On the Casas-Alvero’s Conjecture

Jiaxuan Su*

1 Suffield Academy, Suffield, Connecticut, 06078, America
24js@suffieldacademy.org

Abstract. As we all know, every derivation of a polynomial that is a nth degree of another polynomial has a common factor with this polynomial. Then, Eduardo Casas-Alvero came up with a conjecture that think opposite with the axiom we just talked before. That is to proof this polynomial is a nth degree of another polynomial if every derivation of this polynomial has a common factor with this polynomial. We are going to use different method to attack this conjecture and prove it when n is small.

1. Introduction
Let us talk about some of the background information about Casas-Alvero’s conjecture. It was first appeared in [1] in 2001 by Eduardo Casas-Alvero. At that time, it was a unsolvable problem back then, but between that time and now, people have already solve the the conjecture for infinitely many degree. And we are going to show special cases today.

In Section 2, we are going to give the statement of the conjecture and prove it directly for the case n = 2, 3.

In Section 3, we introduce the Gauss-Lucas theorem and employ this theorem to prove the cases n = 2, 3, 4. We also find a general result concerning the conjecture which can be proved using the Gauss-Lucas theorem.

Theorem 1. Let f(X) be a polynomial with complex coefficients of degree n > 0, whose roots form exactly the vertices of a convex polygon in the complex plane. If furthermore, f(X) satisfies the conditions of the CA conjecture, then f(X) is the form of \( a(X - \alpha) \).

2. Direct proofs for small degrees
In this section, we will present the Casas-Alvero’s conjecture and try to prove it for n = 2, 3.

The Casas-Alvero’s conjecture goes

Conjecture. Let \( f(X) \in \mathbb{C}[X] \) be a polynomial with complex coefficients of degree n > 0. If for any 0 < k < n, \( f(X) \) and \( f^{(k)}(X) \) have a common root, then \( f(X) \) is of the form \( a(X - \alpha)^n \) for some a, \( \alpha \in \mathbb{C} \).

In the rest of this section, we are going to prove this conjecture for n = 2, 3 using direct calculations. The main tool we are going to use is the following theorem:

Theorem 2. Let \( p(X) \) be a polynomial with complex coefficients. Let \( \alpha \) be a root of \( p(X) \). Then \( \alpha \) is a multiple root of \( p(X) \) if and only if \( \alpha \) is a root of \( p'(X) \).

Here, \( \alpha \) is called a multiple root of \( p(X) \), if in the decomposition of \( p(X) \), \( (X - \alpha) \) appears at least twice.
Proof. Assume $\alpha$ is a multiple root of $p(X)$ Since $\alpha$ is a multiple root of $p(X)$, $p(X) = a(X - \alpha)^2 Q(X)$ for some polynomial $Q(X)$. Then, $p'(X) = 2a(X - \alpha)Q(X) + a(X - \alpha)^2 Q'(X)$. Therefore, $p'(\alpha) = 0 + 0 = 0$ so $\alpha$ is a root of $p'(X)$. Now assume $\alpha$ is a root of $p'(X)$. Since $\alpha$ is a root of $p(X)$, we can write $p(X) = a(X - \alpha)(X - \alpha_2)...(X - \alpha_n)$. Then
\[
p'(X) = a((X - \alpha_2)(X - \alpha_3)...(X - \alpha_n)
+ (X - \alpha)(X - \alpha_3)...(X - \alpha_n)
+ (X - \alpha_2)(X - \alpha_3)...(X - \alpha_n))\]
So $p'(\alpha) = a(\alpha_1 - \alpha_2)...(\alpha - \alpha_n)$ but $\alpha$ is a root of $p'(X)$, i.e., $p'(X) = 0$, i.e., at least one of the numbers from $\alpha_2, ..., \alpha_n$ is equal to $\alpha$. Therefore, $\alpha$ is a multiple root of $p(X)$.

2.1. $n = 2$
Now we prove the conjecture for $n = 2$. First, we reformulate the conjecture in this case. Let $f(X) = ax^2 + bx + c (a \neq 0)$ be a complex polynomial of degree 2. If $f'(X)$ and $f(X)$ have a common root, we try to prove that $f(X) = a(X - \alpha)^2$.

In fact, assume $\beta$ is a common root of $f'(X)$ and $f(X)$, by the precedent theorem, we know that $\beta$ is a multiple root of $f(X)$. However, $f(X)$ is of degree 2, so $f(X)$ is necessarily of the form $a(X - \beta)^2$, as desired.

2.2. $n = 3$
Now we prove the conjecture for $n = 3$. First, we reformulate the conjecture in this case. Let $f(X)$ be a polynomial of degree 3. If $f(X)$ and $f'(X)$ have a common root, and if $f(X)$ and $f'(X)$ have a common root, we try to prove that $f(X) = a(X - \alpha)^3$.

In fact, let $\alpha$ be a common root of $f(X)$ and $f'(X)$. According to the precedent theorem, we know that $\alpha$ is a multiple root of $f(X)$. Therefore, $f(X) = a(X - \alpha)^2 (X - \beta)$. Next, we prove $\alpha = \beta$. By direct calculation, $f''(X) = 2a[3X - (2\alpha + \beta)]$. The only root of $f''(X)$ is $\frac{2\alpha + \beta}{3}$, while the only roots of $f(X)$ are $\alpha$, $\beta$. Since $f(X)$ and $f''(X)$ have a common root, we have $\frac{2\alpha + \beta}{3} = \alpha$ or $\frac{2\alpha + \beta}{3} = \beta$. But both solve $\alpha = \beta$. In conclusion, $f(X) = a(X - \alpha)^3$.

Although direct calculations work when degree is small, we find that as the degree augments, the complexity of computations explodes. Specially, $n = 4$ is already very difficult to calculate and we cannot even give a direct proof for $n = 5$. Therefore, we need to find other indirect methods for this conjecture. This is what we are going to do in the following sections.

3. Gauss-Lucas theorem and its application
In this section, we are going to give a useful tool for this conjecture: the Gauss-Lucas theorem. We will use this theorem to give alternative proofs for $n = 2, 3, 4$. Furthermore, we are going to prove a new general result: If the roots of the polynomial form exactly the vertices of a convex polygon, and the polynomial satisfies the condition of the Casas-Alvero’s conjecture [2][4], then the polynomial is necessarily a power of a linear term.

3.1. Gauss-Lucas theorem
The Gauss-Lucas theorem is an important theorem in complex analysis which we can find in any textbook on this subject, for example, in [3][5].


**Theorem 3.** Let \( f(X) \) be a complex polynomial. let \( y \) be the set of roots of \( p(X) \). Then the roots of \( p'(X) \) contain in the convex hall of \( y \).

Proof. Assume by contradiction, \( p'(X) \) has a root lying outside of the convex hall of the roots of \( p(X) \), and let us denote this root \( c \). Then we can find a straight line that separates this root of \( p'(X) \) and the convex hall. And next we are going to prove it is not possible.

After a rotation and a translation, we may assume this straight line is right at the x-axis, and that the roots of \( p(X) \) lie in the upper half plane and the outside part lies in the lower half plane. All roots of \( p(X) \) have imaginary part greater than 0. But the outside part, \( c \), has imaginary part that is less than 0.

\[
p(X) = (X-a_1)...(X-a_n)
\]

\[
p'(X) = a_1(X-a_2)(X-a_3)...(X-a_n)
\]

\[
+ (X-a)(X-a_3)...(X-a_n)
\]

\[
\cdots
\]

\[
+(X-a)(X-a_2)...(X-a_n)
\]

Therefore, \[
\frac{p'(X)}{p(X)} = \frac{1}{X-a_1} + \cdots + \frac{1}{X-a_n} \]

Let \( X = c \) and we get

\[
\text{LHS} = \frac{p'(c)}{p(c)} = \frac{0}{p(c)} = 0
\]

\[
\text{RHS} = \frac{1}{c-a_1} + \cdots + \frac{1}{c-a_n}
\]

But \( \text{Im} \frac{1}{c-a_i} > 0 \) for any \( i \). Therefore, the imaginary part of right-hand side is greater than 0, which contradicts to the fact that 0 = LHS = RHS.

3.2. A new general result

**Theorem 4.** Let \( f(X) \) be a polynomial with complex coefficients of degree \( n > 0 \), whose roots form exactly the vertices of a convex polygon in the complex plane. If furthermore, \( f(X) \) satisfies the conditions of the CA conjecture, then \( f(X) \) is the form of \( a(X-\alpha)^n \).

Proof. Consider \( f^{(n-1)}(X) \). Since degree \( f(X) = n \), degree \( f^{(n-1)}(X) = 1 \), i.e., \( f^{(n-1)} \) has only one root. Since \( f(X) \) and \( f^{(n-1)}(X) \) have a common root, and since the roots of \( f(X) \) form exactly the vertices of a polygon. We may assume the root of \( f^{(n-1)}(X) \) is one of the vertices, for instance \( \alpha \).

Now consider \( f^{(n-2)}(X) \), whose two roots must be symmetric with respect to the root of \( f^{(n-1)}(X) \); but the root of \( f^{(n-2)}(X) \) is \( \alpha \), one vertex of the polygon formed by the roots of \( f(X) \), so by the Gauss-Lucas theorem, all roots of \( f^{(n-2)}(X) \) are the same, which is \( \alpha \). The same procedure shows by induction that for any \( m \), \( f^{(n-m)}(X) \) has only one root \( \alpha \). Specially when \( m = n \), \( f(X) \), which can also be shown like \( f^0(X) \) has only one root, which means that \( f(X) \) is the form of \( a(X-\alpha)^n \).

4. On analyzing a conjecture about univariate polynomials and their roots by using Maple

4.1. The general setting

Let \( P(x) \) be a monic polynomial in \( K[x] \) of degree \( n \). It is assumed that \( \text{deg} (\text{gcd}(P, P(i))) = 1 \) for any \( i \) such that \( 1 \leq i \leq n \). Let \( n \) be the root of \( P(x) \) which is also a root of \( P(i)(x) = P(i)(i) = 0 \). Next the easiest cases are shown to be true: degrees two and three. If \( P(x) = x^2 + bx + c \) is a monic polynomial in \( K[x] \) of degree 2 with a common root, 1, with \( P(1)(x) = 1 = b/2 \) \( K \) and \( P(x) = (x-1)^2 \). For degree 3,
the conditions on \( P(x) \) are the following: there exist 1 and 2 such that: \( P(2) = P(1)(2) = 0 \) and \( P(1) = P(2)(1) = 0 \). There are two possibilities depending on 1 and 2 are not equal. If 1 = 2 then \( \gcd(P, P(1), P(2)) = 1 \) and \( P(x) = (x - 1)^3 \) with 1 K (since it is the root of \( P(2)(x) \)). If 1 6= 2 then \( P(x) = (x - 2)^2(x - 1) \) and \( P(2)(x) = 6x(142) : hence0 = P(2)(1) = 4(12)6 = 0 \), contradictingtheassumptionthat \( P(2)(1) = 0 \).

**Definition 1.** Let \( f(X) = a_nX^n + \ldots + a_1X + a_0 \), \( g(X) = b_mX^m + \ldots + b_1X + b_0 \) be polynomials in complex coefficients. Define the resultant of \( f \) and \( g \), to be

\[
\text{Res}(f, g) := \begin{vmatrix}
a_n & 0 & \ldots & 0 & b_m & 0 & \ldots & 0 \\
a_{n-1} & a_n & \ldots & 0 & b_{m-1} & b_m & \ldots & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & \ldots & a_0 & 0 & \ldots & 0 & b_0
\end{vmatrix}
\]

**Theorem 5.** Let \( f \), \( g \) be polynomials with complex coefficients. Then \( f \) and \( g \) have common root if and only if \( \text{Res}(f, g) = 0 \).

Proof. Let \( C[X] \) =polynomials of degree \( \leq l-1 \).

4.2. **Gröbner** basis

Assume \( f(X) = (X - \alpha_1)(X - \alpha_2)(X - \alpha_3)(X - \alpha_4) \) and assume that \( \text{Res}(f, f^{(k)}) = 0 \) for any \( 0 < k < 4 \).

We are going to prove that \( \alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 \). We try to calculate the **Gröbner** basis of \( \text{Res}(f, f^{(k)}) = 0 \) for \( 0 < k < 4 \), and judge if some power of \( (\alpha_1 - \alpha_2) \) lie in the ideal generated by \( \text{Res}(f, f^{(k)}) = 0 \).

5. Factorization, Equations for Linear Systems and **Gröbner** basis

**Theorem 6.** \( P(X) \) has a common root with each of its derivatives (i.e. for \( 1 \leq i \leq n \), \( \deg(\gcd(P, P(i))) = 1 \)) if and only if there exists \( \alpha \in K \) such that \( P = (X - \alpha)^n \).

5.1. \( n = 2 \)

Let assume that the polynomial is in the form of \( f(X) = (X - \alpha_1)(X - \alpha_2) \), where \( \alpha_1 \) and \( \alpha_2 \) are different. First, we take the derivative, by calculation, we can get \( f^1(X) = (X - \alpha_1) + (X - \alpha_2) \). If we assume \( \alpha_1 \) as the common root, the polynomial will be \( f^1(\alpha_1) = (\alpha_1 - \alpha_2) \). Since we assume that \( \alpha_1 \neq \alpha_2 \). The root won’t be 0. Same if we choose \( \alpha_2 \) as the common root. So the only possible form is \( f(X) = (X - \alpha_1)^2 \).

5.2. \( n = 2 \)

There are three possible forms for degree 3. \( f(X) = (X - \alpha_1)(X - \alpha_2)(X - \alpha_3) \), \( f(X) = (X - \alpha_1)^2(X - \alpha_2) \), \( f(X) = (X - \alpha_1)^3 \).

5.2.1. \( f(X) = (X - \alpha_1)(X - \alpha_2)(X - \alpha_3) \)

By direct calculation, \( f^1(X) = (X - \alpha_1)(X - \alpha_2) + (X - \alpha_1)(X - \alpha_3) + (X - \alpha_2)(X - \alpha_3) \). Because \( f^1(X) \neq 0 \) when \( X = \alpha_1, \alpha_2, \alpha_3 \). None of the root can be the common root. This form doesn’t fit the conjecture.
5.2.2. \( f(X) = (X - \alpha_1)^2(X - \alpha_2) \)

By direct calculation, \( f^2(X) = 2(X - \alpha_1) + 2(X - \alpha_2) \). Because \( f^2(X) \neq 0 \) when \( X = \alpha_1, \alpha_2 \) Still, none of the root can be the common root. This form doesn’t fit the conjecture.

5.2.3. \( f(X) = (X - \alpha_1)^3 \)

By direct calculation, we can find that every derivative has a common root with the original polynomial, which is \( X = \alpha_1 \). Making \( f(X) = (X - \alpha_1)^3 \) fits the conjecture.

5.3. \( n = 4 \)

For degree 4, there are 5 possible form for the polynomial.
\[
\begin{align*}
    f(X) &= (X - \alpha_1)(X - \alpha_2)(X - \alpha_3)(X - \alpha_4), \\
    f(X) &= (X - \alpha_1)^2(X - \alpha_2)(X - \alpha_3), \\
    f(X) &= (X - \alpha_1)^2(X - \alpha_2)^2, \\
    f(X) &= (X - \alpha_1)^3(X - \alpha_2), \\
    f(X) &= (X - \alpha_1)^4.
\end{align*}
\]

5.3.1. \( f(X) = (X - \alpha_1)(X - \alpha_2)(X - \alpha_3)(X - \alpha_4) \)

By direct calculation,
\[
\begin{align*}
    f^1(X) &= (X - \alpha_1)(X - \alpha_2)(X - \alpha_3) + (X - \alpha_1)(X - \alpha_2)(X - \alpha_4) \\
    &+ (X - \alpha_1)(X - \alpha_3)(X - \alpha_4) + (X - \alpha_2)(X - \alpha_3)(X - \alpha_4)
\end{align*}
\]

Because \( f^1(X) \neq 0 \) when \( X = \alpha_1, \alpha_2, \alpha_3, \alpha_4 \). None of the root can be the common root. This form doesn’t fit the conjecture.

5.3.2. \( f(X) = (X - \alpha_1)^3(X - \alpha_2)(X - \alpha_3) \)

By direct calculation, \( f^2(X) = (X - \alpha_1)^2 + 2(X - \alpha_1)(X - \alpha_2) + 2(X - \alpha_1)(X - \alpha_3) + 2(X - \alpha_2)(X - \alpha_3) \). Because \( f^2(X) \neq 0 \) when \( X = \alpha_1, \alpha_2, \alpha_3 \). None of the root can be the common root. This form doesn’t fit the conjecture.

5.3.3. \( f(X) = (X - \alpha_1)^2(X - \alpha_2)^2 \)

By direct calculation, \( f^2(X) = 2(X - \alpha_1)^2 + 4(X - \alpha_1)(X - \alpha_2) + 2(X - \alpha_2)^2 \). Because \( f^2(X) \neq 0 \) when \( X = \alpha_1, \alpha_2 \). None of the root can be the common root. This form doesn’t fit the conjecture.

5.3.4. \( f(X) = (X - \alpha_1)^3(X - \alpha_2) \)

By direct calculation, \( f^3(X) = 6(X - \alpha_1) + 6(X - \alpha_2) \). Because \( f^3(X) \neq 0 \) when \( X = \alpha_1, \alpha_2 \). None of the root can be the common root. This form doesn’t fit the conjecture.

5.3.5. \( f(X) = (X - \alpha_1)^4 \)

By direct calculation, we can find that every derivative has a common root with the original polynomial, which is \( X = \alpha_1 \). Making \( f(X) = (X - \alpha_1)^4 \) fits the conjecture.

5.4. \( n = 5 \)

For the fifth degree, there are seven possibilities about the formation of the polynomial.
\[
\begin{align*}
    f(X) &= (X - \alpha_1)(X - \alpha_2)(X - \alpha_3)(X - \alpha_4)(X - \alpha_5), \\
    &+ \ldots
\end{align*}
\]
\[ f(X) = (X - \alpha_1)^2(X - \alpha_2)(X - \alpha_3)(X - \alpha_4), \]
\[ f(X) = (X - \alpha_1)^2(X - \alpha_2)^2(X - \alpha_3), \]
\[ f(X) = (X - \alpha_1)^3(X - \alpha_2)(X - \alpha_3), \]
\[ f(X) = (X - \alpha_1)^3(X - \alpha_2)^2, \]
\[ f(X) = (X - \alpha_1)^4(X - \alpha_2) \]
\[ f(X) = (X - \alpha_1)^5. \]

Here, we assume \( \alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5 \) are different.

5.4.1. \( f(X) = (X - \alpha_1)(X - \alpha_2)(X - \alpha_3)(X - \alpha_4)(X - \alpha_5) \)

In the this formula, if we seek for the first derivative, we can find that there isn’t a common root between \( P(X) \) and \( P'(X) \). Which is contradict to the setting of the conjecture.

5.4.2. \( f(X) = (X - \alpha_1)^2(X - \alpha_2)(X - \alpha_3)(X - \alpha_4) \)

As we calculate out the answer for the second derivative of this formula, we get
\[
2(X - \alpha_2)(X - \alpha_3)(X - \alpha_4) + 4(X - \alpha_1)(X - \alpha_3)(X - \alpha_4)
+ 4(X - \alpha_1)(X - \alpha_2)(X - \alpha_4) + 4(X - \alpha_1)(X - \alpha_2)(X - \alpha_3)
+ 2(X - \alpha_1)^2(X - \alpha_2) + 2(X - \alpha_1)^2(X - \alpha_3) + (X - \alpha_1)^2(X - \alpha_4)
\]

According to the conjecture, we know that this formula must contain a common root with the original formula. However, whichever common root we chose that the \( X \) is equal to, the second derivative formula won’t be zero. Which prove that this form doesn’t work.

5.4.3. \( f(X) = (X - \alpha_1)^2(X - \alpha_2)^2(X - \alpha_3) \)

This formation can be dealt with the last solution, after we take the second derivative, we can find that the original formula and the second derivative formula can’t have one common root. which infer this form also doesn’t work.

5.4.4. \( f(X) = (X - \alpha_1)^3(X - \alpha_2)(X - \alpha_3) \)

A third derivative can perfectly solve this form, as we find the third derivative we can notice that
\[ f^3(X) = a(X - \alpha_2)(X - \alpha_3) + b(X - \alpha_1)(X - \alpha_3) + c(X - \alpha_1)(X - \alpha_2) + d(X - \alpha_1)^3, \quad (a, b, c, d \in \mathbb{Z}) \] .

If we assume the common root is \( \alpha_1 \), the formula will be \( f(\alpha_1) = a(X - \alpha_2)(X - \alpha_3) + 0 + 0 + 0 \), which is unequal to 0, since \( (X - \alpha_1) \neq (X - \alpha_2) \neq (X - \alpha_3) \).

5.4.5. \( f(X) = (X - \alpha_1)^3(X - \alpha_2)^2 \)

A third derivative can still solve the problem. if we calculate the third derivative, it will occur in a form \( f(X) = a(X - \alpha_2)^2 + b(X - \alpha_1)(X - \alpha_2) + c(X - \alpha_2)^2, \quad (a, b, c \in \mathbb{Z}) \) same as last one, none of the root can fit the common root.

5.4.6. \( f(X) = (X - \alpha_1)^4(X - \alpha_2) \)

If we take the fourth derivative, we can notice that \( f^4(X) = a(X - \alpha_1) + b(X - \alpha_2) \). If we choose \( \alpha_1 \) for the common root, \( f^4(\alpha_1) = 0 + b(\alpha_1 - \alpha_2) \). Since \( \alpha_1 \neq \alpha_2 \), the root won’t be 0. Same will happen if we choose \( \alpha_2 \) as the common root. Implying this form doesn’t work.
5.4.7. \( f(X) = (X - \alpha_i)^5 \)

Now, we have the last possible form. In this form, no matter how many times we take its derivative, there will always be a common root, \( X = \alpha_i \). So this form is the only form that fits the conjecture.

5.5. \( n = 7 \)

For the seventh degree, there are fifteen possibilities about the formation of the polynomial.

\[
f(X) = (X - \alpha_1)(X - \alpha_2)(X - \alpha_3)(X - \alpha_4)(X - \alpha_5)(X - \alpha_6)(X - \alpha_7)
\]

\[
f(X) = (X - \alpha_1)^2(X - \alpha_2)(X - \alpha_3)(X - \alpha_4)(X - \alpha_5)(X - \alpha_6)(X - \alpha_7)
\]

\[
f(X) = (X - \alpha_1)^3(X - \alpha_2)(X - \alpha_3)(X - \alpha_4)(X - \alpha_5)(X - \alpha_6)(X - \alpha_7)
\]

\[
f(X) = (X - \alpha_1)^4(X - \alpha_2)(X - \alpha_3)(X - \alpha_4)(X - \alpha_5)(X - \alpha_6)(X - \alpha_7)
\]

\[
f(X) = (X - \alpha_1)^5(X - \alpha_2)(X - \alpha_3)(X - \alpha_4)(X - \alpha_5)(X - \alpha_6)(X - \alpha_7)
\]

\[
f(X) = (X - \alpha_1)^6(X - \alpha_2)(X - \alpha_3)(X - \alpha_4)(X - \alpha_5)(X - \alpha_6)(X - \alpha_7)
\]

Here, we assume \( \alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6, \alpha_7 \) are different.

5.5.1. \( f(X) = (X - \alpha_1)(X - \alpha_2)(X - \alpha_3)(X - \alpha_4)(X - \alpha_5)(X - \alpha_6)(X - \alpha_7) \)

If we seek for the first derivative, we can find that there isn’t a common root between \( P(X) \) and \( P'(X) \).

5.5.2. \( f(X) = (X - \alpha_1)^2(X - \alpha_2)(X - \alpha_3)(X - \alpha_4)(X - \alpha_5)(X - \alpha_6) \)

By direct calculation,

\[
f^2(X) = 2(X - \alpha_2)(X - \alpha_3)(X - \alpha_4)(X - \alpha_5)(X - \alpha_6)
\]

\[
+ 2(X - \alpha_1)(X - \alpha_2)(X - \alpha_3)(X - \alpha_4)(X - \alpha_6)
\]

\[
+ 2(X - \alpha_1)(X - \alpha_2)(X - \alpha_3)(X - \alpha_4)(X - \alpha_5)
\]

\[
+ 2(X - \alpha_1)^2(X - \alpha_2)(X - \alpha_3)(X - \alpha_4)
\]

\[
+ (X - \alpha_1)^2(X - \alpha_2)(X - \alpha_3)(X - \alpha_5)
\]

\[
+ ... + (X - \alpha_1)^2(X - \alpha_4)(X - \alpha_5)(X - \alpha_6)
\]

Because \( f^2(X) \neq 0 \) when \( X = \alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6 \). None of the root can be the common root. This form doesn’t fit the conjecture.

5.5.3. \( f(X) = (X - \alpha_1)^2(X - \alpha_2)^2(X - \alpha_3)(X - \alpha_4)(X - \alpha_5) \)

By direct calculation,
\[ f^2(X) = 2(X - \alpha_2)^2(X - \alpha_3)(X - \alpha_4)(X - \alpha_5) + 2(X - \alpha_1)(X - \alpha_2)^2(X - \alpha_3)(X - \alpha_4) + (X - \alpha_1)^2(X - \alpha_2)^2(X - \alpha_3)^2 + (X - \alpha_2)^2(X - \alpha_3)^2(X - \alpha_4) + 2(X - \alpha_1)^2(X - \alpha_2)(X - \alpha_3)(X - \alpha_4) + 2(X - \alpha_1)(X - \alpha_2)(X - \alpha_3)(X - \alpha_4)(X - \alpha_5) + 2(X - \alpha_1)^2(X - \alpha_2)^2(X - \alpha_3)^2(X - \alpha_4) + 2(X - \alpha_1)^2(X - \alpha_2)^2(X - \alpha_3)(X - \alpha_4) + 2(X - \alpha_1)(X - \alpha_2)(X - \alpha_3)^2(X - \alpha_4) + 4(X - \alpha_1)(X - \alpha_2)(X - \alpha_3)(X - \alpha_4)(X - \alpha_5) + 4(X - \alpha_1)(X - \alpha_2)(X - \alpha_3)^2(X - \alpha_4) + 4(X - \alpha_1)(X - \alpha_2)^2(X - \alpha_3)(X - \alpha_4) + 4(X - \alpha_1)(X - \alpha_2)^2(X - \alpha_3)^2(X - \alpha_4) \]

Because \( f^2(X) \neq 0 \) when \( X = \alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5 \). None of the root can be the common root. This form doesn’t fit the conjecture.

5.5.4. \( f(X) = (X - \alpha_1)^2(X - \alpha_2)^2(X - \alpha_3)^2(X - \alpha_4) \)

By direct calculation,
\[
\begin{align*}
  f^2(X) &= 2(X - \alpha_1)^2(X - \alpha_2)^2(X - \alpha_3)^2 + 2(X - \alpha_1)(X - \alpha_2)^2(X - \alpha_3)^2 + 2(X - \alpha_1)^2(X - \alpha_2)^2(X - \alpha_3)^2 + 2(X - \alpha_2)^2(X - \alpha_3)^2(X - \alpha_4) + 2(X - \alpha_1)^2(X - \alpha_2)(X - \alpha_3)(X - \alpha_4) + 2(X - \alpha_1)(X - \alpha_2)(X - \alpha_3)(X - \alpha_4)(X - \alpha_5) + 2(X - \alpha_1)^2(X - \alpha_2)^2(X - \alpha_3)(X - \alpha_4) + 2(X - \alpha_1)^2(X - \alpha_2)^2(X - \alpha_3)^2(X - \alpha_4) + 4(X - \alpha_1)(X - \alpha_2)(X - \alpha_3)^2(X - \alpha_4) + 4(X - \alpha_1)(X - \alpha_2)^2(X - \alpha_3)(X - \alpha_4) + 4(X - \alpha_1)(X - \alpha_2)^2(X - \alpha_3)^2(X - \alpha_4) \end{align*}
\]

Because \( f^2(X) \neq 0 \) when \( X = \alpha_1, \alpha_2, \alpha_3, \alpha_4 \). None of the root can be the common root. This form doesn’t fit the conjecture.

5.5.5. \( f(X) = (X - \alpha_1)^3(X - \alpha_2)^2 \)

By direct calculation,
\[
\begin{align*}
  f^3(X) &= 6(X - \alpha_2)(X - \alpha_3)(X - \alpha_4)(X - \alpha_5) + 6(X - \alpha_1)(X - \alpha_2)(X - \alpha_3)(X - \alpha_4) + 6(X - \alpha_1)(X - \alpha_2)(X - \alpha_3)(X - \alpha_4) \end{align*}
\]

Because \( f^3(X) \neq 0 \) when \( X = \alpha_1, \alpha_2 \). None of the root can be the common root. This form doesn’t fit the conjecture.

5.5.6. \( f(X) = (X - \alpha_1)^3(X - \alpha_2)^2(X - \alpha_3)(X - \alpha_4) \)

By direct calculation,
\[ f^3(X) = 6(X - \alpha_2)^2(X - \alpha_3)(X - \alpha_4) + 6(X - \alpha_1)(X - \alpha_2)^2(X - \alpha_3) + 6(X - \alpha_1)(X - \alpha_2)^2(X - \alpha_3) + 12(X - \alpha_1)(X - \alpha_2)(X - \alpha_3)(X - \alpha_4) + 12(X - \alpha_1)(X - \alpha_2)(X - \alpha_3)(X - \alpha_4) + 6(X - \alpha_1)^2(X - \alpha_2)(X - \alpha_3)(X - \alpha_4) + 6(X - \alpha_1)^2(X - \alpha_2)^2(X - \alpha_3)(X - \alpha_4) + 6(X - \alpha_1)^2(X - \alpha_2)^2(X - \alpha_3)(X - \alpha_4) + (X - \alpha_1)^3(X - \alpha_2)^2 + 2(X - \alpha_1)^3(X - \alpha_2)(X - \alpha_3) + 2(X - \alpha_1)^3(X - \alpha_2)(X - \alpha_3) + 2(X - \alpha_1)^3(X - \alpha_2)(X - \alpha_3) \]

Because \( f^3(X) \neq 0 \) when \( X = \alpha_1, \alpha_2, \alpha_3, \alpha_4 \). None of the root can be the common root. This form doesn’t fit the conjecture.

5.5.7. \( f(X) = (X - \alpha_1)^3(X - \alpha_2)^2(X - \alpha_3)^2 \)

By direct calculation,
\[
\begin{align*}
f^3(X) &= a(X - \alpha_2)^3(X - \alpha_3) + b(X - \alpha_1)(X - \alpha_2)(X - \alpha_3)^2 + c(X - \alpha_1)(X - \alpha_2)^2(X - \alpha_3) + d(X - \alpha_1)^2(X - \alpha_2)^2 + e(X - \alpha_1)^3(X - \alpha_2) + f(X - \alpha_1)^3(X - \alpha_2)(X - \alpha_3) + g(X - \alpha_1)^3(X - \alpha_3) + h(X - \alpha_1)^3(X - \alpha_2) + (a,b,c,d,e,f,g,h \in N) \end{align*}
\]

Because \( f^3(X) \neq 0 \) when \( X = \alpha_1, \alpha_2, \alpha_3 \). None of the root can be the common root. This form doesn’t fit the conjecture.

5.5.8. \( f(X) = (X - \alpha_1)^3(X - \alpha_2)(X - \alpha_3) \)

By direct calculation,
\[
\begin{align*}
f^3(X) &= a(X - \alpha_2)^3(X - \alpha_3) + b(X - \alpha_1)(X - \alpha_2)^2(X - \alpha_3) + c(X - \alpha_1)(X - \alpha_2)^2(X - \alpha_3) + d(X - \alpha_1)^2(X - \alpha_2)(X - \alpha_3) + e(X - \alpha_1)^2(X - \alpha_2)^2 + f(X - \alpha_1)^3(X - \alpha_3) + g(X - \alpha_1)^3(X - \alpha_2) + (a,b,c,d,e,f,g \in N) \end{align*}
\]

Because \( f^3(X) \neq 0 \) when \( X = \alpha_1, \alpha_2, \alpha_3 \). None of the root can be the common root. This form doesn’t fit the conjecture.
5.5.9. \( f(X) = (X - \alpha_1)^4(X - \alpha_2)^3(X - \alpha_3)(X - \alpha_4) \)

By direct calculation,

\[
\begin{align*}
f^4(X) &= a(X - \alpha_2)(X - \alpha_3)(X - \alpha_4) \\
&\quad + b(X - \alpha_1)(X - \alpha_3)(X - \alpha_4) \\
&\quad + c(X - \alpha_1)(X - \alpha_2)(X - \alpha_4) \\
&\quad + d(X - \alpha_1)(X - \alpha_2)(X - \alpha_3) \\
&\quad + e(X - \alpha_1)^2(X - \alpha_2) \\
&\quad + f(X - \alpha_1)^2(X - \alpha_3) \\
&\quad + g(X - \alpha_1)^2(X - \alpha_4) \\
&\quad + h(X - \alpha_1)^3 \\
(a, b, c, d, e, f, g, h \in N)
\end{align*}
\]

Because \( f^4(X) \neq 0 \) when \( X = \alpha_1, \alpha_2, \alpha_3, \alpha_4 \). None of the root can be the common root. This form doesn’t fit the conjecture.

5.5.10. \( f(X) = (X - \alpha_1)^4(X - \alpha_2)^2(X - \alpha_3) \)

By direct calculation,

\[
\begin{align*}
f^4(X) &= a(X - \alpha_2)^2(X - \alpha_3) \\
&\quad + b(X - \alpha_1)(X - \alpha_2)(X - \alpha_3) \\
&\quad + c(X - \alpha_1)(X - \alpha_2)^2 \\
&\quad + d(X - \alpha_1)^2(X - \alpha_2) \\
&\quad + e(X - \alpha_1)^2(X - \alpha_3) \\
&\quad + f(X - \alpha_1)^3 \\
(a, b, c, d, e, f \in N)
\end{align*}
\]

Because \( f^4(X) \neq 0 \) when \( X = \alpha_1, \alpha_2, \alpha_3 \). None of the root can be the common root. This form doesn’t fit the conjecture.

5.5.11. \( f(X) = (X - \alpha_1)^4(X - \alpha_2)^3 \)

By direct calculation,

\[
\begin{align*}
f^4(X) &= a(X - \alpha_2)^3 \\
&\quad + b(X - \alpha_1)(X - \alpha_2)^2 \\
&\quad + c(X - \alpha_1)^2(X - \alpha_2) \\
&\quad + d(X - \alpha_1)^3 \\
(a, b, c, d \in N)
\end{align*}
\]

Because \( f^4(X) \neq 0 \) when \( X = \alpha_1, \alpha_2, \alpha_3, \alpha_4 \). None of the root can be the common root. This form doesn’t fit the conjecture.

5.5.12. \( f(X) = (X - \alpha_1)^5(X - \alpha_2)(X - \alpha_3) \)

By direct calculation,
Because \( f^5(X) \neq 0 \) when \( X = \alpha_1, \alpha_2, \alpha_3 \). None of the root can be the common root. This form doesn’t fit the conjecture.

5.5.13. \( f(X) = (X - \alpha_1)^5(X - \alpha_2)^2 \)

By direct calculation,
\[
f^5(X) = a(X - \alpha_2)^2 + b(X - \alpha_1)(X - \alpha_2) + c(X - \alpha_1)^2 \quad (a,b,c \in N)
\]

Because \( f^5(X) \neq 0 \) when \( X = \alpha_1, \alpha_2 \). None of the root can be the common root. This form doesn’t fit the conjecture.

5.5.14. \( f(X) = (X - \alpha_1)^6(X - \alpha_2) \)

By direct calculation,
\[
f^6(X) = a(X - \alpha_2) + b(X - \alpha_1) \quad (a,b \in N)
\]

Because \( f^6(X) \neq 0 \) when \( X = \alpha_1, \alpha_2 \). None of the root can be the common root. This form doesn’t fit the conjecture.

5.5.15. \( f(X) = (X - \alpha_1)^7 \)

By direct calculation, we can find that every derivative has a common root with the original polynomial, which is \( X = \alpha_1 \). Enabling \( f(X) = (X - \alpha_1)^7 \) to fit the conjecture.

6. Conclusion
If the polynomial has a common root with each of its derivative, we can always prove those polynomials that is not \( a(X - \alpha_1)^n (a,n \in R) \) wrong. We just need to take the derivative by the time of the index of the leading term. If there is more than one variables, the original polynomial won’t have a common root with every of it derivative. So the polynomial must stays in a form of \( f(X) = a(X - \alpha)^n \) where \( a,n \in R \).

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