The deformation-stability fundamental length and deviations from c

R. Vilela Mendes∗†

Abstract

The existence of a fundamental length (or fundamental time) has been conjectured in many contexts. However, the "stability of physical theories principle" seems to be the one that provides, through the tools of algebraic deformation theory, an unambiguous derivation of the stable structures that Nature might have chosen for its algebraic framework. It is well-known that $1/c$ and $\hbar$ are the deformation parameters that stabilize the Galilean and the Poisson algebra. When the stability principle is applied to the Poincaré-Heisenberg algebra, two deformation parameters emerge which define two length (or time) scales. In addition there are, for each of them, a plus or minus sign possibility in the relevant commutators. One of the deformation length scales, related to non-commutativity of momenta, is probably related to the Planck length scale but the other might be much larger. In this paper this is used as a working hypothesis to compute deviations from $c$ in speed measurements of massless wave packets.

PACS: 11.10.Nx, 14.60.Lm, 06.20.Jr

1 Introduction

The idea of modifying the algebra of the space-time components $x_{\mu}$ in such a way that they become non-commuting operators has appeared many times

∗CMAF, Instituto para a Investigação Interdisciplinar, Av. Gama Pinto 2, 1649-003 Lisboa, Portugal, vilela@cii.fc.ul.pt, [http://label2.ist.utl.pt/vilela/]
†IPFN - EURATOM/IST Association, Instituto Superior Técnico, Av. Rovisco Pais 1, 1049-001 Lisboa, Portugal
in the physical literature ([1] [2] [3] [4] [7], [6] [7] [8] [9] [10] [11] [12] [13] [14] [15] [16] [17] [18] [19] [20], etc.). The aim of most of these proposals was to endow space-time with a discrete structure, to be able, for example, to construct quantum fields free of ultraviolet divergences. Sometimes a non-zero commutator is simply postulated, in some other instances the motivation is the formulation of field theory in curved spaces. String theories [21] [22] and quantum relativity [23] [24] have also provided hints concerning the non-commutativity of space-time at a fundamental level.

A somewhat different point of view has been proposed in [25] [26]. There the space-time noncommutative structure is arrived at through the application of the stability of physical theories principle (SPT). The rationale behind this principle is the fact that the parameters entering in physical theories are never known with absolute precision. Therefore, robust physical laws with a wide range of validity can only be those that do not change in a qualitative manner under a small change of parameters, that is, stable (or rigid) theories. The stable-model point of view originated in the field of non-linear dynamics, where it led to the notion of structural stability [27] [28]. Later on, Flato [29] and Faddeev [30] have shown that the same pattern occurs in the fundamental theories of Nature, namely the transition from non-relativistic to relativistic and from classical to quantum mechanics, may be interpreted as the replacement of two unstable theories by two stable ones. The stabilizing deformations lead, in the first case, from the Galilean to the Lorentz algebra and, in the second one, from the algebra of commutative phase-space to the Moyal-Vey algebra (or equivalently to the Heisenberg algebra). The deformation parameters are $\frac{1}{c}$ (the inverse of the speed of light) and $h$ (the Planck constant). Except for the isolated zero value, the deformed algebras are all equivalent for non-zero values of $\frac{1}{c}$ and $h$. Hence, relativistic mechanics and quantum mechanics might have been derived from the conditions for stability of two mathematical structures, although the exact values of the deformation parameters cannot be fixed by purely algebraic considerations. Instead, the deformation parameters are fundamental constants to be obtained from experiment and, in this sense, not only is deformation theory the theory of stable theories, it is also the theory that identifies the fundamental constants.

The SPT principle is related to the idea that physical theories drift towards simple algebras [31] [32] [33], because all simple algebras are stable, although not all stable algebras are simple.

When the SPT principle is applied to the algebra of relativistic quantum
mechanics (the Poincaré-Heisenberg algebra)

\[
\begin{align*}
[M_{\mu \nu}, M_{\rho \sigma}] &= i(M_{\mu \sigma} \eta_{\nu \rho} + M_{\nu \rho} \eta_{\mu \sigma} - M_{\nu \sigma} \eta_{\mu \rho} - M_{\mu \rho} \eta_{\nu \sigma}) \\
[M_{\mu \nu}, p_\lambda] &= i(p_\mu \eta_{\nu \lambda} - p_\nu \eta_{\mu \lambda}) \\
[M_{\mu \nu}, x_\lambda] &= i(x_\mu \eta_{\nu \lambda} - x_\nu \eta_{\mu \lambda}) \\
[p_\mu, p_\nu] &= 0 \\
[x_\mu, x_\nu] &= 0 \\
[p_\mu, x_\nu] &= i \eta_{\mu \nu} 1
\end{align*}
\]

\(\eta_{\mu \nu} = (1, -1, -1, -1), \ c = h = 1,\) it leads \[25\] to

\[
\begin{align*}
[M_{\mu \nu}, M_{\rho \sigma}] &= i(M_{\mu \sigma} \eta_{\nu \rho} + M_{\nu \rho} \eta_{\mu \sigma} - M_{\nu \sigma} \eta_{\mu \rho} - M_{\mu \rho} \eta_{\nu \sigma}) \\
[M_{\mu \nu}, p_\lambda] &= i(p_\mu \eta_{\nu \lambda} - p_\nu \eta_{\mu \lambda}) \\
[M_{\mu \nu}, x_\lambda] &= i(x_\mu \eta_{\nu \lambda} - x_\nu \eta_{\mu \lambda}) \\
[p_\mu, p_\nu] &= -i \frac{\ell'}{\ell R} M_{\mu \nu} \\
[x_\mu, x_\nu] &= -i \ell^2 M_{\mu \nu} \\
[p_\mu, x_\nu] &= i \eta_{\mu \nu} \mathbb{3} \\
[p_\mu, \mathbb{3}] &= -i \frac{\ell'}{\ell R} x_\mu \\
[x_\mu, \mathbb{3}] &= i \ell^2 p_\mu \\
[M_{\mu \nu}, \mathbb{3}] &= 0
\end{align*}
\]

The stabilization of the Poincaré-Heisenberg algebra has been further studied and extended in \[34\] \[35\] \[36\]. The essential message from \[2\] or from the slightly more general form obtained in \[34\] is that from the unstable Poincaré-Heisenberg algebra \(\{M_{\mu \nu}, p_\mu, x_\nu\}\) one obtains a stable algebra with two deformation parameters \(\ell\) and \(\frac{1}{R}\). In addition there are two undetermined signs \(\epsilon\) and \(\epsilon'\) and the central element of the Heisenberg algebra becomes a non-trivial operator \(\mathbb{3}\). The existence of two continuous deformation parameters when the algebra is stabilized is a novel feature of the deformation point of view, which does not appear in other noncommutative space-time approaches. These deformation parameters may define two different length scales. Of course, once one of them is identified as a fundamental constant, the other will be a pure number.

Being associated to the noncommutativity of the generators of space-time translations, the parameter \(\frac{1}{R}\) may be associated to space-time curvature and therefore might not be relevant for considerations related to the tangent space. It is, of course, very relevant for quantum gravity studies \[36\]. Already in the past, some authors \[30\], have associated the noncommutativity of translations to gravitational effects, the gravitation constant being
the deformation parameter. Presumably then $\frac{1}{R}$ might be associated to the Planck length scale. However $\ell$, the other deformation parameter, defines a completely independent length scale which might be much closer to laboratory phenomena. This will be the working hypothesis to be explored in this paper. Therefore when $\frac{1}{R}$ is assumed to be very small the deformed algebra may be approximated by

\[
\begin{align*}
[M_{\mu\nu}, M_{\rho\sigma}] &= i(M_{\mu\sigma} \eta_{\nu\rho} + M_{\nu\rho} \eta_{\mu\sigma} - M_{\nu\sigma} \eta_{\mu\rho} - M_{\mu\rho} \eta_{\nu\sigma}) \\
[M_{\mu\nu}, p_\lambda] &= i(p_\mu \eta_{\nu\lambda} - p_\nu \eta_{\mu\lambda}) \\
[M_{\mu\nu}, x_\lambda] &= i(x_\mu \eta_{\nu\lambda} - x_\nu \eta_{\mu\lambda}) \\
[p_\mu, p_\nu] &= 0 \\
[x_\mu, x_\nu] &= -i\epsilon \ell^2 M_{\mu\nu} \\
[p_\mu, \Im] &= i\eta_{\mu\nu} \Im \\
[p_\mu, 3] &= 0 \\
[x_\mu, 3] &= i\epsilon \ell^2 p_\mu \\
[M_{\mu\nu}, 3] &= 0
\end{align*}
\tag{3}
\]

Notice that in addition to the space-time non-commutative structure, there is also a new non-trivial operator $\Im$ which replaces the central element of the Heisenberg algebra. In particular this operator corresponds to an additional component in the most general connections compatible with (3) \[26\]. For future reference this algebra will be denoted $\mathcal{R}_{\ell,\infty}$. Notice that in relation to the more general deformation obtained in \[34\], we are also considering $\alpha_3 = 0$ (or $\beta = 0$ in \[36\]). The nature of the sign $\epsilon$ has physical consequences. If $\epsilon = +1$ time will have a discrete spectrum, whereas if $\epsilon = -1$ it is when one the space coordinates is diagonalized that discrete spectrum is obtained. In this sense if $\epsilon = +1$, $\ell$ might be called ”the fundamental time” and ”the fundamental length” if $\epsilon = -1$.

General (noncommutative) geometry properties of the algebra (3) have been studied before \[26\] as well as some other consequences \[37\] \[38\] \[39\] \[40\] \[41\]. Here the emphasis will be on the hypothesis that $\ell$ defines a length scale much larger than Planck’s and on its consequences for deviations from $c$ in speed measurements of massless wave packets.

In the Appendix, some explicit representations of the space-time algebra are collected, which are useful for the calculations.
2 Measuring the speed of wave-packets

In the noncommutative context, because the space and the time coordinates cannot be simultaneously diagonalized, speed can only be defined in terms of expectation values, for example

\[ v^i_\psi = \frac{1}{\langle \psi_t, \psi_t \rangle} \frac{d}{dt} \langle \psi_t, x^i \psi_t \rangle \]  

(4)

Here, one considers a normalized state \( \psi \) with a small dispersion of momentum around a central value \( p \). At time zero

\[ \psi_0 = \int |k^0 \vec{k} \alpha \rangle f_p(k) d^3k \]  

(5)

where \( k^0 = \sqrt{|\vec{k}|^2 + m^2} \), \( \alpha \) standing for the quantum numbers associated to the little group of \( k \) and \( f_p(k) \) is a normalized function peaked at \( k = p \).

To obtain \( \psi_t \) one should apply to \( \psi_0 \) the time-shift operator. However this is not \( p^0 \) because

\[ e^{-iap^0} e^{iap^0} = t + a3 \]  

(6)

follows from

\[ [p^0, t] = i3 \]  

(7)

whereas a time-shift generator \( \Gamma \) should satisfy

\[ [\Gamma, t] = i1 \]  

(8)

In order \( O(\ell^4) \) one has

\[ \Gamma = p^03^{-1} - \frac{\ell^2}{3} (p^0)^3 3^{-3} \]  

(9)

because

\[ [\Gamma, t] = i \left( 1 - \ell^4 (p^0)^4 3^{-4} \right) \]  

(10)

To obtain this result, use was made of \( [t, 3^{-1}] = -i\ell^2 p^0 3^{-2} \), which follows from \( [t, 33^{-1}] = 0 \).

Now use a basis where the set \( (p^\mu, 3) \) is diagonalized and define

\[ \tilde{p}^\mu = \frac{p^\mu}{3} \]  

(11)
\(\tilde{p}^\mu\) is the momentum in units of \(\Im\).

Therefore, in the same \(O(\ell^4)\) order

\[
\psi_t = \int \exp \left( -it \left( \tilde{p}^0 - \frac{\epsilon}{3} \ell^2 \left( \tilde{p}^0 \right)^3 \right) \right) \left| \tilde{k}^0 \tilde{k}^\alpha \right> f_\nu (\tilde{k}) \, d^3 \tilde{k} \tag{12}
\]

To compute the expectation value of \(x^i\) one notices that from (22)

\[
x^\mu = i \left( \epsilon \ell^2 p^\mu \frac{\partial}{\partial \Im} - \Im \frac{\partial}{\partial p^\mu} \right) \tag{13}
\]

using \(\frac{\partial}{\partial \Im} = -p^\nu \frac{\partial}{\partial p^\nu}\) one obtains

\[
x^\mu = -i \left( \frac{\partial}{\partial \tilde{p}^\mu} + \epsilon \ell^2 \left\{ p^\mu p^\nu \frac{\partial}{\partial \tilde{p}^\nu} \right\}_S \right) \tag{14}
\]

\(\{}_S\) meaning symmetrization of the operators.

Now the expectation value of this operator in the state \(\psi_t\) is computed and taking the time derivative one obtains for the wave packet speed in order \(\ell^2\)

\[
v_\psi = \frac{\tilde{p}^0}{p^0} \left( 1 - \epsilon \ell^2 \left( \tilde{p}^0 \right)^2 \right) - \epsilon \ell^2 \left( \tilde{p} \tilde{p}^0 + \left( \tilde{p}^0 \right)^2 \frac{\tilde{p}}{p^0} \right) \tag{15}
\]

\(\ell^2\) being small, this deviation from \(\frac{\tilde{p}^0}{p^0}\) may be difficult to detect for massive particles given the uncertainty on the values of the masses. However, for massless particles the deviation from \(c (= 1)\)

\[
\Delta v_\psi = -3 \epsilon \ell^2 \left( \tilde{p}^0 \right)^2 \tag{16}
\]

might already be possible to detect accurately with present experimental means. Such deviation above or below the speed \(c\) (depending on the sign of \(\epsilon\)) would not imply any modification of the relativistic deformation constant \(\left( \frac{1}{c} \right)\), nor a breakdown of relativity.

The deviation formula (16) will now be compared with the velocity data of neutrino packets. The OPERA experiment [42] finds

\[
\frac{v - c}{c} = (2.37 \pm 0.32) \times 10^{-5} \tag{17}
\]
for \( p^0 = 17 \) GeV neutrinos. Comparing with (16) implies

\[
\ell = 3.26^{+0.21}_{-0.23} \times 10^{-18} \text{cm} \tag{18}
\]

and \( \epsilon = -1 \). On the other hand the MINOS \cite{43} result

\[
\frac{v - c}{c} = (5.1 \pm 2.9) \times 10^{-5} \tag{19}
\]

using the peak energy at 3 GeV would imply \( \ell = 2.7 \times 10^{-17} \text{cm} \). However this data has a long high energy tail, hence this result is not reliable.

On the other hand the analysis of the SN1987A supernova \cite{44} \cite{45}, taking into account the fact that the outburst of visible light begins later than the neutrino burst, when the cooler envelope is blown way, led to the bound \cite{46}

\[
\left| \frac{v - c}{c} \right| \lesssim 2 \times 10^{-9} \tag{20}
\]

With the SN1987A neutrinos in the range 20 – 40 MeV, one computes from (16) with \( p^0 = 40 \) MeV and \( \ell = 3.2 \times 10^{-18} \text{cm} \)

\[
\Delta v_\psi = 1.26 \times 10^{-10}
\]

compatible with the bound (20). Incidentally, the shift for the visible light is much smaller, namely \( \Delta v_\psi = 4.9 \times 10^{-25} \) for \( p^0 = 2.5 \) eV.

In conclusion: the formula (16) is consistent with all existing data and the most probable value for \( \ell \) is around \( 3 \times 10^{-18} \text{cm} \) (or \( 10^{-28} \text{s} \)) and \( \epsilon = -1 \).

3 Remarks and conclusions

1) The most relevant point of the stability approach to noncommutative space-time is the emergence of two deformation parameters, which might define different length scales. This led to the conjecture that one of them might be much larger than the Planck length and therefore already detectable with contemporary experimental means.

2) The deviation from \( c \) when measuring the velocity of wavepackets of massless (or near massless) particles, does not implies any violation of relativity nor does it imply a modification of the value of the deformation
parameter $\frac{1}{c}$. What it perhaps implies is that $c$ should not be called the speed of light.

3) Other effects arising from the deformation-stability noncommutative structure are explored in Ref. [47]. Both the effects explore here and in [47] are rather conservative in the sense that they explore well-known physical observables. A more speculative aspect of the noncommutative structure concerns the physical relevance of the extra derivation $\partial_4$ in the geometrical structure. This includes new fields associated to gauge interactions which may lead to effective mass terms for otherwise massless particles (see [26] for more details)

4 Appendix A: Representations of the deformed algebra and its subalgebras

For explicit calculations of the consequences of the non-commutative space-time algebra [2] (with $\epsilon' = 0$) it is useful to have at our disposal functional representations of this structure. Such representations on the space of functions defined on the cone $C^4 (\epsilon = -1)$ or $C^3, 1 (\epsilon = +1)$ have been described in [26]. Here one collects a few other useful representations of the full algebra and some subalgebras.

1 - As differential operators in a 5-dimensional commutative manifold $M_5 = \{\xi_\mu\}$ with metric $\eta_{aa} = (1, -1, -1, -1, \epsilon)$

\[
\begin{align*}
  p_\mu &= i \frac{\partial}{\partial x_\mu} \\
  \Im &= 1 + i \ell \frac{\partial}{\partial \xi_4} \\
  M_{\mu\nu} &= i(\xi_\mu \frac{\partial}{\partial \xi_\nu} - \xi_\nu \frac{\partial}{\partial \xi_\mu}) \\
  x_\mu &= \xi_\mu + i\ell(\xi_\mu \frac{\partial}{\partial \xi_4} - \epsilon \xi_4 \frac{\partial}{\partial \xi_\mu})
\end{align*}
\]

(21)

2 - Another global representation is obtained using the commuting set $(p^\mu, \Im)$, namely

\[
\begin{align*}
  x_\mu &= i \left( \epsilon \ell^2 p_\mu \frac{\partial}{\partial \xi_4} - \Im \frac{\partial}{\partial p_\mu} \right) \\
  M_{\mu\nu} &= i \left( p_\mu \frac{\partial}{\partial p_\nu} - p_\nu \frac{\partial}{\partial p_\mu} \right)
\end{align*}
\]

(22)

3 - Representations of subalgebras

Because of non-commutativity only one of the coordinates can be diagonalized. Here, consider the restriction to one space dimension, namely the algebra of $\{p^0, \Im, p^1, x^0, x^1\}$. 

8
For $\epsilon = +1$ define hyperbolic coordinates in the plane $(p^1, \Im)$ and polar coordinates in the plane $(p^0, \Im)$. Then, from it follows from (22)

\begin{align*}
  p^1 &= \frac{r}{\gamma} \sinh \mu \\
  p^0 &= \frac{r}{\gamma} \sin \theta \\
  \Im &= r \cosh \mu = \gamma \cos \theta \\
  x^1 &= i \ell \frac{\partial}{\partial \mu} \\
  x^0 &= -i \ell \frac{\partial}{\partial \theta}
\end{align*}

(23)

For $\epsilon = -1$ with polar coordinates in the plane $(p^1, \Im)$ and hyperbolic coordinates in the plane $(p^0, \Im)$,

\begin{align*}
  p^1 &= \frac{r}{\gamma} \sin \theta \\
  p^0 &= \frac{r}{\gamma} \sinh \mu \\
  \Im &= \gamma \cosh \mu = r \cos \theta \\
  x^1 &= i \ell \frac{\partial}{\partial \theta} \\
  x^0 &= -i \ell \frac{\partial}{\partial \mu}
\end{align*}

(24)

References

[1] H. S. Snyder; Phys. Rev. 71 (1947) 38-41; 72 (1947) 68-71.
[2] C. N. Yang; Phys. Rev. 72 (1947) 874.
[3] V. G. Kadyshevsky; Nucl. Phys. B141 (1978) 477-496.
[4] V. G. Kadyshevsky and D. V. Fursaev; Teor. Mat. Phys. 83 (1990) 474-481.
[5] M. Banai; Int. J. Theor. Phys. 23 (1984) 1043-1063.
[6] A. Das; J. Math. Phys. 7 (1966) 52-60.
[7] D. Atkinson and M. B. Halpern; J. Math. Phys. 8 (1967) 373-387.
[8] S. P. Gudder; SIAM J. Appl. Math. 16 (1968) 1011-1019.
[9] D. Finkelstein; Phys. Rev. D9 (1974) 2219-2231.
[10] M. Dineykhan and K. Namsrai; Int. J. Theor. Phys. 24 (1985) 1197-1215.
[11] T. D. Lee; Phys. Lett. 122B (1983) 217-220.

[12] E. Prugovecki; *Stochastic Quantum Mechanics and Quantum Spacetime*, Reidel Pub. Comp., Dordrecht, 1983.

[13] D. I. Blokhintsev; Teor. Mat. Phys. 37 (1978) 933-937.

[14] A. Schild; Phys. Rev. 73 (1948) 414-415.

[15] G. Veneziano; CERN preprint TH 5581/89.

[16] G. ’t Hooft; in *Themes in Contemporary Physics*, vol. II, page 77, S. Deser and R. J. Finkelstein (Eds.), World Scientific, Singapore 1990.

[17] J. C. Jackson; J. Phys. A: Math. Gen. 10 (1977) 2115-2122.

[18] J. Madore; Ann. Phys. 219 (1992) 187-198.

[19] M. Maggiore; Phys. Lett. B319 (1993) 83-86.

[20] A. Kempf and G. Mangano; Phys. Rev. D55 (1997) 7909-7920.

[21] E. Witten; Nucl. Phys. B268 (1986) 253-294.

[22] N. Seiberg and E. Witten; J. High Energy Phys. 09 (1999) 032.

[23] S. Doplicher, K. Fredenhagen and J. E. Roberts; Commun. Math. Phys. 172 (1995) 187-220.

[24] M. Dubois-Violette, R. Kerner and J. Madore; J. Math. Phys. 39 (1998) 730-738.

[25] R. Vilela Mendes; J. Phys. A: Math. Gen. 27 (1994) 8091-8104.

[26] R. Vilela Mendes; J. Math. Phys. 41 (2000) 156-186.

[27] A. Andronov and L. Pontryagin; Dokl. Akad. Nauk. USSR 14 (1937) 247.

[28] S. Smale; Am. J. Math. 88 (1966) 491; Bull. Am. Math. Soc. 73 (1967) 747.

[29] M. Flato; Czech J. Phys. B32 (1982) 472.
[30] L. D. Faddeev; Asia-Pacific Physics News 3 (1988) 21 and in "Frontiers in Physics, High Technology and Mathematics" (ed. Cerdeira and Lundqvist) pp.238-246, World Scientific, 1989.

[31] I. E. Segal; Duke Math. Journal 18 (1951) 221-265.

[32] D. R. Finkelstein; Int. J. Theor. Phys. 45 (2006) 1399-1429.

[33] A. A. Galiautdinov and D. R. Finkelstein; J. Math. Phys. 43 (2002) 4741-4752.

[34] C. Chryssomalakos and E. Okon; Int. J. Mod. Phys. D 13 (2004) 2003-2034.

[35] D. V. Ahluwalia-Khalilova; Int. J. Mod. Phys. 14 (2005) 2151-2165.

[36] D. V. Ahluwalia-Khalilova; Class. Quantum Grav. 22 (2005) 1433-1450.

[37] R. Vilela Mendes; Phys. Lett. A 210 (1996) 232-240.

[38] E. Carlen and R. Vilela Mendes; Physics Letters A 290 (2001) 109-114.

[39] R. Vilela Mendes; Eur. J. Phys C 42 (2005) 445-452.

[40] V. Dzhunushaliev and R. Myrzakulov; Int. J. Mod. Phys. D 16 (2007) 755-761.

[41] G. A. Goldin and S. Sarkar; J. of Phys. A - Math. and Gen. 39 (2006) 2757-2772.

[42] OPERA Collaboration, T. Adam et al., arXiv:1109.4897.

[43] MINOS Collaboration, P. Adamson et al.; Phys. Rev. D76 (2007) 072005.

[44] K. Hirata et al.; Phys. Rev. Lett. 58 (1987) 1490-1493.

[45] R. M. Bionta et al.; Phys. Rev. Lett. 58 (1987) 1494-1496.

[46] M. J. Longo; Phys. Rev. D36 (1987) 3276-3277.

[47] R. Vilela Mendes; arXiv:1111.5576.