Trade-off relations between measurement dependence and hiddenness for separable hidden variable models

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The Bell theorem is investigated as a trade-off relation between assumptions for the underlying hidden variable model. Considering the introduction of a set of hidden variables itself to be one of the essential assumptions, we introduce a measure of hiddenness, a quantity that expresses the degree to which hidden variables are needed. We derive novel relaxed Bell-Clauser-Horne-Shimony-Holt (Bell-CHSH) inequalities for separable models, which are hidden variable models that only satisfy the locality but not the measurement independence condition, in terms of their measurement dependence and hiddenness. The derived relations can be interpreted as trade-off relations between the measurement dependence and hiddenness for separable models in the CHSH scenario. It is also revealed that the relation gives a necessary and sufficient condition for the measures to be realized by a separable model.

I. INTRODUCTION

In the history of physics, the violation of Bell inequality \([1, 2]\) is one of the most striking results since it dramatically changed our world view completely different from the classical one. In a nutshell, we should give up naive local realism to explain nature. Beyond a mere philosophy of science, this result has been investigated from various perspectives \([3, 4]\). In foundational aspects, how it is related to uncertainty relations was revealed \([5]\), and its extensions to generalized probabilistic theories (GPTs) \([6–8]\) were given \([9, 10]\). In practical aspects, the existence of correlations that violate Bell inequality was used to prove the device independent security for key distribution \([11, 12]\), and also for randomness expansion \([13]\) and randomness extraction \([14]\).

In a conventional approach, a local hidden variable model is adopted when discussing the Bell theorem. There is introduced a set of “hidden” variables that satisfies a certain locality condition to explain the empirically observed statistics. In the original work by Bell \([1]\), determinism was also assumed. Under this assumption, the locality condition is equivalent to the parameter independence (no-signaling), which assumes that local statistics are not influenced instantaneously (or superluminally) by remote (spacelike separated) measurement contexts. However, in a general hidden variable model admitting indeterminism, it requires not only the parameter independence but also the outcome independence, the assumption that local statistics are independent of the remote outcomes \([15]\). Moreover, in a local hidden variable model, there is another indispensable assumption called the measurement independence (also referred to as a free-choice assumption or the existence of free will), which indicates that the choice of measurements is independent of the underlying hidden variables. In summary, the assumptions behind Bell inequality are three independencies: (i) parameter independence, (ii) outcome independence, and (iii) measurement independence for the hidden variables. We call a hidden variable model a Bell-local model if all these assumptions are satisfied, and a separable model (following the terminology in \([16]\)) if only the former two assumptions (i) and (ii) are satisfied relaxing the measurement independence. The violation of Bell inequality thus implies that at least one of the underlying assumptions is violated. However, it does not tell anything about how much each assumption should be violated.

With this background, Hall introduced quantitative measures for each assumption and derived relaxed Bell inequalities as trade-off relations between these measures \([17–19]\). More precisely, considering a deterministic hidden variable model, he introduced measures for indeterminism, signaling, and measurement dependence, and obtained a relaxed Bell-Clauser-Horne-Shimony-Holt (Bell-CHSH) inequality in the CHSH scenario. Since then, much research has been done with the same line of thought \([20–24]\). For instance, a relaxed Bell inequality was obtained for a given separable model in \([23]\) with its upper and lower bounds given by the conditional probabilities of the measurement contexts for the underlying variables. In \([24]\), there was found a relaxed Bell inequality as a trade-off relation between the guessing probability and a free will parameter.

In this work, we consider the introduction of hidden variables itself as one of the essential assumptions behind Bell inequality, and ask how much we need such variables to explain the empirical statistics. In the previous study

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Consider two spacelike separated parties, Alice and Bob, who perform measurements $x$ and $y$ on each subsystem to obtain outcomes $a$ and $b$ respectively. In the CHSH scenario, there are two choices of measurements $x, y \in \{0, 1\}$ with two outcomes $a, b \in \{-1, 1\}$ for each system. The state, called a behavior $[1]$, is a set of probabilities $s = \{p(a, b|x, y)\}_{a,b \in \{-1, 1\}, x,y \in \{0,1\}}$, where $p(a, b|x, y)$ denotes the joint probability of observing outcomes $a$ and $b$ when measurements $x$ and $y$ are performed by Alice and Bob respectively. We say that a behavior $s = \{p(a, b|x, y)\}_{a,b \in \{-1, 1\}, x,y \in \{0,1\}}$ admits a hidden variable model (HV model) if there exists a set $\Lambda$ of “hidden” variables through which each joint probability is given as the marginal distribution

$$p(a, b|x, y) = \sum_{\lambda} p(a, b, \lambda|x, y),$$

(1)

where $p(a, b, \lambda|x, y)$ is a joint probability of $a, b$ and a hidden variable $\lambda \in \Lambda$ conditioned on $x, y$. The equation (1) can be rewritten by using the conditional probabilities as

$$p(a, b|x, y) = \sum_{\lambda} p(\lambda|x, y)p(a, b|x, y, \lambda).$$

(2)

Although in this paper we basically treat a discrete set of hidden variables, one can consider continuous cases as well just by replacing the summation by an integration. In the following, we sometimes refer to the sets $I = \{p(\lambda|x, y)\}_{x, \lambda}$ and $O = \{p(a, b|x, y, \lambda)\}_{a,b,x,y,\lambda}$ as the input and output of the underlying HV model respectively. We note that once two sets of probabilities $I = \{p(\lambda|x, y)\}_{x, \lambda}$ and $O = \{p(a, b|x, y, \lambda)\}_{a,b,x,y,\lambda}$ are given, an HV model $s = \{p(a, b|x, y)\}_{a,b,x,y}$ can be defined through (2). Thus we write a behavior $s$ also as $s = (I, O)$ by means of its input $I$ and output $O$.

In this paper, we consider separable models (following the terminology used in [16]). An HV model $s = (I, O)$ is called separable if its output $O$ is written as

$$p(a, b|x, y, \lambda) = p(a|x, \lambda)p(b|y, \lambda)$$

(3)

in terms of local distributions $\{p(a|x, \lambda)\}_{x, \lambda}$ and $\{p(b|y, \lambda)\}_{y, \lambda}$. We remark that separable models have been investigated in several studies [23, 26] and that the condition (3) is equivalent to the so-called parameter independence (or no-signaling) and the outcome independence [15, 27]. If the input $I$ of $s$ in addition satisfies the measurement independence between every measurement context $(x, y)$ and underlying variable $\lambda$, i.e.,

$$p(\lambda|x, y) = p(\lambda),$$

(4)

then the model is called Bell-local. Hence the underlying assumptions for Bell-local models are three independencies: (i) parameter, (ii) outcome, and (iii) measurement, independencies of the underlying (hidden) variables. Bell-local models are the local HV models that have been adopted traditionally when discussing the Bell theorem.

For a behavior $s = \{p(a, b|x, y)\}_{a,b \in \{-1, 1\}, x,y \in \{0,1\}}$, its CHSH value $S(s)$ of is defined as

$$S(s) = \max \left\{ |\langle 00 \rangle + \langle 01 \rangle + \langle 10 \rangle - \langle 11 \rangle |, |\langle 00 \rangle + \langle 01 \rangle - \langle 10 \rangle + \langle 11 \rangle |, \\ |\langle 00 \rangle - \langle 01 \rangle + \langle 10 \rangle + \langle 11 \rangle |, | - \langle 00 \rangle + \langle 01 \rangle + \langle 10 \rangle + \langle 11 \rangle | \right\}$$

(5)

[25], we introduced a measure called hiddenness to quantify the influence of introducing hidden variables, and derived a relaxed Bell inequality as a trade-off relation between the measurement dependence and the hiddenness for any separable model. However, the adopted measure there for hiddenness is discrete, and does not precisely reflect the amount of underlying variables. We remedy this problem by introducing a refined quantity for hiddenness taking the distribution of hidden variables into consideration, and find a novel relaxed Bell inequality (more precisely, a relaxed Bell-CHSH inequality as in the previous studies) for any separable model in the CHSH scenario (Theorem III.1). Moreover, we demonstrate that the set of inequalities that we found completely characterizes separable models: any point satisfying the inequality can be realized by a suitably chosen separable model (Theorem III.2). Our results give a different evaluation of the CHSH value of a separable model, which makes it possible in another way to interpret a violation of the usual Bell-CHSH inequality in terms of the measurement dependence and the introduction of hidden variables itself.

This paper is organized as follows. In section II, we introduce fundamental notions on the CHSH scenario, and make a brief review on the relaxed Bell inequalities obtained by Hall. In section III, after introducing a measure for hiddenness, we show novel relaxed Bell inequalities for separable models, and discuss its physical interpretation. We conclude this paper in section IV.
with

\[ \langle xy \rangle = \sum_{a, b \in \{-1, 1\}} ab p(a, b|x, y) \quad (x, y \in \{0, 1\}). \]  

Notice that the CHSH value \( S(s) \) can be directly accessed by experiments. There is a trivial upper bound

\[ S(s) \leq 4 \]

since \(|\langle xy \rangle| \leq 1\) for any \(x, y \in \{0, 1\}\). It is known that for a Bell-local behavior \( s = (I, O) \)

\[ S(s) = S(I, O) \leq 2 \]  

(7)

holds, which is the famous Bell-CHSH inequality [1, 2]. The violation of this inequality thus implies that at least one of the assumptions (i), (ii), or (iii) of a Bell-local model is violated. The most important theory that does not admit the Bell-CHSH inequality is quantum theory. There, by means of a suitable choice of measurements and an entangled state, the Bell-CHSH inequality is violated [2, 28] (in fact it can reach the Tsirelson bound \(2\sqrt{2}\) [4]), and thus quantum behaviors cannot be described by Bell-local models.

Let us focus on the CHSH value \( S(s) \) of a separable model \( s = \{p(a, b|x, y)\}_{a, b, x, y} = (I, O) \) that is not necessarily measurement independent (i.e., not necessarily Bell-local). For the CHSH value, Hall proved a simple evaluation in terms of the measurement dependence of the model defined as \([17, 19]\)

\[ M(s) = \max_{x, y, x', y'} \sum_{\lambda \in \Lambda} |p(\lambda|x, y) - p(\lambda|x', y')|. \]  

(8)

It is easy to see that \( 0 \leq M(s) \leq 2 \) and \( M = 0 \) if and only if the model is measurement independent. Introducing measures of indeterminism and signaling as well, Hall obtained relaxed Bell inequalities (relaxed Bell-CHSH inequalities) that give trade-off relations among these measures and CHSH values [19]. In particular, for the separable model \( s \), the inequality gives

\[ S \leq 2 + \min[3M, 2]. \]  

(9)

Note that here and in the following \( S(s) \) and \( M(s) \) are denoted simply as \( S \) and \( M \) respectively when the behavior of interest is obvious. The inequality (9) can be regarded as a generalization the Bell-CHSH inequality. In fact, if \( M = 0 \) holds for a separable model, then the model reduces to a Bell-local model, and the Bell-CHSH inequality is reproduced as expected.

### III. RESULTS

Although a broad class of HV models has been investigated and several relaxed Bell inequalities were derived in the previous studies [17–24], there was not considered the very assumption of introducing hidden variables. Motivated by this observation, in this part we derive new relaxed Bell inequalities for separable models in terms of measurement dependence and hiddenness, which is a measure expressing how hidden variables are introduced. The expression of hiddenness used in our argument is a refined version of the one defined originally in the previous paper [25], and thus contributes to obtaining refined inequalities. The new inequalities are also examined as trade-off relations between measurement dependence and hiddenness. We remark that in our argument we only consider the case where the number of the hidden variables is finite, i.e., the set of the hidden variables is written as \( \Lambda = \{\lambda_1, \ldots, \lambda_n\} \) \((n: \text{finite})\) as well as its limiting case \( n \to \infty \).

We begin with reviewing the previous result in [25]. There was defined a measure of hiddenness simply by the number (the cardinality) of hidden variables \( \#(\Lambda) \). That is, for an HV model \( s \) with a set of hidden variables \( \Lambda \), its hiddenness \( H' \) is defined as

\[ H' = \#(\Lambda) - 1. \]  

(10)

We note that here we again follow the notation in (9) and write \( H'(s) \) simply as \( H' \). By means of this \( H' \) and the measurement dependence \( M \) given by (8), a refined version of (9) for a separable model \( s \) was derived in [25] as

\[ S \leq 2 + \min[3M, 2]. \]  

(11)
We remark that this inequality gives a trade-off relation between the measurement dependence and the hiddenness: As would be naturally guessed, the less the measurement dependence \( M \) is, the more the hiddenness \( H' \) is, and vice versa. In [25], there was also demonstrated that any triple \((M, H', S) \in \mathbb{R}^3\) satisfying the inequality (11) is realized by a separable model. 

While the above result enables us to interpret a violation of the Bell-CHSH inequality for a separable model in terms of its hiddenness and measurement dependence, the measure (10) would not be appropriate for quantifying hiddenness. Two reasons can be given to show this: First, \( H' \) is a discrete (integer-valued) quantity. Second, it does not necessary represent properly the amount of the underlying hidden variables that essentially contribute to the statistics. To understand this, consider a separable model where the set of hidden variables \( \Lambda \) has large number of elements but the probability \( p(\lambda) \) for each hidden variable is negligible for almost all \( \lambda \)'s except for small numbers of elements. In this case, although \( H' \) is inevitably estimated to be large, it is more natural to assume that there are fewer hidden variables used intrinsically. To reflect these considerations, we introduce a new measure \( H \) for hiddenness by

\[
H = 1 - \max_{\lambda \in \Lambda} p(\lambda) = 1 - \frac{1}{4} \max_{\lambda \in \Lambda} \left( \sum_{x,y} p(\lambda|x,y) \right).
\]

(12)

Note here that the unbiased settings are concerned where measurements \( x \in \{0, 1\} \) and \( y \in \{0, 1\} \) are chosen randomly: \( p(x,y) = p(x)p(y) = \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4} \). This quantity \( H = 1 - \max_{\lambda \in \Lambda} p(\lambda) \) can be regarded as describing the “width” of the probability distribution \( \{p(\lambda)\}_\Lambda \). It is easy to see that \( 0 \leq H \) and \( H = 0 \) if and only if there is \( \lambda^* \in \Lambda \) such that \( p(\lambda^*) = 1 \), which essentially corresponds to the case without the introduction of hidden variables. In our finite model with arbitrary \( \#(\Lambda) = n \), it holds that \( H \leq 1 - 1/n \) (\( < 1 \)), and the upper bound is attained if and only if \( p(\lambda) = 1/n \) for all \( \lambda \in \Lambda \). Note that the measurement dependence \( M \) and the hiddenness \( H \) are functionals of the input \( I = \{\{p(\lambda|x,y)\}_\lambda\}_{x,y} \) of the underlying HV model. They are not completely independent of each other but intrinsically satisfy

\[
H \geq \frac{M}{8} \quad (13)
\]

(the proof for this relation is given in Appendix B). This means that large measurement dependence requires large hiddenness, and also \( H = 0 \) implies \( M = 0 \). Now our main finding of relaxed Bell inequalities for separable models is presented as follows.

**Theorem III.1**

For any separable model, a relaxed Bell inequality

\[
S \leq 2 + \frac{3}{4} M + 2H
\]

(14)

holds.

This inequality is a novel trade-off relation between \( M \) and \( H \) for \( S \) that refines the previously obtained inequality (11). Combining this theorem with (11), we obtain the tightest trade-off relation.

**Theorem III.2**

The measurement dependence \( M \), the hiddenness \( H \), and the CHSH value \( S \) for any separable model satisfy

\[
S \leq 2 + \min \left[ 3M, \frac{3}{4} M + 2H, 2 \right].
\]

(15)

Conversely, any triple \((M, H, S) \in \mathbb{R}^3\) that satisfies the inequality (15) together with the intrinsic relations

\[
0 \leq M \leq 2, \quad \frac{M}{8} \leq H < 1, \quad S \geq 0
\]

(16)

is realized by a separable model.

This theorem gives a necessary and sufficient condition for a triple \((M, H, S)\) to be realized by a separable model. Fig. 1 (a) shows the allowed region of \((M, H, S)\) for all separable models determined by (15) and (16), which forms a polyhedron with 10 vertices in \( \mathbb{R}^3 \). The proofs of Theorems III.1 and III.2 are given in Appendix A. To illustrate
FIG. 1: (a) The allowed region of \((M, H, S)\) in \(\mathbb{R}^3\) determined by (15) and (16), which is a polyhedron with 10 vertices. The faces indicated by (15) is expressed in red and the others by (16) in yellow. (b) The allowed regions (light blue polygone) of \((M, H)\) in \(\mathbb{R}^2\) with \(S = k + 2\) fixed: \(k = 2\) (the maximal violation of the Bell-CHSH inequality), \(0 < k < 2\) (a violation of the Bell-CHSH inequality including the quantum case, i.e., the Tsirelson bound), and \(k \leq 0\) (no violation of the Bell-CHSH inequality) respectively.

how (15) gives a trade-off relation between \(M\) and \(H\), assume that a CHSH value \(S = 2 + k (-2 \leq k \leq 2)\) is observed in an experiment. Then, from Theorem III.2, we have

\[
k \leq \min \left[ 3M, \frac{3}{4}M + 2H, 2 \right].
\]

(17)

It follows that the allowed region \(W_k\) of \((M, H)\) is given by

\[
W_k = \left\{ (M, H) \mid \frac{k}{3} \leq M, -\frac{3}{8}M + \frac{k}{2} \leq H, 0 \leq M \leq 2, \frac{M}{8} \leq H < 1 \right\}.
\]

(18)

In Fig. 1 (b), the allowed regions \(W_k\) for the cases \(k = 2\) (the maximal violation of the Bell-CHSH inequality), \(0 < k < 2\) (a violation of the Bell-CHSH inequality including the quantum case, i.e., the Tsirelson bound), and \(k \leq 0\) (no violation of the Bell-CHSH inequality) are plotted in light blue. It implies that if the Bell-CHSH inequality is not violated \((k \leq 0)\), then there are no constraints between \(H\) and \(M\) but the intrinsic relations (16). On the other hand, if the Bell-CHSH inequality is violated \((k > 0)\), then there appears a trade-off relation between \(M\) and \(H\) given by (14): the less the measurement dependence \(M\) is, the more the hiddenness \(H\) is, and vice versa. In this case, we can also see that there is a lower bound for the measurement dependence \(M\) given by (17), which means that sufficiently large measurement dependence is needed.

IV. CONCLUSION

In this study, we introduced a measure of hiddenness where the distribution of hidden variables is taken into account. We showed a novel relaxed Bell-CHSH inequality for a given separable model in the CHSH scenario (Theorem III.1),
which gives a non-trivial trade-off relation between the measurement dependence and the hiddenness for that model. The result can be considered as another version of the inequality in [25] in terms of another measure for hiddenness. Combining our novel inequality with the known trade-off relations in [19, 25], we also derived the tightest relations between measurement dependence, hiddenness, and CHSH values (Theorem III.2) in the sense that they give necessary and sufficient conditions for these measures to take arbitrary values realized by a separable model. Future studies will be needed to reveal how our inequalities can be re-expressed when non-separable HV models are considered. It may also be interesting to rewrite our inequalities by means of other measures such as entropic quantities.

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Appendix A: Proofs of Theorems III.1 and III.2

In this appendix, we present proofs for Theorem III.1 in section A 1 and Theorem III.2 in section A 2.

1. Proof of Theorem III.1

To prove Theorem III.1, it is convenient to introduce the optimal CHSH value $S_{\text{opt}}$ for separable models [25] defined by

$$S_{\text{opt}}(I) = \sup_{O: \text{separable}} S(I, O). \quad (A1)$$

There was also proven that for $I = \{\{p(\lambda|x, y)\}\}_{x,y}$

$$S_{\text{opt}}(I) = 4 - 2 \sum_{\lambda \in \Lambda} \min_{x,y} p(\lambda|x, y) \quad (A2)$$

holds and there exists an output $O^*$ such that $S(I, O^*) = S_{\text{opt}}(I)$ (see Lemma 1 in [25]). Then, because $S(s) = S(I, O) \leq S_{\text{opt}}(I)$ holds for any separable model $s = (I, O)$, our proof for Theorem III.1 proceeds by showing that the inequality

$$S_{\text{opt}}(I) \leq 2 + \frac{3}{4} M(I) + 2H(I) \quad (A3)$$

holds for any input $I = \{\{p(\lambda|x, y)\}\}_{x,y \in \{0,1\}}$ (note that the quantities $M$ and $H$ depend only on $I$, and thus here we write them by $M(I)$ and $H(I)$ respectively). In the following, we represent the set $\Lambda$ of hidden variables as $\Lambda = \{\lambda_1, \lambda_2, \ldots, \lambda_n\}$ with $n = \#(\Lambda) \in \mathbb{N}$ (including the limiting case $n \to \infty$), and use the relabeling $i \in \{1, 2, 3, 4\}$ for the measurement contexts $(x, y) \in \{(0,0), (0,1), (1,0), (1,1)\}$ in this order. We also write the input $I = \{\{p(\lambda|x, y)\}\}_{x,y \in \{0,1\}}$ simply as $I = \{p(\lambda|i)\}_{\lambda \in \Lambda, i \in \{1,\ldots,4\}}$. By the definitions (8) and (12) as well as the optimal CHSH value (A2), the inequality (A3) is rewritten as

$$2\tilde{K}(\{p(\lambda|i)\}_{\lambda,i}) + \frac{3}{4} \tilde{M}(\{p(\lambda|i)\}_{\lambda,i}) - \frac{1}{2} \tilde{H}(\{p(\lambda|i)\}_{\lambda,i}) \geq 0, \quad (A4)$$

where

$$\tilde{K}(\{p(\lambda|i)\}_{\lambda,i}) = \sum_{\lambda} \min_{i} p(\lambda|i),$$

$$\tilde{M}(\{p(\lambda|i)\}_{\lambda,i}) = \max_{i,j} \left( \sum_{\lambda} |p(\lambda|i) - p(\lambda|j)| \right), \quad (A5)$$

$$\tilde{H}(\{p(\lambda|i)\}_{\lambda,i}) = \max_{\lambda} \left( \sum_{i} p(\lambda|i) \right).$$
Thus, if we introduce
\[ F(\{p(\lambda|i)\}_{\lambda,i}) := 2\tilde{K}(\{p(\lambda|i)\}_{\lambda,i}) + \frac{3}{4}\tilde{M}(\{p(\lambda|i)\}_{\lambda,i}) - \frac{1}{2}\tilde{H}(\{p(\lambda|i)\}_{\lambda,i}), \]  
then the inequality (A4) reduces to the positivity of \( F \):
\[ F(\{p(\lambda|i)\}_{\lambda,i}) \geq 0. \]  
(A7)

Note that \( \tilde{K}, \tilde{M}, \tilde{H} \), and hence \( F \) are all invariant under any permutation of the label \( i \) as well as the label \( \lambda \). In the following, we shall prove that (A7) holds for any set of conditional probabilities \( \{p(\lambda|i)\}_{\lambda \in \Lambda, i \in \{1, \ldots, 4\}} \).

The cases \( n = 1 \) and \( 2 \) are easily shown as follows. When \( n = 1 \), noting that \( p(\lambda|i) = 1 \) for all \( i \), one has \( \tilde{K} = 1, \tilde{M} = 0, \) and \( \tilde{H} = 4 \). Thus we have
\[ F(\{p(\lambda|i)\}_{\lambda,i}) = 2 \cdot 1 - \frac{1}{2} \cdot 4 = 0, \]
which implies that (A7) is satisfied as an equality. When \( n = 2 \), any input \( \{p(\lambda|i)\}_{\lambda \in \Lambda, i \in \{1, \ldots, 4\}} \) is of the form
\[ \{p(\lambda|i)\}_{\lambda,i} = \begin{bmatrix} \lambda_1 & p_1 & p_2 & p_3 & p_4 \\ \lambda_2 & 1-p_1 & 1-p_2 & 1-p_3 & 1-p_4 \end{bmatrix}. \]

By the invariance of the function \( F \) under permutations of the labels \( i \) and \( \lambda \), we assume without loss of generality that \( 0 \leq p_1 \leq p_2 \leq p_3 \leq p_4 \leq 1 \) and \( \sum_i p_i = 4 - \sum p_i \). Direct calculations of the functions (A5) show
\[ \tilde{K}(\{p(\lambda|i)\}_{\lambda,i}) = p_1 + (1-p_4), \quad \tilde{M}(\{p(\lambda|i)\}_{\lambda,i}) = 2(p_4-p_1), \quad \tilde{H}(\{p(\lambda|i)\}_{\lambda,i}) = 4 - \sum p_i, \]
and thus we obtain
\[ F(\{p(\lambda|i)\}_{\lambda,i}) = \frac{1}{2}(2p_1 + p_2 + p_3) \geq 0. \]

For general \( n \geq 3 \), we give an inductive proof. Assume that (A7) holds for any input with \( n = k(\geq 2) \), and consider an input \( \{p(\lambda|i)\}_{\lambda \in \{\lambda_1, \ldots, \lambda_{k+1}\}, i \in \{1, \ldots, 4\}} \) with \( n = k + 1 \). By the invariance of \( F \) under any permutation of \( \lambda \), we can assume without loss of generality that
\[ \max_{\lambda \in \Lambda} \left( \sum_{i=1}^{4} p_i(\lambda) \right) = \sum_{i=1}^{4} p_i(\lambda_3), \]

i.e., \( \tilde{H}(\{p(\lambda|i)\}_{\lambda,i}) = \sum_{i=1}^{4} p_i(\lambda_3) \) holds. Let \( p = \min\{p_i(\lambda)\}_{(\lambda,i) \in \Gamma} \), where \( \Gamma := \{(\lambda, i) \mid \lambda \in \{\lambda_1, \lambda_2\}, i \in \{1, 2, 3, 4\}\} \), and assume that \( p(\lambda_1|1) = p \). We note that the latter assumption is verified due to the invariance of \( F \) under any permutation of \( i \), and that it does not affect the assumption \( \tilde{H}(\{p(\lambda|i)\}_{\lambda,i}) = \sum_{i=1}^{4} p_i(\lambda_3) \).

To prove the positivity of \( F(\{p(\lambda|i)\}_{\lambda,i}) \), let us introduce the following three transformations for the input \( I = \{p(\lambda|i)\}_{\lambda,i} \). The first transformation \( I = \{p(\lambda|i)\}_{\lambda,i} \rightarrow I' = \{q(\lambda|i)\}_{\lambda,i} \) is given by the rule
\[ q(\lambda|i) = \begin{cases} p(\lambda|i) - p & (\lambda = \lambda_1) \\ p(\lambda|i) + p & (\lambda = \lambda_3) \\ p(\lambda|i) & \text{(otherwise)} \end{cases} \]  
(A8)

With \( \Gamma' := \Gamma \setminus \{(\lambda, i) = (\lambda_1, 1)\} \), \( \lambda_1 = \lambda_2 \), the second transformation \( I' = \{q(\lambda|i)\}_{\lambda,i} \rightarrow I'' = \{r(\lambda|i)\}_{\lambda,i} \) is described by the following algorithm (A)-(C).

(A) If there exists \( i \in \{1, 2, 3, 4\} \) such that \( q(\lambda|i) = q(\lambda_2|i) = 0 \), then let \( r(\lambda|i) = q(\lambda|i) \), and end the algorithm. Otherwise, go to (B).

(B) Let \( q := \min\{q(\lambda|i)\}_{\lambda,i \in \Gamma'} = q(\lambda^*|i^*) (\{(\lambda^*, i^*) \in \Gamma'\} \setminus I) \), \( I := \{i \in \{1, 2, 3, 4\} \mid q(\lambda^*|i) = 0 \} \), and \( I^c := \{1, 2, 3, 4\} \setminus I \). Introduce a new input \( \{q'(\lambda|i)\}_{\lambda,i} \) by
\[ q'(\lambda|i) = q'(\lambda|i) = \begin{cases} q(\lambda|i) - q & (\lambda = \lambda^*, i \in I^c) \\ q(\lambda|i) - q & (\lambda = \lambda_2, i \in I) \\ q(\lambda|i) + q & (\lambda = \lambda_3) \\ q(\lambda|i) & \text{(otherwise)} \end{cases} \]  
(A9)

and go to (C) (here we note that if \( I = \emptyset \), then \( q'(\lambda^*|i) = q(\lambda^*|i) - q \) and \( q'(\lambda_2^*|i) = q(\lambda_2^*|i) \) for all \( i \)).
(C) Let \( \{ q(\lambda | i) \}_{\lambda, i} = \{ q'(\lambda | i) \}_{\lambda, i} \) and \( \Gamma' = \Gamma' \setminus \{(\lambda^*, i^*)\} \), and return to (A).

We can see that the algorithm is designed to make the newly obtained input \( I'' = \{ r(\lambda | i) \}_{\lambda, i} \) satisfy \( r(\lambda_1 | 1) = r(\lambda_2 | 1) = 0 \). The final transformation \( I'' = \{ r(\lambda | i) \}_{\lambda, i} \rightarrow I''' = \{ r'(\lambda | i) \}_{\lambda, i} \) is given by the rule

\[
r'(\lambda | i) = \begin{cases} 
  r(\lambda_1 | i) + r(\lambda_2 | i) & (\theta = 1) \\
  r(\lambda_\theta + 1 | i) & (\theta = 2, \ldots, k).
\end{cases}
\]

(A10)

Note that through the transformations, \( I', I'' \) and \( I''' \) are all valid probabilities, which means, for example, that \( \sum_\lambda r(\lambda | i) = 1 \) holds for all \( i \) in \( I'' \). Moreover, the function \( F \) is monotonically decreasing through the transformations:

\[
F(\{ p(\lambda | i) \}_{\lambda, i}) \geq F(\{ q(\lambda | i) \}_{\lambda, i}) \geq F(\{ r(\lambda | i) \}_{\lambda, i}) \geq F(\{ r'(\lambda | i) \}_{\lambda, i}).
\]

(A11)

Let us show these by examining each transformation one by one. For the first transformation, it is easy to see that

\[
\tilde{K}(\{ q(\lambda | i) \}_{\lambda, i}) = \tilde{K}(\{ p(\lambda | i) \}_{\lambda, i}), \quad \tilde{M}(\{ q(\lambda | i) \}_{\lambda, i}) = \tilde{M}(\{ p(\lambda | i) \}_{\lambda, i}),
\]

and

\[
\tilde{H}(\{ q(\lambda | i) \}_{\lambda, i}) = \tilde{H}(\{ p(\lambda | i) \}_{\lambda, i}) + 4p
\]

hold. By the definition (A6) and the non-negativity of \( p \), we have \( F(\{ p(\lambda | i) \}_{\lambda, i}) \geq F(\{ q(\lambda | i) \}_{\lambda, i}) \). For the second transformation, it is enough to confirm that \( F(\{ q'(\lambda | i) \}_{\lambda, i}) \leq F(\{ q(\lambda | i) \}_{\lambda, i}) \) holds in step (B). We prove this by considering the following three cases: (i) \( q = 0 \), (ii) \( q \neq 0 \) and \( I = \emptyset \), (iii) \( q \neq 0 \) and \( I \neq \emptyset \). In case (i), \( \{ q(\lambda | i) \}_{\lambda, i} = \{ q(\lambda | i) \}_{\lambda, i} \) holds clearly, and thus \( F(\{ q'(\lambda | i) \}_{\lambda, i}) = F(\{ q(\lambda | i) \}_{\lambda, i}) \) is concluded. In case (ii), we can demonstrate easily that

\[
\tilde{K}(\{ q'(\lambda | i) \}_{\lambda, i}) = \tilde{K}(\{ q(\lambda | i) \}_{\lambda, i}), \quad \tilde{M}(\{ q'(\lambda | i) \}_{\lambda, i}) = \tilde{M}(\{ q(\lambda | i) \}_{\lambda, i}), \quad \tilde{H}(\{ q'(\lambda | i) \}_{\lambda, i}) = \tilde{H}(\{ q(\lambda | i) \}_{\lambda, i}) + 4q
\]

hold. It follows that \( F(\{ q'(\lambda | i) \}_{\lambda, i}) = F(\{ q(\lambda | i) \}_{\lambda, i}) - 2q \leq F(\{ q(\lambda | i) \}_{\lambda, i}) \). For case (iii), we need a slightly complicated argument. Let us first investigate \( \tilde{K}(\{ q'(\lambda | i) \}_{\lambda, i}) \). Since

\[
\min_i q'(\lambda^*|i) = \min_i q(\lambda^*|i) = 0, \\
\min_i q'(\overline{\lambda}^*|i) \leq \min_i q(\overline{\lambda}^*|i), \\
\min_i q'(\lambda_3|i) = \min_i q(\lambda_3|i) + q, \\
\min_i q'(\lambda|i) = \min_i q(\lambda|i) \quad (\lambda \in \{ \lambda_4, \ldots, \lambda_{k+1} \})
\]

we have \( \tilde{K}(\{ q'(\lambda | i) \}_{\lambda, i}) \leq \tilde{K}(\{ q(\lambda | i) \}_{\lambda, i}) + q \). We next focus on \( \tilde{M}(\{ q'(\lambda | i) \}_{\lambda, i}) \). The following observations are important (remember that \( q(\lambda^*|i) = 0 \) for \( i \in I \)):

\[
| q'(\lambda^*|i) - q(\lambda^*|j) | = \begin{cases} | q(\lambda^*|i) - q(\lambda^*|j) | & (i, j \in I \text{ or } i, j \in I^c) \\
| q(\lambda^*|i) - q(\lambda^*|j) | - q & (i \in I \text{ and } j \in I^c)
\end{cases},
\]

\[
| q'(\overline{\lambda}^*|i) - q(\overline{\lambda}^*|j) | = \begin{cases} | q(\overline{\lambda}^*|i) - q(\overline{\lambda}^*|j) | & (i, j \in I \text{ or } i, j \in I^c) \\
| q(\overline{\lambda}^*|i) - q(\overline{\lambda}^*|j) | + q & (i \in I \text{ and } j \in I^c)
\end{cases},
\]

\[
| q'(\lambda|i) - q(\lambda|j) | = \min_\lambda \left( | q(\lambda|i) - q(\lambda|j) | \right) \quad (\text{for all } i, j \text{ and } \lambda \in \{ \lambda_3, \ldots, \lambda_{k+1} \}).
\]

Because \( | q(\lambda|i) - q(\lambda|j) | \leq | q(\lambda|i) - q(\lambda|j) | + q \), we can see from these observations that

\[
\sum_{\lambda \in A} | q'(\lambda|i) - q(\lambda|j) | \leq \sum_{\lambda \in A} | q(\lambda|i) - q(\lambda|j) |
\]

holds for all \( i, j \), which implies \( \tilde{M}(\{ q'(\lambda | i) \}_{\lambda, i}) \leq \tilde{M}(\{ q(\lambda | i) \}_{\lambda, i}) \). For \( \tilde{H}(\{ q'(\lambda | i) \}_{\lambda, i}) \), we obtain easily \( \tilde{H}(\{ q'(\lambda | i) \}_{\lambda, i}) = \tilde{H}(\{ q(\lambda | i) \}_{\lambda, i}) + 4q \). Therefore, we can conclude that \( F(\{ q'(\lambda | i) \}_{\lambda, i}) \leq F(\{ q(\lambda | i) \}_{\lambda, i}) \) holds also in case (iii). The
third inequality in (A11) as well as the positivity of \( F(\{r'(\lambda|i)\}_{\lambda,i}) \) is shown as follows. First, it is easily verified

\[
\sum_{\lambda' \in \Lambda'} \min_{i} r'(\lambda'|i) = \sum_{\theta=1}^{k+1} \sum_{i} \min_{\lambda} r(\lambda|\theta) = \sum_{\lambda \in \Lambda} \min_{i} r(\lambda|i),
\]

and

\[
\max_{\lambda' \in \Lambda'} \left( \sum_{i} r'(\lambda'|i) \right) \geq \max_{\lambda \in \Lambda} \left( \sum_{i} r(\lambda|i) \right),
\]

which implies

\[
\tilde{K}(\{r'(\lambda'|i)\}_{\lambda',i}) = \tilde{K}(\{r(\lambda|i)\}_{\lambda,i}),
\]

\[
\tilde{H}(\{r'(\lambda'|i)\}_{\lambda',i}) \geq \tilde{H}(\{r(\lambda|i)\}_{\lambda,i}).
\]

In addition, since

\[
|r'(\lambda'|i) - r'(\lambda|\theta) - (r(\lambda_1|i) + r(\lambda_2|i) - (r(\lambda_1|j) + r(\lambda_2|j)))| \quad (\theta = 1)
\]

\[
|\{r(\lambda_1|i) + r(\lambda_2|i) - (r(\lambda_1|j) + r(\lambda_2|j))\}| \quad (\theta = 2, \ldots, k),
\]

and

\[
|(r(\lambda_1|i) + r(\lambda_2|i) - (r(\lambda_1|j) + r(\lambda_2|j)))| \leq |r(\lambda_1|i) - r(\lambda_1|j)| + |r(\lambda_2|i) - r(\lambda_2|j)|,
\]

we have

\[
\tilde{M}(\{r'(\lambda'|i)\}_{\lambda',i}) \leq \tilde{M}(\{r(\lambda|i)\}_{\lambda,i}),
\]

and thus \( F(\{r'(\lambda'|i)\}_{\lambda',i}) \leq F(\{r(\lambda|i)\}_{\lambda,i}) \). This inequality, together with the assumption \( F(\{r'(\lambda'|i)\}_{\lambda',i}) \geq 0 \) for the table \( \{r'(\lambda'|i)\}_{\lambda',i} \) with \( k \) rows, implies \( F(\{r(\lambda|i)\}_{\lambda,i}) \geq 0 \). Because \( F(\{r(\lambda|i)\}_{\lambda,i}) \leq F(\{p(\lambda|i)\}_{\lambda,i}) \) and \( F(\{q(\lambda|i)\}_{\lambda,i}) \leq F(\{p(\lambda|i)\}_{\lambda,i}) \), we can conclude \( F(\{p(\lambda|i)\}_{\lambda,i}) \geq 0 \).

Finally, we prove that the claim \( F(\{p(\lambda|i)\}_{\lambda,i}) \geq 0 \) holds also for \( \{p(\lambda|i)\}_{\lambda \in \Lambda, i = 1, \ldots, 4} \) with countably infinite \( \Lambda = \{\lambda_1, \lambda_2, \ldots\} \) (note that in this case the definition of \( H \) is modified as \( H = 1 - \sup_{\lambda \in \Lambda} p(\lambda) \)). To see this, we first confirm that all the quantities in (A5) have definite values for this \( \{p(\lambda|i)\}_{\lambda \in \Lambda, i = 1, \ldots, 4} \). In fact, for example, we can see that the right hand side of

\[
\tilde{K}(\{p(\lambda|i)\}_{\lambda,i}) = \sum_{\lambda \in \Lambda} \min_{i} p(\lambda|i)
\]

is definite because \( \sum_{\lambda \in \Lambda} \min_{i} p(\lambda|i) \leq \sum_{\lambda \in \Lambda} p(\lambda|i) = 1 \). Now we construct a family of tables \( \{\{p_\alpha(\lambda|i)\}_{\lambda \in \Lambda, i = 1, \ldots, 4}\}_{\alpha = 1, 2, \ldots} \) by

\[
p_\alpha(\lambda|i) = \begin{cases} p(\lambda_\alpha|i) & (\lambda = \lambda_1, \ldots, \lambda_{\alpha-1}) \\ 1 - \sum_{\beta=1}^{\alpha-1} p_\alpha(\lambda_\beta|i) & (\lambda = \lambda_\alpha) \\ 0 & (\lambda = \lambda_{\alpha+1}, \lambda_{\alpha+2}, \ldots). \end{cases}
\]

We note that each \( \{p_\alpha(\lambda|i)\}_{\lambda,i} \) is essentially a finite table, and thus \( F(\{p_\alpha(\lambda|i)\}_{\lambda,i}) \geq 0 \). Then, because

\[
\tilde{K}(\{p(\lambda|i)\}_{\lambda,i}) = \sum_{\lambda \in \Lambda} \min_{i} p(\lambda|i)
\]

holds for example, we have \( \lim_{\alpha \to \infty} F(\{p_\alpha(\lambda|i)\}_{\lambda,i}) = F(\{p(\lambda|i)\}_{\lambda,i}) \), and \( F(\{p(\lambda|i)\}_{\lambda,i}) \geq 0 \) is concluded.
2. Proof of Theorem III.2

Since the inequality (15) is derived easily from (11) and (14), we prove the existence of a separable model \( s \) such that

\[
M (s) = M, \ H (s) = H, \ S (s) = S
\]

(A12)

holds for any triple \((M, H, S) \in \mathbb{R}^3\) that satisfies the inequality (15) and (16), i.e.,

\[
0 \leq b_1(S) := 4 - S, \quad 0 \leq b_2(M, S) := 3M + 2 - S, \\
0 < b_3(H) := 1 - H, \quad 0 \leq b_4(M, H, S) := H - \frac{1}{2}S + \frac{3}{8}M + 1, \\
0 \leq S, \quad 0 \leq M, \quad 0 \leq 2 - M, \quad 0 \leq H - M/8
\]

(see Fig. 1 (a)).

\[\text{a. The notations of values}\]

In this part, we introduce several values that are needed to prove the claim. Let \((M, H, S) \in \mathbb{R}^3\) be a triple that satisfies (A13). We first introduce a slightly larger CHSH value \( \bar{S} \geq S \) by

\[
\bar{S} := S + \min \{ b_1(S), b_2(M, S), 2b_4(M, H, S) \} \tag{A14}
\]

so that the inequalities

\[
0 \leq b_1(\bar{S}), \quad 0 \leq b_2(M, \bar{S}), \quad 0 \leq b_4(M, H, \bar{S})
\]

in (A13) that are concerning to \( \bar{S} \) are satisfied still and in addition at least one of the equalities holds among them. Then, since another auxiliary inequality

\[
0 \leq b_5 (M, \bar{S}) := \bar{S} - M - 2
\]

holds due to (A13) and (A14) and natural numbers \( n, n_0 \in \mathbb{N} \) such that

\[
n \geq \frac{1}{1 - \bar{H}} \quad \left( \Leftrightarrow b_3 (H) - \frac{1}{n} \geq 0 \right), \quad n = 4n_0
\]

always exist, we can introduce successfully non-negative numbers

\[
u := \frac{b_3(H) - \frac{1}{n}}{b_4(M, H, \bar{S}) + b_3(H) - \frac{1}{n}}, \quad \bar{u} := \frac{b_3(M, H, \bar{S})}{b_4(M, H, \bar{S}) + b_3(H) - \frac{1}{n}},
\]

\[
t_1 := \frac{b_1(\bar{S})}{2}, \quad t_2 := \frac{b_2(M, \bar{S})}{4}, \quad t_3 := \frac{3b_5(M, \bar{S})}{4},
\]

\[y := 1/(12n_0),\]

which satisfy \( u + \bar{u} = 1, \ t_1 + t_2 + t_3 = 1, \ 12n_0y = 1. \)
By means of the values above, let us define an input $I^{M,H,S} = \{p^{M,H,S}(\lambda|i)\}_{\lambda,i}$ by

\[
p^{M,H,S}(\lambda|1) := \begin{cases} 
  ut_1 + 3t_1 \bar{u}y & (l = 1) \\
  3t_1 \bar{u}y & (1 < l \leq n_0) \\
  3u t_1 y + 12t_2 y + 4t_3 y & (n_0 < l \leq 2n_0) , \\
  3u t_1 y + 4t_3 y & (2n_0 < l \leq 3n_0) \\
  3u t_1 y + 4t_3 y & (3n_0 < l \leq 4n_0)
\end{cases}
\]

\[
p^{M,H,S}(\lambda|2) := \begin{cases} 
  u(1 - \frac{2u}{3}) + \bar{u}y(3t_1 + 12t_2 + 4t_3) & (l = 1) \\
  \bar{u}y(3t_1 + 12t_2 + 4t_3) & (1 < l \leq n_0) \\
  3u t_1 y + 4t_3 y & (n_0 < l \leq 2n_0) , \\
  3u t_1 y & (2n_0 < l \leq 3n_0) \\
  4t_3 y + \bar{u}y(3t_1 + 12t_2) & (3n_0 < l \leq 4n_0)
\end{cases}
\]

\[
p^{M,H,S}(\lambda|3) := \begin{cases} 
  u(1 - \frac{2u}{3}) + \bar{u}y(3t_1 + 4t_3) & (l = 1) \\
  \bar{u}y(3t_1 + 4t_3) & (1 < l \leq n_0) \\
  3u t_1 y + 4t_3 y & (n_0 < l \leq 2n_0) , \\
  3u t_1 y & (2n_0 < l \leq 3n_0) \\
  4t_3 y + \bar{u}y(3t_1 + 12t_2) & (3n_0 < l \leq 4n_0)
\end{cases}
\]

\[
p^{M,H,S}(\lambda|4) := \begin{cases} 
  u(1 - \frac{2u}{3}) + \bar{u}y(3t_1 + 4t_3) & (l = 1) \\
  \bar{u}y(3t_1 + 4t_3) & (1 < l \leq n_0) \\
  3u t_1 y + 4t_3 y & (n_0 < l \leq 2n_0) , \\
  3u t_1 y & (2n_0 < l \leq 3n_0) \\
  4t_3 y & (3n_0 < l \leq 4n_0)
\end{cases}
\]

Note that due to (A13) and (A15), this input is definitely a set of probability distributions, that is, all components are non-negative and for each $i = 1, \ldots, 4$, they sum up to 1. Now we can prove that a behavior $s$ whose input is given by $I^{M,H,S}$ realizes $M(s) = M$ and $H(s) = H$ in (A12). In fact, since $M(s)$ and $H(s)$ are independent of the output of $s$ and

\[
\max_{i,j} \sum_{l=1}^{n} |p^{M,H,S}(\lambda|i) - p^{M,H,S}(\lambda|j)| = \sum_{l=1}^{n} |p^{M,H,S}(\lambda|1) - p^{M,H,S}(\lambda|2)| ,
\]

\[
\max_{l=1,2,\ldots,n} \sum_{i=1}^{4} p^{M,H,S}(\lambda|i) = \sum_{i=1}^{4} p^{M,H,S}(\lambda_1|i)
\]

hold (see (A15)), direct calculations show

\[
M(s) = \max_{i,j} \sum_{l=1}^{n} |p^{M,H,S}(\lambda|i) - p^{M,H,S}(\lambda|j)| = \sum_{l} |p^{M,H,S}(\lambda|1) - p^{M,H,S}(\lambda|2)|
\]

\[
= [ut_2 + ut_3/3 + 12yut_2 + 4yut_3] + (n_0 - 1) [12yut_2 + 4yut_3]
\]

\[
+ n_0 [12t_2 y + 4t_3 y] + n_0 [0] + n_0 [0]
\]

\[
= (t_2 + t_3/3) (1 + u + \bar{u})
\]

\[
= M/2 \cdot 2 = M,
\]
and
\[ H(s) = 1 - \frac{1}{4} \max_{i=1,2,\ldots,n} \sum_{i=1}^{4} p^{M,H,S}(\lambda_i|i) \]
\[ = 1 - \frac{1}{4} \sum_{i=1}^{4} p^{M,H,S}(\lambda_i|i) \]
\[ = 1 - \frac{1}{4} \left[ u t_1 + 3 t_1 \bar{u} y + u \left( 1 - \frac{2 t_3}{3} \right) + \bar{u} y (3 t_1 + 12 t_2 + 4 t_3) \right] + u \left( 1 - \frac{2 t_3}{3} \right) + \bar{u} y (3 t_1 + 4 t_3) \]
\[ = 1 - u (b_4 + b_3) + \bar{u}/n_0 \]
\[ = 1 - b_3 = H. \]

\[ c. \text{ The output} \]

In the argument above, we showed that any behavior \( s \) whose input is \( I^{M,H,S} = \{ p^{M,H,S}(\lambda|i) \}_{\lambda,i} \) defined in (A15) gives \( M(s) = M \) and \( H(s) = H \). In this part, we prove the existence of a separable output \( O^{M,H,S} \) such that the behavior \( s = (I^{M,H,S}, O^{M,H,S}) \) realizes the remaining condition \( S(s) = S \). We first consider auxiliary outputs
\[ O^{M,H,S} := \left\{ \left\{ p^{M,H,S}(a|x,\lambda) \right\}_a, \left\{ p^{M,H,S}(b|y,\lambda) \right\}_b \right\}, \]
\[ \bar{O}^{M,H,S} := \left\{ \left\{ p^{M,H,S}(a|x,\lambda) \right\}_a, \left\{ p^{M,H,S}(b|y,\lambda) \right\}_b \right\}, \]
which are designed to have the CHSH values
\[ S \left( I^{M,H,S}, O^{M,H,S} \right) = 0, \]
\[ S \left( I^{M,H,S}, \bar{O}^{M,H,S} \right) = \bar{S} \] (A16)
respectively. The first auxiliary output \( O^{M,H,S} \) is constructed via
\[ p(a|x,\lambda) = 1/2, \quad p(b|y,\lambda) = 1/2, \]
\[ \forall a, b = \pm 1, \forall x \in \{0,1\}, \forall \lambda = \lambda_1, \ldots, \lambda_n, \]
because it lets all the expectation values in (6) vanish. On the other hand, since we have
\[ \min_i \left( p^{M,H,S}(\lambda_i|i) \right) = \begin{cases} p^{M,H,S}(\lambda_l|1) & (1 \leq l \leq n_0) \\ p^{M,H,S}(\lambda_l|2) & (n_0 < l \leq 2n_0) \\ p^{M,H,S}(\lambda_l|3) & (2n_0 < l \leq 3n_0) \\ p^{M,H,S}(\lambda_l|4) & (3n_0 < l \leq 4n_0) \end{cases} \]
and thus
\[ S_{\text{opt}} \left( I^{M,H,S} \right) = 4 - 2 \sum_{l=1}^{n} \min_i \left( p^{M,H,S}(\lambda_i|i) \right) \]
\[ = 4 - 2 \sum_{l=1}^{n_0} \sum_{i=1}^{4} p^{M,H,S}(\lambda_{i(n_0-1)+l}|i) \]
\[ = 4 - 2 [u t_1 + 3 t_1 \bar{u} y + (n_0 - 1) (3 t_1 \bar{u} y)] - 2n_0 [3 \cdot 3 t_1 \bar{u} y] \]
\[ = 4 - 2 t_1 + 4 - 2 \cdot \frac{\bar{S}}{2} = \bar{S} \]
by virtue of (A2), the existence of \( O^{M,H,S} = \left\{ \left\{ p^{M,H,S}(a|x,\lambda) \right\}_a, \left\{ p^{M,H,S}(b|y,\lambda) \right\}_b \right\} \) satisfying the second equation of (A16) is verified according to the previous result in [25] (see the description after (A2)). Now let us consider a
family \{O_{t_A,t_B}^{M,H,S}\}_{(t_A,t_B)\in[0,1]^2} of separable outputs defined by

\[ O_{t_A,t_B}^{M,H,S} = \left\{ \left\{ p_{t_A}^{M,H,S}(a|x,\lambda) \right\}_a, \left\{ p_{t_B}^{M,H,S}(b|y,\lambda) \right\}_b \right\}, \]

where the two local distributions are convex combinations of the corresponding two auxiliary outputs:

\[ p_{t_A}^{M,H,S}(a|x,\lambda) = (1 - t_A) p_A(a|x,\lambda) + t_A p_{AB}^{M,H,S}(a|x,\lambda), \]
\[ p_{t_B}^{M,H,S}(b|y,\lambda) = (1 - t_B) p_B(b|y,\lambda) + t_B p_{AB}^{M,H,S}(b|y,\lambda). \]

Based on this family of outputs, we can introduce a function \( S(\left( I^{M,H,S}, O_{t_A,t_B}^{M,H,S} \right)) =: S(t_A, t_B) \), which is continuous at any \((t_A, t_B) \in [0,1]^2\) and satisfies \(0(= S(0,0)) \leq S(t_A, t_B) \leq S(1,1))\). Then the intermediate-value theorem [31] guarantees that there exists \((t_A, t_B^*) \in [0,1]^2\) for every \(S \in [0, \bar{S}]\) such that \(S(t_A^*, t_B^*) = S\), and thus, by letting \(O_{t_A,t_B}^{M,H,S} = O_{t_A^*,t_B^*}^{M,H,S}\), we obtain a separable behavior \(s = s(I^{M,H,S}, O^{M,H,S})\) that realizes all of the conditions (A12).

**Appendix B: Proof of (13)**

In this appendix, under the same notation as Appendix A, we prove that

\[ H \left( \{ p(\lambda|i) \}_{\lambda,i} \right) \geq \frac{1}{8} M \left( \{ p(\lambda|i) \}_{\lambda,i} \right) \]  

(B1)

holds for any input \( \{ p(\lambda|i) \}_{\lambda,i} \). If \( n := \#(\Lambda) = 1 \), then, as we have seen in Appendix A, \( H = M = 0 \) holds, and thus (B1) is verified. The proof for \( n \geq 2 \) is again given by induction for the number of the hidden variables \( n \). We note that the claim (B1) can be rewritten explicitly as

\[ \frac{1}{4} \max_{\lambda} \sum_i p(\lambda|i) + \frac{1}{8} \max_{i,j} \sum_{\lambda} |p(\lambda|i) - p(\lambda|j)| \leq 1, \]  

(B2)

and it is easy to see that the quantities \( \max_{\lambda} \sum_i p(\lambda|i) \) and \( \max_{i,j} \sum_{\lambda} |p(\lambda|i) - p(\lambda|j)| \) are invariant under permutations of the labels \( i \) and \( \lambda \). We first examine the case \( n = 2 \). Any input \( I = \{ p(\lambda|i) \}_{\lambda=\lambda_1,\lambda_2,i=1,2,3,4} \) is expressed as

\[ \{ p(\lambda|i) \}_{\lambda,i} = \begin{bmatrix} i = 1 & i = 2 & i = 3 & i = 4 \\ \lambda_1 & p_1 & p_2 & p_3 & p_4 \\ \lambda_2 & 1-p_1 & 1-p_2 & 1-p_3 & 1-p_4 \end{bmatrix}. \]

In this expression, we can assume \( p_1 \geq p_2, \sum_i p_i \leq 4 - \sum_i p_i, \) and \( \max_{i,j} \sum_{\lambda} |p(\lambda|i) - p(\lambda|j)| = \sum_{\lambda} |p(\lambda|1) - p(\lambda|2)| \) without loss of generality. Then, since

\[ \sum_{\lambda} |p(\lambda|1) - p(\lambda|2)| = |p_1 - p_2| + |p_1 - p_2| = 2(p_1 - p_2) \]

holds, we have

\[ \frac{1}{4} \max_{\lambda} \sum_i p(\lambda|i) + \frac{1}{8} \max_{i,j} \sum_{\lambda} |p(\lambda|i) - p(\lambda|j)| = \left( 1 - \frac{1}{4} \sum_i p_i \right) + \frac{1}{4} (p_1 - p_2) \]
\[ = 1 - \frac{1}{4} (2p_2 + p_3 + p_4) \leq 1. \]

We next suppose that (B2) holds for any inputs with \( n = k \) (\( \geq 2 \)), and consider an input \( \{ p(\lambda|i) \}_{\lambda,i} \) with \( n = k + 1 \). By means of suitable permutations, this input \( \{ p(\lambda|i) \}_{\lambda,i} \) can be assumed to satisfy \( \max_{i,j} \sum_{\lambda} |p(\lambda|i) - p(\lambda|j)| = \sum_{\lambda} |p(\lambda|1) - p(\lambda|2)|, \) and \( p(\lambda_1|1) \geq p(\lambda_2|1) \) and \( p(\lambda_3|1) \geq p(\lambda_3|2) \). We note that the latter condition is valid because \( n = k + 1 \geq 3 \). Now we define an input \( \{ p'(\lambda'|i) \}_{\lambda',i=1,\ldots,k'} \) with \( k' = \{ k'_1, \ldots, k'_k \} (n = k) \) by

\[ p'(\lambda'_0|i) = \begin{cases} p(\lambda_1|i) + p(\lambda_2|i) & (\theta = 1) \\ p(\lambda_{\theta+1}|i) & (\theta = 2, \ldots, k). \end{cases} \]
It follows easily from this definition of \( \{p'(\lambda'\mid i)\}_{\lambda',i} \) that
\[
\max_{\lambda' \in \Lambda'} \sum_i p(\lambda'\mid i) \geq \max_{\lambda \in \Lambda} \sum_i p(\lambda\mid i). \tag{B3}
\]

To evaluate the second term of the l.h.s. of (B2), it should be noted that
\[
\sum_{\lambda' \in \Lambda'} |p(\lambda'\mid i) - p(\lambda'\mid j)| = |p(\lambda'\mid i) - p(\lambda'\mid j)| + \sum_{\lambda' \in \Lambda' \setminus \{\lambda'\}} |p(\lambda'\mid i) - p(\lambda'\mid j)|
\leq |p(\lambda_1\mid 1) - p(\lambda_1\mid 2)| + |p(\lambda_2\mid 1) - p(\lambda_2\mid 2)| + \sum_{\lambda \in \Lambda \setminus \{\lambda_1, \lambda_2\}} |p(\lambda\mid i) - p(\lambda\mid j)|
= \sum_{\lambda \in \Lambda} |p(\lambda\mid i) - p(\lambda\mid j)|
\leq \sum_{\lambda \in \Lambda} |p(\lambda\mid 1) - p(\lambda\mid 2)|
\]
holds for any \((i, j)\), where the first inequality follows from the triangle inequality and the second one from the assumption imposed on \( \{p(\lambda_0\mid i)\}_{\lambda, i} \). While this implies
\[
\max_{i, j} \sum_{\lambda' \in \Lambda'} |p(\lambda'\mid i) - p(\lambda'\mid j)| \leq \sum_{\lambda \in \Lambda} |p(\lambda\mid 1) - p(\lambda\mid 2)|,
\]
we can show
\[
\sum_{\lambda' \in \Lambda'} |p(\lambda'\mid 1) - p(\lambda'\mid 2)| = \sum_{\lambda \in \Lambda} |p(\lambda\mid 1) - p(\lambda\mid 2)|,
\]
i.e.,
\[
\max_{i, j} \sum_{\lambda' \in \Lambda'} |p(\lambda'\mid i) - p(\lambda'\mid j)| = \max_{i, j} \sum_{\lambda \in \Lambda} |p(\lambda\mid i) - p(\lambda\mid j)|. \tag{B4}
\]

In fact, because
\[
|p(\lambda'_1\mid 1) - p(\lambda'_1\mid 2)| = |(p(\lambda_1\mid 1) + p(\lambda_2\mid 1)) - (p(\lambda_1\mid 2) + p(\lambda_2\mid 2))| = |(p(\lambda_1\mid 1) - p(\lambda_1\mid 2)) + (p(\lambda_2\mid 1) - p(\lambda_2\mid 2))| = |p(\lambda_1\mid 1) - p(\lambda_1\mid 2)| + |p(\lambda_2\mid 1) - p(\lambda_2\mid 2)|
\]
holds due to the initial assumption \( p(\lambda_1\mid 1) \geq p(\lambda_1\mid 2) \) and \( p(\lambda_2\mid 1) \geq p(\lambda_2\mid 2) \), we obtain
\[
\sum_{\lambda' \in \Lambda'} |p(\lambda'\mid 1) - p(\lambda'\mid 2)| = |p(\lambda'_1\mid 1) - p(\lambda'_1\mid 2)| + \sum_{\lambda' \in \Lambda' \setminus \{\lambda'_1\}} |p(\lambda'\mid 1) - p(\lambda'\mid 2)|
= |p(\lambda_1\mid 1) - p(\lambda_1\mid 2)| + |p(\lambda_2\mid 1) - p(\lambda_2\mid 2)| + \sum_{\lambda \in \Lambda \setminus \{\lambda_1, \lambda_2\}} |p(\lambda\mid i) - p(\lambda\mid j)|
= \sum_{\lambda \in \Lambda} |p(\lambda\mid 1) - p(\lambda\mid 2)|.
\]

Now (B3) and (B4) prove
\[
\frac{1}{4} \max_{\lambda \in \Lambda} \sum_i p(\lambda\mid i) + \frac{1}{8} \max_{i, j} \sum_{\lambda \in \Lambda} |p(\lambda\mid i) - p(\lambda\mid j)| \leq \frac{1}{4} \max_{\lambda' \in \Lambda'} \sum_i p(\lambda'\mid i) + \frac{1}{8} \max_{i, j} \sum_{\lambda \in \Lambda} |p(\lambda'\mid i) - p(\lambda'\mid j)| \leq 1.
\]
