Stability of Einstein-Kalb-Ramond Wormholes

Paul H. Cox\textsuperscript{1,}\textsuperscript{*} and Benjamin C. Harms\textsuperscript{2,}\textsuperscript{†}

\textsuperscript{1}Department of Physics and Geosciences, Texas A\&M University-Kingsville, M.S.C. 175, Kingsville, TX 78363-8202, USA
\textsuperscript{2}Department of Physics and Astronomy, The University of Alabama, Box 870324, Tuscaloosa, AL 35487-0324, USA

This paper investigates a particular type of wormhole. The wormholes studied are the result of sewing together two Reissner-Nordstrom-type black hole metrics at their horizons. By requiring the stress-energy tensor associated with this geometry to be that of a Kalb-Ramond field, we obtain the mass and Kalb-Ramond charge of the wormholes in terms of the parameters describing the mass density, tension and pressure. We show by direct calculation of the action of these wormholes that they can be made quasi-stable against tunnelling to the vacuum via gravitational instantons by a suitable choice of the parameters.

PACS numbers: 04.20.Jb, 04.40.Nr, 04.60.Bc

I. INTRODUCTION

Wormhole solutions of the Einstein equations were first studied by Einstein and Rosen \cite{1}, who called such solutions 'bridges'. The solutions in \cite{1} were offered as models for the elementary particles known at the time (the proton and the electron). Wormhole models of neutral and charged particles were also studied in \cite{2,3}, where the relation between the mass and the charge which is characteristic of such models was first obtained \cite{13}. These and subsequent attempts to identify elementary particles as wormholes have failed. Since wormholes can be constructed which join two separate space-times or places in space-time, in a quantum theory of gravity they allow the formation of 'baby universes' \cite{3}. Wormholes have also been proposed as a means of interstellar travel \cite{6}, although such objects would require some form of 'exotic' matter in order to allow interstellar travellers to safely pass through. The goal of the present paper is to show that in a quantum theory of gravity wormholes can be made quasi-stable by the addition of a Kalb-Ramond (K-R) field \cite{7} to the background space-time. The wormholes studied in this paper are obtained by sewing together two 'charged' black hole metrics at their horizons. The resulting space-time consists of two congruent regions which are geometrically connected but are causally disconnected. The stability of these wormholes is analyzed by calculating the Euclidian action obtained for the metric, whose specific form is determined by the condition that the stress tensor is that of a K-R field.

II. CHARGED WORMHOLES

A. Metric Tensor Elements

Morris and Thorne \cite{6} investigated traversable wormholes, finding that they require 'exotic' matter, with negative energy density (at least as seen by some observers) as well as significant levels of tension (negative pressure). Some types of fields allow tension, and Rahaman, Kalam, and Ghosh \cite{8} showed that certain static, spherically symmetric solutions of the K-R field \cite{7} are wormholes (though they are not traversable). We study such (non-traversable) wormholes with a metric of the spherically symmetric form \cite{8} [with $c = 1$; we also frequently use $G = 1$]

$$ds^2 = -e^{2\Phi}dt^2 + dr^2 / (1 - b(r)/r) + r^2 (d\theta^2 + \sin^2 \theta d\phi^2), \quad (1)$$

where the singularity at the root $r = b_0$ of $b(r) = r$ is handled by sewing together two congruent copies of the region $r \geq b_0$. (Continuity at this junction can be seen by embedding the manifold in a five-dimensional spatially flat space-time with an additional coordinate $z(r)$ defined by \cite{6}

$$\frac{dz}{dr} = \pm \left(\frac{r}{b(r)} - 1\right), \quad (2)$$

\textsuperscript{*}Electronic address: phcox@tamuk.edu
\textsuperscript{†}Electronic address: bharms@bama.ua.edu
where \( z(b_0) = 0 \); the +/- sign distinguishes the two copies.

Many relationships among local properties look more familiar in a local orthonormal coordinate basis, the proper reference frame of observers at fixed \( r, \theta, \phi \):

\[
e^\hat{t} = e^{-\Phi} e_t, \quad e^\hat{\phi} = (1 - b(r)/r)^{1/2} e_r, \\
e^\hat{\theta} = e_\theta, \quad e^\hat{\phi} = e_\phi.
\]  (3)

In such a basis, pressure \( p \) is given by the equal \( \theta \theta \) and \( \phi \phi \) components of the stress-energy tensor, while energy density \( \rho \) and tension \( \tau \) are given by the negatives of the \( \hat{t} \hat{t} \) and \( \hat{r} \hat{r} \) components respectively. We assume the stress-energy tensor is that of a K-R field; for a static spherical-symmetric solution the K-R field will have only one independent non-zero component which then gives \( \rho c^2 = \tau = p \). The Einstein gravitational field equations then require \( g_{00} = -1/g_{11} \) and

\[
b(r) = b_0(1 + A) - \frac{b_0^2 A}{r},
\]  (4)

where \( A \) is a dimensionless parameter. With this, Eq. (1) is easily seen to describe exterior Reissner-Nordstrom solutions, with mass proportional to \( b_0 (1 + A) \) and squared charge proportional to \( b_0^2 A \). The energy density, tension, and pressure are related to \( b(r) \) by

\[
\rho(r) c^2 = \tau(r) = \frac{b(r)'}{r^2} = \frac{b_0^2 A}{r^4},
\]  (5)

where the prime indicates differentiation with respect to \( r \). Evidently \( A \geq 0 \) is required by the K-R field contribution, while \( A \geq 1 \) is excluded since such values give unacceptable metric singularities in the region \( r \geq b_0 \). When substituted into Einstein’s equations, the conditions in Eq. (5) lead to a metric of the form

\[
ds^2 = -(1 - \frac{b(r)}{r}) dt^2 + \frac{1}{(1 - \frac{b(r)}{r})} dr^2 + r^2 d\theta^2 + r^2 \sin^2(\theta) d\phi^2.
\]  (6)

This metric describes two Reissner-Nordstrom-type black holes glued together at their horizons (Fig. 1) to form a wormhole of radius \( b_0 \) with a causal boundary at

\[
r_+ = M + \sqrt{M^2 - Q^2} = b_0.
\]  (7)

where \( M = b_0 (1 + A) \) is the ADM mass, and \( Q = \pm \sqrt{b_0^2 A} \) is the K-R ‘charge’.

The total K-R charge for the space-time described by Eq. (6) is zero. However, a distant observer would be on either one side of the causal boundary or the other and would therefore detect a charge of \( Q = \pm \sqrt{b_0^2 A} \).

**B. Einstein-Kalb-Ramond Action**

The metric in Eq. (6) is part of a solution of the field equations obtained from the Einstein-Kalb-Ramond action on a manifold \( \mathcal{M} \)

\[
S = \frac{1}{8 \pi G} \int_{\mathcal{M}} \sqrt{-g} \left( R - \frac{1}{4} H_{\mu\nu\lambda} H^{\mu\nu\lambda} \right) d^4x
\]  (8)

where \( g \) is the metric tensor determinant, \( R \) is the Ricci scalar, and \( H_{\mu\nu\lambda} \) is the totally antisymmetric K-R field. Our assumptions of a static solution with spherical symmetry lead to a K-R field with only one independent non-zero component, \( H_{023} \), which is normalized as

\[
H_{\mu\nu\lambda} H^{\mu\nu\lambda} = -f(r)^2 = -\frac{b_0^2 A}{r^4}.
\]  (9)

\( H_{\mu\nu\lambda} \) satisfies the field equation

\[
\frac{1}{\sqrt{-g}} \partial_\mu \left( \sqrt{-g} H^{\mu\nu\lambda} \right) = 0
\]  (10)
FIG. 1: The throat region of a wormhole obtained by sewing together two Reissner-Nordstrom-type black holes at their horizons.

To analyze the stability of wormholes described by the metric in Eq. (6), the action is calculated using the procedure of [9]. Although the Ricci scalar vanishes for our solution, the result we invoke requires the value of an action that is first order in field derivatives [9]; this requires an integration by parts, leading to a non-trivial boundary term,

$$ S_B = \frac{1}{8 \pi G} \int_{\partial M} \sqrt{-h} K d^3x $$

where $h$ is the induced metric on the boundary $\partial M$ and $K$ is the trace of the second fundamental form which is

$$ K_{\mu \nu} = \nabla_\mu n_\nu - n_\lambda n^\lambda n_\mu n_\sigma \nabla_\nu n_\sigma. $$

(12)

The vector $n_\mu$ is a unit vector normal to the boundary $\partial M$: in the present case $n_\mu = (0, 1/\sqrt{1 - b(r)/r}, 0, 0)$. To evaluate the action on the boundary we transform to Kruskal coordinates which for the metric in Eq. (6) can be expressed as

$$ T = F(r) \sinh(\alpha t) $$

$$ \rho = F(r) \cosh(\alpha t). $$

(13)

with

$$ F(r) = b_0 \sqrt{\frac{r}{b_0}} - 1 \left( \frac{r}{b_0} - A \right)^{-A^2/2} \exp \left( \frac{(1 - A) r}{2 b_0} \right) $$

$$ \alpha = \frac{(1 - A)}{2 b_0} $$

(14)

The action in Eq. (11) diverges when the radial distance becomes large, so it is renormalized by subtracting the flat-space value of the trace of the second fundamental form. Eq. (11) then becomes

$$ S_B = \frac{1}{8 \pi G} \int_{\partial M} \sqrt{-h [K]} d^3x $$

(15)
where \( [K] = K - K^F \) and \( K^F \) is the second fundamental form for flat space. Our goal is to determine the stability of the wormholes described by Eq. (6). Therefore we need the Euclidian action, which can be obtained from Eq. (15) by Wick rotating the time coordinate \( t \rightarrow -iT \). With this rotation it is evident that the Kruskal coordinates multiply cover the Euclidian manifold, due to the periodicity of the sine and cosine factors in Eq. (13), so integration over the physical manifold thus corresponds to integration over only a single period \( 0 \leq T \leq T = 4\pi b_0/(1 - A) \). The contribution to the Euclidian action from the surface term is

\[
S_E^B = i\pi b_0 (1 + A) \frac{1}{4G}
\]  

or, substituting for \( T \),

\[
S_E^B = i\pi b_0^2 (1 + A) = \frac{G}{(1 - A)}
\]  

The contribution to the Euclidian action from the K-R term is, similarly,

\[
S_{KR}^E = -i\pi b_0^2 A = \frac{2G}{(1 - A)}
\]  

and the total Euclidian action is

\[
S^E = i\pi b_0^2 (1 + A/2) = \frac{G}{(1 - A)}
\]  

C. Wormhole Stability

The stability of a wormhole can be analyzed within the context of quantum gravity by evaluating the partition function obtained from the summation over all histories

\[
Z = \int [dg] \exp(iS[g])
\]  

in units where \( \hbar = c = 1 \). The decay probability of a body is determined by the probability of the gravitational instantons [14] tunnelling to the vacuum. The probability can be written as [11, 12]

\[
P \sim e^{iS_E}
\]  

where \( S_E \) is the Euclidian action, which for our EKR wormholes is given by Eq. (19). The action in (19) is inversely dependent on \( 1 - A \) and can be made arbitrarily large by letting \( A \) approach 1. This means that wormholes produced with \( A \sim 1 \) are quasi-stable, even wormholes with small radii and therefore small masses. The tension is \( \tau(r) = b_0^2 A/r^4 \), so that large tension is therefore highly localized to the region of the wormhole, due to its \( r^{-4} \) falloff. The possibility of forming quasi-stable, microscopic wormholes is due to the presence of the K-R field with ‘charge’ \( |Q| = b_0 \sqrt{A} \). Thus quasi-stable wormholes can have large values of \( A \) but small values of \( Q \), provided that \( b_0 \) is small enough. Quasi-stability requires that

\[
\frac{\pi b_0^2 (1 + A)}{G(1 - A)} \gg 1
\]  

or in terms of the Planck length \( l_p \)

\[
b_0 \gg \frac{l_p}{\pi} \sqrt{\frac{1 - A}{1 + A}}.
\]  

which means that \( b_0 \) can be of the order of the Planck length or smaller, if \( A \) is near 1, without violating the stability condition. The Planck length is usually considered to be the length at which general relativity breaks down. However, this may be a prejudice which has grown out of the perceived limit on the geometrical sizes which particle accelerators can probe without creating black holes. Our results suggest the possibility that non-gravitationally collapsed objects with dimensions smaller than \( l_p \) may exist or can be created.
III. CONCLUSIONS

Since wormholes may exist or perhaps can be created in nature as particles, baby universes or mechanisms for time travel, an analysis of their stability is essential. We have shown in the above analysis that a particular type of wormhole can be made quasi-stable by a suitable adjustment of the parameters of the system. The ‘charge’ associated with the K-R field is determined by the radius of the wormhole, so that in our model of K-R particles the ‘charge’ is a geometrical property of such particles. Although the geometry of the wormholes considered here is that of two Reissner-Nordstrom black holes sewn together at their horizons, the charge giving rise to the gauge field cannot be an electrical charge if the wormholes are to be created in four dimensions. A wormhole created in four dimensions with even a single electronic charge, e, would have to have a mass within a few percent of the Planck mass, $m_p$, in order to satisfy the condition that $r_+$ in Eq. (7) must be real. Since the K-R ‘charge’ coupling strength is not constrained to match that of observed electrical charges, the mass of the wormhole could be very small without violating the reality condition on $r_+$. For example, if the wormholes described in this paper can be considered as a semiclassical description of K-R axions, whose masses are in the range (depending on the interactions allowed with other particles) $10^{-2}$ eV to 25 MeV, the geometrical radius of the wormhole, the geometrical radius of the wormhole, $b_o$, would be much less than $l_p$, since the mass is much less than $M_p$, where $M_p$ is the Planck mass. Such wormholes could be created by high energy cosmic rays or in an accelerator, providing a realization of the original attempt of Einstein and Rosen to describe a particle as a wormhole.

Acknowledgments

We thank Allen Stern and Roberto Casadio for discussions on the work described in this paper. This work was supported in part by the U.S. Department of Energy under Grant No. DE-FG02-10ER41714.

[1] A. Einstein and N. Rosen, Phys. Rev. 48, 73(1935).
[2] R. Arnowitt, S. Deser, C.W. Misner, Phys. Rev. Lett. 4, 375(1960).
[3] R. Arnowitt, S. Deser, C.W. Misner, Phys. Rev. 120, 313(1960).
[4] R. Casadio, R. Garrattini, F. Scardigli, Phys. Lett. B 679, 156 (2009).
[5] S. Hawking, Phys. Rev. D37, 904(1988).
[6] M. S. Morris and K. S. Thorne, Am. J. Phys. 56, 395(1988).
[7] M. Kalb and P. Ramond, Phys. Rev. D9, 2273(1974).
[8] F. Rahaman, M. Kalam, and A. Ghosh, Nuovo Cim. B121, 303(2006).
[9] G. Gibbons and S. Hawking, Phys. Rev. D15, 2752(1977).
[10] G. Gibbons and S. Hawking, Commun. Math. Phys. 66, 291(1979).
[11] R. Jackiw, C. Rebbi, Phys. Lett. 37, 172 (1976).
[12] C. Callan, R. Dashen, D. Gross, Phys. Lett. 63B, 334 (1976).
[13] For a more recent investigation of a wormhole model of neutral particles see [4].
[14] For a classification of gravitational instantons see [10].