Do stochastic inhomogeneities affect dark-energy precision measurements?

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Estimating the existence of dark energy and determining its parameters is one of the central issues in modern cosmology. Evidence for a sizable dark-energy component in the cosmic fluid comes from different sources: CMB anisotropies, models of large-scale-structure formation and, most directly, the luminosity redshift relation of Type Ia supernovae, used as standard candles.

In this latter case, on which we concentrate our attention, the analysis is usually made in the simplified context of a homogeneous and isotropic (FLRW) cosmology. The issue has then been raised about whether inhomogeneities may affect the conclusion of such a naive analysis. Inhomogeneous models in which we occupy a privileged position in the Universe, for instance, can mimic dark energy (as first pointed out in [1]), but look both unrealistic and highly fine-tuned. More interestingly, we should address this question in the presence of stochastically isotropic and homogeneous perturbations of the kind predicted by inflation. We present here the main ideas and results of such a study, while its detailed derivation and discussion is presented in [2] and in a forthcoming paper [3].

There is by now general agreement that super-horizon perturbations cannot mimic dark-energy effects [4]. By contrast, the impact of sub-horizon perturbations is still unsettled [5,6] owing to the appearance of ultraviolet divergences [7] while computing their “backreaction” on certain classes of large-scale averages [6,7]. The possibility that these effects may simulate a substantial fraction of dark energy, or that they may at least play some role in the context of near-future precision cosmology, has to be seriously considered.

In order to address these issues we have studied the luminosity-redshift relation in a spatially-flat ΛCDM model, perturbed by a stochastic background of inhomogeneities. The luminosity distance $d_L$ now depends on the redshift $z$ as well as on the angular coordinates of the sources, and must be inserted in an appropriate light-cone and ensemble average [9,10]. Unlike the analyses in [6,7], we find a result always free from ultraviolet divergences and with no significant infrared contributions either. As a consequence, corrections are typically small, certainly too small to mimic a sizeable fraction of dark energy. However, interestingly enough, both their size and their $z$-dependence strongly depend on the particular function of $d_L$ being averaged.

We find, in particular, that the energy flux $\Phi \sim d_L^{-2}$ is practically unaffected by inhomogeneities, while the most commonly used variables (like the distance modulus $\mu \sim 5 \log_{10} d_L$) may receive much larger corrections. This creates (at least in principle) intrinsic ambiguities in the measure of the dark-energy parameters, unless the back-reaction of stochastic inhomogeneities is properly taken into account. Actually, the advantages of flux averaging for minimizing biases on dark-energy parameters was first pointed out in [11], where it was shown how the binning of data in appropriate redshift intervals can reduce the bias due to systematic effects such as weak lensing. It is intriguing that the preferred role played by the flux variable also comes out in this paper where we perform a completely different averaging procedure, at fixed redshift. Our conclusions are not due to a binning of data,
but to an application of our covariant space-time average to different functions of the luminosity distance.

Let us start by recalling the standard expression for the luminosity distance in an unperturbed flat ΛCDM model, with present fractions of critical density Ω_m and Ω_Λ:

\[
d_L^{FLRW}(z) = \frac{1 + z}{H_0} \int_0^z \frac{dz'}{[\Omega_m + (1 + z')^3]^{1/2}}. \tag{1}
\]

Consider now the expression for \(d_L\) in the corresponding perturbed geometry. Combining light-cone and ensemble averages (denoted, respectively, by brackets and over-bars), we can write the averaged result in the form:

\[
\langle d_L \rangle(z) = d_L^{FLRW} (1 + f_d(z)), \tag{2}
\]

where \(f_d(z)\) represents the “backreaction” on \(d_L\) due to inhomogeneities. For consistency, \(d_L\) has to be computed (at least) up to the second perturbative order since ensemble averages of first-order quantities are vanishing for stochastic perturbations. In particular, backreaction terms arise also from correlations between the inhomogeneities present in the averaged variable and in the covariant integration measure. Therefore, a consistent perturbative calculation requires the inclusion of linear second-order contributions, since they are of the same order as the above quadratic first-order terms (see also [10], Sect. 4). A detailed computation of \(f_d(z)\) would thus enable to extract the “true” value of the dark-energy parameters from the measurement of \(\langle d_L \rangle(z)\) after taking the correction into account.

However, as already stressed in [10], given the covariant (light-cone) average of a perturbed (inhomogeneous) observable \(S\) the average of a generic function of this observable differs, in general, from the function of its average, i.e. \(\langle F(S) \rangle \neq F(\langle S \rangle)\). Expanding the observable to second order as \(S = S_0 + S_1 + S_2 + \cdots\), one finds:

\[
\langle F(S) \rangle = F(S_0) + F'(S_0)\langle S_1 + S_2 \rangle + F''(S_0)\langle S_1^2/2 \rangle \tag{3}
\]

where \(\langle S_1 \rangle \neq 0\) as a consequence of the “induced backreaction” terms (see [10], Sect. 4). Thus different functions of the luminosity distance are differently affected by the inhomogeneities, and require different “subtraction” procedures. Finding the function that minimizes the backreaction will help of course for a precision estimate of the cosmological parameters. One of the main claims of this paper is the identification of such an optimal observable with the energy flux \(\Phi = L/(4\pi d_L^2)\) received from a standard candle of luminosity \(L\) located on the observer’s past light-cone. We now illustrate how we have performed such a calculation.

The average value of \(\Phi\), obviously controlled by the average of \(d_L^2\), has to be carried out on the past light-cone of the observer, at a fixed redshift \(z\), using the gauge-invariant prescription introduced in [9]. This is most conveniently done [9] [10] in the so-called geodesic light-cone gauge (GLC), where the metric depends on six arbitrary functions \((\Upsilon, U^a, \gamma_{ab}, a, b = 1, 2)\), and the line-element takes the form (with \(\tilde{\theta}^2 = \tilde{\phi}^2 = \tilde{\rho}^2\)):

\[
ds^2 = \Upsilon^2 d\tau^2 - 2\Upsilon d\tau dt + \gamma_{ab}(d\tilde{\theta}^a - U^a d\tau)(d\tilde{\theta}^b - U^b d\tau). \tag{4}
\]

The correspondence between the GLC gauge and the spatially flat FLRW geometry is [9]: \(\tau = t, w = \tau + \eta, \Upsilon = \alpha(t), U^a = 0\) and \(\gamma_{ab}d\tilde{\theta}^a d\tilde{\theta}^b = a^2 r^2 (d\tilde{\theta}^2 + \sin^2 \tilde{\theta} d\tilde{\varphi}^2)\), where \(\eta\) is the conformal-time coordinate (\(dt/d\alpha\)).

In the GLC gauge the past light-cone is defined by the condition \(w = w_0 = \text{const}\), and the redshift is given by:

\[
1 + z = \frac{\Upsilon(w_0, \tau_0, \tilde{\theta}^a)}{\Upsilon(w_0, \tau, \tilde{\theta}^a)}. \tag{5}
\]

Furthermore, the luminosity distance of the source is simply expressed as [10] \(d_L = (1 + z)^2\gamma^{-1/2}(\sin \tilde{\theta})^{-1/2}\), yielding the following exact result [2]:

\[
\langle d_L^2 \rangle(z, w_0) = \frac{4\pi(1 + z)^{-4}}{\int d\tilde{\theta}^a \sqrt{\gamma(w_0, \tau, z, \tilde{\theta}^a, \tilde{\varphi}^a)}}, \tag{6}
\]

where \(\gamma = \det \gamma_{ab}\), and \(\tau(z, \tilde{\theta}^a)\) is obtained by solving Eq. [5]. The above expression has a simple physical interpretation: the averaged flux, for a given \(z\), is inversely proportional to the proper area (computed with respect to the metric [1]) of the surface lying on our past light-cone at the given value of \(z\). Flux conservation is probably at the basis of the particular simplicity of this average and of its minimal deviation from the homogeneous value.

To compute this quantity in the perturbed geometry of our interest, we need to express it in a gauge where the stochastic background of cosmological perturbations is explicitly known up to second order. To this purpose, we can use the standard Poisson gauge where we include first and second-order scalar perturbations, neglecting their tensor and vector counterparts (see [2] for a discussion of this point). Performing the relevant transformations to second order we arrive at the following analogue of [2]:

\[
\langle d_L^2 \rangle = (d_L^{FLRW})^{-2} \langle I_\Phi(z) \rangle^{-1} = (d_L^{FLRW})^{-2} [1 + f_\Phi(z)], \tag{7}
\]

where \(I_\Phi\) has in general the following structure:

\[
I_\Phi(z) = \int \frac{d\tilde{\theta} d\tilde{\varphi} \sin \tilde{\theta}}{4\pi} \left[ 1 + I_1 + I_{1,1} + I_2 \right] (\tilde{\theta}, \tilde{\varphi}, z). \tag{8}
\]

Here \(I_1, I_{1,1}, I_2\) are, respectively, the first-order, quadratic first-order, and genuine second-order contributions of our stochastic fluctuations. After solving the relevant perturbation equations [12] they can all be expressed in terms of the first-order Bardeen potential \(\Psi(x, \eta)\). Using the stochastic properties of this perturbation, and expanding in Fourier modes \(\Psi_k(\eta)\), we can then obtain an expression for \(I_\Phi^{-1}\) where first-order contributions drop out because of the ensemble average, and the scalar perturbations only appear through
the so-called dimensionless power spectrum, \( \mathcal{P}(k, \eta) = (k^2/2\pi^2)\Psi(k(\eta))^2 \).

Unfortunately, \((k_d)^{-1}\) contains integrals over null geodesics lying on the past light-cone of the given observer (see [10], Sect. 3.2), which get intertwined with the time-dependence of \(\mathcal{P}\), forcing us to proceed with an approximate numerical integration. This will be done below, after inserting (as an instructive example) an illustration of the limiting CDM case, where all integrals but the one over \(k\) can be done analytically thanks to the time-independence of \(k\) but the one over \(\eta\) can be done analytically thanks to the time-independence of \(\mathcal{P}\) ([10], Sect. 5).

In that case the result can be written in the form

\[
    f_\Phi(z) = \int_0^\infty \frac{dk}{k} \mathcal{P}(k) \left[ f_{1,1}(k, z) + f_2(k, z) \right],
\]

where \(f_{1,1}\) and \(f_2\) are complicated but known analytic functions of their arguments [3]. Furthermore, the leading contribution in the region of \(z\) relevant for dark-energy phenomenology comes from terms of the type \(f(k, z) \sim (k/H_0)^2 \tilde{f}(z)\), where \(H_0\) is the present Hubble scale. We can then write, to a very good accuracy,

\[
    f_\Phi(z) \approx \left[ \tilde{f}_{1,1}(z) + \tilde{f}_2(z) \right] \int_0^\infty \frac{dk}{k} \left( k/H_0 \right)^2 \mathcal{P}(k),
\]

where an explicit calculation gives [3]:

\[
    \tilde{f}_{1,1}(z) = \frac{10 - 12\sqrt{1+z} + 5z (2 + \sqrt{1+z})}{27(1+z) (\sqrt{1+z} - 1)^2},
\]

\[
    \tilde{f}_2(z) = \frac{1}{189} \left[ 2 - 2\sqrt{1+z} + z (9 - 2\sqrt{1+z}) \right].
\]

The absolute value (and sign) of \(f_\Phi(z)\) are illustrated in Fig. 1, showing the accuracy of the leading order terms adapted to \(\Lambda\)CDM.

and confirming that the backreaction of a realistic spectrum of stochastic perturbations induces negligible corrections to the averaged flux at large \(z\) (the larger corrections at small \(z\), due to “Doppler terms”, has been discussed in [10]). In addition, it shows that, in any case, such corrections have the wrong \(z\)-dependence (in particular change sign at some \(z\)) to simulate even a tiny dark-energy component. For the considered spectrum (behaving as \(k^{n_s-5}\log^2 k\) at large \(k\), see [14]) the spectral integral is convergent and very weakly sensitive to the chosen value of the UV cutoff [10] representing here the limit of validity of our perturbative approach.

We now come to the more realistic \(\Lambda\)CDM case, where the \(f_\Phi\) correction should be obtained by a full numerical integration of Eqs. ([7], [8]). For simplicity, we will only take into account those terms giving the leading \((k^2\text{-enhanced})\) contributions in the CDM case. For \(\Lambda\)CDM we can generally expect a smaller correction due to the fact that the spectrum is now suppressed, at large \(k\), by a lower value of the equality scale \(k_{\text{eq}}\) [14]. This is confirmed by the explicit numerical result for \(|f_\Phi|\) presented in Fig. 2. The small value of \(|f_\Phi|\) at large \(z\) leads us to conclude that the averaged flux is a particularly appropriate quantity for extracting from the observational data the “true” cosmological parameters. As we are going to see now, the situation is somewhat different for other functions of \(d_L\).

Indeed, let’s apply the general result [3] to the flux variable, \(S = \Phi\), and consider two important examples: \(F(\Phi) = \Phi^{-1/2} \sim d_L\), and \(F(\Phi) = -2.5 \log_{10} \Phi + \text{const} \sim \mu\) (the distance modulus). For the luminosity distance, following the notations of Eq. ([2]) and using the general result [3], we obtain:

\[
    f_d = -(1/2)f_\Phi + (3/8)((\Phi_1/\Phi_0)^2). \tag{13}
\]
Similarly, for the distance modulus we obtain:
\[
\overline{(\mu)} - \mu^{FLRW} = -1.25(\log_{10} e) \left[ 2f_\Phi - \frac{((\Phi_1/\Phi_0)^2)}{\langle((\Phi_1/\Phi_0)^2)\rangle} \right],
\]
where \(f_\Phi\) is defined in Eq. (7).

As clearly shown by the two above equations, the corrections to the averaged values of \(d_L\) and \(\mu\) are qualitatively different from those of the flux (represented by \(f_\Phi\)), because of the extra contribution (inevitable for any non-linear function of the flux) proportional to the square of the first-order fluctuations. As mentioned before, the averaged flux corrections have leading spectral contributions of the type \(k^2P(k)\); on the contrary, the new corrections to \(d_L\) and \(\mu\) are due to the so-called "lensing effect", they dominate at large \(z\), and have leading spectral contributions of the type \(k^3P(k)\) (as already discussed in [10]). The explicit numerical integration, reported in Fig. 2, confirms that, as a result, \(|f_\Phi| \ll f_d\) at large \(z\). We stress that even the \(k^3\)-enhanced contributions are UV-finite for the case under consideration.

We also stress that our results concerning the effects of lensing are in good agreement with previous estimates of the bias on supernova observables [15] and other cosmological parameters [16] induced by weak-lensing magnification effects. Unlike in those papers, however, our general approach automatically includes (and estimates the effects of) all possible corrections due to the stochastic fluctuations of the cosmological background, to second order, for all given functions of the flux (or of \(d_L\)). In fact, as discussed in detail in [2,9], the fractional correction \(f_d\) includes, besides the lensing effect, also Doppler, Sachs-Wolfe, integrated Sachs-Wolfe, frame-dragging effects, etc.

Let us now briefly discuss to what extent the enhanced corrections due to the squared first-order fluctuations can affect the determination of the dark-energy parameters if quantities other than the flux are used in the fits. To this purpose we consider the much used (average of the) distance modulus given in Eq. (14), referred as usual to the homogeneous Milne model with \(\mu^M = 5 \log_{10}[(2+z)/2H_0]\). In Fig. 3 we compare the averaged value \(\overline{(\mu)} - \mu^M\) with the corresponding expression in a homogeneous \(\Lambda CDM\) model with different values of \(\Omega_\Lambda\). We also show the expected dispersion around the averaged result, represented by the square root of the variance [10]. The latter is given by:
\[
\sqrt{\langle\mu^2\rangle - \langle\mu\rangle^2} = 2.5(\log_{10} e)\sqrt{\langle((\Phi_1/\Phi_0)^2)\rangle};
\]
while for the flux we simply find:
\[
\sqrt{\langle((\Phi/\Phi_0)^2)\rangle - \langle\Phi/\Phi_0\rangle^2} = \sqrt{\langle((\Phi_1/\Phi_0)^2)\rangle}.\]

As illustrated in Fig. 3, we find that, even for the distance modulus, the effect of inhomogeneities on the average only affects the determination of \(\Omega_\Lambda\) at the third decimal figure (see also Fig. 2), at least for the inflationary power spectrum with the \(\Lambda CDM\) transfer function of [14]: in that case, the curves for \(\overline{(\mu)}\) and \(\mu^{FLRW}\) are practically coincident at large \(z\). We have considered other spectra which take into account non-linear effects and have more power at short scales, like those obtained following [17]. Using such spectra only affects very mildly the \(k^2\)-enhanced terms (hence the flux) while they increase the corrections wherever the \(k^3\)-enhanced lensing terms play a major role. In particular, the variance due to the fluctuations, which is already at the few-% level at large \(z\) for the power spectrum of [14] (see Fig. 3), can be further increased [3]. Note that, even for these improved spectra, all our integrals are still free of UV divergences since, in any case, \(P\) falls faster than \(k^{-3}\) (i.e. the matter density constrain spectrum grows slower than \(k\)).

Our main conclusions can be summarized as follows:

(1) Dealing directly with the experimentally measured luminosity-redshift relation within a gauge-independent approach leads to results for the fractional corrections to the averaged variables and the corresponding variances which are automatically free from UV (and IR) divergences for any function of the luminosity distance. This can be contrasted with the case of more formal space-like averages [8,7] for which the physical interpretation of the results may have no direct relation with the observed cosmic acceleration (first reference in [5]) and, as shown in [7], the accidental cancellation of UV divergences is strongly dependent on the observable considered.

(2) The actual value of the backreaction strongly depends on the quantity being averaged. It turns out to be minimal for the flux \(\Phi\), which is also practically insensitive to the short-distance behaviour of the power spectrum. Therefore, the flux stands out as the safest observ-
able for precision cosmology. For other observables, such as the distance modulus, the backreaction is considerably larger and is more sensitive to the spectrum used.

3. The dispersion due to stochastic fluctuations is much larger than the backreaction itself, implying an irreducible scatter of the data that may limit to the percent level (see Fig. 3) the precision attainable on cosmological parameters because of the present limited statistics.

4. We calculated here the full second order effect of stochastic perturbations and concluded that they cannot simulate a substantial fraction of dark energy. Possible contributions coming from the non-perturbative regime on length scales much smaller than 1 Mpc have still to be taken in consideration before final conclusions can be drawn.

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[1] M. -N. Celerier, Astron. Astrophys. 353, 63 (2000).
[2] I. Ben-Dayan, G. Marozzi, F. Nugier and G. Veneziano, JCAP 11, 045 (2012).
[3] I. Ben-Dayan, M. Gasperini, G. Marozzi, F. Nugier and G. Veneziano, in preparation.
[4] E. E. Flanagan, Phys. Rev. D 71, 103521 (2005);
G. Geshnizjani, D. J. H. Chung and N. Afshordi, Phys. Rev. D 72, 023517 (2005);
C. M. Hirata and U. Seljak, Phys. Rev. D 72, 083501 (2005).
[5] A. Ishibashi, and R. M. Wald, Class. Quant. Grav. 23, 235 (2006);
A. Paranjape and T. P. Singh, Phys. Rev. Lett. 101, 181101 (2008).
[6] E. W. Kolb, Class. Quant. Grav. 28, 164009 (2011).
[7] C. Clarkson and O. Umehr, Class. Quant. Grav. 28, 164010 (2011).
[8] G. Marozzi and J. -P. Uzan, Phys. Rev. D 86, 063528 (2012).
[9] M. Gasperini, G. Marozzi, F. Nugier and G. Veneziano, JCAP 07, 008 (2011).
[10] I. Ben-Dayan, M. Gasperini, G. Marozzi, F. Nugier and G. Veneziano, JCAP 04, 036 (2012).
[11] Y. Wang, Astrophys. J. 536, 531 (2000).
[12] N. Bartolo, S. Matarrese and A. Riotto, JCAP 05, 010 (2006).
[13] E. Komatsu et al., Astrophys. J. Suppl. 192, 18 (2011).
[14] D. J. Eisenstein, W. Hu, Astrophys. J. 496, 605 (1998).
[15] Y. Wang, D. E. Holz and D. Munshi, Astrophys. J. 572, L15 (2002).
[16] D. Sarkar, A. Amblard, D. F. Holz and A. Cooray, Astrophys. J. 678, 1 (2008).
[17] R. E. Smith et al., Mon. Not. R. Astron. Soc. 341, 1311 (2003).