Nonlinear effects at elastic deformation of cubic materials

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Abstract. A variant of the relations of nonlinear elasticity is considered for anisotropic materials which are supposed to be crystals of cubic syngony with respect to the type of elastic symmetry. The proposed model takes into account the physical nonlinearity in the behavior of such materials under the condition of small deformations. Based on the representation of the elastic potential in the form of a tensor polynomial in third-order strains, relations for stresses with elastic constants of the second and third orders are obtained. On the basis of the concept of the elastic eigenstates of materials, in the case of a cubic material, representations for the elastic tensors of the fourth and sixth ranks in eigentensor bases are obtained. The proposed variant of constitutive relations takes into account the mutual influence of the processes occurring in various eigenspaces of cubic material.

1. Introduction

Anisotropic materials, which are the crystals of cubic syngony with respect to the type of elastic symmetry, are the most common both among crystals and among artificially created composites. The elastic properties of cubic materials satisfy the symmetry conditions inherent in the point group of a volumetric-centered or face-centered cube. The cubic crystal is characterized by point groups 23, m3, 432, 43m, m3m in the designations of Newnham [1]. These groups are characterized by the presence of three four-fold symmetry axes (\vec{n}_i), four three-fold symmetry axes (\vec{l}_j), and six two-fold symmetry axes (\vec{d}_k). The relationship between the unit vectors determining the position of these axes is given by the relations [2]:

\[ \vec{l}_j = \frac{1}{\sqrt{3}} (\delta_1 \vec{n}_1 + \delta_2 \vec{n}_2 + \delta_3 \vec{n}_3), \quad (\delta_1, \delta_2, \delta_3) = (1, 1, 1), (-1, 1, 1), (1, -1, 1), (1, 1, -1), \]

\[ \vec{d}_k = \frac{1}{\sqrt{2}} (\vec{n}_i \pm \vec{n}_j), \quad (i, j) = (1, 2), (2, 3), (3, 1). \]

Cubic materials are similar to isotropic materials by their properties. As it is known from [1] a sphere of anisotropic material generally becomes an ellipsoid in a hydrostatic compression experiment. In the cases of isotropic and cubic materials under the influence of hydrostatic pressure, the spheres remain spheres, which does not allow to distinguish isotropic and cubic materials from this experiment.
Linear elastic cubic materials are described in [1,3–8] in the framework of the generalized Hooke’s law. Structural representations of fourth-rank elastic tensors invariant with respect to the described symmetry groups are obtained within these papers. The elastic tensors written in an arbitrary (laboratory) coordinate system have, in the general case, 21 non-zero not independent components. An analysis of the dependence of the elastic moduli of the cubic material and the Poisson’s ratio on the direction of tension of the specimen was made in [4–7]. Within the framework of the linear relationship between stresses and strains, the manifestation of auxetic properties in some crystals of cubic syngony was revealed in [3,7]. However, a coordinate system for a cubic material in which the tensor of elastic properties has three non-zero independent constants [1–8] can be defined. Such a coordinate system was called canonical in [9].

Nonlinear models of the behavior of cubic materials can take into account either geometric nonlinearity or physical nonlinearity. In the most complex models, it is necessary to take into account both geometrical and physical nonlinearity at the same time. The construction of models of cubic materials that take into account physical nonlinearity at finite strains is the subject of [10]. In this article, the author constructs nonlinear constitutive relations for a cubic material based on nine tensor generators constructed on the base of Cauchy–Green finite strain tensor and obtained in [2]. The author of [10] deals only with the algebraic aspects of constructing invariant tensor bases and does not consider the mechanical manifestations of the nonlinear behavior of cubic materials.

If non-linear effects are observed in the materials under consideration during elastic deformation, even in the region of small deformations, then physically nonlinear constitutive relations are required. The articles [11–13] are devoted to the solution of this problem. In these works, the expansion of the elastic potential into a series with the retention of terms of the second and third degrees with respect to deformations is used to write nonlinear constitutive relations. However, the authors of the works do not analyze the obtained constitutive relations from a physical point of view. In these works, there is no analysis of the possible effects of nonlinear deformation of the materials under consideration. In [11–13], the structure of elasticity tensors of the sixth rank is determined, which in the canonical coordinate system contain six non-zero independent constants. A geometrically and physically nonlinear model of cubic material was proposed in [13].

In this article, one of the possible general approaches to the construction of physically nonlinear elasticity relations proposed in [14,15] is methodologically applied to cubic materials. A detailed analysis of the obtained constitutive relations from the point of view of the described nonlinear effects in cubic materials is carried out.

2. Representation of the elastic tensors of a cubic material in canonical tensor bases
We assume that the cubic material under consideration is hyperelastic. In this case, it is possible to build an elastic potential. We use the specific (referred to the volume) potential strain energy \( W(\varepsilon) \) as such a potential, in which the strain tensor \( \varepsilon \) and the Cauchy true stress tensor \( S \) are energetically conjugated. This means that the differential of the specific potential energy can be represented in the form \( dW = S : d\varepsilon \), where the colon means the double dot product of tensors. It follows the possibility of determining stresses when concretizing the type of specific potential energy of deformations by the formula:

\[
S = \frac{\partial W}{\partial \varepsilon}. \tag{1}
\]

We represent the tensor function \( W(\varepsilon) \) in the form of a series in which we hold only the first two non-zero terms:
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The structure of tensors $N$ and $L$ for cubic material is also known [1, 2, 13]. The tensor $N$ contains three independent elastic constants of the second order, and the tensor $L$ contains six independent elastic constants of the third order. The elastic tensors have the smallest number of independent non-zero constants in the system of canonical axes of anisotropy of the material [9, 15]. The elastic tensors of the cubic material have an arbitrary form in an arbitrary (laboratory) coordinate system. The main axes of anisotropy according to V. V. Novozhilov may not coincide with the canonical axes. Then it becomes necessary to determine the mutual orientation of the laboratory coordinate system with an orthonormal basis $\overrightarrow{k}$ and the system of canonical coordinate axes with a basis $\overrightarrow{a}_i$.

Since the experiment on the comprehensive compression of the sample is difficult, it was proposed in [15] to replace this experiment with three experiments on the compression of a cubic sample in three mutually perpendicular directions. In these experiments, it is mandatory to measure all the components of the strain tensor in the laboratory coordinate system. The equivalence of such experiments is a consequence of the linear relationship between strains and stresses in the region of small strains. In this case, we are talking about determining the initial position of the axes of anisotropy of the material.

Let experiments on the compression of cubic samples be carried out in a laboratory coordinate system with an orthonormal basis $\overrightarrow{k}$, the identities hold:

$$q_{11}^2 + q_{21}^2 + q_{31}^2 = 1, \quad q_{12}^2 + q_{22}^2 + q_{32}^2 = 1, \quad q_{13}^2 + q_{23}^2 + q_{33}^2 = 1,$$

$$q_{11}q_{12} + q_{21}q_{22} + q_{31}q_{32} = 0, \quad q_{11}q_{13} + q_{21}q_{23} + q_{31}q_{33} = 0, \quad q_{12}q_{13} + q_{22}q_{23} + q_{32}q_{33} = 0.$$

It was shown in [15] that two experiments on the compression of cubic samples are sufficient to determine the position of the canonical axes of anisotropy in a cubic material. In the first experiment, the stress tensor is defined as $\overline{S}_1 = -t \overrightarrow{k} \overrightarrow{k}_1$. Let $C$ be the elastic compliance tensor inverse to the elastic tensor $N$. The measured strains are expressed in terms of compliance constants and tensor $Q$ components as follows:

$$\overline{e}_{11} = -t \left[ Q_{11} (C_{1111} - (C_{1122} + 2C_{1212})) + C_{1122} + 2C_{1212} \right].$$
\[ \varepsilon_{22} = -t \left[ Q_4 \left( C_{111} - (C_{1122} + 2C_{1212}) \right) + C_{1122} \right], \]  
\[ \varepsilon_{33} = -t \left[ Q_5 \left( C_{1111} - (C_{1122} + 2C_{1212}) \right) + C_{1122} \right], \]  
\[ \varepsilon_{13} = -tQ_8 \left( C_{1111} - (C_{1122} + 2C_{1212}) \right), \]  
\[ \varepsilon_{23} = -tQ_9 \left( C_{1111} - (C_{1122} + 2C_{1212}) \right), \]

where indicated

\[ Q_1 = q_1^4 + q_1^4 + q_1^4, \quad Q_4 = q_1^4 + q_{12}^2 + q_{13}^2 + q_{23}^2, \quad Q_5 = q_1^4 + q_{12}^2 + q_{13}^2 + q_{23}^2. \]

\[ Q_7 = q_2q_1q_3 + q_2q_4q_3 + q_2q_5q_3, \quad Q_8 = q_3q_1q_2^3 + q_3q_4q_3^3 + q_3q_5q_3^3, \quad Q_9 = q_1q_2q_3q_4 + q_2q_3q_4q_1 + q_3q_4q_1^2. \]

In the second experiment, the stress tensor is \( S_2 = -t\vec{k}\vec{k}^2 \), and the measured components of the strain tensor are also expressed in terms of compliance constants and tensor components \( Q \):

\[ \varepsilon_{11} = -t \left[ Q_4 \left( C_{111} - (C_{1122} + 2C_{1212}) \right) + C_{1122} + 2C_{1212} \right], \]
\[ \varepsilon_{22} = -t \left[ Q_2 \left( C_{111} - (C_{1122} + 2C_{1212}) \right) + C_{1122} \right], \]
\[ \varepsilon_{33} = -t \left[ Q_6 \left( C_{111} - (C_{1122} + 2C_{1212}) \right) + C_{1122} \right], \]
\[ \varepsilon_{13} = -tQ_{10} \left( C_{1111} - (C_{1122} + 2C_{1212}) \right), \]
\[ \varepsilon_{23} = -tQ_{11} \left( C_{1111} - (C_{1122} + 2C_{1212}) \right), \]

where indicated

\[ Q_2 = q_1^4 + q_2^4 + q_3^4, \quad Q_6 = q_1^4q_2^3 + q_2^4q_3^3 + q_3^4q_2^3, \quad Q_{10} = q_1^4q_2^3 + q_2^4q_3^3 + q_1^4q_3^3, \]

\[ Q_{11} = q_1^4q_2^3 + q_3^3q_3^3 + q_3^4q_2^3, \quad Q_{12} = q_1^4q_2^3q_3^3 + q_2^4q_3^3q_2^3 + q_1^4q_3^3q_2^3. \]

To find the nine components of the tensor \( Q \), we use six relations (5) and four independent relations from (6), (7):

\[ -tQ_7 \left( C_{1111} - (C_{1122} + 2C_{1212}) \right) = \varepsilon_{12}, \quad -tQ_8 \left( C_{1111} - (C_{1122} + 2C_{1212}) \right) = \varepsilon_{13}, \]
\[ -tQ_9 \left( C_{1111} - (C_{1122} + 2C_{1212}) \right) = \varepsilon_{23}, \quad -tQ_{10} \left( C_{1111} - (C_{1122} + 2C_{1212}) \right) = \varepsilon_{12}. \]

Eliminating the multiplier \( (C_{1111} - (C_{1122} + 2C_{1212})) \) from them, we obtain three equations for the components of the tensor \( Q \):

\[ q_1^3q_{11} + q_3^3q_{12} + q_3^3q_{13} = \frac{\varepsilon_{13}}{\varepsilon_{12}} \left( q_2q_{11} + q_2q_{12} + q_3q_{13} \right), \]
\[ q_1q_2q_{11} + q_2q_3q_{12} + q_3q_3q_{13} = \frac{\varepsilon_{23}}{\varepsilon_{12}} \left( q_1q_{11} + q_2q_{12} + q_3q_{13} \right), \]
\[ q_1^3q_{21} + q_2^3q_{22} + q_3^3q_{23} = \frac{\varepsilon_{12}}{\varepsilon_{12}} \left( q_1q_{21} + q_2q_{22} + q_3q_{23} \right). \]

The numerical solution of the system of equations (5), (8) allows us to find the components of the tensor \( Q \), i.e., to determine the orientation of the canonical coordinate system in the cubic material relative to the laboratory coordinate system from the strains measured in experiments.

The calculations showed that in the case of cubic crystals, the canonical axes of anisotropy coincide with their crystallographic axes [100], [101] and [001], and for composite materials or wood their position coincides with the preferred structural directions.
We introduce the tensor basis formed by the dyads of the basis vectors of the canonical axes of anisotropy of the cubic axes $\vec{a}_i$:

$$A^1 = \vec{a}_1 \vec{a}_1, \quad A^2 = \vec{a}_2 \vec{a}_2, \quad A^3 = \vec{a}_3 \vec{a}_3, \quad A^4 = \frac{1}{\sqrt{2}} (\vec{a}_1 \vec{a}_2 + \vec{a}_2 \vec{a}_1), \quad A^5 = \frac{1}{\sqrt{2}} (\vec{a}_2 \vec{a}_3 + \vec{a}_3 \vec{a}_2), \quad A^6 = \frac{1}{\sqrt{2}} (\vec{a}_3 \vec{a}_1 + \vec{a}_1 \vec{a}_3).$$ (9)

The basis (9) is normalized by the relation: $A^i : A^j = \delta^{ij}$.

Along with basis (9), we consider the Il’yushin tensor basis $I^\alpha (\alpha = 0, 1, \ldots, 5)$ with basic tensors [9, 14, 15]:

$$I^0 = \frac{1}{\sqrt{3}} (\vec{a}_1 \vec{a}_1 + \vec{a}_2 \vec{a}_2 + \vec{a}_3 \vec{a}_3), \quad I^1 = \frac{1}{\sqrt{6}} (2\vec{a}_2 \vec{a}_3 - \vec{a}_1 \vec{a}_1 - \vec{a}_2 \vec{a}_2), \quad I^2 = \frac{1}{\sqrt{2}} (\vec{a}_1 \vec{a}_1 - \vec{a}_2 \vec{a}_2),$$

$$I^3 = \frac{1}{\sqrt{2}} (\vec{a}_1 \vec{a}_2 + \vec{a}_2 \vec{a}_1), \quad I^4 = \frac{1}{\sqrt{2}} (\vec{a}_2 \vec{a}_3 + \vec{a}_3 \vec{a}_2), \quad I^5 = \frac{1}{\sqrt{2}} (\vec{a}_3 \vec{a}_1 + \vec{a}_1 \vec{a}_3).$$ (10)

The tensor basis (10) is also normalized: $I^\alpha : I^\beta = \delta^{\alpha\beta}$.

The strain tensor $\varepsilon = \varepsilon_{ij} \vec{a}_i \vec{a}_j$ as a symmetric tensor of the second rank can be expanded in terms of (9) and (10). These expansions are of the form:

$$\varepsilon = \varepsilon_{11} A^1 + \varepsilon_{22} A^2 + \varepsilon_{33} A^3 + \sqrt{2} \varepsilon_{12} A^4 + \sqrt{2} \varepsilon_{23} A^5 + \sqrt{2} \varepsilon_{31} A^6$$ (11)

and

$$\varepsilon = \varepsilon_0 I^0 + \varepsilon_1 I^1 + \varepsilon_2 I^2 + \varepsilon_3 I^3 + \varepsilon_4 I^4 + \varepsilon_5 I^5,$$

where $\varepsilon_\alpha = \varepsilon : I^\alpha$. (12)

There is a relation between the expansion coefficients of (11) and (12):

$$\varepsilon_0 = \frac{1}{\sqrt{3}} (\varepsilon_{11} + \varepsilon_{22} + \varepsilon_{33}), \quad \varepsilon_1 = \frac{1}{\sqrt{6}} (2\varepsilon_{33} - \varepsilon_{11} - \varepsilon_{22}), \quad \varepsilon_2 = \frac{1}{\sqrt{2}} (\varepsilon_{11} - \varepsilon_{22}),$$

$$\varepsilon_3 = \sqrt{2} \varepsilon_{12}, \quad \varepsilon_4 = \sqrt{2} \varepsilon_{23}, \quad \varepsilon_5 = \sqrt{2} \varepsilon_{31}.$$

The inverse relation has the form:

$$\varepsilon_{11} = \frac{1}{\sqrt{3}} \varepsilon_0 - \frac{1}{\sqrt{6}} \varepsilon_1 + \frac{1}{\sqrt{2}} \varepsilon_2, \quad \varepsilon_{22} = \frac{1}{\sqrt{3}} \varepsilon_0 - \frac{1}{\sqrt{6}} \varepsilon_1 - \frac{1}{\sqrt{2}} \varepsilon_2, \quad \varepsilon_{33} = \frac{1}{\sqrt{3}} \varepsilon_0 + \frac{\sqrt{2}}{\sqrt{3}} \varepsilon_1,$$

$$\varepsilon_{12} = \frac{1}{\sqrt{2}} \varepsilon_3, \quad \varepsilon_{23} = \frac{1}{\sqrt{2}} \varepsilon_4, \quad \varepsilon_{31} = \frac{1}{\sqrt{2}} \varepsilon_5.$$

Using tensors of the second rank (9), we form tensors of the fourth and sixth ranks, possessing internal symmetry (9):

$$A^{\alpha\beta} = \frac{1}{2} \left( A^\alpha A^\beta + A^\beta A^\alpha \right),$$

$$A^{\alpha\beta\gamma} = \frac{1}{6} \left( A^\alpha A^\beta A^\gamma + A^\beta A^\alpha A^\gamma + A^\gamma A^\alpha A^\beta + A^\alpha A^\gamma A^\beta + A^\beta A^\gamma A^\alpha + A^\gamma A^\beta A^\alpha \right),$$ (13)

where $\alpha, \beta = 1, 2, \ldots, 6$. 

5
From the works [1, 2, 13], expansions of elasticity tensors $N$ and $L$ in bases (13) are known. For cubic material, these decompositions are of the form:

$$N = n_{11} \left( A^{11} + A^{22} + A^{33} \right) + n_{12} \left( A^{12} + A^{23} + A^{13} \right) + n_{44} \left( A^{44} + A^{55} + A^{66} \right),$$  
$$L = c_1 \left( A^{111} + A^{222} + A^{333} \right) + c_2 \left( A^{155} + A^{266} + A^{344} \right) +$$
$$+ c_3 \left( A^{112} + A^{113} + A^{122} + A^{133} + A^{223} + A^{233} \right) +$$
$$+ c_4 \left( A^{144} + A^{166} + A^{255} + A^{244} + A^{355} + A^{366} \right) + c_5 A^{123} + A_6 A^{456},$$

where $n_{11}, n_{12}, n_{44}$ are non-zero components of the elastic tensor $N$, i.e., second order elastic constants; $c_1, c_2, c_3, c_4, c_5, c_6$ are third order elastic constants.

Using tensors (10), we construct bases consisting of tensors of the fourth ($I^{\alpha \beta}$) and sixth ($I^{\alpha \beta \gamma}$) ranks:

$$I^{\alpha \beta} = \frac{1}{2} \left( I^{\alpha} I^{\beta} + I^{\beta} I^{\alpha} \right), \quad (16)$$
$$I^{\alpha \beta \gamma} = \frac{1}{6} \left( I^{\alpha} I^{\beta} I^{\gamma} + I^{\beta} I^{\gamma} I^{\alpha} + I^{\gamma} I^{\alpha} I^{\beta} + I^{\alpha} I^{\gamma} I^{\beta} + I^{\beta} I^{\gamma} I^{\alpha} + I^{\gamma} I^{\beta} I^{\alpha} \right), \quad (17)$$

where $\alpha, \beta = 0, 1, \ldots, 5$.

Bases (16) and (17) are normalized by the relations:

$$I^{\alpha \beta} \cdot I^{\gamma \epsilon} = \frac{1}{2} \left( \delta^{\alpha \gamma} \delta^{\beta \epsilon} + \delta^{\alpha \epsilon} \delta^{\beta \gamma} \right),$$
$$I^{\alpha \beta \gamma} \cdot I^{\delta \epsilon \zeta} = \frac{1}{6} \left( \delta^{\alpha \delta} \delta^{\beta \epsilon} \delta^{\gamma \zeta} + \delta^{\alpha \delta} \delta^{\beta \zeta} \delta^{\gamma \epsilon} + \delta^{\alpha \delta} \delta^{\gamma \epsilon} \delta^{\beta \zeta} + \delta^{\alpha \epsilon} \delta^{\beta \delta} \delta^{\gamma \zeta} + \delta^{\alpha \epsilon} \delta^{\beta \zeta} \delta^{\gamma \delta} + \delta^{\alpha \epsilon} \delta^{\gamma \delta} \delta^{\beta \zeta} \right).$$

In [9, 15], the expansion of the elastic tensor $N$ by the basis (16) was obtained in the form:

$$N = n^{(1)} I^{00} + n^{(2)} \left( I^{11} + I^{22} \right) + n^{(3)} \left( I^{33} + I^{44} + I^{55} \right). \quad (18)$$

The coefficients in the representation (18) are related to the second-order elasticity constants (4) by the expressions:

$$n^{(1)} = n_{11} + 2n_{12}, \quad n^{(2)} = n_{11} - n_{12}, \quad n^{(3)} = n_{44}.$$

The tensor $L$ of elastic constants of the third order can be expanded by the basis (17). However, it should contain in its expansion only such combinations of tensors $I^{\alpha \beta \gamma}$ that are invariant with respect to the group of orthogonal transformations of cubic syngony. Using the direct verification method [16], it is found that for the cubic material there are six invariant combinations of tensors (17):

$$B^{(1)} = I^{000}, \quad B^{(2)} = I^{011} + I^{022}, \quad B^{(3)} = I^{033} + I^{044} + I^{055}, \quad B^{(4)} = \frac{1}{\sqrt{6}} \left( I^{111} - 3 I^{122} \right), \quad B^{(5)} = \frac{1}{\sqrt{6}} \left( I^{144} + I^{155} - 2 I^{133} \right) + \frac{1}{\sqrt{2}} \left( I^{255} - I^{244} \right), \quad B^{(6)} = \frac{1}{\sqrt{2}} I^{345}. \quad (19)$$

The expansion of the tensor $L$ in the basis (19) has the form:

$$L = \sum_{s=1}^{6} b_s B^{(s)}.$$  

(20)
In the expression (20), the coefficients $b_s$ are related to the third-order elastic constants of the cubic material (15) by the expressions

$$
b_1 = \frac{1}{\sqrt{3}} (c_1 + 6c_3 + 2c_5), \quad b_2 = \sqrt{3} (c_1 - c_5), \quad b_3 = 2\sqrt{3} (c_2 + 2c_4), \quad b_4 = A_1 - 3A_3 + 2A_5, \quad b_5 = 6 (A_4 - A_2), \quad b_6 = 24A_6.
$$

3. Physically nonlinear relations for cubic material

If we restrict ourselves to a second-order term only in the representation for the specific potential strain energy (2), then relations (3) take the form of a generalized Hooke’s law:

$$
S = N : \varepsilon.
$$

(21)

We take into account the representation (18) in relations (21) and obtain

$$
S = n^{(1)} I^{00} : \varepsilon + n^{(2)} (I^{11} + I^{22}) : \varepsilon + n^{(3)} (I^{33} + I^{44} + I^{55}) : \varepsilon
$$
or

$$
S = n^{(1)} \varepsilon_{(1)} + n^{(2)} \varepsilon_{(2)} + n^{(3)} \varepsilon_{(3)}.
$$

(22)

It follows from relations (18) and (22) that the tensor basis (16) is an eigentensor for a cubic material. The concept of eigentensors and eigenstates was introduced in the works [17, 18]. Eigentensors for cubic material were obtained in [9, 10, 19]. The strain tensors $\varepsilon_{(1)}$, $\varepsilon_{(2)}$, $\varepsilon_{(3)}$ are elastic eigenstates of a cubic material and belong to three eigensubspaces [9, 10, 19]:

1D (Volumetric) Subspace: $\varepsilon_{(1)} = \varepsilon_0 I^0$,

2D (Deviatoric) Subspace: $\varepsilon_{(2)} = \varepsilon_1 I^1 + \varepsilon_2 I^2$,

3D (Shear) Subspace: $\varepsilon_{(3)} = \varepsilon_3 I^3 + \varepsilon_4 I^4 + \varepsilon_5 I^5$.

(23)

The first eigensubspace (one-dimensional) is purely volumetric deformation $\varepsilon_{(1)}$, the second eigensubspace (two-dimensional) corresponds to deformations of shape change associated only with a change in the lengths of material fibers, $\varepsilon_{(2)}$, and the third eigensubspace (three-dimensional) corresponds to pure shear deformations $\varepsilon_{(3)}$.

We project the tensor $S$ into the same eigensubspaces:

$$
S_{(1)} = S_0 I^0, \quad S_{(2)} = S_1 I^1 + S_2 I^2, \quad S_{(3)} = S_3 I^3 + S_4 I^4 + S_5 I^5,
$$

(24)

where $S_\alpha = S : I^\alpha$.

In accordance with (22), Hooke’s law can be written as

$$
S = \sum_{k=1}^{3} S_{(k)} = \sum_{k=1}^{3} n^{(k)} \varepsilon_{(k)}.
$$

(25)

It follows from (25) that, within the framework of linear elasticity, stress tensors $S_{(\alpha)}$ and strain tensors $\varepsilon_{(\alpha)}$ in each eigensubspace are coaxial and proportional.

Let us concretize the relations (3) for the cubic material. To do this, we find out the properties of the tensors $B^{(s)}$ included in the expansion (20) of the tensor $L$. These properties are manifested in the calculation of convolution of tensors $B^{(s)}$ with strain eigentensors (23):

$$
\varepsilon_{(\alpha)} \cdot B^{(s)} : \varepsilon_{(\beta)} = \varepsilon_{(\beta)} \cdot B^{(s)} : \varepsilon_{(\alpha)}.
$$

(26)
Nine of the products (26) are non-zero:
\[
\begin{align*}
\varepsilon_{(1)} \cdot B^{(1)} : \varepsilon_{(1)} &= 3J_1(\varepsilon_{(1)}) I^0, \\
\varepsilon_{(1)} \cdot B^{(2)} : \varepsilon_{(2)} &= \frac{1}{3\sqrt{3}} J_1(\varepsilon_{(1)}) \varepsilon_{(2)}, \\
\varepsilon_{(2)} \cdot B^{(2)} : \varepsilon_{(2)} &= J_1(\varepsilon_{(2)}) I^0, \\
\varepsilon_{(1)} \cdot B^{(3)} : \varepsilon_{(3)} &= \frac{1}{3\sqrt{3}} J_1(\varepsilon_{(1)}) \varepsilon_{(3)}, \\
\varepsilon_{(3)} \cdot B^{(3)} : \varepsilon_{(3)} &= J_1(\varepsilon_{(3)}) I^0, \\
\varepsilon_{(2)} \cdot B^{(4)} : \varepsilon_{(2)} &= Q_{(2)2}, \\
\varepsilon_{(2)} \cdot B^{(5)} : \varepsilon_{(3)} &= \frac{2}{3} P_{(3)}, \\
\varepsilon_{(3)} \cdot B^{(5)} : \varepsilon_{(3)} &= \frac{2}{3} Q_{(3)2}, \\
\varepsilon_{(3)} : B^{(6)} : \varepsilon_{(3)} &= \frac{1}{3} Q_{(3)3}.
\end{align*}
\]

(27)

In the relations (27) \( J_1(A) = A : E \) is the first invariant of the tensor \( A \); \( Q_{(2)2} \) is the projection of the tensor into the second eigensubspace:
\[
\varepsilon_{(2)}^2 = \frac{1}{\sqrt{3}} (\varepsilon_1^2 + \varepsilon_2^2) I^0 + Q_{(2)2}, \\
Q_{(2)2} = \frac{1}{\sqrt{6}} (\varepsilon_1^2 - \varepsilon_2^2) I^1 - \sqrt{\frac{2}{3}} \varepsilon_1 \varepsilon_2 L^2.
\]

(28)

\( Q_{(3)2} \) and \( Q_{(3)3} \) are the projections of the tensor into the second and third eigensubspaces:
\[
\begin{align*}
\varepsilon_{(3)}^2 &= \frac{1}{\sqrt{3}} (\varepsilon_3^2 + \varepsilon_4^2 + \varepsilon_5^2) I^0 + Q_{(3)2} + Q_{(3)3}, \\
Q_{(3)2} &= \frac{1}{\sqrt{6}} (2\varepsilon_3^2 - \varepsilon_1^2 - \varepsilon_2^2) I^1 - \frac{1}{2\sqrt{2}} (\varepsilon_4^2 - \varepsilon_5^2) L^2, \\
Q_{(3)3} &= \frac{1}{\sqrt{2}} \varepsilon_4 \varepsilon_5 L^3 + \frac{1}{\sqrt{2}} \varepsilon_3 \varepsilon_5 L^4 + \frac{1}{\sqrt{2}} \varepsilon_3 \varepsilon_4 L^5, \\
P_{(3)} &= \frac{1}{\sqrt{6}} \varepsilon_1 \varepsilon_3 L^3 + \frac{1}{\sqrt{6}} \left( \frac{1}{\sqrt{2}} \varepsilon_1 - \frac{1}{\sqrt{2}} \varepsilon_2 \right) \varepsilon_4 L^4 + \frac{1}{2} \left( \frac{1}{\sqrt{6}} \varepsilon_1 + \frac{1}{\sqrt{6}} \varepsilon_2 \right) \varepsilon_5 L^5.
\end{align*}
\]

(29)

(30)

In this model, the expression for the specific potential strain energy has the form
\[
W = \frac{1}{2} \left( 3n^{(1)} J_2(\varepsilon_{(1)}) + n^{(2)} J_2(\varepsilon_{(2)}) + n^{(3)} J_2(\varepsilon_{(3)}) \right) + \\
+ \frac{1}{6\sqrt{3}} J_1(\varepsilon_{(1)}) \left( 9b_1 J_2(\varepsilon_{(1)}) + b_2 J_2(\varepsilon_{(2)}) + b_3 J_2(\varepsilon_{(3)}) \right) + \\
+ \frac{1}{6} J_3(\varepsilon_{(2)}) \left( \sqrt{3}b_4 - 2b_5 \right) + \frac{1}{36} J_3(\varepsilon_{(3)}) (b_6 - 12b_5) + \frac{1}{3} b_5 J_3(\varepsilon_{(2)} + \varepsilon_{(3)}),
\]

where the notation for the second and third invariants of the second-rank tensor is introduced: \( J_2(A) = A : A \); \( J_3(A) = \det A \).

We write the expressions for the invariants of the tensors in relations (27) and (31) in terms of coefficients \( \varepsilon_\alpha \) of the expansion (12). The invariants of tensors \( \varepsilon_{(1)} \), \( \varepsilon_{(2)} \), \( \varepsilon_{(3)} \) are as follows:
\[
\begin{align*}
J_1(\varepsilon_{(1)}) &= \frac{1}{\sqrt{3}} \varepsilon_0, \\
J_2(\varepsilon_{(1)}) &= \frac{1}{3} \varepsilon_0^2, \\
J_3(\varepsilon_{(1)}) &= \frac{1}{3\sqrt{3}} \varepsilon_0^3, \\
J_1(\varepsilon_{(2)}) &= 0, \\
J_2(\varepsilon_{(2)}) &= \varepsilon_1^2 + \varepsilon_2^2, \\
J_3(\varepsilon_{(2)}) &= \frac{1}{3\sqrt{6}} \varepsilon_1 (\varepsilon_1^2 - 3\varepsilon_2^2), \\
J_1(\varepsilon_{(3)}) &= 0, \\
J_2(\varepsilon_{(3)}) &= \varepsilon_3^2 + \varepsilon_4^2 + \varepsilon_5^2, \\
J_3(\varepsilon_{(3)}) &= \frac{1}{\sqrt{2}} \varepsilon_3 \varepsilon_4 \varepsilon_5.
\end{align*}
\]

(32)
The invariants of tensors \( \varepsilon^2_{(1)}, \varepsilon^2_{(2)}, \varepsilon^2_{(3)} \) are as follows:

\[
J_1(\varepsilon^2_{(1)}) = \frac{1}{3} \varepsilon^2_0, \quad J_1(\varepsilon^2_{(2)}) = \frac{1}{3} (\varepsilon^2_1 + \varepsilon^2_2), \quad J_1(\varepsilon^2_{(3)}) = \frac{1}{3} (\varepsilon^2_3 + \varepsilon^2_4 + \varepsilon^2_5).
\]

(33)

The mixed invariant of tensors \( \varepsilon_{(2)}, \varepsilon_{(3)} \) is:

\[
J_3(\varepsilon_{(2)} + \varepsilon_{(3)}) = \varepsilon_{(2)} : Q_{(3)2} + J_3(\varepsilon_{(2)}) + J_3(\varepsilon_{(3)}) = \varepsilon_{(2)} : P_{(3)} + J_3(\varepsilon_{(2)}) + J_3(\varepsilon_{(3)}) = \\
= \frac{\varepsilon_1}{3\sqrt{6}} (\varepsilon^2_1 - 3\varepsilon^2_2) + \frac{1}{\sqrt{2}} \varepsilon_3 \varepsilon_4 \varepsilon_5 + \frac{\varepsilon_1}{2\sqrt{6}} (\varepsilon^2_4 + \varepsilon^2_5 - 2\varepsilon^2_3) + \frac{\varepsilon_2}{2\sqrt{2}} (\varepsilon^2_4 - \varepsilon^2_5).
\]

(34)

The presence of the mixed invariant (34) in the expression for the specific potential strain energy allows one to take into account the mutual influence of processes occurring in the eigenspaces 2D and 3D. Note that the expressions for the mixed invariant (34) are simplified in cases where there are no deformations in the second or third eigenspaces, since in this case

\( \varepsilon_{(2)} : Q_{(3)2} = \varepsilon_{(3)} : P_{(3)} = 0. \)

Moreover,

- if \( \varepsilon_{(3)} = 0 \), then \( J_3(\varepsilon_{(2)} + \varepsilon_{(3)}) = J_3(\varepsilon_{(2)}) = \frac{\varepsilon_1}{3\sqrt{6}} (\varepsilon^2_1 - 3\varepsilon^2_2), \)
- if \( \varepsilon_{(2)} = 0 \), then \( J_3(\varepsilon_{(2)} + \varepsilon_{(3)}) = J_3(\varepsilon_{(3)}) = \frac{\varepsilon_2}{\sqrt{2}} \varepsilon_3 \varepsilon_4 \varepsilon_5. \)

Substituting (18) and (20) into the constitutive relations (3), we obtain the following form of the relationship between stresses and strains:

\[
S_{(1)} = \left( n^1 + b_1 \varepsilon_0 \right) \varepsilon_0 + \frac{1}{3} b_2 (\varepsilon^2_1 + \varepsilon^2_2) + \frac{1}{3} b_3 (\varepsilon^2_3 + \varepsilon^2_4 + \varepsilon^2_5) \right) I^0,
\]

\[
S_{(2)} = \left( n^2 + \frac{1}{3} b_2 \varepsilon_0 \right) \varepsilon_0 + b_4 Q_{(2)2} + 2b_5 Q_{(3)2}, \quad (35)
\]

\[
S_{(3)} = n^3 \varepsilon_{(3)} + \frac{2}{3} b_5 P_{(3)} + \frac{1}{3} b_6 Q_{(3)3}.
\]

Relations (35) contain strains in the second degree and are physically non-linear relationships.

4. Analysis of nonlinear effects described by the model

In accordance with physically nonlinear relations (35) in the non-one-dimensional eigenspaces 2D and 3D, the stress and strain tensors \( S_{(2)} \) and \( \varepsilon_{(2)}, S_{(3)} \) and \( \varepsilon_{(3)} \) cease to be coaxial. In the second subspace, the deviation from the coaxiality of the tensors \( S_{(2)} \) and \( \varepsilon_{(2)} \) is related both to the appearance of the stress component along the tensor \( Q_{(2)2} \) and to the component along \( Q_{(3)2} \). In the third subspace, the deviation from the coaxiality of the tensors \( S_{(3)} \) and \( \varepsilon_{(3)} \) is related to the appearance of the stress components along the tensors and the latter vanishes, if \( \varepsilon_{(2)} = 0 \).

The analysis shows that the obtained relations (35) do not satisfy the generalization to anisotropic materials of partial Il’yushin’s postulate formulated in [9, 15], and take into account the mutual influence of processes occurring in various eigenspaces.

Consider the deformation process, located entirely in the first eigensubspace: \( \varepsilon_{(1)} = \varepsilon_0 I^0, \quad \varepsilon_{(2)} = \varepsilon_{(3)} = 0. \) This process is a process of purely volumetric deformation. According to (35), in response to such deformations, stresses \( S_{(1)} = (n^1 + b_1 \varepsilon_0) \varepsilon_0 I^0 \) appear in cubic materials, which are hydrostatic and nonlinearly dependent on volumetric strains \( \varepsilon_0 \). Shear stresses in the canonical axes of anisotropy do not appear.

Let the deformation process be entirely located in the second eigensubspace \( \varepsilon = \varepsilon_{(2)} = \varepsilon_1 I^1 + \varepsilon_2 I^2, \quad \varepsilon_{(1)} = \varepsilon_{(3)} = 0. \) Such a process corresponds to a change in the lengths of fibers
located along the canonical axes of anisotropy without shears. In this case, in accordance
with (35), the emerging stresses have the form $S = S_{(1)} + S_{(2)}$, and

$$S_{(1)} = \frac{1}{3} b_2 (\varepsilon_1^2 + \varepsilon_2^2) I^0, \quad S_{(2)} = \left( n^2 + \frac{1}{3} b_2 \varepsilon_0 \right) \varepsilon_{(2)} + b_4 Q_{(2)}^2.$$  

These relations describe the nonlinear dependence of stresses on strains. In the process of
forming, hydrostatic stresses appear. As in the first case, shear stresses $S_{(3)}$ do not appear in
such a process.

If the deformation process $\xi = \xi_{(3)} = \varepsilon_3 I^3 + \varepsilon_4 I^4 + \varepsilon_5 I^5$, $\xi_{(1)} = \xi_{(2)} = 0$ consists in pure shifts
in at least one of the planes containing the canonical axes of anisotropy, then, in accordance
with (35), stresses $S = S_{(1)} + S_{(2)} + S_{(3)}$ arise, and

$$S_{(1)} = \frac{1}{3} b_3 (\varepsilon_3^2 + \varepsilon_4^2 + \varepsilon_5^2) I^0, \quad S_{(2)} = 2 b_5 Q_{(3)}^3, \quad S_{(3)} = n^3 \xi_{(3)} + \frac{1}{3} b_6 Q_{(3)}^3.$$  

Relations (35) describe the nonlinear dependence of shear stresses $S_{(3)}$ on shear strains $\xi_{(3)}$ and predict the occurrence of normal stresses, including hydrostatic ones.

5. Conclusion
An expansion of the elastic tensors of the fourth and sixth ranks by tensor bases in eigensubspaces
is obtained for a cubic material. Stress-strain relations containing the second degree of strains
are obtained from the conditions for the existence of an elastic potential (specific potential
strain energy). The expressions for stresses in each of the eigensubspaces of the cubic material
are written.

The analysis of nonlinear effects that occur during deformation of cubic materials, which can
be described by the obtained version of the constitutive relations, is carried out. All effects are
second-order effects, they are similar to the Poynting and Kelvin effects, which are known for an
isotropic material. For example, when defining deformations of the shape change, the proposed
model predicts the appearance of hydrostatic stresses. When describing the shear deformations
of a cubic sample, the appearance of normal stresses on its faces, including hydrostatic ones,
which depend on the second degree of shear deformations, is predicted. If the elasticity constants
of the third order are zeroed in the proposed relations, the last will be reduced to linear relations
of the Hooke’s law for the cubic material.

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