Symmetry Restoring Phase Transitions at High Density in a 4D Nambu–Jona-Lasinio Model with a Single Order Parameter

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Abstract High density phase transitions in a 4-dimensional Nambu–Jona-Lasinio model containing a single symmetry breaking order parameter coming from the fermion-antifermion condensates are researched and expounded by means of both the gap equation and the effective potential approach. The phase transitions are proven to be second-order at a high temperature $T$; however at $T = 0$ they are first- or second-order, depending on whether $\Lambda/m(0)$, the ratio of the momentum cutoff $\Lambda$ in the fermion-loop integrals to the dynamical fermion mass $m(0)$ at zero temperature, is less than 3.387 or not. The former condition cannot be satisfied in some models. The discussions further show complete effectiveness of the critical analysis based on the gap equation for second order phase transitions including determination of the condition of their occurrence.

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1 Introduction

Nambu–Jona-Lasinio (NJL) model[1] is a good laboratory to research symmetry restoring phase transition at finite temperature $T$ and finite chemical potential $\mu$.[2] It could simulate the phase transitions in Quantum Chromodynamics (QCD)[3] and is directly related to some dynamical schemes of electroweak symmetry breaking.[4,5] It is known that in the NJL model with a single order parameter, which comes from fermion-antifermion ($ff$) condensates, the phase transitions at a high critical temperature $T_c$ for fixed $\mu$ are second order.[6] On the other hand, with regard to the phase transitions in this class of models at a high chemical potential for a fixed $T$, especially in the $T \to 0$ case, less deep-going researches have been made and the conclusions fail to be made simple and clear. Therefore, a more careful investigation of this problem is certainly interesting. On the high density phase transitions at $T = 0$ in this class of NJL models, the authors of Ref. [7] derived the effective potential from the imaginary-time formalism of thermal field theory and briefly outlined the results. However, it seems that some more detailed demonstrations are necessary for understanding these results and in addition, some of the results there need to be clarified further. Actuated by the above motivates, in this paper, we will make a thorough examination of the phase transitions at high $\mu$ in this class of 4D NJL model. The discussions will be made in the real-time formalism of thermal field theory and involve both $T \neq 0$ and $T = 0$ cases. For second-order phase transitions, we will make critical analysis of the order parameter of symmetry breaking based on both the gap equation and the zero temperature effective potential. For first order phase transitions we will explicitly give the equations determining the critical curves. The condition which distinguishes between second and first order phase transitions will be indicated definitely.

The Lagrangian of the model will be expressed by

$$\mathcal{L}(x) = \sum_{k=1}^{N} \bar{\psi}(x)i\gamma^{\mu}\partial_{\mu}\psi(x) + \frac{g}{2} \sum_{k=1}^{N} [\bar{\psi}(x)\psi(x)]^{2},$$

(1)

where $\psi(x)$ are the fermion fields with $N$ “color” components and $g$ is the four-fermion coupling constant. It is indicated that the four-fermion interactions in Eq. (1) can lead to only the fermion-antifermion condensates $\langle \bar{\psi}\psi \rangle$. The conclusions reached from Eq. (1) are also essentially applicable to more complicated models with a single order parameter coming from the condensates $\langle \bar{\psi}\psi \rangle$. The discussions will be made in the fermion bubble diagram approximation, which is equivalent to the leading order of the 1/N expansion. Since when $D = 4$ the Lagrangian (1) is not renormalizable, we will regard it only as a low energy effective field theory and examine its physical results. We will analyze second order high $\mu$ phase transitions based on the gap equation in Sec. 2, and examine second- and first-order high $\mu$ phase transitions by means of the zero temperature effective potential in Secs. 3 and 4, respectively. Section 5 is our conclusion.

2 Second-Order High $\mu$ Phase Transitions — Gap Equation Analysis

Assume that the dynamical fermion mass can be generated by the fermion condensates induced by the four-fermion interactions in Eq. (1) at $T = \mu = 0$ and at some finite $T$ and $\mu$, then the corresponding mass term
will spontaneously break the discrete chiral symmetry \( \chi_D \): \( \psi_k(x) \xrightarrow{\chi_{D, \mu}} \gamma_5 \psi_k(x) \) and the special parities \( \mathcal{P}_j \): 
\[ \psi_k(t, \ldots, x^j, \ldots) \xrightarrow{\mathcal{P}_j} \gamma^j \psi_k(t, \ldots, -x^j, \ldots) \] (\( j = 1, 2, 3 \)).

After the gap equation at \( T = \mu = 0 \) is substituted, the gap equation at \( T \neq 0 \) can be transformed to \[ \frac{1}{2} m^2(0) \ln \left[ \frac{\Lambda^2}{m^2(0)} + 1 \right] = \frac{1}{2} m^2 \ln \left[ \frac{\Lambda^2}{m^2} + 1 \right] \]

with

\[ I_3(y, \mp r) = \frac{1}{2} \int_0^\infty \frac{dx}{\sqrt{x^2 + y^2}} \exp \left( \frac{x^2}{\sqrt{x^2 + y^2} \mp r} \right) + 1, \]

where \( \gamma \) is the Euler constant. Equation (5) indicates that for high \( T \), the phase transition at \( \mu \rightarrow 0 \) will be restored. Setting \( m = 0 \) in Eq. (2), we will obtain the critical equations

\[ \frac{1}{2} m^2(0) \ln \left[ \frac{\Lambda^2}{m^2(0)} + 1 \right] = 2T^2 c \int_0^\infty dx \left[ -\frac{x}{e^{-\mu c / T c} + 1} + (-\mu c - \mu c) \right]. \]

When taking the limit \( m \rightarrow 0 \) and obtaining Eq. (5), we do not encounter any singularity. This means that \( m \) may go to zero continuously, thus equation (5) will be the critical equation of a second-order phase transition. In fact, for a given high temperature \( T \), substituting Eq. (5) with \( T \) replaced by \( T = T_c \) into Eq. (2), then using the high temperature expansion of \( F_3(T, \mu, m, \frac{[6]}{}) \) we can obtain the critical behavior of \( m^2 \) near \( T_c \) at high \( T \),

\[ m^2 = (\mu_c^2 - \mu^2) \sum_{\nu=1}^{\infty} \ln(\Lambda / T \pi) - 1/2 + \gamma - h(T, \mu), \]

with \( \bar{c} = \mu_c / T \) and \( h(T, \mu) = 7\zeta(3) \left( \frac{r}{2\pi} \right)^2 - 31\zeta(5) \left( \frac{r}{2\pi} \right)^4 + 127\zeta(7) \left( \frac{r}{2\pi} \right)^6 \),

where \( \zeta(s) \) \((s = 3, 5, 7)\) is the Riemann zeta function and \( \gamma \) is the Euler constant. Equation (6) indicates that for high \( T \), the phase transition at \( \mu_c \) is second order. For proving the phase transitions at \( \mu_c \) for low \( T \), we first consider the limit case of \( T \rightarrow 0 \). In this case \((r \rightarrow \infty)\), the critical equation (5) with \( \mu_c \rightarrow \mu \) and \( T \rightarrow T_c \) is reduced to

\[ \mu^2 = \frac{1}{2} m^2(0) \ln \left[ \frac{\Lambda^2}{m^2(0)} + 1 \right] \equiv \mu^2 \omega, \text{ when } T \rightarrow 0. \]

Through changing the integral variable by \( z = (x^2 + y^2)^{1/2} \) we may rewrite

\[ F_3(T, \mu, m) = 2m^2 \int_1^\infty dz \left[ \frac{\sqrt{z^2 - 1}}{\sqrt{z^2 - m^2}} + (-\alpha \rightarrow \alpha) \right]. \]

with \( \alpha = \mu / m \), and obtain

\[ F_3(T = 0, \mu, m) = \theta(\mu - m) \left[ \mu \sqrt{\mu^2 - m^2} - m^2 \ln \left( \frac{\mu + \sqrt{\mu^2 - m^2}}{m} \right) \right]. \]

By using Eqs. (8) and (10), we may reduce the \( T \rightarrow 0 \) limit of the gap equation (2) to

\[ m = m(0), \text{ when } \mu \leq m(0). \]

\[ m^2 - \mu^2 \sqrt{1 - \frac{m^2}{\mu^2}} = m^2 \ln \left( \frac{\Lambda^2 + m^2}{(\mu + \sqrt{\mu^2 - m^2})^2} \right), \text{ when } \mu > m(0). \]

From Eq. (11) it can be proven that

\[ \lim_{T \rightarrow 0} \frac{\partial m}{\partial \mu} = \frac{-2 \sqrt{\mu^2 - m^2}}{m \sqrt{\mu^2 - m^2}} \left[ \frac{m^2 \sqrt{\mu^2 - m^2} - m^2 \Lambda^2 / 2(\Lambda^2 + m^2)}{[\mu^2 - \mu \sqrt{\mu^2 - m^2} - m^2 \Lambda^2 / 2(\Lambda^2 + m^2)]} \right] \]

\leq 0, \text{ when } m(0) \leq \mu < \mu_c, \]

and

\[ \lim_{T \rightarrow 0} \frac{\partial^2 m}{\partial \mu^2} < 0, \text{ when } m(0) < \mu < \mu_c. \]
Equation (12) shows that in the $T \to 0$ limit, when $\mu = m(0)$, $\partial m/\partial \mu = 0$ and when $\mu = \mu_c$, where $m = 0$, $\partial m/\partial \mu = -\infty$. From these results and Eq. (13), we can deduce that the $m-\mu$ curve at $T = 0$ is concave downward in the region $m(0) \leq \mu \leq \mu_c$. Near the critical point $\mu \sim \mu_c$, we find out from Eq. (11) that

$$m^2 \approx \frac{(\mu^2 - \mu_c^2)}{[\ln(\Lambda/2\mu) - 1/2]} \quad \text{when} \quad T \to 0.$$  

Equation (14) indicates that in the $D = 4$ NJL model with a single order parameter, when $T = 0$, as same as when $T$ is high, the symmetry restoration phase transition at $\mu_c$ could be second order. This is different from the high density phase transition when $T = 0$ in $D = 2$ and $D = 3$ GN model where they are only first order.$^{[8-10]}$

However, for consistence of Eq. (14) i.e., when $\mu < \mu_c$, $m^2 \geq 0$, we must have $\ln(\Lambda/2\mu) - 1/2 \gtrsim \ln(\Lambda/2\mu_c) - 1/2 \geq 0$ and this implies that

$$\mu_c^2 \leq \frac{\Lambda^2}{4e},$$  

which is the condition in which a second-order phase transition could occur at $T = 0$.

3 Second-Order Phase Transitions — Effective Potential Analysis

The above conclusion that when the condition (15) is satisfied the high density phase transition at $T = 0$ is second order reached only by the gap equation can be verified by an effective potential analysis. It has been proven$^{[11]}$ that the extreme value condition of the thermal effective potential $V_{\text{eff}}^{(4)}(T, \mu, m)$ is

$$\frac{\partial V_{\text{eff}}^{(4)}(T, \mu, m)}{\partial m} = 0$$  

with

$$\frac{\partial V_{\text{eff}}^{(4)}(T, \mu, m)}{\partial m} = m \left\{ \frac{m^2}{2} \frac{\Lambda^2}{m^2} + 1 + \frac{m^2(0)}{2} \ln \left[ \frac{\Lambda^2}{m^2(0)} + 1 \right] + \frac{\Lambda^2}{\Lambda^2} \ln \left[ 1 + \frac{\Lambda^2}{\Lambda^2} \right] - \frac{m^2(0)}{2} \ln \left[ \frac{\Lambda^2}{m^2(0)} + 1 \right] + \frac{m^2(0)}{2} \ln \left[ \frac{\Lambda^2}{m^2(0)} + 1 \right] \right\}.$$  

Hence equation (16) just corresponds to the gap equation (2) multiplied by $m$. From Eqs. (17) and (10) we can find out that the effective potential $V_{\text{eff}}^{(4)}(T = 0, \mu, m)$ at $T = 0$ satisfying the condition $V_{\text{eff}}^{(4)}(T = 0, \mu, m) = 0$ can be expressed by

$$V_{\text{eff}}^{(4)}(T = 0, \mu, m) = \frac{1}{2\pi^2} \left\{ \frac{m^4}{8} \ln \left[ \frac{\Lambda^2}{m^2} + 1 \right] + \frac{m^2(0)}{2} \ln \left[ \frac{\Lambda^2}{m^2(0)} + 1 \right] + \frac{m^4}{8} \ln \left[ \frac{\Lambda^2}{m^2} + 1 \right] + \frac{m^2(0)}{2} \ln \left[ \frac{\Lambda^2}{m^2(0)} + 1 \right] \right\}.$$  

Equation (18) is essentially identical to the effective potential in Ref. [7] derived from the imaginary-time thermal field theory, except for the extra “normalization” condition $V_{\text{eff}}^{(4)}(T = 0, \mu, m) = 0$. From Eqs. (10) and (17) we can directly obtain the extreme value condition of $V_{\text{eff}}^{(4)}(T = 0, \mu, m)$,

$$\frac{\partial V_{\text{eff}}^{(4)}(T = 0, \mu, m)}{\partial m} = m \left\{ \frac{m^2}{2} \frac{\Lambda^2}{m^2} + 1 + \frac{m^2(0)}{2} \ln \left[ \frac{\Lambda^2}{m^2(0)} + 1 \right] + \frac{m^2(0)}{2} \ln \left[ \frac{\Lambda^2}{m^2(0)} + 1 \right] \right\} = 0.$$  

and

$$\frac{\partial^2 V_{\text{eff}}^{(4)}(T = 0, \mu, m)}{\partial m^2} = \frac{1}{2\pi^2} \left\{ \frac{3}{2} \frac{m^2}{2} \ln \left[ \frac{\Lambda^2}{m^2} + 1 \right] + m^2 - \frac{m^4}{\Lambda^2 + m^2} - \frac{m^2(0)}{2} \ln \left[ \frac{\Lambda^2}{m^2(0)} + 1 \right] + \frac{m^2(0)}{2} \ln \left[ \frac{\Lambda^2}{m^2(0)} + 1 \right] \right\}.$$  

Variation of $V_{\text{eff}}^{(4)}(T = 0, \mu, m)$ as $\mu$ increases may be discussed by Eqs. (18) $\sim$ (20).

First we note that when $\mu = m$, for any $\Lambda^2/m^2(0) \neq 0$, $V_{\text{eff}}^{(4)}(T = 0, \mu, m)$ will have a maximum point $m = 0$ and a minimum point $m = m(0)$ and this shows spontaneous symmetry breaking at $T = 0$.

If the symmetries will be restored at high $\mu$ through second-order phase transitions, then the extreme value equation of $V_{\text{eff}}^{(4)}(T = 0, \mu, m)$ with $m \neq 0$, i.e. the $T \to 0$ limit of the gap equation (2), should have the only solution and it must correspond to a minimum point of $V_{\text{eff}}^{(4)}(T = 0, \mu, m)$. We will examine this case in the condition expressed by Eq. (15). This condition, by Eq. (8), is

$$\frac{m^2(0)}{2} \ln \left[ \frac{\Lambda^2}{m^2(0)} + 1 \right] \leq \frac{\Lambda^2}{4e},$$  

No. 6 Symmetry Restoring Phase Transitions at High Density in a 4D Nambu-Jona-Lasinio Model with · · · 671
which amounts to $\Lambda/m(0) \geq 3.387$ and will lead to $\mu_{c0} > m(0)$. Variation of $V^{(4)}_{\text{eff}}(T = 0, \mu, m)$ as $\mu$ increases is as follows.

i) $0 < \mu < m(0)$. In this case, $V^{(4)}_{\text{eff}}(T = 0, \mu, m)$ will have a maximum point $m = 0$ (when $m < \mu$) and a minimum point $m = m(0)$ (when $m > \mu$). It can be proven that the other extreme value equation contained in Eq. (19) when $m < \mu$

$$\frac{1}{2}m^2(0)\ln\left[\frac{\Lambda^2}{m^2(0)} + 1\right] = \frac{m^2}{2} \ln\frac{\Lambda^2 + m^2}{(\mu + \sqrt{\mu^2 - m^2})^2} + \mu\sqrt{\mu^2 - m^2} \tag{21}$$

has no solution for $m < \mu < m(0)$ when $(1/2)\ln[\Lambda^2/m^2(0) + 1] \geq 1$. In fact, if we set $\mu = \alpha m(0)$ and $m = \beta \mu$, then equation (21) can be changed into

$$\frac{1}{2} \ln\left[\frac{\Lambda^2}{m^2(0)} + 1\right] = \frac{\beta^2\alpha^2}{2} - \ln\frac{\Lambda^2/m^2(0) + \alpha^2\beta^2}{\alpha^2(1 + \sqrt{1 - \beta^2})^2} + \alpha^2\sqrt{1 - \beta^2}. \tag{22}$$

It is easy to check by numerical solution that equation (22) has no solution for $\alpha < 1$ and $\beta < 1$ when $(1/2)\ln[\Lambda^2/m^2(0) + 1] \geq 1$ or $\Lambda^2/m^2(0) \geq e^2 - 1$. Our prerequisite $\Lambda/m(0) \geq 3.387$ or $\Lambda^2/m^2(0) \geq 15.11$ apparently satisfies this condition. Thus equation (21) does have no solution. As a result, $m = m(0)$ becomes the only minimum point of $V^{(4)}_{\text{eff}}(T = 0, \mu, m)$ and we will have the same spontaneous symmetry breaking as one at $T = \mu = 0$.

ii) $m(0) \leq \mu < \mu_{c0}$. In this case, when $m > \mu$, $V^{(4)}_{\text{eff}}(T = 0, \mu, m)$ has no extreme value point and when $m < \mu$, $V^{(4)}_{\text{eff}}(T = 0, \mu, m)$ will have a maximum point $m = 0$ and a minimum point $m_1$, which is determined by Eq. (21). Noting that in the present case equation (21) may have the solution $m = m_1$ when $(1/2)\ln[\Lambda^2/m^2(0) + 1] \geq 1$ and

$$\frac{\partial^2 V^{(4)}_{\text{eff}}(T = 0, \mu, m)}{\partial m^2}\bigg|_{m=m_1} = \frac{1}{\pi^2} \left(\mu_{\text{eff}}^2 - \mu \sqrt{\mu^2 - m_1^2} + \frac{m_1^2}{2} \frac{\Lambda^2}{\Lambda^2 + m_1^2}\right) > 0, \quad \text{when } \mu < \mu_{c0}. \tag{23}$$

In fact, equation (21) has the solutions

$$m_1 = m(0), \quad \text{when } \mu = m(0), \tag{24}$$

$$m_1 < m(0), \quad \text{when } \mu > m(0).$$

Noting that equation (21) obeyed by $m_1$ is just the second formula in Eq. (11), hence we can obtain from Eq. (12) that $\partial m_1/\partial \mu \leq 0$. Equation (23) indicates that when $\mu = m(0)$, we will go back to the same case as $m < \mu(0)$ and equations (21) and (24) show that as $\mu$ increases from $m(0)$ further, the minimum point $m_1$ will become smaller and smaller and finally it will go to zero continuously at a critical chemical potential. Correspondingly, the global minimum of $V^{(4)}_{\text{eff}}(T = 0, \mu, m)$ at $m = m_1$ will go up as $\mu$ increases from a negative value to zero, since it can be proven that

$$\frac{dV^{(4)}_{\text{eff}}(T = 0, \mu, m)}{d\mu}\bigg|_{m=m_1} = \frac{1}{12\pi^2} \left[3m_1^2 \sqrt{\mu^2 - m_1^2} + 4\mu^3 - (\mu^2 - m_1^2)^{3/2} - 3\mu^2 \sqrt{\mu^2 - m_1^2}\right] > 0.$$

The critical chemical potential can be determined by taking $m = 0$ in Eq. (21) and the result is precisely $\mu_{c0}$ given by Eq. (8). Since equation (21) is exactly the second formula in Eq. (11) and the critical behavior (14) of $m^2$ follows. Equation (8) expresses a critical curve $C_2$ of second-order phase transitions in $\mu(m(0))$ plane.

iii) $\mu = \mu_{c0}$. Now $V^{(4)}_{\text{eff}}(T = 0, \mu, m)$ has the only extreme value point $m = 0$ and it is found out that

$$\frac{\partial^n V^{(4)}_{\text{eff}}(T = 0, \mu, m)}{\partial m^n}\bigg|_{m=0} = \begin{cases} 0, & \text{when } n = 2, 3, 5, \\ \frac{3}{2\pi^2} \left(\frac{\Lambda^2}{4\mu_{c0}^2} - 1\right), & \text{when } n = 4, \\ \frac{3}{2\pi^2} \left(\frac{4}{\Lambda^2} + \frac{5}{\mu_{c0}^2}\right), & \text{when } n = 6. \end{cases} \tag{25}$$

Equation (25) implies that in the condition $\ln[\Lambda^2/(4\mu_{c0}^2)] - 1 \geq 0$ or $\mu_{c0}^2 \leq \Lambda^2/4e$, $m = 0$ will be the only minimum point of $V^{(4)}_{\text{eff}}(T = 0, \mu, m)$ and the broken symmetries will be restored.

iv) $\mu > \mu_{c0}$. In this case, it can be proven that the extreme value equation (21) has no solution when $m < \mu$. This can be seen most simply from Eq. (14), which is the approximation of Eq. (21) when $\mu \sim \mu_{c0}$ and $m \approx 0$. More rigorously, we can set $\mu = \gamma \mu_{c0}$, $m = \beta \mu = \beta \mu_{c0}$, then equation (21) can be changed into

$$\frac{1}{2} \beta^2 \gamma^2 \left\{\ln\left[\frac{\Lambda^2}{\mu_{c0}^2} + \beta^2 \gamma^2\right] - 2\ln[\gamma(1 + \sqrt{1 - \beta^2})]\right\} + \gamma^2 \sqrt{1 - \beta^2} = 1. \tag{26}$$

It may be checked that equation (26) has no solution with $\gamma > 1$ ($\mu > \mu_{c0}$) and $\beta < 1$ ($m < \mu$) when $\mu_{c0}^2 \leq \Lambda^2/4e$. Consequently, $V^{(4)}_{\text{eff}}(T = 0, \mu, m)$ will have the only minimum point $m = 0$ in this case and this fact further indicates restoration of the symmetries that are broken at $T = 0$ and $\mu < \mu_{c0}$ through a second-order phase transition when equation (15) is satisfied.

Up to now we have proven that the phase transitions at $\mu_c$ are second order for a high $T$ and for $T = 0$ when $\mu_{c0}^2 \leq \Lambda^2/4e$. Considering these results and the fact that equation (5) is a critical equation of second-order phase
transitions for any finite $T_e$ and $\mu_c$, we can conclude that the phase transitions at $\mu_c$ for any $T$ including low $T$ are second-order when $\mu_0^2 < \Lambda^2/4\epsilon$.

4 First-Order Phase Transition at $T = 0$

A first-order phase transition could generally occur in the case where the extremum value equation of $V_{\text{eff}}(T=0, \mu, m)$ has two or more solutions with $m \neq 0$. We will prove that this is the case when $\mu_0^2 > \Lambda^2/4\epsilon$ and $\mu > \mu_\text{c0}$. First we indicate that when $\mu > \mu_\text{c0}$, $m = 0$ is always a minimum point of $V_{\text{eff}}(T=0, \mu, m)$ and when $\mu_0^2 > \Lambda^2/4\epsilon$, equation (26) will have the solutions with $\gamma > 1$ and $\beta < 1$, i.e. the extreme value equation (21) will always have solutions for $\mu > \mu_\text{c0} > \Lambda/2\epsilon^{1/2}$ and $m < \mu$. This conclusion can also be checked by examining Eq. (14). Since in this case $m = 0$ is a minimum point, the solutions of Eq. (21) must correspond to at least a maximum and a minimum point of $V_{\text{eff}}(T=0, \mu, m)$ and this could lead to a first-order phase transition. Assume the maximum point is $m_2$ and the minimum point is $m_1$, then a first-order phase transition curve should be determined by the equations

\[
V_{\text{eff}}^{(4)}(T=0, \mu, m) = V_{\text{eff}}^{(4)}(T=0, \mu, m = 0) = 0, \\
\frac{\partial V_{\text{eff}}^{(4)}(T=0, \mu, m)}{\partial m} \bigg|_{m=m_1 \neq 0} = 0, \\
\frac{\partial^2 V_{\text{eff}}^{(4)}(T=0, \mu, m)}{\partial m^2} \bigg|_{m=0} = 0,
\]

whose explicit or equivalent forms are

\[
m_1 \ln \left( \frac{\Lambda^2}{m_1^2} + 1 \right) + \Lambda^2 m_1^2 - \Lambda^4 \ln \left( 1 + \frac{m_1^2}{\Lambda^2} \right) - 4m_1^2 \mu^2 + \frac{4}{3} \mu^4 + \theta(\mu - m_1)
\]

\[
\times \left[ 2\mu m_1 \sqrt{\mu^2 - m_1^2} - \frac{4}{3} \mu (\mu^2 - m_1^2)^{3/2} - m_1^2 \ln \frac{\mu + \sqrt{\mu^2 - m_1^2}}{m_1} \right] = 0, \]

\[
\mu_\text{c0} = m_1^2 \ln \left( \frac{\Lambda^2}{m_1^2} + 1 \right) + \theta(\mu - m_1) \left[ \mu \sqrt{\mu^2 - m_1^2} - m_1^2 \ln \frac{\mu + \sqrt{\mu^2 - m_1^2}}{m_1} \right],
\]

where $m_1 \neq 0$, and $\mu > \mu_\text{c0}$.

When $\mu \leq m_1$, from Eqs. (31) and (8) we obtain $m_1 = m(0)$. It may be pointed out that the condition $\mu_\text{c0} > \Lambda^2/4\epsilon$ means that $\Lambda/m(0) < 3.387$ thus we can have either $\mu_\text{c0} > m(0)$ or $\mu_\text{c0} < m(0)$, so even if $\mu > \mu_\text{c0}$, it is still possible for $\mu \leq m(0)$. Substituting $\mu \leq m_1 = m(0)$ into Eq. (30), we are led to that

\[
\frac{4}{3} \mu^4 = m(0)^2 \ln(a + 1) - a + a^2 \ln \left( 1 + \frac{1}{a} \right), \quad a = \frac{\Lambda^2}{m(0)^2},
\]

which expresses a first-order phase transition curve $C_1$ in the $\mu-m(0)$ plane. Equation (32) is valid only if $\mu \leq m_1 = m(0)$, and this will lead to the constraint $\ln(a + 1) - a + a^2 \ln(1 + 1/a) \leq 4/3$ thus $\Lambda/m(0) \leq 2.21$. So the curve $C_1$ must be in the region where $m(0) \geq \Lambda/2.21$.

When $\mu > m(0)$, equations (30) and (31) can be reduced to

\[
\frac{4}{3} \mu^4 - 4 \mu (\mu^2 - m_1^2)^{3/2} = 2m_1^2 \mu_\text{c0}^2 - \Lambda^2 m_1^2 - \Lambda^4 \ln \left( 1 + \frac{m_1^2}{\Lambda^2} \right),
\]

\[
\mu_\text{c0} = m_1^2 \ln \frac{\Lambda^2 + m_1^2}{(\mu + \sqrt{\mu^2 - m_1^2})^2} + \mu \sqrt{\mu^2 - m_1^2}.
\]

Equations (33) and (34) together with $m_1 \neq 0$, $\mu > \mu_\text{c0}$ will be the equations to determine the first-order phase transition curve $C'_1$. When $\mu \rightarrow m(0)$, we may obtain from Eq. (34) that $m_1 \rightarrow m(0)$, as a result, equation (33) becomes

\[
\frac{4}{3} \mu^4 \ln \left[ \mu \rightarrow m(0) \right] = m(0)^4 \ln(a + 1) - a + a^2 \ln \left( 1 + \frac{1}{a} \right),
\]

which coincides with Eq. (32) of the curve $C_1$ at the point $\mu = m(0)$. This means that the two-first-order phase transition curves $C'_1$ and $C_1$ meet at the point $\mu = m(0)$. On the other hand, we note that $\mu \rightarrow \mu_\text{c0}$ from $\mu > \mu_\text{c0}$ and $m_1 \rightarrow 0$ is a limiting solution of Eqs. (33) and (34) and that the curve $C'_1$ should be in the region $\mu_\text{c0}^2 > \Lambda^2/4\epsilon$, so it must intersect the second-order phase transition curve $C_2$ at $\mu_\text{c0}^2 = \Lambda^2/4\epsilon$. Consequently, in the $\mu-m(0)$ plane the curve $C'_1$ will start from the point $\mu_\text{c0}^2 = \Lambda^2/4\epsilon$, extend itself in the region of $\mu > \mu_\text{c0}$, and end at $\mu = m(0)$ where it meets the curve $C_1$. The curve $C'_1$ must be in the region where $\mu > \mu_\text{c0}$, since for the opposite case of $\mu \leq \mu_\text{c0}$, it may be seen from Eqs. (20) and (25) that when $\mu_\text{c0}^2 > \Lambda^2/4\epsilon$, $m = 0$ will be a maximum point of $V_{\text{eff}}^{(4)}(T=0, \mu, m)$, and this only corresponds to the phase of symmetries being broken. The above discussions show that as far as symmetry restoring phase transition is concerned, $\mu^2 = \mu_\text{c0}^2 = \Lambda^2/4\epsilon$ will be a tricritical point in the $\mu-m(0)$ plane. We have noted that the similar results were obtained in Ref. [7], however, no detailed demonstration
leading to the results, especially no distinction or relation between the first-order phase transition curves \( C_1 \) and \( C'_1 \) were given there.

5 Conclusions

In this paper, we have researched the phase transitions at high density in a 4D NJL model with a single order parameter of symmetry breaking coming from fermion-antifermion condensates by means of the gap equation and the zero temperature effective potential. We have proven that for high \( T \) the symmetry restoring phase transitions are always second-order; and for \( T = 0 \), depending on whether \( \mu^2_0 \) is bigger than \( \Lambda^2/4e \) or not, or equivalently, whether \( \Lambda/m(0) \) is less than 3.387 or not, the phase transitions will be first- or second-order. Hence whether \( \Lambda/m(0) \) are always second-order; and for high density in a 4D NJL model with a single order parameter coming from the fermion-antifermion condensates by means of the gap equation and the effective potential. We have proven that for high \( T \) the symmetry restoring phase transitions are always second-order; and for \( T = 0 \), depending on whether \( \mu^2_0 \) is bigger than \( \Lambda^2/4e \) or not, or equivalently, whether \( \Lambda/m(0) \) is less than 3.387 or not, the phase transitions will be first- or second-order. Hence \( \mu^2_0 = \Lambda^2/4e \) is a tricritical point of high-density phase transitions at \( T = 0 \).

We have also further deduced that the phase transitions at \( \mu \) for any \( T \) are second-order when \( \Lambda/m(0) \geq 3.387 \).

We note that in this class of models, first-order phase transitions occur only if the dynamical fermion mass \( m(0) \) at \( T = 0 \) is close to and has the same order of magnitude as the momentum cutoff \( \Lambda \) of the loop integrals. However, this condition is generally not natural or cannot be satisfied in some low-energy effective theories of NJL-form. For instance, in the top-quark condensate scheme of electroweak symmetry breaking, \( \Lambda/m(0) \) must be as large as \( \sim 10^{11} \) owing to the constraint from the standard electroweak model. Even if in the four-generation fermion extension of such scheme, \( \Lambda/m(0) \) could go down greatly, it must be still bigger than 10. Therefore, in some physical 4D NJL models, it can be assumed that first-order phase transition in fact does not occur and even if at \( T = 0 \), there are only second-order phase transitions.

It is seen from the above discussions that as far as analysis of a second-order phase transition is concerned, the gap equation approach and the effective potential approach have the same effectiveness, including determination of the occurrence condition of a second-order phase transition. However, the critical analysis of the order parameter \( m \) based on the gap equation is apparently more simple and direct than the ones based on the effective potential.

The total conclusions in this paper are obtained only in the model (1) with a single order parameter coming from the fermion-antifermion condensates \( \langle \bar{\psi}\psi \rangle \) and they could have substantial change when one transfers to the model with both the fermion-antifermion condensates \( \langle \bar{\psi}\psi \rangle \) and the fermion-antifermion condensates \( \langle \bar{\psi}\bar{\psi} \rangle \). The latter model may be a better simulation to QCD and deserves to be researched further.

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