THE NUMBER OF REALIZATIONS OF A LAMAN GRAPH

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ABSTRACT. Laman graphs model planar frameworks that are rigid for a general choice of distances between the vertices. There are finitely many ways, up to isometries, to realize a Laman graph in the plane. Such realizations can be seen as solutions of systems of quadratic equations prescribing the distances between pairs of points. Using ideas from algebraic and tropical geometry, we provide a recursive formula for the number of complex solutions of such systems.

INTRODUCTION

For a graph $G$ with edges $E$, we consider the set of all its realizations in the plane, such that the lengths of the edges coincide with some prescribed edge labeling $\lambda : E \rightarrow \mathbb{R}_{\geq 0}$. Edges and vertices are allowed to overlap in such a realization. For example, suppose that $G$ is the complete graph on four vertices minus one edge. Figure 1 shows all possible realizations of $G$ up to rotations and translations, for a particular given edge labeling.

![Figure 1. Realizations of a graph up to rotations and translations.](image)

We say that a property holds for a general edge labeling if it holds for all edge labelings belonging to the complement of a proper algebraic subset of the set of all edge labelings. In this paper we address the following problem:

For a given graph, determine the number of realizations, up to rotations and translations, for a general edge labeling.

Key words and phrases. Laman graph, Minimally rigid graph, Tropical geometry, Euclidean embedding, Puiseux series, Graph realization, Graph embedding.

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The realizations of a graph can be considered as structures in the plane, which are comprised of rods connected by rotational joints. If a graph with an edge labeling admits infinitely (finitely) many realizations up to rotations and translations, then the corresponding planar structure is flexible (rigid), see Figure 2.

![Figure 2](image)

(a) flexible  (b) rigid  (c) rigid (overdetermined)

**Figure 2.** Graphs and their state of rigidity

**Historical notes.** The study of rigid structures, also called frameworks, was originally motivated by mechanics and architecture, and goes back as early as the 19th century to the works of James Clerk Maxwell, August Ritter, Karl Culmann, Luigi Cremona, August Föppl, and Lebrecht Henneberg. Nowadays, there is still a considerable interest in rigidity theory \[GSS93, Con93\] due to various applications in natural science and engineering; for an exemplary overview, see the conference proceedings “Rigidity Theory and Applications” \[TD02\]. Let us just highlight three application areas that are covered there: In materials science the rigidity of crystals, non-crystalline solids, glasses, silicates, etc. is studied; among the numerous publications in this area we can mention \[BS13, JH97\]. In biotechnology one is interested in possible conformations of proteins and cyclic molecules \[JRKT01\], in particular to the enumeration of such conformations \[LML+14, EM99\]. In robotics, one aims at computing the configurations of mechanisms, such as 6R chains or Stewart-Gough platforms. For the former, the 16 solutions of the inverse kinematic problem have been found by using very elegant arguments from algebraic geometry \[Sel05, Section 11.5.1\]. For the latter, the 40 complex assembly modes have been determined by algebraic geometry \[RV95\] or by computer algebra \[FL95\]; Dietmaier \[Die98\] showed that there is also an assignment of the parameters such that all 40 solutions are real. Recently, connections between rigidity theory and incidence problems have been established \[Raz17\].

**Pollaczek-Geiringer’s and Laman’s characterization.** A graph is called generically rigid (or isostatic) if a general edge labeling yields a rigid realization. No edge in a generically rigid graph can be removed without losing rigidity. This is why such graphs are also called minimally rigid in the literature. Note that the graph in Figure 2c is not generically rigid, while the one in Figure 2b is. Hilda Pollaczek-Geiringer \[Pol27\] characterized this property in terms of the number of edges and vertices of the graph and its subgraphs. The same characterization can be found in a paper of Gerard Laman \[Lam70\] more than 40 years later. Unfortunately, the results of Pollaczek-Geiringer have been unnoticed until recently. Nowadays, these objects are known as Laman graphs; since this terminology is well-known, we stick to it in this paper.
State of the art. All realizations of a Laman graph with an edge labeling can be recovered as the solution set of a system of algebraic equations, where the edge labels can be seen as parameters. Here, we are interested in the number of complex solutions of such a system, up to an equivalence relation coming from direct planar isometries; this number is the same for any general choice of parameters, so we call it the Laman number of the graph. For some graphs up to 8 vertices, this number has been computed using random values for the parameters [JO12] — this means that it is very likely, but not absolutely certain, that these computations give the true numbers. Upper and lower bounds on the maximal Laman number for graphs with up to 10 vertices were found by analyzing the Newton polytopes of the equations and their mixed volumes [ETV09] using techniques from [ST10]. It has been proven [BS04] that the Laman number of a Laman graph with \( n \) vertices is at most \( \frac{2^{n-4}}{n-2} \).

Our contribution. Our main result is a combinatorial algorithm that computes the number of complex realizations of any given Laman graph. This is much more efficient than just solving the corresponding nonlinear system of equations.

We found it convenient to see systems of equations related to Laman graphs as special cases of a slightly more general type of systems, determined by bigraphs. Roughly stated, a bigraph is a pair of graphs whose edges are in bijection. Every graph can be turned into a bigraph by duplication and it is possible to extend the notion of Laman number also to bigraphs. The majority of these newly introduced systems do not have geometric significance: they are merely introduced to have a suitable structure to set up a recursive strategy. Our main result (Theorem 4.7) is a recursive formula expressing the Laman number of a bigraph in terms of Laman numbers of smaller bigraphs. Using this formula we succeeded in computing the exact Laman numbers of graphs with up to 18 vertices — a task that was absolutely out of reach with the previously known methods.

The idea for proving the recursive formula is inspired by tropical geometry (see [MS15] or [Stu02, Chapter 9]): we consider the system of equations over the field of Puiseux series, and the inspection of the valuations of the possible solutions allows us to endow every bigraph with some combinatorial data that prescribes how the recursion should proceed. This gives, therefore, a recursive formula for the right hand side of Corollary 3.6.16 in [MS15] in our particular case. Notice that the Laman number of a graph can be understood as the base degree (as defined at the end of Section 1 of [Ros14]) of the algebraic matroid associated to the variety parametrized by the square distances of the pairs of points prescribed by the edges of the Laman graph.

Structure of the paper. Section 1 contains the statement of the problem and a proof of the equivalence of generic rigidity and Laman’s condition in our setting. This section is meant for a general mathematical audience and requires almost no prerequisite. Section 2 analyzes the system of equations defined by a bigraph, and Section 3 provides a general formulation for a recursive formula for the number of solutions of the system. Here, we employ some standard techniques in algebraic geometry, so the reader should be acquainted with the basic
In this area. In Section 4, we specialize the general result provided at the end of Section 3 and we give a recursive formula for the Laman number. It leads to an algorithm that is employed in Section 5 to derive some new results on the number of realizations of Laman graphs. These last two sections are again meant for a general audience, and they require only the knowledge of the objects and the results in Sections 2 and 3, but not of the proof techniques used there. For a condensed and streamlined version of this paper, we refer to the extended abstract [CGG+17].

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1. Laman graphs

In this section, by a graph we mean a finite, connected, undirected graph without self-loops or multiple edges. We write $G = (V,E)$ to denote a graph $G$ with set of vertices $V$ and set of edges $E$. An (unoriented) edge $e$ between vertices $u$ and $v$ is denoted by $\{u,v\}$.

**Definition 1.1.** A labeling of a graph $G = (V,E)$ is a function $\lambda : E \rightarrow \mathbb{R}$; the pair $(G,\lambda)$ is called a labeled graph. A realization of $G$ is a function $\rho : V \rightarrow \mathbb{R}^2$. We say that a realization $\rho$ is compatible with a labeling $\lambda$ if for each edge $e \in E$ the Euclidean distance between its endpoints agrees with its label:

$$\lambda(e) = \|\rho(u) - \rho(v)\|^2,$$

where $e = \{u,v\}$.

A labeled graph $(G,\lambda)$ is realizable if and only if there is a realization compatible with $\lambda$.

**Definition 1.2.** We say that two realizations $\rho_1$ and $\rho_2$ of a graph $G$ are equivalent if and only if there exists a direct Euclidean isometry $\sigma$ of $\mathbb{R}^2$ such that $\rho_1 = \sigma \circ \rho_2$; a direct Euclidean isometry is an affine-linear map $\mathbb{R}^2 \rightarrow \mathbb{R}^2$ that preserves distance and orientation in $\mathbb{R}^2$.

**Definition 1.3.** A labeled graph $(G,\lambda)$ is called rigid if it satisfies the following properties:

- $(G,\lambda)$ is realizable;
- there are only finitely many realizations compatible with $\lambda$, up to equivalence.

Our main interest is to count the number of realizations of generically rigid graphs, namely graphs for which almost all realizable labelings induce rigidity. Unfortunately, in the real setting, this number is not well-defined, since it may depend on the actual labeling and not only on the graph. In order to define a number that depends only on the graph, we switch to the complex setting. By this we mean that we allow complex labelings $\lambda : E \rightarrow \mathbb{C}$ and complex realizations $\rho : V \rightarrow \mathbb{C}^2$. In this case, the compatibility condition Equation (1) becomes

$$\lambda(e) = \langle \rho(u) - \rho(v), \rho(u) - \rho(v) \rangle,$$

where $e = \{u,v\}$,
where \( (x, y) = x_1y_1 + x_2y_2 \). Moreover, we consider “direct complex isometries”, namely maps
\[
(\begin{pmatrix} x \\ y \end{pmatrix}) \mapsto A(\begin{pmatrix} x \\ y \end{pmatrix}) + b, \quad A \in \mathbb{C}^{2 \times 2} \text{ and } b \in \mathbb{C}^2,
\]
where \( A \) is an orthogonal matrix with determinant 1. Here, the word “isometries” is an abuse of language, since in this case \( \langle \cdot, \cdot \rangle \) is not an inner product. Notice that if we are given a labeling \( \lambda : E \to \mathbb{R} \) for a graph \( G \) and two realizations of \( G \) into \( \mathbb{R}^2 \) that are not equivalent under real direct isometries, then they are also not equivalent under complex isometries. This means that counting the number of non-equivalent realizations in \( \mathbb{C}^2 \) delivers an upper bound for the number of non-equivalent realizations in \( \mathbb{R}^2 \).

**Terminology.** Given a graph \( G = (V, E) \), the set of possible labelings \( \lambda : E \to \mathbb{C} \) forms a vector space, that we denote by \( \mathbb{C}^E \). In this way we are able to address the components of a vector \( \lambda \) in \( \mathbb{C}^E \) by edges \( e \in E \), namely by writing \( \lambda = (\lambda_e)_{e \in E} \). Since \( \mathbb{C}^E \) is a vector space, it is meaningful to speak about properties holding for a *general* labeling: a property \( \mathcal{P} \) holds for a general labeling if the set
\[
\{ \lambda \in \mathbb{C}^E : \mathcal{P}(\lambda) \text{ does not hold} \}
\]
is contained in a proper algebraic subset of \( \mathbb{C}^E \), i.e. a subset strictly contained in \( \mathbb{C}^E \) and defined by polynomial equations.

**Definition 1.4.** A graph \( G \) is called *generically realizable* if for a general labeling \( \lambda \) the labeled graph \( (G, \lambda) \) is realizable. A graph \( G \) is called *generically rigid* if for a general labeling \( \lambda \) the labeled graph \( (G, \lambda) \) is rigid.

**Remark 1.5.** If a graph \( G \) is generically realizable, then every subgraph \( G' \) of \( G \) is generically realizable. Every general labeling for \( G' \) can be extended to a general labeling for \( G \). Since by hypothesis \( G \) has a compatible realization, the subgraph \( G' \) admits such a realization as well.

**Definition 1.6.** A *Laman graph* is a graph \( G = (V, E) \) such that \( |E| = 2|V| - 3 \), and for every subgraph \( G' = (V', E') \) it holds \( |E'| \leq 2|V'| - 3 \).

We are going to see (Theorem 1.8) that Laman graphs are exactly the generically rigid ones. Many different characterizations of this property have appeared in the literature, for example by construction steps [Hen03] (see Theorem 1.8), or in terms of spanning trees after doubling one edge [LY82] or after adding an edge [Rec84], or in terms of three trees such that each vertex of the graph is covered by two trees [Cra06]. These characterizations can be used for decision algorithms on the minimal rigidity of a given graph [Ber05, JH97, DK09, GHT10].

For any graph \( G = (V, E) \), there is a natural map \( r_G \) from the set \( \mathbb{C}^{2|V|} \) of its realizations to the set \( \mathbb{C}^E \) of its labelings:
\[
r_G : \mathbb{C}^{2|V|} \to \mathbb{C}^E, \quad (x_v, y_v)_{v \in V} \mapsto ((x_u - x_v)^2 + (y_u - y_v)^2)\{u,v\} \in E.
\]
Each fiber of \( r_G \), i.e. a preimage \( r_G^{-1}(p) \) of a single point \( p \in \mathbb{C}^E \), is invariant under the group of direct complex isometries. We define a subspace \( \mathbb{C}^{2|V|-3} \subseteq \mathbb{C}^{2|V|} \) as follows: choose two...
(a) The first Henneberg rule: given any two vertices \( u \) and \( v \) (which may be connected by an edge or not), we add a vertex \( t \) and the two edges \( \{u, t\} \) and \( \{v, t\} \).

(b) The second Henneberg rule: given any three vertices \( u \), \( v \), and \( w \) such that \( u \) and \( v \) are connected by an edge, we remove the edge \( \{u, v\} \), we add a vertex \( t \) and the three edges \( \{u, t\} \), \( \{v, t\} \), and \( \{w, t\} \).

**Figure 3.** Henneberg rules

distinguished vertices \( \bar{u} \) and \( \bar{v} \) with \( \{\bar{u}, \bar{v}\} \in E \), and consider the linear subspace defined by the equations \( x_{\bar{u}} = y_{\bar{u}} = 0 \) and \( x_{\bar{v}} = 0 \). In this way, the subspace \( \mathbb{C}^{2|V| - 3} \) intersects every orbit of the action of isometries on a fiber of \( r_G \) in exactly two points: in fact, the equations do not allow any further translation or rotation; however, for any labeling \( \lambda: E \rightarrow \mathbb{C} \) and for every realization in \( \mathbb{C}^{2|V| - 3} \) compatible with \( \lambda \) there exists another realization, obtained by multiplying the first one by \(-1\), which is equivalent, but gives a different point in \( \mathbb{C}^{2|V| - 3} \). The restriction of \( r_G \) to \( \mathbb{C}^{2|V| - 3} \) gives the map

\[
h_G: \mathbb{C}^{2|V| - 3} \rightarrow \mathbb{C}^E.
\]

The following statement follows from the construction of \( h_G \); notice that the choice of \( \bar{u} \) and \( \bar{v} \) has no influence on the result. Recall that a map \( f: X \rightarrow Y \) between algebraic sets is called dominant if \( Y \setminus f(X) \) is contained in an algebraic proper subset of \( Y \).

**Lemma 1.7.** A graph \( G \) is generically rigid if and only if \( h_G \) is dominant and a general fiber of \( h_G \) is finite. This is equivalent to saying that \( h_G \) is dominant and \( 2|V| - 3 - |E| \).

**Proof.** It is enough to notice that if \( h_G \) is dominant, then the dimension of the general fiber is \( 2|V| - 3 - |E| \). \( \square \)

We state Laman’s theorem characterizing generically rigid graphs. A proof, which closely follows Laman’s original argument in his paper \([Lam70]\), can be found in Appendix A. For our purposes, we need a result that implies the existence of only a finite number of complex realizations, while the original statement deals with the real setting and proves that a given realization does not admit infinitesimal deformations.

**Theorem 1.8.** Let \( G \) be a graph. Then the following three conditions are equivalent:

(a) \( G \) is a Laman graph;

(b) \( G \) is generically rigid;

(c) \( G \) can be constructed by iterating the two Henneberg rules (see Figures 3a and 3b), starting from the graph that consists of two vertices connected by an edge.
Given a Laman graph, we are interested in the number of its realizations in \( \mathbb{C}^2 \) that are compatible with a general labeling, up to equivalence. As we have already pointed out, the degree of the map \( h_G \) (namely, the cardinality of a fiber \( h_G^{-1}(p) \) over a general point \( p \)) is twice the number of realizations of \( G \) compatible with a general labeling, up to equivalence. Lemma 1.7 and Theorem 1.8 imply that the map \( h_G \) is dominant and its degree is finite.

We now construct a map whose degree is exactly the number of equivalence classes. For this purpose, we employ a different way than in \( h_G \) to get rid of complex “translations” and “rotations”: first, for the translations, we take a quotient of vector spaces, which can be interpreted as setting \( x_u = y_u = 0 \) as for \( h_G \), or alternatively as moving the barycenter of a realization to the origin; second, we use projective coordinates to address the rotations. More precisely, in order to study the system of equations

\[
(x_u - x_v)^2 + (y_u - y_v)^2 = \lambda_{uv} \quad \text{for all } \{u, v\} \in E,
\]

which defines a realization of a Laman graph, we can regard the vectors \((x_u)_{u \in V}\) and \((y_u)_{u \in V}\) as elements of the space \( \mathbb{C}^V / (x_u = x_v \text{ for all } u, v \in V) \). In this way, we are allowed to add arbitrary constants to all components \( x_u \) or to all components \( y_u \) without changing the representative in the quotient; hence these vectors are invariant under translations. Moreover, if one performs the change of variables

\[
(x'_v)_{v \in V}, (y'_v)_{v \in V} \quad \mapsto \quad (x'_v := x_v + i y_v)_{v \in V}, \quad (y'_v := x_v - i y_v)_{v \in V},
\]

then the previous system of equations becomes

\[
(x'_u - x'_v)(y'_u - y'_v) = \lambda_{uv} \quad \text{for all } \{u, v\} \in E
\]

and the action of a complex rotation turns into the multiplication of the \( x'_u \)-coordinates by a scalar in \( \mathbb{C} \), and of the \( y'_u \)-coordinates by its inverse. Thus, by considering \((x'_u)_{u \in V}\) and \((y'_u)_{u \in V}\) as coordinates in two different projective spaces, the points we obtain are invariant under complex rotations. In order to employ these two strategies, we define

\[
P^{|V|-2}_\mathbb{C} := P\left( \mathbb{C}^V / \langle (1, \ldots, 1) \rangle \right) = P\left( \mathbb{C}^V / \langle (x_v)_{v \in V} : x_u = x_w \text{ for all } u, w \in V \rangle \right)
\]

and the map

\[
f_G: \quad P^{|V|-2}_\mathbb{C} \times P^{|V|-2}_\mathbb{C} \quad \rightarrow \quad P^{|E|-1}_\mathbb{C}
\]

\[
[(x_v)_{v \in V}], [(y_v)_{v \in V}] \quad \mapsto \quad \left( (x_u - x_v)(y_u - y_v) \right)_{\{u, v\} \in E},
\]

where \([ \cdot, \cdot] \) denotes the point in \( P^{|V|-2}_\mathbb{C} \) determined by a vector in \( \mathbb{C}^V \). Notice that the map \( f_G \) is well-defined, because the quantities \( x_u - x_v \) depend, up to scalars, only on the points \([(x_v)_{v \in V}]\), and not on the particular choice of representatives (and similarly for \( y_u - y_v \)). Note that \( f_G \) may not be defined everywhere, which is conveyed by the notation \( \rightarrow \).

**Lemma 1.9.** For any Laman graph \( G \) the equality \( \deg(h_G) = 2 \deg(f_G) \) holds.

**Proof.** Recall that the degree is computed by counting the number of preimages of a general point in the codomain. Let therefore \( \lambda \in \mathbb{C}^E \) be a general labeling and let \( \{u, v\} \) be the edge
used to define $h_G$, so in particular we can suppose $\lambda_{\{u,v\}} \neq 0$. We show that there is a 2:1 map $\eta$ from $h_G^{-1}(\lambda)$ to $f_G^{-1}(\lambda)$, where $\lambda \in \mathbb{P}_C^{V-1}$ is the point defined by the values of $\lambda \in \mathbb{C}^E$ as projective coordinates. The map $\eta$ is defined according to the change of variables Equation (2):

$$\eta: h_G^{-1}(\lambda) \rightarrow f_G^{-1}(\lambda), \quad (x_v)_{v \in V}, (y_v)_{v \in V} \mapsto [(x_v + i y_v)_{v \in V}, (x_v - i y_v)_{v \in V}].$$

In other words, we just take the coordinates of the embedded vertices as projective coordinates and make a complex coordinate transformation, namely one that diagonalizes the linear part of the isometries. The map $\eta$ is well-defined, since the quantities $(x_v + i y_v)_{v \in V}$ and $(x_v - i y_v)_{v \in V}$ are never all zero because of the definition of the map $h_G$. For $q \in \mathbb{P}_C^{V-2} \times \mathbb{P}_C^{V-2}$ of the form $q = \{([x_v]_{v \in V}, ([y_v]_{v \in V})\}$ and such that $\tilde{x}_u \neq \tilde{x}_v$ and $\tilde{y}_u \neq \tilde{y}_v$, we choose coordinates $(\tilde{x}_v)_{v \in V}, (\tilde{y}_v)_{v \in V}$ such that $\tilde{x}_u = \tilde{x}_v = 0$, $\tilde{y}_u = 1$, and $\tilde{y}_v = -1$. This is possible because we can add a constant vector to any of $(\tilde{x}_v)_{v \in V}$ or $(\tilde{y}_v)_{v \in V}$ without changing the point in $\mathbb{P}_C^{V-2} \times \mathbb{P}_C^{V-2}$. When $q \in f_G^{-1}(\lambda)$, every point in $\eta^{-1}(q)$ is of the form $\{(x_v)_{v \in V}, (y_v)_{v \in V}\}$, where $x_u = y_u = 0$ and $x_v = 0$ (recall the definition of the map $h_G$). By definition of $\eta$, we have that for all $v \in V$:

$$\begin{cases} x_v + i y_v = c \tilde{x}_v, \\ x_v - i y_v = d \tilde{y}_v, \end{cases}$$

for some constants $c, d \in \mathbb{C}$. Thus, for $v = \tilde{v}$, we get the equation $0 = c - d$, which in turn implies that every point in $\eta^{-1}(q)$ determines a realization of the form

$$\rho: V \rightarrow \mathbb{C}^2, \quad v \mapsto \left(\frac{\tilde{x}_v + \tilde{y}_v}{2}, \frac{\tilde{x}_v - \tilde{y}_v}{2i}\right),$$

that must be compatible with $\lambda$. By construction, the constant $c$ must satisfy $\lambda_{\{\tilde{u}, \tilde{v}\}} = (\rho(\tilde{u}) - \rho(\tilde{v}), \rho(\tilde{u}) - \rho(\tilde{v}))$. There are exactly two such numbers $c$, and this proves the statement. $\square$

**Corollary 1.10.** The number of realizations of a Laman graph, compatible with a general labeling and counted up to equivalence, is equal to the degree of the map $f_G$.

**2. Bigraphs and their equations**

In this section we introduce the main concept of the paper, the one of bigraph. Bigraphs are pairs of graphs whose edges are in bijection. Every graph determines a bigraph by simply duplicating it and considering the natural bijection between the edges. It is possible to associate to any bigraph a rational map as we did with the map $f_G$ in Equation (3). The reason for this duplication is that, in order to set up a recursive formula for the degree of $f_G$, we want to be able to handle independently the two factors $(x_u - x_v)$ and $(y_u - y_v)$ that appear in its specification. To do this, we have to allow disconnected graphs with multiple edges.

Notice that if we allow graphs with multiedges, then we have to give away the possibility to encode an edge via an unordered pair of vertices. Instead, we consider the sets $V$ and $E$ of vertices and edges, respectively, to be arbitrary sets, related by a function $\tau: E \rightarrow \mathcal{P}(V)$, where $\mathcal{P}$ denotes the power set, assigning to each edge its corresponding vertices. The image of an element $e \in E$ via $\tau$ can be either a set of cardinality two, when $e$ connects two distinct
vertices, or a singleton, when $e$ is a self-loop. This way of encoding graphs allows to use the same set for the edges of two graphs; this realizes formally the idea of prescribing a bijection between the edges of two graphs.

**Definition 2.1.** A *bigraph* is a pair of finite undirected graphs $(G, H)$ — allowing several components, multiple edges and self-loops — where $G = (V, E)$ and $H = (W, E)$. We denote by $\tau_G : E \to \mathcal{P}(V)$ and $\tau_H : E \to \mathcal{P}(W)$ the two maps assigning to each edge its vertices. The set $E$ is called the set of *biedges*. For technical reasons, we need to order the vertices of edges in $G$ or $H$; therefore, we assume that there is a total order $\prec$ given on the sets of vertices $V$ and $W$. An example of a bigraph is provided in Figure 5.

Notice that a single graph $G = (V, E)$ can be turned into a bigraph by considering the pair $(G, G)$, and by taking the set of biedges to be $E$; the total order $\prec$ is obtained by fixing any total order on $V$ and duplicating it. Next, we extend a weakened version of the Laman condition to bigraphs.

**Definition 2.2.** For a graph $G = (V, E)$ we define the *dimension* of $G$ as

$$\dim(G) := |V| - |\{\text{connected components of } G\}|.$$

**Remark 2.3.** Since a Laman graph is connected by assumption, the condition $2|V| = |E| + 3$ can be rewritten as $2\dim(G) = |E| + 1$.

**Definition 2.4.** Let $B = (G, H)$ be a bigraph with biedges $E$, then we say that $B$ is *pseudo-Laman* if

$$\dim(G) + \dim(H) = |E| + 1.$$

It follows from Remark 2.3 that for a Laman graph $G$ the bigraph $(G, G)$ is pseudo-Laman.

We introduce two operations that can be performed on a graph, starting from a subset of its edges: the subtraction of edges and the quotient by edges. We are going to use these constructions several times in our paper: subtraction is first used at the end of this section, while the quotient operation is mainly utilized starting from Section 3.

**Definition 2.5.** Let $G = (V, E)$ be a graph, and let $E' \subseteq E$. We define two new graphs, denoted $G / E'$ and $G \setminus E'$, as follows. An example for the operations is provided in Figure 4.

- Let $G'$ be the subgraph of $G$ determined by $E'$. We define $G / E'$ to be the graph obtained as follows. Its vertices are the equivalence classes of the vertices of $G$ modulo the relation dictating that two vertices $u$ and $v$ are equivalent if there exists a path in $G'$ connecting them. Its edges are determined by edges in $E \setminus E'$, more precisely an edge $e$ in $E \setminus E'$ such that $\tau_G(e) = \{u, v\}$ defines an edge in the quotient connecting the equivalence classes of $u$ and $v$ if and only if $e$ is not an edge of $G'$.
- Let $\bar{V}$ be the set of vertices of $G$ that are endpoints of some edge in $E \setminus E'$. Define $G \setminus E' = (\bar{V}, E \setminus E')$.

Via Definitions 2.6 and 2.7 we associate to each bigraph $B$ a rational map $f_B$, as we did in Section 1 for graphs.
Definition 2.6. Let $B = (G, H)$ be a bigraph, where $G = (V, \mathcal{E})$ and $H = (W, \mathcal{E})$. We set
\[ \mathbb{P}_C^{\dim(G)-1} := \mathbb{P}(\mathbb{C}^V / L_G), \quad \mathbb{P}_C^{\dim(H)-1} := \mathbb{P}(\mathbb{C}^W / L_H), \]
where
\[ L_G := \left\{ (x_v)_{v \in V} : x_u = x_t \text{ if and only if } u \text{ and } t \right\}, \]
\[ L_H := \left\{ (y_w)_{w \in W} : y_u = y_t \text{ if and only if } u \text{ and } t \right\}, \]
and $(x_v)_{v \in V}$ are the standard coordinates of $\mathbb{C}^V$ and similarly for $(y_w)_{w \in W}$.

Definition 2.7. Let $B = (G, H)$ be a bigraph, where $G = (V, \mathcal{E})$ and $H = (W, \mathcal{E})$. Define
\[ f_B: \mathbb{P}_C^{\dim(G)-1} \times \mathbb{P}_C^{\dim(H)-1} \to \mathbb{P}_C^{\dim(\mathcal{E})-1} \]
\[ [(x_v)_{v \in V}], [(y_w)_{w \in W}] \mapsto \left( (x_u - x_v)(y_t - y_w) \right)_{e \in \mathcal{E}}, \]
where $\{u, v\} = \tau_G(e)$, $u < v$, and $\{t, w\} = \tau_H(e)$, $t < w$, with $\tau_G$ and $\tau_H$ as in Definition 2.1. Here and in the rest of the paper, if $e$ is a self-loop say in $G$, then the corresponding polynomial in the definition of $f_B$ is considered to be $x_u - x_u = 0$. As in Section 1, the square brackets $[\cdot]$ denote points in $\mathbb{P}_C^{\dim(G)-1}$ or $\mathbb{P}_C^{\dim(H)-1}$ determined by vectors in $\mathbb{C}^V$ or $\mathbb{C}^W$. As for the map $f_G$, the map $f_B$ is well-defined because the quantities $(x_u - x_v)$ and $(y_t - y_w)$ depend only, up to scalars, on points in $\mathbb{P}_C^{\dim(G)-1}$ and $\mathbb{P}_C^{\dim(H)-1}$, and not on the chosen representatives. We call the map $f_B$ the rational map associated to $B$.

In Definition 2.7 we impose $u < v$ and $t < w$ in the equations defining the map $f_B$. The reason for this is that we want $f_B$, when $B$ is of the form $(G, G)$, to coincide with $f_G$ defined at the end of Section 1. If we do not specify the order in which the vertices appear in the expressions $(x_u - x_v)$ and $(y_t - y_w)$, we could end up with a map $f_B$ for which one component is of the form $(x_u - x_v)(y_v - y_u)$, and not $(x_u - x_v)(y_u - y_v)$ as we would expect. As in Section 1, we are mainly interested in the degree of the rational map associated to a bigraph.

Definition 2.8. Let $B$ be a bigraph. If $f_B$ is dominant, we define the Laman number of $B$, Lam($B$), as $\deg(f_B)$, which can hence be either a positive number, or $\infty$. Otherwise we set Lam($B$) to zero.
Remark 2.9. Notice that if $B$ is pseudo-Laman and $\text{Lam}(B) > 0$, then $\text{Lam}(B) \in \mathbb{N} \setminus \{0\}$.

If a bigraph has a self-loop or it is particularly simple, then its Laman number is zero or one, as shown by the following proposition.

Proposition 2.10. Let $B = (G, H)$ be a bigraph.

▷ If $G$ or $H$ has a self-loop, then $\text{Lam}(B) = 0$.

▷ If both $G$ and $H$ consist of a single edge that joins two vertices, then $\text{Lam}(B) = 1$.

Proof. If $G$ or $H$ has a self-loop, a direct inspection of the map $f_B$ shows that the defining polynomial corresponding to the self-loop is zero, hence $f_B$ cannot be dominant. If both $G$ and $H$ consist of a single edge that joins two vertices, then the map $f_B$ reduces to the map $\mathbb{P}_C^0 \times \mathbb{P}_C^0 \to \mathbb{P}_C^0$, which has degree 1. □

By simply unraveling the definitions, we see that the number of realizations of a Laman graph, up to equivalence, can be expressed as a Laman number.

Proposition 2.11. Let $G$ be a Laman graph, then the Laman number of the bigraph $(G, G)$ — where biedges are the edges of $G$ — is equal to the number of different realizations compatible with a general labeling of $G$, up to direct complex isometries.

Due to Proposition 2.11, the problem we want to address in this work is a special instance of the problem of computing the Laman number of a bigraph. Notice, however, that the Laman number of an arbitrary bigraph does not have an immediate geometric interpretation.

Remark 2.12. Let $B$ be a bigraph with biedges $\mathcal{E}$ such that $\text{Lam}(B) > 0$ and fix a biedge $\bar{e} \in \mathcal{E}$. Since $f_B$ is a rational dominant map between varieties over $\mathbb{C}$, there is a Zariski open subset $U \subseteq \mathbb{P}_C^{\mathcal{E}}$ such that the preimage of any point $p \in U$ under $f_B$ consists of $\text{Lam}(B)$ distinct points. In particular, we can suppose that $p$ is of the form $(\lambda_e)_{e \in \mathcal{E}}$ with $\lambda_{\bar{e}} = 1$ and $(\lambda_e)_{e \in \mathcal{E} \setminus \{\bar{e}\}}$ a general point of $\mathbb{C}^{\mathcal{E} \setminus \{\bar{e}\}}$.

In the following we find it useful to work in an affine setting: this is why in Definition 2.14 we introduce the sets $Z_B^\mathcal{E}$. We are going to use the language of affine schemes, mainly to be able to manipulate the equations freely without being concerned about the reducedness of the ideal they generate. The reader not acquainted with scheme theory can harmlessly think about classical affine varieties, and indeed we are going to prove that the ideals we are concerned with are reduced. We first need to set some notation.

Definition 2.13. Let $B = (G, H)$ be a bigraph, where $G = (V, \mathcal{E})$ and $H = (W, \mathcal{E})$. Define

\[ P := \{(u, v) \in V^2 : \{u, v\} \in \tau_G(\mathcal{E}), u \neq v\}, \]
\[ Q := \{(t, w) \in W^2 : \{t, w\} \in \tau_H(\mathcal{E}), t \neq w\}. \]

Notice that the elements of $P$ and $Q$ are ordered pairs (and this is conveyed also by the different notation used). In particular, from the definition we see that if $(u, v) \in P$, then also $(v, u) \in P$, and similarly for $Q$. Moreover, we require the two elements in each pair to be different, and this is crucial in view of Definition 3.6.
Figure 5. A bigraph that consists of two copies of the only Laman graph with 4 vertices. Edges on the left and on the right bearing the same label are associated to the same biedge.

**Definition 2.14.** Let $B = (G, H)$ be a bigraph with biedges $E$ without self-loops. Fix a biedge $e \in E$. For a general point $(\lambda_e)_{e \in E \setminus \{e\}}$ in $\mathbb{C}^E \setminus \{e\}$, we define $Z^B_C$ as the subscheme of $\mathbb{C}^P \times \mathbb{C}^Q$ defined by

$$
\begin{align*}
  x_{uv} = y_{tw} &= 1, \quad \bar{u} \prec \bar{v}, \quad \bar{t} \prec \bar{w}, \\
  x_{uv} y_{tw} &= \lambda_e, \quad \text{for all } e \in E \setminus \{e\}, \quad u < v, \quad t < w, \\
  \sum_{e \in \mathcal{C}} x_{uv} &= 0, \quad \text{for all cycles } \mathcal{C} \text{ in } G, \\
  \sum_{e \in \mathcal{D}} y_{tw} &= 0, \quad \text{for all cycles } \mathcal{D} \text{ in } H,
\end{align*}
$$

where we take $(x_{uv})_{(u,v)\in P}$ and $(y_{tw})_{(t,w)\in Q}$ as coordinates and where

$$
\begin{align*}
  \{\bar{u}, \bar{v}\} &= \tau_G(e), \\
  \{u, v\} &= \tau_G(e), \\
  \{\bar{t}, \bar{w}\} &= \tau_H(e), \\
  \{t, w\} &= \tau_H(e).
\end{align*}
$$

Here and in the following, when we write $\sum_{e \in \mathcal{C}} x_{uv}$ for a cycle $\mathcal{C} = (u_0, u_1, \ldots, u_n = u_0)$ in $G$ we mean the expression $x_{u_0u_1} + \cdots + x_{u_{n-1}u_0}$ (and similarly for cycles in $H$). Notice that in cycles we allow repetitions of edges. In particular, if $(u, v) \in P$, one can always consider the cycle $(u, v, u)$, which implies the relation $x_{uv} = -x_{vu}$. We drop the dependence of $Z^B_C$ on $\bar{e}$ and $(\lambda_e)_{e \in E \setminus \{e\}}$ in the notation, since in the following it is clear from the context.

**Example 2.15.** Consider the bigraph $(G, G)$ with set of biedges $E$ as in Figure 5, that consists of two copies of the only Laman graph with 4 vertices. Fix the biedge $e$ to be the one associated to the two edges connecting 2 and 3. If $(\lambda_e)_{e \in E \setminus \{e\}}$ is a general point, then the scheme $Z^B_C$ is defined by the following equations:

$$
\begin{align*}
  x_{23} y_{23} &= 1, \\
  x_{12} y_{12} &= \lambda_r, \\
  x_{12} + x_{21} &= x_{13} + x_{31} = x_{23} + x_{32} = x_{24} + x_{42} = x_{34} + x_{43} = 0, \\
  x_{13} y_{13} &= \lambda_g, \\
  y_{12} y_{21} &= y_{13} + y_{31} = y_{23} + y_{32} = y_{24} + y_{42} = y_{34} + y_{43} = 0, \\
  x_{24} y_{24} &= \lambda_o, \\
  x_{12} + x_{23} + x_{31} &= y_{12} + y_{23} + y_{31} = 0, \\
  x_{34} y_{34} &= \lambda_b, \\
  x_{24} + x_{43} + x_{32} &= y_{24} + y_{43} + y_{32} = 0.
\end{align*}
$$

Note that we did not include redundant equations coming from cycles such as $(1, 2, 4, 3, 1)$.

In the following lemma we show that the sets $Z^B_C$ can be used to compute the degree of $f_B$. 
Lemma 2.16. Let \( B = (G, H) \) be a bigraph with biedges \( E \) without self-loops. Fix a biedge \( \vec{e} \in E \). Let \( p \in \mathbb{P}^{E-1}_{C} \) be given by \( p_{e} = 1 \) and \( p_{\bar{e}} = \lambda_{e} \) for all \( e \in E \setminus \{ \vec{e} \} \). Then the schemes \( f_{B}^{-1}(p) \) and \( Z_{C}^{B} \) are isomorphic. In particular, \( Z_{C}^{B} \) consists of \( \text{Lam}(B) \) distinct points.

Proof. Write \( \tau_{G}(\vec{e}) = \{ \vec{u}, \vec{v} \} \) with \( \vec{u} \prec \vec{v} \) and \( \tau_{H}(\vec{e}) = \{ \vec{t}, \vec{w} \} \) with \( \vec{t} \prec \vec{w} \). We define a morphism from \( f_{B}^{-1}(p) \) to \( Z_{C}^{B} \) by sending a point

\[
\left( [(x_{u})_{u \in V}], [(y_{w})_{w \in W}] \right) \in f_{B}^{-1}(p)
\]

to the point whose \( uv \)-coordinate is \( (x_{u} - x_{v})/(x_{u} - x_{v}) \), where \( u < v \), for all \( (u, v) \in P \), and whose \( tw \)-coordinate is \( (y_{t} - y_{w})/(y_{t} - y_{w}) \), where \( t < w \), for all \( (t, w) \in Q \).

We define a morphism from \( Z_{C}^{B} \) to \( f_{B}^{-1}(p) \) as follows. For every component \( C \) of \( G \), fix a rooted spanning tree \( T_{C} \) and denote its root by \( r(C) \); similarly for \( H \). We send a point \( \left( [(x_{uv})_{(u, v) \in P}, (y_{tw})_{(t, w) \in Q}] \right) \in Z_{C}^{B} \) to the point \( \left( [(x_{u})_{u \in V}], [(y_{t})_{t \in W}] \right) \in f_{B}^{-1}(p) \) such that if a vertex \( u \in V \) belongs to the connected component \( C \), then \( x_{u} = \sum_{i=0}^{u-1} x_{u_{i+1}}, \) where \( (r(C) = u_{0}, \ldots, u_{n} = u) \) is the unique path in \( T_{C} \) from \( r(C) \) to \( u \), and similarly for the vertices \( t \in W \). A direct computation shows that both maps are well-defined, and are each other’s inverse. From this the statement follows. \( \square \)

We conclude the section by proving a few results about the Laman number of a special kind of bigraph, that are used in Section 4 to obtain the final algorithm.

Definition 2.17. Let \( G \) be a graph and let \( e \) be an edge of \( G \). We say that \( e \) is a bridge if removing \( e \) increases the number of connected components of \( G \).

Lemma 2.18. Let \( B = (G, H) \) be a pseudo-Laman bigraph with biedges \( E \) without self-loops and fix \( \vec{e} \in E \). If \( \vec{e} \) is a bridge in both \( G \) and \( H \), then \( \text{Lam}(B) = 0 \).

Proof. Suppose for a contradiction \( \text{Lam}(B) > 0 \). Consider the equations defining \( Z_{C}^{B} \). Since \( \vec{e} \) is a bridge in both \( G \) and \( H \), the variables \( x_{\vec{u}\vec{v}} \) and \( y_{\vec{t}\vec{w}} \), where \( \{ \vec{u}, \vec{v} \} = \tau_{G}(\vec{e}) \) and \( \{ \vec{t}, \vec{w} \} = \tau_{H}(\vec{e}) \), do not appear in any of the equations defined by cycles in \( G \) or in \( H \) except for the equations \( x_{\vec{u}\vec{v}} = -x_{\vec{v}\vec{u}} \) and \( y_{\vec{t}\vec{w}} = -y_{\vec{w}\vec{t}} \). Hence, the system of equations

\[
\begin{align*}
x_{uv} y_{tw} &= \lambda_{e}, & &\text{for all } e \in E \setminus \{ \vec{e} \}, u < v, t < w, \\
\sum_{e \in E} x_{uv} &= 0, & &\text{for all cycles } C \in G \setminus \{ \vec{e} \}, \\
\sum_{e \in E} y_{tw} &= 0, & &\text{for all cycles } D \in H \setminus \{ \vec{e} \}
\end{align*}
\]

defines an affine scheme \( \mathcal{Z} \) isomorphic to \( Z_{C}^{B} \). One notices, however, that if \( (x_{uv}, y_{tw}) \) is a point in \( \mathcal{Z} \), then for every \( \eta \in C \setminus \{ 0 \} \) also the point \( (\eta x_{uv}, \eta y_{tw}) \) is in \( \mathcal{Z} \). This implies that \( Z_{C}^{B} \) has infinite cardinality, which contradicts the pseudo-Laman assumption on \( B \). \( \square \)

Lemma 2.19. Let \( B = (G, H) \) be a pseudo-Laman bigraph with biedges \( E \) without self-loops and fix \( \vec{e} \in E \). If \( \vec{e} \) is a bridge in \( G \), but not in \( H \), then

\[
\text{Lam}(B) = \text{Lam}(\langle (G \setminus \{ \vec{e} \}), (H \setminus \{ \vec{e} \}) \rangle).
\]
Proof. Consider another biedge $\tilde{e}$ and use it to define the scheme $Z^B_C$. Its equations are:

$$Z^B_C: \begin{cases} x_{\tilde{u}\tilde{v}} = y_{\tilde{u}\tilde{v}} = 1, & \tilde{u} < \tilde{v}, \tilde{t} < \tilde{w}, \\ x_{uv} y_{tw} = \lambda_e, & \text{for all } e \in E \setminus \{\tilde{e}\}, u < v, t < w, \\ \sum_\xi x_{\xi uv} = 0, & \text{for all cycles } \xi \text{ in } G, \\ \sum_\varnothing y_{tw} = 0, & \text{for all cycles } \varnothing \text{ in } H. \end{cases}$$

Now consider the bigraph $\tilde{B} = (G \setminus \{\tilde{e}\}, H \setminus \{\tilde{e}\})$. Notice that we can still use $\tilde{e}$ to define the scheme $Z^B_C$. Its equations are:

$$Z^B_C: \begin{cases} x_{\tilde{u}\tilde{v}} = y_{\tilde{u}\tilde{v}} = 1, & \tilde{u} < \tilde{v}, \tilde{t} < \tilde{w}, \\ x_{uv} y_{tw} = \lambda_e, & \text{for all } e \in E \setminus \{\tilde{e}, \tilde{\bar{e}}\}, u < v, t < w, \\ \sum_\xi x_{\xi uv} = 0, & \text{for all cycles } \xi \setminus \{\tilde{e}\} \text{ in } G, \\ \sum_\varnothing y_{tw} = 0, & \text{for all cycles } \varnothing \setminus \{\tilde{e}\} \text{ in } H. \end{cases}$$

We are going to prove that $Z^B_C$ and $Z^B_C$ are isomorphic, concluding the proof. Since $\tilde{e}$ is a bridge in $G$, the coordinate $x_{\tilde{u}\tilde{v}}$ appears in the equations of $Z^B_C$ only in $x_{\tilde{u}\tilde{v}} y_{\tilde{u}\tilde{w}} = \lambda_{\tilde{e}}$, and in $x_{\tilde{u}\tilde{v}} = -x_{\tilde{v}\tilde{u}}$. This means that the image of $Z^B_C$ under the projection from the coordinates $x_{\tilde{u}\tilde{v}}$, $x_{\tilde{u}\tilde{w}}$, $y_{\tilde{u}\tilde{w}}$ and $y_{\tilde{u}\tilde{v}}$ coincides with $Z^B_C$. Moreover, the projection is an isomorphism on $Z^B_C$: in fact, the $y_{\tilde{u}\tilde{w}}$-coordinate can be recovered by a cycle condition (recall that $\tilde{e}$ is not a bridge in $H$, so it appears in a cycle different from the trivial cycle $(\tilde{t}, \tilde{w}, \tilde{t})$). Then the $x_{\tilde{u},\tilde{v}}$-coordinate can be recovered from the equation $x_{\tilde{u}\tilde{v}} y_{\tilde{u}\tilde{w}} = \lambda_{\tilde{e}}$. 

**Definition 2.20.** Let $B = (G, H)$ be a bigraph with biedges $E$ without self-loops and let $\tilde{e} \in E$ be fixed. Suppose that the graph $G$ splits into disconnected subgraphs $G'_1, G'_2$ and that $H$ splits into disconnected subgraphs $H'_1, H'_2$. Suppose further that $E = E_1 \cup E_2 \cup \{\tilde{e}\}$ decomposes into three disjoint subsets such that

$$G'_1 = (V'_1, E_1 \cup \{\tilde{e}\}), \quad G'_2 = (V'_2, E_2) \quad \text{and} \quad H'_1 = (W'_1, E_1), \quad H'_2 = (W'_2, E_2 \cup \{\tilde{e}\}).$$

Under these assumptions we say that the bigraph $B$ **untangles** via $\tilde{e}$ into bigraphs

$$B_1 := (G'_1 \setminus \{\tilde{e}\}, H'_1), \quad B_2 := (G'_2, H'_2 \setminus \{\tilde{e}\}).$$

See Figure 6b for an example of a bigraph that untangles via an edge (the gray vertical one).

**Proposition 2.21.** Suppose that a bigraph $B = (G, H)$ with biedges $E$ without self-loops untangles via $\tilde{e} \in E$ into bigraphs $B_1$ and $B_2$, where $\tilde{e}$ is neither a bridge in $G$ nor in $H$, then

$$\text{Lam}(B) = \text{Lam}(B_1) \cdot \text{Lam}(B_2).$$

**Proof.** We use the notation from **Definition 2.20.** The hypothesis implies that

$$\dim(G) = \dim(G'_1) + \dim(G'_2) \quad \text{and} \quad \dim(H) = \dim(H'_1) + \dim(H'_2).$$
Set \( \{\bar{u}, \bar{v}\} = \tau_G(\bar{e}) \) and \( \{\bar{t}, \bar{w}\} = \tau_H(\bar{e}) \). Fix a biedge \( e_1 \in E_1 \) and let \( \{t_1, w_1\} = \tau_H(e_1) \). Similarly, fix a biedge \( e_2 \in E_2 \) and let \( \{u_2, v_2\} = \tau_G(e_2) \). We consider the following three rational maps:

\[
\begin{align*}
\mathbb{P}_C^{\dim(G)-1} &\to \mathbb{P}_C^{\dim(G_1)'-1} \times \mathbb{P}_C^{\dim(G_2)'-1} \times \mathbb{P}_C^1 \\
([x_v]_{v \in V}) &\mapsto (\Lbrack ([x_v]_{v \in V'}, \{x_{\bar{u}} - x_{u_2} - x_{v_2}\}) \\
\mathbb{P}_C^{\dim(H)-1} &\to \mathbb{P}_C^{\dim(H_1)'-1} \times \mathbb{P}_C^{\dim(H_2)'-1} \times \mathbb{P}_C^1 \\
([y_w]_{w \in W}) &\mapsto (\Lbrack ([y_w]_{w \in W'}, \{y_{\bar{t}} - y_{\bar{w}} - x_{t_1} - x_{w_1}\}) \\
\mathbb{P}[\mathcal{E}]^{-1} &\to \mathbb{P}[\mathcal{E}_1]^{-1} \times \mathbb{P}[\mathcal{E}_2]^{-1} \times \mathbb{P}_C^1 \\
(z_e)_{e \in \mathcal{E}} &\mapsto (\Lbrack (z_{\bar{e}})_{\bar{e} \in \mathcal{E}_1}, (z_{\bar{e}})_{\bar{e} \in \mathcal{E}_2}, (z_{\bar{e}} : z_{e_1}), (z_{\bar{e}} : z_{e_2}))
\end{align*}
\]

One can check that these maps are birational. We define the rational map \( \bar{f} \) so that the following diagram is commutative:

\[
\begin{array}{ccc}
\mathbb{P}_C^{\dim(G)-1} \times \mathbb{P}_C^{\dim(H)-1} & \xrightarrow{\bar{f}_B} & \mathbb{P}_C^{\dim(G_1)'-1} \times \mathbb{P}_C^{\dim(G_2)'-1} \times \mathbb{P}_C^1 \\
\mathbb{P}[\mathcal{E}]^{-1} & \xrightarrow{\bar{f}} & \mathbb{P}[\mathcal{E}_1]^{-1} \times \mathbb{P}[\mathcal{E}_2]^{-1} \times \mathbb{P}_C^1 \times \mathbb{P}_C^1
\end{array}
\]

It follows that \( \deg(f_B) = \deg(\bar{f}) \). Denote \( ([x_v]_{v \in V'}) \) by \( [X_i] \) for \( i \in \{1, 2\} \), and denote \( ([y_w]_{w \in W'}) \) by \( [Y_i] \) for \( i \in \{1, 2\} \). An explicit computation shows that \( \bar{f} \) sends a point

\[
\left( ([X_1], [X_2], (\mu_G : \nu_G)), (\Lbrack [Y_1], [Y_2], (\mu_H : \nu_H)) \right)
\]

to the point

\[
\left( f_B([X_1], [Y_1]), f_B([X_2], [Y_2]), (\mu_G \delta_G([X_1]) : \nu_G \delta_G([X_1])), (\mu_H \delta_H([Y_2]) : \nu_H \delta_H([Y_2])) \right),
\]

where \( \delta_G: \mathbb{P}_C^{\dim(G_1)'-1} \to \mathbb{C} \) and \( \delta_H: \mathbb{P}_C^{\dim(H_1)'-1} \to \mathbb{C} \) are some rational functions. From the explicit form of \( \bar{f} \) we see that \( \deg(\bar{f}) = \deg(\bar{f}_1) \cdot \deg(\bar{f}_2) \), where the map \( \bar{f}_1 \) is given by

\[
\begin{array}{ccc}
\mathbb{P}_C^{\dim(G_1)'-1} \times \mathbb{P}_C^{\dim(H_1)'-1} \times \mathbb{P}_C^1 & \to & \mathbb{P}_C[\mathcal{E}_1]^{-1} \times \mathbb{P}_C^1 \\
\Lbrack (X_1), [Y_1], (\mu_G : \nu_G) \mapsto f_B([X_1], [Y_1]), (\mu_G \delta_G([X_1]) : \nu_G \delta_G([X_1]))
\end{array}
\]

and similarly for \( \bar{f}_2 \). Note that for both \( i \in \{1, 2\} \), the map \( \bar{f}_i \) is the restriction to a suitable open set of the map \( f_B \times \text{id}_{\mathbb{P}_C^1} \), since the rational maps \( \delta_G \) and \( \delta_H \) do not have any other influence than restricting the domain of the map. This means that \( \deg(\bar{f}_i) = \deg(f_B) \) for both \( i \in \{1, 2\} \), which concludes the proof.

\[\square\]

**Lemma 2.22.** If a pseudo-Laman bigraph \( B = (G, H) \) without self-loops untangles via \( \bar{e} \in \mathcal{E} \) into bigraphs \( B_1 \) and \( B_2 \) such that \( \bar{e} \) is a bridge in \( G \) but not in \( H \), then \( \text{Lam}(B) = 0 \).
Lemma 2.19. We see that if $\text{Lam}(B) = \text{Lam}(\tilde{B})$, where $\tilde{B} = (G \setminus \{\bar{e}\}, H \setminus \{\bar{e}\})$. It follows that $\tilde{B}$ is the disjoint union of $B_1$ and $B_2$. Using the same technique adopted in Lemma 2.18 we see that if $\text{Lam}(\tilde{B})$ were positive, then we could scale the points in $\mathbb{Z}^B$ by arbitrary scalars $\eta \in \mathbb{C} \setminus \{0\}$, obtaining an infinite Laman number. This would contradict the pseudo-Laman hypothesis, so the statement is proved. \hfill \Box

Lemma 2.23. If a pseudo-Laman bigraph $B = (G, H)$ without self-loops untangles via $e \in E$ into bigraphs $B_1$ and $B_2$, where $e$ is a bridge in $G$, then either $B_1$ or $B_2$ is not pseudo-Laman.

Proof. Suppose that both $B_1$ and $B_2$ are pseudo-Laman; we show that this leads to a contradiction. Using the hypothesis one sees that $\dim(G_1') = \dim(G_1) - 1$. Moreover, if $\bar{e}$ is also a bridge, then $\dim(H_2' \setminus \{\bar{e}\}) = \dim(H_2') - 1$, otherwise we have $\dim(H_2' \setminus \{\bar{e}\}) = \dim(H_2')$. Since $\dim(G) = \dim(G'_1) + \dim(G'_2)$ and $\dim(H) = \dim(H'_1) + \dim(H'_2)$, the pseudo-Lamanity of $B_1$ and $B_2$ implies

$$\dim(G'_1) - 1 + \dim(G'_2) + \dim(H'_1) + \dim(H'_2) \geq |E_1| + |E_2| + 2,$$

where $E_1$ and $E_2$ are the biedges of $B_1$ and $B_2$, respectively (the inequality $\geq$ takes into account the fact that $\bar{e}$ may or may not be a bridge). Since $|E| = |E_1| + |E_2| + 1$, the previous equation in turn implies

$$\dim(G) + \dim(H) \geq |E| + 2,$$

contradicting the hypothesis that $B$ is pseudo-Laman. \hfill \Box

3. Bidistances and quotients

Tropical geometry is a technique that allows us to transform systems of polynomial equations into systems of piecewise linear equations. This is possible if one works over the field of Puiseux series. An algebraic relation between Puiseux series implies a piecewise linear relation between their orders (which are rational numbers). One hopes that the piecewise linear system is easier to solve; if so, one has candidates for the orders of solutions of the initial system, and sometimes this is enough to obtain the desired information. This technique has been successfully used, amongst others, by Mikhalkin [Mik05] to count the number of algebraic curves with some prescribed properties.

We use a similar idea for computing the Laman number of a pseudo-Laman bigraph. As we pointed out in the Introduction, this amounts to compute the base degree of the algebraic matroid associated to the variety parametrizing distances between pairs of points; however, we do not use the matroid formalism in our work. For each pseudo-Laman bigraph $B$, we need to know the number of solutions of the system defining $\mathbb{Z}^B$. This number coincides with the number of solutions of a “perturbed” system over the Puiseux field (Lemma 3.3). The orders of each solution of the new system satisfy piecewise linear conditions (Definition 3.6). We prove (Lemmas 3.18, 3.20 and 3.21) that the Puiseux series solutions sharing the same orders are in bijection with the complex solutions of another system of equations of a certain “quotient bigraph”. This yields a first recursive scheme (Theorem 3.23).
Notation. Denote by \( \mathbb{K} \) the field \( \mathbb{C}\{\{s\}\} \) of Puiseux series with coefficients in \( \mathbb{C} \). Recall that \( \mathbb{K} \) is of characteristic zero and is algebraically closed. The field \( \mathbb{K} \) is equipped with a valuation \( \nu: \mathbb{K} \setminus \{0\} \rightarrow \mathbb{Q} \) associating to an element \( \sum_{i=k}^{\infty} c_i s^{in} \) the rational number \( k/n \), where \( k \in \mathbb{Z} \) and \( c_k \neq 0 \). Recall that \( \nu(a \cdot b) = \nu(a) + \nu(b) \) and \( \nu(a + b) \geq \min\{\nu(a),\nu(b)\} \).

Definition 3.1. Let \( B = (G,H) \) be a bigraph. Define \( f_{B,\mathbb{K}} \) to be the map obtained as the extension of scalars, via the natural inclusion \( \mathbb{C} \hookrightarrow \mathbb{K} \), of the rational map \( f_B \) associated to \( B \) (see Definition 2.6). This means that, with the notation as in Definition 2.7, we define
\[
\mathbb{P}^{\dim(G)-1}_\mathbb{K} := \mathbb{P}\left(\mathbb{K}^V/(L_G \otimes \mathbb{K})\right), \quad \mathbb{P}^{\dim(H)-1}_\mathbb{K} := \mathbb{P}\left(\mathbb{K}^W/(L_H \otimes \mathbb{C} \otimes \mathbb{K})\right),
\]
and then \( f_{B,\mathbb{K}}: \mathbb{P}^{\dim(G)-1}_\mathbb{K} \times \mathbb{P}^{\dim(H)-1}_\mathbb{K} \rightarrow \mathbb{P}^{[\mathcal{E}]-1}_\mathbb{K} \) is given by the same equations as \( f_B \).

Remark 3.2. By construction, \( \deg(f_B) \) is defined if and only if \( \deg(f_{B,\mathbb{K}}) \) is defined, and in that case they coincide. In fact, \( f_B \) is dominant if and only if \( f_{B,\mathbb{K}} \) is so. In this case, let \( Y_C \) be the open subset where \( f_B \) is defined. Because \( f_B: Y_C \rightarrow \mathbb{P}^{[\mathcal{E}]-1}_\mathbb{C} \) is a dominant morphism between complex varieties, there exists an open subset \( \mathcal{U}_C \subseteq \mathbb{P}^{[\mathcal{E}]-1}_\mathbb{C} \) such that the fiber of \( f_B \) over any point of \( \mathcal{U}_C \) consists of \( \deg(f_B) \) distinct points. Since \( f_{B,\mathbb{K}} \) is the extension of scalars of \( f_B \), it follows that also the fiber of \( f_{B,\mathbb{K}} \) over any point in \( \mathcal{U}_K := \mathcal{U}_C \times_{\text{Spec}(\mathbb{C})} \text{Spec}(\mathbb{K}) \) consists of \( \deg(f_B) \) distinct points. In fact, for every \( q_\kappa \in \mathcal{U}_K \) we have \( f_{B,\mathbb{K}}^{-1}(q_\kappa) \cong f_B^{-1}(q_\kappa) \times_{\text{Spec}(\mathbb{C})} \text{Spec}(\mathbb{K}) \), where \( q_\kappa \) is the image of \( q_\kappa \) under the natural morphism \( \mathcal{U}_K \rightarrow \mathcal{U}_C \). Hence the cardinality of \( f_{B,\mathbb{K}}^{-1}(q_\kappa) \) is equal to the cardinality of \( f_B^{-1}(q_\kappa) \) and therefore \( \deg(f_{B,\mathbb{K}}) = \deg(f_B) \).

The fact that the map \( f_{B,\mathbb{K}} \) is defined over \( \mathbb{C} \), and not over \( \mathbb{K} \), gives us a lot of freedom concerning the valuation of the general point whose fiber we consider. More precisely:

Lemma 3.3. Let \( B \) be a bigraph such that \( \text{Lam}(B) > 0 \). Fix a vector \( wt = (\lambda_e)_{e \in \mathcal{E}} \in \mathbb{Q}^{[\mathcal{E}]} \). Then \( \deg(f_{B,\mathbb{K}}) \) coincides with the cardinality of the fiber of \( f_{B,\mathbb{K}} \) over any point \( p \in \mathbb{P}^{[\mathcal{E}]-1}_\mathbb{K} \) of the form \( p = (\lambda_e s^{wt(e)})_{e \in \mathcal{E}} \), where \( (\lambda_e)_{e \in \mathcal{E}} \) is a general point in \( \mathbb{C}^{[\mathcal{E}]} \).

Proof. Consider the rational map \( f_B: \mathbb{P}^{\dim(G)-1}_\mathbb{C} \times \mathbb{P}^{\dim(H)-1}_\mathbb{C} \rightarrow \mathbb{P}^{[\mathcal{E}]-1}_\mathbb{C} \), which is dominant by hypothesis. To prove the statement it is enough to show that a point \( p \) satisfying the hypothesis lies in the set \( \mathcal{U}_K \) defined in Remark 3.2. Suppose by contradiction that \( p \notin \mathcal{U}_K \). Since \( \mathcal{U}_K \) is Zariski open, it is defined by a disjunction of polynomial inequalities with coefficients in \( \mathbb{C} \). Let \( g \neq 0 \) be one of these inequalities: by assumption \( g(p) = 0 \), but this implies that \( \tilde{g}(\lambda_e)_{e \in \mathcal{E}} = 0 \) for some non-zero polynomial \( \tilde{g} \) over \( \mathbb{C} \), contradicting the generality of \( \lambda_e \). \( \square \)

Let \( B \) be a bigraph such that \( \text{Lam}(B) > 0 \). Fix a vector \( wt = (\lambda(e))_{e \in \mathcal{E}} \in \mathbb{Q}^{[\mathcal{E}]} \) and a bidegree \( \bar{e} \in \mathcal{E} \). Arguing as in Remark 2.12, we see that it is enough to consider fibers of \( f_{B,\mathbb{K}} \) over points \( p \) of the form \( p_e = 1 \), while \( p_e = \lambda_e s^{wt(e)} \) for a general point \( (\lambda_e)_{e \in \mathcal{E} \setminus \{\bar{e}\}} \) in \( \mathbb{C}^{\mathcal{E} \setminus \{\bar{e}\}} \). This is why we formulate the following assumption, which is used throughout this section.

Assumption. Let \( B \) be a pseudo-Laman bigraph with bides \( \mathcal{E} \) such that \( \text{Lam}(B) > 0 \). Notice that by Proposition 2.10 this implies that \( B \) has no self-loops. Fix a bidegree \( \bar{e} \in \mathcal{E} \), fix \( wt \in \mathbb{Q}^{\mathcal{E} \setminus \{\bar{e}\}} \) and let \( (\lambda_e)_{e \in \mathcal{E} \setminus \{\bar{e}\}} \) be a general point in \( \mathbb{C}^{\mathcal{E} \setminus \{\bar{e}\}} \). Let \( p \in \mathbb{P}^{[\mathcal{E}]-1}_\mathbb{K} \) be such that \( p_{\bar{e}} = 1 \) and \( p_e = \lambda_e s^{wt(e)} \) for all bides \( e \in \mathcal{E} \setminus \{\bar{e}\} \).
Remark 3.4. Let $B = (G, H)$ be a bigraph with bidges $E$ and use Section 3. Following Lemma 2.16 one can prove that $f^{1}_{B,K}(p)$ is isomorphic to

$$Z^{B}_{K} := \text{Spec} \left( \frac{\mathbb{K}^{\mathbb{P}} \times \mathbb{K}^{Q}}{\mathbb{P}^{x}, \mathbb{Q}^{y}} \right)$$

where the notation is as in Definition 2.14.

Example 3.5. We continue with Example 2.15: if we fix the vector $wt$ to be $(1)_{E \setminus \{e\}}$, then the scheme $Z^{B}_{K}$ is defined by the equations

$$x_{23} = 1, \quad x_{12} y_{12} = \lambda_{r} s, \quad x_{24} y_{24} = \lambda_{o} s,$$

$$y_{23} = 1, \quad x_{13} y_{13} = \lambda_{g} s, \quad x_{34} y_{34} = \lambda_{b} s,$$

and by the equations coming from the cycles (they are the same as in Example 2.15).

If $p$ is a point of the form $(\bar{\lambda}_{v} s^{\text{wt}(e)})$ in the codomain of the map $f^{1}_{B,K}$, then for every point $q \in f^{1}_{B,K}(p)$ we can consider the vector of the valuations of its coordinates. In terms of tropical geometry, this means that we take the tropicalization of the preimage $f^{1}_{B,K}(p)$. In Definition 3.6 we associate to each such point $q$ a discrete object, which we call bidistance (see Definition 3.8). We then partition the set $f^{1}_{B,K}(p)$ according to the bidistances that are determined by its points.

Definition 3.6. Let $B = (G, H)$ be a bigraph with bidges $E$ and use Section 3. Fix $q \in f^{-1}_{B,K}(p)$. Then $q = \left( [(x_{e})_{e \in V}, [(y_{e})_{e \in W}] \right)$ and by construction

$$\frac{x_{u} - x_{v}}{x_{u} - y_{u}} \cdot \frac{y_{t} - y_{w}}{y_{t} - y_{w}} = \lambda_{e} s^{\text{wt}(e)} \quad \text{for all } e \in E \setminus \{\bar{e}\},$$

$$\{u, \bar{v}\} = \tau_{G}(e), \quad \bar{u} \prec \bar{v}, \quad \{u, v\} = \tau_{G}(e), \quad u \prec v,$$

$$\{\bar{t}, \bar{w}\} = \tau_{H}(e), \quad \bar{t} \prec \bar{w}, \quad \{t, w\} = \tau_{H}(e), \quad t \prec w.$$ 

We define two functions $d_{V}: \mathbb{P} \rightarrow \mathbb{Q}$ and $d_{W}: \mathbb{Q} \rightarrow \mathbb{Q}$, with $P$ and $Q$ as in Definition 2.13:

$$d_{V}(u, v) := \nu \left( \frac{x_{u} - x_{v}}{x_{u} - y_{u}} \right) \quad \text{for all } (u, v) \in P,$$

$$d_{W}(t, w) := \nu \left( \frac{y_{t} - y_{w}}{y_{t} - y_{w}} \right) \quad \text{for all } (t, w) \in Q.$$ 

Notice that the definition of $P$ and $Q$ implies that $x_{u} - x_{v}$ and $y_{t} - y_{w}$ are always nonzero. Moreover, both $d_{V}$ and $d_{W}$ depend on $q$, but not on the representatives $(x_{e})_{e \in V}$ and $(y_{e})_{e \in W}$.

Lemma 3.7. With the notation and assumptions as in Definition 3.6, the two functions $d_{V}: \mathbb{P} \rightarrow \mathbb{Q}$ and $d_{W}: \mathbb{Q} \rightarrow \mathbb{Q}$ satisfy:

- $d_{V}(u, v) = d_{V}(v, u)$ for all $(u, v) \in P$, and similarly for $d_{W}$;
- $d_{V}(u, v) + d_{W}(t, w) = \text{wt}(e)$ for all $e \in E \setminus \{\bar{e}\}$, where $\{u, v\} = \tau_{G}(e)$ and $\{t, w\} = \tau_{H}(e)$;
- $d_{V}(\bar{u}, \bar{v}) = d_{W}(\bar{t}, \bar{w}) = 0$, where $\{\bar{u}, \bar{v}\} = \tau_{G}(\bar{e})$ and $\{\bar{t}, \bar{w}\} = \tau_{H}(\bar{e})$;
\[ \text{for every cycle } \mathcal{C} \text{ in } G, \text{ the minimum of the values of } d_V \text{ on the pairs of vertices } (u, v) \text{ appearing in } \mathcal{C} \text{ is attained at least twice, and similarly for } d_W. \]

**Proof.** The statement follows from the definitions and the properties of the valuation, see [Bou98, Section VI.3.1, Definition 1 and Corollary to Proposition 1]. In particular, we use that \( \nu(a) = \nu(-a) \) and \( \nu(a \cdot b) = \nu(a) + \nu(b) \) for all nonzero \( a \) and \( b \). The fourth property follows from \( \sum_{e} (x_u - x_v) / (x_u - x_v) = 0 \) if \( \mathcal{C} \) is a cycle in \( G \) (and similarly for cycles in \( H \)): we employ the fact that if the sum of finitely many elements is zero, then the minimum of their valuations is achieved at least twice. Notice that the values \( d_V(\bar{u}, \bar{v}) \) and \( d_W(\bar{f}, \bar{w}) \) are defined because \( \bar{e} \) is not a self-loop by Section 3. \( \square \)

**Definition 3.8.** Let \( B \) be a bigraph with biedges \( \mathcal{E} \) without self-loops, let \( \bar{e} \) be a fixed biedge, and let \( wt \in \mathbb{Q}^\mathcal{E} \setminus \{\bar{e}\} \). A **bidistance** \( d \) on \( B \) compatible with \( wt \) is a pair \((d_V, d_W)\) of functions \( d_V: P \to \mathbb{Q} \) and \( d_W: Q \to \mathbb{Q} \) such that the conditions of Lemma 3.7 are satisfied. If the weight vector is clear from the context, we omit the clause “compatible with \( wt \)”.  

**Remark 3.9.** Let \( B \) be a bigraph and use Section 3. Then any \( q \in f_{B,K}^{-1}(p) \) defines a bidistance \( d \) on \( B \), and via the isomorphism provided by Remark 3.4 also any point in \( Z_K^B \) defines a bidistance. 

As mentioned before, we are going to count the number of points in a general fiber of \( f_{B,K} \) that determine a fixed bidistance. We do so by computing the Laman number of a “smaller” bigraph, obtained via a quotient operation as explained in Definition 3.10.

**Definition 3.10.** Let \( B = (G, H) \) be a bigraph with set of biedges \( \mathcal{E} \) and without self-loops, and fix a bidistance \( d = (d_V, d_W) \) on \( B \). We define a new bigraph \( B_d \) as follows: For every \( \alpha \in \text{im}(d_V) \), define the graphs \( G_{\geq \alpha} \) and \( G_{> \alpha} \) to be the subgraphs of \( G \) determined by all edges with endpoints \( u \) and \( v \) such that \( d_V(u, v) \geq \alpha \) and \( d_V(u, v) > \alpha \), respectively. Similarly, for every \( \beta \in \text{im}(d_W) \), define \( H_{\geq \beta} \) and \( H_{> \beta} \). Let 

\[
G_{d_V} := \bigcup_{\alpha \in \text{im}(d_V)} G_{\geq \alpha} / G_{\geq \alpha} \quad \text{and} \quad H_{d_W} := \bigcup_{\beta \in \text{im}(d_W)} H_{\geq \beta} / H_{\geq \beta}.
\]

Here by \( G_{\geq \alpha} / G_{\geq \alpha} \) and \( H_{\geq \beta} / H_{\geq \beta} \) we mean the quotients of graphs as described in Definition 2.5, followed by removing singleton components without edges. The union symbol \( \bigcup \) indicates the disjoint union of graphs.

There is a natural bijection between edges of \( G \) and edges of \( G_{d_V} \), sending each edge \( e \) in \( G \) to the corresponding edge in the quotient \( G_{2d_V(e)}(\tau_G(e)) / G_{2d_V(e)}(\tau_G(e)) \). We define \( B_d \) to be the bigraph \((G_{d_V}, H_{d_W})\) with set of biedges \( \mathcal{E} \) inherited from \( B \). Moreover, we fix any total order on the vertices in \( B_d \).

**Remark 3.11.** Notice that in Definition 3.10 we did not use any of the properties of bidistances. This means that the definition of \( B_d \) makes sense also for bigraphs \( B \) and pairs of functions \( d_V: P \to \mathbb{Q} \) and \( d_W: Q \to \mathbb{Q} \). This is important and useful in Section 4.
Lemma 3.12. If \( B = (G, H) \) is a pseudo-Laman bigraph without self-loops and \( d \) is a bidistance on \( B \), then the quotient graph \( B_d \) is also pseudo-Laman.

Proof. We first prove that for any graph \( G = (V, E) \) and for any subgraph \( G' \subseteq G \) the following equation holds:

\[
\dim(G) = \dim(G') + \dim(G / G').
\]

Let \( G = \bigcup_{i=1}^{k} G_i \) be the decomposition of \( G \) into connected components. Write \( G' = \bigcup_{i=1}^{k} G'_i \), where \( G'_i \) is the part of \( G' \) belonging to \( G_i \). Let \( V_i \) and \( V'_i \) be the set of vertices of \( G_i \) and \( G'_i \), respectively. Now, \( G'_i \) itself may be disconnected, so let \( n_i \) be the number of connected components of \( G'_i \). Contraction of edges of \( G_i \) does not introduce new components, thus \( G'_i \) consists of one connected component. Moreover, each connected component of \( G'_i \) will correspond to one vertex in \( G_i / G'_i \). It follows that

\[
\dim(G_i) = |V_i| - 1, \quad \dim(G'_i) = |V'_i| - n_i, \quad \dim(G_i / G'_i) = (|V_i| - |V'_i| + n_i) - 1,
\]

and therefore \( \dim(G_i) = \dim(G'_i) + \dim(G_i / G'_i) \) for all \( i \). Now the claim follows, because \( \dim\left( \bigcup_{i=1}^{k} G_i \right) = \sum_{i=1}^{k} \dim(G_i) \). If \( B = (G, H) \) is a bigraph and \( d \) is a bidistance on it then

\[
\dim(G) = \dim(G_{d'}) \quad \text{and} \quad \dim(H) = \dim(H_{d'}).
\]

We prove only the first equality, the second one follows analogously. Let \( \pi \) be the minimum value attained by \( d' \). Then \( G = G_{2\pi} \) and

\[
\dim(G) = \dim(G_{2\pi}) = \dim(G_{\pi}) + \dim(G_{2\pi} / G_{\pi}).
\]

By repeating this argument, considering one by one all values in \( \im(d') \) in increasing order, we prove the asserted equality. The proof is concluded by noticing that the number of bieedges of \( B_d \) equals the number of bieedges of \( B \) by construction. \( \square \)

Example 3.13. Continuing Example 3.5, we fix the following bidistance \( d = (d_V, d_W) \):

\[
d_V(1, 2) = 0, \quad d_V(1, 3) = 1, \quad d_V(2, 3) = 0, \quad d_V(2, 4) = 1, \quad d_V(3, 4) = 0,
\]

\[
d_W(1, 2) = 1, \quad d_W(1, 3) = 0, \quad d_W(2, 3) = 0, \quad d_W(2, 4) = 0, \quad d_W(3, 4) = 1.
\]

The bidistance \( d \) is illustrated in Figure 6a and the resulting bigraph \( B_d \) is shown in Figure 6b. The scheme \( Z_{\bar{C}}^{B_d} \) associated to \( B_d \) is defined by the following equations:

\[
\begin{align*}
x_{13|24} &= 1, \quad x_{13|24} y_{1|2} = \lambda_r, \quad x_{2|4} y_{12|34} = \lambda_{\omega}, \\
y_{12|34} &= 1, \quad x_{1|3} y_{12|34} = \lambda_r, \quad x_{13|24} y_{3|4} = \lambda_{\omega}, \\
x_{13|24} + x_{24|13} &= x_{1|3} + x_{3|1} = x_{2|4} + x_{4|2} = 0, \\
y_{12|34} + y_{34|12} &= y_{1|2} + y_{2|1} = y_{3|4} + y_{4|3} = 0.
\end{align*}
\]

To link the points in a general fiber of \( f_{B_d} \) that define a given bidistance \( d \) with the Laman number of \( B_d \), we introduce in Definition 3.15 a family of varieties \( \bar{A}_{\bar{C}}^{B_d} \), parametrized by a parameter \( \sigma \). This family has the property that a general element is isomorphic to \( Z_{\bar{K}}^{B} \), while
Given a bidistance \( d \) and \( 0 \)\( \text{Remark 3.14} \)

Example 3.13

Section 3

Lemmas 3.20

\( \text{Lemma 3.21} \)

\( \text{Lemma 3.20} \)

\( \text{Remark 3.14} \)

\( \text{Definition 3.15} \)

\( \text{Figure 6. A bigraph } B \text{ with bidistance } d \text{ and the corresponding bigraph } B_d \)

a special element is isomorphic to \( Z^B_C \). To prove this, we establish in \( \text{Lemmas 3.20 and 3.21} \) the following two bijections:

\[
\left\{ \text{points in } Z^B_C \text{ that determine the bidistance } d \right\} \leftrightarrow \left\{ \text{points in } A^d_C \right\} \leftrightarrow \left\{ \text{points in } Z^{B_d}_C \right\}
\]

\( \text{Remark 3.14} \). Notice that, for a fixed bigraph \( B \) with biedges \( E \) and a fixed choice of a vector \( wt \in \mathbb{Q}^E \setminus \{\bar{e}\} \) and of a bidistance \( (d_V, d_W) \), we can suppose that all entries of the vector \( wt \) and all values \( d_V(u, v) \) and \( d_W(t, w) \) are integers. Indeed, consider the subgroup of \( \mathbb{Q} \) that is generated by the rational numbers in \( \{wt(e)\}_{e \in E \setminus \{\bar{e}\}} \cup \text{im}(d_V) \cup \text{im}(d_W) \): such a group is of the form \( \mathbb{Z}^{2n} \), and so we can apply the automorphism of \( \mathbb{K} \) that sends \( s \) to \( s^{n/m} \).

\( \text{Definition 3.15} \). Let \( B \) be a bigraph with biedges \( E \) and use Section 3. Given a bidistance \( d \) on \( B \), we can suppose by \( \text{Remark 3.14} \) that \( wt \), \( d_V \) and \( d_W \) take integer values. We define the scheme \( \widetilde{A}^d_C \) in \( \mathbb{C}^P \times \mathbb{C}^Q \times \mathbb{C} \), with coordinates \((\bar{x}_{uv})_{(u,v) \in P}, (\bar{y}_{tw})_{(t,w) \in Q} \) and \( \sigma \):

\[
\widetilde{A}^d_C := \text{Spec} \left( \frac{\mathbb{C}[\bar{x}_{uv}, \bar{y}_{tw}]}{\begin{align*}
\bar{x}_{uv} \bar{y}_{tw} &= \lambda_e \quad \text{for all } e \in E \setminus \{\bar{e}\}, \quad u < v, \quad t < w \\
\sum_{\mathcal{C}} \bar{x}_{uv} \sigma^{d_V(u,v)-m(\mathcal{C})} &= 0 \quad \text{for all cycles } \mathcal{C} \text{ in } G \\
\sum_{\mathcal{D}} \bar{y}_{tw} \sigma^{d_W(t,w)-m(\mathcal{D})} &= 0 \quad \text{for all cycles } \mathcal{D} \text{ in } H
\end{align*}} \right),
\]

where \( m(\mathcal{C}) \) denotes the minimum value attained by the function \( d_V \) on the cycle \( \mathcal{C} \), and similarly for \( d_W \). Since the differences \( d_V(u, v) - m(\mathcal{C}) \) and \( d_W(t, w) - m(\mathcal{D}) \) are non-negative integers, we see that all equations are indeed polynomial in \( \bar{x}, \bar{y} \) and \( \sigma \). Moreover, we define

\[
A^d_C := \widetilde{A}^d_C \cap \{\sigma = 0\},
\]

where \( \{\sigma = 0\} \) denotes the hyperplane defined by the equation \( \sigma = 0 \).
Example 3.16. Continuing Example 3.13, the scheme $\tilde{A}_C^d$ is defined by the equations:
\[
\begin{align*}
\tilde{x}_{23} = \tilde{y}_{23} &= 1, \\
\tilde{x}_{12} \tilde{y}_{12} &= \lambda_r, \\
\tilde{x}_{12} + \tilde{x}_{21} &= \tilde{x}_{13} + \tilde{x}_{31} + \tilde{x}_{23} + \tilde{x}_{32} + \tilde{x}_{24} = \tilde{x}_{34} = 0, \\
\tilde{x}_{13} \tilde{y}_{13} &= \lambda_g, \\
\tilde{y}_{12} + \tilde{y}_{21} &= \tilde{y}_{13} + \tilde{y}_{31} + \tilde{y}_{23} + \tilde{y}_{32} + \tilde{y}_{24} + \tilde{y}_{42} = \tilde{y}_{34} + \tilde{y}_{43} = 0, \\
\tilde{x}_{24} \tilde{y}_{24} &= \lambda_v, \\
\tilde{x}_{24} + \tilde{x}_{34} &= \lambda_b, \\
\tilde{x}_{24} \tilde{x}_{34} &= \tilde{x}_{24} + \tilde{x}_{34} = \tilde{x}_{24} + \tilde{x}_{34} = 0.
\end{align*}
\]

In Lemma 3.20 we use a special set of generators for the ideals defining $A_C^d$ and $Z_C^{B_\ast}$. To describe this set of generators we need the concept of spanning forest for a bigraph.

**Definition 3.17.** Let $B = (G, H)$ be a bigraph. A spanning forest $F$ for $B$ is a pair $(F_G, F_H)$ of spanning forests for $G$ and $H$ respectively. A spanning forest for a graph is a tuple of spanning trees, one for each connected component of the graph.

An auxiliary result describing how to obtain a special system of generators for the ideal of $Z_C^B$, once a spanning forest for $B$ is fixed is given in the next lemma.

**Lemma 3.18.** Let $B$ be a bigraph. Use Section 3 and define $Z_C^B$ according to Definition 2.14. Then $Z_C^B$ is a complete intersection, and every choice of a spanning forest for $B$ determines a set of codim($Z_C^B$) generators for the ideal of $Z_C^B$.

**Proof.** Notice that the dimension of the ambient affine space of $Z_C^B$ is $|P| + |Q|$, where $P$ and $Q$ are as in Definition 2.13. Moreover $Z_C^B$ is zero-dimensional, since Lam($B$) is defined. We are going to exhibit a system consisting of $|P| + |Q|$ equations defining $Z_C^B$.

Let $F = (F_G, F_H)$ be a spanning forest for the bigraph $B = (G, H)$ with biedges $E$. For every $(u, v) \in P$, where $u$ and $v$ are not connected by an edge of $F_G$, we consider the equation
\[
x_{uv} - \sum_{i=0}^{n-1} x_{uiu_{i+1}} = 0,
\]
where $(u_0 = u, \ldots, u_n = v)$ is the unique path in $F_G$ from $u$ to $v$. Similarly, we construct equations for each $(t, w) \in Q$ for which $t$ and $w$ are not connected by an edge of $F_H$. We claim that these equations generate the same ideal as the equations coming from all cycles in $G$ and in $H$. It is enough to show this for every connected component of $G$, so we can suppose that $G$ is connected. We show the claim by induction on the number of edges of $G$: when this number is minimal, the graph $G$ is a tree and so there is nothing to prove since there are no cycles. Suppose now that the statement holds for $G$, and add an edge to $G$ obtaining $G'$; suppose that this edge connects the vertices $u'$ and $v'$. Consider an equation $\sum_{E'} x_{uv}$ coming from a cycle $C'$ in $G'$: if it does not involve the edge $(u', v')$, then by induction hypothesis it is a linear combination of the equations coming from the spanning tree. Otherwise, the cycle $C'$ is of the form $(u = u_0, \ldots, u_i, u', v', u_{i+1}, \ldots, u_n = u)$. If we add to the equation $\sum_{E'} x_{uv}$ the equation $x_{u'v'} - \sum_{j=0}^{m-1} x_{v_jv_{j+1}}$, we obtain the equation induced by the cycle $(u = u_0, \ldots, u_i, v_0, \ldots, v_m, u_{i+1}, \ldots, u_n = u)$, which is completely contained in $G$. So
by induction hypothesis the sum is a linear combination of the equations coming from the spanning tree; this concludes the proof of the claim.

The number of equations coming from edges not in the spanning forest is

\[ |P| - |\{\text{edges of } \mathcal{F}_G\}| + |Q| - |\{\text{edges of } \mathcal{F}_G\}| = |P| - \dim(G) + |Q| - \dim(H). \]

The above equations, together with

\[
\begin{cases}
  x_{\bar{a}\bar{b}} = y_{\bar{b}\bar{a}} = 1, & \bar{u} < \bar{v}, \bar{t} < \bar{w}, \\
  x_{uv} y_{tw} = \lambda_e, & \text{for all } e \in \mathcal{E} \setminus \{\bar{v}\}, \quad u < v, \quad t < w,
\end{cases}
\]

define \( Z^B \). Therefore, the total number of equations is \( |\mathcal{E}| + 1 + (|P| - \dim(G)) + (|Q| - \dim(H)) \), which equals \(|P| + |Q|\) since \( B \) is pseudo-Laman and has no self-loops by Section 3. In particular, \( Z^B \) is a complete intersection. \( \square \)

**Remark 3.19.** To proceed, we need spanning forests with an additional property: For a bigraph \( B = (G, H) \), we consider spanning forests \( \mathcal{F}_G \) and \( \mathcal{F}_H \) for \( G \) and \( H \), respectively, such that for any edge in \( G \) with vertices \( u, v \), the value \( d_V(u, v) \) is equal to the minimum of the values of \( d_V \) in the unique path in \( \mathcal{F}_G \) connecting \( u \) and \( v \), and similarly for \( \mathcal{F}_H \).

The construction of such forests can be achieved by iteratively removing non-bridges (see Definition 3.17) with endpoints \( u \) and \( v \) such that \( d_V(u, v) \) is minimal within the non-bridges of the graph in the current iteration. This construction can be proven to be correct using the loop invariant \( \delta: (V \times V) \setminus \Delta \to \mathbb{Q} \), where \( \Delta = \{(v, v) : v \in V\} \) and \( \delta(u, v) \) is defined as follows. We consider all paths \( v_0 = u, v_1, \ldots, v_n = v \) from \( u \) to \( v \) and take the minimum of the values \( \{d_V(v_i, v_{i+1}) : i \in \{0, \ldots, n\}\} \) for each of them. Then \( \delta(v, u) \) is the maximum of all these values. Note that if \( u \) and \( v \) are connected by an edge in the graph, then \( \delta(u, v) = d_V(u, v) \): in fact, by assumption the path \((u, v)\) connects \( u \) and \( v \), and so \( d_V(u, v) \) appears as a minimum of a path from \( u \) to \( v \); moreover, for every path \( v_0 = u, v_1, \ldots, v_n = v \) from \( u \) to \( v \) the minimum of the values \( \{d_V(v_i, v_{i+1}) : i \in \{0, \ldots, n-1\}\} \) cannot be bigger than \( d_V(u, v) \), because this would contradict the property of bidistances that the minimum in the cycle \( v_0 = u, v_1, \ldots, v_n = v, u \) occurs twice; hence \( d_V(u, v) \) is the maximum of these minima, and so it coincides with \( \delta(u, v) \).

The map \( \delta \) is indeed a loop invariant, since if we are about to delete an edge with endpoints \( u \) and \( v \), then this edge has to be a non-bridge of minimal \( d_V(u, v) \). Hence, there is a cycle containing both \( u \) and \( v \) and the endpoints of another edge with the same \( d_V \)-value, since the minimum \( d_V \)-value occurs at least twice in every cycle. Therefore, there is still a path from \( u \) to \( v \) with the same minimum, so that the set of minima in the definition of \( \delta \) does not change at all. In a similar way one argues that all other values of \( \delta \) are not changed either.

The forests constructed in this way share a useful property, namely if we consider the set of edges in \( B_d \) that correspond to \( \mathcal{F}_G \) and \( \mathcal{F}_H \), then such a set forms a spanning forest for \( B_d \).

**Lemma 3.20.** Let \( B \) be a bigraph. Use Section 3 and fix a bidistance \( d \) on \( B \). Suppose that \( B_d \) (Definition 3.10) satisfies \( \text{Lam}(B_d) > 0 \). Then the scheme \( A^d_C \) (Definition 3.15) can be defined by \(|P| + |Q|\) equations. Furthermore, the scheme \( A^d_C \) is isomorphic to \( Z^B \), so in particular it consists of \( \text{Lam}(B_d) \) distinct points and is defined by \( \text{codim}(A^d_C) \) equations.
Lemma 3.18, we can give a starting from, we may achieve that the equations are of the form

\[ \tilde{x}_{uv} - \sum_{i=0}^{n-1} \tilde{x}_{u_i u_{i+1}} \sigma_{d_V(u_i, u_{i+1})} - d_V(u, v) = 0 \]

together with

\[
\begin{align*}
\tilde{x}_{\tilde{u} \tilde{v}} &= \tilde{y}_{\tilde{u} \tilde{v}} = 1, & \tilde{u} < \tilde{v}, & \tilde{t} < \tilde{w}, \\
\tilde{x}_{uv} \tilde{y}_{tw} &= \lambda_e, & \text{for all } e \in \mathcal{E} \setminus \{\tilde{e}\}, & u < v, & t < w.
\end{align*}
\]

The number of these equations is hence \(|P| + |Q|\). We obtain a set of equations for \(A_d^C\) by setting \(\sigma = 0\) in the previous ones. Note that we could not have obtained this kind of equations if we started from an arbitrary spanning forest for \(B\).

Let \(P_d\) and \(Q_d\) be the sets as in Definition 2.13 starting from \(B_d\). The elements of \(P_d\) are of the form \(\left(\left[u]\alpha, [v]_\alpha\right)\), where \([u]\alpha\) is the class of the vertex \(u \in V\) in the set of vertices of \(G_{2\alpha}/G_{\alpha}\), and \(\alpha\) is a value in the image of \(d_V\). In the following we simply write \([u]\) and \([v]\) for such classes (and similarly for \(Q_d\)).

We define two maps \(\varphi: Z_C^{B_d} \to A_d^C\) and \(\psi: A_d^C \to Z_C^{B_d}\) as follows. For a point \(q = (x_{[u][v]})([u][v])_{P_d}, (y_{t}[w])_{(t)[w]} \in Z_C^{B_d}\), let \(\varphi(q)\) be the point whose \(x_{uv}\)-coordinate is \(x_{[u][v]}\) and whose \(y_{tw}\)-coordinate is \(y_{[t][w]}\). For a point \(\tilde{q} = (\tilde{x}_{uv})(\tilde{u}, \tilde{v})_{P}, (\tilde{y}_{tw})(\tilde{t}, \tilde{w})_{Q}\) in \(A_d^C\), define \(\psi(\tilde{q})\) to be the point whose \(x_{[u][v]}\)-coordinate equals \(\tilde{x}_{uv}\), and whose \(y_{[t][w]}\)-coordinate equals \(\tilde{y}_{tw}\). We have to show that \(\varphi\) and \(\psi\) are well-defined. It is then a direct consequence of the definitions that they are isomorphisms.

To show that \(\varphi\) is well-defined, we need to prove that \(\varphi(q) \in A_d^C\). Notice that the coordinates of \(\varphi(q)\) satisfy the equations determined by the bides of \(B\) because the coordinates of \(q\) do so by construction. Consider now an equation of \(A_d^C\) obtained by setting \(\sigma = 0\) in an equation of \(A_d^C\) determined by a cycle \(\mathcal{C}\) in \(G\) (analogous considerations can be done for cycles in \(H\)). Let \(\alpha\) be the minimum value attained by \(d_V\) along the cycle \(\mathcal{C}\). Such an equation is of the form \(\sum \tilde{x}_{uv} = 0\), where the subscript in \(\mathcal{C}_\alpha\) indicates that the sum is taken over the pairs \((u, v)\) in \(P\) appearing in the cycle \(\mathcal{C}\) and satisfying \(d_V(u, v) = \alpha\). On the other hand, such a cycle determines a cycle in \(G_{2\alpha}/G_\alpha\), which defines an equation of the same form, namely \(\sum \tilde{x}_{uv} = 0\), satisfied by the coordinates of \(q\). Hence \(\varphi(q) \in A_d^C\).

To show that \(\psi\) is well-defined we need to first prove that if \([u] = [u']\) and \([v] = [v']\) for two pairs \((u, v), (u', v') \in P\) such that \(d_V(u, v) = d_V(u', v')\), then the coordinates of the point \(\tilde{q}\) satisfy \(\tilde{x}_{uv} = \tilde{x}_{u'v'}\). This is true since by hypothesis there is a cycle in \(G\) involving two edges between \(u\) and \(v\) and \(u'\) and \(v'\), respectively, such that every other edge in the cycle has endpoints whose \(d_V\)-value is strictly greater than \(d_V(u, v)\). The definition of \(A_d^C\) implies that such an equation holds for points in \(A_d^C\). Secondly, we should prove that \(\psi(\tilde{q}) \in Z_C^{B_d}\), and here we argue as in the previous paragraph. \(\square\)
The following result can be considered as a particular instance (in the zero-dimensional case) of the so-called tropical lifting lemma, see [Kat09, Lemma 4.15] and [Pay09] for a result over more general fields. We report here the proof for self-containedness and since it does not require results from tropical geometry, being essentially a consequence of the implicit function theorem for power series.

**Lemma 3.21.** With the notation as in Lemma 3.20, so in particular a bidistance $d$ is fixed, there is a bijection between $A^d_C$ and the set of points in $Z^B_K$ that determine the bidistance $d$.

**Proof.** We know from Remark 3.4 and from Lemma 3.20 that both $A^d_C$ and $Z^B_K$ consist of finitely many points. Let $q \in Z^B_K$ be a point determining the bidistance $d$: this means that $q = ((x_{uv})_{(u,v) \in P}, (y_{tw})_{(t,w) \in Q})$ with $\nu(x_{uv}) = d_V(u, v)$ and $\nu(y_{tw}) = d_W(t, w)$. We can write $x_{uv} = \tilde{x}_{uv} s^{d_V(u,v)}$ and $y_{tw} = \tilde{y}_{tw} s^{d_W(t,w)}$, where the elements $\tilde{x}_{uv}$ and $\tilde{y}_{tw}$ have zero valuation. Therefore $\tilde{q} = ((\tilde{x}_{uv})_{(u,v) \in P}, (\tilde{y}_{tw})_{(t,w) \in Q})$ is a point of $s^d \cdot Z^B_K$, which is the scheme in $K^P \times K^Q$ defined by the equations

$$
\begin{align*}
\tilde{x}_{uv} = \tilde{y}_{tw} &= 1, & u < \bar{v}, \bar{t} < w, \\
\tilde{x}_{uv} \tilde{y}_{tw} &= \lambda_c, & \text{for all } e \in \mathcal{E} \setminus \{e\}, \ u < v, \ t < w, \\
\sum_{e < v} \tilde{x}_{uv} s^{d_V(u,v)-m(e)} &= 0, & \text{for all cycles } \mathcal{C} \text{ in } G, \\
\sum_{e < w} \tilde{y}_{tw} s^{d_W(t,w)-m(e)} &= 0, & \text{for all cycles } \mathcal{D} \text{ in } H,
\end{align*}
$$

where the notation is as in Definition 3.15. Since all coordinates of $\tilde{q}$ have valuation equal to zero, we can define $\tilde{x}_{uv} := \tilde{x}_{uv} \mod (s)$, obtaining $\tilde{x}_{uv} \in K$, and similarly for $\tilde{y}_{tw}$. It follows that the point $\tilde{q} = ((\tilde{x}_{uv})_{(u,v) \in P}, (\tilde{y}_{tw})_{(t,w) \in Q})$ satisfies the equations of $A^d_C$. In this way we obtain a map from the set of points in $Z^B_K$ that determine the bidistance $d$ to $A^d_C$.

Let now $\tilde{q}$ be a point of $A^d_C$. From Lemma 3.20 we know that $A^d_C$ is a complete intersection and that it is defined by $\text{codim}(A^d_C)$ equations $g_i = 0$ of the form

$$
g_i((x_{uv})_{(u,v) \in P}, (y_{tw})_{(t,w) \in Q}) = \tilde{g}_i((\tilde{x}_{uv})_{(u,v) \in P}, (\tilde{y}_{tw})_{(t,w) \in Q}, 0),
$$

where the equations $\tilde{g}_i = 0$ define $A^d_{\tilde{C}}$. Since $A^d_C$ is smooth by Lemma 3.20, we know that the Jacobian determinant

$$
\det \left( \left( \frac{\partial \tilde{g}_i}{\partial \tilde{x}_{uv}} \right), \left( \frac{\partial \tilde{g}_i}{\partial \tilde{y}_{tw}} \right) \right)_{\tilde{q}}
$$

evaluated at $\tilde{q}$ is non-zero. By the implicit function theorem for formal power series (see [Bou03, A.IV.37, Corollary]) applied to the system of equations $\tilde{g}_i = 0$, there exists a unique point $\tilde{q} \in K[[\sigma]]^P \times K[[\sigma]]^Q$ such that $\tilde{g}_i(\tilde{q}, \sigma) = 0$ and the constant terms of the coordinates of $\tilde{q}$ equal the coordinates of $\tilde{q}$. The point $\tilde{q}$ determines in turn a point in $s^d \cdot Z^B_K$ whose coordinates have valuation equal to zero, and therefore a point in $Z^B_K$ whose coordinates have valuation prescribed by $d$. We get a map from $A^d_C$ to the set of points in $Z^B_K$ determining the bidistance $d$.

Suppose now that there were two points $q$ and $q'$ in $Z^B_K$ determining the bidistance $d$ and specializing to the same point in $A^d_C$. After applying a suitable automorphism of $K$ of the form $s \mapsto s^m/n$, we can suppose that both the points in $s^d \cdot Z^B_K$ corresponding to $q$ and $q'$ were
given by power series in $s$. This would contradict the uniqueness of the power series solution provided by the implicit function theorem. Therefore, the two maps we have just specified provide the desired bijection. □

**Remark 3.22.** If $B$ is a pseudo-Laman bigraph with $\text{Lam}(B) = 0$ and $d$ is a bidistance on $B$, then the definition of $Z^K_B$ makes still sense, as well as the definitions of $\bar{A}_C^d$, $A_C^d$ and $Z^B_C$. In this case, the scheme $Z^K_B$ is nothing but the empty set, and the proof of Lemma 3.20 shows that the schemes $A_C^d$ and $Z^B_C$ are isomorphic. To conclude that $Z^B_C$ is also the empty set we argue as in Lemma 3.21: if $Z^B_C$ were not empty, then one could construct a point in $Z^K_B$, contradicting the hypothesis.

**Theorem 3.23.** Let $B$ be a pseudo-Laman bigraph with biedges $E$ without self-loops. Fix a biedge $e \in E$, and fix $\text{wt} \in \mathbb{Q}^E \setminus \{e\}$. Then we have

$$\text{Lam}(B) = \sum_d \text{Lam}(B_d),$$

where $d$ runs over all bidistances on $B$ compatible with $\text{wt}$.

**Proof.** When $\text{Lam}(B) > 0$, the statement follows directly by combining Lemma 3.20 and Lemma 3.21. The case $\text{Lam}(B) = 0$ is covered by Remark 3.22. □

4. A Formula for the Laman Number

In this section we develop a formula for the Laman number of a bigraph from Theorem 3.23. We fix a very special weight vector, namely the vector $(-1, \ldots, -1)$: with this choice it is easy to determine which bidistances are compatible with $\text{wt}$ (Lemma 4.2); the bigraphs $B_d$ that one obtains are complicated, but it is possible to use this approach recursively (Theorem 4.7), translating any situation to a limited number of simple base cases (Proposition 2.10). First we show in Lemmas 4.1 and 4.2 that the bidistances that are compatible with $(-1, \ldots, -1)$ can take only values in $\{0, -1\}$.

**Lemma 4.1.** Let $B$ be a pseudo-Laman bigraph with biedges $E$. Suppose that $\text{Lam}(B) > 0$. Pick $e \in E$ and fix $\text{wt} \in \mathbb{Z}^E \setminus \{e\}$. Let $d = (d_V, d_W)$ be a bidistance for $B$ and suppose that $\text{Lam}(B_d) \in \mathbb{N} \setminus \{0\}$. Then both $d_V$ and $d_W$ take values in $\mathbb{Z}$.

**Proof.** Suppose by contradiction that the images of $d_V$ and $d_W$ are not contained in $\mathbb{Z}$. We are going to construct an infinite family $\{d^\kappa : \kappa \in (0, 1] \cap \mathbb{Q}\}$ of different bidistances for $B$ that satisfies $B_{d^\kappa} = B_d$ for every $\kappa \in (0, 1] \cap \mathbb{Q}$. Lemma 3.20 together with Lemma 3.21 imply then that $Z^K_B$ consists of infinitely many points, contradicting the hypothesis, because every quotient $B_{d^\kappa}$ contributes nontrivially to $\text{Lam}(B)$. For every $\kappa \in (0, 1] \cap \mathbb{Q}$, define

$$d^\kappa_V := \kappa \cdot d_V + (1 - \kappa) \cdot \lfloor d_V \rfloor, \quad d^\kappa_W := \kappa \cdot d_W + (1 - \kappa) \cdot \lceil d_W \rceil,$$

where $\lfloor \cdot \rfloor$ and $\lceil \cdot \rceil$ denote the ceiling and the floor functions, respectively. Since $\text{im}(d_V) \cup \text{im}(d_W) \notin \mathbb{Z}$, the family

$$\{d^\kappa = (d^\kappa_V, d^\kappa_W) : \kappa \in (0, 1] \cap \mathbb{Q}\}$$

is nothing but the empty set, and the proof of $K$ consists of infinitely many points, contradicting the hypothesis, because every quotient $B_{d^\kappa}$ contributes nontrivially to $\text{Lam}(B)$. For every $\kappa \in (0, 1] \cap \mathbb{Q}$, define

$$d^\kappa_V := \kappa \cdot d_V + (1 - \kappa) \cdot \lfloor d_V \rfloor, \quad d^\kappa_W := \kappa \cdot d_W + (1 - \kappa) \cdot \lceil d_W \rceil,$$

where $\lfloor \cdot \rfloor$ and $\lceil \cdot \rceil$ denote the ceiling and the floor functions, respectively. Since $\text{im}(d_V) \cup \text{im}(d_W) \notin \mathbb{Z}$, the family

$$\{d^\kappa = (d^\kappa_V, d^\kappa_W) : \kappa \in (0, 1] \cap \mathbb{Q}\}$$

is nothing but the empty set, and the proof of $K$ consists of infinitely many points, contradicting the hypothesis, because every quotient $B_{d^\kappa}$ contributes nontrivially to $\text{Lam}(B)$.
has infinitely many elements. We show that each $d^\kappa$ is a bidistance for $B$. Since by hypothesis

$$d_V(u,v) + d_W(t,w) = \text{wt}(e) \in \mathbb{Z} \quad \text{for all } e \in \mathcal{E} \setminus \{\bar{e}\},$$

it follows that $[d_V] + [d_W] = d_V + d_W$. Hence, for all $\kappa \in (0,1] \cap \mathbb{Q}$

$$d^\kappa_V(u,v) + d^\kappa_W(t,w) = \text{wt}(e) \quad \text{for all } e \in \mathcal{E} \setminus \{\bar{e}\}.$$ 

By construction it follows that $d^\kappa_V(\bar{u},\bar{v}) = d^\kappa_W(\bar{t},\bar{w}) = 0$, where $\{\bar{u},\bar{v}\} = \tau_G(\bar{e})$ and $\{\bar{t},\bar{w}\} = \tau_H(\bar{e})$. Next, note that the two functions

$$x \mapsto \kappa \cdot x + (1 - \kappa) \cdot [x], \quad x \mapsto \kappa \cdot x + (1 - \kappa) \cdot [x]$$

are strictly increasing for every $\kappa \in (0,1] \cap \mathbb{Q}$. This implies that also the last property stated in Lemma 3.7 is preserved. Hence, $d^\kappa$ is a bidistance. Note that

$$\{(u,v) \in \mathcal{P} : d_V(u,v) \geq \alpha\} = \{(u,v) \in \mathcal{P} : d^\kappa_V(u,v) \geq \kappa \alpha + (1 - \kappa) \lceil \alpha \rceil\}$$

and similarly for $> \alpha$ and for $d_W$. Recall from Definition 3.10 that $B_\delta$ is a disjoint union of graphs of the form $G_{> \alpha} / G_{\geq \alpha}$ and $H_{> \beta} / H_{\geq \beta}$. By what we noticed, these graphs do not change when we pass from $d$ to $d^\kappa$ and therefore $B_{d^\kappa} = B_\delta$ for each $\kappa$.

**Lemma 4.2.** Let $B$ be a pseudo-Laman bigraph with bidges $\mathcal{E}$. Suppose that $\text{Lam}(B) > 0$. Pick $\bar{e} \in \mathcal{E}$ and fix $\text{wt} = (-1)_{\mathcal{E} \setminus \{\bar{e}\}}$. Let $d = (d_V,d_W)$ be a bidistance for $B$ and suppose that $\text{Lam}(B_\delta) \in \mathbb{N} \setminus \{0\}$. Then both $d_V$ and $d_W$ take values in $\{0,-1\}$.

**Proof.** Suppose by contradiction that the claim does not hold. Then (after possibly swapping the roles of $d_V$ and $d_W$) we can suppose that $d_V(u,v) < -1$ for some $u$ and $v$ that are vertices of an edge. We construct an infinite family $\{d^\kappa : \kappa \in \mathbb{N}\}$ of bidistances for $B$ such that $B_{d^\kappa} = B_\delta$. Then the same argument as in the proof of Lemma 4.1 gives a contradiction.

Let $\bar{\alpha}$ be the minimum of the values in $\text{im}(d_V)$. By Lemma 4.1 the value of $\bar{\alpha}$ is integer, so we have $\bar{\alpha} \leq -2$. For any $\kappa \in \mathbb{N}$ define

$$d^\kappa_V(u,v) := \begin{cases} d_V(u,v) - \kappa, & \text{if } d_V(u,v) = \bar{\alpha}, \\ d_V(u,v), & \text{otherwise}; \end{cases}$$

$$d^\kappa_W(t,w) := \begin{cases} d_W(t,w) + \kappa, & \text{if } d_W(t,w) = -1 - \bar{\alpha}, \\ d_W(t,w), & \text{otherwise}. \end{cases}$$

The family $\{d^\kappa = (d^\kappa_V,d^\kappa_W) : \kappa \in \mathbb{N}\}$ consists of infinitely many elements. From the construction it follows that each $d^\kappa$ is a bidistance. Furthermore,

$$\{(u,v) \in \mathcal{P} : d^\kappa_V(u,v) \geq \bar{\alpha}\} = \{(u,v) \in \mathcal{P} : d^\kappa_V(u,v) \geq \bar{\alpha} - \kappa\}$$

and by construction similar equalities hold for all other cases. Here we use that $-1 - \bar{\alpha}$ is the maximal value attained for $d_W$. Therefore, $B_{d^\kappa} = B_\delta$ by the same argument as in Lemma 4.1. Notice that if both $d_V$ and $d_W$ take values in $\{0,-1\}$, then the previous argument does
Proposition 2.21

by saying that if

Lemma 4.3

and taking into account

Lemma 2.19

to a recursive formula. By what we just said and by unraveling the notions

Definition 4.4

Lemma 4.3

Definition 4.4

that

Lemma 4.3

Definition 4.4

, the special shape of the bidistances compatible with the weight

vector \((-1, \ldots, -1)\) allows to split the problem of computing the Laman number of a bigraph of the form \(B_d\) into the computation of the Laman numbers of two smaller bigraphs.

Lemma 4.3. Let \(B = (G, H)\) be a bigraph with biedges \(\mathcal{E}\) and fix a bids\(e \in \mathcal{E}\). Fix a bids\(d = (d_V, d_W)\) such that \(d_V\) and \(d_W\) take values only in \(\{-1, 0\}\). If \(B\) is pseudo-Laman such that

\[\triangleright \text{ the bids\(e\) is compatible with } wt = (-1)_{\mathcal{E}\setminus\{e\}},\]

\[\triangleright \text{ \(e\) is neither a bridge in } G \text{ nor a bridge in } H,\]

\[\triangleright \text{ neither } d_V \text{ nor } d_W \text{ is the zero map},\]

then the quotient bigraph \(B_d\) untangles via \(\bar{e} \in \mathcal{E}\) into bigraphs \(B_{d,1}\) and \(B_{d,2}\).

Proof. Recall from Definition 3.10 that \(B\) and \(B_d\) have the same set of biedges. We define two sets \(\mathcal{E}_1, \mathcal{E}_2 \subseteq \mathcal{E}\) as the biedges in \(B_d\) corresponding to the following sets of biedges in \(B\):

\[
\left\{
\begin{array}{ll}
    e \in \mathcal{E} : & d_V(u, v) = 0, \quad \text{where } \{u, v\} = \tau_G(e) \\
    & d_W(t, w) = -1, \quad \text{where } \{t, w\} = \tau_H(e)
\end{array}
\right\},
\]

\[
\left\{
\begin{array}{ll}
    e \in \mathcal{E} : & d_V(u, v) = -1, \quad \text{where } \{u, v\} = \tau_G(e) \\
    & d_W(t, w) = 0, \quad \text{where } \{t, w\} = \tau_H(e)
\end{array}
\right\}.
\]

By hypothesis we have that both \(\mathcal{E}_1\) and \(\mathcal{E}_2\) are non-empty, and that \(\mathcal{E} = \mathcal{E}_1 \cup \mathcal{E}_2 \cup \{\bar{e}\}\) is a partition, because the maps \(d_V\) and \(d_W\) take values only in \(\{0, -1\}\). In \(G_{d_V}\) and \(H_{d_W}\), edges with different values of \(d_V\) and \(d_W\), respectively, are in different components. Hence the statement is proved.

In order to state the final formula we introduce some notation; we then express the bigraphs obtained from Lemma 4.3 in terms of this new notation.

Definition 4.4. Let \(B = (G, H)\) be a bigraph, where \(G = (V, \mathcal{E})\) and \(H = (W, \mathcal{E})\). Given \(\mathcal{M} \subseteq \mathcal{E}\), we define two bigraphs \(\mathcal{M}B = (G/\mathcal{M}, H\setminus\mathcal{M})\) and \(B^\mathcal{M} = (G\setminus\mathcal{M}, H/\mathcal{M})\), with the same set of biedges \(\mathcal{E}' = \mathcal{E} \setminus \mathcal{M}\). For both constructions we fix a total order on the vertices of the resulting bigraphs.

We can re-interpret Lemma 4.3 in the light of Definition 4.4 by saying that if \(d = (d_V, d_W)\) is a bids\(d\) such that both \(d_V\) and \(d_W\) take values in \(\{0, -1\}\), and \(\mathcal{M}\) and \(\mathcal{N}\) are defined as in Lemma 4.3, then \(B_d\) untangles via \(\bar{e} \in \mathcal{E}\) into \(\mathcal{M}B\) and \(B^\mathcal{N}\). This allows us to specialize Theorem 3.23 to a recursive formula. By what we just said and by unraveling the notions introduced in Definition 4.4 and taking into account Lemma 4.3 and Lemma 2.19 we get the following characterization.
Definition 4.4. We have Lam

Lemma 3.12. Lam

Lemma 4.3. Lemma 2.18. B

We have the equality Lam

since dim

is a bridge in G

concludes the first case; the second is proved analogously.

Here dim

is a bridge in H

as shown in the proof of Lemma 3.12. Since dim(\{e\}) = 1 we have

\[
\dim(H/\{e\}) + \dim(\{\bar{e}\}) = \dim(H) - 1 + \dim(G) + 1 = |E| + 1 = |E|.
\]

Here \(\dim(G \setminus \bar{e})\) = \(\dim(G) + 1\) because removing a bridge increases the dimension by 1. This concludes the first case; the second is proved analogously.

Suppose now that neither \(d_V\) nor \(d_W\) is the zero map, and \(\bar{e}\) is neither a bridge in \(G_{d_V}\) nor a bridge in \(H_{d_W}\). Then by Lemma 4.3 \(B_d\) untangles, and the two bigraphs \(B_{d,1}\) and \(B_{d,2}\) described in that lemma coincide with \(\mathcal{M}B\) and \(B^N\). If \(B_d\) does not contain self-loops then by Proposition 2.21 we have \(\Lam(B_d) = \Lam(\mathcal{M}B) \cdot \Lam(B^N)\). If \(B_d\) contains a self-loop which is different from \(\bar{e}\), then by construction it is also a self-loop in \(\mathcal{M}B\) or \(B^N\). Then by Proposition 2.10 \(\Lam(\mathcal{M}B) \cdot \Lam(B^N) = 0 = \Lam(B_d)\). Note that \(\bar{e}\) might never be a loop in \(B_d\). This is because in our case \(B_d\) is

\[
(G / G_{\geq 1} \cup G_{\geq 0}, H_{\geq 0} \cup H / H_{\geq 1})
\]

and \(\bar{e}\) only appears in \(G_{\geq 0}\) and \(H_{\geq 0}\). \(\square\)

Figure 7. Example of the construction in Definition 4.4.
Lemma 4.6. Let $B = (G, H)$ be a pseudo-Laman bigraph with biedges $E$ without self-loops. Pick $\bar{e} \in E$, and fix $wt = (-1)_E \cdot \langle \bar{e} \rangle$. Suppose that $d = (d_V, d_W) \in \mathbb{R}^P$ is a pair of functions $d_V: P \mapsto \{0, -1\}$ and $d_W: Q \mapsto \{0, -1\}$ that satisfy the first three conditions of Lemma 3.7, but not the last one. Then $^{\mathcal{M}}B$ or $^{\mathcal{N}}B$ has a self-loop, where the sets $\mathcal{M}$ and $\mathcal{N}$ are as in Proposition 4.5.

Proof. By assumption, $d$ is not a bidistance, and it must happen that there exists a cycle in $G$ or in $H$ such that $d_V$ or $d_W$ attains its minimum only once. Let us suppose that there is a cycle $\mathcal{C}$ in $G$ such that $d_V$ attains its minimum only on the pair $(u, v)$, which is part of $\mathcal{C}$. If we set $\alpha = d_V(u, v)$, then we get that $G_{\geq \alpha} / G_{\geq \alpha}$ has a self-loop. Since by definition $G_{\geq \alpha} / G_{\geq \alpha}$ is a union of components of the graphs in either $^{\mathcal{M}}B$ or $^{\mathcal{N}}B$, the proof is completed. \qed

Proposition 2.10 gives the two base cases for the computation of the Laman number of a bigraph: if the bigraph has a self-loop, then its Laman number is zero, and if the bigraph is constituted of two copies of a single edge, then its Laman number is one. They are going to be used in combination with the formula in Theorem 4.7 to obtain a recursive algorithm. We are now able to state the formula for the computation of the Laman number of a bigraph.

Theorem 4.7. Let $B = (G, H)$ be a pseudo-Laman bigraph with biedges $E$ without self-loops. Let $\bar{e}$ be a fixed biedge, then

\begin{equation}
\text{Lam}(B) = \text{Lam}(\langle \bar{e} \rangle B) + \text{Lam}(\langle \bar{e} \rangle \bar{e}) + \sum_{(\mathcal{M}, \mathcal{N})} \text{Lam}^{\mathcal{M}}B \cdot \text{Lam}^{\mathcal{N}}B,
\end{equation}

where each pair $(\mathcal{M}, \mathcal{N}) \in E^2$ satisfies:

\begin{itemize}
  \item $\mathcal{M} \cup \mathcal{N} = E$;
  \item $\mathcal{M} \cap \mathcal{N} = \{\bar{e}\}$;
  \item $|\mathcal{M}| \geq 2$ and $|\mathcal{N}| \geq 2$;
  \item both $^{\mathcal{M}}B$ and $^{\mathcal{N}}B$ are pseudo-Laman.
\end{itemize}

Proof. From Theorem 3.23 we know that $\text{Lam}(B) = \sum d \text{Lam}(B_d)$, where $d$ runs over all bidistances on $B$ compatible with $wt = (-1)_E \cdot \langle \bar{e} \rangle$. We distinguish two cases.

Suppose $\text{Lam}(B) > 0$. Let $d = (d_V, d_W)$ be a pair of functions as in Lemma 4.6, and let $\mathcal{M}, \mathcal{N} \in E$ be the two sets of biedges defined by $d$. If $d$ is not a bidistance, then by Lemma 4.6 either $^{\mathcal{M}}B$ or $^{\mathcal{N}}B$ has a self-loop, and so by Proposition 2.10 the contribution $\text{Lam}^{\mathcal{M}}B \cdot \text{Lam}^{\mathcal{N}}B$ is zero. If $d$ is a bidistance and $\bar{e}$ is neither a bridge in $G_{d_V}$ nor a bridge in $H_{d_W}$, then by Proposition 4.5 the contribution of $\text{Lam}(B_d)$ appears on the right-hand side of Equation (4). If instead $\bar{e}$ is a bridge in either $G_{d_V}$ or in $H_{d_W}$, then by Lemma 2.22 we conclude that $\text{Lam}(B_d) = 0$; at the same time by Lemma 2.23 either $^{\mathcal{M}}B$ or $^{\mathcal{N}}B$ is not pseudo-Laman so there is no contribution to the right-hand side of Equation (4). A similar argument works in the case $\bar{e}$ is a bridge in both $G_{d_V}$ and $H_{d_W}$ using Lemma 2.18.

It remains to settle the case $\text{Lam}(B) = 0$. In this case, $\text{Lam}(B_d) = 0$ for all bidistances compatible with $wt$. We have to prove that the right hand side of Equation (4) is zero, too. By Proposition 4.5, if $d_V$ is the zero map then $\text{Lam}(B_d) = \text{Lam}(\langle \bar{e} \rangle B)$. Hence, the first summand of the right-hand side of Equation (4) is zero. For the second summand, the
Table 1. Number of Laman graphs with \( n \) vertices; this sequence of numbers is A227117 in the OEIS [Slo]. There the sequence originally ended with \( n = 8 \), whose value was erroneously given as 609; we corrected and complemented this OEIS entry accordingly.

| \( n \) | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 |
|-------|---|---|---|---|---|---|---|---|----|----|----|
| \#    | 1 | 1 | 1 | 3 | 13| 70| 608|7222|110132|2039273|44176717|

situation is similar. For the other summands, let us fix \( \mathcal{M} \) and \( \mathcal{N} \) as in the hypothesis. Define \( d_V(\vec{u}, \vec{v}) = d_W(\vec{t}, \vec{w}) = 0 \), and define \( d_V(u, v) = -1 \) if there is an edge \( e \) in \( \mathcal{M} \) such that \( \tau_G(e) = \{u, v\} \), or \( d_V(u, v) = 0 \) if there is no such edge; similarly for \( d_W \). If \( d = (d_V, d_W) \) is not a bidistance, then by Lemma 4.6 one of the bigraphs \( \mathcal{M}_B \) or \( \mathcal{B}_N \) has a self-loop and by Proposition 2.10 the summand is zero. If \( d = (d_V, d_W) \) is a bidistance, then the argument follows the same way as in the case of \( \text{Lam}(B) > 0 \). \( \Box \)

5. Computational results

Theorem 4.7, together with Proposition 2.10, translates naturally into a recursive algorithm, which has exponential complexity since it has to loop over all subsets of \( \mathcal{E} \setminus \{\overline{e}\} \). We have implemented this algorithm [CGG+16] in the computer algebra system Mathematica and in C++. Despite its exponential runtime, it is a tremendous improvement over the naive approach, which is to determine the number of solutions via a Gröbner basis computation. For example, to compute the Laman number 880 of the Laman graph with 10 vertices (see Figure 8), our recursive algorithm took 1.7s in Mathematica and 0.18s with C++, while the Gröbner basis approach took about 2353s in Mathematica and 45s using the FGb library in Maple [Fan10]. Note also that the latter is feasible in practice only after replacing the parameters \( \lambda_e \) by random integers, which turns it into a probabilistic algorithm. Moreover, for speed-up, we compute the Gröbner basis only modulo a prime number so that the occurrence of large rational numbers is avoided. In contrast, our combinatorial algorithm computes the Laman number with certainty. As a consistency check, we computed the Laman numbers of all 118,051 Laman graphs with at most 10 vertices, using both approaches, and found that the results match perfectly.

For this purpose we generated lists of Laman graphs. In principle this is a simple task, by applying the two Henneberg rules in all possible ways. In practice, it becomes demanding since one has to identify and eliminate duplicates, which leads to the graph isomorphism problem. Using our implementation we constructed all Laman graphs up to 12 vertices, see Table 1.

Recently, there has been large interest [BS04, ETV09, ETV13, ST10, JO12] in the maximal Laman number that a Laman graph with \( n \) vertices can have. By applying our algorithm to all Laman graphs with \( n \) vertices, we determined the maximal Laman number for \( 6 \leq n \leq 12 \), which previously was only known for \( n = 6 \) and \( n = 7 \); the results are given in Table 2. For
Table 2. Minimal and maximal Laman number among all Laman graphs with $n$ vertices; the minimum is $2^{n-2}$ and it is achieved, for example, on Laman graphs obtained by applying only the first Henneberg rule (see Theorem 1.8).

| $n$ | 6  | 7  | 8  | 9  | 10 | 11 | 12 |
|-----|----|----|----|----|----|----|----|
| min | 16 | 32 | 64 | 128| 256| 512| 1024|
| max | 24 | 56 | 136| 344| 880| 2288| 6180|

For $n = 12$ this was a quite demanding task: computing the Laman numbers of more than 44 million graphs with 12 vertices took 56 processor days using our fast C++ implementation.

Appendix A. Proof of Theorem 1.8

Proof of Theorem 1.8. $(b) \implies (a)$: Assume that $G = (V, E)$ is generically rigid. Then every subgraph $G' = (V', E')$ is generically realizable (see Remark 1.5), and so the map $h_{G'}$ is dominant. Therefore, the dimension of the codomain is bounded by the dimension of the domain, which says $2|V'| - 3 \geq |E'|$. The equality in the previous formula for the whole graph $G$ follows from Lemma 1.7.

$(a) \implies (c)$: We prove the statement by induction on the number of vertices. The induction base with two vertices is clear. Assume that $G$ is a Laman graph with at least 3 vertices. By [Lam70, Proposition 6.1], the graph $G$ has a vertex of degree 2 or 3. If $G$ has a vertex of degree 2, then the subgraph $G'$ obtained by removing this vertex and its two adjacent edges is a Laman graph by [Lam70, Theorem 6.3]. By induction hypothesis, $G'$ can be constructed by Henneberg rules, and then $G$ can be constructed from $G'$ by the first Henneberg rule. Assume now that $G$ has a vertex $v$ of degree 3. By [Lam70, Theorem 6.4], there are two vertices $u$ and $w$ connected with $v$ such that the graph $G'$ obtained by removing $v$ and its three adjacent edges and then adding the edge $\{u, w\}$ is Laman. By induction hypothesis, $G'$ can be constructed by Henneberg rules, and then $G$ can be constructed from $G'$ by the second Henneberg rule.

$(c) \implies (b)$: We prove the statement by induction on the number of Henneberg rules. The induction base is the case of the one-edge graph, which is generically rigid. By induction hypothesis we assume that $G = (V, E)$ is generically rigid. Perform a Henneberg rule on $G$ and let $G'$ be the result. We intend to show that $G'$ is generically rigid, too.

As far as the first Henneberg rule is concerned, we observe that for any realization of $G$, compatible with a general labeling $\lambda$, and for any labeling $\lambda'$ extending $\lambda$ we can always construct exactly two realizations of $G'$ that are compatible with $\lambda'$.

Let us now assume that $G'$ is constructed via the second Henneberg rule. Call $t$ the new vertex of $G'$, and denote the three vertices to which it is connected by $u$, $v$ and $w$. Let $G''$ be the graph obtained by removing from $G$ the same edge $e$ that is removed in $G'$. Without loss of generality we assume $e = \{u, v\}$.
We first show that \( G' \) is generically realizable (see Definition 1.4). Let \( N : \mathbb{C}^2 \to \mathbb{C} \) be the quadratic form corresponding to the bilinear form \( \langle \cdot, \cdot \rangle \).

Fix a general labeling for \( G'' \). We define the algebraic set \( C \subseteq \mathbb{C}^3 \) as the set of all points \((a, b, c)\) such that there is a compatible realization \( \rho \) of \( G'' \) satisfying

\[
N(\rho(u) - \rho(v)) = a, \quad N(\rho(u) - \rho(w)) = c, \quad N(\rho(v) - \rho(w)) = b.
\]

For a general \( a_0 \in \mathbb{C} \), there exist finitely many, up to equivalence, points \((a_0, b, c)\) in \( C \), namely the “lengths” of the triangle \((u, v, w)\) that come from the finitely many realizations of \( G \). It follows that \( \dim(C) \geq 1 \).

A complex version of a classical result in distance geometry (see [ETV13, Theorem 2.4]) states that four points \( p_0, p_1, p_2, p_3 \in \mathbb{C}^2 \) fulfill

\[
N(p_0 - p_1) = x, \quad N(p_0 - p_2) = y, \quad N(p_0 - p_3) = z,
\]

\[
N(p_1 - p_2) = a, \quad N(p_2 - p_3) = b, \quad N(p_1 - p_3) = c,
\]

if and only if the following Cayley-Menger determinant

\[
F(a, b, c, x, y, z) := \begin{vmatrix} 0 & a & c & x & 1 \\ a & 0 & b & y & 1 \\ c & b & 0 & z & 1 \\ x & y & z & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 \end{vmatrix}
\]

vanishes. We define

\[
U := \bigcup_{p=(a,b,c) \in \mathbb{C}} S_p, \quad \text{where} \quad S_p := \{(x, y, z) \in \mathbb{C}^3 : F(x, y, z, a, b, c) = 0\}
\]

and

\[
e_p := (abc : a(a - b - c) : b(b - a - c) : c(c - a - b) : a - b - c : b - a - c : c - a - b : a : b : c) \in \mathbb{P}^0_{\mathbb{C}}.
\]

The point \( e_p \in \mathbb{P}^0_{\mathbb{C}} \) with \( p = (a, b, c) \) has coordinates given by the coefficients of \( F(a, b, c, x, y, z) \), considered as a polynomial in \( x, y \) and \( z \). Because of this, the point \( e_p \) determines \( S_p \) uniquely as a surface. The function \( \mathbb{C}^3 \setminus \{0\} \to \mathbb{P}^0_{\mathbb{C}} \) sending \( p \mapsto e_p \) is injective, and hence the family \((S_p)_{p \in \mathbb{C}}\) of surfaces is not constant. It follows that the algebraic set \( U \) has dimension 3, and thus a general point \((x, y, z) \in \mathbb{C}^3 \) lies in \( U \). If we extend the general labeling of \( G'' \) by assigning a general triple \((x, y, z) \in U\) as labels to the three new edges, then we get at least one realization of \( G' \). It follows that \( G' \) is generically realizable.

Since \(|V'| = |V| + 1\) and \(|E'| = |E| + 2\), it follows that \(2|V'| = |E'| + 3\). Since, as we have just shown, the map \( h_{G'} \) is dominant, the graph \( G' \) is generically rigid by Lemma 1.7. \( \square \)
Appendix B. Laman graphs with maximal Laman number

Figure 8. Laman graphs with \(6 \leq n \leq 12\) vertices; for each \(n\) the graph with the largest Laman number is shown.

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