Optimal asymmetric phase-covariant and real state cloning in $d$ dimensions

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Abstract. We generalize three symmetric cloners to the asymmetric cases and present three explicit cloning transformations as well as their corresponding fidelity distributions in $d$ dimensions. The three asymmetric cloners, including the optimal asymmetric phase-covariant cloning (APCC) and the suboptimal asymmetric economical phase-covariant cloning (AEPCC) working without ancilla and the optimal asymmetric real state cloning (ARSC), together with the optimal asymmetric universal quantum cloning (AUQC) construct a generic cloning, where the quantum information of initial systems of different pure input states in $d$ dimensions with their information not completely known can be optimally distributed to different final systems. By comparison of the fidelity distributions of the four asymmetric cloners, some interesting results can be obtained.

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1. Introduction

The no-cloning theorem [1] states that one cannot copy an arbitrary quantum state perfectly, which guarantees the security of the quantum key distribution (QKD) protocols [2], such as the Bennett–Brassard 1984 (BB84) protocol [3]. Since exact copies cannot be made, much more attention has been paid to approximate quantum cloning, a concept that was introduced by Bužek and Hillery [4]. For a pure state in a high-dimensional system with the form of $|\psi\rangle = \sum_{i=0}^{d-1} a_i e^{i\phi_i} |i\rangle$ ($\varphi_i \in [0, 2\pi]$ and $a_i \in [0, 1]$ satisfying $\sum_{i=0}^{d-1} a_i^2 = 1$), if the information about an input state is completely known (i.e. $\varphi_i$ and $a_i$ are completely known), it can be cloned (or prepared) perfectly in principle. However, in the case where the information about different input states is not completely known, one is unable to clone them perfectly. If different input states are confined as pure ones, the quantum cloners can be divided into three types of cloners depending on different input states. For instance, if $\varphi_i$ and $a_i$ are not completely known, the cloner is called universal quantum cloning (UQC) [5, 6]. If $\varphi_i$ are unknown and $a_i$ are known ($a_i = 1/\sqrt{d}$ for example), the cloner is defined as phase-covariant cloning (PCC) [7, 8]. If $a_i$ are unknown ($a_i \in [-1, 1]$) and $\varphi_i$ are known ($\varphi_i = 0$ for example), the cloner is called real state cloning (RSC) [9, 10]. Therefore, the three types of quantum cloners make up a generic cloning. These cloners have useful applications in practice. Furthermore, investigation of generic cloning is also in the foundation of quantum mechanics.

In earlier investigations, optimal symmetric cloning was extensively studied. The term ‘symmetric’ means the fidelity of all clones is the same, while ‘optimal’ implies that the fidelity of each clone should be maximal. Lately, [11]–[13] proposed the optimal asymmetric universal quantum cloning (AUQC), where the quantum cloning machines can be regarded as the well-defined distributors of quantum information from certain initial systems to the different final ones after quantum evolutions. In the case that two clones are produced, the term ‘optimal asymmetric’ means the optimal trade-off between two fidelities [11], i.e., by fixing the fidelity of one clone the fidelity of the other one should be maximal. Thus, three types of cloners have different functions. For the AUQC, how to distribute initial information on the phase factors and
the amplitudes in a pure state to different final systems is studied. For the asymmetric phase-

covariant cloning (APCC) (or the asymmetric economical phase-covariant cloning (AEPCC) working without ancilla [14]–[17]), the distribution of the phase information is investigated in distributed systems. And for the asymmetric real state cloning (ARSC), the distribution of the amplitudes is focused upon. By analysis of this generic cloning, maybe some function characterized by the phase factors and the amplitudes can quantify the quantum information of a pure state, and it further facilitates study of whether quantum information is conservative or not in the terms of the fidelity or other quantities. Recently, some cloners have been realized experimentally [18]–[21].

In this paper, we present three explicit cloning transformations of the one to two cloners, including the APCC and the AEPCC and ARSC in \( d \) dimensions as well as their corresponding fidelity distributions. By comparison of the fidelity distributions of the four asymmetric cloners (the asymmetric universal quantum cloning (AUQC), the APCC, the AEPCC and the ARSC), our results lead to some interesting observations. If the fidelities of clone 1 of the four asymmetric cloners are equal, the fidelity of clone 2 of the AUQC is the lowest, i.e. if \( F_{1}^{(AUQC)}(d) = F_{1}^{(AEPCC)}(d) = F_{1}^{(APCC)}(d) = F_{1}^{(ARSC)}(d) \) one can observe that the inequality \( F_{2}^{(AUQC)}(d) < F_{2}^{(AEPCC)}(d), F_{2}^{(APCC)}(d), F_{2}^{(ARSC)}(d) \) holds for all \( d \). The inequality suggests that the more information about input states is obtained, the higher the fidelity that can be reached. The form of a pure input state of the APCC and the AEPCC is the same, but the AEPCC works without ancilla (the ancillary system). If the fidelities of clone 1 of the APCC and the AEPCC are equal, i.e. \( F_{1}^{(AEPCC)}(d) = F_{1}^{(APCC)}(d) \), the fidelity of clone 2 of the AEPCC is slightly smaller than that of the APCC, i.e. \( F_{2}^{(AEPCC)}(d) < F_{2}^{(APCC)}(d) \) (equality holds when \( d = 2 \)). The inequality shows that introduction of a suitable ancillary system can indeed improve the fidelity of clones. For a pure state in \( d \) dimensions, the independent parameters of the amplitudes and the phase factors are both \( d - 1 \) ones. By comparison of the fidelity distributions of the APCC and the ARSC, equality \( F_{1}^{(APCC)}(d) = F_{1}^{(ARSC)}(d) \) can be obtained with \( F_{2}^{(APCC)}(d) \leq F_{2}^{(ARSC)}(d) \) (equality holds when \( d = 2 \)). This result implies that the phase factor contains more information than the amplitude does for a pure input state in \( d \) dimensions when \( d > 3 \). Essentially, all of the four asymmetric cloners are information distributors from different initial systems depending on different pure input states to final ones. Therefore it can be expected that they will have promising applications in quantum information processing. Moreover, investigations of the role of ancillary systems of certain initial physical systems to improve quantum information processing and studies of the phase factor containing more information in the pure state are in the foundation of quantum mechanics.

This paper is constructed as follows. In section 2, we briefly review some previous contributions, mainly including AUQC, symmetric phase-covariant cloning (SPCC), symmetric economical phase-covariant cloning (SEPCC) and symmetric real state cloning (SRSC) in \( d \) dimensions. While in section 3, we generalize three symmetric cloners to the asymmetric cases and present three explicit cloning transformations of the APCC and the AEPCC and the ARSC as well as their corresponding fidelity distributions. By comparison of the fidelity distributions of four asymmetric cloners, we derive some interesting results. The paper ends with a conclusion in section 4.
2. Previous contributions

2.1. Optimal AUQC

If one wants to clone a completely unknown state in the form of $|\varphi\rangle_1 = a_1 |0\rangle + a_2 e^{i\varphi} |1\rangle$, the explicit cloning transformation of the optimal symmetric $1 \rightarrow 2$ universal quantum cloning (SUQC) in 2 dimensions is given by [4]

$$
|00\rangle_{1,2,A} \rightarrow \sqrt{\frac{2}{3}} |00\rangle_{1,2,A} + \sqrt{\frac{1}{3}} (|01\rangle_{1,2} + |10\rangle_{1,2}) |1\rangle_A,
$$

$$
|10\rangle_{1,2,A} \rightarrow \sqrt{\frac{2}{3}} |11\rangle_{1,2,A} + \sqrt{\frac{1}{3}} (|10\rangle_{1,2} + |01\rangle_{1,2}) |0\rangle_A,
$$

where particles 1, 2 and A stand for the input state, the blank copy and the ancilla. The fidelity of two clones is $F_{1(2)}^{(SUQC)} (2) = 5/6$. In $d$ dimensions, a completely unknown input state takes the form of $|\varphi\rangle_1 = \sum_{i=0}^{d-1} a_i e^{i\varphi_i} |i\rangle (\varphi_i \in [0, 2\pi])$ and $a_i \in [0, 1]$ satisfying $\sum_{i=0}^{d-1} a_i^2 = 1$ and the explicit cloning transformation of the optimal asymmetric $1 \rightarrow 2$ universal quantum cloning (AUQC) can be expressed as [11, 12]

$$
|i0\rangle_{1,2,A} \rightarrow \frac{1}{\sqrt{N}} \left[ |ii\rangle + \left( \sum_{j=1}^{d-1} p |i\rangle_1 |j\rangle_2 + q |j\rangle_1 |i\rangle_2 \right) |j\rangle_A \right],
$$

where $N = 1 + (d - 1) (p^2 + q^2)$ is a normalization factor and $p, q \geq 1$ satisfies $p + q = 1$. The corresponding fidelity distributions of two clones read [12]

$$
F_{1(2)}^{(AUQC)} (d) = \left[ 1 + (d - 1) p^2 \right]/N, F_{2(2)}^{(AUQC)} (d) = \left[ 1 + (d - 1) q^2 \right]/N.
$$

In figure 1, we plot a series of curves showing the fidelity distribution of the AUQC given by equation (3) when $d = 2, 3, 4, 5, 8, 10$.

From figure 1, one can observe that the two endpoints of the curves are $(1, 1/d)$ and $(1/d, 1)$. The first endpoint ($F_{1(2)}^{(AUQC)} (d) = 1, F_{2(2)}^{(AUQC)} (d) = 1/d$) means the information of the pure input state does not transfer to the clone 2. The second ($F_{1(2)}^{(AUQC)} (d) = 1/d, F_{2(2)}^{(AUQC)} (d) = 1$) implies the information of the unknown input state transfer to the clone 2 completely. One can also observe that the APCC and the AEPCC and the ARSC have such properties in figures 2, 3 and 6.

2.2. Optimal SPCC

If partial information about an input state is obtained, such as the state $|\varphi\rangle_2 = (|0\rangle + e^{i\varphi} |1\rangle)/\sqrt{2}$ with $\varphi \in [0, 2\pi)$ is unknown, the explicit cloning transformation of the optimal asymmetric $1 \rightarrow 2$ phase-covariant cloning (APCC) in 2 dimensions is defined by [22]

$$
|00\rangle_{1,2,A} \rightarrow \frac{1}{\sqrt{2}} \left[ |00\rangle_{1,2,A} + (\cos \theta |01\rangle_{1,2} + \sin \theta |10\rangle_{1,2}) |1\rangle_A \right],
$$

$$
|10\rangle_{1,2,A} \rightarrow \frac{1}{\sqrt{2}} \left[ |11\rangle_{1,2,A} + (\cos \theta |10\rangle_{1,2} + \sin \theta |01\rangle_{1,2}) |0\rangle_A \right],
$$

where $\theta \in [0, \pi/2]$. The corresponding fidelities of two clones can be expressed as

$$
F_{1(2)}^{(APCC)} (2) = (1 + \cos \theta)/2, F_{2(2)}^{(APCC)} (2) = (1 + \sin \theta)/2.
$$
Figure 1. Distribution of the fidelities of the AUQC when \( d = 2, 3, 4, 5, 8, 10 \) (from outside to inside).

For an pure input state in \( d \) dimensions \( |\psi\rangle_2 = \frac{1}{\sqrt{d}} \sum_{i=0}^{d-1} e^{i\varphi_i} |i\rangle \) with \( \varphi_i \in [0, 2\pi) \) is unknown, the explicit cloning transformation of the optimal symmetric \( 1 \rightarrow 2 \) phase-covariant cloning (SPCC) takes the following form [8]

\[
|i00\rangle_{1,2,A} \rightarrow p |i\rangle_{1,2} |i\rangle_A + q \sum_{j=0}^{d-1} \sum_{i \neq j} (|ij\rangle + |ji\rangle)_{1,2} |j\rangle_A,
\]

where

\[
p^2 = \frac{1}{2} \left( 1 - \frac{d - 2}{\sqrt{d^2 + 4d - 4}} \right), \quad q^2 = \frac{1}{4 (d - 1)} \left( 1 + \frac{d - 2}{\sqrt{d^2 + 4d - 4}} \right).
\]

The corresponding fidelity of two clones reads

\[
F^{(\text{SPCC})}_{1(2)}(d) = \frac{1}{d} + \frac{1}{4d} \left( d - 2 + \sqrt{d^2 + 4d - 4} \right).
\]

Up to now, the explicit cloning transformation of the APCC in \( d \) dimensions has not been determined. We will show it in the next section. Our result can cover all the results in this subsection.

2.3. Suboptimal SEPCC

In equation (4) one can observe that the ancilla is necessary. Another phase-covariant cloner works without ancilla, called economical PCC (EPCC). In the case of 2 dimensions, the explicit cloning transformation of the optimal \( 1 \rightarrow 2 \) AEPCC is given by [23]

\[
|00\rangle_{1,2} \rightarrow |00\rangle_{1,2}, \quad |10\rangle_{1,2} \rightarrow \cos \theta \ |10\rangle_{1,2} + \sin \theta \ |01\rangle_{1,2},
\]

\[\]

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\[\]

\[\]
where $\theta \in [0, \pi/2]$. Another independent transformation can be expressed as [24]

$$
|00\rangle_{1,2} \rightarrow (\cos \theta \ |01\rangle_{1,2} + \sin \theta \ |10\rangle_{1,2}), \quad |10\rangle_{1,2} \rightarrow |11\rangle_{1,2}.
$$

The corresponding fidelities of two clones have the same expressions as equation (5), i.e.

$$
F^{(\text{AEPCC})}_1(2) = (1 + \cos \theta)/2, \quad F^{(\text{AEPCC})}_2(2) = (1 + \sin \theta)/2.
$$

While in $d$ dimensions, the fidelities of the SEPCC and the SPCC are not equal. The fidelity of the SEPCC in $d$ dimensions is [16]

$$
F^{(\text{SEPCC})}_1(d) = \frac{1}{2d^2} \left[d - 1 + \left(d - 1 + \sqrt{2}\right)^2\right].
$$

Numerical calculations can be derived $F^{(\text{SEPCC})}_1(d) - F^{(\text{SEPCC})}_1(d) \leq 60.0045$ when $d \geq 3$ [16]. This is a reason that the SEPCC was originally called suboptimal. In this paper, we present the explicit transformation of the AEPCC and the corresponding fidelity distribution in $d$ dimensions. All the results in this subsection can be covered by our new result.

2.4. Optimal SRSC

If an input state takes the form of $|\psi\rangle_3 = \sum_{i=0}^{d-1} a_i |i\rangle$ (the unknown amplitudes $a_i$ satisfy $|a_i| \leq 1$ and $\sum_{i=0}^{d-1} a_i^2 = 1$), the cloner is then nominated as the RSC [9]. For the input state in 2 dimensions $|\varphi\rangle_3 = (a_1 |0\rangle + a_2 |1\rangle)$, the explicit cloning transformation of the optimal $1 \rightarrow 2$ ARSC is defined by [10]

$$
|000\rangle_{1,2,A} \rightarrow (\alpha |00\rangle + \beta |11\rangle)_1,2,00,10 \rightarrow (\gamma |01\rangle + \chi |10\rangle)_1,2,00,10,
$$

with $\alpha, \beta, \gamma, \chi \geq 0$ satisfying the normalization condition $\alpha^2 + \beta^2 + \gamma^2 + \chi^2 = 1$. The cloning coefficients which are functions of the fidelity of clone 1 $F^{(\text{ARSC})}_1(2)$ (denoted as $F_1$ for convenience) can be expressed as

$$
\alpha = \sqrt{\frac{F_1 - 2 \sqrt{F_1^3 - F_1^4}}{2F_1 (2F_1 - 1)}}; \quad \beta = \sqrt{\frac{F_1^3 - F_1^4}{2F_1^2 (2F_1 - 1)}};
$$

$$
\gamma = \sqrt{\frac{F_1 (2F_1 - 1) (F_1^2 - F_1^4 + \sqrt{F_1^3 - F_1^4})}{2 \sqrt{F_1 - 2 \sqrt{F_1^3 - F_1^4} (F_1^2 + \sqrt{F_1^3 - F_1^4})}}},
$$

$$
\chi = \sqrt{\frac{F_1 (2F_1 - 1) (F_1^2 - F_1^4 - \sqrt{F_1^3 - F_1^4})}{2 \sqrt{F_1 - 2 \sqrt{F_1^3 - F_1^4} (F_1^2 - \sqrt{F_1^3 - F_1^4})}}}.
$$

The fidelity of the clone 2 $F^{(\text{ARSC})}_2(2)$ (denoted as $F_2$ for convenience) is

$$
F_2 = 1/2 + \sqrt{F_1 (1 - F_1)},
$$

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where $F_{1}^{(ARSC)}(2)$, $F_{2}^{(ARSC)}(2) \in [1/2, 1]$. The distribution of the fidelities of the ARSC is the same as that of the APCC and the AEPC (see equations (5) and (10)). In $d$ dimensions, the explicit cloning transformation of the optimal $1 \rightarrow 2$ SRSC is expressed as [10]

$$
|i00\rangle_{1,2,A} \rightarrow \left( \alpha |ii\rangle + \beta \sum_{j=0}^{d-1} |jj\rangle \right)_{1,2} + \gamma \sum_{j=0}^{d-1} (|ij\rangle + |ji\rangle)_{1,2} |j\rangle_{A}. \quad (15)
$$

The cloning coefficients are determined by

$$
\alpha = \left( 4 + d + \sqrt{d^2 + 4d + 20} \right) \left[ 20 + 24d + 5d^2 + d^3 + (6 + 3d + d^2) \sqrt{d^2 + 4d + 20} \right]^{-1/2};
$$

$$
\beta = 2 \left[ 20 + 24d + 5d^2 + d^3 + (6 + 3d + d^2) \sqrt{d^2 + 4d + 20} \right]^{-1/2};
$$

$$
\gamma = \frac{1}{2} \left( 2 + d + \sqrt{d^2 + 4d + 20} \right) \left[ 20 + 24d + 5d^2 + d^3 + (6 + 3d + d^2) \sqrt{d^2 + 4d + 20} \right]^{-1/2}. \quad (16)
$$

The corresponding fidelity of two clones takes the form [10]

$$
F_{1(2)}^{(SRSC)}(d) = 1/2 + \left( \sqrt{d^2 + 4d + 20} - d + 2 \right)/4 (d + 2). \quad (17)
$$

By comparison of the fidelities of four symmetric cloners given by equations (3), (7), (11) and (17), one can easily calculate that a series of the following inequalities hold

$$
F_{1(2)}^{(SUQC)}(d) < F_{1(2)}^{(SEPCC)}(d) \leq F_{1(2)}^{(SPCC)}(d) \leq F_{1(2)}^{(SRSC)}(d), \quad (18)
$$

where equality holds when $d = 2$, i.e.

$$
F_{1(2)}^{(SEPCC)}(2) = F_{1(2)}^{(SPCC)}(2) = F_{1(2)}^{(SRSC)}(2) = (1 + 1/\sqrt{2})/2 > 5/6 = F_{1(2)}^{(SUQC)}(2).
$$

In the next section, we will present the optimal $1 \rightarrow 2$ ARSC and the corresponding fidelity distribution in $d$ dimensions.

3. Three asymmetric cloners in $d$ dimensions

In this section, we generalize three symmetric cloners (the SPCC, the SEPCC and the SRSC) to asymmetric cases (the APCC, the AEPC and the ARSC) and present their explicit cloning transformations as well as their corresponding fidelity distributions in $d$ dimensions. The three asymmetric cloners we define in this section together with the AUQC can structure a generic cloning for pure input state in $d$ dimensions. By comparison of the fidelity distributions of four asymmetric cloners, some interesting results can be obtained.

3.1. Optimal APCC

If an input state takes the form of

$$
|\psi\rangle = \frac{1}{\sqrt{d}} \sum_{i=0}^{d-1} e^{i\psi} |i\rangle
$$

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(\varphi_i \in [0, 2\pi]) is unknown, we determine the explicit cloning transformation of the optimal 1 \rightarrow 2 APCC in \(d\) dimensions as follows

\[
|i\rangle_1 |0\rangle_2 |0\rangle_A \rightarrow P \left[ |i\rangle_{1,2} |i\rangle_A + Q \sum_{\substack{j = 0 \\ i \neq j}}^{d-1} \left( \cos \theta |ij\rangle + \sin \theta |ji\rangle \right)_{1,2} |j\rangle_A \right],
\]

where \(\theta \in [0, \pi/2]\), the parameters \(P\) and \(Q\) are defined by

\[
P = \left( \frac{\sqrt{d^2 + 4d - 4} - d + 2}{2\sqrt{d^2 + 4d - 4}} \right)^{1/2}, \quad Q = \left[ \frac{\sqrt{d^2 + 4d - 4} + d - 2}{(d - 1) (2 - d + \sqrt{d^2 + 4d - 4})} \right]^{1/2}.
\]

It can be easily calculated that equation (19) can be reduced to equation (4) when \(d = 2\) and to equation (6) when \(\theta = \pi/4\), respectively. The fidelities of two clones take the form

\[
F_1^{(APCC)}(d) = \frac{P^2}{d} \left\{ (d - 1) Q \left[ (d - 1) Q \cos^2 \theta + 2 \cos \theta + Q \sin^2 \theta \right] + 1 \right\}
\]

\[
F_2^{(APCC)}(d) = \frac{P^2}{d} \left\{ (d - 1) Q \left[ (d - 1) Q \sin^2 \theta + 2 \sin \theta + Q \cos^2 \theta \right] + 1 \right\}.
\]

It also can be testified that equation (21) can be reduced to equation (5) when \(d = 2\) and to equation (7) when \(\theta = \pi/4\), respectively. The reducibility of the APCC sufficiently confirms the correctness of our result.

In figure 2, we plot a series of curves showing the fidelity distribution of two clones of the APCC given by equation (21) when \(d = 2, 3, 4, 5, 8, 10\). From figure 2, one can clearly observe that the all curves cross through two endpoints of \((1, 1/d)\) and \((1/d, 1)\).

**Figure 2.** Distribution of the fidelities of the APCC when \(d = 2, 3, 4, 5, 8, 10\) (from outside to inside).
3.2. Suboptimal AEPCC

For an input state taking the form of

$$|\psi_2\rangle = \frac{1}{\sqrt{d}} \sum_{i=0}^{d-1} e^{i\phi_i} |i\rangle$$

($\phi_i \in [0, 2\pi]$ is unknown), if the ancilla is not required, we determine the explicit cloning transformation of the suboptimal 1 → 2 AEPCC in $d$ dimensions as follows

$$|i0\rangle_{1,2} \rightarrow |ii\rangle_{1,2}, \quad |j0\rangle_{1,2} \rightarrow (\cos \theta |ji\rangle + \sin \theta |jj\rangle)_{1,2},$$

where $i, j = 0, 1, \ldots, d - 1 (j \neq i)$. For any one fixed number $i$, there exists one cloning transformation. Therefore, this kind of cloner has $d$ independent transformations, which can cover equations (8) and (9) when $d = 2$. The fidelities of two clones can be expressed as

$$F_1^{(AEPCC)}(d) = \frac{1}{d^2} \left[ d + 2 \cos \theta (d - 1) + \cos^2 \theta (d^2 - 3d + 2) \right]$$

$$F_2^{(AEPCC)}(d) = \frac{1}{d^2} \left[ d + 2 \sin \theta (d - 1) + \sin^2 \theta (d^2 - 3d + 2) \right],$$

where $\theta \in [0, \pi/2]$. One can observe that equation (23) can be reduced to equation (10) when $d = 2$ and to equation (11) when $\theta = \pi/4$, respectively.

In figure 3, we plot a series of curves showing the relation of the fidelity distribution of the AEPCC given by equation (23) as $d = 2, 3, 4, 5, 8, 10$.  

![Figure 3. Distribution of the fidelities of the AEPCC when $d = 2, 3, 4, 5, 8, 10$ (from outside to inside).](http://www.njp.org/)

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From figure 3, one can clearly observe that all of the endpoints of the curves are \((1, 1/d)\) and \((1/d, 1)\). In figure 4, we make comparisons between the AEPCC given by equation (23) and the AUQC given by equation (3) when \(d = 2, 3\).

In comparison of fidelities of different asymmetric cloners, we exclude the comparison of two special points \((1, 1/d)\) and \((1/d, 1)\). From figure 4, for any value of \(F_1^{(AUQC)}(d) = F_1^{(AEPCC)}(d)\), one can clearly observe \(F_2^{(AUQC)}(d) < F_2^{(AEPCC)}(d)\) \((d \geq 2)\).

Since the curves of the fidelity distributions of the AEPCC and the APCC are very close, in figure 5, we plot part of three pairs of curves to compare the fidelity distributions of the AEPCC given by equation (23) and the APCC given by equation (21) when \(d = 3, 4, 5\).

The form of pure input states of the APCC and the AEPCC is the same, but the AEPCC works without ancilla (the ancillary system). If the fidelities of clone 1 of the APCC and the AEPCC are equal, the fidelity of clone 2 of the AEPCC is slightly smaller than that of the APCC, i.e., in the case of \(F_1^{(AEPCC)}(d) = F_1^{(APCC)}(d)\), the inequality \(F_2^{(AEPCC)}(d) \leq F_2^{(APCC)}(d)\) can be intuitively observed from figure 5 (equality holds when \(d = 2\)). The inequality shows that introduction of a suitable ancillary system can improve quantum information processing, for example, improvement of the fidelity of clones in quantum cloning. Therefore, the investigation of the role of ancillary systems as a necessary part of the whole system is also an important task in quantum information processing.

3.3. Optimal ARSC

If an input state takes the form of

\[
|\psi\rangle_3 = \sum_{i=0}^{d-1} a_i |i\rangle
\]
Figure 5. Comparison of the distributions of the fidelities of the APCC and the AEPCC when \(d = 3, 4, 5\). Red lines denote the APCC and green lines denote the AEPCC.

The unknown amplitudes \(\alpha_i\) satisfy \(|\alpha_i| \leq 1\) and \(\sum_{i=0}^{d-1} a_i^2 = 1\), we determine the explicit cloning transformation of the optimal \(1 \rightarrow 2\) ARSC as

\[
|00\rangle_{1,2,A} \rightarrow \alpha |ii\rangle + \beta \sum_{j=0}^{d-1} |jj\rangle_{A} |i\rangle_{1,2} + \sum_{j=0}^{d-1} (\chi |ji\rangle + \gamma |ij\rangle)_{1,2} |j\rangle_{A}.
\]

(24)

We first give some constants as follows

\[
\Delta = (d + 2 + \sqrt{d^2 + 4d + 20}) \left[ d^3 + 5d^2 + 24d + 20 + (d^2 + 3d + 6) \sqrt{d^2 + 4d + 20} \right]^{-1/2},
\]

\[
a = \left[ \sqrt{2d + \sqrt{d^2 + 4d + 20}} \right] \left( \Delta - \frac{1}{\sqrt{d}} \right) ; \quad b = a + \frac{\Delta}{\sqrt{d}}.
\]

(25)

The cloning coefficients can be expressed as

\[
\gamma = \frac{(a^2 + 1) \chi - \sqrt{2b} + \sqrt{2a^2 - 2a^4 + 2a^2b^2 - 4\sqrt{2a^2b}\chi + 4a^2\chi^2}}{a^2 - 1},
\]

\[
\alpha = \left[ (d - 1)(\gamma + \chi) + \sqrt{(\gamma + \chi)^2 - d(d\gamma^2 + 2\gamma\chi + d\chi^2 - 1)} \right] / d,
\]

\[
\beta = \left[ \sqrt{(\gamma + \chi)^2 - d(d\gamma^2 + d\chi^2 + 2\gamma\chi - 1)} - \gamma - \chi \right] / d.
\]

(26)
The cloning coefficients only contain one parameter $\chi \in [0, 1/\sqrt{d}]$. The two fidelities take the following form

$$F_1^{(ARSC)}(d) = \beta^2 + \gamma^2 + (d - 2) \chi^2 + 2 (\alpha \chi + \beta \gamma),$$

$$F_2^{(ARSC)}(d) = \beta^2 + \chi^2 + (d - 2) \gamma^2 + 2 (\alpha \gamma + \beta \chi).$$

(27)

In the case of $\gamma = \chi = \frac{1}{2} \left(2 + d + \sqrt{d^2 + 4d + 20}\right) \left[20 + 24d + 5d^2 + d^3 + (6 + 3d + d^2) \sqrt{d^2 + 4d + 20}\right]^{-1/2},

(28)

Equation (27) can be reduced to equation (17). Equation (27) can also be reduced to equation (14) when $d = 2$, which can be observed from figure 6. In figure 6, we plot a series of curves showing the distribution of the fidelities of the ARSC given by equation (27) when $d = 2, 3, 4, 5, 8, 10$. From figure 6, one can clearly observe that the two endpoints of the curves are $(1, 1/d)$ and $(1/d, 1)$.

In figure 7, we plot three pairs of curves to make comparisons between the fidelity distribution of the ARSC given by equation (27) and the APCC given by equation (21) when $d = 3, 4, 5$.

For a pure state of discrete variables in $d$ dimensions, the independent parameters of the amplitudes and the phase factors are both $d - 1$ in mathematics. By comparison of the fidelity distributions of the APCC and the ARSC from figure 7, taking $F_1^{(APCC)}(d) = F_1^{(ARSC)}(d)$ one can derive $F_2^{(APCC)}(d) \leq F_2^{(ARSC)}(d)$ (equality holds when $d = 2$). In the case of $d = 2$, the inequality means that the phase factor and the amplitudes are independent and equivalent in characterizing the quantum information of a pure state. While in the case of $d \geq 3$, the phase
factor contains more information than the amplitude does. This is an interesting issue not only in quantum information theory but also in quantum mechanics. For instance, the optimal universal quantum cloning has intimate connection to (completely unknown) state estimation [25, 26] and the optimal phase-covariant cloning to optimal estimation of multiple phases [27]. Obviously, the optimal real state cloning has a link to optimal estimation of multiple amplitudes, which would be further investigated.

By comparison of the fidelity distributions of the AUQC and the other three asymmetric cloners from figures 4, 5 and 7, it can be easily obtained that if the fidelity of clone 1 of the four cloners are equal, the fidelity of clone 2 of the AUQC is the lowest, i.e. when $F_1^{(AUQC)}(d) = F_1^{(AEPC)}(d) = F_1^{(APCC)}(d) = F_1^{(ARSC)}(d)$ the inequality $F_2^{(AUQC)}(d) < F_2^{(AEPC)}(d), F_2^{(APCC)}(d), F_2^{(ARSC)}(d)$ holds for all $d$. The inequality suggests that the more information about input states is obtained, the higher the fidelity that can be reached. Furthermore, three asymmetric cloners together with the AUQC construct a generic cloning for different pure input states of discrete variables in $d$ dimensions. By analysis of this generic cloning, maybe some function characterized by the phase factors and the amplitudes can quantify the quantum information of a pure state, and it further facilitates the study of whether quantum information is conservative or not in the merit of the fidelity or other quantities.

4. Conclusion

In summary, we have presented three explicit cloning transformations of the asymmetric cloners of one to two in $d$ dimensions, including optimal APCC and suboptimal asymmetric economical phase-covariant cloning working without ancilla and the optimal ARSC as well as their corresponding fidelity distributions. The three asymmetric cloners presented in this paper
together with the optimal AUQC construct a generic cloning in which the quantum information of an arbitrary state in \( d \) dimensions with its information not completely known can be optimally distributed in different systems. By comparison of the fidelity distributions of four asymmetric cloners, we derive some interesting results.

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