Un-unzippable Convex Caps

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Abstract
An unzipping of a polyhedron \( P \) is a cut-path through its vertices that unfolds \( P \) to a non-overlapping shape in the plane. It is an open problem to decide if every convex \( P \) has an unzipping. Here we show that there are nearly flat convex caps that have no unzipping. A convex cap is a “top” portion of a convex polyhedron; it has a boundary, i.e., it is not closed by a base.

1 Introduction
We define an unzipping of a polyhedron \( P \) in \( \mathbb{R}^3 \) to be a non-overlapping, single-piece unfolding of the surface to the plane that results from cutting a continuous path \( \gamma \) through all the vertices of \( P \). The cut-path \( \gamma \) need not follow the edges of \( P \), nor even be polygonal, but it must include every vertex of \( P \), passing through \( n - 2 \) vertices and beginning and ending at the other two vertices, where \( n \) is the total number of vertices. If \( \gamma \) does follow edges of \( P \), we call it an edge-unzipping.

Edge-unzippings are special cases of edge-unfoldings, where the cuts follow a tree of edges that span the \( n \) vertices. The interest in edge-unfoldings stems largely from what has become known as Dürer’s problem [DO07] [O’R13]: Does every convex polyhedron have an edge-unfolding? The emphasis here is on a non-overlapping result, what is often called a net for the polyhedron. This question was first formally raised by Shephard in [She75]. In that paper, he already investigated the special case where the cut edges form a Hamiltonian path of the 1-skeleton of \( P \): Hamiltonian unfoldings. These are exactly what I’m calling edge-unzippings. Shephard noted that the rhombic dodecahedron does not have an edge-unzipping because its 1-skeleton has no Hamiltonian path.

The attractive “zipping” terminology stems from the paper [DDL+10], which defined zipper unfoldings to be what I’m shortening to unzippings. They showed that all the Platonic and the Archimedean solids have edge-unzippings. And they posed a fascinating question:

*The title mimics that of the paper [BDE+03]: “Ununfoldable Polyhedra with Convex Faces”

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Open Problem: Does every convex polyhedron have an unzipping?

1.1 Nonconvex Polyhedra

First we note that not every nonconvex polyhedron has an unzipping. This has been a “folk theorem” for years, but has apparently not been explicitly stated in the literature. In any case, it is not difficult to see.

Consider the polyhedron illustrated in Fig. 1. The central vertex $v$ has more than $4\pi$ incident surface angle. In fact, it has well more than $8\pi$ incident angle, but we only need $> 4\pi$. An unzipping cut-path $\gamma$ cannot terminate at $v$, because the neighborhood of $v$ in the unfolding has more than $2\pi$ incident angle, and so would overlap in the planar development. Nor can $\gamma$ pass through $v$, because partitioning the $> 4\pi$ angle would leave more than $2\pi$ to one side or the other, again forcing overlap in the neighborhood of at least one of the two planar images of $v$. Therefore, no polyhedron with a vertex with more than $> 4\pi$ incident angle has an unzipping. Indeed, as Stefan Langerman observed, similar reasoning shows that for any degree $\delta$ there is a polyhedron that cannot be unfolded without overlap by a cut tree of maximum degree $\delta$. The polyhedron in Fig. 1 requires degree $> 4$ at $v$ to partition the more than $8\pi$ angle into $< 2\pi$ pieces.

Figure 1: A polyhedron that cannot be unzipped. Based on Fig. 24.14, p.370 in [DO07].

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1 The closest is [BDF+03], which notes that “the neighborhood of a negative-curvature vertex... requires two or more cuts to avoid self-overlap.”
2 Personal communication, Aug. 2017
1.2 Open Problem: Conjecture

This negative result for nonconvex polyhedra increases the interest in the open problem for convex polyhedra. In [O’R15] I conjectured the answer is no, but it seems far from clear how to settle the problem. For that reason, here we turn to a very special case.

1.3 Convex Caps

The special case is unzipping “convex caps.” I quote the definition from [O’R17b]:

“Let $P$ be a convex polyhedron, and let $\phi(f)$ be the angle the normal to face $f$ makes with the $z$-axis. Let $H$ be a halfspace whose bounding plane is orthogonal to the $z$-axis, and includes points vertically above that plane. Define a convex cap $C$ of angle $\Phi$ to be $C = P \cap H$ for some $P$ and $H$, such that $\phi(f) \leq \Phi$ for all $f$ in $C$. [...] Note that $C$ is not a closed polyhedron; it has no “bottom,” but rather a boundary $\partial C$."

The result of this note is:

**Theorem 1** For any $\Phi > 0$, there is a convex cap $C$ that has no unzipping.

Because this holds for any $\Phi > 0$, there are arbitrarily flat convex caps that cannot be unzipped. ($\Phi$ will not otherwise play a role in the proof.)

2 Proof of Theorem 1

The convex caps used to prove the theorem are all variations on the cap shown in Fig. 2. The base $\partial C = (b_1, b_2, b_3)$ forms a unit side-length equilateral triangle in the $xy$-plane. The three vertices $a_1, a_2, a_3$ are also the corners of an equilateral triangle, lifted a small amount $z_a$ above the base. In projection to the the $xy$-plane, the “apron” of quadrilaterals between $\triangle b_1b_2b_3$ and $\triangle a_1a_2a_3$ has width $\varepsilon > 0$. The vertex $c$, at height $z_c > z_a$, sits over the centroids of the equilateral triangles. The shape of the cap is controlled by three parameters: $\varepsilon, z_a, z_b$. Keeping $\varepsilon$ fixed and varying $z_a$ and $z_c$ permits controlling the curvatures $\omega_a$ at $a_i$ and $\omega_c$ at $c$. In Fig. 2 $\varepsilon = 0.1$ and $z_a, z_c = 0.02, 0.1$ leads to $\omega_a = 1.9^\circ$ and $\omega_c = 5.6^\circ$.

A typical attempt at an unzipping (of a variant of Fig. 2) is shown in Fig. 3

In general we will only display what are labeled $L$ and $R$ in this figure, rather than the full unfolding. From now on we will illustrate cut-paths and unzippings in the plane, starting from Fig. 4 (and not always repeating all the labels).

2.1 Constraints on the cut-path

Any point $p$ in the relative interior of $\gamma$ (i.e., not an endpoint) develops in the plane to two points $p'$ and $p''$, with right and left incident surface angles $\rho = \rho(p)$
Figure 2: A convex cap $C$ that has no unzipping.

Figure 3: An overlapping unfolding of a convex cap (a variant of Fig. 2) from cut-path $\gamma = (c, a_2, a_3, a_1, b_1)$. Compare Fig. 9 ahead.
and $\lambda = \lambda(p)$. If $p$ is not at a vertex of $\mathcal{C}$, then $\lambda + \rho = 2\pi$. If $p$ is at a vertex of curvature $\omega = \omega(p)$, then $\lambda + \rho + \omega = 2\pi$. We will show the development $\mathcal{C}$ as cut by $\gamma$ by drawing two directed paths $R$ and $L$, each determined by the $\rho$ and $\lambda$ angles, which deviate by $\omega(v)$ at each vertex $v \in \gamma$. The surface of $\mathcal{C}$ is right of $R$ and left of $L$ (see Fig. 3), but not explicitly depicted in subsequent figures.

The constraints on $\gamma$ to be an unzipping are:

1. $\gamma$ must be a path, by definition of unzipping.
2. $\gamma$ must start at one of the vertices \{c, a_1, a_2, a_3\} and terminate on $\partial \mathcal{C}$.
3. $\gamma$ does not have to include any of the vertices \{b_1, b_2, b_3\}, it just needs to exit $\mathcal{C}$ at some point of $\partial \mathcal{C}$.
4. $\gamma$ can only touch $\partial \mathcal{C}$ at one point, for if it touches at two or more points, the unfolding would be disconnected into more than one piece.
5. Between vertices, $\gamma$ can follow any path on $\mathcal{C}$, as long as $\gamma$ does not self-cross, which would again result in more than one piece.
6. And of course, the developments of $R$ and $L$ must not cross in the plane, for $R/L$ crossings imply overlap.\textsuperscript{3}

We think of $\gamma$ as directed from its root start vertex to $\partial \mathcal{C}$; the path opens from the root to its boundary exit. The main constraint we exploit is item 4: $\gamma$ can only touch $\partial \mathcal{C}$ at one point. We will see that only by leaving $\mathcal{C}$ and returning could the unzipping avoid overlap.

Due to the symmetry of $\mathcal{C}$—in particular, the equivalence of \{a_1, a_2, a_3\}—there are only four combinatorially distinct possible cut-paths $\gamma$, where we use $b$ to represent any point on $\partial \mathcal{C}$:

\textsuperscript{3}The reverse is not always true: It could be that $R$ and $L$ do not cross, but other portions of the surface away from the cut $\gamma$ are forced to overlap by, for example, large curvature openings.
1. \( \gamma = (c, a_1, a_2, a_3, b) = caaab. \)
2. \( \gamma = (a_1, c, a_2, a_3, b) = acaab. \)
3. \( \gamma = (a_1, a_2, c, a_3, b) = aacab. \)
4. \( \gamma = (a_1, a_2, a_3, c, b) = aaacb. \)

We abbreviate the path structure with strings acaab and so on, with the obvious meaning. It turns out that the location of \( b \), the point at which \( \gamma \) exits \( C \), plays little role in the proof.

We will display the structure of \( \gamma \) and the developments of \( R \) and \( L \) as in Fig. 5. Here \( \gamma \) is shown following straight segments between vertices, and the developments overlap substantially. But as per item 5 above, \( \gamma \) can follow potentially any (non-self-intersecting) curve between vertices. However, the developed images of the vertices are independent of the shape of the path between vertices, a condition we exploit in the proof. So once the combinatorial structure of the cut \( \gamma \) is fixed, the developed locations of the vertex images are determined. We will continue to use \( \{\omega_a, \omega_c\} = \{5^\circ, 10^\circ\} \) for illustration, although any smaller curvatures also work in the proofs.

**2.2 Radial Monotonicity: Intuition**

Before beginning the proof details, we provide the intuition behind it. That intuition depends on the notion of a “radially monotone” curve, a concept used in [O’R16] and [O’R17b]. A directed polygonal chain \( P \) in the plane with vertices \( u_1, u_2, \ldots, u_k \) is radially monotone with respect to \( u_1 \) if the distance from \( u_1 \) to every point \( p \in P \) increases monotonically as \( p \) moves out along the chain. \( P \) is radially monotone if it is radially monotone with respect to each vertex \( u_i \); concentric circles centered on each \( u_i \) are crossed just once by the chain beyond \( u_i \).

If both the \( R \) and \( L \) developments are radially monotone, then \( L \) and \( R \) do not intersect except at their common “root” vertex, a fact proved in the
cited papers. This suggests that \( \gamma \) should be chosen so that \( R \) and \( L \) are radially monotone. However, if \( R \) or \( L \) or both are not radially monotone, they do not necessarily overlap: radial monotonicity is sufficient for non-overlap but not necessary. Nevertheless, striving for radial monotonicity makes sense. The sharp turns necessary to span the vertices of \( C \) (visible in Fig. 5) should be avoided, for they violate radial monotonicity. (Any angle \( \angle u_{i-1}, u_i, u_{i+1} \) smaller than 90° implies non-monotonicity at \( u_i \) with respect to \( u_{i-1} \).) Avoiding these sharp turns forces \( \gamma \) to exit \( C \) before spanning the vertices. Although radial monotonicity is not used in the proofs to follow, it is the intuition behind the proofs.

2.3 Lemmas 1,2,3,4

Of the four possible types of \( \gamma \), \( acaab \) is the “closest” to being unzippable, so we start with this type.

**Lemma 1** For sufficiently small \( \omega_a \), \( \omega_c \), and \( \varepsilon \), any cut-path \( \gamma \) of type \( acaab \) must leave and reenter \( C \) to avoid overlap. Therefore, \( C \) cannot be unzipped with this type of cut-path.

**Proof:** We have already seen in Fig. 5 that straight connections between the vertices leads to overlap. Fig. 6(a) repeats the set-up of that figure, with added notation. Let \( R_1, R_2, R_3 \) be the portions of the right development \( R \) between vertices, and similarly for \( L_i \). We now imagine that \( R_i \) and \( L_i \) are arbitrary cuts between their vertex endpoints. We concentrate on \( R_3 \) and \( L_3 \).

From the fact that the images of the vertices, and in particular, \( a_2 \), are in their correct developed planar locations, we can derive constraints on the shape of the \( R_3 \) and \( L_3 \) paths. The shape of \( R_i \) determines \( L_i \) and vice versa, because for all non-vertex points of \( \gamma \), \( \rho + \lambda = 2\pi \). Thus \( R_i \) and \( L_i \) are congruent as curves, but rigidly rotated differently by the curvatures along \( \gamma \).

There are only two topological possibilities for \( R_3 \) and \( L_3 \) to avoid crossing earlier portions of \( R \) and \( L \), illustrated in Fig. 7. In (a) of the figure, \( R_3 \) passes right of \( a''_2 \) on its way counterclockwise to \( a'_3 \), and in (b), \( L_3 \) passes right of \( a'_2 \) on its way clockwise to \( a''_3 \). The situations are analogous in the neighborhood of \( a_2 \), and we concentrate only on the former more direct route.

Knowing that \( R_3 \) passes to the right of \( a''_2 \) determines the vector displacement of the tightest possible prefix \((a'_2, a''_2)\) of \( R_3 \), but not the shape of that prefix. This vector displacement forms an effective angle \( \angle c', a'_2, a''_2 \) of much larger than the near-30° necessary to stay on the narrow \( \varepsilon \)-apron. Fig. 6(a) and (b) show that this angle is nearly 70°, well beyond 30°. (And the angle is larger if \( R_3 \) passes further to the right of \( a''_2 \) ) This 70° turn implies an effective surface angle \( \rho = 290° \) to the right of \( \gamma \) on \( C \) at \( a_2 \), “effective” because the exact shape of \( \gamma \) is unknown. The exact angle and length of vector displacement depend on \( \{\omega_a, \omega_c\} \), but for any given curvatures, we can choose an \( \varepsilon \) small enough so that the prefix steps \( \gamma \) exterior to \( C \). Thus \( \gamma \) must leave \( C \) to avoid overlap before it

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There are some curvature bound assumptions to this claim that are not relevant here.
completes its tour of the vertices. Although this proves the lemma, we continue the analysis below to reveal a deeper structure.

Figure 6: Analysis of $\gamma$ of type $acaab$. (a) Opening at $a_1$ and $c$ causes $R_3/L_3$ overlap. (b) $R_3$ bends around $a_2'$, (c) $L_3$ complements $R_3$, which again intersects $L_3$. (d) $L_3$ complements $R_3$. $R_3$ is following the arc centered on $x$.

Knowing this constraint just derived on the prefix of $R_3$, we know $L_3$ must complement $R_3$ on that prefix, which leads to Fig. 6(c). Again there is an overlap intersection further along $R_3$. Altering $R_3$ to again bend around $L_3$ leads to Fig. 6(d). Continuing this process of incrementally determining constraints on $R_3$ that are mirrored in $L_3$, leads to the conclusion that $R_3$ must follow (or be outside of) the arc of a circle centered at $x$, where $x$ is the combined center of rotation of the rotations at $a_1$ and at $c$. This is why we can be sure the angle at $a_2$ is well beyond $30^\circ$ independent of $\{\omega_a, \omega_c\}$: it is determined by (an approximation to) the tangent to this circle. Finally, $R_3$ and $L_3$ are opened further by the curvature $\omega_a$ at $a_2$, which does not alter the previous analysis.

Here we pause to discuss the “combined center of rotation” just used. Any pair of rotations about two distinct points is equivalent to a single rotation about a combined center. In our situation, the two rotations are $\omega_a$ about $a_1$ and $\omega_c$ about $c$. For small rotations, they are equivalent to a rotation by $\omega_a + \omega_c$ about the weighted center

$$x = \frac{\omega_a a_1 + \omega_c c}{\omega_a + \omega_c}.$$

This point is indicated in Figs. 6(a) and (d). This result on combining rotations
Figure 7: Possible paths from $a_2$ to $a_3$. (a) $R_3$ skirts $a'''_2$. (b) $L_3$ skirts $a''_2$.

is proved in both [O’R17a] and [BG17] (and likely elsewhere).

The above analysis suggests that $C$ can be unzipped if the apron were large enough to include the circle arc that $\gamma$ must follow from $a_2$ to $a_3$. And indeed Fig. 8 shows that this is true. An interesting consequence of this unzipping is that, even with a small apron, if we close the convex cap $C$ by adding an equilateral triangle base to form a closed convex polyhedron $P$, then $P$ does have an unzipping. Follow the path shown in Fig. 8 and complete it by extending $\gamma$ to cut $(b_3, b_1, b_2)$, leaving $b_2b_3$ uncut. Then the arc illustrated would lie on the unfolding of the base $\triangle b_1b_2b_3$.

Figure 8: An acaab unzipping of $C$ extending outside $\partial C$. Right: The $R$ and $L$ developments do not cross.

We now turn to the other three types of cut-paths $\gamma$. The proof for type caaab is similar to Lemma 1 and so will only be sketched.

**Lemma 2** For sufficiently small $\omega_a$, $\omega_c$, and $\varepsilon$, any cut-path $\gamma$ of type caaab must leave and reenter $C$ to avoid overlap. Therefore, $C$ cannot be unzipped with this type of cut-path.
Proof: The cut-path with straight segments overlaps at two spots in development, as shown in Fig. 9 (cf. Fig. 3). Using the same reasoning as in Lemma 1, except that the rotations are centered on \( c \) (rather than on both \( a_1 \) and \( c \)), leads to the conclusion that \( R_2 \) and \( R_3 \) must both deviate from the 30° turn at \( a_2 \) and the 60° turn at \( a_3 \) needed to stay on an arbitrarily thin apron. In fact, \( R_2 \) and \( R_3 \) must follow circle arcs centered on \( c \). Doing so would in fact allow \( C \) to be unzipped if apron were large enough, as shown in Fig. 10. But for an arbitrarily thin \( \epsilon \)-apron, \( \gamma \) must exit \( C \) before visiting all vertices, and so cannot be unzipped with this type of cut-path.

![Figure 9: The cut-path type caaab leads to overlap with straight segments.](image)

![Figure 10: An caaab unzipping of \( C \) extending outside \( \partial C \). Right: The \( R \) and \( L \) developments do not cross.](image)

The third type of cut-path, \( aacab \) (Fig. 11), is different in that not even following arcs outside of \( C \) would suffice to unzip it without overlap.

**Lemma 3** For sufficiently small \( \omega_a, \omega_c, \) and \( \epsilon \), any cut-path \( \gamma \) of type \( aacab \) cannot visit all vertices without overlap in the development. Therefore, \( C \) cannot be unzipped with this type of cut-path.

**Proof:** We analyze the constraints on \( R_2 \) in Fig. 12. The rotation at \( a_1 \) determines the prefix of \( R_2 \) following the same reasoning as in Lemma 1 and again
already in Fig. 12 b) we have an angle at $a_2$ much larger than the $30^\circ$ turn required to reach $c$. This already establishes the cut-path cannot be an unzipping. But in fact, it is clear that $R_2$ must follow the circle arc shown in Fig. 12 d), centered on $a_1$. Following this arc makes it impossible for $R_2$ to reach $c$: the path is forced toward $a_3$ instead.

The last combinatorial cut-path type, $aaacb$, mixes themes in the others: first, $\gamma$ must go outside $C$, which already establishes there is no unzipping, and second, even if the apron were large enough, $\gamma$ cannot reach $c$. We rely just on the first impediment.

**Lemma 4** For sufficiently small $\omega_a$, $\omega_c$, and $\epsilon$, any cut-path $\gamma$ of type $aaacb$ cannot visit all vertices without leaving and re-entering $C$. Therefore, $C$ cannot be unzipped with this type of cut-path.

**Proof:** Fig. 13 shows there is overlap when $\gamma$ is composed of straight segments. By now familiar reasoning, the portion of $\gamma$ from $a_2$ to $a_3$ must follow a circular arc centered on $a_1$. This is illustrated in Fig. 14 and already steps outside and $\epsilon$-thin apron in the neighborhood of $a_2$, where it makes an angle of approximately $90^\circ$ rather than the necessary $60^\circ$. This establishes the claim of the lemma.

We restate Theorem 1 in more detail:

**Theorem 1** Convex caps $C$ with sufficiently small $\{\omega_a, \omega_c, \epsilon\}$, as depicted in Fig. 2 have no unzipping: they are un-unzippable. Thus there are arbitrarily flat convex caps that cannot be unzipped.

**Proof:** We argued that only four combinatorial types of cut-paths $\gamma$ are possible on $C$. Lemmas 1, 2, 3, 4 established that for sufficiently small curvatures $\{\omega_a, \omega_c\}$ and a sufficiently thin $\epsilon$-apron, each of these cut-path types fails to unzip $C$. Because the arguments are independent of the exact values of $\{\omega_a, \omega_c\}$, only requiring a sufficiently small $\epsilon$ to match, the claim holds for arbitrarily flat convex caps.
Figure 12: Analysis of $\gamma$ of type aacab. (a) Opening at $a_1$ causes $R_2/L_2$ overlap. (b) $R_2$ bends around $a_2''$. (c) $L_2$ complements $R_2$, which again intersects $L_2$. (d) $L_2$ complements $R_2$. $R_2$ is following the arc centered on $a_1$.

Figure 13: The cut-path type aaacb leads to $L/R$ overlap with straight segments.
Figure 14: The cut-path type $aaacb$ extends outside $C$ on the $(a_2, a_3)$ arc (and still $R$ crosses $L$ near $a_3$).

3 Discussion

It is tempting to hope that the negative result of Theorem 1 can somehow be used to address the open problem for convex polyhedra. However, as mentioned earlier (Sec. 2.3), closing the convex cap in Fig. 2 by adding a base creates a polyhedron that can in fact be unzipped. Perhaps this is not surprising, as the proofs rely crucially on the fact that $C$ has a boundary $\partial C$. I have also explored using several un-unzippable convex caps to tile a closed convex polyhedron, but so far to no avail. The open problem from DDL+10 quoted in Sec. 1 remains open.
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