TWISTED GROUP RINGS 
WHOSE UNITS FORM AN FC-GROUP

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ABSTRACT. Let $U(K,G)$ be the group of units of the infinite twisted group algebra $K\cdot G$ over a field $K$. We describe the FC-centre $\Delta U$ of $U(K,G)$ and give a characterization of the groups $G$ and fields $K$ for which $U(K,G) = \Delta U$. In the case of group algebras we obtain the Cliff-Sehgal-Zassenhaus theorem.

1. Introduction. Let $G$ be a group, $K$ a field and $\lambda: G \times G \to U(K)$ a 2-cocycle of $G$ with respect to the trivial action of $G$. Then the twisted group algebra $K\cdot G$ of $G$ over the field $K$ is an associative $K$-algebra with basis $\{u_g \mid g \in G\}$ and with multiplication defined for all $g,h \in G$ by

$$u_g u_h = \lambda_{g,h} u_{gh}, \quad (\lambda_{g,h} \in \lambda)$$

and using distributivity.

Let $U(K,G)$ be the group of units of $K\cdot G$ and let $\Delta U$ be its subgroup consisting of all elements with finitely many conjugates in $U(K,G)$. This subgroup $\Delta U$ is called the FC-centre of $U(K,G)$. Clearly, if $\Delta U = U(K,G)$, then $U(K,G)$ is an FC-group (group with finite conjugacy classes).

The problem to study the group of units of group rings with FC property was posed by S. K. Sehgal and H. J. Zassenhaus [1]. For a field $K$ of characteristic 0 they described all groups $G$ without subgroups of type $p^\infty$ for which the group of units of the group algebra of $G$ over $K$ is an FC-group. This was spelling for arbitrary groups by H. Cliff and S. K. Sehgal [2].

In this paper we describe the subgroup $\Delta U$ when $K\cdot G$ is infinite. Let $t(\Delta U)$ be the group of all elements of finite order of $\Delta U$. Then $\Delta U$ is a solvable group of length at most 3 and the subgroup $t(\Delta U)$ is nilpotent of class at most 2. This is new even for group algebras. We use this result for the characterization of those cases when $U(K,G)$ has FC property, and obtain a generalization of the Cliff-Sehgal-Zassenhaus theorem for twisted group algebras.
2. The FC-centre of $U(K_A)$. By a theorem of B. H. Neumann [3] the elements of finite order in $\Delta U$ form a normal subgroup which we denote by $t(\Delta U)$, and the factorgroup $\Delta U/t(\Delta U)$ is a torsion free abelian group. Evidently, $\tilde{G} = \{\kappa u_a \mid \kappa \in U(K), a \in G\}$ is a subgroup in $U(K_A(G))$, while $U(K)$ is a normal subgroup in $\tilde{G}$, with factorgroup $\tilde{G}/U(K)$ isomorphic to $G$.

If $x$ is a nilpotent element of the ring $K_A(G)$ then the element $y = 1 + x$ is a unit in $K_A(G)$ and is referred to as a unipotent element of $U(K_A(G))$.

Let $\zeta(G)$ be the centre of the group $G$ and $[g, h] = g^{-1}h^{-1}gh (g, h \in G)$.

**Lemma 1.** Let $K_A(G)$ be an infinite twisted group algebra. Then all unipotent elements of the subgroup $\Delta U$ are central in $\Delta U$.

**Proof.** Let $y = 1 + x$ be a unipotent element of $\Delta U$ and $v \in \Delta U$. Then for a positive integer $k$ we have $x^k = 0$ and by induction on $k$ we will prove $vx = xv$.

The subgroup $\tilde{G} = \{\kappa u_a \mid \kappa \in U(K), a \in G\}$ is infinite and by Poincaré’s theorem the centralizer $S$ of the subset $\{v, y\}$ of $\tilde{G}$ is a subgroup of finite index in $\tilde{G}$. Since $\tilde{G}$ is infinite, $S$ is infinite and $fy = yf$ for all $f \in S$. Then $xf$ is nilpotent and $1 + xf$ is a unit in $K_A(G)$. We can see easily that the set $\{(1 + xf)^{-1}v(1 + xf) \mid f \in S\}$ is finite. Let $v_1, \ldots, v_t$ be the elements of this set and $W_i = \{f \in S \mid (1 + xf)^{-1}v(1 + xf) = v_i\}$. Then $S = \bigcup W_i$ and there exists an index $j$ such that $W_j$ is infinite. Fix an element $f \in W_j$. Any element $q \in W_j, q \neq f$ satisfies

$$(1 + xf)^{-1}v(1 + xf) = (1 + xq)^{-1}v(1 + xq)$$

and

$$v(1 + xf)(1 + xq)^{-1} = (1 + xf)(1 + xq)^{-1}v.$$ 

Then

$$v((1 + xq) + (xf - xq))(1 + xq)^{-1} = ((1 + xq) + (xf - xq))(1 + xq)^{-1}v,$$

$$v(1 + x(f - q)(1 + xq)^{-1}) = (1 + x(f - q)(1 + xq)^{-1})v$$

and

$$vx(f - q)(1 + xq)^{-1} = x(f - q)(1 + xq)^{-1}v.$$ 

Now we use the induction mentioned above. For $k = 1$ the statement is trivial; so we suppose it is true for all $1 \leq n < k$, where $k \geq 2$ is any given integer.

If $m \geq 2$, then by induction hypothesis $x^m v = vx^m$ for all $v \in \Delta U$. Clearly, if $i \geq 1$ then

$$x(f - q)x^i q^i v = (f - q)x^{i+1} q^i v = (f - q)v x^{i+1} q^i = vx(f - q)x^i q^i.$$ 

From (1) we have

$$vx(f - q)(1 - xq + x^2 q^2 + \cdots + (-1)^{k-1} x^{k-1} q^{k-1}) = x(f - q)(1 - xq + x^2 q^2 + \cdots + (-1)^{k-1} x^{k-1} q^{k-1})v.$$
So \((f - q)(v_x - x_v) = 0\).

Now suppose \(v_x \neq x_v\). The element \(q^{-1}f \in \tilde{G}\) can be written as \(\lambda u_h \ (\lambda \in U(K), h \in G)\). By \(v_x - x_v = \sum_{i=1}^{s} \alpha_i u_{g_i} \neq 0\) we have

\[
\sum_{i=1}^{s} \lambda \alpha_i u_{g_i} - \sum_{i=1}^{s} \alpha_i u_{g_i} = 0.
\]

If \(h \in G\) satisfies this equation, then \(g_1 = h g_i\) for some \(j\), and the number of such elements \(h\) is finite. Since \(W_j = \{\lambda u_h \mid \lambda \in U(K)\}\) is an infinite set, there exist \(h\) and different elements \(\lambda_1, \lambda_2 \in K\) such that \(\lambda_1 u_h, \lambda_2 u_h \in W_j\). Then \((\lambda_i u_h - 1)(v_x - x_v) = 0, \ (i = 1, 2)\) and we obtain \((\lambda_1 u_h - \lambda_2 u_h)(v_x - x_v) = 0\). This condition is satisfied only if \(v_x = x_v\) but does not hold.

**Lemma 2.** Let \(K, G\) be an infinite twisted group algebra, \(H\) a finite subgroup of \(\Delta U\) and \(L_H\) the subalgebra of \(K, G\) generated by \(H\). Then the group of units \(U(L_H)\) of the algebra \(L_H\) is contained in \(\Delta U\), and the factorgroup \(U(L_H)/(1 + J(L_H))\) is abelian.

**Proof.** If \(H\) is a finite subgroup of \(\Delta U\) and \(L_H\) is the subalgebra of \(K, G\) generated by \(H\), then \(L_H\) is an algebra of finite rank over \(K\) and its radical \(J(L_H)\) is nilpotent. Then \(U(L_H)\) is a subgroup of \(\Delta U\) and by Lemma 1 all unipotent elements of \(U(L_H)\) are central in \(\Delta U\). Therefore \(1 + J(L_H)\) is a central subgroup of \(\Delta U\) and \(J(L_H) \subset \zeta(L_H)\), where \(\zeta(L_H)\) is the centre of \(L_H\). Then by Theorem 48.3 in [4] (p. 209)

\[
L_H = L_H e_1 \oplus \cdots \oplus L_H e_n \oplus N,
\]

where \(L_H e_i\) is a semiprime algebra (i.e. \(L_H e_i / J(L_H e_i)\) is a division ring), \(N\) is a commutative artinian radical algebra, \(e_1, \ldots, e_n\) are pairwise orthogonal idempotents. By Lemma 13.2 in [4] (p. 57) any idempotent \(e_i\) is central in \(L_H\) and \(U(L_H e_i)\) is isomorphic to the subgroup \((1 - e_i + ze_i \mid z \in U(L_H))\) of \(U(L_H)\).

Since \(U(L_H e_i)\) is a subgroup of the FC-group \(\Delta U\) it is an FC-group, too. As \(J(L_H e_i)\) is nilpotent (see [5]),

\[
U(L_H e_i)/(1 + J(L_H e_i)) \cong U(L_H e_i / J(L_H e_i)).
\]

By Scott’s theorem [7], in the skewfield \(L_H e_i / J(L_H e_i)\) every nonzero element is either central or its conjugacy class is infinite. Thus the FC-group \(U(L_H e_i)/(1 + J(L_H e_i))\) is abelian.

Decomposition (2) implies

\[
L_H / J(L_H) \cong L_H e_1 / J(L_H e_1) \oplus \cdots \oplus L_H e_n / J(L_H e_n)
\]

and

\[
U(L_H) / (1 + J(L_H)) \cong U(L_H / J(L_H)) \cong U(L_H e_1 / J(L_H e_1)) \times \cdots \times U(L_H e_n / J(L_H e_n)).
\]

Therefore \(U(L_H) / (1 + J(L_H))\) is abelian.

\[\square\]
**Theorem 1.** Let $K\Lambda G$ be an infinite twisted group algebra and $t(\Delta U)$ the subgroup of $\Delta U$ consisting of all elements of finite order in $\Delta U$. Then all elements of the commutator subgroup of $t(\Delta U)$ are unipotent and central in $\Delta U$.

**Proof.** Let $H$ be a finite subgroup of $t(\Delta U)$ and $L_H$ the subalgebra of $K\Lambda G$, generated by $H$. Then the elements of the subgroup $H_1 = H \cap (1 + J(L_H))$ are unipotent and (by Lemma 1) central in $\Delta U$. The subgroup $H(1 + J(L_H))$ is contained in $U(L_H)$ and

$$H / H_1 = H / \left( H \cap (1 + J(L_H)) \right) \cong \left( H(1 + J(L_H)) \right) / (1 + J(L_H)).$$

By Lemma 2 the factorgroup $U(L_H)/(1 + J(L_H))$ is abelian. So $H / H_1$ is abelian and the commutator subgroup of $H$ is contained in $H_1$ and consists of unipotent elements.

Since the commutator subgroup of $t(\Delta U)$ is the union of the commutator subgroups of the finite subgroups of $t(\Delta U)$, all elements of the commutator subgroup of $t(\Delta U)$ are unipotent and central in $\Delta U$.

**Theorem 2.** Let $K\Lambda G$ be an infinite twisted group algebra where $\text{char}(K)$ does not divide the order of any element of $\Delta G$. Then $t(\Delta U)$ is abelian.

**Proof.** Let $H$ be a finite subgroup of the commutator subgroup of $t(\Delta U)$. Then (by Theorem 1) $H$ is contained in the centre of $\Delta U$. The set $\{u_g^{-1}Hu_g \mid g \in \Delta G\}$ contains only a finite number of subgroups $H_1, H_2, \ldots, H_r$. The subgroup $L = H_1H_2 \cdots H_r$ is finite and is invariant under inner automorphism $f_g(x) = u_g^{-1}xu_g$ of the ring $K\Lambda \Delta G$, where $g \in \Delta G$. Let $x_1, \ldots, x_r$ be all elements of $L$. Then $y_i = x_i - 1$ is a nilpotent element, and in the commutative ring $L$ the elements $y_1, \ldots, y_r$ commute. Therefore

$$J \cong \left\{ \sum_{i=1}^s \alpha_i y_i \mid \alpha_i \in K, x_i = y_i + 1 \in L \right\}$$

is a nilpotent subring. Let

$$F = \left\{ \sum_{i=1}^s \alpha_i y_i z_i \mid \alpha_i \in K, x_i = y_i + 1 \in L, z_i \in K\Lambda \Delta G \right\}.$$

Let us prove that $F$ is a nilpotent right ideal of $K\Lambda \Delta G$. If $z = \sum_j \beta_j u_{g_j} \in K\Lambda \Delta G$ then $y_i z = \sum_j \beta_j u_{g_j} u_{g_j}^{-1} y_i u_{g_j}$, and $u_{g_j}^{-1} y_i u_{g_j}$ equals one of the elements $y_1, \ldots, y_z$. This and the nilpotency of the ring $J$ imply that $F$ is a nilpotent ring. By Passman’s theorem [6], if $\text{char}(K)$ does not divide the order of any element of $\Delta G$ then $K\Lambda \Delta G$ does not contain nilideals. Therefore $F = 0$, $L = 1$ and the commutator subgroup $t(\Delta U)$ is trivial so $t(\Delta U)$ is abelian.

**Corollary.** Let $K\Lambda \Delta G$ be an infinite twisted group algebra. Then $\Delta U$ is a solvable group of length at most 3, and the subgroup $t(\Delta U)$ is nilpotent of class at most 2.
3. The FC property of $U(K, G)$.

**Lemma 3.** Let $L$ be a subfield of the twisted group algebra $K \times G$, where $K$ is a subfield of $L$, $g \in G$ an element of order $n$ and

$$
\lambda_g = u^n_g = \lambda_{rg}^2 \lambda_{rg}^2 \cdots \lambda_{rg}^n.
$$

If $\alpha^g \neq \lambda_g$ for some $\alpha \in L$ and $\alpha u_g = u_g \alpha$ then $u_g - \alpha$ is a unit in $K \times G$. Furthermore, if $L$ is an infinite field then the number of such units is infinite.

**Proof.** Let $\alpha \in L$, $\alpha^g \neq \lambda_g$ and $u_g \alpha = \alpha u_g$. Then $\lambda_g - \alpha^g$ is a nonzero element of $L$ and

$$(\alpha^{n-1} + \alpha^{n-2} u_g + \cdots + \alpha u_g^{n-2} + u_g^{n-1})(\lambda_g - \alpha^n)^{-1}$$

is the inverse of $u_g - \alpha$. We know that the number of solutions of the equation $x^n - \lambda_g = 0$ in $L$ does not exceed $n$. Thus in an infinite field $L$ there are infinitely many elements not satisfying the equation $x^n - \lambda_g = 0$.

**Lemma 4.** Let $G$ be an infinite locally finite group where char$(K)$ does not divide the order of any element of $G$. If $U(K, G)$ is an FC-group then $G$ is abelian and $K \times G$ is commutative.

**Proof.** Let $W$ be a finite subgroup of $G$. Then the subalgebra $K \times W$ is a semiprime artinian ring and by the Wedderburn-Artin theorem

$$K \times W = M(n_1, D_1) \oplus \cdots \oplus M(n_t, D_t),$$

where each $D_k$ is a skewfield and $M(n_k, D_k)$ is a full matrix algebra. Let $e_{ij}, e_{ji}$ be matrix units in $M(n_k, D_k)$ and $i \neq j$. Then the unipotent elements $1 + e_{ij}, 1 + e_{ji}$ are central in $K \times G$ (see Theorem 1) which is impossible if $i \neq j$. Thus $n_k = 1$ and $K \times W$ is a direct sum of skewfields, $K \times W = D_1 \oplus D_2 \oplus \cdots \oplus D_t$ and

$$U(K \times W) = U(D_1) \times U(D_2) \times \cdots \times U(D_t).$$

By Scott’s theorem [7] any nonzero element of a skewfield is either central or has an infinite number of conjugates. Therefore $K \times W$ is a direct sum of fields and $W$ is abelian. Since $G$ is a locally finite group, $G$ is abelian and $K \times G$ is a commutative algebra.

**Lemma 5.** Let $K \times G$ be infinite and char$(K)$ does not divide the order of any element of the normal torsion subgroup $L$ of $G$. If $U(K \times G)$ is an FC-group then all idempotents of $K \times L$ are central in $K \times G$.

**Proof.** Let the idempotent $e \in K \times L$ be noncentral in $K \times G$. Then there exists $g \in G$ such that $e u_g \neq u_g e$. The subgroup $H = \langle g^{-1} \text{supp}(e) g | i \in \mathbb{Z} \rangle$ is finite and for any $a \in G$ the subalgebra $K \times H$ of $K \times L$ is invariant under the inner automorphism $\phi(x) = u_a^{-1} x u_a$. It is easy to see (by Lemma 4) that $K \times H$ is a commutative semisimple $K$-algebra of finite rank and the idempotent $e \in K \times H$ is a sum of primitive idempotents. Consequently, there exists a primitive idempotent $f$ of $K \times H$ which does not commute with $u_g$. Then the idempotents $f$ and $u_g^{-1} f u_g$ are orthogonal and $(u_g f)^2 = u_g f u_g f = u_g^2 u_g^{-1} f u_g f = 0$. By Theorem 1 the unipotent element $1 + u_g f$ commutes with $u_g$ and $(1 + u_g f) u_g = u_g (1 + u_g f)$ implies $u_g f = f u_g$, which is impossible. Thus, all idempotents of $K \times L$ are central in $K \times G$. 

LEMMA 6. Let $U(K,G)$ be an FC-group and $t(G)$ the set of all elements of finite order in $G$. Then
\begin{enumerate}
    \item $G$ is an FC-group;
    \item if there exists an infinite subfield $L$ in the centre of $K,G$ such that $L \supseteq K$ then $t(G)$ is central in $G$ and \( \lambda_{g,h} = \lambda_{h,g} \) \( h \in t(G), g \in G \).
\end{enumerate}

PROOF. If $U(K,G)$ is an FC-group then $G = \{ Xg \mid X \in U(K), g \in G \}$ is an FC-group. Clearly $U(K)$ is normal in $G$ and $G \cong \tilde{G} / U(K)$. We conclude that $G$ is an FC-group as it is a homomorphic image of the FC-group $\tilde{G}$.

Let $L$ be an infinite field which satisfies condition 2 of the lemma. Then by Lemma 3 for any $h \in t(G)$ there exists a countable set $S = \{ \alpha_i \mid i \in \mathbb{Z} \}$ such that $u_h - \alpha_i$ is a unit for all $i \in \mathbb{Z}$. Suppose that $u_g u_h = u_h u_g$ for some $g \in G$. Next we observe that the equality
\[(u_h - \alpha_i)u_g(u_h - \alpha_i)^{-1} = (u_h - \alpha_j)u_g(u_h - \alpha_j)^{-1}\]
holds only in case $\alpha_i = \alpha_j$. Since
\[(u_h - \alpha_i)(u_h - \alpha_j)^{-1} = 1 + (\alpha_j - \alpha_i)(u_h - \alpha_j)^{-1},\]
we obtain $(\alpha_i - \alpha_j)(u_g u_h - u_h u_g) = 0$ and $\alpha_i = \alpha_j$. It follows that the set
\[\{(u_h - \alpha_j)u_g(u_h - \alpha_j)^{-1} \mid i \in \mathbb{Z}\}\]
is infinite which contradicts the condition that $U(K,G)$ is an FC-group. Then $u_g u_h = u_h u_g$, therefore $[g, h] = 1$, $t(G) \subseteq \z(\tilde{G})$ and $\lambda_{g,h} = \lambda_{h,g}$ \( h \in t(G), g \in G \).

LEMMA 7. Let $G$ be an abelian torsion group, $K,G$ a commutative semisimple algebra and $v$ an idempotent of $K,G$. If $K,Gv$ contains a finite number of idempotents then $K,Gv$ is a direct sum of finitely many fields.

PROOF. If $e_1, \ldots, e_s$ are all the idempotents of $K,Gv$, then
\[L = \langle \text{supp}(e_1), \ldots, \text{supp}(e_s) \rangle\]
is a finite subgroup in $G$ and $K,Lv$ is a direct sum of finitely many fields,
\[K,Lv = (K,Lv)f_1 \oplus \cdots \oplus (K,Lv)f_s,\]
where $f_1, \ldots, f_s$ are orthogonal primitive idempotents of $K,Lv$. The corresponding direct sum in $K,Gv$ is
\[K,Gv = (K,Gv)f_1 \oplus \cdots \oplus (K,Gv)f_s.\]

We show that the element $0 \neq x \in (K,Gv)f_i$ is a unit. $R = \langle L, \text{supp}(x) \rangle$ is a finite subgroup and $K,Rv$ is a direct sum of finitely many fields,
\[K,Rv = (K,Rv)l_1 \oplus \cdots \oplus (K,Rv)l_s,\]
and each idempotent $f_i$ is either equal to an idempotent $l_j$ or is a sum of these idempotents. If $f_i = l_j$ then $x f_i \in (K,Rv)f_i$ and $x$ is a unit in $(K,Lv)f_i$. If $f_i = l_{i_1} + l_{i_2}$ \( (l_{i_1}, l_{i_2} \in K,Lv) \) then $(K,Lv)f_i = (K,Lv)l_{i_1} \oplus (K,Lv)l_{i_2}$, but this does not hold.
THEOREM 3. Let $K\gamma G$ be an infinite twisted group algebra of $\text{char}(K\gamma G) = p$, such that $t(G)$ contains a $p$-element and either the field $K$ is perfect or for any element $g \in G$ of order $p^k$, the element $u_g^p$ is algebraic over the prime subfield of $K$. Then $U(K\gamma G)$ is an FC-group if and only if $G$ is an FC-group and satisfies the following conditions:

1. $p = 2$ and $|G'| = 2$;
2. $t(G)$ is central in $G$ and $t(G) = G' \times H$, where $H$ is abelian, and has no 2-elements;
3. $K\gamma H$ is a direct sum of a finite number of fields;
4. $\{\lambda^{-1}_{h, h^{-1}, g} \lambda_{h^{-1}, g, h} \mid h \in H\}$ is a finite set for all $g \in G$.

PROOF (NECESSITY). By Lemma 6 $G$ is an FC-group. Let $g$ be an element of order $p^k$. Then $u_g^p = \lambda_g \in U(K)$ and in the perfect field $K$ we can take a $p^k$-th root of $\lambda_g$ which we denote by $\mu$. If $K_0$ is the prime subfield of $K$ and $\lambda_g$ is algebraic over $K_0$ then $K_0(\lambda_g)$ is a finite field and so it is perfect. Thus $u_g - \mu$ is nilpotent and $1 + \mu - u_g$ and (by Theorem 1) $1 - (u_g - \mu)u_a$ is central in $U(K\gamma G)$. Then for any $b \in G$ by

$$u_b(1 - (u_g - \mu)u_a) = (1 - (u_g - \mu)u_a)u_b$$

implies

(4) $$u_bu_gu_a - \mu u_bu_a - u_gu_a + \mu u_a u_b = 0.$$

Each $u_g$ can be written in the form $\mu + (u_g - \mu)$ and so $\mu^{-1}u_g = 1 + \mu^{-1}(u_g - \mu)$. Thus $\mu^{-1}u_g$ is an unipotent element and it commutes with $u_b$ and $u_a$. Then (4) can be written as

(5) $$u_gu_bu_a - u_gu_a u_b - \mu u_bu_a + \mu u_a u_b = 0.$$

If $[a, b] = 1$ then, by (5), we have $(\lambda_{a, b} - \lambda_{b, a})(u_g - \mu) = 0$. From this equation we get that the coefficient of $u_g$ must be zero and $\lambda_{a, b} = \lambda_{b, a}$. Thus, $u_g u_b = u_a u_b$.

Let $[a, b] \neq 1$. Then by (5), $u_g u_b u_a = -\mu u_a u_b$ and $u_g u_a u_b = -\mu u_b u_a$. So

(6) $$\begin{cases} u_g = -\mu [u_a^{-1}, u_b^{-1}]^{-1}, \\ u_g = -\mu [u_a^{-1}, u_b^{-1}]. \end{cases}$$

Consequently $u_g^2 = \mu^2$ and $(u_g\mu^{-1})^2 = 1$. Note that in (6) $g$ may be any $p$-element, further $a$ and $b$ may be any noncommuting elements of $G$. This is possible only if $p = 2$. Then the commutator subgroup $\hat{G}'$ of group $\hat{G} = \{\kappa u_a \mid \kappa \in U(K), a \in G\}$ is of order 2 and coincides with the Sylow 2-subgroup of $\hat{G}$. Thus $\hat{G}' \subseteq \zeta(\hat{G})$ and $\hat{G}$ is a nilpotent group of class at most 2. Let

$$L = \langle \mu u_h \mid \mu \in U(K), h \in t(G) \rangle.$$

Then $L / U(K)$ is nilpotent torsion group and its 2-Sylow subgroup is of order 2. Here $L$ is abelian because $\hat{G}'$ is of order 2 and it is a subgroup in $L$. Therefore $t(G)$ is abelian and

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1 If $K, H$ is a group ring then $H$ is a finite abelian group.
$t(G) = S \times H$, where $S = \langle g \mid g^2 = 1 \rangle$ is the Sylow 2-subgroup of $t(G)$ and all elements of $H$ are of odd order.

We show that $K_\lambda H$ is central in $K_\lambda G$. Let $h \in H, a \in G$ and $[u_a, u_h] \neq 1$. Then

\[ [u_a, u_h] = \mu u_g \]

and

\[ \lambda u_{a^{-1}h^{-1}ah} = \mu u_g. \]

It is clear that $[a, h] \in H$ and the order of $[a, h]$ is odd because $H$ is normal in $G$. Since $g$ is a 2-element, (7) does not hold. Thus $K_\lambda H$ is central in $K_\lambda G$ and $t(G) \subseteq \zeta(G)$.

Let us prove that $K_\lambda H$ contains only a finite number of idempotents. Suppose $K_\lambda H$ contains an infinite number of idempotents $e_1, e_2, \ldots$. If $d, b \in G$ and $[b, d] = g \neq 1$ then $g^2 = 1$ and (by Lemma 5) $1 - e_i + u_d e_i$ is a unit. Clearly,

\[ (1 - e_i + u_d e_i)^{-1} u_b (1 - e_i + u_d e_i) = u_b (1 - e_i + \mu u_g e_i), \]

where $\mu = \lambda_{d, d}^{-1} \lambda_{b, b}^{-1} \lambda_{d^{-1}b, d^{-1}b}^{-1}$.

If $i \neq j$ then $1 - e_i + \mu u_g e_i \neq 1 - e_j + \mu u_g e_j$. Indeed, if $1 - e_i + \mu u_g e_i = 1 - e_j + \mu u_g e_j$, then $(e_i - e_j)(\mu u_g - 1) = 0$. Since $e_i, e_j \in K_\lambda H$ and $u_g \notin K_\lambda H$, the last equality is true only in case $i = j$. Therefore if $i \neq j$ then $1 - e_i + \mu u_g e_i \neq 1 - e_j + \mu u_g e_j$ and $u_b$ has an infinite number of conjugates, which does not hold. Thus $K_\lambda H$ contains a finite number of idempotents $e_1, e_2, \ldots, e_t$, and (by Lemma 7) $K_\lambda H$ is a direct sum of a finite number of fields.

Since \( \{u_g^{-1} u_h u_g \mid g \in G\} \) is a finite set, we obtain condition 4 of the theorem.

**Sufficiency.** Let the conditions of the theorem be satisfied. We prove that $U(K_\lambda G)$ is an FC-group.

Let $G' = \langle a \mid a^2 = 1 \rangle$ be the commutator subgroup of $G$ and $\mu^2 = \lambda_{a, a}$. Thus the ideal $\mathfrak{F} = K_\lambda G(u_a - \mu)$ is nilpotent.

In $K_\lambda G$ we choose a new basis $\{w_g \mid g \in G\}$,

\[ w_g = \begin{cases} u_g, & \text{if } g \in G \setminus \langle a \rangle, \\ \mu^{-1} u_g, & \text{if } g \in \langle a \rangle. \end{cases} \]

Let $G = \cup b \langle a \rangle$ be the decomposition of the group $G$ by the cosets of $\langle a \rangle$. Any element $x + \mathfrak{F} \in K_\lambda G / \mathfrak{F}$ can be written as

\[ x + \mathfrak{F} = \sum_i \alpha_i w_{b_i} + \sum_i \beta_i w_{b_i} w_a + \mathfrak{F} \]

\[ = \sum_i \alpha_i w_{b_i} + \sum_i \beta_i w_{b_i} (w_a - 1) + \sum_i \beta_i w_{b_i} + \mathfrak{F} = \sum_i (\alpha_i + \beta_i) w_{b_i} + \mathfrak{F}. \]

We show that $K_\lambda G / \mathfrak{F}$ is commutative. Indeed

\[(w_g + \mathfrak{F})(w_h + \mathfrak{F}) = w_g w_h + \mathfrak{F} = w_h w_g + \mathfrak{F} = w_h w_g + \mathfrak{F},\]

and the commutator $[w_g, w_h]$ is either 1 or $w_a$. If $[w_g, w_h] = w_a$ then

\[ w_g w_h + \mathfrak{F} = w_h w_g w_a + \mathfrak{F} = w_h w_g (w_a - 1) + w_h w_g + \mathfrak{F} = w_h w_g + \mathfrak{F}. \]
We will construct the twisted group algebra $K_{\mu}H$ of the group $H = G/\langle a \rangle$ over the field $K$ with the system of factors $\mu$.

Let $R(H) = R(G/\langle a \rangle)$ be a fixed set of representatives of all left cosets of the subgroup $\langle a \rangle$ in $G$ and $H = \langle h_i = b_i\langle a \rangle \mid b_i \in R(G/\langle a \rangle) \rangle$. Let $t_h$ denote element $w_h \in \mathbb{Z}$ if $h_i, h_j = h_k$, then $b_i b_j = b_k a^s \ (s = \{0, 1\})$, and

\[ t_i t_j = w_{b_i} w_{b_j} + \mathbb{Z} = \lambda_{b_i, b_j} w_{b_i a^w} + \mathbb{Z} = \lambda_{b_i, b_j} \lambda_{b_i, a}^{-1} w_{b_i a^w} + \mathbb{Z}. \]

Let $\mu_{b_i, b_j} = \lambda_{b_i, b_j} \lambda_{b_i, a}^{-1}$ and $\mu = \{\mu_{a, b} \mid a, b \in H\}$. Let $\{t_h \mid h \in H\}$ be a basis of the twisted group algebra $K_{\mu}H$ with the system of factors $\mu$. Clearly $t_0 t_0 = \mu_{b_0, b_0} = I$. Let $t(H)$ be the set of elements of finite order of $H$ and $H = \cup c_i t(H)$ the decomposition of the group $H$ by the cosets of the subgroup $t(H)$. Then $x, x^{-1} \in U(K_{\mu}H)$ can be written as

\[ x = \sum_{i=1}^{t} \alpha_i t_{c_i} \quad \text{and} \quad x^{-1} = \sum_{i=1}^{s} \beta_i t_{d_i}, \]

where $\alpha_i, \beta_j$ are nonzero elements of $K_{\mu}t(H)$. The subgroup

\[ L = \langle \text{supp}(\alpha_1), \ldots, \text{supp}(\alpha_t), \text{supp}(\beta_1), \ldots, \text{supp}(\beta_s) \rangle \]

is finite and $K_{\mu}L$ is a direct sum of fields

\[ (8) \quad K_{\mu}L = e_1 K_{\mu}L \oplus \cdots \oplus e_n K_{\mu}L. \]

Let $x e_k = \sum_{i=1}^{n} \gamma_i t_{c_i}$ and $x^{-1} e_k = \sum_{i=1}^{m} \delta_i t_{d_i}$, where $\gamma_i, \delta_j$ are nonzero elements of the field $K_{\mu}L e_k$.

We know [8], that a torsion free abelian group is orderable. Therefore we can assume that

\[ c_i \ t(H) < c_{i+1} t(H) < \cdots < c_n t(H) \]

and

\[ d_i \ t(H) < d_{i+1} t(H) < \cdots < d_m t(H). \]

Then $c_i, d_j t(H)$ is called the least and $c_i, d_j t(H)$ is called the greatest among the elements of the form $c_i, d_j t(H)$. It is easy to see that $c_i, d_j t(H) < c_i, d_j t(H)$ if $n > 1$ or $m > 1$. Therefore $\gamma_i d_{i+1} c_{i} t_{c_{i}} \neq \gamma_{m} d_{m} e_{l_{m}} t_{d_{m}}$. Since $x^{-1} e_k x e_k = e_k$, we have $n = m = 1$, $x e_k = \gamma t_{c_i}$ and $x^{-1} e_k = \gamma^{-1} t_{c_i}^{-1}$. Thus $x$ and $x^{-1}$ can be written as

\[ x = \sum_{i=1}^{t} \gamma_i t_{c_i} \quad \text{and} \quad x^{-1} = \sum_{i=1}^{t} \gamma_i^{-1} t_{c_i}^{-1}, \]

where $\gamma_1, \ldots, \gamma_t$ are orthogonal elements.

Let $\phi: K_2 G/\langle a \rangle \rightarrow K_{\mu}H$ be an isomorphism of these algebras. If $x \in U(K_2 G)$ then $\phi(x + \mathbb{Z}) = \sum_{i=1}^{t} \gamma_i t_{c_i}$ where $\gamma_i \in K_{\mu}L e_i$. It is easy to see that there exists an abelian subgroup $L$ of $G$ such that $L = L/\langle a \rangle$. The algebra $K_2 L$ is commutative and its radical is a nilpotent ideal equal to $\mathbb{Z} \cap K_2 L$. Since $K_{\mu}L/\langle 3 \cap K_2 L \rangle \cong K_2 L$, the classical method
of lifting idempotents yields idempotents \( f_1, \ldots, f_t \) in \( K/L \) such that \( f_1 + \cdots + f_t = 1 \) and \( f_1 + 3 = e \). Then \( x = xf_1 + \cdots + xf_t \) and \( \phi(xf_1 + 3) = \gamma_i t_i \), where \( h_i = b_i(a), b_i \in G \). There exists an element \( v_i \in K/L \) such that \( \phi(v_i + 3) = \gamma_i t_i \). We can find an element \( r \in 3 \) such that \( xf_i = (v_i + rf_i)w_{g_i} \).

Clearly \( s_i = v_i + rf_i \) is a unit in \( K/L \) and is central in \( K/G \). So \( s_1, \ldots, s_t \) are orthogonal and \( x = \sum_{i=1}^t s_i w_{g_i} x^{-1} = \sum_{i=1}^t s_i x w_{g_i}^{-1} \). Since \( s_i \in \zeta(K/G), x^{-1} w_{g_i} x = \sum_{i=1}^t w_{g_i}^{-1} w_{g_i} w_{g_i} \) for any \( g \in G \). By condition 4 our theorem \( w_{g_i} \) has a finite number of conjugates, because \( G \) is an FC-group. Thus \( U(K/G) \) is an FC-group.

**Lemma 8.** Let \( K \) be a field such that \( \text{char}(K) \) does not divide the order of any element of \( t(G) \), \( K/t(G) \) a commutative algebra that does not contain a minimal idempotent. Then for any idempotent \( e \in K/t(G) \) there exists an infinite set of idempotents \( e_1 = e, e_2, \ldots \) such that

\[
(9) \quad e_k e_{k+1} = e_{k+1} \quad (k \in \mathbb{N}).
\]

**Proof.** Suppose \( K/t(G) \) does not contain a minimal idempotent. First we prove that for any idempotent there exists an infinite set of idempotents \( e_1, e_2, \ldots \) in \( K/t(G) \) satisfying condition (9).

Let \( e_1 \) be an idempotent of \( K/t(G) \) and \( H_1 = \langle \text{supp}(e_1) \rangle \). Then the ideal \( K/t(G)e_1 \) is not minimal and so contains a proper ideal \( \mathfrak{Z}_1 \) of \( K/t(G) \). Let \( 0 \neq x_1 \in \mathfrak{Z}_1 \) and \( H_2 = \langle H_1, \text{supp}(x_1) \rangle \). Then \( \mathfrak{Z}_1 = \mathfrak{Z}_1 \cap K/t(G)H_2 \) is an ideal of \( K/t(G) \) and \( \mathfrak{Z}_1 \) is generated by the idempotent \( e_2 \) because \( H_2 \) is a finite subgroup of \( t(G) \) and the commutative algebra \( K/t(G)H_2 \) is semiprime. It is easy to see that \( e_1 = e_2 + f, f \neq 0 \) and \( e_1 e_2 = e_2 \). Indeed, if \( f = 0 \), then \( e_1 = e_2 \) and \( K/t(G)e_1 = K/t(G)e_2 \subset \mathfrak{Z}_1 \), which does not hold. The ideal \( K/t(G)e_2 \) contains a proper ideal \( \mathfrak{Z}_2 \) of \( K/t(G) \). We choose a nonzero element \( 0 \neq x_2 \in \mathfrak{Z}_2 \) and consider the subgroup \( H_3 = \langle H_2, \text{supp}(x_2) \rangle \). The ideal \( \mathfrak{Z}_2 = \mathfrak{Z}_2 \cap K/t(G)H_3 \) is generated by the idempotent \( e_3 \) and \( e_2 e_3 = e_3 \neq e_2 \). This method enables us to construct an infinite number of idempotents \( e_1, e_2, \ldots \) satisfying condition (9), which completes the proof.

**Lemma 9.** Let \( K \) be a field such that \( \text{char}(K) \) does not divide the order of any element of \( t(G) \), and \( U(K/G) \) an FC-group. If the commutative algebra \( K/t(G) \) contains an infinite number of central idempotents \( f_i, f_2, \ldots \) and \( g = [a, b] (a, b \in G) \) is an element of order \( n \) then the commutators \( [u_a, u_b] \) and \( [a, b] \) have the same order and

\[
(10) \quad (f_i - f_j)(1 - [u_a, u_b]) = 0
\]

for some \( i \neq j \).

**Proof.** Let \( g = [a, b] \neq 1 \) where \( a, b \in G \). By B. H. Neumann's theorem \( G/t(G) \) is abelian, thus \( g \in t(G) \) and \( 1 - f_i + u_b f_i \) is a unit in \( K/G \). The element \( u_a \) has a finite number of conjugates in \( U(K/G) \) and

\[
(1 - f_i + u_b^{-1} f_i)u_a (1 - f_i + u_b f_i) = u_a (1 - f_i + [u_a, u_b] f_i).
\]
Consequently there exist \( i \) and \( j \) \((i < j)\), such that
\[
1 - f_i + [u_a, u_b]f_i = 1 - f_j + [u_a, u_b]f_j
\]
and
\[
(f_i - f_j)(1 - [u_a, u_b]) = 0.
\]
If \( n \) is the order of \( g = [a, b] \) then
\[
[u_a, u_b]^n = \gamma \in U(K).
\]
Then by (11) we have \( \gamma(f_i - f_j) = f_i - f_j \). So \( \gamma = 1 \) and
\[
[u_a, u_b]^n = 1.
\]

**Theorem 4.** Let \( K_\lambda G \) be an infinite twisted group algebra, and \( \text{char}(K) \) does not divide the order of any element of \( t(G) \). If \( K_\lambda t(G) \) contains only a finite number of idempotents then \( U(K_\lambda G) \) is an FC-group if and only if \( G \) is an FC-group and the following conditions are satisfied:
1. all idempotents of \( K_\lambda t(G) \) are central in \( K_\lambda G \);
2. \( \{\lambda^{-1}_{h,h^{-1}g,h} \mid h \in H\} \) is a finite set for every \( g \in G \);
3. \( K_\lambda t(G) \) is a direct sum of a finite number of fields;
4. if \( K_\lambda t(G) \) is infinite then it is central in \( K_\lambda G \).

**Proof (Necessity).** By Lemmas 4, 6 and 7 \( K_\lambda t(G) \) is commutative, \( G \) is an FC-group and all idempotents of \( K_\lambda t(G) \) are central in \( K_\lambda G \). Since \( \{u_g^{-1} u_g \mid g \in G\} \) is a finite set, condition 2 of the theorem is satisfied.

Since \( K_\lambda t(G) \) contains only a finite number of idempotents (by Lemma 7) \( K_\lambda t(G) \) is a direct sum of a finite number of fields. Let \( K_\lambda t(G) \) be infinite and \( K_\lambda t(G) e_i \) a field in this direct decomposition of \( K_\lambda t(G) \). Lemma 5 implies that \( K_\lambda t(G) e_i \) is invariant under the inner automorphism \( \psi(x) = u_g^{-1} x u_g \) for any \( g \in G \). Since \( \langle u_g, K_\lambda t(G) e_i \setminus \{0\} \rangle \) is an FC-group there exists a finite subfield \( L_y \) of \( K_\lambda t(G) e_i \) such that \( y u_g = u_g y \) for every \( y \in L \). Let \( H = \langle g, t(G) \rangle \). Then \( K_\lambda H \) is subalgebra of \( K_\lambda G \) and (by Lemma 6) \( K_\lambda t(G) \) is central in \( K_\lambda H \).

**Sufficiency.** Let \( K_\lambda t(G) \) be a direct sum of fields,
\[
K_\lambda t(G) = F_1 \oplus F_2 \oplus \cdots \oplus F_l.
\]
Then \( F_i = K_\lambda t(G) e_i \), where \( e_i \) is a central idempotent in \( K_\lambda G \). It is easy to see that \( K_\lambda G \) is a direct sum of ideals
\[
K_\lambda G = K_\lambda G e_1 \oplus \cdots \oplus K_\lambda G e_l.
\]
Let us prove that \( K_\lambda G e_q \) is isomorphic to a crossed product \( F_q \ast H \) of the group \( H = G / t(G) \) and the field \( F_q \).
Let $R_1(G/t(G))$ be a fixed set of representatives of all left cosets of the subgroup $t(G)$ in $G$. Any element $x \in K \cdot Ge_q$ can be written as

$$x = e_q u_{c_i} \gamma_1 + \cdots + e_q u_{c_i} \gamma_s,$$

where $\gamma_k \in K \cdot t(G), c_k \in R_1(G/t(G))$. If $c_k c_j = c_k h$ ($h \in t(G)$) then

$$u_{c_i} u_{c_j} = u_{c_i} \gamma_{c_i, c_j} = u_{c_i} \lambda_{c_i, c_j} = u_{c_i} u_h \lambda_{c_i, c_j}.$$

We will construct the crossed product $F_q \ast H$, where

$$H = \{ h \in c_i t(G) \mid c_i \in R_1(G/t(G)) \}.$$

Let $\alpha \in F_q$ and $\sigma$ be a map from $H$ to the group of automorphism $\text{Aut}(F_q)$ of the field $F_q$ such that $\sigma(h)(\alpha) = u_i^{-1} \alpha u_i$ and let $\mu h_h = u_h \lambda_{c_i, c_j}^{-1} \lambda_{c_i, c_j}$.

Clearly, the set $\mu = \{ \mu a, b \in U(F_q) \mid a, b \in H \}$ of nonzero elements of the field $F_q$ satisfies

$$\mu_{a, b} \mu_{h, c} = \mu_{a, b} \mu_{h, c},$$

where $\alpha \in F_q$ and $a, b, c \in H$.

Then $F_q \ast H = \{ \sum_{h \in H} w_h \alpha_h \mid \alpha_h \in F_q \}$ is a crossed product of the group $H$ and the field $F_q$ and we have $w_d w_d = w_d \mu_{d, d}$ and $\alpha w_d = w_d \alpha^{\sigma(d)}$.

Clearly, $F_q \ast H$ and $K \cdot Ge_q$ are isomorphic because

$$u_{c_i} \alpha u_{c_j} = u_{c_i} u_{c_j} (u_i^{-1} \alpha u_i) = u_{c_i} u_{c_j} \alpha^{\sigma(c_i)}.$$

We know [5] that the group of units of the crossed product $K \ast H$ of the torsion free abelian group $H$ and the field $K$ consists of the elements $w_h \alpha$, where $\alpha \in U(K), h \in H$.

By (12), for every $y \in U(K \cdot G)$,

$$y = u_i \gamma_1 + \cdots + u_i \gamma_t$$

and

$$y^{-1} = u_i^{-1} \gamma_1^{-1} + \cdots + u_i^{-1} \gamma_t^{-1},$$

where $\gamma_1, \ldots, \gamma_t$ are orthogonal elements.

Let $x = \delta_1 u_{d_1} + \cdots + \delta_l u_{d_l} \in U(K \cdot G).$ Then

$$y x y^{-1} = u_i \gamma_1 \delta_1 u_{d_1} u_i^{-1} \gamma_1^{-1} + \cdots + u_i \gamma_l \delta_l u_{d_l} u_i^{-1} \gamma_l^{-1}.$$

If $K \cdot t(G)$ is infinite then $K \cdot t(G) \subseteq \zeta(K \cdot G)$ and

$$y x y^{-1} = \sum_{i=1}^l \delta_i u_{c_i} u_{d_i} u_i^{-1} = \sum_{i=1}^l \delta_i \lambda_{c_i, d_i}^{-1} \lambda_{c_i, d_i} \lambda_{c_i, d_i} u_{c_i, d_i}^{-1} u_{c_i, d_i}^{-1}.$$

Since $G$ is an FC-group, by condition 2 of the theorem, $x$ has a finite number of conjugates, so $U(K \cdot G)$ is an FC-group.

If $K \cdot t(G)$ is finite then $F_q$ is a finite field and

$$y^{-1} x y = \sum_{i=1}^l \gamma_i^{-1} u_{c_i}^{-1} \delta_i u_{d_i} u_i \gamma_i = \sum_{i=1}^l \lambda_{c_i, c_i}^{-1} \lambda_{c_i, d_i} \lambda_{c_i, d_i}^{-1} \gamma_i^{-1} \delta_i \gamma_i \lambda_{c_i, c_i}^{-1} u_{c_i, d_i}^{-1} u_{c_i, d_i}.$$

Since $G$ is an FC-group and $F_q$ is a finite field, $x$ has a finite number of conjugates, so $U(K \cdot G)$ is an FC-group.
THEOREM 5. Let $KzG$ be infinite and $\text{char}(K)$ does not divide the order of any element of $t(G)$. If $K\lambda t(G)$ contains an infinite number of idempotents then $U(K\lambda G)$ is an FC-group if and only if $G$ is an FC-group and the following conditions are satisfied:

1. $K\lambda t(G)$ is central in $KzG$ and contains a minimal idempotent;
2. $\{\lambda_{h,1}^{-1}, \lambda_{h,2}^{-1}, g \lambda_{h,1}^{-1}, g \lambda_{h,2}^{-1}, h \mid h \in H\}$ is a finite set for any $g \in G$;
3. the commutator subgroups of $G$ and of $\tilde{G} = \{\kappa u_a \mid \kappa \in U(K), a \in G\}$ are isomorphic and $G'$ is either a finite group or isomorphic to the group $\mathbb{Z}(q^\infty)$ ($q \neq p$), and there exists an $n \in \mathbb{N}$, such that the field $K$ does not contain the primitive $q^n$-th root of $1$;
4. for every finite subgroup $H$ of the commutator subgroup of $G$ the element $e_H = \frac{1}{|H|} \sum_{h \in H} h$ is an idempotent of $K\lambda t(G)$, and $K\lambda t(G)(1 - e_H)$ is a direct sum of a finite number of fields.

PROOF (NECESSITY). By Lemmas 4, 6 and 7 $K\lambda t(G)$ is commutative, $G$ is an FC-group and all idempotents of $K\lambda t(G)$ are central in $KzG$.

Let us prove that $K\lambda t(G)$ contains a minimal idempotent. Suppose the contrary. Let $a, b \in G$ and $1 \neq [a, b] = g$. Since $g$ is an element of finite order $n$, by Lemma 9, $[u_a, u_b]^n = 1$ and

$$f = \frac{1}{n}(1 + [u_a, u_b] + [u_a, u_b]^2 + \cdots + [u_a, u_b]^{n-1})$$

is an idempotent. By Lemma 11, for $1 - f$ one can construct an infinite sequence of idempotents $e_1 = 1 - f, e_2, \ldots$, satisfying (9). By Lemma 9,

$$(1 - [u_a, u_b])(e_i - e_j) = 0,$$

where $i < j$. Consequently $([u_a, u_b])^k(e_i - e_j) = (e_i - e_j)$ for all $k$ and $f(e_i - e_j) = e_i - e_j$. This implies $(1 - f)(e_i - e_j) = 0$. Since $e_1 = 1 - f$, $e_1(e_i - e_j) = 0$. If we multiply this equality from the right by the elements $e_2, \ldots, e_{i-1}$, by (9) we obtain $e_{i-1} - e_j = 0$. Now we arrived at a contradiction, which proves that $K\lambda t(G)$ contains a minimal idempotent.

It is easy to see that $t(G)$ is infinite, otherwise $K\lambda t(G)$ would contain a finite number of idempotents. $K\lambda t(G)$ contains a minimal idempotent $e$, and so there exist only a finite number of elements $g \in t(G)$, such that $eug = e$. Consequently $K\lambda t(G)e$ is an infinite field and contains $K$ as a subfield. Then as in the proof of Theorem 4, $K\lambda t(G)$ is central in $KzG$.

Since $\{u_g^{-1}u_hu_g \mid g \in G\}$ is a finite set, we obtain condition 2 of the theorem.

Suppose $c \in G'$ and

$$c = [a_1, b_1][a_2, b_2] \cdots [a_n, b_n].$$

Since $K\lambda t(G)$ is central in $KzG$ and $1 - e_i + e_iu_h \in U(K\lambda t(G))$ we have

$$\prod_{k=1}^n(1 - e_i + e_iu_h^{-1})u_{a_k}(1 - e_i + e_iu_h) = \prod_{k=1}^n(u_{a_k}(1 - e_i + e_i[u_{a_k}, u_h])) = \prod_{k=1}^n(u_{a_k})(\prod_{k=1}^n(1 - e_i + e_i[u_{a_k}, u_h]))$$

If $KzG$ is a group ring, then 1 and 3 imply 4 (see [6] p. 690, Lemma 4.3, also [10]).
for all $i \in \mathbb{N}$. Since each $u_{a_1}, u_{a_2}, \ldots, u_{a_n}$ has a finite number of conjugates, there are only a finite number of different elements of the form $\prod_{k=1}^{n}(1 - e_t + e_l[u_{a_k}, u_{b_k}])$. These elements will be denoted by $w_1, \ldots, w_t$. Let

$$W_r(c) = \left\{ i \in \mathbb{N} \mid \prod_{k=1}^{n}(1 - e_t + e_l[u_{a_k}, u_{b_k}]) = w_r \right\}.$$ 

It is easy to see that the set of natural numbers $\mathbb{N}$ is the union of the subsets $W_i(c)$ ($i = 1, \ldots, r$), of which at least one is infinite. If $W_i(c)$ is infinite and $i, j \in W_i(c)$ then

$$(e_i - e_j)\left(1 - \prod_{k=1}^{n}[u_{a_k}, u_{b_k}]\right) = 0.$$ 

This implies that if

$$\prod_{k=1}^{n}[u_{a_k}, u_{b_k}] = \gamma \in U(K)$$

then $\gamma = 1$.

Now we prove that the commutator subgroups of $G$ and of $\tilde{G} = \{ \kappa u_a \mid \kappa \in U(K), a \in G \}$ are isomorphic. It is easy to see that the map $\tau(\lambda u_a) = g \left( \lambda \in U(K), g \in G \right)$ is a homomorphism from $\tilde{G}$ to $G$. Every element $h \in \tilde{G}$ can be written as

$$h = [u_{a_1}, u_{b_1}][u_{a_2}, u_{b_2}] \cdots [u_{a_n}, u_{b_n}].$$

As we have shown above, if $h = \lambda \in U(K)$ then $\lambda = 1$. Thus, $\tau$ is an isomorphism from $\tilde{G}$ to $G'$.

Let $H$ be a finite subgroup of $\tilde{G}'$. Then $e_H = \frac{1}{|H|} \sum_{h \in H} h$ is an idempotent of $K \lambda(t(G))$. Suppose that $K \lambda(t(G))(1 - e_H)$ contains an infinite number of idempotents $e_1, e_2, \ldots$. If $H = \{h_1, h_2, \ldots, h_s\}$, then, as it is shown above, for every $h_j \in H$,

$$\mathbb{N} = W_{1}(h_1) \cup \cdots \cup W_{r_j}(h_j),$$

where $j = 1, 2, \ldots, s$, and for every $k \neq l$, $W_{k}(h_l)$ and $W_{l}(h_k)$ have empty intersection.

It is clear that there exists an infinite subset $M = W_{i_1}(h_1) \cap \cdots \cap W_{i_s}(h_s)$. If $i, j \in M$, then by (13), we have $(e_i - e_j)(1 - h_r) = 0$ for any $r$. Then

$$e_i - e_j = \frac{1}{|H|} \sum_{r=1}^{s}(h_r(e_i - e_j)) = e_H(e_i - e_j).$$

Since $e_i - e_j \in K \lambda(t(G))(1 - e_H)$, by (14),

$$(e_i - e_j)(1 - e_H) = (e_i - e_j) - e_H(e_i - e_j) = 0.$$ 

Thus, $K \lambda(t(G))(1 - e_H)$ contains a finite number of idempotents, and by Lemma 7, it can be given as a direct sum of a finite number of fields.

Let us prove that there exists only finitely many elements of prime order in $G'$.

Suppose the contrary. If $a, b \in G$ then $1 \neq [a, b] = g \in t(G)$. As we have seen above, if $h \in G'$, then there exists $\mu \in U(K)$ such that the order of the element $\mu u_h$ equals the
order of $h$. Then there exists a countably infinite subgroup $S$, generated by elements of prime order, such that $\langle g \rangle \cap S = 1$. By Prüfer’s theorem [9] $S$ is a direct product of cyclic subgroups $S = \prod_i \langle a_i \rangle$. If $q_j$ is the order of the element $a_j$, then

$$e_j = \frac{1}{q_j} \left( 1 + \mu a_j + (\mu a_j)^2 + \cdots + (\mu a_j)^{q_j-1} \right)$$

is a central idempotent and $x_i = 1 - e_i + e_i \mu a_i \in U(K, G)$. By Lemma 9, $(e_i - e_j)(1 - \mu a_i) = 0$. Since $g \notin S$, we have $i = j$, which does not hold. Consequently $G'$ contains only a finite number of elements of prime order and satisfies the minimum condition for subgroups (see [8]). Then

$$G' \cong P_1 \times P_2 \times \cdots \times P_t \times H,$$

where $P_i = \mathbb{Z}(q^{\infty})$ and $|H| < \infty$. Let us prove that either $G' = \mathbb{Z}(q^{\infty})$ or $|G'|$ is finite.

Let $a, b \in G$ and $1 \neq [a, b] = g \in t(G)$. Suppose there exists $l$ such that $g \notin P_l = \langle a_1, a_2, \ldots \mid a_l^q = 1, a_{j+1} = a_j \rangle$. Then

$$e_k = \frac{1}{q_l} \left( 1 + \mu a_k + (\mu a_k)^2 + \cdots + (\mu a_k)^{q_l-1} \right)$$

is an idempotent, and $(e_i - e_j)(1 - \mu a_k) = 0$. This is true only for $i = j$, if $g \notin P_l$, which is impossible. Thus, $G' \cong \mathbb{Z}(q^{\infty})$ or $G'$ is a finite subgroup.

Let $K$ be a field which contains a primitive $q^n$-th root $e$ of 1 for all $n$ and

$$P_1 = \langle a_1, a_2, \ldots \mid a_l^q = 1, a_{j+1} = a_j \rangle.$$

Put

$$e_j = \frac{1}{q_j} \left( 1 + e_j \mu a_j + (e_j \mu a_j)^2 + \cdots + (e_j \mu a_j)^{q_j-1} \right).$$

If $i \neq j$ then the element $(e_l - e_j)(1 - \mu a_k) \neq 0$ and by Lemma 9 this is impossible. Thus there exists $n \in \mathbb{N}$ such that $K$ does not contain a primitive $q^n$-th root $e_n$ of 1.

**Sufficiency.** Let us prove that any element $u_g (g \in G)$ has a finite number of conjugates in $U(K, G)$.

Let $G = \{ ku_a \mid k \in U(K), a \in G \}$. We prove that $H = \langle [u_g, \bar{G}] \rangle$ is a finite subgroup in $G'$. If $G'$ is finite, it is obvious. If $G'$ is infinite then it is isomorphic to a subgroup of the group $\mathbb{Z}(q^{\infty})$. Any element of $\bar{G}$ is of the form $\mu u_h$ ($\mu \in U(K), h \in G$) and

$$[u_g, \mu u_h] = \lambda_{g,h}^{-1} \lambda_{h,g}^{-1} \lambda g^{-1} h^{-1} \lambda g^{-1} h^{-1} g \lambda g^{-1} h^{-1} g h u_g^{-1} h^{-1} g h.$$

Since $G$ is an FC-group, and for a fixed element $g$ the set $\{ \lambda_{g,h}^{-1} \lambda g^{-1} h^{-1} \mid h \in H \}$ is finite, the number of commutators $[u_g, \mu u_h]$ is finite. These commutators generate a finite cyclic subgroup $H$ of $\mathbb{Z}(q^{\infty})$. The element $e_H = \frac{1}{|H|} \sum_{h \in H} h$ is an idempotent in $K, t(G)$ and by condition 4 of the theorem, $K, t(G)(1 - e_H)$ is a direct sum of a finite number of fields $K, t(G)(1 - e_H) f_i (i = 1, \ldots, s)$.

In $K, t(G)$ we have the decomposition

$$K, t(G) = K, t(G) e_H \oplus K, t(G) f_1 \oplus \cdots \oplus K, t(G) f_s.$$
Then

\[ K \times G = K \times G_f \oplus K \times G_f \oplus \cdots \oplus K \times G_f. \]

If \( x \in U(K \times G) \) then

\[ x = x e_H + x f_1 + \cdots + x f_t \]

and

\[ x^{-1} = x^{-1} e_H + x^{-1} f_1 + \cdots + x^{-1} f_t. \]

Consequently

\[ x^{-1} u_g x = x^{-1} e_H u_g x e_H + x^{-1} f_1 u_g x f_1 + \cdots + x^{-1} f_t u_g x f_t. \]

We show that the element \( x e_H \) is central in \( U(K \times G) \). If \( x = \alpha_1 u_{h_1} + \cdots + \alpha_t u_{h_t} \), then

\[ u_g x e_H = \alpha_1 u_{g h_1} e_H + \cdots + \alpha_t u_{g h_t} e_H = \alpha_1 u_{h_1} u_g [u_g, u_{h_1}] e_H + \cdots + \alpha_t u_{h_t} u_g [u_g, u_{h_t}] e_H \]

and \( [u_g, u_{h_k}] \in H \). Clearly, \( [u_g, u_{h_k}] e_H = e_H \) and

\[ u_g x e_H = \alpha_1 u_{h_1} u_g e_H + \cdots + \alpha_t u_{h_t} e_H = x e_H u_g. \]

\( K \times G_f \) is a crossed product \( F \ast H \) of the group \( H = G / n(G) \) and the field \( F = K \times t G_f \).

We know (see [5]) that the group of units of the crossed product \( F \ast H \) of a torsion free abelian group \( H \) and a field \( F \) consists of the elements \( \alpha u_{h} \) (\( \alpha \in U(F), h \in H \)). The unit \( x f_i \) can be given as \( \alpha_i u_{h_i} \) (\( h_i \in G \)), where \( \alpha_i \) is central in \( U(K \times G_f) \). Thus

\[ x^{-1} f_i u_g x f_i = u_{h_i}^{-1} \alpha_i^{-1} u_g \alpha_i u_{h_i} = u_{h_i}^{-1} u_g u_{h_i} = \lambda_{h_i^{-1} h_i, h_i}^{-1} \lambda_{h_i^{-1} g h_i, h_i}^{-1} \lambda_{h_i^{-1} g h_i, h_i}^{-1} u_{h_i}^{-1} g h_i. \]

Therefore

\[ x^{-1} u_g x = u_g + \sum_{i=1}^{t} \lambda_{h_i^{-1} h_i, h_i}^{-1} \lambda_{h_i^{-1} g h_i, h_i}^{-1} \lambda_{h_i^{-1} g h_i, h_i}^{-1} u_{h_i}^{-1} g h_i. \]

Since \( G \) is an FC-group, by condition 2 of the theorem, \( u_g \) has a finite number of conjugates in \( U(K \times G) \). \( \blacksquare \)

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