CGAL Made More Accessible

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Abstract

We introduce Python bindings that enable the convenient, efficient, and reliable use of software modules of CGAL (The Computational Geometry Algorithms Library), which are written in C++, from within code written in Python. There are different tools that facilitate the creation of such bindings. We present a short study that compares three main tools, which leads to the tool of choice. The implementation of algorithms and data structures in computational geometry presents tremendous difficulties, such as obtaining robust software despite the use of (inexact) floating point arithmetic, found in standard hardware, and meticulous handling of all degenerate cases, which typically are in abundance. The code of CGAL extensively uses function and class templates in order to handle these difficulties, which implies that the programmer has to make many choices that are resolved during compile time (of the C++ modules). While bindings take effect at run time (of the Python code), the type of the C++ objects that are bound must be known when the bindings are generated, that is, when they are compiled. The types of the bound objects are instances (instantiated types) of C++ function and class templates. The number of object types that can potentially be bound, in implementation of generic computational-geometry algorithms, is enormous; thus, the generation of the bindings for all these types in advance is practically impossible. For example, the programmer needs to choose among a dozen types of curves (e.g., line segments, circular arcs, geodesic arcs on a sphere, or polylines of any curve type) to yield a desired arrangement type; often there are several choices to make, resulting in a prohibitively large number of combinations. We present a system that rapidly generates bindings for desired object types according to user prescriptions, which enables the convenient use of any subset of bound object types concurrently. After many years, in which the usage of these packages was restricted to C++ experts, the introduction of the bindings made them easily accessible to newcomers and practitioners in non-computing fields, as we report in the paper. Additional information can be found at http://acg.cs.tau.ac.il/cgal-python-bindings. The bindings software can be found at https://bitbucket.org/taucgl/cgal-python-bindings.

1 Introduction

CGAL (The Computational Geometry Algorithms Library) is an open source software project that provides access to efficient and reliable implementations of geometric algorithms in the form of a C++ library [31]. CGAL has evolved through the years and now represents the state of the art in computational geometry software. CGAL is used in a diverse range of domains requiring geometric computation, such as computer graphics, scientific visualization, computer-aided design and modeling, geographic information systems, molecular biology, and medical imaging. CGAL provides a large number of components that cover a wide range of areas. The bindings for most components are supported now. The principles for the bindings described in this paper applies to all packages of CGAL (and perhaps to other generic C++ libraries as well), and can be easily adopted to support the bindings of the few components that are still missing.

There is a large community of users of CGAL packages in academia and the industry, and these packages seem to be helpful in many diverse projects. In all the projects that we are aware of, the packages have been used by C++ experts or by users who have been strongly supported by such experts from the CGAL developer community, including GeometryFactory.1 In our experience, incorporating CGAL in education or in academic projects lacking C++ experts has been rugged. We attribute the difficulties that arise in such settings to the highly sophisticated usage of C++ in CGAL. The binding project that we describe in this paper is meant to mitigate these difficulties and make all CGAL packages more accessible to a wider audience of programmers. We already see, and report below on, first signs that the project is on a good trajectory toward achieving this goal.

The software modules of CGAL rigorously adhere to the generic programming paradigm—a discipline that consists of the gradual lifting of concrete algorithms abstracting over details, while retaining the algorithm semantics and efficiency. The software modules of CGAL also follow the exact geometric-computation paradigm, which simply amounts to ensuring that errors in predicate evaluations do not occur; it guarantees robustness of the applied algorithms. As a result, the software of CGAL is robust despite the use of (inexact) floating point arithmetic, found in standard hardware, it is complete as it handles all degenerate cases, which typically are in abundance, and it is efficient—all at the same time.

1GeometryFactory is the company established to market and support the usage of CGAL packages.
Generic programming identifies an abstraction that consists of a formal hierarchy of polymorphic abstract (syntactic and semantic) requirements on data types, referred to as concepts, and a set of classes that conform precisely to the specified requirements, referred to as models [2]. A hierarchy of related concepts can be viewed as a directed acyclic graph, where a node of the graph represents a concept and an arc represents a refinement relation. An arc directed from concept $A$ to concept $B$ indicates that concept $B$ refines concept $A$. A model of concept $B$ is inherently also a model of concept $A$. When a class template is instantiated, each one of its template parameters is substituted with a model of one or more concepts associated with the template parameter.

1.1 Python Bindings for C++ Modules

Python, like many other languages, was originally defined by a reference implementation. The original reference implementation of the Python programming language is CPython. To date, CPython is still the default and commonly used reference implementation of the language, although newcomers are emerging, see, e.g., PyPy. As the Python language evolved, its definition has tightened and a well-defined specification has materialized.

Bindings are essentially wrapper libraries that bridge two programming languages, so that a software module developed in one language can be used in code written in another language, exploiting (i) the strengths of both languages, and (ii) the availability of modules developed in the former language. C++ Python bindings enable (i) the invocation of C++ functions, and (ii) the access to C++ variables from Python code, taking advantage of Python’s quick development cycles and C++’s high performance.

Python is sufficiently efficient for many tasks; nevertheless low-level code written in Python tends to be too slow, largely because Python is dynamically typed. In particular, low-level computational loops are simply infeasible. In addition, the magnitude of existing and well-tested code in Fortran, C, or C++, is enormous. The Python/C API, the module that enables the usage of external modules developed in Fortran, C, or C++, from Python code, or embedding Python code in code written in other languages, is very rich. Indeed, the Python/C API module of CPython exposes a vast amount of details of CPython internals. It enables the efficient exploitation of existing code in Fortran, C, or C++, as well as the replacement of critical sections written in Python when speed is essential. Wrapping existing code has traditionally been the domain of Python experts due to the steep learning curve, which characterizes its Python/C API. Although using such wrappers is possible without ever knowing their internals, providing such wrappers creates a sharp line between developers using Python and developers using C or C++ with the Python/C API. Several tools that facilitate the creation of such wrappers have been introduced; a sample of such tools is listed in Section 2.

When running Python code that uses bindings for some C++ modules, one or more libraries that provide the bindings must be accessible. A software module in C++ that adheres to the generic programming paradigm consists of function and class templates; these templates are instantiated at compile time of the binding libraries. In other words, the types of the C++ objects that are bound, that is, instances (instantiated types) of C++ function and class templates, must be known when the bindings are generated.

1.2 CMAKE

CMAKE [27] is an open-source, cross-platform suite of tools designed to build, test, and package software. The suite of CMAKE tools were created to address the need for cross-platform build environments for open-source projects. However, since it was conceived over two decades ago, it has grown and become powerful, rich, and flexible. It is now considered a modern tool; it is widely spread and used by major commercial software products for their build and test environments.

CMAKE, for the most part, is a cross-platform build system generator. It is used to control the software build process using simple platform and compiler independent configuration files, and it generates native build environments of the user choice. Users build a project by using CMAKE to generate a build system for native tools on their platforms. CMAKE supports an interpreted, functional, scripting language. The build process of a project is specified with platform-independent CMAKE text files, named CMakeLists.txt, written in the CMAKE language, included in several directories of the source tree of the project. For example, once the CMAKE configuration files are in place for a certain project, CMAKE can be used to generate Makefile files required by the make utility of the GNU suite to compile a project written in C++ on a Unix system using the g++ compiler. The command-line interface of the cross-platform build system generator of CMAKE is an executable called cmake. (It has a counterpart called cmake−gui, which includes a graphical user interface.) In order to generate native project build files, the user invokes cmake in a terminal and specifies the directory that contains the root CMakeLists.txt file.

A key feature of CMAKE is the ability to (optionally) place compiler outputs (such as object files) outside the source tree. This enables multiple and concurrent builds from the same source tree and cross-compilation. It also

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2PyPy is a replacement for CPython; see https://www.pypy.org.
keeps the source tree clean and enables the removal of entire build directories without impairing the source files. We exploit this feature, and the scripting capabilities to conveniently generate the C++ Python bindings for several CGAL objects concurrently.

1.3 Bindings for CGAL and Other Geometric Software

Instantiated types in CGAL are characterized by long instantiation chains of C++ class templates. An instantiated type is a template that has one or more template parameters and every parameter is substituted by another type, which is typically an instance of another template. While the number of models of most concepts is small, the number of potential types of objects that could be bound, and thus must be supported by the bindings, is enormous. Offering bindings for all these types in advance is practically impossible. Moreover, in some cases models need to be extended with types provided by the user, which may not be available when the bindings are generated.

CGAL uses CMAKE [27] for the build process. Since version 5.0, CGAL is header-only by default, which means that an application that depends on CGAL only depends on (i) the source code of CGAL that resides in header files, (the names of which typically end with the suffix .hpp), and (ii) native configuration files generated by CMAKE (in particular, the execution of cmake). In other words, there is no need to compile any library before an application that depends on CGAL and written in C++ is built. On the other hand, when running an application written in Python that depends on CGAL, the libraries that contain the bindings must be accessible. Typically, those bindings are generated ahead of time with little knowledge about the application itself. This would impose a severe limitation in cases like ours, where the number of objects to be bound is large. In cases where the types of the objects to be bound are closely tied with the application itself, generating bindings ahead of time is impractical. Our system enables easy and convenient (re)generation of bindings, thus alleviating the burden caused by these problems.

The Shapely Python package supports planar geometric algorithms provided by GEOS [17]. The common functionality exposed by Shapely and our bindings is small. The scikit-geometry Python package supports geometric algorithms derived from CGAL. The API of scikit-geometry is detached from CGAL. While the API is Pythonic in its nature, the package exposes only a small fraction of the rich API of CGAL. C++ Python bindings are also available for a certain subset of CGAL packages as part of an experimental project that uses SWIG. While the project has been conceived more than a decade ago, it contains bindings only for a small set of type and function instances of a limited subset of CGAL packages.

1.4 Contribution

We introduce C++ Python bindings that enable the convenient, efficient, and reliable use of CGAL software modules in Python code. There are different tools that facilitate the creation of C++ Python bindings. We present a short study that compares several tools, which leads to the method of choice. The binding themselves are implemented in C++ and their implementation exploits advanced features in C++, which we explicate. This results in elegant and compact code that is easy to maintain and extend with additional bindings. We present a system that rapidly generates bindings of desired objects according to user prescriptions, which enables the convenient use of any subset of bound object types concurrently. With our system it is possible to generate a single library that contains bindings for instances of different CGAL templates, e.g., an instance of the 2D arrangement class template and an instance of the 2D triangulation class template, or several libraries (which can be used concurrently), such that distinct libraries contain different instances of the same template.

1.5 Outline

The rest of this paper is organized as follows. A study that compares several methods for implementing C++ Python bindings is presented in Section 2. General instructions for rapidly generating bindings with our system are presented in Section 3. The description of the bindings generated by our system for various modules are given in Section 4. A sample of highlighted modern C++ techniques and idioms used in the binding implementation are described in Section 5. The tight coupling between concepts and binding generation is described in Section 6. Applications of the new bindings, which are geared toward making the CGAL code more accessible are presented in Section 7.

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3Some dependencies of CGAL need to be installed in advance.
4See https://github.com/scikit-geometry/scikit-geometry.
5See http://www.swig.org/.
1.6 Conventions

The paper is packed with code samples written either in the CMake, Python, or C++ languages. We use several conventions to improve readability. Code excerpt written in C++ is always numbered, while short code excerpts written in Python is sometimes preceded with `>>>`. The C++ identifier `py` is an alias for the namespace `nanobind`.

2 Python Bindings Tool Comparison

Several tools that facilitate binding generation have been introduced throughout the years. The following is by no means an exhaustive list of such tools; however, it gives a taste of the rich possibilities one faces when there is a need to make C++ code accessible to Python developers. One goal that all these tools share is transforming Python/C API into a relatively more user-friendly interface. Most listed tools are designed to wrap C++ interfaces non-intrusively; that is, the C++ code of an entity does not change when it is wrapped.

2.1 Binding Tools

c**types** is a foreign function module for Python that provides C compatible data types.\(^6\) It supports the loading of shared libraries and marshaling data between Python and C.\(^7\) The `ctypes` module is part of the Python standard library; thus, it can be used to wrap shared libraries in pure Python. Using the `ctypes` module requires manual programming labour and detailed knowledge of the `ctypes` interface. However, it does not require writing C code (in addition to existing C code that needs to be wrapped).

**cffi** is another Python module that generates Python bindings (it stands for C Foreign Function Interface). It interacts with almost any C code from Python, based on C-like declarations that can often be copy-pasted from header files. It is compatible with both PyPy and CPython; Moreover, in the PyPy (see 1.1) reference implementation it is a first class citizen (and provided by default), thus, exhibits outstanding performance. Compared to `ctypes`, the `cffi` module takes a more automated approach to generate Python bindings and it minimizes the extra bits of API that need to be mastered; thus, it scales better to larger projects than `ctypes`. Like `ctypes`, `cffi` directly interfaces C libraries only. Wrapping C++ libraries with any one of these modules requires the provision of another layer of C wrapper around the C++ code.

**cppyy** is yet another external module for generating bindings.\(^8\) It is an automatic, run-time, C++ Python bindings generator, for calling C++ from Python and Python from C++.\(^9\) Run-time generation enables detailed specialization for higher performance, lazy loading for reduced memory use in large scale projects, Python-side cross-inheritance and callbacks for working with C++ frameworks, run-time template instantiation, automatic object downcasting, exception mapping, and interactive exploration of C++ libraries. `cppyy` delivers this without any language extensions, intermediate languages, or the need for boiler-plate hand-written code. For design and performance, see [25], albeit that the `cppyy` performance has been vastly improved since. Like `cffi`, `cppyy` was designed to ease binding development for the Python programmer, minimizing the need for extra C++ code; it is also compatible with both CPython and PyPy.

A completely different approach is adopted by Cython [4], which refers to a programming language and to an optimising static compiler for the Cython language.\(^10\) The Cython language is a superset of the Python language. It extends Python with explicit type declarations of native C/C++ types. Cython is a compiled language used to generate CPython extension modules. The Cython compiler generates C/C++ code from Cython code, which in turn compiles with any C/C++ compiler, and is automatically wrapped in interface code, producing extension modules that can be loaded and used by regular Python code using the `import` statement, but with significantly less computational overhead at run time. Cython also facilitates wrapping independent C or C++ code into python-importable modules. With Cython it is possible to call back and forth between Python code, Cython code, and native library code originally developed in Fortran, C, or C++.

**SIP** is a collection of tools that makes it easy to create Python bindings for C and C++ libraries.\(^11\) It was originally developed in 1998 to create PyQt, the Python bindings for the Qt toolkit, but can be used to create bindings for any C or C++ library. SIP comprises a set of build tools and a module called sip. The build tools process a set of specification files and generates C or C++ code, which is then compiled to create the bindings extension module.

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\(^{6}\)See https://docs.python.org/3/library/ctypes.html#module-ctypes.

\(^{7}\)Marshaling is the process of transforming the memory representation of an object into a data format suitable for storage or transmission; see https://en.wikipedia.org/wiki/Marshalling_(computer_science).

\(^{8}\)See https://cppyy.readthedocs.io.

\(^{9}\)See https://cppyy.readthedocs.io.

\(^{10}\)See https://cython.org/.

\(^{11}\)See https://pypsi.org/project/sip/.
Several extension modules may be installed in the same Python package. Extension modules can be built so that they are independent of the version of Python being used.

SWIG is a software development tool that connects programs written in C and C++ with a variety of high-level programming languages.\(^\text{12}\) It is different than any of the others tools listed here. SWIG is used to create bindings to C and C++ code for different types of target languages, and not only Python, including common scripting languages such as Javascript, Perl, PHP, Python, Tcl, or Ruby. The list of supported languages also includes non-scripting languages such as C#, D, Go language, Java including Android, Lua, OCaml, Octave, Scilab and R. Also several interpreted and compiled Scheme implementations (Guile, MzScheme/Racket) are supported. SWIG is most commonly used to create high-level interpreted or compiled programming environments, user interfaces, and as a tool for testing and prototyping C/C++ software. SWIG is typically used to parse C/C++ interfaces and generate the ‘glue code’ required for the above target languages to call into the C/C++ code. SWIG can also export its parse tree in the form of XML. SWIG is free software and the code that SWIG generates is compatible with both commercial and non-commercial projects.

Boost is a large and sophisticated suite of utility libraries that works with almost every C++ compiler in existence. Boost.Python \(^\text{13}\) is a member of the Boost suit that enables seamless interoperability between C++ and the Python programming language. Similar to cppyy, Boost.Python focuses at C++; unlike cppyy it uses C++ to specify and build bindings, taking advantage of the metaprogramming tools and polymorphism in C++. The library includes support for:

- References and shared pointers (which are handled extremely gracefully)
- Globally Registered Type Coercions
- Automatic Cross-Module Type Conversions
- Efficient Function Overloading
- C++ to Python Exception Translation
- Default Arguments
- Keyword Arguments
- Manipulating Python objects in C++
- Exporting C++ Iterators as Python Iterators
- Documentation Strings

Nanobind \(^\text{14}\) is a lightweight header-only library that exposes C++ types in Python and vice versa, mainly to create Python bindings of existing C++ code.\(^\text{14}\) It is the evolution of PyBind11.\(^\text{15}\) Its goal, like the goal of Boost.Python, is to minimize boilerplate code in traditional extension modules by inferring type information using compile-time introspection. Its syntax is also similar to the syntax of Boost.Python. In fact, it is a tiny self-contained version of Boost.Python with everything stripped away that isn’t relevant for binding generation. This compact implementation was possible thanks to some of the new C++11 language features (specifically, tuples, lambda functions and variadic templates). Since its creation, these libraries have grown beyond Boost.Python in several ways, leading to simpler binding code in many common situations.

We have concentrated on the three most propitious tools, namely SWIG, Nanobind, and Boost.Python and conducted several experiments to compare the performance of the bindings generated using the selected tools; see Section 2.2.

### 2.2 Experiments

Limited Python bindings for CGAL are available as part of an experimental project that uses SWIG; see Section 2.1. Naturally, embarking on the existing venture might have saved time in developing new bindings, on the one hand, and in exploiting new bindings by our users on the other. Another advantage of using SWIG, and probably the reason for choosing SWIG in the first place in the aforementioned project, is the support for multiple target languages. Once bindings have been developed for one language, generating bindings for additional languages does not require additional development resources. However, as our experiments show, this profiteering has its cost—the generated bindings are not as time or space efficient as their counterparts generated with other tools. Both Boost.Python and PyBind11 (the origin of Nanobind) are both widely spread and tested, and actively maintained and enhanced. They are developed using advanced features of C++11 and higher versions. These two tools have similar interfaces; that is, the C++ code used to generate the bindings using both tools is similar and in some cases even identical. As we are very much familiar with developing code that exploits advanced feature of C++, developing bindings using these tools does not present a major hurdle for us. On the other hand, developing bindings using SWIG, requires writing code that describes the bindings in a proprietary language. Some developers may avoid using Boost.Python, because it requires dealing with

12See http://www.swig.org/.
13See https://www.boost.org/doc/libs/1_70_0/libs/python/doc/html/index.html.
14See https://github.com/wjakob/nanobind.
15See https://github.com/pybind/pybind11.
a large and complex system. This does not present a drawback for us, because CGAL itself depends on several other libraries of Boost; thus, we are familiar with Boost and must deal with it anyway. The compatibility of Boost with almost every C++ compiler in general has its cost; arcane template tricks and workarounds are necessary to support the oldest and buggiest of compiler specimens. Now that C++11-compatible compilers are widely available, the heavy machinery required to support the wide range of compilers may take some toll. However, our main concern was the efficiency of the generated bindings in terms of execution time and memory space.

We have conducted two sets of experiments. In the first set we compared the time it took to access C++ objects from Python code via bindings to the c++ shared pointers of the objects, and in the second set we compared the time it took to call C++ functions from Python code. The first set was further divided into two subsets. In the first subset we compared the time it took to access many objects and in the second subset we compared the time it took to access a single object many times; see Table 1 for the results. The second set was further divided into too subsets as well. In the first subset we compared the time it took to call many small functions, and in the second subset we compared the time it took to call few large functions. We also used the second set of experiments to measure the time it took to compute the various tasks in native Python; see Table 2 for the results. All benchmark programs were executed on a machine equipped with an Intel Core i5 clocked at 3.5GHz with 16GB of RAM. The results we obtained left nothing in doubt. We picked Nanobind for its high efficiency in every category.

### 3 Binding Generation

The most simple form of building an application written in C++ that depends on CGAL consists of the stages below. Let $APP\_SRC\_DIR$ point at the root directory of the application source tree.

1. Obtain the sources of CGAL; let $CGAL\_SRC\_DIR$ point at the root directory of the CGAL source tree.
2. Create a build directory for CGAL; let $CGAL\_DIR$ point at this build directory.
3. Change directory to $CGAL\_DIR$ and run `cmake [ options ] $CGAL\_SRC\_DIR`.
4. Create a build directory for the application; let $APP\_DIR$ point at this build directory.
5. Change directory to $APP\_DIR$ and run `cmake [ options ] $APP\_SRC\_DIR` followed by make.

In the procedure above, options refer to optional arguments passed to cmake, which govern the generation of the native build environment. When running an application written in Python that depends on CGAL instead of building the application, the user must build the bindings. Here, $APP\_SRC\_DIR$ needs to point at the root directory of the binding source tree, and $APP\_DIR$ needs to point at the build directory for the bindings. The optional arguments passed to cmake in stage 5 mainly consist of variables that determine (i) the object types to be bound and the unique identifier of each such object, and (ii) the name of the libraries that comprise the bindings, in case the default library name is overridden.

#### 3.1 Bindings Components

Our binding code is divided into modules, which correspond to CGAL packages. Each module has a long (and meaningful) name and a short name; see Table 3 for the list of relevant modules. Each module is associated with a CMAKE Boolean flag that determines whether to generate bindings for that particular module. Each module may be associated with zero or more additional flags that specify selections for bindings of entities of the corresponding CGAL packages. For example, the long name of the `2D Arrangements` module is `ARRANGEMENT_ON_SURFACE_2` its short name is `AOS2` `CGALPY_ARRANGEMENT_ON_SURFACE_2_BINDINGS` is the CMAKE Boolean flag that indicates whether to generate bindings for this module. This module is associated with the Boolean flag `CGALPY_AOS2_POINT_LOCATION_BINDINGS`, which indicates whether to generate bindings for point location queries supported by the

|                  | Nanobind | pybind11 | boost | SWIG |
|------------------|----------|----------|-------|------|
| Many references to many objects | 0.995    | 2.632    | 2.306 | 7.874 |
| Many references to a single object | 1.122    | 2.300    | 1.478 | 5.095 |

|                  | Nanobind | pybind11 | boost | SWIG | Native |
|------------------|----------|----------|-------|------|--------|
| Many small tasks | 6.231    | 11.532   | 7.545 | 9.112| 9.264  |
| One large task   | 0.353    | 0.353    | 0.353 | 0.353| 8.581  |
2D Arrangements package. This module is also associated with the CGALPY_AOS2_GEOMETRY_TRAITS_NAME string flag, which specify the name of the geometry traits and used to instantiate the `Arrangement_2<GeometryTraits_2 → Dcel>` class template. The Boolean flags CGALPY_AOS2_VERTEX_EXTENDED, CGALPY_AOS2_HALFEDGE_EXTENDED, and CGALPY_AOS2_FACE_EXTENDED specify whether to extend the vertex, halfedge, or face records of the DCEL, respectively. See Section 4.3 for more details on the 2D Arrangements module.

### Table 3: Module names and short names

| Module                                      | Name                        | Short Name |
|---------------------------------------------|-----------------------------|------------|
| 2D and 3D Geometry Kernel                   | KERNEL                      | KER        |
| dD Geometry Kernel                          | KERNEL_D                    | KERD       |
| 2D Arrangements                             | ARRANGEMENT_ON_SURFACE_2    | AOS2       |
| 2D Alpha Shapes                              | ALPHA_SHAPE_2               | AS2        |
| 3D Alpha Shapes                              | ALPHA_SHAPE_3               | AS3        |
| 2D Regularized Boolean Set-Operations        | BOOLEAN_SET_OPERATIONS_2    | BSO2       |
| Bounding Volumes                             | BOUNDING_VOLUMES            | BV         |
| 2D Convex Hulls and Extreme Points           | CONVEX_HULL_2               | CH2        |
| 3D Convex Hulls                              | CONVEX_HULL_3               | CH3        |
| 2D Polygons                                  | POLYGON_2                   | POL2       |
| 2D Polygon Partitioning                     | POLYGON_PARTITIONING        | PP         |
| 2D Minkowski Sums                            | MINKOWSKI_SUM_2             | MS2        |
| dD Spatial Searching                         | SPATIAL_SEARCHING           | SS         |
| 2D Triangulations                            | TRIANGULATION_2             | TR12       |
| 3D Triangulations                            | TRIANGULATION_3             | TR13       |
| Surface Mesh                                 | SURFACE_MESH                | SM         |
| 3D Polyhedral Surfaces                       | POLYHEDRON_3                | POL3       |
| Polygon Mesh Processing                      | POLYGON_MESH_PROCESSING     | PMP        |

The exposed names of the bound entities of different modules are gathered in distinct Python namespaces to prevent name conflicts. The namespace is the short name of the module in lower-case except for the first letter (which is in upper-case); for example, the namespaces of the bound types and free functions of the 2D and 3D Geometry Kernel and the 2D Regularized Boolean Set-Operations packages are Ker and Bso2, respectively. Each of these two namespaces has the attribute `intersection` (i.e., Ker.intersection() and Bso2.intersection()). We introduce as many (Python) namespaces as modules, and nest every exposed name of a bound entity in the appropriate (Python) namespace according to the module of the entity. We provide implementation details in the next section.

The nesting of constructs in C++ is reflected in Python code. For example, assume that the binding module is called CGALPY and it includes bindings for an instance of the class template `Arrangement_2<GeometryTraits, Dcel>`, where the GeometryTraits template parameter is substituted with a traits class that handles segments, and the Dcel template parameter is substituted with the default DCEL; see Section 4.3. The following code excerpt constructs the segment `(0, 0), (1, 0)`.

```python
>>> import CGALPY
>>> Aos2 = CGALPY.Aos2  # define the module namespace
>>> Arr = Aos2.Arrangement_2  # define the arrangement type
>>> Point = Arr.Point_2  # define the point type
>>> Segment = Arr.Curve_2  # define the curve type
>>> seg = Segment(Point(0, 0), Point(1, 0))
```

The code that binds all functions and classes of different modules is nested in distinct C++ namespaces (even if the C++ CGAL module does not have a namespace). It prevents name conflicts between bound entities in different modules. Within a (C++) namespace we first set the Python namespace to match the name of a module. Then, we bind all functions and classes of CGAL packages associated with the module. The following code excerpt shows how it is done for the 2D and 3D Geometry Kernel module:
For every module there exists a function called export_<name>, where <name> is the module name. This function contains the binding code for the essential functions and classes of the module. The 2D and 3D Geometry Kernel module is associated with the CMAKE KERNEL_INTERSECTION_BINDINGS flag (see Table 6), which determines whether to generate bindings for intersections; hence, the compile-time conditional calls to the function export_kernel_intersections(), which contains the binding code for intersections. This code presents a certain challenge described in the Section 5.2.

3.2 Binding Library Name

By default, the base name of the generated library is CGALPY. However, the user can override the name with a generated string that maps to the set of bound types. This is imperative when more than one instance of the same template must be bound. Each execution of cmake followed by make generates a single library. The CMAKE flag CGALPY_FIXED_LIBRARY_NAME determines whether the base name of the generated library is CGALPY or not. If not, it has the prefix CGALPY_ followed by as many as substrings as modules, the bindings of which are enabled, separated by an underscore (_). Each such substring starts with the short name of the module in lower case followed by strings that map to the selections for bindings within the module. Each such string is a single word that starts with a capital letter. For example, the name GALPY_kernelEpecInt_Aos2SegPl of a generated library, names a library that contains bindings for (i) the Exact-Predicate-Exact-Construction (EPEC) 2D and 3D Geometry Kernel module and intersections, and (ii) the 2D Arrangements module, where the Arrangement_2<> class template is instantiated with a traits class that handles segments and the default DCEL (see Section 4.3), and point location queries.

The generated library is dynamically linked—it must be so. However, the library itself can be compiled of either static or dynamic (dependent) libraries. If you intend to generate and use just a single library that contains the bindings, you have the freedom to choose between generating a library compiled of static libraries or dynamic libraries. However, if you intend to generate several libraries and use them all in a single Python module, it is recommended using binding libraries compiled of dynamic libraries. The attributes of a kernel type, such as Kernel::Point_2, in different bindings libraries compiled of dynamic libraries refer to the same object. Otherwise, objects of the same bound type cannot be interchanged. (Also, different generated libraries might be compiled of conflicting static libraries.) The CMAKE flag CGALPY_USE_SHARED_LIBS indicates whether the generated library is compiled of static or dynamic libraries; it is true by default.

The content of the library and its name are governed by flags provided to cmake. All flags have the prefix CGALPY_. In Tables 4, 5, 6, 8, 11, 13, 15, 16, 20, and 22 this prefix is omitted.

### Table 4: General Arguments

| Name                  | Type    | Default | Description                                               |
|-----------------------|---------|---------|-----------------------------------------------------------|
| USE_SHARED_LIBS       | Boolean | true    | Determines whether to compile shared libraries             |
| BUILD_SHARED_LIBS     | Boolean | true    | Determines whether to generate shared libraries            |
| FIXED_LIBRARY_NAME    | Boolean | true    | Determines whether the library name is fixed cgalpy.so or set based on other selections |

4 Binding Modules

We describe the bindings generated by our system for various CGAL packages and exemplify the use of the generated bindings. When developing code in Python that uses the bindings, the statement that imports the binding library, that is,

```python
>>> import CGALPY
```
| Name                                | Type       | Default | Description                                                                 |
|-------------------------------------|------------|---------|-----------------------------------------------------------------------------|
| KERNEL_BINDINGS                     | Boolean    | true    | Determines whether to generate bindings for 2D and 3D Kernel types          |
| KERNEL_D_BINDINGS                   | Boolean    | false   | Determines whether to generate bindings for dD Kernel types                 |
| ARRANGEMENT_ON_SURFACE_2_BINDINGS   | Boolean    | false   | Determines whether to generate bindings for 2D arrangement instances        |
| ALPHA_SHAPE_2_BINDINGS              | Boolean    | false   | Determines whether to generate bindings for 2D Alpha shape instances        |
| ALPHA_SHAPE_3_BINDINGS              | Boolean    | false   | Determines whether to generate bindings for 3D Alpha shape instances        |
| BOOLEAN_SET_OPERATIONS_2_BINDINGS   | Boolean    | false   | Determines whether to generate bindings for 2D Boolean set operation instances |
| BOUNDING_VOLUMES_BINDINGS           | Boolean    | false   | Determines whether to generate bindings for bounding volume instances       |
| CONVEX_HULL_2_BINDINGS              | Boolean    | false   | Determines whether to generate bindings for 2D convex hull instances        |
| CONVEX_HULL_3_BINDINGS              | Boolean    | false   | Determines whether to generate bindings for 3D convex hull instances        |
| POLYGON_2_BINDINGS                  | Boolean    | false   | Determines whether to generate bindings for 2D polygon instances            |
| POLYGON_PARTITIONING_BINDINGS       | Boolean    | false   | Determines whether to generate bindings for 2D polygon partitioning instances |
| MINKOWSKI_SUM_2_BINDINGS            | Boolean    | false   | Determines whether to generate bindings for 2D Minkowski sum instances      |
| SPATIAL_SEARCHING_BINDINGS          | Boolean    | false   | Determines whether to generate bindings for spatial searching instances     |
| TRIANGULATION_2_BINDINGS            | Boolean    | false   | Determines whether to generate bindings for 2D triangulation instances      |
| TRIANGULATION_3_BINDINGS            | Boolean    | false   | Determines whether to generate bindings for 3D triangulation instances      |
| SURFACE_MESH_BINDINGS               | Boolean    | false   | Determines whether to generate bindings for surface mesh instances          |
| POLYHEDRON_3_BINDINGS               | Boolean    | false   | Determines whether to generate bindings for 3D polyhedron instances         |
| POLYGON_MESH_PROCESSING_BINDINGS    | Boolean    | false   | Determines whether to generate bindings for polygon mesh processing instances |
(assuming the default binding library name is retained) must precede any statement that use the binding. This statement is omitted in all examples hereafter.

4.1 Two- and Three-Dimensional Kernel Bindings

The 2D and 3D Geometry Kernel package [8] of CGAL consists of constant-size non-modifiable geometric primitive objects and operations on these objects. The objects are sets of points in d-dimensional affine Euclidean space, where \( d = 2, 3 \). Each point is uniquely represented either by Cartesian coordinates or by homogeneous coordinates. An object type can be defined either precatively as a member of a kernel type or imperatively as a global class-template parameterized by a kernel type, defined in the C++ CGAL namespace. For example, assume that Kernel is a kernel type; the type that represents a two-dimensional point of this kernel is either Kernel::Point_2 or CGAL::Point_2<Kernel>. The generated bindings better reflects the latter types; that is, defining a two-dimensional point in Python amounts to the code below.\(^{16}\)

```python
>>> Ker = CGALPY.Ker
# define the module namespace
class Point_2 = Ker.Point_2
# define the point type in the module namespace
>>> p = Point_2(0, 0);
```

Table 6 lists the CMAKE flags associated with the 2D and 3D Geometry Kernel module. The kernel type is an instance of a chain of C++ class templates. For convenience, CGAL provides the following predefined types of generally useful kernels:

1. Exact_predicates_inexact_constructions_kernel—provides exact geometric predicates, but geometric constructions may be inexact due to round-off errors.
2. Exact_predicates_exact_constructions_kernel—provides exact geometric constructions, in addition to exact geometric predicates.
3. Exact_predicates_exact_constructions_kernel_with_sqrt—same as Exact_predicates_exact_constructions_kernel, but the number type is a model of the concept that requires operations that perform square roots, namely FieldWithSqrt.\(^{17}\)
4. Exact_predicates_exact_constructions_kernel_with_kth_root—same as Exact_predicates_exact_constructions_kernel, but the number type is a model of the concept that requires operations that perform \(k\)-th roots, namely FieldWithKthRoot.\(^{18}\)
5. Exact_predicates_exact_constructions_kernel_with_root_of—same as Exact_predicates_exact_constructions_kernel, but the number type is a model of the concept that requires operations that computes the root of univariate polynomial, namely FieldWithRootOf.\(^{19}\)

| Name                          | Type       | Default | Description                      |
|-------------------------------|------------|---------|----------------------------------|
| KERNEL_NAME                   | String     | epic    | The kernel type used             |
| KERNEL_INTERSECTION_BINDINGS  | Boolean    | true    | Determines whether to generate bindings for intersection functions |

All the predefined types are of Cartesian kernels. The CMAKE flag CGALPY_KERNEL_NAME specifies which kernel type should be used for the generated bindings. Currently, only three predefined types and two specific types are supported; see Table 7.

The kernel type determines the underlying number type used to represent coefficients and coordinates of kernel objects and for evaluating mathematical expressions that involve these coefficients and coordinates. The Python attribute FT, nested in the Python namespace Ker, exposes the C++ underlying field number type when it is not a primitive data type; that is, when the selected kernel is neither Filtered_kernel<Simple_cartesian<double>>. (In both cases the underlying number type is double.) Similar to the Python code above, the code below defines a two-dimensional point; here, the coordinates are explicitly converted to the underlying number type.

\(^{16}\)When writing code in C++ the precative style is advantageous, as it enables the extension of kernel object types, which is irrelevant when generating bindings.

\(^{17}\)See https://doc.cgal.org/latest/Algebraic_foundations/classFieldWithSqrt.html.

\(^{18}\)See https://doc.cgal.org/latest/Algebraic_foundations/classFieldWithKthRoot.html.

\(^{19}\)See https://doc.cgal.org/latest/Algebraic_foundations/classFieldWithRootOf.html.
### Table 7: Kernel name options.

| KERNEL_NAME         | Predefined Type                                                               |
|---------------------|--------------------------------------------------------------------------------|
| epic                | Exact_predicates_inexact_constructions_kernel                                 |
| epec                | Exact_predicates_exact_constructions_kernel                                   |
| epecws              | Exact_predicates_exact_constructions_kernel_with_sqrt                         |

| KERNEL_NAME                               | Predefined Type                                               |
|-------------------------------------------|---------------------------------------------------------------|
| filteredSimpleCartesianDouble             | NT = double Filtered_kernel<Simple_cartesian<NT>>              |
| filteredSimpleCartesianLazyGmpq           | NT = Lazy_exact_nt<Gmpq> Filtered_kernel<Simple_cartesian<NT>>|

Listing 1: CMAKE flag settings used to generate bindings for the exact-predicates and exact-constructions kernel.

```cmake
set (CMAKE_BUILD_TYPE "Release" CACHE STRING "build type" FORCE)
set (CGALPY_USE_SHARED_LIBS ON CACHE BOOL "use shared libs" FORCE)
set (CGALPY_FIXED_LIBRARY_NAME ON CACHE BOOL "use name CGALPY" FORCE)
set (CGALPY_KERNEL_NAME "epec" CACHE STRING "use EPEC kernel" FORCE)
set (CGALPY_KERNEL_INTERSECTION_BINDINGS ON CACHE BOOL "with intersections" FORCE)
```

```python
>>> p = Point_2(Ker.FT(0), Ker.FT(0))
```

The code excerpt shown in Listing 1, in the CMAKE language, sets the CMAKE flags for our first example. When cmake is applied with these settings followed by the native build commands (e.g., make on Linux platforms) a library called CGALPY is generated. This library consists of the bindings necessary to run the Python example shown in Listing 2, which (i) determines whether two segments intersects, and (ii) computes their intersection point, using the generated binding. Observe that bindings of the 2D and 3D Geometry Kernel module are generated by default, whereas bindings for all other modules must be specifically requested.

#### 4.2 d-Dimensional Kernel Bindings

CGAL includes a separate package that consists of constant-size non-modifiable geometric primitive objects in arbitrary dimensions, and operations on these objects, called dD Geometry Kernel [29]. Similar to the two- and three-dimensional kernels, the objects of the d-dimensional kernels are sets of points in some d-dimensional affine Euclidean space, where the dimension d is either static across a kernel type or dynamic; see below for more details. Each point is uniquely represented either by Cartesian coordinates or by homogeneous coordinates. For convenience, CGAL provides the following predefined types of generally useful kernels:

1. Epick_d<DimensionTag>—provides exact geometric predicates, but geometric constructions may be inexact due to round-off errors.
2. Epeck_d<DimensionTag>—provides exact geometric constructions, in addition to exact geometric predicates.

Listing 2: Computing the intersection between two line segments.

```python
Ker = CGALPY.Ker
Point_2 = Ker.Point_2
Segment_2 = Ker.Segment_2
s1 = Segment_2(Point_2(0, 1), Point_2(1, 0))
s2 = Segment_2(Point_2(0, 0), Point_2(1, 1))
b = Ker.do_intersect(s1, s2)
print(b)
p = Ker.intersection(s1, s2)
print(p)
```
Table 8 lists the CMAKE flags associated with the \textit{dD Geometry Kernel} module.

| Name                     | Type     | Default | Description                                    |
|--------------------------|----------|---------|------------------------------------------------|
| KERNEL_D_NAME            | String   | epicd   | The kernel type used                           |
| KERNEL_D_DIMENSION_TAG   | String   | dynamic | Determines whether the dimension is dynamic    |
| KERNEL_D_DIMENSION       | Integer  | 2       | The dimension of the ambient Euclidean space   |

The CMAKE flag \texttt{CGALPY\_KERNEL\_D\_NAME} specifies which kernel type should be used for the generated bindings; see Table 9.

| KERNEL_D_NAME             | Predefined Type                      |
|--------------------------|--------------------------------------|
| epicd                    | Epick\_d<DimensionTag>               |
| epecd                    | Epeck\_d<DimensionTag>               |
| cartesiandDouble         | Cartesian\_d<\texttt{double}>        |
| cartesiandLazyGmpq       | Cartesian\_d<Lazy\_exact\_nt<Gmpq>> |

When either \texttt{Epick\_d<DimensionTag>} or \texttt{Epeck\_d<DimensionTag>} are instantiated, the template parameter must be substituted with a type that represents the dimension of the ambient Euclidean space. It may be either \texttt{Dimension\_tag\_d} where \(d\) is an integer or \texttt{Dynamic\_dimension\_tag}. In the latter case, the dimension of the space is specified for each point when it is constructed, so it does not need to be known at compile-time of the bindings. The CMAKE \texttt{CGALPY\_KERNEL\_D\_DIMENSION\_TAG} flag specifies whether the dimension is static or dynamic. If it is static, the dimension is extracted from the CMAKE \texttt{CGALPY\_KERNEL\_D\_DIMENSION} CMAKE flag; see Table 10.

| KERNEL_D_DIMENSION_TAG   | Predefined Type                      |
|--------------------------|--------------------------------------|
| static                  | Dimension\_tag\_d                   |
| dynamic                 | Dynamic\_dimension\_tag              |

The kernel type determines the underlying number type. It is possible to have different underlying number types for the 2D and 3D Geometry Kernel and the dD Geometry Kernel models. However, when the number types differ, expensive conversions might be necessary to combine operations from both kernels, or it may not be possible at all using binding code developed thus far.

The code excerpt in the CMAKE language shown in Listing 3, sets the CMAKE flags for our second example. When \texttt{cmake} is applied with these settings followed by the native build commands, a library called \texttt{CGALPY\_kerdCdlgDynamic} is generated. This library consists of the bindings necessary to run the Python example shown in Listing 4, which determines whether two segments in four dimensions intersect using the generated binding.

4.3 2D Arrangement Bindings

The Cgal arrangements packages constitute a large component of the Cgal library. This component is particularly intricate, partly due to the interplay between combinatorial algorithms and algebra [15]. Arrangements are space subdivisions induced by curves and surfaces, which have been intensively studied in discrete and computational geometry [20], and have applications in various domains, from robotics [19] and assembly planning [18] through Geographic Information Systems (GIS) [32] to protein structure determination [26], to mention just a few uses. The arrangements packages of Cgal have been developed since the early days of Cgal, first for planar arrangements and maps [14, 35],
Listing 3: CMake flag settings used to generate bindings for the d-dimensional kernel.

```cmake
set (CMAKE_BUILD_TYPE "Release" CACHE STRING "build type" FORCE)
set (CGALPY_USE_SHARED_LIBS ON CACHE BOOL "" FORCE)
set (CGALPY_FIXED_LIBRARY_NAME OFF CACHE BOOL "" FORCE)
set (CGALPY_KERNEL_BINDINGS OFF CACHE BOOL "Disable kernel" FORCE)
set (CGALPY_KERNEL_D_BINDINGS ON CACHE BOOL "Enable dD kernel" FORCE)
set (CGALPY_KERNEL_D_NAME "cartesiandLazyGmpq" CACHE STRING "use dD kernel" FORCE)
```

Listing 4: Determining whether two segments in 4D intersect.

```python
Kerd = CGALPY.Kerd
if hasattr(Kerd, 'FT'):
    FT = Kerd.FT
else:
    FT = float
Point_d = Kerd.Point_d
Segment_d = Kerd.Segment_d
p11 = Point_d(4, [FT(n) for n in [0, 0, 0, 0]])
p12 = Point_d(4, [FT(n) for n in [1, 1, 1, 1]])
s1 = Segment_d(p11, p12)
p21 = Point_d(4, [FT(n) for n in [1, 0, 1, 0]])
p22 = Point_d(4, [FT(n) for n in [0, 1, 0, 1]])
s2 = Segment_d(p21, p22)
print(Kerd.do_intersect(s1, s2))
```

Boolean operations, and Minkowski sums [3]. Then, envelopes of surfaces in three-dimensions have been added [28]. Finally, a major effort has been undertaken to support two-dimensional arrangements on (not necessarily planar) surfaces [5, 6].

Given a surface \(S\) in \(\mathbb{R}^3\) and a set \(C\) of curves embedded in this surface, the curves subdivide \(S\) into cells of dimension 0 (vertices), 1 (edges), and 2 (faces). This subdivision is the arrangement \(A(C)\) induced by \(C\) on \(S\) [15]. Arrangements embedded in curved surfaces in \(\mathbb{R}^3\) are generalizations of arrangements embedded in the plane. The 2D Arrangements package [34] can be used to construct, maintain, alter, and display 2D arrangements embedded in ruled curved surfaces, such as, spheres, ellipses, tori, cones, paraboloids, and the plane. It also supports queries on such arrangements, such as point location and vertical ray shooting. One of the main components of the 2D Arrangements package is the Arrangement_2<Traits, Dcel> class template. An instance of this template is used to represent an arrangement embedded in the plane. Table 11 lists the CMAKE flags associated with the 2D Arrangements module. A description of the two template parameters of this class template follows.

Table 11: 2D Arrangements module flags

| Name                        | Type      | Default | Description                                               |
|-----------------------------|-----------|---------|-----------------------------------------------------------|
| AOS2_GEOMETRY_TRAITS_NAME   | String    | segment | The basic geometry traits                                  |
| AOS2_EXTEND_VERTEX          | Boolean   | false   | Determines whether to extend the vertex type              |
| AOS2_EXTEND_HALFEDGE        | Boolean   | false   | Determines whether to extend the halfedge type             |
| AOS2_EXTEND_FACE            | Boolean   | false   | Determines whether to extend the face type                 |
| AOS2_POINT_LOCATION_BINDINGS| Boolean   | true    | Determines whether to generate bindings for point location and vertical ray shooting queries |

- The Traits template-parameter determines the family of curves that induce the arrangement. The parameter should be substituted with a model of the basic arrangement traits concept or one or more concepts that refine the basic concept. A model of the basic traits concept defines the types of x-monotone curves and two-dimensional points and supports basic geometric predicates on them. A rather large directed acyclic graph is required to capture the entire hierarchy of the geometry traits-class concepts; therefore, we typically use subgraphs to
describe the refinement relations among closely related concepts, and refer to these subgraph as clusters. Figure 1 depicts four clusters. The list of supported traits class templates follows. For each class template we describe the family of curves it handles.

1. Arr_non_caching_segment_basic_traits_2—handles segments.
2. Arr_segment_traits_2—handles segments, where each segment is represented a by its supporting line in addition to its two endpoints.
3. Arr_linear_traits_2—handles linear curves, i.e., segments, rays, and lines.
4. Arr_polyline_traits_2—handles polylines.
5. Arr_circle_segment_traits_2—handles segments and circular arcs.
6. Arr_conic_traits_2—handle conic arcs.
7. Arr_rational_function_traits_2—handle rational functions.
8. Arr_Bezier_curve_traits_2—handles Bézier curves of arbitrary degrees.
9. Arr_algebraic_segment_traits_2—handles algebraic curves of arbitrary degrees.
10. Arr_polycurve_traits_2—handle polycurves, which are piecewise curves that are not necessarily linear.

The **CMake** flag CGALPY_AOS2_GEOMETRY_TRAITS_NAME specifies which geometry traits should be used for the generated bindings; see Table 12. Observe that instances of the class templates General_polygon_set_2 and Polygon_set_2 (see Section 4.4) employ 2D arrangement types. Bindings for instances of Polygon_set_2 and General_polygon_set_2 are enabled as part of the bindings for the **2D Regularized Boolean Set-Operations** package. When bindings for the **2D Regularized Boolean Set-Operations** package are enabled, bindings of the **2D Arrangements** package must be explicitly enabled as well. The traits type that substitutes the traits parameter determines the type of curves that bound the polygons or generalized polygons, and the type must be explicitly indicated too (unless the segment type is selected, which is the default). The traits type must also model the GeneralPolygonSetTraits_2 concept; see Figure 2b for the relevant traits-concept cluster. The final traits type that is used for the bindings is automatically extended to satisfy this requirement. The details of this extension is given in Section A.1 of the appendix.

- The Dcel template-parameter should be substituted with a type that models the **ArrangementDcel** concept, which is used to represent the topological layout of the arrangement. This layout is, in particular, represented by a doubly-connected edge list data-structure (DCEL for short), which consists of containers of vertices, edges and faces and maintains the incidence relations among these objects. We substitute this type with an instance of the template CGAL:: Arr_dcel_base<V, H, F>, where V, H, and F are models of the concepts ArrangementDcelVertex, ArrangementDcelHalfedge, and ArrangementDcelFace, respectively; by default they are substituted with Arr_vertex_base<Traits::Point_2>, Arr_halfedge_base<Traits::X_monotone_curve_2>, and Arr_face_base, respectively. In many applications it is necessary to extend the types of the DCEL main features. This is governed by three **CMake** Boolean flags as follows. If any one of the **CMake** variables CGALPY_AOS2_VERTEX_EXTENDED, CGALPY_AOS2_HALFEDGE_EXTENDED, or CGALPY_AOS2_FACE_EXTENDED is set to true, the corresponding template parameter, V, H, or F, is substituted with instances of Arr_extended_vertex<Vb, VertexData>, Arr_extended_halfedge<Hb, HalfedgeData>, or Arr_extended_face<Fb, FaceData>, respectively, where Vb, Hb, and Fb are the basic types above. It is impossible to define a custom C++ type from Python code. Therefore, when the bindings are generated each one of the VertexData, HalfedgeData, and FaceData template parameters must be substituted with a C++ type known at the time the bindings were implemented. Flexibility is nevertheless retained by substituting every one of these parameters with the generic Python object py::object when the respective cell is extended.

The code excerpt shown in Listing 5 in the **CMake** language sets the **CMake** flags for our next example shown in Listing 6. This example constructs an arrangement with two faces. The arrangement is induced by line segments and its face type is extended. The properties of the bounded face and the unbounded face are initialized with Python integer objects '0' and '1', respectively.

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20 See https://doc.cgal.org/latest/Arrangement_on_surface_2/classArrangementDcel.html.
21 For more information on Python objects, see, e.g., https://docs.python.org/3/library/functions.html?highlight=object#object.
Figure 1: (a) The central cluster. (b) The landmark cluster. (c) The open boundary cluster. (d) The spherical boundary cluster.

| AOS2_GEOMETRY_TRAITS_NAME          | Type                                      |
|-----------------------------------|-------------------------------------------|
| nonCachingSegment                 | Arr_non_caching_segment_basic_traits_2<Kernel> |
| segment                           | Arr_segment_traits_2<Kernel>              |
| linear                            | Arr_linear_traits_2<Kernel>               |
| conic                             | Arr_conic_traits_2<RatKernel, AlgKernel, NtTraits> |
| circleSegment                     | Arr_circle_segment_traits_2<Kernel>       |
| algebraic                         | Arr_algebraic_segment_traits_2<Coefficient> |
Listing 5: CMAKE flag settings used to generate bindings for 2D arrangements with their faces extended.

```cmake
set (CMAKE_BUILD_TYPE "Release" CACHE STRING "build type" FORCE)
set (CGALPY_USE_SHARED_LIBS ON CACHE BOOL "use shared libs" FORCE)
set (CGALPY_FIXED_LIBRARY_NAME OFF CACHE BOOL "use name CGALPY" FORCE)
set (CGALPY_KERNEL_NAME "epec" CACHE STRING "use EPEC kernel" FORCE)
set (CGALPY_ARRANGEMENT_ON_SURFACE_2_BINDINGS ON CACHE BOOL "2D AOS" FORCE)
set (CGALPY_AOS2_GEOMETRY_TRAITS_NAME "segment" CACHE STRING "use segments" FORCE)
set (CGALPY_AOS2_FACE_EXTENDED ON CACHE BOOL "extend face type" FORCE)
set (CGALPY_AOS2_POINT_LOCATION_BINDINGS ON CACHE BOOL "" FORCE)
```

Listing 6: Constructing an arrangement with its faces extended and initializing their data.

```python
Aos2 = CGALPY.Aos2
Arrangement_2 = Aos2.Arrangement_2
Point_2 = Arrangement_2.Geometry_traits_2.Point_2
Curve_2 = Arrangement_2.Geometry_traits_2.Curve_2
arr = Arrangement_2()
c1 = Curve_2(Point_2(0, 0), Point_2(1, 0))
c2 = Curve_2(Point_2(1, 0), Point_2(0, 1))
c3 = Curve_2(Point_2(0, 1), Point_2(0, 0))

# Insert the curves into the arrangement
Aos2.insert(arr, [c1, c2, c3])
print("Number of vertices in the arrangement:", arr.number_of_vertices())
print("Number of halfedges in the arrangement:", arr.number_of_halfedges())
print("Number of faces in the arrangement:", arr.number_of_faces())

unb_face = arr.unbounded_face()
unb_face.set_data(0)
next(next(unb_face.inner_ccbs())).twin().face().set_data(1)
for f in arr.faces(): print(f.is_unbounded(), f.data())
```
Listing 7: CMake flag settings used to generate bindings for 2D regularized Boolean operations on generalized polygons bounded by line segments and circular arcs.

```cmake
set(CMAKE_BUILD_TYPE "Release" CACHE STRING "build type" FORCE)
set(CGALPY_USE_SHARED_LIBS ON CACHE BOOL "" FORCE)
set(CGALPY_FIXED_LIBRARY_NAME OFF CACHE BOOL "" FORCE)
set(CGALPY_KERNEL_NAME "epec" CACHE STRING "use EPEC kernel" FORCE)
set(CGALPY_ARRANGEMENT_ON_SURFACE_2_BINDINGS ON CACHE BOOL "" FORCE)
set(CGALPY_AOS2_GEOMETRY_TRAITS_NAME "circleSegment" CACHE STRING "use circular arcs \langle k \text{ segments}\rangle" FORCE)
set(CGALPY_BOOLEAN_SET_OPERATIONS_2_BINDINGS ON CACHE BOOL "" FORCE)
```

4.4 2D Regularized Boolean Set Operation Bindings

The CGAL package *2D Regularized Boolean Set-Operations* consists of the implementation of regularized Boolean set-operations, intersection predicates, and point containment predicates on point sets bounded by weakly x-monotone curves in two-dimensional Euclidean space [16]. The *2D Regularized Boolean Set-Operations* module is not associated with CMake flags (besides the flag CGALPY_BOOLEAN_SET_OPERATIONS_2_BINDINGS, which indicates whether to generate bindings for this module). The Boolean set operations supported by this package depend on the *2D Arrangements* package. If the operations are applied on (linear) polygons they also depend on the *2D Polygons* package. Recall, that both further depend on the *2D and 3D Geometry Kernel* package. Therefore, bindings for these packages must be explicitly enabled as well when bindings for the *2D Regularized Boolean Set-Operations* package are enabled.

The code excerpt shown in Listing 7 in the CMake language sets the CMake flags for our next example. Applying these settings followed by the native build commands generates a library, the basename of which is `CGALPY_kerEpec_aos2Cs_bso2_pol2`. It supports bindings for the types below and operations on these types, but nothing else.

- kernel types,
- line segment and circular arc,
- arrangement in the plane induced by curves of the above types,
- generalized polygon and generalized polygon with holes bounded by curves of the above types, and
- Boolean operations on generalized polygons of the above types.

This library can be used to execute the Python example shown in Listing 8; the example constructs a general polygon-set that represents the point set depicted in Figure 2a. It is the result of the union of four disjoint circles and four rectangles. Each circle is represented as a generalized polygon bounded by two x-monotone circular arcs. The union is computed incrementally, resulting with a single generalized polygon with a single hole. Note that as the four circles are disjoint, their union is computed with the `insert()` function, while the union with the rectangles is computed with the `join()` function.

![A generalized polygon with holes bounded by circular arcs and line segments.](image1)

![The general point-set cluster of concepts.](image2)

Figure 2: (a) A generalized polygon with holes bounded by circular arcs and line segments. (b) The general point-set cluster of concepts.

An arrangement data structure is internally used to represent the point set maintained by a general polygon-set object; it is possible to obtain it and apply further operations on it, as demonstrated by the code excerpt shown in Listing 9.

The bindings for the *2D Arrangements* package includes bindings for the geometry traits and the DCEL suitable for Boolean operations. In particular, the traits must model the concept GeneralPolygonSetTraits_2; see Figure 2b.
Listing 8: Constructing a general polygon-set that represents the point set depicted in Figure 2a.

```python
def ctr_circle_polygon(circle):
    # Subdivide the circle into two x-monotone arcs.
    cv = curve(circle)  # circle orientation is counterclockwise
    traits = Traits()
    make_x_monotone = traits.make_x_monotone_2_object()
    objects = make_x_monotone(cv)
    assert (len(objects) == 2)
    return Polygon([objects[0], objects[1]])

def ctr_quad_polygon(p1, p2, p3, p4):
    c1 = X_monotone_curve(p1, p2);
    c2 = X_monotone_curve(p2, p3)
    c3 = X_monotone_curve(p3, p4)
    c4 = X_monotone_curve(p4, p1)
    return Polygon([c1, c2, c3, c4])

S = General_polygon_set()
S.insert(ctr_circle_polygon(Circle(Point(1, 1), 1)))
S.insert(ctr_circle_polygon(Circle(Point(5, 1), 1)))
S.insert(ctr_circle_polygon(Circle(Point(5, 5), 1)))
S.insert(ctr_circle_polygon(Circle(Point(1, 5), 1)))
S.join(ctr_quad_polygon(Point(1, 0), Point(5, 0), Point(5, 2), Point(1, 2)))
S.join(ctr_quad_polygon(Point(1, 4), Point(5, 4), Point(5, 6), Point(1, 6)))
S.join(ctr_quad_polygon(Point(0, 1), Point(2, 1), Point(2, 5), Point(0, 5)))
S.join(ctr_quad_polygon(Point(4, 1), Point(6, 1), Point(6, 5), Point(4, 5)))

assert (S.number_of_polygons_with_holes() == 1)
pwhs = S.polygons_with_holes()
print(pwhs)
```

Listing 9: Extracting the arrangement from a point set.

```python
arr = S.arrangement()
print("# Cells: ", arr.number_of_vertices(), arr.number_of_edges(),
        arr.number_of_faces())
```


4.5 2D Minkowski Sums

Given two sets of points $A, B \subseteq \mathbb{R}^d$, their Minkowski sum, denoted by $A \oplus B$, is their point-wise sum, namely the set $\{a + b | a \in A, b \in B\}$. The CGAL package 2D Minkowski Sums contains functions that compute the planar Minkowski sum of two polygons and the planar Minkowski sum of a simple polygon and a disc—an operation also referred to as offsetting or dilating a polygon. The package also supports inner offsetting a polygon (also referred to as insetting), which is equivalent to the complement of the offset of (i) a disk and (ii) the complement of a polygon [33].

The 2D Minkowski Sums module is not associated with any CMAKE flags (besides the flag CGALPY_MINKOWSKI_SUM_2_BINDINGS, which indicates whether to generate bindings for this module). Similar to the case of generating bindings of Boolean operations on (linear) polygons supported by the 2D Regularized Boolean Set-Operations package, the operations supported by this package depend on the 2D and 3D Geometry Kernel, 2D Arrangements, and 2D Polygons packages. Therefore, bindings for these packages must be explicitly enabled as well when bindings for the 2D Minkowski Sums package are enabled. The result of the insetting and offset operations is a generalized polygon bounded by line segments and circular arc. Binding for a type that represents such polygons, that is, an instance of CGAL:: General_polygon_with_holes_2<> is generated as well.

The function CGAL:: minkowski_sum() is extremely overloaded. It can be used to compute the Minkowski sum of two polygons either applying the convex-decomposition approach or the reduced-convolution approach. When applying the convex-decomposition approach we first decompose each summand into convex sub-polygons. The function template CGAL:: minkowski_sum__by__full_convolution_2() applies a third approach, namely full convolution. For more information on the various approaches refer to the manual.

There are two overloaded function templates CGAL:: minkowski_sum() that apply the reduced-convolution approach; both accepts the two input polygons as input; one also accepts a specific geometry traits as input (while the other constructs and uses a default traits object). Each type of the summands must represent either a simple polygon or a polygon with holes. The signature of the former is shown in Listing 10, where each of PolygonType1 and PolygonType2 can be substituted either with an instance of Polygon_2<Kernel, Container> or with an instance of Polygon_with_holes_2<Kernel, Container>. Thus, given a specific kernel and container types, we get eight overloaded instances that apply the reduced-convolution approach in total. (The Container type determines the representation of the polygon’s extreme points in memory.)

There are two sets of function templates CGAL:: minkowski_sum() that apply the convex-decomposition approach; all functions accept two polygons as input; functions in one set also accept a specific geometry traits as input. As with the functions that apply the reduced-convolution approach, each type of the summands must be substituted either with a type that represents a simple polygon or a type that represents a polygon with holes. Each set consists of two function templates; one is parameterized with the type that represents a single decomposition strategy that should be applied to both summands and another one that is parameterized with two types that represent two decomposition strategies that should be applied to the two summands, respectively. The package provides four types of decomposition strategies; however, only two can be applied to a polygon with holes. Listing 23 shows the signatures of the two function templates that do not accept a traits parameter. We get 10 overloaded instances of functions that apply the convex-decomposition approach, do not accept a traits parameter, and are parameterized with a single decomposition strategy. We get 36 overloaded instances of functions that apply the convex-decomposition approach, do not accept a traits parameter, and are parameterized with two decomposition strategies. Thus, given a specific kernel and container types, we get $2 \times (10 + 36) = 92$ overloaded instances of functions that apply the convex-decomposition approach.

The approximated_inset_2(P, r, eps, oi) function template accepts a polygon $P$, an inset radius $r$, (a floating-point number) $\epsilon > 0$, and an output iterator $oi$; dereferencing the iterator must yield an instance of the class template CGAL:: Gps_circle_segment_traits_2<Kernel>::Polygon_2. It constructs an approximation of the inset of $P$ by the radius $r$, where the approximation error is bounded by $\epsilon$. The function returns the polygons that approximate the inset polygon through the output iterator $oi$.

The code excerpt shown in Listing 11 demonstrates the construction of an approximated inner offset; see Figure 3.
Listing 11: Computing the approximated inner offset.

```python
filename = sys.argv[2] if len(sys.argv) > 2 else "tight.dat"
with open(filename, 'r') as inp:
P = read_polygon(inp)

# Approximate the offset polygon.
tic = time.perf_counter()
radius = Ker.FT(1)
inset_polygons = Ms2.approximated_inset_2(P, radius, 0.00001)
toc = time.perf_counter()

print("The inset comprises", len(inset_polygons), "polygon(s).")
for pgn in inset_polygons: print(pgn)
print(f"Inset computation took {{toc - tic:.4f}} seconds")
```

Figure 3: The inset (yellow) of a polygon (blue) with a tight corridor consists of two generalized polygons bounded by line segments and circular arcs.

4.6 2D Triangulation Bindings

Triangulation is perhaps the most common term in the lexicon of computational geometry. Triangulations are ubiquitous geometric data structures, which are used in numerous areas, such as, GIS, robotics, geometric modeling and meshing to name a few; see, e.g., [7] for a survey on triangulations. The triangulation packages of CGAL are integral parts of the library and have constantly improved and enhanced since the early days of CGAL. In two-dimensions CGAL offers basic, Delaunay and regular triangulations, as well as constrained triangulations and constrained Delaunay triangulations. In three dimensions CGAL offers basic, Delaunay, and regular triangulations. CGAL also offers periodic triangulations both in the plane and in space [10].

A triangulation of a set of points $\mathcal{P}$ in $\mathbb{R}^2$ is a partition of the convex hull of $\mathcal{P}$ into triangles whose vertices are the points of $\mathcal{P}$. Together with the unbounded face having the convex hull boundary as its frontier, the triangulation forms a partition of $\mathbb{R}^2$. Any two facets (2-face) are either disjoint or share a common edge (1-face) or vertex (0-face). A triangulation can be described as a simplicial complex. The binding module 2D Triangulations consists of bindings of types provided by the CGAL packages 2D Triangulations and 2D Periodic Triangulations. Table 13 lists the CMAKE flags associated with the 2D Triangulations module.

The 2D Triangulations package [36] provides several class templates, instances of which can be used to represent a variety of 2D triangulations. In particular, the package provides the following class templates:

1. Triangulation_2<Traits, Tds>,
2. Regular_triangulation_2<Traits, Tds>,
3. Delaunay_triangulation_2<Traits, Tds>,
4. Constrained_triangulation_2<Traits, Tds, Itag>,
5. Constrained_Delaunay_triangulation_2<Traits, Tds, Itag>, and
6. Triangulation_hierarchy_2<Triangulation_2>

Instances of the Delaunay_triangulation_2<> and Regular_triangulation_2<> class templates can be used to represent Delaunay and regular triangulation, respectively. In a regular triangulation points have an associated weight, and some points can be hidden and do not result in vertices of the triangulation. The class template
Table 13: 2D Triangulations module flags.

| Name                  | Type  | Default | Description                                      |
|-----------------------|-------|---------|--------------------------------------------------|
| TRI2_NAME             | String| plain   | The 2D triangulation type                        |
| TRI2_VERTEX_WITH_INFO | Boolean| false  | Determines whether the vertex type is extended  |
| TRI2_FACE_WITH_INFO   | Boolean| false  | Determines whether the face type is extended     |
| TRI2_INTERSECTION_TAG_NAME | String | ncirc | The intersection tag                             |
| TRI2_HIERARCHY        | Boolean| false  | Determines whether to generate the binding for a hierarchy triangulation |

Table 14: 2D triangulation name options

| TRI2_NAME | Type                                           |
|-----------|------------------------------------------------|
| plain     | Triangulation_2<Traits , Tds>                  |
| regular   | Regular_triangulation_2<Traits , Tds>          |
| delaunay  | Delaunay_triangulation_2<Traits , Tds>         |
| constrained | Constrained_triangulation_2<Traits , Tds , Itag> |
| constrainedDelaunay | Constrained_Delaunay_triangulation_2<Traits , Tds , Itag> |
| periodicPlain | Periodic_2_triangulation_2<Traits , Tds> |
| periodicDelaunay | Periodic_2_Delaunay_triangulation_2<Traits , Tds> |

Triangulation_hierarchy_2 enables fast point location queries. When instantiated its template parameter must be substituted with an instance of any other triangulation class template.

The 2D Periodic Triangulations package [24] supports triangulations of sets of points in the two-dimensional flat torus [10]. This package provides the class templates

1. Periodic_2_triangulation_2<Traits , Tds>,
2. Periodic_2_Delaunay_triangulation_2<Traits , Tds>, and
3. Periodic_2_triangulation_hierarchy_2<TPeriodicTriangulation>

The CMake flag CGALPY_TRI2_NAME specifies the particular type of triangulation, bindings for which should be generated; see Table 14.

When any template above is instantiated the template parameter Traits must be substituted with a model of a suitable geometric traits concept; this model is referred to as the geometric traits class; it provides the type of points to use as well as elementary operations on points of the indicated type. The type of traits used for the generated bindings is determined based on the selection of the triangulation type as explained below. It is conveniently defined in C++ as Dt::Geom_traits, where Dt is a triangulation instance. This type is exposed as a Python attribute with the same name under Triangulation_2 (which in turn is nested under the Python namespace Tri2). Figure 4 depicts the 2D triangulation traits concept hierarchy. Any kernel instance is a model of any non-periodic traits concept; thus, when one of the templates

1. Triangulation_2<Traits , Tds>,
2. Regular_triangulation_2<Traits , Tds>,
3. Delaunay_triangulation_2<TTraits , Tds>,
4. Constrained_triangulation_2<TTraits , Tds , Itag>, or
5. Constrained_Delaunay_triangulation_2<TTraits , Tds , Itag>

is instantiated, the Traits parameter is substituted with the Kernel type (the selected kernel; see Section 4.1). When one of the templates

1. Periodic_2_triangulation_2<TTraits , Tds> or
2. Periodic_2_Delaunay_triangulation_2<TTraits , Tds>

is instantiated, the Traits parameter is substituted with Periodic_2_triangulation_traits_2<Kernel> or Periodic_2_Delaunay_triangulation_traits_2<Kernel>, respectively. Observe that when a Delaunay
The triangulation instance is used to define an alpha shape type (see Section 4.7), the traits parameter must be substituted with a traits class that models the AlphaShapeTraits_2 concept. Similarly, when a regular triangulation instance is used to define a fixed alpha shape type (see Section 4.7), the traits parameter must be substituted with a traits class that models the WeightedAlphaShapeTraits_2 concept. Also in these cases the selected kernel serves as the traits.

![Triangulation traits concept hierarchy](image)

Figure 4: The 2D triangulation traits concept hierarchy. Triangulation, Periodic_2 and Traits_2 are abbreviated as Tri, Pr2, and Tr2, respectively.

The CMake Boolean flag CGALPY_TRI2_HIERARCHY indicates whether the type of triangulation, bindings for which should be generated, is an instance of one of the triangulation hierarchy templates, namely,

- Triangulation_hierarchy_2<Triangulation_2>
- Periodic_triangulation_hierarchy_2<PeriodicTriangulation_2>

The template parameter in both cases is substituted with the triangulation selected via the CMake flag CGALPY_TRI2_NAME, which also determines whether to use the non-periodic or periodic version above.

The type that substitutes the Tds parameter models the concept TriangulationDataStructure_2. An object of this type stores the combinatorial structure of the triangulation; it is an instance of the class template Triangulation_data_structure_2<V, F>. The types that substitute the V and F template parameters when the template Triangulation_data_structure_2<V, F> is instantiated represent the type of the vertex and the type of the face of the triangulation, respectively; they must model the concepts TriangulationDSVertexBase_2 and TriangulationDSFaceBase_2, respectively. If the binding is generated for a periodic triangulation, these parameters must be substituted with types that also model the concepts Periodic_2TriangulationDSVertexBase_2 and Periodic_2TriangulationDSFaceBase_2, respectively. If the triangulation is used to define an alpha shape type, these parameters must be substituted with types that also model the concepts AlphaShapeVertex_2 and AlphaShapeFace_2, respectively; see Section 4.7. Finally, if the binding is generated for a hierarchy triangulation, e.g., an instance of the template parameter Triangulation_hierarchy_2<Triangulation_2>, the V template parameter must be substituted with a model of the concept TriangulationHierarchyVertexBase_2. (Observe that there is no special requirements on the type that substitutes the F parameter in this case.) Similar to the 2D arrangement data structure, it is possible to extend the vertex and face types of the triangulation. This is governed by two CMake Boolean flags as follows. If any one of the CMake variables CGALPY_TRI2_VERTEX_WITH_INFO or CGALPY_TRI2_FACE_WITH_INFO is set to true, the corresponding template parameter, V or F, is substituted with instances of CGAL::Triangulation_vertex_base_with_info_2<py::object, Traits, Vb> or CGAL::Triangulation_face_base_with_info_2<py::object, Traits, Fb>, respectively, where Vb and Fb are types that model the concepts above. The final vertex type is selected accordingly; it is explained in details in Section A.3 of the appendix.

The final vertex type is conveniently defined in C++ as Dt::Vertex, where Dt is the triangulation instance. This type is exposed as a Python attribute with the same name under Triangulation_2. The final face type is conveniently defined in C++ as Dt::Face, where Dt is the triangulation instance. This type is also exposed as a Python attribute with the same name under Triangulation_2.

4.7 2D Alpha Shape Bindings

The alpha shape (a.k.a. alpha complex) of a set of points is one of several notions of a shape formed by the set. Given a set of points sampled in a 2D body, an alpha shape is demarcated by a frontier, which is a linear approximation of the original boundary of the body. A two-dimensional alpha shape object maintains an underlying triangulation of a set of input points in the plane. There are two distinguished versions of alpha shapes as follows. Basic alpha shapes are based on the Delaunay triangulation and weighted alpha shapes are based on its generalization, the regular triangulation, where the euclidean distance is replaced by the power to weighted points. The package 2D Alpha Shapes [11] provides the class template Alpha_shape_2<Dt, ExactAlphaComparisonTag>. In a 2D alpha shape object represented
by an instance of this class template each $k$-simplex of the underlying triangulation is associated with an interval that specifies for which values of $\alpha$ the $k$-simplex belongs to the alpha shape. Table 15 lists the CMake flags associated with the 2D Alpha Shapes module.

| Name                        | Type    | Default | Description                                |
|-----------------------------|---------|---------|--------------------------------------------|
| AS2_EXACT_COMPARISON        | Boolean | false   | Determines whether to apply exact comparisons |

### 4.8 3D Triangulation Bindings

A triangulation of a set of points $P$ in $\mathbb{R}^3$ is a partition of the convex hull of $P$ into tetrahedra whose vertices are the points of $P$. Similar to the two-dimensional triangulation, together with the unbounded cell having the convex hull boundary as its frontier, the triangulation forms a partition of $\mathbb{R}^3$. Any two cells (3-face) are either disjoint or share a common facet (2-face), edge (1-face) or vertex (0-face). The binding module 3D Triangulations consists of bindings of types provided by the CGAL packages 3D Triangulations and 3D Periodic Triangulations. Table 16 lists the CMake flags associated with the 3D Triangulations module.

| Name                        | Type    | Default | Description                                |
|-----------------------------|---------|---------|--------------------------------------------|
| TRI3_NAME                   | String  | plain   | The 3D triangulation type                  |
| TRI3_CONCURRENCY_NAME       | String  | sequential | The concurrency method                    |
| TRI3_LOCATION_POLICY_NAME   | String  | compact | The location policy                        |
| TRI3_HIERARCHY              | Boolean | false   | Determines whether to generate the binding for a hierarchy triangulation |

The 3D Triangulations package [23] provides several class templates, instances of which can be used to represent a variety 3D triangulations. In particular, the package provides the following class templates:

1. Triangulation_3<Traits , Tds, Slds>,
2. Regular_triangulation_3<Traits , Tds, Slds>, and
3. Delaunay_triangulation_3<Traits , Tds, LocationPolicy , Slds>
4. Triangulation_hierarchy_3<Triangulation_3>

Instance of the Delaunay_triangulation_3<> and Regular_triangulation_3<> class templates can be used to represent Delaunay and regular triangulation, respectively. In a regular triangulation points have an associated weight, and some points can be hidden and do not result in vertices in the triangulation. The template Triangulation_hierarchy_3 enables fast point location queries. When instantiated its template parameter must be substituted with an instance of any other triangulation class template.

The 3D Periodic Triangulations package [9] supports triangulations of sets of points in the three-dimensional flat torus [10]. This package provides the class templates

1. Periodic_3_triangulation_3<Traits , Tds>,
2. Periodic_3_regular_triangulation_3<Traits , Tds>, and
3. Periodic_3_Delaunay_triangulation_3<Traits , Tds>
4. Periodic_3_triangulation_hierarchy_3<PeriodicTriangulation>

The CMake flag CGALPY_TRI3_NAME specifies the particular types of triangulation, bindings for which should be generated; see Table 17.

When any template above is instantiated the template parameter Traits must be substituted with a model of a suitable geometric traits concept; this model is referred to as the geometric traits class; it provides the type of points to use as well as elementary operations on points of the indicated types. The type of traits used for the generated bindings is determined based on the selection of the triangulation type as explained below. It is conveniently defined in C++ as Dt::Geom_traits, where Dt is the triangulation instance. This type is exposed as a Python attribute with the same name under Triangulation_3 (which in turn is nested under the Python namespace Tri3). Figure 5 depicts
Table 17: 3D triangulation name options

| TR3_NAME      | Type                                                   |
|---------------|--------------------------------------------------------|
| plain         | Triangulation_3<Traits, Tds, Slds>                     |
| regular       | Regular_triangulation_3<Traits, Tds, Slds>            |
| delaunay      | Delaunay_triangulation_3<Traits, Tds, LocationPolicy, Slds> |
| periodicPlain | Periodic_3_triangulation_3<Traits, Tds>               |
| periodicRegular | Periodic_3_regular_triangulation_3<Traits, Tds>     |
| periodicDelaunay | Periodic_3_Delaunay_triangulation_3<Traits, Tds> |

the 3D triangulation traits concept hierarchy. Any kernel instance is a model of any non-periodic traits concept; thus, when one of the templates
1. Triangulation_3<Traits, Tds>,
2. Regular_triangulation_3<Traits, Tds>, and
3. Delaunay_triangulation_3<Traits, Tds>
is instantiated, the Traits parameter is substituted with the Kernel type (the selected kernel; see Section 4.1). When one of the templates
1. Periodic_3_triangulation_3<Traits, Tds>,
2. Periodic_3_regular_triangulation_3<Traits, Tds>, or
3. Periodic_3_Delaunay_triangulation_3<Traits, Tds>
is instantiated, the Traits parameter is substituted with Periodic_3_triangulation_traits_3<Kernel>, Periodic_3_regular_triangulation_traits_3<Kernel>, or Periodic_3_Delaunay_triangulation_traits_3<Kernel>, respectively. Observe that when a Delaunay triangulation instance is used to define either a plain or a fixed alpha shape type (see Section 4.9), the traits parameter must be substituted with a traits class that models either the AlphaShapeTraits_3 or the FixedAlphaShapeTraits_3 concept, respectively. Similarly, when a regular triangulation instance is used to define either a plain or a fixed alpha shape type (see Section 4.9), the traits parameter must be substituted with a traits class that models either the WeightedAlphaShapeTraits_3 or the FixedWeightedAlphaShapeTraits_3 concept, respectively. Also in these cases the selected kernel serves as the traits.

![Diagram of the 3D triangulation traits concept hierarchy](image)

Figure 5: The 3D triangulation traits concept hierarchy. Triangulation, AlphaShape, Traits_3 and Periodic_3 are abbreviated as Tri, AS, Pr3, and Tr3, respectively.

The CMake Boolean flag CGALPY_TRI3_HIERARCHY indicates whether the type of triangulation, bindings for which should be generated, is an instance of one of the triangulation hierarchy templates, namely,
- Triangulation_hierarchy_3<Triangulation_2> and
- Periodic_triangulation_hierarchy_3<PeriodicTriangulation_3>.
The template parameter in both cases is substituted with the triangulation selected via the CMake flag CGALPY_TRI3_NAME, which also determines whether to use the non-periodic or periodic version above.
The type that substitutes the Tds parameter models the concept TriangulationDataStructure_3. An object of this type stores the combinatorial structure of the triangulation; it is an instance of the class template Triangulation_data_structure_3<V, C, ConcurrencyTag>. The types that substitute the V and C template parameters when the Triangulation_data_structure_3<V, C> template is instantiated represent the type of the vertex and the type of the cell of the triangulation, respectively; they must model the TriangulationDS_VERTEXBase_3 and TriangulationDSCellBase_3 concepts, respectively. If the binding is generated for a
periodic triangulation, these parameters must be substituted with types that also model the concepts Periodic_3 and Periodic_3TriangulationDSVertexBase_3 and Periodic_3TriangulationDSCellBase_3, respectively. If the triangulation is used to define a plain alpha shape type, these parameters must be substituted with types that model the concepts AlphaShapeVertex_3 and AlphaShapeCell_3, respectively; see Section 4.9. If the triangulation is used to define a fixed alpha shape type, these parameters must be substituted with types that model the concepts FixedAlphaShapeVertex_3 and FixedAlphaShapeCell_3, respectively. Finally, if the binding is generated for a hierarchy triangulation, e.g., an instance of the template parameter Triangulation_hierarchy_3<Triangulation_3>, the V template parameter must be substituted with a model of the concept TriangulationHierarchyVertexBase_3. (Observe that there is no special requirements on the type that substitutes the C parameter in this case.) Similar to the 2D arrangement and 2D triangulation data structures, it is possible to extend the vertex and cell types of the triangulation. This is governed by two CMake Boolean flags as follows. If any one of the CMAKE variables CGALPY_TRI3_VERTEX_WITH_INFO or CGALPY_TRI3_CELL_WITH_INFO is set to true, the corresponding template parameter, V or C, is substituted with instances of CGAL::Triangulation_vertex_base_3<py::object, Traits, Vb> or CGAL::Triangulation_cell_base_3<py::object, Traits, Cb>, respectively, where Vb and Cb are types that model the concepts above. The final vertex type is selected accordingly; it is explained in details in Section A.4 of the appendix.

The final vertex type is conveniently defined in C++ as Dt::Vertex, where Dt is the triangulation instance. This type is exposed as a Python attribute with the same name under Triangulation_3. The final face type is conveniently defined in C++ as Dt::Face, where Dt is the triangulation instance. This type is also exposed as a Python attribute with the same name under Triangulation_3.

The template parameter ConcurrencyTag is substituted with either Sequential_tag or Parallel_tag when the template Triangulation_data_structure_3<V, C, ConcurrencyTag is instantiated. It enables the use of a concurrent container to store vertices and cells. The CMAKE flag CGALPY_TRI3_CONCURRENCY_NAME determines the selection; see Table 18.

### Table 18: 3D triangulation concurrency options

| TRI3_CONCURRENCY_NAME | Type            |
|----------------------|-----------------|
| sequential           | Sequential_tag  |
| parallel             | Parallel_tag    |

The template parameter LocationPolicy is substituted with either Fast_location or Compact_location when the template Delaunay_triangulation_3<Traits, Tds, LocationPolicy> is instantiated. It enables a faster point location at the account of memory space. This is useful when performing point locations or random point insertions in large data sets. The CMAKE flag CGALPY_TRI3_LOCATION_POLICY_NAME determines the selection; see Table 19.

### Table 19: 3D triangulation location policy options

| TRI3_LOCATION_POLICY_NAME | Type          |
|--------------------------|---------------|
| fast                     | Fast_location |
| compact                  | Compact_location |

The code excerpt in the CMAKE language shown in Listing 12 sets the CMAKE flags for our next example in Python shown in Listing 13. The example constructs a three-dimensional Delaunay triangulation from six points and verifies that the triangulation is valid.

### 4.9 3D Alpha Shape Bindings

Given a set of points sampled in a 3D body, an alpha shape (a.k.a. alpha complex) is demarcated by a frontier, which is a linear approximation of the original boundary of the body. Similar to the 2D alpha shape object, A 3D alpha shape object maintains an underlying 3D triangulation of a set of input points. Basic alpha shapes are based on the Delaunay triangulation and weighted alpha shapes are based on regular triangulation, where the euclidean distance is replaced
Listing 12: CMake flag settings used to generate bindings for 3D triangulations.

```cmake
set(CMAKE_BUILD_TYPE "Release" CACHE STRING "build type" FORCE)
set(CGALPY_USE_SHARED_LIBS ON CACHE BOOL "use shared libs" FORCE)
set(CGALPY_FIXED_LIBRARY_NAME OFF CACHE BOOL "use name CGALPY" FORCE)
set(CGALPY_KERNEL_NAME "epic" CACHE STRING "use EPIC kernel" FORCE)
set(CGALPY_TRIANGULATION_3_BINDINGS ON CACHE BOOL "3D Triangulations" FORCE)
set(CGALPY_TRI3_NAME "delaunay" CACHE STRING "Use Delaunay" FORCE)
```

Listing 13: Constructing a 3D triangulations of six vertices.

```python
Ker = CGALPY.Ker
Tri3 = CGALPY.Tri3
Point_3 = Ker.Point_3
p1 = Point_3(0, 0, 0)
p2 = Point_3(1, 0, 0)
p3 = Point_3(0, 1, 0)
p4 = Point_3(0, 0, 1)
p5 = Point_3(2, 2, 2)
p6 = Point_3(-1, 0, 1)
dt = Tri3.Triangulation_3([p1, p2, p3, p4, p5, p6])
print(dt.is_valid(False, 0))
```

by the power to weighted points. The package 3D Alpha Shapes [12] provides the class templates Alpha_shape_3 ↩ <Dt, ExactAlphaComparisonTag> and Fixed_alpha_shape_3<Dt>. Instances of both templates can be used to represent a large variety of alpha shapes for a given set of points. In a plain alpha shape each k-face of this triangulation is associated with an interval specifying for which values of α the face belongs to the alpha complex. In a fixed alpha shape each k-face is associated with a classification that specifies its status in the alpha complex, alpha being fixed. Table 20 lists the CMake flags associated with the 3D Alpha Shapes module.

| Name            | Type         | Default | Description                         |
|-----------------|--------------|---------|-------------------------------------|
| AS3_NAME        | String       | plain   | The 3D Alpha shape type             |
| AS3_EXACT_COMPARISON | Boolean    | false   | Determines whether to apply exact comparisons |

The CMake flag AS3_NAME specifies which alpha shape should be used for the generated bindings; see Table 21

| AS3_NAME | Type                                      |
|----------|-------------------------------------------|
| plain    | Alpha_shape_3<Tri, Ec>                    |
| fixed    | Fixed_alpha_shape_3<Tri>                  |

The template parameter DT must be substituted with an instance of one of the following class templates that can be used to represent a triangulation:
1. Delaunay_triangulation_3,
2. Regular_triangulation_3,
3. Periodic_3_Delaunay_triangulation_3, or
4. Periodic_3_regular_triangulation_3;
Listing 14: CMake flag settings used to generate bindings for 3D alpha shapes.

```cmake
set (CMAKE_BUILD_TYPE "Release" CACHE STRING "build type" FORCE)
set (CGALPY_USE_SHARED_LIBS ON CACHE BOOL "use shared libs" FORCE)
set (CGALPY_FIXED_LIBRARY_NAME OFF CACHE BOOL "use name CGALPY" FORCE)
set (CGALPY_KERNEL_NAME "epic" CACHE STRING "use EPIC kernel" FORCE)
set (CGALPY_TRIANGULATION_3_BINDINGS ON CACHE BOOL "3D Triangulations" FORCE)
set (CGALPY_TRI3_NAME "delaunay" CACHE STRING "Use Delaunay" FORCE)
set (CGALPY_TRI3_LOCATION_POLICY_NAME "compact" CACHE STRING "Preserve space" FORCE)
set (CGALPY_TRI3_CONCURRENCY_NAME "parallel" CACHE STRING "Concurrent" FORCE)
set (CGALPY_ALPHA_SHAPE_3_BINDINGS ON CACHE BOOL "3D Alpha Shapes" FORCE)
```

Listing 15: Constructing a 3D alpha shape object.

```python
Ker = CGALPY.Ker
As3 = CGALPY.As3
Point_3 = Ker.Point_3
Alpha_shape_3 = As3.Alpha_shape_3
Mode = As3.Mode
p1 = Point_3(492, 291, 677)
p2 = Point_3(493, 314, 533)
p3 = Point_3(494, 326, 462)
p4 = Point_3(493, 303, 605)
alphaShape = Alpha_shape_3([p1, p2, p3, p4])
print("Alpha shape computed in REGULARIZED mode by default")
alphaSolid = alphaShape.find_alpha_solid() # Find alpha solid
print("Smallest alpha value to get a solid through data points is ", alphaSolid)
opthandle = alphaShape.find_optimal_alpha(1) # find optimal alpha value
optAlpha = next(opthandle)
print("Optimal alpha value to get one connected component is ", optAlpha)
alphaShape.set_alpha(optAlpha)
um = alphaShape.number_of_solid_components()
print("# solid components: ", num)
```

see Section 4.8. Note that Dt::Geom_traits must be model of a suitable alpha-shape traits concept, and Dt::

Vertex and Dt::Face must be models suitable alpha-shape vertex and cell concepts, respectively; see Section 4.8.

The following code excerpt in the CMake language shown in Listing 14 sets the CMake flags for our next Python example shown in Listing 15. The examples constructs an alpha shape object.

### 4.10 Spatial Searching Bindings

The dD Spatial Searching package [30] implements exact and approximate distance browsing by providing implementations of algorithms supporting

1. both nearest and furthest neighbor searching,
2. both exact and approximate searching,
3. (approximate) range searching,
4. (approximate) k-nearest and k-furthest neighbor searching,
5. (approximate) incremental nearest and incremental furthest neighbor searching, and
6. query items representing points and spatial objects.

In these searching problems a set \( P \) of data points in \( d \)-dimensional space is given. The points in \( P \) are preprocessed into a tree data structure, so that given any query item \( q \) the points of \( P \) can be browsed efficiently. The approximate dD Spatial Searching package is designed for data sets that are small enough to store the search structure in main memory (in contrast to approaches from databases that assume that the data reside in secondary storage).

The code excerpt in the CMake language shown in Listing 16, sets the CMake flags for our last example shown in Listing 17. The example constructs a kD-tree of 2D points and applies various queries on the tree.
Table 22: dD Spatial Searching module flags

| Name                        | Type   | Default | Description                                               |
|-----------------------------|--------|---------|-----------------------------------------------------------|
| SPATIAL_SEARCHING_DIMENSION | Integer| 2       | The dimension of spatial searching related classes        |

Listing 16: CMake flag settings used to generate bindings for spatial searching types.

```
set (CMAKE_BUILD_TYPE "Release" CACHE STRING "build type" FORCE)
set (CGALPY_USE_SHARED_LIBS ON CACHE BOOL "use shared libs" FORCE)
set (CGALPY_FIXED_LIBRARY_NAME OFF CACHE BOOL "use name CGALPY" FORCE)
set (CGALPY_KERNEL_BINDINGS OFF CACHE BOOL "2,3 Kernel" FORCE)
set (CGALPY_KERNEL_D_BINDINGS ON CACHE BOOL "dD Kernel" FORCE)
set (CGALPY_KERNEL_D_NAME "cartesiandLazyGmpq" CACHE STRING "used dD kernel" FORCE)
set (CGALPY_KERNEL_D_DIMENSION_TAG "static" CACHE STRING "fixed dimension" FORCE)
set (CGALPY_KERNEL_D_DIMENSION 2 CACHE STRING "2 dimensions" FORCE)
set (CGALPY_SPATIAL_SEARCHING_BINDINGS ON CACHE BOOL "Spatial Searching" FORCE)
set (CGALPY_SPATIAL_SEARCHING_DIMENSION 2 CACHE STRING "2 dimensions" FORCE)
```

Observe that the kD-tree dimension (and the dimension of related data structures) is set via the CMake flag CGALPY_SPATIAL_SEARCHING_DIMENSION during compile time. The code in the listings assumes that the dimension is 2. In order to assert the dimension the bindings were compiled with (see Line 10 in the listing), we have introduced and exposed a free function called get_spatial_searching_dimension() that returns the dimension as an int.

5 Binding code

Nanobind allows users to expose C++ classes and functions to Python using nothing more than a C++ compiler. It exploits metaprogramming techniques to implement a rich set of features and high-level user-friendly interface. Nanobind extracts as much as information as possible from the source code to be wrapped. This approach is referred to as user guided wrapping. Additional information that cannot be automatically deduced is explicitly supplied by the user. The interface specification is written in the same full-featured C++ language as the code being exposed, which has the potential of generating efficient binding. In this section we describe some of the techniques we use to ensure that.

5.1 Return value policy

Python and C++ use fundamentally different ways of managing the memory and lifetime of objects. This can lead to issues when creating bindings for functions that return a non-trivial type. Just by looking at the type, it is unclear whether Python should take charge of the returned value and eventually free its resources, or whether this is handled by the C++ code. When writing code in C++, it is usually considered a good practice to use smart pointers, which exactly describe ownership semantics. Still, even good C++ interfaces use raw references and pointers sometimes. In some cases, in order to assure proper memory management of a return value from a function, explicitly specifying a return value policy is needed. The policy may depend on whether the returned object is a newly created object, a reference to some already existing (internal) object, or a const reference to some existing object. Additionally, in some cases we need to tie the lifetime of the result to the lifetime of the arguments, or the other way around.

For some functions of kernel objects the kind of return value depends on the kernel type; thus, the return value policy needs to be set accordingly. For example, when Exact_predicates_exact_constructions_kernel is the kernel type, the return type of some functions that return a coordinate or a coefficient is a const reference, while when the kernel type is Exact_predicates_inexact_constructions_kernel the same functions return an object by value. This is why for these functions a Kernel_return_value_policy type is being passed as the return value policy. The type is defined differently based on the kernel being exposed.
Listing 17: Applying various queries on a kD-tree of 2D points.

```python
Kerd = CGALPY.Kerd
Ss = CGALPY.Ss

if hasattr(Kerd, 'FT'): FT = Kerd.FT
else: FT = float
Point_d = Ss.Point_d

# Verify that the bindings are generated with the CMake flag
# CGALPY_SPATIAL_SEARCHING_DIMENSION set to 2
assert (Ss.get_spatial_searching_dimension() == 2)

k = 3
points = [Point_d(2, [FT(n) for n in [4, 0]]), Point_d(2, [FT(n) for n in [-4, 0]]),
          Point_d(2, [FT(n) for n in [40, 0]]), Point_d(2, [FT(n) for n in [-40, 0]]),
          Point_d(2, [FT(n) for n in [1, 0]])]
tree = Ss.Kd_tree(points)

all_points = tree.points()
for x in all_points: print(x)
query = Point_d(2, [FT(n) for n in [0, 0]])
eps = FT(0.0)  # 0.0 for exact NN, otherwise approximate NN
search_nearest = True  # set to False to search farthest
sort_neighbors = False  # set to True to obtain the neighbors sorted by distance
distance = Ss.Euclidean_distance()  # The distance metric to use

starttime = timeit.default_timer()
search = Ss.K_neighbor_search(tree, query, k, eps, search_nearest, distance, sort_neighbors)
lst = search.k_neighbors()
print("Neighbors search took ", timeit.default_timer() - starttime)

print("Found", len(lst), "neighbors")
for pair in lst:
    print("Neighboring point is: ", pair[0])
    print("Squared distance from query is: ", pair[1])

# Search for points inside a sphere
s = Ss.Fuzzy_sphere(Point_d(2, [FT(0), FT(0)]), FT(5), FT(0))
res = tree.search(s)
print("Points within distance 5 from (0,0):")
for p in res: print(p)

# Search for points inside a box
s = Ss.Fuzzy_iso_box(Point_d(2, [FT(-1), FT(-1)]), Point_d(2, [FT(1), FT(1)]), FT(0))
res = tree.search(s)
print("Points with no coordinate exceeding absolute value of 1:")
```
5.2 Intersection Detection

The function template CGAL::do_intersect (const T1& o1, const T2& o2) determines whether two geometric objects intersect. The function is overloaded with several implementations that handle different combinations of types of arguments provided by the 2D and 3D Geometry Kernel package (see Section 4.1). However, not every combination of two types is implemented in this package. For example, the intersection of a line segment and a circle in the general case is a point with algebraic coordinates, and thus not supported by the package.\textsuperscript{22} We use substitution failure is not an error (SFINAE) to generate bindings for all supported combinations while avoiding getting compilation errors for unsupported combinations. SFINAE refers to a situation in C++ where an invalid substitution of template parameters should not be treated as an error. In particular, it may occur during a process called overload resolution. In order to compile a function call, the compiler creates a set of candidate functions the names of which match the call and that can be accessed by the caller. Then, it reduces the candidate set to a set of viable functions that includes all function instances the parameters of which match the call arguments. Finally, the compiler selects the best match among the viable functions according to the C++ standard;\textsuperscript{23} a flowchart that describes the process is shown in Figure 6. When the substitution of an explicitly specified or deduced type for the template parameter fails, the specialization is discarded from the overload set instead of causing a compilation error.\textsuperscript{24}

Our entry point is the function export_intersections_2(py::module_& m); see Line 8 in Listing 18. It accepts a wrapper for Python extension modules, which is used to wrap the CGAL::do_intersect() functions (Line 7 in Listing 20). It also serves as the entry point for the generation of bindings for the function CGAL::intersection\rightarrow(const T1& o1, const T2& o2); see Section 5.3. Let \( \mathcal{T}_2 \) and \( \mathcal{T}_3 \) denote the complete sets of types of two-dimensional and three-dimensional objects, respectively, supported by the 2D and 3D Geometry Kernel package; see Listing 18 Lines 9–12. Our goal is the automatic instantiation of the class template Wrapper (Listing 18 Line 1) for every pair (T1, T2) in the Cartesian products \( \mathcal{T}_2 \times \mathcal{T}_2 \) and \( \mathcal{T}_3 \times \mathcal{T}_3 \), where the template parameter pack \( \ldots \) Types is substituted with the pair T1, T2. In particular, we seek for the evaluation of the expression Wrapper<T1, T2>::operator() (m) for each instance, which in turn, leads to the evaluation of the expression bind_do_intersect\rightarrow<T1, T2>(m, true) for every such pair. The expansion of the constant expression in Line 13 of Listing 18 as part of the compilation process results in the evaluation of the expression Wrapper<T1, T2>::operator() (m) for each (T1, T2) \( \in \mathcal{T}_2 \times \mathcal{T}_2 \). Similarly, the expansion of the constant expression in Line 14 of Listing 18 results in the evaluation of the expression Wrapper<T1, T2>::operator() (m) for each (T1, T2) \( \in \mathcal{T}_3 \times \mathcal{T}_3 \).

The function template cartesian_product (see Listing 19 Line 17) can be used to evaluate expressions for each member of the Cartesian product of a given sequence of sets of types. The code in Listing 19 was presented to us, so we merely provide the code. When the template function is instantiated the Wrap template parameter must be substituted with a class template that has an operator() member function. The template class that substitutes the Wrap template parameter is instantiated for every member of the Cartesian product of the sets of types that are used to define the types of the type_lists argument pack. Observe that the type of each object in this pack is an instance of the class template Type_list, which in turn is defined by substituting a template parameter pack with

\textsuperscript{22}For a complete list of valid combinations of do_intersect () argument types refer to the reference manual at https://doc.cgal.org/latest/Kernel_23/group__do__intersect__linear__grp.html.
\textsuperscript{23}For the full specification of the C++ standard see https://isocpp.org/std/the-standard.
\textsuperscript{24}For more information on SFINAE see, e.g., https://en.cppreference.com/w/cpp/language/sfinae.
Listing 18: Entry points for the code that automatically generates binding for `CGAL::do_intersect()` and `CGAL::intersection()` with arguments that represent planar geometric objects.

```cpp
template <typename Arg, typename ... Types> struct Wrapper {
    void operator() (Arg& arg) {
        bind_do_intersect<Types... >(arg, true);
        bind_intersection<Types... >(arg, true);
    }
};

void export_intersections_2(py::module_& m) {
    CGALPY::Type_list<Iso_rectangle_2, Line_2, Ray_2, Segment_2, Triangle_2,
        Point_2, Circle_2> type_list_2;
    CGALPY::Type_list<Iso_cuboid_3, Line_3, Ray_3, Segment_3, Tetrahedron_3,
        Triangle_3, Point_3, Sphere_3> type_list_3;
    cartesian_product<Wrapper>(m, type_list_2, type_list_2);
    cartesian_product<Wrapper>(m, type_list_3, type_list_3);
}
```

a sequence of types. The arg argument is propagated all the way and passed to the `operator()` member function of the wrapper instantiated object. Finally, the expression `Wrap<T1, T2>::operator() (arg)` is evaluated for every member of the Cartesian product. The function `cartesian_product` is implemented with the help of the (compile-time) recursive function template `cartesian_product_recursive()`, which exploits a unary right fold expression; see Listing 19 Line 13.\textsuperscript{25} Let \(E\) denote an expression that contains a variadic template pack \(T\). Consider a block of code that contains a unary right fold expression \((E \text{ op...})\), and assume that the template pack \(T\) is substituted with the sequence of types \(T_1, T_2, \ldots, T_n\), when the code block is instantiated. The unary right fold expression \((E \text{ op...})\) expands to \((E_1 \text{ op...} (E_n-1 \text{ op } E_n))\), where \(E_i\) is the expression \(E\) with the template pack \(T\) substituted with \(T_i\). In our case (lines 13-14 in the listing) (i) \text{ op} is the (binary) comma operator,\textsuperscript{26} (ii) \(E\) is the recursive call to `cartesian_product_recursive()`, and (iii) the variadic template pack is \(\ldots\) \(\ldots\) \(\ldots\) \(Types\). Let `type_lists_1, type_lists_2, \ldots, type_lists_m` denote the sequence of objects that comprise the `type_lists` argument pack. In all evaluations of the recursive expression at level \(i\) of the recursion tree, that represents the iterated recurrence, the template pack \(\ldots\) \(\ldots\) \(\ldots\) is substituted with the sequence of types that are used to define the type of the `type_lists_i`.

Consider two types `Type1` and `Type2`, and assume that the `CGAL::do_intersect(const Type1& o1, const Type2& o2)` overloaded version is not implemented but all other three combinations in the Cartesian product of \{`Type1`, `Type2`\} × \{`Type1`, `Type2`\} are, and refer to Listing 20. The `bind_do_intersect()` function template is overloaded with two implementations (Line 5 and 7). As a consequence, assuming that the code compiles, one overload of `bind_do_intersect()` is selected for every member of the Cartesian product of \{`Type1`, `Type2`\} × \{`Type1`, `Type2`\}. The primary implementation of `bind_do_intersect()` (Line 1) uses `variadic arguments` and serves as a fall-through when the evaluation of the second implementation fails during the resolution process.\textsuperscript{27} The type of the second parameter of the second implementation is defined as the return type of `CGAL::do_intersect<Kernel>(T1(), T2())`. When `T1` and `T2` are substituted with `Type1` and `Type2`, respectively, the evaluation fails, and the blank implementation is selected, as this is the only candidate left. In all other three cases the evaluation succeeds and the second implementation is selected, as functions that accept variadic arguments have a lower rank for the purpose of overload resolution. Observe that in the call to `bind_do_intersect()` (Line 2) we pass a `bool` argument of value `true`. This value can be `false` just as well, but its type, that is `bool`, must match the return type of `CGAL::do_intersect<Kernel>(T1(), T2())` when defined. The result is not only compact code that is easy to maintain, but also code that supports bindings for every potential `do_intersect()` overload that might be implemented in the future.

\textsuperscript{25}For more information about fold expressions see https://en.cppreference.com/w/cpp/language/fold.

\textsuperscript{26}For more information about the comma operator see https://en.cppreference.com/w/cpp/language/operator_other#Built-in_function_call_operator.

\textsuperscript{27}For more information on `variadic arguments` see https://en.cppreference.com/w/cpp/language/variadic_arguments.
Listing 19: Automatic expansion of expressions with types that are members of the Cartesian product of sets of types.

```cpp
template <typename... Types> class Type_list {};

template <typename... Types> class Call_args {};

template <template <typename ...> class Wrapper, typename Arg, typename... Types>
void cartesian_product_recursive(Arg& arg, Call_args<Types...>){
    Wrapper<Arg, Types...>()(arg);
}

template <template <typename ...> class Wrapper, typename Arg, typename... CallTypes, typename... Types, typename... TypeLists>
void cartesian_product_recursive(Arg& arg, Call_args<CallTypes...>, Type_list<Types...>, TypeLists... type_lists){
    (cartesian_product_recursive<Wrapper>(arg, Call_args<CallTypes..., Types>(), type_lists...), ...);
}

template <template <typename ...> class Wrapper, typename Arg, typename... TypeLists>
void cartesian_product(Arg& arg, TypeLists... type_lists){
    cartesian_product_recursive<Wrapper>(arg, Call_args<>(), type_lists...);
}
```

Listing 20: Automatic generation of binding for CGAL::do_intersect() exploiting SFINAE.

```cpp
template <typename, typename> void bind_do_intersect(py::module_&, ...) {}

template <typename T1, typename T2>
void bind_do_intersect(py::module_& m, decltype(CGAL::do_intersect<Kernel>(T1(), T2()))) {
    using Do_intersect = bool(*)(const T1&, const T2&);
    m.def("do_intersect", static_cast<Do_intersect>(&CGAL::do_intersect<Kernel>));
}
```
5.3 Intersection Computation

The situation presented by the overloaded function template CGAL::intersection (const T1& o1, const T2& o2) that computes the intersection of two geometric objects is a bit more complicated. The various implementations are also provided by the 2D and 3D Geometry Kernel package. Similar to the CGAL::do_intersect() function templates, different implementations handle different combination of types of arguments and not all combinations are supported. The intersection of two geometric objects can be either empty, a single object, or several points.

Unlike the CGAL::do_intersect() function templates, the return types of which are all simple bool, the return type of each CGAL::intersection() function template is a polymorphic type that either implicitly converts to false if the intersection is empty or represents the intersection and depends on the types of the input arguments.

The mechanism introduced in the previous section results in the evaluation of the expression bind_intersection (p, true) for every pair (T1, T2) in the Cartesian products T2 × T2 ∪ T3 × T3; see Listing 18 Line 4. See Line 1 and Line 3 in Listing 21 for the primary and specialized implementations, respectively, of the function template bind_intersection (). Given a combination of two specific types, T1 and T2, we still evaluate the return value of the function template CGAL::intersection (const T1& o1, const T2& o2) as a mean to determine whether the function is defined, but we do not try to match this type to a type of an argument. Instead, we introduce an unnamed template parameter with a default value that evaluates to the return type of CGAL::intersection (const T1& o1, const T2& o2); see Line 4 in the listing. Observe that the outcome of the evaluation is irrelevant. The only thing that matters is whether the evaluation succeeds—it succeeds only if the function template CGAL::intersection (const T1& o1, const T2& o2) is defined. If it is defined, the specialized implementation is selected by the overload-resolution process, and the function cgalpy_intersection (const T1& t1, const T2& t2), which actually computes the intersection, is exposed; see Listing 22.

The exposed Python function that computes the intersection returns None if the intersection is empty; see Line 17 in Listing 22. If the intersection consists of a single geometric object, the function returns a single Python object the type of which is the exposed CGAL C++ type of the geometric object; see Line 4. Finally, if several points comprise the intersection, the return type is a list of Python objects (see Line 10), where the type of each element in the list is the exposed type of Kernel::Point_2. For more information of functions that return collections of object see Section 5.7.3.

5.4 Minkowski Sum Construction

The situation presented by the function template CGAL::minkowski_sum_2() that applies the convex-decomposition approach (see Section 4.5) is even more complicated. The function template shown in Line 1 of Listing 23 has 10 instances of valid combinations of argument types. There is another set, of the same cardinality, of function templates that also accept a fourth traits parameter. The number of instances of the function template shown in Line 8 is 36, and also here there is another set, of the same cardinality, of function templates that accept an additional traits parameter. Consider the function template shown in Line 1. We need to generate bindings for instances of this function only for valid combinations of PolygonType1, PolygonType2, and PolygonConvexDecomposition_2 types. Hereafter and in the code listing we use the respective short names PT1, PT2, and PP instead. A valid combination exists only if PP is a type of a polygon-partition unary function that can be applied to a polygon of type PT1 and to a polygon of type PT2. Some supported polygon-partition functions cannot be applied to polygon with holes, hence invalid combinations. In particular, if pp is a polygon-partition function, the calls pp(pgn1, it) and pp(pgn2, it) must be valid, where pgn1 and pgn2 are input polygons of types PT1 and PT2, respectively, and it is an output iterator of a container of the resulting polygons. Dereferencing the iterator must yield a type that can represent a simple polygon. If PT1 is identical to Polygon_2, then std::list<PT1>::iterator can serve as an output iterator; otherwise, PT1 must represent a polygon with holes, and in this case std::list<PT1::Polygon_2>::iterator
 Listing 22: Implementation of the binding for \texttt{CGAL::intersection()}.  

class Intersection_visitor : public boost::static_visitor&lt;py::object&gt; {

public:

    template&lt;typename T&gt;
    py::object operator()(T& operand) const { return py::cast(operand); }

   // Handle vector of points
    py::object operator()(std::vector<Point_2>& operand) const {
        py::list lst;
        for (const auto& p : operand) lst.append(p);
        return lst;
    }
};

template &lt;typename T1, typename T2&gt;
py::object cgalpy_intersection(const T1& t1, const T2& t2) {
    auto result = CGAL::intersection&lt;Kernel&gt;(t1, t2);
    if (!result) return py::object(); // no intersection
    return boost::apply_visitor(Intersection_visitor(), *result);
}

can serve as an output iterator; the same holds for PT2. We use SFINAE yet again to make this distinction,\textsuperscript{28} and we use SFINAE one more time to generate binding for \texttt{CGAL::minkowski_sum_2()} only for valid combinations of argument types.

Our entry point is the function \texttt{void export_minkowski_sum_2()}; see Line 11 in Listing 24. We need to capture valid combinations of three types and valid combinations of four types. Recall that the code in Listing 20 captures valid combinations of two types. Following the method presented in the previous section, we introduce two wrapper template classes with \texttt{operator()} members, which call \texttt{bind_mink_sum_one_strategy()} and \texttt{bind_mink_sum_two_strategies()}, respectively; see Line 3 and Line 8 in Listing 24.

First, we introduce a class template called \texttt{target}; see Line 3 in Listing 25. It is parameterized with a named parameter \texttt{T} and an unnamed parameter that defaults to \texttt{void}. It delegates the type \texttt{std::list&lt;T::Polygon_2>::iterator}. We also introduce a specialization (Line 6) that delegates the type \texttt{std::list&lt;T::Polygon_2&gt;::iterator}. The second parameter is substituted with a type provided by a utility template called \texttt{void_t}<> (Line 1). If the type \texttt{T::Polygon_2} is undefined, the evaluation of \texttt{void_t&lt;T::Polygon_2&gt;} fails, and in turn the \texttt{Target}<> specialization is discarded from the overloaded resolution set. Otherwise, \texttt{void_t&lt;T::Polygon_2&gt;} successfully evaluates (to \texttt{void}). In this case the \texttt{Target}<> specialization remains a viable option, and it is selected over the primary implementation, since specializations are ranked higher. Observe that the type delegated by the \texttt{void_t}<> template and the type of the unnamed template parameter of the primary implementation of \texttt{target}<> must match for the ordering to apply. These types are arbitrarily chosen to be \texttt{void}.

We need to capture valid combinations of three types for Minkowski-sum computation using a single convex decomposition strategy and valid combinations of four types for Minkowski-sum computation using two convex decomposition strategies. Recall that the code in the listings of the previous sections captures valid combinations of two types. Following the method presented in the previous sections, we introduce a wrapper template class with an \texttt{operator()} member, which calls \texttt{bind_mink_sum()}; see Lines 10,13 in Listing 25 for the primary and specialized implementations, respectively. The polygon-partition function returns an output iterator that points to the element, which is next to the last element of the container. Passing an argument, the type of which matches the returned value of the polygon-partition function, to the function \texttt{bind_mink_sum()} is complicated, as this type is not a simple type (e.g., \texttt{bool}) anymore. Instead, we introduce two unnamed template parameters with default values that evaluate to return types of the polygon-partition function when applied to a polygon of type PT1 or PT2, respectively; see Lines 14–15. Observe that the outcome of an evaluation is irrelevant. The only thing that matters is whether the evaluation succeeds—it succeeds only if the matching function is defined.

We exploit SFINAE to generate bindings of other types as described in the following sections.

\textsuperscript{28}This usage is similar to the example shown in \url{https://en.wikipedia.org/wiki/Substitution_failure_is_not_an_error#C++_11_simplification}. 

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Listing 23: Signatures of function templates that compute the 2D Minkowski sum using the convex-decomposition approach.

```cpp
template <typename Kernel, typename Container, 
          typename PolygonConvexDecomposition_2>
Polygon_with_holes_2<Kernel, Container>
minkowski_sum_2(const PolygonType1<Kernel, Container>& p, 
                 const PolygonType2<Kernel, Container>& q, 
                 const PolygonConvexDecomposition_2& decomp)

template <typename Kernel, typename Container, 
          typename PolygonConvexDecompositionP_2, 
          typename PolygonConvexDecompositionQ_2>
Polygon_with_holes_2<Kernel, Container>
minkowski_sum_2(const PolygonType1<Kernel, Container>& p, 
                 const PolygonType2<Kernel, Container>& q, 
                 const PolygonConvexDecompositionP_2& decomp_p, 
                 const PolygonConvexDecompositionQ_2& decomp_q)
```

Listing 24: Automatic expansion of expressions that generate bindings for Minkowski-sum computation.

```cpp
template <typename Arg, typename ... Types> struct Wrapper_one_strategy {
    void operator() (Arg& arg)
    { bind_mink_sum_decomp_one_strategy<Types... >(arg , true); }
};

template <typename Arg, typename ... Types> struct Wrapper_two_strategies {
    void operator() (Arg& arg)
    { bind_mink_sum_decomp_two_strategies<Types... >(arg , true); }
};

void export_minkowski_sum_2(py::module_& m) {
    using Pnp = ms2::Polygon_nop_decomposition_2;
    using Pvd = pp2::Polygon_vertical_decomposition_2;
    using Ptd = pp2::Polygon_triangulation_decomposition_2;
    using Ssabd = pp2::Small_side_angle_bisector_decomposition_2;

    CGALPY::Type_list<Polygon_2, Polygon_with_holes_2> polygon_types;
    CGALPY::Type_list<Pnp, Pvd, Ptd, Ssabd> strategy_types;

    CGALPY::cartesian_product<Wrapper_one_strategy>(m, polygon_types, polygon_types, 
                                                    strategy_types);
    CGALPY::cartesian_product<Wrapper_two_strategies>(m, polygon_types, polygon_types, 
                                                   strategy_types, strategy_types);
}
```
template <typename... Ts> using void_t = void;

template <typename T, typename = void> struct target {
    using type = std::list<T>::iterator;
};

template <typename T>
struct target<T, void_t<typename T::Polygon_2>> {
    using type = std::list<typename T::Polygon_2>::iterator;
};

void bind_mink_sum(py::module_& m, bool) {
    m.def("minkowski_sum_2", &CGAL::minkowski_sum_2<Kernel, Point_2_container, PP>);
}

5.5 Arrangement Extension

In many applications it is necessary to extend the type that represents certain Cgal data structures with some types that represent new properties. In Cgal it is more convenient, and thus more common, to extend the types that represent the topological features of the data structures (rather than the types that represent the geometric features), e.g., the DCEL of the arrangement data structure; see Section 4.3. When developing code in pure C++, there is no restriction on the property types used to extend the data structure. However, when the data structure interface is wrapped with Python bindings, the property type must be known when the bindings are compiled. Nevertheless, we support extensions with generic Python objects. If a certain feature must be extended with a certain property, the type of which is defined in the C++ code, it is always possible to use an external property map, where the properties are indexed by a Python object, such as an integer or a string. When generating the bindings the user must set the flag that enables the extension; see Table 11. Attaching an index to and extracting an index from a feature is done via the set_data() and data() exposed functions. Each cell type, namely, Vertex, Halfedge, or Face, can be extended independently, which implies that there are eight plausible combinations that the user can select from to form the final DCEL type of the arrangement.

Extending the arrangement features is useful for many applications, but essential for applications that compute the overlay of arrangements as discussed in the next Section.

5.6 Arrangement Overlay Traits

The map overlay of two arrangements \(A_1\) and \(A_2\), conveniently referred to as the red and blue arrangements, is a third arrangement \(A\), such that there is a cell \(c\) in \(A\) iff there are cells \(c_1\) and \(c_2\) in \(A_1\) and \(A_2\), respectively, and \(c\) is a maximal connected component of \(c_1 \cap c_2\). Computing the overlay of two arrangements is useful for many applications, including Boolean operations; see Section 4.4. Indeed, the 2D Arrangements package provides function templates that computes the overlay of two arrangements, namely overloaded CGAL::overlay(). In particular, the call CGAL::overlay\(\rightarrow\) (arr_r, arr_b, arr) computes the overlay of the arrangements arr_r and arr_b and stores the result in the arrangement arr. All three types must be instances of the Arrangement_2<Traits, Dcel> class template; see Section 4.3. When the function is used in pure C++ code, the geometry-traits classes that substitute the Traits\(\rightarrow\) template parameters of the input arrangements must be convertible to the geometry-traits class of the resulting arrangement. We only wrap instances of the overlay function with Python bindings, where all three arrangements use the exact same geometry-traits classes and the same DCEL structures, and thus the types of the three arrangements of the exposed CGAL::overlay() function are identical.

The CGAL::overlay() function template is overloaded with a variant that accepts four arguments. The last argument, referred to as the overlay traits, enables the update of cells of the output arrangement, and in particular their properties. Using the overlay traits is typically accompanied with the extension of one or more types of the
arrangement cells that hold the properties. The overlay traits must model the OverlayTraits concept, which requires the provision of ten functions that handle all possible overlapping cases as listed below. Let \( v_r, e_r, \) and \( f_r \) denote input red cells, i.e., a vertex, an edge, and a face, respectively, \( v_b, e_b, \) and \( f_b \) denote input blue cells, and \( v, e, \) and \( f \) denote output cells.

1. A new vertex \( v \) is induced by coinciding vertices \( v_r \) and \( v_b \).
2. A new vertex \( v \) is induced by a vertex \( v_r \) that lies on an edge \( e_b \).
3. An analogous case of a vertex \( v_b \) that lies on an edge \( e_r \).
4. A new vertex \( v \) is induced by a vertex \( v_b \) that is contained in a face \( f_r \).
5. An analogous case of a vertex \( v_b \) contained in a face \( f_b \).
6. A new vertex \( v \) is induced by the intersection of two edges \( e_r \) and \( e_b \).
7. A new edge \( e \) is induced by the (possibly partial) overlap of two edges \( e_r \) and \( e_b \).
8. A new edge \( e \) is induced by an edge \( e_r \) that is contained in a face \( f_b \).
9. An analogous case of an edge \( e_b \) contained in a face \( f_r \).
10. A new face \( f \) is induced by the overlap of two faces \( f_r \) and \( f_b \).

Evidently a custom C++ overlay traits cannot be defined in Python; more precisely, the traits type must be known when the bindings are compiled. We introduce and expose with Python bindings two models of the OverlayTraits concept. The most general between the two, called Arr_overlay_traits, defines ten functions that correspond to the list above, each accepting three arguments, namely, two objects that represent input cells, and one object that represents an output cell. By default these functions do nothing. A user (that is, a Python programmer) can override any subset of these functions. One constructor of this model accepts all the ten functions at once. Another constructor accepts a single function, which corresponds to Function (10) in the list above.

In the following we assume that the face type of the arrangement is extended. The code excerpt in Listing 26 constructs two arrangements with two faces each. For each arrangement the properties of the unbounded face and the bounded face are initialized with Python integer objects \( 0 \) and \( 1 \), respectively. The code excerpt in Listing 27 computes the overlay of the two arrangements constructed by the code in Listing 26 using the overlay traits. The property of each face of the resulting arrangement is updated as part of the overlay computation to indicate the number of overlapping bounded faces; see Figure 7a. Observe that while the computation of the overlay is carried out by the compiled C++ code, the summation is carried out by a Python lambda function, which accepts as the third argument the face to be updated. By default, arguments are copied into new Python objects. We pass the new face by reference, overriding the default, to enforce persistent updates.

![Figure 7](image.png)

Figure 7: (a) The arrangement faces are extended with integers that indicate the number of overlapping bounded faces, shown in brackets. (b) The arrangement vertices are extended with integers that indicate the weighted degree of the vertices.

In many cases, such as in the example above, the property of the new cell depends solely on the properties of the overlapping two cells. To this end we introduce (and expose) a second model of the OverlayTraits concept called Arr_overlay_function_traits. This model also defines ten functions that correspond to the list above. However, here each function accepts the two Python objects that extend the overlapping cells as input arguments and returns the Python object that extends the resulting cell. The code excerpt in Listing 28 has the same effect the code in Listing 27 has, but is more compact.

Each one of the ten functions supported by the Arr_overlay_function_traits model returns a Python object that is passed back to the compiled C++ code. Then, the object is stored with the new cell. Thus, there is no need to pass arguments by reference (even though it could be more efficient in certain cases); however, the implementation of this model presents a new coding challenge—if a certain cell is not extended, attempting to access its non-existing extension would cause a compilation error. Assume that only the vertex type of the Arrangement_2 instance is
Listing 26: Constructing two arrangements with their faces extended and initializing their data.

```python
Aos2 = CGALPY.Aos2
Arrangement_2 = Aos2.Arrangement_2
Point_2 = Arrangement_2.Geometry_traits_2.Point_2
Curve_2 = Arrangement_2.Geometry_traits_2.Curve_2

arr1 = Arrangement_2()
c1 = Curve_2(Point_2(0, 0), Point_2(2, 0))
c2 = Curve_2(Point_2(2, 0), Point_2(2, 2))
c3 = Curve_2(Point_2(2, 2), Point_2(0, 2))
c4 = Curve_2(Point_2(0, 2), Point_2(0, 0))
Aos2.insert(arr1, [c1, c2, c3, c4])

arr2 = Arrangement_2()
c1 = Curve_2(Point_2(1, 1), Point_2(3, 1))
c2 = Curve_2(Point_2(3, 1), Point_2(3, 3))
c3 = Curve_2(Point_2(3, 3), Point_2(1, 3))
c4 = Curve_2(Point_2(1, 3), Point_2(1, 1))
Aos2.insert(arr2, [c1, c2, c3, c4])

# Set the data for the faces. The data can be any python object
for arr in [arr1, arr2]:
    for f in arr.faces():
        f.set_data(0) if f.is_unbounded() else f.set_data(1)
```

Listing 27: Computing the map overlay of two arrangements using the general overlay traits.

```python
traits = Aos2.Arr_overlay_traits(lambda f1, f2, f: f.set_data(f1.data()+f2.data()))
Aos2.overlay(arr1, arr2, result, traits)
```

Listing 28: Computing the map overlay of two arrangements using the functional overlay traits.

```python
traits = Aos2.Arr_overlay_function_traits(lambda x, y: x+y)
os2.overlay(arr1, arr2, result, traits)
```
Listing 29: Constructing two arrangements with their vertices extended and computing their map overlay using the overlay function traits.

```python
Aos2.insert(arr1, [Curve(Circle(Point(3,4),25)), Curve(Circle(Point(-3,-4),25))])
Aos2.insert(arr2, [Curve(Circle(Point(3,4),25)), Curve(Circle(Point(3,4),25))])
for arr in [arr1, arr2]:
    for v in arr.vertices():
        v.set_data(v.degree())
traits = Aos2.Arr_overlay_function_traits()
traits.set_vv_v(lambda x, y: 2*x+y)
traits.set_ve_v(lambda x, y: 2*x+2)
traits.set_vf_v(lambda x, y: 2*x)
traits.set_ev_v(lambda x, y: 4+y)
traits.set_fv_v(lambda x, y: y)
traits.set_ee_v(lambda x, y: 6)
Aos2.overlay(arr1, arr2, result, traits)
```

extended. The Python code excerpt in Listing 29 computes the overlay of two arrangements and updates the property of each vertex of the resulting arrangement to indicate the weighted degree of the vertex, where blue incident edges weigh twice as much as red incident edges; see Figure 7b. We set all the six functions that update output vertices in the Arr_overlay_function_traits object using dedicated setters. The name of a setter matches the pattern set_<c>_v(), where each of a, b, and c can be substituted with v, e, or f. cr and cb determine the input cell type, respectively, and c determines the output cell type.

Consider a generic function called apply() that accepts two objects that represent input cells, namely r and b, one object that represents an output cell, namely o, and a function called f. The objective of apply() is to apply f to the properties of the cells r and b and store the return value as the property of the cell o. Recall that a property of a cell c is obtained and set via the calls c.data() and c.set_data(), respectively. If the type of the output cell is not extended, however, apply() should become idle. Similarly, if the type of an input cell is not extended, the None Python object (represented by py::object) should be passed to f instead. We use SFINAE yet again, twice, to address the above. First, we introduce an overload of apply() that is idle and serves as a fall-through; see Line 9 in Listing 30. When a call to apply() is made, the idle overload is selected by the overload resolution process if the output cell does not have a member called set_data(). If the output cell does have this member the other implementation is selected and calls data(r) and data(b) to obtain the properties of the red and blue input cells, respectively. Second, we introduce an overload of data() that serves as a fall-through (Line 2). It does nothing but return the None Python object. It is selected if the corresponding cell does not have a member called data(). If the cell does have this member the other implementation is selected; it returns the property of the cell.

5.7 Miscellaneous

In this section we list additional techniques used by the binding code.

5.7.1 Handle, iterators, and Circulators

The Handle concept, provided by the Handles and Circulators package [13], describes types that are akin to pointers to objects. A handle provides the dereference operator (operator*()) and the member access operator (->()). Since Python does not have any kind of pointers, both a handle to an object in CGAL and the object itself are converted to the same Python object. For example, both types Arrangement_2::Vertex and Arrangement_2::Vertex_handle are converted to the Python type Vertex, which is an attribute of the Python type Arrangement_2.

The Iterator concept and its refinements describe types that can be used to identify and traverse the elements of a container that contains sequential data. An iterator provides the increment or decrement operators. In CGAL quite often an iterator is convertible to a handle, for example, Arrangement_2::Vertex_iterator is convertible to Arrangement_2::Vertex_handle. Therefore, also Arrangement_2::Vertex_iterator is converted to the python class Vertex. Every converted iterator is supplied with both magic functions __iter__ and __next__, thus made a Python iterator. The __iter__ function simply returns the input object, and the __next__ function

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29For more information about STL iterators, see, e.g., https://en.cppreference.com/w/cpp/iterator.
30For more information about Python iterators, see, e.g., https://docs.python.org/3/c-api/iterator.html.
returns the next object in the traversal order of the corresponding iterator.

Similar to the Iterator concept and its refinements, the Circulator concept and its refinements, also provided by the Handles and Circulators package, describe types that can be used to identify and traverse the elements of a container that contains circular data, for example, the halfedges incident to a face or the halfedges incident to a vertex in an arrangement. A circulator object does not have a past-the-end value. Instead, the range \([a,b)\) of two circulators \(a\) and \(b\) denotes either the empty range, or the sequence of all elements in the container. nanobind supports several features that aid in the wrapping of iterators. Circulators are artificially converted to iterators to leverage on those features. In addition, while traversing a circulator, an stop-iteration exception is thrown when a circular refers to an item that has been traversed already.

5.7.2 Functions Accepting Collections of Elements

The Standard Template Library (STL) provides various type-safe containers for storing collections of related objects. The Container concept describes types of objects that store collections of other objects (their elements).\(^{31}\) Container models are implemented as class templates, which enables great flexibility in the elements types and in the implemented algorithms that operate on container objects. Traditionally, implementations of algorithms in C++ manipulate iterators pointing into the containers they operate on. To date it is popular to pass a collection of elements to a function via two generic input iterators, where the first points at the first element of a container and the second points past the end of the container. An input iterator object, the type of which is a model of the InputIterator concept,\(^{32}\) supports reading from the location obtained via the dereference operator and can be pre- and post incremented. The Boost.Range library provides an alternative modern method, which uses ranges, for applying algorithms on collections of objects. In particular, it utilizes a new family of concepts that refine a basic concept called Range. The Range concept is similar to the Container concept; it requires the provision of iterators for accessing a half-open range of elements and provides information about the number of elements in the range. The new method results in code that is more efficient and more expressive (and thus more comprehensible).\(^{33}\) Soon, (C++20) ranges will become part of the standard.\(^{34}\) The Handles and Circulators package of CGAL provides a variant of a Range concept suitable for CGAL. Many functions in CGAL accept as input collections of elements, and some of them already use this concept. The use of ranges is expected to grow. For every function in CGAL that operates on one or more collections of elements, and regardless of the interface of the function, that is, whether a pair of generic input iterators, or a range, is used to represent each collection, we introduce and expose a wrapper function that accepts a Python list (of type \texttt{py::list}) as an argument;\(^{35}\) the wrapper function calls the original function, properly passing the input collection; see Listing 31 for an example of such a function that uses iterators; the wrapper function is shown in Listing 32; an excerpt code in

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\(^{31}\)For more information about STL containers, see, e.g., https://en.cppreference.com/w/cpp/container.

\(^{32}\)For more information about input iterators, see, e.g., https://en.cppreference.com/w/cpp/iterator/input_iterator.

\(^{33}\)See https://www.boost.org/doc/libs/1_78_0/libs/range/doc/html/index.html for the documentation of Boost.Range.

\(^{34}\)For more information about STL ranges, see, e.g., https://en.cppreference.com/w/cpp/range.

\(^{35}\)For more information about Python lists, see, e.g., https://docs.python.org/3/tutorial/datastructures.html#more-on-lists.
Listing 31: The signature of the free function that inserts a collection of curves into an arrangement.

```cpp
template<typename InputIterator>
CGAL::insert(Arrangement_2& arr, InputIterator first, InputIterator past_the_end)
```

Listing 32: A function that wraps the function shown in Listing 31 and accepts a Python list that contains the input curves.

```cpp
void insert_curves(Arrangement_2& arr, py::list& lst) {
    if (!lst) return;
    if (py::isinstance<X_monotone_curve_2>(lst[0]).check()) {
        auto begin = stl_input_iterator<X_monotone_curve_2>(lst);
        auto end = stl_input_iterator<X_monotone_curve_2>();
        CGAL::insert(arr, begin, end);
    } else if (py::isinstance<Curve_2>(lst[0]).check()) {
        auto begin = stl_input_iterator<Curve_2>(lst);
        auto end = stl_input_iterator<Curve_2>();
        CGAL::insert(arr, begin, end);
    }
}
```

Python code that exploits the above is shown in Listing 33. The function `insert_curves()` (in Listing 32) exploits an iterator adapter called `stl_input_iterator`.

5.7.3 Functions Resulting in Collections of Elements

Functions may compute collections of elements as results. Using output iterators enables an efficient and flexible method for passing output collections from functions. An output iterator object, the type of which is a model of the `OutputIterator` concept,

36 supports writing to the location obtained by the dereference operator and can be pre- and post-incremented. A function that computes a collection of elements accepts an output iterator as an argument and populates the underlying collection. Modern C++ compilers support the efficient transfer of resources from one object of a certain type to another object of the same type, referred to as move semantics. If the underlying collection supports move semantics, a function that computes a collection of elements can simply return the collection, resulting in an elegant yet efficient code. For every function in CGAL that results in a collection of elements, and regardless of the interface of the function, that is, whether an output iterator is used or the collection is returned, we introduce and expose a wrapper function that returns a Python list (of type `py::list`); the wrapper function calls the original function, properly populating a Python list with the output collection, and finally it returns the list.

The free function template `CGAL::decompose()` accepts an arrangement \(A\) and computes a collection of polymorphic elements via an output iterator. For each vertex \(v\) of \(A\) the output collection contains a pair of features—one that directly lies below \(v\) and another that directly lies above \(v\). Let \(v\) be a vertex of \(A\). The feature above (respectively below) \(v\) may be one of the following:

- Another vertex \(u\) having the same \(x\)-coordinate as \(v\).
- An arrangement edge associated with an \(x\)-monotone curve that contains \(v\) in its \(x\)-range.

36 For more information of output iterators, see, e.g., [https://en.cppreference.com/w/cpp/iterator/output_iterator](https://en.cppreference.com/w/cpp/iterator/output_iterator).

37 For more information of move semantics, see, e.g., [https://en.cppreference.com/w/cpp/language/move_constructor](https://en.cppreference.com/w/cpp/language/move_constructor).

Listing 33: A code sample in Python that inserts three curves into an arrangement.

```python
arr = Arrangement_2()
c1 = X_monotone_curve_2(Point_2(0, 0), Point_2(1, 0))
c2 = X_monotone_curve_2(Point_2(1, 0), Point_2(0, 1))
c3 = X_monotone_curve_2(Point_2(0, 1), Point_2(1, 1))
Aos2.insert(arr, [c1, c2, c3])
```
Listing 34: The signature of the free function that decomposes an arrangement into pseudo trapezoids.

```cpp
template <typename OutputIterator>
CGAL::decompose(Arrangement_2& arr, OutputIterator oi);
```

Listing 35: A function that wraps the function shown in Listing 34 and returns a Python list that represents a vertical decomposition.

```cpp
py::list decompose(Arrangement_2& arr) {
    using Decompose_result = std::pair<
        Arrangement_2::Vertex_const_handle,
        std::pair<boost::optional<variant>,
        boost::optional<variant>>>;
    // The argument type of boost::function_output_iterator (UnaryFunction) must be Assignable and Copy Constructible; hence the application of std::ref().
    auto it = boost::make_function_output_iterator(std::ref(op));
    CGAL::decompose(arr, it);
    return lst;
}
```

- An unbounded face in case \( v \) is incident to an unbounded face, and there is no curve lying above (respectively below) it.
- An empty object, in case \( v \) is the lower (respectively upper) endpoint of a vertical edge in the arrangement.

Listing 34 shows the signature of the function. Dereferencing the output iterator must yield an object the type of which is Decompose_result shown in Line 2 of Listing 35.

The wrapper of the CGAL::decompose() template function, shown in Listing 35, returns a list of Python elements; each element is a Python tuple of two items; the first wraps a C++ vertex and the second wraps another Python tuple of two items; the first is the cell above the vertex or py::Object if none exists and the second is the cell below the vertex or py::Object is none exists. Implementing such a wrapper can be done, for example, by introducing an intermediate standard container, say container, and populating it with the elements, of the Decompose_result polymorphic type, computed by the CGAL::decompose() function in C++ (by passing an object of type std::back_inserter<Decompose_result> obtained by the call std::back_inserter(container), as the output iterator argument of the function), and then transforming the container elements into Python elements. Clearly, this is inefficient. Instead, we use the Boost c++ function_output_iterator<> adapter to directly populate a Python list instead of an intermediate container. The adapter (i) utilizes a helper function called decompose_helper() (not listed), which is applied to every resulting element transforming it to the corresponding bounded Python object, and (ii) appends the latter to the output Python list.

The Python code example shown in Listing 36 exploits the above and computes the vertical decomposition shown in Figure 8.

Figure 8: An arrangement of four line segments and its vertical decomposition into pseudo trapezoids, as constructed by the Python code shown in Listing 36. The segments of the arrangement are drawn in solid blue lines and the segments of the vertical decomposition are drawn in dark dotted lines.
Listing 36: A code sample in Python that decomposes an arrangement.

```python
Aos2 = CGALPY.Aos2
Arrangement_2 = Aos2.Arrangement_2
Point_2 = Arrangement_2.Geometry_traits_2.Point_2
Curve_2 = Arrangement_2.Geometry_traits_2.Curve_2

arr = Arrangement_2()
c1 = Curve_2(Point_2(0, 0), Point_2(3, 3))
c2 = Curve_2(Point_2(3, 3), Point_2(6, 0))
c3 = Curve_2(Point_2(2, 0), Point_2(5, 3))
c4 = Curve_2(Point_2(5, 3), Point_2(8, 0))
Aos2.insert(arr, [c1, c2, c3, c4])
d = Aos2.decompose(arr)

for pair in d:
    v0 = pair[0]
    print(v0.point())
    for obj in pair[1]:
        if type(obj) is Arrangement_2.Vertex:
            print("", obj.point())
        elif type(obj) is Arrangement_2.Halfedge:
            print("", obj.curve())
        elif type(obj) is Arrangement_2.Face:
            print(" unbounded face")
        else:
            print(" none")
```

6 Concepts-Binding Coupling

Generic programming enables the implementation of generic algorithms, which work on collections of different types, can be easily maintained, extended, and customized, and are type safe and easier to read. As mentioned in the introduction, CGAL rigorously adheres to the generic programming paradigm. As a consequence, most components of CGAL are either class or function templates. Many of the parameters of these templates are described in terms of models of concepts. When a class or a function template is instantiated, each one of its template parameters is substituted with a model of one or more concepts associated with the template parameter. Close to 750 concepts can be identified in CGAL at the time this article is written. Most hierarchy graphs of concepts are small. Few graphs, such as the graph of concepts of the geometry traits of the 2D Arrangements package, are quite large with intricate refinement relations. We use clusters of closely related concepts to describe the refinement relations among them; see, e.g., Figure 1. We have introduced tight coupling between concepts and (i) binding generations and (ii) type annotation. We describe these relations in the following sections.

6.1 Following Generic Concepts

For each concept we have introduced a template function that generates bindings for all the type and function members required by the concept. We call this function for every model of the concept. This systematic approach guarantees that all the documented functions and types, but nothing else, are exposed. Consider for example the cluster of geometry traits concepts depicted in Figure 1a. The template functions export_AosBasicTraits_2(), export_AosXMonotoneTraits_2(), and export_AosTraits_2() accept binding class objects for particular geometry traits models and populate it with all attributes that correspond to the requirements of the concepts AosBasicTraits_2, AosXMonotoneTraits_2, and AosTraits_2, respectively. If a concept B refines a concept A, then the function export_B() calls the function export_A(). The bindings of different traits models are generated in separate compilation units. As nodes in the concept refinement graphs may have more than a single parent, we ensure that a function that corresponds to a concept is not invoked more than once in a given compilation unit; see Lines 3–4 in the definition of the function export_AosBasicTraits_2() in Listing 37.

Typically, a concept in our case requires the provision of nested types that are themselves models of other concepts. For example, the concept AosBasicTraits_2 requires the provision of the nested type Point_2 used to represent a two-dimensional point. The type Point_2 must be DefaultConstructible and CopyConstructible. Let p be an object of type Point_2 required by the AosBasicTraits_2 concept. Most traits models also support the calls p.x() and p.y(); they return the x and y coordinates of the points p, respectively. The number

---

38A binding class object of type py:class_<T> is used to expose the C++ type T to Python.
Listing 37: Exposing attributes that correspond to the concept AosBasicTraits_2.

```cpp
template <typename GeometryTraits_2, typename ClassObject, typename Concepts>
void export_AosBasicTraits_2(ClassObject c, Concepts& concepts) {
    // Sentinel
    static bool exported = false;
    if (exported) return;

    // Expose attributes that correspond to the traits
    auto& classes = concepts.m_basic_traits_classes;
    classes.m_point_2 = py::class_<Point_2>(m, "Point_2")
        .def(py::init<>())
        .def(py::init<const Point_2&>())
        .def(py::init<const Point_2&>());
}
```

type used to represent the coordinates depends on the traits instance; for example, the type Kernel::FT is used for the Arr_segment_traits<Kernel> traits instance. However, the Point_2 type nested in the traits model Arr_Bezier_curve_traits_2 does not maintain an exact representation of the coordinates. Thus, it instead supports the call p. approximate(), which returns a pair of floating point numbers that only approximate the real coordinates. The function export_AosBasicTraits_2() constructs a class object for the Point_2 type, but adds nothing but Python attributes that expose the default and copy constructors of this type; see Lines 10–11 in the definition of the function export_AosBasicTraits_2() in the listing. It also saves the class object of the Point_2 type (see Line 8), so that later on the functions responsible for the bindings of the various traits models can add additional Python attributes. For example, the function that exposes the Arr_segment_traits<Kernel> instance adds the attributes x and y, which expose the member functions x(), y(), respectively, to the Python object that exposes the type Point_2. Similarly, the function that exposes the Arr_Bezier_curve_traits_2 instance adds the attribute approximate, which exposes the member function approximate() to the Python object that exposes the type Point_2.

The concept AosHorizontalSideTraits_2 requires the provision of the Compare_x_on_boundary_2 functor\textsuperscript{39} that has two implementations of operator() as follows; one overload accepts two ends of open or unbounded curves (refer to the manual for the precise encoding of a curve end) and another overload accepts a point and an end of a curve that is open or unbounded at that end. The former compares the x-coordinates of the curves at their respective limits or ends and the latter compares the x-coordinates of the point and the x-coordinates of the curve at its respective limit or end. The concept AosIdentifiedHorizontalTraits_2 indirectly refines the concept AosHorizontalSideTraits_2 mentioned above. (The former would be used to define a traits model that handles arcs on a sphere with poles on the left and right sides; Figure 1d depicts a cluster of concepts for models that handle arcs on a sphere with poles on the top and bottom sides.) In addition to the two implementations of the overloaded members operator() of the functor Compare_x_on_boundary_2 it requires the provision of a third implementation that accepts two points and compares their x coordinates. Similar to the case described in the previous paragraph, where the class object that handles the bindings for the Point_2 type must be available in two separate functions, also here the class object that handles the bindings for the functor Compare_x_on_boundary_2 must be available in both functions export_AosHorizontalSideTraits_2() and export_AosIdentifiedHorizontalTraits_2(). Saving the class object in one function so that it can be reused latter on in the other addresses this necessity.

6.2 Type Annotation

CGAL is a large library to start with. It is divided into approximately 150 packages with countless function and class templates. The vast number of function and type instances that can be defined using these templates can be overwhelming. The development of non critical code that is based on CGAL can be expedited using Python and CGAL Python bindings. Using code completion (which is based on type annotations) offered by several Python IDEs can accelerate the development process much further. Type annotation in Python refers to annotations of Python functions, arguments, and variables in a way that can be used by various tools. It is an optional feature of Python that has been introduced in Version 3.5 and enhanced in successive versions. With this feature implemented a type checker, a tool separate from the Python interpreter, can be used to statically analyze code base to find bugs during

\textsuperscript{39}A functor is a class that has one or more operator() members; thus, it acts as a function.
Annotating a Python type that wraps a C++ type, which is a model of some concepts is guided by the concepts. Similar to the C++ code that generates the bindings, we have introduced a tight coupling between concepts and Python type annotations; see Section 6.1. As opposed to the C++ binding code, we have introduced a framework for the automatic generation of the annotation stubs. The framework includes a Python script called generate.py that accepts as input static data files in json format that describe the concepts, their refinement relations, the exposed C++ classes, and their modeling relations with concepts. The script generates annotation stubs for the bindings of modules selected by the user. As mentioned in Section 3, CMAKE is used to generate the native build environment. We have augmented the scope of CMAKE, and now CMAKE is also used to generate the annotation stubs. Before we delve into the details of our annotation stub generation system, we describe in the following two paragraphs an unsuccessful attempt, which resembles the C++ binding code. It reached a dead-end because of limitations of Python.

Imagine that for each C++ concept we introduce a Python annotation class that annotates all the requirements of the concept. The annotated Python classes do not exist; these annotation classes are introduced to facilitate the creation of the annotation classes of real Python classes. Every annotation-class that annotates a certain model type, say T, inherits from all the annotation-classes that correspond to the concepts that T models. For example, the type annotation for the concept AosBasicTraits_2 contains the annotations the nested type Point_2, which in turn, contains annotations for the default constructor and copy constructor of the type Point_2 defined by every model of the concept AosBasicTraits_2; see Listing 39 for partial annotations. Observe that the annotated Python class does not exist in reality, thus the _ prefix in the name of the annotation class _AosBasicTraits_2 in the listing. Assume that bindings are generated for the Arrangement_2<GeometryTraits_2, Dcel> type instantiated with a geometry traits type that is an instance of the Arr_segment_traits_2 class template. This type models several concepts; see, e.g., Figure 1a. Partial erroneous Python type annotation for the type Arr_segment_traits_2 is shown in Listing 40.

Unfortunately, the code presented in Listing 39 and Listing 40 does not work. Python and static type checkers for Python (e.g., mypy)\(^40\) have made significant progress since Python was conceived. However, even with the introduction of the Typing module, and in particular the Generic class within,\(^41\) there are still barriers that cannot be bridged. The barrier that we encountered is related to deep nesting. The culprit lies in Line 7 and Line 9 of Listing 39. In Line 7, for example, the class Point_2 nested in the utility class _AosBasicTraits_2 is referred to from a method of the class Equal_2, which is nested in _AosBasicTraits_2. An object of type Arr_segment_traits_2 → .Equal_2 was not recognized (probably because Arr_segment_traits_2.Equal_2 is not substituted by _AosBasicTraits_2.Equal_2 even though Arr_segment_traits_2 is derived from _AosBasicTraits_2.) and our ambitious attempt failed flat.

Assume that two users select the 2D Arrangements module for binding generation. In particular, one user chooses

\(^40\)See, e.g., https://mypy.readthedocs.io/en/stable/.

\(^41\)See https://docs.python.org/3/library/typing.html#module-typing.

Listing 38: Type annotation of the overloaded overlay function.

```python
@overload
def overlay(r: Arrangement_2, b: Arrangement_2, o: Arrangement_2) -> None: pass
@overload
def overlay(r: Arrangement_2, b: Arrangement_2, o: Arrangement_2, t: Arr_overlay_traits) -> None: pass
@overload
def overlay(r: Arrangement_2, b: Arrangement_2, o: Arrangement_2, t: Arr_overlay_function_traits) -> None: pass
```
Listing 39: Erroneous partial type annotation for the attributes that correspond to the concept `AosBasicTraits_2`.

```python
class _AosBasicTraits_2( ):
    class Point_2( ):
        def __init__( ):
        def __init__(const Point_2&):
    class Equal_2( ):
        @overload
        def __call__(self, p: _AosBasicTraits_2.Point_2, q: _AosBasicTraits_2.Point_2) -> bool:
        @overload
        def __call__(self, p: _AosBasicTraits_2.X_monotone_curve_2, q: _AosBasicTraits_2.X_monotone_curve_2) -> bool:
        def equal_2_object(self) -> _AosBasicTraits_2.Equal_2:
```

Listing 40: Erroneous partial Python type annotation for the type `Arr_segment_traits_2`.

```python
class Arr_segment_traits_2(AosBasicTraits_2._AosBasicTraits_2,
    AosXMonotoneTraits_2._AosXMonotoneTraits_2,
    AosTraits_2._AosTraits_2,
    AosLandmarkTraits_2._AosLandmarkTraits_2,
    AosVerticalSideTraits_2._AosVerticalSideTraits_2,
    AosVerticalSideTraits_2._AosHorizontalSideTraits_2):
    class Point_2(AosBasicTraits_2._AosBasicTraits_2.Point_2):
        def x(self) -> Ker.FT:
        def y(self) -> Ker.FT:
```

to generate bindings for the `Arrangement_2<GeometryGaits_2, Dcel>` type instantiated, where the template parameter `GeometryGaits_2` is substituted with an instance of the `Arr_segment_traits_2` class template and the other user chooses to generate bindings for the `Arrangement_2<GeometryGaits_2, Dcel>` type, where `GeometryGaits_2` is substituted with an instance of the `Arr_Bezier_curve_traits_2` class template; see Listing 41 and Listing 42 for the corresponding partial valid Python type annotations generated by our system. The input data files describe the concepts and the classes. More specifically, for each concept `C`, there exists a record that lists the requirements of the concept `C` and the concepts the `C` refines. For each type `T`, there exists a record that lists the concepts that `T` models and the additional members and types nested in `T` that are not covered by the concepts. Type annotations are, naturally, generated only for the types. An important feature of the system is the ability to merge methods of a class that appears both as a nested type of another class and as a type required by a concept. An example of such a scenario is the `Point_2` type required by the concept `Aosbasictraits_2` and nested in every geometry traits class, e.g., `Arr_segment_traits_2` and `Arr_Bezier_curve_2`.

While using a code-generation approach adds a level of complexity to our binding system, it achieves the goal, it is flexible, and it posses an additional advantage. Once the input data files for specific concepts and models are in place, they can be used to automatically generate the documentation for those models and concept; we certainly plan to pursue this goal in the future. The use of the type-annotation subsystem, and in particular the `generate.py` script or the script that generates the documentation that we intend to develop are not limited to CGAL—they can be used by any binding generation system that generates Python bindings for generic C++ code.

7 Applications

The CGAL library, and in particular the CGAL arrangement packages have been used intensively at the Computational Geometry Lab at Tel Aviv University for over twenty years now, not only for research, where the researchers typically (though not always—see Section 7.2) have good knowledge of generic programming in C++, but also for teaching. We have been using CGAL for projects in the Computational Geometry course, for assignments and final projects in the courses “Algorithmic Robotics and Motion Planning” and “Algorithms for 3D printing and other Manufacturing Processes”, and for the implementation of a variety of robot algorithms in guided software workshops. Using CGAL for teaching has persistently been a pain-point in these courses especially due to the intricate utilization of C++ in the implementation of the library. This has recently dramatically changed with the introduction of the Python bindings presented in this paper, as we describe in two examples below.
Listing 41: Valid Python type annotation for the type `Arr_segment_traits_2`.

```python
from typing import Iterator, overload
class Arr_segment_traits_2():
    class Point_2():
        @overload
def __init__(self) -> None: ...  
        @overload
        def __init__(self, p: Point_2) -> None: ...
        @overload
        def __init__(self, x: float, y: float) -> None: ...
        def x(self) -> FT: ...
        def y(self) -> FT: ...

class Equal_2():
    def __call__(self, p: Point_2, Q: Point_2) -> Boolean: ...
    def equal_2_object(self) -> Equal_2: ...
```

Listing 42: Valid Python type annotation for the type `Arr_segment_traits_2`.

```python
from typing import Iterator, overload
class Arr_Bezier_curve_traits_2():
    class Point_2():
        @overload
def __init__(self) -> None: ...  
        @overload
        def __init__(self, p: Point_2) -> None: ...
        def approximate(self) -> list[float]: ...

class Equal_2():
    def __call__(self, p: Point_2, Q: Point_2) -> Boolean: ...
    def equal_2_object(self) -> Equal_2: ...
```
7.1 A Multi Robot Motion Planning Platform

We developed a program called DISCOPYGAL written in Python that provides visualization and verification for multi-robot motion planning in 2D. The program uses Python bindings of CGAL arrangements, CGAL Minkowski sums, and other components of CGAL needed for the implementation of fundamental motion-planning algorithms and enables the dynamic loading of motion-planning algorithms in the form of Python scripts during run time. It allows users to easily swap and compare between different motion-planning algorithms, and even modify an existing implementation of an algorithm while the program is running. (There is no need to recompile the code or restart the program.)

We were invited to give a crash course in multi-robot motion planning in a summer school geared primarily toward engineering doctoral students, where the organizers particularly asked for hands-on algorithm implementation and experimentation session. After two hours of lecture including fifteen minutes of presentation of DISCOPYGAL, students installed it and were able to interact with it. Testimonies of students and organizers affirmed that the system was friendly and easy-to-use, and fruitful in assisting the students to absorb the algorithmic ideas. The students only needed to access and write Python code, while still employing heavy-duty CGAL procedures in the background.

In last year’s Algorithmic Robotics course, the experimental assignments are based on DISCOPYGAL. This was the first time that the introduction of the software into the assignment passed completely smoothly, without compromising the required level of algorithmic sophistication. The system allowed the students to focus on algorithmic issues using Python, a programming language they are all familiar with, freeing them from many of the past points of hardship raised by working directly with CGAL, and in particular with the advanced generic-programming mechanisms that CGAL uses.

A snapshot of the DISCOPYGAL application window is shown in Figure 9. The window is divided into three parts; informative messages are displayed on the console on the left, a rich menu resides to the right of the console; the workspace is drawn on the right. The particular workspace in the figure consists of two disc robots. The objective is to swap their positions. The path of each robot planned by the system is drawn in the color of the robot. For more information visit the page http://acg.cs.tau.ac.il/discopygal.

![Figure 9: A snapshot of the DISCOPYGAL application window.](http://disc.tudelft.nl/education/summer-school/disc-summer-school-2021/)

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42http://disc.tudelft.nl/education/summer-school/disc-summer-school-2021/
7.2 Neural Collision Detection

In collaboration with neuroscientists, we helped developing a simulation system for the interaction between neurons and blood vessels in the brain [21]. The need arose to use three-dimensional alpha shapes. The CGAL offerings in this respect looked a perfect fit, with the caveat that the main tools of the simulation are developed in Python and the most convenient programming language for that part of the project is Python. We developed Python bindings for the CGAL alpha-shape package, and their usage was immediately and almost effortlessly integrated into the overall system by the neuroscientists.

References

[1] D. Abrahams and R. W. Grosse-Kunstleve. Building hybrid systems with boost.python. C/C++ Users Journal, pages 31–39, 2003.

[2] M. H. Austern. Generic Programming and the STL. Addison-Wesley, Boston, MA, USA, 1999.

[3] A. Baram, E. Fogel, D. Halperin, M. Hemmer, and S. Morr. Exact Minkowski sums of polygons with holes. Computational Geometry: Theory and Applications, 1:01–01, 2015.

[4] S. Behmel, R. Bradshaw, C. Citro, L. Dalcin, D. S. Seljebotn, and K. Smith. Cython: The best of both worlds. Computational Sciences Engineering, 13(2):31–39, 2011.

[5] E. Berberich, E. Fogel, D. Halperin, M. Kerber, and O. Setter. Arrangements on parametric surfaces II: Concretizations and applications. Mathematics in Computer Science, 4:67–91, 2010.

[6] E. Berberich, E. Fogel, D. Halperin, K. Mehlhorn, and R. Wein. Arrangements on parametric surfaces I: General framework and infrastructure. Mathematics in Computer Science, 4:45–66, 2010.

[7] J.-D. Boissonnat and M. Yvinec. Triangulations. In Algorithmic Geometry. Cambridge University Press, 1998. Translated by Hervé Brönnimann.

[8] H. Brönnimann, A. Fabri, G.-J. Giezeman, S. Hert, M. Hoffmann, L. Kettner, S. Pion, and S. Schirra. 2D and 3D linear geometry kernel. In CGAL User and Reference Manual. CGAL Editorial Board, 5.5 edition, 2022.

[9] M. Caroli, A. Pellé, M. Rouxel-Labbé, and M. Teillaud. 3D periodic triangulations. In CGAL User and Reference Manual. CGAL Editorial Board, 5.5 edition, 2022.

[10] M. Caroli and M. Teillaud. Computing 3d periodic triangulations. In Proceedings of the 17th Annual European Symposium on Algorithms (ESA), volume 5757 of LNCS, pages 37–48. Springer-Verlag, 2009.

[11] T. K. F. Da. 2D alpha shapes. In CGAL User and Reference Manual. CGAL Editorial Board, 5.5 edition, 2022.

[12] T. K. F. Da, S. Loriot, and M. Yvinec. 3D alpha shapes. In CGAL User and Reference Manual. CGAL Editorial Board, 5.5 edition, 2022.

[13] O. Devillers, L. Kettner, S. Pion, M. Seel, and M. Yvinec. Handles and circulators. In CGAL User and Reference Manual. CGAL Editorial Board, 5.5 edition, 2022.

[14] E. Flato, D. Halperin, I. Hanniel, O. Nechushtan, and E. Ezra. The design and implementation of planar maps in CGAL. ACM J. Exp. Algorithmics, 5:13, 2000.

[15] E. Fogel, D. Halperin, and R. Wein. CGAL Arrangements and Their Applications, A Step by Step Guide. Springer-Verlag, Berlin Heidelberg, Germany, 2012.

[16] E. Fogel, O. Setter, R. Wein, G. Zucker, B. Zukerman, and D. Halperin. 2D regularized boolean set-operations. In CGAL User and Reference Manual. CGAL Editorial Board, 5.5 edition, 2022.

[17] GEOS contributors. GEOS coordinate transformation software library. Open Source Geospatial Foundation, 2021.

[18] D. Halperin, J.-C. Latombe, and R. H. Wilson. A general framework for assembly planning: The motion space approach. Algorithmica, 26:577–601, 2000.

[19] D. Halperin and M. Sharir. Arrangements and their applications in robotics: Recent developments. In K. Goldberg, D. Halperin, J.-C. Latombe, and R. Wilson, editors, International Workshop on Algorithmic Foundations of Robotics, pages 495–511. A.K. Peters, Ltd., Boston, MA, 1995.
A Extenders

In several places in our binding code we are given a type that models certain concepts and based on several conditions, this type must be replaced with a new type that models additional concepts. Instead of using old style code in the form of \#if, \#elif, \#else, \#endif directives, we exploit generic programming and implement these extenders using modern code that is type safe. In the following we present four such scenarios.
// Traits extender
template <bool b, typename Base> struct Tr {};

// Specialization for the case where bindings for Bso2 are not generated
template <typename Base>
struct Tr<false, Base> { typedef Base type; };

// Specialization for the general case where bindings for Bso2 are generated
template <typename Base>
struct Tr<true, Base> { typedef CGAL::Gps_traits_2<Base> type; };

// Specialization for the segment traits
template <>
struct Tr<true, CGAL::Arr_segment_traits_2<Kernel>> {
  typedef CGAL::Gps_segment_traits_2<Kernel, Point_2_container> type;
};

// Specialization for the circle segment traits
template <>
struct Tr<true, CGAL::Arr_circle_segment_traits_2<Kernel>> {
  typedef CGAL::Gps_circle_segment_traits_2<Kernel> type;
};

typedef Tr<boolean_set_operations_2_bindings(), BGT>::type Geometry_traits;

A.1 2D Arrangement Traits Extender

When the user enables the generation of bindings for the 2D Arrangements package, the binding for an instance of the class template Arrangement_2<Traits, Dcel> is generated. When this class template is instantiated the Traits template parameter must be substituted with a specific traits model. By default an instance of the Arr_segment_traits_2<Kernel> is used; however, the user can override this selection and choose any one of the supported geometry traits. We refer to this traits as the basic geometry traits. If the user also enables the generation of bindings for the 2D Regularized Boolean Set-Operations package, the basic traits must be extended, because it must also model the GeneralPolygonSetTraits_2 concept (see Figure 2b); see Section 4.3. The extension however differs for different basic traits. It is automatically done via a dedicated class template and four specializations shown in Listing 43. The final traits is obtained using this extender as shown in Listing 44. Here, boolean_set_operations_2_bindings() is a constant binary function (evaluated at compile time) that determines whether the generation of bindings for the 2D Regularized Boolean Set-Operations package is enabled, and BGT is the basic geometry traits.

A.2 2D Arrangement DCEL Cell Extenders

When the user enables the generation of bindings for the 2D Arrangements package, the binding for an instance of the class template Arrangement_2<Traits, Dcel> is generated. When this class template is instantiated the Dcel template-parameter should be substituted with a type that models the ArrangementDcel concept. We substitute this parameter with an instance of the template CGAL::Arr_dcel_base<V, H, F>, where V, H, and F must model the concepts ArrangementDcelVertex, ArrangementDcelHalfedge, and ArrangementDcelFace, respectively. By default, we substitute these parameters with the instances Arr_vertex_base<Traits::Point_2>, Arr_halfedge_base<Traits::X_monotone_curve_2>, and Arr_face_base, respectively, referred to as the basic types of cells. If the user chooses to extend the vertex type, then we substitute V with the instance Arr_extended_vertex<Vb, py::object>, where Vb is the basic vertex type above. This conditional extension is carried out via a dedicated class template and two specializations shown in Listing 45. The final vertex type is
Listing 45: 2D Arrangement DCEL user extended vertex type extender.

```cpp
// Vertex extender
template <bool b, typename Vb> struct Vertex_extender {};

// Specialization
template <typename Vb> struct Vertex_extender<false, Vb> { typedef Vb type; };

// Specialization
template <typename Vb> struct Vertex_extender<true> { typedef CGAL::Arr_extended_vertex<Vb, py::object> type; };
```

Listing 46: The final 2D Arrangement vertex type definition.

```cpp
CGAL::Arr_vertex_base<Geometry_traits::Point_2> Vb;
typedef Vertex_extended<is_vertex_extended(), Vb, py::object>::type V;
```

obtained using this extender as shown in Listing 46. Here, is_vertex_extended() is a constant binary function that determines whether the user extension of vertices is enabled.

If the user also enables the generation of bindings for the 2D Regularized Boolean Set-Operations package, the template parameters H and F must be substituted with types that also model the GeneralPolygonSetDcelHalfedge and GeneralPolygonSetDcelFace concepts, respectively; see Section 4.4. In addition, similar to the case of vertex extension, if the user chooses to extend the halfedge or face types, then we substitute V with the instances Arr_extended_halfedge<Hb, py::object> or Arr_extended_face<Fb, py::object>, respectively, where Hb and Fb are the types above. The conditional extensions of the halfedge type are carried out via dedicated class templates and corresponding specializations shown in Listing 47 and Listing 48. The final halfedge type is obtained using these extenders as shown in Listing 49. Here, boolean_set_operations_2_bindings() is a constant binary function that determines whether the generation of bindings for the 2D Regularized Boolean Set-Operations package is enabled, and is_halfedge_extended() is a constant binary function that determines whether the user extension of halfedges is enabled.

Similarly, the conditional extensions of the face type are carried out via dedicated class templates and corresponding specializations shown in Listing 50 and Listing 51. The final face type is obtained using these extenders as shown in Listing 52.

A.3 2D Triangulation Cell Extenders

When the user enables the generation of bindings for the 2D Triangulations package, the binding for an instance of one of the 2D triangulation class templates is generated. Each of these class templates has a template parameter that must be substituted with a model of the concept TriangulationDataStructure_2 when the triangulation class template is instantiated. We substitute this parameter with an instance of the class template Triangulation_data_structure_2<V, F>.

```
Listing 47: 2D Arrangement DCEL Halfedge extender for 2D Regularized Boolean set operations.
```

```cpp
// Basic halfedge extender
template <bool b> struct Halfedge_gps {};

// Specialization
template <> struct Halfedge_gps<false> { typedef CGAL::Arr_halfedge_base<Geometry_traits::X_monotone_curve_2> type; };

// Specialization
template <> struct Halfedge_gps<true> { typedef CGAL::Gps_halfedge_base<Geometry_traits::X_monotone_curve_2> type; };
```
Listing 48: 2D Arrangement DCEL user extended halfedge type extender.

```cpp
// Halfedge extender
template <bool b, typename Hb> struct Halfedge_extender {}
;

// Specialization
template <typename Hb> struct Halfedge_extender<false, Hb> { typedef Hb type; }
;
// Specialization
template <typename Hb> struct Halfedge_extender<true, Hb>
{ typedef CGAL::Arr_extended_halfedge<Hb, py::object> type; }
```

Listing 49: The final 2D Arrangement DCEL halfedge type definition.

```cpp
typedef Halfedge_gps<boolean_set_operations_2_bindings () >::type Hb;
typedef Halfedge_extended<is_halfedge_extended (), Hb, py::object >::type H;
```

Listing 50: 2D Arrangement DCEL Face extender for 2D Regularized Boolean set operations.

```cpp
// Basic face extender
template <bool b> struct Face_gps {}
;

// Specialization
template <> struct Face_gps<false> { typedef CGAL::Arr_face_base type; }
;
// Specialization
template <> struct Face_gps<true> { typedef CGAL::Gps_face_base type; }
```

Listing 51: 2D Arrangement DCEL user extended face type extender.

```cpp
// Face extender
template <bool b, typename Fb> struct Face_extender {}
;

// Specialization
template <typename Fb> struct Face_extender<false, Fb> { typedef Fb type; }
;
// Specialization
template <typename Fb> struct Face_extender<true, Fb>
{ typedef CGAL::Arr_extended_face<Fb, py::object> type; }
```

Listing 52: The final 2D Arrangement DCEL face type definition.

```cpp
typedef Face_gps<boolean_set_operations_2_bindings () >::type Fb;
typedef Face_extended<is_face_extended (), Fb, py::object >::type F;
```
Listing 53: The 2D Triangulation base vertex type extender.

```cpp
// Vertex base selection
template <bool b, int i> struct Vertex_base_name {}
;

// Specialization
template <int i> struct Vertex_base_name<false, i>
{ typedef CGAL::Triangulation_vertex_base_2<Traits> type; };

// Specialization
template <> struct Vertex_base_name<false, CGALPY_TRI2_REGULAR>
{ typedef CGAL::Regular_triangulation_vertex_base_2<Traits> type; };

// Specialization
template <int i> struct Vertex_base_name<true, i>
{ typedef CGAL::Periodic_2_triangulation_vertex_base_2<Traits> type; };
```

Listing 54: The 2D Triangulation user extended vertex type extender.

```cpp
// Vertex with info
template <bool b, typename Vb> struct Vertex_with_info {}
;

// Specialization
template <typename Vb> struct Vertex_with_info<false, Vb> {
  typedef Vb type;
};

// Specialization
template <typename Vb> struct Vertex_with_info<true, Vb>
{ typedef CGAL::Triangulation_vertex_base_with_info_2<py::object, Traits, Vb> type; };
```

The type that substitutes the V template parameter when the class template `Triangulation_data_structure_2<V, F>` is instantiated must model the `TriangulationDSVertexBase_2` concepts. If the binding is generated for a periodic triangulation, this parameter must be substituted with a type that also models the concept `Periodic_2TriangulationDSVertexBase_2`. The extension is automatically done via a dedicated class template and three specializations shown in Listing 53. Here `Traits` is the geometry traits type; see Section 4.6 for an explanation on how this type is determined.

If the user chooses to extend the vertex type, the type that substitutes the V template parameter is further extended. This is automatically done via yet another dedicated class template and two specializations shown in Listing 54.

If the triangulation is used to define an alpha shape type, the template parameter `V` must be substituted with a type that also models the concept `AlphaShapeVertex_2`; see Section 4.7. This is automatically done via yet another class template shown in Listing 55.

Finally, if the binding is generated for a hierarchy triangulation, e.g., an instance of the template parameter `Triangulation_hierarchy_2<Triangulation_2>`, the V template parameter must be substituted with a type that also models the concept `TriangulationHierarchyVertexBase_2`. (Observe that there is no special requirements on the type that substitutes the F parameter in this case.) This is automatically done via yet another class template shown in Listing 56.

The type `Vb` is defined as the vertex base type. If bindings are generated for a periodic triangulation, the type `Vb` is defined as `Periodic_2_triangulation_vertex_base_2<Traits>;` otherwise, if bindings are generated for an instance of `Regular_triangulation_2<Traits, Tds>`, the type `Vb` is defined as `Regular_triangulation_vertex_base_2<Traits>;` otherwise, the type `Vb` is defined as `Triangulation_vertex_base_2<Traits>;`:

If the user enables vertex-type extension, the type `Vbi` is defined as `Triangulation_vertex_base_with_info_2<py::object, Traits, Vb>;` otherwise the type `Vbi` is defined as `Vb`. If the triangulation is used to define an alpha shape type, the type `Vbi` is defined as `Alpha_shape_vertex_base_2<Traits, Vb, ExactComparison>;` where `ExactComparison` is substituted with either `true` or `false` based on the setting of the CMAKE flag `CGALPY_AS2_EXACT_COMPARISON`; see Table 15. Otherwise, `V` is defined as `Vbi`. If the binding is
Listing 55: The 2D Triangulation for alpha shapes vertex type extender.

```cpp
// Vertex alpha shape
template <bool b, typename Vb, typename ExactComparison>
struct Vertex_alpha_shape {};

// Specialization
template <typename Vb, typename ExactComparison>
struct Vertex_alpha_shape<false, Vb, ExactComparison> {
    typedef Vb type;
};

// Specialization
template <typename Vb, typename ExactComparison>
struct Vertex_alpha_shape<true, Vb, ExactComparison>
{
    typedef CGAL::Alpha_shape_vertex_base_2<Traits, Vb, ExactComparison> type;
};
```

Listing 56: The 2D Triangulation hierarchy vertex type extender.

```cpp
// Vertex triangulation hierarchy
template <bool b, typename Vb> struct Vertex_hierarchy {};

// Specialization
template <typename Vb> struct Vertex_hierarchy<false, Vb> {
    typedef Vb type;
};

// Specialization
template <typename Vb> struct Vertex_hierarchy<true, Vb>
{
    typedef CGAL::Triangulation_hierarchy_vertex_base_2<Vb> type;
};
```

generated for a hierarchy triangulation, the final vertex type V is defined as Triangulation_hierarchy_vertex_base_2<>; otherwise the final vertex type V is defined as Vbia; see Listing 57. Here, is_periodic(), vertex_with_info(), alpha_shape_2Bindings(), and hierarchy() are constant binary functions. is_periodic() that determines whether the triangulation, binding for which is generated, is periodic. vertex_with_info() determines whether the user extension of vertices is enabled. alpha_shape_2Bindings() indicates whether bindings for the 2D Alpha Shapes package. hierarchy() indicates whether the binding is generated for a hierarchy triangulation.

The final face type is determined in a similar fashion. The only difference between the selections of the vertex and face types is that the setting of the CMAKE flag CGALPY_TRI2_HIERARCHY does not have an effect on the face type selection.

A.4 3D Triangulation Cell Extenders

When the user enables the generation of bindings for the 3D Triangulations package, the binding for an instance of one of the 3D triangulation class templates is generated; see Section 4.8. The scenario here is similar to the scenario of the 2D Triangulations package described in Section A.3. We skip the description of the extenders and jump to the description of their application.

If bindings are generated for a periodic triangulation, the type Vbp is defined as Periodic_3_triangulation_ds_vertex_base_3<>; otherwise, Vbp is defined as Triangulation_ds_vertex_base_3<>. If bindings are generated for an instance of either Regular_triangulation_vertex_base_3<Kernel> or Periodic_3_

Listing 57: The final 2D Triangulation vertex type definition.

```cpp
typedef Vertex_base_name<is_periodic(), CGALPY_TRI2>::type Vb;
typedef Vertex_with_info<vertex_with_info(), Vb>::type Vbi;
typedef Vertex_alpha_shape<alpha_shape_2_bindings(), Vbi, Ec>::type Vbia;
typedef Vertex_hierarchy<hierarchy(), Vbia>::type V;
```
typedef Vertex_periodic<is_periodic () >::type Vb;
typedef Vertex_regular<is_regular () , Vb>::type Vbr;
typedef Vertex_with_info<vertex_with_info () , Vbr>::type Vbri;
typedef Vertex_alpha_shape<alpha_shape_3_bindings() , Vbri , Ec>::type Vbria;
typedef Vertex_hierarchy<hierarchy () , Vbria >::type V;

→ Regular_triangulation_vertex_base_3<Kernel>, the type Vb is defined as Regular_triangulation_vertex_base_3<Traits , Vbp>; otherwise, the type Vb is defined as Triangulation_vertex_base_3<Traits , Vbp>. If the vertex type is extended, the type Vbi is defined as Triangulation_vertex_base_with_info_3<py::object , Traits , Vb>; otherwise, the type Vbi is defined as Vb. If the binding is generated for a hierarchy triangulation, the type Vbih is defined as Triangulation_hierarchy_vertex_base_3<Vbi>; otherwise the type Vbih is defined as Vbi. If the triangulation is used to define an alpha shape type, the final type V (that substitutes the V parameter) is defined as either Alpha_shape_vertex_base_3<Traits , Vbih , ExactComparison> or Fixed_Alpha_shape_vertex_base_3<Traits , Vbih , ExactComparison> based on the selection of the alpha shape class (either fixed or plain); see Table 20; ExactComparison is substituted with either true or false based on the user selection of exact comparisons; see Table 20. Otherwise, V is defined as Vbih; see Listing 58. (Observe that while 3D periodic regular triangulations are supported in the form of the class template Periodic_3_regular_triangulation_3, corresponding 2D triangulations are not; thus the difference between the structures of the code in Listing 57 and Listing 58.

The final cell type is determined in a similar fashion. The only difference between the selections of the vertex and cell types is that the setting of the CMAKE flag CGALPY_TR3_HIERARCHY does not have an effect on the cell type selection.