Abstract. We compute explicitly the $A_\infty$-structure on the algebra $\text{Ext}^\ast(\mathcal{O}_C \oplus L, \mathcal{O}_C \oplus L)$, where $L$ is a line bundle of degree 1 on an elliptic curve $C$. The answer involves higher derivatives of Eisenstein series.

Introduction

The bounded derived category $D^b(X)$ of coherent sheaves is an important invariant of an algebraic variety $X$ (see [1] for a survey). This category can be described in a purely algebraic way if one has a generator, i.e., an object $G$ generating $D^b(X)$ as a triangulated category. With such an object one can associate a graded algebra $E_G = \oplus_{n \in \mathbb{Z}} \text{Ext}^n(G, G)$. However, in order to recover $D^b(X)$ from $E_G$ one has to take into account certain higher products which fit together into a structure of an $A_\infty$-algebra on $E_G$ (see [3] for an introduction into $A_\infty$-algebras). Namely, one can realize $E_G$ as the cohomology of a dg-algebra and then apply a general algebraic construction that gives an $A_\infty$-structure on such cohomology (see [7]). This $A_\infty$-structure is minimal in the sense that $m_1 = 0$, and is canonical up to $A_\infty$-equivalence. Now the category $D^b(X)$ can be shown to be equivalent to the derived category of perfect $A_\infty$-modules over $A$ (see [4], Thm. 3.1 or [6], 7.6).

Thus, it is of interest to compute explicitly higher products on algebras of the form $\text{Ext}^\ast(G, G)$ as above. In this paper we solve this problem in the case when $X$ is a complex elliptic curve and $G = \mathcal{O}_X \oplus L$, where $L$ is a line bundle of degree 1. Namely, we compute the $A_\infty$-structure arising from the harmonic representatives (with respect to natural metrics) in the Dolbeault complex computing $\text{Ext}^\ast(G, G)$. The resulting formulas involve higher derivatives of Eisenstein series (see Theorem 2.5.1). More precisely, we have to use the well-known non-holomorphic (but modular) modification $e^*_{2k}$ of the standard Eisenstein series $e_2$ along with all the higher Eisenstein series $e_2$ and $e_{2k}$, $k \geq 2$.

It is interesting that Eisenstein series appear not in their usual form but rather as some rapidly decreasing series, similar to those considered in [9] (see Theorem 1.2.1). The $A_\infty$-constraint gives rise to some quadratic relations involving derivatives of Eisenstein series, some of them well known (see Proposition 2.6.1).

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1. Eisenstein series

1.1. Definitions. Let us recall some basic definitions and facts (see [10], ch. III). We consider $C^\infty$-functions $F(\omega_1, \omega_2)$ on the space of all oriented bases of $\mathbb{C} = \mathbb{R}^2$. Recall that such a function is said to be of weight $k \in \mathbb{Z}$ if

$$F(\lambda \omega_1, \lambda \omega_2) = \lambda^{-k} F(\omega_1, \omega_2).$$

Such $F$ is called modular (with respect to $\text{SL}(2, \mathbb{Z})$) if it is invariant with respect to $\text{SL}(2, \mathbb{Z})$ base changes of $(\omega_1, \omega_2)$ and $F(1, \tau) = f(e^{2\pi i \tau})$, where $f(q)$ is meromorphic at $q = 0$. The Eisenstein series $e_{2k}$ for $k \geq 2$ is defined by

$$e_{2k}(\omega_1, \omega_2) = \sum_{\omega \in L \setminus \{0\}} \frac{1}{\omega^{2k}}$$

where $L = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2$. The function $e_{2k}$ is modular of weight $2k$. One can also consider the analogous series for $k = 1$ using Eisenstein’s summation rule:

$$e_2(\omega_1, \omega_2) = \sum_{m, n \neq 0 \text{ if } m = 0} \frac{1}{(m\omega_2 + n\omega_1)^2}.$$

The function $e_2$ is not modular but admits a simple non-holomorphic correction that makes it $\text{SL}(2, \mathbb{Z})$-invariant. Namely, let us set

$$e^*_2(\omega_1, \omega_2) = e_2(\omega_1, \omega_2) - \frac{\pi}{a(L)} \frac{\omega_1}{\omega_1^2},$$

where $a(L) = \text{Im}(\overline{\omega_1}\omega_2)$ is the area of $\mathbb{C}/L$. Then $e^*_2(\omega_1, \omega_2)$ is $\text{SL}(2, \mathbb{Z})$-invariant (and of weight 2). For convenience we set also $e^*_2 = e_{2k}$ for $k \geq 2$. Eisenstein series appear as coefficients in the expansion of the Weierstrass zeta-function (cf. [10], ch. III, formula (9)):

$$\zeta(z; \omega_1, \omega_2) = \frac{1}{z} - \sum_{k \geq 2} e_{2k}(\omega_1, \omega_2)z^{2k-1}.$$

Following [2], sec. 1.5, we consider the normalized Weil operator

$$W = -\frac{\pi}{a(L)} \left( \overline{\omega_1} \frac{\partial}{\partial \omega_1} + \overline{\omega_2} \frac{\partial}{\partial \omega_2} \right).$$

This operator is $\text{SL}(2, \mathbb{Z})$-invariant and is of weight two. Slightly modifying the definition in [10], ch.VI, for a pair of integers $b > a \geq 0$ of different parity we set

$$g_{a,b} = g_{b,a} = (b-a)!W^a(e^*_b - e^*_{a+1}).$$

Note that $(a(L)/\pi)^a \cdot g_{a,b}$ differs by a rational factor from Weil’s $e^*_{a,b+1}$. As shown in [10], sec. VI.5, $g_{a,b}$ is a polynomial in $e^*_2, e_4, \ldots, e_{a+b+1}$ with rational coefficients.

1.2. Presentation by rapidly decreasing series. For $m, n \in \mathbb{Z}$ and a lattice $L \subset \mathbb{C}$ we set

$$f_{m,n}(L) = \left( \frac{\pi}{a(L)} \right)^m \sum_{\omega \in L \setminus \{0\}} \frac{\omega^m}{\omega^n} \exp\left( -\frac{\pi}{a(L)} |\omega|^2 \right).$$

Note that $f_{m,n} = 0$ unless $m + n$ is even (as one can see making the substitution $\omega \mapsto -\omega$).
Theorem 1.2.1. For \( n = 2k \geq 2 \) one has

\[
e^*_n = \frac{2}{(n-1)!} f_{n-1,1} + \sum_{m=2}^{n} \frac{1}{(n-m)!} f_{n-m,m} = \sum_{\omega \neq 0} P_{n-1}(\frac{\pi}{a(L)} |\omega|^2) \exp(-\frac{\pi}{a(L)} |\omega|^2),
\]

where

\[
P_{n-1}(z) = \frac{2z^{n-1}}{(n-1)!} + \sum_{m=2}^{n} \frac{z^{n-m}}{(n-m)!}.
\]

Proof. This follows easily from Theorem 1 of \([9]\) stating that

\[
\zeta(z;\omega_1,\omega_2) - z_1 \eta_1 - z_2 \eta_2 = \sum_{\omega \in L} \exp(-\frac{\pi}{a(L)} |\omega + z|^2) - \sum_{\omega \in L \setminus \{0\}} \exp(-\frac{\pi}{a(L)} |\omega|^2 + 2\pi i \text{Im}(\pi z))
\]

where \( z = z_1 \omega_1 + z_2 \omega_2 \) with \( z_1, z_2 \in \mathbb{R} \). Indeed, we can use the expansion \([1.1.1]\) to check the assertion for \( n \geq 4 \): one has to subtract \( 1/z \) from both parts of the above identity, apply \((\frac{\partial}{\partial z})^{n-1}\) and then evaluate at \( z = 0 \). The case \( n = 2 \) is slightly different: we again subtract \( 1/z \) from both parts, then apply \( \frac{\partial}{\partial z_k} \) (taking \( z_1, z_2 \) as independent variables) and evaluate at \( z = 0 \). The required formula follows the fact that \( e_2 = -\eta_1/\omega_1 \) (see e.g., \([2]\), sec. 1.2).

Remark. The fact that \( e^*_n \) is holomorphic in \((\omega_1,\omega_2)\) for \( n > 2 \) is equivalent to the identity

\[
2 f_{n-1,-1} = (n-1) f_{n-2,0}.
\]

For \( n = 2 \) we have instead \( 2 f_{1,-1} = f_{0,0} + 1 \). These identities can be derived from the Poisson summation formula and Fourier self-duality of \( \exp(-\frac{\pi}{a(L)} |z|^2) \) (see \([9]\), sec. 1.1, Remark 1; for \( n > 2 \) one also has to use differentiation).

One can immediately check that

\[
W(f_{m,n}) = f_{m+2,n} + nf_{m+1,n+1}.
\]

Hence, from Theorem [1.2.1] we get the following formula for \( g_{a,b} \).

Corollary 1.2.2. For a pair of integers \( a, b \geq 0 \) of different parity one has

\[
g_{a,b} = \sum_{k \geq 0} k! \left( \begin{array}{c} a \\ k \end{array} \right) \left( \begin{array}{c} b \\ k \end{array} \right) f_{a+k,b+k+1}
\]

2. Minimal \( A_\infty \)-algebra of an elliptic curve

2.1. General construction. Let us first recall the general construction of the \( A_\infty \)-structure on the cohomology of a dg-algebra \((A,d)\) equipped with a projector \( \Pi : A \rightarrow B \) onto a subspace of \( \ker(d) \) and a homotopy operator \( Q \) such that \( 1 - \Pi = dQ + Qd \). Merkulov’s formula for this \( A_\infty \)-structure (see \([7]\)) was rewritten in \([5]\) as a sum over trees:

\[
m_n(b_1,\ldots,b_n) = -\sum_T \epsilon(T)m_T(b_1,\ldots,b_n).
\]

Here \( T \) runs over all oriented planar rooted 3-valent trees with \( n \) leaves (different from the root) marked by \( b_1,\ldots,b_n \) left to right, and the root marked by \( \Pi \) (we draw the tree in such
a way that leaves are above, and every vertex has two edges coming from above and one from below. The expression \( m_T(b_1, \ldots, b_n) \) is obtained by going down from leaves to the root, applying multiplication in \( A \) at every vertex and applying the operator \( Q \) at every inner edge (see [4], sec. 6.4, for details). The sign \( \epsilon(T) \) has form

\[
\epsilon(T) = \prod_v (-1)^{|e_1(v)| + (|e_2(v)| - 1) \deg(e_1(v))},
\]

where \( v \) runs through vertices of \( T \), \( (e_1(v), e_2(v)) \) is the pair of edges above \( v \), for an edge \( e \) we denote by \(|e|\) the total number of leaves above \( e \) and by \( \deg(e) \) the sum of degrees of all leaves above \( e \) (recall that leaves are marked by \( b_i \)).

**Lemma 2.1.1.** Assume in addition that \( \Pi Q = Q \Pi = Q^2 = 0 \). Let \( (b_1, \ldots, b_n) \) be a collection of elements in \( B \), where \( n \geq 3 \), such that \( b_i = 1 \) for some \( i \). Then \( m_n(b_1, \ldots, b_n) = 0 \).

**Proof.** It is convenient to use Merkulov’s original formula

\[
m_n(b_1, \ldots, b_n) = \Pi \lambda_n(b_1, \ldots, b_n),
\]

where \( \lambda_n : A^{\otimes n} \to A \) are defined for \( n \geq 2 \) by the following recursion: \( \lambda_2(a_1, a_2) = a_1 a_2 \),

\[
\lambda_n(a_1, \ldots, a_n) = \pm Q(\lambda_{n-1}(a_1, \ldots, a_{n-1})) \cdot a_n \pm a_1 \cdot Q(\lambda_{n-1}(a_2, \ldots, a_n)) + \sum_{k+l=n, k,l \geq 2} \pm Q(\lambda_k(a_1, \ldots, a_k)) \cdot Q(\lambda_l(a_{k+1}, \ldots, a_n)).
\]

Since, \( \Pi Q = 0 \), it is enough to prove that \( \lambda_n(b_1, \ldots, b_n) \in Q(A) \). Let us use induction in \( n \). In the case \( n = 3 \) we have

\[
\lambda_3(b_1, b_2, b_3) = Q(b_1 b_2) b_3 \pm b_1 Q(b_2 b_3)
\]

and the assertion follows immediately from the fact that \( Q(B) = 0 \). Suppose now that \( n \geq 4 \) and the assertion holds for all \( n' < n \). Since \( Q^2 = 0 \), the induction assumption easily implies that the first two terms in the recursive formula for \( \lambda_n \) belong to \( Q(A) \). Similarly, all the remaining terms vanish if \( n \geq 5 \). In the case \( n = 4 \) the term \( Q(b_1 b_2) \cdot Q(b_3 b_4) \) also vanishes because either \( b_1 b_2 \in B \) or \( b_3 b_4 \in B \) and \( Q(B) = 0 \).

**2.2. The case of an elliptic curve.** Let \( C = \mathbb{C}/(\mathbb{Z} \oplus \mathbb{Z} \tau) \) be a complex elliptic curve. We denote by \( L \) the holomorphic line bundle of degree 1 on \( C \), such that the theta-function \( \theta(z, \tau) \) descends to a global section of \( L \). We consider the Dolbeault dg-algebra

\[
A = (\Omega^{0,*} \otimes \mathcal{E}nd(\mathcal{O}_C \oplus L), \bar{\partial}).
\]

Its cohomology \( B \) is the direct sum of the following components:

(i) \( \text{Hom}(\mathcal{O}, \mathcal{O}) \) and \( \text{Hom}(L, L) \), both generated by identity maps;

(ii) \( \text{Hom}(\mathcal{O}, L) \), one-dimensional space;

(iii) \( \text{Ext}^1(L, \mathcal{O}) \), one-dimensional space;

(iv) \( \text{Ext}^1(\mathcal{O}, \mathcal{O}) \) and \( \text{Ext}^1(L, L) \), both isomorphic to the one-dimensional space \( H^1(\mathcal{O}) \).
To construct the homotopy operator $Q$, as in [S], we use the flat metric on $C$ and the hermitian metric on $L$ given by

$$(f, g) = \int_C f(z)\overline{g(z)} \exp(-2\pi \frac{\text{Im}(z)^2}{\text{Im}(\tau)}) dxdy,$$

where $z = x + iy$. Then we set $Q = \overline{\partial}^* G$, where $G$ is the Green operator corresponding to the Laplacian $\overline{\partial}^* \overline{\partial} + \overline{\partial} \partial^*$. Then $B \subset A$ is exactly the space of harmonic forms, and $\Pi : A \to B$ is the orthogonal projection. Let us fix the following harmonic generators in the above components:

(i) $\text{id}_\mathcal{O}$ and $\text{id}_L$;
(ii) $\theta = \theta(z, \tau)$ viewed as a holomorphic section of $L$;
(iii) $\eta := \sqrt{2\text{Im}(\tau)} \cdot \theta(z, \tau) \exp(-2\pi \frac{\text{Im}(z)}{\text{Im}(\tau)}) d\overline{z}$ viewed as a $(0, 1)$-form with values in $L^{-1}$;
(iv) $\xi = d\overline{z}$. When it is viewed as an element of $\text{Ext}^1(L, L)$ we write $\xi_L$.

Note that we have a natural symmetric bilinear pairing on $A = A^0 \oplus A^1$ given by

$$\langle \alpha, \beta \rangle = \frac{1}{2i \text{Im}(\tau)} \cdot \int_C \text{Tr}(\alpha \circ \beta) \wedge dz,$$

where $\alpha$ and $\beta$ are homogeneous elements such that $\deg(\alpha) + \deg(\beta) = 1$. The normalization is chosen in such a way that

$$\langle \xi, 1 \rangle = \frac{1}{2i \text{Im}(\tau)} \cdot \int_C d\overline{z} \wedge dz = 1.$$

By Serre duality, the induced pairing between $B^0$ and $B^1$ is nondegenerate. Also, by Theorem 1.1 of [S], the $A_\infty$-structure on $B$ satisfies the following cyclic symmetry:

$$(2.2.1) \quad \langle m_n(\alpha_1, \ldots, \alpha_n), \alpha_{n+1} \rangle = (-1)^{\text{deg}(\alpha_1)+1} \langle \alpha_1, m_n(\alpha_2, \ldots, \alpha_{n+1}) \rangle.$$

The product $m_2$ on $B$ is just the induced product on cohomology. The only interesting products are $m_2(\theta, \eta) \in \text{Ext}^1(\mathcal{O}, \mathcal{O})$ and $m_2(\eta, \theta) \in \text{Ext}^1(L, L)$. Both are proportional to the generator $\xi$. To find the coefficient of proportionality it is enough to compute

$$\langle m_2(\theta, \eta), \text{id}_\mathcal{O} \rangle = \langle m_2(\eta, \theta), \text{id}_L \rangle = \frac{1}{2i \text{Im}(\tau)} \cdot \int_C \theta \cdot \eta \wedge dz.$$

The above integral is well known:

$$\sqrt{2\text{Im}(\tau)} \cdot \int_C \theta(z, \tau)\overline{\theta(z, \tau)} \exp(-2\pi \frac{\text{Im}(z)}{\text{Im}(\tau)^2}) d\overline{z} \wedge dz = 1.$$

Thus, we obtain

$$(2.2.2) \quad m_2(\theta, \eta) = \xi, \quad m_2(\eta, \theta) = \xi_L.$$

By Lemma 2.1, every higher product $m_n$ containing $\text{id}_\mathcal{O}$ or $\text{id}_L$ vanishes. Together with the cyclic symmetry (2.2.1) this implies that the only potentially nonzero higher products are of the following types:

(I) $m_n((\xi)^a, \theta, (\xi_L)^b, \eta, (\xi)^c, \theta, (\xi_L)^d) \in \text{Hom}(\mathcal{O}, L)$,

(II) $m_n((\xi_L)^a, \eta, (\xi)^b, \theta, (\xi_L)^c, \eta, (\xi)^d) \in \text{Ext}^1(L, \mathcal{O})$,

(III) $m_n((\xi)^a, \theta, (\xi_L)^b, \eta, (\xi)^c, \theta, (\xi_L)^d, \eta, (\xi)^e) \in \text{Hom}(\mathcal{O}, \mathcal{O})$,

(IV) $m_n((\xi_L)^a, \eta, (\xi)^b, \theta, (\xi_L)^c, \eta, (\xi)^d, \theta, (\xi_L)^e) \in \text{Hom}(L, L)$,
where we denote by $(\xi)^a$ the string $(\xi, \ldots, \xi)$ with $\xi$ repeated $a$ times.

By the cyclic symmetry [2.2.1], we have

$$
\langle m_n((\xi_L)^a, \eta, (\xi^b, \theta, (\xi_L)^c, \eta, (\xi^d, \theta, (\xi_L)^a), \eta),
\langle m_n((\xi)^a, \theta, (\xi_L^b, \eta, (\xi)^c, \theta, (\xi_L)^d), \eta),
\langle m_n((\xi_L)^a, \eta, (\xi)^b, \theta, (\xi_L)^c, \eta, (\xi)^d, \theta, (\xi_L)^e), \xi_L) = \langle m_n((\xi)^b, \theta, (\xi_L)^c, \eta, (\xi)^d, \theta, (\xi_L)^a), \eta).
$$

Hence, it is enough to compute the products of type (I), i.e., the coefficients

$$
\langle m_n((\xi)^a, \theta, (\xi_L)^b, \eta, (\xi)^c, \theta, (\xi_L)^d), \eta).
$$

2.3. Calculation I: combinatorial part. We start by computing some signs $\epsilon(T)$. For a pair of oriented planar rooted 3-valent trees $T_1, T_2$ let us denote by $\text{join}(T_1, T_2)$ the tree (of the same type) obtained by joining together the roots of $T_1$ and $T_2$ and adding a root to the obtained new vertex (we keep $T_1$ on the left from $T_2$ in the plane).

**Lemma 2.3.1.** Let $T = \text{join}(T_1, T_2)$, where each $T_i$ has $n_i + 1$ leaves ($i = 1, 2$), exactly one of which is marked by a degree 0 element, and the rest marked by degree 1 elements. Assume also that in each $T_i$ no two leaves of degree 1 can be attached to the same vertex. Then

$$
\epsilon(T) = (-1)^{(n_1 + n_2 + 2) + n_2}.
$$

**Proof.** By definition,

$$
\epsilon(T) = \epsilon(T_1)\epsilon(T_2) \cdot (-1)^{(n_1 + 1) + n_1 n_2},
$$

so it remains to compute $\epsilon(T_i)$ for $i = 1, 2$. Note that under our assumptions each tree $T_i$ has a very simple structure: it has the main stem from the root to the leaf of degree 0, to which leaves of degree 1 can attach on the left and on the right:

Suppose we have a pair of consecutive vertices $v$ and $w$ on the stem (with $v$ above $w$) such that there is a degree 1 leaf attaching to $v$ on the left and a degree 1 leaf attaching to $w$ on the right. Let $a$ be the number of degree 1 leaves above $v$. Then the contribution of $v$ into the
product defining $\epsilon(T_i)$ equals $(-1)^{a+1}$, while the contribution of $w$ equals $(-1)^{a+2}$. Hence, the contribution of both $v$ and $w$ is $-1$. It is easy to check that if $T'$ is the tree obtained from $T$ by reversing the order of attaching these two leaves at vertices $v$ and $w$ then the contribution of these vertices into $\epsilon(T')$ will still be $-1$. Assume that $T_i$ has exactly $a$ (resp., $b$) leaves to the left (resp., to the right) of the degree 0 leaf, and let $v_1, \ldots, v_a$ (resp., $w_1, \ldots, w_b$) be the vertices on the stem to which they attach. By the above observation, it is enough to consider the case when all the vertices $v_1, \ldots, v_a$ are above $w_1, \ldots, w_b$:

Then one can easily calculate that

$$\epsilon(T_i) = (-1)^{\binom{a+b+1}{2}}.$$  

Hence,

$$\epsilon(T) = (-1)^{\binom{n_1+1}{2} + \binom{n_2+1}{2} + n_1 + n_2} = (-1)^{\binom{n_1+n_2+2}{2} + n_2}.$$  

Next, we consider the terms $m_T((\xi)^a, \theta, (\xi_L)^b, \eta, (\xi)^c, \theta, (\xi_L)^d)$. Note that if two leaves of degree 1 in $T$ are attached to the same vertex then $m_T$ vanishes because it will involve taking products of two elements of degree 1 in $A$. Henceforward, we assume that no two leaves of degree 1 can be attached to the same vertex. It is convenient to introduce the operator

$$H_L : C^\infty(L) \to C^\infty(L) : s \mapsto Q(s \cdot d\bar{z})$$

and a similar operator $H_O$ on $C^\infty$-functions. The next result follows immediately from the definition.

**Lemma 2.3.2.** (i) Let $T = \text{join}(T_1, T_2)$, where the leaves of $T_1$ are marked (from left to right) by

$$(\xi)^a, \theta, (\xi_L)^b, \eta, (\xi)^c_1,$$
while the leaves of $T_2$ are marked by 

$$(\xi)^{c_2}, \theta, (\xi_L)^d.$$ 

Let $a = a_1 + a_2$, where $a_1$ leftmost leaves of $T_1$ get attached to the main stem of $T_1$ below the vertex where $\eta$ attaches to the main stem, and the next $a_2$ leaves get attached to the stem above this vertex:

![Diagram](attachment:image.png)

Then

$$m_T((\xi)^a, \theta, (\xi_L)^b, \eta, (\xi)^{c_1+c_2}, \theta, (\xi_L)^d) = \prod \left( [H_{O}^{a_1+c_1}Q(H_{L}^{a_2+b}(\theta \cdot \eta))] \cdot H_{L}^{c_2+d}(\theta) \right).$$

(ii) Let $T = \text{join}(T_1, T_2)$, where the leaves of $T_1$ are marked by 

$$(\xi)^a, \theta, (\xi_L)^{b_1},$$

while the leaves of $T_2$ are marked by 

$$(\xi_L)^{b_2}, \eta, (\xi)^c, \theta, (\xi_L)^d.$$ 

Let $d = d_1 + d_2$, where exactly $d_2$ rightmost leaves of $T_2$ get attached to the main stem of $T_2$ below the vertex where $\eta$ attaches to the main stem. Then

$$m_T((\xi)^a, \theta, (\xi_L)^{b_1+b_2}, \eta, (\xi)^c, \theta, (\xi_L)^d) = \prod \left( [H_{O}^{b_2+d_2}Q(H_{L}^{c+d_1}(\theta \cdot \eta))] \cdot H_{L}^{a+b_1}(\theta) \right).$$

Combining Lemmas 2.3.1 and 2.3.2 we arrive at the following expression for $m_n$.

**Lemma 2.3.3.** One has

$$\langle m_n((\xi)^a, \theta, (\xi_L)^b, \eta, (\xi)^c, \theta, (\xi_L)^d), \eta \rangle =$$

$$(-1)^{\binom{n}{2}} \cdot \sum_{a=a_1+a_2:b=c_1+c_2} \frac{(a_2+b)}{a_2} \left( \frac{a_1+c_1}{a_1} \right) \left( \frac{c_2+d}{c_2} \right) \phi(a_2+b, a_1+c_1+c_2+d) +$$

$$(-1)^{\binom{n}{2}+n+1} \sum_{b=b_1+b_2:d=d_1+d_2} \frac{(c+d_1)}{c} \left( \frac{b_2+d_2}{b_2} \right) \left( \frac{a+b_1}{a} \right) \phi(c+d_1, b_2+d_2, a+b_1),$$

where $a, b, c, d, a_1, a_2, b_1, b_2, c_1, c_2, d_1, d_2, n$ are non-negative integers.
where
\[ \phi(m, n, p) = (-1)^p \langle \Pi ([H^m_0 Q(H^m_0(\theta) \cdot \eta)] \cdot H^p_L(\theta)), \eta \rangle. \]

2.4. Calculation II: analytic part. We will use real coordinates \( u, v \) on \( \mathbb{C} \) such that \( z = u + v\tau \). Let us also set \( a = \text{Im}(\tau) \).

Lemma 2.4.1. Consider the differential operator \( D = -\frac{a}{\pi} \frac{\partial}{\partial z} - 2iav \). Then for \( k \geq 0 \) one has:

\[
(2.4.1) \quad H^k_L(\theta) = \frac{1}{k!} \cdot D^k \theta = \frac{(-2ia)^k}{k!} \sum_{n \in \mathbb{Z}} (n + v)^k \exp(\pi i n^2 + 2\pi i nz),
\]

\[
(2.4.2) \quad \sqrt{2a} (D^k \theta) \cdot \theta \exp(-2\pi av^2) = \sum_{(m,n) \in \mathbb{Z}^2} (-1)^{mn} (m\overline{\tau} - n)^k \exp(-\frac{\pi}{2a} |m\tau - n|^2 + 2\pi i (mu + nv)).
\]

Proof. The case \( k = 0 \) of the identity \(2.4.2\) is well-known (see [8], eq. (2.2)). The general case follows easily by applying \( D^k \). Next, let us prove \(2.4.1\) by induction in \( k \). Recall that the operator \( Q : \Omega^{0,1} \rightarrow \Omega^{0,0} \) is uniquely determined by the following two properties: \( \mathcal{J} \circ Q = \text{id} \) and the image of \( Q \) is orthogonal to \( \theta \). Thus, \( H^k_L(\theta) \) equals the unique function \( f \) such that \( \frac{\partial f}{\partial z} = D^k \theta \) and \( (f, \theta) = 0 \). Thus, it is enough to check the identities

\[
(2.4.3) \quad D^k \theta = (-2ia)^k \cdot \sum_{n \in \mathbb{Z}} (n + v)^k \exp(\pi i n^2 + 2\pi i nz),
\]

\[
(2.4.4) \quad \frac{\partial}{\partial z} D^k \theta = k D^{k-1} \theta,
\]

for \( k > 0 \). The latter follows immediately from the Fourier expansion \(2.4.2\), since \( (D^k \theta, \theta) \) is proportional to the Fourier coefficient of this expansion corresponding to \( (m, n) = (0, 0) \). To check \(2.4.3\) one can apply \( D \) to the similar expansion for \( D^{k-1} \theta \). Finally, one checks \(2.4.4\) by applying \( \frac{\partial}{\partial z} \) to the right-hand side of \(2.4.3\). \(\square\)

Now we can calculate the expressions \( \phi(m, n, p) \) (see Lemma 2.3.3).

Lemma 2.4.2. One has

\[
\phi(k, l, p) = \frac{1}{k! l!} \left( \frac{a}{\pi} \right)^{l+1} \cdot \sum_{\omega \in Z+2\tau \setminus \{0\}} \frac{\exp(-\frac{\pi}{a} |\omega|^2)}{\omega^{l+1}} = \frac{1}{k! l!} \left( \frac{a}{\pi} \right)^{k+l+1} f_{k+l+1}(Z + Z\tau).
\]

Proof. It is easy to see that the operator \( Q : \Omega^{0,1} \rightarrow \Omega^{0,0} \) is given by

\[
Q(\exp(2\pi i (mu + nv)) d\overline{z}) = \begin{cases} 
\frac{a}{\pi (m\tau - n)} \exp(2\pi i (mu + nv)), & (m, n) \neq (0, 0), \\
0, & (m, n) = (0, 0).
\end{cases}
\]
Hence, using Lemma 2.4.1 we obtain

\[
Q(H^k_L(\theta) \cdot \eta) = \frac{1}{k!} Q(\sqrt{2a}(D^k\theta) \cdot \bar{\theta} \exp(-2\pi av^2) = \\
\frac{a}{k!\pi} \cdot \sum_{(m,n) \in \mathbb{Z}^2 \setminus \{(0,0)\}} (-1)^{mn} \frac{(m\tau - n)^k}{(m\tau - n)} \exp(-\frac{\pi}{2a}|m\tau - n|^2 + 2\pi i(mu + nv)).
\]

Therefore,

(2.4.5)

\[
H^l_O Q(H^k_L(\theta) \cdot \eta) = \frac{1}{k! l!} (\frac{a}{\pi})^{l+1} \cdot \sum_{(m,n) \in \mathbb{Z}^2 \setminus \{(0,0)\}} (-1)^{mn} \frac{(m\tau - n)^k}{(m\tau - n)^{l+1}} \exp(-\frac{\pi}{a}|m\tau - n|^2 + 2\pi i(mu + nv)).
\]

Next, comparing the formulas for \(\eta\) and for the metric on \(L\) we observe that for a \(C^\infty\)-section \(f\) of \(L\) one has \((f,\theta) = 0\) if and only if \((f,\eta) = 0\). Hence, for \(f \in \mathcal{C}^\infty(L)\) one has

\[
\langle \Pi(f), \eta \rangle = \langle f, \eta \rangle
\]

(since \(\Pi\) is the orthogonal projection onto \(\mathbb{C}\theta\)). Therefore,

\[
\phi(k, l, p) = (-1)^p \langle [H^l_O Q(H^k_L(\theta) \cdot \eta)] \cdot H^p_L(\theta) \cdot \eta, f, \eta \rangle = (-1)^p \langle H^l_O Q(H^k_L(\theta) \cdot \eta, H^p_L(\theta) \cdot \eta, f, \eta \rangle.
\]

Now the right-hand side can be computed using the Fourier expansion for \(H^p_L(\theta)\) from Lemma 2.4.1 and the Fourier expansion (2.4.5):

\[
\phi(k, l, p) = \frac{1}{k! l!} (\frac{a}{\pi})^{l+1} \cdot \sum_{(m,n) \in \mathbb{Z}^2 \setminus \{(0,0)\}} \frac{(m\tau - n)^{k+p}}{(m\tau - n)^{l+1}} \exp(-\frac{\pi}{a}|m\tau - n|^2).
\]

\[
\square
\]

2.5. Calculation III: conclusion. It remains to put everything together. Substituting the expressions for \(\phi(k, l, p)\) found in Lemma 2.4.2 into the formula of Lemma 2.3.3 we get

\[
\langle m_n((\xi)^a, \theta, (\xi_L)^b, \eta, (\xi)^c, \theta, (\xi_L)^d), \eta \rangle = \\
(-1)^{\binom{n}{2}}+\left(\frac{\text{Im}(\tau)}{\pi}\right)^{n-2} \cdot \sum_{a=a_1+a_2,c=c_1+c_2} (a_1 + c_1)! a_1! c_1! e_1! e_2! d_1! d_2! f_{a_2+c_2+b+d,a_1+c_1+1} \\
\langle m_n((\xi)^a, \theta, (\xi_L)^b, \eta, (\xi)^c, \theta, (\xi_L)^d), \eta \rangle = \\
(-1)^{\binom{n}{2}+n+1} \left(\frac{\text{Im}(\tau)}{\pi}\right)^{n-2} \cdot \sum_{b=b_1+b_2,d=d_1+d_2} (b_2 + d_2)! b_1! b_2! d_1! d_2! a! c! f_{b_1+d_1+a+c,b_2+d_2+1}.
\]

Since \(f_{k,l} = 0\) unless \(k + l\) is even, this immediately implies that \(m_n = 0\) for odd \(n\). Now assuming that \(n\) is even we can rewrite the above equation as follows (denoting \(k = a_1 + c_1\) and \(l = b_2 + d_2\)):

\[
(-1)^{\binom{n}{2}+1} \left(\frac{\text{Im}(\tau)}{\pi}\right)^{n-2} \cdot \langle m_n((\xi)^a, \theta, (\xi_L)^b, \eta, (\xi)^c, \theta, (\xi_L)^d), \eta \rangle = \\
\frac{1}{b!d!} \cdot \sum_{k \geq 0} C(a, c, k) f_{a+c-k+b+d,k+1} + \frac{1}{a!c!} \cdot \sum_{l \geq 0} C(b, d, l) f_{b+d-l+a+c,l+1},
\]

\[\square\]
where

\[
C(a, c, k) = \sum_{a_1 + c_1 = k, a_1 \leq c} \frac{k!}{a_1!c_1!(a - a_1)!(c - c_1)!} = \frac{k!}{a_1!c_1!} \sum_{a_1 + c_1 = k} \binom{a}{a_1} \binom{c}{c_1} = \frac{k!}{a!c!} \binom{a + c}{k}
\]

Thus, our formula for \( \langle m_n((\xi)^a, \theta, (\xi_L)^b, \eta, (\xi)^c, \theta, (\xi_L)^d), \eta \rangle \) takes form

\[
(-1)^{n+1} \frac{1}{\text{Im}(\tau)^{n-2}} \cdot \langle m_n((\xi)^a, \theta, (\xi_L)^b, \eta, (\xi)^c, \theta, (\xi_L)^d), \eta \rangle = \frac{1}{a!b!c!d!} \cdot \sum_{k \geq 0} k! \left( \binom{a + c}{k} + \binom{b + d}{k} \right) f_{n-3-k,k+1}.
\]

Taking into account Corollary 1.2.2 we obtain for even \( n \)

\[
(2.5.1) \quad \langle m_n((\xi)^a, \theta, (\xi_L)^b, \eta, (\xi)^c, \theta, (\xi_L)^d), \eta \rangle = (-1)^{\frac{n}{2}} \frac{1}{a!b!c!d!} \cdot \left( \frac{\text{Im}(\tau)}{\pi} \right)^{n-2} \cdot g_{a+c,b+d}.
\]

Let us summarize our calculations. We set

\[
(2.5.2) \quad M(a, b, c, d) := (-1)^{\frac{a+b+c+d+1}{2}} \cdot \frac{1}{a!b!c!d!} \cdot \left( \frac{\text{Im}(\tau)}{\pi} \right)^{a+b+c+d+1} \cdot g_{a+c,b+d}.
\]

**Theorem 2.5.1.** The only non-trivial higher products \( m_n \) of the \( A_\infty \)-structure on \( B = \text{Ext}^*(\mathcal{O} \oplus L, \mathcal{O} \oplus L) \) are of the form

\[
m_n((\xi)^a, \theta, (\xi_L)^b, \eta, (\xi)^c, \theta, (\xi_L)^d) = M(a, b, c, d) \cdot \theta,
\]

\[
m_n((\xi_L)^a, \eta, (\xi)^b, \theta, (\xi_L)^c, \eta, (\xi_L)^d) = M(a, b, c, d) \cdot \eta,
\]

\[
m_n((\xi)^a, \theta, (\xi_L)_b, \eta, (\xi)^c, \theta, (\xi_L)^d, \eta, (\xi)^c) = M(a + e + 1, b, c, d) \cdot id_{\mathcal{O}},
\]

\[
m_n((\xi_L)^a, \eta, (\xi)^b, \theta, (\xi_L)^c, \eta, (\xi)^d, \theta, (\xi_L)^c) = M(a + e + 1, b, c, d) \cdot id_L.
\]

All products \( m_n \) with odd \( n \) vanish.

**Remarks.**

1. Assume that \( \tau \) belongs to a ring of integers of an imaginary quadratic field (so that our elliptic curve admits complex multiplication). Set \( \varpi = 2\pi |\eta(q)|^2 \), where \( \eta(q) \) is the Dedekind’s \( \eta \)-function, and \( q = \exp(2\pi i \tau) \). Then the numbers \( \varpi^{-a-b-1} \cdot g_{a,b} \) are algebraic over \( \mathbb{Q} \) (see [10], sec. VI.6). Hence, if we multiply our basis elements of degree 1 (\( \eta, \xi \) and \( \xi_L \)) by the factor \( |\eta(q)|^{-2} \) then the structure constants of our \( A_\infty \)-structure with respect to the new basis will be algebraic over \( \mathbb{Q} \).

2. Another meaningful rescaling is obtained if we multiply our basis elements of degree 1 by the factor \( \pi / \text{Im}(\tau) \). Then the new structure constants \( M'(a, b, c, d) \) will all have limit at the cusp \( \text{Im}(\tau) \to +\infty \). Namely, using the well known \( q \)-series for the Eisenstein series, one can easily check that the only nonzero limiting values will be

\[
M'(i, 0, j, 0) = M'(0, i, 0, j) \to (-1)^{\frac{i+j+1}{2}} \cdot \left( \binom{i+j}{i} \right) \cdot 2\zeta(i + j + 1),
\]

where \( i + j \) is odd.
2.6. A_∞-constraint. The A_∞-axiom (we follow [4], 3.1 for sign conventions) gives certain quadratic equations on coefficients \((M(a, b, c, d))\) and hence leads to identities for \(g_{m,n}\). For example, applying this axiom to the string

\[
(\xi)^a, \theta, (\xi_L)^b, \eta, (\xi)^c, \theta, (\xi_L)^d, \eta, (\xi)^e, \theta, (\xi_L)^f,
\]

where \(a, b, c, d, e, f\) are positive, we get

\[
\sum_{a=a_1+a_2} (-1)^{(a_2+b+c+a_1+1)(a_1+d_2+e+f)+a_1} M(a_2, b, c, d_1) M(a_1, d_2, e, f) + \sum_{b=b_1+b_2} (-1)^{(b_2+c+d+e_1+1)(a+b_2+e_2+f+1)+a+b_1+1} M(b_2, c, d, e_1) M(a, b_1, c_2, f) + \sum_{c=c_1+c_2} (-1)^{(c_2+d+e+f_1+1)(a+b+c_1+1+f_2)+a+b+c_1} M(c_2, d, e, f_1) M(a, b, c_1, f_2) = 0.
\]

When one of \(a, b, c, d, e, f\) is zero, additional terms will arise due to the presence of double products. For example, for the string

\[(\xi)^a, \theta, \eta, \theta, \eta, \theta, (\xi_L)^b\]

we get

\[
\sum_{a=a_1+a_2} (-1)^{(a_2+1)(a_1+b)+a_1} M(a_2, 0, 0, 0) M(a_1, 0, 0, b) + \sum_{b=b_1+b_2} (-1)^{(b_1+1)(a+b_2+1)+a} M(0, 0, 0, b_1) M(a, 0, 0, b_2) + (-1)^a [M(a + 1, 0, 0, b) - M(a, 1, 0, b) + M(a, 0, 1, b) - M(a, 0, 0, b + 1)] + \delta_{b,0} (-1)^a M(a + 1, 0, 0, 0) - \delta_{a,0} M(b + 1, 0, 0, 0) = 0.
\]

Substituting into the above identities the expressions for \(M(a, b, c, d)\) from (2.5.2) we arrive at the following result.

**Proposition 2.6.1.** (i) For positive integers \(a, b, c, d, e, f\) one has

\[
\sum_{a=a_1+a_2} (-1)^{c+d_1+1} \binom{a}{a_1} \binom{d}{d_1} g_{a_2+c,b+d_1} g_{a_1,e,d_2+f} + \sum_{b=b_1+b_2} (-1)^{b_2+1} \binom{b}{b_1} \binom{c}{c_1} g_{b_2+d,c+e_1} g_{a_1+e_2,b_1+f} + \sum_{c=c_1+c_2} (-1)^{c_1} \binom{c}{c_1} \binom{f}{f_1} g_{c_2+e,d+f_1} g_{a+c_1,b+f_2} = 0.
\]

(ii) For integers \(a, b \geq 0\) one has

\[
\sum_{a=a_1+a_2} \binom{a}{a_1} g_{a_1,a_2} g_{a_2,b} - \frac{a+2+\delta_{b,0}}{a+1} g_{a+1,b} = \sum_{b=b_1+b_2} \binom{b}{b_1} g_{b_1,b_2} g_{b_0,b} - \frac{b+2+\delta_{a,0}}{b+1} g_{a,b+1}.
\]
Note that the identity in (ii) gives a recursive formula for $g_{a+1,b}$ in terms of all $g_{a',b'}$ with $a' \leq a$. Since $g_{0,n} = n!e^*_n$, we recover the fact that all $g_{a,b}$ are polynomials in $(e^*_n)$ with rational coefficients. For example, in the case $a = 0$ the obtained identity (for even $n$)

$$2g_{1,n} = - \sum_{n=m+k} \binom{n}{m} g_{0,m}g_{0,k} + \frac{n+3}{n+1}g_{n+1,0}$$

is equivalent to the formula

$$\frac{1}{n} We^*_n = - \sum_{n=m+k} e^*_{m+1}e^*_{k+1} + (n+3)e^*_{n+2}$$

found in [10], VI.5.

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