The parity-violating asymmetry in the $^3\text{He}(\bar{n},p)^3\text{H}$ reaction

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The longitudinal asymmetry induced by parity-violating (PV) components in the nucleon-nucleon potential is studied in the charge-exchange reaction $^3\text{He}(\bar{n},p)^3\text{H}$ at vanishing incident neutron energies. An expression for the PV observable is derived in terms of $T$-matrix elements for transitions from the $^{2S+1}L_J = 1^S_0$ and $^3S_1$ states in the incoming $n^3\text{He}$ channel to states with $J = 0$ and 1 in the outgoing $p^3\text{H}$ channel. The $T$-matrix elements involving PV transitions are obtained in first-order perturbation theory in the hadronic weak-interaction potential, while those connecting states of the same parity are derived from solutions of the strong-interaction Hamiltonian with the hyperspherical-harmonics method. The coupled-channel nature of the scattering problem is fully accounted for. Results are obtained corresponding to realistic or chiral two- and three-nucleon strong-interaction potentials in combination with either the DDH or pionless EFT model for the weak-interaction potential. The asymmetries, predicted with PV pion and vector-meson coupling constants corresponding (essentially) to the DDH “best values” set, range from $-9.44$ to $-2.48$ in units of $10^{-8}$, depending on the input strong-interaction Hamiltonian. This large model dependence is a consequence of cancellations between long-range (pion) and short-range (vector-meson) contributions, and is of course sensitive to the assumed values for the PV coupling constants.

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I. INTRODUCTION, RESULTS, AND CONCLUSIONS

A number of experiments aimed at studying parity violation in nuclei are currently being planned, including measurements of the neutron spin rotation in $\bar{n}$-$p$ radiative capture at thermal neutron energies. In the next phase, we have studied the spin rotation in $\bar{n}$-$d$ [9] and $\bar{n}$-$\alpha$ [10] scattering at cold neutron energies.

Measurements are available for the following PV observables: the longitudinal analyzing power in $\bar{n}$-$p$ [11]–[14] and $\bar{n}$-$\alpha$ [15] scattering, the photon asymmetry and photon circular polarization in, respectively, the $^1\text{H}(\bar{n}, \gamma)^2\text{H}$ [16]–[17] and $^1\text{H}(n, \gamma)^2\text{H}$ [18] radiative captures, and the neutron spin rotation in $\bar{n}$-$\alpha$ scattering [19, 20]. There is also a set of experiments which are currently being planned, including measurements of the neutron spin rotation in $\bar{n}$-$p$ [19] and $\bar{n}$-$d$ [21]...
Further discussion of this aspect of the calculations is in functions in order to achieve numerically stable results.

At vanishing neutron energies, the only channels entering the incoming $n$-$^3$He scattering state have quantum numbers $2S+1L_J = ^1S_0$ and $^3S_1$. In the outgoing $p$-$^3$H scattering state, the relevant channels are: $2S+1L_J = ^1S_0$, $^3S_1$, $^3P_1$ with positive parity, and $^3P_0$, $^1P_1$, $^2P_1$ with negative parity. We show (in Sec. II) that the PV observable in this process, i.e. the longitudinal analyzing power $A_z$, reads

$$A_z = a_z \cos \theta,$$

where $\theta$ is the angle between the proton momentum and the neutron beam direction, and the coefficient $a_z$ can be expressed in terms of products of $T$-matrix elements involving (three) parity-conserving (PC) and (three) PV transitions as

$$a_z = -\frac{4}{\Sigma} \text{Re} \left( \sqrt{3} T_{21,0}^{21,1} - T_{00,0}^{21,0} - T_{01,0}^{21,1} + \sqrt{2} T_{00,11}^{21,0} T_{01,01}^{21,1} + \sqrt{2} T_{01,11}^{21,1} T_{01,01}^{21,1} + \sqrt{3} T_{01,11}^{21,1} T_{01,21}^{21,1} \right),$$

and

$$\Sigma = \left| T_{00,0}^{21,0} \right|^2 + 3 \left| T_{01,0}^{21,1} \right|^2 + 3 \left| T_{01,21}^{21,1} \right|^2. \quad (3)$$

In $T_{LS,L'S'}^{J\ell}$, the label $J$ specifies the total angular momentum, the superscripts $21$ denote the charge-exchange transition $n$-$^3$He to $p$-$^3$H (as opposed, for example, to the elastic transition, which would be denoted by the superscripts $22$), the subscripts $LS (L'S')$ are the relative orbital angular momentum and channel spin of the $n$-$^3$He ($p$-$^3$H) clusters, and lastly the overline is to note the inclusion of a convenient phase factor—see Eq. (16) below. The PC (PV) $T$-matrix elements have $L + L'$ even (odd), and the sum $\Sigma$ in Eq. (3) is proportional to the $^3\text{He}(n,p)^3\text{H}$ cross section. We observe that $a_z$ vanishes if only the channels $^1S_0$ and $^3P_0$ (with $J = 0$) are retained.

The $T$-matrix elements are related to the (real) $R$-matrix elements (Sec. III and Appendix A), and the latter for PC transitions are calculated via the Kohn variational principle with the hyperspherical-harmonics (HH) method [23, 24] (Sec. V). We use strong Hamiltonian models, consisting of the Argonne $\nu\delta$ (AV18) [25] or chiral (N3LO) [26] two-nucleon potential in combination with the Urbana IX (UIX) [27] or chiral (N2LO) [28] three-nucleon potential. The HH calculation is a challenging one, for two reasons: the first is the coupled-channel nature of the scattering problem: even at vanishing energies for the incident neutron, the elastic $n$-$^3$He and charge-exchange $p$-$^3$H channels are both open. The second is the presence of a $J^\pi = 0^+$ resonant state (of zero total isospin) between the $p$-$^3$H and $n$-$^3$He thresholds, which slows down the convergence of the expansion, and requires a large number of HH basis functions in order to achieve numerically stable results. Further discussion of this aspect of the calculations is in Sec. V, where we also present current predictions for the $n$-$^3$He scattering lengths corresponding to the Hamiltonian models mentioned earlier. They are in very good agreement with the measured values.

The $R$-matrix elements involving PV transitions are computed in first-order perturbation theory with Quantum Monte Carlo techniques (Sec. VI). We adopt as PV potential the meson-exchange (DDH) model of Desplanques et al. [29] as well as the pionless effective-field-theory (EFT) model recently derived in Refs. [30, 31] (Sec. IV), and present results for the various components of the DDH and EFT potentials in combination with the AV18, AV18/UIX, N3LO, and N3LO/N2LO Hamiltonians in Sec. VII. Additional results for the $R$- and $T$-matrix elements, and combinations thereof entering the PV observable, are listed (for the AV18/UIX) in Appendix B for completeness. For the DDH model only, we also present predictions for $a_z$ corresponding essentially—but see Sec. IV for further details—to the “best values” of the $\pi$, $\rho$, and $\omega$-meson weak-interaction coupling constants [29]. These predictions range from $-9.44$ to $-2.48$ in units of $10^{-8}$ depending on whether the N3LO/N2LO or AV18/UIX Hamiltonian is considered, and thus exhibit a significant model dependence due to cancellations (or lack thereof) between the pion and vector-meson contributions.

It is useful to express the asymmetry as

$$a_z = h_\pi^1 C_\pi^1 + h_\rho^0 C_\rho^0 + h_\rho^1 C_\rho^1 + h_\omega^2 C_\omega^2 + h_\omega^0 C_\omega^0 + h_\omega^1 C_\omega^1,$$

where the $h_i^j$’s, $\alpha = \pi, \rho, \omega$ and $i = 0, 1, 2$, denote the PV coupling constants in the DDH model along with the isospin content of the corresponding interaction. The coefficients $C_\alpha^i$ are listed in Table I, and depend on the input Hamiltonian used to generate the continuum wave functions, as well as on the assumed values for the PC pion- and vector-meson coupling constants and associated cutoffs (see Table IV).
The coefficients $C_{i}^{\alpha}$ follow from the linear combination given in Eq. (2). Isotensor $\rho$-exchange ($C_{\rho}^{2}$) is negligible. The isoscalar and isovector vector-meson exchanges give contributions of the same magnitude, both of which are smaller than OPE. However, the OPE contribution seems to be significantly suppressed. For example, in the case of the neutron spin rotation in $\bar{n}d$ scattering this contribution is calculated to be at least a factor of $\sim 30$ larger than that of any of the $\rho$ and $\omega$ exchanges, which is not the case for the process under consideration. This may be due to the predominant isoscalar character of the $1S_{0}$ and $3P_{0}$ channels—see discussion in Appendix B. The N3LO/N2LO results should be considered as preliminary, since the HH solution for the $0^{+}$ wave function has not yet fully converged (at least as far as the singlet scattering length is concerned, see Sec. V). This fact may explain why the inclusion of a three-nucleon potential like N2LO [28] should reduce $C_{4}^{2}$ by almost a factor of two relative to the other models. This point will be discussed in Secs. V and VI. Finally we note that the “best values” for the PV couplings constants of the pion and $\rho$-meson are (in units of $10^{-7}$) respectively $+4.56$ and $-16.4$, and this leads to the large cancellation (and consequent model dependence) in the values predicted for $a_{z}$ and referred to earlier.

We conclude by observing that the EFT analysis presented in this work could be improved by employing chiral potentials in both the strong- and weak-interaction sectors. At order $Q/{\Lambda}_{\chi}$, where $Q$ is the low energy/momentum scale that characterizes the particular process of interest, and $\Lambda_{\chi} \simeq 1$ GeV is the chiral-symmetry-breaking scale, the PV potential contains 7 low-energy constants (LECs), 5 of which are associated with four-nucleon contact terms, and the remaining 2 with long-range OPE components [30]. When electromagnetic interactions are also introduced, another (unknown) LEC must be included—it is needed to fix the strength of a PV two-body current operator of pion range [30]. One can envisage, at least in principle, a suite of experiments involving $A = 2-5$ systems, which would constrain, in fact over-constrain, these eight LECs. Some of these have been mentioned above, additional ones include, for example, measurements of the photon asymmetries in the radiative captures $^{2}H(\bar{n}, \gamma)^{3}H$ and $^{3}He(\bar{n}, \gamma)^{4}He$. These processes are strongly suppressed:

The experimental values for the corresponding (PC) cross sections [32, 33] are, respectively, almost 3 and 4 orders of magnitude smaller than measured in $^{3}H(u, \gamma)^{3}H$. One would naively expect relatively large PV asymmetries in these cases, possibly orders of magnitude larger than in the $A=2$ system. Clearly, accurate theoretical estimates for them could be useful in motivating our experimental colleagues to carry out these extremely challenging measurements.

From a theoretical perspective, most of the methodological and technical developments needed to carry out the calculations are already in place. We have recently reported results [34] for the $A = 3$ and 4 (PC) captures, using wave functions obtained from the N3LO/N2LO Hamiltonian and electromagnetic currents derived in chiral EFT up to one loop [35], which are in excellent agreement with data. However, there is one aspect in the computation of the proposed PV threshold captures, which still needs to be addressed: the determination of the small admixtures induced by the PV potential into the bound and continuum wave functions. Even a first-order perturbative treatment of those admixtures requires construction of the full Green’s function for the strong (PC) Hamiltonian, an impractical task. However, it may be possible to generate them using correlated basis methods, similar to those employed in Ref. [36].

### II. THE PARITY-VIOLATING OBSERVABLE

The neutron energies in the reaction $^{3}He(\bar{n}, p)^{3}H$ of interest here are in the meV range, and at these energies only two channels are open: the $n^{3}He$ elastic channel and the $p^{3}H$ charge-exchange channel. In the following, the index $\gamma=1$ (2) is used to identify the $p^{3}H$ ($n^{3}He$) clusters in the final (initial) state. In the absence of strong and weak interactions, the wave function in channel $\gamma$ is written as

$$\Phi_{\gamma}^{m_{1} m_{1}} = \frac{1}{\sqrt{4}} \sum_{p=1}^{4} \Psi_{\gamma}^{m_{1}}(i,j,k) \chi_{\gamma}^{m_{1}}(l) \phi_{q_{p}}(y_{p})$$

$$= \frac{1}{\sqrt{4}} \sum_{p=1}^{4} \Phi_{\gamma}^{m_{1}}(i,j,k),$$

(5)
where \( \Psi_{\gamma}^{m_3} \) is the (antisymmetrized) trinucleon bound-state wave function in spin projection \( m_3 \), \( \chi_{\gamma}^{m_1} \) is the trinucleon spin-isospin state with spin and isospin projections \( m_1 \) and \( p \) for \( \gamma = 1 \) or \( n \) for \( \gamma = 2 \), respectively, and \( \phi \) is the inter-cluster wave function, i.e., a Coulomb wave function for \( \gamma = 1 \) or simply a plane wave \( e^{i\mathbf{q}_p \cdot \mathbf{r}_p} \) for \( \gamma = 2 \).

The separation between the center-of-mass positions of the two clusters is denoted by \( \mathbf{y}_p \) with \( \mathbf{y}_p = \mathbf{r}_i - \mathbf{R}_{ijk} \), and their relative momentum is specified by \( \mathbf{q}_\gamma \), so that the energy \( E \) is given by

\[
E = -B_\gamma + \frac{q_\gamma^2}{2\mu_\gamma}, \quad \frac{1}{\mu_\gamma} = \frac{1}{m_\gamma} + \frac{1}{M_\gamma}.
\]  

(6)

Here \( B_\gamma \) and \( M_\gamma \) are the binding energy and mass of \(^3\text{He}\) for \( \gamma = 1 \) \((^3\text{He})\), and \( m_\gamma \) is the proton (neutron) mass for \( \gamma = 1 \) \((^3\text{He})\). Lastly, the wave functions in Eq. (5) are antisymmetrized by summing over the four permutations \( p \) with \((ijk,l) \equiv (123,4), (124,3), (134,2), \) and \((234,1)\).

It is useful to expand the wave functions in Eq. (5) in partial waves as

\[
\Psi_{\gamma}^{m_3 m_1} = \frac{1}{\sqrt{4}} \sum_{p=1}^{4} \sum_{LSJ} i^L Z_{\gamma m_3 m_1}^{L0SJJ} \Omega_{\gamma LSJ p} \rho L(q_\gamma y_p),
\]  

(7)

and contains outgoing spherical waves in the \( n-^3\text{He} \) elastic channel \((\gamma = 2)\) as well as in the \( p-^3\text{He} \) charge-exchange channel \((\gamma = 1)\) multiplied by corresponding \( T \)-matrix elements \( T_{LSJ}^{\gammaijJ} \). Here \( \gamma = \alpha m_1/\mu_1 \), where \( \alpha \) is the fine structure constant and \( \mu_1 \) is the \(^3\text{He} \) reduced mass defined above, and \( \sigma_L \) is the Coulomb phase shift. Thus Coulomb distortion in the \(^3\text{He} \) outgoing state is fully accounted for.

The probability amplitude \( M_{m_3' m_1', m_3 m_1} \) to observe a \( p-^3\text{He} \) final state with spin projections \( m_1' \) and \( m_3' \), respectively, is obtained from

\[
\langle \Psi_{\gamma=1, p=1}^{m_3' m_1'} | \Psi_{\gamma=2}^{m_3 m_1} \rangle = \frac{1}{\sqrt{4}} \sum_{p=1}^{4} \sum_{LSJ} i^L Z_{m_3 m_1}^{L0SJJ} \rho_{LSJ p} (q_\gamma y_p) e^{i[q_\gamma y - q_1 y + \ln(2q_1 y)]},
\]  

(11)

where we have assumed that the \( p-^3\text{He} \) state is in partition \((123,4)\) corresponding to permutation \( p = 1 \), namely the bound cluster consists of particles 123 and the proton is particle 4. For brevity, we have also set \( y \equiv y_{p=1} \). Using the orthonormality of the channel functions \( \Omega_{\gamma LSJ p} \), we find

\[
M_{m_3' m_1', m_3 m_1} = \frac{1}{\sqrt{4\pi}} \sum_{LSJ} \rho_{LSJ p} (q_\gamma y_p) e^{i[q_\gamma y - q_1 y + \ln(2q_1 y)]} Z_{m_3 m_1}^{L0SJJ} \rho_{LSJ p} (q_\gamma y_p) e^{i[q_\gamma y - q_1 y + \ln(2q_1 y)]}
\]  

(12)

where the Clebsch-Gordan coefficients require \( J_z = S_z = m_3 + m_1, S'_z = m'_3 + m'_1, \) and \( M' = J_z - S'_z = m_3 + m_1 - (m'_3 + m'_1) \).

The spin-averaged cross section follows from

\[
\sigma_0 \equiv \frac{d\sigma}{d\Omega} = \frac{1}{4} \frac{\mu_2}{\mu_3} \sum_{m_3 m_1 m_3' m_1'} |M_{m_3' m_1', m_3 m_1}|^2 d\Omega,
\]  

(13)

since \((1/4)(q_1/\mu_1)|M_{m_3' m_1', m_3 m_1}|^2 d\Omega\) is the flux of outgoing particles in the solid angle \( d\Omega \equiv d\mathbf{y} \), and \((1/4)(q_2/\mu_2)\) is the incident flux, where the factors 1/4...
 originate from the normalization factors $1/\sqrt{2}$ in Eqs. (5) and (11). These cancel out in Eq. (13), leaving an extra $1/4$ coming from the average over the initial polarizations. The longitudinal asymmetry $A_z$ is defined as

$$
\sigma_0 A_z = \frac{1}{2} \mu \frac{q_1}{q_2} \sum \sum \frac{|M_{m_3 m_1}^* m_3 m_1 + 1/2|^2}{|M_{m_3 m_1}^* m_3 m_1 - 1/2|^2}.
$$

(14)

At meV energies it suffices to keep only $L = 0$ in the entrance channel, so that

$$
M_{m'_3 m'_1, m_3 m_1} = \sum \sum (1/2, m_3; 1/2, m_1 | J, J_z)
$$

$$
\times \frac{T_{0J, J_z}^{21, J}}{2L + 1} 2_{m'_3 m'_1}^{J, J_z} Y^{L'M'}_1 (\hat{y}) ,
$$

(15)

where we have defined

$$
T_{0J, J_z}^{21, J} = (-i)^L e^{i\sigma} T_{21, J}^{21, J} .
$$

(16)

After inserting the expression for $Z_{m_3 m'_3, J_z}^{L}M'(1)M'$ and carrying out the sums over $m_1, m_3$ and $m'_1, m'_3$, we find the unpolarized cross section to be given by

$$
\sigma_0 = \frac{1}{4} \mu \frac{q_1}{q_2} \sum \sum \left( 2J + 1 \right) \left| T_{00, 00}^{21, J} \right|^2
$$

$$
= \frac{1}{4} \mu \frac{q_1}{q_2} \sum \sum \left[ T_{21, 00, 00}^{21, J} \right]^2 + 3 \left| T_{21, 01, 01}^{21, J} \right|^2 + 3 \left| T_{21, 11, 11}^{21, J} \right|^2 ,
$$

(17)

where in the second line we have ignored $T$-matrix elements involving transitions to odd parity final states (and hence parity violating), since these are induced by hadronic weak interactions and consequently are much smaller than the parity-conserving $T$-matrices associated with strong interactions. We observe that the matrix elements $T_{21, J}$ and $T_{21, -J}$ are finite in the limit $q_2 = 0$, and therefore $\sigma_0$ is divergent as $q_2$ goes to zero, as expected for a neutron capture reaction.

The asymmetry $A_z$ can be written as

$$
\sigma_0 A_z = \frac{1}{2} \mu \frac{q_1}{q_2} \sum \sum \sum \sum \sigma_{L_1 L_2 S}^{21, J} M_{L_1 L_2 S}^{21, J} (\theta) P_{L_1}^{|M|} (\theta) P_{L_2}^{|M|} (\theta) ,
$$

(18)

where the $P_{L_1}^{|M|} (\theta)$’s are associated Legendre functions, $\theta$ is the angle of the outgoing proton momentum relative to the direction of the incident beam, the $C_{L_1, L_2 S}^{J_1 J_2} (|M|)$’s denote combinations of Clebsch-Gordan coefficients, defined as

$$
C_{L_1, L_2 S}^{J_1 J_2} (|M|) = \frac{1}{2\pi} \sum_{L_{z_1}, L_{z_2}} \sum_{M_{L_1, L_2}} \left[ \frac{(L_1 - |M|)(L_2 - |M|)}{(L_1 + |M|)(L_2 + |M|)} \right]
$$

$$
\times \frac{Z_{L_{z_1} S_{1}, J_{1}} L_{1} M_{1}}{2^{L_{1} - m_{1}} m_{1} + 1/2} \frac{Z_{L_{z_2} S_{2}, J_{2}} L_{2} M_{2}}{2^{L_{2} - m_{2}} m_{2} + 1/2} ,
$$

(19)

and lastly the phase factor $\epsilon_{L_1 L_2}$,

$$
\epsilon_{L_1 L_2} = \frac{1 - (-1)^{L_1 + L_2}}{2} ,
$$

(20)

ensures that either $L_1$ or $L_2$ must be odd, which in turn implies that either $T_{0J, L_1 S}^{21, J}$ or $T_{0J, L_2 S}^{21, J}$ involves a parity-violating transition, i.e. a transition from an incoming positive parity $p^{-3}\text{He}$ state to an outgoing negative parity $p^{-3}\text{H}$ state. The non-vanishing $C$’s for the relevant channels are listed in Table II, and evaluation of the sums in Eq. (18) allows one to express the parity-violating asymmetry as in Eqs.(1)–(3).

### III. T-MATRIX ELEMENTS

The calculation proceeds in two steps: we first determine, via the Kohn variational principle, the $R$-matrix elements, and then relate these to the $T$-matrix elements. The wave function describing a scattering state with total angular momentum $J J_z$ in channel $\gamma L S$ is written as

$$
\Psi_{\gamma, L S} = \Psi_{\gamma, L S}^C + \sum_{\gamma' L' S'} R_{\gamma' L' S'}^{\gamma, L S} \Psi_{\gamma' L' S'}^{G, J J_z} ,
$$

(21)

where the asymptotic wave functions $\Psi_{\gamma, L S}^C$ with $\lambda = F, G$ are defined as

$$
\Psi_{\gamma, L S}^{\lambda, J J_z} = \frac{D}{\sqrt{V}} \sum_{p=1}^{4} O_{\gamma J_z L S}^{\lambda} \frac{F_{\lambda}(q_x, y_p)}{q_x y_p} ,
$$

(22)

and the superscript $\lambda$ is $F$ or $G$, depending on whether the regular or irregular radial function is considered, namely

$$
\gamma = 1 \quad F_{\lambda}(x) = F_L(q_x, x) \quad F_{\lambda}(x) = G_L(q_x, x) ,
$$

(23)

$$
\gamma = 2 \quad F_{\lambda}(x) = x^{\lambda} L(x) \quad F_{\lambda}(x) = -x \tilde{y} L(x) ,
$$

(24)
and \( F_L/G_L \) and \( j_L/y_L \) denote the regular/irregular Coulomb and spherical Bessel functions, respectively. The tilde over \( G_L \) and \( y_L \) indicates that they have been multiplied by short-range cutoffs in order to remove the singularity at the origin. Thus \( F_L \) is well-behaved in all space. The normalization factor \( D_\gamma \),

\[
D_\gamma = \sqrt{\frac{2\mu_j q_j}{\kappa^3}}
\]  

(25)

and \( \kappa = \sqrt{3/2} \), is introduced for convenience—\( \kappa \) is a numerical factor relating the inter-cluster separation \( y_p \) to the Jacobi variable \( x_{1,p} \), i.e. \( x_{1,p} = \kappa y_p \).

The wave functions \( \Psi_{\gamma,LS}^{C,J_2} \) vanish in the asymptotic region, and describe the dynamics of the interacting nucleons when they are close to each other, while the \( R_{LS,L'S'}^{1/2,1/2} \)'s are the \( R \)-matrix elements. The latter, as well as the coefficients entering the expansion of \( \Psi_{\gamma,LS}^{C,J_2} \) in terms of hyperspherical-harmonics functions, are determined via the Kohn variational principle

\[
\left[ R_{LS,L'S'}^{\gamma,J} \right] = \left[ \Psi_{\gamma,LS}^{C,J_2} | H - E | \Psi_{\gamma,L'S'}^{C,J_2} \right],
\]  

(26)

where \( \Psi^C \) is a combination of internal parts of no interest here. We now require \( \Psi \) to consist, in the asymptotic region, of a plane wave in channel \( \gamma=2 \) (or \( n^{-3}\text{He} \)) and of a purely outgoing wave in channel \( \gamma=1 \) (or \( p^{-3}\text{H} \)). These requirements are satisfied by demanding that

\[
\begin{align*}
U - i \left( UR^{11} + VR^{21} \right) &= 0, \\
V - i \left( UR^{12} + VR^{22} \right) &= I,
\end{align*}
\]  

(31)

(32)

where \( I \) is the identity matrix. Comparing the resulting \( \Psi \) with the wave function given in Eq. (10), specifically as discussed in Sec. V.

The next step consists in relating the \( R \)- to the \( T \)-matrix elements. To this end, it is convenient to simplify the notation by dropping the superscripts \( JJ_2 \) and by introducing a single label \( \alpha \) to denote the channel quantum numbers \( LS \), so that the wave functions in Eq. (21) corresponding to \( \gamma = 1 \) and 2 are written as

\[
\begin{align*}
\Psi_{1,\alpha} &= \Psi_{1,\alpha}^C + \Psi_{1,\alpha}^F + \sum_{\alpha'} R_{\alpha\alpha'}^{11} \Psi_{1,\alpha'}^C + \sum_{\alpha'} R_{\alpha\alpha'}^{12} \Psi_{1,\alpha'}^G, \\
\Psi_{2,\alpha} &= \Psi_{2,\alpha}^C + \Psi_{2,\alpha}^F + \sum_{\alpha'} R_{\alpha\alpha'}^{21} \Psi_{1,\alpha'}^C + \sum_{\alpha'} R_{\alpha\alpha'}^{22} \Psi_{1,\alpha'}^G.
\end{align*}
\]  

(27)

(28)

From these we form the linear combination

\[
\Psi = \sum_\alpha \left( U_{\alpha,\alpha'} \Psi_{1,\alpha} + V_{\alpha,\alpha'} \Psi_{2,\alpha} \right),
\]  

(29)

where the matrices \( U \) and \( V \) are determined below. Inserting the expressions above for \( \Psi_{\gamma,\alpha} \) and rearranging terms lead to

\[
\begin{align*}
\Psi &= \Psi^C + \sum_{\alpha'} \left[ U - i (UR^{11} + VR^{21}) \right]_{\alpha,\alpha'} \Psi_{1,\alpha'}^C + \sum_{\alpha'} (UR^{11} + VR^{21})_{\alpha,\alpha'} \left( \Psi_{1,\alpha'}^G + i \Psi_{1,\alpha'}^F \right) \\
&\quad + \sum_{\alpha'} \left[ V - i (UR^{12} + VR^{22}) \right]_{\alpha,\alpha'} \Psi_{2,\alpha'}^C + \sum_{\alpha'} (UR^{12} + VR^{22})_{\alpha,\alpha'} \left( \Psi_{2,\alpha'}^G + i \Psi_{2,\alpha'}^F \right),
\end{align*}
\]  

(30)

its component in channel \( LSJ \), allows one to express the \( T \)-matrix as

\[
T_{LS,L'S'}^{21,J} = \frac{D_1}{D_2 q_1} \left( U^J R^{11,J} + V^J R^{21,J} \right)_{LS,L'S'}
\]  

\[
= -i \frac{D_1}{D_2 q_1} U^J_{LS,L'S'},
\]  

(33)

where we have reinstated the \( LSJ \) labels. Finally the matrix \( U \) is obtained by solving the system in Eq. (32):

\[
T_{LS,L'S'}^{21,J} = \frac{D_1}{D_2 q_1} \left[ I - i R^{22,J} + R^{21,J} (I - i R^{11,J})^{-1} R^{12,J} \right]^{-1} R^{21,J} (I - i R^{11,J})^{-1}_{LS,L'S'}.
\]  

(34)

In fact, we compute the \( R \)-matrix elements at zero energy, i.e. in the limit \( q \to 0 \), and define

\[
R_{LS,L'S'}^{21,J} = \frac{R_{LS,L'S'}^{12,J}}{q_2^{J+1/2}}, \quad R_{LS,L'S'}^{21,J} = \frac{R_{LS,L'S'}^{21,J}}{q_2^{J+1/2}}.
\]
TABLE III: Components of the DDH and EFT models for the parity-violating potential. The vector operators $X_{ij}^{(n)}$ and functions $f_x(r)$, $x = \pi, \rho, \omega, \mu$, are defined in Eqs. (39)–(40) and Eqs. (41)–(42), respectively. Only 5 operators and low-energy constants enter the pioneer EFT interaction at the leading order, and in this paper they have been chosen to correspond to the rows 1, 4, 6, 8 and 9.

$$\frac{R_{LS,L'S'}^{22,J}}{q_2^{1/2}} = \frac{R_{LS,L'S'}^{22,J}}{q_2^{1/2}}$$

and it can be shown that the $R$-matrix elements are finite in this limit. In particular, we note that the factor $q_2^{1/2}$ follows from the small argument expansion of the spherical Bessel function $j_L$ in $\Psi_{J,L}$, while the extra $q_2^{1/2}$ is due to the normalization $D_2$. At zero energy, we have

$$\left[I - i R^{22,J} + R^{21,J}(I - i R^{11,J})^{-1} R^{12,J}\right] \rightarrow I,$$

since $R^{22,J}$ and the product $R^{21,J} R^{12,J}$ are proportional to $q_2$ or higher powers of $q_2$. Furthermore, the relevant $T$-matrix elements entering the expression for the asymmetry $A_2$ are those with quantum number $L = 0$ in channel $\gamma = 2$, and hence

$$T_{0,L,L'S'}^{21,J} = \frac{1}{\sqrt{2}} \sum_{L',S'} T_{0,L',L'S'}^{21,J} (I - i R^{11,J})^{-1} R_{0,L'L'S'}^{11,J},$$

with $J = 0, 1$. Note that we have neglected the difference in the $n^3$He and $p^3$H reduced masses.

IV. THE PARITY-VIOLATING POTENTIAL

Two different models of the PW weak-interaction potentials are adopted in the calculations reported below.

One is the model developed thirty years ago by Desplanques et al. [29] (and known as DDH): it is parametrized in terms of $\pi$, $\rho$, and $\omega$-meson exchanges, and involves in practice six weak pion and vector-meson coupling constants to the nucleon [37]. These were estimated within a quark model approach incorporating symmetry arguments and current algebra requirements [29, 38]. Due to the inherent limitations of such an analysis, however, the coupling constants determined in this way have rather wide ranges of allowed values.

The other model for the PV potential considered in the present work is that formulated by Zhu et al. [30] in 2005, and revised by Girlanda [31] in 2008, within an effective-field-theory (EFT) approach in which only nucleon degrees of freedom are retained explicitly. At lowest order $Q/\Lambda_{QCD}$, where $Q$ is the small momentum scale characterizing the low-energy PV process and $\Lambda_{QCD} \approx 1$ GeV is the scale of chiral symmetry breaking, it is parametrized by a set of five contact four-nucleon terms.

The DDH and EFT PV two-nucleon potentials are conveniently written as

$$n_{ij}^{(n)} = \sum_{n=1}^{12} c_{ij}^{(n)} O_{ij}^{(n)}, \quad \alpha = \text{DDH or EFT},$$

where the parameters $c_{ij}^{(n)}$ and operators $O_{ij}^{(n)}$, $n = 1, \ldots, 12$, are listed in Table III. In this table the vector operators $X_{ij,k}^{(n)}$ are defined as
\[ X_{ij,+}^{(n)} = [p_{ij}, f_n(r_{ij})]_+ , \]  
\[ X_{ij,-}^{(n)} = i [p_{ij}, f_n(r_{ij})]_- , \]

where \([\ldots, \ldots, \ldots]_\pm\) denotes the commutator (-) or anti-commutator (+), and \(p_{ij}\) is the relative momentum operator, \(p_{ij} \equiv (\vec{p}_i - \vec{p}_j)/2\). In the DDH model, the functions \(f_x(r), x = \pi, \rho, \omega\), are Yukawa functions, suitably modified by the inclusion of monopole form factors,

\[ f_x(r) = \frac{1}{4\pi r} \left\{ e^{-m_{x} r} - e^{-\Lambda_{x} r} \left[ 1 + \frac{\Lambda_{x} r}{2} \left( 1 - \frac{m_{x}^2}{\Lambda_{x}^2} \right) \right] \right\} . \]  

In the EFT model, however, the short-distance behavior is described by a single function \(f_\mu(r)\), which is itself taken as a Yukawa function with mass parameter \(\mu\),

\[ f_\mu(r) = \frac{1}{4\pi r} e^{-\mu r} , \]

with \(\mu \simeq m_\pi\) as appropriate in the present formulation, in which pion degrees of freedom are integrated out.

In the potential \(v^{\text{DDH}}_{ij}\), the strong-interaction coupling constants of the \(\pi, \rho, \omega\)-meson to the nucleon are denoted as \(g_\pi, g_\rho, \kappa_\rho, g_\omega, \kappa_\omega\), while the weak-interaction ones as \(h_1^\pi, h_0^\rho, h_1^\rho, h_2^\rho, h_0^\omega, h_1^\omega\), where the superscripts 0, 1, and 2 specify the isoscalar, isovector, and isotensor content of the corresponding interaction components. In the EFT model, the five low-energy constants \(C_1, C_1, C_2 + C_4, C_5\) and \(C_6\) completely characterize \(v^{\text{EFT}}\), to lowest order \(Q/\Lambda_x\).

| \[g^_/4\pi\] | \(\kappa_\rho\) | \(10^\dagger \times h_0^\pi\) | \(10^\dagger \times h_1^\pi\) | \(10^\dagger \times h_2^\pi\) | \(\Lambda_\pi\) (GeV/c) |
|---|---|---|---|---|---|
| \(\pi\) | 13.2 | 4.56 | 4.56 | 4.56 | 1.72 |
| \(\rho\) | 0.840 | 6.1 | -16.4 | -2.77 | -13.7 | 1.31 |
| \(\omega\) | 20.0 | 0.0 | 3.23 | 1.94 | 1.50 |

TABLE IV: Values used for the strong- and weak-interaction coupling constants and short-range cutoff parameters of the \(\pi, \rho, \omega\)-meson in the DDH potential.

The values for the coupling constants and short-range cutoffs in the DDH model are listed in Table IV, while the mass \(\mu\) in the EFT model is taken to be \(m_\pi\). These values for coupling constants and cutoffs were also used in the DDH-based calculations of PV two-nucleon observables in Refs. [7, 8] and neutron spin rotation in \(n\bar{d}\) scattering [9]. In particular, we note that the linear combination of \(p\)- and \(\omega\)-meson weak coupling constants corresponding to \(pp\) states has been taken from an earlier analysis of \(\vec{p}\vec{p}\) elastic scattering experiments [7]. The remaining couplings are the “best value” estimates, suggested in Ref. [29].

In the analysis of the \(a_z\) observable to follow, we will report results for the coefficients \(I_n^{\text{DDH}}\) and \(I_n^{\text{EFT}}\) in the expansion

\[ a_z = \sum_{n=1}^{12} c_n^{\sigma} I_n^{\sigma} . \]  

Thus we will not need to consider specific values (or range of values) for the strength parameters \(c_n^{\sigma}\). However, the \(I_n^{\sigma}\) depend on the masses (and short-range cutoffs \(\Lambda_x\) for the DDH model) occurring in the Yukawa functions. Note that the coefficients \(c_n^{\sigma}\) entering Eq. (4) are obtained from the \(I_n^{\text{DDH}}\)’s and \(I_n^{\text{EFT}}\)’s listed in Table III via

\[ C_1^{\sigma} = \frac{g_{\pi}^{\sigma}}{2\sqrt{2}m} I_1^{\text{DDH}} , \]  
\[ C_0^{\sigma} = \frac{g_{\rho}^{\sigma}}{m} I_0^{\text{DDH}} - \frac{g_{\rho}^{\sigma}(1 + \kappa_\rho)}{m} I_3^{\text{DDH}} , \]  
\[ C_1^{\rho} = -\frac{g_{\rho}^{\rho}}{2m} I_1^{\text{DDH}} - \frac{g_{\rho}^{\rho}(1 + \kappa_\rho)}{2m} I_5^{\text{DDH}} + \frac{g_{\rho}^{\rho}}{2m} I_1^{\text{DDH}} , \]  
\[ C_1^{\omega} = -\frac{g_{\omega}^{\omega}}{m} I_1^{\text{DDH}} - \frac{g_{\omega}^{\omega}(1 + \kappa_\omega)}{m} I_9^{\text{DDH}} , \]  
\[ C_1^{\omega} = -\frac{g_{\omega}^{\omega}}{2m} I_1^{\text{DDH}} - \frac{g_{\omega}^{\omega}(1 + \kappa_\omega)}{2m} I_1^{\text{DDH}} - \frac{g_{\omega}^{\omega}}{2m} I_1^{\text{DDH}} . \]

V. THE HH WAVE FUNCTIONS

The “internal” wave function \(\Psi_{C, J, \mathcal{J}}^{I, I_F}\) seen Eq. (21), is expanded in the HH basis. For four equal mass particles, a suitable choice for the Jacobi vectors is

\[ x_{1p} = \sqrt{\frac{3}{2}} \left( r_l - r_j + r_k \right) , \]  
\[ x_{2p} = \sqrt{\frac{4}{3}} \left( r_k - r_l + r_j \right) , \]  
\[ x_{3p} = r_j - r_l , \]

where \(p\) specifies a given permutation corresponding to the ordering \((ijkl)\). By definition, the permutation \(p = 1\) is chosen to correspond to \((1234)\).

For the given Jacobi vectors, the hyperspherical coordinates include the so-called hyperradius \(\rho\), defined by

\[ \rho = \sqrt{x_{1p}^2 + x_{2p}^2 + x_{3p}^2} \quad \text{(independent of } p) , \]

and a set of angular variables which in the Zemnike and Brinkman [39, 40] representation are (i) the polar angles \(x_{p} = (\theta_{1p}, \phi_{1p})\) of each Jacobi vector, and (ii) the two additional “hyperspherical” angles \(\phi_2p\) and \(\phi_3p\), defined as

\[ \cos \phi_2p = \frac{x_{2p}}{\sqrt{x_{1p}^2 + x_{2p}^2}} , \]  
\[ \cos \phi_3p = \frac{x_{3p}}{\sqrt{x_{1p}^2 + x_{2p}^2 + x_{3p}^2}} , \]

where \(x_{jp}\) is the magnitude of the Jacobi vector \(x_{jp}\). The set of angular variables \(x_{1p}, x_{2p}, x_{3p}, \phi_2p, \phi_3p\), and \(x_{jp}\) is denoted hereafter as \(\Omega_p\). A generic HH function reads
\[ \mathcal{N}_{\ell_1\ell_2\ell_3L_{2n_2n_3}}(\Omega_p) = N_{n_2n_3}^{\ell_1\ell_2\ell_3} \left[ Y_{\ell_1}(\hat{x}_{1p}) \otimes Y_{\ell_2}(\hat{x}_{2p}) \right] \otimes Y_{\ell_3}(\hat{x}_{3p}) \sin^{\ell_1} \phi_{2p} \cos^{\ell_2} \phi_{2p} \sin^{\ell_1+\ell_2+2n_2} \phi_{3p} \cos^{\ell_3} \phi_{3p} \times P_{\ell_2+1/2, \ell_1+1/2}^{n_2+1/2, \ell_3+1/2}(\cos 2\phi_{2p}) P_{\ell_3+1/2, \ell_2+2n_2+1/2, \ell_1+1/2}(\cos 2\phi_{3p}) , \] (48)

where \( P^{a,b}_{c,d} \) are Jacobi polynomials, and the coefficients \( N_{n_2n_3}^{\ell_1\ell_2\ell_3} \) are normalization factors. The quantity \( K = \ell_1 + \ell_2 + \ell_3 + 2(n_2 + n_3) \) is the so-called grand angular quantum number. The HH functions are the eigenfunctions of the hyperangular part of the kinetic energy operator. Another important property is that \( \rho^K \mathcal{N}_{\ell_1\ell_2\ell_3L_{2n_2n_3}}(\Omega_p) \) are homogeneous polynomials of the particle coordinates of degree \( K \).

A set of antisymmetrized hyperangular-spin-isospin states of grand angular quantum number \( K \), total orbital angular momentum \( \Lambda \), total spin \( \Sigma \), and total isospin \( T \) (for the given values of total angular momentum \( J \) and parity \( \pi \)) can be constructed as follows:

\[ \Psi^K_{\Lambda \Sigma T} = \sum_{p=1}^{12} \Phi^K_{\Lambda \Sigma T}(ijkl) , \] (49)

where the sum is over the 12 even permutations \( p \equiv i j k l \), and

\[ \Phi^K_{\Lambda \Sigma T}(ijkl) = \left[ \mathcal{N}_{\ell_1\ell_2\ell_3L_{2n_2n_3}}(\Omega_p) \otimes \left[ [\chi_i \otimes \chi_j] S_a \otimes \chi_k \right] [\chi_j \otimes \chi_i] S_b \otimes \chi_l \right]_{\Sigma J J_z} \left[ [\chi_i \otimes \chi_k] T_a \otimes \chi_i \right] [\chi_i \otimes \chi_k] T_b \otimes \chi_l \right]_{T T_z} \]. (50)

Here, \( \chi_i (\xi_i) \) denotes the spin (isospin) state of particle \( i \). The total orbital angular momentum \( \Lambda \) of the HH function is coupled to the total spin \( \Sigma \) to give the total angular momentum \( J J_z \), whereas \( \pi = (-1)^{\ell_1+\ell_2+\ell_3} \). The quantum number \( T \) specifies the total isospin of the state, and \( \mu \) labels the possible choices of hyperangular, spin and isospin quantum numbers, namely

\[ \mu = \{ \ell_1, \ell_2, \ell_3, L_2, n_2, n_3, S_a, S_b, T_a, T_b \} \], (51)

compatible with the given values of \( K, \Lambda, \Sigma, T, J, \) and \( \pi \). Another important classification scheme for the states is to group them in “channels”: states belonging to the same channel have the same values of angular \( (\ell_1, \ell_2, \ell_3, L_2, \Lambda) \), spin \( (S_a, S_b, \Sigma) \), and isospin \( (T_a, T_b, T) \) quantum numbers, but different values of \( n_2 \) and \( n_3 \).

Each state \( \Phi^K_{\Lambda \Sigma T} \) entering the expansion of the four-nucleon wave function must be antisymmetric under the exchange of any pair of particles. Consequently, it is necessary to consider states such that

\[ \Phi^K_{\Lambda \Sigma T}(ijkl) = -\Phi^K_{\Lambda \Sigma T}(jikl) , \] (52)

which is fulfilled when the condition

\[ \ell_3 + S_a + T_a = \text{odd} , \] (53)

is satisfied.

The number \( M_{\Lambda \Sigma T} \) of antisymmetrized functions \( \Phi^K_{\Lambda \Sigma T} \) having given values of \( K, \Lambda, \Sigma, \) and \( T, \) but different combinations of quantum numbers \( \mu \)—see Eq.(51)—is in general very large. In addition to the degeneracy
that in the present case (\(n\)) a superposition of class with \(K\) channels with total isospin \(T\) of these HH functions depends only on \(\cos \phi\), in particular for the N3LO/N2LO interaction model. The second difficulty is the slow convergence of the HH expansion.

We include in the first class the \(n_2 = 0\) HH functions belonging to some special \(L \leq 2\) channels, for which the convergence has been found to be critical. The radial part of these HH functions depends only on \(\cos \theta = r_{ij}/\rho\) and thus these states take into account two-body correlations, see Eq. (48). The HH functions belonging to these channels and having \(n_2 > 0\) are included in the second class, together with the rest of HH functions having \(L \leq 2\). The other classes are then defined simply by grouping HH functions belonging to channels with an increasing value of \(L\). The convergence of these classes is less critical, and consequently, only HH functions with lower values of grand angular quantum number \(K\) need be considered. Moreover, the convergence with \(L\) is quite fast. In particular, we have found that the contribution of HH states with \(L > 5\) is quite small.

To exhibit the convergence pattern, we report in Table V the calculated \(n^3\)He scattering lengths. As is evident from Eq. (10), they are defined as

\[
a_J = - \lim_{q_2 \to 0} T_{0J,0J}^{02,J},
\]

with both incoming and outgoing \(n^3\)He clusters in relative S-wave. Note that in general this scattering length is complex, since at this energy the channel \(p^3\)H is always open, and therefore the unitarity condition only imposes that \(\text{Im} a_J < 0\), since the total cross section is proportional to \(\sum_{J=0,1} (2J + 1) \text{Im} T_{0J,0J}^{22,J}\). The results obtained for the singlet (\(J = 0\)) and triplet (\(J = 1\)) scattering lengths are reported in Table V, for all four potential models used in this work. The calculated \(n^3\)He scattering lengths are compared with experimental values and the results of other calculations available in the literature.

A more complete study of the convergence will be presented elsewhere [24]. Here, we have only reported the scattering lengths obtained with two sets of values of \(K_1, \ldots, K_4\), where, as already mentioned, we have included all the HH functions belonging to a given class \(i\) with grand angular quantum number \(K \leq K_i\). Inspection of the Table shows that the convergence for the triplet scattering length is very good, and that there is reasonable agreement with available experimental values, and the results of other calculations, in particular those of the AGS method. In the case of the singlet scattering length, the situation is more delicate, since in the channel \(J^p = 0^+\) the \(n^3\)He interaction is attractive and the wave function must be orthogonal to the \(^3\)He bound state. Consequently, the convergence is more problematic, in particular for the N3LO/N2LO interaction model. In the row labelled “EXT”, we have reported the extrapolated values for this quantity obtained by analyzing the convergence pattern [24]. For the AV18, N3LO, and AV18/UIX interaction models we observe reasonable agreement with the results of other calculations and the experimental data. The N3LO/N2LO values are significantly different from those obtained with the other interaction models, which is presumably related to the slow convergence observed in this case. At this stage we are unclear about the origin of problem for the N3LO/N2LO interaction model in the \(0^+\) channel. A complete study of the \(n^3\)He scattering lengths is in progress [24].

Recently, there has been a new measurement [48] for the quantity \(\Delta a = \text{Re}(a_1 - a_0) = -4.20(3)\) fm. The calculated values of \(\Delta a\) with the AV18, N3LO, AV18/UIX, and N3LO/N2LO models are \(-4.13, -4.11, -4.50,\) and \(-2.65\) fm, respectively. Again the N3LO/N2LO value stands out: it is off that obtained with the other interaction models and the measured value.

The convergence for the negative-parity waves is similar to that discussed above. For the \(0^-\) wave, there is a close resonant state and the convergence is slow as in the \(0^+\) case. For the \(1^-\) wave, the resonance is far and we observe good convergence, as for the \(1^+\) wave. Note, however, that in these cases the N3LO/N2LO convergence pattern is not different from that observed with the other models.
Monte Carlo (QMC) techniques are employed to evaluate elements relevant for the calculation of the asymmetry (simplified by retaining only linear terms in the PV elements via Eq. (37). This latter equation can be further simplified by the set of Jacobi variables $(x_1, x_2, x_3)$ are expanded on a basis of $16 \times 6$ spin-isospin states for the four nucleons as
\[
\psi(x_1, x_2, x_3) = \sum_{a=1}^{96} \psi_a(x_1, x_2, x_3) |a\rangle,
\]
where the components $\psi_a(x_1, x_2, x_3)$ are generally complex functions, and the basis states $|a\rangle = |(n \uparrow \downarrow 1)(p \downarrow 2)(n \uparrow \downarrow 3)(p \downarrow 1)\rangle$, $|(n \uparrow \downarrow 1)(n \downarrow \uparrow 2)(p \downarrow 1)\rangle$, and so on. Matrix elements of the PV potential components are written schematically as
\[
\langle f | O | i \rangle = \sum_{a,b=1}^{96} \int dx_1 dx_2 dx_3 \psi^*_f,a(x_1, x_2, x_3) \times [O(x_1, x_2, x_3)]_{ab} \psi_i,b(x_1, x_2, x_3),
\]
where $[O(x_1, x_2, x_3)]_{ab}$ denotes the matrix representing in configuration space any of the components in Table III. Note that the operators $X^{(n)}_{j_1,\pm}$ occurring in $v_{ij}^{PV}$ are conveniently expressed as
\[
X^{(n)\pm}_{j_1} = -i [2 f_n(r_{ij}) \nabla_{ij} + \hat{r}_{ij} f'_n(r_{ij})].
\]
where the gradient operator \( \nabla_{ij} = (\nabla_i - \nabla_j)/2 \) acts on the right (initial) wave function, and \( f'(x) = df(x)/dx \). Gradients are discretized as

\[
\nabla_{i,a} \psi(x_1, x_2, x_3) \approx \left[ \psi(\ldots, r_i + \delta e_a, \ldots) - \psi(\ldots, r_i - \delta e_a, \ldots) \right]/(2 \delta),
\]

where \( \delta \) is a small increment and \( e_a \) is a unit vector in the \( a \)-direction. Matrix multiplications in the spin-isospin space are performed exactly with the techniques developed in Ref. [49]. The problem is then reduced to the evaluation of the spatial integrals, which is efficiently carried out by a combination of MC and standard quadratures techniques. We write

\[
\langle f | O | i \rangle = \int dx_1 dx_2 dx_3 F(x_1, x_2, x_3) \frac{1}{N_c} \sum_{c=1}^{N_c} \frac{F(c)}{W(c)},
\]

where the \( c \)'s denote collectively (uniformly sampled) directions \( x_i \) and Jacobi coordinates \( (x_2, x_3) \), and the probability density \( W(c) = |\Psi(x_2, x_3)|^2 / (4\pi) - \Psi(x_2, x_3) \) is the triton bound-state wave function normalized to one—sampled via the Metropolis algorithm. For each such configuration \( c \) (total number \( N_c \)), the function \( F \) is obtained by Gaussian integrations over the \( x_1 \) variable, i.e.

\[
F(c) = \sum_{a,b=1}^{96} \int_0^\infty dx_1 x_1^2 \psi_{f,a}^* (x_1, x_2, x_3) \times \left[ O(x_1, x_2, x_3) \right]_{ab} \psi_{i,b}^* (x_1, x_2, x_3).
\]

Convergence in the \( x \) integrations requires of the order of 50 Gaussian points, distributed over a non-uniform grid extending beyond 20 fm, while \( N_c \) of the order of a hundred thousand is sufficient to reduce the statistical errors in the MC integration on the PV \( T \)-matrix elements at the few percent level. In this respect, we note that these errors are computed directly, by accumulating, in the course of the random walk, values—and their squares—for the appropriate linear combinations of \( R \)-matrix elements, as given in Eqs. (A5) and (A14)–(A15) of Appendix A. Because of correlations, the errors on the \( T \)-matrix elements obtained in this way are much smaller than those that would be inferred from the \( R \)-matrix elements by naive error propagation.

The present method turns out to be computationally intensive, particularly because of the large number of wave functions (and their derivatives) that have to be generated at each configuration \( (x_1, x_2, x_3) \). The computer codes have been successfully tested by carrying out a calculation based on Gaussian wave functions for the initial and final states, as described in the following subsection.

### A. Test calculation

In order to test the computer programs based on QMC techniques, we carried out a preliminary calculation using wave functions for which it is possible to evaluate the matrix elements of the PV potential also analytically. These (antisymmetric) wave functions are written as

\[
\Psi^{J_J_z}_{\gamma,LS} = \frac{1}{4\pi} \sum_{p=1}^{4} e^{-\beta p^2} y_p^{L+2n_\beta} \times \left[ Y_L(\hat{\mathbf{r}}_p) \otimes \phi_\gamma(ijk) \otimes \chi_\gamma(l) \right]_{JJ_z},
\]

where \( \phi_\gamma (\chi) \) represents a three-nucleon (single-nucleon) spin-isospin one-half state with isospin projection \(-1/2\) for \( \gamma = 1 \) (\( p^3 \)H channel) and \( +1/2 \) for \( \gamma = 2 \) (\( n^3 \)He channel). Thus, as in the realistic case, the wave functions above do not have a definite total isospin \( T \) but, rather, are combinations of \( T = 0 \) and \( T = 1 \) states (having, of course, \( T_z = 0 \)). The whole radial dependence is given by the factor \( y_p^{L+2n_\beta} e^{-\beta p^2} \), where \( \rho \) is the hyperradius. The nonnegative integer \( n_\beta \) and the real parameter \( \beta \) can be varied so as to obtain a family of wave functions. For the purpose of computing matrix elements of two-body operators, it is convenient to express the pieces in Eq. (66) corresponding to permutations \( p \neq 1 \) in terms of quantities relative to the permutation \( p = 1 \) or \((123, 4)\). This can be accomplished by making use of the properties of Wigner coefficients:

\[
\Psi^{J_J_z}_{\gamma,LS} = e^{-\beta p^2} \sum_\mu C^{LJSJ}_{\gamma,n_\beta,\mu} x_1^{n_1} x_2^{n_2} x_3^{n_3} \left[ \left[ Y_L(\hat{\mathbf{r}}_3) \otimes \chi_1 \otimes \chi_2 \right]_{T_3} \otimes \left[ Y_L(\hat{\mathbf{r}}_2) \otimes \chi_3 \right]_{T_2} \otimes \left[ Y_L(\hat{\mathbf{r}}_1) \otimes \chi_4 \right]_{T_1} \right]_{JJ_z},
\]

where \( \chi_i \) and \( \xi_i \) are the spin and isospin states of nucleon \( i \), \( x_j \) are the Jacobi vectors corresponding to the permutation \( p = 1 \) and \( n_1 + n_2 + n_3 = L + 2 n_\beta \), and the \( C \)'s denote combinations of Wigner coefficients. It is now relatively simple to evaluate the matrix of the PV potential \( \sum_{i<j} v_{ij}^{PV} \) [9], by expressing the wave
functions as in Eq. (67).

As an example, we report here the results obtained for two \( J = 0 \) wave functions. In Table VI, we list the values of the quantum numbers \( LSJ \), and parameters \( n_\beta \) and \( \beta \), used in the actual calculation. The ket \( |\gamma\rangle \) with \( \gamma = 1 \; (2) \) describes a \( \rho^{-2}\text{H} \) (\( \text{“n}^{-3}\text{He} \)) state. We compute the matrix elements in two ways: i) by performing the analytical calculation via the transformation of Eq. (67), and ii) by using the QMC techniques discussed earlier.

| State | \( J \) | \( L \) | \( S \) | \( J_\pi \) | \( \beta \) |
|-------|------|-----|------|-------|-----|
| \( |1\rangle \) | 0    | 0   | 0    | 0.25  | 0   |
| \( |2\rangle \) | 1    | 1   | 0    | 1.62  | 0.25|

**TABLE VI:** Values of the quantum numbers and parameters for some of the test wave functions used in this work. See text for explanation.

The values for the matrix elements \(-\langle 1|O_{12}^{(n)}|2\rangle\) corresponding to the 12 components of the DDH potential (see Table III) are reported in Table VII. There is good agreement between the results of the two calculations. For the latter calculation, the statistical uncertainties are reported in parentheses, and correspond to a (rather modest) set of 5,000 samples. The operators \( O_{12}^{(n)} \) are those of the DDH potential, listed in Table III. The ket \( |\gamma\rangle \) with \( \gamma = 1 \; (2) \) describes a \( \rho^{-2}\text{H} \) (\( \text{“n}^{-3}\text{He} \)) state. We compute the matrix elements in two ways: i) by performing the analytical calculation via the transformation of Eq. (67), and ii) by using the QMC techniques discussed earlier.

The results for the coefficients \( I_n^\text{DDH} \) in Eq. (43), obtained with the (zero energy) \( n^{-3}\text{He} \) continuum wave functions corresponding to the AV18, AV18/UIX, N3LO, and N3LO/N2LO strong-interaction Hamiltonians, are reported for the DDH and pionless EFT PV potentials in Tables VIII and X, respectively. The subscript \( n \) in \( I_n^\text{DDH} \) specifies the operators as listed in Table III, and the set of cutoff parameters entering the modified Yukawa functions are given in Table IV.

| \( n \) | AV18 | AV18/UIX | N3LO | N3LO/N2LO |
|-------|------|---------|------|-----------|
| 1     | -0.186E+00 | 0.189E+00 | 0.203E+00 | 0.113E+00 |
| 2     | -0.826E-02  | -0.577E-02 | -0.608E-02 | -0.622E-02 |
| 3     | +0.811E-02  | +0.864E-02 | +0.333E-02 | -0.693E-02 |
| 4     | -0.620E-02  | -0.794E-02 | -0.970E-02 | -0.753E-02 |
| 5     | -0.800E-02  | -0.976E-02 | -0.102E-01 | -0.781E-02 |
| 6     | -0.359E-03  | -0.170E-03 | -0.942E-03 | +0.322E-03 |
| 7     | +0.631E-03  | +0.115E-02 | -0.641E-04 | +0.703E-03 |
| 8     | +0.605E-02  | +0.404E-02 | -0.899E-03 | -0.794E-02 |
| 9     | +0.314E-02  | +0.289E-02 | -0.171E-02 | -0.577E-02 |
| 10    | -0.689E-02  | -0.887E-02 | -0.115E-01 | -0.902E-02 |
| 11    | -0.930E-02  | -0.113E-01 | -0.123E-01 | -0.940E-02 |
| 12    | -0.801E-02  | -0.979E-02 | -0.115E-01 | -0.606E-02 |

**TABLE VII:** Results for the real part of the (adimensional) matrix element \(-\langle 1|O_{12}^{(n)}|2\rangle\) calculated analytically and by using the QMC code. For the latter calculation, the statistical uncertainties are reported in parentheses, and correspond to a (rather modest) set of 5,000 samples. The operators \( O_{12}^{(n)} \) are those of the DDH potential, listed in Table III.

A quick glance at Table VIII makes it clear that i) the contribution of the long-range component of the DDH potential due to pion exchange is at least a factor 15 larger than that of any of the short-range components induced by vector-meson exchanges, and ii) among the vector-meson exchange contributions the isoscalar (\( n = 2, 3 \) and \( n = 5, 9 \)) and isovector (\( n = 4, 5 \) and \( n = 8–12 \)) ones are comparable in magnitude and much larger than those due to isoscalar \( \rho \)-meson exchanges (\( n = 6, 7 \)). It is also clear that the pion-exchange contribution is fairly insensitive to the choice of input strong-interaction Hamiltonians (with or without the inclusion of a three-nucleon potential), used to generate the \( n^{-3}\text{He} \) even and odd parity states with \( J = 0 \) and 1. However, the N3LO/N2LO model stands out: the pion-range contribution is (in magnitude) substantially smaller than that calculated for the other models. Moreover, the isoscalar \( \rho \)-meson (\( \omega \)-meson) contribution corresponding to \( n = 3 \) (\( n = 8 \)) has opposite sign than obtained for the other (AV18 and AV18/UIX) models.

To investigate the stability of the AV18/UIX and N3LO/N2LO results with respect to convergence in the internal part of the wave function, we present in Table IX the coefficients \( C_\alpha \) entering the PV observable \( a_z \) in Eq. (4) for two different choices of wave functions.
The results labeled “wf2” were listed earlier in Table I, except that those relative to the N3LO/N2LO model are based here on a smaller number of configurations. These results are obtained by including in the expansion of the internal parts of the 0\(^{\pm}\) and 1\(^{\pm}\) wave functions the maximum number of HH functions we have considered in the present work. The results corresponding to the row “wf1” are obtained by reducing this number for each of the classes \(K_1, \ldots, K_4\) (see discussion in Sec. V). Note also that the Monte Carlo calculation of the “wf1” co-

TABLE IX: The coefficients \(C_n^3\) entering the PV observable \(a_z\), corresponding to the AV18/UIX and N3LO/N2LO strong-interaction Hamiltonians for two sets of wave functions (see text for details). The statistical errors due to the Monte Carlo integrations are indicated in parentheses.

| Model          | \(C_0^3\)       | \(C_1^3\)       | \(C_2^3\)       | \(C_3^3\)       | \(C_4^3\)       | \(C_5^3\)       |
|----------------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|
| AV18/UIX-wf1   | -0.2077(281)    | -0.0433(116)    | +0.0242(29)     | -0.0011(2)      | -0.0232(77)     | +0.0490(30)     |
| AV18/UIX-wf2   | -0.1853(150)    | +0.0389(70)     | +0.0268(18)     | -0.0011(1)      | -0.0231(56)     | +0.0500(20)     |
| N3LO/N2LO-wf1  | -0.1118(29)     | +0.0639(25)     | +0.0200(8)      | -0.0009(1)      | +0.0390(23)     | -0.0402(12)     |
| N3LO/N2LO-wf2  | -0.1050(35)     | +0.0455(33)     | +0.0189(9)      | 0.0008(1)       | +0.0454(31)     | -0.0417(12)     |

Therefore, the differences found between the N3LO/N2LO and the other models are presumably due to the fact that the HH expansion for the N3LO/N2LO wave functions (specifically the 0\(^{\pm}\) wave function) has not fully converged. Consequently, in the following we restrict our discussion to the results obtained with the AV18, N3LO, and AV18/UIX models. In reference to the pion contribution, the calculated \(C_0^3\) is rather insensitive to the choice of strong Hamiltonian. However, there is still a considerable model dependence in the results obtained for the individual contributions due to vector-meson exchanges. This model dependence, in turn, impacts very significantly predictions for the PV asymmetry \(a_z\), as it can be surmised from Table XI. Of course, this is so under the assumption that the values for the strong- and weak-interaction coupling constants characterizing the DDH potential are those listed in Table IV. For example, the combination of coupling constants corresponding to pion-exchange (\(n = 1\)) and isoscalar \(\rho\)-meson exchange (\(n = 2\) and 3) are, respectively, \(c_{DDH}^3 = (4.48 \times 10^{-7})\) fm, \(c_{DDH}^2 = (11.2 \times 10^{-7})\) fm and \(c_{DDH}^1 = (79.5 \times 10^{-7})\) fm—note that \(c_3^{DDH} = (1 + \kappa_\rho)c_0^{DDH}\) and \(\kappa_\rho = 6.1\) is the value adopted here for the tensor coupling of the \(\rho\)-meson to the nucleon [50]. Consequently, the contribution

\[ c_3^{DDH} \times I_3^{DDH} \] is comparable in magnitude and opposite in sign to the pion-exchange contribution \(c_1^{DDH} \times I_1^{DDH}\). In this respect, we note that the asymmetry \(a_z\) changes roughly from \(-27 \times 10^{-8}\) to \(+13 \times 10^{-8}\) as the six PV weak coupling constants entering the DDH model are varied over their respective allowed ranges determined

| \(I_n^{EFT}\) | \(n\) | AV18/UIX | N3LO/N2LO |
|-------------|------|--------|----------|
| 1           | 1    | -0.195E+00 | -0.119E+00 |
| 4           | -0.606E+00 | -0.391E+00 |
| 6           | -0.639E-02 | +0.179E-01 |
| 8           | +0.608E+00 | -0.515E-01 |
| 9           | +0.301E+00 | +0.426E-01 |

TABLE X: The coefficient \(I_n^{EFT}\) corresponding to the pionless EFT potential components \(O^{(n)}\) in combination with the AV18/UIX and N3LO/N2LO strong interaction Hamiltonians. Note that there are no potential components with \(n = 2, 3, 5, 7, 10, 11,\) and 12. The statistical Monte Carlo errors are not shown, but are typically less than 5%. The \(I_n^{EFT}\) are in units of fm\(^{-1}\).

The coefficients \(I_n^{EFT}\) for the operators entering the pionless EFT PV potential, that is \(n = 1, 4, 6, 8,\) and 9, are

in Ref. [29]. Thus, \(a_z\) could potentially be large enough to make its measurement (relatively) easy.
reported in Table X. The coefficients $I_n^{\text{EFT}}$ for $n=1, 4, 8, \text{and} 9$, corresponding to isoscalar and isovector structures, are all of the same order of magnitude, while that for $n=6$ with isotensor character is much smaller. Note that the radial functions are taken to be the same for all $n$, $f_n^{\text{EFT}}(r) = f_n(r)$. Of course, the $I_n^{\text{EFT}}$'s will depend significantly on the value of the mass $\mu$—either $\mu = m_0$, as appropriate in the present pionless EFT formulation, or $\mu = 1$ GeV, the scale of chiral symmetry breaking, as appropriate in the formulation in which pion degrees of freedom are explicitly retained. Indeed, in this latter formulation the leading order component of $v^{\text{PV}}$ has the same form as the pion-exchange term in DDH.

Finally, rough estimates have been made for the range of values allowed for the low-energy constants $C_1, C_2 + C_4, C_5, C_1$, and $C_6$ in Ref. [30]. However, at the present time a systematic program for their determination is yet to be carried out. In view of this, we refrain here from making EFT-based predictions for the longitudinal asymmetry.

TABLE XI: Cumulative contributions to $a_2$ and associated errors (rows 1–12), obtained for the DDH PV potential with values for the coupling constants as listed in Table IV. The four columns correspond to the different combinations of strong-interaction Hamiltonians adopted in the calculations. The last row shows the minimum and maximum (central) values that $a_2$ can attain, as the PV couplings are varied over the allowed ranges in the original DDH formulation [29].

| $n$ | $a_{18}$ | $a_{18}/110$ | $a_{30}$ | $a_{30}/N2LO$ |
|-----|---------|-------------|----------|--------------|
| 1   | -8.33±0.35 | -8.45±0.69 | -9.07±0.40 | -5.06±0.34  |
| 2   | -9.26±0.35 | -9.09±0.70 | -9.75±0.40 | -5.76±0.38  |
| 3   | -2.80±0.68 | -2.22±1.34 | -7.10±0.89 | -11.3±0.98  |
| 4   | -2.86±0.68 | -2.30±1.34 | -7.20±0.89 | -11.3±0.98  |
| 5   | -3.40±0.68 | -2.95±1.34 | -7.88±0.89 | -11.9±0.98  |
| 6   | -3.41±0.68 | -2.95±1.34 | -7.90±0.89 | -11.9±0.98  |
| 7   | -3.32±0.68 | -2.80±1.34 | -7.91±0.89 | -11.8±0.98  |
| 8   | -3.97±0.69 | -3.23±1.35 | -7.83±0.90 | -10.9±0.99  |
| 9   | -4.31±0.69 | -3.55±1.35 | -7.65±0.90 | -10.3±0.99  |
| 10  | -4.09±0.69 | -3.26±1.35 | -7.28±0.90 | -10.0±0.99  |
| 11  | -3.79±0.69 | -2.89±1.35 | -6.88±0.90 | -9.70±0.99  |
| 12  | -3.45±0.69 | -2.48±1.35 | -6.40±0.90 | -9.44±0.99  |

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**APPENDIX A: FROM $R$- TO $T$-MATRICES**

Consider the case with $J = 0$ first. For the parity-conserving (PC) $T$-matrix we have:

$$T_{00,00}^{21,0} = \frac{1}{\sqrt{q_1}} \left[ R_{00,00}^{21,0} (I - i R_{11,0}^{11,0})^{-1} + R_{00,11}^{21,0} (I - i R_{11,0}^{11,0})^{-1} \right],$$

(A1)

where $I - i R_{11}^{11}$ is a $2 \times 2$ matrix with very small off-diagonal elements, *i.e.*

$$I - i R_{11}^{11} = \begin{pmatrix} a & \epsilon \\ \epsilon & b \end{pmatrix}, \quad a = 1 - i R_{00,00}^{11,0},$$

$$\epsilon = -i R_{00,11}^{11,0}, \quad b = 1 - i R_{11,11}^{11,0},$$

(A2)

with $|a|, |b| \gg |\epsilon|$. To first order in $\epsilon$, we approximate

$$(I - i R_{11,0}^{11,0})^{-1} = \begin{pmatrix} 1/a & -\epsilon/ab \\ -\epsilon/ab & 1/b \end{pmatrix},$$

(A3)

and hence

$$T_{00,00}^{21,0} = \frac{1}{\sqrt{q_1}} \frac{R_{00,00}^{21,0}}{a}.$$  

(A4)

Similarly, for the parity-violating (PV) $T$-matrix element we find:

$$T_{00,00}^{21,0} = \frac{1}{\sqrt{q_1}} \frac{R_{00,00}^{21,0}}{a}.$$
The case $J = 1$ is somewhat more involved since the matrices are now $4 \times 4$. The matrix $(I - i R^{11,1})^{-1}$ is written as

$$\begin{align*}
I - i R^{11,1} &= \begin{pmatrix} A & \epsilon \\ \epsilon^T & B \end{pmatrix},
\end{align*}$$

where $A$, $\epsilon$, and $B$ are $2 \times 2$ matrices,

$$\begin{align*}
A &= \begin{pmatrix} 1 - i R_{01,01}^{11,1} & -i R_{01,21}^{11,1} \\ -i R_{21,01}^{11,1} & 1 - i R_{21,21}^{11,1} \end{pmatrix},
\end{align*}$$

$$\begin{align*}
B &= \begin{pmatrix} 1 - i R_{10,10}^{11,1} & -i R_{10,11}^{11,1} \\ -i R_{11,10}^{11,1} & 1 - i R_{11,11}^{11,1} \end{pmatrix},
\end{align*}$$

$$\begin{align*}
\epsilon &= \begin{pmatrix} -i R_{01,10}^{11,1} & -i R_{01,11}^{11,1} \\ -i R_{21,10}^{11,1} & -i R_{21,11}^{11,1} \end{pmatrix}.
\end{align*}$$

Note that $A$ and $B$, as well as their inverse $A^{-1}$ and $B^{-1}$, are symmetric. To first order in $\epsilon$, it follows that

$$(I - i R^{11,1})^{-1} = \begin{pmatrix} A^{-1} & C \\ CT & B^{-1} \end{pmatrix},$$

where the $2 \times 2$ matrix $C$ and its transpose are defined as

$$C = -A^{-1} \epsilon B^{-1}, \quad CT = -B^{-1} \epsilon^T A^{-1}.$$  

This shows that $(I - i R^{11,1})^{-1}$ is also symmetric in this approximation. The PC $T_{01,01}^{21,1}$ and $T_{01,21}^{21,1}$ and PV $T_{01,10}^{21,1}$ and $T_{01,11}^{21,1}$ matrix elements entering Eq. (37) are then given by

$$\begin{align*}
T_{01,01}^{21,1} &= \frac{1}{\sqrt{q_1}} \left[ R_{01,01}^{21,1} (A^{-1})_{01,01} \\
& \quad + R_{21,01}^{21,1} (A^{-1})_{21,01} \right],
\end{align*}$$

$$\begin{align*}
T_{01,21}^{21,1} &= \frac{1}{\sqrt{q_1}} \left[ R_{01,01}^{21,1} (A^{-1})_{01,21} \\
& \quad + R_{21,01}^{21,1} (A^{-1})_{21,21} \right],
\end{align*}$$

$$\begin{align*}
T_{01,10}^{21,1} &= \frac{1}{\sqrt{q_1}} \left[ R_{01,01}^{21,1} C_{01,10} + R_{01,21}^{21,1} C_{21,10} + R_{01,10}^{21,1} \\
& \times (B^{-1})_{10,10} + R_{21,11}^{21,1} (B^{-1})_{11,10} \right],
\end{align*}$$

$$\begin{align*}
T_{01,11}^{21,1} &= \frac{1}{\sqrt{q_1}} \left[ R_{01,01}^{21,1} C_{01,11} + R_{01,21}^{21,1} C_{21,11} + R_{01,10}^{21,1} \\
& \times (B^{-1})_{10,11} + R_{21,11}^{21,1} (B^{-1})_{11,11} \right].
\end{align*}$$

### APPENDIX B: NUMERICAL VALUES FOR R- AND T-MATRIX ELEMENTS

The set of Tables XII–XV are all relative to the AV18/UIX+DDH model, and present results for the $R$-matrix elements involving PV transitions between states with $J = 0$ and $J = 1$, the corresponding $T$-matrix elements which follow from them and the parity-conserving (PC) $R$-matrix elements via Eqs. (A5) and (A14)–(A15), and lastly the coefficients $d_i^{(n)}$,

| $n$ | $R_{01,01}^{21,0}$ | $R_{01,11}^{21,0}$ |
|-----|--------------------|--------------------|
| 1   | +0.198E+01         | +0.275E+01         |
| 2   | +0.126E+00         | +0.305E+00         |
| 3   | -0.149E+00         | -0.373E+00         |
| 4   | +0.533E-01         | +0.530E-01         |
| 5   | +0.632E-01         | +0.609E-01         |
| 6   | +0.156E-02         | +0.154E-02         |
| 7   | +0.129E-02         | +0.163E-02         |
| 8   | -0.211E+00         | +0.523E+00         |
| 9   | -0.797E-01         | -0.203E+00         |
| 10  | +0.589E-01         | +0.588E-01         |
| 11  | +0.720E-01         | +0.784E-01         |
| 12  | +0.154E-01         | +0.134E-01         |

TABLE XII: The parity-violating $R$-matrix elements for $J = 0$ corresponding to the DDH potential components $O^{(n)}$ in combination with the AV18/UIX strong interaction potentials at vanishing $n^3$-He energy. The statistical Monte Carlo errors are not shown, but are typically $\sim 1–2\%$ for the largest values, and less than $10\%$ for the smallest. The $R$-matrix element without (with) an overline is in units of fm$^{-1}$ (fm$^{-1/2}$), see text for explanation.

$$\begin{align*}
d_1^{(n)} &= T_{01,00}^{21,1}(n) T_{00,00}^{21,0},
& d_2^{(n)} = T_{00,11}^{21,0}(n) T_{01,01}^{21,1},
& d_3^{(n)} = T_{01,00}^{21,0}(n) T_{01,01}^{21,1},
& d_4^{(n)} = T_{01,11}^{21,0}(n) T_{01,01}^{21,1},
& d_5^{(n)} = T_{01,11}^{21,0}(n) T_{01,01}^{21,1},
\end{align*}$$

where the $T$-matrix elements are defined as in Eq. (16), and the label $(n)$ on those involving PV transitions refers to the operator component $O^{(n)}$ in Table III. The $I_n$'s discussed earlier follow from

$$\begin{align*}
I_n &= -\frac{4}{\Sigma} \Re \left[ \sqrt{3} d_1^{(n)} - d_2^{(n)} + \sqrt{2} d_3^{(n)} \right. \\
& \left. + \sqrt{6} d_4^{(n)} + \sqrt{3} d_5^{(n)} \right],
\end{align*}$$

where $\Sigma$ has been defined in Eq. (3). A few words on units: since the operators $O^{(n)}$ do not include the $c_n$'s, i.e. the combinations of nucleon mass and strong-
and weak-interaction coupling constants, the resulting $R$-matrix ($T$-matrix) elements involving PV transitions are in units of fm$^{-1}$ (adimensional)—they would otherwise be adimensional (in units of fm). Further, because of the definition in Eq. (35), the $T$-matrix elements only differ by a phase factor, and hence the former are also adimensional.

| $n$ | $R_{01,00}$ | $R_{01,10}$ | $R_{01,11}$ | $R_{02,10}$ | $R_{02,11}$ | $R_{03,10}$ | $R_{03,11}$ |
|-----|-------------|-------------|-------------|-------------|-------------|-------------|-------------|
| 1   | 0.160E+01   | -0.930E+01  | +0.199E+02  | +0.365E+02  | -0.535E+01  | -0.106E+01  |
| 2   | +0.614E-03  | +0.131E-02  | -0.216E-04  | -0.627E-04  | -0.339E-02  | +0.584E-02  |
| 3   | -0.837E-03  | -0.198E-02  | -0.204E-04  | +0.941E-04  | +0.528E+02  | -0.604E-03  |
| 4   | -0.188E-03  | +0.782E-03  | -0.643E-04  | -0.958E-05  | -0.179E-02  | +0.161E-02  |
| 5   | -0.317E-03  | +0.918E-03  | -0.853E-04  | -0.133E-04  | -0.232E-02  | +0.186E-02  |
| 6   | +0.116E-03  | +0.159E-02  | -0.404E-04  | -0.777E-04  | -0.257E-03  | +0.427E-02  |
| 7   | -0.186E-04  | +0.191E-02  | -0.713E-04  | -0.870E-04  | +0.617E-04  | +0.518E-02  |
| 8   | -0.769E-03  | -0.181E-02  | -0.217E-04  | +0.860E-04  | +0.506E-02  | +0.812E-03  |
| 9   | -0.364E-03  | -0.852E-03  | -0.274E-04  | +0.382E-04  | +0.260E-02  | +0.448E-02  |
| 10  | -0.211E+03  | +0.867E+03  | -0.716E+03  | -0.107E+03  | -0.201E+02  | +0.179E-02  |
| 11  | -0.367E+03  | +0.105E+03  | -0.985E+03  | -0.151E+03  | -0.270E+02  | +0.214E-02  |
| 12  | -0.543E+03  | -0.144E+03  | -0.699E+03  | +0.636E+03  | -0.258E+02  | +0.102E-03  |

TABLE XIII: Same as in Table XII, but for $J = 1$.

| $n$ | $T_{00,11}$ | $T_{01,10}$ | $T_{01,11}$ | $T_{02,10}$ | $T_{02,11}$ | $T_{03,10}$ | $T_{03,11}$ |
|-----|-------------|-------------|-------------|-------------|-------------|-------------|-------------|
| 1   | -0.104E+01  | -0.302E+01  | -0.133E+00  | -0.134E-02  | -0.316E-01  | -0.168E-01  |
| 2   | +0.219E+00  | -0.996E-01  | -0.830E-02  | +0.143E-03  | +0.210E-01  | +0.123E-02  |
| 3   | -0.289E+00  | +0.108E+00  | +0.129E-01  | -0.269E-03  | +0.136E-02  | -0.293E-03  |
| 4   | -0.767E-01  | -0.971E-01  | -0.442E-02  | +0.335E-05  | +0.401E-02  | +0.330E-03  |
| 5   | -0.771E-01  | -0.111E+00  | -0.573E-02  | -0.682E-05  | +0.463E-02  | +0.385E-03  |
| 6   | -0.226E-02  | -0.285E-02  | -0.625E-03  | +0.453E-04  | -0.104E-01  | -0.224E-03  |
| 7   | -0.110E-02  | -0.210E-02  | +0.150E-03  | +0.194E-04  | -0.126E-01  | -0.277E-03  |
| 8   | -0.393E+00  | +0.159E+00  | +0.124E-01  | -0.253E-03  | +0.189E-02  | -0.239E-03  |
| 9   | -0.161E+00  | +0.559E+01  | +0.637E-02  | -0.144E-03  | +0.110E-01  | +0.364E-03  |
| 10  | -0.843E-01  | -0.107E+00  | -0.495E-02  | +0.375E-05  | +0.445E-02  | +0.367E-03  |
| 11  | -0.887E-01  | -0.126E+00  | -0.667E-02  | -0.757E-05  | +0.533E-02  | +0.443E-03  |
| 12  | -0.265E-01  | -0.295E-01  | -0.640E-02  | -0.285E-04  | -0.340E-03  | -0.245E-03  |

TABLE XIV: The parity-violating $T$-matrix elements corresponding to the DDH potential components $O^{(n)}$ in combination with the AV18/UIX strong interaction potentials at vanishing $n^3$He energy. The statistical Monte Carlo errors are not shown, but are typically less than 10%. The $T$-matrix elements are adimensional, see text for explanation.

The $R$-matrix elements in $J = 1$ states (Table XIII) are typically two orders of magnitude smaller than those in $J = 0$ states (Table XII). Among the former, those with
TABLE XV: The real parts of the coefficients $d_i$ (i = 1, …, 5), and the coefficients $I^{\text{DDH}}_n$, corresponding to the DDH potential components $G^{(i)}_n$ in combination with the AV18/UIX strong interaction potentials. The statistical Monte Carlo errors are not shown, but are typically less than 10%. The $d_i$ are adimensional, while $I^{\text{DDH}}_n$ is in units of fm$^{-1}$.

| n  | Re $d_1^{(i)}$ | Re $d_2^{(i)}$ | Re $d_3^{(i)}$ | Re $d_4^{(i)}$ | Re $d_5^{(i)}$ | $I^{\text{DDH}}_n$ |
|----|---------------|---------------|---------------|---------------|---------------|-----------------|
| 1  | +0.617E+00    | +0.349E-01    | +0.107E-01    | -0.414E-02    | +0.616E-04    | -0.189E+00      |
| 2  | +0.384E-01    | +0.436E-01    | +0.333E-03    | +0.345E-02    | -0.597E-05    | -0.577E-02      |
| 3  | -0.355E-01    | -0.356E-03    | +0.205E-01    | +0.205E-01    | +0.926E-06    | +0.864E-02      |
| 4  | +0.205E-01    | +0.614E-02    | +0.347E-03    | +0.651E-03    | -0.147E-05    | -0.794E-02      |
| 5  | +0.356E-01    | +0.527E-02    | +0.395E-03    | +0.751E-03    | -0.171E-05    | -0.976E-02      |
| 6  | +0.356E-01    | -0.183E-03    | +0.102E-04    | +0.159E-03    | +0.159E-05    | -0.170E-03      |
| 7  | -0.356E-01    | -0.378E-04    | -0.748E-05    | -0.210E-02    | +0.195E-05    | +0.115E-02      |
| 8  | -0.573E-01    | -0.769E-01    | -0.528E-03    | +0.333E-03    | +0.694E-06    | +0.404E-02      |
| 9  | -0.573E-01    | -0.309E-01    | -0.184E-03    | +0.181E-02    | -0.213E-05    | +0.289E-02      |
| 10 | +0.230E-01    | +0.672E-02    | +0.383E-03    | +0.722E-03    | -0.163E-05    | -0.887E-02      |
| 11 | +0.310E-01    | -0.612E-02    | +0.451E-03    | +0.864E-03    | -0.197E-05    | -0.113E-01      |
| 12 | +0.297E-01    | -0.241E-02    | +0.106E-03    | -0.401E-04    | +0.888E-06    | -0.979E-02      |

orbital angular momentum $L = 2$ in channel $p^3\text{He}$ is much suppressed at the low energies of interest in the present work. Inspection of Table XII also shows that the (isovector) pion-exchange interaction ($n = 1$) is dominant, which suggests that the $J^P = 0^+$ and $0^-$ states in both $n^3\text{He}$ and $p^3\text{He}$ are not purely isoscalar, but rather have significant admixtures of isospin components $T > 0$.

In order to compute the $d_i$’s in Table XV, one needs, in addition to the $T$-matrix elements listed in Table XIV, also the $T$-matrix elements associated with FC transitions. These have been calculated to (at zero $n^3\text{He}$ energy): $T_{00,00}^{21,0} = (-1.356 + i \cdot 4.482)$ fm, $T_{01,01}^{21,1} = (0.1679 - i \cdot 0.6937)$ fm, and $T_{01,21}^{21,1} = (0.003497 - i \cdot 0.0003535)$ fm. We conclude by noting that the $d_1^{(1)}$ and $d_2^{(1)}$ combinations give the leading contributions to $I_n$ and that, in the case of pion exchange, $d_1^{(1)}$ is in fact dominant. This fits in well with the expectation that the $1^S_0 \rightarrow 3^P_0$ transition entering $T_{00,11}^{21,0}$ in $d_2^{(1)}$ is predominantly isoscalar, while $d_1^{(1)}$ involves the transition $3^S_1 \rightarrow 1^P_1$ in $T_{01,10}^{21,1}$, which presumably has both isoscalar and isovector character. Indeed, the contributions of isoscalar $\rho$- and $\omega$-exchange interactions are comparable in $d_1^{(2)}$, $d_2^{(3)}$ and $d_1^{(8)}$, $d_2^{(9)}$, respectively.
In principle, there is one additional parity-violating coupling constant, $h_3^{1'}$, multiplying an isovector operator of $\rho$-meson range. However, theoretical estimates [38] indicate that its value is much smaller than the values obtained for the other coupling constants in the original DDH reference [29].