ON THE ANALYTICITY OF THE TRAJECTORIES OF THE PARTICLES IN THE PLANAR PATCH PROBLEM FOR SOME ACTIVE SCALAR EQUATIONS

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Abstract. We prove analyticity in time of the particle trajectories associated with the solutions of some transport equations when the initial condition is the characteristic function of a regular bounded domain. These results are obtained from a detailed study of the Beurling transform, that represents a derivative of the velocity field. The precise estimates obtained for the solutions of an equation satisfied by the Lagrangian flow, are a key point in the development.

1. Introduction. A classical partial differential equation related with biological systems is a continuity equation named aggregation equation

\[
\begin{aligned}
\partial_t \rho(x, t) + \text{div}(\rho v)(x, t) &= 0, \\
v(x, t) &= K * \rho(x)(t)(x) \\
\rho(x, 0) &= \rho_0(x),
\end{aligned}
\]

(A)

where \( v \) represents a velocity field and \( \rho \) the density of mass of an irrotational inviscid and compressible fluid. The vector-valued kernel \( K \) is

\[ K(x) = \frac{x}{|x|^2}. \]

The velocity field \( v \) provides a flow through the equation

\[
\begin{aligned}
\frac{\partial \psi}{\partial t}(z, t) &= v(\psi(z, t), t) \\
\psi(z, 0) &= z.
\end{aligned}
\]

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The main result of this paper is the following fact, contained in Theorem 1.3 below:

**Main result.** Let $\Omega \subset \mathbb{R}^2$ be a bounded domain with smooth boundary and $\psi(x, t)$ the flow corresponding to the solution $(v, \rho)$ of the equation (A) when the initial condition is $\rho_0(x) = \chi_\Omega(x)$. Then, for each $x \in \mathbb{R}^2$, the function $\psi(x, t)$, is real analytic in the time variable $t$ in its interval of existence.

A non-linear rescaling of the time variable and the values of the function $\rho$ in the equation (A) bring into the arena the transport equation (see [4])

$$
\begin{aligned}
\partial_t \rho(x, t) + v(x, t) \cdot \nabla \rho(x, t) &= 0, \quad (x, t) \in \mathbb{R}^2 \times (0, \infty) \\
v(x, t) &= K * \rho(\cdot, t)(x) \\
\rho(x, 0) &= \rho_0(x),
\end{aligned}
$$

which is dual to the equation (A) in the weak sense.

The techniques introduced in this paper also provide a prove of the analyticity in time of the flow associated to the solution of the equation, (\(\tilde{A}\)), as well as to some other transport equations, in the case of the initial condition being the characteristic function of a smooth bounded domain. The most remarkable one is the Euler equation (Euler, 1755), describing the evolution of an incompressible and inviscid fluid, that in the two dimensional case can be written, in vorticity form, as

$$
\begin{aligned}
\partial_t \omega(x, t) + v(x, t) \cdot \nabla \omega(x, t) &= 0, \quad (x, t) \in \mathbb{R}^2 \times (-\infty, \infty) \\
v(x, t) &= \omega(\cdot, t) * K(\cdot)(x) \\
\omega(x, 0) &= \omega_0(x),
\end{aligned}
$$

where $\omega$ is a scalar function representing the vorticity, $v$ is the velocity field of the fluid and

$$
K(\cdot)(x) = \frac{x^\perp}{|x|^2},
$$

for $x \in \mathbb{R}^2 \setminus \{0\}$ and $t \in \mathbb{R}$.

Concerning the known related facts, for the equation (E), a classical theorem due to Yudovich asserts that if the initial datum $\omega_0$ is a function in $(L^\infty \cap L^p)(\mathbb{R}^2)$, where $1 < p < \infty$, then there exists a unique weak solution of the equation (E) (see [19], [9], [20]). Also existence and uniqueness of solutions for the equation (A), with initial datum in $(L^\infty \cap L^p)(\mathbb{R}^2)$, where $1 < p < \infty$, are proved in [4] and references therein. In both cases the solution depends continuously on the time variable and is as regular as the datum in the space variables.

In the case of equation (E), the regularity in time of the flow has been largely studied:

In [7] the classical Euler’s equation for the velocity of an incompressible inviscid fluid is studied for arbitrary space dimension, and the $C^\infty$ regularity in time of the flow is established in the case of initial velocity being $C^r$-regular for $r > 1$.

In [15] a conceptual argument is developed for proving analyticity of the flow associated to the velocity solution of some equations or systems related to very general fluid models. The method relies on ideas already introduced by Serfati [25] and consists in complexifying the time and considering an operator describing the time derivative of the flow in terms of the initial condition and the flow itself. Then the local boundedness and preservation of the analyticity of this operator, in appropriated Banach spaces, implies the analyticity of the flow. The method covers, among other situations, the vortex patch problem for the Euler equation in
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\( \mathbb{R}^2 \) in the case of \( C^2 \) boundary, but, as far as we know, it does not cover an identical problem for the aggregation equation.

In [27] the analyticity of the flow for the Euler equation is proven in the case of the torus \( T^3 \) and the initial datum in a Sobolev space contained in \( C^1 \). The key point consists in considering the flow as a geodesic, image of the exponential map, in the group of volume preserving diffeomorphisms of the space.

In [12] the authors prove the analyticity of the flow in a sufficiently small interval of time, in the case of 3D Euler equations for the vorticity in a torus \( T^3 \) and for the initial vorticity in a Hölder class. The (a priori) method consists in the study of the size of the Taylor coefficients of the time-development of an hypothetical analytic solution. It is elementary enough to make possible the use of a relatively simple technology. This allows a great generality. It has inspired our procedure in this paper.

In [26] (see also [23] and [22]) the author proves analyticity of the flow for Euler equation in the case of the initial velocity being in a subspace of \( C^{1+s} \), for \( 0 < s < 1 \), for some interval of time, that turns to be \( \mathbb{R} \) in the 2-dimensional case. The method consists in considering a second order ordinary differential operator in complex time, with values in a Banach space and where the unknown variable is a family of maps from \( \mathbb{R}^n \) to itself, whose Jacobian map is close to the identity. The key point in the proof is the fact that the equation associated to this operator has a solution admitting a power series development.

Nevertheless, as far as we know, the time-analyticity in the case of the aggregation equation \((A)\) is uncovered in the literature.

Let us briefly sketch the procedure of our proof: Since our equation is supported in \( \mathbb{R}^2 \times I \), we write the problem in \( \mathbb{C} \times I \) and use complex coordinates for the space variables. It turns out that the flow \( \psi \) provided by the velocity field associated to the solution of equation \((A)\), satisfies an equation

\[
\begin{align*}
P[\psi](z, t) &= \varpi(z, t), \\
\psi(z, 0) &= z,
\end{align*}
\]

(2)

where \( P \) is a non-linear second order differential operator (see 5) and \( \varpi(z, t) \simeq \frac{\rho_0(z)}{1 - \rho_0(z) t} \).

Then we study equation (2) in a more general setting where \( \varpi \) is a complex valued function, analytic in \( t \) in some interval, and we prove that (2) has an analytic solution on some interval too. It is worth to remark that both intervals are related but no necessarily the same, and also that in general the solution of (2) is not unique. It turns out that the solution we obtain is also a time-parametrized family of homeomorphisms of \( \mathbb{C} \), so it is a flow.

In the case of the datum \( \rho_0 \) of \((A)\) be a patch, the flow we obtain as a solution of equation (2), provides a velocity field \( u \) and then a density, \( \Xi \). The couple \( (u, \Xi) \) satisfies the equation \((\tilde{A})\), the dual equation of \((A)\), so it is a weak solution of \((A)\). The uniqueness of solution of \((A)\) implies that both flows coincide in the aforementioned interval.

Using the persistence of the boundary regularity (see [3]) and the topological properties and regularity results (see [4]) of solutions of the equation \((\tilde{A})\) for patches, global analyticity in time is proven.

Just for completeness we have also considered the case of the Euler equation, already solved in [15] in the case of initial patch with \( C^2 \) boundary, that can be
treated by the same method. Now, since the initial condition is transported by
the flow, we consider equation (2) where \( \varpi(z, t) \) is essentially \( i^2 \) times the initial
vorticity (initial datum), so \( \varpi \) is an analytic function in \( t \) and purely imaginary.
That makes the development easier than in the case of \((A)\). Now the uniqueness
of the solution of equation \((E)\) provided by Yudovich’s theorem, its self-duality
and the persistence of the regularity given in a theorem of Chemin for vortex patches,
[8], imply that this solution is global in time.

There are several reasons for the introduction of complex notation in our ap-
proach. Among them, the presence of some objects that are in fact intrinsic of
complex analysis, the Beurling transform is the most remarkable one, and also the
fact that geometric objects and properties in the plane are nicely captured by this
notation. Also, this notation simplifies many arguments and formulations. Never-
thless, the problem belongs to the realm of real analysis.

1.1. Basic notation. We will consider in \( \mathbb{C} \) the standard coordinate \( z = x + iy \).
Then
\[
\frac{\partial}{\partial z} = \frac{1}{2} \left( \frac{\partial}{\partial x} - i \frac{\partial}{\partial y} \right), \quad \frac{\partial}{\partial \bar{z}} = \frac{1}{2} \left( \frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \right)
\]
and we have, identifying the field \( v = (v_1, v_2) \) with \( v = v_1 + iv_2 \), that
\[
\text{curl } v = 2 \Im \left( \frac{\partial v}{\partial z} \right) = \frac{1}{i} \left( \frac{\partial v}{\partial z} - \frac{\partial \bar{v}}{\partial \bar{z}} \right)
\]
and
\[
\text{div } v = 2 \Re \left( \frac{\partial v}{\partial z} \right) = \frac{\partial v}{\partial z} + \frac{\partial \bar{v}}{\partial \bar{z}}.
\]

We will denote by \( m \) the Lebesgue measure in \( \mathbb{R}^2 \), associated to \( \frac{1}{2\pi i} d\bar{z} \wedge dz \), the
standard volume form in \( \mathbb{C} \).

The conjugate Cauchy transform is the operator inverse of \( \frac{\partial}{\partial z} \) and will be denoted
by
\[
\bar{C}[\varphi](z) = \frac{1}{2\pi i} \int_{\mathbb{C}} \frac{\varphi(\zeta)}{\zeta - z} d\bar{z} \wedge dz = \frac{1}{\pi} \int_{\mathbb{C}} \frac{\varphi(\zeta)}{\zeta - z} dm,
\]
defined for suitable functions \( \varphi \). Then
\[
\frac{\partial}{\partial z} \bar{C}[\varphi] = \varphi
\]
and the derivative
\[
\bar{B}[\varphi](z) \overset{\text{def}}{=} \left( \frac{\partial}{\partial \bar{z}} \bar{C}[\varphi] \right)(z)
= \text{p. v. } \frac{i}{2\pi} \int_{\mathbb{C}} \frac{\varphi(\zeta)}{(\zeta - z)^2} d\bar{z} \wedge dz = \text{p. v. } \frac{-1}{\pi} \int_{\mathbb{C}} \frac{\varphi(\zeta)}{(\zeta - z)^2} dm,
\]
is the conjugate Beurling transform.

A defining function for an open regular subset in \( \mathbb{C} \) is a real-valued function
defined in \( \mathbb{C} \), as regular as the boundary of the set, positive outside and negative
inside and whose gradient does not vanish at the points of the boundary. There are
many of these functions and some choices are used to capture the geometry of the
boundary. For more information on defining functions see [5].

In order to have functions defined at all points in \( \mathbb{C} \), we introduce a notion of
density of a distribution. Let \( \phi \in \mathcal{C}^\infty(\mathbb{R}^2) \) be a (test) radial function supported in
the unit ball, whose integral is equal to 1 and such that \( \phi(0) = 1 \). Let us consider, for a distribution \( T \), the limit
\[
\lim_{\epsilon \to 0} \langle T, \phi_{x_0, \epsilon} \rangle, \tag{3}
\]
where \( \phi_{x_0, \epsilon}(x) = \frac{1}{\epsilon^d} \phi\left(\frac{x-x_0}{\epsilon}\right) \).

If this limit exists at some point \( x_0 \) and is independent of the choice of \( \phi \) we call it the density function of \( T \) at \( x_0 \) and denote it by \( \Theta(T, x_0) \).

**Lemma 1.1.** Let \( \Omega \subset \mathbb{C} \) be a bounded domain such that \( \partial \Omega \in C^1 \). If \( T_{\chi_{\Omega}} \) is the distribution given by \( \chi_{\Omega} \), we have
\[
\Theta(T_{\chi_{\Omega}}, z_0) = \begin{cases} 
1 & \text{if } z_0 \in \Omega, \\
\frac{1}{2} & \text{if } z_0 \in \partial \Omega, \\
0 & \text{if } z_0 \in \overline{\Omega}^c.
\end{cases} \tag{4}
\]

**Proof.** Let \( \rho \) be a defining function for \( \Omega \). Take \( x_0 \in \partial \Omega \) and consider
\[
\kappa(x; x_0) = (\nabla \rho(x_0), x - x_0)
\]
and
\[
\omega(x; x_0) = \rho(x) - \kappa(x; x_0).
\]
If \( B_\epsilon(x_0) \) is the ball of radius \( \epsilon \) centered in \( x_0 \), taking the function \( \phi_{x_0, \epsilon} \) used in the definition of \( (3) \), we have
\[
\int_{B_\epsilon(x_0) \cap \Omega} \phi_{x_0, \epsilon}(x) \, dm(x) = \int_{B_\epsilon(x_0) \cap \{ \rho < 0 \}} \phi_{x_0, \epsilon}(x) \, dm(x) \tag{\ast},
\]
and since
\[
\{ \rho < 0 \} = (\{ \rho < 0 \} \cap \{ \kappa < 0 \}) \cup (\{ \rho < 0 \} \cap \{ \kappa \geq 0 \}),
\]
then
\[
(\ast) = \int_{B_\epsilon(x_0) \cap \{ \kappa < 0 \}} \phi_{x_0, \epsilon}(x) \, dm(x) \nonumber
\]
\[
+ \left\{ \int_{B_\epsilon(x_0) \cap \{ \rho < 0 \} \cap \{ \kappa > 0 \}} \phi_{x_0, \epsilon}(x) \, dm(x) - \int_{B_\epsilon(x_0) \cap \{ \rho > 0 \} \cap \{ \kappa > 0 \}} \phi_{x_0, \epsilon}(x) \, dm(x) \right\} = (I) + (II).
\]
So, after a rotation
\[
(I) = \lim_{\epsilon \to 0} \int_{B_\epsilon(x_0) \cap \{ \kappa < 0 \}} \phi_{x_0, \epsilon}(x) \, dm(x)
\]
\[
= \lim_{\epsilon \to 0} \int_{B_\epsilon(x_0) \cap \mathbb{R}^d_+} \phi_{x_0, \epsilon}(x) \, dm(x) = \frac{1}{2}.
\]
On the other hand,
\[
| (II) | = \left| \int_{B_\epsilon(x_0) \cap \{ \rho < 0 \} \cap \{ \kappa > 0 \}} \phi_{x_0, \epsilon}(x) \, dm(x) - \int_{B_\epsilon(x_0) \cap \{ \rho > 0 \} \cap \{ \kappa > 0 \}} \phi_{x_0, \epsilon}(x) \, dm(x) \right|
\]
\[
\leq \frac{1}{\epsilon} \left\{ m(B_\epsilon(x_0) \cap \{ \rho < 0 \} \cap \{ 0 \leq \kappa \}) + m(B_\epsilon(x_0) \cap \{ \rho > 0 \} \cap \{ \kappa < 0 \}) \right\},
\]
and since the set \( \{ \rho < 0 \} \cap B_\epsilon(x_0) \) can be described as the graph of a \( C^1 \) function over \( \{ \kappa = 0 \} \) vanishing at \( x_0 \) at the order 1, then both
\[
m(B_\epsilon(x_0) \cap \{ \rho < 0 \} \cap \{ 0 \leq \kappa \})
\]
and

\[ m(B_\epsilon(x_0) \cap \{ \rho \geq 0 \} \cap \{ \kappa < 0 \}) \]

are of size \( o(\epsilon) \), so \( (II) \rightarrow \epsilon \rightarrow 0 \), and the lemma is proved.

As a consequence of this fact we have that for \( f \in C^\infty(\mathbb{C}) \), we have

\[ \Theta(T f \chi_{\Omega}, z) = f(z) \Theta(T \chi_{\Omega}, z). \]

The space \( C^{k,\gamma}(U) \), where \( U \) is an open subset of the plane, \( k \) is a non-negative integer and \( 0 < \gamma < 1 \) is the space of functions with continuous derivatives up to the order \( k \) such that each derivative of order \( k \) extends to a \( \gamma \)-Hölder function in the closure of \( U \).

In this paper we will mainly use the spaces \( C^{k,\gamma}(U) \) for \( k = 0, 1 \), equipped with the norms

\[ \| f \|_{\gamma} = \| f \|_\infty + \sup_{z \neq w; z, w \in U} \frac{|f(z) - f(w)|}{|z - w|^\gamma} \]

and

\[ \| f \|_{1,\gamma} = \| f \|_\infty + \| \nabla f \|_\gamma \simeq \| f \|_\infty + \left\| \frac{\partial f}{\partial \overline{z}} \right\|_\gamma + \left\| \frac{\partial f}{\partial z} \right\|_\gamma. \]

We use the notation \( C^\omega \) for the space of real analytic functions and \( \mathcal{D}(A) \) for compactly supported \( C^\infty \) functions whose support is contained in a closed set \( A \subset \mathbb{R}^2 \).

1.2. Statement of results. Let \( \Omega \subset \mathbb{C} \) be a bounded domain such that \( \partial \Omega \in C^{1,\gamma} \) for \( \gamma \in (0, 1) \).

We will consider

\[ \varpi_0(z) = \Theta(T, z), \]

where \( T \) is the distribution given by a multiple of the characteristic function of \( \Omega \), as the initial datum in a Cauchy problem (it corresponds to the density in the case of aggregation equation and to the vorticity in the case of Euler equation) and then we define

\[ v_0 = \mathcal{C}[\varpi_0] \]

as the initial velocity. We have (see [1, section 4.3.2]) that \( v_0 \in \text{Lip}(1, \mathbb{C}) \cap C^\infty(\mathbb{C} \setminus \partial \Omega) \).

We will establish the analyticity in time of the flow associated to the problems \((A)\) (\( \tilde{A} \)) and \((E)\), in the case in which the initial datum is \( \varpi_0 \) above, representing \( \rho_0 \) or \( \omega_0 \). This problem is known in the current literature as the patch problem.

The main point for establishing analyticity is solving an equation that relates the flow with the vorticity or the density for a fluid satisfying a transport or continuity equation. This is summarized in the next theorem:

**Theorem 1.2.** Let \( \varpi_0 \) as in (4), \( T_0 \in (0, \infty) \) and \( a: \mathbb{C} \times (-T_0, T_0) \rightarrow \mathbb{C} \) a function in \( C^\infty((\mathbb{C} \setminus \partial \Omega) \times (-T_0, T_0)) \) and such that

\[ t \rightarrow a(z, t) \]

is analytic in \((-T_0, T_0)\) for every \( z \in \mathbb{C} \).

Then the problem

\[ \begin{cases} \left( \frac{\partial^2 \psi}{\partial \zeta \partial \bar{\zeta}} - \frac{\partial^2 \psi}{\partial \zeta \partial \bar{z}} \right)(z, t) = \varpi_0(z)(1 + a(z, t) t), \\
\psi(z, 0) = z, \end{cases} \]

(5)
has a solution $\psi$ in $\mathbb{C} \times (-T_0, T_0)$, such that for every $z \in \mathbb{C}$, the mapping $t \rightarrow \psi(z, t)$ is in $C^\omega((-T_0, T_0))$.

Equation (5) is used in an a priori treatment of the flow when the datum (the term in the right hand side of the equation) is analytic in time and we will prove (in Lemma 2.1) that the trajectories of the particles in the problems $(A)$, $(\tilde{A})$ and $(E)$ satisfy equation (5). A recursive procedure provides a solution of (5) that is analytic too.

**Remark 1.** It is worth to observe that for a general function $a$ in the statement of Theorem 1.2, the uniqueness of the solution of (5) is far from being granted.

Let $v$ the velocity field associated to this solution through the equation (1).

In the case of the aggregation equation, the solution $(v, \rho)$ of the patch problem satisfies, written in complex form,

$$v(t) = \rho(t) * \left( \frac{x + iy}{|z|^2} \right) = \rho(t) * \left( \frac{1}{\bar{z}} \right) = \pi \mathcal{C}[\rho(t)],$$

so $\frac{\partial v}{\partial \bar{z}} = \rho$, the density of mass (curl $v = 0$).

In the case of the Euler equation, the solution $(v, \omega)$ of the patch problem satisfy

$$v(t) = \omega(t) * \left( -\frac{y + ix}{|z|^2} \right) = (i \omega(t)) * \left( \frac{1}{\bar{z}} \right) = \pi \mathcal{C}[i \omega(t)],$$

so $\frac{\partial v}{\partial z} = \omega$, the vorticity (div $v = 0$). In each of these cases the velocity field $v$ provides a flow,

$$\psi(z, t) = z + \int_0^t v(\psi(z, \tau), \tau) \, d\tau,$$

in $\mathbb{C} \times I$, for some interval of time $I$.

Then uniqueness of solution of the equations $(A)$ or $(E)$ concludes the argument, and we have

**Theorem 1.3.** Let $\Omega \subset \mathbb{C}$ a bounded domain such that $\partial \Omega \in C^{1,\gamma}$ for $\gamma \in (0, 1)$. If $\psi$ is the flow corresponding to the solution of the equations $(A)$, $(\tilde{A})$ or $(E)$, with initial condition $\chi_\Omega$, then for any $z \in \mathbb{C}$, the function

$$t \rightarrow \psi(z, t),$$

is in $C^\omega(I)$, where $I$ is the interval of existence of the flow.

It is worth noticing that the proof in the case of equation $(\tilde{A})$ comes as a bonus from the proof for equation $(A)$.

1.3. **Plan of the paper.** The remaining of the paper is devoted to the proof of Theorems 1.2 and 1.3 above in three sections.

In section 2 we give a proof of Theorem 1.3 using Theorem 1.2. Since the density in the case of the continuity equation $(A)$ and the vorticity in case of Euler’s equation $(E)$ are transported by the corresponding flow, then Theorem 1.2 applies locally in time, providing a family of homeomorphisms of the whole plane, parametrized by the time $t$. This family, that we call $\phi(z, t)$, is regular away of the boundary of $\Omega$, for any fixed $t$, and depends analytically on $t$ in a neighbourhood of $t = 0$. In each case, the time derivative of $\phi$ gives rise to a velocity field whose $z$-derivative away of $\partial \Omega$ provides respectively a new density or vorticity that satisfies $(A)$ or $(E)$ respectively in these points.
The new density satisfies \((\tilde{A})\) and since equation \((\tilde{A})\) is the dual equation of \((A)\), then it also satisfies \((A)\) in a weak sense. By uniqueness of the solution of \((A)\) in this case, this new density coincides with the one given initially. As a consequence, since \(\partial \Omega\) is regular, also the velocities coincide, and then the function obtained in section 1.3, is in fact the flow corresponding to \((A)\). This shows that this flow is analytic in time, locally. For the equation \((E)\) the arguments are similar, and even simpler because \((E)\) is self-dual.

The persistence of the regularity, established in [3] for the solution of the aggregation equation and (previously) in [8], [2] and [24] for the solution of the Euler’s equation, allows the extension of this solution to all values of \(t\) in the interval of existence. Then the uniqueness of the solution, established in [4] in the aggregation case or in [19] (see also [Che1] or [2] in the Euler case shows that the flow is analytic globally in each case.

In section 3 we prove Theorem 1.2 following a standard a priori method based in power series developments for \(\psi\) in the equation

\[
\frac{\partial^2 \psi}{\partial t \partial z} \frac{\partial \psi}{\partial z} - \frac{\partial^2 \psi}{\partial t \partial \bar{z}} \frac{\partial \psi}{\partial \bar{z}} = \varpi_0(z) e^{2 \Re \{ \varpi_0(z) \} t},
\]

where \(\varpi_0\) is essentially \(i \omega_0\) or \(\rho_0\), that changes the PDE problem in a system of functional equations. The use of densities makes possible the extension of these equations to all values of \(z \in \mathbb{C}\). Our procedure is inspired in the paper [12], showing analyticity of flows in a different context.

Section 4 is devoted to some technical facts. The proof of Theorem 1.2 heavily relies on some formulas and precise bounds for the (conjugate) Beurling transform on domains with bounded regular boundary.

2. **Proof of Theorem 1.3.** First of all, the following lemma provides a link between the flow obtained in the local study of aggregation and Euler equations and Theorem 1.2.

**Lemma 2.1.** Let \(V \subset \mathbb{C}\) an open subset, \(\alpha: V \to (0, \infty)\). Let us consider

\[
U = \{(z, t) : z \in V, t \in (-\alpha(z), \alpha(z))\},
\]

and a complex valued function \(a\), such that there exist functions \(v\) and \(\psi\) defined in \(U\) and regular enough such that

\[
\frac{\partial v}{\partial z}(z, t) = a(z, t)
\]

and

\[
\frac{\partial \psi}{\partial t}(z, t) = v(\psi(z, t), t),
\]

then

\[
\left( \frac{\partial^2 \psi}{\partial t \partial z} \frac{\partial \psi}{\partial z} - \frac{\partial^2 \psi}{\partial t \partial \bar{z}} \frac{\partial \psi}{\partial \bar{z}} \right)(z, t) = a(\psi(z, t), t) \left( \left| \frac{\partial \psi}{\partial z} \right|^2 - \left| \frac{\partial \psi}{\partial \bar{z}} \right|^2 \right)(z, 0) e^{2 \int_0^t \Re \{a(\psi(z, \tau), \tau)\} d\tau}.
\]
Proof. Taking derivatives with respect to $z$ in (6) we have
\[ \frac{\partial^2 \psi}{\partial t \partial z}(z, t) = \frac{\partial v}{\partial \zeta}(\psi(z, t), t) \frac{\partial \psi}{\partial z}(z, t) + \frac{\partial v}{\partial \zeta}(\psi(z, t), t) \frac{\partial \bar{\psi}}{\partial z}(z, t) \]
and multiplying by $\frac{\partial \bar{\psi}}{\partial z}$ we have
\[ \frac{\partial^2 \psi}{\partial t \partial z}(z, t) \frac{\partial \bar{\psi}}{\partial z}(z, t) = a(\psi(z, t), t) \left| \frac{\partial \psi}{\partial z} \right|^2(z, t) + \frac{\partial v}{\partial \zeta}(\psi(z, t), t) \frac{\partial \bar{\psi}}{\partial z}(z, t). \]  
(7)

Also taking derivatives with respect to $\bar{z}$ in (6), we have
\[ \frac{\partial^2 \psi}{\partial t \partial \bar{z}}(z, t) = \frac{\partial v}{\partial \zeta}(\psi(z, t), t) \frac{\partial \psi}{\partial \bar{z}}(z, t) + \frac{\partial v}{\partial \zeta}(\psi(z, t), t) \frac{\partial \bar{\psi}}{\partial \bar{z}}(z, t) \]
and multiplying by $\frac{\partial \bar{\psi}}{\partial \bar{z}}$ we have
\[ \frac{\partial^2 \psi}{\partial t \partial \bar{z}}(z, t) \frac{\partial \bar{\psi}}{\partial \bar{z}}(z, t) = a(\psi(z, t), t) \left| \frac{\partial \psi}{\partial \bar{z}} \right|^2(z, t) + \frac{\partial v}{\partial \zeta}(\psi(z, t), t) \frac{\partial \bar{\psi}}{\partial \bar{z}}(z, t). \]  
(8)

Subtracting (8) from (7), and taking advantage of a cancellation we conclude that
\[ \left( \frac{\partial^2 \psi}{\partial t \partial z} \frac{\partial \bar{\psi}}{\partial z} - \frac{\partial^2 \psi}{\partial t \partial \bar{z}} \frac{\partial \bar{\psi}}{\partial \bar{z}} \right)(z, t) = a(\psi(z, t), t) \left( \left| \frac{\partial \psi}{\partial z} \right|^2 - \left| \frac{\partial \psi}{\partial \bar{z}} \right|^2 \right). \]

Moreover, we have
\[ \frac{\partial}{\partial t} \left( \left| \frac{\partial \psi}{\partial z} \right|^2 - \left| \frac{\partial \psi}{\partial \bar{z}} \right|^2 \right)(z, t) = 2 \Re \left\{ \left( \frac{\partial^2 \psi}{\partial t \partial z} \frac{\partial \bar{\psi}}{\partial z} - \frac{\partial^2 \psi}{\partial t \partial \bar{z}} \frac{\partial \bar{\psi}}{\partial \bar{z}} \right)(z, t) \right\} \]
\[ = 2 \Re \left\{ a(\psi(z, t), t) \right\} \left( \left| \frac{\partial \psi}{\partial z} \right|^2 - \left| \frac{\partial \psi}{\partial \bar{z}} \right|^2 \right)(z, t), \]
and if $\left( \left| \frac{\partial \psi}{\partial z} \right|^2 - \left| \frac{\partial \psi}{\partial \bar{z}} \right|^2 \right)(z, t)$ never vanishes on $U$, then
\[ \left( \left| \frac{\partial \psi}{\partial z} \right|^2 - \left| \frac{\partial \psi}{\partial \bar{z}} \right|^2 \right)(z, t) = \left( \left| \frac{\partial \psi}{\partial z} \right|^2 - \left| \frac{\partial \psi}{\partial \bar{z}} \right|^2 \right)(z, 0) e^2 \int_0^t \Re \{ a(\psi(z, \tau), \tau) \} d\tau. \]

Now
1. In the case of the vortex patch problem, we start by considering, for the case of the Euler equation, the purely imaginary valued function defined on $\mathbb{C}$ by
\[ \varpi_0 = i \epsilon \left\{ \chi + \frac{1}{2} \chi \partial \varphi \right\}, \]
where $\epsilon \in \mathbb{R}$. Then
\[ v_0(z) = \tilde{C}[\varpi_0](z) \]
and by Yudovich’s theorem there exist functions $\varpi$ and $v$ defined in $\mathbb{C} \times \mathbb{R}$ such that $\varpi \in L^\infty(\mathbb{C} \times \mathbb{R}) \cap L^\infty_{\text{loc}}(\mathbb{R}; L^p(\mathbb{C}))$ for $1 < p < \infty$ and takes its values in $i \mathbb{R}$, and $v$ satisfies
$$\frac{\partial v}{\partial z}(z,t) = \frac{1}{2} \varpi(z,t)$$
in the distributions sense, and the couple $(\varpi, v)$ satisfy the Euler equation $(E)$ in the weak sense.

Moreover, there exists a unique function $\psi \in C(\mathbb{C} \times \mathbb{R}; \mathbb{C})$ such that
$$\psi(z,t) = z + \int_0^t v(\psi(z,\tau),\tau) \, d\tau,$$
and there is a constant $C > 0$ such that for any $t \in \mathbb{R}$,
$$\psi(\cdot, t) - I \in C_{e^{-Ct\|w_0\|_{L^p \cap L^\infty}}}^{\text{loc}}(\mathbb{C}).$$

Then, from the particular shape of $\varpi_0$, we have that if $U \subset \Omega$ or $U$ is a bounded subset of $\mathbb{C} \setminus \bar{\Omega}$, then $v_0 \in C^\infty(U)$ and then, using for instance Proposition 8.3 in [20], we can conclude that for any $t \in \mathbb{R}$, $\varpi(\cdot, t) \in C^\infty(\psi(U, t))$ and $v(\cdot, t) \in C^\infty(\psi(U, t))$.

Moreover, both the velocity and the flow given by the function $\psi$ in the theorem are globally defined with respect to the time variable and in general are regular beyond the continuity in the $z$ variable.

The flow $\psi$ also inherits the local space regularity (after derivation under the integral sign). Moreover, using Theorem 1.3.1 of Chapter 1 in [18], we have that $\psi \in C^1(U \times [0,\infty))$ and $\frac{\partial \psi}{\partial z}(\cdot, t) \in C^1(U)$.

Then the incompressibility of the fluid can be written in terms of the Jacobian of $\psi$ as
$$J_\psi = \left| \frac{\partial \psi}{\partial z} \right|^2 - \left| \frac{\partial \psi}{\partial \bar{z}} \right|^2 = 1,$$
also the vorticity is constant along the flow lines.

From these facts and Lemma 2.1 we get the relationship between the flow and the vorticity
$$\left( \frac{\partial^2 \psi}{\partial t \partial z} \frac{\partial \psi}{\partial z} - \frac{\partial^2 \psi}{\partial t \partial \bar{z}} \frac{\partial \psi}{\partial \bar{z}} \right)(z,t) = \frac{1}{2} \varpi_0(z),$$
in the region of $\mathbb{C} \times \mathbb{R}$ where it makes sense.

The formula above falls in the so called Lagrangian approach. This approach was introduced by A. Cauchy in [6] and used by several authors since then (e. g. [12] or [27]).

2. For the case of the aggregation equation, we consider the real valued function defined on $\mathbb{C}$ by
$$\varpi_0 = c \left\{ \chi_\Omega + \frac{1}{2} \chi_{\partial \Omega} \right\},$$
where $c \in \mathbb{R}$.

Again
$$v_0(z) = \tilde{C}[\varpi_0](z)$$
and, as proven in [4, Theorems 2.3, 2.4 and 3.1], there exists a constant $T = T(c)$ and functions $\varpi$ and $v$ defined in $\mathbb{C} \times [0, T)$ such that $\varpi \in L^\infty(\mathbb{C} \times$
\[ \partial v/\partial z(z,t) = \frac{1}{2} \varpi(z,t) \]

in weak sense. The functions \( \varpi \) and \( v \) are unique, solving the equation

\[
\begin{cases}
\frac{\partial \varpi}{\partial t} + 2 \Re \left( \frac{\partial (\varpi v)}{\partial z} \right) = 0, \\
\varpi(\cdot,0) = \varpi_0.
\end{cases}
\] (A)

All this implies that \( v_0 \in \text{Lip}(1, \mathcal{C}) \cap \mathcal{C}(0, T); \mathcal{C} \)), and since \( v(z,t) = \bar{C}[\varpi(\cdot, t)](z) \)

also (cf. [1])

\[ v \in L^\infty(\mathcal{C} \times [0, T)) \cap \mathcal{C}(0, T); \text{Lip}(1, \mathcal{C})). \]

Then (cf. [9, Theorem 5.2.1]), there exists a unique function \( \psi \in \mathcal{C}(\mathcal{C} \times [0, T); \mathcal{C}) \) such that

\[ \psi(z,t) = z + \int_0^t v(\psi(z, \tau), \tau) \, d\tau, \]

and there exists a constant \( C > 0 \) such that for any \( t \in \mathbb{R} \),

\[ \psi(\cdot, t) - I \in \mathcal{C}^{-C \|\varpi_0\|_{L^p \cap L^\infty}(\mathcal{C})} \mathcal{C}(\mathcal{C}). \]

It is then clear that

\[
\left( \frac{1}{2} \left| \frac{\partial \varpi}{\partial z} \right|^2 - \left| \frac{\partial \varpi}{\partial \bar{z}} \right|^2 \right)(z, 0) = 1.
\]

As in the case of the Euler equation, for each \( t \in [0, T) \) and \( U \subset \mathcal{C} \setminus \partial \Omega \) and bounded, we have that \( \psi(U, t) \subset \mathcal{C} \setminus \psi(\partial \Omega, t) \) and bounded. Then, from the particular shape of \( \varpi_0 \), we have that if \( U \subset \Omega \) or \( U \subset \mathcal{C} \setminus \bar{\Omega} \) and bounded, then \( v_0 \in \mathcal{C}^\infty(U) \) and, using Proposition 8.3 in [20], we can conclude that for any \( t \in [0, T), \varpi(\cdot, t) \in \mathcal{C}(\mathcal{C}, \psi(U, t)) \) and \( v(\cdot, t) \in \mathcal{C}^\infty(\psi(U, t)). \)

Moreover, both the velocity and the flow given by the function \( \psi \) in the theorem are globally defined with respect to the time variable and in general are regular beyond the continuity in the \( z \) variable. This is because the local regularity is improved by the Cauchy transform, and so the regularity of initial conditions is propagated by solutions of ordinary differential equations.

The flow \( \psi \) also inherits the local space regularity (after derivation under the integral sign). Moreover, using Theorem 1.3.1 of Chapter 1 in [18], we have that \( \psi \in \mathcal{C}^1(U \times [0, T)) \) and \( \frac{\partial \psi}{\partial t}(\cdot, t) \in \mathcal{C}^1(U). \)

Then we have that (cf. [4]) the transport of the density by the flow satisfies

\[ \varpi(\psi(z,t), t) = \frac{\varpi_0(z)}{1 - \varpi_0(z) t} \]

in \( \mathcal{C} \setminus \partial \Omega \).

These facts allow us to formulate the relationship between the flow and the density, in the region of \( \mathcal{C} \times [0, T) \) where it makes sense, as

\[
\left( \frac{\partial \varpi}{\partial t} \frac{\partial \psi}{\partial \bar{z}} + \frac{\partial \varpi}{\partial \bar{z}} \frac{\partial \psi}{\partial z} \right)(z, t) = \frac{1}{2} \frac{\varpi_0(z)}{1 - \varpi_0(z) t} \int_0^t \frac{\varpi_0(\tau)}{1 - \varpi_0(\tau) t} \, d\tau,
\] (10)
and in fact, in $\mathbb{C} \setminus \partial \Omega$ we have
\[
\left( \frac{\partial^2 \psi}{\partial t \partial z} \frac{\partial \psi}{\partial z} - \frac{\partial^2 \bar{\psi}}{\partial t \partial \bar{z}} \frac{\partial \bar{\psi}}{\partial \bar{z}} \right) (z, t) = \frac{c}{2(1 - ct)^3} \chi_\Omega(z).
\] (11)

Let $\phi$ be the solution of the equation (9) or (11) provided by Theorem 1.2. We have that $\phi(z, t)$ extends to a function in $C(\mathbb{C} \setminus \partial \Omega)$.

Moreover

**Proposition 1.** The solution $\phi$ of the equation (9) or (11) obtained using Theorem 1.2 satisfies, for $t \in [-T_0, T_0]$, that

(a) $\phi(\cdot, t): \mathbb{C} \to \mathbb{C}$ is an homeomorphism.

(b) $\phi(\cdot, t) \in C^\infty(\mathbb{C} \setminus \partial \Omega)$.

This result will be proved in section 3.2.

As we have seen, the (unique) flow $\psi$ corresponding to the equation $(E)$ or $(A)$ with the initial condition $\omega_0(z)$ or $\rho_0(z)$, also satisfies equation (9) or (11). Now we prove that $\phi(\cdot, t) \equiv \psi(\cdot, t)$ for $t \in [-T_0, T_0]$:

We have that $\psi$ and $\phi$ generate vector fields $v$ and $u$, namely
\[
v(\zeta, t) = \frac{\partial \psi}{\partial t}(\psi(\cdot, t)^{-1}(\zeta), t)
\]
and
\[
u(\zeta, t) = \frac{\partial \phi}{\partial t}(\phi(\cdot, t)^{-1}(\zeta), t).
\]

On the other hand, we have that
\[
\frac{\partial v}{\partial \zeta}(\zeta, t) = \varpi(\zeta, t) = \frac{1}{2} \omega_0(\psi(\cdot, t)^{-1}(\zeta)) b(t)
\]
and
\[
\frac{\partial u}{\partial \zeta}(\zeta, t) \overset{\text{def}}{=} \Xi(\zeta, t) = \frac{1}{2} \omega_0(\phi(\cdot, t)^{-1}(\zeta)) b(t)
\]
in $\mathbb{C} \setminus \partial \Omega$, where $b(t) = 1$ in the case of equation $(E)$ and $b(t) = \frac{1}{(1 - ct)^3}$ in the case of equation $(A)$.

Now, we will prove that $\Xi$ is a weak solution of equation $(E)$ or equation $(A)$, and then, by uniqueness $\varpi \equiv \Xi$ in the weak sense, so $\varpi = \Xi$ almost everywhere. Then
\[
\frac{\partial(u - v)}{\partial \zeta}(\zeta, t) = 0
\]
for $\zeta$ outside $\partial \Omega$, a subset of $\mathbb{C}$ whose continuous analytic capacity (see [28, pg 38], also [13] or [14]) is equal to 0. Since $u$ and $v$ are bounded and vanish at infinity, by Liouville’s theorem we conclude that $u \equiv v$. This finishes the argument.

Let us prove that $\Xi$ is a weak solution of equation $(E)$ or equation $(A)$. It is worth to remark that the equation $(E)$ is self-dual, but the dual equation of $(A)$ is $(A)$. 

For $\varphi \in C^1([0, T]; C^1_0(\mathbb{C}))$

$$\int_0^T \int_{\mathbb{C}} \Xi(\zeta, t) \frac{D\varphi}{Dt}(\zeta, t) \, dm(\zeta) \, dt$$

$$= \int_0^T \int_{\mathbb{C}} \omega_0(\phi(\zeta, t)^{-1}(\zeta)) b(t) \frac{D\varphi}{Dt}(\zeta, t) \, dm(\zeta) \, dt$$

$$= \int_0^T \int_{\mathbb{C}} \omega_0(\zeta) \left. \frac{D\varphi}{Dt}(\phi(\zeta, t), t) \right| J_z \varphi(\zeta, t) \, dm(\zeta) \, dt$$

$$= \int_0^T \int_{\mathbb{C}} \omega_0(\zeta) \{ \varphi(\phi(\zeta, T), T) - \varphi(\zeta, 0) \} \, dm(\zeta),$$

in the second equality we have used that $b(t) J_z \varphi(\zeta, t) \equiv 1$.

Finally

(a) In the Euler’s case there exists a number $T_1 > 0$ and a function $\psi: \mathbb{C} \rightarrow \mathbb{C}$, such that $t \rightarrow \psi(z, t)$ is real analytic in $(-T_1, T_1)$.

If $T_1 = +\infty$ there is nothing to say. Otherwise, at time $\pm T_1$, from the theorem of persistence of regularity ([8], [2], [24]), we have that $\partial \Omega_{\pm T_1} \in C^{1,\gamma}$ and we can iterate the procedure and use the uniqueness of the solution to obtain $t$-analyticity in $(-T_2, T_2)$ for $T_2 > T_1$. This implies that the set of analyticity is open and closed in $\mathbb{R}$, so it is $\mathbb{R}$.

(b) For the equations ($A$) and ($\tilde{A}$), the procedure is similar. We only have to take in account that now $t > 0$. For the equation $\tilde{A}$, and for $\tilde{A}$ in the case of $c < 0$, the analyticity will occur for $t \in [0, \infty)$. For the equation $A$ and $c > 0$, the analyticity will occur for $t \in [0, \frac{1}{2c})$. The only necessary ingredient in the proof is the persistence of the regularity.

In fact it is enough to have persistence in the case ($A$). The rescaling

$$s(t) = \ln \left( \frac{1}{1 - ct} \right)$$

and

$$\tilde{\omega}(z, s) = \frac{1 - ct(s)}{c} \omega(z, t(s))$$

transforms the problem

$$\begin{cases}
\frac{\partial \tilde{\omega}}{\partial t} + 2 \Re \left( \frac{\partial (\tilde{\omega} v)}{\partial z} \right) = 0, \\
v(z, t) = -C[\tilde{\omega}(., t)](z), \\
\tilde{\omega}(., 0) = \omega_0,
\end{cases}$$

in the problem

$$\begin{cases}
\frac{\partial \tilde{v}}{\partial t} + \Re (\tilde{v} \frac{\partial \tilde{v}}{\partial z}) = 0, \\
\tilde{v}(z, s) = -C[\tilde{\omega}(., s)](z), \\
\tilde{\omega}(., 0) = \chi_{\Omega},
\end{cases}$$

which is a transport equation with initial datum the indicator function of $\Omega$, a region with $C^{1,\gamma}$ boundary.

For the problem ($\tilde{A}$) it is proven in [3] that for every $s \in [0, \infty)$, if $\tilde{\psi}$ is the related flow, then $\tilde{\psi}(\partial \Omega, s)$ is a $C^{1,\gamma}$ embedded submanifold of $\mathbb{C}$ of real
Since the rescalings above do not affect the $z$ variable, the same regularity is true in the case of $(A)$, for $t \in \left[0, \frac{1}{c}\right)$.

From Theorem 1.2 there exists a number $T_1 > 0$ and a function $\psi: \mathbb{C} \to \mathbb{C}$, such that $t \to \psi(z, t)$ is real analytic in $(-T_1, T_1)$.

If $T_1 = +\infty$ there is nothing else to say. Otherwise, after a time $T_1$, we have

$$\varpi(z, T_1) = \frac{c}{(1 - c T_1)^3} \varpi_0(z),$$

that must be used as initial density to iterate the procedure, because the boundary $\partial\Omega_{T_1}$ is $C^1$, getting a new $T_2 > T_1$ and analyticity in $(-T_1, T_2)$.

Again an argument of connectivity and the uniqueness conclude that the flow is analytic in $[0, \frac{1}{c})$.

3. Proof of Theorem 1.2 and consequences. The proof is divided in two steps.

In the first one we construct an analytic local solution of $(5)$. In the second one we prove that this solution is, in fact, a flow, $C^\infty$ regular in $z$ outside of $\partial\Omega$.

3.1. Construction of the solution. The analyticity in $t$ of $a$ implies the local existence of functions $a^{(s)}: \mathbb{C} \to \mathbb{C}$, for each $s \in \mathbb{N} \setminus \{0\}$ such that

$$\varpi_0(z) (1 + a(z, t) t) = \varpi_0(z) + \sum_{s=1}^{\infty} a^{(s)}(z) t^s,$$

for $t \in (-T_0, T_0)$.

It is worth to remark that $a^{(s)}(z) = 0$ for every $z \in \mathbb{C} \setminus \bar{\Omega}$.

Assume that there exists a family of functions $\xi^{(s)}: \mathbb{C} \to \mathbb{C}$, $s \in \mathbb{N}$, having first order derivatives with respect to $z$ and $\bar{z}$ at each point of $\mathbb{C} \setminus \partial\Omega$ and such that

$$\psi(z, t) = \sum_{s=0}^{\infty} \xi^{(s)}(z) t^s,$$

then

$$\left( \frac{\partial^2 \psi}{\partial \bar{z} \partial z} - \frac{\partial^2 \psi}{\partial \partial z \partial \bar{z}} \right)(z, t) = \sum_{s=0}^{\infty} \left\{ \sum_{k=0}^{s} (k+1) \left( \frac{\partial \xi^{(k+1)}}{\partial z}(z) \frac{\partial \xi^{(s-k)}}{\partial z}(z) - \frac{\partial \xi^{(k+1)}}{\partial \bar{z}}(z) \frac{\partial \xi^{(s-k)}}{\partial \bar{z}}(z) \right) t^s.\right.$$

If $\psi$ is a solution analytic in $t$ in a neighbourhood of $\mathbb{C} \times \{0\}$ of the problem $(5)$, then

$$\xi^{(0)}(z) \equiv z$$

and if $z \in \mathbb{C} \setminus \partial\Omega$, then

$$\left\{ \begin{array}{ll}
\frac{\partial \xi^{(1)}}{\partial z}(z) = \varpi_0(z) \overset{\text{def}}{=} a^{(0)}(z), \\
\frac{\partial \xi^{(s+1)}}{\partial z}(z) = \frac{a^{(s)}(z)}{s+1} - \frac{1}{s+1} \sum_{k=0}^{s-1} (k+1) \left( \frac{\partial \xi^{(k+1)}}{\partial z} \frac{\partial \xi^{(s-k)}}{\partial z} - \frac{\partial \xi^{(k+1)}}{\partial \bar{z}} \frac{\partial \xi^{(s-k)}}{\partial \bar{z}} \right)(z),
\end{array} \right. s \in \mathbb{N} \setminus \{0\}.\right.
$$
The system (13) provides for each \( s \in \mathbb{N} \setminus \{0\} \) the \( z \)-derivative of \( \xi^{(s)} \) in terms of the first order derivatives of the functions \( \xi^{(l)} \), where \( 1 \leq l < s \), if \( s > 1 \), and in terms of \( \varpi_0 \) in the case \( s = 1 \).

Let us choose
\[
\xi^{(1)}(z) = \mathcal{C}[\varpi_0](z).
\]

We have

**Proposition 2.** The function \( \mathcal{C}[\varpi_0](z) \) is in \( C^\infty(\mathbb{C}\setminus\bar{\Omega})\cap\text{Lip}(1,\mathbb{C}) \) and has a decay at the infinity of the type
\[
\frac{1}{\max\{R_0, d(z,\partial\Omega)\}},
\]
for some fixed \( R_0 > 0 \) related to the geometry of \( \partial\Omega \) (cf. lemma 4.1 in section 4).

Moreover, if \( T^{(1)} \) is the distribution corresponding to \( \frac{\partial\varpi^{(1)}}{\partial z} \), then
\[
\Theta(T^{(1)}, z) = \varpi_0(z)
\]
for all \( z \in \mathbb{C} \).

**Proof.** The first part is the result of a direct estimate.

The last part is a consequence of the fact that
\[
\frac{\partial T_{\mathcal{C}[\varpi_0]}}{\partial z} = \varpi_0
\]
in the sense of distributions, and then, since \( \partial\Omega \in C^{1,\gamma} \), the use of (4) we have that \( \Theta(\varpi_0, z) = \varpi_0(z) \) for every \( z \in \mathbb{C} \). \( \square \)

In the remaining we use the following theorem on the boundedness of the conjugate Beurling transform

**Theorem 3.1.** Let \( \Omega \) be a domain such that \( \partial\Omega \in C^{1,\gamma} \), where \( \gamma \in (0,1) \) and \( g \) a function defined at every \( z \in \mathbb{C} \), \( g \in C^\infty(\mathbb{C}\setminus\partial\Omega) \) and \( \chi_{\Omega}g \) extends to a (unique) function \( g_- \in \text{Lip}(\gamma,\bar{\Omega}) \), \( \chi_{\Omega} \) extending to a (unique) function \( g_+ \in \text{Lip}(\gamma,\mathbb{C}\setminus\Omega) \) and for every \( z \in \partial\Omega \), \( g(z) = \frac{1}{2}\{g_+(z) + g_-(z)\} \).

We also assume that there is a constant \( R_0 > 0 \) depending only on \( \Omega \) such that if
\[
U_{R_0} = \{z \in \mathbb{C} : d(z,\Omega) < R_0\},
\]
then there is a constant \( C(g) > 0 \) such that for every \( z \in \mathbb{C} \setminus U_{R_0} \),
\[
|g(z)| \leq \frac{C(g)}{\max\{R_0^2, d(z,\partial\Omega)^2\}}.
\]

Then \( \bar{B}[g](z) \) is well defined for each \( z \in \mathbb{C} \) and
\[
\chi_{\Omega}\bar{B}[g] \in C^\infty(\Omega) \cap \text{Lip}(\gamma,\bar{\Omega}),
\]
\[
\chi_{\Omega}\bar{B}[g] \in C^\infty(\mathbb{C}\setminus\Omega) \cap \text{Lip}(\gamma,\mathbb{C}\setminus\Omega),
\]
and for any \( z \in \partial\Omega \) we have
\[
\bar{B}[g](z) = \frac{1}{2}\left\{\lim_{w \to z; w \in \Omega} \chi_{\Omega}\bar{B}[g](w) + \lim_{w \to z; w \in \mathbb{C}\setminus\Omega} \chi_{\mathbb{C}\setminus\Omega}\bar{B}[g](w)\right\}.
\]

Moreover, there exists a constant \( K = K(\gamma,\Omega, R_0) > 0 \) such that
\[
\|\bar{B}[g]\|_{C^\infty(\mathbb{C})} \leq K\|g\|_{\gamma},
\]
\[
\|\bar{B}[g]\|_{\gamma,\Omega} \leq K\|g\|_{\gamma},
\]
\[
\|\bar{B}[g]\|_{\gamma,\Omega} \leq K\|g\|_{\gamma},
\]
\[
\|\bar{B}[g]\|_{\gamma,\Omega} \leq K\|g\|_{\gamma},
\]
and

\[ \| B[g] \|_{\gamma, C\setminus \Omega} \leq K \| g \|_{\gamma}. \]

Moreover, for any \( z \in (U_{R_0} \cup \Omega)^c \),

\[ |B[g](z)| \leq K (1 + C(g)) \frac{1 + \ln d(z, \Omega)}{\max\{R_0^2, d(z, \partial \Omega)^2\}} \| g \|_{\gamma}. \quad (15) \]

Using Theorem 3.2 we have that

**Proposition 3.** If \( \tilde{T}^{(1)} \) is the distribution corresponding to \( \frac{\partial \xi}{\partial \bar{z}} \), then

\[ \Theta(\tilde{T}^{(1)}, z) = B[\omega_0](z) \]

for all \( z \in \mathbb{C} \).

So \( \tilde{T}^{(1)} \) is given by a function whose decay at \( \infty \) is of type

\[ \frac{1 + \ln d(z, \Omega)}{\max\{R_0^2, d(z, \partial \Omega)^2\}}. \quad (16) \]

**Proof.** For the first part, since

\[ \frac{\partial T[z_0]}{\partial \bar{z}} = \frac{i}{2\pi} \int_{\mathbb{C}} \left\{ \frac{\hat{\omega}_2(z)}{\zeta - \bar{z}} \right\} \omega_0(\zeta) d\zeta = B[\omega_0](z), \]

that is well defined at every point \( z \in \mathbb{C} \). Then we have that, by 4, as \( \partial \Omega \in C^{1,\gamma} \), then

\[ \Theta(\frac{\partial T[z_0]}{\partial \bar{z}}, z) = B[\omega_0](z) \]

The decay estimate (16) is a consequence of (15) in Theorem 3.2. \( \square \)

Now, for \( s > 1 \) we will define \( T^{(s)} \) and \( \tilde{T}^{(s)} \), the distributions corresponding to \( \frac{\partial \xi^{(s)}}{\partial \bar{z}} \) and \( \frac{\partial \xi^{(s)}}{\partial \bar{z}} \) respectively, and then \( \Theta(T^{(s)}, z) = \theta^{(s)}(z) \) and \( \Theta(\tilde{T}^{(s)}, z) = \eta^{(s)}(z) \). All these objects exist, by a direct application of the previous proposition to the cases \( s' < s \).

So, taking densities, the system (13) becomes

\[
\begin{aligned}
\theta^{(1)}(z) &= \omega_0(z), \\
\eta^{(1)}(z) &= B[\omega_0](z), \\
\theta^{(s+1)}(z) &= a^{(s)}(z) + \frac{1}{s+1} \sum_{k=0}^{s-1} (k + 1) \\
&\quad \times \left\{ \theta^{(s-k)} \eta^{(s-k)} - \eta^{(s-k)} \theta^{(s-k)} \right\}(z), \text{ for } s \in \mathbb{N} \setminus \{0\}.
\end{aligned}
\]

(17)

Then we consider, in \( \mathbb{C} \), the decomposition

\[ \theta^{(i)} = \chi_\Omega \theta^{(i)} + \chi_{\Omega^c} \theta^{(i)} + \chi_{\partial \Omega} \theta^{(i)} + \phi^{(i)} + \psi^{(i)} + 1^{(i)}. \quad (18) \]

So
\( (s + 1) (\phi^{(s+1)} + \psi^{(s+1)} + \imath^{(s+1)}) (z) \)

\[
= a^{(s)} (z) - \sum_{k=0}^{s-1} (k + 1) \left\{ (\phi^{(k+1)} + \psi^{(k+1)} + \imath^{(k+1)}) \bar{\phi^{(s-k)}} + \bar{\psi^{(s-k)}} + \bar{\imath^{(s-k)}} \right\} \\
- B[\phi^{(k+1)} + \psi^{(k+1)} + \imath^{(s-k)}] B[\phi^{(s-k)} + \psi^{(s-k)} + \imath^{(s-k)}] (z)
\]

\[
= a^{(s)} (z) - \sum_{k=0}^{s-1} (k + 1) \left\{ (\phi^{(k+1)} + \psi^{(k+1)} + \imath^{(k+1)}) \bar{\phi^{(s-k)}} + \bar{\psi^{(s-k)}} + \bar{\imath^{(s-k)}} \right\} \\
- B[\phi^{(k+1)} + \psi^{(k+1)}] B[\phi^{(s-k)} + \psi^{(s-k)}] (z),
\]

consequently, if we define

\[
\Phi[\gamma] = \chi_{\Omega} \bar{B}[\gamma], \quad \Psi[\gamma] = \chi_{\Omega^c} \bar{B}[\gamma], \quad \text{and} \quad \Gamma[\gamma] = \chi_{\partial\Omega} \bar{B}[\gamma],
\]

we have

\[
(s + 1) \phi^{(s+1)} = a^{(s)} - \sum_{k=0}^{s-1} (k + 1) \left\{ \phi^{(k+1)} \bar{\phi^{(s-k)}} - \Phi[\phi^{(k+1)}] \Phi[\bar{\phi^{(s-k)}}] \right\} \\
+ \sum_{k=0}^{s-1} (k + 1) \left\{ \Phi[\phi^{(k+1)}] \Phi[\bar{\psi^{(s-k)}}] + \Phi[\bar{\psi^{(s-k)}}] \Phi[\bar{\phi^{(s-k)}}] \\
+ \Phi[\bar{\psi^{(k+1)}}] \Phi[\bar{\phi^{(s-k)}}] \right\},
\]

in \( \Omega \),

\[
(s + 1) \psi^{(s+1)} = - \sum_{k=0}^{s-1} (k + 1) \left\{ \psi^{(k+1)} \bar{\psi^{(s-k)}} - \Psi[\psi^{(k+1)}] \Psi[\bar{\psi^{(s-k)}}] \right\} \\
+ \sum_{k=0}^{s-1} (k + 1) \left\{ \Psi[\phi^{(k+1)}] \Psi[\bar{\psi^{(s-k)}}] + \Psi[\bar{\psi^{(s-k)}}] \Psi[\bar{\phi^{(s-k)}}] \\
+ \Psi[\phi^{(k+1)}] \Psi[\bar{\phi^{(s-k)}}] \right\},
\]

in \( \Omega^c \), and

\[
(s + 1) \imath^{(s+1)} = a^{(s)} - \sum_{k=0}^{s-1} (k + 1) \left\{ \imath^{(k+1)} \bar{\imath^{(s-k)}} - \Gamma[\phi^{(k+1)}] \Gamma[\bar{\imath^{(s-k)}}] \right\} \\
+ \sum_{k=0}^{s-1} (k + 1) \left\{ \Gamma[\phi^{(k+1)}] \Gamma[\bar{\psi^{(s-k)}}] + \Gamma[\bar{\psi^{(s-k)}}] \Gamma[\bar{\imath^{(s-k)}}] \\
+ \Gamma[\phi^{(k+1)}] \Gamma[\bar{\imath^{(s-k)}}] \right\},
\]

in \( \partial\Omega \).

Now we can determine inductively the functions \( \phi \), \( \psi \) and \( \imath \).

**Proposition 4.** If \( \phi^{(1)} = \chi_{\Omega} \varpi_0 \), \( \psi^{(1)} = 0 \) and \( \imath^{(1)} = \chi_{\partial\Omega} \varpi_0 \), then for any \( s \in \mathbb{N} \setminus \{0, 1\} \), we have that \( \phi^{(s)} \in \text{Lip} (\gamma, \Omega) \), \( \psi^{(s)} \in \text{Lip} (\gamma, \Omega^c) \) and \( \imath^{(s)} \in \text{Lip} (\gamma, \partial\Omega) \).
Proof. The formulas above allow, for any $s \geq 2$, the control of $\phi^{(s+1)}$ and $\psi^{(s+1)}$ in terms of $\phi^{(l)}$ and $\psi^{(r)}$, for all $1 \leq r, l \leq s$.

In the case of $s = 1$, Theorem 3.2 gives the desired estimate.

For $s > 1$, we need the existence and estimates of terms of type $B[B[f^{(l)}]B[f^{(r)}]]$ for $1 \leq r, l \leq s$, where $f^{(s)}$ states for any of the functions in the statement.

If $\{d(z, \partial \Omega) > 1\}$ then, using (15) in Theorem 3.2 we have that

$$B[f^{(l)}]B[f^{(r)}] = O\left(\frac{\ln d(z, \partial \Omega)}{d(z, \partial \Omega)^4}\right),$$

so the product is integrable and satisfies the hypotheses of Theorem 3.2 with the corresponding estimates and we can perform iteration.

We also have that

$$|\psi^{(s)}(z)| \leq C \frac{\ln d(z, \partial \Omega)}{\max\{1, d(z, \partial \Omega)^2\}},$$

with $C$ independent of $s$. \hfill \Box

Let us control now the Hölder norm and some $L^p$ of $\phi$, $\psi$ and $1$.

If $\| \|$ denotes

$$\| \|_{\gamma, \tilde{\Omega}} + \| \|_{L^p(\tilde{\Omega})} + \| \|_{L^q(\tilde{\Omega})},$$

or

$$\| \|_{\gamma, \Omega^c} + \| \|_{L^p(\Omega^c)} + \| \|_{L^q(\Omega^c)},$$

where $p = \frac{2}{1-\gamma}$ and $q = \frac{2}{1+\gamma}$ is the dual exponent, then

$$\|\phi^{(1)}\| = \|\varpi_0\| \overset{\text{def}}{=} \frac{1}{2} \alpha,$$

and

$$\|\psi^{(1)}\| = 0.$$

Using that $\| \|$ is a multiplicative norm, the remaining terms have the control

$$\|\phi^{(s+1)}\| \leq \frac{1}{s+1} \|\varpi_0\| \left(\frac{2\|\varpi_0\|}{s!}\right)^s$$

$$+ \frac{1}{s+1} \sum_{k=0}^{s-1} (k+1) \left\{\|\phi^{(k+1)}\| \|\phi^{(s-k)}\| + \|\Phi[\phi^{(k+1)}]\| \|\Phi[\phi^{(s-k)}]\|\right\}$$

$$+ \frac{1}{s+1} \sum_{k=0}^{s-1} (k+1) \left\{\|\Phi[\phi^{(k+1)}]\| \|\Phi[\psi^{(s-k)}]\| \right.$$}

$$+ \|\Phi[\psi^{(k+1)}]\| \|\Phi[\phi^{(s-k)}]\| + \|\Phi[\psi^{(k+1)}]\| \|\Phi[\psi^{(s-k)}]\| \right\},$$
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\[ \|\psi^{(s+1)}\| \leq \frac{1}{s+1} \sum_{k=0}^{s-1} (k+1) \left\{ \|\psi^{(s-1)}\| \|\psi^{(s-2)}\| + \|\Psi[\psi^{(s-1)}]\| \|\Psi[\psi^{(s-2)}]\| \right\} \\
+ \frac{1}{s+1} \sum_{k=0}^{s-1} (k+1) \left\{ \|\Psi[\phi^{(s-1)}]\| \|\Psi[\phi^{(s-2)}]\| \\
+ \|\Psi[\phi^{(s-1)}]\| \|\Psi[\phi^{(s-2)}]\| + \|\Psi[\phi^{(s-1)}]\| \|\Psi[\phi^{(s-2)}]\| \right\}. \]

If \( f = \phi^{(s)} \) or \( \psi^{(s)} \), using Theorem 3.2, we have that \( \|\Phi[f]\| \) and \( \|\Psi[f]\| \leq K \|f\| \), and then, using the notation \( \alpha_p = \|\phi^{(p)}\| \) and \( \beta_p = \|\psi^{(p)}\| \),

\[ \alpha_{s+1} \leq \frac{1}{s+1} \frac{\alpha^s}{s!} + \frac{1}{s+1} \sum_{k=0}^{s-1} (k+1) \alpha_{k+1} \alpha_{s-k} \\
+ K^2 \frac{1}{s+1} \sum_{k=0}^{s-1} (k+1) \{ \alpha_{k+1} \beta_{s-k} + \beta_{k+1} \alpha_{s-k} + \alpha_{k+1} \beta_{s-k} \}, \]

and

\[ \beta_{s+1} \leq (1 + K^2) \frac{1}{s+1} \sum_{k=0}^{s-1} (k+1) \beta_{k+1} \beta_{s-k} \\
+ K^2 \frac{1}{s+1} \sum_{k=0}^{s-1} (k+1) \{ \alpha_{k+1} \beta_{s-k} + \beta_{k+1} \alpha_{s-k} + \alpha_{k+1} \alpha_{s-k} \}. \]

Now, we have that if we define

\[ A_s = \sum_{k=0}^{s-1} (k+1) \alpha_{k+1} \alpha_{s-k}, \]
\[ B_s = \sum_{k=0}^{s-1} (k+1) \beta_{k+1} \beta_{s-k} \]

and

\[ C_s = \sum_{k=0}^{s-1} (k+1) \{ \alpha_{k+1} \beta_{s-k} + \beta_{k+1} \alpha_{s-k} \}, \]

then we have

\[ (s+1) \alpha_{s+1} \leq \frac{1}{2} \frac{\alpha^{s+1}}{s!} + (1 + K^2) A_s + K^2 (C_s + B_s) \]

and

\[ (s+1) \beta_{s+1} \leq (1 + K^2) B_s + K^2 (C_s + A_s). \]
Let us consider, next, the polynomic functions

\[ f_1(\xi) = \alpha_2 \xi, \]
\[ g_1(\xi) = 0, \]
\[ f_N(\xi) = \sum_{p=0}^{N} \alpha_p \xi^p, \]

and

\[ g_N(\xi) = \sum_{q=0}^{N} \beta_q \xi^q, \]

where \( \alpha_0 = \beta_0 = 0. \)

Then the previous inequalities imply that, for \( \xi \in [0, \infty), \)

\[ f'_N(\xi) + 1 \leq \frac{1}{2} \sum_{s=0}^{N} (s+1) \alpha_{s+1} \xi^s \]
\[ \leq \frac{1}{2} \sum_{s=0}^{N} \alpha_{s+1} \xi^s + (1 + K^2) \sum_{s=0}^{N} A_s \xi^s + K^2 \sum_{s=0}^{N} (C_s + B_s) \xi^s \]

and

\[ g'_N(\xi) = \sum_{s=0}^{N} (s+1) \beta_{s+1} \xi^s \leq (1 + K^2) \sum_{s=0}^{N} B_s \xi^s + K^2 \sum_{s=0}^{N} (C_s + A_s) \xi^s. \]

Then, we have that

\[ \sum_{s=0}^{N} A_s \xi^s \leq f'_N f_N, \]

because

\[ f'_N(\xi) f_N(\xi) = \left( \sum_{q=0}^{N-1} (q+1) \alpha_{q+1} \xi^q \right) \left( \sum_{p=0}^{N} \alpha_p \xi^p \right) = \sum_{q=0}^{N-1} \sum_{p=0}^{N} (q+1) \alpha_{q+1} \alpha_p \xi^{q+p} \]
\[ = \sum_{s=0}^{2N-1} \left( \sum_{q=0}^{N-1} (q+1) \alpha_{q+1} \alpha_{s-q} \right) \xi^s \geq \sum_{s=0}^{N} \left( \sum_{k=0}^{s-1} (k+1) \alpha_{k+1} \alpha_{s-k} \right) \xi^s \]
\[ = \sum_{s=0}^{N} A_s \xi^s \]

with a similar argument we get

\[ \sum_{s=0}^{N} B_s \xi^s \leq g'_N g_N \]

and

\[ \sum_{s=0}^{N} C_s \xi^s \leq f'_N g_N + f_N g'_N, \]
so

\[ f_{N+1}(\xi) \leq \frac{1}{2} \sum_{s=0}^{N} \frac{\alpha^{s+1}}{s!} \xi^s + (1 + K^2) \left( \frac{1}{2} f_N^2 \right)' + K^2 \left( \left( \frac{1}{2} g_N^2 \right)' + (f_N g_N)' \right) \]

and

\[ g_{N+1}(\xi) \leq (1 + K^2) \left( \frac{1}{2} g_N^2 \right)' + K^2 \left( \left( \frac{1}{2} f_N^2 \right)' + (f_N g_N)' \right), \]

and integrating

\[ f_{N+1}(\xi) \leq c_N + \frac{1}{2} \sum_{s=0}^{N} \frac{\alpha^{s+1}}{(s+1)!} \xi^{s+1} + \frac{1 + K^2}{2} f_N^2 + K^2 \left( \frac{1}{2} g_N^2 + f_N g_N \right) \]

and

\[ g_{N+1}(\xi) \leq d_N + \frac{1 + K^2}{2} g_N^2 + K^2 \left( \frac{1}{2} f_N^2 + f_N g_N \right). \]

Adding these two inequalities and considering that \( f_N(0) = g_N(0) = 0 \), we have that

\[ (f_{N+1} + g_{N+1})(\xi) \leq \frac{\alpha}{2} e^{\alpha \xi} + \frac{1 + 2 K^2}{2} (f_N + g_N)^2. \]

Then, if \( h_N = f_N + g_N \), we have

\[ h_{N+1} \leq \frac{\alpha}{2} e^{\alpha \xi} + \frac{1 + 2 K^2}{2} h_N^2, \quad (19) \]

and then

**Proposition 5.** For any \( N \in \mathbb{N} \setminus \{0\} \), let \( h_N \) a sequence of positive functions satisfying (19) and \( h_N(0) = 0 \), we have

\[ h_N(\xi) \leq \frac{1}{1 + 2 K^2} \]

for \( \xi \in \left[ 0, \frac{1}{\alpha(1 + 2 K^2)} e^{-\frac{2}{1 + 2 K^2}} \right] \).

**Proof.** We proceed by induction in \( N \).

Assume that \( \xi \in [0, M] \) for some \( M > 0 \).

We have that \( h_1(\xi) = \frac{\alpha}{2} \xi \leq \frac{\alpha}{2} M \), so the choice of \( M = \frac{2}{\alpha(1 + 2 K^2)} \) implies the result for \( h_1 \).

From now on we consider this choice of \( M \). This means that \( \xi = \nu \frac{2}{\alpha(1 + 2 K^2)} \), for \( \nu \in [0, 1] \). Then, if we assume that for some given \( N \), \( h_N(\xi) \leq \frac{1}{1 + 2 K^2} \), the inequality 19, implies that

\[ h_{N+1} \leq \frac{\alpha}{2} e^{\nu \frac{2}{\alpha(1 + 2 K^2)}} e^{\frac{1 + 2 K^2}{2} (\frac{\nu}{1 + 2 K^2})^2} + \frac{1 + 2 K^2}{2} \left( \frac{1}{1 + 2 K^2} \right)^2 \]

\[ = \frac{\nu}{1 + 2 K^2} e^{\frac{2 \nu}{1 + 2 K^2}} + \frac{1}{2 (1 + 2 K^2)} \]

\[ = \frac{1}{1 + 2 K^2} \left( \nu e^{\frac{2 \nu}{1 + 2 K^2}} + \frac{1}{2} \right), \]

and if \( \nu \in [0, \frac{1}{2} e^{-\frac{2}{1 + 2 K^2}}] \) then \( \nu e^{\frac{2 \nu}{1 + 2 K^2}} \leq \frac{1}{2} \) and this proves the result. \( \square \)
Back to the decomposition (18) and the fact that \( \alpha = 2 \| \varpi_0 \| \), the previous proposition implies that

\[
\sum_{s=0}^{N} \left\| \frac{\partial \xi(s)}{\partial z} \right\| \tau^s \leq h_N(\tau) \leq \frac{2}{1 + 2K^2}
\]

for every \( N \) and \( \tau \in [0, \frac{1}{\alpha(1 + 2K^2)} e^{-\frac{4}{1 + 2K^2}}) \).

Since the functions \( \frac{\partial \xi(s)}{\partial z} \) are in \( \mathcal{C}^{1, \gamma} \) and, as the recurrence shows, they exhibit a decay at \( \infty \) of order \( \frac{1}{|z|^2} \), then

\[
\xi(s) = \tilde{C} \left[ \frac{\partial \xi(s)}{\partial z} \right]
\]

is in \( \mathcal{C}^{1, \gamma} \) (cf. Theorems 4.3.11 and 4.3.12 in [1]). Moreover

\[
\|\xi(s)\|_{L^\infty} \leq K_0 \left\| \frac{\partial \xi(s)}{\partial z} \right\|_{\gamma}.
\]

Consequently, for any \( N \),

\[
\left| \sum_{s=0}^{N} \xi(s)(z) t^s \right| \leq \sum_{s=0}^{N} \|\xi(s)\|_{L^\infty} |t|^s \leq K_0 \sum_{s=0}^{N} \left\| \frac{\partial \xi(s)}{\partial z} \right\| |t|^s
\]

and the last term is uniformly bounded in \( N \), for \( |t| \leq \frac{1}{\| \varpi_0 \| (1 + 2K^2)} e^{-\frac{4}{1 + 2K^2}} \).

This implies the existence of an analytic solution of the equation (5), providing the proof of Theorem 1.2.

### 3.2. Proof of Proposition 1

We have seen that

\[
\phi(z, t) = z + \sum_{s=1}^{\infty} \xi(s)(z) t^s \overset{\text{def}}{=} z + A(z, t)
\]

and

\[
\xi(s)(z) = \tilde{C} \theta(s)(z).
\]

Let us prove first an auxiliary lemma.

**Lemma 3.2.** There exists \( T_0 > 0 \) and \( K > 0 \) such that for \( |t| \leq T_0 \), we have

\[
\|A(\cdot, t)\|_{L^\infty(\mathbb{C})} \leq K
\]

and

\[
\|\nabla_z A(\cdot, t)\|_{L^\infty(\mathbb{C})} \leq \frac{1}{2}.
\]

**Proof.** Since

\[
\sum_{s=1}^{N} \left\| \xi(s) \right\|_{L^\infty(\mathbb{C})} \tau^s = \sum_{s=1}^{N} \left\| \tilde{C} \theta(s) \right\|_{L^\infty(\mathbb{C})} \tau^s
\]

and

\[
\|\tilde{C} \theta(s)\|_{L^\infty(\mathbb{C})} \leq C \{ \|\theta(s)\|_{L^\infty(\mathbb{C})} + \|\theta(s)\|_{L^p(\mathbb{C})} \}
\]

with \( C \) independent of \( s \) and \( N \), and \( 1 < p < 2 \), then by Proposition 5, we obtain

\[
\|A(\cdot, t)\|_{L^\infty(\mathbb{C})} \leq K.
\]
On the other hand, using Theorem, we have
\[
\sum_{s=1}^{N} \| \nabla z(t) \|_{L^\infty(C)} \, \tau^s \leq \sum_{s=1}^{N} \{ \| \theta(t) \|_{L^\infty(C)} + \| B(\theta(t)) \|_{L^\infty(C)} \} \, \tau^s
\]
\[
\leq K \sum_{s=1}^{N} \| \theta(t) \|_{\gamma} \, \tau^s
\]
where \( K \) is independent of \( s \) and \( N \).

3.2.1. \( \phi \) is a flow.

**Proposition 6.** The function \( \phi \) defined in (20) satisfies
1. \( \phi(C, t) \) is closed for every \( t \).
2. \( \phi(C, t) \) is one to one.
3. \( \phi(C, t) = C \).
4. \( \phi(C, t) \) is an homeomorphism and in \( C^\infty(C \setminus \partial \Omega) \).

**Proof.**
1. If \( w_1 = \phi(z_1, t) = z_1 + A(z_1, t) \) is a sequence such that \( w_1 \to w_0 \), then
   \[
   |z| \leq |w_1| + |A(z_1, t)| \leq M + \| A(\cdot, t) \|_{L^\infty},
   \]
   where \( M \) is a bound for \( w_1 \). Then there exists a partial sequence \( z_1, \ldots, z_n \to z_0 \) and since \( \phi(\cdot, t) \) is a continuous map then \( w_0 = \phi(z_0, t) \).
2. Since
   \[
   \phi(z, t) - \phi(z', t) = z - z' + A(z, t) - A(z', t),
   \]
   and \( \nabla A(\cdot, t) \) exists almost everywhere in \( C \) and \( \| \nabla A(\cdot, t) \|_{L^\infty} < 1 \) if \( |t| \leq T_0^m \), then by Rademacher's theorem \( A(\cdot, t) \in \text{Lip}(1, C) \), so if \( \phi(z, t) - \phi(z', t) = 0 \), then \( |z - z'| < |z - z'| \). That's a contradiction.
3. By the theorem of invariance of domains in \( C \) (it is in fact true for \( \mathbb{R}^n \), for every dimension \( n \) (cf. [10])), the injectivity of \( \phi \) implies that \( \phi(\mathbb{R}^n) \) is an open set. Since it is also closed, we have that \( \phi(C) = C \) and then \( \phi \) is an homeomorphism.
4. For some \( \tilde{T} \in (0, T_0^m] \) we have \( \| D_x A(z, t) \| < \frac{1}{3} \) for \( (x, t) \in (C \setminus \partial \Omega) \times [\tilde{T}, \tilde{T}] \), and \( \phi(C, t) \in C^\infty(C \setminus \partial \Omega) \), then \( \phi(C \setminus \partial \Omega, t) \) is open and \( \phi(C, t) \) is a diffeomorphism onto the image, because of the inverse function theorem and the injectivity.

4. **Proof of Theorem 3.2.** This is the most technical section of the paper. We divide it in several subsections, where we exhibit some structural and geometric facts about the Beurling transform, necessary for our purposes, and then we perform the uniform and Lipschitz estimates, necessary for the proof of Theorem 1.2.

Let us consider a function \( f \) defined on a domain \( W \subset \mathbb{C} \) with bounded \( C^1, \gamma \)-regular boundary. We will also denote by \( f \) the extension of \( f \) by 0 to \( \bar{W} \), and \( f(z) = \lim_{w \to z} f(w) \) if \( z \in \partial W \).

Whenever it be necessary, we will specialize \( f = \phi \) and then \( W = \text{supt}(\phi) \), the support of \( \phi \), or \( f = \psi \) and \( W = \text{supt}(\psi) \), respectively.

4.1. **The geometric lemma.** The metric and geometric properties of \( \partial \Omega \) play a crucial role in the behaviour of the Beurling transform. The next lemma is a synthesis of these properties.
Lemma 4.1. Let \( \Omega \subset \mathbb{C} \) be a bounded domain such that \( \partial \Omega \in C^{1,\gamma} \), defined by a function \( \rho \).

There exists \( 0 < R_0 < 1 \) such that if
\[
U_{R_0}(\partial \Omega) := \bigcup_{z \in \partial \Omega} B_{R_0}(z),
\]
then there exists \( R_1 \) such that for any \( z_0 \in U_{R_0} \) the level set \( \{ \rho = \rho(z_0) \} \cap B_{R_1}(z_0) \) coincides with the graph of a function \( \varphi_{z_0} \).

Moreover \( \varphi_{z_0} \) is \( C^{1,\gamma} \) and \( \varphi_{z_0}(0) = \varphi'_{z_0}(0) = 0 \).

Remark 2. The function \( \varphi_{z_0} \) is defined on a segment of the tangent line to the level set of \( \rho \) across \( z_0 \).

Then
\[
\rho(z_0 + s \eta(z_0) + \varphi_{z_0}(s) \eta(z_0)) = \rho(z_0).
\]

Proof. Let \( \rho \in C^{1,\gamma} \) be the defining function for \( \Omega \), as in [5]. Since \( \partial \Omega \) is a compact set, let
\[
m = \min\{ \| \nabla \rho(z) \| : z \in \partial \Omega \}.
\]

Since \( \nabla \rho \in \text{Lip}(\gamma) \), then for
\[
R = \left( \frac{m}{2 \sqrt{2} \| \nabla \rho \|_\gamma} \right)^{\frac{1}{\gamma}},
\]
we define \( U_R = \bigcup_{\zeta \in \partial \Omega} B_R(\zeta) \). It is an open neighbourhood of \( \partial \Omega \), and for every \( z \in U_R \),
\[
\| \nabla \rho(z) \| \geq \frac{m}{2}.
\]

Fix \( z_0 \in U_R \). The vectors \( \eta(z_0) = \frac{\nabla \rho(z_0)}{\| \nabla \rho(z_0) \|} \) and \( \tau(z_0) = \eta(z_0) \perp \) form an orthonormal basis of \( T_{z_0}(\mathbb{R}^2) \), and the map
\[
\Xi: \mathbb{R}^2 \to \mathbb{C}
\]
given by
\[
\Xi(s, t) = z_0 + s \tau(z_0) + t \eta(z_0)
\]
is a rigid movement.

The function
\[
r(s, t) = \rho \circ \Xi(s, t)
\]
satisfies
\[
\nabla r(0, 0) = \| \nabla \rho(z_0) \| e_2.
\]

Let us consider the map
\[
\Psi(s, t) = \left( s, \frac{r(s, t) - r(0, 0)}{\| \nabla \rho(z_0) \|} \right).
\]

We have that \( \Psi(0, 0) = (0, 0) \) and
\[
\det J\Psi(0, 0) = \frac{\partial r(z_0)}{\partial \tau}(\xi) \left( \| \nabla \rho(z_0) \| \right).
\]

Next, \( \Psi(s, t) = \Psi(s', t') \) if \( s = s' \) and \( r(s, t) = r(s', t') \), then, for some \( \xi \) in the open interval delimited by \( t \) and \( t' \), we have
\[
0 = r(s, t) - r(s, t') = \frac{\partial r}{\partial t}(s, \xi)(t - t')
\]
\[
= \frac{\partial r}{\partial t}(0, 0)(t - t') + \left( \frac{\partial r}{\partial t}(s, \xi) - \frac{\partial r}{\partial t}(0, 0) \right)(t - t'),
\]
where \( \xi \) is the point where the partial derivative is evaluated.


so

\[ 0 \geq \left| \frac{\partial r}{\partial t}(0,0) \right| |t-t'| - \left\| \frac{\partial r}{\partial t} \right\|_{\text{Lip}(\gamma)} |t-t'|^{1+\gamma} \]

\[ = \left\{ \left\| \nabla \rho(z_0) \right\| - \left\| \frac{\partial r}{\partial t} \right\|_{\text{Lip}(\gamma)} |t-t'|^{\gamma} \right\} |t-t'|, \]

and if we choose \( R_1 \) in such a way that if \( z_0 \in \overline{U}_{R_1} \), then \( B_{R_1}(z) \subset U_{\frac{R}{2}} \) and \( R_1 \leq \left( \frac{\left\| \nabla \rho(z_0) \right\|}{\left\| \frac{\partial r}{\partial t} \right\|_{\text{Lip}(\gamma)}} \right)^{\frac{1}{\gamma}} \), then

\[ 0 \geq \frac{1}{2} \left\| \nabla \rho(z_0) \right\| |t-t'|. \]

So \( \Psi \) is one-to-one in \( B_{R_1}(0,0) \).

Choose, now, \( 0 < R_2 < R_1 \). Then, if \( s^2 + t^2 = R_2^2 \), we have

\[ \| \Psi(s,t) \|^2 = s^2 + \frac{(r(s,t) - r(0,0))^2}{\left\| \nabla \rho(z_0) \right\|^2} \]

\[ = s^2 + \left( t + \frac{(\nabla r(s',t') - \nabla r(0,0), (s,t))}{\left\| \nabla \rho(z_0) \right\|} \right)^2 \]

\[ \geq R_2^2 \left\{ 1 + \left( \frac{(\nabla r(s',t') - \nabla r(0,0), u)}{\left\| \nabla \rho(z_0) \right\|} \right)^2 \right. \]

\[ - 2 \left| (u, e_2) \| (\nabla r(s',t') - \nabla r(0,0), u) \right| \right\} = (*), \]

where \( u \) is a unitary vector. Moreover

\[ (* \geq R_2^2 \left\{ 1 - 2 \left\| \nabla \rho \right\|_{\text{Lip}(\gamma)} \right\}, \]

and if

\[ R_2 \leq \left( \frac{\left\| \nabla \rho(z_0) \right\|}{4 \left\| \nabla r \right\|_{\text{Lip}(\gamma)}} \right)^{\frac{1}{\gamma}}, \]

then

\[ d(\Psi(\partial B_{R_2}(0,0)), (0,0)) \geq \frac{R_2}{\sqrt{2}}. \]

This implies that for \( (p,q) \in B_{R_2}(0,0) \), the function \( g(s,t) = \| \Psi(s,t) - (p,q) \|^2 \)

has a local minimum at some point \( (s_0, t_0) \in B_{R_2}(0,0) \), and then

\[ \left\{ 2(s_0 - p) + 2\left( \frac{r(s_0,t_0) - r(0,0)}{\left\| \nabla \rho(z_0) \right\|} - q \right) \frac{\partial r}{\partial t}(s_0,t_0) = 0, \right. \]

\[ 2\left( \frac{r(s_0,t_0) - r(0,0)}{\left\| \nabla \rho(z_0) \right\|} - q \right) \frac{\partial r}{\partial s}(s_0,t_0) = 0. \]

And we can change \( R_2 \) by \( R_3 \) so that \( \frac{\partial r}{\partial t} \) does not vanish at \( B_{R_3}(0,0) \) and the other properties still hold.

So, there is a new choice of the region \( U \) such that \( \| \nabla \rho \| \) is uniformly bounded from below on \( U \), and then there are choices of \( R', R'' > 0 \) such that for every \( z_0 \in U \) the corresponding map \( \Psi \) composed with the rigid movement describe above map \( C^{1,\gamma} \)-diffeomorphically \( B_{R'}(z_0) \) in \( B_{R''}(z_0) \).
If \( \Phi = \Psi^{-1} \), then \( \Phi \) is a \( C^{1,\gamma} \)-diffeomorphism such that \( \Phi(0, 0) = (0, 0) \). Moreover \( \Psi_1(s, t) = s \) and
\[
\Psi_2\{(s, t) : r(s, t) = r(0, 0)\} = 0,
\]
so the level set \( \{r(s, t) = r(0, 0)\} \cap B_{R^2}(0, 0) \) coincides with the set \( \{t = \Phi_2(s, 0)\} \).

Taking a convenient rectangle inside the balls, we have the function \( \varphi \) in the statement of the theorem.

Finally, since essentially \( \varphi(s) = \Phi_2(s, 0) \), we have that
\[
\varphi'(0) = \frac{\partial \Phi_2}{\partial p}(0, 0),
\]
and since
\[
J\Phi(0, 0) = (J\Phi(0, 0))^{-1}
\]
we have \( \varphi'(0) = 0 \).

Using the rigid movement above, we can take the construction backward to \( \mathbb{C} \) and get the theorem.

\( \square \)

4.2. Decomposition of the singularities. Set, for any \( z \in \mathbb{C} \), \( d(z) = d(z, \partial \Omega) \) and \( \delta(z) = \max\{d(z), \frac{R_0}{\epsilon}\} (R_0 \text{ as in Lemma 4.1}) \).

**Proposition 7.** Let \( \Omega \subset \mathbb{C} \) be a bounded domain with boundary of class \( C^{1,\gamma} \) and let \( W \) denote \( \Omega \) or \( \mathbb{C} \setminus \Omega \), and if \( z \in \partial \Omega \) and \( \eta(z) \) is the normal vector exterior to \( \Omega \), then \( \kappa_2(\zeta) = (\zeta - z)\eta(z) \) (the polynomial of first degree defining the tangent line to \( \partial \Omega \) at the point \( z \)).

If \( f \in C^r(W) \cap L^p(W) \) with \( p > 1 \) and we identify \( f \) with its extension by 0 outside \( W \), then there exists
\[
\bar{B}[f](z) = \lim_{\epsilon \to 0} B_\epsilon[f](z)
\]
for every \( z \in \mathbb{C} \).

Moreover
\[
\bar{B}[f](z) = Q[f](z) + L[f](z) + f(z) \Theta_\frac{\eta}{\kappa_2}(z),
\]
where
\[
Q[f](z) = \int_{\mathbb{C} \setminus B_{d(z)}(z)} f(\zeta) \frac{1}{(\zeta - z)^2} dm(\zeta),
\]
\[
L[f](z) = \int_{B_{d(z)}(z)} \frac{f(\zeta) - f(z)}{|\zeta - z|^\gamma} \frac{|\zeta - z|^\gamma}{(\zeta - z)^2} dm(\zeta),
\]
and
\[
\Theta_\frac{\eta}{\kappa_2}(z) = \begin{cases} 
0 & \text{if } d(z) > 0, \\
\{\int_{B_{\frac{\eta}{|\kappa_2|}(z) \cap W \cap \{\kappa_2 > 0\}}} - \int_{B_{\frac{\eta}{|\kappa_2|}(z) \cap W \cap \{\kappa_2 < 0\}}} \} \frac{dm(\zeta)}{(\zeta - z)^2} & \text{if } d(z) = 0.
\end{cases}
\]

**Remark 3.** The term \( \Theta_\frac{\eta}{\kappa_2}(z) \) is an intrinsic geometric object. Sometimes we will also use it in the form
\[
\int_{B_{\frac{\eta}{|\kappa_2|}(z) \cap W}} \frac{dm(\zeta)}{(\zeta - z)^2}.
\]

**Remark 4.** In order to avoid notation, from now on we will use in the proofs the notation \( b_\zeta(\zeta) \) instead of \( \frac{dm(\zeta)}{(\zeta - z)^2} \).
Remark 5. Since the Beurling transform is a classical Calderón–Zygmund operator, the existence of the principal value is well known for functions in many different classes (See [11, Corollary 5.8]), nevertheless, we need here the existence of the principal value at each point in the plane, for functions in the aforementioned class.

Proof. Let $z \in \mathbb{C}$.

- If $d(z) > 0$, then for $0 < \epsilon < d(z)$ we have

$$B_\epsilon[f](z) = \int_{\mathbb{C} \setminus B_\epsilon(z)} f b_z = \int_{\mathbb{C} \setminus B(d(z))} f b_z + \int_{B(d(z)) \setminus B_\epsilon(z)} f b_z.$$

- If $z \notin W$, the second term is 0 and we have

$$B[f](z) = \lim_{\epsilon \to 0} B_\epsilon[f](z) = \int_{\mathbb{C} \setminus B(d(z))} f b_z.$$

- If $z \in W$, we have

$$B_\epsilon[f](z) = \int_{\mathbb{C} \setminus B(d(z))} f b_z + \int_{B(d(z)) \setminus B_\epsilon(z)} (f - f(z)) b_z$$

$$= \int_{\mathbb{C} \setminus B(d(z))} f b_z + \int_{B(d(z)) \setminus B_\epsilon(z)} \frac{f(\zeta) - f(z)}{|\zeta - z|^\gamma} \frac{|\zeta - z|}{(\zeta - \bar{z})^2} dm(\zeta),$$

and in the second term we have an integrable function, so the limit exists

$$B[f](z) = \lim_{\epsilon \to 0} B_\epsilon[f](z)$$

$$= \int_{\mathbb{C} \setminus B(d(z))} f b_z + \int_{B(d(z))} \frac{f(\zeta) - f(z)}{|\zeta - z|^\gamma} \frac{|\zeta - z|}{(\zeta - \bar{z})^2} dm(\zeta).$$

If $d(z) \leq \frac{R_0}{2}$, then

$$\int_{\mathbb{C} \setminus B(d(z))} f b_z = \int_{\mathbb{C} \setminus B \frac{R_0}{2}(z)} f b_z + \int_{B \frac{R_0}{2}(z) \setminus B(d(z))} f b_z,$$

and applying again the cancellation lemma (below), we have that the second integral is

$$\int_{C \frac{R_0}{2}(z)} \frac{f(\zeta) - f(z)}{|\zeta - z|^\gamma} \frac{|\zeta - z|}{(\zeta - \bar{z})^2} dm(\zeta),$$

where $C \frac{R_0}{2}(z) = B \frac{R_0}{2}(z) \setminus B(d(z))$.

Implementing these facts in the formula for $B[f](z)$ above, we have the result.

- If $d(z) = 0$, then we for any $\epsilon < \frac{R_0}{2}$ we have

$$B_\epsilon[f](z) = \int_{\mathbb{C} \setminus B \frac{R_0}{2}(z)} f b_z + \int_{C \frac{R_0}{2}(z)} f b_z.$$
Lemma 4.2 (Beurling-Geometric lemma). If $\Omega \subset \mathbb{C}$ and $\partial \Omega$ is compact and $C^{1,\gamma}$, then there exists

$$\lim_{\epsilon \to 0} \int_{C_{\epsilon}^{\infty}} \frac{dm(\zeta)}{(\zeta - z)^2} = \left\{ \int_{B_{\frac{\rho}{2}}(z) \cap \Omega \cap \{\kappa_\gamma > 0\}} - \int_{B_{\frac{\rho}{2}}(z) \cap \Omega \cap \{\kappa_\gamma < 0\}} \right\} \frac{dm(\zeta)}{(\zeta - z)^2}.$$
Lemma 4.3 (Cancellation lemma). If $z \in B_{R_0}(w)$ and $\epsilon < R_0 - |z - w|$, then
\[
\int_{B_{R_0}(w) \setminus B_{\epsilon}(z)} \frac{dm(\zeta)}{(\zeta - \overline{z})^2} = 0.
\]

Proof. In complex coordinates, the integral is
\[
\frac{1}{2\pi} \int_{\partial B_{R_0}(w) \setminus \partial B_{\epsilon}(z)} \frac{d\zeta \wedge d\zeta}{(\zeta - \overline{z})^2} = \frac{i}{2} \left( \int_{\partial B_{R_0}(w)} - \int_{\partial B_{\epsilon}(z)} \right) \frac{d\zeta}{\zeta - \overline{z}},
\]
and
\[
\int_{\partial B_{\epsilon}(z)} \frac{d\zeta}{\zeta - \overline{z}} = \int_0^{2\pi} i e^{2i\theta} d\theta = 0,
\]
and
\[
\int_{\partial B_{R_0}(w)} \frac{d\zeta}{\zeta - \overline{z}} = \int_0^{2\pi} \frac{i R_0 e^{i\theta} d\theta}{R_0 e^{-i\theta} + w - \overline{z}} = R_0 \int_{\partial B_{\epsilon}(0)} \frac{\tau d\tau}{R_0 + (w - z) e^{i\theta}} = 0,
\]
because $\frac{\tau}{R_0 + (w - z) e^{i\theta}}$ is a function holomorphic in $\tau$, as $|\frac{R_0}{w - z}| > 1$.

4.3. The jump formula for $\tilde{B}$. Now we prove the jump formula
\[
\tilde{B}[g](z) = \frac{1}{2} \left\{ \lim_{w \to z; w \in \Omega} \chi_{\Omega} \tilde{B}[g](w) + \lim_{w \to z; w \in \mathbb{C} \setminus \Omega} \chi_{\mathbb{C} \setminus \Omega} \tilde{B}[g](w) \right\}
\]
that is identity (14) in Theorem 3.2.

Remark 6. Jump formulas of this type, for Calderón–Zygmund operators in potential theory, appear in [16]. For the special case of the conjugate Beurling transform we give a simple proof in order to keep the exposition more self-contained.

As we will see in section 4.4 and 4.5 we have that $\chi_{\Omega} \tilde{B}[g] \in \text{Lip}_p(\Omega)$ and $\chi_{\mathbb{C} \setminus \Omega} \tilde{B}[g] \in \text{Lip}_p(\mathbb{C} \setminus \Omega)$, and then the limits $\lim_{w \to z; w \in \Omega} \chi_{\Omega} \tilde{B}[g](w)$ and $\lim_{w \to z; w \in \mathbb{C} \setminus \Omega} \chi_{\mathbb{C} \setminus \Omega} \tilde{B}[g](w)$ both exist and we can choose $w = z \pm \eta(z)$, where $\eta(z)$ is the unit vector, normal exterior to $\partial \Omega$ at $z$. 
Also, if \( g_{\pm} \) are the Lipschitz extensions of \( g \) to \( \Omega^c \) and \( \bar{\Omega} \), respectively, we have, for \( w \in B_{2 |z-w|}(z) \), the following facts

- If \( w \in \bar{\Omega} \), then
  \[
  \bar{B}[g_-](w) = \int_{\Omega} (g_- - g_- (w)) b_w + g_- (w) \int_{\Omega} b_w \\
  = \int_{\Omega \setminus B_{2 |z-w|}(z)} (g_- - g_- (w)) b_w \\
  + \int_{\Omega \cap B_{2 |z-w|}(z)} (g_- - g_- (w)) b_w + g_- (w) \int_{\Omega} b_w \\
  = (I)(w) + (II)(w) + g_- (w) (III)(w).
  \]

For the integral \( (I)(w) \), we have immediately that for any fixed \( \zeta \),

\[
\chi_{\Omega \setminus B_{2 |z-w|}(z)}(\zeta) (g_- (\zeta) - g_- (w)) b_w (\zeta) \to_{w \to z} \chi_{\Omega}(\zeta) (g_- (\zeta) - g_- (z)) b_z (\zeta),
\]

and also

\[
|\chi_{\Omega \setminus B_{2 |z-w|}(z)}(\zeta) (g_- (\zeta) - g_- (w)) b_w (\zeta)| \leq \|g_-\|_{\text{Lip}(\gamma, \bar{\Omega})} \frac{2}{\gamma} |\zeta - z|^{2-\gamma},
\]

and by the dominated convergence theorem,

\[
(I)(w) \to_{w \to z} \int_{\Omega} (g_- - g_- (z)) b_z.
\]

The term \( (II)(w) \) is controlled by

\[
\|g_-\|_{\text{Lip}(\gamma, \bar{\Omega})} \int_{\Omega \cap B_{2 |z-w|}(z)} \frac{1}{|\zeta - w|^{2-\gamma}} dm(\zeta)
\]

and the last integral is bounded by

\[
\int_{\Omega \cap B_{2 |z-w|}(z)} \frac{1}{|\zeta - w|^{2-\gamma}} dm(\zeta) \leq 2\pi \int_0^{3 |z-w|} \frac{1}{r^{1-\gamma}} dr = 2\pi 3^\gamma |z-w|^{\gamma},
\]

and then

\[
(II)(w) \to_{w \to z} 0.
\]

Also, if \( \varrho \in \mathcal{D}(\mathbb{C}) \) such that \( \varrho \equiv 1 \) in a ball containing \( \bar{\Omega} \), then

\[
\bar{B}[g_+] (w) = \bar{B}[(1 - \varrho) g_+] (w) + \bar{B}[\varrho g_+] (w)
\]

\[
= (IV)(w) + \int_{\mathcal{C}_q(\Omega \cup B_{2 |z-w|}(z))} \varrho (g_+ - g_+ (z)) b_w \\
+ \int_{\bar{\Omega} \cup B_{2 |z-w|}(z)} \varrho (g_+ - g_+ (z)) b_w + g_+ (z) \int_{\bar{\Omega} \cup B_{2 |z-w|}(z)} \varrho b_w
\]

\[
= (IV)(w) + (V)(w) + (VI)(w) + g_+ (z) (VII)(w).
\]

It is an immediate fact that

\[
\lim_{w \to z} (IV)(w) = \bar{B}[(1 - \varrho) g_+](z).
\]

By similar arguments to those used for \( (I)(w) \), we have that

\[
\lim_{w \to z} (V)(w) = \int_{\bar{\Omega} \cup B_{2 |z-w|}(z)} \varrho (g_+ - g_+ (z)) b_z,
\]
and, analogously to \((II)(w)\), we have that
\[
\lim_{w \to z} (VI)(w) = 0.
\]

So, since all limits exist, we have
\[
\lim_{w \to z; \; w \in \Omega} \chi_{\Omega} \bar{B}[g](w)
\]
\[
= \int_{\Omega} (g_+ - g_-(z)) b_z + \bar{B}[(1 - \varrho) g_+](z)
\]
\[
+ \int_{\Omega^c} \rho (g_+ - g_+(z)) b_z + g_-(z) \lim_{w \to z} \int_{\Omega} b_w + g_+(z) \lim_{w \to z} \int_{\Omega^c} \rho b_w
\]
\[
= \bar{B}[g_-](z) + \bar{B}[g_+](z) + g_-(z) \left\{ \lim_{w \to z} \int_{\Omega} b_w - \int_{\Omega^c} \rho b_w \right\}
\]
\[
+ g_+(z) \left\{ \lim_{w \to z} \int_{\Omega^c} \rho b_w - \int_{\Omega} \rho b_z \right\}.
\]

- If \(w \not\in \bar{\Omega}\), then, in a similar way, we have
\[
\lim_{w \to z; \; w \in \Omega^c} \chi_{\Omega^c} \bar{B}[g](w) = \bar{B}[g](z) + g_-(z) \left\{ \lim_{w \to z} \int_{\Omega} b_w - \int_{\Omega^c} \rho b_z \right\}
\]
\[
+ g_+(z) \left\{ \lim_{w \to z} \int_{\Omega^c} \rho b_w - \int_{\Omega} \rho b_z \right\}.
\]

- For any \(z \in \partial \Omega\) we have
\[
\frac{1}{2} \left\{ \lim_{w \to z; \; w \in \Omega} \chi_{\Omega} \bar{B}[g](w) + \lim_{w \to z; \; w \in \Omega^c} \chi_{\Omega^c} \bar{B}[g](w) \right\} = \bar{B}[g](z)
\]
\[
+ \frac{1}{2} g_-(z) \left\{ \lim_{w \to z; \; w \in \Omega} \int_{\Omega} b_w - \int_{\Omega^c} \rho b_z \right\}
\]
\[
+ \frac{1}{2} g_+(z) \left\{ \lim_{w \to z; \; w \in \Omega^c} \int_{\Omega^c} \rho b_w - \int_{\Omega} \rho b_z \right\}
\]
\[
+ \frac{1}{2} g_-(z) \left\{ \lim_{w \to z; \; z \in \partial \Omega} \int_{\Omega^c} b_w - \int_{\Omega} \rho b_z \right\}
\]
\[
+ \frac{1}{2} g_+(z) \left\{ \lim_{w \to z; \; z \in \partial \Omega} \int_{\Omega} \rho b_w - \int_{\Omega^c} \rho b_z \right\}
\]
\[
= \bar{B}[g](z) + g_-(z) \left( \frac{1}{2} \left\{ \lim_{w \to z; \; w \in \Omega} \int_{\Omega} b_w + \lim_{w \to z; \; z \in \partial \Omega} \int_{\Omega} b_w \right\} - \int_{\Omega^c} \rho b_z \right)
\]
\[
+ g_+(z) \left( \frac{1}{2} \left\{ \lim_{w \to z; \; w \in \Omega} \int_{\Omega^c} \rho b_w + \lim_{w \to z; \; z \in \partial \Omega} \int_{\Omega} \rho b_w \right\} - \int_{\Omega} \rho b_z \right).
\]
Proof. First of all, for \( w \neq \partial W \) and \( \eta = \eta(z) \) be the normal exterior vector at \( z \). Consider the points \( w = z \pm \lambda \eta \).

For \( h \in \mathcal{D}(\mathbb{C}) \) we have

\[
\frac{1}{2} \left\{ \lim_{\lambda \to 0; \: w = z - \lambda \eta} \int_{W} h b_{w} + \lim_{\lambda \to 0; \: w = z + \lambda \eta} \int_{W} h b_{w} \right\} = \int_{W} h b_{z}.
\]

Lemma 4.4. Let \( W \subset \mathbb{C} \) a domain with compact \( C^{1,\gamma} \) boundary.

Let \( z \in \partial W \) and \( \eta = \eta(z) \) be the normal exterior vector at \( z \). Consider the points \( w = z \pm \lambda \eta \).

After a translation and a rotation,

\[
\int_{\partial W \setminus B_{2 |z-w|}(z)} h(\zeta) \frac{d\zeta \wedge d\zeta}{\zeta - \bar{w}} = \int_{\partial W \cap B_{2 |z-w|}(z)} h(\zeta) \frac{d\zeta}{\zeta - \bar{w}}
\]

and

\[
\int_{\partial W \cap B_{2 |z-w|}(z)} \frac{d\zeta}{\zeta - \bar{w}} = \int_{\{z = 0\} \cap B_{2 |z-w|}(z)} \frac{d\zeta}{\zeta - \bar{w}} + \int_{\{z = 0\} \cap B_{2 |z-w|}(z)} \frac{d\zeta}{\zeta - \bar{w}}
\]

and again by the Stokes formula the second term is

\[
(II) = - \left\{ \int_{\{z < 0\} \cap W \cap \partial B_{2 |z-w|}(z)} + \int_{\{z > 0\} \cap W \cap \partial B_{2 |z-w|}(z)} \right\} \frac{d\zeta \wedge d\zeta}{(\zeta - \bar{w})^{2}}
\]

and

\[
(II) = - \left\{ \int_{\{z < 0\} \cap W \cap \partial B_{2 |z-w|}(z)} + \int_{\{z > 0\} \cap W \cap \partial B_{2 |z-w|}(z)} \right\} \frac{d\zeta}{\zeta - \bar{w}}.
\]

All the integrals are well defined for \( w = z \pm \lambda \eta \) and the geometric cancellation lemma and the facts that both \( \{z > 0\} \cap W \cap \partial B_{2 |z-w|}(z) \) and \( \{z < 0\} \cap W \cap \partial B_{2 |z-w|}(z) \) tend to 0 as \( r \) tends to 0, imply that

\[
\lim_{w \to z} (II) = 0.
\]

After a translation and a rotation,

\[
(I) = \int_{-2 |z-w|}^{2 |z-w|} \frac{dx}{x - i\lambda},
\]

and then, by a direct computation

\[
\lim_{w \to z} (I) = i \pi.
\]
Then,

\[ \frac{1}{2} \left\{ \lim_{\lambda \to 0; w = z - \lambda \eta} \int_W h b_w + \lim_{\lambda \to 0; w = z + \lambda \eta} \int_W h b_w \right\} \]

\[ = \frac{1}{2} i \left\{ \int_W \frac{\partial h}{\partial \bar{\zeta}}(\zeta) \frac{d\zeta \wedge d\bar{\zeta}}{\zeta - w} - \frac{1}{2} \lim_{\lambda \to 0} \int_{\partial W \setminus B_{2 \lambda}(z)} h(\zeta) \frac{d\zeta}{\zeta - z - i \lambda} \right\} \]

\[ + \lim_{\lambda \to 0} \int_{\partial W \setminus B_{2 \lambda}(z)} h(\zeta) \frac{d\zeta}{\zeta - z + i \lambda} \right\} \]

\[ = \frac{1}{2} \frac{i}{2} \int \frac{\partial h}{\partial \bar{\zeta}}(\zeta) \frac{d\zeta \wedge d\bar{\zeta}}{\zeta - w} \]

because in the second term

\[ \lim_{w \to z}(I) = -i \pi. \]

Then

\[ (**) = \frac{1}{2} i \left\{ \int_W \frac{\partial h}{\partial \bar{\zeta}}(\zeta) \frac{d\zeta \wedge d\bar{\zeta}}{\zeta - w} \right\} \]

On the other hand,

\[ \int_W h b_z = \lim_{\lambda \to 0} \int_{W \setminus B_{2 \lambda}(z)} h(\zeta) \frac{d\zeta}{(\zeta - z)^2} \]

and by Cauchy–Green’s formula (cf. [17])

\[ I_\lambda = \frac{1}{2} i \left\{ - \int_{\partial W \setminus B_{2 \lambda}(z)} h(\zeta) \frac{d\zeta}{\zeta - z} - \int_{\partial B_{2 \lambda}(z) \cap W} h(\zeta) \frac{d\zeta}{\zeta - z} \right\} \]

and as in the previous developments

\[ I_\lambda \to \lambda \to 0 \frac{1}{2} i \left\{ - \text{p. v.} \int_{\partial W} h(\zeta) \frac{d\zeta}{\zeta - \bar{z}} + \int_W \frac{\partial h}{\partial \bar{\zeta}}(\zeta) \frac{d\zeta \wedge d\bar{\zeta}}{\zeta - \bar{z}} \right\} \]

where the fact that

\[ \int_0^\pi e^{2i \theta} \, d\theta = 0 \]

is used.

\[ \square \]

4.4. Uniform estimates. Assume that \( R_0 \leq 1 \) is as given in Lemma 4.1, and also that \( 0 \in \Omega \setminus \bar{U}_{R_0} \).

The parameter

\[ \Delta(z) = \max\{|z|^2, d(z)^2\}, \]

for \( z \in (\Omega \setminus \bar{U}_{R_0})^c \) plays an important role in the estimates.

We have
Proposition 8. Let $\phi \in C^\infty(\Omega) \cap \text{Lip}(\gamma, \bar{\Omega})$. Let $\psi \in C^\infty(C \setminus \bar{\Omega}) \cap \text{Lip}(\gamma, C \setminus \Omega)$, satisfying that for a fixed constant $C(\psi)$, and set $\delta(z) = \max\{d(z), \frac{R_0}{2}\}$

$$|\psi(z)| \leq \frac{C(\psi)}{\delta(z)^2},$$

for $z \in (\Omega \cup U_{R_0})^c$.

Then there exists a constant $K = K(\gamma, \Omega, R_0)$ such that for $f = \phi$ or $\psi$, we have

1. $\|\chi_{\Omega} B[f]\|_\infty \leq K \|f\|_\gamma$,

2. If $z \in \Omega^c \cap U_{R_0}$,

$$|\chi_{\gamma \setminus \bar{\Omega}} B[f](z)| \leq K \|f\|_\gamma.$$

3. If $z \in C \setminus (\Omega \cup U_{R_0})^c$,

$$|\chi_{C \setminus \bar{\Omega}} B[\phi](z)| \leq K \frac{\|\phi\|_\gamma}{\delta(z)^2}.$$

4. If $z \in C \setminus (\Omega \cup U_{R_0})^c$,

$$|\chi_{C \setminus \bar{\Omega}} B[\psi](z)| \leq K(1 + \|\psi\|_\gamma) C(\psi) \left\{ \frac{1}{\delta(z)^2} + \frac{1}{\Delta(z)^2} (1 + \ln(\Delta(z))) \right\}.$$

Remark 7. In particular, if $f = \chi_{\Omega}$ then

$$\|\chi_{\Omega} B[\chi_{\Omega}]\|_\infty \leq K$$

and

$$|\chi_{\gamma \setminus \bar{\Omega}} B[\chi_{\Omega}](z)| \leq K \left( \chi_{U_{R_0}}(z) + \frac{\chi_{C \setminus \bar{U}_{R_0} \cup \Omega}(z)}{\delta(z)^2} \right).$$

4.4.1. Proof of Proposition 8: Using the decompostion given in Proposition 7, we have to estimate $Q[f](z)$ and $L[f](z)$ in different situations depending on the support of $f$ and the position of the point $z$. For doing so, we have

1) Estimates for $L$:

Proposition 9. If $z \in \Omega \setminus U_{R_0}$, there exists a constant $C = C(\gamma, \Omega) > 0$ such that

$$|L[\phi](z)| \leq C \|\phi\|_\gamma$$

and

$$L[\psi](z) = 0.$$

Proof. We have now that $z \in \Omega$ and $\delta(z) = d(z) > R_0$.

For the first part,

$$|L[\phi](z)| \leq \|\phi\|_\gamma 2\pi \int_0^{d(z)} r^{\gamma-1} dr = \|\phi\|_\gamma 2\pi \frac{d(z)\gamma}{\gamma} \leq C \|\phi\|_\gamma.$$

For the second part, we have $L[\psi](z) = 0$. \qed

Proposition 10. If $z \in \overline{U_{R_0}}$, there exists a constant $C = C(\gamma, \Omega, R_0) > 0$ such that both for $f = \phi, \psi$, we have

$$|L[f](z)| \leq C \|f\|_\gamma.$$
Proof. For $z \in \overline{U_{R_0}}$, let $\tau \in \partial \Omega$ such that $d(z) = |z - \tau|$ and $\kappa_\tau$ defining the half space determined by the tangent line to $\partial \Omega$ across $\tau$, and containing the inward normal vector to $\partial \Omega$ in $\tau$. Then

$$L[f](z) = \left( \int_{B_\delta(z) \cap \{ \kappa_\tau \leq \rho \}} + \int_{B_\delta(z) \cap \{ \kappa_\tau \geq \rho \}} \right) \frac{f(\zeta) - f(z)}{|\zeta - z|^\gamma} \frac{|\zeta - z|^\gamma}{(\zeta - \bar{z})^2} dm(\zeta)$$

$$= \left( \int_{B_\delta(z) \cap \{ \kappa_\tau \leq \rho \}} + \int_{B_\delta(z) \cap \{ \kappa_\tau \geq \rho \}} \right) \frac{f(\zeta) - f(z)}{|\zeta - z|^\gamma} \frac{|\zeta - z|^\gamma}{(\zeta - \bar{z})^2} dm(\zeta)$$

$$+ \left( \int_{B_\delta(z) \cap \{ \kappa_\tau \geq \rho \}} + \int_{B_\delta(z) \cap \{ \kappa_\tau \geq 0 \} \cap \{ \rho > 0 \}} \right) \frac{f(\zeta) - f(z)}{|\zeta - z|^\gamma} \frac{|\zeta - z|^\gamma}{(\zeta - \bar{z})^2} dm(\zeta).$$

If $f = \phi$, in the case of $z \in \Omega$, we have

$$L[\phi](z) = \int_{B_\delta(z) \cap \{ \kappa_\tau \leq \rho \} \cap \{ \rho \leq 0 \}} \frac{\phi(\zeta) - \phi(z)}{|\zeta - z|^\gamma} \frac{|\zeta - z|^\gamma}{(\zeta - \bar{z})^2} dm(\zeta)$$

$$- \phi(z) \int_{B_\delta(z) \cap \{ \kappa_\tau \leq \rho \} \cap \{ \rho > 0 \}} \frac{1}{(\zeta - \bar{z})^2} dm(\zeta)$$

$$+ \int_{B_\delta(z) \cap \{ \kappa_\tau \geq 0 \} \cap \{ \rho \leq 0 \}} \frac{\phi(\zeta) - \phi(z)}{|\zeta - z|^\gamma} \frac{|\zeta - z|^\gamma}{(\zeta - \bar{z})^2} dm(\zeta)$$

$$- \phi(z) \int_{B_\delta(z) \cap \{ \kappa_\tau \geq 0 \} \cap \{ \rho > 0 \}} \frac{1}{(\zeta - \bar{z})^2} dm(\zeta)$$

$$= (I) + (II) + (III) + (IV).$$

The integral

$$|(I)| \leq C \| \phi \|_\gamma$$

as in the previous proposition.

Concerning the terms (II) and (III), the integrals there are extended to the subsets of $B_\delta(z)$ located between the tangent line to $\partial \Omega$ at $\tau$ and $\partial \Omega$ itself. Also $z$ is not in the domain of integration. Then, using the Lemma 4.5 below, we have

$$|(II)| \leq \| \phi \|_\infty \frac{4 \delta(z)^\gamma}{\gamma} \leq C \| \phi \|_\infty.$$

And the integral

$$|(III)| \leq \| \phi \|_\gamma \frac{4 \delta(z)^\gamma}{\gamma} \leq C \| \phi \|_\gamma.$$

Finally, the integral in (IV) can be written as

$$\left( \int_{B_\delta(z) \cap \{ \kappa_\tau \geq 0 \} \cap \{ \rho \leq 0 \}} - \int_{B_\delta(z) \cap \{ \kappa_\tau \geq 0 \} \cap \{ \rho \geq 0 \}} \right) b_z,$$

and using Lemmas 4.5 and 4.6 we have the estimate

$$|(IV)| \leq \| \phi \|_\infty \left( \frac{\pi}{2} + \frac{4 \delta(z)^\gamma}{\gamma} \right).$$

And in this case $\frac{R_0}{4} \leq \delta(z) \leq R_0$. \hfill $\square$

The case of $f = \psi$ is completely similar.
Lemma 4.5. Under the conditions and notation of Proposition 10, the integrals
\[ \int_{B_\delta(z) \cap \{ \kappa \tau \leq 0 \} \cap \{ \rho > 0 \}} b_z \]
and
\[ \int_{B_\delta(z) \cap \{ \kappa \tau \geq 0 \} \cap \{ \rho \leq 0 \}} b_z \]
are bounded by
\[ \frac{4 R_0^2}{\gamma} \].

Proof. The integrals above are extended to the subsets of \( B_\delta(z) \) located between the tangent line to \( \partial \Omega \) at \( \tau \) and \( \partial \Omega \) itself.

Then, we can take coordinates centred at \( \tau \) given by the frame \( \eta(\tau) = 2 \bar{\partial} \rho(\tau) / \| \nabla \rho(\tau) \| \), \( \vartheta(\tau) = i \eta(\tau) \), so any point \( w \in \mathbb{C} \) is of the form
\[ w = \tau + \lambda \eta(\tau) + \mu \vartheta(\tau) \]
and
\[ \kappa_\tau(w) = \lambda \| \nabla \rho(\tau) \| \]
and
\[ \rho(w) = \kappa_\tau(z) + \omega(\sqrt{\lambda^2 + \mu^2}) \sqrt{\lambda^2 + \mu^2}, \]
where \( \omega \) is the modulus of continuity of the derivatives of \( \rho \), at \( \tau \).

Then the isometric map \( P: \mathbb{C} \to \mathbb{C} \)
given by
\[ \phi(\mu + i \lambda) = \tau + \lambda \eta(\tau) + \mu \vartheta(\tau) \]
transforms 0 in \( \tau, -i d(z) \) in \( z \), and the real and the imaginary axis in the lines across \( \tau \) directed by \( \eta(\tau) \) and \( \vartheta(\tau) \) respectively. Also
\[ P(B_\delta(z)(-i d(z))) = B_\delta(z). \]

Then, if
\[ A_1 = \{ \mu + i \lambda \in \mathbb{C} : |\mu| \leq \sqrt{R_0^2 - d(z)^2}, \lambda \| \nabla \rho(\tau) \| + \omega(\sqrt{\lambda^2 + \mu^2}) \sqrt{\lambda^2 + \mu^2} \leq 0 \} \]
and
\[ A_2 = \{ \mu + i \lambda \in \mathbb{C} : |\mu| \leq \sqrt{R_0^2 - d(z)^2}, \lambda \| \nabla \rho(\tau) \| + \omega(\sqrt{\lambda^2 + \mu^2}) \sqrt{\lambda^2 + \mu^2} > 0 \}, \]
we have that the first integral in the statement of the lemma is equal to
\[ \int_{B_\delta(z)(-i d(z)) \cap \{ \lambda > 0 \} \cap A_1} \frac{1}{(\mu - i \lambda + i d(z))^2} \ d m(\mu, \lambda) \]
\[ = \int_{-\sqrt{R_0^2 - d(z)^2}}^{\sqrt{R_0^2 - d(z)^2}} d \mu \int_0^{\varphi_+(\mu)} \frac{1}{( \mu - i \lambda + i d(z))^2} \ d \lambda = (\ast) \]
where
\[ \varphi_+(\mu) = \min\{ \varphi(\mu), \sqrt{R_0^2 - d(z)^2} - \mu^2 \}. \]

Then
\[ (\ast) = \int_{-\sqrt{R_0^2 - d(z)^2}}^{\sqrt{R_0^2 - d(z)^2}} \frac{\varphi_+(\mu)}{\mu - i \varphi_+(\mu) + i d(z)} (\mu + i d(z)) \ d \mu, \]
and is bounded by
\[
\left| (II) \right| \leq C \int_{\mathbb{R}_0^2 - d(z)^2} \left| \varphi_+ (\mu) \right| d\mu
\]
where
\[
\varphi_+ (\mu) = \left| \mu \right|^{1+\gamma} \sqrt{\mu^2 + (d(z) - \varphi_+ (\mu))^2} \sqrt{\mu^2 + d(z)^2}
\]
which is identical for the second integral.

**Lemma 4.6.**
\[
\left| \int_{B_{\delta}(z) \cap \{ \kappa \geq 0 \}} b_z \right| \leq \frac{\pi}{2}.
\]

**Proof.** It is clear that if \( \delta(z) \geq \frac{R_0}{2} \) or if \( z = \tau \), then
\[
\int_{B_{\delta}(z) \cap \{ \kappa \geq 0 \}} b_z = 0
\]
by the cancellation property. So we assume that \( 0 < d(z) < \frac{R_0}{2} \) and then our integral is
\[
\int_{B_{\frac{R_0}{2}}(z) \cap \{ \kappa \geq 0 \}} b_z.
\]
Also by the cancellation property, and after a rigid movement, our integral is
\[
\int_{B_{\frac{R_0}{2}}(0) \cap \{ \exists (\zeta) \geq 0 \}} b_z = \int_{\alpha}^{R_0} \frac{dr}{r} \int_{\arcsin(\frac{\alpha}{r})}^{\pi-\arcsin(\frac{\alpha}{r})} e^{2i\theta} d\theta
\]
which is identical for the second integral.

**Proposition 11.** If \( z \in \mathbb{C} \setminus U_{R_0} \cup \Omega \), then
Proof. A) It is immediate.
B) There exists a constant \( C = C(\gamma, \Omega, R_0) \) such that

\[
|L[\psi](z)| \leq C \left( 1 + \|\psi\|_\gamma \right) C(\psi) \left\{ \frac{1}{\max\{R_0^2, d(z)^2\}} \right. \\
\left. + \frac{1}{\max\{d(z), |z|\}^2} \{1 + \ln(\max\{d(z), |z|\}) \} \right\}.
\]

If \( \alpha(z) > d(z) \), we have

\[
L[\psi](z) = \int_{B_d(z)} \frac{\psi(z) - \psi(z)}{|z - \bar{z}|^\gamma} \, dm(\zeta).
\]

If \( \alpha(z) \leq d(z) \), then

\[
L[\psi](z) = \left( \int_{B_{\alpha(z)}(z)} + \int_{B_{d(z)}(z) \setminus B_{\alpha(z)}(z)} \right) \frac{\psi(z) - \psi(z)}{|z - \bar{z}|^\gamma} \, dm(\zeta)
\]

\[
= (I)_{\alpha(z)} + (II)_{\alpha(z)}.
\]

Since

\[
|\psi(\zeta)| \leq |\psi(z)| + \|\psi\|_\gamma |\zeta - z|^\gamma
\]

then for \( \zeta \in B_{\alpha(z)}(z) \) we have

\[
|\psi(\zeta)| \leq |\psi(z)| + \|\psi\|_\gamma \alpha(z)^\gamma \leq C \left( 1 + \|\psi\|_\gamma \right) C(\psi) \frac{1}{\max\{R_0^2, d(z)^2\}}.
\]

Therefore

\[
|I)_{\alpha(z)}| \leq \|\psi\|_\gamma \int_{B_{\alpha(z)}(z)} \frac{dm(\zeta)}{|z - \bar{z}|^{2-\gamma}}
\]

\[
= 2\pi \|\psi\|_\gamma \int_0^{\alpha(z)} \frac{dr}{r^{1-\gamma}}
\]

\[
\leq 2\pi \|\psi\|_\gamma \frac{\alpha(z)^\gamma}{\gamma} = \frac{2\pi}{\gamma} \max\{|\psi(z)|, \frac{\|\psi\|_\gamma}{\max\{R_0^2, d(z)^2\}}\}.
\]
Using the notation $C_{\alpha(z)}^{d(z)}(z) = B_{d(z)}(z) \setminus B_{\alpha(z)}(z)$, we have

$$(II)_{\alpha(z)} = \left( \int_{C_{\alpha(z)}^{d(z)}(z) \cap U_{R_0}} + \int_{C_{\alpha(z)}^{d(z)}(z) \setminus U_{R_0}} \right) \frac{\psi(\zeta) - \psi(z)}{|\zeta - z|^{\gamma}} \frac{|\zeta - z|^{\gamma}}{(\zeta - z)^2} \, dm(\zeta)$$

$$= (III)_{\alpha(z)} + (IV)_{\alpha(z)}.$$ 

And

$$(III)_{\alpha(z)} = \left( \int_{C_{\alpha(z)}^{d(z)}(z) \cap U_{R_0}} + \int_{C_{\alpha(z)}^{d(z)}(z) \setminus (U_{R_0} \setminus U_{\frac{d(z)}{2}})} \right) \frac{\psi(\zeta) - \psi(z)}{|\zeta - z|^{\gamma}} \frac{|\zeta - z|^{\gamma}}{(\zeta - z)^2} \, dm(\zeta)$$

$$= (V)_{\alpha(z)} + (VI)_{\alpha(z)}.$$ 

Since for $\zeta \in U_{R_0}$, we have $|\zeta - z| \geq \frac{d(z)}{2}$, then

$$|(V)_{\alpha(z)}| \leq \int_{B_{d(z)}(z) \cap U_{\frac{d(z)}{2}}} \frac{\psi(\zeta) - \psi(z)}{|\zeta - z|^{\gamma}} \frac{|\zeta - z|^{\gamma}}{(\zeta - z)^2} \, dm(\zeta)$$

$$\leq \|\psi\|_{\infty} \frac{4 m(U_R)}{\max\{R_0^2, d(z)^2\}}.$$ 

The integral of the term $(VI)_{\alpha(z)}$, extends to $\zeta \in U_{R_0} \setminus U_{\frac{d(z)}{2}}$. Then, if $d(z) \geq \frac{3R_0}{2}$ we have, for $\zeta \in \Omega$ such that

$$|\zeta - \zeta| = d(\zeta, \partial \Omega)$$

$$|\zeta - z| \geq ||\zeta - \zeta| - |\zeta - z|| \geq d(z) - R_0 \geq \frac{d(z)}{3}$$

so, as in the case of the term $(V)_{\alpha(z)}$,

$$|(VI)_{\alpha(z)}| \leq \|\psi\|_{\infty} \frac{36 m(U_R)}{\max\{R_0^2, d(z)^2\}}.$$ 

If $R_0 \leq d(z) < \frac{3R_0}{2}$, then

$$|(VI)_{\alpha(z)}| \leq \|\psi\|_{\gamma} 2\pi \int_0^{\frac{d(z)}{2}} \frac{dr}{r^{1-\gamma}} \leq \|\psi\|_{\gamma} \frac{2\pi}{\gamma} d(z) \leq \frac{\|\psi\|_{\gamma} C(\gamma, R_0)}{\max\{R_0^2, d(z)^2\}}.$$ 

The term $(IV)_{\alpha(z)}$ in the case $\frac{d(z)}{2} > \alpha(z)$, is equal to

$$\left( \int_{C_{\alpha(z)}^{d(z)}(z) \cap U_{R_0}} + \int_{C_{\alpha(z)}^{d(z)}(z) \setminus U_{R_0}} \right) \psi b_z = (VII)_{\alpha(z)} + (VIII).$$

In the case of $\frac{d(z)}{2} \leq \alpha(z)$,

$$|(IV)_{\alpha(z)}| \leq \int_{C_{\alpha(z)}^{d(z)}(z) \setminus U_{R_0}} |\psi b_z|$$

and has the same estimate as $(VIII)$. 

Since for $\zeta \in B_{d(z)}(z)$, we have $d(\zeta) > \frac{d(z)}{2}$, because $d(\zeta) = |\zeta - \tau|$ for some $\tau \in \Omega$ and

$$|\zeta - \tau| \geq |z - \tau| - |z - \zeta| \geq d(z) - \frac{d(z)}{2},$$

$$\|\psi\|_{\infty} \leq \frac{\|\psi\|_{\gamma} C(\gamma, R_0)}{\max\{R_0^2, d(z)^2\}}.$$
we have

\[ |(VII)_{\alpha(z)}| \leq C(\psi) \int_{C_{\alpha(z)}} \frac{dm(\zeta)}{d(\zeta)^2 |\zeta - z|^2} \]

\[ \leq C(\psi) \frac{4}{d(z)^2} \int_{C_{\alpha(z)}} \frac{dm(\zeta)}{|\zeta - z|^2} = C(\psi) \frac{8\pi}{d(z)^2} \int_{\alpha(z)} \frac{d(\zeta)}{r} dr \]

\[ = C(\psi) \frac{8\pi}{d(z)^2} \ln \left( \frac{d(z)}{2\alpha(z)} \right), \]

and if \( \frac{\|\psi(z)\|}{\|\psi\|_\gamma} \geq \frac{1}{\max\{R_0, d(z)^2\}} \), then

\[ |(VII)_{\alpha(z)}| \leq C(\psi) \frac{8\pi}{d(z)^2} \left( \ln(d(z) \|\psi\|_\gamma) + \ln(2 |\psi(z)|^\gamma) \right), \]

and in the opposite case we have

\[ |(VII)_{\alpha(z)}| \leq C(\psi) \frac{8\pi}{d(z)^2} \left( \ln(d(z)) + \ln(2 \max\{R_0^2, d(z)^2\}^\gamma) \right). \]

For the term \((VIII)\), we consider

\[ C_{\frac{d(z)}{2}}(z) \setminus U_{R_0} = E_1(z) \cup E_2(z), \]

where \(E_1(z)\) and \(E_2(z)\) are the intersection of the domain of integration with the set

\[ \{ \zeta \in \mathbb{C} : d(\zeta) \geq |\zeta - z| \} \]

or its complement, respectively. Then

\[ (VIII) = \left( \int_{E_1(z)} + \int_{E_2(z)} \right) \psi b_z = (IX) + (X). \]

Since \( |\psi(\zeta)| \leq C(\psi) \frac{\|\psi\|_\gamma}{d(\zeta)^2}, \)

\[ |(IX)| \leq C(\psi) \int_{C \setminus B_{\frac{d(z)}{2}}(z)} \frac{dm(\zeta)}{|\zeta - z|^2} \leq C(\psi) 2\pi \int_{\frac{d(z)}{2}}^\infty \frac{dr}{r^3} \leq C(\psi) \frac{8\pi}{d(z)^2}. \]

Let

\[ M = \max\{|w| : w \in (U \cup \Omega)\}. \]

The term

\[ (X) = \left( \int_{E_2(z) \setminus B_{2M}(0)} + \int_{E_2(z) \setminus B_{2M}(0)} \right) \psi b_z = (XI) + (XII) \]

since we have that if \( |\zeta| \geq 2M, \) then \( d(\zeta) \geq \frac{|\zeta|}{2} \) and then

\[ |(XII)| \leq C(\psi) 4 \int_{C \setminus B_{\frac{d(z)}{2}}(z) \setminus B_{2M}(0)} \frac{dm(\zeta)}{|\zeta|^2 |\zeta - z|^2}. \]

If \( |z| \geq 5M, \) then \( d(z) \geq 3M \) the balls \( B_{2M}(0) \) and \( B_{\frac{d(z)}{2}}(z) \) are mutually disjoint and we consider then the decomposition in disjoint sets

\[ C_{\frac{d(z)}{2}}(z) \setminus B_{2M}(0) = A_1 \cup A_2 \cup A_3, \]
where
\[ A_1 = C_{d(z)}^{\mathcal{E}(z)}(z) \setminus B_{2|z|}(0), \]
\[ A_2 = \{ \zeta \in B_{2|z|}(0) \setminus B_{2M}(0) : |\zeta| \leq |\zeta - z| \} \cap C_{d(z)}^{\mathcal{E}(z)}(z) \]
and
\[ A_3 = \{ \zeta \in B_{2|z|}(0) \setminus B_{2M}(0) : |\zeta| \geq |\zeta - z| \} \cap C_{d(z)}^{\mathcal{E}(z)}(z). \]

If \(|\zeta| > 2|z|\), then \(\frac{1}{2} |\zeta| \leq |\zeta - z|\), so the integral over \(A_1\) is bounded by
\[
\int_{\mathcal{C}_{d(z)}^{\mathcal{E}(z)}(z) \setminus B_{2|z|}(0)} \frac{4 \, dm(\zeta)}{|\zeta|^4} = \frac{4\pi}{|z|^2}. \]

Since for \(\zeta \in A_2\), \(|\zeta - z| \geq \frac{|z|}{2}\), the integral over \(A_2\) is bounded by
\[
\frac{4}{|z|^2} \int_{B_{2|z|}(0) \setminus B_{d(z)}(z)} \frac{dm(\zeta)}{|\zeta - z|^2} \leq 8\pi \frac{\int_{2M}^{2|z|} dr}{|z|^2} = \frac{8\pi}{|z|^2} \ln \left(\frac{|z|}{M}\right). \]

Finally, for \(\zeta \in A_3\) we have that \(|\zeta| \geq \frac{|z|}{2}\) and the integral in \(A_3\) is bounded by
\[
\frac{4}{|z|^2} \int_{B_{2|z|}(0) \setminus B_{d(z)}(z)} \frac{dm(\zeta)}{|\zeta - z|^2} \leq \frac{4}{|z|^2} \int_{B_{d(z)}(z) \setminus B_{d(z)}(z)} \frac{dm(\zeta)}{|\zeta - z|^2} = \frac{8\pi}{|z|^2} \ln \left(\frac{8|z|}{d(z)}\right). \]

If \(|z| \leq 5M\), then the set
\[ C_{d(z)}^{\mathcal{E}(z)}(z) \setminus B_{2M}(0) \]
decomposes in a disjoint union of the resulting intersection with the set
\[ \{|\zeta| > |\zeta - z|\} \]
and its complement, namely \(G_1\) and \(G_2\). Then
\[
\int_{G_1} \frac{dm(\zeta)}{|\zeta|^2 |\zeta - z|^2} \leq \int_{C \setminus B_{d(z)}(z)} \frac{dm(\zeta)}{|\zeta|^4} = \frac{16\pi}{d(z)^2}, \]
and
\[
\int_{G_2} \frac{dm(\zeta)}{|\zeta|^2 |\zeta - z|^2} \leq \int_{C \setminus B_{2M}(0)} \frac{dm(\zeta)}{|\zeta|^4} = \frac{16\pi}{4M^2} \leq \frac{16\pi 25}{|z|^2}. \]

Since if \(\zeta \in E_2(z) \cap B_{2M}(0)\) then \(\frac{d(z)}{2} \leq |\zeta - z| \leq d(z)\) and \(R_0 \leq |\zeta| \leq 2M\), we have that the integral
\[
|XI| \leq \int_{E_2(z) \cap B_{2M}(0)} |\psi_b| \leq \psi(z) \int_{B_{2M}(0)} \frac{dm(\zeta)}{R_0^2 |\zeta - z|^2} \leq \psi(z) \frac{C(R_0, M)}{d(z)^2}. \]

2) Estimates for \(Q\):
Proposition 12. There exists a constant $C = C(\Omega, R_0)$ such that if $z \in \Omega \cup U_{R_0}$, we have
\[
|Q[\phi](z)| \leq C \|\phi\|_\infty
\]
and
\[
|Q[\psi](z)| \leq C \left(\|\psi\|_\infty + C(\psi)\right).
\]

Proof. If $z \in \Omega \setminus U_{R_0}$, we have $\delta(z) = d(z) \geq R_0$ and then
\[
Q[\phi](z) = \int_{\Omega \setminus B_\delta(z)} \phi b_z
\]
so
\[
|Q[\phi](z)| \leq \|\phi\|_\infty \frac{m(\Omega)}{R_0^2}.
\]

Also
\[
Q[\psi](z) = \int_{C \setminus B_\delta(z)} \psi b_z = \int_{C \setminus \Omega} \psi b_z
\]
\[
= \int_{(C \setminus \Omega) \setminus U_{R_0}} \psi b_z + \int_{(C \setminus \Omega) \setminus U_{R_0}} \psi b_z = (I) + (II),
\]
and $(I)$ has the same control as $Q[\phi](z)$. The integral
\[
|(II)| \leq C(\psi) \int_{C \setminus (\Omega \cup U_{R_0})} \frac{dm(\zeta)}{\max\{R_0^2, d(\zeta)^2\}|\zeta - z|^2},
\]
and for $\zeta \in \mathbb{C} \setminus (\Omega \cup U_{R_0})$ we have that $d(\zeta) \geq |\zeta - z|$. Then the integral above is
\[
\int_{C \setminus (\Omega \cup U_{R_0})} \frac{dm(\zeta)}{d(\zeta)^2|\zeta - z|^2} \leq \int_{C \setminus (\Omega \cup U_{R_0})} \frac{dm(\zeta)}{\zeta - z|^4} \leq \frac{2\pi}{3 R_0^2}.
\]
If $z \in U_{R_0}$, we have
\[
Q[f](z) = \int_{C \setminus U_{R_0}(z)} f b_z = \left(\int_{C \setminus (B_{R_0}(z) \cup (\Omega \cup U_{R_0}))} + \int_{(U_{R_0} \setminus \Omega) \setminus B_{R_0}(z)}\right) f b_z
\]
\[
= (I) + (II).
\]
In the case $f = \phi$, the integral
\[
(I) = 0.
\]
and in the case of $f = \psi$,
\[
|(I)| \leq C(\psi) \int_{C \setminus (B_{R_0}(z) \cup (\Omega \cup U_{R_0}))} \frac{dm(\zeta)}{\max\{R_0^2, d(\zeta)^2\}|\zeta - z|^2} \leq C(\psi) \frac{C}{R_0^2}.
\]
The integral
\[
|(II)| \leq \|f\|_\infty \frac{m(\Omega \cup U_{R_0})}{R_0^2}
\]
in all cases. \hfill \Box

Proposition 13. There exists a constant $C = C(\Omega, R_0)$ such that if $z \in \mathbb{C} \setminus (\Omega \cup U_{R_0})$, then
\[
|Q[\phi](z)| \leq C \|\phi\|_\infty \frac{1}{d(z)^2}
\]
and
\[ |Q[\psi](z)| \leq C \left\{ \|\psi\|_\infty \frac{1}{d(z)^2} + C(\psi) \frac{1}{\max\{\|z\|^2, d(z)^2\}} \left( 1 + \ln \max\{|z|, d(z)\} \right) \right\}. \]

**Proof.** If \( z \in \mathbb{C} \setminus (\Omega \cup U_{R_0}) \), then
\[ Q[\phi](z) = \int_{\Omega \setminus B_{d(z)}(z)} \phi b_z \]
and is bounded by
\[ \|\phi\|_\infty \frac{m(\Omega)}{d(z)^2}, \]
leading to the first statement.

On the other hand
\[ Q[\psi](z) = \int_{\mathbb{C} \setminus (\Omega \cup B_{d(z)}(z))} \psi b_z = \int_{\mathbb{C} \setminus (\Omega \cup U_{R_0} \cup B_{d(z)}(z))} \psi b_z + \int_{U_{R_0} \setminus (\Omega \cup B_{d(z)}(z))} \psi b_z = (I) + (II). \]

The integral
\[ |(II)| \leq \|\psi\|_\infty \frac{m(U_{R_0})}{d(z)^2}. \]

The term
\[ |(I)| \leq C(\psi) \int_{\mathbb{C} \setminus (\Omega \cup U_{R_0} \cup B_{d(z)}(z))} \frac{dm(\zeta)}{d(z)^2 |\zeta - z|^2}, \]
and we consider
\[ \mathbb{C} \setminus (\Omega \cup B_{d(z)}(z)) = E_1(z) \cup E_2(z), \]
where \( E_1(z) \) and \( E_2(z) \) are the intersection of the domain of integration with the set
\[ \{ \zeta \in \mathbb{C} : d(\zeta) \geq |\zeta - z| \} \]
or its complement. Then the integral in \( (I) \) can be decomposed as
\[ \left( \int_{E_1(z)} + \int_{E_2(z)} \right) \psi b_z = (III) + (IV), \]
and the term
\[ |(III)| \leq \int_{\mathbb{C} \setminus B_{d(z)}(z)} \frac{dm(\zeta)}{|\zeta - z|^2} \leq 2\pi \int_{d(z)}^\infty \frac{dr}{r^2} \leq \frac{\pi}{d(z)^2}, \]
and
\[ (IV) = \left( \int_{E_2(z) \cap B_{2M}(0)} + \int_{E_2(z) \setminus B_{2M}(0)} \right) \psi b_z = (V) + (VI). \]

The integral
\[ |(V)| \leq \int_{E_2(z) \cap B_{2M}(0)} |\psi b_z| \leq \int_{B_{2M}(0)} \frac{dm(\zeta)}{R_0^2 |\zeta - z|^2} \leq \frac{8\pi M^2}{R_0^2} \frac{4}{d(z)^2}. \]

For \( (VI) \), let us consider \( M = \text{diam}(U_{R_0} \cup \Omega) \).

Since we have that if \( |\zeta| \geq 2M \), then \( d(\zeta) \geq \frac{|\zeta|}{2M} \) then
\[ |(VI)| \leq 4 \int_{\mathbb{C} \setminus (B_{2M}(0) \cup B_{d(z)}(z))} \frac{dm(\zeta)}{|\zeta|^2 |\zeta - z|^2}. \]
If \(|z| \geq 5M\), then \(d(z) \geq 4M\) and if \(|\zeta - z| < \frac{d(z)}{2}\) then \(|\zeta| \geq 5M - \frac{d(z)}{2} \geq 3M\) so the balls \(B_{2M}(0)\) and \(B_{\frac{d(z)}{2}}(0)\) are mutually disjoint. On the other hand, \(B_{2M}(0) \cup B_{\frac{d(z)}{2}}(z) \subset B_{2|z|}(0)\) and we may consider then the decomposition in disjoint sets
\[
C \setminus (B_{2M}(0) \cup B_{\frac{d(z)}{2}}(z)) = A_1 \cup A_2 \cup A_3,
\]
where
\[
A_1 = C \setminus B_{2|z|}(0),
\]
\[
A_2 = \{ \zeta \in B_{2|z|}(0) \setminus (B_{2M}(0) \cup B_{\frac{d(z)}{2}}(z)) : |\zeta| \leq |\zeta - z| \},
\]
and
\[
A_3 = \{ \zeta \in B_{2|z|}(0) \setminus (B_{2M}(0) \cup B_{\frac{d(z)}{2}}(z)) : |\zeta| \geq |\zeta - z| \}.
\]

If \(|\zeta| > 2|z|\), then \(\frac{1}{2} |\zeta| \leq |\zeta - z|\), so the integral over \(A_1\) is bounded by
\[
\int_{C \setminus B_{2|z|}(0)} \frac{4 \, dm(\zeta)}{|\zeta|^4} = \frac{4\pi}{|z|^2}.
\]

Next, \(A_2 \subset \{ \zeta \in B_{2|z|}(0) \setminus B_{2M}(0) : |\zeta| \leq |\zeta - z| \}\), and for \(\zeta\) in this set, we have \(|\zeta - z| \geq \frac{|z|}{2}\), so the integral over \(A_2\) is bounded by
\[
\frac{4}{|z|^2} \int_{C_{2|z|}(0)} \frac{dm(\zeta)}{|\zeta|^2} \leq \frac{8\pi}{|z|^2} \int_{2M} \frac{dr}{r} = \frac{8\pi}{|z|^2} \ln \left( \frac{|z|}{M} \right).
\]

Finally, \(A_3 \subset \{ \zeta \in B_{2|z|}(0) \setminus B_{\frac{d(z)}{2}}(z) : |\zeta| \geq |\zeta - z| \}\), and for \(\zeta\) in this set we have \(|\zeta| \geq \frac{|z|}{2}\) an the integral over \(A_3\) is bounded by
\[
\frac{4}{|z|^2} \int_{B_{2|z|}(0) \setminus B_{\frac{d(z)}{2}}(z)} \frac{dm(\zeta)}{|\zeta - z|^2} \leq \frac{4}{|z|^2} \int_{B_{d(z)}(0) \setminus B_{\frac{d(z)}{2}}(z)} \frac{dm(\zeta)}{|\zeta - z|^2} = \frac{8\pi}{|z|^2} \ln \left( \frac{8|z|}{d(z)} \right).
\]

If \(|z| \leq 5M\), then the set
\[
C \setminus (B_{2M}(0) \cup B_{\frac{d(z)}{2}}(z))
\]
decomposes in a disjoint union of the resulting intersection with the set
\[
\{ |\zeta| > |\zeta - z| \}
\]
and its complement, namely \(G_1\) and \(G_2\). Then
\[
\int_{G_1} \frac{dm(\zeta)}{|\zeta|^2 |\zeta - z|^2} \leq \int_{C \setminus B_{\frac{d(z)}{2}}(z)} \frac{dm(\zeta)}{|\zeta - z|^4} = \frac{16\pi}{d(z)^2},
\]
and
\[
\int_{G_2} \frac{dm(\zeta)}{|\zeta|^2 |\zeta - z|^2} \leq \int_{C \setminus B_{2M}(0)} \frac{dm(\zeta)}{|\zeta|^4} = \frac{16\pi}{4M^2} \leq \frac{16\pi}{|z|^2}.
\]

\(\square\)
4.5. Lipschitz estimates. We assume the same hypotheses, definitions and notation of the previous sections and subsections. Let us assume that $R_0 \leq 1$ is as given in Lemma 4.1, and also that $0 \in \Omega \setminus U_{R_0}$.

Proposition 14. There exists a constant $K = K(\gamma, \Omega, R_0)$ such that for $\phi$ and $\psi$ satisfying the conditions of Proposition 8 and $f = \phi$ or $\psi$, we have, that, if both $z, w \in \Omega$, or both $z, w \in \mathbb{C} \setminus \Omega$, then

$$\frac{|B[f](z) - B[f](w)|}{|z - w|^{\gamma}} \leq K_0 \|f\|_{\gamma}.$$

Remark 8. As we shall see, this proposition implies that $B[\phi]$ has a Lipschitz extension to $\Omega$ and $B[\psi]$ has an extension to $\mathbb{C} \setminus \Omega$. These extensions differ in a jump along $\partial \Omega$.

4.5.1. Proof of Proposition 14: We only need to estimate the quotients in the left hand side of the inequalities in the statement above, only in the case of $|z - w| \leq \frac{R_0}{4}$.

Also, most of the proof goes along for $\phi$ or $\psi$ with no distinction, so unless it be necessary, we will use $f$ for $\phi$ or $\psi$.

The goal is to estimate $\frac{|Q[f](z) - Q[f](w)|}{|z - w|^{\gamma}}$ for $z, w \in \Omega$ and $z, w \in \mathbb{C} \setminus \Omega$, so we will estimate

$$\frac{|Q[f](z) - Q[f](w)|}{|z - w|^{\gamma}} = (I)$$

and

$$\frac{|L[f](z) - L[f](w)|}{|z - w|^{\gamma}} = (II).$$

The term $Q[f](z) - Q[f](w) = \int_{\mathbb{C}\setminus B_{\delta}(z)} f_b z - \int_{\mathbb{C}\setminus B_{\delta}(w)} f_b w$

$$= \int_{\mathbb{C}\setminus B_{\delta}(z) \cup B_{\delta}(w)} f_b (z - w)$$

and the term $L[f](z) - L[f](w) = \int_{B_{\delta}(z)} \frac{f(\zeta) - f(z)}{|\zeta - z|^{\gamma}} \frac{|\zeta - z|^2}{(\zeta - z)^2} \, dm(\zeta)$

$$- \int_{B_{\delta}(w)} \frac{f(z) - f(\zeta)}{|\zeta - w|^{\gamma}} \frac{|\zeta - w|^2}{(\zeta - w)^2} \, dm(\zeta)$$

$$= \int_{B_{\delta}(w) \cap B_{\delta}(z)} \left[ \frac{f(\zeta) - f(z)}{|\zeta - z|^{\gamma}} \frac{|\zeta - z|^2}{(\zeta - z)^2} - \frac{f(z) - f(\zeta)}{|\zeta - w|^{\gamma}} \frac{|\zeta - w|^2}{(\zeta - w)^2} \right] \, dm(\zeta)$$. 


A direct computation shows that

\[ b_z(\zeta) - b_w(\zeta) = - (\bar{z} - \bar{w}) (\bar{z} + \bar{w} - 2\bar{\zeta}) b_z(\zeta) b_w(\zeta), \quad (21) \]

so

\[ (1) = - (\bar{z} - \bar{w}) \int f(\zeta) (\bar{z} + \bar{w} - 2\bar{\zeta}) b_z(\zeta) b_w(\zeta), \]

and if we define \( \tau = \frac{\bar{z} + \bar{w}}{2} \) and \( a = \tau - z = \frac{w - z}{2} \), then \( \tau - w = \frac{z - w}{2} = -a \) and

\[ (1) = 2 (\bar{z} - \bar{w}) \int f(\zeta) \left( \frac{\bar{\zeta} - \bar{\tau}}{(\zeta - \bar{\tau} + \bar{\tau} - \bar{z})^2 (\bar{\zeta} - \bar{\tau} + \bar{\tau} - \bar{w})^2} \right) dm(\zeta) \]

\[ = 2 (\bar{z} - \bar{w}) \int f(\zeta) \left( \frac{\bar{\zeta} - \bar{\tau}}{(\zeta - \bar{\tau})^2 - \bar{a}^2} \right) dm(\zeta) \]

\[ \overset{\text{def}}{=} 2 (\bar{z} - \bar{w}) K[f](z, w). \]

If

\[ \lambda = \left( \frac{R_0}{8} \right)^\gamma |a|^{1-\gamma} + |a|, \]

then we have that \( B_\lambda(\tau) \subset B_{\delta(z)}(z) \cap B_{\delta(w)}(w) \) because \( \zeta \in B_\lambda(\tau) \) implies that \( |\zeta - z| \leq |\zeta - \tau| + |\tau - z| < \lambda + |a| \) and, similarly, \(|\zeta - w| < \lambda + |a|\), but

\[ \lambda + |a| = \left( \frac{R_0}{8} \right)^\gamma |a|^{1-\gamma} + 2|a| \leq \left( \frac{R_0}{8} \right)^\gamma \left( \frac{R_0}{8} \right)^{1-\gamma} + 2 \frac{R_0}{8} < \min\{\delta(z), \delta(w)\}. \]

Consequently, using polar coordinates and the Schwartz inequality

\[ |K[f](z, w)| \leq \int_{C \setminus B_\lambda(\tau)} \left| f(\zeta) \frac{\bar{\zeta} - \bar{\tau}}{(\zeta - \bar{\tau})^2 - \bar{a}^2} \right| dm(\zeta) \]

\[ \leq \|f\|_\infty \int_\lambda^{\infty} \int_0^{2\pi} \frac{r^2}{|r^2 e^{2i\theta} - a|^2} \, dr \, d\theta \]

\[ \leq 2\pi \|f\|_\infty \int_\lambda^{\infty} \frac{r^2}{r^4 - |a|^4} \, dr \leq 2\pi \|f\|_\infty \int_\lambda^{\infty} \frac{1}{t^2 - |a|^2} \, dt \]

\[ = \pi \|f\|_\infty \ln \frac{t - |a|}{t + |a|} \bigg|_\lambda^{\infty} = \pi \|f\|_\infty \ln(\lambda + |a|) - \ln(\lambda - |a|) \]

\[ = \pi \|f\|_\infty \frac{1}{\lambda - |a|} = \frac{\pi \|f\|_\infty}{\left( \frac{R_0}{8} \right)^\gamma |a|^{1-\gamma}}. \]

So

\[ |(1)| \leq \frac{16\pi \|f\|_\infty}{R_0^\gamma} |z - w|^\gamma. \]
For the terms (2), (3) and (4) in the decomposition of $Q$ and $L$, the absolute and mutual positions of $z$ and $w$, specially related to $\partial \Omega$, play an important role. For this purpose, we will consider the following situations:

1) If $B_{\delta(w)}(w) \cap B_{\delta(z)}(z) = \emptyset$, equivalent to the fact that $\delta(w) + \delta(z) \leq |z - w|$ can never happen because then $\frac{R_0}{2} + \frac{R_0}{2} \leq \frac{R_0}{4}$.

2) The cases of $B_{\delta(w)}(w) \subset B_{\delta(z)}(z)$, or $B_{\delta(z)}(z) \subset B_{\delta(w)}(w)$, corresponding respectively to the facts that $\delta(w) + |z - w| \leq \delta(z)$ or $\delta(z) + |z - w| \leq \delta(w)$, so is

$$\frac{|z - w|}{\delta(z) + \delta(w)} \leq 1 - \frac{2 \delta(w)}{\delta(z) + \delta(w)}$$

or

$$\frac{|z - w|}{\delta(z) + \delta(w)} \leq 1 - \frac{2 \delta(z)}{\delta(z) + \delta(w)},$$

respectively.

The situations are completely symmetric and in each case only one term in (2) survives.

3) The case where the conditions

$$\begin{cases} B_{\delta(w)}(w) \cap B_{\delta(z)}(z) \neq \emptyset \\ B_{\delta(w)}(w) \cap B_{\delta(z)}(z)^c \neq \emptyset \\ B_{\delta(w)}(w)^c \cap B_{\delta(z)}(z) \neq \emptyset, \end{cases}$$

are satisfied.

In this case, we have

$$\begin{cases} \delta(w) + \delta(z) \geq |z - w|, \\ \delta(z) \leq |z - w| + \delta(w), \\ \delta(w) \leq |z - w| + \delta(z) \end{cases}$$

or

$$1 - \frac{2 \delta(w)}{\delta(z) + \delta(w)} \leq \frac{|z - w|}{\delta(z) + \delta(w)} \leq 1,$$

and

$$1 - \frac{2 \delta(z)}{\delta(z) + \delta(w)} \leq \frac{|z - w|}{\delta(z) + \delta(w)} \leq 1.$$
by the cancellation Lemma 4.3, and since $B_{\delta(w)}(w) \subset \Omega$ or $B_{\delta(w)}(w) \subset \Omega^c$, we have clearly that

$$|(2)| \leq \|f\|_{\gamma} 2\pi \int_{d(z)} d(w) \frac{dr}{r^{1-\gamma}} = \|f\|_{\gamma} \frac{2\pi}{\gamma} \{d(w) - d(z)\}$$

$$\leq \|f\|_{\gamma} 2\pi \frac{d(w) - d(z)}{d(z)^{1-\gamma}} \leq \|f\|_{\gamma} \frac{2\pi R_0^{2-\gamma}}{R_0^{1-\gamma}} |w - z|.$$ 

- If $z \in U_{R_0}$, we have that $\delta(z) = \frac{R_0}{2}$ and

$$\delta(w) \geq |z - w| + \delta(z) = |z - w| + \frac{R_0}{2} \geq \frac{R_0}{2},$$

then

$$(2) = \int_{B_{\delta(w)}(w) \setminus B_{\frac{R_0}{2}(z)}} \frac{f(\zeta) - f(z)}{|\zeta - z|^{\gamma}} \frac{|\zeta - z|^\gamma}{(\zeta - z)^2} dm(\zeta),$$

so

$$|(2)| \leq \|f\|_{\gamma} 2\pi \int_{\frac{R_0}{2}}^{d(w) + |z-w|} \frac{dr}{r^{1-\gamma}}$$

$$= \|f\|_{\gamma} \frac{2\pi}{\gamma} \left\{ (d(w) + |z - w|)^{\gamma} - \left( \frac{R_0}{2} \right)^{\gamma} \right\}$$

$$\leq \|f\|_{\gamma} \frac{2\pi \gamma^{2-\gamma}}{R_0^{1-\gamma}} \left( d(w) - \frac{R_0}{2} + |z - w| \right)$$

$$\leq \|f\|_{\gamma} \frac{2\pi \gamma^{2-\gamma}}{R_0^{1-\gamma}} (d(w) - d(z) + |z - w|),$$

having the same estimate as above.

In the situation of the case 3),

$$(2) = \int_{B_{\delta(w)}(w) \setminus B_{\delta(z)}(z)} f b_z - \int_{B_{\delta(z)}(z) \setminus B_{\delta(w)}(w)} f b_w.$$ 

- If $z, w \notin U_{\frac{R_0}{4}}$, since $|z - w| \leq \frac{R_0}{4}$, we have that $z, w \in \Omega$ or $z, w \in \Omega^c$ are the only possibilities, and also that

$$(2) = \int_{B_{\delta(w)}(w) \setminus B_{\delta(z)}(z)} f b_z - \int_{B_{\delta(z)}(z) \setminus B_{\delta(w)}(w)} f b_w.$$ 

Since $B_{d(w)}(w) \subset B_{d(z) + |z-w|}(z)$, then the first term

$$\left| \int_{B_{\delta(w)}(w) \setminus B_{\delta(z)}(z)} f b_z \right|$$

$$\leq \|f\|_{L^\infty} \frac{1}{d(z)^2} m(C_{d(z) + |z-w|}(z))$$

$$= \pi \|f\|_{L^\infty} \frac{d(z) + |z - w| + d(z)}{d(z)^2} (d(z) + |z - w| - d(z))$$

$$\leq 4\pi \frac{\|f\|_{L^\infty}}{R_0} \frac{|z - w|}{d(z)} 2 \frac{d(z) + |z - w|}{d(z)} \leq 10\pi \frac{\|f\|_{L^\infty}}{R_0} |z - w|.$$
because \( d(w) \leq d(z) + |z - w| \).

The estimate for the other term is similar.

- If \( w \notin U_{R_0} \), but \( z \in U_{R_0} \), we have that

\[
(2) = \int_{B_{d(w)}(w) \setminus B_{R_0}(z)} f b_z - \int_{B_{R_0}(z) \setminus B_{d(w)}(w)} f b_w,
\]

and we consider several subcases, working in each one \( f = \phi \) and \( f = \psi \) separately.

- If \( z, w \in \Omega \), then (2) is

\[
\int_{B_{d(w)}(w) \setminus B_{R_0}(z)} \phi b_z - \int_{B_{R_0}(z) \setminus B_{d(w)}(w)} \phi b_w = (21)_\phi = (211)_\phi - (212)_\phi,
\]

or

\[
- \int_{B_{R_0}(z) \setminus B_{R_0}(w)} \psi b_w = (21)_\psi.
\]

Now,

\[
| (211)_\phi | \leq \| \phi \|_\infty \int_{B_{d(w)}(w) \setminus B_{R_0}(z)} \frac{1}{|\zeta - z|^2} \, dm
\]

\[
\leq \frac{4 \| \phi \|_\infty}{R_0^2} m(B_{d(w)}(w) \setminus B_{R_0}(z))
\]

\[
= \frac{4\pi \| \phi \|_\infty}{R_0^2} ((d(w) + |z - w|)^2 - (\frac{R_0}{2})^2)
\]

\[
= \frac{4\pi \| \phi \|_\infty}{R_0^2} ((d(w) + |z - w|)^2 - d(z)^2)
\]

\[
\leq \frac{16\pi \| \phi \|_\infty}{R_0^2} (d(w) + |z - w| + d(z))(d(w) - d(z) + |z - w|)
\]

\[
\leq \frac{16\pi \| \phi \|_\infty}{R_0^2} (2 \text{diam}(\Omega) + R_0) |w - z|,
\]

for the distance function to a closed subset of \( \mathbb{C} \) is a Lipschitz function.

Also

\[
| (212)_\phi | \leq \frac{4 \| \phi \|_\infty}{R_0^2} m(B_{R_0}(z) \setminus B_{d(w)}(w))
\]

\[
\leq \frac{4 \| \phi \|_\infty}{R_0^2} m(C_{d(w)}^{R_0 + |z - w|}(w))
\]

\[
\leq \frac{4\pi \| \phi \|_\infty}{R_0^2} \left( \left( \frac{R_0}{2} + |z - w| - d(w) \right) \left( \frac{R_0}{2} + |z - w| + d(w) \right) \right)
\]

\[
\leq \frac{16\pi \| \phi \|_\infty}{R_0^2} (2 \text{diam}(\Omega) + R_0) |w - z|,
\]

for \( |d(w) - R_0| \leq |d(w) - d(z)| \).
The term
\[
|\varphi(21)| \leq \|\varphi\|_\infty \int_{B_{R_0}(z) \cap \Omega} |b_w| \leq \|\varphi\|_\infty \frac{4}{R_0} m(C_{d(z)}(z))
\]
\[
\leq \|\varphi\|_\infty \frac{4\pi}{R_0^2} \left( \left( \frac{R_0}{2} \right)^2 - d(z)^2 \right)
\]
\[
\leq \|\varphi\|_\infty \frac{2\pi}{R_0^2} \left( \text{diam}(\Omega) + R_0 \right) (d(w) - d(z)),
\]
because \(d(w) \geq \frac{R_0}{2}\).

- If \(z, w \in \Omega^c\), then (2) is
\[
- \int_{B_{R_0}(z)} \phi b_w = (22)\phi,
\]
or
\[
\int_{B_{d(w)}(w) \setminus B_{R_0}(z)} \psi b_z - \int_{B_{R_0}(z) \setminus B_{d(w)}(w)} \psi b_w = (22)\psi.
\]
The term \((22)\phi\) is analogous to \((21)\psi\), so
\[
|\varphi(22)| \leq \|\phi\|_\infty \frac{2\pi}{R_0^2} \left( \text{diam}(\Omega) + R_0 \right) (d(w) - d(z)).
\]
Also the term \((22)\psi = (221)\psi - (222)\psi\)
is analogous to \((21)\phi\) and
\[
|\varphi(221)| \leq \frac{4\pi}{R_0^2} \left( (d(w))^2 - \left( \frac{R_0}{2} \right)^2 \right)
\]
\[
\leq \frac{4\pi}{R_0^2} \left( (d(w) - \frac{R_0}{2}) (d(w) + \frac{R_0}{2}) \right)
\]
\[
\leq \frac{8\pi}{R_0^2} \left( \text{diam}(\Omega) + R_0 \right) \|\phi\|_\infty (d(w) - d(z)),
\]
for \(d(w) \leq |z - w| \leq d(z) \leq \frac{R_0}{2}\).

The term \((222)\psi\) is similar to the previous one.

- If \(z, w \in U_{\frac{R_0}{2}}\), we have that
\[
(2) = \int_{B_{R_0}(w) \setminus B_{R_0}(z)} f b_z - \int_{B_{R_0}(z) \setminus B_{R_0}(w)} f b_w,
\]
and then
\[
|(2)| \leq 2 \frac{4}{R_0^2} \|f\|_{L^\infty} m(B_{R_0}(w) \setminus B_{R_0}(z))
\]
and the previous procedure applies.

3. Now we consider the term
\[
(3) = \int_{B_{d(w)}(w) \cap B_{d(z)}(z)} \left[ \frac{f(\zeta) - f(z)}{|\zeta - z|^\gamma} \left( \frac{|\zeta - z|}{(\zeta - z)^2} - \frac{|f(\zeta) - f(w)}{|\zeta - w|^\gamma}} \left( \frac{|\zeta - w|}{(\zeta - w)^2} \right) \right] d\mu(\zeta)
\]
in the decomposition of (II).
Since \(|z - w| < \frac{R_0}{4}\), then \(z, w \in B_{\frac{R_0}{4}}(w) \cap B_{\frac{R_0}{2}}(z)\), we have \(d(z, \partial B_{\frac{R_0}{4}}(w)) = \delta(w) - |z - w|\) and \(d(w, \partial B_{\frac{R_0}{4}}(z)) = \delta(z) - |z - w|\), so the maximum radius of a ball centred at \(z\) and contained in \(B_{\delta(z)}(z) \cap B_{\delta(w)}(w)\) is equal to \(\min\{\delta(z), \delta(w) - |z - w|\}\), and for \(w\) this maximum radius is \(\min\{\delta(w), \delta(z) - |z - w|\}\). Also \(\min\{\delta(z), \delta(w)\} \geq \frac{R_0}{2} \geq 2|z - w|\).

Let us define

\[
\begin{align*}
  k^2_\gamma(\zeta) &= |\zeta - z|^{\gamma}, \\
  \Delta^2_\gamma(\zeta) &= \frac{f(\zeta) - f(z)}{k^2_\gamma(\zeta)},
\end{align*}
\]

If \(B_{\delta(w)}(w) \cap B_{\delta(z)}(z) \subset W\), then

\[
(3) = \int_{B_{\delta(w)}(w) \cap B_{\delta(z)}(z)} [\Delta^2_\gamma k^2_\gamma b_z - \Delta^2_\gamma k^2_\gamma b_w]
\]

\[
= \left\{ \begin{array}{c}
  \int_{B_{\frac{z - w}{2}}(z) \cap W} \Delta^2_\gamma k^2_\gamma b_z - \int_{B_{\frac{z - w}{2}}(w) \cap W} \Delta^2_\gamma k^2_\gamma b_w \\
  + \int_{B_{\delta(w)}(w) \cap B_{\delta(z)}(z) \cap W \setminus B_{\frac{z - w}{2}}(z)} \Delta^2_\gamma k^2_\gamma b_z \\
  - \int_{B_{\delta(w)}(w) \cap B_{\delta(z)}(z) \cap W \setminus B_{\frac{z - w}{2}}(w)} \Delta^2_\gamma k^2_\gamma b_w \\
  + \int_{B_{\frac{z - w}{2}}(z) \cap W^c} \Delta^2_\gamma k^2_\gamma b_z - \int_{B_{\frac{z - w}{2}}(w) \cap W^c} \Delta^2_\gamma k^2_\gamma b_w \\
  + \int_{B_{\delta(w)}(w) \cap B_{\delta(z)}(z) \cap W^c \setminus B_{\frac{z - w}{2}}(z)} \Delta^2_\gamma k^2_\gamma b_z \\
  - \int_{B_{\delta(w)}(w) \cap B_{\delta(z)}(z) \cap W^c \setminus B_{\frac{z - w}{2}}(w)} \Delta^2_\gamma k^2_\gamma b_w
\end{array} \right\}
\]

\[
= (31) + (32) + (33) + (34).
\]

Then

\[
|\Delta^2_\gamma(\zeta)| \leq 2||f||_\gamma 2\pi \int_0^{\frac{|z - w|}{2}} \frac{dr}{r^{\gamma - 1}} = \frac{4\pi ||f||_\gamma}{\gamma 2^{\gamma - 1}} |z - w|^{\gamma}.
\]

Since for \(\tau = \frac{z + w}{2}\) we have \(B_{\frac{z + w}{2}}(z) \cup B_{\frac{z + w}{2}}(w) \subset B_{\frac{z - w}{2}}(\tau)\), then

\[
(32) = \left\{ \begin{array}{c}
  \int_{B_{\frac{z - w}{2}}(\tau) \cap W \setminus B_{\frac{z - w}{2}}(z)} \Delta^2_\gamma k^2_\gamma b_z - \int_{(B_{\frac{z - w}{2}}(\tau) \cap W \setminus B_{\frac{z - w}{2}}(w)) \setminus B_{\frac{z - w}{2}}(z)} \Delta^2_\gamma k^2_\gamma b_z \\
  + \int_{B_{\frac{z - w}{2}}(\tau) \cap W^c \setminus B_{\frac{z - w}{2}}(z)} \Delta^2_\gamma k^2_\gamma b_z - \int_{(B_{\frac{z - w}{2}}(\tau) \cap W^c \setminus B_{\frac{z - w}{2}}(w)) \setminus B_{\frac{z - w}{2}}(z)} \Delta^2_\gamma k^2_\gamma b_z
\end{array} \right\}
\]

Then
and the term

\[ |(321)| \leq 2 \|f\|_\gamma \frac{4\pi}{\gamma} \left( \frac{3}{2}^\gamma - \frac{1}{2}^\gamma \right) |z - w|^\gamma. \]

For the term (322), we have

- If \( z \notin U_{B_0} \), then, as \( |z - w| < \frac{R_0}{4} \), we have that \( w, \tau = \frac{z + w}{2} \in B_{\frac{R_0}{4}}(z) \subset W \), and then, using the decomposition

\[
\Delta^\gamma_\tau k^\gamma_\tau (b_z - b_w) = \frac{1}{2}\left( \frac{\tau - z}{a} \right)^\gamma - \Delta^\gamma_{\frac{z+w}{2}} \left( k^\gamma_{\frac{z+w}{2}} (b_z - b_w) \right) - (f(z) - f(\tau))(b_z - b_w)(\tau) + (f(w) - f(\tau))(b_w - b_z)(\tau),
\]

we have

\[
(322) = \int_{(B_{\delta(w)}(w) \cap B_{\delta(z)}(z)) \setminus B_{|z - w|}(\tau)} \frac{\Delta^\gamma_\tau k^\gamma_\tau (b_z - b_w)}{B_{|z - w|}(\tau)} + \left\{ (f(w) - f(\tau)) \int_{(B_{\delta(w)}(w) \cap B_{\delta(z)}(z)) \setminus B_{|z - w|}(\tau)} b_w \right. \\
- (f(z) - f(\tau)) \int_{(B_{\delta(w)}(w) \cap B_{\delta(z)}(z)) \setminus B_{|z - w|}(\tau)} b_z \bigg\}
= (3221) + (3222).
\]

For the integral (3221) we use the change of variables

\[
\zeta \to s = \frac{\zeta - \tau}{a} = \vartheta(\zeta),
\]

where \( a = \frac{z - w}{2} \), then \( \vartheta(\tau) = 0 \) and \( J\vartheta = |a|^{-2} \).

We have

\[
k^\gamma_\tau (\zeta) (b_z - b_w)(\zeta) = |\zeta - \tau|^\gamma \left\{ \frac{1}{(\zeta - z)^2} - \frac{1}{(\zeta - w)^2} \right\}
= |a|^\gamma \left\{ \frac{1}{a^2 (s + 1)^2} - \frac{1}{a^2 (s - 1)^2} \right\}
= \frac{4s^4}{a^2 (s + 1)^2 (s - 1)^2}.
\]

And then, since \( \vartheta((B_{\delta(w)}(w)) = B_{\frac{a\delta}{\sqrt{\gamma}}}(-1), \vartheta((B_{\delta(z)}(z)) = B_{\frac{a\delta}{\sqrt{\gamma}}}(1) \) and \( \vartheta((B_{|z - w|}(\tau)) = B_2(0) \), we have that

\[
(3221) = -\frac{|a|^{\gamma+2}}{a^2} \times \int_{(B_{\delta(w)}(-1) \cap B_{\delta(z)}(1)) \setminus B_2(0)} \frac{|s|^\gamma 4s^4}{(s + 1)^2 (s - 1)^2} dm(s),
\]

so

\[
|(3221)| \leq |a|^\gamma \|f\|_\gamma \int_{C \setminus B_2(0)} \frac{4|s|^{1+\gamma}}{(s + 1)^2 (s - 1)^2} dm(s).
\]
Lemma 4.7. If $z \notin U_{\frac{R_0}{4}}$ and $|z - w| < \frac{R_0}{4}$, then
\[
\left| \int_{B_{\delta}(w) \cap B_{\delta}(z)} b_z \right| \leq 16\pi.
\]

Proof. Using Stokes formula,
\[
\int_{(B_{\delta}(w) \cap B_{\delta}(z)) \setminus B_{|z - w|}(\tau)} b_w = \frac{1}{2i} \left\{ \int_{\partial B_{\delta}(w) \cap B_{\delta}(z)} + \int_{B_{\delta}(w) \setminus \partial B_{\delta}(z)} - \int_{\partial B_{|z - w|}(\tau)} \right\} \frac{d\zeta}{\zeta - \bar{z}}
\]
\[
= (I) + (II) - (III).
\]

Then using the change $\zeta = \tau + 2|a| e^{i\theta}$, we have
\[
(III) = \int_{\partial B_{2|a|}(\tau)} \frac{d\zeta}{\zeta - \bar{a}} = 2\pi \frac{|a|}{2|a| - e^{i\theta} - \bar{a}} = 2\pi \frac{|a|}{2|a| - e^{i\theta}} \int_{S_{\delta}(0)} \frac{s ds}{2|a| - s} = 0,
\]
by the Cauchy formula.

The integral
\[
|\langle II \rangle| \leq \frac{1}{\delta(z)} \int_{\partial B_{\delta}(z)} |d\zeta| = 2\pi.
\]

In the same way, in our situation,
\[
|\langle I \rangle| \leq \frac{1}{\delta(z) - |z - w|} \int_{\partial B_{\delta}(w)} |d\zeta| = 2\pi \frac{\delta(w)}{\delta(w) - |z - w|}.
\]

If $d(w) \leq \frac{R_0}{2}$, then
\[
|\langle I \rangle| \leq 2\pi \frac{R_0}{2} \frac{\frac{R_0}{2} - \frac{R_0}{4}} = 4\pi.
\]

If $d(w) > \frac{R_0}{2}$, then
\[
|\langle I \rangle| \leq 2\pi \frac{d(w)}{d(w) - \frac{R_0}{4}} \leq 2\pi \frac{d(w)}{\frac{d(w)}{2}} = 4\pi.
\]

The lemma implies that
\[
|\langle 3222 \rangle| \leq \frac{32\pi}{2\gamma} |z - w|^\gamma \|f\|_\gamma.
\]

The situation is completely equivalent to the case of $w \notin U_{\frac{R_0}{2}}$.

This discussion above covers the completely the cases of $z \notin U_{\frac{R_0}{2}}$ or $w \notin U_{\frac{R_0}{2}}$, for then $(B_{\delta}(w) \cap B_{\delta}(z)) \cap W_c = \emptyset$, so
\[
(33) = (34) = 0.
\]
If $z, w \in U_{\frac{R_0}{2}}$, then $\delta(z) = \delta(w) = \frac{R_0}{2}$ and if $\tau'$ is a point in $\bar{W}$ at minimizing the distance to $\tau$,

\[
(322) = \int_{(B_{\frac{R_0}{2}}(w) \cap B_{\frac{R_0}{2}}(z) \cap W) \setminus B_{|z-w|}(\tau)} [(f(\zeta) - f(z)) b_z(\zeta) - (f(\zeta) - f(w)) b_w(\zeta)]
\]

\[
= \int_{(B_{\frac{R_0}{2}}(w) \cap B_{\frac{R_0}{2}}(z) \cap W) \setminus B_{|z-w|}(\tau)} (f(\zeta) - f(\tau')) [b_z(\zeta) - b_w(\zeta)]
\]

\[
+ \left\{ (f(\tau') - f(z)) \int_{(B_{\frac{R_0}{2}}(w) \cap B_{\frac{R_0}{2}}(z) \cap W) \setminus B_{|z-w|}(\tau)} b_z(\zeta)
\right.
\]

\[
- (f(\tau') - f(w)) \int_{(B_{\frac{R_0}{2}}(w) \cap B_{\frac{R_0}{2}}(z) \cap W) \setminus B_{|z-w|}(\tau)} b_z(\zeta) \right\}
\]

\[
= (3223) + (3224).
\]

Now, using the identity (21), we have

\[
(3223) = - (\bar{z} - \bar{w}) \int_{(B_{\frac{R_0}{2}}(w) \cap B_{\frac{R_0}{2}}(z) \cap W) \setminus B_{|z-w|}(\tau)} \left( \frac{f(\zeta) - f(\tau')}{|\zeta - \tau'|^\gamma} \right)
\]

\[
\times k_{\gamma,\tau}(\zeta) (\bar{z} + \bar{w} - 2 \bar{\zeta}) b_z(\zeta) b_w(\zeta).
\]

Since $\tau' \in \partial W$ then,

\[
|\tau' - \tau| \leq |\tau - z| = \frac{|z - w|}{2} = |a|,
\]

so

\[
|z - \tau'| \leq |z - \tau| + |\tau - \tau'| \leq 2 |a|
\]

and

\[
|\zeta - \tau'| \leq |\zeta - \tau| + |\tau - \tau'| \leq |\zeta - \tau| + |a|,
\]

so

\[
|z - \tau'| \gamma \leq \left( |z - \tau| + |a| \right) \gamma \leq \frac{\gamma |\zeta - \tau|}{|a|^{1-\gamma}},
\]

and then

\[
|3223| \leq \frac{\gamma |z - w|}{|a|^{1-\gamma}} \|f\|_\gamma
\]

\[
\times \int_{(B_{\frac{R_0}{2}}(w) \cap B_{\frac{R_0}{2}}(z) \cap W) \setminus B_{|z-w|}(\tau)} \frac{|\zeta - \tau|^\gamma (|z - \zeta| + |\zeta - w|)}{|\zeta - z|^2 |\zeta - w|^2} dm(\zeta)
\]

and the estimate is identical to the case of (3221).

Also, a repetition of the arguments shows that the term (3224) has the same estimate as the term (3222).
The remaining terms in this situation do not vanish necessarily and, for convenience, we write them as

\[(33) + (34) = \int_{B_{\frac{R_0}{2}}(w) \cap B_{\frac{R_0}{2}}(z) \cap W^c} [f(w)b_w(\zeta) - f(z)b_z(\zeta)] = (35)\]

\[= (f(w) - f(z)) \int_{B_{\frac{R_0}{2}}(w) \cap B_{\frac{R_0}{2}}(z) \cap W^c} b_w(\zeta)\]

\[+ f(z) \int_{B_{\frac{R_0}{2}}(w) \cap B_{\frac{R_0}{2}}(z) \cap W^c} (b_w - b_z)(\zeta)\]

\[= (351) + (352).\]

Now,

\[|\text{(351)}| \leq ||f||_\gamma |z - w|^7 \int_{B_{\frac{R_0}{2}}(w) \cap B_{\frac{R_0}{2}}(z) \cap W^c} b_w(\zeta)|\]

and

**Lemma 4.8.** If \( z, w \in U_{\frac{R_0}{2}} \cap W \), there exist \( C(n, R_0) > 0 \) such that

\[|\int_{B_{\frac{R_0}{2}}(w) \cap B_{\frac{R_0}{2}}(z) \cap W^c} b_w(\zeta)| \leq C(n, R_0).\]

**Proof.** Since

\[B_{\frac{R_0}{2} - |z - w|}(w) \subset B_{\frac{R_0}{2}}(w) \cap B_{\frac{R_0}{2}}(z) \subset B_{\frac{R_0}{2} + |z - w|}(w),\]

we have

\[\int_{B_{\frac{R_0}{2}}(w) \cap B_{\frac{R_0}{2}}(z) \cap W^c} b_w(\zeta) = \int_{B_{\frac{R_0}{2} - |z - w|}(w) \cap W^c} b_w(\zeta)\]

\[+ \int_{B_{\frac{R_0}{2}}(w) \cap B_{\frac{R_0}{2}}(z) \cap W^c \setminus B_{\frac{R_0}{2} - |z - w|}(w)} b_w(\zeta) = (A) + (B),\]

and

\[|(B)| \leq \left( \frac{R_0}{2} + |z - w| \right)^2 \pi \leq 9 \pi.\]

Also, if \( w_0 \in \partial W \) is a point minimizing the distance of \( w \) to \( W^c \), then

\[(A) = \int_{B_{\frac{R_0}{2} - |z - w|}(w) \cap \{ \rho > 0 \}} b_w(\zeta) = \int_{B_{\frac{R_0}{2} - |z - w|}(w) \cap \{ \kappa w_0 > 0 \}} b_w\]

\[+ \left\{ \int_{B_{\frac{R_0}{2} - |z - w|}(w) \cap \{ \rho > 0, \kappa w_0 < 0 \}} b_w - \int_{B_{\frac{R_0}{2} - |z - w|}(w) \cap \{ \rho < 0, \kappa w_0 > 0 \}} b_w \right\} = (A.1) + (A.2)\]

The term \((A.2)\) is bounded by the Beurling geometric lemma (lemma 4.2). Using polar coordinates, the term

\[(A.1) = \int_{\frac{R_0}{2} - |z - w|}^{\pi} \frac{dr}{r} \int_{\arcsin \frac{d(w)}{\rho}}^{\pi - \arcsin \frac{d(w)}{\rho}} e^{2i\theta} d\theta\]
and the integral in $d\theta$ is equal to

$$-2 \sin(\arcsin \frac{d(w)}{r}) = -2 \frac{d(w)}{r} \sqrt{1 - \left(\frac{d(w)}{r}\right)^2}.$$ 

Using the change of variable $s = \frac{d(w)}{r}$, we have

$$(A.1) = -2 \int_{d(w)}^{R_0 - |z-w|} ds \sqrt{1 - \left(\frac{d(w)}{r}\right)^2} = -2 \int_{\frac{1}{|z-w|}}^{1} \sqrt{1 - s^2} ds,$$ 

that is bounded in terms of $R_0$. 

Finally

$$|(352)| \leq \|f\|_{\infty} \int_{B_{R_0}(w) \cap B_{R_0}(z) \cap W^c} (b_w - b_z)(\zeta),$$

and

**Lemma 4.9.** If $z, w \in U_{R_0} \cap W$, there exist $C(n, R_0) > 0$ such that

$$|\int_{B_{R_0}(w) \cap B_{R_0}(z) \cap W^c} (b_w - b_z)(\zeta)| \leq C(n, R_0) |z - w|^\gamma.$$ 

**Proof.** If $\tau = \frac{z+w}{2}$ and $a = \frac{z-w}{2}$, then $B_{R_0}(w) \cap B_{R_0}(z) \subset B_{\frac{R_0}{\sqrt{2}}}(\tau)$ and

$$\int_{B_{R_0}(w) \cap B_{R_0}(z) \cap W^c} (b_w - b_z)(\zeta) = \int_{B_{\frac{R_0}{\sqrt{2}}}(\tau) \cap W^c} (b_w - b_z)(\zeta)$$

$$- \int_{(B_{\frac{R_0}{\sqrt{2}}}(\tau) \setminus B_{R_0}(w)) \cap W^c} (b_w - b_z)(\zeta) = (I) - (II) - (III).$$

Using the fact that $b_\zeta(\zeta) - b_\zeta(\zeta) = (\bar{z} - \bar{w}) (\bar{\zeta} - \bar{z} + \bar{z} - \bar{w}) b_\zeta(\zeta) b_w(\zeta)$ and the fact that $|\bar{\zeta} - \bar{z}|$ and $|\bar{\zeta} - \bar{w}|$ have positive lower bounds in the corresponding domains, we have the estimate for $(II)$ and $(III)$.

To estimate $(I)$, we observe that the boundary of the open set $B_{\sqrt{(\frac{R_0}{2^2})^2 - (\frac{|a|}{2^2})^2}}(\tau) \cap W^c$ is contained in the union of the circle $t \rightarrow \tau + \sqrt{\left(\frac{R_0}{2}\right)^2 - \left(\frac{|a|}{2}\right)^2} e^{it}$ and a $C^{1+\gamma}$ Jordan curve describing $B_{\sqrt{(\frac{R_0}{2^2})^2 - (\frac{|a|}{2^2})^2}}(\tau) \cap \partial W$. If $A$ is the intersection set of these two arcs, then we can decompose

$$B_{\sqrt{(\frac{R_0}{2^2})^2 - (\frac{|a|}{2^2})^2}}(\tau) \cap W^c$$

in two open disjoint sets: one containing $A$ in the boundary and where $|\bar{\zeta} - \bar{z}|$ and $|\bar{\zeta} - \bar{w}|$ have positive lower bounds independent of $|z - w|$, and the other having boundary of class $C^{1+\gamma}$ and satisfying the geometrical properties described in Proposition 7 with the same constants, also independent of $|z - w|$. 


Finally, the term
\[ (4) = \int_{B_\delta(z) \setminus B_\delta(w)(z)} f(\zeta) - f(z) \, \frac{|\zeta - z|^\gamma}{|\zeta - \bar{z}|^{\gamma + 2}} \]
\[ - \int_{B_\delta(w)(z) \setminus B_\delta(z)} f(\zeta) - f(w) \, \frac{|\zeta - w|^\gamma}{|\zeta - \bar{w}|^{\gamma + 2}} \, dm(\zeta) \]
and since \(|z - w| \leq \frac{R_\delta}{4} < \delta(w), \delta(z)|\), then we have that \(z \in B_\delta(w)(z)\). Moreover, if \(|\zeta - z| < \delta(w) - |z - w|\), then \(\delta(w) > |\zeta - z| + |z - w| \geq |\zeta - w|\). This implies that \(B_{\delta(w) - |z - w|}(\zeta) \subset B_\delta(w)(z)\)
and then, since the \(\delta(w) - |z - w| < \delta(z)|\), otherwise the domain of integration is empty, the previous integral is bounded by
\[ \int_{\partial_\delta(w) - |z - w|} \frac{dm(\zeta)}{|\zeta - z|^{1 - \gamma}} \]
\[ = 2\pi \int_{\partial_\delta(w) - |z - w|} \frac{dr}{r^{1 - \gamma}} \]
\[ = \frac{2\pi}{\gamma} \{ \delta(z)^\gamma - (\delta(w) - |z - w|)^\gamma \} \]
\[ = \frac{2\pi}{\delta(w) - |z - w| + \lambda (\delta(z) - (\delta(w) - |z - w|))^{1 - \gamma}} \]
\[ \leq \frac{2\pi}{\delta(w) - |z - w| + \lambda |z - w|} \]
where \(\lambda \in (0, 1)\).

The term (42) is symmetric to (41) and admits an analogous treatment.

With the previous arguments we have that the conjugate Beurling transforms of \(\phi\) and \(\psi\) are Hölder-\(\gamma\) at \(\Omega\) or \((\Omega)^c\). The following lemma completes the case of \(\partial\Omega^c\).

**Lemma 4.10.** If \(W = \Omega\) or \(W = (\Omega)^c\) and \(f \in \text{Lip}(\gamma, W)\), then \(f\) extends to a Lipschitz function on \(W\), with the same Lipschitz norm.

**Proof.** Since \(\partial W\) is compact, then \(f\) is uniformly continuous in \(\bar{U}_{\frac{R_\delta}{4}}\).

If \(z, w \in \partial\Omega \) and \(|z - w| < \frac{R_\delta}{4}\), for \(U_{\frac{R_\delta}{4}} \cap W\) we have
\[ f(z) - f(w) = f(z) - f(z') + f(z') - f(w') + f(w') - f(w) = (1) + (2) + (3). \]
If \(|z' - z|, |w' - w| < \min\{\delta, \frac{|z - w|}{3}\}\), then
\[ (1), (3) \leq M |z - w|^\gamma. \]
Also
\[ |z' - w'| \leq \frac{5}{3} |z - w|, \]
so
\[ (2) \leq M |z - w|^\gamma. \]

Finally, we can apply this lemma to the last term in the decomposition given in proposition 7 and we have

**Corollary 1.** There exists a constant $C_1$, depending only on $R_0$, $\nu_0$, $\text{diam}(\Omega)$, $\gamma$, such that
\[ |f(z) \Theta_{\Omega}^{R_0}(z) - f(w) \Theta_{\Omega}^{R_0}(w)| \leq |z - w|^\gamma \|f\| C_1. \]

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