A characteristic map for the holonomy groupoid of a foliation

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Abstract
We prove a generalisation of Bott’s vanishing theorem for the full transverse frame holonomy groupoid of any transversely orientable foliated manifold. As a consequence we obtain a characteristic map encoding both primary and secondary characteristic classes. Previous descriptions of this characteristic map are formulated for the Morita equivalent étale groupoid obtained via a choice of complete transversal. By working with the full holonomy groupoid we obtain novel geometric representatives of characteristic classes. In particular we give a geometric, non-étale analogue of the codimension 1 Godbillon–Vey cyclic cocycle of Connes and Moscovici in terms of line integrals of the curvature form of a Bott connection.

1 Introduction

Characteristic classes for foliation groupoids have been studied in an étale context in [10,13–15,17,20,21,31,32]. In this paper, we give analogous constructions in the context of the non-étale, or “full”, holonomy groupoid of a foliation. In working with such groupoids we stay close to Bott’s Chern–Weil construction of the characteristic classes of a foliation [3], and obtain a novel, global geometric interpretation of the Godbillon–Vey cyclic cocycle of Connes and Moscovici [15, Proposition 19]. Moreover the methods of this paper are more likely to carry over to singular foliations, where reduction to an étale groupoid is no longer possible.

Associated to any (regular) foliated manifold \((M,F)\), of codimension \(q\), with leafwise tangent bundle \(TF\) and normal bundle \(N = TM/TF\), are characteristic classes associated to \(N\) living in the de Rham cohomology \(H^*_dR(M)\) of \(M\). Among these classes are the usual Pontryagin classes for the real vector bundle \(N\), as well as certain secondary classes, such as the Godbillon–Vey invariant, which is tied to the dynamical behaviour of the foliation \(F\) [24].

Classically, representatives of these classes are obtained either via Gelfand–Fuks cohomology [4] or via Chern–Weil theory [3]. The Chern–Weil theory in particular makes critical use of connections on \(N\) that, in any foliated coordinate system, coincide with the trivial connection on \(N\) along leaves. Such connections are called Bott connections, named after their originator Bott [2]. Bott connections give rise to Bott’s vanishing theorem, which states
that elements of degree greater than $2q$ in the Pontryagin ring of $N$ must vanish. It is precisely this vanishing phenomenon that guarantees the existence of the secondary classes.

To any such foliated manifold $(M, F)$ is associated its holonomy groupoid $\mathcal{G}$. As a space, $\mathcal{G}$ may be thought of as a quotient of the space of all smooth paths in leaves of $F$, with two such paths identifying in $\mathcal{G}$ if and only if their germinal parallel transport maps on local transversals coincide. The holonomy groupoid $\mathcal{G}$ carries a natural, locally Hausdorff differentiable structure, and is thus a Lie groupoid [35]. Moreover the normal bundle $N$ carries a natural action of $\mathcal{G}$, and one may therefore be interested in extending the “static” characteristic classes appearing in $H^*_dR(M)$ to “dynamic” characteristic classes for the holonomy groupoid $\mathcal{G}$.

In order to obtain such classes, it has become standard practice in the literature [10, 13–15, 17, 20, 21, 31, 32] to “étalify” the holonomy groupoid as follows. Letting $q$ be the codimension of $(M, F)$, one takes any $q$-dimensional submanifold $T \subset M$ which intersects each leaf of $F$ at least once, and which is everywhere transverse to $F$ in the sense that $T_xT \oplus T_xF = T_xM$ for all $x \in T$. Such a submanifold is called a complete transversal for $(M, F)$. Having chosen such a complete transversal $T$ we consider the subgroupoid $G^T := \{ u \in G : r(u), s(u) \in T \}$

of $\mathcal{G}$. The subgroupoid $G^T$ inherits from $\mathcal{G}$ a differential topology for which it is a (generally non-Hausdorff) étale Lie groupoid [16, Lemma 2]—that is, a Lie groupoid whose range (and therefore source) are local diffeomorphisms. For any choice of complete transversal $T$, the $C^*$-algebras of the groupoids $\mathcal{G}$ and $G^T$ are Morita equivalent [1, 25], so are the same as far as $K$-theory is concerned. Moreover, the groupoids $\mathcal{G}$ and $G^T$ are themselves Morita equivalent [16, Lemma 2], and consequently they are (co)homologically identical [16, 17] also.

This paper provides an analogous characteristic map to those defined in [14, 17] in the context of the full holonomy groupoid $\mathcal{G}$ of a foliated manifold. Our characteristic map will be constructed in a Chern–Weil fashion from Bott connections for $N$, staying as close as possible to the classical geometric approach. Section 2 briefly reviews well-known background on differential graded algebras, Weil algebras, classical characteristic classes for foliations, and Chern–Weil theory for Lie groupoids as developed in [28]. This background material is necessary to access the Pontryagin classes of $N$, for which a choice of connection form $\alpha$ on the positively oriented frame bundle $Fr^+(N)$ of $N$ determines a characteristic map $\psi_{\alpha}^{\mathcal{G}}$ sending the invariant polynomials on $gl(q, \mathbb{R})$ to the de Rham cohomology $\Omega^*(\mathcal{G}_1^{(s)})$ of the action groupoid $\mathcal{G}_1 := \mathcal{G} \rtimes Fr^+(N)$.

In order to access the secondary classes of the foliation, such as the Godbillon–Vey invariant, in Sect. 3 we introduce and prove the following generalisation of Bott’s vanishing theorem.

**Theorem 1.1** Let $\alpha^b$ be the connection form associated to a Bott connection on $N$. Then the image of $\psi_{\alpha^b}^{\mathcal{G}}$ in $\Omega^*(\mathcal{G}_1^{(s)})$ vanishes in total degree greater than $2q$.

The vanishing of certain cocycles implied by Theorem 1.1 enables us to refine $\psi_{\alpha^b}$ to a characteristic map of the truncated Weil algebra so as to obtain secondary classes, in a manner entirely analogous to the classical setting. More precisely we have the following theorem.

**Theorem 1.2** Let $\mathcal{W}O_q$ denote the truncated Weil algebra. If $\alpha^b$ is the connection form associated to any Bott connection on $N$, then we obtain a characteristic map

$$\phi_{\alpha^b}^{\mathcal{G}} : \mathcal{W}O_q \to \Omega^*(\mathcal{G}_1^{(s)}/SO(q, \mathbb{R})).$$
Theorems 1.1 and 1.2 should be thought of as the non-étale analogues of [17, Theorem 2 (iv)] and [14, Lemma 17] respectively. Theorem 1.2 is followed by proving in Theorem 3.4 that the characteristic map considered in an étale setting by Crainic and Moerdijk in [17] factors through that of Theorem 1.2. Section 3 is concluded with a discussion of how the lack of an invariant Euclidean structure on $\mathcal{N}$ obstructs the naïve construction of a characteristic map for the de Rham complex $\Omega^*(\mathcal{G}^{(e)})$ associated to $\mathcal{G}$.

Finally in Sect. 4, we restrict ourselves to the codimension 1 case, and use the explicit formulae provided by Theorem 1.2 to derive a Godbillon–Vey cyclic cocycle for the convolution algebra $C^\infty_c(\mathcal{G}_1; \Omega^1_2)$ of smooth leafwise half-densities $\Omega^1_2$ associated to $\mathcal{G}_1$. Our cocycle should be thought of as a non-étale analogue of the Connes–Moscovici formula [15, Proposition 19]. In contrast with the Connes–Moscovici formula obtained in the étale setting, the cocycle we obtain in this paper has a novel geometric interpretation in terms of line integrals of the curvature form associated to a Bott connection. These facts can be summarised as follows.

**Theorem 1.3** Let $R^b$ be the curvature form on $Fr^+(N)$ associated to a Bott connection on $N$, and let $R^G$ denote the differential 1-form on $\mathcal{G}_1$ defined by

$$R^G_u := \int_{\gamma} R^b, \quad u \in \mathcal{G}_1$$

where $\gamma$ is any leafwise path in $Fr^+(N)$ representing $u$. Assume that $Fr^+(N) \cong M \times \mathbb{R}_+$ has been trivialised by a choice of trivialisation for $N$. Then the formula

$$\varphi_{gv}(a^0, a^1) := \int_{(x,t) \in Fr^+(N)} \int_{u \in (\mathcal{G}_1)_{(x,t)}} a^0(u^{-1}) a^1(u) \frac{dt}{t} \wedge R^G_u, \quad a^0, a^1 \in C^\infty_c(\mathcal{G}_1; \Omega^1_2)$$

defines a cyclic cocycle $\varphi_{gv}$ for the convolution algebra $C^\infty_c(\mathcal{G}_1; \Omega^1_2)$.

We conclude the paper by demonstrating that the cyclic cocycle $\varphi_{gv}$ coincides with that obtained as Chern character of a semifinite spectral triple constructed using groupoid equivariant $KK$-theory in [29, Section 4.3]. In doing so we give a (non-étale) geometric interpretation for the off-diagonal term appearing in the triangular structures considered by Connes [10, Lemma 5.2] and Connes–Moscovici [12, Part I], in terms of the integrated curvature of Equation (1).

Let us stress that the approach taken in this paper has the advantage of being *intrinsically geometric*, giving representatives of cohomological data that are expressed in terms of *global* geometric data for $(M, \mathcal{F})$. For instance, the Godbillon–Vey cyclic cocycle obtained in this paper has a completely novel interpretation in terms of line integrals of the Bott curvature over paths in $\mathcal{F}$ representing elements of $\mathcal{G}$ (see Proposition 4.1). This is to be contrasted with the approaches taken in the étale context, in which the geometry of $M$ has necessarily been lost by “chopping up” $\mathcal{G}$ into $\mathcal{G}_T^T$. In the étale context, explicit formulae have so far tended to be obtained by tracking the displacement of *local* geometric data (trivial connections in local transversals) [15,17, Section 5.1, p. 47] under the action of $\mathcal{G}_T^T$, which will in general not be easily relatable to the global geometry of $M$.

### 2 Background

For the entirety of this section, denote by $G$ a Lie group with Lie algebra $\mathfrak{g}$. “Differential graded” will be abbreviated to DG.
2.1 The Weil algebra

Denote by $S(g^\ast)$ and $\Lambda(g^\ast)$ the symmetric and exterior algebras respectively on the dual $g^\ast$ of $g$. The Weil algebra of $g$ is the $G$-DG algebra [27, Definition 3.12]

$$W(g) := S(g^\ast) \hat{\otimes} \Lambda(g^\ast),$$

where for $\xi \in S^k(g^\ast)$ and $\eta \in \Lambda^l(g^\ast)$, $\xi \otimes \eta \in W(g)$ is given the grading $2k + l$. For $X \in g$, we denote by $i_X$ and $L_X$ the contraction and Lie derivative operators respectively on $W(g)$. For a Lie subgroup $H$ of $G$ with Lie algebra $\mathfrak{h}$, we denote by $W(g, H)$ the DG subalgebra of $H$-basic elements in $W(g)$, that is all those elements which are $H$-invariant and which are annihilated by $i_X$, $X \in \mathfrak{h}$.

By a connection in a $G$-DG algebra $A$, we mean an element $\alpha \in A^1 \otimes g$ which is $G$-invariant, and for which $(i_X \otimes id)\alpha = X$ for all $X \in g$. The Weil algebra has the following universal property for connections in $G$-DG algebras.

**Theorem 2.1** [23, Theorem 3.3.1] Let $(A, d)$ be a $G$-differential graded algebra, and suppose that $\alpha \in A^1 \otimes g$ is a connection on $A$. Then there exists a unique homomorphism $\phi_\alpha : W(g) \to A$ of $G$-differential graded algebras such that $\phi_\alpha(\omega) = \alpha$. Moreover if $\alpha_0, \alpha_1$ are two different connections on $A$, the corresponding maps $\phi_{\alpha_0}$ and $\phi_{\alpha_1}$ are $G$-cochain homotopic. \hfill \Box

In this paper we will mostly be interested in the case $G = GL^+(q, \mathbb{R})$ of invertible $q \times q$ matrices with positive determinant, and where $K = SO(q, \mathbb{R})$. Let $\xi$ denote the $q \times q$ matrix of canonical basis elements of $gl(q, \mathbb{R})^\ast$, and let $\Omega := \xi \otimes 1$ and $\omega := 1 \otimes \xi$ be the corresponding matrices of basis elements of $S^1(gl(q, \mathbb{R})^\ast) \otimes \Lambda^0(gl(q, \mathbb{R})^\ast)$ and $S^0(gl(q, \mathbb{R})^\ast) \otimes \Lambda^1(gl(q, \mathbb{R})^\ast)$ respectively. Using this matrix notation, the differential on $W(gl(q, \mathbb{R}))$ acts by the simple formulae

$$d\omega = \Omega - \omega^2, \quad d\Omega = \Omega \omega - \omega \Omega.$$

It is well-known [22, p. 187] that for $1 \leq i \leq q$, the elements $c_i \in W(gl(q, \mathbb{R}))$ defined by

$$c_i := Tr(\Omega^i)$$

are all cocycles and are all $GL^+(q, \mathbb{R})$-basic. With respect to the decomposition $gl(q, \mathbb{R}) = so(q, \mathbb{R}) \oplus s(q, \mathbb{R})$ of all $q \times q$ matrices into antisymmetric matrices and symmetric matrices respectively, our elements $\omega$ and $\Omega$ defined above decompose as

$$\omega = \omega_o + \omega_s, \quad \Omega = \Omega_o + \Omega_s.$$

Here the subscript $o$ denotes the antisymmetric part, while the subscript $s$ denotes the symmetric part. It can then be shown [22, Proposition 5] that for $1 \leq i \leq q$, the elements

$$h_i := iTr\left(\int_0^1 \omega_s(t\Omega_s + \Omega_o + (t^2 - 1)\omega_s^2) i^{-1} dt\right)$$

of $W(gl(q, \mathbb{R}))$ satisfy $dh_i = c_i$, and are $SO(q, \mathbb{R})$-basic for $i$ odd. The $c_i$ and $h_i$ generate a differential graded subalgebra $WO_q$ of $W(gl(q, \mathbb{R}), SO(q, \mathbb{R}))$ which can be used for the following refinement of Theorem 2.1.

**Corollary 2.2** Let $(A, d)$ be a $GL^+(q, \mathbb{R})$-differential graded algebra, and suppose that $\alpha \in A \otimes gl(q, \mathbb{R})$ is a connection on $A$, with curvature $R := d\alpha + \frac{1}{2}[\alpha, \alpha]$. Decompose $\alpha = \alpha_o + \alpha_s$. \hfill \Box Springer
and $R = R_o + R_s$ into their antisymmetric and symmetric components respectively. Then the formulae

$$c_i \mapsto Tr(R^i), \quad \text{for } 1 \leq i \leq q,$$

$$h_i \mapsto iTr\left( \int_0^1 \alpha_s(tR_s + R_o + (i^2 - 1)\alpha_s^2)^i-1 dt \right), \quad \text{for } 1 \leq i \leq q, \text{ } i \text{ odd},$$

define a homomorphism $\psi_\alpha : W_{O_q} \rightarrow A_{SO(q,\mathbb{R})}$ of differential graded algebras. Moreover if $\beta$ is any other choice of connection on $A$, the maps induced by $\psi_\alpha$ and $\psi_\beta$ on cohomology coincide.

\[\square\]

### 2.2 The classical Chern–Weil homomorphism for foliations

Recall that a foliated manifold $(M, \mathcal{F})$ of codimension $q$ is transversely orientable if its normal bundle $\pi_N : N := TM/T\mathcal{F} \rightarrow M$ is an orientable vector bundle. Given such a foliated manifold, we can mimic the classical Chern–Weil construction of Bott [3] using Corollary 2.2 as follows. Let $\pi_{Fr^+(N)} : Fr^+(N) \rightarrow M$ denote the positively oriented transverse frame bundle of $N$, a principal $GL^+(q, \mathbb{R})$-bundle whose fibre $Fr^+(N)_x$ over $x \in M$ consists of all positively oriented linear isomorphisms $\phi : \mathbb{R}^q \rightarrow N_x$. Letting $\mathcal{G}$ denote the holonomy groupoid [35] of $(M, \mathcal{F})$, recall [29, Section 2.2] that there is a natural action $\mathcal{G} \times_{s,\pi_N} N \rightarrow N$ of $\mathcal{G}$ on $N$ by linear isomorphisms, which we denote

$$\mathcal{G} \times_{s,\pi_N} N \ni (u, n) \mapsto u*n \in N.$$

We obtain an induced action $\mathcal{G} \times_{s,\pi_{Fr^+(N)}} Fr^+(N) \rightarrow Fr^+(N)$ of $\mathcal{G}$ on $Fr^+(N)$ defined by

$$u * \phi := u_\ast \circ \phi : \mathbb{R}^q \rightarrow N_{\pi(u)}, \quad (u, \phi) \in \mathcal{G} \times_{s,\pi_{Fr^+(N)}} Fr^+(N).$$

By associativity of composition, this action of $\mathcal{G}$ commutes with the canonical right action of $GL^+(q, \mathbb{R})$ on the principal $GL^+(q, \mathbb{R})$-bundle $Fr^+(N)$. Moreover, the orbits of $\mathcal{G}$ in $Fr^+(N)$ define a foliation $\mathcal{F}_{Fr^+(N)}$ of $Fr^+(N)$ for which the differential of the projection $\pi_{Fr^+(N)}$ maps $T\mathcal{F}_{Fr^+(N)}$ fibrewise-isomorphically onto $T\mathcal{F}$. Amongst all principal connections on $Fr^+(N)$ are those which are adapted to the foliation $\mathcal{F}_{Fr^+(N)}$ in the sense of [27, Definition 1.34].

**Definition 2.3** A principal connection form $\omega^b \in \Omega^*(Fr^+(N); gl(q, \mathbb{R}))$ is called a Bott connection form if $T\mathcal{F}_{Fr^+(N)} \subset \ker(\omega^b)$.

Let us justify this terminology. First, let $p : TM \rightarrow N$ denote the projection onto the normal bundle, and recall that a connection $\nabla^b : \Gamma^\infty(M; N) \rightarrow \Gamma^\infty(M; T^*M \otimes N)$ for $N$ is called a Bott connection if it satisfies

$$\nabla^b_X p(Y) = p[X, Y], \quad Y \in \Gamma^\infty(M; TM),$$

whenever $X$ is a leafwise vector field. The next result establishes the relationship between Bott connections in the sense of Eq. (3) and Bott connection forms in the sense of Definition 2.3. Although essentially classical (cf. [27]) we give a proof of this result, as it will be referred back to several times in the constructions and results of Sects. 3 and 4.

**Proposition 2.4** Bott connections $\nabla^b$ on $N$ are in bijective correspondence with Bott connection forms $\omega^b$ on $Fr^+(N)$. Moreover, any Bott connection form $\omega^b$ on $Fr^+(N)$ canonically determines a Bott connection $\nabla^{Fr^+(N)}$ for the foliated manifold $(Fr^+(N), \mathcal{F}_{Fr^+(N)})$. 
Proof. For the first part, let $\nabla$ be a connection on $N$ and let $\alpha$ be the connection form on $Fr^+(N)$ determined by $\nabla$. Notice that in foliated coordinates $(x^1, \ldots, x^p; z^1, \ldots, z^q)$ over $U \subset M$, defining a local section $\chi_U : U \to Fr^+(N)|_U$ of $\pi_{Fr^+(N)}$, $\nabla$ can be written

$$\nabla = d + \alpha_U$$

where $\alpha_U := \chi_U^* \alpha \in \Omega^1(U; \mathfrak{gl}(q, \mathbb{R}))$. Let $\sigma = \sigma^i \frac{\partial}{\partial z^i}$ be a normal vector field over $U$, and let $X$ be a leafwise vector field over $U$. Then

$$\nabla_X \sigma = d\sigma^i(X) \frac{\partial}{\partial z^i} + \alpha_U(X)\sigma = p[X, \sigma] + \alpha_U(X)\sigma$$

is equal to $p[X, \sigma]$ if and only if $\alpha_U(X)$ vanishes. Thus a connection $\nabla$ on $N$ is a Bott connection if and only if its local connection form $\alpha_U$ vanishes on leafwise vectors in any foliated coordinate neighbourhood $U$. Since $\pi_{Fr^+(N)}$ maps $T\mathcal{F}_{Fr^+(N)}$ fibrewise isomorphically to $\mathcal{T} \mathcal{F}$ we see that every $\alpha_U$ vanishes on $T\mathcal{F}$ if and only if $\alpha$ vanishes on $T\mathcal{F}_{Fr^+(N)}$. Consequently $\nabla = \nabla^\flat$ is a Bott connection on $N$ if and only if its associated connection form $\alpha = \alpha^\flat$ on $Fr^+(N)$ is a Bott connection form.

For the second part, suppose that $\alpha^\flat$ is a Bott connection form on $Fr^+(N)$. By hypothesis we have $T\mathcal{F}_{Fr^+(N)} \subset \ker(\alpha^\flat)$, and since $\ker(\alpha^\flat)$ projects fibrewise-isomorphically onto $TM$, the quotient bundle $H := \ker(\alpha^\flat)/T\mathcal{F}_{Fr^+(N)}$ projects fibrewise-isomorphically onto $N = TM/T\mathcal{F}$. Consequently, $H$ admits a tautological trivialisation $H \cong Fr^+(N) \times \mathbb{R}^q$ defined by

$$H_{\phi} \ni h \mapsto \phi^{-1}(d\pi_{Fr^+(N)}(h)) \in \mathbb{R}^q, \ \phi \in Fr^+(N).$$

The vertical bundle $V := \ker(d\pi_{Fr^+(N)})$ over $Fr^+(N)$ is canonically trivialised by the fundamental vector fields, so we obtain the canonical trivialisation

$$N_{Fr^+(N)} := TFr^+(N)/T\mathcal{F}_{Fr^+(N)} = V \oplus H \cong Fr^+(N) \times (\mathbb{R}^q \oplus \mathbb{R}^q)$$

of the normal bundle for the foliated manifold $(Fr^+(N), \mathcal{F}_{Fr^+(N)})$. With respect to this global trivialisation, the vanishing of $\alpha^\flat$ on $T\mathcal{F}_{Fr^+(N)}$ implies that

$$\nabla^{Fr^+(N)} := d + id_{\mathbb{R}^q} \oplus \alpha^\flat$$

defines a Bott connection on $N_{Fr^+(N)}$. \qed

Bott connections are important because of the following vanishing result, known as Bott’s vanishing theorem [2, p. 34]. Its proof follows from an easy local coordinate calculation.

Theorem 2.5 (Bott’s vanishing theorem) Let $\alpha^\flat \in \Omega^1(Fr^+(N); \mathfrak{gl}(q, \mathbb{R}))$ be a Bott connection form, and let $R^\flat := d\alpha^\flat + \alpha^\flat \wedge \alpha^\flat$ be its curvature. Then any polynomial of degree greater than $q$ in the components of $R^\flat$ vanishes. \qed

Letting $WO_q$ denote the quotient of $WO_q$ by the differential ideal of elements $\mathbb{R}[c_1, \ldots, c_q]$ that are of degree greater than $2q$, Bott’s vanishing theorem together with Corollary 2.2 has the following consequence.

Theorem 2.6 A choice of Bott connection form $\alpha^\flat$ determines a homomorphism $\psi_{\alpha^\flat} : WO_q \to \Omega^*(Fr^+(N)/SO(q, \mathbb{R}))$ of differential graded algebras, that factors through the truncated Weil algebra $\overline{WO}_q$. That is, letting $p : WO_q \to \overline{WO}_q$ denote the projection, there

\[ \square \] Springer
is a homomorphism $\phi_{\alpha^\flat} : WO_q \to \Omega^*(Fr^+(N)/SO(q, \mathbb{R}))$ of differential graded algebras such that the diagram

$$
\begin{array}{ccc}
WO_q & \xrightarrow{\psi_{\alpha^\flat}} & \Omega^*(Fr^+(N)/SO(q, \mathbb{R})) \\
\downarrow^p & & \downarrow \\
WO_q & \xrightarrow{\phi_{\alpha^\flat}} & \Omega^*(M)
\end{array}
$$

commutes. The maps on cohomology induced by $\psi_{\alpha^\flat}$ and $\phi_{\alpha^\flat}$ do not depend on the Bott connection chosen. \qed

Given a choice of Bott connection form $\alpha^\flat$ on $Fr^+(N)$, those classes determined by the range of $\phi_{\alpha^\flat}$ that are not contained in the Pontryagin ring $[\phi_\alpha(\mathbb{R}[c_1, \ldots, c_q])] \subset H^*_d(M)$ for $N$ are called secondary characteristic classes. In particular, the Godbillon–Vey class is the class $[\phi_{\alpha^\flat}(h_1 c_1^q)] \in H^{2q+1}(Fr^+(N)/SO(q, \mathbb{R}))$.

**Remark 2.7** The fibre $GL^+(q, \mathbb{R})/SO(q, \mathbb{R})$ of $Fr^+(N)/SO(q, \mathbb{R})$ is contractible, so the total space of $Fr^+(N)/SO(q, \mathbb{R})$ has the same cohomology as $M$. More specifically, a choice of Euclidean metric on $N$ determines a smooth section $\sigma : M \to Fr^+(N)/SO(q, \mathbb{R})$, which, together with a choice of Bott connection $\alpha^\flat$ on $Fr^+(N)$, determines a characteristic map $\sigma^\ast \circ \phi_{\alpha^\flat} : WO_q \to \Omega^*(M)$ for $M$. The arguments of [22, Remarque (c)] show that this characteristic map agrees on the level of differential forms with that defined by Bott [3].

### 2.3 Chern–Weil homomorphism for Lie groupoids

The groupoid Chern–Weil material we present in this subsection is sourced primarily from the paper [28], whose historical antecedents are to be found in the papers [5, 18].

Just as the classical Chern–Weil theory can be simplified and systematised by using principal $G$-bundles, Chern–Weil theory at the level of Lie groupoids is most easily studied using principal bundles over groupoids. For the entirety of this section we let $G$ be a (not necessarily Hausdorff) Lie groupoid, with unit space $G^{(0)}$ and range and source maps $r, s$ respectively. Associated to the groupoid $G$ are the face maps $\epsilon^k_i : G^{(k)} \to G^{(k-1)}$, $0 \leq i \leq k$, defined as in [28, p. 445], which give to $G^\bullet = (G^{(k)})_{k \geq 0}$ the structure of a semisimplicial manifold [28, p. 445] called the nerve of $G$. For each $k$, the alternating sum

$$
\partial := \sum_{i=0}^k (-1)^i (\epsilon^k_i)^\ast
$$

of pullbacks defines a map $\partial : \Omega^*(G^{(k)}) \to \Omega^*(G^{(k+1)})$ which squares to zero. One thus obtains a double complex

$$
\begin{array}{ccccccc}
\cdots & \xrightarrow{d} & \Omega^1(G^{(0)}) & \xrightarrow{\partial} & \Omega^1(G^{(1)}) & \xrightarrow{\partial} & \Omega^1(G^{(2)}) & \xrightarrow{\partial} & \cdots \\
\cdots & \xrightarrow{d} & \Omega^0(G^{(0)}) & \xrightarrow{\partial} & \Omega^0(G^{(1)}) & \xrightarrow{\partial} & \Omega^0(G^{(2)}) & \xrightarrow{\partial} & \cdots
\end{array}
$$
called the Bott-Shulman-Stasheff complex [5]. The associated total complex is given by
\[ \text{Tot}^* \Omega(G) := \bigoplus_{n+m=*} \Omega^n(G^{(m)}), \quad \delta_{\Omega^*(G^{(m)})} := (-1)^m d + \partial, \]
and its cohomology \( H^*_dR(G) \) is called the groupoid de Rham cohomology of \( G \). It carries a
natural ring structure induced by a cup product on the Bott-Shulman-Stasheff complex [28, p. 461].

To construct cocycles in the Bott–Shulman–Stasheff complex one uses Dupont’s simplicial
differential forms [18]. For \( k \in \mathbb{N} \), let \( \Delta^k \) denote the standard \( k \)-simplex
\[ \Delta^k := \left\{ (t_0, t_1, \ldots, t_k) \in [0, 1]^{k+1} : \sum_{i=1}^k t_i = 1 \right\}. \tag{4} \]
There are face maps \( \bar{e}_i^k : \Delta^{k-1} \rightarrow \Delta^k \) defined for all \( k > 1 \) and \( 1 \leq i \leq k \) by the formulae
\[ \bar{e}_i^k(t_0, \ldots, t_{k-1}) := (t_0, \ldots, t_{i-1}, 0, t_i, \ldots, t_{k-1}). \]
and for \( i = 0 \) by simply
\[ \bar{e}_0^k(t_0, \ldots, t_{k-1}) := (0, t_0, \ldots, t_{k-1}). \]

**Definition 2.8** For \( l \in \mathbb{N} \), a **simplicial \( l \)-form** on \( G \) is a sequence \( \omega = \{ \omega^{(k)} \}_{k \in \mathbb{N}} \) of differential
\( l \)-forms \( \omega^{(k)} \in \Omega^l(\Delta^k \times G^{(k)}) \) such that
\[ (\epsilon_i^k \times id)^* \omega^{(k)} = (id \times \epsilon_i^k)^* \omega^{(k-1)} \in \Omega^l(\Delta^{k-1} \times G^{(k)}) \]
for all \( i = 0, \ldots, k \) and for all \( k \in \mathbb{N} \). We denote the space of all simplicial \( l \)-forms on \( G \) by
\( \Omega^l_\Delta(G) \).

By definition, simplicial differential forms on \( G \) can be thought of as differential forms on
the fat realisation [33] of the classifying space \( BG \) of \( G \). Under the component-wise exterior derivative and wedge product, \( \Omega^*_\Delta(G) \) is a DG algebra, and is related to the Bott–Shulman-Stasheff double complex by an integration map.

**Proposition 2.9** [18, Theorem 2.3, Theorem 2.14] The map \( I : \Omega^*_\Delta(G) \rightarrow \text{Tot}^* \Omega(G) \) defined by
\[ I(\omega) := \sum_{l \in \mathbb{N}} \int_{\Delta^l} \omega^{(l)} \]
is a map of cochain complexes. Moreover the map determined by \( I \) on cohomology is a homomorphism of rings, where the ring structure on \( H^*(\Omega^*_\Delta(G)) \) is induced by the wedge product and where the ring structure on \( H^*_dR(G) \) is induced by the cup product. \( \square \)

**Remark 2.10** When \( G \) is Hausdorff, the map \( I : \Omega^*_\Delta(P/K) \rightarrow \text{Tot}^* \Omega(P/K) \) descends to an isomorphism on cohomology [18, Theorem 2.3]. Thus for Hausdorff \( G \) the double complex \( \Omega^*(G^{(0)}) \) computes the cohomology of the classifying space \( BG \).

The construction of characteristic classes in groupoid de Rham cohomology now proceeds
following [28]. Let \( \pi : P^{(0)} \rightarrow G^{(0)} \) be a \( G \)-principal \( G \)-bundle; that is, a principal \( G \)-bundle
over \( G^{(0)} \) carrying a left action \( G \times_{s, \pi} P^{(0)} \rightarrow P^{(0)} \) of \( G \) that commutes with the right
action of $G$. Then $\mathcal{P} := G \times_s \pi \mathcal{P}^{(0)}$ is itself a Lie groupoid. In order to make notation less cumbersome, for $(u_1, \ldots, u_k) \in \mathcal{G}^{(k)}$ and $p \in \mathcal{P}^{(0)}_{\pi(u_k)}$ we will denote the composable $k$-tuple 

\[(u_1, (u_2 \cdots u_k) \cdot p), (u_2, (u_3 \cdots u_k) \cdot p), \ldots, (u_k, p) \]  

by simply 

\[(u_1, \ldots, u_k) \cdot p. \]

It is then simple to verify that the projections $\mathcal{P}^{(k)} \to \mathcal{G}^{(k)}$ are principal $G$-bundles and that the action of $G$ on the $\mathcal{P}^{(k)}$ commutes with all face maps, so that the algebra of simplicial forms $\Omega^*_\Delta(\mathcal{P})$ on $\mathcal{P}$ is a $G$-DG algebra. Moreover if $K$ is a Lie subgroup of $G$, then the $K$-basic forms in $\Omega^*_\Delta(\mathcal{P})$ are isomorphic with the simplicial forms $\Omega^*(\mathcal{P}/K)$ on the groupoid $\mathcal{P}/K$.

For each $k > 0$, and each $0 \leq i \leq k$, define $p^k_i : \mathcal{P}^{(k)} \to \mathcal{P}^{(0)}$ by the formulae

\[p^k_0((u_1, \ldots, u_k) \cdot p) := (u_1 \cdots u_k) \cdot p, \quad p^k_i((u_1, \ldots, u_k) \cdot p) := p, \]  

for $i = 0, k$ respectively, and by

\[p^k_i((u_1, \ldots, u_k) \cdot p) := (u_{i+1} \cdots u_k) \cdot p \]  

for $1 \leq i \leq k - 1$. If $\alpha \in \Omega^1(\mathcal{P}^{(0)}; g)$ is a principal connection, then for each $k \in \mathbb{N}$ define $\alpha^{(k)} \in \Omega^1(\Delta^k \times \mathcal{P}^{(k)}) \otimes g$ by the formula

\[\alpha^{(k)}_{(t_0: \ldots: t_k; (u_1, \ldots, u_k) \cdot p)} := \sum_{i=0}^{k} t_i ((p^k_i)^* \alpha)_{(u_1, \ldots, u_k) \cdot p}. \]  

By [28, Proposition 5.3], the sequence $\alpha^* := \{\alpha^{(k)}\}$ determines a connection on the $G$-DG algebra $\Omega^*_\Delta(\mathcal{P})$, and one thus has the following consequence of Theorem 2.1 and Proposition 2.9.

**Theorem 2.11** A choice of connection form $\alpha \in \Omega^1(\mathcal{P}^{(0)}; g)$ determines, for any Lie subgroup $K$ of $G$, a homomorphism

\[\Psi^G_\alpha : W(g, K) \to \Omega^*_\Delta(\mathcal{P}/K)\]

of differential graded algebras, hence a cochain map

\[\Psi^G_\alpha = \text{id} \circ \Psi^G_\alpha : W(g, K) \to \text{Tot}^* \Omega(\mathcal{P}/K)\]

of total complexes. The induced map on cohomology is a homomorphism of graded rings and does not depend on the connection chosen.

\[\square\]

### 3 Characteristic map for foliated manifolds

We will now use the background material of Sect. 2 to prove new results on the secondary characteristic classes for the transverse frame holonomy groupoid of a foliation. Let us consider a transversely orientable foliated manifold $(M, \mathcal{F})$ of codimension $q$ with holonomy groupoid $\mathcal{G}$. As we have already discussed in Sect. 2.2, the principal $GL^+(q, \mathbb{R})$-bundle $\pi_{Fr^+}(N) : Fr^+(N) \to M$ of positively oriented frames for $N$ carries a left action of the holonomy groupoid $\mathcal{G}$ that commutes with the canonical right action of $GL^+(q, \mathbb{R})$. 
Definition 3.1 We will denote by $G_1 = G \times Fr^+(N)$ the action groupoid over $G$ corresponding to the $G$-principal $GL^+(q, \mathbb{R})$-bundle $\pi_{Fr^+(N)} : Fr^+(N) \to M$.

Given a connection form $\alpha \in \Omega^1(Fr^+(N); \mathfrak{gl}(q, \mathbb{R}))$, the characteristic map $\psi^G_\alpha$ of Theorem 2.11 composes with the inclusion $WO_q \hookrightarrow W(\mathfrak{gl}(q, \mathbb{R}), SO(q, \mathbb{R}))$ to give a cochain map $\psi^G_\alpha : WO_q \to \Omega^*(G_1^{(1)}/SO(q, \mathbb{R}))$, whose induced map on cohomology does not depend on the connection $\alpha$. The image of $\psi^G_\alpha$ in the first-column subcomplex $\Omega^*(Fr^+(N)/SO(q, \mathbb{R}))$ of $\Omega^*(G_1^{(1)}/SO(q, \mathbb{R}))$ coincides with that of the characteristic map $\psi_\alpha$ of Theorem 2.6. More generally the image of $\psi^G_\alpha$ in $H^*_d(G_1/SO(q, \mathbb{R}))$ consists of the Pontryagin classes of the groupoid $G_1/SO(q, \mathbb{R})$, accessed in the same manner as in [28]. In order to construct secondary characteristic classes for the groupoid $G_1/SO(q, \mathbb{R})$, we must prove an analogue of Bott’s vanishing theorem - that is, we must prove that the Pontryagin characteristic classes for the groupoid $G_1/SO(q, \mathbb{R})$ vanish in total degree greater than $2q$. To this end, we present the following generalisation of Bott’s vanishing theorem, which is the non-étale analogue of [17, Theorem 2 (iv)]. Regard elements of the subalgebra $I^*_q(\mathbb{R}) := \mathbb{R}[c_1, \ldots, c_q] \subset WO_q$ as invariant polynomials as in Eq. (2).

Theorem 3.2 (Bott’s vanishing theorem for $G_1$) Let $\alpha^b \in \Omega^1(Fr^+(N); \mathfrak{gl}(q, \mathbb{R}))$ be a Bott connection form. If $P \in I^*_q(\mathbb{R})$ is an invariant polynomial of degree $\deg(P) > q$ (so that its degree in $I^*_q(\mathbb{R})$ is greater than $2q$), then $\psi_{\alpha^b}(P) = 0 \in \Omega^*(G_1^{(1)}/SO(q, \mathbb{R}))$.

Proof For each $k \in \mathbb{N}$, let $(R^b)^{(k)} = d(\alpha^b)^{(k)} + (\alpha^b)^{(k)} \wedge (\alpha^b)^{(k)}$ denote the curvature of the connection form $(\alpha^b)^{(k)}$ on $\Delta^k \times G_1^{(1)}$ obtained as in Eq. (7). Let $P \in I^*_q(\mathbb{R})$. The cochain $\psi_{\alpha^b}(P)$ in $\Omega^*(G_1^{(1)}/SO(q, \mathbb{R}))$ identifies with the $SO(q, \mathbb{R})$-basic cochain

$$\sum_k \int_{\Delta^k} P((R^b)^{(k)}), \quad (8)$$

in $\Omega^*(G_1^{(1)}/SO(q, \mathbb{R}))$. Thus it suffices to show that the cochain in Equation (8) is zero.

The form $(R^b)^{(k)}$ is by construction of degree at most 1 in the $\Delta^k$ variables due to the $d(\alpha^b)^{(k)}$, and therefore $P((R^b)^{(k)})$ is of degree at most $\deg(P)$ in the $\Delta^k$ variables. Thus $\int_{\Delta^k} P((R^b)^{(k)})$ vanishes when $\deg(P) < k$, implying that $\psi_{\alpha^b}(P)$ vanishes in $\Omega^*(G_1^{(1)})$ for $k > \deg(P)$.

Let us assume therefore that $k \leq \deg(P)$. We will show that $\int_{\Delta^k} P((R^b)^{(k)}) = 0$ as a differential form on $\Omega^2\deg(P) - k(G_1^{(1)})$. On $\Delta^k \times G_1^{(1)}$, using Equation (7), we compute

$$(R^b)^{(k)} = \sum_{i=0}^k dt_i \wedge (p_i^k)^*\alpha^b + \sum_{i=0}^k t_i(p_i^k)^*d\alpha^b + \left(\sum_{i=0}^k t_i(p_i^k)^*\alpha^b\right) \wedge \left(\sum_{i=0}^k t_i(p_i^k)^*\alpha^b\right), \quad (9)$$

with the $p_i^k : G_1^{(1)} \to Fr^+(N)$ defined as in Eqs. (5) and (6). To proceed further, we must consider a local coordinate picture.

About a point $(u_1, \ldots, u_k) \cdot \phi \in G_1^{(1)}$, consider a local coordinate chart for $G_1^{(1)}$ of the form

$$\left(\chi^1_{j=1}^{\dim(F)}; \ldots; \chi^q_{j=1}^{\dim(F)}; \varepsilon_{j=1}^q; g\right) \in B_1 \times \cdots \times B_k \times V \times GL^+(q, \mathbb{R}),$$

where the $B_1, \ldots, B_k$ are open balls in $\mathbb{R}^{\dim(F)}$ corresponding to plaques in foliated charts $U_1, \ldots, U_k$ in $(M, F)$, and where $V$ is an open ball in $\mathbb{R}^q$ such that $B_k \times V \cong U_k$. For
\[ \tilde{u}_{j+1} = u_{j+1} \cdots u_k \in G \] we let \((h_{\tilde{u}_{j+1}})^i : V \rightarrow \mathbb{R}\) denote the \(i^{th}\) component function of some holonomy transformation \(h_{\tilde{u}_{j+1}}\) representing \(\tilde{u}_{j+1}\). Then in these coordinates the maps \(p_i^k : G^{(k)} \rightarrow Fr^+(N)\) take the form

\[
p_i^k(\left(x^i_1,j_d\dim(F)\right), \ldots ; \left(x^i_k,j_d\dim(F)\right); \left(z^j_1,q\right), \ldots , \left(z^j_j,q\right)) := \left(\left(x^i_1,j_d\dim(F)\right), \left(h_{\tilde{u}_{j+1}})^i_1, \ldots , \left(0^9\right))_j,q\right) ; g.\]

To write \((R^b)^{(k)}\) in these local coordinates, consider the chart \(U_i \times GL^+(q, \mathbb{R})\) of \(Fr^+(N)\). In the foliated chart \(U_i\), the local connection form \(\alpha_i \in \Omega^1(U_i; gl(q, \mathbb{R}))\) corresponding to the Bott connection \(\nabla^b\) vanishes on plaquewise \(^1\) tangent vectors (cf. Proposition 2.3). Letting \(\pi_1 : U_i \times GL^+(q, \mathbb{R}) \rightarrow U_i\) and \(\pi_2 : U_i \times GL^+(q, \mathbb{R}) \rightarrow GL^+(q, \mathbb{R})\) denote the projections, over \(U_i \times GL^+(q, \mathbb{R})\) the form \(\omega^b\) can be written

\[
\omega^b_{(x,g)} = Ad^{-1}(\pi_1^* \alpha_i)_{(x,g)} + (\pi_2^* \omega^{MC})_{(x,g)}, \quad (x, g) \in U_i \times GL^+(q, \mathbb{R})
\]

where \(\omega^{MC}\) is the Maurer–Cartan form on \(GL^+(q, \mathbb{R})\) [30, Section 2.4 (b)]. For simplicity, let us abuse notation in letting \(\alpha_i\) denote the form \(Ad^{-1}(\pi_1^* \alpha_i)\) on \(U_i \times GL^+(q, \mathbb{R})\). Then since \(\alpha_i\) is defined by a Bott connection, its matrix components can all be written in terms of the differentials of the transverse coordinates \(z^j\) in \(U_i\) (cf. Proposition 2.3). Consequently, in coordinates we can write

\[
(p_i^k)^* \alpha^b = (p_i^k)^* \alpha_i + \pi_k^* \omega^{MC},
\]

where \(\pi_k : B_1 \times \cdots \times B_k \times V \times GL^+(q, \mathbb{R}) \rightarrow GL^+(q, \mathbb{R})\) is the projection and where \((p_i^k)^* \alpha_i\) is a \(gl(q, \mathbb{R})\)-valued 1-form in the coordinate differentials \((dz^j)_j^q\).

Let us now rewrite the expression (9) for \((R^b)^{(k)}\) in coordinates. The first term on the right hand side can be written

\[
\sum_{i=0}^k dt_i \wedge (p_i^k)^* \alpha^b = \sum_{i=0}^k dt_i \wedge (p_i^k)^* \alpha_i + \left(\sum_{i=0}^k dt_i\right) \wedge \pi_k^* \omega^{MC} = \sum_{i=0}^k dt_i \wedge (p_i^k)^* \alpha_i \tag{10}
\]

since \(\sum_{i=0}^k t_i = 1\). The middle term on the right hand side of (9) can be written

\[
\sum_{i=0}^k t_i (p_i^k)^* d \omega^b = \sum_{i=0}^k t_i (p_i^k)^* \alpha_i + \left(\sum_{i=0}^k t_i\right) \pi_k^* d \omega^{MC} = \sum_{i=0}^k t_i (p_i^k)^* \alpha_i + \pi_k^* d \omega^{MC}, \tag{11}
\]

while the last term on the right hand side of (9) can be written

\[
\left(\sum_{i=0}^k t_i (p_i^k)^* \alpha^b\right) \wedge \left(\sum_{i=0}^k t_i (p_i^k)^* \alpha^b\right) = \left(\sum_{i=0}^k t_i (p_i^k)^* \alpha_i\right) \wedge \left(\sum_{i=0}^k t_i (p_i^k)^* \alpha_i\right)
\]

\[
+ \left(\sum_{i=0}^k t_i (p_i^k)^* \alpha_i\right) \wedge \pi_k^* \omega^{MC}
\]

\[+ \pi_k^* \omega^{MC} \wedge \left(\sum_{i=0}^k t_i (p_i^k)^* \alpha_i\right) + \pi_k^* (\omega^{MC} \wedge \omega^{MC}). \tag{12}\]

\(^1\) i.e. locally leafwise
Adding the expressions (10), (11) and (12) and using the fact that the Maurer-Cartan form satisfies \( d\omega^{MC} + \omega^{MC} \wedge \omega^{MC} = 0 \) [30, Equation 2.46], we find that

\[
(R^b)^{(k)} = \sum_{i=0}^{k} dt_i \wedge (p_i^k)^*\alpha_i + \sum_{i=0}^{k} t_i(p_i^k)^*d\alpha_i + \left( \sum_{i=0}^{k} t_i(p_i^k)^*\alpha_i \right) \wedge \left( \sum_{i=0}^{k} t_i(p_i^k)^*\alpha_i \right).
\]

(13)

For a summand of \( \int_{\Delta^k} P((R^b)^{(k)}) \) to be nonzero, it must contain precisely \( k \) factors of the first term appearing in (13). Thus, in our coordinates, due to the \((p_i^k)^*\alpha_i \) appearing in this first term of (13) each summand of \( \int_{\Delta^k} P((R^b)^{(k)}) \) contains a string of wedge products of at least \( k \) of the \( dz^i \)'s. This consideration accounts for \( 2k \) of the coordinate differentials that appear in each summand of \( \int_{\Delta^k} P((R^b)^{(k)}) \), and we must concern ourselves now with the \( 2 \deg(P) - 2k \) coordinate differentials that remain.

Now each of the final four terms in (13) is a matrix of 2-forms, and contains either an \( \alpha_i \) or a \( da^i \) as a factor. Consequently, all the components of each such matrix must contain at least one \( dz^i \) as a factor. Therefore, of the remaining \( 2 \deg(P) - 2k \) coordinate differentials in each summand of \( \int_{\Delta^k} P((R^b)^{(k)}) \), at least \( \deg(P) - k \) more must be \( dz^i \)'s. Thus in our local coordinate system for \( \mathcal{G}^{(k)}_1 \), each summand in \( \int_{\Delta^k} P((R^b)^{(k)}) \) contains a string of wedge products of at least \( k + (\deg(P) - k) = \deg(P) > q \) of the \( dz^i \), and must therefore be zero by dimension count. \( \Box \)

Bott’s vanishing theorem at the level of the holonomy groupoid enables us to refine the characteristic map of Theorem 2.11 in a way entirely analogous to the classical case.

**Theorem 3.3** If \( \omega^b \in \Omega^1(Fr^+(N); \mathfrak{gl}(q, \mathbb{R})) \) is a Bott connection on \( N \), the cochain map \( \psi^G_{\omega^b} : \mathcal{W}O_q \to \Omega^*(\mathcal{G}^{(s)}_1/SO(q, \mathbb{R})) \) descends to a cochain map

\[
\phi^G_{\omega^b} : \mathcal{W}O_q \to \Omega^*(\mathcal{G}^{(s)}_1/SO(q, \mathbb{R}))
\]

whose induced map on cohomology is independent of the Bott connection chosen. \( \Box \)

We now relate the characteristic map of Theorem 3.3 to that constructed by Crainic and Moerdijk [17]. Recall from [17] that for a codimension \( q \) foliation \((M, \mathcal{F})\), a transversal section is an embedded, \( q \)-dimensional submanifold \( U \subset M \) which is everywhere transverse to \( \mathcal{F} \), and an embedding \( U \to V \) of transversal sections induced by a holonomy transformation is called a holonomy embedding. A transversal basis is then a family \( \mathcal{U} = \{U_j\}_{j \in J} \) of transversal sections with the property that if \( y \) is any point in \( M \) and \( V \subset M \) is any transversal section through \( y \), then there exists \( U_j \in \mathcal{U} \) and a holonomy embedding \( h : U_j \to V \) such that \( y \in h(U_j) \). Given such a transversal basis \( \mathcal{U} \), Crainic and Moerdijk consider [17, Corollary 1] the associated Čech–de Rham double complex

\[
\tilde{C}^k(\mathcal{U}, \Omega^1(Fr^+(N)/SO(q, \mathbb{R}))) := \prod_{U_{j_0} \to h_1 \cdots \to h_k \to U_{j_k}} \Omega^1(Fr^+(N)/SO(q, \mathbb{R})|_{U_{j_0}}),
\]

(14)

where the product is taken over all composable \( k \)-tuples of holonomy embeddings.

Crainic and Moerdijk give a characteristic map \( \mathcal{W}O_q \to \text{Tot}^*\tilde{C}(\mathcal{U}, \Omega(Fr^+(N)/SO(q, \mathbb{R}))) \) which may be reinterpreted in the following fashion. Observe that associated to the transversal basis \( \mathcal{U} \) is a Čech groupoid \( \check{\mathcal{C}}\mathcal{U} \) with morphism set
\[ \mathcal{CU}^{(1)} = \coprod_{U_{j_0} \xrightarrow{h} U_{j_1}} U_{j_1}, \text{ unit space } \mathcal{CU}^{(0)} := \coprod_{j \in J} U_j, \text{ range and source given by } \]

\[
(x, U_{j_0} \xrightarrow{h} U_{j_1}) \mapsto (x, j_0) \quad \text{and} \quad (x, U_{j_0} \xrightarrow{h} U_{j_1}) \mapsto (h(x), j_1) \text{ respectively, and multiplication given by } \]

\[
(x, U_{j_0} \xrightarrow{h_1} U_{j_1}) \cdot (h(x), U_{j_1} \xrightarrow{h_2} U_{j_2}) := (x, U_{j_0} \xrightarrow{h_2 h_1} U_{j_2}).
\]

Equipping the unit space with its canonical \(q\)-manifold structure, the groupoid \(\mathcal{CU}\) is then equipped with the weakest topology for which the range and source are local homeomorphisms, under which it is an étale Lie groupoid. We let \(\mathcal{CU}_t\) be the corresponding groupoid built from the \(Fr^+(N)|_{U_j}\), whose Bott–Shulman–Stasheff complex identifies canonically with the \(\check{C}\)ech–de Rham complex of Equation (14). Equipping each \(U_j \in \mathcal{U}\) with a connection form \(\alpha_j \in \Omega^1(Fr^+(N)|_{U_j}; gl(q,\mathbb{R}))\) as in [17] amounts to equipping the unit space of \(\mathcal{CU}_t\) with a connection \(\alpha = \{\alpha_j\}_{j \in J}\). The Crainic–Moerdijk characteristic map is then precisely the cochain map \(\phi_{\mathcal{CU}} : \mathcal{W}O_q \rightarrow \text{Tot}^*\Omega(\mathcal{CU}_t/\text{SO}(q,\mathbb{R}))\) arising from Theorem 2.11 as a consequence of the generalisation [17, Theorem 2, (iv)] of Bott’s vanishing theorem.

Now observe that the canonical morphism \(\iota : \mathcal{CU} \rightarrow \mathcal{G}\) defined by

\[
\iota : (x, U_{j_0} \xrightarrow{h} U_{j_1}) \mapsto \text{germ}_{h(x)}(h^{-1})
\]

is a morphism of Lie groupoids, and induces a morphism \(\iota_1 : \mathcal{CU}_t \rightarrow \mathcal{G}_1\) of Lie groupoids which commutes with the respective \(GL^+(q,\mathbb{R})\) actions. Since the characteristic map of Theorem 2.11 is functorial [28, Theorem A], we have therefore shown the following.

**Theorem 3.4** For any Bott connection form \(\alpha^b\) on \(Fr^+(N)\), one has a commuting diagram

\[
\begin{array}{ccc}
\mathcal{W}O_q & \xrightarrow{\phi_{\alpha^b}} & \text{Tot}^*\Omega(\mathcal{G}_1/\text{SO}(q,\mathbb{R})) \\
\downarrow \psi_{\mathcal{CU}} & & \downarrow \psi_1 \\
\text{Tot}^*\Omega(\mathcal{CU}_t/\text{SO}(q,\mathbb{R})) & \xrightarrow{\iota_{\alpha^b}} & \mathcal{G}_1
\end{array}
\]

of cochain maps, with the top being the map of Theorem 3.3 and the bottom being the Crainic–Moerdijk characteristic map. \(\square\)

Recall now the characteristic map \(\phi_{\alpha^b} : \mathcal{W}O_q \rightarrow \Omega^*(Fr^+(N)/\text{SO}(q,\mathbb{R}))\) of Theorem 2.6, and for any \(b \in \Omega^*(\mathcal{G}_1^{(s)}/\text{SO}(q,\mathbb{R}))\) let \(b_0\) denote its component in \(\Omega^*(Fr^+(N)/\text{SO}(q,\mathbb{R}))\). Then by construction we have

\[
(\phi_{\alpha^b}^\mathcal{G}(a))_0 = \phi_{\alpha^b}(a), \quad \text{for all } a \in \mathcal{W}O_q.
\]

Thus \(\phi_{\alpha^b}\) should be thought of as encoding the “static” transverse geometric information that can be accessed via classical Chern–Weil theory, while the “larger” characteristic map \(\phi_{\alpha^b}^\mathcal{G}\) encodes both the static and dynamic information pertaining to the relationship of the groupoid action with transverse geometry. As discussed in Remark 2.7, one can pull back the static information encoded by \(\phi_{\alpha^b}\) to \(\Omega^*(M)\) through the choice of a Euclidean structure for \(N\), so one might hope that it is also possible to pull back all the dynamical information encoded by \(\phi_{\alpha^b}^\mathcal{G}\) to the double complex \(\Omega^*(\mathcal{G}^{(s)})\) in the same way.

Indeed it is claimed, with some vagueness, by Crainic and Moerdijk in [17, Section 3.4] (who work with an étalified, \(\check{C}\)ech version of the double complex \(\Omega^*(\mathcal{G}^{(s)})\), as in the discussion preceding Theorem 3.4) that the contractibility of the fibres of \(Fr^+(N)/\text{SO}(q,\mathbb{R})\) allows one...
to pull all of $\phi^G_{\alpha^q}(WQq)$ down to $\Omega^*(G^{(s)})$ “as in” the static case. While it is unclear exactly what Crainic and Moerdijk mean by this, let us point out here that one is prevented from naïvely extending the cochain map $\sigma^*: \Omega^*(Fr^+(N)/SO(q, \mathbb{R})) \to \Omega^*(M)$ to a cochain map $\Omega^*(G_1^{(s)}/SO(q, \mathbb{R})) \to \Omega^*(G^{(s)})$ precisely by the lack of invariance of the Euclidean structure on $N$ defining $\sigma$ under the action of $G$. More precisely, we have the following proposition.

**Proposition 3.5** Define $\sigma^{(k)}: G^{(k)} \to G_1^{(k)}/SO(q, \mathbb{R})$ by the formula

$$
\sigma^{(k)}(u_1, \ldots, u_k) := (u_1, \ldots, u_k) \cdot \sigma(s(u_k)), \quad (u_1, \ldots, u_k) \in G^{(k)}.
$$

Pulling back by the $\sigma^{(k)}$ defines a cochain map $\Omega^*(G_1^{(s)}/SO(q, \mathbb{R})) \to \Omega^*(G^{(s)})$ if and only if $\sigma$ is invariant under the action of $G$.

**Proof** The $\sigma^{(k)}$ define a cochain map if and only if $\sigma^{(k-1)} \circ \epsilon^k_i = \epsilon^k_i \circ \sigma^{(k)}$ for all $i \leq k$. However, we see that

$$
\epsilon^k_i(\sigma^{(k)}(u_1, \ldots, u_k)) = (u_1, \ldots, u_k) \cdot (u_k \cdot \sigma(s(u_k)))
$$

while

$$
\sigma^{(k-1)}(\epsilon^k_i(u_1, \ldots, u_k)) = (u_1, \ldots, u_{k-1}) \cdot \sigma(s(u_{k-1}))
$$

for all $(u_1, \ldots, u_k) \in G^{(k)}$. Consequently we have $\epsilon^k_i \circ \sigma^{(k)} = \sigma^{(k-1)} \circ \epsilon^k_i$ if and only if $u \cdot \sigma(s(u)) = \sigma(r(u))$ for all $u \in G$, which occurs if and only if the Euclidean structure $\sigma$ on $N$ is preserved by the action of $G$. \hfill \Box

An invariant section $\sigma: M \to Fr^+(N)/SO(q, \mathbb{R})$ is the same thing as a $G$-invariant Euclidean structure on $N$, which is not always guaranteed to exist. Moreover in any situation where such an invariant Euclidean structure does exist, it induces via its determinant an invariant transverse volume form. In this case, well-known results [26, Theorem 2] state that all generalised Godbillon–Vey classes (that is, those classes determined by cocycles $h_1h_1c_{IJ} \in W\Omega_{q^*}$ for multi-indices $I$ and $J$ with $\deg(c_{IJ}) = 2q$ and with $h_1 = 1$ permitted) vanish in $H^dR(M)$. In particular, whenever $(M, \mathcal{F})$ has nonvanishing Godbillon–Vey invariant, in order to probe the algebraic topology of $G$ using the characteristic map $\phi^G_{\alpha^q}$ we need a more sophisticated method of getting from $\Omega^*(G_1^{(s)}/SO(q, \mathbb{R}))$ to $\Omega^*(G^{(s)})$ which takes into account the lack of invariance of Euclidean structures on $N$ under the action of $G$. Giving such a construction constitutes an interesting research question, which we leave to a future paper.

### 4 The codimension 1 Godbillon–Vey cyclic cocycle

Connes and Moscovici [14, Section 4] use the étale picture of a foliation groupoid to obtain an analogue of Theorem 3.3. More specifically, they replace $G_1$ with the groupoid $FX \rtimes \Gamma_X$ of germs of local diffeomorphisms of an $q$-manifold $X$, lifted to the frame bundle $FX$ of $X$. Then they obtain a characteristic map from $H^*(WQ_q)$ to the cyclic cohomology of the algebra $C^\infty_c(FX) \rtimes \Gamma_X$. While unfortunately the lack of an easily-defined “transverse exterior derivative” prevents a complete replication of the Connes–Moscovici construction in the non-étale case, we can use Theorem 3.3 to give, in codimension 1, a cyclic cocycle for the Godbillon–Vey invariant on the algebra $C^\infty_c(G_1; \Omega^{1,1}_M)$ (recall from [9] that $C^\infty_c(G_1; \Omega^{1,1}_M)$.
is the convolution algebra spanned by leafwise half-densities that are smooth with compact support in some Hausdorff open subset of the locally Hausdorff Lie groupoid $\mathcal{G}_1$.

Let us begin with a preliminary calculation. Suppose that $(M, \mathcal{F})$ is of codimension 1 (in which case $SO(1, \mathbb{R})$ is the trivial group so we need not concern ourselves with basic elements), and let $\alpha \in \Omega^1(Fr^+(N))$ correspond to a Bott connection on $N$ (we have dropped the $b$ superscript for notational simplicity). We obtain the corresponding connection forms

$$\alpha(0) = \alpha \otimes \mathcal{G}_1^0 = Fr^+(N)$$

and $\alpha(1)$ on $\Delta^1 \times \mathcal{G}_1^1$ defined by

$$\alpha'(t; u) := t(p_0^1)^* \alpha + (1 - t)(p_1^1)^* \alpha = tr^* \alpha + (1 - t) s^* \alpha$$

for $(t; u) \in \Delta^1 \times \mathcal{G}_1^1$. For simplicity let us denote $(p_i^1)^* \alpha$ by simply $\alpha_i$, $i = 0, 1$. Then since $q = 1$, the curvature of $\alpha'(1)$ is given simply by

$$R_{(t; u)}^{(1)} := dt \wedge (\alpha_0 - \alpha_1) + td\alpha_0 + (1 - t)d\alpha_1.$$ 

Now in $\mathcal{WO}_1$ the Godbillon–Vey invariant is given by the cocycle $h_{1c_1}$, which is mapped via the $\phi_W^{(1)}$ of Theorem 3.3 to the simplicial differential form

$$\alpha'(1) \wedge R^{(1)} = (t\alpha_0 + (1 - t)\alpha_1) \wedge (dt \wedge (\alpha_0 - \alpha_1) + td\alpha_0 + (1 - t)d\alpha_1)$$

$$= -dt \wedge (t\alpha_0 + (1 - t)\alpha_1) \wedge (\alpha_0 - \alpha_1) + (t\alpha_0 + (1 - t)\alpha_1) \wedge (td\alpha_0 + (1 - t)d\alpha_1)$$

on $\Delta^1 \times \mathcal{G}_1^1$. Integration over $\Delta^1$ then produces the form

$$\int_0^1 \alpha'(1) \wedge R^{(1)} = -\int_0^1 tdt \wedge \alpha_0 \wedge (\alpha_0 - \alpha_1) - \int_0^1 (1 - t)dt \wedge \alpha_1 \wedge (\alpha_0 - \alpha_1)$$

$$= -\frac{1}{2}(\alpha_0 + \alpha_1) \wedge (\alpha_0 - \alpha_1)$$

(15)

on $\mathcal{G}_1$. Equation (15) is geometrically opaque, and our immediate task now is to elucidate its geometric content. First, we will prove that the factor $\alpha_0 - \alpha_1$ has an interpretation as a line integral of the Bott curvature form $R$.

**Proposition 4.1** Let $(M, \mathcal{F})$ be codimension $q$, and let $\alpha \in \Omega^1(Fr^+(N); \mathfrak{gl}(q, \mathbb{R}))$ correspond to a Bott connection on $N$ with associated curvature $R \in \Omega^2(Fr^+(N); \mathfrak{gl}(q, \mathbb{R}))$. For $u \in \mathcal{G}_1$, let $\gamma : [0, 1] \to Fr^+(N)$ be any smooth path in a leaf of $\mathcal{F}_{Fr^+(N)}$ that represents $u$. Letting $p : TFr^+(N) \to N_{Fr^+(N)}$ denote the projection, for any $X \in T_u \mathcal{G}_1$ choose a smooth vector field $\tilde{X} \in \Gamma^\infty(\gamma([0, 1]); TFr^+(N))$ along $\gamma$ for which

1. $d\alpha u X = \tilde{X}_{\gamma(0)}$ and $d\rho u X = \tilde{X}_{\gamma(1)}$, and
2. the projection $Z = p\tilde{X} \in \Gamma^\infty(\gamma([0, 1]); N_{Fr^+(N)})$ of $\tilde{X}$ to a normal vector field is parallel along $\gamma$ with respect to the Bott connection $\nabla Fr^+(N)$ (see Proposition 2.4) for the foliation $\mathcal{F}_{Fr^+(N)}$ determined by $\alpha$.

Then

$$(\alpha_0 - \alpha_1)_u(X) = \int_{\gamma} R(\gamma, \tilde{X}).$$

(16)

In particular, the integral on the right hand side does not depend on the choices of $\gamma$ and $\tilde{X}$.

**Proof** That such a vector field $\tilde{X}$ can be chosen is a consequence of the surjectivity of the projection $p$ together with the definition of the parallel transport map for $N_{Fr^+(N)}$ along $\gamma$. More precisely, since $\mathcal{G}$ acts on $N$ by parallel transport with respect to $\nabla^p$, the projections of
Proposition 4.2 Let $\pi_{\mathcal{G}_1} : \mathcal{G}_1 \to \mathcal{G}$ be the projection induced by $\pi_{Fr^+(N)} : Fr^+(N) \to M$, and let $T_r \mathcal{G}_1$ and $T_s \mathcal{G}_1$ denote the tangent bundles to the range and source fibres of $\mathcal{G}_1$ respectively, so that the differentials of $r \circ \pi_{\mathcal{G}_1}$ and $s \circ \pi_{\mathcal{G}_1}$ define fibrewise isomorphisms

$N_1 := T\mathcal{G}_1 / (T_r \mathcal{G}_1 \oplus T_s \mathcal{G}_1 \oplus \ker(d\pi_{\mathcal{G}_1})) \to N$.

Then the formula in Eq. (16) depends only on the class $[X] \in (N_1)_u$ determined by $X$, and not on the choices of $\gamma$ and $\tilde{X}$.

Proof To see that $(\alpha_0 - \alpha_1)_u(X)$ depends only on the class of $X$ in $N_1$ we consider a perturbation $X' = X + Y$ of $X$ where $Y \in \ker(d\pi_{\mathcal{G}_1})$. Identify $\ker(d\pi_{\mathcal{G}_1})$ with $\mathcal{G} \times (\ker(d\pi_{Fr^+(N)}) = \mathcal{G} \times (Fr^+(N) \times \mathfrak{gl}(q, \mathbb{R}))$. By commutativity of the action of $\mathcal{G}$ on $Fr^+(N)$ with that of $GL^+(q, \mathbb{R})$, the action of $\mathcal{G}$ on the $\mathfrak{gl}(q, \mathbb{R})$ factor is by the identity, and we have $dr_u(Y) = ds_u(Y) = Y$. Since $\alpha$ is a connection form we have $\alpha(Y) = Y$ and therefore

$$(\alpha_0 - \alpha_1)_u(X + Y) = (\alpha_0 - \alpha_1)_u(X) + Y - Y = (\alpha_0 - \alpha_1)_u(X).$$

Now suppose that $X' = X + Z$, where $Z \in T\mathcal{G}_1^{(u)} \oplus T((\mathcal{G}_1)_{s(u)})$. Then both $dr_u Z$ and $ds_u Z$ are contained in $T \mathcal{F}_{Fr^+(N)}$ and therefore are annihilated by $\alpha$. Hence

$$(\alpha_0 - \alpha_1)_u(X + Z) = (\alpha_0 - \alpha_1)_u(X)$$

as required. \qed
In the paper [28], the differential form
\[
\partial \alpha = (p_0^1 \ast \alpha - (p_1^1 \ast \alpha = (e_0^1) \ast \alpha - (e_1^1) \ast \alpha = r^* \alpha - s^* \alpha \in \Omega^1(G_1; gl(q, \mathbb{R}))
\]
determined by a connection form \( \alpha \) appears as a measure of the failure of the connection form \( \alpha \) to be invariant under the action of \( G_1 \). In light of Proposition 4.1 we give any such differential form arising from a Bott connection a special name.

**Definition 4.3** Given a Bott connection form \( \alpha \in \Omega^1(Fr^+(N); gl(q, \mathbb{R})) \), we refer to the 1-form
\[
R^G := \partial \alpha = (r^* \alpha - s^* \alpha) \in \Omega^1(G_1; gl(q, \mathbb{R}))
\]
as the **integrated curvature** of \( \alpha \).

**Remark 4.4** Line integrals of differential forms such as in Proposition 4.1 are already of great use in determining de Rham representatives for loop space cohomology [6,7,19,34]. Since the holonomy groupoid is really a coarse sort of “path space”, in light of Proposition 4.1 it may be possible to obtain a characteristic map for the holonomy groupoid defined in terms of iterated line integrals. This would provide an exciting new geometric window into the existing theory of foliations, and has the potential to open up links with loop space theory. We leave this question to a future paper.

Let us now come back to the Godbillon–Vey invariant of a codimension 1 foliated manifold \((M, \mathcal{F})\). Denoting \( \alpha^G := -\frac{1}{2}(r^* \alpha + s^* \alpha) \) for notational simplicity, the differential form in (15) can now be written
\[
\int_0^1 \alpha^{(1)} \wedge R^{(1)} = \alpha^G \wedge R^G \in \Omega^2(G_1^{(1)}).
\]
Thus we have reconciled the Chern–Weil description of the Godbillon–Vey invariant, as “Bott connection wedge curvature”, with the image of the Godbillon–Vey invariant arising from the characteristic map of Theorem 3.3. Proposition 4.2 now allows us to integrate against \( \alpha^G \wedge R^G \) in the following way.

**Lemma 4.5** Let \((M, \mathcal{F})\) be a transversely orientable foliated n-manifold of codimension 1, and let \( a^0, a^1 \in C^\infty_c(G_1; \Omega^2) \). Then, setting \( x = \pi_{Fr^+(N)}(\phi) \) for \( \phi \in Fr^+(N) \), the formula
\[
g_v(a^0, a^1) := \int_{u \in G_1} a^0(u^{-1}, u \cdot \phi)a^1(u, \phi)((\alpha^G \wedge R^G)_{(u, \phi)})
\]
defines a compactly supported 1-density \( g_v(a^0, a^1) \in \Gamma(|Fr^+(N)|) \).

**Proof** Fix \( \phi \in Fr^+(N) \). For each \((u, \phi) \in G_1 \), we have
\[
a^0(u^{-1}, u \cdot \phi)a^1(u, \phi) \in |\mathcal{F}_{Fr^+(N)}|_{\phi} \otimes |\mathcal{F}_{Fr^+(N)}|_{u \cdot \phi},
\]
while \((\alpha^G \wedge R^G)_{(u, \phi)} \in \Lambda^2(T^u_{(\phi)}G_1) \), which we now describe using coordinates.

Consider a chart \( B_1 \times B_2 \times \mathbb{V} \times \mathbb{R}^+_t \) for \( G_1 \) about \((u, \phi)\), where \( B_1, B_2 \) are open balls in \( \mathbb{R}^{dim(F)} \) and \( V \) an open ball in \( \mathbb{R} \) such that \( B_2 \times V \cong U_2 \subset M \) is a foliated chart about \( \pi_{Fr^+(N)}(\phi) \) and \( B_1 \times h_\phi(V) \cong U_1 \) is a foliated chart about \( \pi_{Fr^+(N)}(u \cdot \phi) \), where \( h_\phi : V \rightarrow h_\phi(V) \subset \mathbb{R} \) is a holonomy diffeomorphism representing \( u \). By Proposition 4.2, in the local coordinates \((x_1^i)_{i=1}^{dim(F)}; (x_2^i)_{i=1}^{dim(F)}; z; t) \in B_1 \times B_2 \times \mathbb{V} \times \mathbb{R}^+_t \) we have that
\[ R^G = f_1 dz \] for some \( f_1 \) defined on \( B_1 \times B_2 \times V \times \mathbb{R}_+^+ \). Moreover, with \( t^{-1} dt \) the Maurer-Cartan form on \( \mathbb{R}_+^+ \), \( a^{G} \) is of the form \( f_2 dz + t^{-1} dt \) for some smooth \( f_2 \) defined on \( B_1 \times B_2 \times V \). Consequently, \( a^{G} \wedge R^G \) is of the form \( f t^{-1} dt \wedge dz \) for some smooth function \( f \) defined on \( B_1 \times B_2 \times V \times \mathbb{R}_+^+ \).

Since the coordinate differentials \( dz \) and \( dt \) span \( T_0^* Fr^+(N) \oplus T_0^* \mathcal{F}_{Fr^+(N)} \), at each point \((u, \phi) \in \mathcal{G}_1 \) we have that

\[
a^0(u^{-1}, u \cdot \phi) a^1(u, \phi) (a^{G} \wedge R^G)_{(u, \phi)} \in |Fr^+(N)|_\phi \otimes |T\mathcal{F}_{Fr^+(N)}|_{u \cdot \phi}.
\]

Then by compact support of \( a^0 \) and \( a^1 \), the integral

\[
gv(a^0, a^1) = \int_{u \in \mathcal{G}_2} a^0(u^{-1}, u \cdot \phi) a^1(u, \phi) (a^{G} \wedge R^G)_{(u, \phi)} \in |Fr^+(N)|_\phi
\]

is well-defined and \( gv(a^0, a^1) \) is a compactly supported density on \( Fr^+(N) \).

**Theorem 4.6** Let \((M, \mathcal{F})\) be a transversely orientable foliated \( n \)-manifold of codimension 1. Then for \( a^0, a^1 \in C_c^\infty(\mathcal{G}_1; \Omega^2_2) \) the formula

\[
\varphi_{gv}(a^0, a^1) := \int_{\phi \in Fr^+(N)} gv(a^0, a^1) \phi = \int_{(u, \phi) \in \mathcal{G}_1} a^0(u^{-1}, u \cdot \phi) a^1(u, \phi) (a^{G} \wedge R^G)_{(u, \phi)}
\]

defines a cyclic 1-cocycle \( \varphi_{gv} \) on the convolution algebra \( C_c^\infty(\mathcal{G}_1; \Omega^2_2) \).

**Proof** We work with Connes’ \( \lambda \)-complex [11]. For notational simplicity we denote elements of \( \mathcal{G}_1 \) by \( v_i \), and we use the notation \( \int_{v_0 v_1 v_2 \in Fr^+(N)} \) to mean the iterated integral over all triples \((v_0, v_1, v_2) \in \mathcal{G}_1^{(3)} \) for which \( v_0 v_1 v_2 \in Fr^+(N) \), followed by an integral over \( Fr^+(N) \). For \( a^0, a^1, a^2 \in C_c^\infty(\mathcal{G}_1; \Omega^2_1) \), we calculate

\[
\varphi_{gv}(a^0 a^1, a^2) = \int_{v_0 v_1 v_2 \in Fr^+(N)} a^0(v_0) a^1(v_1) a^2(v_2) (a^{G} \wedge R^G)_{v_2}
\]

\[
= \int_{v_0 v_1 v_2 \in Fr^+(N)} a^0(v_0) a^1(v_1) a^2(v_2) (\epsilon^2_0)^*(a^{G} \wedge R^G)_{(v_1, v_2)},
\]

\[
\varphi_{gv}(a^0 a^1 a^2) = \int_{v_0 v_1 v_2 \in Fr^+(N)} a^0(v_0) a^1(v_1) a^2(v_2) (a^{G} \wedge R^G)_{v_1 v_2}
\]

\[
= \int_{v_0 v_1 v_2 \in Fr^+(N)} a^0(v_0) a^1(v_1) a^2(v_2) (\epsilon^2_1)^*(a^{G} \wedge R^G)_{(v_1, v_2)}
\]

and

\[
\varphi_{gv}(a^2 a^0, a^1) = \int_{v_0 v_1 v_2 \in Fr^+(N)} a^2(v_2) a^0(v_0) a^1(v_1) (a^{G} \wedge R^G)_{v_1}
\]

\[
= \int_{v_0 v_1 v_2 \in Fr^+(N)} a^2(v_2) a^0(v_0) a^1(v_1) (\epsilon^2_2)^*(a^{G} \wedge R^G)_{(v_1, v_2)}.
\]

Because \( h_{11} c_1 \in WO_1 \) is closed under \( d \), the component \( a^{G} \wedge R^G \in \Omega^2(\mathcal{G}_1^{(1)}) \) of its image under the cochain map \( \psi_\alpha : WO_1 \to \Omega^*(\mathcal{G}_1^{(s)}) \) of Theorem 3.3 is closed under
\[ \partial : \Omega^2(G^{(1)}_1) \to \Omega^2(G^{(2)}_1). \]  

Thus

\[ b\varphi_{gv}(a^0, a^1, a^2) = \varphi_{gv}(a^0 a^1, a^2) - \varphi_{gv}(a^0, a^1 a^2) + \varphi_{gv}(a^2 a^0, a^1) \]

\[ = \int_{v_0 v_1 v_2 \in Fr^+(N)} a^0(v_0) a^1(v_1) a^2(v_2) \partial(\alpha^G \wedge R^G)_{(v_1,v_2)} \]

\[ = 0 \]

making \( \varphi_{gv} \) a Hochschild cocycle.

It remains only to check that \( \varphi_{gv}(a^0, a^1) = -\varphi_{gv}(a^1, a^0) \). For this, we observe that by definition \( R^G_\phi = \alpha_\phi - \alpha_\phi = 0 \) for any unit \( \phi \in G_1 \), hence

\[ 0 = \partial(\alpha^G \wedge R^G)_{(v^{-1},v)} = (\alpha^G \wedge R^G)_v - (\alpha^G \wedge R^G)_{v^{-1}} + (\alpha^G \wedge R^G)_{v^{-1}} \]

\[ = (\alpha^G \wedge R^G)_{v^{-1}} + (\alpha^G \wedge R^G)_v. \]

Therefore

\[ \varphi_{gv}(a^0, a^1) = \int_{v \in G_1} a^0(v^{-1}) a^1(v) (\alpha^G \wedge R^G)_v = - \int_{v^{-1} \in G_1} a^1(v) a^0(v^{-1})(\alpha^G \wedge R^G)_{v^{-1}} \]

\[ = - \varphi_{gv}(a^1, a^0) \]

making \( \varphi_{gv} \) a cyclic cocycle. \( \square \)

**Definition 4.7** We refer to the cyclic cocycle \( \varphi_{gv} \) on \( C^\infty_c(G_1; \Omega^2) \) given in Theorem 4.6 as the **Godbillon–Vey cyclic cocycle**.

**Remark 4.8** The Godbillon–Vey cyclic cocycle for \( C^\infty_c(G_1; \Omega^2) \) is the analogue of the Connes–Moscovici formula [15, Proposition 19] for the crossed product of a manifold by a discrete group action. Note that in contrast with the étale setting of Connes and Moscovici, the differential form \( \alpha^G \wedge R^G \) on \( G_1 \) with respect to which \( \varphi_{gv} \) is defined has, by Proposition 4.1, an explicit interpretation in terms of the integral of the Bott curvature along paths representing elements in \( G_1 \). Such a geometric interpretation is novel, and is completely lost in the étale setting that has been almost exclusively used in studying the secondary characteristic classes of foliations using noncommutative geometry.

Note that for the Godbillon–Vey cyclic cocycle to pair with the \( K \)-theory of the reduced \( C^* \)-algebra \( C^*_r(G_1) \), it is sufficient to know that \( C^\infty_c(G_1; \Omega^2) \) is smooth, in the sense that it is Fréchet (complete with jointly continuous multiplication with respect to a topology induced by a countable family of seminorms) and closed under the holomorphic functional calculus as a subalgebra of \( C^*_r(G_1) \) [8, Section 3]. While this is likely true, the author does not know of a reference. On the other hand, as will be shown below, the Godbillon–Vey cyclic cocycle is the Chern character of a semifinite spectral triple over \( C^\infty_c(G_1; \Omega^2) \), which does pair with the \( K \)-theory of \( C^*_r(G_1) \).

Extension of this construction to higher codimension, for arbitrary Gel’fand-Fuks classes, is not clear. This is due primarily to the lack of a “transverse exterior derivative” in the non-étale case, which prevents a simple extension of Lemma 4.5. The author expects that extension of this construction must instead proceed using Connes’ transverse fundamental class [11, Chapter 3, Section 7]; this is left to future work.

One has the following immediate corollary of Proposition 4.1 which, while completely unsurprising, is novel due to our non-étale perspective that incorporates the global transverse geometry of \( (M, \mathcal{F}) \).
Corollary 4.9  If \((M, \mathcal{F})\) is a codimension 1, transversely orientable foliated manifold with a flat Bott connection, then the Godbillon–Vey cyclic cocycle vanishes. \(\square\)

Our final task is to demonstrate that the Godbillon–Vey cyclic cocycle coincides with the cocycle obtained from the local index formula for the semifinite spectral triple considered in [29, Section 4.3]. For this purpose, it will be convenient to have a formula for the Godbillon–Vey cyclic cocycle in terms of a transverse volume form \(\omega \in \Omega^1(M)\) (that is, a form \(\omega\) that is nonvanishing and is identically zero on leafwise tangent vectors). By the final statement of Proposition 4.2, we know that we can write

\[
R_G = \delta (s \circ \pi^{(1)})^* \omega
\]

for some smooth function \(\delta : G_1 \to \mathbb{R}\), where \(s : G \to M\) is the source and where \(\pi^{(1)} : G_1 \to G\) is the projection. Since \(r^* \omega\) and \(s^* \omega\) both annihilate the tangents to the range and source fibres we can formulate the following definition.

Definition 4.10  Given a transverse volume form \(\omega \in \Omega^1(M)\), the smooth homomorphism \(\Delta : G \to \mathbb{R}_+\) defined by the equation

\[
R^\Delta = \Delta \delta s^* \omega
\]

is called the modular function or Radon-Nikodym derivative associated to \(\omega\).

Note that the face maps \(\epsilon^2_1 : G^{(2)} \to G\) of \(G\) satisfy

\[
s \circ e^2_0(u_1, u_2) = s \circ e^2_1(u_1, u_2) = s(u_2), \quad s \circ e^2_2(u_1, u_2) = r \circ e^2_0(u_1, u_2) = r(u_2)
\]

for all \((u_1, u_2) \in G^{(2)}\). Via a mild abuse of notation let us also denote the face maps of \(G_1\) by \(e^2_1\). Then the fact that \(R^\Delta = \partial \alpha\) gives \((e^2_0)^* R^\Delta - (e^2_1)^* R^\Delta + (e^2_2)^* R^\Delta = \partial^2 \alpha = 0\). Therefore, letting \(\pi^{(2)} : G_1^{(2)} \to G^{(2)}\) denote the projection, we have

\[
0 = \delta(u_2)(s \circ \pi^{(1)} \circ e^2_0)^* \omega_{(u_1, u_2)} - \delta(u_1 u_2)(s \circ \pi^{(1)} \circ e^2_1)^* \omega_{(u_1, u_2)}
\]

\[
+ \delta(u_1)(s \circ \pi^{(1)} \circ e^2_2)^* \omega_{(u_1, u_2)}
\]

\[
= \delta(u_2)(s \circ e^2_0 \circ \pi^{(2)})^* \omega_{(u_1, u_2)} - \delta(u_1 u_2)(s \circ e^2_1 \circ \pi^{(2)})^* \omega_{(u_1, u_2)}
\]

\[
+ \delta(u_1)(s \circ e^2_2 \circ \pi^{(2)})^* \omega_{(u_1, u_2)}
\]

\[
= (\delta(u_1) - \delta(u_1 u_2))(s \circ e^2_0 \circ \pi^{(2)})^* \omega_{(u_1, u_2)} + \delta(u_1)(r \circ e^2_0 \circ \pi^{(2)})^* \omega_{(u_1, u_2)}
\]

\[
= (\delta(u_2) - \delta(u_1 u_2) + \delta(u_1) \Delta(u_2))(s \circ e^2_0 \circ \pi^{(2)})^* \omega_{(u_1, u_2)}
\]

for all \((u_1, u_2) \in G^{(2)}\). Hence

\[
\delta(u_1 u_2) = \delta(u_2) + \delta(u_1) \Delta(u_2), \quad (u_1, u_2) \in G^{(2)}.
\]

Now the choice of \(\omega\) determines a trivialisation \(Fr^+ (N) \cong M \times \mathbb{R}_+\) in which we can write elements of \(G_1\) as \((u, x, t) \in G \times (M \times \mathbb{R}_+)\). By the arguments of the second paragraph in the proof of Lemma 4.5 we can now write

\[
(\alpha^G \wedge R_G)_{(u, x, t)} = \frac{\delta(u)}{t} dt \wedge \omega_x, \quad (u, x, t) \in G \times (M \times \mathbb{R}_+)
\]

so that our Godbillon–Vey cyclic cocycle becomes

\[
\varphi_{xv}(a^0, a^1) = - \int_{(x, t) \in M \times \mathbb{R}_+} \int_{u \in G_v} a^0(u^{-1}, \Delta(u)t) a^1(u, t) \frac{\delta(u)}{t} \omega_x \wedge dt
\]

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for \( a^0, a^1 \in C^\infty_c(\mathcal{G}_1; \Omega^1) \). In order to compare our formula with that in [29], we will want to assume that \( \mathcal{G}_1 \) is of the form \( \mathcal{G}_1 = Fr^+(N) \times \mathcal{G} \) rather than \( \mathcal{G} \times Fr^+(N) \). Now these groupoids are of course isomorphic via the map \( Fr^+(N) \times \mathcal{G} \ni (\phi, u) \mapsto (u, u^{-1} \cdot \phi) \in \mathcal{G} \times Fr^+(N) \), and a function \( a \in C^\infty_c(\mathcal{G} \times Fr^+(N); \Omega^1) \) identifies under this map with \( \tilde{a} \in C^\infty_c(Fr^+(N) \times \mathcal{G}; \Omega^1) \) given by

\[
\tilde{a}(\phi, u) := a(u, u^{-1} \cdot \phi), \quad (\phi, u) \in Fr^+(N) \times \mathcal{G}.
\]

With these identifications, and using the notational convention

\[
a_u(\phi) := a(\phi, u) \quad (\phi, u) \in Fr^+(N) \times \mathcal{G}
\]

for \( a \in C^\infty_c(Fr^+(N) \times \mathcal{G}; \Omega^1) \), we see that the Godbillon–Vey cyclic cocycle is defined for \( a^0, a^1 \in C^\infty_c(Fr^+(N) \times \mathcal{G}; \Omega^1) \) by the formula

\[
\varphi_{gv}(a^0, a^1) = -\int_{[x,t) \in M \times \mathbb{R}^+} \int_{u \in \mathcal{G}^x} a^0_u(x, t) a^1_{u^{-1}}(u^{-1} \cdot x, \Delta(u^{-1})t) \frac{\delta(u^{-1})}{t} \omega_x \wedge dt. \quad (19)
\]

Let us now recall the cocycle \( \varphi_1 \) obtained via the local index formula in [29, Section 4.3]. In the coordinates we have chosen in this paper, \( \varphi_1 \) is given by the equation

\[
\begin{align*}
\varphi_1(a^0, a^1) &= -(2\pi i)^{\frac{1}{2}} \int_{[x,t) \in M \times \mathbb{R}^+} \int_{u \in \mathcal{G}^x} a^0_u(x, t) a^1_{u^{-1}}(u^{-1} \cdot x, \Delta(u^{-1})t) \frac{\partial \log \Delta(u^{-1})}{t} \omega_x \wedge dt \\
&= (20)
\end{align*}
\]

for \( a^0, a^1 \in C^\infty_c(\mathcal{G}_1; \Omega^1) \). Thus in order to conclude that the index formula \( \varphi_1 \) of [29] and the cyclic cocycle \( \varphi_{gv} \) of Eq. (19) coincide (up to the constant multiple \((2\pi i)^{\frac{1}{2}}\)), we need only show that \( \delta = \partial \log \Delta \). This will be a consequence of the following fact, which holds for foliations of arbitrary codimension and gives a geometric interpretation (in the non-étale setting) for the off-diagonal term appearing in the triangular structures considered by Connes [10, Lemma 5.2] and Connes–Moscovici [12, Part I].

**Proposition 4.11** Let \((M, \mathcal{F})\) be a transversely orientable foliated manifold of codimension \( q \), and let \( \alpha^x \in \Omega^1(Fr^+(N); gl(q, \mathbb{R})) \) be a Bott connection form. Let \( H := \ker(a^\alpha)/T F_r^+(N) \) be the horizontal normal bundle determined by \( \alpha^x \), let \( V := \ker(d\pi_{Fr^+(N)}) \) be the vertical tangent bundle, and let

\[
N_{Fr^+(N)} = V \oplus H \cong Fr^+(N) \times (gl(q, \mathbb{R}) \oplus \mathbb{R}^q)
\]

be the corresponding decomposition of \( N_{Fr^+(N)} \), with \( V \) and \( H \) trivialised as in the proof of Proposition 2.4. With respect to this decomposition, for \( u \in \mathcal{G} \) and for any \( \phi \in Fr^+(N)_{\pi(u)} \), the action \( u_{Fr^+(N)} : (N_{Fr^+(N)}_\phi \to (N_{Fr^+(N)}_{\pi(u)}) \) can be written

\[
u_{Fr^+(N)} = \begin{pmatrix} id_{gl(q, \mathbb{R})} & R_u^\mathcal{G} \\ 0 & id_{\mathbb{R}^q} \end{pmatrix}.
\]

**Proof** That the top left corner is \( id_{gl(q, \mathbb{R})} \) follows from the commutativity of the left action of \( \mathcal{G} \) on \( Fr^+(N) \) with the right action of \( GL^+(q, \mathbb{R}) \), and the bottom left entry is zero for the same reason. For the bottom right corner, we note that equivariance of the map \( \pi_{Fr^+(N)} : Fr^+(N) \to M \) with respect to the action of \( \mathcal{G} \) implies that the induced fibrewise isomorphisms \( \pi_\phi : H_\phi \to N_{\pi_{Fr^+(N)}(\phi)} \) are also equivariant. Thus if \( [v_\phi] \in H_\phi \), denoting
\[ u_{22} \cdot [v_\phi] := \proj_H (u_* [v_\phi]) \]
and letting \( u_* : N_{s(u)} \to N_{r(u)} \) denote the action of \( u \) on \( N \), we have
\[
((u \cdot \phi)^{-1} \circ \pi_{u \cdot \phi}) (u_{22} \cdot [v_\phi]) = (\phi^{-1} \circ u_*^{-1} \circ u_\phi \circ \pi_\phi ([v_\phi])) = (\phi^{-1} \circ \pi_\phi)(v_\phi)
\]
giving the bottom right entry of (21).

Finally we come to the top right entry. Since a connection form maps vertical vectors to themselves, the top right entry of (21) is the map which sends \([v_\phi] \in H\) to
\[
\alpha^b(u_{Fr} + N_\ast)[v_\phi]
\]
where \( u_{Fr} + N_\ast \) is any element of \( TF_{Fr} + (N) \). Since \( v_\phi \) is contained in \( \ker(\alpha^b) \), we can equally regard the top right entry as the map which sends \([v_\phi] \in \ker(\alpha^b)/TF_{Fr} + (N)\) to
\[
\alpha^b(u_{Fr} + N_\ast)v_\phi - \alpha^b(v_\phi),
\]
which by (16) in Proposition 4.1 coincides with \( R_G^u(v_\phi) \), the well-definedness of which is due to Proposition 4.2.

Coming back to our codimension 1 foliation \((M, \mathcal{F})\), recall [29, p. 23] that the function \( \partial \log \Delta \) on \( \mathcal{G} \) is by definition the top right corner of the matrix in Eq. (21) that gives the action of \( \mathcal{G} \) on \( N_{Fr} + (N) \). Therefore
\[
\delta = \partial \log \Delta
\]
as required, proving the following result.

\textbf{Theorem 4.12} The Godbillon–Vey cyclic cocycle of Eq. (19) coincides with the local index formula cocycle in Eq. (20) for the semifinite spectral triple considered in [29, Section 4.3].

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