The definition of holonomic measures

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Abstract

We consider certain Borel measures on a direct sum of several copies of the tangent bundle of a smooth manifold. We show that those that can be approximated by closed submanifolds coincide with those whose integrals of exact differential forms vanish.

1 Introduction

In this paper we consider certain Borel measures on the direct sum $T^nM$ of $n$ copies of the tangent bundle of a smooth manifold $M$. We distinguish two classes of such measures. First, those for which the integrals of exact forms vanish. Second, those that can be approximated by measures induced by embeddings of closed submanifolds (or, more precisely, by embeddings of parameterized CW-complexes without boundary). Our main result Theorem 1 which states that these two classes coincide. We give precise definitions in Section 2, where we also state our result. Section 3 is devoted to the proof of the theorem.

The $n = 1$ case of this result was proved by Bangert [1] and Bernard [2]. The author saw a letter by Mather [6] in which an idea similar to Bangert’s was sketched. Our proof of that case is different to theirs.

As explained in Section 2.1 our proof can be adapted to prove a similar statement in which the submanifolds are allowed to have a boundary contained in certain subsets of $M$.

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2 Setting and statement of results

Riemannian structure. Throughout, we fix a compact, oriented $C^\infty$ manifold $M$, without boundary, of dimension $d \geq 1$. Denote by $TM$ its tangent bundle, and by $T^nM$ the direct sum bundle

$$T^nM = TM \oplus \cdots \oplus TM.$$ 

The dimension of $T^nM$ is $d(n + 1)$. We will refer to its elements as

$$(x, v_1, v_2, \ldots, v_n),$$

where $x \in M$ and $v_i \in T_x M$. Sometimes for brevity we will write $v$ instead of $(v_1, \ldots, v_n)$.

We will use the word smooth to refer to $C^\infty$ functions. The space of smooth, real-valued, compactly supported functions on $T^nM$ will be denoted $C^\infty_c(T^nM)$.

We fix a Riemannian metric $g$ on $M$, together with its Levi-Civita connection. We denote $|v| = \sqrt{g(v, v)}$ for $v \in T_x M$ and we extend this norm to $T^nM$ by letting

$$|(v_1, v_2, \ldots, v_n)| = \sqrt{|v_1|^2 + |v_2|^2 + \cdots + |v_n|^2}.$$ 

We will denote by $\Omega^k(M)$ the space of smooth differential $k$-forms on $M$. We will often consider these forms as smooth functions on $T^nM$. We also define the projection $\pi : T^nM \to M$ by

$$\pi(x, v_1, \ldots, v_n) = x.$$ 

Mild measures. We let $\mathcal{P}_n$ be the space of subpower functions, that is, the space of real-valued, continuous functions $f \in C^0(T^nM)$ such that

$$\sup_{(x,v)\in T^nM} \frac{|f(x,v)|}{1 + |v|^n} < +\infty.$$ 

Note that all differential $n$-forms on $M$ belong to $\mathcal{P}_n$ when regarded as functions on $T^nM$. We endow $\mathcal{P}_n$ with the supremum norm and its induced topology.

A signed, Borel measure $\mu$ on $T^nM$ is mild if

$$\int_{T^nM} 1 + |(v_1, \ldots, v_n)|^n \, d|\mu| < +\infty,$$
where $|\mu| = \mu^+ + \mu^-$ is the absolute value of the measure with Hahn decomposition $\mu = \mu^+ - \mu^-$, for positive measures $\mu^+$ and $\mu^-$. We denote the space of mild measures by $\mathcal{M}_n$. We define the mass $M(\mu)$ of $\mu \in \mathcal{M}_n$ to be

$$M(\mu) = \int_{T^nM} \text{vol}_n(v_1, v_2, \ldots, v_n) d|\mu|(x, v_1, \ldots, v_n).$$

The space $\mathcal{M}_n$ is naturally embedded in the dual space $\mathcal{P}_n^*$ and we endow it with the topology induced by the weak* topology on $\mathcal{P}_n^*$. This topology is metrizable on $\mathcal{M}_n$. We can give a metric by picking a sequence of functions $\{f_i\}_{i \in \mathbb{N}} \subset \mathcal{C}_c^\infty(T^nM)$ that are dense in $\mathcal{P}_n$ and letting

$$\text{dist}_{\mathcal{M}_n}(\mu_1, \mu_2) = M(\mu_1 - \mu_2) + \sum_{k=1}^{\infty} \frac{1}{2^k \sup |f_k|} \left| \int |f_k| d\mu_1 - \int |f_k| d\mu_2 \right|. \quad (1)$$

**Cellular complexes.** An $n$-dimensional cell (or $n$-cell) $\gamma$ is a smooth map

$$\gamma : D \subseteq \mathbb{R}^n \rightarrow M,$$

where $D$ is a subset of $\mathbb{R}^n$ homeomorphic to a closed ball, together with a choice of coordinates $t = (t_1, t_2, \ldots, t_n)$ on $D$. A chain of $n$-cells is a formal linear combination of the form

$$a_1 \gamma_1 + a_2 \gamma_2 + \cdots + a_k \gamma_k$$

for real numbers $a_1, a_2, \ldots, a_k$ and $n$-cells $\gamma_1, \gamma_2, \ldots, \gamma_k$.

Let $\gamma : D \subseteq \mathbb{R}^n \rightarrow M$ be an $n$-cell. Denote by $d\gamma$ the differential map associating, to each element in $D$, an element in $T^nM$. Explicitly, if we have coordinates $t = (t_1, t_2, \ldots, t_n)$ on $D$, then

$$d\gamma(t) = \left( \gamma(t), \frac{\partial \gamma}{\partial t_1}(t), \frac{\partial \gamma}{\partial t_2}(t), \ldots, \frac{\partial \gamma}{\partial t_n}(t) \right).$$

This map depends on our choice of coordinates $t$.

To an $n$-cell $\gamma$, we associate a measure $\gamma^\ast$ on $T^nM$ defined by

$$\int_{T^nM} f \, d\gamma^\ast = \int_D f(d\gamma(t)) \, dt,$$

where $dt = dt_1 \wedge \cdots \wedge dt_n$. Similarly, to a chain of $n$-cells $\alpha = \sum_{i=1}^k a_i \gamma_i$, we associate the measure $\alpha^\ast$ given by

$$\alpha^\ast = \sum_{i=1}^k a_i \gamma_i^\ast.$$
The measure \( \lambda \alpha \) is an element of \( \mathcal{M}_n \). We will say that a chain \( \alpha \) is a cycle if for all forms \( \omega \in \Omega^{n-1}(M) \),

\[
\int_{T^nM} d\omega \, d\lambda \alpha = 0.
\]

**Theorem 1.** Assume that \( 1 \leq n \leq d \). Let \( \mu \in \mathcal{M}_n \) be a positive mild measure. Then the following conditions are equivalent:

(Hol) For all forms \( \omega \in \Omega^{n-1}(M) \),

\[
\int_{T^nM} d\omega \, d\mu = 0.
\]

(Cyc) There exists a sequence \( \{\alpha_k\}_{k \in \mathbb{N}} \) of cycles such that the induced measures \( \lambda \alpha_k \rightarrow \mu \) as \( k \rightarrow \infty \) in the topology induced by the distance \( (1) \).

Most of the rest of the paper will be devoted to proving this result. A probability measure \( \mu \in \mathcal{M}_n \) that satisfies Conditions (Hol) and (Cyc) is said to be holonomic. The space of all holonomic measures is convex.

### 2.1 Relative holonomic measures

Since our proof of Theorem 1 relies on triangulations, it is easy to modify it in order to prove

**Theorem 2.** Assume that \( 1 \leq n \leq d \). Let \( \mu \in \mathcal{M}_n \) and \( U \subset M \) be a closed set diffeomorphic to a union of simplices of a smooth triangulation of \( M \). Then the following conditions are equivalent:

1. For all forms \( \omega \in \Omega^{n-1}(M) \) such that \( \omega|_U = 0 \),

\[
\int_{T^nM} d\omega \, d\mu = 0.
\]

2. There exists a sequence \( \{\alpha_k\}_{k \in \mathbb{N}} \) of chains such that the boundaries \( \partial \alpha_k \) are contained in \( U \), and such that the induced measures \( \lambda \alpha_k \rightarrow \mu \) as \( k \rightarrow \infty \) in the topology induced by the distance \( (1) \).

**Remark 3.** The boundaries \( \partial \alpha_k \) can either be defined as in singular homology (see for example [5, §2.1]), or alternatively one can interpret the condition that \( \partial \alpha_k \) be contained in \( U \) as meaning that

\[
\int d\omega \, d(\lambda \alpha_k) = 0
\]

for all \( \omega \in \Omega^n(M) \) such that \( \omega \) vanishes on \( U \).
A probability measure $\mu \in \mathcal{M}_n$ that satisfies the conditions in Theorem 2 is said to be holonomic relative to $U$. The space of all these measures is again convex.

3 Proof

This section is devoted to the proof of Theorem 1, which will be given in Section 3.6.

The idea of the proof is the following. The fact that Condition (Cyc) implies Condition (Hol) is an easy consequence of Stokes’s theorem, so we concentrate in the other implication.

We start with a positive measure $\mu$ that satisfies Condition (Hol). We prove in Section 3.1 that we may assume that the measure $\mu$ is a smooth density. In Section 3.2 we specify a family of triangulations $T_k$ on $M$ for $k \in \mathbb{N}$. Then in Section 3.3.1 we construct ‘base measures’ $\bar{\mu}_k$, which are approximations to our smooth density that are (in a sense) constant on each simplex of $T_k$; this is analogous to approximating a smooth function on $\mathbb{R}$ with simple functions. In Section 3.3.2 we construct an $n$-chain $\beta_k$ that is again (in a sense) constant on each simplex of $T_k$.

In Section 3.4 we derive a condition on the $(d-n)$-dimensional skeleton of $T_k$ that in Section 3.5.1 allows us to construct cycles that contain the chains $\beta_k$, and whose mass $M$ can be estimated. We work on the estimates for the mass in Sections 3.5.2 and 3.5.3. Finally, we put everything together in Section 3.6.

3.1 Smoothing

Lemma 4. Any measure $\mu$ in $\mathcal{M}_n$ can be approximated arbitrarily well using a smooth density on $T^n M$. If $\mu$ is a probability measure that satisfies Condition (Hol) then it can be approximated by smooth probability densities that also satisfy Condition (Hol).

Proof. Denote the exponential map by $\exp_x : T_x M \to M$.

A mollifier $\psi \in C_0^\infty(\mathbb{R})$ is a function such that $\psi(x) = \psi(-x)$, $\int \psi = 1$, and $\psi \geq 0$.

Fix a set of smooth vector fields $F_1, F_2, \ldots, F_\ell$ on $M$ such that for each $x \in M$ the vectors $F_1(x), \ldots, F_\ell(x)$ span all of $T_x M$. Note that $\ell \geq d = \dim M$. 

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Denote by $\phi^i: M \times \mathbb{R} \rightarrow M$ the flow of $F_i$:

$$\phi^i_0(x) = 0, \quad \frac{d\phi^i_s(x)}{ds} = F_i(\phi^i_s(x)), \quad s \in \mathbb{R}.$$

For fixed $s \in \mathbb{R}$, denote the derivative of the diffeomorphism $\phi^i_s$ by $d\phi^i_s: TM \rightarrow TM$. Extend it to a map $d\phi^i_s: T^n M \rightarrow T^n M$ by setting

$$d\phi^i_s(x, v_1, v_2, \ldots, v_n) = (\phi^i_s(x), d\phi^i_s v_1, \ldots, d\phi^i_s v_n).$$

For $f \in C_\infty_c(T^n M)$, we will denote by $P_i(f)$ the function given by

$$P_i(f)(x, v_1, v_2, \ldots, v_n) = \int_{T^n M} f \circ d\phi^i_s(x, v_1, \ldots, v_n) \psi(s) ds.$$

This is a convolution in the horizontal direction $F_i$. Also, for $f \in C_\infty_c(T^n M)$ we let $V(f)$ be the convolution in the vertical direction,

$$V(f)(x, v_1, \ldots, v_n) = \int_{T^n M} dw_1 \psi(|w_1 - v_1|) \int_{T^n M} dw_2 \psi(|w_2 - v_2|) \ldots \int_{T^n M} dw_n \psi(|w_n - v_n|) f(x, w_1, w_2, \ldots, w_n).$$

For $f \in C_\infty_c(T^n M)$, we will denote

$$\psi \ast f = P_1 P_2 \cdots P_l V(f).$$

Note that $\psi \ast f$ is a $C^\infty$ function even if $f$ is only measurable. Moreover, if the diameter of the support of $\psi$ is sufficiently small, and if $f$ is an exact form on $M$, i.e. $f(x, v_1, \ldots, v_n) = d\omega_\psi(v_1, \ldots, v_n)$ for some $\omega \in \Omega_{n-1}^\omega(M)$, then $\psi \ast d\omega$ is the exact form $d(\psi \ast \omega)$. To see this, note first that by linearity of $\omega$ on each entry $V(d\omega) = d\omega$. Also, for $s$ small enough, $\phi^i_s$ is a diffeomorphism and hence

$$P_i(d\omega) = \int \psi(s) d\phi^i_s d\omega ds = d \left[ \int \psi(s) d\phi^i_s \omega ds \right] = d(P_i \omega).$$

Now let $\mu$ be a probability measure on $T^n M$. We define the convolution $\psi \ast \mu$ by duality, setting

$$\int_{T^n M} f d(\psi \ast \mu) = \int_{T^n M} (\psi \ast f) d\mu.$$

Then $\psi \ast \mu$ is a smooth density (see for example [4, §5.2]), and in the topology of $\mathcal{M}_n$,

$$\psi \ast \mu \rightarrow \mu \quad \text{as} \quad \text{diam supp } \psi \rightarrow 0.$$
Also, if $\mu$ satisfies Condition (Hol), then

$$\int_{T^n M} d\omega \, d(\psi \ast \mu) = \int_{T^n M} d(\psi \ast \omega) \, d\mu = 0,$$

so $\psi \ast \mu$ also satisfies Condition (Hol).

### 3.2 Triangulations

A triangulation $T = (K, h)$ of $M$ is a simplicial complex $K$ homeomorphic to $M$ together with a homeomorphism $h : K \to M$. When talking about such a triangulation $T$, we will speak indistinctly of a simplex $U \subseteq K$ and of its image $h(U) \subseteq M$. In other words, we will ignore $K$ as a topological space, and we will instead think of the triangulation as being ‘drawn’ directly on $M$.

We fix a sequence of triangulations $\{T_k\}_{k \in \mathbb{N}}$ on $M$ such that:

- **T1. (Successive refinements)** For $k > 1$, $T_k$ is a refinement of $T_{k-1}$.
  
  For each simplex $V$ in $T_k$, $k \geq 1$, we denote by $U(V)$ the simplex of dimension $d$ of $T_1$ in which $V$ is contained. (This is ambiguous for the simplices of dimension less than $d$, but any choice will work, so we assume that this choice has been made for each simplex $V$ once and for all.)

- **T2. (Finite)** $T_k$ has finitely many simplices.

- **T3. (Charted)** For each simplex $U$ of dimension $d$ of $T_1$, there is a chart $\varphi_U : N_U \subseteq M \to \mathbb{R}^d$ (for $N_U$ some neighborhood of $U$) such that the image $\varphi_U(U)$ is the standard simplex with vertices at the origin and at the vectors of the standard basis of $\mathbb{R}^d$.
  
  For brevity, we will denote $\varphi_U(V)$ by $\varphi_V$ for all simplices $V$ in the triangulations $T_k$, $k \geq 1$.

- **T4. (Affine)** For every simplex $V$ in $T_k$, $\varphi_V(V)$ is contained in a translate of a vector space $Y(V) \subset \mathbb{R}^d$ of dimension $\dim V$.

- **T5. (Nondegeneracy)** All simplices of $T_k$ are non-degenerate. In other words, if a simplex $V$ has dimension $m$, then also $\text{vol}_m V > 0$.

- **T6. (Vanishing diameter)**
  $$\lim_{k \to \infty} \text{diam } T_k = 0.$$
Existence of triangulations on manifolds is discussed in great detail for example in [8]. A triangulation \( T_1 \) satisfying T2–T5 always exists. To obtain all other refinements \( T_k \) of \( T_1 \), one successively refines the standard simplex \( \varphi_U(U) \) (for \( U \) a simplex in \( T_1 \)) making sure that the rules T2–T5 are respected every time. It can be seen by induction on \( k \) that this is possible. One can take a refinement that respects T2–T5. Ensuring overall compliance with T6 is easy. Then one pulls the resulting triangulation back to \( M \) using the charts \( \varphi_U \).

We will denote by \( E^k_m \) the \( m \)-dimensional skeleton of the triangulation \( T_k \).

### 3.3 The base measure and its approximation

#### 3.3.1 Construction of the base measure

In Section 3.2 we specified the triangulations \( T_k, k \in \mathbb{N} \), and we introduced the notation \( \varphi_V \).

Let \( \mu \) be a smooth density in \( \mathcal{M}_n \). We will define base measures \( 0 \leq \bar{\mu}_k \leq \mu \) depending on the triangulations \( T_k \) such that \( \bar{\mu}_k \to \mu \) as \( k \to \infty \). Roughly speaking, the measure \( \bar{\mu}_k \) is the largest density, constant on a constant section of \( T^n M \) in the interior of each \( d \)-dimensional simplex \( U \) of \( T_k \). Our goal here is not to produce measures that satisfy Condition (Hol).

For a simplex \( V \) of dimension \( d \) in the triangulation \( T_k \), we take the chart \( \varphi_V \) and extend it to a trivialization of \( T^n M \), \( d\varphi_V : T^n M \to \mathbb{R}^{(n+1)d} \), by setting

\[
d\varphi_V(x,v_1,v_2,\ldots,v_n) = (\varphi_V(x),d\varphi_V(v_1),\ldots,d\varphi_V(v_n)).
\]

Let \( m \) denote Lebesgue measure on \( \mathbb{R}^{(n+1)d} \) and let \( \rho \) be the Radon-Nikodym derivative of the pushforward measure \( (d\varphi_V)_*\mu = \rho m \) on \( \mathbb{R}^{(n+1)d} \).

For \( (x,v) \in \mathbb{R}^{d(n+1)} \) with \( x \in \varphi_V(V) \), we let

\[
\tilde{\rho}_k(x,v) = \inf_{y \in \varphi_V(V)} \rho(y,v).
\]

Note that \( v \) is the same on both sides of the equation, and the dependence of the right-hand-side on \( x \) comes from the choice of \( V \). Also, this is ambiguous when \( x \) lies in a simplex of dimension \( < d \). This ambiguity happens only on a set of \( m \)-measure zero, so we may just ignore it, as it will not affect the rest of our argument. We let

\[
\bar{\mu}_k|_{T^n V} = d\varphi_V^*(\tilde{\rho}_k m).
\]
This completely determines $\bar{\mu}_k$ on the whole bundle $T^n M$. Also, $\rho_k \to \rho$ uniformly on compact sets, because $\rho$ is smooth and $\text{diam} \, T_k \to 0$ by $T6$.

Similarly, $M(\bar{\mu}_k - \mu) \to 0$. Hence $\text{dist} \, M(\bar{\mu}_k, \mu) \to 0$ as $k \to \infty$.

### 3.3.2 Construction of the approximation

For each $k \in \mathbb{N}$, we will construct a chain $\beta_k$ whose induced measure $\bar{\omega}_k$ will approximate the base measure $\bar{\mu}_k$ very well. We do this in the following steps.

**Step 1.** On each $d$-dimensional simplex $V$ of $T_k$, we sample the distribution $\bar{\rho}_k m$ to get a finite sequence of points $p^V_1, \ldots, p^V_{\ell_V} \in \mathbb{R}^d(\mathbb{N} + 1)$. We may assume that the following conditions are true for these points:

**A1.** Each point $p^V_i$ is in the interior of $\varphi_V(V)$.

**A2.** Write $p^V_i$ as $(x, v_1, \ldots, v_n) \in \mathbb{R}^d \times \cdots \times \mathbb{R}^d = (\mathbb{R}^d)^{n+1}$. Let $\Pi^V_i$ be the plane

$$
\Pi^V_i = \{x + t_1v_1 + t_2v_2 + \cdots + t_nv_n : t_i \in \mathbb{R}\} \subseteq \mathbb{R}^{(n+1)}.
$$

We assume that $\Pi^V_i$ intersects all the simplices $W \subseteq \partial \varphi_V(V)$ of dimension $\text{dim} W \geq d - n$ transversally.

**A3.** For a $(d - n)$-dimensional simplex $W \subset E^{k}_{d-n}$, let $V_1$ and $V_2$ be two $d$-dimensional simplices adjacent to $W$. Let $A_i, i = 1, 2$, be the set of points of the form $\Pi^V_i \cap W$. We assume that there is a finite partition of $W$ by disjoint, convex sets $U_1, \ldots, U_m$ with $\text{diam} \, U_i < a(k)$ such that each of them contains at least one point in $A_1$, and

$$
\frac{|\bar{\mu}_k(V_2) - \#U_i \cap A_2|}{|\bar{\mu}_k(V_1) - \#U_i \cap A_1|} < a(k), \tag{2}
$$

where $a : \mathbb{N} \to \mathbb{R}$ is an asymptotically-vanishing function that will be specified at the end of Section 3.3.2.

Note that the measure

$$
\sum_{V \subset E^{k}_{d}} \frac{1}{\ell_V} \sum_{i} \varphi^+_V \delta_{p^V_i} \tag{3}
$$

is a good approximation of $\bar{\mu}_k$. Compliance with item A3 can be achieved by increasing the number of points, thus making the sample more dense.

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Step 2. Let $V$ be a $d$-dimensional simplex in $T_k$. Let $\gamma_i^V : D_i^V \subseteq \mathbb{R}^n \to \mathbb{R}^d$ be the solution to the equations

$$\gamma_i^V (0, 0, \ldots, 0) = x, \quad \frac{\partial \gamma_i^V}{\partial t_j} = v_j, \quad i = 1, \ldots, n.$$  \hspace{1cm} (4)

Assume that the domain $D_i^V$ of $\gamma_i^V$ is the largest closed subset of $\mathbb{R}^n$ such that $\gamma_i^V$ remains within $\varphi_V(V)$. Note that image $\gamma_i^V = \gamma_i^V(D_i^V) \subset \Pi_i^V$, so by A2 this image also intersects the simplices in the boundary of the standard simplex $\partial(\varphi_V(V))$ transversally.

We let

$$\beta_k = \sum_{V \subseteq E_d} \frac{1}{\ell_V} \sum_i \frac{1}{|D_i^V|} (d\varphi_V)^* \gamma_i^V.$$  

When we consider the measure $\beta_k$, this is like taking the measure in equation (3) and spreading the mass of each point along a simplex determined by its velocity vectors $v_1, \ldots, v_n$. Since $\mu_k|_V$ is ‘constant’ for each such set of velocity vectors, $\beta_k$ is in fact a very natural approximation to $\mu_k$. Note that we divide by the $n$-dimensional Lebesgue measure of the domain, $|D_i^V|$, in order to normalize and obtain the correct weights.

3.4 Conditions on the boundary

We say that a sequence of simplices $V_1, \ldots, V_\ell$ of a triangulation is properly nested if $V_i \subset \partial V_{i-1}$ and $\dim V_i = d - i$.

Let $V$ be a simplex in a triangulation $T$ of $M$. For $x$ in $V$, let

$$u_V(x) = \text{dist}(x, \partial V).$$

If the triangulation $T$ is reasonably nice, $u_V$ can then be extended to all of $M$ in such a way that $u_V$ will be smooth on the interiors of the simplices of $\partial V$. In our case, this can be done because the triangulation satisfies T3–T5. There is some ambiguity in the choice of the extension, but it is immaterial in our argument.

Let, for $\varepsilon > 0$,

$$u_V^\varepsilon(x) = \begin{cases} 
  u_V(x)/\varepsilon, & \text{if } |u_V(x)| < \varepsilon, \\
  -1, & \text{if } u_V(x) < -\varepsilon \\
  1, & \text{if } u_V(x) > \varepsilon
\end{cases}$$

Finally, let $\bar{u}_V^\varepsilon$ be a smoothed version of $u_V^\varepsilon$, such that the amount of smoothing tends to 0 as $\varepsilon \to 0$. This can be obtained, for example, by
convolving as in Section 3.1 and ensuring that one uses mollifiers $\psi$ such that $\text{diam supp } \psi < \varepsilon^2$.

Let $C = \{V_1 \supset \cdots \supset V_n\} \subseteq T_k$ be a set of $n$ properly nested simplices. Observe that the form

$$\omega_\varepsilon = d\bar{u}_V \wedge d\bar{u}_V \wedge \cdots \wedge d\bar{u}_V$$

is exact.

Let $\nu$ be a measure on $T^n M$. Let $C = \{V_1 \supset V_2 \supset \cdots \supset V_\ell\}$ be properly nested simplices in some triangulation of $M$. Let

$$B_\varepsilon(C) = \{x \in M : |u_V(x)| \leq \varepsilon, i = 1, 2, \ldots, \ell\}.$$

Define the measure $\nu^C$ by

$$\int f \, d\nu^C = \lim_{\varepsilon \to 0^+} \frac{1}{\varepsilon^\ell} \int_{B_\varepsilon(C)} f \, d\nu, \quad (5)$$

where $f \in C_C^\infty(T^n M)$.

Observe that

$$\lim_{\varepsilon \to 0^+} \int \omega_\varepsilon \, d\mu = \int d\bar{u}_V \wedge d\bar{u}_V \wedge \cdots \wedge d\bar{u}_V \, d\mu^C.$$

Since the left-hand-side vanishes when $\mu$ satisfies Condition (Hol), we get

**Lemma 5.** If the smooth density $\mu \in \mathcal{M}_n$ satisfies Condition (Hol), then for every $k \in \mathbb{N}$ and for every properly nested sequence of simplices $C = \{V_1 \supset V_2 \supset \cdots \supset V_n\}$ of the triangulation $T_k$, we have

$$\int_{T^n M} d\bar{u}_V \wedge d\bar{u}_V \wedge \cdots \wedge d\bar{u}_V \, d\mu^C = 0. \quad (6)$$

### 3.5 Closing up the base measure

#### 3.5.1 Inductive construction of cycles

In this section we inductively construct $n$-dimensional cycles $\eta_k$ that contain the chains $\beta_k$ that approximate the base measure $\bar{\mu}_k$. Our starting point will be a measure $\nu_0^C$ corresponding to a fictitious $n$-chain $\eta_0^C$ that will help us guess what the 0-dimensional intersections of $\eta_k$ with the skeleton $E^{k}_{d-n}$ should be.
The 0-dimensional chain. Recall that the chain $\beta_k$ was constructed in Section 3.3.2. It is a linear combination of $n$-cells $\varphi_i^*\gamma_i^Y$, determined by the equations (4). For each $k > 0$, we let $\tilde{\beta}_k$ be the chain that results from extending the domain of the $n$-cell $\gamma_i^Y$ (still respecting (4)) to an open set very slightly larger than its original domain $D_i^Y$, so that it now intersects the skeleton $E_{d-1}^k$ of $T^k$. By property A2, this intersection is transversal.

Then, for properly-nested simplices $C = \{V_1 \supset \cdots \supset V_\ell\}$ the measure $\wedge \tilde{\beta}_k^C$ defined in equation (5) reflects the way the boundary of $\beta_k$ intersects $\partial V_\ell$.

For a point $p$ in $T^n M$ such that $\pi(p) \in V_\ell$, and for a set of $n$ properly nested simplices $C = \{V_1 \supset \cdots \supset V_n\}$ let

$$W(p, C) = du_{V_1} \wedge du_{V_2} \wedge \cdots \wedge du_{V_n}(p),$$

where the functions $u_{V_i}$ are as in Section 3.4. Observe that if $C$ and $C'$ are two sets of $n$ properly nested simplices that differ only in the $\ell$th simplex, $\ell < n$, and the corresponding simplices $V_\ell$ and $V'_\ell$ are adjacent, then

$$W(p, C) = -W(p, C')$$

because $du_{V_\ell} = -du_{V'_\ell}$ at $p$.

For each $k$, we pick a finite set of points $\{p_i^k\} \subset T^n M$, and weights $r_i^k \in \mathbb{R}_+$ such that Conditions U1–U4 below are true. We want to construct a measure $\eta_0^k$ that will capture the way in which our cycles will ultimately intersect the skeleton $E_{d-n}^k$. This measure will be the starting point for the full construction of the cycles. Crucially, at each point in its support $\eta_0^k$ carries information about the $k$-dimensional subspace that will eventually turn out to be the intersection of our cycles $\eta^n_k$ with the skeleton $E_{d-n}^k$. We will imagine that there is an $n$-chain whose (degenerate) cells are the points $\{\pi(p_i)\} \subseteq M$, so that $\eta_0^k$ is given by

$$\eta_0^k = \sum_i r_i^k \pi(p_i^k),$$

and parameterized so that

$$\wedge \eta_0^k = \sum_i r_i^k \delta_{p_i^k}.$$  

Strictly speaking, such a chain $\eta_0^k$ does not exist, but the measure $\wedge \eta_0^k$ does, and this is the object we need.

The conditions are:
U1. The projection $\pi(p_k^i)$ of each point $p_k^i$ on $M$ is contained in the $(d-n)$-dimensional skeleton $E_{d-n}^k$ of the triangulation $T_k$.

U2. We require the points in the support of $\tilde{\beta}_k^C$ to be contained in $\{p_k^i\}_i$, and the corresponding weights $r_k^i$ to be at least as large as the weights these points have in the measure $\tilde{\beta}_k^C$.

U3. For each set of $n$ properly nested simplices $C = \{V_1 \supset \cdots \supset V_n\} \subseteq T^k$,

$$\sum_i W(p_i^k, C) r_i^k = 0,$$

where the sum is taken over all $i$ such that $\pi(p_i)$ is in $V_n$.

U4. The measure $\eta_{k}^0$ approximates the restriction of $\mu$ to the skeleton $E_{d-n}^k$:

$$\text{dist}_{\#} \left( \sum_C \mu^C, \sum_C \eta_{k}^0 C \right) \leq \frac{1}{k},$$

where the sums are taken over all sets $C$ of $n$ properly nested simplices of $T^k$.

The idea is that $\{p_k^i\}_i \cap \pi^{-1}(V_n)$ should be a very good sample of the measure $\mu^C$. The set of points and weights can be found as follows. Start with the points in the support of $\tilde{\beta}_k^C$, with the weights they inherit from $\beta_k$. Then by further sampling the measure $\mu^C$, and invoking the fact that it satisfies the conclusion of Lemma 5, a solution for the condition in item U3 is guaranteed to exist. Note that the condition in item U3 is essentially a rephrasing of the conclusion of Lemma 5 adapted to $\eta_{k}^0 C$. Taking a sufficiently large sample of $\mu^C$, one can also guarantee that item U4 will be satisfied.

The higher-dimensional chains. For every set of $n+1$ properly nested simplices $C = \{V_1 \supset \cdots \supset V_{n+1}\}$, we let $\eta_k^C$ denote the 0-dimensional chain

$$\eta_k^C = \sum_i (\text{sgn} W(p_i^k, C)) r_i^k \pi(p_i^k)$$

where the sum is taken over all indices $i$ such that $p_i^k$ is contained in $V_{n+1}$. 

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For every set $C = \{ V_1 \supset \cdots \supset V_{n-j} \} \subseteq T_k$ of $n-j$ properly nested simplices, $1 \leq j < n$, $\tilde{\beta}_k$ induces an $j$-dimensional chain $\beta^C_k$ on $\partial V_{n-j}$ that satisfies, for all $\omega \in \Omega^j(M)$,
\[
\int_{\beta^C_k} \omega = \int_{\partial V_{n-j}} \omega \wedge du_{V_1} \wedge du_{V_2} \wedge \cdots \wedge du_{V_{n-j}} \, d\tilde{\beta}_k.
\]
Observe that the chain $\beta^C_k$ is in general not unique, but any choice will do for our purposes. We also let $\beta^\emptyset_k = \beta_k$.

For sets of properly nested simplices
\[
C' = \{ V_1 \supset \cdots \supset V_{n-j-1} \} \subset C = \{ V_1 \supset \cdots \supset V_{n-j} \},
\]
we refine the chain $\beta^C'_{k}$ so that each of its $(j+1)$-dimensional cells intersects only one of the $(d-n+j+1)$-dimensional simplices of the boundary $\partial V_{n-j-1}$. We then let $\beta^C_k$ be the part of $\beta^C'_{k}$ that is contained in $V_{n-j}$. In other words,
\[
\beta^C_k = \sum_{V \subset \partial V_{n-j}} \beta^C'_{k \cup \{ V \}}.
\]

We proceed to construct, inductively on $j = 0, 1, \ldots, n-1$, $(j+1)$-dimensional cycles $\eta^C_k$ corresponding to each set of $n-j$ properly nested simplices $C = \{ V_1 \supset \cdots \supset V_{n-j} \} \subseteq T_k$, such that:

E1. The cells of $\eta^C_k$ are contained in $V_{n-j} \subseteq E^k_{d-n+j+1} \subseteq M$.

E2. We require that $\beta^C_k$ be contained in $\eta^C_k$, in the sense that all the cells of $\beta^C_k$ appear in $\eta^C_k$ with coefficients of magnitude greater or equal to those they have in $\beta^C_k$.

If $j = n-1$, $C = \{ V_1 \}$ and $\eta^C_k$ contains precisely the cells of $\beta_k$ that are contained in $V_1$, and with exactly the same parameterization for each cell.

E3. We have
\[
\partial \eta^C_k = \sum_{V \subset \partial V_{n-j}} \eta^C_{k \cup \{ V \}},
\]

where the sum is taken over all simplices in the boundary of $V_{n-j}$.

E4. If $C$ and $C'$ are sets of $n-j$ properly nested simplices of $T_k$ that only differ in the $\ell$-th simplex, $1 \leq \ell < n-j$, and the corresponding simplices $V_\ell$ and $V'_\ell$ are adjacent, then
\[
\eta^C_k = -\eta^{C'}_k.
\]
This should hold in the sense that the induced functionals on $\Omega^{j+1}(M)$ (i.e., the induced currents) must be equal.

E5. If $C' = \{ V_1 \supset \cdots \supset V_{n-j-1} \} \subseteq T_k$ is not empty,
\[
\sum_{V \subset \partial V_{n-j-1}} \partial \eta_{C' \cup \{V\}} = 0,
\]
where the sum is taken over all simplices in the boundary of $V_{n-j-1}$. If $C'$ is empty, then the same equation should hold, but now taking the sum over all simplices $V$ of dimension $d$ in $T_k$.

E6. The cells of $\eta_k^C$ that are not inherited from $\tilde{\beta}_C$ are almost $M$-mass minimizing, in a sense that will be specified at the end of Section 3.5.3.

E7. If $j = n - 1$, the cells of $\eta_k^C$ that are not inherited from $\tilde{\beta}_C$ are parameterized with very high speed (and thus the induced total measure $\int_{C'(T^n M)}$ is very small), in a sense that will be specified in Section 3.6.

First we show how to create the 1-chain $\eta_k^C$ corresponding to the case in which $C$ contains $n$ properly nested simplices. We start with $\tilde{\beta}_C$, which will provide for compliance with item E2. By U2, the boundary of $\tilde{\beta}_C$ is also contained in $\sum_{V \subset \partial V_{n-1}} \eta_{C' \cup \{V\}}$. So what we do, in order to comply with E1 and E3, is that we connect the remaining dots in $\sum_{V \subset \partial V_{n-1}} \eta_{C' \cup \{V\}}$ with curves contained in $V_{n-1}$ in the way prescribed by the weights of the dots; because of property U3 this is possible. By taking very short curves, we ensure compliance with E6. Because of identity (7), the construction of $\eta_{C' \cup \{V\}}$ ($V \subset \partial V_{n-1}$) immediately implies E4. Property E5 also follows from the identity (7).

Now assume that we have $\eta_k^C$ for $j = m - 1$, and let us construct it for $j = m$, $m > 1$. Let $C = \{ V_1 \supset \cdots \supset V_{n-m} \} \subseteq T_k$. For each simplex $V \subset \partial V_{n-m}$, we are assuming that there exists $\eta_k^{C \cup V}$ that satisfies E1. To close these up, we again start with $\tilde{\beta}_C$ (whence complying with E2) and we add cells of dimension $m+1$ contained in $V_{n-m}$ (complying with E1) so that property E3 will hold; this is possible because $V_{n-m}$ has trivial homology and because $\sum_{V \subset \partial V_{n-1}} \eta_{C' \cup \{V\}}$ is a cycle as it satisfies E5. Properties E4 and E6 for $j = m$ follow from property E3 for $j = m - 1$. Compliance with property E6 can be attained by choosing an almost mass-minimizing set of $(m+1)$-cells. Property E7 can be achieved by adjusting the parameterization of the cells involved.

Write $\eta_k = \eta_k^\emptyset$. We have proved:
Lemma 6. There is a sequence of cycles $\eta_k$ that contain $\beta_k$ and such that

$$M(\eta_k) - M(\beta_k)$$  \hspace{1cm} (8)

is as almost minimal (in the sense of $H[0]$, while respecting

$$\text{dist}_n \left( \sum C \mu^C, \sum C \eta_k^C \right) \leq \frac{1}{k},$$  \hspace{1cm} (9)

where the sums are taken over all sets $C$ of $n$ properly nested simplices of $T_k$. Also, the part of $\eta_k^C$ that comes inherited from $\beta_k$ satisfies $A[3]$.

By construction, equation (9) is exactly the same as the condition in $U[4]$.

3.5.2 Density lemma

For each set $C = \{V_1 \supset \cdots \supset V_{n-1}\} \subset T_k$ of properly nested simplices, in Section 3.5.1 the 1-dimensional chain $\eta_k^C$ was constructed. Our goal in this section is to estimate the asymptotic behavior of its mass $M(\eta_k^C)$ as $k \to \infty$.

For a set $U \subset \mathbb{R}^m$, the diameter of $U$ within $U$ is defined to be

$$\text{diam}_U U = \sup_{x,y \in U} \inf_{\gamma} \text{arclength}(\gamma)$$

where the infimum is taken over all absolutely-continuous curves $\gamma$ parameterized on all intervals $[a,b] \subseteq \mathbb{R}$ and such that $\gamma(a) = x$ and $\gamma(b) = y$.

Lemma 7. Let $U$ be a path-connected, bounded open set in $\mathbb{R}^m$, $m \geq 1$. There is a number $\varepsilon_0 > 0$ such that if $0 < \varepsilon < \varepsilon_0$ and $A$ and $B$ are two finite subsets of $U$ of equal cardinality, then the following is true. Assume that there is a finite partition of $U$ by disjoint, path-connected sets $U_1, \ldots, U_m$ with $\text{diam}_{U_i} U_i < \varepsilon$ such that each of them contains at least one point of $A$, and

$$\left| 1 - \frac{\# U_i \cap B}{\# U_i \cap A} \right| < \varepsilon^2.$$  \hspace{1cm} (10)

Then there is a 1-dimensional chain $\theta$ such that $\#\theta(U) = 1$, $M(\theta) < 2\varepsilon$, and

$$\partial \theta = \frac{1}{\# A} \left( \sum_{x \in A} x - \sum_{y \in B} y \right).$$  \hspace{1cm} (11)

Remark 8. Equation (11) should be interpreted in the sense of boundaries of singular homology (cf. [5, §2.1]).
Proof. Let
\[ \varepsilon_0 = \frac{1}{2 \text{diam}_U U}. \]
Condition (10) implies that at least \((1 - \varepsilon^2)\# U_i \cap A\) points of \(A\) can be joined to points of \(B\) within \(U_i\). Since \(\text{diam}_U U_i < \varepsilon\), this can be done using curves \(\gamma\) of length smaller than \(\varepsilon\). Let \(\lambda_1\) be the chain formed by all those curves \(\gamma\), each parameterized at the right speed that its induced measure will be a probability, \(\|\gamma\|(TU) = 1\). The remaining \(\sim \varepsilon^2 \# U_i \cap A\) points of \(A\) (and a similar amount of points of \(B\)) need to be paired with points outside \(U_i\). Since \(\# A = \# B\), this is always possible, and it can be done using curves of length \(\leq \text{diam}_U U\). Let \(\lambda_2\) be the chain corresponding to these longer curves, again parameterized at a speed that will make the induced measure a probability.

We let \(\theta = (\lambda_1 + \lambda_2)/\# A\). It is clear then that \(\|\theta\|\) is a probability, and that (11) holds. We estimate
\[
M(\|\theta\|) = \frac{\text{arclength}(\lambda_1) + \text{arclength}(\lambda_2)}{\# A} \\
\leq \frac{\varepsilon(1 - \varepsilon^2)\# A + (\text{diam}_U U)\varepsilon^2 \# A}{\# A} \leq 2\varepsilon. \]

Let \(k \geq 1\) and let \(C\) be a set of properly nested simplices in \(T_k\). Decompose the chain \(\eta_k^C\) into the part of it that comes from \(\bar{\beta}_k^C\) and a remainder \(\zeta_k^C\),
\[
\eta_k^C = \bar{\beta}_k^C + \zeta_k^C.
\]

Fix \(k \in \mathbb{N}\) and a set \(C\) of \(n\) properly nested simplices. From the construction of \(\eta_k^C\), it follows that \(\zeta_k^C\) is formed from two types of components:

- Curves joining two points in the 0-chains \(\beta_k^C\); call the corresponding chain \(\zeta_{\text{short}}\).
- Curves joining points in various 0-chains \(\eta_k^{C'} (C' \supset C)\) that are not both already in \(\beta_k^C\); call the corresponding chain \(\zeta_{\text{long}}\).

Observe that as \(k \to \infty\), the first quotient in (2) behaves as
\[
\frac{\bar{\mu}_k(V_2)}{\mu_k(V_1)} \to 1
\]
since the triangulations \(T_k\) satisfy (16) and \(\mu\) is assumed to be a smooth density. So (2) tends to look like (10). It follows that if \(k\) is large, we can
apply Lemma 7 to a large subset of the points of $\partial \zeta_{\text{short}}$, with the conclusion that the part of $\zeta_{\text{short}}$ joining them has very small mass $M$. What remains of $\partial \zeta_{\text{short}}$ tends to have 0 weight, so the mass of the corresponding part of $\zeta_{\text{short}}$ also vanishes asymptotically.

Similarly, since $\bar{\mu}_k \rightarrow \mu$ as $k \rightarrow \infty$, and since the points $\{p_i^k\}_i$ are a sample of $\mu|_{E_k^{d-n}}$ (they satisfy (11)), the weight of $\partial \zeta_{\text{long}}$ vanishes asymptotically, and hence so does the mass of $\zeta_{\text{long}}$.

We let the function $a$ in $A^3$ decrease rapidly enough that the following lemma will hold as per the preceding argument.

**Lemma 9.** As $k \rightarrow \infty$,

$$\sum_C M(\zeta^C_k) \rightarrow 0 \quad \text{and} \quad \frac{\sum_C M(\eta^C_k)}{\sum_C M(\beta^C_k)} \rightarrow 1,$$

where the sums are taken over all sets $C$ of $n-1$ properly nested simplices in $T_k$.

### 3.5.3 Isoperimetric inequality

In this section we want to find an upper bound for the mass difference (8).

Recall the isoperimetric inequality:

**Proposition 10** (Federer [3 §4.2.10], [7 §5.3]). There is a constant $C_4 > 1$ such that if $\theta$ is an $m$-chain with $\partial \theta = 0$ and contained in a simplex $V$ of some triangulation $T_k$ and of diameter $\text{diam}_V V < 1$, then there exists an $(m+1)$-chain $\sigma$ with $\partial \sigma = \theta$ contained in $V$ and with mass bounded by

$$M(\sigma) \leq C_4 M(\theta)^{\frac{k+1}{k}}.$$

The original proposition is valid for chains $\theta$ in $\mathbb{R}^d$. It is true as stated because when we pullback a chain from $\mathbb{R}^d$ to $M$ via any of the functions $\varphi_V$, the modulus of continuity of these mappings is globally bounded. This in turn is true because there are only finitely many of them, and they have compact domains.

Let $k \geq 1$ and let $V_1$ be a $d$-dimensional simplex in $T_k$. Recall that the chains $\zeta^C_k$ were defined in Section 3.5.2. It follows from Lemma 9 and
Proposition 10 that we can take the cells in \( \zeta_k \) to be such that, as \( k \to \infty \),
\[
M(\zeta_k^{\{V_1\}}) \leq C_4 \sum_{V_2 \subset \partial V_1} M(\zeta_k^{\{V_1, V_2, V_3\}})^2 + \varepsilon_k^2 \\
\leq C_4^{1+\frac{2}{3}} \sum_{V_3 \subset \partial V_2} \sum_{V_2 \subset \partial V_1} M(\zeta_k^{\{V_1, V_2, V_3\}})^3 + \varepsilon_k^3 \\
\leq \cdots \leq C_4^{q_n} \sum_{V_{n-1} \subset \partial V_{n-2}} \cdots \sum_{V_2 \subset \partial V_1} M(\zeta_k^{\{V_1, V_2, \ldots, V_n\}})^{n-1} + \varepsilon_k^n \to 0,
\]
where \( q_n > 1 \) is some number depending only on \( n \), \( \varepsilon_k^\ell \) is arbitrarily small (it is the error we may get from not taking exactly the cell provided by Proposition 10 but one with slightly larger mass; we thus specify property E6 to mean that \( \varepsilon_k^\ell \to 0 \) as \( k \to \infty \) for all \( \ell \)), and the sums are taken over all simplices in the corresponding boundaries. We conclude

**Lemma 11.**
\[
|M(\eta_k) - M(\beta_k)| \to 0 \quad \text{as} \quad k \to \infty.
\]

### 3.6 Conclusion

**Proof of Theorem 1.** Let \( \mu \in \mathcal{M}_n \) be a positive measure. If \( \mu \) satisfies Condition (Cyc), it follows from Stokes's theorem that it also satisfies Condition (Hol).

To prove the other direction, assume that \( \mu \) satisfies Condition (Hol). By Lemma 4, we can assume that \( \mu \) is smooth. We can thus construct for \( k \geq 1 \) triangulations \( T_k \) as in Section 3.2, base measures \( \bar{\mu}_k \) as in Section 3.3.1, chains \( \beta_k \) approximating these as in Section 3.3.2, and cycles \( \eta_k \) as in Section 3.5.1 that contain \( \beta_k \). We have
\[
\text{dist}_\mathcal{M}(\mu, \eta_k) \leq \text{dist}_\mathcal{M}(\mu, \bar{\mu}_k) + \text{dist}_\mathcal{M}(\bar{\mu}_k, \lambda_k) + \text{dist}_\mathcal{M}(\lambda_k, \eta_k).
\]
The first two summands on the right-hand-side vanish asymptotically by construction. The last term, as per the definition of \( \text{dist}_\mathcal{M} \) in equation (1), has two parts: the mass difference, which tends to zero by Lemma 11, and the one involving the functions \( f_i \). The second one can be arranged to tend to zero by having the cells of \( \eta_k \) not present in \( \beta_k \) be parameterized at very high speeds, thus specifying property E7. We conclude that the measures induced by the cycles \( \eta_k \) indeed approximate \( \mu \), so \( \mu \) satisfies Condition (Cyc).

\[ \square \]
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