Combinatorial proofs of inverse relations and log-concavity for Bessel numbers

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November 14, 2018

Abstract

Let the Bessel number of the second kind $B(n, k)$ be the number of set partitions of $[n]$ into $k$ blocks of size one or two, and let the Bessel number of the first kind $b(n, k)$ be the coefficient of $x^{n-k}$ in $-y_{n-1}(-x)$, where $y_n(x)$ is the $n$th Bessel polynomial. In this paper, we show that Bessel numbers satisfy two properties of Stirling numbers: The two kinds of Bessel numbers are related by inverse formulas, and both Bessel numbers of the first kind and the second kind form log-concave sequences. By constructing sign-reversing involutions, we prove the inverse formulas. We review Krattenthaler’s injection for the log-concavity of Bessel numbers of the second kind, and give a new explicit injection for the log-concavity of signless Bessel numbers of the first kind.

1 Introduction

1.1 Main Results

For nonnegative integer $n$ and $k$, let $B(n, k)$ be the set of partitions of $[n] := \{1, \ldots, n\}$ into $k$ blocks of size one or two. Let $B(n, k)$ denote the number of elements in $B(n, k)$. The Bessel polynomials \cite{1, 2, 3, 5} are the polynomial solutions $y_n(x)$ of the second-order differential equations

$$x^2 y_{n}'' + (2x + 2) y_n' = n(n+1)y_n, \quad y_n(0) = 1.$$ 

Let $a(n, k)$ denote the coefficient of $x^{n-k}$ in $y_{n-1}(x)$. In fact, $a(n, k)$ equals the cardinality of the set $B(2n - k - 1, n - 1)$. Set $b(n, k) := (-1)^{n-k} a(n, k)$.

In this paper we deduce following two identities.

$$\sum_{k=0}^{n} B(n, k) b(k, l) = \delta_{n,l} \quad \text{and} \quad \sum_{k=0}^{n} b(n, k) B(k, l) = \delta_{n,l}.$$ 

(1)

These identities are reminiscent for the inverse formulas of Stirling numbers \cite{9} Prop 1.4.1. By constructing sign-reversing involutions, we give combinatorial proofs for both identities.

Second, we show the log-concavity of $\{ a(n, k) \}_{1 \leq k \leq n}$, i.e., that

$$a(n, k - 1) \cdot a(n, k + 1) \leq a(n, k)^2,$$

which is an analogue of the log-concavity of the signless Stirling numbers of the first kind. We prove this result by constructing an explicit injection from the set $B(2n - k, n - 1) \times B(2n - k - 2, n - 1)$ to the set $B(2n - k - 1, n - 1) \times B(2n - k - 1, n - 1)$, which provides a combinatorial proof of the log-concavity of the sequence.

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# 1.2 Notations and Backgrounds

We review some standard terminology, which will be used throughout the paper.

**Stirling numbers**

For a nonnegative integer $n$, let $[n]$ denote the set $\{1, 2, \ldots, n\}$, and let $[0] = \emptyset$. A partition of $[n]$ is a collection of nonempty pairwise disjoint subsets whose union is $[n]$. These disjoint subsets of $[n]$ are called blocks of the set partition. The number of partitions of $[n]$ with $k$ blocks $B_1, \ldots, B_k$ is called the Stirling number of the second kind $S(n, k)$. By convention, we put $S(0, 0) = 1$. We denote by $S_n$ the set of all permutations of $[n]$. Let $c(n, k)$ be the number of permutations in $S_n$ that have $k$ cycles. Define $s(n, k)$ to be $(-1)^{n-k} c(n, k)$. This number $s(n, k)$ is called the Stirling number of the first kind. It is well-known that Stirling numbers are related to one another by the following inverse formulas [9, Prop 1.4.1]:

$$
\sum_{k=0}^{n} S(n, k) s(k, l) = \delta_{n,l} \quad \text{and} \quad \sum_{k=0}^{n} s(n, k) S(k, l) = \delta_{n,l}.
$$

Moreover, both Stirling numbers satisfy the log-concavity property, i.e.,

$$
c(n, k-1) \cdot c(n, k+1) \leq c(n, k)^2 \quad \text{and} \quad S(n, k-1) \cdot S(n, k+1) \leq S(n, k)^2.
$$

There are many proofs for (2) and (3) with both algebraic and combinatorial ways. See [9, 10, 11].

**Bessel numbers**

Recall that the Bessel polynomials are the polynomial solutions $y_n(x)$ of the second-order differential equations

$$
x^2 y_n'' + (2x + 2) y_n' = n(n + 1) y_n
$$

satisfying the initial condition $y_n(0) = 1$. Bochner [11] seems first to have realized that these polynomials are closely related to the Bessel functions. Krall and Frink [5] have considered the system of Bessel polynomials in connection with certain solutions of the wave equations. Moreover, Burchnall [2] has developed more properties of the Bessel polynomials in detail. Recently, Choi and Smith [3] have investigated coefficients of Bessel polynomials from a combinatorial point of view.

From the differential equation (4) one can easily derive the formula:

$$
y_n(x) = \sum_{k=0}^{n} \frac{(n+k)!}{2^k k! (n-k)!} x^k.
$$

Let $a(n, k)$ denote the coefficient of $x^{n-k}$ in $y_{n-1}(x)$. Set $b(n, k) := (-1)^{n-k} a(n, k)$, i.e.,

$$
b(n, k) := \begin{cases} 
(-1)^{n-k} \frac{(2n-k-1)!}{2^n k! (n-k)! (k-1)!}, & \text{if } 1 \leq k \leq n, \\
0, & \text{if } 1 \leq n < k.
\end{cases}
$$

We call the number $b(n, k)$ a Bessel number of the first kind, and $a(n, k)$ a signless Bessel number of the first kind. By convention, we put $a(0, k) = b(0, k) = \delta_{0,k}$.

For nonnegative integers $n$ and $k$, define $B(n, k)$ to be the set of partitions of $[n]$ into $k$ nonempty blocks of each size one or two. A block $B$ is called a singleton if $|B| = 1$, and a pair if $|B| = 2$. Note that if $\pi \in B(n, k)$, then $\pi$ has exactly $2k-n$ singletons and $n-k$ pairs. Set $B(n, k) := |B(n, k)|$. To choose $n-k$ pairs, we should choose $2n-2k$ elements from $[n]$ and pairing the chosen $2n-2k$ elements, so the number $B(n, k)$ is given by

$$
B(n, k) := \begin{cases} 
n!, & \text{if } \lfloor n/2 \rfloor \leq k \leq n, \\
2^{n-k} (n-k)! (2k-n)!, & \text{if } k = \lfloor n/2 \rfloor - 1, \text{ and otherwise.}
\end{cases}
$$

We call the number $B(n, k)$ a Bessel number of the second kind.

**Remark 1.** It is easily checked that $a(n, k) = B(2n - k - 1, n - 1)$. So we can consider $a(n, k)$ as the cardinality of the set $B(2n - k - 1, n - 1)$. 

[11]
Matching number

Let $G$ be a loopless graph with $n$ vertices. A $k$-matching in a graph $G$ is a set of $k$ edges, no two of which have a vertex in common. A matching $\alpha$ saturates a vertex $x$, or $x$ is said to be saturated under $\alpha$, if an edge in $\alpha$ is incident with $x$; otherwise, $x$ is unsaturated under $\alpha$. The complete graph $K_n$ with $n$ vertices is a simple graph with each pair of whose distinct vertices is adjacent. Let $\mathcal{M}(n, k)$ denote the set of $k$-matchings in complete graph $K_n$ and $m(n, k)$ be the cardinality of $\mathcal{M}(n, k)$. We call $m(n, k)$ a matching number. From now on, assume that all vertices of $K_n$ are labeled with the integers in $\{1, \ldots, n\}$. Under this assumption, a partition of $\{1, \ldots, n\}$ into $k$ nonempty blocks, each of size at most two, can be identified with an $(n - k)$-matching in $K_n$ canonically. For example, the partition $\pi = \{\{1\}, \{2, 6\}, \{3\}, \{4, 5\}\}$ in $B(6, 4)$, corresponds to the matching $\alpha = \{\{2, 6\}, \{4, 5\}\}$ in $M(6, 2)$. Hence $B(n, k) = m(n, n - k)$. Whether $\{a, b\}$ is an edge or a pair, we will write it such that $a < b$.

It is well known (see [4, 6, 7]) that the matching numbers $\{m(n, k)\}_{0 \leq k \leq n}$ form a log-concave sequence, i.e., that

$$m(n, k - 1) \cdot m(n, k + 1) \leq m(n, k)^2,$$

which implies that the Bessel numbers of the second kind $B(n, k)$ is also log-concave. In particular, Krattenthaler [6] has given a neat combinatorial proof of (7). We review Krattenthaler’s proof in Section 4.1.

2 Inverse formulas for the Bessel numbers

In this section we deduce the inverse formulas for the Bessel numbers [11] by the generating function technic. For indeterminates $x$ and $t$, let

$$\sum_{n=0}^{\infty} f_n(x) \frac{t^n}{n!} := \left(1 + t + \frac{t^2}{2!}\right)^x.$$

Note that $\{f_n(x)\}$ is a sequence of polynomials of binomial type, i.e.,

$$f_n(x + y) = \sum_{k=0}^{n} \binom{n}{k} f_k(x) f_{n-k}(y).$$

For details on polynomials of binomial type, see [8] pp. 167–213].

Lemma 2.1. For any nonnegative integer $n$, we have

$$f_n(x) = \sum_{k=0}^{n} B(n, k) \cdot [x]_k,$$

where $[x]_0 = 1$ and $[x]_k = x(x-1)\cdots(x-k+1)$, for $k \geq 1$.

Proof. We have

$$f_n(x) = \left[ \frac{t^n}{n!} \right] \left(1 + t + \frac{t^2}{2!}\right)^x$$

$$= \left[ \frac{t^n}{n!} \right] \sum_{k \geq 0} \binom{n}{k} \left(1 + \frac{t^2}{2!}\right)^k$$

$$= \sum_{k \geq 0} \left[ \frac{t^n}{n!} \right] \left(1 + \frac{t^2}{2!}\right)^k \cdot [x]_k$$

$$= \sum_{k \geq 0} B(n, k) \cdot [x]_k.$$

The last equation holds due to the exponential formula [11] p. 81].
Lemma 2.2. For any nonnegative integer \( n \), we have

\[
[x]_n = \sum_{k=0}^{n} b(n, k) f_k(x).
\]

Proof. From the formula (6) and Lemma 2.1, we have

\[
f_n(x) = \sum_{k=0}^{n} 2^{n-k} \frac{n!}{(n-k)! (2k-n)!} [x]_k.
\]

Multiplying both sides of this equation by \( 2^n/n! \) yields

\[
\frac{2^n}{n!} f_n(x) = \sum_{k=0}^{n} \frac{1}{(n-k)! (2k-n)!} 2^k [x]_k
\]

\[= \sum_{k=0}^{n} \left( \frac{k}{n-k} \right) \frac{2^k}{k!} [x]_k.
\]

Wilf [11, p. 169] shows, using the Lagrange Inversion Formula, that if two sequences \( \{a_n\} \) and \( \{b_n\} \) are related by

\[b_n = \sum_k \left( \frac{k}{n-k} \right) a_k,
\]

then we have the inversion

\[n a_n = \sum_k \left( \frac{2n-k-1}{n-k} \right) (-1)^{n-k} k b_k.
\]

So we have

\[n \frac{2^n}{n!} [x]_n = \sum_k \left( \frac{2n-k-1}{n-k} \right) (-1)^{n-k} k \frac{2^k}{k!} f_k(x).
\]

Multiplying both sides of this equation by \( 2^{-n} (n-1)! \) yields that

\[[x]_n = \sum_k (-1)^{n-k} \frac{(2n-k-1)!}{2^{n-k} (n-k)! (k-1)!} f_k(x).
\]

From the formula (6), we have

\[[x]_n = \sum_k b(n, k) f_k(x),
\]

which completes the proof.

Note that both \([x]_n\) and \(f_n(x)\) are monic polynomials of degree \( n \). By Lemma 2.1 and 2.2, we have the following inverse formulas for the Bessel numbers.

Theorem 2.3. Let \( b(n, k) \) be the Bessel number of the first kind and \( B(n, k) \) be the Bessel number of the second kind. Then two Bessel numbers are related to as follow s:

\[
\sum_{k=0}^{n} B(n, k) b(k, l) = \delta_{n,l}, \tag{8}
\]

\[
\sum_{k=0}^{n} b(n, k) B(k, l) = \delta_{n,l}. \tag{9}
\]

In Section 3 we prove these two inverse formulas (8) and (9) combinatorially.
3 Two involutions for the inverse formulas

Recall that a partition of \([n]\) into \(k\) blocks of size one or two can be identified with an \((n-k)\)-matching in \(K_n\). So we can describe the Bessel numbers in terms of the matching numbers \(m(n,k)\) as follows:

\[
B(n,k) = m(n, n-k) \quad \text{and} \quad b(n,k) = (-1)^{n-k} a(n,k) = (-1)^{n-k} m(2n-k-1, n-k).
\]

3.1 An involution for the first inverse formula

Theorem 3.1. For all nonnegative integers \(n\) and \(l\), we have

\[
\sum_{k=0}^{n} B(n,k) b(k,l) = \delta_{n,l}.
\]

Proof. Consider the complete graph \(K_{2n-l-1}\). For arbitrary \(k\), let \(U\) be the set of ordered pairs \((\alpha, \beta)\) such that

1. \(\alpha\) is an \((n-k)\)-matching in \(K_{2n-l-1}\) whose saturated vertices are in \([n]\); and

2. \(\beta\) is a \((k-l)\)-matching in \(K_{2n-l-1}\) such that \(\alpha\) and \(\beta\) saturate no vertex in common.

Define the sign of an element \((\alpha, \beta) \in U\) by \(\text{sgn}(\alpha, \beta) = (-1)^{|\beta|}\), where \(|\beta|\) denotes the number of edges in \(\beta\). Then we have

\[
\sum_{(\alpha, \beta) \in U} \text{sgn}(\alpha, \beta) = \sum_{k} (-1)^{k-l} m(n, n-k) m(2k-l-1, k-l) = \sum_{k} B(n,k) b(k,l).
\]

To prove the theorem, it suffices to find a sign-reversing involution \(I_1\) on \(U\) having exactly one fixed point with \(\text{sgn} +1\).

Consider the union of \(\alpha\) and \(\beta\). Define a linear order on \(\alpha \cup \beta\) as follows: For edges \(\{a, b\}\) and \(\{c, d\}\) in \(\alpha \cup \beta\), \(\{a, b\} < \{c, d\}\) if and only if \(b < d\). Note that this order is the same as the colex order (see [10] p. 8).

With respect to this order, take the smallest edge \(e\) of \(\alpha \cup \beta\) and move \(e\) from \(\alpha\) to \(\beta\), if \(e\) is in \(\alpha\); otherwise, move \(e\) from \(\beta\) to \(\alpha\). This defines the needed involution \(I_1\) on \(U\).

Since there are \(n-l\) edges in \(\alpha \cup \beta\), for the smallest edge \(\{a, b\}\) in \(\alpha \cup \beta\), the number of vertices greater than \(b\) in \([2n-l-1]\) is at least \(n-l-1\). This implies that \(b \leq (2n-l-1) - (n-l-1) = n\). So \(I_1\) is well-defined. Obviously, \(I_1\) changes the number of elements of \(\beta\) by one, i.e., \(I_1\) is sign-reversing and double applications of \(I_1\) on \((\alpha, \beta)\) will clearly restore \((\alpha, \beta)\). This construction cannot be accomplished, if both \(\alpha\) and \(\beta\) are empty sets, i.e., \(n = k = l\). Thus we obtain the desired result.

Example 1. Given \(n = 7, k = 5, l = 2\), let \(\alpha = \{\{2,3\}, \{4,7\}\}\) and \(\beta = \{\{1,10\}, \{5,11\}, \{8,9\}\}\). Then \(\alpha\) and \(\beta\) are matchings in \(K_{11}\) satisfying \((\alpha, \beta) \in U\). Take the smallest edge \(e = \{2,3\}\) in \(\alpha \cup \beta\). Since \(e\) is an element of \(\alpha\), the image of \((\alpha, \beta)\) under \(I_1\) is \((\alpha', \beta')\), where \(\alpha' = \alpha \setminus \{e\} = \{\{4,7\}\}\) and \(\beta' = \beta \cup \{e\} = \{\{1,10\}, \{2,3\}, \{5,11\}, \{8,9\}\}\). It is easy to check that \(I_1(\alpha', \beta') = (\alpha, \beta)\). (See Figure [I])
3.2 An involution for the second inverse formula

**Theorem 3.2.** For all nonnegative integers $n$ and $l$, we have

$$
\sum_{k=0}^{n} b(n,k) B(k,l) = \delta_{n,l}.
$$

**Proof.** Consider the complete graph $K_{2n-k}$. Let $V_k$ be the set of ordered pairs $(\alpha, \beta)$ such that

1. $\alpha$ is an $(n-k)$-matching in $K_{2n-k}$ in which the vertex $2n-k$ is unsaturated under $\alpha$; and
2. $\beta$ is a $(k-l)$-matching in $K_{2n-k}$ such that $\alpha$ and $\beta$ saturate no vertex in common.

Set $V = \bigcup V_k$. Define the sign of an element $(\alpha, \beta) \in V$ by $\text{sgn}(\alpha, \beta) = (-1)^{|\alpha|}$. Then we get

$$
\sum_{(\alpha, \beta) \in V} \text{sgn}(\alpha, \beta) = \sum_{k} (-1)^{n-k} m(2n-k-1, n-k) m(k, k-l) = \sum_{k} b(n,k) B(k,l).
$$

To prove the theorem, it suffices to find a sign-reversing involution $I_2$ on $V$, which has exactly one fixed point with sign $+1$.

Suppose that $(\alpha, \beta) \in V_k$. Now consider the largest edge $e$ of $\alpha \cup \beta$ according to the colex order in $\alpha \cup \beta$. Define the map $I_2$ as follows:

- If $e \in \alpha$, then set $I_2(\alpha, \beta) = (\alpha \setminus \{e\}, \beta \cup \{e\})$. Since $e \in \alpha$, neither $\alpha$ nor $\beta$ saturates the vertex $2n-k$. Moreover, the matching $\alpha \setminus \{e\}$ cannot saturate the vertex $2n-k-1$. So $(\alpha \setminus \{e\}, \beta \cup \{e\})$ belongs to $V_{k+1}$.
- If $e \in \beta$, then set $I_2(\alpha, \beta) = (\alpha \cup \{e\}, \beta \setminus \{e\})$. Regard $(\alpha \cup \{e\}, \beta \setminus \{e\})$ as a pair of matchings in $K_{2n-k+1}$. Clearly, the matching $\alpha \cup \{e\}$ cannot saturate the vertex $2n-k+1$. Thus $(\alpha \cup \{e\}, \beta \setminus \{e\})$ belongs to $V_{k-1}$.

Evidently $(I_2 \circ I_2)(\alpha, \beta) = (\alpha, \beta)$ and $I_2$ is a sign-reversing map, whenever $\alpha \cup \beta \neq \emptyset$. The unique fixed point of $I_2$ is $(\alpha, \beta)$, where $\alpha$ and $\beta$ are both empty, i.e., $n = k = l$. This completes the proof. \qed

**Example 2.** Given $n=10$, $k=8$, $l=5$, let $\alpha = \{2,3\}, \{4,11\}$ and $\beta = \{1,7\}, \{5,10\}, \{8,9\}$. Then $\alpha$ and $\beta$ are matchings in $K_{12}$ satisfying $(\alpha, \beta) \in V_8$. Take the largest edge $e = \{4, 11\}$ in $\alpha \cup \beta$. Since $e$ is an element of $\alpha$, the image of $(\alpha, \beta)$ under $I_2$ is $(\alpha', \beta') \in V_9$, where $\alpha' = \alpha \setminus \{e\} = \{2,3\}$ and $\beta' = \beta \cup \{e\} = \{1,7\}, \{4,11\}, \{5,10\}, \{8,9\}$. It is easy to check that $I_2(\alpha', \beta') = (\alpha, \beta)$. (See Figure 2.)

4 Log-concavity of the Bessel numbers

In this section, we show that both Bessel numbers of the second kind $B(n,k)$ and signless Bessel numbers of the first kind $a(n,k)$ form log-concave sequences. In particular, we will check each result by constructing an explicit injection.
Figure 3: The injection $I_K$.  

4.1 Log-concavity of the Bessel numbers of the second kind

**Theorem 4.1.** The Bessel numbers of the second kind \( \{ B(n, k) \}_{k \geq 0} \) form a log-concave sequence:

\[
B(n, k - 1) \cdot B(n, k + 1) \leq B(n, k)^2 \quad \text{for all } k \geq 1.
\]

**Proof.** In fact, this theorem is a special case of the log-concavity of the sequence of matching numbers \([1, 4, 10]\). For reader’s convenience, we sketch here Krathenttaler’s idea \([6]\). Since \( B(n, k) = m(n-n-k) \), it suffices to show that

\[
m(n-n-k+1) \cdot m(n-n-k-1) \leq m(n-n-k)^2.
\]

We will construct an injection \( I_K \) from the set of all pairs \((\alpha_1, \alpha_2)\) of an \((n-k-1)\)-matching \(\alpha_1\) in \(K_n\) and an \((n-k)\)-matching \(\alpha_2\) in \(K_n\) into the set of all pairs \((\beta_1, \beta_2)\) of an \((n-k)\)-matching \(\beta_1\) in \(K_n\) and an \((n-k-1)\)-matching \(\beta_2\) in \(K_n\). Let an \((n-k+1)\)-matching \(\alpha_1\) in \(K_n\) and an \((n-k-1)\)-matching \(\alpha_2\) in \(K_n\) be given. Color the edges of \(\alpha_1\) and \(\alpha_2\) in blue and red, respectively.

Let \(G = G[\alpha_1 \cup \alpha_2]\) denote the subgraph of \(K_n\) induced by all edges of \(\alpha_1\) and \(\alpha_2\). Consider the connected components of \(G\). Since \(\alpha_1\) and \(\alpha_2\) are matchings, each connected component of \(G\) is either a cycle or a path. Note that all edges in the components of \(G\) are colored alternately. So all cycles and paths of even length have an equal number of blue and red edges. Meanwhile, in case a path of odd length, the number of blue edges in the path differs from the number of red edges by one. We call a path of odd length a blue path, if it has more blue edges than red edges, and a red path, otherwise.

Let \(R\) be the set of all red paths of \(G\) and let \(r\) denote the cardinality of \(R\). Since \(G\) has two more blue edges than red, the number of all blue paths of \(G\) should be \(r + 2\). Consider the set \(C\) of all blue and red paths of \(G\). Label the paths of \(C\) from 1 to \(2r + 2\) in the increasing order of the largest vertices in the paths.

Let \(A = \{a_1, \ldots, a_r\} \subset [2r + 2]\), where \(a_1 < \cdots < a_r\). Define \(\phi(A) = A \cup \{a_j + 1\}\), where \(j\) is the largest \(i\) for which \(a_i - 2i\) is minimal (assume \(a_0 = 0\)). Then the map \(\phi\) is an injection from all \(r\)-element subsets of \([2r + 2]\) into \((r + 1)\)-element subsets of \([2r + 2]\). (See \([11]\) pp. 33–39 and p. 54.)

Thus given the \(r\)-element subset \(S\) of \([2r + 2]\) corresponding to \(R\), we have an \((r + 1)\)-element subset \(\phi(S) = S \cup \{t\}\). Note that the element \(t\) corresponds to a blue path, say \(P\). If we exchange the colors in \(P\), then \(P\) becomes a red path, say \(P'\). Now we consider the graph \(G' = (G \setminus P) \cup P'\). Obviously, \(G'\) has \(n-k\) blue edges and \(n-k\) red edges.

Finally set \(\beta_1\) be the blue edges in \(G'\) and \(\beta_2\) be the red edges in \(G'\). Let \(I_K(\alpha_1, \alpha_2) = (\beta_1, \beta_2)\). It is easy to check that this defines the desired injection.

**Example 3.** Let \(\alpha_1 = \{1, 2\}, \{3, 4\}, \{6, 11\}, \{7, 12\}, \{13, 14\}, \{16, 17\}, \{19, 20\}, \{21, 22\}, \{23, 24\}\) and \(\alpha_2 = \{2, 3\}, \{6, 7\}, \{8, 9\}, \{11, 12\}, \{14, 15\}, \{16, 21\}, \{19, 24\}\). Then \(\alpha_1\) and \(\alpha_2\) are matchings in \(K_{25}\). Color the edges of \(\alpha_1\) in blue (−) and the edges of \(\alpha_2\) in red (· · ·). There are three blue paths \(1-2-3-4, 17-16-\cdots-21-22\) and \(20-19-\cdots-24-23\) and one red path \(8\cdots 9\), which are labeled by \(1, 3, 4\) and \(2\), respectively. So we have \(r = 1\) and \(S = \{2\}\). Since \(\phi(\{2\}) = \{2, 3\}\), the blue path \(P = 17-16-\cdots-21-22\) is changed to the red path \(P' = 17-16-\cdots-21-22\). From the graph \(G' = (G \setminus P) \cup P'\), we can extract two 8-matchings in \(K_{25}\) : \(\beta_1 = \{1, 2\}, \{3, 4\}, \{6, 11\}, \{7, 12\}, \{13, 14\}, \{16, 21\}, \{19, 20\}, \{23, 24\}\) and \(\beta_2 = \{2, 3\}, \{6, 7\}, \{8, 9\}, \{11, 12\}, \{14, 15\}, \{16, 17\}, \{19, 24\}, \{21, 22\}\). (See Figure 3.)
4.2 Log-concavity of the signless Bessel numbers of the first kind

**Theorem 4.2.** The sequence of signless Bessel numbers of the first kind \( \{a(n,k)\}_{k \geq 0} \) is log-concave:

\[
a(n,k-1) \cdot a(n,k+1) \leq a(n,k)^2 \quad \text{for all } k \geq 1.
\]

**Proof.** Since \( a(n,k) = B(2n-k-1,n-1) = m(2n-k-1,n-k) \), this theorem is equivalent to the log-concavity of the sequence \( m(2n-k-1,n-k) \), for \( k = 0, \ldots, n \). So it suffices to find an injection from the set of all pairs \((A_1,A_2)\) of an \((n-k-1)\)-matching \(A_1\) in \(K_{2n-k}\) and an \((n-k)\)-matching \(A_2\) in \(K_{2n-k-2}\) into the set of all pairs \((B_1,B_2)\) of an \((n-k)\)-matching \(B_1\) in \(K_{2n-k-1}\) and an \((n-k)\)-matching \(B_2\) in \(K_{2n-k-2}\).

Now we construct a new injection \(I_S\) motivated by Krattenthaler’s injection. Let an \((n-k+1)\)-matching \(A_1\) in \(K_{2n-k}\) and an \((n-k)\)-matching \(A_2\) in \(K_{2n-k-2}\) be given. Assign the colors blue to the edges of \(A_1\) and red to the edges of \(A_2\), respectively. Then there are two possible cases:

(a) The vertex \(2n-k\) is unsaturated under \(A_1\).

(b) The vertex \(2n-k\) is saturated under \(A_1\).

In case (a), if we disregard the vertex \(2n-k\) in \(K_{2n-k}\) and regard \(K_{2n-k-2}\) as the subgraph of \(K_{2n-k-1}\) induced by the vertices in \([2n-k-2]\), then Krattenthaler’s injection \(I_K\) yields a blue \((n-k)\)-matching \(B_1\) in \(K_{2n-k-1}\) and a red \((n-k)\)-matching \(B_2\) in \(K_{2n-k-1}\). If the vertex \(2n-k-1\) is saturated under \(A_1\), then the blue path containing the vertex \(2n-k-1\) should be labeled with the largest number \(2r+2\), where \(r\) is the number of red paths in \(G[A_1 \cup A_2]\). By the property of the map \(\phi\) [10 Chap. 2, ex. 25], the largest number \(2r+2\) is not contained in \(\phi(S) \setminus S\) for any \(r\)-subset \(S\) of \([2r+2]\). So a blue path containing the vertex \(2n-k-1\) is not selected to change into a red path. Moreover, the vertex \(2n-k-1\) cannot be saturated under \(A_2\). Therefore \(B_2\) cannot saturate the vertex \(2n-k-1\).

In case (b), we introduce a new mapping as follows. Consider the sets \(X\) and \(Y\) of all unsaturated vertices under \(A_1\) and \(A_2\), respectively. Let \(x\) be the vertex which is adjacent to the vertex \(2n-k\). Choose the element \(y\) in \(Y\) with the same relative order as \(x\) in \(X \cup \{x\}\). Since the cardinality of \(X \cup \{x\}\) is \(k-1\) and the cardinality of \(Y\) is \(k\), the selection \(y\) is well-defined. Cut the blue edge \(e_b = \{x,2n-k\}\) and join two vertices \(y\) and \(2n-k-1\) by the red edge \(e_r\). Then \(A_1 \setminus \{e_b\}\) becomes a blue \((n-k)\)-matching \(B_1\) in \(K_{2n-k-1}\) and \(A_2 \cup \{e_r\}\) becomes a red \((n-k)\)-matching \(B_2\) in \(K_{2n-k-1}\). Define \(I_N(A_1,A_2) = (B_1,B_2)\). In this case, the vertex \(2n-k-1\) is saturated under \(B_2\). Let \(I_S\) be the map \(I_K \cup I_N\), i.e.,

\[
I_S(A_1,A_2) = \begin{cases} 
I_K(A_1,A_2), & \text{if } A_1 \text{ does not saturate } 2n-k, \\
I_N(A_1,A_2), & \text{if } A_1 \text{ saturates } 2n-k.
\end{cases}
\]

Since both \(I_K\) and \(I_N\) are injective and the saturating condition of the vertex \(2n-k-1\) makes that the image of \(I_K\) and the image of \(I_N\) are disjoint, the map \(I_S\) is a desired injection. \(\square\)

**Example 4.** Let \(A_1 = \{1,2\}, \{3,4\}, \{5,10\}, \{6,11\}, \{7,12\}, \{13,14\}, \{18,19\}, \{20,25\}, \{23,24\}\} \in \mathcal{M}(25,9)\) and \(A_2 = \{2,3\}, \{6,7\}, \{8,9\}, \{11,12\}, \{14,15\}, \{17,18\}, \{19,20\}\} \in \mathcal{M}(23,7)\). Color the edges of \(A_1\) in blue \((-\)) and the edges of \(A_2\) in red \((\cdot\cdot\cdot)\). Since the vertex \(25 = 2n-k\) is saturated under \(A_1\), we should apply the injection \(I_N\). In this case, \(X = \{8,9,15,16,17,21,22\}\), \(x = 20\) and \(Y = \{1,4,5,10,13,16,21,22,23\}\). Since \(20\) is the sixth smallest element in \(X \cup \{x\}\), we should choose \(16\) for \(y\), which is the sixth smallest element in \(Y\). Now cut the blue edge \(\{20,25\}\) and join \(16\) and \(24\) with a red edge. From this, we can extract two 8-matchings in \(K_{24}\): \(B_1 = \{1,2\}, \{3,4\}, \{5,10\}, \{6,11\}, \{7,12\}, \{13,14\}, \{18,19\}, \{23,24\}\) and \(B_2 = \{2,3\}, \{6,7\}, \{8,9\}, \{11,12\}, \{14,15\}, \{17,18\}, \{19,20\}, \{16,24\}\). (See Figure 4.)

**Remark 2.** Theorems 1.1 and 1.2 imply the matching numbers \(\{m(n,k)\}\) are log-concave for both coordinates:

\[
m(n,k-1) \cdot m(n,k+1) \leq m(n,k)^2 \quad \text{for all } k \geq 1,
\]

\[
m(n-1,k) \cdot m(n+1,k) \leq m(n,k)^2 \quad \text{for all } n \geq 1.
\]
Acknowledgements

The authors are grateful to their advisor Dongsu Kim for his advice and encouragement. The second author thanks Ira M. Gessel for his helpful remarks and suggestions.

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