ZEROS OF ANALYTIC FUNCTIONS,
WITH OR WITHOUT MULTIPLEITIES

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Abstract. The classical Mason–Stothers theorem deals with nontrivial polynomial solutions to the equation \( a + b = c \). It provides a lower bound on the number of distinct zeros of the polynomial \( abc \) in terms of \( \deg a \), \( \deg b \) and \( \deg c \). We extend this to general analytic functions living on a reasonable bounded domain \( \Omega \subset \mathbb{C} \), rather than on the whole of \( \mathbb{C} \). The estimates obtained are sharp, for any \( \Omega \), and a generalization of the original result on polynomials can be recovered from them by a limiting argument.

1. Introduction

Given a polynomial \( p \) (in one complex variable), write \( \deg p \) for the degree of \( p \) and \( \tilde{N}(p) = \tilde{N}_C(p) \) for the number of its distinct zeros in \( \mathbb{C} \). The so-called \( abc \) theorem, also known as Mason’s or Mason–Stothers’ theorem, reads as follows.

\textbf{Theorem A.} Suppose \( a, b \) and \( c \) are relatively prime polynomials, not all constants, satisfying \( a + b = c \). Then

\begin{equation}
\max\{\deg a, \deg b, \deg c\} < \tilde{N}(abc).
\end{equation}

Of course, the two sides of (1.1) are positive integers, so (1.1) is equivalent to saying that the left-hand side does not exceed \( \tilde{N}(abc) - 1 \).

More general variants of this result can be found in Mason’s book [15], while the current version is essentially due to Stothers [19]. Various approaches to and consequences of Theorem A are discussed in [10, 11, 14, 17], the most impressive corollary being probably Fermat’s Last Theorem for polynomials. The argument leading from \( abc \) to Fermat is delightfully simple and elegant, so we take the liberty of reproducing it here.

To prove that there are no nontrivial polynomial solutions to the Fermat equation \( P^n + Q^n = R^n \) for \( n \geq 3 \), apply the \( abc \) theorem with \( a = P^n \), \( b = Q^n \) and \( c = R^n \). Write

\[ d = \max\{\deg P, \deg Q, \deg R\} \]
and note that the left-hand side of (1.1) equals $nd$. As to the right-hand side, $$\tilde{N}(abc),$$ we now have $$\tilde{N}(abc) = \tilde{N}(PQR) \leq \deg(PQR) \leq 3d,$$ whence it follows that $n < 3$.

The importance of Theorem A is also due to the fact that it served as a prototype (under the classical analogy between polynomials and integers) for the famous abc conjecture in number theory. The conjecture, as formulated by Masser and Oesterlé in 1985, states that for every $\varepsilon > 0$ there is a constant $K(\varepsilon)$ with the following property: whenever $a$, $b$ and $c$ are relatively prime positive integers satisfying $a + b = c$, one has

$$c \leq K(\varepsilon) \cdot \{\text{rad}(abc)\}^{1+\varepsilon}.$$

Here, we write $\text{rad}(\cdot)$ for the radical of the integer in question, defined as the product of the distinct primes that divide it. See, e.g., [10, 14] for a discussion of the abc conjecture and its potential applications. So far the conjecture remains wide open.

Going back to the polynomial case, let us point out the following “abc...xyz theorem” (or “$n$-theorem”), which generalizes Theorem A to sums with any finite number of terms.

**Theorem B.** Let $p_0, p_1, \ldots, p_n$ be linearly independent polynomials. Put $p_{n+1} = p_0 + \cdots + p_n$ and assume that the zero-sets $p_0^{-1}(0), \ldots, p_{n+1}^{-1}(0)$ are pairwise disjoint. Then

$$(1.2) \quad \max\{\deg p_0, \ldots, \deg p_{n+1}\} \leq n\tilde{N}(p_0p_1\ldots p_{n+1}) - \frac{n(n + 1)}{2}.$$ 

When $n = 1$, this reduces to Theorem A. In fact, the assumption on the zero-sets can be relaxed to $\bigcap_{j=0}^{n+1} p_j^{-1}(0) = \emptyset$, in which case the quantity $n\tilde{N}(p_0p_1\ldots p_{n+1})$ gets replaced by $\sum_{j=0}^{n+1} \tilde{N}(p_j)$. The latter variant was given by Gundersen and Hayman in [11, Sect. 3], along with a far-reaching generalization from polynomials to entire functions; in addition, it was shown there that (1.2) is asymptotically sharp as $n \to \infty$.

In connection with Theorem B, we also mention Brownawell and Masser’s early work [2], as well as subsequent extensions to polynomials on $\mathbb{C}^m$ (see [13, 16]) and their conjectural analogs in number theory. It should be noted that those extensions – at least the stronger result in [13] – relied heavily on Nevanlinna’s value distribution theory of meromorphic functions, supplemented with some recent developments [22]. The Nevanlinna theory (or, more precisely, Cartan’s version thereof) was also the main tool in [14] when proving the appropriate version of Theorem B for entire functions. The analogy between Nevanlinna’s value distribution theory and Diophantine approximations in number theory was unveiled and explored by Vojta [21].

In this paper, we are concerned with “local” versions of the abc theorem – and more generally, of Theorem B – for analytic functions, a topic not encountered (to the best of our knowledge) in the existing literature. This time, the functions will live on a bounded – and reasonably nice – simply connected domain $\Omega \subset \mathbb{C}$ rather than on the whole of $\mathbb{C}$. The role of $\deg p$, the degree of a polynomial, will of
course be played by $\Omega(f)$, the number of zeros (counted with their multiplicities) that $f$ has in $\Omega$. Another quantity involved will be $\tilde{\Omega}(f)$, the number of the function’s distinct zeros in $\Omega$. The Nevanlinna value distribution theory, which was crucial to earlier approaches in the “global” setting, will now be replaced by the Riesz–Nevanlinna factorization theory on the disk (transplanted, if necessary, to $\Omega$). Specifically, Blaschke products will be repeatedly employed.

In what follows, we distinguish two cases. First, we assume that the functions involved have finitely many zeros in $\Omega$ (which enables us to count the zeros, with or without multiplicities, and compare the quantities that arise). Of course, this is automatic for functions that are analytic on some larger domain containing $\Omega \cup \partial \Omega$, and we actually begin by imposing this stronger assumption. We then weaken the hypotheses, this time taking $\Omega$ to be the unit disk $D$, by allowing that the functions be analytic on $D$ and nicely behaved on $T := \partial D$ but still requiring that each of them have at most finitely many zeros in $D$. The results pertaining to this “finitely many zeros” situation are stated and discussed in Section 2, and then proved in Section 3 below.

Secondly, we consider the case of infinitely many zeros (the functions being again analytic on $D$ and suitably smooth up to $T$). Now it makes no sense to count the zeros, but the appropriate substitutes for $N_D(\cdot)$ and $\tilde{N}_D(\cdot)$ are introduced and dealt with. This is done in Section 4.

Our method can be roughly described as a mixture of algebraic and analytic techniques. The algebraic part is elementary and mimics the reasoning that leads to the classical $abc$ theorem, as presented, e.g., in [10]. The analytic component involves certain estimates from [3, 5, 20] that arose when studying the canonical (Riesz–Nevanlinna) factorization in various classes of “smooth analytic functions”. In connection with this last topic, which has a long history, let us also mention the seminal paper [12], the monograph [18], and some further developments in [4, 6, 7].

A special case of our current results from Section 2 was announced in [8], in quite a sketchy form.

We conclude this introduction by asking if our local $abc$-type theorems might suggest, by analogy, any number-theoretic results or conjectures. So far, none have occurred to us.

## 2. FINITELY MANY ZEROS: RESULTS AND DISCUSSION

Throughout the rest of the paper, $\Omega$ is a bounded simply connected domain in $C$ such that $\partial \Omega$ is a rectifiable Jordan curve. We write $dA$ for area measure and $ds$ for arc length, and we endow the sets $\Omega$ and $\partial \Omega$ with the measures $dA/\pi$ and $ds/(2\pi)$, respectively. The normalizing factors are chosen so as to ensure that the former (resp., latter) measure assigns unit mass to the disk $D := \{z \in C : |z| < 1\}$ (resp., to the circle $T := \partial D$). The $L^p$-spaces (and norms) on $\Omega$ and $\partial \Omega$ are then defined in the usual way, with respect to the appropriate measure.

In this section, we mainly restrict ourselves to functions that are analytic on $\text{clos} \Omega := \Omega \cup \partial \Omega$, i.e., analytic on some open set containing $\text{clos} \Omega$. Clearly, such functions (when non-null) can only have finitely many zeros in $\Omega$, if any.
Let \( f \) be analytic on \( \text{clos} \Omega \), and suppose that \( a_1, \ldots, a_l \) are precisely the distinct zeros of \( f \) in \( \Omega \), of multiplicities \( m_1, \ldots, m_l \) respectively. The quantity \( N_\Omega(f) \), defined as the total number of zeros for \( f \) in \( \Omega \), equals then \( m_1 + \cdots + m_l \); the corresponding number of distinct zeros, \( \tilde{N}_\Omega(f) \), is obviously \( l \). Next, we fix a conformal map \( \varphi \) from \( \Omega \) onto \( \mathbb{D} \) and write

\[
B(z) := \prod_{k=1}^{l} \left( \frac{\varphi(z) - \varphi(a_k)}{1 - \varphi(a_k) \varphi(z)} \right)^{m_k}, \quad z \in \Omega,
\]

for the (finite) Blaschke product built from \( f \). The zeros of \( B \) in \( \Omega \), counted with multiplicities, are thus the same as those of \( f \). In addition, \( B \) is continuous up to \( \partial \Omega \) (because \( \varphi \) is, by Carathéodory’s theorem) and satisfies \( |B(z)| = 1 \) for all \( z \in \partial \Omega \).

Now, given Blaschke products \( B_1, \ldots, B_s \), we write \( \text{LCM}(B_1, \ldots, B_s) \) for their least common multiple, defined in the natural way: this is the Blaschke product whose zero-set is \( B_1^{-1}(0) \cup \cdots \cup B_s^{-1}(0) =: Z \), the multiplicity of a zero at \( a \in Z \) being \( \max_{1 \leq j \leq s} m(a, B_j) \), where \( m(a, B_j) \) is the multiplicity of \( a \) as a zero of \( B_j \).

Further, for a Blaschke product \( B \), we let \( \text{rad}(B) \) denote the radical of \( B \); the latter is defined (by analogy with the number-theoretic situation) as the Blaschke product with zero-set \( B^{-1}(0) \) whose zeros are all simple. In other words, if \( B \) is given by (2.1), then \( \text{rad}(B) \) is obtained by replacing each \( m_k \) with \( 1 \). Observe that \( N_\Omega(\text{rad}(B)) = \tilde{N}_\Omega(B) \).

Finally, we use the notation \( W(f_0, \ldots, f_n) \) for the Wronskian of the (analytic) functions \( f_0, \ldots, f_n \), so that

\[
W(f_0, \ldots, f_n) := \begin{vmatrix} f_0 & f_1 & \cdots & f_n \\ f'_0 & f'_1 & \cdots & f'_n \\ \vdots & \vdots & \ddots & \vdots \\ f^{(n)}_0 & f^{(n)}_1 & \cdots & f^{(n)}_n \end{vmatrix}.
\]

We are now in a position to state the main results of this section.

**Theorem 2.1.** Let \( f_j \) \((j = 0, 1, \ldots, n)\) be analytic on \( \text{clos} \Omega \), and suppose that the Wronskian \( W := W(f_0, \ldots, f_n) \) vanishes nowhere on \( \partial \Omega \). Let

\[
f_{n+1} = f_0 + \cdots + f_n.
\]

Further, write

\[
B := \text{LCM}(B_0, \ldots, B_{n+1}) \quad \text{and} \quad \mathcal{B} := \text{rad}(B_0B_1 \cdots B_{n+1}),
\]

where \( B_j \) is the (finite) Blaschke product associated with \( f_j \). Then

\[
N_\Omega(B) \leq \lambda^2 + n\mu^2 N_\Omega(\mathcal{B}),
\]

where

\[
\lambda = \lambda_\Omega(W) := \|W'\|_{L^2(\Omega)} \|1/W\|_{L^\infty(\partial \Omega)}
\]

and

\[
\mu = \mu_\Omega(W) := \|W\|_{L^\infty(\partial \Omega)} \|1/W\|_{L^\infty(\partial \Omega)}.
\]
In addition to [2.5], we provide an alternative estimate on $N_B(B)$. The factor in front of $N_B(B)$ will now be reduced from $\mu^2$ to $\mu$ (note that $\mu \geq 1$), while $\lambda^2$ will be replaced by another, possibly larger, quantity.

**Theorem 2.2.** Under the hypotheses of Theorem 2.1, we have

$$N_B(B) \leq \kappa + \mu N_B(B),$$

where

$$\kappa = \kappa_B(W) := \|W'\|_{L^1(\partial \Omega)} \|1/W\|_{L^\infty(\partial \Omega)}$$

and $\mu = \mu_B(W)$ is defined as in (2.7).

It should be noted that if the zero-sets of $f_0, \ldots, f_{n+1}$ are pairwise disjoint, then $B = \prod_{j=0}^{n+1} B_j$ and $B = \prod_{j=0}^{n+1} \operatorname{rad}(B_j)$, in which case

$$N_B(B) = \sum_{j=0}^{n+1} N_B(f_j) \quad \text{and} \quad N_B(B) = \sum_{j=0}^{n+1} \tilde{N}_B(f_j).$$

The example below shows that the estimates in Theorems 2.1 and 2.2 are both sharp, for all $\Omega$ and $n$, in the sense that equality may occur in (2.5) and (2.8).

**Example 1.** We may assume, without loss of generality, that $0 \in \Omega$. Let $\Delta$ denote the diameter of $\Omega$, and let $\varepsilon$ be a number with $0 < \varepsilon < e^{-\Delta}$. Consider the functions

$$f_0(z) = 1, \quad f_j(z) = \varepsilon^{j}\frac{z^j}{j!} \quad (j = 1, \ldots, n)$$

and set $f_{n+1} = f_0 + \cdot \cdot \cdot + f_n$. This done, observe that $f_{n+1}$ has no zeros in $\Omega$. Indeed,

$$\sum_{j=1}^{n} |f_j(z)| \leq \varepsilon (e^{\varepsilon} - 1) \leq \varepsilon (e^{\varepsilon} - 1) \leq 1 - \varepsilon, \quad z \in \Omega,$$

and so

$$|f_{n+1}(z)| \geq 1 - \sum_{j=1}^{n} |f_j(z)| \geq \varepsilon, \quad z \in \Omega.$$

Letting $\varphi : \Omega \to \mathbb{D}$ be a conformal map with $\varphi(0) = 0$ (say, the one with $\varphi'(0) > 0$), we see that the Blaschke products $B_j$ associated with the $f_j$’s are given by

$$B_0(z) = B_{n+1}(z) = 1 \quad \text{and} \quad B_j(z) = \varphi^j(z) \quad (j = 1, \ldots, n).$$

Using the notation from (2.4), we have then $B(z) = \varphi^n(z)$ and $B(z) = \varphi(z)$, whence $N_B(B) = n$ and $N_B(B) = 1$. On the other hand, one easily finds that $W := W(f_0, \ldots, f_n) = \varepsilon^n$ (the Wronskian matrix being upper triangular), so that $\lambda_B(W) = \kappa_B(W) = 0$ and $\mu_B(W) = 1$. Consequently, equality holds in both (2.5) and (2.8).

The next example, which is a slight modification of the previous one, tells us that equality may also occur – at least on the disk – in the “less trivial” case where $W \not\equiv \text{const}$ (or equivalently, when $\lambda \not\equiv 0$ and $\kappa \not\equiv 0$).

**Example 2.** Let $\Omega = \mathbb{D}$ and define $f_0, \ldots, f_{n-1}$ as in (2.9), with some $\varepsilon \in (0, 1/e)$. Then put $f_n(z) = \varepsilon z^n/m!$, where $m$ is a fixed integer with $m > n$, and finally write $f_{n+1} = f_0 + \cdot \cdot \cdot + f_n$. As before, $f_{n+1}$ is zero-free on $\mathbb{D}$. For $j = 0, \ldots, n - 1$, the
Blaschke product associated with \( f_j \) is \( z^j \), while the Blaschke products corresponding to \( f_n \) and \( f_{n+1} \) are \( z^m \) and 1, respectively. It follows that \( B(z) = z^m \) and \( B(z) = z \), whence \( N_B(B) = m \) and \( N_B(B) = 1 \). The Wronskian \( W = W(f_0, \ldots, f_n) \) now equals a constant times \( z^{m-n} \); a simple calculation then yields \( \lambda_2^2(W) = \kappa_D(W) = m - n \) and \( \mu_D(W) = 1 \). Therefore, equality is again attained in both (2.5) and (2.8), this time with no zero terms on the right.

Now let us recall that the functions \( f_j \) in Theorems 2.1 and 2.2 were supposed to be analytic on \( \Omega \). In fact, this can be relaxed to the hypothesis that the functions be merely analytic on \( \Omega \) and suitably smooth up to \( \partial \Omega \). This time, we should explicitly assume that the \( f_j \)'s have finitely many zeros in \( \Omega \), so as to ensure \( N_\Omega(B) < \infty \) and \( N_\Omega(B) < \infty \). The next proposition contains the appropriate versions of the two theorems, specialized (for the sake of simplicity) to the case where \( \Omega = \mathbb{D} \). When stating it, we write \( \mathcal{D} = \mathcal{D}(\mathbb{D}) \) for the Dirichlet space of the disk, defined as the set of all analytic \( g \) on \( \mathbb{D} \) with \( g' \in L^2(\mathbb{D}) \), and we use the standard notation \( H^p \) for the Hardy spaces on \( \mathbb{D} \); see [9, Chapter II].

**Proposition 2.3.** (a) Suppose \( f_j \) \((j = 0, 1, \ldots, n)\) are analytic functions on \( \mathbb{D} \) satisfying

\[
(2.10) \quad f_j^{(n)} \in \mathcal{D} \cap H^\infty \quad \text{and} \quad 1/W \in L^\infty(\mathbb{T}), \quad \text{where} \quad W = W(f_0, \ldots, f_n). \quad \text{Further, put} \quad f_{n+1} = f_0 + \cdots + f_n \quad \text{and assume that}
\]

\[
(2.11) \quad N_\mathbb{D}(f_j) < \infty \quad \text{for} \quad 0 \leq j \leq n + 1.
\]

Then

\[
(2.12) \quad N_\mathbb{D}(B) \leq \lambda^2 + n\mu^2 N_\mathbb{D}(B),
\]

where \( B \) and \( B \) are defined as in Theorem 2.1, \( \lambda = \lambda_\mathbb{D}(W) \), and \( \mu = \mu_\mathbb{D}(W) \).

(b) Replacing (2.10) by the stronger hypothesis that

\[
(2.13) \quad f_j^{(n+1)} \in H^1
\]

for all \( j \), while retaining the other assumptions above, one has

\[
(2.14) \quad N_\mathbb{D}(B) \leq \kappa + n\mu N_\mathbb{D}(B)
\]

with \( \kappa = \kappa_\mathbb{D}(W) \) and \( \mu = \mu_\mathbb{D}(W) \).

We conclude this section by showing that our “local” theorems imply Theorem B, as stated above, and hence the original \( abc \) theorem for polynomials. We shall deduce the required “global” result from Theorem 2.2 by a limiting argument. An alternative route via Theorem 2.1 would be equally successful.

**Deduction of Theorem B.** Suppose \( p_0, \ldots, p_n \) are linearly independent polynomials and \( p_{n+1} = \sum_{j=0}^n p_j \). Assume also that the zero-sets \( p_j^{-1}(0) \) are pairwise disjoint, so that \( p_j^{-1}(0) \cap p_k^{-1}(0) = \emptyset \) whenever \( 0 \leq j < k \leq n + 1 \).
An application of Theorem 2.2 with \( \Omega = R^D = \{ z : |z| < R \} \) gives
\[
N_{RD}(p_0) + \cdots + N_{RD}(p_{n+1}) \leq \kappa_{RD}(W) + n\mu_{RD}(W)\tilde{N}_{RD}(p_0p_1\ldots p_{n+1}).
\]
Here,
\[
\kappa_{RD}(W) = \left( \frac{1}{2\pi} \int_{|z|=R} |W'(z)| \, |dz| \right) \cdot \left( \min_{|z|=R} |W(z)| \right)^{-1}
\]
and
\[
\mu_{RD}(W) = \left( \max_{|z|=R} |W(z)| \right) \cdot \left( \min_{|z|=R} |W(z)| \right)^{-1}
\]
with \( W = W(p_0, \ldots, p_n) \). Now if \( R \) is sufficiently large, then
\[
N_{RD}(p_j) = N_{C}(p_j) = \deg p_j =: d_j, \quad 0 \leq j \leq n + 1,
\]
and
\[
\tilde{N}_{RD}(p_0p_1\ldots p_{n+1}) = \tilde{N}_{C}(p_0p_1\ldots p_{n+1}) =: \tilde{d}.
\]
On the other hand, \( W \) is a (non-null) polynomial, so that
\[
W(z) = c_m z^m + \text{lower order terms},
\]
where \( m = \deg W \) and \( c_m \neq 0 \). The asymptotic behavior of \( \kappa_{RD}(W) \) and \( \mu_{RD}(W) \) as \( R \to \infty \) is governed by the leading term, \( c_m z^m \), whence
\[
\lim_{R \to \infty} \kappa_{RD}(W) = m \quad \text{and} \quad \lim_{R \to \infty} \mu_{RD}(W) = 1.
\]
We therefore deduce from (2.15), upon letting \( R \to \infty \), that
\[
d_0 + \cdots + d_{n+1} \leq m + n\tilde{d}.
\]
To get a bound on \( m \), we now recall that \( W \) is the sum of \((n + 1)!\) products of the form
\[
\pm p_0^{(k_0)} p_1^{(k_1)} \cdots p_n^{(k_n)},
\]
where \((k_0, \ldots, k_n)\) runs through the permutations of \((0, \ldots, n)\). And since
\[
\deg p_j^{(k_j)} = d_j - k_j,
\]
it follows that \( m \), the degree of \( W \), satisfies
\[
m \leq d_0 + \cdots + d_n - 1 - \cdots - n = d_0 + \cdots + d_n - \frac{n(n+1)}{2}.
\]
Finally, we put \( d := \max_{0 \leq j \leq n+1} d_j \) and observe that at least two of the polynomials involved must be of degree \( d \). We may assume that this happens for \( p_n \) and \( p_{n+1} \), so that \( d_n = d_{n+1} = d \). The above estimate for \( m \) now reads
\[
m \leq d_0 + \cdots + d_{n-1} + d - \frac{n(n+1)}{2},
\]
while the left-hand side of (2.16) takes the form \( d_0 + \cdots + d_{n-1} + 2d \). Consequently, (2.16) yields
\[
d \leq n\tilde{d} - \frac{n(n+1)}{2},
\]
or equivalently,

\[ \max_{0 \leq j \leq n+1} \deg p_j \leq nN_C(p_0p_1 \ldots p_{n+1}) - \frac{n(n+1)}{2}, \]

as required.

3. Finitely many zeros: proofs

Let \( D(\Omega) \) denote the Dirichlet space on \( \Omega \), i.e., the set of all analytic functions \( f \) on \( \Omega \) for which the quantity

\[ \| f \|_{D(\Omega)}^2 := \| f' \|_{L^2(\Omega)}^2 = \frac{1}{\pi} \int_{\Omega} |f'(z)|^2 dA(z) \]

is finite. A bounded analytic function \( \theta \) on \( \Omega \) is said to be inner if its nontangential boundary values have modulus 1 almost everywhere on \( \partial \Omega \) (with respect to arc length). The following result will be needed.

**Lemma 3.1.** Let \( f \in D(\Omega) \) and let \( \theta \) be an inner function on \( \Omega \). Then

\[ \| f\theta \|_{D(\Omega)}^2 = \| f \|_{D(\Omega)}^2 + \frac{1}{2\pi} \int_{\partial \Omega} |f|^2 |\theta'| ds. \]

In the case where \( \Omega \) is the unit disk, \( \mathbb{D} \), the above lemma follows from Carleson’s formula in \[3\]; see also [5] for an alternative (operator-theoretic) approach. The general case is then established by means of a conformal mapping. Indeed, the Dirichlet integral (3.1) is conformally invariant, and so is the last term in (3.2).

The derivative \( \theta' \) in (3.2) should be interpreted as angular derivative. Anyhow, we shall only use formula (3.2) when \( \theta \) is a finite Blaschke product, so that \( \theta = B \) for some \( B \) of the form (2.1). In this situation, \( B' \) is sure to have nontangential boundary values almost everywhere on \( \partial \Omega \), since this is the case for \( \varphi' \). Now, applying (3.2) to such a \( B \) and letting \( f \equiv 1 \), we get

\[ \| B \|_{D(\Omega)}^2 = \frac{1}{2\pi} \int_{\partial \Omega} |B'| ds. \]

Moreover, the common value of the two sides in (3.3) is actually \( N_\Omega(B) \). This is clear from the geometric interpretation of the two quantities in terms of area and length, combined with the fact that \( B \) is an \( N \)-to-1 mapping between \( \Omega \) and \( \mathbb{D} \), where \( N = N_\Omega(B) \).

**Proof of Theorem 2.1.** The first step will be to verify that \( B \) divides \( WB^n \), in the sense that \( WB^n/B \) is analytic on \( \Omega \).

Clearly, we should only be concerned with those zeros of \( B \) whose multiplicity exceeds \( n \). So let \( z_0 \in \Omega \) be a zero of multiplicity \( k \), \( k > n \), for \( B \). Then there is an index \( j \in \{0, \ldots, n+1\} \) such that \( B_j \) vanishes to order \( k \) at \( z_0 \), and so does \( f_j \).

Expanding the determinant (2.22) along the column that contains \( f_j, \ldots, f_j^{(n)} \), while noting that \( f_j^{(l)} \) vanishes to order \( k-l \) at \( z_0 \), we see that \( W \) has a zero of multiplicity \( \geq k-n \) at \( z_0 \). (In case \( j = n+1 \), one should observe that, by (2.23), the determinant remains unchanged upon replacing any one of its columns by \( (f_{n+1}, \ldots, f_{n+1}^{(n)})^T \).)
And since $B$ has a zero at $z_0$, it follows that $WB^n$ vanishes at least to order $k$ at that point.

We conclude that $WB^n$ is indeed divisible by $B$. In other words, we have

$$WB^n = F B,$$

where $F$ is analytic on $\Omega$. This $F$ is also continuous on $\text{clos} \Omega$ because $W$, $B$ and $B^n$ enjoy this property and because $|B| = 1$ on $\partial\Omega$.

Next, we are going to compute – and estimate – the Dirichlet integral $\| \cdot \|^2_{D(\Omega)}$ for each of the two sides of (3.4). On the one hand, an application of Lemma 3.1 yields

$$\|WB^n\|^2_{D(\Omega)} = \|FB\|^2_{D(\Omega)}$$

$$= \|F\|^2_{D(\Omega)} + \frac{1}{2\pi} \int_{\partial\Omega} |F|^2 |B'| ds$$

$$\geq \frac{1}{2\pi} \int_{\partial\Omega} |F|^2 |B'| ds$$

$$\geq \left( \min_{\partial\Omega} |F| \right)^2 \cdot \frac{1}{2\pi} \int_{\partial\Omega} |B'| ds$$

$$= \|1/W\|^2_{L^\infty(\partial\Omega)} N_\Omega(B).$$

Here, the last step relies on the fact that $|F| = |W|$ everywhere on $\partial\Omega$, an obvious consequence of (3.4). Therefore, the minimum of $|F|$ over $\partial\Omega$ coincides with that of $|W|$, i.e., with $\|1/W\|_{L^\infty(\partial\Omega)}^{-1}$. We have also used the equality $(2\pi)^{-1} \int_{\partial\Omega} |B'| ds = N_\Omega(B)$, which holds by the discussion following (3.3).

On the other hand, by Lemma 3.1 again,

$$\|WB^n\|^2_{D(\Omega)} = \|W\|^2_{D(\Omega)} + \frac{1}{2\pi} \int_{\partial\Omega} |W|^2 |(B^n)'| ds$$

$$\leq \|W\|^2_{D(\Omega)} + n \|W\|^2_{L^\infty(\partial\Omega)} \cdot \frac{1}{2\pi} \int_{\partial\Omega} |B'| ds$$

$$= \|W'\|^2_{L^2(\Omega)} + n \|W\|^2_{L^\infty(\partial\Omega)} N_\Omega(B).$$

Comparing the resulting inequalities from (3.5) and (3.6), we obtain

$$\|1/W\|^2_{L^\infty(\partial\Omega)} N_\Omega(B) \leq \|W'\|^2_{L^2(\Omega)} + n \|W\|^2_{L^\infty(\partial\Omega)} N_\Omega(B),$$

which proves (2.5).

To prove Theorem 2.2, we need another lemma. Before stating it, we recall that an analytic function $f$ on $\Omega$ is said to be in the Hardy space $H^p(\Omega)$ if $(f \circ \psi) \cdot (\psi')^{1/p}$ is in $H^p$ of the disk, for some (or any) conformal map $\psi : \mathbb{D} \to \Omega$.

**Lemma 3.2.** Let $f \in H^\infty(\Omega)$ and let $\theta$ be an inner function on $\Omega$ with $(f\theta)' \in H^1(\Omega)$. Then

$$\| (f\theta)' \|^2_{L^1(\partial\Omega)} \geq \frac{1}{2\pi} \int_{\partial\Omega} |f| |\theta'| ds.$$
For \( \Omega = \mathbb{D} \), this estimate is due to Vinogradov and Shirokov \[20\]. The full statement follows by conformal transplantation. Indeed, the class \( \{ f : f' \in H^1(\Omega) \} \) is conformally invariant, and so are the two sides of (3.7).

**Proof of Theorem 2.2.** Proceeding as in the proof of Theorem 2.1, we arrive at (3.4), where \( F \) is analytic on \( \Omega \) and continuous up to \( \partial \Omega \). Together with Lemma 3.2, this yields

\[
\| (W^B)'' \|_{L^1(\partial \Omega)} = \| F'B' \|_{L^1(\partial \Omega)} \\
\geq \frac{1}{2\pi} \int_{\partial \Omega} |F' | \, |B'| \, ds
\]

(3.8)

On the other hand,

\[
\| (W^B)'' \|_{L^1(\partial \Omega)} \leq \| W'' B'' \|_{L^1(\partial \Omega)} + \| W \cdot (B')'' \|_{L^1(\partial \Omega)} \\
\leq \| W'' \|_{L^1(\partial \Omega)} + n \| W \|_{L^\infty(\partial \Omega)} \| B'' \|_{L^1(\partial \Omega)} \\
= \| W'' \|_{L^1(\partial \Omega)} + n \| W \|_{L^\infty(\partial \Omega)} P_N(B).
\]

(3.9)

Finally, a juxtaposition of (3.8) and (3.9) gives

\[
\| 1/W \|_{L^{-1}_\infty(\partial \Omega)} P_N(B) \leq \| W'' \|_{L^1(\partial \Omega)} + n \| W \|_{L^\infty(\partial \Omega)} P_N(B),
\]

which proves (2.8).

**Proof of Proposition 2.3.** It is easy to check that if either (2.10) or (2.13) holds, then the derivatives \( f_j^{(k)} \) with \( 0 \leq k \leq n \) are all in \( H^\infty \). It follows that, in either case, \( W \in H^\infty \). Next, note that the derivative \( W' \) of the Wronskian \( W = W(f_0, \ldots, f_n) \) is given by

\[
W' = \begin{vmatrix}
    f_0 & f_1 & \ldots & f_n \\
    f'_0 & f'_1 & \ldots & f'_n \\
    \vdots & \vdots & \ddots & \vdots \\
    f_0^{(n-1)} & f_1^{(n-1)} & \ldots & f_n^{(n-1)} \\
    f_0^{(n+1)} & f_1^{(n+1)} & \ldots & f_n^{(n+1)}
\end{vmatrix}
\]

(3.10)

Expanding this determinant along its last row, one therefore deduces that \( W \in D \) in case (a), while \( W' \in H^1 \) in case (b). These observations show that the quantities \( \lambda_D(W), \mu_D(W) \) and \( \kappa_D(W) \) appearing in (2.12) and (2.14) are finite under the stated conditions.

This said, the two estimates are proved in the same way as their counterparts in Theorems 2.1 and 2.2 above. Namely, one arrives at (3.4) as before (with a suitable analytic function \( F \) on \( \mathbb{D} \)) and then essentially rewrites the ensuing norm estimates, with \( \Omega = \mathbb{D} \), based on the (original) disk versions of Lemmas 3.1 and 3.2 as contained in \[3\] and \[20\].

One minor modification is that, in case (a), the functions \( W \) and \( F \) no longer need to be continuous on \( T \). However, they both belong to \( \mathcal{D} \cap H^\infty \), and the equality
\[ |F| = |W| \] holds almost everywhere on \( T \), rather than everywhere. Accordingly, the quantity \( \min_{\partial \Omega} |F| \) appearing in (3.3) should be replaced by the essential infimum of \( |F| \) over \( T \), which still coincides with \( \|1/W\|_{L^\infty(T)}^{-1} \).

4. INFINITELY MANY ZEROS

Given \( 0 < \alpha \leq 1 \), we write \( \mathcal{D}_\alpha \) for the space of all analytic functions \( f \) on \( \mathbb{D} \) with

\[ \|f\|_{\mathcal{D}_\alpha}^2 := \sum_{k \geq 1} k^\alpha |\hat{f}(k)|^2 < \infty, \]

where \( \hat{f}(k) := f^{(k)}(0)/k! \). A calculation shows that

\[ (4.1) \quad \|f\|_{\mathcal{D}_\alpha}^2 \asymp \frac{1}{\pi} \int_{\mathbb{D}} |f'(z)|^2 (1 - |z|)^{1-\alpha} dA(z), \]

where the notation \( U \asymp V \) means that the ratio \( U/V \) lies between two positive constants depending only on \( \alpha \). When \( \alpha = 1 \), (4.1) reduces to an identity, and \( \mathcal{D}_1 \) is just the Dirichlet space \( \mathcal{D} = \mathcal{D}(\mathbb{D}) \).

Earlier, when proving Theorem 2.1 and Proposition 2.3 (a), we made use of the fact that the total number of zeros of a (finite) Blaschke product \( B \) coincides with its Dirichlet integral \( \|B\|_{\mathcal{D}}^2 \). In this section, we shall be concerned with functions living on \( \mathbb{D} \) that are allowed to have infinitely many zeros therein. (Our functions will, of course, be analytic on \( \mathbb{D} \) and appropriately smooth up to \( T \).) The associated Blaschke products are thus, in general, infinite products of the form

\[ (4.2) \quad B(z) = z^m \prod_k \left( \frac{\overline{a_k} a_k - z}{|a_k| 1 - \overline{a_k} z} \right)^{m_k}, \quad z \in \mathbb{D}; \]

here \( a_k \) are the function’s distinct zeros in \( \mathbb{D} \setminus \{0\} \) of respective multiplicities \( m_k \), so that \( \sum_k m_k (1 - |a_k|) < \infty \), and \( m \geq 0 \) is the multiplicity of its zero at the origin.

While there are no infinite Blaschke products in \( \mathcal{D} = \mathcal{D}_1 \), the spaces \( \mathcal{D}_\alpha \) with \( 0 < \alpha < 1 \) do contain such products. (For instance, any Blaschke product (4.2) satisfying \( \sum_k m_k (1 - |a_k|)^{1-\alpha} < \infty \) will be in \( \mathcal{D}_\alpha \); see [1, Theorem 4.2] for this and other membership criteria.) Therefore, when looking for a reasonable abc-type theorem in the current setting, one might expect to arrive at a fairly natural formulation by comparing the \( \mathcal{D}_\alpha \)-norms of the Blaschke products \( B \) and \( B' \) rather than counting their zeros.

Here, it is understood that \( B \) and \( B' \) are built from the given functions \( f_j \) exactly as before. There is no problem about that, since the notions of the least common multiple (LCM) and the radical are perfectly meaningful for infinite Blaschke products as well. In particular, if \( B \) is defined by (4.2) with \( m \geq 0 \) and \( m_k \geq 1 \), then \( \text{rad}(B) \) stands for the product obtained by replacing \( m \) with \( \min\{m, 1\} \) and each of the \( m_k \)'s with 1.

**Theorem 4.1.** Let \( 0 < \alpha < 1 \) and suppose \( f_j \) \((j = 0, 1, \ldots, n)\) are analytic functions on \( \mathbb{D} \) with

\[ (4.3) \quad f_j^{(m)} \in \mathcal{D}_\alpha \cap H^\infty. \]
Assume also that the Wronskian $W := W(f_0, \ldots, f_n)$ satisfies $1/W \in L^\infty(\mathbb{T})$. Put $f_{n+1} = f_0 + \cdots + f_n$.

Finally, write

\begin{equation}
(4.4) \quad \mathcal{B} := \text{LCM}(B_0, \ldots, B_{n+1}) \quad \text{and} \quad \mathcal{B} := \text{rad}(B_0B_1 \cdots B_{n+1}),
\end{equation}

where $B_j$ is the Blaschke product associated with $f_j$. Then there exists a constant $c_\alpha > 0$ depending only on $\alpha$ such that

\begin{equation}
(4.5) \quad c_\alpha \|\mathcal{B}\|_{D_\alpha}^2 \leq \lambda_\alpha^2 + n\mu^2\|\mathcal{B}\|_{D_\alpha}^2,
\end{equation}

with

\begin{equation}
(4.6) \quad \lambda_\alpha = \lambda_{\alpha,\mathbb{D}}(W) := \|W\|_{D_\alpha} \|1/W\|_{L^\infty}(\mathbb{T})
\end{equation}

and

\begin{equation}
(4.7) \quad \mu = \mu_{\mathbb{D}}(W) := \|W\|_{L^\infty} \|1/W\|_{L^\infty}(\mathbb{T}),
\end{equation}

where $\| \cdot \|_{\infty}$ stands for $\| \cdot \|_{L^\infty}(\mathbb{T})$.

The proof hinges on the following result, which can be found in [5, Section 4].

**Lemma 4.2.** Let $0 < \alpha < 1$. If $f \in \mathcal{D}_\alpha$ and $\theta$ is an inner function on $\mathbb{D}$, then the quantity

\[
R_\alpha(f, \theta) := \|f\theta\|_{D_\alpha}^2 - \|f\|_{D_\alpha}^2
\]

is nonnegative and satisfies

\[
R_\alpha(f, \theta) \asymp \int_\mathbb{D} |f(z)|^2 \frac{1 - |\theta(z)|^2}{(1 - |z|^2)^{1+\alpha}} dA(z).
\]

In particular,

\[
\|\theta\|_{D_\alpha}^2 \asymp \int_\mathbb{D} \frac{1 - |\theta(z)|^2}{(1 - |z|^2)^{1+\alpha}} dA(z).
\]

The inequality $R_\alpha(f, \theta) \geq 0$, when rewritten in the form

\begin{equation}
(4.8) \quad \|f\theta\|_{D_\alpha} \geq \|f\|_{D_\alpha},
\end{equation}

is actually true under the *a priori* assumption that $f \in H^2$ and $\theta$ is inner. This is a refinement of the well-known fact that division by inner factors preserves membership in $\mathcal{D}_\alpha$; see [12, 18] and [7, Section 2] for a discussion of a similar phenomenon in various smoothness classes.

Yet another piece of notation will be needed. Namely, given a nonnegative measurable function $h$ on $\mathbb{T}$ with $\log h \in L^1(\mathbb{T})$, we shall write $\mathcal{O}_h$ for the *outer function* with modulus $h$, so that

\[
\mathcal{O}_h(z) := \exp \left( \frac{1}{2\pi} \int_\mathbb{T} \frac{\zeta + z}{\zeta - z} \log h(\zeta) |d\zeta| \right), \quad z \in \mathbb{D}.
\]

**Proof of Theorem 4.1.** First of all, the assumption (4.3) implies that $W \in \mathcal{D}_\alpha \cap H^\infty$. Indeed, the inclusion $W \in H^\infty$ is immediate from the fact that $f_j^{(k)} \in H^\infty$ whenever $0 \leq j, k \leq n$. To check that $W \in \mathcal{D}_\alpha$, we recall (3.10) and expand the determinant in that formula along its last row. Since the derivatives $f_j^{(n+1)}$ are square integrable
shall also assume that
against the measure \( d\nu_\alpha(z) := (1 - |z|^{1-\alpha})dA(z) \), while the lower order
derivatives are bounded, we infer that \( W'' \in L^2(d\nu_\alpha) \)
and hence indeed \( W \in D_\alpha \). Thus, the
hypotheses of the theorem guarantee that the quantities \( \lambda_\alpha \) and \( \mu \) are finite. We
shall also assume that \( B \in D_\alpha \), since otherwise \( \|B\|_{D_\alpha} = \infty \) and there is nothing to
prove.

Arguing as in the proof of Theorem \[2.1\] we verify that \( B \) divides \( WB^n \), so that
\[3.4\] holds with some \( F \in H^\infty \). Factoring \( F \) canonically (see \[9\] Chapter II), we
write \( F = \mathcal{O}\mathcal{I} \), where \( \mathcal{O} \) is outer and \( \mathcal{I} \) is inner. Furthermore, since \( |F| = |W| \) almost
everywhere on \( T \), the outer factor \( \mathcal{O} = \mathcal{O}|_F \) coincides with \( \mathcal{O}|_W \). An application of
\[4.8\] with \( f = \mathcal{O}|_W B \) and \( \theta = \mathcal{I} \) now shows that
\[
\|WB^n\|_{D_\alpha} = \|FB\|_{D_\alpha} = \|\mathcal{O}|_W\mathcal{I}B\|_{D_\alpha} \geq \|\mathcal{O}|_W B\|_{D_\alpha}.
\]
From this and Lemma \[4.2\] it follows that
\[
\|WB^n\|_{D_\alpha}^2 \geq \|\mathcal{O}|_W B\|_{D_\alpha}^2 \geq \mathcal{R}_\alpha(\mathcal{O}|_W, B)
\geq c_1(\alpha) \int_D \left|\mathcal{O}|_W(z)\right|^2 \frac{1 - |B(z)|^2}{(1 - |z|^2)^{1+\alpha}}dA(z)
\geq c_1(\alpha) \cdot \left(\inf_{z \in \overline{D}} \left|\mathcal{O}|_W(z)\right|\right)^2 \cdot \int_D \frac{1 - |B(z)|^2}{(1 - |z|^2)^{1+\alpha}}dA(z)
\geq c_2(\alpha) \cdot \left(\inf_{z \in \overline{D}} \left|\mathcal{O}|_W(z)\right|\right)^2 \cdot \|B\|_{D_\alpha}^2,
\]
where \( c_1(\alpha) \) and \( c_2(\alpha) \) are positive constants depending only on \( \alpha \). Now let us
observe that
\[
1/\|\mathcal{O}|_W\| = O_{1/|W|} \in H^\infty
\]
(because \( 1/W \in L^\infty(T) \)) and
\[
\sup_{z \in \overline{D}} \left|\mathcal{O}|_W(z)\right|^{-1} = \|1/\mathcal{O}|_W\|_\infty = \|1/W\|_\infty,
\]
whence
\[
\inf_{z \in \overline{D}} \left|\mathcal{O}|_W(z)\right| = \|1/W\|_\infty^{-1}.
\]
Substituting this into \[4.9\], we obtain
\[4.10\]
\[
\|WB^n\|_{D_\alpha}^2 \geq c_2(\alpha)\|1/W\|^{-2}_{\infty}\|B\|_{D_\alpha}^2.
\]
Another application of Lemma \[1.2\] in conjunction with the elementary inequality
\[1-t^n \leq n(1-t) \], valid for \( 0 \leq t \leq 1 \), yields
\[
\mathcal{R}_\alpha(W, B^n) \leq C_1(\alpha) \int_D |W(z)|^2 \frac{1 - |B(z)|^{2n}}{(1 - |z|^2)^{1+\alpha}}dA(z)
\leq C_1(\alpha) \cdot n\|W\|_\infty^2 \int_D \frac{1 - |B(z)|^2}{(1 - |z|^2)^{1+\alpha}}dA(z)
\leq C_2(\alpha) \cdot n\|W\|_\infty^2 \|B\|_{D_\alpha}^2,
\]
with suitable constants $C_1(\alpha)$ and $C_2(\alpha)$. Consequently,

$$\|W^B_n\|_{D^\alpha}^2 \leq \|W\|_{D^\alpha}^2 + C_2(\alpha) \cdot n \|W\|_{\infty}^2 \|B\|_{D^\alpha}^2. \tag{4.11}$$

Finally, a juxtaposition of (4.10) and (4.11) gives

$$c_2(\alpha) \|1/W\|_{\infty}^2 \|B\|_{D^\alpha}^2 \leq \|W\|_{D^\alpha}^2 + C_2(\alpha) \cdot n \|W\|_{\infty}^2 \|B\|_{D^\alpha}^2 \leq C_2(\alpha) \cdot \left( \|W\|_{D^\alpha}^2 + n \|W\|_{\infty}^2 \|B\|_{D^\alpha}^2 \right)$$

(we may assume $C_2(\alpha) \geq 1$). This implies (4.3), with

$$c_\alpha = c_2(\alpha)/C_2(\alpha),$$

and completes the proof. \(\square\)

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