Research Article

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Using Krasnoselkii's theorem to investigate the Cauchy and neutral fractional $q$-integro-differential equation via numerical technique

Abstract: This article discusses the stability results for solution of a fractional $q$-integro-differential problem via integral conditions. Utilizing the Krasnoselkii's, Banach fixed point theorems, we demonstrate existence and uniqueness results. Based on the results obtained, conditions are provided to ensure the generalized Ulam and Ulam–Hyers–Rassias stabilities of the original system. The results are illustrated by two examples.

Keywords: nonlinear fractional integro-differential equation, neutral differential equation, Cauchy differential equation, existence and stability

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1 Introduction and formulation of the problem

The fractional derivative can be considered as a global operator that has greater degrees of flexibility as compared to integer-order derivative, because a classical derivative with integer-order could be a nearby operator. A few researchers have demonstrated that fractional-order derivatives play a noteworthy part in electrochemical analysis to explain the mechanistic behavior of the concentration of a substrate at the electrode surface to the current [1–13]. Some interesting applications of fractional calculus in science and engineering have been discussed [14–18].

It is interesting to study solution to fractional $q$-integro-differential problem with integral conditions, which will allow a generalized stability. It is shown in [4] that, in a real $k$-dimensional Euclidean space, the local and global solutions exist for the following Cauchy problem:

$$
\left\{\begin{array}{l}
{^{C}D}^\sigma_{0+}y(t)=h(t, y(t))+\int_0^t \Theta(t, \xi, y(\xi))d\xi, y(0) = \eta_0,
\end{array}\right. (1.1)
$$

where $0 < \sigma \leq 1$, $h \in C(\bar{I} \times \mathbb{R}^k, \mathbb{R}^k)$, $\Theta \in C(\bar{I} \times \mathbb{R}^k, \mathbb{R}^k)$, $\bar{I} = [0, 1]$, and $^{C}D^\sigma_{0+}$ is the Caputo fractional operator. A class of abstract delayed fractional neutral integro-differential equations was introduced in [19], for $\sigma \in (1, 2)$,

$$
\left\{\begin{array}{l}
{D}^\rho_{t^+}y(t) = a_t\mathcal{N}(y(t)) + \int_0^t a_{\xi}(t-\xi)\mathcal{N}(y(\xi))d\xi \\
+ h(t, y(\rho(t,y))), \\
y(0) = \eta, \quad y'(0) = 0.
\end{array}\right. (1.2)
$$

Using the Leray–Schauder alternative fixed point theorem, the existence results were obtained (for more details, see [3]). Recently, much attention has been paid to the study of differential equations with fractional derivatives
Note that in [22], the authors introduced and studied a related problem. Shah et al. [8] investigated the following problem under delay differential equations involving Caputo fractional derivative and under nonlocal initial condition with non-monotone term as
\[
\begin{aligned}
&\mathbb{R}^\text{L} D^\sigma_0 g(y(t)) = h(t, y(t), D^\sigma_0 g(y(at))), \quad t \in [0, q], \quad \sigma \in (0, 1), \\
&y(0) = y_0 + \psi(t, y(t)),
\end{aligned}
\]
where \(\mathbb{R}^\text{L} D^\sigma_0 g(y(t))\) represent Riemann–Liouville fractional derivative of order \(\sigma \in (0, 1)\) and \(h \in C([0, q] \times \mathbb{R}^2, \mathbb{R}), \psi \in C([0, q] \times \mathbb{R}, \mathbb{R}).\) Ruzhansky et al. studied particularly the existence for the following problem:
\[
\begin{aligned}
&\mathbb{C} D^\sigma_0 \mathbb{I} D^\sigma_0 g[y(t) + h(t, y(t))] = h_0(t, y(t)), \quad t \in [0, 1], \\
y(0) = \sum_{j=1}^{n} \eta_j y(h_j), \\
&cy(1) = \alpha \int_{0}^{1} y(\xi) dH(\xi) + \sum_{j=1}^{n} a_j \int_{\tau_{j-1}}^{\tau_j} y(\xi) d\xi,
\end{aligned}
\]
where \(0 < \tau_0 < \tau_1 < \ldots < \tau_n < 1, \quad \sigma, \nu \in \mathbb{I} = (0, 1), \quad \eta_j, a_j \in \mathbb{R}\) and \(\mathbb{C} D^\sigma_0 \mathbb{I} D^\sigma_0 g\) is the CFD of order \(\sigma, h_0, h_1, h_2\) are given continuous functions. By using the classical tools of fixed point theory, the existence and uniqueness results are obtained. On an arbitrary domain, in [21], the authors studied an FDE with non-conjugate Riemann–Stieltjes integro-multpoint boundary conditions by using new tools on function analysis. For some more related works, refer to [26,27]. Shah et al. considered the following system is investigated under Atangana, Baleanu, and Caputo fractional order derivative
\[
\begin{aligned}
&\mathbb{A} \mathbb{B} \mathbb{C} D^\sigma_0 g(y_1(t)) = h_1(t, y(t), y_1(t), y_2(t)), \quad y_1(0) = \eta_1, \\
&\mathbb{A} \mathbb{B} \mathbb{C} D^\sigma_0 g(y_2(t)) = h_2(t, y(t), y_2(t), y_3(t)), \quad y_2(0) = \eta_2, \\
&\mathbb{A} \mathbb{B} \mathbb{C} D^\sigma_0 g(y_3(t)) = h_3(t, y(t), y_3(t), y_4(t)), \quad y_3(0) = \eta_3,
\end{aligned}
\]
where \(t \in [0, q], \sigma \in (0, 1), h_i: [0, q] \times \mathbb{R}^3 \rightarrow \mathbb{R}, \quad (i = 1, 2, 3)\) are continuous functions [28]. Derbazi et al. determined the existence criteria of extremal solutions for the following \(\theta\)-Caputo-type fractional differential equation in a Caputo sense under nonlinear boundary conditions
\[
\begin{aligned}
&\mathbb{C} D^\sigma_0 a \mathbb{D}^\sigma_0 b[y(t)] = h(t, y(t)), \quad t \in [a, b], \quad \sigma \in (0, 1), \\
&h_2(y(a), y(b)) = 0,
\end{aligned}
\]
where \(\mathbb{C} D^\sigma_0 a \mathbb{D}^\sigma_0 b\) is the \(\theta\)-fractional operator of order \(\sigma \in (0, 1)\) in the Caputo sense and \(h_1 \in C([a, b] \times \mathbb{R}, \mathbb{R}), \quad h_2 \in C(\mathbb{R}^2, \mathbb{R})\) [29]. Abbas [7] investigated the following Langevin equation with the generalized proportional fractional derivatives with respect to another function
\[
\begin{aligned}
&D^{\sigma, \beta}_a \mathbb{D}^{\sigma, \beta}_a [D^{\sigma, \beta}_a \mathbb{D}^{\sigma, \beta}_a + \lambda] y(t) = h(t, y(t)), \quad t \in [a, b], \lambda \in \mathbb{R}, \\
&D^{\sigma, \beta}_a \mathbb{D}^{\sigma, \beta}_a y(a) = y_0, \quad y(b) + \int_{c}^{b} \psi(t, y(t)) dt, \quad a < c < b, \psi \in C([a, b] \times \mathbb{R}, \mathbb{R}), \quad \lambda > 0,
\end{aligned}
\]
where \(\theta > 0, D^{\sigma, \beta}_a \mathbb{D}^{\sigma, \beta}_a\) are the generalized proportional fractional derivative and integral with respect to another continuous function \(\theta\) of order \(\alpha \in (0, 1)\) and \(1 - \alpha\), respectively, and \(h_1 \in C([a, b] \times \mathbb{R}, \mathbb{R})\) is the given function. The authors in [30] studied qualitative aspects of the system of fractional differential equations via Caputo–Hadamard derivative given as
\[
\begin{aligned}
&\mathbb{C} D^\sigma_0 y(t) + h_0(t, y(t), z(t)) = 0, \quad t \in [1, e], \\
&\mathbb{C} D^\sigma_0 z(t) + h_0(t, y(t), z(t)) = 0,
\end{aligned}
\]
under boundary conditions
\[
\begin{aligned}
y(1) = z(1) = y'(1) = z'(1) = 0 = y'(e) = z'(e),
\end{aligned}
\]
and
\[
y(e) = \phi_1(y), \quad z(e) = \phi_2(z) \text{ with } \sigma \in (3, 4), \quad a \in (0, 1), \quad \phi_1, \phi_2 : \mathbb{Y} \rightarrow \mathbb{R}
\]
are continuous functions, where \(\mathbb{Y}\) is complete norm space.

The authors in [2], considered the problem for the system (1.9), and we generalized the system in the \(q\)FDE which is not explicitly presented, and therefore it makes sense to consider for \(t \in \mathbb{I}, \sigma, \nu \in \mathbb{I}\), the problem for system as follows:
\[
\begin{aligned}
&\mathbb{C} D^q_0 y(t) = h_0(t, y(t)) + \int_{t}^{\tau} \Theta(t, \xi, y(\xi)) d\xi, \\
&y(0) = \eta \int_{0}^{\tau^*} y(\xi) d\xi, \quad (\tau^* \in \mathbb{I}),
\end{aligned}
\]
where \(\eta\) is a real constant, \(\mathbb{C} D^q_0\) is the Caputo fractional \(q\)-derivative of order \(\sigma + \nu\), \(I^q_0\) denotes the left-sided Riemann–Liouville fractional \(q\)-integral of order \(\sigma\), and \(h_1 : \mathbb{I} \times \mathbb{I} \rightarrow \mathbb{I}, \Theta : \mathbb{I}^2 \times \mathbb{I} \rightarrow \mathbb{I}\), are appropriate functions satisfying some conditions which will be stated later. \(\mathbb{I}\) is a Banach space equipped with the norm \(\|\cdot\|\).

Here, this study is focused on the question of existence and uniqueness in Section 3. In addition, Section 4 is devoted to show a generalized stability. Note that this representation also allows us to generalize the results obtained recently in the literature. The article is ended by two examples illustrating our results.
2 Notations and preliminaries

We recall some essential preliminaries that are used for the results of the subsequent sections. Let \( t_0 \in \mathbb{R} \) and \( q \in \mathbb{I} \). The time scale \( \mathbb{T}_0 \) is defined by
\[
\mathbb{T}_0 = \{0\} \cup \{ t : t = t_0 q^n, \; \forall n \in \mathbb{N} \}.
\]
If there is no confusion concerning \( t_0 \) we shall denote \( \mathbb{T}_0 \) by \( \mathbb{T} \). Let \( s \in \mathbb{R} \). Define \( s_l = (1 - q^s)/(1 - q) \) (see [31]).

The \( \nu \)-factorial function \( (y - z)^{(n)}_\nu \) is defined by
\[
(y - z)^{(n)}_\nu = \prod_{k=0}^{n-1} (y - zq^k), \quad n \in \mathbb{N}_0,
\]
and \( (y - z)^{(0)}_\nu = 1 \), where \( y, z \in \mathbb{R} \) and \( \mathbb{N}_0 = \{0\} \cup \mathbb{N} \) (see [32]). Also, we have
\[
(y - z)^{(n)}_\nu = y^n \prod_{k=0}^{\infty} \frac{y - zq^k}{y - zq^{\nu + k}}, \quad \sigma \in \mathbb{R}, \; s \neq 0.
\]

Algorithms 1 and 2 simplify \( \nu \)-factorial functions \( (y - z)^{(n)}_\nu \) and \( (y - z)^{(s)}_\nu \), respectively. In [33], the authors proved
\[
(y - z)^{(n)}_\nu(y - q^s z)_\nu^s, \quad (sy - sz)^{(s)}_\nu = s^s (y - z)^{(s)}_\nu.
\]

If \( z = 0 \), then it is clear that \( y^{(0)} = y^0 \). The \( q \)-gamma function is given by [31].
\[
\Gamma_q(y) = (1 - q)^{1-y} \frac{(1 - q)^{(1 - y)}(1 - q)^{(y - 1)}}{1 - q^{1 - y} - 1}, \quad (y \in \mathbb{R} \setminus \{-1, 0\}).
\]

In fact, by using (2.2), we have
\[
\Gamma_q(y) = (1 - q)^{1-y} \prod_{k=0}^{\infty} \frac{1 - q^{k+1}}{1 - q^{k+y+1}}.
\]

Algorithm 3 shows the MATLAB lines for calculation of \( \Gamma_q(y) \) which we tend \( n \) to infinity in it. Note that, \( \Gamma_q(y + 1) = \Gamma_q(y) \) [33, Lemma 1]. For any positive numbers \( \sigma \) and \( v \), the \( q \)-beta function is defined by
\[
B_q(\sigma, v) = \frac{\Gamma_q(\sigma)\Gamma_q(v)}{\Gamma_q(\sigma + v)}.
\]

For a function \( w : \mathbb{T} \to \mathbb{R} \), the \( q \)-derivative of \( w \), is
\[
D_q[w](t) = \left( \frac{d}{dt}_q \right) w(t) = \frac{w(qt) - w(t)}{qt - t},
\]
for all \( t \in \mathbb{T} \setminus \{0\} \), and \( D_q[w](0) = \lim_{t \to 0} D_q[w](t) \) (see [32]). Also, the higher order \( q \)-derivative of the function \( y \) is defined by
\[
D^n_q[y](t) = D_q[D^{n-1}_q[y]](t), \quad \forall n \geq 1,
\]
where \( D^n_q[y](t) = y(t) \) (see [32]). In fact,
\[
D^n_q[y](t) = \frac{1}{t^n(1 - q^n)} \sum_{k=0}^{n} \frac{(1 - q^{-n})^{(k)}}{(1 - q^{-n})^{(k)}} q^k y(q^k),
\]
for \( t \in \mathbb{T} \setminus \{0\} \) (see ref. [27]).

**Remark 2.1.** [9] By using Eq. (2.1), we can change Eq. (2.6) as follows:
\[
D^n_q[y](t) = \frac{1}{t^n(1 - q^n)} \sum_{k=0}^{n} \frac{(1 - q^{-n})^{(k)}}{(1 - q^{-n})^{(k)}} q^k y(q^k).
\]

Algorithms 4 and 5 show the MATLAB codes for calculation of Eqs. (2.5) and (2.7), respectively. The \( q \)-integral of the function \( y \) is defined by
\[
I_q[y](t) = \int_0^1 y(t) dt dq_q = t(1-q) \sum_{k=0}^{\infty} q^k y(q^k),
\]
for \( 0 \leq t \leq b \), provided the series absolutely converges (see [32]). By using Algorithm 6, we can obtain the numerical results of \( I_q[y](t) \) when \( n \to \infty \). If \( s \in [0, b] \), then
\[
I_q[y](t) = \int_0^b y(t) dt dq_q = \frac{1}{q^n} \sum_{k=0}^{\infty} q^k [by(bq^k) - sy(sq^k)],
\]
whenever the series exists. The operator \( I_q^n \) is given by \( I_q^n[y](t) = y(t) \) and [32]
\[
I_q^n[y](t) = I_q[I_q^{n-1}[y]](t), \quad \forall n \geq 1, \; y \in \mathbb{C}([0, b]).
\]

It has been proved that
\[
D_q[I_q^n[y]](t) = y(t), \quad I_q[D_q[y]](t) = y(t) - y(0),
\]
whenever the function \( y \) is continuous at \( t = 0 \) (see [32]). The fractional Riemann–Liouville-type \( q \)-integral of the function \( y \) is defined by
\[
I_q^n[y](t) = \int_0^t (t - \xi)^{(n-1)} \frac{y(\xi)}{\Gamma_q(n)} dq_q, \quad n \geq 1,
\]
for \( t \in [0, 1] \) and \( \sigma > 0 \) (see refs [27, 34]).

**Remark 2.2.** [9] By using Eqs. (2.2), (2.3), and (2.8), we have
\[
I^{[a]}_q[y](t) = t^a(1 - q)^a \lim_{n \to \infty} \sum_{k=0}^{n} q^k \sum_{i=0}^{n} q^k \frac{(1 - q^{-n})^{(k)}}{(1 - q^{-n})^{(k)}} q^k y(q^k),
\]
for \( t \in \mathbb{T} \setminus \{0\} \) (see [27]).
Algorithm 7 shows the MATLAB codes of numerical technique. The Caputo fractional $q$-derivative of the function $y$ is defined by

$$
\mathcal{D}_q^\sigma y(t) = t^{-\sigma} \frac{d}{dt} \left( t^{\sigma-1} \frac{d^{[\sigma]} y(t)}{d [\sigma-1] q} \right)
$$

for $t \in [0, 1]$ and $\sigma > 0$ (see [34,35]). It has been proved that

$$
\mathcal{I}_q^n \mathcal{D}_q^\sigma y(t) = \mathcal{I}_q^{\sigma+n} y(t),
$$

and $\mathcal{D}_q^\sigma \mathcal{I}_q^n y(t) = y(t)$, where $\sigma, \nu \geq 0$ [34]. Also, (see [34])

$$
\mathcal{I}_q^n \mathcal{D}_q^\sigma y(t) = \mathcal{D}_q^n \mathcal{I}_q^\sigma y(t) - \frac{1}{n!} \sum_{k=0}^{n-1} \Gamma(\sigma + k - n + 1) q^k y(0), \quad \sigma > 0, \ n \geq 1.
$$

**Remark 2.3.** From Eq. (2.3), Remark 2.1, and Eq. (2.10) in Remark 2.2, we obtain

$$
\mathcal{D}_q^\sigma y(t) = \frac{1}{t^{\sigma}} \lim_{n \to \infty} \frac{n}{n} \prod_{i=0}^{n-1} \left( 1 - q^{i+1} \right) \mathcal{I}_q^n \mathcal{D}_q^\sigma y(t)
$$

(2.12)

$$
\times \left( \prod_{i=0}^{n} \left( 1 - q^i \right) \right) \left( \prod_{i=0}^{n} \left( 1 - q^{\sigma + 1 + k} \right) \right)
$$

Algorithm 8 shows the MATLAB codes of numerical technique. One can find other algorithms in [36]. Now, we introduce some basic definitions, lemmas, and theorems, which are used in the subsequent sections.

**Lemma 2.4.** [37] Let $y \in AC^q[t_1, t_2]$. Then, one has

1. For $\sigma, \delta > 0$,

$$
\mathcal{I}_q^{-\delta} \mathcal{D}_q^\sigma y(t) = y(t) + \sum_{i=0}^{n-1} \mathcal{C}_i (t - t_i)^i, \ (c_0, c_1, ..., c_{n-1} \in \mathbb{R}),
$$

for $n - 1 < \sigma \leq n, \ n \in \mathbb{N}$;

2. $\mathcal{I}_q^n \mathcal{I}_q^m y(t) = \mathcal{I}_q^{n+m} y(t) = \mathcal{I}_q^n y(t)$;

3. $\mathcal{D}_q^n \mathcal{I}_q^m y(t) = y(t)$.

**Lemma 2.5.** [37] Let $n - 1 < \sigma \leq n, \ n \in \mathbb{N}$, and $y \in C[t_1, t_2]$. Then for all $t \in [t_1, t_2]$, we have $\mathcal{D}_q^\sigma \mathcal{I}_q^n y(t) = y(t)$.

**Lemma 2.6.** [37] Let $\sigma \in (0, 1)$. Then for each $y \in AC[0, 1]$, $\mathcal{I}_q^{[\sigma]} y(t) = y(t)$ for a.e. $t \in [0, 1]$, where

$$
\mathcal{D}_q^\sigma y(t) = \frac{d}{d t} \frac{1}{\Gamma(1 - \sigma)} \int_0^t (t - \xi)^{1-\sigma} \mathcal{D}_q^\sigma y(\xi) \ d\xi.
$$

**Lemma 2.7.** ([38], Banach fixed point theorem) Let $\mathcal{B}$ be a non-empty complete metric space and $\mathcal{T} : \mathcal{B} \to \mathcal{B}$ is a contraction mapping. Then, there exists a unique point $y \in \mathcal{B}$ such that $\mathcal{T}(y) = y$.

**Lemma 2.8.** ([38], Krasnoselskii fixed point theorem) Let $\mathcal{B}$ be bounded, closed, and convex subset in a Banach space $\mathcal{B}$. If $\mathcal{T}_1, \mathcal{T}_2 : \mathcal{B} \to \mathcal{B}$ are two applications satisfying the following conditions: (A1) $\mathcal{T}_1(y) + \mathcal{T}_2(z) \in \mathcal{B}$ for every $y, z \in \mathcal{B}$; (A2) $\mathcal{T}_1$ is a contraction; (A3) $\mathcal{T}_2$ is compact and continuous. Then there exists $v \in \mathcal{B}$ such that $\mathcal{T}_1(v^*) + \mathcal{T}_2(v^*) = v^*$.

### 3 Existence results

Before presenting our main results, we need the following auxiliary lemma.

**Lemma 3.1.** Let $\sigma + \nu \in 1$ and $\eta \nu + 1$. Assume that $h_1, h_2, \text{ and } \Theta$ are three continuous functions. If $y \in C(I, \mathcal{I})$, then $y$ is solution of (1.9) if $y$ satisfies the IE

$$
y(t) = \int_0^1 (t - \xi)^{\sigma + 1} \frac{\Gamma(\sigma + \nu)}{\Gamma(1 - \sigma)} h(\xi, y(\xi)) \ d\xi + \int_0^1 \Theta(\xi, s, y(s)) \ d\xi
$$

(3.1)

**Proof.** Let $y \in C(I, \mathcal{I})$ be a solution of system (1.9). First, we show that $y$ is a solution of integral Eq. (3.1). By Lemma 2.4 and using boundary condition, we obtain
By substituting from integral boundary condition of our problem with using Fubini's theorem and after some computations, we obtain

\[
y(0) = \eta \int_0^\xi y(\xi) d\xi
\]

From integral boundary condition of our problem with using Fubini's theorem and after some computations, we obtain

\[
y(0) = \eta \int_0^\xi y(\xi) d\xi
\]

that is,

\[
y(0) = \frac{\eta}{1 - \eta^*} \int_0^{\tau^*} \int_0^s (s - r)^{\nu v - 1} \frac{h_2(s, y(r))}{\Gamma_q(\sigma + v)} dr d\xi
\]

Finally, by substituting (3.5) in (3.4), we find (3.1). Conversely, from Lemma 3.1 and by applying the operator \( D^\nu v \) on both sides of (3.1), we find
\[ C_D^{\alpha+\nu}y(t) \]
\[ = C_D^{\alpha+\nu}\left[ q(t, y(t)) + \int_0^t \Theta(t, \xi, y(\xi))d\xi \right] + h_0(t, y(t)) \]
\[ + I_D^\nu h(t, y(t)) \]
\[ = h_0(t, y(t)) + I_D^\nu h(t, y(t)) + \int_0^t \Theta(t, \xi, y(\xi))d\xi. \]  

This means that \( y \) satisfies the equation in problem (1.9). Furthermore, by substituting \( t \) by 0 in integral Eq. (3.1), we are clear that the integral boundary condition in (1.9) holds. Therefore, \( y \) is solution of problem (1.9), which completes the proof.

In order to prove the existence and uniqueness of solution for problem (1.9) in \( C(I, Y) \), we use two fixed point theorem. First, we transform the system (1.9) into fixed point problem as \( y = Ly \), where \( L : (I, Y) \rightarrow (I, Y) \) is an operator defined by following

\[ Ly(t) = \int_0^t \left[ (t - \xi)^{\alpha-1}_D \right] \left[ h(\xi, y(\xi)) + \int_0^\xi \Theta(\xi, s, y(s))d\xi \right] + \right. \]
\[ + \int_0^\xi (\xi - s)^{\alpha-1}_D h_0(s, y(s))d\xi \]
\[ + \frac{\eta}{1 - \eta^\nu} \int_0^\nu (\tau^\nu - s)^{\alpha+\nu}_D h_0(s, y(s))d\xi \]
\[ + \left. \int_0^s \Theta(s, r, y(r))dr \right] + \int_0^s (s - r)^{\alpha-1}_D h_0(r, y(r))d\xi. \]  

**3.1 Existence result by Krasnoselskii’s fixed point**

**Theorem 3.2.** Consider continuous functions \( h_1, h_2 : I \times Y \rightarrow Y \) and \( \Theta : I^2 \times Y \rightarrow Y \) such that satisfying: (H1) the inequalities

\[ \| (t, y(t)) - h_1(z(t)) \| \leq \mu \| y(t) - z(t) \|, \quad j = 1, 2, \]

and

\[ \| \Theta(t, s, z(s)) \| \leq \mu \| y(s) - z(s) \|, \]

where \( \mu, H_j \geq 0, (j = 1, 2) \) with \( \mu = \max \{ \mu_1, \mu_2, \mu^* \} \). (H2) there exist three functions \( g^*, \varrho_j \in L_\infty(I, R^+), (j = 1, 2) \), such that

\[ \| h_j(t, y(t)) \| \leq \varrho_j(t) \| y(t) \|, \quad j = 1, 2, \]

and

\[ \| \Theta(t, s, y(s)) \| \leq g^*(t) \| y(s) \|, \]

\( \forall t \in I, y, z \in Y \) and \( (t, s) \in G = \{(t, s) : 0 \leq s \leq t \leq 1 \} \). If \( \lambda \leq 1 \) and \( \mu \lambda^* \leq 1 \), then problem (1.9) has at least one solution on \( I \), where

\[ \lambda = \frac{|\varrho_1|_{L_\infty} + |\varrho_2|_{L_\infty}}{\Gamma_\nu(\sigma + \nu + 1) + \Gamma_\sigma(\sigma + 1)\Gamma_\nu(\sigma + \nu + 1)} \]
\[ + \frac{|\eta|_{L_\infty}}{|1 - \eta^\nu| \Gamma_\nu(\sigma + \nu + 2) + |\eta|_{L_\infty} \Gamma_\sigma(\sigma + 1)\Gamma_\nu(\sigma + \nu + 1)} \]
\[ + \frac{2\lambda^* \nu^{\alpha+\nu}}{|1 - \eta^\nu| \Gamma_\nu(\sigma + \nu + 2)} \]
\[ + \frac{\nu^{\alpha+\nu}\nu(\sigma + 1)\Gamma(\sigma + \nu + 1)}{\Gamma_\sigma(\sigma + 1)\Gamma_\nu(\sigma + \nu + 1)}. \]

**Proof.** For any function \( y \in C(I, Y) \), we define the norm

\[ \| y \|_{L_{\infty}} = \max \{ e^{-\lambda t} \| y(t) \| : t \in I \}, \]

and consider the closed ball

\[ B_\lambda = \{ y \in C(I, Y) : \| y \|_{L_{\infty}} \leq \lambda \}. \]

Next, let us define the operators \( L_1, L_2 \) on \( B_\lambda \) as follows:

\[ L_1 y(t) = \int_0^t \left[ (t - \xi)^{\alpha-1}_D \right] \left[ h_1(\xi, y(\xi)) + \int_0^\xi \Theta(\xi, s, y(s))d\xi \right] + \right. \]
\[ + \int_0^\xi (\xi - s)^{\alpha-1}_D h_0(s, y(s))d\xi \]
\[ + \left. \int_0^\nu (\tau^\nu - s)^{\alpha+\nu}_D h_0(s, y(s))d\xi \right| + \int_0^s \Theta(s, r, y(r))dr \]
\[ + \int_0^s (s - r)^{\alpha-1}_D h_0(r, y(r))d\xi. \]  

and
\[
\mathcal{U}_t y(t) = \frac{\eta}{1 - \eta \tau^q} \int_0^\tau (\tau - s)^{q-1} \mathcal{G}_t(\sigma + s + 1) \bigg[ h_t(s, y(s)) + \int_0^s \Theta_t(s, r, y(r))dr \bigg] ds + \int_0^\tau (s - r)^{q-1} \mathcal{G}_t(\sigma + s + 1) h_t(r, y(r))dr ds.
\]

(3.11)

For \( y, z \in B_t \), \( t \in I \) and by the assumption (H2), we find

\[
\|\mathcal{U}_t y(t) + \mathcal{U}_t z(t)\|
\leq \int_0^\tau \left( 1 - \xi \tau^q \right) \left[ \|h_t(\xi, y(\xi))\| + \int_0^\xi \|\Theta_t(\xi, s, y(s))\|ds \right. \\
+ \int_0^\tau \left( \xi - s \right)^{q-1} \mathcal{G}_t(\sigma + s + 1) h_t(\xi, y(\xi))ds \bigg] d\xi \\
+ \frac{\xi}{1 - \eta \tau^q} \int_0^\tau (\tau - s)^{q-1} \mathcal{G}_t(\sigma + s + 1) \bigg[ h_t(s, z(s)) + \int_0^s \Theta_t(s, r, z(r))dr \bigg] ds \\
+ \int_0^\tau \left( s - r \right)^{q-1} \mathcal{G}_t(\sigma + s + 1) h_t(r, z(r))dr ds \bigg] d\xi \\
\leq \int_0^\tau \left( 1 - \xi \tau^q \right) \left[ \|h_t(\xi, y(\xi))\| + \int_0^\xi \|\Theta_t(\xi, s, y(s))\|ds \right. \\
+ \int_0^\tau \left( \xi - s \right)^{q-1} \mathcal{G}_t(\sigma + s + 1) h_t(\xi, y(\xi))ds \bigg] d\xi \\
+ \frac{\xi}{1 - \eta \tau^q} \int_0^\tau (\tau - s)^{q-1} \mathcal{G}_t(\sigma + s + 1) \bigg[ h_t(s, z(s)) + \int_0^s \Theta_t(s, r, z(r))dr \bigg] ds \\
+ \int_0^\tau \left( s - r \right)^{q-1} \mathcal{G}_t(\sigma + s + 1) h_t(r, z(r))dr ds \bigg] d\xi \\
\leq \int_0^\tau \left( 1 - \xi \tau^q \right) \left[ \|h_t(\xi, y(\xi))\| + \int_0^\xi \|\Theta_t(\xi, s, y(s))\|ds \right. \\
+ \int_0^\tau \left( \xi - s \right)^{q-1} \mathcal{G}_t(\sigma + s + 1) h_t(\xi, y(\xi))ds \bigg] d\xi \\
+ \frac{\xi}{1 - \eta \tau^q} \int_0^\tau (\tau - s)^{q-1} \mathcal{G}_t(\sigma + s + 1) \bigg[ h_t(s, z(s)) + \int_0^s \Theta_t(s, r, z(r))dr \bigg] ds \\
+ \int_0^\tau \left( s - r \right)^{q-1} \mathcal{G}_t(\sigma + s + 1) h_t(r, z(r))dr ds \bigg] d\xi \\
\leq \int_0^\tau \left( 1 - \xi \tau^q \right) \left[ \|h_t(\xi, y(\xi))\| + \int_0^\xi \|\Theta_t(\xi, s, y(s))\|ds \right. \\
+ \int_0^\tau \left( \xi - s \right)^{q-1} \mathcal{G}_t(\sigma + s + 1) h_t(\xi, y(\xi))ds \bigg] d\xi \\
+ \frac{\xi}{1 - \eta \tau^q} \int_0^\tau (\tau - s)^{q-1} \mathcal{G}_t(\sigma + s + 1) \bigg[ h_t(s, z(s)) + \int_0^s \Theta_t(s, r, z(r))dr \bigg] ds \\
+ \int_0^\tau \left( s - r \right)^{q-1} \mathcal{G}_t(\sigma + s + 1) h_t(r, z(r))dr ds \bigg] d\xi \\
\leq \int_0^\tau \left( 1 - \xi \tau^q \right) \left[ \|h_t(\xi, y(\xi))\| + \int_0^\xi \|\Theta_t(\xi, s, y(s))\|ds \right. \\
+ \int_0^\tau \left( \xi - s \right)^{q-1} \mathcal{G}_t(\sigma + s + 1) h_t(\xi, y(\xi))ds \bigg] d\xi \\
+ \frac{\xi}{1 - \eta \tau^q} \int_0^\tau (\tau - s)^{q-1} \mathcal{G}_t(\sigma + s + 1) \bigg[ h_t(s, z(s)) + \int_0^s \Theta_t(s, r, z(r))dr \bigg] ds \\
+ \int_0^\tau \left( s - r \right)^{q-1} \mathcal{G}_t(\sigma + s + 1) h_t(r, z(r))dr ds \bigg] d\xi.
\]

Therefore,

\[
\|\mathcal{U}_t y + \mathcal{U}_t z\|
\leq \int_0^\tau \left( 1 - \xi \tau^q \right) \left[ \|h_t(\xi, y(\xi))\| + \int_0^\xi \|\Theta_t(\xi, s, y(s))\|ds \right. \\
+ \int_0^\tau \left( \xi - s \right)^{q-1} \mathcal{G}_t(\sigma + s + 1) h_t(\xi, y(\xi))ds \bigg] d\xi \\
+ \frac{\xi}{1 - \eta \tau^q} \int_0^\tau (\tau - s)^{q-1} \mathcal{G}_t(\sigma + s + 1) \bigg[ h_t(s, z(s)) + \int_0^s \Theta_t(s, r, z(r))dr \bigg] ds \\
+ \int_0^\tau \left( s - r \right)^{q-1} \mathcal{G}_t(\sigma + s + 1) h_t(r, z(r))dr ds \bigg] d\xi \\
\leq \int_0^\tau \left( 1 - \xi \tau^q \right) \left[ \|h_t(\xi, y(\xi))\| + \int_0^\xi \|\Theta_t(\xi, s, y(s))\|ds \right. \\
+ \int_0^\tau \left( \xi - s \right)^{q-1} \mathcal{G}_t(\sigma + s + 1) h_t(\xi, y(\xi))ds \bigg] d\xi \\
+ \frac{\xi}{1 - \eta \tau^q} \int_0^\tau (\tau - s)^{q-1} \mathcal{G}_t(\sigma + s + 1) \bigg[ h_t(s, z(s)) + \int_0^s \Theta_t(s, r, z(r))dr \bigg] ds \\
+ \int_0^\tau \left( s - r \right)^{q-1} \mathcal{G}_t(\sigma + s + 1) h_t(r, z(r))dr ds \bigg] d\xi \\
\leq \epsilon \lambda \leq \epsilon.
\]

This implies that \((\mathcal{U}_t y + \mathcal{U}_t z) \in B_t\). Here we used the computations

\[
\int_0^\tau (1 - \xi \tau^q) \xi^q \mathcal{G}_t(\xi, y(\xi))d\xi = \beta_0(\xi, y(\xi))
\]

and the estimations:
In this step, we show that $\mathcal{U}_1$ is a contraction mapping. Let $y, z \in \mathcal{S}$, $t \in [0, 1]$. We have

$$
\|\mathcal{U}_1 y(t) - \mathcal{U}_1 z(t)\| \\
\leq \frac{|\eta|}{|1 - \eta t^r|} \int_0^r t^r (t^r - s^q_y)^{\sigma + v} \frac{\Gamma_q(\sigma + v + 1)}{\Gamma_q(\sigma)} ds \\
\times \left[ \|h_1(s, y(s)) - h_1(s, z(s))\| \\
+ \int s^r \|h_2(s, y'()) - h_2(s, z'())\| ds \\
+ \int s^r \|\Theta(s, y(s)) - \Theta(s, z(s))\| ds \right] ds
$$

which implies that

$$
\|\mathcal{U}_1 y\| \leq \frac{A}{|1 - \eta t^r|} \int_0^r t^r (t^r - s^q_y)^{\sigma + v} \frac{\Gamma_q(\sigma + v + 1)}{\Gamma_q(\sigma)} ds \\
\times \left[ \|h_1(s, y(s))\| + \int s^r \|h_2(s, y'())\| ds \\
+ \int s^r \|\Theta(s, y(s))\| ds \right] ds
$$

Thus,

$$
\|\mathcal{U}_1 y - \mathcal{U}_1 z\| \leq \frac{|\eta|}{|1 - \eta t^r|} \int_0^r t^r (t^r - s^q_y)^{\sigma + v} \frac{\Gamma_q(\sigma + v + 1)}{\Gamma_q(\sigma)} ds \\
\times \left[ \|y - z\| \left\| \frac{e^s}{e^{s-1}} - e^s \right\| \right] \left[ (e^{s-1} - 1) \right] \\
\times \left[ \int s^r \|h_2(s, y'()) - h_2(s, z'())\| ds \\
+ \int s^r \|\Theta(s, y(s)) - \Theta(s, z(s))\| ds \right] ds
$$

Then since $\mu t \leq 1$, $\mathcal{U}_2$ is a contraction mapping. The continuity of the functions $h_1, h_2, \Theta$ implies that $\mathcal{U}_1$ is continuous and $\mathcal{U}_1 B_t \subset B_t$, for each $y \in B_t$, i.e., $\mathcal{U}_1$ is uniformly bounded on $B_t$ as

$$
\|\mathcal{(U_1 y)}(t)\| \leq \frac{t}{\Gamma_q(\sigma + v)} \left[ \left( t - \xi^q_{y'} - \frac{1}{\Gamma_q(\sigma + v + 1)} \right) \\
\times \|h_1(\xi, y(\xi))\| + \int_0^\xi \|\Theta(\xi, s, y(s))\| ds \right]
$$

which implies that

$$
\|\mathcal{U}_1 y\| \leq \frac{t}{\Gamma_q(\sigma + v)} \left[ \left( t - \xi^q_{y'} - \frac{1}{\Gamma_q(\sigma + v + 1)} \right) \\
\times \|h_1(s, y(s))\| + \int_0^\xi \|\Theta(s, y(s))\| ds \right]
$$

Finally, we will show that $(\mathcal{U}_1 B_t)$ is equi-continuous. For this end, we put

$$
\mathcal{H}_j = \sup_{(t, y(t)) \in B_t} \|h_1(t, y(t))\|, \quad \mathcal{H}_j = \sup_{(t, y(t)) \in B_t} \|\Theta(t, y(t))\| ds
$$

Let for any $y \in B_t$ and for each $t_1, t_2 \in I$ with $t_1 \leq t_2$, we have

$$
\|\mathcal{U}_1 y(t_2) - \mathcal{U}_1 y(t_1)\| \\
\leq \int_{t_1}^{t_2} (t_2 - \xi^q_{y'}) \frac{1}{\Gamma_q(\sigma + v + 1)} ds \\
\times \left[ \|h_1(\xi, y(\xi))\| + \int_0^\xi \|\Theta(\xi, s, y(s))\| ds \right]
$$

Then since $\mu t \leq 1$, $\mathcal{U}_2$ is a contraction mapping. The continuity of the functions $h_1, h_2, \Theta$ implies that $\mathcal{U}_1$ is continuous and $\mathcal{U}_1 B_t \subset B_t$, for each $y \in B_t$, i.e., $\mathcal{U}_1$ is uniformly bounded on $B_t$ as
\[
\begin{align*}
&\leq \int_{t_1}^{t_2} \frac{(t_2 - \xi q)^{\alpha - 1}}{\Gamma(\sigma + v)} \left[ \mathcal{H}_1 + \mathcal{Y} + \int_{0}^{\xi} \frac{\mathcal{H}_2(\mathcal{H}_1 - s)^{\alpha - 1}}{\Gamma(\sigma)} ds \right] \, dq_1 \\
&\quad + \frac{1}{\Gamma(\sigma + v)} \int_{t_1}^{t_2} \left[ (t_1 - \xi q)^{\alpha - 1} - (t_2 - \xi q)^{\alpha - 1} \right] \\
&\quad \times \left[ \mathcal{H}_1 + \mathcal{Y} + \int_{0}^{\xi} \frac{\mathcal{H}_2(\mathcal{H}_1 - s)^{\alpha - 1}}{\Gamma(\sigma)} ds \right] \, dq_1 \\
&\quad \leq \int_{t_1}^{t_2} \frac{(t_2 - \xi q)^{\alpha - 1}}{\Gamma(\sigma + v)} \left[ \mathcal{H}_1 + \mathcal{Y} + \int_{0}^{\xi} \frac{1}{\Gamma(\sigma + v + 1)} [\mathcal{H}_1 + \mathcal{Y}] + \frac{\mathcal{H}_2}{\Gamma(\sigma + v + 1)} \right] (t_2 - t_1)^{\alpha - v + 1} + (t_1 - \xi q)^{\alpha - v} - (t_2 - \xi q)^{\alpha - v}].
\end{align*}
\]

The RHS of the last inequality is independent of \( y \) and tends to zero when \( |t_2 - t_1| \to 0 \), this means that \( \mathcal{U} \mathcal{y}(t_2) - \mathcal{U} \mathcal{y}(t_1) \to 0 \), which implies that \( \mathcal{U} \mathcal{B} \) is equicontinuous, then \( \mathcal{U} \) is relatively compact on \( \mathcal{B} \). Hence, by the Arzelà–Ascoli theorem, \( \mathcal{U} \) is compact on \( \mathcal{B} \). Now, all hypotheses of Theorem 3.2 hold; therefore, the operator \( \mathcal{U} \mathcal{I} + \mathcal{U} \) has a fixed point on \( \mathcal{B} \). So problem (1.9) has at least one solution on \( \mathcal{I} \). This proves the theorem.

### 3.2 Existence and uniqueness result

**Theorem 3.3.** Assume that \( (H_i) \) holds. If \( \mu \lambda < 1 \), then the boundary value problems (1.9) has a unique solution on \( \mathcal{I} \).

**Proof.** Define \( m = \max\{m_1, m_2, m^*\} \), where \( m_1 \) and \( m^* \) are positive numbers such that

\[
m_j = \sup_{t \in \mathcal{I}} \|h(t, 0)\|, \quad (j = 1, 2),
\]

\[
m^* = \sup_{(t, s) \in \mathcal{I} \times \mathcal{I}} \|\Theta(t, s, 0)\|.
\]

We fix \( \ell \geq \frac{m^*}{\mu \lambda} \) and we consider

\[
\mathcal{N}_\ell = \{y \in C(\mathcal{I}, \mathcal{S}) : \|y\| \leq \ell\}.
\]

Then, in view of the assumption \( (H_i) \), we have

\[
\|\mathcal{H}_2(t, y(t))\| = \|h(t, y(t)) - h(t, 0) + h(t, 0)\| \\
\leq \|h(t, y(t)) - h(t, 0)\| + \|h(t, 0)\| \\
\leq \mu \|y\| + m_1,
\]

\[
\|\mathcal{H}_2(t, y(t))\| \leq \mu \|y\| + m^*,
\]

and \( \|\Theta(t, s, y(s))\| \leq m^* \|y\| + m^* \). In the first step, we show that \( \mathcal{U} \mathcal{N}_\ell \subset \mathcal{N}_\ell \). For each \( t \in \mathcal{I} \) and for any \( y \in \mathcal{N}_\ell \),

\[
\|\mathcal{U} \mathcal{y}(t)\| \leq \int_{0}^{\xi} \left[ (t - \xi q)^{\alpha - v + 1} + (t_1 - \xi q)^{\alpha - v} - (t_2 - \xi q)^{\alpha - v} \right] \, dq_1 \\
\times \left[ \|h(t, y(t))\| + \int_{0}^{\xi} \|\Theta(t, s, y(s))\| \, ds \right] \\
\times \left[ \|h(t, y(t))\| + \int_{0}^{\xi} \|\Theta(t, s, y(s))\| \, ds \right] \\
\times \left[ \|h(t, y(t))\| + \int_{0}^{\xi} \|\Theta(t, s, y(s))\| \, ds \right] \\
\times \left[ \|h(t, y(t))\| + \int_{0}^{\xi} \|\Theta(t, s, y(s))\| \, ds \right] \\
\leq \mu \|y\| + m^* \lambda \leq \ell.
\]

Hence, \( \mathcal{U} \mathcal{N}_\ell \subset \mathcal{N}_\ell \). Now, in the second step, we shall show that \( \mathcal{U} : \mathcal{N}_\ell \to \mathcal{N}_\ell \) is a contraction. From the assumption \( (H_i) \) we have for any \( y, z \in \mathcal{N}_\ell \) and each \( t \in \mathcal{I} \)

\[
\|\mathcal{U} \mathcal{y}(t) - \mathcal{U} \mathcal{z}(t)\| \\
\leq \int_{0}^{\xi} \left[ (t - \xi q)^{\alpha - v + 1} \right] \, dq_1 \\
\times \left[ \|h(t, y(t))\| - \|h(t, z(t))\| \right] \\
\times \left[ \|h(t, y(t))\| - \|h(t, z(t))\| \right] \\
\times \left[ \|h(t, y(t))\| - \|h(t, z(t))\| \right] \\
\times \left[ \|h(t, y(t))\| - \|h(t, z(t))\| \right] \\
\leq \mu \|y - z\|.
\]

Since \( \mu \lambda < 1 \), it follows that \( \mathcal{U} \) is a contraction. All assumptions of Lemma (3.1) are satisfied, then there exists
and problem (1.9) is called Ulam–Hyers–Rassias stable with respect to \( \varrho \in C(\bar{I}, R) \) if
\[
\|z(t)\| \leq \varrho(t), \quad t \in \bar{I},
\]
and there exist a real number \( \gamma > 0 \) and a solution \( z \in C(\bar{I}, \mathcal{S}) \) of problem (1.9) such that
\[
\|y(t) - z(t)\| \leq \gamma \varrho(t), \quad t \in \bar{I},
\]
where \( \gamma \) is a positive real number depending on \( \varepsilon \).

**Theorem 4.1.** Under assumption (H2) in Theorem 3.1, with \( \mu \lambda < 1 \), problem (1.1) is both Ulam–Hyers and generalized Ulam–Hyers stable.

**Proof.** Let \( y \in C(\bar{I}, \mathcal{S}) \) be a solution of problem (1.9), satisfying (3.1) in the sense of Theorem 3.2. Let \( z \) be any solution satisfying (4.1). Lemma 2.4 implies the equivalence between the operators \( \mathcal{P} \) and \( \mathcal{T} - \mathcal{I} \) (where \( \mathcal{I} \) is the identity operator) for every solution \( z \in C(\bar{I}, \mathcal{S}) \) of problem (1.9) satisfying \( \mu \lambda < 1 \). Therefore, we deduce by the fixed-point property of the operator \( \mathcal{T} \) that
\[
\|z(t)\| = \|z(t)Tz(t) + Tz(t)y(t)\| \\
= \|Tz(t)\| \|z(t)\| + \|Tz(t)y(t)\| \\
\leq \|Tz(t)\| \|y(t)\| + \|Tz(t)z(t)\| \\
\leq \mu \lambda \|z(t)\| + \varepsilon,
\]

because \( \mu \lambda < 1 \) and \( \varepsilon > 0 \), we find
\[
\|u - v\| \leq \frac{\varepsilon}{1 - \mu \lambda}.
\]
Fixing \( \varepsilon = \frac{\varepsilon}{1 - \mu \lambda} \) and \( \gamma = 1 \), we obtain the Ulam–Hyers stability condition. In addition, the generalized Ulam–Hyers stability follows by taking \( \varrho(\varepsilon) = \frac{\varepsilon}{1 - \mu \lambda} \).

**Theorem 4.2.** Assume that (H2) holds with \( \mu < \lambda \), and there exists a function \( \varrho \in C(\bar{I}, R^+ \) satisfying the condition 4.2. Then problem (1.9) is Ulam–Hyers–Rassias stable with respect to \( \varrho \).

**Proof.** We have from the proof of Theorem 4.1,
\[
\|y(t) - z(t)\| \leq \varepsilon \varrho(t), \quad \forall t \in \bar{I},
\]
where \( \varepsilon \), \( \frac{\varepsilon}{1 - \mu \lambda} \), and so the proof is completed.

**5 Illustrative of our outcome**

First we present Example 5.1, for illustrative our main result.
Example 5.1. Consider the following fractional integro-differential problem:

\[
\begin{align*}
C^d_D^{6\frac{5}{17}}[y](t) &= \left(\frac{15 - 2t}{25} y(t) + \frac{5}{43} \int_0^t (5 - t) \sin(y(t)) \, dt\right) \\
&\quad + \int_0^t y(\xi) \exp(-(t + \xi)) \, d\xi,
\end{align*}
\]

with boundary condition

\[y(0) = -\frac{15}{2} \int_0^{0.6} y(\xi) \, d\xi, \quad \forall t \in \mathbb{I}.
\]

Clearly \(\sigma + v = \frac{68}{77}, \sigma = \frac{5}{17}, \tau^* = 0.6, \) and \(\eta = -\frac{15}{2}.\) To illustrate our results in Theorems 3.2 and 4.1, we take for \(y, z \in \mathcal{H} = \mathbb{R}^+\) and \(t \in [0, 1]\) the following continuous functions:

\[
h_1(t, y(t)) = \frac{(15 - 2t) y(t)}{25},
\]

\[
h_2(t, y(t)) = \frac{(5 - t) \sin(y(t))}{43},
\]

and

\[
\Theta(t, s, y(s)) = \frac{y(s) \exp(-(t + s))}{20}.
\]

Now, for \(y, z \in \mathcal{H},\) we have

\[
\|h_1(t, y(t)) - h_1(t, z(t))\| = \left\|\frac{(15 - 2t) y(t) - (15 - 2t) z(t)}{25}\right\| = \frac{15 - 2t}{25} \|[y(t) - z(t)]\| \
\]

\[
\leq \frac{3}{5} \|[y(t) - z(t)]\|,
\]

\[
\|h_2(t, y(t)) - h_2(t, z(t))\| = \left\|\frac{(5 - t) \sin(y(t)) - (5 - t) \sin(z(t))}{43}\right\| = \frac{5 - 2t}{43} \|[y(t) - z(t)]\| \
\]

\[
\leq \frac{5}{43} \|[y(t) - z(t)]\|,
\]

and

\[\Theta(t, s, y(s)) - \Theta(t, s, z(s))\| = \left\|\frac{y(s) \exp(-(t + s)) - y(s) \exp(-(t + s))}{20}\right\| = \frac{\exp(-(t + s))}{20} \|[y(s) - z(s)]\| \
\]

\[\leq \frac{1}{20} \|[y(s) - z(s)]\|,
\]

for each \(t, s \in \mathbb{I}\) and \((t, s) \in G.\) Hence, \(\mu_1 = \frac{17}{25}, \mu_2 = \frac{17}{43}, \mu' = \frac{1}{20},\) and so

\[\mu = \max\{\mu_1, \mu_2, \mu'\} = \frac{17}{25}.
\]

Also, we obtain

\[
\|h_1(t, y(t))\| = \left\|\frac{(15 - 2t) y(t)}{25}\right\| \leq \frac{15 - 2t}{25} \|[y(t)]\|,
\]

\[
\|h_2(t, y(t))\| = \left\|\frac{(5 - t) \sin(y(t))}{43}\right\| \leq \frac{5 - 2t}{43} \|[y(t)]\|,
\]

\[
\|\Theta(t, s, y(s))\| \leq \frac{\exp(-(t + s))}{20} \|[y(s)]\|,
\]

for each \(t, s \in \mathbb{I}.\) Hence,\n
\[
q_{1}(t) = \frac{15 - 2t}{25}, \quad q_{2}(t) = \frac{5 - 2t}{43}, \quad q'(t) = \frac{\exp(-t)}{20},
\]

for all \(t \in \mathbb{I}, y, z \in \mathcal{H}\) and \((t, s) \in G.\) By the above, we find that

\[
\lambda = \left\|q_{1} \hat{I}^{\sigma\nu}_{\alpha} + q_{2} \hat{I}^{\sigma\nu}_{\alpha} + \frac{\mu_{1}}{\mu_{2}}B_{\alpha}(\sigma + 1, \sigma + v)\hat{I}_{\alpha}(\sigma + 1, \sigma + v) + \frac{\mu_{1}}{\mu_{2}}B_{\alpha}(\sigma + 1, \sigma + v + 1)\hat{I}_{\alpha}(\sigma + 1, \sigma + v + 1) + \frac{\mu_{1}}{\mu_{2}}B_{\alpha}(\sigma + 1, \sigma + v + 1)\hat{I}_{\alpha}(\sigma + 1, \sigma + v + 1)\right\|
\]

\[
= \frac{1}{5} + \frac{1}{30} \left|\alpha\right| B_{\alpha}(\frac{5}{7} + 1, \frac{5}{7} + \frac{1}{7}) \hat{I}_{\alpha}(\frac{5}{7} + 1, \frac{5}{7} + \frac{1}{7}) + \frac{15}{20} \times 0.6 \hat{I}_{\alpha}(\frac{5}{7} + 1, \frac{5}{7} + \frac{1}{7}) + \frac{\mu_{1}}{\mu_{2}} - 0.6 \hat{I}_{\alpha}(\frac{5}{7} + 1, \frac{5}{7} + \frac{1}{7})
\]

\[
= \frac{15}{20} \times 0.6 \hat{I}_{\alpha}(\frac{5}{7} + 1, \frac{5}{7} + \frac{1}{7}) + \frac{15}{20} \times 0.6 \hat{I}_{\alpha}(\frac{5}{7} + 1, \frac{5}{7} + \frac{1}{7})
\]

\[
= \frac{15}{20} \times 0.6 \hat{I}_{\alpha}(\frac{5}{7} + 1, \frac{5}{7} + \frac{1}{7}) + \frac{15}{20} \times 0.6 \hat{I}_{\alpha}(\frac{5}{7} + 1, \frac{5}{7} + \frac{1}{7})
\]

and

\[
\frac{\hat{I}_{\alpha}(\frac{5}{7} + 1, \frac{5}{7} + \frac{1}{7})}{\hat{I}_{\alpha}(\frac{5}{7} + 1, \frac{5}{7} + \frac{1}{7}) + \hat{I}_{\alpha}(\frac{5}{7} + 1, \frac{5}{7} + \frac{1}{7})}.
\]
\[ \lambda^* = \left| \eta \right| \left[ \frac{2^{n+1/s+1}}{\Gamma(\sigma + 1, \sigma + 1)} + \frac{\Gamma(\sigma + 1)}{\Gamma(\sigma + 1, \sigma + v + 1)} \right] \\
= \left[ \frac{15}{2} \right] \left[ \frac{2 \times 0.6^{n+1/2+1}}{\Gamma\left(\frac{5}{n} + \frac{3}{2} + 1\right)} + \frac{\Gamma\left(\frac{5}{n} + 1, \frac{5}{n} + \frac{3}{2} + 1\right)}{\Gamma\left(\frac{5}{n} + 1\right)} \right]. \]  

(5.3)

Considering \( q = \frac{3}{8}, \frac{1}{2}, \frac{8}{9} \), we can see the results of \( \lambda \) and \( \lambda^* \) in Table 1. These results are plotted in Figure 1.

Then, we obtain

\[ \lambda_j \approx \begin{cases} 0.95547, & q_j = \frac{3}{8} \\ 0.76172, & q_j = \frac{1}{2} \\ 0.16793, & q_j = \frac{8}{9} \end{cases} < 1, \]

\[ \lambda^*_j \approx \begin{cases} 1.41986, & q_j = \frac{3}{8} \\ 1.13395, & q_j = \frac{1}{2} \\ 0.25096, & q_j = \frac{8}{9} \end{cases} < 1. \]

\[ \mu \lambda^*_j \approx \begin{cases} 0.9655, & q_j = \frac{3}{8} \\ 0.7711, & q_j = \frac{1}{2} \\ 0.1707, & q_j = \frac{8}{9} \end{cases} < 1. \]

Using Krasnoselkii’s theorem for investigation.

---

**Table 1:** Numerical results of \( \lambda \) and \( \lambda^* \) for \( q = \frac{3}{8}, \frac{1}{2}, \frac{8}{9} \) in Example 5.1

| \( n \) | \( q = \frac{3}{8} \) | \( q = \frac{1}{2} \) | \( q = \frac{8}{9} \) |
|-------|-------------|-------------|-------------|
| \( \lambda \) | \( \lambda^* \) | \( \lambda \) | \( \lambda^* \) | \( \lambda \) | \( \lambda^* \) |
| 1 | 0.93177 | 1.34571 | 0.7630 | 0.99360 | 0.11402 | 0.07701 |
| 2 | 0.94654 | 1.39205 | 0.78335 | 1.06376 | 0.12377 | 0.11354 |
| 3 | 0.95212 | 1.40943 | 0.78502 | 1.09885 | 0.12828 | 0.12878 |
| 4 | 0.95222 | 1.41595 | 0.78598 | 1.11640 | 0.13242 | 0.14232 |
| 5 | 0.95500 | 1.41840 | 0.78885 | 1.12518 | 0.13618 | 0.15436 |
| 6 | 0.95530 | 1.41931 | 0.79629 | 1.12957 | 0.13957 | 0.16506 |
| 7 | 0.95541 | 1.41966 | 0.79610 | 1.13176 | 0.14262 | 0.17458 |
| 8 | 0.95545 | 1.41978 | 0.79613 | 1.13286 | 0.14356 | 0.18304 |
| 9 | 0.95546 | 1.41983 | 0.79615 | 1.13341 | 0.14781 | 0.19057 |
| 10 | 0.95547 | 1.41985 | 0.79616 | 1.13368 | 0.15001 | 0.19727 |
| 11 | 0.95547 | 1.41986 | 0.79619 | 1.13382 | 0.15197 | 0.20322 |
| 12 | 0.95547 | 1.41986 | 0.79619 | 1.13389 | 0.15372 | 0.20852 |
| 13 | 0.95547 | 1.41986 | 0.79622 | 1.13392 | 0.15528 | 0.21323 |
| 14 | 0.95547 | 1.41986 | 0.79622 | 1.13394 | 0.15667 | 0.21741 |
| 15 | 0.95547 | 1.41986 | 0.79623 | 1.13395 | 0.15791 | 0.22114 |
| 16 | 0.95547 | 1.41986 | 0.79623 | 1.13396 | 0.15901 | 0.22445 |
| 17 | 0.95547 | 1.41986 | 0.79623 | 1.13396 | 0.16000 | 0.22739 |
| 18 | 0.95547 | 1.41986 | 0.79623 | 1.13396 | 0.16000 | 0.22739 |
| ... | ... | ... | ... | ... | ... | ... |
| 76 | 0.95547 | 1.41986 | 0.79617 | 1.13396 | 0.16792 | 0.25095 |
| 77 | 0.95547 | 1.41986 | 0.79617 | 1.13396 | 0.16792 | 0.25096 |
| 78 | 0.95547 | 1.41986 | 0.79617 | 1.13396 | 0.16793 | 0.25096 |
| 79 | 0.95547 | 1.41986 | 0.79617 | 1.13396 | 0.16793 | 0.25096 |
| 80 | 0.95547 | 1.41986 | 0.79617 | 1.13396 | 0.16793 | 0.25096 |
All assumptions of Theorem 3.2 are satisfied. Hence, there exists at least one solution for problem (5.1) on I. One can use Algorithm 9 to obtain these results. By taking the same functions, we result the assumption

\[
\mu_l = \begin{cases} 
0.6497, & q_l = \frac{3}{8}, \\
0.5180, & q_l = \frac{1}{2}, \\
0.1142, & q_l = \frac{8}{9}, 
\end{cases} < 1,
\]

then system (5.1) is Ulam–Hyers stable, then it is generalized Ulam–Hyers stable if there exists a continuous and positive function \( q_j \in C(I, R^+) \) such that

\[
\|y(t) - z(t)\| \leq \epsilon_j q(t) = \frac{\epsilon_j q(t)}{1 - \mu_l j},
\]

which it satisfies in assumption of Theorem 4.2.

In the next example, we review and check Theorem 3.3 numerically.

**Example 5.2.** Consider the following fractional integro-differential problem:

\[
^{c}D_{t}^{29/45} y(t) = \left( 16 - \sqrt{t} \tan^{-1}(y(t)) + \frac{\sqrt{t}}{75} \left[ 2t \sin^{-1}(y(t)) \right] \right) + \int_{0}^{t} y(\xi) \exp \left( -3t + \xi \right) d\xi,
\]

with boundary condition

**Figure 1:** Graphical representation of \( \lambda, \lambda^* \), and \( \mu \lambda, \mu \lambda^* \) for \( q = \frac{3}{8}, \frac{1}{2}, \frac{8}{9} \) in Example 5.1. (a) \( \lambda \), Eq. (3.8). (b) \( \mu \lambda \). (c) \( \lambda^* \), Eq. (3.9). (d) \( \mu \lambda^* \).
Clearly $\sigma + v = \frac{29}{45}$, $\sigma = \frac{4}{9}$, $r^* = 0.95$, and $\eta = \frac{5}{2}$. To illustrate our results in Theorem 3.3, we take for $y, z \in \bar{S}$ the following continuous functions:

$$h_1(t, y(t)) = \frac{(16 - \sqrt{t}) \tan^{-1}(y(t))}{75},$$
$$h_2(t, y(t)) = \frac{2t \sin^{-1}(y(t))}{21},$$

and

$$\Theta(t, s, y(s)) = \frac{y(s) \exp(-(3t + s))}{10}.$$ 

Now, for $y, z \in \bar{S}$, we have

$$\|h_1(t, y(t)) - h_1(t, z(t))\| = \frac{(16 - \sqrt{t}) \tan^{-1}(y(t))}{75} - \frac{(16 - \sqrt{t}) \tan^{-1}(z(t))}{75}$$
$$= \frac{16 - \sqrt{t}}{75} |y(t) - z(t)|$$
$$\leq \frac{17}{75} \|y(t) - z(t)\|.$$

Using Krasnoselskii's theorem for investigation.

### Table 2: Numerical results of $\lambda$ and $\mu \lambda$ for $q = \frac{2}{7}, \frac{1}{2}, \frac{9}{11}$ in Example 5.2

| $n$ | $q = \frac{2}{7}$ | $q = \frac{1}{2}$ | $q = \frac{9}{11}$ |
|-----|-----------------|-----------------|-----------------|
|     | $\lambda$      | $\mu \lambda$  | $\lambda$      | $\mu \lambda$  | $\lambda$      | $\mu \lambda$  |
| 1   | 0.81214         | 0.55225         | 0.54150         | 0.36822         | 0.15811         | 0.10752         |
| 2   | 0.81764         | 0.55600         | 0.55200         | 0.37876         | 0.16610         | 0.11295         |
| 3   | 0.81923         | 0.55708         | 0.56491         | 0.38414         | 0.17332         | 0.11785         |
| 4   | 0.81969         | 0.55739         | 0.56890         | 0.38685         | 0.17947         | 0.12204         |
| 5   | 0.81982         | 0.55748         | 0.57090         | 0.38821         | 0.18462         | 0.12554         |
| 6   | 0.81986         | 0.55750         | 0.57190         | 0.38889         | 0.18887         | 0.12843         |
| 7   | 0.81987         | 0.55751         | 0.57240         | 0.38923         | 0.19238         | 0.13082         |
| 8   | 0.81987         | 0.55751         | 0.57265         | 0.38940         | 0.19526         | 0.13278         |
| 9   | 0.81987         | 0.55751         | 0.57278         | 0.38949         | 0.19763         | 0.13439         |
| 10  | 0.81987         | 0.55751         | 0.57284         | 0.38953         | 0.19996         | 0.13570         |
| 11  | 0.81987         | 0.55751         | 0.57287         | 0.38955         | 0.20115         | 0.13678         |
| 12  | 0.81987         | 0.55751         | 0.57289         | 0.38956         | 0.20245         | 0.13767         |
| 13  | 0.81987         | 0.55751         | 0.57290         | 0.38957         | 0.20352         | 0.13839         |
| 14  | 0.81987         | 0.55751         | 0.57290         | 0.38957         | 0.20439         | 0.13898         |
| 15  | 0.81987         | 0.55751         | 0.57290         | 0.38957         | 0.20510         | 0.13947         |
| ... | ...             | ...             | ...             | ...             | ...             | ...             |
| 43  | 0.81987         | 0.55751         | 0.57290         | 0.38957         | 0.20830         | 0.14164         |
| 44  | 0.81987         | 0.55751         | 0.57290         | 0.38957         | 0.20830         | 0.14165         |
| 45  | 0.81987         | 0.55751         | 0.57290         | 0.38957         | 0.20830         | 0.14165         |
| 46  | 0.81987         | 0.55751         | 0.57290         | 0.38957         | 0.20830         | 0.14165         |
| 47  | 0.81987         | 0.55751         | 0.57290         | 0.38957         | 0.20831         | 0.14165         |
for each $t, s \in J$. Hence,

$$q_0(t) = \frac{16 - \sqrt{i}}{75}, \quad q_5(t) = \frac{2t}{21}, \quad q^*(t) = \frac{\exp(-3t)}{10},$$

for all $t \in I, y, z \in S_2$, and $(t, s) \in G$. By the above, we find that

$$\lambda = \frac{\|q\|_{L^\infty} + \|q^*\|_{L^\infty}}{\Gamma_{\delta}(\sigma + \nu + 1)} + \frac{\|q\|_{L^\infty} B_0(\sigma + 1, \sigma + \nu)}{\Gamma_{\delta}(\sigma + 1) \Gamma_{\delta}(\sigma + \nu)}$$

$$+ \left| \eta \|q\|_{L^\infty} \Gamma_{\delta}(\sigma + \nu + 2) \right| \Gamma_{\delta}(\sigma + 1) \Gamma_{\delta}(\sigma + \nu + 1)$$

$$= \frac{16}{75} + \frac{1}{10} + \frac{2}{21} B_0(\frac{4}{9} + 1, \frac{4}{9} + \frac{1}{5})$$

$$+ \frac{\Gamma_{\delta}(\frac{4}{9} + 1) \Gamma_{\delta}(\frac{4}{9} + \frac{1}{5})}{[2.5] \times \frac{16}{75} + \frac{1}{10} + \frac{2}{21} \times 0.95^{\frac{3}{5} + \frac{1}{4} + 1} + \frac{2.5}{10} \times 0.95^{\frac{3}{5} + \frac{1}{4} + 1}}$$

$$+ \frac{[2.5] \times \frac{16}{75} + \frac{1}{10} + \frac{2}{21} \times 0.95^{\frac{3}{5} + \frac{1}{4} + 1}}{[1 - 2.5 \times 0.95^{\frac{3}{5} + \frac{1}{4} + 1} + \frac{2.5}{10} \times 0.95^{\frac{3}{5} + \frac{1}{4} + 1}] \times \frac{\Gamma_{\delta}(\frac{4}{9} + 1) \Gamma_{\delta}(\frac{4}{9} + \frac{1}{5})}{1 - 2.5 \times 0.95^{\frac{3}{5} + \frac{1}{4} + 1}}}. \tag{5.5}$$

Considering $q = \frac{2}{7}, \frac{1}{7}, \frac{9}{11}$, we can see the results of $\lambda$ and $\lambda^*$ in Table 2. These results are plotted in Figure 2. Then, we obtain

$$\lambda = \begin{cases} 0.81987, & q_1 = \frac{3}{8}, \\ 0.57290, & q_1 = \frac{1}{2}, \\ 0.20831, & q_1 = \frac{8}{9}, \\ 0.55751, & q_1 = \frac{3}{8}, \\ 0.38957, & q_1 = \frac{1}{2}, \\ 0.14165, & q_1 = \frac{8}{9}, \end{cases} < 1. \tag{5.4}$$

All assumptions of Theorem 3.3 are satisfied. Hence, there exists at least one solution for problem (5.4) on $I$.

### 6 Conclusion

Determining the answer of differential equations from the order of fractions in the discrete state simplifies many problems. The $q$-integro-differential boundary equations and their applications have attracted several researchers’ interests in the field of fractional $q$-calculus and its applications in various phenomena from science and technology. $q$-Integro-differential boundary value problems occur in the mathematical modeling of a variety of physical operations. Using the Krasnoselskii’s, Banach fixed point theorems, we prove existence and uniqueness results.
Based on the results obtained, conditions are provided to ensure the generalized Ulam stability of the original system. The results of Eq. (1.9) investigation on a time scale are illustrated by two numerical examples.

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Appendix

Algorithm 1: MATLAB lines for calculation $q$-factorial function $(x - y)_q^{(m)}$

**Input:** $y, z, q, m$
**Output:** $H$

1. If $m = 0$
   1. Set totalout = 1
2. Else
   1. Set totalout = 1;
   2. For $n = 0 : m - 1$
      1. Set totalout = totalout * $(y - z * q^n)$;
      2. $H$ = totalout;

Algorithm 2: MATLAB lines for calculation $y - z_{q}^{(a)}$.

**Input:** $y, z, q, \sigma, m$
**Output:** $H$

1. If $m = 0$
   1. Set $H$ = 1
2. Else
   1. Set totalout = 1;
   2. For $n = 0 : m - 1$
      1. Set totalout = totalout * $(y - z * q^n)/(y - z * q^{\sigma+n})$;
      2. $H$ = totalout * $y^{\sigma+n}$;

Algorithm 3: MATLAB lines for calculation $\Gamma_q(y)$.

**Input:** $q, y, m$
**Output:** $H$

1. For $n = 0 : m$
   1. Set totalout = totalout * $(1 - q^{n+1})/(1 - q^{n+k-1})$;
   2. $H$ = totalout * $(1 - q)^{(1-y)}$;

Algorithm 4: MATLAB lines for calculation $D_q[y](t)$.

**Input:** $q, s, \text{fun}$
**Output:** $H$

1. If $s = 0$
   1. Set $H$ = limit(($\text{subs(fun,s)} - \text{subs(fun,q*s)})/((1-q)*s),s,0)$;
2. Else
   1. Set $H$ = (eval($\text{subs(fun,s)}$) - eval($\text{subs(fun,q*s)})/((1-q)*s)$);
Algorithm 5: MATLAB lines for calculation $D_{q|t}|y|(t)$.

**Input**: $q$, $s$, $m$, $func$
**Output**: $H$

```
totalout=0;
for n = 0 : m do
    V = 1;
    for i = 0 : m - 1 do
        V = V * (1 - $q^{(i-m)})/(1 - q^{(i+1)})$;
        V = V * $q^n$ * eval(subs(func,s * $q^n$));
        totalout=totalout + V;
    totalout=totalout/(s^m * (1 - $q^m$));
```

Algorithm 6: MATLAB lines for calculation $I_q[y](t)$.

**Input**: $q$, $s$, $m$, $func$
**Output**: $H$

```
totalout=1;
for n = 0 : m do
    totalout = totalout + $q^n$*eval(subs(funx, totalout*$q^n$));
H = s*(1-q)*totalout;
```

Algorithm 7: MATLAB lines for calculation $I^0_q[y](t)$.

**Input**: $q$, $sigma$, $s$, $m$, $func$
**Output**: $H$

```
totalout=0;
for n = 0 : m do
    V = 1;
    for i = 0 : m do
        V = V * (1 - $q^{(sigma+i-1)})*(1 - q^{(sigma+n+i-1)})$;
        totalout=totalout + $q^k$ * V * eval(subs(func,s * $q^n$));
    H = round(totalout *($s^{sigma}$) * (1 - $q^{sigma}$, 6));
```
Algorithm 8: MATLAB lines for calculation $^C\mathcal{D}_q^\sigma[y](t)$.

**Input:** q, sigma, s, m, func

**Output:** H

Tootalout=0;

for $n = 0 : m$ do

$V = 1$;

for $i = 0 : m$ do

$V = V \times (1 - q^{(\text{floor}(\text{sigma}) - \text{sigma} + i - 1)})$

$(1 - q^{(k + i)}) / ((1 - q^{(i + 1)})$

$(1 - q^{(\text{floor}(\text{sigma}) - \text{sigma} + k + i - 1)})$;

Tootalout2=0;

for $k = 0 : \text{floor}(\text{sigma})$ do

$V2 = 1$;

for $i = 0 : k - 1$ do

$V2 = V2 \times (1 - q^{(\text{floor}(\text{sigma}))}) / (1 - q^{(i + 1)})$;

$V2 = V2 \times \text{eval}(\text{subs}(\text{fun}, s, q^{(m + k)}))$;

Tootalout2=Tootalout2 + V2;

Tootalout=Tootalout + V*Tootalout2;

end

end

H = round(tootalout * (s*sigma) + (1 - q)*sigma, 6);
Algorithm 9: MATLAB lines for calculating values of $\lambda$ and $\lambda^*$ in Example ?? for $q = \frac{3}{\pi}, \frac{1}{2}, \frac{8}{\pi}$.

**Input:** $q$, sigma, nu, eta, taustar  
**Output:** H

1. clear;
2. format long;
3. column=1
4. for $s = 1 : yq$ do
5.     for $n = 1 : k$
6.         paramsmatrix(n, column)=n;
7.         C1=qGamma(q(s),sigma+1,n);
8.         C2=qGamma(q(s),sigma+nu,n);
9.         C3=qGamma(q(s),sigma+nu+1,n);
10.        C4=qGamma(q(s),2*sigma+nu+1,n);
11.       C5=qGamma(q(s),sigma+nu+2,n);
12.      C6=qGamma(q(s),2*sigma+nu+2,n);
13.     end
14.     paramsmatrix(n, column+1) = round
15. end
16.  ( (normvarrho1 + normvarhostar)/C3 + normvarrho2/C4
17.  +( abs(eta)*normvarrho1 * tausta
18.  r (sigma+ nu+1), abs(eta)
19.  * normvarrhostar * taustar (sigma+ nu+1))
20.  * (abs(1 - eta*taustar)*C5) + abs(eta)
21.  * normvarrho2 * taustar(2* sigma+ nu+1)/C6, 6);
22.  paramsmatrix(n, column + 2)
23.  = round(abs( eta)abs(1 - eta*taustar)*
24.  2*taustar (sigma+ nu+1) / C5
25.  + taustar(2* sigma+ nu+1) / C6, 6);
26. column=column +3;