MTH ROOTS OF H-SELFADJOINT MATRICES OVER THE QUATERNIONS

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Abstract. The complex matrix representation for a quaternion matrix is used in this paper to find necessary and sufficient conditions for the existence of an $H$-selfadjoint $m$th root of a given $H$-selfadjoint quaternion matrix. In the process, when such an $H$-selfadjoint $m$th root exists, its construction is also given.

Key words. Quaternion matrices, Roots of matrices, Indefinite inner product, $H$-selfadjoint matrices, Canonical forms

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1. Introduction. Denote the skew-field of real quaternions by $\mathbb{H}$. The basic theory of quaternion linear algebra can be found in various books and papers, see for example the book by Rodman, [10], and [13, 14].

Let $m$ be any positive integer and let $H$ be a square quaternion matrix which is invertible and Hermitian. We focus on the class of selfadjoint matrices relative to the indefinite inner product generated by $H$. In the general sense of the definition a square quaternion matrix $A$ is said to be an $H$-selfadjoint matrix if $HA = A^*H$.

If $B$ is a square $H$-selfadjoint matrix with quaternion entries, we seek to find necessary and sufficient conditions for the existence of an $H$-selfadjoint matrix $A$ such that $A^m = B$. This matrix $A$ is referred to as an $H$-selfadjoint $m$th root of the matrix $B$.

It is well known that there exists a complex matrix representation for a matrix with quaternion entries, that is, there exists an isomorphism $\omega_n$ between the real algebra of all $n \times n$ quaternion matrices and a subalgebra $\Omega_{2n}$ of the algebra of all $2n \times 2n$ complex matrices. This isomorphism $\omega_n$ maps an $n \times n$ quaternion matrix $A = A_1 + jA_2$, where $A_1, A_2 \in \mathbb{C}^{n\times n}$, to a $2n \times 2n$ matrix

$$\begin{bmatrix}
A_1 & 2A_2 \\
-A_2 & A_1
\end{bmatrix}.$$ 

This fact simplifies our problem. Therefore we find necessary and sufficient conditions for the existence of an $H$-selfadjoint $m$th root in $\Omega_{2n}$ of an $H$-selfadjoint matrix in $\Omega_{2n}$.

Quaternion matrices in indefinite inner product spaces, and specifically different canonical forms, were studied in [1, 4, 9, 11]. Not much research has been done on roots of quaternion matrices, although [7] does give a formula for obtaining $m$th roots of quaternion matrices of a particular form. On the other hand, in the complex case $H$-selfadjoint $m$th roots of $H$-selfadjoint matrices have been studied extensively. Necessary
and sufficient conditions for the existence of such roots can be found in [5]. For the case of $H$-selfadjoint square roots and applications to polar decompositions of $H$-selfadjoint matrices, see [2], and for the case of square roots of $H$-nonnegative matrices, see [8]. We refer to the introduction of a previous paper [5], for an overview of $m$th roots of matrices in general.

Note that in the process of finding an $H$-selfadjoint $m$th root $A \in \Omega_{2n}$ of $B \in \Omega_{2n}$, i.e. $A^m = B$, the functional analytic approach via the Cauchy integral

$$A := \frac{1}{2\pi i} \int_{\Gamma} \sqrt[m]{\lambda} (\lambda I - B)^{-1} d\lambda,$$

which can be found for example in [6], would be sufficient in the case where there are no eigenvalues of $B$ in $(-\infty, 0]$. However, in this paper we prefer to take a more direct approach also in the case where the eigenvalues do not lie on the negative real line.

**2. Preliminaries.** We recap the basic theory for quaternions and matrices with quaternion entries as found in the book by Leiba Rodman, [10]. Every element in $\mathbb{H}$ is of the form

$$x = x_0 + x_1 i + x_2 j + x_3 k,$$

where $x_0, x_1, x_2, x_3 \in \mathbb{R}$ and the elements $i, j, k$ satisfy the following formulas

$$i^2 = j^2 = k^2 = -1, \quad ij = -ji = k, \quad jk = -kj = i, \quad ki = -ik = j.$$

It is important to keep in mind the fact that multiplication in $\mathbb{H}$ is not commutative, i.e. in general $xy \neq yx$ for $x, y \in \mathbb{H}$. Let $\bar{x} = x_0 - x_1 i - x_2 j - x_3 k$ denote the conjugate quaternion of $x$. For a quaternion matrix $A$, let $\bar{A}$ denote the matrix in which each entry is the conjugate of the corresponding entry in $A$.

Let $A$ be an $n \times n$ quaternion matrix, i.e. $A \in \mathbb{H}^{n \times n}$, and write $A$ as $A = A_1 + jA_2$ where $A_1, A_2 \in \mathbb{C}^{n \times n}$. The map $\omega_n : \mathbb{H}^{n \times n} \to \mathbb{C}^{2n \times 2n}$ is defined by

$$\omega_n(A) = \begin{bmatrix} A_1 & \bar{A}_2 \\ -A_2 & \bar{A}_1 \end{bmatrix}.$$ 

Then $\omega_n$ is an isomorphism of the real algebra $\mathbb{H}^{n \times n}$ onto the real unital subalgebra

$$\Omega_{2n} := \left\{ \begin{bmatrix} A_1 & \bar{A}_2 \\ -A_2 & \bar{A}_1 \end{bmatrix} \mid A_1, A_2 \in \mathbb{C}^{n \times n} \right\}$$

of $\mathbb{C}^{2n \times 2n}$. The following properties can be found in [10, Section 3.4] and [13]. Let $X, Y \in \mathbb{H}^{n \times n}$ and $s, t \in \mathbb{R}$ be arbitrary, then

(i) $\omega_n(I_n) = I_{2n}$;
(ii) $\omega_n(XY) = \omega_n(X)\omega_n(Y)$;
(iii) $\omega_n(sX + tY) = s\omega_n(X) + t\omega_n(Y)$;
(iv) $\omega_n(X^*) = (\omega_n(X))^*$;
(v) $\omega_n(X^{-1}) = (\omega_n(X))^{-1}$ if $X$ is invertible.

Note that the matrix $X^* \in \mathbb{H}^{m \times n}$ is obtained from $X \in \mathbb{H}^{m \times n}$ by replacing each entry with its conjugate quaternion and then taking the transpose. This isomorphism between $\mathbb{H}^{n \times n}$ and $\Omega_{2n}$ ensures that results
for matrices in $\mathbb{H}^{n \times n}$ that are purely algebraic, are also true for matrices in the subalgebra $\Omega_{2n}$, since we can apply $\omega_n$, and vice versa as long as we stay within the subalgebra $\Omega_{2n}$. All definitions that follow could also be made with respect to matrices in $\Omega_{2n}$.

An $n \times n$ quaternion matrix $A$ has left eigenvalues and right eigenvalues but since we only work with right eigenvalues and right eigenvectors, we will refer to them as eigenvalues and eigenvectors.

**Definition 2.1.** A nonzero vector $v \in \mathbb{H}^n$ is called an eigenvector of a matrix $A \in \mathbb{H}^{n \times n}$ corresponding to the eigenvalue $\lambda \in \mathbb{H}$ if the equality $Av = v\lambda$ holds.

The spectrum of $A$, denoted by $\sigma(A)$, is the set of all eigenvalues of $A$. Note that $\sigma(A)$ is closed under similarity of quaternions, i.e., if $v$ is an eigenvector of $A$ corresponding to the eigenvalue $\lambda$, then $v\alpha$ is an eigenvector of $A$ corresponding to the eigenvalue $\alpha^{-1}\lambda\alpha$, for all nonzero $\alpha \in \mathbb{H}$. From [13] we see that $A$ has exactly $n$ eigenvalues which are complex numbers with nonnegative imaginary parts and the Jordan normal form of $A$ has exactly these numbers on the diagonal. Let $\mathbb{C}_+ = \{a + ib \mid a \in \mathbb{R}, b > 0\}$.

Let a single Jordan block of size $n \times n$ at the eigenvalue $\lambda$ be denoted by $J_n(\lambda)$. The $n \times n$ matrix with ones on the main anti-diagonal and zeros elsewhere, called a standard involutary permutation (sip) matrix, is denoted by $Q_n$.

Recall the following definition from [12].

**Definition 2.2.** Let $A$ be a square quaternion matrix with Jordan blocks $\bigoplus_{i=1}^{r} J_n(\lambda)$ at the eigenvalue $\lambda$ in its Jordan normal form and assume that $n_1 \geq n_2 \geq n_3 \geq \ldots \geq n_r > 0$. The Segre characteristic of $A$ corresponding to the eigenvalue $\lambda$ is defined as the sequence

$$n_1, n_2, n_3, \ldots, n_r, 0, 0, \ldots.$$ 

We will use this definition mostly in the case where $\lambda$ is equal to zero unless indicated otherwise and therefore sometimes will simply use Segre characteristic to refer to the Segre characteristic of $A$ corresponding to the eigenvalue 0. Note that the Jordan normal form of matrices in the subalgebra $\Omega_{2n}$ can be found from the Jordan normal form of matrices in $\mathbb{H}^{n \times n}$ since $\omega_n(J_n(\lambda)) = J_n(\lambda) \oplus J_n(\bar{\lambda})$, for $\lambda \in \mathbb{C}$. Then it is easy to see the following result.

**Corollary 2.3.** If a nilpotent matrix $A$ is in $\Omega_{2n}$, then each number in the Segre characteristic of $A$ occurs twice.

This actually holds for the Segre characteristic corresponding to any real eigenvalue in the case where $A \in \Omega_{2n}$ has real numbers in its spectrum. However, only the nilpotent case will be used later.

A matrix $X \in \mathbb{H}^{n \times n}$ is said to be Hermitian if $X^{*} = X$. Let $H \in \mathbb{H}^{n \times n}$ be an invertible Hermitian matrix. We consider the indefinite inner product $[\cdot, \cdot]$ generated by $H$:

$$[x, y] = \langle Hx, y \rangle = y^{*}Hx, \quad x, y \in \mathbb{H}^n,$$

where $\langle \cdot, \cdot \rangle$ denotes the standard inner product. A matrix $A \in \mathbb{H}^{n \times n}$ is called $H$-selfadjoint if $HA = A^{*}H$.

Two pairs of matrices $(A_1, H_1)$ and $(A_2, H_2)$ are said to be unitarily similar if there exists an invertible quaternion matrix $S$ such that $S^{-1}A_1S = A_2$ and $S^{*}H_1S = H_2$ hold.

As in the complex case, there exists a canonical form for the pair $(A, H)$ where $A \in \mathbb{H}^{n \times n}$ is an $H$-selfadjoint matrix. This is given in, for example [1, Theorem 4.1], [9] and [10, Theorem 10.1.1].
THEOREM 2.4. Let $H \in \mathbb{H}^{n \times n}$ be an invertible Hermitian matrix and $A \in \mathbb{H}^{n \times n}$ an $H$-selfadjoint matrix. Then there exists an invertible matrix $S \in \mathbb{H}^{n \times n}$ such that

$$ S^{-1}AS = J_{k_1}(\lambda_1) \oplus \cdots \oplus J_{k_n}(\lambda_n) $$

(2.1)

$$ + \begin{bmatrix} J_{k_{\alpha+1}}(\lambda_{\alpha+1}) & 0 & \cdots & 0 \\ 0 & J_{k_{\alpha+1}}(\lambda_{\alpha+1}) & & \\ & & \ddots & \ddots \\ & & & J_{k_{\beta}}(\lambda_{\beta}) \end{bmatrix}, $$

where $\lambda_i \in \sigma(A) \cap \mathbb{R}$ for all $i = 1, \ldots, \alpha$, $\lambda_i \in \sigma(A) \cap \mathbb{C}_+^{\text{for all $i = \alpha + 1, \ldots, \beta$,}}$ and

$$ S^*HS = \eta_1Q_{k_1} \oplus \cdots \oplus \eta_\alpha Q_{k_\alpha} \oplus Q_{2k_{\alpha+1}} \oplus \cdots \oplus Q_{2k_\beta}, $$

(2.2)

where $\eta_i = \pm 1$. The form $(S^{-1}AS, S^*HS)$ in (2.1) and (2.2) is uniquely determined by the pair $(A, H)$, up to a permutation of diagonal blocks.

We refer to the pair $(S^{-1}AS, S^*HS)$ in (2.1) and (2.2) as the canonical form of the pair $(A, H)$ of quaternion matrices.

The following result from [5] holds for quaternion matrices as well, but by applying $\omega_n$ it also holds for matrices in the subalgebra $\Omega_{2n}$. We use it in the proofs throughout this paper.

LEMMA 2.5. Let $X$ and $Y$ be $n \times n$ quaternion matrices such that the pair $(X, H_X)$ is unitarily similar to the pair $(Y, H_Y)$ where $H_X \in \mathbb{H}^{n \times n}$ and $H_Y \in \mathbb{H}^{n \times n}$ are invertible Hermitian matrices, i.e. there exists an invertible matrix $P \in \mathbb{H}^{n \times n}$ such that

$$ P^{-1}XP = Y \quad \text{and} \quad P^*H_XP = H_Y. $$

Let the matrix $J \in \mathbb{H}^{n \times n}$ be an $H_X$-selfadjoint $m$th root of $X$, i.e. $J^m = X$. Then the matrix $A := P^{-1}JP$ is an $H_Y$-selfadjoint $m$th root of $Y$.

From the properties of the map $\omega_n$ we see that a matrix $A$ is invertible if and only if $\omega_n(A)$ is invertible. Also, from Proposition 3.4.1 in [10] $A$ is Hermitian if only if $\omega_n(A)$ is Hermitian.

To find a canonical form for a pair $(\hat{A}, \hat{H})$ where $\hat{A} \in \Omega_{2n}$ is $\hat{H}$-selfadjoint, and $\hat{H} \in \Omega_{2n}$ is invertible and Hermitian, we apply $\omega_n$ to the equations (2.1) and (2.2). Denote the right-hand side of (2.1) by $J$ and the right-hand side of (2.2) by $Q$ and then we have

$$ (\omega_n(S))^{-1}\omega_n(A)\omega_n(S) = \omega_n(J), \quad \omega_n(S)^*\omega_n(H)\omega_n(S) = \omega_n(Q). $$

The uniqueness follows from the fact that the canonical form of $(A, H)$ is unique (Theorem 2.4) and that $\omega_n$ is an isomorphism. Therefore the canonical form of a pair of matrices in $\Omega_{2n}$ is as follows.

THEOREM 2.6. Let $H \in \Omega_{2n}$ be an invertible Hermitian matrix and $\hat{A} \in \Omega_{2n}$ an $\hat{H}$-selfadjoint matrix. Then there exists an invertible matrix $S \in \Omega_{2n}$ such that

$$ S^{-1}\hat{A}S = J_{k_1}(\lambda_1) \oplus \cdots \oplus J_{k_n}(\lambda_n) $$

(2.3)

$$ + \begin{bmatrix} J_{k_{\alpha+1}}(\lambda_{\alpha+1}) & 0 & \cdots & 0 \\ 0 & J_{k_{\alpha+1}}(\lambda_{\alpha+1}) & & \\ & & \ddots & \ddots \\ & & & J_{k_{\beta}}(\lambda_{\beta}) \end{bmatrix}, $$

where $\lambda_i \in \sigma(\hat{A}) \cap \mathbb{R}$ for all $i = 1, \ldots, \alpha$, $\lambda_i \in \sigma(\hat{A}) \cap \mathbb{C}_+^{\text{for all $i = \alpha + 1, \ldots, \beta$,}}$ and

$$ S^*\hat{H}S = \eta_1Q_{k_1} \oplus \cdots \oplus \eta_\alpha Q_{k_\alpha} \oplus Q_{2k_{\alpha+1}} \oplus \cdots \oplus Q_{2k_\beta} $$

(2.4)

$$ + \begin{bmatrix} J_{k_{\alpha+1}}(\lambda_{\alpha+1}) & 0 & \cdots & 0 \\ 0 & J_{k_{\alpha+1}}(\lambda_{\alpha+1}) & & \\ & & \ddots & \ddots \\ & & & J_{k_{\beta}}(\lambda_{\beta}) \end{bmatrix}. $$
where $\eta_i = \pm 1$. The form $(S^{-1} \tilde{A} S, S^* \tilde{H} S)$ in (2.3) and (2.4) is uniquely determined by the pair $(\tilde{A}, \tilde{H})$ up to a permutation of diagonal blocks.

We note at this stage that the canonical form of the nonreal part of an $H$-selfadjoint matrix $A$ must be of dimensions a multiple of 4 due to the direct sums of $J_{k_i}(\lambda_i)$ and $J_{k_i}(\bar{\lambda}_i)$ occurring twice. It is crucial to ensure that all matrices throughout the proofs are in $\Omega_{2n}$, and for this reason we give the following result to explain why we can study different Jordan blocks separately.

**Lemma 2.7.** Let $H_1 = Q_1 \oplus \bar{Q}_1, B_1 = J_1 \oplus \bar{J}_1 \in \Omega_{2n}$, and $H_2 = Q_2 \oplus \bar{Q}_2, B_2 = J_2 \oplus \bar{J}_2 \in \Omega_{2p}$, where $B_1$ is $H_1$-selfadjoint and $B_2$ is $H_2$-selfadjoint. Let $A_1 \in \Omega_{2n}$ be an $H_1$-selfadjoint $m$th root of $B_1$ and $A_2 \in \Omega_{2p}$ an $H_2$-selfadjoint $m$th root of $B_2$, and let their entries be as follows

$$A_1 = \begin{bmatrix} A_{11}^{(1)} & A_{12}^{(1)} \\ -A_{12}^{(1)} & A_{11}^{(1)} \end{bmatrix} \quad \text{and} \quad A_2 = \begin{bmatrix} A_{11}^{(2)} & A_{12}^{(2)} \\ -A_{12}^{(2)} & A_{11}^{(2)} \end{bmatrix}.$$ 

Let $\hat{B} = J_1 \oplus J_2 \oplus \bar{J}_1 \oplus \bar{J}_2 \in \Omega_{2(n+p)}$ and $\hat{H} = Q_1 \oplus Q_2 \oplus \bar{Q}_1 \oplus \bar{Q}_2 \in \Omega_{2(n+p)}$. Then $\hat{B}$ is $\hat{H}$-selfadjoint and the matrix

$$\hat{A} = \begin{bmatrix} A_{11}^{(1)} & 0 & \bar{A}_{12}^{(1)} & 0 \\ 0 & A_{11}^{(1)} & 0 & \bar{A}_{12}^{(1)} \\ -A_{12}^{(1)} & 0 & \bar{A}_{11}^{(1)} & 0 \\ 0 & -A_{12}^{(1)} & 0 & \bar{A}_{11}^{(1)} \end{bmatrix} \in \Omega_{2(n+p)}$$

is an $\hat{H}$-selfadjoint $m$th root of the matrix $\hat{B}$.

**Proof.** Let $P$ be the following permutation matrix

$$P = \begin{bmatrix} I_n & 0 & 0 & 0 \\ 0 & 0 & I_p & 0 \\ 0 & I_n & 0 & 0 \\ 0 & 0 & 0 & I_p \end{bmatrix}.$$ 

This $P$ produces a map from $\Omega_{2n} \oplus \Omega_{2p}$ to $\Omega_{2(n+p)}$ which satisfies $P(A_1 \oplus A_2)P^{-1} = \hat{A}$. We also then have $P(B_1 \oplus B_2)P^{-1} = \hat{B}$ and $P(H_1 \oplus H_2)P^{-1} = \hat{H}$. Therefore

$$\hat{A}^m = (P(A_1 \oplus A_2)P^{-1})^m = P(A_1 \oplus A_2)^m P^{-1} = P(B_1 \oplus B_2)P^{-1} = \hat{B}.$$ 

Now, by noting that $P^* = P^{-1}$ and using the facts that $B_1$ and $A_1$ are $H_1$-selfadjoint and $B_2$ and $A_2$ are $H_2$-selfadjoint, it follows that $\hat{H} \hat{B} = \hat{B}^* \hat{H}$ and $\hat{H} \hat{A} = \hat{A}^* \hat{H}$. Therefore $\hat{A}$ is an $\hat{H}$-selfadjoint $m$th root of the $\hat{H}$-selfadjoint matrix $\hat{B}$. \[\square\]

The matrix $J_i$ in Lemma 2.7 can be the Jordan normal form $J_k(\lambda)$ in the case where $\lambda$ is real or the Jordan normal form $J_k(\lambda) \oplus J_k(\bar{\lambda})$ in the case where $\lambda$ is nonreal.

In general, let the permutation matrix $P$ be a $2t \times 2t$ block matrix where the block in the $i$th row and $j$th column is defined by

$$(2.5) \quad P_{ij} = \begin{cases} I_{2k_i} & \text{if } j = 2i - 1, i \leq t; \\ I_{2k_{i-t}} & \text{if } j = 2(i-t), i > t; \\ 0 & \text{otherwise}. \end{cases}$$
Then we have, for example, that
\[
P \bigoplus_{j=1}^{t} \left( Q_{k_j} \oplus -Q_{k_j} \right) = \bigoplus_{j=1}^{t} \left( Q_{k_j} \oplus -Q_{k_j} \right).
\]

3. Existence of \(\text{mth roots}\). We first present a very handy tool for working with quaternion matrices.

**Lemma 3.1.** Let \(H\) be an \(n \times n\) quaternion matrix which is invertible and Hermitian, and let \(B\) be an \(n \times n\) \(H\)-selfadjoint quaternion matrix. There exists an \(H\)-selfadjoint quaternion matrix \(A\) such that \(A^m = B\) if and only if there exists an \(\tilde{H} = \omega_n(H)\)-selfadjoint matrix \(\tilde{A} = \omega_n(A)\) such that \(\tilde{A}^m = \tilde{B}\), where \(\tilde{B} = \omega_n(B)\) is an \(\tilde{H}\)-selfadjoint matrix in the subalgebra \(\Omega_{2n}\).

**Proof.** From the properties of the map \(\omega_n\) and the fact that it is an isomorphism from \(\mathbb{H}^{n \times n}\) to \(\Omega_{2n}\) we see that \(A^m = B\) if and only if \((\omega_n(A))^m = \omega_n(B)\), and \(A\) is \(H\)-selfadjoint if and only if \(\omega_n(A)\) is \(\omega_n(H)\)-selfadjoint. \(\square\)

Because of this lemma, necessary and sufficient conditions for the existence of an \(H\)-selfadjoint \(m\)th root of an \(H\)-selfadjoint matrix \(B\) are the same as the necessary and sufficient conditions for the existence of an \(H\)-selfadjoint \(m\)th root of an \(H\)-selfadjoint matrix \(\tilde{B}\) where \(\tilde{B}, \tilde{H} \in \Omega_{2n}\). If we could work in \(\mathbb{C}^{2n \times 2n}\) we would now be done by simply referring to [5]. However, since \(\omega_n\) is an isomorphism between \(\mathbb{H}^{n \times n}\) and the subalgebra \(\Omega_{2n}\) of \(\mathbb{C}^{2n \times 2n}\), we have to be more careful.

We now first present in Theorem 3.2 our main theorem for the existence of \(H\)-selfadjoint \(m\)th roots of \(H\)-selfadjoint matrices in \(\Omega_{2n}\). The proof of this theorem may be split into the following separate parts due to Lemma 2.7, viz. the case where \(\tilde{B}\) has only positive eigenvalues, the case where \(\tilde{B}\) has only nonreal eigenvalues, the case where \(\tilde{B}\) has only negative eigenvalues (separated into two cases for \(m\) even and for \(m\) odd), and lastly, the case where \(\tilde{B}\) has only zero as an eigenvalue. These we state and prove in Theorems 3.3, 3.4, 3.6, 3.7 and 3.9.

**Theorem 3.2.** Let \(\tilde{B}, \tilde{H}\) be matrices in the subalgebra \(\Omega_{2n}\) such that \(\tilde{H}\) is invertible and Hermitian, and \(\tilde{B}\) is \(\tilde{H}\)-selfadjoint. Then there exists an \(\tilde{H}\)-selfadjoint matrix in \(\Omega_{2n}\), say \(\tilde{A}\), such that \(\tilde{A}^m = \tilde{B}\) if and only if the canonical form of \((\tilde{B}, \tilde{H})\) has the following properties:

1. The part of the canonical form corresponding to negative eigenvalues when \(m\) is even, say \((\tilde{B}_-, \tilde{H}_-), \) is given by

\[
\tilde{B}_- = \bigoplus_{j=1}^{t} \left( J_{k_j} (\lambda_j) \oplus -J_{k_j} (\lambda_j) \right) \oplus \bigoplus_{j=1}^{t} \left( J_{k_j} (\lambda_j) \oplus -J_{k_j} (\lambda_j) \right),
\]

\[
\tilde{H}_- = \bigoplus_{j=1}^{t} \left( Q_{k_j} \oplus -Q_{k_j} \right) \oplus \bigoplus_{j=1}^{t} \left( Q_{k_j} \oplus -Q_{k_j} \right),
\]

where \(\lambda_j < 0\).

2. The part of the canonical form corresponding to zero eigenvalues, say \((\tilde{B}_0, \tilde{H}_0)\), is given by

\[
\tilde{B}_0 = \bigoplus_{j=1}^{t} \left( \bigoplus_{i=1}^{r_j} J_{n_j+1}(0) \oplus \bigoplus_{i=r_j+1}^{m} J_{n_j}(0) \right) \oplus \bigoplus_{j=1}^{t} \left( \bigoplus_{i=1}^{r_j} J_{n_j+1}(0) \oplus \bigoplus_{i=r_j+1}^{m} J_{n_j}(0) \right),
\]

where \(r_j \geq 0\).
and
\[
\hat{H}_0 = \bigoplus_{j=1}^{t} \left( \bigoplus_{i=1}^{r_j} \varepsilon_{i}^{(j)} Q_{a_{j+1}} \bigoplus_{i=r_{j+1}}^{m} \varepsilon_{i}^{(j)} Q_{a_{j}} \right) \bigoplus_{j=1}^{t} \left( \bigoplus_{i=1}^{r_j} \varepsilon_{i}^{(j)} Q_{a_{j+1}} \bigoplus_{i=r_{j+1}}^{m} \varepsilon_{i}^{(j)} Q_{a_{j}} \right),
\]
for some \(a_{j}, r_{j} \in \mathbb{Z}\) with \(0 < r_{j} \leq m\). The signs are as follows, given in terms of \(\eta_{j}\), where \(\eta_{j}\) could be either 1 or -1: If \(r_{j}\) (respectively \(m - r_{j}\)) is even, half of \(\varepsilon_{i}^{(j)}\) for \(i = 1, \ldots, r_{j}\) (respectively for \(i = r_{j} + 1, \ldots, m\)) is equal to \(\eta_{j}\) and the other half is equal to \(-\eta_{j}\). If \(r_{j}\) (respectively \(m - r_{j}\)) is odd, there is one more of \(\varepsilon_{i}^{(j)}\) for \(i = 1, \ldots, r_{j}\) (respectively for \(i = r_{j} + 1, \ldots, m\)) equal to \(\eta_{j}\) than those equal to \(-\eta_{j}\).

Note that it follows from Lemma 2.5 (which also holds for matrices in \(\Omega_{2n}\)) that it is sufficient to assume that the pair \((\tilde{B}, \tilde{H})\) is in canonical form.

For the positive eigenvalue case, we now prove the following:

**Theorem 3.3.** Let \(\tilde{B}, \tilde{H} \in \Omega_{2n}\), where \(\tilde{H}\) is invertible and Hermitian, and \(\tilde{B}\) is \(\tilde{H}\)-selfadjoint with a spectrum consisting of only positive real numbers. Then there exists an \(\tilde{H}\)-selfadjoint matrix in \(\Omega_{2n}\), say \(\hat{A}\), such that \(\hat{A}^{m} = \tilde{B}\).

**Proof.** Let \(\tilde{B} \in \Omega_{2n}\) be \(\tilde{H}\)-selfadjoint with only positive real eigenvalues, where \(\tilde{H} \in \Omega_{2n}\) is invertible and Hermitian. We can assume that \(\tilde{B} = J_{n}^{*}(\lambda) \oplus J_{n}(\lambda)\), where \(\lambda\) is a positive real number, and that \(\tilde{H} = \varepsilon_{Q_{n}} \oplus \varepsilon_{Q_{n}}\), \(\varepsilon = \pm 1\). To construct an \(m\)th root of \(\tilde{B}\) which is \(\tilde{H}\)-selfadjoint and also in \(\Omega_{2n}\), we let \(J = J_{n}(\mu) \oplus J_{n}(\mu)\) where \(\mu\) is the positive real \(m\)th root of \(\lambda\). Then both \(J\) and \(J^{m} = (J_{n}(\mu))^{m} \oplus (J_{n}(\mu))^{m}\) are \(\tilde{H}\)-selfadjoint, and the Jordan normal form of \(J^{m}\) is equal to \(\tilde{B}\). We now wish to find an invertible matrix \(P \in \Omega_{2n}\) such that equations
\[
P^{-1} J^{m} P = \tilde{B} \quad \text{and} \quad P^{*} \tilde{H} P = \tilde{H}
\]
hold. To ensure that the first equation holds, let \(P = P_{1} \oplus P_{2}\) with
\[
P_{1} = \left[ (J_{n}(\mu))^{m} - \lambda I \right] y \cdots \left( (J_{n}(\mu))^{m} - \lambda I \right) y \quad y,
\]
\[
P_{2} = \left[ (J_{n}(\mu))^{m} - \lambda I \right] z \cdots \left( (J_{n}(\mu))^{m} - \lambda I \right) z \quad z,
\]
where \(y, z \in \text{Ker} \left( (J_{n}(\mu))^{m} - \lambda I \right)^{n} = \mathbb{C}^{n}\) but \(y, z \notin \text{Ker} \left( (J_{n}(\mu))^{m} - \lambda I \right)^{n-1}\). From the choices of \(y\) and \(z\) it is clear that \(P_{1}\) and \(P_{2}\) are invertible \(n \times n\) matrices and hence the \(2n \times 2n\) matrix \(P\) is invertible. Let \(z = \tilde{y}\), so that \(P_{2} = \tilde{P}_{1}\); then \(P\) is in \(\Omega_{2n}\). Note that \(P^{*} \tilde{H} P = \tilde{H}\) if and only if \(P_{1}^{*} Q_{n} P_{1} = Q_{n}\) and since \(P_{1}^{*} Q_{n} P_{1}\) is a lower anti-triangular Hankel matrix, \(P^{*} \tilde{H} P = \tilde{H}\) holds if and only if
\[
\phi_{n,j}(y) = [p_{j}, p_{n}] = y^{*} Q_{n} ( (J_{n}(\mu))^{m} - \lambda I )^{n-j} y = \begin{cases} 1 & \text{if } j = 1, \\ 0 & \text{if } j = 2, \ldots, n, \end{cases}
\]
where \(\phi_{n,j}(y)\) denotes the entries in the matrix \(P_{1}^{*} Q_{n} P_{1}\) and \(p_{i}\) denotes the columns in the matrix \(P_{1}\). As illustrated in [5], one can easily find one solution to these equations by assuming that \(y\) is real. Therefore there exists a matrix \(P \in \Omega_{2n}\) satisfying the equations (3.6). Then by using Lemma 2.5, the matrix \(\hat{A} := P^{-1} J P\) is an \(\tilde{H}\)-selfadjoint \(m\)th root of \(\tilde{B}\), and \(\hat{A}\) is also in \(\Omega_{2n}\) since \(P\) is.

Now, for the nonreal eigenvalue case.
Theorem 3.4. Let \( \tilde{B}, \tilde{H} \in \Omega_{4n} \), where \( \tilde{H} \) is invertible and Hermitian, and \( \tilde{B} \) is \( \tilde{H} \)-selfadjoint with a spectrum consisting of only nonreal numbers. Then there exists an \( \tilde{H} \)-selfadjoint matrix in \( \Omega_{4n} \), say \( \tilde{A} \), such that \( \tilde{A}^m = \tilde{B} \).

Proof. Let \( \tilde{B} \in \Omega_{4n} \) be \( \tilde{H} \)-selfadjoint with only nonreal eigenvalues, where \( \tilde{H} \in \Omega_{4n} \) is invertible and Hermitian. We can assume that \( \tilde{B} = J_n(\lambda) \oplus J_n(\bar{\lambda}) \oplus J_n(\lambda) \oplus J_n(\bar{\lambda}) \), where \( \lambda \) is a nonreal number, and that \( \tilde{H} = Q_{2n} \oplus Q_{2n} \). To construct an \( m \)th root of \( \tilde{B} \), let \( J = J_n(\mu) \oplus J_n(\bar{\mu}) \oplus J_n(\mu) \oplus J_n(\bar{\mu}) \) where \( \mu \) is any \( m \)th root of \( \lambda \). Then the Jordan normal form of \( J^m \) is equal to \( \tilde{B} \), and both \( J \) and \( J^m \) are \( \tilde{H} \)-selfadjoint. Let \( P = P_1 \oplus \tilde{P}_1 \oplus \tilde{P}_1 \oplus P_1 \in \Omega_{4n} \) where
\[
P_1 = \begin{pmatrix}
((J_n(\mu))^m - \lambda I)^{n-1} y & \cdots & ((J_n(\mu))^m - \lambda I) y & y
\end{pmatrix},
\]
with \( y \in \text{Ker}((J_n(\mu))^m - \lambda I)^n = \mathbb{C}^n \) but \( y \notin \text{Ker}((J_n(\mu))^m - \lambda I)^{n-1} \). Then \( P^* \tilde{H} P = \tilde{H} \) if and only if \( P_1^T Q_n P_1 = Q_n \), and according to the proof of Theorem 2.4 in [5], there exists a solution to the latter equation. Hence, there exists an invertible matrix \( P \in \Omega_{4n} \) such that the equations
\[
P^{-1} J^m P = \tilde{B} \quad \text{and} \quad P^* \tilde{H} P = \tilde{H}
\]
hold. Once again, by Lemma 2.5, the matrix \( \tilde{A} := P^{-1} J P \) is an \( \tilde{H} \)-selfadjoint \( m \)th root of \( \tilde{B} \) and is in \( \Omega_{4n} \). \( \Box \)

Before stating the results for the negative case, we give the following result which was obtained by applying \( \omega_n \) to all matrices in Lemma 2.12 of [5].

Lemma 3.5. Let \( T \in \Omega_{2n} \) be a diagonal block matrix consisting of an upper triangular Toeplitz matrix with diagonal entries \( t_1, \ldots, t_n \) and its complex conjugate. Let the diagonal entries \( \lambda = t_1 \) be real, \( t_2 \) nonzero and \( B = T \oplus \bar{T} \). Then \( B \) is \((Q_{2n} \oplus Q_{2n})\)-selfadjoint, and the pair \((B, Q_{2n} \oplus Q_{2n})\) is unitarily similar to
\[
(J_n(\lambda) \oplus J_n(\lambda) \oplus J_n(\lambda) \oplus J_n(\lambda), Q_n \oplus -Q_n \oplus Q_n \oplus -Q_n).
\]

Turning to the case of the negative eigenvalue for a matrix \( \tilde{B} \), we first point out that if \( m \) is even, any \( m \)th root \( \tilde{A} \) will necessarily have only complex eigenvalues, so in order for \( \tilde{A} \) to be \( \tilde{H} \)-selfadjoint, by Theorem 2.6, it (and so also \( \tilde{B} \)) would have to be in \( \Omega_{4n} \). We now first prove a result for \( m \) even.

Theorem 3.6. Let \( \tilde{H} \in \Omega_{4n} \) be an invertible Hermitian matrix and let \( \tilde{B} \in \Omega_{4n} \) be an \( \tilde{H} \)-selfadjoint matrix with a spectrum consisting of only negative real numbers. Then, for an even positive integer \( m \), there exists an \( \tilde{H} \)-selfadjoint matrix in \( \Omega_{4n} \), say \( \tilde{A} \), such that \( \tilde{A}^m = \tilde{B} \) if and only if the canonical form of \((B, H)\) is given by
\[
(3.7) \quad S^{-1} \tilde{B} S = \bigoplus_{j=1}^t (J_{k_j}(\lambda_j) \oplus J_{k_j}(\lambda_j)) \oplus \bigoplus_{j=1}^t (J_{k_j}(\lambda_j) \oplus J_{k_j}(\lambda_j)),
\]
and
\[
(3.8) \quad S^* \tilde{H} S = \bigoplus_{j=1}^t (Q_{k_j} \oplus -Q_{k_j}) \oplus \bigoplus_{j=1}^t (Q_{k_j} \oplus -Q_{k_j}),
\]
where \( \lambda_j < 0 \) and \( S \) is some invertible matrix in \( \Omega_{4n} \).

Proof. Let \( \tilde{B} \) be an \( \tilde{H} \)-selfadjoint matrix where both \( \tilde{B} \) and \( \tilde{H} \) are in \( \Omega_{4n} \) and \( \tilde{B} \) has only negative eigenvalues. Assume that there exists an \( \tilde{H} \)-selfadjoint \( m \)th root \( \tilde{A} \in \Omega_{4n} \) of \( \tilde{B} \), i.e. \( \tilde{A}^m = \tilde{B} \). Denote
the eigenvalues of $\tilde{B}$ by $\lambda_j$, and let $\mu_j$ be any $m$th root of $\lambda_j$. Since $m$ is even, $\mu_j$ is nonreal. Thus from Theorem 2.6 we know that the canonical form of $(A, \tilde{H})$ is $(J, Q)$ where

$$J = \bigoplus_{j=1}^{t} (J_k_j(\mu_j) \oplus J_k_j(\bar{\mu}_j)) \oplus \bigoplus_{j=1}^{t} (J_k_j(\bar{\mu}_j) \oplus J_k_j(\mu_j))$$

and

$$Q = \bigoplus_{j=1}^{t} Q_{2k_j} \oplus \bigoplus_{j=1}^{t} Q_{2k_j},$$

for some $t$. Hence there exists an invertible matrix $P \in \Omega_{4n}$ such that the equations $P^{-1}\tilde{A}P = J$ and $P^*\tilde{H}P = Q$ hold. Consider

$$P^{-1}\tilde{B}P = (P^{-1}\tilde{A}P)^m = J^m = \bigoplus_{j=1}^{t} ((J_k_j(\mu_j))^m \oplus (J_k_j(\bar{\mu}_j))^m) \oplus \bigoplus_{j=1}^{t} ((J_k_j(\bar{\mu}_j))^m \oplus (J_k_j(\mu_j))^m),$$

and note that this matrix is $P^*\tilde{H}P$-selfadjoint. Next, by taking

$$T_j = (J_k_j(\mu_j))^m \oplus (J_k_j(\bar{\mu}_j))^m$$

for $j = 1, \ldots, t$, and applying Lemma 3.5 we have that $(T_j \oplus T_j, Q_{2k_j} \oplus Q_{2k_j})$ is unitarily similar to

$$(J_k_j(\lambda_j) \oplus J_k_j(\bar{\lambda}_j) \oplus J_k_j(\bar{\lambda}_j) \oplus J_k_j(\lambda_j), Q_{k_j} \oplus -Q_{k_j} \oplus Q_{k_j} \oplus -Q_{k_j})$$

for all $j = 1, \ldots, t$. Therefore by using the permutation matrix defined in (2.5), we see that the canonical form of $(\tilde{B}, \tilde{H})$ is given by (3.7) and (3.8).

Conversely, let $\tilde{B}$ be $\tilde{H}$-selfadjoint and such that the canonical form of $(\tilde{B}, \tilde{H})$ is given by (3.7) and (3.8), and let $m$ be even. Assume that

$$\tilde{B} = J_n(\lambda) \oplus J_n(\lambda) \oplus J_n(\lambda) \oplus J_n(\lambda) \quad \text{and} \quad \tilde{H} = Q_n \oplus -Q_n \oplus Q_n \oplus -Q_n,$$

where $\lambda < 0$. Then let $J = J_n(\mu) \oplus J_n(\bar{\mu}) \oplus J_n(\bar{\mu}) \oplus J_n(\mu)$ where $\mu$ is any $m$th root of $\lambda$. The number $\mu$ is nonreal and therefore $J$ is $Q$-selfadjoint where $Q = Q_{2n} \oplus Q_{2n}$. Note that the matrix $J^m$ has $\lambda$ on its main diagonal and satisfies the conditions of $T \oplus T$ as in Lemma 3.5. Thus it follows that the pair $(J^m, Q_{2n} \oplus Q_{2n})$ is unitarily similar to the pair $(\tilde{B}, \tilde{H})$, i.e., there exists an invertible matrix $P \in \Omega_{4n}$ such that the equations $P^{-1}J^mP = \tilde{B}$ and $P^*QP = \tilde{H}$ hold. Finally, from Lemma 2.5 the matrix $\tilde{A} := P^{-1}JP \in \Omega_{4n}$ is an $\tilde{H}$-selfadjoint $m$th root of $\tilde{B}$. \hfill \Box

Next we give the negative eigenvalue case where $m$ is odd. This case is similar to the positive eigenvalue case.

**Theorem 3.7.** Let $\tilde{H} \in \Omega_{2n}$ be an invertible Hermitian matrix and $\tilde{B} \in \Omega_{2n}$ an $\tilde{H}$-selfadjoint matrix with a spectrum consisting of only negative real numbers. Then, for $m$ odd, there exists an $\tilde{H}$-selfadjoint matrix in $\Omega_{2n}$, say $\tilde{A}$, such that $\tilde{A}^m = \tilde{B}$.

**Proof.** Let $\tilde{B} \in \Omega_{2n}$ be $\tilde{H}$-selfadjoint with only negative real eigenvalues, where $\tilde{H} \in \Omega_{2n}$ is invertible and Hermitian, and let $m$ be odd. Assume that $\tilde{B} = J_n(\lambda) \oplus J_n(\lambda)$ and $\tilde{H} = \varepsilon Q_n \oplus \varepsilon Q_n$, where $\lambda < 0$ and...
\( \varepsilon = \pm 1 \). Since \( m \) is odd, we can take \( \mu \) to be the real \( m \)-th root of \( \lambda \) and let \( J = J_a(\mu) \oplus J_n(\mu) \). Then the Jordan normal form of \( J^m \) is equal to \( \tilde{B} \), and both \( J \) and \( J^m \) are \( \tilde{H} \)-selfadjoint. Similarly to Theorem 3.3, we can construct an invertible matrix \( P \in \Omega_{2n} \) such that the equations

\[
P^{-1}J^mP = \tilde{B} \quad \text{and} \quad P^*\tilde{H}P = \tilde{H}
\]

hold. Therefore, from Lemma 2.5, the matrix \( \tilde{A} := P^{-1}JP \in \Omega_{2n} \) is an \( \tilde{H} \)-selfadjoint \( m \)-th root of \( \tilde{B} \).

In the case where the matrix \( \tilde{B} \) has only the number zero as an eigenvalue, we instantly notice a necessary condition for the existence of an \( m \)-th root \( \tilde{A} \) of \( \tilde{B} \). Note that \( \tilde{A} \) is not necessarily \( \tilde{H} \)-selfadjoint. Since each number in the Segre characteristic of \( \tilde{A} \) corresponding to the zero eigenvalue occurs twice (see Corollary 2.3) and from the way \( m \)-th roots are formed for nilpotent matrices, see for example [3, 5], the \( m \)-tuples in the Segre characteristic (or some reordering thereof) corresponding to the zero eigenvalue of \( \tilde{B} \) have to exist in pairs, i.e. there are two of each \( m \)-tuple. For example, with \( m = 4 \), if the nonzero part of the Segre characteristic of a nilpotent matrix \( \tilde{B} \in \Omega_{20} \) is \((3, 3, 2, 2), (3, 3, 2, 2)\), then the nonzero part of the Segre characteristic of any \( m \)-th root \( \tilde{A} \in \Omega_{20} \) is \((10, 10)\).

Here we need another result from [5] which was obtained by applying \( \omega_n \) to all matrices:

**Lemma 3.8.** Let \( A \) be equal to \( J_n(0) \oplus J_n(0) \). Then \( A^m \) has Jordan normal form

\[
\bigoplus_{i=1}^{r} J_{a+1}(0) \oplus \bigoplus_{i=1}^{m-r} J_a(0) \oplus \bigoplus_{i=1}^{r} J_{a+1}(0) \oplus \bigoplus_{i=1}^{m-r} J_a(0),
\]

where \( n = am + r \), for \( a, r \in \mathbb{Z}, 0 < r \leq m \).

We now give the result for this last case.

**Theorem 3.9.** Let \( \tilde{H} \in \Omega_{2n} \) be an invertible Hermitian matrix and \( \tilde{B} \in \Omega_{2n} \) an \( \tilde{H} \)-selfadjoint matrix with a spectrum consisting of only the number zero. Then there exists an \( \tilde{H} \)-selfadjoint matrix in \( \Omega_{2n} \), say \( \tilde{A} \), such that \( \tilde{A}^m = \tilde{B} \) if and only if the following properties hold:

1. There exists a reordering of the Segre characteristic of \( \tilde{B} \) such that each \( m \)-tuple occurs twice and the difference between any two numbers in each \( m \)-tuple is at most one.
2. By using a reordering satisfying the first property, the canonical form of \( (\tilde{B}, \tilde{H}) \) is given by \( (J_B \oplus J_B, H_B \oplus H_B) \) where

\[
J_B = \bigoplus_{j=1}^{t} \left( \bigoplus_{i=1}^{r_j} J_{a_j+1}(0) \oplus \bigoplus_{i=r_j+1}^{m} J_{a_j}(0) \right),
\]

and

\[
H_B = \bigoplus_{j=1}^{t} \left( \bigoplus_{i=1}^{r_j} \varepsilon_i^{(j)} Q_{a_j+1} \oplus \bigoplus_{i=r_j+1}^{m} \varepsilon_i^{(j)} Q_{a_j} \right),
\]

for some \( a_j, r_j \in \mathbb{Z} \) with \( 0 < r_j \leq m \), and the signs are as follows, given in terms of \( \eta_j \), where \( \eta_j \) could be either 1 or -1: If \( r_j \) (respectively \( m - r_j \)) is even, half of the \( \varepsilon_i^{(j)} \) for \( i = 1, \ldots, r_j \) (respectively for \( i = r_j + 1, \ldots, m \)) are equal to \( \eta_j \) and the other half are equal to \( -\eta_j \). If \( r_j \) (respectively \( m - r_j \))
is odd, there is one more of the $\varepsilon_i^{(j)}$ for \(i = 1, \ldots, r_j\) (respectively for \(i = r_j + 1, \ldots, m\)) equal to $\eta_j$ than to $-\eta_j$.

Proof. Let $B \in \Omega_{2n}$ be $\tilde{H}$-selfadjoint with only zero in its spectrum, where $\tilde{H} \in \Omega_{2n}$ is invertible and Hermitian. Assume there exists an $\tilde{H}$-selfadjoint $m$th root $\tilde{A} \in \Omega_{2n}$, i.e. $\tilde{A}^m = B$. From Theorem 2.6 there exists an invertible matrix $S \in \Omega_{2n}$ such that the canonical form of $(\tilde{A}, \tilde{H})$ is given by

$$S^{-1} \tilde{A} S = \bigoplus_{j=1}^{t} J_{k_j}(0) \oplus \bigoplus_{j=1}^{t} J_{k_j}(0),$$

and the permutation matrix defined in (3.10) for some $t, k_j$ and signs $\eta_j = \pm 1$. Consider

$$S^{-1} \tilde{B} S = (S^{-1} \tilde{A} S)^m = \bigoplus_{j=1}^{t} (J_{k_j}(0))^m \oplus \bigoplus_{j=1}^{t} (J_{k_j}(0))^m.$$

Therefore by using Lemma 3.8 and the permutation matrix defined in (2.5), the matrix $\tilde{B}$ has Jordan normal form

$$(3.10) \bigoplus_{j=1}^{t} \left( \bigoplus_{i=1}^{r_j} J_{a_i+1}(0) \oplus \bigoplus_{i=1}^{m-r_j} J_{a_i}(0) \right) \oplus \bigoplus_{j=1}^{t} \left( \bigoplus_{i=1}^{r_j} J_{a_i+1}(0) \oplus \bigoplus_{i=1}^{m-r_j} J_{a_i}(0) \right),$$

where $k_j = a_j m + r_j$, $a_j, r_j \in \mathbb{Z}$ and $0 < r_j \leq m$. Hence from (3.10) we see that there exists a reordering of the Segre characteristic of $\tilde{B}$ in which each $m$-tuple occurs twice and where the difference of any two numbers in each $m$-tuple is at most one.

Since the Jordan normal form of $\tilde{B}$ is given in (3.10), by Theorem 2.6 the corresponding matrix in the canonical form is given by $H_B \oplus H_B$ where

$$H_B = \bigoplus_{j=1}^{t} \left( \bigoplus_{i=1}^{r_j} \varepsilon_i^{(j)} Q_{a_i+1} + \bigoplus_{i=r_j+1}^{m} \varepsilon_i^{(j)} Q_{a_i} \right),$$

for $\varepsilon_i^{(j)} = \pm 1$, $i = 1, \ldots, m$, $j = 1, \ldots, t$. From the assumption that an $\tilde{H}$-selfadjoint $m$th root exists, we know that the two properties given in [5, Theorem 2.5] hold. The second property is used to find the signs of $H_B$. Thus it follows: if $r_j$ (respectively $m - r_j$) is even, half of the $\varepsilon_i^{(j)}$ for $i = 1, \ldots, r_j$ (respectively for $i = r_j + 1, \ldots, m$) are equal to $\eta_j$ and the other half are equal to $-\eta_j$. If $r_j$ (respectively $m - r_j$) is odd, there is one more of the $\varepsilon_i^{(j)}$ for $i = 1, \ldots, r_j$ (respectively for $i = r_j + 1, \ldots, m$) equal to $\eta_j$ than to $-\eta_j$.

Conversely, let $\tilde{B}$ be an $\tilde{H}$-selfadjoint matrix and suppose that the two properties in the theorem hold. Assume from the first property that $\tilde{B} = B_1 \oplus B_1 \in \Omega_{2n}$ where

$$B_1 = \bigoplus_{j=1}^{t} \left( \bigoplus_{i=1}^{r_j} J_{a_i+1}(0) \oplus \bigoplus_{i=1}^{m-r_j} J_{a_i}(0) \right),$$

and assume that $\tilde{H} = H_1 \oplus H_1$ where $H_1 = H_B$ is as in (3.9). Let

$$J = \bigoplus_{j=1}^{t} J_{t_j}(0) \oplus \bigoplus_{j=1}^{t} J_{t_j}(0),$$

for some $t, t_j$ and signs $\eta_j = \pm 1$. Consider

$$S^{-1} \tilde{B} S = (S^{-1} \tilde{A} S)^m = \bigoplus_{j=1}^{t} (J_{k_j}(0))^m \oplus \bigoplus_{j=1}^{t} (J_{k_j}(0))^m.$$
where $t_j$ is the sum of the sizes of the blocks in $\tilde{B}$ which correspond to one $m$-tuple, i.e. $t_j = r_j(a_j + 1) + (m - r_j)(a_j) = a_jm + r_j$. Thus by using Lemma 3.8 and the permutation matrix defined by (2.5), the Jordan normal form of $J^m$ is equal to $\tilde{B}$. Also, let $Q = \bigoplus_{j=1}^{t} \varepsilon_j Q_{t_j} \oplus \bigoplus_{j=1}^{t} \varepsilon_j Q_{t_j}$ where $\varepsilon_j = \eta_j$ is obtained from the signs of $H_B$, then $J$ is $Q$-selfadjoint. According to [5, Theorem 2.5] there exists an invertible matrix $P_1$ such that

$$P_1^{-1}\left(\bigoplus_{j=1}^{t} J_{t_j}(0)\right)^m P_1 = B_1 \quad \text{and} \quad P_1^*\left(\bigoplus_{j=1}^{t} \varepsilon_j Q_{t_j}\right)P_1 = H_1.$$ 

Let $P = P_1 \oplus \tilde{P}_1$, then $P$ is an invertible matrix in $\Omega_{2n}$ and the equations

$$P^{-1}J^m P = \tilde{B} \quad \text{and} \quad P^*QP = \tilde{H}$$

hold. From Lemma 2.5 the matrix $\tilde{A} := P^{-1}JP \in \Omega_{2n}$ is an $\tilde{H}$-selfadjoint $m$th root of $\tilde{B}$.

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