Balanced Truncation of Networked Linear Passive Systems

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Abstract

This paper studies model order reduction of multi-agent systems consisting of identical linear passive subsystems, where the interconnection topology is characterized by an undirected weighted graph. Balanced truncation based on a pair of specifically selected generalized Gramians is implemented on the asymptotically stable part of the full-order networked model, which leads to a reduced-order system preserving the passivity of each subsystem. To restore the network structure, we then apply a coordinate transformation to convert the resulting reduced-order model to a state-space model of Laplacian dynamics. The proposed method simultaneously reduces the complexity of the network structure and individual agent dynamics. Moreover, it preserves the passivity of the subsystems and allows for the \textit{a priori} computation of a bound on the approximation error. Finally, the feasibility of the method is demonstrated by an example.

Key words: Model reduction; Balanced truncation; Passivity; Laplacian matrix; Network topologies.

1 Introduction

Multi-agent systems, or network systems, recently have become a rapidly evolving area of research with a tremendous amount of applications, including power grids, cooperative robots, biology and chemical reaction networks (see, e.g. [20,28] for an overview). A multi-agent system captures the behaviors of multiple dynamical subsystems which are interacting through a network. Due to this characteristic, a multi-agent system may easily become high-dimensional, i.e., when large-scale networks and complex agent dynamics are considered. However, the full-order complex network models, in most cases, are neither practical nor necessary for controller design, system simulation and validation. Hence, it is desirable to apply model order reduction techniques to derive a lower-order approximation of the original network system with an acceptable accuracy.

The interaction of agents is usually characterized by the structure of the Laplacian matrix, which represents the communication graph. Since synchronization and stability of networks are analyzed in the context of Laplacian dynamics (see e.g. [18,22]), it is quite a natural requirement to preserve the algebraic structure of the Laplacian matrix in order to inherit a network interpretation of the reduced-order model.

Although conventional reduction techniques, including balanced truncation, Hankel-norm approximation, and Krylov subspace methods, provide systematic procedures to generate a lower dimensional approximation for linear dynamical systems (see e.g. [1] for an overview), the direct application of these methods to multi-agent systems potentially leads to the loss of specific properties such as the synchronization of networks and the structure of the subsystems. Namely, such methods do not explicitly take the interconnection structure into account in deriving the reduced-order models.

Towards the model order reduction with the preservation of network structure, mainstream methodologies are focusing on graph clustering. From the results of networked single integrators in [31,23,24,4], we have observed that the clustering-based approaches naturally maintain the spatial structure of networks and show an insightful physical interpretation for the reduction process. Further extensions to directed and second-order networks can be found in [14,8,6]. Nevertheless, the approximation accuracy of these methods highly relies on the selection of node clusters, and generally finding a reduced network with the best approximation is an NP-hard problem, see [15]. A combination of the Krylov subspace method with graph clustering is proposed by [21].

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where a reduced-order model is firstly found by the Iterative Rational Krylov Algorithm (IRKA), and then the partition of network nodes is obtained by the QR decomposition with column pivoting on the projection matrix. However, no error bound is provided for the network approximation. Differently, the work [2] considers so-called edge dynamics of networks with a tree topology, and the importance of edges is characterized by generalized edge controllability and observability Gramians. Nodes linked by the less important edges are clustered, and an a priori bound for the approximation error is then computed based on the generalized singular values of the edge dynamics. Nonetheless, the application of this approach is still restrictive since the reduction process and error bound are heavily reliant on the tree topology of the studied network. Another attempt to simplify the complexity of network structure is developed based on singular perturbation approximation, which is mainly applied to electrical grids and chemical reaction networks (see e.g., [9,3,26] and references therein). The network structure is preserved as the Schur complement of the Laplacian matrix of the original network is again a Laplacian matrix, which represents a smaller-scale network. Despite the simplicity, it is challenging to implement this approach for multi-agent systems with higher-order dynamics as the Laplacian matrix is coupled with agent systems in this case.

The other direction in model order reduction of multi-agent systems is to lower the dimension of the individual subsystem while keeping the interconnection topology untouched. Related methods can be found in [22,29] which are developed based on the generalization of balanced truncation, and can be interpreted as structure-preserving model reduction procedures.

In this paper, we aim to find a technique that can reduce the complexity of network structures and individual agent dynamics simultaneously, extending preliminary results in [7]. This problem setting has been absent from the literature so far. In particular, this paper considers multi-agent systems composed of identical higher-order linear passive subsystems, where the interconnection topology is characterized by an undirected weighted graph. It is remarked that passive systems are natural candidates to model many types of real physical systems and the passivity property benefits the synchronization and stability analysis of network systems [19,32,16,33]. The core step in the proposed reduction technique for networked passive systems is balancing the asymptotically stable part based on generalized Gramians. After truncating the balanced model, we obtain a reduced-order system that has a lower dimension and has preserved the passivity of the subsystems. Although the network structure is not necessarily preserved in this step, we show that there exists a set of coordinates in which the reduced-order model can again be interpreted as a network system. Specifically, the interconnection matrix of the reduced-order system can always be transformed to one which has a Laplacian structure. Thus, the network structure is restored in the reduced multi-agent system, which also admits the energy dissipation of the lower-order agent dynamics and thus preserves synchronization in the reduced network. Furthermore, our method also guarantees the a priori computation of a bound on the approximation error with respect to external inputs and outputs. Compared to the existing results, the proposed method has the following advantages.

First, the balanced truncation is based on two generalized Gramians that are selected to serve the double purpose of reducing the subsystem dimensions and of simplifying the communication network topology. The resulting reduced-order model is verified to preserve the network structure as well as the passivity of the agent dynamics.

Secondly, the reduced-order multi-agent system resulting from the proposed method generally achieves a smaller approximation error than the ones found by clustering-based approaches, at the cost of constructing a new simplified network that is less correlated with the original one. This difference is mainly due to the different projections. Clustering-based methods adopt the characteristic matrix of a graph partition to project the original network model to a reduced space (see [6]), while our method does not enforce a restriction on the structure of projection matrix since balanced truncation is applied. For that reason, the model error bound can be determined a priori.

Third, the studied models of the multi-agent dynamics are relatively general in terms of the network topology and the structures of the input and output matrices. Specifically, the underlying communication graph, in contrast to [2], can be formed with various topologies, and there are no restrictions on the input and output distributions. Unlike the clustering-based approaches in e.g. [14,5,6], the effort of state observability is also considered, and no special structure of the output matrix is required as in [23,21].

The remainder of this paper is organized as follows. Section 2 provides the preliminaries regarding passivity and formulates the model reduction problem of networked passive systems. Then, the approximation procedure based on the balanced truncation approach is presented in Section 3, which provides the main results of this paper. Finally, the proposed method is illustrated by means of an example in Section 4 and some concluding remarks are summarized in Section 5.

Notation: The symbol $\mathbb{R}$ denotes the set of real numbers, whereas $I_n$ and $I_1$ represent the identity matrix of size $n$ and all-ones vector of $n$ entries, respectively. The subscript $n$ is omitted when no confusion arises. The Kronecker product of matrices $A \in \mathbb{R}^{m \times n}$ and $B \in \mathbb{R}^{p \times q}$ is...
denoted by $A \otimes B \in \mathbb{R}^{mp \times nq}$. Next, $\Sigma$ represents a linear system, and the operation $\Sigma_1 + \Sigma_2$ means the parallel interconnection of two linear systems by summing their transfer functions. The $\mathcal{H}_\infty$-norm of the transfer function of a linear system $\Sigma$ is denoted by $\|\Sigma\|_{\mathcal{H}_\infty}$.

2 Preliminaries and Problem Formulation

Consider a network of $N$ vertices, and the dynamics on each vertex is described by the following linear time-invariant model

$$\Sigma_i : \begin{cases} \dot{x}_i = Ax_i + Bv_i, \\ \eta_i = Cx_i, \end{cases}$$

where $x_i \in \mathbb{R}^n$, $v_i \in \mathbb{R}^m$ and $\eta_i \in \mathbb{R}^m$ are the states, control inputs and measured outputs of agent $i$, respectively.

Here, we assume that the system realization in (1) is minimal and passive. Passivity is a natural property of many real physical systems, including mechanical systems, power networks, and thermodynamical systems (see [33,16,13]). It is defined as follows, see e.g. [34,19,30].

Definition 1 The system $\Sigma_i$ in (1) is passive if there exists a differentiable storage function $S : \mathbb{R}^n \to \mathbb{R}$ with $S(0) = 0$ and $S(x_i) \geq 0$ for any $x_i$, such that

$$\dot{S}(x_i) := \frac{\partial S(x_i)}{\partial x_i} \dot{x}_i \leq v_i^T \eta_i$$

for all solution trajectories $(v_i(\cdot), x_i(\cdot), \eta_i(\cdot))$ of the system (1). The system $\Sigma_i$ is called lossless if the equality holds, and strictly passive if $\dot{S}(x_i) < v_i^T \eta_i$, $\forall x_i \neq 0$.

Furthermore, for minimal linear system, there exists a quadratic storage function $H(x) = x^T P x_i$ (with $P > 0$), leading to the following version of the Kalman-Yakubovich-Popov (KYP) condition [34]:

Lemma 2 A minimal system $\Sigma_i$ in (1) is passive if and only if there exists a positive definite matrix $P$ such that

$$A^T P + PA \leq 0, \ C = B^T P.$$  

Equality holds if $\Sigma_i$ is lossless. If $\Sigma_i$ is strictly passive, we have $A^T P + PA < 0$ and $C = B^T P$.

In a multi-agent system, all the agents are interacting through a weighted undirected connected graph $G$ containing $N$ vertices. For agent $i$, the static communication protocol is implemented as

$$v_i = - \sum_{j=1,j \neq i}^N w_{ij} (\eta_i - y_j) + \sum_{j=1}^P f_{ij} u_j,$$

where $w_{ij} \in \mathbb{R} \geq 0$ stands for the intensity of the coupling between vertices $i$ and $j$. Besides, $u_j \in \mathbb{R}^m$ with $j = \{1,2,\cdots,p\}$ are external control signals acting on the agents, and $f_{ij} \in \mathbb{R}$ represents the amplification of the $j$-the input acting on agent $i$, which is zero when $u_j$ has no effect on vertex $i$. Similarly, $y_i \in \mathbb{R}^m$ is the $i$-th external output, which is introduced as

$$y_i = \sum_{j=1}^N h_{ij} y_j, \ i = 1,2,\cdots,q, \ (5)$$

where $h_{ij} \in \mathbb{R}$. Combining (1), (4), and (5), we obtain the total multi-agent system in a compact form as

$$\Sigma : \begin{cases} \dot{x} = (I_N \otimes A - L \otimes BC)x + (F \otimes B)u, \\ y = (H \otimes C)x. \end{cases} \ (6)$$

Here, $F \in \mathbb{R}^{N \times p}$ and $H \in \mathbb{R}^{q \times N}$ are the collections of $f_{ij}$ and $h_{ij}$, respectively, and

$$x := [x_1^T, x_2^T, \cdots, x_N^T]^T \in \mathbb{R}^{Nn},$$
$$u := [u_1^T, u_2^T, \cdots, u_p^T]^T \in \mathbb{R}^{pm},$$
$$y := [y_1^T, y_2^T, \cdots, y_q^T]^T \in \mathbb{R}^{q}\,$$

are the combined state vector, external control inputs and measured outputs, respectively. Furthermore, $L \in \mathbb{R}^{N \times N}$ is the Laplacian matrix of the underlying graph $G$ with the $(i,j)$-th entry as

$$L_{ij} = \begin{cases} \sum_{j=1,j \neq i}^N w_{ij}, \text{if } i = j, \\ -w_{ij}, \text{otherwise.} \end{cases} \ (7)$$

This paper assumes that the underlying graph $G$ is undirected and connected, such that the Laplacian matrix $L$ has the following properties, see e.g. [6].

Remark 3 For a connected undirected graph, the Laplacian matrix $L$ fulfills the following structural conditions:

- $1^T L = 0$, and $L1 = 0$;
- $L_{ij} \leq 0$ if $i \neq j$, and $L_{ii} > 0$;
- $L$ is positive semi-definite with a single zero eigenvalue.

The Laplacian $L$ is the matrix representation of the graph $G$. Conversely, a real square matrix can be interpreted as a Laplacian matrix representing a connected undirected graph, if it satisfies the above structural conditions.

Here, we address the model order reduction problem for multi-agent systems of the form (6) as follows.

Problem 4 Given a multi-agent system $\Sigma$ as in (6), find
a reduced-order model
\[
\hat{\Sigma} : \begin{cases}
\dot{\hat{x}} = (I_k \otimes \hat{A} - \hat{L} \otimes \hat{B}C)\hat{x} + (\hat{F} \otimes \hat{B})u, \\
\hat{y} = (\hat{H} \otimes \hat{C})\hat{x},
\end{cases}
\tag{8}
\]
such that the following objectives are achieved:

- \( \hat{L} \in \mathbb{R}^{k \times k} \), with \( k \leq N \), is an undirected graph Laplacian satisfying the structural conditions in Remark 3.
- The lower-order approximation of the agent dynamics
\[
\hat{\Sigma}_i : \begin{cases}
\dot{\hat{x}}_i = \hat{A} \hat{x}_i + \hat{B}u_i, \\
\hat{y}_i = \hat{C} \hat{x}_i,
\end{cases}
\tag{9}
\]
with the reduced state vector \( \hat{x}_i \in \mathbb{R}^r \) (\( r \leq n \)), is passive, i.e., satisfies the KYP condition in Lemma 2.
- The overall approximation error \( \| \Sigma - \hat{\Sigma} \|_{\mathcal{H}_\infty} \) is small.

Remark 3. We therefore consider the following spectral decomposition
\[
L = TAT^T = \begin{bmatrix} T_1 & T_2 \end{bmatrix} \begin{bmatrix} \hat{A} & 0 \\
0 & \hat{T}_2^T \end{bmatrix},
\tag{10}
\]
where \( T_2 = 1/\sqrt{N} \in \mathbb{R}^N \) by the first condition in Remark 3, and
\[
\hat{A} := \text{diag}(\lambda_1, \lambda_2, \cdots, \lambda_{N-1})
\tag{11}
\]
with \( \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_{N-1} > 0 \) the nonzero eigenvalues of \( L \).

Then, applying a coordinate transformation \( x = (T \otimes I)z \), to \( \Sigma \) yields
\[
\begin{cases}
\dot{z} = (I \otimes A - \hat{A} \otimes BC)z + (T^T F \otimes B)u, \\
y = (HT \otimes C)z.
\end{cases}
\tag{12}
\]

Observe that (10) implies the structure
\[
I \otimes A - \hat{A} \otimes BC = \text{bldiag}(I \otimes A - \hat{A} \otimes BC, A).
\tag{13}
\]

Since \( A \) in (1) is not necessarily Hurwitz, meaning that the overall system \( \Sigma \) may not be asymptotically stable, a direct application of the balanced truncation method to \( \Sigma \) is not feasible. To overcome this difficulty, we split the system \( \Sigma \) into two independent components, namely, an average module
\[
\Sigma_a : \begin{cases}
\dot{z}_a = A z_a + \frac{1}{\sqrt{N}} (1_N^T F \otimes B)u, \\
y_a = \frac{1}{\sqrt{N}} (H 1_N \otimes C)z_a,
\end{cases}
\tag{14}
\]
with \( z_a \in \mathbb{R}^n \), and
\[
\Sigma_s : \begin{cases}
\dot{z}_s = (I_{N-1} \otimes A - \hat{A} \otimes BC)z_s + (\hat{F} \otimes B)u, \\
y_s = (H \otimes C)z_s,
\end{cases}
\tag{15}
\]
where \( z_s \in \mathbb{R}^{(N-1) \times n} \), \( F = T_1^T F \), and \( H = HT_1 \). The asymptotic stability of \( \Sigma_a \) is proven in the following lemma.

Lemma 6 If the agent dynamics in (1) is observable and passive, and \( \hat{A} \) is diagonal positive definite, then the system \( \Sigma_a \) in (15) is asymptotically stable.

\[\text{PROOF.}\] If the passive system in (1) is observable, then any negative feedback \( u_i(t) = \lambda_\eta_i(t) = -\lambda BCx_i \), with \( \lambda > 0 \), asymptotically stabilizes the origin \( x_i = 0 \) (see
Then, a closed-loop system of (1) defined by
\[ \dot{x} = (A - \lambda_i BC)x \quad (16) \]
is asymptotically stable for any \( i = 1, 2, \ldots, N - 1 \). Note that
\[ I \otimes A - \hat{A} \otimes BC = \text{blkdiag}(A - \lambda_1 BC, \ldots, A - \lambda_{N-1} BC). \]
Hence, \( \Sigma_s \) is asymptotically stable.

Note that the above proof only requires the observability of the agent system \( \Sigma_i \) in (1). Besides, the asymptotic stability of \( \Sigma_s \) equivalently means that \( A - \lambda_i BC \) is Hurwitz for any \( i = 1, 2, \ldots, N - 1 \), which is exactly the necessary and sufficient condition for synchronization of the multi-agent system \( \Sigma \) (see e.g. [22,35]). Therefore, we have the following proposition.

**Lemma 7** The network system \( \Sigma \) in (6) is synchronized for \( u = 0 \), i.e.,
\[ \lim_{t \to \infty} [x_i(t) - x_j(t)] = 0, \quad \forall i, j \in \{1, 2, \ldots, N\}, \quad (17) \]
if the underlying graph \( G \) is connected, and each subsystem \( \Sigma_i \) in (1) is observable.

Furthermore, \( \Sigma_s \) can be balanced and truncated to generate a lower-order approximation \( \tilde{\Sigma}_s \). Correspondingly, the reduced subsystems result in a reduced-order average module \( \Sigma_s \). Then, combining \( \tilde{\Sigma}_s \) with \( \Sigma_s \) formulates a reduced-order model \( \tilde{\Sigma} \) whose input-output behavior is similar to that of the original multi-agent system \( \Sigma \). However, at this stage, the network structure is not necessarily preserved by \( \tilde{\Sigma} \). Therefore, in the second step, a particular coordinate transformation, we re-model \( \tilde{\Sigma} \) as \( \Sigma \), which restores the algebraic structure of a Laplacian matrix. The whole procedure is summarized in Fig. 1, and the detailed implementations are discussed in the following subsections.

### 3.2 Balanced Truncation by Generalized Gramians

Following [11], the generalized Gramians of the asymptotically stable system \( \Sigma_s \) are defined.

**Definition 8** Consider the stable system \( \Sigma_s \), and denote \( \Phi := I \otimes A - \hat{A} \otimes BC \). Two positive definite matrices \( \mathcal{X} \) and \( \mathcal{Y} \) are said to be the generalized controllability and observability Gramians of \( \Sigma_s \), respectively, if they satisfy
\[ \mathcal{X} \Phi + \Phi^T \mathcal{X} + (\mathcal{F} \otimes B)(\mathcal{F}^T \otimes B^T) \leq 0, \quad (18a) \]
\[ \Phi^T \mathcal{Y} + \mathcal{Y} \Phi + (\mathcal{H}^T \otimes C^T)(\mathcal{H} \otimes C) \leq 0. \quad (18b) \]

Moreover, a balanced realization is achieved when \( \mathcal{X} = \mathcal{Y} > 0 \) are diagonal. The diagonal entries are called generalized Hankel singular values (GHSVs).

To find a pair of generalized Gramians, we first consider the following accompanying system of \( \Sigma_s \), which only contains the information of the network configuration:
\[ \dot{z} = -\tilde{A} z + \tilde{F} u, \quad \tilde{y} = \tilde{H} z, \quad (19) \]
where \( \tilde{A} \) is defined in (11). Assume \( \tilde{A} \) in (11) has \( s \) distinct diagonal entries ordered as: \( \lambda_1 > \lambda_2 > \cdots > \lambda_s \). We then rewrite it as
\[ \tilde{A} = \text{blkdiag}(\tilde{\lambda}_1 I_{m_1}, \tilde{\lambda}_2 I_{m_2}, \cdots, \tilde{\lambda}_s I_{m_s}), \quad (20) \]
where \( m_i \) is the multiplicity of \( \tilde{\lambda}_i \), and \( \sum_{i=1}^s m_i = N - 1 \).

In order to guarantee that the reduced-order model will satisfy the desired properties in Problem 4, we define the generalized controllability and observability Gramians of (19) as the solutions \( \mathcal{X} \) and \( \mathcal{Y} \) to the following Lyapunov equation and inequality, respectively:
\[ -\tilde{A} \mathcal{X} - \mathcal{X} \tilde{A}^T + \tilde{F} \tilde{F}^T = 0, \quad (21a) \]
\[ -\tilde{A} \mathcal{Y} - \mathcal{Y} \tilde{A}^T + \tilde{H} \tilde{H}^T = 0. \quad (21b) \]
Here, \( \mathcal{X} = \mathcal{X}^T > 0 \) and \( \mathcal{Y} \) is
\[ \mathcal{Y} := \text{blkdiag}(Y_1, Y_2, \cdots, Y_s), \quad (22) \]
with \( Y_i = Y_i^T > 0 \) and \( Y_i \in \mathbb{R}^{m_i \times m_i} \), for \( i = 1, 2, \ldots, s \). The block-diagonal structure of \( \mathcal{Y} \) in (22) is crucial as it will guarantee that the reduced-order model obtained by performing balanced truncation on the basis of \( \mathcal{X} \) and \( \mathcal{Y} \) can again be interpreted as a network system, see Lemma 15 and Theorem 17. Compared with our former notation in [7], the definition of the observability Gramian is more general, since it is not necessary to be strictly diagonal.
Remark 9. There exist a variety of networks, especially symmetric ones such as stars, circles, chains or complete graphs, whose Laplacian matrices have repeated eigenvalues. Particularly, when $L$ refers to a complete graph with identical weights, all the eigenvalues in (11) are equal. Then, $Y$ becomes a full matrix, and (21b) is specialized to an equality.

The existence of the solutions $X$ and $Y$ in (21a) and (21b) are guaranteed, as $A > 0$ is positive diagonal and has the structure as given in (20). Furthermore, in practice, the generalized observability Gramian is obtained by minimizing the trace of $Y$, see e.g., [2, 29].

Next, based on $X$ and $Y$, we further define a pair of generalized Gramians for the stable system $\Sigma_s$, and therefore a balancing transformation can be applied in the following theorem.

Theorem 10. Consider $X$, $Y$ as the generalized Gramians of the accompanying system in (19), and let $K_m > 0$ and $K_M > 0$ be the minimum and maximum solutions of

$$A^TK + KA \leq 0, \quad C = BT^TK. \quad (23)$$

Then, the matrices

$$X := X \otimes K_M^{-1} and \quad Y := Y \otimes K_m$$

characterize generalized Gramians of the asymptotically stable system $\Sigma_s$, i.e., satisfying the inequalities in (18a) and (18b), respectively.

Moreover, there exists a nonsingular matrix $T$ such that $\Sigma_s$ is balanced, i.e.,

$$TXY^T = T^{-T}YT^{-1} = \Sigma_D \otimes \Sigma_D. \quad (25)$$

Here, $\Sigma_D := \text{diag}\{\sigma_1, \sigma_2, \ldots, \sigma_N\}$, and $\Sigma_D := \text{diag}\{\tau_1, \tau_2, \ldots, \tau_n\}$, where $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_N$, and $\tau_1 \geq \tau_2 \geq \cdots \geq \tau_n$ are corresponding to the square roots of the spectrum of $XY$ and $K_M^{-1}K_m$, respectively.

Proof. The LMI (23) follows from the KYP condition in Lemma 2 as $\Sigma_i$ in (1) is passive. Since $K_m$ and $K_M$ are two solutions of (23), we have

$$AK_m^{-1} + K_M^{-1}A^T \leq 0, \quad BCK_m^{-1} = BB^T. \quad (26)$$

and

$$A^TK_m + K_mA \leq 0, \quad C^TB^TK_m = C^TC. \quad (27)$$

Let $\Phi = I \otimes A - A \otimes BC$. Then from (26) and (21a), we obtain

$$\begin{align*}
\Phi X + X \Phi^T + (F \otimes B)(F^T \otimes B^T) & = X \otimes (AK_m^{-1} + K_M^{-1}A^T) \\
& \quad + (-AX - XA + F\Phi^T) \otimes BB^T \leq 0,
\end{align*}$$

Similarly, with (27) and (21b), it is verified that

$$\begin{align*}
\Phi^TY + Y \Phi^T + (\bar{H} \otimes C^T)(\bar{H} \otimes C) & = Y \otimes (A^TK_m + K_mA) \\
& \quad + (-AY - YA + \bar{H} \bar{H}^T) \otimes C^TC \leq 0.
\end{align*}$$

Therefore, by Definition (8), $X$ and $Y$ in (24) are characterized as the generalized Gramians of $\Sigma_s$. We note that a similar construction was performed in [2, 17].

Next, by the standard balancing transformation [1], we obtain a nonsingular matrix $T_D$ such that

$$T_DXT_D^T = \Sigma_D = T_D^{-T}YT_D^{-1}. \quad (30)$$

Thus,

$$T_DXYT_D^{-1} = T_DXT_D^TT_D^{-T}YT_D^{-1} = \Sigma_D^2. \quad (31)$$

Analogously, there exists a nonsingular matrix $T_M$ satisfying

$$T_MK_M^{-1}T_M^T = \Sigma_M = T_M^{-T}K_MT_M^{-1}, \quad (32)$$

which leads to

$$T_MK_M^{-1}K_mT_M^T = T_MK_M^{-1}T_M^TT_M^{-T}K_mT_M^{-1} = \Sigma_M^2. \quad (33)$$

Therefore, the transformation for the overall system $\Sigma_s$ is given by

$$T = T_D \otimes T_M, \quad (34)$$

which satisfies (25) due to (31) and (33). That completes the proof. □

There exist multiple choices of generalized Gramians as the solutions of (18a) and (18b). But this paper specifically selects the pair of Gramians in (24) such that the structure of the balancing transformation $T$ as in (34) corresponds with the network topology and agent dynamics. Essentially, the balanced truncation can be implemented independently on the topology part and each subsystem, allowing the resulting reduced-order model to preserve a network interpretation as well as the passivity of subsystems.

Remark 11. In [34], the solutions of (23) have been discussed in detail. Particularly, any $K > 0$ satisfying (23) lies between the two extremal solutions, i.e.
0 < K_m \leq K \leq K_M, and \( \frac{1}{2}(x,Kx) \) is a quadratic storage function as in (2), since

\[
d \frac{1}{dt} \frac{1}{2}(x,Kx) = \frac{1}{2}(x,(A^T K + KA)x) + (u,y) \leq (u,y).
\]

Then, \( \frac{1}{2}(x,K_m x) \) and \( \frac{1}{2}(x,K_M x) \) are called the available storage and the required supply [34], respectively.

**Remark 12** The inequality in (23) characterizes the passivity of a linear system without a direct feedthrough. Therefore, there does not exist a corresponding Riccati equation for computing \( K_m \) and \( K_M \). Instead, the LMI tool is applied in this paper. Besides, we discuss under what conditions the solution of (23) is unique, i.e., when \( K_M = K_m \). Lossless systems treated in [30] are examples. However, a system does not need to be lossless to yield \( K_M = K_m \). For instance, when \( B \) is nonsingular, \( K_M B = C^T = K_m B \) leads to \( K_M = K_m \). When the solution of (23) is unique, i.e., \( K_M = K_m \), we have \( \Sigma_D = I_n \), meaning that the subsystems are not suitable for reduction. If \( K_M \neq K_m \), it can be verified that the diagonal entries of \( \Sigma_D \) in (25) satisfy \( \tau_i \leq 1, \forall i = 1,2, \ldots, n \).

In the balanced system of \( \Sigma_s \), the diagonal entries of \( \Sigma \) are ordered in a descending order accordingly, thus allowing the following matrix partitions:

\[
T_D A T_D^{-1} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \quad T_G A T_G^{-1} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix},
\]

\[
T_D^{-1} B = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}, \quad T_G^{-1} F = \begin{bmatrix} F_1 \\ F_2 \end{bmatrix},
\]

\[
C T_D = \begin{bmatrix} C_1 \\ C_2 \end{bmatrix}, \quad \hat{H} T_G = \begin{bmatrix} \hat{H}_1 \\ \hat{H}_2 \end{bmatrix},
\]

where \( A_{11} \in \mathbb{R}^{r \times r_0}, B_1 \in \mathbb{R}^{r \times m}, C_1 \in \mathbb{R}^{m \times r_0}, A_{11} \in \mathbb{R}^{(k-1) \times (k-1)}, F_1 \in \mathbb{R}^{(k-1) \times p}, \) and \( \hat{H}_1 \in \mathbb{R}^{q \times (k-1)} \).

The reduced-order agent dynamics is denoted by the minimal realization of the triplet \((A_{11}, B_1, C_1)\):

\[
\hat{\Sigma}_i := (\hat{A}, \hat{B}, \hat{C}),
\]

(36)

with \( \hat{A} \in \mathbb{R}^{r \times r}, \hat{B} \in \mathbb{R}^{r \times m}, \hat{C} \in \mathbb{R}^{m \times r}, \) and \( r \leq r_0 \). Consequently, the reduced-order model of the stable system \( \Sigma_s \) in (15) is presented as

\[
\hat{\Sigma}_s : \begin{cases}
\dot{\tilde{z}}_a = (I_{k-1} \otimes \hat{A} - A_{11} \otimes \hat{B} \hat{C}) \tilde{z}_a + (F_1 \otimes \hat{B}) u, \\
\dot{\tilde{y}}_a = (\hat{H}_1 \otimes \hat{C}) \tilde{z}_a.
\end{cases}
\]

(37)

Furthermore, the reduced-order subsystem \( \hat{\Sigma}_i \) yields a lower-dimensional average module as

\[
\hat{\Sigma}_a : \begin{cases}
\dot{\tilde{z}}_a = \hat{A} \tilde{z}_a + \frac{1}{\sqrt{N}}(1_T \otimes \hat{F} \otimes \hat{B}) u, \\
\dot{\tilde{y}}_a = \frac{1}{\sqrt{N}}(H_1 T \otimes \hat{C}) \tilde{z}_a.
\end{cases}
\]

(38)

**Remark 13** Note that the truncated triplet \((A_{11}, B_1, C_1)\) is not necessarily minimal. Actually, \((A_{11}, B_1, C_1)\) is minimal when the original subsystem \( \Sigma_i \) is strictly passive, see [12, Thm. 3.11]. However, we can always replace \( \hat{\Sigma}_i \) by its minimal realization as in [25], and it can be verified that this replacement does not change the transfer functions of \( \hat{\Sigma}_s \) and \( \hat{\Sigma}_a \).

As the whole stable system \( \Sigma_s \) is balanced based on the generalized Gramians (24), the accompanying system (19) reflecting the interconnection topology is generalized balanced. Meanwhile, the subsystems \( \Sigma_i \) are essentially positive real balanced, since the maximum and minimal solutions of (23), \( K_M \) and \( K_m \), are used to compute the balancing transformation for the subsystems. See more details about positive real balancing in e.g., [1,27,12]. Hence, the passivity of the reduced-order agent dynamics are potentially preserved, which is proven in the following lemma.

**Lemma 14** The obtained minimal reduced-order subsystem \( \hat{\Sigma}_i = (\hat{A}, \hat{B}, \hat{C}) \) is passive.

**Proof.** Note that a coordinate transformation does not change the passivity of a system. Thus, we have

\[
(T_D A T_D^{-1})^T \Sigma_D + \Sigma_D (T_D A T_D^{-1}) \leq 0,
\]

(39)

which is expanded as

\[
\begin{bmatrix}
A_{11}^T \Sigma_D^1 + \Sigma_D^1 A_{11} & A_{21}^T \Sigma_D^1 + \Sigma_D^1 A_{21} \\
A_{21}^T \Sigma_D^1 + \Sigma_D^1 A_{21} & A_{22}^T \Sigma_D^2 + \Sigma_D^2 A_{22}
\end{bmatrix} \leq 0,
\]

(40)

with \( \Sigma_D^1 := \text{diag}(\tau_1, \ldots, \tau_r), \) \( \Sigma_D^2 := \text{diag}(\tau_{r+1}, \ldots, \tau_n). \) Therefore, we obtain

\[
A_{11}^T \Sigma_D^1 + \Sigma_D^1 A_{11} \leq 0, \quad \Sigma_D^2 B_1 = C_1^T.
\]

(41)

Following [25], the minimal realization \( \hat{\Sigma}_i = (\hat{A}, \hat{B}, \hat{C}) \), obtained from the Kalman decomposition of the truncated model \((A_{11}, B_1, C_1)\), satisfies the KYP condition in (3). It then follows from the minimality and Lemma 2 that \( \hat{\Sigma}_i \) is passive.
Next, by combining the average module $\hat{\Sigma}_a$ and the obtained $\hat{\Sigma}_s$, a lower-dimensional approximation of the overall system $\Sigma$ is formulated as

$$\hat{\Sigma}: \begin{cases} \dot{z} = (I_k \otimes \hat{A} - N \otimes \hat{B}\hat{C})z + (\mathcal{F} \otimes \hat{B})u, \\ \dot{y} = (H \otimes \hat{C})z. \end{cases}$$

(42)

where

$$N = \begin{bmatrix} A_{11} & 0 \\ 0 & 0 \end{bmatrix}, \mathcal{F} = \begin{bmatrix} \hat{F}_1 & \frac{1}{\sqrt{N}} 1^T F \end{bmatrix}, H = \begin{bmatrix} \hat{H}_1 & \frac{1}{\sqrt{N}} H^T 1 \end{bmatrix}.$$

Notice that $N$ is not yet a Laplacian matrix, which prohibits the interpretation of $\hat{\Sigma}$ as a network system. This provides the motivation for studying the properties of $N$ in the following lemma.

**Lemma 15** The matrix $N$ in (42) has only one zero eigenvalue at the origin and all the other eigenvalues are positive real.

**PROOF.** Recall the balancing transformation in (30), we obtain from [1] that

$$T_G = \Sigma_G^{-1/2} V_G^T Y^{1/2},$$

(43)

where $U_G \Sigma_G V_G^T$ is the singular value decomposition of $X^{1/2}Y^{1/2}$, and $X, Y$ are generalized Gramians in (21). Then, using the structures of $\hat{A}$ in (20) and $Y$ in (22), we verify that $\hat{A}$ and $Y$ commute, namely $\hat{A}Y = Y\hat{A}$, and $Y^{1/2} \hat{A} Y^{-1/2} = \hat{A}$. Therefore,

$$\hat{A}^b := T_G \hat{A} T_G^{-1} = \Sigma_G^{-1/2} V_G^T Y^{1/2} \hat{A} Y^{-1/2} V_G \Sigma_G^{1/2} = \Sigma_G^{-1/2} V_G^T \hat{A} V_G \Sigma_G^{1/2},$$

(44)

Observe that $V_G^T \hat{A} V_G$ is positive definite, and all the principal submatrices are all positive definite. Due to the diagonal matrix $\Sigma_G$, each principal submatrix of $\hat{A}^b$, including $A_{11}$, is similar to a positive definite matrix, which leads to the positivity and realness of its eigenvalues. That gives the spectrum of $N$. \hfill \blacksquare

**Remark 16** Generally, balanced truncation does not preserve the realness of eigenvalues. Lemma 15 is the result of using a generalized observability Gramian $Y$ with a special structure as in (22).

### 3.3 Network Realization

Now we show that the reduced-order model $\hat{\Sigma}$ in (42) can be interpreted as a network system again. This result is due to the following Theorem.

**Theorem 17** A real square matrix $\mathcal{N}$ is similar to a Laplacian matrix $\mathcal{L}$ associated with an undirected connected graph, if and only if $\mathcal{N}$ is diagonalizable and has exactly one zero eigenvalue while all the other eigenvalues are real positive.

The proof is provided in the Appendix A, because it is rather lengthy. The proof of Theorem 17 provides a procedure to construct a Laplacian matrix $\mathcal{L}$ for a given matrix $\mathcal{N}$. Here, we illustrate this procedure by means of an example in 4-dimension.

**Example 1** Given a diagonalizable matrix $\mathcal{N}$, whose eigenvalues are $\lambda_1 \geq \lambda_2 \geq \lambda_3 > \lambda_4 = 0$. The goal is to find an undirected graph Laplacian matrix $\mathcal{L}$ whose spectrum exactly matches the given one. Let $w_{ij}$ be the weight of the edge linking agents $i$ and $j$, such that the Laplacian matrix can be explicitly expressed as

$$\mathcal{L} = \begin{bmatrix} \alpha_1 & -w_{12} & -w_{13} & -w_{14} \\ -w_{12} & \sigma_2 & -w_{23} & -w_{24} \\ -w_{13} & -w_{23} & \alpha_3 & -w_{34} \\ -w_{14} & -w_{24} & -w_{34} & \alpha_4 \end{bmatrix},$$

(45)

where $w_{ij} = w_{ji} \geq 0$ and $\alpha_i = \sum_{j=1,j\neq i}^4 w_{ij}$. Furthermore, the eigenvalues of $\mathcal{L}$ are computed as the roots of the equation

$$|\mathcal{L} - \lambda I_4| = 0.$$  

(46)

Following (A.5), the algebraic manipulation of (46) then leads to

$$\begin{bmatrix} \alpha_1 + w_{14} - \lambda & w_{14} - w_{12} & w_{14} - w_{13} & -w_{14} \\ w_{24} - w_{12} & \alpha_2 + w_{24} - \lambda & w_{24} - w_{23} & -w_{24} \\ w_{34} - w_{13} & w_{34} - w_{23} & \alpha_3 + w_{34} - \lambda & -w_{34} \\ 0 & 0 & 0 & -\lambda \end{bmatrix} = 0.$$

The strategy is to let the lower triangular part to be zero and use the diagonal entries to match the desired eigenvalues. Precisely, we have

$$\begin{cases} w_{24} - w_{12} = 0, \\ w_{34} - w_{13} = 0, \\ w_{12} + w_{13} + 2w_{14} = \lambda_1, \\ w_{12} + w_{23} + 2w_{24} = \lambda_2, \\ w_{13} + w_{23} + 2w_{34} = \lambda_3, \end{cases}$$

(47)

which yields

$$\begin{cases} w_{34} = w_{23} = w_{13} = \frac{1}{4}\lambda_3, \\ w_{24} = w_{12} = \frac{1}{3}\left(\lambda_2 - \frac{1}{4}\lambda_3\right), \\ w_{14} = \frac{1}{2}\left(\lambda_1 - \frac{1}{3}\lambda_2 - \frac{1}{6}\lambda_3\right). \end{cases}$$

(48)
For instance, when $\lambda_1 = 3$, $\lambda_2 = \lambda_3 = 2$, $\lambda_4 = 0$, the Laplacian matrix is given by

$$L = \begin{bmatrix}
2 & -0.5 & -0.5 & -1 \\
-0.5 & 1.5 & -0.5 & -0.5 \\
-0.5 & -0.5 & 1.5 & -0.5 \\
-1 & -0.5 & -0.5 & 2
\end{bmatrix}. \quad (49)$$

Finally, we note that the obtained $L$ satisfies all the properties in Remark 3 as expected.

**Remark 18** Note that reduced Laplacian matrices obtained by the procedure in Theorem 17 represent undirected complete graphs, since all the weights are strictly positive. However, the matrix $N$ in (42) may very well be similar to a Laplacian matrix of a incomplete graph, though there are examples where $N$ can only be similar to a complete graph Laplacian. We illustrate this by an example of 3 dimension. Suppose vertex 2 is not adjacent to vertex 3, i.e., $w_{23} = 0$. Then, the Laplacian matrix is given by

$$L = \begin{bmatrix}
w_{12} + w_{13} & -w_{12} & -w_{13} \\
-w_{12} & w_{12} & 0 \\
-w_{13} & 0 & w_{13}
\end{bmatrix} \quad (50)$$

whose characteristic polynomial is

$$|\lambda I_3 - L| = \lambda \left[ \lambda^2 - 2(w_{12} + w_{13})\lambda + 3w_{12}w_{13} \right].$$

Let $\lambda_1 \geq \lambda_2 > \lambda_3 = 0$ be the desired eigenvalues. We then obtain

$$2(w_{12} + w_{13}) = \lambda_1 + \lambda_2, \quad 3w_{12}w_{13} = \lambda_1 \lambda_2. \quad (51a, b)$$

Expressing $w_{12}$ as a function of $w_{13}$ using (51a) and substitution of the result in (51b) gives

$$w_{13}^2 - \frac{1}{2}(\lambda_1 + \lambda_2)w_{13} + \frac{1}{3}\lambda_1 \lambda_2 = 0$$

$$\Leftrightarrow \left[ w_{13} - \frac{1}{4}(\lambda_1 + \lambda_2) \right]^2 - \frac{\lambda_1^2 + \lambda_2^2}{16} + \frac{5\lambda_1 \lambda_2}{24} = 0. \quad (52)$$

Obviously, (52) has a real solution if and only if

$$3(\lambda_1^2 + \lambda_2^2) \leq 10\lambda_1 \lambda_2. \quad (53)$$

Therefore, when $\lambda_1 \leq 3\lambda_2$, we can find suitable weights $w_{12}$ and $w_{13}$ such that the eigenvalues of the incomplete graph Laplacian $L$ in (50) match the given real spectrum $\lambda_1$, $\lambda_2$, and $\lambda_3$. However, if $\lambda_1 > 3\lambda_2$, then it is impossible to find a set of suitable weights.

In this paper, we can only guarantee to find a network realization of the system $\hat{\Sigma}$ with a complete graph topology. Observe that $N$ in (42) is diagonalizable, and Lemma 15 implies that it has only one eigenvalue at the origin and all the other poles are real and strictly positive. Therefore, by Theorem 17, there exists a Laplacian matrix $\hat{L}$ which has the same spectrum as $N$. In other words, we can find a nonsingular matrix $T_n$ such that

$$\hat{L} = T_n^{-1}NT_n. \quad (54)$$

Here, $\hat{L}$ is a Laplacian matrix representing a reduced connected undirected graph $\hat{G}$, which contains $k$ nodes.

Applying a coordinate transform $\hat{z} = (T_n \otimes I_k)\hat{x}$ to the system $\hat{\Sigma}$ in (42) yields a reduced-order network model

$$\hat{\Sigma}: \begin{cases} \dot{\hat{x}} = (I_k \circ \hat{A} - \hat{L} \circ \hat{B}\hat{C})\hat{x} + (\hat{F} \circ \hat{B})u, \\ \hat{y} = (\hat{H} \circ \hat{C})\hat{x}, \end{cases}$$

with $\hat{F} = T_n^{-1}F$ and $\hat{H} = HT_n$. We show in the following theorem that the reduced-order network $\hat{\Sigma}$ is also synchronized for $u = 0$.

**Theorem 19** The reduced networked passive system $\hat{\Sigma}$ preserves the synchronization, i.e., when $u = 0$, for any initial condition, it holds that

$$\lim_{t \to \infty} [\hat{x}_i(t) - \hat{x}_j(t)] = 0, \ \forall i, j \in \{1, 2, \cdots, k\}. \quad (56)$$

**PROOF.** Due to the connectedness of reduced graph $\hat{G}$ and the passivity and minimality of the reduced subsystems $(A, \hat{B}, \hat{C})$ from Lemma 14, we obtain the following result simply by applying Lemma 7.

### 3.4 Error Analysis

Following the separation of the multi-agent system $\Sigma$ in subsection 3.1, we analyze the approximation error for the overall system as follows.

$$\|\Sigma - \hat{\Sigma}\|_{\mathcal{H}_\infty} = \|\Sigma_{a} - \hat{\Sigma}_{a} - (\hat{\Sigma}_{a} - \hat{\Sigma}_{a})\|_{\mathcal{H}_\infty} \leq \|\Sigma_{a} - \hat{\Sigma}_{a}\|_{\mathcal{H}_\infty} + \|\Sigma_{a} - \hat{\Sigma}_{a}\|_{\mathcal{H}_\infty}. \quad (57)$$

The overall approximation error can be evaluated based on the reduction results of the stable system $\Sigma_{a}$ and the average module $\Sigma_{a}$.

First, the *a priori* bound on the approximation error of the stable part is provided as follows.
Lemma 20 Consider the original stable system $\Sigma_s$ in (15) and its truncated model $\hat{\Sigma}_s$ in (9). The approximation error has an upper bound as
\[
\|\Sigma_s - \hat{\Sigma}_s\|_{\infty} \leq \gamma_s, \tag{58}
\]
where
\[
\gamma_s = 2 \sum_{i=k}^{N-1} \sum_{j=1}^{n} \sigma_i \tau_j + 2 \sum_{i=1}^{k-1} \sum_{j=r+1}^{n} \sigma_i \tau_j. \tag{59}
\]
with $\sigma_i$ and $\tau_i$ the diagonal entries of $\Sigma_G$ and $\Sigma_D$ in (25), respectively.

**PROOF.** The GHSVs of the balanced system of $\Sigma_s$ are ordered and located in the diagonal of $\Sigma_G \otimes \Sigma_D$, forming the structure as
\[
\Sigma_G \otimes \Sigma_D = \text{blkdiag}
\begin{pmatrix}
\tau_1 & & \\
& \ddots & \\
& & \tau_n
\end{pmatrix}
\begin{pmatrix}
\sigma_1 & & \\
& \ddots & \\
& & \sigma_{N-1}
\end{pmatrix}
\begin{pmatrix}
\tau_1 & & \\
& \ddots & \\
& & \tau_n
\end{pmatrix}
\tag{60}
\]
Then, recall the standard error bound for balanced truncation based on generalized Gramians in [11], we obtain a $\mathcal{H}_\infty$ bound as
\[
\|\Sigma_s - \hat{\Sigma}_s\|_{\infty} \leq \gamma_s. \tag{61}
\]
The constant $\gamma_s$ is computed as (58) since we truncate the system according to the block diagonal structure of $\Sigma_G \otimes \Sigma_D$ as in (60).

The approximation error on the average module of the network system is then discussed. From (14) and (38), we write the transfer function of $\Sigma_a - \hat{\Sigma}_a$ as
\[
\Delta_a(s) = \frac{1}{N} (H1_N \otimes G)(sI_n - A)^{-1}(1_N F \otimes \hat{B})
- \frac{1}{N} (H1_N \otimes \hat{G})(sI_r - \hat{A})^{-1}(1_N F \otimes \hat{B}).
\tag{62}
\]
Using the properties of Kronecker products, reshuffling (62) then leads to
\[
\Delta_a(s) = \frac{H1_N 1_N^T F}{N} \otimes \Delta_i(s), \tag{63}
\]
where $\Delta_i(s) := C(sI_n - A)^{-1}B - \hat{C}(sI_r - \hat{A})^{-1}\hat{B}$ is the transfer function of $\Sigma_i - \hat{\Sigma}_i$. Hence, the approximation error on the average module is actually bounded if and only if the error between the original and reduced agent dynamics is bounded. However, since the agent system in (1) is not necessarily asymptotically stable, $\Delta_i(s)$ may not have an $\mathcal{H}_\infty$-norm bound, so does $\Delta_a(s)$. If $\|\Delta_i(s)\|_{\mathcal{H}_\infty}$ exists, we can take the $\mathcal{H}_\infty$-norm of $\Delta_a(s)$ and obtain
\[
\|\Sigma_a - \hat{\Sigma}_a\|_{\mathcal{H}_\infty} \leq \frac{\gamma_a}{N} \|\Sigma_i - \hat{\Sigma}_i\|_{\mathcal{H}_\infty} \tag{64}
\]
by triangular inequality of norms, where $\gamma_a := \|H1_N 1_N^T F\|_{\mathcal{H}_\infty}$. Note that $\Sigma_i$ is essentially obtained from positive real balancing of $\Sigma_i$. Generally, there does not exist an a priori bound on $\|\Sigma_i - \hat{\Sigma}_i\|_{\mathcal{H}_\infty}$. Nevertheless, a posteriori bound can be obtained, see [12].

In the rest of this section, several special cases are discussed where a priori error bounds on $\|\Sigma - \hat{\Sigma}\|_{\mathcal{H}_\infty}$ in (57) can be obtained.

The first case is when we only reduce the dimension of the network while the agent dynamics are untouched as in [2,21]. Then, we have the following result.

**Theorem 21** Consider the network system $\Sigma$ with $N$ agents and its reduced-order model $\hat{\Sigma}$ with $k$ agents. If the agent system $\Sigma_i$ is not reduced, the error bound
\[
\|\Sigma - \hat{\Sigma}\|_{\mathcal{H}_\infty} = \|\Sigma_a - \hat{\Sigma}_a\|_{\mathcal{H}_\infty} \leq \frac{2}{N} \sum_{i=k}^{N-1} \sum_{j=1}^{n} \sigma_i \tau_j, \tag{65}
\]
holds, where $\sigma_i$ and $\tau_i$ are defined in Theorem 10.

**PROOF.** If the agent dynamics are untouched, we have $\|\Sigma_a - \hat{\Sigma}_a\|_{\mathcal{H}_\infty} = 0$ due to (64). Then, the error bound straightforwardly follows from (57) and (58). Even though the agent dynamics are retained, $\tau_j$, $j = 1, 2, \cdots, n$ still show up because of (60).

The second case is $H1 = 0$ or $1^T F = 0$, which happens when the average module is not observable from the outputs of the overall system $\Sigma$, or it is uncontrollable by the external inputs. In application, it means that we only observe or control the differences between the agents due to the coupling. Such differences usually play a crucial role in distributed control of networks, since they indicate whether two nodes or two clusters of nodes are synchronized as time evolves. A special case can be found in [22,23,21] where $H$ in (6) is the incidence matrix of the underlying network. We consider the more general case when $H1 = 0$ or $1^T F = 0$, which yields the following result immediately.

**Corollary 22** Consider the network system $\Sigma$ with $N$ agents and its reduced-order network model $\hat{\Sigma}$ with $k$
agents. If $H1_N = 0$ or $1^T_N F = 0$, the approximation between $\Sigma$ and $\hat{\Sigma}$ is bounded by
\[
\| \Sigma - \hat{\Sigma} \|_{\infty} = \| \Sigma_s - \hat{\Sigma}_s \|_{\infty} \leq \gamma,
\]
where $\gamma$ is given in (58).

**PROOF.** When $H1_N = 0$ or $1^T_N F = 0$, we have $\| \Sigma_a - \hat{\Sigma}_a \|_{\infty} = 0$ from (63). Therefore, (57) gives
\[
\| \Sigma - \hat{\Sigma} \|_{\infty} = \| \Sigma_s - \hat{\Sigma}_s \|_{\infty},
\]
which is bounded as (58).

The third case is when $K^{-1}_M$ in (27) and $K_m$ in (26) can be interpreted as the generalized Gramians of $\Sigma_a$, i.e., $A$ in (1) is Hurwitz, and the following two LMIs hold.
\[
\begin{align*}
AK^{-1}_M + K^{-1}_M A^T + \frac{1}{N}1^T \Sigma_s^T F \Sigma_s 1^T B^T B & \leq 0, \\
A^T K_m + K_m A + \frac{1}{N}1^T H^T H1 \cdot C^T C & \leq 0.
\end{align*}
\]
(67)

Consequently, an *a priori* bound on $\| \Sigma_a - \hat{\Sigma}_a \|_{\infty}$ in (57) can be evaluated by the GHSVs of $\Sigma_a$, namely $\tau_i$. Then the following result is obtained.

**Corollary 23** Consider the network system $\Sigma$ with $N$ agents and its reduced-order network model $\hat{\Sigma}$ with $k$ agents. If $A$ in (1) is Hurwitz, and both $K_M$ and $K_m$ satisfy
\[
A^T K + KA + \beta C^T C \leq 0
\]
with $\beta := 1/N \cdot \min \{1^T F \Sigma_s^T F 1, 1^T H^T H1 \}$, then the approximation between $\Sigma$ and $\hat{\Sigma}$ is bounded by
\[
\| \Sigma - \hat{\Sigma} \|_{\infty} \leq \gamma_s + 2 \sum_{i=r+1}^n \tau_i,
\]
(69)

where $\gamma_s$ is given in (58).

**PROOF.** It is easy to verify that the LMIs in (67) hold as $K_M$ and $K_m$ satisfy (68) and $B^T K_M = C$. Since $A$ is Hurwitz, then $K^{-1}_M$ and $K_m$ are regarded as the generalized controllability and observability Gramians of the average system $\Sigma_a$. Then the balanced truncation based on $K^{-1}_M$ and $K_m$ is applied to $\Sigma_a$, leading to the reduced-order average module $\hat{\Sigma}_a$. Therefore, the following error bound is obtained from [11].
\[
\| \Sigma_a - \hat{\Sigma}_a \|_{\infty} \leq 2 \sum_{i=r+1}^n \tau_i,
\]
(70)

which yields (69) from (58) and (57).

## 4 Illustrative Example

To demonstrate the feasibility of the proposed method, we consider networked robotic manipulators as a multi-agent system example.

Following [10], the dynamics of each rigid robot manipulator is described as a standard mechanical system in the form (1) with
\[
A = \begin{bmatrix} 0 & M^{-1} \\ -I & -DM^{-1} \end{bmatrix}, \quad C = B^T \begin{bmatrix} I & 0 \\ 0 & M^{-1} \end{bmatrix},
\]
(71)

where $D \geq 0$ and $M > 0$ are the system damping and mass-inertia matrices, respectively. By Lemma 2, each manipulator agent is passive since there exists a positive definite matrix $P := \text{blkdiag}(I, M^{-1})$ satisfying (3).

In this example, the system parameters in (71) are specified as
\[
M = \frac{1}{2} I_4, \quad D = \begin{bmatrix} 2 & -1 & 0 & 0 \\ -1 & 4 & -2 & 0 \\ 0 & -2 & 4 & -1 \\ 0 & 0 & -1 & 2 \end{bmatrix},
\]
(72)

\[
B = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix}^T,
\]

which yields the dynamics of each individual agent with state and input dimensions as $n = 8$ and $m = 1$.

Furthermore, the agents communicate according to an undirected cyclic graph depicted in Fig. 2a, which contains $N = 6$ agents. Suppose that nodes 1 and 2 are actuated, and the output error between the nodes 1 and 3 are the external measurement. Then, the Laplacian matrix and external input and output matrices are given by
\[
L = \begin{bmatrix} 2 & -1 & 0 & 0 & 0 & -1 \\ -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 & 0 \\ 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 2 \\ -1 & 0 & 0 & 0 & -1 & 2 \end{bmatrix}, \quad F = \begin{bmatrix} 0.5 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix},
\]
(73)

\[
H = \begin{bmatrix} 1 & 0 & -1 & 0 & 0 \end{bmatrix}.
\]

It can be verified that the subsystems $\Sigma_i$ is minimal. Thus, the original network system is synchronized as time evolves by Lemma 7.

Note that the nonzero eigenvalues of $L$ are $\lambda_1 = 4, \lambda_2 = \lambda_3 = 3, \lambda_4 = \lambda_5 = 1$. Solving the LMI (21b) by minimizing the trace of $Y$, we obtain the generalized observabil-
Fig. 2. (a) and (b) illustrate the original and reduced communication graph, respectively. 

Fig. 3. The frequency responses of the original and reduced multi-agent systems, which are represented by the solid and dashed lines in the plot respectively.

original network is well approximated by the reduced-order model. This conclusion can be seen from the plots of both systems in Fig. 3 as well.

5 Conclusion

In this paper, we have developed a structure-preserving model reduction method for networked passive systems. The identical agents are assumed to be linear time-invariant systems, and the communication topology is undirected and connected. The observability and passivity of each agent guarantee the synchronization of networks. Balanced truncation based on generalized Gramians is applied to reduce the dimension of the asymptotically stable component. The resulting model can be converted to a new representation of Laplacian dynamics, which again has a network interpretation. Therefore, the proposed method can reduce the dimension of each subsystem and the scale of the network simultaneously. Moreover, an a priori error bound on the multi-agent system has been provided. Finally, the proposed model reduction scheme was applied to a numerical example. The simulation results indicate that the reduced-order model approximates the original one with a reasonable accuracy. Our future work considers the extensions to nonlinear agent dynamics and communication protocols.

A Proof of Theorem 17

PROOF. The “only if” part can be seen from Remark 3. The rest of the proof shows the “if” part. Therefore, let $N \in \mathbb{R}^{n \times n}$ be diagonalizable, and denote its eigenvalues as

$$
\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_{n-1} > \lambda_n = 0. \quad \text{(A.1)}
$$

Then, there exists a spectral decomposition $N = T_1 D_1 T_1^{-1}$ with $D = \text{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n)$. 

On the other hand, any undirected graph Laplacian $L$
can be written in the form of
\[
\mathbf{L} = \begin{bmatrix}
\alpha_1 - w_{1,2} & \cdots & - w_{1,n} \\
-w_{2,1} & \alpha_2 & \cdots & - w_{2,n} \\
\vdots & \vdots & \ddots & \vdots \\
-w_{n,1} & - w_{n,2} & \cdots & \alpha_n
\end{bmatrix},
\] (A.2)

where \( w_{i,j} = w_{j,i} \geq 0 \) denotes the weight of edge \((i, j)\), which is the same with \( w_{i,j} \) in (4). The sum of the off-diagonal entries in the \( i \)-th row (or column) of \( \mathbf{L} \) is denoted by \( \alpha_i, \) i.e.,
\[
\alpha_i = \sum_{j=1, j \neq i}^{n} w_{i,j}.
\] (A.3)

There exists a spectral decomposition \( \mathbf{L} = \mathbf{T}_0 \mathbf{D} \mathbf{T}_0^{-1} \). If \( \mathbf{D}_1 = \mathbf{D}_2 \), then we have the following similarity transformation
\[
\mathbf{L} = (\mathbf{T}_0 \mathbf{T}_1^{-1}) \mathbf{N} (\mathbf{T}_0 \mathbf{T}_1^{-1})^{-1}.
\] (A.4)

Hence, it is sufficient to prove that there always exists a set of weights \( w_{i,j} \) such that the resulting Laplacian matrix \( \mathbf{L} \) in (A.2) and \( \mathbf{N} \) have the same eigenvalues (A.1).

Consider the characteristic polynomial of \( \mathbf{L} \), i.e.,
\[
|\mathbf{L} - \lambda \mathbf{I}| =
\[
\begin{vmatrix}
\alpha_1 - \lambda & -w_{1,2} & \cdots & - w_{1,n-1} & - w_{1,n} \\
-w_{2,1} & \alpha_2 - \lambda & \cdots & - w_{2,n-1} & - w_{2,n} \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
-w_{n,1} & - w_{n,2} & \cdots & \alpha_{n-1} - \lambda & - w_{n-1,n} \\
-w_{1,n-1} & - w_{2,n-1} & \cdots & \alpha_{n-1} - \lambda & \alpha_n - \lambda
\end{vmatrix}.
\]

As elementary row operations do not change the determinant, we sum all rows to the final row to obtain
\[
|\mathbf{L} - \lambda \mathbf{I}| =
\[
\begin{vmatrix}
\alpha_1 - \lambda & -w_{1,2} & \cdots & - w_{1,n-1} & - w_{1,n} \\
-w_{2,1} & \alpha_2 - \lambda & \cdots & - w_{2,n-1} & - w_{2,n} \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
-w_{n,1} & - w_{n,2} & \cdots & \alpha_{n-1} - \lambda & - w_{n-1,n} \\
-\lambda & - \lambda & \cdots & - \lambda & - \lambda
\end{vmatrix}.
\]

where the expression in (A.3) is applied.

Using a similar argument, adding the last column to all other columns then leads to (A.5). Note that the eigenvalues of \( \mathbf{L} \) are determined by the roots of \( |\mathbf{L} - \lambda \mathbf{I}| = 0 \), and we can assign the eigenvalues of \( \mathbf{L} \) by manipulating the weights \( w_{i,j} \).

When \( n = 2 \), we have a special case, and therefore it is considered separately. Equation (A.5) becomes
\[
|\mathbf{L} - \lambda \mathbf{I}_2| = \begin{vmatrix}
\alpha_1 + w_{1,2} - \lambda & -w_{1,2} & \cdots & 0 \\
0 & -\lambda & \cdots & 0 \\
\end{vmatrix}.
\]

To match the eigenvalues 0, \( \lambda_1 \), we let \( w_{1,2} = 0.5 \lambda_1 \), which yields a Laplacian matrix as
\[
\mathbf{L} = \begin{bmatrix}
0.5 \lambda_1 & -0.5 \lambda_1 \\
-0.5 \lambda_1 & 0.5 \lambda_1
\end{bmatrix},
\] (A.6)

and proves the desired result for \( n = 2 \).

Now we continue the proof for the case \( n > 2 \). To match the eigenvalues of \( \mathbf{L} \) with the desired ones in (A.1), we let the off-diagonal entries in the lower triangular part of the determinant in (A.5) be zero and use the diagonal entries to match the eigenvalues \( \lambda_i \) \((i = 1, 2, \cdots, n)\). Specifically, the weights \( w_{i,j} \) in (A.2) need to satisfy
\[
\begin{align*}
w_{2,n} &= w_{1,2} \\
w_{3,n} &= w_{1,3} = w_{2,3} \\
w_{4,n} &= w_{1,4} = w_{2,4} = w_{3,4} \\
&\vdots \\
w_{n-1,n} &= w_{1,n-1} = w_{2,n-1} = \cdots = w_{n-2,n-1}.
\end{align*}
\] (A.7)

and
\[
\alpha_i + w_{i,n} = \lambda_i, \ \forall i \in \{1, 2, \cdots, n-1\}.
\] (A.8)

Hereafter we prove that the equations (A.7) and (A.8) produce a unique set of non-negative real weights \( w_{i,j} \), which is an essential property of a Laplacian matrix, see Remark 3.

For simplicity, we denote
\[
a_l = w_{n-l,n}, \ l = 1, 2, \cdots, n-1.
\] (A.9)

For any \( 1 \leq l \leq n-2 \), it follows from (A.7) and the symmetry of \( \mathbf{L} \) that
\[
a_l = w_{l,n-l} = w_{n-l,k}, \ \forall k \in \{1, \cdots, n-l-1\}.
\] (A.10)

Furthermore, denote the sum of the above series as
\[
S_l := \sum_{k=1}^{l} a_k, \ l = 1, 2, \cdots, n-1.
\] (A.11)

From the equation (A.8) and the expression (A.3), we
\[ |\mathcal{L} - \lambda I| = \begin{vmatrix}
\alpha_1 + w_{1,n} - \lambda & w_{1,n} - w_{1,2} & \cdots & w_{1,n} - w_{1,n-1} & -w_{1,n} \\
w_{2,n} - w_{1,2} & \alpha_2 + w_{2,n} - \lambda & \cdots & w_{2,n} - w_{2,n-1} & -w_{2,n} \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
w_{n-1,n} - w_{1,n-1} & w_{n-1,n} - w_{2,n-1} & \cdots & \alpha_{n-1} + w_{n-1,n} - \lambda & -w_{n-1,n} \\
0 & 0 & \cdots & 0 & -\lambda
\end{vmatrix} . \quad (A.5) \]

have
\[
\lambda_i = (w_{i,1} + \cdots + w_{i,i-1}) \\
+ (w_{i,i+1} + \cdots + w_{i,n-1}) + 2w_{i,n} \\
= (i-1)a_{n-i} + (a_{n-i-1} + \cdots + a_1) + 2a_{n-i} \\
= (i+1)a_{n-i} + S_{n-i-1}, \quad (A.12)
\]
for \( i = 1, 2, \cdots, n - 2 \). Here, the first equality follows from (A.10) (with \( i = n - l \) for the first term) and the definition (A.9). The latter equation is the result of the definition (A.11).

Rewriting (A.12) for \( l = n - i \) leads to
\[
a_l = \frac{1}{n-l+1} (\lambda_{n-1} - S_{l-1}). \quad (A.13)
\]

Now, we prove that \( a_l > 0, \ \forall l \in \{1, 2, \cdots, n-1\} \) To do so, we consider the cases \( l = 1 \) and \( l = 2 \) explicitly and then proceed by induction.

For \( l = 1 \), it follows from (A.3) and the last equation in (A.7) that (A.8) can be written as \( nw_{n-1,n} = \lambda_{n-1} \), which leads to
\[
a_1 = \frac{\lambda_{n-1}}{n} = S_1 > 0, \quad (A.14)
\]
by the definitions in (A.9) and (A.11).

For \( l = 2 \), (A.13) gives
\[
a_2 = \frac{1}{n-1} (\lambda_{n-2} - S_1) \\
\geq \frac{1}{n-1} (\lambda_{n-1} - S_1) = \frac{\lambda_{n-1}}{n} > 0, \quad (A.15)
\]
where the inequality follows from the ordering of the eigenvalues in (A.1). Then, using the definition (A.11), it follows that
\[
S_2 = S_1 + a_2 = \frac{\lambda_{n-2}}{n-1} + \frac{(n-2)\lambda_{n-1}}{n(n-1)} . \quad (A.16)
\]

Note that
\[
\frac{1}{n-m} + \frac{m(n-m-1)}{n(n-m)} = \frac{m+1}{n}, \ \forall m \neq n, n \neq 0. \quad (A.17)
\]
Using the above equation with \( m = 1 \) and the inequality \( \lambda_{n-2} \geq \lambda_{n-1} \), we show bounds on \( S_2 \) as
\[
S_2 \geq \frac{1}{n-1} + \frac{(n-2)}{n(n-1)} \lambda_{n-1} = \frac{2\lambda_{n-1}}{n}, \quad (A.18)
\]
To proceed with induction on \( l \) for \( l > 2 \), we assume both \( a_l > 0 \) and
\[
\frac{l\lambda_{n-l}}{n} \leq S_l \leq \frac{l\lambda_{n-l}}{n}, \quad (A.19)
\]
for \( 2 < l < n - 1 \). Then, we obtain from (A.13) and (A.19) that
\[
a_{l+1} = \frac{1}{n-l} (\lambda_{n-l-1} - S_l) \\
\geq \frac{1}{n-l} (\lambda_{n-l-1} - \frac{l\lambda_{n-l}}{n}) \geq \frac{\lambda_{n-l}}{n} > 0, \quad (A.20)
\]
after which the first line in (A.20) yields
\[
S_{l+1} = S_l + a_{l+1} = \frac{\lambda_{n-l-1}}{n-l} + \frac{(n-l-1)S_l}{n-l}. \quad (A.21)
\]
The upper and lower bounds on \( S_{l+1} \) are implied by (A.19) as
\[
S_{l+1} \geq \frac{\lambda_{n-l-1}}{n-l} + \frac{l(n-l-1)\lambda_{n-l}}{(n-l)n}, \\
S_{l+1} \leq \frac{\lambda_{n-l-1}}{n-l} + \frac{l(n-l-1)\lambda_{n-l}}{(n-l)n}. \quad (A.22)
\]
Using the relation \( \lambda_{n-l-1} \geq \lambda_{n-l} \geq \lambda_{n-1} \) and the equation (A.17) with \( m = l \), we obtain
\[
\frac{(l+1)\lambda_{n-l}}{n} \leq S_{l+1} \leq \frac{(l+1)\lambda_{n-l-1}}{n}. \quad (A.23)
\]
 Consequently, by induction, we now verify that \( a_l > 0, \ \forall l \in \{1, 2, \cdots, n-1\} \). As the parameters \( a_l \) uniquely
characterize all the the weights $w_{i,j}$ in (A.2) through (A.9) and (A.10, it follows that $w_{i,j} > 0$ for all $(i,j)$.

In summary, there always exist a set of weights $w_{i,j} > 0$ such that $L$ in (A.2) has the eigenvalues matching the desired spectrum $\lambda_1 \geq \cdots \geq \lambda_{n-1} > \lambda_n = 0$. The matrix $L$ satisfies all properties stated in Remark 3 and can indeed be regarded as the Laplacian matrix of an undirected graph. Therefore, we conclude that if $\mathcal{N}$ is diagonalizable and has a single zero eigenvalue while all the other eigenvalues are real positive, then there always exists a similarity transformation between $\mathcal{N}$ and a Laplacian matrix. This finalizes the proof of Theorem 17. ■

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