A saturated model of an unsuperstable theory of cardinality greater than its theory has the small index property

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Abstract

A model $M$ of cardinality $\lambda$ is said to have the small index property if for every $G \subseteq \text{Aut}(M)$ such that $[\text{Aut}(M) : G] \leq \lambda$ there is an $A \subseteq M$ with $|A| < \lambda$ such that $\text{Aut}_A(M) \subseteq G$. We show that if $M^*$ is a saturated model of an unsuperstable theory of cardinality $> Th(M)$, then $M^*$ has the small index property.

1 Introduction

Throughout the paper we work in $\mathfrak{C}^e_{eq}$, and we assume that $M^*$ is a saturated model of $T$ of cardinality $\lambda$. We denote the set of automorphisms of $M^*$ by $\text{Aut}(M^*)$ and the set of automorphisms of $M^*$ fixing $A$ pointwise by $\text{Aut}_A(M^*)$. $M^*$ is said to have the small index property if whenever $G$ is a subgroup of $\text{Aut}(M^*)$ with index not larger than $\lambda$ then for some $A \subseteq M^*$ with $|A| < \lambda$, $\text{Aut}_A(M^*) \subseteq G$. The main theorem of this paper is the following result of Shelah: If $M^*$ is a saturated model of cardinality $\lambda > |T|$ and there is a tree of height some uncountable regular cardinal

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\(\kappa \geq \kappa_\tau(T)\) with \(\mu > \lambda\) many branches but at most \(\lambda\) nodes, then \(M^*\) has the small index property, in fact

\[|\text{Aut}(M^*) : G| \geq \mu\]

for any subgroup \(G\) of \(\text{Aut}(M^*)\) such that for no \(A \subseteq M^*\) with \(|A| < \lambda\) is \(\text{Aut}_A(M^*) \subseteq G\). By a result of Shelah on cardinal arithmetic this implies that if \(\text{Aut}(M^*)\) does not have the small index property, then for some strong limit \(\mu\) such that \(\text{cf } \mu = \aleph_0\),

\[\mu < \lambda < 2^\mu\]

So in particular, if \(T\) is unsuperstable, \(M^*\) has the small index property.

In the paper “Uncountable Saturated Structures have the Small Index Property” by Lascar and Shelah, the following result was obtained:

**Theorem 1.1** Let \(M^*\) be a saturated model of cardinality \(\lambda\) with \(\lambda > |T|\) and \(\lambda^{<\lambda} = \lambda\). Then if \(G\) is a subgroup of \(\text{Aut}(M^*)\) such that for no \(A \subseteq M^*\) with \(|A| < \lambda\) is \(\text{Aut}_A(M^*) \subseteq G\) then \(|\text{Aut}(M^*) : G| = \lambda^\lambda\).

**Proof** See [L Sh].

**Corollary 1.2** Let \(M^*\) be a saturated model of cardinality \(\lambda\) with \(\lambda > |T|\) and \(\lambda^{<\lambda} = \lambda\). Then \(M^*\) has the small index property.

**Theorem 1.3** \(T\) has a saturated model of cardinality \(\lambda\) iff \(\lambda = \lambda^{<\lambda} + D(T)\) or \(T\) is stable in \(\lambda\).

**Proof** See [Sh c] chp. VIII.

So we can assume in the rest of this paper that \(T\) is stable in \(\lambda\).

**Theorem 1.4** \(T\) is stable in \(\mu\) iff \(\mu = \mu_0 + \mu^{<\kappa(T)}\) where \(\mu_0\) is the first cardinal in which \(T\) is stable.

**Proof** See [Sh c] chp. III.

Since \(T\) is stable in \(\lambda\), we must have \(\lambda = \lambda^{<\kappa(T)}\), so \(\text{cf } \lambda \geq \kappa(T)\).

Since the first cardinal \(\kappa\), such that \(\lambda^\kappa > \lambda\) is regular, we also know that \(\text{cf } \lambda \geq \kappa_\tau(T)\).
Definition 1.5 Let $Tr$ be a tree. If $\eta, \nu \in Tr$, then $\gamma[\eta, \nu] =$ the least $\gamma$ such that $\eta(\gamma) \neq \nu(\gamma)$ or else it is $\min(\text{height}(\eta), \text{height}(\nu))$.

Notation 1.6 Let $Tr$ be a tree. If $h \in \text{Aut}(M^*)$ and $\alpha < \text{height}(Tr)$, $\eta, \nu \in Tr$, then

$$h^{\eta(\alpha)} < \nu(\alpha) = h$$

if $\eta(\alpha) < \nu(\alpha)$ and $\text{id}_{M^*}$ otherwise.

Lemma 1.7 Let $\{C_i \mid i \in I\}$ be independent over $A$ and let $\{D_i \mid i \in I\}$ be independent over $B$. Suppose that for each $i \in I$, $\text{tp}(C_i/A)$ is stationary. Let $f$ be an elementary map from $A$ onto $B$, and let for each $i \in I$, $f_i$ be an elementary map extending $f$ which sends $C_i$ onto $D_i$. Then

$$\bigcup_{i \in I} f_i$$

is an elementary map from $\bigcup C_i$ onto $\bigcup D_i$.

**Proof** Left to the reader.

Lemma 1.8 Let $|T| < \lambda$. Let $Tr$ be a tree of height $\omega$ with $\kappa_n$ nodes of height $n$ for some $\kappa_n < \lambda$. Let $n < \omega$ and let $\langle M_i \mid i \leq n \rangle$ be an increasing chain of models. Let $M_n \subseteq N_0 \subseteq N_1 \subseteq M^*$ with $|N_1| < \lambda$. Suppose $\langle h_i \mid i \leq n \rangle$ are automorphisms of $M^*$ such that

1. $h_i = \text{id}_{M_i}$
2. $h_i[N_j] = N_j$ for $j \leq 1$
3. $h_i[M_k] = M_k$ for $k \leq n$

For each $\nu \in Tr \upharpoonright \text{level}(n+1)$ let $m_\nu, l_\nu$ be automorphisms of $N_0$. Let $\eta \in Tr \upharpoonright \text{level}(n+1)$. Suppose $g_\eta \in \text{Aut}(N_0)$ such that for all $\nu \in Tr \upharpoonright \text{level}(n+1),$

$$g_\eta m_\eta(m_\nu)^{-1}(g_\eta)^{-1} = l_\eta(l_\nu)^{-1}h^{\eta(\gamma[\eta, \nu]) < \nu(\gamma[\eta, \nu])}$$

Let $m_\nu^+, l_\nu^+$ be extensions of $m_\nu$ and $l_\nu$ to automorphisms of $N_1$ for all $\nu \in Tr \upharpoonright \text{level}(n+1)$. Then there exists a model $N_2 \subseteq M^*$ containing
Let $g^+_η$ be a map with domain $N_1$ such that $g^+_η(N_1) \cup_{N_0} N_1$, $g^+_η(N_1) \subseteq M^∗$ and $g^+_η$ extends $g_η$. Let $g^{++}_η$ be a map extending $g_η$ such that the domain of $(g^{++}_η)^{-1}$ is $N_1$, $(g^{++}_η)^{-1}(N_1) \subseteq M^*$ and $(g^{++}_η)^{-1}(N_1) \cup_{N_0} N_1$. So $g^+_η \cup g^{++}_η$ is an elementary map. Let $l''_η$ and $m''_η$ be an extensions of $l^+_η$ and $m^+_η$ to an automorphisms of $M^*$. Let

$$m^{++}_η = (g^{++}_η)^{-1}(h_η, ν)^{-1}l''_η(l''_η)^{-1}g^{++}_ηm''_η \cup (m''_η)^{-1}[(g^{++})^{-1}[N_1]]$$

Note that $m^{++}_η \cup m^{++}_η$ is an elementary map. Let

$$l^{++}_η = (l''_η)^{-1}g''_ηm^+_η(l^+_η)^{-1}g^+_ηl''_η \cup h_η, ν[g^+_η[N_1]]$$

Note that $l^{++}_η \cup l^{++}_η$ is an elementary map. Let $g''_η$, $m''_η$, $l''_η$ be elementary extensions to $M^*$ of $g''_η \cup g^{++}_η$, $m''_η \cup m^{++}_η$, and $l''_η \cup l^{++}_η$. Let $N_2$ be a model of size $|N_1| + |T| + κ_{n+1}$ containing $N_1$ such that $N_2$ is closed under $m''_η$, $g''_η$, $l''_η$ all the $h_η, ν$ and $m''_η$, $l''_η$. Let $m''_η$, $l''_η$, $g''_η$, $h_η, ν$, $m''_η$, $l''_η$ be the restrictions to $N_2$ of the $m''_η$, $l''_η$, $g''_η$, $h_η, ν$, $m''_η$, $l''_η$.

**Theorem 1.9** If $λ > |T|$, cf $λ = ω$, $M^*$ is a saturated model of cardinality $λ$ and if $G$ is a subgroup of $Aut(M^*)$ such that for no $A \subseteq M^*$ with $|A| < λ$ is $Aut(A(M^*)) \subseteq G$ then $|Aut(M^*) : G| = λ^ω$.

**Proof** Suppose not. Let $\{κ_i \mid i < ω\}$ be an increasing sequence of cardinals each greater than $|T|$ with $sup = λ$. Let $Tr = \{η ∈ <ωλ \mid η(i) < κ_i\}$. Let $M^* = \bigcup_{i<ω} B_i$ with $|B_i| \leq κ_i$. By induction on $n < ω$ for every $η \in Tr \upharpoonright level n$ we define models $N_n \subset M^*$ and $h_n \in Aut_{N_n}(M^*) - G$ such that $B_n \subseteq N_n$ and $|N_n| \leq κ_n$, and automorphisms $g_η, m_η, l_η$ of $N_n$ such that if $ρ ≠ ν$ then $l_ρ \neq l_η$ and

$$g_ρm_ρ(m_ρ)^{-1}(g_ρ)^{-1} = l_ρ(l_η)^{-1}h_η^{ρ(γ[ρ, ν])}h_ρ^{ρ(γ[ρ, ν])}$$

Suppose we have defined the $g_η, m_η, l_η$ for $height(η) ≤ m$, and $N_j$ for $j ≤ m$. If $n = m + 1$, for each $i < κ_n$ we define models $N_{n,i}$ such that
\[ B_n \subseteq N_{n,i}, \ N_m \subseteq N_{n,i}, \ \langle N_{n,i} \mid i < \kappa_n \rangle \text{ is increasing continuous, and for some } \eta_i \in Tr \upharpoonright \text{level } n, \ g_{\eta_i} \in Aut(N_{n,i}) \text{ such that for each } \eta \in Tr \upharpoonright \text{level } n, \ \eta = \eta_i \text{ cofinally many times in } \kappa_n, \text{ and for every } \nu \in Tr \upharpoonright \text{level } n, \ m^i_\nu \neq l^i_\nu \in Aut(N_{n,i}) \text{ such that} \]
\[ g_{\eta_i} m^i_\eta (m^i_\nu)^{-1}(g_{\eta_i})^{-1} = l^i_\eta (l^i_\nu)^{-1} h^{\eta_i}(\gamma[\eta_i, \nu]) < \nu(\gamma[\eta_i, \nu]) \]

The \( g_{\eta_i}, \ m^i_\nu, \ l^i_\nu \) are easily defined by induction on \( i < \kappa_n \) using lemma 8 so that if \( i_1 < i_2 \) then \( m^i_\nu \subseteq m^{i_1}_\nu, \ l^i_\nu \subseteq l^{i_1}_\nu \), and if \( \eta_1 = \eta_2 \) then \( g_{\eta_1} \subseteq g_{\eta_2} \). Then if we let \( g_{\eta} = \bigcup \{ g_{\eta_i} \mid \eta_i = \eta \}, \ m_\eta = \bigcup \ m^i_\eta, \ l_\eta = \bigcup \ l^i_\eta \),

\[ N_n = \bigcup_{i < \kappa_n} N_{n,i} \text{ and } h_\eta \in Aut_{N_n}(M^*) - G \text{ we have finished. Let } B_r \text{ be the set of branches of } Tr \text{ of height } \omega. \text{ For } \rho \in Br \text{ let } g_\rho = \bigcup \{ g_{\eta} \mid \eta < \rho \}, \ m_\rho = \bigcup \{ m_\eta \mid \eta < \rho \}, \text{ and } l_\rho = \bigcup \{ l_\eta \mid \eta < \rho \}. \text{ If } \rho \neq \nu, \ g_\rho \neq g_\nu \text{ since without loss of generality } \rho(\gamma[\rho, \nu]) < \nu(\gamma[\rho, \nu]), \text{ and} \]
\[ g_\rho m_\rho (m_\nu)^{-1}(g_\nu)^{-1} = l_\rho (l_\nu)^{-1} h^{(\rho(\gamma[\rho, \nu]))^< \nu(\gamma[\rho, \nu])} \]

and
\[ g_\nu m_\nu (m_\rho)^{-1}(g_\rho)^{-1} = l_\nu (l_\rho)^{-1} \]

implies
\[ g_\rho (g_\nu)^{-1} l_\rho (l_\nu)^{-1} g_\nu (g_\rho)^{-1} = l_\rho (l_\nu)^{-1} h^{\rho(\gamma[\rho, \nu])^< \nu(\gamma[\rho, \nu])} \]

So if \( g_\rho = g_\nu \) this would imply \( h^{\rho(\gamma[\rho, \nu])^< \nu(\gamma[\rho, \nu])} = id_{M^*} \) a contradiction. If
\[ [Aut(M^*) : G] < \lambda^\omega \]

then for some \( \rho, \nu \in Br \) we must have \( l_\rho (l_\nu)^{-1} \in G \) and \( g_\rho (g_\nu)^{-1} \in G \), but then we get a contradiction as \( g_\rho (g_\nu)^{-1} l_\rho (l_\nu)^{-1} g_\nu (g_\rho)^{-1} \in G \) and \( l_\rho (l_\nu)^{-1} \in G \), but \( h^{\rho(\gamma[\rho, \nu])^< \nu(\gamma[\rho, \nu])} \not\in G \).

**Corollary 1.10** If \( \lambda > |T|, \ cf \lambda = \omega \) and \( M^* \) is a saturated model of cardinality \( \lambda \) then \( M^* \) has the small index property.

So we will assume in the remainder of the paper that in addition to \( T \) being stable, \( cf \lambda \geq \kappa_r(T) + R_1 \) and \( T, M^*, \text{ and } \lambda \) are constant.
2 Constructing $M^*$ as a chain from $K_\delta$

**Definition 2.1** Let $\delta < \lambda^+$, cf $\delta \geq \kappa_r(T)$.

$$K^*_\delta = \left\{ N | N = \langle N_i \mid i \leq \delta \rangle, N_i \text{ is increasing continuous, } |N_i| = \lambda, N_0 \text{ is saturated, } N_\delta = M^*, \text{ and } (N_{i+1}, c)_{c \in N_i} \text{ is saturated} \right\}$$

For $\mu > \aleph_0$,

$$K^\mu_\delta = \left\{ \bar{A} | \bar{A} = \langle A_i \mid i \leq \delta \rangle, A_i \text{ is increasing continuous, } |A_\delta| < \mu, \text{ acl } A_i = A_i \right\}$$

If $\bar{A} \in K^\delta_\lambda$, then $f \in \text{Aut}(\bar{A})$ if $f$ is an elementary permutation of $A_\delta$ and if $i \leq \delta$, then $f \upharpoonright A_i$ is a permutation of $A_i$.

**Definition 2.2** Let $\bar{A}^0, \bar{A}^1 \in K^\mu_\delta$. Then $\bar{A}^0 \leq \bar{A}^1$ iff $\bigwedge_{i \leq \delta} A^0_i \subseteq A^1_i$ and $i < j \leq \delta \Rightarrow A^1_i \cup A^0_j$.

**Lemma 2.3**

1. $(K^\mu_\delta, \leq)$ is a partial order.

2. Let $\bar{A}^\zeta \in K^\mu_\delta$ for $\zeta < \zeta(*)$ and let $\xi < \zeta \Rightarrow \bar{A}^\xi \leq \bar{A}^\zeta$. If we let $A_i = \bigcup_{\zeta < \zeta(*)} A^\zeta_i$, and $\left| \bigcup_{\zeta < \zeta(*)} A^\zeta_i \right| < \mu$, then

$$\bar{A} = \langle A_i \mid i \leq \delta \rangle \in K^\mu_\delta$$

and for every $\zeta < \zeta(*)$, $\bar{A}^\zeta \leq \bar{A}$.

3. If $\bar{A}^\zeta \leq \bar{A}^\ast$ for $\zeta < \zeta(*)$, and $\bar{A}$ is as above, then $\bar{A} \leq \bar{A}^\ast$

**Proof**

1. By the transitivity of nonforking.

2. By the finite character of forking.

3. By the finite character of forking.
Definition 2.4 Let \( A \subseteq M \), with \( |A| < \kappa_r(T) \) and let \( p \in S(\text{acl} A) \). Then \( \dim(p, M) \) is the minimal cardinality of a maximal independent set of realizations of \( p \) inside \( M \). If \( M \) is \( \kappa^c(T) \)-saturated (\( \kappa^c \)-saturated means \( \kappa \)-saturated if \( \kappa(T) = \aleph_0 \) and \( \kappa(T) \) saturated otherwise) then by [Sh c] III 3.9. \( \dim(p, M) \) is the cardinality of any maximal independent set of realizations of \( p \) inside \( M \).

Lemma 2.5 Let \( |M| = \lambda \) and assume that \( M \) is \( \kappa^c(T) \)-saturated. Then \( M \) is saturated if and only if for every \( A \subseteq M \), with \( |A| < \kappa_r(T) \) and \( p \in S(\text{acl} A) \), \( \dim(p, M) = \lambda \).

Proof. See [Sh c] III 3.10.

Lemma 2.6 Let \( \langle \bar{A}^\alpha \mid \alpha < \lambda \rangle \) be an increasing continuous sequence of elements of \( K^\lambda_\delta^+ \) such that \( \forall \gamma < \delta, \forall A \subseteq \bigcup_{\alpha < \lambda} A^\alpha_\gamma \) if \( |A| < \kappa_r(T) \) and \( p \in S(\text{acl} A) \) then for \( \lambda \) many \( \alpha < \lambda \),

1. \( A^\alpha_\zeta = A^{\alpha+1}_\zeta \) \( \forall \zeta \leq \gamma \)

2. There exists \( a \in A^{\alpha+1}_\gamma \) such that the type of \( a/A^\alpha_\gamma \) is the stationary of \( p \)

then

\[ (N_\gamma \mid \gamma < \delta) \in K^\delta_\delta \]

where \( N_\gamma = \bigcup_{\alpha < \lambda} A^\alpha_\gamma \).

Proof. It is enough to show \( \forall \gamma < \delta \) that \( (N_{\gamma+1}, c)_{c \in N_\gamma} \) is saturated. For this by lemma 2.5 it is enough to show \( \forall A \subseteq N_{\gamma+1} \) such that \( |A| < \kappa_r(T) \) and for every type \( p \in S(\text{acl} A \cup N_\gamma) \),

\[ \dim(p, N_{\gamma+1}) = \lambda \]

By the assumption of the lemma, there exists \( \{a_i \mid i < \lambda\} \) realizations of \( p \upharpoonright \text{acl} A \) and \( \langle A^\alpha_{\gamma+1} \mid i < \lambda \rangle \) such that for each \( i < \lambda \), \( a_i \in A^{\alpha_{i+1}}_{\gamma+1}, A^{\alpha_{i+1}}_{\gamma+1} = A^{\alpha_i}_{\gamma} \), and

\[ a_i \bigcup_{A} A^\alpha_{\gamma+1} \text{ and } a_i A^\alpha_{\gamma+1} \bigcup_{A^\alpha_{\gamma}} N_\gamma \]
which implies

\[ a_i \bigcup_{A^{\alpha+1}} N_\gamma \quad \text{and} \quad a_i \bigcup_{A} N_\gamma \]

Since \( cf \lambda \geq \kappa_r(T) \) without loss of generality \( A \subseteq A^{\alpha_0+1}_\gamma \). We must show the \( \langle a_i \mid i < \lambda \rangle \) are independent over \( N_\gamma \cup A \). By induction on \( i < \lambda \), we show that

\[ \langle a_j \mid j \leq i \rangle \]

are independent over \( A \cup \{ A^{\alpha_j}_\gamma \mid j \leq i \} \). This is enough as

\[ \{a_j \mid j \leq i\} \bigcup_{A \cup \{A^{\alpha_j}_\gamma \mid j \leq i\}} N_\gamma \]

Since \( \langle a_j \mid j < i \rangle \) are independent over \( A \cup \{ A^{\alpha_j}_\gamma \mid j < i \} \), and

\[ \{a_j \mid j < i\} \bigcup_{A \cup \{A^{\alpha_j}_\gamma \mid j < i\}} A^{\alpha_i}_\gamma \]

\( \langle a_j \mid j < i \rangle \) are independent over \( A \cup A^{\alpha_i}_\gamma \). Since \( a_i \bigcup_{A \cup A^{\alpha_i}_\gamma} \) we have

\[ a_i \bigcup_{A \cup A^{\alpha_i}_\gamma} \{a_j \mid j < i\} \]

**Lemma 2.7** Let \( \langle \tilde{N}^\alpha_\gamma \mid \alpha < \delta \rangle \) be an increasing continuous sequence of elements of \( K^{\mu^+}_\delta \) such that \( \bigcup_{\alpha<\delta} N^\alpha_\gamma = M^* \) and for every \( \gamma < \delta \), and \( \alpha < \delta \),

\( (N^{\alpha+1}_{\gamma+1}, c)_{c \in N^{\alpha}_{\gamma+1} \cup N^{\alpha+1}_{\gamma+1}} \)

and

\( (N^{\alpha+1}_0, c)_{c \in N^{\alpha}_0} \)

are saturated of cardinality \( \lambda \). Then

\[ \langle N_\gamma \mid \alpha < \delta \rangle \in K^{\mu^+}_\delta \]

where \( N_\gamma = \bigcup_{\alpha<\delta} N^\alpha_\gamma \).

**Proof** Similar to the proof of the previous lemma.
Lemma 2.8 Let \( \text{cf} \delta \geq \kappa_r(T) + \aleph_1 \). Let \( \bar{M} \in K^*_\delta \). Let \( A_\delta \subseteq M^* \) such that \( |A_\delta| < \lambda \) and \( A_\delta \subseteq \bigcup_{i < \delta} A_i \) where \( \langle A_i \mid i < \delta \rangle \) is an increasing continuous chain. Suppose \( \forall \beta < \delta, \) and \( \forall i < \delta, \)

\[
M_\beta \bigcup_{A_i} A_i \\
A_i \cap M_\beta
\]

Let \( a \subseteq M_\beta \) such that \( |a| < \kappa_r(T) \). Then there exists a continuous increasing sequence \( \langle A'_i \mid i < \delta \rangle \) and a set \( B \) such that \( |B| < \kappa_r(T) \), \( A_i \subseteq A'_i \), \( a \subseteq \bigcup A'_i = A'_\delta \), \( |A'_\delta| < \lambda \), for some non-limit \( i^* < \delta \), \( A'_i = A_i \) if \( i < i^* \), and \( A'_i = A_i \cup B \) if \( i^* \leq i \) and \( \forall i, \beta < \delta, \)

\[
M_\beta \bigcup_{A'_i \cap M_\beta} A'_i
\]

and \( \forall i, \beta < \delta, \)

\[
M_\beta \cup (M_{\beta+1} \cap A_\delta) \bigcup_{A'_i \cap M_\beta} A'_i \cap M_{\beta+1}
\]

and

\[
A_\delta \bigcup_{A'_i} A'_i
\]

PROOF First by induction on \( n \in \omega \), we define \( \langle B_n \mid n < \omega \rangle \) such that \( B_0 = a, \) \( |B_n| < \kappa_r(T) \) and \( \forall i < \delta, \forall \beta < \delta, \)

\[
B_n \bigcup_{(M_\beta \cap (A_i \cup B_{n+1})) \cup A_i} M_\beta \cup A_i
\]

So suppose \( B_n \) has been defined. By induction on \( m < \omega \) we define subsets \( C_1 \) and \( C_2 \) of \( \delta \) such that \( 0 \in C_i, \) \( |C_i| \leq \kappa_r(T) \) and such that if \( (a_1, b_1), (a_2, b_1), (a_1, b_2), (a_2, b_2) \) are four neighboring points in \( C_1 \times C_2 \) with \( a_1 < a_2 \) and \( b_1 < b_2 \), then for all \( i, j \) such that \( a_1 \leq i < a_2 \) and \( b_1 \leq j < b_2 \)

\[
B_n \bigcup_{M_{a_1+i} \cup A_{b_1+j}} M_{a_1+i} \cup A_{b_1+j}
\]

So it is enough to find \( |B_{n+1}| < \kappa_r(T) \) such that for every \( (a, b) \in C_1 \times C_2, \)

\[
B_n \bigcup_{(M_a \cap (A_b \cup B_{n+1})) \cup A_b} M_a \cup A_b
\]
As $|C_1 \times C_2| < \kappa_r(T)$ this is possible. Let $B = \bigcup_{n \in \omega} B_n$. (If $\kappa_r(T) = \aleph_0$ then without loss of generality we can define the $B_n$ such that for some $k < \omega$, $\bigcup_{n \in \omega} B_n = \bigcup_{n \in k} B_n$.) It is enough to prove the following statement.

There exists a non-limit $i^* < \delta$ such that if $A'_i = A_i$ for $i < i^*$, and $A'_i = A_i \cup B$ for $i \geq i^*$ then the conditions of the theorem hold.

**Proof** \:

For all $\beta < \delta$, if $A'_i = A_i \cup B$, then since $B \bigcup |M_{\beta} \cap (A_i \cup B)) \cup A_i$

we have $A'_i \bigcup M_{\beta}$

Let $i^{**} < \delta$ such that for all $i \geq i^{**}$, $A_i \bigcup A'_i$

It is enough to find $i^{**} \leq i^* < \delta$ such that \forall $\beta < \delta$,

$B \bigcup \bigcap_{\beta} M_{\beta} \cup (M_{\beta+1} \cap A_i)$

Let $\langle \beta_\alpha | \alpha \in \gamma \rangle$ where $\gamma < \kappa_r(T)$ be the set of all places such that $B \bigcup \bigcap_{\beta} M_{\beta} \cup (M_{\beta+1} \cap A_i)$

For each $\beta \in \langle \beta_\alpha | \alpha \in \gamma \rangle$ let $i_\alpha$ be such that $B \bigcup \bigcap_{\beta} M_{\beta} \cup (M_{\beta+1} \cap A_i)$

Let $i_\gamma$ be such that $B \bigcup \bigcap_{\beta} M_{\beta} \cup (M_{\beta+1} \cap A_i)$

Let $i^* = \sup\{i_\alpha | \alpha \in \gamma + 1\} + 1 + i^{**}$. As $|B| < \kappa_r(T)$ and $cf \delta \geq \kappa_r(T)$, $i^* < \delta$, so there is no problem.
Lemma 2.9 Let $\bar{M} \in K_\delta^s$. Let $A \subseteq M^*$ such that $|A| < \lambda$ and $A = \bigcup_{i<\delta} A_i$ where $\langle A_i \mid i < \delta \rangle$ is increasing continuous, each $A_i$ is algebraically closed and $\forall i < \delta, \forall \beta < \delta$,

$$M_\beta \bigcup_{A_i \cap M_\beta} A_i$$

Let $i^*$ be a successor $< \delta$, $\beta^* < \delta$, $\beta^*$ a successor, and let $p \in S(A_i \cap M_{\beta^*})$. (Or even $a < \lambda$ type over $A_i \cap M_{\beta^*}$.) Let $p' \in S((A_i \cap M_{\beta^*}) \cup M_{\beta^*+1})$ such that $p'$ does not fork over $p$. Then there exists an $a \in M_{\beta^*}$ such that $a$ realizes $p'$,

$$A \bigcup_{M_{\beta^*} \cap A_i} a$$

and if $A'_i = A_i \cup \{a\}$ for $i \geq i^*$ and $A'_i = A_i$ for $i < i^*$, then $\forall \beta < \delta, \forall i < \delta$,

$$M_\beta \bigcup_{A'_i \cap M_\beta} A'_i$$

PROOF Let $B \subseteq M_{\beta^*}$ such that $|B| < \lambda$, $A_i^* \cap M_{\beta^*} \subseteq B$, and

$$M_{\beta^*} \bigcup_{M_{\beta^*+1} \cup B} A$$

Let $a \in M_{\beta^*}$ such that $a$ realizes $p$ and

$$a \bigcup_{A_i^* \cap M_{\beta^*}} B \cup M_{\beta^*+1}$$

Since

$$M_{\beta^*} \bigcup_{A \cap M_{\beta^*+1} \cup B} A$$

we have

$$a \bigcup_{M_{\beta^*+1} \cup B} A$$

which implies

$$a \bigcup_{A_i^* \cap M_{\beta^*}} M_{\beta^*+1} \cup A$$

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Since for all $i \geq i^*$,
\[ a \bigcup_{A_i} M_{\beta^* - 1} \cup A \]
we have for all $\gamma < \beta^*$,
\[ a \bigcup_{A_i} M_{\gamma} \cup A \]
which implies
\[ a \cup A_i \bigcup_{A_i \cap M_{\gamma}} M_{\gamma} \]

Since $a \subseteq M_{\beta^*}$ we also have $\forall \gamma \geq \beta^*$,
\[ a \cup A_i \bigcup_{(\alpha \cup A_i) \cap M_{\gamma}} M_{\gamma} \]

**Lemma 2.10** Let $\bar{M} \in K^\delta$. Let $A \subseteq M^\ast$ such that $|A| < \lambda$ and $A = \bigcup_{i < \delta} A_i$ where $\langle A_i \mid i < \delta \rangle$ is increasing continuous, each $A_i$ is algebraically closed and $\forall i < \delta, \forall \beta < \delta$,
\[ M_{\beta} \bigcup_{M_{\beta} \cap A_i} A_i \]
Let $i^* < \delta$, $\beta^* < \delta$, $i^*$ successors, and let $p \in S(A_i \cap M_{\beta})$. Let $p' \in S((A_i \cap M_{\beta^*}) \cup M_{\beta^* - 1})$ such that $p'$ does not fork over $p$. Let $f \in \text{Aut}(A)$ such that $\forall i < \delta, f[A_i] = A_i$. Then there exists $\{a_i \mid i \in \mathbb{Z}\} \subseteq M^\ast$ and an extension $f'$ of $f$ with domain $A \cup \{a_i \mid i \in \mathbb{Z}\}$ such that $a_0$ realizes $p'$, $a_0 \in M_{\beta^*}$, and $\forall i \in \mathbb{Z} \cup \mathbb{Y}(\mathbb{Z}) = \mathbb{Z}_{2 + \lambda}$ and if $A'_{i} = A_i \cup \{a_i \mid i \in \mathbb{Z}\}$ for $i \geq i^*$ and $A'_{i} = A_i$ for $i < i^*$, then for all $\beta < \delta$,
\[ M_{\beta} \bigcup_{M_{\beta} \cap A'_i} A'_i \]
\[ A_{\delta} \bigcup_{A_i} A'_i \]
and
\[ M_{\beta - 1} \cup (M_{\beta} \cap A) \bigcup_{M_{\beta - 1} \cup (M_{\beta} \cap A_i)} M_{\beta} \cap A'_i \]
PROOF We define \( \{a_i \mid i \in -n, \ldots, 0, \ldots, n\} \) by induction on \( n \) such that if \( A'_i = acl(A_i \cup \{a_i \mid i \in -n, \ldots, 0, \ldots, n\}) \) if \( i \geq i^* \) and \( A'_i = A_i \) if \( i < i^* \), then \( \forall i < \delta, \forall \beta < \delta \),

\[
M_\beta \bigcup_{M_\beta \cap A'_i} A'_i \\
A_\delta \bigcup_{A_i} A'_i
\]

and

\[
M_{\beta-1} \cup (M_\beta \cap A) \bigcup_{M_{\beta-1} \cup (M_\beta \cap A)} M_\beta \cap A'_i
\]

and \( f_n = f \cup \{(a_i, a_{i+1}) \mid -n \leq i < n\} \) is an elementary map. In addition we define a sequence of successor ordinals \( \langle \beta_i \mid i \in \mathbb{Z} \rangle \) such that \( \beta_i < \beta_j \) if \( |i| < |j| \), and \( \beta_n < \beta_{-n} \) such that

\[
a_{n+1} \bigcup M_{\beta_{n+1}-1} \cup A \cup \{a_n, \ldots, a_0, \ldots, a_n\}
\]

and

\[
a_{-(n+1)} \bigcup M_{\beta_{-(n+1)}} \cup A \cup \{a_n, \ldots, a_0, \ldots, a_n, a_{n+1}\}
\]

Define \( a_0 \) as in the previous lemma. Suppose that \( \{a_{-n}, \ldots, a_0, \ldots, a_n\} \) and \( \beta_i \) for \( -n \leq i \leq n \) have been defined satisfying the conditions. Let \( C = acl \ C \) such that for some \( B \subseteq C \) with \( |B| < \kappa_r(T) \), \( aclB = C \), \( C \subseteq M_{\beta_{-n}} \cap A_i^* \) and

\[
a_n \bigcup C \cup \{a_{-n}, \ldots, a_0, \ldots, a_{n-1}\}
\]

Let \( \beta_{n+1} > \beta_{-n} \) be a successor such that \( f(C) \subseteq M_{\beta_{n+1}} \cap A_i^* \). Let \( a_{n+1} \in M_{\beta_{n+1}} \) realize

\[
f_n\left(tp(a_n/A \cup \{a_{-n}, \ldots, a_0, \ldots, a_{n-1}\})\right)
\]

and in addition

\[
a_{n+1} \bigcup M_{\beta_{n+1}} \cap A_i^* \cup M_{\beta_{n+1}-1}
\]

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Similarly for \( a_{-(n+1)} \). Now as in the proof of the previous lemma, all the conditions of the induction hold.

**Lemma 2.11** Let \( \delta \) be an ordinal less than \( \lambda^+ \) such that \( \text{cf} \delta \geq \aleph_1 + \kappa_r(T) \). Let \( f \in \text{Aut}_E(M^*) \) with \( |E| < \lambda \). Let \( M \in K_\delta^1 \). Then there exists \( N^1, N^2 \in K_\delta^2 \), \( f_1 \in \text{Aut}_E(N^1) \), \( f_2 \in \text{Aut}_E(N^2) \) with \( E \subseteq N^1_0 \), \( E \subseteq N^2_0 \) such that

1. \( f = f_2 f_1 \)
2. \( \forall i, \beta < \delta, \forall l \in \{0, 1\}, \quad M_\beta \bigcup_{M_\beta \cap N^l_1} N^l_1 \)
3. \( \forall i, \beta < \delta, \forall l \in \{0, 1\}, \quad (N^l_{i+1} \cap M_{\beta+1}, c)_{c \in (N^l_{i+1} \cap M_{\beta}) \cup (N^l_1 \cap M_{\beta+1})} \)
   is saturated of cardinality \( \lambda \)
4. \( (N^l_{i+1} \cap M_0, c)_{c \in N^l_1 \cap M_0} \) is saturated of cardinality \( \lambda \)

**Proof** Without loss of generality \( E = \emptyset \). By induction on \( \alpha < \lambda \) we build increasing continuous sequences \( \langle A^\alpha_i \mid i \leq \delta \rangle \), \( \langle B^\alpha_i \mid i \leq \delta \rangle \), \( \langle f^\alpha_1 \mid \alpha < \lambda \rangle \), \( \langle f^\alpha_2 \mid \alpha < \lambda \rangle \) such that

1. \( M^* = \bigcup_{\alpha < \lambda} A^\alpha_0 = \bigcup_{\alpha < \lambda} B^\alpha_0 \)
2. \( N^1_i = \bigcup_{\alpha < \lambda} A^\alpha_i \quad N^2_i = \bigcup_{\alpha < \lambda} B^\alpha_i \)
3. \( f^\alpha_1 \in \text{Aut}(A^\alpha_0) \) such that \( f^\alpha_1[A^\alpha_0] = A^\alpha_i \)
4. \( f^\alpha_2 \in \text{Aut}(B^\alpha_0) \) such that \( f^\alpha_2[B^\alpha_0] = B^\alpha_i \)
5. \( f[A^\alpha_0] = A^\alpha_i, \quad f[B^\alpha_0] = B^\alpha_i \)
6. \( |A^\alpha_0| < |\alpha|^+ + \kappa_r(T) + \aleph_1 \)
7. \( |B^\alpha_0| < |\alpha|^+ + \kappa_r(T) + \aleph_1 \)
8. \( A^\alpha_0 = B^\alpha_0 \)
9. $f_\alpha^2 f_\alpha^1 = f \upharpoonright A_\delta^\alpha$

10. $\forall \beta < \delta, \forall i < \delta, \forall \alpha < \lambda,$
    
    $$M_\beta \bigcup_{M_\beta \cap A_i^\alpha} A_i^\alpha$$

11. $\forall \beta < \delta, \forall i < \delta, \forall \alpha < \lambda,$
    
    $$M_\beta \bigcup_{M_\beta \cap B_i^\alpha} B_i^\alpha$$

12. $\forall i, \beta < \delta, \forall l \in \{0, 1\},$
    
    $$(N_i^l \cap M_{\beta+1}, c)_{c \in (N_i^l \cap M_\beta) \cup (N_i^l \cap M_{\beta+1})}$$
    
    is saturated of cardinality $\lambda$

13. $(N_i^l \cap M_0, c)_{c \in N_i^l \cap M_0}$ is saturated of cardinality $\lambda$

14. $\forall i < \delta, \forall \alpha < \lambda,$
    
    $$A_\delta^\alpha \bigcup_{A_i^\alpha} A_i^{\alpha+1}$$

15. $\forall i < \delta, \forall \alpha < \lambda,$
    
    $$B_\delta^\alpha \bigcup_{B_i^\alpha} B_i^{\alpha+1}$$

16. $\forall \beta < \delta, \forall i < \delta, \forall \alpha < \lambda,$
    
    $$M_\beta \cup (M_{\beta+1} \cap A_\delta^\alpha) \bigcup_{M_\beta \cup (M_{\beta+1} \cap A_i^\alpha)} M_{\beta+1} \cap A_i^{\alpha+1}$$

17. $\forall \beta < \delta, \forall i < \delta, \forall \alpha < \lambda,$
    
    $$M_\beta \cup (M_{\beta+1} \cap B_\delta^\alpha) \bigcup_{M_\beta \cup (M_{\beta+1} \cap B_i^\alpha)} M_{\beta+1} \cap B_i^{\alpha+1}$$
At limit stages we take unions. Let $\alpha$ be even. Let $M^* = \langle m_\alpha \mid \alpha < \lambda \rangle$. 
In the induction we define $\langle p_\alpha \mid \alpha$ is even and $\alpha < \lambda \rangle$ such that each $p_\alpha \in S((M_{\beta+1} \cap A^\alpha_{i+1}) \cup M_\beta)$ for some $i, \beta < \delta$ and such that $\forall i < \delta, \forall \beta < \delta, \forall A \subseteq M^*$ such that $|A| < \kappa(T)$, $\forall p \in S(acl A)$ there exists $\lambda$ many $p_\alpha \in S((M_{\beta+1} \cap A^\alpha_{i+1}) \cup M_\beta)$, $p_\alpha$ is a nonforking extension of $p$, $p_\alpha$ is realized in $A^\alpha_{i+1} \cap M_{\beta+1}$, and $\forall j \leq i, A^\alpha_j = A^\alpha_{j+1}$. By the proof of lemma 2.6 this insures 12. and 13. holds for $l = 1$ when we finish our construction. So let $i^*, \beta^* < \delta$ such that $p_\alpha \in S((M_{\beta+1} \cap A^\alpha_{i+1}) \cup M_\beta)$. By lemma 2.10 we can find an extensions $(A^\alpha_i)'$ of $A^\alpha_i$ with $(A^\alpha_i)' = A^\alpha_i$ for $i \leq i^*$ and extension $f'_1$ of $f_1$ such that $f'_1[(A^\alpha_i)'] = (A^\alpha_i)'$, $p_\alpha$ is realized in $M_{\beta+1} \cap (A^\alpha_{i+1})'$ and $\forall \beta < \delta, \forall i < \delta$, 

$$\text{if } |M_{\beta-1} \cup (M_\beta \cap A^\alpha_i) | = |M_{\beta-1} \cup (M_\beta \cap A^\alpha_i) |$$

$$\text{then } |M_{\beta-1} \cup (M_\beta \cap A^\alpha_i) |$$

and 

$$\text{if } |M_{\beta} \cup (M_\beta \cap (A^\alpha_i)') | = |M_{\beta} \cup (M_\beta \cap (A^\alpha_i)') |$$

and 

$$\text{if } |M_{\beta} \cup (M_\beta \cap A^\alpha_i) | = |M_{\beta} \cup (M_\beta \cap A^\alpha_i) |$$

Let $F'_1$ be an extension of $f'_1$ to an automorphism of $M^*$. By iterating $\omega$ times the procedure in the proof of lemma 2.8 we can find $D \subseteq M^*$ such that $|D| < \kappa(T) + \omega_1$, if $m$ is the least element of $\langle m_\alpha \mid \alpha < \lambda \rangle$ then $m \in D$, $D$ is closed under $f, f^{-1}, F'_1, (F'_1)^{-1}$ and for some $i^*, i^{**} < \delta$ if $A^\alpha_{i+1} = (A^\alpha_i)' \cup D$, for $i \geq i^*$ and $(A^\alpha_i)'$ for $i < i^*$ and if $B^\alpha_{i+1} = B^\alpha_i \cup D$, for $i \geq i^{**}$ and $B^\alpha_i$ for $i < i^{**}$ then 

$$\text{if } |M_{\beta} \cup (M_\beta \cap A^\alpha_i) | = |M_{\beta} \cup (M_\beta \cap A^\alpha_i) |$$

$$\text{then } |M_{\beta} \cup (M_\beta \cap A^\alpha_i) |$$

and 

$$\text{if } |M_{\beta} \cup (M_\beta \cap A^\alpha_i) | = |M_{\beta} \cup (M_\beta \cap A^\alpha_i) |$$

and 

$$\text{if } |M_{\beta} \cup (M_\beta \cap A^\alpha_i) | = |M_{\beta} \cup (M_\beta \cap A^\alpha_i) |$$
and
\[ M_\beta \cup (M_{\beta+1} \cap B_\delta^\alpha) \bigcup_{i=1}^{M_{\beta+1} \cap B_i^\alpha} \bigcup_{i=1}^{B_\delta^\alpha} B_i^{\alpha+1} \]

and
\[ M_\beta \cup B_\delta^\alpha \bigcup_{i=1}^{B_i^\alpha} B_i^{\alpha+1} \]

Similarly for $\alpha$ odd. Let $f_{\alpha+1}^1 = F_1 \upharpoonright A_\alpha^{\alpha+1}$ and $f_{\alpha+1}^2 = f(f_{\alpha+1}^1)^{-1}$.

3 The proof of the small index property

**Definition 3.1** Let $\delta$ be a limit ordinal and let $\bar{N} \in K_\delta^s$. Then $f \in Aut^*(\bar{N})$ if and only if $f \in Aut(M^*)$ and for some $n \in \omega$, $f[N_\alpha] = N_\alpha$ for every $\alpha$ such that $n \leq \alpha \leq \delta$. $Aut_A^*(\bar{N}) = \{ f \in Aut^*(\bar{N}) \mid f \upharpoonright A = id_A \}$.

**Definition 3.2** Let $\delta$ be a limit ordinal and let $\bar{N} \in K_\delta^s$. Let $B \subseteq N_0$ as in the above definition. If for every $f \in Aut(M^*)$

\[ (f \in Aut^*(\bar{N}) \land f \upharpoonright B = id_B) \Rightarrow f \in G \]

then we define

\[ E = \{ C \subseteq B \mid f \in Aut^*(\bar{N}) \land f \upharpoonright C = id_C \Rightarrow f \in G \} \]

**Lemma 3.3** Let $\delta$ be a limit ordinal and let $\bar{N} \in K_\delta^s$. Let $B \subseteq N_0$ such that $(N_0, c)_{c \in B}$ is saturated. Let $C = acl C$, $C \subseteq B$, and $g$ an elementary map with $\text{dom} g = B$, $g \upharpoonright C = id_C$, $(N_0, c)_{c \in B \cup g[B]}$ is saturated, and

\[ B \bigcup_{C} g(B) \]

Then the following are equivalent.

1. $C \in E$
2. All extensions of $g$ in $\text{Aut}^*(\bar{N})$ are in $G$

3. Some extension of $g$ in $\text{Aut}^*(\bar{N})$ is in $G$

**Proof**  
1. $\Rightarrow$ 2. is trivial.

2. $\Rightarrow$ 3. We just need to prove $g$ has some extension in $\text{Aut}^*(\bar{N})$. But this follows easily by the saturation for every $j < \delta$ of $(N_{j+1}, c)_{c \in N_{j'}}$.

3. $\Rightarrow$ 1. Let $f \in \text{Aut}^*(\bar{N})$ such that $f \upharpoonright C = \text{id}_C$. Let $n \in \omega$ and $g^* \in \text{Aut}^*(\bar{N})$ such that $g^* \supseteq g$, $f, g^* \in \text{Aut}(\bar{N} \upharpoonright [n, \delta])$, and $g^* \in G$. Let $B' \subseteq N_{n+1}$ such that $B' \bigcup N_n$ and $\text{tp}(B'/C) = \text{tp}(B/C)$. Let $g_1 \in \text{Aut}(\bar{N} \upharpoonright [n+2, \delta])$ such that $g_1$ maps $g(B)$ onto $B'$ and $g_1 \upharpoonright B = \text{id}_B$. Since $g_1 \upharpoonright B = \text{id}_B$, $g_1 \in G$. Let $g_2 = g_1g^*(g_1)^{-1}$. Again $g_2 \in G$, $g_2 \upharpoonright C = \text{id}_C$, and $g_2[B] = B'$. As

$$B' \bigcup N_n$$

$f \in \text{Aut}(\bar{N} \upharpoonright [n, \delta])$ and $f \upharpoonright C = \text{id}_C$, clearly

$$f(B') \bigcup N_n$$

Therefore there exists $g_3 \in \text{Aut}(\bar{N} \upharpoonright [n+2, \delta])$ such that $g_3 \upharpoonright B' = f \upharpoonright B'$ and $g_3 \upharpoonright N_n = \text{id}_{N_n}$, hence $g_3 \in G$. $(g_3)^{-1}f \upharpoonright B' = \text{id}_{B'}$ so $(g_2)^{-1}(g_3)^{-1}fg_2 = \text{id}_B$ hence $(g_2)^{-1}(g_3)^{-1}fg_2 \in G$. But this implies $f \in G$.

**Theorem 3.4** Let $|T| < \lambda$. Let $\bar{M} \in K^*$. Let $G \subseteq \text{Aut}^*(\bar{M})$. If

$$f \in \text{Aut}^*_{M_0}(\bar{M}) \Rightarrow f \in G$$

but for no $C \subseteq M_0$ with $|C| < \lambda$ does

$$f \in \text{Aut}^*_{C}(\bar{M}) \Rightarrow f \in G$$

then

$$[\text{Aut}(M^*) : G] > \lambda$$

**Proof** Suppose not. Let $(h_i \mid i < \lambda)$ be a list of the representatives of the left $G$ cosets of $\text{Aut}(\bar{M} \upharpoonright [1, \delta])$ possibly with repetition. Let $\lambda = \bigcup_{\zeta < \sigma \lambda} \lambda_{\zeta}$
with \( \langle \lambda_{\zeta} \mid \zeta < cf \lambda \rangle \) increasing continuous and \( |T| \leq |\lambda_0| \leq |\lambda_{\zeta}| < \lambda \). Let \( M_0 = \bigcup_{\zeta < cf \lambda} M^0_{\zeta} \) and \( M_1 = \bigcup_{\zeta < cf \lambda} M^1_{\zeta} \) with each being a continuous chain such that \( |M^i_{\zeta}| \leq |\lambda_{\zeta}| \).

Now we define by induction on \( \zeta < cf \lambda \), \( N_{0,\zeta}, N_{1,\zeta}, f_{\zeta}, B_{\zeta}, \) and \( h_{j,\zeta} \) for \( j < \lambda_{\zeta} \) such that

1. \( f_{\zeta} \) is an automorphism of \( N_{1,\zeta} \)
2. \( \langle f_{\zeta} \mid \zeta < cf \lambda \rangle \) is increasing continuous
3. If \( j < \lambda_{\zeta} \) and there is an \( h \in Aut(\bar{M} \upharpoonright [1, \delta)) \) such that
   (a) \( h \) extends \( f_{\zeta} \)
   (b) \( hG = h_jG \)
   then \( h_{j,\zeta} \) satisfies a. and b.
4. \( B_{\zeta} \) is a subset of \( N_{1,\zeta} \) of cardinality \( \leq |\lambda_{\zeta}| \)
5. \( M^1_{\zeta} \subseteq B_{\zeta} \)
6. \( N_{0,\zeta} \subseteq B_{\zeta+1} \) and \( B_{\zeta+1} \) is closed under \( h_{j,\epsilon} \) and \( h_{j,\epsilon}^{-1} \) for \( j < \lambda_{\epsilon} \) and \( \epsilon \leq \zeta \)
7. \( f_{\zeta+1}^{-1}(B_{\zeta+1}) \bigcup_{N_{0,\zeta}} N_{0,\zeta+1} \)
8. \( N_{1,\zeta} \bigcup_{N_{0,\zeta}} M_0 \)
9. \( M_1 = \bigcup_{\zeta < cf \lambda} N_{1,\zeta} \)
   \( M_0 = \bigcup_{\zeta < cf \lambda} N_{0,\zeta} \)
10. \( |N_{0,\zeta}| \leq |\lambda_{\zeta}| \)
11. \( (N_{1,\zeta+1}, c)_{c \in N_{1,\zeta}} \) is saturated of cardinality \( \lambda \)
12. \( (M_1, c)_{c \in M_0 \cup N_{1,\zeta}} \) is saturated of cardinality \( \lambda \)

For \( \zeta = 0 \) let \( B_0 \) be empty, let \( N_{0,0} \) be a submodel of \( M_0 \) of cardinality \( |\lambda_0| \), let \( N_{1,0} \) be a saturated submodel of \( M_1 \) of cardinality \( \lambda \) such that \( N_{1,0} \bigcup_{N_{0,0}} M_0 \) and let \( f_{\zeta} = id_{N_{1,0}} \). At limit stages take
unions. If $\zeta = \epsilon + 1$, let $B_\zeta$ be as in 4, 5, 6. Let $N_{0, \zeta} \subseteq M_0$ such that $B_\zeta \cup M_0$, $N_{0, \epsilon} \subseteq N_{0, \zeta}$, $M_0^0 \subseteq N_{0, \zeta}$, $|N_{0, \zeta}| \leq \lambda_\zeta$. Let $N_{1, \zeta} \subseteq M_1$ such that $B_\zeta \subseteq N_{1, \zeta}$, $N_{1, \zeta} \cup M_0$, $(N_{1, \zeta}, c)_{c \in N_{1, \epsilon}}$ is saturated of cardinality $\lambda$, and $(M_1, c)_{c \in M_0 \cup N_{1, \zeta}}$ is saturated of cardinality $\lambda$. Let $f_\zeta$ be an extension of $f_\epsilon|N_{1, \epsilon}$ to an automorphism of $N_{1, \zeta}$ so that

$$f_\zeta^{-1}(B_\zeta) \bigcup_{N_{1, \epsilon}} N_{0, \zeta}$$

Since

$$N_{0, \zeta} \bigcup_{N_{0, \epsilon}} N_{1, \epsilon}$$

we have

$$f_\zeta^{-1}(B_\zeta) \bigcup_{N_{0, \epsilon}} N_{0, \zeta}$$

Let $f$ be an extension of $\bigcup f_\zeta$ to an element of $Aut(\bar{M} | [1, \delta])$. We have defined $f$ so that

1. (By nonforking calculus) $\forall \zeta < cf \lambda, \forall j < \lambda_\zeta,$

$$f^{-1}h_{j, \zeta}(M_0) \bigcup_{N_{0, \zeta}} M_0$$

2. $f^{-1}h_{j, \zeta} | N_{0, \zeta} = id$

By lemma 3.3 none of the $f^{-1}h_{j, \zeta}$ are in $G$, a contradiction as for some $j < \lambda$, $fG = h_jG$ so for some $\zeta$, $j < \lambda_\zeta$, $h_jG = h_{j, \zeta}G = fG$.

**Lemma 3.5** Let $|T| < \lambda$. Let $cf \delta \geq \kappa_T(T) + \aleph_1$. Suppose $[Aut(M^*) : G] \leq \lambda$ and assume that for no $A \subseteq M^*$ with $|A| < \lambda$ is $Aut_A(M^*) \subseteq G$. Then for some $\bar{N} \in K_\delta^*$,

$$\bigwedge_{\alpha < \delta} Aut_{\bar{N}_\alpha}(\bar{N}) \not\subseteq G$$

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Proof. Suppose not. Let \( \bar{M} \in K_\delta^s \). Then there exists an \( \alpha < \delta \) such that \( \text{Aut}_{M_\alpha}(M) \subseteq G \). Without loss of generality \( \alpha = 0 \). By lemma 3.4 there exists \( E \subseteq M_0 \) such that \( |E| < \lambda \) and \( \text{Aut}_E(\bar{M}) \subseteq G \). Let \( f \in \text{Aut}_E(M^*) \setminus G \). By lemma 2.11 we can find \( \bar{N}^1, \bar{N}^2 \in K_\delta^s \) and automorphisms \( f_1 \in \text{Aut}_E(\bar{N}^1) \) and \( f_2 \in \text{Aut}_E(\bar{N}^2) \) such that

1. \( E \subseteq N_1^0, E \subseteq N_0^2 \)
2. \( f = f_2 f_1 \)
3. \( f_1 \upharpoonright E = f_2 \upharpoonright E = \text{id}_E \)
4. \( \forall \alpha, \beta < \delta, \)
   a. \( N_1^1 \cup M_\beta \)
   b. \( N_2^2 \cup M_\beta \)
   c. \( (N_{\alpha+1}^1 \cap M_{\beta+1}, c)_{c \in (N_{\alpha+1}^1 \cap M_{\beta+1}) \cup (N_{\alpha+1}^2 \cap M_{\beta+1})} \) is saturated of cardinality \( \lambda \)
   d. \( (N_{\alpha+1}^2 \cap M_{\beta+1}, c)_{c \in (N_{\alpha+1}^2 \cap M_{\beta+1}) \cup (N_{\alpha+1}^2 \cap M_{\beta+1})} \) is saturated of cardinality \( \lambda \)
   e. \( (N_{\alpha+1}^1 \cap M_0)_{c \in N_{\alpha+1}^1 \cap M_0} \) is saturated of cardinality \( \lambda \)
   f. \( (N_{\alpha+1}^2 \cap M_0)_{c \in N_{\alpha+1}^2 \cap M_0} \) is saturated of cardinality \( \lambda \)

Since \( f \notin G \) we can assume without loss of generality that \( f_1 \notin G \). Also, by the hypothesis of suppose not we can assume there is a \( F \subseteq N_0^1 \) such that \( (N_0^1, c)_{c \in F} \) is saturated and \( \text{Aut}_F(\bar{N}^1) \subseteq G \). By lemma 3.4 we can assume that \( |F| < \lambda \) and without loss of generality \( E \subseteq F \). Let for \( \alpha < \delta \),

\[ F_\alpha = F \cap M_\alpha \]

By the lemma 3.4 we can find a sequence \( \langle F'_\alpha \mid \alpha < \delta \rangle \) such that for each \( \alpha \), \( F_\alpha \subseteq F'_\alpha \) with \( |F'_\alpha| < \lambda \) and for each \( \beta < \alpha \) \( F'_\alpha \cap M_\beta = F'_\beta \) and if \( F' = \bigcup_{\alpha < \delta} F'_\alpha \) then

\[ M_\alpha \cap N_0^1 \bigcup_{F'_\alpha} F' \]

We define by induction on \( \alpha < \delta \) a map \( g_\alpha \) an automorphism of \( M_\alpha \cap N_0^1 \) such that
1. \( \forall \beta, \alpha < \delta, \beta < \alpha \Rightarrow g_{\beta} \subseteq g_{\alpha} \)

2. If \( \alpha \) is a limit then \( g_{\alpha} = \bigcup_{\beta < \alpha} g_{\beta} \)

3. \( g_{\alpha}(F'_\alpha) \cup \bigcup_{\beta < \alpha} F'_\alpha \)

4. \( g_{\alpha} \mid E = id_E \)

Let \( \alpha = \beta + 1 \) and suppose \( g_{\beta} \) has been defined. Let \( X \subseteq M_\alpha \cap N_{10}^1 \) such that \( X \backslash g_{\beta}(F'_\beta) \equiv F'_\alpha \backslash F'_\beta \) by \( h_{\beta} \) an extension of \( g_{\beta} \mid F'_\beta \) and

\[
X = \bigcup_{g_{\beta}(F'_\beta)} F'_\alpha \cup (M_\beta \cap N_{10}^1)
\]

Let \( g'_\alpha = g_{\beta} \cup h_{\beta} \). Since \( X = \bigcup_{g_{\beta}(F'_\beta)} g_{\beta}(M_\beta \cap N_{10}^1) \) and \( F'_\alpha \cup M_\beta \cap N_{10}^1 \), \( g'_\alpha \) is an elementary map. Now let \( g_{\alpha} \) be an extension of \( g'_\alpha \) to an automorphism of \( M_\alpha \cap N_{10}^1 \). Let \( g' = \bigcup_{\alpha < \delta} g_{\alpha} \). \( g' \) is an automorphism of \( N_{10}^1 \) such that for every \( \alpha < \delta \),

\[
g'(M_\alpha \cap N_{10}^1) = [M_\alpha \cap N_{10}^1]
\]

By the saturation and independence of the \( N_{10}^1, M_\beta \) we can find an extension \( g \) of \( g' \) such that \( g \in Aut(\bar{N}_1) \) and \( g \in Aut(M) \). This gives a contradiction since \( g(F) \cup E \subseteq F' \) and \( g \in Aut(\bar{N}_1) \) implies \( g \notin G \), but \( g \in Aut(M) \) and \( g \mid E = id_E \) implies \( g \in G \).

**Lemma 3.6** Let \( \bar{M} = (M_\beta \mid \beta \leq \delta) \in K_\delta^s \). Let \( F \subseteq M^* \) with \( |F| < \lambda \). Then there exists a set \( F' \) such that \( |F'| < \lambda \), \( F \subseteq F' \), and \( \forall \beta < \delta \),

\[
* M_\beta \bigcup_{F' \cap M_\beta} F'
\]

**Proof** Let \( w \subseteq F \) be finite. There are less than \( \kappa(T) \) many \( \alpha < \delta \) such that

\[
w \cup_{M_\alpha} M_{\alpha+1}
\]

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Let \( a_w \) be the set of such \( \alpha \). For each \( \alpha \in a_w \) let \( w_\alpha \subseteq M_\alpha \) such that \( |w_\alpha| < \kappa_r(T) \), and

\[
w = \bigcup_{\alpha \in a_w} M_\alpha
\]

Let \( w^1 = \bigcup_{\alpha \in a_w} w_\alpha \). Let \( F^1 = \bigcup_{w \subseteq F} w^1 \) and repeat this procedure \( \omega \) times with \( F^n \) relating to \( F^{n+1} \) as \( F \) is related to \( F^1 \). Let \( F' = \bigcup_{n \in \omega} F^n \). \( F' \) satisfies \( \ast \).

**Lemma 3.7** Let \( Tr \) be a tree of infinite height. Let \( \alpha < \text{height}(Tr) \) and let \( \eta \in Tr \upharpoonright \text{level}(\alpha + 1) \). Let \( \langle M_\beta \mid \beta \leq \alpha \rangle \) be an increasing chain of models such that for all \( \beta < \alpha \), \( (M_{\beta+1}, c_{\beta+1})_{c \in M_\beta} \) is saturated. Let \( M_\alpha \subseteq N_0 \subseteq N_1 \subseteq N_2 \subseteq N_3 \) with \( (N_{i+1}, c_{i+1})_{c \in N_i} \) saturated for \( i \leq 2 \).

 Suppose \( \langle h_\beta \mid \beta \leq \alpha \rangle \) are such that

1. \( h_\beta = id_{M_\beta} \)
2. \( h_\beta[N_i] = N_i \) for \( i \leq 3 \)
3. \( h_\beta[M_\gamma] = M_\gamma \) for \( \gamma \leq \alpha \)

For each \( \nu \in Tr \upharpoonright \text{level}(\alpha + 1) \) let \( m_\nu, l_\nu \) be automorphisms of \( N_0 \). Suppose \( g_\eta \in \text{Aut}(N_0) \) such that for all \( \nu \in Tr \upharpoonright \text{level}(\alpha + 1) \),

\[
g_\eta m_\eta (m_\nu)^{-1}(g_\eta)^{-1} = l_\eta (l_\nu)^{-1} h_\eta \gamma [\gamma, \nu]^\eta < \nu(\gamma, \nu)
\]

Let \( m^+_\nu, l^+_\nu \) be extensions of \( m_\nu \) and \( l_\nu \) to automorphisms of \( N_1 \) for all \( \nu \in Tr \upharpoonright \text{level}(\alpha + 1) \). Then there exists a \( g'_\eta \in \text{Aut}(N_3) \) extending \( g_\eta \) and for all \( \nu \in Tr \upharpoonright \text{level}(\alpha + 1) \) automorphisms of \( N_3 \), \( m'_\nu \) and \( l'_\nu \) extending \( m^+_\nu \) and \( l^+_\nu \) respectively such that

\[
g'_\eta m'_\eta (m'_\nu)^{-1}(g'_\eta)^{-1} = l'_\eta (l'_\nu)^{-1} h'_\eta \gamma [\gamma, \nu]^\eta < \nu(\gamma, \nu)
\]

**Proof** Similar to the proof of lemma 1.8.

**Theorem 3.8** Let \( |T| < \lambda \). Let \( M^* \) be a saturated model of cardinality \( \lambda \), and let \( G \subseteq \text{Aut}(M^*) \). Suppose that for no \( A \subseteq M \) with \( |A| < \lambda \) is \( \text{Aut}_A(M^*) \subseteq G \). Suppose \( Tr \) is a tree of height \( \kappa \), where \( \kappa \) is a regular cardinal \( \geq \kappa_r(T) + \aleph_1 \) such that each level of \( Tr \) is of size at most \( \lambda \), but \( Tr \) having more than \( \lambda \) branches. Then

\[
[\text{Aut}(M^*) : G] > \lambda
\]
PROOF  Suppose not. Then by lemma 3.3 there is a $\bar{N} \in K^*_\lambda \times \kappa$, such that

$$\bigwedge_{\alpha<\lambda\times\kappa} Aut^*_\alpha(\bar{N}) \not\subseteq G$$

By thinning $\bar{N}$ if necessary we can assume for each $\alpha < \kappa$ there exists an automorphism $h_\alpha \in Aut_{N_\alpha}(\bar{N})$ such that $h_\alpha \not\in G$. By induction on $\alpha < \kappa$ for every $\eta \in Tr \upharpoonright level \alpha$ we define automorphisms $g_\eta, m_\eta, l_\eta$ of $N_\alpha$ such that if $\rho \neq \nu$ then $l_\rho \neq l_\nu$ and

$$g_\rho m_\rho (m_\nu)^{-1} (g_\rho)^{-1} = l_\rho (l_\nu)^{-1} h_{\gamma(\rho, \nu)}^{\rho(\gamma(\rho, \nu)) < \nu(\gamma(\rho, \nu))}$$

At limit steps we take unions. If $\alpha = \beta + 1$, for each $i < \lambda$ we define for some $\eta_i \in Tr \upharpoonright level \alpha$, $g_{\eta_i} \in Aut(N_\alpha \times \beta + 3i)$ such that for each $\eta \in Tr \upharpoonright level \alpha$, $\eta = \eta_i$ cofinally many times in $\lambda$, and for every $\nu \in Tr \upharpoonright level \alpha$, $m^{\iota}_\nu \neq l^{\iota}_\nu \in Aut(N_\alpha \times \beta + 3i)$ such that

$$g_{\eta_i} m^{\iota}_\eta (m^{\iota}_\nu)^{-1} (g_{\eta_i})^{-1} = l^{\iota}_{\eta_i} (l^{\iota}_\nu)^{-1} h_{\gamma(\eta_i, \nu)}^{\rho(\gamma(\eta_i, \nu)) < \nu(\gamma(\eta_i, \nu))}$$

The $g_{\eta_i}$, $m^{\iota}_\nu$, $l^{\iota}_\nu$ are easily defined by induction on $i < \lambda$ using lemma 3.7. Then if we let $g_\eta = \bigcup \{ g_{\eta_i} \mid \eta_i = \eta \}$, $m_\eta = \bigcup m^{\iota}_\eta$ and $l_\eta = \bigcup l^{\iota}_\eta$ we have finished. Let $Br$ the set of branches of $Tr$ of height $\kappa$. For $\rho \in Br$ let $g_\rho = \bigcup \{ g_\eta \mid \eta < \rho \}$, $m_\rho = \bigcup \{ m_\eta \mid \eta < \rho \}$, and $l_\rho = \bigcup \{ l_\eta \mid \eta < \rho \}$. If $\rho \neq \nu$, $g_\rho \neq g_\nu$ since without loss of generality $\rho(\gamma(\rho, \nu)) < \nu(\gamma(\rho, \nu))$ and

$$g_\rho m_\rho (m_\rho)^{-1} (g_\rho)^{-1} = l_\rho (l_\rho)^{-1} h_{\gamma(\rho, \nu)}^{\rho(\gamma(\rho, \nu)) < \nu(\gamma(\rho, \nu))}$$

and

$$g_\nu m_\nu (m_\nu)^{-1} (g_\nu)^{-1} = l_\nu (l_\nu)^{-1}$$

implies

$$g_\rho (g_\nu)^{-1} l_\rho (l_\nu)^{-1} g_\nu (g_\rho)^{-1} = l_\rho (l_\nu)^{-1} h_{\gamma(\rho, \nu)}^{\rho(\gamma(\rho, \nu)) < \nu(\gamma(\rho, \nu))}$$

So if $g_\rho = g_\nu$ this would imply $h_{\gamma(\rho, \nu)}^{\rho(\gamma(\rho, \nu)) < \nu(\gamma(\rho, \nu))} = id_{M^*}$ a contradiction. If

$$[Aut(M^*) : G] \leq \lambda$$

then for some $\rho, \nu \in Br$ we must have $l_\rho (l_\nu)^{-1} \in G$ and $g_\rho (g_\nu)^{-1} \in G$, but then we get a contradiction as $g_\rho (g_\nu)^{-1} l_\rho (l_\nu)^{-1} g_\nu (g_\rho)^{-1} \in G$ and $l_\rho (l_\nu)^{-1} \in G$, but $h_{\gamma(\rho, \nu)}^{\rho(\gamma(\rho, \nu)) < \nu(\gamma(\rho, \nu))} \not\in G$. 

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Corollary 3.9 Let $G \subseteq \text{Aut}(M^*)$. Suppose that for no $A \subseteq M$ with $|A| < \lambda$ is $\text{Aut}_A(M^*) \subseteq G$. Suppose $|T| < \lambda$ and $M^*$ does not have the small index property. Then

1. There is no tree of height an uncountable regular cardinal $\kappa$ with at most $\lambda$ nodes, but more than $\lambda$ branches.

2. For some strong limit cardinal $\mu$, $\text{cf} \, \mu = \aleph_0$ and $\mu < \lambda < 2^\mu$.

3. $T$ is superstable.

PROOF

1. By the previous theorem

2. By 1. and [Sh 430, 6.3]

3. If $T$ is stable in $\lambda$, then $\lambda = \lambda^{<\kappa_r(T)}$, so if $\kappa_r(T) > \aleph_0$ we can let $\kappa$ from the previous theorem be the least $\kappa$ such that $\lambda < \lambda^\kappa$. 

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