z_1^m z_2^n \)
where
(3.1) \( \hat{\Phi}_{j,k}(m,n) = \left( \frac{\sqrt{2}}{2} \right)^{2(j+k)-m-n} \gamma^{(j+k)!} \gamma_{m+n+1}^{(j+k-m-n)!} \gamma_{j+k+2}^{(j-m)! (k-n)!} \)
a representation formula for optimal approximants follows from Lemma 4.

**Theorem 5.** For \( \gamma > 0 \) fixed, we have
\[ p_n^*(z_1, z_2) = \sum_{(j,k) \leq (n_1, n_2)} \Phi_{j,k}(z_1, z_2) \]
where \((n_1, n_2)\) is the bidegree of the polynomial \( p_n^* \).

Explicitly, then,
\[ p_0^* = \Phi_{0,0}, \quad p_1^* = \Phi_{0,0} + \Phi_{1,0}, \quad p_2^* = \Phi_{0,0} + \Phi_{1,0} + \Phi_{0,1}, \]
\[ p_3^* = \Phi_{0,0} + \Phi_{1,0} + \Phi_{0,1} + \Phi_{2,0} \quad p_4^* = \Phi_{0,0} + \Phi_{1,0} + \Phi_{0,1} + \Phi_{2,0} + \Phi_{1,1}, \]
and so on. Some \( p_n^* \)'s for \( \gamma = 1 \) (the Drury-Arveson space) are written out in [12, Section 6.1]. The first few optimal approximants for the Hardy space \( H^2(\mathbb{B}^2) \) (\( \gamma = 2 \)) are as follows:
\[ p_0^* = \frac{2}{3}, \quad p_1^* = \frac{3}{4} + \frac{1}{4} \sqrt{2} z_1, \quad p_2^* = \frac{5}{6} + \frac{\sqrt{2}}{4} (z_1 + z_2), \quad p_3^* = \frac{17}{20} + \frac{3\sqrt{2}}{10} z_1 + \frac{\sqrt{2}}{4} + \frac{1}{5} z_2, \]
\[ p_4^* = \frac{53}{60} + \frac{7\sqrt{2}}{10} z_1 + \frac{3\sqrt{2}}{10} + \frac{1}{5} z_1 + \frac{2}{5} z_2, \]
while the first optimal approximants in the Bergman space \( A^2(\mathbb{B}^2) \) (\( \gamma = 3 \)) have the form
\[ p_0^* = \frac{3}{4}, \quad p_1^* = \frac{33}{40} + \frac{3\sqrt{2}}{10} z_1, \quad p_2^* = \frac{9}{10} + \frac{3\sqrt{2}}{10} (z_1 + z_2), \quad p_3^* = \frac{73}{80} + \frac{7\sqrt{2}}{20} z_1 + \frac{3\sqrt{2}}{10} z_2 + \frac{1}{4} z_2, \]
\[ p_4^* = \frac{15}{16} + \frac{2\sqrt{2}}{5} z_1 + \frac{7\sqrt{2}}{20} z_2 + \frac{1}{4} z_2 + \frac{1}{2} z_1 z_2. \]
The symmetric form of \( p_2^* \) above is explained in [12, Section 6].
4. An application

Our results can be applied to study the cyclicity properties of the function $f = 1 - \frac{1}{\sqrt{2}}(z_1 + z_2)$. Recall that $f$ is said to be cyclic in $\mathcal{H}_\gamma$ if the closure of the invariant subspace $\text{span}\{z_1^j z_2^k f : j, k \in \mathbb{N}\}$ equals $\mathcal{H}_\gamma$.

Define the optimal distance $\nu_n^2(f, \mathcal{H}_\gamma) = \|p_n^* f - 1\|_{\mathcal{H}_\gamma}$: then $f$ is cyclic if and only if $\nu_n(f, \mathcal{H}_\gamma) \to 0$ as $n \to \infty$. Combining [3, Corollary 5.3] with our explicit formulas, we obtain the following.

Corollary 6. We have

$$\nu_n^2(f, \mathcal{H}_\gamma) = 1 - \gamma^2 \sum_{(j,k) < (n_1,n_2)} 2^{-(j+k)} \frac{(j+k)!}{(\gamma)_{j+k+2}} \left( \begin{array}{c} j+k \\ j \end{array} \right),$$

where $(n_1,n_2)$ is the bidegree of $p_n^*$.

The function $f$ was already known to be cyclic in all $\mathcal{H}_\gamma$, but the above gives a precise description of how quickly the finite-dimensional subspaces $f \cdot \mathcal{P}_n$ fill up $\mathcal{H}_\gamma$. (The trick used to prove [12, Proposition 23] combined with [8, Proposition 3.10] applied to the weight sequence $\omega(k) = k!/\gamma_k \approx k^{\gamma-1}$ shows that the optimal distances have power law decay with exponent $-\gamma$.)

5. Closing remarks

As was highlighted in the course of the proof of Theorem 2, the cancellation in (2.9) simplifies the structure of the orthogonal polynomials, giving rise to a relatively simple recursive relation that in turn allows us to write down an explicit formula for their coefficients; this phenomenon of course reflects the fact that the target function $f = 1 - \frac{1}{\sqrt{2}}(z_1 + z_2)$ is well-adapted to the structure of $\mathcal{H}_\gamma$ (viz. also [12, Proposition 23]).

In [12], optimal approximants to $1/f$ for the similar function $f = 1 - \frac{1}{2}(z_1 + z_2)$ were examined for the family of Dirichlet-type spaces $\mathcal{D}_\alpha$ on the bidisk $\mathbb{D}^2 = \{ (z_1, z_2) \in \mathbb{C}^2 : |z_1| < 1, |z_2| < 1 \}$, as were the corresponding orthogonal polynomials. While an analog of Lemma 1 holds in that setting, cancellation fails for the orthogonal polynomials. Indeed, as is pointed out in [12, Section 6], coefficients appearing in the orthogonal polynomials and optimal approximants in $\mathcal{D}_\alpha$ in the bidisk exhibit sign changes and other complications, suggesting that obtaining a closed formula as well as precise estimates on optimal distances might be a harder problem than for the ball.

Returning to $\mathbb{B}^2$, we note that an analog of Lemma 1 for the target function $g = \left(1 - \frac{1}{\sqrt{2}}(z_1 + z_2)\right)^2$, and indeed for other powers of $f$, is readily obtained. One can then proceed as we have done here in order to analyze orthogonal polynomials associated with the weight $g$. The computations quickly become more involved, but in principle one could attempt to obtain a recursive relation analogous to that in Corollary 6, and then extract a closed formula for coefficients of orthogonal polynomials. As a sample, we invite the reader to verify that for $\gamma = 1$ (the Drury-Arveson space), the
orthogonal polynomials for the weight $g = f^2$ satisfy the relation

$$
\phi_{j,k} = z_1^j z_2^k + \frac{\sqrt{2}}{j + k + 2} (k\phi_{j,k-1} + j\phi_{j-1,k}) - \frac{1}{(j + k + 1)(j + k + 2)} \left( \frac{k(k - 1)}{2} \phi_{j,k-2} + jk\phi_{j-1,k-1} + \frac{j(j - 1)}{2} \phi_{j-2,k} \right).
$$

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