PRIMITIVE ORTHOGONAL IDEMPOTENTS OF BRANDT SEMIGROUP ALGEBRAS AND BRANDT SEMIGROUP CODES

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ABSTRACT. In this paper, we construct a complete set of primitive orthogonal idempotents for any finite Brandt semigroup algebra. As applications, we define a new class of codes called Brandt semigroup codes and compute the Cartan matrices of some Brandt semigroup algebras. We also study the supports, Hamming distances, and minimum weights of Brandt semigroup codes.

Key words: Brandt semigroup algebras; primitive orthogonal idempotents; coding theory

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1. INTRODUCTION

In the past few decades, semigroup algebras are studied intensively, see for example, [Put88], [Okn91], [Bro00], [Ren01], [JO06], [Ste06], [Sal07], [Sch08], [Ste08], [GX09], [BBBS11], [DHST11], [MS12], [EL15].

Berg, Bergeron, Bhargava and Saliola [BBBS11] found a complete system of primitive orthogonal idempotents of any $R$-trivial monoid algebra. Denton, Hivert, Schilling and Thiery [DHST11] gave a construction of a system of primitive orthogonal idempotents for any $J$-trivial monoid algebra.

Steinberg [Ste06], [Ste08] studied the arbitrary finite inverse semigroups $S$ and computed the primitive central idempotent associated to an irreducible representation of $S$. Let $S$ be a finite inverse semigroup, where $E(S)$ is the semilattice of idempotents of $S$ and the maximal subgroup at $e$ is denoted by $H_e$ for all $e \in E(S)$. Suppose that $K$ is a field such that $\text{char}(K) \nmid |H_e|$. Let $\mu$ be the Möbius function of $S$ and $\chi$ an irreducible character of $S$ coming from a $D$-class $D$. 

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Then the primitive central idempotents corresponding to $\chi$ are given by
\[
e_{\chi} = \sum_{e \in E(D)} \left( \frac{\chi(e)}{|He|} \sum_{s \in He} \chi(s) \sum_{t \leq s} t^{-1} \mu(t, s) \right),
\]
see Theorem 5.1 in [Ste06].

In this paper, we construct a complete set of primitive orthogonal idempotents for any finite Brandt semigroup algebra. Brandt semigroups are not $R$-trivial and in particular, they are not $J$-trivial. Let $G$ denote any finite group with identity $e$ and $S = B(G, n)$ a Brandt semigroup. Suppose that $G$ has $r$ conjugate classes. Then a complete set of primitive orthogonal idempotents of $\mathbb{C}S$ is (Theorem 3.3)
\[
\left\{ e_{ij} = \frac{\chi_i(e)}{|G|} \sum_{g \in G} \chi_i(g^{-1})(j, g, j) \mid 1 \leq i \leq r, 1 \leq j \leq n \right\}.
\]

In general, the primitive orthogonal idempotents are not primitive central idempotents, see Remark 3.5.

We apply primitive orthogonal idempotents of Brandt semigroup algebras to study the coding theory.

Abelian codes were defined as the ideals in finite abelian group algebras which were firstly introduced by Berman [Ber67a], [Ber67b] and MacWilliams [Mac70]. In general, a group (left) code was defined as an (left) ideal in a finite group algebra [BRC60]. Group code are studied intensively in the past few years, see for example, [Mil79], [Aro97], [AP99], [KP01], [BR02], [BR03], [SRDR04], [FM07], [BRS08], [KAB14].

A group code is called minimal if the corresponding ideal is minimal in the set of ideals of the group algebra. The papers [Aro97], [AP99], [FM07] computed the number of simple components of a semisimple finite abelian group algebra and determined all cases where this number was minimal. This result is used to compute idempotent generators of minimal abelian codes.

We define a new class of codes called Brandt semigroup codes and obtain the set of mutually orthogonal idempotent generators of minimal Brandt cyclic codes and minimal Brandt abelian codes. We also study the supports, Hamming distances, and minimum weights of Brandt semigroup codes.

By using complete sets of primitive orthogonal idempotents, we compute the Cartan matrices of some Brandt semigroup algebras.

The paper is structured as follows. In Section 2 we recall some background information about semigroup algebras, finite group characters, and group algebra codes. In Section 3 we construct a complete set of primitive orthogonal idempotents for any finite Brandt semigroup algebra and prove our main theorem (Theorem 3.3) given in Section 3.1 In Section 4 we give the Cartan matrices of some Brandt semigroup algebras. In Section 5 we define and study a new class of codes called Brandt semigroup codes. In Section 6 we give some examples of complete sets of primitive orthogonal idempotents of Brandt semigroup algebras.

2. Preliminaries

In this section, we recall the definitions of semigroup algebras, Brandt semigroups, a complete set of primitive orthogonal idempotents, finite group characters, Schur orthogonality relations, Cartan matrices, and group algebra codes.
2.1. Semigroup algebras. Let $K$ be a algebraically closed field, $(S, \cdot)$ be a finite semigroup with identity element $e$ and $A$ be a $K$-algebra. The semigroup algebra of $S$ with coefficients in $A$ is the $K$-vector space $AS$ consisting of all the formal sums $\sum_{g \in S} \lambda_g g$, where $\lambda_g \in A$, with the multiplication defined by the formula

\[
(\sum_{g \in S} \lambda_g g) \cdot (\sum_{h \in S} \mu_h h) = \sum_{f=gh \in S} \lambda_g \mu_h f.
\]  

Then $AS$ is a $K$-algebra and the element $e = 1e$ is the identity of $AS$, where $1$ is the identity element of $A$, see [Okn91]. If $A = K$, then the elements $g \in S$ form a basis of $KS$ over $K$. In this paper we study the Brandt semigroup algebra $\mathbb{C}S$, where $\mathbb{C}$ is the field of complex numbers.

2.2. Brandt semigroups. Let $G$ be a group with identity element $e$, and $I, \Lambda$ be nonempty sets. Let $P = (p_{\lambda i})$ be a $\Lambda \times I$ matrix with entries in the $0$-group $G^0 = G \cup \{0\}$, and suppose that $P$ is regular, in the sense that no row or column of $P$ consists entirely of zeros. Formally, 

\[
(\forall i \in I) \ (\exists \lambda \in \Lambda) \ p_{\lambda i} \neq 0, \tag{2.2}
\]

\[
(\forall \lambda \in \Lambda) \ (\exists i \in I) \ p_{\lambda i} \neq 0. \tag{2.3}
\]

Let $S = (I \times G \times \Lambda) \cup \{0\}$, and define a multiplication on $S$ by

\[
(i, a, \lambda)(j, b, \mu) = \begin{cases} 
(i, ab\lambda j b, \mu), & \text{if } p_{\lambda i} \neq 0, \\
0, & \text{if } p_{\lambda i} = 0,
\end{cases} \tag{2.4}
\]

\[
(i, a, \lambda)0 = 0(i, a, \lambda) = 00 = 0. \tag{2.5}
\]

Then $S$ is called a completely $0$-simple semigroup, see [How95].

If $P = E$, is a identity matrix, whose diagonal elements are identity of the group $G$ and $I = \Lambda$. Then the semigroup $S$ is called a Brandt semigroup, denoted by $B(G, n)$, where $n = |I|$.

2.3. A complete set of primitive orthogonal idempotents. Let $A$ be a $K$-algebra with an identity (denoted by $1$), see I.4 in [ASS06]. A set of nonzero elements $\{e_i\}_{i \in I}$ of $A$ is called a complete set of primitive orthogonal idempotents for $A$, if it satisfies the following four properties:

(i) every element $e_i$ is idempotent, namely, $e_i^2 = e_i$ for all $i \in I$;
(ii) each of the two elements are orthogonal: $e_i e_j = e_j e_i = 0$ for all $i, j \in I$ with $i \neq j$;
(iii) every element $e_i$ is primitive: $e_i$ cannot be written as a sum, that is, if $e_i = x + y$, then $x = 0$ or $y = 0$, where $x$ and $y$ are orthogonal idempotents in $A$;
(iv) the set $\{e_i\}_{i \in I}$ is complete: $\sum_{i \in I} e_i = 1$.

Remark 2.1 (Remark 3.2, [BBBS11]). If $\{e_i\}_{i \in I}$ is a maximal set of nonzero elements satisfying conditions (i) and (ii), then $\{e_i\}_{i \in I}$ is a complete system of primitive orthogonal idempotents (that is, (iii) and (iv) also hold).

2.4. Finite group characters. Let $V$ be a finite dimensional vector space over a field $K$ and $\rho : G \to GL(V)$ a representation of a group $G$ on $V$. The character of $\rho$ is the function $\chi_\rho : G \to K$ given by

\[
\chi_\rho(g) = \text{Tr}(\rho(g)), \tag{2.6}
\]

where $\text{Tr}$ is the trace.
A character $\chi_\rho$ is called irreducible if $\rho$ is an irreducible representation. The number of conjugacy classes of $G$ is equal to the number of irreducible characters of $G$ and equals the number of isomorphism classes of irreducible $KG$-modules. The degree of the character $\chi$ is the dimension of $\rho$ and this is equal to the value $\chi(e)$, see [Isa76].

2.5. Schur orthogonality relations. Schur orthogonality relations, see [Isa76], express a central fact about representations of finite groups. The space of complex valued class functions of a finite group $G$ has a natural inner product

$$\langle \alpha, \beta \rangle = \frac{1}{|G|} \sum_{g \in G} \overline{\alpha(g)} \beta(g),$$

(2.7)

where $\overline{\beta(g)}$ is the complex conjugate of the value of $\beta$ on $g$.

With respect to this inner product, the irreducible characters form an orthogonal basis for the space of class functions, and this yields the orthogonality relation for the rows of the character table

$$\langle \chi_i, \chi_j \rangle = \begin{cases} 0, & \text{if } i \neq j, \\ 1, & \text{if } i = j. \end{cases}$$

(2.8)

For $g, h \in G$, the orthogonality relation for columns is as follows

$$\sum_{\chi_i} \chi_i(g) \chi_i(h) = \begin{cases} |C_G(g)|, & \text{if } g, h \text{ are conjugate,} \\ 0, & \text{otherwise}, \end{cases}$$

(2.9)

where $|C_G(g)|$ denotes the cardinality of the centralizer of $g$.

2.6. Cartan Matrices. Let $A$ be a finite dimensional $K$-algebra with a complete set $\{e_1, \ldots, e_n\}$ of primitive orthogonal idempotents. The Cartan matrix of $A$ is the $n \times n$ matrix, see [ASS06],

$$C_A = \begin{pmatrix} c_{11} & \cdots & c_{1n} \\ \vdots & \ddots & \vdots \\ c_{n1} & \cdots & c_{nn} \end{pmatrix}_{n \times n},$$

(2.10)

where $c_{ji} = \dim_K \text{Hom}_A(Ae_j, Ae_i) = \dim_K e_i Ae_j$ is called the Cartan invariants of $A$, for $i, j = 1, \ldots, n$.

By the definition, we can see that the Cartan matrix is defined with respect to a given complete set $\{e_1, e_2, \ldots, e_n\}$ of primitive orthogonal idempotents of $A$. We know that

$$e_i Ae_j \cong \text{Hom}_A(P(j), P(i)) \cong \text{Hom}_A(I(j), I(i)).$$

(2.11)

Then the Cartan matrix of $A$ records the number of homomorphisms between the indecomposable projective $A$-modules and the number of homomorphisms between the indecomposable injective $A$-modules.
2.7. Group algebra codes. Let $G$ be a finite group, and let $K$ be an arbitrary field such that $\text{char}(K) \nmid |G|$ or $\text{char}(K) = 0$. The group algebra $KG$ of the group $G$ over the field $K$ is defined as the algebra over the field $K$ consisting of all possible linear combinations

$$\alpha = \sum_{g \in G} \alpha_g g, \text{ where } \alpha_g \in K. \quad (2.12)$$

Any ideal $V$ of the algebra $KG$ is called a $G$-code. If the group $G$ is abelian, the $G$-code $V$ is abelian. Cyclic $(n, k)$-codes over the field $K$ are in one to one correspondence with the ideals of the factor ring

$$A = K[x]/J, \quad (2.13)$$

where $J = (x^n - 1)$ is the principal ideal, see [Ber67b].

Let $\alpha = \sum_{g \in G} \alpha_g g, \beta = \sum_{g \in G} \beta_g g$. Then

$$\alpha = \beta \text{ if and only if } \alpha_g = \beta_g, \quad (2.14)$$

for all $g \in G$.

The support of an element $\alpha \in KG$ is the set of elements of $G$ effectively appearing in $\alpha$

$$\text{supp}(\alpha) = \{ g \in G \mid \alpha_g \neq 0 \}. \quad (2.15)$$

Given $\alpha = \sum_{g \in G} \alpha_g g \in KG$, the number of elements in its support is called the weight of $\alpha$, namely

$$\omega(\alpha) = |\{ g \mid \alpha_g \neq 0 \}|. \quad (2.16)$$

3. A Complete Set of Primitive Orthogonal Idempotents for $B(G, n)$

In this section, we compute a complete set of the primitive orthogonal idempotents of any finite Brandt semigroup algebra.

3.1. Constructing primitive orthogonal idempotents. Suppose that $G$ is any finite group with identity $e$ and has $r$ conjugacy classes. Let $S = B(G, n)$ be a finite Brandt semigroup and $\chi_i$ an irreducible complex character of $G$. For $1 \leq i \leq r, 1 \leq j \leq n$, we define

$$e_{ij} = \frac{\chi_i(e)}{|G|} \sum_{g \in G} \chi_i(g^{-1})(j, g, j). \quad (3.1)$$

We note that the element $\sum_{i=1}^{n} (i, e, i)$ is the identity of Brandt semigroup algebra $\mathbb{C}S$, denoted by $1$.

Lemma 3.1 (Corollary 2.7, [Isa76]). Let $G$ be a finite group with the identity $e$ and $\text{Irr}(G)$ the set of all irreducible complex character of $G$. Then $|\text{Irr}(G)|$ equals the number of conjugacy classes of $G$ and

$$\sum_{\chi \in \text{Irr}(G)} \chi(e)^2 = |G|. \quad (3.2)$$

By Schur orthogonality relations, we have the following lemma.
Lemma 3.2 (Theorem 2.13, [Isa76]). Let $G$ be a finite group with the identity $e$ and $\chi_i$ an irreducible complex character of $G$. Then the following holds for every $h \in G$,

$$\frac{1}{|G|} \sum_{g \in G} \chi_i(gh)\chi_j(g^{-1}) = \delta_{ij} \frac{\chi_i(h)}{\chi_i(e)}, \quad (3.3)$$

where

$$\delta_{ij} = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j. \end{cases} \quad (3.4)$$

Our first main result is the following theorem.

Theorem 3.3. Let $G$ denote any finite group with identity $e$ and $S = B(G, n)$ a Brandt semigroup. Suppose that $G$ has $r$ conjugacy classes. Then the elements

$$e_{ij} = \frac{\chi_i(e)}{|G|} \sum_{g \in G} \chi_i(g^{-1})(j, g, j), \quad (3.5)$$

where $1 \leq i \leq r$, $1 \leq j \leq n$, form a complete set of primitive orthogonal idempotents of the Brandt semigroup algebra $\mathbb{C}S$.

Proof. Firstly, we prove the set $\{e_{ij}\}_{1 \leq i \leq r, \ 1 \leq j \leq n}$ is complete.

Since $G$ has $r$ conjugacy classes, $G$ has $r$ irreducible representations. For any $1 \leq j \leq n$, we have

$$\sum_{i=1}^{r} e_{ij} = \frac{1}{|G|} \sum_{i=1}^{r} \sum_{g \in G} \chi_i(e)\chi_i(g^{-1})(j, g, j)$$

$$= \frac{1}{|G|} \sum_{i=1}^{r} \chi_i(e)\left(\chi_i(e^{-1})(j, e, j) + \sum_{g \in G \setminus \{e\}} \chi_i(g^{-1})(j, g, j)\right)$$

$$= \frac{1}{|G|} \sum_{i=1}^{r} \chi_i(e)\chi_i(e^{-1})(j, e, j) + \frac{1}{|G|} \sum_{g \in G \setminus \{e\}} \sum_{i=1}^{r} \chi_i(e)\chi_i(g^{-1})(j, g, j).$$

Obviously, $e$ and $g \in G \setminus \{e\}$ are not conjugate. According to Schur orthogonality relations (2.9), we have

$$\sum_{i=1}^{r} \chi_i(e)\chi_i(g) = 0,$$

where $g \neq e$.

Hence

$$\sum_{i=1}^{r} \chi_i(e)\chi_i(g^{-1}) = 0.$$

Thus

$$\sum_{i=1}^{r} e_{ij} = \frac{1}{|G|} \sum_{i=1}^{r} \chi_i(e)\chi_i(e^{-1})(j, e, j) = \frac{1}{|G|} \sum_{i=1}^{r} (\chi_i(e))^2(j, e, j).$$
By Lemma 3.1, we have
\[ \sum_{i=1}^{r} (\chi_{i}(e))^2 = |G|. \]

Then
\[ \sum_{i=1}^{r} e_{ij} = (j, e, j). \]

Thus
\[ \sum_{j=1}^{n} \sum_{i=1}^{r} e_{ij} = \sum_{j=1}^{n} (j, e, j) = 1. \]

Secondly, we prove that the elements of \( \{e_{ij}\}_{1 \leq i \leq r, 1 \leq j \leq n} \) are pairwise orthogonal. For any \( j_1 \neq j_2, 1 \leq j_1, j_2 \leq n, 1 \leq i_1, i_2 \leq r, \)
\[ e_{i_1j_1}e_{i_2j_2} = \left( \frac{\chi_{i_1}(e)}{|G|} \sum_{g' \in G} \chi_{i_1}(g'^{-1})(j_1, g', j_1) \right) \left( \frac{\chi_{i_2}(e)}{|G|} \sum_{g \in G} \chi_{i_2}(g^{-1})(j_2, g, j_2) \right) \]
(3.6)
\[ = 0. \]

For any \( j_1 = j_2, i_1 \neq i_2, \)
\[ e_{i_1j_1}e_{i_2j_2} = \left( \frac{\chi_{i_1}(e)}{|G|} \sum_{g' \in G} \chi_{i_1}(g'^{-1})(j_1, g', j_1) \right) \left( \frac{\chi_{i_2}(e)}{|G|} \sum_{g \in G} \chi_{i_2}(g^{-1})(j_2, g, j_2) \right) \]
(3.8)
\[ = \frac{\chi_{i_1}(e)\chi_{i_2}(e)}{|G|^2} \sum_{g' \in G} \sum_{g \in G} \chi_{i_1}(g'^{-1})\chi_{i_2}(g^{-1})(j_1, g'g, j_1) \]
(3.9)
\[ = \frac{\chi_{i_1}(e)\chi_{i_2}(e)}{|G|^2} \sum_{g'' \in G} \sum_{g \in G} \chi_{i_1}(g''^{-1})\chi_{i_2}(g^{-1})(j_1, g'', j_1). \]
(3.10)

By Lemma 3.2
\[ \sum_{g \in G} \chi_{i_1}(g''^{-1})\chi_{i_2}(g^{-1}) = \delta_{i_1i_2} \frac{\chi_{i_1}(g''^{-1})}{\chi_{i_1}(e)}. \]
(3.11)

Then, for any \( i_1 \neq i_2, \)
\[ e_{i_1j_1}e_{i_2j_1} = 0. \]
(3.12)

Hence, the elements of \( \{e_{ij}\}_{1 \leq i \leq r, 1 \leq j \leq n} \) are pairwise orthogonal.

In the following, we prove that each element of \( \{e_{ij}\}_{1 \leq i \leq r, 1 \leq j \leq n} \) is idempotent. Let \( e_{st} \) be any element of the set \( \{e_{ij}\}_{1 \leq i \leq r, 1 \leq j \leq n}, \) where \( 1 \leq s \leq r, 1 \leq t \leq n. \) Then
\[ e_{st}^2 = e_{st} \left( \sum_{j=1}^{n} \sum_{i=1}^{r} e_{ij} \right) = e_{st} 1 = e_{st}. \]
(3.13)
Finally, we prove that each element of \( \{ e_{ij} \}_{1 \leq i \leq r, 1 \leq j \leq n} \) is primitive. Suppose that \( f \in \mathbb{C}S \), \( f^2 = f \), \( f \neq 0 \), \( f \notin \{ e_{ij} \}_{1 \leq i \leq r, 1 \leq j \leq n} \), and \( f \) is orthogonal to every \( e_{ij} \), \( 1 \leq i \leq r \), \( 1 \leq j \leq n \). Then

\[
f = f1 = f \left( \sum_{j=1}^{n} \sum_{i=1}^{r} e_{ij} \right) = 0.
\]

This contradicts the assumption that \( f \neq 0 \). Therefore \( \{ e_{ij} \}_{1 \leq i \leq r, 1 \leq j \leq n} \) is a maximal set of nonzero elements satisfying conditions (i) and (ii) in Section 2.3. By Remark 2.1, each element of \( \{ e_{ij} \}_{1 \leq i \leq r, 1 \leq j \leq n} \) is primitive. \( \square \)

**Corollary 3.4.** Let \( G = \langle a \mid a^{k} = e \rangle \) be a finite cyclic group and \( S \) a Brandt semigroup with \( |\Lambda| = |I| = n \). Then for \( 1 \leq p \leq k \), \( 0 \leq q \leq n - 1 \), the elements

\[
e_{p+qk(n+1)} = \frac{1}{k} \sum_{j=1}^{k} \omega^{-jp}(q + 1, a^j, q + 1),
\]

where \( \omega = e^{\frac{2\pi i}{k}} \), \( i = \sqrt{-1} \), form a complete set of primitive orthogonal idempotents of the Brandt semigroup algebra \( \mathbb{C}S \).

**Proof.** This follows directly from Theorem 3.3. \( \square \)

**Remark 3.5.** Let \( S = B(G, n) \) be a finite Brandt semigroup. If \( n = 1 \), the primitive orthogonal idempotents of \( \mathbb{C}S \) are precisely primitive central idempotents. If \( n \geq 2 \), in general, the primitive orthogonal idempotents are not primitive central idempotents. We can see the following example.

Let \( G = \{ e \} \), \( I = \{ 1, 2 \} \). Then we have the Brandt semigroup \( S = B_2 = \langle a, b \mid a^2 = b^2 = 0, aba = a, bab = b \rangle \).

We also have the identity

\[
1 = \sum_{i=1}^{2} (i, e, i).
\]

By Theorem 3.4, the elements

\[
e_1 = (1, e, 1), \quad e_4 = (2, e, 2)
\]

form a complete set of primitive orthogonal idempotents in the semigroup algebra \( \mathbb{C}S \).

Let \( \mathbb{C}S = \{ \sum_{i=1}^{4} k_is_i \mid k_i \in \mathbb{C}, s_i \in S \} \). Then we have

\[
e_1 \mathbb{C}S = \{ k_1(1, e, 1) + k_2(1, e, 2) \mid k_1, k_2 \in \mathbb{C} \},
\]

\[
\mathbb{C}Se_1 = \{ k_1(1, e, 1) + k_3(2, e, 1) \mid k_1, k_2 \in \mathbb{C} \}.
\]

Obviously,

\[
e_1 \mathbb{C}S \neq \mathbb{C}Se_1.
\]

Then the primitive orthogonal idempotents are not primitive central idempotents in the semigroup algebra \( \mathbb{C}S \).
4. The Cartan Matrices of some Brandt semigroup algebras

In this section, we compute the Cartan matrices of some Brandt semigroup algebras. We have the following theorems.

**Theorem 4.1.** The Cartan matrix of the Brandt semigroup algebra $\mathbb{C}B(\{e\}, n)$ is

$$
\mathbf{C}_{\mathbb{C}B(\{e\}, n)} = \begin{pmatrix}
1 & 1 & \cdots & 1 \\
1 & 1 & \cdots & 1 \\
\vdots & \vdots & \ddots & \vdots \\
1 & 1 & \cdots & 1
\end{pmatrix}_{n \times n}.
$$

**Proof.** By Corollary 3.4, a complete set of primitive orthogonal idempotents of Brandt semigroup algebra $\mathbb{C}B(\{e\}, n)$ is

$$
\{e_i = (i, e, i) \mid 1 \leq i \leq n\}.
$$

For any $1 \leq \lambda, \mu \leq n$, we have

$$
\mathbb{C}B(\{e\}, n) = \left\{ \sum_{\mu=1}^{n} \sum_{\lambda=1}^{n} k_{\lambda \mu}(\lambda, e, \mu) \mid k_{\lambda \mu} \in \mathbb{C} \right\}.
$$

Hence

$$
\mathbb{C}B(\{e\}, n)e_1 = \left\{ \sum_{\mu=1}^{n} \sum_{\lambda=1}^{n} k_{\lambda \mu}(\lambda, e, \mu)(1, e, 1) \mid k_{\lambda \mu} \in \mathbb{C} \right\}
$$

$$
= \left\{ \sum_{\lambda=1}^{n} k_{\lambda 1}(\lambda, e, 1) \mid k_{\lambda 1} \in \mathbb{C} \right\}.
$$

Therefore, for any $1 \leq i \leq n$,

$$
e_i \mathbb{C}B(\{e\}, n)e_1 = \left\{ k_{i1}(i, e, 1) \mid k_{i1} \in \mathbb{C} \right\}.
$$

Thus

$$
dim_{\mathbb{C}} e_i \mathbb{C}B(\{e\}, n)e_1 = 1.
$$

By a similar argument, for any $1 \leq i, j \leq n$,

$$
dim_{\mathbb{C}} e_i \mathbb{C}B(\{e\}, n)e_j = 1.
$$

**Theorem 4.2.** Let $G_2 = \langle a \mid a^2 = e \rangle$ be a cyclic group. Then the Cartan matrix of the Brandt semigroup algebra $\mathbb{C}B(G_2, n)$ is

$$
\mathbf{C}_{\mathbb{C}B(G_2, n)} = \begin{pmatrix}
E & E & \cdots & E \\
E & E & \cdots & E \\
\vdots & \vdots & \ddots & \vdots \\
E & E & \cdots & E
\end{pmatrix}_{2n \times 2n},
$$

where $E$ is the identity matrix of order 2.
Proof. By Corollary [3.4], a complete set of primitive orthogonal idempotents of Brandt semigroup algebra \( \mathbb{C}B(G_2, n) \) is
\[
\{ e_{2i-1}, e_{2i} \mid 1 \leq i \leq n \}
\]
where
\[
e_{2i-1} = \frac{1}{2}(i, e, i) - \frac{1}{2}(i, a, i), \quad e_{2i} = \frac{1}{2}(i, e, i) + \frac{1}{2}(i, a, i).
\]
(4.3)
For any \( 1 \leq \lambda, \mu \leq n \), we have
\[
\mathbb{C}B(G_2, n) = \left\{ \sum_{\lambda=1}^{n} \sum_{\mu=1}^{n} \sum_{j=1}^{2} k_{\lambda j} \mu \left( (\lambda, e, 1) - (\lambda, a, 1) \right) \mid k_{\lambda j} \mu \in \mathbb{C} \right\}.
\]
Hence
\[
\mathbb{C}B(G_2, n)e_1 = \sum_{\lambda=1}^{n} \sum_{\mu=1}^{n} \sum_{j=1}^{2} k_{\lambda j} \mu \left( (\lambda, e, 1) - (\lambda, a, 1) \right) k_{\lambda j} \mu \in \mathbb{C}.
\]
(4.4)
Therefore, for any \( 1 \leq i \leq n \),
\[
e_{2i} \mathbb{C}B(G_2, n)e_1 = 0,
\]
(4.6)
e_{2i-1} \mathbb{C}B(G_2, n)e_1 = \sum_{\lambda=1}^{n} \sum_{j=1}^{2} k_{\lambda j} (i, e, 1) - (i, a, 1) e_{2i} \mathbb{C}B(G_2, n)e_1 = 0.
\]
(4.7)
Thus
\[
\dim_{\mathbb{C}} e_{2i-1} \mathbb{C}B(G_2, n)e_1 = 1, \quad \dim_{\mathbb{C}} e_{2i} \mathbb{C}B(G_2, n)e_1 = 0.
\]
(4.8)
Similarly, for any \( 1 \leq i, j \leq n \),
\[
\dim_{\mathbb{C}} e_{2i-1} \mathbb{C}B(G_2, n)e_{2j-1} = 1, \quad \dim_{\mathbb{C}} e_{2i} \mathbb{C}B(G_2, n)e_{2j-1} = 0.
\]
(4.9)
By a similar argument, for any \( 1 \leq i, j \leq n \),
\[
\dim_{\mathbb{C}} e_{2i-1} \mathbb{C}B(G_2, n)e_{2j} = 0, \quad \dim_{\mathbb{C}} e_{2i} \mathbb{C}B(G_2, n)e_{2j} = 1.
\]
(4.10)
\[\square\]

Theorem 4.3. Let \( G = \langle a \mid a^k = e, k \geq 3 \rangle \) be a cyclic group. Then the Cartan matrix of the Brandt semigroup algebra \( \mathbb{C}B(G, n) \) is
\[
\mathbf{C}_{\mathbb{C}B(G, n)} = \begin{pmatrix}
E & E & \cdots & E \\
E & E & \cdots & E \\
\vdots & \vdots & \ddots & \vdots \\
E & E & \cdots & E
\end{pmatrix}_{kn \times kn},
\]
(4.11)
where \( E \) is the identity matrix of order \( k \).
Proof. By Corollary 3.4, a complete set of primitive orthogonal idempotents of Brandt semigroup algebra $\mathbb{C}B(G, n)$ is

$$\left\{ \frac{1}{k} \sum_{j=1}^{k} \omega^{-jp}(q + 1, a^j, q + 1) \mid 1 \leq p \leq k, 0 \leq q \leq n - 1 \right\}. \tag{4.12}$$

We list all them by the following matrix:

$$
\begin{pmatrix}
\frac{n}{k} \sum_{j=1}^{k} \omega^{-j}(1, a^j, 1) & \cdots & \frac{n}{k} \sum_{j=1}^{k} \omega^{-j}(n, a^j, n) \\
\frac{n}{k} \sum_{j=1}^{k} \omega^{-2j}(1, a^j, 1) & \cdots & \frac{n}{k} \sum_{j=1}^{k} \omega^{-2j}(n, a^j, n) \\
\vdots & \ddots & \vdots \\
\frac{n}{k} \sum_{j=1}^{k} \omega^{-kj}(1, a^j, 1) & \cdots & \frac{n}{k} \sum_{j=1}^{k} \omega^{-kj}(n, a^j, n)
\end{pmatrix}_{n \times k}
$$

For any $1 \leq \lambda, \mu \leq n$, we have

$$\mathbb{C}B(G, n) = \left\{ \sum_{\lambda=1}^{n} \sum_{\mu=1}^{n} k_{\lambda j} \mu (\lambda, a^j, \mu) \mid k_{\lambda j} \mu \in \mathbb{C} \right\}.$$ 

Hence

$$\mathbb{C}B(G, n)e_1 = \left\{ \frac{1}{k} \sum_{\lambda=1}^{n} \sum_{j=1}^{k} k_{\lambda j} (\lambda, a^j, 1) \sum_{j'=1}^{k} \omega^{-j'}(1, a^{j'}, 1) \mid k_{\lambda j} \in \mathbb{C} \right\}$$

$$= \left\{ \frac{1}{k} \sum_{\lambda=1}^{n} \left( (k_{\lambda 11} + k_{\lambda 22} \omega^{-(k-1)}) + k_{\lambda 33} \omega^{-(k-2)} + \cdots + k_{\lambda k1} \omega^{-1} \right)(\lambda, a, 1) \\
+ (k_{\lambda 11} \omega^{-1} + k_{\lambda 22} + k_{\lambda 33} \omega^{-(k-1)} + \cdots + k_{\lambda k1} \omega^{-2})(\lambda, a^2, 1) \\
+ \cdots \\
+ (k_{\lambda 11} \omega^{-(k-1)} + k_{\lambda 22} \omega^{-(k-2)} + \cdots + k_{\lambda k1})(\lambda, b, 1) \mid k_{\lambda j} \in \mathbb{C} \right\}.$$
Therefore

\[ e_{k+1}CB(G, n)e_1 = \left\{ \frac{1}{k^2} \left( \omega^{-1}(k_{211}^{(k-1)} + k_{221}^{(k-2)} + \cdots + k_{2k1}^{(k-1)}) + \omega^{-2}(k_{211}^{(k-2)} + k_{221}^{(k-3)} + \cdots + k_{2k1}^{(k-1)}) + \cdots + \omega^{-k}(k_{211}^{(k-1)} + k_{221}^{(k-2)} + \cdots + k_{2k1}^{(k-1)})) \right\} (2, a, 1) + (\omega^{-1}(k_{211}^{(k-2)} + k_{221}^{(k-3)} + \cdots + k_{2k1}^{(k-1)})) (2, a, 1) + \cdots + \omega^{-k}(k_{211}^{(k-1)} + k_{221}^{(k-2)} + \cdots + k_{2k1}^{(k-1)})) (2, e, 1) \mid k_{2j1} \in \mathbb{C} \right\} \]

\[ = \left\{ \frac{1}{k} \left( (k_{211} + k_{221}^{(k-1)} + k_{231}^{(k-2)} + \cdots + k_{2k1}^{(k-1)})) (2, a, 1) + (k_{211}^{(k-1)} + k_{221} + k_{231}^{(k-1)} + \cdots + k_{2k1}^{(k-2)})) (2, a, 1) + \cdots + (k_{211}^{(k-1)} + k_{221}^{(k-2)} + \cdots + k_{2k1}^{(k-1)}) (2, e, 1) \mid k_{2j1} \in \mathbb{C} \right\} \}

\[ e_2CB(G, n)e_1 = \left\{ \frac{1}{k^2} (1 + \omega^{-1} + \omega^{-2} + \cdots + \omega^{-(k-1)}) \sum_{j=1}^{k} k_{1j1}(1, a^j, 1) \mid k_{1j1} \in \mathbb{C} \right\} \]

\[ = 0. \]

Similarly, for any \( 0 \leq q \leq n - 1, \)

\[ e_{qk+1}CB(G, n)e_1 = \left\{ \frac{1}{k} \left( (k_{(q+1)11} + k_{(q+1)21}^{(k-1)} + k_{(q+1)31}^{(k-2)} + \cdots + k_{(q+1)k1}^{(k-1)})) (q + 1, a, 1) + (k_{(q+1)11}^{(k-1)} + k_{(q+1)21} + k_{(q+1)31}^{(k-2)} + \cdots + k_{(q+1)k1}^{(k-1)}) (q + 1, a^2, 1) + \cdots + (k_{(q+1)11}^{(k-1)} + k_{(q+1)21}^{(k-2)} + \cdots + k_{(q+1)k1}) (q + 1, e, 1) \mid k_{(q+1)j1} \in \mathbb{C} \right\} \]

For any \( 0 \leq q \leq n - 1, 2 \leq l \leq k, \)

\[ e_{qk+l}CB(G, n)e_1 = 0. \]
Thus, for any \( k \geq 3 \),
\[
\dim \mathbb{C} e_{qk+1} CB(G, n) e_1 = k, \quad \dim \mathbb{C} e_{qk+1} CB(G_2, n) e_1 = 0.
\]
The other cases are similar. \( \Box \)

5. Brandt semigroup Codes

In this section, we introduce a new class of codes called Brandt semigroup codes, and study the supports, Hamming distances, and minimum weights of Brandt semigroup codes.

We introduce a new class of codes called Brandt semigroup codes as follows.

**Definition 5.1.** Let \( S = B(G, n) \) be a Brandt semigroup. Then a Brandt semigroup algebra code \( C \) in \( CS \), or a Brandt semigroup code for short, is a one side ideal of \( \mathbb{C} S \). A code \( C \) is called a Brandt abelian code if the finite group \( G \) is an abelian group and the index \( n = \{1\} \). Otherwise, the code \( C \) is called a Brandt non-abelian code.

**Definition 5.2.** Let \( C_1 = \sum_{(j,g,j) \in S} \alpha_{(j,g,j)}(j,g,j) \in \mathbb{C} S \), \( C_2 = \sum_{(j,g,j) \in S} \beta_{(j,g,j)}(j,g,j) \in \mathbb{C} S \), where \( \alpha_{(j,g,j)}, \beta_{(j,g,j)} \in \mathbb{C} \), be two Brandt semigroup codes. Then the number of the two elements of the support in which the coefficients differ is called the Hamming distance of \( C_1 \) and \( C_2 \), namely,
\[
d(C_1, C_2) = \left| \{ (j,g,j) \mid \alpha_{(j,g,j)} \neq \beta_{(j,g,j)}, (j,g,j) \in S \} \right|.
\]
For a code \( C \) of \( \mathbb{C} S \), we define the minimum weight of \( C \) as:
\[
\omega(C) = \min \{ \omega(\alpha) \mid \alpha \in C, \alpha \neq 0 \}. \tag{5.2}
\]

**Theorem 5.3.** Let \( S = B(G, 1) \) be a finite Brandt semigroup and \( G = \langle a \mid a^k = e \rangle \) a finite cyclic group. Then
\[
\left\{ e_p = \frac{1}{k} \sum_{j=1}^{k} \omega^{-jp}(1, a^j, 1) \mid 1 \leq p \leq k, \omega = e^{2\pi i} \right\} \tag{5.3}
\]
is a set of mutually orthogonal idempotent generators of minimal Brandt cyclic codes of \( \mathbb{C} B(G, 1) \).

Let \( G \) be a finite abelian group with identity \( e \) of conjugate classes \( r \). Then
\[
\left\{ e_i = \frac{\chi_i(e)}{|G|} \sum_{g \in G} \chi_i(g^{-1})(1, g, 1) \mid 1 \leq i \leq r \right\} \tag{5.4}
\]
is a set of mutually orthogonal idempotent generators of minimal Brandt abelian codes of \( \mathbb{C} B(G, 1) \).

**Proof.** This result follows from Theorem 3.3 in Section 3 and Theorem 2 in [Sab93]. \( \Box \)

**Lemma 5.4** (Theorem 2.5.11, [MS02]). Let \( A = \bigoplus_{i=1}^{t} A_i \) be a decomposition of a semisimple algebra as a direct sum of minimal left ideals. Then, there exists a family of elements \( \{ e_1, \ldots, e_t \} \) which consist a complete set of primitive orthogonal idempotents of \( A \).

Conversely, if there exists a family of idempotents \( \{ e_1, \ldots, e_t \} \) satisfying the four properties of the Definition 2.5.3, then the left ideals \( A_i = Ae_i \) are minimal and \( A = \bigoplus_{i=1}^{t} A_i \).

**Theorem 5.5.** Let \( S = B(G, n) \) be a finite Brandt semigroup. Suppose that \( n \geq 2 \) and \( \{ e_{ij} \mid 1 \leq i \leq r, 1 \leq j \leq n \} \) is a complete set of primitive orthogonal idempotents of \( \mathbb{C} S \). Then \( \{ \mathbb{C} S e_{ij} \mid 1 \leq i \leq r, 1 \leq j \leq n \} \) are minimal Brandt non-abelian codes.
Proof. By Corollary 4.5 in [Ste06], the Brandt semigroup algebra \( \mathbb{C}S \) is semisimple. By Lemma 5.4 and Definition 5.1, \( \mathbb{C}Se_{ij}, 1 \leq i \leq r, 1 \leq j \leq n \), are minimal Brandt non-abelian codes. \( \square \)

Proposition 5.6 (Theorem 2.5.10, [MS02]). Let \( S \) be a finite Brandt semigroup. Then every left ideal \( L \) of \( \mathbb{C}S \) is of the form \( L = \mathbb{C}Se \), where \( e \in \mathbb{C}S \) is an idempotent.

Lemma 5.7 (Proposition 4.5, [ASS06]). Let \( B = A/\text{rad}\,A \), where \( A \) is a \( K \)-algebra. Then every left ideal \( I \) of \( B \) is a direct sum of simple right ideals of the form \( Be \), where \( e \) is a primitive idempotent of \( B \).

Theorem 5.8. Let \( S \) be a finite Brandt semigroup. Then every Brandt semigroup code \( C \) of \( \mathbb{C}S \) is a direct sum of minimal Brandt semigroup codes of the form \( \mathbb{C}Se \), where \( e \) is a primitive idempotent of \( \mathbb{C}S \).

Proof. This follows directly from Lemma 5.7. \( \square \)

We are mostly interested in that how to get all Brandt semigroup codes by a complete set of primitive orthogonal idempotents of \( \mathbb{C}S \). Using the basic group table matrix of \( G \), W. S. Park [Par97] found all the idempotents of the group algebra \( KG \), where \( K \) is an algebraically closed field of characteristic 0 and \( G \) is a cyclic group of order \( n \). Suppose that the index set of Brandt semigroup \( S \) is trivial, then we can get all the Brandt semigroup codes by a complete set of primitive orthogonal idempotents of \( \mathbb{C}S \).

Theorem 5.9. Let \( S = B(G, 1) \) be a finite Brandt semigroup and \( G \) a cyclic group of order \( n \). Then every Brandt semigroup code \( C \) of \( \mathbb{C}S \) is a direct sum of minimal Brandt semigroup codes of the form \( \mathbb{C}Se_{i1} \), where \( \{e_{i1}\}_{1 \leq i \leq n} \) is a complete set of primitive orthogonal idempotents of \( \mathbb{C}S \).

Proof. By Theorem 3.1 in [Par97], if \( r_0 = 1/n \), the idempotents \( \sum_{j=0}^{n-1} r_j(1, g^j, 1) \) are precisely form a complete set of primitive orthogonal idempotents of \( \mathbb{C}S \) which are denoted \( e_{i1} \), where \( 1 \leq i \leq n \). Then \( \mathbb{C}Se_{i1} \) are all minimal Brandt semigroup codes. Since all the other idempotents can be represented by \( e_{i1}, 1 \leq i \leq n \), by combination. Then every Brandt semigroup code \( C \) of \( \mathbb{C}S \) is a direct sum of some minimal Brandt semigroup codes. \( \square \)

Example 5.10. Let \( G_4 = \langle a \mid a^4 = e \rangle \) be a cyclic group, \( \Lambda = I = \{1\} \), and \( S = B(G_4, 1) \) a Brandt semigroup. Then the elements

\[
e_{11} = \frac{1}{4} \left((1, e, 1) + (1, a, 1) + (1, a^2, 1) + (1, a^3, 1)\right),
\]
\[
e_{21} = \frac{1}{4} \left((1, e, 1) - (1, a, 1) + (1, a^2, 1) - (1, a^3, 1)\right),
\]
\[
e_{31} = \frac{1}{4} \left((1, e, 1) + i(1, a, 1) - (1, a^2, 1) - i(1, a^3, 1)\right),
\]
\[
e_{41} = \frac{1}{4} \left((1, e, 1) - i(1, a, 1) - (1, a^2, 1) + i(1, a^3, 1)\right),
\]

We are mostly interested in that how to get all Brandt semigroup codes by a complete set of primitive orthogonal idempotents of \( \mathbb{C}S \). Using the basic group table matrix of \( G \), W. S. Park [Par97] found all the idempotents of the group algebra \( KG \), where \( K \) is an algebraically closed field of characteristic 0 and \( G \) is a cyclic group of order \( n \). Suppose that the index set of Brandt semigroup \( S \) is trivial, then we can get all the Brandt semigroup codes by a complete set of primitive orthogonal idempotents of \( \mathbb{C}S \).
form a complete set of primitive orthogonal idempotents of $CS$. The following codes

$$C_{1000} = CSS_{111} = \left\{ \frac{k_1 + k_2 + k_3 + k_4}{4} (1, e, 1) + (1, a, 1) + (1, a^2, 1) + (1, a^3, 1) \right\} | k_1, k_2, k_3, k_4 \in \mathbb{C} \},$$

$$C_{0100} = CSS_{31} = \left\{ \frac{k_1 - k_2 + k_3 - k_4}{4} (1, e, 1) + (1, a, 1) + (1, a^2, 1) - (1, a^3, 1) \right\} | k_1, k_2, k_3, k_4 \in \mathbb{C} \},$$

$$C_{0010} = CSS_{21} = \left\{ \frac{k_1 + k_2 - k_3 + k_4}{4} (1, e, 1) - (1, a, 1) - (1, a^2, 1) + (1, a^3, 1) \right\} | k_1, k_2, k_3, k_4 \in \mathbb{C} \},$$

$$C_{0001} = CSS_{41} = \left\{ \frac{k_1 - k_2 - k_3 + k_4}{4} (1, e, 1) - (1, a^2, 1) - k_1 k_2 - k_3 - k_4 \right\} | k_1, k_2, k_3, k_4 \in \mathbb{C} \}. $$

are minimal Brandt cyclic codes, where $CS = \{ \sum_{i=1}^{4} k_i s_i | k_i \in \mathbb{C}, s_i \in S \}$. We list all the Brandt cyclic codes of $CS$ and their dimensions and weights in Table\[Table 1\] In Table\[Table 1\] $C_{0000}, C_{1000}, C_{0100}, C_{0010}, C_{0001}$ are minimal Brandt cyclic codes and $C_{1100}, C_{0011}, C_{1010}, C_{1001}, C_{0110}, C_{0101}, C_{1110}, C_{1011}, C_{0111}, C_{1111}$ are non-minimal Brandt cyclic codes.

| Brandt semigroup code | Dimension | Minimum weight |
|------------------------|-----------|----------------|
| $C_{0000} = 0$          | 0         | 0              |
| $C_{1000}$              | 3         | 4              |
| $C_{0100}$              | 3         | 4              |
| $C_{0000}$              | 2         | 4              |
| $C_{1100}$              | 2         | 4              |
| $C_{0001}$              | 2         | 4              |
| $C_{0101}$              | 4         | 4              |
| $C_{1010}$              | 4         | 4              |
| $C_{0110}$              | 4         | 4              |
| $C_{1110}$              | 4         | 4              |
| $C_{0011}$              | 4         | 4              |
| $C_{1011}$              | 4         | 4              |
| $C_{1111}$              | 4         | 4              |

Table 1. Brandt semigroup codes in $CB(G_4, 1)$.

If the index set of Brandt semigroup $S$ is not trivial, the complete set of primitive orthogonal idempotents of $CS$ is not unique. We may not get all the idempotents by a complete set of primitive orthogonal idempotents of $CS$ by combination. See the following example.
Example 5.11. Let $G_2 = \langle a \mid a^2 = e \rangle$ be a cyclic group, $\Lambda = I = \{1, 2\}$, and $S = B(G_2, 2)$ a Brandt semigroup. By Corollary 3.4, the elements

\[
e_1 = \frac{1}{2} \sum_{j=1}^{k} \omega^{-j}(1, a^j, 1) = \frac{1}{2}(1, e, 1) - \frac{1}{2}(1, a, 1),
\]
\[
e_2 = \frac{1}{2} \sum_{j=1}^{k} \omega^{-2j}(1, a^j, 1) = \frac{1}{2}(1, e, 1) + \frac{1}{2}(1, a, 1),
\]
\[
e_3 = \frac{1}{2} \sum_{j=1}^{k} \omega^{-j}(2, a^j, 2) = \frac{1}{2}(2, e, 2) - \frac{1}{2}(2, a, 2),
\]
\[
e_4 = \frac{1}{2} \sum_{j=1}^{k} \omega^{-2j}(2, a^j, 2) = \frac{1}{2}(2, e, 2) + \frac{1}{2}(2, a, 2),
\]

form a complete set of primitive orthogonal idempotents of $C_S$. The following codes

\[
C_{1000} = \mathbb{C}S e_1 = \left\{ \frac{k_1 - k_2}{2}((1, e, 1) - (1, a, 1)) + \frac{k_3 - k_4}{2}((2, e, 1) - (2, a, 1)) \mid k_1, k_2, k_3, k_4 \in \mathbb{C} \right\},
\]
\[
C_{0100} = \mathbb{C}S e_2 = \left\{ \frac{k_1 + k_2}{2}((1, e, 1) + (1, a, 1)) + \frac{k_3 + k_4}{2}((2, e, 1) + (2, a, 1)) \mid k_1, k_2, k_3, k_4 \in \mathbb{C} \right\},
\]
\[
C_{0010} = \mathbb{C}S e_3 = \left\{ \frac{k_5 - k_6}{2}((1, e, 2) - (1, a, 2)) + \frac{k_7 - k_8}{2}((2, e, 2) - (2, a, 2)) \mid k_5, k_6, k_7, k_8 \in \mathbb{C} \right\},
\]
\[
C_{0001} = \mathbb{C}S e_4 = \left\{ \frac{k_5 + k_6}{2}((1, e, 2) + (1, a, 2)) + \frac{k_7 + k_8}{2}((2, e, 2) + (2, a, 2)) \mid k_5, k_6, k_7, k_8 \in \mathbb{C} \right\},
\]

are minimal Brandt non-abelian codes, where $\mathbb{C}S = \{ \sum_{i=1}^{8} k_i s_i \mid k_i \in \mathbb{C}, s_i \in S \}$. We can easily check that the element $e_z = (1, e, 1) + z(2, e, 1) + z(2, a, 1)$ is a idempotent of $\mathbb{C}S$, where $z \in \mathbb{C}$. However,

\[
\mathbb{C}Se_z = (k_1 + k_5 z + k_6 z)(1, e, 1) + (k_2 + k_5 z + k_6 z)(1, a, 1)
\]
\[
+ (k_3 + k_7 z + k_8 z)(2, e, 1) + (k_4 + k_7 z + k_8 z)(2, a, 1)
\]

(5.5)

(5.6)

can not be written as a direct sum of minimal Brandt semigroup codes.

6. Examples of primitive orthogonal idempotents of Brandt semigroup algebras

In this section, we give some examples of complete sets of primitive orthogonal idempotents of Brandt semigroup algebras.
6.1. Example 1. Let $G_3 = \langle a \mid a^3 = e \rangle$ be a cyclic group and $\Lambda = I = \{2\}$. Then the elements

$$e_1 = \frac{1}{3} \sum_{j=1}^{3} \omega^{-3j}(1, a^j, 1) = \frac{1}{3}(1, e, 1) + \frac{1}{3}(1, a, 1) + \frac{1}{3}(1, a^2, 1), \quad (6.1)$$

$$e_2 = \frac{1}{3} \sum_{j=1}^{3} \omega^{-2j}(1, a^j, 1) = \frac{1}{3}(1, e, 1) - \left(\frac{1}{6} - \frac{\sqrt{3}i}{6}\right)(1, a, 1) - \left(\frac{1}{6} + \frac{\sqrt{3}i}{6}\right)(1, a^2, 1), \quad (6.2)$$

$$e_3 = \frac{1}{3} \sum_{j=1}^{3} \omega^{-j}(1, a^j, 1) = \frac{1}{3}(1, e, 1) - \left(\frac{1}{6} + \frac{\sqrt{3}i}{6}\right)(1, a, 1) - \left(\frac{1}{6} - \frac{\sqrt{3}i}{6}\right)(1, a^2, 1), \quad (6.3)$$

$$e_4 = \frac{1}{3} \sum_{j=1}^{3} \omega^{-3j}(2, a^j, 2) = \frac{1}{3}(2, e, 2) + \frac{1}{3}(2, a, 2) + \frac{1}{3}(2, a^2, 2), \quad (6.4)$$

$$e_5 = \frac{1}{3} \sum_{j=1}^{3} \omega^{-2j}(2, a^j, 2) = \frac{1}{3}(2, e, 2) - \left(\frac{1}{6} - \frac{\sqrt{3}i}{6}\right)(2, a, 2) - \left(\frac{1}{6} + \frac{\sqrt{3}i}{6}\right)(2, a^2, 2), \quad (6.5)$$

$$e_6 = \frac{1}{3} \sum_{j=1}^{3} \omega^{-j}(2, a^j, 2) = \frac{1}{3}(2, e, 2) - \left(\frac{1}{6} + \frac{\sqrt{3}i}{6}\right)(2, a, 2) - \left(\frac{1}{6} - \frac{\sqrt{3}i}{6}\right)(2, a^2, 2), \quad (6.6)$$

where $w = e^{\frac{2\pi i}{3}} = \cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}$, $i = \sqrt{-1}$, form a complete set of primitive orthogonal idempotents of $\mathbb{C}B(G_3, 1)$.

6.2. Example 2. Let $G = \langle a \rangle \times \langle b \rangle \cong C_2 \times C_2$ be an abelian group of order 4. The character table of $G$ is as follows.

| $G$ | $e$ | $a$ | $b$ | $ab$ |
|-----|-----|-----|-----|-----|
| $\chi_1$ | 1   | 1   | 1   | 1   |
| $\chi_2$ | 1   | -1  | 1   | -1  |
| $\chi_3$ | 1   | 1   | -1  | -1  |
| $\chi_4$ | 1   | -1  | -1  | 1   |

Table 2. Character table of abelian group $G$.

Let $\Lambda = I = \{1\}$. Then the Brandt semigroup algebra $\mathbb{C}B(G, 1)$ has four primitive orthogonal idempotents:

$$e_{11} = \frac{1}{4}((1, e, 1) + (1, a, 1) + (1, b, 1) + (1, ab, 1)),$$

$$e_{21} = \frac{1}{4}((1, e, 1) - (1, a, 1) + (1, b, 1) - (1, ab, 1)),$$

$$e_{31} = \frac{1}{4}((1, e, 1) + (1, a, 1) - (1, b, 1) - (1, ab, 1)),$$

$$e_{41} = \frac{1}{4}((1, e, 1) - (1, a, 1) - (1, b, 1) + (1, ab, 1)).$$
6.3. **Example 3.** Let $G = S_3$. This group has 6 elements:
\[ 1, (12), (13), (23), (123), (132), \]
where $\sim$ means the elements are conjugate. There are three conjugacy classes. Next, we have the character table of $S_3$.

| $S_3$ | 1   | (12) | (123) |
|-------|-----|------|-------|
| $\chi_1$ | 1   | 1    | 1     |
| $\chi_2$ | 1   | -1   | 1     |
| $\chi_3$ | 2   | 0    | -1    |

**Table 3.** Character table of $S_3$.

Let $\Lambda = I = \{1, 2\}$. Then the Brandt semigroup algebra $\mathbb{C}B(S_3, 2)$ has six primitive orthogonal idempotents:
\[
e_{11} = \frac{1}{6}((1, 1, 1) + (1, (123), 1) + (1, (132), 1) + (1, (12), 1) + (1, (13), 1) + (1, (23), 1)),
\]
\[
e_{21} = \frac{1}{6}((1, 1, 1) + (1, (123), 1) + (1, (132), 1) - (1, (12), 1) - (1, (13), 1) - (1, (23), 1)),
\]
\[
e_{31} = \frac{1}{6}(2^2(1, 1, 1) - 2(1, (123), 1) - 2(1, (132), 1)),
\]
\[
e_{12} = \frac{1}{6}((2, 1, 2) + (2, (123), 2) + (2, (132), 2) + (2, (12), 2) + (2, (13), 2) + (2, (23), 2)),
\]
\[
e_{22} = \frac{1}{6}((2, 1, 2) + (2, (123), 2) + (2, (132), 2) - (2, (12), 2) - (2, (13), 2) - (2, (23), 2)),
\]
\[
e_{32} = \frac{1}{6}(2^2(2, 1, 2) - 2(2, (123), 2) - 2(2, (132), 2)).
\]

These primitive orthogonal idempotents are obtained from Theorem 3.3. For example,
\[
e_{31} = \frac{\chi_3(1)}{6}(\chi_3(1)^{-1}(1, 1, 1) + \chi_3(123)^{-1}(1, (123), 1) + \chi_3(132)^{-1}(1, (132), 1))
\]
\[
= \frac{1}{6}(2^2(1, 1, 1) - 2(1, (123), 1) - 2(1, (132), 1)).
\]

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