A Surprising Property of Multidimensional Hamiltonian Systems; Application to Semiclassical Quantization of Phase Space

Maurice de Gosson
Blekinge Institute of Technology
371 79 Karlskrona, Sweden
e-mail: mdg@bth.se

October 27, 2018

Introduction

In the mid 1980’s the mathematician Gromov discovered a very unexpected property of canonical transformations (and hence of Hamiltonian motion). That property, often dubbed the “principle of the symplectic camel”, seems at first sight to be in conflict with the common conception of Liouville’s theorem; it actually only shows that canonical transformations have a far more “rigid” behavior than usual volume-preserving mappings. Here is one description of Gromov’s result (another will be given a moment). Let us cut a circular hole with radius \( r \) in any of the conjugate coordinate planes \( q_j, p_j \) in phase-space \( \mathbb{R}^{2N} \), and consider a phase-space ball with radius \( R \) larger than \( r \). Clearly, we cannot push the ball through the hole: it is too big. However, we can always squeeze it through the hole by deforming it using volume-preserving transformations: conservation of volume does not imply conservation of shape, and we will be able to turn the ball into, for instance, a long ellipsoid with smallest half-axis inferior to \( r \) and pass it through the hole. Now, Hamiltonian flows are volume-preserving because of Liouville’s theorem, and we could therefore hope to use such a flow to perform the squeezing of the ball. But Gromov proved that this is impossible because such a squeezing can never be done if one limits oneself to canonical transformations (and hence, in particular, to Hamiltonian flows): the proverbial camel will never pass through the eye of the needle if it is symplectic!
Here is an alternative statement of the principle of the symplectic camel; it has a strong quantum mechanical flavor – although everything is expressed in classical terms. Consider again a phase-space ball with radius \( R \). Its orthogonal projection on any of the phase-space planes \( q_i, p_j, q_i, q_j \), or \( p_i, p_j \) is obviously a circle with area \( \pi R^2 \). Suppose now we let a Hamiltonian flow act on the ball. It will start distorting; after some time it will perhaps occupy a very large region of phase-space. Some part of this “blob” will thus have to become very thin, and one can hence expect that the areas of at least some of the orthogonal projections on the coordinate planes will shrink and become very small. However Gromov’s theorem implies that the areas of the projections of the distorted ball on the conjugate planes \( q_j, p_j \) will never decrease below their original value \( \pi R^2 \)! This is of course strongly reminiscent of Heisenberg’s uncertainty principle\(^1\): suppose that we choose \( R = \sqrt{\hbar} \); then the area of the projection of the deformed ball on the conjugate planes will always remain \( \geq \frac{1}{2} \hbar \) as time elapses (see \(^1\) for an explicit derivation of a “classical” equivalent form Heisenberg inequalities).

Gromov had stumbled onto something big: his theorem has led to a thriving development of a new field of mathematics, *symplectic topology*, which is related to the theory of periodic orbits for Hamiltonian systems (see \(^1\) and the references therein). The purpose of this Letter is to show that his principle of the symplectic camel also can be used in physics to cast some new light on semiclassical quantization, in particular *EBK* quantization. This is because the principle of the symplectic camel allows the definition of a new quantity, the *symplectic area*, which coincides with the usual notion of area in the two-dimensional case and can be viewed as a generalization of the notion of *action*. We will show that symplectic area is a better candidate than volume for the quantization of phase space; because it allows to recover the usual *EBK* quantum levels for integrable systems by merely assuming that phase-space is subdivided in “quantum blobs” with symplectic area \( (n + \frac{1}{2})\hbar \).

### 1 Symplectic Area

Let \( \Omega \) be a subset of \( \mathbb{R}^{2N} \) (in our applications it will be the interior of an energy shell \( H(q, p) = E \)). We call *symplectic radius* of \( \Omega \), and denote by \( R_\Omega \), the supremum of all \( R \geq 0 \) such that the phase-space ball

\[
B(R) : |q - q_0|^2 + |p - p_0|^2 \leq R^2
\]

\(^1\) Especially since the property ceases to hold if \( q_j, p_j \) is replaced by any pair of nonconjugate coordinates.
can be sent inside $\Omega$ using arbitrary canonical transformations (not just those arising from Hamiltonians). By definition, the \textit{symplectic area} of $\Omega$ is the number $\text{SA}(\Omega) = \pi R_{\Omega}^2$. Notice that it can happen that $\text{SA}(\Omega) = 0$ (no ball can be sent inside $\Omega$) or that $\text{SA}(\Omega) = +\infty$ (every ball, no matter its radius, can be sent inside $\Omega$). Symplectic area coincides with the usual area in two-dimensional phase-space $\mathbb{R}^2$: the radius of the largest disk that can be sent inside a surface with area $\pi R^2$ using canonical transformations is precisely $R$ (because canonical transformations are just the area preserving mappings when $N = 1$, as already remarked before), hence the symplectic area of this region is just its ordinary area $\pi R^2$. However, in higher dimensions symplectic area is not generally directly linked to volume (see however \footnote{Some authors call it ”symplectic capacity” but we find this denomination slightly misleading in our context.} below). For instance, the symplectic area of a ball $B(R)$ in $\mathbb{R}^{2N}$ is obviously

$$\text{SA}(B(R)) = \pi R^2$$

while its volume is

$$\text{Vol} B(R) = \frac{\pi^N R^{2N}}{N!} = \frac{1}{N!}(\text{SA}(B(R)))^N.$$

An essential observation is that symplectic area is a \textit{symplectic invariant}: it is conserved by canonical transformations, that is

$$\varphi \text{ canonical} \implies \text{SA}(\varphi(\Omega)) = \text{SA}(\Omega).$$

One can thus say that while volume is the natural invariant for volume-preserving mappings, symplectic area is the natural invariant for canonical transformations.

A noticeable feature of the symplectic area of a ball with given radius is that it is independent of the dimension of the ambient phase-space (as opposed to its volume). As the number of degrees of freedom increases, the volume of a ball $B(R)$ decreases towards zero, while its symplectic area remains equal to $\pi R^2$. Also, sets with infinite volume can have finite symplectic areas. Consider for instance the phase-space cylinder $Z_j(R)$ with radius $R$ and based on the conjugate coordinate plane $q_j, p_j$: a point $(q, p)$ is inside (or on) $Z_j(R)$ if and only if its $j$-th coordinates $q_j$ and $p_j$ satisfy $q_j^2 + p_j^2 \leq R^2$. The cylinder $Z_j(R)$ has infinite volume if $N > 1$ (if $N = 1$ it is just a circle in phase-plane), but its symplectic area is $\pi R^2$. This readily follows from Gromov’s theorem: suppose we could deform, using canonical transformations, a ball with radius $R' > R$ so that it becomes a volume $\Omega$ fitting inside $Z_j(R)$. \footnote{Some authors call it ”symplectic capacity” but we find this denomination slightly misleading in our context.}
We could then also let \( \Omega \) "fall through the hole" \( q_j^2 + p_j^2 \leq R^2 \) in the plane \( q_j, p_j \); since translations are canonical transformations, and the compose of two canonical transformations still is canonical, we would thus have violated the principle of the symplectic camel. More generally, using the fact that symplectic area is an increasing function of size, we see that any subset \( \Omega \) of phase-space containing a ball and which is itself contained in a cylinder \( Z_j \) with same radius as the ball will have symplectic area \( \pi R^2 \):

\[
B(R) \subset \Omega \subset Z_j(R) \implies \text{SA}(\Omega) = \pi R^2.
\] (4)

We said in the Introduction that symplectic area generalizes the notion of action. Here is why. Suppose that \( \Omega = \Omega(E) \) is bounded by an energy shell \( \Sigma(E) : H(q, p) = E \) for some smooth Hamiltonian \( H \). One shows (\([2, 7]\)) that when \( \Omega(E) \) is both compact and convex, then its symplectic area is equal to the lower limit of all the action integral calculated along periodic Hamiltonian orbits on the energy shell \( \Sigma(E) \):

\[
\text{SA}(\Omega(E)) = \inf \oint_{\gamma} pdq \quad (\gamma \text{ p.o. on } \Sigma(E))
\] (5)

with \( pdq \equiv p_1 dq_1 + \cdots + p_N dq_N \). Moreover, there exists a minimal periodic orbit \( \gamma_{\min} \) for which equality effectively occurs:

\[
\text{SA}(\Omega(E)) = \oint_{\gamma_{\min}} pdq.
\] (6)

This property (the proof of which is far from being trivial; see \([2, 7]\)) does not hold in general if one does not assume \( \Omega(E) \) is compact and connected. Consider for instance a long "Bordeaux bottle" fitting exactly inside a cylinder \( Z_j(R) \), and whose neck has a smaller radius \( r < R \). The capacity of the bottle is \( \pi R^2 \) in view of (4), but the action of a closed orbit encircling its neck is \( \pi r^2 < \pi R^2 \), contradicting formula (6).

2 "Quantum Blobs" vs. Quantum Cells

The use of \( h^N \) as the volume of a "quantum cell" in phase-space in statistical quantum mechanics is justified by inference from a few special cases. It turns out that the consideration of the same particular cases justifies the following definition:

\[3\text{In fact every hypersurface } \Sigma \text{ in phase-space can be viewed as an energy shell for some Hamiltonian: just choose } H \text{ equal to } E \text{ near } \Sigma.\]
**Definition.** A quantum blob is any subset of phase space with symplectic area \((n + \frac{1}{2})\hbar\), where \(n\) is an integer \(\geq 0\).

**Remark 1.** Since symplectic area is a symplectic invariant, a quantum blob will remain a quantum blob in any system of canonical coordinates.

**Remark 2.** Notice that dimension does not matter in this definition: the symplectic area of a quantum blob is independent of which \(\mathbb{R}^{2N}\) we choose to be a host, as it always is \((n + \frac{1}{2})\hbar\). Also, as opposed to a quantum cell, a quantum blob can have infinite volume. For instance, any ball \(B(\sqrt{(2n+1)\hbar})\) is a quantum blob, and so are the cylinders \(Z_j(\sqrt{(2n+1)\hbar})\) for \(j = 1, \ldots, N\); the latter have infinite volume as soon as \(N > 1\).

We now make the following “Quantum Blob Ansatz” (QBA):

**Ansatz:** The only admissible semiclassical motions are those who take place on the boundary of a quantum blob.

We are going to see that this Ansatz leads, in spite of its simplicity, to the correct semiclassical quantum levels for all integrable systems. We begin by showing, as a first application, that it leads to the correct quantum features of an ensemble of \(N\) linear harmonic oscillators with collective Hamiltonian

\[
H = \frac{1}{2m}(|p|^2 + m^2\omega^2|q|^2) = \sum_{j=1}^{N} \frac{1}{2m}(p_j^2 + m^2\omega^2 q_j^2). \tag{7}
\]

The energy shell \(\Sigma(E) : H(q, p) = E\) is the boundary of the ellipsoid

\[
\frac{1}{2m}(|p|^2 + m^2\omega^2|q|^2) = E \tag{8}
\]

with interior \(\Omega(E)\). Performing the symplectic change of variables

\[
(q_j, p_j) \mapsto ((m\omega)^{-1/2}q_j, (m\omega)^{1/2}p_j) \quad (j = 1, \ldots, N)
\]

in (8) the ellipsoid \(\Sigma(E)\) becomes the ball

\[
\frac{\omega}{2}(|p|^2 + |q|^2) = E
\]

with radius \(\sqrt{2E/\omega}\). Since canonical transformations do not affect symplectic areas (see [3]) we have

\[
SA(\Omega(E)) = \text{Cap} B(\sqrt{2E/\omega}) = \frac{2\pi E}{\omega}
\]
and the requirement that Ω(E) should be a quantum blob is thus equivalent to

\[ \frac{2\pi E}{\omega} = (n + \frac{1}{2})\hbar. \]

It follows that the energy can only take the values

\[ E = (n + \frac{1}{2})\omega \frac{\hbar}{2\pi} = (n + \frac{1}{2})\hbar \omega \]

and we have thus recovered the energy levels predicted by quantum mechanics. The volume of the ball \( B(\sqrt{2E/\omega}) \) being related to its symplectic area by formula (2) we have

\[ \text{Vol} B(\sqrt{2E/\omega}) = \frac{1}{N!} \left( \text{SA}(B(\sqrt{2E/\omega})) \right)^N = \frac{1}{N!} \left( \frac{E}{\hbar\omega} \right)^N \]

and we hence recover the density of states predicted by statistical mechanics:

\[ g(E) = \frac{\partial \text{Vol}(E)}{\partial E} = \left( \frac{1}{\hbar\omega} \right)^N \frac{E^{N-1}}{(N-1)!} \]

where \( \text{Vol}(E) \) is the volume of \( B(\sqrt{2E/\omega}) \).

### 3 Quantum Blobs and EBK Quantization

Let us now broaden the discussion to the more general case of arbitrary \( N \)-dimensional integrable systems; the Hamiltonian is \( H = H(q,p) \). We claim that the “quantum blob Ansatz” leads to the semiclassical energy levels predicted by semiclassical mechanics. We are in fact going to show more, namely that the Lagrangian manifolds (“invariant tori”) \( T \) associated to \( H \) automatically satisfy the EBK quantum condition

\[ \frac{1}{\hbar} \oint_{\gamma} pdq - \frac{1}{4} \mu(\gamma) \text{ is an integer } \geq 0 \]

for all loops \( \gamma \) drawn on \( T \) (\( \mu(\gamma) \) is the Maslov index of \( \gamma \); see e.g [4, 6]).

The semiclassical approximation to the quantum energy is then obtained by calculating the energy along the classical trajectories of \( H \) satisfying (9). The first step of the proof consists in introducing action-angle variables \( (\phi, I) = \)

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4 It is sometimes mistakenly believed that the \( \gamma \) have to be periodic orbits of the Hamiltonian itself.
\((\phi_1, ..., \phi_N; I_1, ..., I_N)\). Hamilton’s function becomes \(K(I) = H(p, q)\) and the corresponding equations of motion are

\[
\dot{\phi}_j = \frac{\partial K(I)}{\partial I} \equiv \omega_j(I) \quad, \quad \dot{I}_j = 0 \quad, \quad j = 1, ..., N.
\]

In these coordinates \(\mathbb{T}\) becomes a new Lagrangian manifold \(\mathbb{T}_{\phi,I}\) defined by the condition \(I = I(0) = \text{constant}\), and can be parametrized by the angles \(\phi_j\). The next step is to define a new Hamiltonian function by

\[
K_0(I) = \omega_1(0)I_1 + \cdots + \omega_N(0)I_N
\]

where we have set \(\omega_j(0) = \omega_j(I(0))\) for \(1 \leq j \leq N\). The trajectory for \(K_0\) passing through \((\phi(0), I(0))\) at time \(t = 0\) will lie on \(\mathbb{T}_{\phi,I}\). We now make a new canonical change of variables \((\phi, I) \mapsto (Q, P)\) where the \(Q_j, P_j\) are obtained from the \(\phi_j, I_j\) as the modified polar coordinates

\[
Q_j = \sqrt{2I_j} \cos \phi \quad, \quad P_j = \sqrt{2I_j} \sin \phi.
\]

This transformation brings the Hamiltonian \(K_0\) into the form

\[
H_0(Q, P) = \frac{\omega_1(0)}{2}(P_1^2 + Q_1^2) + \cdots + \frac{\omega_N(0)}{2}(P_N^2 + Q_N^2) \tag{10}
\]

and \(\mathbb{T}_{\phi,I}\) becomes in these coordinates the torus \(\mathbb{T}_{Q,P} = C_1 \times \cdots \times C_N\), where the \(C_j\) are the circles

\[
C_j : P_j^2 + Q_j^2 = \omega_j(0)I_j(0)
\]

lying in the \(Q_j, P_j\) plane. Let now \(\gamma\) be an orbit for the Hamiltonian \(H_0\), starting from some point of \(\mathbb{T}_{Q,P}\). That orbit will not only wind around \(\mathbb{T}_{Q,P}\) but also around each cylinder \(Z_j(\sqrt{\omega_j(0)I_j(0)})\); in view of the quantum blob Ansatz that cylinder must be a quantum blob if the motion is semiclassically admissible, and hence

\[
\pi \omega_j(0)I_j(0) = (n_j + \frac{1}{2})h \tag{11}
\]

for some integer \(n_j \geq 0\). Choose a topological basis \(\epsilon'_1, ..., \epsilon'_N\) of \(\mathbb{T}_{Q,P}\) consisting of the circles \(C_j\) parametrized as \(Q_{jk}(t) = P_{jk}(t) = 0\) if \(k \neq j\) and

\[
Q_{jj}(t) = \omega_j(0)I_j(0) \cos t \quad, \quad P_{jj}(t) = \omega_j(0)I_j(0) \sin t \quad(0 \leq t \leq 2\pi)
\]

for \(j = 1, ..., N\); we have

\[
\oint_{\epsilon'_j} P dQ = \oint_{\epsilon'_j} P_j dQ_j = (n_j + \frac{1}{2})h. \tag{12}
\]
Returning to the original coordinates $q, p$ and denoting by $\epsilon_1, \ldots, \epsilon_N$ the topological basis $\epsilon'_1, \ldots, \epsilon'_N$ expressed in these coordinates, we finally get

$$\oint_{\epsilon_j} pdq = \oint_{\epsilon'_j} PdQ = (n_j + \frac{1}{2})h$$

since action integrals around loops are invariant under canonical transformations. Hence

$$\oint_{\gamma} pdq = \sum_{j=1}^{N} \nu_j \oint_{\epsilon_j} pdq = h \sum_{j=1}^{N} \nu_j n_j + \frac{1}{2}h \sum_{j=1}^{N} \nu_j; \quad (13)$$

since the Maslov index is, by definition,

$$\mu(\gamma) = 2 \sum_{j=1}^{N} \nu_j$$

formula (13) implies that

$$\frac{1}{h} \oint_{\gamma} pdq = \frac{1}{4} \mu(\gamma) + \text{integer}$$

which is precisely the announced EBK condition ($\exists$).

### 4 Concluding Remarks

All the existing proofs of Gromov’s theorem are difficult and make use of sophisticated mathematical techniques. It is probably the reason why it has taken two hundred years after Lagrange wrote down “Hamilton’s equations” to discover the principle of the symplectic camel! We refer to [2, 3, 7, 9] for detailed proofs; a heuristic justification is given in [4].

In the present state of the art symplectic areas are very difficult to calculate explicitly, outside a few particular cases (one of which being the situation of the double inclusion (4)). The notions introduced in this Letter might therefore be for the moment of a more theoretical than practical value. We hope that they might provide some insights in related fields of current research such as Bose-Einstein condensation (in which we obtain a single quantum state: for a BEC the notion of “number of particles” does not make sense, so it might be viewed as a gigantic quantum blob immersed

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5We recall that Lagrange proposed the letter $H$ in his Mécanique Analytique to honor Huygens—not Hamilton who still was in his early childhood at that time!
in some (indefinite dimensional) phase space), or in a better understanding of the Casimir effect (which is related to renormalization questions for the ground energy levels). A natural, less grandiose, application would of course be to find out whether the quantum blob Ansatz would be of some utility in the study of semiclassical quantization of non-integrable Hamiltonian systems. This is plausible, because symplectic area seems to be related to the notion of adiabatic invariance, and one could then envisage applying it to the method of adiabatic switching (see [8]; also [1] in the context of ergodicity).

References

[1] Brown, R., Ott, E. and Grebogi, C. The Goodness of Ergodic Adiabatic Invariants, J. Stat. Phys. 49(3/4), 1987, 511–550

[2] Ekeland, I. and Hofer, H. Symplectic topology and Hamiltonian dynamics, I and II, Math. Zeit. 200, 355–378 and 203 (1990), 553–567

[3] Gromov M. Pseudoholomorphic curves in symplectic manifolds, Invent. Math. 82 (1985), 307–47

[4] de Gosson, M. The Principles of Newtonian and Quantum Mechanics (Imperial College Press, 2001).

[5] de Gosson, M. The symplectic camel and phase space quantization, J. Phys. A: Math. Gen. 34 (2001) 10085–100096

[6] Gutzwiller, M.C. Chaos in Classical and Quantum Mechanics (Springer-Verlag, 1990).

[7] Hofer, H. and Zehnder, E. Symplectic Invariants and Hamiltonian Dynamics, Birkhäuser Advanced Texts (Basler Lehrbücher, Birkhäuser Verlag, 1994)

[8] Skodje, R.T., Borondo,F., and Reinhardt, W.P. J. The semiclassical quantization of nonseparable systems using the method of adiabatic switching. Chem. Phys. 82(10), 1985, 4611–4632

[9] Viterbo, C. Symplectic topology as the geometry of generating functions, Math. Ann., 292 (1992), 685–710.