Compensated Compactness Method on Non-isentropic Polytropic Gas Flow

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Abstract. In this paper, we are concerned with a model of polytropic gas flow, which consists the mass equation, the momentum equation and a varying entropy equation. First, a new technique, to set up a relation between the Riemann invariants of the isentropic system and the entropy variable $s$, coupled with the maximum principle, is introduced to obtain the a-priori $L^\infty$ estimates for the viscosity-flux approximation solutions. Second, the convergence framework from the compensated compactness theory on the system of isentropic gas dynamics is applied to prove the pointwise convergence of the approximation solutions and the global existence of bounded entropy solutions for the Cauchy problem of the system with bounded initial data. Finally, as a by-product, we obtain a non-classical bounded generalized solution $(\rho, u, s)$, of the original non-isentropic polytropic gas flow, which satisfies the mass equation and the momentum equation, the entropy equation with an extra nonnegative measure in the sense of distributions.

1. Introduction

The Euler system for compressible polytropic gas flow in one-space dimension is the following system of three conservation laws

$$\begin{cases}
\rho_t + (\rho u)_x = 0, \\
(\rho u)_t + (\rho u^2 + k\rho^\gamma e^s)_x = 0, \\
\left(\frac{1}{2}\rho u^2 + \frac{k}{\gamma-1}\rho^\gamma e^s\right)_t + (u\left(\frac{1}{2}\rho u^2 + \frac{k}{\gamma-1}\rho^\gamma e^s\right))_x = 0,
\end{cases}$$

where $\rho$ is the density of gas, $u$ the velocity, $s$ the entropy and $\gamma > 1$ corresponds to the adiabatic exponent, $c > 0$ is the specific heat at constant volume, $k > 0$ can be any constant under scaling. Without any loss of generality, we may choose $k = \frac{\alpha^2}{\gamma}$, $\theta = \frac{\gamma - 1}{2}$ and $c = \frac{1}{2}$ for the simplicity.

For a smooth solution, the third equation in (1.1) is equivalent to

$$\left(\rho s\right)_t + (\rho u s)_x = 0,$$

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which, united with the first and the second equations in (1.1), forms the following system of three conservation laws

\[
\begin{aligned}
&\rho_t + (\rho u)_x = 0, \\
&(\rho u)_t + (\rho u^2 + \frac{\theta^2}{\gamma}\rho \gamma e^{2s})_x = 0, \\
&(\rho s)_t + (\rho us)_x = 0.
\end{aligned}
\] (1.3)

System (1.3) was derived by relaxation from an isentropic two-phase mixture, and its global weak solutions were studied in [1], under the assumption of the uniform boundedness of the viscosity solutions. In this paper, we shall give a rigorous proof of the uniform bound of the viscosity solutions and obtain the global bounded entropy solutions of the Cauchy problem (1.3) with the following bounded initial data

\[(1.4) \quad (\rho, u, s)|_{t=0} = (\rho_0(x), u_0(x), s_0(x)), \quad \rho_0(x) \geq 0.
\]

When \(s\) is a constant, both (1.1) and (1.3) are reduced to the following isentropic gas dynamics system of two conservation laws

\[
\begin{aligned}
&\rho_t + (\rho u)_x = 0, \\
&(\rho u)_t + (\rho u^2 + \frac{\theta^2}{\gamma}\rho \gamma e^{2s})_x = 0.
\end{aligned}
\] (1.5)

Numerous papers deal with the analysis of weak solutions of the Cauchy problem (1.5). The first existence theorem for large initial data of locally finite total variation was proved in [2] for \(\gamma = 1\) and in [3] for \(\gamma \in (1, 1 + \delta)\) in Lagrangian coordinates, where \(\delta\) is small. The Glimm scheme [4] was used in these papers.

The ideas of compensated compactness developed in [5, 6] were used in [7] to establish a global existence theorem for the Cauchy problem (1.5) with large initial data for \(\gamma = 1 + \frac{2}{N}\), where \(N \geq 5\) odd, with the use of the viscosity method. The convergence of the Lax-Friedrichs scheme and the existence of a global solution in \(L^\infty\) for large initial data with adiabatic exponent \(\gamma \in (1, \frac{5}{2})\) were proved in [8, 9]. In [10], the global existence of a weak solution was proved for \(\gamma \geq 3\) with the use of the kinetic setting in combination with the compensated compactness method. The method in [10] was finally improved in [11] to fill the gap \(\gamma \in (\frac{5}{2}, 3)\), and a new proof of the existence of a global solution for all \(\gamma > 1\) was given there. Later on, a new application of the method in [10] was obtained in [12] on the study of the Euler equations of one-dimensional, compressible fluid flow, where the linear combinations of weak and strong entropies were invented to replace the weak entropies used in [7, 8, 9, 10, 11]. The isothermal case \(\gamma = 1\) with the vacuum was studied in [13].

Thus, the problem on the existence of a generalized solution of the Cauchy problem (1.5) with bounded initial data (1.4) has been completely solved in the case of a polytropic gas.

For the case of a non-isentropic polytropic gas, namely \(s \neq 0\), the existence theorem of (1.1) for small initial data, away from the vacuum, of locally finite total variation was proved in [14] for \(\gamma \in (1, \frac{5}{3})\) in Lagrangian coordinates, where the proof is based on the finite difference scheme of Glimm [4].
How to obtain the global existence, for the equations of non-isentropic gas dynamics (1.1) (or the simplified system (1.3)) with arbitrarily large initial data (1.4) including the vacuum, is still a challenging open problem.

Our aim in this paper is to apply the convergence framework from the compensated compactness theory on the system of isentropic gas dynamics (1.5), to prove the pointwise convergence of the approximation solutions of (1.3), and to obtain the global existence of bounded entropy solutions for the Cauchy problem (1.3) with the bounded initial data (1.4). As a by-product, we obtain a non-classical bounded generalized solution (ρ, u, s), of the non-isentropic polytropic gas flow (1.1), which satisfies the mass equation and the momentum equation, the entropy equation with an extra nonnegative measure in the sense of distributions.

Substituting the first equation in (1.3) into the second and the third, we have the following system about the variables (ρ, u, s),

\[
\begin{align*}
\rho_t + u \rho_x + \rho u_x &= 0, \\
u_t + uu_x + \theta^2 \rho \gamma - 2 \rho e^{2s} \rho_x + \frac{2\theta^2 \rho \gamma - 1}{\gamma} e^{2s} s_x &= 0, \\
s_t + us_x &= 0,
\end{align*}
\]

which, for smooth solutions, is equivalent to system (1.3) as well as system (1.1).

Let the matrix \(dF(U)\) of (1.6) be

\[
dF(U) = \begin{pmatrix}
u & \rho & 0 \\
\theta^2 \rho \gamma - 2 \rho e^{2s} & u & \frac{2\theta^2 \rho \gamma - 1}{\gamma} e^{2s} \\
0 & 0 & u
\end{pmatrix}.
\]

Then three eigenvalues of (1.3) are

\[
\lambda_1 = u - \theta \rho^\theta e^s, \quad \lambda_2 = u + \theta \rho^\theta e^s, \quad \lambda_3 = u
\]

with corresponding right eigenvectors

\[
r_1 = (1, -\theta \rho^{\theta - 1} e^s, 0)^T, \quad r_2 = (1, \theta \rho^{\theta - 1} e^s, 0)^T, \quad r_3 = (0, 0, 1)^T.
\]

The Riemann invariants of (1.3) are functions \(w_1(\rho, u, s), w_2(\rho, u, s)\) and \(w_3(\rho, u, s)\) satisfying the equations

\[
(\rho, u, s) \cdot dF = \lambda_i (w_{i\rho}, w_{i\mu}, w_{is}), \quad i = 1, 2, 3.
\]

Since the system (1.10) is not well defined, we consider \(s\) to be a constant, then the Riemann invariants of the isentropic system (1.5),

\[
(\rho, u, s) \cdot dF = \lambda_i (w_{i\rho}, w_{i\mu}, w_{is}, w_{is}), \quad i = 1, 2, 3.
\]

satisfy the first two equations of (1.10).

By simple calculations,

\[
\nabla \lambda_1 \cdot r_1 = -\theta (1 + \theta) \rho^{\theta - 1} e^s, \quad \nabla \lambda_2 \cdot r_2 = \theta (1 + \theta) \rho^{\theta - 1} e^s, \quad \nabla \lambda_3 \cdot r_3 = 0.
\]

Therefore it follows from (1.8) that system (1.3) is strictly hyperbolic in the domain \(\{(x, t) : \rho(x, t) > 0\}\), while it is hyperbolically degenerate in the domain \(\{(x, t) : \rho(x, t) = 0\}\), since \(\lambda_1 = \lambda_2 = \lambda_3\) when \(\rho = 0\). From (1.12), the first two characteristic fields in (1.3) are genuinely nonlinear if the adiabatic exponent \(\gamma \in (1, 3]\), while the system is no longer genuinely nonlinear at \(\rho = 0\) if the adiabatic
exponent \( \gamma > 3 \); and the third characteristic field is always linearly degenerate, or of the Temple type \([15]\).

It is well known that, in order to prove the existence of solutions by using the compensated compactness theory, we should first obtain the a-priori \( L^p \) estimate, \( 1 < p \leq \infty \), of the approximate solutions, and look for enough entropy-entropy flux pairs. Then, we may obtain the measure equations by applying the div-curl lemma and the representation of weak limit of solution in terms of Young measure. Finally we must show the reduction of Young measure to a Dirac measure.

Unfortunately, in the face of nonlinear hyperbolic systems of more than two conservation laws, we meet the difficulties in all the above three steps.

In fact, except the scalar equation, even if for systems of two equations, not all pairs of entropy-entropy flux \((\eta, q)\) could be used to reduce the Young measure to be the Dirac mass since \( \eta_t + q_x \), where \((\eta, q)\) is a pair of entropy-entropy flux, must be compact in \( H^{-1}_{\text{loc}}(R \times R^+) \) when one applies the div-curl lemma of Tartar \([5]\).

When we study the system \((1.3)\), the crucial difficulty is how to obtain the a-priori \( L^p \), \( 1 < p \leq \infty \), estimates of the approximate solutions because the invariant regions theory \([16]\), in general, does not work.

In \([17]\), the authors studied the following system

\[
\begin{align*}
\rho_t + (\rho u)_x &= 0, \\
(\rho(1 + s)u)_t + (\rho(1 + s)u^2 + P(\rho, s))_x &= 0, \\
(\rho s)_t + (\rho us)_x &= 0,
\end{align*}
\]

where \( P(\rho, s) = \frac{a^2 s^\gamma}{1 - \rho} \) and \( a \) is a constant. The same as system \((1.3)\), two characteristic fields of \((1.13)\) are genuinely nonlinear and the third one is of the Temple type. Under the assumption of a uniform bound on the \( L^\infty \) norm of the viscosity approximate solutions and other several technical assumptions, the global entropy solution of the Cauchy problem \((1.13)\) with bounded initial data was studied with the help of the compactness framework of DiPerna on \( 2 \times 2 \) strictly hyperbolic, genuinely nonlinear systems \([18]\). Later, the author in \([19]\) showed the existence of invariant regions for the Riemann problem and obtained global existence using the Glimm scheme \([4]\).

In \([20]\), the classical smooth solution of \((1.13)\) was obtained when \( P(\rho, s) \) is fixed as \( e^{-\frac{s}{\rho}} \), where the characteristic fields are assumed to be nondecreasing. System \((1.13)\) with this special pressure is interesting because it can be diagonalized.

In \([21]\), the authors studied the following model of polytropic gas flow with diffusive entropy

\[
\begin{align*}
\rho_t + (\rho u)_x &= 0, \\
(\rho u)_t + (\rho u^2 + P(\rho, s))_x &= 0, \\
((\rho s)_t + (\rho us)_x &= (\frac{1}{\rho^2} s)_x,
\end{align*}
\]

where \( P(\rho, s) = e^{(\gamma - 1)s} \rho^\gamma \), \( \gamma > 1 \). With the help of the diffusive term \( (\frac{1}{\rho^2} s)_x \), a skill was used to obtain the a-priori \( L^\infty \) estimates of the approximate solutions constructed by the Lax-Friedrichs or Godunov schemes, and the pointwise convergence
of the approximate solutions was proved by using the compactness framework on $2 \times 2$ polytropic gas flow \cite{8, 9, 10, 11}.

The main contribution of this paper is to obtain the a-priori, $L^\infty$ estimates of the viscosity solutions of (1.3).

The classical vanishing viscosity method is to add the viscosity terms to the right-hand side of system (1.3) and consider the Cauchy problem for the following related parabolic system

\begin{equation}
\begin{cases}
\rho_t + (\rho u)_x = \varepsilon \rho_{xx}, \\
(\rho u)_t + (\rho u^2 + P(\rho)e^{2\xi})_x = \varepsilon (\rho u)_{xx}, \\
(\rho s)_t + (\rho us)_x = \varepsilon (\rho s)_{xx}
\end{cases}
\end{equation}

(1.15)

with bounded initial data

\begin{equation}
(\rho^\varepsilon, u^\varepsilon, s^\varepsilon)|_{t=0} = (\rho_0^\varepsilon(x), u_0^\varepsilon(x), s_0^\varepsilon(x)), \quad \rho_0^\varepsilon(x) \geq \varepsilon > 0,
\end{equation}

(1.16)

where $\rho_0^\varepsilon(x) = (\rho_0(x) + \varepsilon) * G^\varepsilon$, $u_0^\varepsilon(x) = u_0(x) * G^\varepsilon$, $s_0^\varepsilon(x) = s_0(x) * G^\varepsilon$ are the smooth approximations of $\rho_0(x)$, $u_0(x)$, $s_0(x)$ and $G^\varepsilon$ is a mollifier. However, if we consider $\rho, m, \Upsilon$, where $m = \rho u, \Upsilon = \rho s$ as three independent variables in (1.15), then the terms $\rho u^2 = \frac{\rho u^2}{\rho}, \rho us = \frac{\rho u}{\rho} \Upsilon$ are singular near the line $\rho = 0$.

Compared with the previous results on (1.13) and (1.14) introduced above, we mainly need to resolve the following three difficulties when we study the Cauchy problem for System (1.3).

**Difficulty I.** How to obtain the positive, lower bound of the viscosity solutions $\rho^\varepsilon$ for the Cauchy problem (1.15) and (1.16)?

To overcome this difficulty, instead of the classical viscosity approximation, we use again the flux approximation introduced in \cite{22, 23} and consider the following parabolic system

\begin{equation}
\begin{cases}
\rho_t + ((\rho - 2\delta)u)_x = \varepsilon \rho_{xx}, \\
(\rho u)_t + (\rho u^2 - \delta u^2 + P_1(\rho, \delta)e^{2\xi})_x = \varepsilon (\rho u)_{xx}, \\
(\rho s)_t + ((\rho - 2\delta)us)_x = \varepsilon (\rho s)_{xx}
\end{cases}
\end{equation}

(1.17)

with bounded initial data

\begin{equation}
(\rho^\varepsilon, u^\varepsilon, s^\varepsilon)|_{t=0} = (\rho_0^\varepsilon(x), u_0^\varepsilon(x), s_0^\varepsilon(x)), \quad \rho_0^\varepsilon(x) \geq 2\delta > 0,
\end{equation}

(1.18)

where $\varepsilon, \delta$ are positive small perturbation constants, the perturbation pressure

\begin{equation}
P_1(\rho, \delta) = \int_0^\rho \frac{t - 2\delta}{t} P'(t) dt = k \rho^\gamma - 2\delta k \frac{\gamma}{\gamma - 1} \rho^{\gamma - 1},
\end{equation}

(1.19)

and $\rho_0^\varepsilon(x) = (\rho_0(x) + \delta) * G^\varepsilon$, $u_0^\varepsilon(x) = u_0(x) * G^\varepsilon$, $s_0^\varepsilon(x) = s_0(x) * G^\varepsilon$ are the smooth approximations of $\rho_0(x)$, $u_0(x)$ and $s_0(x)$, satisfying $\lim_{|x| \to 0} \frac{\partial}{\partial x} (s_0^\varepsilon(x)) = 0, i = 1, 2$. Since $\rho_0^\varepsilon(x) \geq 2\delta$, applying the maximum principle to the first equation in (1.17), we may obtain the uniformly positive lower bound $\rho^\varepsilon(x, t) \geq 2\delta$, which grateets that $\rho u^2 = \frac{\rho u^2}{\rho}, \rho us = \frac{\rho u}{\rho} \Upsilon$ in (1.17) are regular. Besides, the flux approximation given in (1.17) has the following advantage.
When we consider s as a parameter or s = 0, (1.17) is deduced to the following system of two equations

\[
\begin{aligned}
\rho_t + ((\rho - 2\delta)u)_x &= \varepsilon \rho_{xx}, \\
(\rho u)_t + (\rho u^2 - \delta u^2 + P_{1}(\rho, \delta)e^{2s})_x &= \varepsilon (\rho u)_{xx}.
\end{aligned}
\]

(1.20)

The uniformly lower bound \(\rho^{s,\delta}(x, t) \geq 2\delta\) helped us to obtain the proof of the \(H^{-1}\) compactness of \(\eta_t + q_x\) for any weak entropy-entropy flux pair \((\eta, q)\), and for general pressure function \(P(\rho)\) (cf. [22]).

Moreover, system (1.20) has the same Riemann invariants and the entropy equation like system (1.5).

These special behaviors, of system (1.17) as well as (1.20), will help us to obtain the uniformly upper bound estimate of \((\rho^{s,\delta}, u^{s,\delta})\) and to overcome the following difficulty.

**Difficulty II.** How to obtain the uniformly, upper bound \(\rho^{s,\delta} \leq M\) and \(|u^{s,\delta}| \leq M\)?

The outline to overcome the above difficulty is as follows. First, substituting the first equation in (1.17) into the third, we may rewrite the third equation in (1.17) as

\[
s_t + \frac{(\rho - 2\delta)}{\rho} \rho u s_x = \varepsilon s_{xx} + 2\varepsilon \frac{\rho_x}{\rho} s_x.
\]

(1.21)

By using the technique given in [24, 25] (see also [26]), we can easily prove that \(s^{s,\delta}_x\) is uniformly bounded in \(L^1(R)\).

Second, multiplying \((w_{\rho}, w_m, w_s), (z_{\rho}, z_m, z_s)\) to system (1.17), where \(m = \rho u\) and \(w, z\) are given in (1.11), we obtain

\[
w_t + \lambda^s_{1} w_x - \left(\frac{2\rho_{s}}{P_{1}} u + \theta(\rho - 2\delta)\rho^{\theta - 1} e^{s}\right)\rho^{\theta} e^{s} s_x
\]

\[
= \varepsilon w_{xx} + \frac{2\varepsilon}{\rho} \rho w_x - \varepsilon e^{s} \rho^{\theta - 2}(\theta(\theta + 1)\rho_{s}^{2} + 2 \theta \rho \rho_{s} s_x + \rho^{2} s_{s}^{2})
\]

and

\[
z_t + \lambda^s_{2} z_x - \left(\frac{2\rho_{s}}{P_{1}} u - \theta(\rho - 2\delta)\rho^{\theta - 1} e^{s}\right)\rho^{\theta} e^{s} s_x
\]

\[
= \varepsilon z_{xx} + \frac{2\varepsilon}{\rho} \rho z_x - \varepsilon e^{s} \rho^{\theta - 2}(\theta(\theta + 1)\rho_{s}^{2} + 2 \theta \rho \rho_{s} s_x + \rho^{2} s_{s}^{2}),
\]

(1.23)

where

\[
\lambda^s_{1} = u - \frac{\rho - 2\delta}{\rho} \rho\theta e^{s}, \quad \lambda^s_{2} = u + \frac{\rho - 2\delta}{\rho} \rho\theta e^{s}
\]

are first two eigenvalues of the left-hand side of system (1.17).

Compared with the case of \(s = 0\), when we intend to apply the maximum principle to (1.22) and (1.23), the functions \(\left(\frac{2\rho_{s}}{P_{1}} u + \theta(\rho - 2\delta)\rho^{\theta - 1} e^{s}\right)\rho^{\theta} e^{s} s_x\) in (1.22) and \(\left(\frac{2\rho_{s}}{P_{1}} u - \theta(\rho - 2\delta)\rho^{\theta - 1} e^{s}\right)\rho^{\theta} e^{s} s_x\) in (1.23) are two major stumbling blocks.

Fortunately, since \(s_{\rho}\) is uniformly bounded in \(L^1(R)\), we might choose a suitable nonnegative, bounded function \(\beta(x,t), \beta_{x}(x,t) \geq 0\), and make the transformation of variables

\[
w = v_1 + M + \beta(x,t), \quad z = v_2 + M - \beta(x,t)
\]

(1.25)
holds for all test function \( \phi \) |

**Difficulty III.** How to prove the pointwise convergence of \((\rho^{\varepsilon, \delta}, u^{\varepsilon, \delta}, s^{\varepsilon, \delta})\) as \( \varepsilon, \delta \) go to zero?

First, as introduced in [24], the pointwise convergence of \( s^{\varepsilon, \delta} \) can be obtained by using the div-curl lemma to some special pairs of functions \((c, F(s))\), where \( c \) is a constant and \( F(s) \) is a suitable function of \( s \) since the \( L^1(R) \) estimate of \( s^{\varepsilon, \delta}(-, t) \) and the compactness of \( c_t + F(s^{\varepsilon, \delta})_x \) in \( H^{-1}_{loc} \).

Second, to prove the pointwise convergence of \((\rho^{\varepsilon, \delta}, u^{\varepsilon, \delta})\) as \( \varepsilon, \delta \) go to zero, we may fix the variable \( s \) or think of \( s \) as a parameter, with the help of the compactness framework \([8, 9, 10, 11]\) on the \( 2 \times 2 \) polytropic gas dynamic system \((1.20)\), we may prove the pointwise convergence of \((\rho^{\varepsilon, \delta}, u^{\varepsilon, \delta})\) and obtain the global existence of solutions.

The main results of this paper are listed in the following Theorem 1.1.

**Theorem 1.1.** Let the initial data \((\rho_0(x), u_0(x), s_0(x))\) be bounded in \( L^\infty (\mathbb{R}) \); \( |s_0(x)| \leq N, |s_0|_{L^1(\mathbb{R})} \leq c_0 < 1 \), for two positive constants \( N, c_0 \). Then, (I) for fixed \( \varepsilon, \delta \), the global smooth solution, \((\rho^{\varepsilon, \delta}, u^{\varepsilon, \delta}, s^{\varepsilon, \delta})\) of the Cauchy problem \((1.17)\) and \((1.18)\), exists and satisfies

\[
|s^{\varepsilon, \delta}| \leq N, \ |s_x^{\varepsilon, \delta}(\cdot, t)|_{L^1(\mathbb{R})} \leq c_0 < 1,
\]

and

\[
\begin{cases}
    z(\rho^{\varepsilon, \delta}, u^{\varepsilon, \delta}, s^{\varepsilon, \delta}) \leq M - c \int_{-\infty}^{x} |s_x^{\varepsilon, \delta}| dx \leq M, \\
    w(\rho^{\varepsilon, \delta}, u^{\varepsilon, \delta}, s^{\varepsilon, \delta}) \leq M + c \int_{-\infty}^{x} |s_x^{\varepsilon, \delta}| dx \leq M + cc_0,
\end{cases}
\]

where \( z, w \) are the Riemann invariants, of isentropic system \((1.22)\), given in \((1.11)\) and \( c, M \) are two suitable large constants, satisfying \( M \leq c \) and \( 0 < cc_0 < M \).

(II) There exists a subsequence of \((\rho^{\varepsilon, \delta}(x, t), u^{\varepsilon, \delta}(x, t), s^{\varepsilon, \delta}(x, t))\), which converges pointwise, on the set \( \rho_+ = \{(x, t) : \rho(x, t) > 0\} \), to a set of bounded functions \((\rho(x, t), u(x, t), s(x, t))\) as \( \varepsilon, \delta \) tend to zero, and the limit is a weak entropy solution of the Cauchy problem \((1.3)\)–\((1.4)\).

**Definition 1.** A set of bounded functions \((\rho(x, t), u(x, t), s(x, t))\) is called a weak entropy solution of the Cauchy problem \((1.3)\)–\((1.4)\) if

\[
\begin{align*}
    \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \rho \phi_t + \rho u \phi_x dx dt + \int_{-\infty}^{\infty} \rho_0(x) \phi(0) dx &= 0, \\
    \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \rho u \phi_t + (\rho u^2 + \Theta \rho^\gamma e^{2s}) \phi_x dx dt + \int_{-\infty}^{\infty} \rho_0(x) u(0) \phi(0) dx &= 0, \\
    \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \rho s \phi_t + \rho u s \phi_x dx dt + \int_{-\infty}^{\infty} \rho_0(x) s(0) \phi(0) dx &= 0,
\end{align*}
\]

holds for all test function \( \phi \in C_0^1(\mathbb{R} \times \mathbb{R}^+) \) and

\[
\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \eta(\rho, m, \Sigma) \phi_t + q(\rho, m, \Sigma) \phi_x dx dt \geq 0
\]
holds for any nonnegative test function \( \phi \in C^\infty_0(R \times R^+ - \{ t = 0 \}) \), where \( m = \rho u, \ U = ps + (\eta, q) \), \( \eta(0, m, U) = 0 \), is a pair of convex, weak entropy-entropy flux of system (1.3).

In the next section, we will introduce a technique from the maximum principle, to set up a relation between the Riemann invariants of isentropic system and the solutions of the parabolic system (1.17) with the initial data (1.18). Under the conditions in Theorem 1.1, we may obtain the estimates (1.26), (1.27) on \( \Upsilon = \rho(x, t) \). Then, based on these estimates, we obtain the pointwise convergence of \( (\rho^\varepsilon(x, t), u^\varepsilon(x, t), s^\varepsilon(x, t)) \) by applying the compensated compactness theory on the polytropic gas dynamics (1.5) (the first two equations in (1.13)).

The study of (1.3) could be considered as the beginning of a study of the non-isentropic polytropic gas flow (1.1).

In fact, beside the standard viscosity terms, if we add the extra perturbation function \( A(x, t) \) to (1.1) and consider the following parabolic system

\[
\rho_t + (\rho u)_x = \varepsilon \rho_{xx},
\]

\[
(\rho u)_t + (\rho u^2 + k \rho \gamma e^{2s})_x = \varepsilon (\rho u)_{xx},
\]

\[
\left( \frac{1}{2} \rho u^2 + \frac{k}{\gamma - 1} \rho \gamma e^{2s} \right)_t + (u(\frac{1}{2} \rho u^2 + \frac{k}{\gamma - 1} \rho \gamma e^{2s}))_x = \varepsilon \left[ \frac{1}{2} \rho u^2 + \frac{k}{\gamma - 1} \rho \gamma e^{2s} \right]_{xx} - \varepsilon A(x, t),
\]

(1.30)

where

\[
A(x, t) = \rho u^2 + k \rho \gamma^{-2} e^{2s} - \frac{4}{\gamma - 1} \rho^2 s^2 + 4 \rho \rho_x s_x,
\]

(1.31)

then we may prove that (1.15) and (1.30) are completely same.

To prove the equivalent of (1.15) and (1.30), we substitute the first equation in (1.30) into the second to obtain

\[
u_t + uu_x + k \gamma \rho \gamma^{-2} e^{2s} \rho_x + 2k \rho \gamma^{-1} e^{2s} s_x = \varepsilon u_{xx} + 2 \varepsilon e^2 \rho u_x.
\]

(1.32)

By simple calculations,

\[
\left( \frac{1}{2} \rho u^2 + \frac{k}{\gamma - 1} \rho \gamma e^{2s} \right)_t + (u(\frac{1}{2} \rho u^2 + \frac{k}{\gamma - 1} \rho \gamma e^{2s}))_x
\]

\[
= \frac{1}{2} u^2 \rho_t + \rho uu_t + \frac{k}{\gamma - 1} \rho \gamma^{-1} e^{2s} \rho_t + \frac{2k}{\gamma - 1} \rho \gamma e^{2s} \rho_t
\]

\[
+ \frac{1}{2} u^2 (\rho u)_x + \rho u (uu_x) + \rho u (k \gamma \rho \gamma^{-2} e^{2s} \rho_x)
\]

\[
+ \frac{k}{\gamma - 1} \rho \gamma^{-1} e^{2s} (\rho u)_x + \frac{2k}{\gamma - 1} \rho \gamma e^{2s} s_x
\]

\[
= \frac{1}{2} u^2 (\rho_t + (\rho u)_x) + \rho u (u_t + uu_x + k \gamma \rho \gamma^{-2} e^{2s} \rho_x + 2k \rho \gamma^{-1} e^{2s} s_x)
\]

\[
+ \frac{k}{\gamma - 1} \rho \gamma^{-1} e^{2s} (\rho_t + (\rho u)_x) + \frac{2k}{\gamma - 1} \rho \gamma e^{2s} (s_t + us_x) = B(x, t)
\]

(1.33)
and
$$
\varepsilon (\frac{1}{2} \rho u^2 + \frac{k}{\gamma - 1} \rho^2 e^{2s})_{xx} = \frac{1}{2} u^2 \varepsilon \rho_{xx} + \varepsilon (2 u \rho x u_x + \rho u u_{xx}) + \varepsilon \rho u_x^2
$$
(1.34)

$$
+ \frac{k \gamma}{\gamma - 1} \rho \gamma^{-1} e^{2s} \varepsilon \rho_{xx} + \frac{2k}{\gamma - 1} \rho \gamma e^{2s} \varepsilon s_{xx} + \varepsilon k \gamma \rho \gamma^{-2} e^{2s} \rho_x^2
$$

$$
+ \varepsilon \frac{4k}{\gamma - 1} \rho \gamma^{-1} e^{2s} \rho_x s_x + \varepsilon \frac{4k}{\gamma - 1} \rho \gamma e^{2s} s_x^2 = C(x, t).
$$

From the third equation in (1.30), we have
$$
\rho_u = \frac{1}{2} x, t
$$
(1.35)

where we used the first equation in (1.30) and (1.32).

Summing up (1.35) and (1.36), we have
$$
\frac{2k}{\gamma - 1} \rho \gamma^{-1} e^{2s} (s t + s(\rho u)_x) = \varepsilon \rho_{xx} \frac{2k}{\gamma - 1} \rho \gamma^{-1} e^{2s} s.
$$
(1.36)

From the first equation in (1.30), we have
$$
\frac{2k}{\gamma - 1} \rho \gamma^{-1} e^{2s} ((\rho s)_t + (\rho u s)_x) = \frac{2k}{\gamma - 1} \rho \gamma^{-1} e^{2s} \varepsilon s_{xx}
$$
(1.37)

which deduces the third equation in (1.15).

Similarly, we may prove the following parabolic system
$$
\rho_t + ((\rho - 2 \delta) u)_x = \varepsilon \rho_{xx},
$$
(1.38)

$$
(\rho u)_t + (\rho u^2 - \delta u^2 + P_t (\rho, \delta) e^{2s})_x = \varepsilon (\rho u)_{xx},
$$

$$
(\frac{1}{2} \rho u^2 + \frac{k}{\gamma - 1} \rho \gamma e^{2s})_t + (u (\frac{1}{2} \rho u^2 + \frac{k}{\gamma - 1} \rho \gamma e^{2s}))_x
$$

$$
= \varepsilon (\frac{1}{2} \rho u^2 + \frac{k}{\gamma - 1} \rho \gamma e^{2s})_{xx} - \varepsilon A(x, t) + \delta D(x, t),
$$

is equivalent to the flux-viscosity approximate system (1.17), where
$$
D(x, t) = \frac{1}{3} (u^3)_x + \frac{4k}{\gamma - 1} \rho \gamma^{-1} e^{2s} u s_x + 2k \gamma^{1 - 1} (\rho \gamma^{-1} e^{2s} u)_x.
$$
(1.39)

In fact, we rewrite the first and the second equations in (1.38) as follows
$$
\rho_t + (\rho u)_x - 2 \delta u_x = \varepsilon \rho_{xx}
$$
(1.40)

and
$$
(\rho u)_t + (\rho u^2 + k \gamma \gamma^{-1} e^{2s})_x - 2 \delta u u_x - 2 \delta k \frac{\gamma}{\gamma - 1} (\rho \gamma^{-1} e^{2s})_x = \varepsilon (\rho u)_{xx}.
$$
(1.41)

Accordingly, (1.32) is rewritten as follows
$$
\frac{u_t + uu_x + k \gamma \gamma^{-1} e^{2s} \rho_x + 2k \rho \gamma^{-1} e^{2s} s_x - 2 \delta k \frac{\gamma}{\gamma - 1} (\rho \gamma^{-1} e^{2s})_x}{(\rho - \delta u)_x} = \varepsilon u_{xx} + 2 \varepsilon \frac{2k}{\rho} u_x.
$$
(1.42)
By using the calculations in (1.33)-(1.34), the equations (1.40)-(1.42), and the third equation in the system (1.38), we obtain

\[
\frac{1}{2}u^2(2\delta u_x) + \rho u(2\delta k \frac{\gamma}{(\gamma-1)\rho}(\rho^{-1}e^{2s})_x) + k \frac{\gamma}{(\gamma-1)}\rho^{-1}e^{2s}(2\delta u_x)
\]

(1.43)

\[
+ \frac{2k}{\gamma-1}\rho\gamma^2(\varepsilon_1 + us_x) = \varepsilon \rho u^2 + \frac{2k}{\gamma-1}\rho\gamma^2\varepsilon s_{xx} + \varepsilon k\gamma\rho^{-1}e^{2s}\rho_x^2
\]

\[
+ \varepsilon \frac{4k}{\gamma-1}\rho\gamma^2\rho_x s_x + \varepsilon \frac{4k}{\gamma-1}\rho\gamma^2s_x^2 - \varepsilon A(x,t) + \delta D(x,t).
\]

Multiplying \(\frac{2k}{\gamma-1}\rho\gamma^{-1}e^{2s}\) to the first equation in (1.38), we have

\[
\frac{2k}{\gamma-1}\rho\gamma^{-1}e^{2s}(s\rho t + (\rho - 2\delta)u)_x = \varepsilon \rho u_x\frac{2k}{\gamma-1}\rho\gamma^{-1}e^{2s}s_x.
\]

(1.44)

Since the following terms in (1.43)

\[
\rho u(2\delta k \frac{\gamma}{(\gamma-1)\rho}(\rho^{-1}e^{2s})_x) + k \frac{\gamma}{(\gamma-1)}\rho^{-1}e^{2s}(2\delta u_x)
\]

(1.45)

\[
= 2\delta k \frac{\gamma}{(\gamma-1)}(\rho^{-1}e^{2s}u)_x,
\]

\[
\frac{2k}{\gamma-1}\rho\gamma^2(s_x + us_x)
\]

(1.46)

\[
= \frac{2k}{\gamma-1}\rho\gamma^{-1}e^{2s}(\rho s_t + (\rho - 2\delta)u_s)_x + \delta \frac{4k}{\gamma-1}\rho\gamma^{-1}e^{2s}u_s
\]

and

\[
\frac{1}{2}u^2(2\delta u_x) = \frac{\delta}{3}(u^3)_x,
\]

(1.47)

we may obtain the following equation by summing up (1.43) and (1.44)

\[
\frac{\delta}{3}(u^3)_x + \delta \frac{4k}{\gamma-1}\rho\gamma^{-1}e^{2s}u_s + 2\delta k \frac{\gamma}{(\gamma-1)}(\rho^{-1}e^{2s}u)_x
\]

(1.48)

\[
+ \frac{2k}{\gamma-1}\rho\gamma^{-1}e^{2s}(\rho s_t + (\rho - 2\delta)u_s)_x = \frac{2k}{\gamma-1}\rho\gamma^{-1}e^{2s}A(x,t) + \delta D(x,t)
\]

which deduces the third equation in (1.17).

Remark 1. From the analysis above, under the conclusions given in Theorem 1.1, since \(|s_x|\) is bounded in \(L^1_\text{loc}(R \times R^+)\), we may prove from (1.38) that

\[
\varepsilon A(x,t) = \varepsilon \rho u^2 + \varepsilon k\rho^{-2}e^{2s}(\gamma \rho_x^2 + \frac{4}{\gamma-1}\rho^2 s_x^2 + 4\rho \rho_x s_x)
\]

(1.49)

is bounded in \(L^1_\text{loc}(R \times R^+)\), and hence converges weakly to a nonnegative integrable measure \(\mu(x,t) \geq 0\), as \(\varepsilon, \delta \to 0\), and the limit \((\rho, u, s)\), given in Theorem 1.1, satisfies

\[
\left\{
\begin{array}{l}
\int_0^\infty \int_\infty \rho \phi_t + \rho u \phi_x \phi dxd t + \int_\infty \rho_0(x) \phi(x,0) dx = 0,
\\
\int_0^\infty \int_\infty \rho \phi_t + (\rho u^2 + \frac{\rho^2}{\gamma-1}\rho \gamma e^{2s}) \phi_x dxd t + \int_\infty \rho_0(x) u_0(x) \phi(x,0) d x = 0,
\end{array}
\right.
\]

(1.50)
for all test function $\phi \in C_0^1(R \times R^+)$, and
\begin{equation}
\int_0^\infty \int_{-\infty}^\infty \left(\frac{1}{2} \rho u^2 + \frac{k}{\gamma - 1} \rho^\gamma e^{2s}\right) \phi_t + \left(u\left(\frac{1}{2} \rho u^2 + \frac{k}{\gamma - 1} \rho^\gamma e^{2s}\right)\right) \phi_x \, dx \, dt
\end{equation}
\begin{equation}
\int_{-\infty}^\infty \left(\frac{1}{2} \rho u_0(x)(u_0(x))^2 + \frac{k}{\gamma - 1} (\rho_0(x))^{\gamma} e^{2s_0(x)}\right) \phi(x,0) \, dx
\end{equation}
\begin{equation}
= \int_0^\infty \int_{-\infty}^\infty \mu(x,t) \phi \, dx \, dt \geq 0,
\end{equation}
for any nonnegative test function $\phi \in C_0^1(R \times R^+)$. This means that, if we could prove $\mu(x,t) = 0$, then the global existence of the classical entropy solutions, in the sense of Definition 1, for the system of non-isentropic polytropic gas flow (1.1) could be obtained by using the vanishing viscosity method introduced in (1.15) or (1.17). Clearly, for classical smooth solutions, Systems (1.1) and (1.3) are equivalent because $\mu(x,t) = 0$.

**Remark 2.** The inequality (1.51) is natural because it is the entropy inequality, for the physical entropy-entropy flux pair $(\eta_0, q_0)$ of the isentropic gas dynamics (1.5), in the sense of (1.29), where $s$ is considered as a parameter, and
\begin{equation}
(\eta_0, q_0) = \left(\frac{1}{2} \rho u^2 + \frac{k}{\gamma - 1} \rho^\gamma e^{2s}, \left(\frac{1}{2} \rho u^2 + \frac{k}{\gamma - 1} \rho^\gamma e^{2s}\right)_x\right).
\end{equation}
So, we obtain the following definition of a non-classical generalized solution.

**Definition 2.** A set of bounded functions $(\rho(x,t), u(x,t), s(x,t))$ is called a non-classical generalized solution of the Cauchy problem (1.1) and (1.4) if the pair of functions $(\rho(x,t), u(x,t))$ satisfies the mass and the momentum equations in (1.1) in the sense of distributions, and the function $s(x,t)$ satisfies the entropy equation in (1.4) with a nonnegative integrable measure $\mu(x,t)$ in the sense of (1.51).

Based on the conclusions in Theorem 1.1, we have the following existence of solutions for the Cauchy problem (1.1) and (1.4).

**Theorem 1.2.** Under the conditions of the initial data in Theorem 1.1, the Cauchy problem of the non-isentropic polytropic gas flow (1.1) and the initial data (1.4) has a non-classical generalized solution.

### 2. Proof of Theorem 1.1.

In this section we shall prove Theorem 1.1. First, following the standard theory of semilinear parabolic systems, the local existence result of the Cauchy problem (1.17), (1.18) can be easily obtained by applying the contraction mapping principle to an integral representation for a solution. Second, we have the following lemma about the estimates on $s^{\varepsilon, \delta}(x,t)$.

**Lemma 2.1.** If $|s_0(x)| \leq N$ and $|s_0(x)|_{L^1(\mathbb{R})} \leq c_0$, then the functions $s^{\varepsilon, \delta}(x,t)$ obtained from the Cauchy problem (1.17) and (1.18) satisfy
\begin{equation}
|s^{\varepsilon, \delta}(x,t)| \leq N, \quad |s^{\varepsilon, \delta}(x,t)|_{L^1(\mathbb{R})} \leq c_0.
\end{equation}
Moreover,
\begin{equation}
\int_{-\infty}^x |\omega|_t \, dx + \frac{(\rho - 2\delta)}{\rho} u|\omega| = \varepsilon |\omega|_x + 2\tau \frac{\rho_x}{\rho} |\omega|,
\end{equation}
where $\omega(x,t) = s^{\varepsilon, \delta}_x(x,t)$ and $N, c_0 < 1$ are two positive constants.
Proof of Lemma 2.1. The first assertion in (2.1) can be obtained by applying the maximum principle to (1.21) directly.

Differentiating Equation (1.21) with respect to $x$, we have

$$\omega_i + \frac{(\rho - 2\delta)}{\rho} u_x \omega = \omega_x + (\frac{\rho - 2\delta}{\rho} u_x - 2(\frac{\rho - 2\delta}{\rho}) \omega_x.$$  

(2.3)

The second assertion in (2.1) can be proved by the methods given in [1, 24, 26].

For any fixed $t$, since $s^{\epsilon, \delta}(x, t)$ is of bounded total variation, $s^{\epsilon, \delta}(x, t)$ changes the sign at most the countable points $x_i, i = 1, 2, 3, \ldots$. Thus the following equality is true except at these points $(x_i, t)$

$$|\omega|_t + \frac{(\rho - 2\delta)}{\rho} |\omega| = \varepsilon |\omega|_x + 2\varepsilon \frac{\partial x}{\rho} |\omega|_x.$$  

(2.4)

Integrating (2.4) in $(-\infty, x)$, we get the proof of (2.2). Lemma 2.1 is proved.

To obtain the a-priori upper estimates of $w^{\epsilon, \delta}(x, t)$ and $z^{\epsilon, \delta}(x, t)$ given in (1.22), we first prove (1.22) and (1.23).

The proof of (1.22) and (1.23). By simple calculations,

$$w_p = \theta \rho \varepsilon e - \frac{m}{\rho^2}, \quad w_m = \frac{1}{\rho}, \quad w_s = \rho \varepsilon e,$$

$$w_{p} = \theta/(\theta - 1) \rho \varepsilon e + 2 \frac{m}{\rho^2}, \quad w_{pm} = -\frac{1}{\rho^2}, \quad w_{ps} = \rho \varepsilon e,$$

$$w_{mm} = 0, \quad w_{sm} = 0, \quad w_{ss} = \rho \varepsilon e,$$

$$z_p = \theta \rho \varepsilon e - \frac{m}{\rho^2}, \quad z_m = -\frac{1}{\rho}, \quad z_s = \rho \varepsilon e,$$

$$z_{pp} = \theta/(\theta - 1) \rho \varepsilon e - 2 \frac{m}{\rho^2}, \quad z_{pm} = \frac{1}{\rho^2}, \quad z_{ps} = \rho \varepsilon e,$$

$$z_{mm} = 0, \quad z_{sm} = 0, \quad z_{ss} = \rho \varepsilon e.$$

Multiplying $(w_p, w_m)$ to the first two equations in system (1.17), and $w_s$ to (1.21), then adding the results, we obtain an equality whose left-hand side is

$$L = w_t + \lambda \delta w_x - \lambda \delta w_s s_x + \frac{\varepsilon - 2\delta}{\rho} u w_s s_x$$

$$= w_t + \lambda \delta w_x - (\frac{2\delta}{\rho} u + \theta(\rho - 2\delta) \rho \varepsilon e + \rho \varepsilon e) s_x$$

and the right-hand side is

$$R = \varepsilon w_{xx} - \varepsilon(w_{pp} \rho_s^2 + 2w_{pm} \rho_s m_s + w_{mm} m_s^2 + w_{ss} s_s^2 + 2w_{sm} s_s m_s$$

$$+ 2w_{ps} \rho_s s_x) + 2\varepsilon \frac{\delta}{\rho} w_s s_x$$

$$= \varepsilon w_{xx} + \frac{2\delta}{\rho} \rho_s w_x - \varepsilon \rho \varepsilon s_s^2 \theta(\rho - 1) \rho_s^2 + 2\theta \rho \rho_s s_x + \rho \varepsilon e^2,$$

so we proved (1.22). Similarly, if we multiply $(z_p, z_m)$ to the first two equations in system (1.17), $z_s$ to (1.21) and add the results, we may obtain the proof of (1.23).

Lemma 2.2. Make the transformation of variables

$$w = v_1 + M + c \int_{-\infty}^{x} [s_x]dx, \quad z = v_2 + M - c \int_{-\infty}^{x} [s_x]dx.$$  

(2.5)
where $M, c$ are suitable large constants, satisfying $0 < M \leq c$, $cc_0 < M$. Then the new variables $v_1, v_2$ satisfy the following system of two inequalities

$$
\begin{align*}
&v_{1t} + a_1(x,t)v_{1x} + b_1(x,t)v_1 + c_1(x,t)v_2 \leq \varepsilon v_{1xx}, \\
&v_{2t} + a_2(x,t)v_{2x} + b_2(x,t)v_2 + c_2(x,t)v_1 \leq \varepsilon v_{2xx},
\end{align*}
$$

(2.6)

where $a_i(x,t), b_i(x,t), c_i(x,t) \leq 0, i = 1, 2$, are suitable functions.

**Proof of Lemma 2.2.** Using the transformation (2.5), we have from (1.22) that

$$
v_{1t} + c \int_{-\infty}^{x} |s_x| dt + \lambda_2^\delta v_{1x} + \lambda_2^\delta c |s_x| - \left( \frac{2\delta}{\rho} u + \theta(\rho - 2\delta) - e^s \right) \rho^\delta e^s s_x
$$

(2.7)

$$
= \varepsilon v_{1xx} + c |s_x|_{x} + 2c \rho_x v_{1x} + 2c \rho_x c |s_x| - \varepsilon e^s \rho^\delta - 2(\theta(\rho + 1) + 2\theta \rho_x s_x + \rho^2 s_x^2)
$$

$$
\leq \varepsilon v_{1xx} + c |s_x|_{x} + 2c \rho_x v_{1x} + 2c \rho_x c |s_x|.
$$

From (2.2), we have

$$
c \int_{-\infty}^{x} |s_x|_{x} dt - \varepsilon |s_x|_{x} - 2c c \rho_x |s_x| = -\left( \frac{\rho - 2\delta}{\rho} \right) u |s_x|.
$$

(2.8)

Since, we finally obtain the estimates $v_1 \leq 0, v_2 \leq 0$, which deduce from (2.5) that

$$
w \leq M + c \int_{-\infty}^{x} |s_x|_{x} dx, \ z \leq M - c \int_{-\infty}^{x} |s_x|_{x} dx, \ \rho^\delta e^s = \frac{1}{2}(w + z) \leq M,
$$

thus it is enough if we may prove Lemma 2.2 in the region $\rho^\delta e^s \leq M$.

By simple calculations,

$$
\lambda_2^\delta |s_x| - \left( \frac{2\delta}{\rho} u + \theta(\rho - 2\delta) - e^s \right) \rho^\delta e^s s_x - c \left( \frac{\rho - 2\delta}{\rho} \right) u |s_x|
$$

(2.10)

$$
= (u + e^{2\delta} \rho(\rho^\delta e^s)|c|s_x| - \left( \frac{2\delta}{\rho} u + \theta(\rho - 2\delta) - e^s \right) \rho^\delta e^s s_x - c \left( \frac{\rho - 2\delta}{\rho} \right) u |s_x|,
$$

which we write as $I_1(x, t)$. First, at the points $(x, t)$, where $s_x \geq 0$, we have $s_x = |s_x|$ and

$$
\begin{align*}
I_1(x, t) &= (u + e^{2\delta} \rho(\rho^\delta e^s)|c|s_x| - \left( \frac{2\delta}{\rho} u + \theta(\rho - 2\delta) - e^s \right) \rho^\delta e^s |s_x| - c \left( \frac{\rho - 2\delta}{\rho} \right) u |s_x|
\end{align*}
$$

(2.11)

$$
= \theta e^{2\delta} \rho^\delta e^s |s_x| (c - \rho^\delta e^s) + \frac{2\delta}{\rho} |s_x| u(c - \rho^\delta e^s) \geq \frac{2\delta}{\rho} |s_x| u(c - \rho^\delta e^s)
$$

$$
= \frac{2\delta}{\rho} |s_x| (c - \rho^\delta e^s) \frac{1}{2} (v_1 - v_2 + 2c \int_{-\infty}^{x} |s_x|_{x} dx) \geq \frac{2\delta}{\rho} |s_x| (c - \rho^\delta e^s) \frac{1}{2} (v_1 - v_2),
$$

where we used $\rho^\delta e^s \leq M \leq c$. Similarly, at the points $(x, t)$, where $s_x \leq 0$, we have $s_x = -|s_x|$ and

$$
\begin{align*}
I_1(x, t) &= (u + e^{2\delta} \rho(\rho^\delta e^s)|c|s_x| + \left( \frac{2\delta}{\rho} u + \theta(\rho - 2\delta) - e^s \right) \rho^\delta e^s |s_x| - c \left( \frac{\rho - 2\delta}{\rho} \right) u |s_x|
\end{align*}
$$

(2.12)

$$
= \theta e^{2\delta} \rho^\delta e^s |s_x| (c + \rho^\delta e^s) + \frac{2\delta}{\rho} |s_x| u(c + \rho^\delta e^s) \geq \frac{2\delta}{\rho} |s_x| u(c + \rho^\delta e^s)
$$

$$
= \frac{2\delta}{\rho} |s_x| (c + \rho^\delta e^s) \frac{1}{2} (v_1 - v_2 + 2c \int_{-\infty}^{x} |s_x|_{x} dx) \geq \frac{2\delta}{\rho} |s_x| (c + \rho^\delta e^s) \frac{1}{2} (v_1 - v_2),
$$

Thus, we have from (2.7), (2.8), (2.11) and (2.12) that

$$
v_{1t} + (\lambda_2^\delta - \frac{2\delta}{\rho} \rho_x) v_{1x} + \frac{2\delta}{\rho} |s_x| (c - sgn(s_x) \rho^\delta e^s) \frac{1}{2} (v_1 - v_2) \leq \varepsilon v_{1xx},
$$

(2.13)
which gives us the proof of the first inequality in (2.6).

Similarly, we have from (1.23) and the transformation (2.5) that

\[(2.14)\]

\[v_{2x} - c \int_{-\infty}^{x} |s_x| dx + \lambda_1^0 v_{2x} - \lambda_1^0 c |s_x| - \left( \frac{2\delta}{\rho} u - \theta (\rho - 2\delta) \rho^{\theta - 1} e^s \right) \rho^\theta e^s s_x\]

\[= \varepsilon v_{2xx} - \varepsilon c |s_x| + \frac{2\delta}{\rho} \rho_x v_{2x} - \frac{2\delta}{\rho} \rho_x c |s_x| - \varepsilon \rho^\theta \rho^{\theta - 2} (\theta + 1) \rho_x^2 + 2\theta \rho \rho_x s_x + \rho^2 s_x^2\]

\[\leq \varepsilon v_{2xx} - \varepsilon c |s_x| + \frac{2\delta}{\rho} \rho_x v_{2x} - \frac{2\delta}{\rho} \rho_x c |s_x|\]

By simple calculations,

\[(2.15)\]

\[-\lambda_1^0 c |s_x| - \left( \frac{2\delta}{\rho} u - \theta (\rho - 2\delta) \rho^{\theta - 1} e^s \right) \rho^\theta e^s s_x + c \left( \frac{\rho - 2\delta}{\rho} \right) u |s_x|\]

\[= (-u + \frac{\rho - 2\delta}{\rho} \theta^\rho e^s) c |s_x| - \left( \frac{2\delta}{\rho} u - \theta (\rho - 2\delta) \rho^{\theta - 1} e^s \right) \rho^\theta e^s s_x + c \left( \frac{\rho - 2\delta}{\rho} \right) u |s_x| = I_2(x, t)\]  

At the points \((x, t)\), where \(s_x \geq 0\), we have \(s_x = |s_x|\) and

\[(2.16)\]

\[I_2(x, t) = \theta \frac{\rho - 2\delta}{\rho} \theta^\rho e^s |s_x|(c + \rho^\theta e^s) - \frac{2\delta}{\rho} |s_x| u(c + \rho^\theta e^s) = I(x, t) |s_x|(c + \rho^\theta e^s),\]

where

\[I(x, t) = \theta \frac{\rho - 2\delta}{\rho} \theta^\rho e^s - \frac{2\delta}{\rho} u = \theta \frac{\rho - 2\delta}{\rho} \theta^\rho e^s + \frac{2\delta}{\rho} (z - \rho^\theta e^s)\]

\[(2.17)\]

\[= \theta \frac{\rho - 2\delta}{\rho} \theta^\rho e^s + \frac{2\delta}{\rho} (v_2 + M - c \int_{-\infty}^{x} |s_x| dx - \rho^\theta e^s)\]

\[= (\theta - (\theta + 1) \frac{2\delta}{\rho} \theta^\rho e^s + \frac{2\delta}{\rho} (M - c \int_{-\infty}^{x} |s_x| dx) + \frac{2\delta}{\rho} v_2.\]  

Similarly, at the points \((x, t)\), where \(s_x \leq 0\), we have \(s_x = -|s_x|\) and

\[(2.18)\]

\[I_2(x, t) = \theta \frac{\rho - 2\delta}{\rho} \theta^\rho e^s |s_x|(c - \rho^\theta e^s) - \frac{2\delta}{\rho} |s_x| u(c - \rho^\theta e^s) = I(x, t) |s_x|(c - \rho^\theta e^s).\]

Now, we analyze the function \(I(x, t)\). First, at the points \((x, t)\), where \(\theta - (\theta + 1) \frac{2\delta}{\rho} v_2 \geq 0\), we have \(I(x, t) \geq \frac{2\delta}{\rho} v_2\) due to \(c |s_x|_{L_1} \leq cc_0 < M\).

Second, at the points \((x, t)\), where \(\theta - (\theta + 1) \frac{2\delta}{\rho} v_2 \leq 0\) or \(\rho(x, t) \leq 2\delta \frac{\rho + 1}{\rho}\), we have

\[(2.19)\]

\[\theta - (\theta + 1) \frac{2\delta}{\rho} \theta^\rho e^s + \frac{2\delta}{\rho} (M - c \int_{-\infty}^{x} |s_x| dx)\]

\[\geq (\theta - (\theta + 1) \frac{2\delta}{\rho} (2\theta^{\frac{\rho + 1}{\rho}}) \theta^\rho e^s + \frac{2\delta}{\rho} (M - c \int_{-\infty}^{x} |s_x| dx)\]

\[\geq -\frac{2\delta}{\rho} (2\theta^{\frac{\rho + 1}{\rho}}) \theta^\rho e^s + \frac{2\delta}{\rho} (M - c \int_{-\infty}^{x} |s_x| dx)\]

\[\geq \frac{2\delta}{\rho} (M - c \int_{-\infty}^{x} |s_x| dx) - (2\delta \frac{\rho + 1}{\rho} \theta^\rho e^s) \geq 0\]

if we choose \(\delta\) to be sufficiently small. Therefore we obtain the second inequality in (2.6) from (2.14), (2.2), (2.15)-(2.19), and complete the proof of Lemma 2.2.

Since the initial data are bounded, then at \(t = 0\), \(v_1(x, 0) = w - M - c \int_{-\infty}^{x} |s_x| dx \leq 0\), \(v_2(x, 0) = w - M + c \int_{-\infty}^{x} |s_x| dx \leq 0\) for large \(M\). Applying the maximum principle to (2.6), we have \(v_1(x, t) \leq 0\), \(v_2(x, t) \leq 0\) for any time \(t > 0\). Thus we have
where we used the estimates in (2.20) and so the estimates on the viscosity solutions $(\rho^{\varepsilon, \delta}(x, t), u^{\varepsilon, \delta}(x, t), s^{\varepsilon, \delta}(x, t))$ of the Cauchy problem (1.17) and (1.18):

\begin{equation}
2\delta \leq \rho^{\varepsilon, \delta}(x, t) \leq N, \quad |u^{\varepsilon, \delta}(x, t)| \leq N, \quad |s^{\varepsilon, \delta}(x, t)| \leq N, \quad |s^{\varepsilon, \delta}_x(\cdot, t)|_{L^1} \leq c_0 < 1,
\end{equation}

where $N$ is a positive constant depending only the bound of the initial data, but being independent of $\varepsilon$ and $\delta$.

With the uniformly bounded estimates in (2.20), we may extend the local solution of the Cauchy problem (1.17) and (1.18) step by step, until an arbitrary large time $T$ and obtain the global existence of solution. So, the part (I) in Theorem 1.1 is proved.

To obtain the pointwise convergence of a subsequence of $(\rho^{\varepsilon, \delta}, u^{\varepsilon, \delta}, s^{\varepsilon, \delta})$, we first have the following lemma

**Lemma 2.3.** There exists a subsequence (still labelled $s^{\varepsilon, \delta}(x, t)$) such that

\begin{equation}
s^{\varepsilon, \delta}(x, t) \to s(x, t)
\end{equation}

almost everywhere on the set $\rho_+ = \{(x, t): \rho(x, t) > 0\}$, where $\rho(x, t)$ is the weak-star limit of $\rho^{\varepsilon, \delta}(x, t)$.

**Proof of Lemma 2.3.** Since $s^{\varepsilon, \delta}_x$ and $((s^{\varepsilon, \delta})^2)_x$ are uniformly bounded in $W^{1, \infty}_0(R \times R^+)$, then

\begin{equation}
c_t + s^{\varepsilon, \delta}_x, \quad c_t + ((s^{\varepsilon, \delta})^2)_x \quad \text{are compact in} \quad H^{-1}_0(R \times R^+)
\end{equation}

by Murat’s Lemma 6, where $c$ is a constant.

Multiplying $\left(\frac{\partial \rho u}{\partial \rho} \frac{\partial u}{\partial \rho} \frac{\partial \rho}{\partial \rho}ight)$ to the first two equations in System (1.17), $\frac{\partial \mu}{\partial \mu}$ to (1.21), where $\eta_0$ is given in (1.52), then adding the results, we have

\begin{equation}
\left(\frac{4}{\gamma} \rho u^2 + k \frac{1}{\gamma - 1} \rho^\gamma e^{2s} \right)_t + u \left(\frac{4}{\gamma} \rho u^2 + k \frac{1}{\gamma - 1} \rho^\gamma e^{2s} \right)_x = \varepsilon \left(\frac{4}{\gamma} \rho u^2 + k \frac{1}{\gamma - 1} \rho^\gamma e^{2s} \right)_{xx}
\end{equation}

\begin{equation}
\left(\frac{4}{\gamma} \rho u^2 + k \rho^\gamma e^{2s} \left(\gamma \rho_x^2 + \frac{4}{\gamma - 1} \rho^2 s_x^2 + 4 \rho \rho_x s_x \right) \right) + \varepsilon \left(\frac{4}{\gamma} \rho u^2 + k \rho^\gamma e^{2s} \right)_{xx}.
\end{equation}

Let $K \subset R \times R^+$ be an arbitrary compact set and choose $\phi \in C_0^\infty(R \times R^+)$ such that $\phi K = 1, 0 \leq \phi \leq 1$.

Multiplying Equation (2.23) by $\phi$ and integrating over $R \times R^+$, we may obtain

\begin{equation}
\int_0^\infty \int_{-\infty}^\infty \varepsilon \left(\rho u^2 + k \rho^\gamma e^{2s} \left(\gamma \rho_x^2 + \frac{4}{\gamma - 1} \rho^2 s_x^2 + 4 \rho \rho_x s_x \right) \right) \phi dx dt \leq M(\phi),
\end{equation}

where we used the $L^1$ local integrability of $S_x$, and hence that

\begin{equation}
\varepsilon \rho u^2, \varepsilon \rho^\gamma e^{2s}, \varepsilon \rho^\gamma \rho_x^2 \quad \text{are bounded in} \quad L^1_{loc}(R \times R^+),
\end{equation}

due to

\begin{equation}
\gamma \rho_x^2 + \frac{4}{\gamma - 1} \rho^2 s_x^2 + 4 \rho \rho_x s_x \geq c (\rho_x^2 + \rho^2 s_x^2)
\end{equation}

for a suitable constant $c > 0$.
For any \( \varphi \in H^1_0(R \times R^+) \), we have from the estimates in (2.23) that
\[
\left| \int_0^\infty \int_{-\infty}^\infty \varepsilon \rho \varphi_x dxdt \right| = \left| \int_0^\infty \int_{-\infty}^\infty \varepsilon \rho \varphi_x dxdt \right|
\leq (\int_0^\infty \int_{-\infty}^\infty \varepsilon \rho^{-1-2} \rho^2 |\varphi_x| dxdt)^{\frac{1}{2}} (\int_0^\infty \int_{-\infty}^\infty \varepsilon \rho^{-2-\gamma} |\varphi_x| dxdt)^{\frac{1}{2}}
\leq M (\int_0^\infty \int_{-\infty}^\infty \varepsilon \rho^{-2-\gamma} |\varphi_x| dxdt)^{\frac{1}{2}} \to 0
\]
and
\[
\left| \int_0^\infty \int_{-\infty}^\infty \varepsilon (\rho s)_{xx} \varphi_x dxdt \right| = \left| \int_0^\infty \int_{-\infty}^\infty \varepsilon (\rho s + \rho s_x) \varphi_x dxdt \right|
\leq (\int_0^\infty \int_{-\infty}^\infty \varepsilon \rho^{-1-2} \rho^2 |\varphi_x| dxdt)^{\frac{1}{2}} (\int_0^\infty \int_{-\infty}^\infty \varepsilon \rho^{-2-\gamma} s^2 |\varphi_x| dxdt)^{\frac{1}{2}}
\quad + (\int_0^\infty \int_{-\infty}^\infty \varepsilon \rho^{-1} |\varphi_x| dxdt)^{\frac{1}{2}} (\int_0^\infty \int_{-\infty}^\infty \varepsilon \rho^{-2-\gamma} |\varphi_x| dxdt)^{\frac{1}{2}}
\leq M (\int_0^\infty \int_{-\infty}^\infty \varepsilon \rho^{-2-\gamma} |\varphi_x| dxdt)^{\frac{1}{2}} \to 0,
\]
because we may choose \( \varepsilon \) to go zero much faster than \( \delta \) such that \( \varepsilon \rho^{-2-\gamma} \to 0 \) as \( \varepsilon, \delta \) go to zero. Then we have from the first and the third equations in (1.17) that
\[
\rho_t \varphi_t + ((\rho \varphi - 2\delta)u \varphi)_x \quad \text{and} \quad (\rho \varphi s \varphi t + ((\rho \varphi - 2\delta)u \varphi s \varphi t)_x
\]
are compact in \( H^1_0(R \times R^+) \).

Thus we may apply the div-curl lemma to the pairs of functions
\[
(c, s \varphi), \quad (\rho \varphi, (\rho \varphi - 2\delta)u \varphi)
\]
and
\[
(c, s \varphi), \quad (\rho \varphi s \varphi, (\rho \varphi - 2\delta)u \varphi s \varphi)
\]
respectively to obtain
\[
(\rho \varphi \varphi, s \varphi \varphi), \quad (\rho \varphi s \varphi s \varphi, (\rho \varphi - 2\delta)u \varphi s \varphi s \varphi)
\]
where \( f(\theta \varphi \varphi) \) denotes the weak-star limit of \( f(\theta \varphi \varphi) \).

Let \((\rho \varphi \varphi, s \varphi \varphi) = (\rho, s)\). We have from (2.32) that
\[
\rho \varphi \varphi (s \varphi \varphi - s)^2 = \rho \varphi \varphi (s \varphi \varphi)^2 - 2s \rho \varphi \varphi s \varphi \varphi + \rho s^2 = 0.
\]
Furthermore, we may apply the div-curl lemma to the pair of functions
\[
(c, (s \varphi)^2), \quad (\rho \varphi \varphi, (\rho \varphi - 2\delta)u \varphi \varphi)
\]
to obtain
\[
(\rho \varphi \varphi \varphi, (s \varphi)^2)^2 = \rho \varphi \varphi \varphi (s \varphi)^2.
\]
Using (2.32), (2.33) and (2.35), we have
\[
\rho (s \varphi - s)^2 = \rho (s \varphi)^2 - 2 s \rho s s \varphi \varphi + \rho s^2
\]
\[
= \rho \varphi \varphi (s \varphi)^2 - 2s \rho \varphi \varphi s \varphi \varphi + \rho s^2 = 0.
\]
Then
\[
\rho (s \varphi - s)^2 \to 0, \quad \text{a.e.,}
\]
which deduces the proof of Lemma 2.3.
After we have the pointwise convergence of $s^\varepsilon$, we may consider $s$ as a constant (or a parameter), and study the following system

\[
\begin{cases}
\rho_t + ((\rho - 2\delta)u)_x = 0, \\
(\rho u)_t + (\rho u^2 - \delta u^2 + P_1(\rho, \delta)e^{2s})_x = 0.
\end{cases}
\] (2.38)

For smooth solutions, system (2.38) is equivalent to the following system

\[
\begin{cases}
\rho_t + (-2\delta u + \rho u)_x = 0, \\
u_t + (\frac{1}{2}u^2 + \int_0^t (t - 2\delta)P'(t) dt e^{2s})_x = 0,
\end{cases}
\] (2.39)

and particularly, both systems have the same entropy-entropy flux pairs. Thus any entropy-entropy flux pair $(\eta(\rho, u, s), q(\rho, u, s))$ of system (2.39) satisfies the additional system

\[
\eta_{\rho} = \frac{(\rho - 2\delta)P'(\rho)e^{2s}}{\rho^2} \eta_u, \quad q_u = (\rho - 2\delta)\eta_{\rho} + \eta_u.
\] (2.40)

Eliminating the $q$ from (2.40), we have

\[
\eta_{\rho \rho} = \theta^2 \rho^{-3} e^{2s} \eta_{uu}.
\] (2.41)

Therefore, system (2.38) has the same entropy equation, as system (1.5), given in [6].

An entropy $\eta(\rho, u, s)$ of system (2.38) is called a weak entropy if $\eta(0, u, s) = 0$, that is, a solution of Equation (2.41) with the special initial conditions:

\[
\begin{cases}
\eta(\rho = 0, u, s) = 0, \\
\eta_{\rho}(\rho = 0, u, s) = f(u, 0) = g(u),
\end{cases}
\] (2.42)

where $g(u)$ is an arbitrary given function of $u$. The solution of (2.41)-(2.42) is given by the following lemma:

**Lemma 2.4.** For $\rho \geq 0, u, w \in R$, let

\[
G(\rho, s, w) = (\rho^{-1}e^{2s} - w^2)^\lambda_+, \quad \lambda = \frac{3 - \gamma}{2(\gamma - 1)},
\] (2.43)

where the notation $x_+ = \sup(0, x)$. Then we have

\[
\eta(\rho, u, s) = \int_R g(\xi)G(\rho, s, \xi - u) d\xi
\] (2.44)

and the weak entropy flux $q(\rho, u, s)$ of system (2.38) associated with $\eta(\rho, u, s)$ is

\[
q(\rho, u, s) = \rho \int_0^1 \tau(1 - \tau)^\lambda g(u + \rho^0 e^s - 2\rho^0 e^s\tau) d\tau;
\]

\[
q_u = \rho \int_0^1 \tau(1 - \tau)^\lambda g(u + \rho^0 e^s - 2\rho^0 e^s\tau)(u + \theta(1 - 2\tau)\rho^0 e^s) d\tau
\] (2.45)

where $G(y) = \int^y g(x) dx$. 


Proof of Lemma 2.4. The weak entropy formula (2.44) is given in [10] (See also [27] or Lemma 8.2.1 in [28]).

Using the second equation in (2.40) and the weak solution formula (2.41), we have

\[ q_{u}(\rho, u, s) = \eta + \theta \int_{0}^{1} \lambda g'(u + \rho^\theta e^s - 2\rho^\theta e^s \tau)(1 - 2\tau)\rho^{\theta + 1}e^s d\tau \]

\[ + u\rho \int_{0}^{1} \lambda g'(u + \rho^\theta e^s - 2\rho^\theta e^s \tau)d\tau \]

\[ -2\delta \int_{0}^{1} \lambda g(u + \rho^\theta e^s - 2\rho^\theta e^s \tau)d\tau \]

\[ -2\delta \theta \int_{0}^{1} \lambda g(u + \rho^\theta e^s - 2\rho^\theta e^s \tau)(1 - 2\tau)\rho^\theta e^s d\tau. \]

Since

\[ \int_{0}^{1} u g'(u + \rho^\theta e^s - 2\rho^\theta e^s \tau)du \]

\[ = ug(u + \rho^\theta e^s - 2\rho^\theta e^s \tau) - \int_{0}^{1} g(u + \rho^\theta e^s - 2\rho^\theta e^s \tau)du, \]

we get from (2.40) that

\[ q(\rho, u, s) = u\eta + \theta \int_{0}^{1} \lambda g(u + \rho^\theta e^s - 2\rho^\theta e^s \tau)(1 - 2\tau)\rho^{\theta + 1}e^s d\tau \]

\[ -2\delta \int_{0}^{1} \lambda G(u + \rho^\theta e^s - 2\rho^\theta e^s \tau)d\tau \]

\[ -2\delta \theta \int_{0}^{1} \lambda g(u + \rho^\theta e^s - 2\rho^\theta e^s \tau)(1 - 2\tau)\rho^\theta e^s d\tau \]

\[ = q_1(\rho, u, s), \]

where \( G(y) = \int_{0}^{y} g(x)dx, \)

\[ q_1(\rho, u, s) = u\eta + \theta \int_{0}^{1} \lambda g(u + \rho^\theta e^s - 2\rho^\theta e^s \tau)(1 - 2\tau)\rho^{\theta + 1}e^s d\tau \]

\[ = \rho \int_{0}^{1} \lambda g(u + \rho^\theta e^s - 2\rho^\theta e^s \tau)(u + \theta(1 - 2\tau)\rho^\theta e^s)d\tau \]

and

\[ q_2(\rho, u, s) = \int_{0}^{1} \lambda G(u + \rho^\theta e^s - 2\rho^\theta e^s \tau)d\tau \]

\[ + \theta \int_{0}^{1} \lambda g(u + \rho^\theta e^s - 2\rho^\theta e^s \tau)(1 - 2\tau)\rho^\theta e^s d\tau. \]

To complete the proof of Lemma 6, we still need to prove that \( q(\rho, u, s), \) given in (2.48), satisfies the first equation in (2.40), namely

\[ q_{1\rho} = u\eta_\rho + \theta^2 \rho^\gamma - 2\rho^\gamma e^{2s} \eta_u, \]

\[ -2\delta q_{2\rho} = -2\delta \theta^2 \rho^{\gamma - 3} e^{2s} \eta_u. \]
By simple calculations, (2.52)
\[
q_{1\rho}(\rho, u, s) = u\eta_\rho + \theta^2 \int_0^1 [\tau(1 - \tau)]^\lambda g'(u + \rho^\theta e^s - 2\rho^\theta e^s\tau)(1 - 2\tau)^2 \rho^{2\theta} e^{2s} \rho^{2\theta} e^{2s} d\tau \\
+ \theta(\theta + 1) \int_0^1 [\tau(1 - \tau)]^\lambda g(u + \rho^\theta e^s - 2\rho^\theta e^s\tau)(1 - 2\tau)\rho^\theta e^s d\tau \\
= u\eta_\rho + \theta^2 \rho^{2\theta} e^{2s} \int_0^1 [\tau(1 - \tau)]^\lambda g'(u + \rho^\theta e^s - 2\rho^\theta e^s\tau) d\tau \\
+ \theta^2 \int_0^1 [\tau(1 - \tau)]^\lambda g(u + \rho^\theta e^s - 2\rho^\theta e^s\tau)(1 - 2\tau)\rho^\theta e^s d\tau \\
+ \theta(\theta + 1) \int_0^1 [\tau(1 - \tau)]^\lambda g(u + \rho^\theta e^s - 2\rho^\theta e^s\tau)(1 - 2\tau)\rho^\theta e^s d\tau \\
= u\eta_\rho + \theta^2 \rho^{2\theta} e^{2s} \eta_u + 2\theta^2 \rho^\theta e^s \int_0^1 [\tau(1 - \tau)]^\lambda + 1 d(g(u + \rho^\theta e^s - 2\rho^\theta e^s\tau)) \\
+ \theta(\theta + 1) \int_0^1 [\tau(1 - \tau)]^\lambda g(u + \rho^\theta e^s - 2\rho^\theta e^s\tau)(1 - 2\tau)\rho^\theta e^s d\tau \\
= u\eta_\rho + \theta^2 \rho^{2\theta} e^{2s} \eta_u \\
due to \theta(\theta + 1) = 2\theta(\lambda + 1). Moreover, 
\[
q_{2\rho}(\rho, u, s) = \theta \int_0^1 [\tau(1 - \tau)]^\lambda g(u + \rho^\theta e^s - 2\rho^\theta e^s\tau)(1 - 2\tau)\rho^{\theta - 1} e^s d\tau \\
+ \theta^2 \int_0^1 [\tau(1 - \tau)]^\lambda g'(u + \rho^\theta e^s - 2\rho^\theta e^s\tau)(1 - 2\tau)^2 \rho^{2\theta - 1} e^{2s} d\tau \\
+ \theta^2 \int_0^1 [\tau(1 - \tau)]^\lambda g(u + \rho^\theta e^s - 2\rho^\theta e^s\tau)(1 - 2\tau)\rho^{\theta - 1} e^s d\tau \\
= (\theta + \theta^2) \int_0^1 [\tau(1 - \tau)]^\lambda g(u + \rho^\theta e^s - 2\rho^\theta e^s\tau)(1 - 2\tau)\rho^{\theta - 1} e^s d\tau \\
+ \theta^2 \int_0^1 [\tau(1 - \tau)]^\lambda g'(u + \rho^\theta e^s - 2\rho^\theta e^s\tau)(1 - 2\tau)^2 \rho^{2\theta - 1} e^{2s} d\tau. \\
\]
(2.53)
Since
\[
\theta^2 \int_0^1 [\tau(1 - \tau)]^\lambda g'(u + \rho^\theta e^s - 2\rho^\theta e^s\tau)(1 - 2\tau)^2 \rho^{2\theta - 1} e^{2s} d\tau \\
= \theta^2 \rho^{2\theta - 1} e^{2s} \int_0^1 [\tau(1 - \tau)]^\lambda g'(u + \rho^\theta e^s - 2\rho^\theta e^s\tau) d\tau \\
- 4\theta^2 \int_0^1 [\tau(1 - \tau)]^\lambda + 1 g'(u + \rho^\theta e^s - 2\rho^\theta e^s\tau)\rho^{2\theta - 1} e^{2s} d\tau \\
= \theta^2 \rho^{2\theta - 1} e^{2s} \eta_u - 2\theta^2 (\lambda + 1)\rho^{\theta - 1} e^s \int_0^1 [\tau(1 - \tau)]^\lambda g(u + \rho^\theta e^s - 2\rho^\theta e^s\tau) d\tau, \\
\]
we have the second equality in (2.51) by summing up (2.53) and (2.54),
(2.55)
\[
q_{2\rho} = \theta^2 \rho^{2\theta - 1} e^{2s} \eta_u, \\
\]
and hence obtain the proof of Lemma 2.4.

**Lemma 2.5.**
(2.56) \(\eta(\rho^\gamma, \eta^\gamma, s^\gamma) + q_{1\rho}(\rho^\gamma, u^\gamma, s^\gamma)\) are compact in \(H^{-1}_{loc}(R \times R^+)\), with respect to the viscosity solutions \((\rho^\gamma, u^\gamma, s^\gamma)\) of the Cauchy problem (1.17) and (1.18), where \(\eta, q_1\) are given in (2.44) and (2.49).
Proof of Lemma 2.5. Let $\Phi(\rho, u, s, \tau) = u + \rho^\theta e^s - 2\rho^\theta e^\tau$. By simple calculations, we have from (2.44) that

\begin{equation}
\eta_\rho = \int_0^1 \tau(1-\tau)g(\Phi(\rho, u, s, \tau))d\tau \\
+\theta\rho^\theta e^s \int_0^1 \tau(1-\tau)g'\Phi(\rho, u, s, \tau)(1-2\tau)d\tau, \\
\eta_u = \rho \int_0^1 \tau(1-\tau)g'(\Phi(\rho, u, s, \tau))d\tau, \\
\eta_s = \rho^\theta + 1 e^s \int_0^1 \tau(1-\tau)g''(\Phi(\rho, u, s, \tau))(1-2\tau)d\tau,
\end{equation}

and

\begin{equation}
\eta_{pp} = (\theta + \theta^2)\rho^\theta - 1 e^s \int_0^1 \tau(1-\tau)g'(\Phi(\rho, u, s, \tau))(1-2\tau)d\tau \\
+\theta^2 \rho^\theta - 1 e^{2s} \int_0^1 \tau(1-\tau)g''(\Phi(\rho, u, s, \tau))(1-2\tau)^2d\tau, \\
\eta_{pu} = \int_0^1 \tau(1-\tau)g'(\Phi(\rho, u, s, \tau))d\tau \\
+\theta\rho^\theta e^s \int_0^1 \tau(1-\tau)g''(\Phi(\rho, u, s, \tau))(1-2\tau)d\tau, \\
\eta_{uu} = \rho \int_0^1 \tau(1-\tau)g''(\Phi(\rho, u, s, \tau))d\tau,
\end{equation}

Multiplying the first equation in (1.17) by $\eta_\rho$, the equation (1.42) by $\eta_m$, and (1.21) by $\eta_s$, then adding the result, we have (for simplicity, we omit the superscripts $\varepsilon$ and $\delta$)

\begin{align}
\eta(\rho, u, s)_t + (q_1(\rho, u, s) - 2\delta q_2(\rho, u, s))_x \\
-(q_1(\rho, u, s) - 2\delta q_2(\rho, u, s))_s s_x + \frac{(\rho - 2\delta)}{\rho} u_s \eta_s(\rho, u, s) \\
= \varepsilon \eta(\rho, u, s)_{xx} + 2\varepsilon \frac{u}{\rho} s_x \eta_s + 2\varepsilon \frac{u}{\rho} u_x \eta_u \\
- \varepsilon(\rho, u, s) \cdot \nabla^2 \eta(\rho, u, s) \cdot (\rho, u, s)^T,
\end{align}
where

\[(2.61) \quad \nabla^2 \eta(\rho, u, s) = \begin{pmatrix} \eta_{\rho\rho} & \eta_{\rho u} & \eta_{\rho s} \\ \eta_{\rho u} & \eta_{u u} & \eta_{u s} \\ \eta_{\rho s} & \eta_{u s} & \eta_{s s} \end{pmatrix}.\]

Let

\[(2.62) \quad 2\varepsilon \frac{\partial x}{\rho} s_x \eta_s + 2\varepsilon \frac{\partial x}{\rho} u_x \eta_u - \varepsilon (\rho, u, s) \cdot \nabla^2 \eta(\rho, u, s) \cdot (\rho, u, s)^T = I_1 + I_2\]

where \(I_1, I_2\) be the sets of all functions appeared in (2.57)-(2.59) with \(g'(\Phi)\) and \(g''(\Phi)\) respectively. Then

\[(2.63) \quad I_1 = 2\varepsilon \frac{\partial x}{\rho} s_x \eta_s + 2\varepsilon \frac{\partial x}{\rho} u_x \eta_u - \varepsilon \left((\theta + \theta^2)\rho^0 e^s \rho_x^2 + 2(1 + \theta)\rho^0 e^s \rho_x s_x + \rho^{1+\theta} e^s s_x^2\right)\]

\[\cdot \int_0^1 [\tau(1 - \tau)]^\lambda g'(\Phi(\rho, u, s, \tau))(1 - 2\tau)d\tau\]

\[-2\varepsilon \rho_x u_x \int_0^1 [\tau(1 - \tau)]^\lambda g'(\Phi(\rho, u, s, \tau))d\tau\]

\[-\varepsilon \left((\theta + \theta^2)\rho^0 e^s \rho_x^2 + 2(1 + \theta)\rho^0 e^s \rho_x s_x + \rho^{1+\theta} e^s s_x^2\right)\]

\[\cdot \int_0^1 [\tau(1 - \tau)]^\lambda g'(\Phi(\rho, u, s, \tau))(1 - 2\tau)d\tau\]

because

\[(2.64) \quad \int_0^1 [\tau(1 - \tau)]^\lambda g'(\Phi(\rho, u, s, \tau))(1 - 2\tau)d\tau = \frac{1}{\lambda + 1} \int_0^1 g'(\Phi(\rho, u, s, \tau))d[\tau(1 - \tau)]^{\lambda + 1}\]

\[= \frac{2}{\lambda + 1} \rho^0 e^s \int_0^1 [\tau(1 - \tau)]^{\lambda + 1} g''(\Phi(\rho, u, s, \tau))d\tau\]
and

\[ I_2 = -\varepsilon \left( \theta^2 \rho^{2\theta-1} e^{2s} \rho_x^2 + 2 \theta \rho^{2\theta} e^{2s} \rho_x s_x + \rho^{2\theta+1} e^{2s} s_x^2 \right) \]

\[ \cdot \int_0^1 [\tau(1 - \tau)]^{\lambda+1} g''(\Phi(\rho, u, s, \tau))(1 - 2\tau)^2 d\tau \]

(2.65)

\[-2\varepsilon \theta \rho^\theta e^s \int_0^1 [\tau(1 - \tau)]^{\lambda+1} g''(\Phi(\rho, u, s, \tau))(1 - 2\tau) d\tau \rho_x u_x \]

\[-\varepsilon \rho \int_0^1 [\tau(1 - \tau)]^{\lambda+1} g''(\Phi(\rho, u, s, \tau)) d\tau u_x^2 \]

\[-2\varepsilon \rho^{1+\theta} e^s \int_0^1 [\tau(1 - \tau)]^{\lambda+1} g''(\Phi(\rho, u, s, \tau))(1 - 2\tau) d\tau u_x s_x. \]

Since

(2.66) \[ \varepsilon \rho^\theta |\rho_x u_x| \leq \varepsilon \rho^{\gamma-2} \rho_x^2 + \varepsilon \rho^2 u_x^2, \quad \varepsilon \rho^{1+\theta} |u_x s_x| \leq \varepsilon \rho^\gamma s_x^2 + \varepsilon \rho u_x^2, \]

then the last three terms in (2.65)

\[ E(x, t) = 2\varepsilon \theta \rho^\theta e^s \int_0^1 [\tau(1 - \tau)]^{\lambda+1} g''(\Phi(\rho, u, s, \tau))(1 - 2\tau) d\tau \rho_x u_x \]

(2.67)

\[-\varepsilon \rho \int_0^1 [\tau(1 - \tau)]^{\lambda+1} g''(\Phi(\rho, u, s, \tau)) d\tau u_x^2 \]

\[-2\varepsilon \rho^{1+\theta} e^s \int_0^1 [\tau(1 - \tau)]^{\lambda+1} g''(\Phi(\rho, u, s, \tau))(1 - 2\tau) d\tau u_x s_x \]

are uniformly bounded in \( L^1_{\text{loc}}(R \times R^+) \) due to the estimates in (2.60).

Moreover, the terms on the left-hand side of (2.66)

(2.68) \[ F(x, t) = -(q_1(\rho, u, s) - 2\delta q_2(\rho, u, s))_x s_x + \frac{(\rho - 2\delta)}{\rho} u_x e u_s \eta_s(\rho, u, s) \]

are uniformly bounded in \( L^1_{\text{loc}}(R \times R^+) \).

Thus we have from (2.60), (2.63), (2.64) and (2.67) - (2.68) that

\[ \varepsilon \int_0^1 [\tau(1 - \tau)]^{\lambda+1} g''(\Phi(\rho, u, s, \tau)) d\tau \]

\[ \int_0^1 [\tau(1 - \tau)]^{\lambda+1} g''(\Phi(\rho, u, s, \tau))(1 - 2\tau)^2 d\tau \]

(2.69)

\[ = \varepsilon \eta(\rho, u, s)_x x + E(x, t) - F(x, t) \]

\[-\eta(\rho, u, s)_t - (q_1(\rho, u, s) - 2\delta q_2(\rho, u, s))_x. \]

Multiplying the right-hand side (we write it as \( R(x, t) \)) of Equation (2.69) by \( \phi \), where \( \phi \) is given in (2.24), and integrating over \( R \times R^+ \), we may obtain

(2.70)

\[ |\int_0^\infty \int_{-\infty}^\infty R(x, t) \phi(x, t) dx dt| = |\int_0^\infty \int_{-\infty}^\infty \eta(\rho, u, s) \phi(x, t) dx dt| + (q_1(\rho, u, s) - 2\delta q_2(\rho, u, s)) \phi(x, t)_x + (E(x, t) - F(x, t)) \phi(x, t) dx dt| \leq M(\phi). \]
Then, if we choose $g$, on the left-hand side of (2.69), to be strictly convex, $g''(\Phi) \geq c > 0$ for a constant $c$, we may obtain from (2.69) and (2.71) that
\begin{equation}
\varepsilon \left( (\theta + \theta^2) \rho^2 u_x^2 + 2 \theta^2 \rho^2 \rho s_x + \rho^{1+2\theta} s^2_x \right) \text{ are bounded in } L^1_{\text{loc}}(R \times R^+),
\end{equation}
which deduce that, for any smooth function $f$, $I_1 + I_2$ in (2.62) are bounded in $L^1_{\text{loc}}(R \times R^+)$. Furthermore, for any $\varphi \in H^1_0(R \times R^+)$, we have
\[ | \int_0^\infty \int_{-\infty}^\infty 2\delta q_{2x} \varphi dx dt | = | \int_0^\infty \int_{-\infty}^\infty 2\delta q_{2x} \varphi dx dt | \to 0 \]
as $\delta$ goes to zero, and from the estimates in (2.23) that
\begin{equation}
\left| \int_0^\infty \int_{-\infty}^\infty \varepsilon \eta_{xx} \varphi dx dt \right| = \left| \int_0^\infty \int_{-\infty}^\infty \varepsilon (\eta_{xu_x} + \eta_{u_x}) \varphi dx dt \right|
\leq M \left( \int_0^\infty \int_{-\infty}^\infty \varepsilon \rho^\gamma \rho^2 | \varphi_x | dx dt \right)^\frac{1}{\gamma} \left( \int_0^\infty \int_{-\infty}^\infty \varepsilon \rho^\gamma | \varphi_x | dx dt \right)^{\frac{1}{\gamma}}
\end{equation}
if we let $\varepsilon$ go to zero much faster than $\delta$. Therefore we obtain the proof of Lemma 2.5.
If we apply the div-curl lemma to any two pairs of weak entropy-entropy flux given in Lemma 2.5
\begin{equation}
\left( \eta^{(i)}(\rho^{\varepsilon, \delta}, u^{\varepsilon, \delta}, s^{\varepsilon, \delta}), q_1^{(i)}(\rho^{\varepsilon, \delta}, u^{\varepsilon, \delta}, s^{\varepsilon, \delta}) \right),
\end{equation}
where $i = 1, 2$ corresponds to $g_i(u)$ in (2.42), we have the following weak limit equations [5]
\begin{equation}
\eta^{(1)}(\rho^{\varepsilon, \delta}, u^{\varepsilon, \delta}, s^{\varepsilon, \delta}) \cdot q_1^{(2)}(\rho^{\varepsilon, \delta}, u^{\varepsilon, \delta}, s^{\varepsilon, \delta}) - \eta^{(2)}(\rho^{\varepsilon, \delta}, u^{\varepsilon, \delta}, s^{\varepsilon, \delta}) \cdot q_1^{(1)}(\rho^{\varepsilon, \delta}, u^{\varepsilon, \delta}, s^{\varepsilon, \delta})
= \eta^{(1)}(\rho^{\varepsilon, \delta}, u^{\varepsilon, \delta}, s^{\varepsilon, \delta}) \cdot q_1^{(2)}(\rho^{\varepsilon, \delta}, u^{\varepsilon, \delta}, s^{\varepsilon, \delta}) - \eta^{(2)}(\rho^{\varepsilon, \delta}, u^{\varepsilon, \delta}, s^{\varepsilon, \delta}) \cdot q_1^{(1)}(\rho^{\varepsilon, \delta}, u^{\varepsilon, \delta}, s^{\varepsilon, \delta}).
\end{equation}
Using the conclusion in Lemma 2.3, we may replace $s^{\varepsilon, \delta}$ in (2.74) by $s$ and have the following weak limit equations
\begin{equation}
\eta^{(1)}(\rho^{\varepsilon, \delta}, u^{\varepsilon, \delta}, s) \cdot q_1^{(2)}(\rho^{\varepsilon, \delta}, u^{\varepsilon, \delta}, s) - \eta^{(2)}(\rho^{\varepsilon, \delta}, u^{\varepsilon, \delta}, s) \cdot q_1^{(1)}(\rho^{\varepsilon, \delta}, u^{\varepsilon, \delta}, s)
= \eta^{(1)}(\rho^{\varepsilon, \delta}, u^{\varepsilon, \delta}, s) \cdot q_1^{(2)}(\rho^{\varepsilon, \delta}, u^{\varepsilon, \delta}, s) - \eta^{(2)}(\rho^{\varepsilon, \delta}, u^{\varepsilon, \delta}, s) \cdot q_1^{(1)}(\rho^{\varepsilon, \delta}, u^{\varepsilon, \delta}, s),
\end{equation}
where $s$ is the weak-star limit of $s^{\varepsilon, \delta}$.

**Proof of Lemma 2.6.** First, using the estimate (2.83), we have also
\begin{equation}
| \rho^{\varepsilon, \delta} | s^{\varepsilon, \delta} - s | = 0.
\end{equation}
Second, by the entropy-entropy flux formulas given in (2.44) and (2.45), we have
\begin{equation}
| f(\rho^{\varepsilon, \delta}, u^{\varepsilon, \delta}, s^{\varepsilon, \delta}) - f(\rho^{\varepsilon, \delta}, u^{\varepsilon, \delta}, s) | \leq M(\rho^{\varepsilon, \delta})^\theta | s^{\varepsilon, \delta} - s | \leq M_1 | s^{\varepsilon, \delta} - s |,
\end{equation}
and hence the proof of Lemma 2.6, where \( f \) is any one of the weak entropies \( \eta^{(i)} \) and the weak entropy fluxes \( q^{(i)} \).

**Proof of (II) in Theorem 1.1.** Paying attention to the special structure of the weak entropy-entropy flux formulas given in (2.44) and (2.45), and letting \( \omega^{\varepsilon, \delta} = \rho^{\varepsilon, \delta} e^{\Phi^\delta} \), we have the following weak limit equations from (2.75) and Lemma 2.6,

\[
\begin{align*}
\eta^{(1)}(\omega^{\varepsilon, \delta}, u^{\varepsilon, \delta}) - q^{(2)}(\omega^{\varepsilon, \delta}, u^{\varepsilon, \delta}) &= \eta^{(1)}(\omega^{\varepsilon, \delta}, u^{\varepsilon, \delta}) - q^{(2)}(\omega^{\varepsilon, \delta}, u^{\varepsilon, \delta}) - \eta^{(1)}(\omega^{\varepsilon, \delta}, u^{\varepsilon, \delta}) - q^{(2)}(\omega^{\varepsilon, \delta}, u^{\varepsilon, \delta}) \\
&= \eta^{(1)}(\omega^{\varepsilon, \delta}, u^{\varepsilon, \delta}) - q^{(2)}(\omega^{\varepsilon, \delta}, u^{\varepsilon, \delta}) - \eta^{(1)}(\omega^{\varepsilon, \delta}, u^{\varepsilon, \delta}) - q^{(2)}(\omega^{\varepsilon, \delta}, u^{\varepsilon, \delta})
\end{align*}
\]

(2.78)

where

\[
\eta(\omega, u) = \int_R g(\xi)G(\omega, \xi - u) d\xi
\]

(2.79)

and

\[
q(\omega, u) = \omega \int_0^1 [\tau(1 - \tau)]^\lambda g(u + \omega^\theta - 2\omega^\theta \tau)d\tau
\]

(2.80)

is a pair of weak entropy-entropy flux of the following isentropic gas dynamics system

\[
\begin{align*}
\omega_t + (\omega u)_x &= 0, \\
(\omega u)_t + (\omega u^2 + \frac{\rho^2}{\gamma} \omega^\gamma)_x &= 0.
\end{align*}
\]

(2.81)

Using the Young measure representation theorem from the compensated compactness theory, we may select a subsequence (still labelled) \( (\omega^{\varepsilon, \delta}, u^{\varepsilon, \delta}) \), and a family of positive measures \( \nu(x, t) \in M(R^2) \), depending measurably on \( (x, t) \in K \subset R \times R^+ \), such that

\[
\int_K \eta^{(1)}(\lambda)q^{(2)}(\lambda) - \eta^{(1)}(\lambda)q^{(2)}(\lambda) d\nu(x, t)(\lambda)
\]

(2.82)

\[
\int_K \eta^{(1)}(\lambda) d\nu(x, t)(\lambda) \cdot \int_K q^{(2)}(\lambda) d\nu(x, t)(\lambda)
\]

\[
- \int_K \eta^{(2)}(\lambda) d\nu(x, t)(\lambda) \cdot \int_K q^{(1)}(\lambda) d\nu(x, t)(\lambda).
\]

With the help of the measure equations (2.82) and the results given in [7, 8, 9, 44], we may deduce that, for any fixed point \( (x, t) \in R \times R^+ \), the Young measure \( \nu(x, t) \) is either wholly contained in the line \( \omega(x, t) = 0 \) or concentrated in one point \( (\omega_0(x, t), u_0(x, t)) \), and hence, \( (\omega^{\varepsilon, \delta}(x, t), u^{\varepsilon, \delta}(x, t)) \rightarrow (\omega(x, t), u(x, t)) \) almost everywhere on the set \( \omega_+ = \{(x, t) : \omega(x, t) > 0\} \), where \( \omega(x, t) \) is the weak-star limit of \( \omega^{\varepsilon, \delta}(x, t) \). Then, \( (\rho^{\varepsilon, \delta}(x, t), u^{\varepsilon, \delta}(x, t), s^{\varepsilon, \delta}(x, t)) \rightarrow (\rho(x, t), u(x, t), s(x, t)) \) almost everywhere on the set \( \rho_+ = \{(x, t) : \rho(x, t) > 0\} \) since \( \omega^{\varepsilon, \delta} = \rho^{\varepsilon, \delta} e^{\Phi^\delta} \) and \( \rho^\theta > 0 \).

Since the variables \( (\rho, \rho u, \rho s) \) in (1.3), and the corresponding fluxes \( (\rho u, \rho u^2 + \frac{\rho^2}{\gamma} \rho^\gamma, \rho u) \) are all zero at the line \( \rho = 0 \), we may prove that the set of functions \( (\rho, u, s) \) satisfies (1.28) and (1.29) by letting \( \varepsilon, \delta \) in (1.17) go to zero. Thus we complete the proof of Theorem 1.1.
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