Metrics with conical singularities on the sphere and sharp extensions of the theorems of Landau and Schottky

Daniela Kraus, Oliver Roth and Toshiyuki Sugawa

Abstract. An explicit formula for the generalized hyperbolic metric on the thrice--punctured sphere \( \mathbb{P}\{z_1, z_2, z_3\} \) with singularities of order \( \alpha_j \leq 1 \) at \( z_j \) is obtained in all possible cases \( \alpha_1 + \alpha_2 + \alpha_3 > 2 \). The existence and uniqueness of such a metric was proved long time ago by Picard [26] and Heins [11], while explicit formulas for the cases \( \alpha_1 = \alpha_2 = 1 \) were given earlier by Agard [2] and recently by Anderson, Sugawa, Vamanurthy and Vuorinen [5]. We also establish precise and explicit lower bounds for the generalized hyperbolic metric. This extends work of Hempel [12] and Minda [24]. As applications, sharp versions of Landau– and Schottky–type theorems for meromorphic functions are obtained.

1 Introduction

Let \( \mathbb{P} \) denote the Riemann sphere endowed with its canonical complex structure and let \( \Omega \subseteq \mathbb{P} \) be a subdomain. We say a conformal Riemannian metric \( \lambda(z) \, |dz| \) on \( \Omega \setminus \{p\} \) has a singularity of order \( \alpha \leq 1 \) at the point \( p \in \Omega \), if, in local coordinates,

\[
\log \lambda(z) = \begin{cases} 
-\alpha \log |z - p| + O(1) & \text{if } \alpha < 1 \\
- \log |z - p| - \log (- \log |z - p|) + O(1) & \text{if } \alpha = 1
\end{cases}
\]

as \( z \to p \). Geometrically, the singular surface \( (\Omega, \lambda(z) \, |dz|) \) looks like an ice–cream cone at \( p \) if \( \alpha < 1 \). If \( \alpha = 1 \), then \( (\Omega, \lambda(z) \, |dz|) \) has a cusp at \( p \). We therefore call \( p \) a conical singularity or corner of order \( \alpha \) if \( \alpha < 1 \) and a cusp if \( \alpha = 1 \). It is also customary to say that a conformal Riemannian metric with a conical singularity \( p \) of order \( \alpha < 1 \) has the angle \( 2\pi(1 - \alpha) \) at the point \( p \), see [5]. Conical singularities are typical for conformal metrics. For instance, if the curvature of \( \lambda(z) \, |dz| \) is bounded below and above by negative constants, then \( \lambda(z) \, |dz| \) only has corners or cusps as point singularities (see [11, 22, 15]). For nonnegatively curved metrics with finite energy only corners occur (see [28, 16]).

It is well–known (see [26, 11]) that for \( n \geq 3 \) distinct points \( z_1, \ldots, z_n \in \mathbb{P} \) and real parameters \( \alpha_1, \ldots, \alpha_n \in (-\infty, 1] \) there exists a conformal metric on the \( n \)–punctured sphere \( \mathbb{P}\{z_1, \ldots, z_n\} \) with constant curvature \(-1\) and singularities of order \( \alpha_j \) at \( z_j \) if and only if

\[
\sum_{j=1}^{n} \alpha_j > 2.
\]

In this case, this metric is uniquely determined and will be called generalized hyperbolic metric with singularities of order \( \alpha_j \) at \( z_j \).

2000 Mathematics Subject Classification: Primary 30F45

D.K. and O.R. were supported by a DFG grant (RO 3462/3–1). T.S. was supported in part by JSPS Grant–in–Aid for Scientific Research (B), 17340039 and for Exploratory Research, 19654027. To appear: Math. Z.
We are primarily interested in the case of the thrice–punctured sphere \(\mathbb{P}\{z_1, z_2, z_3\}\). Note that in this case

\[
0 < \alpha_1 \leq 1, \quad 0 < \alpha_2 \leq 1, \quad 0 < \alpha_3 \leq 1, \quad \alpha_1 + \alpha_2 + \alpha_3 > 2.
\] (1.2)

Using a Möbius transformation, which sends \(z_1\) to 0, \(z_2\) to 1 and \(z_3\) to \(\infty\), we may henceforth assume that \(z_1 = 0\), \(z_2 = 1\) and \(z_3 = \infty\) and shall denote by \(\lambda_{\alpha_1, \alpha_2, \alpha_3}(z) |dz|\) the generalized hyperbolic metric with conical singularities of order \(\alpha_1\), \(\alpha_2\) and \(\alpha_3\) at \(z_1 = 0\), \(z_2 = 1\) and \(z_3 = \infty\). In this situation, the Riemannian metric \(\lambda_{\alpha_1, \alpha_2, \alpha_3}(z) |dz|\) on \(\mathbb{P}\{0, 1, \infty\}\) can be described in terms of a single density function \(\lambda_{\alpha_1, \alpha_2, \alpha_3}\) defined on the twice–punctured plane \(\mathbb{C}'' := \mathbb{C}\{0, 1\}\) (see Section 3 below for details). We call \(\lambda_{\alpha_1, \alpha_2, \alpha_3}\) the generalized hyperbolic density of order \((\alpha_1, \alpha_2, \alpha_3)\) on \(\mathbb{C}''\).

Explicit and very useful formulas for \(\lambda_{1, 1, \alpha_3}(z)\) have been obtained by Agard [2] for \(\alpha_3 = 1\) and recently by Anderson, Sugawa, Vamanamurthy and Vuorinen [5] for \(\alpha_3 \in (0, 1]\). Hempel [12] (see also Minda [24]) proved a sharp, explicit and easy–to–use lower bound for the standard hyperbolic density \(\lambda_{1,1,1}(z)\). In combination with Agard’s formula for \(\lambda_{1,1,1}(z)\) this has led to precise bounds in the classical theorems of Landau and Schottky [1] for analytic functions in the open unit disk \(\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}\) omitting the values 0 and 1 (see Ahlfors [3], Hayman [9], Hempel [12, 13], Jenkins [14] and e.g. Li & Qi [18]).

In this note, we extend the above results to the generalized hyperbolic metric and provide sharp extensions of theorems of Landau and Schottky type for meromorphic functions not necessarily omitting the values 0, 1 and \(\infty\).

The paper is organized in the following way. The main results are described and discussed in Section 2. Section 3 contains a quick review of the necessary background material about conformal metrics, while Section 4 is devoted to the proofs of the results. We start in [2.1] with Theorem [2.1] which provides an explicit formula for the generalized hyperbolic density \(\lambda_{\alpha_1, \alpha_2, \alpha_3}(z)\) in all possible cases (see [1.2]). This generalizes the results of Agard [2] and Anderson, Sugawa, Vamanamurthy and Vuorinen [5], which are easily seen to be special cases of Theorem [2.1] to the most general situation. Our method of proof differs from that in [2.6] as we base our proof on Liouville’s representation formula (Theorem 3.2) for constantly curved conformal Riemannian metrics. The use of Liouville’s theorem will also facilitate proving sharpness of (most of) our results. In [2.2] we give a sharp and explicit lower bound for the generalized hyperbolic metric, see Theorem [2.2]. This extends the earlier work of Hempel [12] and Minda [24], which deals with the special case of the standard hyperbolic metric, to the generalized hyperbolic metric. Our method is based on a new device, which we call the Gluing lemma (see Lemma 1.9) and which allows a rather quick proof of Theorem [2.2].

---

\footnote{We refer the reader to the monographs [4] and [10] for an introduction to Landau’s and Schottky’s theorem and to the recent paper [3] for connections of Schottky’s theorem with quasiconformal maps and modular equations.}
These new information about the generalized hyperbolic metric, which are perhaps also interesting in their own right, are then applied to study value distribution properties of functions meromorphic in the unit disk. For that purpose it is sufficient to consider the cases $\alpha_1 = 1 - 1/j$, $\alpha_2 = 1 - 1/k$, $\alpha_3 = 1 - 1/l$, where $j, k, l \geq 2$ are integers (or $= \infty$) such that according to (1.2)

$$\frac{1}{j} + \frac{1}{k} + \frac{1}{l} < 1.$$ 

In this way, we are led to sharp extensions of the theorems of Landau and Schottky for meromorphic functions belonging to the classes

$$\mathcal{M}_{j,k,l} := \{ f \text{ meromorphic in } \mathbb{D} \text{ such that (i) all zeros of } f \text{ have order } \geq j, \text{ (ii) all zeros of } f - 1 \text{ have order } \geq k \text{ and (iii) all poles of } f \text{ have order } \geq l \}. \quad (1.3)$$

These results, which are discussed in Paragraph 2.3 and proved in Section 4, generalize the results in [3, 9, 12, 13, 14, 18], which deal with the particular case of analytic functions in $\mathbb{D}$ omitting the values 0 and 1, i.e., the class $\mathcal{M}_{\infty, \infty, \infty}$, to the much wider classes $\mathcal{M}_{j,k,l}$.

## 2 Results

### 2.1 Explicit formulas

The explicit formula for the generalized hyperbolic density $\lambda_{\alpha_1, \alpha_2, \alpha_3}(z)$, which will be stated momentarily, is necessarily a bit technical, so we first need to introduce some notation. Let $\alpha_1, \alpha_2, \alpha_3$ be real parameters satisfying condition (1.2). We define

$$\alpha := \frac{\alpha_1 + \alpha_2 - \alpha_3}{2}, \quad \beta := \frac{\alpha_1 + \alpha_2 + \alpha_3 - 2}{2}, \quad \gamma := \alpha_1. \quad (2.1)$$

Then $0 < \beta \leq \alpha$ and $\alpha + \beta \leq \gamma \leq 1$. We also consider the hypergeometric functions

$$\varphi_1(z) := F(\alpha, \beta, \gamma; z), \quad \varphi_2(z) := F(\alpha, \beta, \alpha + \beta - \gamma + 1; 1 - z).$$

Note that $\varphi_1$ is analytic in $\mathbb{C}\setminus[1, +\infty)$ and $\varphi_2$ is analytic in $\mathbb{C}\setminus[-\infty, 0]$.

**Theorem 2.1 (Corners at $z = 0$ and $z = 1$)**

Let $0 < \alpha_1, \alpha_2 < 1$ and $0 < \alpha_3 \leq 1$ such that $\alpha_1 + \alpha_2 + \alpha_3 > 2$. Then

$$\lambda_{\alpha_1, \alpha_2, \alpha_3}(z) = \frac{1}{|z|^{\alpha_1}|1 - z|^{\alpha_2} K_1|\varphi_1(z)|^2 + K_2|\varphi_2(z)|^2 + 2 \text{ Re}(\varphi_1(z)\varphi_2(\overline{z}))},$$

where

$$K_1 := \frac{\Gamma(\gamma - \alpha)\Gamma(\gamma - \beta)}{\Gamma(\gamma)\Gamma(\gamma - \alpha - \beta)}, \quad K_2 := \frac{\Gamma(\alpha + 1 - \gamma)\Gamma(\beta + 1 - \gamma)}{\Gamma(1 - \gamma)\Gamma(\alpha + \beta + 1 - \gamma)},$$

$$K_3 := \sqrt{\frac{\sin(\pi \alpha)\sin(\pi \beta)}{\sin(\pi(\gamma - \alpha))\sin(\pi(\gamma - \beta))}} \frac{\Gamma(\alpha + \beta + 1 - \gamma)\Gamma(\gamma)}{\Gamma(\alpha)\Gamma(\beta)}, \quad (2.2)$$

and $\alpha, \beta, \gamma$ are defined as in (2.1).

The previously known formulas for $\lambda_{1,1,1}(z)$ (see [2]) and for $\lambda_{1,1,\alpha_3}(z)$, $0 < \alpha_3 \leq 1$, (see [5]) can easily be obtained from Theorem 2.1 by letting $\alpha_j \to 1$ for $j = 1, 2$. We omit the details.
2.2 Sharp lower bounds

The aim of this section is to provide a sharp lower bound for $\lambda_{\alpha_1,\alpha_2,\alpha_3}(z)$. This generalizes the work of Hempel [12] and Minda [24], who give a precise lower bound for $\lambda_{1,1,1}(z)$, to the general case \([12].\)

**Theorem 2.2 (A sharp explicit lower bound)**

Let $\alpha_1, \alpha_2, \alpha_3$ be real parameters satisfying condition \([12]\) and let

$$C_1 := \frac{1}{1 - \alpha_1} \arcsin \left( \frac{1 - \alpha_1}{\lambda_{\alpha_1,\alpha_2,\alpha_3}(-1)} \right), \quad C_3 := \frac{1}{1 - \alpha_3} \arcsinh \left( \frac{1 - \alpha_3}{\lambda_{\alpha_1,\alpha_2,\alpha_3}(-1)} \right). \quad (2.3)$$

Then

$$\lambda_{\alpha_1,\alpha_2,\alpha_3}(z) \geq \begin{cases} 
\frac{1 - \alpha_1}{|z| \sinh \left[ (1 - \alpha_1)(C_1 - \log |z|) \right]} & \text{if } |z| \leq 1, \ z \neq 0,1 \\
\frac{1 - \alpha_3}{|z| \sinh \left[ (1 - \alpha_3)(C_3 + \log |z|) \right]} & \text{if } |z| > 1.
\end{cases} \quad (2.4)$$

Equality holds if and only if $z = -1$.

**Remark 2.3 (Limit cases $\alpha_1 \neq 1$ and $\alpha_2 \neq 1$)**

If $\alpha_1 = 1$ and/or $\alpha_3 = 1$, then the formulas for $C_1$ and $C_3$ as well as the lower bounds for $\lambda_{\alpha_1,\alpha_2,\alpha_3}$ are to be understood in the limit sense $\lim_{\alpha_1 \to 1-} \text{ resp. } \lim_{\alpha_3 \to 1-}.$

**Remark 2.4 (Computation of $\lambda_{\alpha_1,\alpha_2,\alpha_3}(-1)$)**

The sharp lower bound \((2.3)\) for the generalized hyperbolic density requires the computation of the particular value $\lambda_{\alpha_1,\alpha_2,\alpha_3}(-1)$. Using the Möbius transformation $T(z) = z/(z - 1)$, which fixes $z = 0$ and interchanges $z = 1$ with $z = \infty$, and the easily verified fact $\lambda_{\alpha_1,\alpha_2,\alpha_3}(z) = \lambda_{\alpha_1,\alpha_3,\alpha_2}(T(z)) |T'(z)|$, we get

$$\lambda_{\alpha_1,\alpha_2,\alpha_3}(-1) = \frac{\lambda_{\alpha_1,\alpha_3,\alpha_2}(1/2)}{4}.$$ 

In view of Theorem \([12]\) the computation of $\lambda_{\alpha_1,\alpha_2,\alpha_3}(-1)$ is thereby essentially reduced to the evaluation of two hypergeometric functions at the point $z = 1/2$, which can effectively be achieved using the rapidly converging hypergeometric series.

We wish to single out the special case $\alpha_3 = \alpha_1$ of Theorem \([2.2]\) because then not only $C_1 = C_3$ holds, but also the value of $\lambda_{\alpha_1,\alpha_2,\alpha_1}(-1)$ can explicitly be computed in terms of the Gamma function.

**Corollary 2.5 (The case $\alpha_3 = \alpha_1$)**

Let $\alpha_1, \alpha_2 \in (0,1]$ such that $2\alpha_1 + \alpha_2 > 2$. Then

$$\lambda_{\alpha_1,\alpha_2,\alpha_1}(-1) = 2 \sqrt{\frac{\tan \left( \frac{\pi}{2} \left( \frac{\alpha_1}{2} + \alpha_1 \right) \right)}{\tan \left( \frac{\pi}{2} \left( \frac{\alpha_1}{2} - \alpha_1 \right) \right)}} \frac{\Gamma \left( \frac{\alpha_1}{2} - \frac{\alpha_2}{2} + \frac{1}{2} \right) \Gamma \left( \frac{\alpha_1}{2} + \frac{\alpha_2}{2} \right)}{\Gamma \left( \frac{\alpha_1}{2} + \frac{\alpha_2}{2} \right) \Gamma \left( \frac{\alpha_1}{2} - \frac{\alpha_2}{2} - \frac{1}{2} \right)} \quad (2.5)$$

and

$$\lambda_{\alpha_1,\alpha_2,\alpha_1}(z) \geq \frac{1 - \alpha_1}{|z| \sinh \left[ (1 - \alpha_1)(C_1 + \log |z|) \right]}$$

for all $z \in \mathbb{C}$ with equality if and only if $z = -1$. Here, $C_1$ is given by \((2.3)\) with $\alpha_3 = \alpha_1$. 


For $\alpha_1 = \alpha_2 = \alpha_3 = 1$, Corollary 2.5 further reduces to the sharp bound
\[
\lambda_{1,1,1}(z) \geq \frac{1}{|z| \left(1/\lambda_{1,1,1}(-1) + \log |z|\right)} \quad \text{with} \quad \lambda_{1,1,1}(-1) = \frac{4\pi^2}{\Gamma(1/4)^4} \approx 0.228473,
\]
which was first proved by Hempel [12] (see also Minda [24]).

### 2.3 Applications: Theorems of Landau and Schottky type

Hempel [12, 13] (see also Jenkins [14]), Minda [24] and Li & Qi [18] proved sharp Landau and Schottky type theorems for functions in $M_{\infty,\infty,\infty}$, i.e., analytic functions in $\mathbb{D}$ omitting 0 and 1, with the help of the standard hyperbolic metric $\lambda_{1,1,1}(z) |dz|$. Using the explicit formula and the sharp lower bounds for the generalized hyperbolic metric obtained in the previous sections we now generalize these results by proving sharp versions of Landau and Schottky type theorems for functions belonging to the much larger classes $M_{j,k,l}$ (see (1.3)). Here $j, k, l \geq 2$ are integers (or $= +\infty$) such that
\[
\frac{1}{j} + \frac{1}{k} + \frac{1}{l} < 1
\]
(with the convention $1/\infty := 0$). The extremal functions we shall encounter are obtained in the following way. For $j, k, l$ as above, it is well–known [3] Vol. I, p. 72] that there exists a hyperbolic triangle $\Delta$ in the unit disk $\mathbb{D}$ with interior angles $\pi/j$, $\pi/k$ and $\pi/l$. The triangle is moreover uniquely determined up to a motion of the hyperbolic plane. The conformal map from $\Delta$ onto the upper halfplane $\mathbb{H} = \{w \in \mathbb{C} : \text{Im} w > 0\}$, which maps the vertex with angle $\pi/j$ to 0, the vertex with angle $\pi/k$ to 1 and the vertex with angle $\pi/l$ to $\infty$ is uniquely determined. By Schwarz reflection, this conformal map can be analytically continued to a meromorphic function $f$ on $\mathbb{D}$ such that all zeros of $f$ have exact order $j$, all zeros of $f - 1$ have exact order $k$ and all poles of $f$ have exact order $l$, i.e., $f \in M_{j,k,l}$. We call every such meromorphic function a triangle map of order $(j,k,l)$. Note that a triangle map of order $(\infty,\infty,\infty)$ is a universal covering from $\mathbb{D}$ onto $\mathbb{C}^u$.

**Theorem 2.6 (Landau–type theorem)**

Let $j, k, l \geq 2$ be integers (or $= \infty$) such that $1/j + 1/k + 1/l < 1$ and let
\[
C_1 := j \arcsinh \left(\frac{1}{j \cdot \lambda_{1-1/j,1-1/k,1-1/l}(-1)}\right), \\
C_3 := l \arcsinh \left(\frac{1}{l \cdot \lambda_{1-1/j,1-1/k,1-1/l}(-1)}\right).
\]

Then for every $f \in M_{j,k,l}$ with $a_0 := f(0) \neq \infty$, we have for $a_1 := f'(0)$ the sharp estimate
\[
|a_1| \leq \begin{cases} 
2j |a_0| \sinh \left[\frac{C_1 + \log |a_0|}{j}\right] & \text{if } |a_0| \leq 1, \\
2l |a_0| \sinh \left[\frac{C_3 + \log |a_0|}{l}\right] & \text{if } |a_0| \geq 1.
\end{cases}
\]

Equality holds if and only if $f$ is a triangle map of order $(j,k,l)$ with $f(0) = -1$. 


Corollary 2.7
Let \( j, k \geq 2 \) be integers (or \( = \infty \)) such that \( 2/j + 1/k < 1 \). Then for every \( f \in \mathcal{M}_{j,k,j} \) with \( a_0 := f(0) \neq \infty \), we have for \( a_1 := f'(0) \) the sharp estimate

\[
|a_1| \leq 2 j |a_0| \sinh \left[ \frac{C_1}{j} \right].
\]

Here, \( C_1 \) is as in (2.6) with \( l = j \), where \( \lambda_{1-1/j,1-1/k,1-1/j}(-1) \) is given by (2.5) with \( \alpha_1 = 1 - 1/j, \alpha_2 = 1 - 1/k, \alpha_3 = 1 - 1/j \). Equality holds if and only if \( f \) is a triangle map of order \((j,k,j)\) with \( f(0) = -1 \).

If \( j = k = \infty \), then the corollary reduces to the well–known sharp version of Landau’s theorem due to Hempel [12],

\[
|a_1| \leq 2 |a_0| \left( \left| \log |a_0| \right| + L \right), \quad L = \frac{1}{\lambda_{1,1,1}(-1)} = \frac{1}{4\pi^2} \cdot \Gamma \left( \frac{1}{4} \right)^4,
\]

which holds for every analytic function \( f(z) = a_0 + a_1 z + \cdots \) in \( \mathbb{D} \) omitting 0 and 1. Equality occurs if and only if \( f \) is a universal covering from \( \mathbb{D} \) onto \( \mathbb{C}' \) with \( f(0) = -1 \).

Theorem 2.8 (Schottky–type theorem)
Let \( j, k, l \geq 2 \) be integers (or \( = \infty \)) such that \( 1/j + 1/k + 1/l < 1 \). Then for every \( f \in \mathcal{M}_{j,k,l} \) the sharp estimate

\[
\tanh \left( \frac{\tilde{C}_1 + \log^+ |f(z)|}{2l} \right) \leq \tanh \left( \frac{\tilde{C}_1 + \log^+ |f(0)|}{2l} \right) \frac{1 + |z|}{1 - |z|}, \quad z \in \mathbb{D},
\]

holds, where \( \log^+ x = \max \{ \log x, 0 \} \) for \( x > 0 \) and

\[
\tilde{C}_1 = l \arcsinh \left( \frac{1}{l \cdot \lambda_{1-1/j,1-1/k,1-1/l}(-1)} \right).
\]

In particular,

\[
\log |f(z)| \leq 2 l \arctanh \left[ \tanh \left( \frac{\tilde{C}_1 + \log^+ |f(0)|}{2l} \right) \frac{1 + |z|}{1 - |z|} \right] - \tilde{C}_1
\]

for all

\[
|z| < \exp \left( - \frac{\tilde{C}_1 + \log^+ |f(0)|}{l} \right).
\]

Remark 2.9
The estimate (2.7) is sharp in the following sense: if \( M > 0 \) is a constant such that

\[
\tanh \left( \frac{M + \log |f(z)|}{2l} \right) \leq \tanh \left( \frac{M + \log^+ |f(0)|}{2l} \right) \frac{1 + |z|}{1 - |z|}
\]

holds for all \( z \in \mathbb{D} \) and all meromorphic functions \( f \in \mathcal{M}_{j,k,l} \), then one can show \( M \geq \tilde{C}_1 \).

Remark 2.10
In the situation of Theorem 2.8, we see that if \( f(0) \neq \infty \), then

\[
f(z) \neq \infty \text{ for all } |z| < \exp \left( - \left( \tilde{C}_1 + \log^+ |f(0)| \right) /l \right) \).
\]
If there are no poles \((l = \infty)\) one gets a sharp Schottky–type result on the entire unit disk:

**Corollary 2.11**

Let \(j, k \geq 2\) be integers (or \(\infty\)) such that \(1/j + 1/k < 1\). Then for every \(f \in \mathcal{M}_{j,k,\infty}\),

\[
\log |f(z)| \leq \left[ C + \log^+ |f(0)| \right] \frac{1+|z|}{1-|z|} - C, \quad z \in \mathbb{D},
\]

where \(C = 1/\lambda_{1-1/j,1-1/k,1}(-1)\).

**Corollary 2.12**

Let \(k \geq 2\) be an integer (or \(\infty\)) and let

\[
L_k := \frac{1}{4\pi^2} \cdot \Gamma \left( \frac{1+1/k}{4} \right) \cdot \Gamma \left( \frac{1-1/k}{4} \right) \cdot \cos \left( \frac{\pi}{2k} \right).
\]

If \(f\) is analytic and zero-free in \(\mathbb{D}\) such that \(f(z) - 1\) has only zeros of order \(\geq k\), then

\[
\log |f(z)| \leq \left[ L_k + \log^+ |f(0)| \right] \frac{1+|z|}{1-|z|} - L_k, \quad z \in \mathbb{D}.
\]

A remark similar to Remark 2.9 applies to Corollary 2.11 as well as to Corollary 2.12. Thus Corollary 2.11 and Corollary 2.12 are in some sense best possible. The special case \(k = \infty\) of Corollary 2.12 is the recent result of Li and Qi [18].

### 3 Preliminaries

We first recall a number of facts about conformal pseudo–metrics. Some of the material is discussed in more detail in [11, 23].

If \(G\) is a domain in the complex plane \(\mathbb{C}\), then we can identify a conformal pseudo–metric \(\lambda(z) \, |dz|\) with its conformal density, that is the function \(\lambda : G \to [0, +\infty)\), which represents the pseudo–metric \(\lambda(z) \, |dz|\) in local coordinates when using the identity map as a chart. For instance, if \(\lambda_{\alpha_1, \alpha_2, \alpha_3}(z) \, |dz|\) is the generalized hyperbolic metric on \(\mathbb{P} \setminus \{0, 1, \infty\}\) of order \((\alpha_1, \alpha_2, \alpha_3)\), then the associated generalized hyperbolic density \(\lambda_{\alpha_1, \alpha_2, \alpha_3}\) is a positive function on \(\mathbb{C}^\prime = \mathbb{C} \setminus \{0, 1\}\).

We call an upper semicontinuous pseudo–metric \(\lambda(z) \, |dz|\) on \(G \subset \mathbb{C}\) an SK–metric if its (generalized) Gauss curvature \(\kappa_\lambda(z)\), defined by

\[
\kappa_\lambda(z) := -\liminf_{r \to 0} \frac{4}{r^2} \left( \frac{1}{2\pi} \int_0^{2\pi} \log \lambda(z + re^{it}) \, dt - \log \lambda(z) \right),
\]

is bounded above by \(-1\) at every \(z \in G\) with \(\lambda(z) > 0\). Note, if \(\lambda(z) \, |dz|\) is a regular conformal metric, i.e., \(\lambda\) is twice continuously differentiable and strictly positive on \(G\), then \(\kappa_\lambda(z) = -\Delta \log \lambda(z)/\lambda(z)^2\), where \(\Delta\) denotes the usual Laplace operator.

The Fundamental Theorem about SK–metrics is Ahlfors’ lemma [3, 11]. It says that the hyperbolic metric \(\lambda_D(z) \, |dz|\) on the unit disk \(\mathbb{D}\),

\[
\lambda_D(z) \, |dz| := \frac{2 \, |dz|}{1-|z|^2},
\]

\(^*\)Heins [11] introduced the concept of SK–metrics and established a theory of such metrics. Note that he used the upper bound \(-4\) in his definition of SK–metrics instead of \(-1\) as we do.
is the maximal SK–metric on $D$, i.e., $\mu(z) \leq \lambda_D(z)$ for all $z \in D$ and every SK–metric $\mu(z)\,dz$ on $D$. Actually, $\lambda_D(z)\,dz$ is the unique maximal SK–metric on $D$. This follows from the following result.

**Lemma 3.1 (Heins [11])**

Let $\mu(z)\,dz$ be an SK–metric on a domain $G \subseteq \mathbb{C}$ and $\lambda(z)\,dz$ a regular conformal metric on $G$ with constant curvature $-1$ such that $\mu \leq \lambda$. Then either $\mu < \lambda$ or $\mu \equiv \lambda$.

By definition, the generalized hyperbolic metric $\lambda_{\alpha_1,\alpha_2,\alpha_3}(z)\,dz$ is a regular conformal metric on $G$ with constant curvature $-1$. In general, conformal metrics with constant curvature play a distinctive role. This comes in part from the well–known and easily verified fact that the Schwarzian $S_\lambda$ of a regular conformal metric $\lambda(z)\,dz$ on a domain $G \subseteq \mathbb{C}$,

$$S_\lambda(z) := 2 \left[ \frac{\partial^2 \log \lambda}{\partial z^2}(z) - \left( \frac{\partial \log \lambda}{\partial z}(z) \right)^2 \right], \quad (3.1)$$

is a holomorphic function in $G$ if and only if $\lambda(z)\,dz$ has constant curvature there. The following classical fact tells us that locally every regular metric with constant curvature $-1$ comes from the hyperbolic metric $\lambda_D(z)\,dz$ on the unit disk $\mathbb{D}$:

**Theorem 3.2 (Liouville [19])**

Let $G \subseteq \mathbb{C}$ be a simply connected domain and $\lambda(z)\,dz$ a regular conformal metric on $G$ with constant curvature $-1$. Then the following are true.

(a) There exists a holomorphic function $\varphi : G \to \mathbb{D}$ such that

$$\lambda(z) = \frac{2 |\varphi'(z)|}{1 - |\varphi(z)|^2}, \quad z \in G. \quad (3.2)$$

The function $\varphi$ can be found among all solutions $\Psi$ to the Schwarzian differential equation

$$\left( \frac{\psi''(z)}{\psi'(z)} \right)' - \frac{1}{2} \left( \frac{\psi''(z)}{\psi'(z)} \right)^2 = S_\lambda(z). \quad (3.3)$$

(b) An analytic function $g : G \to \mathbb{D}$ satisfies

$$\lambda(z) = \frac{2 |g'(z)|}{1 - |g(z)|^2}, \quad z \in G,$$

if and only if $g = T \circ \varphi$, where $T$ is an automorphism of $\mathbb{D}$.

Our derivation of the explicit formula for $\lambda_{\alpha_1,\alpha_2,\alpha_3}$ in Theorem 2.1, which will be given in Section 4, depends in an essential way on Liouville’s theorem. Part (b) will also be used to show that the theorems of Landau and Schottky–type stated in 2.3 are best possible.

### 4 Proofs

#### 4.1 The explicit formula for the generalized hyperbolic metric

The proof of Theorem 2.1 is based on the following lemmas.
Lemma 4.1
Let \(0 < \alpha_1, \alpha_2 < 1\) and \(0 < \alpha_3 \leq 1\) such that \(\alpha_1 + \alpha_2 + \alpha_3 > 2\) and define \(\alpha, \beta, \gamma\) by (2.1). Then the following representation formulas are valid.

(a) In the slit disk \(D^- = \mathbb{D} \setminus (-1, 0]\) we have

\[
\lambda_{\alpha_1, \alpha_2, \alpha_3}(z) = \frac{2|\varphi'(z)|}{1 - |\varphi(z)|^2}, \quad z \in D^-
\]

with

\[
\varphi(z) = c_0 \frac{z^{1-\gamma}F(\alpha - \gamma + 1, \beta - \gamma + 1, 2 - \gamma; z)}{F(\alpha, \beta, \gamma; z)}
\]

for some constant \(c_0 > 0\).

(b) In the slit disk \(K_1^+(1) = \{z \in \mathbb{C} : |z - 1| < 1\} \setminus [1, 2)\) we have

\[
\lambda_{\alpha_1, \alpha_2, \alpha_3}(z) = \frac{2|g'(z)|}{1 - |g(z)|^2}, \quad z \in K_1^+(1),
\]

with

\[
g(z) = c_1 \frac{(1 - z)^{\gamma - \alpha - \beta}F(\gamma - \beta, \gamma - \alpha, \gamma - \alpha - \beta + 1; 1 - z)}{F(\alpha, \beta, \alpha + \beta - \gamma + 1; 1 - z)}
\]

for some constant \(c_1 > 0\).

Lemma 4.2
The constant \(c_0\) in Lemma 4.1 has the value

\[
\sqrt{\frac{\Gamma(1 - \alpha) \Gamma(1 - \beta) \Gamma(\alpha + 1 - \gamma) \Gamma(\beta + 1 - \gamma)}{\Gamma(\alpha) \Gamma(\beta) \Gamma(\gamma - \alpha) \Gamma(\gamma - \beta)}} \cdot \frac{\Gamma(\gamma)}{\Gamma(2 - \gamma)}.
\]

Remark 4.3
The proof of Lemma 4.1 will show that the functions \(\varphi\) and \(g\) in Lemma 4.1 can be analytically continued along any path in \(\mathbb{C}'\). Thus the representation formulas for \(\lambda_{\alpha_1, \alpha_2, \alpha_3}(z)\) in Lemma 4.1 clearly hold for any \(z \in \mathbb{C}'\).

Corollary 4.4
Let \(0 < \alpha_1, \alpha_2 < 1\) and \(0 < \alpha_3 \leq 1\) such that \(\alpha_1 + \alpha_2 + \alpha_3 > 2\) and define \(\alpha, \beta, \gamma\) by (2.1). Then

\[
\lambda_{\alpha_1, \alpha_2, \alpha_3}(z) = \frac{2c(1 - \alpha_1)}{|z|^{\alpha_1} |1 - z|^{\alpha_2} \left\{|F(\alpha, \beta, \gamma; z)|^2 - c^2 |1 - z|^{2-2\alpha_1} |F(\alpha - \gamma + 1, \beta - \gamma + 1, 2 - \gamma; z)|^2\right\}}
\]

with \(c\) given by (4.1).

In order to prove Lemma 4.1 we first need to recall some well-known facts. Let \(\lambda(z) |dz|\) be a regular conformal metric on a domain \(G \subseteq \mathbb{C}\) with constant curvature \(-1\). If an isolated boundary point \(z_j \in \mathbb{C}\) of \(G\) is a singularity of order \(\alpha_j\) of \(\lambda(z) |dz|\), then \(S_\lambda\) has a pole of order 2 at \(z_j\) and

\[
S_\lambda(z) = \frac{(2 - \alpha_j) \alpha_j}{2(z - z_j)^2} + \frac{c_j}{z - z_j} + O(1) \quad \text{as} \quad z \to z_j.
\]
See, for instance, [25, 15, 16]. Now let $z_j = \infty$ be an isolated singularity of order $\alpha_j$ of $\lambda(z)\,|dz|$. This means, by definition, that $\mu(z)\,|dz| = \lambda(1/z)\,|dz|/|z|^2$ has an isolated singularity of order $\alpha_j$ at $z = 0$. Hence $S_\lambda(z) = S_\mu(1/z)/z^4$ and we thus see that

$$\lim_{z \to -\infty} z^2 S_\lambda(z) = \lim_{z \to 0} z^2 S_\mu(z) = \frac{(2 - \alpha_j) \alpha_j}{2}.$$

This observation leads to the following lemma.

**Lemma 4.5**

Let $z_1, \ldots, z_{n-1}$ and $z_n = \infty$ distinct points on $\mathbb{P}$ and let $\lambda(z)\,|dz|$ be a regular conformal metric on $\mathbb{C}\setminus\{z_1, \ldots, z_{n-1}\}$ with constant curvature $-1$ and singularities of order $\alpha_j$ at $z_j$. Then

$$S_\lambda(z) = \sum_{j=1}^{n-1} \left( \frac{(2 - \alpha_j) \alpha_j}{2 (z - z_j)^2} + \frac{\beta_j}{z - z_j} \right)$$

and

$$S_\lambda(z) = \frac{(2 - \alpha_n) \alpha_n}{2 z^2} + \frac{\beta_n}{z^3} + O(1/z^4), \quad z \to \infty,$$

with complex numbers $\beta_1, \ldots, \beta_n$.

The numbers $\beta_1, \ldots, \beta_n$ are called the *accessory parameters* of $\lambda(z)\,|dz|$. In view of the asymptotic behavior of $S_\lambda(z)$ at $z = \infty$, the accessory parameters are related by

$$\sum_{j=1}^{n-1} \beta_j = 0, \quad \sum_{j=1}^{n-1} ((2 - \alpha_j) \alpha_j + 2 \beta_j z_j) = (2 - \alpha_n) \alpha_n, \quad \sum_{j=1}^{n-1} ((2 - \alpha_j) \alpha_j z_j + \beta_j z_j^2) = \beta_n.$$

In case of three singularities, these relations determine the accessory parameters completely. Thus, if $\lambda_{\alpha_1,\alpha_2,\alpha_3}(z)$ is the generalized hyperbolic density of order $(\alpha_1, \alpha_2, \alpha_3)$ on $\mathbb{C}^\alpha$, then $S_{\lambda_{\alpha_1,\alpha_2,\alpha_3}}$ is a rational function with poles of order 2 at $z = 0$ and $z = 1$ and

$$S_{\lambda_{\alpha_1,\alpha_2,\alpha_3}}(z) = \frac{1}{2} \left[ 1 - \theta_1^2 \frac{1 - \theta_2^2}{z^2} + \frac{1 - \theta_1^2 - \theta_2^2 + \theta_3^2}{z (1 - z)} \right], \quad (4.2)$$

with $\theta_j = 1 - \alpha_j$, $j = 1, 2, 3$. Hence in this case the Schwarzian $S_{\lambda_{\alpha_1,\alpha_2,\alpha_3}}(z)$ is explicitly determined by $\alpha_1, \alpha_2, \alpha_3$. In order to determine $\lambda_{\alpha_1,\alpha_2,\alpha_3}(z)$ from $\alpha_1, \alpha_2, \alpha_3$, we therefore need to recover the metric from its Schwarzian. Away from the singularities one can use Theorem [3.2] for this purpose. Thus we have to examine the Schwarzian differential equation

$$\left( \frac{\psi'(z)}{\psi(z)} \right)' - \frac{1}{2} \left( \frac{\psi''(z)}{\psi(z)} \right)^2 = \frac{1}{2} \left[ \frac{1 - \theta_1^2}{z^2} + \frac{1 - \theta_2^2}{(1 - z)^2} + \frac{1 - \theta_1^2 - \theta_2^2 + \theta_3^2}{z (1 - z)} \right], \quad (4.3)$$

and use the following classical fact (see [5], p. 116 ff.).

**Lemma 4.6**

Let $u_1, u_2$ be two linearly independent solutions of the hypergeometric differential equation

$$z(1 - z) u'' + [\gamma - (\alpha + \beta + 1) z] u' - \alpha \beta u = 0 \quad (4.4)$$

with $\alpha$, $\beta$ and $\gamma$ determined by (2.7). Then the solutions $\psi$ of the Schwarzian differential equation (4.3) have the form $\psi(z) = T(u_2(z)/u_1(z))$ where $T$ is an arbitrary Möbius transformation.
Proof of Lemma 4.1. We only prove part (a). The proof of part (b) is similar and is left to the reader. We consider (4.4) in \( \mathbb{D}^- \) and note that
\[
 u_1^0(z) = F(\alpha, \beta, \gamma; z), \quad u_2^0(z) = z^{1-\gamma}F(\alpha - \gamma + 1, \beta - \gamma + 1, 2 - \gamma; z)
\]
are two linearly independent solutions to (4.4). In view of Lemma 4.6 and Theorem 6.2 (a), we know that in \( \mathbb{D}^- \)
\[
 \lambda(z) := \lambda_{a_1, a_2, a_3}(z) = \frac{2|\phi'(z)|}{1 - |\phi(z)|^2} \quad \text{with} \quad \phi(z) = \frac{au_2^0(z) + bu_1^0(z)}{cu_2^0(z) + du_1^0(z)}
\]
for appropriate constants \( a, b, c, d \in \mathbb{C} \) with \( ad - bc \neq 0 \). If we let
\[
 h(z) := F(\alpha - \gamma + 1, \beta - \gamma + 1, 2 - \gamma; z)/F(\alpha, \beta, \gamma; z),
\]
then a straightforward computation gives
\[
 |z|^\gamma \lambda(z) = \frac{2|ad - bc| |(1 - \gamma)h(z) + h'(z)|}{||c|^2 - |a|^2||z|^{2-2\gamma}|h(z)|^2 + |[d]^2 - |b|^2| + 2 \Re \left((a\bar{b} - c\bar{d}) z^{1-\gamma}h(z)\right)}.
\] (4.5)
Since \( h \) is analytic at \( z = 0 \) with \( h(0) = 1 \) and \( |z|^\gamma \lambda(z) \) is single–valued in \( \mathbb{D} \setminus \{0\} \), a glance at (4.5) shows that \( a\bar{b} = c\bar{d} \). Since \( |z|^\gamma \lambda(z) \) is strictly positive and \( 0 < \gamma < 1 \), we can then deduce from (4.5) that \( |d| \leq |b| \). Moreover, we can exclude the case \( |d| = |b| \), since \( \lambda(z) |dz| \) has a corner of order \( \gamma \) at \( z = 0 \), so \( |z|^\gamma \lambda(z) \) is bounded at \( z = 0 \). Thus, \( |b| < |d| \). Postcomposing \( \phi \) with a unit disk automorphism \( T \) which sends \( b/d \in \mathbb{D} \) to 0 and using Theorem 3.2 (b), we can hence assume that \( b = 0 \) and thus also \( c = 0 \). This proves part (a) with \( c_0 = a/d \). Note that we can take \( c_0 > 0 \) by multiplying \( \phi \) with an appropriate complex number of absolute value one. Thus \( \phi(z) = c_0 u_2^0(z)/u_1^0(z) \) as claimed. \[\text{■}\]

Proof of Lemma 4.2. Let again
\[
 u_1^0(z) = F(\alpha, \beta, \gamma; z), \quad u_2^0(z) = z^{1-\gamma}F(\alpha - \gamma + 1, \beta - \gamma + 1, 2 - \gamma; z)
\]
be a fundamental system of (4.4) in \( \mathbb{D} \) and let
\[
 u_1^1(z) = F(\alpha, \beta, \alpha + \beta - \gamma + 1; 1 - z), \quad u_2^1(z) = (1 - z)^{\gamma - \alpha - \beta}F(\gamma - \beta, \gamma - \alpha, \gamma - \alpha - \beta + 1; 1 - z)
\]
be a fundamental system of (4.4) in \( K_1(1) \). Note that the above fundamental systems are connected by the transition relations
\[
 u_1^0(z) = Au_1^1(z) + Bu_2^1(z), \quad u_2^0(z) = Cu_1^1(z) + Du_2^1(z), \quad \text{where}
\]
\[
 A = \frac{\Gamma(\gamma) \Gamma(\alpha - \alpha - \beta)}{\Gamma(\gamma - \alpha) \Gamma(\gamma - \beta)}, \quad B = \frac{\Gamma(\gamma) \Gamma(\alpha + \beta - \gamma)}{\Gamma(\alpha) \Gamma(\beta)},
\]
\[
 C = \frac{\Gamma(2 - \gamma) \Gamma(\gamma - \alpha - \beta)}{\Gamma(1 - \alpha) \Gamma(1 - \beta)}, \quad D = \frac{\Gamma(2 - \gamma) \Gamma(\alpha + \beta - \gamma)}{\Gamma(\alpha - \gamma + 1) \Gamma(\beta - \gamma + 1)},
\]
see [8] Vol. II, p. 141]. By part (b) of Liouville’s Theorem 3.2 and Lemma 4.1, we get
\[
 \phi(z) = \eta \frac{g(z) - z_0}{1 - \overline{z}_0 g(z)}.
\] (4.7)
Here, \( \varphi = c_0 u_0^0 / u_0^1 \) and \( g = c_1 u_1^0 / u_1^1 \) are the functions of Lemma 4.1. Inserting these expressions into (4.7) and using the transition relations (4.6), we obtain

\[
c_1 e^{it} \frac{u_0^1(z) - \varphi(z)}{1 - z_0 c_1 u_0^1(z)} = \eta (g(z) - z_0) - \varphi(z) = c_0 \frac{u_0^0(z)}{u_0^1(z)} = c_0 \frac{C u_1^1(z) + D u_1^1(z)}{A u_0^1(z) + B u_0^1(z)} = c_0 D \frac{u_0^1(z)}{u_1^1(z)} + \frac{D}{A}.
\]

This leads to

\[
c_1 e^{it} = \frac{c D}{A}, \quad -\frac{z_0}{c_1} = \frac{C}{D}, \quad -\overline{z_0} c_1 = \frac{B}{A}
\]

and therefore we get

\[
c_0 = \sqrt{\frac{AB}{CD}}.
\]

An easy computation finally yields (4.1).

**Proof of Corollary 4.4** For

\[
\varphi(z) = c_0 z^{1-\gamma} F(\alpha - \gamma + 1, \beta - \gamma + 1, 2 - \gamma; z)
\]

we have ([8, Vol. II, p. 147])

\[
\frac{\varphi'(z)}{z \alpha_1 (1 - z)^{\alpha_2} F(\alpha, \beta, \gamma; z)^2},
\]

which proves the assertion of Corollary 4.4.

**Proof of Theorem 2.1** In order to prove Theorem 2.1 we just use the representation formula of Corollary 4.4 and express \( u_0^1(z) = z^{1-\gamma} F(\alpha - \gamma + 1, \beta - \gamma + 1, 2 - \gamma; z) \) in terms of \( u_0^0(z) = F(\alpha, \beta, \gamma; z) \) and \( u_1^1(z) = F(\alpha, \beta, \alpha - \gamma + 1, 1 - z) \) with the help of the transition formulas (4.6).

We end this section with the following mapping properties of the function \( \varphi \) in Lemma 4.1 (a).

**Remark 4.7**

Let \( \alpha_1, \alpha_2, \alpha_3 \) be real parameters satisfying condition (1.2) and define \( \alpha, \beta, \gamma \) by (2.1). Then the function

\[
\varphi(z) = c_0 z^{1-\gamma} F(\alpha - \gamma + 1, \beta - \gamma + 1, 2 - \gamma; z)
\]

with \( c_0 \) given by (4.1) has an analytic continuation to \( \mathbb{H} \), which maps \( \mathbb{H} \) conformally onto a hyperbolic triangle \( \Delta \subseteq \mathbb{D} \) with interior angles \( \pi(1 - \alpha_1), \pi(1 - \alpha_2) \) and \( \pi(1 - \alpha_3) \) in such a way that 0, 1 and \( \infty \) are mapped to the vertices of \( \Delta \). We refer to [8, Vol. II, p. 116 ff.] for details.

### 4.2 Sharp lower bounds for the generalized hyperbolic density

In order to prove the sharp lower bound (2.4) for the generalized hyperbolic density \( \lambda_{\alpha_1, \alpha_2, \alpha_3} \) we first state a simple, but important extremality property of \( \lambda_{\alpha_1, \alpha_2, \alpha_3} \).

**Lemma 4.8**

Let \( \alpha_1, \alpha_2, \alpha_3 \) be real parameters satisfying (1.2). If \( \mu(z) |dz| \) is an SK–metric on \( \mathbb{C}^n \) with singularities of order \( \beta_1 \leq \alpha_1, \beta_2 \leq \alpha_2 \) and \( \beta_3 \leq \alpha_3 \) at \( z = 0, z = 1 \) and \( z = \infty \), then \( \mu \leq \lambda_{\alpha_1, \alpha_2, \alpha_3} \). Moreover, \( \lambda_{\alpha_1, \alpha_2, \alpha_3}(z) |dz| \) is the unique conformal metric on \( \mathbb{C}^n \) with constant curvature \(-1\) and singularities of order \( \alpha_1, \alpha_2 \) and \( \alpha_3 \) at \( z = 0, z = 1 \) and \( z = \infty \).
Proof. Let \( \lambda(z) \, |dz| \) be a conformal metric on \( \mathbb{C}'' \) with constant curvature \(-1\) and singularities of order \( \alpha_1, \alpha_2 \) and \( \alpha_3 \) at \( z = 0, \, z = 1 \) and \( z = \infty \). Then the function \( s(z) = \log^+ (\mu(z)/\lambda(z)) \) is subharmonic on \( \mathbb{C}'' \) in view of the curvature assumptions on \( \mu \) and \( \lambda \). Moreover, \( s \) is bounded above at \( z = 0 \) and \( z = 1 \), so it has a subharmonic extension to \( \mathbb{C} \). Since \( u \) is also bounded above at \( \infty \), we see that \( s \equiv c \) for some nonnegative constant. If \( c > 0 \), then \( \mu(z) = e^c \lambda(z) \), so \( \kappa_\mu = e^{-2c} \kappa_\lambda > -1 \), which violates the fact that \( \mu(z) \, |dz| \) is an SK–metric. Hence \( c = 0 \), so \( \mu(z) \leq \lambda(z) \) for all \( z \in \mathbb{C}'' \). Choosing \( \lambda = \lambda_{\alpha_1, \alpha_2, \alpha_3} \) proves the first part of Lemma 4.8 and choosing \( \mu = \lambda_{\alpha_1, \alpha_2, \alpha_3} \) proves the second part. \( \blacksquare \)

Hence \( \lambda_{\alpha_1, \alpha_2, \alpha_3}(z) \, |dz| \) is maximal among all SK–metrics on \( \mathbb{C}'' \) with singularities of order \( \beta_1 \leq \alpha_1, \beta_2 \leq \alpha_2 \) and \( \beta_3 \leq \alpha_3 \) at \( z = 0, 1 \) and \( \infty \). In order to make use of this maximality, we need the following simple “gluing lemma” which we state for general SK–metrics.

Lemma 4.9 (Gluing Lemma)
Let \( \lambda(z) \, |dz| \) be an SK–metric on a domain \( G \subset \mathbb{C} \) and let \( \mu(z) \, |dz| \) be an SK–metric on a subdomain \( U \) of \( G \) such that the “gluing condition”

\[
\limsup_{U \ni z \to \xi} \mu(z) \leq \lambda(\xi)
\]

holds for all \( \xi \in \partial U \cap G \). Then \( \sigma(z) \, |dz| \) defined by

\[
\sigma(z) := \begin{cases} 
\max\{\lambda(z), \mu(z)\} & \text{for } z \in U, \\
\lambda(z) & \text{for } z \in G \setminus U
\end{cases}
\]

is an SK–metric on \( G \).

Proof. The gluing condition guarantees that \( \sigma \) is upper semicontinuous on \( G \) and it is easy to see that \( \max\{\lambda(z), \mu(z)\} \) is the density of an SK–metric on \( U \). Hence the curvature of \( \sigma(z) \, |dz| \) is bounded above by \(-1\) at each \( z \in U \). If \( z \in G \setminus U \) then \( \kappa_\sigma(z) \leq -1 \). This is clear if \( z \notin \partial U \cap G \). For \( z \in \partial U \cap G \), this follows from \( \sigma \geq \lambda \). \( \blacksquare \)

We now combine the maximality of \( \lambda_{\alpha_1, \alpha_2, \alpha_3}(z) \, |dz| \) with this gluing lemma.

Theorem 4.10 (Strict Monotonicity)
Let \( \alpha_1, \alpha_2, \alpha_3 \) be real parameters satisfying condition (1.2). Then \( \lambda_{\alpha_1, \alpha_2, \alpha_3}(re^{it}) \) is strictly decreasing for \( 0 < t < \pi \) and strictly increasing for \(-\pi < t < 0\) for any fixed \( r \in (0, +\infty) \).

We note that the case \( \alpha_1 = \alpha_2 = 1 \) of Theorem 4.10 was proved before by Hempel [12] if \( \alpha_3 = 1 \) and by Anderson, Sugawa, Vamanamurthy and Vuorinen [5] if \( \alpha_3 < 1 \). The proofs in [12, 5] are based on an a–priori knowledge of the asymptotic behaviour of the metric at the corners, whereas the following proof is solely based on the gluing lemma and the maximality of the generalized hyperbolic metric.

Proof of Theorem 4.10. For \( \eta \in \partial \mathbb{D} \) let \( \lambda_{\eta}(z) \, |dz| := \lambda_{\alpha_1, \alpha_2, \alpha_3}(\overline{\eta} \, z) \, |dz| \). Then \( \lambda_{\eta}(z) \, |dz| \) is the maximal SK–metric on \( \mathbb{C} \setminus \{0, \eta\} \) with singularities of order \( \alpha_1, \alpha_2 \) and \( \alpha_3 \) at \( z = 0, \, z = \eta \) and \( z = \infty \). In a first step we show that \( \lambda_{\eta}(z) = \lambda_{\eta}(\overline{z}) \) for all \( z \in \mathbb{C} \setminus \{0, \eta\} \). For this we note that \( \lambda_{\eta}(z) \, |dz| \) is an SK–metric on \( \mathbb{C} \setminus \{0, \overline{\eta}\} \) with singularities of order \( \alpha_1, \alpha_2 \) and \( \alpha_3 \) at \( z = 0, \, z = \overline{\eta} \) and \( z = \infty \). Thus by maximality

\[
\lambda_{\eta}(\overline{z}) \leq \lambda_{\eta}(z), \quad z \in \mathbb{C} \setminus \{0, \overline{\eta}\}.
\]
Hence $\lambda_\Psi(z) \leq \lambda_\eta(z)$ for all $z \in \mathbb{C}\setminus\{0, \eta\}$ which implies
\[
\lambda_\Psi(z) \leq \lambda_\eta(z), \quad z \in \mathbb{C}\setminus\{0, \eta\}.
\] (4.9)

Combining (4.8) and (4.9) gives the desired result.

Second, we prove that $\lambda_\Psi(z) < \lambda_\eta(z)$ for all $z \in \mathbb{H}$ if $\text{Im} \eta > 0$. To check this assertion we note that $\lambda_\Psi(z) = \lambda_\eta(z)$ for all $z \in \mathbb{R}\setminus\{0\}$. Thus by the gluing lemma (Lemma 4.9)

\[
\sigma(z) := \begin{cases} 
\max\{\lambda_\Psi(z), \lambda_\eta(z)\}, & z \in \mathbb{H}\setminus\{\eta\} \\
\lambda_\eta(z), & z \in \mathbb{C}\setminus(\mathbb{H} \cup \{0\})
\end{cases}
\]

induces an SK–metric on $\mathbb{C}\setminus\{0, \eta\}$ with singularities of order $\alpha_1$, $\alpha_2$ and $\alpha_3$ at $z = 0$, $z = \eta$ and $z = \infty$. Hence $\sigma \leq \lambda_\eta$ and so $\lambda_\Psi(z) \leq \lambda_\eta(z)$ for all $z \in \mathbb{H}$ if $\text{Im} \eta > 0$ and Lemma 3.1 shows that $\lambda_\Psi(z) < \lambda_\eta(z)$ for all $z \in \mathbb{H}$ if $\text{Im} \eta > 0$.

Finally we derive the strict monotonicity of $\lambda(z) |dz| := \lambda_{\alpha_1,\alpha_2,\alpha_3}(z) |dz|$. Choose $\varphi_1, \varphi_2 \in (-\pi, 0)$ with $\varphi_2 > \varphi_1$ and set $\eta_1 := e^{-i\varphi_1/2}$ and $\eta_2 := e^{i\varphi_2/2}$. Then we have
\[
\lambda(-re^{i\varphi_1}) = \lambda(-r\eta_1^2) = \lambda_{\eta_1}(-r\eta_1 \eta_2 \eta_2) = \lambda_{\eta_1, \eta_2}(-r\eta_1 \eta_2) > \lambda_{\eta_1 \eta_2}(-r\eta_1 \eta_2) = \lambda(-r\eta_2^2) = \lambda(-re^{i\varphi_2})
\]

We are now prepared to prove Theorem 2.2. We shall use Theorem 4.10 and one more time the gluing lemma.

**Proof of Theorem 2.2.** For $\alpha \leq 1$ and $R > 0$ let
\[
\lambda_{\alpha,R}(z) := \frac{2(1-\alpha)R^{1-\alpha}|z|^{-\alpha}}{R^{2(1-\alpha)} - |z|^2(1-\alpha)} = \frac{1 - \alpha}{|z| \sinh \left( (1-\alpha) \log \frac{R}{|z|} \right)}.
\]

Here again, for the case $\alpha = 1$ this formula has to be interpreted in the limit sense $\alpha \searrow 1$, i.e.,
\[
\lambda_{1,R}(z) = \lim_{\alpha \searrow 1} \lambda_{\alpha,R}(z) = \frac{1}{|z| \log \frac{R}{|z|}}.
\]

Then $\lambda_{\alpha,R}(z) |dz|$ is a conformal metric on the punctured disk $0 < |z| < R$ with constant curvature $-1$ and singularity of order $\alpha$ at $z = 0$. In point of fact, $\lambda_{\alpha,R}(z) |dz|$ is the maximal conformal metric on $0 < |z| < R$ with those properties.

We now write $\lambda(z) |dz| := \lambda_{\alpha_1,\alpha_2,\alpha_3}(z) |dz|$. Note that $\lambda(z) \geq \lambda(-1)$ for all $|z| = 1$ by Theorem 4.10. If we choose $R_1$ such that $\lambda_{\alpha_1,R_1}(z) = \lambda(-1)$ for $|z| = 1$, i.e., $R_1 := e^{C_1} > 1$, then
\[
\lambda(z) \geq \lambda(-1) = \frac{1 - \alpha_1}{\sinh \left( (1-\alpha_1) C_1 \right)} = \frac{1 - \alpha_1}{|z| \sinh \left( (1-\alpha_1) \log \frac{R_1}{|z|} \right)} = \lambda_{\alpha_1,R_1}(z) \quad \text{for all } |z| = 1.
\]

So the gluing lemma (Lemma 4.9) ensures that
\[
\sigma(z) := \begin{cases} 
\max\{\lambda(z), \lambda_{\alpha_1,R_1}(z)\}, & \text{if } 0 < |z| < 1, \\
\lambda(z), & \text{if } |z| \geq 1,
\end{cases}
\]
induces an SK–metric on \( C'' \) with singularities of order \( \alpha_1, \alpha_2 \) and \( \alpha_3 \) at \( z = 0, 1 \) and \( \infty \). The maximality of \( \lambda(z) |dz| \) implies \( \sigma(z) \leq \lambda(z) \) for all \( z \in C'' \). In particular,

\[
\lambda(z) \geq \lambda_{\alpha_1,R_1}(z) = \frac{1 - \alpha_1}{|z| \sinh \left( (1 - \alpha_1) \log \frac{R_1}{|z|} \right)} = \frac{1 - \alpha_1}{|z| \sinh \left( (1 - \alpha_1)(C_1 - \log |z|) \right)}
\]

for all \( |z| \leq 1, z \neq 0, 1, \) with equality for \( z = -1 \). In a similar way, one can prove

\[
\lambda(z) \geq \frac{1 - \alpha_3}{|z| \sinh \left( (1 - \alpha_3) \log (R_3|z|) \right)} = \frac{1 - \alpha_3}{|z| \sinh \left( (1 - \alpha_3)(C_3 + \log |z|) \right)}
\]

for all \( |z| \geq 1 \) with equality for \( z = -1 \).

Assume now there is \( z_0 \in C'' \) such that equality holds in (2.4). If \( z_0 \in \mathbb{D} \), then \( \lambda(z_0) = \lambda_{\alpha_1,R_1}(z_0) \) and, as we have seen above, \( \lambda(z) \geq \lambda_{\alpha_1,R_1}(z) \) for all \( 0 < |z| < 1 \). Hence \( \lambda(z) = \lambda_{\alpha_1,R_1}(z) \) for all \( 0 < |z| < 1 \) by Lemma 3.1. This however contradicts

\[
\lim_{z \to 1} \lambda_{\alpha_1,R_1}(z) < +\infty = \lim_{z \to 1} \lambda(z).
\]

In the same way, we can exclude the case \( |z_0| > 1 \). Thus \( |z_0| = 1 \), so \( \lambda(z_0) = \lambda_{\alpha_1,R_1}(z_0) = \lambda(-1) \). Now Theorem 4.10 tells us that \( z_0 = -1 \). \( \blacksquare \)

**Proof of Corollary 2.5.** Clearly, \( C_1 = C_3 \) if \( \alpha_1 = \alpha_3 \), so we only need to compute the value \( \lambda(-1) = \lambda_{\alpha_1,\alpha_2,\alpha_1}(1/2) = \lambda_{\alpha_1,\alpha_1,\alpha_2}(1/2)/4 \). Also, Theorem 2.1 gives

\[
\lambda_{\alpha_1,\alpha_2,\alpha_1}(1/2) = \frac{K_3}{1 + K_1} \frac{\Gamma(\alpha_1)\Gamma(\alpha_2 - \alpha_1)\Gamma(\frac{\alpha_1 + \alpha_2}{2} - \frac{\alpha_2}{2})}{\Gamma(\alpha_1 - \frac{\alpha_2}{2})\Gamma(\alpha_1 + \frac{\alpha_2}{2} - 1)}.
\]

Applying [11 15.1.24] and the duplication formula [11 6.1.18] for the Gamma function, a straightforward computation gives

\[
F \left( \alpha_1 - \frac{\alpha_2}{2}, \alpha_1 - 1 + \frac{\alpha_2}{2}, \alpha_1; \frac{1}{2} \right) = 2^{2\alpha_1} \frac{\Gamma(\alpha_1)\Gamma(\frac{\alpha_2}{2} - \frac{\alpha_1}{2})\Gamma(\frac{\alpha_1 + \alpha_2}{2} - \frac{\alpha_2}{2})}{\Gamma(\alpha_1 - \frac{\alpha_2}{2})\Gamma(\alpha_1 + \frac{\alpha_2}{2} - 1)}.
\]

On the other hand, using the reflection formula [11 6.1.17] in the expressions (2.2) for \( K_1 \) and \( K_3 \) and elementary trigonometric manipulation lead to

\[
\frac{K_3}{1 + K_1} = \frac{\Gamma(\alpha_1)^2}{\Gamma(\alpha_1 - \frac{\alpha_2}{2})\Gamma(\alpha_1 + \frac{\alpha_2}{2} - 1)}.
\]

Combining the last two identities, we arrive at

\[
\lambda_{\alpha_1,\alpha_2,\alpha_1}(1/2) = \frac{\lambda_{\alpha_1,\alpha_1,\alpha_2}(1/2)}{4} = \frac{\Gamma(\alpha_1)\Gamma(\frac{\alpha_1 + \alpha_2}{2} + \frac{\alpha_2}{2})\Gamma(\alpha_1 + \frac{\alpha_2}{2} - 1)}{\Gamma(\alpha_1 - \frac{\alpha_2}{2})\Gamma(\alpha_1 + \frac{\alpha_2}{2} - 1)}.
\]

Finally, making again use of the duplication formula [11 6.1.18] for the Gamma function for \( z = \frac{\alpha_1}{2} - \frac{\alpha_2}{2} \) and \( z = \frac{\alpha_1}{2} + \frac{\alpha_2}{2} - \frac{1}{2} \) in the last numerator, we deduce (2.5). \( \blacksquare \)
4.3 Schottky and Landau–type theorems

We need the following variant of the Ahlfors–Schwarz lemma.

**Lemma 4.11 (Ahlfors–Schwarz)**

Let \( j, k, l \geq 2 \) be integers (or \( \omega = \infty \)) such that \( 1/j + 1/k + 1/l < 1 \) and let \( \lambda(z) |dz| \) be the generalized hyperbolic density on \( \mathbb{C}'' \) of order \((1 - 1/j, 1 - 1/k, 1 - 1/l)\). If \( f \in \mathcal{M}_{j,k,l} \), then \( \lambda(f(z)) |f'(z)| |dz| \) is a regular conformal pseudo–metric of constant curvature \(-1\) on \( \mathbb{D} \), so

\[
\lambda(f(z)) |f'(z)| \leq \frac{2}{1 - |z|^2}, \quad z \in \mathbb{D}.
\]

Equality for one point \( z \in \mathbb{D} \) holds if and only if \( f \) is a triangle map of order \((j, k, l)\).

**Proof.** Let \( S := f^{-1}(\{0, 1, \infty\}) \cap \mathbb{D} \). If \( z_0 \in \mathbb{D} \setminus S \), then \( \mu(z) |dz| := \lambda(f(z)) |f'(z)| |dz| \) is clearly a regular conformal pseudo–metric in a neighborhood of \( z_0 \) with constant curvature \(-1\) there. Moreover, \( \mu \) is continuous at any point \( z_0 \in S \). In order to check this for the case \( f(z_0) = 0 \), we note that the remainder function \( r \) in

\[
\log \lambda(w) = \begin{cases}
-(1 - 1/j) \log |w| + r(w) & \text{if } 2 \leq j \leq \infty \\
- \log |w| - \log(-\log |w|) + r(w) & \text{if } j = \infty.
\end{cases}
\]

as \( w \to 0 \) is continuous at \( w = 0 \). This follows e.g. from the results in [15]. Since \( f \) has a zero of order at least \( j \geq 2 \) at \( z = z_0 \), we easily deduce that \( \mu(z) = \lambda(f(z)) |f'(z)| \) is continuous at \( z = z_0 \). The cases \( f(z_0) = 1 \) and \( f(z_0) = \infty \) are similar. Hence \( \mu \) is continuous on \( \mathbb{D} \). By [24], \( \mu(z) |dz| \) is actually regular on \( \mathbb{D} \setminus \{z \in \mathbb{D} : \mu(z) = 0\} \) with curvature \(-1\) there. In particular, \( \lambda(f(z)) |f'(z)| = \mu(z) \leq \lambda_D(z) \) for any \( z \in \mathbb{D} \) by the Ahlfors–Schwarz lemma. Note that if \( f \) is a triangle map of order \((j, k, l)\), then by Remark [4.4], \( \lambda(f(z)) |f'(z)| = \lambda_D(z) \) for all \( z \in \mathbb{D} \). Conversely, if \( \lambda(f(z)) |f'(z)| = \lambda_D(z) \) for some point \( z \in \mathbb{D} \), then \( \lambda(f(z)) |f'(z)| \equiv \lambda_D(z) \) in \( \mathbb{D} \) by Lemma [3.1]. Now pick a point \( z_0 \in \mathbb{D} \) with \( w_0 = f(z_0) \in f(\mathbb{D}) \setminus \{0, 1, \infty\} \). Then \( f'(z_0) \neq 0 \), so \( f \) has a local inverse \( h \) in some disk \( K_r(w_0) \) such that \( h(K_r(w_0)) \subset \mathbb{D} \). Hence \( \lambda(w) = \lambda_D(h(w)) |h'(w)| \) for all \( w \in K_r(w_0) \). Shrinking \( r > 0 \) if necessary, we also have \( \lambda(w) = \lambda_D(\varphi_0(w)) |\varphi_0'(w)| \) in \( K_r(w_0) \) where \( \varphi_0 \) is a local inverse of a triangle map \( f_0 \) of order \((j, k, l)\). By Theorem [2.2] (b), we get \( \varphi_0 = T \circ h \) for some disk automorphism \( T \), so \( f = f_0 \circ T \), i.e., \( f \) is a triangle map of order \((j, k, l)\).

**Proof of Theorem 2.6.** Let \( \lambda(w) |dw| \) be the generalized hyperbolic density on \( \mathbb{C}'' \) with singularities of order \((1 - 1/j, 1 - 1/k, 1 - 1/l)\). Then \( \lambda(f(z)) |f'(z)| \leq \lambda_D(z) \) for each \( z \in \mathbb{D} \) by Lemma [4.1]. For \( z = 0 \), we get \( |a_1| = |f'(0)| \leq 2/\lambda(f(0)) = 2/\lambda(a_0) \). Now employing the lower bound for \( \lambda \) provided by Theorem [2.2] gives the estimate of Theorem [2.6].

To handle the case of equality, we note Lemma [4.1] and Lemma [3.1] show that \( f: \mathbb{D} \to \mathbb{C} \) is a triangle map of order \((j, k, l)\) if and only if \( \lambda(a_0) |a_1| = 2 \). By Theorem [2.2] we have

\[
\lambda(a_0) = \frac{1}{j|a_0|} \sinh \left( \frac{1}{2} (C_1 + |\log |a_0||) \right) = \frac{1}{l|a_0|} \sinh \left( \frac{1}{7} (C_3 + |\log |a_0||) \right)
\]

if and only if \( a_0 = -1 \). This finishes the proof.

**Proof of Theorem 2.8.** Let \( g := 1/f \in \mathcal{M}_{j,k,l} \) and let \( \lambda(z) \) denote the generalized hyperbolic density on \( \mathbb{C}'' \) of order \((1 - 1/l, 1 - 1/k, 1 - 1/j)\). Then Lemma [4.1] gives

\[
\lambda(g(z)) |g'(z)| \leq \frac{2}{1 - |z|^2} \quad \text{for all } z \in \mathbb{D},
\]
Pick a point \( z_0 \in \mathbb{D} \) such that \( |g(z_0)| < 1 \) and consider the curve \( \gamma(t) := g(t \eta) \) for \( t \in [0, |z_0|] \) and \( z_0 = |z_0| \eta \). If \( \gamma \subset \mathbb{D} \), then Theorem 2.22 applied for \( z = t \eta \) and (4.10) lead to

\[
\frac{|g'(t \eta)|}{t |g(t \eta)| \sinh \left( (\tilde{C}_1 - \log |g(t \eta)|)/t \right)} \leq \frac{2}{1 - t^2}, \quad t \in [0, |z_0|].
\]

(4.11)

Integrating over \([0, |z_0|]\) using \( \frac{4}{\pi t} |g(t \eta)| \leq |g'(t \eta)| \) yields

\[
\int_{|g(0)|}^{|g(z_0)|} \frac{|ds|}{l s \sinh \left( (\tilde{C}_1 - \log s)/l \right)} \leq \log \frac{1 + |z_0|}{1 - |z_0|}.
\]

Hence

\[
\left| \log \left( \frac{\tanh \left( \frac{\tilde{C}_1 - \log |g(z_0)|}{2l} \right)}{\tanh \left( \frac{\tilde{C}_1 - \log |g(0)|}{2l} \right)} \right) \right| \leq \log \frac{1 + |z_0|}{1 - |z_0|}.
\]

(4.12)

If \( \gamma \not\subset \mathbb{D} \), then a similar argument using the “last” point \( \gamma(t^*) \) of \( \gamma \) outside \( \mathbb{D} \) and integrating (4.11) from \( t^* \) to \( |z_0| \) gives

\[
\tanh \left( \frac{\tilde{C}_1 - \log |g(z_0)|}{2l} \right) \leq \left[ \tanh \left( \frac{\tilde{C}_1}{2l} \right) \right] \cdot \frac{1 + |z_0|}{1 - |z_0|}.
\]

(4.13)

Thus in both cases, \( \gamma \subset \mathbb{D} \) and \( \gamma \not\subset \mathbb{D} \), we get by the monotonicity of \( \tanh \)

\[
\tanh \left( \frac{\tilde{C}_1 - \log |g(z_0)|}{2l} \right) \leq \left[ \tanh \left( \frac{\tilde{C}_1 + \log^+ \frac{1}{|g(0)|}}{2l} \right) \right] \cdot \frac{1 + |z_0|}{1 - |z_0|}.
\]

(4.14)

If \( |g(z_0)| \geq 1 \), then (4.14) is trivially true. Finally going back to \( f = 1/g \) finishes the proof. \( \blacksquare \)

**Proof of Corollary 2.12.** Choosing \( j = l = \infty \) in Theorem 2.8 gives \( L_k = 1/\lambda_{1,1-1/k,1}(-1) \).

Equation (2.5) shows

\[
\lambda_{1,1-1/k,1}(-1) = 2 \frac{\Gamma(3/4 + 1/(4k)) \Gamma(3/4 - 1/(4k))}{\Gamma(1/4 + 1/(4k)) \Gamma(1/4 - 1/(4k))}.
\]

Applying [1] 6.1.18 for \( z = 1/4 + 1/(4k) \) and \( z = 1/4 - 1/(4k) \) in the numerator and then using [1] 6.1.17 gives the desired result. \( \blacksquare \)

**References**

[1] M. Abramowitz, I. A. Stegun, *Pocketbook of Mathematical Functions*, Harri Deutsch, 1984.

[2] S. Agard, Distortion theorems for quasiconformal mappings, *Ann. Acad. Sci. Fenn. Ser. A I* (1968), 413.

[3] L. Ahlfors, An extension of Schwarz’s lemma, *Trans. Amer. Math. Soc.* (1938), 43, 359–364.
[4] G. D. Anderson, S.–L. Qiu, M. Vuorinen, Modular equations and distortion functions, *The Ramanujan Journal* (2009), **18** No. 2, 147–169.

[5] G. D. Anderson, T. Sugawa, M. K. Vamanamurthy, M. Vuorinen, Twice punctured sphere with a conical singularity and generalized elliptic integrals, to appear in Math. Z., [http://arxiv.org/abs/0903.1761v1/](http://arxiv.org/abs/0903.1761v1/)

[6] L. Bieberbach, $\Delta u = e^u$ und die automorphen Funktionen, *Math. Ann.* (1916), **77**, 173–212.

[7] R. B. Burckel, *An introduction to classical complex analysis*, Birkhäuser, Basel 1979.

[8] C. Carathéodory, *Funktionentheorie I und II*, Birkhäuser, Basel 1950.

[9] W. K. Hayman, Some remarks on Schottky’s theorem, *Proc. Cambridge Philos. Soc.* (1947), **43**, 442–454.

[10] W. K. Hayman, *Subharmonic functions Vol. 2*, Academic Press, London 1989.

[11] M. Heins, On a class of conformal metrics, *Nagoya Math. J.* (1962), **21**, 1–60.

[12] J. A. Hempel, The Poincaré metric on the twice punctured plane and the theorems of Landau and Schottky, *J. Lond. Math. Soc., II. Ser.* (1979), **20**, 435–445.

[13] J. A. Hempel, Precise bounds on the theorems of Schottky and Picard, *J. London Math. Soc.* **21**, 279–286 (1980).

[14] J. Jenkins, On explicit bounds in Landau’s theorem II, *Can. J. Math.* (1981), **33**, 559–562.

[15] D. Kraus and O. Roth, The behaviour of solutions of the Gaussian curvature equation near an isolated boundary point, *Math. Proc. Cambr. Phil. Soc.* **145**, 643–667, 2008.

[16] D. Kraus and O. Roth, On the isolated singularities of the solutions of the Gaussian curvature equation for nonnegative curvature, *J. Math. Anal. Appl.* **345** No. 2, 628–631, 2008.

[17] D. Kraus and O. Roth, *Conformal Metrics*, [http://arxiv.org/abs/0805.2235/](http://arxiv.org/abs/0805.2235/)

[18] Z. Li and Y. Qi, A remark on Schottky’s theorem, *Bull. London Math. Soc.* **39**, 242–246, 2007.

[19] J. Liouville, Sur l’équation aux différences partielles $\frac{d^2 \log \lambda}{du^2} + \frac{\lambda}{2a^2} = 0$, *J. de Math.* (1853), **18**, 71–72.

[20] G. J. Martin, The distortion theorem for quasiconformal mappings, Schottky’s theorem and holomorphic motions, *Proc. Amer. Math. Soc.* **125**, 1095–1103, (1995).

[21] R. C. McOwen, Point singularities and conformal metrics on Riemann surfaces, *Proc. Am. Math. Soc.* (1988), **103**, No. 1, 222–224.

[22] R. C. McOwen, Prescribed Curvature and Singularities of Conformal Metrics on Riemann Surfaces, *J. Math. Anal. Appl.* (1993), **177** no. 1, 287–298.

[23] D. Minda, Bloch constants, *J. D’Anal. Math.* (1982), **41**, 54–84.
[24] D. Minda, A reflection principle for the hyperbolic metric with applications to geometric function theory, *Compl. Var.* (1987), 8, 129–144.

[25] J. Nitsche, Über die isolierten Singularitäten der Lösungen von $\Delta u = e^u$, *Math. Z.* (1957), 68, 316–324.

[26] E. Picard, De l’integration de l’équation differentielles $\Delta u = e^u$ sur une surface de Riemann fermée, *J. Reine Angew. Math.* 130 (1905) 243–258.

[27] M. Troyanov, Prescribing curvature on compact surfaces with conical singularities, *Trans. Amer. Math. Soc.* (1990), 324 no. 2, 793–821.

[28] Y. Yunyan, Local estimates of singular solution to Gaussian curvature equation, *J. Partial Diff. Eqs.* (2003), 16, 169–185.