Inverse Polynomial Images which Consists of Two Jordan Arcs – An Algebraic Solution*

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Abstract

Inverse polynomial images of $[-1, 1]$, which consists of two Jordan arcs, are characterised by an explicit polynomial equation for the four endpoints of the arcs.

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1 Introduction

Let $\mathbb{P}_n$ be the set of polynomials with complex coefficients of degree $n$ and let $P_n \in \mathbb{P}_n$. Let $P_n^{-1}([-1, 1])$ be the inverse image of $[-1, 1]$ under the polynomial mapping $P_n$, i.e.,

$$P_n^{-1}([-1, 1]) = \{ z \in \mathbb{C} : P_n(z) \in [-1, 1] \}. \quad (1)$$

In general, $P_n^{-1}([-1, 1])$ consists of $n$ Jordan arcs, on which $P_n$ is strictly monotone increasing from $-1$ to $+1$, see [13]. If there is a point $z_1 \in \mathbb{C}$, for which $P_n(z_1) \in \{-1, 1\}$ and $P_n'(z_1) = 0$, then two Jordan arcs can be combined into one Jordan arc. This combination of arcs can be seen in a very good way from the inverse image of the classical Chebyshev polynomial $T_n(z) = \cos(n \arccos(z))$. The inverse image $T_n^{-1}([-1, 1])$ is just $[-1, 1]$, i.e. one Jordan arc, since there are $n - 1$ points $z_j$ with the property $T_n(z_j) \in \{-1, 1\}$ and $T_n'(z_j) = 0$. Note that $T_n$ is, up to a linear transformation, the only polynomial mapping with that property. In this paper, we are interested in polynomials for which the inverse image of $[-1, 1]$ consists of two Jordan arcs. This property is equivalent to the existence of a certain quadratic equation for the corresponding polynomial.

Definition 1. The set $\{a, b, c, d\}$ of four complex numbers is called a $T_n$-tuple if there exist polynomials $T_n \in \mathbb{P}_n$ and $U_{n-2} \in \mathbb{P}_{n-2}$ such that a quadratic equation (sometimes called Pell-equation or Abel-equation) of the form

$$T_n^2(z) - H(z)U_{n-2}^2(z) = 1 \quad (2)$$

holds, where

$$H(z) := (z - a)(z - b)(z - c)(z - d). \quad (3)$$

Note that $T_n$ and $U_{n-2}$ are unique up to sign.

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In [16, Theorem 3], it was proved that a polynomial \(T_n \in \mathbb{P}_n\) satisfies equation (2) if and only if \(T_n^{-1}([-1, 1])\) consists of two Jordan arcs with endpoints \(a, b, c, d\).

The investigation of quadratic equations of the form (2) goes back to Abel [1] and Chebyshev [5]. Their work was continued by Zolotarev [21, 22] and Ahieser [2, 3], and in recent times by Peherstorfer [9, 10, 11, 12], Lebedev [6] and Peherstorfer and the present author [16]. For a more detailed history on the subject and some pictures of \(T_n^{-1}([-1, 1])\) in several interesting cases, see [16].

Most of the above cited papers make extensive use of elliptic functions and integrals. Nevertheless, our approach for characterising \(T_n\)-tuples is purely algebraic and is based on the following result of Peherstorfer and the author [14, Lemma 2.1].

**Lemma 1** (Peherstorfer and Schiefermayr [14]).

(i) Let \(n = 2m + 1\) be an odd degree. The set \(\{a, b, c, d\}\) is a \(\mathbb{T}_n\)-tuple if and only if it satisfies the following system of equations:

\[
\begin{align*}
(x_1^k + \ldots + x_m^k) - (y_1^k + \ldots + y_m^k) + \frac{1}{2}(-a^k - b^k - c^k + d^k) &= 0, \\
&\text{if } k = 1, 2, \ldots, 2m.
\end{align*}
\]

The corresponding polynomial \(T_n\) is given by

\[
T_n(z) = 1 - \frac{2(z-d)\prod_{j=1}^{m}(z-x_j)^2}{(a-d)\prod_{j=1}^{m}(a-x_j)^2} = -1 + \frac{2(z-a)(z-b)(z-c)\prod_{j=1}^{m-1}(z-y_j)^2}{(d-a)(d-b)(d-c)\prod_{j=1}^{m-1}(d-y_j)^2}.
\]

Note that \(T_n(x_j) = T_n(d) = 1\) and \(T_n(y_j) = T_n(a) = T_n(b) = T_n(c) = -1\).

(ii) Let \(n = 2m + 2\) be an even degree. The set \(\{a, b, c, d\}\) is a \(\mathbb{T}_n\)-tuple if and only if it satisfies one of the following two systems of equations:

\[
\begin{align*}
(x_1^k + \ldots + x_m^k) - (y_1^k + \ldots + y_m^k) + \frac{1}{2}(-a^k - b^k + c^k + d^k) &= 0, \\
&\text{if } k = 1, \ldots, 2m + 1.
\end{align*}
\]

or

\[
\begin{align*}
(x_1^k + \ldots + x_{m+1}^k) - (y_1^k + \ldots + y_{m-1}^k) + \frac{1}{2}(-a^k - b^k - c^k - d^k) &= 0, \\
&\text{if } k = 1, \ldots, 2m + 1.
\end{align*}
\]

The corresponding polynomial \(T_n\) for the solution of (5) is given by

\[
T_n(z) = 1 - \frac{2(z-c)(z-d)\prod_{j=1}^{m}(z-x_j)^2}{(c-a)(c-b)\prod_{j=1}^{m}(c-y_j)^2} = -1 + \frac{2(z-a)(z-b)\prod_{j=1}^{m}(z-y_j)^2}{(a-c)(a-b)\prod_{j=1}^{m}(a-y_j)^2}.
\]

If \(\{a, b, c, d\}\) satisfies system (5), i.e., is a \(\mathbb{T}_n\)-tuple, then \(\{a, b, c, d\}\) is also a \(\mathbb{T}_{\frac{n}{2}}\)-tuple and for the corresponding polynomial we have \(T_n(z) = 2T^2_{\frac{n}{2}}(z) - 1\).

Note that the \(x_j\) and \(y_j\) of equation (4), (5), and (6), are the zeros of the corresponding polynomial \(U_{n-2}\) and are therefore exactly the extremal points of the corresponding polynomial \(T_n\) on \(T_n^{-1}([-1, 1])\) (which consists of two Jordan arcs). As usual, a point \(z_0 \in C\), \(C \subseteq \mathbb{C}\) compact, is called an extremal point of \(P_n \in \mathbb{P}_n\) on \(C\) if \(|P_n(z_0)| = \max_{z \in C} |P_n(z)|\).
Further, we want to remark that the above definition of a $T_n$-tuple also includes the case of one interval (if two of the four points $a, b, c, d$ are equal).

The main purpose of the present paper is to modify the polynomial systems (4) and (5) in the following way: With the help of the recent paper [20], in Theorem 1 and Theorem 2, we give one polynomial equation in terms of $a, b, c, d$, which is equivalent to (4) and (5), respectively. In other words, for every degree $n$, we can explicitly give a polynomial in four variables $p(a, b, c, d)$, whose zeros $\{a, b, c, d\}$ are $T_n$-tuples. Moreover, a simple equation for computing the extremal points $x_j$ and $y_j$ is derived. Algebraic solutions of the quadratic equation (2) with the help of Jacobi’s elliptic functions can be found in [17, 18] (in the real case), and [16, Section 4]. In [9, Section 5], an algebraic solution (in the real case) is given with the help of orthogonal polynomials. In [15, Chapter 7], an algebraic solution was found by simplifying the above system of equations with the help of GRÖBNER-Basis.

The paper is organised as follows. In section 2, the fundamental lemma based on [20] is given, from which the simplifications of (4) and (5), proved in section 3, can be deduced. Moreover, the maximum number of $T_n$-tuples is given explicitly assuming that 3 of the 4 points $a, b, c, d$ are fixed. In section 4.1, the polynomial equations for $a, b, c, d$ and for $x_j$ and $y_j$ are explicitly written down for the smallest degrees $n \in \{2, 3, 4\}$. Finally, a brief look at the special case of Zolotarev polynomials is taken in Section 4.2.

2 Auxiliaries

Let us define $F_0 := 1$ and, for $k = 1, 2, \ldots$,

$$F_k \equiv F_k(s_1, s_2, \ldots, s_k) := \frac{(-1)^k}{k!} \det \left( \begin{array}{cccc} s_1 & 1 & 0 & 0 \ldots 0 \ldots 0 \\ s_2 & s_1 & 2 & 0 \ldots 0 \ldots 0 \\ s_3 & s_2 & s_1 & 3 \ldots 0 \ldots 0 \\ & & & \ddots \ddots \ddots \ddots \ddots \\ s_{k-1} & s_{k-2} & s_{k-3} & s_{k-4} \ldots 1 & k-1 \\ s_k & s_{k-1} & s_{k-2} & s_{k-3} \ldots s_2 & s_1 \end{array} \right). \quad (7)$$

For negative indices $k = -1, -2, -3, \ldots$, we define $F_k := 0$.

The next lemma, which is fundamental for our considerations, can be extracted from [20].

**Lemma 2.** For given $s_1, s_2, \ldots, s_{\nu+\mu} \in \mathbb{C}$, consider the system of equations

$$(u_1^k + \ldots + u_\nu^k) - (v_1^k + \ldots + v_\mu^k) = s_k, \quad k = 1, 2, \ldots, \nu + \mu. \quad (8)$$

If $\{u_1, \ldots, u_\nu, v_1, \ldots, v_\mu\}$ is a nontrivial solution of (8), i.e., the sets $\{u_1, \ldots, u_\nu\}$ and $\{v_1, \ldots, v_\mu\}$ are disjoint, then this solution is unique (up to permutations of the $u_j$ and $v_j$) and the values $v_1, v_2, \ldots, v_\mu$ are exactly the solution of the equation

$$v^\mu + \Lambda_1 v^{\mu-1} + \Lambda_2 v^{\mu-2} + \ldots + \Lambda_{\nu-1} v + \Lambda_{\nu} = 0, \quad (9)$$
where $\Lambda_1, \Lambda_2, \ldots, \Lambda_\mu$ are the solution of the following regular linear system of equations:

\[
\sum_{i=1}^{\mu} F_{\nu+1-i} \Lambda_i = -F_{\nu+1} \\
\sum_{i=1}^{\mu} F_{\nu+2-i} \Lambda_i = -F_{\nu+2} \\
\vdots \\
\sum_{i=1}^{\mu} F_{\nu+\mu-i} \Lambda_i = -F_{\nu+\mu}
\] (10)

By Cramer’s rule, the solution $\Lambda_{1}, \Lambda_{2}, \ldots, \Lambda_{\mu}$ of system (10) may be written in the form (note that system (10) is regular and therefore $\det F \neq 0$)

\[\Lambda_i = \frac{\det F_i}{\det F}, \quad i = 1, 2, \ldots, \mu,\] (11)

where

\[
F := \begin{pmatrix}
F_\nu & F_{\nu-1} & F_{\nu-2} & \cdots & F_{\nu+1-\mu} \\
F_{\nu+1} & F_\nu & F_{\nu-1} & \cdots & F_{\nu+2-\mu} \\
F_{\nu+2} & F_{\nu+1} & F_\nu & \cdots & F_{\nu+3-\mu} \\
\vdots & \vdots & \vdots & \cdots & \vdots \\
F_{\nu+\mu-1} & F_{\nu+\mu-2} & F_{\nu+\mu-3} & \cdots & F_\nu
\end{pmatrix} \in \mathbb{R}_\mu^{\mu}
\] (12)

and

\[
F_i := F \text{ and the } i\text{-th column of } F \text{ is replaced by } \begin{pmatrix}
-F_{\nu+1} \\
-F_{\nu+2} \\
-F_{\nu+3} \\
\vdots \\
-F_{\nu+\mu}
\end{pmatrix}.
\] (13)

By (11), equation (9) may be written in the form

\[v^\mu \det F + v^{\mu-1} \det F_1 + v^{\mu-2} \det F_2 + \ldots + v \det F_{\mu-1} + \det F_\mu = 0.\] (14)

### 3 Main Results

Let $n = 2m + 1$ be an odd degree. Starting point is system of equations (4), which may be written in the form

\[(y_1^k + \ldots + y_{m-1}^k) - (x_1^k + \ldots + x_m^k + d^k) = s_k, \quad k = 1, 2, \ldots, 2m,\] (15)

where

\[s_k := \frac{1}{2}(-a^k - b^k - c^k - d^k), \quad k = 1, 2, \ldots, 2m.\] (16)

Then, by Lemma2 the above polynomial system reduces to

\[d^{m+1} \det F + d^m \det F_1 + d^{m-1} \det F_2 + \ldots + d \det F_m + \det F_{m+1} = 0,\]

where $s_k, F_k, F,$ and $F_i$ is defined in (16), (7), (12), and (13), respectively, and $\nu = m - 1$ and $\mu = m + 1$. Note that if $\{y_1, \ldots, y_{m-1}, x_1, \ldots, x_m, d\}$ is a solution of (15) then the
sets \{y_1, \ldots, y_{m-1}\} and \{x_1, \ldots, x_m, d\} are disjoint, thus, by Lemma 2 the corresponding linear system (10) is regular and \(\det F \neq 0\).

In order to get a polynomial equation for the \(y_i\), we write system (4) in the form

\[
(x_1^k + \ldots + x_m^k + d^k) - (y_1^k + \ldots + y_{m-1}^k) = s_k, \quad k = 1, 2, \ldots, 2m,
\]

where

\[
s_k := \frac{1}{2}(a^k + b^k + c^k + d^k), \quad k = 1, 2, \ldots, 2m.
\]

Then, again by Lemma 2 the above polynomial system reduces to

\[
y^{m-1} \det F + y^{m-2} \det F_1 + y^{m-3} \det F_2 + \ldots + y \det F_{m-2} + \det F_{m-1} = 0,
\]

where \(s_k, F_k, F,\) and \(F_i\) is defined in (18), (7), (12), and (13), respectively, and \(\nu = m + 1, \mu = m - 1\). By the same argument as above, \(\det F \neq 0\).

For a polynomial equation for the \(x_i\), we write system (4) in the form

\[
(y_1^k + \ldots + y_{m-1}^k + a^k) - (x_1^k + \ldots + x_m^k) = s_k, \quad k = 1, 2, \ldots, 2m,
\]

where

\[
s_k := \frac{1}{2}(a^k - b^k - c^k + d^k), \quad k = 1, 2, \ldots, 2m.
\]

Then, again by Lemma 2 the above polynomial system reduces to

\[
x^m \det F + x^{m-1} \det F_1 + x^{m-2} \det F_2 + \ldots + x \det F_{m-1} + \det F_m = 0,
\]

where \(s_k, F_k, F,\) and \(F_i\) is defined in (20), (7), (12), and (13), respectively, and \(\nu = m, \mu = m\). By the same argument as above, \(\det F \neq 0\).

We collect the above results in the following theorem.

**Theorem 1.** Let \(n = 2m + 1\).

(i) The set \(\{a, b, c, d\}\) is a \(T_n\)-tuple if and only if \(a, b, c, d\) satisfies the polynomial equation

\[
p := a^{m+1} \det F + a^m \det F_1 + a^{m-1} \det F_2 + \ldots + a \det F_m + \det F_{m+1} = 0,
\]

where \(s_k, F_k, F,\) and \(F_i\) is defined in (16), (7), (12), and (13), respectively, and \(\nu = m - 1, \mu = m + 1\).

(ii) The values \(y_1, y_2, \ldots, y_{m-1}\) are exactly the zeros of the polynomial

\[
y^{m-1} \det F + y^{m-2} \det F_1 + y^{m-3} \det F_2 + \ldots + y \det F_{m-2} + \det F_{m-1},
\]

where \(s_k, F_k, F,\) and \(F_i\) is defined in (18), (7), (12), and (13), respectively, and \(\nu = m + 1, \mu = m - 1\).

(iii) The values \(x_1, x_2, \ldots, x_m\) are exactly the zeros of the polynomial

\[
x^m \det F + x^{m-1} \det F_1 + x^{m-2} \det F_2 + \ldots + x \det F_{m-1} + \det F_m,
\]

where \(s_k, F_k, F,\) and \(F_i\) is defined in (20), (7), (12), and (13), respectively, and \(\nu = m, \mu = m\).

**Corollary 1.** Let \(n = 2m + 1\).
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(i) The polynomial $p \equiv p(a, b, c, d)$ in Theorem 1(i) is a homogeneous polynomial of $a, b, c, d$ with rational coefficients and degree $m^2 + m = (n^2 - 1)/4$.

(ii) Let 3 of the 4 points $a, b, c, d \in \mathbb{C}$ be fixed, then there exist at most $(n^2 - 1) \mathbb{T}_n$-tuples containing these 3 points.

Proof. (i) By the definitions (16), (7), (12), and (13), the following statements concerning the degree of $p$ hold (note that $\nu = m - 1, \mu = m + 1$):

- $F_k$ is a homogeneous polynomial of $a, b, c, d$ with degree $k$.
- $\det F$ is a homogeneous polynomial of $a, b, c, d$ with degree $m^2 - 1$.
- $\det F_i$ is a homogeneous polynomial of $a, b, c, d$ with degree $m^2 - 1 + i$.

From these statements, the assertion follows.

(ii) By the special form of equation (11), there are 4 different possibilities to fix 3 of the 4 points $a, b, c, d$ in (11). These 4 possibilities multiplied with the degree $(n^2 - 1)/4$ of the homogeneous polynomial $p(a, b, c, d)$ gives the maximum number of different solutions.

Finally, we give the analogous results for even degree. Since the proofs run along the same lines as those for odd degree, we omit them.

Theorem 2. Let $n = 2m + 2$.

(i) The set $\{a, b, c, d\}$ is a $\mathbb{T}_n$-tuple but not a $\mathbb{T}_{m^2}$-tuple if and only if $a, b, c, d$ satisfies the polynomial equation

$$p := a^{m+1} \det F + a^m \det F_1 + a^{m-1} \det F_2 + \ldots + a \det F_m + \det F_{m+1} = 0,$$

where

$$s_k := \frac{1}{2}(-a^k + b^k - c^k - d^k), \quad k = 1, 2, \ldots, 2m + 1,$$

and $F_k, F, F_i$ is defined in (7), (12), and (13), respectively, and $\nu = m, \mu = m + 1$.

(ii) The values $y_1, y_2, \ldots, y_m$ are exactly the zeros of the polynomial

$$y^m \det F + y^{m-1} \det F_1 + y^{m-2} \det F_2 + \ldots + y \det F_{m-1} + \det F_m,$$

where

$$s_k := \frac{1}{2}(a^k + b^k + c^k - d^k), \quad k = 1, 2, \ldots, 2m + 1,$$

and $F_k, F, F_i$ is defined in (7), (12), and (13), respectively, and $\nu = m + 1, \mu = m$.

(iii) The values $x_1, x_2, \ldots, x_m$ are exactly the zeros of the polynomial

$$x^m \det F + x^{m-1} \det F_1 + x^{m-2} \det F_2 + \ldots + x \det F_{m-1} + \det F_m,$$

where

$$s_k := \frac{1}{2}(a^k - b^k + c^k + d^k), \quad k = 1, 2, \ldots, 2m + 1,$$

and $F_k, F, F_i$ is defined in (7), (12), and (13), respectively, and $\nu = m + 1, \mu = m$. 
Corollary 2. Let $n = 2m + 2$.

(i) The polynomial $p \equiv p(a, b, c, d)$ in Theorem 2(i) is a homogeneous polynomial of $a, b, c, d$ with rational coefficients and degree $(m + 1)^2 = n^2/4$.

(ii) Let 3 of the 4 points $a, b, c, d \in \mathbb{C}$ be fixed, then there exist at most $3n^2/4$ $T_n$-tuples containing these 3 points.

4 Addenda

4.1 Equations for Small Degrees

By Theorems 1 and 2 the computation of a $T_n$-tuple $\{a, b, c, d\}$ and the extremal points $x_i$ and $y_i$ of the corresponding polynomial $T_n$ can be managed in 4 steps:

1. Given 3 of the 4 points $a, b, c, d$, compute the 4th point by equation (21) and (24), respectively.
2. Compute the $y_j$ by equation (22) and (26), respectively.
3. Compute the $x_j$ by equation (23) and (28), respectively.
4. Compute the corresponding polynomial $T_n$ by Lemma 1.

In the following, we give the equations for $a, b, c, d$, for the $y_j$, and for the $x_j$, in case of the simplest degrees $n = 2, 3, 4$. For greater degrees, the equations get very bulky.

- $n = 2$:  
  \[ a + b - c - d = 0 \]

- $n = 3$:  
  \[
  a^2 - 2ab + b^2 - 2ac - 2bc + c^2 + 2ad + 2bd + 2cd - 3d^2 = 0 \\
  x(-4a + 4b + 4c - 4d) + (a^2 + 2ab - 3b^2 + 2ac - 2bc - 3c^2 - 2ad + 2bd + 2cd + d^2) = 0
  \]

- $n = 4$:  
  \[
  a^4 + 4a^3b - 10a^2b^2 + 4ab^3 + b^4 - 4a^3c + 4a^2bc + 4ab^2c - 4b^3c + 6a^2c^2 \\
  - 4abc^2 + 6b^2c^2 - 4ac^3 - 4bc^3 + c^4 - 4a^3d + 4a^2bd + 4ab^2d - 4b^3d - 4a^2cd \\
  - 8abcd - 4b^2cd + 4ac^2d + 4bc^2d + 4c^3d + 6a^2d^2 - 4abd^2 + 6b^2d^2 \\
  + 4acd^2 + 4bcd^2 - 10c^2d^2 - 4ad^3 - 4bd^3 + 4cd^3 + d^4 = 0 \\
  \\
  y(-2a^2 + 4ab - 2b^2 + 4ac + 4bc - 2c^2 - 4ad - 4bd + 4cd + 6d^2) \\
  + (a^3 - a^2b - ab^2 + b^3 - a^2c + 2abc - b^2c - ac^2 - bc^2 + c^3 + a^2d \\
  - 2abd + b^2d - 2acd - 2bcd + c^2d + 3ad^2 + 3bd^2 + 3cd^2 - 5d^3) = 0
  \]

\[
  x(-2a^2 - 4ab + 6b^2 + 4ac - 4bc - 2c^2 + 4ad - 4bd - 4cd + 2d^2) \\
  + (a^3 + a^2b + 3ab^2 - 5b^3 - a^2c - 2abc + 3b^2c - ac^2 + bc^2 + c^3 \\
  - a^2d - 2abd + 3b^2d + 2acd - 2bcd - c^2d - ad^2 + bd^2 - cd^2 + d^3) = 0
  \]
4.2 Special Case: Zolotarev Polynomial

A special case for which the inverse polynomial image consists of two arcs is the so-called Zolotarev polynomial, which has also applications in signal processing \[19\]. Given \( \sigma > 0 \), the Zolotarev polynomial \( Z_n(x) \) solves the following approximation problem [4, Addendum E]:

\[
\min_{a_j \in \mathbb{C}} \max_{x \in [-1,1]} |x^n - n\sigma x^{n-1} + a_{n-2}x^{n-2} + \ldots + a_1x + a_0| = \max_{x \in [-1,1]} |Z_n(x)| =: L_n
\]

It is well known that, for \( \sigma \leq \tan^2 \frac{\pi}{2n} \), the Zolotarev polynomial \( Z_n(x) \) is simply a suitable linear transformed classical Chebyshev polynomial \( T_n(x) \) and, for \( \sigma > \tan^2 \left( \frac{\pi}{2n} \right) \), the inverse image of \( Z_n(x) \) consists of the two arcs \([-1, 1] \cup [\alpha, \beta] \), where \( 1 < \alpha < \beta \) and \( Z_n \) has \( n \) and \( 2 \) extremal points in \([-1, 1] \) and \([\alpha, \beta] \), respectively. The connection of the parameter \( \sigma \) with the extremal points of \( Z_n \) follows from the well known theorem of Vieta applied to the polynomial \( Z_n(x) + L_n \), which in our notation (put \( a = \alpha, b = 1, c = -1, d = \beta \)) reads as follows:

\[
2 \sum_{j=1}^{m-1} y_j + \alpha = n\sigma \quad (n = 2m + 1) \tag{30}
\]
\[
2 \sum_{j=1}^{m} y_j + \alpha + 1 = n\sigma \quad (n = 2m + 2) \tag{31}
\]

We summarize the results in the following corollary.

**Corollary 3.** Given \( n \in \mathbb{N} \) and \( \sigma > \tan^2 \left( \frac{\pi}{2n} \right) \), the two endpoints \( \alpha \) and \( \beta \) of the inverse polynomial image of the corresponding Zolotarev polynomial can be computed by the following two polynomial equations:

(i) \( n = 2m + 1 \):

\[
\beta^{m+1} \det F + \beta^m \det F_1 + \beta^{m-1} \det F_2 + \ldots + \beta \det F_m + \det F_{m+1} = 0 \tag{32}
\]
\[
-2 \det F_1 + (\alpha - n\sigma) \det F = 0 \tag{33}
\]

where \( F_k, F, F_i \) is defined in (7), (12), (13), respectively, and

\[
\nu = m - 1, \mu = m + 1, \quad s_k \triangleq \frac{1}{2}(-\alpha^k - 1 - (-1)^k - \beta^k) \quad \text{in (32)}
\]
\[
\nu = m + 1, \mu = m - 1, \quad s_k \triangleq \frac{1}{2}(\alpha^k + 1 + (-1)^k + \beta^k) \quad \text{in (33)}
\]

(ii) \( n = 2m + 2 \):

\[
\alpha^{m+1} \det F + \alpha^m \det F_1 + \alpha^{m-1} \det F_2 + \ldots + \alpha \det F_m + \det F_{m+1} = 0 \tag{34}
\]
\[
-2 \det F_1 + (\alpha + 1 - n\sigma) \det F = 0 \tag{35}
\]

where \( F_k, F, F_i \) is defined in (7), (12), (13), respectively, and

\[
\nu = m, \mu = m + 1, \quad s_k \triangleq \frac{1}{2}(-\alpha^k + 1 - (-1)^k - \beta^k) \quad \text{in (34)}
\]
\[
\nu = m + 1, \mu = m, \quad s_k \triangleq \frac{1}{2}(\alpha^k + 1 + (-1)^k - \beta^k) \quad \text{in (35)}
\]

**Proof.** Equation (32) and (34) follow immediately by Theorem (1(i)) and Theorem (2(i)), respectively. Equation (33) and (35) follow from (30) and (31) together with Theorem (1(ii)) and Theorem (2(ii)), respectively, using Vieta’s root theorem. \(\square\)
Remark 1.

(i) Let \( n \in \mathbb{N} \) and \( \sigma > \tan^2\left(\frac{\pi}{2n}\right) \). With the equations (32), (33), and (34), (35), respectively, one can determine \( \alpha, \beta \) uniquely such that \( 1 < \alpha < \beta \).

(ii) Once \( \alpha, \beta \) are determined, by Theorem 1(ii),(iii) and Theorem 2(ii),(iii), respectively, one can easily compute the extremal points \( x_j \) and \( y_j \) (which are all in \([-1,1]\)).

(iii) With \( \alpha, \beta, \) the \( x_j, \) and the \( y_j, \) the corresponding Zolotarev polynomial is given, for \( n = 2m + 1 \) odd,

\[
Z_n(x) = (x - \beta) \prod_{j=1}^{m} (x - x_j)^2 + \frac{1}{2} (\beta - \alpha) \prod_{j=1}^{m} (\alpha - x_j)^2
\]

\[
= (x - \alpha)(x^2 - 1) \prod_{j=1}^{m-1} (x - y_j)^2 - \frac{1}{2} (\beta - \alpha)(\beta^2 - 1) \prod_{j=1}^{m-1} (\beta - y_j)^2,
\]

and, for \( n = 2m + 2 \) even,

\[
Z_n(x) = (x + 1)(x - \beta) \prod_{j=1}^{m} (x - x_j)^2 + \frac{1}{2} (\alpha + 1)(\beta - \alpha) \prod_{j=1}^{m} (\alpha - x_j)^2
\]

\[
= (x - 1)(x - \alpha)(x^2 - 1) \prod_{j=1}^{m-1} (x - y_j)^2
\]

\[
- \frac{1}{2} (\beta - 1)(\beta - \alpha)(\beta^2 - 1) \prod_{j=1}^{m-1} (\beta - y_j)^2.
\]

(iv) For a different approach to an algebraic solution of the Zolotarev problem, see [7, 8].

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