$p$-integrable solutions to multidimensional BSDEs and degenerate systems of PDEs with logarithmic nonlinearities.

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Abstract

We study multidimensional backward stochastic differential equations (BSDEs) which cover the logarithmic nonlinearity $u \log u$. More precisely, we establish the existence and uniqueness as well as the stability of $p$-integrable solutions ($p > 1$) to multidimensional BSDEs with a $p$-integrable terminal condition and a super-linear growth generator in the both variables $y$ and $z$. This is done with a generator $f(y, z)$ which can be neither locally monotone in the variable $y$ nor locally Lipschitz in the variable $z$. Moreover, it is not uniformly continuous. As application, we establish the existence and uniqueness of Sobolev solutions to possibly degenerate systems of semilinear parabolic PDEs with super-linear growth generator and an $p$-integrable terminal data. Our result cover, for instance, certain (systems of) PDEs arising in physics.

1 Introduction

The logarithmic nonlinearity $u \log u$ appears in certain differential equations arising in physics (see e. g. [13, 14, 22, 35, 54]) and in the theory of continuous-state branching processes (see e. g. [12, 32, 33]). For instance, the Cauchy problem

$$
\begin{cases}
\frac{\partial u}{\partial t} - \Delta u + u \log u = 0 \quad \text{on} \quad (0, \infty) \times \mathbb{R}^d \\
u(0^+) = \varphi > 0
\end{cases}
$$

(1.1)

is related to super processes with Neveu’s branching mechanism, see e. g [32]. On the other hand, the logarithmic nonlinearity is also interesting in itself since it is neither locally monotone nor uniformly continuous. In this paper, we give a BSDEs approach which allows to cover this kind of nonlinearity.

Let $(W_t)_{0 \leq t \leq T}$ be a $r$-dimensional Wiener process defined on a complete probability space $(\Omega, \mathcal{F}, P)$. $(\mathcal{F}_t)_{0 \leq t \leq T}$ denote the natural filtration of $(W_t)$ such that $\mathcal{F}_0$ contains all $P$-null sets of $\mathcal{F}$, and $\xi$ be an $\mathcal{F}_T$-measurable $d$-dimensional random variable. Let $f$ be an $\mathbb{R}^d$-valued function defined on $[0, T] \times \Omega \times \mathbb{R}^d \times \mathbb{R}^{d \times r}$ such that for every $(y, z) \in \mathbb{R}^d \times \mathbb{R}^{d \times r}$, the map $(t, \omega) \rightarrow f(t, \omega, y, z)$ is $\mathcal{F}_t$-progressively measurable. The BSDEs under consideration are of the form,

$$(E(\xi, f))$$

$$Y_t = \xi + \int_t^T f(s, Y_s, Z_s) \, ds - \int_t^T Z_s \, dW_s \quad 0 \leq t \leq T$$

The data $\xi$ and $f$ are respectively called the terminal condition and the coefficient or generator.

The present paper is a development of [?], and it constitute a natural continuation of our previous works [1, 2, 3]. To begin with, we give a summarized historic on BSDEs : the linear version of equation $(E(\xi, f))$ has appeared long time ago, both as the equation for the adjoint process in stochastic control (see e.g. [16]), as well as the model behind the Black and Scholes formula for the pricing and hedging of options in mathematical finance, see e.g. [17, 15]. Since the paper [51], where the existence and
uniqueness of solutions have been established for the equation \((E^{(\xi,f)})\) with a uniformly Lipschitz generator \(f\) and a square integrable terminal data \(\xi\), the theory of BSDE has found further important applications and has become a powerful tool in many fields such as financial mathematics, optimal control and stochastic game, non-linear PDEs ... etc. The collected texts \([27]\) give a useful introduction to the theory of BSDEs and some of their applications. See also \([8,9,10,19,28,36,17,48,49,50,55]\) and the references therein for more discussions on BSDEs and their relations with PDEs and mathematical finances. Many authors have attempted to improve the result of \([51]\) by weakening the Lipschitz continuity of the coefficient \(f\) (see e.g \([1,2,3,44]\)) and/or the \(L^2\)-integrability of the initial data \(\xi\) \((18,28)\). Another direction in the BSDEs theory has been developed by introducing the notion of weak solutions, i.e. a solution which can be not adapted to the filtration generated by the driver processes (see e.g \([6,20,21]\)). A step forward has been done in the paper \([20]\) where the Meyer-Zheng topology has been used, to prove the existence of weak solutions for BSDE with continuous generator. More recently, the link between the solution of BSDEs and the "\(L_p\)-viscosity solution" for PDEs with discontinuous coefficients, has been established in \([7]\).

The essential difficulty, to establish the existence of strong (i.e. \(\mathcal{F}_t^W\)-adapted) solutions to BSDEs with local conditions on the generator \(f\), is due to the fact that the control variable \(Z\) is known implicitly, by Itô's martingale representation theorem as the integrand of a Brownian stochastic integral. Actually, we need more information on the variable \(Z\). Consequently, the usual localization procedure (by stopping times) does not work. On the other hand, the methods used to study the existence and/or uniqueness of strong solutions to one-dimensional BSDEs are mainly based on comparison techniques and therefore do not work for multidimensional equations. We cite only a few articles in this area (for instance \([11,29,31,37]\)) sending the reader to the references therein again because neither do we deal with one-dimensional BSDEs nor use the results of these papers. Note however that, although we are focused in the multidimensional equations, our uniqueness result is new even in the one-dimensional case. The first results which deal with the existence and uniqueness as well as the stability of strong solutions for multidimensional BSDEs with local assumptions on the coefficient \(f\) have been established in \([1,2,3]\).

The present work constitute a natural development of \([1,2,3,7]\). To begin, let \(\xi\) be \(p\)-integrable with \(p > 1\), \(K\) be a positive constant, and consider the following example of \(d\)-dimensional BSDE with logarithmic nonlinearity,

\[
Y_t = \xi - \int_t^T K Y_s \log |Y_s| \, ds - \int_t^T Z_s \, dW_s \quad 0 \leq t \leq T
\]

(1.2)

It is worth noting that the coefficient \(f(y) := -K y \log |y|\) of equation (1.2), is not locally monotone and hence not locally Lipschitz. Moreover, its growth is big power than \(y\). In our knowledge, when \(\xi\) is \(p\)-integrable with \(1 < p < 2\), there is no results on multidimensional BSDEs which cover this interesting example. To explain how the BSDE (1.2) follows naturally from \([1,2]\), consider the BSDE \((E^{(\xi,f)})\) with square integrable \(\xi\) and, assume for the simplicity that the generator \(f\) does not depend on the variable \(z\). Let \(f\) be \(L_N\)-locally Lipschitz and with sublinear growth. It has been established in \([1,2]\) that if \(L_N\) behaves as \(\log N\), then the BSDE \((E^{(\xi,f)})\) has a unique strong solution which is \(L^2\)-stable. Now, if we drop the sublinear growth condition on \(f\), then the condition \(L_N \sim \log N\) implies that \(|f(y)| \leq K (1 + |y| \log |y|)\) for some positive constant \(K\). Hence, the following questions arise : could the BSDE with generator \(f(y) = -y \log |y|\) has a strong solution? If yes, what happens about the uniqueness and the stability of solutions? These questions are positively solved in this paper, as particular examples.

The first main purpose of the present work consists to establish a result on the existence and uniqueness as well as the stability of strong solutions to BSDE \((E^{(\xi,f)})\) which cover equation (1.2) as well as, other interesting examples which are, in our knowledge, not covered by the previous works. For instance, we establish the existence and uniqueness of a strong solution to BSDE \((E^{(\xi,f)})\) in the case where the terminal data \(\xi\) is merely \(p\)-integrable (with \(p > 1\)) and the coefficient \(f\) could be neither locally monotone in \(y\) nor locally Lipschitz in \(z\). Moreover, \(f\) can has a super-linear growth in its two variables \(y\) and \(z\). For example, \(f\) can take the form \(f(y,z) = -y \log |y| + g(y)(h(z)\sqrt{\log |z|})\) for some functions \(g : \mathbb{R}^d \rightarrow \mathbb{R}^d\) and \(h : \mathbb{R}^d \times \mathbb{R}^d \rightarrow \mathbb{R}^d\). The assumptions which we impose on \(f\) are local in \(y, z\) and also in \(\omega\). This enables us to cover certain BSDEs with stochastic monotone generators. Our
uniqueness result is new even in the one-dimensional case.

The BSDEs with $p$--integrable terminal data $\xi$ (with $1 < p < 2$), have been studied in [28] in the case where the coefficient $f$ is uniformly Lipschitz in their two variables $(y, z)$, and in [13] in the case where $f$ is uniformly Lipschitz in the variable $z$ and uniformly monotone in the variable $y$. It should be noted that our result cover those of [13, 28] with new proofs. Our method allows, for instance, to treat simultaneously the existence and uniqueness as well as, the $L^p$--stability of solutions by using the same computations.

The techniques which is usually developed in BSDEs consist to applying Itô’s formula to the function $h(y) = |y|^2$ or $h(t, y) = |y|^2 \exp(\alpha t)$ with $\alpha > 0$ in order to estimate the difference between two solutions by the difference between their respective data. Such estimates are not possible in our situation since our assumptions on the generator are merely local. Moreover, due to the super-linear growth and the singularity of the generator, the techniques used in [1, 2] can not be easily extended to our situation. Our proofs mainly consist to establish a non standard a priori estimate between two solutions by applying Itô’s formula to an appropriate function. The existence (of solutions) is then deduced by using a suitable approximation $(\xi_n, f_n)$ of $(\xi, f)$ and an appropriate localization procedure which is close to those given in [1, 2, 3]. However, in contrast to [3], we don’t use the $L^2$-weak compactness of the approximating sequence $(Y^n, Z^n)$. We directly show that the sequence $(Y^n, Z^n)$ strongly converges in some $L^q$ space ($1 < q < 2$) and, the limit satisfies the BSDE $(E(\xi, f))$. The uniqueness as well as the stability of solutions are then deduced by using the same estimates. The results are first established for a small time, and next, for an arbitrarily prescribed time duration by using a continuation procedure.

To deal with the PDEs part, we consider the Markovian version of the BSDE (1.2), which is defined for $0 \leq t \leq s \leq T$ by the system of SDE-BSDE,

\[
\begin{align*}
X_t = x + \int_t^s b(X_r)dr + \int_t^s \sigma(X_r)dB_r, \\
Y_t = H(X_t) - \int_t^T KY_r \log |Y_r|dr - \int_t^T Z_r dB_r,
\end{align*}
\]

(1.3)

where $\sigma : \mathbb{R}^k \mapsto \mathbb{R}^d$, $b : \mathbb{R}^k \mapsto \mathbb{R}^d$, $H : \mathbb{R}^k \mapsto \mathbb{R}^d$ are measurable functions and $K$ is a real positive number.

The system of PDEs associated to the SDE-BSDE (1.3) is then given by

\[
\frac{\partial u(t, x)}{\partial t} + Lu(t, x) - Ku(t, x) \log |u(t, x)| = 0 , \quad u(T, x) = g(x)
\]

(1.4)

where $L := \frac{1}{2} \sum_{i,j} (\sigma \sigma^\ast)_{ij} \partial_{x_{ij}} + \sum_i b_i \partial_{x_i}$ and $g$ is a given measurable function.

The logarithmic nonlinearities $Ku \log |u|$ of the equation (1.4) appear in some PDEs related to physics, see e.g. [13, 14, 22, 35, 54]. In the mathematical point of view, as indicated in [22], the nonlinear term $u \log |u|$ is not continuous on a reasonable functions space. This induces a supplementary difficulty which makes no efficient some standard arguments (local existence and global estimates) to prove existence of solutions. On the other hand, it should be noted that, due to the degeneracy of the diffusion coefficient, the solutions will not be smooth enough, and therefore the uniqueness is rather hard to establish.

In the second part of this paper, we are concerned with the probabilistic approach to Sobolev solutions of semilinear PDEs associated with the Markovian version of BSDE $(E(\xi, f))$. The links between strong solutions of BSDEs and Sobolev solutions of semilinear PDEs were firstly established in [10]. Similar result was established in [8], for the relations between Backward Doubly SDEs (BDSDEs) and SPDEs. The common of these two papers is that the nonlinear term $f$ is at least uniformly Lipschitz with sub-linear growth.

The second main purpose of this paper consists to establish a result on the existence and uniqueness of Sobolev solutions for the (possibly degenerate) system of PDEs associated to BSDE $(E(\xi, f))$. Our result cover equation (1.4) and many other examples. We develop a method which allows to prove the uniqueness of the PDE by means of the uniqueness of its associated BSDE: we first prove the existence and uniqueness in the class of solutions which are representable by BSDEs, and next we show that any solution is unique. To do this, we first prove that $0$ is the unique solution to the homogeneous PDE, and
next we use the BSDEs to establish an equivalence between the uniqueness for the non-homogeneous semilinear PDE and the uniqueness for its associated homogeneous linear PDE. More precisely, denoting by $\mathcal{L}$ the second order parabolic operator associated to a given $\mathbb{R}^d$-diffusion process, we prove that the system of semilinear PDEs
\[
\begin{aligned}
\frac{\partial u(t, x)}{\partial t} + \mathcal{L} u(t, x) + f(t, x, u(t, x), \nabla u(t, x)) &= 0, & t \in [0, T[, \ x \in \mathbb{R}^k \\
\ u(T, x) &= g(x), & x \in \mathbb{R}^k
\end{aligned}
\]
has a unique solution if and only if $0$ is the unique solution of the linear system
\[
\begin{aligned}
\frac{\partial u(t, x)}{\partial t} + \mathcal{L} u(t, x) &= 0, & t \in [0, T[, \ x \in \mathbb{R}^k \\
\ u(T, x) &= 0, & x \in \mathbb{R}^k
\end{aligned}
\]
This seems to be new in the BSDEs framework. Not also that, in order to prove the uniqueness of the above homogeneous linear PDEs, a uniform gradient estimate for some possibly degenerate PDEs is established by a probabilistic method, which is interesting itself.

We mention some others considerations which have motivated the present work.

- The growth conditions on the nonlinearity constitute a critical case in the sense that, for any $\varepsilon > 0$, the solutions of the ordinary differential equation $X_t = x + \int_0^t X_1^{1+\varepsilon} ds$ explode at a finite time.
- The logarithmic nonlinearities appear in some PDEs arising in physics, see e.g. [13], [14], [22], [35], [54], [57]. For instance, in [13] the construction of nonlinear wave quantum mechanics, based on Schrödinger-type equation, is with nonlinearity $-ku \ln(|u|^2)$. This nonlinearity is selected by assuming the factorization of wave functions for composed systems. Its most attractive features are: existence of the lower energy bound. Moreover, it is the only one nonlinearity satisfying the validity of Planck’s relation $E[\psi] = \hbar \psi$ for stationary states $\psi$.

- In terms of continuous-state branching processes, the logarithmic nonlinearity $u \log u$ corresponds to the Neveu branching mechanism. This process was introduced by Neveu in [46], and further studied in [12], [32], [33]. For instance, the super-process with Neveu’s branching mechanism constructed in [32] is related to the Cauchy problem,
\[
\begin{aligned}
\frac{\partial u}{\partial t} - \Delta u + u \log u &= 0 \quad \text{on} \ (0, \infty) \times \mathbb{R}^d \\
\ u(0^+) &= \varphi > 0
\end{aligned}
\]
Hence, our result can be seen as an alternative approach to the PDEs (1.5), and cover the case where the diffusion part is possibly degenerate.

- Since the system of PDEs associated to the Markovian version of the BSDE $(E(\xi, f))$ can be degenerate, our result also covers certain systems of first order PDEs.

- Thanks to the possible degeneracy of the diffusion coefficient, our proposition 4.2 cover for instance the PDE studied in [57] which arises in studying the motion of a particle acting under a force perturbed by noise.

- The method, which we develop to study the system of semilinear PDEs, is based on BSDEs and, both our results as well as their proofs are new, particularly the proof of the uniqueness.

- The BSDEs as well as the PDEs which we consider are interesting in themselves since the nonlinear part $f(t, y, z)$ can be neither locally monotone in $y$ nor locally Lipschitz in $z$. Moreover, $f$ can be big power than $y$ and $z$, and therefore it is not uniformly continuous in $(y, z)$.

- It is worth noting that our condition on the coefficient $f$ is new even for the classical Itô’s forward SDEs. For instance, we do not know whether or not the following equation (1.6) possesses a pathwise unique solution.
\[
X_s = x + \int_0^s X_r \log |X_r| dr + \int_0^s |X_r| \sqrt{\log |X_r|} dW_r, \quad 0 \leq s \leq T
\]
It should be noted that the SDE (1.6) is not covered by [30]. We think that the method developed in the present paper may be used to solve this question. We are currently working on the SDEs (1.6) since the stochastic flows of homeomorphisms defined by these type of SDEs seem be related to the construction of a metric in the Hlder-Sobolev space $H^{1, \mathbb{R}}$, see [43].
The paper is organized as follows. In section 2, we present the assumptions and the main result of the first part. We also give some examples. Section 3 is devoted to the proof of the main result. In section 4, we deal with PDEs: we study the existence and uniqueness of a weak (Sobolev) \( p \)-integrable solution to systems of degenerate semilinear PDEs whose nonlinearities are big power than \( u \) and \( \nabla u \). We also establish, in this section, an equivalence between the uniqueness for non-homogeneous semi-linear PDEs and the uniqueness for its associated homogeneous linear PDEs.

2 Definition, assumptions, main result and examples.

Throughout this paper, \( p > 1 \) is an arbitrary fixed real number and all the considered processes are \( (\mathcal{F}_t) \)-predictable.

2.1 Definition.

A solution of equation \( (E(\xi,f)) \) is an \( (\mathcal{F}_t) \)-adapted and \( \mathbb{R}^{d+dr} \)-valued process \( (Y,Z) \) such that

\[
\mathbb{E}\left( \sup_{t \leq T} |Y_t|^p + \left( \int_0^T |Z_s|^2 ds \right)^{\frac{p}{2}} + \int_0^T |f(s,Y_s,Z_s)|ds \right) < +\infty
\]

and satisfies \( (E(\xi,f)) \).

2.2 Assumptions

We consider the following assumptions on \( (\xi,f) \):

There exist \( M \in L^0(\Omega;L^1([0,T];\mathbb{R}^+)) \), \( K \in L^0(\Omega;L^2([0,T];\mathbb{R}^+)) \) and \( \gamma \in \langle 0, \frac{1 \wedge (p-1)}{2} \rangle \), such that (with \( \lambda_s := 2M_s + \frac{K^2}{2\gamma} \)) we have,

\[ (H.0) \quad \mathbb{E} |\xi|^p e^{\frac{1}{2} \int_0^T \lambda_s ds} < \infty, \]

\[ (H.1) \quad f \text{ is continuous in } (y,z) \text{ for almost all } (t,\omega) \]

\[ (H.2) \quad \text{There exist } \eta \text{ and } f^0 \in L^0(\Omega \times [0,T];\mathbb{R}^+) \text{ satisfying } \mathbb{E} \left( \int_0^T e^{\int_0^s \lambda_r dr} \eta_r ds \right)^{\frac{p}{2}} < \infty, \quad \mathbb{E} \left( \int_0^T e^{\int_0^s \lambda_r dr} f^0_r ds \right)^p < \infty \]

and such that:

\[ \text{for every } t, y, z, \quad (y, f(t,y,z)) \leq \eta_t + f^0_t |y| + M_t |y|^2 + K_t |y| |z| \]

\[ (H.3) \quad \text{There exist } \underline{\eta} \in L^q(\Omega \times [0,T];\mathbb{R}^+) \text{ for some } q > 1 \text{ and } \alpha \in [1,p], \alpha' \in [1,p \wedge 2] \text{ such that:} \]

\[ \text{for every } t, y, z, \quad |f(t,\omega, y, z)| \leq \underline{\eta}_t + |y|^\alpha + |z| |y|^{\alpha'} \]

\[ (H.4) \quad \text{There exist } \underline{\eta} \in L^q(\Omega \times [0,T];\mathbb{R}^+) \text{ for some } q' > 0 \text{ and } K' \in \mathbb{R}^+ \text{ such that for every } N \in \mathbb{N} \text{ and every } y, y', z, z' \text{ satisfying } \|y\|, \|y'\|, \|z\|, \|z'\| \leq N \]

\[ (y - y', f(t,\omega, y, z) - f(t,\omega, y', z')) \mathbb{I}_{\{\nu(\omega) \leq N\}} \leq K' |y - y'|^2 \log A_N + \sqrt{K' \log A_N} \|y - y'\| \|z - z'\| + K' \frac{\log A_N}{A_N} \]

where \( A_N \) is an increasing sequence and satisfies \( A_N > 1, \lim_{N \to \infty} A_N = \infty \) and \( A_N \leq N^\mu \) for some \( \mu > 0 \).
2.3 The main result

**Theorem 2.1.** Assume that (H.0)-(H.4) hold. Then, \((E(\xi,f))\) has a unique solution \((Y,Z)\) which satisfies,

\[
\mathbb{E} \sup_t |Y_t|^p e^{\frac{p}{\varepsilon} \int_0^t |\lambda|^p ds} + \mathbb{E} \left[ \int_0^T e^{\frac{p}{\varepsilon} \int_0^s |\lambda|^p ds} |Z_s|^2 ds \right]^{\frac{p}{2}} \\
\leq C \left\{ \mathbb{E} \left[ |\xi|^p e^{\frac{p}{\varepsilon} \int_0^T |\lambda|^p ds} + \mathbb{E} \left( \int_0^T e^{\frac{p}{\varepsilon} \int_0^s |\lambda|^p ds} ds \right)^{\frac{p}{2}} + \mathbb{E} \left( \int_0^T e^{2 \frac{p}{\varepsilon} \int_0^s |\lambda|^p ds} ds \right)^{\frac{p}{2}} \right] \right\}
\]

for some constant \(C\) depending only on \(p\) and \(\gamma\).

We shall give some examples of BSDEs which satisfy the assumptions of Theorem 2.1. In our knowledge, these examples are not covered by the previous works in multidimensional BSDEs.

2.4 Examples.

**Example 1.** Let \(f(y) := -y \log y\) then for all \(\xi \in L^p(\mathcal{F}_T)\) the following BSDE has a unique solution

\[
Y_t = \xi - \int_t^T Y_s \log Y_s ds - \int_t^T Z_s dW_s.
\]

Indeed, \(f\) satisfies (H.1)-(H.3) since \(|y,f(y)| \leq 1\) and \(|f(y)| \leq 1 + \frac{1}{\varepsilon} |y|^{1+\varepsilon}\) for all \(\varepsilon > 0\). In order to verify (H.4), thanks to triangular inequality, it is sufficient to treat separately the two cases:

- \(0 \leq |y|, |y'| \leq \frac{1}{N}\) and \(\frac{1}{N} \leq |y|, |y'| \leq N\).

In the first case, since the map \(x \mapsto -x \log x\) increases for \(x \in [0,e^{-1}]\), we obtain for \(N > e\)

\[
|f(y) - f(y')| \leq |f(y)| + |f(y')| \\
\leq 2 \frac{\log N}{N}
\]

In the second case, the finite increments theorem applied to \(f\) shows that

\[
|f(y) - f(y')| \leq (1 + \log N) |y - y'|.
\]

Hence (H.4) is satisfied for every \(N > e\) with \(v_0 = 0\) and \(A_N = N\).

**Example 2.** Let \(g(y) := y \log \frac{|y|}{1+|y|}\) and \(h \in C(\mathbb{R}^d;\mathbb{R}) \cap C^1(\mathbb{R}^d - \{0\};\mathbb{R})\) be such that

\[
h(z) = \begin{cases} 
|z| \sqrt{-\log |z|} & \text{if } |z| < 1 - \varepsilon_0 \\
|z| \sqrt{\log |z|} & \text{if } |z| > 1 + \varepsilon_0 
\end{cases}
\]

where \(\varepsilon_0 \in [0,1]\). Finally, we put \(f(y,z) := g(y)h(z)\). Then for every \(\xi \in L^p(\mathcal{F}_T)\) the following BSDE has a unique solution

\[
Y_t = \xi + \int_t^T f(Y_s,Z_s)ds - \int_t^T Z_s dW_s.
\]

It is not difficult to see that \(f\) satisfies (H1). We shall prove that \(f\) satisfies (H2)-(H4).

(i) Since \(g\) is continuous, \(g(0) = 0\) and \(|g(y)|\) tends to 1 as \(|y|\) tends to \(\infty\), we deduce that \(g\) is bounded. Moreover, \(g\) satisfied \((y - y',g(y) - g(y')) \leq 0\). Indeed, in one dimensional case it is not difficult to show that \(g\) is a decreasing function. Since, \(-\langle y,y' \rangle \log \frac{|y|}{1+|y|} \leq -|y||y'| \log \frac{|y|}{1+|y|}\) (because
Consider the BSDE
\[
\langle c, T, k \rangle \text{ such that }
\]
and some positive constant 
\[
\log \frac{|y|}{1 + |y|} \leq 0,
\]
we can reduce the multidimensional case to the one dimension case by developing the inner product as follows,
\[
\langle y - y', g(y) - g(y') \rangle \leq |y|^2 \log \frac{|y|}{1 + |y|} + |y'|^2 \log \frac{|y'|}{1 + |y'|} - |y||y'| (\log \frac{|y|}{1 + |y|} + \log \frac{|y'|}{1 + |y'|})
\]
\[
= (|y| - |y'|) (|y| \log \frac{|y|}{1 + |y|} - |y'| \log \frac{|y'|}{1 + |y'|})
\]
\[
= (|y| - |y'|, g(|y|) - g(|y'|))
\]
\[
\leq 0
\]

(ii) The function \( h(z) \) satisfies for all \( \varepsilon > 0 \)
\[
0 \leq h(z) \leq M + \frac{1}{\sqrt{2\varepsilon}} \left| z \right|^{1+\varepsilon}, \quad \text{where } M = \sup_{\left| z \right| \leq 1+\varepsilon_0} |h(z)|
\]
The last inequality follows since \( \sqrt{2\varepsilon \log |z|} = \sqrt{\log |z|^\varepsilon} \leq |z|^\varepsilon \) for each \( \varepsilon > 0 \) and \( |z| > 1 \). (H3) follows now directly from the previous observations (i) and (ii). (H2) is satisfied since \( \langle y, f(y, z) \rangle = \langle y, g(y) \rangle h(z) \leq 0 \). To verify (H.4) it is enough to show that for every \( z, z' \) such that \( |z|, |z'| \leq N \)
\[
\left| h(z) - h(z') \right| \leq c \left( \sqrt{\log N} |z - z'| + \frac{\log N}{N} \right)
\]
for \( N \) large enough and some positive constant \( c \). This can be proved by considering separately the following five cases, \( 0 \leq |z|, |z'| \leq \frac{1}{N} \), \( \frac{1}{N} \leq |z|, |z'| \leq 1 - \varepsilon_0 \), \( 1 - \varepsilon_0 \leq |z|, |z'| \leq 1 + \varepsilon_0 \), and \( 1 + \varepsilon_0 \leq |z|, |z'| \leq N \).

In the first case (i.e. \( 0 \leq |z|, |z'| \leq \frac{1}{N} \)), since the map \( x \mapsto x\sqrt{\log x} \) increases for \( x \in \left[ 0, \frac{1}{\sqrt{\varepsilon}} \right] \), we obtain
\[
\left| h(z) - h(z') \right| \leq |h(z)| + |h(z')| \leq 2 \frac{1}{N} \sqrt{-\log \frac{1}{N}} \leq 2 \frac{1}{N} \log N \quad \text{for } N > \sqrt{\varepsilon}.
\]
The other cases can be proved by using the finite increments theorem.

**Example 3.** Let \( (X_t)_{t \leq T} \) be an \( \mathcal{F}_t \)-adapted and \( \mathbb{R}^k \)-valued process satisfying the forward stochastic differential equation
\[
X_t = X_0 + \int_0^t b(s, X_s) ds + \int_0^t \sigma(s, X_s) dW_s
\]
where \( X_0 \in \mathbb{R}^k \) and \( \sigma, b : [0, T] \times \mathbb{R}^k \rightarrow \mathbb{R}^{kr} \times \mathbb{R}^k \) are measurable functions such that \( \|\sigma(s, x)\| \leq c \) and \( |b(s, x)| \leq c(1 + |x|) \), for some constant \( c \).

It is known from the forward SDE's theory that there exist \( \kappa > 0 \) and \( C > 0 \) depending only on \( c, T, k \) such that
\[
\mathbb{E} \exp \left( \kappa \sup_{t \leq T} |X_t|^2 \right) \leq C \exp (C \ | X_0 |^2).
\]
Consider the BSDE
\[
Y_t = g(X_T) + \int_t^T |X_s| \bar{Y}_s - Y_s \log |Y_s| ds - \int_t^T Z_s dW_s.
\]
where \( \bar{Y} \epsilon [0, 2] \) and \( g \) is a measurable function satisfying \( |g(x)| \leq c \exp c |x|^\bar{Y} \), for some constants \( c > 0, \bar{Y} \epsilon [0, 2] \).

The previous BSDE has a unique solution \( (Y, Z) \) which satisfies: for every \( p > 1 \) there exists a positive constant \( C \) such that
\[
\mathbb{E} \sup_{t \leq T} |Y_t|^p + \mathbb{E} \left[ \int_0^T |Z_s|^2 ds \right]^{\bar{Y}} \leq C \exp (C \ | X_0 |^2).
\]

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Indeed, one can show that
i) \( < y, f(t,y) > \leq 1 + |X_t|^2 y^2 \)
ii) Using Young inequality we obtain, for every \( \epsilon > 0 \) there is a constant \( c_\epsilon > 0 \) such that
\[
| f(t, y) | \leq c_\epsilon (1 + |X_t|^{2\epsilon} + |y|^{1+\epsilon})
\]
iii) \( f \) satisfies assumption (H.4) with \( v_s = \exp |X_s|^7 \) and \( A_N = N \).

The following example shows that our assumptions enable to treat BSDEs with stochastic monotone coefficient

**Example 4.** Let \( (\xi, f) \) satisfying (H.0)-(H.3) and

\[
(H'.4) \quad \begin{cases}
\langle y - y', f(t,\omega, y, z) - f(t,\omega, y', z') \rangle \leq K' | y - y' |^2 \{C_t(\omega) + |\log | y - y' || \}
+ K' | y - y' | z - z' | \sqrt{C_t(\omega) + |\log | z - z' ||}.
\end{cases}
\]

In particular we have for all \( z, z' \)
\[
| f(t,\omega, y, z) - f(t,\omega, y, z') | \leq K' | z - z' | \sqrt{C_t(\omega) + |\log | z - z' ||}.
\]

Therefore, the following BSDE has a unique solution
\[
Y_t = \xi + \int_t^T f(s, Y_s, Z_s) ds - \int_t^T Z_s dW_s.
\]
To check (H.4), it is enough to show that for some constant \( c \) we have
\[
\langle y - y', f(t, y, z) - f(t, y', z) \rangle \leq c \log N \left( \frac{1}{2N} \right) | y - y' |^2 + \frac{1}{N}
\]
\[
| f(t, y, z) - f(t, y', z') | \leq c \log N \left( \frac{1}{2N} \right) + \frac{1}{N}
\]
whenever \( v_s := e^{C_t} \leq N \). These two inequalities can be respectively proved by considering the following cases
\[
| y - y' | \leq \frac{1}{2N}, \quad 2N \leq | y - y' | \leq 2N.
\]
and
\[
| z - z' | \leq \frac{1}{2N}, \quad 2N \leq | z - z' | \leq 2N.
\]

**Example 5.** Let \( (X_t)_{t \leq T} \) and \( \xi \) be as in example 3, let \( F(t, x, y, z) \) be such that
i) \( F(t, x, \cdot) \) is continuous
ii) \( |F(t, x, y, 0)| \leq C \exp(C | x |^q) + | y |^q \), for some \( q, \alpha \in [0, 2] \) and \( C > 0 \),
iii) \( (F(t, x, y, z) - F(t, x, y', z'), y - y') \leq K' | y - y' |^2 + K' | y - y' | z - z' \).

Let \( \overline{\eta}, \overline{\eta}', \overline{\eta}'' \geq 0 \) such that \( \overline{\eta} + \overline{\eta}' < 2 \) and \( \overline{\eta} + \overline{\eta}'' < 1 \), the following BSDE has a unique solution
\[
Y_t = \xi + \int_t^T |X_s|^{\overline{\eta}} F(s, X_s, |X_s|^{\overline{\eta}} Y_s, |X_s|^{\overline{\eta}} Z_s) ds - \int_t^T Z_s dW_s.
\]
3 Proof of Theorem 2.1

We first give some a priori estimates from which we derive a stability result for BSDEs and next we use a suitable approximation of \((\xi, f)\) to complete the proof. The difficulty comes from the fact that the generator \(f\) can be neither locally monotone in the variable \(y\) nor locally Lipschitz in the variable \(z\) and moreover, it also may have a super-linear growth in its two variables \(y\) and \(z\).

3.1 Estimates for the solutions of equation \((E(\xi, f))\).

In the first step, we give estimates for the processes \(Y\) and \(Z\).

Proposition 3.1. Let \(\Lambda = |\mathrm{Y}_t|^2 \mathrm{e}_t + 2 \int_0^t e_s \eta_s ds + (\int_0^t e_s^2 f_s^0 ds)^2\) and \(e_t := \exp \int_0^t \lambda_s ds\).

Assume that \((H.2)\) hold and \(\mathbb{E}(\sup_{0 \leq s \leq T} |\mathrm{Y}_t|^p e_t^\beta) < \infty\).

Then, there exists a positive constant \(C(\beta, \gamma)\) such that

\[
\mathbb{E} \sup_{0 \leq s \leq T} \Lambda^\frac{\beta}{\gamma} + \mathbb{E} \left( \int_0^T \mathbb{E} \sup_{0 \leq s \leq T} \left| \mathrm{Z}_s \right|^2 ds \right)^{\frac{\beta}{2}} \leq C(\beta, \gamma) \mathbb{E} \Lambda^\frac{\beta}{\gamma}.
\]

To prove this proposition we need some lemmas.

Lemma 3.1. For every \(\epsilon > 0\), every \(\beta > 1\) and every positive functions \(h\) and \(g\) we have

\[
\int_0^T (h(s))^{\beta-1} g(s) ds \leq \epsilon \sup_{0 \leq s \leq T} h(s) |\sup_{0 \leq s \leq T} g(s) ds|^{\beta-1} + \epsilon^{1-\beta} \left( \int_0^T g(s) ds \right)^{\beta}.
\]

Proof. Let \(\epsilon > 0\) and \(\beta > 1\). Using Young’s inequality we get for every \(\delta\) and \(\delta'\) such that \(\frac{1}{\beta} + \frac{1}{\delta'} = 1\)

\[
\int_0^T (h(s))^{\beta-1} g(s) ds \leq \frac{1}{\delta} \epsilon^{(\beta-1)\delta} \sup_{0 \leq s \leq T} h(s) \left( \frac{(\beta-1)\delta}{\delta'} \right) + \frac{\epsilon^{1-\beta}}{\delta'} \left( \int_0^T g(s) ds \right)^{\delta'}.
\]

We now choose \(\delta = \frac{\beta}{\beta-1}\) and use the fact that \(\delta', \delta' > 1\).\(\blacksquare\)

Lemma 3.2. If \((H.2)\) holds then for every \(\beta > 1 + 2\gamma\) there exist positive constants \(C_1(\beta, \gamma), C_2(\beta, \gamma)\) such that for every \(\epsilon > 0\), every stopping time \(\tau \leq T\) and every \(t \leq \tau\)

\[
\Lambda^\frac{\beta}{\gamma} + \int_t^\tau \Lambda^\frac{\beta}{\gamma-1} \mathbb{E} \sup_{s \leq \tau} \left| \mathrm{Z}_s \right|^2 ds \leq \epsilon \sup_{t \leq s \leq \tau} \Lambda^\frac{\beta}{\gamma} + \epsilon^{1-\beta} C_1(\beta, \gamma) \Lambda^\frac{\beta}{\gamma} - C_2(\beta, \gamma) \int_t^\tau \Lambda^\frac{\beta}{\gamma-1} \mathbb{E} \mathbb{E} \left( \int_0^\tau f_s^0 ds \right)^{\beta} ds.
\]

Proof. Without loss of generality, we assume that \(\eta\) and \(f^0\) are strictly positives.

It follows by using Itô’s formula that for every \(t \in [0, \tau]\),

\[
|\mathrm{Y}_t|^2 \mathbb{E}_t + \int_t^\tau |\mathrm{Z}_s|^2 \lambda_s \mathbb{E}_s ds = e_t |\mathrm{Y}_t|^2 + 2 \int_t^\tau \mathbb{E}_s(\mathbb{E}_s f(s, \mathrm{Y}_s, \mathrm{Z}_s)) ds - \int_t^\tau \mathbb{E}_s |\mathrm{Z}_s|^2 ds - 2 \int_t^\tau \mathbb{E}_s(\mathbb{E}_s \mathrm{Z}_s d\mathbb{W}_s).
\]

Again Itô’s formula, applied to the process \(\Lambda\), shows that

\[
\Lambda^\frac{\beta}{\gamma} + \beta \int_t^\tau \Lambda^\frac{\beta}{\gamma-1} \left( \frac{1}{2} |\mathrm{Y}_s|^2 \lambda_s e_s + e_s \eta_s + f_s^0 e_s^2 \int_0^s f_r^0 e_r^2 dr \right) ds = \Lambda^\frac{\beta}{\gamma} + \beta \int_t^\tau \Lambda^\frac{\beta}{\gamma-1} (e_s \mathrm{Y}_s, f(s, \mathrm{Y}_s, \mathrm{Z}_s)) ds - \beta \int_t^\tau \Lambda^\frac{\beta}{\gamma-1} |\mathrm{Z}_s|^2 e_s ds - \beta \int_t^\tau e_s \Lambda^\frac{\beta}{\gamma-1} \mathrm{Y}_s d\mathbb{W}_s - \beta(\frac{\beta}{2} - 1) \int_t^\tau e_s^2 \Lambda^\frac{\beta}{\gamma-2} \left( \sum_{\lambda=1}^d \left( \sum_{i=1}^d \mathrm{Y}^i_s \mathbb{E} \left( \int_0^\tau \mathrm{Y}^j_s \mathbb{E} \int_0^\tau \mathrm{Y}^k_s \mathbb{E} ds \right) ds \right) \right) ds.
\]
Observe that \( \sum_{j=1}^{r} \left( \sum_{i=1}^{d} Y_{s_i}^{j} Z_{s_i}^{j} \right)^2 \leq |Y_{s}|^2 |Z_{s}|^2 \leq e_{s}^{-1} \Lambda_{s} |Z_{s}|^2 \) then use the assumption (H.2) to get
\[
\Lambda_{s}^\beta + \frac{\beta}{2} (1 - 2\gamma - (2 - \beta)^+) \int_{t}^{T} \Lambda_{s}^{\gamma - 1} e_{s}|Z_{s}|^2 ds
\leq \Lambda_{s}^\beta + \beta \int_{t}^{T} \Lambda_{s}^{\gamma - 2} f_{s}^{0} e_{s}^4 ds - \beta \int_{t}^{T} \Lambda_{s}^{\gamma - 1} (e_{s} Y_{s}, Z_{s} dW_{s}).
\]
It follows from Lemma 3.3 with \( g(s) = f_{s}^{0} e_{s}^4 \), since \( \left( \int_{t}^{T} f_{s}^{0} e_{s}^4 ds \right)^\beta \leq \Lambda_{s}^\beta \), that for every \( \varepsilon > 0 \)
\[
\int_{t}^{T} \Lambda_{s}^{\gamma - 2} f_{s}^{0} e_{s}^4 ds \leq \varepsilon \sup_{0 \leq s \leq T} \Lambda_{s}^\beta + e^{1-\beta} \Lambda_{s}^\beta.
\]
Since \( \beta > 1 + 2\gamma \) implies that \( 1 - 2\gamma - (2 - \beta)^+ > 0 \), Lemma 3.2 is proved.

**Lemma 3.3.** Let (H2) be satisfied and assume that \( \mathbb{E}(\sup_{0 \leq s \leq T} |Y_t|^p e_{s}^{\frac{p}{2}}) < \infty \).
Then,
1) There exists a positive constant \( C^{(p,\gamma)} \) such that for every \( \varepsilon > 0 \), we have
\[
\mathbb{E} \int_{0}^{T} \Lambda_{s}^{\frac{p-2}{2}} e_{s}|Z_{s}|^2 ds \leq \varepsilon \mathbb{E}(\sup_{0 \leq s \leq T} \Lambda_{s}^\beta) + \varepsilon^{(1-p)} C^{(p,\gamma)} \mathbb{E}(\Lambda_{T}^\beta).
\]
2) There exists a positive constant \( C^{(p,\gamma)} \) such that
\[
\mathbb{E}(\int_{0}^{T} e_{s}|Z_{s}|^2 ds)^\frac{p}{2} \leq C^{(p,\gamma)} \mathbb{E}(\sup_{0 \leq s \leq T} \Lambda_{s}^\beta).
\]

**Proof.** The first assertion follows by a standard martingale localization procedure. To prove the second assertion, we successively use Lemma 3.2 (with \( \varepsilon = 1 \) and \( \beta = 2 \)), the Burkholder-Davis-Gundy inequality, the fact that \( e_{s}|Y_t|^2 \leq \Lambda_{s} \) and Young’s inequality to obtain
\[
\mathbb{E}(\int_{0}^{T} e_{s}|Z_{s}|^2 ds)^\frac{p}{2} \leq C^{(p,\gamma)} \mathbb{E}(\sup_{0 \leq s \leq T} \Lambda_{s}^\beta) + C^{(p,\gamma)} \mathbb{E}(\int_{0}^{T} e_{s}(Y_{s}, Z_{s} dW_{s}))^\frac{p}{2}
\leq C^{(p,\gamma)} \mathbb{E}(\sup_{0 \leq s \leq T} \Lambda_{s}^\beta) + C^{(p,\gamma)} \mathbb{E}(\int_{0}^{T} e_{s}|Y_{s}|^2 |Z_{s}|^2 ds)^\frac{p}{2}
\leq C^{(p,\gamma)} \mathbb{E}(\sup_{0 \leq s \leq T} \Lambda_{s}^\beta) + C^{(p,\gamma)} \mathbb{E}(\int_{0}^{T} \Lambda_{s} e_{s}|Z_{s}|^2 ds)^\frac{p}{2}
\leq C^{(p,\gamma)} \mathbb{E}(\sup_{0 \leq s \leq T} \Lambda_{s}^\beta) + C^{(p,\gamma)} \mathbb{E}[\sup_{0 \leq s \leq T} \Lambda_{s}^\gamma (\int_{0}^{T} e_{s}|Z_{s}|^2 ds)^\frac{p}{2}]
\leq [C^{(p,\gamma)} + 2(C^{(p,\gamma)} \gamma)] \mathbb{E}(\sup_{0 \leq s \leq T} \Lambda_{s}^\beta) + 1 \mathbb{E}[\int_{0}^{T} e_{s}|Z_{s}|^2 ds]^\frac{p}{2}
\leq [2C^{(p,\gamma)} + 4(C^{(p,\gamma)} \gamma)^2] \mathbb{E}(\sup_{0 \leq s \leq T} \Lambda_{s}^\beta).
\]
Lemma 3.3 is proved.

**Lemma 3.4.** Let the assumptions of Lemma 3.3 be satisfied. Then, there exists a constant \( C^{(p,\gamma)} \) such that
\[
\mathbb{E}(\sup_{0 \leq s \leq T} \Lambda_{s}^\beta) \leq C^{(p,\gamma)} \mathbb{E}(\Lambda_{T}^\beta).
\]
Proposition 3.2 is proved.

Proof. Lemma 3.2 and the Burkholder-Davis-Gundy inequality show that there exists a universal positive constant $c$ such that for every $\varepsilon > 0$ and $t \leq T$

$$
\mathbb{E} \sup_{0 \leq s \leq T} \Lambda_T^\varepsilon \leq \varepsilon \mathbb{E} \sup_{0 \leq s \leq T} \Lambda_T^\varepsilon + \varepsilon^{(1-p)} C_1^{(p,\gamma)} \mathbb{E}(\Lambda_T^\varepsilon)
$$

$$
+ c C_2^{(p,\gamma)} \mathbb{E}\left( \int_0^T \Lambda_T^{\varepsilon - 2}(|Y_s|^2 e_s |Z_s|^2 ds)^{\frac{1}{p}} \right).
$$

Young’s inequality gives, for every $\varepsilon' > 0$,

$$
\mathbb{E} \sup_{0 \leq s \leq T} \Lambda_T^\varepsilon \leq \varepsilon \mathbb{E} \sup_{0 \leq s \leq T} \Lambda_T^\varepsilon + C \varepsilon^{1-p} C_1^{(p,\gamma)} \mathbb{E}(\Lambda_T^\varepsilon)
$$

$$
+ c C_2^{(p,\gamma)} \mathbb{E}\left( \int_0^T \Lambda_T^{\varepsilon - 2}(|Y_s|^2 e_s |Z_s|^2 ds)^{\frac{1}{p}} \right).
$$

Applying Lemma 3.3 we get for every $\varepsilon'' > 0$

$$
\mathbb{E} \sup_{0 \leq s \leq T} \Lambda_T^\varepsilon \leq (\varepsilon + \varepsilon' + \frac{C_2^{(p,\gamma)}}{\varepsilon'} \varepsilon'') \mathbb{E} \sup_{0 \leq s \leq T} \Lambda_T^\varepsilon
$$

$$
+ (\varepsilon^{1-p} C_1^{(p,\gamma)} + \frac{C_2^{(p,\gamma)}}{\varepsilon'} C_1^{(p,\gamma)} (\varepsilon''(1-p)) E(\Lambda_T^\varepsilon).
$$

A suitable choice of $\varepsilon, \varepsilon', \varepsilon''$ allows to conclude the proof.

Proof of Proposition 3.1 It follows from Lemma 3.3 and Lemma 3.4.

Proposition 3.2. If (H.3) holds then,

$$
\mathbb{E}\int_0^T |f(s, Y_s, Z_s)|^\beta ds \leq 9^{p+q}(1 + T) \left(1 + \mathbb{E}\int_0^T \Pi ds + \mathbb{E}\sup_{0 \leq s \leq T} |Y_s|^p + \mathbb{E}\int_0^T |Z_s|^2 ds \right)^\frac{\beta}{2}
$$

where $\beta = \frac{2}{\alpha'} \wedge \frac{p}{\alpha'} \wedge \frac{p}{\alpha} \wedge q$.

Proof. We successively use Assumption (H.3), Young’s inequality and Hölder’s inequality to show that

$$
\mathbb{E}\int_0^T |f(s, Y_s, Z_s)|^\beta ds \leq \mathbb{E}\int_0^T (\Pi^\beta + |Y_s|^{\alpha \beta} + |Z_s|^{\alpha' \beta}) ds
$$

$$
\leq 3^{\beta} \mathbb{E}\int_0^T (\Pi^\beta + |Y_s|^{\alpha \beta} + |Z_s|^{\alpha' \beta}) ds
$$

$$
\leq 3^{\beta} \mathbb{E}\int_0^T ((1 + \Pi_s)^{\beta} + (1 + |Y_s|)^{\alpha \beta} + (1 + |Z_s|)^{\alpha' \beta}) ds
$$

$$
\leq 3^{\beta} \mathbb{E}\int_0^T ((1 + \Pi_s)^{\beta} + (1 + |Y_s|)^p + (1 + |Z_s|)^{p \wedge 2}) ds
$$

$$
\leq 3^{\beta} 3^{p+q} \mathbb{E}\int_0^T (1 + \Pi_s)^{2p+q} |Y_s|^p + |Z_s|^{p \wedge 2} ds
$$

$$
\leq 3^{\beta} 3^{p+q} [T + \mathbb{E}\int_0^T \Pi ds + TE \sup_{0 \leq s \leq T} |Y_s|^p + T \frac{2^{(p \wedge 2)}}{p \wedge 2} \mathbb{E}(\int_0^T |Z_s|^2 ds^\frac{p}{2})]
$$

$$
\leq 9^{p+q}(1 + T) \left(1 + \mathbb{E}\int_0^T \Pi ds + \mathbb{E}\sup_{0 \leq s \leq T} |Y_s|^p + \mathbb{E}(\int_0^T |Z_s|^2 ds^\frac{p}{2}) \right).
$$

Proposition 3.2 is proved.
3.2 Estimate of the difference between two solutions.

The next proposition gives an estimate which is a key tool in the proofs.

**Lemma 3.5.** Let \((\xi, f_i)_{i=1,2}\) satisfy (H3) (with the same \(\bar{\tau}, \alpha\) and \(\alpha'\)) and let \((Y^i, Z^i)\) be a solution of \((E^{(\xi, f_i)})\). Then, there exist \(\beta = \beta(p, q, \alpha, \alpha') \in [1, p \wedge 2]\), \(r = r(p, q, \alpha, \alpha', K', \mu, q') > 0\) and \(\alpha = a(p, q, \alpha, \alpha', K', \mu, q') > 0\) such that for every \(u \in [0, T], u' \in [u, T \wedge (u + r)], N > 0\) and every function \(f\) satisfying (H4)

\[
\mathbb{E}(\sup_{u \leq t \leq u'} |Y^1_t - Y^2_t|^\beta) + \mathbb{E} \int_u^{u'} \frac{|Z^1_t - Z^2_t|^2}{(1 + |Y^1_t - Y^2_t|^2)^{1+\gamma}} ds \\
\leq N^{1+\gamma} \left[ \mathbb{E}(|Y^1_0 - Y^2_0|^\beta) + \mathbb{E} \int_0^T \rho_N(f_1 - f)_s + \rho_N(f_2 - f)_s ds \right] \\
+ \frac{1}{A_N} \left[ 1 + \Theta^1_p + \Theta^2_p + \mathbb{E} \int_0^T \tau_i ds + \mathbb{E} \int_0^T v_i ds \right].
\]

where

\[
\rho_N(f_i - f)(t, \omega) := \sup_{|u|, |v| \leq N} |f(t, \omega, y, z) - f_i(t, \omega, y, z)|
\]

and

\[
\Theta^i_p := \mathbb{E}(\sup_{t \in [0, T]} |Y^i_t|^p) + \mathbb{E} \left( \int_0^T |Z^i_s|^2 ds \right)^{\frac{\beta}{2}}.
\]

**Proof.** Let \(q\) be the number defined in assumption (H3) and \(q', K', \mu\) those defined in assumption (H4). Let \(\tau > 0\) be such that \(1 + 2\tau < \tilde{\beta} := \frac{2}{\alpha'} \wedge \frac{p}{\alpha} \wedge q\) and set \(K := K' + \frac{K'}{2\tau}\). Let \(\beta \in [1 + 2\tau, \tilde{\beta}]\) and \(\nu \in [0, (1 - \frac{2}{\tilde{\beta}})(1 \wedge q')].\) Let \(r \in [0, \frac{\nu}{\mu \beta K^{\nu}} \wedge \frac{1}{2K'} \wedge 1]\).

For \(N \in \mathbb{N}\), we set

\[
\bar{\tau}_i := (A_N)^{2K^{\nu}(t-u)} \text{ and } \Delta_i := \{|Y^1_t - Y^2_t|^2 + (A_N)^{-1}\bar{\tau}_i|.
\]

Using Itô’s formula, we show that for every stopping time \(\tau \in [u, u']\) and every \(t \in [u, \tau]\)

\[
\Delta^\frac{\beta}{2} + 2 \log(A_N)K^\nu \int_t^\tau \tau_i \Delta^\frac{\beta}{2} ds + \frac{\beta}{2} \int_t^\tau \tau_i \Delta^\frac{\beta}{2} |Z^i_s - Z^i_s|^2 ds \\
= \Delta^\frac{\beta}{2} - \beta \int_t^\tau \tau_i \Delta^\frac{\beta}{2} (Y^1_s - Y^2_s, (Z^1_s - Z^2_s) dW_s) \\
+ \beta \int_t^\tau \tau_i \Delta^\frac{\beta}{2} (Y^1_s - Y^2_s, f_1(s, Y^1_s, Z^1_s) - f_2(s, Y^2_s, Z^2_s)) ds
\]

(3.1)

\[
= \Delta^\frac{\beta}{2} - \beta \int_t^\tau \tau_i \Delta^\frac{\beta}{2} \sum_{i=1}^d (Y^1_{i,s} - Y^2_{i,s}) (Z^1_{i,s} - Z^2_{i,s})^2 ds \\
= \Delta^\frac{\beta}{2} - \beta \int_t^\tau \tau_i \Delta^\frac{\beta}{2} (Y^1_s - Y^2_s, (Z^1_s - Z^2_s) dW_s) + \beta I_1 - \beta(\frac{\beta}{2} - 1) I_2,
\]

where

\[
I_1 := \int_t^\tau \tau_i \Delta^\frac{\beta}{2} (Y^1_s - Y^2_s, f_1(s, Y^1_s, Z^1_s) - f_2(s, Y^2_s, Z^2_s)) ds
\]

and

\[
I_2 := \int_t^\tau \tau_i \Delta^\frac{\beta}{2} (Y^1_s - Y^2_s, (Z^1_s - Z^2_s) dW_s) + \beta I_1 - \beta(\frac{\beta}{2} - 1) I_2.
\]
and

\[ I_2 := \int_t^T \tau \Delta_{i}^{2 \gamma - 2} \sum_{j=1}^{r} \left( \sum_{i=1}^{d} (Y_{i,s}^1 - Y_{i,s}^2)(Z_{i,j,s}^1 - Z_{i,j,s}^2) \right)^2 ds. \]

In order to complete the proof of Lemma 3.5 we need to estimate \( I_1 \) and \( I_2 \).

**Estimate of \( I_1 \).** Let \( \Phi(s) := |Y_s^1| + |Y_s^2| + |Z_s^1| + |Z_s^2| + v_s \). Since \( \mathbb{1}_{\{\Phi_s \leq N\}} \leq \mathbb{1}_{\{v_s \leq N\}} \) and \( f \) satisfies (H4), then a simple computation shows that

\[
\langle Y_s^1 - Y_s^2, f_1(s, Y_s^1, Z_s^1) - f_2(s, Y_s^2, Z_s^2) \rangle \\
\leq \frac{\epsilon^2}{\tau} \Delta_{s}^{\frac{2 \gamma - 2}{\gamma - 1}} |f_1(s, Y_s^1, Z_s^1) - f_2(s, Y_s^2, Z_s^2)| \mathbb{1}_{\{\Phi_s > N\}} \tau \\
+ 2N(\rho_N(f_1 - f)_s + \rho_N(f_2 - f)_s) \mathbb{1}_{\{v_s \leq N\}} \tau \\
+ [K^\gamma \log(A_N) \Delta_{s}^{-1} \Delta_{s} + \tau |Z_s^1 - Z_s^2|^2] \mathbb{1}_{\{\Phi_s \leq N\}} \tau
\]

Therefore, using Lemma 3.1 with \( h_s = \Delta_{s} \), we get

\[
I_1 \leq \int_t^T \tau \frac{1}{\tau} \Delta_{s}^{\frac{2 \gamma - 2}{\gamma - 1}} |f_1(s, Y_s^1, Z_s^1) - f_2(s, Y_s^2, Z_s^2)| \mathbb{1}_{\{\Phi_s > N\}} ds \\
+ 2N \int_t^T \tau \Delta_{s}^{\frac{2 \gamma - 2}{\gamma - 1}} |\rho_N(f_1 - f)_s + \rho_N(f_2 - f)_s| \mathbb{1}_{\{v_s \leq N\}} ds \\
+ \int_t^T \tau \Delta_{s}^{\frac{2 \gamma - 2}{\gamma - 1}} [K^\gamma \log(A_N) \Delta_{s}^{-1} \Delta_{s} + \tau |Z_s^1 - Z_s^2|^2] \mathbb{1}_{\{\Phi_s \leq N\}} ds
\]

\[ \leq \epsilon \sup_{s \in [u,v']} \Delta_{s}^{\frac{2 \gamma - 2}{\gamma - 1}} \\
+ \epsilon^{(1-\beta)\frac{2 \gamma - 2}{\gamma - 1}} \int_u^v |f_1(s, Y_s^1, Z_s^1) - f_2(s, Y_s^2, Z_s^2)|^\beta \mathbb{1}_{\{\Phi_s > N\}} ds \\
+ 2N \int_t^T \tau \Delta_{s}^{\frac{2 \gamma - 2}{\gamma - 1}} |\rho_N(f_1 - f)_s + \rho_N(f_2 - f)_s| \mathbb{1}_{\{v_s \leq N\}} ds \\
+ \int_t^T \tau \Delta_{s}^{\frac{2 \gamma - 2}{\gamma - 1}} [K^\gamma \log(A_N) \Delta_{s}^{-1} \Delta_{s} + \tau |Z_s^1 - Z_s^2|^2] \mathbb{1}_{\{\Phi_s \leq N\}} ds
\]

**Estimate of \( I_2 \).** Since

\[
\sum_{j=1}^{r} \left( \sum_{i=1}^{d} (Y_{i,s}^1 - Y_{i,s}^2)(Z_{i,j,s}^1 - Z_{i,j,s}^2) \right)^2 \leq |Y_s^1 - Y_s^2|^2 |Z_s^1 - Z_s^2|^2 \leq \pi_{s}^{-1} \Delta_s |Z_s^1 - Z_s^2|^2
\]

then

\[ I_2 \leq \int_t^T \tau \Delta_{s}^{\frac{2 \gamma - 2}{\gamma - 1}} |Z_s^1 - Z_s^2|^2 ds. \]
Now, coming back to equation (3.1) and taking into account the above estimates we get for every \( \varepsilon > 0 \),

\[
\Delta^\beta \tau + \frac{\beta}{2} (\beta - 1 - \tau^\beta) \int_t^\tau \tau_s \Delta^\beta - 1 |Z_s^1 - Z_s^2|^2 \, ds \\
\leq \varepsilon^\beta |Y_t^1 - Y_t^2|^\beta + \frac{\varepsilon^\beta}{A_N^\beta} + \beta \varepsilon \sup_{s \in [u,u']} \Delta^\beta \tau \\
+ \beta \varepsilon (1 - \beta) \int_u^{u'} \left| f_1(s, Y_s^1, Z_s^1) - f_2(s, Y_s^2, Z_s^2) \right| \beta \mathbb{I}_{\{f_1 > N\}} \, ds \\
+ 2N \beta \varepsilon \int_u^{u'} \left( \rho_N(f_1 - f)_s + \rho_N(f_2 - f)_s \mathbb{I}_{\{v_s \leq N\}} \right) \, ds \\
- \beta \int_u^{u'} \tau_s \Delta^\beta - 1 (Y_s^1 - Y_s^2, (Z_s^1 - Z_s^2) \, dW_s). 
\]

For a given \( h > 1 \), let \( \tau_h \) be the stopping time defined by

\[
\tau_h := \inf\{ s \geq u, |Y_s^1 - Y_s^2|^2 + \int_u^s |Z_t^1 - Z_t^2|^2 \, dt \geq h \} \wedge u',
\]

Choose \( \tau = \tau_h \), \( t = u \), then pass to the expectation in equation (3.2) to obtain, when \( h \to \infty \),

\[
\frac{\beta}{2} (\beta - 1 - \tau^\beta) \mathbb{E} \int_u^{u'} \tau_s \Delta^\beta - 1 |Z_s^1 - Z_s^2|^2 \, ds \\
\leq \varepsilon^\beta \mathbb{E}(|Y_u^1 - Y_u^2|^\beta) + \frac{\varepsilon^\beta}{A_N^\beta} + \beta \varepsilon \mathbb{E} \left( \sup_{s \in [u,u']} \Delta^\beta \tau \right) \\
+ \beta \varepsilon (1 - \beta) \int_u^{u'} \mathbb{E} \left[ f_1(s, Y_s^1, Z_s^1) - f_2(s, Y_s^2, Z_s^2) \right] \beta \mathbb{I}_{\{f_1 > N\}} \, ds \\
+ 2N \beta \varepsilon \mathbb{E} \left[ A_N^\beta \right] \int_u^{u'} \mathbb{E} \left( \rho_N(f_1 - f)_s + \rho_N(f_2 - f)_s \mathbb{I}_{\{v_s \leq N\}} \right) \, ds.
\]

Return back to (3.2) and use the Burkholder-Davis-Gundy inequality to show that there exists a universal constant \( c \) such that

\[
\mathbb{E} \left( \sup_{u \leq t \leq T} \Delta^\beta \tau \right) \leq \mathbb{E} \left( \sup_{u \leq t \leq T} \Delta^\beta \tau \right) |Y_u^1 - Y_u^2|^\beta + \frac{\varepsilon^\beta}{A_N^\beta} + \beta \varepsilon \mathbb{E} \left( \sup_{s \in [u,u']} \Delta^\beta \tau \right) \\
+ \beta \varepsilon (1 - \beta) \mathbb{E} \left[ f_1(s, Y_s^1, Z_s^1) - f_2(s, Y_s^2, Z_s^2) \right] \beta \mathbb{I}_{\{f_1 > N\}} \, ds \\
+ 2N \beta \varepsilon \mathbb{E} \left[ A_N^\beta \right] \int_u^{u'} \mathbb{E} \left( \rho_N(f_1 - f)_s + \rho_N(f_2 - f)_s \mathbb{I}_{\{v_s \leq N\}} \right) \, ds \\
+ c \beta \mathbb{E} \left( \int_u^{T} \tau_s^\beta \Delta^\beta - 1 \sum_{j=1}^d \sum_{i=1}^d (Y_s^{1,i} - Y_s^{2,i})(Z_s^{1,j,i} - Z_s^{2,j,i})^2 \, ds \right)^{\frac{1}{2}}.
\]
But, there exists a positive constant $C_\beta$ depending only on $\beta$ such that

$$c_\beta \mathbb{E}\left(\int_u^{u'} \tau_s \Delta_s^{\beta/2} - 2 \sum_{j=1}^{d} \sum_{i=1}^{r} (Y_{i,j,s} - \mathbb{E}Y_{i,j,s})^2 ds\right) + C_{\beta} \mathbb{E}\left(\int_u^{u'} \tau_s \Delta_s^{\beta/2} |Z_s^1 - Z_s^2|^2 ds\right) \leq \frac{1}{4} \mathbb{E}\left(\sup_{u \leq t \leq u'} \Delta_t^{\beta/2}\right) + C_{\beta} \mathbb{E}\left(\int_u^{u'} \tau_s \Delta_s^{\beta/2} |Z_s^1 - Z_s^2|^2 ds\right).$$

Use (3.3) and take $\varepsilon$ small enough to obtain the existence of a positive constant $C = C(\beta, \gamma)$ such that

$$\mathbb{E}\left(\sup_{u \leq t \leq u'} \Delta_t^{\beta/2}\right) + \mathbb{E}\left(\int_u^{u'} \tau_s \Delta_s^{\beta/2} |Z_s^1 - Z_s^2|^2 ds\right) \leq C \left[\tau_u \mathbb{E}|Y_u^1 - Y_u^2|^\beta + \frac{\tau_u}{A_N^2} + \sup_i \mathbb{E}\left(\int_u^{u'} |f_i(s, Y_s^1, Z_s^1)|^\beta \mathbb{1}_{\{\Phi_i > N\}} ds\right) \right] + N \tau_u A_N^{-\frac{1}{2}} \mathbb{E}\left(\int_u^{u'} \rho N (f_1 - f)_s + \rho N (f_2 - f)_s \mathbb{1}_{\{v_s \leq N\}} ds\right).$$

We shall estimate $J := \sup_i \mathbb{E}\left(\int_u^{u'} |f_i(s, Y_s^1, Z_s^1)|^\beta \mathbb{1}_{\{\Phi_i > N\}} ds\right)$, $i = 1, 2$.

Using the fact that $\mathbb{1}_{\{\Phi_i > N\}} \leq \mathbb{1}_{\{\nu + 5 > N\}} + \mathbb{1}_{\{\nu + 5 > N\}} + \mathbb{1}_{\{\nu + 5 > N\}} + \mathbb{1}_{\{\nu + 5 > N\}} + \mathbb{1}_{\{\nu + 5 > N\}}$ and $\mathbb{1}_{\{a > b\}} \leq \frac{u^\beta}{b}$ for every $a, b, \nu > 0$, we show that for every $N > 1$

$$J \leq \left(\frac{5}{N}\right)^{\nu} \sup_i \mathbb{E}\left(\int_u^{u'} |f_i(s, Y_s^1, Z_s^1)|^\beta \mathbb{1}_{\{\Phi_i > N\}} ds\right) + \left(\frac{5}{N}\right)^{\nu} \sup_i \mathbb{E}\left(\int_u^{u'} |f_i(s, Y_s^1, Z_s^1)|^\beta |Y_s^1|^\nu ds\right) + \left(\frac{5}{N}\right)^{\nu} \sup_i \mathbb{E}\left(\int_u^{u'} |f_i(s, Y_s^1, Z_s^1)|^\beta |Z_s^1|^\nu ds\right) + \left(\frac{5}{N}\right)^{\nu} \sup_i \mathbb{E}\left(\int_u^{u'} |f_i(s, Y_s^1, Z_s^1)|^\beta |Z_s^1|^\nu ds\right) + \left(\frac{5}{N}\right)^{\nu} \sup_i \mathbb{E}\left(\int_u^{u'} |f_i(s, Y_s^1, Z_s^1)|^\beta |Z_s^1|^\nu ds\right).$$

using Young’s inequality, one can prove that there exists a positive constant $C$ such that for every $N > 1$

$$J \leq \frac{C}{N^\nu} \left\{1 + \Theta_p^1 + \Theta_p^2 + \sup_i \mathbb{E}\left(\int_u^{u'} |f_i(s, Y_s^1, Z_s^1)|^\beta \left(\int_0^T |Z_s|^2 ds\right)^\frac{\nu}{2} \mathbb{1}_{\{\Phi_i > N\}} ds + \mathbb{E}\int_u^{u'} v_s^\nu ds\right)\right\}.$$

where $\Theta_p^1 := \mathbb{E}(\sup_{t} |Y_t^1|^p) + \mathbb{E}\left(\int_0^T |Z_s|^2 ds\right)^\frac{\nu}{2}.$

Using Proposition 3.2 we get (since $\beta(\frac{\nu'}{d+1} + \frac{1}{2} + \frac{\nu}{d+1}) \leq \beta$)

$$J \leq \frac{C}{N^\nu} \left\{1 + \Theta_p^1 + \Theta_p^2 + \mathbb{E}\int_0^T |\mathcal{A}_s|^q ds + \mathbb{E}\int_u^{u'} v_s^\nu ds\right\}.$$
Hence, for \( a := \left( \frac{\partial}{\partial t} \right) - \beta r K^n \) and \( N \) large enough we get (since \( A_N \leq N^\mu \) by assumption (bf(H.4)),

\[
\begin{align*}
E \sup_{u \leq t \leq u'} \Delta_t^{0.5} + E \int_u^{u'} \bar{\tau}_s \Delta_s^{0.5-1}|Z_s^1 - Z_s^2|^2 ds \\
\leq N A_N^{1 + \frac{2}{1 - \nu}} \left[ E|Y_t^1 - Y_t^2|^3 + E \int_0^T \rho_N(f_1 - f)_s + \rho_N(f_2 - f)_s \mathbb{I}\{v_s \leq N\} ds \right] \\
+ \frac{1}{A_N} [1 + \Theta_p^1 + \Theta_p^2 + E \int_0^T \bar{\tau}_s^1 ds + E \int_0^T v'_s^2 ds].
\end{align*}
\]

Lemma 3.5 is proved. \( \blacksquare \)

As a consequence of lemma 3.5, we have

**Lemma 3.6.** Let \( (\xi^1, f_1)_i=1,2 \) satisfies (H.3) (with the same \( \pi, \alpha \) and \( \alpha' \)) and let \( (Y^1, Z^1) \) be a solution of \( (E(\xi^1, f_1)) \). Then, there exists \( \beta = \beta(p, q, \alpha, \alpha') \in [1, p \wedge 2] \) such that for every \( \epsilon > 0 \) there is an integer \( N_\epsilon = N_\epsilon(p, q, \alpha, \alpha', K^\mu, q', \epsilon, \text{(AN)}_N) \) such that for every function \( f \) satisfying \( \text{(H.4)} \)

\[
E( \sup_{0 \leq t \leq T} |Y_t^1 - Y_t^2|^3) + E \int_0^T \frac{|Z_s^1 - Z_s^2|^2}{(1 + |Y_s^1 - Y_s^2|^2)^{1-\nu}} ds \\
\leq N_\epsilon \left[ E(|\xi^1 - \xi^2|^3) + E \int_0^T \rho_{N_\epsilon}(f_1 - f)_s + \rho_{N_\epsilon}(f_2 - f)_s ds \right] \\
+ \epsilon [1 + \Theta_p^1 + \Theta_p^2 + E \int_0^T \bar{\tau}_s^1 ds + E \int_0^T v'_s^2 ds].
\]

**Proof.** Let \( (u_0 = 0 < ... < u_{\ell+1} = T) \) be a subdivision of \([0, T]\) such that for every \( i \in \{0, ..., \ell\} \)

\[u_{i+1} - u_i \leq r\]

From lemma 3.5 we have : for all \( \epsilon > 0 \) there is an integer \( N_\epsilon \) such that for every function \( f \) satisfying \( \text{(H.4)} \)

\[
E( \sup_{u \leq t \leq u'} |Y_t^1 - Y_t^2|^3) + E \int_u^{u'} \frac{|Z_s^1 - Z_s^2|^2}{(1 + |Y_s^1 - Y_s^2|^2)^{1-\nu}} ds \\
\leq N_\epsilon \left[ E(|\xi^1 - \xi^2|^3) + E \int_0^T \rho_{N_\epsilon}(f_1 - f)_s + \rho_{N_\epsilon}(f_2 - f)_s ds \right] \\
+ \epsilon [1 + \Theta_p^1 + \Theta_p^2 + E \int_0^T \bar{\tau}_s^1 ds + E \int_0^T v'_s^2 ds].
\]

Assume that for some \( i \in \{0, ..., \ell\} \) we have for all \( \epsilon > 0 \) there is an integer \( N_\epsilon \) such that for every function \( f \) satisfying \( \text{(H.4)} \)

\[
E( \sup_{u_{i+1} \leq t \leq u} |Y_t^1 - Y_t^2|^3) + E \int_{u_{i+1}}^{u} \frac{|Z_s^1 - Z_s^2|^2}{(1 + |Y_s^1 - Y_s^2|^2)^{1-\nu}} ds \\
\leq N_\epsilon \left[ E(|\xi^1 - \xi^2|^3) + E \int_0^T \rho_{N_\epsilon}(f_1 - f)_s + \rho_{N_\epsilon}(f_2 - f)_s ds \right] \\
+ \epsilon [1 + \Theta_p^1 + \Theta_p^2 + E \int_0^T \bar{\tau}_s^1 ds + E \int_0^T v'_s^2 ds].
\]
Then, for every \( \varepsilon' > 0 \) there is an integer \( N_{\varepsilon'} \) such that for every function \( f \) satisfying (H.4)

\[
\mathbb{E} \left( \sup_{u_i \leq t \leq T} |Y_t^1 - Y_t^2|^{\beta} \right) + \mathbb{E} \int_{u_i}^T \frac{|Z_s^1 - Z_s^2|^2}{(1 + |Y_s^1 - Y_s^2|^2)^{1 - \frac{\alpha}{2}}} ds \\
\leq \mathbb{E} \left( \sup_{u_i \leq t \leq u_{i+1}} |Y_t^1 - Y_t^2|^{\beta} \right) + \mathbb{E} \int_{u_i}^{u_{i+1}} \frac{|Z_s^1 - Z_s^2|^2}{(1 + |Y_s^1 - Y_s^2|^2)^{1 - \frac{\alpha}{2}}} ds \\
+ N_{\varepsilon'} \left[ \mathbb{E}(|\xi^1 - \xi^2|^{\beta}) + \mathbb{E} \int_0^T \rho_{N_{\varepsilon'}}(f_1 - f)_s + \rho_{N_{\varepsilon'}}(f_2 - f)_s ds \right] \\
+ \varepsilon' \left[ 1 + \Theta_p^1 + \Theta_p^2 + \mathbb{E} \int_0^T \overline{\pi}_0^2 ds + \mathbb{E} \int_0^T v_s' ds \right].
\]

Using Lemma 3.5 we obtain; for every \( \varepsilon', \varepsilon'' > 0 \) there exist \( N_{\varepsilon'} > 0 \) and \( N_{\varepsilon''} > 0 \) such that for every function \( f \) satisfying (H.4)

\[
\mathbb{E} \left( \sup_{u_i \leq t \leq T} |Y_t^1 - Y_t^2|^{\beta} \right) + \mathbb{E} \int_{u_i}^T \frac{|Z_s^1 - Z_s^2|^2}{(1 + |Y_s^1 - Y_s^2|^2)^{1 - \frac{\alpha}{2}}} ds \\
\leq N_{\varepsilon'} \left[ \mathbb{E}(|\xi^1 - \xi^2|^{\beta}) + \mathbb{E} \int_0^T \rho_{N_{\varepsilon'}}(f_1 - f)_s + \rho_{N_{\varepsilon'}}(f_2 - f)_s ds \right] \\
+ N_{\varepsilon'} \left[ 1 + \Theta_p^1 + \Theta_p^2 + \mathbb{E} \int_0^T \overline{\pi}_0^2 ds + \mathbb{E} \int_0^T v_s' ds \right] \\
\leq N_{\varepsilon'}N_{\varepsilon''} \left[ \mathbb{E}(|\xi^1 - \xi^2|^{\beta}) + (N_{\varepsilon'}N_{\varepsilon''} + 2N_{\varepsilon''}) \mathbb{E} \int_0^T \rho(N_{\varepsilon'},y_{N_{\varepsilon'}})(f_1 - f)_s + \rho(N_{\varepsilon'},y_{N_{\varepsilon'}})(f_2 - f)_s ds \\
\right. \\
\left. + (2\varepsilon' + \varepsilon'') N_{\varepsilon''} \left[ 1 + \Theta_p^1 + \Theta_p^2 + \mathbb{E} \int_0^T \overline{\pi}_0^2 ds + \mathbb{E} \int_0^T v_s' ds \right]. \right]
\]

For \( \varepsilon > 0 \), let \( \varepsilon' := \varepsilon/4 \) and \( \varepsilon'' := \varepsilon/(2N_{\varepsilon'}) \), we then deduce that there exists an integer \( N_{\varepsilon} \) such that for every function \( f \) satisfying (H.4)

\[
\mathbb{E} \left( \sup_{u_i \leq t \leq T} |Y_t^1 - Y_t^2|^{\beta} \right) + \mathbb{E} \int_{u_i}^T \frac{|Z_s^1 - Z_s^2|^2}{(1 + |Y_s^1 - Y_s^2|^2)^{1 - \frac{\alpha}{2}}} ds \\
\leq N_{\varepsilon} \left[ \mathbb{E}(|\xi^1 - \xi^2|^{\beta}) + \mathbb{E} \int_0^T \rho_{N_{\varepsilon'}}(f_1 - f)_s + \rho_{N_{\varepsilon'}}(f_2 - f)_s ds \right] \\
+ \varepsilon \left[ 1 + \Theta_p^1 + \Theta_p^2 + \mathbb{E} \int_0^T \overline{\pi}_0^2 ds + \mathbb{E} \int_0^T v_s' ds \right].
\]

We complete the proof by induction \( \Box \)

**Proposition 3.3.** Let \( (\xi^1, f_1)_{t=1}^{T} \) satisfies (H.3) (with the same \( \overline{\pi}, \alpha \) and \( \alpha' \)) and let \( (Y^1, Z^1) \) be a solution of \( (E(\xi^1, f_1)) \). Then, there exists \( \beta = \beta(p, q, \alpha, \alpha') \in [1, p \wedge 2] \) such that for every \( \varepsilon > 0 \) there is
an integer $N_\epsilon = N_\epsilon(p,q,\alpha,\alpha',K',\mu,q',\epsilon,(A_N)_N)$ such that for every function $f$ satisfying $(H.4)$

$$
\begin{align*}
E(\sup_{0 \leq t \leq T} |Y_t^1 - Y_t^2|^3) + E \left( \int_0^T |Z_s^1 - Z_s^2|^2 ds \right)^{\frac{3}{2}}
\leq N_\epsilon \left[ E((\xi^1 - \xi^2)^3) + E \int_0^T \rho_{N_\epsilon}(f_t - f)_s + \rho_{N_\epsilon}(f_2 - f)_s ds \right]
+ \epsilon \left[ 1 + \Theta_1^p + \Theta_2^p + E \int_0^T \mathbb{P}_s ds + E \int_0^T v_s ds \right],
\end{align*}
$$

where $\Theta_p^p := E(\sup_t |Y_t|^p) + E \left( \int_0^T |Z_s|^2 ds \right)^{\frac{p}{2}}$.

**Proof.** Using Hölder’s inequality, Young’s inequality and the fact that $\frac{\beta}{2} < 1$, we obtain for all $\epsilon' > 0$

$$
\begin{align*}
E \left( \int_0^T |Z_s^1 - Z_s^2|^2 ds \right) \leq
\epsilon' + (1 + \epsilon'^{\frac{\beta-2}{2}}) \left[ E(\sup_{0 \leq t \leq T} |Y_t^1 - Y_t^2|^2) + E \int_0^T \frac{|Z_s^1 - Z_s^2|^2}{(1 + |Y_s^1 - Y_s^2|^2)^{1-\frac{2}{\beta}}} ds \right].
\end{align*}
$$

Use lemma 3.5 to conclude that for every $\epsilon', \epsilon'' > 0$

$$
\begin{align*}
E \left( \int_0^T |Z_s^1 - Z_s^2|^2 ds \right) \leq
\epsilon' + (1 + \epsilon''^{\frac{\beta-2}{2}}) N_\epsilon \left[ E((\xi^1 - \xi^2)^2) + E \int_0^T \rho_{N_\epsilon}(f_t - f)_s + \rho_{N_\epsilon}(f_2 - f)_s ds \right]
+ \epsilon''(1 + \epsilon''^{\frac{\beta-2}{2}}) \left[ 1 + \Theta_1^p + \Theta_2^p + E \int_0^T \mathbb{P}_s ds + E \int_0^T v_s ds \right].
\end{align*}
$$

Letting $\epsilon' = \frac{\epsilon}{2}$ and $\epsilon'' = \frac{\epsilon}{2(1 + (\frac{\beta-2}{2}) \epsilon''^2)}$, we finish this proof of proposition 3.3.

**Remark 3.1.** The uniqueness of equation $(E(\xi,f))$ follows by letting $f_1 = f_2 = f$ and $\xi_1 = \xi_2 = \xi$ in Proposition 3.3.

The following stability result follows from propositions 3.3, 3.2 and 3.1.
Proposition 3.4. Let \((\xi, f)\) satisfies (H.0)-(H.4) and \((\xi_n, f_n)\) satisfies (H.0)-(H.3) uniformly on \(n\). Assume moreover that

(a) \(\xi^n \to \xi\) a.s. and \(\sup_n \mathbb{E}(|\xi_n|^p \exp(\frac{\xi}{T} \lambda_s ds)) < \infty\)

(b) For every \(N \in \mathbb{N}^\ast\), \(\lim_n \rho_N(f_n - f) = 0\) a.e.

(c) for every \(n \in \mathbb{N}^\ast\), the BSDE \((E(\xi^n, f_n))\) has a solution \((Y^n, Z^n)\) which satisfies,

\[
\sup_n \mathbb{E}(\text{sup}_{t \leq T} |Y_t^n| e^{\xi_T^n \lambda_s ds}) < \infty.
\]

Then, there exists \((Y, Z) \in \mathbb{L}^p(\Omega; \mathbb{C}([0, T]; \mathbb{R}^d)) \times \mathbb{L}^p(\Omega; \mathbb{L}^2([0, T]; \mathbb{R}^d))\) such that

i) \(\mathbb{E}(\text{sup}_t |Y_t|^p e^{\xi_T \lambda_s ds} + \mathbb{E} \left[ \int_0^T e^{\xi_T \lambda_s dr} | Z_s |^2 ds \right]^\frac{p}{2} \leq C^p \mathbb{E}(|\xi|^p e^{\xi_T \lambda_s ds} + \mathbb{E} \left( \int_0^T e^{\xi_T \lambda_s dr} \eta_s ds \right)^\frac{p}{2} + \mathbb{E} \left( \int_0^T e^{\xi_T \lambda_s dr} f_0^0 ds \right)^p) \)

\[
\leq C^p \mathbb{E}(|\xi|^p e^{\xi_T \lambda_s ds} + \mathbb{E} \left( \int_0^T e^{\xi_T \lambda_s dr} \eta_s ds \right)^\frac{p}{2} + \mathbb{E} \left( \int_0^T e^{\xi_T \lambda_s dr} f_0^0 ds \right)^p) \leq D. \]

ii) for every \(p < p, (Y^n, Z^n) \to (Y, Z)\) strongly in \(\mathbb{L}^p(\Omega; \mathbb{C}([0, T]; \mathbb{R}^d)) \times \mathbb{L}^p(\Omega; \mathbb{L}^2([0, T]; \mathbb{R}^d)).

iii) for every \(\beta < \frac{2}{\alpha^2} \wedge \frac{P}{\alpha^4} \wedge q\), \(\lim_{n \to \infty} \mathbb{E} \int_0^T |f_n(s, Y^n_s, Z^n_s) - f(s, Y_s, Z_s)|^\beta ds = 0\)

Moreover, \((Y, Z)\) is the unique solution of \((E(\xi, f))\).

Proof. From Proposition 3.2 Proposition 3.2 and Proposition 3.3 we have

\(a')\) \(\mathbb{E}(\text{sup}_t |Y_t|^p e^{\xi_T \lambda_s ds}) + \mathbb{E} \left( \int_0^T e^{\xi_T \lambda_s dr} | Z_s |^2 ds \right)^\frac{p}{2} \leq C^p \mathbb{E}(|\xi|^p e^{\xi_T \lambda_s ds}) + \mathbb{E} \left( \int_0^T e^{\xi_T \lambda_s dr} \eta_s ds \right)^\frac{p}{2} + \mathbb{E} \left( \int_0^T e^{\xi_T \lambda_s dr} f_0^0 ds \right)^p) \)

\[\leq C^p \sup_n \mathbb{E}(|\xi|^p e^{\xi_T \lambda_s ds} + \mathbb{E} \left( \int_0^T e^{\xi_T \lambda_s dr} \eta_s ds \right)^\frac{p}{2} + \mathbb{E} \left( \int_0^T e^{\xi_T \lambda_s dr} f_0^0 ds \right)^p) \]

\[\leq D. \]

\(b')\) \(\int_0^T |f_n(s, Y^n_s, Z^n_s)|^\beta ds \leq C(1 + D) \int_0^T \eta^2 ds\).

\(c')\) There exists \(\beta > 1\) such that for every \(\varepsilon > 0\) there exists \(N_\varepsilon > 0:\)

\(\mathbb{E}(\text{sup}_t |Y_t^n - Y^n|^\beta) + \mathbb{E} \left( \int_0^T |Z^n_s - Z^n_s|^2 ds \right)^\frac{p}{2} \leq N_\varepsilon \mathbb{E} \left[|\xi^n - \xi^n|^\beta + \int_0^T \rho_{N_\varepsilon}(f_n - f)_s + \rho_{N_\varepsilon}(f_m - f)_s ds \right] \)

\[\leq N_\varepsilon \mathbb{E} \left[|\xi^n - \xi^n|^\beta + \int_0^T \rho_{N_\varepsilon}(f_n - f)_s + \rho_{N_\varepsilon}(f_m - f)_s ds \right] + \varepsilon \left[ 1 + 2D + \mathbb{E} \int_0^T \eta^2 ds + \mathbb{E} \int_0^T \nu^2 ds \right]. \]

We deduce the existence of \((Y, Z) \in \mathbb{L}^p(\Omega; \mathbb{C}([0, T]; \mathbb{R}^d)) \times \mathbb{L}^p(\Omega; \mathbb{L}^2([0, T]; \mathbb{R}^d))\) such that

\(i)\) \(\mathbb{E}(\text{sup}_t |Y_t|^p e^{\xi_T \lambda_s ds} + \mathbb{E} \left( \int_0^T e^{\xi_T \lambda_s dr} | Z_s |^2 ds \right)^\frac{p}{2} \leq C^p \mathbb{E}(|\xi|^p e^{\xi_T \lambda_s ds} + \mathbb{E} \left( \int_0^T e^{\xi_T \lambda_s dr} \eta_s ds \right)^\frac{p}{2} + \mathbb{E} \left( \int_0^T e^{\xi_T \lambda_s dr} f_0^0 ds \right)^p) \)

\[\leq C^p \mathbb{E}(|\xi|^p e^{\xi_T \lambda_s ds} + \mathbb{E} \left( \int_0^T e^{\xi_T \lambda_s dr} \eta_s ds \right)^\frac{p}{2} + \mathbb{E} \left( \int_0^T e^{\xi_T \lambda_s dr} f_0^0 ds \right)^p) \leq D. \]

\(ii)\) for all \(p' < p, (Y^n, Z^n) \to (Y, Z)\) strongly in \(\mathbb{L}^p(\Omega; \mathbb{C}([0, T]; \mathbb{R}^d)) \times \mathbb{L}^p(\Omega; \mathbb{L}^2([0, T]; \mathbb{R}^d)).

Let us prove \(iii)\). Set \(a := \lim_{n \to \infty} \mathbb{E} \int_0^T |f(s, Y^n_s, Z^n_s) - f(s, Y_s, Z_s)|^\beta ds\). Consider a subsequence \(n'\) of \(n\) such that \(a := \lim_{n' \to \infty} \mathbb{E} \int_0^T |f(s, Y^n_s, Z^n_s) - f(s, Y_s, Z_s)|^\beta ds\) and, \((Y^n' , Z^n') \to (Y, Z)\) a.e. 

Assumption (H.3) and the continuity of \(f\) ensure that \(a = 0\). It remains to prove that
\[
\lim_{n \to \infty} \mathbb{E} \int_0^T |f_n(s, Y^n_s, Z^n_s) - f(s, Y^*_s, Z^*_s)|^\beta ds = 0
\]

We use Hölder’s inequality, the previous claim b’), Proposition 3.2 and Chebychev’s inequality to get

\[
\begin{align*}
\mathbb{E} \int_0^T |f_n(s, Y^n_s, Z^n_s) - f(s, Y^*_s, Z^*_s)|^\beta ds & \\
& \leq \mathbb{E} \int_0^T \rho_N(f_n - f)^\beta ds + (\mathbb{E} \int_0^T |f_n(s, Y^n_s, Z^n_s) - f(s, Y^*_s, Z^*_s)|^r ds)^{\frac{\beta}{r}} (\mathbb{E} \int_0^T \mathbb{1}_{|Y^n_s| + |Z^n_s| \geq N} ds)^{\frac{r}{r'}} \\
& \leq \mathbb{E} \int_0^T \rho_N(f_n - f)^\beta ds + \frac{C(r)}{N^{1 - \frac{\alpha}{\alpha'} + \frac{\beta}{r'}}},
\end{align*}
\]

for some reel \( r > 1 \) such that \( r\beta < \frac{2}{\alpha'} \wedge \frac{p}{\alpha'} \wedge q. \)

We successively let \( n \to \infty \) and \( N \to \infty \) to derive assertion iii). Proposition 3.4 is proved. \( \blacksquare \)

### 3.3 Approximation

We shall construct a sequence \((\xi^n, f_n)\) which converges in a suitable sense to \((\xi, f)\) and which has good properties. With the help of this approximation, we can construct a solution \((Y, Z)\) to the BSDE \((E(\xi, f))\) by Proposition 3.4.

Let \( h_1 \) be a predictable process such that \( 0 < h_1 \leq 1 \) and set \( \overline{\mathfrak{A}}_t := \eta_t + \Pi_t + M_t + K_t + \frac{1}{h_1} \)

**Proposition 3.5.** Assume that \((\xi, f)\) satisfies (H.0)–(H.3). Then there exists a sequence \((\xi^n, f_n)\) such that

1. For each \( n \), \( \xi^n \) is bounded, \( |\xi^n| \leq \xi \) and \( \xi^n \) converges to \( \xi \) a.s.
2. For each \( n \), \( f_n \) is uniformly Lipschitz in \((y, z)\).
3. \( |f_n(t, \omega, y, z)| \leq \mathbb{E}(|\overline{\mathfrak{A}}_t| + n \cdot |y| + |z| + 2p h_1) \leq 2p + 3n p \).
4. \( f_n(t, \omega, y, z) > \mathbb{E}(\overline{\mathfrak{A}}_t) \leq n \cdot (\eta_t + f_0^0 |y| + M_t |y|^2 + K_t |y| |z| + 10h_1) \).
5. For every \( N \), \( \rho_N(f_n - f)(t, \omega) \to 0 \) as \( n \to \infty \) a.e. \((t, \omega)\).
6. For every \( N \), \( \rho_N(f_n - f)(t, \omega) \leq \frac{2(\Pi_t + N \cdot N + N \cdot 2p h_1)}{N \cdot N} \).

**Proof.** Let \( \psi : \mathbb{R} \to [0, \frac{\exp(-1)}{c_1}] \) defined by:

\[
\psi(x) := \begin{cases} 
    c_1^{-1} \exp(-\frac{1}{1 - x^2}) & \text{if } |x| < 1 \\
    0 & \text{else}
\end{cases}
\]

where \( c_1 = \int_{-1}^1 \exp(-\frac{1}{1 - x^2}) dx \).

Let \( m := \frac{n}{h_1} \), the sequence \((\xi^n, f_n)\) defined by: \( \xi^n := \xi_{|[\xi| \leq n]} \)

\[
f_n(t, y, z) := (c_1 e^2) \mathbb{1}_{\overline{\mathfrak{A}}_t} \cdot \psi(n^{-2} |y|^2) \psi(n^{-2} |z|^2) \times m^{d+dr} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} f(t, y - u, z - v) \Pi_{i=1}^{d} \psi(mu_i) \Pi_{i=1}^{d} \Pi_{j=1}^{r} \psi(mu_{ij}) dudv,
\]

satisfies the required properties. Indeed, (a) is obvious. (c) follows from the definition of \( f_n \). (f) follows from assumption (H.3) and assertion (c). We shall prove assertions (b), (c) and (d).

(b) For a fixed \( t \) and \( \omega \), \( f_n(t, \omega, \ldots) \) is smooth and with compact support in \([-n, n]^{d+dr} \). Moreover

\[
|\nabla_y z f_n(t, \omega, y, z)| \leq Cn^{2n+2},
\]

where \( \nabla \) denotes the gradient and \( C \) is a positive constant.
We define, \( H.2 \)
\[ L := \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} | f(t, y - u, z - v) | \Pi_{i=1}^d \psi(m u_i) \Pi_{j=1}^d \psi(m v_{ij}) d u d v \]
\[ \leq \mathcal{T}_t | y |^\alpha + | z |^\alpha' + \int_{\mathbb{R}^d} \left( | y - u |^\alpha - | y |^\alpha \right) \Pi_{i=1}^d \psi(m u_i) d u \]
\[ + m^{d^2} \int_{\mathbb{R}^d} \left( | z - v |^\alpha' - | z |^\alpha' \right) \Pi_{i=1}^d \Pi_{j=1}^d \psi(m v_{ij}) d v \]
\[ \leq \mathcal{T}_t | y |^\alpha + | z |^\alpha' + \alpha(n + h_t) \alpha' - 1 - h_t n^{2p} + \alpha'(n + h_t) \alpha' - 1 - h_t n^{2p} \]
\[ \leq \mathcal{T}_t + | y |^\alpha + | z |^\alpha' + 2ph_t \]

(d) For all \((t, \omega, y, z)\) such that \( \mathcal{X}_t \leq n, \ | y | \leq n \) and \( | z | \leq n \) we obtain, by using assumptions \( H.2 \) - \( H.3 \), that
\[ \langle y, f_n(t, y, z) \rangle \leq (c_1 e)^2 (n^{-2} | y |^2) | y | n^{-2} | z |^2 \times \]
\[ m^{(d^2)} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \langle f(t, y - u, z - v), y - u \rangle \Pi_{i=1}^d \psi(m u_i) \Pi_{j=1}^d \psi(m v_{ij}) d u d v \]
\[ + m^{(d^2)} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} | f(t, y - u, z - v) | | u | \Pi_{i=1}^d \psi(m u_i) \Pi_{j=1}^d \psi(m v_{ij}) d u d v \]
\[ \leq \eta + f_T | y | + M_T | y |^2 + K_T | y | | z | + 10h_T \]

\[ \text{Remark 3.2. Theorem 2.1 follows now from Proposition 3.4 and Proposition 3.5.} \]

4 Application to partial differential equations (PDEs)

In this section, we consider the system of semilinear PDEs associated to the Markovian version of the BSDE \((E^{\xi, f})\), for which we establish the existence and uniqueness of a weak (Sobolev) solution. In particular, we give a new feature which consists to prove, by using BSDEs techniques, that the uniqueness for a nonhomogeneous system of semilinear PDE follows from the uniqueness of its associated homogeneous system of linear PDE.

4.1 Formulation of the problem.

Let \( \sigma : \mathbb{R}^k \rightarrow \mathbb{R}^{kr}, \ b : \mathbb{R}^k \rightarrow \mathbb{R}^k, \ g : \mathbb{R}^k \rightarrow \mathbb{R}^k \), and \( F : [0, T] \times \mathbb{R}^k \times \mathbb{R}^d \times \mathbb{R}^{d r} \rightarrow \mathbb{R}^d \) be measurable functions. Consider the system of semilinear PDEs

\[ (P(\sigma, F)) \quad \left\{ \begin{array}{ll}
\partial u(t, x) + Lu(t, x) + F(t, x, u(t, x), \sigma \nabla u(t, x)) = 0 & t \in [0, T], \ x \in \mathbb{R}^k \\
u(T, x) = g(x) & x \in \mathbb{R}^k
\end{array} \right. \]

where \( L := \frac{1}{2} \sum_{i,j} (\sigma \sigma^*)_{ij} \partial_{x_i}^2 + \sum_i b_i \partial_i \).

The diffusion process associated to the operator \( L \) satisfies,
\[ X^{t,x}_s = x + \int_t^s b(X^{r,x}_r) d r + \int_t^s \sigma(X^{r,x}_r) d W_r, \quad t \leq s \leq T \]

We assume throughout this section that \( \sigma \in C^3_b(\mathbb{R}^k, \mathbb{R}^{kr}), \) and \( b \in C^3_b(\mathbb{R}^k, \mathbb{R}^k) \).

We define,
\[ \mathcal{H}^+ := \bigcup_{\delta \geq 0, \beta > 1} \left\{ v \in C([0, T]; L^\beta(\mathbb{R}^k, e^{-\delta |x|} d x; \mathbb{R}^d)) : \int_0^T \int_{\mathbb{R}^k} |\sigma \nabla v(s, x)|^\beta e^{-\delta |x|} d x d s < \infty \right\} \]
Definition 4.1. A (weak) solution of \((P^{(g,F)})\) is a function \(u \in \mathcal{H}^{1+}\) such that for every \(t \in [0,T]\) and \(\varphi \in C_{c}^{1}([0,T] \times \mathbb{R}^{d})\)

\[
\int_{t}^{T} < u(s), \frac{\partial \varphi(s)}{\partial s} > ds + < u(t), \varphi(t) > = < g, \varphi(T) > + \int_{t}^{T} < F(s, u(s), \sigma^{*} \nabla u(s)), \varphi(s) > ds + \int_{t}^{T} < Lu(s), \varphi(s) > ds
\]

where \(f(s, h(s)) = \int_{\mathbb{R}^{d}} f(s, x)h(x)dx\).

Observe that an integrating by part shows that,

\[
< Lu(s), \varphi(s) > = - \int_{\mathbb{R}^{d}} \frac{1}{2} (\sigma^{*} \nabla u(s, x); \sigma^{*} \nabla \varphi(s, x)) dx - < u(s), \text{div}(\tilde{b} \varphi)(s) >
\]

where \(\tilde{b} := b_{i} - \frac{1}{2} \sum_{j} \partial_{j}(\sigma \sigma^{*})_{ij}\)

4.2 Assumptions

Consider the following assumptions:

There exist \(\delta \geq 0\) and \(\overline{p} > 1\) such that

(A.0) \(g(x) \in L^{p}([\mathbb{R}^{k}, e^{-\delta|x|}]; \mathbb{R}^{d})\)

(A.1) \(F(t, x, ..., )\) is continuous for a.e. \((t, x)\)

(A.2) \(\begin{cases}
\text{There are } \eta' \in L^{p_{1}}([0, T] \times \mathbb{R}^{k}, e^{-\delta|x|}dtdx; \mathbb{R}^{+}),
\text{and } M, M' \in \mathbb{R}^{+} \text{ such that}
\langle y, F(t, x, y, z) \rangle \leq \eta'(t, x) + f^{0'}(t, x)y + (M + M'|x|)|y|^{2} + \sqrt{M + M'|x|}||y||z|
\end{cases}\)

(A.3) \(\begin{cases}
\text{There are } \mathcal{F} \in L^{q}([0, T] \times \mathbb{R}^{k}, e^{-\delta|x|}dtdx; \mathbb{R}^{+}) \text{ (for some } q > 1), \alpha \in [1, \overline{p}] \text{ and } \alpha' \in [1, \overline{p} \wedge 2] \text{ such that}
|F(t, x, y, z)| \leq \mathcal{F}(t, x) + |y|^{\alpha} + |z|^{\alpha'}
\end{cases}\)

(A.4) \(\begin{cases}
\text{There are } K, r \in \mathbb{R}^{+} \text{ such that for every } N \in \mathbb{N} \text{ and every } x, y, y', z, z' \text{ satisfying: } e^{r|x|}, |y|, |y'|, |z|, |z'| \leq N,
\langle y - y'; F(t, x, y, z) - F(t, x, y', z') \rangle \leq K \log N \left(\frac{1}{N} + |y - y'|^{2}\right) + \sqrt{K \log N} |y - y|| z - z'|.
\end{cases}\)

4.3 Existence and uniqueness for \((P^{(g,F)})\)

Theorem 4.1. Let \(p \in [\alpha \vee \alpha', \overline{p}]\) if \(M' > 0\) and \(p = \overline{p}\) if \(M' = 0\). Under assumption (A.0)-(A.4) we have

1) The PDE \((P^{(g,F)})\) has a unique (weak) solution \(u\) on \([0, T]\)
2) For every \(t \in [0, T]\) there exists \(D_{t} \subset \mathbb{R}^{k}\) such that
   i) \(\int_{\mathbb{R}^{k}} 1_{D_{t}} dx = 0\), where \(D_{t} := \mathbb{R}^{k} \setminus D_{t}\).
   ii) for every \(t \in [0, T]\) and every \(x \in D_{t}\), the BSDE \((E^{(\xi^{t,x}, f^{t,x})})\) has a unique solution \((Y^{t,x}, Z^{t,x})\) on \([t, T]\)
   where \(\xi^{t,x} := g(X^{t,x}_{T})\) and \(f^{t,x}(s, y, z) := 1_{(s > t)} F(s, X^{t,x}_{s-}, y, z)\)
3) For every $t \in [0, T]$

$$
(u(s, X^{t,x}_s), \sigma^* \nabla u(s, X^{t,x}_s)) = (Y^{t,x}_s, Z^{t,x}_s) \quad \text{a.e.}(s, x, \omega)
$$

4) There exists a positive constant $C$ depending only on $\delta, M, M', p, \bar{p}, |\sigma|_\infty, |b|_\infty$ and $T$ such that

$$
\sup_{0 \leq t \leq T} \int \mathbb{R}^k |u(t, x)|^p e^{-\delta'|x|} dx + \int_0^T \int \mathbb{R}^k |\sigma^* \nabla u(t, x)|^{p\wedge 2} e^{-\delta'|x|} dtdx \\
\leq C \left( \mathbb{I}_{[M' \not= 0]} + \int \mathbb{R}^k |g(x)|^\bar{p} dx + \int_0^T \int \mathbb{R}^k \eta'(s, x) \bar{p}^\wedge 1 dsdx + \int_0^T \int \mathbb{R}^k f^\wedge(s, x) \bar{p} dsdx \right)
$$

where $\delta' = \delta + \kappa' + \mathbb{I}_{[M' \not= 0]}$ and $\kappa' := \frac{\eta M'T}{(\bar{p} - p)} \sup(4, \frac{2p}{p-1})$.

4.4 Proof of Theorem 4.1

A) Existence.

**Lemma 4.1.** 1) There exists $\kappa > 0$ depending only on $|\sigma|_\infty, |b|_\infty$ and $T$ such that

$$
\sup_{t \leq s \leq T} \mathbb{E}[\exp(\kappa \sup_{t \leq s \leq T} |X^{t,x}_s - x|^2)] < \infty. \quad (4.0)
$$

In particular, for every $r > 0$ there is a constant $C(r, \kappa)$ such that for each $(t, x)$

$$
\mathbb{E}[\exp(r \sup_{t \leq s \leq T} |X^{t,x}_s - x|)] \leq C(r, \kappa) \exp(r |x|)
$$

2) For every $\delta \geq 0$ there exists a constant $C_{\delta, T} > 1$ such that for every $\varphi \in L^0(\mathbb{R}^k)$, $t \in [0, T]$ and $s \in [t, T]$

$$
C_{\delta, T}^{-1} \int \mathbb{R}^k |\varphi(x)| e^{-\delta|x|} dx \leq \mathbb{E} \int \mathbb{R}^k |\varphi(X^{t,x}_s)| e^{-\delta|x|} dx \leq C_{\delta, T} \int \mathbb{R}^k |\varphi(x)| e^{-\delta|x|} dx. \quad (4.2)
$$

Moreover for every $\delta \geq 0$ there exists a constant $C_{\delta, T} > 1$ such that for every $\psi \in L^0([0, T] \times \mathbb{R}^k)$, $t \in [0, T]$ and $s \in [t, T]$

$$
C_{\delta, T}^{-1} \int_0^T \int_t^T |\psi(s, x)| dse^{-\delta|x|} ds \leq \mathbb{E} \int_0^T \int_t^T |\psi(s, X^{t,x}_s)| dse^{-\delta|x|} ds \leq C_{\delta, T} \int_0^T \int_t^T |\psi(s, x)| dse^{-\delta|x|} ds.
$$

**Proof.** The first assertion is well known. Its particular case follows by using triangular and Young’s inequalities. Indeed

$$
\mathbb{E}[\exp(r \sup_{t \leq s \leq T} |X^{t,x}_s - x|)] \leq \exp(r |x|) \mathbb{E}[\exp(\kappa \sup_{t \leq s \leq T} |X^{t,x}_s - x|)]
$$

$$
\leq \exp(r |x|) \mathbb{E}[\exp(\frac{r}{\sqrt{\kappa}} \sup_{t \leq s \leq T} |X^{t,x}_s - x|)]
$$

$$
\leq \exp(\frac{r^2}{\kappa}) \exp(r |x|) \mathbb{E}[\exp(\kappa \sup_{t \leq s \leq T} |X^{t,x}_s - x|^2)].
$$

For the second assertion, see [8] Proposition 5.1.
Lemma 4.2. Let $p \in [\alpha \wedge \alpha', \overline{p}]$ if $M' > 0$ and $p = \overline{p}$ if $M' = 0$. Let $t \in [0, T]$. There exists $D_t \subset \mathbb{R}^k$ such that

i) $\int_{D_t} 1 \, dx = 0$

ii) for every $x \in D_t$

\[
\mathbb{E}(\{g(X_T^t, x) \mid p \in \mathcal{F}_t \}) \leq \mathbb{E}(\int_t^T \eta'(s, X_s^t, x) e^\int_s^t \lambda_s^t \, ds \, dr) + \mathbb{E}(\int_t^T \eta'(s, X_s^t, x) e^\int_s^t \lambda_s^t \, ds \, ds) + \mathbb{E}(\int_t^T \eta'(s, X_s^t, x)) + \mathbb{E}(\int_t^T \eta'(s, X_s^t, x)) < +\infty,
\]

where $\lambda_s^t := (M + M'|X_t^x|) sup(4, \frac{2p}{p-1})$.

Proof. Using H"{o}lder's inequality, Young's inequality and Lemma 1.1 we get

\[
\mathbb{E}(\{g(X_T^t, x) \mid p \in \mathcal{F}_t \}) \leq \mathbb{E}(\int_t^T \eta'(s, X_s^t, x) e^\int_s^t \lambda_s^t \, ds \, dr) + \mathbb{E}(\int_t^T \eta'(s, X_s^t, x) e^\int_s^t \lambda_s^t \, ds \, ds) + \mathbb{E}(\int_t^T \eta'(s, X_s^t, x)) + \mathbb{E}(\int_t^T \eta'(s, X_s^t, x)) < +\infty
\]

for some constant $C$ depending only on $M, M', p, \overline{p}, |\sigma|_\infty, |b|_\infty$ and $T$.

Using Lemma 1.12 and assumptions (A.0)-(A.4), one can show that

\[
\int_{\mathbb{R}^k} e^{-\delta'|x|} dx < \infty
\]

where $\delta' = \delta + \kappa' + c$. The set $D_t := \{x: \Gamma_t^t < \infty\}$. Lemma 1.2 is proved.

Lemma 4.3. Assume (A.0)-(A.4). Let $p \in [\alpha \wedge \alpha', \overline{p}]$ if $M' > 0$ and $p = \overline{p}$ if $M' = 0$. Then, for every $t \in [0, T]$ and every $x \in D_t$, the BSDE $(E(X^{t,x}, f^{t,x}))$ has a unique solution $(Y^{t,x}, Z^{t,x})$ which satisfies, for every $t \in [0, T]$ and every $x \in D_t$

\[
\mathbb{E}(\sup_{t \leq s \leq T} |Y_s^t|^p + \mathbb{E}(\int_t^T |Z_s^t|^2 ds) < +\infty
\]

for some constant $C$ depending only on $M, M', p, \overline{p}, |\sigma|_\infty, |b|_\infty$ and $T$.

Proof. For every $t \in [0, T]$ and $x \in D_t$, $(E(X^{t,x}, f^{t,x})$ satisfies (H.0)-(H.4) with $\gamma = \inf\{1, \frac{p-1}{4}\}$,

\[
M_s = M + M'|X_s^x|,\ K_s = \sqrt{M + M'|X_s^x|},\ \eta_s = \eta'(s, X_s^x),\ f_0^s = f_0'(s, X_s^x),\ \overline{\eta}_s = \overline{\eta}'(s, X_s^x),\ v_s = \exp(|\sigma|X_s^x)\text{ and } A_N = N.
\]

Hence, Lemma 1.3 follows from Theorem 2.1 and Lemma 1.2.
\[ \xi_t^{t,x} := g_n(X_t^{t,x}) \]

and

\[ f_n^{t,x}(s,y,z) := \mathbb{I}_{\{s > t\}} F_n(s, X_t^{t,x}, y, z). \]

It is not difficult to see that the sequence \((g_n, F_n)\) satisfies (A.0)-(A.3) uniformly in \(n\). Hence \((\xi_n^{t,x}, f_n^{t,x})\) satisfies (H.0)-(H.3) uniformly in \(n\). Moreover, for every \(n \in \mathbb{N}^*\), \((\xi_n^{t,x}, f_n^{t,x})\) is bounded and \(f_n^{t,x}\) is globally Lipschitz.

Let \((Y^{t,x,n}, Z^{t,x,n})\) be the unique solution of BSDE \((E(\xi_n^{t,x}, f_n^{t,x}))\). Let \(p \in [\alpha, \alpha', \bar{p}]\) if \(M' > 0\) and \(p = \bar{p}\) if \(M' = 0\). Arguing as in Lemma 4.3 we show that for every \(t, x \in D_t\) and every \(n \in \mathbb{N}^*\)

\[
\mathbb{E}\left( \sup_{t \leq s \leq T} |Y^{t,x,n}_s|^p \right) + \mathbb{E}\left( \int_t^T |Z^{t,x,n}_s|^2 \, ds \right)^{\frac{p}{2}} \leq C \left( \mathbb{E}\left( \int_t^T e^{-\gamma(|X^{t,x}_s|)} \, ds \right) + \mathbb{E}\left( \int_t^T |g(X^{t,x}_s)| \, ds \right) \right) + \mathbb{E}\left( \int_t^T \eta'(s, X^{t,x}_s, Z^{t,x}_s) \, ds \right)
\]

for some constant \(C = C(\bar{p})\) not depending on \((t, x, n)\). To see this, use Proposition 3.5 (with \(h_s := e^{-\gamma(|X^{t,x}_s|)}\), Proposition 3.1 and the proof of proposition 3.4a).

According to [8] (see also [10]) we have

Lemma 4.4. There exists a unique solution \(u^n\) to the problem,

\[
(P(g_n, F_n)) \begin{cases} \frac{\partial u^n(t,x)}{\partial t} + Lu^n(t,x) + F_n(t,x,u^n(t,x),\sigma^n\nabla u^n(t,x)) = 0, & t \in [0,T], x \in \mathbb{R}^k \\
\frac{\partial u^n(T,x)}{\partial t} = g_n(x), & x \in \mathbb{R}^k 
\end{cases}
\]

such that for every \(t\)

\[ u^n(s, X_s^{t,x}) = Y_s^{t,x,n} \quad \text{and} \quad \sigma^n \nabla u^n(s, X_s^{t,x}) = Z_s^{t,x,n} \quad \text{a.e.} (s, \omega, x). \]

From Proposition 3.4(ii) we have

Lemma 4.5. (Stability) For every \(t \in [0,T], x \in D_t\) and \(p' < \bar{p}\),

\[ \lim_{n \to \infty} \left[ \mathbb{E}\left( \sup_{0 \leq s \leq T} |Y_s^{t,x,n} - Y_s^{t,x}|^p \right) + \mathbb{E}\left( \int_t^T |Z_s^{t,x,n} - Z_s^{t,x}|^2 \, ds \right)^{\frac{p}{2}} \right] = 0. \]

Using Lemma 4.1, inequality (4.4), Lemma 4.4, Lemma 4.5 and the Lebesgue dominated convergence theorem, we obtain

Lemma 4.6. (Convergence of PDE) For every \(p' < \bar{p}\),

\[ \lim_{n,m} \sup_{0 \leq t \leq T} \int_{\mathbb{R}^k} |u^n(t,x) - u^m(t,x)|^{p'} e^{-\delta|x|} \, dx = 0 \]

\[ \lim_{n,m} \int_0^T \int_{\mathbb{R}^k} |\sigma^n \nabla u^n(t,x) - \sigma^m \nabla u^m(t,x)|^{p''} e^{-\delta|x|} \, dx \, dt = 0. \]

Using Lemma 3.1, Lemma 4.9 and the fact that \(\mathcal{H}^{1+}\) is complete, we prove that exists \(u \in \mathcal{H}^{1+}\) such that for every \(p' < \bar{p}\),

i) \(\sup_{0 \leq t \leq T} \int_{\mathbb{R}^k} |u(t,x)|^{p'} e^{-\delta|x|} \, dx + \int_0^T \int_{\mathbb{R}^k} |\sigma^n \nabla u(t,x)|^{p''} e^{-\delta|x|} \, dx \, dt < \infty\)

ii) \(\lim_{n} \sup_{0 \leq t \leq T} \int_{\mathbb{R}^k} |u^n(t,x) - u(t,x)|^{p'} e^{-\delta|x|} \, dx = 0\)

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Lemma 4.7.

Uniqueness. B) has a unique solution \( \phi \) estimate for a regularized degenerate PDE.

In another hand, from Proposition 3.2 and Proposition 3.4 we respectively have for every \( t \in [0, T] \) and \( x \in D_t 
\]

\[
\mathbb{E} \int_t^T |F_n(s, X_s^{t,x}, u^n(s, X_s^{t,x}), \sigma \nabla u^n(s, X_s^{t,x}))|^{\frac{3}{2}} ds \leq C \left( 1 + \Theta_p^{t,x,n} + \mathbb{E} \int_t^T |\eta(s, X_s^{t,x})|^q ds \right)
\]

and

\[
\lim_{n} \mathbb{E} \int_t^T |F_n(s, X_s^{t,x}, u^n(s, X_s^{t,x}), \sigma \nabla u^n(s, X_s^{t,x})) - F(s, X_s^{t,x}, u(s, X_s^{t,x}), \sigma \nabla u(s, X_s^{t,x})))|^{\frac{3}{2}} ds = 0
\]

\( \beta \) is some real in \([1, \infty[ \), \( C \) is some constant not depending on \( (t, x, n) \) and

\[
\Theta_p^{t,x,n} = \mathbb{E} \sup_{s \in [0, T]} |V_s^{t,x,n}|^p + \mathbb{E} \left( \int_0^T |Z_s^{t,x,n}|^2 ds \right)^{\frac{p}{2}}.
\]

We deduce from Lemma 4.1 the Lebesgue dominated convergence theorem and inequality (4.4) that

\[
\lim_{n} \int_0^T \int_{\mathbb{R}^d} |F_n(s, x, u^n(s, x), \sigma \nabla u^n(s, x)) - F(s, x, u(s, x), \sigma \nabla u(s, x))|^{\frac{3}{2}} e^{-\left(1+\varepsilon\right)|\xi|} dx ds = 0.
\]

As a consequence of Lemma 4.3 and the proof of Proposition 3.4 we get the following existence result for the problem \((P^{(g,F)})\).

**Proposition 4.1.** Under assumptions \((A.0)-(A.4)\), the PDE \((P^{(g,F)})\) has a unique solution \( u \) such that \( u(s, X_s^{t,x}) = Y_s^{t,x} \) and \( \sigma \nabla u(s, X_s^{t,x}) = Z_s^{t,x} \). Moreover, letting \( p \in [a \vee \alpha', \bar{p}] \) if \( M > 0 \) and \( p = \bar{p} \) if \( M = 0 \), then there is a constant \( C \) depending only on \( \delta', M, \bar{p}, \bar{p}, |\sigma|_{\infty}, |b|_{\infty} \) and \( T \) such that

\[
\sup_{0 \leq t \leq T} \int_{\mathbb{R}^d} |u(t, x)|^p e^{-\delta'|x|} dx + \int_0^T \int_{\mathbb{R}^d} |\sigma \nabla u(t, x)|^p e^{-\delta'|x|} dx dt \leq C \left( 1 + \int_{\mathbb{R}^d} |g(x)|^p dx + \int_0^T \int_{\mathbb{R}^d} |\eta(s, x)|^{2\bar{p}+1} ds dx + \int_0^T \int_{\mathbb{R}^d} |f^{\bar{p}}(s, x)|^{2\bar{p}+1} ds dx \right)
\]

where \( \delta' = \delta + \kappa' + 1 \) and \( \kappa' := \frac{\bar{p}M'M}{(\bar{p}-\bar{p})} \sup_{(4, \frac{2p}{p-1})} \).

**B) Uniqueness.**

Due to the degeneracy of the diffusion coefficient, the solution of the homogeneous linear PDEs is not sufficiently smooth and hence we can not use it as a test function. In order to construct a suitable test function, we need the following lemma. This lemma is interesting in itself since it gives a uniform estimate for a regularized degenerate PDE.

Let \( W^{1,2}_q([0, T] \times \mathbb{R}^d) \) denotes the Sobolev space of all functions \( u(t, x) \) defined on \( \mathbb{R}_+ \times \mathbb{R}^d \) such that both \( u \) and all the generalized derivatives \( D_t u, D_x u, \text{ and } D_{xx}^2 u \) belong to \( L^q([0, T] \times \mathbb{R}^d) \).

**Lemma 4.7.** Let \( \varepsilon \in [0, 1], g \in C_0^{\infty}([0, T] \times \mathbb{R}^d) \). Then, the PDE

\[
(P_\varepsilon(g)) \left\{ \begin{array}{l}
\frac{\partial \phi^\varepsilon(t, x)}{\partial t} - \frac{1}{2} \text{div}(\sigma \nabla \phi^\varepsilon) - \varepsilon \Delta \phi^\varepsilon(t, x) + (\bar{b}(t); \nabla \phi^\varepsilon(t, x)) = g(t, x) \\
\phi^\varepsilon(0, x) = 0 \quad x \in \mathbb{R}^d \end{array} \right.
\]

has a unique solution \( \phi^\varepsilon \) which satisfies:

(i) \( \phi^\varepsilon \in \bigcap_{q > \frac{2}{1+\varepsilon}} W^{1,2}_q([0, T] \times \mathbb{R}^d; \mathbb{R}) \cap C^{1,2}([0, T] \times \mathbb{R}^d; \mathbb{R}) \)

(ii) \( \sup_{(t,x)} \left\{ \left| \frac{\partial \phi^\varepsilon}{\partial t}(t, x) \right| + |\nabla \phi^\varepsilon(t, x)| + |\phi^\varepsilon(t, x)| \right\} < \infty. \)
Proof. The existence and uniqueness of the solution $\phi^\varepsilon$, follow from [39] (p. 318 and pp. 341–342). We shall prove an uniform estimates for $\phi^\varepsilon$ and for their first derivatives. These estimates can be established by adapting the proofs given in Krylov [38] pp. 330–344. However, we give here a probabilistic proof which is very simple. We assume that the dimension $k$ is 1. Let $X^\varepsilon_t(x)$ denotes the diffusion process associated to the problem $\left( P_\varepsilon(g) \right)$. For simplicity, we assume that $g$ does not depend from $t$ and the drift coefficient of $X^\varepsilon_t(x)$ is zero. The process $X^\varepsilon_t(x)$ is then the unique (strong) solution of the following SDE

$$X^\varepsilon_t(x) = x + \int_0^t \sigma^\varepsilon(X^\varepsilon_s(x))dW_s, \quad 0 \leq t \leq T$$

Let $M := \sup_{(t,x)} |\sigma^\varepsilon(X^\varepsilon_t(x))| + |\sigma(t,x)| + |\sigma'(t,x)|$. Since the coefficients $\sigma^\varepsilon$ is smooth and uniformly elliptic, then the solution $\phi^\varepsilon$ belongs to $C^{1,2}$. Hence, Itô’s formula shows that,

$$\phi^\varepsilon(t,x) = -\mathbb{E} \int_t^T g(X^\varepsilon_s(x))ds.$$ 

Since $g \in C^{\infty}_c$, we immediately get

$$\sup_{(t,x)} \left\{ \frac{\partial \phi^\varepsilon}{\partial t}(t,x) + |\phi^\varepsilon(t,x)| \right\} < \infty.$$ 

Since $\sigma^\varepsilon \in C^3_c$, we can show that

$$|\frac{\partial \phi^\varepsilon}{\partial x}(t,x)| \leq M \mathbb{E} \int_t^T |\frac{\partial X^\varepsilon_s(x)}{\partial x}|ds$$

It remains to show that $\sup_{(t,x)} \mathbb{E}(\frac{\partial X^\varepsilon_s(x)}{\partial x}) < \infty$.

Since $|\sigma^\varepsilon(t,x)| \leq |\sigma'(t,x)| \leq \sup_{(t,x)} |\sigma'(t,x)| \leq M$, we have

$$\mathbb{E}(\frac{\partial X^\varepsilon_s(x)}{\partial x})^2 \leq 1 + \mathbb{E} \int_0^t |\sigma^\varepsilon_s(X^\varepsilon_s(x))|^2|\frac{\partial X^\varepsilon_s(x)}{\partial x}|^2ds$$

$$\leq 1 + M^2 \mathbb{E} \int_0^t \left| \frac{\partial X^\varepsilon_s(x)}{\partial x} \right|^2ds$$

The Gronwall Lemma gives now the desired result.

In multidimensional case, the proof can be performed similarly since it is based on the fact that the first derivative of $\sigma^\varepsilon$ is bounded uniformly in $\varepsilon$, which is valid in multidimensional case also, see Freidlin [34], III § 3.2, pp. 188-193. Lemma 1.1.7 is proved.

Remark 4.1. (i) According to Krylov estimate (because $\sigma^\varepsilon$ is uniformly elliptic), the previous proof (in dimension one) remains valid also when the coefficients $\sigma$ and $b$ are Lipschitz only.

(ii) Since in our situation $\sigma \in C^3_\varepsilon(\mathbb{R}^k, \mathbb{R}^k)$ and $b \in C^2_\varepsilon(\mathbb{R}^k, \mathbb{R}^k)$, we can estimate also the second derivative of $\phi^\varepsilon$. More precisely we have

$$\sup_{(t,x)} \left\{ \phi^\varepsilon(t,x) + |\frac{\partial \phi^\varepsilon}{\partial t}(t,x)| + |\nabla \phi^\varepsilon(t,x)| + |D^2 \phi^\varepsilon(t,x)| \right\} < \infty.$$ 

Proof of Remark 4.1. Let $B_t$ be a $d$-dimensional Wiener process stochastically independent of $W_t$ and consider the SDE:

$$X^\varepsilon_t(s) = x + \int_s^t b(X^\varepsilon_r(s))dr + \int_s^t \sigma(X^\varepsilon_r(s))dW_r + \sqrt{2}\varepsilon(B_s - B_t), \quad t \leq s \leq T$$

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where \( \bar{b}(x) := \bar{b}(x) - \frac{1}{2} \sum_j \partial_j(\sigma\sigma^*)_j(x) = b(x) - \sum_j \partial_j(\sigma\sigma^*)_j(x) \)

Itô’s formula shows that,

\[
\phi^\varepsilon(T - t, x) = \mathbb{E} \int_t^T g(r, X_r^{t,x}(\varepsilon))dr
\]

Then

\[
\partial_t \phi^\varepsilon(T - t, x) = \mathbb{E} \int_t^T \langle \nabla g(r, X_r^{t,x}(\varepsilon)), \partial_t X_r^{t,x}(\varepsilon) \rangle dr
\]

and

\[
\partial^2_{ij} \phi^\varepsilon(T - t, x) = \mathbb{E} \int_t^T \langle \nabla^2 g(r, X_r^{t,x}(\varepsilon)), \partial^2_{ij} X_r^{t,x}(\varepsilon) \rangle + \langle D^2 g(r, X_r^{t,x}(\varepsilon)) \partial_t X_r^{t,x}(\varepsilon), \partial_j X_r^{t,x}(\varepsilon) \rangle dr
\]

On other hand,

\[
\partial_t (X_r^{t,x})_k(x) = \delta_{ik} + \int_t^s \langle \nabla \bar{b}_k(X_r^{t,x}(\varepsilon)), \partial_t X_r^{t,x}(\varepsilon) \rangle dr + \sum_n \int_t^s \langle \nabla \sigma_{kn}(X_r^{t,x}(\varepsilon)), \partial_t X_r^{t,x}(\varepsilon) \rangle dW^n_r
\]

and

\[
\partial^2_{ij} (X_r^{t,x})_k(x) = \int_t^s \langle \nabla \bar{b}_{ij}(X_r^{t,x}(\varepsilon)), \partial_{ij} X_r^{t,x}(\varepsilon) \rangle dr + \sum_n \int_t^s \langle \nabla \sigma_{kn}(X_r^{t,x}(\varepsilon)), \partial_{ij} X_r^{t,x}(\varepsilon) \rangle dW^n_r + \int_t^s \langle \nabla \sigma_{kn}(X_r^{t,x}(\varepsilon)), \partial_{ij} X_r^{t,x}(\varepsilon) \rangle dW^n_r
\]

Itô’s formula gives

\[
\mathbb{E} |\partial_t (X_r^{t,x})_k(x)|^4 \leq \delta_{ik} + 4 \mathbb{E} \int_t^s \langle \nabla \bar{b}_k(X_r^{t,x}(\varepsilon)), \partial_t X_r^{t,x}(\varepsilon) \rangle (\partial_t (X_r^{t,x})_k(x))^3 dr + 6 \sum_n \mathbb{E} \int_t^s |\langle \nabla \sigma_{kn}(X_r^{t,x}(\varepsilon)), \partial_t X_r^{t,x}(\varepsilon) \rangle|^2 (\partial_t (X_r^{t,x})_k(x))^2 dr
\]

and

\[
\mathbb{E} |\partial^2_{ij} (X_r^{t,x})_k(x)|^2 \leq 2 \mathbb{E} \int_t^s \langle \nabla \bar{b}_{ij}(X_r^{t,x}(\varepsilon)), \partial_{ij} X_r^{t,x}(\varepsilon) \rangle (\partial_t (X_r^{t,x})_k(x))^3 dr + \sum_n \mathbb{E} \int_t^s |\langle \nabla \sigma_{kn}(X_r^{t,x}(\varepsilon)), \partial_{ij} X_r^{t,x}(\varepsilon) \rangle|^2 dr + 2 \mathbb{E} \int_t^s \langle \nabla \sigma_{kn}(X_r^{t,x}(\varepsilon)), \partial_{ij} X_r^{t,x}(\varepsilon) \rangle (\partial_t X_r^{t,x}(\varepsilon))^2 (\partial_t (X_r^{t,x})_k(x))^2 dr
\]

\[
\leq \sup_x (2 |\nabla \bar{b}_k(x)| + 2 |D^2 \bar{b}_k(x)| + \sum_n |\nabla \sigma_{kn}(x)|^2) \mathbb{E} \int_t^s (\partial_t X_r^{t,x}(\varepsilon))^4 + (\partial_t X_r^{t,x}(\varepsilon))^4 dr
\]

\[
+ \sup_x (|D^2 \bar{b}_k(x)| + \sum_n |D^2 \sigma_{kn}(x)|^2) \int_t^s \mathbb{E} |\partial_{ij} X_r^{t,x}(\varepsilon)|^4 + \mathbb{E} |\partial_{ij} X_r^{t,x}(\varepsilon)|^4 dr
\]
We deduce that

\[ \mathbb{E} |\partial_t (X^{t,x}_s)(\varepsilon)|^4 \leq k^2 + k^2 \sum_{j} \sup_{x} (2|\nabla b_\varepsilon(x)| + \sum_{n} |\nabla \sigma_{jn}(x)|^2) \int_t^\infty \mathbb{E} |\partial_t (X^{t,x}_s)(\varepsilon)|^4 dr \]
\[ \leq k^2 e^{k^2 T \sum_{j} \sup_{x} (2|\nabla b_\varepsilon(x)| + \sum_{n} |\nabla \sigma_{jn}(x)|^2)} \quad \text{(Gronwall's Lemma)} \]

and

\[ \mathbb{E} |\partial_{ij}^2 (X^{t,x}_s)(\varepsilon)|^2 \leq k \sup_{x} (2|\nabla b_\varepsilon(x)| + 2|D^2 \tilde{b}(x)| + \sum_{n} |\nabla \sigma_{kn}(x)|^2) \mathbb{E} \int_t^\infty |\partial_{ij}^2 (X^{t,x}_s)(\varepsilon)|^2 dr \]
\[ + k^2 T \sup_{x} (|D^2 \tilde{b}(x)| + \sum_{n} |D^2 \sigma_{kn}(x)|^2) k^2 e^{k^2 T \sum_{j} \sup_{x} (2|\nabla b_\varepsilon(x)| + \sum_{n} |\nabla \sigma_{jn}(x)|^2)} \]
\[ \leq k^2 T \sup_{x} (|D^2 \tilde{b}(x)| + \sum_{n} |D^2 \sigma_{kn}(x)|^2) k^2 e^{k^2 T \sum_{j} \sup_{x} (2|\nabla b_\varepsilon(x)| + \sum_{n} |\nabla \sigma_{jn}(x)|^2)} \quad \text{(Gronwall's Lemma)} \]

Since \( g \in C_0^\infty, \sigma \in C_b^3(\mathbb{R}^k, \mathbb{R}^k) \) and \( b \in C_0^2(\mathbb{R}^k, \mathbb{R}^k) \) we get

\[ \sup_{t \leq T} \{ \partial^\beta \phi^\varepsilon(t, x) \} < \infty. \]

Lemma 4.7 is proved.

**Lemma 4.8.** 0 is the unique solution of the PDE

\[ \begin{cases} \partial_t w(t, x) + LW(t, x) + \text{div}(\tilde{b}(x))w(t, x) = 0 & t \in [0, T], x \in \mathbb{R}^k \\ w(T, x) = 0 & x \in \mathbb{R}^k \end{cases} \]
satisfying for some \( \beta > 0 \)

\[ \sup_{t \leq T} \int_{\mathbb{R}^k} |w(t, x)|^\beta + |w(t, x)| \, dx + \int_0^T \int_{\mathbb{R}^k} |\sigma^* \nabla w(t, x)|^\beta + |\sigma^* \nabla w(t, x)| \, dt \, dx < \infty. \quad (4.1) \]

**Proof.** Let \( w \) be a solution of \((P(0, -\text{div}(\tilde{b}(x)y)))\) satisfying (1.1) and consider \( w_n \in C_0^\infty(\mathbb{R}^k) \) such that

\[ \int_0^T \int_{\mathbb{R}^k} |w(s, x) - w_n(s, x)| \, dx \, ds + \int_0^T \int_{\mathbb{R}^k} |\sigma^* \nabla w(s, x) - w_n(s, x)| \, dx \, ds \to 0. \]

Let \( \varepsilon \in [0, 1], \; g \in C_0^\infty([0, T] \times \mathbb{R}^k; \mathbb{R}) \) and consider the unique solution \( \phi^\varepsilon \in \cap_{\gamma > \frac{1}{2}} W^{1,2}_{\gamma, \alpha}([0, T] \times \mathbb{R}^k; \mathbb{R}) \cap C^{1,2}([0, T] \times \mathbb{R}^k; \mathbb{R}) \) of the following problem

\[ (P_\varepsilon(g)) \quad \begin{cases} \partial \phi^\varepsilon(t, x) - \frac{1}{2} \text{div}(\sigma^* \nabla \phi^\varepsilon) - \varepsilon \Delta \phi^\varepsilon(t, x) + \tilde{b}(x) \nabla \phi^\varepsilon(t, x) = g(t, x) \end{cases} \]

The existence and uniqueness of \( \phi^\varepsilon \) follows from Lemma 4.7.

Let \( (\psi_i)_{i \in \mathbb{N}} \subset C_0^\infty(\mathbb{R}^k) \) be such that \( \psi_i \in [0, 1], \; \psi_i \to 1 \) uniformly on every compact set and \( \nabla \psi_i \to 0 \) uniformly on \( \mathbb{R}^k \). By considering \( \phi^\varepsilon \psi_i \) as a test function, we have

\[ \int_0^T \int_{\mathbb{R}^k} \left[ \frac{\partial \phi^\varepsilon}{\partial t} + \frac{1}{2} \langle \sigma^* \nabla w; \sigma^* \nabla \phi^\varepsilon \rangle + w(\tilde{b}; \nabla \phi^\varepsilon) \right] \psi_i \, dx \, dt + \int_0^T \int_{\mathbb{R}^k} \left[ \frac{1}{2} \langle \sigma^* \nabla w; \sigma^* \nabla \psi_i \rangle + w(\tilde{b}; \nabla \psi_i) \right] \phi^\varepsilon \, dx \, dt = 0. \]
Introducing \( w_n \) and integrating by part we obtain
\[
\int_0^T \int_{\mathbb{R}^k} w_n \psi_1 \left[ \frac{\partial \phi^\varepsilon}{\partial t} - \frac{1}{2} \text{div}(\sigma \sigma^* \nabla \phi^\varepsilon) + \langle \tilde{b}; \nabla \phi^\varepsilon \rangle \right] dt \, dx = \chi_{1,n}^\varepsilon(n) + \chi_{2,n}^\varepsilon(i),
\]
where
\[
\chi_{1,n}^\varepsilon(n) := -\int_0^T \int_{\mathbb{R}^k} \left[ (w - w_n) \frac{\partial \phi^\varepsilon}{\partial t} + \frac{1}{2} \sigma^* \nabla (w - w_n) \cdot \sigma^* \nabla \phi^\varepsilon + (w - w_n) \langle \tilde{b}; \nabla \phi^\varepsilon \rangle \right] \psi_1 \, dx \, dt
\]
and
\[
\chi_{2,n}^\varepsilon(i) := -\int_0^T \int_{\mathbb{R}^k} \frac{1}{2} \phi^\varepsilon \sigma^* \nabla w + \phi^\varepsilon w \tilde{b} - \frac{1}{2} w_n \sigma^* \nabla \phi^\varepsilon \cdot \nabla \psi_1 \, dx \, dt.
\]
From Lemma 4.7 we have
\[
\sup \sup \left\{ \frac{\partial \phi^\varepsilon}{\partial t}(t, x) + |\nabla \phi^\varepsilon(t, x)| + |\phi^\varepsilon(t, x)| \right\} < \infty.
\]
Hence
\[
\sup_{\varepsilon,i} |\chi_{1,n}^\varepsilon(n)| \longrightarrow_{n \to \infty} 0
\]
and
\[
\sup_{\varepsilon,i} |\chi_{2,n}^\varepsilon(i)| \longrightarrow_{i \to \infty} 0.
\]
Observe that an integrating by part shows that
\[
\int_0^T \int_{\mathbb{R}^k} w_n \psi_1 \phi^\varepsilon \, dx \, dt = -\int_0^T \int_{\mathbb{R}^k} \nabla (w_n \psi_1) \nabla \phi^\varepsilon \, dx \, dt,
\]
then use the Lebesgue dominated convergence theorem to deduce that
\[
\int_0^T \int_{\mathbb{R}^k} w g(t, x) \, dx \, dt = \lim_{n} \lim_{i} \int_0^T \int_{\mathbb{R}^k} w_n \psi_1 (g(t, x) + \varepsilon \nabla \phi^\varepsilon) \, dx \, dt = \lim_{n} \lim_{i} (\chi_{1,n}^\varepsilon(n) + \chi_{2,n}^\varepsilon(i)) = 0.
\]
Lemma 4.8 is proved.

**Proof of uniqueness for \( (\mathcal{P}(g,F)) \).** The proof is divided into three steps.

**Step 1.** 0 is the unique solution of \( (\mathcal{P}(0,0)) \) satisfying the inequality (77) Lemma 4.8.

Let \( w_1 \) be a solution of \( (\mathcal{P}(0,0)) \) satisfying the inequality (77) Lemma 4.8. Then, by Lemma 4.8 it is also the unique solution of \( (\mathcal{P}(0,\text{div}(\sigma_x) - \text{div}(\sigma_x) w_1(t,x))) \) satisfying the inequality (77) Lemma 4.8. Indeed, if \( u \) is a solution of \( (\mathcal{P}(0,\text{div}(\sigma_x) - \text{div}(\sigma_x) w_1(t,x))) \), then \( u - w_1 \) is a solution of \( (\mathcal{P}(0,\text{div}(\sigma_x))) \) and hence \( u - w_1 = 0 \) by Lemma 4.8.

From Proposition 4.1 the process \( (w_1(s, X_{1,x}^s), \sigma^* \nabla w_1(s, X_{1,x}^s)) \) is the unique solution of BSDE \((\mathcal{E}(0,\text{div}(\sigma_x) - \text{div}(\sigma_x) w_1(t,x)))\). Thanks to the uniqueness of this BSDE and Lemma 4.2, we get \( w_1 = 0 \).

**Step 2.** 0 is the unique solution of \( (\mathcal{P}(0,0)) \).

Let \( w_1 \) be a solution of \( (\mathcal{P}(0,0)) \). Since \( w_1 \in \mathcal{H}^{1+} \), then there exist \( \beta' > 1, \delta' \geq 0 \) such that
\[
\sup_{0 \leq t \leq T} \int_{\mathbb{R}^k} |w_1(t, x)|^{\beta'} e^{-\delta'|x|} \, dx + \int_0^T \int_{\mathbb{R}^k} |\sigma^* \nabla w_1(t, x)|^{\beta'} e^{-\delta'|x|} \, dx \, dt < \infty.
\]
Let \( \delta > \delta' \) and set \( \tilde{w}_1 := w_1 f(x) \) where \( f \in C^2(\mathbb{R}^k; \mathbb{R}^+_* \text{ such that }) f(x) = e^{-\delta|x|} \) if \( |x| > 1 \).

By Lemma 4.8, \( \tilde{w}_1 \) is the unique solution to the PDE
Proposition 4.2. \( \exists \) there exists a positive constant \( C \) such that, if \( H \) and \( \mathbf{H} \) are some bounded and continuous functions. Proposition 4.1 implies that \( \tilde{w}_1(s, X^{t,x}_s) = \sigma^* \nabla \tilde{w}_1(s, X^{t,x}_s) \) is the unique solution of the BSDE \( E(0, \tilde{w}_1(s, X^{t,x}_s); \mathcal{F} - \mathcal{F}
abla \tilde{w}_1(s, X^{t,x}_s))) \). Hence \( \tilde{w}_1 = 0 \), which implies that \( w_1 = 0 \).

Step 3. \( (\mathcal{P}^{(y,F)}) \) has a unique solution if and only if 0 is the unique solution of \( (\mathcal{P}^{(0,0)}) \).

By Proposition 4.1 there exists a unique solution \( u \) of the problem \( (\mathcal{P}^{(y,F)}) \) such that, \( u(s, X^{t,x}_s) = Y^{t,x}_s \) and \( \sigma^* \nabla u(s, X^{t,x}_s) = Z^{t,x}_s \). Let \( u' \) be another solution of \( (\mathcal{P}^{(y,F)}) \) and set
\[
\hat{F}(t, x) = F(s, x, u(s, x), \sigma^* \nabla u(s, x)) - F(s, x, u'(s, x), \sigma^* \nabla u'(s, x)).
\]
The function \( w := u - u' \) is then a solution of the problem
\[
(\mathcal{P}^{(0,F)}) \quad \begin{cases} \frac{\partial w(t, x)}{\partial t} + Lw(t, x) + \hat{F}(t, x) = 0 & , \quad t \in [0, T], \ x \in \mathbb{R}^k \\ w(T, x) = 0 & , \quad x \in \mathbb{R}^k \end{cases}
\]
In other hand, since \( (0, \hat{F}) \) satisfies assumptions \( (A.0)-(A.4) \), then Proposition 4.1 ensures the existence of a unique solution \( \hat{w} \) to the problem \( (\mathcal{P}^{(0,F)}) \) such that, \( \hat{w}(s, X^{t,x}_s) = \hat{Y}^{t,x}_s \) and \( \sigma^* \nabla \hat{w}(s, X^{t,x}_s) = \hat{Z}^{t,x}_s \), where \( (\hat{Y}^{t,x}_s, \hat{Z}^{t,x}_s) \) is the unique solution of
\[
\hat{Y}^{t,x}_s = \int_s^T \hat{F}(r, X^{t,r}_r) dr - \int_s^T \hat{Z}^{t,x}_r dW_r.
\]
The uniqueness of \( (\mathcal{P}^{(0,F)}) \) (which follows from step 2) allows us to deduce that
\[
u'(s, X^{t,x}_s) = Y^{t,x}_s - \hat{Y}^{t,x}_s \quad \text{and} \quad \sigma^* \nabla u'(s, X^{t,x}_s) = Z^{t,x}_s - \hat{Z}^{t,x}_s.
\]
This implies that \( u'(t, X^{t,x}_s) \) is a solution to BSDE \( E(0,F) \). The uniqueness of this BSDE shows that \( u'(t, X^{t,x}_s) = u(t, X^{t,x}_s) \). We get that \( u(t, x) = u'(t, x) \) a.e. by using Lemma 4.12. Theorem 4.3 is proved.

As consequence, we have : Let \( g \in L^p([0, T] \times \mathbb{R}^k, e^{-\delta|x|} dx; \mathbb{R}^d) \) for some \( p > 1 \) and \( \delta > 0 \). Let \( A : [0, T] \times \mathbb{R}^k \to \mathbb{R}^{d \times d}, \ B : [0, T] \times \mathbb{R}^k \to (\mathbb{R}^d)^{dr} \) and \( C : [0, T] \times \mathbb{R}^k \to \mathbb{R}^{d \times d} \) be measurable functions which satisfy :

- There exists a positive constant \( K > 0 \) such that for all \( (t, x) \)
  \[ \|A(t, x)\| + \|B(t, x)\|^2 \leq K(1 + |x|), \quad \|C(t, x)\| \leq K \quad \text{and} \quad C(t, x) \geq 0. \]
  We then have

Proposition 4.2. Let \( g \in L^p([0, T] \times \mathbb{R}^k, e^{-\delta|x|} dx; \mathbb{R}^d) \) for some \( p > 1 \) and \( \delta > 0 \). Let \( A : [0, T] \times \mathbb{R}^k \to \mathbb{R}^{d \times d}, \ B : [0, T] \times \mathbb{R}^k \to (\mathbb{R}^d)^{dr} \) and \( C : [0, T] \times \mathbb{R}^k \to \mathbb{R}^{d \times d} \) be measurable functions. Assume that there exists a positive constant \( K > 0 \) such that for every \( (t, x) \), \( 0 \leq C(t, x) \leq K \) and

- Then, the PDE
  \[
  \begin{cases}
  \frac{\partial w(t, x)}{\partial t} + Lw(t, x) + A(t, x)w(t, x) + \langle (B(t, x); \sigma^* \nabla w(t, x)) \rangle - C(t, x)w(t, x) \log |w(t, x)| = 0, \\
  w(T, x) = g(x) & , \quad x \in \mathbb{R}^k 
  \end{cases}
  \]
  has a unique solution \( w \) and \( (w(s, X^{t,x}_s), \sigma^* \nabla w(s, X^{t,x}_s)) \) is the unique solution of
  \[
  E \left( g(X^{t,x}_s), A(s, X^{t,x}_s) + \langle B(s, X^{t,x}_s) \rangle - C(s, X^{t,x}_s) \log |y| \right),
  \]
  where \( \langle (B; z) \rangle := \sum_{i=1}^d \sum_{j=1}^p B_{ij} Z_{ij}. \)
Set $F(t, x, y, z) := A(t, x)y + \langle B(t, x); z \rangle - C(t, x)y \log |y|$. Arguing as in the introductory examples, we show the following claims 1)–3). The claim 2) follows by using Young’s inequality. 

1) $(y, F(t, x, y, z)) \leq K + (K + K|x|)|y|^2 + \sqrt{K + K|x||y||z|}$ 

2) for all $\varepsilon > 0$ there is a constant $C_\varepsilon$ such that 

$|F(t, x, y, z)| \leq C_\varepsilon(1 + |x|^{C_\varepsilon} + |y|^{1+\varepsilon} + |z|^{1+\varepsilon})$ 

3) for every $N > 3$ and every $x, y, y', z, z'$ satisfying $e^{|x|}, |y|, |y'|, |z|, |z'| \leq N$:

$(y - y'; F(t, x, y, z) - F(t, x, y', z')) \leq K' \log N \left( \frac{1}{N} + |y - y'|^2 \right) + \sqrt{K' \log N} |y - y'| |z - z'|$, 

where $K' := 1 + 4Kd + K^2$.

So assumptions (A.0)-(A.4) are satisfied for $(g, F)$. 

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