Analytical Solutions to the Navier-Stokes
Equations with Density-dependent Viscosity and
with Pressure

LING HEI YEUNG*
Department of Mathematics, The Hong Kong Baptist University,
Kowloon Tong, Hong Kong

YUEN MANWAI†
Department of Applied Mathematics, The Hong Kong Polytechnic University,
Hung Hom, Kowloon, Hong Kong

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Abstract

This article is the continued version of the analytical solutions for the pressureless Navier-
Stokes equations with density-dependent viscosity [9]. We are able to extend the similar
solutions structure to the case with pressure under some restriction to the constants γ and θ.

Key words: Navier-Stokes Equations, Analytical Solutions, Radial Symmetry, Density-
dependent Viscosity, With Pressure

*E-mail address: lightisgood2005@yahoo.com.hk
†E-mail address: nevetsyuen@hotmail.com
1 Introduction

The Navier-Stokes equations can be formulated in the following form:

\[
\begin{aligned}
\rho_t + \nabla \cdot (\rho u) &= 0, \\
(\rho u)_t + \nabla \cdot (\rho u \otimes u) + \delta \nabla P &= \text{vis}(\rho, u).
\end{aligned}
\] (1)

As usual, \( \rho = \rho(x,t) \) and \( u(x,t) \) are the density, the velocity respectively. \( P = P(\rho) \) is the pressure.

We use a \( \gamma \)-law on the pressure, i.e.

\[ P(\rho) = K \rho^\gamma, \]

with \( K > 0 \), which is a universal hypothesis. The constant \( \gamma = c_P/c_v \geq 1 \), where \( c_P \) and \( c_v \) are the specific heats per unit mass under constant pressure and constant volume respectively, is the ratio of the specific heats. \( \gamma \) is the adiabatic exponent in (2). In particular, the fluid is called isothermal if \( \gamma = 1 \). It can be used for constructing models with non-degenerate isothermal fluid. \( \delta \) can be the constant 0 or 1. When \( \delta = 0 \), we call the system is pressureless; when \( \delta = 1 \), we call that it is with pressure. And \( \text{vis}(\rho, u) \) is the viscosity function. When \( \text{vis}(\rho, u) = 0 \), the system (1) becomes the Euler equations. For the detailed study of the Euler and Navier-Stokes equations, see \[1\] and \[3\]. Here we consider the density-dependent viscosity function as follows:

\[ \text{vis}(\rho, u) = \triangledown (\mu(\rho) \triangledown \cdot u). \]

where \( \mu(\rho) \) is a density-dependent viscosity function, which is usually written as \( \mu(\rho) = \kappa \rho^\theta \) with the constants \( \kappa, \theta > 0 \). For the study of this kind of the above system, the readers may refer \[6\], \[8\].

The Navier-Stokes equations with density-dependent viscosity in radial symmetry can be expressed by:

\[
\begin{aligned}
\rho_t + u_r \rho_r + \rho u_r + \frac{N-1}{r} \rho u &= 0, \\
\rho (u_t + uu_r) + \nabla K \rho^\gamma = (\kappa \rho^\theta)_r \left( \frac{N-1}{r^2} u + u_r \right) + \kappa \rho^\theta (u_{rr} + \frac{N-1}{r^2} u_r - \frac{N-1}{r^2} u),
\end{aligned}
\] (3)

Recently, Yuen’s results \[9\] showed that there exists a family of the analytical solutions for the pressureless Navier-Stokes equations with density-dependent viscosity:
Theorem 1 For the $N$-dimensional Navier-Stokes equations in radial symmetry [3], there exists a family of solutions, those are:

for $\theta = \gamma = 1$,

$$\begin{aligned}
\rho(t, r) &= \frac{a^2}{v(t)} - \frac{a(t)}{a(t)} r^2 + C, \\
u(t, r) &= \frac{a(t)}{a(t)} r, \\
n(t) &= \frac{\lambda a(t)}{a(t)^{N+2}}, a(0) = a_0 > 0, \dot{a}(0) = a_1,
\end{aligned}$$

where $A \geq 0$, $B$ and $C$ are constants.

for $\theta = \gamma > 1$,

$$\begin{aligned}
\rho(t, r) &= \left\{ \begin{array}{ll}
\frac{y(t)}{a(t)}, & \text{for } y(t) \geq 0 \\
0, & \text{for } y(t) < 0
\end{array} \right., \\
u(t, r) &= \frac{a(t)}{a(t)} r, \\
n(t) &= \frac{\lambda a(t)}{a(t)^{N+2}}, a(0) = a_0 > 0, \dot{a}(0) = a_1,
\end{aligned}$$

where $a_0$, $a_1$ and $\alpha > 0$ are constants;
for $\frac{2}{2} + \frac{1}{2} - \frac{1}{N} = \theta \geq 1 - \frac{1}{N}$,

$$
\rho(t, r) = \begin{cases} 
\frac{y(r/a(t))}{a(t)^N}, & \text{for } y(r/a(t)) \geq 0, \\
0, & \text{for } y(r/a(t)) < 0
\end{cases}, \quad u(t, r) = \frac{a(t)}{a(t)^N}, \quad a(t) = \sigma(mt + n)^s, 0 < s = \frac{2}{\gamma N - N + 2} \leq 1
$$

(8)

where $m, n > 0, \sigma > 0$ and $\alpha$ are constants.

## 2 Separation Method of Self-Similar Solutions

Before we present the proof of Theorem 1, Lemmas 3 and 12 of [9] are quoted here.

**Lemma 2 (Lemma 3 of [9])** For the equation of conservation of mass in radial symmetry:

$$
\rho_t + u \rho_r + \rho u_r + \frac{N - 1}{r} \rho u = 0,
$$

(9)

there exist solutions,

$$
\rho(t, r) = \frac{f(r/a(t))}{a(t)^N}, \quad u(t, r) = \frac{a(t)}{a(t)^N} r,
$$

(10)

with the form $f \geq 0 \in C^1$ and $a(t) > 0 \in C^1$.

**Lemma 3 (Lemma 12 of [9])** For the ordinary differential equation

$$
\begin{cases} 
\dot{y}(z)y(z)^n - \xi x = 0, \\
y(0) = \alpha > 0, n \neq -1,
\end{cases}
$$

(11)

where $\xi$ and $n$ are constants,

we have the solution

$$
y(z) = \sqrt[n+1]{\frac{1}{2}(n+1)\xi z^2 + \alpha^{n+1}},
$$

(12)

where the constant $\alpha > 0$.

At this stage, we can show the proof of Theorem 1.

**Proof.** Our solutions (6), (7) and (8) fit the mass equation (3) by Lemma (2). Next, for the equation (8), we plug our solutions to check that.
For \( \theta = \gamma = 1 \), we get

\[
\rho(u_t + u \cdot u_r + K [\rho \gamma]_r - (\kappa \rho)_r) = (N - 1) \rho u + u_r - \kappa \rho_r (u_{rr} + N - 1 \frac{u_r}{r} - \frac{N - 1}{r^2} u) \tag{13}
\]

\[
\frac{AeB(\pi \gamma)^2 + C}{a(t)^3} \frac{\dot{a}(t)}{a(t)} + \frac{K \alpha B(\pi \gamma)^2 + C}{a(t)^4} a(t) B \left[ \frac{-2r}{\alpha(t)} \right] - \frac{AN\kappa eB(\pi \gamma)^2 + C}{a(t)^4} a(t) B \left[ \frac{-2r}{\alpha(t)} \right] \frac{\dot{a}(t)}{a(t)} \tag{14}
\]

\[
= 0,
\]

where the function \( a(t) \) is required to be

\[
\frac{\dot{a}(t)}{a(t)} - \frac{2BK}{a(t)} + \frac{BN\kappa a(t)}{a(t)^2} = 0, \quad a(0) = a_0 > 0, \quad \dot{a}(0) = a_1,
\tag{17}
\]

where \( a_0 \) and \( a_1 \) are constants.

For \( \theta = \gamma > 1 \), we have:

\[
\rho(u_t + u \cdot u_r + K [\rho \gamma]_r - (\kappa \rho)_r) = (N - 1) \rho u + u_r - \kappa \rho_r (u_{rr} + N - 1 \frac{u_r}{r} - \frac{N - 1}{r^2} u) \tag{15}
\]

\[
= \frac{y(\frac{r}{\pi(\gamma - 1)}) \dot{a}(t)}{a(t)^N a(t)} + \frac{K \theta y(\frac{r}{\pi(\gamma - 1)})^{\gamma - 1} \dot{y}(\frac{r}{\pi(\gamma - 1)})^{\gamma - 1} \dot{y}(\frac{r}{\pi(\gamma - 1)})}{a(t)^N} a(t) \tag{19}
\]

By defining \( z := r/a(t) \), and requiring

\[
y(z) - y(z)^{\gamma - 1} \dot{y}(z) = 0, \tag{20}
\]

\[
z - y(z)^{\gamma - 2} \dot{y}(z) = 0, \tag{21}
\]

\[\text{[19]}\]

becomes

\[
y(z) \left[ \frac{\dot{a}(t)}{a(t)^N} + \frac{K \gamma y(a(t)^{\gamma - 1} \dot{a}(t))}{a(t)^{N+1}} - \frac{N \kappa \theta a(t)^{\gamma - 1} \dot{a}(t)^{\gamma - 2}}{a(t)^{\gamma N+2}} \right] = 0, \tag{22}
\]

where the function \( a(t) \) is required to be

\[
\frac{\dot{a}(t)}{a(t)^N} + \frac{K \gamma}{a(t)^{N+1}} - \frac{N \kappa \theta a(t)^{\gamma - 1}}{a(t)^{\gamma N+2}} = 0. \tag{23}
\]

Therefore, the equation \([32]\) is satisfied.

With \( n := \theta - 2 \) and \( \xi := 1 \), in Lemma \([32] \tag{21}\) can be solved by

\[
y(z) = \sqrt{\frac{1}{2} \left( \frac{1}{(\theta - 1)} \right) z^2 + \alpha^{\theta - 1}, \tag{24}
\]

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where $\alpha > 0$ is a constant.

For the case of $\frac{7}{2} + \frac{1}{2} - \frac{1}{4} = \theta \geq 1 - \frac{1}{N}$, we have,

$$
\rho(u_t + u \cdot u_r) + K \left[ \rho \gamma \right]_r - (\kappa \rho \gamma)_r \left( \frac{N-1}{r} u + u_r \right) - \kappa \rho \gamma (u_{rr} + \frac{N-1}{r^2} u) = \frac{y(\gamma_\alpha)}{a(t)^N} \left( \frac{N-1}{r} u + u_r \right) + \frac{K \theta y(\gamma_\alpha)}{a(t)^{\gamma N+1}} - \frac{N \kappa \theta y(\gamma_\alpha)}{a(t)^{\gamma N+1}} \frac{y(\gamma_\alpha)}{a(t)^{\gamma N+1}} a(t).
$$

(25)

By letting $a(t) = \sigma(m t + n)^s$, we have

$$
\begin{align*}
\frac{y(\gamma_\alpha)}{a(t)^N} &\cdot \frac{s(s-1)(mt+n)^{s-2}}{\sigma^N (mt+n)^s} \frac{m^2 \sigma r}{a(t)} + \frac{K \theta y(\gamma_\alpha)}{a(t)^{\gamma N+1}} - \frac{sm \kappa \theta y(\gamma_\alpha)}{a(t)^{\gamma N+1}} - \frac{sm \kappa \theta y(\gamma_\alpha)}{a(t)^{\gamma N+1}} - \frac{sm \kappa \theta y(\gamma_\alpha)}{a(t)^{\gamma N+1}} - \frac{sm \kappa \theta y(\gamma_\alpha)}{a(t)^{\gamma N+1}} - \frac{sm \kappa \theta y(\gamma_\alpha)}{a(t)^{\gamma N+1}} - \frac{sm \kappa \theta y(\gamma_\alpha)}{a(t)^{\gamma N+1}} - \frac{sm \kappa \theta y(\gamma_\alpha)}{a(t)^{\gamma N+1}} - \frac{sm \kappa \theta y(\gamma_\alpha)}{a(t)^{\gamma N+1}} - \frac{sm \kappa \theta y(\gamma_\alpha)}{a(t)^{\gamma N+1}} - \frac{sm \kappa \theta y(\gamma_\alpha)}{a(t)^{\gamma N+1}} - \frac{sm \kappa \theta y(\gamma_\alpha)}{a(t)^{\gamma N+1}} - \frac{sm \kappa \theta y(\gamma_\alpha)}{a(t)^{\gamma N+1}} - \frac{sm \kappa \theta y(\gamma_\alpha)}{a(t)^{\gamma N+1}} - \frac{sm \kappa \theta y(\gamma_\alpha)}{a(t)^{\gamma N+1}} - \frac{sm \kappa \theta y(\gamma_\alpha)}{a(t)^{\gamma N+1}} - \frac{sm \kappa \theta y(\gamma_\alpha)}{a(t)^{\gamma N+1}} - \frac{sm \kappa \theta y(\gamma_\alpha)}{a(t)^{\gamma N+1}} - \frac{sm \kappa \theta y(\gamma_\alpha)}{a(t)^{\gamma N+1}}.
\end{align*}
$$

(27)

Here we require that

$$
\begin{align*}
sN - s + 2 = s(\gamma N + 1), \\
s(\gamma N + 1) = s(\theta N + 1) + 1.
\end{align*}
$$

(30)

That is

$$
0 < s = \frac{1}{(\gamma - \theta)N} = \frac{2}{\gamma N - \theta N + 2} \leq 1.
$$

(31)

In the solutions (26), it fits to the conditions (31) by setting $\frac{7}{2} + \frac{1}{2} - \frac{1}{N} = \theta \geq 1 - \frac{1}{N} > 0$ and $s = \frac{2}{\gamma N - \theta N + 2}$. Additionally by defining $z := r/a(t)$, the equation (29) becomes

$$
\begin{align*}
\frac{y(z)}{(mt+n)^{sN-s+2}} \left[ (s-1)m^2 \sigma r \right] + K \gamma \frac{y(z)}{s \sigma^{\gamma N+1}} y(z) - m \kappa \theta y(z) - \frac{m \kappa \theta y(z)}{\sigma^{\theta N+1}} y(z) = \frac{(1-s)m^2}{\sigma^{\gamma N+1}}.
\end{align*}
$$

(32)

Here we require that

$$
\left[ \frac{K \gamma}{s \sigma^{\gamma N+1}} y(z) - \frac{m \kappa \theta y(z)}{\sigma^{\theta N+1}} y(z) \right] y(z) = \frac{(1-s)m^2}{\sigma^{\gamma N+1}} z.
$$

(33)

The proof is completed. ■

In the corollary can be followed immediately:

**Corollary 4** For $m < 0$, the solutions (26) blowup at the finite time $T = -m/n$. 
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