Noncoplanar magnetic orders and gapless chiral spin liquid in the $J_1$-$J_2$-$J_3$ model on the kagome lattice

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Time-reversal-symmetry-breaking three-spin interactions can suppress long-range magnetic order and stabilize quantum spin liquid states in frustrated lattices. We combine a classical approach, parton mean-field theory and variational Monte Carlo methods to study a spin-1/2 model with staggered three-spin interaction $J_3$ on the kagome lattice. In addition, we consider Heisenberg exchange couplings $J_1$ on nearest-neighbor bonds and $J_d$ across the diagonals of the hexagons. In the regime of dominant $J_3$, the phase diagram exhibits a gapless chiral spin liquid with a line Fermi surface. As we increase the exchange couplings, we find a variety of noncoplanar magnetic orders, including a phase that interpolates between cuboc-1 and cuboc-2 states. Our results show that the competition between induced staggered chirality and Heisenberg exchange interactions can give rise to unusual ground states of spin systems.

I. INTRODUCTION

In quantum magnets, the combination of competing exchange interactions and geometric frustration can lead to novel phases of matter [1]. Among the most exotic strongly correlated phenomena, quantum spin liquids (QSLs) occupy a special position [2–4]. QSLs are nontrivial states of localized spins which avoid symmetry breaking order down to zero temperature, but feature fractionalized excitations and a high level of entanglement. Broadly speaking, we divide QSLs into two types: gapped or gapless. The former have topological ground-state degeneracy and anyonic excitations [5, 6]. In the latter, the gapless modes are usually described by emergent fermions coupled to gauge fields [7–10].

Of particular interest are QSLs with broken time reversal symmetry. Chiral spin liquids (CSLs) [11–13] have been extensively studied since Kalmeyer and Laughlin proposed a spin wave function analogous to fractional quantum Hall states [14, 15]. The Kalmeyer-Laughlin CSL can be stabilized as the ground state of spin-1/2 models containing chiral three-spin interactions that drive a uniform scalar spin chirality [16–21]. In spin-1 models, the same interactions give rise to a non-Abelian CSL analogous to the Moore-Read state [22–24]. On the other hand, three-spin interactions that induce staggered spin chiralities on the kagome lattice favor gapless spin liquids with spinon Fermi surfaces [25–27]. CSLs can also arise from spontaneous breaking of time reversal symmetry. Examples have been found numerically in extended Heisenberg models with up to third-neighbor interactions on the kagome [28–31] and triangular [32] lattices, and also in the triangular lattice Hubbard model [33].

A gapless CSL was proposed for the material kapellasite [34–36], described by an extended Heisenberg model in which the dominant antiferromagnetic exchange coupling $J_d$ is present across the hexagons of the kagome lattice. The established model for kapellasite also includes nearest- and next-nearest-neighbor ferromagnetic couplings $J_1$ and $J_2$ [37]. The phase diagram [34, 36, 38] contains non-coplanar ordered states known as cuboc-1 and cuboc-2 [39, 40], and a CSL might be expected to arise from the quantum melting of the magnetic order while preserving a chirality pattern that breaks reflection and time reversal symmetries. However, density-matrix renormalization group simulations suggest that the intermediate phase between cuboc-1 and cuboc-2 phases in the $J_1$-$J_2$-$J_d$ model is a valence bond crystal rather than a CSL [41]. Current estimates for the exchange couplings in kapellasite, with $|J_1| > |J_2|$, place it in the cuboc-2 phase [37], but the proximity to a possible quantum paramagnetic phase was noted in Ref. [38].

Motivated by the competition between the time-reversal-symmetry-breaking interactions and the Heisenberg exchange interactions, in this work we investigate the spin-1/2 $J_1$-$J_d$ model supplemented by staggered three-spin interaction $J_3$ on the kagome lattice. For simplicity, we neglect the next-nearest-neighbor interaction $J_2$, which is subleading in kapellasite. In the limit $J_3 > |J_1|, |J_d|$, the ground state is expected to be a gapless CSL with a symmetry-protected line Fermi surface [25–27]. For $J_d, |J_1| > J_3$, the cuboc-2 state must prevail. We start by mapping out the classical phase diagram of the model. Coming from the limit of dominant $J_d > 0$, we find that increasing $J_1$ leads to a continuous transition to a noncoplanar state that smoothly interpolates between the cuboc-2 and cuboc-1 states. This intermediate phase, which we call the AFMd phase, contains an octahedral state [40] as a special point at which the staggered spin chirality in the triangles of the kagome lattice is maximized. In the parameter regime where the chiral three-spin interaction dominates, we observe a clas-
Finally, we summarize our results in section VI. We compare our VMC results and the phase diagram obtained by analyzing the spinon spectrum. In section V, we show the phase diagram around the pure-J(H) point studied in Ref. [27].

The paper is organized as follows. In section II, we present the J1−Jd−Jχ model on the kagome lattice. In section III, we explore the classical phase diagram and find novel ordered phases for both signs of Jd. Section IV is devoted to the parton mean-field ansatz and the stability of the proposed CSL against order-inducing perturbations. We find that the gapless CSL persists in a sizeable region in the phase diagram around the pure-Jχ phase. To test this idea, we develop a parton mean-field theory [3, 4] of a U(1) CSL with a line Fermi surface. Wecompute the energy of the trial wave function using variational Monte Carlo (VMC) methods, compare it with the energy of competing classical states and test the stability of the proposed CSL against order-inducing perturbations. We find that the gapless CSL persists in a sizeable region in the phase diagram around the pure-Jχ point studied in Ref. [27].

The model and symmetries

We consider an SU(2)-symmetric spin-1/2 model on the kagome lattice described by the Hamiltonian

\[ H = H_0 + H_\chi, \]

where

\[ H_0 = \sum_{ij} J_{ij} \mathbf{S}_i \cdot \mathbf{S}_j, \]

\[ H_\chi = J_\chi \sum_{ijk \in \Delta} \mathbf{S}_i \cdot (\mathbf{S}_j \times \mathbf{S}_k) - J_\chi \sum_{ijk \in \nabla} \mathbf{S}_i \cdot (\mathbf{S}_j \times \mathbf{S}_k). \]

The nonzero exchange couplings in \( H_0 \) are \( J_{ij} = J_1 \) for nearest-neighbor bonds and \( J_{ij} = J_d \) for bonds across the diagonals of the hexagons, see Fig. 1. In Eq. (3), the sites \( i, j, k \) belong to an up-pointing (△) or down-pointing (▽) triangle and are oriented counterclockwise. The relative minus sign between the two terms in Eq. (3) induces a staggered scalar spin chirality. Without loss of generality, hereafter we set \( J_\chi > 0 \).

Besides breaking time reversal symmetry, the chiral three-spin interaction lowers the point group symmetry of the Hamiltonian in comparison with the Heisenberg model on the kagome lattice. The rotational symmetry around the centers of the hexagons is reduced from sixfold to threefold. In addition, we can define reflections about two independent axes, indicated by \( \sigma \) and \( \sigma' \) in Fig. 1. The staggered chirality pattern breaks the reflection symmetry generated by \( \sigma' \), but preserves \( \sigma \).

Let us highlight two important limits of the model. For \( J_d > 0 \) and \( J_d \gg J_\chi, |J_1| \), the Hamiltonian describes three sets of weakly coupled antiferromagnetic spin-1/2 chains rotated by 120° with respect to each other [26, 41]. The low-energy physics of critical spin-1/2 chains is described by the SU(2)\(_1\) Wess-Zumino-Witten (WZW) model [12]. However, the fixed-point of decoupled spin chains, \( J_1 = J_\chi = 0 \), is unstable against interchain couplings and an arbitrarily small \( J_1 < 0 \) drives the system to the cuboc-2 phase [41]. In the limit \( J_d = J_1 = 0 \) and \( J_\chi > 0 \), there is compelling numerical evidence [25, 27] that the ground state corresponds to a gapless CSL with a line Fermi surface protected by reflection symmetry \( \sigma \) [13]. A signature of this gapless CSL is that spin correlations decay with distance \( r \) as a power law \( r^{-2} \) in the directions perpendicular to the Fermi surface lines [26].

Classical phase diagram

To study the ordered phases for the model in Eq. (1), we start from the classical limit, and treat the spins as classical vectors of size \( S \). Our main goal is to identify novel phases stabilized by the chiral interaction \( J_\chi \). Because the chiral term contains a three-spin interaction, we cannot employ the usual Luttinger-Tisza method [43]. Instead, we numerically minimize Eq. (1) using a gradient descent algorithm. Given a spin \( \mathbf{S}_i \), we anti-align it with respect to the gradient of the Hamiltonian: \( \mathbf{S}_i^{m+1} = (1 - \gamma) \mathbf{S}_i^m - \gamma \nabla_i \mathcal{H}(\mathbf{S}_i^m) \), with the step size \( 0 \leq \gamma \leq 1 \) and \( \nabla_i \mathcal{H} = \partial \mathcal{H} / \partial \mathbf{S}_i \) [44]. We consider \( N_{\text{ef}} \in \{100, 200\} \) distinct initial random spin configurations, and we sweep over the lattice locally minimizing each spin. We stop the algorithm when the overall change in the spin configuration after the \( m \)-th iteration is smaller than a given tolerance, which we typically set to \( 10^{-10} \). Our ground state is given by the spin configuration with the lowest energy in the final set. This procedure is realized on a kagome lattice with periodic boundary conditions and system size \( N = 3 \times L \times L \) (see Sec. IV for further de-
Fourier transform the classical ground state spin configuration, we compute its tails of the direct and reciprocal lattices. For a given point \( J_1 = J_d = 0 \), we encounter no incommensurate spiral orders, but we cannot rule out their existence. Interestingly, we also uncover the existence of an extended classically disordered region in the regime \( J_1 \gtrsim |J_d| \), a new phase appears, the FM-stripe, Fig. 3(c). As we reduce the absolute value of \( J_1 \), a new phase appears, the FM-stripe, Fig. 3(c). In this state, ferromagnetic spin chains appear along a single diagonal in the hexagons, with the spins in the other two diagonals displaying a small angle between them. Importantly, this phase possesses a finite spin chirality in the triangles, indicating an energetic trade-off between \( J_1 \) and \( J_\chi \). By symmetry, there are two other equivalent spin configurations, with the ferromagnetic chains running along one of the other two diagonals. As in the AFMd case, the relative intensity of the Bragg peaks varies with \( J_1 \).

If one starts from the limit \( J_1 = 0 \), a negative \( J_d \) favors ferromagnetic chains along the diagonals of the hexagons. This gives rise to the FMd phase, Fig. 3(e), in analogy to the AFMd. The magnetic unit cell, however, contains 48 spins as opposed to the 12 spins in the AFMd, and the relative intensity of the Bragg peaks also depends on \( J_1 \). We leave a more detailed characterization of the magnetically ordered phases for future work.

Finally, we address the classically disordered region. In all its extent, the static spin structure factor shows neither Bragg peaks nor sharp feature and its weight is distributed over the entire Brillouin zone, Fig. 3(f). A classically disordered region is tied to the presence of massive degenerate states, and usually occurs at isolated points in the phase diagram, for instance at the boundaries between two ordered phases. Its extended nature in the present problem may be traced back to the frustrating nature of the kagome lattice. To see this, take a spin \( S_j \). To maximize the chiral interaction, we place the nearest neighbors of \( S_j \) in a plane perpendicular to it, and also at right angles. This causes an accidental degeneracy under relative rotations of these nearest-neighbor spins about the axis defined by \( S_j \). Although quantum fluctuations may lift this degeneracy via the order-by-disorder mechanism [47–50], the presence of an extended classically disordered region in the regime \( J_\chi \gg |J_1|, |J_d| \) is a promising sign that a CSL might be stable for \( S = 1/2 \).
Figure 3. Classical spin configurations (left) and corresponding structure factor (right) for the phases in Fig. 2 with $J_\chi = 1$. The magnetic unit cell is marked by the blue shaded region. (a) cuboc-2; (b) AFMd ($J_1 = -0.05$, $J_d = 0.3$); (c) FM; (d) FM-stripe ($J_1 = -0.13$, $J_d = -0.16$); (e) FMd ($J_1 = -0.05$, $J_d = -0.3$); (f) classically disordered region.

Figure 4. Order parameter squared (normalized with respect to its value in the cuboc-2 state) as a function of $J_1$, indicating the continuous transition between the cuboc-2 and AFMd phases for $J_\chi = 1$ and $J_d > 0$. The red (light-blue) curve shows the normalized Bragg peak intensity at the cuboc-2(1) ordering wave vectors. At the point $J_1 = 0$, an octahedral state maximizing the staggered chirality emerges. The phase transition to the cuboc-1 is slower and takes place in the vicinity of $J_1 = 1$ (not shown). Inset: structure factor for the cuboc-1 phase.

IV. PARTON MEAN-FIELD THEORY FOR GAPLESS CHIRAL SPIN LIQUID

As discussed in Sec. III, the classical phase diagram features a disordered region which may support a CSL ground state for $S = 1/2$. To describe this state, we employ a parton construction in which we fractionalize the spin operator into fermionic spinons, also called Abrikosov fermions [4, 51]. We introduce charge-neutral spin-1/2 fermions $f_{i\alpha}$, with $\alpha = \uparrow, \downarrow$, which satisfy the algebra $\{f_{i\alpha}, f_{j\beta}^\dagger\} = \delta_{ij} \delta_{\alpha\beta}$, $\{f_{i\alpha}, f_{j\beta}\} = 0$. The spin operator at site $i$ is written as

$$S_i^a = \frac{1}{2} \sum_{\alpha, \beta} f_{i\alpha}^\dagger (\sigma^a)_{\alpha\beta} f_{i\beta},$$

where $\sigma^a$ are Pauli matrices. Following the standard parton mean-field decoupling of the Heisenberg interactions in Eq. (1) [51], we obtain

$$H_0^{MF} = -\sum_{\alpha, \beta} \frac{J_{i\alpha}^f}{2} (\xi_{ij} f_{i\alpha}^\dagger f_{j\alpha} + h.c. ) + \sum_{ij} \frac{J_{ij}}{2} |\xi_{ij}|^2,$$

with $\xi_{ij} = \sum_{\alpha} \langle f_{i\alpha}^\dagger f_{j\alpha} \rangle$ a mean-field parameter that specifies the QSL ansatz. This description leads to an SU(2) gauge redundancy [52]. In order to recover physical states, we must impose the single-occupancy constraint locally

$$\sum_{\alpha} n_{i\alpha} = \sum_{\alpha} f_{i\alpha}^\dagger f_{i\alpha} = 1, \ \forall i.$$
The mean-field amplitudes are taken as imaginary numbers. For a fermion with spin $\alpha$ on a lattice site, the mean-field results in No. 1 of Table IX of our ansatz, we consider a U(1) CSL given by a staggered flux phase classified as No. 11 in Table IX of our ansatz. The lattice vectors are defined as an up-pointing triangle, see Fig. 5(a). The arrows on the solid lines indicate the bond direction in which $\xi_i = i\xi_1$. Likewise, $\xi_i = i\xi_d$ in the direction indicated by the arrow on the dashed line. The red arrow represents the nearest-neighbor vector $\delta_i/2$. The dashed line indicates the nearest-neighbor bond, see Fig. 5(a). We obtain zero flux for $\xi_1$ and $\xi_d$ have the same sign and $\pi$ flux otherwise.

We can diagonalize the mean-field Hamiltonian by taking the Fourier transform of the fermion operators:

$$f_{s\alpha}(R) = \sqrt{3/N} \sum_k e^{i k(R + a_\alpha)} f_{s\alpha}(k),$$

where $a_1 = 0$, $a_2 = \delta_3/2$ and $a_3 = -\delta_2/2$ are the relative positions within the unit cell and $N$ is the total number of sites. The mean-field Hamiltonian takes the form

$$H_{\text{MF}} = \sum_k \sum_\alpha \psi^\dagger_{k\alpha} H(k) \psi_{k\alpha} + NJ_0 \xi_1^2 + NJ_1 \xi_0^2 + NJ_2 \xi_3^2,$$

with the spinor $\psi_{k\alpha} = (f_{1\alpha}(k), f_{2\alpha}(k), f_{3\alpha}(k))^T$. The Bloch Hamiltonian $H(k)$ is given by

$$H(k) = - \left( \begin{array}{ccc} \kappa_d \sin(k_1) & \kappa_1 \sin(k_3/2) & \kappa_1 \sin(k_2/2) \\ \kappa_1 \sin(k_3/2) & \kappa_d \sin(k_2) & \kappa_1 \sin(k_1/2) \\ \kappa_1 \sin(k_2/2) & \kappa_1 \sin(k_1/2) & \kappa_d \sin(k_3) \end{array} \right),$$

where $k_i = k \cdot \delta_i$ and we define the effective hopping amplitudes $\kappa_1 = \frac{3}{2}J_0 \xi_1 - J_1 \xi_0$ and $\kappa_d = J_d \xi_d$. For $J_X > 0$ and $J_1 < 0$, we have $\kappa_1 > 0$. For $J_d > 0$, the sign of $\kappa_d$ depends on $\xi_d$, which is related to the gauge flux on the peaks. Diagonalizing $H(k)$, we obtain the dispersion relations $\varepsilon_\lambda(k)$, where $\lambda = 1, 2, 3$ is a band index. Due to particle-hole symmetry, $\varepsilon_\lambda(k) = -\varepsilon_{4-\lambda}(-k)$, the chemical potential must be set to $\mu = 0$ to satisfy the half-filling condition $\sum_\alpha (f_{i\alpha}^\dagger f_{i\alpha}) = 1$. The mean-field ground state $|\Psi_{\text{MF}}\rangle$ is identified with a Fermi sea in which all negative-energy single-particle states are occupied.

Figure 5(b) shows the spectrum for $\kappa_1 > 0$ and $\kappa_d/\kappa_1 = 1.2$, which is representative of the parameter regime $0 < \kappa_d < 2\kappa_1$. In this case, the lower and upper bands exhibit a Dirac cone at the $\Gamma$ point of the Brillouin zone. The middle band shows gapless lines along the $\Gamma$-M directions, see Fig. 5(c). The Fermi surface lines are robust against variations of the ratio $\kappa_d/\kappa_1$ and their location is fixed by the reflection symmetry $\sigma$. However, as we increase $\kappa_d/\kappa_1$, the energy gap for the lower and
Figure 6. Spinon dispersion for the lower and upper bands, $\varepsilon_1(k)$ and $\varepsilon_3(k)$, along the $\Gamma$-$K$ direction, for different values of the ratio $\kappa_d/\kappa_1$. For $\kappa_d/\kappa_1 < 2$, the bands are gapped at the $K$ and $K'$ points. At the critical value $\kappa_d/\kappa_1 = 2$, the bands cross the Fermi level at the $K$ and $K'$ points, forming Fermi pockets for $\kappa_d/\kappa_1 > 2$.

upper bands at the $K$ and $K'$ points decreases. Precisely at the critical value $\kappa_d/\kappa_1 = 2$, these bands cross the Fermi level with a quadratic dispersion at the $K$ and $K'$ points, as represented in Fig. 6. The crossing of the Fermi level and subsequent formation of Fermi pockets around $K$ and $K'$ for $\kappa_d/\kappa_1 > 2$ signals a nesting instability [54] of our CSL at large $\kappa_d$. We have also looked at the spectrum for $\kappa_d < 0$. In this case, a possible instability of the CSL is indicated by a flattening of the middle band around $\kappa_d/\kappa_1 = -1$. For either sign of $\kappa_d$, the instabilities at large $|\kappa_d|$ can be associated with the regime of dominant $J_d$ interaction, where we expect the gapless CSL to be replaced by the ordered phases discussed in Sec. III. This analysis shows that the range of mean-field parameters where we may consider a CSL wave function must be restricted to $-1 < \kappa_d/\kappa_1 < 2$.

V. VARIATIONAL MONTE CARLO RESULTS

With the CSL ansatz at hand, we can construct a free fermion wave function that is invariant under the symmetries of the system once the single-site occupancy, Eq. (6), is enforced. However, the ansatz tells us nothing about the energy of these wave functions. To obtain reliable energy estimates, we carry out a variational analysis based on the projection of the mean-field wave function in the region $-1 < \kappa_d/\kappa_1 < 2$. Specifically, we enforce the constraint in Eq. (6) considering a Gutzwiller projection

$$\hat{P}_G = \prod_i (n_{i\uparrow} - n_{i\downarrow})^2.$$  \hspace{1cm} (14)

To compare the energy of our CSL state to that of the ordered states discussed in Sec. II, we rewrite Eq. (12) as

$$\hat{H}_{\text{MF}} = \sum_{\alpha,ij} \kappa_{ij} f_{i\alpha}^\dagger f_{j\alpha} + h \sum_i \vec{M}_i \cdot \vec{S}_i.$$  \hspace{1cm} (15)

Besides the oriented hopping structure encoded in $\kappa_{ij}$, as discussed in Sec. IV, we include a Zeeman term. Here $h$ controls the strength of the Zeeman coupling, and $\vec{M}_i$ is a classical spin configuration corresponding to one of the ordered states in Fig. 2. It suffices to consider $h \geq 0$. The vector $\vec{M}_i$ effectively acts as a staggered magnetic field and magnetic order can be induced on top of the CSL state if $h \neq 0$ variationally [55, 56]. In this situation, the spinon spectrum is gapped and we interpret the resulting state as adiabatically connected to a conventional magnetically ordered one.

We performed VMC simulations [57] and measured the ground state energy, $E = \langle \Psi | H | \Psi \rangle$, with $H$ given in Eq. (1) and

$$|\tilde{\Psi}_G\rangle = \hat{P}_G |\tilde{\Psi}_{\text{MF}}\rangle.$$  \hspace{1cm} (16)

Here, $|\tilde{\Psi}_{\text{MF}}\rangle$ is the ground state of Eq. (15) at half-filling, with the Gutzwiller projector $\hat{P}_G$ given by Eq. (14). Including local correlations in our variational state, by adding Jastrow factors, will in general reduce the energy of the ordered states further, but we refrain to do so here to limit the number of variational parameters. Even with this simplification, we have a flexible ansatz containing energetically competitive magnetic states in addition to the CSL.

In the VMC simulations, we randomly place each spinon spin flavor on $N/2$ sites of our lattice. Our VMC moves consist of exchanging a random pair of sites of distinct spin flavors. The exchanges are accepted or rejected according to the Metropolis-Hastings algorithm [58]. The probability of each configuration is proportional to the square of the wave function. A number $N$ of exchanges attempts define a VMC sweep. After $N_{\text{warm}} \sim 10^4$ sweeps for thermalization, we calculate our observables considering further $N_{\text{meas}} \sim 10^4$ sweeps. We take $\kappa_d$ and $h$ as our variational parameters, setting $\kappa_1 = 1$ as an inconsequential energy scale. Besides the state discussed in Sec. IV, we also considered an ansatz with $\pi/2$-flux on the trapezoids. This extra case corresponds to ansatz No. 9 in Table IX of Ref. [13]. We find its energy not to be competitive and we refrain from discussing it further. We consider periodic boundary conditions for $H$ and work with systems sizes up to $L = 14$. To mitigate
finite-size effects, we implement mixed boundary conditions for $H_{\text{MF}}$ [8, 9].

The VMC result for the CSL limit ($h = 0$) is shown in Fig. 7 as a function of $\kappa_d$. For large $|\kappa_d|$, our ansatz recovers the energy of the antiferromagnetic Heisenberg chain in the limit $J_d \gg J_\chi, |J_1|$ [59]. As we discussed in Sec. II, we expect the CSL to be unstable in this limit. Inside the stability range of the CSL ansatz, $-1 < \kappa_d < 2$, we observe that the minimum value of the energy occurs for $\kappa_d \approx -0.1$. The energy, per spin, of the CSL state is then given by

$$E_{\text{CSL}}/N = -0.392 (1) J_1 - 0.015 (1) J_d - 0.131 (1) J_\chi,$$  \hspace{1cm} (17)

in the limit $J_\chi \gg |J_d|, |J_1|$. An alternative competitive ansatz for the CSL comes from a parton construction in terms of Majorana fermions [25]. We find that the energy of this state is the same as the one in Eq. (17) as long as one does not include a BCS-like $p$-wave pairing in $\tilde{\Psi}_{\text{MF}}$. For the sake of simplicity, we did not pursue this possibility.

We are now in position to explore the stability of the CSL with respect to the magnetically ordered phases present in Fig. 2. In our variational language, we say that a given ordered state is selected if the energy is minimized for $h \neq 0$. To complement the characterization of the ordered phases, we compute the square of the sublattice magnetization $m$

$$m^2 = \lim_{|i-j| \to \infty} \langle S_i \cdot S_j \rangle.$$ \hspace{1cm} (18)

This observable estimates the spin-spin correlation at maximum distance for two spins belonging to the same magnetic sublattice and gives the square of the staggered magnetization. We then have $m = 0$ in the CSL and $m > 0$ in the corresponding magnetically ordered phase. In Fig. 8 we show the phase transition between the CSL and the AFMd phase as we vary $J_d$ for $J_1 = -0.01$. The finite size-scaling allows us to estimate the transition taking place at $J_d = 0.08(2)$, showing that the CSL is stable around the classically disordered region.

In practice, we construct our phase diagram mainly considering a fixed system size ($L = 12$) due to complex nature of the ordered phases present. Since the classical spin configurations of the AFMd, FMd and FM-stripe phases depend on $J_1$, but not on $J_d$, we compare the energy of the different phases by fixing $J_1$ and varying $J_d$ in the VMC simulation. The resulting phase diagram for $S = 1/2$, in the vicinity of the CSL, is displayed in Fig. 9. The error bars in Fig. 9 come mainly from the presence of magnetization plateaus, due to finite size effects, which hamper a precise determination of the phase boundaries. Overall, the CSL is expanded with respect to the classically disordered region, towards both the AFMd and FM-stripe phases, strongly suggesting the selection of a CSL by a dominant $J_d$.

The position of the transition between the magnetically ordered phases is also altered. Quantum fluctuations reduce the width of the FM-stripe phase with respect to the FMd phase, in the vicinity of the CSL where $|J_d| > |J_1|$. The FMd phase has three coupled FM chains along the diagonal of the hexagons [Fig. 3(e)], as opposed to a single FM chain in the FM-stripe [Fig. 3(e)], which could explain its relative stability despite the larger unit cell.
Figure 9. Phase diagram of the model in Eq. (1) on the kagome lattice obtained via VMC for $J_\chi = 1$ and $S = 1/2$. We focus on the region around the chiral spin liquid (CSL) phase. The classical representation of the magnetically ordered phases is shown in Fig. 4.

VI. DISCUSSION

We investigated the rich phase diagram that stems from the competition between staggered three-spin interactions and frustrated Heisenberg interactions on the extended kagome lattice. In the regime of dominant three-spin interactions, our results support the existence of the gapless chiral spin liquid phase identified in Refs. [25, 27]. The classically disordered region that we observed in this regime is consistent with the analysis of trial wave functions by variational Monte Carlo, which shows that the chiral spin liquid state has lower energy and is stable against order-inducing perturbations.

Increasing the strength of the Heisenberg interactions, we found a number of noncoplanar magnetic states beyond the previously reported cuboc phases. The AFMd and FMd phases can be pictured as coupled spin chains with Néel or ferromagnetic order, respectively, running along the diagonals of the hexagons in the kagome lattice. The angle between the magnetization in different sets of crossing chains varies continuously with the nearest-neighbor exchange coupling, and the cuboc-1, cuboc-2 and octahedral states can be viewed as particular limits of the AFMd phase. We found a continuous transition from the AFMd phase to the cuboc-2 phase which is manifested in the relative intensity of Bragg peaks in the spin structure factor. This continuous transition is reminiscent of the transition from canted antiferromagnetism to the fully polarized state driven by an external magnetic field [60], but here it is driven by a compromise between the frustrated exchange couplings and the three-spin interaction. In addition, we identified a FM-stripe phase at intermediate couplings. This phase breaks the $C_3$ lattice rotational symmetry as the spins select one out of the three diagonals of the hexagons to form ferromagnetically ordered spin chains.

As an extension of this work, it would be interesting to further characterize the novel magnetic phases, in particular by investigating the effects of thermal and quantum fluctuations [50]. Another important question pertains to the nature of the quantum phase transitions from the gapless chiral spin liquid to the magnetically ordered phases. For the topological chiral spin liquid with uniform scalar spin chirality, numerical evidence indicates that exotic continuous transitions may take place as a result of quantum melting of the noncoplanar order [19]. In contrast, transitions from the gapless chiral spin liquid with staggered spin chirality remain largely unexplored.

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