Abstract

We express a matrix version of the self-induced transparency (SIT) equations in the bidifferential calculus framework. An infinite family of exact solutions is then obtained by application of a general result that generates exact solutions from solutions of a linear system of arbitrary matrix size. A side result is a solution formula for the sine-Gordon equation.

1. Introduction. The bidifferential calculus approach (see [1] and the references therein) aims to extract the essence of integrability aspects of integrable partial differential or difference equations (PDDEs) and to express them, and relations between them, in a universal way, i.e. resolved from specific examples. A powerful, though simple to prove, result [1–3] (see section 6) generates families of exact solutions from a matrix linear system. In the following we briefly recall the basic framework and then apply the latter result to a matrix generalization of the SIT equations.

2. Bidifferential calculus. A graded algebra is an associative algebra Ω over C with a direct sum decomposition Ω = ⨁₀≥₀ Ωᵣ into a subalgebra A := Ω₀ and A-bimodules Ωᵣ, such that Ωᵣ Ωₛ ⊆ Ωᵣ+s. A bidifferential calculus (or bidifferential graded algebra) is a unital graded algebra Ω equipped with two (C-linear) graded derivations d, d̄ : Ω → Ω of degree one (hence dΩᵣ ⊆ Ωᵣ₊₁, d̄Ωᵣ ⊆ Ωᵣ₊₁), with the properties

\[ d_z^2 = 0 \quad \forall z \in \mathbb{C}, \quad \text{where} \quad d_z := d - z d, \]  

and the graded Leibniz rule \( d_z(\chi \chi') = (d_z \chi) \chi' + (-1)^r \chi d_z \chi' \), for all \( \chi \in \Omega^r \) and \( \chi' \in \Omega \).

3. Dressing a bidifferential calculus. Let (Ω, d, d̄) be a bidifferential calculus. Replacing \( d_z \) in (1) by \( D_z := d - A - z d \) with a 1-form \( A \in \Omega^1 \) (in the expression for \( D_z \) to be regarded as a multiplication operator), the resulting condition \( D_z^2 = 0 \) (for all \( z \in \mathbb{C} \)) can be expressed as

\[ dA = 0 = dA - AA. \]  

If (2) is equivalent to a PDDE, we have a bidifferential calculus formulation for it. This requires that \( A \) depends on independent variables and the derivations \( d, d̄ \) involve differential or difference
operators. Several ways exist to reduce the two equations (2) to a single one:

1. We can solve the first of (2) by setting $A = d\phi$. This converts the second of (2) into

$$\bar{d}d\phi = d\phi d\phi. \tag{3}$$

2. The second of (2) can be solved by setting $A = (\bar{d}g)g^{-1}$. The first equation then reads

$$d\left((\bar{d}g)g^{-1}\right) = 0. \tag{4}$$

3. More generally, setting $A = [\bar{d}g - (dg)\Delta]g^{-1}$, with some $\Delta \in A$, we have $\bar{d}A - AA = (dA)\Delta g^{-1} + (dg)\Delta g^{-1}$. As a consequence, if $\Delta$ is chosen such that $\bar{d}\Delta = (d\Delta)\Delta$,

$$d\left([\bar{d}g - (dg)\Delta]g^{-1}\right) = 0. \tag{5}$$

With the choice of a suitable bidifferential calculus, (3) and (4), or more generally (5), have been shown to reproduce quite a number of integrable PDDEs. This includes the self-dual Yang-Mills equation, in which case (3) and (4) correspond to well-known potential forms [1]. Having found a bidifferential calculus in terms of which e.g. (3) is equivalent to a certain PDDE, it is not in general guaranteed that also (4) represents a decent PDDE. Then the generalization (5) has a chance to work (cf. [1]). In such a case, the Miura transformation

$$[\bar{d}g - (dg)\Delta]g^{-1} = d\phi \tag{6}$$

is a hetero-Bäcklund transformation relating solutions of the two PDDEs.

Bäcklund, Darboux and binary Darboux transformations can be understood in this general framework [1], and there is a construction of an infinite set of (generalized) conservation laws. Exchanging $d$ and $\bar{d}$ leads to what is known in the literature as ‘negative flows’ [3].

4. A matrix generalization of SIT equations and its Miura-dual. $A = \text{Mat}(n,n,C^\infty(\mathbb{R}^2))$ denotes the algebra of $n \times n$ matrices of smooth functions on $\mathbb{R}^2$. Let $\Omega = A \otimes \Lambda^1(\mathbb{C}^2)$ with the exterior algebra $\Lambda^1(\mathbb{C}^2)$ of $\mathbb{C}^2$. In terms of coordinates $x,y$ of $\mathbb{R}^2$, a basis $\zeta_1, \zeta_2$ of $\Lambda^1(\mathbb{C}^2)$, and a constant $n \times n$ matrix $J$, maps $d$ and $\bar{d}$ are defined as follows on $A$,

$$df = \frac{1}{2}[J,f] \otimes \zeta_1 + f_y \otimes \zeta_2, \quad \bar{d}f = f_x \otimes \zeta_1 + \frac{1}{2}[J,f] \otimes \zeta_2$$

(see also [4]). They extend in an obvious way (with $d\zeta_i = \bar{d}\zeta_i = 0$) to $\Omega$ such that $(\Omega,d,\bar{d})$ becomes a bidifferential calculus. We find that (3) is equivalent to

$$\phi_{xy} = \frac{1}{2}[[J,\phi],\phi_y - \frac{1}{2}J]. \tag{7}$$

Let $n = 2m$ and $J = \text{block-diag}(I,-I)$, where $I = I_m$ denotes the $m \times m$ identity matrix. Decomposing $\phi$ into $m \times m$ blocks, and constraining it as follows,

$$\phi = \begin{pmatrix} p & q \\ q & -p \end{pmatrix}, \tag{8}$$

(7) splits into the two equations

$$p_{xy} = (q^2)_y, \quad q_{xy} = q - p_y q - q p_y. \tag{9}$$
We refer to them as matrix-SIT equations (see section 5), not purporting that they have a similar physical relevance as in the scalar case. The Miura transformation \([6]\) (with \(\Delta = 0\)) now reads

\[
g_x g^{-1} = \frac{1}{2} [J, \phi], \quad \frac{1}{2} [J, g] g^{-1} = \phi_y .
\] (10)

Writing

\[
g = \begin{pmatrix} a & b \\ c & d \end{pmatrix},
\]

with \(m \times m\) matrices \(a, b, c, d\), and assuming that \(a\) and its Schur complement \(S(a) = d - ca^{-1}b\) is invertible (which implies that \(g\) is invertible), \([10]\) with \([8]\) requires

\[
b = -ca^{-1}d, \quad a_x = -ca^{-1}c, \quad d_x = -ca^{-1}ca^{-1}d .
\] (11)

The last equation can be replaced by \(d_x d^{-1} = a_x a^{-1}\). Invertibility of \(S(a)\) implies that \(d\) and \(I + r^2\) are invertible, where \(r := ca^{-1}\). The conditions \([11]\) are necessary in order that the Miura transformation relates solutions of \([9]\) to solutions of its ‘dual’

\[
(g_x g^{-1})_y = \frac{1}{2} [gJg^{-1}, J],
\] (12)

obtained from \([6]\). Taking \([11]\) into account, the Miura transformation reads

\[
q = -ca^{-1} = -r_x - ra_x a^{-1}, \quad q_y = -r(I + r^2)^{-1}, \quad p_y = I - (I + r^2)^{-1} .
\] (13)

As a consequence, we have

\[
q_y^2 + p_y^2 = p_y .
\] (14)

Furthermore, the second of \([11]\) and the first of \([13]\) imply \(a_x a^{-1} = qr\). Hence we obtain the system

\[
r_x = -q - qr, \quad q_y = -r(I + r^2)^{-1},
\] (15)

which may be regarded as a matrix or ‘noncommutative’ generalization of the sine-Gordon equation. There are various such generalizations in the literature. The first equation has the solution \(q = -\sum_{k=0}^{\infty} (-1)^k r^k r_x r^k\), if the sum exists. Alternatively, we can express this as \(q = -(I + r L r R)^{-1}(r_x)\), where \(r_L\) (\(r_R\)) denotes the map of left (right) multiplication by \(r\). This can be used to eliminate \(q\) from the second equation, resulting in

\[
((I + r L r R)^{-1}(r_x))_y = r(I + r^2)^{-1} .
\] (16)

If \(r = \tan(\theta/2) \pi\) with a constant projection \(\pi\) (i.e. \(\pi^2 = \pi\)) and a function \(\theta\), then \([16]\) reduces to the sine-Gordon equation

\[
\theta_{xy} = \sin \theta .
\] (17)

\([15]\) can be obtained directly from \([12]\) as follows, by setting

\[
g = \begin{pmatrix} a & -c \\ c & a \end{pmatrix} = \begin{pmatrix} I & -r \\ r & I \end{pmatrix} a, \quad \text{hence} \quad g^{-1} = a^{-1} \begin{pmatrix} I & r \\ -r & I \end{pmatrix} (I + r^2)^{-1} .
\]

This leads to

\[
((r_x r + r \rho r + \rho)(I + r^2)^{-1})_y = 0, \quad ((r_x + r \rho - \rho r)(I + r^2)^{-1})_y = r(I + r^2)^{-1} ,
\]

where \(\rho := a_x a^{-1}\). Setting an integration ‘constant’ to zero, the first equation integrates to \(\rho = -r_x r - r \rho r\). With its help, the second can be written as \((r_x + r \rho)_y = r(I + r^2)^{-1}\). Since \(q = -(ra)_x a^{-1} = -r_x - r \rho\), this is the second of \([15]\). The first follows noting that \(qr = \rho\).
5. Sharp line SIT equations and sine-Gordon. We consider the scalar case, i.e. \( m = 1 \). Introducing \( E = 2 \sqrt{\alpha} q \) with a positive constant \( \alpha \), \( P = 2q_y, N = 2p_y - 1 \), and new coordinates \( z, t \) via \( x = \sqrt{\alpha}(z - t) \) and \( y = \sqrt{\alpha}z \), the system \([9]\) is transformed into

\[
P_t = E N, \quad N_t = -E P,
\]

and the relation between \( E \) and \( P \) takes the form

\[
E_z + E_t = \alpha P.
\]

These are the sharp line self-induced transparency (SIT) equations \([5–7]\). We note that \( P^2 + N^2 \) is conserved. Indeed, as a consequence of \((14)\), we have

\[
P \text{ constant}.
\]

These are precisely the equations that result from the Miura transformation \((10)\) (or from \((13)\)). As a consequence of the above relations, \( q \) and \( p \) depend as follows on \( \theta \),

\[
q = -\frac{1}{2} \theta_x, \quad q_y = -\frac{1}{2} \sin \theta, \quad p_y = \frac{1}{2}(1 - \cos \theta).
\]

These are precisely the equations that result from the Miura transformation \((10)\) (or from \((13)\)), choosing

\[
g = \begin{pmatrix} \cos \frac{\theta}{2} & -\sin \frac{\theta}{2} \\ \sin \frac{\theta}{2} & \cos \frac{\theta}{2} \end{pmatrix},
\]

and \((12)\) becomes the sine-Gordon equation \((17)\) (cf. \([6]\)). As a consequence of the above relations, \( q \) and \( p \) depend as follows on \( \theta \),

\[
q = -\frac{1}{2} \theta_x, \quad q_y = -\frac{1}{2} \sin \theta, \quad p_y = \frac{1}{2}(1 - \cos \theta).
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These are precisely the equations that result from the Miura transformation \((10)\) (or from \((13)\)), choosing

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\]

and \((12)\) becomes the sine-Gordon equation \((17)\). The conditions \((11)\) are identically satisfied as a consequence of the form of \( g \).

6. A universal method of generating solutions from a matrix linear system.

**Theorem 1.** Let \((\Omega, d, \bar{d})\) be a bidifferential calculus with \( \Omega = A \otimes \wedge(C^2) \), where \( A \) is the algebra of matrices with entries in some algebra \( B \) (where the product of two matrices is defined to be zero if the sizes of the two matrices do not match). For fixed \( N, N' \), let \( X \in \text{Mat}(N, N, B) \) and \( Y \in \text{Mat}(N', N, B) \) be solutions of the linear equations

\[
dX = (dX) P, \quad dY = (dY) P, \quad RX - XP = -QY,
\]

with \( d\)- and \( \bar{d}\)-constant matrices \( P, R \in \text{Mat}(N, N, B) \), and \( Q = \bar{V} \bar{U} \), where \( \bar{U} \in \text{Mat}(n, N', B) \) and \( \bar{V} \in \text{Mat}(N, n, B) \) are \( d\)- and \( \bar{d}\)-constant. If \( X \) is invertible, the \( n \times n \) matrix variable

\[
\phi = \bar{U}YX^{-1} \bar{V} \in \text{Mat}(n, n, B)
\]

solves \( \bar{d}\phi = (d\phi) \phi + d\theta \) with \( \theta = \bar{U}YX^{-1} R \bar{V} \), hence (by application of \( d \)) also \((3)\).

There is a similar result for \((5)\) \((3)\). The Miura transformation is a corresponding bridge.

7. Solutions of the matrix SIT equations. From Theorem \([4]\) we can deduce the following result, using straightforward calculations \([8]\), analogous to those in \([2]\) (see also \([3]\)).

**Proposition 2.** Let \( S \in \text{Mat}(M, M, \mathbb{C}) \) be invertible, \( U \in \text{Mat}(m, M, \mathbb{C}) \), \( V \in \text{Mat}(M, m, \mathbb{C}) \), and \( K \in \text{Mat}(M, M, \mathbb{C}) \) a solution of the Sylvester equation

\[
SK + KS = VU.
\]

Then, with \( \Xi = e^{-Sx - S^{-1}y} \) and any \( p_0 \in \text{Mat}(m, m, \mathbb{C}) \) (more generally \( x \)-dependent),

\[
q = U \Xi (I + (K \Xi)^2)^{-1} V, \quad p = p_0 - U \Xi K \Xi (I + (K \Xi)^2)^{-1} V
\]

(assuming the inverse exists) is a solution of \((9)\).
If the matrix $S$ satisfies the spectrum condition

$$\sigma(S) \cap \sigma(-S) = \emptyset$$

(21)

(where $\sigma(S)$ denotes the set of eigenvalues of $S$), then the Sylvester equation $[19]$ has a unique solution $K$ (for any choice of the matrices $U,V$), see e.g. [9].

By a lengthy calculation [8] one can verify directly that the solutions in Proposition 2 satisfy (14). Alternatively, one can show that these solutions actually determine solutions of the Miura transformation (cf. [3]), and we have seen that (14) is a consequence.

There is a certain redundancy in the matrix data that determine the solutions (20) of (9). This requires that $K$ is invertible. More generally, such a reflexion can be applied to any Jordan block of $S$ and then changes the sign of its eigenvalue [8] (see also [2,10]). The Jordan normal form can be restored afterwards via a similarity transformation.

The following result is easily verified [8].

Proposition 3. Let $S,U,V$ be as in Proposition 2 and $T \in \text{Mat}(M,M,\mathbb{C})$ invertible.

(1) Let $T$ be Hermitian (i.e. $T^\dagger = T$) and such that $S^\dagger = TST^{-1}$, $U = V^\dagger T$. Let $K$ be a solution of (19), which can then be chosen such that $K^\dagger = TKT^{-1}$. Then $q$ and $p$ given by (20) with $p_0 = p_0$ are both Hermitian and thus solve the Hermitian reduction of (9).

(2) Let $T = T^{-1}$ (where the bar means complex conjugation) and $S = TST^{-1}$, $U = UT^{-1}$ and $V = TV$. Let $K$ be a solution of (19), which can then be chosen such that $K = TKT^{-1}$. Then $q$ and $p$ given by (20) with $p_0 = p_0$ satisfy $\bar{q} = q$ and $\bar{p} = p$, and thus solve the complex conjugation reduction of (9).

8. Rank one solutions. Let $M = 1$. We write $S = s$, $U = u$, $V = v^\dagger$, $K = k$ (where $^\dagger$ means the transpose) and $\Xi = \xi = e^{−sx−s^−y}$. Then (19) yields $k = (v^\dagger u)/(2s)$. From (20) we obtain

$$q = \frac{2sk\xi}{1+(k\xi)^2} \pi, \quad p = \bar{p}_0 + \frac{2s}{1+(k\xi)^2} \pi, \quad \bar{p}_0 := p_0 - 2s \pi, \quad \pi := \frac{uv^\dagger}{v^\dagger u}.$$ 

The Miura transformation (13) implies $r = -q_y (I-p_y)^{-1}$, and we obtain

$$r = -\frac{2k\xi}{1-(k\xi)^2} \pi,$$

which is singular. But $\theta = -2 \arctan(2k\xi/|1-(k\xi)^2|)$ is the single kink solution of the sine-Gordon equation (17).
9. Solutions of the scalar (sharp line) SIT equations. We rewrite \( p \) in (20), where now \( m = 1 \), as follows,

\[
p = p_0 - \text{tr} \left( (SK + KS) \Xi K \Xi (I_M + (K \Xi)^2)^{-1} \right)
= p_0 + \text{tr} \left( (I_M + (K \Xi)^2) x (I_M + (K \Xi)^2)^{-1} \right)
= p_0 + \left( \log \left| I_M + (K \Xi)^2 \right| \right)_x,
\]

using (19) and the identity \( (\det M)_x = \text{tr}(M_x M^{-1}) \det M \) for an invertible matrix function \( M \). \( q \) in (20) can be expressed as

\[
q = 2 \text{tr} \left( SK \Xi (I_M + (K \Xi)^2)^{-1} \right).
\]

In particular, if \( S \) is diagonal with eigenvalues \( s_i, i = 1, \ldots, M \), and satisfies (21), then the solution \( K \) of the Sylvester equation (19), which now amounts to \( \text{rank}(SK + KS) = 1 \), is the Cauchy-type matrix with components \( K_{ij} = u_i u_j / (s_i + s_j) \), where \( u_i, v_i \in \mathbb{C} \). Figs. 1 and 2 show plots of two examples from the above family of solutions.

\[\text{Figure 1: A scalar 2-soliton solution with } S = \text{diag}(1, 2) \text{ and } u_i = v_i = 1.\]

\[\text{Figure 2: A scalar breather solution with } S = \text{diag}(1 + i, 1 - i) \text{ and } u_i = v_i = 1.\]

10. A family of solutions of the real sine-Gordon equation. Via the Miura transformation (18), Proposition 2 determines a family of sine-Gordon solutions (see also e.g. [6,11–16] for related results obtained by different methods).

**Proposition 4.** Let \( S \in \text{Mat}(M, M, \mathbb{C}) \) be invertible and \( K \in \text{Mat}(M, M, \mathbb{C}) \) such that \( \text{rank}(SK + KS) = 1 \), \( \det(I_M + (K \Xi)^2) \in \mathbb{R} \) with \( \Xi = e^{-Sx - S^{-1}y} \), and \( \text{tr}(SK \Xi (I_M + (K \Xi)^2)^{-1}) \notin i\mathbb{R} \) (where \( i \) is the imaginary unit). Then

\[
\theta = 4 \arctan \left( \frac{\sqrt{\beta}}{1 + \sqrt{1 - \beta}} \right) \quad \text{with} \quad \beta := (\log |\det(I_M + (K \Xi)^2)|)_{xy}
\]

solves the sine-Gordon equation \( \theta_{xy} = \sin \theta \) in any open set of \( \mathbb{R}^2 \) where \( \det(I_M + (K \Xi)^2) \neq 0. \)
Proof: Let \( p \) be given by (22). Due to the assumption \( \det(I_M + (K\Xi)^2) \in \mathbb{R} \), \( p_y \) is real, hence (14) implies \( |1 - 2p_y|^2 = 1 - 4q_y^2 \). It follows that \( q_y^2 \) is real. Since another of our assumptions excludes that \( q_y \) is imaginary, it follows that \( |1 - 2p_y| \leq 1 \). Hence the equation \( \cos \theta = 1 - 2p_y \) (second of (18)) has a real solution \( \theta \). Inserting the expression (22) for \( p \), we arrive at \( \cos \theta = 1 - 2 \left( \log \det(I_M + (K\Xi)^2) \right)_{xy} \). Moreover, (14) shows that \( p_y \geq 0 \) and thus \( 0 \leq p_y \leq 1 \). Using identities for the inverse trigonometric functions, we find (23), where \( \beta = p_y \). □

Proposition 3 yields sufficient conditions on the matrix data for which the last two assumptions in Proposition 4 are satisfied.

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