Lipschitzian Estimates in Discrete-Time Constrained Stochastic Optimal Control

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Abstract

This paper is devoted to the analysis of a finite horizon discrete-time stochastic optimal control problem, in presence of constraints. We study the regularity of the value function which comes from the dynamic programming algorithm. We derive accurate estimates of the Lipschitz constant of the value function, by means of a regularity result of the multifunction that defines the admissible control set.

In the last section we discuss an application to an optimal asset-allocation problem.

Key words. Optimal control, Dynamic Programming, State constraints, Lipschitz regularity, asset-allocation, Multifunctions.

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This paper is devoted to the analysis of a general Finite Horizon Discrete-Time Stochastic Optimal Control Model, with inequality constraints. The aim of this paper is to give a method to estimate the Lipschitz constant of the value function obtained via the classical dynamic programming algorithm.

The regularity of the value function is related to the regularity of the marginal function \cite{1}, \cite{22}, and, as is proved in \cite{1}, the regularity of the marginal function is connected to the regularity of the multifunction which defines the set of admissible controls. Therefore, our main result concerns the Lipschitz regularity, with respect to the Hausdorff metric \cite{3}, of this multifunction.
Set-valued maps are widely used in optimal control, differential games and their applications to mathematical economics and finance, see [8], and [19]. In many cases these multifunctions are defined by means of inequality constraints for a set of functions defined over a manifold that represents the control space. Unfortunately this manifold is usually non-regular, as in the financial application presented in Section 5. Furthermore, the constraint functions may lose the regularity or be dependent on each other, at some point.

To overcome these difficulties, we allow for Lipschitz manifolds and Lipschitz constraint functions, provided that the set where either the manifold or the constraints are not regular, or the constraints are dependent on each other, can be approximated by points where both the manifold and the constraint function are regular and the constraints are independent on each other.

We use a quantitative formulation of the Implicit Function Theorem that provides an estimate of the neighborhoods where the implicit function is defined.

Our study is carried out in the discrete-time case, because of its high computational relevance, since the regularity properties of the value function can be used to derive a-priori error estimates and convergence results of numerical schemes.

The outline of the paper is as follows. Section 1 introduces the general framework of the optimization model and recalls the classical discrete-time DP algorithm. Section 2 provides the definition of the Hausdorff metric and some related results, Section 3 provides basic notations and definitions, Section 4 establishes the main regularity results about multifunctions and a Lipschitzian estimate for the value function related to the dynamic programming algorithm. Finally, Section 5 applies these results to an optimal asset-allocation problem with regulatory constraints.

1 The Dynamic Programming Algorithm

In this section we present the model which is the subject of our study in next sections. We consider the following discrete-time controlled dynamical system:

\[
\begin{align*}
    x_{k+1} &= f_k(x_k, u_k, y_k), \quad k = 0, \ldots, N - 1 \\
    x_0 &= x \in X_0,
\end{align*}
\]  

(1.1)

where

\[
    f_k : X_k \times M_k \times Y_k \to X_{k+1},
\]

\[
    X_k, X_N \subset \mathbb{R}^m, \quad M_k \subset \mathbb{R}^n,
\]

(1.2)

for every \( k = 0, \ldots, N - 1 \). Here \( x_k \) is the state space, \( u_k \) the control and \( y_k \) the random disturbance. For every \( k \), we are given the following constraint functions:

\[
    c_1^k, \ldots, c_j^k : \mathcal{M}_k \times A_k \to \mathbb{R}, \quad j_k < n,
\]

(1.3)

where \( A_k \) is an open subset of \( \mathbb{R}^m \), with \( X_k \subset A_k \).

The set of admissible controls at time \( t_k \) for the point \( x \in X_k \) is defined as follows:

\[
    U_k(x) = \{ u \in \mathcal{M}_k : c_i^k(u, x) \leq 0, \ i = 1, \ldots, j_k \}.
\]

(1.4)

We assume that \( U_k(x) \) is non empty, for every \( x \in X_k \) and that the random disturbance \( y_k \) is a measurable function over a Probability Space \((\Omega, \mathcal{F}, P)\) with values in a measurable space \((Y_k, \mathcal{E}_k)\),
where $\mathcal{E}_k$ is a sigma-field over $\mathcal{Y}_k$.

The disturbance $y_k$ is characterized by a probability law $p_k(\cdot)$, which we assume independent of $(x_k, u_k)$ and of prior disturbances $y_{k-1}, \ldots, y_0$. We call an admissible control law, a set $\phi = \{u_0, \ldots, u_{N-1}\}$ of functions $u_k : X_k \to M_k$ such that $u_k(x) \in U_k(x)$, for every $x \in X_k$. We denote by $\mathcal{U}$ the set of admissible control laws.

Given an initial state $x \in X_0$, the optimization problem consists in finding an admissible control law $\phi \in \mathcal{U}$ which maximizes the cost functional

$$J_\phi(x) = \mathbb{E}[g(x_N)]$$

where $\mathbb{E}$ denotes the expected value taken over $(\Omega, \mathcal{F}, P)$, and $x_N$ is the value at time $t_N$ of the state $x$, according to (1.1).

The real-valued function $g : X_N \to \mathbb{R}$ is called stopping cost or utility function.

We want to point out that, eventhough we limit ourselves to this simple r model for the sake of simplicity, the discussion in this paper can be easily extended to include running costs or probability measures $p_k$ depending also on the state and control variable.

The Classical Dynamic Programming Algorithm consists in solving the problem (1.5) by means of the following sequence of one-step optimization problems:

$$\left\{ \begin{array}{ll}
    J_k(x) &= \sup_{u \in U_k(x)} \mathbb{E}_k[J_{k+1}(f_k(x, u, y_k))], & \forall x \in X_k, \ 0 \leq k < N \\
    J_N(x) &= g(x), & \forall x \in X_N.
\end{array} \right. \quad (1.6)$$

The value function at time $t_k$, $J_k$ is defined over the state space $X_k$, and $\mathbb{E}_k$ denotes the expected value taken w.r.t. the measure $p_k$ over $\mathcal{Y}_k$. For any given initial state $x_0 \in X_0$, the value $J_0(x_0)$ computed by the algorithm equals the optimal cost

$$\max_{\phi \in \mathcal{U}} J_\phi(x_0) \quad (1.7)$$

and the optimal control policy $\phi^*$ can be obtained by $\phi^* = \{u_0^*, \ldots, u_{N-1}^*\}$ where $u_k^*, k = 0, \ldots, N-1$ maximizes the right-hand side of (1.6). See [4] for a detailed description of the Dynamic Programming Algorithm in the discrete-time case.

## 2 The Hausdorff Metric

We introduce in this section the Hausdorff metric over the space of all compact subsets of a given metric space. This metric is used in Section 3 to estimate the Lipschitz constant of the multifunction $x \mapsto U_k(x)$.

To study the regularity of the value function $J_k$ defined in (1.5), since the admissible control set $U_k(x)$ depends on the state variable, one needs to measure the distance between the admissible control sets corresponding to different states of the system.

We need, therefore, to introduce a distance between sets in order to show some regularity property of the function

$$A \mapsto \max_A f$$
with \( f \) continuous and \( A \) a subset of a separable metric space \( (M, d) \). Since we only consider compact control sets, we can limit ourselves to introducing the Hausdorff metric on the class of all compact subsets of \( M \), denoted by \( \text{Comp}(M) \).

For every \( K_1, K_2 \in \text{Comp}(M) \backslash \{\emptyset\} \), let

\[
d_H(K_1, K_2) = \inf \{ \varepsilon > 0 : K_1 \subset K_1^\varepsilon, \text{ and } K_2 \subset K_2^\varepsilon \}
\]

(2.8)

where, for any set \( A \subset X \), \( A^\varepsilon = \{ y : d(y, A) < \varepsilon \} \) denotes the open ball of radius \( \varepsilon \) around \( A \).

It is easy to verify that \( d_H \) is a metric on \( \text{Comp}(M) \). Furthermore if two points \( x, y \) of \( X \) are regarded as the single point sets \( \{x\} \) and \( \{y\} \) in \( \text{Comp}(M) \), then

\[
d_H(\{x\}, \{y\}) = d(x, y).
\]

(2.9)

That is to say, \( M \) is isometrically embedded in \( \text{Comp}(M) \). See [3] for the properties of the Hausdorff distance.

The following Proposition concerns the Lipschitz regularity of a real-valued map defined over the space of all compact subsets of a metric space.

**Proposition 2.1** Let \( f \) be a Lipschitz continuous function over the space \( M \), and

\[
\hat{f} : \text{Comp}(M) \to \mathbb{R},
\quad K \mapsto \max_K f.
\]

(2.10)

Then \( \hat{f} \) is a Lipschitz continuous map over the metric space \( (\text{Comp}(M), d_H) \) and its Lipschitz constant equals the Lipschitz constant of \( f \) over \( M \).

**Proof of Proposition 2.1.** Let \( K_1, K_2 \) be compact subsets of \( M \), then there exist \( x_i \in K_i \), such that \( \hat{f}(K_i) = f(x_i) \), for \( i = 1, 2 \). By the definition (2.8), for a fixed \( \delta > 0 \), there exists \( \varepsilon > 0 \), such that

\[
\varepsilon < d_H(K_1, K_2) + \delta,
\]

\[
K_1 \subset K_1^\varepsilon,
\]

\[
K_2 \subset K_2^\varepsilon.
\]

(2.11)

Therefore there exist \( y_i \in K_i \) \( i = 1, 2 \), such that,

\[
d(x_1, y_2) < d_H(K_1, K_2) + \delta,
\]

\[
d(y_1, x_2) < d_H(K_1, K_2) + \delta.
\]

(2.12)

Hence, by the definition of \( x_1, x_2 \) and the Lipschitz regularity of \( f \) it follows,

\[
\hat{f}(K_1) - \hat{f}(K_2) \leq f(x_1) - f(y_2) \leq \text{Lip}(f)d(x_1, y_2)
\]

\[
\leq \text{Lip}(f)(d_H(K_1, K_2) + \delta),
\]

\[
\hat{f}(K_1) - \hat{f}(K_2) \geq f(y_1) - f(x_2) \geq -\text{Lip}(f)d(y_1, x_2)
\]

\[
\geq -\text{Lip}(f)(d_H(K_1, K_2) + \delta).
\]

(2.13)

The previous inequalities, being \( \delta \) arbitrary, proves

\[
|\hat{f}(K_1) - \hat{f}(K_2)| \leq \text{Lip}(f)d_H(K_1, K_2).
\]

(2.14)
Therefore $\tilde{f}$ is a Lipschitz continuous map over $\text{Comp}(M)$, and $\text{Lip}(\tilde{f}) \leq \text{Lip}(f)$.

On the contrary, if $x_1, x_2 \in X$, then by (2.9), we have
\[
|f(x_1) - f(x_2)| = |\tilde{f}(\{x_1\}) - \tilde{f}(\{x_2\})| \leq \text{Lip}(\tilde{f}) d(x_1, x_2)
\]
that proves $\text{Lip}(f) \leq \text{Lip}(\tilde{f})$.

3 Main Notations and Definitions

In this section we introduce the main notations which are used in this paper.

1. Let $f$ be some real valued, Lipschitz continuous function over the domain $D \subset \mathbb{R}^m$. We refer to the Lipschitz constant of $f$ as to
\[
\text{Lip}(f) = \sup_{(x, x') \in D \times D} \frac{|f(x) - f(x')|}{|x - x'|}.
\]
(3.16)

2. We recall the definition of norm of an operator $S : \mathbb{R}^{p_1} \to \mathbb{R}^{p_2}$
\[
\|S\| := \max_{x \in \mathbb{R}^{p_1}} |Sx|.
\]
(3.17)

3. If $c \in \mathbb{R}^j$ we say $c \leq 0$ if and only if $c_i \leq 0$ for every $i = 1, \ldots, j$.

4. Let $d \geq j$ be two integers, and $\Pi$ be the set of all multi-indexes $\pi = (i_1, \ldots, i_j)$, with $1 \leq i_1 < i_2 < \ldots < i_j \leq d$. Then for every $\pi \in \Pi$ and $u \in \mathbb{R}^d$, let $u_\pi$ denote the projection of $u$ over the coordinates specified by $\pi$, i.e. $u_\pi = (u_{i_1}, \ldots, u_{i_j}) \in \mathbb{R}^j$.

For every differentiable function $f : V \times A \to \mathbb{R}^j$, with $V \subset \mathbb{R}^d$ and $A \subset \mathbb{R}^m$ open sets, and for any $\pi \in \Pi$, let $\frac{\partial f}{\partial u_\pi}$ denote the Jacobian matrix of $f$ w.r.t. the coordinates of $v \in V$ specified by $\pi$, i.e.
\[
\left(\frac{\partial f}{\partial u_\pi}\right)_{hl} = \frac{\partial f_h}{\partial u_{i_l}}, \quad h, l = 1, \ldots, j
\]

5. Given $\pi \in \Pi$, $d \geq j$, we define the map
\[
Z^\pi : \mathbb{R}^d \times \mathbb{R}^j \to \mathbb{R}^d
\]
(3.18)

where, for every $u \in \mathbb{R}^d$, $v \in \mathbb{R}^j$,
\[
(Z^\pi(u, v))_\pi = v, \quad Z^\pi(u, v) = u_i, \quad \forall \ i \notin \pi, \ \forall \ i = 1, \ldots, d.
\]
(3.19)

In other words $Z^\pi(u, v)$ is a obtained from $u$, by substituting the vector $v$ to the components of $u$ corresponding to $\pi$. Obviously if $j = d$, we have the only $\pi = (1, 2, \ldots, d)$ and $Z^\pi(u, v) = v$. 5
6. For every \( \pi = (i_1, \ldots, i_j) \in \Pi \), let denote by \( T_\pi \), the matrix

\[
(T_\pi)_{h,l} = \begin{cases} 
1 & h = i_l \\
0 & \text{otherwise}
\end{cases}
\]  

(3.20)

for every \( 1 \leq h \leq d \), and \( 1 \leq l \leq j \).

7. We call \textit{regular arc} a function \( \gamma : I \to \mathbb{R}^m \), where \( I \) is a compact interval, which is piecewise differentiable on \( I \), with \( \gamma' \) bounded and \( \gamma'(t) \neq 0 \), for every \( t \in I \) where the derivative exists.

**Definition 3.1** A non empty subset \( X \subset \mathbb{R}^m \) satisfies the property (CON), if \( X \) is connected and there exists a characteristic number \( a(X) > 0 \), such that for every pair of distinct points \( x_1, x_2 \in X \), there exists a regular arc \( \gamma : [w_1, w_2] \to X \) such that the following conditions hold true:

\[
\gamma(w_i) = x_i, \ i = 1, 2, \\
l(\gamma) \leq a(X)|x_1 - x_2|,
\]  

(3.21)

where \( l(\gamma) := \int_{w_1}^{w_2} |\gamma'(t)| dt \), denotes the length of \( \gamma \).

Obviously, if \( X \) is a convex subset of \( \mathbb{R}^m \), then the property (CON) holds with \( a(X) = 1 \). Also, it can be proved that any connected compact submanifold of \( \mathbb{R}^m \) satisfies the property (CON).

**Definition 3.2** A set \( M \subset \mathbb{R}^n \) is called a \( d \)-dimensional \textit{Lipschitz manifold} if, for every \( u \in M \), there exists an open neighborhood \( W \) of \( u \) in \( M \) and an homeomorphism \( \psi : W \to V \), where \( V \) is an open subset of \( \mathbb{R}^d \) and such that \( \psi \) and \( \psi^{-1} \) are Lipschitz continuous. The couple \((W, \psi)\) is called a local chart for \( u \).

**Definition 3.3** An atlas on a compact Lipschitz manifold \( M \subset \mathbb{R}^n \) is a finite collection of charts \( \{(W_\alpha, \psi_\alpha)\}_{\alpha \in A} \) such that \( \{W_\alpha\}_{\alpha \in A} \) is a covering of \( M \).

**Definition 3.4** Let \( M \) be a Lipschitz manifold, we define \( \mathcal{N}R(M) \) to be the set of nonregular points of \( M \), that is, for every point \( u \in M \setminus \mathcal{N}R(M) \) there exists a local chart \((W, \psi)\) for \( u \) whose inverse is \( C^1 \) in a neighborhood of \( \psi(u) \). Such a chart will be called a \textit{regular chart} for \( u \).

Up to a suitable change of the atlas, in the reminder of the paper we make the following assumption:

**Assumption 3.5** If \( u \in M \setminus \mathcal{N}R(M) \), every local chart \((W, \psi)\) for \( u \) is regular.

**Remark 3.6** With Assumption 3.5, the Definition 3.4 becomes:

\[
M \setminus \mathcal{N}R(M) = \{u \in M : \text{every local chart } (W, \psi) \text{ for } u \text{ is a regular chart for } u\}
\]

**Remark 3.7** Let \( M \subset \mathbb{R}^n \) be a Lipschitz manifold with dimension \( d \).
1. We denote by $T_u(M)$ the tangent space to $M$ at $u \in M$. $T_u(M)$ exists $\mathcal{H}^d_M$-almost everywhere on $M$ by Rademacher’s Theorem, where $\mathcal{H}^d_M$ is the $d$-dimensional Hausdorff measure on $M$. Since $M \subset \mathbb{R}^n$, we can view $T_u(M)$ as a linear subspace of $\mathbb{R}^n$. In particular, if $u \notin N\mathcal{R}(M)$ and $\phi : V \subset \mathbb{R}^d \rightarrow \phi(V) \subset M$, $\phi(v) = u$, is the inverse of a regular chart for $u$, the $n$-dimensional vectors

$$\frac{\partial \phi}{\partial v_1}(v), \ldots, \frac{\partial \phi}{\partial v_d}(v),$$

(3.22)

form a basis of $T_u(M)$.

2. A function $f : M \rightarrow \mathbb{R}^l$ is differentiable at $u \notin N\mathcal{R}(M)$ (resp. $C^1$ in a neighborhood of $u$) if for every chart $\psi$ defined in a neighborhood of $u$ with an inverse differentiable at $\psi(u)$ (resp. $C^1$ in a neighborhood of $\psi(u)$), the function $f \circ \psi^{-1}$ is differentiable at $\psi(u)$ (resp. $C^1$ in a neighborhood of $\psi(u)$) in the usual sense.

3. Let $f : M \rightarrow \mathbb{R}^l$ be differentiable at $u \notin N\mathcal{R}(M)$. The **differential of $f$ at $u$** is a linear operator

$$d_u f : T_u(M) \rightarrow \mathbb{R}^l$$

(3.23)

defined as follows. Let $\phi$ be the inverse of a regular chart for $u$, as in 1., and let be given the basis (3.22) of the tangent space $T_u(M)$. Then, for any $w \in T_u(M)$,

$$w = \sum_{i=1}^d w_i \frac{\partial \phi}{\partial v_i}(v), \quad w_i \in \mathbb{R}, \ i = 1, \ldots, d,$$

The differential of $f$ at $u$ is defined as

$$d_u f(w) = \sum_{i=1}^d w_i \frac{\partial f \circ \phi}{\partial v_i}(v) \in \mathbb{R}^l.$$ 

(3.24)

It can be proved that this definition is well-posed in that it does not depend on the choice of the chart $\phi^{-1}$.

Given the immersion of $T_u(M)$ in $\mathbb{R}^n$, we can view the differential of a map as a linear operator over a $d$-dimensional linear subspace of $\mathbb{R}^n$ and therefore we can consider its norm. In other words $M$ is a Riemannian manifold.

The same definition of differential holds if $f$ depends also on a variable $x \in A \subset \mathbb{R}^m$. In this case definition (3.24) applies to $f(\cdot, x)$ for every $x \in A$, and its differential is denoted by $d_u f(\cdot, x)$.

**Definition 3.8** Let $M \subset \mathbb{R}^n$ be a Lipschitz manifold and $f : M \times A \rightarrow \mathbb{R}^l$, $A$ an open subset of $\mathbb{R}^m$, $X \subset A$. We define $N\mathcal{R}(f)$ to be the set of nonregular points of $f$, i.e.

$$N\mathcal{R}(f) = \{ u \in M \backslash N\mathcal{R}(M) : \exists x \in A : f \text{ is non-differentiable at } (u,x) \}. $$

(3.25)

$$N\mathcal{S}(f,X) = \{ u \in M \backslash (N\mathcal{R}(M) \cup N\mathcal{R}(f)) : \exists x \in X : f(u,x) \leq 0 \text{ and } d_u f(\cdot, x) \text{ is not surjective} \}$$

(3.26)

and

$$D(M,X,f) = N\mathcal{R}(M) \cup N\mathcal{R}(f) \cup N\mathcal{S}(f,X)$$

(3.27)
Definition 3.9 For every $u \in M \setminus N R(M)$, we choose, once and for all, a regular chart for $u$, whose inverse will be denoted by $\phi_u$, $\phi_u(v) = u$. We define, for any $\pi \in \Pi$, the linear operator

$$T^u_\pi : \mathbb{R}^j \rightarrow T_u(M)$$

whose associated matrix, w.r.t. the canonical basis of $\mathbb{R}^j$ and the basis

$$\frac{\partial \phi_u}{\partial v_1}(v), \ldots, \frac{\partial \phi_u}{\partial v_d}(v)$$

of $T_u(M)$, is the matrix $T_\pi$ defined in (3.20).

4 The Lipschitz Regularity of the Value Function

In order to prove that the value function is Lipschitz continuous we need to prove that the map $U_k(\cdot)$ defined in (1.4) is Lipschitz continuous with respect to the Hausdorff distance $d_H$ in $\text{Comp}(X_k)$. We present some regularity results for multifunctions which have the same structure as the set of admissible controls (1.4).

Theorem 4.1 Let $c = (c_1, \ldots, c_j)$, with $c_i$, $i = 1, \ldots, j$, $j \leq n$, real valued maps defined on $M \times A$, where $M \subset \mathbb{R}^n$ and $A \subset \mathbb{R}^m$ is open. Let $X$ be a non empty subset of $A$ which has the property (CON), and let $M$ be a compact Lipschitz $d$-dimensional manifold, with $j \leq d$. We further assume that

i) For every $x \in X$ the set

$$U(x) = \{u \in M : c(u, x) \leq 0\}$$

is non empty.

ii) $D(M, X, c)$ is closed and one of the following assumptions holds:

(A) The set $\{(u, x) : x \in X, u \in U(x) \cap D(M, X, c)\} = \emptyset$.

(B) $D(M, X, c) \neq \emptyset$ and every $u \in U(x) \cap D(M, X, c)$, with $x \in X$, is of adherence for $U(x) \setminus D(M, X, c)$.

iii) The function $c := (c_1, \ldots, c_j)$ is $C(M \times A) \cap C^1(M \setminus D(M, X, c) \times A)$.

iv) For every $x \in X$ and $u \in U(x) \setminus D(M, X, c)$,

$$\tau := \sup_{x \in X} \sup_{u \in U(x) \setminus D(M, X, c)} T(u, x) < \infty .$$

with

$$T(u, x) := \max_{\pi \in \Pi(u, x)} \left\| (d_u c(\cdot, x) \circ T^u_\pi)^{-1} \circ d_x c(\cdot, \cdot) \right\| ,$$

and

$$\Pi(u, x) := \{\pi \in \Pi : d_u c(\cdot, x) \circ T^u_\pi \text{ is invertible}\} .$$
Then the map

\[ x \in X \mapsto U(x) \]  

(4.33)

is

if \( \text{ii)-(A) holds} \): \( d_H \)-Lipschitz continuous and its Lipschitz constant can be estimated by \( a(X)\tau L_{\text{Lip}} M \), where \( a(X) \) denotes the characteristic number of \( X \) as in Definition 3.1 and

\[ L_{\text{Lip}} M := \sup \{ L_{\text{Lip}}(\phi) : \phi^{-1} \text{ is a chart over } M \}. \]  

(4.34)

if \( \text{ii)-(B) holds} \): \( d_H \)-uniformly continuous on every compact subset of \( X \) which has the property \( (\text{CON}) \).

Next result strengthens the regularity assumptions on the constraint function \( c \) in order to obtain Lipschitz regularity of the value function even in the case \( \text{ii)-(B)} \).

**Theorem 4.2** In the hypotheses \( \text{i), ii)-(B), iii)} \) of Theorem 4.1, we assume that

iv) there exist \( \mu, \lambda, r > 0 \) such that for every \( x \in X \) and \( u \in U(x) \setminus D(M, X, c) \), and \( \pi \in \Pi(u, x) \),

\[ \left\| \left( d_u c(\cdot, x) \circ T_x^u \right)^{-1} \right\| \leq \lambda, \]

\[ \| d_y c(u, \cdot) \| \leq \mu, \forall y \in A, |y - x| \leq r \]

and that

v) if \( r \) is chosen as in iv), there exists \( L > 0 \) such that for every \( u \in U(x) \setminus D(M, X, c) \) with \( x \in X \), and for every regular chart \((W, \psi)\) for \( u \) it holds:

\[ \left\| \frac{\partial \phi}{\partial v}(v, y) - \frac{\partial \phi}{\partial v}(v_u, x) \right\| \leq L (|v - v_u| + |y - x|) \]  

(4.36)

for every \( v \) in a suitable neighborhood of \( v_u, y \in A, |y - x| \leq r, \) with \( \phi(\cdot, \cdot) = c(\psi^{-1}(\cdot), \cdot) \), \( v_u = \psi(u) \).

Let

\[ \tau = \lambda \mu, \]  

(4.37)

then \( x \in X \mapsto U(x) \) is \( d_H \)-Lipschitz continuous, with constant \( a(X)\tau L_{\text{Lip}} M \).

**Remark 4.3** If \( \text{(A)} \) holds true, then for every compact set \( K \subset X \), we have

\[ \{(u, x) : x \in K, u \in U(x) \cap D(M, X, c)\} = \emptyset, \]

hence, by compactness, there exists \( \sigma^* > 0 \) such that \( U(x) \cap (D(M, X, c))^\sigma = \emptyset \), for any \( x \in K \).

Using these results, we prove a regularity result for the value function \((1.6)\). Then we prove Theorem 4.1 and Theorem 4.2.

**Theorem 4.4** Let be given \((1.1)-(1.4)\) and the related optimization algorithm \((1.7)\). Suppose that:
1) for every $k = 0, \ldots, N - 1$, the triplet $(\mathcal{M}_k, c^k, X_k)$ satisfies the assumptions of Theorem 4.1 under the condition ii)-(A), or the hypotheses of Theorem 4.2.

2) for every $k = 0, \ldots, N - 1$, there exists a nonnegative, $\mathcal{E}_k$-measurable and $p_k$-integrable function $y \in \mathcal{Y}_k \mapsto V_k(y)$, i.e.:

$$\int_{\mathcal{Y}_k} V_k(y) dp_k(y) < \infty,$$

such that for $p_k - a.e. y \in \mathcal{Y}_k$,

$$|f_k(x, u, y) - f_k(x', u', y)| \leq V_k(y)|x - x', u - u'|,$$

$\forall \ x, x' \in X_k, \ \forall \ u, u' \in \mathcal{M}_k.$

If $g$ is Lipschitz continuous over $X_N$ then, for every $k$, $J_k$ is Lipschitz continuous over $X_k$ and the following estimate holds:

$$\left\{ \begin{array}{l}
Lip(J_k) \leq L(J_{k+1}) \mathbb{E}_k[V_k](1 + a_k \tau_k Lip(M_k)), \ k = 0, \ldots, N - 1 \\
Lip(J_N) = Lip(g).
\end{array} \right.$$  \hspace{1cm} (4.40)

with $a_k$ the characteristic number of $X_k$, given in Definition 3.1, and $\tau_k$ defined in (4.30) or (4.37) according to the two alternatives of assumption 1).

We prove this result first, then we prove Theorems 4.1 and 4.2.

**Proof of Theorem 4.4.** We proceed by induction over $k$. For $k = N$, the function $J_N$ is the utility function $g$, which is supposed Lipschitz continuous over $X_N$. We assume $J_{k+1}$ Lipschitz continuous over $X_{k+1}$, for $k \leq N - 1$. Therefore, the function

$$\Psi_k(x, u) = \mathbb{E}[J_{k+1}(f_k(x, u, y_k))], \ (x, u) \in X_k \times \mathcal{M}_k.$$

is Lipschitz continuous over $X_k \times \mathcal{M}_k$, and an easy computation yields

$$Lip(\Psi_k) \leq Lip(J_{k+1}) \mathbb{E}_k[V_k].$$  \hspace{1cm} (4.41)

Moreover, for any $k = 0, \ldots, N - 1$, the assumption 1) implies that the set-valued map $x \in X_k \mapsto U_k(x)$ is $d_H$-Lipschitz continuous, by Theorem 4.1 or Theorem 4.2, with Lipschitz constant estimated by $a_k \tau_k Lip(M_k)$. Hence we can apply Proposition 2.1 to deduce that the marginal function defined on $X_k$ by

$$x \in X_k \mapsto \max_{u \in U_k(x)} \Psi_k(x, u)$$

is Lipschitz continuous. The DP-algorithm (1.6), implies that this map is exactly $J_k$. Moreover, by Proposition 2.1, its Lipschitz constant is estimated by $Lip(\Psi_k)(1 + Lip(U_k))$. Therefore we have

$$Lip(J_k) \leq Lip(J_{k+1}) \mathbb{E}_k[V_k](1 + a_k \tau_k Lip(M_k)),$$

which proves the assertion.
In order to prove Theorem 4.1, we have to estimate the $d_H$-distance between $U(x)$ and $U(y)$ i.e. we need to prove, by the definition of $d_H$, an inclusion of the type $U(x) \subseteq (U(y))^{\delta}$, where $\delta > 0$.

We need therefore to take an admissible control $u$ for $x$ and to show that we can “perturb” it to an admissible control for $y$ whose distance from $u$ is less than $\delta$. The idea of the proof is to show that this property holds true locally and that the radius of the neighborhood where the property holds true is independent of the point $x$ and of the control $u$. This independence will allow us to prove Theorem 4.1 by “iteration”, i.e. by covering the arc between $x$ and $y$ with a finite number of balls of constant radius where the property holds true.

**Lemma 4.5** In the same hypotheses of Theorem 4.1 and according to the two alternatives of assumption ii), we have:

(A) for every regular arc $\gamma : [w_1, w_2] \to X$, there exists $r_0 > 0$ such that, for every $w_1 \leq t < s \leq w_2$ which satisfy

$$\gamma([t, s]) \subseteq B_{r_0}(\gamma(t)),$$

the following inclusion holds:

$$U(\gamma(t)) \subseteq (U(\gamma(s)))^{\tau'}, \quad \forall \tau' > \tau_{t,s} \text{Lip}_M,$$

with

$$\tau_{t,s} := \tau l(\gamma; t, s) := \tau \int_{t}^{s} |\gamma'(\xi)| d\xi,$$

and $\tau$ is given by (4.30).

(B) For every compact set $K \subset X$ which has the property (CON), and for any $\sigma > 0$, there exists $r_\sigma \in (0, \sigma]$ such that, for every $x, y \in K$ such that

$$|x - y| < r_\sigma,$$

we have

$$U(x) \subseteq (U(y))^{\tau' + \sigma}, \quad \forall \tau' > a(K)\tau \text{Lip}_M|x - y|,$$

with $a(K)$ the characteristic constant for $K$, given in Definition 3.7.

Let’s assume for the time being that Lemma 4.5 holds true. Using this result it is straightforward proving Theorem 4.1:

**Proof of Theorem 4.1.** If $X$ is a single point set, then we do not need to prove anything. Let’s assume that $X$ contains at least a pair of distinct points $x_1, x_2$; we want to estimate the Hausdorff distance between the corresponding admissible control sets, i.e.

$$d_H(U(x_1), U(x_2)).$$

By the property (CON), there exists a regular arc $\gamma : [w_1, w_2] \to X$ which connects $x_1$ to $x_2$ and which satisfies (3.21). We have to distinguish between the two alternatives in assumption ii)- (B):

**Case (A):** For every $t \in [w_1, w_2]$ let denote by $B_t$ the open ball centered in $\gamma(t)$ with radius $r_0 > 0$, obtained by applying Lemma 4.5 case (A), to the arc $\gamma$. We can assume that the curve does not
have self-intersections, otherwise we could define a new arc connecting the same points and having smaller length. We introduce the following sequence
\[
\begin{align*}
t_0 &:= w_1 \\
t_{i+1} &:= \sup\{w_2 \geq t \geq t_i : |\gamma(t_i) - \gamma(s)| \leq \frac{\delta}{2^i}, \forall s \in [t_i, t]\}, \quad \text{if } i \geq 0,
\end{align*}
\]
that is increasing in the interval \([w_1, w_2]\). It is straightforward proving that \(t_p = w_2\) for some integer \(p > 0\).

Therefore \([t_0, t_1, \ldots, t_{p-1}, t_p]\) is a finite covering of \([w_1, w_2]\). Moreover \(\gamma([t_i, t_{i+1}]) \subset B_{t_i}\).

By (4.43), we obtain
\[
U(\gamma(t_i)) \subset (U(\gamma(t_{i+1})))^{\tau_i}, \quad i = 0, \ldots, p - 1,
\]
where \(\tau_i = \tau_{\text{Lip}_M} \int_{t_i}^{t_{i+1}} |\gamma(\xi)| d\xi + \frac{\varepsilon}{p^i}, \varepsilon\) is an arbitrary positive number. By iterating the previous inclusion (4.43), we obtain
\[
U(x_1) \subset (U(x_2))^{\tau_{\text{Lip}_M(\gamma)} + \varepsilon}.
\]

By switching the role of \(x_1\) and \(x_2\) and by (3.21), the definition of the metric \(d_H\) and the arbitrary choice of \(\varepsilon\), we can estimate (4.47) with \(a(X)\tau_{\text{Lip}_M}|x_1 - x_2|\).

**Case (B):** Let \(K\) be a compact subset of \(X\) which satisfies the condition \((\text{CON})\). By applying Lemma 4.3 in the case \((\text{B})\), for any \(\sigma > 0\), there exists \(r_\sigma \in (0, \sigma]\), such that the inclusion (4.46) holds, whenever \(x, y \in K\) satisfy (4.45). This implies
\[
d_H(U(x), U(y)) \leq (1 + a(X)\tau_{\text{Lip}_M})\sigma, \quad \forall \ |x - y| < r_\sigma,
\]
and therefore \(U(\cdot)\) is uniformly continuous over \(K\), with respect to \(d_H\).

We turn now to the proof of Lemma 4.6. In the case \((\text{B})\), we need to approximate the controls which lie in a neighborhood of \(D(M, X, c)\).

**Lemma 4.6** Let \(M\) be a manifold as in Theorem 4.4, \(K \subset X\) compact, and suppose that assumptions i), ii)-(\text{B}) and iii) hold true.

For every \(\sigma > 0\), there exists \(\rho > 0\) such that for every \(x \in K\) and \(u \in U(x)\), there exists \(u_\sigma \in U(x)\), such that
\[
d(u_\sigma, D(M, X, c)) \geq \rho, \quad \text{and} \quad |u_\sigma - u| \leq \sigma.
\]

**Proof.** We prove the result for the controls \(u \in U(x) \cap (D(M, X, c))^{\rho}\), otherwise it suffices choosing \(u_\sigma = u\).

Suppose by contradiction that there exists \(\overline{\sigma} > 0\) such that for \(\rho = \frac{1}{h}\), there exist \(u_h \in U(x_h), x_h \in K\) with
\[
d(u_h, D(M, X, c)) < \frac{1}{h},
\]
such that, for any \(u \in U(x_h)\),
\[
d(u, D(M, X, c)) < \frac{1}{h}, \quad \text{or} \quad |u - u_h| > \overline{\sigma}.
\]
Without loss of generality we may assume that \( x_h \to \overline{x}, \ u_h \to \overline{u} \), as \( h \to \infty \), for some \( \overline{x} \in K \) and \( \overline{u} \in D(M, X, c) \), since \( K \) is compact and \( D(M, X, c) \) is closed by assumption \( ii) \) of Theorem 4.1. The continuity of \( c \) over \( M \times A \), implies \( \overline{u} \in U(\overline{x}) \cap D(M, X, c) \) and, by \( ii) \), \( \overline{u} \) is of adherence for \( U(\overline{x}) \setminus D(M, X, c) \), therefore there exists \( \tilde{u} \in U(\overline{x}) \setminus D(M, X, c) \) such that

\[
|\tilde{u} - \overline{u}| < \frac{\overline{\sigma}}{2}.
\]

Let \( \pi \in \Pi(\tilde{u}, x) \) and \( \phi = \phi_{\tilde{u}} \) be the map, relative to \( \tilde{u} \), fixed in Definition 3.9, with \( \phi(\tilde{v}) = \tilde{u}, \ \tilde{v} \in \mathbb{R}^d \). Let \( Z^\pi \) be the function introduced in Section 3, and

\[
F(w, y) := c(\phi(Z^\pi(\tilde{v}, w)), y) - c(\tilde{u}, \overline{x}) \in \mathbb{R}^j, \ \forall (w, y) \in \mathbb{R}^j \times A,
\]

which is of class \( C^1 \) in a neighborhood of \( ((\tilde{v})_{\pi}, \overline{x}) \). We observe that

\[
F((\tilde{v})_{\pi}, \overline{x}) = c(\phi(\tilde{v}), \overline{x}) - c(\tilde{u}, \overline{x}) = 0
\]

and

\[
\frac{\partial F}{\partial w} ((\tilde{v})_{\pi}, \overline{x}) = \frac{\partial c(\phi(\cdot), \overline{x})}{\partial v}(\tilde{v}, \overline{x}), \frac{\partial Z^\pi}{\partial w},
\]

where the first matrix on the right-hand side is \( j \times d \) and the second one is \( d \times j \). It is easy to verify that if \( \pi = (i_1, \ldots, i_j), \ 1 \leq i_1 < i_2 < \ldots < i_j \leq d \), and if \( 1 \leq i \leq d, \ 1 \leq l \leq j \), we have

\[
(T_\pi)_{i,l} = \left( \frac{\partial Z^\pi}{\partial w} \right)_{i,l} = \begin{cases} 1 & i = i_l \\ 0 & \text{otherwise} \end{cases}
\]

By the definition of \( \pi \) we deduce the invertibility of the Jacobian matrix \( (4.57) \). With \( (4.56) \) this allows for the application of the classical Implicit Function Theorem which implies the existence of a continuous map \( q : B \to \mathbb{R}^j \), where \( B \subset A \) is an open neighborhood of \( \overline{x} \) in \( \mathbb{R}^m \), such that

\[
q(\overline{x}) = (\tilde{v})_{\pi}
\]

and

\[
F(q(y), y) = 0, \ \forall y \in B.
\]

Since \( c(\tilde{u}, \overline{x}) \leq 0 \), using \( (4.55) \) and \( (4.60) \), we get

\[
\hat{q}(y) := \phi(Z^\pi(\hat{v}, q(y))) \in U(y), \ \forall y \in B.
\]

For large values of \( h \), we have \( x_h \in B \) and since \( \hat{q}(x_h) \to \hat{u} \) and \( u_h \to \overline{u} \) as \( h \to \infty \), by \( (4.54) \), we also may assume

\[
|\hat{q}(x_h) - u_h| < \frac{\overline{\sigma}}{2}.
\]

Hence by \( (4.61) \) and \( (4.53) \) with \( u = \hat{q}(x_h) \), we infer

\[
d(\hat{q}(x_h), D(M, X, c)) < \frac{1}{h},
\]

for large \( h \). Letting \( h \to \infty \) in \( (4.62) \), we obtain

\[
\hat{u} = \hat{q}(\overline{x}) \in D(M, X, c),
\]

which is a contradiction.
The main difficulty in the proof of Lemma 4.5 is in building the radius $r_0$ independent of $t, s$ and of the controls in $U(\gamma(t))$. We use a quantitative version of the classical Implicit Function Theorem, which provides an estimate of the radius of the balls where the implicit map is defined, see \[7\]. Using the implicit function theorem, we are able to build a map that, for every state $\gamma(s)$ "near" $\gamma(t)$, prescribes how to modify the control $u \in U(\gamma(t))$ to obtain an admissible control for $\gamma(s)$.

**Theorem 4.7** Let $F : \mathcal{O} \to \mathbb{R}^j$ be a map defined in the open set $\mathcal{O} \subset \mathbb{R}^j \times \mathbb{R}^m$. Let $r_1, r_2 > 0$ be such that, if $B_1 = \{ v \in \mathbb{R}^j : |v - v_0| \leq r_1 \}$ and $B_2 = \{ y \in \mathbb{R}^m : |y - y_0| \leq r_2 \}$, then $B_1 \times B_2 \subset \mathcal{O}$ and the following hypotheses hold:

\[
F, \frac{\partial F}{\partial v} \text{ are continuous on } B_1 \times B_2, \tag{4.63}
\]

\[
F(v_0, y_0) = 0, \quad \det \frac{\partial F}{\partial v}(v_0, y_0) \neq 0, \tag{4.64}
\]

\[
\begin{align*}
&\sup_{y \in B_2} |F(v_0, y)| \leq \frac{r_1}{2\|T_0\|}, \\
&\sup_{B_1 \times B_2} \|I_j - T_0 \frac{\partial F}{\partial v}\| \leq \frac{1}{2},
\end{align*} \tag{4.65}
\]

where $I_j$ is the identity matrix of order $j$. Then there exists a unique function $q \in C(B_2; B_1)$ which satisfies

\[
q(y_0) = v_0, \tag{4.66}
\]

and for every $(v, y) \in B_1 \times B_2$, it holds

\[
F(v, y) = 0 \iff v = q(y). \tag{4.67}
\]

Next Proposition provides an expression for the Jacobian matrix of the implicit function $q$.

**Proposition 4.8** Let $D_1$, $D_2$ be two open balls of $\mathbb{R}^j$ and $\mathbb{R}^m$, respectively. Suppose that $F \in C^1(D_1 \times D_2; \mathbb{R}^j)$, and $q \in C(D^2_2; D_1)$ satisfies $F(q(y), y) = 0$, for every $y \in D_2$. If $\frac{\partial F}{\partial v}$ is invertible in $D_1 \times D_2$, then $q \in C^1(D_2)$ and

\[
\frac{\partial q}{\partial y}(y) = -\left(\frac{\partial F}{\partial v}(q(y), y)\right)^{-1} \frac{\partial F}{\partial v}(q(y), y), \text{ for } y \in D_2. \tag{4.68}
\]

We turn finally to the proof of Lemma 4.5.

**Proof of Lemma 4.5.** We prove the assertions in steps. We derive an approximation of $U(x)$ by the controls of $U(y)$ for $x, y$ which lie in a compact connected subset $K$ of $X$, then we specialize the discussion according to the assumptions ii)-(A) and ii)-(B).

(Construction of a covering). Let $K$ be a nonempty compact connected subset of $X$, $\bar{\sigma} > 0$ such that

\[
K^{\bar{\sigma}} \subset A. \tag{4.69}
\]
Let \( \sigma^* \) be chosen as in Remark 5.8 and \( \sigma < \pi \). We introduce

\[
\rho := \begin{cases} 
\sigma^* & \text{if (A) holds} \\
\rho_\sigma & \text{if (B) holds},
\end{cases}
\]

(4.70)

where \( \rho_\sigma \) is related to \( \sigma \) and to the compact set \( K \subset X \) via Lemma 4.6. Let

\[
K_{\gamma, \rho} = \{(u, x) : x \in K, \ u \in U(x) \text{ and } d(u, D(M, X, c)) \geq \rho\}.
\]

(4.71)

By Remark 4.3, in the case (A), and Lemma 4.4, in the case (B), for small \( \sigma \), this set is a non empty, compact subset of \( M \times K \). For every \((u, x) \in K_{\gamma, \rho}\), let \( \pi(u, x) \in \Pi(u, x) \) and \( \varphi_u \) the map given in Definition 3.4, for the point \( u \), \( \varphi_u : V_u \rightarrow M \). We may assume, without loss of generality, that \( \varphi \in C^1(V_u) \), furthermore, since \( D(M, X, c) \) is closed, we may also suppose \( \varphi_u(V_u) \cap D(M, X, c) = \emptyset \).

By the assumption \( iii \), the map \( \varphi_u \), defined by

\[
\varphi_u(v, y) = c(\varphi_u(v), y), \quad \forall (v, y) \in V_u \times A,
\]

is \( C^1 \). Furthermore, the matrix

\[
R(w, y; u, x) := \frac{\partial \varphi_u}{\partial v_{x(u, x)}} (\varphi_u^{-1}(w), y), \quad \forall (w, y) \in \varphi_u(V_u) \times A,
\]

(4.72)

is invertible at \((u, x)\).

By the continuity of the function (4.72), there exists \( \delta' = \delta'(u, x) > 0 \) such that

\[
z(u, x) := \inf \{|\det R(w, y; u, x)| : (w, y) \in (B_{\delta'}(u) \cap M) \times B_{\delta'}(x)\} > 0,
\]

(4.73)

where \( B_{\delta'}(u) \subset \mathbb{R}^n \) is the open ball of radius \( \delta' \) centered at \( u \), with \( B_{\delta'}(u) \cap M \subset \varphi_u(V_u) \), and \( B_{\delta'}(x) \subset K^{\pi/2} \) is the open ball in \( \mathbb{R}^m \) of radius \( \delta' \) centered at \( x \). By the continuity of \( \varphi_u^{-1} \) there exists \( \delta(u, x) \leq \delta'(u, x) \) such that

\[
|\varphi_u^{-1}(w) - \varphi_u^{-1}(u)| \leq \frac{\zeta_u}{2}, \quad \forall \ w \in B_{\delta(u, x)}(u) \cap M.
\]

(4.74)

with \( \zeta_u > 0 \) such that the closed ball in \( \mathbb{R}^d \) centered at \( \varphi_u^{-1}(u) \) with radius \( \zeta_u \) is contained in \( V_u \).

Such a ball is denoted by \( N_u \).

The collection

\[
\left\{ B_{\frac{\delta}{2}}(u) \times B_{\frac{\delta}{2}}(x) : \delta = \delta(u, x), \ (u, x) \in K_{\gamma, \rho} \right\}
\]

is an open covering of \( K_{\gamma, \rho} \), therefore we can extract a finite covering corresponding to some points \((u_1, x_1), \ldots, (u_p, x_p) \in K_{\gamma, \rho}\). To simplify the remainder of the proof let us define:

\[
\phi_i := \varphi_{u_i}, \quad \pi_i := \pi(u_i, x_i), \quad R_i(\cdot, \cdot) := R(\cdot, \cdot; u_i, x_i), \quad T_i := T_{\pi_i}, \quad L_i := Lip(\phi_i),
\]

\[
\pi_i := \pi(u_i, x_i), \quad z_i := z(u_i, x_i), \quad \delta_i := \delta(u_i, x_i), \quad \zeta_i := \zeta_{u_i},
\]

\[
N_i := N_{u_i}, \quad O_i(\delta) := (B_\delta(u_i) \cap M) \times B_\delta(x_i), \quad \forall \delta > 0, \quad i = 1, \ldots, p,
\]

and

\[
\delta^* := \min_{1 \leq i \leq p} \delta_i, \quad z^* := \min_{1 \leq i \leq p} z_i, \quad \theta^* := \min_{1 \leq i \leq p} \frac{\delta_i}{L_i}, \quad \zeta^* := \min_{1 \leq i \leq p} \zeta_i
\]

(4.75)

By (4.73), we have

\[
|\det R_i(w, y)| \geq z^* > 0, \quad \forall (w, y) \in O_i(\delta_i), \quad \forall i = 1, \ldots, p.
\]

(4.76)
Now suppose \((u, x) \in K_{\gamma, \rho}\), then \((u, x) \in O_i(\delta_i^2)\) for some \(i = 1, \ldots, p\). Let
\[
v_i := \phi_i^{-1}(u) \in \mathbb{R}^d,
\]
and
\[
F : \mathbb{R}^j \times A \rightarrow \mathbb{R}^j
\]
given by
\[
F(v, y) := \varphi_i(Z_i^*(v, v), y) - c(u, x)
\]
which is defined and \(C^1\) in the open ball in \(\mathbb{R}^d\) centered at \((v_i)\) and with radius \(\zeta_i^2\), since \((4.74)\) implies
\[
\left| Z_i^*(v, v) - \phi_i^{-1}(u_i) \right| < \zeta_i
\]
that implies \(Z_i^*(v, v) \in V_u\).

In order to apply Theorem 4.7 to \(F\) with \((v_0, y_0) = ((v_i)_{\pi_i}, x)\), let us assume for the time being that \(r_1, r_2\) are chosen as prescribed by Theorem 4.7. We observe that \(F(v_0, y_0) = 0\) and
\[
\frac{\partial F}{\partial v} ((v_i)_{\pi_i}, x) = R_i(u, x),
\]
is invertible by \((4.76)\). By Theorem 4.7, there exists a continuous map
\[
q : B_2 \rightarrow B_1
\]
with \(B_1 \times B_2 \subset \mathbb{R}^j \times A\), such that \((4.66)\) and \((4.67)\) hold true. The function \(q\) is a feedback function that allows us to build an admissible control for \(y\) starting from the admissible control \(u\) for \(x\); as in the proof of Lemma 4.6 (see \((4.61)\)), we have
\[
\tilde{q}(y) := \phi_i(Z_i^*(v, q(y))) \in U(y) \setminus D(M, X, c),
\]
for every \(y \in B_2\).

(Construction of \(r_1\) and \(r_2\)). In order to construct \(r_1, r_2\) as in Theorem 4.7, we consider
\[
\lambda := \max_{1 \leq i \leq p} \sup_{(w, y) \in O_i(\delta_i^2)} \left\| (R_i(w, y))^{-1} \right\|,
\]
which is finite since for every \(l\)
\[
\left\| (R_l(w, y))^{-1} \right\| \leq \frac{\text{Const}}{z^*},
\]
and \(\text{Const}\) is a constant that depends only on the supremum of the norm of \(\frac{\partial \phi_i}{\partial v}\) over a compact subset of \(\mathbb{R}^d \times \mathbb{R}^m\). The regularity assumption \(iii)\) allows us to define
\[
\mu := \sup \left\{ \left\| \frac{\partial c}{\partial x}(u', x') \right\| : u' \in M, \ d(u', D(M, X, c)) \geq \rho, \ x' \in K^{\rho/2} \right\},
\]
and
\[
\omega(h, k) := \max_{1 \leq l \leq p} \sup \left\{ \left\| \frac{\partial \phi_i}{\partial v}(v, y) - \frac{\partial \phi_i}{\partial v}(v', y') \right\| : |v - v'| \leq h, |y - y'| \leq k, \ v, v' \in N_l, \ y, y' \in K^{\rho/2} \right\}, \ h, k \geq 0.
\]
Let us fix

\[
\begin{cases}
\ r_2 = \beta r_1, \ \beta = \frac{1}{2 \mu}, \\
\ r_1 < \min \left( \frac{\mu_1}{\rho_i}, \frac{\mu_2}{\eta_i}, \frac{\mu_3}{\xi_i}, \frac{\mu_4}{\tau_i} \right).
\end{cases}
\] (4.82)

Notice that the modulus \( \omega \) defined in (4.81) is a decreasing function of \( h \) and \( k \) and its limit for \( h, k \to 0 \) is zero, so we may choose \( r_1 \) so that

\[
\omega(r_1, \beta r_1) \leq \frac{1}{2 \lambda \sqrt{d_j}}.
\] (4.83)

Let us verify the inequalities in (4.65) with \((v_0, y_0) = ((v_i)_x, x)\):

\[
\sup_{y \in B_2} |F(v_0, y)| = \sup_{y \in B_2} |c(u, y) - c(u, x)| \leq \mu r_2 = \frac{r_1}{2 \lambda} \leq \frac{r_1}{2 \parallel T_0 \parallel}, \quad T_0 = (R_i(u, x))^{-1}
\] (4.84)

where we used \( d(u, D(M, X, c)) \geq \rho \) and \( x + \eta(y - x) \in K^{\sigma/2} \), for every \( y \in B_2 \) and \( 0 \leq \eta \leq 1 \). Furthermore for any \((v, y) \in B_1 \times B_2\) we have

\[
\begin{align*}
\left\| I - T_0 \frac{\partial F}{\partial v}(v, y) \right\| &\leq \| T_0 \parallel \frac{\partial \phi_i}{\partial v}(v_i, x) - \frac{\partial \phi_i}{\partial v}(Z^{\phi_i}(v_i, v), y) \| T_i \| \\
&\leq \lambda \sqrt{d_j} \omega(r_1, \beta r_1) \leq \frac{1}{2}.
\end{align*}
\] (4.85)

This inequality follows by the definition of \( \omega \) and by (4.83): in fact \( v_i, Z^{\phi_i}(v_i, v) \in N_i \), and \( r_2 < \frac{\sigma}{2} \) implies \( x, y \in K^{\sigma/2} \). This proves (4.65) and justifies the application of Theorem 4.7.

The radius \( r_2 \) depends only on \( K \) and \( \sigma \), and it does not depend on the particular choice of \((u, x)\).

**(Approximation of the control \( u \)).** Again we need to distinguish between the alternatives (A) and (B) in ii):

**Case (A).** Let \( K \) be the image of the arc \( \gamma \) and \( r_0 = r_2(K, \sigma^*) \). We consider \( s > t \) as in (4.43), where \( x = \gamma(t) \), and we have to approximate \( u \) by an admissible control for the state \( \gamma(s) \). To this purpose we apply Proposition 4.8 to the pair \( F, q \) obtained in the previous step. With the choice (4.82) for \( r_1 \), let

\[
D_1 := \left\{ v \in \mathbb{R}^j : |v - (v_i)_x| < 2r_1 \right\}
\] (4.86)

\( D_2 \) is the interior of \( B_2 \).

For any \( v \in D_1 \) and \( y \in D_2 \),

\[
|\phi_i(Z^{\phi_i}(v_i, v)) - u_i| \leq |\phi_i(Z^{\phi_i}(v_i, v)) - \phi_i(v_i)| + |u - u_i| < 2L_r r_1 + \frac{\delta_i}{2} \leq \delta_i,
\]

\[
|y - x_i| \leq r_2 + |x - x_i| < \frac{\delta_i}{2} + \frac{\delta_i}{2} \leq \delta_i.
\]

This implies

\[
(\phi_i(Z^{\phi_i}(v_i, v)), y) \in O_i(\delta_i), \quad \forall (v, y) \in D_1 \times D_2.
\] (4.87)

Therefore by (4.74), the matrix

\[
\frac{\partial F}{\partial v}(v, y) = R_i(\phi_i(Z^{\phi_i}(v_i, v)), y)
\] (4.88)
is invertible over $D_1 \times D_2$, and $q(D_2) \subset B_1 \subset D_1$. By applying Proposition \ref{prop:invertibility} we deduce that $q \in C^1(D_2)$. Since $\gamma([t, s]) \subset D_2$, for any $\xi \in [s, t]$, by \eqref{eq:linearization}

$$\tilde{q}(\gamma(\xi)) \in U(\gamma(\xi)) \backslash \mathcal{D}(\mathcal{M}, X, c).$$

(4.89)

The assumption \textit{iv)} implies

$$\left\| \frac{\partial \tilde{q}}{\partial y}(\gamma(\xi)) \right\| \leq \left\| \left[ R_t(\tilde{q}(\gamma(\xi)), \gamma(\xi)) \right] ^{-1} \frac{\partial c}{\partial x}(\tilde{q}(\gamma(\xi)), \gamma(\xi)) \right\|

= \left\| \left( d_{\tilde{q}(\gamma(\xi))} c(\cdot, \gamma(\xi)) \circ T_{x, \xi}^{\tilde{q}(\gamma(\xi))} \right)^{-1} \circ d_{\gamma(\xi)} c(\tilde{q}(\gamma(\xi)), \cdot) \right\|

\leq \mathcal{T}(\tilde{q}(\gamma(\xi)), \gamma(\xi)) \leq \tau.$$  

(4.90)

In fact the matrix associated to the linear operator $d_{\tilde{q}(\gamma(\xi))} c(\cdot, \gamma(\xi)) \circ T_{x, \xi}^{\tilde{q}(\gamma(\xi))}$, with respect to the canonical basis of $\mathbb{R}^2$, is $R_t(\tilde{q}(\gamma(\xi)), \gamma(\xi))$. Using (4.90) we obtain

$$|\tilde{q}(\gamma(s)) - u| \leq \text{Lip}_M|q(\gamma(s)) - q(\gamma(t))|

\leq \tau \text{Lip}_M \int_t^s |\gamma'(\xi)| d\xi \leq \tau_t \text{Lip}_M.$$  

(4.91)

This inequality proves that

$$u \in (U(\gamma(s)))^{\tau'}, \quad \forall \tau' > \text{Lip}_M \tau_{t,s}.$$  

(4.92)

Since $r_2$ is independent of $(u, x)$, we have

$$U(\gamma(t)) \backslash (\mathcal{D}(\mathcal{M}, X, c)))^\rho \subset (U(\gamma(s)))^{\tau'}.$$  

(4.93)

\textbf{Case (B).} Let $K$ be a compact subset of $X$, which has the property (\textit{CON}), and $r_\sigma = \min(\sigma, \frac{r_2(K, \sigma)}{a(K)})$. Let $x, y \in K$ be such that $|x - y| < r_\sigma$. Then we consider a regular arc $\gamma : [w_1, w_2] \to K$ as in the Definition \ref{def:regulararc}, which connects $x$ to $y$ and lies in $K$. For every $\xi \in [w_1, w_2]$, we have

$$|\gamma(\xi) - \gamma(w_1)| \leq l(\gamma) \leq a(K)|x_1 - x_2| < r_2,$$

which implies

$$\gamma([w_1, w_2]) \subset B_{r_2}(\gamma(w_1)).$$

Hence by the same arguments developed in the previous case, we deduce

$$U(x) \backslash (\mathcal{D}(\mathcal{M}, X, c)))^\rho \subset (U(y))^{\tau'}, \quad \forall \tau' > a(K)\tau \text{Lip}_M|x_1 - x_2|.$$  

(4.94)

\textbf{(Conclusions).}

\textbf{(A)} Using the inclusion (4.93) and the definition of $\rho$ in (4.70), we infer

$$U(\gamma(t)) \subset (U(\gamma(s)))^{\tau'}.$$  

(4.95)

In fact, in this case, $\rho = \sigma^*$, which is defined in Remark 5.8 so that $U(x) \subset (\mathcal{D}(\mathcal{M}, X, c))^\sigma^*$ for every $x \in X$. 

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(B) Let \( x, y \in K \), \( u \in U(x) \), with \( d(u, D(M, X, c)) < \rho = \rho_\sigma \). By Lemma 4.6, we find \( u_\sigma \in U(x) \), with \( d(u_\sigma, D(M, X, c)) \geq \rho \), such that
\[
|u - u_\sigma| \leq \sigma.
\]
Hence, using (4.94), there exists \( v_\sigma = U(y) \) such that
\[
d(u_\sigma, D(M, X, c)) \geq \rho,
\]
and
\[
U(x) \subset (U(y))^{\tau' + \sigma}.
\]
This implies
\[
\delta(u, W, \psi) = \max_{(W, \psi): u \in W} \delta(u, W, \psi).
\]
Since every local chart \( \psi \) is Lipschitz continuous and \( M \) is compact in \( \mathbb{R}^n \), \( \delta(u) \) is finite for every \( u \in M \). We prove that the map \( \delta(\cdot) \) is uniformly bounded from below. Since \( M \) is compact, it suffices proving that \( \delta(\cdot) \) is lower semi-continuous, i.e. that the set
\[
\{ u \in M : \delta(u) > \alpha \}
\]
is open in \( M \), for any \( \alpha > 0 \). Suppose, for the time being, that the set
\[
\{ u \in W : \delta(u, W, \psi) > \alpha \}
\]
is open in \( M \) for any local chart \((W, \psi)\) in the atlas of \( M \) and for any \( \alpha > 0 \). Then if \( u \) is such that \( \delta(u) > \alpha \), there exists a local chart \((W, \psi)\) for \( u \) such that \( \delta(u, W, \psi) > \alpha \). Therefore, there exists \( \eta > 0 \) such that \( \delta(w, W, \psi) > \alpha \) for any \( w \in B_\eta(u) \cap W \). Hence \( \delta(w) > \alpha \) for any \( w \in B_\eta(u) \cap W \). Since \( W \) is open in \( M \), \( B_\eta(u) \cap W \) is open in \( M \), and therefore \( \delta(\cdot) \) is lower semi-continuous over \( M \). To prove that the set (4.100) is open in \( M \), let \( u \in W \) be such that \( \delta(u, W, \psi) > \alpha \) and \( \xi \in (0, \frac{\delta(u, W, \psi) - \alpha}{Lip(\psi)} \) and \( \eta \in (\alpha, \delta(u, W, \psi) - Lip(\psi) \xi) \), then
\[
B_\eta(\psi(w)) \subset \psi(W) \quad \forall w \in B_\xi(u) \cap W.
\]
This implies
\[
\delta(w, W, \psi) \geq \eta > \alpha \quad \forall w \in B_\xi(u) \cap W.
\]
Therefore, since \( W \) is open in \( M \), the assertion is proved.

Let
\[
\bar{\delta} = \min_{u \in M} \delta(u) > 0
\]
and \( \Gamma \) be the image of a regular arc \( \gamma : [w_1, w_2] \to X \), which connects two distinct points \( x_1, x_2 \in X \) and such that the property (CON) holds. Let \( \sigma > 0 \) be such that
\[
\Gamma^{\sigma} \subset A.
\]
Let \( x \in \Gamma \) and \( u \in U(x) \setminus \mathcal{D}(\mathcal{M}, X, c) \), \( \pi \in \Pi(u, x) \) and \( \phi_u : V \to \mathcal{M} \), the map fixed for \( u \) in Definition 3.9. We may assume, up to modifying from the beginning the choice of the local charts \( \{ \phi_u \}_{u \in M} \) that appear in Definition 3.9, that \( \phi_u \) is such that
\[
B_2(\phi_u^{-1}(u)) \subset V.
\]

Let \( \phi = \phi_u \), \( W = \phi(V) \) and \( F \) the map
\[
F(w, y) := \varphi(Z^\pi(v_u, w), y) - c(u, x) \in \mathbb{R}^j, \quad \forall (w, y) : Z^\pi(v_u, w) \in V, y \in A,
\]
with \( \varphi(\cdot, \cdot) := c(\phi(\cdot, \cdot)) \in C^1(V \times A) \) and \( \phi(v_u) = u \), then
\[
\frac{\partial F}{\partial w}(w, y) = \frac{\partial \varphi}{\partial v^\pi}(Z^\pi(v_u, w), y).
\]

Since \( F((v_u)_\pi, x) = 0 \) and using (4.103), the assumptions (4.64) are satisfied. Therefore we can apply Theorem 4.7, with \( (v_0, y_0) := ((v_u)_\pi, x) \). The inequalities in (4.65) can be proved as follows, by assumptions iv) and v): let \( r_1 \) and \( r_2 \) be given by
\[
\begin{aligned}
& r_2 = \beta r_1, \quad \beta = \frac{1}{2\lambda_2} \\
& r_1 < \min\left( \frac{r_2}{\lambda_2}, \frac{1}{2L(1+\beta)\sqrt{d_\lambda}} \right)
\end{aligned}
\]
then since the matrix associated to the linear operator \( d_u c(\cdot, x) \circ T^u_\pi \), w.r.t. the canonical base of \( \mathbb{R}^j \), is
\[
T^{-1}_0 = \frac{\partial \varphi}{\partial v^\pi}(v_u, x),
\]
we obtain
\[
\begin{aligned}
\sup_{y \in B_2} |F(v_0, y)| &= \sup_{y \in B_2} |c(u, y) - c(u, x)| \\
\sup_{y \in B_2} ||d_y c(\cdot, \cdot)|| r_2 &\leq \mu r_2 = \frac{r_1}{2\lambda_2} \leq \frac{r_1}{2||T_0||}.
\end{aligned}
\]
As to the second inequality in (4.65), since
\[
|Z^\pi(v_u, w) - v_u| \leq r_1 \quad \forall w \in B_1
\]
and
\[
|y - x| \leq r_2 < r \quad \forall y \in B_2
\]
we have
\[
\|I_j - T_0 \frac{\partial F}{\partial w}(w, y)\| \leq \lambda \sqrt{d_j} \left\| \frac{\partial \varphi}{\partial v}(v_u, x) - \frac{\partial \varphi}{\partial v}(Z^\pi(v_u, w), y) \right\| \leq \lambda \sqrt{d_j} L(r_1 + r_2) \leq \frac{1}{2}
\]
Moreover, by Proposition 1.8, the implicit function \( q : D_2 \to D_1 \) is \( C^1(D_2) \), where \( D_1 \) and \( D_2 \) are defined as in 1.86. Let \( t > s \), such that \( x = \gamma(t) \), and \( \gamma([t, s]) \subset D_2 \), we derive the inequality (4.91), as in the proof of Lemma 4.7. Since \( r_2 \) and \( r_1 \) do not depend on \( u \), we deduce the inclusion
\[
U(\gamma(t)) \setminus \mathcal{D}(\mathcal{M}, \Gamma, c) \subset (U(\gamma(s)))^{\tau'}, \quad \forall \tau' > \tau_{t,s} \text{ Lip}_\mathcal{M}.
\]
Taking the closure, by assumption (B), we conclude that
\[
U(\gamma(t)) \subset (U(\gamma(s)))^{\tau'}, \quad \forall \tau' > \tau_{t,s} \text{ Lip}_\mathcal{M}.
\]
In the case (A), we can repeat the argument used in the proof of Theorem 4.1 substituting \( r_2 \) for \( r_0 \). This brings to the inequality (4.50) which, as before, implies

\[
d_H(U(x_1), U(x_2)) \leq a(X)\tau Lip_M|x_1 - x_2|.
\]

This proves the Lipschitz regularity of \( U(\cdot) \).

\[\]

5 An Application to a Financial Problem

We discuss here an application of the results in the previous sections to an optimal asset-allocation problem. More precisely we consider an optimal asset-liability management model in presence of constraints: the company can manage the investment coming from the policy-holders’ payments, in order to satisfy several regulatory and solvency constraints and to achieve a given objective. The company can decide, at each time step, how to distribute the total wealth between the available assets to achieve its goal, but he has to obey a number of constraints. Furthermore, each portfolio adjustment entails transaction costs, since it means selling part of an asset to provide either liquidity or a different asset (see [17], [18] for a detailed description of the financial model).

We assume, for the sake of simplicity, that the manager can choose at each time step \( t_0, \ldots, t_{N-1} \), between a riskless and a risky investment, denoted by \( B \) and \( S \), respectively, though the procedure described applies also to the more general case of \( n \) possible investments characterized by different values of yield and volatility.

We assume that the transaction costs of moving wealth between the sections are paid only on buying and not on selling and that these transaction costs are linearly proportional to the size of the transaction. The evolution equations for the amounts invested in stocks and bonds are

\[
\begin{align*}
S_{k+1} &= S_k(1-u_k) + B_kv_k(1-\lambda^s)Y_k^s, \quad S_0 = s_0 > 0 \\
B_{k+1} &= B_k(1-v_k) + S_ku_k(1-\lambda^b)Y_k^b, \quad B_0 = b_0 > 0,
\end{align*}
\]

where for every \( 0 \leq k \leq N-1 \), \( (u_k, v_k) \in M_k \subset [0,1] \times [0,1] \) represent the percentage of risky investment that is moved to riskless investment and viceversa, at time \( t_k = k\Delta t \), \( Y_k^s, Y_k^b : \Omega \rightarrow (0, \infty) \) are random variables and \( 0 < \lambda^s, \lambda^b < 1 \) represent the transaction cost coefficients.

The model is self-financed in that it does not require the use of cash to perform the transactions. After each portfolio adjustment, the investments evolve according to the stochastic yields \( Y_k^s, Y_k^b \).

We assume that the joint process \( \{Y_k^s, Y_k^b\}_k \) is a discrete Markov chain which takes values in a sequence of finite discrete subspaces of \((0, \infty)^2\), \( \mathcal{Y}_k \). The chain is characterized by the density function:

\[
p_k(y^s, y^b) = P(Y_k^s = y^s, Y_k^b = y^b), \quad \forall (y^s, y^b) \in \mathcal{Y}_k.
\]

The company has to satisfy at each time-step a regulatory constraint that imposes a limit on the percentage of wealth invested in risky assets. More precisely, the adjustment must be such that the fund after the adjustment satisfies:

\[
\frac{B_k(1-v_k) + S_ku_k(1-\lambda^b)}{B_k(1-v_k) + S_ku_k(1-\lambda^s) + S_k(1-u_k) + B_kv_k(1-\lambda^s)} \geq \alpha
\]

(5.108)
i.e. the percentage of wealth invested in riskless assets, after the portfolio adjustment, is bigger than \( \alpha \). Furthermore, since \( Y_k \) is finite, we may require that the adjustment is such that the constraint is satisfied also at time \( t_{k+1} \),

\[
\frac{B_{k+1}}{B_{k+1} + S_{k+1}} \geq \alpha
\]

with \((S_{k+1}, B_{k+1})\) given by \((5.106)\).

It can be shown that this constraint is equivalent to the following inequality:

\[
c_k(u_k, v_k, S_k, B_k) := S_k(1-u_k) + v_k B_k(1-\lambda^s) - q_k[B_k(1-v_k) + u_k S_k(1-\lambda^b)] \leq 0. \tag{5.110}
\]

with

\[
q_k = \frac{1-\alpha}{\alpha} \min \left( 1, \min_{(y', y) \in Y_k} \frac{y^b}{y} \right).
\]

We would like to avoid that any of the two investments becomes null at some time. To this aim, we observe that, because of the structure \((5.106)\), if the initial allocation \((s_0, b_0)\) is such that \((s_0, b_0) \in X_0\), with

\[
X_0 = \{(S, B) : S, B \geq \Delta_0, \quad S + B \leq D_0\}, \tag{5.111}
\]

with \(D_0 > \Delta_0 > 0\) it is possible to define \( \Delta_k, D_k \) and \( \delta_k \) such that if the control space at time \( t_k \) is

\[
\mathcal{M}_k = [0, \delta_k] \times [0, \delta_k], \tag{5.112}
\]

then \((S_k, B_k) \in X_k\) with

\[
X_k := \{(S, B) : S, B \geq \Delta_k, \quad S + B \leq D_k\} \tag{5.113}
\]

and the admissible control space at time \( t_k \),

\[
\mathcal{U}_k(S, B) := \{(u, v) \in \mathcal{M}_k : c_k(u, v, S, B) \leq 0\} \tag{5.114}
\]

is nonempty for any \((S, B) \in X_k\).

By the structure \((5.110)\) of the constraints, if \( \mathcal{U}_k(S, B) \neq \emptyset \), then the control \((\delta, 0)\) must belong to \( \mathcal{U}_k(S, B) \) for some \( \delta \in (0, 1) \). Therefore, let \( \delta_k \in (0, 1) \) be such that

\[
c_k(\delta_k, 0, S, B) = S(1-\delta_k) - q_k[B + \delta_k S(1-\lambda^b)] \leq 0, \quad \forall S, \ B > 0,
\]

and therefore

\[
\delta_k = \sup_{S, B > 0} \frac{S - q_k B}{S(1-\lambda^b) + 1} = \frac{1}{q_k(1-\lambda^b) + 1} \in (0, 1), \quad \forall k \geq 0. \tag{5.115}
\]

Assuming \((S_k, B_k) \in X_k\) we can recursively compute \( \Delta_{k+1} \) and \( D_{k+1} \) that define \( X_{k+1} \). Since \( S_k, B_k \geq \Delta_k > 0 \) and \( u, v \in \mathcal{M}_k \), we have

\[
S_{k+1} \geq \Delta_k(1-\delta_k)y_k
\]

\[
B_{k+1} \geq \Delta_k(1-\delta_k)y_k^b,
\]

where

\[
y_k := \min \{(\min(y^s, y^b) : (y^s, y^b) \in \mathcal{Y}_k) \}
\]

Therefore we define

\[
\Delta_{k+1} := \Delta_k(1-\delta_k)\min(y_k^s, y_k^b).
\]
To obtain $D_{k+1}$ as a function of $D_k$, we observe that by equations (5.106), if $S_k + B_k \leq D_k$, for every possible choice of the controls in $M_k$, we have

$$S_{k+1} + B_{k+1} \leq D_k \overline{y}_k,$$

where

$$\overline{y}_k := \max \{ \max(y^a, y^b) : (y^a, y^b) \in Y_k \},$$

that is

$$D_{k+1} = D_k \overline{y}_k, \quad \forall k \geq 0.$$

We have proved that if $(s_0, b_0) \in X_0$, with $X_0$ as in (5.111), equations (5.106) map $X_k \times M_k \times Y_k$ into $X_{k+1}$, being $X_k$ defined as in (5.113). Furthermore the admissible control set defined in (5.114) is non empty for every $(S, B) \in X_k$.

As in section 1, we want to find an optimal investment strategy which maximizes the expected value of a given utility function $g$ at time $t_N$.

Since $M_k = [0, \delta_k] \times [0, \delta_k]$ is not a Lipschitz manifold in the sense specified by Definition 3.2, we decompose it as the union of infinitely many subsets, which are 1-dimensional Lipschitz manifolds. We can apply Theorem 4.2 to these manifolds and then extend the result to $M_k$.

**Proposition 5.1** The map (5.114) is $d_H$-Lipschitz continuous over $X_k$, for every $k \geq 0$. Moreover, if $g : [0, +\infty) \times [0, +\infty) \to \mathbb{R}$ is Lipschitz continuous, the value function $J_k(S, B)$, obtained via the DP algorithm (1.6) is Lipschitz continuous over $X_k$, for any $k = 0, \ldots, N$.

**Proof.** We consider, for every $0 \leq \delta < \delta_k$, the rectangle $R_{\delta,k} = [\delta, \delta_k] \times [0, \delta_k]$, and we define

$$M_{\delta,k} = \partial R_{\delta,k},$$

and

$$U_{\delta,k}(S, B) := U_k(S, B) \cap M_{\delta,k},$$

for every $(S, B) \in X_k$. We prove, by applying Theorem 4.2, that the multifunctions (5.117) are Lipschitz continuous over $X_k$ and that their Lipschitz constant does not depend on the parameter $\delta$.

Since the space $X_k$ is convex, $a_k = a(X_k) = 1$; furthermore $M_{\delta,k}$ is obviously a Lipschitz manifold in $\mathbb{R}^2$ of dimension $d = 1 = j$.

We want to apply Theorem 4.2 to derive the Lipschitz regularity of $U_{\delta,k}$. The assumption i) in Theorem 4.1 is a consequence of the fact that $U_{\delta,k}(S, B)$ contains the control $(\delta, 0)$, for every $(S, B) \in X_k$ and $\delta$. Now let us observe that

$$D(M_{\delta,k}, X_k, c_k) = \{(\delta, 0), (\delta_k, 0), (\delta_k, \delta_k), (\delta, \delta_k)\}. \quad (5.118)$$

Let $(S, B) \in X_k$ and $(u, v) \in M_{\delta,k} \backslash N\mathcal{R}(M_{\delta,k})$, the inverse of a local chart for $(u, v)$ is

$$\phi(t) := \begin{cases} 
(t, 0) & \delta < t < \delta_k, \text{ if } v = 0 \\
(\delta_k, t) & 0 < t < \delta_k, \text{ if } u = \delta_k \\
(t, \delta_k) & \delta < t < \delta_k, \text{ if } v = \delta_k \\
(\delta, t) & 0 < t < \delta_k, \text{ if } u = \delta.
\end{cases} \quad (5.119)$$
The constraint function $c_k$ is regular ($\mathcal{N}(c_k) = \emptyset$) and the Jacobian matrix at $(u, v)$ of the restriction of $c_k$ to the manifold $M_{\bar{\delta}, \bar{k}}$ is, by (5.111),

$$J_{c_k}(u, v) = \begin{cases} -S[1 + q_k(1 - \lambda^s)] & \text{if } v = 0 \text{ or } v = \delta_k \\ B(1 - \lambda^s + q_k) & \text{if } u = \delta \text{ or } u = \delta_k \end{cases}.$$ (5.120)

Therefore $\Pi(u, v, S, B)$ is non empty. This proves the assertion (5.118).

Now, $D(M_{\bar{\delta}, \bar{k}}, X_k, c_k)$ is closed and, if $(u, v) \in D(M_{\bar{\delta}, \bar{k}}, X_k, c_k)$ is an admissible control for $(S, B)$, it can be approximated by

$$(u_\varepsilon, v_\varepsilon) = \begin{cases} (\delta + \varepsilon, 0) & \text{if } (u, v) = (\delta, 0) \\ (\delta_k, \delta_k - \varepsilon) & \text{if } (u, v) = (\delta_k, \delta_k) \\ (\delta + \varepsilon, \delta_k) & \text{if } (u, v) = (\delta, \delta_k) \end{cases}$$

for $0 < \varepsilon < \delta_k, \delta_k - \delta$. In fact $(u_\varepsilon, v_\varepsilon) \in U_{\delta, k}(S, B)\mathcal{D}(M_{\delta, k}, X_k, c_k)$, as follows by the monotonicity of the constraint function $c_k$, w.r.t. $u$ and $v$. If $u = \delta_k$ and $v = 0$, then for every $(S, B) \in X_k$, by (5.117) it holds

$$c_k(u, v, S, B) = -q_k B \leq -q_k \Delta_k < 0,$$

therefore, by continuity, we can approximate $(\delta, 0)$ with admissible controls of $U_{\delta, k}(S, B)\mathcal{D}(M_{\delta, k}, X_k, c_k)$.

In other words the assumption $ii) - (B)$ of Theorem 4.1 holds true.

To prove that the assumptions $iv), v)$ of Theorem 4.2 hold true, we observe that

$$\frac{\partial c_k}{\partial S} \in [0, 1], \quad \frac{\partial c_k}{\partial B} \in [-q_k - \delta_k(q_k + 1 - \lambda^s)].$$ (5.121)

Therefore, the assumption $iv)$ follows by choosing

$$\lambda_k := \min \left( \frac{\delta_k}{\Delta_k}, \frac{1}{\Delta_k(1 - \lambda^s + q_k)} \right)$$ (5.122)

$$\mu_k := \sqrt{1 + \max (q_k, |\delta_k(q_k + 1 - \lambda^s) - q_k|) ^2}$$ (5.123)

and for any $r > 0$.

The assumption $v)$, follows by the linear dependence on the state of the system of (5.121). We can construct an atlas over $M_{\delta, k}$ by taking the parametrization (5.119) for the points which are not vertices and by defining compatible charts on the vertices so that

$$\text{Lip}_{M_{\delta, k}} \leq 1.$$

Therefore, Theorem 4.2 implies that $U_{\delta, k}(\cdot, \cdot)$ is Lipschitz continuous over $X_k$ and its Lipschitz constant is estimated by $\lambda_k \mu_k$. Now observe that

$$U_k(S, B) = \bigcup_\delta U_{\delta, k}(S, B),$$

for every $(S, B) \in X_k$. Therefore by the definition of the metric $d_H$, we have

$$U_{\delta, k}(S, B) \subset (U_{\delta, k}(S', B'))^c \subset (U_k(S', B'))^c,$$

which yields

$$U_k(S, B) \subset (U_k(S', B'))^c,$$
for every \((S, B), (S', B') \in X_k\) and \(\sigma > \lambda_k \mu_k |(S - S', B - B')|\). This implies the Lipschitz regularity of \(U_k(\cdot, \cdot)\).

To prove the Lipschitz regularity of \(J_k\), it suffices proving that assumption 2) in Theorem 4.4 holds true. In fact we can repeat the proof of Theorem 4.4 by replacing assumption 1) by the Lipschitz regularity of \(U_k(\cdot, \cdot)\).

By (5.106), we have

\[
f_k(S, B, u, v, y^s, y^b) = \left( |S(1 - u) + Bu(1 - \lambda^s)|y^s, |B(1 - v) + Su(1 - \lambda^b)|y^b \right),
\]

for each \((S, B, u, v, y^s, y^b) \in X_k \times M_k \times Y_k\). Using the linearity and (5.113), it can be proved that \(f_k(\cdot, \cdot, \cdot, \cdot, \cdot, y^s, y^b)\) is Lipschitz continuous over \(X_k \times M_k\) and its Lipschitz constant is given by

\[
\sqrt{1 + D_k^2 + (1 - \min(\lambda^s, \lambda^b))^2 \max(y^s, y^b)},
\]

which is \(p_k\)-integrable.

\[\Box\]

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