A remark on the generalized Franchetta conjecture for K3 surfaces

Arnaud Beauville

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Abstract
A family of K3 surfaces \( X \rightarrow B \) has the Franchetta property if the Chow group of 0-cycles on the generic fiber is cyclic. The generalized Franchetta conjecture proposed by O’Grady asserts that the universal family \( X_g \rightarrow \mathcal{F}_g \) of polarized K3 of degree \( 2g - 2 \) has the Franchetta property. While this is known only for small \( g \) thanks to [7], we prove that for all \( g \) there is a hypersurface in \( \mathcal{F}_g \) such that the corresponding family has the Franchetta property.

1 Introduction

In 1954, Franchetta stated that the only line bundles defined on the generic curve of genus \( g \geq 2 \) are the powers of the canonical bundle [3]. Since the proof was insufficient, the result became known as the Franchetta conjecture; it was proved by Harer in [5], see also [1].

In [6], O’Grady proposed an analogue of this result for 0-cycles on K3 surfaces. Recall that the Chow group \( \text{CH}^2(X) \) of 0-cycles on a K3 surface \( X \) contains a canonical class \( o_X \), the class of any point lying on some rational curve in \( X \); for any divisors \( D \) and \( D' \) on \( X \), the product \( D \cdot D' \) in \( \text{CH}^2(X) \) is a multiple of \( o_X \) [2]. Let \( \rho : \mathcal{X} \rightarrow B \) be a map of smooth varieties whose general fiber is a K3 surface. We say that the family \( \mathcal{X} \rightarrow B \) has the Franchetta property if for every smooth fiber \( X \) of \( \rho \) the image of the restriction map \( \text{CH}^2(\mathcal{X}) \rightarrow \text{CH}^2(X) \) is contained in \( \mathbb{Z} \cdot o_X \). Equivalently, the Chow group \( \text{CH}^2(\mathcal{X}_\eta) \) of the generic fiber is cyclic.

For \( g \geq 2 \), let \( \mathcal{X}_g \rightarrow \mathcal{F}_g \) be the universal family of polarized K3 surfaces of degree \( 2g - 2 \). The generalized Franchetta conjecture of O’Grady is the assertion that this family has the Franchetta property. \(^1\) It is proved for \( g \leq 10 \) and some higher values of \( g \) in [7]; the general case seems far out of reach. We prove in this note a much weaker (and much easier) statement:

\(^1\) Here one can view \( \mathcal{F}_g \) as a stack, or restrict to the open subset parametrizing K3 with trivial automorphism group.

Pour Olivier – 40 ans déjà...

Arnaud Beauville
arnaud.beauville@unice.fr

\(^1\) CNRS-Laboratoire J.-A. Dieudonné, Université Côte d’Azur, Parc Valrose, 06108 Nice Cedex 2, France
Theorem  There exists for every g a hypersurface in $\mathcal{F}_g$ such that the corresponding family satisfies the Franchetta property.

The key point of the proof is the construction, for each g, of a 18-dimensional family of polarized K3 surfaces of degree $2g - 2$, which can be realized as complete intersections in $\mathbb{P}^1 \times \mathbb{P}^n$ for $n = 2, 3$ or 4 (Sect. 3). Then a simple argument, already used in [7], shows that these families have the Franchetta property (Sect. 2). Here the crucial property of our families is that they are parameterized by a linear space (in particular, they give unirational hypersurfaces in $\mathcal{F}_g$ for every g); thus there is no chance of extending the method to the whole moduli space $\mathcal{F}_g$, which is of general type for g large enough [4].

2 The method

We use the method of [7], based on the following result. Let $P$ be a smooth complex projective variety, $E$ a vector bundle on $P$, globally generated by a subspace $V$ of $H^0(E)$. Consider the subvariety $\mathcal{X} \subset \mathbb{P}(V) \times P$ of pairs $(\mathbb{C}s, x)$ with $s(x) = 0^2$; let $p, q$ be the projections onto $\mathbb{P}(V)$ and $P$. For $s \in V \setminus \{0\}$, the fiber $p^{-1}(\mathbb{C}s)$ is the zero locus of $s$ in $P$; for $x \in P$, the fiber $q^{-1}(x)$ is the space of lines $\mathbb{C}s \subset V$ such that $s(x) = 0$. Since $V$ generates $E$, the projection $q : \mathcal{X} \to P$ is a projective bundle (in particular, $\mathcal{X}$ is smooth).

Proposition For any smooth fiber $X$ of $p$, the image of the restriction map $\text{CH}(\mathcal{X}) \to \text{CH}(X)$ is equal to the image of $\text{CH}(P)$.

Proof Let $h \in \text{CH}^1(\mathbb{P}(V))$ be the class of a hyperplane section. The class $p^*h \in \text{CH}^1(\mathcal{X})$ induces the hyperplane class on a general fiber of $q$; since $q$ is a projective bundle, it follows that $\text{CH}(\mathcal{X})$ is generated by $q^*\text{CH}(P)$ and the powers of $p^*h$. But $p^*h$ vanishes on the fibers of $p$, hence the result.

Corollary Assume that the smooth fibers of $p$ are K3 surfaces, and that the multiplication map $m_P : \text{Sym}^2 \text{CH}^1(P) \to \text{CH}^2(P)$ is surjective. Then the family $\mathcal{X} \to \mathbb{P}(V)$ has the Franchetta property.

Proof Let $X$ be a smooth fiber of $p$. The commutative diagram

$$
\begin{array}{ccc}
\text{Sym}^2 \text{CH}^1(P) & \longrightarrow & \text{Sym}^2 \text{CH}^1(X) \\
m_P \downarrow & & \downarrow m_X \\
\text{CH}^2(P) & \longrightarrow & \text{CH}^2(X)
\end{array}
$$

shows that the image of $\text{CH}^2(P) \to \text{CH}^2(X)$ is contained in the image of $m_X$, hence in $\mathbb{Z} \cdot o_X$. \qed

3 Proof of the theorem

Since $\text{dim } \mathcal{F}_g = 19$, we must construct for every g a family of polarized K3 surfaces $(S, L)$ with $(L)^2 = 2g - 2$ satisfying the Franchetta property, and depending on 18 moduli (this

2 Here $\mathbb{P}(V)$ is the space of lines in $V$. 

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implies our Theorem, see [7, §2, Remark (i)]). We will need three different constructions in order to cover every $g \geq 8$ (the small genus cases follow from [7]). We will apply the Corollary with $P = \mathbb{P}^1 \times \mathbb{P}^n$ for $n = 2, 3$ or 4 — note that the surjectivity of $m_\mathcal{P}$ is trivially satisfied. For $i, j \in \mathbb{N}$, we put $\mathcal{O}(i, j) := \mathcal{O}(\mathbb{P}_1(i)) \boxtimes \mathcal{O}(\mathbb{P}_n(j))$; the vector bundle $E$ will be a direct sum of $n - 1$ line bundles of this type, so $S$ is a complete intersection of $n - 1$ hypersurfaces in $P$. In order for $S$ to be a K3 surface we must have $\det(E) = K_P^{-1} = \mathcal{O}(2, n + 1)$. We will always take $V = H^0(E)$.

The polarization $L$ on our K3 surface $S$ will be the restriction of the very ample line bundle $\mathcal{O}(a, 1)$ on $P$, for $a \geq 1$. Let $p, h \in \operatorname{CH}^1(P)$ be the pull back of the class of a point in $\mathbb{P}^1$ and of the hyperplane class in $\mathbb{P}^n$. Then

$$2g - 2 = (L)^2 = (ap + h)^2 \cdot [S] = (2a(p \cdot h) + h^2) \cdot [S].$$

**Case I:** $n = 2$, $E = \mathcal{O}(2, 3)$, hence

$$2g - 2 = (2a(p \cdot h) + h^2) \cdot (2p + 3h) = 2(3a + 1).$$

**Case II:** $n = 3$, $E = \mathcal{O}(1, 1) \oplus \mathcal{O}(1, 3)$, hence

$$2g - 2 = (2a(p \cdot h) + h^2) \cdot (p + h)(p + 3h) = 2(3a + 2).$$

**Case III:** $n = 4$, $E = \mathcal{O}(0, 3) \oplus \mathcal{O}(1, 1) \oplus \mathcal{O}(1, 1)$, hence

$$2g - 2 = (2a(p \cdot h) + h^2) \cdot 3h(p + h)^2 = 2(3a + 3).$$

Thus we get all values of $g \geq 8$.

It remains to prove that the three families just constructed depend on 18 moduli. The exact sequence

$$0 \to T_S \to T_{P|S} \to N_{S/P} \to 0$$

gives rise to an exact sequence

$$0 \to H^0(T_{P|S}) \to H^0(N_{S/P}) \to \mathcal{O}(S, T_S) \to 0$$

the image of $\partial$ describes, inside the space of first order deformations of $S$, those which come from our family. Thus we want to prove $\dim \partial = 18$, or equivalently $h^0(N_{S/P}) - h^0(T_{P|S}) = 18$.

We have $T_P = \operatorname{pr}_1^* T_{\mathbb{P}^1} \oplus \operatorname{pr}_2^* T_{\mathbb{P}^n}$; from the Euler exact sequence we get $h^0((\operatorname{pr}_1^* T_{\mathbb{P}^1})|_{S}) = h^0(\operatorname{pr}_1^* T_{\mathbb{P}^1})$, and similarly for $\operatorname{pr}_2^* T_{\mathbb{P}^n}$. Thus $h^0(T_{P|S}) = h^0(T_{P|S}) + h^0(T_{\mathbb{P}^n}) = 3 + n(n + 2)$.

Let us denote by $d_S$ the restriction to $S$ of a class $d \in \operatorname{Pic}(P)$. Using $d_S \cdot d'_S = d \cdot d' \cdot [S]$, we find

$$p_S^2 = 0, \quad p_S h_S = 3, \quad h_S^2 = 2n - 2.$$

By Riemann–Roch, we have $h^0(\mathcal{O}(i, j)) = 2 + \frac{1}{2}(ip_S + jh_S)^2 = 2 + 3ij + j^2(n - 1)$.

**Case I:** $h^0(N_{S/P}) = h^0(\mathcal{O}(2, 3)) = 29, h^0(T_{P|S}) = 11$.

**Case II:** $h^0(N_{S/P}) = h^0(\mathcal{O}(1, 1)) + h^0(\mathcal{O}(1, 3)) = 9 + 29 = 36, h^0(T_{P|S}) = 18$.

**Case III:** $h^0(N_{S/P}) = 2h^0(\mathcal{O}(1, 1)) + h^0(\mathcal{O}(0, 3)) = 2 \cdot 8 + 29 = 45, h^0(T_{P|S}) = 27$.

In each case we find $h^0(N_{S/P}) - h^0(T_{P|S}) = 18$ as required. □

**Remarks.** 1) In fact, for $S$ very general in each family, Pic$(S)$ is generated by $p_S$ and $h_S$; this follows from the Noether–Lefschetz theory, see [8, Thm. 3.33]. Therefore Pic$(S)$ is the rank 2 lattice with intersection matrix

$$\begin{pmatrix} 0 & 3 \\ 3 & 2n - 2 \end{pmatrix}.$$
2) Our 3 families admit actually a simple geometric description. In what follows we consider a general surface $S$ in each family. We fix homogeneous coordinates $U, V$ on $\mathbb{P}^1$.

**Case I:** $S$ is given by an equation $U^2A + 2UVB + V^2C = 0$ in $P = \mathbb{P}^1 \times \mathbb{P}^2$, with $A, B, C$ cubic forms on $\mathbb{P}^2$. Projecting onto $\mathbb{P}^2$ gives a double covering $S \rightarrow \mathbb{P}^2$ branched along the sextic plane curve $\Gamma : B^2 - AC = 0$. Let $\alpha$ and $\gamma$ be the divisors on $\Gamma$ defined by $A = B = 0$ and $C = B = 0$; then $2\alpha, 2\gamma$ and $\alpha + \gamma$ are induced by the cubic curves $A = 0, C = 0$ and $B = 0$ respectively, hence belong to the canonical system $|K_\Gamma|$. It follows that $\alpha$ and $\gamma$ are linearly equivalent theta-characteristics, hence belong to a half-canonical $\frac{g_1}{2}$, that is, a vanishing thetanull on $\Gamma$. Conversely, it is easy to see that a smooth plane sextic with a vanishing thetanull has an equation of the above form. We conclude that the surfaces in Case I are the double covers of $\mathbb{P}^2$ branched along a sextic curve with a vanishing thetanull.

**Case II:** The equations of $S$ in $P = \mathbb{P}^1 \times \mathbb{P}^3$ have the form $UL + VM = UA + VB = 0$, where $L, M; A, B$ are forms of degree 1 and 3 on $\mathbb{P}^3$. The projection $S \rightarrow \mathbb{P}^3$ is an isomorphism onto the quartic surface $LB - MA = 0$; this is the equation of a general quartic containing a line. Thus the surfaces in Case II are the quartic surfaces containing a line.

**Case III:** The equations of $S$ in $P = \mathbb{P}^1 \times \mathbb{P}^4$ are of the form $UA + VB = UC + VD = F = 0$, where $A, B, C, D; F$ are forms of degree 1 and 3 on $\mathbb{P}^3$. The projection $S \rightarrow \mathbb{P}^4$ is an isomorphism onto the surface $AD - BC = F = 0$, that is, the intersection of a quadric cone (with one singular point) and a cubic in $\mathbb{P}^4$. Thus the surfaces in Case III are the complete intersections of a quadric cone and a cubic in $\mathbb{P}^4$.

Note that one sees easily from this description that each family depends indeed on 18 moduli.

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