AN OPTIMAL DISTRIBUTED EDGE-BICONNECTIVITY ALGORITHM

DAVID PRITCHARD

Abstract. We describe a synchronous distributed algorithm which identifies the edge-biconnected components of a connected network. It requires a leader, and uses messages of size $O(\log |V|)$. The main idea is to preorder a BFS spanning tree, and then to efficiently compute least common ancestors so as to mark cycle edges. This algorithm takes $O(Diam)$ time and uses $O(|E|)$ messages. Furthermore, we show that no correct singly-initiated edge-biconnectivity algorithm can beat either bound on any graph by more than a constant factor. We also describe a near-optimal local algorithm for edge-biconnectivity.

1. Introduction

The edge-biconnectivity problem is to partition the vertices of a graph into maximal subsets called components such that the subgraph induced by each component remains connected after the deletion of any one edge. An edge whose deletion disconnects a graph is called a bridge; identification of all bridges in a graph is roughly equivalent to computing its components.

Here is a simple application of edge-biconnectivity. Given a (connected) communication network, let us compute its components. Two members of the network would be able to communicate despite the failure of any one communication link if and only if they are in the same component. Furthermore, for a given link, some members need that link to communicate if and only if that link is a bridge. This problem also has more sophisticated applications to module dependency [24] and efficient fault-tolerant broadcast [17].

Connectivity is also important as a tool in graph theory. For example, vertex connectivity plays a major role in the theories of excluded minors and embeddings [20] [21]. There are sequential algorithms for $k$-connectivity that are optimally efficient — having $O(|V| + |E|)$ time complexity — when $k$ is small [20] [15] [26]. In this paper we give an optimal edge-biconnectivity ($k = 2$) algorithm for distributed networks. Although this has been claimed before [8], our algorithm is optimal in a stronger sense: no correct distributed algorithm can outperform it on any graph.

A recurring point will be that the “optimality” of an algorithm depends on what preliminary assumptions are made. This seems to come up in distributed algorithms more than in sequential algorithms due to the variety of models. Our
claims assume a synchronous network with a leader, and that the algorithm is event-driven, starting with a single initiator. These are the same assumptions made by the previous best edge-biconnectivity algorithm [28]. However, we also discuss how our algorithm performs under different assumptions.

2. Preliminaries

Throughout this paper, we write $G = (V, E)$ for a connected graph, with $n = |V|$ and $m = |E|$. We write $\text{Diam}$ for the maximum distance between any two vertices of $G$. We take $G$ to model a computer network, with nodes representing computers and edges representing two-way, reliable, communication links.

Let us precisely state what edge-biconnectivity means. Define a relation $\sim$ on $V(G)$ by $x \sim y$ if, despite the removal of any one edge from $G$, there remains a path from $x$ to $y$. It is easy to show that this is an equivalence relation, and we define the edge-biconnected components to be its equivalence classes. An edge is a bridge if its deletion causes $G$ to become disconnected. The connection between components, bridges, and cycles is shown by the following lemma, whose proof we postpone until Section 7.

**Lemma 2.1.** For a connected graph $G$ and $(x, y) \in E(G)$, the following are equivalent:

1. $x$ and $y$ are in the same biconnected component,
2. $(x, y)$ is not a bridge,
3. $x$ and $y$ lie in a simple cycle.

We will give a distributed algorithm for computing the edge-biconnected components of any graph. Where there is no confusion, we write “biconnectivity” instead of “edge-biconnectivity.” At the end of the algorithm, each node will store a label corresponding to its biconnected component. Thus, by Lemma 2.1, a given edge is a bridge if and only if its two endpoints store different labels.

We will assume that the network is synchronous. This common assumption is validated by synchronizers [2], by which our synchronous algorithm can be efficiently made into an asynchronous algorithm. We also assume that the network initially contains a distinguished node called the leader. If we do not make this assumption, then we instead use an existing leader election algorithm [4] at an overhead of $O(n)$ time and $O(m + n \log n)$ messages.

This biconnectivity algorithm communicates, for the most part, along a spanning tree of the network. The key point is that we can use a tree to efficiently identify which edges lie in cycles, and then apply Lemma 2.1. We distributively compute a pre-ordering of $V$, and using “least common ancestors” we efficiently identify all cycle edges. Then it is straightforward to identify the bridges. Finally, we label the components using the following lemma.

**Lemma 2.2.** If we delete all bridges of $G$ from a spanning tree of $G$, then the forest of resulting trees is a set of spanning trees for the biconnected components of $G$. 

The important point is that, in comparison to DFS-based distributed biconnectivity algorithms such as [14], [8], our algorithm works on any tree. Whereas DFS seems to require \( \Omega(n) \) time, using BFS in our algorithm keeps the time complexity low. The idea of using an arbitrary tree for bridge-finding was first published in a 1974 paper by Tarjan [25], and was generalized somewhat in [29].

All messages in the algorithm are \( O(\log n) \) bits long, and thus it meets the \( CONGEST \) model of [23]. Our algorithm takes \( \Theta(Diam) \) time and sends \( \Theta(m) \) messages. The previous best time complexity for a biconnectivity algorithm is \( O(Diam + n^{0.614}) \) from [28]. Given that this is the same as our bound when \( Diam > n^{0.614} \), one may wonder whether we have really improved the situation. Happily, we can show that this new algorithm is \textit{universally optimal}: any event-driven biconnectivity algorithm with a single initiator must take \( \Omega(Diam(G)) \) time and use \( \Omega(m(G)) \) messages for every possible graph \( G \), or else the algorithm is incorrect on some graphs.

For reference, we note other previous work on biconnectivity. The distributed algorithm of [16] is the most similar to ours, as it takes an arbitrary tree, but it uses \( O(mn) \) messages. The parallel algorithms of [27], [30] also use an arbitrary tree. The distributed algorithm of [19] uses an “ear decomposition” of the network and takes \( O(n) \) time. Several others [8], [7], [6] use messages of size \( \Omega(n) \) bits, including an incremental algorithm [24] and a self-stabilizing algorithm [18]. Another self-stabilizing algorithm [9] takes \( O(n^2) \) time.

3. Overview of the Algorithm

We use a rooted tree \( T \) in our algorithm. An edge of \( G \) which does not lie in \( T \) is called a \textit{cross edge}. Let \( h(T) \) denote the height of \( T \), and \( \text{desc}(v) \) denote the descendants of \( v \) in \( T \), including \( v \) itself. Let \( C \) denote the union of all simple cycles,

\[ C := \{ e \in E \mid e \text{ lies within some simple cycle of } G \} . \]

The algorithm operates in five phases, as follows:

1. Construct a rooted breadth-first search (BFS) tree \( T \).
2. At each node \( v \), compute \( \#\text{desc}(v) \), the number of descendants of \( v \) in \( T \).
3. Compute a preorder labeling of \( V(G) \) with respect to \( T \).
4. By sending messages from cross edges up to the root, mark each edge in \( C \).
5. By downcasting, label the nodes according to their biconnected components.

4. Tree Construction and Preorder Labeling

First, we need a rooted spanning tree \( T \). The algorithm runs fastest when \( T \) is a BFS tree, but for the purposes of correctness any tree will do. We assumed that there is a leader in the network, and given this leader, it is straightforward to construct a rooted spanning tree. This is a well-studied problem, see for example [23]. In what follows, we use the term \textit{downcasting} to mean that the root sends a message to all of its children, each of which sends a message to each of its children, and so forth. \textit{Convergecasting} [23] means an inverse process, where messages are
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propagated from leaves to their parents, and so on up the tree to the root; however, each node waits to hear from all of its children before reporting to its parent, so only \( n - 1 \) messages are sent in total.

The computation of \( \text{#desc}(v) \) at each node in Phase 2 can be accomplished in \( 2h(T) \) time steps. First, the root node sends “Compute \( \text{#desc} \) of yourself” to each of its children, and this message is downcasted through all of \( T \). Each leaf \( v \) determines immediately that \( \text{#desc}(v) = 1 \). Once any non-root node computes its \( \text{#desc} \) value, it sends a message to its parent indicating that value. Each non-leaf node \( v \) aggregates its children’s values in order to compute \( \text{#desc}(v) \); namely, if \( v \) has children \( c_1, \ldots, c_k \), then \( v \) may compute

\[
\text{#desc}(v) := 1 + \sum_{i=1}^{k} \text{#desc}(c_i).
\]

In Phase 3, a preorder labeling of \( T \) is computed by using another downcast. The root node starts by setting its own \( \text{PreLabel} \) field to 1. Whenever a node \( v \) sets its \( \text{PreLabel} \) field to \( \ell \), it orders its children in \( T \) arbitrarily as \( c_1, c_2, \ldots \). Then \( v \) sends the message “Set your \( \text{PreLabel} \) field to \( \ell_i \)” to each \( c_i \), where \( \ell_i \) is computed by \( v \) as

\[
\ell_i = \ell + 1 + \sum_{j<i} \text{#desc}(c_j)
\]

After \( h(T) \) time steps, we will have computed a preordering of \( T \).

5. Least Common Ancestors

In order to simplify the presentation, we hereafter refer to nodes simply by their preorder labels. The preordering allows us to reduce congestion in Phase 4 of the algorithm, using the following properties.

**Lemma 5.1.** The descendants of a node \( v \) in the tree \( T \) are precisely

\[
\text{desc}(v) = \{ u \mid v \leq u < v + \text{#desc}(v) \}.
\]

Let \( \text{LCA}(u_1, u_2, \ldots) \) denote the lowest (by position, not value) common ancestor of nodes \( u_1, u_2, \ldots \) in the tree \( T \). In other words, \( \text{LCA}(u_1, u_2, \ldots) \) is an ancestor of each \( u_i \), but no strict descendant of \( \text{LCA}(u_1, u_2, \ldots) \) is an ancestor of all \( u_i \)s.

**Theorem 5.2.** If \( v_1 \leq v_2 \leq v_3 \), then \( \text{LCA}(v_1, v_3) \) is an ancestor of \( v_2 \).

**Proof.** Let \( a = \text{LCA}(v_1, v_3) \). By Lemma 5.1 \( a \leq v_1 \leq v_3 < a + \text{#desc}(a) \). Thus \( a \leq v_2 < a + \text{#desc}(a) \), and by Lemma 5.1 \( v_2 \) must also be a descendant of \( a \). \( \square \)

**Corollary 5.3.** \( \text{LCA}(u_1, u_2, \ldots, u_k) = \text{LCA}(\min_i(u_i), \max_i(u_i)) \).

**Corollary 5.4.** If \( u_i \leq v_i \) for all \( i \), then

\[
\text{LCA}(\text{LCA}(u_1, v_1), \text{LCA}(u_2, v_2), \ldots, \text{LCA}(u_k, v_k)) = \text{LCA}(\min_i(u_i), \max_i(v_i)).
\]
6. Marking Cycle Edges

The goal of Phase 4 is to determine which edges lie in \( \mathcal{C} \). When \( v' \) is an ancestor of \( v \) in \( T \), let \( \text{Chain}(v', v) \) denote the set of edges on the path from \( v' \) to \( v \) in \( T \).

The cross edges with respect to \( T \) permit a simple formula for \( \mathcal{C} \):

**Lemma 6.1.**

\[
\mathcal{C} = \bigcup_{(u, v) \in \mathcal{G} - T} \{(u, v)\} \cup \text{Chain}(\text{LCA}(u, v), u) \cup \text{Chain}(\text{LCA}(u, v), v).
\]

**Proof.** Note that each set \( \{(u, v)\} \cup \text{Chain}(\text{LCA}(u, v), u) \cup \text{Chain}(\text{LCA}(u, v), v) \) is a simple cycle. It remains to show that this union formula contains all edges appearing in simple cycles. Suppose otherwise, that the above formula missed some edge \((u, v)\) belonging to a simple cycle \( K \) of \( \mathcal{G} \). Since Equation (1) includes all edges of \( \mathcal{G} - T \), we can assume that \((u, v) \in T\), without loss of generality \( u \) the parent of \( v \).

Let the cycle \( K \) contain, in order, the nodes \( (k_0 = v, k_1, k_2, \ldots, k_{m-1} = u, k_m = v) \). If \( k_i \) is the first element of this list not in \( \text{desc}(v) \), then \((k_{i-1}, k_i)\) is a cross edge. But then we have

\[
(u, v) \in \text{Chain}(\text{LCA}(k_{i-1}, k_i), k_{i-1}),
\]

so in fact \((u, v)\) is counted by \( \mathcal{C} \).

Thus, to mark the edges of \( \mathcal{C} \), it suffices to just mark chains going up from each cross edge to its endpoints’ LCA.

We could distributively mark the edges in \( \text{Chain}(\text{LCA}(u, v), v) \) as follows:

- For each cross edge \((u, v)\),
  
  Send a message from \( v \) to \( u \) which states “If you are an ancestor of both \( u \) and \( v \), then ignore this message. Otherwise, pass this message up to your parent, and mark the edge joining you to your parent as being in \( \mathcal{C} \).”

  Send the same message from \( u \) to \( v \).

Checking the ancestry condition is accomplished using Lemma 5.1. We will abbreviate the message “If you are an ancestor of both \( u \) and \( v \), . . . ” as “Mark up to \( \text{LCA}(u, v) \)”.

Without loss of generality, we will send our messages so that \( u \leq v \).

Sending these messages naively leads to congestion. When a node receives many at once that must all be forwarded, not all can be immediately sent to its parent if the \( O(\log n) \) bound on message sizes is to be respected. The following Forwarding Rule fixes this congestion:

- If a node \( w \) receives several messages “Mark up to \( \text{LCA}(u_i, v_i) \)” for \( i = 1 \ldots k \), it should compute \( u_{\text{min}} = \min_i u_i \) and \( v_{\text{max}} = \max_i v_i \). If \( w \) is an ancestor of both \( u_{\text{min}} \) and \( v_{\text{max}} \), then no message is sent up. Otherwise, \( w \) should send “Mark up to \( \text{LCA}(u_{\text{min}}, v_{\text{max}}) \)” to its parent, and mark the edge connecting \( w \) to its parent as being in \( \mathcal{C} \).
Theorem 6.2. The Forwarding Rule correctly marks \( \text{Chain}(\text{LCA}(u, v), v) \) for each cross edge \((u, v)\).

Proof. Suppose that \( w \), as described, is asked to propagate messages so that all edges in \( \bigcup_i \text{Chain}(\text{LCA}(u_i, v_i), w) \) become marked. They must all lie on the unique path between \( w \) and the root of \( T \), so we only need to mark the longest chain. The highest LCA is equal to \( \text{LCA}(\text{LCA}(u_1, v_1), \text{LCA}(u_2, v_2), \ldots) \) and by Corollary 5.4 this is \( \text{LCA}(u_{\min}, v_{\max}) \), so the propagated message (if any) is correct. \( \square \)

7. Biconnected Decomposition

Lemma 7.1. For a connected graph \( G \) and \( x, y \in V(G) \), \( x \) and \( y \) are in the same biconnected component if and only if \( x \) and \( y \) both lie in some cycle of \( G \) that has no repeated edges.

Proof. Note that \( x \not\sim y \) if and only some edge's deletion separates \( x \) from \( y \); by the maxflow-mincut theorem such an edge exists if and only if there are not two edge-disjoint paths between \( x \) and \( y \). Those paths' union is precisely a cycle with no repeated edges, and likewise such a cycle can be broken into two edge-disjoint \( x-y \) paths. \( \square \)

We now prove the Lemma introduced in Section 2.

Lemma 2.1. For a connected graph \( G \) and \( (x, y) \in E(G) \), the following are equivalent:

1. \( x \) and \( y \) are in the same biconnected component,
2. \((x, y)\) is not a bridge,
3. \( x \) and \( y \) lie in a simple cycle.

Proof. Let \( e \) denote the edge \((x, y)\).

(3) \( \Rightarrow \) (2) : Let \( C \) be a cycle containing \( x \) and \( y \). For any \( u-v \) path containing \( e \), we can use \( C-e \) to connect \( u \) and \( v \) in \( G-e \), and thus \( e \) is not a bridge.

\( \neg(1) \Rightarrow \neg(2) \) : If \( x \not\sim y \), then the deletion of some edge from \( G \) causes \( x \) and \( y \) to become separated. But \( x \) remains connected to \( y \) by \( e \) unless \( e \) is the deleted edge. Thus the deletion of \( e \) from \( G \) separates \( x \) from \( y \), so \( e \) is a bridge.

(1) \( \Rightarrow \) (3) : Apply Lemma 7.1 obtaining cycle \( C \) with no repeated edges. If \( e \notin C \), then we take any simple \( x-y \) path in \( C \) and adjoin \( e \) to create a simple cycle containing \( x \) and \( y \). If \( e \in C \), then partition \( C \) into simple cycles, and take the one containing \( e \). \( \square \)

Note that only tree edges can be bridges, for each cross-edge induces a cycle with \( T \). Thus, if each node stores a boolean variable indicating whether the edge to its parent is a bridge, then this suffices to identify all bridges.
Figure 1. Distributed algorithm for edge-biconnectivity, given a rooted spanning tree.
[Algorithm specification omitted, as arxiv does not support the algpseudocode class].

To group the nodes according to their biconnected component (Phase 5), we need to broadcast an identifier along each component. The following claim means that a simple downcast along the edges of $T$ will suffice.

Lemma 2.2. If we delete all bridges of $G$ from $T$, then the forest of resulting trees is a set of spanning trees for the biconnected components of $G$.

Proof. Suppose otherwise, that there are two nodes $u \sim v$ such that the unique path $P$ between $u$ and $v$ in $T$ contains a bridge $e$. Since $u \sim v$, there is a simple $u$-$v$ path $P'$ in $G - e$. But then the symmetric difference $P \triangle P'$ has even degree at every node and contains $e$, and so $P \triangle P'$ contains some simple cycle containing $e$. By Lemma 2.1 we have a contradiction.  

8. Correctness and Complexity of the Algorithm

The main parts of the algorithm, Phases 4 and 5, are shown in Algorithm 1. Its correctness follows from Lemma 6.1, Theorem 6.2 and Lemma 2.2. Note that we specify “null” messages which we have not yet been justified; this is in order for the “Mark up to” messages to be properly synchronized in a convergecast. This way, a node simply waits to hear from all of its non-parent neighbours before reporting to its parent.

Phases 2–4 have total message complexity $O(n)$ and time complexity $O(h(T))$, even in an asynchronous setting. Thus, those phases would be optimized when $T$ is as short as possible. Note that a BFS tree has height at most $\text{Diam}(G)$, and that no spanning tree has height less than $\text{Diam}(G)/2$. Thus, it is essentially optimal to construct a BFS tree in Phase 1. Assuming synchrony, and that we are given a leader at the beginning of the algorithm, it is well-known that a BFS tree can be constructed greedily in $\Theta(\text{Diam})$ time and using $O(m)$ messages. Thus, the biconnectivity algorithm’s total complexity is $O(\text{Diam})$ time and $O(m)$ messages.

9. Optimality

We claim that the performance of this algorithm cannot be improved beyond constant factors. To be precise, we argue that any deterministic, singly-initiated, event-driven protocol for bridge-finding must always send at least $m$ messages and take at least $\text{Diam}/2$ time, or else the protocol will not work on all graphs. By singly-initiated, we mean that there is a single node in the graph which begins computing spontaneously, and by event-driven we mean that every other node must receive a message before it can perform any action. These lower bounds are similar in nature, and both depend on the fact that the whole network must be explored.
First we describe the lower bound for messages. Suppose that, when the protocol is executed on some graph $G$, there is an edge $(u, v)$ along which no messages are sent. Let $G'$ be a graph obtained from $G$ by adding a new node $w$ and dividing $(u, v)$ into two edges $(u, w)$ and $(w, v)$. We also attach some cycles and bridges to $w$, as shown in Figure 2. When we run the protocol on $G'$, assuming that the algorithm is deterministic, no messages are sent along $(u, w)$ or $(w, v)$ and so no messages reach the new nodes and edges. Consequently, the algorithm cannot correctly determine whether the new edges are bridges or not.

The lower bound on the time complexity is similar. If an algorithm uses takes less than $\text{Diam}(G)/2$ steps on some graph $G$, then there are parts of the graph which no messages reach. Consequently, we can modify $G$ so that the algorithm operates incorrectly.

Note that these lower bounds apply to all graphs. In comparison, a $O(n)$-time algorithm of [8] was called “optimal” because some graphs require $O(n)$ time to find their bridges. The $O(n)$ algorithm is existentially optimal, since there exist some instances on which the protocol is optimal, and our $O(\text{Diam})$ algorithm is universally optimal, since it has optimal running time on all instances. The different types of optimality were first observed by [22] and [5] in the context of leader election, and further discussion appears in [13] and [10]. Universal optimality allows us to precisely state that the inherent complexity of the (singly-initiated, event-driven) biconnectivity problem is $\Theta(\text{Diam})$ time and $\Theta(m)$ messages. Finally, although we state these bounds for deterministic algorithms, similar bounds can be proved for randomized ones, using essentially the same argument.

10. A Near-Optimal Local Algorithm

For now, let us forget the problem of labeling nodes according to their biconnected component, and only worry about identifying all of the bridges in a graph. We consider initiating all nodes at the same time, and want to know how long it will be before all edges are correctly identified as bridges or non-bridges. By removing the assumption of a single initiator, we can beat the lower time bound of $\text{Diam}$.

Suppose we remove the restriction on the message size. Then, each node can broadcast everything it knows about its local topology after each step, and so after $t$ steps each node will know its own $t$-neighbourhood. Here is an algorithm for bridge-finding. Initially, each edge is assumed to be a bridge; whenever a node learns of

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure2}
\caption{Modification of $G$ into $G'$, upon which a biconnectivity algorithm fails.}
\end{figure}
a cycle in its neighbourhood, it informs all of the edges in that cycle that they are not bridges. In this way we distributively determine $C$, the union of all cycles in $G$. For a general graph, we cannot be sure that we’re done until $Diam$ steps have elapsed. However, for certain graphs, we can safely terminate in $o(Diam)$ rounds. An algorithm whose time complexity is $o(Diam)$ is often called local.

Now, let us determine the time before this algorithm has correctly identified the non-bridges. A non-bridge $e$ will be identified as soon as a cycle containing $e$ is known by a node; we call such a cycle a witness for $e$. We need each edge to be identified by a witness in order for the algorithm to be correct. Define the cycle-witness radius of $G$, denoted $\Upsilon(G)$, by

$$\Upsilon(G) := \max_{e \in C} \min_{K \ni e} \min_{v \in V(G)} \max_{u \in K} dist_G(u, v).$$

Then the cycle-witness radius is the minimum time needed to identify all of the non-bridges (and, it will take another $\Upsilon(G)$ rounds to notify those edges). Further, it can be shown that $\Upsilon(G)$ is a lower bound on the number of rounds before all non-bridges can be correctly identified.

A $(\log n, \Upsilon)$-neighbourhood cover of $G$ is a collection of connected vertex sets called clusters such that

1. For each vertex $v$, the $\Upsilon$-neighborhood of $v$ is entirely contained in some cluster.
2. The subgraph of $G$ induced by each cluster has diameter $O(\Upsilon \log n)$.
3. Each node belongs to $O(\log n)$ clusters.

See [23, Ch. 21] for a good exposition on this subject. The full version of [10] gives a randomized local algorithm for computing sparse neighborhood covers which, with high probability, runs in $O(\Upsilon \log^3 n)$ time and uses $O(m \log^2 n)$ messages on a synchronous network.

We can use neighbourhood covers to search brute-force for all small cycles, without using large messages as described above. Suppose we have a $(\log n, \Upsilon)$-neighbourhood cover as described above. Each edge has a witness that is entirely contained within one cluster. If we run Algorithm 1 separately on each cluster, each non-bridge will be witnessed in some cluster. Since each node may be in $O(\log n)$ clusters, there will be congestion when we process all clusters in parallel; however, this will only increase the time of the biconnectivity algorithm by a factor of $O(\log n)$, since each node can rotate between participating in its containing clusters. The resulting local algorithm takes $O(\Upsilon \log^3 n)$ time and $O(m \log^2 n)$ messages to construct the clusters, then a further $O(\Upsilon \log n)$ time and $O(m \log n)$ messages to determine the non-bridges.

Finally, it is unlikely that $\Upsilon$ can be computed efficiently and/or locally. However, an algorithm can successively “guess” $\Upsilon = 1, 2, 4, 8, \ldots$, and run the local algorithm for each value in turn. Once the guess is larger than the actual value of $\Upsilon$, all edges will be correctly classified; this algorithm becomes correct within $O(\Upsilon \log^3 n)$ rounds. We note that this is essentially a will-maintaining algorithm as defined by Elkin in [10].
11. OTHER EXTENSIONS

With a small modification, the algorithm of this chapter can also be used to compute the strongly-connected components of a graph. We require that all directed edges function as 2-way communication channels. We compute a directed DFS tree $T$ of the network. It is easy to show that an analog of Equation (1) holds in this case, with the cross edges replaced by the back edges $\{(u, v) \mid u \in \text{desc}(v)\}$. The resulting algorithm takes $O(h(T))$ time and $O(m)$ messages, identifies the edges that belong to cycles, and labels all nodes according to their strongly-connected component. Using Awerbuch’s DFS algorithm from [3] gives a total of $O(n)$ time and $O(m)$ communication complexity, and note that it works on asynchronous networks.

If a short DFS tree could be identified in sub-linear time, then we might be able to get a sub-linear algorithm for identifying strongly connected components.

**Question 11.1.** Does there exist a $(O(\text{Diam}) + o(n))$-time distributed DFS tree construction algorithm, using messages of size $O(\log n)$?

Also, there is a divide-and-conquer algorithm [11] for strongly connected components which might lend itself to distributed implementation in sub-linear time.

In order to ensure reliability in networks, one may want to ensure the 2-vertex connectivity of a network, for example [17]. It does not seem that our algorithm can be easily modified to determine vertex biconnectivity. In contrast, the $\Theta(n)$ time DFS-based biconnectivity algorithms of [1] and [14] can determine vertex biconnectivity.

We might also try to determine the triconnected [15] components of a graph. There are efficient parallel algorithms for this problem [12]. The following lemma might be useful in designing a fast distributed triconnectivity algorithm.

**Claim 11.2.** Let $G$ be a graph with no bridges. Define the relation $\sim_D$ on the edges of $G$ by $x \sim_D y$ if the graph $G - x - y$ is not connected. Then $\sim_D$ is an equivalence relation.

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Department of Combinatorics and Optimization, Waterloo, Canada.