A new formulation for the streamwise turbulence intensity distribution

P. Henrik Alfredsson, Ramis Örlü & Antonio Segalini
Linné FLOW Centre, KTH Mechanics, SE-100 44 Stockholm, Sweden
E-mail: hal@mech.kth.se

Abstract. Numerical and experimental data from zero pressure-gradient turbulent boundary layers over smooth walls have been analyzed by means of the so called diagnostic plot introduced by Alfredsson & Örlü [Eur. J. Fluid Mech. B/Fluids, 42, 403 (2010)]. In the diagnostic plot the local turbulence intensity is shown as a function of the local mean velocity normalized with a reference velocity scale. In the outer region of the boundary layer a universal linear decay of the turbulence intensity is observed independent of Reynolds number. The deviation from this linear region appears in the buffer region and seems to be universal when normalized with the friction velocity. Therefore, a new empirical fit for the streamwise velocity turbulence intensity distribution is proposed and the results are compared with up to date reliable high-Reynolds number experiments and extrapolated towards Reynolds numbers relevant to atmospherical boundary layers.

1. Introduction

The scaling and behaviour of the fluctuating streamwise velocity component \( u \) in two-dimensional wall bounded flows have recently attracted new attention. The variance of \( u \) has a maximum close to the wall at around \( y^+ = 15 \), near the position where also the production of turbulence is at its maximum. The general view among researchers today is that if the maximum is scaled with the friction velocity the maximum increases with increasing Reynolds number, however if scaled with the free stream velocity it decreases. In the viscous sublayer (\( 0 < y^+ < 5 \)) the relative turbulence intensity level is fairly constant and close to 0.40, a value that increases slightly with increasing Reynolds number. These statements are well supported through the emergence of new, spatially well resolved experimental data as well as direct numerical simulation data at numerically high Reynolds numbers (Örlü & Schlatter, 2011).

There has been a few attempts to model the rms or variance distribution of \( u \), one early attempt to model the near wall behaviour was that of Haritonidis (1989) through mixing length ideas and some empirical data regarding turbulent bursts. However, the model gave as a result that the near-wall peak in the variance profile (at \( y^+ \) around 15) decreased with increasing Reynolds number. Today, however, this view has been abandoned, and it is generally accepted that the near-wall peak grows with increasing Reynolds number (DeGraaff & Eaton, 2000; Hutchins et al., 2009). Other more recent examples that are more in line with the present understanding on the fluctuation distribution are the works by Marusic et al. (1997) and Marusic & Kunkel (2003) which partly are based on physical arguments related to the so called attached eddy hypothesis. Another approach was taken by Smits (2010) who undertook an
attempt to obtain the variance through a modeling the spectrum, as a sum of contributions from basic eddy motions of different scales. Although all these models to some extent use physical reasoning, they rely to a large extent on empirical data for their final formulation.

However a new controversial result about the behaviour of $u$ has recently been reported, namely that a second, outer maximum of the variance of $u$ appears for high Reynolds numbers, which none of the above models have indicated. Such results have been shown both for high Reynolds number pipe (Morrison et al., 2004) and boundary layer flows (Fernholz & Finley, 1996), but are under debate since for instance poor spatial resolution of hot-wire probes have been shown to produce an erroneous second maximum. In a recent paper Alfredsson et al. (2011) suggested that the outer maximum would actually appear in turbulent boundary layers if the friction Reynolds number exceeded 15000. It is the purpose of the present paper to present a new formulation that describes the velocity standard deviation ($u'$) distribution over the outer part of the boundary layer (from the buffer region outwards) based on these new results. The new formulation also gives an indication on the distribution of the variance of the streamwise fluctuations at very high Reynolds numbers.

2. Scaling arguments and modeling

Traditional scaling arguments are based on the functional dependence of the streamwise velocity standard deviation ($u'$) on the wall-normal position ($y$):

$$u' = u_\tau f_1 (y^+; Re_\tau) = u_\tau f_2 \left( \frac{y}{\delta}; Re_\tau \right), \quad (1)$$

where $u_\tau$ is the friction velocity (such that the wall shear stress $\tau_w = \rho u_\tau^2$), $\delta$ is the boundary layer thickness and $Re_\tau = u_\tau \delta / \nu = \delta^+$ is the friction Reynolds number (or Karman number). Here and in the followings the ($^+$) superscript indicates viscous scaling (namely $u_\tau$ and $\nu / u_\tau$ for the velocity and length scale, respectively). Note that $Re_\tau$ should be viewed as a parameter and not as a variable in the above expression.

If the Reynolds number is high enough, viscous effects should be limited to a small region close to the wall (the inner region), while most of the boundary layer should be unaffected. So, in the limit of high $Re_\tau$, the viscous scaled variance should be a function of just $y^+$ close to the wall and of $y / \delta$ in the outer region. However, available high quality, experimental data show that Reynolds number effects are small but not negligible. An example is the continuous growth of the near wall peak intensity which is logarithmically dependent on the Reynolds number (Marusic & Kunkel, 2003). Different scaling relationships have been proposed in order to improve experimental data collapse (such as the mixed scaling of DeGraaff & Eaton, 2000) or to match orders of magnitude for the normal Reynolds stress transport equation (George & Castillo, 1997).

The logarithmic dependence of the maximum in $u'$ implies that higher Reynolds number experiments than the ones available today are needed in order to verify the scaling accuracy. The determination of a proper scaling law becomes complicated since measurement uncertainties related to the data quality (like spatial resolution, temporal resolution, measurement issues, etc.) are at hand, but also to the accurate determination of the friction velocity and wall position (Örlü et al., 2010). Numerical simulations are still far from the high Reynolds number limit, but their results must be scrutinized with the same caution of the experimental ones (Schlatter & Örlü, 2010) since they can be affected by several numerical issues as well.

In order to remove the uncertainty on the wall position determination from experimental data, a diagnostic plot has recently been introduced by Alfredsson & Örlü (2010) where the streamwise velocity standard deviation is plotted against the local mean velocity, both scaled with the free-stream velocity ($U_\infty$). The advantage of this approach is that $u'/U_\infty = g_1 \left( U/U_\infty; Re_\tau \right)$
which does not involve neither the wall distance nor the friction velocity. This is meaningful because the local mean velocity is an one-to-one function of the wall distance for a fixed Reynolds number (Monkewitz et al., 2007). By means of any skin friction relationship, the same functional argument as in (1) leads to the functional dependence:

\[
\frac{u'}{U_{\infty}} = g_2 \left( \frac{U}{U_{\infty}}; \frac{U_{\infty}}{u_{\tau}} \right),
\]

which states a relationship between outer and inner scaled mean velocity by means of the use of the friction and free-stream velocity. It is expected that the scaling with \(U_{\infty}\) is able to scale only the outer region of the boundary layer, while the use of the friction velocity \(u_{\tau}\) appears to be more suited for the inner region description. The aim (or rather hope) of this approach is to include as much as possible of the complexities, encountered up to now in the traditional description of the variance as a function of the wall normal position, into such dependence on the local mean velocity.

Figure 1 reports experimental data from DeGraaff & Eaton (2000) at \(Re_{\theta} = 1430, 2900, 5160,\) and \(13000,\) which will be utilized throughout the remainder of the paper. Figure 1(a) reports the mean velocity profiles in inner scaled variables, where the collapse of the profiles is evident in the near wall region. Figure 1(b) shows the velocity variance scaled in viscous variables and, as known, the data show an increasing trend for increasing \(Re_{\tau}.\) The same data are then represented in the diagnostic plot in figure 1(c), showing now a good collapse of both the inner \((U/U_{\infty} < 0.2)\) and outer region \((U/U_{\infty} > 0.7)\). The same plot can be slightly modified by plotting the data as \(U/U_{\infty}\) versus \(u'/U\) instead, which is the local streamwise turbulence intensity (figure 1(d)). The latter representation is equivalent to the diagnostic plot (equation (2)) times \(U/U_{\infty}\). Therefore the conclusions that can be argued are essentially the same with the advantage that the abscissa is now independent of the near wall velocity scaling (c.f. \(u_{\tau}\)) or outer one \((U_{\infty}).\) Also, a linear behavior of \(u'/U\) is observed for \(U/U_{\infty} > 0.7\) with lower limit deviation function of the Reynolds number: the higher the Reynolds number, the lower \(U/U_{\infty}\) where the deviation takes place. On the other side, the traditional value of \(u'/U \approx 0.4\) is observed in the near wall region (Alfredsson et al., 1988).

3. Description of the outer region

According to the data plotted in figure 1(d), the streamwise turbulence intensity in the outer region appears to collapse on a straight line function of \(U/U_{\infty}\) independently of the Reynolds number as was proposed by Alfredsson et al. (2011). This empirical observation suggests a fitting expression like:

\[
\frac{u'}{U} = a + b \frac{U}{U_{\infty}}.
\]

It is important to note that a weak Reynolds number dependence of the coefficients \(a\) and \(b\) is possible. The determination of the fitting constant has been done by using the DNS data of Schlatter & Örlü (2010) with \(1410 \leq Re_{\theta} \leq 4060,\) the experimental data of DeGraaff & Eaton (2000) with \(1400 \leq Re_{\theta} \leq 30000\) and the high \(Re\) data of Fernholz et al. (1995) with \(21000 \leq Re_{\theta} \leq 61000,\) giving a ratio of almost 40 between the utilized Reynolds numbers. It is expected that (at least in the outer region where the fit works) spatial resolution effects should be negligible (Örlü & Alfredsson, 2010). The data are plotted in figure 2(a) in grey symbols. According to the data available, it is not clear if a Reynolds number trend is present. Indeed, in order to keep the model simple enough, the parameters \(a\) and \(b\) have been supposed as constants. The values of such coefficients in the linear range are \(a = 0.2909\) and \(b = -0.2598.\) In the same plot the DNS data of Schlatter & Örlü (2010) at \(Re_{\theta} = 670\) and \(Re_{\theta} = 1000\) are
Figure 1. Plot of the statistics measured by DeGraaff & Eaton (2000) (a) Mean velocity in inner variable (b) Viscous scaled variance (c) Diagnostic plot (d) Turbulence intensity against $U/U_\infty$.

shown, suggesting that a linear behavior is still present but with a sensible deviation from the rest of the data set, probably due to low Reynolds number effects. In the determination of $a$ and $b$ only data with $U/U_\infty \geq 22.5$ have been used.

The upper $U/U_\infty$ value where the line fits the data is $U/U_\infty \approx 0.9$ after which a decay of the turbulent intensity follows in the wake region. An ansatz can be introduced now to model such a decay as:

$$\left(\frac{u'}{U}\right)_{outer} = \left( a + b \frac{U}{U_\infty} \right) Q\left( \frac{U}{U_\infty} \right).$$

(4)

Since $U \to U_\infty$ much faster than $u' \to 0$ in the free stream region, an infinite derivative of the $Q$ function is expected at $U/U_\infty = 1$ making the present approach ill-posed. However, the decay can still approximately be described by an exponential fitting expression such as:

$$Q\left( \frac{U}{U_\infty} \right) = 1 - \exp \left[ -\gamma \left( 1 - \frac{U}{U_\infty} \right) \right].$$

(5)
Figure 2. (a) Streamwise turbulence intensity plotted against $U/U_\infty$. The black line is the proposed fit $u'/U = a + bU/U_\infty$. (b) Wake function $Q$ and the exponential fit (black line). The grey dots are numerical and experimental results from Schlatter & Örlü (2010), DeGraaff & Eaton (2000) and Fernholz et al. (1995).

Figure 3. (a) Streamwise turbulence intensity against $U^+$. (b) Inner function $\Delta$ and the proposed fit (black line). The grey lines are numerical and experimental results from Schlatter & Örlü (2010) (dashed line) and DeGraaff & Eaton (2000) (solid line).

where $\gamma = 64$. The agreement of the fitting expression (5) is shown in figure 2(b) where the same data set has been used.

4. Description of the inner region

The deviation of the turbulent intensity from the straight line and its Reynolds number dependence shown in figure 1(d) suggests that, when the inner region is approached, the scaling with $U/U_\infty$ must be replaced with $U/u_\tau$. Indeed figure 3(a) reports the turbulence intensity as function of $U^+$: It is possible to see that the shape of the inner region seems to be similar but
quantitatively different. On the other hand the function

$$\Delta = \frac{u'}{U} - \left( a + b \frac{U}{U_\infty} \right),$$

which is the difference between the turbulence intensity and equation (3), seems to be an universal function of the mean velocity in viscous units, as evident from figure 3(b) where DNS data of Schlatter & Örlü (2010) and experimental data of DeGraaff & Eaton (2000) have been used. No further datasets have been used in this region because spatial resolution effects and measurements uncertainties will add further scatter in the data.

This function $\Delta(U^+)$ can be fitted with a polynomial function in the near wall region and with a corner function at $U^+ \approx 17$ and the details are given in Appendix A.

By summing now the near wall and outer contribution, the proposed complete fitting function for the streamwise standard deviation becomes:

$$\frac{u'}{U_\infty} = \frac{U}{U_\infty} \left[ \Delta \left( \frac{U}{U_\infty}, U^+ \right) + \left( a + b \frac{U}{U_\infty} \right) Q \left( \frac{U}{U_\infty} \right) \right],$$

and figure 4 shows a comparison with one of the measured profiles of DeGraaff & Eaton (2000). The agreement is remarkable everywhere and it is comparable to the more complicated fit proposed by Marusic & Kunkel (2003) based on the attached eddy hypothesis.

5. Discussion

In the present paper we have used the finding reported in Alfredsson et al. (2011) that in the outer region (composed by the logarithmic and wake regions) the rms-distribution of the streamwise velocity, if scaled with the local mean velocity, is seen to be universal if plotted against $U/U_\infty$. In particular, it was found that in this scaling there is a linear decrease of $u'/U$ over a certain range of $U/U_\infty$, a range that extends to lower values of $U/U_\infty$ when the Reynolds number increases. Here we have shown that the point where the $u'/U$ distribution deviates from
Figure 5. Mean and fluctuating velocity profiles at high Reynolds numbers, using the present formulation (7) to estimate the latter. The mean velocity profile has been estimated with the expression reported by Monkewitz et al. (2007). (a) Mean velocity in inner variables, (b) Viscous scaled variance, (c) Outer scale plot of the variance profiles, (d) Diagnostic plot. As can be observed the outer maximum should become larger than the inner maximum for a Reynolds number around $1.2 \cdot 10^6$.

The linear trend corresponds to a fixed value of $U^+ = 17$ independent of $Re$. This value of $U^+$ corresponds to approximately $y^+ = 120$, i.e. in the lower part of the logarithmic region. If scaled in inner variables it has been shown that the profiles for the present collection of well resolved data collapse nicely in the region below $U^+ = 17$. This gives us the possibility to express the distribution of $u'$ as a composite profile, one part that describes the outer region and one that describes the inner, much in the same way as has been done for the mean velocity distribution (see e.g. Monkewitz et al., 2007).

With the present formulation of the $u'$ distribution it is possible to attempt an extrapolation to even higher Reynolds numbers (see figure 5). The formulation suggest that a second “outer” maximum appears for $Re_0$ higher than about 45000 (corresponding to about $Re_\tau$ of 15000), hence it is shown that the second maximum does not appear until a large enough scale separation is
at hand. It also shows that the amplitude of this maximum becomes larger than the inner one for $Re_\theta > 1.2 \cdot 10^6$. This and the emergence of an outer peak in itself is in contrast to the model by Marusic & Kunkel (2003) or the one by Smits (2010) which do not give an outer maximum. Furthermore, figure 5(c) clearly shows that previous attempts to scale $u'$ with the friction velocity in the outer region (i.e. plotted against $y/\delta$) is futile. Chauhan et al. (2009) on the other hand showed that $(u'/U_\infty)^2$ is almost constant in the range $10000 < Re_\theta < 60000$ at $y^+ = 400$. This Reynolds number range corresponds to $0.7 > U/U_\infty > 0.6$ where the diagnostic plot shows an almost constant value (figure 5(d)), further supporting the present scaling.

An interesting finding by Hutchins & Marusic (2007) is that in the spectra a second outer maxima appears for high enough $Re$. A question is if this maximum is connected to some change in the turbulence structure or if it is just a natural development when the Reynolds number becomes larger? It should also be noted that the $y^+$ position of this maximum scales approximately the same way as the maximum in $-\overline{uv}$ with Reynolds number. One possibility is that the balance between production, dissipation and transport in the outer region changes with $Re$ and this may give such an effect. To substantiate such a claim we may assume that the flow becomes more isotropic as $Re$ increases, which would indicate that the correlation $-\overline{uv}/u'v'$ decreases (Priyadarshana & Klewicki, 2004). However $-\overline{uv}$ is coupled to the mean velocity distribution (a direct relation for channel flows, an approximate one for boundary layers) which means that $u'$ and/or $v'$ need to increase in order for the correlation to decrease.

The behaviour of $u'$ for high Reynolds numbers suggested in this paper should be viewed as a hypothesis that should be tested when high Reynolds number data will become available in the future. It should be noted that the atmospheric data of Metzger et al. (2007) at Reynolds numbers of the order of one million show an outer maximum, however these data may be affected by surface roughness and the freestream velocity is not well defined. Therefore it is hard to make a quantitative comparison, however qualitatively the behaviour seems to abide to the present formulation. On the other hand, the present formulation could be used to determine an effective free stream velocity for atmospheric boundary layer data.

Acknowledgement

Part of these ideas was formulated and discussed during the NORDITA program on Turbulent Boundary Layers in Stockholm in April 2010.

References

ALFREDSSON, P. H., JOHANSSON, A. V., HARITONIDIS, J. H. & ECKELMANN, H. 1988 The fluctuating wall-shear stress and the velocity field in the viscous sublayer. Phys. Fluids 31, 1026–1033.

ALFREDSSON, P. H. & ÖRLÜ, R. 2010 The diagnostic plot–a litmus test for wall-bounded turbulence data. Eur. J. Mech. B-Fluid 29, 403–406.

ALFREDSSON, P. H. SEGALINI, A. & ÖRLÜ, R. 2011 A new scaling for the streamwise turbulence intensity in wall-bounded turbulent flows and what it tells us about the “outer” peak. Phys. Fluids 23, 041702.

CHAUHAN, K. A., MONKEWITZ, P. A. & NAGIB, H. M. 2009 Criteria for assessing experiments in zero pressure gradient boundary layers. Fluid Dyn. Res. 41, 021404.

DEGRAAFF, D. B. & EATON, J. K. 2000 Reynolds-number scaling of the flat-plate turbulent boundary layer. J. Fluid Mech. 422, 319–346.

FERNHOLZ, H. H. & FINLEY, P. J. 1996 The incompressible zero-pressure-gradient turbulent boundary layer: An assessment of the data. Prog. Aero. Sci. 32, 245–311.
Fernholz, H. H., Krause, E., Nockemann, M. & Schober, M. 1995 Comparative measurements in the canonical boundary layer at $Re_{\delta_2} \leq 6 \times 10^4$ on the wall of the German–Dutch windtunnel. *Phys. Fluids* **7**, 1275–1281.

George, W. K. & Castillo, L. 1997 Zero-pressure-gradient turbulent boundary layer. *Appl. Mech. Rev.* **50**, 689–730.

Haritonidis, J. H. 1989 A model for near-wall turbulence. *Phys. Fluids* **1**, 302–306.

Hutchins, N. & Marusic, I. 2007 Evidence of very long meandering features in the logarithmic region of turbulent boundary layers. *J. Fluid Mech.* **579**, 1–28.

Hutchins, N., Nickels, T. B., Marusic, I. & Chong, M. S. 2009 Hot-wire spatial resolution issues in wall-bounded turbulence. *J. Fluid Mech.* **635**, 103–136.

Marusic, I., Uddin, A. K. M. & Perryl, A. E. 2003 Similarity law for the streamwise turbulence intensity in zero-pressure-gradient turbulent boundary layers. *Phys. Fluids* **9**, 3718–3726.

Marusic, I. & Kunkel, G. J. 2003 Streamwise turbulence intensity formulation for flat-plate boundary layers. *Phys. Fluids* **15**, 2461–2464.

Metzger, M. M., McKeon, B. & Holmes, H. 2007 The near-neutral atmospheric surface layer: turbulence and non-stationarity. *Phil. Trans. R. Soc. A* **365**, 859–876.

Monkewitz, P. A., Chauhan, K. A. & Nagib, H. M. 2007 Self-consistent high-Reynolds-number asymptotics for zero-pressure-gradient turbulent boundary layers. *Phys. Fluids* **19**, 115101.

Morrison, J. F., McKeon, B., Jiang, W. & Smits, A. J. 2004 Scaling of the streamwise velocity component in turbulent pipe flow. *J. Fluid Mech.* **508**, 99–131.

Örlü, R. & Alfredsson, P. H. 2010 On spatial resolution issues related to time-averaged quantities using hot-wire anemometry. *Exp. Fluids* **49**, 101–110.

Örlü, R., Fransson, J. H. M. & Alfredsson, P. H. 2010 On near wall measurements of wall bounded flows—the necessity of an accurate determination of the wall position. *Prog. Aero. Sci.* **46**, 353–387.

Priyadarshana, P. J. A & Klewicki, J. C. 2004 Study of the motions contributing to the Reynolds stress in high and low Reynolds number turbulent boundary layers. *Phys. Fluids* **16**, 4586–4600.

Schlatter, P. & Örlü, R. 2010 Assessment of direct numerical simulation data of turbulent boundary layers. *J. Fluid Mech* **659**, 116–126.

Smits, A. J. 2010 High Reynolds Number Wall-Bounded Turbulence and a Proposal for a New Eddy-Based Model. *Turbulence and Interactions. M. Deville, T.H. Le, and P. Sagaut (Eds.), Springer-Verlag Berlin Heidelberg* pp. 51–62.

**Appendix A: The $\Delta$ function**

The inner fitting function is:

$$
\Delta (U^+) = \begin{cases} 
  c_0 + c_1 U^+ + c_2 (U^+)^2 + c_3 (U^+)^3 & (U^+ \leq 11.3) \\
  0.075 \left(1 + \chi \left[(11.3 - U^+)/4.05, 12\right]\right) & (U^+ > 11.3),
\end{cases}
$$

(8)

with additional fitting constants $c_0 = 0.1289$, $c_1 = 9.914 \cdot 10^{-3}$, $c_2 = -1.4 \cdot 10^{-3}$ and $c_3 = 8.932 \cdot 10^{-6}$. 

The $\chi(x, \alpha)$ function is strictly speaking a corner function with adjustable sharpness depending on the $\alpha$ parameter. The mathematical definition is:

$$
\chi(x; \alpha) = \frac{1}{F_\infty(\alpha)} \int_0^x \frac{1}{1 + |\eta|^\alpha} d\eta, \quad (9)
$$

where:

$$
F_\infty(\alpha) = \int_0^{+\infty} \frac{1}{1 + \eta^\alpha} d\eta = 1 - 2 \sum_{n=1}^{+\infty} \frac{(-1)^n}{(n\alpha - 1)(n\alpha + 1)}. \quad (10)
$$

It is immediate to see that the corner is located approximately at $|x| \approx 1$. One of the properties is that $\chi(x; \alpha)$ is an odd function. By using Maclaurin expansions and integration by parts, it is possible to derive the asymptotic approximations:

$$
\chi(x; \alpha) \approx \frac{x}{F_\infty(\alpha)} \left[ 1 - \sum_{n=1}^{+\infty} \frac{(-1)^n}{n\alpha + 1} x^{n\alpha} \right] (0 < x < 1), \quad (11)
$$

$$
\chi(x; \alpha) \approx 1 + \frac{x}{F_\infty(\alpha)} \sum_{n=1}^{+\infty} \frac{(-1)^n}{n\alpha - 1} x^{n\alpha} \quad (x > 1), \quad (12)
$$

very useful to calculate $\chi$, and consequently $\Delta$ in equation (8), in real applications.