A geometric realization of the periodic discrete Toda lattice and its tropicalization

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Received 6 June 2013, in final form 2 October 2013
Published 1 November 2013
Online at stacks.iop.org/JPhysA/46/465203

Abstract

An explicit formula concerning curve intersections equivalent to the time evolution of the periodic discrete Toda lattice (pdTL) is presented. First, the time evolution is realized as a point addition on a hyperelliptic curve, which is the spectral curve of the pdTL, then the point addition is translated into curve intersections. Next, it is shown that the curves which appear in the curve intersections are explicitly given by using the conserved quantities of the pdTL. Finally, the formulation is lifted to the framework of tropical geometry and a tropical geometric realization of the periodic box–ball system is constructed via tropical curve intersections.

PACS numbers: 02.30.Ik, 05.45.Yv, 87.17.−d

1. Introduction

In recent years, many studies on ultradiscrete integrable systems or integrable cellular automata have intensively been made by using various mathematical tools, such as combinatorics [3, 24, 31, 37], ultradiscretization procedure [14, 27, 34, 35], crystal bases of quantum groups [7–9, 11] and so forth. In particular, among many studies on geometric aspects of ultradiscrete integrable systems, several remarkable works have successively been done by Kimijima–Tokihiro [21], Inoue–Takenawa [12, 13] and Iwao [17] in terms of tropical geometry and ultradiscretization procedure. They have solved the initial value problems to box–ball systems (BBS) with periodic boundary conditions (periodic BBS (pBBS)) by using tropical hyperelliptic curves, their tropical Jacobians, ultradiscrete theta functions and ultradiscretization of Abelian integrals. Through their works, we have gradually recognized the advantage in applying the method of tropical geometry to ultradiscrete integrable systems. Therefore, when we focused our interest on the geometric aspect of ultradiscrete integrable systems, we usually chose the framework of tropical geometry as the analysis tool [11, 19, 20, 28, 29].

In this paper, we will go a step further and try to realize a pBBS in the framework of tropical geometry more precisely. Here a ‘precise’ geometric realization means that we will give the time evolution of the pBBS not by linear flows on the tropical Jacobian but...
by tropical curve intersections. If we try to do so, we immediately note that the task is not so easy, because to find tropical curve intersections equivalent to the time evolution of the pBBS is essentially the same as to obtain the general solution to a simultaneous system of piecewise linear equations. (Note that we do not have a Cramer-type formula for such a system of equations.) However, preceding studies on ultradiscrete systems shed light on us and we found a short cut. We first establish a geometric realization of the periodic discrete Toda lattice (pdTL), whose ultradiscretization gives the pBBS, by using curve intersections. We then apply the ultradiscretization procedure to the members in the curve intersections and finally obtain a geometric realization of the pBBS in terms of tropical curve intersections.

To carry out our plan, this paper is organized as follows. We first review the basic results concerning addition of points on hyperelliptic curves in section 2. In section 3, we then introduce the pdTL and investigate its spectral curve and the conserved quantities. In section 4, we establish curve intersections equivalent to the time evolution of the pdTL. We then focus on ultradiscrete systems. In section 5, we introduce the pdTL and investigate its spectral curve and the conserved quantities. In section 4, we establish curve intersections equivalent to the time evolution of the pdTL. Finally, in section 6, we apply the procedure of ultradiscretization to the rational functions (pdTL) and its spectral curve, and review addition of points on tropical hyperelliptic curves. (If we take the initial values of the UD-pTL in positive integers then we obtain the pBBS.)

2. Addition on hyperelliptic curves

We first present a brief review of addition of points on hyperelliptic curves (see for example [26]).

2.1. Hyperelliptic curves

Let \( h(x) \) be the monic polynomial of degree \( 2g + 2 \geq 4 \):

\[
h(x) = x^{2g+2} + a_{2g+1}x^{2g+1} + a_{2g}x^{2g} + \ldots + a_1x + a_0,
\]

where \( a_0, a_1, \ldots, a_{2g+1} \in \mathbb{C} \). Consider the affine curves \( H_0 \) on \( \mathbb{C}^2_{(x,y)} \) and \( H_\infty \) on \( \mathbb{C}^2_{(u,v)} \):

\[
H_0 := \left( y^2 - h(x) = 0 \right), \quad H_\infty := \left( u^2 - u^{2g+2}h \left( \frac{1}{u} \right) = 0 \right),
\]

where \((x, y)\) and \((u, v)\) are the standard coordinates of \( \mathbb{C}^2_{(x,y)} \) and \( \mathbb{C}^2_{(u,v)} \), respectively. Assume that \( h(x) \) has no multiple root, and hence both the affine curves are non-singular. Let the projections on the curves be

\[
\pi_0 : H_0 \to \mathbb{C}_x; \quad (a, b) \mapsto a, \quad \pi_\infty : H_\infty \to \mathbb{C}_u; \quad (\alpha, \beta) \mapsto \alpha.
\]

By gluing \( H_0 \) with \( H_\infty \) in terms of the bi-holomorphic map \( H_0 \setminus \pi_0^{-1}(0) \to H_\infty \setminus \pi_\infty^{-1}(0) \):

\[
(x, y) \mapsto (u, v) = \left( \frac{1}{x}, \frac{y}{x^{g+1}} \right),
\]

we obtain the hyperelliptic curve \( H := H_0 \cup H_\infty \) of genus \( g \).

Substitute \( x = 1/t \) into \( y^2 - h(x) = 0 \), then we obtain

\[
(t^{g+1}y)^2 = 1 + a_{2g+1}t + a_{2g}t^2 + \ldots + a_1t^{2g+1} + a_0t^{2g+2}.
\]

If \( t \) goes to 0 this equation reduces to \( (t^{g+1}y)^2 \sim 1 \). Hence, we obtain \((x, y) \sim (1/t, \pm 1/t^{g+1})\), or equivalently \((u, v) \sim (t, \pm 1)\) for sufficiently small \( t \). Therefore, the curve \( H \) can be expressed as

\[
H = H_0 \cup \{ P_\infty, P_\infty' \},
\]

where \( P_\infty = (0, 1), P_\infty' = (0, -1) \in \mathbb{C}^2_{(u,v)} \) are the points at infinity.
Consider the involution $\iota : H \to H; P = (x, y) \mapsto P' = (x, -y)$. We call $P'$ the conjugate of $P$.

Let $D(H)$ be the divisor group of $H$. If $D \in D(H)$ then $D$ has the form $D = \sum_{P \in H} a_P P$, where $a_P \in \mathbb{Z}$ and $a_P = 0$ except for finite $P$. The number $\sum_{P \in H} a_P$ is called the degree of $D$ and is denoted by $\deg D$. If $a_P \geq 0$ for all $P \in H$, the divisor $D$ is called effective and is denoted by $D > 0$.

2.2. Canonical maps

Let $D_0(H) := \{ D \in D(H) \mid \deg D = 0 \}$ be the group of divisors of degree 0 on $H$. Also let $D_1(H) := \{(k) \mid k \in \mathbb{C}(H)\}$ be the group of principal divisors of rational functions on $H$, where $\mathbb{C}(H)$ is the field of rational functions on $H$ and $(k)$ denotes the principal divisor of $k \in \mathbb{C}(H)$.

We define the Picard group $\text{Pic}^0(H)$ to be the residue class group $\text{Pic}^0(H) := D_0(H)/D_1(H)$.

Let $D^+_g(H) := \{ D \in D(H) \mid D > 0, \deg D = g \}$ be the set of effective divisors of degree $g$ on $H$. For simplicity, we denote the element $P_1 + P_2 + \cdots + P_e$ of $D^+_g(H)$ by $D_P$.

Fix an element $D^*$ of $D^+_g(H)$. Define the canonical map $\Phi : D^+_g(H) \to \text{Pic}^0(H)$ to be

$$\Phi(A) := A - D^* \pmod{D_1(H)} \quad \text{for } A \in D^+_g(H).$$

The following theorem is easily shown.

**Theorem 1.** The canonical map $\Phi$ is surjective. In particular, $\Phi$ is bijective if $g = 1$.

2.3. Addition on symmetric products

By using the surjection $\Phi$, we induce addition of points on the $g$th symmetric product $\text{Sym}^g(H) := H^g/\mathbb{S}_g$ from $\text{Pic}^0(H)$.

There exists a bijection

$$\mu : D^+_g(H) \to \text{Sym}^g(H); \quad D_P \mapsto \mu(D_P) = \{P_1, P_2, \ldots, P_g\}.$$ 

We denote $\mu(D_P) \in \text{Sym}^g(H)$ by $d_P$. Put $\hat{\Phi} := \Phi \circ \mu^{-1} : \text{Sym}^g(H) \to \text{Pic}^0(H)$. Note that we have $\text{Pic}^0(H) = \{ \hat{\Phi}(d) \mid d \in \text{Sym}^g(H) \}$ because $\hat{\Phi}$ is surjective. For $d_P, d_Q \in \text{Sym}^g(H)$, we define $d_P \oplus d_Q$ to be an element in the subset

$$\hat{\Phi}^{-1}(\hat{\Phi}(d_P) + \hat{\Phi}(d_Q)) \subset \text{Sym}^g(H).$$

Since $\hat{\Phi}$ is surjective, $\hat{\Phi}^{-1}(\hat{\Phi}(d_P) + \hat{\Phi}(d_Q)) \neq \emptyset$ holds, and hence $d_P \oplus d_Q$ exists. However, since $\hat{\Phi}$ is not necessarily injective, $d_P \oplus d_Q$ is not always determined uniquely. We choose $o = \mu(D)$ for $D \in \ker \Phi$ as the unit of addition on $\text{Sym}^g(H)$.

Let $d_P, d_Q$ and $d_R$ be the elements of $\text{Sym}^g(H)$ satisfying

$$d_P \oplus d_Q \oplus d_R = o.$$ (1)

This can be written by the divisors on $\text{Pic}^0(H)$:

$$D_P + D_Q + D_R - 3D^* \equiv 0 \pmod{D_1(H)}.$$ 

There exists a rational function $k \in L(3D^*)$ whose 3g zeros are $P_1, \ldots, P_g, Q_1, \ldots, Q_g, R_1, \ldots, R_g$. Let $C$ be the curve defined by $k : C = (k(x, y) = 0)$. Then the zeros of $k$ are points on $C$. Since these points are on $H$ by definition, these points are the intersection points of $H$ and $C$. Thus, (1) is realized by using the intersection of $H$ and $C$. 


2.4. Kernel of $\Phi$

Hereafter, we fix $D^*$ as follows:

$$D^* = \begin{cases} \frac{g}{2} (P_\infty + P'_\infty) & \text{for even } g, \\ \frac{g - 1}{2} (P_\infty + P'_\infty) + P_\infty & \text{for odd } g. \end{cases}$$

Put $I_n := \{1, 2, \ldots, n\}$ for $n \in \mathbb{N}$. We have the following theorems concerning the kernel of the surjective canonical map $\Phi$.

**Theorem 2.** If $g$ is an even number, we have

$$\ker \Phi = \left\{ \sum_{i=1}^{g} P_i \mid \forall i \in I_g ^3 \exists j \in I_g \text{ s.t. } P_j = P'_i, j \neq i \right\}.$$

If $g$ is an odd number, we have

$$\ker \Phi = \left\{ \sum_{i=1}^{g-1} P_i + P_\infty \mid \forall i \in I_{g-1} ^3 \exists j \in I_{g-1} \text{ s.t. } P_j = P'_i, j \neq i \right\}.$$

**Proof.** See appendix A. □

If $g = 1$ the surjection $\Phi$ is also injective (see theorem 1). On the other hand, for $g \geq 2$, $\Phi$ is ‘almost’ injective.

**Theorem 3.** For any $g$, we have

$$\Phi(D_P) = \Phi(D_Q) \iff \exists D_R \in D^*_g (H) \text{ s.t. } D_P + D_Q' = D_R + D_R'.$$

**Proof.** See appendix B. □

In particular, if $g = 2$ we easily see that the surjection $\Phi$ is injective except for its kernel.

**Corollary 1.** If $g = 2$, we have

$$\Phi(D_P) = \Phi(D_Q) \text{ and } D_P \neq D_Q \iff D_P, D_Q \in \ker \Phi.$$

3. Discrete Toda lattice with a periodic boundary condition

Let us consider the pdTL [10], which is a map $\chi : \mathbb{C}^{2g+2} \rightarrow \mathbb{C}^{2g+2}$ given by

$$\chi : (I_1, \ldots, I_{g+1}, V_1, \ldots, V_{g+1}) \mapsto (\tilde{I}_1, \ldots, \tilde{I}_{g+1}, \tilde{V}_1, \ldots, \tilde{V}_{g+1}),$$

where $\tilde{I}_i$ and $\tilde{V}_i$ ($i = 1, 2, \ldots, g + 1$) are defined to be

$$\tilde{I}_i = V_i + I_i \frac{1 - \frac{V_1 V_{i+1}}{V_{i+1} I_i}}{1 + \frac{V_2 V_{i+1}}{V_{i+1} I_i} + \frac{V_3 V_{i+1}}{V_{i+1} I_i} + \cdots + \frac{V_{g+1} V_{i+1}}{V_{i+1} I_i}}.$$

$$\tilde{V}_i = \frac{I_{i+1} V_i}{\tilde{I}_i}.$$

If we assume $0 < \prod_{i=1}^{g+1} V_i < \prod_{i=1}^{g+1} I_i$ these evolution equations are equivalent to the following [21]:

$$\tilde{I}_i + \tilde{V}_{i-1} = I_i + V_i, \quad \tilde{V}_i \tilde{I}_i = I_{i+1} V_i.$$

(2)
For $t = 0, 1, \ldots$, we use the notation

$$(I'_1, \ldots, I'_{g+1}, V'_1, \ldots, V'_{g+1}) := \chi \circ \chi \circ \cdots \circ \chi (I_1, \ldots, I_{g+1}, V_1, \ldots, V_{g+1}).$$

The Lax form of (2) is given by

$$\tilde{L}M = ML,$$

where $L$ and $M$ are the following $(g+1) \times (g+1)$ matrices\(^1\)

$$L := \begin{bmatrix} I_2 + V_1 & 1 & (-1)^{g} I_1 V_1 / y \\ I_2 V_2 & I_3 + V_2 & 1 \\ \vdots & \vdots & \vdots \\ I_{g+1} V_{g+1} & I_{g+1} V_{g+1} + V_{g+1} & 1 \\ I_1 & 1 \\ \vdots & \vdots & \vdots \\ (-1)^{g} y & V_1 & 1 \\ \vdots & \vdots & \vdots \\ I_{g} & 1 \\ (-1)^{g} y & V_{g} & 1 \\ \vdots & \vdots & \vdots \\ I_{1} & 1 \end{bmatrix},$$

and $\tilde{L}$ is obtained from $L$ by replacing $I_i$ and $V_i$ with $\bar{I}_i$ and $\bar{V}_i$ ($i = 1, 2, \ldots, g+1$), respectively.

Here, $y \in \mathbb{C}$ is the spectral parameter.

### 3.1. Conserved quantities

Let $\Lambda$ be the set $\{1, 2, \ldots, g+1\}$. For $\Lambda$, we define the $(g+1) \times (g+1)$ triple diagonal matrix $L_\Lambda$ to be

$$L_\Lambda := \begin{bmatrix} I_2 + V_1 & 1 & (-1)^{g} I_1 V_1 / y \\ I_2 V_2 & I_3 + V_2 & 1 \\ \vdots & \vdots & \vdots \\ I_{g+1} V_{g+1} & I_{g+1} V_{g+1} + V_{g+1} & 1 \\ I_1 & 1 \\ \vdots & \vdots & \vdots \\ (-1)^{g} y & V_1 & 1 \\ \vdots & \vdots & \vdots \\ I_{g} & 1 \\ (-1)^{g} y & V_{g} & 1 \\ \vdots & \vdots & \vdots \\ I_{1} & 1 \end{bmatrix}.$$

Note that $L_\Lambda$ is the Lax matrix of the discrete Toda molecule [10]. We denote the $(i, j)$-entry of $L_\Lambda$ by $\lambda_{ij}$. For a subset $\Omega \subset \Lambda$, we define $L_\Omega := (\lambda_{ij})_{i, j \in \Omega}$ as well.

Let us consider the polynomial $\tilde{f}(x, y)$ in $x$ and $y$

$$\tilde{f}(x, y) := y |xI + L|,$$

where $I$ is the identity matrix of degree $g+1$. The eigenpolynomial of $L$ can be expanded as follows:

$$|xI + L| = y + |xI + L_\Lambda| - I_1 V_1 |xI + L_\Lambda| + \frac{c_{-1}}{y},$$

where we put $\tilde{\Lambda} := \Lambda \setminus \{1, g+1\} = \{2, 3, \ldots, g\} \subset \Lambda$ and

$$c_{-1} = \prod_{i=1}^{g+1} I_i V_i.$$

\(^1\) These matrices $L$ and $M$ are for $g \geq 2$. For the case of $g = 1$, $L$ and $M$ are given in appendix C.
Note the formula

\[ |x| + L_x = x^{g+1} + \sum_{k=1}^{g+1} \sum_{\Omega \subseteq \Lambda} |L_{\Omega}| x^{g+1-k}, \]

where \( \Omega \) ranges over all subsets of \( \Lambda \) consisting of \( k \) elements. We find

\[ |x| + L = y + x^{g+1} + \sum_{k=1}^{g+1} c_{g+1-k} x^{g+1-k} + \frac{c-1}{y}, \]

where we put

\[ c_g = \sum_{i=1}^{g+1} (I_i + V_i), \quad (5) \]

\[ c_{g+1-k} = \sum_{\Omega \subseteq \Lambda} \sum_{|\Omega| = k} |L_{\Omega}| - I_i V_i \sum_{|\Omega| = k} |L_{\Omega}| \quad (k = 2, 3, \ldots, g+1). \quad (6) \]

Thus, the coefficients \( c_{g+1-k} \) is a homogeneous polynomial in \( I_i \) and \( V_j \) of degree \( k \). A subset \( \Omega \subseteq \Lambda \), such that \( |\Omega| = k \) is a union of sets of consecutive numbers:

\[ \Omega = \bigcup_{s=1}^{f} \Omega_s, \quad \Omega_s = \{ \omega_s, \omega_s + 1, \ldots, \omega_s + |\Omega_s| - 1 \}, \]

where we assume \( \omega_s < \omega_t \) for \( s < t \). Therefore, the matrix \( L_\Omega \) is block diagonal. This implies \( |L_{\Omega}| = \prod_{s=1}^{f} |L_{\Omega_s}| \). The determinant \( |L_{\Omega_x}| \) is given by

\[ |L_{\Omega_x}| = \prod_{m=0}^{m-1} \prod_{j=0}^{j+1} \prod_{p=0}^{p+1} \prod_{q=0}^{q+1} V_{j+p+q}, \]

where we assume \( \prod_{m=0}^{m-1} I_{m+1} + I_{m+0} \prod_{j=0}^{j+1} V_{j+0} = 1 \).

Now assume \( 1, g+1 \in \Omega \) for \( \Omega = \bigcup_{s=1}^{f} \Omega_s \), such that \( |\Omega| = k \). Then, by definition, we find \( 1 \in \Omega_1 \) and \( g+1 \in \Omega_f \). Put \( \Omega_1 := \Omega \setminus \{1\} \) and \( \Omega_f := \Omega \setminus \{g+1\} \). Then, we have

\[ |L_{\Omega_1}| = \prod_{i=2}^{g+1} I_i + V_1 |L_{\Omega_1}|, \quad |L_{\Omega_f}| = \prod_{j=g+1}^{g+2} V_j + I_1 |L_{\Omega_f}|, \]

where we use the fact \( \omega_0 + |\Omega_1| - 1 = g+1 \). Moreover, set \( \Omega := \bigcup_{s=2}^{f} \Omega_s, \quad \bar{\Omega} := \bar{\Omega}_1 \cup \bar{\Omega}_2 \cup \bar{\Omega}_1, \quad \Omega_1 := \Omega \cup \bar{\Omega}_1 \) and \( \Omega_f := \Omega_1 \cup \bar{\Omega}_f \). Then, for \( \Omega = \bigcup_{s=1}^{f} \Omega_s \), such that \( 1 \in \Omega_1 \) and \( g+1 \in \Omega_f \), we have

\[ |L_{\Omega}| = \prod_{i=2}^{g+1} I_i |L_{\Omega_1}| + \prod_{j=g+1}^{g+2} V_j |L_{\Omega_1}| + \prod_{i=1}^{g+1} I_i |L_{\Omega_f}| + I_1 V_1 |L_{\Omega_f}|. \]
Noting $\tilde{\Omega} \subset \tilde{\Lambda}$ and $|\tilde{\Omega}| = k - 2$, we obtain

$$c_{g+1-k} = \sum_{\Omega \subset \Lambda} \left| L_{\Omega} \right| + \sum_{\Omega \subset \Lambda, \left| \Omega \right| = k} \left( \prod_{i=2}^{g+1} L_i \prod_{j=m_0}^{g+2} V_j |L_{\Omega_i}| + \prod_{j=m_0}^{g+2} L_i |L_{\Omega_j}| + \prod_{j=m_0}^{g+2} V_j |L_{\Omega_j}| \right).$$

where the first sum is taken over all $\Omega' \subset \Lambda$ which does not contain both 1 and $g + 1$, and the second sum is taken over all $\Omega \subset \Lambda$ which contains both 1 and $g + 1$. We assume $|L_{\Omega}| = 1$. Thus, we obtain the following.

**Proposition 1.** The coefficient $c_{g+1-k}$ of $x^{g+1-k}$ in $\overline{f}(x, y)$ is a subtraction-free homogeneous polynomial in $I_1, \ldots, I_{g+1}, V_1, \ldots, V_{g+1}$ of degree $k$ for $k = 1, 2, \ldots, g + 1$.

**Example 1.** Put $k = g + 1$. Then there exists no $\Omega'$ which does not contain both 1 and $g + 1$ and we find $\Omega = \Lambda$ with $|\Omega| = g + 1$. This implies $\Omega_1 = \Omega$, $\Omega_2 = \emptyset$. Therefore, we obtain

$$c_0 = \prod_{i=1}^{g+1} I_i + \prod_{j=1}^{g+1} V_j.$$

Put $k = g$. Then we find $\Omega'' = \tilde{\Lambda} := \Lambda \setminus \{ g + 1 \} = \{ 1, 2, \ldots, g \}$ or $\Omega' = \{ 2, \ldots, g, g + 1 \}$. We also find $\Omega = \Omega_1 \cup \Omega_2$, $\Omega_1 = \{ 1, 2, \ldots, n - 1 \}$ and $\Omega_2 = \{ n + 1, n + 2, \ldots, g + 1 \}$ for $n = 2, 3, \ldots, g$. Noting $|\Omega_1| = n - 1$ and $|\Omega_2| = n + 1$, we obtain

$$c_1 = \sum_{m=0}^{g} \left( \prod_{j=m+2}^{g+1} I_j \prod_{j=m+3}^{g+2} V_j + \prod_{j=m+2}^{g+1} I_j \prod_{j=m+3}^{g+2} V_j \right) + \sum_{n=2}^{g+1} \left( \prod_{i=1}^{n} I_i \prod_{j=1+n}^{g+1} V_j + \prod_{j=1+n}^{g+1} V_j \sum_{m=0}^{g-2} \prod_{j=m+2}^{g+1} I_j \prod_{j=m+3}^{g+1} V_j \right).$$

\[ (8) \]

### 3.2. Spectral curves

Let us consider the affine curves on $\mathbb{C}^2_{(x, y)}$ and $\mathbb{C}^2_{(u, v)}$, respectively

$$\tilde{\gamma}_0 := (\tilde{f}(x, y) = 0),$$
$$\tilde{\gamma}_\infty := (v^2 + v(1 + c_0 u + \cdots + c_1 u^{g+1} + c_{-1} u^{2g+2}) = 0).$$

These affine curves are non-singular under certain conditions for $c_i$s. Let the projections on the curves be

$$\pi_0 : \tilde{\gamma}_0 \to \mathbb{C}^2_1; \quad (a, b) \mapsto (a), \quad \pi_\infty : \tilde{\gamma}_\infty \to \mathbb{C}^2_{u}; \quad (\alpha, \beta) \mapsto \alpha.$$

By gluing $\tilde{\gamma}_0$ with $\tilde{\gamma}_\infty$ in terms of the bi-holomorphic map $\tilde{\gamma}_0 \setminus \pi_0^{-1}(0) \to \tilde{\gamma}_\infty \setminus \pi_\infty^{-1}(0)$;

$$(x, y) \mapsto (u, v) = \left( \frac{1}{x}, x^{g+1} \right),$$

we obtain the hyperelliptic curve $\tilde{\gamma} = \tilde{\gamma}_0 \cup \tilde{\gamma}_\infty$ of genus $g$.

Substitute $x = 1/y$ into $\tilde{f}(x, y) = 0$. We then find

$$(t^{g+1} y^2 + (t^{g+1} y)(1 + c_0 t + \cdots + c_1 t^{g+1} + c_{-1} t^{2g+2}) + c_{-1} t^{2g+2} = 0.$$
For sufficiently small $t$, this equation reduces to $(t^{g+1}y)^2 + (t^{g+1}y) \sim 0$. Hence, we obtain $(x, y) \sim (1/t, 0)$, $(1/t, -1/t^{g+1})$, or equivalently $(u, v) \sim (t, 0)$, $(t, -1)$. Thus, the curve $\tilde{y}$ can be expressed as

$$\tilde{y} = \tilde{y}_0 \cup \tilde{P}_\infty,$$

where $\tilde{P}_\infty := (0, 0)$, $\tilde{P}_\infty := (0, -1) \in C(x, y)$ are the points at infinity.

Now we consider the rational map $\rho : C^2(x, y) \rightarrow C^2(x, y)$:

$$\rho : (x, y) \mapsto (u, v) = \left( x, y - \frac{c_{g-1}}{y} \right).$$

By applying $\rho$ to points on $\tilde{y}_0$, we obtain the affine curve $\gamma_0$ on $C^2(x, y)$:

$$\gamma_0 := \{(f(u, v) = 0) = \{(u, v) = \rho(x, y) \mid \tilde{f}(x, y) = 0\}, \quad f(u, v) := v^2 - (u^{g+1} + c_g u^{g+1} + \cdots + c_1 u + c_0)^2 + 4c_{g-1}.$$  

In the same manner as in section 2, by gluing $\gamma_0$ with the affine curve $\gamma_\infty := \{(y^2 - (1 + c_x x + \cdots + c_1 x^{g+1} + c_0 x^{g+1} + 4c_{g-1} x^{2g+2}) = 0\}$, we obtain the canonical hyperelliptic curve $\gamma = \gamma_0 \cup \gamma_\infty$ of genus $g$. Remember that the curve $\gamma$ can be given by

$$\gamma = \gamma_0 \cup \{P_\infty, P_\infty\}.$$  

where $P_\infty = (0, 1)$, $P_\infty' = (0, -1) \in C(x, y)$ are the points at infinity.

Let the solutions to the quadratic equation $\tilde{f}(1/t, y) = 0$ in $y$ be $y_+$ and $y_-$ which tend to 0 and $-1/t^{g+1}$ for sufficiently small $t$, respectively. Apply the rational map $\rho$ to the points $(1/t, y_+), (1/t, y_-) \in \tilde{y}_0$. Then, for sufficiently small $t$, we find

$$\rho \left( \frac{1}{t}, y_+ \right) \sim \left( \frac{1}{t^{g+1}}, \frac{1}{t^{g+1}} \right), \quad \rho \left( \frac{1}{t}, y_- \right) \sim \left( \frac{1}{t^{g+1}}, -\frac{1}{t^{g+1}} \right).$$

Therefore, the map $\rho$ is naturally extended to the hyperelliptic curve $\tilde{y}$:

$$\rho(\tilde{P}_\infty) = P_\infty, \quad \rho(\tilde{P}_\infty') = P_\infty'.$$

We also denote the extension by $\rho$. The hyperelliptic curves $\tilde{y}$ and $\gamma$ are called the spectral curves of the pdTL.

### 3.3. Eigenvector maps

Let the phase space of the pdTL be $U := \{(I_1, \ldots, I_{g+1}, V_1, \ldots, V_{g+1}) \mid 0 < \prod_{i=1}^{g+1} V_i < \prod_{i=1}^{g+1} I_i \}$. Also let the moduli space of $\gamma$ be $C := \{(c_{-1}, c_0, \ldots, c_g)\}$. Consider the map

$$\psi : U \rightarrow C; \quad (I_1, \ldots, I_{g+1}, V_1, \ldots, V_{g+1}) \mapsto (c_{-1}, c_0, \ldots, c_g)$$

defined by (4)–(6). We define the isolevel set $U_c$ of the pdTL to be

$$U_c := \psi^{-1}(c_{-1}, c_0, \ldots, c_g) \subset U.$$  

The isolevel set $U_c$ is isomorphic to the affine part of the Jacobian $J(\gamma)$ of $\gamma$, and the time evolution (2) is linearized on it [1, 16, 21].

Let $\varphi(x, y) = (\varphi_1, \ldots, \varphi_g, -\varphi_{g+1})^T$ be the eigenvector of the Lax matrix $L$ associated with the eigenvalue $x$. Consider the eigenvalue equation of $L$

$$(xI + L)\varphi(x, y) = 0.$$  

(9)
Let the \((i, j)\)-entry of the matrix \(L\) be \(l_{ij}\). By applying the Cramer formula, each element of \(\varphi(x, y)\) is explicitly given by

\[
\varphi_i(x, y) = \begin{vmatrix}
I_{l_{11}} + x & \cdots & I_{l_{1,i-1}} & l_{1,i+1} & \cdots & l_{1g} \\
\vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
l_{1j} & \cdots & I_{l_{j,i-1}} & l_{j,i+1} & \cdots & l_{jg} + x
\end{vmatrix}
\]

for \(i = 1, 2, \ldots, g\) and

\[
\varphi_{g+1}(x) = |xI + L_{\lambda}|.
\]

Thus, we see that \(\varphi_{g+1}(x)\) is the eigenpolynomial of \(L_{\lambda}\).

We introduce another expression of \(\varphi_{g+1}(x)\). Expand the eigenpolynomial \(|xI + L|\) with respect to the last row:

\[
|xI + L| = y\varphi_1(x, y) - I_{g+1}V_g\varphi_g(x, y) + (x + I_1 + V_{g+1})\varphi_{g+1}(x).
\]

(10)

To show several properties of the spectral curves, we will set the variables \(x, y\) and parameters \(I_1, I_{g+1}, V_1, V_{g+1}\) specially. For this purpose, we denote \(|xI + L|\) by \(\theta(x, y; I_1, V_1, I_{g+1}, V_{g+1})\). By setting \(I_1 = V_{g+1} = 0\) in (10), we obtain an expression of \(\varphi_{g+1}(x)\)

\[
\varphi_{g+1}(x) = \frac{\theta(x, y; 0, I_{g+1}, V_1, 0) - y}{x}.
\]

Now we define the eigenvector map \([16, 39]\). Each equation in (9) is a three-term relation among \(\varphi_{i-1}, \varphi_i,\) and \(\varphi_{i+1}\). Therefore, if a point \((x, y)\) satisfies \(\varphi_1(x, y) = \varphi_2(x, y) = 0\), then \(\varphi_i(x, y) = 0\) holds for all \(i = 1, 2, \ldots, g + 1\). We can easily see that there exist exactly \(g\) points \((x, y)\) satisfying \(\varphi_1(x, y) = \varphi_2(x, y) = 0\), counting multiplicities. Denote these points by \(\tilde{P}_1, \tilde{P}_2, \ldots, \tilde{P}_g\). By eliminating \(y\) from \(\varphi_1(x, y) = 0\) and \(\varphi_2(x, y) = 0\), we obtain \(\varphi_{g+1}(x) = |xI + L_{\lambda}| = 0\). Thus, the \(x\)-component of \(\tilde{P}_i\) \((i = 1, 2, \ldots, g)\) is the eigenvalue of \(L_{\lambda}\). Moreover, we see that all \(\tilde{P}_i\)s are on the spectral curve \(\tilde{\gamma}\) because the \((g + 1)\)th equation

\[
(-1)^g y\varphi_1(x, y) + I_{g+1}V_g\varphi_g(x, y) - (I_1 + V_{g+1} + x)\varphi_{g+1}(x) = 0
\]

in (9) is equivalent to \(\tilde{f}(x, y) = 0\), the defining equation of \(\tilde{\gamma}\). We choose \(D_P = P_1 + \cdots + P_g\), where \(P_i := \rho(\tilde{P}_i)\), as a representative of \(\text{Pic}^e(\gamma) := \mathcal{D}_e(\gamma)/\mathcal{D}_1(\gamma)\). We define the eigenvector map

\[
\phi : U_c \to \text{Pic}^e(\gamma); \ U \mapsto \phi(U) \equiv D_P \pmod{\mathcal{D}_1(\gamma)}.
\]

Let the subset \(\mathcal{D}\) of \(\mathcal{D}_e^+(\gamma)\) be

\[
\mathcal{D} := \{D_P \in \mathcal{D}_e^+(\gamma) \mid \phi(U) \equiv D_P \pmod{\mathcal{D}_1(\gamma)} \text{ for } U \in U_c\}.
\]

**Theorem 4.** The reduced map \(\Phi : \mathcal{D} \to \text{Pic}^0(\gamma)\) is injective.

**Proof.** If \(\Phi(D_P) \equiv \Phi(D_Q)\) then, by theorem 3, we have

\[
D_P + D_Q = D_R + D_R
\]

(11)

for some \(D_R \in \mathcal{D}_e^+(\gamma)\). It immediately follows

\[
D_P + D_Q = D_R + D_R.
\]

(12)

Subtract (12) from (11), we then obtain

\[
D_P - D_P + D_Q - D_Q = 0.
\]

(13)
Let \(D_P = \sum_{i=1}^{g} P_i\) be the representative of \(\text{Pic}^0(\gamma)\), such that \(\phi(U) \equiv D_P \pmod{\mathcal{D}_1(\gamma)}\) for \(U \in \mathcal{U}_c\). Let \(P_1, \ldots, P_k\) be the bifurcation points and \(P_{k+1}, \ldots, P_g\) ordinary points. Then, by construction of \(D_P\), we have
\[
P_i = P'_i \quad i = 1, \ldots, k,
\]
(14)

\[
\{P_{k+1}, \ldots, P_g\} \cap \{P'_{k+1}, \ldots, P'_g\} = \emptyset.
\]
(15)

Also, let \(D_Q = \sum_{i=1}^{g} Q_i\) be the representative of \(\text{Pic}^0(\gamma)\) such that \(\phi(V) \equiv D_Q \pmod{\mathcal{D}_1(\gamma)}\) for \(V \in \mathcal{U}_c\). Let \(Q_1, \ldots, Q_l\) be the bifurcation points and \(Q_{l+1}, \ldots, Q_g\) ordinary points. Then, we have
\[
Q_i = Q'_i \quad i = 1, \ldots, l,
\]
(16)

\[
\{Q_{l+1}, \ldots, Q_g\} \cap \{Q'_{l+1}, \ldots, Q'_g\} = \emptyset.
\]
(17)

By (13), (14) and (16), we find
\[
D_P - D_P' + D_Q' - D_Q = \sum_{i=k+1}^{g} P_i - \sum_{i=l+1}^{g} P'_i + \sum_{i=l+1}^{g} Q'_i - \sum_{i=l+1}^{g} Q_i = 0.
\]

By (15) and (17), we further obtain
\[
\sum_{i=k+1}^{g} P_i = \sum_{i=l+1}^{g} Q_i, \quad \sum_{i=k+1}^{g} P'_i = \sum_{i=l+1}^{g} Q'_i.
\]

This implies \(k = l\) and \(D_P \equiv D_Q\).

3.4. Time evolution

In the linearization of time evolution of the pdTL on the Jacobian \(J(\gamma)\), we choose the element \(D_P \equiv \phi(U) \pmod{\mathcal{D}_1(\gamma)}\) of \(\mathcal{D}_1^+(\gamma)\) as the representative of an element \(U\) of the isospectral set \(\mathcal{U}_c\). Theorem 4 shows that the subset \(\mathcal{D}\) of \(\mathcal{D}_1^+(\gamma)\) consisting of such representatives is injectively mapped into \(\text{Pic}^0(\gamma)\) by the canonical map \(\Phi\). Therefore, the time evolution of pdTL is also linearized on \(\text{Pic}^0(\gamma)\) through the injection
\[
\mathcal{U}_c \rightarrow \text{Pic}^0(\gamma); \quad U \mapsto \Phi(\Phi \circ \phi)(U) = \Phi(D_P).
\]

It should be noted that the following theorem concerning the linearization of time evolution of the pdTL is shown by Iwao [16, 18].

**Theorem 5** (Proposition 2.16 in [16] with \(N = g + 1\) and \(M = 1\)). Let \(D\) be the divisor \(D = A - P_{\infty}\) where \(A = (0, \prod_{i=1}^{g+1} V_i - \prod_{i=1}^{g+1} I_i)\). Then, the following diagram is commutative:

\[
\begin{array}{ccc}
\mathcal{U}_c & \xrightarrow{\phi} & \text{Pic}^0(\gamma) \\
\downarrow & & \downarrow +D \\
\mathcal{U}_c & \xrightarrow{\phi} & \text{Pic}^0(\gamma).
\end{array}
\]

By theorem 5, there exists a rational function \(h\) on \(\gamma\) such that \(D_{\tilde{P}} = D_P + D + (h)\). It immediately follows
\[
D_{\tilde{P}} - D^* \equiv D_P - D^* + D \pmod{\mathcal{D}_1(\gamma)}.
\]

Note that this is an addition formula on \(\text{Pic}^0(\gamma)\).
Consider the divisor
\[ T := D + D^* = \begin{cases} 
A + \frac{g-2}{2} (P_\infty + P'_\infty) + P_\infty & \text{for even } g \\
A + \frac{g-3}{2} (P_\infty + P'_\infty) + 2P_\infty & \text{for odd } g.
\end{cases} \]
We observe that \( T \in \mathcal{D}_g^+(\gamma) \) for \( g \geq 2 \), and hence we have
\[ \Phi_1(T) \equiv D \pmod{D_l(\gamma)}. \]

Since \( \mu : \mathcal{D}_g^+(\gamma) \to \text{Sym}^g(\gamma) \) is bijective, we obtain the following theorem.

**Theorem 6.** Let \( \tau := \mu(T) \in \text{Sym}^g(\gamma) \) for \( g \geq 2 \). Then, the following diagram is commutative:
\[
\begin{array}{ccc}
\mathcal{U}_c & \xrightarrow{\mu \circ \phi} & \text{Sym}^g(\gamma) \\
(2) \downarrow & & \downarrow \oplus \tau \\
\mathcal{U}_c & \xrightarrow{\mu \circ \phi} & \text{Sym}^g(\gamma).
\end{array}
\]

Thus, the following addition formula on \( \text{Sym}^g(\gamma) \) is equivalent to the time evolution \( (2) \) of the pdTL:
\[ d_{\bar{p}} = d_p \oplus \tau. \]

**4. A geometric realization of pdTL**

In this section, we realize the time evolution \( (18) \) of the pdTL on \( \text{Sym}^g(\gamma) \) by using curve intersections.

**4.1. Linear spaces of rational functions**

The addition formula \( (18) \) on \( \text{Sym}^g(\gamma) \) reduces to
\[ -d_{\bar{p}} \oplus d_p \oplus \tau = 0, \]
where \( -d_{\bar{p}} \) is the inverse of \( d_{\bar{p}} \) with respect to \( \oplus \). This can be written by the divisors on \( \text{Pic}^0(\gamma) \):
\[ D_Q + D_p + A - P'_\infty - 2D^* \equiv 0 \pmod{D_l(\gamma)}, \]
where we put \( D_Q := \mu^{-1}(-d_{\bar{p}}) \). This implies that there exists a rational function \( h \) on \( \gamma \), such that \( (h) = D_Q + D_p + A - P'_\infty - 2D^* \) and \( h \in L(P'_\infty + 2D^*) \). The curve given by \( h \) passes through the points \( A, P_1, \ldots, P_g, Q_1, \ldots, Q_g \) on \( \gamma \).

In order to obtain the curves given by rational functions in \( L(P'_\infty + 2D^*) \), we first establish the linear space \( L(P'_\infty + 2D^*) \). For a while, we assume that \( g \) is an even number. We then have \( L(P'_\infty + 2D^*) = L(gP_\infty + (g+1)P'_\infty) \). The principal divisors of the coordinate functions \( x \) and \( y \) are
\[
(x) = P_0 + P'_0 - P_\infty - P'_\infty \quad \text{and} \quad (y) = \sum_{i=1}^{2g+2} P_{a_i} - (g+1)(P_\infty + P'_\infty),
\]

\[ ^2 \] For the case of \( g = 1 \), see appendix C.
Hence, we consider the linear space whose form is given by the divisors on Pic₀\(\mathcal{P}\)

\[\langle h \rangle\]

Therefore, the rational function \(x^g\) has a pole at \(P'_\infty\) whose order is smaller than or equal to \(g\). Also let the local parameter at \(P'_\infty\) be \(s\). Then, we also find

\[x^{g+1} - y = 2s^{-g-1} - c_g s^{-g} - c_{g-1} s^{-g+1} + o(s^{-g+1}).\]

Thus, the rational function \(x^{g+1} - y\) has a pole at \(P'_\infty\) whose order is exactly \(g + 1\). Therefore, we find \(x^{g+1} - y \in L(P'_\infty + 2D^*)\). By the Riemann–Roch theorem, we have

\[\dim L(P'_\infty + 2D^*) = g + 2.\]

Thus, we obtain \(L(P'_\infty + 2D^*) = \langle 1, x, \ldots, x^g, x^{g+1} - y \rangle\).

In a similar manner, we also obtain \(L(P'_\infty + 2D^*) = \langle 1, x, \ldots, x^g, x^{g+1} + y \rangle\) for an odd number \(g\).

Next, consider the addition formula on \(\text{Sym}^2(\mathcal{O})\)

\[-d_P \oplus d_P = o.\]

This can be written by the divisors on Pic₀\(\mathcal{O}\):

\[D_Q + D_P - 2D^* \equiv 0 \pmod{D}_l(\mathcal{O}).\]

Hence, we consider the linear space \(L(2D^*)\). For an odd number \(g\), we find \(L(2D^*) = \langle 1, x, \ldots, x^g, c_x x^g + x^{g+1} + y \rangle\). Thus, we obtain the following proposition.

**Proposition 2.** For any even \(g\), we have

\[L(P'_\infty + 2D^*) = \langle 1, x, \ldots, x^g, x^{g+1} - (-1)^g y \rangle.\]

Moreover, for any odd number \(g\), we have

\[L(2D^*) = \langle 1, x, \ldots, x^{g-1}, c_x x^g + x^{g+1} + y \rangle.\]

### 4.2. A curve passing through given points

Now, we construct a curve given by a rational function in \(L(P'_\infty + 2D^*)\) and passing through \(g + 1\) given points on \(\gamma\). We see from (20) that there exists a rational function \(h \in L(P'_\infty + 2D^*)\), such that \(h) = D_Q + D_P + A - P'_\infty - 2D^*\). By proposition 2, we can assume that \(h\) has the form \(h(x, y) = \beta_0 + \beta_1 x + \cdots + \beta_g x^g + x^{g+1} - (-1)^g y\), where \(\beta_0, \beta_1, \ldots, \beta_g \in \mathbb{C}\). By definition of \(h\), we have \(h(A) = 0\). Hence, we find \(\beta_0 = (-1)^g \langle \prod_{j=1}^{g+1} V_j - \prod_{l=1}^{g+1} 4 \rangle\).

We show an explicit formula which gives \(\beta_1, \beta_2, \ldots, \beta_g\) by using the coefficients \(c_1, c_2, \ldots, c_g\) of \(\gamma\). Introduce the involutions \(s_j\) and \(t_k\) \((j, k = 1, 2, \ldots, g + 1)\) on the phase space \(\mathcal{U}\) of the pTTL:

\[s_j : (I_1, \ldots, I_j, \ldots, V_{g+1}) \mapsto (I_1, \ldots, -I_j, \ldots, V_{g+1}).\]

\[t_k : (I_1, \ldots, V_k, \ldots, V_{g+1}) \mapsto (I_1, \ldots, -V_k, \ldots, V_{g+1}).\]

These involutions act naturally on the moduli space \(\mathcal{C}\) of \(\gamma\):

\[s_j c_i := c_i(s_j^{-1}(I_1, \ldots, I_j, \ldots, V_{g+1})) = c_i(I_1, \ldots, -I_j, \ldots, V_{g+1}),\]

\[t_k c_i := c_i(t_k^{-1}(I_1, \ldots, V_k, \ldots, V_{g+1})) = c_i(I_1, \ldots, -V_k, \ldots, V_{g+1}).\]
for \( i = -1, 0, \ldots, g \) and \( j, k = 1, 2, \ldots, g + 1 \). We then find

\[
\beta_0 = \begin{cases} 
  s_1 c_0 < 0 & \text{for even } g, \\
  t_{g+1} c_0 > 0 & \text{for odd } g,
\end{cases}
\]

(22)

where we use the assumption \( 0 < \prod_{i=1}^{g+1} V_j < \prod_{i=1}^{g+1} I_i \).

Let \( P_1, P_2, \ldots, P_g \) be the points on \( \gamma \) given by the eigenvector map: \( \sum_{i=1}^{g} P_i \equiv (\rho \circ \phi)(U) \) (mod \( D_i(\gamma) \)) for \( U \in U_c \). Assume that these points are in generic position. (The generic condition for \( P_i = (p_i, q_i) \) is \( p_i \neq p_j \) for \( i \neq j \in \{1, 2, \ldots, g\} \).) We have the following theorem.

Theorem 7. Let \( h_1 \) be a rational function in \( L(P_\infty + 2D^*) \). Then the curve \( \kappa_1 = (h_1(u, v) = 0) \) passing through the points \( A = (0, \prod_{i=1}^{g+1} V_i - \prod_{i=1}^{g+1} I_i) \) and \( P_1, P_2, \ldots, P_g \) are given by the formula

\[
h_1(u, v) = \begin{cases} 
  \sum_{i=0}^{g} s_1 c_i u^i + u^{g+1} - v & \text{for even } g, \\
  \sum_{i=0}^{g} t_{g+1} c_i u^i + u^{g+1} + v & \text{for odd } g.
\end{cases}
\]

(23)

Proof. Remember that \( |x \bar{I} + L| \) is denoted by \( \theta(x, y; I_1, I_{g+1}, V_1, V_{g+1}) \). By replacing \( y \) and \( I_1 \) in \( \theta(x, y; I_1, I_{g+1}, V_1, V_{g+1}) \) with \( -y \) and \( -I_1 \) respectively, we obtain (see (7))

\[
\theta(x, -y; -I_1, I_{g+1}, V_1, V_{g+1}) = -y + \frac{c_{-1}}{y} + x^{g+1} + \sum_{i=0}^{g} s_1 c_1 x^i.
\]

Similarly, by replacing \( V_{g+1} \) in \( \theta(x, y; I_1, I_{g+1}, V_1, V_{g+1}) \) with \( -V_{g+1} \), we obtain

\[
\theta(x, y; I_1, I_{g+1}, V_1, -V_{g+1}) = y - \frac{c_{-1}}{y} + x^{g+1} + \sum_{i=0}^{g} t_{g+1} c_i x^i.
\]

Here, we use the fact \( s_1 c_{-1} = t_{g+1} c_{-1} = -c_{-1} \). By applying the rational map \( \rho \) to these rational functions, we obtain \( h_1 \) defined by (23). Clearly, the rational function \( h_1 \) is in \( L(P_\infty + 2D^*) \) (see proposition 2).

Note that the curve \( \kappa_1 = (h_1(u, v) = 0) \) passes through the point \( A \in \gamma \) because of the fact (22). Let \( \tilde{P}_i = (\tilde{p}_i, \tilde{q}_i) \) (\( i = 1, 2, \ldots, g \)) be the points on \( \tilde{\gamma} \) given by the eigenvector map. Then, we have \( \psi_i(\tilde{p}_i, \tilde{q}_i) = \cdots = \psi_g(\tilde{p}_g, \tilde{q}_g) = 0 \) and \( \psi_{g+1}(\tilde{p}_i) = 0 \). Moreover, we have (see (10))

\[
\theta(x, -y; -I_1, I_{g+1}, V_1, V_{g+1}) = -y\psi_1(x, y) - I_{g+1} V_{g+1} \psi_1(x, y) + (x - I_1 + V_{g+1}) \psi_{g+1}(x),
\]

\[
\theta(x, y; I_1, I_{g+1}, V_1, -V_{g+1}) = y\psi_1(x, y) + I_{g+1} V_{g+1} \psi_1(x, y) + (x + I_1 - V_{g+1}) \psi_{g+1}(x).
\]

Therefore, we find that \( \theta(\tilde{p}_i, \tilde{q}_i; -I_1, I_{g+1}, V_1, V_{g+1}) = 0 \) and \( \theta(\tilde{p}_i, \tilde{q}_i; I_1, I_{g+1}, V_1, -V_{g+1}) = 0 \) hold for all \( i \). Hence, we finally show that \( \kappa_1 \) passes through \( \tilde{P}_i = \rho(\tilde{P}_i) \) (\( i = 1, 2, \ldots, g \)). If the points \( P_1, P_2, \ldots, P_g \) are in generic position, then the curve \( \kappa_1 \) is uniquely determined. \( \square \)

By construction, \( \kappa_1 \) also passes through \( Q_1, Q_2, \ldots, Q_g \) satisfying (20). Thus the points \( P_1, P_2, \ldots, P_g, Q_1, Q_2, \ldots, Q_g \) are on both \( \gamma \) and \( \kappa_1 \). Therefore, the addition formula (19) can be realized by using the intersection of \( \gamma \) and \( \kappa_1 \). In order to realize the time evolution of the pdTL by using curve intersections, we have only to give the inverse \( d_\rho \) of \( -d_\rho = \mu(D_Q) \) in terms of a curve intersection.
4.3. Inverse elements

The inverse $d_{\tilde{p}} \in \text{Sym}^{\gamma}(\gamma)$ of $-d_p = \mu(D_Q) \in \text{Sym}^{\gamma}(\gamma)$ is given by the following theorem.

**Theorem 8.** Let $Q_1, Q_2, \ldots, Q_g$ be the points on $\gamma$ satisfying (20) and (21) (see theorem 7). Also let $\bar{P}_1, \bar{P}_2, \ldots, \bar{P}_g$ be the points on $\gamma$ satisfying (21). If $g$ is an even number, we have $\bar{P}_i = Q'_i$ for $i = 1, 2, \ldots, g$.

If $g$ is an odd number, then $\bar{P}_1, \bar{P}_2, \ldots, \bar{P}_g$ are the intersection points of $\gamma$ and the curve $\kappa_2 = (h_2(u, v) = 0)$ given by

$$h_2(u, v) = \sum_{i=0}^{g-1} (s_1 c_i + 2I \dot{c}_{i+1}) u^i + c_g u^g + u^{g+1} + v,$$

where $\dot{c}_i$ is obtained from $c_i$ by setting $I_1 = V_1 = 0$ for $i = 1, 2, \ldots, g$.

**Proof.** If $g$ is an even number, we can easily see $\bar{P}_i = Q'_i$ for $i = 1, 2, \ldots, g$. This implies the first part.

Next, assume that $g$ is an odd number. Note that the eigenpolynomial of $L$ is invariant with respect to the time evolution. We then have

$$\theta(x, y; I, I_{g+1}, V_1, V_{g+1}) = y\bar{\psi}_i(x, y) - I_1V_{g+1}\bar{\psi}_i(x, y) + (x + I_1 + V_1)\bar{\psi}_{g+1}(x).$$

Here, $\bar{\psi}_i$ is the $i$th element of the eigenvector $\bar{\psi}$ of $\bar{L}$ for $i = 1, 2, \ldots, g + 1$. By setting $I_1 = V_1 = 0$, we obtain

$$\bar{\psi}_{g+1}(x) = \frac{\theta(x, y; 0, I_{g+1}, 0, V_{g+1}) - y}{x} = x^g + \sum_{i=0}^{g-1} \dot{c}_{i+1} x^i,$$

where $\dot{c}_i$ is obtained from $c_i$ by setting $I_1 = V_1 = 0$. Note that the $x$-component $\bar{\psi}_i$ of $\bar{P}_i$ solves $\bar{\psi}_{g+1}(x) = 0$ by definition. Therefore, we have

$$\dot{c}_{i+1} = (-1)^{g-i} \sum_{\Omega \subset \tilde{A}} \prod_{j \in \Omega} \bar{p}_j,$$  \hspace{1cm} (25)

where $\tilde{A} = \{1, 2, \ldots, g\}$.

From (21), there exists a rational function $h_2 \in L(2D^*)$, such that

$$(h_2) = D_Q + D_{\tilde{p}} - 2D^*.$$  \hspace{1cm} (26)

Since $h_2 \in L(2D^*)$, it can be written by $h_2(x, y) = \delta_0 + \delta_1 x + \cdots + \delta_{g-1} x^{g-1} + c_g x^g + x^{g+1} + y$ for $\delta_0, \delta_1, \ldots, \delta_{g-1} \in \mathbb{C}$ (see proposition 2). The coefficients are determined by solving the system of linear equations

$$h_2(\bar{P}_i) = 0 \quad i = 1, 2, \ldots, g.$$  \hspace{1cm} (27)

By using the Cramer formula, we find that if and only if the condition $\bar{p}_i \neq \bar{p}_j$ for $i, j = 1, 2, \ldots, g, i \neq j$ holds, then the following $\delta_i (i = 0, 2, \ldots, g - 1)$ solve equation (27)

$$\delta_i = s_1 c_i + (-1)^{g-i}(c_g - s_1 c_g) \sum_{\Omega \subset \tilde{A}} \prod_{j \in \Omega} \bar{p}_j = s_1 c_i + 2I \dot{c}_{i+1}.$$  

Here, we use the fact $c_g - s_1 c_g = 2I_i$ and (25).
Thus, we see that the rational function $h_2$ satisfying (26) is given by (24). Therefore, the curve $\kappa_2 = (h_2(u, v) = 0)$ passes through the points $Q_1, Q_2, \ldots, Q_g$ and $\tilde{P}_1, \tilde{P}_2, \ldots, \tilde{P}_g$ on $\gamma$.

Thus, the addition formula (18) on Sym$^g(\gamma)$ equivalent to the time evolution (2) of the pdTL is realized by using the intersection of $\gamma, \kappa_1$ and $\kappa_2$.

**Example 2.** Put $g = 3$. The spectral curve $\gamma$ is given by
\[
 f(u, v) = v^4 - (u^4 + c_3u^3 + c_2u^2 + c_1u + c_0)^2 + 4c_{-1},
\]
\[
c_1 = \sum_{i=1}^{4} (U_i + V_i), \quad c_2 = \sum_{1 \leq i < j \leq 4} (I_i, I_j) + V_i V_j + \sum_{i=1}^{4} I_i (V_{i+1} + V_{i+2}),
\]
\[
c_0 = \prod_{i=1}^{4} U_i + \prod_{i=1}^{4} V_i \quad \text{for odd}
\]
\[
c_0 = \prod_{i=1}^{4} I_i \quad \text{for even}
\]

The curve $\kappa_1$ passing through $P_1, P_2, P_3$ and $A = (0, \prod_{i=1}^{4} V_i - \prod_{i=1}^{4} l_i)$ is given by
\[
h_1(u, v) = t_4 c_0 + t_4 c_1 u + t_4 c_2 u^2 + t_4 c_3 u^3 + u^4 + v.
\]

The curve $\kappa_2$ passing through $Q_1, Q_2, Q_3$, the intersection points of $\gamma$ and $\kappa_1$, is given by
\[
h_2(u, v) = c_0 + 2h_1 V_i (I_2 I_3 + I_2 V_2 + V_2 V_3) + \{c_1 + 2h_1 V_i (I_2 + I_3 + V_2 + V_3)\} u + (c_2 + 2h_1 V_i) u^2 + c_3 u^3 + u^4 + v.
\]

Figure 1 shows an example of the intersection $\gamma, \kappa_1$ and $\kappa_2$.

### 4.4. Discrete motion of curves

We can interpret the time evolution of the pdTL as discrete motion of affine curves.

Let $C_1$ be the moduli space of the curve $\kappa_1 = (h_1(u, v) = 0)$:
\[
 C_1 = \begin{cases}
  \{(s_1 c_{i_0}, s_1 c_{i_1}, \ldots, s_1 c_{i_g})\} & \text{for even } g, \\
  \{(t_{g+1} c_{i_0}, t_{g+1} c_{i_1}, \ldots, t_{g+1} c_{i_g})\} & \text{for odd } g.
\end{cases}
\]

Define a map $\tilde{\psi} : U_c \rightarrow C_1$ to be
\[
 \tilde{\psi} : (I_1, \ldots, I_{g+1}, V_1, \ldots, V_{g+1}) \mapsto \begin{cases}
  (s_1 c_{i_0}, \ldots, s_1 c_{i_g}) & \text{for even } g, \\
  (t_{g+1} c_{i_0}, \ldots, t_{g+1} c_{i_g}) & \text{for odd } g.
\end{cases}
\]

Also define a map $\nu$ on $C_1$ to be
\[
 \nu : \begin{cases}
  (s_1 c_{i_0}, \ldots, s_1 c_{i_g}) \mapsto (s_1 \tilde{c}_{i_0}, \ldots, s_1 \tilde{c}_{i_g}) & \text{for even } g, \\
  (t_{g+1} c_{i_0}, \ldots, t_{g+1} c_{i_g}) \mapsto (t_{g+1} \tilde{c}_{i_0}, \ldots, t_{g+1} \tilde{c}_{i_g}) & \text{for odd } g,
\end{cases}
\]
so that the following diagram is commutative:
\[
 \begin{array}{ccc}
 U_c & \xrightarrow{\tilde{\psi}} & C_1 \\
 \downarrow (2) & & \downarrow \nu \\
 U_c & \xrightarrow{\psi} & C_1.
\end{array}
\]
Figure 1. The solid curve is $\gamma$, the dashed one is $\kappa_1$ and the dotted one is $\kappa_2$. The intersection point of $\gamma$ and $\kappa_1$ on the vertical axis is $A$. The triple intersection points of $\gamma$, $\kappa_1$, and $\kappa_2$ are $Q_1$, $Q_2$ and $Q_3$. The intersection points of $\gamma$ and $\kappa_1$ other than $Q_i$ are $P_1$, $P_2$, and $P_3$. Here, we set $I_1 = 1$, $I_2 = 3$, $I_3 = I_4 = 2$, $V_1 = 2$ and $V_2 = V_3 = V_4 = 1$.

Then, $\upsilon$ induces the discrete motion of curves

$$\kappa_1^0 \to \kappa_1^1 \to \kappa_1^2 \to \cdots,$$

where $\kappa_1^i = (h_1^i(u, v) = 0)$ and

$$h_1^i(u, v) = \begin{cases} \sum_{i=0}^{g} s_i c_i u^{i+1} + u^{i+1} - v & \text{for even } g, \\ \sum_{i=0}^{g} t_{i+1} c_i u^{i+1} + u^{i+1} + v & \text{for odd } g. \end{cases}$$

Figure 2 shows an example of the discrete motion of $\kappa_1^i$.

5. Ultradiscrete periodic Toda lattice and tropical hyperelliptic curves

In this section, we tropicalize the geometric framework concerning the time evolution of the pdTL shown above. We then present a tropical geometric realization of the UD-pTL, which is the ultradiscretization of the pdTL.

5.1. UD-pTL and its spectral curve

Suppose that the pdTL has a one parameter family of positive solutions $I_i(\epsilon) > 0$ and $V_i(\epsilon) > 0$ for $i = 1, 2, \ldots, g + 1$, where $\epsilon$ is a positive number. Also suppose that the limits $\lim_{\epsilon \to 0^+} -\epsilon \log I_i(\epsilon) = J_i \in \mathbb{R}$ and $\lim_{\epsilon \to 0^+} -\epsilon \log V_i(\epsilon) = W_i \in \mathbb{R}$ exist. Then, $J_i$ and $W_i$ ($i = 1, 2, \ldots, g + 1$) solve the difference equation
The discrete motion of the curves $\kappa_t^1$ ($t = 0, 1, \ldots, 7$) induced by $\upsilon$. The solid curves are $\kappa_t^1$ and the dashed one is $\gamma$. The figures are sorted in time increasing order from left to right and top to bottom. Here, we set $I_0^1 = 1, I_0^2 = 3, I_0^3 = I_0^4 = 2, V_0^1 = 2$ and $V_0^2 = V_0^3 = V_0^4 = 1$.

![Figure 2](image)

### Equation (28)

\[
\begin{align*}
\bar{J}_i &= [W_i, X_i + J_i], \\
\bar{W}_i &= J_{i+1} + W_i - \bar{J}_i,
\end{align*}
\]

where we define

\[
\lfloor A, B, \ldots \rfloor := \min\{A, B, \ldots\}, \quad \lceil A, B, \ldots \rceil := \max\{A, B, \ldots\}
\]

for $A, B, \ldots \in \mathbb{R}$. Here, we assume that $\sum_{i=1}^{g-1} (J_{i-1} - W_{i-1}) = 0$ and

\[
\sum_{i=1}^{g+1} J_i < \sum_{i=1}^{g+1} W_i.
\]

We denote the map $\mathbb{R}^{2g+2} \rightarrow \mathbb{R}^{2g+2}$;

\[
(J_1, \ldots, J_{g+1}, W_1, \ldots, W_{g+1}) \mapsto (\bar{J}_1, \ldots, \bar{J}_{g+1}, \bar{W}_1, \ldots, \bar{W}_{g+1})
\]

by $\zeta$ and use the notation

\[
(J_1', \ldots, J_{g+1}', W_1', \ldots, W_{g+1}') := \underbrace{\zeta \circ \cdots \circ \zeta}_{t}(J_1, \ldots, J_{g+1}, W_1, \ldots, W_{g+1})
\]

for $t = 0, 1, \ldots, 7$.

We call the dynamical system generated by (28) the UD-pTL. In particular, if the variables $J_1, \ldots, J_{g+1}$ and $W_1, \ldots, W_{g+1}$ take positive integer values then the dynamical system is called the pBBS [21, 38]. The pBBS is obtained from the BBS, which was introduced by Takahashi and Satsuma as a soliton cellular automaton [33], by imposing a periodic boundary condition. The procedure which reduces (2) to (28) is called the ultradiscretization [35].

Now, we introduce the spectral curve of the UD-pTL by using the procedure of ultradiscretization. Note first that all coefficients $c_{-1}, c_0, \ldots, c_g$ in $f$, the defining polynomial of the spectral curve $\gamma$ of the pTL, are subtraction free (see proposition 1). Hence, we can apply the ultradiscretization procedure to them; suppose that the positive numbers $x, y$ and $c_i$ are parametrized with $\epsilon > 0$ and the limits $\lim_{\epsilon \rightarrow 0} -\epsilon \log x = X, \lim_{\epsilon \rightarrow 0} -\epsilon \log y = Y$ and $\lim_{\epsilon \rightarrow 0} -\epsilon \log c_i = C_i$ exist ($i = -1, 0, \ldots, g$). Then, $f$ reduces to the following tropical polynomial $F$ in the limit $\epsilon \rightarrow 0$:
Assume $C_1$ \((j)\) \(J. Phys. A: Math. Theor.\ (2013) 465203\) A Nobe

\[ F(X, Y) := [2Y, Y + [(g + 1)X, C_g + gX, \ldots, C_1 + X, C_0], C_{-1}]. \]

By definition, the coefficient $C_i$ \((i = -1, 0, \ldots, g)\) is a tropical polynomial in $J_f$ and $W_k$ \((j, k = 1, 2, \ldots, g + 1)\). Therefore, the correspondence defines a piecewise linear map

\[ \psi : \mathbb{R}^{2g+1} \to \mathbb{R}^{g+2}, \ (J_1, \ldots, J_{g+1}, W_1, \ldots, W_{g+1}) \mapsto (C_{-1}, C_0, \ldots, C_g). \]

We may use the notation $C_t := C_i(J_1^t, \ldots, J_{g+1}^t, W_1^t, \ldots, W_{g+1}^t) \ (t = 0, 1, \ldots)$. In general, $C_t$ has a complicated form because it is the ultradiscretization of $c_t$ (see (8)). For small or large $t$, however, $C_t$ has a relatively simple form:

\[ C_g = [J_i, W_i]_{1 \leq i \leq g+1}, \]
\[ C_{g-1} = \left[ [J_i + J_j, W_i + W_j], [J_i + W_j] \right]_{1 \leq i, j \leq g+1, j \neq i, i-1}, \]
\[ C_0 = \sum_{i=1}^{g+1} J_i, \]
\[ C_{-1} = \sum_{i=1}^{g+1} (J_i + W_i). \]

Here, we use assumption (29).

One can show that the coefficients $C_{-1}, C_0, \ldots, C_g$ in the tropical polynomial $F$ are the conserved quantities of the UD-pTL. These conserved quantities can also be constructed via paths on a two-dimensional array of boxes \([23]\).

Consider the tropical curve $\tilde{\Gamma}$ defined by the tropical polynomial $F$ \([4, 15, 32]\):

\[ \tilde{\Gamma} := \{ P \in \mathbb{R}^2 \mid F \text{ is not differentiable at } P \}. \]

Assume $C_{-1} > 2C_0$, $C_{g-1} > 2C_g$ and $C_i + C_{i+2} > 2C_{i+1}$ for $i = 0, 1, \ldots, g - 2$. Then, $\tilde{\Gamma}$ is a tropical hyperelliptic curve of genus $g$ \([6, 25]\). The tropical curve $\tilde{\Gamma}$ is often referred to as the tropicalization of the hyperelliptic curve $\tilde{\gamma}$ or $\gamma$. By removing all half rays from $\tilde{\Gamma}$, we obtain the compact tropical curve denoted by $\Gamma$ (see figure 3). The tropical hyperelliptic curves $\tilde{\Gamma}$ and $\Gamma$ are called the spectral curves of the UD-pTL.

Consider the involution $\iota : \tilde{\Gamma} \rightarrow \tilde{\Gamma}, P = (X, Y) \mapsto P' = (X, C_{-1} - Y)$. We call $P'$ the conjugate of $P$. Note that $\tilde{\Gamma}$ is symmetric with respect to the line $Y = C_{-1}/2$.

We denote the $2g + 2$ vertices of $\Gamma$ by $V_i$ and $V_i'$ for $i = 0, 1, \ldots, g$.

\[ V_i = (C_{g-i} - C_{g-i+1}, C_{-1} - (g-i+1)C_{g-i} + (g-i)C_{g-i+1}), \]
\[ V_i' = (C_{g-i} - C_{g-i+1}, (g-i+1)C_{g-i} - (g-i)C_{g-i+1}). \]

The cycle connecting $V_i, V_{i-1}, V_{i-1}'$ and $V_i'$ in a counterclockwise direction is denoted by $\gamma_i$ for $i = 1, 2, \ldots, g$.

5.2. Addition on tropical hyperelliptic curves

We briefly review addition of points on tropical hyperelliptic curves \([30, 36]\).

Denote the divisor group of the tropical hyperelliptic curve $\Gamma$ by $\mathcal{D}(\Gamma)$ as in the non-tropical case. (To reduce symbols, we often use the same ones as in the non-tropical case.) A rational function on $\Gamma$ is a continuous function $f : \Gamma \rightarrow \mathbb{R} \cup \{ \pm \infty \}$, such that its restriction to any edge is piecewise linear with integral slope \([4]\). The order of $f$ at $P \in \Gamma$ is the sum of the outgoing slopes of all segments emanating from $P$ and is denoted by $\text{ord}_P f$. If $\text{ord}_P f < 0$,
Figure 3. The tropical hyperelliptic curve $\tilde{\Gamma}$. The compact subset $\Gamma_1$ is drawn by solid lines and the half rays by dotted lines.

then $P$ is called the zero of $f$ of order $|\text{ord}_P f|$. If $\text{ord}_P f > 0$ then $P$ is called the pole of $f$ of order $\text{ord}_P f$. This is because we choose the tropical semi-field $\mathbb{T} = \mathbb{R} \cup \{\infty\}$ equipped with the operations $\min$ and $\,+\,$. The principal divisor $(f)$ of $f$ is defined to be $(f) := \sum_{P \in \tilde{\Gamma}} (\text{ord}_P f) P$. We then find $\deg(f) = 0$.

Define the Picard group of $\tilde{\Gamma}$ to be the residue class group $\text{Pic}^0(\tilde{\Gamma}) := \mathcal{D}_0(\tilde{\Gamma})/\mathcal{D}_l(\tilde{\Gamma})$, where $\mathcal{D}_0(\tilde{\Gamma})$ is the group of divisors of degree 0 on $\tilde{\Gamma}$ and $\mathcal{D}_l(\tilde{\Gamma})$ is the group of principal divisors of rational functions on $\tilde{\Gamma}$. We also define $\text{Pic}^0(\Gamma) := \mathcal{D}_0(\Gamma)/\mathcal{D}_l(\Gamma)$ for the compact subset $\Gamma$ of $\tilde{\Gamma}$. We then find the following $[5, 30, 36]$:

$$\text{Pic}^0(\tilde{\Gamma}) = \text{Pic}^0(\Gamma).$$

Let us define $L(D) := \{ k \in \mathbb{R}(\Gamma) \mid (k) + D > 0 \}$ for $D \in \mathcal{D}(\Gamma)$, where $\mathbb{R}(\Gamma)$ is the field of rational functions on $\Gamma$ $[2, 5]$. Also define the rank of $L(D)$ to be the maximal integer $k$ such that for all choices of (not necessarily distinct) points $P_1, P_2, \ldots, P_k \in \Gamma$ we have $L(D - P_1 - P_2 - \cdots - P_k) \neq \emptyset$ $[5]$. We denote it by $\text{rank}(L(D))$. The following theorem (a corollary of the tropical Riemann–Roch theorem) is useful $[2, 5, 25]$.

**Theorem 9.** For any divisor $D$ on a tropical hyperelliptic curve $\Gamma$ of genus $g$, such that $\deg D > 2g - 2$, we have

$$\text{rank} L(D) = \deg D - g.$$  

As in the non-tropical case, we define the canonical map $\Phi : \mathcal{D}_g^+(\Gamma) \rightarrow \text{Pic}^0(\Gamma)$ to be

$$\Phi(A) := A - D^* \pmod{\mathcal{D}_l(\Gamma)},$$

for $A \in \mathcal{D}_g^+(\Gamma)$, where $\mathcal{D}_g^+(\Gamma)$ is the group of effective divisors of degree $g$ on $\Gamma$ and $D^* \in \mathcal{D}_g^+(\Gamma)$ is a fixed element. We then have the following theorem.

**Theorem 10** ([30]). The canonical map $\Phi$ is surjective. In particular, $\Phi$ is bijective if $g = 1$. 

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By using the surjection $\Phi$, we induce addition of points on the $g$th symmetric product $\text{Sym}^g(\Gamma) := \Gamma^g / \Sigma_g$ from $\text{Pic}^0_0(\Gamma)$. Put $\tilde{\Phi} := \Phi \circ \mu^{-1} : \text{Sym}^g(\Gamma) \to \text{Pic}^0_0(\Gamma)$, where $\mu : D_k^+(\Gamma) \to \text{Sym}^g(\Gamma)$; $D_P = P_1 + P_2 + \cdots + P_k \mapsto d_P := \mu(D_P) = \{P_1, P_2, \ldots, P_k\}$. For $d_P, d_Q \in \text{Sym}^g(\Gamma)$, we define $d_P \oplus d_Q$ to be an element in the subset
$$\tilde{\Phi}^{-1}(\Phi(d_P) + \Phi(d_Q)) \subset \text{Sym}^g(\Gamma).$$

Put $\alpha_{ij} := \alpha_i \cap \alpha_j \setminus \{\text{end points of } \alpha_i \cap \alpha_j\}$ for the cycles $\alpha_i$ ($i = 1, 2, \ldots, g$). We define the subset $\tilde{D}$ of $D^+_k(\Gamma)$ to be
$$\tilde{D} := \left\{ D_P \in D^+_k(\Gamma) \mid \exists i, \text{ such that } D_P \cap \alpha_i \neq \emptyset \text{ and there exists at most one point on } \alpha_{ij} \right\}.$$ We then have the following theorem.

**Theorem 11** ([12]). The reduced map $\Phi|\tilde{D} : \tilde{D} \sim \text{Pic}^0_0(\Gamma)$ is bijective.

Hereafter, we fix $D^*$ as follows:
$$D^* = \begin{cases} \frac{g}{2}(V_0 + V'_0) & \text{for even } g, \\ \frac{g-1}{2}(V_0 + V'_0) + V_0 & \text{for odd } g. \end{cases}$$

Define the element $o \in \text{Sym}^g(\Gamma)$ to be
$$o := \begin{cases} \bigcup_{i=1}^{g/2}[V_{2i-1}, V'_{2i-1}] & \text{for even } g, \\ [V_0] \cup \left( \bigcup_{i=1}^{(g-1)/2}[V_{2i}, V'_{2i}] \right) & \text{for odd } g. \end{cases}$$

Also, define
$$\mathcal{O} := \mu^{-1}(o) = \begin{cases} \sum_{i=1}^{g/2}(V_{2i-1} + V'_{2i-1}) & \text{for even } g, \\ V_0 + \sum_{i=1}^{(g-1)/2}(V_{2i} + V'_{2i}) & \text{for odd } g. \end{cases}$$

One can show that $\mathcal{O} \in \tilde{D} \cap \ker \Phi$ holds [30]. Therefore, the element $o$ is the unit of addition of the group $\mu(\tilde{D}) \simeq \text{Pic}^0_0(\Gamma)$.

Let $d_P, d_Q$ and $d_R$ be the elements of $\mu(\tilde{D})$ satisfying the addition formula
$$d_P \oplus d_Q \oplus d_R = o.$$ This can be written by the divisors on $\text{Pic}^0_0(\Gamma)$:
$$D_P + D_Q + D_R - 3D^* \equiv 0 \pmod{D^+_1(\Gamma)}.$$ There exists a rational function $k \in L(3D^*)$ whose $3g$ zeros are $P_1, \ldots, P_k$, $Q_1, \ldots, Q_s$, $R_1, \ldots, R_r$. If the rational function $k$ is given by a tropical polynomial, then we can define a tropical curve $C$ as the set of points at which the tropical polynomial is not differentiable. The above addition is realized by using the intersection of $\Gamma$ and $C$ [30].


5.3. Tropical Jacobians

Tropical Jacobians were introduced by Mikhalkin and Zharkov in 2006 [25]. Let $\mathcal{E}(\Gamma)$ be the set of edges of $\Gamma$. Define the weight $w : \mathcal{E}(\Gamma) \to \mathbb{R}_{\geq 0}$ by

$$w(e) = \frac{\|e\|}{\|\xi_e\|},$$

where $\xi_e$ is the primitive tangent vector of $e \in \mathcal{E}(\Gamma)$ and $\|\|$ denotes the Euclidean norm in $\mathbb{R}^2$. We define a symmetric bilinear form $Q$ on the space of paths in $\Gamma$ as follows. For a non-self-intersecting path $\varpi$, set $Q(\varpi, \varpi) := \text{(length of } \varpi \text{ with respect to } w)$, and extend it to any pairs of paths bilinearly.

**Definition 1.** The tropical Jacobian of $\Gamma$ is a $g$-dimensional real torus defined to be

$$J(\Gamma) := \mathbb{R}^g / \Lambda \mathbb{Z}^g,$$

where $\Lambda = (\Lambda_{ij})$ is the $g \times g$ real matrix given by

$$\Lambda_{ij} = Q \left( \sum_{k=1}^{i} \alpha_k, \sum_{i=1}^{j} \alpha_i \right) = C_{-1} + r_i \delta_{ij} + 2[\lambda_i, \lambda_j],
\lambda_i = C_{g-i} - C_{g-i+1},
\lambda_j = C_{g-j} - C_{g-j+1},
\delta_{ij} = \sum_{k=1}^{g} [\lambda_i, \lambda_k]$$

for $i, j = 1, 2, \ldots, g$.

Fix a point $P_0$ on $\Gamma$. Let the path from $P_0$ to $P_i$ on $\Gamma$ be $\varpi_i$. We define the Abel–Jacobi map $\eta : D^+_g(\Gamma) \to J(\Gamma)$ to be

$$\eta : D_P = P_1 + \cdots + P_g \mapsto \sum_{i=1}^{g} (Q(\varpi_i, \alpha_1), \ldots, Q(\varpi_i, \alpha_g)).$$

5.4. Time evolution

Let the phase space of the UD-pTL be $\mathcal{T} := \{(J_1, \ldots, J_{g+1}, W_1, \ldots, W_{g+1}) \mid \sum_{i=1}^{g+1} J_i < \sum_{i=1}^{g+1} W_i\}$ and the moduli space of $\Gamma$ be $\mathcal{C} := \{(C_{-1}, C_0, \ldots, C_g)\}$. Consider the map $\psi : \mathcal{T} \to \mathcal{C}$ defined by (30) and set

$$\mathcal{T}_c := \psi^{-1}(C_{-1}, C_0, \ldots, C_g) \subset \mathcal{T}.$$

Let $\phi : \mathcal{T} \to D^+_g(\Gamma)$ be the tropical eigenvector map [12]. Then the UD-pTL is linearized on the tropical Jacobian $J(\Gamma)$.

**Theorem 12** ([12, 13]). Define the transformation operator $\nu$ to be

$$\nu : J(\Gamma) \to J(\Gamma) : z \mapsto z + (\lambda_1, \lambda_2 - \lambda_1, \ldots, \lambda_g - \lambda_{g-1}).$$

Then the following diagram is commutative:

$$\begin{array}{ccc}
\mathcal{T}_c & \xrightarrow{\psi \phi} & J(\Gamma) \\
\downarrow & & \downarrow \nu \\
\mathcal{T}_c & \xrightarrow{\psi \phi} & J(\Gamma).
\end{array}$$
Define $T \in \mathcal{D}_g(\Gamma)$ to be
\[
T = \begin{cases} 
V_0 + V_g' + \sum_{i=1}^{(g-2)/2} (V_{2i} + V_{2i}') & \text{for even } g, \\
B + V_1 + V_g' + \sum_{i=2}^{(g-1)/2} (V_{2i-1} + V_{2i-1}') & \text{for odd } g
\end{cases}
\] (31)
for $g \geq 2$. Here, $B = (C_g, C_{g-1} - C_g - (g - 1)C_g) \in \bar{V}_0V_0' \subset \alpha_i$ is the unique point, such that $B + V_1 - 2V_0$ is the principal divisor of a rational function on $\Gamma$. Note that $Q(\bar{V}_0B, \bar{V}_0B) = \lambda_1$ and $T \in \bar{D}$. For an even number $g$, we find
\[
\Phi(T) = V_g' - V_0' + \sum_{i=0}^{(g-2)/2} (V_{2i} + V_{2i}') - \frac{g}{2} (V_0 + V_0') \\
\equiv V_g' - V_0' \pmod{D_1(\Gamma)}.
\]
For an odd number $g$, we also find
\[
\Phi(T) = V_g' - V_0' + \sum_{i=0}^{(g-1)/2} (V_{2i} + V_{2i}') - \frac{g - 3}{2} (V_0 + V_0') \\
\equiv V_g' - V_0' \pmod{D_1(\Gamma)}.
\]

For $g = 1$, we define $T$ to be the point $(C_1, C_0) \in \bar{D}$ on the edge $\bar{V}_0V_0'$. Since there exists a rational function whose principal divisor is $V_0 + V_1' - V_0' - T$, we find
\[
\Phi(T) = T - V_0 \equiv V_1' - V_0' \pmod{D_1(\Gamma)}.
\]
Thus, we obtain the following proposition.

**Proposition 3.** For any $g$, we have
\[
\Phi(T) \equiv V_g' - V_0' \pmod{D_1(\Gamma)}.
\]

We obtain the following theorem concerning the time evolution of the UD-pTL and an addition on $\text{Sym}^g(\Gamma)$.

**Theorem 13.** Set $\tau = \mu(T) \in \text{Sym}^g(\Gamma)$. Then, for any $g$, the following diagram is commutative:

\[
\begin{array}{ccc}
\mathcal{T}_C & \xrightarrow{\mu \circ \phi} & \text{Sym}^g(\Gamma) \\
\downarrow{(28)} & & \downarrow{\oplus \tau} \\
\mathcal{T}_C & \xrightarrow{\mu \circ \phi} & \text{Sym}^g(\Gamma).
\end{array}
\]

**Proof.** Let the paths from $V_i'$ to $V_{i-1}'$, $V_i$, and $V_{i+1}'$ be $\sigma_i^1$, $\sigma_i^2$, and $\sigma_i^3$ as in figure 4, respectively ($i = 1, 2, \ldots, g - 1$). Also let the paths from the fixed point $P_0$ to $V_i$, $V_i$, and $V_{i+1}$ via $\sigma_i^1$, $\sigma_i^2$, and $\sigma_i^3$ be $\tilde{\sigma}_i^1$, $\tilde{\sigma}_i^2$, and $\tilde{\sigma}_i^3$, respectively.

Let $\Pi_S$ be the set of paths on $\Gamma$ emanating from $S \in \Gamma$. Define $i S : \Pi_S \rightarrow \mathbb{R}^S : \omega \mapsto i_S(\omega) = (Q(\omega, \alpha_i))_{1 \leq i \leq g}$. For $D_p = P_1 + \cdots + P_g \in \bar{D}$, we see $\eta(D_p) \equiv \sum_{i=1}^{g} i_p(\omega_i)$.
Thus, we obtain

\[ T = B + V_1 + V_g' + \sum_{i=2}^{(g-1)/2}(V_{2i-1} + V'_{2i-1}). \]

Next, assume that \( g \) is an odd number. Then, we have

\[ \eta(T) - \eta(O) = \sum_{i=1}^{g/2}((\lambda_{2i-1} - \lambda_{2i-2})e_{2i-1} + (\lambda_{2i} - \lambda_{2i-1})e_{2i}) \]

\[ = \sum_{i=1}^{g}(\lambda_i - \lambda_{i-1})e_i. \]
We find
\[ \eta(\mathcal{O}) = \sum_{i=1}^{(g-1)/2} (\lambda\eta(\mathcal{a}^2_1) + \eta(\mathcal{b}^2_1)) \]
\[ = \sum_{i=1}^{(g-1)/2} r_{2i}(e_{2i} - e_{2i+1}) - C_{-1}e_1 + 2 \sum_{i=1}^{(g-1)/2} p_{2i} + p_0 \]
and
\[ \eta(T) = \sum_{i=1}^{(g-1)/2} (\lambda\eta(\mathcal{a}^2_1) + \eta(\mathcal{b}^2_1)) + \eta(\mathcal{a}^2_0) + p_0 \]
\[ = \sum_{i=1}^{(g-1)/2} [(r_{2i} + \lambda_2 - \lambda_{2i-1})e_{2i} - (r_{2i} - \lambda_{2i+1} + \lambda_2)e_{2i+1}] \]
\[ - (C_{-1} - \lambda_1)e_1 + 2 \sum_{i=1}^{(g-1)/2} p_{2i} + p_0. \]
It follows that we have
\[ \eta(T) - \eta(\mathcal{O}) = \sum_{i=1}^{(g-1)/2} [(\lambda_2 - \lambda_{2i-1})e_{2i} + (\lambda_{2i+1} - \lambda_2)e_{2i+1}] + \lambda_1 e_1 \]
\[ = \sum_{i=1}^{(g-1)/2} (\lambda_i - \lambda_{i-1})e_i. \]
Since \( \mu : D_g^+(\Gamma) \rightarrow \text{Sym}^g(\Gamma) \) and \( \eta|\tilde{D} : \tilde{D} \rightarrow J(\Gamma) \) are bijective, this completes the proof. \( \square \)

6. A geometric realization of UD-pTL

6.1. Ultradiscretization of rational functions

In order to tropicalize the geometric framework of the pdTL constructed in section 4, we first introduce a rational map which maps the defining polynomials of the curves appearing in the geometric realization into subtraction-free ones. Then, we apply the ultradiscretization procedure to them.

Let \( \sigma : \mathbb{C}^2_{(u,v)} \rightarrow \mathbb{C}^2_{(x,y)} \) be the rational map
\[ (x, y) = \sigma(u, v) = \left( u, \frac{v - u^{g+1} - c_1 u^g - \cdots - c_1 u - c_0}{2} \right). \]
If we apply \( \sigma \) to the points on \( \gamma \), then we recover \( \gamma \). Therefore, \( \sigma \) is the inverse of \( \rho \) on \( \gamma \). Note that the defining function \( f \) of \( \gamma \) is subtraction free.

Consider the curve \( \kappa_1 = (h_1(u, v) = 0) \) introduced in section 4.2. Applying \( \sigma \) to the points on \( \kappa_1 \), we obtain an affine curve
\[ \tilde{k}_1 := (\tilde{h}_1(x, y) = 0) = \{(x, y) = \sigma(u, v) \mid h_1(u, v) = 0\}, \]
where
\[ \tilde{h}_1(x, y) = \begin{cases} \sum_{i=0}^{g} (s_i c_i - c_i) x^i - 2y & \text{for even } g, \\ \sum_{i=0}^{g} (t_{g+1} c_i + c_i) x^i + 2x^{g+1} + 2y & \text{for odd } g. \end{cases} \]
We see that $\hat{h}_1$ for an odd number $g$ is subtraction free as well as $\hat{f}$. Similarly, for an even number $g$, $-\hat{h}_1$ is subtraction free. Therefore, we can simultaneously apply the procedure of ultradiscretization (or tropicalization with trivial valuation [22]) to the rational functions $\hat{f}$ and $\hat{h}_1$. Thus, we recover the intersection of $γ$ and $K_1$ in terms of the tropical curves $Γ$ and $K_1$, defined below, in the framework of tropical geometry.

Now, we define $S_iC_i$ and $T_iC_i$ for $i = -1, 0, \ldots, g$ and $j, k = 1, 2, \ldots, g + 1$ to be

\[ S_iC_i := \lim_{\epsilon \to 0} -\epsilon \log(c_i - s_jC_i) = C_i(\infty, \ldots, \infty, J_j, \infty, \ldots, \infty), \]

\[ T_iC_i := \lim_{\epsilon \to 0} -\epsilon \log(t_kC_i + c_i) = C_i(J_1, \ldots, W_{k-1}, \infty, W_{k+1}, \ldots, W_{g+1}). \]

Note that $S_iC_i$ eliminates the terms in $C_i$ not containing $J_j$ and $T_iC_i$ the terms in $C_i$ containing $W_k$. By applying the ultradiscretization procedure to $\hat{h}_1$, we obtain the tropical polynomial

\[ H_1(X, Y) = \begin{cases} [S_iC_i + iX, Y] & \text{for even } g, \\ [T_iC_i + iX, (g+1)X, Y] & \text{for odd } g. \end{cases} \tag{33} \]

In order to consider non-degenerate curves, we assume

\[ \begin{cases} S_iC_i + S_iC_{i+2} > 2S_iC_{i+1} & \text{for even } g, \\ T_{i+1}C_i + T_{i+1}C_{i+2} > 2T_{i+1}C_{i+1} & \text{for odd } g \end{cases} \]

for $i = 0, 1, \ldots, g - 2$.

We define the tropical curve $K_1$ to be the set of points at which $H_1$ is not differentiable:

\[ K_1 := \{ P \in \mathbb{R}^2 \mid H_1 \text{ is not differentiable at } P \}. \]

### 6.2. Tropical curve intersections

We show that the intersection of $Γ$ and $K_1$ realizes an addition on $\text{Sym}^g(Γ)$.

**Lemma 1.** The restriction $H_1|Γ$ of $H_1$ on $Γ$ satisfies $H_1|Γ \in L(V_0' + 2D^*)$ for any $g$.

**Proof.** Note first that we have

\[ L(V_0' + 2D^*) = \begin{cases} L(gV_0 + (g+1)V_0') & \text{for even } g, \\ L((g+1)V_0 + gV_0') & \text{for odd } g. \end{cases} \]

By the tropical Riemann–Roch theorem (theorem 9), we find

\[ \text{rank } L(V_0' + 2D^*) = g + 1. \]

Also note that the principal divisors of the coordinate functions $x$ and $y$ are given by

\[ (x) = V_g + V_g' - V_0 - V_0', \quad (y) = (g+1)V_0 - (g+1)V_0'. \]

If $g$ is an even number, then the $g + 2$ rational functions $1, x, \ldots, g, y$ on $Γ$ are in $L(V_0' + 2D^*)$. Similarly, if $g$ is an odd number then the $g + 2$ rational functions $1, x, \ldots, g, \{(g+1)x, y\}$ on $Γ$ are in $L(V_0' + 2D^*)$. Here we use the fact that the rational function $\{(g+1)x, y\}$ has a pole of order $g$ at $V_0'$ and of order $g + 1$ at $V_0$. We denote the tropical module spanned by these rational functions by $M_1$ [25]:

\[ M_1 := \begin{cases} \{1, x, \ldots, g, y\} & \text{for even } g, \\ \{1, x, \ldots, g, \{(g+1)x, y\}\} & \text{for odd } g. \end{cases} \]

Then, $M_1 \subseteq L(V_0' + 2D^*)$. Since the rank of $M_1$ is $g + 2$ as a tropical module and

\[ \text{rank } L(V_0' + 2D^*) = g + 1, \]

$M_1$ is the maximal tropical module in $L(V_0' + 2D^*)$. Clearly, $H_1|Γ \in M_1$ holds for any $g$. \qed

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Figure 5. The tropical curve drawn by solid lines is $K_1$ for an odd number $g$. The curve drawn by dotted lines is the tropical hyperelliptic curve $\Gamma_1$.

Lemma 2. The tropical curve $K_1$ passes through the vertex $V'_g = (C_0 - C_1, C_0)$ of $\Gamma$.

Proof. Note that we have

$$c_0 - s_1 c_0 = 2 \prod_{i=1}^{g+1} I_i,$$

and

$$t_{g+1} c_0 + c_0 = 2 \prod_{i=1}^{g+1} I_i.$$

This implies that we have

$$S_1 C_0 = T_{g+1} C_0 = C_0 = \sum_{i=1}^{g+1} J_i.$$

Now assume that $g$ is an even number. There exists an infinite edge $E$ of $K_1$ defined by $S_1 C_0 = Y$. The edge $E$ is emanating rightward from the vertex $(S_1 C_0 - S_1 C_1, S_1 C_0)$. Since $S_1 C_1 = C_1 (J_1, \infty, \ldots, \infty)$, we find $C_1 \leq S_1 C_1$. This implies $C_0 - C_1 \geq S_1 C_0 - S_1 C_1$. Thus, the vertex $V'_g$ of $\Gamma$ is on $E$. For an odd number $g$, the statement is similarly shown. □

Figure 5 shows the tropical curves $K_1$ and $\Gamma_1$. If $g$ is an even number, the vertical edges of $K_1$ are given by $X = S_1 C_i - S_1 C_{i+1}$ for $i = 0, 1, \ldots, g-1$. If $g$ is an odd number, the vertical edges of $K_1$ are given by $X = T_{g+1} C_i - T_{g+1} C_{i+1}$ for $i = 0, 1, \ldots, g-1$ and $X = T_{g+1} C_g$.

If we assume that $g$ is an odd number then $K_1$ passes through the vertex $V'_g$ because $C_g \leq T_{g+1} C_g \leq C_{g-1} - C_g$ holds. Moreover, there exists at least one intersection point of $\Gamma$ and $K_1$ on the upper half of $\alpha_1 \setminus \alpha_{1,2}$.

Let us consider the addition formula on $\text{Sym}^g(\Gamma)$ equivalent to the time evolution of the UD-pTL (see theorem 13):

$$d_{\tilde{p}} = d_P \oplus \tau \iff -d_{\tilde{p}} \oplus d_P \oplus \tau = 0. \quad (34)$$

Since $\phi(T) \equiv V'_g - V'_0 \pmod{D_l(\Gamma)}$ for any $g$ (see proposition 3), the addition formula (34) is written by the divisors on $\text{Pic}^0(\Gamma)$:

$$D_Q + D_P + V'_g - V'_0 - 2D^\tau \equiv 0 \pmod{D_l(\Gamma)}, \quad (35)$$

where $D_Q = \mu^{-1}(-d_P)$. Hereafter, we assume that the divisor $D_P$ is in $\tilde{D}$. We have the following proposition.
Proposition 4. Assume that the points \( P_1, P_2, \ldots, P_g \in \Gamma \) are on the tropical curve \( K_1 \). Then, the rational function \( H_1|\Gamma \) on \( \Gamma \) satisfies
\[
(H_1|\Gamma) = D_Q + D_P + V'_g - V'_0 - 2D^*.
\] (36)

Proof. By lemma 2, the curve \( K_1 \) passes through \( V'_g \). By the assumption of proposition, \( K_1 \) passes through \( P_1, P_2, \ldots, P_g \) as well. These facts suggest that the rational function \( H_1|\Gamma \) on \( \Gamma \) has \( g + 1 \) zeros at \( P_1, P_2, \ldots, P_g, V'_g \). Moreover, by lemma 1, \( H_1|\Gamma \) is in the maximal tropical module \( \mathcal{M}_1 \) in \( L(V'_0 + 2D^*) \). Therefore, \( H_1|\Gamma \) satisfies (36) and is uniquely determined up to an additive constant.

Thus, the points \( P_1, P_2, \ldots, P_g, Q_1, Q_2, \ldots, Q_g, V'_g \) satisfying (35) are on both \( \Gamma \) and \( K_1 \). Therefore, the addition formula (34) can be realized by using the intersection of \( \Gamma \) and \( K_1 \). Note that we have \( D_Q \in \mathcal{D} \) because \( D_P, D_T \in \mathcal{D} \) and \( \mathcal{D} \simeq \text{Pic}^0(\Gamma) \). In order to realize the time evolution of the UD-pTL by using tropical curve intersections, we have only to give the inverse \( d_P \) of \( -d_P = \mu(D_Q) \) in terms of a tropical curve intersection.

6.3. Inverse elements

Consider the addition formula on \( \text{Sym}^g(\Gamma) \)
\[
-d_P \otimes d_P = 0.
\]
This can be written by the divisors on \( \text{Pic}^0(\Gamma) \):
\[
D_Q + D_P - 2D^* \equiv 0 \quad (\text{mod } \mathcal{D}(\Gamma)).
\]
Therefore, the tropical curve whose intersection with \( \Gamma \) gives the inverse \( d_P \) of \( -d_P = \mu(D_Q) \) is given by a rational function in \( L(2D^*) \). In the following, we show that the ultradiscretization of the rational function \( h_2 \) gives such a tropical curve.

Assume that \( g \) is an odd number. Consider the curve \( \kappa_2 = (h_2(u, v) = 0) \) introduced in section 4.3. Applying \( \sigma \) to the points on \( \kappa_2 \), we obtain an affine curve
\[
\tilde{h}_2 := (\tilde{h}_2(x, y) = 0) = \{(x, y) = \sigma(u, v) \} | h_2(u, v) = 0 |,
\]
where
\[
\tilde{h}_2(x, y) = \sum_{i=0}^{g-1} (c_i + s_1 c_i + 2t_i c_{i+1})x^i + 2c_g x^g + 2x^{g+1} + 2y.
\]
Remember that \( \tilde{c}_i \) is obtained from \( c_i \) by setting \( I_1 = V_i = 0 \) for \( i = 1, 2, \ldots, g \). Since the polynomial \( \tilde{h}_2 \) is subtraction free, by applying the procedure of ultradiscretization, we obtain the tropical polynomial
\[
H_2(X, Y) = [\hat{C}_0, \hat{C}_1 + X, \ldots, \hat{C}_{g-1} + (g - 1)X, C_g + gX, (g + 1)X, Y],
\]
where \( \hat{C}_i = \lim_{x \to 0} -\epsilon \log(c_i + s_1 c_i + 2t_i c_{i+1}) \) for \( i = 0, 1, \ldots, g - 1 \). In order to consider non-degenerate curves, we assume \( \hat{C}_i + \hat{C}_{i+2} > 2\hat{C}_{i+1} \) for \( i = 0, 1, \ldots, g - 3 \).

We define the tropical curve \( K_2 \) to be the set of points at which \( H_2 \) is not differentiable:
\[
K_2 := \{ P \in \mathbb{R}^2 | H_2 \text{ is not differentiable at } P \}.
\]
Figure 6 shows the tropical curve \( K_2 \).

Note that the terms \( (g + 1)X \) and \( Y \) are never dominant in \( H_2 \) on \( \Gamma \). Therefore, the restriction of \( H_2 \) on \( \Gamma \), which is denoted by \( H_2|\Gamma \), is given by the formula
\[
H_2|\Gamma(X, Y) = [\hat{C}_0, \hat{C}_1 + X, \ldots, \hat{C}_{g-1} + (g - 1)X, C_g + gX].
\]
We define \( \tilde{Z} \) where \( Z \) is the unique point on \( \Gamma \) such that \( Z \) satisfies condition (38).

\[
\frac{\alpha}{2} \Gamma_1 = \sum_{i=1}^{g} (D_i + R_i + Q_i) - 2D^*,
\]

where \( \Gamma_1 \) is the principal divisor of a rational function on \( \Gamma \) satisfying (37). Then, for any odd number \( g \), we have

\[
Q(V_0 R_1, V'_0 R'_1) = Q(V_0 Z_0, V'_0 Z'_0).
\]
Note that we also have
\[ (\tilde{H}_2) = DR + Z_0 + R'_2 + \cdots + R'_g - 2D^*. \]
We then find
\[ (H + \tilde{H}_2) = Z_1 + DQ + Q'_2 + \cdots + Q'_g - 2D^* \equiv 0 \pmod{D_1(\Gamma)} \]
and \( H + \tilde{H}_2 \in L(2D^*) \). It is clear that \( Z_1 + Q'_2 + \cdots + Q'_g \in \tilde{D} \). \( \square \)

If \( g \) is an even number, we have (see [30])
\[ D_\bar{P} = DQ. \]
If \( g \) is an odd number, proposition 5 implies
\[ D_\bar{P} = Z_1 + Q'_2 + \cdots + Q'_g. \]
Thus, we obtain the following theorem.

**Theorem 14.** Let \( D_\bar{P} \) and \( D_Q \) be elements of \( \tilde{D} \) satisfying (36). Also, let \( T \) be the element of \( \tilde{D} \) given by (31). Put \( d_\bar{P} = \mu(D_\bar{P}) \in \text{Sym}^g(\Gamma) \) and \( \tau = \mu(T) \in \text{Sym}^g(\Gamma) \). Then, the element \( d_\bar{P} \in \text{Sym}^g(\Gamma) \) defined by the addition
\[ d_\bar{P} = d_\bar{P} \oplus \tau \]
is explicitly given by the formula
\[ d_\bar{P} = \begin{cases} \{Q'_1, Q'_2, \ldots, Q'_g\} & \text{for even } g, \\ \{Z_1, Q'_2, \ldots, Q'_g\} & \text{for odd } g. \end{cases} \]

Thus, the time evolution of the UD-pTL is realized by using the intersection of the tropical curves \( \Gamma, K_1 \) and \( K_2 \).

**Example 3.** Put \( g = 3 \). The spectral curve \( \Gamma \) is given by
\[ F(X, Y) = [2X, Y + [4X, C_3 + 3X, C_2 + 2X, C_1 + X, C_0], C_{-1}], \]
\[ C_3 = [J_i, W_{1,i} | 1 \leq i \leq 4], \quad C_2 = \left[ \left[ J_i + J_j, W_i + W_j \right] \right]_{1 \leq i < j \leq 4}, \]
\[ C_1 = \left[ \left[ J_i + J_j + J_k, W_i + W_j + W_k \right] \right]_{1 \leq i < j < k \leq 4}, \]
\[ C_0 = \sum_{i=1}^{4} J_i, \quad C_{-1} = \sum_{i=1}^{4} (J_i + W_i). \]
The tropical curves \( K_1 \) and \( K_2 \) are respectively given by
\[ H_1(X, Y) = [Y, 4X, T_3C_3 + 3X, T_4C_2 + 2X, T_4C_1 + X, T_4C_0], \]
\[ H_2(X, Y) = [Y, 4X, C_3 + 3X, \hat{C}_2 + 2X, \hat{C}_1 + X, \hat{C}_0]. \]

Figure 7 shows an example of the intersection of \( \Gamma, K_1 \) and \( K_2 \).
6.4. Discrete motion of tropical curves

We can interpret the time evolution of the UD-pTL as discrete motion of tropical curves. Let $C_1$ be the moduli space of the tropical curve $K_1$

$$C_1 = \begin{cases} \{(S_1C_0, S_1C_1, \ldots, S_1C_g)\} & \text{for even } g, \\ \{(T_{g+1}C_0, T_{g+1}C_1, \ldots, T_{g+1}C_g)\} & \text{for odd } g. \end{cases}$$

Define a map $\tilde{\psi}: T_C \to C_1$ to be

$$\tilde{\psi}: (J_1, \ldots, J_{g+1}, W_1, \ldots, W_{g+1}) \mapsto \begin{cases} (S_1C_0, \ldots, S_1C_g) & \text{for even } g, \\ (T_{g+1}C_0, \ldots, T_{g+1}C_g) & \text{for odd } g. \end{cases}$$

Also, define a map $\upsilon$ on $C_1$ to be

$$\upsilon: \begin{cases} (S_1C_0, \ldots, S_1C_g) \mapsto (S_1\tilde{C}_0, \ldots, S_1\tilde{C}_g) & \text{for even } g, \\ (T_{g+1}C_0, \ldots, T_{g+1}C_g) \mapsto (T_{g+1}\tilde{C}_0, \ldots, T_{g+1}\tilde{C}_g) & \text{for odd } g, \end{cases}$$

so that the following diagram is commutative:

$$\begin{align*}
T_C \xrightarrow{\tilde{\psi}} C_1 \\
\downarrow^{(28)} \quad \quad \quad \downarrow^{\upsilon} \\
T_C \xrightarrow{\tilde{\psi}} C_1.
\end{align*}$$

Then, $\upsilon$ induces the discrete motion of tropical curves

$$K_1^0 \to K_1^1 \to K_1^2 \to \cdots.$$
Figure 8. The discrete motion of the tropical curves $K^1_t$ ($t = 0, 1, \ldots, 12$) induced by $\nu$. The solid curves are $K^1_1$ and the dashed one is $/Gamma_1$. The figures are sorted in time increasing order from left to right and top to bottom. Note that $K^1_1 = K^1_2 = \cdots = K^1_6$ and the second figure shows them. (This fact does not suggest that the intersection points of $/Gamma_1$ and $K^1_t$ ($t = 1, 2, \ldots, 6$) are fixed.) Here, we set $J^0_1 = 3, J^0_2 = 5, J^0_3 = 7, J^0_4 = 2, W^0_1 = 3, W^0_2 = 6, W^0_3 = 10, W^0_4 = 0$.

where $K^1_1$ is given by the tropical polynomial

$$H^1_t(X, Y) = \begin{cases} [S_t C^t + iX], Y & \text{for even } g, \\ [T^g C^t + iX], (g + 1)X, Y & \text{for odd } g. \end{cases}$$

Figure 8 shows an example of the discrete motion of $K^1_1$.

7. Concluding remarks

We establish a geometric realization of the pdTL via the curve intersections of its spectral curve $\gamma$ and two affine curves $\kappa_1$ and $\kappa_2$. Namely, the linear flow on the Jacobian $J(\gamma)$ equivalent to the time evolution of the pdTL is translated into the curve intersections of $\gamma$, $\kappa_1$ and $\kappa_2$. The rational functions $f$, $h_1$ and $h_2$ which respectively define $\gamma$, $\kappa_1$ and $\kappa_2$ are explicitly given by using the conserved quantities $c_{-1}, c_0, \ldots, c_g$ of the pdTL. In addition, these rational functions can simultaneously be mapped into subtraction-free ones by the rational transformation $\sigma$ on $\mathbb{C}^2$. Therefore, we can naturally apply the procedure of ultradiscretization to them and obtain the tropical hyperelliptic curve $\Gamma$ and two tropical curves $K_1$ and $K_2$ intersecting each other. The tropical hyperelliptic curve $\Gamma$ thus obtained is nothing but the spectral curve of the UD-pTL. Hence, the coefficients $C_{-1}, C_0, \ldots, C_g$ of its defining tropical polynomial $F$ are the conserved quantities of the UD-pTL and the time evolution of the UD-pTL is linearized on its tropical Jacobian $J(\Gamma)$. Moreover, two tropical curves $K_1$ and $K_2$ thus obtained are explicitly given by using the conserved quantities of the UD-pTL. We then show that the tropical curve intersections of $\Gamma$, $K_1$ and $K_2$ give the linear flow on $J(\Gamma)$ equivalent to the time evolution of the UD-pTL. Thus, we also establish a tropical geometric realization of the UD-pTL via the tropical curve intersections of $\Gamma$, $K_1$ and $K_2$. 

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It should be noted that the time evolutions of the pdTL and UD-pTL lead to discrete motions of (tropical) curves $\kappa_1$ and $K_1$ which appear in their (tropical) geometric realizations, respectively. The time dependence of the moving curves $\kappa_1$ and $K_1$ is derived from a simple action on the phase space $\mathcal{M}$ of the pdTL which changes the sign of $I_1$ or $V_{p+1}$. Thus, the moduli spaces of $\kappa_1$ and $K_1$ are easily obtained from those of $\gamma$ and $\Gamma$ by applying such actions, respectively. We do not know the reason why we can obtain these moduli spaces in such a way. To investigate the time evolutions of the pdTL and UD-pTL as actions on the moduli spaces of the (tropical) hyperelliptic curves is a further problem.

Acknowledgments

This work was partially supported by JSPS KAKENHI grant number 22740100. The author is grateful to two anonymous referees for their helpful comments and suggestions.

Appendix A. Proof of theorem 2

Assume that $g$ is an even number. If $D_P = P_1 + P_2 + \cdots + P_g \in \ker \Phi$, there exists a rational function $h \in L(D^*)$ on $H$, such that $(h) = D_P - D^*$. By the Riemann–Roch theorem, we have $\dim L(0) = -g + 1 + \dim L(W)$. Noting $\dim L(0) = 1$, we obtain $\dim L(W) = g$, where $W$ is a canonical divisor on $H$. Now choose $W = (dx/y) = (g - 1)(P_\infty + P'_\infty)$. Again, by the Riemann–Roch theorem, we have

$$\dim L(P_\infty + P'_\infty) = -g + 3 + \dim L((g - 2)(P_\infty + P'_\infty)).$$

Let $x$ be the coordinate function:

$$x(P) = p \quad \text{for } P = (p, q) \in H.$$ We then find $(x - c) = P_1 + P'_1 - P_\infty - P'_\infty$, where $P_i$ is the point whose $x$-component is $c$. This implies $(1, x) \subset L(P_\infty + P'_\infty)$. It immediately follows $\dim L(P_\infty + P'_\infty) \geq 2$. Therefore, we obtain $\dim L((g - 2)(P_\infty + P'_\infty)) \geq g - 1$. Moreover, since $L((g - 2)(P_\infty + P'_\infty)) \supseteq L(W)$ and $\dim L(W) = g$, we find $\dim L((g - 2)(P_\infty + P'_\infty)) = g - 1$. Thus, we inductively obtain $\dim L(i(P_\infty + P'_\infty)) = i + 1 \quad (i = 1, 2, \ldots, g - 1)$.

In particular, for $i = g/2$, we have

$$\dim L(D^*) = \dim L \left( \frac{g}{2}(P_\infty + P'_\infty) \right) = \frac{g}{2} + 1.$$

Since we have $(1, x, x^2, \ldots, x^2) \subset L(D^*)$ and $\dim (1, x, x^2, \ldots, x^2) = g/2 + 1$, we obtain $(1, x, x^2, \ldots, x^2) = L(D^*)$.

A rational function $h \in L(D^*)$ can be expressed as follows:

$$h = a_0 + a_1x + \cdots + a_{g/2}x^{g/2} \quad (a_i \in \mathbb{C}).$$

If $a_{g/2} \neq 0$, the equation $h = 0$ of degree $g/2$ in $x$ has exactly $g/2$ solutions $c_1, c_2, \ldots, c_{g/2}$ counting multiplicities. Thus, the principal divisor of $h$ is

$$(h) = (x - c_1) + (x - c_2) + \cdots + (x - c_{g/2}) = \sum_{i=1}^{g/2} (P_{c_i} + P'_{c_i}) - D^*.$$

Therefore, we obtain $D_P = \sum_{i=1}^{g/2} (P_{c_i} + P'_{c_i})$ as desired.

If $a_{g/2} = 0$ and $a_{g/2-1} \neq 0$, the equation $h = 0$ has exactly $g/2 - 1$ solutions $c_1, c_2, \ldots, c_{g/2-1}$, counting multiplicities. Hence, we have

$$(h) = \sum_{i=1}^{g/2-1} (P_{c_i} + P'_{c_i}) + P_\infty + P'_\infty - D^*.$$
This is equivalent to \( D_P = \sum_{i=1}^{g/2-1} (P_i + P_c') + P_\infty + P'_\infty \). Thus, we inductively obtain the desired result.

For an odd number \( g \), the statement is similarly shown. \( \Box \)

**Appendix B. Proof of theorem 3**

If \( \Phi(D_P) = \Phi(D_Q) \), we have
\[
D_P - D^* \equiv D_Q - D^* \iff D_P - D_Q \equiv 0,
\]
where the equivalence is considered modulo \( D_l(H) \). Let the \( x \)-component of \( Q_i \) be \( q_i \) for \( i = 1, 2, \ldots, g \). We then have \( (x - q_i) = Q_i + Q_i' - P_\infty - P'_\infty \). Hence, we obtain
\[
D_Q + D_Q' - g(P_\infty + P'_\infty) \equiv 0 \pmod{D_l(H)}. \tag{B.2}
\]

By (B.1) and (B.2), we have
\[
D_P + D_Q' - g(P_\infty + P'_\infty) \equiv 0 \pmod{D_l(H)}. \tag{B.3}
\]

Therefore, there exists a rational function \( h \) on \( H \), such that
\[
(h) = (D_P + D_Q' - g(P_\infty + P'_\infty)). \tag{B.4}
\]

Since \( h \in \mathbb{L}(g(P_\infty + P'_\infty)) \) and \( L(g(P_\infty + P'_\infty)) = \{1, x, x^2, \ldots, x^g\} \) (see the proof of theorem 2), the rational function \( h \) can be expressed as \( h = \sum_{i=1}^{g} a_i x^i \) (\( a_i \in \mathbb{C} \)).

Let \( j \) be a number such that
\[
a_0 = a_{g-1} = \cdots = a_{g-j+1} = 0, \quad a_{g-j} \neq 0 \quad \text{and} \quad 0 \leq j \leq g.
\]

Then the rational function \( h \) is factorized as \( h = \prod_{i=1}^{g-j} (x - c_i) \), where \( c_1, c_2, \ldots, c_{g-j} \) are the solutions of \( h = 0 \). The principal divisor of \( h \) is
\[
(h) = \sum_{i=1}^{g-j} (P_i + P_c') - (g - j)(P_\infty + P'_\infty). \tag{B.5}
\]

Comparing (B.4) and (B.5), we find
\[
D_P + D_Q' = \sum_{i=1}^{g-j} (P_i + P_c') + j(P_\infty + P'_\infty).
\]

The right-hand side can be expressed as \( D_R + D_R' \) for \( D_R = \sum_{i=1}^{g-j} P_i + jP_\infty \in D_g^+ \).

The converse is easily shown. \( \Box \)

**Appendix C. A geometric realization of the pdTL for \( g = 1 \)**

For \( g = 1 \), we choose the Lax matrices as follows:
\[
L = \begin{pmatrix} I_1 + V_2 & 1 - I_1 V_3 / y \\ I_2 V_2 - y & I_2 + V_1 \end{pmatrix}, \quad M = \begin{pmatrix} -I_2 & 1 \\ -y & -I_1 \end{pmatrix}.
\]

The spectral curve \( \gamma \) is given by
\[
f(u, v) = v^2 - (u^2 + c_1 u + c_0)^2 + 4c_{-1},
\]
\[
c_1 = \sum_{i=1}^{2} (I_i + V_i), \quad c_0 = \sum_{i=1}^{2} \frac{I_i}{I_i} + \sum_{i=1}^{2} V_i, \quad c_{-1} = \sum_{i=1}^{2} I_i V_i.
\]

By solving the linear equations
\[
\psi_1(x, y) = I_2 V_2 - y = 0, \quad \psi_2(x) = x + I_1 + V_2 = 0,
\]

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we obtain the eigenvector map

\[ P_1 = \phi(I_1, I_2, V_1, V_2) = (-I_1 - V_2, I_2V_2 - I_1V_1). \]

The time evolution of the pdTL is given by the addition formula on Sym\(^1\) (γ) ≃ γ:

\[ \tilde{P}_1 = P_1 \oplus T, \quad (C.1) \]

where \( T \) is the unique point on γ, such that \( T + P'_\infty - A - P_\infty \) is the principal divisor of a rational function on γ and \( A = (0, V_1V_2 - I_1I_2) \).

To obtain \( T \) explicitly, note first that \( l \in L(A + P_\infty) \) for \( (l) = T + P'_\infty - A - P_\infty \) and \( \dim L(A + P_\infty) = 2 \). We can easily see that the rational function

\[ y + \sqrt{\frac{c_0^2 - 4c_{-1}}{x}} + x + c_1 \quad (C.2) \]

is expanded as follows

\[
\begin{align*}
\frac{2}{t_1} + 2c_1 + \left( c_0 + \sqrt{\frac{c_0^2 - 4c_{-1}}{x}} \right) t_1 + o(t_1) & \quad \text{at } P_\infty, \\
\left( -c_0 + \sqrt{\frac{c_0^2 - 4c_{-1}}{x}} \right) t_2 + o(t_2) & \quad \text{at } P'_\infty, \\
\frac{2\sqrt{\frac{c_0^2 - 4c_{-1}}{x}}}{t_3} + \frac{c_0c_1}{\sqrt{\frac{c_0^2 - 4c_{-1}}{x}}} + c_1 + o(1) & \quad \text{at } A, \\
\end{align*}
\]

where \( t_1, t_2 \) and \( t_3 \) are the local parameters at \( P_\infty, P'_\infty \) and \( A \), respectively. The rational function (C.2) has poles of order 1 at \( P_\infty, P'_\infty \) and \( A \), and has a zero of order 1 at \( P'_\infty \). Therefore, the rational function \( l \) is nothing but (C.2). By solving \( l = 0 \) and \( f = 0 \), we obtain

\[ T = (-c_1, -\sqrt{\frac{c_0^2 - 4c_{-1}}{x}}). \]

The addition (C.1) can be written by the divisors

\[ (-\tilde{P}_1) + P_1 + A - P'_\infty - 2P_\infty \equiv 0 \quad (\text{mod } D_l(\gamma)), \]

where \(-\tilde{P}_1\) is the inverse of \( \tilde{P}_1 \). This implies that the rational function \( h \), such that \( (h) = (-\tilde{P}_1) + P_1 + A - P'_\infty - 2P_\infty \), is in \( L(P'_\infty + 2P_\infty) = \langle 1, x, x^2 + y \rangle \). Thus, we obtain the curve \( \kappa_1 = (h_1 = 0) \) passing through \( P_1 \) and \( A \):

\[ h_1(u, v) = (I_1I_2 - V_1V_2) + (I_1 + I_2 - V_1 + V_2)u + u^2 + v. \]

The third intersection point of γ and \( \kappa_1 \) is \( O_1 = (-I_2 - V_2, I_1V_2 - I_2V_1) \). We successively obtain the curve \( \kappa_2 = (h_2(u, v) = 0) \) passing through \( O_1 \):

\[ h_2(u, v) = I_1I_2 + V_1V_2 + 2I_2V_1 + u^2 + c_1u + v. \]

The second intersection point of γ and \( \kappa_2 \) is

\[ (-I_1 - V_1, I_1V_2 - I_2V_1) = (-\tilde{I}_1 - \tilde{V}_2, \tilde{I}_2\tilde{V}_2 - \tilde{I}_1\tilde{V}_1) = \tilde{P}_1. \]

Thus the time evolution of the pdTL is realized by using the intersection of the spectral curve γ and the affine curves \( \kappa_1 \) and \( \kappa_2 \).
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