MORE GRAVITATIONAL ANYONS*

C. Aragone
Departamento de Física, Universidad Simón Bolívar
Apartado 89000, Caracas 10800-A, Venezuela
e-mails address: aragone@usb.ve
and
P. J. Arias
Departamento de Física, Universidad Simón Bolívar
Apartado 89000, Caracas 10800-A, Venezuela
parias@usb.ve

ABSTRACT

The anyonic behaviour of massive spinning point particles coupled to linearized massive vector Chern-Simons gravity is studied. This model constitutes the uniform spin-2 generalization of the vector model formed by coupling charged point particles to the topological massive Maxwell-CS action. It turns out that, for this model, the linearized first order triadic Chern-Simons term is the source of the anyonic behaviour we found.

This is in contrast with the third order topologically massive gravity, where the anyonic behaviour does not stem in its third-order Lorentz-Chern-Simons term, the second order Einstein’s action.

*To appear in the proceedings of SILARG VIII, Aguas de Lindoia, July 1993
1. Introduction

In 2+1 dimensions, particles with arbitrary real spin and non standard statistics appear naturally. This fact arises, respectively, from the abelian structure of the two dimensional rotation group, and from the topology of the many identical particles configuration space\(^1\). As a consequence, in 2+1 dimensions, besides the usual occurrence of bosons and fermions, we can have anyons\(^2\), or particles that have spin and statistics which are neither bosonic nor fermionic. This ideas can be dynamically implemented when we consider theories of “bare” bosons (or fermions) with an appropriate interaction between them. In this theories the particles behave effectively as ideal anyons i.e. as localized magnetic flux-points\(^3\).

This implementation of exotic statistics is commonly done minimally coupling point particles to electrodynamics with a Chern-Simons term. In these models the statistics can be understood in terms of the Aharanov-Bohm effect because the first-order Chern-Simons term induces a magnetic flux-point in the position of each particle\(^3,4\). This picture asymptotically persists in the presence of a (second order) Maxwell term.

There is a close analogy between the action of a charged point particle in an electromagnetic field and the linearized action of a massive point particle in a weak gravitational field\(^5\). So we can define the gravitational analog of the magnetic flux-point and expect that there should be a dynamical mechanism to assign them to particles. This was recently proposed\(^6,7\) in the context of linearized Topological Massive (TMG) gravity and the anyonic behaviour found is, for certain circumstances, just like the one obtained for the corresponding vector theory.

In this note we present a dynamical implementation of exotic statistics in the context of linearized Massive Vector Chern-Simons (VCSG) gravity. We compute the anyonic behaviour for a spinning point particle coupled to this linearized gravity theory. These results are in direct correspondence with those of the vector analog.

We show that the (linearized) triadic Chern-Simons term works for spin-2 just like the vector Chern-Simons term does for spin-1 models.

2. Anyonic behaviour in vector models

The simplest model in which exotic statistics is dynamically implemented corresponds to consider the quantum mechanics of two identical particles with the interaction term\(^9\)

\[
S = \langle \frac{\theta}{\pi} d\varphi >
\]  

where \(\varphi\) is the relative angle between the two particles, and \(\theta \in [0, \pi]\) is a numerical parameter, called the statistics parameter. \(\theta\) is 0 or \(\pi\) if one is willing to consider, respectively, two boson or two fermions, and ranges in the open segment, \(0 < \theta < \pi\), if one wishes to consider anyons. The induced angular momentum for each particle will be \(\theta/2\pi\).

The interaction term can be rewritten as

\[
S_I = \langle q d\vec{r}_{rel} \cdot \vec{A}(\vec{r}_{rel}) > ,
\]
with
\[ A_i = -(q\pi)^{-1} \theta \varepsilon_{ij} \partial_j \ln r_{rel}. \]  
(3)

This corresponds to assigning a magnetic flux-point at the origin of the relative frame. The flux of the "magnetic" field is
\[ \Phi = q^{-1} 2 \theta. \]  
(4)

In this picture each particle "sees" the other as a magnetic flux-point, and the Aharanov-Bohm phase will be $q\Phi$.

Let us define the anyonic behaviour parameter by the circulation
\[ \alpha \equiv q \oint dx^i A_i, \]  
(5)
whose relation with the statistics parameter is $\theta = \alpha/2$.

From this naive non relativistic picture we can go further and consider the second order action
\[ S = -4^{-1} F_{mn} F^{mn} - 2^{-1} \mu \varepsilon^{mnl} \partial_n A_l + A_m J^m >, \]  
(6)
where $F_{mn} = \partial_m A_n - \partial_n A_m$ and $J^m$ is the conserved particle's source. This action might possess other terms related with the particles dynamics, here we consider the simplest minded model. Pure Chern-Simons vector coupling is obtained in the large $\mu$ limit. Independent variations of $A_l$ in action (6) yield the equations of motion
\[ \partial_m F^{mn} - \mu \varepsilon^{mnl} \partial_n A_l = J^m. \]  
(7)

Our conventions are $\eta_{mn} = (- + +), \varepsilon^{012} = 1, \varepsilon^{oij} \equiv \varepsilon_{ij}$.

We take a circularly symmetric static point current as the source of Eq. 7
\[ J^0 = q \delta^{(2)}(\vec{r}), \quad J^i = g \varepsilon_{ij} \partial_j \delta^{(2)}(\vec{r}), \]  
(8)
where we have included, for later comparison, a tranverse spatial dipole current. In order to obtain $A_r$ we make the anzatz
\[ A_0 = a(r), \quad A_i = \varepsilon_{ij} \partial_j V(r) + \partial_i \lambda. \]  
(9)

The static solution turns out to be
\[ A_0 = -(q + \mu g) Y(\mu r), \]  
\[ A_i = \varepsilon_{ij} \partial_j (-(\mu^{-1} q + g) Y(\mu r) + \mu^{-1} q C(r)) \]  
(10)
where $Y(\mu r)$ and $C(r)$ are respectively, the Yukawa and Coulomb Green's functions
\[ (-\Delta + \mu^2) Y(\mu r) = \delta^{(2)}(\vec{r}), \quad (-\Delta) C(r) = \delta^{(2)}(\vec{r}), \]  
(11)
and
\[ Y(\mu r) = (2\pi)^{-1} K_0(\mu r), \quad C(r) = -(2\pi)^{-1} \ln \mu r. \]  
(12)
For the pure Chern-Simons theory (which is the large $\mu$ limit of (7)), the solutions are
\[ A_0 = -\mu^{-1}g\delta^{(2)}(\vec{r}), \quad A_i = \mu^{-1}q\varepsilon_{ij}\partial_j C(r), \] (13)
for which the anyonic behaviour parameter (5) is
\[ \alpha_{CS} = \mu^{-1}q^2, \] (14)
where we are thinking in terms of two equal charged particles exchanged. The dipole current strength $g$ does not affect this result.

For the full theory, with the Maxwell term present, using the fact that the modified Bessel function behaves asymptotically as
\[ K_0(x) \sim x^{-1/2}e^{-x}, \] (15)
we see that
\[ A_0 \sim 0, \quad A_i \sim \mu^{-1}q\varepsilon_{ij}\partial_j C(r), \] (16)
just as it happens with the pure Chern-Simons solution. Its anyonic behaviour parameter, for a circle of radius $R$, is
\[ \alpha_{TM}(R) = \mu^{-1}q^2 + q(q + \mu g)RK_1(\mu R), \] (17)
which differs from (14) unless $q + \mu g = 0$. Asymptotically the two anyonic behaviours for both theories coincide, as expected, due to the fact that the first-order Chern-Simons action is the term which provides the dominant contribution in this limit.

3. Anyonic behaviour for linearized MVCS gravity

The possibility of implementing exotic statistics in linearized gravity can be studied by considering the action of massive monopolar point particle moving in an external gravitational field
\[ S_p = -m \int d\tau (-g_{mn}\dot{x}^m\dot{x}^n)^{1/2} \] (18)
and then taking the limit for small velocities, going to the weak field approximation $g_{mn} = \eta_{mn} + \kappa h_{mn}$, $h_{mn} = h_{nm}$. In this approximation the Hamiltonian density takes the form
\[ H = (2m)^{-1}(\vec{p} - q\vec{A})^2 + qV, \] (19)
with
\[ qA_i = \kappa mh_{i0}, \quad qV = -2^{-1}\kappa mh_{00}. \] (20)

Notice that the expression for $H$ in Eq. 19 corresponds to the Hamiltonian density of a charged particle in an electromagnetic field whose vector potential is $A_\mu = (V, A_i)$. In this weak field approximation the linear coupling with matter can be seen easily to be
\[ S_C = 2^{-1}\kappa m < h_{mn}T^{mn} >, \] (21)
where $T^{mn}$ is the particle’s flat energy momentum tensor. If we introduce the dreibeins $e_m^a$ with $g_{mn} = e_m^ae_n^b\eta_{ab}$, and make the linearization $e_m^a = \delta_m^a + \kappa h_m^a$, then

$$h_{mn} = h_{mn} + h_{nm}, \quad (22)$$

($h_{mn}$ is not symmetric) and the linear coupling with matter that arises will be $\kappa m h_{mn}$ $T^{mn}$ with $T^{mn} = T^{nm}$ as in (21).

We now consider the minimally coupled action

$$S_2 = 2^{-1} < h_{pa}(2^{-1} \epsilon_{pma} \epsilon_{srb} - \epsilon^{pmb} \epsilon_{sra})\partial_m \partial_r h_{sb}> +$$

$$+ 2^{-1} \mu < h_{pa} \epsilon_{prs} \partial_r h_a^s > + \kappa < h_{mn} T^{mn} >$$

$$\equiv S_E^L + S_{TCS}^L + S_C, \quad (23)$$

where $S_E^L + S_{TCS}^L$ is the quadratic approximation of the curved second-order action of massive vector Chern-Simons gravity

$$S_{MVCS} = (2\xi)^{-1} < e_{pa} \epsilon_{pma} R^{sa}_{mn}(e) > + \mu e_{pa} \epsilon_{pma} \partial_m e_a^s, \quad (24)$$

The first term in (24) is Einstein’s action with $R^{sa}_{mn} = \partial_m \omega_n^a - \partial_n \omega_m^a - \epsilon_{abc} \omega_c^b \omega_m^c$ where $\omega_m^a(e)$ is the value of the torsionless connection given in terms of the dreibeins i.e. such that $\epsilon_{pma} (\partial_m e_a^s - \epsilon_{bca} \omega_c^b e_a^s) = 0$. The second term in Eq. 24 is the Triadic Chern-Simons (TCS) term and its quadratic part, as we will show, constitutes the gravitational analog of the Chern-Simons vector term which allow the dynamical implementation of statistics.

$S_E^L + S_{TCS}^L$ was introduced as an intermediate action which interpolates between first-order self-dual gravity and the master action for linearized gravity theories. It propagates one excitation of helicity $\pm 2\mu/|\mu|$, depending on the sign of $\mu$. This action is invariant under the gauge transformations $\delta h_{mn} = n \epsilon_{mn}$ and is equivalent, as a free theory, both to self-dual gravity and to linearized TMG. This equivalence will not persist, as it will be shown, in this interaction picture, if the theories are coupled to point particles under the same footings. So let us pass to the dynamical analysis.

The equation of motion which follows from (23) is

$$(2^{-1} \epsilon_{pml} \epsilon_{srm} - \epsilon_{pmm} \epsilon_{sr} )\partial_m \partial_r h_{sn} + \mu \epsilon_{pms} \partial_s h_{r} = -kT^{pl}. \quad (25)$$

We take the static source

$$T^{00} = m \delta^{(2)}(\tau), \quad T^{0i} = 2^{-1} \sigma \epsilon_{ij} \partial_j \delta^{(2)}(\tau), \quad T^{ij} = 0, \quad (26)$$

which corresponds to a massive ($\int T^{\infty} d^2 x = m$), spinning ($\int \epsilon_{ij} x_i T^{0j} d^2 x = \sigma$) point particle. Looking for static solutions, we choose the natural $T + L$ decomposition (after projection of the unsymmetric linearized dreibein into its symmetric plus antisymmetric parts $h_{mn} = 2^{-1}h_{mmn} + \epsilon_{mnlt} v^l$)

$$h_{00} = n(r),$$

$$h_{0i} = \epsilon_{ij} \partial_j (n^T - V^L) + \partial_i (n^L + V^T),$$

$$h_{i0} = \epsilon_{ij} \partial_j (n^T - V^L) + \partial_i (n^L + V^T),$$

$$h_{ij} = (\delta_{ij} \Delta - \delta_i \partial_j) h^T + (\epsilon_{ik} \partial_k \partial_j + \epsilon_{jk} \partial_k \partial_i) h^{TL} + \partial_i \partial_j h^T + \epsilon_{ij} V. \quad (27)$$
We take the gauge $V = V^T = 0$, $h^L = h^T$ and insert the structures (26) (27) into Eq. 25 in order to determine $h^T$, $h^{TL}$, $n^T$, $n^L$, $n$.

We obtain for the linearized metric ($h_{mn} = h_{mn} + h_{nm}$)

$$h_{00} = \kappa(m + \mu \sigma)Y(\mu r),$$
$$h_{0i} = -\varepsilon_{ij}\partial_j(\kappa^{-1}m + \sigma)Y(\mu r) + \mu^{-1}kmC(r),$$
$$h_{ij} = \delta_{ij}\kappa(m + \mu \sigma)Y(\mu r).$$

(28)

The large $\mu$ limit of (25) represents pure TCS theory. Its solutions are, for the linearized metric

$$h_{00} = \mu^{-1}\kappa\sigma\delta^2(\tau'), \quad h_{0i} = \mu^{-1}\kappa m\varepsilon_{ij}\partial_jC(r), \quad h_{ij} = \mu^{-1}\kappa\sigma\delta^2(\tau').$$

(29)

It is the gravitational generalization of (13) provided we make the identification

$$\kappa m \leftrightarrow q, \quad \kappa \sigma \leftrightarrow g, \quad h_{00} \leftrightarrow -A_0, \quad h_{0i} \leftrightarrow A_i.$$ 

(30)

The anyonic behaviour parameter is

$$\alpha_{\text{TCS}} = \kappa m \oint dx^i h_{0i} = \mu^{-1}\kappa^2 m^2,$$ 

(31)

independently of any contour, as expected.

For the, coupled, linearized MVCS gravity theory, we note that the static solutions, (28) correspond to the generalization of (10) with the above mentioned identifications. The asymptotic behaviour of the solutions is

$$h_{00} \sim 0, \quad h_{0i} \sim -(2\pi \mu)^{-1}\kappa m\varepsilon_{ij}\partial_j\ln r, \quad h_{ij} \sim 0,$$

confirming this assertions. Its anyonic behaviour parameter for a circle of radius $R$, as expected from (17), is given by

$$\alpha_{\text{MVCS}}(R) = \mu^{-1}(\kappa^2 m^2) + (km)\kappa(m + \mu \sigma)RK_1(\mu R).$$

(32)

For the special case that $m + \mu \sigma = 0$ it is identical to Eq. 31. It also asymptotically coincides with it.

We see, then, that massive point particles independently of its intrinsic angular momentum look like “flux-points” when the interaction term between them is a linearized TCS term. This picture persists at finite distances and also, if the linearized Einstein term is present, in the special case that $m + \mu \sigma = 0$. For any other value of $\sigma$ the results remains true asymptotically. So massive point particles behave as anyons with statistic parameter $2^{-1}\mu^{-1}\kappa^2 m^2$.

4. Conclusions

We presented a model that dynamically implements anyonic statistics in the context of linearized gravity models. Coupling massive spinning point particles with a
linearized TCS term constitute the exact analog of the pure Chern-Simons vector theory for charged particles. In the former model massive particles behave as anyons with statistic parameter \(2^{-1}\mu^{-1}\kappa^2m^2\) and induced angular momentum \((4\pi\mu)^{-1}\kappa^2m^2\).

The analogy goes over in the presence of Einstein’s term. Linearized massive vector-CS gravity is the uniform spin-2 generalization of Maxwell-CS theory. This is not the case for linearized topological massive gravity\(^6\), because its linearized version only works as an analog of MCS if there is no intrinsic angular momentum \(\sigma\), and no dipole current strength \(g\). In the case that \(m + \mu\sigma = 0\) there is no induced angular momentum for TMG. In contrast, for the MVCS model, it behaves just as the pure TCS model, exactly as it happens when you consider the vector analog with \(q + \mu g = 0\) and compare with the pure Chern-Simons theory.

We expect that the calculus of the induced spin from the generators of rotations will have close similarities with the vector model calculations\(^6\). These calculations are now in progress and will be reported elsewhere.

5. References

1. J. M. Leinaas and J. Myrheim, Nuovo Cimento B\textbf{37} (1977) 1; B. Binegar, J. Math. Phys. \textbf{23} (1982) 1511.
2. F. Wilczek, Phys. Rev. Lett. \textbf{48} (1982); ibid. \textbf{49} (1982) 957.
3. P. Arovas, R. Schrieffer, F. Wilczek and A. Zee, Nucl. Phys. B\textbf{251} (1985) 117; R. Mackenzie and F. Wilczek, Int. J. Mod. Phys. A\textbf{3} (1988) 2827, and references there in.
4. S. Rao, \textit{An anyon premier}, Preprint TIFR/TH/92-18; A. Khare, \textit{Quantum Mechanics and Statistics Mechanics of Anyons}, in the centenary issue of Holkar Science College, Indore, India.
5. B. De Witt, Phys. Rev. Lett. \textbf{16} (1966) 1092.
6. S. Deser, Phys. Rev. Lett. \textbf{64} (1990) 611; S. Deser and J. G. McCarthy, Nucl. Phys. B\textbf{344} (1990) 747; S. Deser, Class. Quantum Grav. \textbf{9} (1992) 61.
7. M. E. Ortiz, Nucl. Phys. B \textbf{1} (1991).
8. C. Aragone, P. J. Arias and A. Khoudeir, \textit{Massive Vector Chern-Simons gravity}, Preprint SB/F-192/92.
9. Mackenzie and F. Wilczek, in ref. 3.
10. C. Aragone and A. Khoudeir, Phys. Lett. B\textbf{173} (1986) 141; C. Aragone and A. Khoudeir, \textit{Quantum Mechanics of Fundamental Systems}, eds. C. Teitelboim and J. Zanelli. p. 17.
11. S. Deser and J. G. McCarthy, Phys. Lett. B\textbf{245} (1990) 441; (Addendum) B\textbf{248} (1990) 473.