THE $n$-LINEAR EMBEDDING THEOREM FOR DYADIC RECTANGLES

HITOSHI TANAKA AND KÔZO YABUTA

Abstract. Let $\sigma_i$, $i = 1, \ldots, n$, denote reverse doubling weights on $\mathbb{R}^d$, let $\mathcal{DR}(\mathbb{R}^d)$ denote the set of all dyadic rectangles on $\mathbb{R}^d$ (Cartesian products of usual dyadic intervals) and let $K : \mathcal{DR}(\mathbb{R}^d) \to [0, \infty)$ be a map. In this paper we give the $n$-linear embedding theorem for dyadic rectangles. That is, we prove the $n$-linear embedding inequality for dyadic rectangles

$$\sum_{R \in \mathcal{DR}(\mathbb{R}^d)} K(R) \prod_{i=1}^n \left| \int_R f_i \, d\sigma_i \right| \leq C \prod_{i=1}^n \|f_i\|_{L^{p_i}(\sigma_i)}$$

can be characterized by simple testing condition

$$K(R) \prod_{i=1}^n \sigma_i(R) \leq C \prod_{i=1}^n \sigma_i(R)^{\frac{1}{p_i}} \quad R \in \mathcal{DR}(\mathbb{R}^d),$$

in the range $1 < p_i < \infty$ and $\sum_{i=1}^n \frac{1}{p_i} > 1$. As a corollary to this theorem, for reverse doubling weights, we verify a necessary and sufficient condition for which the weighted norm inequality for the multilinear strong positive dyadic operator and for strong fractional integral operator to hold.

1. Introduction

The purpose of this paper is to prove the $n$-linear embedding theorem for dyadic rectangles. We will denote by $\mathcal{DQ}(\mathbb{R}^d)$ the family of all dyadic cubes $Q = 2^{-k}(m + [0,1)^d)$, $k \in \mathbb{Z}$, $m \in \mathbb{Z}^d$. We will denote by $\mathcal{DR}(\mathbb{R}^d)$ the family of all dyadic rectangles on $\mathbb{R}^d$. The dyadic rectangle we mean that the Cartesian product of the dyadic intervals $\mathcal{DQ}(\mathbb{R})$. Throughout this paper $n$ stands for an integer which is greater than one.

Through a series of works [2, 3, 5, 8, 10, 11, 12, 13], one perfectly characterizes the $n$-linear embedding inequality for dyadic cubes. Let $\sigma_i$, $i = 1, \ldots, n$, denote positive Borel measures on $\mathbb{R}^d$ and let $K : \mathcal{DQ}(\mathbb{R}^d) \to [0, \infty)$ be a map. The $n$-linear embedding inequality for dyadic cubes

$$\sum_{Q \in \mathcal{DQ}(\mathbb{R}^d)} K(Q) \prod_{i=1}^n \left| \int_Q f_i \, d\sigma_i \right| \leq C \prod_{i=1}^n \|f_i\|_{L^{p_i}(d\sigma_i)}$$

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can be characterized in the full range $1 < p_i < \infty$. The $n$-linear embedding theorem (1.1), either can be reduced to the (localized) $(n - 1)$-linear embedding theorems, or characterized by certain $n$-weight discrete Wolff potential conditions. The division line is whether the exponents $p_1, \ldots, p_n$ are in the super-dual range $\sum_{i=1}^n \frac{1}{p_i} \geq 1$ or in the strictly sub-dual range $\sum_{i=1}^n \frac{1}{p_i} < 1$. The inner workings of each range seem to be rather different (see [12]). The main technique used to is that of “parallel corona” decomposition from the work of Lacey et al. [7] on the two-weight boundedness of the Hilbert transform. However, this powerful technique deeply depends on the structure of dyadic cubes and can not apply dyadic rectangles. It is natural to consider what happens in the case $\mathcal{DR}(\mathbb{R}^d)$ and the partial answer is given in this paper.

By weights we will always mean nonnegative, locally integrable functions which are positive on a set of positive measure. Given a measurable set $E$ and a weight $\omega$, we will use $\omega(E)$ to denote $\int_E \omega \, dx$. By $1_E$ we stand for the characteristic function of $E$.

Let $1 \leq p < \infty$ and $\omega$ be a weight. We define the weighted Lebesgue space $L^p(\omega)$ to be a Banach space equipped with the norm

$$
\|f\|_{L^p(\omega)} = \left( \int_{\mathbb{R}^d} |f|^p \, d\omega \right)^{\frac{1}{p}},
$$

where we have used $d\omega := \omega \, dx$. Given $1 < p < \infty$, $p' = \frac{p}{p-1}$ will denote the conjugate exponent of $p$.

Let $\mathcal{R}(\mathbb{R}^d)$ denote the set of all rectangles in $\mathbb{R}^d$ with sides parallel to the coordinate axes. We say that a weight $\omega$ is “reverse doubling weight” if there is a constant $\beta > 1$ such that $\beta \omega(R') \leq \omega(R)$ for any $R', R \in \mathcal{R}(\mathbb{R}^d)$ where $R'$ is the two equal division of $R$. We shall prove the following theorem.

**Theorem 1.1.** Let $1 < p_i < \infty$ and $\sum_{i=1}^n \frac{1}{p_i} > 1$. Let $K : \mathcal{DR}(\mathbb{R}^d) \to [0, \infty)$ be a map and let $\sigma_i, i = 1, \ldots, n$, be reverse doubling weights on $\mathbb{R}^d$. The following statements are equivalent:

(a) The $n$-linear embedding inequality for dyadic rectangles

$$
\sum_{R \in \mathcal{DR}(\mathbb{R}^d)} K(R) \prod_{i=1}^n \left| \int_R f_i \, d\sigma_i \right| \leq c_1 \prod_{i=1}^n \|f_i\|_{L^{p_i}(\sigma_i)}
$$

holds for all $f_i \in L^{p_i}(\sigma_i), i = 1, \ldots, n$;

(b) The testing condition

$$
K(R) \prod_{i=1}^n \sigma_i(R) \leq c_2 \prod_{i=1}^n \sigma_i(R)^{\frac{1}{p_i}}
$$

holds for all dyadic rectangles $R \in \mathcal{DR}(\mathbb{R}^d)$.

Moreover, the least possible constants $c_1$ and $c_2$ are equivalent.
Corollary 1.2. Let $1 < p_i < \infty$, $1 < q < \infty$ and $\sum_{i=1}^{n} \frac{1}{p_i} > \frac{1}{q}$. Let $K : \mathcal{DR}(\mathbb{R}^d) \to [0, \infty)$ be a map and let $\sigma_i$, $i = 1, \ldots, n$, and $\omega$ be reverse doubling weights on $\mathbb{R}^d$. The following statements are equivalent:

(a) The weighted norm inequality for multilinear strong positive operator

\begin{equation}
\|T_K(f_1, \ldots, f_n)\|_{L^q(\omega)} \leq c_1 \prod_{i=1}^{n} \|f_i\|_{L^{p_i}(\sigma_i^{-p_i})}
\end{equation}

holds for all $f_i \in L^{p_i}(\sigma_i^{-p_i})$, $i = 1, \ldots, n$; Here,

$$T_K(f_1, \ldots, f_n) := \sum_{R \in \mathcal{DR}(\mathbb{R}^d)} K(R) 1_{R} \prod_{i=1}^{n} \int_{R} f_i \, d x.$$

(b) The testing condition

\begin{equation}
K(R) \omega(R)^{\frac{1}{q}} \prod_{i=1}^{n} \sigma_i(R) \leq c_2 \prod_{i=1}^{n} \sigma_i(R)^{\frac{1}{p_i}}
\end{equation}

holds for all dyadic rectangles $R \in \mathcal{DR}(\mathbb{R}^d)$. Moreover, the least possible constants $c_1$ and $c_2$ are equivalent.

In the last section we apply Corollary 1.2 to strong fractional integral operator. Two-weight estimates for multilinear fractional strong maximal operator and for strong fractional integral operator see \cite{1,4,9}.

The letter $C$ will be used for constants that may change from one occurrence to another. Constants with subscripts, such as $C_1, C_2$, do not change in different occurrences. By $A \approx B$ we mean that $C^{-1} B \leq A \leq B$ with some positive finite constant $c$ independent of appropriate quantities.

2. LEMMAS

We need two lemmas and we will give their proofs for the sake of completeness.

Lemma 2.1. Given a weight $\sigma$ in $\mathbb{R}^d$ and $1 < p < q < \infty$, the following statements are equivalent:

(a) The Carleson type embedding inequality for dyadic cubes

\begin{equation}
\sum_{Q \in \mathcal{Q}(\mathbb{R}^d)} \sigma(Q)^{\frac{2}{q}} \left( \frac{1}{\sigma(Q)} \int_{Q} f \, d \sigma \right)^{q} \leq c_1 \left( \int_{\mathbb{R}^d} f^{p} \, d \sigma \right)^{\frac{q}{p}}
\end{equation}

holds for all nonnegative function $f \in L^{p}(\sigma)$;

(b) The testing condition

\begin{equation}
\sum_{Q' \in \mathcal{Q}(\mathbb{R}^d)} \sigma(Q')^{\frac{2}{p}} \leq c_2 \sigma(Q)^{\frac{2}{p}}
\end{equation}

holds for all cubes $Q' \in \mathcal{Q}(\mathbb{R}^d)$. Moreover, the least possible constants $c_1$ and $c_2$ are equivalent.
Proof. The necessity \((2.2)\) follows at once if we substitute the test function \(f = 1_Q\) into inequality \((2.1)\). To show that inequality \((2.2)\) is sufficient, we fix a (big enough) dyadic cube \(Q_0 \in \mathcal{D}(\mathbb{R}^d)\) and we prove the inequality

\[
\sum_{Q \in \mathcal{D}(\mathbb{R}^d)} \sigma(Q)^\frac{q}{p} \left( \frac{1}{\sigma(Q)} \int_Q f \, d\sigma \right)^q \leq Cc_2 \left( \int_{Q_0} f^p \, d\sigma \right)^{\frac{q}{p}}.
\]

We define the collection of principal cubes \(\mathcal{F}\) for the pair \((f, \sigma)\). Namely,

\[
\mathcal{F} := \bigcup_{k=0}^{\infty} \mathcal{F}_k,
\]

where \(\mathcal{F}_0 := \{Q_0\}\),

\[
\mathcal{F}_{k+1} := \bigcup_{F \in \mathcal{F}_k} \text{ch}_\mathcal{F}(F)
\]

and \(\text{ch}_\mathcal{F}(F)\) is defined by the set of all maximal dyadic cubes \(Q \subset F\) such that

\[
\frac{1}{\sigma(Q)} \int_Q f \, d\sigma > \frac{2}{\sigma(F)} \int_F f \, d\sigma.
\]

Observe that

\[
\sum_{F' \in \text{ch}_\mathcal{F}(F)} \sigma(F') \leq \left( \frac{2}{\sigma(F)} \int_F f \, d\sigma \right)^{-1} \sum_{F' \in \text{ch}_\mathcal{F}(F)} \int_{F'} f \, d\sigma \leq \left( \frac{2}{\sigma(F)} \int_F f \, d\sigma \right)^{-1} \int_F f \, d\sigma = \frac{\sigma(F)}{2},
\]

and, hence,

\[
\sigma(E_\mathcal{F}(F)) := \sigma \left( F \setminus \bigcup_{F' \in \text{ch}_\mathcal{F}(F)} F' \right) \geq \frac{\sigma(F)}{2},
\]

where the sets in the collection \(\{E_\mathcal{F}(F) : F \in \mathcal{F}\}\) are pairwise disjoint.

We further define the stopping parent, for \(Q \in \mathcal{D}(\mathbb{R}^d)\),

\[
\pi_\mathcal{F}(Q) := \min \{ F \supset Q : F \in \mathcal{F} \}.
\]
Then we can rewrite the series in (2.3) as follows:

\[
\sum_{Q \subset Q_0} \sigma(Q)^{\frac{q}{p}} \left( \frac{1}{\sigma(Q)} \int_Q f \, d\sigma \right)^q
\]

\[
= \sum_{F \in \mathcal{F}} \sum_{Q : \pi_F(Q) = F} \sigma(Q)^{\frac{q}{p}} \left( \frac{1}{\sigma(Q)} \int_Q f \, d\sigma \right)^q
\]

\[
\leq \sum_{F \in \mathcal{F}} \left( \frac{2}{\sigma(F)} \int_F f \, d\sigma \right)^q \sum_{Q : \pi_F(Q) = F} \sigma(Q)^{\frac{q}{p}}
\]

\[
\leq 2^q c_2 \sum_{F \in \mathcal{F}} \left( \frac{1}{\sigma(F)} \int_F f \, d\sigma \right)^q \sigma(F)^{\frac{q}{p}},
\]

where we have used the condition (2.2).

Using \( \| \cdot \|_{l^p} \geq \| \cdot \|_{l^q}, \) for \( 0 < p \leq q < \infty, \) and (2.4) we can proceed further that

\[
\leq C c_2 \left\{ \sum_{F \in \mathcal{F}} \left( \frac{1}{\sigma(F)} \int_F f \, d\sigma \right)^p \sigma(F) \right\}^{\frac{q}{p}}
\]

\[
\leq C c_2 \left\{ \sum_{F \in \mathcal{F}} \left( \frac{1}{\sigma(F)} \int_F f \, d\sigma \right)^p \sigma(E_F(F)) \right\}^{\frac{q}{p}}
\]

\[
\leq C c_2 \left( \int_{Q_0} M^p_{\mathcal{DQ}}[f 1_{Q_0}] \, d\sigma \right)^{\frac{q}{p}}
\]

\[
\leq C c_2 \left( \int_{Q_0} f^p \, d\sigma \right)^{\frac{q}{p}},
\]

where \( M^p_{\mathcal{DQ}} \) stands for the dyadic Hardy-Littlewood maximal operator with respect to the measure \( d\sigma \) and we have used its boundedness. This completes the proof. \[\square\]

We denote by \( P_i, i = 1, \ldots, d, \) the projection on the \( x_i \)-axis. For the dyadic rectangle \( R \in \mathcal{D}R(\mathbb{R}^d), \) the dyadic interval \( I \in \mathcal{D}Q(\mathbb{R}) \) and \( j = 1, \ldots, d, \) we define the dyadic rectangle

\[
[R; I, j] := \left( \prod_{i=1}^{j-1} P_i(R) \right) \times I \times \left( \prod_{i=j+1}^d P_i(R) \right).
\]

**Lemma 2.2.** Given a weight \( \sigma \) in \( \mathbb{R}^d \) and \( 1 < p < q < \infty, \) the following statements are equivalent:

(a) The Carleson type embedding inequality for rectangles

\[
\sum_{R \in \mathcal{D}R(\mathbb{R}^d)} \sigma(R)^{\frac{q}{p}} \left( \frac{1}{\sigma(R)} \int_R f \, d\sigma \right)^q \leq c_1 \left( \int_{\mathbb{R}^d} f^p \, d\sigma \right)^{\frac{q}{p}}
\]

holds for all nonnegative function \( f \in L^p(\sigma); \)
(b) The testing condition

\[(2.6) \sum_{I \in \mathcal{D}(\mathbb{R})} \sum_{I \subset P_j(R)} \sigma([R; I, j])^{\frac{2}{p}} \leq c_2 \sigma(R)^{\frac{2}{p}} \]

holds for all dyadic rectangles \( R \in \mathcal{D}(\mathbb{R}^d) \) and \( j = 1, \ldots, d \).

Moreover, the least possible constants \( c_1 \) and \( c_2 \) enjoy \( c_1 \leq Cc_2^d \) and \( c_2 \leq c_1 \).

**Proof.** The necessity is clear, so we shall prove the sufficiency. We use induction on the dimension \( d \). To do this, we assume that the lemma is true for the case \( d-1 \).

We assume the weight \( \sigma \) in \( \mathbb{R}^d \) satisfies the testing condition (2.6) (\( d \)-dimensional case). We will write \( x = (x_1, \ldots, x_{d-1}, x_d) = (\bar{x}, x_d) \).

We need two observations. First, we verify that, for any dyadic interval \( I_d \in \mathcal{D}(\mathbb{R}) \), if we let

\[ v_{I_d}(\bar{x}) := \int_{I_d} \sigma(\bar{x}, x_d) \, dx_d, \]

then \( v_{I_d}(\bar{x}) \) satisfies the testing condition (2.6) ((\( d-1 \))-dimensional case). Indeed, for any \( \overline{\mathbb{R}} \in \mathcal{D}(\mathbb{R}^{d-1}) \), setting \( R = \overline{\mathbb{R}} \times I_d \), we have that, for \( j = 1, \ldots, d-1 \),

\[ \sum_{I \in \mathcal{D}(\mathbb{R})} \sum_{I \subset P_j(R)} v_{I_d}([\overline{\mathbb{R}}; I, j])^{\frac{2}{p}} = \sum_{I \in \mathcal{D}(\mathbb{R})} \sum_{I \subset P_j(R)} \sigma([R; I, j])^{\frac{2}{p}} \]

\[ \leq c_2 \sigma(R)^{\frac{2}{p}} = c_2 v_{I_d}(\overline{\mathbb{R}})^{\frac{2}{p}}. \]

We next verify that, for a.e. \( \bar{x} \in \mathbb{R}^{d-1} \), if we let

\[ v_{\bar{x}}(x_d) = \sigma(\bar{x}, x_d), \]

then \( v_{\bar{x}}(x_d) \) satisfies the testing condition (2.2) (one-dimensional case). We must prove that the inequality

\[(2.7) \sum_{I \in \mathcal{D}(\mathbb{R})} v_{\bar{x}}(I)^{\frac{2}{p}} \leq c_2 v_{\bar{x}}(I_d)^{\frac{2}{p}} \]

holds for any \( I_d \in \mathcal{D}(\mathbb{R}) \). For a cube \( \overline{\mathbb{Q}} \in \mathcal{D}(\mathbb{R}^{d-1}) \), it follows by setting \( R = \overline{\mathbb{Q}} \times I_d \) that

\[ \sum_{I \in \mathcal{D}(\mathbb{R})} \sigma([R; I, d])^{\frac{2}{p}} \leq c_2 \sigma(R)^{\frac{2}{p}}. \]

Dividing the both sides by the volume \( |\overline{\mathbb{Q}}|^{\frac{2}{p}} \),

\[ \sum_{I \in \mathcal{D}(\mathbb{R})} \left( \frac{1}{|\overline{\mathbb{Q}}|} \int_{\overline{\mathbb{Q}} \times I} \sigma(\bar{x}, x_d) \, dx_d \, d\bar{x} \right)^{\frac{2}{p}} \leq c_2 \left( \frac{1}{|\overline{\mathbb{Q}}|} \int_{\overline{\mathbb{Q}} \times I_d} \sigma(\bar{x}, x_d) \, dx_d \, d\bar{x} \right)^{\frac{2}{p}}. \]
In the both sides of this inequality, considering the Lebesgue point \( y \) with respect to the integral averages over \( Q \), which exists a.e. in \( \mathbb{R}^{d-1} \) because our argument is countable, and shrinking \( Q \) to \( y \), we obtain
\[
\sum_{I \in \mathcal{D}(\mathbb{R})} \left( \int_I \sigma(y, x_d) \, dx_d \right)^{\frac{2}{p}} \leq c_2 \left( \int_{I_d} \sigma(y, x_d) \, dx_d \right)^{\frac{2}{p}},
\]
which means (2.7).

By the use of these two observations we can prove the lemma.

Fix a nonnegative function \( f \in L^p(\sigma) \). We shall evaluate
\[
(i) = \sum_{I_d \in \mathcal{D}(\mathbb{R})} \sum_{R \in \mathcal{D}(\mathbb{R}^{d-1})} v_{I_d}(R)^{\frac{2}{p}} \times \left( \frac{1}{v_{I_d}(R)} \int_R \left( \int_{I_d} f(x, x_d) \sigma(x, x_d) \, dx_d v_{I_d}(x_d)^{-1} \right) v_{I_d}(x) \, dx \right)^{\frac{q}{p}}.
\]

Since \( v_{I_d}(\mathbb{R}) \) satisfies the testing condition (2.6) \((d - 1)\)-dimensional case), by our induction assumption, we have that
\[
\leq C c_2^{d-1} \sum_{I_d \in \mathcal{D}(\mathbb{R})} \left( \int_{\mathbb{R}^{d-1}} \left( \int_{I_d} f(x, x_d) \sigma(x, x_d) \, dx_d v_{I_d}(x_d)^{-1} \right)^{p} v_{I_d}(x) \, dx \right)^{\frac{2}{p}}.
\]

By integral version of Minkowski’s inequality,
\[
\left\{ \sum_{I_d \in \mathcal{D}(\mathbb{R})} \left( \int_{\mathbb{R}^{d-1}} \left( \int_{I_d} f(x, x_d) \sigma(x, x_d) \, dx_d v_{I_d}(x_d)^{-1} \right)^{p} v_{I_d}(x) \, dx \right)^{\frac{2}{p}} \right\}^{\frac{q}{2}}
\]
\[
\leq \int_{\mathbb{R}^{d-1}} \left\{ \sum_{I_d \in \mathcal{D}(\mathbb{R})} \left( \int_{I_d} f(x, x_d) \sigma(x, x_d) \, dx_d v_{I_d}(x_d)^{-1} \right)^{q} v_{I_d}(x) \right\}^{\frac{2}{q}} \, dx \]
\[
= \int_{\mathbb{R}^{d-1}} \left\{ \sum_{I_d \in \mathcal{D}(\mathbb{R})} v_{I_d}(x_d)^{\frac{2}{q}} \left( \frac{1}{v_{I_d}(x_d)} \int_{I_d} f(x, x_d) v_{I_d}(x_d) \, dx_d \right)^{q} \right\}^{\frac{2}{q}} \, dx.
\]

Since \( v_{I_d}(x_d) \) satisfies (2.2) \((d - 1)\)-dimensional case), by Lemma 2.1
\[
\leq c_2^{\frac{q}{2}} \int_{\mathbb{R}^{d-1}} \int_{\mathbb{R}} f(x, x_d)^{p} \sigma(x, x_d) \, dx_d \, dx
\]
\[
= c_2^{\frac{q}{2}} \int_{\mathbb{R}^d} f^p \, d\sigma.
\]
Altogether, we obtain

\[(i) \leq Cc^2 \left( \int_{\mathbb{R}^d} f^p \, d\sigma \right)^{\frac{q}{p}}.\]

This proves the lemma. \(\square\)

3. **Proof of Theorem 1.1**

In what follows we shall prove Theorem 1.1.

We first notice that, if \(\sigma\) is a reverse doubling weight on \(\mathbb{R}^d\) with \(\beta > 1\), then it satisfies the testing condition (2.6). Indeed, for the dyadic rectangles \(R \in \mathcal{D}_R(\mathbb{R}^d)\) and \(j = 1, \ldots, d\), we have that

\[
\sum_{I \in \mathcal{D}_R(R)} \sigma([R; I, j])^{\frac{1}{p}} = \sum_{k=0}^{\infty} \sum_{I \in \mathcal{D}_R(R) \cap P_j(R), |I| = 2^{-k}|P_j(R)|} \sigma([R; I, j])^{\frac{1}{p}} \leq \sum_{k=0}^{\infty} \left( \frac{1}{\beta^k} \right)^{\frac{1}{p}} \sigma(R)^{\frac{1}{p}} \sum_{I \in \mathcal{D}_R(R) \cap P_j(R), |I| = 2^{-k}|P_j(R)|} \sigma([R; I, j])
\]

\[
= \sigma(R)^{\frac{1}{p}} \sum_{k=0}^{\infty} \left( \frac{1}{\beta^k} \right)^{\frac{1}{p}} \sigma(R)^{\frac{1}{p}}
\]

\[
= C\sigma(R)^{\frac{1}{p}}.
\]

The necessity (1.3) follows at once if we substitute the test functions \(f_i = 1_R, i = 1, \ldots, n\), into inequality (1.2). To show that inequality (1.3) is sufficient, we take \(q_i > p_i, i = 1, \ldots, n\), with \(\sum_{i=1}^{n} \frac{1}{q_i} = 1\). This is possible because \(\sum_{i=1}^{n} \frac{1}{p_i} > 1\). It follows from testing condition (1.3) and Hölder’s inequality that

\[
\sum_{R \in \mathcal{D}_R(\mathbb{R}^d)} K(R) \prod_{i=1}^{n} \left| \int_{R} f_i \, d\sigma_i \right|
\]

\[
\leq c_2 \sum_{R \in \mathcal{D}_R(\mathbb{R}^d)} \prod_{i=1}^{n} \sigma_i(R)^{\frac{1}{p_i}} \left( \frac{1}{\sigma_i(R)} \int_{R} |f_i| \, d\sigma_i \right)
\]

\[
\leq c_2 \prod_{i=1}^{n} \left( \sum_{R \in \mathcal{D}_R(\mathbb{R}^d)} \sigma_i(R)^{\frac{q_i}{p_i}} \left( \frac{1}{\sigma_i(R)} \int_{R} |f_i| \, d\sigma_i \right)^{q_i} \right)^{\frac{1}{q_i}}
\]

\[
\leq Cc_2 \prod_{i=1}^{n} \|f_i\|_{L^{p_i}(\sigma_i)},
\]

where we have used Lemma 2.2 by noticing every \(\sigma_i\) satisfies the testing condition (2.6). This completes the proof.
4. Proof of Corollary 1.2

In what follows we shall prove Corollary 1.2. The necessity (1.5) follows at once if we substitute the test functions $f_i = 1_{R \sigma_i}, i = 1, \ldots, n$, into inequality (1.4). To show that inequality (1.5) is sufficient, we notice that the condition

$$\sum_{i=1}^{n} \frac{1}{p_i} > \frac{1}{q}$$

leads the condition

$$\frac{1}{q'} + \sum_{i=1}^{n} \frac{1}{p_i} > 1.$$  

By Theorem 1.1 we have that the inequality

$$\sum_{R \in DR(\mathbb{R}^d)} K(R) \int_R g \, d\omega \prod_{i=1}^{n} \int_R f_i \, d\sigma_i \leq C \|g\|_{L^{q'}(\omega)} \prod_{i=1}^{n} \|f_i\|_{L^{p_i}(\sigma_i)}$$

holds for all nonnegative functions $g \in L^{q'}(\omega)$ and $f_i \in L^{p_i}(\sigma_i)$, provided that the testing condition

$$K(R) \omega(R) \prod_{i=1}^{n} \sigma_i(R) \leq C \omega(R)^{\frac{1}{q'}} \prod_{i=1}^{n} \sigma_i(R)^{\frac{1}{p_i}}$$

holds for all dyadic rectangles $R \in DR(\mathbb{R}^d)$. Since (4.2) is equivalent to our assumption (1.5), the inequality (4.1) is proper. Rewrite $f_i \sigma_i = h_i$ in (4.1), then

$$\sum_{R \in DR(\mathbb{R}^d)} K(R) \int_R g \, d\omega \prod_{i=1}^{n} \int_R h_i \, dx \leq C c_2 \|g\|_{L^{q'}(\omega)} \prod_{i=1}^{n} \|h_i\|_{L^{p_i}(\sigma_i^{1-p_i})}.$$  

This means that

$$\int_{\mathbb{R}^d} g T_K(h_1, \ldots, h_n) \, d\omega \leq C c_2 \|g\|_{L^{q'}(\omega)} \prod_{i=1}^{n} \|h_i\|_{L^{p_i}(\sigma_i^{1-p_i})}$$

and, by duality,

$$\|T_K(h_1, \ldots, h_n)\|_{L^q(\omega)} \leq C c_2 \prod_{i=1}^{n} \|h_i\|_{L^{p_i}(\sigma_i^{1-p_i})},$$

which yields the proof.

5. Remarks

In what follows we give some remarks for strong fractional integral operator.

For a number $c > 0$ and a rectangle $R \in \mathcal{R}$, we will use $cR$ to denote the rectangle with the same center as $R$ but with $c$ times the side-lengths of $R$. Let $f_i, i = 1, \ldots, n$, be locally integrable functions on $R^d$. The
multilinear strong fractional integral operator $I_\alpha(f_1, \ldots, f_n)(x)$, $0 < \alpha < dn$ and $x \in \mathbb{R}^d$, is given by
\[
I_\alpha(f_1, \ldots, f_n)(x) := \int_{y_1, \ldots, y_n \in \mathbb{R}^d} \frac{f_1(y_1) \cdots f_n(y_n) \, dy_1 \cdots dy_n}{\left(\prod_{j=1}^{d} \max_{i=1}^{n} |P_j(x) - P_j(y_i)|\right)^{n-\alpha}},
\]
where $P_j(x)$, $j = 1, \ldots, d$, is the projection on the $x_j$-axis of the point $x \in \mathbb{R}^d$.

We observe that, for $s, t \in \mathbb{R}$ with $s \neq t$, the minimal dyadic interval $I \in \mathcal{DQ}(\mathbb{R})$ such that $I \ni s$ and $3I \ni t$ satisfies
\[
\frac{|I|}{2} < |s - t| < 2|I|.
\]
This observation and a calculus of geometric series enable us that, for any $y_1, \ldots, y_n \neq x$,
\[
\sum_{R \in \mathcal{D}(\mathbb{R}^d)} |R|^{\frac{\alpha}{d} - n} 1_R(x) \prod_{i=1}^{n} 1_{3R}(y_i) \approx \left(\prod_{j=1}^{d} \max_{i=1}^{n} |P_j(x) - P_j(y_i)|\right)^{-\frac{\alpha}{d} - n}.
\]
This equation and Fubini’s theorem yield the precise point-wise relation
\[
(5.1) \quad I_\alpha(f_1, \ldots, f_n)(x) \approx \sum_{R \in \mathcal{D}(\mathbb{R}^d)} |R|^{\frac{\alpha}{d} - n} 1_R(x) \prod_{i=1}^{n} \int_{3R} f_i(y_i) \, dy_i, \quad x \in \mathbb{R}^d.
\]

Since the right-hand of (5.1) can be controlled by the estimate based upon the finite number of the systems of dyadic rectangles (see, for example, [6]), by Corollary 1.2, we have the following.

**Proposition 5.1.** Let $1 < p_i < \infty$, $1 < q < \infty$ and $\sum_{i=1}^{n} \frac{1}{p_i} > \frac{1}{q}$. Let $0 < \alpha < dn$ and let $\sigma_i$, $i = 1, \ldots, n$, and $\omega$ be reverse doubling weights on $\mathbb{R}^d$. The following statements are equivalent:

(a) The weighted norm inequality for multilinear strong fractional integral operator
\[
\|I_\alpha(f_1, \ldots, f_n)\|_{L^q(\omega)} \leq c_1 \prod_{i=1}^{n} \|f_i\|_{L^{p_i}(\sigma_i^{1-p_i})}
\]
holds for all $f_i \in L^{p_i}(\sigma_i^{1-p_i})$, $i = 1, \ldots, n$;

(b) The testing condition
\[
|R|^{\frac{\alpha}{d} - n} \omega(R)^{\frac{1}{d}} \prod_{i=1}^{n} \sigma_i(R) \leq c_2 \prod_{i=1}^{n} \sigma_i(R)^{\frac{1}{p_i}}
\]
holds for all rectangles $R \in \mathcal{R}(\mathbb{R}^d)$.

Moreover, the least possible constants $c_1$ and $c_2$ are equivalent.

Letting $\omega \equiv \sigma_1 \equiv \cdots \equiv \sigma_n \equiv 1$, we have the following Hardy-Littlewood-Sobolev inequality for strong fractional integral operator.
Proposition 5.2. Let $1 < q < \infty$, $1 < p_i < \infty$, $0 < \alpha < dn$ and
\[
\frac{1}{q} = \sum_{i=1}^{n} \frac{1}{p_i} - \frac{\alpha}{d}.
\]
Then the multilinear norm inequality
\[
\|I_\alpha(f_1, \ldots, f_n)\|_{L^q(\mathbb{R}^d)} \leq C \prod_{i=1}^{n} \|f_i\|_{L^{p_i}(\mathbb{R}^d)}
\]
holds for all $f_i \in L^{p_i}(\mathbb{R}^d)$, $i = 1, \ldots, n$.

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