SCATTERING FOR A CLASS OF NON-RADIAL INHOMOGENEOUS BI-HARMONIC HARTREE EQUATIONS

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ABSTRACT. This manuscript proves the energy scattering of global solutions to a repulsive fourth-order generalized Hartree equation with non-radial data in the inter-critical regime. This work uses a new approach due to Dodson-Murphy [4] and extends the previous work [14] by removing the spherically symmetric assumption on the data.

1. Introduction

This work treats the energy scattering theory for the following inhomogeneous focusing Choquard equation

\[
\begin{cases}
  i \dot{u} + \Delta^2 u - (I_\alpha \ast | \cdot |^b |u|^p) |x|^b |u|^{p-2} u = 0; \\
u(0, \cdot) = u_0.
\end{cases}
\]

(1.1)

Here and hereafter \( u : \mathbb{R} \times \mathbb{R}^N \to \mathbb{C} \), for a natural integer \( N \geq 5 \). The singular inhomogeneous term is \( | \cdot |^b \), for a certain \( b < 0 \). The Riesz-potential is defined on \( \mathbb{R}^N \) by

\[
I_\alpha : x \mapsto \frac{\Gamma(\frac{N-\alpha}{2})}{\Gamma(\frac{\alpha}{2}) \pi^{\frac{N}{2}} 2^\alpha |x|^{N-\alpha}}, \quad 0 < \alpha < N.
\]

The limiting case \( b = 0 \) corresponds to the homogeneous fourth-order Schrödinger problem considered first in [8, 9] to take into account the role of small fourth-order dispersion terms in the propagation of intense laser beams in a bulk medium with a Kerr non-linearity.

The equation (1.1) is invariant under the scaling

\[
u_\lambda = \lambda^{\frac{4+2b+\alpha}{2(p-1)}} u(\lambda^4 \cdot, \lambda \cdot), \quad \lambda > 0.
\]

The homogeneous Sobolev norm gives the critical Sobolev index denoted by \( s_c \) as follows

\[
\| u_\lambda \|_{H^s} = \lambda^{\frac{s - \left(\frac{N}{2} - \frac{4+2b+\alpha}{2(p-1)}\right)}{}} \| u(\lambda^4 \cdot) \|_{H^s} : = \lambda^{s - s_c} \| u(\lambda^4 \cdot) \|_{H^s},
\]

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In this note, one considers the inter-critical regime $0 < s_c < 2$, which corresponds to the mass super-critical and energy sub-critical case.

Let us recall give a brief literature about the inhomogeneous bi-harmonic non-linear Schrödinger equation (IBNLS). The finite time blow-up of solutions to the IBNLS for negative energy with a source term $|x|^{-2}|u|^\frac{2}{3}u$ was considered in [3]. The local well-posedness in the energy space was treated in [7]. See also [12], for the well-posedness in $H^s$, $0 < s \leq 2$ and [2] for existence of global and non-global solutions in $\dot{H}^{s_c} \cap \dot{H}^2$. The case of a non-local source term was considered by the author in [5]. The scattering of global spherically symmetric solutions under the ground state threshold was established recently by the author [14]. Moreover, the scattering without any radial assumption for the IBNLS with a local source term was proved very recently [1].

It is the aim of this note to extend [14] to the non-radial regime and [1] to the generalized Hartree equation. The challenge of this work is to deal with the non-local source term with use of a Hardy-Littlewood-Sobolev estimate. The main ingredient here is the use of the decay of the inhomogeneous term $| \cdot |^b$ instead of the spherically symmetric assumption. So, the scattering in the limiting case $b = 0$ is not a consequence of this manuscript.

The rest of the note is organized as follows. In section 2, one gives the main result and some useful estimates. Section three contains a proof of a Morawetz identity. In section four one proves a scattering criterion. The last section proves the main theorem.

Here and hereafter, $C$ denotes a constant which may vary from line to another. Denote the Lebesgue space $L^r := L^r(\mathbb{R}^N)$ with the usual norm $\| \cdot \|_r := \| \cdot \|_{L^r}$ and $\| \cdot \| := \| \cdot \|_2$. The inhomogeneous Sobolev space $H^2 := H^2(\mathbb{R}^N)$ is endowed with the norm

$$\| \cdot \|_{H^2} := \left( \| \cdot \|^2 + \| \Delta \cdot \|^2 \right)^\frac{1}{2}.$$  

Let us denote also $C_T(X) := C([0, T], X)$ and $X_r$ the set of radial elements in $X$. Moreover, for an eventual solution to (1.1), $T^* > 0$ denotes it’s lifespan. Finally, $x^+$ is a real numbers near to $x$ satisfying $x^+ > x$.

2. Background and main results

This section contains the contribution of this paper and some standard estimates needed in the sequel.

2.1. Preliminary. Take for $R > 0$, $\psi_R := \psi(\frac{\cdot}{R})$, where $0 \leq \psi \leq 1$ is a radial smooth function satisfying

$$\psi \in C_0^\infty(\mathbb{R}^N), \quad supp(\psi) \subset \{|x| < 1\}, \quad \psi = 1 \text{ on } \{|x| < \frac{1}{2}\}.$$
The mass-critical and energy-critical exponents for the Choquard problem (1.1) are
\[ p_* := 1 + \frac{\alpha + 4 + 2b}{N} \quad \text{and} \quad p^* = 1 + \frac{4 + 2b + \alpha}{N - 4}. \]

Solutions of the Choquard problem (1.1) satisfy the conservation of the mass and the energy
\[ M[u] := \|u\|^2; \]
\[ E[u] := \|\Delta u\|^2 - \frac{1}{p} \int_{\mathbb{R}^N} (I_\alpha \cdot |b| |u|^p) |x| |u|^p \, dx. \]

Let \( \phi \) a ground state solution to the elliptic problem
\[ \phi + \Delta^2 \phi - (I_\alpha \cdot |b| |\phi|^p) |x| |\phi|^{p-2} \phi = 0, \quad 0 \neq \phi \in H^2, \]
and \( u \) a solution to (1.1). The following scale invariant quantities describe the dichotomy of global/non-global existence of solutions [6].
\[ \mathcal{ME}[u] := \left( \frac{E[u]}{E[\phi]} \right) \left( \frac{M[u]}{M[\phi]} \right)^{\frac{2-\alpha}{2-\alpha_e}}; \]
\[ \mathcal{MG}[u] := \left( \frac{\|\Delta u\|}{\|\Delta \phi\|} \right) \left( \frac{\|u\|}{\|\phi\|} \right)^{\frac{2-\alpha}{2-\alpha_e}}. \]

The local well-posedness of the above problem (1.1) in the energy space for \( 2 \leq p < p^* \) was proved in [5] under the assumptions denoted for simplicity \((N, \alpha, b)\) satisfies (C) if, \( 0 < \alpha < N \) and \( \max\{- (N + \alpha), -4 (1 + \frac{\alpha}{2}), N - 8 - \alpha \} < 2b < 0 \) and \([N \geq 5 \text{ or } 3 \leq N \leq 4 \text{ and } 2\alpha + 4b + N > 0]\). Moreover, the global existence versus finite time blow-up of energy solutions under the ground state threshold was obtained in [14]. Precisely,

**Theorem 2.1.** Let \((N, \alpha, b)\) satisfying (C) and \( \max\{p_*, x_\alpha\} < p < p^* \) such that \( p \geq \max\{2, \frac{3}{2} + \frac{\alpha}{N}\} \). Let \( u_0 \in H^2_r \) satisfying
\[ \max \left\{ \mathcal{ME}[u_0], \mathcal{MG}[u_0] \right\} < 1. \quad (2.2) \]

Take \( u \in C_{T^*}(H^2_r) \) be a maximal solution to (1.1). Then, \( u \) is global and there exists \( u_\pm \in H^2 \) such that
\[ \lim_{t \to \pm \infty} \|u(t) - e^{it\Delta} u_\pm\|_{H^2} = 0. \]

**Remarks 2.2.** Note that
1. \( x_\alpha \) is the positive root of the polynomial
\[ (X - 1)(2X - 1) - \frac{4 + 2b + \alpha}{N - 4}; \]
2. the proof follows a new approach due to [4];
3. the spherically symmetric assumption is essential in the used method.

The following variational estimates [14] are needed in the proof of the scattering of global solutions to the focusing Choquard problem (1.1).
Lemma 2.3. Take \((N, \alpha, b)\) satisfying (C) and \(p_s < p < p^*\) such that \(p \geq 2\). Let \(u_0 \in H^2\) satisfying (2.2) and \(u \in C(\mathbb{R}, H^2)\) be the solution to (1.1). Then,

1. there exists \(0 < \delta < 1\) such that
   \[
   \max \left\{ \sup_{t \in \mathbb{R}} \mathcal{M}E[u(t)], \sup_{t \in \mathbb{R}} \mathcal{M}G[u(t)] \right\} < 1 - \delta.
   \]
2. There exists \(R_0 := R_0(\delta, M(u), \phi) > 0\) such that for any \(R > R_0\),
   \[
   \sup_{t \in \mathbb{R}} \| \psi_R u(t) \|^{2-s_c} \| \Delta(\psi_R u(t)) \|^{s_c} \leq (1 - \delta) \| \phi \|^{2-s_c} \| \Delta \phi \|^{s_c}.
   \]

Moreover, there exists \(\delta' > 0\) such that
\[
\| \Delta(\psi_R u) \|^2 - \frac{B}{2p} \int_{\mathbb{R}^N} (I_\alpha * | \cdot |^b |\psi_R u|^p) |x|^b |\psi_R u|^p \ dx \geq \delta' \| \psi_R u \|^{2p} \frac{2N}{N+\alpha+2b}.
\]

2.2. Main result. The main goal of this manuscript is to prove the following scattering result.

Theorem 2.4. Let \((N, \alpha, b)\) satisfying (C) and \(p_s < p < p^*\) such that \(p \geq 2\) and \(N \geq 5\). Let \(u_0 \in H^2\) satisfying (2.2) and \(u \in C_T(H^2)\) be the maximal solution to (1.1). Then, \(u\) is global and there exists \(u_\pm \in H^2\) such that
\[
\lim_{t \to \pm \infty} \| u(t) - e^{it\Delta} u_\pm \|_{H^2} = 0.
\]

Remarks 2.5. Note that

1. the global existence of solutions under the assumption (2.2) was proved in [5];
2. the proof follows a new approach due to [4] and avoids the concentration-compactness method introduced by [10];
3. the main novelty here is the removal of the spherically symmetric method introduced by [10];
4. one exploits the decay of the inhomogeneous term \(| \cdot |^b\) of the source term, instead of the spherically symmetric assumption;
5. the condition \(N \geq 5\) is used in the scattering criteria;
6. it seems that the scattering for non-radial data in lower dimensions still remains open;
7. in the limiting case \(b = 0\), it seems that the scattering for non-radial data is still open.

2.3. Useful estimates. Let us gather some classical tools needed in the sequel.

Definition 2.6. Take \(N \geq 1\) and \(s \in [0, 2)\). A couple of real numbers \((q, r)\) is said to be s-admissible (admissible for 0-admissible) if
\[
\frac{2N}{N-2s} \leq r < \frac{2N}{N-4}, \quad 2 \leq q, r \leq \infty \quad \text{and} \quad N\left(\frac{1}{2} - \frac{1}{r}\right) = \frac{4}{q} + s.
\]

Denote the set of s-admissible pairs by \(\Gamma_s\) and \(\Gamma := \Gamma_0\). If \(I\) is a time slab, one denotes the Strichartz spaces
\[
S^s(I) := \cap_{(q, r) \in \Gamma_s} L^q(I, L^r).
\]

Recall the Strichartz estimates [13, 6].
Proposition 2.7. Let $N \geq 1$, $0 \leq s < 2$ and $t_0 \in I \subset \mathbb{R}$, an interval. Then,

1. sup_{(q,r)\in \Gamma} \|u\|_{L^q(I,L^r)} \lesssim \|u(t_0)\| + \inf_{(q,r)\in \Gamma} \|iu + \Delta^2 u\|_{L^q(I,L^r)};
2. sup_{(q,r)\in \Gamma} \|\Delta u\|_{L^q(I,L^r)} \lesssim \|\Delta u(t_0)\| + \|iu + \Delta^2 u\|_{L^q(I,L^r)};
3. sup_{(q,r)\in \Gamma} \|u\|_{L^q(I,L^r)} \lesssim \|u(t_0)\|_{H^s} + \inf_{(q,r)\in \Gamma_{\alpha}} \|iu + \Delta^2 u\|_{L^q(I,L^r)}.

Let us recall a Hardy-Littlewood-Sobolev inequality [11, 15].

Lemma 2.8. Take $N \geq 1$.

1. Let $0 < \lambda < N$ and $1 < r, s < \infty$ satisfying $2 = \frac{1}{r} + \frac{s}{s} + \frac{N}{N}$. Thus,
   \[ \int_{\mathbb{R}^N} \frac{u(x)v(y)}{|x - y|^\lambda} \, dx \, dy \leq C_{N,s,\lambda} \|u\|_r \|v\|_s, \forall u \in L^r, \forall v \in L^s. \]
2. Let $0 < \alpha < N$ and $1 < r, s, q < \infty$ satisfying $1 + \frac{\alpha}{N} = \frac{1}{q} + \frac{1}{s} + \frac{1}{\alpha}$. Thus,
   \[ \|I_\alpha * u\|_{r'} \leq C_{N,s,\alpha} \|u\|_s \|v\|_q, \forall u \in L^s, \forall v \in L^q; \]
3. Let $0 < \alpha < N$ and $1 < r, s, q < \infty$ satisfying $1 + \frac{2 - \gamma - \mu}{N} = \frac{1}{q} + \frac{1}{s} + \frac{1}{\alpha}$ and $0 < -\gamma < \frac{N}{s'}, 0 < -\mu < \frac{N}{s'}$. Thus,
   \[ \|(I_\alpha * | \cdot |^\gamma u) \cdot |^\mu v\|_{r'} \leq C_{N,s,\alpha,\gamma,\mu} \|u\|_s \|v\|_q, \forall u \in L^s, \forall v \in L^q. \]

Finally, let us give an abstract result [14].

Lemma 2.9. Let $T > 0$ and $X \in C([0,T], \mathbb{R}_+)$ such that
\[ X \leq a + bX^\theta \text{ on } [0,T], \]
where $a, b > 0$, $\theta > 1$, $a < (1 - \frac{1}{\theta})(\theta b)^{\frac{1}{\theta - 1}}$ and $X(0) \leq (\theta b)^{\frac{1}{\theta - 1}}$. Then
\[ X \leq \frac{\theta}{\theta - 1} a \text{ on } [0,T]. \]

3. Morawetz identity

One adopts the convention that repeated indexes are summed. Recall a classical Morawetz estimate [14] satisfied by the energy global solutions to the inhomogeneous Choquard problem (1.1).

Lemma 3.1. Take $(N, \alpha, b)$ satisfying (C), $p_* < p < p^*$ such that $p \geq 2$ and $u \in C_T(H^2)$ be a local solution to (1.1). Let $a : \mathbb{R}^N \to \mathbb{R}$ be a convex smooth function and the real function defined on $[0,T]$, by
\[ M_a : t \to 2 \int_{\mathbb{R}^N} \nabla a(x) \cdot \Im \left( \nabla u(t,x) \bar{u}(t,x) \right) \, dx. \]
Then, the following equality holds on $[0,T]$,
\[ M'_a = 2 \int_{\mathbb{R}^N} \left( 2\partial_{jk}\Delta a \partial_j u \partial_k u - \frac{1}{2}(\Delta^3 a)|u|^2 - 4\partial_{jk} a \partial_k u \partial_j u + \Delta^2 a |\nabla u|^2 \right) \]
\[ + 2 \left( 1 - \frac{2}{p} \right) \int_{\mathbb{R}^N} \Delta a (I_\alpha * |^p u^p) |x|^p |u|^p \, dx - \frac{2}{p} \int_{\mathbb{R}^N} \partial_k a \partial_k (|x|^p [I_\alpha * |^p u^p]) |u|^p \, dx. \]

Let us write the main result of this section.
**Proposition 3.2.** Take \((N, \alpha, b)\) satisfying \((C)\) and \(p_* < p < p^*\) such that \(p \geq 2\). Let \(u_0 \in H^2\) satisfying \((2.2)\). Then, for any \(T > 0\), one has

\[
\int_0^T \|u(t)\|_{L^{2p} \mathbb{R}^{Np+2x(x|<R)}}^{2p} dt \leq CT^{1/(1-b)}.
\]

**Proof.** Take a smooth real function such that

\[
f : r \rightarrow \begin{cases} r^2, & \text{if } 0 \leq r \leq 1; \\
r, & \text{if } r \geq 2,
\end{cases}
\]

moreover,

\[
\min \{f', f''\} \geq 0, \quad \text{on } [1, 2].
\]

Note that for \(|x| \leq 1\), confusing for \(x \in \mathbb{R}^N\), \(f(x) := f(|x|)\), one has

\[
f_{ij} = \delta_{ij}, \quad \Delta f = N \quad \text{and} \quad \partial^\gamma f = 0 \quad \text{for } |\gamma| \geq 3.
\]

Finally, one denotes for \(R > 0\), the smooth radial function defined on \(\mathbb{R}^N\) by \(f_R := R^2 f(|\cdot|)\) and the real function \(M_R := M_{f_R}\). Using the estimate \(\|\nabla^\gamma f_R\|_\infty \lesssim R^{2-|\gamma|}\), one has

\[
|\int_{\mathbb{R}^N} \Delta^2 f_R |\nabla u|^2 \, dx| + |\int_{\mathbb{R}^N} \partial_{jk} \Delta f_R \partial_j u \partial_k \bar{u} \, dx| \lesssim R^{-2};
\]

\[
|\int_{\mathbb{R}^N} (\Delta^3 f_R) |u|^2 \, dx| \lesssim R^{-4}.
\]

Thus, by Morawetz estimate in Lemma 3.1, one gets

\[
M'_R = - \frac{4}{p} \int_{\mathbb{R}^N} \partial_k f_R \partial_k [I_{\alpha} * | |^b |u|^p] |x|^b |u|^p \, dx + O(R^{-2})
\]

\[
+ 2 \left( N \left( 1 - \frac{2}{p} \right) \int_{\{|x|<R\}} (I_{\alpha} * | |^b |u|^p) |x|^b |u|^p \, dx - 4 \int_{\{|x|<R\}} |\Delta u|^2 \, dx \right)
\]

\[
+ 2 \left( 1 - \frac{2}{p} \right) \int_{\{|x|>R\}} \Delta f_R (I_{\alpha} * | |^b |u|^p) |x|^b |u|^p \, dx - 4 \int_{\{|x|>R\}} \partial_{jk} f_R \partial_k u \partial_{ij} \bar{u} \, dx \right).
\]

Moreover, denoting the radial derivative by \(\partial_r := \nabla \cdot \frac{x}{|x|}\), one writes

\[
\partial_{jk} f_R = \left( \delta_{jk} - \frac{x_j x_k}{r^2} \right) \partial_r f_R + \frac{x_j x_k}{r^2} \partial_r^2 f_R.
\]
\[
\int_{\{x > R\}} \partial_{jk} f_R \partial_{ik} u \partial_{ij} \bar{u} \, dx = \int_{\{x > R\}} \left[ \left( \frac{\delta_{jk}}{r} - \frac{x_j x_k}{r^3} \right) \partial_r f_R + \frac{x_j x_k}{r^2} \partial_r^2 f_R \right] \partial_{ik} u \partial_{ij} \bar{u} \, dx \\
= \sum_{i=1}^N \int_{\{x > R\}} \left( |\nabla u_i|^2 - \frac{|x \cdot \nabla u_i|^2}{|x|^2} \right) \partial_r f_R \, dx \\
+ \sum_{i=1}^N \int_{\{x > R\}} \frac{|x \cdot \nabla u_i|^2}{|x|^2} \partial_r^2 f_R \, dx \\
= \sum_{i=1}^N \int_{\{x > R\}} |\nabla u_i|^2 \partial_r f_R \, dx + \sum_{i=1}^N \int_{\{x > R\}} \frac{|x \cdot \nabla u_i|^2}{|x|^2} \partial_r^2 f_R \, dx,
\]

where the angular gradient is \( \nabla := \nabla - \frac{x \nabla}{|x|^2} x \). Then,

\[
-M_R' \geq \frac{4}{p} \int_{\mathbb{R}^N} \partial_{jk} f_R \partial_{ik} [(I_\alpha * | \cdot |^b |u|^p) |x|^b |u|^p] \, dx + O(R^{-2}) \\
+ 2 \left( 4 \int_{\{x < R\}} |\Delta u|^2 \, dx - N \left( 1 - \frac{2}{p} \right) \int_{\{x < R\}} (I_\alpha * | \cdot |^b |u|^p) |x|^b |u|^p \, dx \right) \\
- \left( 1 - \frac{2}{p} \right) \int_{\{x > R\}} \Delta f_R (I_\alpha * | \cdot |^b |u|^p) |x|^b |u|^p \, dx.
\]

Take the quantity

\[
(A) := \int_{\mathbb{R}^N} \partial_{jk} f_R \partial_{ik} [(I_\alpha * | \cdot |^b |u|^p) |x|^b |u|^p] \, dx \\
= - \left( N - \alpha \right) \int_{\mathbb{R}^N} \nabla f_R \left( \frac{\cdot}{|x|^2} I_\alpha * | \cdot |^b |u|^p \right) |x|^b |u|^p \, dx + b \int_{\mathbb{R}^N} \frac{\nabla f_R \cdot x}{|x|^2} (I_\alpha * | \cdot |^b |u|^p) |x|^b |u|^p \, dx \\
:= -(N - \alpha)(I) + b(II).
\]

With the properties of \( f_R \), one writes

\[
(II) = \int_{\{|x| < R\}} (I_\alpha * | \cdot |^b |u|^p) |x|^b |u|^p \, dx + O \left( \int_{\{|x| > R\}} (I_\alpha * | \cdot |^b |u|^p) |x|^b |u|^p \, dx \right).
\]

With calculus done in [14], one has

\[
(I) = \frac{1}{2} \int_{\{|x| < R\}} \int_{\{|y| < R\}} I_\alpha (x - y) |y|^b |u(y)|^p |x|^b |u(x)|^p \, dx + O \left( \int_{\{|x| > R\}} (I_\alpha * | \cdot |^b |u|^p) |x|^b |u|^p \, dx \right) \\
= \frac{1}{2} \int_{\{|x| < R\}} (I_\alpha * | \cdot |^b |u|^p) |x|^b |u(x)|^p \, dx + O \left( \int_{\{|x| > R\}} (I_\alpha * | \cdot |^b |u|^p) |x|^b |u|^p \, dx \right).
\]

Thus,

\[
(A) = (b - \frac{N - \alpha}{2}) \int_{\{|x| < R\}} (I_\alpha * | \cdot |^b |u|^p) |x|^b |u(x)|^p \, dx + O \left( \int_{\{|x| > R\}} (I_\alpha * | \cdot |^b |u|^p) |x|^b |u|^p \, dx \right).
\]
\[ -M'_R \geq 2 \left( 4 \int_{|x|<R} |\Delta u|^2 dx - N(1 - \frac{2}{p}) \int_{|x|<R} (I_\alpha \ast |\cdot|^b|u|^p)|x|^b|u|^p dx \right) \\
+ \frac{4}{p}(A) + O(R^{-2}) + O\left( \int_{|x|>R} |\Delta f_R(I_\alpha \ast |\cdot|^b|u|^p)|x|^b|u|^p dx \right) \\
\geq 2 \left( 4 \int_{|x|<R} |\Delta u|^2 dx - N(1 - \frac{2}{p}) \int_{|x|<R} (I_\alpha \ast |\cdot|^b|u|^p)|x|^b|u|^p dx \right) \\
+ \frac{4}{p}(b - \frac{N - \alpha}{2}) \int_{|x|<R} (I_\alpha \ast |\cdot|^b|u|^p)|x|^b|u(x)|^p dx \\
+ O(R^{-2}) + O\left( \int_{|x|>R} (I_\alpha \ast |\cdot|^b|u|^p)|x|^b|u|^p dx \right). \]

Let us write
\[
\int_{|x|>R} (I_\alpha \ast |\cdot|^b|u|^p)|x|^b|u|^p dx \leq R^b \int_{\mathbb{R}^N} (I_\alpha \ast |\cdot|^b|u|^p)|u|^p dx \\
\leq R^b \int_{\mathbb{R}^N} \left( \int_{\{|y|<1\}} \int_{\{|y|>1\}} I_\alpha(x-y)|y|^b|u(y)|^p|u(x)|^p dy |u|^p dx \right).
\]

Now, with Hardy-Littlewood-Sobolev estimate, one has
\[
\int_{\mathbb{R}^N} \int_{\{|y|<1\}} I_\alpha(x-y)|y|^b|u(y)|^p|u(x)|^p dy |u|^p dx \ \lesssim \ ||| \cdot|^b||_{L^p(|x|<1)}||u||_{2p}^p,
\]
where
\[
1 + \frac{\alpha}{N} = \frac{1}{a} + \frac{2p}{r}.
\]
Taking \( a < \frac{N}{b} \), one gets \( 1 + \frac{\alpha}{N} - \frac{2p}{r} > \frac{b}{N} \) and equivalently
\[
\frac{2Np}{N + \alpha + b} < r.
\]
Since \( p < p^* \), there is \( r \in [2, \frac{2N}{N-4}] \) satisfying the above estimate. Thus,
\[
\int_{\mathbb{R}^N} \int_{\{|y|<1\}} I_\alpha(x-y)|y|^b|u(y)|^p|u(x)|^p dy |u|^p dx \ \lesssim \ ||u||_{H^2}^{2p}.
\]

Now, with Hardy-Littlewood-Sobolev estimate, one has
\[
\int_{\mathbb{R}^N} \int_{\{|y|>1\}} I_\alpha(x-y)|y|^b|u(y)|^p|u(x)|^p dy |u|^p dx \ \lesssim \ ||| \cdot|^b||_{L^p(|x|>1)}||u||_{2p}^p,
\]
where
\[
1 + \frac{\alpha}{N} = \frac{1}{a} + \frac{2p}{r}.
\]
Taking \( a > \frac{N}{b} \), one gets \( 1 + \frac{\alpha}{N} - \frac{2p}{r} < \frac{b}{N} \) and equivalently
\[
r < \frac{2Np}{N + \alpha + b}.
\]
Since \( p_* < p \), there is \( r \in \left[ 2, \frac{2N}{N-4} \right] \) satisfying the above estimate. Thus,
\[
\int_{\mathbb{R}^N} \int_{\{|y| > 1\}} I_\alpha(x-y)|y|^b|u(y)|^p|u(x)|^p \, dy \, dx \lesssim \|u\|^{2p}_{H^2}.
\]
Thus,
\[
-M'_R \geq 8 \left( \int_{|x| < R} |\Delta u|^2 \, dx - \frac{B}{2p} \int_{|x| < R} (I_\alpha \ast | \cdot |^b|u|^p)|x|^b|u|^p \, dx \right) + O(R^b).
\]
On the other hand, an expansion via the properties of \( \psi \) gives
\[
\|\Delta(\psi_R u)\|^2 \leq \|\psi_R \Delta u\|^2 + C(u_0, \phi)R^{-2}
\]
\[
\leq \int_{|x| < R} |\Delta u|^2 - \int_{\frac{R}{2} < |x| < R} (1 - \psi_R^2)|\Delta u|^2 \, dx + C(u_0, \phi)R^{-2}.
\]
Moreover,
\[
\int_{\mathbb{R}^N} (I_\alpha \ast | \cdot |^b\psi_R u^p)|x|^b\psi_R u^p \, dx - \int_{\mathbb{R}^N} (I_\alpha \ast | \cdot |^b(1 - \psi^p)|u|^p)|x|^b|u|^p \, dx
\]
\[
= \int_{\mathbb{R}^N} (I_\alpha \ast | \cdot |^b|u|^p)|x|^b\psi_R u^p \, dx
\]
\[
= \int_{|x| < R} (I_\alpha \ast | \cdot |^b|u|^p)|x|^b|u|^p \, dx - \int_{\frac{R}{2} < |x| < R} (I_\alpha \ast | \cdot |^b|u|^p)|x|^b(1 - \psi_R^p)|u|^p \, dx.
\]
Then,
\[
\int_{\mathbb{R}^N} (I_\alpha \ast | \cdot |^b\psi_R u^p)|x|^b\psi_R u^p \, dx = \int_{|x| < R} (I_\alpha \ast | \cdot |^b|u|^p)|x|^b|u|^p \, dx + O(R^b).
\]
So, with Lemma 2.3, one gets
\[
\sup_{[0,T]} |M| \geq 8 \int_0^T \left( \int_{\mathbb{R}^N} |\Delta(\psi_R u)|^2 \, dx - \frac{B}{2p} \int_{\mathbb{R}^N} (I_\alpha \ast |\psi_R u|^p)|\psi_R u|^p \, dx \right) dt + O(R^b)T
\]
\[
\geq 8\delta' \int_0^T \|\psi_R u(t)\|^{2p}_{\frac{2Np}{N+\alpha+2p}} dt + O(R^b)T
\]
\[
\geq 8\delta' \int_0^T \|u(t)\|^{2p}_{\frac{2Np}{L^{N+\alpha+2p}}} \|u(t)\|_{L^{N+\alpha+2p}}(\{|x| < R\}) dt + O(R^b)T.
\]
The previous calculus gives
\[
\int_0^T \|u(t)\|^{2p}_{\frac{2Np}{L^{N+\alpha+2p}}(\{|x| < R\})} dt \leq C \left( \sup_{[0,T]} |M| + TR^b \right)
\]
\[
\leq C \left( R + TR^b \right).
\]
Taking \( R = T^{1/(1-b)} \gg 1 \), one gets the requested estimate. For \( 0 < T \ll 1 \), the proof follows with Sobolev injections.
\[
\text{As a consequence, one has the following energy evacuation.}
\]
Lemma 3.3. Take \((N, \alpha, b)\) satisfying (C). Let \(p_* < p < p^*\) such that \(p \geq 2\) and \(u_0 \in H^2\) satisfying (2.2). Then, there exists a sequence of real numbers \(t_n \to \infty\) such that the global solution to (1.1) satisfies

\[
\lim_{n} \int_{\{|x| < R\}} |u(t_n, x)|^2 \, dx = 0, \quad \text{for all } R > 0.
\]

Proof. Take \(t_n \to \infty\). By Hölder estimate

\[
\int_{\{|x| < R\}} |u(t_n, x)|^2 \, dx \leq R^{\frac{2p}{N}} \|u(t_n)\|_{L^{\frac{2Np}{N + \alpha + 2b}}(|x| < R)}^2 \to 0.
\]

Indeed, by the previous Lemma

\[
\|u(t_n)\|_{L^{\frac{2Np}{N + \alpha + 2b}}(|x| < R)} \to 0.
\]

\[
\square
\]

4. Scattering Criterion

In this section one proves the next result.

Proposition 4.1. Take \((N, \alpha, b)\) satisfying (C). Let \(p_* < p < p^*\) such that \(p \geq 2\). Let \(u \in C(\mathbb{R}, H^2)\) be a global solution to (1.1). Assume that

\[
0 < \sup_{t \geq 0} \|u(t)\|_{H^2} := E < \infty.
\]

There exist \(R, \epsilon > 0\) depending on \(E, N, p, b, \alpha\) such that if

\[
\liminf_{t \to +\infty} \int_{|x| < R} |u(t, x)|^2 \, dx < \epsilon,
\]

then, \(u\) scatters for positive time.

Proof. By Lemma 2.3, \(u\) is bounded in \(H^2\). Take \(\epsilon > 0\) near to zero and \(R(\epsilon) >> 1\) to be fixed later. Let us give a technical result [14].

Lemma 4.2. Let \((N, b, \alpha)\) satisfying (C) and \(p_* < p < p^*\) satisfying \(p \geq 2\). Then, there exists \(\theta \in (0, 2p - 1)\) such that the global solution to (1.1) satisfies

\[
\|u - e^{i \Delta u_0}\|_{S^\infty(I)} \lesssim \|u\|_{L^\infty(I, H^2)}^\theta \|u\|_{L^0(I, L^r)}^{2p - 1 - \theta},
\]

for certain \((a, r) \in \Gamma_{sc}\).

The following result is the key to prove the scattering criterion.

Proposition 4.3. Take \((N, \alpha, b)\) satisfying (C). Let \(p_* < p < p^*\) such that \(p \geq 2\). Let \(u_0 \in H^2\) satisfying (2.2). Then, for any \(\epsilon > 0\), there exist \(T, \mu > 0\) such that the global solution to (1.1) satisfies

\[
\|e^{i(-T)\Delta^2} u(T)\|_{L^a(T, \infty, L^r)} \lesssim \epsilon^\mu.
\]
**Proof.** Let $0 < \beta < 1$ and $T > \varepsilon^{-\beta} > 0$. By the integral formula

$$e^{i(-T)\Delta^2}u(T) = e^{i\Delta^2}u_0 + i \int_0^T e^{i(s)\Delta^2} [(I_\alpha * | \cdot |^b|u|^p)|x|^b|u|^{p-2}u] \, ds$$

$$= e^{i\Delta^2}u_0 + i \left( \int_0^{T-\varepsilon^{-\beta}} + \int_0^T \right) e^{i(s)\Delta^2}N \, ds$$

$$= e^{i\Delta^2}u_0 + F_1 + F_2.$$

Take the real numbers

$$a := \frac{2(2p-\theta)}{2-s_c}, \quad d := \frac{2(2p-\theta)}{2+(2p-1-\theta)s_c};$$

$$r := \frac{2N(2p-\theta)}{(N-2s_c)(2p-\theta) - 4(2-s_c)}.$$

The condition $\theta = 0^+$ gives

$$(a, r) \in \Gamma_{s_c}, \quad (d, r) \in \Gamma_{-s_c} \quad \text{and} \quad (2p-1-\theta)d' = a.$$

- The linear term. Since $(a, d) \in \Gamma_{s_c}$, by Strichartz estimate and Sobolev injections, one has

$$\|e^{i\Delta^2}u_0\|_{L^2((T,\infty),L^r)} \lesssim \|\nabla_x e^{i\Delta^2}u_0\|_{L^2((T,\infty),L^r)}^{\frac{s_c}{N+s_{sc}}} \lesssim \|u_0\|_{H^2}.$$

- The term $F_2$. Using Hardy-Littlewood-Sobolev and Hölder inequalities, via the fact that $0 < \psi_R < 1$, one has

$$\|\psi_R N\|_{r'} = \|(I_\alpha * | \cdot |^b|u|^p)|x|^b|u|^{p-2}\psi_R u\|_{r'} \lesssim \|u\|_{r'}^{2p-1-\theta} \|\psi_R u\|_{r_1}^{\theta}, \quad \text{(4.3)}$$

where

$$\frac{2b+\alpha}{N} + \frac{2p-\theta}{r} + \frac{\theta}{r_1}.$$  

Thus,

$$N + \alpha + 2b = \frac{N(2p-\theta)}{r} + \frac{N\theta}{r_1}$$

$$= \frac{(N-2s_c)(2p-\theta) - 4(2-s_c)}{2} + \frac{N\theta}{r_1}$$

$$= N + \alpha + 2b + \frac{N\theta}{r_1} - \theta \frac{4 + 2b + \alpha}{2(p-1)}.$$

Because $p_* < p < p^*$, one gets

$$2 < r_1 = \frac{2N(p-1)}{4 + 2b + \alpha} < \frac{2N}{N-4}. $$
So, denoting $I_2 := (T - \varepsilon^{-\beta}, T)$, it follows that for $\lambda := \lambda_N \in (0, 1)$,
\[
\|\psi R\mathcal{N}\|_{L^\theta(I_2, L^\theta')} \lesssim \|u\|_{L^\theta(I_2, L^\theta')}^2 \|\psi Ru\|_{L^\theta_{r_1}}^\theta \\
\lesssim \frac{\varepsilon^{-2\beta\theta(1-\theta)}}{\alpha} \|\psi Ru\|_{L^\theta_{r_1}}^\theta \\
\lesssim \frac{\varepsilon^{-2\beta\theta(1-\theta)}}{\alpha} \|\psi Ru\|_{\theta^\lambda}.
\]

Now, by the assumptions of the scattering criterion, one has
\[
\int_{\mathbb{R}^N} \psi_R(x)|u(T, x)|^2 \, dx < \varepsilon^2.
\]

Moreover, a computation with use of (1.1) and the properties of $\psi$ give
\[
\left|\frac{d}{dt} \int_{\mathbb{R}^N} \psi_R(x)|u(t, x)|^2 \, dx\right| \lesssim R^{-1}.
\]

Then, for any $T - \varepsilon^{-\beta} \leq t \leq T$ and $R > \varepsilon^{-2-\beta}$, yields
\[
\|\psi_R u(t)\| \leq \left(\int_{\mathbb{R}^N} \psi_R(x)|u(T, x)|^2 \, dx + C\frac{T - t}{R}\right)^{\frac{1}{2}} \leq C\varepsilon.
\]

Then, for small $\beta > 0$, there exists $\eta > 0$ such that
\[
\|\psi R\mathcal{N}\|_{L^\theta(I_2, L^\theta')} \lesssim \varepsilon^{-2\beta\theta(1-\theta)} \|\psi Ru\|_{\theta^\lambda}^\theta \\
\lesssim \varepsilon^{\theta(1-\lambda) - 2\beta\theta(1-\theta)} \\
\lesssim \varepsilon^\eta.
\]

On the other hand, by Hardy-Littlewood-Sobolev inequality
\[
\|\psi_R\mathcal{N}\|_{L^\theta} = \|(I_\alpha * | \cdot |^b|u|^p)| \cdot |^b|u|^{p-2}(1 - \psi_R)u\|_{r'} \\
\lesssim \| |x|^b\|_{L^{p_1}(|x|<1)} \|u\|_{r'}^{2p-1-\theta} \|u\|_{L^{\infty}}^\theta \| |x|^b\|_{L^{p_2}(|x|>\frac{2}{\theta})} \\
+ \| |x|^b\|_{L^{p_1}(|x|>1)} \|u\|_{r'}^{2p-1-\theta} \|u\|_{L^{\infty}}^\theta \| |x|^b\|_{L^{p_2}(|x|>\frac{2}{\theta})} \\
\lesssim R^{N+b\mu_2} \|u\|_{r'}^{2p-1-\theta} \|u\|_{r_1}^\theta + R^{N+b\mu} \|u\|_{r'}^{2p-1-\theta} \|u\|_{r_1}^\theta,
\]

where
\[
N + b\mu < 0; \\
N + b\mu_1 > 0; \\
N + b\mu_2 < 0; \\
1 + \frac{\alpha}{N} = \frac{2p - \theta}{r} + \frac{\theta}{2} + \frac{2}{\mu}; \\
1 + \frac{\alpha}{N} = \frac{2p - \theta}{r} + \frac{\theta(N - 4)}{2N} + \frac{1}{\mu_1} + \frac{1}{\mu_2}.
\]
Compute
\[ N + \alpha = \frac{N(2p - \theta)}{r} + \frac{\theta N}{2} + \frac{2N}{\mu} \]
\[ = \theta(s_c - \frac{N}{2}) - 2(2 - s_c) + 2(p - 1)(\frac{N}{2} - s_c) + 2(\frac{N}{2} - s_c) + \frac{\theta N}{2} + \frac{2N}{\mu} \]
\[ = \theta s_c - 2(2 - s_c) + 4 + 2b + \alpha + 2(\frac{N}{2} - s_c) + \frac{2N}{\mu}. \]

Thus,
\[ \mu = \frac{2N}{-2b - \theta s_c} > -\frac{N}{b}. \]

Moreover,
\[ N + \alpha = \frac{N(2p - \theta)}{r} + \frac{\theta(N - 4)}{2} + N \left( \frac{1}{\mu_1} + \frac{1}{\mu_2} \right) \]
\[ = \theta(s_c - \frac{N}{2}) - 2(2 - s_c) + 2(p - 1)(\frac{N}{2} - s_c) + 2(\frac{N}{2} - s_c) + \theta(\frac{N}{2} - 2) + N \left( \frac{1}{\mu_1} + \frac{1}{\mu_2} \right) \]
\[ = \theta(s_c - 2) - 2(2 - s_c) + 4 + 2b + \alpha + 2(\frac{N}{2} - s_c) + N \left( \frac{1}{\mu_1} + \frac{1}{\mu_2} \right) \]
\[ = \theta(s_c - 2) + 2b + \alpha + N + N \left( \frac{1}{\mu_1} + \frac{1}{\mu_2} \right). \]

Taking \( \epsilon < \theta(2 - s_c) \) and \( \frac{N}{\mu_1} = -b + \epsilon \), one gets
\[ \frac{N}{\mu_2} = -b - \epsilon - \theta(s_c - 2) > -b. \]

Then,
\[ \| (1 - \psi_R)N \|_{L^a(I_2, L^r')} \lesssim (R^{N + b_{\mu_2}} + R^{N + b_{\mu_1}}) \| u \|_{2p - 1 - \theta} \]
\[ \lesssim (R^{N + b_{\mu_2}} + R^{N + b_{\mu_1}}) \| I_2 \|_{2p - 1 - \theta} \]
\[ \lesssim (R^{N + b_{\mu_2}} + R^{N + b_{\mu_1}}) \| I_2 \|^{2p - 1 - \theta} \]
\[ \lesssim (R^{N + b_{\mu_2}} + R^{N + b_{\mu_1}}) \epsilon^{-\beta}. \]

Regrouping the above estimates and choosing \( R > \max\{ [\epsilon^{1 + \beta\frac{2p - 1 - \theta}{\beta}}]^{\frac{1}{\beta + \beta_2}}, [\epsilon^{1 + \beta\frac{2p - 1 - \theta}{\beta}}]^{\frac{1}{\beta + \beta_3}} \} \),
one gets for some \( \lambda > 0 \),
\[ \| F_2 \|_{L^a((T, \infty), L^r')} \lesssim \epsilon^{\lambda}. \]

• The term \( F_1 \). Following lines in [1] via the estimate (4.3), which gives \( \| N \|_{\nu'} \lesssim \| u \|_{L^r(2p - 1)} \), there is a positive real number denoted also by \( \lambda > 0 \) such that one gets
\[ \| F_1 \|_{L^a((T, \infty), L^r')} \lesssim \epsilon^{\lambda}. \]

The proof is closed via the three above points. \( \square \)
Now, one proves the scattering criterion. Taking account of Duhamel formula, there exists $\mu > 0$ such that
$$
\|e^{i\Delta^2}u(T)\|_{L^a((0,\infty),L^r)} = \|e^{i(-T)\Delta^2}u(T)\|_{L^a((T,\infty),L^r)} \lesssim e^{\mu}.
$$
So, with Lemma 4.2 via the absorption result Lemma 2.9, one gets
$$
\|u\|_{L^a((0,\infty),L^r)} < \infty.
$$
With Lemma 4.2, one gets for $u_+ := e^{-iT\Delta^2}u(T) + i \int_T^\infty e^{-is\Delta^2}N \, ds$,
$$
\|u(t) - e^{it\Delta^2}u_+\|_{H^2} = \| \int_t^\infty e^{i(t-s)\Delta^2}N \, ds\|_{H^2} \lesssim \|u\|_{L^h((t,\infty),H^2)} \|u\|_{L^a((t,\infty),L^r)}^{2q-1-\theta} \to 0.
$$
This finishes the proof. \hfill \blacksquare

5. SCATTERING

Theorem 2.4 about the scattering of energy global solutions to the focusing problem (1.1) follows with Proposition 4.1 via Lemma 3.3.

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