Self-organization in a group of mobile autonomous agents *

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Abstract: This paper considers a discrete time swarm model of a group of mobile autonomous agents with a simple attraction and repulsion function for swarm aggregation and investigates its stability properties. In particular, it is proved that the individuals (members) of the swarm will aggregate and form a cohesive cluster of a finite size depending only on the parameters of the swarm model in a finite time, and the swarm system is completely stable.

Keywords: Swarms; stability; cohesion; discrete-time systems

1 Introduction

In recent years the topic of swarms has attracted considerable attention because swarming behavior can be found in many organisms in nature, ranging from simple bacteria to large animals. Generally speaking, “swarm behavior” is a kind of aggregate motion that a variety of organisms have the ability to cooperatively forage for food while trying to avoid predators and other risks. Examples include colonies of ants, flocks of birds, and groups of animals. Understanding the operational principles of such motions in swarms is useful in developing distributed cooperative control, coordination, formation control, and learning strategies for autonomous agent systems such as autonomous multi-robots; satellite group maneuvers; multiple unmanned undersea/aerial vehicles, etc. The increasing interest has been motivated to study the swarming behavior and its applications, and work on modelling of swarming behavior [1]–[4]. Specially, the collective dynamics models have been explored in [5]–[14]. In this paper, we study swarming behavior in terms of aggregation, cohesion, and stability of a group of mobile autonomous agents.

In [15], Gazi and Passino proposed an “individual-based” continuous time model for swarm aggregation in n-dimensional space with the same attraction-repulsion rule and with identical interaction strength among these individuals (or members). They showed that the swarm with such model has the properties of aggregation, cohesion and stability. In this paper we develop a discrete time swarm model of a group of mobile autonomous agents moving in n-dimensional space, and analyze its aggregation and stability properties. Particularly, we will prove that individual agents can form a cohesive swarm with a finite size, which depends only on parameters of the swarm model, in a finite time.

In section 2, we present the swarm model. Then we show that the model can converge to the center and have complete stability behavior (i.e., cohesion)
in section 3 and section 4, respectively. Finally we briefly summarize the results of this paper in section 5.

## 2 Swarm Model

We consider a swarm of $M$ individuals (or members) in an $n$-dimensional Euclidian space. We assume the motion is synchronous without delays, i.e., all these individuals move simultaneously and know the exact position of all the other members. The model is given by

$$x_i(k + 1) = x_i(k) + \sum_{j=1,j\neq i}^{M} g(x_i(k) - x_j(k)) \quad (2.1)$$

where $x_i(k) \in \mathbb{R}^n$ represents the position of individual $i$ at time $k$, $i \in S$, $k \in N$; sets $S = \{1, 2, \cdots, M\}$ and $N = \{0, 1, 2, \cdots\}$: $g(\cdot)$ represents the function of attraction and repulsion between the members. That is, the direction and magnitude of every individual depend on the sum of the attraction and repulsion of the other individuals on it. The attraction-repulsion function $g(\cdot)$ is of the form

$$g(y(k)) = -y(k)\left( a - b \exp \left( -\frac{\|y(k)\|^2}{c} \right) \right) \quad (2.2)$$

where $a, b, c$ are positive constants, and $\|y(k)\|^2 = y(k)^\top y(k)$, $\forall k \in N$.

We can see that this function is attractive for large distances and repulsion for small distances, and the function $g(\cdot)$ changes its sign when $g(\cdot) = 0$ ($b > a$), i.e., at the set of points as

$$\Theta = \left\{ y = 0 \text{ or } \|y(k)\| = \sqrt{\frac{b}{c \ln \left( \frac{b}{a} \right)}} = \delta \right\}.$$

**Definition 1:** The center of the swarm members is

$$\bar{x}(k) = \frac{1}{M} \sum_{i=1}^{M} x_i(k), \forall k \in N.$$

By the symmetry property of function $g(\cdot)$ with respect to the origin, individual $i$ moves to each other individual $j$ just the same amount as $j$ moves to $i$. This implies that the center $\bar{x}(k)$ is stationary for all $k$. Simultaneously, we have

$$\sum_{j=1}^{M} (x_i(k) - x_j(k)) = M(x_i(k) - \bar{x}(k)) = Me_i(k). \quad (2.3)$$

**Lemma 1:** The center $\bar{x}(k)$ of the swarm described in Eqs. (2.1) and (2.2) is stationary for all $k$.

**Proof:** By the symmetry of function $g(\cdot)$, we know that

$$\sum_{i=1}^{M} \sum_{j=1,j\neq i}^{M} g(x_i(k) - x_j(k)) = 0,$$

then for all $k$. It follows that

$$\bar{x}(k + 1) = \frac{1}{M} \sum_{i=1}^{M} x_i(k + 1)$$

$$= \bar{x}(k).$$

Hence the swarm center $\bar{x}(k)$ is stationary.

This lemma says that the swarm described in Eqs. (2.1) and (2.2) is not drifting on average. In the next section, we show that the individuals will move toward the swarm center and form a cohesive cluster around it.

## 3 Swarm Aggregation

This section presents results concerning aggregation properties of the autonomous agents swarm modeled in Eqs. (2.1) and (2.2).

**Definition 2:** A swarm member $i$ is a free agent if, for all $k$,

$$\|x_i(k) - x_j(k)\| > \delta, \forall j \in S, j \neq i.$$

Since the swarm center $\bar{x}(k)$ is stationary, it is noted $\bar{x}$ for simplicity.

**Definition 3:** The error variable is defined as

$$e_i(k) = x_i(k) - \bar{x}, \forall i \in S, \forall k \in N.$$

**Assumption 1:** All the swarm members are free agents at any time $k$.

**Lemma 2:** If a member $i$ of the swarm is described by Eqs. (2.1) and (2.2), under Assumption
1, its distance to the center $\bar{x}$ of the swarm is greater than $\delta$, i.e.,

$$\|e_i(k)\| = \|x_i(k) - \bar{x}\| > \delta.$$  

Then, at any time $k$, it moves toward the center $\bar{x}$; in other words, its motion is in a direction of decrease of $\|e_i(k)\|$.  

**Proof:** Set

$$
\Lambda(i, j; k) = \exp\left(-\frac{\|x_i(k) - x_j(k)\|^2}{c}\right),
$$

$$
\Phi = \exp\left(-\frac{\delta^2}{c}\right),
$$

$$
x_{ij}(k) = x_i(k) - x_j(k),
$$

and by (2.3) and Assumption 1, we have

$$x_i(k+1) = x_i(k) - \sum_{j=1}^{M} x_{ij}(k)(a - b\Lambda(i, j; k))$$

$$= x_i(k) - aM e_i(k) + b \sum_{j=1, j \neq i}^{M} x_{ij}(k)\Lambda(i, j; k),$$

$$= (1 - aM)e_i(k) + b \sum_{j=1, j \neq i}^{M} x_{ij}(k)\Lambda(i, j; k) + \bar{x},$$

thus

$$e_i(k+1) = (1 - aM)e_i(k) + b \sum_{j=1, j \neq i}^{M} x_{ij}(k)\Lambda(i, j; k).$$

Let $V_i(k+1) = \frac{1}{2} e_i^T(k+1)e_i(k+1)$, then

$$\Delta V_i(k) = V_i(k+1) - V_i(k)$$

$$\leq -a\|e_i(k)\|^2 - a(M - 1)\|e_i(k)\|^2$$

$$+ \frac{b}{2}\left( a^2 + b^2\Phi^2 \right)M^2\|e_i(k)\|^2$$

$$+ b\left(1 + aM\right)(M - 1)\delta\Phi\|e_i(k)\|.$$  

So

$$\Delta V_i(k) \leq -a\|e_i(k)\|^2 - a(M - 1)\|e_i(k)\|^2$$

$$\left( a - \frac{a^2 + b^2\Phi^2}{2(M - 1)} \right)\|e_i(k)\|$$

$$- (1 + aM)b\delta\Phi.$$  

If

$$\|e_i(k)\| \geq \frac{1 + aM}{a - \frac{a^2 + b^2\Phi^2}{2(M - 1)}}b\delta\Phi \geq \delta,$$

for $M$ large enough, then

$$\Delta V_i(k) \leq -a\|e_i(k)\|^2 = -2aV_i(k).$$

Thus the result holds.  

**Remark 1:** Lemma 2 does not say that $x_i(k)$ will converge to $\bar{x}$ for all $i$ and $k$.

**Theorem 1:** As time progresses all the members of the swarm described in Eqs. (2.1) and (2.2) will enter into a bounded hyperball in a finite time bound $k$,

$$B_{\varepsilon}(\bar{x}) = \{x_i(k) : \|x_i(k) - \bar{x}\| \leq \varepsilon\}$$

where

$$\varepsilon = \frac{b}{a}\sqrt{c \exp(- \frac{1}{2})},$$

and

$$\bar{k} = \max_{i \in \mathbb{S}} \left[ \frac{V_i(0)}{\varepsilon^2} \right].$$

**Proof:** Let $V_i(k+1) = \frac{1}{2} e_i^T(k+1)e_i(k+1)$, and by Lemma 2, we know that if $\|e_i(k)\| \geq \delta$, then we will have

$$\Delta V_i(k) \leq 0.$$  

Note that function $\Psi = \Lambda(i, j; k)\|x_i(k) - x_j(k)\|$ is a bounded decreasing function of the distance with the maximum occurring at $\|x_i(k) - x_j(k)\| = \delta$, and we can obtain a position independent bound by using its maximum. (i.e., solving the continuous equation $\frac{\partial}{\partial y} \left( y \exp\left(-\frac{y^2}{2}\right) \right) = 0$. We know that it occurs at $\|x_i(k) - x_j(k)\| = \sqrt{\frac{2}{\pi}}$. So we can obtain that $\Delta V_i(k) \leq 0$ as long as

$$\|e_i(k)\| \geq \frac{b}{a}\sqrt{c \exp(- \frac{1}{2})}.$$  

Define $\varepsilon = \frac{b}{a}\sqrt{\frac{c}{\pi}} \exp(- \frac{1}{2})$. This implies that as $k \to \infty$, $e_i(k)$ converges within the ball $B_{\varepsilon}(\bar{x})$. Notice that $i$ is arbitrary, so the result holds for all the swarm members.

Next, the finite time bound will be presented. Since

$$\Delta V_i(k) = V_i(k+1) - V_i(k)$$

and by $\Delta V_i(k) \leq -a\|e_i(k)\|^2 = -2aV_i(k)$, we have

$$V_i(k+1) - V_i(0) = \Delta V_i(0) + \cdots + \Delta V_i(k)$$

$$\leq -2a(V_i(0) + V_i(1) + \cdots + V_i(k)).$$
Since \( V_i(k) = \frac{1}{2} \| e_i(k) \|^2 \), we have, for \( \| e_i(k) \| = \varepsilon, \forall k \in N \),
\[
V_i(0) \geq 2a \frac{1}{2} \varepsilon^2(k+1) + \frac{1}{2} \varepsilon^2 \geq ak \varepsilon^2,
\]
thus for all \( k \),
\[
k \leq \frac{V_i(0)}{a \varepsilon^2},
\]
so we obtain the result,
\[
\bar{k} = \max_{i \in S} \left[ \frac{V_i(0)}{a \varepsilon^2} \right].
\]

**Remark 2:** Notice that this theorem is very important not only because it shows the aggregation of the swarm and gives an explicit bound on the size of the swarm in a finite time bound, but also it says that when \( M \) is large enough, the bound on the swarm size only depends on the model parameters \( a, b, c \) and is almost independent of the number of the swarm members. However, it does not say anything about the motion of the swarm members in the hyperball. Next we will study the issue further.

### 4 Swarm Stability

Now we prove that the swarm system described in Eqs. (2.1) and (2.2) will converge to its equilibrium points, and the configuration of the swarm members converges to a constant arrangement. Now we define the invariable set of equilibrium points of the swarm system as
\[
\Omega_c = \left\{ x : \sum_{j=1,j \neq i}^M g(x_i - x_j) = 0, i \in S \right\}
\]
where \( x = \{ x_1^T, \ldots, x_M^T \}^T \in \mathbb{R}^{Mn} \).

**Theorem 2:** As \( k \to \infty \), all the members of the swarm described in Eqs. (2.1) and (2.2) converge to \( \Omega_c \).

**Proof:** For all \( k \in N \), let the Lyapunov function be
\[
J(x_i(k)) = \frac{1}{2} \sum_{i=1}^{M-1} \sum_{j=i+1}^M \| x_i(k) - x_j(k) \|^2
\]
and we have
\[
\Delta J(x_i(k)) = J(x_i(k+1)) - J(x_i(k)).
\]
In fact, it can be seen that \( \Delta J(x_i(k)) \leq 0 \) as long as
\[
\tilde{\Delta} = \| x_i(k+1) - x_j(k+1) \|^2 - \| x_i(k) - x_j(k) \|^2 \leq 0.
\]
So next we will consider \( \tilde{\Delta} \), and for simplicity, we set
\[
\theta(i, l; k) = \exp \left( - \frac{\| x_i(k) - x_l(k) \|^2}{c} \right),
\]
\[
\phi(j, l; k) = \exp \left( - \frac{\| x_j(k) - x_l(k) \|^2}{c} \right),
\]
\[
x_{ij}(k) = x_i(k) - x_j(k),
\]
\[
x_{il}(k) = x_i(k) - x_l(k),
\]
\[
x_{jl}(k) = x_j(k) - x_l(k),
\]
\[
\Delta = x_i(k+1) - x_j(k+1), \quad \Phi = \exp(-\frac{\delta^2}{c}),
\]
and then
\[
\Delta = (1 - aM)x_{ij}(k) + b \left( \sum_{l=1, l \neq i}^M \theta(i, l; k)x_{il}(k) - \sum_{l=1, l \neq j}^M \phi(j, l; k)x_{jl}(k) \right),
\]
so we have
\[
\| \Delta \|^2 = \Delta^T \Delta 
\leq (1 - aM)^2 \| x_{ij}(k) \|^2 + b^2 \Phi^2 M^2 \| x_{ij}(k) \|^2 
+ 4b \delta (1 + aM)(M - 1) \Phi \| x_{ij}(k) \|.
\]
Then,
\[
\frac{1}{2} \tilde{\Delta} = -a \| x_{ij}(k) \|^2 - (M - 1) \| x_{ij}(k) \| 
\times \left( a - \frac{a^2 + b^2 \Phi^2}{2(M - 1)} \right) \| x_{ij}(k) \| 
- 2b \delta (1 + aM) \Phi .
\]
Under Assumption 1, when \( M \) is large enough, we have
\[
\| x_{ij}(k) \| \geq \delta,
\]
then
\[
\| \Delta \|^2 \leq -a \| x_{ij}(k) \|^2 \leq 0.
\]
From this,
\[ \Delta J(x_i(k)) = J(x_i(k+1)) - J(x_i(k)) \leq 0. \]
Thus by the LaSalle’s Invariance Principle we can obtain the fact that as \( k \to \infty \) the state \( x_i(k) \) converges to the largest invariant subset of the set defined as
\[
\Omega = \{ x : \Delta J(x) = 0 \} = \left\{ x : \sum_{j=1, j \neq i}^{M} g(x_i - x_j) = 0 \right\} = \Omega_e.
\]
Because each point in \( \Omega_e \) is an equilibrium, \( \Omega \) is an invariant set. This result is proved. \( \square \)

**Remark 3:** Notice that the stability of the swarm system implies global convergence to its equilibrium point set. And we know that it does not require the system to have unique equilibrium point. Moreover, note also that in the analysis of the above results, the dimension of the state space \( n \) was not used so that these results hold for any dimension.

5  **Conclusion**

We have shown that the swarm model described in Eqs. (2.1) and (2.2) can exhibit aggregation, cohesion, and global convergence behavior. That is, such properties are of practical interest in formation control of multi-robot systems. All members of the swarm eventually enter into a finite size ball in a finite bounded time and converge to a constant arrangement.

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