Analogy and duality

between

random channel coding and lossy source coding

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Here we write in a unified fashion (using “$R(P, Q, D)$” [1]) the random coding exponents in channel coding and lossy source coding.¹ We derive their explicit forms and show, that, for a given random codebook distribution $Q$, the channel decoding error exponent can be viewed as an encoding success exponent in lossy source coding, and the channel correct-decoding exponent can be viewed as an encoding failure exponent in lossy source coding. We then extend the channel exponents to arbitrary $D$, which corresponds for $D > 0$ to erasure decoding and for $D < 0$ to list decoding. For comparison, we also derive the exact random coding exponent for Forney’s optimum tradeoff decoder [2].

In the case of source coding, we assume discrete memoryless sources with a finite alphabet $\mathcal{X}$ and a finite reproduction alphabet $\hat{\mathcal{X}}$. In the case of channel coding, we assume discrete memoryless channels with finite input and output alphabets $\mathcal{X}$ and $\mathcal{Y}$, such that for any $(x, y) \in \mathcal{X} \times \mathcal{Y}$ the channel probability is positive $P(y | x) > 0$. For simplicity, let $R$ denote an exponential size of a random codebook, such that there exist block lengths $n$ for which $e^{nR}$ is integer. We assume the size of the codebook $M = e^{nR}$ for source coding, and $M = e^{nR} + 1$ for channel coding. Let $Q$ denote the (i.i.d.) distribution, according to which the codebook is generated. We use also the definition:

$$R(T, Q, D) \triangleq \min_{W(\hat{x} | x) : d(T \circ W) \leq D} D(T \circ W \parallel T \times Q),$$

(1)

where $T(x)$ is a distribution over $\mathcal{X}$, $Q(\hat{x})$ is a distribution over $\hat{\mathcal{X}}$, and $d(T \circ W)$ denotes an average distortion measure $d(x, \hat{x})$. We consider $R(T, Q, D) = +\infty$, if the set $\{W(\hat{x} | x) : d(T \circ W) \leq D\}$ is empty.

¹This paper is self-contained, and serves also as an addendum to our paper “Exponential source/channel duality”. The proofs of the formulas with $R(P, Q, D)$ will be given below.
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1. Encoding success exponent (for sources)

**Theorem 1:** For a source \( P(x) \) and distortion constraint \( D \), the exponent in the probability of successful encoding is given by

\[
\lim_{n \to \infty} \left\{ -\frac{1}{n} \ln P_s \right\} = E_s(R, D) \triangleq \min_{T(x)} \left\{ D(T\|P) + |R(T, Q, D) - R|^+ \right\},
\]

(2)

except possibly for \( D = D_{\min} = \min_{x, \hat{x}} d(x, \hat{x}) \), when the RHS is a lower bound.

Note that the exponent (2) is zero for \( R \geq R(P, Q, D) \). This theorem is proved in Section 19.

2. Channel decoding error exponent

For a channel \( P(y \mid x) \), the exponent in the probability of decoding error is given by

\[
E_e(R) = \min_{T(x, y)} \left\{ D(T \parallel Q \circ P) + |R(T, Q, 0) - R|^+ \right\},
\]

(3)

where \( R(T, Q, D = 0) \) is determined with respect to a particular distortion measure defined as

\[
d((x, y), \hat{x}) \triangleq \ln \frac{P(y \mid x)}{P(y \mid \hat{x})}.
\]

(4)

Note that this exponent is zero for \( R \geq R(Q \circ P, Q, 0) = I(Q \circ P) \).

3. Encoding failure exponent (for sources)

**Theorem 2:** For a source \( P(x) \) and distortion constraint \( D \), the exponent in the probability of encoding failure is given by

\[
\lim_{n \to \infty} \left\{ -\frac{1}{n} \ln P_f \right\} = E_f(R, D) \triangleq \begin{cases} 
\min_{T(x): R(T, Q, D) \geq R} D(T\|P), & R \leq R_{\max}(D), \\
+\infty, & R > R_{\max}(D),
\end{cases}
\]

(5)

where \( R_{\max}(D) \triangleq \max_{T(x)} R(T, Q, D) \), with the possible exception of points of discontinuity of the function \( E_f(R, D) \).

This exponent is zero for \( R \leq R(P, Q, D) \). For \( R \) above \( R_{\max}(D) \), the probability of encoding failure tends to zero super-exponentially as \( n \) increases, i.e. the limit of its exponent, as \( n \to \infty \) (which is exactly “the exponent” by definition), is infinity. This theorem is proved in Section 21.

\(^2\)\( \max_{T(x)} R(T, Q, D) \) may be alternatively expressed as \( \max_x \), but it can be \( +\infty \).
4. Channel correct-decoding exponent

For a channel \( P(y \mid x) \), the exponent in the probability of correct decoding is given by

\[
E_c(R) = \min_{T(x, y)} \left\{ D(T \parallel Q \circ P) + |R - R(T, Q, 0)|^+ \right\},
\]  

(6)

where \( R(T, Q, D = 0) \) is determined with respect to the distortion measure \( d((x, y), \hat{x}) \) (4). This exponent coincides with

\[
E_c(R) = \begin{cases}
\min_{T(x, y): R(T, Q, 0) \geq R} D(T \parallel Q \circ P), & R \leq R_1, \\
\min_{T(x, y)} \left\{ D(T \parallel Q \circ P) + R - R(T, Q, 0) \right\}, & R > R_1,
\end{cases}
\]

(7)

where \( R_1 \leq \max_{T(x, y)} R(T, Q, 0) \). The exponent is zero for \( R \leq R(Q \circ P, Q, 0) = I(Q \circ P) \). For \( R > R_1 \), the exponent is a linearly increasing function of \( R \) with constant slope = 1.3

5. Derivation of the explicit encoding success exponent

We start with a derivation of an explicit formula for \( R(T, Q, D) \):

**Lemma 1:**

\[
R(T, Q, D) = \sup_{s \geq 0} \left\{ -\sum_x T(x) \ln \sum_{\hat{x}} Q(\hat{x}) e^{-s[d(x, \hat{x}) - D]} \right\}.
\]

(8)

**Proof:**

\[
R(T, Q, D) \triangleq \min_{W(\hat{x} \mid x): d(T \circ W) \leq D} D(T \circ W \parallel T \times Q)
\]

\[= 4 \min_{W(\hat{x} \mid x)} \sup_{s \geq 0} \left\{ D(T \circ W \parallel T \times Q) + s[d(T \circ W) - D] \right\}
\]

(9)

\[= \sup_{s \geq 0} \min_{W(\hat{x} \mid x)} \left\{ D(T \circ W \parallel T \times Q) + s[d(T \circ W) - D] \right\}
\]

(10)

\[= \sup_{s \geq 0} \min_{W(\hat{x} \mid x)} \left\{ \sum_{x, \hat{x}} T(x)W(\hat{x} \mid x) \ln \frac{W(\hat{x} \mid x)}{Q(\hat{x})} + s \left[ \sum_{x, \hat{x}} T(x)W(\hat{x} \mid x)d(x, \hat{x}) - D \right] \right\}
\]

\[= \sup_{s \geq 0} \min_{W(\hat{x} \mid x)} \left\{ \sum_{x, \hat{x}} T(x)W(\hat{x} \mid x) \ln \frac{W(\hat{x} \mid x)}{Q(\hat{x})}e^{-sd(x, \hat{x})} - sD \right\}
\]

\[= \sup_{s \geq 0} \left\{ -\sum_x T(x) \ln \sum_{\hat{x}} Q(\hat{x}) e^{-sD} - sD \right\},
\]

where (*) follows by the minimax theorem5, since the objective function is convex (\( \cup \)) in \( W(\hat{x} \mid x) \) and concave (linear) in \( s \).

\[5\]If the exponent is for the natural base \( e \), then \( R \) here must be accordingly in natural units (nats).

\[4\]This also includes the case when the set \( \{ W(\hat{x} \mid x): d(T \circ W) \leq D \} \) is empty, then \( R(T, Q, D) = +\infty \).

\[5\]The equality can also be verified directly, for different values of \( D \), using continuity of the minimizing solution \( W_s \) and its limit as \( s \to +\infty \), or, alternatively, showing that \( R(T, Q, D) \) is a convex (\( \cup \)) function of \( D \) and (10) is the lower convex envelope of \( R(T, Q, D) \).
Before we plug the explicit formula for $R(T, Q, D)$ (8) into the expression for the encoding success exponent, we note the following property:

**Lemma 2:** $R(T, Q, D)$ is a convex ($\cup$) function of $(T, D)$.

**Proof:**

$$R(\lambda T + (1 - \lambda)\tilde{T}, Q, \lambda D + (1 - \lambda)\tilde{D})$$

$$= \sup_{s \geq 0} \left\{ -\sum_x (\lambda T(x) + (1 - \lambda)\tilde{T}(x)) \ln \sum_{\hat{x}} Q(\hat{x}) e^{-sd(x, \hat{x})} - s(\lambda D + (1 - \lambda)\tilde{D}) \right\}$$

$$= \sup_{s \geq 0} \left\{ \lambda \left[ -\sum_x T(x) \ln \sum_{\hat{x}} Q(\hat{x}) e^{-sd(x, \hat{x})} - sD \right] +
\quad + (1 - \lambda) \left[ -\sum_x \tilde{T}(x) \ln \sum_{\hat{x}} Q(\hat{x}) e^{-sd(x, \hat{x})} - s\tilde{D} \right] \right\}$$

$$\leq \lambda \sup_{s \geq 0} \left\{ -\sum_x T(x) \ln \sum_{\hat{x}} Q(\hat{x}) e^{-sd(x, \hat{x})} - sD \right\} +
\quad + (1 - \lambda) \sup_{s \geq 0} \left\{ -\sum_x \tilde{T}(x) \ln \sum_{\hat{x}} Q(\hat{x}) e^{-sd(x, \hat{x})} - s\tilde{D} \right\}$$

$$= \lambda R(T, Q, D) + (1 - \lambda) R(\tilde{T}, Q, \tilde{D}).$$

The encoding success exponent can be rewritten as

**Lemma 3:**

$$\min_{T(x)} \left\{ D(T\| P) + |R(T, Q, D) - R|^+ \right\} = \sup_{0 \leq \rho \leq 1} \min_{T(x)} \left\{ D(T\| P) + \rho [R(T, Q, D) - R] \right\}, \quad (11)$$

**Proof:** The expression for the encoding success exponent (2), which is written with the help of the Csiszár-Körner style brackets $| \cdot |^+$ for compactness, translates into the minimum between two exponents:

$$\min_{T(x)} \left\{ D(T\| P) + |R(T, Q, D) - R|^+ \right\} = \min \left\{ E_A(R, D), \ E_B(R, D) \right\} = \quad (12)$$

$$\min \left\{ \min_{T(x): R(T, Q, D) \leq R} D(T\| P), \quad \min_{T(x): R(T, Q, D) \geq R} D(T\| P) + R(T, Q, D) - R \right\}.$$

Using the fact that $R(T, Q, D)$ is convex ($\cup$) in $T$, we can rewrite the left exponent as follows

$$E_A(R, D) = \min_{T(x): R(T, Q, D) \leq R} D(T\| P) = \min_{T(x)} \sup_{\rho \geq 0} \left\{ D(T\| P) + \rho [R(T, Q, D) - R] \right\}$$

$$= \sup_{\rho \geq 0} \min_{T(x)} \left\{ D(T\| P) + \rho [R(T, Q, D) - R] \right\}, \quad (13)$$
where \( (*) \) follows by the minimax theorem\(^6\), because the objective function is convex (\( \cup \)) in \( T(x) \) and concave (linear) in \( \rho \).

For the right exponent we have a lower bound:

\[
E_B(R, D) = \min_{T(x): R(T,Q,D) \geq R} \left\{ D(T\|P) + R(T,Q,D) - R \right\}
\]

\[
\geq \sup_{\rho \geq 0} \min_{T(x): R(T,Q,D) \geq R} \left\{ D(T\|P) + R(T,Q,D) - R + \rho [R - R(T,Q,D)] \right\}
\]

\[
\geq \sup_{\rho \geq 0} \min_{T(x)} \left\{ D(T\|P) + R(T,Q,D) - R + \rho [R - R(T,Q,D)] \right\}
\]

\[
= \sup_{\rho \geq 0} \min_{T(x)} \left\{ D(T\|P) + (1 - \rho) [R(T,Q,D) - R] \right\}
\]

\[
= \sup_{0 \leq \rho \leq 1} \min_{T(x)} \left\{ D(T\|P) + (1 - \rho) [R(T,Q,D) - R] \right\}
\]

\[
= \sup_{0 \leq \rho \leq 1} \min_{T(x)} \left\{ D(T\|P) + \rho [R(T,Q,D) - R] \right\}. \quad (14)
\]

Observe from (8), that if \( d(x, \hat{x}) - D > 0 \) for all \( (x, \hat{x}) \), then \( R(T,Q,D) = +\infty \) for any choice of \( T \). In this case (11) holds trivially.

On the other hand, if there exists at least one pair \( (x, \hat{x}) \), such that \( d(x, \hat{x}) - D \leq 0 \), then there exists \( T \) with finite \( R(T,Q,D) \). In this case, consider the following function of \( T \):

\[
D(T\|P) + R(T,Q,D).
\]

This is a strictly convex (\( \cup \)) function of \( T \), because \( R(T,Q,D) \) is convex and \( D(T\|P) \) is strictly convex. Consequently, there exists a unique \( T_1 \), which attains its minimum:

\[
D(T_1\|P) + R(T_1,Q,D) = \min_{T(x)} \left\{ D(T\|P) + R(T,Q,D) \right\}.
\]

Note that for \( R_1 = R(T_1,Q,D) \) we obtain:

\[
E_A(R_1, D) = \min_{T(x): R(T,Q,D) \leq R_1} D(T\|P) = \min_{T(x)} \left\{ D(T\|P) + R(T,Q,D) - R_1 \right\}.
\]

Since \( E_A(R_1, D) \) is finite, we conclude that for \( R \geq R_1 \) the function \( E_A(R) \) is finite and nonincreasing. It can be seen from (13) that \( E_A(R, D) \) is a convex (\( \cup \)) function of \( R \). We conclude, that for \( R \geq R_1 \), in (13) it is sufficient to take the supremum over \( 0 \leq \rho \leq 1 \):

\[
E_A(R, D) = \sup_{0 \leq \rho \leq 1} \min_{T(x)} \left\{ D(T\|P) + \rho [R(T,Q,D) - R] \right\}, \quad R \geq R_1. \quad (15)
\]

\(^6\)Alternatively, check directly that \( E_A(R, D) \) is convex (\( \cup \)) in \( R \) and observe that (13) is the lower convex envelope of \( E_A(R) \).
Observe further, that for $R \leq R_1$

\[
E_B(R, D) = \min_{T(x): R(T,Q,D) \geq R} \{ D(T\|P) + R(T,Q,D) - R \}
= D(T_1\|P) + R(T_1, Q, D) - R
= \min_{T(x)} \{ D(T\|P) + R(T,Q,D) - R \}
\leq \sup_{0 \leq \rho \leq 1} \min_{T(x)} \{ D(T\|P) + \rho [R(T,Q,D) - R] \}, \quad R \leq R_1. \tag{16}
\]

Comparing (14) and (16), we conclude that the equality holds

\[
E_B(R, D) = \sup_{0 \leq \rho \leq 1} \min_{T(x)} \{ D(T\|P) + \rho [R(T,Q,D) - R] \}, \quad R \leq R_1. \tag{17}
\]

Now, the result of the lemma follows by (12), when we compare (13) with (17) for $R \leq R_1$, and (14) with (15) for $R \geq R_1$, respectively.

Finally, we are ready to prove the following formula:

**Theorem 3:**

\[
E_s(R, D) = \sup_{0 \leq \rho \leq 1} \left\{ - \inf_{s \geq 0} \ln \sum_x P(x) \left[ \sum_{\hat{x}} Q(\hat{x}) e^{-s[D(\hat{x}, \hat{x}) - D]} \right]^\rho - \rho R \right\}. \tag{18}
\]

**Proof:**

\[
\min_{T(x)} \{ D(T\|P) + |R(T,Q,D) - R| \} \overset{(a)}{=} \sup_{0 \leq \rho \leq 1} \min_{T(x)} \{ D(T\|P) + \rho [R(T,Q,D) - R] \}
= \sup_{0 \leq \rho \leq 1} \min_{T(x)} \left\{ \sum_x T(x) \ln \frac{T(x)}{P(x)} + \rho \left[ - \sum_{x} T(x) \ln \sum_{\hat{x}} Q(\hat{x}) e^{-s[D(\hat{x}, \hat{x}) - D]} \right] - R \right\}
= \sup_{0 \leq \rho \leq 1} \min_{T(x)} \sup_{s \geq 0} \left\{ \sum_x T(x) \ln \frac{T(x)}{P(x)} + \rho \left[ - \sum_{x} T(x) \ln \sum_{\hat{x}} Q(\hat{x}) e^{-s[D(\hat{x}, \hat{x}) - D]} \right] - R \right\}
= \sup_{0 \leq \rho \leq 1} \sup_{s \geq 0} \min_{T(x)} \left\{ \sum_x T(x) \ln \frac{T(x)}{P(x)} + \rho \left[ - \sum_{x} T(x) \ln \sum_{\hat{x}} Q(\hat{x}) e^{-s[D(\hat{x}, \hat{x}) - D]} \right] - R \right\}
= \sup_{0 \leq \rho \leq 1} \sup_{s \geq 0} \left\{ - \ln \sum_x P(x) \left[ \sum_{\hat{x}} Q(\hat{x}) e^{-s[D(\hat{x}, \hat{x}) - D]} \right]^\rho - \rho R \right\}
\]

where (a) is by (11), in (b) we insert the identity (8) for $R(T,Q,D)$, and (c) follows by the minimax theorem\(^7\), for the objective function which is convex ($\cup$) in $T(x)$ and concave\(^8\) ($\cap$) in $s$.

\(^7\)Alternatively, the equality can be shown by substituting (9) for $R(T,Q,D)$ and equating a convex function with its lower convex envelope.

\(^8\)The concavity ($\cap$) is apparent from (10), where the function of $s$ is expressed as a minimum of affine functions of $s$. 
Discussion:

Let $T_\rho$ denote the unique solution of the minimum

$$
\min_{T(x)} \{ D(T\|P) + \rho R(T, Q, D) \} = D(T_\rho\|P) + \rho R(T_\rho, Q, D),
$$

for $\rho \geq 0$, and define

$$
R_\rho \triangleq R(T_\rho, Q, D), \quad \rho \geq 0.
$$

Clearly, $T_0 = P$, and consequently, by our definition, $R_0 = R(P, Q, D)$. However, note, that $\lim_{\rho \to 0} R_\rho$ is not necessarily equal to $R_0$. In general, it is less than or equal:

$$
\lim_{\rho \to 0} R_\rho \leq R_0 = R(P, Q, D).
$$

The inequality arises when $R(P, Q, D) = +\infty$ and $\lim_{\rho \to 0} R_\rho$ is still finite. In this case the exponent $E_s(R, D)$ does not decrease all the way to zero, as $R$ increases, but stays strictly above zero, at the height

$$
\lim_{\rho \to 0} D(T_\rho\|P) = \min_{T(x): R(T, Q, D) < +\infty} D(T\|P) > 0.
$$

In this particular case, each one of the straight lines

$$
D(T_\rho\|P) + \rho [R(T_\rho, Q, D) - R_\rho], \quad \rho > 0,
$$

touches the curve $E_s(R)$, except for the line of slope zero: $E = D(T_0\|P)$, which is equal to zero for all $R$ and runs strictly below $E_s(R)$. The range of $D$, for which this behavior occurs, is given by the following

**Proposition 1:**

$$
+\infty > \lim_{R \to \infty} E_s(R, D) > 0 \iff \min_x \min_{\hat{x}} d(x, \hat{x}) \leq D < \sum_x P(x) \min_{\hat{x}} d(x, \hat{x}).
$$

**Proof:** Follows from the relations

$$
R(P, Q, D) = +\infty \iff D < \sum_x P(x) \min_{\hat{x}} d(x, \hat{x}),
$$

$$
R(T, Q, D) = +\infty, \quad \forall T \iff D < \min_x \min_{\hat{x}} d(x, \hat{x}).
$$

$^9$Note that $R(T, Q, D)$ cannot “diverge” to infinity, as a function of $T$, since it is bounded when finite, as divergence is bounded.
Fig. 1. Encoding success exponent (18) vs. $R$, for various $D = \{0, 0.5, -1, -1.7\} \cdot p \cdot \ln \frac{1-p}{p}$. Parameter $p = 0.22$.

Source: $X = \{a, b, c, d\}$, $P(a) = P(d) = \frac{1-p}{2}$, $P(b) = P(c) = \frac{p}{2}$. Reproduction: $\hat{X} = \{0, 1\}$, $Q(0) = Q(1) = \frac{1}{2}$.

Distortion measure: $d(a, 0) = d(b, 0) = d(c, 1) = d(d, 1) = 0$, $d(a, 1) = d(d, 0) = \ln \frac{1-p}{p}$, $d(b, 1) = d(c, 0) = -\ln \frac{1-p}{p}$.

The lowest distortion for which $E_s(R, D)$ decreases to zero as $R$ increases: $D^* = \sum_x P(x) \min_{\hat{x}} d(x, \hat{x}) = -p \ln \frac{1-p}{p}$.

As $D \downarrow D_{\min} = \min_{x, \hat{x}} d(x, \hat{x}) = -\ln \frac{1-p}{p}$, the curves $E_s(R)$ tend to a “135° angle”: $E_s(R, D) \nrightarrow \max \{ \ln \frac{2}{p} - R, \ln \frac{1}{p}\}$.

For $D < D_{\min}$ the encoding success exponent is $+\infty$.

As $D \nearrow D_{\max} = \max_{x, \hat{x}} d(x, \hat{x}) = \ln \frac{1-p}{p}$, the curves $E_s(R)$ tend to 0.

This example corresponds also to the channel error exponent (with the same values of $D$) for the channel BSC($p$) and $Q(0) = \frac{1}{2}$.

6. Derivation of the explicit channel decoding error exponent

We make the following substitutions in (18):

\[
(x, y) \rightarrow x
\]  
\[Q(x)P(y | x) \rightarrow P(x)
\]  
\[d((x, y), \hat{x}) = \ln \frac{P(y | x)}{P(y | \hat{x})} \rightarrow d(x, \hat{x})
\]  
\[0 \rightarrow D
\]
The result is the random coding exponent of Gallager [3]:

\[
E_e(R) = \sup_{0 \leq \rho \leq 1} \left\{ -\inf_{s \geq 0} \ln \sum_{x, y} Q(x) P(y \mid x) \left[ \sum_{\hat{x}} Q(\hat{x}) \left( \frac{P(y \mid \hat{x})}{P(y \mid x)} \right)^{-s}\right]^{\rho} - \rho R \right\}
\]

\[
= \sup_{0 \leq \rho \leq 1} \left\{ -\inf_{s \geq 0} \ln \sum_{x, y} Q(x) P^{1-s\rho}(y \mid x) \left[ \sum_{\hat{x}} Q(\hat{x}) P^{s}(y \mid \hat{x}) \right]^{\rho} - \rho R \right\}
\]

\[
\equiv \sup_{0 \leq \rho \leq 1} \left\{ -\ln \sum_{x, y} Q(x) P^{1+\rho}(y \mid x) \left[ \sum_{\hat{x}} Q(\hat{x}) P^{1+\rho}(y \mid \hat{x}) \right]^{1+\rho} - \rho R \right\}, \tag{23}
\]

where (\#) follows by Hölder’s inequality.\(^1\)

7. Derivation of the explicit encoding failure exponent

Here we derive an explicit expression, which does not always coincide with the encoding failure exponent (5) for all \(R\), but gives the best convex (\(\cup\)) lower bound for (5), for sufficiently lax distortion constraint \(D\).

For the benefit of the next section, we give a number of lemmas first.

\textbf{Lemma 4:} For any \(\rho \geq 0\)

\[
E_f(R, D) \geq \min_{T(x)} \left\{ D(T \parallel P) - \rho [R(T, Q, D) - R] \right\}, \tag{24}
\]

with equality if \(R = R(T_\rho, Q, D)\), where \(T_\rho\) is a solution of the minimum:

\[
\min_{T(x)} \left\{ D(T \parallel P) - \rho R(T, Q, D) \right\} = 11 \; D(T_\rho \parallel P) - \rho R(T_\rho, Q, D).
\]

\textbf{Proof:}

\[
E_f(R, D) = \min_{T(x): \; R(T, Q, D) \geq R} D(T \parallel P) \overset{(*)}{=} D(T(R) \parallel P) \geq \min_{T(x)} \left\{ D(T \parallel P) - \rho [R(T(R), Q, D) - R] \right\}
\]

\[
\geq \min_{T(x)} \left\{ D(T \parallel P) - \rho [R(T, Q, D) - R] \right\}
\]

\(^1\)Together with the derivation of (18) from (2), this is a lengthy derivation of (23). Its purpose is demonstration and a “sanity check”: that the channel decoding error exponent is indeed a special case of the encoding success exponent for sources. A shorter straightforward derivation of the explicit channel decoding error exponent (23) can be made from

\[
\min_{T(x, y), W(\hat{x} \mid x, y)} \left\{ D(T \parallel Q \circ P) + |D(T \circ W \parallel T \times Q) - R|^+ \right\},
\]

which is equivalent to (3) and uses the same distortion measure (4).

\(^{11}\)In general, \(T_\rho\) may be not unique, and as a result, the value of \(R(T_\rho, Q, D)\) may be not unique, while only the difference \(E_0(\rho) \triangleq D(T_\rho \parallel P) - \rho R(T_\rho, Q, D)\) is a function of \(\rho\).
where in (*) we assumed that the set \( \{ T(x) : R(T, Q, D) \geq R \} \) is nonempty. Otherwise \( E_f(R, D) \) is considered to be \(+\infty\) and any lower bound is valid.

If \( R = R(T_\rho, Q, D) \), then by (25) we obtain for this \( R \):

\[
D(T(R) \parallel P) \geq D(T_\rho \parallel P) - \rho [R(T_\rho, Q, D) - R] = D(T_\rho \parallel P) \geq D(T(R) \parallel P),
\]

where the second inequality holds because \( T_\rho \) satisfies (with equality) the minimization constraint \( R \). \( \blacksquare \)

**Lemma 5:** \(^{12}\) If

\[
E_f(R, D) = \min_{T(x)} \{ D(T \parallel P) - \rho [R(T, Q, D) - R] \},
\]

for some \( \rho > 0 \), then necessarily \( R = R(T_\rho, Q, D) \) for some \( T_\rho \), such that

\[
\min_{T(x)} \{ D(T \parallel P) - \rho R(T, Q, D) \} = D(T_\rho \parallel P) - \rho R(T_\rho, Q, D).
\]

**Proof:** Since for \( R \leq R(P, Q, D) \) the exponent \( E_f(R, D) \) is zero, by the lower bound (24) from the previous lemma we conclude that here necessarily \( R \geq R(P, Q, D) \). Note also, that the condition of the lemma implies that \( R \leq \max_{T(x)} R(T, Q, D) < +\infty \). For \( R \geq R(P, Q, D) \) we can write

\[
E_f(R, D) = \min_{T(x): R(T, Q, D) \geq R} D(T \parallel P) = D(T(R) \parallel P)
\]

\[
(25) = D(T(R) \parallel P) - \rho [R(T(R), Q, D) - R]
\]

\[
\geq \min_{T(x)} \{ D(T \parallel P) - \rho [R(T, Q, D) - R] \},
\]

where in (*) the difference \( [R(T(R), Q, D) - R] \) cannot be positive in the case of \( R \geq R(P, Q, D) \), and must be zero, because \( D(T \parallel P) \) is strictly convex and \( R(T, Q, D) \) is a continuous function of \( T \). When we have equality in the above, \( T(R) \) is a solution of the last minimum, i.e. \( T(R) = T_\rho \). \( \blacksquare \)

**Lemma 6:** If \( \max_{T(x)} R(T, Q, D) < +\infty \), then

lower convex envelope \( \{ E_f(R) \} = \sup_{\rho \geq 0} \min_{T(x)} \{ D(T \parallel P) - \rho [R(T, Q, D) - R] \} \).

(26)

If \( \max_{T(x)} R(T, Q, D) = +\infty \), then the right-hand side expression gives zero, which is strictly lower than \( E_f(R) \) for \( R > R(P, Q, D) \).

**Proof:** By (24) of Lemma 4 we have a lower bound:

\[
E_f(R, D) \geq \sup_{\rho \geq 0} \min_{T(x)} \{ D(T \parallel P) - \rho [R(T, Q, D) - R] \}.
\]

(27)

\(^{12}\)This is the “only if” addition to the statement of Lemma 4. This lemma will be needed in an example only.
Observe, that if $\max_{T(x)} R(T, Q, D) = +\infty$, then the minimum in (27) is $-\infty$ for all $\rho > 0$, and for $\rho = 0$ the minimum is $D(P \| P) = 0$. We conclude, that if $\max_{T(x)} R(T, Q, D) = +\infty$, then the lower bound (27) is 0. Apparently, this is not a tight lower bound if $R > R(P, Q, D)$, i.e. no “strong Lagrangian duality” in this case.

On the other hand, if $\max_{T(x)} R(T, Q, D) < +\infty$, then for any $\rho \geq 0$ there exists at least one (i.e. possibly not unique) finite $R_\rho = R(T_\rho, Q, D)$, where by Lemma 4 the curve $E_f(R)$ touches the straight line lower bound (24). We conclude, that the curve $E_f(R)$ touches the straight line lower bounds (24) for each slope value $\rho \geq 0$. Since $E_f(R)$ is a nondecreasing function, it follows, that the supremum of the straight lines over $\rho \geq 0$ (27) is the lower convex envelope of $E_f(R)$.

Lemma 7:

$$D \geq \max_x \min_{\hat{x}} d(x, \hat{x}) \iff \max_{T(x)} R(T, Q, D) < +\infty.$$  

Proof: Observe, that if there exists at least one $x$ for which $\min_{\hat{x}} \{d(x, \hat{x}) - D\} > 0$, then, using the expression for $R(T, Q, D)$ (8), we obtain

$$\max_{T(x)} R(T, Q, D) = \max_x \sup_{s \geq 0} \left\{ -\ln \sum_{\hat{x}} Q(\hat{x}) e^{-s[d(x, \hat{x}) - D]} \right\} = +\infty.$$  

On the other hand, if for every $x$ holds $\min_{\hat{x}} \{d(x, \hat{x}) - D\} \leq 0$, then by the same expression we have

$$\max_{T(x)} R(T, Q, D) < +\infty.$$  

Lemma 8: For $\rho \geq 0$

$$\min_{T(x)} \{ D(T \| P) - \rho R(T, Q, D) \} = -\sup_{s \geq 0} \ln \sum_x P(x) \left[ \sum_{\hat{x}} Q(\hat{x}) e^{-s[d(x, \hat{x}) - D]} \right]^{-\rho}.$$  

If $D \geq \max_x \min_{\hat{x}} d(x, \hat{x})$, then the minimum on the LHS is achieved by some $T^*(x)$ if and only if

$$T^*(x) = \lim_{s \to s^*} T_{p, s}(x), \quad T_{p, s}(x) \propto P(x) \left[ \sum_{\hat{x}} Q(\hat{x}) e^{-s[d(x, \hat{x}) - D]} \right]^{-\rho},$$  

where $s^*$ is any limit (which may be finite or $+\infty$), which achieves the supremum on the RHS.

Proof:

$$D(T \| P) - \rho R(T, Q, D) \overset{(a)}{=} \sum_x T(x) \ln \frac{T(x)}{P(x)} - \rho \sup_{s \geq 0} \left\{ -\sum_x T(x) \ln \sum_{\hat{x}} Q(\hat{x}) e^{-s[d(x, \hat{x}) - D]} \right\}$$  

$$\overset{(b)}{=} \inf_{s \geq 0} \left\{ \sum_x T(x) \ln \frac{T(x)}{P(x)} + \rho \sum_x T(x) \ln \sum_{\hat{x}} Q(\hat{x}) e^{-s[d(x, \hat{x}) - D]} \right\}$$  

$$= \lim_{s \to s^*(T)} \sum_x T(x) \ln \frac{T(x)}{P(x) \left[ \sum_{\hat{x}} Q(\hat{x}) e^{-s[d(x, \hat{x}) - D]} \right]^{-\rho}}$$
\[
\lim_{s \to s^*(T)} \left\{ \sum_x T(x) \ln \frac{T(x) P(x)}{\sum_a P(a) [\sum_{\hat{x}} Q(\hat{x}) e^{-s[d(\hat{x},x)-D]}]^{-\rho}} - \ln \sum_x P(x) \left[ \sum_{\hat{x}} Q(\hat{x}) e^{-s[d(\hat{x},x)-D]} \right]^{-\rho} \right\}
\geq - \lim_{s \to s^*(T)} \ln \sum_x P(x) \left[ \sum_{\hat{x}} Q(\hat{x}) e^{-s[d(\hat{x},x)-D]} \right]^{-\rho} \geq - \lim_{s \to s^*} \ln \sum_x P(x) \left[ \sum_{\hat{x}} Q(\hat{x}) e^{-s[d(\hat{x},x)-D]} \right]^{-\rho}
\]
\[
\inf_{s \geq 0} \left\{ - \ln \sum_x P(x) \left[ \sum_{\hat{x}} Q(\hat{x}) e^{-s[d(\hat{x},x)-D]} \right]^{-\rho} \right\},
\]
where in (a) we use (8) for \( R(T, Q, D) \), and (b) holds for \( \rho \geq 0 \). Observe, that both inequalities above become equalities if
\[
T(x) = \lim_{s \to s^*} T_{\rho,s}(x), \quad T_{\rho,s}(x) \propto P(x) \left[ \sum_{\hat{x}} Q(\hat{x}) e^{-s[d(\hat{x},x)-D]} \right]^{-\rho},
\]
where \( s^* \) is a limit, achieving the infimum in (30). We conclude, that the infimum (30) coincides with the minimum over \( T(x) \), i.e. obtain the desired result (28). Note, however, that (31) is not a necessary condition in the case when the infimum (30) is \(-\infty\).

Note further, that if \( D \geq \max_x \min_{\hat{x}} d(x, \hat{x}) \), then (by Lemma 7) \( \max_{T(x)} R(T, Q, D) < +\infty \) and the infimum in (30) accordingly must be finite (not \(-\infty\)) for any \( \rho \geq 0 \). In this case, the lower bound (30) is attained if and only if \( T(x) \) is given by (31).

Now, by Lemma 6, Lemma 7, and identity (28) of Lemma 8 we have the following

**Theorem 4:** For distortion constraint \( D \geq \max_x \min_{\hat{x}} d(x, \hat{x}) \),

lower convex envelope \( (E_f(R)) = \sup_{\rho \geq 0} \left\{ -\inf_{s \geq 0} \ln \sum_x P(x) \left[ \sum_{\hat{x}} Q(\hat{x}) e^{-s[d(\hat{x},x)-D]} \right]^{-\rho} \right\} \) (32)

For \( D < \max_x \min_{\hat{x}} d(x, \hat{x}) \), the right-hand side expression gives zero, which is strictly lower than \( E_f(R) \), if \( R > R(P, Q, D) \).

8. **Derivation of the explicit channel correct-decoding exponent**

We would like to show that the channel correct-decoding exponent (6) is equivalent to (7), and find an explicit expression for it.

The expression (6), which is written with the help of the Csiszár-Körner style brackets \(|\cdot|_+^*\) for compactness, translates into the minimum between two exponents:
\[
\min_{T(x,y)} \left\{ D(T \mid \| Q \circ P) + \left| R - R(T, Q, 0) \right|^* \right\} = \min \{ E_A(R), E_B(R) \} = \min \left\{ \min_{T(x,y): R(T, Q, 0) \geq R} D(T \parallel Q \circ P), \min_{T(x,y): R(T, Q, 0) \leq R} D(T \parallel Q \circ P) + R - R(T, Q, 0) \right\}.
\]
Fig. 2. Encoding failure exponent (32) vs. \( R \), for \( D = \{0, 0.05, 0.10, 0.15\} \cdot \ln \frac{1-p}{p} \geq 0 = \max_x \min_y d(x, \hat{x}) \). Parameter \( p = 0.22 \).

Source: \( \mathcal{X} = \{a, b, c, d\} \), \( P(a) = P(d) = \frac{1-p}{p} \), \( P(b) = P(c) = \frac{2}{5} \). Reproduction: \( \hat{\mathcal{X}} = \{0, 1\} \), \( Q(0) = Q(1) = \frac{1}{2} \).

Distortion measure: \( d(a, 0) = d(b, 0) = d(c, 1) = d(d, 1) = 0 \), \( d(a, 1) = d(d, 0) = \ln \frac{1-p}{p} \), \( d(b, 1) = d(c, 0) = -\ln \frac{1-p}{p} \).

As \( R \to \max_{T(x)} R(T, Q, D) \), each curve \( E_f(R) \to \log \frac{1}{1-p} \approx 0.249 \).

For \( R > \max_{T(x)} R(T, Q, D) \) the encoding failure exponent is \(+\infty\).

For \( D \geq D_{\max} = \max_x \min_y d(x, \hat{x}) = \ln \frac{1-p}{p} \) the encoding failure exponent is \(+\infty\).

This example corresponds also to the channel correct-decoding exponent (with the same values of \( D \)) for the channel BSC(\( p \)) and \( Q(0) = \frac{1}{2} \).

Note, that the left exponent \( E_A(R) \) is the same as (5), after we make the substitutions (19)-(22). Therefore, in order to characterize this exponent, we can use Lemmas 4, 6, 7, 8, with the substitutions.

With the distortion measure (4) we have

\[
\max_{x,y} \min_{\hat{x}} d((x,y),\hat{x}) = \max_x \max_y \min_{\hat{x}} \ln \frac{P(y|x)}{P(y|\hat{x})} \leq \max_x \max_y \ln \frac{P(y|x)}{P(y|\hat{x})} = 0 = \max_y \min_{\hat{x}} \ln \frac{P(y|\hat{x})}{P(y|\hat{x})} \leq \max_x \max_y \min_{\hat{x}} \ln \frac{P(y|x)}{P(y|\hat{x})},
\]

\[
\max_{x,y} \min_{\hat{x}} d((x,y),\hat{x}) = 0,
\]

which is precisely the condition of Lemma 7 with \( D = 0 \) (satisfied with equality). Therefore, by Lemma 7
The left mode provides a unique solution

Observe, that the difference

Observe, that the difference $d(x, x) - D$ satisfies the condition of Theorem 4 (each row has negative values). Therefore (32) holds.

The lower convex envelope (32) is depicted in the left graph. The envelope has a segment with constant slope $\rho = 0.65$.

The right graph shows, that there are exactly 2 different values $s^*$, achieving the supremum inside $-\ln$ in (32) for $\rho = 0.65$:

$E_0(\rho = 0.65) = \inf_{s \geq 0} E_0(s, \rho = 0.65) = E_0(s^*, \rho = 0.65) = D(T_i \| P) - \rho R(T_i, Q, D), \quad i = 1, 2.$

Note from the right graph, that there are 2 “modes” (local minima) in the curves of $E_0(s, \rho)$ vs. $s$, for each $\rho$.

The left mode provides a unique solution $R_\rho = R(T_{\rho, s^*}, Q, D) < R(T_1, Q, D)$ for $\rho < 0.65$.

The right mode provides a unique solution $R_\rho = R(T_{\rho, s^*}, Q, D) > R(T_2, Q, D)$ for $\rho > 0.65$.

$E_f(R)$ (5) must run strictly above its lower convex envelope (32) for $R(T_1, Q, D) < R < R(T_2, Q, D)$, according to Lemmas 8 and 5.

and Lemma 6, for the left exponent $E_A(R)$ we can write

$$\text{lower convex envelope } (E_A(R)) = \sup_{\rho \geq 0} \min_{T(x, y)} \left\{ D(T \parallel Q \circ P) - \rho [R(T, Q, 0) - R] \right\}. \quad (35)$$

Similarly to the case of the encoding success (and the channel decoding error) exponent, in order to compare between $E_A(R)$ and $E_B(R)$, it is useful to consider the following function of $T$ (this time with a minus before $R(T, Q, 0)$):

$D(T \parallel Q \circ P) - R(T, Q, 0)$. 

Fig. 3. Encoding failure exponent: an example where $E_f(R)$ (5) does not always coincide with its lower convex envelope (32).

Source / reproduction: $|X| = |\hat{X}| = 5$, $P = [0.2923, 0.0142, 0.2673, 0.3210, 0.1051]$, $Q = [0.2573, 0.0908, 0.2437, 0.0294, 0.3787]$.

Distortion measure and constraint: $d(x, \hat{x}) - D = \begin{bmatrix} -0.0799 & 0.1580 & 0.0425 & 0.0673 & -0.3449 \\ 0.0815 & 0.2024 & -0.1511 & 0.1030 & 0.4020 \\ 0.0147 & -0.0079 & 0.7994 & 0.6861 & 0.1450 \\ 0.8545 & 0.9160 & 0.9066 & 0.5624 & -0.0015 \\ -0.2179 & -0.4107 & -0.0435 & -0.2367 & -0.2594 \end{bmatrix}$.
Its minimum is given by identity (28) of Lemma 8:

\[
\min_{T(x, y)} \left\{ D(T \| Q \circ P) - R(T, Q, 0) \right\} = \sup_{s \geq 0} \ln \sum_{x, y} Q(x) P(y \| x) \left[ \sum_{\hat{x}} Q(\hat{x}) \left( \frac{P(y \| x)}{P(\hat{x} \| \hat{x})} \right)^{-s} \right]^{-1}
\]

\[
= \sup_{s \geq 0} \ln \sum_{y} \left[ \sum_{x} Q(x) P^{s}(y \| x) \right] \left( \sum_{\hat{x}} Q(\hat{x}) P^{s}(y \| \hat{x}) \right) P(y \| x)
\]

\[
\geq - \ln \sum_{y} \max_{x} P(y \| x)
\]

\[
= - \lim_{s \to +\infty} \ln \sum_{y} \left[ \sum_{x} Q(x) P^{s}(y \| x) \right] \left( \sum_{\hat{x}} Q(\hat{x}) P^{s}(y \| \hat{x}) \right) P(y \| x)
\]

\[
\geq - \sup_{s \geq 0} \ln \sum_{y} \left[ \sum_{x} Q(x) P^{s}(y \| x) \right] \left( \sum_{\hat{x}} Q(\hat{x}) P^{s}(y \| \hat{x}) \right) P(y \| x)
\]

That is, the supremum is achieved when \( s \to +\infty \):

\[
\min_{T(x, y)} \left\{ D(T \| Q \circ P) - R(T, Q, 0) \right\} = - \lim_{s \to +\infty} \ln \sum_{x, y} Q(x) P(y \| x) \left[ \sum_{\hat{x}} Q(\hat{x}) \left( \frac{P(y \| x)}{P(\hat{x} \| \hat{x})} \right)^{-s} \right]^{-1}
\]

\[
= D(T_1 \| Q \circ P) - R(T_1, Q, 0),
\]

where by (29)

\[
T_1(x, y) \propto \lim_{s \to +\infty} Q(x) P(y \| x) \left[ \sum_{\hat{x}} Q(\hat{x}) \left( \frac{P(y \| x)}{P(\hat{x} \| \hat{x})} \right)^{-s} \right]^{-1}.
\]

By Lemma 4 we conclude, that \( E_A(R) \) touches the line

\[
\min_{T(x, y)} \left\{ D(T \| Q \circ P) - R(T, Q, 0) \right\}
\]

at \( R_1 = R(T_1, Q, 0) \).

For \( 0 \leq \rho < 1 \), by Lemma 8 we obtain

\[
\min_{T(x, y)} \left\{ D(T \| Q \circ P) - \rho R(T, Q, 0) \right\} = - \ln \sup_{s \geq 0} \sum_{x, y} Q(x) P(y \| x) \left[ \sum_{\hat{x}} Q(\hat{x}) \left( \frac{P(y \| x)}{P(\hat{x} \| \hat{x})} \right)^{-s} \right]^{-\rho}
\]

\[
= - \ln \sup_{s \geq 0} \sum_{x, y} Q(x) P^{1+s\rho}(y \| x) \left[ \sum_{\hat{x}} Q(\hat{x}) P^{s\rho}(y \| \hat{x}) \right]^{-s}
\]

\[
\overset{(*)}{=} - \ln \sum_{x, y} Q(x) P^{-1/\rho}(y \| x) \left[ \sum_{\hat{x}} Q(\hat{x}) P^{-1/\rho}(y \| \hat{x}) \right]^{-\rho}
\]

\[
= - \ln \sum_{y} \left[ \sum_{\hat{x}} Q(\hat{x}) P^{-1/\rho}(y \| \hat{x}) \right]^{-\rho}
\]

where \((*)\) follows by Hölder’s inequality. For each \( 0 \leq \rho < 1 \)

\[
\min_{T(x, y)} \left\{ D(T \| Q \circ P) - \rho R(T, Q, 0) \right\} = D(T_\rho \| Q \circ P) - \rho R(T_\rho, Q, 0),
\]

(38)
where by (29)

\[ T_\rho(x,y) \propto Q(x)P(y \mid x) \left[ \sum_{\hat{x}} Q(\hat{x}) \left[ \frac{P(y \mid \hat{x})}{P(y \mid x)} \right]^{-\frac{1}{\rho}} \right]^{-\rho}. \]

By Lemma 4, it appears, that \( E_A(R) \) touches each line

\[ \min_{T(x,y)} \{ D(T \parallel Q \circ P) - \rho[R(T, Q, 0) - R] \}, \quad 0 \leq \rho \leq 1, \]

at \( R_\rho = R(T_\rho, Q, 0) \). Since \( E_A(R) \) is nondecreasing, by continuity of \( R(T_\rho, Q, 0) \) as a function of \( \rho \) for \( 0 \leq \rho \leq 1 \), we conclude that

\[ E_A(R) = \sup_{0 \leq \rho \leq 1} \min_{T(x,y)} \{ D(T \parallel Q \circ P) - \rho[R(T, Q, 0) - R] \}, \quad R \leq R_1. \quad (39) \]

Now we proceed to the right exponent \( E_B(R) \), which is lower-bounded as follows:

\[ E_B(R) = \min_{T(x,y)} D(T \parallel Q \circ P) + R - R(T, Q, 0) \]

\[ \geq \sup_{\rho \geq 0} \min_{T(x,y): R(T, Q, 0) \leq R} \{ D(T \parallel Q \circ P) + R - R(T, Q, 0) + \rho[R(T, Q, 0) - R] \} \]

\[ \geq \sup_{\rho \geq 0} \min_{T(x,y)} \{ D(T \parallel Q \circ P) + R - R(T, Q, 0) + \rho[R(T, Q, 0) - R] \} \]

\[ = \sup_{0 \leq \rho \leq 1} \min_{T(x,y)} \{ D(T \parallel Q \circ P) - (1 - \rho)[R(T, Q, 0) - R] \} \]

\[ \geq \sup_{0 \leq \rho \leq 1} \min_{T(x,y)} \{ D(T \parallel Q \circ P) - (1 - \rho)[R(T, Q, 0) - R] \} \]

\[ = \sup_{0 \leq \rho \leq 1} \min_{T(x,y)} \{ D(T \parallel Q \circ P) - \rho[R(T, Q, 0) - R] \}. \quad (40) \]

Observe further, that for \( R \geq R_1 \)

\[ E_B(R) = \min_{T(x,y): R(T, Q, 0) \leq R} \{ D(T \parallel Q \circ P) + R - R(T, Q, 0) \} \]

\[ = D(T_1 \parallel Q \circ P) + R - R(T_1, Q, 0) \]

\[ = \min_{T(x,y)} \{ D(T \parallel Q \circ P) + R - R(T, Q, 0) \} \]

\[ \leq \sup_{0 \leq \rho \leq 1} \min_{T(x,y)} \{ D(T \parallel Q \circ P) - \rho[R(T, Q, 0) - R] \}, \quad R \geq R_1. \quad (41) \]

Comparing (39) with (40), for \( R \leq R_1 \), and (35) with (40)-(42), for \( R \geq R_1 \), respectively, we ascertain
the validity of (7), and obtain the explicit expression\(^{13}\):

\[
E_c(R) = \sup_{0 \leq \rho \leq 1} \min_{T(x, y)} \left\{ D(T \parallel Q \circ P) - \rho \left[ R(T, Q, 0) - R \right] \right\}
\]

\[
\cong \sup_{0 \leq \rho < 1} \left\{ -\ln \sum_y \left[ \sum_{\hat{x}} Q(\hat{x}) P^{1-\rho}(y | \hat{x}) \right]^{1-\rho} + \rho R \right\},
\]

(43)

where (*\(\)) follows from (36)-(38), the fact that \(T_\rho \rightarrow T_1\), as \(\rho \rightarrow 1\), and by continuity of \(D(T \parallel Q \circ P)\) and \(R(T, Q, 0)\), as functions of \(T\). This, together with the Gallager expression (23) forms a single convex (\(\cup\)) “error/correct-decoding” exponent curve:

\[
E_{e-c}(R) = \sup_{-1 \leq \rho < 1} \left\{ -\ln \sum_y \left[ \sum_{\hat{x}} Q(\hat{x}) P^{1-\rho}(y | \hat{x}) \right]^{1-\rho} + \rho R \right\}.
\]

9. Extension of the channel decoding error exponent to arbitrary \(D\)

The channel decoding error exponent (3) can be written for arbitrary \(D\) as

\[
\lim_{n \rightarrow \infty} \left\{ -\frac{1}{n} \ln P_e \right\} = E_e(R, D) \triangleq \min_{T(x, y)} \left\{ D(T \parallel Q \circ P) + |R(T, Q, D) - R|^+ \right\},
\]

(44)

where \(R(T, Q, D)\) is determined with respect to the distortion measure \(d((x, y), \hat{x})\) (4), with the possible exception of \(D = D_{\text{min}} = \min_{(x, y), \hat{x}} d((x, y), \hat{x})\), when the RHS is a lower bound. This exponent is exactly the same as the encoding success exponent (2), after we make the substitutions (19)-(21).

Note, that the original exponent (2) corresponds to the “encoding success” condition

\[
\sum_{x, \hat{x}} T(x) W(\hat{x} | x) d(x, \hat{x}) \leq D,
\]

(45)

where the joint distribution \(T(x) W(\hat{x} | x)\) represents the joint type of a source sequence \(x\) and a reproduction sequence \(\hat{x}_m\). That is, the encoding success condition (45) is an extension (to the set of all distributions) of the condition on the joint type of sequences of length \(n\):

\[
\sum_{x, \hat{x}} P_{x, \hat{x}_m}(x, \hat{x}) d(x, \hat{x}) \leq D.
\]

This condition, in turn, represents the encoding success event:

\[
\{ \exists m : d(X, \hat{X}_m) \leq nD \}.
\]

\(^{13}\)Starting from (5)-(6), it is a lengthy derivation of the explicit channel correct-decoding exponent. Its purpose is to prove (7), which shows the relation to the encoding failure exponent for sources (5), and a sanity check of (6). A much shorter derivation of the explicit channel correct-decoding exponent can be made from an alternative expression:

\[
\min_{T(x, y)} \left\{ D(T \parallel Q \circ P) + |R - D(T \parallel Q \times T_{y, \hat{y}})|^+ \right\}.
\]

This expression, unlike (6), leads to convex objective functions, and has itself a simple derivation/explanation (alternative to the derivation of (6)) as the channel correct-decoding exponent.
It is obvious from the definition of the encoding success condition, that the decoding error exponent (44) corresponds to a decoding error condition:

\[
\sum_{x, y, \hat{x}} T(x, y) W(\hat{x} | x, y) d((x, y), \hat{x}) \leq D
\]

\[
\sum_{x, y, \hat{x}} T(x, y) W(\hat{x} | x, y) \ln \frac{P(y | x)}{P(y | \hat{x})} \leq D
\]

\[
\sum_{x, y, \hat{x}} T(x, y) W(\hat{x} | x, y) \ln P(y | x) \leq D + \sum_{x, y, \hat{x}} T(x, y) W(\hat{x} | x, y) \ln P(y | \hat{x}),
\]

where \(T(x, y) W(\hat{x} | x, y)\) represents the joint type of a transmitted codeword \(x_m\), a received vector \(y\), and a competing codeword \(x_{m'}\). The decoding error condition represents the decoding error event:

\[
\left\{ \exists m' \neq m : \ln \frac{P(Y | X_m)}{P(Y | X_{m'})} \leq nD \right\}.
\] (46)

A positive \(D\) amounts to a stricter receiver, which requires a confidence distance greater than \(nD\) between the log-likelihoods of the most likely codeword and the second most likely codeword, in order to make a decision. In this case, the decoding error event consists of an erasure and an undetected error.

A negative \(D\) amounts to a list decoder. All codewords with log-likelihoods within a distance of less than \(-nD\) from the most likely codeword are in the list. A decoding error occurs when the transmitted codeword is not in the list.

10. **Explicit channel decoding error exponent with arbitrary** \(D\)

The substitutions (19)-(21) into (18) give an explicit form of (44):

\[
E_e(R, D) = \sup_{0 \leq \rho \leq 1} \left\{ - \inf_{s \geq 0} \ln \sum_{x, y} Q(x) P(y | x) \left[ \sum_{\hat{x}} Q(\hat{x}) \left[ P(y | x) / P(y | \hat{x}) e^{-D} \right]^{-s} \right]^\rho - \rho R \right\}. \] (47)

11. **Extension of the channel correct-decoding exponent to arbitrary** \(D\)

A natural extension is possible with respect to the decoding error event defined by (46). In this case, the correct-decoding exponent is given by

\[
E_{\text{c}}^*(R, D) = \min_{T(x, y): R(T, Q, D) \geq R} D(T \parallel Q \circ P),
\] (48)

with the possible exception of points of discontinuity of this function. This exponent is exactly the same as the encoding failure exponent (5), after we make the substitutions (19)-(21), also in the case \(D = 0\).

The superscript \(^*\) serves to indicate that this exponent is different from (6) or (7), for \(D = 0\), as here the receiver declares an error also when there is only an equality in (46), i.e. no tie-breaking\(^{14}\).

\(^{14}\)This distinction is important in the case of the correct-decoding exponent, but not in the case of the decoding error exponent.
12. Explicit channel correct-decoding exponent with arbitrary $D$

Since (48) is equivalent to the encoding failure exponent (5), we can use Theorem 4 with substitutions (19)-(21) and (34). For distortion constraint $D \geq 0$:

$$\text{lower convex envelope } (E^*_c(R)) = \sup_{\rho \geq 0} \left\{ -\sup_{s \geq 0} \ln \sum_{x, y} Q(x)P(y | x) \left[ \sum_{\hat{x}} Q(\hat{x}) \left[ \frac{P(y | x)}{P(y | \hat{x})} e^{-D} \right]^{-s} \right]^{-\rho} + \rho R \right\}.$$  

For $D < 0$, the right-hand side expression gives zero, which is strictly lower than $E^*_c(R)$, if $R > R(Q \circ P, Q, D)$.

13. Random coding error exponent of Forney’s decoder (lower bound)

In [2] the decoding error event, given that message $m$ is transmitted, is defined as

$$\mathcal{E}_m \triangleq \left\{ \ln \frac{P(Y | X_m)}{\sum_{m' \neq m} P(Y | X_{m'})} < nD \right\}. \quad (49)$$

This is different from the definition of the decoding error event (46) we have used in order to establish duality between channel decoding and source encoding. The sum over $m'$, which appears in Forney’s metric (49), and consists of an exponentially large number $(e^{nR})$ of terms, can be written equivalently as another sum — of a polynomial number of terms — over conditional types of different $X_{m'}$ given a transmitted-received vector pair\(^{15}\) $(X_m, Y)$. Denote these conditional types as $P_{\hat{x} | x, y}$. Then we can write the sum using indicator functions as

$$\sum_{m' \neq m} P(Y | X_{m'}) = \sum_{P_{\hat{x} | x, y}} \sum_{m' \neq m} P(Y | X_{m'}) \cdot \mathbb{1}_{\{X_{m'} \in T(P_{\hat{x} | x, y}, X_m, Y)\}}(m')$$

$$= \sum_{P_{\hat{x} | x, y}} e^{-nE(P_{\hat{x} | x, y}, X_m, Y)}$$

$$\leq \sum_{P_{\hat{x} | x, y}} e^{-nE_{\min}(X_m, Y)} \leq (n + 1)^{|X| \cdot |X| - |Y|} e^{-nE_{\min}(X_m, Y)}, \quad (50)$$

where some of the exponents $E(P_{\hat{x} | x, y}, X_m, Y)$ may have value $+\infty$ (when the corresponding conditional type class $T(P_{\hat{x} | x, y}, X_m, Y)$ is not represented among $X_{m'}$), but not all the exponents are $+\infty$ at the same time, and $E_{\min}(X_m, Y) \triangleq \min_{P_{\hat{x} | x, y}} E(P_{\hat{x} | x, y}, X_m, Y) < +\infty$. Note, that here the exponents

\(^{15}\)Conditioning on a transmitted vector is not necessary for our derivation.
\[ E(P_{x|x,y}, X_m, Y) \] and their minimum over conditional types \( E_{\min}(X_m, Y) \) are random variables, also given \((X_m, Y)\).^{16}

Using types, we can upper-bound the ensemble average probability of error, given that message \( m \) is transmitted, as follows

\[
\Pr \{ E_{m} \} \leq \sum_{P_{x,y}} \Pr \left\{ (X_{m}, Y) \in T(P_{x,y}) \right\} \cdot \Pr \left\{ E(P_{\hat{x}|x,y}, X_{m}, Y) = E_{\min}(X_{m}, Y), E_{m} \mid P_{x,y} \right\}
\]

\[ \leq \sum_{P_{x,y}} \Pr \left\{ (X_{m}, Y) \in T(P_{x,y}) \right\} \cdot \Pr \left\{ E(P_{\hat{x}|x,y}, X_{m}, Y) = E_{\min}(X_{m}, Y), \right. \]

\[
E_{\min}(X_{m}, Y) \leq \frac{|X||X||Y|\ln(n+1)}{n} - \ln P(Y | X_{m}) + D \mid P_{x,y}
\]

\[ \leq \sum_{P_{x,y}} \Pr \left\{ (X_{m}, Y) \in T(P_{x,y}) \right\} \times \]

\[
\Pr \left\{ E(P_{\hat{x}|x,y}, X_{m}, Y) \leq \frac{|X||X'||Y'|\ln(n+1)}{n} - \ln P(Y | X) + D \mid P_{x,y} \right\}
\]

\[ \leq \sum_{P_{x,y}} \Pr \left\{ (X_{m}, Y) \in T(P_{x,y}) \right\} \cdot \Pr \left\{ E(P_{\hat{x}|x,y}, X_{m}, Y) \leq \epsilon_{1} - \ln P(Y | X) + D \mid P_{x,y} \right\}
\]

\[ \leq \sum_{P_{x,y} \text{ : } f(P_{x,y}) \leq R} \Pr \left\{ (X_{m}, Y) \in T(P_{x,y}) \right\} \cdot \Pr \left\{ E(P_{\hat{x}|x,y}, X_{m}, Y) > g(P_{x,y}, x) \right\}, \]

\[
E(P_{\hat{x}|x,y}, X_{m}, Y) \leq \epsilon_{1} - \ln P(Y | X) + D \mid P_{x,y}
\]

\[ + \sum_{P_{x,y} \text{ : } f(P_{x,y}) \leq R} \Pr \left\{ (X_{m}, Y) \in T(P_{x,y}) \right\} \cdot \Pr \left\{ E(P_{\hat{x}|x,y}, X_{m}, Y) \leq g(P_{x,y}, x) \right\}, \]

\[
E(P_{\hat{x}|x,y}, X_{m}, Y) \leq \epsilon_{1} - \ln P(Y | X) + D \mid P_{x,y}
\]

\[ ^{16} \text{It is convenient to think that } E(P_{\hat{x}|x,y}, X_{m}, Y) \text{ is a function of all stochastic matrices } P_{\hat{x}|x,y} \text{ possible for a given block length } n, \text{ regardless of the joint type of } (X_{m}, Y). \text{ If a stochastic matrix } P_{\hat{x}|x,y} \text{ is not compatible with a certain joint type of } (X_{m}, Y), \text{ then simply } T(P_{\hat{x}|x,y}, X_{m}, Y) = \emptyset \text{ and } E(P_{\hat{x}|x,y}, X_{m}, Y) = +\infty, \text{ i.e. the corresponding term in (50) is zero. Given a joint type of } (X_{m}, Y) \text{ (and nothing else), the random variables } E(P_{\hat{x}|x,y}, X_{m}, Y) \text{ and } E_{\min}(X_{m}, Y) \text{ become independent of } (X_{m}, Y), \text{ but their proper definitions still require a reference to } (X_{m}, Y). \]
\[ \sum_{P_{x,y} : f(P_{x,y}) > R} \Pr \{(X_m, Y) \in T(P_{x,y})\} \cdot \Pr \left\{ E(P_{\hat{x}|x,y} ; X_m, Y) > h(P_{x,y}) \right\} \]
\[ + \sum_{P_{x,y} : f(P_{x,y}) > R} \Pr \{(X_m, Y) \in T(P_{x,y})\} \cdot \Pr \left\{ E(P_{\hat{x}|x,y} ; X_m, Y) \leq h(P_{x,y}) \right\} \]
\[ \leq \sum_{P_{x,y} : f(P_{x,y}) \leq R} \Pr \{(X_m, Y) \in T(P_{x,y})\} \cdot \Pr \left\{ g(P_{x,y}) \leq \epsilon_1 - \mathbb{E} \left[ \ln P(Y | X) \right] + D \bigg| P_{x,y} \right\} \]
\[ + \sum_{P_{x,y} : f(P_{x,y}) \leq R} \Pr \{(X_m, Y) \in T(P_{x,y})\} \cdot \Pr \left\{ E(P_{\hat{x}|x,y} ; X_m, Y) \leq g(P_{x,y}) \bigg| P_{x,y} \right\} \]
\[ + \sum_{P_{x,y} : f(P_{x,y}) > R} \Pr \{(X_m, Y) \in T(P_{x,y})\} \cdot \Pr \left\{ E(P_{\hat{x}|x,y} ; X_m, Y) < +\infty \bigg| P_{x,y} \right\} \]
\[ \leq \sum_{P_{x,y} : f(P_{x,y}) \leq R} \Pr \{(X_m, Y) \in T(P_{x,y})\} \times \]
\[ \mathbb{I}\left\{ g(P_{x,y}) \leq \epsilon_1 - \mathbb{E} \left[ \ln P(Y | X) \right] + D \right\} \]
\[ + \sum_{P_{x,y} : f(P_{x,y}) \leq R} \Pr \{(X_m, Y) \in T(P_{x,y})\} \cdot \Pr \left\{ E(P_{\hat{x}|x,y} ; X_m, Y) \leq g(P_{x,y}) \bigg| P_{x,y} \right\} \]
\[ + \sum_{P_{x,y} : f(P_{x,y}) > R} \Pr \{(X_m, Y) \in T(P_{x,y})\} \cdot \Pr \left\{ E(P_{\hat{x}|x,y} ; X_m, Y) < +\infty \bigg| P_{x,y} \right\} \times \]
\[ \mathbb{I}\left\{ h(P_{x,y}) \leq \epsilon_1 - \mathbb{E} \left[ \ln P(Y | X) \right] + D \right\} \]
\[ + \sum_{P_{x,y} : f(P_{x,y}) > R} \Pr \{(X_m, Y) \in T(P_{x,y})\} \cdot \Pr \left\{ E(P_{\hat{x}|x,y} ; X_m, Y) \leq h(P_{x,y}) \bigg| P_{x,y} \right\}, \]

for sufficiently large \( n \), which is needed for (c) to hold. Explanation of steps:

(a) follows by the definition of the error event \( \mathcal{E}_m \) (49) and the bound on Forney’s sum (51);
(b) is an identity, with a notation, given \( \{(X_m, Y) \in T(P_{x,y})\} \):
\[
\frac{\ln P(Y | X_m)}{n} = \sum_{x,y} P_{x,y}(x,y) \ln P(y | x) \triangleq \mathbb{E} [\ln P(Y | X)];
\] (53)

(c) holds for any \( \epsilon_1 > 0 \), for sufficiently large \( n \), such that
\[
\frac{|X||X'||Y| \ln(n+1)}{n} \leq \epsilon_1 ;
\] (54)

(d) is an identity, for arbitrary functions \( f(P_{x,y,\hat{x}}), g(P_{x,y,\hat{x}}), h(P_{x,y,\hat{x}}) \);

(e) uses if-then relations between events:
\[
\left\{ \begin{array}{c} E(P_{\hat{x}|x,y}X_m, Y) > g(P_{x,y,\hat{x}}), E(P_{\hat{x}|x,y}X_m, Y) \leq \epsilon_1 - \mathbb{E} [\ln P(Y | X)] + D \end{array} \right\} \Rightarrow \left\{ \begin{array}{c} g(P_{x,y,\hat{x}}) \leq \epsilon_1 - \mathbb{E} [\ln P(Y | X)] + D \end{array} \right\} ;
\]
\[
\left\{ \begin{array}{c} E(P_{\hat{x}|x,y}X_m, Y) > h(P_{x,y,\hat{x}}), E(P_{\hat{x}|x,y}X_m, Y) \leq \epsilon_1 - \mathbb{E} [\ln P(Y | X)] + D \end{array} \right\} \Rightarrow \left\{ \begin{array}{c} E(P_{\hat{x}|x,y}X_m, Y) < +\infty, h(P_{x,y,\hat{x}}) \leq \epsilon_1 - \mathbb{E} [\ln P(Y | X)] + D \end{array} \right\} ;
\]

(f) is an identity, when the functions \( g(P_{x,y,\hat{x}}) \) and \( h(P_{x,y,\hat{x}}) \) are arbitrary deterministic. In this case, the events
\[
\left\{ \begin{array}{c} g(P_{x,y,\hat{x}}) \leq \epsilon_1 - \mathbb{E} [\ln P(Y | X)] + D \end{array} \right\} ,
\left\{ \begin{array}{c} h(P_{x,y,\hat{x}}) \leq \epsilon_1 - \mathbb{E} [\ln P(Y | X)] + D \end{array} \right\}
\]
are deterministic conditions (i.e., they either hold with probability 1 or with probability 0), and indicator functions can be used in place of probabilities.

The upper bound (52) was devised with something like the following lemma in mind:

**Lemma 9:** Let \( Z_i \sim i.i.d Bernoulli(e^{-nI}) \), \( i = 1, 2, \ldots, e^{nR} \). For \( \epsilon > 0 \), if \( I \leq R + \epsilon \), then
\[
\Pr \left\{ \sum_{i=1}^{e^{nR}} Z_i \geq e^{n(R-I+2\epsilon)} \right\} < \exp \left\{- [e^{n\epsilon} - (e - 1)e^{-n\epsilon}] \right\}, \quad e^{2n\epsilon} \geq e - 1, \quad (55)
\]
if \( I > R \), then
\[
\Pr \left\{ \sum_{i=1}^{e^{nR}} Z_i \geq e^{n\epsilon} \right\} < \exp \left\{- [e^{n\epsilon} - e + 1] \right\}. \quad (56)
\]
Proof: This is an unoptimized Chernoff bound, with the parameter in the exponent \( 1 \):

\[
\Pr \left\{ \sum_{i=1}^{e^{nR}} Z_i \geq e^{n(\Delta + \epsilon)} \right\} = \Pr \left\{ \exp \left\{ \sum_{i=1}^{e^{nR}} Z_i \right\} \geq \exp \left\{ e^{n(\Delta + \epsilon)} \right\} \right\}
\]

\[
\leq \exp \left\{ -e^{n(\Delta + \epsilon)} \right\} \cdot \mathbb{E} \left[ \exp \left\{ \sum_{i=1}^{e^{nR}} Z_i \right\} \right]
\]

\[
= \exp \left\{ -e^{n(\Delta + \epsilon)} \right\} \cdot \prod_{i=1}^{e^{nR}} \mathbb{E} \left[ e^{Z_i} \right]
\]

\[
= \exp \left\{ -e^{n(\Delta + \epsilon)} \right\} \cdot \left[ \left( 1 + (e - 1)e^{-nI} \right)^{e^{nR}} \right]^{(e - 1)e^{-nI}, e^{nR}} < e \]

\[
\leq \exp \left\{ -e^{n(\Delta + \epsilon)} \right\} \cdot \exp \left\{ (e - 1)e^{n(R - I)} \right\}
\]

\[
= \begin{cases} 
\exp \left\{ -e^{n(R - I)} \left[ e^{2ne} - e + 1 \right] \right\}, & \Delta = R - I + \epsilon, \\
\exp \left\{ -e^{ne} + (e - 1)e^{n(R - I)} \right\}, & \Delta = 0,
\end{cases}
\]

where \((a)\) is Markov’s inequality (yielding at this step an unoptimized Chernoff bound with parameter \(1\)), and \((b)\) holds because \((1 + x)^{1/x} < e\).

For the case \(I \leq R + \epsilon\), we take the bound with \(\Delta = R - I + \epsilon\) and obtain

\[
... \leq \exp \left\{ -e^{n(R - I)} \left[ e^{2ne} - e + 1 \right] \right\} \leq \exp \left\{ -e^{ne} - (e - 1)e^{-ne} \right\}, \quad e^{2ne} - e + 1 \geq 0.
\]

For the case \(I > R\), we take the bound with \(\Delta = 0\) and obtain

\[
\exp \left\{ -e^{ne} + (e - 1)e^{n(R - I)} \right\} < \exp \left\{ -e^{ne} + e - 1 \right\}.
\]

In order to use Lemma 9, recall that the probability of a conditional type is bounded from above and below as

\[
\exp \left\{ -nf(P_{x,y}) \right\} \triangleq \exp \left\{ -nD(P_{x,y} \| P_{x,y}(x,y) \cdot Q(\hat{x})) \right\} \geq \Pr \left\{ \mathbf{X}_{m'} \in T(P_{x,y}, \mathbf{X}_m, Y) \mid (\mathbf{X}_m, Y) \in T(P_{x,y}) \right\} \triangleq \exp \{-nI\}
\]

\[
\geq (n + 1)^{-|\mathcal{X}||\mathcal{X}||\mathcal{Y}|} \cdot \exp \left\{ -nf(P_{x,y}) \right\}.
\]

These definitions give

\[
f(P_{x,y}) \leq I \leq f(P_{x,y}) + \frac{|\mathcal{X}||\mathcal{X}||\mathcal{Y}| \ln(n + 1)}{n}.
\]
If \( f(P_{x,y}) \leq R \), then for \( n \) sufficiently large, as in (54), we get \( I \leq R + \epsilon_1 \). If \( n \) satisfies (54), then it is also large enough to satisfy \( e^{2n\epsilon_1} \geq e - 1 \). For such \( n \), the first part of Lemma 9 holds for the following:

\[
\Pr \left\{ E(P_{x|y}, X_m, Y) \leq -\mathbb{E}_{P_{x,y}}[\ln P(Y | \hat{X})] - R + f(P_{x,y}) - 2\epsilon_1 \mid (X_m, Y) \in T(P_{x,y}) \right\}
\]

\[
\leq \Pr \left\{ \sum_{m' \neq m} P(Y | X_{m'}) \cdot 1 \{ X_{m'} \in T(P_{x|y}, X_m, Y) \} (m') \geq \exp \left\{ n(\mathbb{E}_{P_{x,y}}[\ln P(Y | \hat{X})] + R - f(P_{x,y}) + 2\epsilon_1) \right\} \mid (X_m, Y) \in T(P_{x,y}) \right\}
\]

\[
= \Pr \left\{ \sum_{m' \neq m} \mathbb{1} \{ X_{m'} \in T(P_{x|y}, X_m, Y) \} (m') \geq e^{n(R - f(P_{x,y}) + 2\epsilon_1)} \mid (X_m, Y) \in T(P_{x,y}) \right\}
\]

\[
\leq \Pr \left\{ \sum_{m=1}^{e^n R} \mathbb{1} \{ X_{m'} \in T(P_{x|y}, X_m, Y) \} (m') \geq e^{n(R - I + 2\epsilon_1)} \mid (X_m, Y) \in T(P_{x,y}) \right\}
\]

\[
< \exp \left\{ -[e^{n\epsilon_1} - (e - 1)e^{-ne\epsilon_1}] \right\}, \quad \frac{|\mathcal{X}| |\mathcal{X}'| |\mathcal{Y}| \ln(n + 1)}{n} \leq \epsilon_1,
\]

where in \( (a) \) we use the definition of \( E(P_{x|y}, X_m, Y) \) (50), and notation (53) with \( P_{x,y} \); in \( (b) \) we assume the size of the codebook \( M = e^{nR} + 1 \), and use \( I \geq f(P_{x,y}) \); and \( (c) \) holds by (58) and (55) of the lemma. If we choose

\[
g(P_{x,y}) \triangleq -\mathbb{E}_{P_{x,y}}[\ln P(Y | \hat{X})] - R + f(P_{x,y}) - 2\epsilon_1,
\]

then with (60) we obtain that the second sum in (52) is upper-bounded as

\[
\sum_{P_{x,y}: f(P_{x,y}) \leq R} \Pr \left\{ (X_m, Y) \in T(P_{x,y}) \right\} \cdot \Pr \left\{ E(P_{x|y}, X_m, Y) \leq g(P_{x,y}) \mid P_{x,y} \right\}
\]

\[
< \sum_{P_{x,y}: f(P_{x,y}) \leq R} \Pr \left\{ (X_m, Y) \in T(P_{x,y}) \right\} \cdot \exp \left\{ -[e^{n\epsilon_1} - (e - 1)e^{-ne\epsilon_1}] \right\}
\]

\[
\leq (n + 1)^{|\mathcal{X}| |\mathcal{X}'| |\mathcal{Y}|} \cdot \exp \left\{ -[e^{n\epsilon_1} - (e - 1)e^{-ne\epsilon_1}] \right\}, \quad \forall n : \frac{|\mathcal{X}| |\mathcal{X}'| |\mathcal{Y}| \ln(n + 1)}{n} \leq \epsilon_1.
\]

On the other hand, if \( f(P_{x,y}) > R \), then by (59) also \( I > R \), and the second part of Lemma 9 holds for the following:

\[
\Pr \left\{ E(P_{x|y}, X_m, Y) \leq -\mathbb{E}_{P_{x,y}}[\ln P(Y | \hat{X})] - \epsilon_2 \mid (X_m, Y) \in T(P_{x,y}) \right\}
\]

\[
\leq \Pr \left\{ \sum_{m' \neq m} P(Y | X_{m'}) \cdot 1 \{ X_{m'} \in T(P_{x|y}, X_m, Y) \} (m') \geq \exp \left\{ n(\mathbb{E}_{P_{x,y}}[\ln P(Y | \hat{X})] + \epsilon_2) \right\} \mid (X_m, Y) \in T(P_{x,y}) \right\}
\]
\[
= \Pr \left\{ \sum_{m' \neq m} \mathbb{1}\{X_{m'} \in T(P_{\hat{x} | x, y}, X_{m}, Y)\} \geq e^{n \epsilon_2} \mid (X_m, Y) \in T(P_{x, y}) \right\} \\
\overset{(b)}{=} \Pr \left\{ \sum_{m' = 1}^{e^{nR}} \mathbb{1}\{X_{m'} \in T(P_{\hat{x} | x, y}, X_{m}, Y)\} \geq e^{n \epsilon_2} \mid (X_m, Y) \in T(P_{x, y}) \right\} \\
\overset{(c)}{<} \exp \left\{ - [e^{n \epsilon_2} - e + 1] \right\}, \quad \epsilon_2 > 0,
\]

where in (a) we use the definition of \( E(P_{\hat{x} | x, y}, X_m, Y) \) (50), and notation (53) with \( P_{\hat{x}, y} \); in (b) we assume the codebook size \( M = e^{nR} + 1 \); and (c) holds by (58) and (56) of the lemma. If we choose

\[
h(P_{x, y, \hat{x}}) \triangleq - \mathbb{E}_{P_{x,y}}[\ln P(Y \mid \hat{X})] - \epsilon_2,
\]

then with (63) we obtain that the fourth sum in (52) is upper-bounded as

\[
\sum_{P_{x, y, \hat{x}}: f(P_{x,y,\hat{x}}) > R} \Pr \left\{ (X_m, Y) \in T(P_{x,y}) \right\} \cdot \Pr \left\{ E(P_{\hat{x} | x, y}, X_m, Y) \leq h(P_{x,y,\hat{x}}) \mid P_{x,y} \right\} \\
< \sum_{P_{x, y, \hat{x}}: f(P_{x,y,\hat{x}}) > R} \Pr \left\{ (X_m, Y) \in T(P_{x,y}) \right\} \cdot \exp \left\{ - [e^{n \epsilon_2} - e + 1] \right\} \\
\leq (n + 1)^{|X| |Y| |Y|} \cdot \exp \left\{ - [e^{n \epsilon_2} - e + 1] \right\}, \quad \epsilon_2 > 0.
\]

With the definitions (57), (61), (64) at hand, we are ready to bound also the first and the third sums in (52).

The first sum in (52) is upper-bounded as follows:

\[
\sum_{P_{x, y, \hat{x}}: f(P_{x,y,\hat{x}}) \leq R} \Pr \left\{ (X_m, Y) \in T(P_{x,y}) \right\} \times \\
\mathbb{1}\{g(P_{x,y,\hat{x}}) \leq \epsilon_1 - \mathbb{E}_{P_{x,y}}[\ln P(Y \mid X)] + D\} (P_{x,y,\hat{x}})
\]

\[
\leq \sum_{P_{x, y, \hat{x}}: f(P_{x,y,\hat{x}}) \leq R} \exp \left\{ - nD(P_{x,y} \parallel Q \circ P) \right\} \times \\
\mathbb{1}\{g(P_{x,y,\hat{x}}) \leq \epsilon_1 - \mathbb{E}_{P_{x,y}}[\ln P(Y \mid X)] + D\} (P_{x,y,\hat{x}})
\]

\[
\overset{(a)}{=} \sum_{P_{x, y, \hat{x}}} \exp \left\{ - nD(P_{x,y} \parallel Q \circ P) \right\} \times \\
\mathbb{1}\left\{ \mathbb{E}_{P_{x,y}}[\ln P(Y \mid X)] - \mathbb{E}_{P_{x,y}}[\ln P(Y \mid \hat{X})] + f(P_{x,y,\hat{x}}) \leq R + D + 3\epsilon_1 \right\} (P_{x,y,\hat{x}})
\]

\[
\overset{(b)}{=} \sum_{P_{x, y, \hat{x}}} \exp \left\{ - nD(P_{x,y} \parallel Q \circ P) \right\} \times \\
\left\{ \mathbb{E}_{P_{x,y}}[\ln P(Y \mid X)] - \mathbb{E}_{P_{x,y}}[\ln P(Y \mid \hat{X})] + f(P_{x,y,\hat{x}}) \leq R + D + 3\epsilon_1 \right\} (P_{x,y,\hat{x}})
\]
\[ \begin{align*}
(c) \quad & \sum_{P_{x,y,\hat{x}}} \exp \left\{ -n \tilde{E}_1^{\text{types}}(R, D + 3\epsilon_1) \right\} \\
(d) \quad & \sum_{P_{x,y,\hat{x}}} \exp \left\{ -n \tilde{E}_1(R, D + 3\epsilon_1) \right\} \leq (n + 1)^{|X| - |Y|} \cdot \exp \left\{ -n \tilde{E}_1(R, D + 3\epsilon_1) \right\}, \quad (66)
\end{align*} \]

where in

(a) we use the bound

\[ \Pr \left\{ (X_m, Y) \in T(P_{x,y}) \right\} \leq \exp \left\{ -n D(P_{x,y}(x, y) \parallel Q(x) \cdot P(y \mid x)) \right\}; \quad (67) \]

(b) collect all the conditions in the indicator function and substitute the definition of \( g(P_{x,y,\hat{x}}) \) (61);

(c) the minimal exponent \( \tilde{E}_1^{\text{types}}(R, D + 3\epsilon_1) \) is determined by minimization over types \( P_{x,y,\hat{x}} \), corresponding to block length \( n \), subject to the two conditions, which appear in the indicator function:

\[ \tilde{E}_1^{\text{types}}(R, D) \triangleq \min_{P_{x,y,\hat{x}}(x,y,\hat{x})} D(P_{x,y} \parallel Q \circ P) \]

subject to:

\[ \mathbb{E}_{P_{x,y,\hat{x}}} \left[ \ln \frac{P(Y \mid X)}{P(Y \mid \hat{X})} \right] + D(P_{x,y,\hat{x}} \parallel P_{x,y} \times Q) \leq R + D, \]

\[ D(P_{x,y,\hat{x}} \parallel P_{x,y} \times Q) \leq R, \]

where we use also the definition of \( f(P_{x,y,\hat{x}}) \) (57);

(d) the minimal exponent \( \tilde{E}_1^{\text{types}}(R, D + 3\epsilon_1) \) is lower-bounded further by the result of the same minimization, denoted as \( \tilde{E}_1(R, D + 3\epsilon_1) \), performed over all possible joint distributions \( T(x, y) \cdot W(\hat{x} \mid x, y) \) :

\[ \tilde{E}_1(R, D) \triangleq \min_{T(x, y), W(\hat{x} \mid x, y)} D(T \parallel Q \circ P) \]

subject to:

\[ \mathbb{E}_{T \circ W} \left[ \ln \frac{P(Y \mid X)}{P(Y \mid \hat{X})} \right] + D(T \circ W \parallel T \times Q) \leq R + D, \]

\[ D(T \circ W \parallel T \times Q) \leq R. \]

Finally, the third sum in (52) is upper-bounded as follows:

\[ \sum_{P_{x,y,\hat{x}} : f(P_{x,y,\hat{x}}) > R} \Pr \left\{ (X_m, Y) \in T(P_{x,y}) \right\} \cdot \Pr \left\{ E(P_{x \mid x,y}, X_m, Y) < +\infty \middle| P_{x,y} \right\} \times \]

\[ \mathbb{1}_{\{h(P_{x,y,\hat{x}}) \leq \epsilon_1 - \mathbb{E}_{P_{x,y}} \ln P(Y \mid X) + D\}}(P_{x,y,\hat{x}}) \]

\[ \sum_{P_{x,y,\hat{x}} : f(P_{x,y,\hat{x}}) > R} \exp \left\{ -n D(P_{x,y} \parallel Q \circ P) \right\} \cdot \exp \left\{ -n \left[ D(P_{x,y,\hat{x}} \parallel P_{x,y} \times Q) - R \right] \right\} \times \]

\[ \mathbb{1}_{\{h(P_{x,y,\hat{x}}) \leq \epsilon_1 - \mathbb{E}_{P_{x,y}} \ln P(Y \mid X) + D\}}(P_{x,y,\hat{x}}) \]
\begin{align}
&\sum_{P_{x,y,\hat{x}}} \exp \left\{ -n \left[ D(P_{x,y} \mid Q \circ P) + D(P_{x,y,\hat{x}} \mid P_{x,y} \times Q) - R \right] \right\} \times \\
&\left\{ \mathbb{E}_{P_{x,y}} \left[ \ln P(Y \mid X) \right] - \mathbb{E}_{P_{x,y}} \left[ \ln P(Y \mid \hat{X}) \right] \leq D + \epsilon_1 + \epsilon_2 \right\}^{(P_{x,y,\hat{x}})} \\
&\leq \sum_{P_{x,y,\hat{x}}} \exp \left\{ -n E_2^{\text{types}}(R, D + \epsilon_1 + \epsilon_2) \right\} \\
&\leq \sum_{P_{x,y,\hat{x}}} \exp \left\{ -n E_2(R, D + \epsilon_1 + \epsilon_2) \right\} \leq (n + 1)^{|X| \cdot |Y|} \cdot \exp \left\{ -n E_2(R, D + \epsilon_1 + \epsilon_2) \right\},
\end{align}

where

\((a)\) follows by \((67)\) and the union bound

\begin{align}
\Pr \left\{ E(P_{x,y,\hat{x}} \mid X_m, Y) < +\infty \mid P_{x,y} \right\} \leq \exp \left\{ -n \left[ D(P_{x,y,\hat{x}} \mid (x, y, \hat{x}) \mid P_{x,y}(x, y) \cdot Q(\hat{x})) - R \right] \right\};
\end{align}

\((b)\) uses the definitions of \(h(P_{x,y,\hat{x}})\) \((64)\) and \(f(P_{x,y,\hat{x}})\) \((57)\);

\((c)\) the minimal exponent \(E_2^{\text{types}}(R, D + \epsilon_1 + \epsilon_2)\) is determined by minimization over types \(P_{x,y,\hat{x}}\), corresponding to block length \(n\), subject to the two conditions, which appear in the indicator function:

\begin{align}
E_2^{\text{types}}(R, D) \triangleq \min_{P_{x,y,\hat{x}}(x, y, \hat{x})} \left\{ D(P_{x,y} \mid Q \circ P) + D(P_{x,y,\hat{x}} \mid P_{x,y} \times Q) - R \right\} \\
\text{subject to:} \\
\mathbb{E}_{P_{x,y,\hat{x}}} \left[ \ln \frac{P(Y \mid X)}{P(Y \mid \hat{X})} \right] \leq D, \\
D(P_{x,y,\hat{x}} \mid P_{x,y} \times Q) > R;
\end{align}

\((d)\) \(E_2^{\text{types}}(R, D + \epsilon_1 + \epsilon_2)\) is lower-bounded further by the result of the same minimization, denoted as \(E_2(R, D + \epsilon_1 + \epsilon_2)\), performed over all possible joint distributions \(T(x, y) \cdot W(\hat{x} \mid x, y)\):

\begin{align}
E_2(R, D) \triangleq \min_{T(x, y), W(\hat{x} \mid x, y)} \left\{ D(T \mid Q \circ P) + D(T \circ W \mid T \times Q) - R \right\} \\
\text{subject to:} \\
\mathbb{E}_{T \circ W} \left[ \ln \frac{P(Y \mid X)}{P(Y \mid \hat{X})} \right] \leq D, \\
D(T \circ W \mid T \times Q) \geq R.
\end{align}

Comparing the bounds \((62)\), \((65)\), \((66)\), and \((70)\), we conclude, that, for \(n\) sufficiently large, the exponent in the upper bound \((52)\) is lower-bounded by \(\min \left\{ \tilde{E}_1(R, D + 3\epsilon_1), E_2(R, D + \epsilon_1 + \epsilon_2) \right\} - \epsilon_1\). Since \(\epsilon_1\) and \(\epsilon_2\) are arbitrary, they can be replaced with zeros, resulting in the following
Theorem 5:

\[
\lim_{n \to \infty} \inf \left\{ -\frac{1}{n} \ln \Pr \{ \mathcal{E}_m \} \right\} \geq \min \left\{ \tilde{E}_1(R, D), E_2(R, D) \right\}
\]

\[
= \min \left\{ \min_{T(x, y), W(\hat{x} | x, y): \atop d(T \circ W) + D(T \circ W \| T \times Q) \leq D + R} \left\{ D(T \| Q \circ P) \right\}, \min_{T(x, y), W(\hat{x} | x, y): \atop D(T \circ W \| T \times Q) \leq R} \left\{ D(T \| Q \circ P) + D(T \circ W \| T \times Q) - R \right\} \right\}, \tag{74}
\]

where \( d(T \circ W) \triangleq \mathbb{E}_{T \circ W} \left[ d((X, Y), \hat{X}) \right] \triangleq \mathbb{E}_{T \circ W} \left[ \ln \frac{P_Y(X)}{P(Y \mid X)} \right] \).

14. Explicit lower bound on the random coding error exponent of Forney’s decoder

We use Theorem 5 to prove the following\textsuperscript{17}

Theorem 6:

\[
\lim_{n \to \infty} \inf \left\{ -\frac{1}{n} \ln \Pr \{ \mathcal{E}_m \} \right\} \geq \min \left\{ \sup_{\rho \geq 0} \left\{ -\inf_{0 \leq s \leq 1} \ln \sum_{x, y} Q(x) P(y \mid x) \left[ \sum_{\hat{x}} Q(\hat{x}) \left[ \frac{P(y \mid x)}{P(y \mid \hat{x})} e^{-D} \right]^s \right]^\rho \right\}, \sup_{0 \leq \rho \leq 1} \left\{ -\inf_{s \geq 0} \ln \sum_{x, y} Q(x) P(y \mid x) \left[ \sum_{\hat{x}} Q(\hat{x}) \left[ \frac{P(y \mid x)}{P(y \mid \hat{x})} e^{-D} \right]^s \right]^\rho \right\} \right\}.
\]

Proof:

\[
\tilde{E}_1(R, D) = \min_{T(x, y), W(\hat{x} | x, y): \atop d(T \circ W) + D(T \circ W \| T \times Q) \leq D + R} \left\{ D(T \| Q \circ P) \right\}
\]

\[
\geq \sup_{\alpha \geq 0, \beta \geq 0} \min_{T(x, y), W(\hat{x} | x, y): \atop \alpha + \beta \geq 0 \atop d(T \circ W) + D(T \circ W \| T \times Q) \leq D + R} \left\{ D(T \| Q \circ P) \right\}
\]

\[
+ \alpha \left[ D(T \circ W \| T \times Q) + d(T \circ W) - D - R \right]
\]

\[
+ \beta \left[ D(T \circ W \| T \times Q) - R \right].
\]

\textsuperscript{17}This result is redundant, as below we derive the same expression as the true exponent.
\[ \geq \sup_{\alpha \geq 0, \beta \geq 0} \min_{\alpha + \beta > 0} \{ D(T \parallel Q \circ P) + \alpha [D(T \circ W \parallel T \times Q) + d(T \circ W) - D - \bar{R}] + \beta [D(T \circ W \parallel T \times Q) - \bar{R}] \} \]

\[ = \sup_{\alpha \geq 0, \beta \geq 0} \min_{\alpha + \beta > 0} \{ D(T \parallel Q \circ P) - \alpha D - (\alpha + \beta)R \]

\[ + \min_{W[\hat{x} \mid x, y]} [(\alpha + \beta)D(T \circ W \parallel T \times Q) + \alpha d(T \circ W)] \}

\[ = \sup_{\alpha \geq 0, \beta \geq 0} \min_{\alpha + \beta > 0} \{ \sum_{x, y} T(x, y) \ln \frac{T(x, y)}{Q(x)P(y \mid x)} - \alpha D - (\alpha + \beta)R \]

\[ + \min_{W[\hat{x} \mid x, y]} [(\alpha + \beta) \sum_{x, y, \hat{x}} T(x, y)W(\hat{x} \mid x, y) \ln \frac{W(\hat{x} \mid x, y)}{Q(\hat{x})} + \alpha \sum_{x, y, \hat{x}} T(x, y)W(\hat{x} \mid x, y)d((x, y), \hat{x})] \}

\[ = \sup_{\alpha \geq 0, \beta \geq 0} \min_{\alpha + \beta > 0} \{ \sum_{x, y} T(x, y) \ln \frac{T(x, y)}{Q(x)P(y \mid x)} - \alpha D - (\alpha + \beta)R \]

\[ + \min_{W[\hat{x} \mid x, y]} [(\alpha + \beta) \sum_{x, y, \hat{x}} T(x, y)W(\hat{x} \mid x, y) \ln \frac{W(\hat{x} \mid x, y)}{Q(\hat{x})e^{-\frac{\alpha}{\alpha + \beta}d((x, y), \hat{x})}}] \}

\[ = \sup_{\alpha \geq 0, \beta \geq 0} \min_{\alpha + \beta > 0} \left\{ \sum_{x, y} T(x, y) \ln \frac{T(x, y)}{Q(x)P(y \mid x)} \left[ \sum_{\hat{x}} Q(\hat{x})e^{-\frac{\alpha}{\alpha + \beta}d((x, y), \hat{x})} \right]^{\alpha + \beta} - \alpha D - (\alpha + \beta)R \right\} \]

\[ = \sup_{\alpha \geq 0, \beta \geq 0} \left\{ - \ln \sum_{x, y} Q(x)P(y \mid x) \left[ \sum_{\hat{x}} Q(\hat{x})e^{-\frac{\alpha}{\alpha + \beta}d((x, y), \hat{x})} - D \right]^{\alpha + \beta} - (\alpha + \beta)R \right\} \]

\[ = \sup_{\rho > 0} \left\{ - \inf_{0 \leq s \leq 1} \ln \sum_{x, y} Q(x)P(y \mid x) \left[ \sum_{\hat{x}} Q(\hat{x})e^{-s[d((x, y), \hat{x})] - D} \right]^\rho - \rho R \right\}, \]

where we define \( \rho \triangleq \alpha + \beta \) and \( s \triangleq \frac{\alpha}{\alpha + \beta} \). The case \( \rho = 0 \) can also be included, because it gives bound zero, which is always true.

\[ E_2(R, D) = \min_{T(x, y), W[\hat{x} \mid x, y]} \{ D(T \parallel Q \circ P) + D(T \circ W \parallel T \times Q) - R \} \]

\[ d(T \circ W) \leq D \]

\[ D(T \circ W \parallel T \times Q) \geq R \]
\[
\begin{align*}
&\geq \sup_{\alpha \geq 0} \sup_{\beta \geq 0} \min_{T(x, y), W(\hat{x} | x, y) \mid d(T \circ W) \leq D} \left\{ D(T \| Q \circ P) + D(T \circ W \| T \times Q) - R \right\} \\
&\quad + \alpha \left[ d(T \circ W) - D \right] - \beta \left[ D(T \circ W \| T \times Q) - R \right]\} \\
&\geq \sup_{\alpha \geq 0} \sup_{\beta \geq 0} \min_{T(x, y), W(\hat{x} | x, y)} \left\{ D(T \| Q \circ P) + D(T \circ W \| T \times Q) - R \right\} \\
&\quad + \alpha \left[ d(T \circ W) - D \right] - \beta \left[ D(T \circ W \| T \times Q) - R \right]\} \\
&= \sup_{\alpha \geq 0} \sup_{\beta \geq 0} \min_{T(x, y), W(\hat{x} | x, y)} \left\{ D(T \| Q \circ P) \right\} \\
&\quad + \alpha \left[ d(T \circ W) - D \right] + (1 - \beta) \left[ D(T \circ W \| T \times Q) - R \right]\} \\
\end{align*}
\]

where (a) is obtained by the same steps as (75), and in (b) we define \( \rho \triangleq \beta \) and \( s \triangleq \frac{\alpha}{\beta} \). Similarly, the case \( \rho = 0 \) can also be included. It remains to substitute \( d((x, y), \hat{x}) = \ln \frac{P(y \mid x)}{P(\hat{y} \mid \hat{x})} \) into (76) and (77), and combine them for the final result.

15. **Upper bound on the random coding error exponent of Forney’s decoder**

The sum in Forney’s metric (49) can be lower-bounded as follows

\[
\sum_{m' \neq m} P(Y \mid X_{m'}) \geq e^{-nE(P_{\hat{y} \mid x, y}, X_m, Y)} \quad \forall P_{\hat{y} \mid x, y},
\]

where the exponents \( E(P_{\hat{y} \mid x, y}, X_m, Y) \) are defined as in (50).

Using types, we can lower-bound the ensemble average probability of error, given that message \( m \) is transmitted, as follows

\[
\Pr \{ \mathcal{E}_m \} \geq \max_{P_{\hat{y} \mid x, y}} \Pr \{(X_m, Y) \in T(P_{\hat{y} \mid x, y})\} \cdot \Pr \{ \mathcal{E}_m \mid P_{\hat{y} \mid x, y} \}
\]

\[
\geq \max_{P_{\hat{y} \mid x, y}} \Pr \{(X_m, Y) \in T(P_{\hat{y} \mid x, y})\} \cdot \Pr \left\{ E(P_{\hat{y} \mid x, y}, X_m, Y) < -\frac{\ln P(Y \mid X_m)}{n} + D \right\} \lesssim \mathcal{E}_m \]
\[
\begin{align*}
\text{(b)} & \quad \max_{P_{x,y}} \Pr \left\{ \left( X_m, Y \right) \in T(P_{x,y}) \right\} \cdot \Pr \left\{ E(P_{x|y}^{x,y}, X_m, Y) \leq -\mathbb{E} \left[ \ln P(Y | X) \right] + D \mid P_{x,y} \right\} \\
\geq & \quad \max_{P_{x,y,k}} \Pr \left\{ \left( X_m, Y \right) \in T(P_{x,y}) \right\} \cdot \Pr \left\{ E(P_{x|y}^{x,y}, X_m, Y) \leq g(P_{x,y,k}), \quad g(P_{x,y,k}) \leq -\mathbb{E} \left[ \ln P(Y | X) \right] + D \mid P_{x,y} \right\} \\
= & \quad \max_{P_{x,y,k} : f(P_{x,y,k}) \leq R - 2\epsilon_1} \Pr \left\{ \left( X_m, Y \right) \in T(P_{x,y}) \right\} \times \\
& \quad \left[ 1 - \Pr \left\{ E(P_{x|y}^{x,y}, X_m, Y) > g(P_{x,y,k}) \mid P_{x,y} \right\} \right] \times \\
& \quad \mathbb{I} \left\{ g(P_{x,y,k}) < -\mathbb{E} \left[ \ln P(Y | X) \right] + D \right\} (P_{x,y,k}).
\end{align*}
\]

Explanation of steps:
(a) follows by the definition of the error event \( \mathcal{E}_m \) (49) and the lower bound on Forney’s sum (78);
(b) uses notation (53);
(c) holds for any functions \( f(P_{x,y,k}) \) and \( g(P_{x,y,k}) \), because

\[
\left\{ E(P_{x|y}^{x,y}, X_m, Y) \leq g(P_{x,y,k}), \quad g(P_{x,y,k}) \leq -\mathbb{E} \left[ \ln P(Y | X) \right] + D \right\} \Rightarrow \\
\left\{ E(P_{x|y}^{x,y}, X_m, Y) < -\mathbb{E} \left[ \ln P(Y | X) \right] + D \right\}.
\]

We use also another version of (c), written with functions \( f(P_{x,y,k}) \) and \( h(P_{x,y,k}) \) as

\[
\Pr \left\{ \mathcal{E}_m \right\} \geq \\
\max_{P_{x,y,k} : f(P_{x,y,k}) \geq R} \Pr \left\{ \left( X_m, Y \right) \in T(P_{x,y}) \right\} \cdot \Pr \left\{ E(P_{x|y}^{x,y}, X_m, Y) \leq h(P_{x,y,k}), \quad h(P_{x,y,k}) \leq -\mathbb{E} \left[ \ln P(Y | X) \right] + D \mid P_{x,y} \right\} = \\
\max_{P_{x,y,k} : f(P_{x,y,k}) \geq R} \Pr \left\{ \left( X_m, Y \right) \in T(P_{x,y}) \right\} \cdot \Pr \left\{ E(P_{x|y}^{x,y}, X_m, Y) \leq h(P_{x,y,k}) \mid P_{x,y} \right\} \times \\
\mathbb{I} \left\{ h(P_{x,y,k}) < -\mathbb{E} \left[ \ln P(Y | X) \right] + D \right\} (P_{x,y,k}).
\]

The lower bounds (79), (80) were constructed for the use with the following lemma

**Lemma 10:** Let \( Z_i \sim i.i.d \) Bernoulli \( \left( e^{-nI} \right) \), \( i = 1, 2, \ldots, e^{nR} \).

If \( I \leq R - \epsilon \), with \( \epsilon > 0 \), then

\[
\Pr \left\{ \sum_{i=1}^{e^{nR}} Z_i < e^{n(R-I-\epsilon)} \right\} < \frac{e^{-n\epsilon}}{(1 - e^{-n\epsilon})^2}.
\]
If \( I \geq R \), then

\[
\Pr \left\{ \sum_{i=1}^{e^n R} Z_i \geq 1 \right\} > e^{-n(I-R)} \left( 1 - e^{-nR} \right)^{e^n R} \rightarrow 1/e .
\] (82)

**Proof:** For \( I \leq R - \epsilon \)

\[
\Pr \left\{ \sum_{i=1}^{e^n R} Z_i < e^{n(R-I-\epsilon)} \right\} = \Pr \left\{ \sum_{i=1}^{e^n R} (Z_i - e^{-nI}) < e^{n(R-I-\epsilon)} - e^{n(R-I)} \right\}
\]

\[
= \Pr \left\{ \sum_{i=1}^{e^n R} (Z_i - e^{-nI}) < e^{n(R-I)}(e^{-n\epsilon} - 1) \right\}
\]

\[
\leq \Pr \left\{ \left| \sum_{i=1}^{e^n R} (Z_i - e^{-nI}) \right| > e^{n(R-I)}(1 - e^{-n\epsilon}) \right\}
\]

\[
\leq \frac{1 - e^{-nI}}{(1 - e^{-n\epsilon})^2} \cdot e^{-n(R-I)} < \frac{e^{-n(R-I)}}{(1 - e^{-n\epsilon})^2} \leq \frac{e^{-n\epsilon}}{(1 - e^{-n\epsilon})^2},
\]

where \((*)\) is Chebyshev’s inequality.

For \( I \geq R \)

\[
\Pr \left\{ \sum_{i=1}^{e^n R} Z_i \geq 1 \right\} > \sum_{i=1}^{e^n R} \Pr \left\{ Z_i = 1 \right\} \prod_{j \neq i} \Pr \left\{ Z_j = 0 \right\}
\]

\[
= e^{nR}e^{-nI}(1 - e^{-nI})^{e^n R - 1} > e^{nR}e^{-nI}(1 - e^{-nR})^{e^n R} .
\]

Let \( f(P_{x,y,\hat{x}}) \) and \( I \) be defined as in (57) and (58). If \( f(P_{x,y,\hat{x}}) \leq R - 2\epsilon_1 \), then for \( n \) sufficiently large, as in (54), we obtain by (59): \( I \leq f(P_{x,y,\hat{x}}) + \epsilon_1 \leq R - \epsilon_1 \). For such \( n \), the first part of Lemma 10 holds for the following:

\[
\Pr \left\{ E(P_{\hat{x}|x,y, X_m, Y}) > \frac{-\mathbb{E}_{P_{x,y}} [\ln P(Y|\hat{X})] - R + f(P_{x,y,\hat{x}}) + 2\epsilon_1}{g(P_{x,y,\hat{x}})} \bigg| (X_m, Y) \in T(P_{x,y}) \right\}
\]

\[
= \Pr \left\{ \sum_{m' \neq m} P(Y|X_{m'}) \cdot \mathbb{1}\left\{ X_{m'} \in T(P_{\hat{x}|x,y}, X_m, Y) \right\} (m') < \exp \left\{ n(\mathbb{E}_{P_{x,y}} [\ln P(Y|\hat{X})] + R - f(P_{x,y,\hat{x}}) - 2\epsilon_1) \right\} \bigg| (X_m, Y) \in T(P_{x,y}) \right\}
\]

\[
= \Pr \left\{ \sum_{m' \neq m} \mathbb{1}\left\{ X_{m'} \in T(P_{\hat{x}|x,y}, X_m, Y) \right\} (m') < e^{n(R - f(P_{x,y,\hat{x}}) - 2\epsilon_1)} \bigg| (X_m, Y) \in T(P_{x,y}) \right\}
\]

\[
\leq \Pr \left\{ \sum_{m' = 1}^{e^n R} \mathbb{1}\left\{ X_{m'} \in T(P_{\hat{x}|x,y}, X_m, Y) \right\} (m') < e^{n(R - I - \epsilon_1)} \bigg| (X_m, Y) \in T(P_{x,y}) \right\}
\]

\[
\leq \frac{e^{-n\epsilon_1}}{(1 - e^{-n\epsilon_1})^2} \cdot \frac{|\mathcal{X}|/|\mathcal{X}| |Y| \ln(n + 1)}{n} \leq \epsilon_1 ,
\] (83)
where in

(a) the definition of $E(P_{\hat{x},x,y}, X_m, Y)$ (50) is used, and notation (53) with $P_{\hat{x},y}$;

(b) the codebook size is assumed to be $M = e^{nD} + 1$; the RHS of the inequality is increased by substitution of $I \leq f(P_{x,y}, \hat{x}) + \epsilon_1$;

(c) holds by the definition of $I$ (58) and the first statement of the lemma (81).

If we choose

$$g(P_{x,y}, \hat{x}) \triangleq -\mathbb{E}_{P_{x,y}}[\ln P(Y | \hat{X})] - R + f(P_{x,y}, \hat{x}) + 2\epsilon_1,$$

then with (83) the lower bound (79) becomes

$$\max_{P_{x,y}, \hat{x}} \Pr \left\{ (X_m, Y) \in T(P_{x,y}) \right\} \cdot \left[ 1 - \Pr \left\{ E(P_{\hat{x},x,y}, X_m, Y) > g(P_{x,y}, \hat{x}) \mid P_{x,y} \right\} \right] \times$$

$$\left\{ \begin{array}{ll} g(P_{x,y}, \hat{x}) < -\mathbb{E}_{P_{x,y}}[\ln P(Y | X)] + D \\ f(P_{x,y}, \hat{x}) \leq R - 2\epsilon_1 \end{array} \right\} (P_{x,y}, \hat{x})$$

$$\geq \max_{P_{x,y}, \hat{x}} (n + 1)^{-|X|-|Y|} \cdot \exp \left\{ -nD(P_{x,y} \parallel Q \circ P) \right\} \cdot \left[ 1 - \frac{e^{-n\epsilon_1}}{(1 - e^{-n\epsilon_1})^2} \right] \times$$

$$\left\{ \begin{array}{ll} \mathbb{E}_{P_{x,y}}[\ln P(Y | X)] - \mathbb{E}_{P_{x,y}}[\ln P(Y | \hat{X})] + f(P_{x,y}, \hat{x}) < R + D - 2\epsilon_1 \\ f(P_{x,y}, \hat{x}) \leq R - 2\epsilon_1 \end{array} \right\} (P_{x,y}, \hat{x})$$

$$\geq \exp \left\{ -n\hat{E}_1^{\text{types}}(R - 2\epsilon_1, D) + \epsilon_2 \right\} \geq \exp \left\{ -n\hat{E}_1(R - 2\epsilon_1 - \epsilon_3, D) + \epsilon_2 + \epsilon_3 \right\}. \quad (85)$$

Explanation of steps:

(a) follows by the lower bound on the probability of the joint type

$$\Pr \left\{ (X_m, Y) \in T(P_{x,y}) \right\} \geq (n + 1)^{-|X|-|Y|} \cdot \exp \left\{ -nD(P_{x,y} \parallel Q(x) \cdot P(y | x)) \right\}, \quad (86)$$

and (83), (84);

(b) holds for $n$ sufficiently large for a given $\epsilon_2 > 0$, and uses the definition of the minimal exponent similar to (68):

$$\hat{E}_1^{\text{types}}(R, D) \triangleq \min_{P_{x,y}, \hat{x}} \left\{ D(P_{x,y} \parallel Q \circ P) \right\}$$

subject to:

$$\mathbb{E}_{P_{x,y}, \hat{x}} \left[ \ln \frac{P(Y | X)}{P(Y | \hat{X})} \right] + D(P_{x,y}, \hat{x} \mid P_{x,y} \times Q) < R + D,$$

$$D(P_{x,y}, \hat{x} \mid P_{x,y} \times Q) \leq R,$$
and the definition of \( f(P_{x,y,\hat{x}}) \) (57).

(c) Let \( T^* \circ W^* \) denote the joint distribution, achieving \( \tilde{E}_1(R - 2\epsilon_1 - \epsilon_3, D) \), defined by (69), for some \( \epsilon_3 > 0 \). This implies

\[
D(T^* \| Q \circ P) = \tilde{E}_1(R - 2\epsilon_1 - \epsilon_3, D), \tag{87}
\]

\[
\mathbb{E}_{T^* \circ W^*} \left[ \ln \frac{P(Y \mid X)}{P(Y \mid X)} \right] + D(T^* \circ W^* \| T^* \times Q) \leq R + D - 2\epsilon_1 - \epsilon_3,
\]

\[
D(T^* \circ W^* \| T^* \times Q) \leq R - 2\epsilon_1 - \epsilon_3.
\]

Let \( T^*_n \circ W^*_n \) denote a quantized version of the joint distribution \( T^* \circ W^* \) with precision \( \frac{1}{n} \), i.e. a joint type with denominator \( n \). Note, that the divergences, as functions of \( T \circ W \), have bounded derivatives, and also the ratio \( \ln \frac{P_{(y \mid x)}}{P_{(y \mid x)}} \) is bounded. Therefore, for any \( \epsilon_3 > 0 \) there exists \( n \) large enough, such that the quantized distribution \( T^*_n \circ W^*_n \) satisfies

\[
D(T^*_n \| Q \circ P) \leq D(T^* \| Q \circ P) + \epsilon_3, \tag{88}
\]

\[
\mathbb{E}_{T^*_n \circ W^*_n} \left[ \ln \frac{P(Y \mid X)}{P(Y \mid X)} \right] + D(T^*_n \circ W^*_n \| T^*_n \times Q) < R + D - 2\epsilon_1,
\]

\[
D(T^*_n \circ W^*_n \| T^*_n \times Q) \leq R - 2\epsilon_1.
\]

It follows from the last two inequalities that for \( n \) sufficiently large

\[
D(T^*_n \| Q \circ P) \geq \tilde{E}^\text{types}_1(R - 2\epsilon_1, D). \tag{89}
\]

The relations (89), (88), (87) together give

\[
\tilde{E}_1(R - 2\epsilon_1 - \epsilon_3, D) + \epsilon_3 \geq \tilde{E}^\text{types}_1(R - 2\epsilon_1, D).
\]

This explains (c).

Now we return to the bound (80). If \( f(P_{x,y,\hat{x}}) \geq R \), then also \( I \geq R \) by (59), and the second part of Lemma 10 holds for the following:

\[
\Pr \left\{ E(P_{x,y,\hat{x}} | X_m, Y) \leq -\mathbb{E}_{P_{x,y}} [\ln P(Y \mid \hat{X})] \right\} \quad (X_m, Y) \in T(P_{x,y}) \]

\[
\overset{(a)}{=} \Pr \left\{ \sum_{m' \neq m} P(Y | X_{m'}) \cdot \mathbb{1}\{X_{m'} \in T(P_{x,y}, X_m, Y)\} \right\} \geq \exp \left\{ n \mathbb{E}_{P_{x,y}} [\ln P(Y \mid \hat{X})] \right\} \quad (X_m, Y) \in T(P_{x,y}) \]

\[
= \Pr \left\{ \sum_{m' \neq m} \mathbb{1}\{X_{m'} \in T(P_{x,y}, X_m, Y)\} \right\} \geq 1 \quad (X_m, Y) \in T(P_{x,y})
\]
\[ (b) \quad e^{-n(I - R)} \cdot \left( 1 - e^{-nR} \right)^{e^{nR}} \]
\[ (c) \quad \geq e^{-n(f(P_{x,y,\hat{x}}) - R + \epsilon_1)} \cdot \left( 1 - e^{-nR} \right)^{e^{nR}}, \quad \frac{|\mathcal{X}| |\mathcal{X}'| |\mathcal{Y}| \ln(n+1)}{n} \leq \epsilon_1, \quad (90) \]

where \((a)\) follows by the definition \((50)\), \((b)\) follows by \((82)\) of the lemma, \((c)\) holds for sufficiently large \(n\), as in \((54)\), because for such \(n\), according to \((59)\), we obtain \(I \leq f(P_{x,y,\hat{x}}) + \epsilon_1\). If we choose

\[ h(P_{x,y,\hat{x}}) \triangleq -\mathbb{E}_{P_{x,y}}[\ln P(Y | \hat{X})], \quad (91) \]

then with \((90)\) the lower bound \((80)\) becomes

\[
\max_{P_{x,y,\hat{x}}} \Pr \{ (X_m, Y) \in T(P_{x,y}) \} \cdot \Pr \left\{ E(P_{x,y} | x', y', X_m, Y) \leq h(P_{x,y,\hat{x}}) \mid P_{x,y} \right\} \times \\
1 \left\{ h(P_{x,y,\hat{x}}) < -\mathbb{E}_{P_{x,y}}[\ln P(Y | X)] + D \right\}^{(P_{x,y,\hat{x}})}
\]

\[
\geq \max_{P_{x,y,\hat{x}}} (n + 1)^{-|\mathcal{X}||\mathcal{Y}|} \cdot \exp \left\{ -nD(P_{x,y} \parallel Q \circ P) \right\} \cdot \exp \left\{ -n \left( f(P_{x,y,\hat{x}}) - R + \epsilon_1 \right) \right\} \times \\
\left( 1 - e^{-nR} \right)^{e^{nR}} \rightarrow 1/e \times \\
1 \left\{ \mathbb{E}_{P_{x,y}}[\ln P(Y | X)] - \mathbb{E}_{P_{x,y}}[\ln P(Y | \hat{X})] < D \right\}^{(P_{x,y,\hat{x}})}
\]

\[
\geq \exp \left\{ -n \left[ E_2^{\text{types}}(R, D) + \epsilon_1 + \epsilon_4 \right] \right\}
\]

\[
\geq \exp \left\{ -n \left[ E_2(R + \epsilon_5, D - \epsilon_5) + \epsilon_1 + \epsilon_4 + 2\epsilon_5 \right] \right\}. \quad (92)
\]

Explanation of steps:

\((a)\) follows by the lower bound on the probability of the joint type \((86)\) and \((90)\), \((91)\);

\((b)\) holds for \(n\) sufficiently large for a given \(\epsilon_4 > 0\), and uses the definition of \(f(P_{x,y,\hat{x}})\) \((57)\) with the definition of the minimal exponent

\[ E_2^{\text{types}}(R, D) \triangleq \min_{P_{x,y,\hat{x}}(x,y,\hat{x})} \left\{ D(P_{x,y} \parallel Q \circ P) + D(P_{x,y,\hat{x}} \parallel P_{x,y} \times Q) - R \right\} \quad (93) \]

subject to:

\[
\mathbb{E}_{P_{x,y,\hat{x}}} \left[ \ln \frac{P(Y | X)}{P(Y | \hat{X})} \right] < D, \\
D(P_{x,y,\hat{x}} \parallel P_{x,y} \times Q) \geq R,
\]
which differs from \((72)\) by the inequality symbols “\(<\)” and “\(\geq\”).

\((c)\) Let \(T^* \circ W^*\) denote the joint distribution, achieving \(E_2(R + \epsilon_5, D - \epsilon_5)\), defined by \((73)\), for some \(\epsilon_5 > 0\). This implies

\[
D(T^* \| Q \circ P) + D(T^* \circ W^* \| T^* \times Q) - R = E_2(R + \epsilon_5, D - \epsilon_5), \tag{94}
\]

\[
\mathbb{E}_{T^* \circ W^*} \left[ \ln \frac{P(Y | X)}{P(Y | X)} \right] \leq D - \epsilon_5,
\]

\[
D(T^* \circ W^* \| T^* \times Q) \geq R + \epsilon_5.
\]

Let \(T^*_n \circ W^*_n\) denote a quantized version of the joint distribution \(T^* \circ W^*\) with precision \(\frac{1}{n}\), i.e. a joint type with denominator \(n\). Since the divergences, as functions of \(T \circ W\), have bounded derivatives, and also the ratio \(\ln \frac{P(y | x)}{P(y | x)}\) is bounded, for any \(\epsilon_5 > 0\) there exists \(n\) large enough, such that the quantized distribution \(T^*_n \circ W^*_n\) satisfies

\[
D(T^*_n \| Q \circ P) + D(T^*_n \circ W^*_n \| T^*_n \times Q) - \epsilon_5 \leq D(T^* \| Q \circ P) + D(T^* \circ W^* \| T^* \times Q), \tag{95}
\]

\[
\mathbb{E}_{T^*_n \circ W^*_n} \left[ \ln \frac{P(Y | X)}{P(Y | X)} \right] < D,
\]

\[
D(T^*_n \circ W^*_n \| T^*_n \times Q) \geq R.
\]

It follows from the last two inequalities that for \(n\) sufficiently large

\[
D(T^*_n \| Q \circ P) + D(T^*_n \circ W^*_n \| T^*_n \times Q) - R \geq E^\text{types}_2(R, D), \tag{96}
\]

where \(E^\text{types}_2(R, D)\) is defined as in \((93)\). The relations \((96), (95), (94)\) give

\[
E_2(R + \epsilon_5, D - \epsilon_5) + 2\epsilon_5 \geq E^\text{types}_2(R, D).
\]

This explains \((c)\).

The lower bounds on the probability \((79), (80)\) are replaced now by \((85), (92)\), resulting in the upper bound on the error exponent:

\[
\lim sup_{n \to \infty} \left\{ -\frac{1}{n} \ln \Pr \{ \mathcal{E}_m \} \right\} \leq \min \left\{ \bar{E}_1(R - 2\epsilon_1 - \epsilon_3, D) + \epsilon_2 + \epsilon_3, \right. \\
\left. E_2(R + \epsilon_5, D - \epsilon_5) + \epsilon_1 + \epsilon_4 + 2\epsilon_5 \right\}.
\]

Since \(\epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4, \epsilon_5\) are arbitrary, they can be replaced with zeros and limits, as follows
\textbf{Theorem 7:}

\[ \limsup_{n \to \infty} \left\{ -\frac{1}{n} \ln \Pr \{ \mathcal{E}_m \} \right\} \leq \lim_{\epsilon \to 0} \min \{ \widetilde{E}_1(R - \epsilon, D), E_2(R + \epsilon, D - \epsilon) \} \]

\[ = \min \left\{ \inf_{T(x, y), W(\hat{x} | x, y):} \left\{ D(T \| Q \circ P) \right\}, \right. \]

\[ \left. \inf_{T(x, y), W(\hat{x} | x, y):} \left\{ D(T \| Q \circ P) + D(T \circ W \| T \times Q) - R \right\} \right\}, \] (98)

where \( d(T \circ W) = \mathbb{E}_{T \circ W} \left[ d((X, Y), \hat{X}) \right] = \mathbb{E}_{T \circ W} \left[ \ln \frac{P(Y | X)}{\hat{P}(Y | X)} \right]. \]

Note, that the only difference of the upper bound of Theorem 7 from the lower bound of Theorem 5 is that here the conditions are strict inequalities.

16. Derivation of the explicit random coding error exponent of Forney’s decoder

In order to compare the bounds (98) and (74), it is convenient to rewrite (97) and (74), and replace the second argument of \( \min \) with another expression. Observe that the first argument in the minimum of (97) can be upper-bounded as

\[ \widetilde{E}_1(R - \epsilon, D) = \min_{T(x, y), W(\hat{x} | x, y):} \left\{ D(T \| Q \circ P) \right\} \]

\[ \leq \min_{T(x, y), W(\hat{x} | x, y):} \left\{ D(T \| Q \circ P) \right\} \]

\[ = \min_{T(x, y), W(\hat{x} | x, y):} \left\{ D(T \| Q \circ P) \right\} \triangleq E_1(R - \epsilon, D - \epsilon). \] (99)

Therefore, the minimum of (97) becomes

\[ \min \{ \widetilde{E}_1(R - \epsilon, D), E_2(R + \epsilon, D - \epsilon) \} \]

\[ = \min \{ \widetilde{E}_1(R - \epsilon, D), E_1(R - \epsilon, D - \epsilon), E_2(R + \epsilon, D - \epsilon) \} \]

\[ = \min \{ \widetilde{E}_1(R - \epsilon, D), \min \{ E_1(R - \epsilon, D - \epsilon), E_2(R + \epsilon, D - \epsilon) \} \}. \] (100)
Similarly (using $\epsilon = 0$), for the minimum in (74) we obtain

$$
\min \{ \widetilde{E}_1(R, D), E_2(R, D) \} = \min \{ \widetilde{E}_1(R, D), \min \{ E_1(R, D), E_2(R, D) \} \}
$$

$$
\triangleq \tilde{E}_2(R, D)
$$

$$
= \min \{ \widetilde{E}_1(R, D), \tilde{E}_2(R, D) \}.
$$

(101)

The next lemma serves to clarify the relationship between the new right argument in (100) and $\tilde{E}_2(R, D)$ defined in (101), as follows:

Lemma 11: If $D > D_{\min} = \min_{(x, y), \hat{x}} d((x, y), \hat{x})$, then for $R > 0$

$$
\lim_{\epsilon \to 0} \min \{ E_1(R - \epsilon, D), E_2(R + \epsilon, D) \} = \tilde{E}_2(R, D).
$$

Proof: Consider the definition of $\tilde{E}_2(R, D) = \min \{ E_1(R, D), E_2(R, D) \}$:

$$
\tilde{E}_2(R, D) = \min \left\{ \begin{array}{l}
\min_{T(x, y), W(\hat{x} | x, y):}
\begin{cases}
D(T \parallel Q \circ P) \\
D(T \circ W) \leq D \\
D(T \circ W \parallel T \times Q) \leq R
\end{cases}
\end{array} \right\},
$$

$$
\leq \min_{T(x, y), W(\hat{x} | x, y):}
\begin{cases}
D(T \parallel Q \circ P) + D(T \circ W \parallel T \times Q) - R
\end{cases}
\right\}.
$$

Let $T^* \circ W^*$ be the joint distribution, achieving $\tilde{E}_2(R, D)$.

If $D(T^* \circ W^* \parallel T^* \times Q) \neq R$, then for sufficiently small $\epsilon > 0$ the minimum is achieved

$$
\min \{ E_1(R - \epsilon, D), E_2(R + \epsilon, D) \} = \min \{ E_1(R, D), E_2(R, D) \} = \tilde{E}_2(R, D),
$$

and the statement of the lemma holds.

If exactly $D(T^* \circ W^* \parallel T^* \times Q) = R > 0$, then, given the condition of the lemma $D > D_{\min} = \min_{(x, y), \hat{x}} d((x, y), \hat{x})$, there exists $W' \neq W^*$, such that

$$
d(T^* \circ W') \leq D.
$$

(102)

Since $D(T \circ W \parallel T \times Q)$ is strictly convex in $W$, there exists $W_\lambda = \lambda W' + (1 - \lambda)W^*$ such that $D(T^* \circ W_\lambda \parallel T^* \times Q) \neq R$ and is arbitrarily close to $R$. Note also that $d(T^* \circ W_\lambda) \leq D$. The convergence in the lemma follows.

Note, that for $D = D_{\min}$ a distinct $W' \neq W^*$, satisfying (102), may not exist.
Since the limit of the right argument in (100) can be taken in two steps as
\[
\lim_{\epsilon \to 0} \min \{ E_1(R - \epsilon, D - \epsilon), \ E_2(R + \epsilon, D - \epsilon) \} = \lim_{\epsilon_2 \to 0} \lim_{\epsilon_1 \to 0} \min \{ E_1(R - \epsilon_1, D - \epsilon_2), \ E_2(R + \epsilon_1, D - \epsilon_2) \},
\]
Lemma 11 implies, that for any\(^{19} D \)
\[
\lim_{\epsilon \to 0} \min \{ E_1(R - \epsilon, D - \epsilon), \ E_2(R + \epsilon, D - \epsilon) \} = \lim_{\epsilon_2 \to 0} \lim_{\epsilon_1 \to 0} \min \{ \tilde{E}_1(R, D - \epsilon), \ \tilde{E}_2(R, D - \epsilon) \}. \quad (103)
\]
Thus, with the help of (100) and (103), the bound (97) can be rewritten as
\[
\lim_{\epsilon \to 0} \min \{ \tilde{E}_1(R - \epsilon, D), \ E_2(R + \epsilon, D - \epsilon) \} = \lim_{\epsilon \to 0} \min \{ \tilde{E}_1(R - \epsilon, D), \ \tilde{E}_2(R, D - \epsilon) \}. \quad (104)
\]
Thus far, we have obtained the two alternative expressions (101) and (104) for the bounds (74) and (97), respectively, replacing the second argument of \(\min\) in each one of the bounds.

In what follows, first we obtain the explicit formula for the expression (101), which is equivalent to the lower bound (74). Then we conclude about the values of \((R, D)\) for which the bounds (101) and (104) may not agree, which may occur only at the points of discontinuity of the bounds as functions of \((R, D)\).

It is convenient to express the new second argument \(\tilde{E}_2(R, D)\) with the help of \(R(T, Q, D)\), defined in (1), which is written here with substitutions (19) and (21):
\[
R(T, Q, D) \triangleq \min_{W(\hat{x} | x, y): \ d(T \circ W) \leq D} D(T \circ W \parallel T \times Q). \quad (105)
\]
Using \(R(T, Q, D)\), we can write the following two identities:
\[
\tilde{E}_2(R, D) = \min \left\{ E_1(R, D), \ E_2(R, D) \right\} = \min \left\{ \begin{array}{l} \min_{T(x, y), W(\hat{x} | x, y): \ d(T \circ W) \leq D} \left\{ D(T \parallel Q \circ P) \right\}, \\ \min_{D(T \circ W \parallel T \times Q) \leq R} \left\{ D(T \parallel Q \circ P) + D(T \circ W \parallel T \times Q) - R \right\} \end{array} \right\} = \end{align*}
\]
\(^{19}\)For \(D \leq D_{\min}\) both sides of (103) are trivially \(+\infty\).
A similar function, a "coupled" version of the error exponent for the decoding error event defined in (46). Consequently, (108) also equals (47). Recall, that (47) is equivalent to (44), and derives from (44) by exactly the same derivation as (18) from (2), with substitutions (19)-(21).

The first argument $\tilde{E}_1(R, D)$ in the minimum (101) can also be expressed alternatively, with the help of a similar function, a "coupled" version of $R(T, Q, D)$, defined as

$$ R^c(T, Q, D) \triangleq \min_{\hat{W}(\hat{x} | x, y) \text{: } d(T \circ W) + D(T \circ W \parallel T \times Q) \leq D} D(T \parallel W \parallel T \times Q). \tag{109} $$

This definition gives

$$ \tilde{E}_1(R, D) = \min_{\hat{T}(x, y), \hat{W}(\hat{x} | x, y) \text{: } d(T \circ W) + D(T \circ W \parallel T \times Q) \leq D + R} \left\{ D(T \parallel Q \circ P) \right\} = \min_{\hat{T}(x, y) \text{: } R(T, Q, D + R) \leq R} \left\{ D(T \parallel Q \circ P) \right\}. \tag{110} $$

An explicit formula for $R^c(T, Q, D)$ is given by

$$ R^c(T, Q, D) = \sup_{\mu \geq 0} \left\{ -\sum_{x, y} T(x, y) \ln \left[ \sum_{\hat{x}} Q(\hat{x}) e^{\frac{\mu}{4} \left[ d((x, y), \hat{x}) - D \right]} \right]^{1+\mu} \right\}. \tag{111} $$

**Proof:**

$$ R^c(T, Q, D) \triangleq \min_{\hat{W}(\hat{x} | x, y) \text{: } D(T \circ W) \parallel T \times Q \leq D} D(T \circ W \parallel T \times Q) $$
\[ \begin{align*}
= & \min_{W(x|y)} \sup_{\mu \geq 0} \left\{ D(T \circ W \parallel T \times Q) + \mu [d(T \circ W) + D(T \circ W \parallel T \times Q) - D] \right\} \\
\overset{(*)}{=} & \sup_{\mu \geq 0} \min_{W(x|y)} \left\{ D(T \circ W \parallel T \times Q) + \mu [d(T \circ W) + D(T \circ W \parallel T \times Q) - D] \right\} \\
= & \sup_{\mu \geq 0} \min_{W(x|y)} \left\{ (1 + \mu) \sum_{x,y,\hat{x}} T(x,y)W(\hat{x}|x,y) \ln \frac{W(\hat{x}|x,y)}{Q(\hat{x})} \\
& \quad + \mu \left[ \sum_{x,y,\hat{x}} T(x,y)W(\hat{x}|x,y)d((x,y), \hat{x}) - D \right] \right\} \\
= & \sup_{\mu \geq 0} \min_{W(x|y)} \left\{ (1 + \mu) \sum_{x,y,\hat{x}} T(x,y)W(\hat{x}|x,y) \ln \frac{W(\hat{x}|x,y)}{Q(\hat{x})} e^{-\frac{\mu}{1+\mu}d((x,y), \hat{x})} - \mu D \right\} \\
= & \sup_{\mu \geq 0} \left\{ - \sum_{x,y} T(x,y) \ln \left[ \sum_{\hat{x}} Q(\hat{x}) e^{-\frac{\mu}{1+\mu}d((x,y), \hat{x})} \right]^{1+\mu} - \mu D \right\} \tag{113}
\end{align*} \]

where the equality \((*)\) holds because \(R^c(T, Q, D)\) is a convex (\(\cup\)) function of \(D\) (checked directly) and (112) is its lower convex envelope, therefore they must coincide. Alternatively, \((*)\) follows by the minimax theorem, since the objective function is convex (\(\cup\)) in \(W(\hat{x}|x,y)\) and concave (linear) in \(\mu\). \hfill \blacksquare

The last expression (113) helps to recognize the following property of \(R^c(T, Q, D)\):

**Lemma 13:** \(R^c(T, Q, D)\) is a convex (\(\cup\)) function of the pair \((T, D)\).

**Proof:** The same as of Lemma 2. \hfill \blacksquare

This, in turn, results in convexity of \(\tilde{E}_1(R, D)\):

**Lemma 14:** \(\tilde{E}_1(R, D)\) is a convex (\(\cup\)) function of \((R, D)\).

**Proof:** Using (110):

\[
\begin{align*}
\lambda \tilde{E}_1(R_1, D_1) + (1 - \lambda) \tilde{E}_1(R_2, D_2)
= & \lambda \cdot \min_{T(x,y)} \left\{ D(T \parallel Q \circ P) \right\} + (1 - \lambda) \cdot \min_{T(x,y)} \left\{ D(T \parallel Q \circ P) \right\} \\
= & \lambda D(T_1^* \parallel Q \circ P) + (1 - \lambda) D(T_2^* \parallel Q \circ P) \\
\geq & D(\lambda T_1^* + (1 - \lambda)T_2^* \parallel Q \circ P) \\
\geq & \min_{T(x,y)} \left\{ D(T \parallel Q \circ P) \right\} \\
\overset{R^c(T, Q, D_1 + R_1) \leq R_1}{\leq} & R^c(T, Q, \lambda(D_1 + R_1) + (1 - \lambda)(D_2 + R_2)) \\
\overset{R^c(T^* \parallel Q \circ P, \lambda(D_1 + R_1) + (1 - \lambda)(D_2 + R_2) \leq R^c(T^* \parallel Q \circ P, \lambda(D_1 + R_1) + (1 - \lambda)(D_2 + R_2))}{\leq}
\end{align*}
\]

20This also includes the case when the set \(\{W(\hat{x}|x,y) : d(T \circ W) + D(T \circ W \parallel T \times Q) \leq D\}\) is empty, then \(R^c(T, Q, D) = +\infty\).
\[
\min_{T(x, y)} \{ D(T \parallel Q \circ P) \} \geq \min_{T(x, y)} \{ D(T \parallel Q \circ P) \}
\]

\[
R^c(T, Q, \lambda(D_1 + R_1) + (1-\lambda)(D_2 + R_2)) \leq \lambda R^c(T_1, Q, D_1 + R_1) + (1-\lambda)R^c(T_2, Q, D_2 + R_2)
\]

\[
\min_{T(x, y)} \{ D(T \parallel Q \circ P) \} \geq \min_{T(x, y)} \{ D(T \parallel Q \circ P) \}
\]

\[
= \tilde{E}_1(\lambda R_1 + (1-\lambda)R_2, \lambda D_1 + (1-\lambda)D_2),
\]

where \(*) follows by Lemma 13.

Lemma 14 helps to prove the following Lagrangian duality:

**Lemma 15:**

\[
\tilde{E}_1(R, D) = \sup_{\alpha \geq 0, \beta \geq 0} \min_{T(x, y), W(\hat{x} | x, y)} \left\{ D(T \parallel Q \circ P) \right\}
\]

\[
= \min_{T(x, y), W(\hat{x} | x, y)} \left\{ D(T \parallel Q \circ P) \right\}
\]

\[
\geq \min_{T(x, y), W(\hat{x} | x, y)} \left\{ D(T \parallel Q \circ P) \right\}
\]

\[
\alpha \geq 0, \beta \geq 0
\]

\[
\geq \alpha \left[ D(T \circ W \parallel T \times Q) + d(T \circ W) - D - R \right]
\]

\[
+ \beta \left[ D(T \circ W \parallel T \times Q) - R \right]
\]

\[
\leq D(T \circ W \parallel T \times Q) \leq D + R
\]

\[
= D(T \circ W \parallel T \times Q) \leq D + R
\]

\[
= D(T \circ W \parallel T \times Q) \leq D + R
\]

\[
+ \alpha \left[ D(T \circ W \parallel T \times Q) + d(T \circ W) - D - R \right]
\]

\[
+ \beta \left[ D(T \circ W \parallel T \times Q) - R \right]
\]

\[
\alpha \geq 0, \beta \geq 0
\]

\[
\geq \min_{T(x, y), W(\hat{x} | x, y)} \left\{ D(T \parallel Q \circ P) \right\}
\]

\[
\geq \alpha \left[ D(T \circ W \parallel T \times Q) + d(T \circ W) - D - R \right]
\]

\[
+ \beta \left[ D(T \circ W \parallel T \times Q) - R \right]
\]

\[
\leq D(T \circ W \parallel T \times Q) \leq D + R
\]

\[
= D(T \circ W \parallel T \times Q) \leq D + R
\]

\[
\geq \min_{T(x, y), W(\hat{x} | x, y)} \left\{ D(T \parallel Q \circ P) \right\}
\]

\[
\geq \alpha \left[ D(T \circ W \parallel T \times Q) + d(T \circ W) - D - R \right]
\]

\[
+ \beta \left[ D(T \circ W \parallel T \times Q) - R \right]
\]

where \(*) follows by Lemma 13.

**Proof A:** This variant of the proof equates a convex function with its lower convex envelope.

Let \( T^* \circ W^* \) achieve the minimum in (115). Then there exist \( R^* \geq 0 \) and \( D^* \), such that

\[
D(T^* \circ W^* \parallel T^* \times Q) + d(T^* \circ W^*) - D^* - R^* = 0,
\]

\[
D(T^* \circ W^* \parallel T^* \times Q) - R^* = 0.
\]
It follows, that the two-dimensional plane (115) touches $\tilde{E}_1(R, D)$. Since (115) is also a lower bound on $\tilde{E}_1(R, D)$, we conclude, that (115) is a supporting plane of the surface $\tilde{E}_1(R, D)$, for each pair $\alpha \geq 0$, $\beta \geq 0$.

Now consider the other possible pairs $(\alpha, \beta)$, with negative $\alpha$ or $\beta$, or both. Observe by the definition, that $\tilde{E}_1(R, D)$ is a nonincreasing function of $R$ when the sum $D + R$ is kept constant. Consequently, there does not exist a supporting plane for $\tilde{E}_1(R, D)$ given by

$$E_0 - \alpha(D + R) - \beta R,$$

with a negative $\beta$. Similarly, $\tilde{E}_1(R, D)$ is a nonincreasing function of the sum $D + R$ when $R$ is kept constant. Consequently, there does not exist a supporting plane for $\tilde{E}_1(R, D)$ given by (116) with a negative $\alpha$.

We conclude, that the supremum of the two-dimensional planes over $\alpha \geq 0$, $\beta \geq 0$ on the RHS of (114) is the lower convex envelope of $\tilde{E}_1(R, D)$. On the other hand, the LHS of (114) is a convex ($\cup$) function of $(R, D)$ by Lemma 14. Therefore they must coincide. □

**Proof B:** This is a proof by repeated application of the minimax theorem for convex-concave functions.

$$\tilde{E}_1(R, D) = \min_{T(x, y), W(\hat{x} | x, y)} \left\{ D(T \parallel Q \circ P) \right\}$$

$$= \min_{T(x, y)} \min_{W(\hat{x} | x, y)} \sup_{\alpha \geq 0, \beta \geq 0} \left\{ D(T \parallel Q \circ P) + \alpha[D(T \circ W \parallel T \times Q) + d(T \circ W) - D - R] + \beta[D(T \circ W \parallel T \times Q) - R] \right\}$$

$$\overset{(a)}{=} \min_{T(x, y)} \sup_{\alpha \geq 0, \beta \geq 0} \min_{W(\hat{x} | x, y)} \left\{ D(T \parallel Q \circ P) + \alpha[D(T \circ W \parallel T \times Q) + d(T \circ W) - D - R] + \beta[D(T \circ W \parallel T \times Q) - R] \right\}$$

$$\overset{(b)}{=} \sup_{\alpha \geq 0, \beta \geq 0} \min_{T(x, y)} \min_{W(\hat{x} | x, y)} \left\{ D(T \parallel Q \circ P) + \alpha[D(T \circ W \parallel T \times Q) + d(T \circ W) - D - R] + \beta[D(T \circ W \parallel T \times Q) - R] \right\},$$

where

(a) follows by the minimax theorem, because the objective function is convex ($\cup$) in $W(\hat{x} | x, y)$ and
concave (linear) in \((\alpha, \beta)\);

(b) follows by the minimax theorem, because the corresponding objective function

\[
\min_{W(x | y)} \left\{ \text{D}(T \| Q \circ P) + \alpha [\text{D}(T \circ W \| T \times Q) + d(T \circ W) - D - R] \\
+ \beta [\text{D}(T \circ W \| T \times Q) - R] \right\}
\]

\[
= \text{D}(T \| Q \circ P) - \alpha D - (\alpha + \beta)R + \min_{W(x | y)} [(\alpha + \beta) \text{D}(T \circ W \| T \times Q) + \alpha d(T \circ W)]
\]

\[
= \sum_{x, y} T(x, y) \ln \frac{T(x, y)}{Q(x)P(y | x)} - \alpha D - (\alpha + \beta)R
\]

\[
+ \min_{W(x | y)} \left[ (\alpha + \beta) \sum_{x, y, x} T(x, y)W(\hat{x} | x, y) \ln \frac{W(\hat{x} | x, y)}{Q(\hat{x})} + \alpha \sum_{x, y, \hat{x}} T(x, y)W(\hat{x} | x, y)d((x, y), \hat{x}) \right]
\]

\[
= \sum_{x, y} T(x, y) \ln \frac{T(x, y)}{Q(x)P(y | x)} - \alpha D - (\alpha + \beta)R
\]

\[
+ \min_{W(x | y)} \left[ (\alpha + \beta) \sum_{x, y, x} T(x, y)W(\hat{x} | x, y) \ln \frac{W(\hat{x} | x, y)}{Q(\hat{x})e^{-\frac{\alpha}{\alpha + \beta}d((x, y), \hat{x})}} \right]
\]

\[
= \sum_{x, y} T(x, y) \ln \frac{T(x, y)}{Q(x)P(y | x)} - \alpha D - (\alpha + \beta)R - (\alpha + \beta) \sum_{x, y} T(x, y) \ln \sum_{\hat{x}} Q(\hat{x})e^{-\frac{\alpha}{\alpha + \beta}d((x, y), \hat{x})}
\]

(117)

is

1) concave (\(\cap\)) in \((\alpha, \beta)\) as a minimum of affine functions of \((\alpha, \beta),\)

2) convex (\(\cup\)) in \(T(x, y). \Box\)

Continuing (114) with (117) gives

\[
\tilde{E}_1(R, D)
\]

\[
= \sup_{\alpha \geq 0, \beta \geq 0} \min_{T(x, y)} \left\{ \sum_{x, y} T(x, y) \ln \frac{T(x, y)}{Q(x)P(y | x)} \left[ \sum_{\hat{x}} Q(\hat{x})e^{-\frac{\alpha}{\alpha + \beta}d((x, y), \hat{x})} \right]^{\alpha + \beta} - \alpha D - (\alpha + \beta)R \right\}
\]

\[
= \sup_{\alpha \geq 0, \beta \geq 0} \left\{ -\ln \sum_{x, y} Q(x)P(y | x) \left[ \sum_{\hat{x}} Q(\hat{x})e^{-\frac{\alpha}{\alpha + \beta}[d((x, y), \hat{x}) - D]} \right]^{\alpha + \beta} - (\alpha + \beta)R \right\}
\]

\[
= \sup_{\rho \geq 0} \sup_{0 \leq s \leq 1} \left\{ -\ln \sum_{x, y} Q(x)P(y | x) \left[ \sum_{\hat{x}} Q(\hat{x})e^{-s[d((x, y), \hat{x}) - D]} \right]^\rho - \rho R \right\},
\]

(118)

where we define \(\rho \triangleq \alpha + \beta\) and \(s \triangleq \frac{\alpha}{\alpha + \beta}. \quad (21)\)

\(21\)Note also, that the parameter in the explicit formula for \(R^\epsilon(T, Q, D)\) (111) is related to \((\alpha, \beta)\) as \(\mu = \frac{\alpha}{\beta}.\)
With (118) for \(\tilde{E}_1(R, D)\) and (47) for \(\tilde{E}_2(R, D)\), the lower bound (74) now acquires its final form:

\[
\min \left\{ \tilde{E}_1(R, D), \tilde{E}_2(R, D) \right\} = 
\min \left\{ \sup_{\rho \geq 0} \sup_{0 \leq s \leq 1} \left\{ -\ln \sum_{x,y} Q(x)P(y|x) \left[ \sum_{\hat{x}} Q(\hat{x})e^{-s[d((x,y), \hat{x}) - D]} \right]^\rho - \rho R \right\}, \right. \\
\left. \sup_{0 \leq \rho \leq 1} \sup_{s \geq 0} \left\{ -\ln \sum_{x,y} Q(x)P(y|x) \left[ \sum_{\hat{x}} Q(\hat{x})e^{-s[d((x,y), \hat{x}) - D]} \right]^\rho - \rho R \right\} \right\} (119)
\]

\[\triangleq E_{\text{tradeoff decoder}}(Q, R, D),\]

where \(d((x, y), \hat{x}) = \ln \frac{P(y|x)}{P(y|\hat{x})}\).

As can be seen, both \(\tilde{E}_1(R, D)\) and \(\tilde{E}_2(R, D)\) are suprema of affine functions of \((R, D)\) and, as such, are convex (\union) in \((R, D)\). Therefore, both \(\tilde{E}_1(R, D)\) and \(\tilde{E}_2(R, D)\) are, basically, continuous. With the exception of boundary points where they switch to +\(\infty\). In this respect, as can be verified from the expressions above, the functions are lower semi-continuous, i.e. the convex sets of \((R, D)\), on which the functions are finite, are closed sets.

Specifically, the second argument \(\tilde{E}_2(R, D)\) becomes +\(\infty\) for

\[
D < D_{\min} = \min_{(x, y), \hat{x}} d((x, y), \hat{x}) = \min_{x, y, \hat{x}} \ln \frac{P(y|x)}{P(y|\hat{x})}. \quad (120)
\]

The first argument \(\tilde{E}_1(R, D)\), as can be seen from (110), equals +\(\infty\) for

\[
R < \min_{T(x, y)} R^c(T, Q, D + R). \quad (121)
\]

Note, that \(\min_{T(x, y)} R^c(T, Q, D + R)\) itself is a nonincreasing right-continuous function of \(R\) (in fact it is convex (\union) and therefore lower semi-continuous). In particular, given a sufficiently small \(D\), like \(D < D_{\min}\), the function \(f(R) = \min_{T(x, y)} R^c(T, Q, D + R)\) equals +\(\infty\) for small \(R\), then jumps from +\(\infty\) to a finite value and decreases to 0, with increase of \(R\). In any case, we can define

\[
R_{\min}(D) \triangleq \min_{\min_{T(x, y)} R^c(T, Q, D + R) \leq R} \{R\} \geq \min_{T(x, y)} R^c(T, Q, D + R_{\min}(D)). \quad (122)
\]

Thus, \(\tilde{E}_1(R, D)\) becomes +\(\infty\) for

\[
R < R_{\min}(D). \quad (123)
\]

We conclude, that the only possible points, where the expressions (101) and (104) may not be equal, are the points with \(D = D_{\min}\), and the points \((R, D) = (R_{\min}(D), D)\).
Theorem 8:
\[
\lim_{n \to \infty} \left\{ -\frac{1}{n} \ln \Pr \{ \mathcal{E}_m \} \right\} = \min \{ \tilde{E}_1(R, D), \tilde{E}_2(R, D) \}
\]
for all \((R, D)\), with the possible exception of some points \((R, D_{\min})\) and \((R_{\min}(D), D)\), where still
\[
\liminf_{n \to \infty} \left\{ -\frac{1}{n} \ln \Pr \{ \mathcal{E}_m \} \right\} \geq \min \{ \tilde{E}_1(R, D), \tilde{E}_2(R, D) \}
\]
\[
\limsup_{n \to \infty} \left\{ -\frac{1}{n} \ln \Pr \{ \mathcal{E}_m \} \right\} \leq \lim_{\epsilon \to 0} \min \{ \tilde{E}_1(R - \epsilon, D), \tilde{E}_2(R, D - \epsilon) \},
\]
with \(\tilde{E}_1(R, D)\) and \(\tilde{E}_2(R, D)\) given explicitly by (119).

17. Comparison of decoding error exponents for arbitrary \(D\)

For convenience, let us define
\[
E_0(s, \rho, Q, D) \triangleq -\ln \sum_{x, y} Q(x) P(y | x) \left[ \sum_{\hat{x}} Q(\hat{x}) \left[ \frac{P(y | x)}{P(y | \hat{x})} e^{-D} \right]^{-s} \right]^\rho.
\]

We start with the exponent (47), which is the highest, and corresponds to the “source duality” decoding error event defined in (46):
\[
E_e(Q, R, D) = \sup_{0 \leq \rho \leq 1} \sup_{s \geq 0} \{ E_0(s, \rho, Q, D) - \rho R \}
\]
\[
= \min \left\{ \sup_{\rho \geq 0} \sup_{s \geq 0} \{ E_0(s, \rho, Q, D) - \rho R \} \right\}
\]
\[
\geq \min \left\{ \sup_{\rho \geq 0} \sup_{0 \leq s \leq 1} \{ E_0(s, \rho, Q, D) - \rho R \} \right\} = E_{\text{tradeoff decoder}}(Q, R, D)
\]
\[
\geq \min \left\{ \sup_{0 \leq \rho \leq 1} \sup_{0 \leq s \leq 1} \{ E_0(s, \rho, Q, D) - \rho R \} \right\} = E_{\text{bound}}(Q, R, D).
\]

Thus we obtain
\[
E_e(Q, R, D) \geq E_{\text{tradeoff decoder}}(Q, R, D) \geq E_{\text{bound}}(Q, R, D),
\]
where both $E_{\text{tradeoff decoder}}(Q, R, D)$ and $E_{\text{bound}}(Q, R, D)$ denote lower bounds on the random coding error exponent of Forney’s decoder (49). $E_{\text{tradeoff decoder}}(Q, R, D)$ is our tight bound given by Theorem 8, and $E_{\text{bound}}(Q, R, D)$ appears in [2, eq. (24)] (subject to additional maximization over $Q$).

**Lemma 16:** For $D \geq 0$

\[
E_e(Q, R, D) = E_{\text{tradeoff decoder}}(Q, R, D) = E_{\text{bound}}(Q, R, D).
\]

**Proof:**

\[
E_0(s = \frac{1}{1+\rho}, \rho, Q, D) = -\ln \sum_{x,y} Q(x) P(y|x) \left[ \sum_{\hat{x}} Q(\hat{x}) \left( \frac{P(y|x)}{P(y|\hat{x})} \right)^{-\frac{1}{1+\rho}} \right]^\rho - \frac{1}{1+\rho} \cdot \rho D
\]

\[
\geq -\ln \sum_{x,y} Q(x) P(y|x) \left[ \sum_{\hat{x}} Q(\hat{x}) \left( \frac{P(y|x)}{P(y|\hat{x})} \right)^{-s} \right]^\rho - s \cdot \rho D
\]

\[
= E_0(s, \rho, Q, D),
\]

where (*) holds by (23) for $s \geq \frac{1}{1+\rho}$ and $D \geq 0$. We conclude, that

\[
\sup_{0 \leq \rho \leq 1} \sup_{s \geq 0} \left\{ E_0(s, \rho, Q, D) - \rho R \right\} = \sup_{0 \leq \rho \leq 1} \sup_{0 \leq s \leq 1} \left\{ E_0(s, \rho, Q, D) - \rho R \right\}.
\]

18. **Maximization over $Q$ of the random coding error exponent of Forney’s decoder**

When we try to maximize the random coding exponent, given by Theorem 8, over $Q$, straightforward maximization, at first glance, is hampered by the special points where the true exponent is unknown:

\[
(R, D_{\text{min}}(Q)), \quad D_{\text{min}}(Q) = \min_y \min_{x: Q(x) > 0} \min_{\hat{x}: Q(\hat{x}) > 0} \ln \frac{P(y|x)}{P(y|\hat{x})},
\]

\[
(R_{\text{min}}(Q, D), D), \quad R_{\text{min}}(Q, D) = \min_{R(T, Q, D + R) \leq R} \left\{ R \right\}.
\]

The special points of the first kind $(R, D_{\text{min}}(Q))$ can be avoided by simply maximizing for $D \neq D_{\text{min}}(Q)$, leaving the finite set of lines \{ $D = D_{\text{min}}(Q)$ \} (whose size is bounded by the number of all possible subsets of the channel input alphabet $\mathcal{X}$) unaddressed. The second kind of the special points $(R_{\text{min}}(Q, D), D)$ cannot be avoided that simple, but, better still, can be almost completely circumvented, as shown by the next lemmas.

**Lemma 17:**

\[
\min_{T(x,y)} R^c(T, Q, D + R) \leq f^*(R, D) \triangleq \left\{ \begin{array}{ll} 0, & R \geq -D, \\ +\infty, & R < -D. \end{array} \right. \tag{124}
\]
Proof:

$$\min_{T(x, y)} R^c(T, Q, D + R)$$

$$= \min_{T(x, y)} \sup_{\mu \geq 0} \left\{ - \sum_{x, y} T(x, y) \ln \left[ \sum_{\hat{x}} Q(\hat{x}) e^{-\frac{\mu}{1+\mu} \left[ d((x, y), \hat{x}) - D - R \right]} \right]^{1+\mu} \right\}$$

$$\leq \min_{T(x, y)} \sup_{\mu \geq 0} \left\{ - \sum_{x, y} T(x, y) \ln \min_{Q(\hat{x})} \left[ \sum_{\hat{x}} Q(\hat{x}) e^{-\frac{\mu}{1+\mu} \left[ d((x, y), \hat{x}) - D - R \right]} \right]^{1+\mu} \right\}$$

$$= \min_{T(x, y)} \sup_{\mu \geq 0} \left\{ - \sum_{x, y} T(x, y) \ln \left[ e^{-\frac{\mu}{1+\mu} \left[ \max_{\hat{x}} d((x, y), \hat{x}) - D - R \right]} \right]^{1+\mu} \right\}$$

$$= \min_{T(x, y)} \sup_{\mu \geq 0} \left\{ \mu \sum_{x, y} T(x, y) \left[ \max_{\hat{x}} d((x, y), \hat{x}) - D - R \right] \right\}$$

$$= \sup_{\mu \geq 0} \min_{T(x, y)} \left\{ \mu \left[ \min_{x, y} \max_{\hat{x}} d((x, y), \hat{x}) - D - R \right] \right\}$$

$$= \sup_{\mu \geq 0} \{ \mu [-D - R] \} = \left\{ \begin{array}{ll} 0, & R \geq -D, \\ +\infty, & R < -D, \end{array} \right.$$ 

where (a) follows by Lemma 12, (b) follows by the minimax theorem for the objective function convex (linear) in $T(x, y)$ and concave (linear) in $\mu$, and (c) follows by the property of $d((x, y), \hat{x})$, similar to (34):

$$\min_{x, y} \max_{\hat{x}} d((x, y), \hat{x}) = \min_{x} \min_{y} \max_{\hat{x}} \ln \frac{P(y|x)}{P(y|\hat{x})}$$

$$\geq \min_{x, y} \ln \frac{P(y|x)}{P(y|\hat{x})} = 0 = \min_{y} \max_{\hat{x}} \ln \frac{P(y|\hat{x})}{P(y|\hat{x})}$$

$$\geq \min_{x} \min_{y} \max_{\hat{x}} \ln \frac{P(y|x)}{P(y|\hat{x})},$$

$$\min_{x, y} \max_{\hat{x}} d((x, y), \hat{x}) = 0. \quad (125)$$

**Lemma 18:**

$$\max_{Q(\hat{x})} R_{\min}(Q, D) = \max \{0, -D\}. \quad (126)$$

**Proof:**

$$R_{\min}(Q, D) = \min_{T(x, y)} R^c(T, Q, D + R) \leq \min_{f^*(R, D) \leq R} \left\{ R \right\} \leq \min_{f^*(R, D) \leq R} \{ R \} = \max \{0, -D\},$$
where \((\ast)\) follows by Lemma 17. This upper bound is achieved by any degenerate distribution

\[
Q(\hat{x}) = \begin{cases} 
1, & \hat{x} = a, \\
0, & \hat{x} \neq a.
\end{cases}
\]  

(127)

Substitution of such \(Q\) in the explicit formula (119) gives

\[
\tilde{E}_1(Q, R, D) = \sup_{\rho \geq 0} \sup_{0 \leq s \leq 1} \{-\rho s D - \rho R\} = \begin{cases} 
0, & R \geq \max \{0, -D\}, \\
+\infty, & \text{else}.
\end{cases}
\]  

(128)

The conclusion of Lemma 18 is that \(\tilde{E}_1(Q, R, D)\) is finite, and hence continuous in \(R\), for \(R > \max \{0, -D\}\).

Observe also, that substitution of the degenerate distribution (127) in (119) gives

\[
\tilde{E}_2(Q, R, D) = \sup_{0 \leq \rho \leq 1} \sup_{s \geq 0} \{-\rho s D - \rho R\} = \begin{cases} 
0, & D \geq 0, \\
+\infty, & D < 0.
\end{cases}
\]  

(129)

It follows from (128) and (129), that, in the case of negative \(D\) and \(R < -D\), the maximum of the random coding exponent over \(Q\) is

\[
\sup_{Q(x)} \min \{\tilde{E}_1(Q, R, D), \tilde{E}_2(Q, R, D)\} = +\infty, \quad 0 < R < -D.
\]  

(130)

Therefore, we can formulate the following

**Theorem 9:**

\[
\sup_{Q(x)} \lim_{n \to \infty} \left\{ -\frac{1}{n} \ln \Pr \{\mathcal{E}_m\} \right\} = \sup_{Q(x)} \min \{\tilde{E}_1(Q, R, D), \tilde{E}_2(Q, R, D)\},
\]

for all \((R, D)\), with the possible exception of points with \(R = -D\), and points with \(0 > D \in \{D_{\text{min}}(Q)\}_Q\) (for \(R > -D\)), where still

\[
\sup_{Q(x)} \liminf_{n \to \infty} \left\{ -\frac{1}{n} \ln \Pr \{\mathcal{E}_m\} \right\} \geq \sup_{Q(x)} \min \{\tilde{E}_1(Q, R, D), \tilde{E}_2(Q, R, D)\}
\]

\[
\sup_{Q(x)} \limsup_{n \to \infty} \left\{ -\frac{1}{n} \ln \Pr \{\mathcal{E}_m\} \right\} \leq \sup_{Q(x)} \lim_{\epsilon \to 0} \min \{\tilde{E}_1(Q, R - \epsilon, D), \tilde{E}_2(Q, R, D - \epsilon)\},
\]

with \(\tilde{E}_1(Q, R, D)\) and \(\tilde{E}_2(Q, R, D)\) given explicitly by (119).

Now, using Lemma 16 with Theorem 9, we obtain, that the original Forney’s random coding exponent is tight at least for \(D \geq 0\):

**Corollary 1:** For \(D \geq 0\)

\[
\sup_{Q(x)} \lim_{n \to \infty} \left\{ -\frac{1}{n} \ln \Pr \{\mathcal{E}_m\} \right\} = \sup_{Q(x)} E_{\text{bound}}(Q, R, D).
\]
19. **Derivation of the encoding success exponent**

This will lead to both (2) and (44).

Upper bound on the probability of successful encoding:

$$P_s \leq \sum_{P_{x \hat{x}}} \Pr \{ X \in T(P_x) \} \cdot \Pr \{ \exists m : \hat{X}_m \in T(P_{x|X}, X) \mid P_x \} \cdot \mathbb{I}\{d(P_{x \hat{x}}) \leq D\} \mathbb{I}\{d(P_x) \leq D\} \mathbb{I}\{d(P_{x \hat{x}}) \leq D\} \cdot \mathbb{I}\{d(P_{x \hat{x}}) \leq D\} \mathbb{I}\{d(P_{x \hat{x}}) \leq D\}$$

$$\leq \sum_{D(P_{x \hat{x}} \parallel P_x \times Q) \leq R} \Pr \{ X \in T(P_x) \} \cdot \Pr \{ \exists m : \hat{X}_m \in T(P_{x|X}, X) \mid P_x \} \times \mathbb{I}\{d(P_{x \hat{x}}) \leq D\} \mathbb{I}\{d(P_x) \leq D\} \mathbb{I}\{d(P_{x \hat{x}}) \leq D\} \mathbb{I}\{d(P_{x \hat{x}}) \leq D\} \mathbb{I}\{d(P_{x \hat{x}}) \leq D\} \cdot \mathbb{I}\{d(P_{x \hat{x}}) \leq D\} \mathbb{I}\{d(P_{x \hat{x}}) \leq D\} \mathbb{I}\{d(P_{x \hat{x}}) \leq D\} \mathbb{I}\{d(P_{x \hat{x}}) \leq D\} \mathbb{I}\{d(P_{x \hat{x}}) \leq D\} \mathbb{I}\{d(P_{x \hat{x}}) \leq D\}$$

$$= \sum_{P_{x \hat{x}}} \exp \{-nD(P_x \parallel P)\} \cdot \mathbb{I}\{d(P_{x \hat{x}}) \leq D\} \mathbb{I}\{d(P_x) \leq D\} \mathbb{I}\{D(P_{x \hat{x}} \parallel P_x \times Q) \leq R\}$$

$$+ \sum_{P_{x \hat{x}}} \exp \{-n\left[D(P_x \parallel P) + D(P_{x \hat{x}} \parallel P_x \times Q) - R\right]\} \cdot \mathbb{I}\{d(P_{x \hat{x}}) \leq D\} \mathbb{I}\{D(P_{x \hat{x}} \parallel P_x \times Q) \geq R\}$$

$$\leq \sum_{P_{x \hat{x}}} \exp \{-nE_1(R, D)\} + \sum_{P_{x \hat{x}}} \exp \{-nE_2(R, D)\}$$

$$\leq 2(n + 1)^{|X| + |\hat{X}|} \cdot \exp \{-n \min \{E_1(R, D), E_2(R, D)\}\},$$

where (a) uses the bound analogous to (67)

$$\Pr \{ X \in T(P_x) \} \leq \exp \{-nD(P_x \parallel P(x))\}, \quad (131)$$

and the union bound analogous to (71)

$$\Pr \{ \exists m : \hat{X}_m \in T(P_{x|X}, X) \mid P_x \} \leq \exp \{-n \left[D(P_{x \hat{x}}(x, \hat{x}) \parallel P(x) \cdot Q(\hat{x})) - R\right]\}.$$
(b) uses the definitions
\[
E_1(R, D) \triangleq \min_{T(x), W(\hat{x} | x): d(T \circ W) \leq D, D(T \circ W || T \times Q) \leq R} \{ D(T \parallel P) \},
\] (132)
\[
E_2(R, D) \triangleq \min_{T(x), W(\hat{x} | x): d(T \circ W) \leq D, D(T \circ W || T \times Q) \geq R} \{ D(T \parallel P) + D(T \circ W || T \times Q) - R \},
\] (133)

analogous to (99) and (73). Thus, we obtain the lower bound on the encoding success exponent:

**Theorem 10:**
\[
\liminf_{n \to \infty} \left\{ -\frac{1}{n} \ln P_s \right\} \geq \min \{ E_1(R, D), E_2(R, D) \}.
\] (134)

Next, we construct two alternative lower bounds on the probability of successful encoding:

\[
P_s \geq \max_{P_{x, \hat{x}}} \Pr \{ X \in T(P_x) \} \cdot \Pr \left\{ \exists m: \hat{X}_m \in T(P_{\hat{x}|x}, X) \mid P_x \right\} \times
\]
\[
\left\{ \frac{1}{D(P_{x, \hat{x}} \parallel P_x \times Q) \leq R - \epsilon_1} \right\} (P_{x, \hat{x}})
\]
\[
= \max_{P_{x, \hat{x}}} \Pr \{ X \in T(P_x) \} \cdot \Pr \left\{ \sum_m \mathbb{1}_{\hat{X}_m \in T(P_{\hat{x}|x}, X)} (m) \geq 1 \mid P_x \right\} \times
\]
\[
\left\{ \frac{1}{D(P_{x, \hat{x}} \parallel P_x \times Q) \leq R - \epsilon_1} \right\} (P_{x, \hat{x}})
\]
\[
\geq \max_{P_{x, \hat{x}}} \Pr \{ X \in T(P_x) \} \cdot \Pr \left\{ \sum_{i=1}^{e^{nR}} Z_i \geq 1 \mid Z_i \sim \text{Ber}(e^{-nR}) \right\} \cdot \frac{1}{D(P_{x, \hat{x}} \parallel P_x \times Q) \leq R - \epsilon_1} (P_{x, \hat{x}})
\]
\[
\geq \max_{P_{x, \hat{x}}} (n + 1)^{-|X|} \cdot \exp \left\{ -n D(P_x \parallel P) \right\} \cdot \left( 1 - e^{-nR} \right)^{e^{nR}} \times
\]
\[
\left\{ \frac{1}{D(P_{x, \hat{x}} \parallel P_x \times Q) \leq R - \epsilon_1} \right\} (P_{x, \hat{x}})
\]
\[
\geq \exp \left\{ -n \left[ E_1^{\text{types}}(R - \epsilon_1, D) + \epsilon_2 \right] \right\}
\]
\[
\geq \exp \left\{ -n \left[ E_1(R - \epsilon_1 - \epsilon_3, D - \epsilon_3) + \epsilon_2 + \epsilon_3 \right] \right\}.
\] (135)

Explanation of steps:

(a) holds for sufficiently large $n$, when
\[
\Pr \{ \hat{X}_m \in T(P_{\hat{x}|x}, X) \mid X \in T(P_x) \} \geq \exp \left\{ -n \left[ D(P_{x, \hat{x}} \parallel P_x \times Q) + \epsilon_1 \right] \right\} \geq \exp(-nR),
\]
for
\[
Z_i \sim \text{i.i.d Bernoulli} \left( \exp \left\{ -nR \right\} \right).
\]

(b) uses a lower bound on the probability of a type, and the second part of Lemma 10 with \( I = R \).

(c) holds for sufficiently large \( n \), given \( \epsilon_2 > 0 \), with the exponent \( E_1^\text{types} (R - \epsilon_1, D) \) defined as in (132) with types in place of \( T \circ W \).

(d) Analogous to the steps in (85). Let \( T^* \circ W^* \) denote the joint distribution, achieving
\[
E_1 (R - \epsilon_1 - \epsilon_3, D - \epsilon_3),
\]
defined by (132), for some \( \epsilon_3 > 0 \). This implies
\[
D (T^* \parallel P) = E_1 (R - \epsilon_1 - \epsilon_3, D - \epsilon_3),
\]
\[
d(T^* \circ W^*) \leq D - \epsilon_3,
\]
\[
D (T^* \circ W^* \parallel T^* \times Q) \leq R - \epsilon_1 - \epsilon_3.
\]

Let \( T^*_n \circ W^*_n \) denote a quantized version of the joint distribution \( T^* \circ W^* \) with precision \( \frac{1}{n} \), i.e. a joint type with denominator \( n \). Note, that the divergences, as functions of \( T \circ W \), have bounded derivatives, and also the distortion measure \( d(x, \hat{x}) \) is bounded. Therefore, for any \( \epsilon_3 > 0 \) there exists \( n \) large enough, such that the quantized distribution \( T^*_n \circ W^*_n \) satisfies
\[
D (T^*_n \parallel P) \leq D (T^* \parallel P) + \epsilon_3,
\]
\[
d(T^*_n \circ W^*_n) \leq D,
\]
\[
D (T^*_n \circ W^*_n \parallel T^*_n \times Q) \leq R - \epsilon_1.
\]

It follows from the last two inequalities that for \( n \) sufficiently large
\[
D (T^*_n \parallel P) \geq E_1^\text{types} (R - \epsilon_1, D).
\]

The relations (138), (137), (136) result in
\[
E_1 (R - \epsilon_1 - \epsilon_3, D - \epsilon_3) + \epsilon_3 \geq E_1^\text{types} (R - \epsilon_1, D).
\]

This explains (d).
The second bound:

\[ P_s \geq \max_{P_{x,\hat{x}}} \Pr \{ X \in T(P_x) \} \cdot \Pr \left\{ \exists m : \hat{X}_m \in T(P_{\hat{x}|x}, X) \mid P_x \right\} \times \]

\[
\begin{cases} 
1 & \frac{d(P_{x,\hat{x}})}{D(P_{x,\hat{x}})} \leq D \\
D(P_{x,\hat{x}}) \leq P_x \times Q \geq R 
\end{cases}
\]

\[ = \max_{P_{x,\hat{x}}} \Pr \{ X \in T(P_x) \} \cdot \Pr \left\{ \sum_m \hat{X}_m \in T(P_{\hat{x}|x}, X) \mid P_x \right\} \times \]

\[
\begin{cases} 
1 & \frac{d(P_{x,\hat{x}})}{D(P_{x,\hat{x}})} \leq D \\
D(P_{x,\hat{x}}) \leq P_x \times Q \geq R 
\end{cases}
\]

\[ \geq (a) \max_{P_{x,\hat{x}}} \Pr \{ X \in T(P_x) \} \cdot \Pr \left\{ \sum_{i=1}^{e^{nR}} Z_i \geq 1 \right\} \cdot \Pr \left\{ \exists m : \hat{X}_m \in T(P_{\hat{x}|x}, X) \mid P_x \right\} \times \]

\[
\begin{cases} 
1 & \frac{d(P_{x,\hat{x}})}{D(P_{x,\hat{x}})} \leq D \\
D(P_{x,\hat{x}}) \leq P_x \times Q \geq R 
\end{cases}
\]

\[ \geq (b) \max_{P_{x,\hat{x}}} \left( \frac{1}{n+1} \right)^{-|X|} \cdot \exp \left\{ -nD(P_x \parallel P) \right\} \cdot \exp \left\{ -n \left[ D(P_{x,\hat{x}} \parallel P_x \times Q) - R + \epsilon_1 \right] \right\} \times \]

\[
\left( 1 - e^{-nR} \right)^{e^{nR}} \cdot \frac{1}{n} \cdot \Pr \{ X \in T(P_x) \} \geq \frac{e^{nR}}{n} \cdot \exp \left\{ -n \left[ E_2^\text{types}(R, D) + \epsilon_1 + \epsilon_4 \right] \right\} \]

\[ \geq \exp \left\{ -n \left[ E_2(R + \epsilon_5, D - \epsilon_5) + \epsilon_1 + \epsilon_4 + 2\epsilon_5 \right] \right\}. \tag{139} \]

Explanation of steps:

(a) holds for sufficiently large \( n \), when

\[ \Pr \{ X \in T(P_x) \} \geq \exp \left\{ -n \left[ D(P_{x,\hat{x}} \parallel P_x \times Q) + \epsilon_1 \right] \right\}, \]

for

\[ Z_i \sim \text{i.i.d Bernoulli} \left( \exp \left\{ -n \left[ D(P_{x,\hat{x}} \parallel P_x \times Q) + \epsilon_1 \right] \right\} \right). \]

(b) uses the lower bound on the probability of a type, and the second part of Lemma 10.

(c) holds for sufficiently large \( n \), given \( \epsilon_4 > 0 \), with the exponent \( E_2^\text{types}(R, D) \) defined as in (133) with types in place of \( T \circ W \).

(d) parallels the analogous step in (92) with \( E_2(\cdot, \cdot) \) defined in (133).

The two lower bounds on the probability (135) and (139) result in the upper bound on the exponent:

\[ \lim_{\epsilon \to 0} \min \left\{ E_1(R - \epsilon, D - \epsilon), E_2(R + \epsilon, D - \epsilon) \right\} \]
Analogously to (103), this limit can be simplified as
\[
\lim_{\epsilon \to 0} \min \left\{ E_1(R - \epsilon, D - \epsilon), E_2(R + \epsilon, D - \epsilon) \right\} = \lim_{\epsilon \to 0} \min \left\{ E_1(R, D - \epsilon), E_2(R, D - \epsilon) \right\}.
\]

**Theorem 11:**
\[
\limsup_{n \to \infty} \left\{ \frac{-1}{n} \ln P_s \right\} \leq \lim_{\epsilon \to 0} \min \left\{ E_1(R, D - \epsilon), E_2(R, D - \epsilon) \right\}.
\]

In order to combine the lower and upper bounds given by Theorem 10 and Theorem 11, and determine the true exponent of successful encoding, observe, that the lower bound (134) of Theorem 10 can be rewritten, analogously to (108), as the RHS of (2). As we have shown previously, the implicit expression on the RHS of (2) equals the explicit expression (18). As can be seen from (18), it is a convex \((\cup)\) function of \((R, D)\), and therefore it is continuous in \((R, D)\), except for the points where its result switches to the value \(+\infty\). This occurs for \(D = D_{\text{min}} = \min_{x, \hat{x}} d(x, \hat{x})\). For this value of \(D\), the upper bound of Theorem 11 is \(+\infty\), while the lower bound, given by Theorem 10, is finite. For all other values of \(D\) the bounds of Theorem 10 and Theorem 11 coincide. Therefore we have proved Theorem 1.

20. **Maximization over \(Q\) of the decoding error exponent for arbitrary \(D\)**

The decoding error exponent, corresponding to the “source duality” decoder (46), is given by (47), with the possible exception of the points \((R, D_{\text{min}}(Q))\), where

\[
D_{\text{min}}(Q) = \min_y \min_{x: Q(x) > 0} \min_{\hat{x}: Q(\hat{x}) > 0} \ln \frac{P(y | x)}{P(y | \hat{x})}.
\]

Let us assume \(D_{\text{min}}(Q) < 0\) for all \(Q\), except for the degenerate \(Q\) given by (127), for which \(D_{\text{min}}(Q) = 0\). Otherwise, there exist distinct channel inputs with exactly the same \(P(y | x)\), as a function of \(y\), i.e. indistinguishable at the channel output. Such input letters can be merged without loss of generality.

With this assumption, we obtain the following. For \(D > 0\), the maximal random coding exponent over \(Q\) is given by the supremum of (47) over \(Q\). For any \(D < 0\) the expression (47) yields the true exponent for the degenerate distribution (127), which equals \(+\infty\). Therefore, for \(D < 0\) the maximal random coding exponent over \(Q\) is \(+\infty\). For \(D = 0\), the true exponent for the degenerate distribution (127) is 0, which can be inferred directly from the definition of the decoder (46) itself, and the same is given by the expression (47). For all other \(Q\), in the case of \(D = 0\), the true exponent is also given by (47). We conclude, that, for \(D = 0\), the maximal random coding exponent over \(Q\) is given by the supremum of (47) over \(Q\).
To summarize the above, we have

**Theorem 12:**
\[
\sup_{Q(x)} \lim_{n \to \infty} \left\{ -\frac{1}{n} \ln P_e \right\} = \\
\sup_{Q(x)} \sup_{0 \leq \rho \leq 1} \left\{ -\inf_{s \geq 0} \ln \sum_{x,y} Q(x)P(y|x) \left[ \sum_{\hat{x}} Q(\hat{x}) \left[ \frac{P(y|x)}{P(y|\hat{x})} e^{-D} \right]^{-s} \right]^\rho - \rho R \right\}. \tag{140}
\]

Note, that this is equal \(+\infty\) for \(D < 0\).

Now, using Lemma 16 with Theorem 12, we obtain, that the original Forney’s random coding exponent coincides with the “source duality” exponent for \(D \geq 0\):

**Corollary 2:** For \(D \geq 0\)
\[
\sup_{Q(x)} \lim_{n \to \infty} \left\{ -\frac{1}{n} \ln P_e \right\} = \sup_{Q(x)} E_{\text{bound}}(Q, R, D).
\]

21. **Derivation of the encoding failure exponent**

This will lead to both (5) and (48).

Here another generic lemma is needed, similar to Lemma 9 and Lemma 10.

**Lemma 19:** Let \(Z_i \sim \text{i.i.d Bernoulli} \left( e^{-nI} \right), i = 1, 2, ..., e^{nR} \). If \(I \leq R - \epsilon\), with \(\epsilon > 0\), then
\[
\Pr \left\{ \sum_{i=1}^{e^{nR}} Z_i = 0 \right\} < \exp \{ -e^{n\epsilon} \}. \tag{141}
\]

**Proof:**
\[
\Pr \left\{ \sum_{i=1}^{e^{nR}} Z_i = 0 \right\} = \prod_{i=1}^{e^{nR}} \Pr \left\{ Z_i = 0 \right\} = \left[ 1 - e^{-nI} \right]^{e^{nR}} = \left[ (1 - e^{-nI})^{-e^{-nI}, e^{nR}} > e \right]^{(\ast)} < \exp \left\{ -e^{n(R-I)} \right\} \leq \exp \left\{ -e^{n\epsilon} \right\},
\]
where (\ast) holds because \((1 - x)^{-1/x} > e\) for \(0 < x < 1\).
Upper bound on the probability of encoding failure:

$$P_f \leq \sum_{P_x: R^{\text{ypes}}(P_x, Q, D) \leq R - 2\epsilon_1} \Pr \{ X \in T(P_x) \} \times$$

$$\min_{P_x: d(P_x, x) \leq D} \Pr \left\{ \sum_m \text{1}_{\{ \hat{X}_m \in T(P_x|x, X) \}}(m) = 0 \mid P_x \right\}$$

$$+ \sum_{P_x: R^{\text{ypes}}(P_x, Q, D) \geq R - 2\epsilon_1} \Pr \{ X \in T(P_x) \}$$

$$\leq \sum_{P_x: R^{\text{ypes}}(P_x, Q, D) \leq R - 2\epsilon_1} \Pr \{ X \in T(P_x) \} \times$$

$$\min_{P_x: d(P_x, x) \leq D} \sum_{i=1}^{e^{|nR|}} Z_i = 0$$

$$+ \sum_{P_x: R^{\text{ypes}}(P_x, Q, D) \geq R - 2\epsilon_1} \Pr \{ X \in T(P_x) \}$$

$$\leq \sum_{P_x: R^{\text{ypes}}(P_x, Q, D) \leq R - 2\epsilon_1} \exp \left\{ - e^{nR} \right\}$$

$$+ \sum_{P_x: R^{\text{ypes}}(P_x, Q, D) \geq R - 2\epsilon_1} \Pr \{ X \in T(P_x) \}$$

$$\leq (n + 1)^{|X|} \cdot \exp \left\{ - e^{n\epsilon_1} \right\}$$

$$+ \sum_{P_x: R^{\text{ypes}}(P_x, Q, D) \geq R - 2\epsilon_1} \exp \left\{ - nD(P_x \parallel P) \right\}$$

Explanation of steps:
\( (a) \) holds for sufficiently large \( n \), when
\[
\Pr \left\{ \hat{X}_n \in T(P_{\hat{x}|x}, X) \mid X \in T(P_x) \right\} \geq \exp \left\{ -n \left[ D(P_{\hat{x}|x} \| P_x \times Q) + \epsilon_1 \right] \right\},
\]
with
\[
Z_i \sim \text{i.i.d Bernoulli} \left( \exp \left\{ -n \left[ D(P_{\hat{x}|x} \| P_x \times Q) + \epsilon_1 \right] \right\} \right).
\]
\( (b) \) holds for
\[
B_i \sim \text{i.i.d Bernoulli} \left( \exp \left\{ -n \left[ R_{\text{types}}(P_x, Q, D) + \epsilon_1 \right] \right\} \right),
\]
where
\[
R_{\text{types}}(P_x, Q, D) \triangleq \min_{P_{\hat{x}|x} \mid d(P_{\hat{x}|x}) \leq D} D(P_{\hat{x}|x} \| P_x \times Q).
\( (c) \) holds by Lemma 19 for
\[
I = R_{\text{types}}(P_x, Q, D) + \epsilon_1 \leq R - 2\epsilon_1 + \epsilon_1 = R - \epsilon_1.
\]
\( (d) \) uses the upper bound on the probability of a type \( (131) \).
\( (e) \) Let \( W^* \) denote the conditional distribution, achieving \( R(P_x, Q, D - \epsilon_2) < +\infty \) for some \( \epsilon_2 > 0 \). This implies
\[
D(P_x \circ W^* \| P_x \times Q) = R(P_x, Q, D - \epsilon_2),
\]
\[
d(P_x \circ W^*) \leq D - \epsilon_2.
\]
Let \( W_n^* \) denote a quantized version of the conditional distribution \( W^* \) with variable precision \( 1/(nP_x(x)) \), i.e. a set of types with denominators \( nP_x(x) \), such that the joint distribution \( P_x \circ W_n^* \) is a type with denominator \( n \). Observe, that the differences between \( P_x \circ W^* \) and \( P_x \circ W_n^* \) do not exceed \( \frac{1}{n} \). Therefore, since the divergence, as a function of \( P_x \circ W \), has bounded derivatives, and also the distortion measure \( d(x, \hat{x}) \) is bounded, for any \( \epsilon_2 > 0 \) there exists \( n \) large enough, such that the quantized distribution \( W_n^* \) satisfies
\[
D(P_x \circ W_n^* \| P_x \times Q) \leq D(P_x \circ W^* \| P_x \times Q) + \epsilon_2;
\]
\[
d(P_x \circ W_n^*) \leq D.
\]
The last inequality implies
\[
D(P_x \circ W_n^* \| P_x \times Q) \geq R_{\text{types}}(P_x, Q, D).
\( (146) \)
The relations (146), (145), (144) together give

\[ R(P_x, Q, D - \epsilon_2) + \epsilon_2 \geq R_{\text{types}}(P_x, Q, D). \]  

(147)

This explains (e).

(f) uses the definition

\[ E_f^{\text{types}}(R, D) \triangleq \min_{P_x : R(P_x, Q, D) \geq R} D(P_x \| P). \]  

(148)

(g) \( E_f^{\text{types}}(R, D) \) is bounded from below by \( E_f(R, D) \) defined in (5).

We conclude from (142):

**Theorem 13:**

\[ \liminf_{n \to \infty} \left\{ -\frac{1}{n} \ln P_f \right\} \geq \lim_{\epsilon \to 0} E_f(R - \epsilon, D - \epsilon). \]  

(149)

Lower bound on the probability of encoding failure:

\[ P_f \overset{(a)}{=} \max_{P_x} \Pr \{ \mathbf{X} \in T(P_x) \} \times \]

\[ \left[ 1 - \left( \sum_{P_{x|x} : d(P_{x|x}) \leq D} \exp \left\{ -n \left[ D(P_{x|x} \| P_x \times Q) - R \right] \right\} \right) \right] \]

\[ \overset{(b)}{=} \max_{P_x} \Pr \{ \mathbf{X} \in T(P_x) \} \times \]

\[ \left[ 1 - \left( \sum_{P_{x|x} : d(P_{x|x}) \leq D} \exp \left\{ -n \left[ R(P_x, Q, D) - R \right] \right\} \right) \right] \]

\[ \overset{(c)}{=} \max_{P_x} \Pr \{ \mathbf{X} \in T(P_x) \} \cdot \left[ 1 - \left( \sum_{P_{x|x}} \exp \left\{ -n \epsilon_1 \right\} \right) \right] \]

\[ \overset{(d)}{=} \max_{P_x} (n + 1)^{-|X|} \cdot \exp \left\{ -nD(P_x \| P) \right\} \cdot \left[ 1 - (n + 1)^{|X|} \cdot \exp \left\{ -n \epsilon_1 \right\} \right] \]

\[ \overset{(e)}{=} \max_{P_x} \exp \left\{ -n \left[ D(P_x \| P) + \epsilon_2 \right] \right\} \overset{(f)}{=} \exp \left\{ -n \left[ E_f^{\text{types}}(R + \epsilon_1, D) + \epsilon_2 \right] \right\} \]

\[ \overset{(g)}{=} \exp \left\{ -n \left[ E_f(R + \epsilon_1 + \epsilon_3, D) + \epsilon_2 + \epsilon_3 \right] \right\}. \]  

(150)
Explanation of steps:

(a) uses the union bound for the probability of the complementary event of encoding success.

(b) uses the upper bound on the probability of a conditional type

$$\Pr \left\{ \tilde{X}_m \in T(P_{x|X}, X) \mid X \in T(P_x) \right\} \leq \exp \left\{ -nD(P_{x,x} \parallel P_x \times Q) \right\},$$

(c) follows by the definition (143) and the property $R(P_x, Q, D) \leq R_{\text{types}}(P_x, Q, D)$.

(d) uses the lower bound on the probability of a type and the polynomial upper bound on the number of conditional types.

(e) holds for sufficiently large $n$ for a given $\epsilon_2 > 0$.

(f) follows by the definition (148).

(g) Let $T^*$ denote the distribution achieving $E_f(R + \epsilon_1 + \epsilon_3, D) < +\infty$. Then

$$D(T^* \parallel P) = E_f(R + \epsilon_1 + \epsilon_3, D),$$

$$R(T^*, Q, D) \geq R + \epsilon_1 + \epsilon_3.$$ (151)

Let $T^*_n$ denote a quantized version of the distribution $T^*$ with precision $\frac{1}{n}$, i.e. a type with denominator $n$. We note that both $D(T^* \parallel P)$ and $R(T^*, Q, D)$ are convex (∪) functions of $T$, $R(T^*, Q, D)$ is lower semi-continuous. The latter property implies that if $R(T^*, Q, D)$ is $+\infty$, so is $R(T^*_n, Q, D)$ for sufficiently large $n$. Thus for any $\epsilon_3 > 0$ there exists $n$ sufficiently large, such that

$$D(T^*_n \parallel P) \leq D(T^* \parallel P) + \epsilon_3;$$

$$R(T^*_n, Q, D) \geq R + \epsilon_1.$$ (152)

The last inequality implies

$$D(T^*_n \parallel P) \geq E_f^{\text{types}}(R + \epsilon_1, D).$$ (153)

The relations (153), (152), (151) give

$$E_f(R + \epsilon_1 + \epsilon_3, D) + \epsilon_3 \geq E_f^{\text{types}}(R + \epsilon_1, D).$$

This explains (g).

We conclude from (150):

**Theorem 14:**

$$\limsup_{n \to \infty} \left\{ -\frac{1}{n} \ln P_f \right\} \leq \lim_{\epsilon \to 0} E_f(R + \epsilon, D).$$ (154)

The bounds of Theorem 13 and Theorem 14 prove Theorem 2.
22. **Proof of the identity** $R(Q \circ P, Q, 0) = I(Q \circ P)$

**Proposition 2:**

$$R(Q \circ P, Q, 0) = I(Q \circ P).$$ \hspace{1cm} (155)

**Proof:** This proof uses Lemma 1. Alternatively, it can be proved by the method of Lagrange multipliers. We use the explicit expression of Lemma 1:

$$R(Q \circ P, Q, 0) = \sup_{s \geq 0} \left\{ -\sum_{x,y} Q(x)P(y \mid x) \ln \sum_{\hat{x}} Q(\hat{x}) \left[ \frac{P(y \mid x)}{P(y \mid \hat{x})} \right]^{-s} \right\}$$ \hspace{1cm} (156)

$$= \sup_{s \geq 0} \min_{W(\hat{x} \mid x, y)} \{ D((Q \circ P) \circ W \parallel (Q \circ P) \times Q) + s[d((Q \circ P) \circ W) - D] \}. \hspace{1cm} (157)$$

In the above, (156) is the same as (8), and (157) is the same as (10). Observe, that the expression inside the supremum of (156) and (157) is a concave (∩) function of $s$, as a minimum of affine functions of $s$. We conclude, that in order to find the maximum over $s$, it suffices to find such $s$, for which the derivative of the expression in (156) is zero.

Differentiation with respect to $s$ gives:

$$\frac{d}{ds} \left\{ -\sum_{x,y} Q(x)P(y \mid x) \ln \sum_{\hat{x}} Q(\hat{x}) \left[ \frac{P(y \mid x)}{P(y \mid \hat{x})} \right]^{-s} \right\}$$

$$= -\sum_{x,y} Q(x)P(y \mid x) \frac{1}{\sum_{a} Q(a) \left[ \frac{P(y \mid x)}{P(y \mid a)} \right]^{-s}} \sum_{\hat{x}} Q(\hat{x}) \left[ \frac{P(y \mid x)}{P(y \mid \hat{x})} \right]^{-s} \left( -\ln \frac{P(y \mid x)}{P(y \mid \hat{x})} \right)$$

$$= \sum_{x,y} Q(x)P(y \mid x) \frac{1}{\sum_{a} Q(a)P^s(y \mid a)} \sum_{\hat{x}} Q(\hat{x})P^s(y \mid \hat{x}) \ln \frac{P(y \mid x)}{P(y \mid \hat{x})}.$$ 

Now, let us substitute $s = 1$:

$$\sum_{x,y} Q(x)P(y \mid x) \frac{1}{\sum_{a} Q(a)P(y \mid a)} \sum_{\hat{x}} Q(\hat{x})P(y \mid \hat{x}) \ln \frac{P(y \mid x)}{P(y \mid \hat{x})}$$

$$= \sum_{x,y} Q(x)P(y \mid x) \frac{1}{\sum_{a} Q(a)P(y \mid a)} \ln P(y \mid x) - \sum_{y} \left[ \sum_{a} Q(a)P(y \mid a) \sum_{\hat{x}} Q(\hat{x})P(y \mid \hat{x}) \ln P(y \mid \hat{x}) \right]$$

$$= \sum_{x,y} Q(x)P(y \mid x) \ln P(y \mid x) - \sum_{x,y} Q(\hat{x})P(y \mid \hat{x}) \ln P(y \mid \hat{x}) = 0.$$ 

We conclude, that $s^* = 1$. With $s^* = 1$ we obtain:

$$R(Q \circ P, Q, 0) = -\sum_{x,y} Q(x)P(y \mid x) \ln \sum_{\hat{x}} Q(\hat{x}) \left[ \frac{P(y \mid x)}{P(y \mid \hat{x})} \right]^{-1}$$

$$= -\sum_{x,y} Q(x)P(y \mid x) \ln \sum_{\hat{x}} Q(\hat{x})P(y \mid \hat{x}) = I(Q \circ P).$$
Note, that the minimizing $W(\hat{x} \mid x, y)$ in (157) is given by

$$W^*(\hat{x} \mid x, y) = \frac{Q(\hat{x})P(y \mid \hat{x})}{\sum_{\hat{x}} Q(\hat{x})P(y \mid \hat{x})}.$$

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