Linear Statistics of Random Matrix Ensembles and the Airy Kernel

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July 2, 2018

Abstract

In this paper, we continue to study the large $N$ behavior of the moment-generating function (MGF) of the linear statistics of $N \times N$ Hermitian matrices in the Gaussian unitary, symplectic, orthogonal ensembles (GUE, GSE, GOE) and Laguerre unitary, symplectic, orthogonal ensembles (LUE, LSE, LOE). From the finite $N$ Fredholm determinant expression of the MGF of the linear statistics \[16\], we find the large $N$ asymptotics of the MGF associated with the Airy kernel in these Gaussian and Laguerre ensembles. Then we obtain the mean and variance of the suitably scaled linear statistics. We show that there is an equivalence between the large $N$ behavior of the MGF of the scaled linear statistics in Gaussian and Laguerre ensembles, which leads to the statistical equivalence between the mean and variance of suitably scaled linear statistics in Gaussian and Laguerre ensembles. In the end, we use two different methods to obtain the large $N$ behavior of the MGF for another type of linear statistics in GUE. The mean and variance of the linear statistics then follows.

Keywords: Random matrix ensembles; Linear statistics; Moment-generating function; Airy kernel; Mean and variance.

Mathematics Subject Classification 2010: 15B52, 47A53, 33C45
1 Introduction

In random matrix theory, the joint probability density function for the eigenvalues \( \{x_j\}_{j=1}^N \) of \( N \times N \) Hermitian matrices from an unitary ensemble (\( \beta = 2 \)), symplectic ensemble (\( \beta = 4 \)) or orthogonal ensemble (\( \beta = 1 \)) is given by [15]

\[
P^{(\beta)}(x_1, x_2, \ldots, x_N) = C_N^{(\beta)} \prod_{1 \leq j < k \leq N} |x_j - x_k|^\beta \prod_{j=1}^N w(x_j).
\]

Here \( w(x) \) is a weight function or a probability density supported on \([a, b] \), such that all the moments of \( w \), namely, \( \int_a^b x^j w(x) \, dx, \quad j = 0, 1, 2, \ldots \) exist. \( C_N^{(\beta)} \) is the normalization constant so that

\[
\int_{[a,b]^N} P^{(\beta)}(x_1, x_2, \ldots, x_N) \prod_{j=1}^N dx_j = 1.
\]

The unitary ensemble is the simplest one and has been studied extensively. The symplectic ensemble and orthogonal ensemble are much more complicated than the unitary case. For the symplectic ensemble and orthogonal ensemble and also their relations to the unitary ensemble, please refer to [2, 3, 12, 16, 22, 23]. In this paper, we take \( w(x) = e^{-x^2} \), \( x \in \mathbb{R} \) and \( w(x) = x^\alpha e^{-x} \), \( \alpha > -1 \), \( x \in \mathbb{R}^+ \). These are known as the Gaussian unitary, symplectic, orthogonal ensemble and Laguerre unitary, symplectic, orthogonal ensemble, respectively.

The moment-generating function (MGF) of the linear statistics \( \sum_{j=1}^N F(x_j) \) is,

\[
E \left( e^{-\lambda \sum_{j=1}^N F(x_j)} \right) = \frac{\int_{[a,b]^N} \prod_{1 \leq j < k \leq N} |x_j - x_k|^\beta \prod_{j=1}^N w(x_j) \, e^{-\lambda F(x_j)} \, dx_j}{\int_{[a,b]^N} \prod_{1 \leq j < k \leq N} |x_j - x_k|^\beta \prod_{j=1}^N w(x_j) \, dx_j}.
\]

We write it in the following form

\[
G_N^{(\beta)}(f) := C_N^{(\beta)} \int_{[a,b]^N} \prod_{1 \leq j < k \leq N} |x_j - x_k|^\beta \prod_{j=1}^N w(x_j) \left[ 1 + f(x_j) \right] \, dx_j, \tag{1.1}
\]

where \( f(x) = e^{-\lambda F(x)} - 1 \). We assume \( f(x) \) lies in the Schwartz space [19] and \( 1 + f(x) \neq 0 \) over \([a, b] \).

For the unitary ensembles, Tracy and Widom [22] obtained the result for \( G_N^{(2)}(f) \), and can be expressed as a Fredholm determinant.

Lemma 1.1. Let

\[
\varphi_j(x) := P_j(x) \sqrt{w(x)}, \quad j = 0, 1, 2, \ldots,
\]

\[ \text{such that all the moments of} \ w, \text{namely,} \ \int_a^b x^j w(x) \, dx, \quad j = 0, 1, 2, \ldots \text{exist.} \]
where $P_j(x)$, $j = 0, 1, 2, \ldots$ are the polynomials of degree $j$ orthonormal with respect to the weight $w(x)$,

\[ \int_a^b P_j(x)P_k(x)w(x)dx = \delta_{jk}, \ j, k = 0, 1, 2, \ldots. \]

Then

\[ G_N^{(2)}(f) = \det \left( I + K_N^{(2)}f \right), \]

where $K_N^{(2)}$ is an operator on $L^2[a,b]$ with kernel

\[ K_N^{(2)}(x,y) := \sum_{j=0}^{N-1} \varphi_j(x)\varphi_j(y), \]

and $f$ denotes the operator, multiplication by $f$. That is, for a function $g \in L^2[a,b]$,

\[ \left( K_N^{(2)}f \right) g(x) := \int_a^b K_N^{(2)}(x,y)f(y)g(y)dy. \]

Min and Chen [16] expressed $G_N^{(4)}(f)$ and $G_N^{(1)}(f)$ as Fredholm determinants based on the work [12] and [22]. For the $\beta = 1$ case, we take $N$ to be even for simplicity. We state the results as the following two lemmas [16].

**Lemma 1.2.** Define

\[ \psi_j(x) := \pi_j(x)\sqrt{w(x)}, \ j = 0, 1, 2, \ldots, \]

where $\pi_j(x)$ is any polynomial of degree $j$, and

\[ M^{(4)} := \left( \int_a^b \left( \psi_j(x)\psi'_k(x) - \psi'_j(x)\psi_k(x) \right) dx \right)_{j,k=0}^{2N-1} \]

with its inverse denoted by

\[ (M^{(4)})^{-1} =: (\mu_{jk})_{j,k=0}^{2N-1}. \]

Then

\[ \left[ G_N^{(4)}(f) \right]^2 = \det \left( I + 2K_N^{(4)}f - K_N^{(4)}\varepsilon f' \right), \]

where $K_N^{(4)}$ and $\varepsilon$ are integral operators with kernel

\[ K_N^{(4)}(x,y) = -\sum_{j,k=0}^{2N-1} \mu_{jk}\psi_j(x)\psi'_k(y) \]

and

\[ \varepsilon(x,y) := \frac{1}{2} \text{sgn}(x-y), \]

respectively.
Lemma 1.3. We assume that $N$ is even. Let

$$
\tilde{\psi}_j(x) := \pi_j(x)w(x), \ j = 0, 1, 2, \ldots,
$$

where $\pi_j(x)$ is any polynomial of degree $j$, and

$$
M^{(1)} := \left( \int_a^b \tilde{\psi}_j(x) \tilde{\psi}_k(x) dx \right)_{j,k=0}^{N-1}
$$

with its inverse denoted by

$$
(M^{(1)})^{-1} =: (\nu_{jk})_{j,k=0}^{N-1}.
$$

Then

$$
\left[ G_N^{(1)}(f) \right]^2 = \det \left( I + K_N^{(1)}(f^2 + 2f) - K_N^{(1)}\varepsilon f' - K_N^{(1)}f\varepsilon f' \right),
$$

where $K_N^{(1)}$ is an integral operator with kernel

$$
K_N^{(1)}(x, y) = \sum_{j,k=0}^{N-1} \nu_{jk}\varepsilon \tilde{\psi}_j(x)\tilde{\psi}_k(y).
$$

We introduce here some notations, which will be used in the following sections of this paper.

Let $\text{Ai}(x)$ denote the Airy function, the first solution of $y''(x) - xy(x) = 0$ [14] (page 136-137).

Define

$$
B(x) := \int_{-\infty}^{x} \text{Ai}(y)dy - \int_{x}^{\infty} \text{Ai}(y)dy.
$$

(1.2)

We can also write $B(x)$ in another form:

$$
B(x) = \int_{-\infty}^{0} \text{Ai}(y)dy - \int_{0}^{\infty} \text{Ai}(y)dy + 2 \int_{0}^{x} \text{Ai}(y)dy
$$

$$
= \frac{1}{3} + 2 \int_{0}^{x} \text{Ai}(y)dy,
$$

since we have the fact that [11] (page 449)

$$
\int_{-\infty}^{0} \text{Ai}(y)dy = \frac{2}{3}, \quad \int_{0}^{\infty} \text{Ai}(y)dy = \frac{1}{3}.
$$

Let $K(x, y)$ be the Airy kernel

$$
K(x, y) := \frac{\text{Ai}(x)\text{Ai}'(y) - \text{Ai}(y)\text{Ai}'(x)}{x - y}.
$$

(1.3)

When $x = y$,

$$
K(x, x) = (\text{Ai}'(x))^2 - x \text{Ai}^2(x).
$$

(1.4)
The equality (1.4) is obtained by taking the limit \( y \to x \) from (1.3) and using the property \( \text{Ai}''(x) = x \cdot \text{Ai}(x) \).

We then define

\[
L(x, y) := \int_{x}^{\infty} K(y, z)dz - \int_{-\infty}^{x} K(y, z)dz.
\]  

Finally, we mention that \( \chi_J(x) \) is the indicator function defined on the interval \( J \), namely,

\[
\chi_J(x) = \begin{cases} 
1, & x \in J; \\
0, & x \notin J.
\end{cases}
\]

The paper [16] studied the large \( N \) behavior of the MGF of the linear statistics in Gaussian ensembles associated with the sine kernel and Laguerre ensembles associated with the Bessel kernel. This paper continues to study the large \( N \) behavior of the MGF in these Gaussian and Laguerre ensembles associated with the Airy kernel, from which we obtain the mean and variance of the scaled linear statistics. The unitary case is the simplest one among them. We established the relation between the mean and variance of the scaled linear statistics in symplectic, orthogonal and unitary ensembles. We also show that as \( N \to \infty \), the MGF of a suitably scaled linear statistics in the Gaussian ensembles and Laguerre ensembles are the same, which leads to the same mean and variance of the linear statistics between the Gaussian ensembles and Laguerre ensembles. For the problems on the mean and variance of linear statistics in unitary ensembles, see [4, 5, 11, 17] for reference. Finally, we point out that the variance of linear statistics play an important role in the random matrix theory of quantum transport [8, 9].

The rest of this paper is organized as follows. In Sec. 2, we study the large \( N \) behavior of the MGF of the scaled linear statistics in Gaussian unitary, symplectic and orthogonal ensembles, respectively. From this we obtain the mean and variance of the scaled linear statistics in the three Gaussian ensembles. In Sec. 3, we repeat the development of Sec. 2, but for the Laguerre ensembles. In Sec. 4, we use two different methods to consider the large \( N \) behavior of another type of linear statistics in GUE. The mean and variance of the linear statistics are obtained. The conclusion is given in Sec. 5.


2 The Gaussian Ensembles

2.1 Gaussian Unitary Ensemble

In the Gaussian case, \( w(x) = e^{-x^2}, \ x \in \mathbb{R} \). From Lemma 1.1 we have

\[
\varphi_j(x) = \frac{1}{\pi^{\frac{1}{4}} 2^{\frac{j}{2}} \sqrt{j!}} H_j(x) e^{-\frac{x^2}{2}}, \quad j = 0, 1, 2, \ldots
\]

(2.1)

where \( H_j(x) \) are the Hermite polynomials of degree \( j \).

We consider the large \( N \) asymptotics of \( G_N^{(2)}(f) \) in this subsection. It is well known that

\[
\log \det \left( I + K^{(2)}_N f \right) = \text{Tr} \log \left( I + K^{(2)}_N f \right) = \text{Tr} K^{(2)}_N f - \frac{1}{2} \text{Tr} \left( K^{(2)}_N f \right)^2 + \frac{1}{3} \text{Tr} \left( K^{(2)}_N f \right)^3 - \cdots
\]

(2.2)

We state a theorem before our discussion.

**Theorem 2.1.** As \( N \to \infty \),

\[
2^{\frac{1}{2}} N^{-\frac{1}{4}} K^{(2)}_N \left( \sqrt{2N + 2^{\frac{1}{2}} N^{-\frac{1}{4}} x}, \sqrt{2N + 2^{\frac{1}{2}} N^{-\frac{1}{4}} y} \right) = K(x, y) + O(N^{-\frac{3}{4}}),
\]

where \( K(x, y) \) is the Airy kernel defined by (1.3).

**Proof.** From the asymptotic formula of Hermite polynomial \([20]\) (page 201),

\[
e^{-\frac{x^2}{2}} H_n(x) = 2^{\frac{n}{2} + \frac{1}{4}} \pi^{\frac{1}{4}} (n!)^{\frac{1}{2}} n^{-\frac{1}{4}} \left[ \text{Ai}(3^{-\frac{1}{3}} x) + O(n^{-\frac{2}{3}}) \right],
\]

where

\[
x = (2n + 1)^{\frac{1}{4}} - 2^{\frac{1}{2}} 3^{-\frac{1}{3}} n^{-\frac{1}{3}} t,
\]

we have

\[
e^{-\frac{x^2}{2}} H_n(x) = \pi^{\frac{1}{4}} 2^{\frac{n}{2} + \frac{1}{4}} (n!)^{\frac{1}{2}} n^{-\frac{1}{4}} \left[ \text{Ai} \left( 2^{\frac{1}{2}} n^{\frac{1}{3}} \left( x - \sqrt{2n + 1} \right) \right) + O \left( n^{-\frac{2}{3}} \right) \right].
\]

(2.3)

Using the Christoffel-Darboux formula,

\[
K^{(2)}_N(x, y) = \frac{e^{-\frac{x^2}{2}} H_N(x) e^{-\frac{y^2}{2}} H_{N-1}(y) - e^{-\frac{y^2}{2}} H_N(y) e^{-\frac{x^2}{2}} H_{N-1}(x)}{\pi^{\frac{1}{4}} 2^N (N-1)! (x-y)}.
\]

Replacing the variables \( x \) by \( \sqrt{2N + 2^{-\frac{1}{2}} N^{-\frac{1}{4}} x} \) and \( y \) by \( \sqrt{2N + 2^{-\frac{1}{2}} N^{-\frac{1}{4}} y} \), and using (2.3), we obtain the desired result after some elaborate computations.

\[\square\]
Remark. The above result was obtained by [10, 13, 18], but they did not show the order term. See also [21] on the study of this Airy kernel.

We now use Theorem 2.1 to compute (2.2) term by term as \( N \to \infty \). We replace \( f(x) \) by \( f \left( 2^{\frac{1}{2}} N^\frac{1}{6} \left( x - \sqrt{2N} \right) \right) \) in the following computations. The first term reads,

\[
\text{Tr}K_N^{(2)} f = \int_{-\infty}^{\infty} K_N^{(2)}(x, x) f \left( 2^{\frac{1}{2}} N^\frac{1}{6} \left( x - \sqrt{2N} \right) \right) dx
\]

\[
= \int_{-\infty}^{\infty} 2^{-\frac{1}{2}} N^{-\frac{1}{6}} K_N^{(2)} \left( \sqrt{2N + 2^{-\frac{1}{2}} N^{-\frac{1}{6}} x}, \sqrt{2N + 2^{-\frac{1}{2}} N^{-\frac{1}{6}} y} \right) f(x) dx
\]

\[
= \int_{-\infty}^{\infty} K(x, x) f(x) dx + O(N^{-\frac{1}{6}}).
\]

The second term,

\[
\text{Tr} \left( K_N^{(2)} f \right)^2 = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} K_N^{(2)}(x, y) f \left( 2^{\frac{1}{2}} N^\frac{1}{6} \left( y - \sqrt{2N} \right) \right) K_N^{(2)}(y, x) f \left( 2^{\frac{1}{2}} N^\frac{1}{6} \left( x - \sqrt{2N} \right) \right) dxdy
\]

\[
= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} 2^{-\frac{1}{2}} N^{-\frac{1}{6}} K_N^{(2)} \left( \sqrt{2N + 2^{-\frac{1}{2}} N^{-\frac{1}{6}} x}, \sqrt{2N + 2^{-\frac{1}{2}} N^{-\frac{1}{6}} y} \right) f(x) f(y) dxdy
\]

\[
= \int_{-\infty}^{\infty} K^2(x, y) f(x) f(y) dxdy + O(N^{-\frac{1}{6}}).
\]

It follows from (2.2) that

\[
\log \det \left( I + K_N^{(2)} f \right) = \int_{-\infty}^{\infty} K(x, x) f(x) dx - \frac{1}{2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} K^2(x, y) f(x) f(y) dxdy
\]

\[
+ \cdots + O(N^{-\frac{1}{6}}) \tag{2.4}
\]

We proceed to study the mean and variance of the scaled linear statistics \( \sum_{j=1}^{N} F \left( 2^{\frac{1}{2}} N^\frac{1}{6} \left( x_j - \sqrt{2N} \right) \right) \), so we need to obtain the coefficients of \( \lambda \) and \( \lambda^2 \) from (2.4). From the relation of \( f(x) \) and \( F(x) \) we know

\[
f(x) = -\lambda F(x) + \frac{\lambda^2}{2} F^2(x) - \cdots \tag{2.5}
\]

Substituting (2.5) into (2.4), we have

\[
\log \det \left( I + K_N^{(2)} f \right) = -\lambda \int_{-\infty}^{\infty} K(x, x) F(x) dx
\]

\[
+ \frac{\lambda^2}{2} \left[ \int_{-\infty}^{\infty} K(x, x) F^2(x) dx - \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} K^2(x, y) F(x) F(y) dxdy \right] + \cdots + O(N^{-\frac{1}{6}}).
\]

Let \( \mu_N^{(\text{GUE})} \) and \( \nu_N^{(\text{GUE})} \) be the mean and variance of the linear statistics \( \sum_{j=1}^{N} F \left( 2^{\frac{1}{2}} N^\frac{1}{6} \left( x_j - \sqrt{2N} \right) \right) \), and note that \( \log G_N^{(2)}(f) = \log \det \left( I + K_N^{(2)} f \right) \). Then we have the following theorem.
Theorem 2.2. As \( N \to \infty \),
\[
\mu_{N}^{\text{GUE}} = \int_{-\infty}^{\infty} K(x, x)F(x)dx + O(N^{-\frac{1}{3}}),
\]
\[
\nu_{N}^{\text{GUE}} = \int_{-\infty}^{\infty} K(x, x)F^2(x)dx - \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} K^2(x, y)F(x)F(y)dxdy + O(N^{-\frac{1}{3}}),
\]
where \( K(x, y) \) is the Airy kernel defined by (1.3).

2.2 Gaussian Symplectic Ensemble

In this case, \( w(x) = e^{-x^2}, \ x \in \mathbb{R} \). Let
\[
\psi_{2j+1}(x) := \frac{1}{\sqrt{2}} \varphi_{2j+1}(x), \ \psi_{2j}(x) := -\frac{1}{\sqrt{2}} \varepsilon \varphi_{2j+1}(x), \ j = 0, 1, 2, \ldots,
\]
where \( \varphi_j(x) \) is given by (2.1). It follows that \( M^{(4)} \) is the direct sum of the \( N \) copies of
\[
\begin{pmatrix}
0 & 1 \\
-1 & 0
\end{pmatrix}
\]
and \( (M^{(4)})^{-1} = -M^{(4)} \) (see [12, 22]). From Lemma 1.2 we obtain the following result [16].

Theorem 2.3. For the Gaussian symplectic ensemble, we have
\[
\left[ G_N^{(4)}(f) \right]^2 = \det(I + T_{\text{GSE}}),
\]
where
\[
T_{\text{GSE}} := K_{2N+1}^{(2)}f - \frac{1}{2} K_{2N+1}^{(2)} \varepsilon f' + \sqrt{N + \frac{1}{2} (\varepsilon \varphi_{2N+1}) \otimes \varphi_{2N}f} + \frac{1}{2} \sqrt{N + \frac{1}{2} (\varepsilon \varphi_{2N+1}) \otimes (\varepsilon \varphi_{2N})f'},
\]
and \( K_{2N+1}^{(2)} \) is an operator on \( L^2(\mathbb{R}) \) with kernel
\[
K_{2N+1}^{(2)}(x, y) = \sum_{j=0}^{2N} \varphi_j(x)\varphi_j(y).
\]

We also have the following expansion formula,
\[
\log \det(I + T_{\text{GSE}}) = \operatorname{Tr} \log(I + T_{\text{GSE}}) = \operatorname{Tr} T_{\text{GSE}} - \frac{1}{2} \operatorname{Tr} T_{\text{GSE}}^2 + \frac{1}{3} \operatorname{Tr} T_{\text{GSE}}^3 - \cdots. \tag{2.8}
\]

Similarly as Theorem 2.1 we have the following theorem.

Theorem 2.4. As \( N \to \infty \),
\[
2^{-\frac{2}{3}} N^{-\frac{1}{3}} K_{2N+1}^{(2)} \left( \sqrt{4N + 2^{-\frac{2}{3}} N^{-\frac{1}{3}} x}, \sqrt{4N + 2^{-\frac{2}{3}} N^{-\frac{1}{3}} y} \right) = K(x, y) + O(N^{-\frac{1}{3}}).
\]
Theorem 2.5. As $N \to \infty$,

$$\varphi_{2N} \left( \sqrt{4N} + 2^{-\frac{3}{4}} N^{-\frac{1}{6}} x \right) = 2^{\frac{1}{6}} N^{-\frac{1}{12}} \text{Ai}(x) + O(N^{-\frac{3}{4}}),$$  \hfill (2.9)

$$\varphi_{2N+1} \left( \sqrt{4N} + 2^{-\frac{3}{4}} N^{-\frac{1}{6}} x \right) = 2^{\frac{1}{6}} N^{-\frac{1}{12}} \text{Ai}(x) + O(N^{-\frac{3}{4}}),$$  \hfill (2.10)

$$\varepsilon \varphi_{2N} \left( \sqrt{4N} + 2^{-\frac{3}{4}} N^{-\frac{1}{6}} x \right) = 2^{-\frac{3}{2}} N^{-\frac{1}{4}} \text{B}(x) + O(N^{-\frac{11}{12}}),$$  \hfill (2.11)

$$\varepsilon \varphi_{2N+1} \left( \sqrt{4N} + 2^{-\frac{3}{4}} N^{-\frac{1}{6}} x \right) = 2^{-\frac{3}{2}} N^{-\frac{1}{4}} \text{B}(x) + O(N^{-\frac{11}{12}}),$$  \hfill (2.12)

where $B(x)$ is defined by (1.2).

Proof. From the definition (2.1) and the asymptotic formula (2.3), we readily obtain (2.9) and (2.10). It follows from the definition of $\varepsilon$ that

$$\varepsilon \varphi_{2N} \left( \sqrt{4N} + 2^{-\frac{3}{4}} N^{-\frac{1}{6}} x \right) = \frac{1}{2} \left( \int_{-\infty}^{\sqrt{4N+2^{-\frac{3}{4}} N^{-\frac{1}{6}} x}} \varphi_{2N}(y) dy - \int_{\sqrt{4N+2^{-\frac{3}{4}} N^{-\frac{1}{6}} x}}^{\infty} \varphi_{2N}(y) dy \right)$$

$$= 2^{-\frac{3}{2}} N^{-\frac{1}{4}} \left( \int_{-\infty}^{x} \varphi_{2N} \left( \sqrt{4N} + 2^{-\frac{3}{4}} N^{-\frac{1}{6}} y \right) dy - \int_{x}^{\infty} \varphi_{2N} \left( \sqrt{4N} + 2^{-\frac{3}{4}} N^{-\frac{1}{6}} y \right) dy \right)$$

$$= 2^{-\frac{3}{2}} N^{-\frac{1}{4}} \left( \int_{-\infty}^{x} \text{Ai}(y) dy - \int_{x}^{\infty} \text{Ai}(y) dy \right) + O(N^{-\frac{11}{12}})$$

$$= 2^{-\frac{3}{2}} N^{-\frac{1}{4}} \text{B}(x) + O(N^{-\frac{11}{12}}),$$

where use has been made of (2.3).

Similarly, we find

$$\varepsilon \varphi_{2N+1} \left( \sqrt{4N} + 2^{-\frac{3}{4}} N^{-\frac{1}{6}} x \right)$$

$$= \frac{1}{2} \left( \int_{-\infty}^{\sqrt{4N+2^{-\frac{3}{4}} N^{-\frac{1}{6}} x}} \varphi_{2N+1}(y) dy - \int_{\sqrt{4N+2^{-\frac{3}{4}} N^{-\frac{1}{6}} x}}^{\infty} \varphi_{2N+1}(y) dy \right)$$

$$= 2^{-\frac{3}{2}} N^{-\frac{1}{4}} \left( \int_{-\infty}^{x} \varphi_{2N+1} \left( \sqrt{4N} + 2^{-\frac{3}{4}} N^{-\frac{1}{6}} y \right) dy - \int_{x}^{\infty} \varphi_{2N+1} \left( \sqrt{4N} + 2^{-\frac{3}{4}} N^{-\frac{1}{6}} y \right) dy \right)$$

$$= 2^{-\frac{3}{2}} N^{-\frac{1}{4}} \left( \int_{-\infty}^{x} \text{Ai}(y) dy - \int_{x}^{\infty} \text{Ai}(y) dy \right) + O(N^{-\frac{11}{12}})$$

$$= 2^{-\frac{3}{2}} N^{-\frac{1}{4}} \text{B}(x) + O(N^{-\frac{11}{12}}).$$

\qed
Now we use Theorem 2.4 and Theorem 2.5 to compute (2.8) as \( N \to \infty \). We will change \( f(x) \) to \( f \left( 2^{\frac{2}{3}} N^{\frac{1}{2}} \left( x - \sqrt{4N} \right) \right) \) in the following calculations. In this case, \( f'(x) \) becomes \( 2^{\frac{2}{3}} N^{\frac{1}{2}} f' \left( 2^{\frac{2}{3}} N^{\frac{1}{2}} \left( x - \sqrt{4N} \right) \right) \). We consider \( \text{Tr} T_{GSE} \) firstly,

\[
\text{Tr} T_{GSE} = \text{Tr} K_{2N+1} f - \text{Tr} \frac{1}{2} K_{2N+1} \varepsilon f' + \text{Tr} \sqrt{N + \frac{1}{2} \varepsilon \varphi_{2N+1} \otimes \varphi_{2N} f} + \text{Tr} \frac{1}{2} \sqrt{N + \frac{1}{2} \varepsilon \varphi_{2N+1} \otimes \varphi_{2N} f} f'.
\]

The first term reads,

\[
\text{Tr} K_{2N+1}^{(2)} f = \int_{-\infty}^{\infty} K_{2N+1}^{(2)}(x,x) f \left( 2^{\frac{2}{3}} N^{\frac{1}{2}} \left( x - \sqrt{4N} \right) \right) dx
\]

\[
= \int_{-\infty}^{\infty} 2^{-\frac{2}{3}} N^{-\frac{1}{2}} K_{2N+1}^{(2)} \left( \sqrt{4N} + 2^{-\frac{2}{3}} N^{-\frac{1}{2}} x, \sqrt{4N} + 2^{-\frac{2}{3}} N^{-\frac{1}{2}} x \right) f(x) dx
\]

\[
= \int_{-\infty}^{\infty} K(x,x) f(x) dx + O(N^{-\frac{1}{3}}).
\]

The second term,

\[
\text{Tr} \frac{1}{2} K_{2N+1}^{(2)} \varepsilon f' = 2^{-\frac{2}{3}} N^{-\frac{1}{2}} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} K_{2N+1}^{(2)}(x,y) \varepsilon(y,x) f' \left( 2^{\frac{2}{3}} N^{\frac{1}{2}} \left( x - \sqrt{4N} \right) \right) dx dy
\]

\[
= 2^{-\frac{2}{3}} N^{-\frac{1}{2}} \int_{-\infty}^{\infty} \left( \int_{-\infty}^{x} K_{2N+1}^{(2)}(x,y) dy - \int_{-\infty}^{x} K_{2N+1}^{(2)}(x,y) dy \right) f' \left( 2^{\frac{2}{3}} N^{\frac{1}{2}} \left( x - \sqrt{4N} \right) \right) dx.
\]

Let

\[
x = \sqrt{4N} + 2^{-\frac{2}{3}} N^{-\frac{1}{2}} u, \quad y = \sqrt{4N} + 2^{-\frac{2}{3}} N^{-\frac{1}{2}} v.
\]

Then

\[
\text{Tr} \frac{1}{2} K_{2N+1}^{(2)} \varepsilon f' = \frac{1}{4} \int_{-\infty}^{\infty} \left( \int_{u}^{\infty} 2^{-\frac{2}{3}} N^{-\frac{1}{2}} K_{2N+1}^{(2)} \left( \sqrt{4N} + 2^{-\frac{2}{3}} N^{-\frac{1}{2}} u, \sqrt{4N} + 2^{-\frac{2}{3}} N^{-\frac{1}{2}} v \right) dv
\]

\[
- \int_{-\infty}^{u} 2^{-\frac{2}{3}} N^{-\frac{1}{2}} K_{2N+1}^{(2)} \left( \sqrt{4N} + 2^{-\frac{2}{3}} N^{-\frac{1}{2}} u, \sqrt{4N} + 2^{-\frac{2}{3}} N^{-\frac{1}{2}} v \right) dv \right) f'(u) du
\]

\[
= \frac{1}{4} \int_{-\infty}^{\infty} \left( \int_{u}^{\infty} K(u,v) dv - \int_{-\infty}^{u} K(u,v) dv \right) f'(u) du + O(N^{-\frac{1}{3}})
\]

\[
= \frac{1}{4} \int_{-\infty}^{\infty} L(x,x) f'(x) dx + O(N^{-\frac{1}{3}}),
\]

where \( L(x,y) \) is given by (1.5).

The third term,

\[
\text{Tr} \sqrt{N + \frac{1}{2} \varepsilon \varphi_{2N+1} \otimes \varphi_{2N} f} = \int_{-\infty}^{\infty} \sqrt{N + \frac{1}{2} \varepsilon \varphi_{2N+1} (x) \varphi_{2N} f \left( 2^{\frac{2}{3}} N^{\frac{1}{2}} \left( x - \sqrt{4N} \right) \right) dx
\]

\[
= 2^{-\frac{2}{3}} N^{-\frac{1}{2}} \sqrt{N + \frac{1}{2} \int_{-\infty}^{\infty} \varepsilon \varphi_{2N+1} \left( \sqrt{4N} + 2^{-\frac{2}{3}} N^{-\frac{1}{2}} x \right) \varphi_{2N} \left( \sqrt{4N} + 2^{-\frac{2}{3}} N^{-\frac{1}{2}} x \right) f(x) dx
\]

\[
= \frac{1}{4} \int_{-\infty}^{\infty} Ai(x) B(x) f(x) dx + O(N^{-\frac{1}{3}}).
\]
The fourth term,
\[
\text{Tr} \frac{1}{2} \sqrt{N + \frac{1}{2}} (\varphi_{2N+1} \otimes \varphi_{2N}) f' = 2^{-\frac{1}{2}} N^{\frac{1}{2}} \sqrt{N + \frac{1}{2}} \int_{-\infty}^{\infty} \varphi_{2N+1}(x) \varphi_{2N}(x) f' \left( 2^\frac{3}{2} N^{\frac{1}{2}} \left( x - \sqrt{4N} \right) \right) dx
\]
\[
= \frac{1}{2} \sqrt{N + \frac{1}{2}} \int_{-\infty}^{\infty} \varphi_{2N+1} \left( \sqrt{4N + 2^{-\frac{3}{2}} N^{-\frac{1}{2}}} x \right) \varphi_{2N} \left( \sqrt{4N + 2^{-\frac{3}{2}} N^{-\frac{1}{2}}} x \right) f'(x) dx
\]
\[
= \frac{1}{16} \int_{-\infty}^{\infty} B^2(x) f'(x) dx + O(N^{-\frac{1}{2}}).
\]

So we obtain
\[
\text{Tr} T_{\text{GSE}} = \int_{-\infty}^{\infty} K(x, x) f(x) dx - \frac{1}{4} \int_{-\infty}^{\infty} L(x, x) f'(x) dx + \frac{1}{4} \int_{-\infty}^{\infty} \text{Ai}(x) B(x) f(x) dx
\]
\[
+ \frac{1}{16} \int_{-\infty}^{\infty} B^2(x) f'(x) dx + O(N^{-\frac{1}{2}}).
\]  \hfill (2.11)

We proceed to compute \( T_{\text{GSE}}^2 \),
\[
\text{Tr} T_{\text{GSE}}^2 = \text{Tr} K_{2N+1}^{(2)} f K_{2N+1}^{(2)} f - \text{Tr} K_{2N+1}^{(2)} f K_{2N+1}^{(2)} f' + \text{Tr} \sqrt{4N + 2} K_{2N+1}^{(2)} f (\varphi_{2N+1} \otimes \varphi_{2N}) f'
\]
\[
+ \frac{1}{2} \int_{-\infty}^{\infty} K(x, y) f(x) f(y) dxdy + \frac{1}{8} \int_{-\infty}^{\infty} K(x, y) B(x) B(y) f(x) f'(y) dxdy
\]
\[
+ \frac{1}{16} \int_{-\infty}^{\infty} L(x, y) L(x, y) f'(x) f'(y) dxdy - \frac{1}{8} \int_{-\infty}^{\infty} L(x, y) \text{Ai}(x) B(y) f(x) f'(y) dxdy
\]
\[
- \frac{1}{32} \int_{-\infty}^{\infty} L(x, y) B(x) B(y) f'(x) f'(y) dxdy + \frac{1}{16} \int_{-\infty}^{\infty} \text{Ai}(x) \text{Ai}(y) B(x) B(y) f(x) f(y) dxdy
\]
\[
+ \frac{1}{32} \int_{-\infty}^{\infty} \text{Ai}(x) B(x) B^2(y) f'(x) f'(y) dxdy + \frac{1}{256} \int_{-\infty}^{\infty} B^2(x) B^2(y) f'(x) f'(y) dxdy + O(N^{-\frac{1}{2}}).
\]  \hfill (2.12)
Proceeding as in the previous subsection, we replace \( f(x) \) with \(-\lambda F(x) + \frac{\lambda^2}{2} F^2(x)\), then \( f'(x) \) becomes \(-\lambda F'(x) + \lambda^2 F(x) F'(x)\). Substituting these into (2.11) and (2.12), we finally find

\[
\log \det(I + T_{\text{GSE}}) = -\lambda \left\{ \int_{-\infty}^{\infty} K(x, x) F(x) dx - \frac{1}{4} \int_{-\infty}^{\infty} L(x, x) F'(x) dx + \frac{1}{4} \int_{-\infty}^{\infty} \operatorname{Ai}(x) B(x) F(x) dx + \frac{1}{16} \int_{-\infty}^{\infty} B^2(x) F'(x) dx \right\} \\
+ \frac{\lambda^2}{2} \left\{ \int_{-\infty}^{\infty} K(x, x) F^2(x) dx - \frac{1}{2} \int_{-\infty}^{\infty} L(x, x) F(x) F'(x) dx + \frac{1}{4} \int_{-\infty}^{\infty} \operatorname{Ai}(x) B(x) F^2(x) dx \right\} \\
+ \frac{1}{8} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} B^2(x) F(x) F'(x) dx - \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} K^2(x, y) F(x) F(y) dy dx \\
+ \frac{1}{2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} K(x, y) L(x, y) F'(x) F(y) dy dx \\
- \frac{1}{8} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} K(x, y) B(x) B(y) F(x) F'(y) dy dx - \frac{1}{16} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} L(x, y) L(y, x) F'(x) F'(y) dy dx \\
+ \frac{1}{8} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} L(x, y) \operatorname{Ai}(y) B(x) F'(x) F(y) dy dx + \frac{1}{32} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} L(x, y) B(y) B(x) F'(x) F'(y) dy dx \\
- \frac{1}{16} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \operatorname{Ai}(x) \operatorname{Ai}(y) B(x) B(y) F(x) F(y) dy dx \\
- \frac{1}{256} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} B^2(x) B^2(y) F'(x) F'(y) dy dx \right\} + O(N^{-\frac{1}{3}}).
\]

Denoting by \( \mu_N^{(\text{GSE})} \) and \( \nu_N^{(\text{GSE})} \) the mean and variance of the linear statistics \( \sum_{j=1}^{N} F \left( 2^{\frac{3}{2}} N^\frac{1}{8} (x_j - \sqrt{4N}) \right) \), and noting that \( \log G_{N}^{(4)}(f) = \frac{1}{2} \log \det(I + T_{\text{GSE}}) \), we have the following theorem.

**Theorem 2.6.** As \( N \to \infty \),

\[
\mu_N^{(\text{GSE})} = \frac{1}{2} \mu_N^{(\text{GUE})} - \frac{1}{8} \int_{-\infty}^{\infty} L(x, x) F'(x) dx + \frac{1}{8} \int_{-\infty}^{\infty} \operatorname{Ai}(x) B(x) F(x) dx + \frac{1}{32} \int_{-\infty}^{\infty} B^2(x) F'(x) dx + O(N^{-\frac{1}{4}}),
\]

\[
\nu_N^{(\text{GSE})} = \frac{1}{2} \nu_N^{(\text{GUE})} - \frac{1}{4} \int_{-\infty}^{\infty} L(x, x) F(x) F'(x) dx + \frac{1}{8} \int_{-\infty}^{\infty} \operatorname{Ai}(x) B(x) F^2(x) dx + \frac{1}{16} \int_{-\infty}^{\infty} B^2(x) F(x) F'(x) dx \\
+ \frac{1}{4} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} K(x, y) L(x, y) F'(x) F(y) dy dx \\
- \frac{1}{16} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} K(x, y) B(x) B(y) F(x) F'(y) dy dx - \frac{1}{32} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} L(x, y) L(y, x) F'(x) F'(y) dy dx \\
+ \frac{1}{16} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} L(x, y) \operatorname{Ai}(y) B(x) F'(x) F(y) dy dx + \frac{1}{64} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} L(x, y) B(y) B(x) F'(x) F'(y) dy dx \\
- \frac{1}{32} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \operatorname{Ai}(x) \operatorname{Ai}(y) B(x) B(y) F(x) F(y) dy dx \\
- \frac{1}{512} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} B^2(x) B^2(y) F'(x) F'(y) dy dx + O(N^{-\frac{1}{4}}),
\]

where \( \mu_N^{(\text{GUE})} \) and \( \nu_N^{(\text{GUE})} \) are given by (2.6) and (2.7), respectively.
2.3 Gaussian Orthogonal Ensemble

It is convenient in this case to choose \( w(x) \) to be the square root of the Gaussian weight,

\[
w(x) = e^{-\frac{x^2}{2}}, \quad x \in \mathbb{R},
\]

and keep in mind that \( N \) is even. Define

\[
\psi_{2n+1}(x) := \frac{d}{dx} \varphi_{2n}(x), \quad \psi_{2n}(x) := \varphi_{2n}(x), \quad n = 0, 1, 2, \ldots,
\]

where \( \varphi_j(x) \) is given by (2.1). It follows that \( M^{(1)} \) is the direct sum of the \( \frac{N}{2} \) copies of \( \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \) and \( (M^{(1)})^{-1} = -M^{(1)} \) (see [12, 22]). From Lemma 1.3 we obtain the following result [16].

**Theorem 2.7.** For the Gaussian orthogonal ensemble, we have

\[
\left[ G_N^{(1)}(f) \right]^2 = \det(I + T_{\text{GOE}}),
\]

where

\[
T_{\text{GOE}} := K_N^{(2)}(f^2 + 2f) - K_N^{(2)} \varepsilon f' - K_N^{(2)} f \varepsilon f' + \sqrt{\frac{N}{2}} (\varepsilon \varphi_N) \otimes \varphi_{N-1} (f^2 + 2f)
\]

\[
+ \sqrt{\frac{N}{2}} (\varepsilon \varphi_N) \otimes (\varepsilon \varphi_{N-1}) f' - \sqrt{\frac{N}{2}} ((\varepsilon \varphi_N) \otimes \varphi_{N-1}) f \varepsilon f',
\]

and \( K_N^{(2)} \) is an operator on \( L^2(\mathbb{R}) \) with kernel

\[
K_N^{(2)}(x, y) = \sum_{j=0}^{N-1} \varphi_j(x) \varphi_j(y).
\]

We also have

\[
\log \det(I + T_{\text{GOE}}) = \Tr \log(I + T_{\text{GOE}}) = \Tr T_{\text{GOE}} - \frac{1}{2} \Tr T_{\text{GOE}}^2 + \cdots.
\]

Similarly as the previous subsection, we have the following results.

**Theorem 2.8.** As \( N \to \infty \),

\[
\varphi_N \left( \sqrt{2N} + 2^{-\frac{1}{2}} N^{-\frac{1}{8}} x \right) = 2^{\frac{3}{4}} N^{-\frac{1}{4}} \text{Ai}(x) + O(N^{-\frac{3}{4}}),
\]

\[
\varphi_{N-1} \left( \sqrt{2N} + 2^{-\frac{1}{2}} N^{-\frac{1}{8}} x \right) = 2^{\frac{3}{4}} N^{-\frac{1}{4}} \text{Ai}(x) + O(N^{-\frac{3}{4}}),
\]

\[
\varepsilon \varphi_N \left( \sqrt{2N} + 2^{-\frac{1}{2}} N^{-\frac{1}{8}} x \right) = 2^{-\frac{3}{4}} N^{-\frac{1}{4}} B(x) + O(N^{-\frac{1}{16}}),
\]

\[
\varepsilon \varphi_{N-1} \left( \sqrt{2N} + 2^{-\frac{1}{2}} N^{-\frac{1}{8}} x \right) = 2^{-\frac{3}{4}} N^{-\frac{1}{4}} B(x) + O(N^{-\frac{1}{16}}),
\]

where \( B(x) \) is given by (1.2).
In the computations below, we replace $f(x)$ by $f\left(2^{\frac{3}{2}}N^\frac{1}{2}\left(x - \sqrt{2N}\right)\right)$ and $f'(x)$ by $2^{\frac{3}{2}}N^\frac{1}{2}f'\left(2^{\frac{3}{2}}N^\frac{1}{2}\left(x - \sqrt{2N}\right)\right)$. Using Theorem 2.1 and Theorem 2.8 to compute Tr$T_{\text{GOE}}$ and Tr$T_{\text{GOE}}^2$ as $N \to \infty$, we obtain the following results:

\[
\begin{align*}
\text{Tr } T_{\text{GOE}} &= \int_{-\infty}^{\infty} K(x, x)(f^2(x) + 2f(x))dx - \frac{1}{2} \int_{-\infty}^{\infty} L(x, x)f'(x)dx \\
&\quad - \frac{1}{2} \int_{-\infty}^{\infty} dx f'(x) \int_{-\infty}^{\infty} (1 - 2\chi(-\infty, x)(y))K(x, y)f(y)dy \\
&\quad + \frac{1}{4} \int_{-\infty}^{\infty} \text{Ai}(x)B(x)(f^2(x) + 2f(x))dx + \frac{1}{8} \int_{-\infty}^{\infty} B^2(x)f'(x)dx \\
&\quad - \frac{1}{8} \int_{-\infty}^{\infty} dx B(x)f'(x) \int_{-\infty}^{\infty} (1 - 2\chi(-\infty, x)(y))\text{Ai}(y)f(y)dy + O(N^{-\frac{1}{2}}), \quad (2.13)
\end{align*}
\]

\[
\begin{align*}
\text{Tr } T_{\text{GOE}}^2 &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} K^2(x, y)(f^2(x) + 2f(x))(f^2(y) + 2f(y))dxdy \\
&\quad - \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} K(x, y)L(x, y)f'(x)(f^2(y) + 2f(y))dxdy \\
&\quad + \frac{1}{2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} K(x, y)\text{Ai}(x)B(y)(f^2(x) + 2f(x))(f^2(y) + 2f(y))dxdy \\
&\quad + \frac{1}{4} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} K(x, y)B(x)B(y)f'(x)(f^2(y) + 2f(y))dxdy \\
&\quad + \frac{1}{4} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} L(x, y)L(y, x)f'(x)f'(y)dxdy \\
&\quad - \frac{1}{4} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} L(x, y)\text{Ai}(y)B(x)f'(x)(f^2(y) + 2f(y))dxdy \\
&\quad - \frac{1}{8} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} L(x, y)B(x)B(y)f'(x)f'(y)dxdy \\
&\quad + \frac{1}{16} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \text{Ai}(x)\text{Ai}(y)B(x)B(y)(f^2(x) + 2f(x))(f^2(y) + 2f(y))dxdy \\
&\quad + \frac{1}{16} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \text{Ai}(x)B(x)B^2(y)(f^2(x) + 2f(x))f'(y)dxdy \\
&\quad + \frac{1}{64} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} B^2(x)B^2(y)f'(x)f'(y)dxdy + R + O(N^{-\frac{1}{2}}), \quad (2.14)
\end{align*}
\]

where $R$ contains the terms of integrals with integrands consisting of $f$, $f$, $f'$ or $f$, $f'$, $f'$. These lead to at least power 3 of $\lambda$ in the following discussions, and they will not affect the final results, so we need not write down the detailed results of $R$.

Similarly as the previous subsection, we replace $f(x)$ with $-\lambda F(x) + \frac{\lambda^2}{2} F^2(x)$ and $f'(x)$ with
\[-\lambda F'(x) + \lambda^2 F(x)F'(x) \text{.} \] Substituting these into (2.13) and (2.14), and we finally find

\[
\log \det(I + T_{\text{GOE}}) = -\lambda \left\{ 2 \int_{-\infty}^{\infty} K(x, x)F(x)dx - \frac{1}{2} \int_{-\infty}^{\infty} L(x, x)F'(x)dx + \frac{1}{4} \int_{-\infty}^{\infty} A(x)B(x)F(x)dx + \frac{1}{16} \int_{-\infty}^{\infty} B^2(x)F'(x)dx \right\} + \lambda^2 \left\{ \frac{1}{4} \int_{-\infty}^{\infty} K(x, x)F^2(x)dx - \frac{1}{2} \int_{-\infty}^{\infty} L(x, x)F(x)F'(x)dx \
- \int_{-\infty}^{\infty} dx F'(x) \int_{-\infty}^{\infty} (1 - 2\chi_{(-\infty,x)}(y))K(x, y)F(y)dy + \int_{-\infty}^{\infty} A(x)B(x)F^2(x)dx \
+ \frac{1}{4} \int_{-\infty}^{\infty} B^2(x)F(x)F'(x)dx - \frac{1}{4} \int_{-\infty}^{\infty} dx B(x)F'(x) \int_{-\infty}^{\infty} (1 - 2\chi_{(-\infty,x)}(y))A(x)F(y)dy \
- 2 \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} K(x, y)A(x)B(x)F(y)F'(y)dydx - \frac{1}{2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} K(x, y)B(x)B(y)F'(x)F'(y)dydx \
- \frac{1}{4} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} L(x, y)B(x)B(y)F'(x)F'(y)dydx + \frac{1}{2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} L(x, y)A(x)B(x)F'(x)F'(y)dydx \
+ \frac{1}{8} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} A(x)B(x)B^2(y)F(x)F'(y)dydx \right\} + O(N^{-\frac{3}{2}}). \]

Let \( \mu_N^{(\text{GOE})} \) and \( \nu_N^{(\text{GOE})} \) be the mean and variance of the linear statistics \( \sum_{j=1}^N F \left( 2 \frac{j}{N} + \theta \left( x_j - \sqrt{2N} \right) \right) \).

Noting that \( G^{(1)}(f) = \frac{1}{2} \log \det(I + T_{\text{GOE}}) \), we have the following theorem.

**Theorem 2.9.** As \( N \to \infty \),

\[
\mu_N^{(\text{GOE})} = \mu_N^{(\text{GUE})} - \frac{1}{4} \int_{-\infty}^{\infty} L(x, x)F'(x)dx + \frac{1}{4} \int_{-\infty}^{\infty} A(x)B(x)F(x)dx + \frac{1}{16} \int_{-\infty}^{\infty} B^2(x)F'(x)dx + O(N^{-\frac{3}{2}}),
\]

\[
\nu_N^{(\text{GOE})} = 2\nu_N^{(\text{GUE})} - \frac{1}{2} \int_{-\infty}^{\infty} L(x, x)F(x)F'(x)dx - \frac{1}{2} \int_{-\infty}^{\infty} dx F'(x) \int_{-\infty}^{\infty} (1 - 2\chi_{(-\infty,x)}(y))K(x, y)F(y)dy \
+ \int_{-\infty}^{\infty} A(x)B(x)F^2(x)dx - \frac{1}{8} \int_{-\infty}^{\infty} dx B(x)F'(x) \int_{-\infty}^{\infty} (1 - 2\chi_{(-\infty,x)}(y))A(y)F(y)dy \
+ \frac{1}{8} \int_{-\infty}^{\infty} B^2(x)F(x)F'(x)dx - \frac{1}{16} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} A(x)B(x)B^2(y)F'(x)F'(y)dydx \
- \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} K(x, y)A(x)B(y)F(x)F'(y)dydx - \frac{1}{4} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} K(x, y)B(x)B(y)F'(x)F'(y)dydx \
- \frac{1}{8} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} L(x, y)B(x)B(y)F'(x)F'(y)dydx + \frac{1}{2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} L(x, y)A(x)B(x)F'(x)F'(y)dydx \
+ \frac{1}{16} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} L(x, y)B(x)B(y)F'(x)F'(y)dydx - \frac{1}{8} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} A(x)A(y)B(x)B(y)F(x)F(y)dydx \
+ \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} K(x, y)L(x, y)F(x)F'(y)dydx - \frac{1}{128} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} B^2(x)B^2(y)F'(x)F'(y)dydx + O(N^{-\frac{3}{2}}),
\]

where \( \mu_N^{(\text{GUE})} \) and \( \nu_N^{(\text{GUE})} \) are given by (2.6) and (2.7), respectively.
3 Laguerre Ensembles

3.1 Laguerre Unitary Ensemble

In the Laguerre case, $w(x) = x^\alpha e^{-x}$, $\alpha > -1$, $x \in \mathbb{R}^+$. From Lemma 1.1 we have

$$\varphi_j(x) = \sqrt{\frac{\Gamma(j+1)}{\Gamma(j+\alpha+1)}} L_j^{(\alpha)}(x) x^\alpha e^{-\frac{x}{2}}, \quad j = 0, 1, 2, \ldots,$$

where $L_j^{(\alpha)}(x)$ are the Laguerre polynomials of degree $j$.

We state a theorem before our discussion.

**Theorem 3.1.** As $N \to \infty$,

$$2^{\frac{3}{4}} N^{\frac{1}{4}} K_N^{(2)}(4N + 2\alpha + 2 + 2^{\frac{3}{4}} N^{\frac{3}{4}} x, 4N + 2\alpha + 2 + 2^{\frac{3}{4}} N^{\frac{3}{4}} y) = K(x, y) + O(N^{-\frac{1}{3}}),$$

where $K(x, y)$ is the Airy kernel (1.3).

**Proof.** From the asymptotic formula of Laguerre polynomials [20] (page 201),

$$e^{-\frac{t}{2}} L_n^{(\alpha)}(x) = (-1)^n 2^{-\alpha - \frac{1}{2}} n^{-\frac{3}{4}} \text{Ai}(3^{-\frac{1}{4}} t) + O(n^{-1}),$$

where

$$x = 4n + 2\alpha + 2 - 2 \left(\frac{2n}{3}\right)^{\frac{1}{3}} t,$$

we have

$$e^{-\frac{t}{2}} L_n^{(\alpha)}(x) = (-1)^n 2^{-\alpha - \frac{1}{2}} n^{-\frac{3}{4}} \text{Ai}(2^{-\frac{4}{3}} n^{-\frac{1}{2}} (x - 4n - 2\alpha - 2)) + O(n^{-1}). \quad (3.1)$$

Using the Christoffel-Darboux formula,

$$K_N^{(2)}(x, y) = -\frac{N!}{\Gamma(N + \alpha)} \frac{x^{\frac{3}{2}} e^{-\frac{x}{2}} L_N^{(\alpha)}(x) y^{\frac{3}{2}} e^{-\frac{y}{2}} L_N^{(\alpha)}(y) - y^{\frac{3}{2}} e^{-\frac{y}{2}} L_N^{(\alpha)}(y) x^{\frac{3}{2}} e^{-\frac{x}{2}} L_N^{(\alpha)}(x)}{x - y}.$$

Replacing the variables $x$ by $4N + 2\alpha + 2 + 2^{\frac{3}{4}} N^{\frac{3}{4}} x$ and $y$ by $4N + 2\alpha + 2 + 2^{\frac{3}{4}} N^{\frac{3}{4}} y$, and using (3.1) together with Stirling’s formula, we obtain the desired result after some elaborate computations. \qed

**Remark.** The above result was obtained by Forrester [13], but they also did not show the order term.
We now use Theorem 3.1 to compute (2.2) term by term as \( N \to \infty \). We replace \( f(x) \) by 
\[
 f\left(2^{-\frac{4}{3}}N^{-\frac{3}{4}}(x - 4N - 2\alpha - 2)\right)
\]
in the following computations. The first term reads,
\[
 \text{Tr}K_N^{(2)} f = \int_0^\infty K_N^{(2)}(x, x)f\left(2^{-\frac{4}{3}}N^{-\frac{3}{4}}(x - 4N - 2\alpha - 2)\right)dx
\]
\[
 = \int_{-\frac{4}{3}N^{-\frac{3}{4}}(4N+2\alpha+2)}^\infty 2^{\frac{4}{3}}N^{\frac{2}{3}}K_N^{(2)}(4N + 2\alpha + 2 + 2^{\frac{4}{3}}N^{\frac{1}{3}}x, 4N + 2\alpha + 2 + 2^{\frac{4}{3}}N^{\frac{1}{3}}x)f(x)dx
\]
\[
 = \int_{-\infty}^\infty K(x, x)f(x)dx + O(N^{-\frac{1}{3}}).
\]
The second term,
\[
 \text{Tr}\left(K_N^{(2)} f\right)^2 = \int_0^\infty \int_0^\infty K_N^{(2)}(x, y)f\left(2^{-\frac{4}{3}}N^{-\frac{3}{4}}(y - 4N - 2\alpha - 2)\right)K_N^{(2)}(y, x)
\]
\[
 f\left(2^{-\frac{4}{3}}N^{-\frac{3}{4}}(x - 4N - 2\alpha - 2)\right)dx dy
\]
\[
 = \int_{-\infty}^\infty \int_{-\infty}^\infty K^2(x, y)f(x)f(y)dx dy + O(N^{-\frac{1}{3}}).
\]
It follows from (2.2) that
\[
 \log \det \left( I + K_N^{(2)} f \right) = \int_{-\infty}^\infty K(x, x)f(x)dx - \frac{1}{2} \int_{-\infty}^\infty \int_{-\infty}^\infty K^2(x, y)f(x)f(y)dx dy + \cdots + O(N^{-\frac{1}{3}}).
\]

We proceed to study the mean and variance of the scaled linear statistics \( \sum_{j=1}^N F\left(2^{-\frac{4}{3}}N^{-\frac{3}{4}}(x - 4N - 2\alpha - 2)\right) \). From the relation (2.5), we have
\[
 \log \det \left( I + K_N^{(2)} f \right) = -\lambda \int_{-\infty}^\infty K(x, x)F(x)dx
\]
\[
 + \frac{\lambda^2}{2} \left[ \int_{-\infty}^\infty K(x, x)F^2(x)dx - \int_{-\infty}^\infty \int_{-\infty}^\infty K^2(x, y)F(x)F(y)dx dy \right] + \cdots + O(N^{-\frac{1}{3}}).
\]
Let \( \mu_N^{(\text{LUE})} \) and \( V_N^{(\text{LUE})} \) be the mean and variance of the linear statistics \( \sum_{j=1}^N F\left(2^{-\frac{4}{3}}N^{-\frac{3}{4}}(x - 4N - 2\alpha - 2)\right) \). Then we have the following theorem.

**Theorem 3.2.** As \( N \to \infty \),
\[
 \mu_N^{(\text{LUE})} = \int_{-\infty}^\infty K(x, x)F(x)dx + O(N^{-\frac{1}{3}}), \tag{3.2}
\]
\[
 V_N^{(\text{LUE})} = \int_{-\infty}^\infty K(x, x)F^2(x)dx - \int_{-\infty}^\infty \int_{-\infty}^\infty K^2(x, y)F(x)F(y)dx dy + O(N^{-\frac{1}{3}}), \tag{3.3}
\]
where \( K(x, y) \) is the Airy kernel defined by (1.3).

**Remark.** Comparing Sec. 2.1 and Sec. 3.1, we see that the large \( N \) behavior of the MGF of a suitably scaled linear statistics in GUE are the same with a suitably scaled linear statistics in LUE. It follows that as \( N \to \infty \), the mean and variance of the corresponding linear statistics are also the same in GUE and LUE.
3.2 Laguerre Symplectic Ensemble

For the Laguerre symplectic ensemble, \( w(x) = x^\alpha e^{-x}, \ \alpha > 0, \ x \in \mathbb{R}^+ \). Following [16], we let

\[
\psi_{2j+1}(x) := \frac{1}{\sqrt{2}} x^{\varphi_{2j+1}^{(\alpha-1)}(x)}, \ \psi_{2j}(x) := -\frac{1}{\sqrt{2}} \varepsilon^{(\alpha-1)} \varphi_{2j+1}(x), \ j = 0, 1, 2, \ldots,
\]

where \( \varphi_j^{(\alpha-1)}(x) \) is given by

\[
\varphi_j^{(\alpha-1)}(x) := \sqrt{\frac{\Gamma(j+1)}{\Gamma(j+\alpha)}} L_j^{(\alpha-1)}(x) x^{\frac{\alpha-1}{2}} e^{-\frac{x}{2}}, \ j = 0, 1, 2, \ldots. \tag{3.4}
\]

It follows that \( M^{(4)} \) is the direct sum of the \( N \) copies of \( \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \) and \( (M^{(4)})^{-1} = -M^{(4)} \). From Lemma 1.2 we obtain the following results [16].

**Theorem 3.3.** For the Laguerre symplectic ensemble,

\[
\left[ G_N^{(4)}(f) \right]^2 = \det(I + T_{\text{LSE}}),
\]

where

\[
T_{\text{LSE}} := S_N^{(4)} f - \frac{1}{2} S_N^{(4)} \varepsilon f' - \sqrt{\left( N + \frac{1}{2} \right) \left( N + \frac{\alpha}{2} \right) \varepsilon \varphi_{2N+1}^{(\alpha-1)} \otimes (\varepsilon \varphi_{2N}^{(\alpha-1)})} f
- \frac{1}{2} \sqrt{\left( N + \frac{1}{2} \right) \left( N + \frac{\alpha}{2} \right) \varepsilon \varphi_{2N+1}^{(\alpha-1)} \otimes (\varepsilon \varphi_{2N}^{(\alpha-1)})} f', \tag{3.5}
\]

and

\[
S_N^{(4)}(x, y) = \sum_{j=0}^{2N} x^{\varphi_j^{(\alpha-1)}(x)} y^{\varphi_j^{(\alpha-1)}(y)}.
\]

We also have the following expansion formula,

\[
\log \det(I + T_{\text{LSE}}) = \text{Tr} \log(I + T_{\text{LSE}}) = \text{Tr} T_{\text{LSE}} - \frac{1}{2} \text{Tr} T_{\text{LSE}}^2 + \cdots.
\]

Using the similar method in Theorem 3.1, we obtain the following theorem.

**Theorem 3.4.** As \( N \to \infty \),

\[
2^{\frac{5}{3}} N^\frac{1}{3} S_N^{(4)}(8N + 2\alpha + 2^{\frac{5}{3}} N^\frac{2}{3} x, 8N + 2\alpha + 2^{\frac{5}{3}} N^\frac{2}{3} y) = K(x, y) + O(N^{-\frac{1}{3}}),
\]

where \( K(x, y) \) is the Airy kernel (1.3).
Theorem 3.5. As $N \to \infty$, we have

$$\varphi_{2N}^{(\alpha-1)}(8N + 2\alpha + 2^{\frac{5}{2}}N^{\frac{3}{4}}x) = 2^{-\frac{13}{8}} N^{-\frac{5}{8}} \text{Ai}(x) + O(N^{-\frac{7}{8}}),$$

$$\varphi_{2N+1}^{(\alpha-1)}(8N + 2\alpha + 2^{\frac{5}{2}}N^{\frac{3}{4}}x) = -2^{-\frac{13}{8}} N^{-\frac{5}{8}} \text{Ai}(x) + O(N^{-\frac{7}{8}}),$$

$$\varepsilon \varphi_{2N}^{(\alpha-1)}(8N + 2\alpha + 2^{\frac{5}{2}}N^{\frac{3}{4}}x) = -2^{-\frac{3}{2}} N^{-\frac{1}{2}} B(x) + O(N^{-\frac{7}{2}}),$$

$$\varepsilon \varphi_{2N+1}^{(\alpha-1)}(8N + 2\alpha + 2^{\frac{5}{2}}N^{\frac{3}{4}}x) = -2^{-\frac{3}{2}} N^{-\frac{1}{2}} B(x) + O(N^{-\frac{7}{2}}),$$

where $B(x)$ is given by (1.2).

Proof. From the definition (3.4) and the asymptotics (3.1), we have

$$\varphi_{2N}^{(\alpha-1)}(8N + 2\alpha + 2^{\frac{5}{2}}N^{\frac{3}{4}}x) = \sqrt{\frac{\Gamma(2N+1)}{\Gamma(2N+\alpha)}} (8N + 2\alpha + 2^{\frac{5}{2}}N^{\frac{3}{4}}x)^{\frac{3}{2} - 1} \left(2^{-\alpha+\frac{1}{2}} N^{-\frac{5}{8}} \text{Ai}(x) + O(N^{-1})\right)$$

$$= 2^{-\frac{13}{8}} N^{-\frac{5}{8}} \text{Ai}(x) + O(N^{-\frac{7}{8}}),$$

where we have made use of the formula \(\Pi\) (page 257)

$$\frac{\Gamma(n+a)}{\Gamma(n+b)} = n^{a-b}(1 + O(n^{-1})), \ n \to \infty.$$

It follows that

$$\varepsilon \varphi_{2N}^{(\alpha-1)}(8N + 2\alpha + 2^{\frac{5}{2}}N^{\frac{3}{4}}x)$$

$$= \frac{1}{2} \left( \int_{0}^{8N+2\alpha+2^{\frac{5}{2}}N^{\frac{3}{4}}x} \varphi_{2N}^{(\alpha-1)}(y)dy - \int_{8N+2\alpha+2^{\frac{5}{2}}N^{\frac{3}{4}}x}^{\infty} \varphi_{2N}^{(\alpha-1)}(y)dy \right)$$

$$= 2^{\frac{3}{2}} N^{\frac{3}{4}} \left( \int_{-2^{-\frac{7}{8}} \sqrt{8N+2\alpha} N^{-\frac{5}{8}} \text{Ai}(y) + O(N^{-\frac{3}{2}})}^{\infty} \varphi_{2N}^{(\alpha-1)}(8N + 2\alpha + 2^{\frac{5}{2}}N^{\frac{3}{4}}y)dy - \int_{x}^{\infty} \varphi_{2N}^{(\alpha-1)}(8N + 2\alpha + 2^{\frac{5}{2}}N^{\frac{3}{4}}y)dy \right)$$

$$= 2^{\frac{3}{2}} N^{\frac{3}{4}} \left( \int_{-\infty}^{x} \left(2^{-\frac{13}{8}} N^{-\frac{7}{8}} \text{Ai}(y) + O(N^{-\frac{3}{2}})\right)dy - \int_{x}^{\infty} \left(2^{-\frac{13}{8}} N^{-\frac{7}{8}} \text{Ai}(y) + O(N^{-\frac{3}{2}})\right)dy \right)$$

$$= 2^{-\frac{3}{2}} N^{-\frac{1}{2}} B(x) + O(N^{-\frac{7}{2}}).$$

Similarly, we obtain

$$\varphi_{2N+1}^{(\alpha-1)}(8N + 2\alpha + 2^{\frac{5}{2}}N^{\frac{3}{4}}x) = -2^{-\frac{13}{8}} N^{-\frac{5}{8}} \text{Ai}(x) + O(N^{-\frac{7}{8}})$$

and

$$\varepsilon \varphi_{2N+1}^{(\alpha-1)}(8N + 2\alpha + 2^{\frac{5}{2}}N^{\frac{3}{4}}x) = -2^{-\frac{3}{2}} N^{-\frac{1}{2}} B(x) + O(N^{-\frac{7}{2}}).$$

The proof is complete.
Now we use Theorem 3.3 and 3.4 to compute $\text{Tr } T_{LSE}$ and $\text{Tr } T_{LSE}^2$ as $N \to \infty$. We change $f(x)$ to $f(2^{-\frac{3}{4}}N^{-\frac{1}{8}}(x - 8N - 2\alpha))$ in the following computations. Firstly we have

$$
\text{Tr } T_{LSE} = \text{Tr } S_N^{(4)} f - \text{Tr } \frac{1}{2} S_N^{(4)} \varepsilon f' - \text{Tr } \sqrt{\left( N + \frac{1}{2} \right) \left( N + \frac{\alpha}{2} \right)} \left( \varepsilon \varphi_{2N+1}^{(\alpha-1)} \otimes \varphi_{2N}^{(\alpha-1)} \right) f
$$

The first term reads,

$$
\text{Tr } S_N^{(4)} f = \int_0^\infty S_N^{(4)}(x, x) f(2^{-\frac{3}{4}}N^{-\frac{1}{8}}(x - 8N - 2\alpha)) dx
$$

The second term,

$$
\text{Tr } \frac{1}{2} S_N^{(4)} \varepsilon f' = 2^{-\frac{3}{4}}N^{-\frac{1}{8}} \int_0^\infty \int_0^\infty S_N^{(4)}(x, y) \varepsilon(y, x) f'(2^{-\frac{3}{4}}N^{-\frac{1}{8}}(x - 8N - 2\alpha)) dx dy
$$

Let

$$
x = 8N + 2\alpha + 2\frac{3}{4}N^{\frac{3}{4}} u, \quad y = 8N + 2\alpha + 2\frac{3}{4}N^{\frac{3}{4}} v,
$$

then

$$
\text{Tr } \frac{1}{2} S_N^{(4)} \varepsilon f' = \frac{1}{4} \int_{-\infty}^\infty \left( \int_{-\infty}^\infty K(u, v) dv - \int_{-\infty}^u K(u, v) dv \right) f'(u) du + O(N^{-\frac{1}{3}})
$$

The third term,

$$
\text{Tr } \sqrt{\left( N + \frac{1}{2} \right) \left( N + \frac{\alpha}{2} \right)} \left( \varepsilon \varphi_{2N+1}^{(\alpha-1)} \otimes \varphi_{2N}^{(\alpha-1)} \right) f
$$

$$
= \sqrt{\left( N + \frac{1}{2} \right) \left( N + \frac{\alpha}{2} \right)} \int_0^\infty \varepsilon \varphi_{2N+1}^{(\alpha-1)}(x) \varphi_{2N}^{(\alpha-1)}(x) f(2^{-\frac{3}{4}}N^{-\frac{1}{8}}(x - 8N - 2\alpha)) dx
$$

$$
= 2^{\frac{3}{4}}N^{\frac{3}{4}} \sqrt{\left( N + \frac{1}{2} \right) \left( N + \frac{\alpha}{2} \right)} \int_{-2^{-\frac{3}{4}}N^{-\frac{1}{8}}(4N+\alpha)}^\infty \varepsilon \varphi_{2N+1}^{(\alpha-1)}(8N + 2\alpha + 2\frac{3}{4}N^{\frac{3}{4}} x) \varphi_{2N}^{(\alpha-1)}(8N + 2\alpha + 2\frac{3}{4}N^{\frac{3}{4}} x) dx
$$

$$
- \frac{1}{4} \int_{-\infty}^\infty \text{Ai}(x) B(x) f(x) dx + O(N^{-\frac{3}{4}}).
$$
The fourth term,
\[
\text{Tr } \frac{1}{2} \sqrt{\left( N + \frac{1}{2} \right) \left( N + \frac{\alpha}{2} \right) \left( \varphi_{2N+1}^{(\alpha-1)} \right) \otimes \left( \varphi_{2N}^{(\alpha-1)} \right) f'}
\]
\[
= 2^{-\frac{\alpha}{4}} N^{-\frac{1}{4}} \sqrt{\left( N + \frac{1}{2} \right) \left( N + \frac{\alpha}{2} \right) \int_0^\infty \varphi_{2N+1}^{(\alpha-1)}(x) \varphi_{2N}^{(\alpha-1)}(x) f'(2^{-\frac{\alpha}{4}} N^{-\frac{1}{4}} (x - 8N - 2\alpha)) dx}
\]
\[
= -\frac{1}{16} \int_{-\infty}^\infty B^2(x) f'(x) dx + O(N^{-\frac{3}{4}}).
\]

Hence,
\[
\text{Tr } T_{\text{LSE}} = \int_{-\infty}^\infty K(x, x) f(x) dx - \frac{1}{4} \int_{-\infty}^\infty L(x, x) f'(x) dx + \frac{1}{4} \int_{-\infty}^\infty \text{Ai}(x) B(x) f(x) dx + \frac{1}{16} \int_{-\infty}^\infty B^2(x) f'(x) dx + O(N^{-\frac{3}{4}}).
\]  
\[\text{(3.6)}\]

Similarly we obtain the result for \text{Tr } T_{\text{LSE}}^2 after some tedious computations,
\[
\text{Tr } T_{\text{LSE}}^2 = \int_{-\infty}^\infty \int_{-\infty}^\infty K(x, y) f(x) f(y) dxdy - \frac{1}{2} \int_{-\infty}^\infty \int_{-\infty}^\infty K(x, y) L(x, y) f'(x) f(y) dxdy
\]
\[
+ \frac{1}{2} \int_{-\infty}^\infty \int_{-\infty}^\infty K(x, y) \text{Ai}(x) B(y) f(x) f(y) dxdy + \frac{1}{8} \int_{-\infty}^\infty \int_{-\infty}^\infty K(x, y) B(x) B(y) f'(x) f(y) dxdy
\]
\[
+ \frac{1}{16} \int_{-\infty}^\infty \int_{-\infty}^\infty L(x, y) L(x, y) f'(x) f'(y) dxdy - \frac{1}{8} \int_{-\infty}^\infty \int_{-\infty}^\infty L(x, y) \text{Ai}(y) B(x) f'(x) f(y) dxdy
\]
\[
- \frac{1}{32} \int_{-\infty}^\infty \int_{-\infty}^\infty L(x, y) B(x) B(y) f'(x) f'(y) dxdy + \frac{1}{16} \int_{-\infty}^\infty \int_{-\infty}^\infty \text{Ai}(x) \text{Ai}(y) B(x) B(y) f(x) f(y) dxdy
\]
\[
+ \frac{1}{32} \int_{-\infty}^\infty \int_{-\infty}^\infty \text{Ai}(x) B(x) B^2(y) f(x) f'(y) dxdy + \frac{1}{256} \int_{-\infty}^\infty \int_{-\infty}^\infty B^2(x) B^2(y) f'(x) f'(y) dxdy + O(N^{-\frac{3}{4}}).
\]
\[\text{(3.7)}\]

From the above, we find that as \( N \to \infty \),
\[
\text{Tr } T_{\text{LSE}} = \text{Tr } T_{\text{GSE}}, \quad \text{Tr } T_{\text{LSE}}^2 = \text{Tr } T_{\text{GSE}}^2.
\]

It follows that as \( N \to \infty \),
\[
\log \det(I + T_{\text{LSE}}) = \log \det(I + T_{\text{GSE}}).
\]

Let \( \mu_N^{(\text{LSE})} \) and \( \nu_N^{(\text{LSE})} \) be the mean and variance of the linear statistics \( \sum_{j=1}^N F \left( 2^{-\frac{\alpha}{4}} N^{-\frac{1}{4}} (x_j - 8N - 2\alpha) \right) \).

We have the following theorem.
Theorem 3.6. As $N \to \infty$,

$$
\mu_N^{(\text{LSE})} = \frac{1}{2} \mu_N^{(\text{LUE})} - \frac{1}{4} \int_{-\infty}^{\infty} L(x, x)F'(x)dx + \frac{1}{8} \int_{-\infty}^{\infty} \text{Ai}(x)B(x)F(x)dx + \frac{1}{32} \int_{-\infty}^{\infty} B^2(x)F'(x)dx + O(N^{-\frac{1}{2}}),
$$

and

$$
\nu_N^{(\text{LSE})} = \frac{1}{2} \nu_N^{(\text{LUE})} - \frac{1}{4} \int_{-\infty}^{\infty} L(x, x)F'(x)dx + \frac{1}{8} \int_{-\infty}^{\infty} \text{Ai}(x)B(x)F^2(x)dx + \frac{1}{16} \int_{-\infty}^{\infty} B^2(x)F(x)F'(x)dx \\
+ \frac{1}{4} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} K(x, y)L(x, y)F'(x)F'(y)dx dy - \frac{1}{4} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} K(x, y)\text{Ai}(x)B(x)F(x)F(y)dx dy \\
- \frac{1}{16} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} K(x, y)B(x)B(y)F'(x)F'(y)dx dy - \frac{1}{32} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} L(x, y)\text{Ai}(x)B(y)F(x)F'(y)dx dy \\
+ \frac{1}{16} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} L(x, y)\text{Ai}(x)B(y)B(x)F'(x)F'(y)dx dy + \frac{1}{64} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} L(x, y)B(x)B(y)F'(x)F'(y)dx dy \\
- \frac{1}{32} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \text{Ai}(x)\text{Ai}(y)B(x)B(y)F(x)F(y)dx dy - \frac{1}{64} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \text{Ai}(x)B(x)B^2(y)F(x)F'(y)dx dy \\
- \frac{1}{512} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} B^2(x)B^2(y)F'(x)F'(y)dx dy + O(N^{-\frac{1}{2}}),
$$

where $\mu_N^{(\text{LUE})}$ and $\nu_N^{(\text{LUE})}$ are given by (3.2) and (3.3), respectively.

**Remark.** We find that the large $N$ behavior of the MGF of a suitably scaled linear statistics in LSE are the same with a suitably scaled linear statistics in GSE. It follows that as $N \to \infty$, the mean and variance of the corresponding linear statistics are also the same in LSE and GSE.

### 3.3 Laguerre Orthogonal Ensemble

In this subsection, $w(x)$ is taken to be the *square root* of the Laguerre weight, namely,

$$
w(x) = x^{\frac{\alpha}{2}} e^{-\frac{x}{2}}, \quad \alpha > -2, \quad x \in \mathbb{R}^+,
$$

and $N$ is even. Following [16], we let

$$
\psi_{2n+1}(x) := \frac{d}{dx}(x^{\alpha} \varphi_{2n}(x)), \quad \psi_{2n}(x) := \varphi_{2n}(x), \quad n = 0, 1, 2, \ldots,
$$

where $\varphi_j^{(\alpha+1)}(x)$ is given by

$$
\varphi_j^{(\alpha+1)}(x) := \sqrt{\frac{\Gamma(j+1)}{\Gamma(j+\alpha+2)}} L_j^{(\alpha+1)}(x) x^{\frac{\alpha}{2}} e^{-\frac{x}{2}}, \quad j = 0, 1, 2, \ldots.
$$

Note that the definition of $\varphi_j^{(\alpha+1)}(x)$ coincides with the LSE case if we replace $\alpha$ with $\alpha + 2$ there.

It follows that $M^{(1)}$ is the direct sum of the $\frac{N}{2}$ copies of \( \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \) and \( (M^{(1)})^{-1} = -M^{(1)} \). From Lemma 1.3, we obtain the following result [16].
Theorem 3.7. For the Laguerre orthogonal ensemble,

\[ \left[ G^{(1)}_N(f) \right]^2 = \det(I + T_{\text{LOE}}), \]

where

\[
T_{\text{LOE}} : = S^{(1)}_N(f^2 + 2f) - S^{(1)}_N \varepsilon f' - S^{(1)}_N f \varepsilon f' - \frac{1}{2} \sqrt{N} (N + \alpha + 1)(\varepsilon \varphi^{(\alpha+1)}_N) \otimes \varphi^{(\alpha+1)}_{N-1} (f^2 + 2f) \\
- \frac{1}{2} \sqrt{N} (N + \alpha + 1)(\varepsilon \varphi^{(\alpha+1)}_N) \otimes \varepsilon \varphi^{(\alpha+1)}_{N-1} f \varepsilon f',
\]

and \( S^{(1)}_N \) is an integral operator with kernel

\[
S^{(1)}_N(x, y) = \sum_{j=0}^{N-1} x \varphi^{(\alpha+1)}_j(x) \varphi^{(\alpha+1)}_j(y).
\]

We also have the following expansion formula,

\[
\log \det(I + T_{\text{LOE}}) = \text{Tr} \log(I + T_{\text{LOE}}) = \text{Tr} T_{\text{LOE}} - \frac{1}{2} \text{Tr} T_{\text{LOE}}^2 + \cdots.
\]

Similarly as previous subsection, we have the following theorems.

Theorem 3.8. As \( N \to \infty \),

\[
2^{\frac{4}{3}} N^\frac{1}{3} S^{(1)}_N(4N + 2\alpha + 4 + 2^{\frac{4}{3}} N^{\frac{4}{3}} x, 4N + 2\alpha + 4 + 2^{\frac{4}{3}} N^{\frac{4}{3}} y) = K(x, y) + O(N^{-\frac{1}{3}}),
\]

where \( K(x, y) \) is the Airy kernel \( I.3 \).

Theorem 3.9. As \( N \to \infty \), we have

\[
\varphi^{(\alpha+1)}_N(4N + 2\alpha + 4 + 2^{\frac{4}{3}} N^{\frac{4}{3}} x) = 2^{-\frac{4}{3}} N^{-\frac{8}{3}} \text{Ai}(x) + O(N^{-\frac{2}{3}}),
\]

\[
\varphi^{(\alpha+1)}_{N-1}(4N + 2\alpha + 4 + 2^{\frac{4}{3}} N^{\frac{4}{3}} x) = -2^{-\frac{4}{3}} N^{-\frac{8}{3}} \text{Ai}(x) + O(N^{-\frac{2}{3}}),
\]

\[
\varepsilon \varphi^{(\alpha+1)}_N(4N + 2\alpha + 4 + 2^{\frac{4}{3}} N^{\frac{4}{3}} x) = 2^{-1} N^{-\frac{2}{3}} B(x) + O(N^{-\frac{1}{3}}),
\]

\[
\varepsilon \varphi^{(\alpha+1)}_{N-1}(4N + 2\alpha + 4 + 2^{\frac{4}{3}} N^{\frac{4}{3}} x) = -2^{-1} N^{-\frac{1}{3}} B(x) + O(N^{-\frac{1}{3}}).
\]

Now we change \( f(x) \) to \( f(2^{-\frac{4}{3}} N^{-\frac{1}{3}} (x - 4N - 2\alpha - 4)) \) and use Theorem 3.8 and 3.9 to compute \( \text{Tr} T_{\text{LOE}} \) and \( \text{Tr} T_{\text{LOE}}^2 \). We find that as \( N \to \infty \),

\[
\text{Tr} T_{\text{LOE}} = \text{Tr} T_{\text{GOE}}, \quad \text{Tr} T_{\text{LOE}}^2 = \text{Tr} T_{\text{GOE}}^2.
\]
It follows that as $N \to \infty$,

$$\log \det(I + T_{\text{LOE}}) = \log \det(I + T_{\text{GOE}}).$$

Denoting by $\mu^{(\text{LOE})}_N$ and $\mathcal{V}^{(\text{LOE})}_N$ the mean and variance of the scaled linear statistics

$$\sum_{j=1}^N F \left( 2^{\frac{3}{2}} N^{\beta} \left( x_j - \sqrt{2N} \right) \right),$$

we have the following theorem.

**Theorem 3.10.** As $N \to \infty$,

$$\mu^{(\text{LOE})}_N = \mu^{(\text{LUE})}_N - \frac{1}{4} \int_{-\infty}^{\infty} L(x, x) F'(x) dx + \frac{1}{4} \int_{-\infty}^{\infty} \text{Ai}(x) B(x) F(x) dx + \frac{1}{16} \int_{-\infty}^{\infty} B^2(x) F'(x) dx + O(N^{-\frac{1}{4}}),$$

$$\mathcal{V}^{(\text{LOE})}_N = 2 \mathcal{V}^{(\text{LUE})}_N - \frac{1}{2} \int_{-\infty}^{\infty} L(x, x) F'(x) dx - \frac{1}{2} \int_{-\infty}^{\infty} dx F'(x) \int_{-\infty}^{\infty} (1 - 2 \chi_{(-\infty, x)}(y)) K(x, y) F(y) dy$$

$$+ \frac{1}{2} \int_{-\infty}^{\infty} \text{Ai}(x) B(x) F^2(x) dx - \frac{1}{8} \int_{-\infty}^{\infty} dx B(x) F'(x) \int_{-\infty}^{\infty} (1 - 2 \chi_{(-\infty, x)}(y)) \text{Ai}(y) F(y) dy$$

$$+ \frac{1}{8} \int_{-\infty}^{\infty} B^2(x) F(x) F'(x) dx - \frac{1}{16} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \text{Ai}(x) B(x) B^2(y) F(x) F'(y) dy dx$$

$$- \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} K(x, y) \text{Ai}(x) B(y) F(x) F(y) dy dx$$

$$- \frac{1}{8} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} L(x, x) L(y, x) F'(x) F'(y) dy dx$$

$$+ \frac{1}{16} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} L(x, y) B(x) B(y) F'(x) F'(y) dy dx$$

$$- \frac{1}{8} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \text{Ai}(x) \text{Ai}(y) B(x) B(y) F(x) F(y) dy dx$$

$$+ \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} K(x, y) L(x, y) F'(x) F(y) dy dx$$

$$- \frac{1}{128} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} B^2(x) B^2(y) F'(x) F'(y) dy dx + O(N^{-\frac{1}{4}}),$$

where $\mu^{(\text{LUE})}_N$ and $\mathcal{V}^{(\text{LUE})}_N$ are given by (3.2) and (3.3), respectively.

**Remark.** We find that the large $N$ behavior of the MGF of a suitably scaled linear statistics in LOE are the same with a suitably scaled linear statistics in GOE. It follows that as $N \to \infty$, the mean and variance of the corresponding linear statistics are also the same in LOE and GOE.

### 4 Gaussian Unitary Ensemble Continued

For the Gaussian unitary ensemble, if we change $f(x)$ to $f \left( x - \sqrt{2N} \right)$, we can gain a better insight into the mean and variance of the corresponding linear statistics by using the result of Basor and Widom [7]. We see that as $N \to \infty$,

$$\det(I + K^{(2)}_N f) = \det(I + K \tilde{f}),$$
where
\[ \tilde{f}(x) := f \left( \frac{x}{2^{\frac{3}{2}} N^{\frac{1}{6}}} \right). \]

This is because, as \( N \to \infty \),
\[
\begin{align*}
\text{Tr } K_N^{(2)} f &= \int_{-\infty}^{\infty} K_N^{(2)}(x,x)f\left(x - \sqrt{2N}\right)\,dx \\
&= \int_{-\infty}^{\infty} 2^{-\frac{3}{2}} N^{-\frac{1}{6}} K_N^{(2)}(\sqrt{2N} + 2^{-\frac{1}{2}} N^{-\frac{1}{6}} x, \sqrt{2N} + 2^{-\frac{1}{2}} N^{-\frac{1}{6}} x) f \left( \frac{x}{2^{\frac{3}{2}} N^{\frac{1}{6}}} \right) \, dx \\
&= \int_{-\infty}^{\infty} K(x,x)f \left( \frac{x}{2^{\frac{3}{2}} N^{\frac{1}{6}}} \right) \, dx \\
&= \text{Tr } K\tilde{f},
\end{align*}
\]

\[
\text{Tr } \left( K_N^{(2)} f \right)^2 = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} K_N^{(2)}(x,y)f\left(y - \sqrt{2N}\right) K_N^{(2)}(y,x)f\left(x - \sqrt{2N}\right) \, dx \, dy \\
= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} 2^{-\frac{3}{2}} N^{-\frac{1}{6}} K_N^{(2)}(\sqrt{2N} + 2^{-\frac{1}{2}} N^{-\frac{1}{6}} y, \sqrt{2N} + 2^{-\frac{1}{2}} N^{-\frac{1}{6}} x) f \left( \frac{y}{2^{\frac{3}{2}} N^{\frac{1}{6}}} \right) \, dx \, dy \\
&\cdot \quad 2^{-\frac{3}{2}} N^{-\frac{1}{6}} K_N^{(2)}(\sqrt{2N} + 2^{-\frac{1}{2}} N^{-\frac{1}{6}} y, \sqrt{2N} + 2^{-\frac{1}{2}} N^{-\frac{1}{6}} x) f \left( \frac{x}{2^{\frac{3}{2}} N^{\frac{1}{6}}} \right) \, dx \, dy \\
= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} K^2(x,y)f \left( \frac{x}{2^{\frac{3}{2}} N^{\frac{1}{6}}} \right) f \left( \frac{y}{2^{\frac{3}{2}} N^{\frac{1}{6}}} \right) \, dx \, dy \\
&= \text{Tr } \left( K\tilde{f} \right)^2,
\]

and so on.

We now introduce the result of Basor and Widom as the following lemma [7].

**Lemma 4.1.** Let
\[ \hat{f}(x) := f \left( \frac{x}{\gamma} \right), \]
then as \( \gamma \to \infty \),
\[
\log \det(I + K\hat{f}) = c_1 \gamma^{\frac{3}{2}} + c_2 + o(1),
\]
where
\[
c_1 = \frac{1}{\pi} \int_0^{\infty} \sqrt{x} \log(1 + f(-x)) \, dx,
\]
\[
c_2 = \frac{1}{2} \int_0^{\infty} x G^2(x) \, dx
\]
and
\[
G(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ixy} \log(1 + f(-y^2)) \, dy.
\]
From Lemma 4.1 and noting that $\gamma = 2^{1/2}N^{1/2}$ in our problem, we obtain as $N \to \infty$,

$$\log G_N^{(2)}(f) = \frac{2^{1/2}N^{1/2}}{\pi} \int_0^\infty \sqrt{x} \log(1 + f(-x))dx + \frac{1}{2} \int_0^\infty x G^2(x)dx + o(1), \quad (4.1)$$

Substituting $f(x) = e^{-\lambda F(x)} - 1$ into (4.1), we find

$$\log G_N^{(2)}(f) = -\frac{2^{1/2}N^{1/2}\lambda}{\pi} \int_0^\infty \sqrt{x} F(-x)dx + \frac{\lambda^2}{8\pi^2} \int_0^\infty x \left( \int_{-\infty}^\infty e^{i xy} F(-y^2)dy \right)^2 dx + o(1)$$

Hence we have the following theorem.

**Theorem 4.2.** As $N \to \infty$, the mean and variance of the linear statistics $\sum_{j=1}^N F \left( x_j - \sqrt{2N} \right)$ in Gaussian unitary ensemble are given by

$$\mu = \frac{2^{1/2}N^{1/2}}{\pi} \int_0^\infty \sqrt{x} F(-x)dx + o(1)$$

and

$$\nu = \frac{1}{\pi^2} \int_0^\infty x \left( \int_0^\infty \cos(xy) F(-y^2)dy \right)^2 dx + o(1)$$

respectively.

At the end of this section, we use another method, the coulomb fluid approach, to prove the above theorem. We state an important lemma [6].

**Lemma 4.3.** As $N \to \infty$,

$$\mathbb{E} \left( e^{-\lambda \sum_{j=1}^N F(x_j)} \right) \sim \exp(-S_1 - S_2),$$

where

$$S_1 = \frac{\lambda^2}{4\pi^2} \int_a^b \int_a^b \frac{F(x)F(y)}{\sqrt{(b-x)(x-a)}} \frac{\partial}{\partial y} \left( \frac{\sqrt{(b-y)(y-a)}}{x-y} \right) dxdy,$$

$$S_2 = \lambda \int_a^b \sigma(x) F(x)dx,$$

and $\sigma(x)$ is the equilibrium density of the eigenvalues (particles) supported on the interval $(a,b)$.

For the Gaussian unitary ensemble, it is known that [6]

$$\sigma(x) = \frac{\sqrt{b^2 - x^2}}{\pi}, \quad b = -a = \sqrt{2N}.$$
In our case, we replace $F(x)$ by $F\left(x - \sqrt{2N}\right)$. We have as $N \to \infty$,

$$
\mathbb{E}\left(e^{-\lambda \sum_{j=1}^{N} F(x_j - \sqrt{2N})}\right) \sim \exp(-S_1 - S_2),
$$

where

$$
S_1 = \frac{\lambda^2}{4\pi^2} \int_{-\infty}^{\sqrt{2N}} \int_{-\infty}^{\sqrt{2N}} F(x - \sqrt{2N})F(y - \sqrt{2N}) \frac{\partial}{\partial y} \left(\frac{\sqrt{2N} - y^2}{x - y}\right) dxdy,
$$

$$
S_2 = \frac{\lambda}{\pi} \int_{-\sqrt{2N}}^{\sqrt{2N}} \sqrt{2N - x^2} F(x - \sqrt{2N}) dx.
$$

Firstly, we compute $S_1$. Let $x = \sqrt{2N} - u$, $y = \sqrt{2N} - v$,

$$
S_1 = -\frac{\lambda^2}{4\pi^2} \int_{0}^{2\sqrt{2N}} \int_{0}^{2\sqrt{2N}} F(-u)F(-v) \frac{\partial}{\partial v} \left(\frac{\sqrt{v\left(2\sqrt{2N} - v\right)}}{v - u}\right) dudv.
$$

As $N \to \infty$,

$$
S_1 \sim -\frac{\lambda^2}{4\pi^2} \int_{0}^{\infty} \int_{0}^{\infty} F(-u)F(-v) \frac{\partial}{\partial v} \left(\frac{\sqrt{v}}{v - u}\right) dudv.
$$

After the change of variables $u = s^2$, $v = t^2$, we find

$$
S_1 \sim \frac{\lambda^2}{2\pi^2} \int_{0}^{\infty} \int_{0}^{\infty} F(-s^2)F(-t^2) \frac{s^2 + t^2}{(s^2 - t^2)^2} dsdt
$$

$$
= \frac{\lambda^2}{8\pi^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} F(-s^2)F(-t^2) \frac{s^2 + t^2}{(s^2 - t^2)^2} dsdt
$$

$$
= \frac{\lambda^2}{8\pi^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} F(-s^2)F(-t^2) \frac{s^2 - 2st + t^2}{(s^2 - t^2)^2} dsdt
$$

$$
= \frac{\lambda^2}{8\pi^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} F(-s^2)F(-t^2) \frac{1}{(s + t)^2} dsdt.
$$

Noting that

$$
\frac{1}{(s + t)^2} = -\frac{1}{2} \int_{-\infty}^{\infty} |x| \exp(-ix(s + t)) dx.
$$

We have

$$
S_1 \sim -\frac{\lambda^2}{16\pi^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} F(-s^2)F(-t^2) |x| \exp(-ix(s + t)) dxdsdt
$$

$$
= -\frac{\lambda^2}{16\pi^2} \int_{-\infty}^{\infty} |x| \left[ \int_{-\infty}^{\infty} F(-s^2) \exp(-ixs) ds \right]^2 dx
$$

$$
= -\frac{\lambda^2}{2\pi^2} \int_{0}^{\infty} x \left[ \int_{0}^{\infty} F(-s^2) \cos(xs) ds \right]^2 dx.
$$
Now we compute $S_2$. Let $y = \sqrt{2N} - x$,

$$S_2 = \frac{\lambda}{\pi} \int_0^{2\sqrt{2N}} \sqrt{y \left(2\sqrt{2N} - y\right)} F(-y) dy.$$

As $N \to \infty$,

$$S_2 \sim \frac{\sqrt{2\sqrt{2N}\lambda}}{\pi} \int_0^{\infty} \sqrt{y} F(-y) dy = \frac{2^{\frac{3}{2}} N^{\frac{3}{4}} \lambda}{\pi} \int_0^{\infty} \sqrt{y} F(-y) dy.$$

Theorem 4.2 then follows.

5 Conclusion

This paper studies the large $N$ behavior of the MGF of the scaled linear statistics in Gaussian ensembles and Laguerre ensembles, from which we obtain the mean and variance of the corresponding linear statistics. We find that there is an equivalence between the mean and variance of suitably scaled linear statistics in Gaussian and Laguerre ensembles. In addition, we use the results of [7] and [6] to consider another type of linear statistics in GUE and also obtain the mean and variance of the corresponding linear statistics. For the GSE and GOE, we will deal with the corresponding type of linear statistics in the future.

Acknowledgments

Chao Min was supported by the Scientific Research Funds of Huaqiao University under grant number 600005-Z17Y0054. Yang Chen was supported by the Macau Science and Technology Development Fund under grant numbers FDCT 130/2014/A3, FDCT 023/2017/A1 and by the University of Macau under grant numbers MYRG 2014-00011-FST, MYRG 2014-00004-FST.

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