On the occurrence of Hecke eigenvalues in sectors

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Abstract
Let π be a non-self-dual unitary cuspidal automorphic representation not of solvable polyhedral type for GL(2) over a number field. We show that π has a positive upper Dirichlet density of Hecke eigenvalues in any sector whose angle is at least 2.63 radians.

Keywords Automorphic L-functions · Hecke eigenvalues · Automorphic representations

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1 Introduction
Let π be a unitary cuspidal automorphic representation for GL(2) over a number field F. We assume that it is not of solvable polyhedral type, which means that it does not correspond to an Artin representation of dihedral, tetrahedral, or octahedral type. Associated to a finite place v where π is unramified, we have the multiset of Satake parameters \{αv(π), βv(π)\} and their sum is called the Hecke eigenvalue \(a_v(\pi)\) of π at v.

One can ask about the distribution of the sequence \((a_v(\pi))_v\). If one restricts to automorphic representations that correspond to holomorphic forms, then more is known. For example the Sato-Tate conjecture has been proved for a wide range of Hilbert modular forms [1]. In the general case however, much less is known. For example, in an appendix to [9], J.-P. Serre asked if, for self-dual π, it can be shown that there are infinitely many Hecke eigenvalues greater than a given positive constant \(c\) (and similarly, if there are infinitely many Hecke eigenvalues less than a given negative constant \(c'\)). An answer to this was provided by Theorem 1.2 of [11] with \(c = 0.905\) and \(c' = -1.164\).
In the case of when $\pi$ is a non-self-dual, one can extend the question as follows: For what angle $\theta$ can it be shown that there are infinitely many Hecke eigenvalues in any sector of size $\theta$? Furthermore, given any such sector, for what $c$ do we have infinitely many Hecke eigenvalues greater than size $c$?

A consequence of Theorem 1.3 of [11] is that this holds true for $\theta = \pi$ radians, with $c = 0.5$. In this paper, we will improve the value of $\theta$ to 2.63 radians and improve $c$ to 0.595.

**Theorem 1.1** Let $\pi$ be a non-self-dual unitary cuspidal automorphic representation for $\text{GL}(2)/F$, where $F$ is a number field, that is not of solvable polyhedral type. Then for any angle $\phi$ we have that the following set of places

$$\{ v \mid \arg(a_v(\pi)) \in (\phi - 1.314, \phi + 1.314) \}$$

has positive upper Dirichlet density. Furthermore, the subset of such places whose associated Hecke eigenvalue has a size of at least 0.595 also has positive upper Dirichlet density.

2 **Asymptotic properties of certain Dirichlet series**

In this section, we assume that $\pi$ is a cuspidal automorphic representation for $\text{GL}(2)/F$ that is not self-dual and not of solvable polyhedral type.

**Notation** Denote by $X = X(\pi)$ the set of archimedean places as well as places at which $\pi$ is ramified. Values of $k$ will be associated to our examination of the asymptotic behaviour of

$$\sum_{v \notin X} \text{Re}(e^{i\phi} a_v(\pi))^k N_v^{-s}$$

as $s \to 1^+$, for $k = 3, 4, 6, 8$, where $\phi$ is any fixed angle in $[0, 2\pi)$. Let $\omega$ be the central character of $\pi$ and denote the order of this character by $r$. Lastly, we will write $\ell(s) := \log(1/(s - 1))$.

We will repeatedly make use of the bounds towards the Ramanujan conjecture of Kim–Sarnak [4] (in the rational case) and Blomer–Brumley [2] (for number fields). We will also need the functoriality results of Gelbart–Jacquet [3], Kim–Shahidi [5,6], and Kim [4], regarding the symmetric square, cube, and fourth power lifts of cuspidal automorphic representations for $\text{GL}(2)$.

2.1 **$k = 3$**

We consider incomplete $L$-functions of the form $L^X(s, \pi^m \times \overline{\pi^n})$ where $m, n$ are non-negative integers and $m + n = 3$. 

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In the case \((m, n) = (2, 1)\), making use of Clebsch–Gordan decompositions and the unitary of \(\pi\), we obtain

\[
L^X(s, \pi \times \pi \times \pi) = L^X\left(s, \text{Sym}^3\pi \otimes \omega^{-1}\right) L^X(s, \pi)^2,
\]

where \(\omega\) is the central character of \(\pi\). Taking logarithms and using the bounds towards the Ramanujan conjecture [2, 4] we obtain

\[
\sum_{v \notin X} a_v(\pi)^2 a_v(\pi) \frac{1}{N v^s} = O(1)
\]
as \(s \to 1^+\).

Using a similar approach for the other cases, we see that the same asymptotic behaviour occurs for \(\sum_{v \notin X} a_v(\pi)^m a_v(\pi) \bar{a}_v(\pi) N v^{-s}\) for \((m, n) = (3, 0), (1, 2),\) and \((0, 3)\). Therefore, for any \(\phi \in [0, 2\pi)\),

\[
\sum_{v \notin X} \text{Re}\left(\frac{e^{i\phi} a_v(\pi)}{N v^s}\right)^3 = 1
\]

\[
= \frac{1}{23} \left(\sum_{v \notin X} \frac{e^{3i\phi} a_v(\pi)^3}{N v^s} + 3 \sum_{v \notin X} \frac{e^{i\phi} a_v(\pi)^2 a_v(\pi)}{N v^s} + 3 \sum_{v \notin X} \frac{e^{-i\phi} a_v(\pi) a_v(\pi)^2}{N v^s} + \sum_{v \notin X} \frac{e^{-3i\phi} a_v(\pi)^3}{N v^s}\right)
\]

\[
= O(1)
\]
since each of the four series on the right-hand side is bounded as \(s \to 1^+\).

2.2 \(k = 4\)

Using the same approach as in the \(k = 3\) case, we find that

\[
L^X(s, \pi \times \pi \times \pi \times \pi) = L^X\left(s, \text{Sym}^4\pi\right) L^X\left(s, \text{Sym}^2\pi \otimes \omega\right)^3 L^X\left(s, \omega\right)^2.
\]

Therefore, if \(\pi\) has central character of order two,

\[
\sum_{v \notin X} a_v(\pi)^4 N v^{-s} = 2\ell(s) + O(1)
\]
as \(s \to 1^+\). If not, then the series is bounded in that limit.

We also note

\[
L^X(s, \pi \times \pi \times \pi \times \pi) = L^X\left(s, \text{Sym}^4\pi \otimes \omega^{-1}\right) L^X\left(s, \text{Sym}^2\pi\right)^3 L^X(s, \omega)^2,
\]
which then implies
\[
\sum_{v \notin X} a_v(\pi)^3 \overline{a_v(\pi)} N v^{-s} = O(1)
\]
as \( s \to 1^+ \), since \( \pi \) is not self-dual. We similarly obtain
\[
\sum_{v \notin X} a_v(\pi)^2 \overline{a_v(\pi)}^2 N v^{-s} = 2\ell(s) + O(1),
\]
and we conclude
\[
\sum_{v \notin X} \text{Re}(e^{i\phi} a_v(\pi))^4 N v^{-s} = q_4 \cdot \ell(s) + O(1),
\]
where
\[
q_4 = q_4(\pi, \phi) = \begin{cases} 
\frac{3+4\cos 4\phi}{4} & \text{if } r = 2, \\
\frac{3}{4} & \text{if } r \geq 3.
\end{cases}
\]

### 2.3 \( k = 6 \)

Note that the incomplete \( L \)-function \( L^X(s, \pi^m \times \overline{\pi}^n) \), for non-negative integers \( m + n = 6 \), can be expressed as

\[
L^X\left(s, \text{Sym}^3 \pi \times \text{Sym}^3 \pi \otimes \omega^{-n}\right) L_{\chi}(s, \text{Sym}^3 \pi \times \pi \otimes \omega^{1-n})^4 \\
\cdot L^X\left(s, \pi \times \pi \otimes \omega^{2-n}\right)^4
\]

and also as

\[
L^X\left(s, \text{Sym}^4 \pi \times \text{Sym}^2 \pi \otimes \omega^{-n}\right) L_{\chi}(s, \text{Sym}^4 \pi \otimes \omega^{1-n}) \\
\cdot L^X\left(s, \left(\text{Sym}^2 \pi \otimes \omega^{1-n}\right) \times \text{Sym}^2 \pi\right)^3 L_{\chi}(s, \text{Sym}^2 \pi \otimes \omega^{2-n})^5 L^X\left(s, \omega^{3-n}\right)^2.
\]

The first and third \( L \)-functions in Eq. (2.1) are either invertible at \( s = 1 \) or have a simple pole there. The second \( L \)-function is invertible at \( s = 1 \). Therefore, Eq. (2.1) either is invertible at \( s = 1 \), or has a pole of order 1, 4, or 5 there. The third and fifth \( L \)-functions in (2.2) are either invertible at \( s = 1 \) or have a simple pole there. The rest are invertible at \( s = 1 \). Therefore, at \( s = 1 \) Eq. (2.2) is either invertible there or has a pole of order 2, 3, or 5. So \( L^X(s, \pi^m \times \overline{\pi}^n) \) is either invertible at \( s = 1 \) or has a pole of order 5. In the latter case, this holds if and only if \( \omega^{3-n} = 1 \).
If \( r = 2 \), then the incomplete \( \mathcal{L} \)-function \( \mathcal{L}(s, \pi \times m \times \bar{\pi} \times n) \) has a pole of order 5 exactly when \( n = 1, 3, 5 \). If \( r = 3 \), then this \( \mathcal{L} \)-function has a pole of order 5 exactly when \( n = 0, 3, 6 \). If \( r \geq 4 \), then it has a pole of order 5 only when \( n = 3 \).

Denote by \( \alpha_v(\pi) \) and \( \beta_v(\pi) \) the Satake parameters of \( \pi \) at \( v \). Taking logarithms and applying the known bounds on the size of the Satake parameters, we obtain:

\[
\sum_{v \notin X} \sum_{t=1,2} \frac{(\alpha_v(\pi)^t + \beta_v(\pi)^t)^6 \omega_v^{-tn}}{t N v^{st}} = \begin{cases} 
5 \ell(s) + O(1) & \text{if } r = 2 \text{ and } n = 1, 3, 5, \\
& \text{if } r = 3 \text{ and } n = 0, 3, 6, \\
& \text{or if } r \geq 4 \text{ and } n = 3. \\
O(1) & \text{if } r = 2 \text{ and } n = 0, 2, 4, 6, \\
& \text{if } r = 3 \text{ and } n = 1, 2, 4, 5, \\
& \text{or if } r \geq 4 \text{ and } n \neq 3. 
\end{cases}
\]

So for any angle \( \phi \in [0, 2\pi) \),

\[
\frac{1}{26} \sum_{n=0}^{6} C_n \sum_{v \notin X} \sum_{t=1,2} \frac{(\alpha_v(\pi)^t + \beta_v(\pi)^t)^6 \omega_v^{-tn}}{t N v^{st}} e^{i(6-2n)\phi} = \begin{cases} 
\frac{5}{16} (3 \cos 4\phi + 5) \cdot \ell(s) + O(1) & \text{if } r = 2, \\
\frac{5}{32} (\cos 6\phi + 10) \cdot \ell(s) + O(1) & \text{if } r = 3, \\
\frac{25}{16} \ell(s) + O(1) & \text{if } r \geq 4, 
\end{cases}
\] (2.3)

as \( s \to 1^+ \).

We also note that the left-hand side of Eq. (2.3) above is equal to

\[
\frac{1}{26} \sum_{v \notin X} \sum_{t=1}^{2} \frac{(\alpha_v(\pi)^t + \beta_v(\pi)^t)^6}{t N v^{st}} (e^{i\phi} + \omega_v^{-t} e^{-i\phi})^6.
\]

Since

\[
(\alpha_v(\pi)^t + \beta_v(\pi)^t) \left(e^{i\phi} + \omega_v^{-t} e^{-i\phi}\right)
= (\alpha_v(\pi)^t + \beta_v(\pi)^t) e^{i\phi} + (\bar{\alpha}_v(\pi)^t + \bar{\beta}_v(\pi)^t) e^{-i\phi}
\]

we know that

\[
\sum_{v \notin X} \frac{(\alpha_v(\pi)^2 + \beta_v(\pi)^2)^6}{N v^{2s}} (e^{i\phi} + \omega_v^{-2} e^{-i\phi})^6
\]

is non-negative. We conclude

\[
\sum_{v \notin X} \frac{\text{Re} (\alpha_v(\pi))^6}{N v^{s}} \leq q_6 \cdot \ell(s) + O(1). \tag{2.4}
\]
where we can choose
\[ q_6 = q_6(\pi) = \begin{cases} 
\frac{5}{2} & \text{if } r = 2, \\
\frac{55}{32} & \text{if } r = 3, \\
\frac{25}{16} & \text{if } r \geq 4.
\end{cases} \]

2.4 \( k = 8 \):

For non-negative integers \( m + n = 8 \), we have
\[
L^X(s, \pi^m \times \pi^n) = L^X(s, \pi^8 \otimes \omega^{-n}) = L^X(s, \text{Sym}^4\pi \times \text{Sym}^4\pi \otimes \omega^{-n}) L^X(s, \text{Sym}^2\pi \times \text{Sym}^2\pi \otimes \omega^{-2n})^9 \\
\cdot L^X(s, \omega^{4-n})^4 \\
\cdot L^X(s, \text{Sym}^2\pi \times \text{Sym}^2\pi \otimes \omega^{-1-n})^6 L^X(s, \text{Sym}^4\pi \otimes \omega^{-n})^4 \\
\cdot L^X(s, \text{Sym}^2\pi \otimes \omega^{-3-n})^{12}.
\] (2.5)

\( L^X(s, \text{Sym}^4\pi \times \text{Sym}^4\pi \otimes \omega^{-n}) \) has a simple pole at \( s = 1 \) when \( n = 4 \). If it has a pole for other values of \( n \), then either it means that \( \text{Sym}^4\pi \) admits a self-twist, or \( \omega \) has order less than or equal to four. Since there is no known characterisation of when \( \text{Sym}^4\pi \) admits a self-twist, we examine different cases in terms of the possible order of the central character. If we assume that \( L^X(s, \text{Sym}^4\pi \times \text{Sym}^4\pi \otimes \omega^{-n}) \) has a simple pole at \( s = 1 \), then

\[
\text{Sym}^4\pi \otimes \omega^{-n} \simeq \text{Sym}^4\pi.
\]

Considering the central characters of each side, we obtain \( \omega^{10-5n} = \omega^{-10} \) and so \( \omega \) has order dividing \((20 - 5n)\).

At this stage, we consider all the different possible pairs of values of \((r, n)\) for which the incomplete \( L \)-function \( L^X(s, \text{Sym}^4\pi \times \text{Sym}^4\pi \otimes \omega^{-n}) \) may have a (simple) pole. We mention a few cases explicitly here: If \( r = 2 \), then \( L^X(s, \text{Sym}^4\pi \times \text{Sym}^4\pi \otimes \omega^{-n}) \) has a simple pole when \( n \) is even and is invertible otherwise. If \( r = 3 \), then the \( L \)-function has a pole exactly when \( n = 1, 4, 7 \). If \( r = 4 \), there is a pole exactly when \( n = 0, 4, 8 \), and if \( r = 5 \), we cannot rule out the existence of a pole for any value of \( n \).

For \( L^X(s, \text{Sym}^2\pi \times \text{Sym}^2\pi \otimes \omega^{-n}) \), we note that Theorem 2.2.2 of [6] states that for non-dihedral \( \pi \), the adjoint lift of \( \pi \) admits a self-twist if and only if \( \text{Sym}^2\pi \) is not cuspidal. However, we have assumed that \( \pi \) is not of solvable polyhedral type which means that its symmetric cube lift must be cuspidal, so its adjoint lift, and thus its symmetric square lift, cannot admit a non-trivial self-twist. We now consider the cases of the different values of \( r \): If \( r = 2 \), then the \( L \)-function has a pole when \( n \)
is even and is invertible otherwise. If \( r = 3 \), the \( L \)-function has a pole exactly when \( n = 1, 4, 7 \). If \( r = 4 \), the \( L \)-function has a pole exactly when \( n = 0, 4, 8 \). Lastly, if \( r \geq 5 \), then the \( L \)-function only has a pole when \( n = 4 \).

For \( L^X(s, \omega^{4-n}) \), the analysis has the exact same outcomes as in the paragraph directly above.

Finally, we note that the last three \( L \)-functions in Eq. (2.5), namely,

\[
L^X(s, \text{Sym}^4 \pi \times \text{Sym}^2 \pi \otimes \omega^{1-n}), \\
L^X(s, \text{Sym}^4 \pi \otimes \omega^{2-n}), \text{ and} \\
L^X(s, \text{Sym}^2 \pi \otimes \omega^{3-n}),
\]

are all invertible at \( s = 1 \).

We consider

\[
A(n, r) := \sum_{v \not\in X} \sum_{t=1}^{8} \frac{(\alpha_v(\pi)^t + \beta_v(\pi)^t)^8 \omega_v^{-tn}}{tn^{st}}.
\] (2.6)

If \( n = 0 \), then from the discussion above on the possible existence (and order) of poles at \( s = 1 \) of the various \( L \)-functions, we find that Eq. 2.6 is bounded as \( s \to 1^+ \) when \( r \neq 2, 4, 5, 10, 20 \). In the case where \( r = 2 \) or 4, we have

\[
A(n, r) = 14 \cdot \ell(s) + O(1),
\]

and in the case where \( r = 5, 10, \) or 20, we have

\[
A(n, r) \leq \ell(s) + O(1).
\]

We proceed similarly in considering other values of \( n \) and \( r \), recording the asymptotic behaviour of \( A(n, r) \) in the table below:

| \( n \)   | \( r \)  | \( A(n,r) \)          |
|----------|---------|-----------------------|
| 0 or 8   | 2, 4    | \( 14\ell(s) + O(1) \) |
|          | 5, 10, 20 | \( \leq \ell(s) + O(1) \) |
|          | otherwise | \( O(1) \)         |
| 1 or 7   | 3       | \( 14\ell(s) + O(1) \) |
|          | 5, 15   | \( \leq \ell(s) + O(1) \) |
|          | otherwise | \( O(1) \)         |
| 2 or 6   | 2       | \( 14\ell(s) + O(1) \) |
|          | 5, 10   | \( \leq \ell(s) + O(1) \) |
|          | otherwise | \( O(1) \)         |
| 3 or 5   | 5       | \( \leq \ell(s) + O(1) \) |
|          | otherwise | \( O(1) \)         |
| 4        | all     | \( 14\ell(s) + O(1) \) |
We can use the above to establish asymptotic bounds on

\[
\sum_{n=0}^{8} \sum_{v \not\in X}^{8} \sum_{t=1}^{8} C_n \frac{(\alpha_v(\pi)^t + \beta_v(\pi)^t)^8 \omega_v^{-tn}}{tN_v^{st}} e^{i(8-2n)\phi}.
\]

We scale the left-hand side of equation above by \(1/2^8\) and use positivity to obtain

\[
\sum_{v \not\in X}^{8} \frac{(\text{Re} (a_v(\pi)e^{i\phi}))^8}{N_v^s} \leq \sum_{v \not\in X}^{8} \sum_{t=1}^{8} \frac{(\text{Re} (\alpha_v(\pi)^t e^{i\phi} + \beta_v(\pi)^t e^{i\phi}))^8}{tN_v^{st}} e^{i(8-2n)\phi}
\]

\[
\leq q_8 \cdot \ell(s) + O(1), \quad (2.7)
\]
as \(s \to 1^+\), where

\[
2^8 \cdot q_8 = \begin{cases} 
1792 & \text{if } r = 2, \\
1204 & \text{if } r = 3, \\
1008 & \text{if } r = 4, \\
1166 & \text{if } r = 5, \\
1038 & \text{if } r = 10, \\
996 & \text{if } r = 15, \\
982 & \text{if } r = 20, \\
980 & \text{otherwise.}
\end{cases}
\]

Remark 1 These bounds appear to be best possible given current knowledge; in particular, there is no known characterisation for when a symmetric fourth power lift from GL(2) admits a self-twist (in contrast to, say, the symmetric square and cube cases, which are well-understood). For context, if we assumed the Ramanujan conjecture, then for any \(r \geq 6\) with \(r \neq 10, 15, 20\), the left-hand side of equation \((2.7)\) would have a lower bound of \((980/2^8) \cdot \ell(s) + O(1)\).

3 Bounding subsets of Hecke eigenvalues

First we recall that the upper and lower Dirichlet densities of a set \(S\) of places of a number field \(F\) are defined as

\[
\delta(S) = \lim \sup_{s \to 1^+} \frac{\sum_{v \in S} N_v^{-s}}{\log(1/(s - 1))}
\]

and

\[
\delta(S) = \lim \inf_{s \to 1^+} \frac{\sum_{v \in S} N_v^{-s}}{\log(1/(s - 1))},
\]
respectively, and note that these are equal if and only if the set has a Dirichlet density \( \delta(S) \).

### 3.1 Absolute value of Hecke eigenvalues

The following lemma simply arises from adjusting the proof of Theorem 4.1 from [6].

**Lemma 3.1** Let \( Q = r/s \geq 2 \) be a rational number, where \( r, s \) are positive integers. Then, for any unitary cuspidal automorphic representation \( \pi \) for \( GL(2) \) over a number field, we have

\[
\delta\{ v \mid |a_v(\pi)| > Q \} \leq \frac{1}{1 + (Q^2 - 1)^2 + (Q^4 - 3Q^2 + 1)^2}.
\]

**Proof** If \( \pi \) is of solvable polyhedral type, then we know that it corresponds to an Artin representation [7,10] and therefore satisfies the Ramanujan conjecture, so the inequality holds. If \( \pi \) is not of solvable polyhedral type, we know that its adjoint and symmetric fourth power lifts are cuspidal. We construct the following isobaric automorphic representation

\[
\eta = s^4 \alpha \cdot 1 \boxplus s^4 \beta \cdot \text{Ad} \pi \boxplus s^4 \gamma \cdot (\omega^{-2} \otimes \text{Sym}^4 \pi),
\]

where \( \alpha, \beta, \gamma \) are non-negative integers whose values will be determined later. Now

\[
a_v(\eta) = s^4 \alpha + s^4 \beta \left(|a_v(\pi)|^2 - 1\right) + s^4 \gamma \left(|a_v(\pi)|^4 - 3|a_v(\pi)|^2 + 1\right).
\]

If \( |a_v(\pi)| > Q \geq 2 \), then \( a_v(\eta) > s^4(\alpha + \beta(Q^2 - 1) + \gamma(Q^4 - 3Q^2 + 1)) \).

For some automorphic representation \( \mu \) and non-negative real number \( t \), define \( T(\mu, t) \) to be the set of finite places \( v \) at which \( |a_v(\mu)| > t \). From [8] we know that

\[
\delta(T(\mu, t)) \leq \frac{-\text{ord}_{s=1} L(s, \mu \times \bar{\mu})}{t^2}
\]

Therefore, since \( v \in T(\pi, Q) \Rightarrow v \in T(\eta, s^4\alpha + s^4\beta(Q^2 - 1)+s^4\gamma(Q^4 - 3Q^2 + 1)) \), we have

\[
\delta(T(\pi, Q)) \leq \frac{s^8(\alpha^2 + \beta^2 + \gamma^2)}{(s^4\alpha + s^4\beta(Q^2 - 1)+s^4\gamma(Q^4 - 3Q^2 + 1))^2}
\]
Choose $\alpha = 1$, $\beta = Q^2 - 1$, and $\gamma = Q^4 - 3Q^2 + 1$ to get
\[
\bar{\delta}(T(\pi, Q)) \leq \frac{1}{1 + (Q^2 - 1)^2 + (Q^4 - 3Q^2 + 1)^2}.
\]

\[\square\]

3.2 Case of central characters of order at least 6

From here on, we assume that $\pi$ is non-self-dual and not of solvable polyhedral type, and we will fix an angle $\phi \in [0, 2\pi)$. We will also make use of the notations $q_4$, $q_6$, and $q_8$ from Sect. 2.2, 2.3, and 2.4, respectively. Later in the proof we will make the distinction between the cases $r < 6$ and $r \geq 6$.

Let
\[
A = A(\pi) := \left\{ v \notin X \mid \Re\left( a_v(\pi)e^{-i\phi}\right) > 0 \right\},
\]
\[
B = B(\pi) := \left\{ v \notin X \mid \Re\left( a_v(\pi)e^{-i\phi}\right) \leq 0 \right\}.
\]

Given a set $S$ of finite places and a non-negative integer $t$, we establish the notation
\[
ls(S, t) := \limsup_{s \to 1^+} \left( \sum_{v \in S} \left( \frac{\Re\left( a_v(\pi)e^{-i\phi}\right)^t N_v^{-s}}{\log(1/(s-1))} \right) \right)
\]
and similarly
\[
li(S, t) := \liminf_{s \to 1^+} \left( \sum_{v \in S} \left( \frac{\Re\left( a_v(\pi)e^{-i\phi}\right)^t N_v^{-s}}{\log(1/(s-1))} \right) \right).
\]

We also note the following identities that will be referred to later:
Given real-valued functions $f$, $g$ and a point $w \in \mathbb{R}$, we have
\[
\limsup_{s \to w} (f(s) + g(s)) \geq \limsup_{s \to w} f(s) + \liminf_{s \to w} g(s) \geq \liminf_{s \to w} (f(s) + g(s)).
\]

(3.1)

Furthermore, if $f$ and $g$ are non-negative functions, then
\[
\limsup_{s \to w} (f(s) \cdot g(s)) \leq \limsup_{s \to w} f(s) \cdot \limsup_{s \to w} g(s),
\]

(3.2)

and if $f = g$, then Eq. (3.2) is an equality.
From Sect. 2.2 we have that $\text{ls}(\Sigma_F - X, 4) = \text{li}(\Sigma_F - X, 4) = q_4$, where $\Sigma_F$ is the set of places of $F$. Applying Eq. (3.1), we have

$$\text{li}(A, 4) = q_4 - \text{ls}(B, 4).$$

We set $d := \text{ls}(B, 4)$. Define

$$S = S(\beta) := \{ v \in A \mid \left( \text{Re}(a_v(\pi)e^{-i\phi}) \right)^4 > (q_4 - d)\beta \},$$

for some constant $\beta \leq 1$, where we make the assumption that $\delta(S) < 1/m$, for some constant $m$. Note that

$$\text{li}(A - S, 4) \leq (q_4 - d)\beta \cdot \delta(A - S).$$

Using Eq. (3.1),

$$\text{li}(A - S, 4) + \text{ls}(S, 4) \geq \text{li}(A, 4) = q_4 - d,$$

$$\text{ls}(S, 4) \geq (q_4 - d) \left( 1 - \beta \delta(A - S) \right).$$

Applying Eqs. (2.7) and (3.2),

$$\text{ls}(S, 4)^2 \leq \text{ls}(S, 8) \cdot \text{ls}(S, 0),$$

$$(q_4 - d)^2 \left( 1 - \beta \delta(A - S) \right)^2 \leq q_8 \cdot \delta(S)$$

and from (3.1) we have

$$\delta(A - S) \leq \delta(A) - \delta(S),$$

so

$$(q_4 - d)^2 \left( 1 - \beta \left( \delta(A) - \delta(S) \right) \right)^2 \leq q_8 \cdot \delta(S),$$

$$(q_4 - d)^2 \left( 1 - \beta \left( 1 - \delta(S) \right) \right)^2 \leq q_8 \cdot \delta(S).$$

(3.3)

Now define

$$T = T(\alpha) := \{ v \in A \mid (\text{Re}(a_v(\pi)e^{-i\phi}))^3 \geq \alpha d^{5/4} \left( q_8 - (q_4 - d)^2 \right)^{-1/4} \}$$

for some constant $\alpha \leq 1$, and we make the assumption that $\delta(T) < 1/m$. Note that

$$\text{ls}(A - T, 3) \leq \alpha d^{5/4} \left( q_8 - (q_4 - d)^2 \right)^{-1/4} \delta(A - T).$$

Using the method from Sect. 3.1 of [11] applied to our setting, we deduce

$$\text{ls}(A - T, 3) + \text{ls}(T, 3) \geq d^{5/4} \left( q_8 - (q_4 - d)^2 \right)^{-1/4}.$$
Combining the two equations above,
\[
ls(T, 3) \geq d^{5/4} \left( q_8 - (q_4 - d)^2 \right)^{-1/4} \left( 1 - \alpha \delta(A - T) \right),
\]
\[
ls(T, 3)^2 \geq \left( \frac{d^{5/4}}{(q_8 - (q_4 - d)^2)^{1/4}} \right)^2 (1 - \alpha)^2.
\]

From Eq. (2.4), we have
\[
ls(T, 3)^2 \leq ls(T, 6) \cdot ls(T, 0) \leq q_6 \cdot \delta(T),
\]
and so
\[
\left( \frac{d^{5/4}}{(q_8 - (q_4 - d)^2)^{1/4}} \right)^2 (1 - \alpha)^2 \leq q_6 \cdot \delta(T).
\] (3.4)

Given \( \beta \), choose \( \alpha \) such that
\[
((q_4 - d) \beta)^{1/4} = \left( \alpha \frac{d^{5/4}}{(q_8 - (q_4 - d)^2)^{1/4}} \right)^{1/3}.
\]

We now specify \( r \geq 6 \). We therefore can set \( q_4 = 3/4 \), \( q_6 = 25/16 \), and \( q_8 = 519/128 \). If we choose \( \alpha \) and \( \beta \) such that the upper Dirichlet densities of the sets \( S \) and \( T \) are bounded above by 1/234, then the Eqs. (3.3) and (3.4) imply that \( (\beta = 0.4906 \ldots , d = 0.4934 \ldots ) \) is a boundary case. Therefore, there is an upper Dirichlet density of at least 1/234 for the set of places \( v \in A \) such that
\[
\Re(a_v(\pi)e^{-i\phi}) > ((q_4 - d)\beta)^{1/4} - \epsilon = 0.59566 \ldots - \epsilon,
\]
for any \( \epsilon > 0 \).

Recall that Lemma 3.1 states that for \( Q \geq 2 \) we have
\[
\delta \{ v \mid |a_v(\pi)| > Q \} \leq \frac{1}{1 + (Q^2 - 1)^2 + (Q^4 - 3Q^2 + 1)^2}.
\]
The right-hand side is smaller than 1/234 when \( Q > 2.341 \). This implies that there is a positive upper Dirichlet density of places \( v \) where \( a_v(\pi)e^{-i\phi} \) lies in the region
\[
\{ z \in \mathbb{C} \mid \Re(ze^{-i\phi}) > 0.59566, |z| \leq 2.341 \}.
\]
Note that \( \cos^{-1}(0.59566/2.341) = 1.31352 \) radians (which is equal to 75.259 degrees). This means that there is a positive upper Dirichlet density of places \( v \) whose associated Hecke eigenvalues whose argument is in the interval
\[
(-1.31353 - \phi, +1.31353 - \phi).
\]
3.3 Case of central characters of order at most five

We now assume that the central character $\omega$ of the cuspidal automorphic representation $\pi$ is of order less than six.

We are handling these cases separately since our bounds for the asymptotic behaviour of various Dirichlet series from Sect. 2 are less strong, and so would lead to a weaker result if we only relied on the proof for the $r \geq 6$ case in the previous two subsections.

At a finite place $v$ where $\pi$ is unramified, we have the associated multiset of Satake parameters $\{\alpha_v(\pi), \beta_v(\pi)\}$ where their product is equal to some (not necessarily primitive) $r$th root of unity $e^{i\mu}$, and their sum is equal to the Hecke eigenvalue $a_v(\pi)$. We write $\alpha_v(\pi) = \rho e^{i\theta}$ and $\beta_v(\pi) = \rho^{-1} e^{i(-\theta+\mu)}$, for some positive real number $\rho$ and some angle $\theta$.

Unitarity implies that

$$\{\rho e^{-i\theta}, \rho^{-1} e^{i(\theta-\mu)}\} = \{\rho^{-1} e^{-i\theta}, \rho e^{i(\theta-\mu)}\}. \tag{3.5}$$

If $\rho = 1$, then

$$\begin{align*}
\text{Re} (a_v(\pi)) &= (1 + \cos \mu) \cos \theta + \sin \mu \sin \theta, \\
\text{Im} (a_v(\pi)) &= (1 - \cos \mu) \sin \theta + \sin \mu \cos \theta
\end{align*}$$

and

$$\frac{\text{Im} (a_v(\pi))}{\text{Re} (a_v(\pi))} = \frac{\sin \mu}{1 + \cos \mu} = \tan (\mu/2),$$

so $\text{arg}(a_v(\pi)) = \mu/2 + n\pi$, for some integer $n$.

If $\rho \neq 1$, then Eq. (3.5) implies $e^{-i\theta} = e^{i(\theta-\mu)}$, so $\theta = \mu/2 + n\pi$ for some integer $n$. This again means

$$\text{arg} (a_v(\pi)) = \mu/2 + n\pi. \tag{3.6}$$

We also want to apply the method of Sect. 3.2. For each $r$ (and corresponding $q_4$, $q_6$ and $q_8$), we obtain a statement, for any angle $\phi \in [0, 2\pi)$, of the form

$$\bar{\delta} \left( \left\{ v \mid \text{Re} \left( a_v(\pi)e^{-i\phi} \right) > T(r) \right\} \right) > 0. \tag{3.7}$$

For $r = 5$, we set $q_4 = 3/4$, $q_6 = 25/16$, and $q_8 = 583/128$, and obtain $T(5) = 0.679$.

For $r = 4$, we set $q_4 = 3/4$, $q_6 = 25/16$, and $q_8 = 504/128$, and get $T(4) = 0.684$.

For $r = 3$, set $q_4 = 3/4$, $q_6 = 55/32$, and $q_8 = 602/128$, obtaining $T(3) = 0.678$.

In the case of $r = 2$, we set $q_4 = (3 + \cos 4\phi)/4$, $q_6 = 5/2$, and $q_8 = 7$, and obtain, for $\cos 4\phi \geq -0.785$,
\[ \bar{\delta} \left( \left\{ v \mid \text{Re} \left( a_v(\pi) e^{-i\phi} \right) > 0.5956 \right\} \right) > 0. \]

If \( \cos 4\phi < -0.785 \), then we conclude that

\[ \bar{\delta} \left( \left\{ v \mid \text{Re} \left( a_v(\pi) e^{-i\phi} \right) > 0.5723 \right\} \right) > 0 \]

and use of basic geometry in this setting then implies that \( |a_v(\pi)| > 0.702 \).

Applying the results from the above Eqs. (3.6) and (3.7) for suitable values of \( r \) and \( \phi \), we find that any sector of angle greater than \( 4\pi/5 \) radians must contain a positive upper Dirichlet density of Hecke eigenvalues of size greater than 0.5956, which proves Theorem 1.1 for \( r \leq 5 \).

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