CONVERGENCE OF DENSITIES OF SPATIAL AVERAGES OF THE PARABOLIC ANDERSON MODEL DRIVEN BY COLORED NOISE

SEFIKA KUZGUN AND DAVID NUALART

Abstract. In this paper, we present a rate of convergence in the uniform norm for the densities of spatial averages of the solution to the \(d\)-dimensional parabolic Anderson model driven by a Gaussian multiplicative noise, which is white in time and has a spatial covariance given by the Riesz kernel. The proof is based on the combination of Malliavin calculus techniques and the Stein’s method for normal approximations.

Mathematics Subject Classifications (2020): 60H15, 60H07.

Keywords and Phrases: Stochastic heat equation. Malliavin calculus. Stein’s method.

1. Introduction

Consider the parabolic Anderson model
\[
\frac{\partial u}{\partial t} = \frac{1}{2} \Delta u + u \dot{W}, \quad x \in \mathbb{R}^d, \ t > 0,
\]
with initial condition \(u(0, x) = 1\), where \(\dot{W}\) is a noise which is white in time and colored in space. This is to say, informally, that \(\dot{W} = \{(\dot{W}(t, x) : (t, x) \in \mathbb{R}_+ \times \mathbb{R}^d\} is a centered Gaussian random field with covariance
\[
E[\dot{W}(t, x)\dot{W}(s, y)] = \delta_0(t - s)|x - y|^{-\beta},
\]
for \(0 < \beta < \min(2, d)\) where \(\delta_0\) is the Dirac delta measure at zero. The existence and uniqueness of a mild solution \(u(t, x)\) to equation (1.1) has been proved by Dalang in [8], assuming that \(u_0\) is bounded. The one-dimensional equation driven by a space-time white noise was studied in the pioneering work by Walsh [17]. Here and along the paper we will make use of the notation
\[
p_t(x) := \frac{1}{(2\pi t)^{d/2}}e^{-|x|^2/2t}
\]
for \(t > 0\) and \(x \in \mathbb{R}^d\). Fix \(R > 0\) and let \(Q_R = [-R, R]^d\). Consider the corresponding centered and normalized spatial averages defined by
\[
F_{R, t} := \frac{1}{\sigma_{R, t}^2} \left( \int_{Q_R} u(t, x)dx - (2R)^d \right), \quad \text{where } \sigma_{R, t}^2 := \text{Var} \left( \int_{Q_R} u(t, x)dx \right).
\]

In this paper, we will investigate the quantitative rates of convergence corresponding to the normal approximations in density of the above spatial averages. The main result of this paper is as follows:

The work by D. Nualart has been supported by the NSF grants DMS-2054735.
Theorem 1.1. Let \( u = \{ u(t, x) : (t, x) \in \mathbb{R}_+ \times \mathbb{R}^d \} \) be the mild solution to the stochastic heat equation (1.1) with the initial condition \( u_0 = 1 \). Fix \( t > 0 \) and let \( F_{R,t} \) be defined as in (1.2). Then, for all \( R \geq 1 \), \( F_{R,t} \) has a density \( f_{F_{R,t}} \) and
\[
\sup_{z \in \mathbb{R}} |f_{F_{R,t}}(z) - \phi(z)| \leq C_1 R^{-\beta/2},
\]
where \( \phi \) is the density of a standard normal distribution on \( \mathbb{R} \).

2. Preliminaries

In this section, we will recall some preliminary notions and results. Let \( C^\infty_c(\mathbb{R}_+ \times \mathbb{R}^d) \) be the space of infinitely differentiable functions with compact support on \( \mathbb{R}_+ \times \mathbb{R}^d \). Let \( \mathcal{H} \) be the Hilbert space defined as the completion of \( C^\infty_c(\mathbb{R}_+ \times \mathbb{R}^d) \) with respect to the inner product
\[
\langle \varphi, \psi \rangle_{\mathcal{H}} = \int_0^\infty \int_{\mathbb{R}^d} \varphi(s, x) \psi(s, y) |x - y|^{-\beta} dx dy ds.
\]

Suppose \( W \) a Gaussian noise encoded by a centered Gaussian family of random variables \( \{ W(\varphi) : \varphi \in C^\infty_c(\mathbb{R}_+ \times \mathbb{R}^d) \} \), defined in a complete probability space \( (\Omega, \mathcal{F}, \mathbb{P}) \), with the covariance structure
\[
\mathbb{E}[W(\varphi)W(\psi)] = \langle \varphi, \psi \rangle_{\mathcal{H}}.
\]

Let \( (\mathcal{F}_t)_{t \in \mathbb{R}_+} \) be the filtration such that \( \mathcal{F}_t = \mathcal{F}_t^0 \vee \mathcal{N} \) where \( \mathcal{F}_t^0 \) is the \( \sigma \)-field generated by the random variables \( W(\varphi) \) where \( \varphi \) has support in \( [0, t] \times \mathbb{R}^d \), and \( \mathcal{N} \) is the \( \sigma \)-field generated by the \( \mathbb{P} \)-null sets. We say that a random field \( X = \{ X(t, x) : (t, x) \in \mathbb{R}_+ \times \mathbb{R}^d \} \) is adapted if for each \( (t, x) \in \mathbb{R}_+ \times \mathbb{R}^d \) the random variable \( X(t, x) \) is \( \mathcal{F}_t \)-measurable. Then, for any adapted, jointly measurable random field \( X \) such that
\[
\mathbb{E}\left[ \|X\|_{\mathcal{H}}^2 \right] = \int_0^\infty \int_{\mathbb{R}^d} \mathbb{E}[X(s, x)X(s, y)] |x - y|^{-\beta} dx dy ds < \infty,
\]
the stochastic integral
\[
\int_{\mathbb{R}_+ \times \mathbb{R}^d} X(\tau, \xi) W(d\tau, d\xi)
\]
is well-defined in the sense of Walsh and the Itô-Walsh isometry
\[
\mathbb{E}\left[ \int_{\mathbb{R}_+ \times \mathbb{R}^d} X(s, y) W(ds, dy) \right]^2 = \int_0^\infty \int_{\mathbb{R}^d} \mathbb{E}[X(s, x)X(s, y)] |x - y|^{-\beta} dx dy ds
\]
holds. Moreover, the following Burkholder-Davis-Gundy equality is satisfied.

Proposition 2.1. For all \( p \geq 2 \) there exists a constant \( C_p > 0 \) such that for all adapted, jointly measurable random field \( X \) satisfying (2.2) and for all \( t \in [0, \infty) \), we have
\[
\mathbb{E}\left[ \left( \int_{[0,t] \times \mathbb{R}^d} X(s, y) W(ds, dy) \right)^p \right] \leq C_p \mathbb{E}\left[ \left( \int_0^t \int_{\mathbb{R}^d} X(s, x)X(s, y) |x - y|^{-\beta} dx dy ds \right)^{p/2} \right].
\]

Let us recall the following theorem (see [8]) on the existence of a unique mild solution to equation (1.1).
Proposition 2.2. There exists a unique measurable and adapted random field
\[ u = \left\{ u(t, x) : (t, x) \in \mathbb{R}_+ \times \mathbb{R}^d \right\} \]
such that for all \( T > 0 \) and \( p \geq 2 \)
\[ \sup_{(t,x) \in [0,T] \times \mathbb{R}^d} E \left[ |u(t, x)|^p \right] = C_{T,p} < \infty, \tag{2.4} \]
and for all \( t \geq 0 \) and \( x \in \mathbb{R}^d \)
\[ u(t, x) = 1 + \int_{[0,t] \times \mathbb{R}^d} p_{t-s}(x - y)u(s,y)W(ds,dy). \tag{2.5} \]

For the existence of densities and the estimates, we will also need some non-degeneracy conditions. For this we will need the following result, which is a corollary \[4\], Theorem 1.5.

Proposition 2.3. Let \( u \) be the mild solution to the equation (2.5). Then for all \( p > 0 \), and \( t > 0 \):
\[ \sup_{0 \leq s \leq t} E \left[ |u(s, 0)|^{-p} \right] = \kappa_{t,p} < +\infty. \]

2.1. Malliavin Calculus. For a smooth and cylindrical random variable of the form \( F = f(W(\varphi_1), \ldots, W(\varphi_n)) \), with \( \varphi_i \in \mathfrak{F}, 1 \leq i \leq n, \) and \( f \in C^\infty_c(\mathbb{R}^n) \) (\( f \) and its partial derivatives are bounded), we define its Malliavin derivative as the \( \mathfrak{F} \)-valued random variable given by
\[ DF = \sum_{i=1}^n \frac{\partial f}{\partial x_i}(W(\varphi_1), \ldots, W(\varphi_n))\varphi_i. \]

By iteration, we can also define the \( k \)-th derivative \( D^k F \), which is an element in the space \( L^2(\Omega; \mathfrak{F}^\otimes k) \). For any real \( p \geq 1 \) and any integer \( k \geq 1 \), the Sobolev space \( D^{1,p,k} \) is defined as the closure of the space of smooth and cylindrical random variables with respect to the norm \( \| \cdot \|_{k,p} \) defined by
\[ \| F \|^p_{k,p} = E[|F|^p] + \sum_{i=1}^k E \left[ \| D^i F \|^p_{\mathfrak{F}^\otimes i} \right]. \]

We define the divergence operator \( \delta \) as the adjoint in \( L^2 \) of the derivative operator \( D \). Namely, an element \( U \in L^2(\Omega; \mathfrak{F}) \) belongs to the domain of \( \delta \), denoted by \( \text{Dom} \delta \), if there is a constant \( c_U > 0 \) depending on \( U \) satisfying
\[ |E[(DF, U)_\mathfrak{F}]| \leq c_U \| F \|_{L^2(\Omega)} \]
for any \( F \in D^{1,2} \). If \( U \in \text{Dom} \delta \), the random variable \( \delta(U) \) is defined by the duality relationship
\[ E[F \delta(U)] = E[(DF, U)_\mathfrak{F}], \tag{2.6} \]
which is valid for all \( F \in D^{1,2} \).

The Itô-Walsh stochastic integral we introduced in the previous subsection coincides with the divergence operator for adapted processes (see, for instance, [6]). That is to say, for any adapted random field \( U = \{ U(t, x) : (t, x) \in \mathbb{R}_+ \times \mathbb{R} \} \) such that \( E[\|U\|_{\mathfrak{F}}^2] < \infty \), we have that \( U \in \text{Dom} \delta \) and
\[ \delta(U) = \int_{\mathbb{R}_+ \times \mathbb{R}^d} U(s, y)W(ds, dy). \tag{2.7} \]
The operators $D$ and $\delta$ satisfy the following commutation relation
\[ D_{s,y}(\delta(V)) = V(s,y) + \delta(D_{s,y}V), \tag{2.8} \]
for almost all $(s,y) \in \mathbb{R}_+ \times \mathbb{R}^d$, provided $V \in D^{1,2}(\Omega; \mathcal{F})$ is such that for almost all $(s,y) \in \mathbb{R}_+ \times \mathbb{R}^d$, $D_{s,y}V$ belongs to the domain of the divergence in $L^2$ and
\[ \mathbb{E}\left[ \int_{\mathbb{R}_+ \times \mathbb{R}^d} |\delta(D_{s,y}V)||\delta(D_{s,y'}V)||y - y'|^{-\beta} dy dy' ds \right] < \infty \]
(see [15, Proposition 1.3.2]).

The following estimate can be found in [13] (see also [4] for the differentiability of the solution).

**Proposition 2.4.** Let $u$ be the solution to the stochastic heat equation (1.1). Fix $(t,x) \in \mathbb{R}_+ \times \mathbb{R}^d$. Then $u(t,x) \in \cap_{p \geq 2} D^{2,p}$. Moreover,

(a) The derivative $D_{s,y}u(t,x)$ satisfies
\[ D_{s,y}u(t,x) = p_{t-s}(x-y)u(s,y) + \int_{[s,t] \times \mathbb{R}^d} p_{t-r}(x-\xi)D_{s,y}u(r,\xi)W(dr,d\xi) \tag{2.9} \]
if $s < t$ and $D_{s,y}u(t,x) = 0$ if $s > t$. For all $0 < s < t < T$, we have
\[ \|D_{s,y}u(t,x)\|_p \leq C_{T,p}p_{t-s}(x-y), \tag{2.10} \]
where $C_{T,p}$ is a constant that depends on $T$ and $p$.

(b) The second derivative $D_{r,z}D_{s,y}u(t,x)$ satisfies
\[ D_{r,z}D_{s,y}u(t,x) = p_{t-s}(x-y)D_{r,z}u(s,y) \tag{2.11} \]
\[ + \int_{[s,t] \times \mathbb{R}^d} p_{t-r}(x-\xi)D_{r,z}D_{s,y}u(r,\xi)W(dr,d\xi) \]
if $r < s < t$. For all $0 < r < s < t < T$, we have
\[ \|D_{r,z}D_{s,y}u(t,x)\|_p \leq C_{T,p}p_{t-s}(x-y)p_{s-r}(y-z), \tag{2.12} \]
where $C_{T,p}$ is a constant that depends on $T$ and $p$.

### 2.2. Malliavin-Stein Method

In this subsection we recall a result which gives a bound on the uniform distance between the density of a random variable and the density of the standard normal distribution under some assumptions. A version of this result first appeared in [9] and the proof of Theorem 2.5 below can be found in [12]. For the existence of the density, see also [2, Proposition 1].

For $V \in L^2(\Omega; \mathcal{F})$ and $F \in D^{1,2}$ we will make use of the notation $D_VF = \langle DF, V \rangle_{\mathcal{F}}$.

**Theorem 2.5.** Assume that $V \in D^{1,6}(\Omega; \mathcal{H})$ and $F = \delta(V) \in D^{2,6}$ with $\mathbb{E}[F] = 0$, $\mathbb{E}[F^2] = 1$ and $(D_VF)^{-1} \in L^4(\Omega)$. Then, $V/D_VF \in \text{Dom} \delta$, $F$ admits a density $f_F(x)$ and the following inequality holds true
\[ \sup_{x \in \mathbb{R}} |f_F(x) - \phi(x)| \leq (\|F\|_4 \|(D_VF)^{-1}\|_4 + 2) \|1 - D_VF\|_2 \]
\[ + \|(D_VF)^{-1}\|_4^2 \|D_VF\|_2 \tag{2.13}, \]
where $\phi(x)$ is the density of the law $\mathcal{N}(0,1)$.
2.3. Order of variance and total variation distance. Let \( u \) be the mild solution to (1.1) with \( u_0 = 1 \) given in Proposition 2.2. Then, for any fixed \( t > 0 \), the random variable \( F_{R,t} \) defined in (1.2) is given by

\[
F_{R,t} = \frac{1}{\sigma_{R,t}} \left( \int_{Q_R} \int_{[0,t] \times \mathbb{R}^d} p_{t-\tau}(x-\xi)u(\tau,\xi)W(d\tau,d\xi,dx) \right).
\]

Using the stochastic Fubini’s theorem, we can rewrite \( F_{R,t} \) as follows

\[
F_{R,t} = \int_{[0,t] \times \mathbb{R}^d} \varphi_{R,t}(\tau,\xi)u(\tau,\xi)W(d\tau,d\xi),
\]

where \( \varphi_{R,t}(\tau,\xi) := \frac{1}{\sigma_{R,t}} \int_{Q_R} p_{t-\tau}(x-\xi)dx. \) (2.14)

So, using (2.7), we obtain

\[
F_{R,t} = \delta(V_{R,t}),
\]

where \( V_{R,t}(\tau,\xi) = 1_{[0,t]}(\tau)\varphi_{R,t}(\tau,\xi)u(\tau,\xi). \) (2.15)

The following result provides the asymptotic behavior of \( \sigma_{R,t}^2 \) as \( R \to \infty \) and an upper bound for the total variation distance between the law of \( F_{R,t} \) and the one-dimensional standard distribution (see [11, Theorem 1.1]).

**Theorem 2.6.** Let \( u = \{u(t,x) : (t,x) \in \mathbb{R}_+ \times \mathbb{R}^d\} \) be the mild solution to stochastic the heat equation (1.1) with the initial condition \( u_0 = 1 \). Fix \( t > 0 \) and let \( F_{R,t} \) and \( \sigma_{R,t} \) be defined as in (1.2). Then,

\[
\lim_{R \to \infty} \frac{\sigma_{R,t}^2}{R^{2d-\beta}} = k_{\beta} t,
\]

where

\[
k_{\beta} := \int_{Q_1^2} |x_1 - x_2|^{-\beta}dx_1dx_2.
\]

Moreover, for all \( R > 0 \),

\[
d_{TV}(F_{R,t},Z) \leq C_{t,\beta}R^{-\beta/2},
\]

where \( Z \) is a \( N(0,1) \) random variable.

3. Proof of Theorem 1.1

In order to apply Theorem 2.5, we need estimates on the negative moments of \( D_{V_{R,t}}F_{R,t} \). The next proposition provides this type of estimates.

**Proposition 3.1.** Let \( u \) be the mild solution to stochastic the heat equation (1.1) with \( u_0 = 1 \), let \( F_{R,t} \) be defined as in (1.2) and fix \( p \geq 2 \). Then,

\[
\sup_{R \geq 1} \mathbb{E} \left[ |D_{V_{R,t}}F_{R,t}|^{-p} \right] < \infty.
\]

(3.1)
Proof. Consider the Malliavin derivative of $F_{R,t}$ given by
\[
D_{s,y}F_{R,t} = \frac{1}{\sigma_{R,t}} \int_{Q_R} D_{s,y}u(t,x)dx.
\] (3.2)

From (2.15) and (3.2), we can write
\[
D_{V_{R,t}}F_{R,t} = \int_0^t \int_{\mathbb{R}^2d} V_{R,t}(s,\tilde{y}) D_{s,y}F_{R,t}|y - \tilde{y}|^{-\beta} d\tilde{y}dyds
\]
\[
= \frac{1}{\sigma_{R,t}} \int_{Q_R}^t \int_{\mathbb{R}^2d} p_{t-s}(x_1 - y) u(s,\tilde{y}) D_{s,y}u(t,x_2)|y - \tilde{y}|^{-\beta} d\tilde{y} dyds dx_1 dx_2. \tag{3.3}
\]

First note that the integrand in (3.3) is nonnegative since for all $(t,x) \in \mathbb{R}_+ \times \mathbb{R}^d$, $u(t,x) \geq 0$, and for all $(s,y), (t,x) \in \mathbb{R}_+ \times \mathbb{R}^d$, $D_{s,y}u(t,x) \geq 0$ a.s.

Fix $t > \epsilon > 0$ and set $t_\alpha := t - \epsilon^\alpha$ where $\alpha < 1$. Then we can estimate (3.3) as follows:
\[
D_{V_{R,t}}F_{R,t} \geq \frac{1}{\sigma_{R,t}} \int_{Q_R}^t \int_{\mathbb{R}^2d} p_{t-s}(x_1 - y) u(s,\tilde{y}) D_{s,y}u(t,x_2)|y - \tilde{y}|^{-\beta} d\tilde{y} dyds dx_1 dx_2 =: I_{R,\epsilon}.
\]

Using this estimate, we get
\[
P \left( D_{V_{R,t}}F_{R,t} < \epsilon \right) \leq P \left( I_{R,\epsilon} < \epsilon \right).
\]

From (2.9), we obtain
\[
I_{R,\epsilon} = \int_{t_\alpha}^t \int_{\mathbb{R}^2d} \varphi_{R,t}(s,y) \varphi_{R,t}(s,\tilde{y}) u(s,z) u(s,\tilde{y}) |y - \tilde{y}|^{-\beta} d\tilde{y} dyds
\]
\[
+ \int_{t_\alpha}^t \int_{\mathbb{R}^2d} \left( \int_{[s,t] \times \mathbb{R}^d} \varphi_{R,t}(\tau,\xi) D_{s,y}u(\tau,\xi) W(d\tau, d\xi) \right) \varphi_{R,t}(s,\tilde{y}) u(s,\tilde{y}) |y - \tilde{y}|^{-\beta} d\tilde{y} dyds
\]
\[
=: I_1 + I_2.
\]

Taking into account the estimates
\[
P \left( I_1 + I_2 < \epsilon \right) \leq P \left( I_1 < 2\epsilon \right) + P \left( I_1 + I_2 < \epsilon, I_1 \geq 2\epsilon \right)
\]
\[
\leq P \left( I_1 < 2\epsilon \right) + P \left( |I_2| > \epsilon \right), \tag{3.4}
\]

we have
\[
P \left( I_{R,\epsilon} < \epsilon \right) \leq P \left( I_1 < 2\epsilon \right) + P \left( |I_2| > \epsilon \right).
\]

We shall next estimate these probabilities, starting with the first one.

\[
K := P \left( I_1 < 2\epsilon \right) = P \left( \int_{t_\alpha}^t \int_{\mathbb{R}^2d} \varphi_{R,t}(s,y) \varphi_{R,t}(s,\tilde{y}) u(s,y) u(s,\tilde{y}) |y - \tilde{y}|^{-\beta} d\tilde{y} dyds < 2\epsilon \right).
\]

By Chebyshev’s inequality, for any $q \geq 2$ we obtain
\[
K = P \left( \left[ \int_{t_\alpha}^t \int_{\mathbb{R}^2d} \varphi_{R,t}(s,y) \varphi_{R,t}(s,\tilde{y}) u(s,y) u(s,\tilde{y}) |y - \tilde{y}|^{-\beta} d\tilde{y} dyds \right]^{-1} > \frac{1}{2\epsilon} \right)
\]
\[
\leq (2\epsilon)^q \mathbb{E} \left[ \left( \int_{t_\alpha}^t \int_{\mathbb{R}^2d} \varphi_{R,t}(s,y) \varphi_{R,t}(s,\tilde{y}) u(s,y) u(s,\tilde{y}) |y - \tilde{y}|^{-\beta} d\tilde{y} dyds \right)^{-q} \right]. \tag{3.5}
\]
Set
\[ m(t_\alpha, R) := \int_{t_\alpha}^t \int_{\mathbb{R}^d} \varphi_{R,t}(s, y)\varphi_{R,t}(s, \tilde{y})|y - \tilde{y}|^{-\beta} dy dy ds. \] (3.6)

Then, taking into account that the function \( x \to x^{-q} \) is convex and applying Jensen’s inequality, we can write
\[
E \left[ \left( \int_{t_\alpha}^t \int_{\mathbb{R}^d} \varphi_{R,t}(s, y)\varphi_{R,t}(s, \tilde{y})u(s, y)u(s, \tilde{y})|y - \tilde{y}|^{-\beta} dy dy ds \right)^{-q} \right] 
= m(t_\alpha, R)^{-q}E \left[ \left( \frac{1}{m(t_\alpha, R)} \int_{t_\alpha}^t \int_{\mathbb{R}^d} \varphi_{R,t}(s, y)\varphi_{R,t}(s, \tilde{y})u(s, y)u(s, \tilde{y})|y - \tilde{y}|^{-\beta} dy dy ds \right)^{-q} \right] 
\leq m(t_\alpha, R)^{-q-1} \int_{t_\alpha}^t \int_{\mathbb{R}^d} \varphi_{R,t}(s, y)\varphi_{R,t}(s, \tilde{y})E \left[ (u(s, y)u(s, \tilde{y}))^{-q} \right] |y - \tilde{y}|^{-\beta} dy dy ds. \] (3.7)

Using Hölder’s inequality, the stationarity of solution in the space variable (see, for instance, [6, Lemma 7.1]), and finally Proposition 2.3, we get
\[
E \left[ (u(s, y)u(s, \tilde{y}))^{-q} \right] \leq \sqrt{E \left[ u(s, y)^{-2q} \right] E \left[ u(s, \tilde{y})^{-2q} \right]} = E \left[ u(s, 0)^{-2q} \right] \leq \kappa_{t,2q} \] (3.8)
for \( t_\alpha \leq s \leq t \). Substituting the estimate (3.8) into (3.7), we get
\[
E \left[ \left( \int_{t_\alpha}^t \int_{\mathbb{R}^d} \varphi_{R,t}(s, y)\varphi_{R,t}(s, \tilde{y})u(s, y)u(s, \tilde{y})|y - \tilde{y}|^{-\beta} dy dy ds \right)^{-q} \right] 
\leq \kappa_{t,2q}m(t_\alpha, R)^{-q-1} \int_{t_\alpha}^t \int_{\mathbb{R}^d} \varphi_{R,t}(s, y)\varphi_{R,t}(s, \tilde{y})|y - \tilde{y}|^{-\beta} dy dy ds = \kappa_{t,2q}m(t_\alpha, R)^{-q}. \] (3.9)

Therefore, from (3.5) and (3.9), we obtain
\[
K \leq \kappa_{t,q}2(2\varepsilon)^q m(t_\alpha, R)^{-q}. 
\]
Using Lemma A.5, we obtain
\[
K \leq C_{t,q,\beta,d}\varepsilon^{(1-n)q}, \quad (3.10)
\]
where \( C_{t,q,\beta,d} \) is a constant depending on \( t, q, \beta \) and \( d \). Next, we will estimate the probability \( L := P (|I_2| > \varepsilon) \). Using the stochastic Fubini’s theorem and Chebyshev’s inequality, for any \( q \geq 2 \) we have
\[
L \leq \frac{1}{\varepsilon^q} \times E \left[ \left( \int_{t_\alpha}^t \int_{\mathbb{R}^d} \int_{t_\alpha}^t \varphi_{R,t}(s, \tilde{y})\varphi_{R,t}(\tau, \xi)D_{s,y}u(s, \tilde{y})u(s, \tilde{y})|y - \tilde{y}|^{-\beta} dy dy ds dy d\tilde{y} d\tau d\xi \right) W (d\tau, d\xi) \right]^q. 
\]
Then, applying Burkholder-Davis-Gundy inequality (2.3), followed by Minkowski’s inequality, we get

\[
L \leq \frac{C_q}{\varepsilon^q} E \left[ \left\| \int_{t_a}^{t} \int_{R^{6d}} \int_{t_{a}}^{T} \int_{t_a}^{T} \varphi_{R,t}(s_1, \tilde{y}_1) \varphi_{R,t}(s_2, \tilde{y}_2) \varphi_{R,t}(\tau, \xi) D_{s_1, \tilde{y}_1, u(\tau, \xi)} u(s_1, \tilde{y}_1) \right\|_{q/2} \right]
\]

\[
= \frac{C_q}{\varepsilon^q} E \left[ \left\| \int_{t_a}^{t} \int_{t_a}^{T} \int_{t_a}^{T} \int_{R^{6d}} \varphi_{R,t}(s_1, \tilde{y}_1) \varphi_{R,t}(s_2, \tilde{y}_2) \varphi_{R,t}(\tau, \xi) X_{s_1, \tilde{y}_1, s_2, \tilde{y}_2, \tau, \xi, \tilde{\xi}} \right\|_{q/2} \right]
\]

\[
\leq \frac{C_q}{\varepsilon^q} \left( \int_{t_a}^{t} \int_{t_a}^{T} \int_{t_a}^{T} \int_{R^{6d}} \varphi_{R,t}(s_1, \tilde{y}_1) \varphi_{R,t}(s_2, \tilde{y}_2) \varphi_{R,t}(\tau, \xi) X_{s_1, \tilde{y}_1, s_2, \tilde{y}_2, \tau, \xi, \tilde{\xi}} \right)^{\frac{q}{2}},
\]

(3.11)

where

\[
X_{s_1, \tilde{y}_1, s_2, \tilde{y}_2, \tau, \xi, \tilde{\xi}} := D_{s_1, \tilde{y}_1, u(\tau, \xi)} u(s_1, \tilde{y}_1) D_{s_2, \tilde{y}_2, u(\tau, \xi)} u(s_2, \tilde{y}_2).
\]

Using Hölder’s inequality, the estimate (2.10) and the fact that \( \sup_{(r,z) \in [0,t] \times R} \|u(r, z)\|_q < \infty \) for all \( q \geq 2 \), we have

\[
\|X_{s_1, \tilde{y}_1, s_2, \tilde{y}_2, \tau, \xi, \tilde{\xi}}\|_{q/2} \leq C p_{r-s_1}(y_1 - \xi)p_{r-s_2}(y_2 - \tilde{\xi}),
\]

where here and in the rest of the proof \( C \) will denote a generic constant that depends on \( t, q, \beta \) and \( d \). Plugging this bound in the estimate (3.11), we see that

\[
L \leq \frac{C}{\varepsilon^q} \left( \int_{t_a}^{t} \int_{t_a}^{T} \int_{t_a}^{T} \int_{R^{6d}} p_{r-s_1}(y_1 - \xi)p_{r-s_2}(y_2 - \tilde{\xi}) \varphi_{R,t}(s_1, \tilde{y}_1) \varphi_{R,t}(s_2, \tilde{y}_2) \varphi_{R,t}(\tau, \xi) \right)^{2q}.
\]

(3.12)

Then, we can write

\[
L \leq \frac{C}{\varepsilon^q} \left( \int_{Q_R^t} dx_1 dx_2 dx_3 dx_4 \int_{t_a}^{t} \int_{t_a}^{T} E_{R,t}(s_1, s_2, \tau) ds_1 ds_2 d\tau \right)^{\frac{q}{2}},
\]

where

\[
E_{R,t}(s_1, s_2, \tau) := \int_{Q_R^t} dx_1 dx_2 dx_3 dx_4 \int_{R^{6d}} d\tilde{y}_1 d\tilde{y}_2 d\tilde{\xi} dy_1 dy_2 \left| \varphi_{R,t}(x_1 - \xi) \varphi_{R,t}(x_2 - \tilde{\xi}) \varphi_{R,t}(x_3 - \tilde{y}_1) \varphi_{R,t}(x_4 - \tilde{y}_2) \varphi_{R,t}(\tau, \xi) \right|^{\beta}.
\]

(3.13)
Now, using the estimate in Lemma A.6 and (2.16), we get, for $R \geq 1$,
\[
L \leq C \frac{R^{d-3\beta}}{\sigma_{R,t}^4} \left( \int_{t_n}^t \int_{t_n}^\tau \int_{t_n}^\tau ds_1 ds_2 d\tau \right)^{q/2} \leq C \varepsilon^{\left(\frac{q}{2} - 1\right)q}.
\]
(3.14)

Finally, choosing $\alpha = 4/5$ in (3.10) and (3.14), we obtain
\[
\sup_{R \geq 1} \mathbb{P} \left( D_{V_{R,t}} F_{R,t} < \varepsilon \right) \leq C \varepsilon^{q/5}.
\]

Together with Lemma A.7, this estimate completes our proof. \qed

Now, we are ready to prove the main result.

Proof of Theorem 1.1. We will apply Theorem 2.5 to the random variable $F_{R,t} = \delta(V_{R,t})$. Fix $t > 0$. Along the proof $C$ will denote a generic constant that depends on $t$, $q$, $\beta$ and $d$. It has been already proven in [11] that
\[
\left\| 1 - D_{V_{R,t}} F_{R,t} \right\|_2 \leq C R^{-\beta/2}.
\]
(3.15)

Taking into account Proposition 3.1 we are only left to estimate the term $\left\| D_{V_{R,t}} (D_{V_{R,t}} F_{R,t}) \right\|_2$ appearing in (2.13). Recall that

\[
D_{V_{R,t}} F_{R,t} = \frac{1}{\sigma_{R,t}^2} \int_0^t \int_{\mathbb{R}^{2d}} \int_{Q_R} \varphi_{R,t}(s, \tilde{y}) u(s, \tilde{y}) D_{s,y} u(t, x) |y - \tilde{y}|^{-\beta} dxd\tilde{y}dyds.
\]

Taking the Malliavin derivative, we get
\[
D_{r,z} \left( D_{V_{R,t}} F_{R,t} \right) = \frac{1}{\sigma_{R,t}^2} \int_0^t \int_{\mathbb{R}^{2d}} \int_{Q_R} \varphi_{R,t}(s, \tilde{y}) D_{r,z} u(s, \tilde{y}) D_{s,y} u(t, x) |y - \tilde{y}|^{-\beta} dxd\tilde{y}dyds
+ \frac{1}{\sigma_{R,t}^2} \int_0^t \int_{\mathbb{R}^{2d}} \int_{Q_R} \varphi_{R,t}(s, \tilde{y}) u(s, \tilde{y}) D_{r,z} D_{s,y} u(t, x) |y - \tilde{y}|^{-\beta} dxd\tilde{y}dyds,
\]
and hence, using (2.15), we can write
\[
D_{V_{R,t}} \left( D_{V_{R,t}} F_{R,t} \right) = \frac{1}{\sigma_{R,t}^4} \int_0^t \int_{\mathbb{R}^{4d}} \int_{Q_R} \varphi_{R,t}(r, \tilde{z}) \varphi_{R,t}(s, \tilde{y}) u(r, \tilde{z}) D_{r,z} u(s, \tilde{y}) D_{s,y} u(t, x) |y - \tilde{y}|^{-\beta} dxd\tilde{y}dzdydsdr
+ \frac{2}{\sigma_{R,t}^4} \int_0^t \int_{\mathbb{R}^{4d}} \int_{Q_R} \varphi_{R,t}(r, \tilde{z}) \varphi_{R,t}(s, \tilde{y}) u(r, \tilde{z}) u(s, \tilde{y}) D_{r,z} D_{s,y} u(t, x) |y - \tilde{y}|^{-\beta} dxd\tilde{y}dzdydsdr.
\]

Now using (2.9) for $D_{s,y} u(t, x)$ and (2.11) for $D_{r,z} D_{s,y} u(t, x)$, respectively, we have
\[
D_{V_{R,t}} \left( D_{V_{R,t}} F_{R,t} \right) = Y_{R,t}^0 + Y_{R,t}^1 + Y_{R,t}^2,
\]
where:

\[ \mathcal{Y}_{R,t}^0 = 2 \int_0^t \int_{\mathbb{R}^d} \varphi_{R,t}(r, \tilde{z}) \varphi_{R,t}(s, \tilde{y}) \varphi_{R,t}(s, y) D_{r,z} u(s, \tilde{y}) u(r, \tilde{z}) u(s, y) \times |y - \tilde{y}|^{-\beta} |z - \tilde{z}|^{-\beta} d\tilde{y} d\tilde{z} dz dy ds dr, \]

\[ \mathcal{Y}_{R,t}^1 = \int_0^t \int_{\mathbb{R}^d} \varphi_{R,t}(r, \tilde{z}) u(s, \tilde{y}) W(d\tilde{y}, d\xi) \mathcal{F}_{R,t}(r, \tilde{z}) \varphi_{R,t}(s, \tilde{y}) \times u(r, \tilde{z}) D_{r,z} u(s, \tilde{y}) |y - \tilde{y}|^{-\beta} |z - \tilde{z}|^{-\beta} d\tilde{y} d\tilde{z} dz dy ds dr, \]

\[ \mathcal{Y}_{R,t}^2 = 2 \int_0^t \int_{\mathbb{R}^d} \varphi_{R,t}(r, \tilde{z}) \varphi_{R,t}(s, \tilde{y}) W(d\tilde{y}, d\xi) \mathcal{F}_{R,t}(r, \tilde{z}) \varphi_{R,t}(s, \tilde{y}) \times u(r, \tilde{z}) u(s, \tilde{y}) |y - \tilde{y}|^{-\beta} |z - \tilde{z}|^{-\beta} d\tilde{y} d\tilde{z} dz dy ds dr. \]

We will handle the terms \( \mathcal{Y}_{R,t}^1 \) and \( \mathcal{Y}_{R,t}^2 \) simultaneously. To this end, for \( i = 1, 2 \), let

\[ \mathcal{Y}_{R,t}^i = \int_{[0,t] \times \mathbb{R}^d} \left( \int_0^\tau \int_{\mathbb{R}^d} \varphi_{R,t}(r, \tilde{z}) \varphi_{R,t}(s, \tilde{y}) D_{r,z} D_{s,y} u(r, \tilde{z}) \varphi_{R,t}(s, \tilde{y}) \right) \varphi_{R,t}(r, \tilde{z}) \mathcal{F}_{R,t}(r, \tilde{z}) \right) \times u(r, \tilde{z}) u(s, \tilde{y}) |y - \tilde{y}|^{-\beta} |z - \tilde{z}|^{-\beta} d\tilde{y} d\tilde{z} dz dy ds dr, \]

where

\[ Z_{r,z,s,y,\tilde{y}}^i(\tau, \xi) := \left\{ \begin{array}{l} u(r, \tilde{z}) D_{r,z} u(s, \tilde{y}) D_{s,y} u(\tau, \xi), \text{ for } i = 1 \\ u(r, \tilde{z}) u(s, y) D_{r,z} D_{s,y} u(\tau, \xi), \text{ for } i = 2. \end{array} \right. \]

**Estimation of \( \| \mathcal{Y}_{R,t}^0 \|_2 \):** Note that using the estimates (2.4) and (2.10) and Hölder’s inequality we have, for \( r < s \),

\[ \| u(r, \tilde{z}) u(s, y) D_{r,z} u(s, \tilde{y}) \|_2 \leq C p_{s-r}(z - \tilde{y}). \]

As a consequence,

\[ \| \mathcal{Y}_{R,t}^0 \|_2 \leq C \int_0^t \int_{\mathbb{R}^d} \varphi_{R,t}(s, \tilde{y}) \varphi_{R,t}(s, y) \mathcal{F}_{R,t}(s, \tilde{y}) D_{r,z} u(s, \tilde{y}) |y - \tilde{y}|^{-\beta} |z - \tilde{z}|^{-\beta} d\tilde{y} d\tilde{z} dz dy ds dr. \]

Integrating in the variables \( z, y \) and using Lemma A.1 and Lemma A.2, we have

\[ \int_{\mathbb{R}^d} \varphi_{R,t}(s, y) p_{s-r}(z - \tilde{y}) |z - \tilde{z}|^{-\beta} |y - \tilde{y}|^{-\beta} dz dy = \frac{1}{\sigma_{R,t}} \int_{Q_R} \int_{\mathbb{R}^d} p_{s-r}(s - x) p_{s-r}(z - \tilde{y}) |y - \tilde{y}|^{-\beta} |z - \tilde{z}|^{-\beta} dz dy dx \]

\[ \leq C |\tilde{y} - \tilde{z}|^{-\beta} \int_{Q_R} \int_{\mathbb{R}^d} p_{s-r}(s - x) |y - \tilde{y}|^{-\beta} dy dx \leq C |\tilde{y} - \tilde{z}|^{-\beta} \int_{Q_R} |x - \tilde{y}|^{-\beta} dx \leq C |\tilde{y} - \tilde{z}|^{-\beta} \frac{P_{s-r}^{-\beta}}{\sigma_{R,t}}. \]
Using this estimate followed with Lemma A.4 and (2.16), we get, for $R \geq 1$,

$$
\|Y^t_{R,t}\| \leq \frac{C R^{d-\beta}}{\sigma_{R,t}} \int_0^t \int_{\mathbb{R}^d} \varphi_{R,t}(r,\tilde{z}) \varphi_{R,t}(s,\tilde{y}) |\tilde{z} - \tilde{y}|^{-\beta} d\tilde{y} d\tilde{z} ds dr
\leq \frac{R^{d-\beta}C}{\sigma_{R,t}} \leq C R^{-\beta/2}. \tag{3.17}
$$

**Estimation of $\|Y^t_{R,t}\|_2$ for $i = 1, 2$:** Using the Itô-Walsh isometry of the stochastic integral and Cauchy-Schwarz inequality, we obtain

$$
\|Y^t_{R,t}\|_2^2 = \int_0^t \int_{\mathbb{R}^d} E \left[ I(Z^i(\tau,\xi)) I(Z^i(\tau,\tilde{\xi})) \right] \varphi_{R,t}(\tau,\xi) \varphi_{R,t}(\tau,\tilde{\xi}) |\xi - \tilde{\xi}|^{-\beta} d\xi d\tilde{\xi} d\tau
\leq \int_0^t \int_{\mathbb{R}^d} \|I(Z^i(\tau,\xi))\|_2 \|I(Z^i(\tau,\tilde{\xi}))\|_2 \varphi_{R,t}(\tau,\xi) \varphi_{R,t}(\tau,\tilde{\xi}) |\xi - \tilde{\xi}|^{-\beta} d\xi d\tilde{\xi} d\tau
$$

where

$$
I(Z^i(\tau,\xi)) = \int_0^\tau \int_0^\tau \int_{\mathbb{R}^d} \varphi_{R,t}(s,\tilde{y}) \varphi_{R,t}(r,\tilde{z}) Z^i_{r,z,s,y,\tilde{y}}(\tau,\xi) y - \tilde{y}|\tilde{z} - \tilde{y}|^{-\beta} d\tilde{y} d\tilde{z} ds dr.
$$

From the definition (3.16), using Hölder’s inequality and the estimates (2.4), (2.10) and (2.12), we have

$$
\|Z^i_{r,z,s,y,\tilde{y}}(\tau,\xi)\|_2 \leq C \phi^{(i)}_{r,z,s,y,\tilde{y}}(\tau,\xi),
$$

where

$$
\phi^{(i)}_{r,z,s,y,\tilde{y}}(\tau,\xi) = \begin{cases} p_{s-r}(\tilde{y} - z)p_{r-s}(\xi - y), & \text{for } i = 1 \\ p_{r-s}(\xi - y)p_{s-r}(y - z), & \text{for } i = 2. \end{cases} \tag{3.18}
$$

Hence, we get

$$
\|Y^t_{R,t}\|_2 \leq C \int_0^t \int_{\mathbb{R}^d} \varphi_{R,t}(\tau,\xi) \varphi_{R,t}(\tau,\tilde{\xi}) \Phi^{(i)}(\tau,\xi,\Phi^{(i)}(\tau,\tilde{\xi}) |\xi - \tilde{\xi}|^{-\beta} d\xi d\tilde{\xi} d\tau,
$$

where

$$
\Phi^{(i)}(\tau,\xi) = \int_0^\tau \int_0^\tau \int_{\mathbb{R}^d} \varphi_{R,t}(s,\tilde{y}) \varphi_{R,t}(r,\tilde{z}) \phi^{(i)}_{r,z,s,y,\tilde{y}}(\tau,\xi) |z - \tilde{z}|^{-\beta} d\tilde{y} d\tilde{z} ds dr. \tag{3.19}
$$

Using Lemma A.3, we obtain

$$
\|Y^t_{R,t}\|_2 \leq CR^{-2\beta} \int_0^t \int_{\mathbb{R}^d} \varphi_{R,t}(\tau,\xi) \varphi_{R,t}(\tau,\tilde{\xi}) |\xi - \tilde{\xi}|^{-\beta} d\tilde{\xi} d\xi d\tau.
$$

Then, Lemma A.4 implies that

$$
\|Y^t_{R,t}\|_2 \leq CR^{-\beta}. \tag{3.20}
$$

Using the estimates (3.17) and (3.20), we obtain

$$
\|D_{Y^t_{R,t}}(D_{V^t_{R,t},F^t_{R,t}})\|_2 \leq CR^{-\beta/2}. \tag{3.21}
$$

Finally, plugging (3.1), (3.15) and (3.21) into (2.13) we complete the proof. \qed
Lemma A.1. For any $\beta \in (0, d)$ we have
\[
\sup_{t > 0} \int_{\mathbb{R}^d} p_t(x - y)|y|^{-\beta} dy \leq C|x|^{-\beta},
\]
for some constant $C > 0$ that depends on $d$ and $\beta$.

Proof. See, for instance, [11, Lemma 3.1].

Lemma A.2. There is a constant $\beta, d$ depending on $\beta$ and $d$ such that for any $R > 0$ we have
\[
\int_{Q_R} \beta, d - \beta dy \leq C_{\beta, d} R^{d-\beta}.
\]

Proof. Making the change of variables $y = Rz$, yields
\[
\int_{Q_R} \beta, d - \beta dy = R^{d-\beta} \int_{Q_1} \frac{x}{R} - z|^{-\beta} dz.
\]
Then the desired result follows from the fact that
\[
\sup_{w \in \mathbb{R}^d} \int_{Q_1} |z + w|^{-\beta} dz < \infty.
\]

Lemma A.3. Let $\Phi^{(i)}$ be as in (3.19). There is a constant $C_{t, \beta, d}$ depending on $t$, $\beta$ and $t$, such that for $0 < \tau < t$ and $\xi \in \mathbb{R}^d$ we have
\[
\Phi^{(i)}(\tau, \xi) \leq C_{t, \beta, d} R^{-\beta}.
\]

Proof. We will first consider the case $i = 1$. Integrating in $z$ using semigroup property and then integrating in the variables $y$ and $\tilde{z}$, applying Lemma A.1, we have
\[
\int_{\mathbb{R}^d} \varphi_{R, t}(s, \tilde{y}) \varphi_{R, t}(r, \tilde{z}) \phi^{(1)}_{r, z, \tilde{z}, s, y, \beta}(\tau, \xi)|y - \tilde{y}|^{-\beta} |z - \tilde{z}|^{-\beta} dy dz d\tilde{y} d\tilde{z}
\]
\[
= \frac{1}{\sigma_{R, t}^2} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} p_{t-s}(x_1 - \tilde{y}) p_{t-r}(x_2 - z + \tilde{z}) p_{\tau-s} (\xi - y) p_{\tau-s} (\xi - \tilde{y})
\]
\[
\times |y - \tilde{y}|^{-\beta} |z - \tilde{z}|^{-\beta} dx_1 dx_2 dy dz d\tilde{y} d\tilde{z}
\]
\[
= \frac{1}{\sigma_{R, t}^2} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} p_{t-s}(x_1 - \tilde{y}) p_{t-s-2r}(x_2 - \tilde{y} + \tilde{z}) p_{\tau-s} (\xi - y) |y - \tilde{y}|^{-\beta} |z - \tilde{z}|^{-\beta} d\tilde{y} d\tilde{z}
\]
\[
\leq C_{\beta, d} \int_{\mathbb{R}^d} \int_{Q_{R}} p_{t-s}(x_1 - \tilde{y}) |x_2 - \tilde{y}|^{-\beta} |\xi - \tilde{y}|^{-\beta} d\tilde{y} dx_2,
\]
where $C_{\beta, d}$ is a generic constant depending on $\beta$ and $d$. Next we apply Lemma A.2 to get
\[
\int_{\mathbb{R}^d} \varphi_{R, t}(s, \tilde{y}) \varphi_{R, t}(r, \tilde{z}) \phi^{(1)}_{r, z, \tilde{z}, s, y, \beta}(\tau, \xi)|y - \tilde{y}|^{-\beta} |z - \tilde{z}|^{-\beta} dy dz d\tilde{y} d\tilde{z}
\]
\[
\leq \frac{R^{d-\beta} C_{\beta, d}}{\sigma_{R, t}^2} \int_{\mathbb{R}^d} \int_{Q_{R}} p_{t-s}(x_1 - \tilde{y}) |\xi - \tilde{y}|^{-\beta} d\tilde{y} dx_1.
Integrating in the variable \( \tilde{y} \), using Lemma A.1 and Lemma A.2 and (2.16), we finally obtain for \( 0 < \tau < t \)

\[
\Phi^i(\tau, \xi) \leq C_{\beta,d} R^{2d-2\beta} \frac{2^d}{\sigma_{R,t}^2} \int_0^\tau \int_0^s dr ds \leq C_{t,\beta,d} R^{-\beta}.
\]

Similarly for \( i = 2 \), integrating in \( \tilde{y}, \tilde{z} \) using Lemma A.1, and then integrating in \( x_1, x_2 \) using Lemma A.2, we get

\[
\int_{\mathbb{R}^{4d}} \varphi_{R,t}(s, \tilde{y}) \varphi_{R,t}(r, \tilde{z}) q^{(2)}_{r,\tilde{z},s,\tilde{y}}(\tau, \xi) |y - \tilde{y}|^{-\beta} |z - \tilde{z}|^{-\beta} dy dz d\tilde{y} d\tilde{z}
\]

\[
= \frac{1}{\sigma_{R,t}^2} \int_{Q_R^4} \int_{\mathbb{R}^{2d}} p_{t-s}(x_1 - \tilde{y}) p_{t-r}(x_2 - \tilde{z}) q_{r,s,y}(\tau, \xi) |y - \tilde{y}|^{-\beta} |z - \tilde{z}|^{-\beta} dy dz d\tilde{y} d\tilde{z}
\]

\[
\leq \frac{C_{\beta,d}}{\sigma_{R,t}^2} \int_{Q_R^4} \int_{\mathbb{R}^{2d}} p_{t-s}(y - z) q_{r,s,y}(\tau, \xi) |y - \tilde{y}|^{-\beta} |z - \tilde{z}|^{-\beta} dy dz d\tilde{y} d\tilde{z}
\]

\[
\leq \frac{R^{2d-2\beta} C_{\beta,d}}{\sigma_{R,t}^2} \int_{\mathbb{R}^{2d}} p_{t-s}(y - z) |y - \tilde{y}|^{-\beta} dy dz = \frac{R^{2d-2\beta} C_{\beta,d}}{\sigma_{R,t}^2}.
\]

The desired result follows from (2.16) after integrating in time variables.

\[\square\]

**Lemma A.4.** For \( \beta \in (0, d) \) and \( t > 0 \), there is a constant \( C_{t,\beta,d} \) depending on \( t, \beta \) and \( d \) such that for any \( t > 0 \),

\[
\sup_{s, r \in [0, t]} \int_{\mathbb{R}^{2d}} \varphi_{R,t}(s, y) \varphi_{R,t}(r, z) |y - z|^{-\beta} dy dz \leq C_{t,\beta,d}.
\]

**Proof.** We can write

\[
\int_{\mathbb{R}^{2d}} \varphi_{R,t}(s, y) \varphi_{R,t}(r, z) |y - z|^{-\beta} dy dz
\]

\[
= \frac{1}{\sigma_{R,t}^2} \int_{Q_R^4} \int_{\mathbb{R}^{2d}} p_{t-s}(x_1 - y) p_{t-r}(x_2 - z) |y - z|^{-\beta} dy dz dx_1 dx_2
\]

Integrating in the variables \( y \) and \( z \) and applying Lemma A.1, we obtain

\[
\int_{\mathbb{R}^{2d}} \varphi_{R,t}(s, y) \varphi_{R,t}(r, z) |y - z|^{-\beta} dy dz \leq \frac{C}{\sigma_{R,t}^2} \int_{Q_R^4} |x_1 - x_2|^{-\beta} dx_1 dx_2.
\]

The desired result follows from Lemma A.2 and (2.16). \(\square\)

**Lemma A.5.** Let \( m(t_\alpha, R) \) be as in (3.6). Then, there is a constant \( C_{t,\beta,d} \) depending on \( t, \beta \) and \( t \), such that for \( R \geq 1 \) and \( \varepsilon \leq 1 \)

\[
m(t_\alpha, R) \geq C_{t,\beta,d} \varepsilon^\alpha.
\]
Proof. Integrating in \( y \) using semigroup property and then applying the Fourier transform we get

\[
m(t, R) = \frac{1}{\sigma_{R,t}} \int_0^\alpha \int_{Q_R} \int_{\mathbb{R}^d} p_s(x_1 - y)p_s(y - y - x_2)|\vec{y}|^{-\beta}dydydx_1dx_2ds
\]

\[
= \frac{1}{\sigma_{R,t}} \int_0^\alpha \int_{Q_R} \int_{\mathbb{R}^d} p_{2s}(\vec{y} - x_1 + x_2)|\vec{y}|^{-\beta}d\vec{y}dx_1dx_2ds
\]

\[
= \frac{1}{\sigma_{R,t}} \int_0^\alpha \int_{Q_R} \int_{\mathbb{R}^d} e^{-i(x_1 - x_2)\cdot\xi - s|\xi|^2} |\xi|^{-d}d\xi dx_1dx_2ds
\]

\[
= \frac{1}{\sigma_{R,t}} \int_0^\alpha \int_{\mathbb{R}^d} e^{-s|\xi|^2} \left| \int_{Q_R} e^{-ix\cdot\xi}dx \right|^2 |\xi|^{-d}d\xi ds.
\]

Integrating the variable \( x \) and then the variable \( s \) yields

\[
m(t, R) = \frac{1}{\sigma_{R,t}} \int_0^\alpha \int_{\mathbb{R}^d} e^{-s|\xi|^2} \left( \prod_{j=1}^d \frac{\sin(\xi_j R)}{\xi_j} \right)^2 |\xi|^{-d}dsd\xi
\]

\[
= \frac{R^{2d}}{\sigma_{R,t}} \int_{\mathbb{R}^d} \frac{1 - e^{-\epsilon|\xi|^2}}{|\xi|^2} \left( \prod_{j=1}^d \frac{\sin(\xi_j R)}{\xi_j} \right)^2 |\xi|^{-d}d\xi.
\]

After the change of variable \( \eta_j = R\xi_j \), we can rewrite the above integral as

\[
m(t, R) = \frac{R^{2d-\beta}\epsilon^\alpha}{\sigma_{R,t}} \int_{\mathbb{R}^d} \frac{1 - e^{-\epsilon|\eta|^2/R^2}}{e^{\alpha|\eta|^2/R^2}} \left( \prod_{j=1}^d \frac{\sin(\eta_j)}{\eta_j} \right)^2 |\eta|^{-\beta}d\eta.
\]

Using the estimate \((1 - e^{-x})/x \geq 1 - x \) and \(|\sin x/x| \geq c > 0 \) when \( 0 \leq x < 1 \) together with (2.16), we get for \( R \geq 1 \) and \( \epsilon \leq 1 \),

\[
m(t, R) \geq \frac{R^{2d-\beta}\epsilon^\alpha}{\sigma_{R,t}} \int_{B(0,1)} \left( 1 - e^{\alpha|\eta|^2/R^2} \right) \left( \prod_{j=1}^d \frac{\sin(\eta_j)}{\eta_j} \right)^2 |\eta|^{-\beta}d\eta
\]

\[
\geq \frac{c^{2d}R^{2d-\beta}\epsilon^\alpha}{\sigma_{R,t}} \int_{B(0,1)} \left( 1 - e^{\alpha|\eta|^2/R^2} \right) |\eta|^{-\beta}d\eta
\]

\[
\geq C_{t,\beta,d}\epsilon^\alpha \int_0^1 \left( 1 - e^{\alpha r^2/R^2} \right) r^{\beta-1}dr
\]

\[
= C_{t,\beta,d}\epsilon^\alpha \left( \frac{1}{\beta} - \frac{e^{\alpha}}{(\beta + 2)R^2} \right) \geq C_{t,\beta,d}\epsilon^\alpha.
\]

This completes the proof. \( \square \)

Lemma A.6. Let \( E_{R,t}(s_1, s_2, \tau) \) be as defined in (3.13). Then, there is a constant \( C_{t,\beta,d} \) depending on \( t, \beta \) and \( d \), such that

\[
E_{R,t}(s_1, s_2, \tau) \leq C_{t,\beta,d}R^{4d-3\beta}.
\]
Proof. After applying the change of variables $\theta = \xi, \tilde{\theta} = \xi - \tilde{\xi}, \eta_1 = y_1, \eta_2 = y_2, \tilde{\eta}_1 = y_1 - \tilde{y}_1, \tilde{\eta}_2 = y_2 - \tilde{y}_2$, we can rewrite $E_{R,t}$ as follows

$$E_{R,t}(s_1, s_2, \tau) = \int_{Q_R^4} \int_{G^{2d}} p_{t-}\tau(x_1 - \theta)p_{t-}\tau(\theta - \tilde{\theta} - x_2)p_{t-s_1}(\eta_1 - \tilde{\eta}_1 - x_3)$$

$$\times p_{t-s_2}(\eta_2 - \tilde{\eta}_2 - x_4)p_{t-s_1}(\theta - \eta_1)p_{t-s_2}(\theta - \tilde{\theta} - \eta_2)$$

$$\times |\tilde{\theta}|^{-\beta}|\eta_1|^{-\beta}|\tilde{\eta}_2|^{-\beta}d\theta d\tilde{\theta}d\eta_1 d\tilde{\eta}_2 dx_1 dx_2 dx_3 dx_4.$$

Integrating in $\eta_1$ and $\eta_2$ using semigroup property, and using Lemma A.1 for the integrals in $\tilde{\eta}_1, \tilde{\eta}_2$, we obtain

$$E_{R,t}(s_1, s_2, \tau) \leq C_{d,\beta}\int_{Q_R^4} \int_{G^{2d}} p_{t-}\tau(x_1 - \theta)p_{t-}\tau(\theta - \tilde{\theta} - x_2)|x_3 - \theta + \tilde{\theta}|^{-\beta}$$

$$\times |\tilde{\theta}|^{-\beta}d\theta d\tilde{\theta}dx_1 dx_2 dx_3 dx_4.$$

Using Lemma A.2 we can estimate the integrals in $x_3$ and $x_4$ to get

$$E_{R,t}(s_1, s_2, \tau) \leq R^{2d-2\beta}C_{d,\beta}\int_{Q_R^4} \int_{G^{2d}} p_{t-}\tau(x_1 - \theta)p_{t-}\tau(\theta - \tilde{\theta} - x_2)|\tilde{\theta}|^{-\beta}d\theta d\tilde{\theta}dx_1 dx_2.$$

Finally, applying again Lemma A.1 twice we get

$$E_{R,t}(s_1, s_2, \tau) \leq C_{d,\beta}R^{2d-2\beta}\int_{Q_R^4} |x_1 - x_2|^{-\beta}dx_1 dx_2,$$

which allows us to complete the proof. $\square$

**Lemma A.7.** Let $F$ be a nonnegative random variable. Then $E[F^{-p}] < \infty$ for all $p \geq 2$ if and only if for all $q \geq 2$, there exists $C = C(q) > 0$ and $\varepsilon_0 = \varepsilon_0(q) > 0$ such that $P(F < \varepsilon) \leq C\varepsilon^q$ for all $\varepsilon \leq \varepsilon_0$.

**References**

[1] G. Amir, I. Corwin and J. Quastel: Probability distribution of the free energy of the continuum directed random polymer in $1 + 1$ dimensions. *Comm. Pure Appl. Math.* 64 (2011), 466–537.

[2] M. E. Caballero, B. Fernández and D. Nualart: Estimation of densities and applications. *J. Theoret. Probab.* 11 (1998), no. 3, 831–851.

[3] L. Chen and R. C. Dalang: Moments, intermittency, and growth indices for the nonlinear stochastic heat equation with rough initial conditions. *Ann. Probab.* 45 (2017), no. 6, 3006–3051.

[4] L. Chen, Y. Hu and D. Nualart: Regularity and strict positivity of densities for the nonlinear stochastic heat equations. *Mem. Amer. Math. Soc.* 273 (2011), no. 1340, v+102 pp.

[5] L. Chen, D. Khoshnevisan, D. Nualart and F. Pu: Central limit theorems for spatial averages of the stochastic heat equation via Malliavin-Stein’s method. *Stochastics and Partial Differential Equations: Analysis and Computations*. To appear.

[6] L. Chen, D. Khoshnevisan, D. Nualart and F. Pu: Spatial ergodicity for SPDEs via Poincaré-type inequalities. *Electron. J. Probab.* 26 (2021), 1–37.

[7] L. Chen, K. Kim: On comparison principle and strict positivity of solutions to the nonlinear stochastic fractional heat equations. *Ann. Inst. Henri Poincaré Probab. Stat.* 53 (2017), no. 1, 358–388.

[8] R. Dalang: Extending the martingale measure stochastic integral with applications to spatially homogeneous spde’s. *Electron. J. Probab.* 4 (1999), 1–29.

[9] Y. Hu, F. Lu and D. Nualart: Convergence of densities of some functionals of Gaussian processes. *J. Functional Anal.* 266 (2014), no. 2, 814–875.

[10] Y. Huang, D. Nualart and L. Viitasaari: A central limit theorem for the stochastic heat equation. *Stochastic Process. Appl.* 130 (2020), no. 12, 7170–7184.
[11] J. Huang, D. Nualart, L. Viitasaari and Q. Zheng: Gaussian fluctuations for the stochastic heat equation with colored noise. *Stochastics and Partial Differential Equations: Analysis and Computations* **8** (2020), no. 2, 402–421.

[12] S. Kuzgun and D. Nualart: Convergence of densities of spatial averages of stochastic heat equation. Preprint arXiv:2108.09531, 2021.

[13] S. Kuzgun and D. Nualart: Feynman-Kac formula for iterated derivatives of the parabolic Anderson model. *Potential Anal.* To appear.

[14] I. Nourdin and G. Peccati: *Normal approximations with Malliavin calculus: from Stein’s method to universality.* 2012, Cambridge University Press.

[15] D. Nualart: *The Malliavin calculus and related topics.* 2006, Springer.

[16] D. Nualart and L. Quer-Sardanyons: Existence and smoothness of the density for spatially homogeneous SPDEs. *Potential Anal.* **27** (2007), 281–299.

[17] J. B. Walsh: An introduction to stochastic partial differential equations. In: *École d’Été de Probabilités de Saint Flour XIV-1984*, 265–439, 1986, Springer.

**University of Kansas, Department of Mathematics, USA**
*Email address: sefika.kuzgun@ku.edu*

**University of Kansas, Department of Mathematics, USA**
*Email address: nualart@ku.edu*