Enumerating Acyclic Digraphs by Descents

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Abstract

A descent of a labeled acyclic digraph is a directed edge \( x \rightarrow y \) with \( x > y \). In this paper, we find a recurrence for the number of labeled acyclic digraphs with a given number of descents.

Keywords: acyclic digraphs; descents; recurrence

1 Introduction

1.1 Background

An acyclic digraph is a directed graph that has no cycles. It is well-known that the number of acyclic digraphs on \( n \) vertices with labels in \( \{1, 2, \ldots, n\} \) is given by the following recurrence:

\[
a_n = \sum_{k=1}^{n} (-1)^{k+1} \binom{n}{k} 2^{k(n-k)} a_{n-k}.
\]

The first 8 numbers in the sequence can be found in the last row of the table in Figure 1. This result is attributed to Robinson [4, 5] and was recovered in the same year by Stanley [7] who found the following equivalent enumeration for the number of acyclic digraphs:

\[
\sum_{n=0}^{\infty} \frac{a_n}{n!2^{\binom{n}{2}}} x^n = \left( \sum_{n=0}^{\infty} \frac{(-1)^n}{n!2^{\binom{n}{2}}} x^n \right)^{-1}.
\]

The enumeration of these graphs has been refined with respect to many statistics. These include the number of edges [6], the number of sources [3], the number of initially connected components [1], and the joint distributions of edges, sources, and sinks [2]. In this paper, we define descents for acyclic digraphs and enumerate these graphs with respect to this newly-defined statistic.

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1.2 Definitions and notation

Given a labeled acyclic digraph with vertex set \{1, 2, \ldots, n\}, we let \(x \to y\) denote the directed edge from \(x\) to \(y\). If \(x \to y\) is an edge and \(x > y\), we call this edge a descent while if \(x < y\), the edge is an ascent or an increasing edge. For a vertex \(x\) in an acyclic digraph, if there are no edges of the form \(y \to x\) for all vertices \(y\), then we say vertex \(x\) is a source. If there is a directed path from \(x\) to \(y\), we say the vertex \(y\) is reachable from \(x\). Every vertex is considered to be reachable from itself.

We also make use of some standard notation as follows. The set \(\{1, 2, \ldots n\}\) is denoted by \([n]\). Also, we let \(\mathbb{Z}_+\) denote the set of non-negative integers. The Gaussian binomial coefficients are used throughout this paper as well. For \(n, j, i \in \mathbb{Z}_+\), the necessary formulas and notation are:

- \([n]_q! = \frac{1-q^n}{1-q}\),
- \(\binom{n}{j}_q = \begin{cases} [n]_q! \frac{[n-j]_q!}{[j]_q [n-j]_q!} & \text{if } j \leq n \\ 0 & \text{otherwise} \end{cases}\) and
- \(Q_{n,j,i}\) is the coefficient of \(q^i\) in \(\binom{n}{j}_q\).

It is well-known that \(Q_{n,j,i}\) can also be interpreted as the number of partitions of \(i\) into \(n-j\) parts each less than or equal to \(j\). (See for example, [8].)

2 Main Result

Our main result gives a recursive formula for the number of acyclic digraphs on \(n\) vertices with exactly \(k\) descents. For the remainder of the paper, let \(D_{n,k}\) denote the set of acyclic digraphs on \(n\) vertices with \(k\) descents. In order to state our main result, we make use of the following definition.

**Definition 1.** Assume \(n \geq 1, k \geq 0,\) and \(m \geq 2\) are integers.

- Let \(a_{n,k,m}\) denote the number of graphs in \(D_{n,k}\) where one of the descents is \(m \to 1\).
- Let \(b_{n,k,m}\) denote the number of graphs in \(D_{n,k}\) where \(m\) is reachable from 1.
- Let \(c_{n,k,m}\) denote the number of graphs in \(D_{n,k}\) where exactly \(m\) of the descents point to 1.
- Let \(d_{n,k}\) denote the number of graphs in \(D_{n,k}\).

Our main result is the value of \(d_{n,k}\) stated here in terms of \(a_{n,k,m}, b_{n,k,m}\) and \(c_{n,k,m}\), which are addressed in Lemmas 6, 7, and 8, respectively. The values of \(d_{n,k}\) for \(n \leq 8\) can be found in Figure 1.
| $k$ | $n$ | 1  | 2  | 3  | 4  | 5  | 6  | 7  | 8   |
|-----|-----|----|----|----|----|----|----|----|-----|
| 0   | 1   | 1  | 2  | 8  | 64 | 1,024 | 32,768 | 2,097,152 | 268,435,456 |
| 1   | 0   | 1  | 11 | 161 | 3,927 | 172,665 | 14,208,231 | 2,234,357,849 |
| 2   | 0   | 0   | 5  | 167 | 6,698 | 419,364 | 45,263,175 | 8,854,386,165 |
| 3   | 0   | 0   | 1  | 102 | 7,185 | 656,733 | 94,040,848 | 23,016,738,169 |
| 4   | 0   | 0   | 0  | 39  | 5,477 | 757,939 | 145,990,526 | 44,953,824,619 |
| 5   | 0   | 0   | 0  | 1   | 1,329 | 504,084 | 187,742,937 | 94,103,501,133 |
| 6   | 0   | 0   | 0  | 1   | 423  | 305,207 | 165,596,535 | 108,068,923,630 |
| 7   | 0   | 0   | 0  | 0   | 96   | 153,333 | 126,344,492 | 109,265,863,921 |
| 8   | 0   | 0   | 0  | 0   | 14   | 63,789  | 84,115,442 | 98,446,816,132 |
| 9   | 0   | 0   | 0  | 0   | 1    | 21,752  | 49,085,984 | 79,697,456,418 |
| 10  | 0   | 0   | 0  | 0   | 0    | 5,477   | 25,134,230 | 58,293,422,939 |
| 11  | 0   | 0   | 0  | 0   | 0    | 1,267   | 3,927,813 | 38,657,195,560 |
| 12  | 0   | 0   | 0  | 0   | 0    | 197     | 4,403,313 | 23,283,565,343 |
| 13  | 0   | 0   | 0  | 0   | 0    | 20     | 1,486,423 | 12,741,518,134 |
| 14  | 0   | 0   | 0  | 0   | 0    | 1      | 428,139  | 6,328,700,820 |
| 15  | 0   | 0   | 0  | 0   | 0    | 103,345 | 2,846,683,820 |
| 16  | 0   | 0   | 0  | 0   | 0    | 20     | 1,155,387,912 |
| 17  | 0   | 0   | 0  | 0   | 0    | 3,153   | 421,001,237 |
| 18  | 0   | 0   | 0  | 0   | 0    | 1      | 136,799,627 |
| 19  | 0   | 0   | 0  | 0   | 0    | 0      | 428,139  | 6,328,700,820 |
| 20  | 0   | 0   | 0  | 0   | 0    | 0      | 27      | 39,294,726 |
| 21  | 0   | 0   | 0  | 0   | 0    | 0      | 1       | 9,865,371 |
| 22  | 0   | 0   | 0  | 0   | 0    | 0      | 0       | 2,133,019 |
| 23  | 0   | 0   | 0  | 0   | 0    | 0      | 0       | 389,396 |
| 24  | 0   | 0   | 0  | 0   | 0    | 0      | 0       | 58,400 |
| 25  | 0   | 0   | 0  | 0   | 0    | 0      | 0       | 6,913 |
| 26  | 0   | 0   | 0  | 0   | 0    | 0      | 0       | 606 |
| 27  | 0   | 0   | 0  | 0   | 0    | 0      | 0       | 35 |
| 28  | 0   | 0   | 0  | 0   | 0    | 0      | 0       | 1 |

TOTAL  | 1 | 3 | 25 | 543 | 29,281 | 3,781,503 | 1,138,779,265 | 783,702,329,343 |

Figure 1: Values of $d_{n,k}$, the number of acyclic digraphs on $n$ vertices with $k$ descents, for $n \leq 8$. The total is the number of labeled acyclic digraphs on $n$ vertices.
Theorem 2. The number of acyclic digraphs on \( n \) vertices with \( k \) descents, denoted \( d_{n,k} \), is given by the recurrence

\[
d_{n,k} = 2^{n-1}d_{n-1,k} + (n - 1)d_{n,k-1} - \sum_{m=2}^{n} (a_{n,k-1,m} + b_{n,k-1,m}) - \sum_{m=2}^{k} (m - 1)c_{n,k,m}
\]

with initial conditions

\[
d_{n,0} = 2\binom{n}{2} \quad \text{and} \quad d_{0,k} = \begin{cases} 1 & \text{for } k = 0 \\ 0 & \text{for } k \geq 1. \end{cases}
\]

Proof. Consider the base case where \( k = 0 \). The number of acyclic digraphs with 0 descents is found by including any increasing edge. There are a total of \( \binom{n}{2} \) increasing edges, so the result holds. For the remainder of the proof, assume \( k \geq 1 \).

We first note that any acyclic digraph with \( k \) descents either has a descent of the form \( x \to 1 \) or it does not. If it does not have a descent of the form \( x \to 1 \), then vertex 1 is a source. The number of acyclic digraphs with \( k \) descents where 1 is a source is counted recursively by taking any acyclic digraph with \( k \) descents on \( n - 1 \) vertices labeled \( \{2, 3, \ldots, n\} \) and adding the vertex labeled 1. Since 1 is smaller than the labels of all the other vertices, we can add any increasing edge of the form \( 1 \to y \) for any \( y \in \{2, 3, \ldots, n\} \) without creating a descent. Thus, there are a total of

\[
2^{n-1}d_{n-1,k}
\]

acyclic digraphs on \( n \) vertices with \( k \) descents where the vertex labeled 1 is a source.

The remainder of the proof counts the number of acyclic digraphs on \( n \) vertices with \( k \) descents where 1 is not a source. Consider the set \( \mathcal{D}_{n,k-1} \) of acyclic digraphs on \( n \) vertices with only \( k - 1 \) descents. For each graph in \( \mathcal{D}_{n,k-1} \) and for each \( m \) between 2 and \( n \), we want to add the descent \( m \to 1 \) to create a acyclic digraph on \( n \) vertices with \( k \) descents. However, this new graph with the added descent is only in \( \mathcal{D}_{n,k} \) if:

- the graph did not already have the descent \( m \to 1 \) and
- the graph did not have a path from 1 to \( m \).

The second condition is necessary to ensure that our new graph remains acyclic. Thus, we can add the descent \( m \to 1 \) to a total of

\[
d_{n,k-1} - a_{n,k-1,m} - b_{n,k-1,m}
\]

graphs. Summing over all possible \( m \) between 2 and \( n \) yields a total of

\[
(n - 1)d_{n,k-1} - \sum_{m=2}^{n} (a_{n,k-1,m} + b_{n,k-1,m})
\]

graphs formed in this manner.

However, counting the desired graphs in such a way counts some graphs more than once, in particular those with more than one descent of the form \( x \to 1 \). In fact, for any \( \ell \) between 2 and \( k \), graphs with exactly \( \ell \) descents pointing at 1 are counted exactly \( \ell \) times. Subtracting the number of graphs that were counted multiple times yields the desired result. \( \square \)
3 Enumeration Lemmas

The remainder of this paper is then devoted to finding formulas for

\[\sum_{m=2}^{n} a_{n,k,m}, \sum_{m=2}^{n} b_{n,k,m}, \text{ and } \sum_{m=2}^{k} (m-1)c_{n,k,m}.\]

To this end, we define two more values.

**Definition 3.** Assume \( n \geq 1 \) and \( k \geq 0 \) are integers.

- Let \( t_{n,k} \) denote the number of graphs in \( D_{n,k} \) where every vertex is reachable from 1.
- Let \( u_{n,k} \) denote the number of graphs in \( D_{n,k} \) where every vertex is reachable from \( n \).

In order to find formulas for \( t_{n,k} \) and \( u_{n,k} \) we state a brief lemma which will be used later.

**Lemma 4.** There are \( Q_{n,j,i} \) ways to partition \([n]\) into two sets \( X \) and \( Y \) where \(|X| = j\) and with \( i \) pairs \((x,y) \in X \times Y \) with \( x < y \).

**Proof.** Consider a partition of \([n]\) into two sets \( X \) and \( Y \) where \(|X| = j\) and with \( i \) pairs \((x,y) \in X \times Y \) with \( x < y \). Write \( Y = \{y_1, y_2, \ldots, y_{n-j}\} \) where \( y_r < y_{r+1} \) for \( r \in [n-j-1] \).

Notice that \( 0 \leq y_r - r \leq j \) for all \( r \in [n-j] \). Also, the number of pairs \((x,y) \in X \times Y \) with \( y < x \) is

\[i = (y_1 - 1) + (y_2 - 2) + \cdots + (y_{n-j} - n + j).\]

Because each \( y_r - r \) is between 0 and \( j \), this directly corresponds to a partition of \( i \) into \( n-j \) parts each less than or equal to \( j \).

The formulas for \( t_{n,k} \) and \( u_{n,k} \) are described in the following lemma.

**Lemma 5.** Let \( t_{n,k} \) and \( u_{n,k} \) be as defined in Definition 3. Then \( t_{n,k} \) and \( u_{n,k} \) satisfy the following recurrences:

\[
t_{n,k} = \sum_{(j,i,r,s) \in \Omega_t} u_{j,s} t_{n-j,r} \left(c_{i,k-s-r}\right) 2^{(j-1)(n-j)-i} (2^{n-j} - 1) Q_{n-2,j-1,i}
\]

and

\[
u_{n,k} = \sum_{(j,i,r,s) \in \Omega_u} t_{j,s} u_{n-j,r} \left(\binom{i}{k-r-s} - \binom{i-n+j}{k-s-r}\right) 2^{j(n-j)-i} Q_{n-2,j-1,i-n+1}
\]

where

\[
\Omega_t = \{(j,i,r,s) \in \mathbb{Z}_+^4 | 1 \leq j \leq n-1, \ i \leq (j-1)(n-j-1), \ r+s \leq k\}
\]

5
\[ \Omega_u = \{(j, i, r, s) \in \mathbb{Z}_4^4 | 1 \leq j \leq n - 1, \ n - 1 \leq i \leq j(n - j), \ r + s \leq k - 1\}, \]

with initial conditions

\[
t_{n,0} = \begin{cases} 1 & \text{for } n = 0 \\ [n - 1]! & \text{for } n \geq 1 \end{cases}, \quad t_{1,k} = \begin{cases} 1 & \text{for } k = 0 \\ 0 & \text{for } k \geq 1 \end{cases},
\]

and

\[
u_{n,0} = \begin{cases} 1 & \text{for } n = 1 \\ 0 & \text{for } n \neq 1. \end{cases}
\]

Proof. We first consider the formula for \( t_{n,k} \). For the base case where \( k = 0 \), we need to find the number of graphs on \( n \) vertices with 0 descents where every vertex is reachable from 1. The only way the vertex labeled 2 is reachable from 1 is if the edge 1 \( \rightarrow \) 2 is included. Let \( 2 \leq m \leq n \) be another vertex. If vertices \( \{1, 2, \ldots m - 1\} \) are reachable from 1, then vertex \( m \) is reachable from 1 if at least one edge of the form \( m' \rightarrow m \) is included for some \( 1 \leq m' \leq m - 1 \); thus, there are \( 2^{m-1} - 1 \) possible edges that can point to \( m \). Multiplying over all \( m \) between 2 and \( n \) yields a total of

\[ (2^1 - 1)(2^2 - 1) \cdots (2^{n-1} - 1) = [n - 1]! \]

such graphs. For the base case where \( n = 1 \), it is clear that there is only one acyclic digraph and it has 0 descents.

For the remainder of the proof of the formula for \( t_{n,k} \), assume that \( n, k \geq 1 \). The set of all graphs in \( \mathcal{D}_{n,k} \) where every vertex is reachable from 1 can be partitioned based on the number of vertices, \( j \), that are reachable from \( n \). For any such graph, let \( X \) be the set of vertices that are reachable from \( n \) and \( Y = [n] \setminus X \). Thus we want to count how many graphs there are in \( \mathcal{D}_{n,k} \) where every vertex is reachable from 1 that also satisfy the following conditions:

- \( |X| = j \),
- the number of pairs of vertices \((x, y) \in X \times Y \) with \( x < y \) is \( i \),
- the number of descents in the subgraph induced by \( X \) is \( s \), and
- the number of descents in the subgraph induced by \( Y \) is \( r \),

where the values of \( j, i, r, \) and \( s \) satisfy certain conditions. We first notice that \( n \in X \) since \( n \) is reachable from itself. Also, since \( n \) is reachable from 1 and the desired graphs are acyclic, 1 cannot be reachable from \( n \) and thus \( 1 \in Y \). It follows that \( 1 \leq j \leq n - 1 \). Furthermore, the number of pairs of vertices \((x, y) \in X \times Y \) with \( x < y \) is at most \((j - 1)(n - j - 1)\) which occurs when every vertex in \( X \setminus \{n\} \) is smaller than every vertex in \( Y \setminus \{1\} \). Finally, we note that \( 0 \leq r + s \leq k \). Thus, \((j, i, r, s) \in \Omega_t \) as defined in the statement of Lemma 5.
Since $n \in X$ and $1 \in Y$, we now consider the remaining vertices. The number of ways to partition the remaining $n-2$ vertices into two sets $X$ and $Y$ with $|X| = j$ and with $i$ pairs of vertices $(x, y) \in X \times Y$ with $x < y$ is $Q_{n-2,j-1,i}$ by Lemma 4. There are $u_{j,s}$ choices for the subgraph induced by $X$ and $t_{n-j,r}$ choices for the subgraph induced by $Y$. Also, because the graphs must have a total of $k$ descents, the remaining $k-s-r$ descents can be chosen from the $i$ pairs of vertices $(x, y) \in X \times Y$ where $x < y$. Since every vertex is reachable from 1, vertex $n$ must be reachable from 1; thus there must be an edge from some vertex in $Y$ to $n$. There are $n-j$ possible increasing edges from $Y$ to $n$, and at least one must be included yielding a total of $2^{n-j}-1$ possibilities. Finally, there are $(n-j)(j-1)-i$ possible increasing edges from $Y \setminus \{1\}$ to $X \setminus \{n\}$. Because these edges can all be included or not, we multiply our total by $2^{(n-j)(j-1)-i}$. Note that we cannot include any edges from $X$ to $Y$ since all vertices reachable from the vertex labeled $n$ are already in $X$.

We use a similar technique to find a formula for $u_{n,k}$. The set of graphs in $D_{n,k}$ where every vertex is reachable from $n$ can be partitioned based on the number of vertices, $j$, that are reachable from 1. For any such graph, let $X$ be the set of vertices that are reachable from 1 and $Y = [n] \setminus X$. Again, we want to count the number of graphs that satisfy the aforementioned conditions along with the following:

- $|X| = j$
- the number of pairs of vertices $(x, y) \in X \times Y$ with $x < y$ is $i$,
- the number of descents in the subgraph induced by $X$ is $s$, and
- the number of descents in the subgraph induced by $Y$ is $r$,

where $j, i, r,$ and $s$ satisfy certain conditions. Because every vertex is reachable from $n$, and the vertex labeled 1 is reachable from itself, we have that $1 \in X$ and $n \in Y$ and hence $1 \leq j \leq n-1$. The number of pairs of vertices $(x, y) \in X \times Y$ with $x < y$ is at least $n-1$ since there are $j$ such pairs of the form $(x, n)$ and $n-j$ pairs of the form $(1, y)$. The pair $(n, 1)$ is counted twice in this argument and hence $i \geq n-1$. The largest number of pairs occurs when every element in $Y$ is greater than every element in $X$ and thus $i \leq j(n-j)$. Finally, it is clear that $r+s \leq k-1$, since there must be at least one descent of the form $y \to 1$ where $y \in Y$ and $1 \in X$. Thus, we see that $(j,i,r,s) \in \Omega_u$ as defined in Lemma 5.

Consider the number of ways to partition the remaining $n-2$ vertices into $X$ and $Y$ meeting the desired conditions. We know that the number of pairs of vertices $(x, y) \in X \times Y$ with $x < y$ is $i$, but $j$ of these pairs are of the form $(x, n)$ and $n-j$ pairs are of the form $(1, y)$. So there are $i-\binom{n-1}{2}$ pairs of vertices in $(x, y) \in (X \setminus \{1\}) \times (Y \setminus \{n\})$ with $x < y$. Thus, the number of ways to partition the remaining $n-2$ vertices into sets $X$ and $Y$ meeting the desired conditions is $Q_{n-2,j-1,i-\binom{n-1}{2}}$.

The remainder of the terms in our recursive formula for $u_{n,k}$ can be seen in a very similar manner to that of $t_{n,k}$. There are $t_{j,s}$ and $u_{n-j,r}$ choices for the subgraphs induced by $X$ and $Y$ respectively. In order to get a total of $k$ descents, the remaining $k-r-s$ descents can be chosen from the $i$ pairs of vertices $(x, y) \in X \times Y$ with $x < y$. However, because
every vertex is reachable from $n$, at least one of those $k - r - s$ descents must be of the form $y \to 1$. There are $n - j$ pairs of the form $(1, y)$, and hence there are $i - n + j$ pairs that do not contain 1. Thus, the term
\[
\binom{i}{k - r - s} - \binom{i - n + j}{k - s - r}
\]
counts the number of ways the $k - r - s$ descents can be chosen from the $i$ pairs of vertices while still including at least one descent pointing at 1. Finally, there are $j(n - j)$ possible increasing edges from $Y$ to $X$ which can all be included or not which multiplies our total by $2^{j(n - j) - i}$.

\[\square\]

We are now ready to state the formulas needed for our main result, namely
\[
\sum_{m=2}^{n} a_{n,k,m}, \sum_{m=2}^{n} b_{n,k,m}, \text{ and } \sum_{m=2}^{k} (m - 1)c_{n,k,m},
\]
which are found in the lemmas below.

**Lemma 6.** For $2 \leq m \leq n$, let $a_{n,k,m}$ denote the number of graphs in $\mathcal{D}_{n,k}$ where one of the descents is $m \to 1$. Then
\[
\sum_{m=2}^{k} a_{n,k,m} = \sum_{(j,i,r,s,\ell) \in \Omega_a} \ell \cdot d_{n-j,r,s,\ell}(n - j)\binom{i}{k - s - r - \ell}2^{(j-1)(n-j)-i}Q_{n-1,j-1,i}
\]
where
\[
\Omega_a = \{(j,i,r,s,\ell) \in \mathbb{Z}_+^4 \mid 1 \leq j \leq n - 1, \ i \leq (j - 1)(n - j), \ r + s \leq k - 1, \ 1 \leq \ell \leq k - r - s\}
\]

**Proof.** We begin by partitioning the set of all graphs in $\mathcal{D}_{n,k}$ that have at least one descent of the form $y \to 1$ by the number of vertices, $j$, that are reachable from 1. For any such graph, let $X$ be the set of vertices that are reachable from 1 and let $Y = [n] \setminus X$. We will proceed by counting the number of graphs satisfying the stated conditions along with the following:

- $|X| = j$,
- the number of pairs of vertices $(x, y) \in (X \setminus \{1\}) \times Y$ where $x < y$ is $i$,
- the number of descents in the subgraph induced by $X$ is $s$,
- the number of descents in the subgraph induced by $Y$ is $r$, and
- the number of descents pointing at 1 is $\ell$, 

\[\square\]
where \((j, i, r, s, \ell)\) satisfy certain conditions. Notice that 1 is reachable from itself and thus \(1 \in X\). Also, \(Y\) cannot be empty because there must be at least one descent pointing at 1. Thus, \(1 \leq j \leq n - 1, \ r + s \leq k - 1\), and \(1 \leq \ell \leq k - r - s\). Also, the maximum number of pairs of vertices \((x, y) \in (X \setminus \{1\}) \times Y\) where \(x < y\) is \((n - j)(j - 1)\) which occurs when every element in \(Y\) is greater than every element of \(X\). Hence, \((j, i, r, s, \ell) \in \Omega_a\) as defined in the statement of the Lemma.

The number of ways to partition the remaining \(n - 1\) vertices into sets \(X\) and \(Y\) meeting the desired conditions is \(Q_{n - 1, j - 1, i}\) by Lemma 4. There are \(d_{n - j, r}\) and \(t_{j, s}\) choices for the subgraphs induced by \(Y\) and \(X\), respectively, and there are \(\binom{n - j}{\ell}\) ways to choose the \(\ell\) descents pointing at 1. The remaining \(k - s - r - \ell\) descents are chosen from the \(i\) pairs of vertices \((x, y) \in (X \setminus \{1\}) \times Y\) where \(x < y\). Finally, there are \((j - 1)(n - j) - i\) pairs of vertices \((x, y) \in X \times Y\) where \(x > y\); any of these increasing edges can be included. Thus, there are

\[
\sum_{(j, i, r, s, \ell) \in \Omega_a} d_{n - j, r} t_{j, s} \binom{n - j}{\ell} \binom{i}{k - s - r - \ell} 2^{(j - 1)(n - j) - i} Q_{n - 1, j - 1, i}
\]

graphs in \(D_{n, k}\) that that have at least one descent pointing at 1. However, for each \(1 \leq \ell \leq k - r - s\), we have contributed to the sum \(\sum_{m=2}^{n} a_{n, k, m}\) exactly \(\ell\) times. Thus the equality stated in Lemma 6 holds.

\[\square\]

**Lemma 7.** For \(2 \leq m \leq n\), let \(b_{n, k, m}\) denote the number of graphs in \(D_{n, k}\) where \(m\) is reachable from 1. Then,

\[
\sum_{m=2}^{n} b_{n, k, m} = \sum_{(j, i, r, s) \in \Omega_b} (j - 1)d_{n - j, r} t_{j, s} \binom{i}{k - s - r - s} 2^{j(n - j) - i} Q_{n - 1, j - 1, i - n + j}
\]

where

\[
\Omega_b = \{(j, i, r, s) \in \mathbb{Z}_+^4 | 2 \leq j \leq n, \ n - j \leq i \leq j(n - j), \ r + s \leq k\}
\]

**Proof.** We partition the graphs in \(D_{n, k}\) by the number of vertices, \(j\) that are reachable from 1 where \(2 \leq j \leq n\). If we can count the graphs in \(D_{n, k}\) where \(j\) vertices are reachable from 1, then multiplying by \((j - 1)\) gives the number of graphs in \(D_{n, k}\) where \(m\) is reachable from 1 and and there are exactly \(j\) vertices reachable from 1 for all \(2 \leq m \leq n\). Summing over all \(j\) will then give the desired result.

Toward this end, we again let \(X\) be the set of vertices reachable from 1 and let \(Y = [n] \setminus X\). We count the number of graphs with the conditions that:

- \(|X| = j\),
- the number of pairs of vertices \((x, y) \in X \times Y\) where \(x < y\) is \(i\),
- the number of descents in the subgraph induced by \(X\) is \(s\), and
• the number of descents in the subgraph induced by $Y$ is $r$,

where $j, i, r$ and $s$ satisfy certain conditions. In particular, we need $2 \leq j \leq n$ and $0 \leq r + s \leq k$. Also, because all $n - j$ elements in $Y$ are greater than 1, we see that $i \geq n - j$, and the maximum value of $i$ occurs when every element in $X$ is greater than every element in $Y$ which gives $i \leq j(n - j)$. Hence, $(j, i, r, s) \in \Omega_b$ as defined in the statement of Lemma 7.

Consider the number of ways to partition the set of vertices $[n] \setminus \{1\}$ into $X$ and $Y$ meeting the desired conditions. We know that the number of pairs of vertices $(x, y) \in X \times Y$ with $x < y$ is $i$, but $n - j$ pairs are of the form $(1, y)$. So there are $i - (n - 1)$ pairs of vertices in $(x, y) \in (X \setminus \{1\}) \times Y$ with $x < y$. Thus, the number of ways to partition the remaining $n - 1$ vertices into sets $X$ and $Y$ meeting the desired conditions is $Q_{n-1,j-1,i-n+j}$ by Lemma 4. It is clear that the number of choices for the subgraph induced by $X$ is $t_{j,s}$, and that the number of choices for the subgraph induced by $Y$ is $d_{n-j,r}$. The remaining $k$ descents can be chosen from the $i$ pairs of vertices $(x, y) \in X \times Y$ where $x < y$, and the increasing edges from $Y$ to $X$, of which there are $j(n - j) - i$, can be included or not. The result follows.

**Lemma 8.** For $2 \leq m \leq n$, let $c_{n,k,m}$ denote the number of graphs in $D_{n,k}$ where exactly $m$ of the descents are of the form $x \to 1$. Then

$$\sum_{m=2}^{k} (m-1)c_{n,k,m}$$

is equivalent to

$$\sum_{(m,j,i,r,s) \in \Omega_c} (m-1)d_{n-j,r}t_{j,s} \binom{n-j}{m} \binom{i}{k-r-s-m} 2^{j-1(n-j)-i} Q_{n-1,j-1,i}$$

where

$$\Omega_c = \{(m,j,i,r,s) \in \mathbb{Z}_+^5 | 2 \leq m \leq k, \ 1 \leq j \leq n-m, \ i \leq (j-1)(n-j), \ r + s \leq k - m\}.$$  

**Proof.** For a fixed $2 \leq m \leq k$, partition all graphs in $D_{n,k}$ that have exactly $m$ descents pointing at 1 by the number of vertices, $j$, that are reachable from 1. In this case we have $1 \leq j \leq n - m$ as there must be at least $m$ vertices that are not reachable from 1. Let $X$ be the set of vertices reachable from 1 and let $Y = [n] \setminus X$. We count the number of graphs satisfying the following conditions:

• $|X| = j$,

• the number of pairs of vertices $(x, y) \in (X \setminus \{1\}) \times Y$ where $x < y$ is $i$,

• the number of descents in the subgraph induced by $X$ is $s$, and

• the number of descents in the subgraph induced by $Y$ is $r$. 

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where the values \( j, i, s, \) and \( r \) satisfy certain conditions. Since 1 is reachable from itself, clearly \( j \geq 1 \). Since there are \( m \) descents pointing at 1, these \( m \) elements are not reachable from 1, and thus must be elements of \( Y \). Thus, \( j \leq n - m \). As before, there are at most \((j - 1)(n - j)\) possible edges which could be descents of the form \( y \to x \) where \( y \in Y \) and \( x \in (X \setminus 1) \). Finally, since there are \( m \) descents of the form \( y \to 1 \) for some \( y \in Y \), there are at most \( k - m \) descents which occur in the subgraphs induced by \( X \) and by \( Y \), thus \( r + s \leq k - m \). We conclude that \((m, j, i, r, s) \in \Omega_c \).

By Lemma 4, there are \( Q_{n-1,j-1,i} \) ways to partition the vertices \([n] \setminus \{1\}\) with these given conditions. Also, there are \( t_{j,s} \) and \( d_{n-j,r} \) choices for the subgraphs induced by \( X \) and \( Y \) respectively. Of the \( n - j \) vertices in \( Y \), exactly \( m \) of them must point at 1 (giving the \( \binom{n-j}{m} \) term) and the remaining \( k - r - s - m \) descents can be chosen from the \( i \) pairs of vertices \((x, y) \in (X \setminus \{1\}) \times Y \) where \( x < y \) (which gives the \( \binom{k-r-s-m}{i} \) term). Finally, there are \((j - 1)(n - j) - i \) edges from \( Y \) to \( X \) which are increasing that can also be added without introducing any new descents or cycles. The result follows. \( \square \)

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