A semi-canonical reduction for periods of Kontsevich-Zagier

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- What is a period?
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- Open problems and conjectures

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Part I

Introduction
What is a "period"?

- "Most of the important constants in mathematics, coming from algebraic geometry".
- Let $X$ be a smooth variety and $Y$ an closed subvariety of $X$, both defined over $\mathbb{Q}$:

\[ H^\bullet_{\text{sing}}(X; \mathbb{C}), Y; \mathbb{C}) \oplus H^\bullet_{\text{dR}}(X, Y; \mathbb{Q}) \]

Integration via Poincaré duality defines a pairing:

\[ H^\bullet_{\text{sing}}(X; \mathbb{C}), Y; \mathbb{C}) \times H^\bullet_{\text{dR}}(X, Y; \mathbb{Q}) \to \mathbb{C} \]

Tensorizing by $\mathbb{C}$, the previous pairing gives the comparison isomorphism:

\[ H^\bullet_{\text{dR}}(X, Y; \mathbb{Q}) \otimes \mathbb{C} \cong H^\bullet_{\text{B}}(X, Y; \mathbb{Q}) \otimes \mathbb{C} \]

represented taking $\mathbb{Q}$-basis by the period matrix $\Pi = (\int \gamma_i \omega^j)_{i,j=1,...,s}$. 
What is a "period"?

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- Let $X$ be a smooth variety and $Y$ an closed subvariety of $X$, both defined over $\mathbb{Q}$:
  - Betti cohomology: $H^\bullet_B(X, Y; \mathbb{Q}) = \left( H^\bullet_{\text{sing}}(X(\mathbb{C}), Y(\mathbb{C}); \mathbb{Q}) \right)^\vee$
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$$(\gamma, \omega) \mapsto \int_\gamma \omega$$
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- Tensorizing by $\mathbb{C}$, the previous pairing gives the *comparison isomorphism*

$$\text{comp}_{B,dR} : H^\bullet_{\text{dR}}(X, Y; \mathbb{Q}) \otimes \mathbb{C} \sim \rightarrow H^\bullet_B(X, Y; \mathbb{Q}) \otimes \mathbb{C}$$

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**QUESTION:** Could the comparison isomorphism be induced by an isomorphism $H_{dR}^\bullet(X, Y; \mathbb{Q}) \xrightarrow{\sim} H_B^\bullet(X, Y; \mathbb{Q})$?

- No! If $X = \mathbb{A}^1_\mathbb{Q} \setminus \{0\} = \text{Spec} \mathbb{Q}[t, t^{-1}], Y = \emptyset$ and $\gamma = S^1 \subset \mathbb{C}^*$:

  $$H^\bullet_B(\mathbb{C}^*; \mathbb{Q}) = \mathbb{Q} \gamma^*, \quad H^\bullet_{dR}(X; \mathbb{Q}) = \mathbb{Q} \frac{dt}{t}$$

  but $\int_\gamma \frac{dt}{t} = 2\pi i \notin \mathbb{Q}$. 


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“Transcendental” obstrucción, invariant of the pair $(X, Y)$!
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H_B^\bullet(\mathbb{C}^*; \mathbb{Q}) = \mathbb{Q}\gamma^*, \quad H_{dR}^\bullet(X; \mathbb{Q}) = \mathbb{Q}\frac{dt}{t}
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but \( \int_\gamma \frac{dt}{t} = 2\pi i \not\in \mathbb{Q}. \)

\[ \exists \]

“Transcendental” obstrucción, invariant of the pair \((X, Y)\)!
Let $\mathbb{R}_{\text{alg}}$ be the field of algebraic numbers.

A set $S \subset \mathbb{R}^d$ is called $\mathbb{R}_{\text{alg}}$–semi-algebraic if it can be described as finite unions of sets $\{f_1 \star_1 0, \ldots, f_s \star_s 0\}$, where $f_i \in \mathbb{R}_{\text{alg}}[x_1, \ldots, x_d]$ and $\star_i \in \{=, >\}$ for $i = 1, \ldots, s$. 
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**Definition**

A *period of Kontsevich-Zagier* (or *effective period*) is a complex number whose real and imaginary parts are values of absolutely convergent integrals of the form

$$\mathcal{I}(S, P/Q) = \int_S \frac{P(x_1, \ldots, x_d)}{Q(x_1, \ldots, x_d)} \cdot dx_1 \wedge \ldots \wedge dx_d$$

where $S \subset \mathbb{R}^d$ is a $d$–dimensional $\mathbb{R}_{\text{alg}}$–semi-algebraic set and $P/Q \in \mathbb{R}_{\text{alg}}(x_1, \ldots, x_d)$.
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Denote by $\mathcal{P}_{\text{KZ}}$ the set of periods of Kontsevich-Zagier and $\mathcal{P}_{\text{KZ}}^{\mathbb{R}} = \mathcal{P}_{\text{KZ}} \cap \mathbb{R}$. 
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Denote by $\mathcal{P}_{kz}$ the set of periods of Kontsevich-Zagier and $\mathcal{P}^{\mathbb{R}}_{kz} = \mathcal{P}_{kz} \cap \mathbb{R}$. 
Examples of numbers in $\mathcal{P}_{KZ}$

1. **Algebraic numbers:** $\alpha = \int_0^\alpha dx$, $\forall \alpha \in \mathbb{R}_{\text{alg}}$.

2. As a first transcendental number

\[
\pi = \int_{\{x^2+y^2 \leq 1\}} 1 \, dx \, dy = \int_{-\infty}^{\infty} \frac{1}{1 + x^2} \, dx = \int_{\{(1-x^2)y^2 < 1\}} \frac{dx \, dy}{2}
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3. **Logarithms of algebraic numbers**: if $\alpha \in \mathbb{R}_{alg}$ such that $\alpha > 1$,

   $$\log(\alpha) = \int_1^\alpha \frac{dt}{t} = \int_{0 < xy < 1} 1 \, dx \, dy$$
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4. **Multi-zeta values, Elliptic integrals, $\Gamma(p/q)^q$, Feynmann integrals,**...
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Extended inclusion diagram for fields:

\[
\mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R}_{\text{alg}} \subset \overline{\mathbb{Q}} \\
\cap \cap \cap \cap \cap \\
\mathcal{P}_K \subset \mathcal{P}_K \\
\cap \cap \cap \cap \cap \\
\mathbb{R} \subset \mathbb{C}
\]

But, how many transcendental numbers contains $\mathcal{P}_K$?
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But, how many transcendental numbers contains \( \mathbb{P}_{\text{KZ}} \)?

**Theorem**

\( \mathbb{P}_{\text{KZ}} \) forms a countable \( \overline{\mathbb{Q}} \)-algebra.
Extended inclusion diagram for fields:

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\begin{align*}
\mathbb{Z} & \subset \mathbb{Q} \subset \mathbb{R}_{\text{alg}} \subset \overline{\mathbb{Q}} \\
\mathbb{R} \subset \mathbb{C} & \quad \mathcal{P}_{\mathbb{R}} \subset \mathcal{P}_{KZ} \subset \mathcal{P}_{KZ} \\
& \quad \mathbb{R} \subset \mathbb{C}
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But, how many transcendental numbers contains \( \mathcal{P}_{KZ} \)?

**Theorem**

\( \mathcal{P}_{KZ} \) *forms a countable* \( \overline{\mathbb{Q}} \)-*algebra.*

\[ \uparrow \]

Not “a lot”!
Extended inclusion diagram for fields:

\[ \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R}_{\text{alg}} \subset \overline{\mathbb{Q}} \]

\[ \bigcap \mathcal{P}^\mathbb{R}_{\text{kz}} \subset \mathcal{P}_{\text{kz}} \bigcap \]

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Not “a lot”!

**Kontsevich-Zagier:** Conjecturally, \( e, 1/\pi \) or Liouville numbers are not periods.
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Open problems and conjectures

From the foundational paper:

Maxim Kontsevich and Don Zagier. *Periods*, 2001.

Conjecture (Konsevich-Zagier periods conjecture)

If a real period admits two integral representations, then we can pass from one formulation to the other using only three operations (called the KZ–rules):

- integral additions by domains or integrands.
- change of variables.
- Stokes formula.

Moreover, these operations should respect the class of the objects previously defined.
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Conjecture (Equality algorithm)

Determination of an algorithm which allows us to prove if two periods are equal or not.
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**Conjecture (Konsevich-Zagier periods conjecture)**

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**Conjecture (Equality algorithm)**

* Determination of an algorithm which allows us to prove if two periods are equal of not.*
Part II

A semi-canonical reduction for periods

"Periods of Kontsevich-Zagier I: A semi-canonical reduction.", arXiv:1509.01097, 26 pags., (Preprint)
Resolution of poles and compact domains

Main ideas:

- codify all the complexity of a period on the semi-algebraic domain.
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- codify all the complexity of a period on the semi-algebraic domain.
- choose a “good” class of semi-algebraic domains in $\mathbb{R}^d$:
  - Topological properties,
  - Semi-algebraic complexity,
  - ...
Main ideas:

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- obtain this new form from an integral representation of a period in an algorithmic way and only using the three KZ–rules.
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Our principal result:

**Theorem (Semi-canonical reduction)**

Let \( p \in \mathcal{P}_{KZ} \) be non-zero given in an integral form \( \mathcal{I}(S, P/Q) \) in \( \mathbb{R}^d \). Then there exists an effective algorithm satisfying the KZ–rules such that \( \mathcal{I}(S, P/Q) \) can be written as

\[
p = \text{sgn}(p) \cdot \text{vol}_{d+1}(K),
\]

where \( K \subset \mathbb{R}^{d+1} \) is a top-dimensional compact semi-algebraic set.
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**Strategy**: (birational) change of variables + (linear) semi-algebraic partitions!
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2. (Algorithmic) resolution of poles over the boundary: holding local compacity of domains!
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**Theorem (Semi-canonical reduction)**

Let $p \in \mathcal{P}_{KZ}$ be non-zero given in an integral form $I(S, P/Q)$ in $\mathbb{R}^d$. Then there exists an effective algorithm satisfying the KZ–rules such that $I(S, P/Q)$ can be written as

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where $K \subset \mathbb{R}^{d+1}$ is a top-dimensional compact semi-algebraic set.

**Strategy:** (birational) change of variables $+$ (linear) semi-algebraic partitions!

1. Compactification of domains.
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3. We obtain $p = \text{vol}_{d+1}(K_1) - \text{vol}_{d+1}(K_2) \leadsto$ Riemann sums to construct $K$. 
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**Theorem (Semi-canonical reduction)**

Let \( p \in \mathcal{P}_{KZ} \) be non-zero given in an integral form \( \mathcal{I}(S, P/Q) \) in \( \mathbb{R}^d \). Then there exists an effective algorithm satisfying the KZ–rules such that \( \mathcal{I}(S, P/Q) \) can be written as

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**Strategy:** (birational) change of variables + (linear) semi-algebraic partitions!

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3. We obtain \( p = \text{vol}_{d+1}(K_1) - \text{vol}_{d+1}(K_2) \Rightarrow \) Riemann sums to construct \( K \).
We define the *projective closure* of a semi-algebraic set $S \subset \mathbb{R}^d$ by the topological closure of the inclusion of $S \hookrightarrow \mathbb{P}^d_{\mathbb{R}}$.

**Theorem**

$\mathbb{P}^d_{\mathbb{R}}$ can be constructed as the gluing of $C_1, \ldots, C_{d+1}$ affine unit hypercubes through their opposite faces, and such that the Zariski closure of

$$\bigcup_{i,j=0}^{d} (C_i \cap C_j)$$

is the hyperplane arrangement

$$\mathcal{A} = \{x_i^2 - x_j^2 = 0 \mid 0 \leq i < j \leq d\} \subset \mathbb{P}^d_{\mathbb{R}}$$

$\leadsto D = D_1 \sqcup \ldots \sqcup D_{d+1}$ affine compact up to $(d-1)$–dim semi-algebraic sets.
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$$\mathcal{A} = \{x_i^2 - x_j^2 = 0 \mid 0 \leq i < j \leq d\} \subset \mathbb{P}^d$$

$\leadsto D = D_1 \sqcup \ldots \sqcup D_{d+1}$ affine compact up to $(d - 1)$–dim semi-algebraic sets.
We can assume that we are dealing with integrals $\mathcal{I}(S, P/Q)$ with compact domains.

Let $W_0$ be a smooth real algebraic variety defined over $\mathbb{R}_{\text{alg}}$. Let $S \subset W_0$ be a compact semi-algebraic set in $W_0$ and $\omega$ a top differential rational form in $W_0$. Denote by $\partial_z S$ the Zariski closure of $\partial S$ and by $Z(\omega)$ and $P(\omega)$ the real zero and pole locus of $\omega$, respectively.
Resolution of poles

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We use embedded resolution of singularities to send the poles ”far away“ from $\partial S$.

**Proposition (Geometric criterion for convergence)**

The integral $\int_S \omega$ converges absolutely if and only if there exist a finite sequence of blow-ups $\pi = \pi_r \circ \cdots \circ \pi_1 : W_r \to W_0$ over smooth centers such that $\tilde{S} \cap P(\pi^* \omega) = \emptyset$, where $\tilde{S}$ the strict transform of $S$. 
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We can assume that we are dealing with integrals \( \mathcal{I}(S, P/Q) \) with compact domains.

Let \( W_0 \) be a smooth real algebraic variety defined over \( \mathbb{R}_{\text{alg}} \). Let \( S \subset W_0 \) be a compact semi-algebraic set in \( W_0 \) and \( \omega \) a top differential rational form in \( W_0 \). Denote by \( \partial_z S \) the Zariski closure of \( \partial S \) and by \( Z(\omega) \) and \( P(\omega) \) the real zero and pole locus of \( \omega \), respectively.

We use embedded resolution of singularities to send the poles "far away" from \( \partial S \).

**Proposition (Geometric criterion for convergence)**

The integral \( \int_S \omega \) converges absolutely if and only if there exist a finite sequence of blow-ups \( \pi = \pi_r \circ \cdots \circ \pi_1 : W_r \to W_0 \) over smooth centers such that \( \tilde{S} \cap P(\pi^*\omega) = \emptyset \), where \( \tilde{S} \) the strict transform of \( S \).

\( \leadsto \) it suffices to consider the embedded resolution of singularities of \( X = \partial_z S \cup Z(\omega) \cup P(\omega) \).
Resolution of poles

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$\Rightarrow$ it suffices to consider the embedded resolution of singularities of $X = \partial_z S \cup Z(\omega) \cup P(\omega)$. 
Hironaka’s desingularization is effective algorithmic for fields of char. 0 (Villamayor, 89), implemented in Maple and Singular (Bodnár and Schicho, 2000).

A proper birational map $\pi : W \to \mathbb{R}^d$ where $W$ is a closed $d$–dimensional $\mathbb{R}_{\text{alg}}$–subvariety of $\mathbb{R}^d \times \mathbb{P}_\mathbb{R}^m$. 
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**Corollary**

Any real period $p = \mathcal{I}(S, P/Q)$ can be expressed as

$$p = \text{vol}_d(K_1) - \text{vol}_d(K_2),$$

where $K_1, K_2$ are compact $(d + 1)$–dimensional $\mathbb{R}_{\text{alg}}$–semi-algebraic sets, obtained algorithmically and respecting the KZ–rules from $\mathcal{I}(S, P/Q)$. 
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Compact domains in $\mathbb{R}^2$ and tangent cones

This case is more easy to manipulate:

- Blow-ups over points $p \in \partial S$.
- The compacity of the domain can be controlled \textit{a priori} using the tangent cone $T_p(\partial z S)$ at $p$ of $\partial z S$.

\begin{proposition}
Let $p \in \partial S$ and suppose that there exists a line $L$ such that $\overline{S} \cap L = \{p\}$. If $L \notin T_p(\partial z S)$ then there exist a Zariski open $U \subset \widehat{\mathbb{R}}^2$ such that $\widetilde{S^T} \cap U$ is compact.
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\end{align*}

- If $T_p(\partial_z S)$ contains $n \geq 2$ lines: let $X = T_p(X) \cap S$, and $S = X \cup S_1 \cup \ldots \cup S_n$.
- If $T_p(\partial_z S)$ only contains one line: consider $N_p(\partial_z S)$ the normal space of $\partial_z S$ at $p$ and let $X = (T_p(X) \cup N_p(\partial_z S)) \cap S$. We obtain a partition $S = X \cup S_1 \cup S_2$. 

\text{Juan Viu-Sos}
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A semi-canonical reduction for periods of Kontsevich-Zagier  
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A classical way to write $\pi/4$ as an integral is:

$$\frac{\pi}{4} = \int_1^\infty \frac{1}{1+x^2} \, dx = \int_{D} dx dy$$

with $D = \{ x > 1, 0 < y(1+x^2) < 1 \} \subset \mathbb{R}^2$.

By a change of charts given by the inclusion $U_z = \{ [x : y : z] \mid z \neq 0 \} \subset \mathbb{P}^2_\mathbb{R}$, we obtain a diffeomorphism $\varphi$ of $\mathbb{R}^2$ minus a line such that

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\[ I(D,1) = \int_D dx\,dy = \int_{D_1} \frac{dx_1\,dy_1}{x_1^3}. \]

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PART III

PERSPECTIVES AND CONTINUATION
Compact semi-algebraic sets have a PL–manifold structure via triangulations:
- Reduction of the KZ-conjecture in a combinatorial problem?
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We can define a notion of degree for periods with some transcendence consequences:

$$\deg(p) = \min\{d \in \mathbb{N} \mid \exists K \subset \mathbb{R}^d \text{ compact s.a. such that } |p| = \text{vol}_d(K)\}$$
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