A multiplier approach to the Lance-Blecher theorem

M. Frank

Abstract. A new approach to the Lance-Blecher theorem is presented resting on the interpretation of elements of Hilbert C*-module theory in terms of multiplier theory of operator C*-algebras: The Hilbert norm on a Hilbert C*-module allows to recover the values of the inducing C*-valued inner product in a unique way, and two Hilbert C*-modules \{M_1, \langle \cdot, \cdot \rangle_1\}, \{M_2, \langle \cdot, \cdot \rangle_2\} are isometrically isomorphic as Banach C*-modules if and only if there exists a bijective C*-linear map \( S : M_1 \to M_2 \) such that the identity \( \langle \cdot, \cdot \rangle_1 \equiv \langle S(\cdot), S(\cdot) \rangle_2 \) is valid. In particular, the values of a C*-valued inner product on a Hilbert C*-module are completely determined by the Hilbert norm induced from it. In addition, we obtain that two C*-valued inner products on a Banach C*-module inducing equivalent norms to the given one give rise to isometrically isomorphic Hilbert C*-modules if and only if the derived C*-algebras of "compact" module operators are ∗-isomorphic. The involution and the C*-norm of the C*-algebra of "compact" module operators on a Hilbert C*-module allow to recover its original C*-valued inner product up to the following equivalence relation: \( \langle \cdot, \cdot \rangle_1 \sim \langle \cdot, \cdot \rangle_2 \) if and only if there exists an invertible, positive element \( a \) of the center of \( \mathcal{M}(A) \) such that the identity \( \langle \cdot, \cdot \rangle_1 \equiv a \cdot \langle \cdot, \cdot \rangle_2 \) holds.

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One of the crucial problems of the theory of Hilbert C*-modules is the interrelation of three kinds of isomorphisms between them: Banach C*-module isomorphisms, isometric Banach C*-module isomorphisms and unitary C*-module isomorphisms intertwining the C*-valued inner products. Already in 1985 L. G. Brown gave examples of pairs of C*-valued inner products on a certain Banach C*-module which induce equivalent norms, but without any linking isometric Banach C*-module automorphism between these Hilbert C*-modules, [3, Ex. 6.2, 6.3] (cf. [13, Ex. 2.3], [4]). In 1994 E. C. Lance showed that two Hilbert C*-modules \{\mathcal{M}_1, \langle \cdot, \cdot \rangle_1\}, \{\mathcal{M}_2, \langle \cdot, \cdot \rangle_2\} are isometrically isomorphic as Banach C*-modules if and only if there exists a bijective C*-linear map \( S : \mathcal{M}_1 \to \mathcal{M}_2 \) such that the identity \( \langle \cdot, \cdot \rangle_1 \equiv \langle S(\cdot), S(\cdot) \rangle_2 \) is valid on \( \mathcal{M}_1 \times \mathcal{M}_1 \). [11, Th.]. The theorem of E. C. Lance indicates that the Hilbert norm on a Hilbert C*-module might determine the possible values of the related C*-valued inner product(s), in the best case up to uniqueness. Looking for examples one recalls that for every Hilbert space the concrete values of its inner product can be exactly recovered from the Hilbert norm. Surprisingly, the same turns out to be true for general Hilbert C*-modules as D. P. Blecher obtained in 1995, [1, Th. 3.1, 3.2], [2]. He used the point of
view of operator modules over (non-self-adjoint) operator algebras and elements of the
representation theories of C*-algebras and of Hilbert C*-modules to prove this result.

We want to give an alternative purely C*-algebraic proof of this important fact pointing
out the related background in multiplier theory of C*-algebras. The formula (1) telling
how to recover the values of the C*-valued inner product from the Hilbert norm was
partially suggested by the approach of D. P. Blecher. Beside this, our approach has
the advantage that intertwining isomorphisms of C*-valued inner products on a Banach
C*-module which induce equivalent norms to the given one can be expressed in terms of
the related operator C*-algebras: Two Hilbert C*-modules \( \{M_1, \langle \cdot, \cdot \rangle_1\} \), \( \{M_2, \langle \cdot, \cdot \rangle_2\} \) are
isometrically isomorphic as Banach C*-modules if and only if the derived C*-algebras of
"compact" module operators are *-isomorphic. For this aim results on quasi-multipliers
of C*-algebras are involved due to L. G. Brown, [3]. The involution and the C*-norm
of the C*-algebra of "compact" module operators on a Hilbert C*-module allow to
recover its original C*-valued inner product up to the following equivalence relation:
\( \langle \cdot, \cdot \rangle_1 \sim \langle \cdot, \cdot \rangle_2 \) if and only if there exists an invertible, positive element \( a \) of the center
of \( M(A) \) such that the identity \( \langle \cdot, \cdot \rangle_1 \equiv a \cdot \langle \cdot, \cdot \rangle_2 \) holds for arbitrary elements of the given
Hilbert C*-module. If the center of \( M(A) \) is trivial then one has only to fix the Hilbert
norm on one singular non-zero element of the Hilbert C*-module to make the choice
unique.

We start our investigations recalling some definitions and basic facts from the literature,
cf. [8,9,10,12,15]. We consider Hilbert C*-modules \( \{M, \langle \cdot, \cdot \rangle\} \) over general C*-algebras
\( A \), i.e. (left) \( A \)-modules \( M \) together with an \( A \)-valued inner product \( \langle \cdot, \cdot \rangle : \mathcal{M} \times \mathcal{M} \to A \)
satisfying the conditions:

(i) \( \langle x, x \rangle \geq 0 \) for every \( x \in \mathcal{M} \).

(ii) \( \langle x, x \rangle = 0 \) if and only if \( x = 0 \).

(iii) \( \langle x, y \rangle = \langle y, x \rangle^* \) for every \( x, y \in \mathcal{M} \).

(iv) \( \langle ax + by, z \rangle = a \langle x, z \rangle + b \langle y, z \rangle \) for every \( a, b \in A, x, y, z \in \mathcal{M} \).

(v) \( \mathcal{M} \) is complete with respect to the norm \( \|x\| = \|\langle x, x \rangle\|^{1/2}_A \).

We always suppose, that the linear structures of the C*-algebra \( A \) and of the (left)
\( A \)-module \( \mathcal{M} \) are compatible, i.e. \( \lambda(ax) = (\lambda a)x = a(\lambda x) \) for every \( \lambda \in \mathbb{C}, a \in A, \)
\( x \in \mathcal{M} \). A Hilbert C*-module is said to be full if the norm-closed linear span of the
values of the C*-valued inner product coincides with its C*-algebra of coefficients. Let
us denote the \( A \)-dual Banach \( A \)-module of a Hilbert \( A \)-module \( \{M, \langle \cdot, \cdot \rangle\} \) by \( \mathcal{M}' \), where
\( \mathcal{M}' = \{r : \mathcal{M} \to A : r \text{ is } A \text{-linear and bounded}\} \). A Hilbert C*-module \( \{M, \langle \cdot, \cdot \rangle\} \)
is self-dual if the standard isometric C*-linear embedding \( x \in \mathcal{M} \to \langle \cdot, x \rangle \in \mathcal{M}' \) is
surjective. The class of (self-dual) Hilbert W*-modules is of special interest. Many
pathologies can be avoided for them because the C*-valued inner product lifts always
to the C*-dual Banach W*-module turning it into a self-dual Hilbert W*-module, [15].
To each Hilbert C*-module \( \{M, \langle \cdot, \cdot \rangle\} \) over a C*-algebra \( A \) one can assign a standard
Hilbert W*-module over the bidual W*-algebra \( A^{**} \) of \( A \) in the following way, cf. [14,
Def. 1.3], [15, §4]: Form the algebraic tensor product \( A^{**} \otimes M \) which becomes a (left)
\( A^{**} \)-module defining the action of \( A^{**} \) on its elementary tensors by the formula \( ab \otimes x = \)
given one norm fulfills the identity
\[ \langle a(b \otimes x) \rangle = \sum_{i,j} a_i \otimes x_i \otimes \sum_{j} b_j \otimes y_j = \sum_{i,j} a_i \langle x_i, y_j \rangle b_j \]
on finite sums of elementary tensors one obtains a degenerate $A^{**}$-valued inner product. The factorization of $A^{**} \otimes M$ by the set $\{ z \in A^{**} \otimes M : [z, z] = 0 \}$ gives a Hilbert $A^{**}$-module denoted by $M^\#$ in the sequel. It contains $M$ as a $A$-submodule. If $M$ is self-dual then $M^\#$ is self-dual, too, but the converse conclusion is still an open problem. Every bounded $A$-linear operator $T$ on $M$ has a unique extension to a bounded $A^{**}$-linear operator on $M^\#$ preserving the operator norm. In the following we want to consider several kinds of module operators on Hilbert $C^*$-modules. An $A$-linear bounded operator $K$ on a Hilbert $A$-module $\{ M, \langle \cdot, \cdot \rangle \}$ is "compact" if it belongs to the norm-closed linear hull of the set of elementary operators $\{ \theta_{x,y} : \theta_{x,y}(z) = \langle z, x \rangle y, \ x, y \in M \}$, [10,15]. The set of all "compact" operators on $M$ is denoted by $K_A(M)$. A bounded $A$-linear operator on a Hilbert $C^*$-module $M$ is adjointable if the operator $T^*$ defined by the formula $\langle T(x), y \rangle = \langle x, T^*(y) \rangle$ for all $x, y \in M$ is a bounded $A$-linear operator on $M$. By [9,10] the $C^*$-algebra $K_A(M)$ is a two-sided ideal of the set of all bounded, adjointable module operators $End_A^A(M)$ on $M$ which is $*$-isomorphic to its multiplier $C^*$-algebra. To characterize unitary isomorphisms of Hilbert $C^*$-modules we use the following definition:

**Definition 1.** Let $A$ be a fixed $C^*$-algebra. Two Hilbert $A$-modules $\{ M_1, \langle \cdot, \cdot \rangle_1 \}$ and $\{ M_2, \langle \cdot, \cdot \rangle_2 \}$ are said to be isomorphic as Hilbert $C^*$-modules (or equiv., unitarily isomorphic) if and only if there exists a linear bijective mapping $S : M_1 \to M_2$ such that the equalities $S(ax) = aS(x)$ and $\langle x, y \rangle_1 = \langle S(x), S(y) \rangle_2$ are valid for every $a \in A$, every $x, y \in M_1$.

The literature contains some results about the existence of such isomorphisms between Hilbert $C^*$-modules: If a Hilbert $A$-module $\{ M, \langle \cdot, \cdot \rangle \}$ over a given $C^*$-algebra $A$ is self-dual then every $A$-valued inner product $\langle \cdot, \cdot \rangle_2$ on $M$ inducing an equivalent to the given one norm fulfills the identity $\langle \cdot, \cdot \rangle_2 = \langle S(\cdot), S(\cdot) \rangle_1$ on $M \times M$ for a unique positive invertible bounded $A$-linear operator $S$ on $M$, cf. [5, Th. 2.6]. Similarly, E. C. Lance proved for arbitrary Hilbert $A$-modules $M_1, M_2$ over a fixed $C^*$-algebra $A$ that in case of existence of a bounded $A$-linear adjointable operator $T : M_1 \to M_2$ with dense ranges for $T$ and $T^*$ there exists an unitary Banach $C^*$-module isomorphism of $M_1$ and $M_2$, [12, Prop. 3.8]. Countably generated Hilbert $C^*$-modules are isomorphic as Banach $C^*$-modules if and only if they are isometrically isomorphic as Banach $C^*$-modules, [3, Cor. 4.8, Th. 4.9], [6, Th. 3.1]. To explain what kind of general results one could obtain we prefer to rely on multiplier theory of $C^*$-algebras. The fundamental result of H. Lin cited below appears to be very helpful. (In fact, it extends a well-known result of P. Green and G. G. Kasparov.)

**Proposition 2.** ([13, Th. 1.5, 1.6], cf. [9,10] and [17])
Let $A$ be a $C^*$-algebra and let $\{ M, \langle \cdot, \cdot \rangle \}$ be a Hilbert $A$-module. Then the mapping $\phi$ defined by the formula
\[ \phi : End_A(M, M') \to QM(K_A(M)) \right., \theta_{x,y} \phi(T) \theta_{z,t} = \theta(T(t)(x))z,y \]
for every \( s \in \mathcal{M} \) is an isometric isomorphism of involutive Banach spaces.

The restriction of \( \phi \) to \( \text{End}_A(\mathcal{M}) \) induces an isometric algebraic isomorphism to the Banach algebra \( \text{LM}(K_A(\mathcal{M})) \).

Finally, the restriction of \( \phi \) to \( \text{End}_{A*}^*(\mathcal{M}) \) induces a *-isomorphism to the \( C^* \)-algebra \( M(K_A(\mathcal{M})) \).

Note, that every left Hilbert \( A \)-module \( \mathcal{M} \) can be considered as a right Hilbert \( K_A(\mathcal{M}) \)-module fixing another \( K_A(\mathcal{M}) \)-valued inner product \( \langle x, y \rangle_{op} = \theta_{x,y} \). This point of view gives another interpretation of the left actions of \( M(A) \) and of \( \text{LM}(A) \) on full Hilbert \( A \)-modules \( \mathcal{M} \) by Proposition 2.

Our key observation is that every \( A \)-valued inner product \( \langle ., . \rangle \) on a Hilbert \( C^* \)-module \( \mathcal{M} \) over a given \( C^* \)-algebra \( A \) defines a mapping \( T \) from \( \mathcal{M} \) into its \( A \)-dual Banach \( A \)-module \( \mathcal{M}' \) by the formula \( T : x \in \mathcal{M} \to \langle ., x \rangle \in \mathcal{M}' \). The properties of these mappings \( T \) in terms of multiplier \( C^* \)-theory are the following:

**Proposition 3.** Let \( A \) be a \( C^* \)-algebra and let \( \{\mathcal{M}, \langle ., . \rangle_1\} \) be a Hilbert \( A \)-module. Denote by \( \langle ., . \rangle_2 \) a second \( A \)-valued inner product on \( \mathcal{M} \) inducing an equivalent norm to the given one. Then the mapping \( T : x \in \mathcal{M} \to \langle ., x \rangle_2 \in \mathcal{M}' \) can be identified with a uniquely defined invertible positive element of \( QM(K_A(\mathcal{M})) \subset K_A^{(1)}(\mathcal{M})^{**} \). Conversely, every invertible positive element \( T' \in QM(K_A(\mathcal{M})) \subset K_A^{(1)}(\mathcal{M})^{**} \) induces an \( A \)-valued inner product and an equivalent norm on \( \mathcal{M} \) via the formula \( \langle x, y \rangle_2 = (\phi^{-1}(T')(y))(x) \) for \( x, y \in \mathcal{M} \).

**Proof.** Using the identifications of Proposition 2 made by the mapping \( \phi \) one derives the equality

\[
\theta_{x,y}^{(1)} \phi(T) \theta_{z,t}^{(1)} = \theta_{(x,t)2z,y}^{(1)} \in K_A^{(1)}(\mathcal{M}),
\]

which defines \( \phi(T) \in QM(K_A(\mathcal{M})) \) by the right side of this equality. To show the positivity of the quasi-multiplier \( \phi(T) \) one modifies the equality above setting \( x = t, y = z \). Making use of the identity \( \theta_{x,y} = \theta_{y,x}^* \) valid for every \( x, y \in \mathcal{M} \) one obtains

\[
\langle \theta_{(x,x)2t,t}^{(1)}(s), s \rangle_1 = \langle \langle s, t \rangle_1(z, z)_{2t}z, t, s \rangle_1 = \langle s, t \rangle_1(z, z)_{2(t, s)}_1 \geq 0
\]

for every \( s \in \mathcal{M} \). Since \( \phi(T) \in K_A^{(1)}(\mathcal{M})^{**} \) by construction and since the linear span of the "compact" operators of type \( \theta \) is norm-dense inside \( K_A^{(1)}(\mathcal{M}) \) the positivity of \( \phi(T) \) as an element of the \( W^* \)-algebra \( K_A^{(1)}(\mathcal{M})^{**} \) follows.

To show the invertibility of \( \phi(T) \) inside \( K_A^{(1)}(\mathcal{M})^{**} \) we use a standard construction from the introduction. First, build the Hilbert \( A^{**} \)-module \( \mathcal{M}^\# \) from \( \mathcal{M} \). Both the \( A \)-valued inner products on \( \mathcal{M} \) can be extended in a unique way to \( A^{**} \)-valued inner products on \( \mathcal{M}^\# \). Secondly, take the (self-dual) \( A^{**} \)-dual Hilbert \( A^{**} \)-module \( (\mathcal{M}^\#)' \) of \( \mathcal{M}^\# \). Again, both the inner products can be continued (cf. [15, Th. 3.2, 3.6]), and their extensions are connected by an invertible positive operator \( S \) as described in [5, Th. 2.6]. Obviously, the uniquely defined extension of the operator \( T \) inside \( \text{End}_{A^{**}}((\mathcal{M}^\#)') \) equals \( S^* S \). Hence, the (real) spectrum of \( T \) is deleted away from zero by a positive constant, and \( \phi(T) \) is invertible.
Conversely, set $\langle x, y \rangle_2 = (\phi^{-1}(T')(y))(x)$ for $x, y \in \mathcal{M}$ and for a given invertible positive $T' \in \text{QM}(K_A(\mathcal{M})) \subset K_A^{(1)}(\mathcal{M})^{**}$. As can be easily seen by considerations similar to that above $\langle \cdot, \cdot \rangle_2$ is an $A$-valued inner product on $\mathcal{M}$ inducing an equivalent norm to the given one. ■

**Example 4.** Let $A$ be a C*-algebra. Define the action of $A$ on itself by multiplication from the left. Then $A$ becomes a Hilbert $A$-module setting $\langle a, b \rangle_T = aTb^*$ for every $a, b \in A$ and a fixed positive invertible $T \in \text{QM}(A)$. Vice versa, every $A$-valued inner product on $A$ arises in this manner. If $A$ is unital then $T \in A \equiv \text{QM}(A)$.

**Theorem 5.** (E. C. Lance [11, Th.] / D. P. Blecher, [1, Th. 3.1, 3.2])

Let $A$ be a C*-algebra and $\mathcal{M}$ be a left Banach $A$-module the norm of which is known to be generated by an $A$-valued inner product on $\mathcal{M}$ with unknown values. Then this $A$-valued inner product $\langle \cdot, \cdot \rangle$ on $\mathcal{M}$ is unique, and it can be recovered by the formulae

$$\langle x, x \rangle := \sup \{r(x)r(x)^* : r \in \mathcal{M}', \|r\| \leq 1 \},$$

$$\langle x, y \rangle := \frac{1}{4} \sum_{k=0}^{3} i^k \langle x + i^k y, x + i^k y \rangle$$

for every $x, y \in \mathcal{M}$, where the right side of (1) uses the norm of the underlying Banach $A$-module only.

Consequently, every bijective isometric $A$-linear isomorphism of two Hilbert $A$-modules $S : \{\mathcal{M}_1, \langle \cdot, \cdot \rangle_1\} \to \{\mathcal{M}_2, \langle \cdot, \cdot \rangle_2\}$ identifies the two $A$-valued inner products by the formula $\langle \cdot, \cdot \rangle_1 \equiv \langle S(\cdot), S(\cdot) \rangle_2$, and vice versa.

**Proof.** Let $\langle \cdot, \cdot \rangle_1$, $\langle \cdot, \cdot \rangle_2$ be two $A$-valued inner products on $\mathcal{M}$ giving the norm. Applying again the standard construction from the introduction both the $A$-valued inner products can be continued to $A^{**}$-valued inner products on the self-dual Hilbert $A^{**}$-module $(\mathcal{M}^#)'$. Inside $\text{End}_{A^{**}}((\mathcal{M}^#)')$ exists a positive invertible operator $T$ such that the identity $\langle \cdot, \cdot \rangle_2 \equiv \langle T(\cdot), \cdot \rangle_1$ holds for the continued $A^{**}$-valued inner products on $(\mathcal{M}^#)' \times (\mathcal{M}^#)'$, cf. [5, Prop. 2.2]. By construction one has

$$\|x\| \equiv \|\langle T(x), x \rangle_1\|_A \equiv \|\langle x, x \rangle_1\|_A,$$

$$\|x\| \equiv \|\langle T^{-1}(x), x \rangle_2\|_A \equiv \|\langle x, x \rangle_2\|_A,$$

and by a theorem of W. L. Paschke [15, Th. 2.8] one obtains

$$\|\langle T(x), x \rangle_1\|_A \leq \|T^{1/2}\|^2 \cdot \|\langle x, x \rangle_1\|_A,$$

$$\|\langle T^{-1}(x), x \rangle_2\|_A \leq \|T^{-1/2}\|^2 \cdot \|\langle x, x \rangle_2\|_A.$$

This implies $\|T\| = \|T^{-1}\| = 1$, and by the positivity of $T$ and by general spectral properties of elements of C*-algebras $T = \text{id}_{\mathcal{M}}$ yields.

To show the estimation formula (1) of the values of the $A$-valued inner product one recalls that

$$r(x)r(x)^* \leq \|r\|^2 \langle x, x \rangle$$

for every $x \in \mathcal{M}$, every $r \in \mathcal{M}'$ by [15, Th. 2.8]. Since the $A$-valued inner product $\langle \cdot, \cdot \rangle$ induces an isometric $A$-linear embedding of $\mathcal{M}$ into $\mathcal{M}'$ by the formula $x \to \langle \cdot, x \rangle$ one has only to indicate a sequence $\{r_n : n \in \mathbb{N}\}$ of bounded by one $A$-linear functionals on $\mathcal{M}$.
of this special nature such that the expressions \( \{ r_n(x)r_n(x)^* : n \in \mathbb{N} \} \) converge to the value \( \langle x, x \rangle \) in norm from below. This can be done setting \( r_n = \langle ., ((x, x) + 1/n \cdot 1_A)^{-1/2} \rangle \) for \( n \in \mathbb{N} \). Consequently, the supremum really exists. The second formula is obvious.

To show the last statement one has to consider the two \( A \)-valued inner products \( \langle ., . \rangle_1 \) and \( \langle S(\cdot), S(\cdot) \rangle_2 \) on the Hilbert \( A \)-module \( M_1 \) inducing exactly the same norm for every element of \( M_1 \). Therefore, they take identically the same values by the first part of the proof. □

**Proposition 6.** Let \( A \) be a C*-algebra and \( M \) be a Banach \( A \)-module possessing two \( A \)-valued inner products \( \langle ., . \rangle_1, \langle ., . \rangle_2 \) inducing equivalent to the given one norms on \( M \). Suppose, \( 0 < C, D < \infty \) are the minimal real numbers for which the inequality \( \|x\|_1 \leq C \cdot \|x\|_2 \leq D \cdot \|x\|_1 \) is satisfied for every \( x \in M \). Then the inequality

\[
\langle x, x \rangle_1 \leq C \cdot \langle x, x \rangle_2 \leq D \cdot \langle x, x \rangle_1
\]

is valid for every \( x \in M \) and the same real numbers \( C, D \).

**Proof.** We extend both the \( A \)-valued inner products to \( A^{**} \)-valued inner products to the self-dual Hilbert \( A^{**} \)-module \( (M^\#)' \) using the standard construction. Then there exists a positive invertible operator \( T \) inside \( \text{End}_{A^{**}}((M^\#)') \) such that the identity \( \langle ., . \rangle_2 \equiv \langle T(\cdot), . \rangle_1 \) holds for the continued \( A^{**} \)-valued inner products on \( (M^\#)' \times (M^\#)' \), cf. [5]. Applying [15, Th. 2.8] to the operators \( T \) and \( T^{-1} \) on \( (M^\#)' \) one obtains the minimal real numbers \( C = \|T^{-1}\| \) and \( D = \|T\| \cdot \|T^{-1}\| \) for which the inequality (2) is valid, and these constants equal the minimal constants obtained in the comparison inequality of the two norms. □

**Remark 7.** The expression (1) could make sense for more general Banach C*-modules than Hilbert C*-modules. However, if it would be well-defined for every element \( x \) of a Banach, non-Hilbert C*-module \( M \) then it should be non-C*-linear and/or degenerated, anyway.

For more similar results we refer the reader to the work of D. P. Blecher who has treated Hilbert C*-modules as operator spaces and operator modules over (non-self-adjoint) operator algebras using mainly geometric notions like complete contractability and complete boundedness of mappings, for example, [1,2]. The advantage of our approach comes to light in the following statement characterizing isometric isomorphisms of different C*-valued inner products on a fixed Banach C*-module in terms of *-isomorphisms of the related operator C*-algebras.

**Theorem 8.** Let \( A \) be a C*-algebra and let \( \{ M, \langle ., . \rangle_1 \} \) be a Hilbert \( A \)-module. Let \( \langle ., . \rangle_2 \) be another \( A \)-valued inner product on \( M \) inducing an equivalent to the given one norm. The following conditions are equivalent:

(i) The \( A \)-valued inner product \( \langle ., . \rangle_2 \) on \( M \) is generated by an invertible bounded \( A \)-linear operator \( S \) on \( M \) satisfying the identity \( \langle ., . \rangle_2 \equiv \langle S(\cdot), S(\cdot) \rangle_1 \) on \( M \times M \).

(ii) The positive invertible quasi-multiplier \( T \) of \( K_A^{(1)}(M) \) corresponding to the \( A \)-valued inner product \( \langle ., . \rangle_2 \) by Proposition 3 is decomposable as \( T = S^*S \) for at least one invertible left multiplier \( S \) of \( K_A^{(1)}(M) \).
(iii) The C*-algebra $K_A^{(2)}(M)$ of "compact" operators on $M$ corresponding to the $A$-valued inner product $\langle ., . \rangle_2$ is *-isomorphic to the original C*-algebra of "compact" operators $K_A^{(1)}(M)$.

(iv) The C*-algebra $\text{End}_A^{(2)}(M)$ of adjointable bounded $A$-linear operators on $M$ corresponding to the $A$-valued inner product $\langle ., . \rangle_2$ is *-isomorphic to the original C*-algebra of adjointable bounded $A$-linear operators $\text{End}_A^{(1)}(M)$.

**Proof.** The implications (i)$\iff$(ii) follow from Proposition 2 together with the key Proposition 3. Keeping in mind Proposition 3 one adapts L. G. Brown's results on quasi-multipliers [3, Th. 4.2, Prop. 4.4] of (non-unital) C*-algebras to the C*-algebra $K_A^{(1)}(M)$. For a positive invertible quasi-multiplier $T$ of $K_A^{(1)}(M)$ the C*-subalgebra $T^{1/2}K_A^{(1)}(M)T^{1/2}$ of the bidual W*-algebra $K_A^{(1)}(M)^{**}$ is *-isomorphic to $K_A^{(1)}(M)$ if and only if there exists a left multiplier $S$ of $K_A^{(1)}(M)$ such that $T = S^*S$ inside $K_A^{(1)}(M)^{**}$. Thus, one obtains the equivalence of the conditions (ii) and (iii), cf. [4],[6].

The equivalence of the last two statements is shown in [7].

**Example 9.** Every positive invertible quasi-multiplier $T$ of a (non-unital) C*-algebra $A$ is decomposable as $T = S^*S$ for at least one invertible left multiplier $S$ of $A$ if and only if every pair of $A$-valued inner products $\langle ., . \rangle_1, \langle ., . \rangle_2$ on $A$ inducing equivalent norms to the given C*-norm is connected by an isometric Banach $A$-module isomorphism $S$ of the two corresponding (left) Banach $A$-modules $\{A, \| \cdot \|_1\}, \{A, \| \cdot \|_2\}$, cf. [13, Ex. 2.3] for a counterexample.

Note, that the equivalence of the conditions of Theorem 8 does not hold any longer if one considers C*-valued inner products on different Banach C*-modules and *-isomorphisms of corresponding operator C*-algebras, in general. A counterexample can be found in [6,7]. The canonical question arising is whether the original Hilbert norm can be recovered from the C*-norm of the related operator C*-algebras, or not. The answer is given by the following statement.

**Proposition 10.** Let $A$ be a C*-algebra and $\{M, \langle ., . \rangle_1\}$ be a full Hilbert $A$-module possessing a second $A$-valued inner product $\langle ., . \rangle_2$ which induces an equivalent norm to the given one. Suppose, both the $A$-valued inner products define the same bounded $A$-linear operators on $M$ to be "compact", and both they induce the same involution and C*-norm on this algebra of all "compact" $A$-linear operators. Then there exists an invertible positive element $a$ of the center of the multiplier C*-algebra $M(A)$ of $A$ such that the identity $\langle ., . \rangle_1 \equiv a \cdot \langle ., . \rangle_2$ holds on $M \times M$.

If the center of $M(A)$ is trivial then the condition $\| x \| = 1$ for some fixed non-zero $x \in M$ makes the choice of the $A$-valued inner product on $M$ unique.

**Proof.** Since both the related C*-algebras of "compact" operators coincide, i.e. they are *-isomorphic, Theorem 8 applies: The invertible positive quasi-multiplier $T$ corresponding to the $A$-valued inner product $\langle ., . \rangle_2$ is decomposable as $T = S^*S$ for an invertible left multiplier $S$ which can be considered as a bounded $A$-linear operator on $M$ by Proposition 3. In particular, the inequality

$$\langle K(x), x \rangle_2 = \langle (SK)(x), S(x) \rangle_1 \geq 0$$
holds for every positive "compact" operator $K$, every $x \in \mathcal{M}$. Consequently, $S$ commutes with every positive "compact" operator and belongs to the center of the multiplier C*-algebra \( \text{End}_A^*(\mathcal{M}) \) of \( K_A(\mathcal{M}) \). But, \( \text{Z}(\text{End}_A^*(\mathcal{M})) \) consists of the operators \( \{ a \cdot \text{id}_\mathcal{M} : a \in \text{Z}(\mathcal{M}(A)) \} \), and it is \( * \)-isomorphic to \( \text{Z}(\mathcal{M}(A)) \). No further restrictions apply to $S$ and $T = S^*S$ since \( \| x \|_1 = \| a^{-1/2} \cdot x \|_2 \) and \( \| K(x) \|_1 = \| K(a^{-1/2} \cdot x) \|_2 \) for every $x \in \mathcal{M}$, every $K \in K_A(\mathcal{M})$ (where $T = a \in \text{Z}(\mathcal{M}(A))$). 

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