On symplectically rigid local systems of rank four and Calabi-Yau operators

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Abstract
We classify all $Sp_4(\mathbb{C})$-rigid, quasi-unipotent local systems and show that all of them have geometric origin. Furthermore, we investigate which of those having a maximal unipotent element are induced by fourth order Calabi-Yau operators. Via this approach, we reconstruct all known Calabi-Yau operators inducing a $Sp_4(\mathbb{C})$-rigid monodromy tuple and obtain closed formulae for special solutions of them.

1 Introduction

Differential operators of geometric origin are proposed to describe periods of families of complex algebraic varieties and have been studied quite extensively during the last fifty years. A special class of such operators are fourth order differential Calabi-Yau operators which are related to families of Calabi-Yau threefolds having a large complex structure limit and $h^{2,1} = 1$. A conjectural characterization of those operators from a purely differential algebraic point of view, together with a list of most of the known examples is stated in [AESZ10]. The majority of those operators is not constructed from a geometric situation, as only very few examples of this type are known at the moment. Thus it is natural to ask, which of the operators really are of geometric origin and what would be a geometric realization.

It is quite challenging to decide whether a given differential operator has geometric origin or not. The first order ones are exactly those, which have a non trivial algebraic solution, see e.g. [Sim90]. Furthermore, as observed by Y. André in [And89, Chapter II], the class of geometric differential operators is preserved by a multitude of constructions as taking subquotients, direct sums, tensor products and Hadamard products. We call an operator which can be obtained in this way geometrically constructible. An appropriate method to check whether an operator is geometrically constructible or not is provided by the following investigation of local solutions.

Given a differential operator $L$ of degree $n$ with coefficients in $\mathbb{C}(z)$ and singular locus $S$, a classical theorem due to Cauchy states that for each $x \in \mathbb{P}^1 \backslash S$ we find a basis $F = \{f_1, \ldots, f_n\}$ of the $n$-dimensional $\mathbb{C}$-vectorspace $\text{Sol}(L)_x = \{f \in O_{\mathbb{P}^1 \backslash S}(U) \mid L(f) = 0\}$ of holomorphic functions in some disc around $x$. If we chose a closed path $\gamma$ starting at $x$, analytic continuation of $F$ around $\gamma$ yields a different basis $\tilde{F}$ of $\text{Sol}(L)_x$. The change from $F$ to $\tilde{F}$ only depends on the homotopy class of $\gamma$, which reflects the elements of $S$ encircled by $\gamma$. The translation of Cauchy’s theorem into 20th century language thus states the following: the operator $L$ induces a local system $\mathcal{L}$ of rank $n$ on $\mathbb{P}^1 \backslash S$ via

$$\mathcal{L}(U) := \{f \in O_{\mathbb{P}^1 \backslash S}(U) \mid L(f) = 0\}.$$ 

Furthermore, with respect to an arbitrary base point $x_0 \in \mathbb{P}^1 \backslash S$ this local system naturally induces a representation

$$\rho_{\mathcal{L}}: \pi_1(\mathbb{P}^1 \backslash S, x_0) \to \text{GL}(\mathcal{L}_{x_0})$$

of $\pi_1(\mathbb{P}^1 \backslash S, x_0)$, the so called monodromy representation. Its image is called the monodromy group associated to $L$. We may chose a set of generators $\{\gamma_s\}_{s \in S} \subset \pi_1(\mathbb{P}^1 \backslash S, x_0)$, whose elements are just simple loops $\gamma_s$ around each $s \in S$. As $S$ is finite, it can be equipped with an ordering $I$ such that

$$\prod_{i \in I} \gamma_{s_i} = 1 \in \pi_1(\mathbb{P}^1 \backslash S, x_0)$$

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holds. Thus the monodromy group is completely determined by the tuple

\[(T_i)_{i \in I} := (\rho_\gamma(s_i))_{i \in I}\]

of linear maps, which fulfill \(\prod_{i \in I} T_i = \text{id}_{\mathbb{C}^n}\). This tuple \((T_i)_{i \in S}\) is called the monodromy tuple associated to \(L\) and represents the effect of analytic continuation of holomorphic solutions near \(x\) around each singularity of \(L\). We call a monodromy tuple to be of geometric origin, if it is induced by a differential operator of geometric origin.

The constructions preserving the geometric origin of an operator have counterparts on the level fuchsian systems and monodromy tuples, see [Kat96] and [DR07]. Furthermore, taking tensor- and middle Hadamard products with rank one tuples of geometric origin is an invertible operation. Thus a tuple is of geometric origin, if we can produce a tuple of geometric origin out of it, using those invertible operations.

As shown by N. Katz in [Kat96], a subclass of monodromy tuples of geometric origin are the linearly rigid ones, i.e. those, whose elements are quasi-unipotent, generate an irreducible subgroup in \(\text{GL}_n(\mathbb{C})\) and which, are up to simultaneous conjugation, completely determined by the Jordan forms of its elements. In particular, Katz shows that each tuple of this type can be reduced to a geometric tuple of rank one by a sequence of invertible operations introduced above. The most prominent examples of linearly rigid tuples are those induced by generalized hypergeometric differential equations and where studied by Levelt [Lev61] and Beukers and Heckmann [BH89] for instance.

One can extend the notion of rigidity from \(\text{GL}_n(\mathbb{C})\) to any reductive complex algebraic group, but then a reduction a la Katz generally fails. Nevertheless, Simpson conjectured that each tuple of this type is of geometric origin, see [Sim92].

We know that the elements of the monodromy tuples induced by a fourth order differential Calabi-Yau operator lie in \(\text{Sp}_4(\mathbb{C})\). By the discussion above, it seems to be promising to investigate those Calabi-Yau operators inducing an \(\text{Sp}_4(\mathbb{C})\)-rigid monodromy tuple. A bit surprisingly, the classification of all \(\text{Sp}_4(\mathbb{C})\)-rigid monodromy tuples reveals the following

**Existence Theorem (cf. Theorem 3.1)** Each \(\text{Sp}_4(\mathbb{C})\)-rigid tuple consisting of quasi-unipotent elements can be reduced to a tuple of rank one via geometric operations. In particular, it is geometrically constructible using only tuples of rank one and thus of geometric origin.

Section three of this article is devoted to the proof of the existence theorem via explicit constructions of those tuples using rational pullbacks, tensor- and Hadamard products of tuples of rank one. A review of all constructions involved, as well as basic facts concerning rigid monodromy tuples, is given in section two. To construct inducing operators of geometric origin, we translate the constructions to the level of differential operators directly rather than choosing an appropriate cyclic vector of the differential system. This is done in section four. The translation of the construction enables us to compute distinguished solutions of the resulting operators explicitly, which is discussed in section five. Finally, we state an explicit construction of those operators whose induced monodromy tuples have a maximally unipotent element in section six. A further investigation yields the following

**Conjecture** An \(\text{Sp}_4(\mathbb{C})\)-rigid tuple consisting of quasi-unipotent elements and having a maximally unipotent element is induced by a differential Calabi-Yau operator if and only if the elements of its second exterior power lie up to simultaneous conjugation in \(\text{SO}_5(\mathbb{Z})\). Furthermore, the inducing operator is unique.

The construction of differential operators inducing the remaining monodromy tuples will be done in a subsequent article.

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2 Rigidity and the middle convolution

2.1 Rigidity

Here we recall the definition of rigidity in various contexts and state criteria how to read off rigidity via numerical invariants.

**Definition 2.1**

(i) We call \(T\) a tuple of rank \(n\) if there exist an \(r \in \mathbb{N}\) and \(T_i \in \text{GL}_n(\mathbb{C}), i = 1, \ldots, r + 1\) such that \(T = (T_1, \ldots, T_{r+1})\) and \(T_1 \cdots T_{r+1} = 1\). Two tuples are equivalent if they are simultaneously conjugate by an element in \(\text{GL}_n(\mathbb{C})\).

(ii) We call a tuple \(T\) irreducible of rank \(n\) if \(T\) generates an irreducible subgroup \(\langle T \rangle := \langle T_1, \ldots, T_{r+1} \rangle\) of \(\text{GL}_n(\mathbb{C})\).

(iii) We call a tuple \(T\) quasi-unipotent if the eigenvalues of all its elements are roots of unity.

(iv) An irreducible tuple \(T\) is called symplectic, resp. orthogonal, if \(\langle T \rangle\) respects a skew-symmetric, resp. a symmetric bilinear form.

(v) Let \(G \leq \text{GL}_n(\mathbb{C})\) be an irreducible reductive algebraic subgroup and \(\langle T \rangle \leq G\) be irreducible. We say that \(T\) is \(G\)-rigid, if the following dimension formula holds:

\[\sum_{i=1}^{r+1} \text{codim}(C_G(T_i)) = 2(\dim(G) - \dim(Z(G))),\]

where \(C_G(T_i)\) denotes the centralizer of \(T_i\) in \(G\), the codimension is taken relative to \(G\), and \(Z(G)\) denotes the centre of \(G\).

(vi) We call an irreducible tuple \(T\) of rank \(n\) linearly rigid if \(T\) is \(\text{GL}_n(\mathbb{C})\)-rigid and symplectically rigid if \(T\) is \(\text{Sp}_n(\mathbb{C})\)-rigid.

The following lemma in [Sco77] is often helpful to decide whether a tuple \(T\) is irreducible.

**Lemma 2.2 (Scott)** Let \(T\) act on a vector space \(V\). Then

\[\sum_{i=1}^{r+1} \text{rk}(T_i - 1) \geq (\dim(V) - \dim(V^T)) + (\dim(V) - \dim(V^T)),\]

where \(T^\ast\) denotes the tuple corresponding to dual representation of \(T\) and \(V^T\) the fixed space of \(T\).

Moreover, if \(T\) is irreducible of rank \(n\) then we have

\[\sum_{i=1}^{r+1} \text{rk}(T_i - 1) \geq 2n \quad \text{(Scott formula)} \quad \text{and} \quad \sum_{i=1}^{r+1} \dim(C_{\text{GL}_n(\mathbb{C})}(T_i)) \leq (r-1)^2n^2 + 2 \quad \text{(dimension count)}.

**Theorem 2.3**

(i) Let \(T\) be irreducible of rank \(n\). Then \(T\) is linearly rigid if and only if \(T\) is uniquely determined by the Jordan forms of its elements.

(ii) Let \(T\) be an irreducible symplectic tuple of rank \(2m\). If there exist only finitely many tuples \((h_1, \ldots, h_{r+1})\) with \(h_1 \cdots h_{r+1} = 1\) and such that \(h_i\) is conjugate in \(\text{Sp}_{2m}(\mathbb{C})\) to \(T_i\) then \(T\) is \(\text{Sp}_{2m}(\mathbb{C})\)-rigid, i.e., the dimension formula holds.

**Proof:**

(i) The first result goes back to Deligne, Katz and Steenbrink, see e.g. [Kat96].

(ii) This statement can be found in [SV99].
Alternatively one can consider a tuple as a finite dimensional $\mathbb{C}[F_r]$-module. For this let $F_r$ denote the free group on $r$ generators $f_1, \ldots, f_r$. Setting $f_{r+1} = (f_1 \cdots f_r)^{-1}$ we can view an element in $\text{Mod}(\mathbb{C}[F_r])$ as a pair $(T, V)$, where $V$ is a vector space over $\mathbb{C}$ and $T = (T_1, \ldots, T_{r+1})$ is a tuple in $\text{GL}(V)^{r+1}$ such that $f_i$ acts on $V$ via $T_i$ for $i = 1, \ldots, r + 1$. We also assign to $T$ a tuple $s = s_T = (s_1, \ldots, s_r, s_{r+1} = \infty)$, where $s_1, \ldots, s_r$ are pairwise different elements in $\mathbb{C}$ with an ordering $s_i < s_j$ in $s$ if $i < j$.

In a geometric context one can also speak in terms of local systems, as done in the introduction.

2.2 Basic properties of the middle convolution

In this section we recall some of the main properties of the middle convolution functor $\text{MC}$. This functor was introduced by Katz in [Kat96] in the category of perverse sheaves. A down to earth version for Fuchsian systems and their monodromy group generators can be found in [DR07].

We recall the main properties of the middle convolution functor $\text{MC}$. This functor was introduced by Katz in [Kat96] in the category of perverse sheaves. A down to earth version for Fuchsian systems and their monodromy group generators can be found in [DR07]. We recall the main properties of the convolution that are are stated in [DR07, Section 2].

For $(T, V) \in \text{Mod}(\mathbb{C}[F_r])$, where $T = (T_1, \ldots, T_{r+1}) \in \text{GL}(V)^{r+1}$, and $\lambda \in \mathbb{C}^\times$ one can construct an element $(C_\lambda(T), V^r) \in \text{Mod}(\mathbb{C}[F_r])$ as follows. For $k = 1, \ldots, r$, we define $B_k \in \text{GL}(V^r)$ as an element that maps a vector $(v_1, \ldots, v_r)^T \in V^r$ to

\[
\begin{pmatrix}
1 & 0 & \cdots & 0 \\
\vdots & \ddots & & \\
\lambda(T_1 - 1) & \cdots & \lambda(T_{k-1} - 1) & \lambda T_k \\
0 & \cdots & 0 & 1 \\
\end{pmatrix}
\begin{pmatrix}
v_1 \\
\vdots \\
v_k \\
v_r
\end{pmatrix}
\]

Further we set $B_{r+1} = (B_1 \cdots B_r)^{-1}$. The subspaces $K := \bigoplus_{k=1}^r K_k$, where

\[
K_k = \begin{pmatrix}
0 \\
\vdots \\
\ker(T_k - 1) \\
0 \\
\vdots \\
0
\end{pmatrix}
\]

and

\[
L = \bigcap_{k=1}^r \ker(B_k - 1) = \ker(B_1 \cdots B_r - 1).
\]

of $V^r$ are $(B_1, \ldots, B_r)$-invariant. If $\lambda \neq 1$ we have

\[
L = \left\{ \begin{pmatrix} T_2 \cdots T_r v \\
T_3 \cdots T_r v \\
\vdots \\
v \end{pmatrix} \mid v \in \ker(\lambda \cdot T_1 \cdots T_r - 1) \right\}
\]

and

\[
K + L = K \oplus L.
\]

Definition 2.4 Let $(T, V) \in \text{Mod}(\mathbb{C}[F_r])$. 


2.2 Basic properties of the middle convolution

(i) We call the \( \mathbb{C}[F_r] \)-module \( C_\lambda(V) := (C_\lambda(T), V^r) := ((B_1, \ldots, B_{r+1}), V) \) the convolution of \( V \) with \( \lambda \), where \( \mathcal{Z}_\lambda(T) := \mathcal{Z}_T \).

(ii) Let \( MC_\lambda(T) := (B_1, \ldots, B_{r+1}) \in \text{GL}(V^r/(K+L))^{r+1} \), where \( B_k \) is induced by the action of \( B_k \) on \( V^r/(K+L) \). The \( \mathbb{C}[F_r] \)-module \( MC_\lambda(V) := (MC_\lambda(T), V^r/(K+L)) \) is called the middle convolution of \( T \) with \( \lambda \).

**Theorem 2.5** Let \((T, V) \in \text{Mod}(\mathbb{C}[F_r])\) be irreducible. Assume further that if \( \dim(V) = 1 \) that at least two of the \( T_i \), \( i = 1, \ldots, r \), are non trivial. Let \( \lambda \in \mathbb{C}^\times \).

(i) If \( \lambda \neq 1 \) then
\[
\dim(MC_\lambda(V)) = \sum_{k=1}^{r} \text{rk}(T_k) \cdot (|\lambda| - (\dim(V) - \text{rk}(\lambda \cdot T_1 \cdots T_r))).
\]

(ii) If \( \lambda_1, \lambda_2 \in \mathbb{C}^\times \) then
\[
MC_{\lambda_2} \circ MC_{\lambda_1}(V) \cong MC_{\lambda_2 \lambda_1}(V), \quad \text{where} \quad MC_1(V) \cong V.
\]

(iii) \( MC_\lambda(V) \) is irreducible.

Obviously, tensoring a linearly rigid tuple with a rank 1 tuple preserves linearly rigidity. Nevertheless this operation plays an essential role in the study of linear rigid tuples due to Katz’ existence algorithm, see Thm. [2.10]

**Definition 2.6** Let \( (T_k, V_k) \in \text{Mod}(\mathbb{C}[F_r]) \), \( k = 1, 2 \), be semisimple and Set\( (\mathcal{Z}) = \text{Set}(\mathcal{Z}_T) \cup \text{Set}(\mathcal{Z}_T^2), \vert \text{Set}(\mathcal{Z}) \vert = r + 1 \), where an ordering on \( s_i < s_j \) in \( \mathcal{Z} \) is given by the rule: If \( s_i, s_j \in \text{Set}(\mathcal{Z}_T^2) \) then \( s_i < s_j \) in \( \text{Set}(\mathcal{Z}_T^2) \) for \( k = 1, 2 \). Thus we consider \((T_1, V_1)\) and \((T_2, V_2)\) as elements in \( \text{Mod}(\mathbb{C}[F_r]) \), where \( T_{k,j} = I_{V_k} \) if \( s_j \notin \text{Set}(\mathcal{Z}_T^2) \) for \( k = 1, 2 \). Then we call
\[
MT(V_1, V_2) = V_1 \otimes V_2,
\]
\[
MT(T_1, T_2) = MT_{T_1}(T_2) = (T_1 \otimes T_2, \ldots, T_{1, r+1} \otimes T_{2, r+1})
\]
the middle tensor product of \((T_1, V_1)\) and \((T_2, V_2)\).

**Proposition 2.7** Let \((T, V) \in \text{Mod}(\mathbb{C}[F_r])\) be irreducible. Assume further that if \( \dim(V) = 1 \) that at least two of the \( T_i \), \( i = 1, \ldots, r \), are non trivial.

(i) If \( T \) is orthogonal, symplectic resp., then \( MC_{-1}(T) \) is symplectic, orthogonal resp.

(ii) Let \( T \) be orthogonal or symplectic and \( A_1 = (\lambda_1, \lambda_2, (\lambda_1 \lambda_2)^{-1}), A_2 = (\lambda_1 \lambda_2^{-1}, \lambda_1^{-1} \lambda_2, 1) \) be rank 1 tuples such that \( \mathcal{Z}_{\lambda_1} = \mathcal{Z}_{\lambda_2} = (s_i, s_j, s_{r+1}) \). Then
\[
MT_{A_1} \circ MC_{\lambda_1 \lambda_2} \circ MT_{A_2} \circ MC_{(\lambda_1 \lambda_2)^{-1}} \circ MT_{A_1}(T)
\]
is either orthogonal or symplectic.

**Proof:** For (ii) see [DR00, Thm. 5.14]. \( \square \)

**Definition 2.8** Let \( A = (\lambda^{-1}, \lambda), \mathcal{Z}_A = (0, \infty) \), be a rank 1 tuple. Then we call
\[
MH_\lambda(T) := MC_\lambda(MT(T, A))
\]
the middle Hadamard product of \( T \) with \( \lambda \).
The above definition of the middle Hadamard product is motivated by the fact that the convolution of $f$ with $x^\mu$, $\lambda = \exp(2\pi i \mu)$, can formally be written as a Hadamard product

$$\int f(x)(y-x)^\mu \frac{dx}{y-x} = \int f(x)x^\mu \cdot (\frac{x}{x-1})^{\mu-1} \frac{dx}{x}$$

Due to the relation between the convolution and the Hadamard product we can switch between this both operations freely.

**Remark 2.9** Let $T$ be irreducible and $\lambda \in \mathbb{C}^\times$. Let $\Lambda = (\lambda, \lambda^{-1})$, $\varepsilon_\Lambda = (0, \infty)$, be a rank 1 tuple. Then

$$\text{MC}_\Lambda(T) = \text{MH}_\lambda(\text{MT}(T, \Lambda)).$$

The middle convolution yields Katz Existence Theorem, cf. [Kat96].

**Theorem 2.10** Any linearly rigid irreducible tuple $T$ of rank $n$ can be reduced to a rank 1 tuple via a suitable sequence of at most $n - 1$ middle convolutions $\text{MC}_\Lambda$ and middle tensor products $\text{MT}_\Lambda$ with rank one tuples $\Lambda$.

This theorem results in an algorithm to check the existence of a linearly rigid tuple with given Jordan forms of the local monodromy, given by Katz in [Kat96], Chap. 6:

2.3 The numerology of the middle convolution

Those monodromy tuples were first described by Levelt [Lev61]. A detailed study of the monodromy tuples was first described by Beukers and Heckman.

**Example 2.11** The tuple

$$T = (T_0, T_1, T_\infty) := \text{MH}_\beta \circ \text{MH}_{\beta^{-1}} \circ \text{MH}_\alpha (1, \alpha, \alpha^{-1}), \quad \alpha, \beta \in \mathbb{C}^\times \setminus \{1\}$$

is a symplectic tuple of rank 4. Using the methods described in this section we can compute $T$ explicitly. Setting $A = \alpha + \alpha^{-1} - 2$, $B = \beta + \beta^{-1} - 2$ we get

$$T_0 = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad T_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ AB & AB & A + B & 1 \end{pmatrix}.$$
where $k_j$ is determined by

$$\text{rk}(\text{MC}_\lambda(T)) = \sum_{i=1}^r \text{rk}(T_i - 1) + \text{rk}(\lambda^{-1}T_\infty - 1) - n.$$ 

This also shows that the middle convolution $\text{MC}_\lambda$ preserves linear rigidity by Thm. [2.3].

From the definition of the middle Hadamard product and the above proposition we can derive the Jordan forms of $\text{MH}_\lambda(T)$:

**Proposition 2.13** Let $T$ be irreducible of rank $n$ and $\lambda \neq 1$. The transformation of the Jordan forms of its elements under the middle Hadamard product is given by

$$J(\text{MH}_\lambda(T_i)) = \bigoplus_{\rho \in \mathbb{C}\setminus\{1,\lambda^{-1}\}} \lambda\rho J(j)^{v(i,\rho,j)} \bigoplus J(j + 1)^{v(i,\lambda^{-1},j)}$$

for $i \neq 0, r + 1$,

$$J(\text{MH}_\lambda(T_0)) = \bigoplus_{\rho \in \mathbb{C}\setminus\{1,\lambda^{-1}\}} \lambda\rho J(j)^{v(i,\rho,j)} \bigoplus J(j + 1)^{v(i,\lambda^{-1},j)}$$

$$+ \bigoplus_{j \geq 2} \lambda J(j - 1)^{v(i,1,j)} \bigoplus J(1)^{k_i}$$

$$J(\text{MH}_\lambda(T_{r+1})) = \bigoplus_{\rho \in \mathbb{C}\setminus\{1,\lambda^{-1}\}} \rho J(j)^{v(r+1,\rho,j)} \bigoplus J(j - 1)^{v(r+1,1,j)}$$

$$+ \bigoplus_{j \geq 2} \lambda^{r+1} J(j + 1)^{v(r+1,\lambda^{-1},j)} \bigoplus \lambda^{-r-1} J(1)^{k_{r+1}}$$

where $k_j$ is determined by

$$\text{rk}(\text{MH}_\lambda(T)) = \sum_{T_i \neq T_0} \text{rk}(T_i - 1) + \text{rk}(\lambda^{-1}T_0 - 1) - n.$$ 

### 3 Classification of symplectically rigid tuples of rank four

This section is devoted to the classification of symplectically rigid tuples of rank four. In particular we show:

**Theorem 3.1** Let $T$ be a symplectically rigid tuple of rank four consisting of quasi-unipotent elements. Then $T$ is coming from geometry, i.e. $T$ is a monodromy tuple of a factor of a Picard-Fuchs equation. Moreover it can be constructed by a sequence of geometric operations starting with a rank one tuple. These geometric operations include tensor products, rational pullbacks and the middle convolution.

Roughly speaking the proof of Thm. [3.1] is based on the following steps:

**STEP one:** Using Thm. [2.3] (ii) we classify in Table 2 all possible symplectically rigid irreducible tuples $T$ of rank 4 via the tuples

$$P_i := (\dim C_{\text{Sp}_4}(C)(T_1), \ldots, \dim C_{\text{Sp}_4}(C)(T_{r+1}))$$

of the centralizer dimensions of their elements. We list these centralizer dimensions in Table 1. Via Möbius transformations, which are sharply 3-transitive, and more generally the action of the Artin braid group $B_r$ on $T$ that permutes the local monodromies, we can order the entries according to increasing dimensions. Thus we get the finite list $P_1, \ldots, P_5$ in Table 2. Further, we refine these cases by the subcases

$$P_i(\dim C_{\text{GL}_4}(C)(T_1), \ldots, \dim C_{\text{GL}_4}(C)(T_{r+1})).$$
E.g., the $P_5(4, 8, 10, 10)$ case denotes irreducible quadruples $T$ with

$$(\dim C_{Sp_4}(T_1), \ldots, \dim C_{Sp_4}(T_4)) = (2, 6, 6, 6)$$

and

$$(\dim C_{GL_4}(T_1), \ldots, \dim C_{GL_4}(T_4)) = (4, 8, 10, 10).$$

Moreover Table 1 shows that $J(T_1) \in \{ \pm J(4), (-J(2), J(2)), (xJ(2), x^{-1}J(2)), (x, y, y^{-1}, x^{-1}) \}$, $J(T_2) = (-1, -1, 1, 1)$ and $J(T_3) = J(T_4) = (J(2), 1, 1)$.

**STEP two:** The irreducibility condition restricts the possible tuples of Jordan forms via the Scott formula or the dimension count in Lemma 2.2. E.g. there is no rigid tuple of with Jordan forms $(J(4), J(4), (J(2), 1, 1))$ in the $P_5(4, 4, 10)$ case, as $7 = \sum_{i=1} \text{rk}(T_i) - 1 < 2 \cdot 4$.

**STEP three:** We check whether $T$ is linearly rigid using the dimension count by Thm. 2.10. Moreover the algorithm imposes the conditions for the existence of such a $T$ depending on the eigenvalues of the $T_i$.

**STEP four:** Using the operations in Prop. 2.7 we try to construct a tuple $\tilde{T}$ in an orthogonal group of dimension 3, 4, 5 or 6. Due to the exceptional isomorphisms we have

$$\text{Sym}^2 Sp_2(C) = SO_3(C), \quad Sp_2(C) \otimes Sp_2(C) = SO_4(C),$$

$$\Lambda^2 Sp_4(C) = SO_5(C), \quad \Lambda^2 SL_4(C) = SO_6(C),$$

which can again result in linearly rigid tuples. E.g. an orthogonal triple $T$ of rank 3 with $J(T) = (J(3), J(3), J(3))$ yields a linearly rigid triple $\tilde{T}$ of rank 2 with $J(T) = (J(2), J(2), -J(2))$.

It turns out that in all $P_i$ cases we either get contradictions to the irreducibility or we end up with a rank one tuple. In the latter case we obtain a suitable sequence of operations that allows us to construct this symplectically rigid tuple $T$ of rank four, since each operation is invertible. Moreover if the symplectically rigid tuple of rank four is quasi-unipotent it turns out that it can be constructed using only geometric operations cf. [And89, Chap. II].

We begin with Step one and classify the Jordan forms in $Sp_4(C)$ and their centralizer dimensions. Since $\Lambda^2 Sp_4(C) = SO_5(C)$ we also determine the Jordan forms in $SO_5(C)$.

| Jordan form in $Sp_4(C)$ | Jordan form in $SO_5(C)$ | centralizer dimension in $Sp_4(C)$ | GL_4(C) | conditions |
|--------------------------|--------------------------|-----------------------------------|---------|------------|
| $\pm (1, 1, 1, 1)$      | $(1, 1, 1, 1, 1)$         |                                   | 10      |            |
| $\pm (J(2), 1, 1)$      | $(J(2), J(2), 1)$         |                                   | 6       |            |
| $\pm (J(2), J(2))$      | $(J(3), 1, 1)$            |                                   | 4       |            |
| $(1, -1, -1, 1, 1)$     | $(J(3), -1, 1)$           |                                   | 6       |            |
| $(J(2), J(2))$          | $(-J(3), -1, 1)$          |                                   | 4       |            |
| $(x, x, x^{-1}, x^{-1})$| $(x^{-1}, x^{-1}, x^{-1}, x^{-1})$ |                                   | 4       | $x^2 \neq 1$ |
| $(x, 1, 1, x^{-1})$     | $(x, x, x^{-1}, x^{-1})$  |                                   | 4       | $x^2 \neq 1$ |
| $(xJ(2), x^{-1}J(2))$   | $(J(3), x^2, x^{-2})$     |                                   | 2       | $x^2 \neq 1$ |
| $(x, x^{-1}, J(2))$     | $(xJ(2), x^{-1}J(2), 1)$  |                                   | 2       | $x^2 \neq 1$ |
| $(x, y, y^{-1}, x^{-1})$| $(xy, xy^{-1}, 1, x^{-1}y, x^{-1}y^{-1})$ |                                   | 2       | $x^2, y^2 \neq 1$ |

Table 1: The Jordan forms of elements in $Sp_4(C)$ and $SO_5(C) = \Lambda^2 Sp_4(C)$. 
3.1 The $P_1$ case

| case       | subcases          | remarks            |
|------------|-------------------|--------------------|
| $P_1$      | (4,4,10)          | lin. rigid         |
|            | (4,4,8)           | $\Lambda^2$ lin. rigid |
| $P_2$      | (4,6,6)           | lin. rigid         |
|            | (4,6,8)           |                    |
|            | (4,8,8)           | red. (dimension count) |
| $P_3$      | (4,10,10,10)      | red. (Scott)       |
|            | (4,8,10,10)       |                    |
|            | (4,8,8,10)        | $\Lambda^2$ red.  |
|            | (4,8,8,8)         | $\Lambda^2$ red. |
| $P_4$      | (8,8,10,10)       | lin. rigid         |
|            | (6,8,10,10)       |                    |
|            | (6,6,8,10)        | lin. rigid         |
|            | (8,8,8,10)        | $\Lambda^2$ red.  |
|            | (6,8,8,8)         | $\Lambda^2$ red. |
|            | (6,6,8,8)         | $\Lambda^2$ lin. rig. |
| $P_5$      | (10,10,10,10,10)  | lin. rig.          |
|            | (8,10,10,10,10)   | red. (Scott)       |
|            | (8,8,10,10,10)    | red. (Scott)       |
|            | (8,8,8,10,10)     | $\Lambda^2$ lin. rig. |
|            | (8,8,8,8,10)      | $\Lambda^2$ red.  |
|            | (8,8,8,8,8)       | $\Lambda^2$ red. |

Table 2: The centralizer conditions for symplectically rigid tuples

In the following sections we rearrange the order of the centralizer dimensions in Table 2 via Möbius transformations to simplify the proofs. If $T$ is a triple we can assume that $\mathcal{X}_T = \{0, 1, \infty\}$. Thus we also index $T = (T_0, T_1, T_\infty)$. E.g., a linearly rigid tuple in the $P_1(4,10,4)$ case such that $T_0$ is unipotent, can be written as a sequence of 3 Hadamard products starting from a rank 1 tuple, see Ex. 2.11. However in the $P_1(4,4,10)$ case the Katz algorithm requires additional tensor products with rank 1 tuples.

To abbreviate the notations we denote by $J(T)$ the tuple of Jordan forms. Further we write $J_s(T)$ for $(J_s(T_1), \ldots, J_s(T_{r+1}))$, where $J_s(T_i)$ denotes the semisimple part of $J(T_i)$.

3.1 The $P_1$ case

3.1.1 The $P_1(4,10,4)$ case

**Remark 3.2** We omit the linearly rigid $P_1(4,10,4)$ case. This well studied case corresponds to monodromy tuples of generalized hypergeometric differenials equation of order 4 and is settled by Katz’ algorithm. For an example, where $T_0$ is maximally unipotent, see Ex. 2.11.

3.1.2 The $P_1(4,8,4)$ case

**Theorem 3.3** A symplectically rigid tuple $T$ in the case $P_1(4,8,4)$ can be obtained from a rank one tuple using the middle Hadamard product and tensor products. Moreover the tuple $T$ can be written

$$T = \text{MH}_{-1}(\Lambda^2(S)),$$

where $S$ is linearly rigid rank 4 triple containing a transvection.
Proof: By Thm. 2.3 and Cor 2.4 the Hadamard product $\text{MH}_{m}(T)$ yields an irreducible orthogonal triple of rank $m$, where

$$m = \text{rk}(T_0 - 1) + \text{rk}(T_1 - 1) + \text{rk}(T_\infty - 1) - 4 \in \{4, 5, 6\}.$$ 

Hence we can apply one of the identities

$$\Lambda^2 \text{Sp}_4(\mathbb{C}) = \text{SO}_5(\mathbb{C}), \quad \Lambda^3 \text{SL}_4(\mathbb{C}) = \text{SO}_6(\mathbb{C})$$

to obtain a triple of rank 4 containing a transvection, since by Prop. 2.13

$$J(\text{MH}_{m}(T_1)) = (J(2)^2, J(1)^{m-4}).$$

For $m = 4$ we use the natural embedding of $\text{GO}_4(\mathbb{C})$ in $\text{SO}_5(\mathbb{C})$. Thus the triple is linearly rigid the claim follows from Katz’ algorithm. 

Remark 3.4 The construction of $T$ is in general not unique. In the above case one could also get $T$ by using that $\Lambda^2(T)$ yields a linearly rigid tuple and then apply Katz’ algorithm. However in this construction the computation of the matrix representation of $T$ is more complicated.

Corollary 3.5 Let $T$ be as in Thm. 3.3 such that $T_0$ is maximally unipotent and $J_s(T_\infty) = (xy, xy^{-1}, x^{-1}y, (xy)^{-1})$. Then

$$T_0 = \begin{pmatrix}
1 & ab & 0 & (a + b)^2 \\
0 & 1 & 1 & 0 \\
0 & 0 & 1 & -ab \\
0 & 0 & 0 & 1
\end{pmatrix}, \quad T_1 = \begin{pmatrix}
-1 & -2ab & 0 & 0 \\
0 & 1 & 0 & 0 \\
1 & ab & 1 & 0 \\
0 & -1 & 0 & -1
\end{pmatrix},$$

where $a = x + \frac{1}{x}$, $b = y + \frac{1}{y}$, $x, y \in \mathbb{C}^*$ and $ab \neq 0$. The tuple $T$ can be obtained as follows.

$$T = \text{MH}_{1} \circ \text{MT}_{\Lambda}(\Lambda^2 S), \quad S = \text{MH}_{(ix)} \circ \text{MH}_{(ix)^{-1}} \circ \text{MT}_{\Lambda} \circ \text{MH}_{-iy}(A_0)$$

with $A_0 = (1, (iy)^{-1}, iy)$ and $A_1 = (-1, 1, -1)$ are a rank $1$ triples. Further, $\text{MT}_{(i,1,1,1)} S$ is symplectic and linearly rigid of rank 4 with

$$(iJ(S_0), J(S_1), -iJ_s(S_\infty)) = ((iJ(2), -(iJ(2)), (J(2), 1, 1), (x, y, y^{-1}, x^{-1})).$$

Proof: The tuple $T$ can be constructed using the matrices in Section 2.3 according to the given sequence of Hadamard products and tensor products. Prop. 2.13 allows to keep track of the change of Jordan forms under Hadamard product. We demonstrate this for the case, where $x, x^{-1}, y, y^{-1}$ are pairwise different: We start with a rank 1 triple $A_0 = (1, (iy)^{-1}, iy)$ and apply $\text{MH}_{-iy}$. This yields a rank 2 triple with Jordan forms $(J(2), (-1, 1), (-iy^{-1}, -iy))$. Then we proceed with the tensor product $\text{MT}_{\Lambda}$, and so on. Tabulating the operations and the change of the Jordan forms we get

| rk | operation | Jordan | forms |
|-----|-----------|--------|-------|
| 1   | $\text{MH}_{-iy}$ | $(1)$ | $(iy)^{-1}$ |
| 2   | $\text{MT}_{\Lambda}$ | $J(2)$ | $(-1, 1)$ | $(-iy^{-1}, -iy)$ |
| 3   | $\text{MH}_{(ix)^{-1}}$ | $(-J(2), 1)$ | $((ix)^{-1}, 1, 1)$ | $(iy^{-1}, iy, ix^{-1})$ |
| 4   | $\text{MH}_{(ix)}$ | $(-J(2), J(2))$ | $(-1, 1)$ | $(iy^{-1}, iy, ix^{-1}, ix)$ |
| 5   | $\Lambda^2$ | $(-J(3), 1, 1)$ | $(-1, 1)$ | $(-iy^{-1}, xy, x, (xy)^{-1}, x^{-1}y)$ |
| 6   | $\text{MT}_{\Lambda}$ | $(J(3), -1, -1)$ | $(J(2), J(2), 1)$ | $(xy^{-1}, xy, x^{-1}y^{-1}, x^{-1}y)$ |
| 4   | $\text{MH}_{-1}$ | $J(4)$ | $(-1, -1, 1, 1)$ | $(xy^{-1}, xy, x^{-1}y^{-1}, x^{-1}y)$ |

By Prop. 2.7 we know that $\text{MT}_{(i,1,1,1)}(S)$ is symplectic and we use that $\Lambda^2 \text{Sp}_4(\mathbb{C}) = \text{SO}_5(\mathbb{C})$. In the general case the Jordan form of the third element (in each step) is obtained by replacing $k$ equal eigenvalues $z$ by $zJ(k)$. 

The conditions for the irreducibility follow from the fact that the middle Hadamard product has to be non trivial in each step, i.e. $i \neq \pm x, \pm y$ by Thm. 2.4 Thus $ab \neq 0$. 

□
Corollary 3.6 Let $T$ be as in Cor. \ref{cor:3.6}. Then the Zariski closure of $\langle T \rangle$ is $\text{Sp}_4(\mathbb{C})$. Moreover if $ab, a^2 + b^2 \in \mathbb{Z}$ then $\langle T \rangle$ is contained up to conjugation in $\text{Sp}_4(\mathbb{Z})$. Further, if $T$ is quasi-unipotent then the conditions are also necessary.

**Proof:** Since $J(T_1) = (-1, -1, 1, 1)$ the Zariski closure of $\langle T \rangle$ is not $\text{Sym}^3(\text{SL}_2(\mathbb{C}))$ and the first statement follows from Cor. \ref{cor:3.6}. The matrix representation shows that the conditions are sufficient. The necessary condition for the group $\langle T \rangle$ to be contained in $\text{Sp}_4(\mathbb{Z})$ is that all traces of all elements are integers. Hence

$$ \text{tr}(T_\infty) = ab, \quad \text{tr}(T_\infty^2) = (a^2 - 2)(b^2 - 2) = (ab)^2 + 4 - 2(a^2 + b^2) \in \mathbb{Z}. $$

Hence $2(a^2 + b^2) \in \mathbb{Z}$. But if $a, b$ are sums of roots of unity then $2(a^2 + b^2) \in \mathbb{Z}$ implies $(a^2 + b^2) \in \mathbb{Z}$.

3.2 The $P_2$ case

3.2.1 The $P_2(4, 6, 6)$ case

Theorem 3.7 Let $T$ be a symplectically rigid tuple in the case $P_2(4, 6, 6)$, where

$$ J_s(T) = ((z_1 z_2, z_1 z_2^{-1}, z_1^{-1} z_2, (z_1 z_2)^{-1}), (1, -x^2, -x^{-2}, 1), (y^2, -1, -1, y^{-2})), $$

with $x, y, z_1, z_2 \in \mathbb{C}^*$. Then $T$ can be written

$$ T = M H_{-1}(M T(S_1, S_2)), \quad S_i = M T_{A_{2i}}(M H_{z_i x y^{-1}} A_{1i}) $$

with $A_{2i} = (z_i^{-1}, x^{-1}, z_i x), A_{1i} = (z_i^2, z_i^{-1} x y, (z_i x y)^{-1}), i = 1, 2$.

**Proof:** The tuple

$$ S = M T_A \circ M C_{-1}(T), \quad A = (-1, 1, -1), $$

is an orthogonal triple of rank

$$ m = \text{rk}(T_0 - 1) + \text{rk}(T_1 - 1) + \text{rk}(-T_{\infty} - 1) - 4 \in \{3, 4\} $$

by Thm \ref{thm:3.7} and Prop. \ref{prop:3.7} (ii). Using that

$$ \text{SO}_4(\mathbb{C}) = \text{Sp}_2(\mathbb{C}) \otimes \text{Sp}_2(\mathbb{C}), \quad \text{SO}_3(\mathbb{C}) = \text{Sym}^2 \text{Sp}_2(\mathbb{C}) $$

we can write $S$ as $S = S_1 \otimes S_2$ with

$$ (J(S_0), J(S_1), J(S_{\infty})) = ((z_i, z_i^{-1}, (x, x^{-1}, \pm(y, y^{-1})), i = 1, 2. $$

Since $S_1$ and $S_2$ are linearly rigid the claim follows from Katz’ algorithm.

Corollary 3.8 Let $T$ be as in Thm. \ref{thm:3.7}, such that $T_0$ is maximally unipotent. Then

$$ T_0 = \begin{pmatrix} 1 & -a + b & a & -2 \\ 0 & 1 & -2 & b \\ 0 & 0 & 1 & a - b \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad T_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 2 & 1 & -a \\ 2 & a + b & a & -a^2 + 1 \end{pmatrix}, $$

where $a = x + \frac{1}{2}, b = y + \frac{1}{2}$ and $a \neq b$.

The tuple $T$ can be written as

$$ T = M H_{-1}(\text{Sym}^2 S), \quad S = M T_A \circ M H_{x y^{-1}}(A_0), $$

where $A = (1, x^{-1}, x)$ and $A_0 = (1, x y, (x y)^{-1})$ are rank 1 triples with $\zeta_A = \zeta_{A_0} = (0, 1, \infty)$.
Proof: The proof is analogous to the proof of Cor. 3.5.

Corollary 3.9 Let $T$ be as in Cor. 3.8 Then the Zariski closure of $\langle T \rangle$ is $Sp_4(\mathbb{C})$ if and only if $a^2 \neq 1$ and $b^2 \neq 1$. The generated group is up to conjugation contained in $Sp_4(\mathbb{Z})$ if and only if $a^2, b^2, ab \in \mathbb{Z}$.

Proof: By construction there are at most two symplectically rigid tuples with given Jordan forms since $\text{Sym}^2$ does not act bijectively on the Jordan forms. However if $a = -b$ then the Jordan forms determine the tuple $T$ uniquely since a rank 2 triple with Jordan forms $(J(2), (x, x^{-1}), (x, x^{-1}))$ is reducible.

Further if $a^2 = b^2 = 1$ then $x, y$ are sixth roots of unity and $T$ can be also written as $\text{Sym}^3$ of a rank 2 tuple. By uniqueness and Cor. 3.3 the first claim follows.

If the generated group is up to conjugation contained in $Sp_4(\mathbb{Z})$ then the trace condition implies $a^2, b^2 \in \mathbb{Z}$. By construction the middle convolution $\text{MC}_{-1}$ and taking $\text{Sym}^2$ are compatible with the action of a field automorphism. Thus if $ab \not\in \mathbb{Z}$ then there exists a $\sigma \in \text{Gal}(\mathbb{Q}(a, b)/\mathbb{Q})$ such that $\sigma(a) = a$ and $\sigma(b) = -b$. But then we get $T^\sigma = T$ and $S^\sigma \neq S$, a contradiction. The matrix representation shows that these conditions are also sufficient. Namely, if $a, b \not\in \mathbb{Z}$, but $ab \in \mathbb{Z}$ then $a = n_1 \sqrt{d}$ and $b = n_2 \sqrt{d}$. Thus if we conjugate the matrices in Cor. 3.8 by $\text{diag}(\sqrt{d}, 1, 1, \sqrt{d})$ we get a representation in $Sp_4(\mathbb{Z})$.

3.2.2 The $P_2(4, 6, 8)$ case

Since the proofs of the statements in the linearly rigid $P_2(4, 6, 8)$ case are analogous to the proofs before we omit them.

Theorem 3.10 A linearly rigid tuple $T$ in the case $P_2(4, 6, 8)$, where

$$J_4(T) = ((z_1, z_2, z_2^{-1}, z_1^{-1}), (1, 1, y, y^{-1}), (x, x, x^{-1}, x^{-1})),$$

can be obtained as

$$T = MT_{A_3} \circ MH_{x, z_1} \circ MT_{A_2} \circ MH_{(z_2, 1) \circ MT_{A_1}} \circ MH_{y, z_2}(A_0),$$

where $A_3 = (z_1, 1, 1), A_2 = (z_2, 1, z_2^{-1}), A_1 = ((z_1, z_2), y, y, z_1, z_2)$ and $A_0 = (z_2, y, z_1, 1, z_1, 1, 1)$.}

Corollary 3.11 Let $T$ be as in Thm. 3.10 such that $T_0$ is maximally unipotent. Then

$$T_0 = \begin{pmatrix}
1 & -1 & 0 & a - 2 \\
0 & 1 & a - 2 & 0 \\
0 & 0 & 1 & -b + 2 \\
0 & 0 & 0 & 1
\end{pmatrix}, \quad T_1 = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 1 & 1 & b - 2 \\
1 & 0 & 1 & b - 1
\end{pmatrix},$$

where $a = x + \frac{1}{2}, \quad b = y + \frac{1}{2}, \quad x, y \in \mathbb{C} \setminus \{1\}$. The tuple $T$ can be obtained via

$$T = MH_x \circ MH_{y, 1} \circ MT_{A_1} \circ MH_y(A_0),$$

where $A_1 = (1, y, y^{-1})$ and $A_0 = (1, y, y^{-1})$ are rank 1 triples.

Corollary 3.12 Let $T$ be as in Cor. 3.11 Then $\langle T \rangle$ is contained up to conjugation in $Sp_4(\mathbb{Z})$ if and only if $a, b \in \mathbb{Z}$. The Zariski closure of $\langle T \rangle$ is $Sp_4(\mathbb{C})$ if and only if $a \neq 0$ and $b \neq -1$.

3.3 The $P_3$, $P_4$ and $P_5$ cases In this section we show that in the cases $P_3$, $P_4$ and $P_5$ all symplectically rigid tuples $T$ can be reduced via geometric operations to rank 1 tuples. Since we prefer to work with the convolution we index $T = (T_1, \ldots, T_r, T_{r+1} = T_\infty)$. In order to shortcut the following proofs we use without citing that the application of $\text{MC}_{-1}$ changes a symplectical tuple into an orthogonal one by Prop. 2.7(ii) whose rank is given by Thm. 2.5. Moreover, due to Katz’ algorithm it suffices to relate $T$ to a linearly rigid tuple.
3.3 The $P_3$, $P_4$ and $P_5$ cases

3.3.1 The $P_3$ case

**Theorem 3.13** In all the $P_3$ cases a symplectically rigid tuple $T$ can be reduced via middle convolution operations, taking tensor products and rational pullbacks to a rank 1 tuple. Further there exists no $T$ with a maximally unipotent element.

**Proof:**

(i) The case $P_3(4, 10, 10, 10)$ is ruled out by the Scott formula.

(ii) In the case $P_3(4, 8, 10, 10)$ the Scott formula implies that $\text{rk}(T_1 - 1) = \text{rk}(T_1 + 1) = 4$. Let $\Lambda_1 = (\lambda, 1, 1, \lambda^{-1})$ such that $\text{rk}(T_1 \lambda - 1) = 3$. Then

$$T_1 = MC_{\lambda - 1}MT_{\Lambda_1}(T)$$

is a rank 3 tuple. Taking $\Lambda_2 = (\lambda^{-1}, -\lambda, 1, -1)$ and $\Lambda_3 = (-1, \lambda^{-1}, 1, -\lambda)$ we obtain a rank 2 quadruple

$$S = MT_{\Lambda_3} \circ MC_{-\lambda} \circ MT_{\Lambda_2}(T_1)$$

in $GO_2(C)$ by Prop. 2.7 (iii). If $T$ is quasi-unipotent the generated group is finite and therefore a pullback of a linearly rigid monodromy tuple of a Gauss hypergeometric differential equation by a well known result of Klein (cf. [BD79, Thm. 3.4]). In any case a quadratic pullback yields a direct sum of two rank 1 tuples.

(iii) Taking $\Lambda^2$ in the case $P_3(4, 8, 8, 10)$ we obtain a reducible tuple in $SO_5(C)$ by the Scott formula. This excludes $J(T_1) = J(4)$ by Cor. A.3. Let $\Lambda_1 = (\lambda, 1, 1, \lambda^{-1})$ such that $\text{rk}(T_1 \lambda - 1) = 3$. Then

$$T_1 = MC_{\lambda^{-1}}MT_{\Lambda_1}(T)$$

is a rank 4 tuple. Taking $\Lambda_2 = (\lambda^{-1}, -\lambda, 1, -1)$ and $\Lambda_3 = (-1, \lambda^{-1}, 1, -\lambda)$ we obtain a rank 4 quadruple

$$S = MT_{\Lambda_3} \circ MC_{-\lambda} \circ MT_{\Lambda_2}(T_1)$$

in $GO_4(C)$ by Prop. 2.7 (iii). A quadratic pullback yields a 5-tuple $T_2$ with Jordan forms

$$((J(2), J(2)), (J(2), J(2)), (\lambda, \lambda, \lambda^{-1}, \lambda^{-1}), (\lambda, \lambda, \lambda^{-1}, \lambda^{-1}), (\lambda^2, 1, 1, \lambda^2 - 2)), \quad \text{where} \quad \text{rk}(S_1 - \lambda_2) = 3. \quad \text{Hence} \quad T_2 \text{ can be written as a tensor product of two 5-tuples} \quad S_1 \text{ and} \quad S_2 \text{ of rank 2 having two trivial entries. Since the} \quad S_i \text{ are linearly rigid the claim follows.}$$

(iv) We can exclude the case $P_3(4, 8, 8, 8)$. Since $MC_{-1}(T)$ yields an orthogonal tuple of rank $m$, where $m = 2 + \text{rk}(T_1 - 1) \in \{5, 6\}$, we obtain an irreducible quadruple of rank 4 with 3 transvections, using the identities

$$\Lambda^2(\text{Sp}_4(C)) = SO_5(C), \quad \Lambda^2(\text{SL}_4(C)) = SO_6(C).$$

But this contradicts the Scott formula.

\[\square\]

3.3.2 The $P_4$ case

**Theorem 3.14** In all the $P_4$ cases a symplectically rigid tuple $T$ can be reduced via middle convolution operations and taking tensor products and rational pullbacks to a rank 1 tuple.
Proof:

(i) In the case \( P_4(8, 8, 10, 10) \) the dimension count contradicts the irreducibility.

(ii) A tuple \( T \) in the \( P_4(6, 8, 10, 10) \) case is linearly rigid.

(iii) In the case \( P_4(6, 6, 10, 10) \) the irreducibility of \( T \) implies that \( \mathrm{rk}(T_4 + 1) = 1 \). Hence \( S = MC_{-1}(T) \) is an orthogonal rank 2 tuple having two involutions. The claim follows as in the proof (ii) of Thm. 3.13.

(iv) A tuple \( T \) in the \( P_4(8, 8, 8, 10) \) case is linearly rigid.

(v) In the case \( P_4(6, 8, 8, 10) \) the tuple \( S = MC_{-1}(T) \) is an orthogonal rank 5. A suitable sequence as in Prop. 2.7 (iii) yields an orthogonal tuple of rank 2. The claim follows as in the proof (ii) of Thm. 3.13.

(vi) The case \( P_4(6, 6, 10, 10) \) is excluded by the Scott formula.

(vii) In the case \( P_4(8, 8, 8, 8) \) Scott’s lemma shows that \( \Lambda^2(T) \) has a three dimensional orthogonal composition factor. By Cor. A.2 we get that \( T \) is a tensor product of two quadruples of rank 2 containing a trivial element. Hence we are in the linearly rigid case.

(viii) In the case \( P_4(6, 8, 8, 8) \) we get that \( S = \Lambda^2(T) \) is reducible. The Scott formula and Cor. A.2 imply that \( (S_1, S_2, -S_3, -S_4) \) splits into a trivial 1 dimensional component and a 4 dimensional one. Since the rank 4 tuple is linearly rigid the claim follows.

(ix) In the case \( P_4(6, 6, 8, 10) \) the application of \( MC_{-1} \) yields an orthogonal rank 4 tuple in \( \text{SO}_4(\mathbb{C})^4 \), where \( J(T_3) = J(T_4) = (J(2), J(2)) \). Thus we can decompose it into a tensor product of two linearly rigid rank 2 tuples.

\[ \square \]

3.3.3 The \( P_5 \) case

**Theorem 3.15** In all \( P_5 \) cases a symplectically rigid tuple \( T \) can be reduced via middle convolution operations, taking tensor products and rational pullbacks to a rank 1 tuple.

Proof:

(i) In the case \( P_5(10, 10, 10, 10, 10) \) the Scott formula implies that

\[
J(T) = ((J(2), 1, 1), (J(2), 1, 1), (J(2), 1, 1), (J(2), 1, 1), (J(2), -1, -1)).
\]

Thus the tuple is linearly rigid, a so called Jordan-Pochhammer tuple.

(ii) In the \( P_5(8, 10, 10, 10, 10) \) case we get a contradiction to the Scott formula.

(iii) The \( P_5(8, 8, 10, 10, 10) \) case is ruled out by the Scott formula.

(iv) In the case \( P_5(8, 8, 8, 10, 10) \) the application of \( MC_{-1} \) yields an orthogonal rank 4 tuple with Jordan forms

\[
((J(2), J(2)), (J(2), J(2)), (J(2), J(2)), (-1, 1, 1, 1), (-1, 1, 1, 1)).
\]

Hence a quadratic pullback can be written as a tensor product of two linearly rigid six tuples of rank 2 with non trivial Jordan forms \((J(2), J(2), -J(2))\) each.
Remark 3.16 In the case $P_3(8, 8, 8, 8, 10)$. Otherwise $S = \text{MC}_{-1}(T)$ yields an orthogonal rank 5 tuple with Jordan forms $J(S_1) = \ldots = J(S_4) = (J(2), J(2), 1)$ and $J(S_5) = (-1, -1, -1, -1, 1)$. Using $A^2\text{Sp}_4 \cong SO_5$ we get a symplectic rank 4 tuple with Jordan forms 

$$(J(2), 1, 1), \quad (J(2), 1, 1), \quad (J(2), 1, 1), \quad (J(2), 1, 1), \quad (-1, -1, 1, 1).$$

But this contradicts the Scott formula.

(vi) In the case $P_3(8, 8, 8, 8, 8)$ we apply $\text{MC}_{-1}$ and obtain an orthogonal tuple $S$ of rank 6 with Jordan forms $J(S_1) = \ldots = J(S_5) = (J(2), J(2), 1, 1)$. Since $A^2\text{SL}_4(\mathbb{C}) = SO_6(\mathbb{C})$ we get a tuple of rank 4 with Jordan forms 

$$(J(2), 1, 1), \quad (J(2), 1, 1), \quad (J(2), 1, 1), \quad (J(2), 1, 1), \quad \pm(iJ(2), i, i)).$$

The linear rigidity yields the claim.

\[\square\]

\textbf{Remark 3.16} In the $P_3(8, 8, 8, 8, 8)$ case the monodromy group $G = \langle T \rangle$ is a finite 2-group of order 32, where $Z(G) = G'$ and $G/G' \cong \mathbb{Z}_2^4$.

4 Translation to differential operators

Let as usual $\frac{d}{dz}$ be the derivation on $\mathbb{C}[z]$ defined by $\frac{d}{dz}(z) = 1$ and $\mathbb{C}[z, \partial] := \mathbb{C}[z][\partial]$ be the ring of differential operators with respect to $\frac{d}{dz}$. An element $P \in \mathbb{C}[z, \partial]$ with singular locus $S \subset \mathbb{C} \cup \{\infty\}$ can be regarded as a linear homogeneous differential equation on $\mathbb{P}^1 \setminus S$. Thus, we can investigate its induced local system $L$ on $\mathbb{P}^1 \setminus S$ with respect to the following conventions.

**Convention** We fix once and for all an orientation on $\mathbb{P}^1$ and denote the winding number of a closed path $\gamma$ around a point $p \in \mathbb{P}^1 \setminus \text{im}(\gamma)$ by $\nu_\gamma(p)$. Furthermore, we denote the singular locus of a differential operator $L \in \mathbb{C}[z, \partial]$ by $S$, if this leads to no confusion. Having chosen an arbitrary base point $x_0 \in \mathbb{P}^1 \setminus \{p\}$ we attach to each $p \in \mathbb{P}^1$ a loop $\gamma_p$ starting at $b$ with $\nu_\gamma(p) = 1$ and $\nu_\gamma(s) = 0$ for all $s \in S \setminus \{p\}$. Then $\{\gamma_s\}_{s \in S}$ is a set of generators of $\pi_1(\mathbb{P}^1 \setminus S, x_0)$ and we equip $S$ with an ordering $S = \{s_1, \ldots, s_r\}$ such that their composition $\prod_{i=1}^{r+1} \gamma_{s_i}$ is homotopic to the trivial loop. We set the **monodromy tuple** associated to $L$ to be 

$$T := (T_1, \ldots, T_{r+1}) := (\rho_L(\gamma_{s_1}), \ldots, \rho_L(\gamma_{s_{r+1}})) \in \text{GL}([L_{x_0}], r+1).$$

We translate the constructions for monodromy tuples used before to the level of differential operators in an appropriate way. Mainly for computational and aesthetical reasons we use the so called **logarithmic derivation** $z^{1/\partial}$ on $\mathbb{C}[z]$ and the ring of differential operators $\mathbb{C}[z, \partial] := \mathbb{C}[z][\partial]$ with respect to $z^{1/\partial}$, which can naturally be regarded as a subring of $\mathbb{C}[z, \partial]$. We call an operator $L = \sum_{i=0}^{m} a_i \partial^i$ with $a_i \in \mathbb{C}[z]$ reduced, if the greatest common divisor of all its coefficients $a_i$ is a unit. The **degree** $\text{deg}(L)$ of $L$ is the maximal $i$ for which $a_i \neq 0$. Rearranging the coefficients, we also may write $L = \sum_{i=0}^{m} z^i P_i$, with $P_i \in \mathbb{C}[\partial]$. Recall, that $P_0$ is the indicial equation of $L$ at $z = 0$ and the roots of $P_0$ - considering $\partial$ as a formal variable - are the **exponents** $E$ of $L$. For each exponent $e$, we have a formal solution $f \in \mathbb{C}[\partial]^e$ of $L$ at $z = 0$, where $\mu \in (e + N_0) \cap E$. We call $\mu$ the **exponent** of the solution $f$. The indicial equation and the exponents of $L$ at the other points $p \in \mathbb{P}^1$ can be obtained in the same way after having performed the transformation $z \mapsto z + p$ or $z \mapsto z^{1/\partial}$. We call $L$ fuchsian, if the degree of its indicial equation at each point $p \in \mathbb{P}^1$ equals $\text{deg}(L)$. This agrees with the usual definition of a fuchsian operator as given in [PS02, Section 6.2]. As by Deligne’s investigations in [Del70] each operator of geometric origin has to be fuchsian, we will perform all constructions with operators of this type.
All local systems in the constructions done before are built up from local systems of the form
\[ \Lambda_\alpha = (1, \alpha^{-1}, \alpha) \]
for \( a \in \mathbb{Q} \) and \( \alpha = \exp(2\pi ia) \) with respect to the points \( \{0, 1, \infty\} \). Thus the basic operators we are dealing with are those of order one, which induce this monodromy tuple.

**Definition 4.1** Let \( a \in \mathbb{Q} \). We set
\[ L_a := \vartheta - z(\vartheta + a) \in \mathbb{C}[z, \vartheta]. \]

**Remark 4.2** The solution space of \( L_a \) is spanned by the formal expression
\[ f = \frac{1}{(1 - z)^a}, \]
which is algebraic over \( \mathbb{Q}(z) \). Thus \( L_a \) is of geometric origin and its induced monodromy tuple is precisely \( \Lambda_\alpha \). Two operators \( L_a \) and \( L_b \) induce the same monodromy tuple if and only if \( a - b \in \mathbb{Z} \).

**4.1 Tensor product** We state the definition of the tensor product of differential operators as it is given in [PS02, Chapter 2] and investigate some basic properties. Let us briefly recall that there is a universal Picard-Vessiot ring \( \mathcal{F} \) of \( (\mathbb{C}[z], z^\mathfrak{d}) \), i.e., for each \( L \in \mathbb{C}[z, \vartheta] \) the set \( \text{Sol}_L := \{ y \in \mathcal{F} \mid L(y) = 0 \} \) can be regarded as a \( \deg(L) \)-dimensional \( \mathbb{C} \) vectorspace. Therefore we call \( \text{Sol}_L \) the solution space of \( L \).

**Definition 4.3** Let \( L_1, L_2 \in \mathbb{C}[z, \vartheta] \) be reduced. The **tensor product** \( L_1 \otimes L_2 \in \mathbb{C}[z, \vartheta] \) of \( L_1 \) and \( L_2 \) over \( \mathbb{C}[z] \) is the reduced operator of minimal degree, whose solution space contains the set \( \{ y_1 y_2 \mid L_1(y_1) = L_2(y_2) = 0 \} \subset \mathcal{F} \).

**Remark 4.4**
(i) We always have \( L_1 \otimes L_2 \in \mathbb{C}[z, \vartheta] \), as the vector space \( V \subset \mathcal{F} \) spanned by \( \{ y_1 y_2 \mid L_1(y_1) = L_2(y_2) = 0 \} \) is set-wise invariant under the natural action of the differential Galois group \( G \) of \( \mathcal{F} \supset \mathbb{C}[z] \). Thus by [PS02, Lemma 2.17] the solution space of \( L_1 \otimes L_2 \) is exactly \( V \).

(ii) We have \( \deg(L_1 \otimes L_2) \leq \deg(L_1) \deg(L_2) \).

(iii) If \( L_2 \) has order one and its solution space is spanned by \( y \in \mathcal{F} \), the solution space of the tensor product \( L_1 \otimes L_2 \) is spanned by \( \{ yy \mid L_1(y) = 0 \} \subset \mathcal{F} \). Thus we write
\[ L_1^\perp := L_1 \otimes L_2 \in \mathbb{C}[z, \vartheta]. \]

(iv) Symmetric and exterior powers of differential operators are defined similarly. For a reduced \( L \in \mathbb{C}[z, \vartheta] \) we set \( \text{Sym}^n(L) \) to be the reduced operator of minimal degree whose solution space is spanned by the set
\[ \{ y_1 \cdots y_n \mid L(y_i) = 0 \text{ for all } i = 1, \ldots, n \} \subset \mathcal{F} \]
and \( \Lambda^n(L) \) to be the reduced operator of minimal degree whose solution space is spanned by the set
\[ \{ \text{Wr}(y_1, \ldots, y_n) \mid L(y_i) = 0 \text{ for all } i = 1, \ldots, n \} \subset \mathcal{F}, \]
where \( \text{Wr} \) denotes the Wronskian
\[ \text{Wr}(y_1, \ldots, y_n) := \det \begin{pmatrix} y_1 & \cdots & y_n \\ \frac{\mathfrak{d}}{\mathfrak{d}z} y_1 & \cdots & \frac{\mathfrak{d}}{\mathfrak{d}z} y_n \\ \vdots & \ddots & \vdots \\ (\frac{\mathfrak{d}}{\mathfrak{d}z})^{n-1} y_1 & \cdots & (\frac{\mathfrak{d}}{\mathfrak{d}z})^{n-1} y_n \end{pmatrix} \]
with respect to the unique extension of \( z^\mathfrak{d} \) to \( \mathcal{F} \).
4.2 Convolution and Hadamard product

Since the solution space of $L_1 \otimes L_2$ is locally isomorphic to a subspace of the tensor product of the solution spaces of $L_1$ and $L_2$, we have the following

**Proposition 4.5** Let $L_1, L_2 \in \mathbb{C}[z, \vartheta]$ be irreducible with singular loci $S_1, S_2 \in \mathbb{C} \cup \{\infty\}$ and induced monodromy tuples $T_1$ and $T_2$ with respect to $b \in \mathbb{P}^1 \setminus \{S_1 \cup S_2\}$. Then the following hold.

(i) The monodromy tuple induced by $L_1 \otimes L_2$ is a direct summand of $T_1 \otimes T_2$.

(ii) The monodromy tuple induced by $\text{Sym}^n L_1$ is a direct summand of $\text{Sym}^n T_1$.

(iii) The monodromy tuple induced by $\Lambda^n L_1$ is a direct summand of $\Lambda^n T_1$.

We especially get

**Corollary 4.6** Let $L \in \mathbb{C}[z, \vartheta]$ be a monic differential operator with induced monodromy tuple $T$, $a \in \mathbb{Q} \setminus \mathbb{Z}$ and $\alpha = \exp(2\pi ia)$. Then the monodromy tuple induced by $L(1 - z)^a = L \otimes L_a$ is precisely $MT_{\Lambda_\alpha}(T)$.

4.2 Convolution and Hadamard product  

In this section we investigate the Hadamard product with local systems of type $\Lambda_\alpha$, where $\alpha \in S^1$, using relations to the convolution with certain local systems of rank one. We rather work with the Hadamard product than with the convolution on the level of differential operators.

We first define for $a \in \mathbb{Q} \setminus \mathbb{Z}$ the convolution of solutions of a fuchsian operator with $z^a$ and the Hadamard product with $(1 - z)^{-a}$, which spans $\text{Sol}_{L_a}$.

**Definition 4.7** Let $L \in \mathbb{C}[z, \vartheta]$ be fuchsian, $f$ a solution of $L$ and $a \in \mathbb{Q} \setminus \mathbb{Z}$.

(i) For two loops $\gamma_p, \gamma_q$ with $\nu_{\gamma_p}(q) = \nu_{\gamma_q}(p) = 0$ we define the **Pochhammer contour**

$$[\gamma_p, \gamma_q] := \gamma_p^{-1} \gamma_q^{-1} \gamma_p \gamma_q.$$

(ii) For $p \in \mathbb{P}^1$, the expression

$$C^p_a(f) := \int_{[\gamma_p, \gamma_q]} f(x)(z - x)^a \frac{dx}{z - x}$$

is called the **convolution** of $f$ and $z^a$ with respect to the Pochhammer contour $[\gamma_p, \gamma_q]$.

(iii) For $p \in \mathbb{P}^1$, the expression

$$H^p_a(f) := \int_{[\gamma_p, \gamma_q]} f(x)(1 - \frac{z}{x})^{-a} \frac{dx}{x}$$

is called the **Hadamard product** of $f$ and $(1 - z)^{-a}$ with respect to the Pochhammer contour $[\gamma_p, \gamma_q]$.

**Remark 4.8** (i) In the sequel, we will frequently use the following formulae for integrals involving Pochhammer contours for $z \notin S$:

(a) \[ \int_{[\gamma_p, \gamma_q]} f(x)dx = \int_{\gamma_q} f(x)dx + \int_{[\gamma_p, \gamma_q]} \rho_L(\gamma_q)(f)(x)dx. \]

(b) \[ \int_{[\gamma_p, \gamma_q, \gamma_r]} f(x)(\lambda - x)^a \frac{dx}{\lambda - x} = C^q_a(f) + C^p_a(\rho_L(\gamma_q)(f)). \]
Proof: Using Leibniz’s rule for differentiating under the integral sign we get
\[ \int_{[\gamma_p, \gamma_z]} f(x)(z - x)^a \frac{dx}{z - x} = -C_a^p (\rho_L (\gamma_p)^{-1}(f)). \]

(ii) If \( f \in (z - p)^\mu \mathbb{C}[z - p] \) near \( z = p \), we get
\[
C_a^p(f) = (1 - \exp(2\pi i \mu)) \int_{\gamma_p} f(x)(z - x)^a \frac{dx}{z - x} + (\exp(2\pi i a) - 1) \int_{\gamma_p} f(x)(z - x)^a \frac{dx}{z - x},
\]
In particular, we have
\[
\int_{\gamma_p} f(x)(z - x)^a \frac{dx}{z - x} = (1 - \exp(2\pi i a)) \int_{x_0}^z f(x)(z - x)^a \frac{dx}{z - x}
\]
and
\[
\int_{\gamma_p} f(x)(z - x)^a \frac{dx}{z - x} = (1 - \exp(2\pi i \mu)) \int_{x_0}^p f(x)(z - x)^a \frac{dx}{z - x},
\]
if \( \mu \) is not a negative integer. Thus we get
\[
C_a^p(f) = (1 - \exp(2\pi i \mu))(1 - \exp(2\pi i a)) \int_{x_0}^z f(x)(z - x)^a \frac{dx}{z - x}.
\]
Note that the right hand side does not depend on the choice of the base point \( x_0 \in \mathbb{P}^1 \setminus S \) and may be interpreted as a meromorphic function near \( z = p \).

(iii) One checks that the convolution and the Hadamard product for a fixed Pochhammer contour \( [\gamma_p, \gamma_z] \) are related by the following formulae
\[
(a) \quad C_a^p(f) = (-1)^{a-1} H_a^p(z^a f).
(b) \quad H_a^p(f) = (-1)^{-a} C_{1-a}^p (z^{a-1} f).
\]
In order to find differential equations having solutions \( C_a^p(f) \), we investigate some properties of the convolution.

Lemma 4.9 Let \( L \in \mathbb{C}[z, \theta] \) be fuchsian, \( f \) a solution of \( L \), \( a \in \mathbb{Q} \setminus \mathbb{Z} \), \( p \in \mathbb{P}^1 \) and \( [\gamma_p, \gamma_z] \) a fixed Pochhammer contour. We have the following relations
\[
(i) \quad \frac{d}{dz} C_a^p(f) = C_a^p \left( \frac{d}{dz} f \right) = (a - 1) C_a^{p-1}(f).
(ii) \quad C_a^p(z f) = z C_a^p(f) - C_a^{p+1}(f).
(iii) \quad C_a^p \left( \frac{d}{dz} f \right) = (z \frac{d}{dz} - a) C_a^p(f).
(iv) \quad C_a^p(x f) = \prod_{j=0}^{a-1} \left( \frac{x - \theta_j}{\lambda_j} - 1 \right) C_a^{p+i}(f).
\]
Proof: Using Leibniz’s rule for differentiating under the integral sign we get
\[
\frac{d}{dz} \int_{[\gamma_p, \gamma_z]} f(x)(z - x)^{a-1} dx = \int_{[\gamma_p, \gamma_z]} f(x) \frac{d}{dz} (z - x)^{a-1} dx = -\int_{[\gamma_p, \gamma_z]} f(x) \frac{d}{dx} (z - x)^{a-1} dx.
\]
As the monodromy of \( f(x)(z - x)^{a-1} \) along \( [\gamma_p, \gamma_z] \) is trivial, integration by parts yields
\[
-\int_{[\gamma_p, \gamma_z]} f(x) \frac{d}{dx} (z - x)^{a-1} dx = \int_{[\gamma_p, \gamma_z]} \left( \frac{d}{dx} f(x) \right) (z - x)^{a-1} dx
\]
and hence the first result. The other results are obtained by direct computation and the results established before.

Using those properties we get the following
Proposition 4.10 Let $L = \sum_{i=0}^{m} z^i P_i(\vartheta) \in \mathbb{C}[z, \vartheta]$ be fuchsian, $f$ a solution of $L$ and $a \in \mathbb{Q} \setminus \mathbb{Z}$. Then $C_a^b(\vartheta)$ is a solution of

$$C_a(L) := \sum_{i=0}^{m} z^i \prod_{j=0}^{i-1} (\vartheta + i - a - j) \prod_{k=0}^{m-i-1} (\vartheta - k) P_i(\vartheta - a)$$

for each $p \in \mathbb{P}^1$.

Proof: For $0 \leq i \leq m$ and $b \in \mathbb{Q} \setminus \mathbb{Z}$ we have

$$C_{b+i}^b(g) = \frac{1}{\prod_{l=1}^{m-i}(b + m - l)} \left( C_{b+m}^{b}(g) \right)^{(m-i)} = z^{i-m} \frac{z^{m-i}}{\prod_{l=1}^{m-i}(b + m - l)} \left( C_{b+m}^{b}(g) \right)^{(m-i)} = z^{i-m} \prod_{k=0}^{m-i-1}(\vartheta - k) C_{b+m}^{b}(g).$$

for each $g$ which is a solution of some $R \in \mathbb{C}[z, \vartheta]$ by Lemma 4.9. Thus

$$0 = C_{b+i}^b(Lf) = \sum_{i=0}^{m} C_{b+i}^{b}(z^i P_i(\vartheta)f) = \sum_{i=0}^{m} \prod_{j=0}^{i-1} \left( \frac{\vartheta}{b + j} - 1 \right) C_{b+i}^{b}(P_i(\vartheta)f)$$

$$= \sum_{i=0}^{m} \prod_{j=0}^{i-1} \left( \frac{\vartheta}{b + j} - 1 \right) \prod_{k=0}^{m-i-1}(\vartheta - k) C_{b+m}^{b}(P_i(\vartheta)f)$$

$$= \sum_{i=0}^{m} \prod_{j=0}^{i-1} \frac{\vartheta + i - m - b - j}{b + j} \prod_{k=0}^{m-i-1}(\vartheta - k) P_i(\vartheta - (b + m)) C_{b+m}^{b}(f)$$

$$= \frac{1}{z^m \prod_{l=1}^{m-i}(b + l)} \sum_{i=0}^{m} \prod_{j=0}^{i-1} \left( \vartheta + i - m - b - j \right) \prod_{k=0}^{m-i-1}(\vartheta - k) P_i(\vartheta - (b + m)) C_{b+m}^{b}(f).$$

Setting $b = a - m$, we get the desired result. 

An approach via so called Euler-integrals can be found in [KSY91], Chapter II.3 and yields a similar operator in $\mathbb{C}[z, \vartheta]$. We use the relations between the convolution and the Hadamard product to obtain an operator having solutions of the form $H_a^b(f)$.

Corollary 4.11 Let $L = \sum_{i=0}^{m} z^i P_i \in \mathbb{C}[z, \vartheta]$ be fuchsian, $f$ a solution of $L$ and $a \in \mathbb{Q} \setminus \mathbb{Z}$. Then $H_a^b(f)$ is a solution of

$$H_a(L) := \sum_{i=0}^{m} z^i \prod_{j=0}^{i-1} (\vartheta + a + j) \prod_{k=0}^{m-i-1} (\vartheta - k) P_i$$

for each $p \in \mathbb{P}^1$.

Note that for an arbitrary fuchsian operator $L$ the monodromy tuple induced by $C_a(L)$, resp. $H_a(L)$, is a subfactor of $MC_\alpha(T)$, resp. $MH_{\alpha^{-1}}(T)$. To induce the tuple $MC_\alpha(T)$ we will restrict ourselves to operators, for which the expression $f(z)(y - z)^{\alpha-1}$ is free of residues with respect to every $y \in \mathbb{P}^1$. This is guaranteed, if the operator $L$ is positive in the following sense.

Definition 4.12 Let $a \in \mathbb{Q} \setminus \mathbb{Z}$. A differential operator $L \in \mathbb{C}[z, \vartheta]$ is called $a$-positive, if $L$ is fuchsian, has no exponents in $\mathbb{Z}_{<0}$ at each point $p \in \mathbb{C}$ and no exponents in $-a + \mathbb{Z}_{\leq 0}$ at $p = \infty$.

The next proposition justifies, that there is an operator in $\mathbb{C}[z, \vartheta]$, whose solution space is spanned by all $C_a^s(f)$, where $f$ is a solution of an $a$-positive operator $L$ and that this operator induces the desired monodromy tuple. As we have $C_a^s(f) = 0$ if $f$ is holomorphic at $p$ by Remark 4.3 we can concentrate ourselves on the expressions $C_a^s(f)$ for $s \in S$. 
Proposition 4.13 Let $a \in \mathbb{Q} \setminus \mathbb{Z}$, $L \subset \mathbb{C}[z, \bar{z}]$ be irreducible, $a$-positive with $\deg(L) = n$, $S = \{s_1, \ldots, s_r, \infty\}$ and $\alpha = \exp(2\pi i a)$. Let furthermore $\{f_1, \ldots, f_n\}$ be a basis of $\text{Sol}_L$,

$$R = (C^s_a(f_1), \ldots, C^s_a(f_n), \ldots, C^n_a(f_1), \ldots, C^n_a(f_n))$$

and

$$V := \{R \cdot v \mid v \in \mathbb{C}^{nr}\}.$$ 

Then the action of $C_\alpha(T)$ on $V$ as described in Section 2.2 is given by $\text{MC}_\alpha(T)$.

**Proof:** Due to [DR07, Section 4] the vector space $V$ is invariant under the action of the monodromy $C_\alpha(T)$. Let $F = (f_1, \ldots, f_n)$, $K_k$ and $L$ as in Section 2.2 and $v = (v_1, \ldots, v_r)^T$, where $v_i \in \mathbb{C}^n$. Since $L$ is $a$-positive, $F \cdot v_k$ is holomorphic at $s_k$ for $v_k \in \ker(T_{s_k} - \text{id})$ and we get $R \cdot v = 0$ for $v \in K_k$. Thus we have

$$\dim_\mathbb{C}(V) \leq \sum_{s \in S \setminus \{\infty\}} \text{rank}(T_s - \text{id}).$$

We can choose for each $z \in \mathbb{P}^1 \setminus S$ a path $\gamma_z$ fulfilling our conventions such that

$$\gamma_{s_1} \cdots \gamma_{s_r} \gamma_{\infty} = 1.$$ 

With respect to the basis $F$ of $\text{Sol}_L$ and letting $C^L_a(T)$ operate on each component of $F$, the elements of the monodromy group of $L$ operate via

$$C^L_a(\rho_i(\gamma_{s_i})(F \cdot v)) = C^L_a(F) \cdot T_{s_i} v = C^L_a(F \cdot T_{s_i} v),$$

for each $v \in \mathbb{C}^n$ and each $0 \leq i \leq r$. Furthermore, by definition the induced monodromy action of the path $\gamma_z$ on the integrand of $C^L_a(f)$ is just given by multiplication with $\alpha$. Using the rules established before, we have

$$\int_{[\gamma_z]} F \cdot v(z-x)^a \frac{dx}{z-x} = \int_{[\gamma_{s_1} \cdots \gamma_{s_r} \gamma_{s_\infty}]} F \cdot v(z-x)^a \frac{dx}{z-x} = \sum_{i=1}^r C^s_a(F) \cdot T_{s_{i+1}} \cdots T_{s_r} \alpha v$$

on the one hand and

$$\int_{[\gamma_{s_1} \cdots \gamma_{s_r}]} F \cdot v(z-x)^a \frac{dx}{z-x} = -C^\infty_a(F) \cdot \alpha T_{s_i}^{-1} v = -C^\infty_a(F) \cdot \alpha T_{s_1} \cdots T_{s_r} v$$

on the other and thus the relation

$$C^\infty_a(F) \cdot \alpha T_{s_1} \cdots T_{s_r} v = -\sum_{i=1}^r C^s_a(F) \cdot \alpha T_{s_{i+1}} \cdots T_{s_r} v.$$ 

for each $v \in \mathbb{C}^n$. As the left hand side is zero for $v_{\infty} \in \ker(\alpha T_{s_1} \cdots T_{s_r} - \text{id})$, rewriting the right hand side yields $R \cdot v = 0$ for each $v \in L$.

Hence we get

$$\dim_\mathbb{C} V \leq \sum_{s \in S \setminus \{\infty\}} \text{rank}(T_s - \text{id}) - (n - \text{rank}(\alpha T_{s_i}^{-1} - \text{id})).$$

By the definition of $\text{MC}_\alpha(T)$ and comparing the dimensions we get the result. \qed

**Remark 4.14** With the notations used in the proposition above and by the relations between the convolution and the Hadamard product, assuming that $L^{1-a}$ is $(1-a)$-positive and setting

$$\tilde{R} = (H^s_a(f_1), \ldots, H^s_a(f_n), \ldots, H^n_a(f_1), \ldots, H^n_a(f_n))$$

and

$$\tilde{V} = \{R \cdot v \mid v \in \mathbb{C}^{nr}\},$$

the action of $H_{a-1}(T)$ on $\tilde{V}$ is given by $\text{MH}_{a-1}(T)$.
As \(C_n(T)\) and \(H_{n-1}(T)\) are induced by a fuchsian systems, their Zariski closures over \(\mathbb{C}\) are isomorphic to the differential Galois groups of the corresponding systems, see e.g. \cite[Corollary 5.2]{PS02}. By the preceding proposition and \cite[Lemma 2.17]{PS02}, there are non trivial differential operators in \(\mathbb{C}[z, \partial]\) whose solution spaces are exactly \(V\), resp. \(\tilde{V}\). This justifies the following definition.

**Definition 4.15** Let \(a \in \mathbb{Q} \setminus \mathbb{Z}\) and \(L \in \mathbb{C}[z, \partial]\) be irreducible.

(i) If \(L\) is \(a\)-positive, the convolution \(L \star_C (\partial - a)\) of \(L\) and \(\partial - a\) is the non trivial reduced operator of minimal degree in \(\mathbb{C}[z, \partial]\) whose solution space contains the set

\[
\bigcup_{p \in \mathbb{P}^1} \{C_p^a(f) \mid f \text{ is a solution of } L\}.
\]

(ii) If \(L^{1-a}\) is \((1-a)\)-positive, the Hadamard product \(L \star_H L_a\) of \(L\) and \(L_a\) is the non trivial reduced operator of minimal degree in \(\mathbb{C}[z, \partial]\) whose solution space contains the set

\[
\bigcup_{p \in \mathbb{P}^1} \{H_p^a(f) \mid f \text{ is a solution of } L\}.
\]

As a consequence of Proposition 4.13 we get

**Corollary 4.16** Let \(L \in \mathbb{C}[z, \partial]\) be irreducible with \(\deg(L) = n\) and singular locus \(S\). Let furthermore \(S = \{0, s_2, \ldots, s_r, \infty\}\), \(a \in \mathbb{Q} \setminus \mathbb{Z}\) and \(\alpha = \exp(2\pi ia)\).

(i) If \(L\) is \(a\)-positive, \(L \star_C (\partial - a)\) is an irreducible fuchsian right factor of \(C_a(L)\) of degree

\[
\deg(L \star_C (\partial - a)) = \sum_{s \in S \setminus \{\infty\}} \text{rank}(T_s^a - \text{id}) - \left(n - \text{rank}(\alpha^{-1}T_\infty^a - \text{id})\right).
\]

Furthermore, its induced monodromy tuple is \(\text{MC}_{\alpha}(T)\).

(ii) If \(L^{1-a}\) is \((1-a)\)-positive, \(L \star_H L_a\) is an irreducible fuchsian right factor of \(\mathcal{H}_a(L)\) of degree

\[
\deg(L \star_H L_a) = \sum_{s \in S \setminus \{0\}} \text{rank}(T_s - \text{id}) - \left(n - \text{rank}(\alpha T_0 - \text{id})\right).
\]

Furthermore, its induced monodromy tuple is \(\text{MH}_{a-1}(T)\).

The degree of the operator \(C_a(L)\), resp. \(\mathcal{H}_a(L)\), is possibly much higher than the degree of \(L \star_C (\partial - a)\), resp. \(L \star_H L_a\). As we know the degrees of \(L \star_C (\partial - a)\), resp. \(L \star_H L_a\), we can try to find those operators by a factorization of \(C_a(L)\), resp. \(\mathcal{H}_a(L)\), into irreducible operators. Such a factorization is in general not unique, but yields a composition series of the solution space \(W\) of the operator with respect to the action of its differential Galois group \(G\), see e.g \cite[Proposition 2.11]{Sin96}. It will turn out that in our cases we always have a factorization

\[
\mathcal{H}_a(L) = \prod_{i=0}^{l} (\partial + c_i)R,
\]

with \(c_1, \ldots, c_l \in \mathbb{C}\) and \(\deg(R) = \deg(L \star_H L_a) > 1\). As then the only \(\deg(L \star_H L_a)\)-dimensional \(G\)-invariant subspace of \(W\) on which \(G\) acts irredicibly is exactly the solution space of \(R\), we have \(R = L \star_H L_a\). In particular, we have the following quite technical

**Proposition 4.17** Let \(a \in \mathbb{Q} \setminus \mathbb{Z}\), \(L = \sum_{i=0}^{m} \sum_{j=0}^{m-1} P_j \in \mathbb{C}[z, \partial]\) be irreducible and \(\{0, \infty\} \subset S\). Let furthermore \(k_0 \in \mathbb{N}\) maximal such that \(\prod_{i=0}^{k_0-1-j} (\partial + a - 1 - i)\) divides \(P_j\) for all \(0 \leq j \leq k_0 - 1\) and \(k_\infty \in \mathbb{N}\) maximal such that \(\prod_{i=0}^{k_\infty-j} (\partial + 1 + i)\) divides \(P_{m-j}\) for all \(0 \leq j \leq k_\infty - 1\). Then
(i) $\mathcal{H}_a(L) = \prod_{i=0}^{k_a-1}(\vartheta + a - 1 - i) \prod_{j=0}^{k_\vartheta-1}(\vartheta - m + 1 + j) R$, with $R \in \mathbb{C}[z, \vartheta]$.

(ii) If $L_{-a}^{\vartheta}$ is $(1 - a)$-positive, the operator $L \ast_H L_a$ is an irreducible right factor of $R$.

(iii) If $L_{-a}^{\vartheta}$ is $(1 - a)$-positive, $m = \sum_{s \in S \setminus \{0, \infty\}} \text{rank}(T_s - \text{id})$, $\text{rank}(\exp(2\pi i a) T_0 - \text{id}) = n - k_0$ and $\text{rank}(T_\infty - \text{id}) = n - k_\vartheta$, we have $R = L \ast_H L_a$.

**Proof:** By Corollary 4.11 we have

$$\mathcal{H}_a(L) = \sum_{i=0}^{m} z^i \prod_{j=0}^{i-1}(\vartheta + a + j) \prod_{k=0}^{m-i-1}(\vartheta - k) P_i.$$  

Since $\mathcal{H}_a(L)$ has a left factor of the form $\vartheta + c$ with $c \in \mathbb{C}$ if and only if $\vartheta + c + i$ divides $\prod_{j=0}^{i-1}(\vartheta + a + j) \prod_{k=0}^{m-i-1}(\vartheta - k) P_i$ for each $0 \leq i \leq m$, we obtain the first part of the statement by a direct computation. The second part is a direct consequence of Corollary 4.16. To prove the third part, note that we have

$$\deg(R) = n + m - k_0 - k_\vartheta = \sum_{s \in S \setminus \{0, \infty\}} \text{rank}(T_s - \text{id}) + \text{rank}(T_\infty - \text{id}) - (n - \text{rank}(\alpha T_0 - \text{id}))$$

by Corollary 4.16. Now the action of the Galois group on the solution space as discussed above yields the result.

A more general treatment of the factorization of $\mathcal{H}_a(L)$ will be discussed in a subsequent article.

**Example 4.18** Let $a, b \in \mathbb{Q} \setminus \mathbb{Z}$. Recall that the monodromy tuple induced by $L_a$, where the singular locus of $L_b$ is extended by the apparent singularity $z = 0$, is given by $T = (T_0, T_1, T_\infty) = (1, \beta^{-1}, \beta)$, where $\beta = \exp(2\pi i b)$. Thus we have $\deg(L_b \ast_H L_a) = 2$ and

$$\mathcal{H}_b(L_a) = \vartheta^2 - z(\vartheta + b)(\vartheta + a) = L_b \ast_H L_a.$$  

Inductively, one shows that

$$L_{a_1} \ast_H L_{a_2} \ast_H \cdots \ast_H L_{a_n} = \vartheta^n - z \prod_{i=1}^{n}(\vartheta + a_i).$$

In particular, each of those operators is of hypergeometric type.

The situation on local systems suggests, that the operation $\mathcal{H}_a$ is invertible. As we will see in the next lemma, this is not exactly the case.

**Lemma 4.19** Let $L = \sum_{i=0}^{m} z^i P_i \in \mathbb{C}[z, \vartheta]$ and $a \in \mathbb{Q} \setminus \mathbb{Z}$. Then

$$\mathcal{H}_{1-a} \left( \mathcal{H}_a(L)^{z^{-a}} \right) = \prod_{k=1}^{m-1}(\vartheta - k) \prod_{j=0}^{m-1}(\vartheta - a - j) L_{z^{-a} \vartheta}.$$  

**Proof:** As we have

$$L_{z^{-a} \vartheta} = \sum_{i=0}^{m} z^i P_i(\vartheta - a)$$

and

$$\mathcal{H}_a(L)^{z^{-a}} = \sum_{i=0}^{m} z^i \prod_{j=0}^{i-1}(\vartheta + j) \prod_{k=0}^{m-i-1}(\vartheta - a - k) P_i(\vartheta - a),$$

we have

$$\mathcal{H}_{1-a} \left( \mathcal{H}_a(L)^{z^{-a}} \right) = \prod_{k=1}^{m-1}(\vartheta - k) \prod_{j=0}^{m-1}(\vartheta - a - j) L_{z^{-a} \vartheta}.$$
we obtain
\[
\mathcal{H}_{1-a} \left( \mathcal{H}_a(L) z^{-a} \right) = \sum_{i=0}^{m} \prod_{k=0}^{m-i-1} (\vartheta - k) \prod_{j=0}^{i-1} (\varphi + j) \prod_{j=0}^{i-1} (\varphi + 1 - a + j) \prod_{k=0}^{m-i-1} (\vartheta - a - k) P_i(\vartheta - a)
\]
\[
= \sum_{i=0}^{m} z^i \prod_{k=1}^{m} (\vartheta - m + k + i) \prod_{j=0}^{m-1} (\varphi - a - j + i) P_i(\vartheta - a)
\]
and hence the result. \(\square\)

Nevertheless, this lemma turns to be quite useful to determine solutions of \(\mathcal{H}_a(L)\) involving logarithms as we will see in the next section.

## 5 Special solutions

The translation of the constructions appearing in Katz’ algorithm to the level of differential operators enables us to compute certain local solutions of a differential operator produced by those constructions in an explicit way. To be more precise, given a fuchsian operator \(L\) which is constructed by tensor and Hadamard products of differential operators of lower order, we are sometimes able to state closed formulae for the coefficients of a local solution of the form \(f = (z - p)^\mu \sum_{m=0}^{\infty} A_m (z - p)^m \in \{z - p\}^\mu \mathbb{C}[z - p]\) at \(z = p \in \mathbb{C}\), resp. \(f = t^\mu \sum_{m=0}^{\infty} A_m t^m \in \{t\}^\mu \mathbb{C}[t]\) for \(t = z - p\). Those solutions will be called special. As stated in the preceding section, if \(f = (z - p)^\mu \sum_{m=0}^{\infty} A_m (z - p)^m\) is a solution of the differential operator \(L\) at \(z = p\) and \(g = (z - p)^\nu \sum_{m=0}^{\infty} B_m (z - p)^m\) is a solution of the differential operator \(\hat{L}\) at \(z = p\), their Cauchy product
\[
f g = (z - p)^{\mu + \nu} \sum_{m=0}^{\infty} \sum_{k=0}^{m} A_k B_{m-k} (z - p)^m
\]
is a solution of \(L \otimes \hat{L}\) at \(z = p\). Analogously, the self Cauchy product \(f^2\) is a solution of \(\text{Sym}^2 L\) at \(z = p\) and setting \(L = \hat{L}\), the Wronskian
\[
\text{Wr}(f, g) = z(z - p)^{\nu + \mu - 1} \sum_{m=0}^{\infty} \sum_{k=0}^{m} (2k + \mu - \nu - m) A_k B_{m-k} (z - p)^m
\]
is a solution of \(\Lambda^2 L\) at \(z = p\). The situation turns out to be slightly more complicated for the middle Hadamard product \(L \star_H L_a\). Classically one defines the Hadamard product \(\star_H\) of two formal power series \(\sum_{m=0}^{\infty} A_m z^m \in \mathbb{C}[z]\) and \(\sum_{m=0}^{\infty} B_m z^m \in \mathbb{C}[z]\) by term-wise multiplication, i.e.
\[
\sum_{m=0}^{\infty} A_m z^m \star_H \sum_{m=0}^{\infty} B_m z^m := \sum_{m=0}^{\infty} A_m B_m z^m.
\]
As the terminology suggests, given a holomorphic solution \(f = \sum_{m=0}^{\infty} A_m z^m\) of \(L\) near \(z = 0\), the expression
\[
\sum_{m=0}^{\infty} (-1)^m \binom{-a}{m} A_m z^m
\]
should be a solution of \(L \star_H L_a\) near \(z = 0\), as we have
\[
(1 - z)^{-a} = \sum_{m=0}^{\infty} (-1)^m \binom{-a}{m} z^m.
\]
The following more general discussion will recover those solutions.

At \(z = p\) the eigenfunctions of the local monodromy of a fuchsian operator \(L\) are elements of \((z - p)^\mu \mathbb{C}[z - p]^\ast\), where \(\exp(2\pi i \mu)\) is the corresponding eigenvalue. For notational convenience, we use the following
Convention Given $E \subset \mathbb{C}$ and two functions $f, g : E \to \mathbb{C}$ we write

(i) $f \hat{=} g$ if there is a $c \in \mathbb{C}^*$ such that $f(z) = cg(z)$ for all $z \in E$.

(ii) $\int_{t=0}^{t=1} f(x)dx$ for the integral of $f$ along the straight line $[0, 1] \to E, \ t \mapsto (1-t)p + tz$, if it exists.

The relation of $C_p^a(f)$ to the line integral given in Remark 4.8 yields the following

**Lemma 5.1** Let $f$ be an eigenfunction of the local monodromy of $L$ at $z = p \in \mathbb{C} \cup \{\infty\}$ and $\mu$ the exponent of $z^{-\mu-1}f$ at $p$. Then we have

$$H_p^a(f) \hat{=} \begin{cases} \int_{t=0}^{t=1} x^{a-1} f((1-t)p + tz)(z-x)^{-\mu}dx, & \mu \notin \mathbb{Z} \\ 0, & \mu \in \mathbb{N}_0 \end{cases}.$$ 

**Proof:** The statement follows directly from Remark 4.8. ☐

Recalling the well-known Beta function

$$\mathcal{B}(p, q) := \int_0^1 x^{p-1}(1-x)^{q-1}dx = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)},$$

which is assumed to be the analytic continuation of the expression on the very right on $\mathbb{C} \setminus \{p+q \in \mathbb{Z}_{<0}\}$, a direct computation shows

**Lemma 5.2**

(i) Let $f = z^\mu \sum_{m=0}^{\infty} A_m z^m$ be an eigenfunction of the local monodromy of $L$ at $z = 0$ with exponent $\mu \notin \mathbb{Z}_-$. Then

$$C_0^a(f) \hat{=} z^{\mu+a} \sum_{m=0}^{\infty} \mathcal{B}(\mu + 1 + m, a) A_m z^m.$$ 

(ii) Let $t = \frac{1}{z}$ and $f = t^\mu \sum_{m=0}^{\infty} A_m t^m$ be an eigenfunction of the local monodromy of $L$ at $z = \infty$ with exponent $\mu \notin a + \mathbb{Z}_-$. Then

$$C_\infty^a(f) \hat{=} t^{\mu-a} \sum_{m=0}^{\infty} \mathcal{B}(\mu - a + m, a) A_m t^m.$$ 

**Proof:**

(i) By Remark 4.8, we have

$$C_0^a(f) \hat{=} \int_0^z (z-x)^{a-1} f(x)dx = \sum_{m=0}^{\infty} A_m \int_0^z (z-x)^{a-1} x^{\mu+m}dx$$

$$= \sum_{m=0}^{\infty} A_m z^{\mu+m+a-1} \int_0^z \left(1 - \frac{x}{z}\right)^{a-1} \left(\frac{x}{z}\right)^{\mu+m}dx$$

$$= \sum_{m=0}^{\infty} A_m z^{\mu+m+a} \int_0^1 (1-s)^{a-1} s^{\mu+m}ds$$

and thus the result.

(ii) We obtain the result similarly to the first part starting with

$$C_\infty^a(f) \hat{=} \int_0^t x^{-a-1} (1-xz)^{a-1} f \left(\frac{1}{x}\right)dx.$$ 

☐
Combining those statements yields

**Proposition 5.3**  
(i) Let \( f \) be an eigenfunction of the local monodromy of \( L \) at \( z = p \in \mathbb{C} \). Let furthermore \( z^{a-1}f = (z-p)^\mu \sum_{m=0}^{\infty} A_m(z-p)^m \). Then

\[
H^p_a(f) = \begin{cases} 
(z-p)^{\mu+1-a} \sum_{m=0}^{\infty} \mathcal{R}(\mu + 1 + m, 1 - a)A_m(z-p)^m, & \mu \notin \mathbb{Z} \\
0, & \mu \in \mathbb{N}_0
\end{cases}
\]

In particular, if \( L^{z^{-\nu}} \) is \((1-a)\)-positive each \( \mathbb{C}\)-multiple of the right hand side is a solution of \( L \circ H L_a \) near \( z = p \).

(ii) Let \( t = \frac{1}{z} \) and \( f \) be an eigenfunction of the local monodromy of \( L \) at \( z = \infty \). Let furthermore \( t^{1-a}f(t) = t^\mu \sum_{m=0}^{\infty} A_m t^m \). Then

\[
H^\infty_a(f) = \begin{cases} 
t^{\mu+a-1} \sum_{m=0}^{\infty} \mathcal{R}(\mu + 1 + m, 1 - a)A_m t^m, & \mu \notin \mathbb{Z} \\
0, & \mu \in \mathbb{N}_0
\end{cases}
\]

In particular, if \( L^{z^{-\nu}} \) is \((1-a)\)-positive each \( \mathbb{C}\)-multiple of the right hand side is a solution of \( L \circ H L_a \) near \( z = \infty \).

**Proof:**  
As seen before, we have

\[
H^p_a(f) = \int_p^x x^{a-1}f(x)(z-x)^{-a}dx = \int_0^{z-p} (x+p)^{a-1}f(x+p)(z-p-x)^{-a}dx
\]

for \( p \in \mathbb{C} \). Thus the result follows from Lemma 5.2 and Lemma 5.1. The case \( p = \infty \) can be treated similarly. \( \square \)

**Remark 5.4** If \( f \) is holomorphic at \( z = 0 \), one recovers the Hadamard product of formal power series mentioned in the introduction of the section. More generally, if \( L \) has at \( z = 0 \) a solution of the form \( f = z^\nu \sum_{m=0}^{\infty} A_m z^m \) we get the solution

\[
g = z^\nu \sum_{m=0}^{\infty} \mathcal{R}(\nu + a + m, 1 - a)A_m z^m = z^\nu \sum_{m=0}^{\infty} R(m)\mathcal{R}(a + m, 1 - a)A_m z^m
\]

of \( L \circ H L_a \) at \( z = 0 \). Using Stirling’s formula, one can show that \( R(m) \) behaves asymptotically like \( \left( \frac{z^{a+1}+m}{a+m} \right)^{a-1} \).

Proposition 5.3 implies that each special solution \( f \) of \( L \) for which \( z^{a-1}f \) is not a meromorphic eigenfunction near \( z = p \) induces a special solution of \( L \circ H L_a \), while the solutions \( g \) for which \( z^{a-1}g \) is holomorphic at \( z = p \) do not contribute to the solution space of \( L \circ H L_a \). Nevertheless, the following proposition asserts that solutions of the form \( \ln g + r \) with \( r \in \mathbb{C}[z] \) induce certain holomorphic solutions of \( L \circ H L_a \).

**Proposition 5.5**  
(i) Let \( L \) be irreducible and both functions

\[
z^{a-1}f = (z-p)^\mu \sum_{m=0}^{\infty} A_m(z-p)^m
\]

and \( z^{a-1}g \) holomorphic at \( z = p \). Let furthermore \( \ln \) be a branch of the logarithm at \( z = 0 \), \( \ln(z-p)f + g \) a solution of \( L \) at \( z = p \) and \( a \in \mathbb{Q} \setminus \mathbb{Z} \). Then

\[
H^p_a(f) = (z-p)^{\mu+1-a} \sum_{m=0}^{\infty} \mathcal{R}(\mu + 1 + m, 1 - a)A_m(z-p)^m.
\]
(ii) Let \( L \) be irreducible \( t = \frac{1}{z} \) and both functions
\[
\tau^{1-a} f = t^u \sum_{m=0}^{\infty} A_m t^m
\]
and \( \tau^{1-a} g \) holomorphic at \( t = 0 \). Let furthermore \( \ln \) be a branch of the logarithm at \( t = 0 \), \( \ln f + g \) a solution of \( L \) at \( t = 0 \) and \( a \in \mathbb{Q} \setminus \mathbb{Z} \). Then
\[
H_a^\infty(f) = t^{\mu + a - 1} \sum_{m=0}^{\infty} \mathfrak{R}(\mu + 1 + m, 1 - a) A_m t^m.
\]

**Proof:** Let \( \tilde{f} = z^{a-1} f \) and \( \tilde{g} = z^{a-1} g \). As the formal monodromy of \( \ln(z - p) \) around \( \gamma_p \) is given by \( \ln(z - p) + 2\pi i \), evaluating \( H^\infty_a(\ln(z - p)f + g) \) yields
\[
H^\infty_a(\ln(z - p)f + g) = C_{1-a}^p \left( \ln(z - p)\tilde{f} + \tilde{g} \right)
= \int_{[\gamma_p, \gamma_1]} \ln(x - p)\tilde{f}(x)(z - x)^{-a}dx
= -2\pi i \int_{[\gamma_p, \gamma_1]} \tilde{f}(x)(z - x)^{-a}dx + (\exp(-2\pi ia) - 1) \int_{[\gamma_p, \gamma_1]} \ln(x - p)\tilde{f}(x)(z - x)^{-a}dx
= -2\pi i(1 - \exp(-2\pi i a)) \int_{\gamma_p} \tilde{f}(x)(z - x)^{-a}dx - 2\pi i(\exp(-2\pi i a) - 1) \int_{\gamma_1} \tilde{f}(x)(z - x)^{-a}dx
= -2\pi i(1 - \exp(-2\pi i a)) \int_{\gamma_p} \tilde{f}(x)(z - x)^{-a}dx,
\]
hence the result by Lemma 5.1 and Lemma 5.2. The second case can be treated analogously. \( \square \)

Combining this result with Lemma 4.19 we get

**Lemma 5.6** Let \( L \in \mathbb{C}[\vartheta, z] \), \( a \in \mathbb{Q} \setminus \mathbb{Z} \), \( p \not\in \{0, \infty\} \), \( f = (z - p)^\mu \sum_{m=0}^{\infty} A_m (z - p)^m \in \mathbb{C}[z - p] \), \( r \in \mathbb{C}[z - p] \) and \( \ln(z - p)f + r \) a solution of \( H_a(L) \) at \( z = p \). Then

(i) \( h = z^{1-a}\left(z - p\right)^{a+\mu-1} \sum_{m=0}^{\infty} \mathfrak{R}(\mu + 1 + m, a - 1) A_m (z - p)^m \)

is a solution of \( L \) at \( z = p \).

(ii) \( H_a^\infty(h) \equiv f \).

**Proof:** By Proposition 5.5 and Lemma 4.19 the expression
\[
g = (z - p)^{\mu + a} \sum_{m=0}^{\infty} \mathfrak{R}(\mu + 1 + m, a) A_m (z - p)^m
\]
is a solution of \( \prod_{k=1}^{m-1} (\vartheta - k) \prod_{j=0}^{m-1} (\vartheta - a - j) L^{z^-a} \vartheta \) at \( z = p \).
As \( p \) is no singularity of \( \prod_{k=1}^{m-1} (\vartheta - k) \prod_{j=0}^{m-1} (\vartheta - a - j) \) and \( \mu + a \not\in \mathbb{Z} \), we have \( L^{z^-a} \vartheta(g) = 0 \).
Thus
\[
z \frac{d}{dz} g = (z - p)^{\mu + a - 1} \sum_{m=0}^{\infty} \mathfrak{R}(\mu + 1 + m, a - 1) A_m z^m
\]
is a solution of \( L^{z^-a} \) and we obtain the first part of the statement. Setting \( h = z^{1-a} \frac{d}{dz} g \) we get
\[
z^{a-1} h = \frac{d}{dz} g = (z - p)^{a+\mu-1} \sum_{m=0}^{\infty} \mathfrak{R}(\mu + 1 + m, a - 1) A_m z^m.
\]
Thus Proposition 5.3 yields
\[ H^r_n(h) \cong (z-p)^\mu \sum_{m=0}^{\infty} R(\mu + a + m, 1 - a)R(\mu + 1 + m, a - 1)A_m(z-p)^m \]
\[ \cong (z-p)^\mu \sum_{m=0}^{\infty} A_m(z-p)^m = f. \]

Rephrasing the lemma above, at a singular point \( p \not\in \{0, \infty\} \) the special holomorphic solutions \( f \) are those, which induce solutions of the form \( \ln(z-p)^f + r \), where \( r \in \mathbb{C}[z-p] \). In the geometric context solutions of this type turn out to be interesting as indicated in [CilOGP98, Appendix B] or [vEvS04, Chapter 6].

6 Construction of Calabi-Yau operators

In this section, we combine the results of the preceding sections to construct families of irreducible fuchsian differential operators inducing monodromy tuples of type \( P_1 \) and \( P_2 \). We will also compute special solutions of those operators at some of the singular points explicitly. Next, we investigate which of the operators constructed in the first part seem to be Calabi-Yau in the sense of [AESZ10]. As recently uncovered in [GvG10], unlike the definition of a Calabi-Yau operator given in [AESZ10], there are families of Calabi-Yau threefolds, hence also Calabi-Yau operators, having no point of maximally unipotent monodromy. However, we restrict ourselves to the classical case of having such a point. In particular, the families \( P_i \) for \( i \geq 3 \) cannot be induced by an operator corresponding to such a classical family. All operators we find using this method are covered by [AESZ10, Appendix A], but in most of the cases we are unfortunately not able to show, whether the operators are Calabi-Yau.

In the sequel, we will use the notations introduced in the preceding sections. Let furthermore \( t = \frac{1}{2} \). As we have seen before, the construction of monodromy tuples of type \( P_1 \) and \( P_2 \) splits into four cases, each of which we cover by the preceding theorems. Furthermore, we only construct those operators \( L \) for which zero is the only exponent at \( z = 0 \) and choose the singular locus of \( L \) to be \( S = \{0, 1, \infty\} \). We collect the remaining exponents \( \lambda_{1,1}, \ldots, \lambda_{4,1} \) at \( z = 1 \) and \( \lambda_{1,\infty}, \ldots, \lambda_{4,\infty} \) at \( z = \infty \) in its Riemann-scheme

\[
R(L) = \begin{cases}
0 & 1 & \infty \\
0 & \lambda_{1,1} & \lambda_{1,\infty} \\
0 & \lambda_{2,1} & \lambda_{2,\infty} \\
0 & \lambda_{3,1} & \lambda_{3,\infty} \\
0 & \lambda_{4,1} & \lambda_{4,\infty}
\end{cases}
\]

In all occurring cases, the Jordan forms of the local monodromies can be read off directly from the Riemann scheme, as only repeated exponents turn out to induce logarithms. Proofs of those statements which can be obtained directly using the methods established before are omitted. For the sake of clarity, we frequently use well known hypergeometric identities as stated in [Bai35] without any further comment.

**Theorem 6.1 (The \( P_1(4, 10, 4) \) case)** Let \( a, b \in \mathbb{Q} \setminus \mathbb{Z} \). A two parameter family of operators inducing monodromy tuples of type \( P_1(4, 10, 4) \) is given by

\[
P^{(a,b)}_1(4,10,4) := L_a \star_H L_{1-a} \star_H L_{b} \star_H L_{1-b} \]
\[= \partial^4 - z(\partial + a)(\partial + 1 - a)(\partial + b)(\partial + 1 - b).\]
The Riemann scheme reads
\[
\mathcal{R} \left( \mathcal{P}_{\gamma}^{(a,b)}(4,10,4) \right) = \begin{pmatrix} 0 & 1 & \infty \\ 0 & 0 & a \\ 0 & 1 & 1-a \\ 0 & 1 & b \\ 0 & 2 & 1-b \end{pmatrix}.
\]

Special solutions of this operator are \( f = \sum_{m=0}^{\infty} A_m z^m \) at \( z = 0 \), where
\[
A_m = \binom{a+m-1}{m} \binom{b+m-1}{m} \binom{m-b}{m},
\]
\[
g = (z-1) \sum_{m=0}^{\infty} B_m(z-1)^m,
\]
where
\[
B_m = \frac{b-1}{m+1} \binom{1+m-b}{m} \sum_{l=0}^{m} (-1)^l \binom{-b}{m-l} \frac{3F_2 \left( \binom{a-l, 1-a}{1, b-l} ; 1 \right)}{b-l-1}.
\]

and \( h_\gamma = t^\gamma \sum_{m=0}^{\infty} C_m^{(\gamma)} t^m \) at \( z = \infty \), where \( \gamma \in E = \{ a, 1-a, b, 1-b \} \) and
\[
C_m^{(\gamma)} = \mathcal{B}(\gamma + m, 1-a) \mathcal{B}(\gamma + m, a) \mathcal{B}(\gamma + m, 1-b) \mathcal{B}(\gamma + m, b).
\]

Moreover, \( g \) is the conifold-period of \( \mathcal{P}_1^{(a,b)}(4,10,4) \) at \( z = 1 \), i.e. there is an \( r \in (z-1)\mathbb{C}[z-1] \) such that \( \ln(z-1)g + r \) is a solution of \( \mathcal{P}_1^{(a,b)}(4,10,4) \) at \( z = 1 \).

**Proof:** It is clear that \( L_a \star_H L_{1-a} \star_H L_b \star_H L_{1-b} \) induces a monodromy tuple of type \( P_1(4,10,4) \). As in Example 6.13 we get
\[
L_a \star_H L_{1-a} \star_H L_b \star_H L_{1-b} = H_{1-b}(H_b(H_{1-a}(L_a))) = \vartheta^4 - z(\vartheta + a)(\vartheta + 1 - a)(\vartheta + b)(\vartheta + 1 - b).
\]
The formulae for \( A_m, B_m \) and \( C_m^{(\gamma)} \) can be obtained directly using Proposition 6.5 and exchanging the roles of \( a, 1-a, b \) and \( 1-b \) freely. It remains to show, that \( g \) is the conifold-period at \( z = 1 \).

As \( e = 1 \) is an exponent of multiplicity two at \( z = 1 \), the method of Frobenius yields a solution \( \ln(z-1)\tilde{g} + r \) of \( \mathcal{P}_1^{(a,b)}(4,10,4) \) at \( z = 1 \), where \( \tilde{g} \in (z-1)\mathbb{C}_E[z-1] \) and \( r \in (z-1)\mathbb{C}_E[z-1] \). Applying the first statement of Lemma 6.3 yields a solution \( \omega \in (z-1)^{1-b} \mathbb{C}_E[z-1] \) of \( L_a \star_H L_{1-a} \star_H L_b \). As \( 1-b \) is the only exponent of \( L_a \star_H L_{1-a} \star_H L_b \) at \( z = 1 \) lying in \( -b + \mathbb{Z} \), we have \( \omega \equiv H_b^1(H_{1-a}((1-z)^{-a})) \). Applying the second statement of Lemma 6.3 yields
\[
\tilde{g} \equiv H_b^1(-b(\omega) \equiv H_b^1(H_b^1(H_{1-a}((1-z)^{-a}))) \equiv g.
\]

\[
\square
\]

**Theorem 6.2 (The \( P_1(4,8,4) \) case)** Let \( a \in \mathbb{Q} \setminus \left( \frac{1}{4} + \mathbb{Z} \cup \frac{3}{4} + \mathbb{Z} \right) \) and \( b \in \mathbb{Q} \setminus \left( \frac{1}{4} + \mathbb{Z} \cup \frac{3}{4} + \mathbb{Z} \right) \). A two parameter family of operators inducing monodromy tuples of type \( P_1(4,4,8) \) is given by
\[
\mathcal{P}_1^{(a,b)}(4,8,4) := \Lambda^2 \left( L_{a+\frac{1}{2}} \star_H L_{\frac{1}{2}} \right) \star_H L_{b+\frac{1}{2}} \star_H L_{\frac{1}{2}} = 64 \vartheta^4 + z \left( -128 \vartheta^4 - 256 \vartheta^3 + \vartheta^2 (128(a^2 + b^2) - 304) \right)
\]
\[
+ z \left( \vartheta (128(a^2 + b^2) - 176) + 48(a^2 + b^2) + 256 a^2 b^2 - 39 \right)
\]
\[
+ 64z^2 (a + 1 + \vartheta - b) (a + 1 + \vartheta + b) (a - 1 - \vartheta - b) (-1 + a + \vartheta + b).
\]

The Riemann scheme reads

\[
\mathcal{R} \left( \mathcal{P}_1^{(a,b)}(4,8,4) \right) = \begin{pmatrix}
0 & 1 & \infty \\
0 & -\frac{1}{2} & 1 - a - b \\
0 & 0 & 1 + a - b \\
0 & 1 & 1 - a + b \\
0 & \frac{3}{2} & 1 + a + b
\end{pmatrix}.
\]

Special solutions of this operator are given by \( f = \sum_{m=0}^{\infty} A_m z^m \) at \( z = 0 \) with

\[
A_m = \left( \frac{1}{2} + m \right) \sum_{k=0}^{m} \left( 2k - \frac{1}{2} - m \right) \alpha \left( \frac{1}{2}, k \right) \alpha(0, m - k),
\]

where

\[
\alpha(\nu, m) := \mathcal{B} \left( \frac{3}{4} + a + \nu + m, \frac{1}{4} - a \right) \mathcal{B} \left( \frac{3}{4} - a + \nu + m, \frac{1}{4} + a \right)
\]

\[
\mathcal{B} \left( \frac{3}{4} + b + \nu + m, \frac{3}{4} - b \right) \mathcal{B} \left( \frac{3}{4} - b + \nu + m, \frac{3}{4} + b \right)
\]

and \( h(\mu, \nu) = t^{\mu + \nu} \sum_{m=0}^{\infty} C_m^{(\mu, \nu)} t^m \) at \( z = \infty \), where

\[
C_m^{(\mu, \nu)} = \mathcal{B} \left( \nu + \mu + m, \frac{1}{2} \right) \sum_{k=0}^{m} (2k + \mu - \nu - m) \delta(\mu, k) \delta(\nu, m - k),
\]

with

\[
\delta(\mu, k) := \mathcal{B} \left( \mu - \frac{1}{4} + k, \frac{3}{4} - a \right) \mathcal{B} \left( \mu - \frac{1}{4} + k, \frac{3}{4} + a \right)
\]

\[
\mathcal{B} \left( \mu + \frac{1}{4} + k, \frac{1}{4} - b \right) \mathcal{B} \left( \mu + \frac{1}{4} + k, \frac{1}{4} + b \right)
\]

for \( \mu \in \left\{ \frac{1}{2} + a, \frac{1}{2} - a \right\} \) and \( \nu \in \left\{ \frac{1}{2} + b, \frac{1}{2} - b \right\} \).

**Theorem 6.3 (The \( P_2(4,6,6) \) case)** Let \( a, b \in \mathbb{Q} \setminus \mathbb{Z} \). A two parameter family of operators inducing monodromy tuples of type \( P_2(4,6,6) \) is given by

\[
\mathcal{P}_2^{(a,b)}(4,6,6) := \text{Sym}^2 \left( (L_a \ast L_b)^{(1-z)} \overline{\text{Sym}^2 H L_\frac{1}{2}} \right)
\]

\[
= 4 \vartheta^4 - 2 z (2 \vartheta + 1)^2 \left( \vartheta^2 + \vartheta + 2 a b - a + 1 - b \right)
\]

\[
- z^2 (2 \vartheta + 3) (2 \vartheta + 1) (b - 1 - a - \vartheta) (b + 1 - a + \vartheta).
\]

The Riemann scheme reads

\[
\mathcal{R} \left( \mathcal{P}_2^{(a,b)}(4,6,6) \right) = \begin{pmatrix}
0 & 1 & \infty \\
0 & 0 & \frac{1}{2} \\
0 & 1 & \frac{3}{2} \\
0 & -\frac{1}{2} + a + b & 1 + a - b \\
0 & \frac{3}{2} - a - b & 1 - a + b
\end{pmatrix}.
\]
Special solutions of this operator are \( f = \sum_{m=0}^{\infty} A_m z^m \) at \( z = 0 \), where
\[
A_m = \left( m - \frac{1}{2} \right) \sum_{k=0}^{m} \binom{a + k - 1}{k} \binom{b + k - 1}{k} \binom{m - k - a}{k} \binom{m - k - b}{k},
\]
g_{(a,b)} and \( g_{(1-a,1-b)} \), where \( g_{(a,b)} = (z - 1)^{\frac{1}{2} - a - b} \sum_{m=0}^{\infty} B_m^{(a,b)} (z - 1)^m \), with
\[
B_m^{(a,b)} = \mathcal{B} \left( 2 - a - b + m, \frac{1}{2} \right) \sum_{l=0}^{m} \mathcal{B} (1 - b + l, 1 - a) \alpha(l) \left( \frac{-1}{m - l} \right) \left( a - 1 \right)
\]
and
\[
\alpha(l) = 3 \hat{F}_2 \left( \frac{-l}{2} - b, \frac{b}{2} \left( b + l + 1 \right) - 1 \right) \left( \frac{-1}{l} \right) \left( a + b + l + 1 \right).
\]

**Theorem 6.4 (The \( P_2(4,6,8) \) case)** Let \( a, b \in \mathbb{Q} \setminus \mathbb{Z} \). A two parameter family of operators inducing monodromy tuples of type \( P_2(4,6,8) \) is given by
\[
\mathcal{P}_2^{(a,b)}(4,6,8) := (L_a \ast_H L_a)(1-z)^{-a} \ast_H L_b \ast_H L_{1-b}
\]
\[
= \partial^2 - z(\partial + b)(\partial + 1 - b)(2\partial^2 + 2\partial + a^2 - a + 1)
+ z^2(\partial + b)(\partial + 1 - b)(\partial + b + 1)(\partial + 2 - b).
\]

The Riemann scheme reads
\[
\mathcal{R} \left( \mathcal{P}_2^{(a,b)}(4,6,8) \right) = \begin{cases}
0 & 1 & \infty \\
0 & 0 & b \\
0 & 1 & 1 - b \\
0 & a & 1 + b \\
0 & 1 - a & 2 - b
\end{cases}.
\]

Special solutions of this operator are \( f = \sum_{m=0}^{\infty} A_m z^m \) at \( z = 0 \), where
\[
A_m = \left( b + m - 1 \right) \binom{m - b}{m} \sum_{k=0}^{m} \binom{a + m - k - 1}{m - k} \binom{m - k}{k}^2 \binom{k - a}{k}
\]
and \( g_\gamma = (z - 1)^\gamma \sum_{m=0}^{\infty} B_m^{(\gamma)} (z - 1)^m \), where
\[
B_m^{(\gamma)} = \mathcal{B} (1 + \gamma - b + m, b) \sum_{l=0}^{m} (-1)^l \alpha(l) \mathcal{B} (1 - b + l, -\gamma) \left( \frac{-b}{m - l} \right) \left( \frac{l - 1 + \gamma}{\gamma - 1} \right),
\]
with \( \gamma \in \{ a, 1 - a \} \), where
\[
\alpha(l) = 3 \hat{F}_2 \left( \frac{-l}{2} - \frac{1}{2} \left( b + l + 1 \right), \frac{b}{2} \left( b + l + 1 \right) - 1 \right).
\]
Now we investigate which of the operators constructed before are differential Calabi-Yau operators in the spirit of [AESZ10]. We first recall the definition of those objects, which still is quite conjectural and state some of their basic properties. From the geometric point of view, the solutions of a Calabi-Yau operator of order $n$ should correspond to periods of a family of Calabi-Yau manifolds of dimension $n-1$ with Picard number one. In this sense, Calabi-Yau operators should be special Picard-Fuchs operators, which can’t be defined from the differential algebraic point of view in a proper way yet. According to our definition, Calabi-Yau operators respect common conjectures for a differential operator to be Picard-Fuchs, see e.g. [KZ01]. Some of the arithmetic conditions for a differential operator to be Calabi-Yau are basically motivated by approaches of mirror symmetry as discussed in [CdlOGP98], but still seem to be quite mysterious.

**Definition 6.5** For $n \geq 2$, an irreducible operator $L = \partial^n + \sum_{i=0}^{n-1} a_i \partial^i \in \mathbb{Q}(z)[\partial]$ is called a **Calabi-Yau operator** if it satisfies the following conditions. 

(CY-1) The point $z = 0$ is a regular singularity of $L$ and zero is the only exponent at this point.

(CY-2) $L$ has a solution $y_0$ which is $N$-integral at $z = 0$, i.e. at $z = 0$ it is of the form

$$y_0 = 1 + \sum_{m=1}^{\infty} A_m z^m \in \mathbb{Q}[z],$$

with $N^m A_m \in \mathbb{Z}$ for each $m \geq 1$ and a fixed $N \in \mathbb{N}$.

(CY-3) We have $L \alpha = \alpha L^*$ for a non trivial solution $\alpha$ of the differential equation $\omega' = -\frac{2}{n} a_{n-1} \omega$. Here $L^* = \partial^n + \sum_{i=0}^{n-1} (-1)^{n-i} \partial^i a_i \in \mathbb{C}(z)[\partial]$ denotes the dual operator of $L$.

(CY-4) There is a solution $y_1$ linearly independent of $y_0$ given in (CY-3), such that the differential equation

$$\omega' = \left( \frac{y_1}{y_0} \right) \omega$$

has a non trivial solution $q \in z + z^2 \mathbb{Q}[z]$ at $z = 0$ which is $N$-integral. Such a solution is often called the $q$-coordinate or special coordinate of $L$ at $z = 0$.

By the construction done in Theorems 6.1-6.3, we get

**Lemma 6.6** Each of the operators $P_1^{(a,b)}(4,10,4), P_1^{(a,b)}(4,8,4), P_2^{(a,b)}(4,6,8)$ and $P_2^{(a,b)}(4,6,6)$ constructed in Theorem 6.1-6.3 fulfills the properties (CY-1)-(CY-3).

**Proof:** Property (CY-1) can be read off the corresponding Riemann scheme directly. Using [DGS94, Theorem I.4.3] and [DGS94, Formula II.4.6], one shows that the unique solution at $z = 0$ lying in $1 + \mathbb{Q}[z]$ of each operator is $N$-integral. Finally, condition (CY-3) can be obtained by a direct computation. \hfill $\Box$

It remains to investigate which of the operators fulfill property (CY-4). Although there have recently been many improvements in the technique of showing this property, see e.g. [KR11] and [KR08], we are in most of the cases not able to decide whether condition (CY-4) holds or not. Let us point out that for each operator constructed here it is also possible to compute a solution of the form $\ln(z) y_0 + y_1$ by taking $\left. \frac{d}{dn} y_0 \right|_{\mu=0}$ of the holomorphic solution $y_0 = \sum_{\mu=0}^{\infty} f(\mu) z^\mu$ as it
is described in [Inc56, Chapter 16] but that we are often not able to check, whether the criterion
[KR10, Proposition 4.1] holds or the series can be treated by a specialization of [KR08, Theorem 2].

Our investigations lead to the following

**Conjecture** An $\text{Sp}_4(\mathbb{C})$-rigid tuple consisting of quasi-unipotent elements and having a maximally
unipotent element is induced by a differential Calabi-Yau operator if and only if the elements of its
second exterior power lie up to simultaneous conjugation in $\text{SO}_5(\mathbb{Z})$. Furthermore, the inducing
operator is unique.

In the sequel we state which of the cases in each of the families correspond to operators listed
in [AESZ10, Appendix A] and refer to the number of the operator stated there. Note that the
operators constructed here have singular locus $\{0, 1, \infty\}$, so we get the corresponding operators
after having performed a transformation of the form $z \mapsto \lambda z$ with $\lambda \in \mathbb{Q}^*$, which leaves the
properties (CY-1)-(CY-4) untouched and changes the singular locus to $\{0, \frac{1}{\lambda}, \infty\}$. It is remarkable
that after having performed the transformation the coefficients of the $q$-coordinate are minimal
over $\mathbb{Z}$, meaning that they are all lying in $\mathbb{Z}$ and there is no $\alpha \in \mathbb{Z}$ such that $\alpha^m$ divides the
$m$-th coefficient for each $m \in \mathbb{N}$. Furthermore for each series of operators the transformation can be
done uniformly. Let therefore in the sequel for $a = \frac{r}{s}$, where $r \in \mathbb{Z}$ and $s \in \mathbb{N}$ are coprime,

$$
\beta : \mathbb{Q} \setminus \{0\} \to \mathbb{Z}, \quad a \mapsto s \prod_{i=1}^{n} s_i^{-\frac{1}{s_i}},
$$

where $s_1, \ldots, s_n$ denote the distinct prime divisors of $s$.

(i) **The $P_1(4, 10, 4)$ case:**

Having performed the transformation $z \mapsto \beta(a)^2 \beta(b)^2 z$, we get the following Calabi-Yau operators

| a | $\frac{3}{7}$ | $\frac{1}{7}$ | $\frac{3}{11}$ | $\frac{1}{11}$ | $\frac{3}{13}$ | $\frac{1}{13}$ | $\frac{3}{17}$ | $\frac{1}{17}$ | $\frac{3}{19}$ | $\frac{1}{19}$ | $\frac{1}{29}$ |
|---|---|---|---|---|---|---|---|---|---|---|---|
| b | $\frac{1}{7}$ | $\frac{3}{7}$ | $\frac{1}{7}$ | $\frac{3}{7}$ | $\frac{1}{7}$ | $\frac{3}{7}$ | $\frac{1}{7}$ | $\frac{3}{7}$ | $\frac{1}{7}$ | $\frac{3}{7}$ | $\frac{1}{7}$ |
| Number | 3 | 5 | 6 | 14 | 4 | 11 | 8 | 10 | 12 | 13 | 1 |

(ii) **The $P_1(4, 8, 4)$ case:**

To make our observations more transparent, we substitute $c = 2a + \frac{1}{7}$ and $d = 2b + \frac{1}{7}$. Having
performed the transformation $z \mapsto 4\beta(c)^2 \beta(d)^2 z$, we get the following Calabi-Yau operators

| c | $\frac{1}{7}$ | $\frac{1}{7}$ | $\frac{1}{7}$ | $\frac{1}{7}$ | $\frac{1}{7}$ | $\frac{1}{7}$ | $\frac{1}{7}$ | $\frac{1}{7}$ | $\frac{1}{7}$ | $\frac{1}{7}$ | $\frac{1}{7}$ |
|---|---|---|---|---|---|---|---|---|---|---|---|
| d | $\frac{1}{7}$ | $\frac{1}{7}$ | $\frac{1}{7}$ | $\frac{1}{7}$ | $\frac{1}{7}$ | $\frac{1}{7}$ | $\frac{1}{7}$ | $\frac{1}{7}$ | $\frac{1}{7}$ | $\frac{1}{7}$ | $\frac{1}{7}$ |
| Number | 3 | 5 | 6 | 14 | 4 | 11 | 8 | 10 | 12 | 13 | 1 |

where the number $\tilde{i}$ refers to the operators defined in [Alm06]. As shown there, those
operators are equivalent to 206 – 219 in [AESZ10].
Note that the operator

\[ Q_{2}^{c,d} := \left( \Lambda^{2} p_{1}^{c,d}(4,8) \right)^{z^{-1}(1-z)^{-2}} \]

is of hypergeometric type. Its Riemann scheme reads

\[ \mathcal{R}(Q_{2}^{c,d}) = \begin{cases} 
0 & 1 & \infty \\
0 & 0 & \frac{1}{2} \\
0 & 1 & c \\
0 & \frac{3}{2} & d \\
0 & 2 & 1 - c \\
0 & 3 & 1 - d 
\end{cases} . \]

This family contains elements whose induced monodromy group is not a subgroup of \( \text{Sp}_{4}(\mathbb{Z}) \).

(iii) The \( P_{2}(4,6,6) \) case:
Having performed the transformation \( z \mapsto 4 \beta(a) \beta(b) z \), we get the following Calabi-Yau operators

| a   | \( \frac{1}{4} \) | \( \frac{1}{3} \) | \( \frac{1}{2} \) | \( \frac{1}{3} \) | \( \frac{1}{4} \) | \( \frac{1}{5} \) | \( \frac{1}{6} \) | \( \frac{1}{7} \) |
|-----|------------------|------------------|---------------|---------------|---------------|---------------|---------------|---------------|
| b   | \( \frac{1}{4} \) | \( \frac{1}{5} \) | \( \frac{1}{6} \) | \( \frac{1}{7} \) | \( \frac{1}{8} \) | \( \frac{1}{9} \) | \( \frac{1}{10} \) | \( \frac{1}{11} \) |
| Number | 3\ast | 6\ast | 14\ast | 4\ast | 4\ast | 8\ast | 8\ast |

The case \( a = \frac{1}{4} \) and \( b = \frac{1}{4} \) is not listed here, since the corresponding operator is \( \text{Sym}^3 \) of a second order operator.

(iv) The \( P_{2}(4,6,8) \) case:
Having performed the transformation \( z \mapsto \beta(a)^2 \beta(b)^2 z \), we get the following Calabi-Yau operators

| a   | \( \frac{1}{4} \) | \( \frac{1}{5} \) | \( \frac{1}{6} \) | \( \frac{1}{7} \) | \( \frac{1}{8} \) | \( \frac{1}{9} \) | \( \frac{1}{10} \) | \( \frac{1}{11} \) |
|-----|------------------|------------------|---------------|---------------|---------------|---------------|---------------|---------------|
| b   | \( \frac{1}{4} \) | \( \frac{1}{5} \) | \( \frac{1}{6} \) | \( \frac{1}{7} \) | \( \frac{1}{8} \) | \( \frac{1}{9} \) | \( \frac{1}{10} \) | \( \frac{1}{11} \) |
| Number | 111 | 110 | 30 | 112 | 141 | 142 | 196 | 143 |

| a   | \( \frac{1}{4} \) | \( \frac{1}{5} \) | \( \frac{1}{6} \) | \( \frac{1}{7} \) | \( \frac{1}{8} \) | \( \frac{1}{9} \) | \( \frac{1}{10} \) | \( \frac{1}{11} \) |
|-----|------------------|------------------|---------------|---------------|---------------|---------------|---------------|---------------|
| b   | \( \frac{1}{4} \) | \( \frac{1}{5} \) | \( \frac{1}{6} \) | \( \frac{1}{7} \) | \( \frac{1}{8} \) | \( \frac{1}{9} \) | \( \frac{1}{10} \) | \( \frac{1}{11} \) |
| Number | 189 | 194 | 197 | 199 | 190 | 195 | 198 | 61 |
A Appendix: Subgroup structure of the $\text{Sp}_4(\mathbb{C})$

We give an overview of the maximal irreducible subgroups in $\text{Sp}_4(\mathbb{C})$ and their behaviour under taking the exterior product.

**Lemma A.1** The maximal semisimple connected subgroups of $\text{Sp}_4(\mathbb{C})$ are contained in one of the following classes.

(i) $(\text{Sp}_2(\mathbb{C}) \times \text{Sp}_2(\mathbb{C})).2$, 
(ii) $\text{GL}_2(\mathbb{C}).2 \cong \text{Sp}_2(\mathbb{C}) \otimes \text{GO}_2(\mathbb{C})$.
(iii) $\text{Sym}^3\text{SL}_2(\mathbb{C})$, where $2$ denotes a group extension of degree 2.

**Proof:** A maximal connected semisimple subgroup $G$ of $\text{Sp}_4(\mathbb{C})$ can be written as a product $G = G_1 \cdots G_r$ of simple groups $G_i$. Hence $G_i$ is either a torus or $\text{Sp}_2(\mathbb{C})$. Since the Lie-rank of $\text{Sp}_4(\mathbb{C})$ is two we get $r \leq 2$. This gives the claim, cf. [Car85, Chap. 1].

**Corollary A.2** Two classes of the maximal irreducibles subgroups in $\text{Sp}_4(\mathbb{C})$ become reducible in $\text{SO}_5(\mathbb{C})$ taking their antisymmetric square.

\[ \Lambda^2(\text{Sp}_2(\mathbb{C}) \otimes \text{GO}_2(\mathbb{C})) = \langle (A, B) \in \text{GO}_2(\mathbb{C}) \times \text{GO}_2(\mathbb{C}) \mid \det(A) \det(B) = 1 \rangle \]
\[ \Lambda^2(\text{Sp}_2(\mathbb{C}) \times \text{Sp}_2(\mathbb{C})).2 = \text{GO}_4(\mathbb{C}) \],

where $\text{GO}_4(\mathbb{C})$ is naturally embedded into $\text{SO}_5(\mathbb{C})$.

**Proof:** The claims follow from the identities.

\[ \Lambda^2(V_1 \otimes V_2) = \Lambda^2(V_1) \otimes \text{Sym}^2 V_2 \oplus \text{Sym}^2 V_1 \otimes \Lambda^2 V_2, \]
\[ \Lambda^2(V_1 \oplus V_2) = \Lambda^2(V_1) \oplus V_1 \otimes V_2 \oplus \Lambda^2 V_2. \]

**Corollary A.3** Let $H$ be an irreducible proper subgroup of $\text{Sp}_4(\mathbb{C})$. Then the following hold (up to conjugation of $H$).

(i) If $H$ contains a unipotent element with Jordan form $J(4)$ then $H \subseteq \text{Sym}^4\text{Sp}_2(\mathbb{C})$.
(ii) If $H$ contains a transvection then $H \subseteq (\text{Sp}_2(\mathbb{C}) \times \text{Sp}_2(\mathbb{C})).2$.
(iii) If all non trivial unipotent elements in $H$ have the Jordan form $(J(2), J(2))$ then $H \subseteq \text{SL}_2 \otimes \text{GO}_2(\mathbb{C})$.

**Proof:** The claims follow from Lemma A.1.
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