Incoherent coherences

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Abstract

This article explores a generic framework of well-typed and well-scoped syntaxes, with a signature-axiom approach resembling traditional abstract algebra. The boilerplate code needed in defining operations on syntaxes is identified and abstracted away. Some of the frequent boilerplate proofs are also generalized.

This is a literate Agda file, meaning that the code below is all actually checked and automatically typeset by Agda. The complete source code can be found at https://github.com/Trebor-Huang/STLC.

1 Conor’s Exercise

module Conor’s-Exercise where

It all starts when one tries to implement typed $\lambda$-calculus in a language with dependent types.

data $T$ : Set where
  $i$ : $T$
  _$\rightarrow$_ : $T \rightarrow T \rightarrow T$
infixr 15 _$\rightarrow$_

Note that we use the same glyph $\rightarrow$ for the Agda function space and the $\lambda$-calculus function space. The colors are different, but even without the colors, it should be clear from context.

A context (no pun intended) is simply a snoc-list:

Context = List $T$
_ : Context
_ = $\emptyset$ $\bullet$ $\emptyset$ ($t \rightarrow i$)

We use _$\bullet$_ for a visual reminder that it is a snoc-list.

In this way, we can ensure that our variables are well-typed:
Now we can define well-typed terms:

\begin{align*}
\text{infix} & 5 \vdash \\
\text{data } \vdash \colon & \text{Context } \to \mathbb{T} \to \text{Set where} \\
\text{var} : & \Gamma \ni \sigma \to \Gamma \vdash \sigma \\
\text{app} : & \Gamma \vdash \sigma \to \tau \to \Gamma \vdash \sigma \to \Gamma \vdash \tau \\
\text{lam} : & \Gamma \triangleleft \sigma \vdash \tau \to \Gamma \vdash \sigma \to \tau
\end{align*}

The type system of Agda now ensures that every term is well-typed, blurring the distinction between syntax — what terms are well-formed, and semantics — what meanings the terms have. Let’s look at some examples:

\begin{align*}
I : \emptyset & \vdash \sigma \to \sigma \\
I &= \text{lam} (\text{var} z) \\
K : \emptyset & \vdash \sigma \to \tau \to \sigma \\
K &= \text{lam} (\text{lam} (\text{var} (s z)))
\end{align*}

So far so good. But just defining terms is surely not enough, we need to be able to manipulate them. The most fundamental operation on syntax with variable bindings, is substitution:

\begin{align*}
\text{Substitution} : & \text{Context } \to \text{Context } \to \text{Set} \\
\text{Substitution} & \Gamma \Delta = \forall \{ \sigma \} \to \Gamma \ni \sigma \to \Delta \vdash \sigma \\
\text{Transformation} : & \text{Context } \to \text{Context } \to \text{Set} \\
\text{Transformation} & \Gamma \Delta = \forall \{ \sigma \} \to \Gamma \vdash \sigma \to \Delta \vdash \sigma
\end{align*}

Let’s go right into it:

\begin{align*}
\text{subs}_1 : & \text{Substitution } \Gamma \Delta \to \text{Transformation } \Gamma \Delta \\
\text{-- subscript}_1 \text{ for "first attempt"} \\
\text{subs}_1 s (\text{var} i) & = s i \\
\text{subs}_1 s (\text{app} t_1 t_2) & = \text{app} (\text{subs}_1 s t_1) (\text{subs}_1 s t_2)
\end{align*}

The powerful type system helps us write the program: By asking Agda to do a case-split on the term being transformed, we immediately get the required branches. Agda can also automatically generate the correct term to write in the first two branches. But in the third branch there is a problem:

\begin{align*}
\text{subs}_1 s (\text{lam} t) & = \text{lam} (\text{subs}_1 \{! !\} t)
\end{align*}

We need a way to “push in” an extra variable. And here we go:

\begin{align*}
\text{push}_1 : & \text{Substitution } \Gamma \Delta \to \text{Substitution } (\Gamma \triangleleft \sigma) (\Delta \triangleleft \sigma) \\
\text{push}_1 \sigma z & = \text{var} z \\
\text{push}_1 \sigma (s i) & = \{! !\}
\end{align*}
This in turn requires us to weaken a term:

\[\text{weaken}_1 : \Gamma \vdash \sigma \rightarrow \Gamma \triangleright \tau \vdash \sigma\]

\[\text{weaken}_1 (\text{var } i) = \text{var } (s_i)\]

\[\text{weaken}_1 (\text{app } t_1 t_2) = \text{app } (\text{weaken}_1 t_1) (\text{weaken}_1 t_2)\]

\[\text{weaken}_1 (\text{lam } t) = \text{lam } \{! !\}\]

The first two cases are still easy, but the \text{lam} case is problematic. We need to push in yet another variable!

It turns out that we need to do this in two steps. First, we deal with variable renamings only:

\[\text{Renaming} : \text{Context} \rightarrow \text{Context} \rightarrow \text{Set}\]

\[\text{Renaming } \Gamma \Delta = \forall \{\sigma\} \rightarrow \Gamma \triangleright \sigma \rightarrow \Delta \triangleright \sigma\]

\[\text{weaken}_r : \text{Renaming } \Gamma \Delta \rightarrow \text{Renaming } (\Gamma \triangleleft \sigma) (\Delta \triangleleft \sigma)\]

\[\text{weaken}_r z z = z\]

\[\text{weaken}_r (s_i) = s \_ i\]

\[\text{rename} : \text{Renaming } \Gamma \Delta \rightarrow \text{Transformation } \Gamma \Delta\]

\[\text{rename } r (\text{var } i) = \text{var } (r \_ i)\]

\[\text{rename } r (\text{app } t_1 t_2) = \text{app } (\text{rename } r t_1) (\text{rename } r t_2)\]

\[\text{rename } r (\text{lam } t) = \text{lam } (\text{rename } (\text{weaken}_r r) t)\]

The \text{lam} case now goes through. And we can finish off the development:

\[\text{weaken}_s : \text{Substitution } \Gamma \Delta \rightarrow \text{Substitution } (\Gamma \triangleleft \sigma) (\Delta \triangleleft \sigma)\]

\[\text{weaken}_s z z = z\]

\[\text{weaken}_s (s_i) = \text{rename } s \_ (s_i)\]

\[\text{subs} : \text{Substitution } \Gamma \Delta \rightarrow \text{Transformation } \Gamma \Delta\]

\[\text{subs } s \text{ (var } i) = s \_ i\]

\[\text{subs } s \text{ (app } t_1 t_2) = \text{app } (\text{subs } s t_1) (\text{subs } s t_2)\]

\[\text{subs } s \text{ (lam } t) = \text{lam } (\text{subs } (\text{weaken}_s s) t)\]

In retrospect, the reason that we have to do this in two steps, is that \(\_ \triangleright \_\) is defined in two steps: It requires \(\_ \triangleright \_\) in its definition.

However, comparing the two pairs of functions we can see some sort of pattern. It is called \textit{weakening-then-traversal} in exercise 19 of [1]. And let’s do that.

2 Abstraction and Generality

\[\text{module Abstraction (I : Set) where}\]
In this section, we will work with an abstract parameter $I$ instead of $T$ in the previous section. We can start by noticing the similarity in the type signature:

$$\text{Scope} = (\Gamma : \text{List } I) (i : I) \rightarrow \text{Set}$$
$$\text{Morph} = (\Gamma \Delta : \text{List } I) \rightarrow \text{Set}$$

$\vdash$ and $\ni$ both have type $\text{Scope}$. And $\text{Renaming}$, $\text{Substitution}$ and $\text{Transformation}$ all have type $\text{Morph}$. The name “scope” comes from [2].

In the untyped case, $I$ is simply the singleton type (“Untyped is uni-typed”). This, in abstract nonsense, makes $\text{Scope}$ the type of presheafs on the category of renamings. $\text{Morph}$ is then natural transformations between the presheafs.\footnote{Of course, we haven’t imposed the functorial laws yet, so they are better described as raw presheafs and natural transformations. I will not spell out the categorical details, since I’m not going to use the category-theoretic language in an essential way. More can be read at [4].}

Now we define the standard well-typed variables, which can be regarded as the image of the Yoneda embedding:

```haskell
infix 5 _∋_

data _∋_ : Scope where
  z : \Gamma \ni i \ni i
  s_ : \Gamma \ni i \rightarrow \Gamma \ni j \ni i

infixr 100 s_

\ni \ni
```

Next, some combinators that already emerged in the last section.

```haskell
infix 4 _⇒_

⇒ _ : Scope \rightarrow Scope \rightarrow Morph
(C \Rightarrow D) \Gamma \Delta = \forall \{i\} \rightarrow C \Gamma i \rightarrow D \Delta i

[ ] : Morph \rightarrow Set
[ C ] = \forall \{\Gamma\} \rightarrow C \Gamma \Gamma

infixr 3 _⇒⇒_

⇒⇒ _ : Morph \rightarrow Morph \rightarrow Morph
(C \Rightarrow⇒ D) \Gamma \Delta = C \Gamma \Delta \rightarrow D \Gamma \Delta

[ ] : Morph \rightarrow Set
[ C ] = \forall \{\Gamma \Delta\} \rightarrow C \Gamma \Delta
```

With these, we can redefine $\text{Renaming}$, $\text{Substitution}$ and $\text{Transformation}$ uniformly:

```haskell
module _ (C : Scope) where

Renaming Substitution Transformation : Morph

Renaming = \ni \Rightarrow \ni
```

Substitution \( = \mathcal{V} \Rightarrow \mathcal{G} \)
Transformation \( = \mathcal{G} \Rightarrow \mathcal{G} \)

In [2], a notion of “generic syntax” is built, and a datatype \( \text{Desc} \) is used to describe syntaxes in general. This is very much like building a datatype \( \text{Poly} \) to describe polynomial functors, and generating the initial algebras once and for all:

```haskell
module Generic where

data Poly : Set₁ where
  Const : Set → Poly
  Id    : Poly
  _⊕_  : Poly → Poly → Poly
  _⊗_  : Poly → Poly → Poly

-- interprets Poly into Functor (i.e. Set → Set)
interp : Poly → (Set → Set)
interp (Const B) A = B
interp Id A = A
interp (f ⊕ g) A = interp f A + interp g A
interp (f ⊗ g) A = interp f A × interp g A

data Fix (f : Poly) : Set where
    fix : interp f (Fix f) → Fix f
```

One can implement a generic \texttt{cata} on \( \text{Fix} \). This is left as an exercise for the reader. You can find the answer at [3].

In the same spirit, [2] described how to manipulate and reason about generic syntax. In abstract nonsense, this can be recast as initial algebras of functors in the aforementioned presheaf category. But apart from polynomial functors, we get to use more functors related to the binding structure. In particular, the weakening operation induces a new functor that corresponds to \texttt{lam}.

This way of describing things is cool, clean and intuitive. Since we build up a universe of syntaxes inductively, we have absolute control over them. And theorems etc. can be proven by induction on the syntaxes, which is also convenient. However, it is more suited to theorem proving than to practical programming: For the same reason, we almost always prefer using inductive datatypes to using \texttt{Fix}, even though they are equivalent in expressivity!

Therefore, I shall now tread this road less taken. And let us see what plight awaits us!

\(^2\)No, they are not. But non-dependent inductive datatypes can be represented with \texttt{Fix}, once it is appropriately extended to allow infinitary sums and products.

\(^3\)Also, with the Univalence Axiom, we can seamlessly transport all the theorems from there to our homemade syntax (e.g. as defined in our first section), thus taking the best part of both sides.
3 The Road Less Taken

Instead of building a universe on which we have absolute control, let’s choose to place restrictions on a large existing universe so we have minimum control. Now, all we know is that we have a Scope. What more can we say about it?

Well, let’s do weakening first:

```agda
record Weakening (C : Scope) : Set where
  field
    weaken : (V \rightarrow C) \Gamma \Delta \rightarrow \forall i \rightarrow (V \rightarrow C) (\Gamma \triangleleft i) (\Delta \triangleleft i)
open Weakening [] public
```

Note that the \{ \ldots \} tells Agda to treat this record type similarly to type-classes as in Haskell. We can declare instances of this typeclass, which will be automatically used to infer arguments.

A convenient infix:

```
_«_ = weaken
```

Sanity check: \( V \) itself can be weakened:

```
instance
  Vw : Weakening V
  Vw .weaken \rho \ i \ z = z
  Vw .weaken \rho \ i \ (s \ v) = s \ (\rho \ v)
```

Good. Next, we can start to extract the common pattern in the renaming and substitution process. If the code is rewritten with our combinators, the signature of `rename` would be something like \[ \mathbf{V} \Rightarrow \mathbf{C} \Rightarrow \mathbf{C} \]. And `subst` would be of type \[ \mathbf{V} \Rightarrow \mathbf{C} \Rightarrow \mathbf{C} \].

Interesting! So looking at the types only, we would naturally come to the generalization \[ \mathbf{V} \Rightarrow \mathcal{A} \Rightarrow \mathbf{C} \Rightarrow \mathbf{C} \]. Now of course, we can’t define this for arbitrary \( \mathcal{A} \). So what more would we need? We have an assignment of variables to \( \mathcal{A} \), and we are given an expression of \( \mathbf{C} \). We need to replace all the free variables in the expression according to the assignment. So we definitely need a conversion \[ \mathcal{A} \Rightarrow \mathbf{C} \] from the assigned \( \mathcal{A} \)'s to \( \mathbf{C} \)’s.

During the process, we also need to be able to push into binders. Therefore we also need to “weaken”. This brings us to the complete type signature:

```
\{ Weakening \mathcal{A} \} \rightarrow \{ \mathcal{A} \Rightarrow \mathbf{C} \} \rightarrow \{ \mathbf{V} \Rightarrow \mathcal{A} \Rightarrow \mathbf{C} \Rightarrow \mathbf{C} \}
```

Last but not least, we need to ensure that every variable is also a legitimate term, otherwise we wouldn’t have much to work with. Packing all of these up gives us another typeclass:

```agda
record Syntax (C : Scope) : Set where
  field
```
\[
\text{var} : [ \mathcal{V} \Rightarrow \mathcal{C} ]
\]
\[
\text{map} : \{ \text{Weakening} \ \mathcal{A} \} \rightarrow [ \mathcal{A} \Rightarrow \mathcal{C} ] \rightarrow [ \mathcal{V} \Rightarrow \mathcal{A} = \Rightarrow \mathcal{C} \Rightarrow \mathcal{C} ]
\]

Now we can implement renaming and substitution based on the typeclass methods, and when we are using them in practice, we only need to provide the implementation of \text{var} and \text{map}. Sweet!

Renaming is a piece of cake:

\[
\begin{align*}
\text{rename} &: [ \mathcal{V} \Rightarrow \mathcal{V} \Rightarrow \mathcal{C} \Rightarrow \mathcal{C} ] \\
\text{rename} &= \text{map var}
\end{align*}
\]

Note that since we already told Agda that \mathcal{V} has weakening, we don’t need to mention that at all here.

Next, before we start to implement substitution, we need:

\[
\begin{align*}
\text{Syntax}^w &: \text{Weakening} \ \mathcal{C} \\
\text{Syntax}^w . \text{weaken} \ \sigma \ i \ z &= \text{var} \ z \\
\text{Syntax}^w . \text{weaken} \ \sigma \ i \ (s \ v) &= \text{rename} \ s_\_ \ (\sigma \ v)
\end{align*}
\]

For technical reasons, it is not very convenient for us to make it an instance. So for substitution, we will tell Agda about \text{Weakening} manually.

\[
\begin{align*}
\text{subst} &: [ \mathcal{V} \Rightarrow \mathcal{C} \Rightarrow \mathcal{C} \Rightarrow \mathcal{C} ] \\
\text{subst} &= \text{map id} \\
\quad \text{where instance } _\_ &= \text{Syntax}^w
\end{align*}
\]

And there you have it. Finally, since we often need to substitute only one variable (the rightmost one in the context), we make a little helper function:

\[
\begin{align*}
\text{z/\_} &: \mathcal{C} \ \Gamma \ i \rightarrow (\mathcal{V} \Rightarrow \mathcal{C}) \ (\Gamma \circ \ i) \ \Gamma \\
(z/\ t) \ z &= t \\
(z/\ t) \ (s \ v) &= \text{var} \ v \\
\text{infixr} \ 6 \ z/\_
\end{align*}
\]

Let’s see how things goes if we rewrite the simply typed \(\lambda\)-calculus with the tools we just developed. First, we need to show that \mathcal{V} itself is an instance of \text{Syntax}:

\[
\begin{align*}
\text{instance} \\
\mathcal{V}^* &: \text{Syntax} \ \mathcal{V} \\
\mathcal{V}^* . \text{var} &= \text{id} \\
\mathcal{V}^* . \text{map} \ f \ \sigma \ v &= f \ (\sigma \ v)
\end{align*}
\]

Recall that we worked in a parameterized module. Now we instantiate the parameter with the concrete type \(\mathbb{T}\).
open Abstraction T
infix 5 _⊢_
data _⊢_ : Scope where
  _-_ : Γ ⊢ i → j → Γ ⊢ j
  _λ_ : Γ ⊢ i ⊢ j
infixl 20 _·_
infixr 10 _λ_
infixr 100 _v_

 Again, we used λ for both Agda functions and λ-calculus functions. Now we give the implementation for the Syntax structure:

 instance
   _T_ : Syntax _T_
   _T_.var = _v_

 The var case is easy. What about map?

 _T_.map = helper
 where
   helper : { | Weakening _ _ |}
   → [ _ _ ⇒ _ T _ ]
   → [ _ V ⇒ _ _ ⇒ _ _ ⇒ _ _ ⇒ _ T _ ]

 Agda helps us fill in the goals pretty easily:

 helper f σ (v v) = f (σ v)
 helper f σ (t · s) = helper f σ t · helper f σ s
 helper f σ (λ t) = λ helper f (σ « _ _ ) t

 We now get substitution for free! For example:

 A : ⊘ ⊢ i ⊢ i ⊢ i
 A = λ v s z · (v s z · v z)
 B : ⊘ ⊢ i ⊢ i
 B = λ v z

 A[B] = subst (z/ B) A
 _ _ : A[B] ≡ λ (λ v z) · ((λ v z) · v z)
 _ _ = refl
4 Higher-order Homomorphisms

module HoHom (I : Set) where

From this section on, we move from doing things to proving things. For instance, in a classic proof of the Church-Rosser theorem, a technique is used that “colors” specific λ’s to track their behavior in reduction. An indispensable lemma of that proof is that substitution commutes with color erasure. Suppose Λ is the set of uncolored terms, and Λ is the set of terms with color. \( \lfloor \cdot \rfloor : \Lambda \to \Lambda \) erases the colors. Then the lemma to be proved is

\[
\lfloor t(x \mapsto s) \rfloor = \lfloor t \rfloor(x \mapsto \lfloor s \rfloor).
\]

This looks suspiciously similar to homomorphisms in abstract algebra. For example a group homomorphism \( \phi \) satisfies

\[
\phi(x \cdot y) = \phi(x) \cdot \phi(y).
\]

One crucial difference: The definition of group homomorphisms is directly concerned about group multiplication, which is a “primitive” concept; On the other hand, substitution is defined in terms of \( \text{var} \) and \( \text{map} \). So it is clear that we need to come up with an equation in terms of these two primitives.

record Hom (\| \cdot : Syntax C \| | \cdot : Syntax D \}) (f : [C \Rightarrow D]) : Set where

private instance

_ = Syntax \( w \) (\| \cdot \|) = Syntax \( w \) (\| \cdot \|)

field

Hvar : f \{ \Gamma \} \{ i \} (\text{var} \ v) ≡ \text{var} \ v

But there is some difficulty for \( \text{map} \), because it is a higher-order function, acting on other functions. How do we write out the homomorphism requirements for high-order structures? For concreteness, let’s take an example.

We define a \( \text{fixoid} \) to be a set \( X \) together with a functional \( F : (X \to X) \to X \) on it. This has no practical use whatsoever,\(^4\) and is only intended as an example for us to discuss what higher-order homomorphisms should be like.

\(^4\)Actually, if we impose that \( F \) sends functions to their fixpoints:

\[
F(f) = f(F(f)),
\]

then this becomes what is called a \textbf{fixed-point space}. It is trivial for discrete spaces, but once we impose a topology it becomes interesting. For example, any closed interval endowed with the real topology is a fixed-point space.
Take two fixoids, \((X, F)\) and \((Y, G)\). What makes a function \(\phi : X \to Y\) a homomorphism? The first reasonable guess is that we have an equation of the form
\[
\phi(F(?)) = G(?).
\]
What should we put in the arguments? There should be an \(f : X \to X\) and a \(g : Y \to Y\). But surely they can't be arbitrary. What should be the relation between them? Looking at the type signatures, let's go for the obvious:
\[
\forall x, \quad \phi(f(x)) = g(\phi(x)).
\]
So, if we collect everything, we get the definition:

A **homomorphism** between fixoids \((X, F)\) and \((Y, G)\) is a function \(\phi : X \to Y\) such that for every pair of functions \(f : X \to X\) and \(g : Y \to Y\),
\[
\phi \circ f = g \circ \phi \implies \phi(F(f)) = G(g).
\]
In fact, any other "reasonable" conditions we may impose are already implied by this one! The proof is quite combinatorial, and is left as an exercise for the reader.

## 5 Incoherent Coherences

In a similar spirit, after meditating at the type signature for a while, one may come up with such a condition for \texttt{map}:

\[
\texttt{Hnat : } \{ w : \texttt{Weakening } w \} \ (\delta : [ w \Rightarrow c ])
\to \forall \{ \Gamma \Delta \} \ (\sigma : (\mathcal{V} \Rightarrow w) \ \Gamma \ \Delta \ \{i\}) \ (v : c \ \Gamma i)
\to f(\texttt{map } \delta \sigma v) \equiv (\texttt{map } (f \circ \delta) \sigma) (f v)
\]

It looks quite intimidating, but the first two lines are just setup. It is pretty much what is expected, but unfortunately it is not enough. In the degenerate case that \(w = c\), it turns out that we need an additional rule...

\[
\texttt{Hpol : } (\delta : [ c \Rightarrow c ]) \ (\delta' : \mathcal{P} \Rightarrow \mathcal{P})
\to (\texttt{nat : } \forall \{ \Gamma \sigma \} \ (t : c \ \Gamma \sigma)
\to f (\delta t) \equiv \delta' (f t)) \ -- \ !
\to (\texttt{wk : } \forall \{ \Gamma \Delta \} \ (\sigma : (\mathcal{V} \Rightarrow c) \ \Gamma \ \Delta \ \{i\} \ (v : \mathcal{V} \ _i)
\to f ((\sigma \bowtie j) v) \equiv ((f \circ \sigma) \bowtie j) v)
\to \forall \{ \Gamma \Delta \} \ (\sigma : (\mathcal{V} \Rightarrow c) \ \Gamma \ \Delta \ \{i\} \ (t : c \ \Gamma i)
\to f(\texttt{map } \delta \sigma t) \equiv \texttt{map } \delta' (f \circ \sigma) (f t) \ -- \ !
\]

... It’s even more intimidating. But only the two lines marked by comments matters. The first says that the two \(\delta\)'s satisfy a similar condition to the equation for \(f\) and \(g\) in the case of fixoids. The second is slightly different to \texttt{Hnat}, but also reasonable.

The argument \texttt{wk} is actually not necessary, and we will eliminate it. But when implementing this typeclass method, it helps to make inductions go through.
But after some straightforward definitions, we can prove the desired lemma for \texttt{rename}:

\[
\text{Hrename} : \forall \{\Gamma \Delta\} \ (\rho : (\mathcal{V} \Rightarrow \mathcal{V}) \Gamma \Delta) \\
\quad \Rightarrow \forall \{i\} \ (t : \mathcal{C} \Gamma i) \\
\quad \Rightarrow f (\text{rename} \ \rho \ t) \equiv \text{rename} \ \rho \ (f \ t) \\
\text{Hrename} \ \rho \ t \ \text{rewrite} \ \text{symm} \ fHvar = \text{Hnat} \ \var \ \rho \ t
\]

And \texttt{subst}:

\[
\text{Hsubst} : \forall \{\Gamma \Delta\} \ (\sigma : (\mathcal{V} \Rightarrow \mathcal{C}) \Gamma \Delta) \\
\quad \Rightarrow \forall \{i\} \ (t : \mathcal{C} \Gamma i) \\
\quad \Rightarrow f (\text{subst} \ \sigma \ t) \equiv \text{subst} \ (f \circ \sigma) \ (f \ t) \\
\text{Hsubst} \ \sigma \ t = \text{Hmap} \ \text{id} \ \text{id} \ (\lambda _- \rightarrow \text{refl}) \ \sigma \ t
\]

\[
\text{Hsubstz/_} : \forall \{\Gamma \ i \ j\} \ (t : \mathcal{C} \Gamma i) \ (t' : \mathcal{C} (\Gamma \triangleright i) j) \\
\quad \Rightarrow f (\text{subst} \ (z/ \ t) \ t') \equiv \text{subst} \ (z/ \ f \ t) \ (f \ t') \\
\text{Hsubstz/_} \ t \ t' \ \text{rewrite} \ \text{Hsubst} \ (z/ \ t) \ t' \ | \ \text{Hz/} \ t = \ \text{refl}
\]

Note that \texttt{Hmap} is a version of \texttt{Hnat}, but with the \texttt{wk} argument removed. Also, we used the function extensionality axiom here. The details are hidden.

\texttt{open Hom } \{\[\ldots\]} \texttt{ public}

You can see an example of usage in the accompanying repository.

Apart from homomorphisms, we are also interested in the interactions of substitution with itself. This is the celebrated \textbf{substitution lemma}:

\[
t(x \mapsto s_1)(y \mapsto s_2) = t(y \mapsto s_2)(x \mapsto s_1(y \mapsto s_2))
\]

under the condition that \(x\) is not free in \(s_2\). To prove this, we similarly need to prove a version for renamings first. One would naturally ask whether these two versions can be unified.

Here the coherence conditions get really nasty. The final goal is clear:

\[
\text{map} \ (g \circ f) \ \sigma \ (\text{map} \ g \ \delta \ t) \equiv \text{map} \ g \ (\text{map} \ f \ \sigma \circ \delta) \ t
\]

With suitably instantiated \texttt{f} and \texttt{g}, this equation encapsulates both the renaming and substitution lemmas. But it is unclear what conditions should be imposed on \texttt{f} and \texttt{g}. How should we proceed? Is this a dead end, and should we now turn to more conventional methods of manipulating syntax with binding? This is left as an \textit{exercise} for the reader.

It is a question of foundations of mathematics, rather than mathematics itself; or, at least, I hope so. The reply is left to the reader as an exercise. (This phrase always means that the writer cannot do the problem himself.)

\cite{5}, p. 15.
References

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