Topological operators, noninvertible symmetries and decomposition

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In this paper we discuss noninvertible topological operators in the context of one-form symmetries and decomposition of two-dimensional quantum field theories, focusing on two-dimensional orbifolds with and without discrete torsion. As one component of our analysis, we study the ring of dimension-zero operators in two-dimensional theories exhibiting decomposition. From a commutative algebra perspective, the rings are naturally associated to a finite number of points, one point for each universe in the decomposition. Each universe is canonically associated to a representation, which defines a projector, an idempotent in the ring of dimension-zero operators. We discuss how bulk Wilson lines act as defects bridging universes, and how Wilson lines on boundaries of two-dimensional theories decompose, and compute actions of projectors. We discuss one-form symmetries of the rings, and related properties. We also give general formulas for projection operators, which previously were computed on a case-by-case basis. Finally, we propose a characterization of noninvertible higher-form symmetries in this context in terms of representations. In that characterization, non-isomorphic universes appearing in decomposition are associated with noninvertible one-form symmetries.

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1 Introduction

In recent years, there has been a great deal of interest in more general notions of symmetries, such as “one-form symmetries” and other higher symmetries, and more recently, “non-invertible” symmetries, see e.g. [1–12]. Briefly, these symmetries are often described by defects such as topological Gukov-Witten operators [13, 14]. It has been noted that all these notions extend ordinary notions of symmetries, and they have been applied to e.g. the completeness hypothesis [15]. In particular, these symmetries arise in discrete gauge theories with trivially-acting subgroups, and existence of such subgroups is on its face at odds with the statement that all matter representations should appear in the spectrum, see e.g. [2,3,16].

Now, two-dimensional gauge theories in which a subgroup of the gauge group acts trivially (discussed in [17–19]) are examples of theories with (possibly noninvertible) higher-form symmetries. One of the most important properties of such theories is decomposition, the statement that these theories are equivalent to (“decompose into”) disjoint unions of other quantum field theories, known in this context as “universes.” This was first described in [20], and has since been applied in many different areas, including Gromov-Witten theory (see e.g. [21–26]), phases of gauged linear sigma models (see e.g. [27–31]), elliptic genera (see e.g. [32]), and anomaly resolution in orbifolds (see e.g. [33–35]). (See also e.g. [4, 36–40] for other recent work.)

The structure of decomposition is tied to existence of one-form symmetries. Briefly, if the universes are copies of one another, one would naturally label the one-form symmetry invertible, and if some universes are distinct, then for reasons we shall discuss, the one-form symmetry could be naturally called noninvertible. Studying this solely from the topological Gukov-Witten operators (dimension-zero twist fields) can be confusing: in the ring of dimension-zero twist fields, there exist many noninvertible elements, such as e.g. projectors, regardless of whether there is a noninvertible one-form symmetry. The universes of decomposition are associated with representations of the trivially-acting subgroup, and we will propose that those representations can be used to distinguish invertible from noninvertible one-form symmetries. If the representation is nontrivial and one-dimensional, then it signals existence of an (invertible) one-form symmetry and, hand-in-hand, multiple identical universes. If the representation associated with a universe is of higher dimension, then it indicates a noninvertible symmetry, and, except in special cases, implies that the corresponding universe is distinct.

We give a couple of examples in orbifolds (without discrete torsion) below:

1. If the trivially-acting normal subgroup lies within the center (so that one has an ordinary one-form symmetry), the theory decomposes into copies of one other theory (with

\[ \text{Idempotents built from dimension-zero twist fields that project states and operators onto corresponding universes.} \]
possible theta angle or $B$ field shifts). For example, consider an orbifold $[X/\mathbb{Z}_n]$ where the $\mathbb{Z}_n$ acts trivially on $X$. In this case, decomposition says

$$\text{QFT} ([X/\mathbb{Z}_n]) = \text{QFT} \left( \coprod_n X \right). \quad (1.1)$$

Both sides have a $B\mathbb{Z}_n$ (one-form) symmetry, the dimension-zero symmetry generators (twist fields) associated to elements of the trivially-acting $\mathbb{Z}_n$ are invertible, and the representations of $\mathbb{Z}_n$ associated to each copy of $X$ are one-dimensional.

2. If the trivially-acting normal subgroup does not lie within the center, then in general the theory decomposes into multiple different theories, and there are often noninvertible topological Gukov-Witten operators (dimension-zero twist fields) associated to conjugacy classes. For example, consider an orbifold $[X/\mathbb{H}]$ by the eight-element group of quaternions $\mathbb{H}$, in which the subgroup $\langle i \rangle \cong \mathbb{Z}_4$ acts trivially. In this case, decomposition says [20, section 5.4]

$$\text{QFT} ([X/\mathbb{H}]) = \text{QFT} \left( X \coprod [X/\mathbb{Z}_2] \coprod [X/\mathbb{Z}_2] \right). \quad (1.2)$$

Both sides have a $B\mathbb{Z}_2$ symmetry, but not a $B\mathbb{Z}_4$, and instead of four copies of a theory, one gets two copies of one theory plus one more distinct theory. Furthermore, one of the dimension-zero twist fields associated to a conjugacy class is noninvertible, and the representation of $\mathbb{Z}_4$ associated to $X$ is two-dimensional.

We will also see decomposition reflected explicitly in the rings of dimension-zero operators (linear combinations of the twist fields associated to conjugacy classes). These rings possess projection operators, projecting onto one-dimensional spaces associated with the distinct universes. Furthermore, these (commutative) rings also have a geometric interpretation via commutative algebra. They describe complete intersections of hypersurfaces in vector spaces, which are a set of points – one point for each universe – located on a space of order parameter, corresponding to vevs of dimension-zero twist fields. We will explore formal properties of those rings, and also describe residual one-form symmetry actions on these rings.

We begin in section 2 by outlining general aspects of rings of dimension-zero operators in orbifolds with trivially-acting normal subgroups of the orbifold group. After reviewing decomposition and its relation to properties of the trivially-acting subgroup, we compute the fusion algebra of dimension-zero twist fields associated to conjugacy classes, give a systematic construction of projectors onto universes of decomposition, and discuss the geometry implicit in the commutative rings of dimension-zero operators, remarking on the semilocality and

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2To be clear, decomposition is not spontaneous symmetry breaking, and universes are not superselection sectors, although we borrow the language of ‘order parameters’ here. See e.g. [36] for a recent detailed discussion of this point.
semisimplicity of those rings. We also discuss both bulk and boundary Wilson lines (Chan-Paton factors) and how projectors act on them. In particular, mathematically bundles and sheaves on gerbes decompose into bundles and sheaves on universes of decomposition, and by explicitly computing the action of projectors on boundary Wilson lines, we can recover that mathematical decomposition of bundles and sheaves explicitly. We also discuss one-form symmetries (invertible and noninvertible) in this context, and subtleties in characterizing the presence of noninvertible one-form symmetries from the ring of dimension-zero twist fields. We propose a characterization in terms of the representations associated to the universes.

In section 3 we study the example of a sigma model on a disjoint union. This is not (presented as) a gauge theory, but it is a prototype for the results of decomposition, and in particular has an invertible one-form symmetry. As a result, it has a ring of dimension-zero operators, which include projection operators as part of a more general subspace of noninvertible dimension-zero operators.

In section 4 we discuss several examples in orbifolds. In subsection 4.1 we discuss orbifolds with a trivially-acting central $\mathbb{Z}_n$ – banded abelian gerbes, with a $B \mathbb{Z}_n$ symmetry. These are physically equivalent to sigma models on disjoint unions as above, and we discuss the rings of dimension-zero operators in greater detail. In subsection 4.2 we discuss an orbifold with a trivially-acting non-central $\mathbb{Z}_4$. Here the (invertible) one-form symmetry is only $B \mathbb{Z}_2$, not $B \mathbb{Z}_4$, and the decomposition is more complicated than just copies of one space, which is reflected in the existence of a noninvertible symmetry. We discuss the ring of dimension-zero operators in detail, explicitly construct projectors, explicitly compute the action of projectors on Wilson lines, and so forth. In section 4.3 we discuss an orbifold with a trivially-acting non-central $\mathbb{Z}_2 \times \mathbb{Z}_2$, which we analyze in the same fashion. In this example there is no invertible one-form symmetry, only noninvertible symmetries. In section 4.4 we discuss an orbifold with a trivially-acting nonabelian group, analyzing its ring of dimension-zero operators. In section 4.5 we turn on discrete torsion in the same example, and study its effects. Briefly, turning on discrete torsion has the effect of complicating the dictionary between one-form symmetries and invertible operators.

In section 5 we study two-dimensional supersymmetric gauge theories with trivially-acting subgroups, specifically, a family of analogues of the supersymmetric $\mathbb{P}^n$ model and their mirrors. Since the trivially-acting subgroup is in the center, these theories have full one-form symmetries. As in other cases, we discuss rings of dimension-zero operators.

In section 6, we discuss analogous phenomena in four-dimensional theories exhibiting decomposition.

In appendix A we work out technical results concerning the action of twist fields on Wilson lines. In appendix B we collect several character identifies regarding orthogonality and normalization of characters, which are used elsewhere in the text. In appendix C we use induced representations to give some technical results on restrictions and extensions.
of representations of finite groups to subgroups. Finally, in appendix D we collect some
miscellaneous results on group cohomology, to make this paper’s discussion of discrete torsion
self-contained.

In this paper we focus on orbifolds with and without discrete torsion. Now, it was
recently argued in [33–35] that two-dimensional orbifold in which a subgroup of the orbifold
group acts trivially admit additional modular-invariant degrees of freedom, beyond discrete
torsion, which were labelled “quantum symmetries.” A version of decomposition for orbifolds
with quantum symmetries was discussed in [33–35], and we leave further details of quantum
symmetries to future work.

2 General aspects

In this section we discuss several general features of rings of dimension-zero operators appearing
in two-dimensional quantum field theories (typically with one-form symmetries), Wilson
lines, symmetries, and the relation to decomposition. In subsection 2.1 we review decom-
position; in subsection 2.2 we discuss the rings of dimension-zero operators: computation of
the fusion algebra, the basis of projectors, and the geometry. In subsection 2.3 we discuss
the action of those dimension-zero operators on Wilson lines at the boundary. Finally, in
section 2.4 we discuss one-form and noninvertible symmetries of decomposition and their
realization in terms of the dimension-zero operators and Wilson lines. In later sections we
discuss specific examples.

2.1 Dictionary and decomposition

It will be helpful to begin by correlating math and physics nomenclature. To that end,
let us quickly review. Orbifolds and gauge theories with trivially-acting normal subgroups
(and hence one-form symmetries) were first discussed in [17–19], as part of a program of
making sense of string compactifications on certain generalizations of manifolds known as
“stacks.” Briefly, gauge theories correspond to stacks, and gauge theories with trivially-
acting subgroups correspond to special stacks known as “gerbes,” essentially, fiber bundles
whose fibers are ‘groups’ of one-form symmetries. As a result of that geometry, a sigma
model on a gerbe admits a one-form symmetry (or a noninvertible analogue), corresponding
to translations along the fibers.

Such gauge theories have a natural classification:

1. Cases in which the trivially-acting normal subgroup lies within the center of the gauge
group. In these cases, at least in the absence of twisting, the trivially-acting subgroup
Math                      Physics
Stack                      Gauge theory
Gerbe                      Gauge theory with trivially-acting normal subgroup
Banded abelian gerbe       Trivially-acting subgroup is in center
Nonbanded abelian gerbe    Trivially-acting subgroup abelian but not in center
Nonabelian gerbe           Trivially-acting subgroup not abelian

Table 2.1: A dictionary \[18, 20\] between math and physics language for descriptions of two-dimensional gauge theories (including orbifolds).

defines a(n invertible) one-form symmetry.

2. Cases in which the trivially-acting normal subgroup is abelian, but does not lie within the center of the gauge group. Here, the one-form symmetry is obstructed, and as we shall see in examples, this leads to what we will propose to label noninvertible one-form symmetries.

3. Cases in which the trivially-acting normal subgroup is nonabelian. Here, one does not expect a one-form symmetry at all, in the absence of twisting, unless of course that nonabelian group has a center. Here, again, one gets invertible symmetries.

These cases have a mathematical understanding, in terms of a classification of gerbes. As discussed in e.g. \[17–19\], the first case corresponds to ‘banded abelian’ gerbes; the second, to ‘nonbanded abelian’ gerbes; and the third, to nonabelian gerbes. We summarize this dictionary in table 2.1.

In most of this paper, we will focus on orbifolds, so let us elaborate for such theories. Given an orbifold by a finite group \(\Gamma\), say, with trivially-acting subgroup \(K \subset \Gamma\), the orbifold is physically equivalent to a disjoint union, with components of the disjoint union determined as follows.

Let \(K_{i,\omega}\) denote the set of isomorphism classes of irreducible projective representations of \(K\), twisted by the restriction of discrete torsion \(\omega \in H^2(\Gamma, U(1))\) to \(K\) (along the inclusion map \(i : K \hookrightarrow \Gamma\)). The number of components of the orbifold \([X/\Gamma]_\omega\) is given by the number of orbits of the action of \(G = \Gamma/K\) on \(K_{i,\omega}\). We define the action as follows \[20,39\]. Given an isomorphism class \([\rho] \in K_{i,\omega}\) and \(q \in G\), we take

\[
q \cdot [\rho] = [L_q \rho]
\]

(2.1)

where \(L_q \rho\) is another irreducible projective representation of \(K\), also twisted by \(i^* \omega\), defined by

\[
(L_q \rho)(k) = \frac{\omega(s(q)^{-1}k, s(q))}{\omega(s(q), s(q)^{-1}k)} \rho(s(q)^{-1}ks(q))
\]

(2.2)
for $k \in K$, $s : G \to \Gamma$ a section, and products are taken in $\Gamma$. The various components (corresponding to orbits) may also have discrete torsion, determined as described in [20][39].

Briefly, the orbifold $[X/\Gamma]_\omega$ is equivalent to a disjoint union of theories, one for each orbit of $G$ on $\hat{K}_\omega$, with the theory corresponding to a given orbit being an orbifold of $X$ by the stabilizer of the orbit, with some discrete torsion given by an analysis described elsewhere. This is known as “decomposition,” and the various components of the disjoint union, corresponding to orbits of the $G$ action on $\hat{K}_\omega$, are known as “universes.” (See [20][39] for numerous additional details.)

There also exist more modular-invariant phases than just discrete torsion in the case a nontrivial subgroup of $\Gamma$ acts trivially; see [33–35] for more information on such orbifolds and their corresponding decomposition. We leave such more general theories for future work.

In the special case that the $\Gamma$ orbifold does not contain any discrete torsion ($\omega = 0$), then decomposition reduces to [20]

$$\text{QFT} ([X/\Gamma]) = \text{QFT} \left( \left[ \frac{X \times \hat{K}}{G} \right]_{\hat{\omega}} \right),$$

where $\hat{K}$ is the set of ordinary irreducible representations of $K$, and $\hat{\omega}$ is some possible discrete torsion, which is described in detail in [20].

If in addition $K$ is in the center of $\Gamma$, so that the $\Gamma$ orbifold has a $B K$ (one-form) symmetry, then the action of $G$ on $\hat{K}$ is trivial, and the theory decomposes into a disjoint union of identical copies (modulo choices of discrete torsion $\omega$):

$$\text{QFT} ([X/\Gamma]) = \text{QFT} \left( \prod_{k \in \hat{K}} [X/G]_{\hat{\omega}(k)} \right).$$

Here, each $[X/G]_{\hat{\omega}(k)}$ is a universe, so-named because in a string compactification on such an orbifold, each summand $[X/G]_{\hat{\omega}(k)}$ would give rise to its own separate low-energy theory.

If $K$ is not central (but $\omega$ still vanishes), then instead of just copies of $[X/G]$, one will get various covers of $[X/G]$. (The sum of the number of copies and the orders of the covers will always equal the number of elements in $\hat{K}$.)

For example, as we shall see in section 4.2 if $\Gamma = \mathbb{H}$, the eight-element group of unit quaternions, with trivially-acting subgroup $K = \langle i \rangle \cong \mathbb{Z}_4$, then

$$\text{QFT} ([X/\mathbb{H}]) = \text{QFT} \left( X \prod [X/\mathbb{Z}_2] \prod [X/\mathbb{Z}_2] \right).$$

If in the decomposition, there are $n$ universes which are identical (up to choices of discrete torsion / $B$ fields / theta angles), then the theory has a $B\mathbb{Z}_n$ symmetry. If some of the
universes are different, then the theory has a noninvertible symmetry. We can make this more precise as follows. Each universe is associated to an orbit of the \(G\) action on \(\hat{K}_{\mathcal{r} \cdot \omega}\), the set of isomorphism classes of \(\mathcal{r} \cdot \omega\)-projective irreducible representations of \(K\). Given an orbit, if we let \(R_1, \ldots, R_\ell\) denote representations of each isomorphism class appearing in the orbit, then we can associate the universe to the representation

\[
R = R_1 \oplus \cdots \oplus R_\ell,
\]

and so we see that we can also associate universes to (isomorphism classes of) (possibly projective) representations of the trivially-acting part of the gauge group. As we will see in examples, universes related by a \(B\mathbb{Z}_n\) symmetry are associated to one-dimensional irreducible representations \(R\). Universes associated to a higher-rank representation \(R\) are related by a noninvertible symmetry.

In the rest of this paper we will elaborate on that dictionary. We will look at the structure of both the ring of dimension-zero operators as well as its action on Wilson lines and the corresponding symmetries.

### 2.2 Ring of dimension-zero operators

Consider an orbifold \([X/\Gamma]\), perhaps with discrete torsion, in which a normal subgroup \(K \subset \Gamma\) acts trivially. In this section we will study the ring of dimension-zero operators. We will describe two bases for the dimension-zero operators:

- **Twist fields**, constructed from \((\omega\)-compatible) conjugacy classes of trivially-acting group elements, and their products (forming a fusion algebra) in subsection 2.2.1
- **Projectors**, corresponding to (projective) representations of the trivially-acting subgroup \(K\) of the gauge group, in subsection 2.2.2. These project onto states and operators in the universes of decomposition, and we will give general expressions.

Mathematically, the vector space of dimension-zero operators is the center of the (twisted) group algebra, and it is a standard result that the twist field and projector constructions we will review here both form bases for that space. Essentially as a result of the existence of the (complete orthogonal) set of projectors, the vector space of dimension-zero operators is equivalent to \(\mathbb{C}^n\), where \(n\) is the number of universes in the decomposition.

In subsection 2.2.3, we will also describe the corresponding geometry: each ring \(R\) of dimension-zero operators will have the form

\[
\mathbb{C}[x_0, \ldots, x_m]/I,
\]

(2.7)
which describes a complete intersection in $\mathbb{C}^{m+1}$, a space of order parameters, and that complete intersection will always be a set of points, essentially because of the existence of the projectors.

2.2.1 Twist fields and their products

In this section, we describe the topological (dimension zero) twist fields and their products.\footnote{In principle, a mathematically rigorous description of such products in considerably greater generality can be found in \cite{41}. In this section we describe only the special case of trivially-acting group elements.}

Consider an orbifold $[X/\Gamma]$ in which a normal subgroup $K \subset \Gamma$ acts trivially. Suppose that there is discrete torsion in the $\Gamma$ orbifold, represented by a cocycle $\omega$. Formally, let $\tau_g$ denote the operator associated to $g \in K$. (Formally, $\tau_g$ generates a branch cut or topological defect line in the sense of e.g. \cite{42,43}. It need not be a twist field in the usual sense, as it is not gauge invariant and does not commute in general.) These operators obey a multiplication given by

$$\tau_g \tau_h = \omega(g, h) \tau_{gh}. \quad (2.8)$$

Associativity is guaranteed by the cocycle condition on $\omega$. Rescaling the $\tau$’s alters $\omega$ by a coboundary, so this product structure is naturally associated to the cohomology class of $\omega$, rather than $\omega$ itself. (Technically, for any group $G$, the linear combinations of $\tau_g$ for $g \in G$ with this multiplication define a twisted group algebra, denoted $\mathbb{C}[G]_\omega$, see e.g. \cite{45}.)

Now, not all of the $\tau_g$ are relevant: only conjugation-invariant combinations contribute to the pertinent dimension-zero operators, which we label $\sigma_{[g]}$, where $[g]$ denotes a conjugacy class of $g$ (in $\Gamma$). The $\sigma_{[g]}$ are what one would ordinarily call twist fields. The algebra of $\sigma$’s is naturally commutative, which at root follows from the observation:

$$gh = (ghg^{-1})g. \quad (2.9)$$

The left-hand-side is a representative of the product $[g][h]$, and the right-hand-side is a representative of the product $[h][g]$, hence one expects

$$\sigma_{[g]} \sigma_{[h]} = \sigma_{[h]} \sigma_{[g]}, \quad (2.10)$$

so that the $\sigma$’s are commutative, whereas the $\tau$’s are not. (Technically, just as the $\tau_g$ generate the twisted group algebra, the $\sigma_{[g]}$ generate the center of the twisted group algebra, see e.g. \cite{46} section 6.3.)

In detail, under conjugation by an element $h \in \Gamma$,

$$\tau_g \mapsto \tau_h \tau_g \tau_{h^{-1}} = \tau_h \left( \omega(g, h^{-1}) \tau_{gh^{-1}} \right), \quad (2.11)$$

$$= \omega(g, h^{-1}) \omega(h, gh^{-1}) \tau_{gh}^{-1}, \quad (2.12)$$

$$= \omega(h, g) \omega(hg, h^{-1}) \tau_{gh^{-1}}, \quad (2.13)$$
using the cocycle condition. (Note that since $K$ is normal in $\Gamma$, $hgh^{-1} \in K$ for all $g \in K$ and $h \in \Gamma$.) We assume the cocycle is normalized\footnote{See appendix \ref{appendix_d}.} so that
\begin{equation} \omega(1,g) = 1 = \omega(g,1), \tag{2.14} \end{equation}
\begin{equation} \omega(g,g^{-1}) = 1 = \omega(g^{-1},g). \tag{2.15} \end{equation}
The $\sigma_{[g]}$ are then conjugation-invariant combinations of the $\tau_{g'}$ for $g'$ conjugate to $h$.

Furthermore, the reader should note that although the dimension-zero fields arise from elements of $K$, we need to consider conjugation by elements of $\Gamma \supset K$. Therefore, we will discuss conjugacy classes in $\Gamma$, such that the conjugacy classes contain elements in $K$ only.

In addition, we will see that only certain conjugacy classes in $\Gamma$ contribute. Beyond restricting to conjugacy classes whose elements are in $K$, we will also restrict to conjugacy classes (in $\Gamma$) of "$\omega$-regular" elements (of $K$). An $\omega$-regular element of $\Gamma$ is defined to be an element such that, for all $h \in \Gamma$ commuting with $g$,
\begin{equation} \omega(g,h) = \omega(h,g). \tag{2.16} \end{equation}

For any $\omega$-regular conjugacy class $[g]$ whose elements all lie in $K \subset \Gamma$, one defines \footnote{See appendix \ref{appendix_d}.}[47] section 3]
\begin{equation} \sigma_{[g]} = \frac{1}{|\Gamma|} \sum_{h \in \Gamma} \omega(h,g)\omega(hg,h^{-1})\omega(h,h^{-1})\tau_{hgh^{-1}}. \tag{2.17} \end{equation}
This is a dimension-zero twist field. (In fact, in mathematics, this is a standard construction of one of two bases for the center of the twisted group algebra, as discussed in e.g. \cite{47}.) In the case that discrete torsion is trivial, this reduces to
\begin{equation} \sigma_{[g]} = \frac{1}{|[g]|} \sum_{h \in [g]} \tau_h. \tag{2.18} \end{equation}
(The reader should note that additional twist fields of nonzero dimension may exist in a theory, but for our purposes we are only interested in topological operators.)

The factors of $\omega$ make the fact that $\sigma$’s commute with $\tau$’s more obscure, so let us check that more carefully.
\begin{align*}
\sigma_{[g]} \tau_y &= \frac{1}{|\Gamma|} \sum_{h \in \Gamma} \frac{\omega(h,g)\omega(hg,h^{-1})}{\omega(h,h^{-1})} \tau_{hgh^{-1}} \tau_y, \tag{2.19} \\
&= \frac{1}{|\Gamma|} \sum_{h \in \Gamma} \frac{\omega(h,g)\omega(hg,h^{-1})\omega(hgh^{-1},y)}{\omega(h,h^{-1})} \tau_{hgh^{-1}} \tau_y, \tag{2.20} \\
&= \frac{1}{|\Gamma|} \sum_{h \in \Gamma} \frac{\omega(h,g)\omega(hg,h^{-1})\omega(hgh^{-1},y)}{\omega(y,y^{-1}hgh^{-1}y)\omega(h,h^{-1})} \tau_y \tau_{y^{-1}hgh^{-1}y}. \tag{2.21}
\end{align*}
It can be shown that
\[
\omega(h, g)\omega(hg, h^{-1})\omega(hgh^{-1}, y) = \frac{\omega(y^{-1}h, g)\omega(y^{-1}hg, h^{-1}y)}{\omega(h^{-1}y, y^{-1}h)},
\]
by multiplying in
\[
\frac{(d\omega)(y, y^{-1}, hg, h^{-1}y)(d\omega)(hg, h^{-1}, y)(d\omega)(h^{-1}, y, y^{-1}h)}{(d\omega)(y^{-1}, h, h^{-1}y)(d\omega)(y^{-1}, h, g)(d\omega)(y, y^{-1}, h)}.
\]

hence
\[
\sigma[g]\tau_y = \tau_y\sigma[g],
\]
as claimed.

Thus, we see that \(\sigma[g]\) is central in the twisted group algebra, and in fact, it can be shown [47, section 3] that the collection of \(\sigma[g]\) for \(\omega\)-regular conjugacy classes (in \(\Gamma\), for classes whose elements lie in \(K\)) form a basis for the center of the twisted group algebra. The reader should note that the number of irreducible projective representations matches the number of \(\omega\)-regular conjugacy classes [4752]. There is not a canonical isomorphism between the two, but the number of elements of each set is the same.

The definition of \(\sigma[g]\) above is slightly sensitive to the choice of representative \(g\) of the conjugacy class: as described in [47, remark 3.2], changing it will multiply \(\sigma[g]\) by a constant factor. Specifically, if \(g' = hgh^{-1}\) is another representative of the same conjugacy class, then as shown in [47, section 3],
\[
\sigma[hgh^{-1}] = \frac{\omega(gh^{-1}, h)}{\omega(h, gh^{-1})}\sigma[g].
\]

Now, let us make some general observations about the conjugacy classes. First, we will see in examples below that if one drops the \(\omega\)-regularity condition, the conjugacy class will not contribute: the various terms in the sum (2.17) will appear with different signs and cancel out.

Note that since \(\tau_1 \mapsto \omega(g, g^{-1})\tau_1\) under conjugation by \(g\), and we have normalized so that \(\omega(g, g^{-1}) = 1\), the identity should always contribute, and indeed, thanks to the normalization condition, the identity is always an \(\omega\)-regular element.

We have restricted to \(\omega\)-regular conjugacy classes, and mentioned that for other conjugacy classes, the terms in (2.17) may cancel out. As a consistency check, let us verify that the same does not happen in \(\omega\)-regular conjugacy classes. To see this, note that if \(h\) commutes with \(g\), then from our expression above, under conjugation by \(h\),
\[
\tau_g \mapsto \omega(h, g)\omega(hg, h^{-1})\tau_g.
\]
However, from the cocycle condition for \((\delta\omega)(g, h, h^{-1})\) and the normalization condition, we have
\[
\omega(g, h^{-1})\omega(g, h) = 1,
\]
hence
\[
\omega(h, g)\omega(hg, h^{-1}) = \omega(h, g)\omega(gh, h^{-1}),
\]
(2.28)
\[
= \frac{\omega(h, g)}{\omega(g, h)},
\]
(2.29)
which equals one by assumption. Therefore, if for all \( h \) that commute with \( g \), \( \omega(g, h) = \omega(h, g) \), then under conjugation by \( h \), \( \tau_g \) is invariant, and so should contribute to the spectrum of dimension-zero ground states.

The reader should note that the fact that the twist fields \( \sigma_{|g|} \) are counted by \( \omega \)-regular conjugacy classes is in accord [39] with the fact that irreducible projective representations of a finite group \( G \) are also counted by \( \omega \)-regular conjugacy classes, as discussed in [47–52]. Thus, the number of twist fields \( \sigma_{|g|} \) matches the number of irreducible projective representations, as expected.

So far we have discussed the construction of the twist fields \( \sigma_{|g|} \) themselves. Now, let us turn to their products. The \( \tau \)'s obey a group-like multiplication (2.8). Results for products of \( \sigma \)'s can be derived from the product structure on the \( \tau \)'s, as the \( \sigma \)'s are linear combinations of the \( \tau \)'s. However, the resulting products of \( \sigma \)'s are no longer group-like, but rather are more general ring elements, and in fact the products of \( \sigma \)'s define a fusion algebra.

To clarify these remarks, next we compute the \( \sigma \)'s and their products in examples. For our first example, suppose that \( K \) is in the center of \( G \), and there is no discrete torsion. In this case, every element of \( K \) is its own conjugacy class, and so the set of pertinent dimension-zero operators is simply a copy of \( K \), specifically \( \sigma_g = \tau_g \) for all \( g \in K \).

For another example, consider the orbifold \([X/\mathbb{H}]\), where \( \mathbb{H} \) is the eight-element group of unit quaternions, \( \mathbb{Z}_4 \cong \langle i \rangle \) acts trivially, and again without discrete torsion. (This orbifold will be described in greater detail in section 1.2.) Here, \( \pm 1 \) are in the center of \( \mathbb{H} \), and so they are their own conjugacy classes. The elements \( \pm i \), on the other hand, are mapped into one another under conjugation: there exists \( g \in \mathbb{H} \) such that \( g(i)g^{-1} = -i \). Therefore, we have the dimension-zero operators
\[
\sigma_{[+1]} = 1, \quad \sigma_{[-1]} = \tau_{-1}, \quad \sigma_{[i]} = \frac{1}{2} (\tau_i + \tau_{-i}).
\]
(2.30)
For example, since
\[
\tau_{-1}\tau_{\pm i} = \tau_{\mp i},
\]
(2.31)
we have
\[
\sigma_{[-1]}\sigma_{[i]} = \sigma_{[i]}.
\]
(2.32)
Similarly, since \( \tau_{\pm i}^2 = \tau_{-1} \) and \( \tau_{\pm i}\tau_{\mp i} = 1 \), we have
\[
\sigma_{[i]}^2 = \frac{1}{4} \left( \tau_i^2 + \tau_{-i}^2 + \tau_i\tau_{-i} + \tau_{-i}\tau_i \right) = \frac{1}{2} (1 + \sigma_{[-1]}).
\]
(2.33)
In particular, $\sigma_{[g]}$ does not have a group-like multiplication. In this example, the $\sigma_{[g]}$ form a fusion algebra.

For another example, consider the orbifold $\text{point}/\mathbb{Z}_2 \times \mathbb{Z}_2$, this time with discrete torsion. (As $H^2(\mathbb{Z}_2 \times \mathbb{Z}_2, U(1)) = \mathbb{Z}_2$, there is only one nontrivial choice of discrete torsion.) Without discrete torsion, every element of $\mathbb{Z}_2 \times \mathbb{Z}_2$ would be its own conjugacy class, since the group is abelian. With discrete torsion, the story is somewhat more complicated. Write $\mathbb{Z}_2 \times \mathbb{Z}_2 = \langle a, b \rangle$, then under conjugation by $b$, 

$$
\tau_a \mapsto -\tau_a \quad (2.34)
$$

and under conjugation by $a$,

$$
\tau_b \mapsto -\tau_b. \quad (2.35)
$$

In this fashion, using equation (2.17), one can show

$$
\sigma_{[a]} = 0 = \sigma_{[b]} = \sigma_{[ab]]. \quad (2.36)
$$

The only nonzero dimension-zero twist field is the identity itself — the sums defining $\sigma$’s associated to other conjugacy classes cancel out. Indeed, the other conjugacy classes are not $\omega$-regular, so this example confirms our earlier statement that only $\omega$-regular conjugacy classes contribute to twist fields. This is also consistent with the statement that $\mathbb{Z}_2 \times \mathbb{Z}_2$ has only one irreducible projective representation.

Finally, we should comment on the relation to computations of massless spectra with discrete torsion in [53]. There, it was stated that the action of an element $h \in G$ on a $g$-twisted sector is given by multiplying by a phase

$$
\epsilon(g, h) = \frac{\omega(g, h)}{\omega(h, g)}. \quad (2.37)
$$

This defines an action because, as a consequence of the cocycle condition,

$$
\epsilon(g, h_1 h_2) = \epsilon(g, h_1)\epsilon(g, h_2). \quad (2.38)
$$

We can derive this phase factor from the algebra above by interpreting the algebra above in terms of OPEs of topological defect lines, as in e.g. [43][44]. In this context, a trivalent vertex of topological defect lines is defined by group elements $g, h$, and their product $gh$, which in our case is weighted by the cocycle $\omega$. The $g$ action on an $h$-twisted sector is described formally by the intersection of two orthogonal lines, one for $g$ and one for $h$. That intersection can be resolved into a pair of trivalent vertices, each of which contributes a factor of $\omega$, which together give $\epsilon$.

Fusion algebras have also been recently studied from other perspectives in e.g. [3][5][54].
2.2.2 Projectors

So far, we have discussed the dimension-zero twist fields and their (fusion) products. Mathematically, those twist fields form a basis for the center of a twisted group algebra. Now, such group algebras have another basis consisting of idempotents (projectors). In physics, this is a reflection of decomposition. In mathematics, it is ultimately because of Wedderburn’s theorem, see e.g. [53, section XVII.3]. In any event, this means the (twisted) group algebra can be written in the form

$$\mathbb{C}[G] = \oplus_i \text{End}(V_i),$$  \hspace{1cm} (2.39)

where the \( \{V_i\} \) are irreducible (projective) representations of \( G \). Technically because this is a decomposition as an algebra, the center is the subset of elements of the form

$$\sum_i c_i I_{V_i},$$  \hspace{1cm} (2.40)

In particular, the identity matrices \( I_{V_i} \) are a basis for the ring of dimension-zero operators as a vector space. Since these are idempotents in the center, we can write the ring of dimension-zero operators in the form

$$\oplus_i \mathbb{C}I_{V_i}$$ \hspace{1cm} (2.41)

as an algebra. In particular, this ring is isomorphic to a sum of copies of \( \mathbb{C} \), with a complete set of orthogonal projectors. (Geometrically, in subsection 2.2.3 we will see that this means the ring describes a finite set of points.)

In this section we will describe those projectors more explicitly. In fact, expressions for projectors are known in the relevant mathematics literature, as we will review. Projectors are in one-to-one correspondence with (projective) representations, and also coincide with the projectors onto local operators associated with each universe in the decomposition of the quantum field theory.

A general expression for projectors is given abstractly in [46, section 2.6, exercise 6.4], [56, prop. 9.21] (for the case of vanishing discrete torsion), and more generally in [49, chapter 7, theorem 3.1] which is easily adapted to the present situation.

Let \( R \) be a representation of \( K \) associated to a universe – meaning,

$$R = \oplus R_i,$$ \hspace{1cm} (2.42)

for \( R_i \) a set of irreducible (projective) representations of \( K \) spanning the isomorphism classes of \( \hat{K}, \omega \) in a single fixed orbit of the action of \( G \). Then, for each irreducible \( R_i \), define\(^\text{5}\)

$$\Pi_{R_i} = \frac{\dim R_i}{|K|} \sum_{k \in K} \chi_{R_i}(k^{-1}) \omega(k, k^{-1}) \tau_k,$$ \hspace{1cm} (2.43)

\(^5\text{Equivalently, instead of summing over all } \omega, \text{ one might only sum over } \omega\text{-regular } g, \text{ meaning those elements } g \text{ such that } \omega(g, h) = \omega(h, g) \text{ for all } h \text{ commuting with } g. \text{ The resulting sum is equivalent because characters of projective representations vanish for non-}\omega\text{-regular elements.}$$
and then finally define
\[ \Pi_R = \sum_i \Pi_{R_i}. \] (2.44)

(In the case that a representation \( R_i \) is projective, characters \( \chi_R \) exist, but their properties are slightly obscure. We will describe them in more detail later in section 2.3.)

Just as the \( \sigma_{[g]} \) form a basis for the center of the twisted group algebra, so too do the projectors above, see e.g. [26] prop. 9.14, [17] section 3. In particular, although it may not be obvious, each projector \( \Pi_R \) is a linear combination of \( \sigma \)'s, and so can be written in terms of the closed string twist fields.

We will see later in examples that the \( \Pi_R \) also coincide with the projectors onto universes described elsewhere in the literature on decomposition. Specifically, the universe projectors are the \( \Pi_R \) for distinct \( R \) obtained as restrictions to \( K \) of irreducible representations of \( G \). Note that in general, the restriction of an irreducible representation of \( G \) may be a reducible representation of \( K \), as we shall see in examples.

It is straightforward to check that the \( \Pi_R \) are projectors. For example, for any two irreducible representations \( R, S \),
\[
\Pi_R \Pi_S = \frac{(\dim R)(\dim S)}{|K|^2} \sum_{g,h \in K} \chi_R(g^{-1})\chi_S(h^{-1}) \omega(g, h^{-1}) \omega(g, h^{-1}) \tau_{gh},
\] (2.45)
\[
= \frac{(\dim R)(\dim S)}{|K|^2} \sum_{g,k \in K} \chi_S(k^{-1}g)\chi_R(g^{-1}) \omega(g, g^{-1}k) \omega(g, g^{-1}k) \omega(g^{-1}k, k^{-1}g) \tau_k,
\] (2.46)
Using the identity \( \omega(h, h^{-1}) = \omega(h^{-1}, h) \) (see appendix D), the condition \( (d\omega)(k^{-1}, g, g^{-1}k) = 1 \) implies that
\[
\frac{\omega(g, g^{-1}k)}{\omega(g^{-1}k, k^{-1}g)} = \frac{\omega(k^{-1}, g)}{\omega(k, k^{-1})},
\] (2.47)
so we can apply identity (B.13) to find that
\[
\Pi_R \Pi_S = \frac{(\dim R)(\dim S)}{|K|} \sum_{k \in K} \delta_{R,S} \frac{1}{\omega(k, k^{-1})} \frac{\chi_R(k^{-1})}{\dim R} \tau_k,
\] (2.48)
\[
= \delta_{R,S} \Pi_R.
\] (2.49)

Next, we check that these projectors are complete.
\[
\sum_R \Pi_R = \sum_R \frac{\dim R}{|K|} \sum_{k \in K} \chi_R(k^{-1}) \omega(k, k^{-1}) \tau_g,
\] (2.50)
\[
= \frac{1}{|K|} \sum_{k \in K} \left( \sum_R \frac{(\dim R)\chi_R(k^{-1})}{\omega(k, k^{-1})} \right) \tau_g.
\] (2.51)
Now, in a sum over all irreducible (projective) representations of a group $K$, from (B.20), we have

$$
\sum_R \frac{(\dim R) \chi_R(k^{-1})}{\omega(k, k^{-1})} = \sum_R \frac{\chi_R(1) \chi_R(k^{-1})}{\omega(k, k^{-1})},
$$

(2.52)

$$
= \begin{cases} 
0 & k \neq 1, \\
|K| & k = 1. 
\end{cases}
$$

(2.53)

The sum above is a sum over all irreducible representations of $K$, hence

$$
\sum_R \Pi_R = 1.
$$

(2.54)

### 2.2.3 Geometry

So far we have discussed the twist fields associated to conjugacy classes of trivially-acting group elements in orbifolds, as well as projectors. As discussed, these each form a basis for the space of all dimension-zero operators. Next, we will study the geometry and commutative algebra of the ring of such linear combinations, the ring of dimension-zero operators in a given orbifold with a trivially-acting subgroup.

In all cases, the rings of dimension-zero operators geometrically describe a set of points, one for each universe in the decomposition. Schematically, each such ring $\mathcal{R}$ is of the form

$$
\mathbb{C}[x_0, \cdots, x_n]/I,
$$

(2.55)

where $I$ is some ideal, and the locus on which $I$ vanishes will be a set of isolated points. In the language of spontaneous symmetry breaking, we can think of $x_0, \cdots, x_n$ above as being order parameters, and the points at which the ideal $I$ vanishes correspond to vevs of order parameters corresponding to each universe.

If the theory has a one-form symmetry, it is reflected in a symmetry in the ring of zero-dimensional operators as phases of the form

$$
x_i \mapsto \xi x_i,
$$

(2.56)

which leave the ideal $I$ invariant. In orbifolds, these phases look identical to quantum symmetries. We shall discuss this in detail in examples.

In any theory exhibiting a nontrivial decomposition, the ring of dimension-zero operators will always have noninvertible operators – for example, the projectors which project onto

---

6Such language is not entirely appropriate, as it incorrectly suggests that the different universes are mere superselection sectors. In particular, as described in detail in e.g. [32], universes are not just superselection sectors, as for example decomposition exists for finite volumes, not just in an infinite volume limit.
subspaces associated with the universes, as we described in subsection 2.2.2. In general, there will be additional noninvertible elements. The locus of noninvertible operators is always codimension-one in the space of order parameters. This is because we are working in a quantum-mechanical system, and these are operators acting on a finite-dimensional Hilbert space. Such operators can be represented by finite-dimensional matrices, so the noninvertibility criterion is for the determinant to vanish, which always gives a codimension-one condition. This fact will complicate our efforts to characterize noninvertible symmetries in theories admitting a decomposition, as we shall discuss in subsection 2.4.

These rings of dimension-zero operators can be understood conveniently in the mathematical language of commutative algebra (see e.g. [57] for a very readable introduction), which is pertinent to the geometry. Let \( \mathcal{R} \) denote the ring of dimension-zero operators appearing in a two-dimensional conformal field theory, then \( \mathcal{R} \) is a commutative ring with identity. Saying that \( \mathcal{R} \) “is associated to a set of points” can be understood more formally as saying that \( \text{Spec} \, \mathcal{R} \) is a disjoint union of reduced\(^8\) points, or equivalently that \( \mathcal{R} \) has finitely many maximal ideals and no nilpotent elements. (A nilpotent element would correspond to an operator whose eigenvalues all vanish, which seems unlikely, and we assume that each individual universe has a unique dimension-zero operator.)

Furthermore, each universe is supported at a maximal ideal \( m \) such that all the projectors but one are in \( m \). Equivalently,

\[
P_i \in \bigcap_{j \neq i} m_j.
\] (2.57)

In some sense, the ideals \( \Pi_i \) and the maximal ideals \( m_j \) (corresponding to points on spaces of order parameters at which universes are supported) are ‘dual’ to one another. We have already seen that projectors are in the intersections of maximal ideals corresponding to other points. Conversely, given the projectors, the point at which any universe is supported is given by the vanishing locus of all other projectors:

\[
\bigcap_{j \neq i} \{ \Pi_j = 0 \} = \{ \text{i th point} \}. \tag{2.58}
\]

Given any ring \( \mathcal{R} \) and maximal ideal \( m \subset \mathcal{R} \), the quotient \( \mathcal{R}/m \) is a field [57, chapter 1]. In the present circumstances, given the decomposition

\[
\mathcal{R} = \sum_k (\Pi_k), \tag{2.59}
\]

\(^7\)Strictly speaking, we should show that the inverse is in the ring. Let \( T \) be the matrix corresponding to an element of the ring, and \( f \) its characteristic polynomial. By the Cayley-Hamilton theorem, \( f(T) = 0 \). Since we are assuming the determinant is nonzero, the constant term of \( f(T) \) is nonzero, so the inverse of \( T \) is a polynomial in \( T \). Taking the same polynomial in the original ring element gives the inverse of that element. We would like to thank T. Pantev for making this observation.

\(^8\)‘Reduced’ is a technical term in commutative algebra, which distinguishes ordinary points from ‘fat’ points which are collisions of multiple points.
of the ring $\mathcal{R}$, for a maximal ideal $m$ corresponding to a projector $\Pi$, we have that

$$\mathcal{R}/m = \sum_k (\Pi_k)/m = (\Pi)/m \cong \mathbb{C},$$

where we have used (2.57).

This also has a simple understanding in terms of localization (in the sense of commutative algebra). Briefly, $\mathcal{R}_m$ for any prime (here, maximal) ideal $m$ denotes a ring of fractions $S^{-1}\mathcal{R}$ where $S = \mathcal{R} - m$. Let $\Pi_i$ denote the one projector that is not in maximal ideal $m_i$, then for all other projectors, $(\Pi_{k\neq i})m_i = 0$, essentially because $\Pi_i \in S$ and so can be used to project out ideals generated by other projectors. In physics terms, the restriction of $\Pi_j$ to the point corresponding to $m_i$ vanishes if $j \neq i$; only for the point corresponding to $m_i$ can the restriction of $\Pi_i$ be nonzero.

We should mention that the fact that all points are reduced, that the ring has no nilpotents, (a mathematical consequence of the construction reviewed in subsection 2.2.2,) is physically a reflection of cluster decomposition. Specifically, we assume that each universe has a unique (up to scaling) dimension-zero operator, and so the points appearing in the ring should all be reduced.

In passing, since these rings of dimension-zero operators describe isolated points (i.e. have finitely many maximal ideals), they are semi-local rings, see e.g. [58–60], and since there are no nilpotents (reflecting cluster decomposition in each separate universe), the nilradical vanishes, hence since $\mathcal{R}/J(\mathcal{R})$ is semisimple in semi-local rings, where $J(\mathcal{R})$ denotes the Jacobson radical, which coincides with the nilradical for polynomial rings, the rings are semisimple. From another perspective, the Jacobson radical is the intersection of the maximal ideals, so for a ring describing isolated points, $J(\mathcal{R}) = 0$, and the ring is semisimple.

### 2.3 Wilson lines

So far, we have focused on dimension-zero operators. Next, we turn to Wilson lines in these theories. To clarify, there are two types of Wilson lines that one might be interested in: bulk Wilson lines, associated with the (finite) gauge theory, and boundary Wilson lines, also known as Chan-Paton factors in open strings. If there is no discrete torsion, then those Chan-Paton factors are in an ordinary representation of the orbifold group. If there is discrete torsion, on the other hand, then they are in a projective representation, determined by a cocycle $\omega$ representing an element of group cohomology.

The two types of Wilson lines have somewhat different interpretations: bulk Wilson lines act as bridges between universes, whereas boundary Wilson lines can be associated with different universes in the decomposition. (Geometrically, in the latter context, K theory and sheaves on gerbes decompose into K theory and sheaves on universes, as discussed in [20].)
We shall see that the projectors $\Pi_R$ we computed in subsection 2.2.2 project onto Wilson lines associated with the universe corresponding to $R$. We will set up the computational technology in this section, then see that decomposition in examples later.

Formally, several fusion categories of topological line operators have been defined in e.g. [9–12]. Briefly, the objects in one such category are topological line operators (such as Wilson lines), and the morphisms are local operators at junctions of the line operators. In this paper, we will associate such line operators with topological Wilson lines, associated to pairs $(C, \rho)$, where $C$ is a curve and $\rho$ is a representation. For simplicity we focus on open string diagrams with a Wilson line on the boundary and local operator insertions in the bulk, so we will usually omit the curve $C$, and focus on the representation relevant to any given Wilson line.

2.3.1 Bulk Wilson lines: defects bridging universes

In most of this paper, for simplicity we will focus on boundary Wilson lines (Chan-Paton factors). In this section, we will discuss how bulk Wilson lines act as defects spanning universes in a decomposition.

Before discussing orbifolds specifically, it may be useful to consider a related example in a different theory. Specifically, consider $U(1)$ $BF$ theory at level $k$. This theory is known to decompose (see e.g. [76, appendix B]), and the local operators $O_p$ and Wilson lines $W_q$ obey clock-shift commutation relations (see e.g. [76, appendix B])

$$O_p W_q = \xi^{pq} W_q O_p, \quad (2.61)$$

where $\xi = \exp(2\pi i/k)$. Now, the projectors are

$$\Pi_m = \frac{1}{k} \sum_{n=0}^{k-1} \xi^{nm} O_n, \quad (2.62)$$

and it is straightforward to check that the clock-shift commutation relations are algebraically equivalent to

$$\Pi_m W_p = W_p \Pi_{m+p \mod k}. \quad (2.63)$$

In other words, moving a projector through a Wilson line changes the projector – bulk Wilson lines in abelian $BF$ theory act as defects linking different universes of the decomposition.

We shall see that the same effect happens in orbifolds, later in section 2.3.3.

2.3.2 Boundary Wilson lines: Decomposition of bundles and sheaves

Next we turn to boundary Wilson lines (Chan-Paton factors), which most of this paper will focus on. As mentioned previously, in principle sheaves and bundles on an orbifold $[X/\Gamma]_\omega$,
with trivially-acting \( K \subset \Gamma \), are equivalent to sheaves and bundles on the universes of decomposition. Briefly, the association is computed by comparing the (projective) representation of \( K \) associated to the bundle or sheaf with that associated to each universe of decomposition.

Let us walk through that more carefully. A bundle or sheaf on \([X/\Gamma]_\omega\) is a bundle or sheaf on \( X \) with a (projective) \( \Gamma \)-equivariant structure. If we restrict to \( K \subset \Gamma \), then since \( K \) acts trivially on \( X \), the \( \Gamma \)-equivariant structure determines a (projective) representation of \( K \), call it \( \rho \). (In passing, if \( \rho \) is a projective representation associated to a nontrivial element of \( H^2(\Gamma, U(1)) \), then \( \rho \) is necessarily of dimension greater than one.)

Now, as described earlier in section 2.1, we can associate a representation \( R \) of \( K \) to each universe of the decomposition. That representation is of the form

\[
R = R_1 \oplus \cdots \oplus R_\ell,
\]

(2.64)

which each \( R_i \) is an irreducible projective representation of \( K \), and the isomorphism classes \( \{[R_i]\} \) span the orbit of the \( G = \Gamma/K \) action on \( \hat{K} = \Gamma/\rho \).

In order to associate a given bundle or sheaf on \([X/\Gamma]_\omega\) to a bundle or sheaf on one of the universes of decomposition, the idea then is that we take the (projective) representation \( \rho \) of \( K \), derived from the (projective) \( \Gamma \)-equivariant structure, decompose into its irreducible components, and compare to the representation \( R \) associated to each of the universes. If a (sum of) component(s) match, that piece lives in the corresponding universe.

We shall see this correspondence in examples.

The projectors, defined in section 2.2.2, also have the property of projecting Wilson lines into the pertinent universes. After describing general aspects of the relation between twist fields and representations, we will discuss the pairing between the two provided by characters, and determine the action of the projectors on Wilson lines.

### 2.3.3 Pairing between twist fields and representations

In general terms, for a fixed group \( \Gamma \), although (\( \omega \)-regular) conjugacy classes are in one-to-one correspondence with (projective) representations, bijections are not canonical. Instead, the two are more nearly dual, with a canonical pairing defined by characters.

Recall that the character \( \chi_R(g) \) of a (possibly projective) representation \( R \) of a group \( \Gamma \), evaluated on group element \( g \in \Gamma \), is \( \text{Tr} \ T_gR \), where \( T_gR \) is a matrix representing \( g \) in \( R \). In the case of ordinary representations, characters are constant on conjugacy classes.

\(^9\)In fact, unlike conjugacy classes, for projective representations for a fixed \( [\omega] \in H^2(G, U(1)) \), there is no ring structure, as the product does not preserve \( \omega \): if \( R \) is projective with respect to \( \omega_1 \) and \( S \) is projective with respect to \( \omega_2 \) then \( R \otimes S \) is projective with respect to \( \omega_1 \otimes \omega_2 \). [49, section 7.1, prop. 1.2].
(more formally, they are class functions). For projective representations, their properties are somewhat different, as explained in e.g. [49, section 7.2, prop. 2.2]. For example:

- Characters no longer need be constant on conjugacy classes, and so need not be class functions. Instead,
  \[ \chi(g) = \frac{\omega(g, h^{-1})}{\omega(h^{-1}, hgh^{-1})} \chi(hgh^{-1}), \]  
  (2.65)

- If \( g \in \Gamma \) is not \( \omega \)-regular then \( \chi(g) = 0 \),

- If for any \( \omega \)-regular element \( g \in \Gamma \) and for any \( h \in \Gamma \),
  \[ \omega(g, h) = \omega(h, h^{-1}gh), \]  
  (2.66)

then the character is a class function.

The pairing between conjugacy classes and representations, via characters, is reflected in physics. Consider a Wilson loop in representation \( R \) of \( \Gamma \) encircling a twist operator \( \tau_g \) associated to \( g \in \Gamma \). The Wilson loop will cross a \( g \) branch cut, effectively inserting a \( g \) inside the Wilson loop, implemented by a matrix \( T^R_{\tau_g} \) describing \( g \) in representation \( R \). Except in special cases, \( T^R_{\tau_g} \) will not be proportional to the identity, and so the effect of \( \tau_g \) is not to merely multiply \( W^R \) by a phase – \( W^R \) need not be an eigenbrane of \( \tau_g \) in general.

Consider on the other hand the effect of multiplying \( \sigma_{[g]} \) into \( W^R \). From the definition (2.17), this should insert into the Wilson line \( W^R \) the matrix

\[ \frac{1}{|G|} \sum_{h \in G} \frac{\omega(h, g)\omega(hg, h^{-1})}{\omega(h, h^{-1})} T^R_{hgh^{-1}}. \]  
(2.67)

It can be shown that the quantity above commutes with other matrices (see appendix A), just as \( \sigma_{[g]} \) commutes with other local operators, hence the sum above is proportional to the identity, and a short computation demonstrates that

\[ \frac{1}{|G|} \sum_{h \in G} \frac{\omega(h, g)\omega(hg, h^{-1})}{\omega(h, h^{-1})} T^R_{hgh^{-1}} = \frac{\chi^R(g)}{\dim R} I, \]  
(2.68)

(See appendix A for further details.)

As an upshot of this, we then have that

\[ \sigma_{[g]} W^R = \frac{\chi^R(g)}{\dim R} W^R \sigma_{[g]}, \]  
(2.69)

\[ ^{10} \text{For } g \in \Gamma \text{ to be } \omega \text{-regular means that } \omega(g, h) = \omega(h, g) \text{ for all } h \text{ commuting with } g. \]

\[ ^{11} \text{For example, in the special case that } R \text{ is a one-dimensional representation, then the matrix } T^R_{\tau_g} \text{ is just the number } \chi^R(g), \text{ and in that case, } \tau_g W^R = \chi^R(g) W^R. \text{ That will not be the case in general, however.} \]

\[ ^{12} \text{See also e.g. [40, footnote 10] for the special case of one-dimensional representations } R \text{ and no discrete torsion, and also [3, equ’n (2.2)].} \]
for bulk Wilson lines, an analogue of the clock-shift commutation relations of BF theory discussed in section 2.3.1 and
\[ \sigma_{[g]} W_R = \frac{\chi_R(g)}{\dim R} W_R, \]

for boundary Wilson lines, which we will confirm in examples. In particular, in the boundary case, although \( W_R \) is not an eigenbrane of \( \tau_g \), it is an eigenbrane of \( \sigma_{[g]} \).

Before going on, let us check the consistency of this expression. Specifically, if there is nonzero discrete torsion, two of the factors are ambiguous: both \( \sigma_{[g]} \) and \( \chi_R(g) \) depend upon the choice of representative of the conjugacy class, and different representatives give results that differ by phases. Specifically, if \( g' = hgh^{-1} \), then from equation (2.25) and [47, section 3],
\[ \sigma_{[hgh^{-1}]} = \frac{\omega(hgh^{-1}, h)}{\omega(h, gh^{-1})} \sigma_{[g]} \].

Similarly, if there is discrete torsion, so that the representation \( R \) is projective, the characters \( \chi_R \) are not true class functions – they are not constant on conjugacy classes – but instead obey [49, section 7.2, prop. 2.2]
\[ \chi_R(hgh^{-1}) = \frac{\omega(h^{-1}, hgh^{-1})}{\omega(g, h^{-1})} \chi_R(g). \]

Indeed, from the product
\[ (d\omega)(h^{-1}, h, gh^{-1}) (d\omega)(g, h^{-1}, h) = 1, \]
on one finds
\[ \frac{\omega(h^{-1}, hgh^{-1})}{\omega(g, h^{-1})} = \frac{\omega(gh^{-1}, h)}{\omega(h, gh^{-1})}, \]
and so we see that
\[ \sigma_{[hgh^{-1}]} W_R = \frac{\chi_R(hgh^{-1})}{\dim R} W_R, \]
as needed for consistency.

The relations (2.69), (2.70) are similar to the usual relation between ’t Hooft loops and Wilson lines, which is of the schematic form (see e.g. [61, equ’n (3.8)])
\[ T \cdot W = (\text{phase}) W \cdot T \]
in three and four dimensions. As twist fields generate branch cuts, they are natural analogues of ’t Hooft loops in higher dimensions, and so the analogy with ’t Hooft-Wilson line relations is precise.
2.3.4 Action of projectors on Wilson lines

Now, let us apply projectors $\Pi_R$ (defined in subsection 2.2.2) to Wilson lines. We will see that the projectors commute with bulk Wilson lines in such a way as to describe defects bridging universes, and the projectors project boundary Wilson lines into corresponding universes, in the fashion outlined in subsection 2.3.2.

The reader should note that

- $R$ is a representation of the trivially-acting subgroup $K$. For the projector $\Pi_R$ corresponding to any given universe, $R$ is a sum of representatives of the orbit of $G = \Gamma/K$ in $\tilde{K}_r\omega$, and so is of the form
  \[ R = R_1 \oplus \cdots \oplus R_\ell. \]  
  \hspace{1cm} (2.77)

- $S$, on the other hand, is a representation of $\Gamma$. In evaluating the action of $\Pi_R$ on $W_S$, we effectively restrict $S$ to $K$. Even if $S$ itself were irreducible, in general its restriction is reducible, and so $S|_K$ can also be written as a sum of irreducible (projective) representations of $K$.

- We will see in appendix C that all representations of $K$ appear as summands in restrictions of representations of $\Gamma$; however, we will see in examples in e.g. sections 4.1.4, 4.1.5 that some irreducible representations of $K$ only appear as summands and never as the entire restriction of an irreducible representation of $\Gamma$.

First, we consider bulk Wilson lines. Here, for simplicity, we specialize for the moment to the special case of a $G = \mathbb{Z}_k$ orbifold, where all of $G$ acts trivially. Let $\sigma_j$ denote the twist field associated to $g^j$, for $g$ the generator of $G$. Projectors are of the form

$$\Pi_i = \frac{1}{n} \sum_{j=0}^{n-1} \xi^{ij} \sigma_j$$  \hspace{1cm} (2.78)

for $\xi = \exp(2\pi i/k)$. Applying (2.69), we find

$$\Pi_i W_R = \frac{1}{k} \sum_j \xi^{ij} \sigma_j W_R,$$  \hspace{1cm} (2.79)

$$= \frac{1}{k} \chi_R(g^j) W_R \sigma_j;$$  \hspace{1cm} (2.80)

using the fact that for $G = \mathbb{Z}_k$, all irreducible representations have dimension 1. Now, suppose $R$ is the $n$th irreducible representation, meaning $\chi_R(g^j) = \xi^m$, then

$$\Pi_i W_R = W_R \left( \frac{1}{k} \sum_j \xi^{i(j+n)} \sigma_j \right),$$  \hspace{1cm} (2.81)

$$= W_R \Pi_{i+n \mod k},$$  \hspace{1cm} (2.82)
so that again, moving the projector through the Wilson line changes the projector, just as in $BF$ theory as in section 2.3.1. We will see closely analogous relations of this form in less trivial examples later in this paper.

Next, we consider boundary Wilson lines, returning to more general orbifold groups $G$. For simplicity, and without loss of significant generality, let us restrict to a single irreducible (projective) representation $R$ of $K$, and let us denote by $S$ a single summand in the restriction of a representation of $\Gamma$ to $K$.

Then, for boundary Wilson lines, using the pairing (2.70) and the identities (B.2), (B.11), we see that

$$\Pi_R W_S = \frac{\dim R}{|K|} \sum_{k \in K} \frac{\chi_R(k^{-1}) \chi_S(k)}{\omega(k, k^{-1})} \dim S \omega(k, k^{-1}) W_S,$$

$$= \frac{\dim R}{\dim S} \sum_{k \in K} \frac{\chi_R(k) \chi_S(k)}{\omega(k, k^{-1})} W_S,$$

$$= \delta_{R,S} W_S.$$  

In effect, the $W_S$ are eigenbranes of the projectors, with eigenvalues 0, 1.

Walking through the details of the prototypical computation above, the effect of acting on $W_S$ with $\Pi_R$ is to restrict the $\Gamma$ representation $S$ to $K$, and then project into universes based on comparing that projection with the $K$ representations associated with each universe. This is a computational realization of the correspondence described in subsection 2.3.2 and [20] between bundles and sheaves on gerbes and on universes – the projectors implement the mathematical statement that bundles and sheaves on gerbes decompose into bundles and sheaves on constituent universes, by picking out the Wilson lines that exist on each universe. We will see this explicitly in examples later.

Next, we will discuss symmetries of these theories.

2.4 Symmetries

Now that we have thoroughly set up the mechanics of dimension-zero twist fields, projectors, Wilson lines, and corresponding geometries, let us take a moment to describe the (possibly noninvertible) one-form symmetries present in theories obeying decomposition. We will propose that invertibility of the I-form symmetries is correlated to properties of the representations characterizing the universes: one-form symmetries go hand-in-hand with universes corresponding to one-dimensional representations, and propose that noninvertible (one-form) symmetries are related to universes corresponding to higher-dimensional representations.
It is sometimes said that a \( q \)-form symmetry in \( d \) dimensions is generated by an operator

\[
U_g(M_{d-q-1}),
\]

supported on a codimension-(\( q + 1 \)) submanifold \( M \), acting as a ’t Hooft operator. It is invertible if it obeys an (abelian) group law

\[
U_q U_q' = U_{qq'},
\]

while an analogous noninvertible symmetry has a generator on a submanifold of the same codimension, but obeying a fusion algebra instead of a group law:

\[
U_q U_q' = \sum_p N^p_{qq'} U_p.
\]

We are focused on two-dimensional theories, hence our symmetry generators are supported at points. Given that they generate branch cuts, it is natural to identify the \( U_g \) with twist fields (see e.g. \[8\] section III), which in two-dimensional orbifolds play a role analogous to Gukov-Witten operators \[13,14\] in higher dimensions. Specifically, we are interested in twist fields associated to conjugacy classes whose elements lie in the trivially-acting subgroup \( K \).

If the trivially-acting subgroup \( K \) is a subgroup of the center of the orbifold group \( \Gamma \), and there is no discrete torsion – corresponding to a banded abelian gerbe – then each conjugacy class has one element, and the conjugacy classes obey a group law:

\[
\sigma_g \sigma_{g'} = \sigma_{gg'}.
\]

In this case, the twist fields \( \sigma_g \) are invertible, and the universes of decomposition are all identical (modulo theta angle / discrete torsion phases). As a consequence, there is an ordinary one-form symmetry, no noninvertible symmetry.

In more general cases, if \( K \) is not in the center, twist fields are typically associated to conjugacy classes containing multiple elements of \( K \). In such cases, as we have computed in section 2.2.1 (and will see in examples later), the twist fields obey a fusion algebra structure:

\[
\sigma_{[g]} \sigma_{[g']} = \sum_h N^h_{gg'} \sigma_{[h]}
\]

in general. Mathematically, at least in the absence of discrete torsion, these correspond to nonbanded abelian and nonabelian gerbes. In these cases, the universes of decomposition are not all identical, and at least some of the twist fields \( \sigma_{[g]} \) are noninvertible. These would appear to correspond to noninvertible symmetries.

All that said, there is an important subtlety: abstractly in the vector space of dimension-zero operators spanned by the twist fields, in general there is not to our knowledge a natural mathematical way to distinguish twist fields associated to conjugacy classes from general
linear combinations of twist fields (spanning the center of the twisted group algebra). (In fact, some of the group structure is also washed out by the group algebra. For example, $\mathbb{C}[\mathbb{Z}_4] \cong \mathbb{C}[\mathbb{Z}_2 \times \mathbb{Z}_2]$ even though the groups $\mathbb{Z}_4$, $\mathbb{Z}_2 \times \mathbb{Z}_2$ are not isomorphic.) In special cases, there may be a pertinent construction, a means to detect “group-like” elements. For example, in any group ring $\mathbb{C}[G]$ for finite $G$, there is a comultiplication $\Delta$ that maps $g \mapsto g \otimes g$ for $g \in G$, and is extended linearly to other elements of the group ring. (This action of the comultiplication is one of the defining properties of group-like elements of a Hopf algebra.) One can define group-like elements to be those elements $\sigma$ for which $\Delta(\sigma) = \sigma \otimes \sigma$. However, the $\sigma$’s only correspond to $g$’s in a group ring in the special cases of an abelian orbifold group without discrete torsion, and so in general this cannot be used to identify twist fields associated with conjugacy classes in a generic ring of dimension-zero operators. Similarly, in the symmetry categories discussed in e.g. [9], one can define ‘simple’ objects (namely, objects $a$ such that $\text{Hom}(a, a) \cong \mathbb{C}$); however, we do not know of an analogous construction here.

Furthermore, as we have seen, the vector space of dimension-zero operators always include noninvertible operators, regardless of whether $K$ is central. Put another way, even if there are no noninvertible symmetries, even if all the higher symmetries are (invertible) one-form symmetries, then nevertheless the theory still contains noninvertible dimension-zero operators, noninvertible linear combinations of twist fields, that form a codimension-one subspace of the space of all linear combinations of twist fields (the center of the twisted group algebra).

To be clear, the vector space of dimension-zero operators does have a natural basis of twist fields, so one can try to use those elements. For banded abelian gerbes, this appears to work well, as outlined above. However, for nonbanded gerbes, this approach runs into subtleties. For example, for the nonbanded $\mathbb{Z}_4$ gerbe we will discuss in section 4.2 the projection operators onto the two identical universes in the decomposition (which should be related by a $B\mathbb{Z}_2$ symmetry) mix invertible and noninvertible twist fields, and the projection operator onto the third, distinct, universe, involves only invertible twist fields.

This suggests that a more invariant method to identify noninvertible 1-form symmetries may be desirable. To that end, in the spirit of [9,12], we propose instead to associate symmetries with representations classified by representations. One-dimensional representations obey a group-like multiplication. Higher-dimensional representations obey a more general algebra, and so in general are not invertible. For our purposes, we will see in examples that it is more useful to characterize noninvertible symmetries in terms of dimensions of corresponding representations.

In terms of that dictionary, we can make a very explicit connection to decomposition. As

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\[13\] This is in the spirit of topological defect lines. In this language, intersection vertices of topological defect lines carry complex vector spaces of operators – the twist fields $\sigma_{[g]}$. The operators $\tau_g$ are naturally associated with endpoints of topological defect lines.
we noted in section 2.2.2 associated to each universe is a (projective) representation $R$ of $K$ (a direct sum of representatives of the elements of a $G$ orbit on $K_{*}\omega$), which corresponds to a projector $\Pi_R$. Universes for which $R$ is a one-dimensional representation are, in general, identical to one another (modulo theta angle / discrete torsion shifts), and so are related by ordinary one-form symmetries. Universes for which $R$ has higher dimension need not be the same, and are related by noninvertible symmetries. Thus, properties of decomposition naturally tie into this proposed characterization of invertibility of higher-form symmetries: in general terms, copies of universes are associated with (invertible) one-form symmetries, whereas distinct universes are associated with noninvertible symmetries.

3 Example: Sigma model on a disjoint union

Before studying orbifolds with trivially-acting subgroups, let us first consider sigma models whose targets are disjoint unions, to clarify expectations in an example which, by virtue of decomposition, will often be equivalent to the orbifolds we will study later.

Consider first a sigma model on a disjoint union of $n$ copies of the space $X$. For simplicity, we assume $X$ is Calabi-Yau, so that the sigma model is a CFT. As discussed in e.g. [37], this theory has a $\mathbb{Z}_n^{(1)} = B\mathbb{Z}_n$ (one-form) symmetry. The spectrum of this theory contains

- an $n$-dimensional space of dimension-zero operators, with basis given by the identity operators on each component,

- a set of $n$ projection operators $\Pi_k$, each dimension zero, obeying

$$\Pi_k \Pi_\ell = \delta_{k,\ell} \Pi_k, \quad \sum_k \Pi_k = 1, \quad (3.1)$$

and each corresponding to a (projective) representation of $K$.

We can recast this theory in the form of an orbifold of $X$ by a trivially-acting $\mathbb{Z}_n$, as follows. Let $\xi$ generate $k$th roots of unity, and define

$$y = \sum_{k=0}^{n-1} \xi^k \Pi_k. \quad (3.2)$$

Then, it is straightforward to check that

$$y^p = \sum_{k=0}^{n-1} \xi^{pk} \Pi_k, \quad (3.3)$$

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and \( y^p y^q = y^{p+q} \), with identity given by

\[
y^0 = 1 = \sum_{k=0}^{n-1} \Pi_k,
\]

(3.4)

and relation \( y^n = 1 \).

The ring of dimension-zero operators is then given by \( \mathbb{C}[y]/(y^n - 1) \). There is one sum-
mund in the disjoint union, one universe, for every solution of \( y^n = 1 \). In addition to the
projection operators, which are clearly noninvertible, this ring has a codimension-one space
of noninvertible operators. We will explore the structure of this ring in greater detail in
section 4.1, when we find the same ring of dimension-zero operators in orbifolds describing
banded \( \mathbb{Z}_n \) gerbes (i.e., orbifolds with trivially-acting central \( \mathbb{Z}_n \)). (Indeed, from decomposi-
tion, such orbifolds are physically equivalent to sigma models on disjoint unions, so the same
structure is expected.)

As discussed in detail in [37], a sigma model on a disjoint union of \( n \) identical copies of
a space has a \( B\mathbb{Z}_n \) (one-form) symmetry. In the present language, that one-form symmetry
acts by phases:

\[
y \mapsto \xi y,
\]

(3.5)

for \( \xi \) an \( n \)th root of unity. This is a symmetry of the ring, leaving the relation \( y^n = 1 \)
invariant. It does, however, interchange the vevs of \( y \) at which the ring has support, the
values of \( y \) for which \( y^n = 1 \). This structure is reminiscent of spontaneous symmetry breaking,
in that the different vacua are interchanged by the action of the symmetry, but unlike
spontaneous symmetry breaking, here we have a decomposition and not mere superselection
sectors, in this case by explicit construction of the disjoint union.

Next, consider a sigma model with target a disjoint union of two different Calabi-Yau
spaces, say \( X \) and \( Y \). This theory has a \( B\mathbb{Z}_2 \) action, but not a symmetry. The ring of
dimension-zero operators is two-dimensional, from linear combinations of the pair of identity
operators, and has two projectors \( \Pi_X, \Pi_Y \):

\[
\Pi^2_X = \Pi_X, \quad \Pi^2_Y = \Pi_Y, \quad \Pi_X \Pi_Y = 0 = \Pi_Y \Pi_X, \quad 1 = \Pi_X + \Pi_Y.
\]

(3.6)

In principle, we could define the same structure on the ring that we have discussed
previously, defining

\[
z = \Pi_X - \Pi_Y,
\]

(3.7)

so that \( z^2 = 1 \), but the ring would then have a symmetry \( z \mapsto -z \) which is not reflected in
the physics. A little more generally, we could instead define

\[
z = \Pi_X - \alpha \Pi_Y,
\]

(3.8)
for some number $\alpha \neq 0, -1$, so that

$$\Pi_X = \frac{1}{2} (1 + \alpha^{-1} z), \quad \Pi_Y = \frac{1}{1 + \alpha} (1 - z), \quad (3.9)$$

and

$$z^2 = \left( \frac{1 + \alpha + 2\alpha^2}{2(1 + \alpha)} \right) + \left( \frac{1 + \alpha - 2\alpha^3}{2\alpha(1 + \alpha)} \right) z, \quad (3.10)$$

in other words a deformation of the ring relation $z^2 = 1$. We will see an orbifold that gives a ring of this form in section 4.3.

Finally, let us comment on the boundary Wilson lines. If $W_i$ is a brane corresponding in the $i$th universe, then

$$\Pi_i W_j = \delta_{ij} W_i. \quad (3.11)$$

We will see the same structure when discussing Wilson lines in orbifolds.

### 4 Examples in orbifolds

Next we will describe these structures in orbifold examples. In each case, we will explicitly describe the ring of dimension zero operators (twist fields, fusion algebra, projectors, support loci, and loci of noninvertible operators), as well as the corresponding structure of Wilson lines. We will also discuss symmetries in each example, describing both ordinary one-form and noninvertible symmetries, and their correspondence to the structure of the universes.

Many of these examples have appeared previously in the literature, where projectors were often computed, on an ad-hoc basis. We will see that the general formula for projectors in section 2.2.2 correctly predicts the projectors worked out previously.

#### 4.1 Banded $\mathbb{Z}_n$ gerbe

Suppose our banded $\mathbb{Z}_n$ gerbe is an orbifold $[X/\Gamma]$, where $\mathbb{Z}_n$ is a (trivially-acting) subgroup of the center of $\Gamma$. Let $G = \Gamma/K$, then in general terms decomposition predicts

$$\text{QFT} ([X/\Gamma]) = \text{QFT} \left( \prod_K [X/G]_{\hat{\omega}} \right) \quad (4.1)$$

for choices of discrete torsion $\hat{\omega}$.

In this section we begin by considering general aspects of banded $\mathbb{Z}_n$ gerbes. We include some specific concrete examples.
4.1.1 Ring of dimension-zero operators

Let \( g \) denote a generator of \( \mathbb{Z}_n \). In a sigma model on a banded \( \mathbb{Z}_n \) gerbe, meaning an orbifold or a gauge theory with a trivially-acting central \( \mathbb{Z}_n \), there exist dimension-zero twist fields corresponding to the elements of \( \mathbb{Z}_n \), namely

\[
\sigma_1 = 1, \quad \sigma_g, \quad \sigma_{g^2} = \sigma_g^2, \quad \cdots \quad \sigma_{g^{n-1}} = \sigma_g^{n-1},
\]

(4.2)

where \( \sigma_g^n = 1 \).

This ring can be described as \( \mathbb{C}[y]/(y^n - 1) \), where \( y \) is identified with \( \sigma_g \).

The reader should note that

\[
\text{Spec } \mathbb{C}[y]/(y^n - 1) = n \text{ points } = \prod_{n} \text{pt},
\]

(4.3)

which corresponds to the fact that there are \( n \) universes in this example. In fact, the locations of the points, \( y \) such that \( y^n = 1 \), correspond to expectation values of the physical order parameter which distinguish the various universes.

Next, let us compute the projectors. There are \( n \) irreducible representations of \( K = \mathbb{Z}_n \), all one-dimensional and invertible, and in the banded case, because \( K \) is central, the representations are invariant under \( G = \Gamma/K \), so there is a one-to-one correspondence between irreducible representations of \( K \) and universes. Label those irreducible representations / universes as \( \rho_i \), so that \( \rho_0 \) is the identity operator, and \( i \) is counted mod \( n \), then their products are simply

\[
\rho_i \otimes \rho_j = \rho_{i+j}.
\]

(4.4)

If we let \( g \) denote the generator of \( \mathbb{Z}_n \), \( \xi = \exp(2\pi i/n) \), and let representation \( i \) denote the representation with character

\[
\chi_i(g^k) = \xi^{-ik},
\]

(4.5)

then the general expression for projectors (2.44) reduces to

\[
\Pi_k = \frac{1}{n} \sum_{i=0}^{n-1} \xi^{ik} \sigma_g^k = \frac{1}{n} \sum_{i=0}^{n-1} \xi^{ik} y^i,
\]

(4.6)

in the notation above. These projectors – whose derivation we have outlined from (2.44) – are precisely of the expected form. It is straightforward to check that

\[
\Pi_k \Pi_\ell = \delta_{k,\ell} \Pi_k, \quad \sum_k \Pi_k = 1.
\]

(4.7)

Let \( p_i \) be the point \( y = \xi^{-i} \), then it is straightforward to check that

\[
\Pi_k|_{p_i} = \delta_{k,i}.
\]

(4.8)
Now, let us compute the noninvertible elements of these rings. The noninvertible elements will include operators proportional to projectors, but can also include other elements as well.

First, consider the case \( n = 2 \). Here, it is straightforward to compute that

\[
(a + by)^{-1} = \Delta^{-1} (a - by),
\]

(4.9)

for

\[
\Delta = a^2 - b^2.
\]

(4.10)

Thus, the dimension-zero operator \( a + by \) is noninvertible precisely when \( a^2 = b^2 \). In this simple case the projectors onto universes are proportional to \( 1 \pm y \), so we see immediately that for \( n = 2 \), the non-invertible elements are proportional to projectors. (For higher \( n \), only some non-invertible elements will be proportional to projectors.)

One efficient way to do this computation is to identify \( y \) with the matrix

\[
\begin{bmatrix}
0 & 1 \\
1 & 0 
\end{bmatrix},
\]

(4.11)

which encodes the fact that \( y^2 = 1 \). Then,

\[
a + by \sim \begin{bmatrix} a & b \\ b & a \end{bmatrix},
\]

(4.12)

and the inverse above can be read off immediately.

Next, consider the case \( n = 3 \). Here, we identify

\[
y \sim \begin{bmatrix}
0 & 0 & 1 \\
1 & 0 & 0 \\
0 & 1 & 0 
\end{bmatrix}, \quad y^2 \sim \begin{bmatrix}
0 & 1 & 0 \\
0 & 0 & 1 \\
1 & 0 & 0 
\end{bmatrix},
\]

(4.13)

and proceeding as before one finds

\[
(a_0 + a_1 y + a_2 y^2)^{-1} = \Delta^{-1} \left( a_0^2 - a_1 a_2, a_2^2 - a_0 a_1, a_1^2 - a_0 a_2 \right),
\]

(4.14)

where

\[
\Delta = a_0^3 + a_1^3 + a_2^3 - 3a_0 a_1 a_2,
\]

(4.15)

\[
= (a_0 + a_1 + a_2)(a_0^2 - a_0 a_1 + a_1^2 - a_0 a_2 - a_1 a_2 + a_2^2).
\]

(4.16)

Thus, the noninvertible operators lie along the locus \( \{ \Delta = 0 \} \). This locus include operators proportional to the projectors, but as those lie along one-dimensional lines, whereas \( \{ \Delta = 0 \} \) has dimension two. For example, the operator \( (1 + y - 2y^2) \) is not invertible, but is also not proportional to a projector.
While considering the case $n = 3$, let us also look in more detail at the projectors and maximal ideals. The three points at which the order parameters have nonzero vevs are $y = \xi^{i}$, for $\xi$ a third root of unity, so the corresponding maximal ideals are

$$m_0 = (y - 1), \quad m_1 = (y - \xi^{-1}), \quad m_2 = (y - \xi^{-2}), \quad (4.17)$$

and the corresponding projectors are

$$\Pi_0 = \frac{1}{3} (1 + y + y^2), \quad \Pi_1 = \frac{1}{3} (1 + \xi y + \xi^2 y^2), \quad \Pi_2 = \frac{1}{3} (1 + \xi^2 y + \xi y^2). \quad (4.18)$$

It is straightforward to verify that

$$\Pi_0|_{y=1} = 1 = \Pi_1|_{y=\xi^{-1}} = \Pi_2|_{y=\xi^{-2}}, \quad (4.19)$$

with other restrictions vanishing, and furthermore

$$\begin{align*}
(y - \xi^{-1})(y - \xi^{-2}) &= 1 + y + y^2 \propto \Pi_0, \\
\xi^2(y - 1)(y - \xi^{-2}) &= 1 + \xi y + \xi^2 y^2 \propto \Pi_1, \\
\xi(y - 1)(y - \xi^{-1}) &= 1 + \xi^2 y + \xi y^2 \propto \Pi_2,
\end{align*}$$

hence

$$\Pi_0 \in m_1 \cap m_2, \quad \Pi_1 \in m_0 \cap m_2, \quad \Pi_2 \in m_0 \cap m_1. \quad (4.23)$$

Similarly,

$$\{\Pi_0 = 0\} \cap \{\Pi_1 = 0\} = \{y = \xi^{-2}\}, \quad (4.24)$$

and so forth.

Finally, for the case $n = 4$, one can similarly demonstrate that amongst the operators $a_0 + a_1 y + a_2 y^2 + a_3 y^3$, the noninvertible operators lie along the locus $\{\Delta = 0\}$ for

$$\begin{align*}
\Delta &= a_0^4 - a_1^4 + 4a_0a_1^2a_2 - 2a_0^2a_2^2 + a_2^4 - 4a_0^2a_1a_3 - 4a_1a_2a_3 - 2a_1^2a_3^2 + 4a_0a_2a_3^2 - a_3^4, \\
&= (a_0 + a_1 + a_2 + a_3)(a_0 - a_1 + a_2 - a_3)(a_0 + a_1 + a_2 - a_3)(a_0^2 + a_1^2 + a_2^2 + a_3^2). \quad (4.25)
\end{align*}$$

For completeness, the maximal ideals describing order parameters at which the universes have support are

$$m_0 = (y - 1), \quad m_1 = (y - \xi^{-1}), \quad m_2 = (y - \xi^{-2}), \quad m_3 = (y - \xi^{-3}), \quad (4.26)$$

where $\xi$ generates fourth roots of unity, and the projectors are

$$\begin{align*}
\Pi_0 &= \frac{1}{4} (1 + y + y^2 + y^3), \quad \Pi_1 = \frac{1}{4} (1 + \xi y + \xi^2 y^2 + \xi^3 y^3), \\
\Pi_2 &= \frac{1}{4} (1 + \xi^2 y + y^2 + \xi^2 y^3), \quad \Pi_3 = \frac{1}{4} (1 + \xi^3 y + \xi^2 y^2 + \xi y^3).
\end{align*}$$

(4.27)
When restricted to points, the restriction of $\Pi_i$ is nonzero only at $y = \xi^{-i}$, and furthermore, for example,

$$(y - \xi^{-1})(y - \xi^{-2})(y - \xi^{-3}) = 1 + y + y^2 + y^3 \propto \Pi_0,$$

hence

$$\Pi_0 \in m_1 \cap m_2 \cap m_3,$$

and so forth.

### 4.1.2 Wilson lines

Let us return momentarily to the presentation of banded $\mathbb{Z}_k$ gerbes as orbifolds $[X/\Gamma]$ where a central subgroup $\mathbb{Z}_k \subset \Gamma$ acts trivially. As reviewed earlier in section 2.3, bulk Wilson lines act as defect operators linking universes, and boundary Wilson lines are stratified by choices of universe – sheaves and bundles on $[X/\Gamma]$ are equivalent to sheaves and bundles on the constituent universes $[X/G]_{\hat{\omega}}$, for $G = \Gamma/\mathbb{Z}_k$.

In fact, since $\mathbb{Z}_k$ is central, we can make a slightly stronger statement: representations of $\Gamma$ decompose into projective representations of $G$, with projectivity determined by the restriction of $\Gamma$ to $K$, which determines the discrete torsion $\hat{\omega}$.

In principle, the projectors $\Pi_k$ described in the previous subsection should project onto the representations associated with various universes in the decomposition, which we will confirm in examples.

### 4.1.3 Symmetries

Let us briefly make some general comments on one-form symmetries in banded $\mathbb{Z}_n$ gerbes. Here, since $K = \mathbb{Z}_n$ acts trivially, and is both abelian and central, the twist fields all correspond to single elements of $K$: $\sigma_g$ rather than merely $\sigma_{[g]}$. As a result, the twist fields obey a group-like multiplication:

$$\sigma_g \sigma_{g'} = \sigma_{gg'},$$

and so define invertible symmetries.

Hand-in-hand, the representations $R$ associated to the universes are all one-dimensional, because irreducible representations of $K$ are one-dimensional (since $K$ is abelian), and because the action of $G$ is trivial, so orbits of the $G$ action consist of single elements of $\hat{K}$.

We emphasize that even in these examples, there exist non-invertible dimension-zero twist fields, defined by the locus $\{\Delta = 0\}$ computed earlier, even though this theory contains only invertible symmetries.
4.1.4 Specific example: $\Gamma = D_4, K = \mathbb{Z}_2$

Next, let us consider a specific concrete example of a banded $\mathbb{Z}_2$ gerbe, namely an orbifold $[X/D_4]$, where $D_4$ is the eight-element dihedral group, and the center $\mathbb{Z}_2$ acts trivially. (This example has been previously discussed in e.g. [17, section 2.0.1], [20, section 5.2].)

We denote the elements of $D_4$ by

$$\{1, z, a, b, az, bz, ab, ba = abz\}, \quad (4.32)$$

where $z$ generates the center, $a^2 = z^2 = 1$, and $b^2 = z$.

Since the trivially-acting subgroup is (in) the center, this is a banded gerbe, and this theory admits a $B\mathbb{Z}_2$ symmetry.

Most of the analysis of this example proceeds as in the rest of this subsection. For example, there are two irreducible representations of $\mathbb{Z}_2$, call them $\pm$, which are invariant under $D_4/\mathbb{Z}_2 = \mathbb{Z}_2 \times \mathbb{Z}_2$ as $K$ is central. From the general formula (2.44) one quickly computes that the projectors are

$$\Pi_{\pm} = \frac{1}{2} (1 \pm \tau_z), \quad (4.33)$$

where $z$ denotes the generator of the $\mathbb{Z}_2$ center.

Comparing to Wilson lines is more interesting in this example. The group $D_4$ has five irreducible representations, four one-dimensional, and one two-dimensional. The two-dimensional representation can be given explicitly as

$$a = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad b = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \quad z = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}, \quad (4.34)$$

and the character table is

|     | \{1\} | \{z\} | \{a, az\} | \{b, bz\} | \{ab, ba\} |
|-----|-------|-------|----------|----------|----------|
| 1   |  1   |  1   |  1       |  1       |  1       |
| $1_a$ |  1   |  1   |  1       |  1       |  1       |
| $1_b$ |  1   |  1   | -1       |  1       | -1       |
| $1_{ab}$ |  1   |  1   | -1       | -1       |  1       |
| 2   |  2   | -2   |  0       |  0       |  0       |

From the character table, we can see explicitly that each of the one-dimensional representations $1, 1_a, 1_b, 1_{ab}$ restricts to the trivial one-dimensional representation of $\mathbb{Z}_2$. The nontrivial one-dimensional representation of $\mathbb{Z}_2$ is not the restriction of any one-dimensional
representation of $D_4$. However, the two-dimensional representation restricts to the sum of two copies of the nontrivial one-dimensional representation of $Z_2$.

It is straightforward to check that bulk Wilson lines obey

$$\Pi_+ W_{1,1_a,1_b,1_ab} = W_{1,1_a,1_b,1_ab} \Pi_+, \quad \Pi_+ W_2 = W_2 \Pi_-,$$  \hspace{1cm} \text{(4.35)}

$$\Pi_- W_{1,1_a,1_b,1_ab} = W_{1,1_a,1_b,1_ab} \Pi_-, \quad \Pi_- W_2 = W_2 \Pi_+,$$  \hspace{1cm} \text{(4.36)}

so that $W_2$ acts as a defect linking the two universes, and that boundary Wilson lines obey

$$\Pi_+ W_{1,1_a,1_b,1_ab} = W_{1,1_a,1_b,1_ab}, \quad \Pi_+ W_2 = 0,$$  \hspace{1cm} \text{(4.37)}

$$\Pi_- W_{1,1_a,1_b,1_ab} = 0, \quad \Pi_- W_2 = W_2.$$  \hspace{1cm} \text{(4.38)}

As expected, the four one-dimensional representations of $D_4$ correspond to the four one-dimensional ordinary representations of $Z_2 \times Z_2$, and the two-dimensional irreducible representation of $D_4$ corresponds to the single irreducible (two-dimensional) projective representation of $Z_2 \times Z_2$ [49, section 3.7].

Not only does the counting match, but indeed so do the representations themselves. Since the four one-dimensional representations of $D_4$ are trivial on the center generator $z$, they descend to four honest representations of $D_4/Z_2 = Z_2 \times Z_2$, which are easily checked from the character table to be distinct. Similarly, the fact that the two-dimensional representation of $D_4$ is nontrivial on $z$ means that it descends to a projective representation of $D_4/Z_2$. Explicitly, if we let the generators of $Z_2 \times Z_2$ be denoted $\bar{a}, \bar{b}$, and define a representation $\rho$ by

$$\rho(\bar{a}) = \left[ \begin{array}{cc} -1 & 0 \\ 0 & 1 \end{array} \right], \quad \rho(\bar{b}) = \left[ \begin{array}{cc} 0 & -1 \\ 1 & 0 \end{array} \right], \quad \rho(\bar{a}\bar{b}) = \left[ \begin{array}{cc} 0 & 1 \\ 1 & 0 \end{array} \right]$$  \hspace{1cm} \text{(4.39)}

(the images of $az$, $b$, $abz$ under the two-dimensional $D_4$ representation), then we find that $\rho$ is a projective representation of $Z_2 \times Z_2$, coinciding with that given in [49, section 3.7].

Finally, we observe that the projectors $\Pi_\pm$ correctly select out the representations associated with each universe. Specifically, $\Pi_+$ projects onto the four one-dimensional representations of $D_4$, which we have seen descend to honest representations of $Z_2 \times Z_2$, and $\Pi_-$ projects onto the irreducible two-dimensional representation of $D_4$, which descends to a projective representation of $Z_2 \times Z_2$. Thus, the projectors are correctly projecting onto boundary Wilson lines associated with the two universes.

If instead we had used $\Gamma = \mathbb{H}$, the eight-element group of unit quaternions, and picked the trivially-acting subgroup to be the center $K = Z_2$, we would have identical results: the projectors have the same form, $\mathbb{H}$ has five irreducible representations, four of which are one-dimensional, one of which is two-dimensional, and the character table is the same as for $D_4$, so that the four one-dimensional representations all restrict to the trivial representation
of $K = \mathbb{Z}_2$, and the two-dimensional representation restricts to two copies of the nontrivial representation of $\mathbb{Z}_2$.

The banded gerbe $[X/\mathbb{H}]$ has the same decomposition as for $D_4$:

$$QFT ([X/\mathbb{H}]) = QFT \left( [X/\mathbb{Z}_2 \times \mathbb{Z}_2] \coprod [X/\mathbb{Z}_2 \times \mathbb{Z}_2]_{d.t.} \right).$$

The five irreducible representations of $\mathbb{H}$ also naturally decompose into four ordinary irreducible representations of $\mathbb{Z}_2 \times \mathbb{Z}_2$ plus one projective (two-dimensional) irreducible representation of $\mathbb{Z}_2 \times \mathbb{Z}_2$, which are selected by the corresponding projectors, as expected.

### 4.1.5 Specific example: $\Gamma = D_8$, $K = \mathbb{Z}_2$

In this section we consider one more example of a banded gerbe, namely, $[X/D_8]$ with the central $K = \mathbb{Z}_2 \subset D_8$ acting trivially.

The group $D_8$ is the sixteen-element dihedral group generated by $\tilde{a}$, $\tilde{b}$, subject to the relations

$$\tilde{a}^2 = 1, \quad \tilde{b}^8 = 1, \quad \tilde{a}\tilde{b}\tilde{a} = \tilde{b}^{-1} = \tilde{b}^7.$$  

(4.41)

The center is generated by $\tilde{b}^4$.

Computing e.g. the genus-one partition function, one can quickly verify the decomposition

$$QFT ([X/D_8]) = QFT \left( [X/D_4] \coprod [X/D_4]_{d.t.} \right),$$

(4.42)

where $H^2(D_4, U(1)) = \mathbb{Z}_2$, with phases given in [39, appendix D.3], [49, section 3.7].

If we let $z$ denote the central element of $D_8$, then the projectors can be computed in exactly the same fashion as in the previous example, and one finds from the general formula (2.44) that, for the irreducible representations $\pm 1$ of $\mathbb{Z}_2$,

$$\Pi_{\pm 1} = \frac{1}{2} \left( 1 \pm \tau_z \right).$$

(4.43)

The group $D_8$ has seven irreducible representations, four of dimension one and three of dimension two. The conjugacy classes of $D_8$ are

$$\{1\}, \{\tilde{z}\}, \{\tilde{b}, \tilde{b}^7 = \tilde{b}^3\tilde{z}\}, \{\tilde{b}^2, \tilde{b}^5 = \tilde{b}^2\tilde{z}\}, \{\tilde{a}, \tilde{a}\tilde{z}, \tilde{a}\tilde{b}^4, \tilde{a}\tilde{b}^2\tilde{z}\}, \{\tilde{b}\tilde{a} = \tilde{a}\tilde{b}^7, \tilde{a}\tilde{b}, \tilde{a}\tilde{b}^5, \tilde{a}\tilde{b}^3\}.$$

(4.44)

for $\tilde{z} = \tilde{b}^4$.

The character table of ordinary irreducible representations of $D_8$ is
The three ordinary two-dimensional irreducible representations are defined by
\[ \rho_r(b^i a^j) = B_r^i A_r^j \]  
for
\[ A_r = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad B_r = \begin{bmatrix} \xi^r & 0 \\ 0 & \xi^{-r} \end{bmatrix}, \]  
for \( r \in \{1, 2, 3\}, \xi = \exp(2\pi i/8). \)

Applying the projectors \( \Pi_{\pm 1} \) to bulk Wilson lines, we find
\[ \Pi_{\pm 1} W_{1,1_b,1_c,1_d} = W_{1,1_b,1_c,1_d} \Pi_{\pm 1}, \]
\[ \Pi_{\pm 1} W_{2_1} = W_{2_1} \Pi_{\mp 1}, \quad \Pi_{\pm 1} W_{2_2} = W_{2_2} \Pi_{\pm 1}, \quad \Pi_{\pm 1} W_{2_3} = W_{2_3} \Pi_{\mp 1}, \]
so that \( W_{2_1,2_3} \) act as defects linking different universes, and for boundary Wilson lines, we find
\[ \Pi_{+1} W_{1,1_b,1_c,1_d} = W_{1,1_b,1_c,1_d}, \quad \Pi_{+1} W_{2_1,2_3} = 0, \quad \Pi_{+1} W_{2_2} = W_{2_2}, \]
\[ \Pi_{-1} W_{1,1_b,1_c,1_d} = 0, \quad \Pi_{-1} W_{2_1,2_3} = W_{2_1,2_3}, \quad \Pi_{-1} W_{2_2} = 0. \]

Much as in the last example, the nontrivial representation of \( K = \mathbb{Z}_2 \) only appears in the restriction of (two of the) two-dimensional representations, and not as the restriction of a one-dimensional representation.

By comparison, the group \( D_4 \) has five ordinary irreducible representations, four one-dimensional and one two-dimensional, and two irreducible projective representations, both two-dimensional \[49\] section 3.7. This matches the prediction of decomposition: the \( D_8 \) representations are a union of \( D_4 \) representations with and without discrete torsion. The projectors \( \Pi_{\pm 1} \) above correctly identify which representations of \( D_8 \) descend to the twisted and untwisted representations of \( D_4 \): \( \Pi_{+1} \) projects onto ordinary \( D_4 \) representations, and \( \Pi_{-1} \) projects onto projective \( D_4 \) representations. Indeed, the representations selected out by \( \Pi_{+1} \) are invariant under \( \bar{z} \), and so descend to honest representations of \( D_4 \), whereas the representations selected out by \( \Pi_{-1} \) are not invariant under \( \bar{z} \), and so descend to projective representations of \( D_4 \), in essentially the same fashion as we observed in the previous subsection for the example there.
4.2 Nonbanded $\mathbb{Z}_4$ gerbe

Next, consider a nonbanded $\mathbb{Z}_4$ gerbe, described as the orbifold $[X/\mathbb{H}]$, where $\mathbb{H}$ is the eight-element group of unit quaternions, and $\mathbb{Z}_4 \cong \langle i \rangle$ acts trivially. This example decomposes into three universes as

$$\text{QFT}([X/\mathbb{H}]) = \text{QFT} \left( X \coprod [X/\mathbb{Z}_2] \coprod [X/\mathbb{Z}_2] \right). \quad (4.51)$$

(This example is discussed in [17, section 2.0.4] as well as [20, section 5.4], where its decomposition was checked via its spectrum of operators and in multiloop partition functions.)

4.2.1 Ring of dimension-zero operators

Here, as discussed in section 2.2.1, the dimension-zero twist fields are

$$\sigma_{[+1]}, \, \, \sigma_{[-1]}, \, \, \sigma_{[i]}, \quad (4.52)$$

which obey

$$\sigma_{[+1]}^2 = \sigma_{[+1]}, \, \, \sigma_{[+1]} \sigma_{[-1]} = \sigma_{[-1]}, \, \, \sigma_{[+1]} \sigma_{[i]} = \sigma_{[i]}, \quad (4.53)$$

$$\sigma_{[-1]}^2 = \sigma_{[+1]}, \, \, \sigma_{[-1]} \sigma_{[i]} = \sigma_{[i]}, \, \, \sigma_{[i]}^2 = (1/2) (\sigma_{[+1]} + \sigma_{[-1]}). \quad (4.54)$$

Clearly we can identify $\sigma_{[+1]}$ with the identity, and $\sigma_{[-1]}$ generates a $\mathbb{Z}_2$. These are consistent with the fact that, although this is a $\mathbb{Z}_4$ gerbe, it only has a $B\mathbb{Z}_2$ (one-form) symmetry, as only $\mathbb{Z}_2 \subset \mathbb{H}$ is central.

Identifying $x$ with $\sigma_{[-1]}$ and $y$ with $\sigma_{[i]}$, we can write the ring of dimension-zero operators more efficiently as

$$\mathbb{C}[x, y]/ \left( x^2 - 1, xy - y, y^2 - (1/2)(1 + x) \right). \quad (4.55)$$

Geometrically, this describes a complete intersection of three quadrics in $\mathbb{C}^2$, and it is straightforward to see the only solutions are

$$(x, y) = \{ (+1, +1), \, (+1, -1), \, (-1, 0) \}, \quad (4.56)$$

corresponding to three points.

Physically, as discussed in [20, section 5.4], this orbifold decomposes into three pieces:

$$\text{QFT}([X/\mathbb{H}]) = \text{QFT} \left( X \coprod [X/\mathbb{Z}_2] \coprod [X/\mathbb{Z}_2] \right). \quad (4.57)$$

The fact that two of the universes match reflects a $B\mathbb{Z}_2$ (one-form) symmetry in the decomposition, matching the (center) symmetry of the $\mathbb{H}$ orbifold. The fact that the third does not, reflects a noninvertible symmetry.
In terms of the ring of dimension-zero operators, this $B\mathbb{Z}_2$ acts as
\begin{align}
y &\mapsto -y, \tag{4.58} \\
x &\mapsto x \text{ (invariant)}. \tag{4.59}
\end{align}

This symmetry leaves the relations invariant, but interchanges two of the universes (with nonzero $y$ vevs), as one would expect from the physical decomposition. (As remarked elsewhere, this structure is reminiscent of spontaneous symmetry breaking, though the details differ here, as the theory exhibits decomposition and not superselection sectors.)

In passing, let us describe another way to arrive at the ring above. Mathematically, the $[X/\mathbb{H}]$ gerbe is a nonbanded $\mathbb{Z}_4$ gerbe, which means it looks like a fiber bundle with fibers $B\mathbb{Z}_4$, but the transition functions include a nontrivial bundle of outer automorphisms, and as a result, the ring in this case can be derived by taking the $\mathbb{Z}_2$-invariants of the ring of dimension-zero operators of a banded $\mathbb{Z}_4$ gerbe. In terms of the quaternions $\mathbb{H}$ and $\mathbb{Z}_4 \cong \langle i \rangle \subset \mathbb{H}$, we can lift the $\mathbb{Z}_2 = \mathbb{H}/\langle i \rangle$ via a section $s$ which we take to be $s(+1) = 1$, $s(-1) = j$, so that $s(-1)^{-1} = -j$. The action on any element $g \in \mathbb{Z}_4$ is $g \mapsto s^{-1}gs$, for which we find $\pm 1 \in \langle i \rangle$ are invariant but $i \mapsto -i$. In the ring $\mathbb{C}[y]/\langle y^4 - 1 \rangle$ of the banded $\mathbb{Z}_4$ gerbe, this maps $y \mapsto -y$, leaving $y^2$ invariant. Define $\bar{x} = y^2$, $\bar{y} = (1/2)(y + y^3)$, and then the $\mathbb{Z}_2$ invariants are
\begin{equation}
(\mathbb{C}[y]/\langle y^4 - 1 \rangle)^{\mathbb{Z}_2} = \mathbb{C}[ar{x}, \bar{y}]/(\bar{x}^2 - 1, \bar{x}\bar{y} - \bar{y}, \bar{y}^2 - (1/2)(1 + \bar{x})), \tag{4.60}
\end{equation}
recovering the ring of the nonbanded $\mathbb{Z}_4$ gerbe, as desired.

Next, let us compute the noninvertible operators. It is straightforward to show that
\begin{equation}
(a + bx + cy)^{-1} = \frac{1}{2} \left( \frac{1}{a-b} - \frac{(a+b)}{c^2 - (a+b)^2} \right) - \frac{x}{2} \left( \frac{1}{a-b} + \frac{(a+b)}{c^2 - (a+b)^2} \right) + \frac{y}{c^2 - (a+b)^2} \frac{c}{c^2 - (a+b)^2}, \tag{4.61}
\end{equation}

hence we see the operator is noninvertible if it lies along the locus $\{ \Delta = 0 \}$ where
\begin{equation}
\Delta = (a - b) \left( c^2 - (a+b)^2 \right). \tag{4.62}
\end{equation}

Now, let us compute the projectors onto universes. Such projectors are already listed in [20, section 5.4]; let us instead compute them from the general expression (2.44) and then compare to the results in [20]. The (one-dimensional) irreducible representations of $\langle i \rangle \cong \mathbb{Z}_4$ can be characterized by their values on $i$: $\rho_{\pm 1}(i) = \pm 1$, $\rho_{\pm i}(i) = \pm i$. From the discussion above, we see that the action of $\mathbb{H}/\langle i \rangle = Z_2$ on the irreducible representations leaves $\rho_{\pm 1}$ invariant but exchanges $\rho_i \leftrightarrow \rho_{-i}$. Therefore, the universes of decomposition correspond to the representations $\rho_{+1}$, $\rho_{-1}$, and $\rho_{+i} \oplus \rho_{-i}$ of $\mathbb{Z}_4$.

Letting $g$ denote the generator of $\mathbb{Z}_4$, we have the $\mathbb{Z}_4$ characters
\begin{equation}
\chi_{\pm 1}(g) = \pm 1, \quad \chi_{\pm i}(g) = \pm i, \tag{4.63}
\end{equation}

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from which we find from the general formula (2.44) that

\[ \Pi_R = \frac{1}{|Z_4|} \sum_{k=0}^{3} \chi_1(g^{-k})\tau_g^k, \]  

(4.64)

\[ = \frac{1}{4} (1 + x + \tau_i + \tau_{-i}), \]  

(4.65)

\[ = \frac{1}{4} (1 + x + 2y), \]  

(4.66)

\[ \Pi_{R=-1} = \frac{1}{|Z_4|} \sum_{k=0}^{3} \chi_{-1}(g^{-k})\tau_g^k, \]  

(4.67)

\[ = \frac{1}{4} (1 + x - 2y), \]  

(4.68)

Finally, from

\[ \Pi_{\rho_i} = \frac{1}{|Z_4|} \sum_{k=0}^{3} \chi_i(g^{-k})\tau_g^k, \]  

(4.69)

\[ = \frac{1}{4} (1 - i\tau_i - \tau_{-1} + i\tau_{-i}), \]  

(4.70)

\[ \Pi_{\rho_{-i}} = \frac{1}{|Z_4|} \sum_{k=0}^{3} \chi_{-i}(g^{-k})\tau_g^k, \]  

(4.71)

\[ = \frac{1}{4} (1 + i\tau_i - \tau_{-1} - i\tau_{-i}), \]  

(4.72)

we find

\[ \Pi_{R=[2]} = \Pi_{\rho_i} + \Pi_{\rho_{-i}} = \frac{1}{4} (2 - 2\tau_{-1}) = \frac{1}{2} (1 - x). \]  

(4.73)

The projectors computed above match the projectors worked out on an ad-hoc basis in [20, section 5.4], that project operators onto each of the three universes:

\[ \Pi_1 = \frac{1}{4} (1 + x + 2y), \]  

(4.74)

\[ \Pi_{-1} = \frac{1}{4} (1 + x - 2y), \]  

(4.75)

\[ \Pi_2 = \frac{1}{2} (1 - x), \]  

(4.76)

which are easily checked to obey

\[ \Pi_i \Pi_j = \delta_{i,j} \Pi_i, \quad \Pi_1 + \Pi_{-1} + \Pi_2 = 1. \]  

(4.77)
As observed in [20, section 5.4], from looking at twisted sector states, \( \Pi_{\pm 1} \) project onto the universes \([X/\mathbb{Z}_2]\), and \( \Pi_2 \) projects onto universe \( X \). It is also straightforward to check that each of these projectors lies along the locus \( \{ \Delta = 0 \} \), as expected as they are not invertible.

In passing, note that the projectors above have the property that when restricted to the possible vacua, the complete intersection of quadrics, each projector is nonzero on precisely one point:

\[
\Pi_1|_{(x,y)=(+1,+1)} = 1, \quad \Pi_{-1}|_{(x,y)=(+1,-1)} = 1, \quad \Pi_2|_{(x,y)=(-1,0)} = 1, \quad (4.78)
\]

with all other restrictions zero. Phrased another way, the point \((x, y) = (+1, +1)\) is the locus \( \Pi_{-1} = 0 = \Pi_2 \), and so forth.

These statements correspond more formally to statements in commutative algebra. First, the points \((x, y) = (+1, +1), (+1, -1), (-1, 0)\) correspond to the maximal ideals

\[
m_1 = (x - 1, y - 1), \quad m_{-1} = (x - 1, y + 1), \quad m_2 = (x + 1, y). \quad (4.79)
\]

Each projector is in all of the maximal ideals save one. For example,

\[
(x - 1) + 2(y + 1) = 1 + x + 2y \propto \Pi_1, \quad (4.80)
\]

hence \( \Pi_1 \in m_{-1} \), and similarly,

\[
(1 + y)(x + 1) = 1 + x + 2y \propto \Pi_1, \quad (4.81)
\]

hence \( \Pi_1 \in m_2 \) also. However, \( \Pi_1 \not\in m_1 \), as its restriction to the corresponding point is nonzero. Similarly, one can show

\[
\Pi_{-1} \in m_1 \cap m_2, \quad \Pi_2 \in m_1 \cap m_{-1}. \quad (4.82)
\]

As a result, for example,

\[
(\Pi_{-1})_{m_1} = 0 = (\Pi_2)_{m_1} \quad (4.83)
\]

since \( \Pi_1 \not\in m_1 \).

### 4.2.2 Wilson lines

In this section we will consider bulk and boundary Wilson lines on \([X/\mathbb{H}]\), the latter via representation of \( \mathbb{H} \), and how they correspond to defects between and sheaves on universes of the decomposition of \([X/\mathbb{H}]\).

The group \( \mathbb{H} \) has five irreducible representations, four one-dimensional, and one two-dimensional. The two-dimensional representation can be given explicitly as

\[
\rho_2(1) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \rho_2(-1) = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}. \quad (4.84)
\]
\[
\rho_2(i) = \begin{bmatrix}
i & 0 \\
0 & -i
\end{bmatrix}, \quad \rho_2(j) = \begin{bmatrix}
0 & -1 \\
1 & 0
\end{bmatrix}, \quad \rho_2(k) = \begin{bmatrix}
0 & -i \\
-i & 0
\end{bmatrix}.
\]

(4.85)

It will be convenient to refer to the character table, which we give below:

|      | +1 | -1 | \{±i\} | \{±j\} | \{±k\} |
|------|----|----|--------|--------|--------|
| 1 (trivial) | 1  | 1  | 1      | 1      | 1      |
| 1_i    | 1  | 1  | 1      | -1     | -1     |
| 1_j    | 1  | 1  | -1     | 1      | -1     |
| 1_k    | 1  | 1  | -1     | -1     | 1      |
| 2      | 2  | -2 | 0      | 0      | 0      |

Using the character table and also [62], one can show

\[
1^2_i = 1, \\
1_i \otimes 1_j = 1_k \text{ and cyclically,} \\
1_{i,j,k} \otimes 2 = 2, \\
2 \otimes 2 = 1 + 1_i + 1_j + 1_k.
\]

(4.86) \quad (4.87) \quad (4.88) \quad (4.89)

Now, consider the restriction to \( \langle i \rangle \cong \mathbb{Z}_4 \). We characterize the one-dimensional irreducible representations of \( \langle i \rangle \) by their values on \( i \): \( \rho_{\pm1}(i) = \pm 1 \), \( \rho_{\pm1}(i) = \pm i \).

| Rep’ of \( \mathbb{H} \) | Restriction |
|------------------------|-------------|
| 1                      | \( \rho_{+1} \) |
| 1_i                    | \( \rho_{+1} \) |
| 1_j                    | \( \rho_{-1} \) |
| 1_k                    | \( \rho_{-1} \) |
| 2                      | \( \rho_i \oplus \rho_{-i} \) |

Thus, we see that three representations appear in the restriction: \( \rho_{+1}, \rho_{-1}, \), and \( [2] \equiv \rho_i \oplus \rho_{-i} \). It is straightforward to check that multiplication in \( \mathbb{H} \) induces

\[
\rho_{+1} \otimes \rho_{-1} = \rho_{-1}, \quad \rho_{-1} \otimes \rho_{-1} = \rho_{+1}, \quad \rho_{-1} \otimes [2] = [2], \quad [2] \otimes [2] = \rho_{+1} \oplus \rho_{+1} \oplus \rho_{-1} \oplus \rho_{-1},
\]

(4.90) \quad (4.91)

consistent with the restrictions. Clearly, \( \rho_{-1} \) is invertible, but \( [2] \) is not.

Now, let us compute the representations associated with the universes in the decomposition

\[
\left[ \frac{X \times \hat{K}}{G} \right] = X \coprod \left[ X/\mathbb{Z}_2 \right] \coprod \left[ X/\mathbb{Z}_2 \right],
\]

(4.92)
for \( G = \mathbb{H}/\langle i \rangle = \mathbb{Z}_2 \). In the decomposition of \([X/\mathbb{H}]\), \( G \) acts trivially on \( \rho_{\pm 1} \) but exchanges \( \rho_{+i} \leftrightarrow \rho_{-i} \), so the representations associated with the three universes are \( \rho_{+1}, \rho_{-1} \), and \( \rho_{+i} \oplus \rho_{-i} \), which happen to match the representations appearing as restrictions of the irreducible representations of \( \mathbb{H} \). Of these, \( \rho_{+1} \) and \( \rho_{-1} \) each correspond to an \([X/\mathbb{Z}_2]\), whereas \( \rho_{[2]} \) corresponds to

\[
\left[ \frac{X \times \hat{\mathbb{Z}}_2}{\mathbb{Z}_2} \right] = X.
\]  

From the character table and the restrictions to \( \langle i \rangle \subset \mathbb{H} \),

- the representations 1, 1, transform as \( \rho_{+1} \) under \( K = \langle i \rangle \) and so correspond to one copy of \([X/\mathbb{Z}_2]\),
- the representations \( 1_j, 1_k \) transform as \( \rho_{-1} \) under \( K \) and so correspond to another copy of \([X/\mathbb{Z}_2]\),
- the representation 2 transforms as \([2]\) under \( K \) and so corresponds to \( X \).

Next, we will consider the action on bulk and boundary Wilson lines, and in the latter, recover that same classification above from the projectors.

For bulk Wilson lines, if we label Wilson lines by the restrictions of representations of \( \mathbb{H} \) to \( \langle i \rangle \), we have

\[
\sigma_{[-1]} W_1 = W_1 \sigma_{[-1]}, \quad \sigma_{[-1]} W_{-1} = W_{-1} \sigma_{[-1]}, \quad \sigma_{[i]} W_1 = W_1 \sigma_{[i]}, \quad \sigma_{[i]} W_{-1} = -W_{-1} \sigma_{[i]}.
\]  

Since \( W_{[2]} \) involves a higher-dimension representation, we proceed more carefully. Note

\[
\sigma_{[-1]} \rho_{i} = -\rho_{i} \sigma_{[-1]}, \quad \sigma_{[-1]} \rho_{-i} = -\rho_{-i} \sigma_{[-1]}, \quad \sigma_{[i]} \rho_{\pm i} = \pm i \rho_{\pm i} \sigma_{[i]}.
\]  

This implies

\[
\sigma_{[-1]} W_{[2]} = -W_{[2]} \sigma_{[-1]}, \quad \sigma_{[i]} W_{[2]} = 0.
\]

For the projection operators

\[
\Pi_{\pm 1} = \frac{1}{4} \left( 1 + \sigma_{[-1]} \pm 2 \sigma_{[i]} \right), \quad \Pi_2 = \frac{1}{2} \left( 1 - \sigma_{[-1]} \right),
\]  

now labelling them by representations of \( \mathbb{H} \), we find

\[
\Pi_{\pm 1} W_{1,1_i} = W_{1,1_i} \Pi_{\pm 1}, \quad \Pi_2 W_{1,1_i} = W_{1,1_i} \Pi_2,
\]

\[
\Pi_1 W_{1_j,1_k} = W_{1_j,1_k} \Pi_{-1}, \quad \Pi_{-1} W_{1_j,1_k} = W_{1_j,1_k} \Pi_{+1}, \quad \Pi_2 W_{-1} = W_{-1} \Pi_2.
\]
\[ \Pi_1 W_2 = \frac{1}{2} W_2 \Pi_2, \quad \Pi_{-1} W_2 = \frac{1}{2} W_2 \Pi_2, \quad \Pi_2 W_2 = W_2 (\Pi_1 + \Pi_{-1}). \] (4.102)

Thus, we see that \( W_{1,j,k} \) acts as a defect linking universes \( \pm 1 \) on either side, and \( W_2 \) acts as a defect linking universe 2 to either of universes \( \pm 1 \).

Next, we repeat this analysis for boundary Wilson lines. From the pairing \( \tau_g W_\rho = \chi_\rho(g) W_\rho \), if we label Wilson lines by the restrictions of representations of \( H \) to \( \langle i \rangle \), we have

\[ \sigma_{[-1]} W_1 = W_1, \quad \sigma_{[-1]} W_{-1} = W_{-1}, \] (4.103)
\[ \sigma_{[i]} W_1 = W_1, \quad \sigma_{[i]} W_{-1} = -W_{-1}. \] (4.104)

Since \( W_2 \) involves a higher-dimension representation, we should be more careful. First, note

\[ \sigma_{[-1]} \rho_i = -\rho_i, \quad \sigma_{[-1]} \rho_{-i} = -\rho_{-i}, \] (4.105)
\[ \sigma_{[i]} \rho_i = 0 = \sigma_{[i]} \rho_{-i}. \] (4.106)

Then, the effect of \( \sigma_{[-1]} \) is to insert in \( W_2 \) the diagonal matrix \(-I\), so that

\[ \sigma_{[-1]} W_2 = -W_2 \] (4.107)

and the effect of \( \sigma_{[i]} \) is to insert in \( W_2 \) the diagonal matrix 0, so that

\[ \sigma_{[i]} W_2 = 0. \] (4.108)

Note that in both cases, the eigenvalue corresponding to \( W_2 \) is precisely \( \chi_2(g) / \text{dim}[2] \), as expected.

For the projection operators

\[ \Pi_{\pm 1} = \frac{1}{4} \left( 1 + \sigma_{[-1]} \pm 2 \sigma_{[i]} \right), \quad \Pi_2 = \frac{1}{2} \left( 1 - \sigma_{[-1]} \right), \] (4.109)

now labelling them by representations of \( H \), we find

\[ \Pi_1 W_{1,1,i} = W_{1,1,i}, \quad \Pi_{-1} W_{1,1,i} = 0 = \Pi_2 W_{1,1,i}, \] (4.110)
\[ \Pi_1 W_{1,j,1,k} = 0, \quad \Pi_{-1} W_{1,j,1,k} = W_{1,j,1,k}, \quad \Pi_2 W_{1,j,1,k} = 0, \] (4.111)
\[ \Pi_{\pm 1} W_2 = 0, \quad \Pi_2 W_2 = W_2. \] (4.112)

As expected, the projectors project onto Wilson lines associated with the corresponding universes.
4.2.3 Symmetries

Now, let us turn to the symmetries of the \([X/H]\) orbifold. There is a \(B\mathbb{Z}_2\) symmetry, corresponding to the fact that a \(\mathbb{Z}_2\) subgroup of the trivially-acting \(\langle i \rangle\) is the center of \(H\). In addition, there is a noninvertible symmetry, corresponding to the fact that one of the three universes is not like the others.

The theory has twist fields

\[
x = \sigma_{[-1]}, \quad y = \sigma_{[i]}. \tag{4.113}
\]

The twist field \(x\) obeys \(x^2 = 1\), and is invertible: \(x^{-1} = x\). The twist field \(y\) obeys a fusion rule \(y^2 = (1/2)(1 + x)\), and lies along the locus \(\{\Delta = 0\}\), so it is not invertible.

As mentioned earlier, the one-form symmetry \(B\mathbb{Z}_2\) acts on the twist fields as \(y \mapsto -y\), leaving \(x\) invariant. Note that although \(y\) is not invertible, \(B\mathbb{Z}_2\) acts on it nontrivially.

The three universes correspond to the three representations \(R = \pm 1, 2\). Of these, \(R = \pm 1\) correspond to the two copies of the universe \([X/\mathbb{Z}_2]\), while \(R = 2\) corresponds to the universe \(X\). Note that the projectors \(\Pi_{R=\pm 1}\), onto the two universes \([X/\mathbb{Z}_2]\) exchanged by the one-form symmetry, both involve \(x\) and \(y\), even though \(y\) is not invertible, whereas the projector \(\Pi_{R=2}\) onto the distinct universe \(X\) only involves the invertible twist field \(x\). Thus, the relation between invertibility of twist fields and universes of decomposition is nontrivial – as stated elsewhere, conjugacy classes and representations are merely dual, and not canonically isomorphic, as this example illustrates.

In terms of the representations, the two identical universes \([X/\mathbb{Z}_2]\) both correspond to one-dimensional representations, as expected for the one-form symmetry, while the distinct universe \(X\) is associated to a two-dimensional representation, as appropriate for a noninvertible symmetry.

4.3 Nonbanded \(\mathbb{Z}_2 \times \mathbb{Z}_2\) gerbe

In this section we consider the orbifold \([X/A_4]\), where \(A_4\) is the group of alternating permutations of four elements, and \(K = \mathbb{Z}_2 \times \mathbb{Z}_2 \subset A_4\) acts trivially. This example exhibits decomposition, but has no one-form symmetry at all.

In this case, \(A_4/K = \mathbb{Z}_3\), and it was argued in [20, section 5.5] that

\[
\text{QFT}([X/A_4]) = \text{QFT} \left( X \coprod [X/\mathbb{Z}_3] \right). \tag{4.114}
\]
4.3.1 Ring of dimension-zero operators

As elements of $A_4$, the elements of $K$ form two conjugacy classes:

$$\{1\}, \ \{(12)(34), (13)(24), (14)(23)\},$$

hence there are corresponding dimension-zero twist fields

$$\sigma_1 = 1, \ \sigma = \frac{1}{3} \left( \tau_{(12)(34)} + \tau_{(13)(24)} + \tau_{(14)(23)} \right),$$

which obey

$$\sigma^2 = \frac{1}{3} + \frac{2}{3} \sigma. \quad (4.117)$$

The ring of dimension-zero operators is then

$$\mathbb{C}[\sigma]/(\sigma^2 - (1/3) - (2/3)\sigma),$$

so the corresponding universes have support at

$$\langle \sigma \rangle = 1, \ -1/3. \quad (4.119)$$

The fact this has support at two points is consistent with the statement of [20, section 5.5] that this example decomposes into two universes.

Next, let us compute the noninvertible operators. It is straightforward to show that

$$(a + b\sigma)^{-1} = \Delta^{-1} \left( a + \frac{2}{3} b - b\sigma \right), \quad (4.120)$$

where

$$\Delta = a^2 + \frac{2}{3} ab - \frac{b^2}{3}, \quad (4.121)$$

hence we see the operator $(a + b\sigma)$ is noninvertible if it lies along the locus $\{\Delta = 0\}$.

In this theory, $\sigma$ itself is invertible, with inverse

$$\sigma^{-1} = -2 + 3\sigma. \quad (4.122)$$

Thus, we see that even in this case, where all of the non-identity elements of $\mathbb{Z}_2 \times \mathbb{Z}_2$ are collected into a single operator $\sigma$, $\sigma$ is invertible, so, roughly speaking, nonbanded does not necessarily imply that the twist fields associated to conjugacy classes are noninvertible.

Next, let us compute the projectors onto the two universes from equation (2.44). Write $K = \mathbb{Z}_2 \times \mathbb{Z}_2 = \langle a, b \rangle$, where $a^2 = 1 = b^2$, and label the four one-dimensional irreducible representations of $K$ as $1, \rho_a, \rho_b, \rho_{ab}$, with character table
It is straightforward to compute that the action of $A_4/K = \mathbb{Z}_3$ on the representations is to permute $\rho_a$, $\rho_b$, $\rho_{ab}$, while leaving 1 invariant. Thus, the universes correspond to the representations

\begin{align*}
1, \quad \rho_a \oplus \rho_b \oplus \rho_{ab}
\end{align*}

of $K$. Applying the definition (2.44), we have

\begin{align*}
\Pi_{R=1} &= \frac{1}{4} \left(1 + \tau_{(12)(34)} + \tau_{(13)(24)} + \tau_{(14)(23)}\right), \quad (4.124) \\
&= \frac{1}{4} (1 + 3\sigma), \quad (4.125)
\end{align*}

and from

\begin{align*}
\Pi_a &= \frac{1}{|\mathbb{Z}_4|} \sum_{g \in \mathbb{Z}_2 \times \mathbb{Z}_2} \chi_{\rho_a} (g^{-1}) \tau_g, \quad (4.126) \\
&= \frac{1}{4} (1 + \tau_a - \tau_b - \tau_{ab}), \quad (4.127) \\
\Pi_b &= \frac{1}{|\mathbb{Z}_4|} \sum_{g \in \mathbb{Z}_2 \times \mathbb{Z}_2} \chi_{\rho_b} (g^{-1}) \tau_g, \quad (4.128) \\
&= \frac{1}{4} (1 - \tau_a + \tau_b - \tau_{ab}), \quad (4.129) \\
\Pi_{ab} &= \frac{1}{|\mathbb{Z}_4|} \sum_{g \in \mathbb{Z}_2 \times \mathbb{Z}_2} \chi_{\rho_{ab}} (g^{-1}) \tau_g, \quad (4.130) \\
&= \frac{1}{4} (1 - \tau_a - \tau_b + \tau_{ab}), \quad (4.131)
\end{align*}

we find

\begin{align*}
\Pi_{R=a\oplus b\oplus ab} &= \Pi_a + \Pi_b + \Pi_{ab} = \frac{1}{4} \left(3 - \tau_{(12)(34)} - \tau_{(13)(24)} - \tau_{(14)(23)}\right), \quad (4.132) \\
&= \frac{3}{4} (1 - \sigma). \quad (4.133)
\end{align*}

Thus, the projectors onto the two universes are

\begin{align*}
\Pi_1 &\equiv \Pi_{R=1} = \frac{1}{4} (1 + 3\sigma), \quad \Pi_2 \equiv \Pi_{R=a\oplus b\oplus ab} = \frac{3}{4} (1 - \sigma). \quad (4.134)
\end{align*}
Specifically, $\Pi_1$ projects onto $[X/\mathbb{Z}_3]$, and $\Pi_2$ projects onto $X$. It is straightforward to check that
\[ \Pi_i \Pi_j = \delta_{ij} \Pi_i, \quad \Pi_1 + \Pi_2 = 1, \quad (4.135) \]
and that they lie along the non-invertible locus $\{ \Delta = 0 \}$.

Note also that
\[ \Pi_1|_{\sigma=1} = 1, \quad \Pi_1|_{\sigma=-1/3} = 0, \quad (4.136) \]
\[ \Pi_1|_{\sigma=1} = 1, \quad \Pi_2|_{\sigma=-1/3} = 0, \quad (4.137) \]
so we see that $\Pi_1$ projects onto states associated with the universe at $\sigma = 1$, and $\Pi_2$ projects onto states associated with the universe at $\sigma = -1/3$.

### 4.3.2 Wilson lines

Now, let us turn to Wilson lines, and see how bulk Wilson lines provide defects linking universes, and from boundary Wilson lines, how bundles and sheaves on $[X/A_4]$ decompose into bundles and sheaves on the universes $X, [X/\mathbb{Z}_3]$.

The group $A_4$ has four conjugacy classes, namely
\[
\{1\}, \{(12)(34), (13)(24), (14)(23)\}, \{(123), (421), (243), (341)\}, \{(132), (412), (234), (314)\},
\]
(4.138)
of which the first two form $K$, hence four irreducible representations. From [63, section 2.3], three of those representations are one-dimensional, and the other is three-dimensional, with character table

|        | 1    | (12)(34) | (123) | (132) |
|--------|------|----------|------|------|
| 1      | 1    | 1        | 1    | 1    |
| $1_a$  | 1    | $\omega$ | $\omega^2$ |
| $1_b$  | 1    | $\omega^2$ | $\omega$ |
| 3      | 3    | −1       | 0    | 0    |

where $\omega = \exp(2\pi i/3)$.

From the character table, we can read off the products
\[ l_a^2 = l_b, \quad l_b^2 = l_a, \quad l_a l_b = 1, \quad (4.139) \]
\[ l_{a,b} \otimes 3 = 3, \quad 3 \otimes 3 = 3 \oplus 3 \oplus 1 \oplus l_a \oplus l_b. \quad (4.140) \]
Clearly, the one-dimensional representations are invertible, but not the three-dimensional representation.
Now, let us look at restrictions of these representations. In particular, mathematically we expect that any bundle or sheaf on $[X/A_4]$ such that the restriction of the $A_4$-equivariant structure to $K$ is the representation 1, descends to universe $[X/Z_3]$, whereas those that restrict to $\rho_a \oplus \rho_b \oplus \rho_{ab}$, descend to universe $S$.

From the character table, it is straightforward to see that the restrictions of the irreducible representations of $A_4$ to $K$ are as follows:

| Representation of $A_4$ | Restriction to $K$ |
|------------------------|--------------------|
| 1                      | 1                  |
| $1_a$                  | 1                  |
| $1_b$                  | 1                  |
| 3                      | $\rho_a \oplus \rho_b \oplus \rho_{ab}$ |

Thus, in the restriction, only two representations of $K$ appear, corresponding to the two universes in the decomposition.

Using those restrictions, let us compute the action of the projectors. We begin with bulk Wilson lines. Trivially,

$$\sigma W_1 = W_1 \sigma,$$

and the effect of $\sigma$ on $W_{a+b+ab}$ is to insert a matrix

$$\frac{1}{3} \text{diag} (1 - 1 - 1, -1 + 1 - 1, -1 - 1 + 1) = -\frac{1}{3} I,$$

so we see that

$$\sigma W_{a+b+ab} = -\frac{1}{3} W_{a+b+ab} \sigma.$$  

(4.143)

Then, in terms of Wilson lines in representations of $A_4$, we compute

$$\Pi_1 W_{1,1_a,1_b} = W_{1,1_a,1_b} \Pi_1, \quad \Pi_2 W_{1,1_a,1_b} = W_1 \Pi_2,$$

$$\Pi_1 W_3 = \frac{1}{3} W_3 \Pi_2, \quad \Pi_2 W_3 = W_3 \left( \Pi_1 + \frac{2}{3} \Pi_2 \right).$$

(4.144)

(4.145)

In particular, $W_3$ acts as a defect bridging universes.

Next, we turn to boundary Wilson lines (Chan-Paton factors). For the moment, we label the Wilson lines by representations of $K$ (obtained as restrictions of representations of $A_4$). Trivially,

$$\sigma W_1 = W_1.$$  

(4.146)

As before, the effect of $\sigma$ on $W_{a+b+ab}$ is to insert a matrix

$$\frac{1}{3} \text{diag} (1 - 1 - 1, -1 + 1 - 1, -1 - 1 + 1) = -\frac{1}{3} I,$$

(4.147)
so we see that
\[ \sigma W_{a+b+ab} = -\frac{1}{3} W_{a+b+ab}. \] (4.148)

Note that the eigenvalue corresponding to \( W_{a+b+ab} \) is \( \chi_3(g)/\dim 3 \), as expected.

Then, in terms of boundary Wilson lines in representations of \( A_4 \), we compute
\[ \Pi_1 W_{1,1_a,1_b} = W_{1,1_a,1_b}, \quad \Pi_2 W_{1,1_a,1_b} = 0, \] (4.149)
\[ \Pi_1 W_3 = 0, \quad \Pi_2 W_3 = W_3, \] (4.150)
so that each projector selects out Wilson lines in the fashion predicted mathematically, as expected.

### 4.3.3 Symmetries

In this example, since the two constituent universes are distinct, we expect no ordinary one-form symmetries, only noninvertible symmetries.

This is reflected at several levels. At a group theoretic level, the alternating group \( A_4 \) has no center (beyond the identity), so no one-form symmetry is expected there. Similarly, the ring of dimension-zero operators does not have any multiplicative symmetries, again in accord with a lack of one-form symmetries.

Perhaps surprisingly in this case, the single twist field \( \sigma \) is invertible – but as we have seen, twist fields are merely dual to representations, not canonically isomorphic, and there is no canonical method to distinguish a single twist field from a general linear combination, which always contains noninvertible elements. In terms of representations, the universe \([X/\mathbb{Z}_3]\) is associated with a one-dimensional representation but the other universe \((X)\) is associated with a three-dimensional representation of \(\mathbb{Z}_2 \times \mathbb{Z}_2\). Since only one universe is associated with a one-dimensional representation, there is no reason to expect an invertible one-form symmetry, and since there exists a universe associated with a higher-dimension representation, one does expect a noninvertible symmetry.

### 4.4 Nonabelian \( D_4 \) gerbe

In this section we consider the orbifold [point]/\( D_4 \), where all of the eight-element dihedral group \( D_4 \) acts trivially.

Geometrically, this is a \( D_4 \)-gerbe (over a point). Since \( D_4 \) is nonabelian, only its center
(\mathbb{Z}_2) \text{ defines a one-form symmetry } (B\mathbb{Z}_2). \text{ From decomposition } [20,39],

\[
\text{QFT}[\text{point}/D_4] = \text{QFT}\left(\prod_{5} \text{point}\right),
\]

\[
= \text{QFT}\left([\text{point}/\mathbb{Z}_2 \times \mathbb{Z}_2] \prod [\text{point}/\mathbb{Z}_2 \times \mathbb{Z}_2]_{\text{d.t.}}\right). \tag{4.152}
\]

For use here, we present the elements of $D_4$ as

\[
D_4 = \{1, z, a, b, az, bz, ab, ba\}, \tag{4.153}
\]

with the relations

\[
a^2 = 1 = b^4, \quad b^2 = z, \quad ba = abz, \tag{4.154}
\]

in the same notation as [39], where $z$ generates the center.

### 4.4.1 Ring of dimension-zero operators

First, let us consider the twist fields, corresponding to conjugacy classes of group elements. Following appendix 2.2.1, the conjugation-invariant dimension-zero twist fields are

\[
\begin{align*}
\sigma_{[+1]} &= 1, \quad \sigma_{[-1]} = \tau_z, \\
\sigma_{[a]} &= (1/2) (\tau_a + \tau_{az}), \quad \sigma_{[b]} = (1/2) (\tau_b + \tau_{bz}), \quad \sigma_{[ab]} = (1/2) (\tau_{ab} + \tau_{ba}).
\end{align*} \tag{4.156}
\]

Using the multiplication rule

\[
\tau_g \tau_h = \tau_{gh} \tag{4.157}
\]

as in section 2.2.1 one quickly finds

\[
\begin{align*}
\sigma_{[+1]}^2 &= \sigma_{[+1]}, \quad \sigma_{[+1]} \sigma_{[-1]} = \sigma_{[-1]}, \quad \sigma_{[+1]} \sigma_{[a,b,ab]} = \sigma_{[a,b,ab]}, \\
\sigma_{[-1]}^2 &= \sigma_{[+1]}, \quad \sigma_{[-1]} \sigma_{[a,b,ab]} = \sigma_{[a,b,ab]}, \quad \sigma_{[a,b,ab]}^2 = (1/2) (\sigma_{[+1]} + \sigma_{[-1]}), \\
\sigma_{[a]} \sigma_{[b]} &= \sigma_{[ab]}, \quad \sigma_{[b]} \sigma_{[ab]} = \sigma_{[a]}, \quad \sigma_{[ab]} \sigma_{[a]} = \sigma_{[b]}.
\end{align*} \tag{4.158}
\]

Identifying $x$ with $\sigma_{[-1]}$ and $y_{1,2,3}$ with $\sigma_{[a,b,ab]}$, we can write the ring of dimension-zero operators more efficiently as

\[
\mathbb{C}[x, y_1, y_2, y_3]/(x^2 - 1, xy_1 - y_1, y_1^2 - (1/2)(1 + x), y_1y_2 - y_2, y_2y_3 - y_1, y_3y_1 - y_2). \tag{4.161}
\]

Clearly, $x$ is invertible but none of $\{y_1, y_2, y_3\}$ are invertible.
Geometrically, this ring describes five points in \( \mathbb{C}^4 \), as many points as the number of components of the decomposition, at the locations

\[
(x, y_1, y_2, y_3) = \{ (+1, +1, +1, +1), \; (+1, +1, -1, -1), \; (+1, -1, +1, -1), \\
(+1, -1, -1, +1), \; (-1, 0, 0, 0) \} . \tag{4.162}
\]

Physically,

\[
\text{QFT} ([\text{point}/D_4]) = \text{QFT} \left( \prod_5 \text{point} \right) , \tag{4.163}
\]
a disjoint union of five points, corresponding to the number of conjugacy classes. We can equivalently decompose this theory using the fact that the center of \( D_4 \) is \( \mathbb{Z}_2 \), with quotient \( D_4/\mathbb{Z}_2 = \mathbb{Z}_2 \times \mathbb{Z}_2 \), and write the decomposition as

\[
\text{QFT} ([\text{point}/D_4]) = \text{QFT} \left( [\text{point}/\mathbb{Z}_2 \times \mathbb{Z}_2] \prod [\text{point}/\mathbb{Z}_2 \times \mathbb{Z}_2]_{dt} \right) . \tag{4.164}
\]
As explained in [39, section 4.1],

\[
\text{QFT} ([\text{point}/\mathbb{Z}_2 \times \mathbb{Z}_2]) = \text{QFT} (\text{four points}) , \tag{4.165}
\]
\[
\text{QFT} ([\text{point}/\mathbb{Z}_2 \times \mathbb{Z}_2]_{dt}) = \text{QFT} (\text{one point}) , \tag{4.166}
\]
so again we get a total of five points, but this alternative description may make the role of the central \( \mathbb{Z}_2 \) more clear. Note that this structure is reflected in the order parameters, the points on \( \mathbb{C}^4 \) where the decomposition has support:

- the single point at \( x = -1 \) corresponds to the \( \mathbb{Z}_2 \times \mathbb{Z}_2 \) orbifold with discrete torsion,
- the four points at \( x = +1 \) correspond to the \( \mathbb{Z}_2 \times \mathbb{Z}_2 \) orbifold without discrete torsion,

as will be confirmed from the representations associated to the universes.

Now, let us compute the projectors from (2.44). For this we need some facts about the representation theory of \( D_4 \). This group has five irreducible representations, corresponding to the five conjugacy classes

\[
\{1\}, \{z\}, \{a, az\}, \{b, bz\}, \{ab, ba = abz\} . \tag{4.167}
\]
It has five irreducible representations: four one-dimensional, and one two-dimensional.

The two-dimensional representation can be given explicitly as

\[
a = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} , \quad b = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} , \quad z = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} , \tag{4.168}
\]
and the group has character table
Here, since $K = \Gamma$, there is no quotient group to interchange the representations, and so there is one universe for each representation. Applying (2.44), we find

\[
\Pi_{R=1} = \frac{1}{|G|} \sum_{g \in G} \chi_1 (g^{-1}) \tau_g, \quad (4.169)
\]

\[
= \frac{1}{8} (1 + \tau_z + \tau_a + \tau_{az} + \tau_b + \tau_{bz} + \tau_{ab} + \tau_{ba}), \quad (4.170)
\]

\[
\Pi_{R=1_a} = \frac{1}{|G|} \sum_{g \in G} \chi_a (g^{-1}) \tau_g, \quad (4.172)
\]

\[
= \frac{1}{8} (1 + \sigma_{[-1]} + 2(\sigma_{[a]} + \sigma_{[b]} + \sigma_{[ab]})), \quad (4.173)
\]

\[
\Pi_{R=1_b} = \frac{1}{|G|} \sum_{g \in G} \chi_b (g^{-1}) \tau_g, \quad (4.174)
\]

\[
= \frac{1}{8} (1 + \sigma_{[-1]} + 2(\sigma_{[a]} + \sigma_{[b]} - \sigma_{[ab]})), \quad (4.175)
\]

\[
\Pi_{R=1_{ab}} = \frac{1}{|G|} \sum_{g \in G} \chi_{ab} (g^{-1}) \tau_g, \quad (4.176)
\]

\[
= \frac{1}{8} (1 + \sigma_{[-1]} + 2(\sigma_{[a]} - \sigma_{[b]} + \sigma_{[ab]})), \quad (4.177)
\]

\[
\Pi_{R=2} = \frac{2}{|G|} \sum_{g \in G} \chi_2 (g^{-1}) \tau_g, \quad (4.178)
\]

\[
= \frac{2}{8} (2 - 2\sigma_{[-1]}) = \frac{1}{2} (1 - \sigma_{[-1]}), \quad (4.179)
\]
Thus, the projectors are

\[
\Pi_{++} \equiv \Pi_{R=1} = \frac{1}{8} (1 + x + 2(y_1 + y_2 + y_3)), \quad (4.180)
\]

\[
\Pi_{+-} \equiv \Pi_{R=1a} = \frac{1}{8} (1 + x + 2(y_1 - y_2 - y_3)), \quad (4.181)
\]

\[
\Pi_{-+} \equiv \Pi_{R=1b} = \frac{1}{8} (1 + x + 2(-y_1 - y_2 + y_3)), \quad (4.182)
\]

\[
\Pi_{-} \equiv \Pi_{R=2} = \frac{1}{2} (1 - x). \quad (4.184)
\]

It is straightforward to check that

\[
\Pi_i \Pi_j = \delta_{i,j} \Pi_i, \quad \sum_i \Pi_i = 1. \quad (4.185)
\]

It is also straightforward to check that each projector is nonzero at exactly one of the points where the order parameter vevs are nonzero:

\[
\Pi_{++}\big|_{(+1, +1, +1, +1)} = 1 = \Pi_{+-}\big|_{(+1, +1, -1, -1)} = \Pi_{-+}\big|_{(+1, -1, -1, +1)} = \Pi_{-}\big|_{(+1, -1, +1, -1)},
\]

\[
\Pi_{5}\big|_{(-1, 0, 0, 0)} = 1, \quad (4.186)
\]

with other restrictions of projectors to points above vanishing.

As the center of \(D_4\) is \(\mathbb{Z}_2\), one would expect that the orbifold \([\text{point}/D_4]\) would have a \(BC_2\) (one-form) symmetry. However, in fact, this algebra is actually consistent with a \(B(\mathbb{Z}_2 \times \mathbb{Z}_2)\) symmetry, generated by

\[
\begin{align*}
    y_{1,2} &\mapsto -y_{1,2}, \\
    x, y_3 &\mapsto x, y_3 \text{ (invariant)}, \\
    y_{1,3} &\mapsto -y_{1,3}, \\
    x, y_2 &\mapsto x, y_2 \text{ (invariant)}. \quad (4.188)
\end{align*}
\]

This permutes the points at which the order parameters have vevs, which is reminiscent of spontaneous symmetry breaking, though again we observe that decomposition is a stronger statement. Existence of a \(B(\mathbb{Z}_2 \times \mathbb{Z}_2)\) is consistent with the decomposition under the \(BC_2\) into a pair of \(\mathbb{Z}_2 \times \mathbb{Z}_2\) orbifolds of points discussed above.

In passing, the orbifold \([\text{point}/\mathbb{H}]\) is nearly identical: the ring of dimension-zero fields is the same, with \(y_{1,2,3}\) corresponding to

\[
    \sigma_{[i]} = (1/2) (\tau_i + \tau_{-i}), \quad \sigma_{[j]} = (1/2) (\tau_j + \tau_{-j}), \quad \sigma_{[k]} = (1/2) (\tau_k + \tau_{-k}),
\]

and it also decomposes into five points:

\[
\text{QFT (}[\text{point}/\mathbb{H}]\) = QFT \left([\text{point}/\mathbb{Z}_2 \times \mathbb{Z}_2] \coprod [\text{point}/\mathbb{Z}_2 \times \mathbb{Z}_2]_{\text{or}}\right) = \text{QFT (5 points).} \quad (4.190)
\]

One difference is that the group \(\mathbb{H}\) does not admit discrete torsion: \(H^2(\mathbb{H}, U(1)) = 0\), unlike \(D_4\) for which \(H^2(D_4, U(1)) = \mathbb{Z}_2\) \cite[appendix D.3]{ref}. 

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4.4.2 Wilson lines

As described in the previous section, the group $D_4$ has five irreducible representations: four one-dimensional, and one two-dimensional, which we label as $1, 1_a, 1_b, 1_{ab}, 2$.

From the character table (see also [62]), we can read off

$$1_a \otimes 1_b = 1_{ab}, \quad 1_a^2 = 1_b^2 = 1_{ab}^2 = 1, \quad (4.191)$$

$$1_{a,b,ab} \otimes 2 = 2, \quad 2 \otimes 2 = 1 + 1_a + 1_b + 1_{ab}. \quad (4.192)$$

We begin with bulk Wilson lines. Letting for example $W_a$ denote a Wilson line associated to representation $1_a$, it is straightforward to compute

$$\Pi_{\pm\pm\pm}W_1 = W_1\Pi_{\pm\pm\pm}, \quad \Pi_5W_1 = W_1\Pi_5, \quad (4.193)$$

$$\Pi_{+++}W_a = W_a\Pi_{+++}, \quad \Pi_{+-}W_a = W_a\Pi_{-+-}, \quad (4.194)$$

$$\Pi_{-+-}W_a = W_a\Pi_{-+-}, \quad \Pi_{-++}W_a = W_a\Pi_{-++}, \quad (4.195)$$

$$\Pi_{++-}W_b = W_b\Pi_{++-}, \quad \Pi_{+-}W_b = W_b\Pi_{+-}, \quad (4.196)$$

$$\Pi_{-+-}W_b = W_b\Pi_{-+-}, \quad \Pi_{-++}W_b = W_b\Pi_{-++}, \quad (4.197)$$

$$\Pi_{++-}W_{ab} = W_{ab}\Pi_{++-}, \quad \Pi_{+-}W_{ab} = W_{ab}\Pi_{+-}, \quad (4.198)$$

$$\Pi_{-+-}W_{ab} = W_{ab}\Pi_{-+-}, \quad \Pi_{-++}W_{ab} = W_{ab}\Pi_{-++}, \quad (4.199)$$

$$\Pi_{\pm\pm\pm}W_2 = \frac{1}{4}W_2\Pi_5, \quad \Pi_5W_2 = W_2 (\Pi_{+++} + \Pi_{+-} + \Pi_{-++} + \Pi_{-+-}). \quad (4.200)$$

We see that the $W_{a,b,ab}$ act as defects bridging the $\pm\pm\pm$ universes, and $W_2$ acts as a defect bridging the universe 2 with the $\pm\pm\pm$ universes.

Next, we turn to boundary Wilson lines (Chan-Paton factors). As before, sheaves and bundles on $[\text{point}/D_4]$ are sheaves and bundles on the universes of the decomposition. Since all of $D_4$ acts trivially, we do not restrict to a subgroup, and consider the action of projectors on the Wilson lines, which are both associated to representations of $D_4$. Thus, 1, $1_a$, $1_b$, $1_{ab}$ correspond to sheaves on $[\text{point}/\mathbb{Z}_2 \times \mathbb{Z}_2]$ (which itself decomposes into four points, one for each of those one-dimensional representations), since their restriction to the center is trivial, and 2 corresponds to a sheaf on $[\text{point}/\mathbb{Z}_2 \times \mathbb{Z}_2]_{d,t}$, since its restriction to the center is nontrivial.

Letting for example $W_a$ denote a Wilson line associated to representation $1_a$, it is straightforward to compute

$$\Pi_{+++}W_1 = W_1, \quad \Pi_{+-}W_a = W_a, \quad \Pi_{-+-}W_b = W_b, \quad \Pi_{-++}W_{ab} = W_{ab}, \quad (4.201)$$

$$\Pi_5W_2 = W_2, \quad (4.202)$$

with other projectors annihilating other Wilson lines. This is consistent with the identification above of projectors with irreducible representations of $D_4$. 

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4.4.3 Symmetries

Since all of the orbifold group acts trivially, this is a rather degenerate special case, and as such, there are unexpected symmetries.

From the group theory alone, as the center of $D_4$ is only $\mathbb{Z}_2$, one would expect in general only a $B\mathbb{Z}_2$ symmetry. On the other hand, since the decomposition is into five points (naturally grouped into two sets, one of four points and the other of one), one could reasonably expect a larger one-form symmetry.

Amongst the twist fields associated to conjugacy classes, only one twist field ($x$) is invertible, while the others ($y_{1,2,3}$) are noninvertible; however, as noted elsewhere, that is somewhat ambiguous.

Abstractly, the ring of dimension-zero operators has three $\mathbb{Z}_2$ symmetries, under which $x$ and one of the $y_i$ is invariant while flipping the signs of the other two $y$’s.

In terms of the representations associated to the universes, four of the universes are associated to one-dimensional representations of $D_4$, while one is associated to the irreducible two-dimensional representation of $D_4$. The four one-dimensional representations form the group $\mathbb{Z}_2 \times \mathbb{Z}_2$, so this is consistent with a $B(\mathbb{Z}_2 \times \mathbb{Z}_2)$ symmetry, plus a noninvertible symmetry relating to the fifth universe.

Of the projectors corresponding to these universes, the projectors onto the four universes associated to one-dimensional representations involve both $x$ and $y$’s; only the projectors onto the universe associated to a higher-dimensional representation involves only the invertible twist field $x$. None of the projectors is itself invertible, and as remarked elsewhere, twist fields and representations are merely dual, not canonically bijective.

4.5 Nonabelian $D_4$ gerbe with discrete torsion

Next, consider the orbifold [point/$D_4$] with discrete torsion. (Here, we use the fact that $H^2(D_4, U(1)) = \mathbb{Z}_2$.) In general terms, we will see the same phenomena as in previous examples.

It will be handy in this section to have an explicit cocycle representing the nontrivial element of $H^2(D_4, U(1))$. Following the conventions of [19 section 3.7], we take the discrete torsion cocycle $\omega$ to be
where \( \xi = \exp(2\pi i/4) = i \), so that \( \xi^4 = 1 \). (As a consistency test, the invariant ratios \( \omega(g,h)/\omega(h,g) \) match those in e.g. \cite{39} table D.4.) The cocycles can also be described as \cite{49} section 3.7

\[
\omega(b_i, b_j^a) = 1, \quad \omega(b_i^a, b_j^a) = \xi^j.
\]  

(4.203)

It is straightforward to check that \( D_4 \) has two irreducible projective representations, hence decomposition implies \cite{20,39}

\[
\text{QFT (point}/D_4\text{)} = \text{QFT (point } \bigcup \text{ point).}
\]  

(4.204)

4.5.1 Ring of dimension-zero operators

Using the methods of section \cite{2.2.1} and the cocycle for the nontrivial element of \( H^2(D_4, U(1)) \) given above we find that the only conjugation-invariant dimension-zero twist fields are

\[
\sigma_1 = 1, \quad \sigma_{[b]} = (1/2) (\tau_b + i \tau_{bz}).
\]  

(4.205)

Consistent with expectations, these are the only two conjugacy classes that correspond to irreducible projective representations of \( D_4 \) \cite{39} appendix D.3]. Since the cocycle is trivial on \( \langle b \rangle \subset D_4 \), we use the fact that

\[
\tau_b \tau_b = \tau_z, \quad \tau_{bz} \tau_{bz} = \tau_z, \quad \tau_b \tau_{bz} = 1 = \tau_{bz} \tau_b
\]  

(4.206)

to derive that

\[
\sigma^2_{[b]} = i/2.
\]  

(4.207)

Identifying \( \sigma_{[b]} \) with \( y \), the ring of dimension-zero operators is then given by

\[
\mathbb{C}[y]/(y^2 - (i/2)),
\]  

(4.208)

which corresponds to a pair of points, supported at

\[
y_{\pm} = \pm \frac{1}{\sqrt{2}} \exp(\pm \pi i/4) = \pm \frac{1}{2} (1 + i).
\]  

(4.209)
Each point corresponds to one of the two universes appearing in the decomposition of \([X/D_4]_4\). The reader should note that in this ring, \(y \sim \sigma b\) is invertible:

\[
y^{-1} = -2iy. \tag{4.210}
\]

Before turning on discrete torsion, in the \(D_4\) orbifold we discussed in the previous section, the twist field built from the conjugacy class \(\{b, bz\}\) was not invertible.

For completeness,

\[
(a + by)^{-1} = \Delta^{-1}(a - by), \tag{4.211}
\]

for

\[
\Delta = a^2 - (i/2)b^2, \tag{4.212}
\]

so we see that noninvertible ring elements lie along the locus \(\{\Delta = 0\}\).

Next, we compute the projectors from (2.44), for which we need the projective representations of \(D_4\). As discussed in e.g. [39, appendix D.3], [47, example 3.12], [49, section 3.7], there are two irreducible projective representations of \(D_4\), and they both have dimension two. In the conventions of [49, section 3.7], they are given by

\[
\rho_r(b'a^j) = B_r^j A_r^i, \tag{4.213}
\]

for

\[
A_r = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad B_r = \begin{bmatrix} \xi^r & 0 \\ 0 & \xi^{1-r} \end{bmatrix}, \tag{4.214}
\]

for \(r \in \{1, 2\}\) indexing the two representations. For example, it is straightforward to check that

\[
\rho_r(a) \rho_r(b) = \omega(a, b) \rho_r(ab). \tag{4.215}
\]

Since there is no quotient group \(G = \Gamma/K\), as \(\Gamma = K\), there is nothing to interchange the two representations, so there are two universes, corresponding to each of those two irreducible representations.

In order to apply (2.44), we need the character table, which we compute next. To that end, we note that of the five conjugacy classes of \(D_4\), namely

\[
\{1\}, \ \{z\}, \ \{a, az\}, \ \{b, bz\}, \ \{ab, ba\}, \tag{4.216}
\]

only two of them \((\{1\}, \{b, bz\})\) are \(\omega\)-regular, meaning that for all elements \(x\) of the conjugacy class, \(\omega(x, g) = \omega(g, x)\) for all \(g\) commuting with \(x\). For example, \(\omega(z, a) \neq \omega(a, z)\), hence neither \(\{z\}\) nor \(\{a, az\}\) can be \(\omega\)-regular, and similarly since \(\omega(ab, z) \neq \omega(z, ab)\), \(\{ab, ba\}\) also cannot be \(\omega\)-regular.
The fact that there are two \( \omega \)-regular conjugacy classes correctly matches the number of irreducible projective representations.

From the explicit form of the representations above, given the definition \( \chi_r(g) = \text{Tr} \rho_r(g) \), we find that the characters are given by

\[
\begin{array}{c|cccccc}
\chi & 1 & z & a & az & b & bz & ab & ba \\
\hline
r = 1 & 2 & 0 & 0 & 0 & 1 + i & 1 - i & 0 & 0 \\
r = 2 & 2 & 0 & 0 & 0 & -1 - i & -1 + i & 0 & 0 \\
\end{array}
\]

As expected, characters of elements of non-\( \omega \)-regular conjugacy classes vanish. Also as expected, the characters are not class functions, but instead obey (2.65). For example, using the fact that \( bz = aba^{-1} \) and

\[
\frac{\omega(b, a^{-1})}{\omega(a^{-1}, aba^{-1})} = \frac{\omega(b, a)}{\omega(a, bz)} = \xi = i,
\]

we can confirm

\[
\begin{align*}
\chi_1(b) & = 1 + i = i(1 - i) = i\chi_1(bz), \\
\chi_2(b) & = -1 - i = i(-1 + i) = i\chi_2(bz),
\end{align*}
\]

as predicted by (2.65).

Now, we can compute projectors. From equation (2.44), we have that

\[
\Pi_r = \frac{2}{|D_4|} \sum_{g \in D_4} \frac{\chi_r(g^{-1})}{\omega(g, g^{-1})} \tau_g,
\]

from which we find

\[
\begin{align*}
\Pi_{r=1} & = \frac{1}{2} \left[ 1 + \frac{1}{2}(1 - i)\tau_b + \frac{1}{2}(1 + i)\tau_{bz} \right], \\
& = \frac{1}{2} \left[ 1 + \sqrt{2} \exp(-\pi i/4)\sigma_{[b]} \right], \\
\Pi_{r=2} & = \frac{1}{2} \left[ 1 - \frac{1}{2}(1 - i)\tau_b - \frac{1}{2}(1 + i)\tau_{bz} \right], \\
& = \frac{1}{2} \left[ 1 - \sqrt{2} \exp(-\pi i/4)\sigma_{[b]} \right].
\end{align*}
\]

It is straightforward to check that they obey

\[
\Pi_r \Pi_s = \delta_{r,s} \Pi_r, \quad \Pi_1 + \Pi_2 = 1,
\]

and are easily checked to lie along the locus \( \{ \Delta = 0 \} \).
4.5.2 Wilson lines

For simplicity, in this section, we will only compute the action of the projectors on the boundary Wilson lines. Since the entire orbifold group acts trivially, both the projectors and the Wilson lines are associated to projective representations of $D_4$.

Using the relation
\[ \sigma_{[g]} W_R = \frac{\chi_R(g)}{\dim R} W_R, \]
(4.226)
it is straightforward to compute that
\[ \Pi_r W_s = \delta_{r,s} W_r. \]
(4.227)
This is precisely as expected – since all of $D_4$ acts trivially, and each universe is associated to an irreducible projective representation of $D_4$, the Wilson lines obey the same decomposition.

4.5.3 Symmetries

Much as in the last example, since this is an orbifold of a point, it is a rather degenerate special case, and so can have unexpected symmetries.

Here, there are two (\(\omega\)-regular) conjugacy classes, including \{1\}. The twist field associated to the nontrivial conjugacy class is invertible.

Since the theory decomposes into two copies of a point, one expects a $B\mathbb{Z}_2$ symmetry.

On the other hand, the representations associated to each universe are both two-dimensional, suggesting that this theory only has noninvertible symmetries (which in this case exchange identical copies).

5 Examples in supersymmetric gauge theories

The rings of dimension-zero operators that we have described are also visible in two-dimensional supersymmetric gauge theories, as we shall now describe. In these examples, the one-form symmetry will always be a center symmetry, corresponding to a banded abelian gerbe, so the structure we derive will coincide with that of the banded abelian examples discussed previously.

First, consider a $U(1)$ gauge theories with nonminimal charges. Theories of this form were first discussed in [17,19], and include variations of the supersymmetric $\mathbb{P}^n$ model. Here, let us briefly consider a family of variations of the $\mathbb{P}^n$ model, discussed in e.g. [19], corresponding
to sigma models on the $\mathbb{Z}_k$ gerbes over $\mathbb{P}^n$. These are described by $U(1)^2$ gauge theories with chiral superfields $x_0, \ldots, n, z$ of charges

$$
\begin{array}{cc}
x_i & z \\
1 & -m \\
0 & k.
\end{array}
$$

The $x_i$ act as homogeneous coordinates on $\mathbb{P}^n$, and $z$ is a coordinate along the fibers of a $\mathbb{C}^\times$ bundle over $\mathbb{P}^n$. The second $U(1)$ ‘overgauges’ the $\mathbb{C}^\times$, so that the fibers become $B\mathbb{Z}_k$, hence a $\mathbb{Z}_k$ gerbe (with a one-form translation symmetry along the fibers). The characteristic class of the gerbe is $-m \mod k$. The case $m = 1$ is equivalent to a $\mathbb{P}^n$ model in which charges of the fields are multiplied by $k$.

The quantum cohomology ring one derives from the Coulomb branch of this gauged linear sigma model is \cite{19}

$$
\mathbb{C}[x, y]/(x^k - 1, y^{n+1} - x^m q).
$$

The dimension-zero operators are encoded in

$$
\mathbb{C}[x]/(x^k - 1),
$$

the same structure we have previously seen in sigma models on disjoint unions of $k$ copies of a space, and in banded $\mathbb{Z}_k$ gerbes.

As discussed in \cite{19}, this structure also arises in the mirror. Computing the mirror ala \cite{64}, one finds Landau-Ginzburg models with superpotentials of the form

$$
W = \exp(-X_1) + \cdots + \exp(-X_n) + q\Upsilon^{-m}\exp(+X_1 + X_2 + \cdots + X_n),
$$

where $\Upsilon$ is a field valued in $k$th roots of unity. Here, $\Upsilon$ is a dimension-zero field, the mirror of the “$x$” appearing in the quantum cohomology ring, obeying the same relation ($\Upsilon^k = 1$) that we have already discussed. (This description of the mirror implicitly encodes decomposition: the path integral’s sum over values of $\Upsilon$ is a finite sum that can be pulled out of the path integral, making it clear that the theory is equivalent to a disjoint union of ordinary Landau-Ginzburg models with staggered complex structures).

Next, let us turn to an $SU(2)$ gauge theory with matter invariant under the central $\mathbb{Z}_2$, so that the theory has a $B\mathbb{Z}_2$ (one-form) symmetry. Here, decomposition predicts \cite{65}, schematically,

$$
SU(2) = SO(3)_+ + SO(3)_-,
$$

where the $\pm$ indicate discrete theta angles.
Mirrors to such theories and their generalizations were discussed in [66–71]. For example, the mirror to the pure $SU(2)$ theory is described by the superpotential [66, equ’n (12.3)]

$$W = \sigma \ln \left( \frac{X_{12}}{X_{21}} \right)^2 + X_{12} + X_{21}. \quad (5.5)$$

Taking the square root, this could be equivalently described as a theory with a $\mathbb{Z}_2$-valued field $\Upsilon$ and superpotential

$$W = \sigma \ln \left( \Upsilon \frac{X_{12}}{X_{21}} \right) + X_{12} + X_{21}. \quad (5.6)$$

Such a theory is equivalent to a disjoint union of two Landau-Ginzburg models, with either value of $\Upsilon = \pm 1$.

For comparison, the mirror to the pure $SO(3)_+$ theory is described by the superpotential [66, equ’n (12.9)]

$$W = \sigma \ln \left( \frac{X_{12}}{X_{21}} \right) + X_{12} + X_{21}, \quad (5.7)$$

which has no vacua (corresponding to the fact that the $SO(3)_+$ theory dynamically breaks supersymmetry), and the mirror to the pure $SO(3)_-$ theory is described by the superpotential [66, equ’n (12.14)]

$$W = \sigma \ln \left( -\frac{X_{12}}{X_{21}} \right) + X_{12} + X_{21}, \quad (5.8)$$

which does have supersymmetric vacua.

In any event, we now see that the two constituent theories of the $SU(2)$ mirror, at either value of $\Upsilon = \pm 1$, are precisely the $SO(3)_\pm$ mirrors, recovering the decomposition statement

$$SU(2) = SO(3)_+ + SO(3)_-. \quad (5.9)$$

Furthermore, $\Upsilon$ acts as a dimension-zero field, with a ring relation $\Upsilon^2 = 1$, the same structure we have seen previously in sigma models on disjoint unions and in other banded abelian gerbes. Similar structures arise in mirrors to other two-dimensional gauge theories with center-invariant matter, see for example [66–70]. As the stories are closely related, we will not describe them explicitly here.

6 Four-dimensional analogues

The paper [36] considered four-dimensional versions of decomposition. Specifically, beginning with ordinary bosonic Yang-Mills theory in four dimensions, with action

$$S = \frac{1}{2g^2} \int \text{Tr} F \wedge *F, \quad (6.1)$$
they construct a modified theory with a restriction on instanton numbers, with action of the form

\[
S = \frac{1}{2g^2} \int \text{Tr} F \wedge *F \\
+ i \int B \left( \frac{1}{8\pi^2} \text{Tr} F \wedge F - \frac{k}{2\pi} dC^{(3)} \right) + \frac{i\hat{\theta}}{2\pi} \int dC^{(3)},
\]

(6.2)

where \(k\) is an integer, \(B\) is a scalar field of periodicity \(2\pi\), and \(C^{(3)}\) is a three-form gauge field. As they discuss, the equations of motion for \(B\) are

\[
\frac{1}{8\pi^2} \text{Tr} F \wedge F = \frac{k}{2\pi} dC^{(3)},
\]

(6.3)

which implies that this theory has a restriction on instantons (to instanton numbers divisible by \(k\)), and exhibits a decomposition. Rather than study Gukov-Witten operators, we will look at a different class of (nonlocal) operators in this section.

Briefly, hand-in-hand with existence of a decomposition, this theory has projection operators, of the form

\[
\Pi_n = \frac{1}{k} \sum_{m=0}^{k-1} \xi^{nm} \exp \left[ im \int_{M_3} \left( C^{(3)} - \frac{1}{k} \omega_{\text{CS}}(A) \right) \right],
\]

(6.4)

where

\[
d\omega_{\text{CS}}(A) = \frac{1}{8\pi^2} \text{Tr} F \wedge F.
\]

(6.5)

and \(\xi\) is a \(k\)th root of unity. (\(C\) absorbs the lack of gauge-invariance of \((1/k)\omega_{\text{CS}}\)) These operators are volume operators, defined on three-manifolds \(M_3\), and obey

\[
\Pi_n \Pi_m = \delta_{n,m} \Pi_n,
\]

(6.6)

using the equations of motion (6.3) in the form

\[
\exp \left[ ik \int_{M_3} \left( C^{(3)} - \frac{1}{k} \omega_{\text{CS}}(A) \right) \right] = 1.
\]

(6.7)

These operators project onto particular universes in the decomposition.

In passing, since \(B\) can take on only finitely many values, inserting \(\Pi_n\) into the path integral is equivalent to coupling the original Yang-Mills theory to the TFT to restrict instanton numbers, as the path integral over \(B\) is equivalent to the sum over \(m\), and the \(\xi^{nm}\) term corresponds to \(\hat{\theta} = 2\pi n\).

Physically, these projection operators form domain walls that are also projectors: “end-of-the-world projectors” that separate distinct universes.
In any event, given a set of projection operators, we can proceed precisely as we have previously in this paper, for e.g. sigma models on disjoint unions and banded $\mathbb{Z}_k$ gerbes, and take linear combinations of projectors so that for any three-submanifold $M_3$, we have a ring of operators given by $\mathbb{C}[x]/(x^k - 1)$, which includes the projectors as a subset, and which describes a set of $k$ points (one for each universe in the decomposition).

7 Conclusions

In this paper we have discussed dimension-zero operators and Wilson lines in two-dimensional theories, especially orbifolds, exhibiting decomposition. We have described the computation of fusion algebras of twist fields, given a systematic construction of projectors onto universes, and used those tools to verify that the projectors project Wilson lines onto universes in the fashion predicted in [20].

We have also discussed the geometries underlying these computations. These algebras of twist fields are commutative algebras, and the methods of commutative ring theory give one a perspective on their properties – for example, the rings describe a set of points, as many as universes in decomposition.

We have also discussed the symmetries of these theories – both ordinary one-form and noninvertible symmetries – and their descriptions in terms of twist fields (Gukov-Witten operators).

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A Casimirs

In this section we will justify the fact that the action of the twist fields $\sigma_{[g]}$ on Wilson lines is to multiply by a factor proportional to a character.

Briefly, each $\sigma_{[g]}$ is a linear combination of operators $\tau_g$, and the effect of any $\tau_g$ on a
Wilson line $W_R$ in representation $R$ is to insert the matrix $T^R_g$ into the Wilson line. We will demonstrate that the linear combination of $T^R_g$’s provided by $\sigma_{[g]}$ commutes with all other matrices in that representation, and so (by e.g. Schur’s lemma) is proportional to the identity. This is equivalent to multiplying the Wilson line by a numerical factor.

We first discuss the pertinent claim for ordinary representations, and then discuss the more complicated case of projective representations subsequently. Let $G$ be a finite group, $R$ an (ordinary) irreducible representation of $G$, and let $T^R_g$ denote a matrix representing $g \in G$ in representation $R$. Let $T_{[g]}$ be the matrix inserted in Wilson lines $W_R$ by the branch cut induced by the twist field $\sigma_{[g]}$. Specifically, in this case (2.18)

$$\sigma_{[g]} = \frac{1}{|[g]|} \sum_{x \in [g]} \tau_x,$$

we have

$$T_{[g]} = \frac{1}{|[g]|} \sum_{x \in [g]} T^R_x,$$

where $[g]$ is a conjugacy class of $G$ containing $g$, $|[g]|$ is the number of elements in that conjugacy class.

We first demonstrate that

$$T_{[g]} = \frac{\chi_R(g)}{\dim R} I,$$

where $I$ is the identity matrix. This is a standard result, but we pause to prove it en route to establishing an analogue for projective representations. The key result is that for any $y \in G$, $T_{[g]} T^R_y = T^R_y T_{[g]}$, which we verify as follows:

$$T_{[g]} T^R_y = \frac{1}{|[g]|} \sum_{x \in [g]} T^R_x T^R_y = \frac{1}{|[g]|} \sum_{x \in [g]} T^R_y T^R_{yx^{-1}} = T^R_y T_{[g]}.$$  

(A.4)

Since $T_{[g]}$ commutes with all the $T^R_y$, Schur’s lemma then implies that $T_{[g]}$ must be proportional to the identity, and the normalization can be easily checked. The desired result follows.

Next, we consider the analogue of this statement for projective representations. To that end, let $\omega \in H^2(G, U(1))$ be a normalized cocycle, meaning

$$\omega(1, g) = 1 = \omega(g, 1)$$

(A.5)

for all $g \in G$, and consider an irreducible projective representation $R$, representing elements of $G$ by matrices $T^R_g$ such that

$$T^R_g T^R_h = \omega(g, h) T^R_{gh}.$$  

(A.6)
From equation (2.17),

\[
\sigma_{[g]} = \frac{1}{|G|} \sum_{h \in G} \frac{\omega(h, g)\omega(hg, h^{-1})}{\omega(h, h^{-1})} \tau_{hgh^{-1}},
\]

(A.7)

so the effect of \( \sigma_{[g]} \) is to insert into a Wilson line \( W_R \) the matrix

\[
T_{[g]} = \frac{1}{|G|} \sum_{h \in G} \frac{\omega(h, g)\omega(hg, h^{-1})}{\omega(h, h^{-1})} T_{hgh^{-1}}^R.
\]

(A.8)

We now check that \( T_{[g]} T_{y}^R = T_{y}^R T_{[g]} \):

\[
T_{[g]} T_{y}^R = \frac{1}{|G|} \sum_{h \in G} \frac{\omega(h, g)\omega(hg, h^{-1})}{\omega(h, h^{-1})} T_{hgh^{-1}}^R T_{y}^R,
\]

(A.9)

\[
= \frac{1}{|G|} \sum_{h \in G} \frac{\omega(h, g)\omega(hg, h^{-1})\omega(hgh^{-1}, y)}{\omega(h, h^{-1})} T_{hgh^{-1}y}^R,
\]

(A.10)

\[
= \frac{1}{|G|} \sum_{h \in G} \frac{\omega(h, g)\omega(hg, h^{-1})\omega(hgh^{-1}, y)}{\omega(h, h^{-1})\omega(y, y^{-1}hgh^{-1}y)} T_{y}^R T_{y}^R T_{hgh^{-1}y}^R.
\]

(A.11)

It can be shown that

\[
\frac{\omega(h, g)\omega(hg, h^{-1})\omega(hgh^{-1}, y)}{\omega(h, h^{-1})\omega(y, y^{-1}hgh^{-1}y)} = \frac{\omega(y^{-1}h, g)\omega(y^{-1}hg, h^{-1}y)}{\omega(y^{-1}h, h^{-1}y)},
\]

(A.12)

by multiplying in

\[
\frac{(d\omega)(y, y^{-1}, hgh^{-1}, y)(d\omega)(hg, h^{-1}, y)(d\omega)(h^{-1}, y, y^{-1}h)}{(d\omega)(y^{-1}, hg, h^{-1}y)(d\omega)(y^{-1}, h, g)(d\omega)(y, y^{-1}, h)} = 1,
\]

(A.13)

and using the fact that for normalized cocycles, \( \omega(h, h^{-1}) = \omega(h^{-1}, h) \) for all \( h \). As a result, the effect of multiplying in \( y \) is to (potentially) interchange elements of the sum, and so we have

\[
T_{[g]} T_{y}^R = T_{y}^R T_{[g]},
\]

(A.14)

confirming that \( T_{[g]} \) is central.

Now, we claim that \( T_{[g]} \) is proportional to the identity matrix. We have shown that \( T_{[g]} \) commutes with other representation matrices, and for honest non-projective representations, Schur’s lemma implies the desired result.

Now, in principle, Schur’s lemma only applies to honest representations, not projective representations, so we need to recast this as a question about honest representations of a central extension \( \Gamma \) of \( G \):

\[
1 \rightarrow U(1) \rightarrow \Gamma \rightarrow G \rightarrow 1.
\]
To that end, we pick a splitting of $\Gamma$, and write $\gamma \in \Gamma$ as pairs $\gamma = (x, \lambda)$ for $x \in G$, $\lambda \in U(1)$, with multiplication
\[(x, \lambda) \cdot (y, \mu) = (xy, \lambda\mu \omega(x, y)).\] (A.16)

Then, given $g \in G$, we lift to $(g, 1) \in \Gamma$. Conjugating $[(g, 1)]$ in $\Gamma$ by $(h, \lambda)$ gives a pair of the form
\[(hgh^{-1}, \omega(h, g)\omega(hg, h^{-1})).\] (A.17)

This pair acts on a representation as
\[\omega(h, g)\omega(hg, h^{-1})T_{hgh^{-1}}^R,\] (A.18)
so we see that the combination
\[\sum_{hgh^{-1} \in [g]} \omega(h, g)\omega(hg, h^{-1})T_{hgh^{-1}}^R\] (A.19)
is a sum over lifts to $\Gamma$, and hence involves honest representations, so we can apply Schur’s lemma again.

Doing so, and checking the normalization, one finds
\[T_{[g]} = \frac{\chi_R(g)}{\dim R}I.\] (A.20)

In any event, given that the action of $\sigma_{[g]}$ on a Wilson line $W_R$ is to insert a matrix proportional to the identity, it is now clear that
\[\sigma_{[g]} W_R = \frac{\chi_R(g)}{\dim R} W_R.\] (A.21)

In passing, in the presence of discrete torsion, changing the representative of the conjugacy class $[g]$ multiplies both sides of the expression above by a phase. We check in section 2.3.3 that the phases are identical, so that the expression above is consistent.

B Character identities

We collect here some character identities for finite groups $G$, to make this paper self-contained. See e.g. [46 section 2], [49 section 7.3], [72 chapter V], [73 chapter 2.1], [74 section 1.12].

Let $\omega$ denote a cocycle representing an element of $H^2(G, U(1))$, corresponding to twisting, and we assume that all representations are unitary (projective) representations. The
identities we will use are sometimes written in terms of complex conjugates of characters, which for an irreducible representation $R$ twisted by $\omega$, are related by \cite[section 2]{[47]}

$$\chi_R(g^{-1}) = \omega(g, g^{-1})\chi_R(g),$$

(B.1)

which for trivial $\omega$ specializes to

$$\chi_R(g^{-1}) = \chi_R(g).$$

(B.2)

Let $T^R(g)$ denote a matrix representing $g \in G$ in irreducible representation $R$, which obeys

$$T^R(g)T^R(h) = \omega(g, h)T^R(gh).$$

(B.3)

Then, the source of the identities we will primarily use is \cite{[14]}

$$\frac{1}{|G|} \sum_{g \in G} T^R(g)T^S(g^{-1})_{ik} \omega(g, g^{-1}) = \frac{\delta_{R,S}}{\dim R} \delta_{jk}\delta_{ui}.$$  

(B.4)

(See e.g. \cite[section 31.1]{[72]}, \cite[exercise 2.1.1, p. 58]{[73]}, \cite[section 1.12]{[74]} for a version without discrete torsion.)

Let us quickly outline a proof of the assertion above, as this identity may seem obscure. Following \cite[section 31]{[72]}, for any $(\dim R) \times (\dim S)$ matrix $C$, define

$$T = \sum_{g \in G} \frac{T^R(g)CT^S(g^{-1})}{\omega(g, g^{-1})},$$

(B.5)

then one can show that $T$ intertwines the representations $R$, $S$, meaning for any $h \in G$,

$$T^R(h)T = TT^S(h).$$

(B.6)

To establish that, the key step is the identity

$$\frac{\omega(h, h^{-1}g)}{\omega(h^{-1}g, g^{-1}h)} = \frac{\omega(g^{-1}, h)}{\omega(g, g^{-1})},$$

(B.7)

which can be established by multiplying by the coboundaries

$$\frac{(d\omega)(h, h^{-1}, g)(d\omega)(g^{-1}, h, h^{-1})(d\omega)(h^{-1}, g, g^{-1}h)(d\omega)(g, g^{-1}h, h^{-1})}{\omega(h, h^{-1}, h)} = 1,$$

(B.8)

Experts should note that the sum is over all elements of $G$, not just the $\omega$-regular elements (meaning, elements $g \in G$ such that $\omega(g, h) = \omega(h, g)$ for all $h$ commuting with $g$). Sums over the latter sometimes arise in character identities for projective representations, simply because characters of non-regular elements vanish. It is straightforward to check in examples that the identity (B.4) above is only valid when one sums over all $g \in G$, not just $\omega$-regular elements of $G$.

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and using the normalization condition. Then, given that intertwining, \( T \propto \delta_{R,S}I \), and the normalization is straightforward to compute. Taking \( C \) to be the identity, the result (B.4) follows.

As a consistency test, note that if we contract \( u \) and \( i \), and set \( S = R \), this identity reduces to
\[
\frac{1}{|G|} \sum_{g \in G} T^R(1)_{jk} = \delta_{jk},
\]
(B.9)
which is trivially correct.

As another consistency test, if we contract the pair \((u, j)\) and the pair \((i, k)\), this identity reduces to
\[
\frac{1}{|G|} \sum_{g \in G} \chi_R(g) \chi_S(g^{-1}) \omega(g, g^{-1}) = \delta_{R,S},
\]
(B.10)
or more simply,
\[
\frac{1}{|G|} \sum_{g \in G} \chi_R(g) \chi_S(g) = \delta_{R,S},
\]
(B.11)
which can be found in e.g. [46, section 2.3], [47, section 2], [49, chapter 7.3].

If we multiply factors of \( T^R(a), T^S(b) \) into expression (B.4), we find
\[
\frac{1}{|G|} \sum_{g \in G} \omega(a, g) \omega(g^{-1}, b) \omega(g, g^{-1}) T^R(ag)_{\ell u} T^S(g^{-1}b)_{im} = \frac{\delta_{R,S}}{\dim R} \omega(a, b) T^R(ab)_{\ell m} \delta_{ui},
\]
(B.12)
whose traces gives
\[
\frac{1}{|G|} \sum_{g \in G} \frac{\omega(a, g) \omega(g^{-1}, b) \omega(g, g^{-1})}{\omega(g, g^{-1})} \chi_R(ag) \chi_S(g^{-1}b) = \frac{\delta_{R,S}}{\dim R} \omega(a, b) \chi_R(ab).
\]
(B.13)

Similarly, if we take \( S = R \) and multiply factors of \( T^R(a), T^R(b) \) into expression (B.4), we find
\[
\frac{1}{|G|} \sum_{g \in G} \frac{\omega(g, a) \omega(g^{-1}, b) \omega(ga, g^{-1}b)}{\omega(g, g^{-1})} T^R(gag^{-1}b)_{jm} = \frac{1}{\dim R} \chi_R(a) T^R(b)_{jm},
\]
(B.14)
whose trace gives
\[
\frac{1}{|G|} \sum_{g \in G} \frac{\omega(g, a) \omega(g^{-1}, b) \omega(ga, g^{-1}b)}{\omega(g, g^{-1})} \chi_R(gag^{-1}b) = \frac{1}{\dim R} \chi_R(a) \chi_R(b).
\]
(B.15)

\(^{15}\)Because characters of non-\(\omega\)-regular elements vanish, these identities are sometimes equivalently written as a sum over only \(\omega\)-regular elements, see e.g. [49, section 7.3], instead of all elements of \(G\). For simplicity, we have chosen to write these in terms of sums over all elements of \(G\).
The expressions (B.13), (B.15) may seem somewhat exotic, but in another context they may look more familiar. Specifically, if we specialize to the case of vanishing discrete torsion, they reduce to

\[ \frac{1}{|G|} \sum_{g \in G} \chi_R(ag)\chi_S(g^{-1}b) = \delta_{R,S} \frac{\chi_R(ab)}{\dim R}, \] (B.16)

\[ \frac{1}{|G|} \sum_{g \in G} \chi_R(gag^{-1}b) = \frac{\chi_R(a)\chi_R(b)}{\dim R}. \] (B.17)

which are simply finite-group analogues of perhaps more familiar versions from Lie groups of nonzero dimension:

\[ \int dV \chi_R(XV)\chi_S(V^\dagger Y) = \frac{\delta_{RS}}{\dim R} \chi_R(XY), \] (B.18)

\[ \int dU \chi_R(AUBU^\dagger) = \frac{1}{\dim R} \chi_R(A)\chi_R(B), \] (B.19)

(see e.g. [73]).

Another useful orthogonality relation arises from summing over all the irreducible (projective) representations of a group [46, prop. 2.7], [49, section 7.3, theorem 3.2]:

\[ \sum_R \chi_R(g)\chi_R(h^{-1}) \omega(g, h^{-1}) = \begin{cases} 0 & g, h \text{ not conjugate}, \\ \frac{|G|}{|[g]|} & g, h \text{ conjugate}, \end{cases} \] (B.20)

where \([g]\) denotes the number of elements in a conjugacy class containing \(g\). For applications, it may be helpful to note that the number of elements in the centralizer of \(g\) (the set of elements that commute with \(g\)) equals \(|G|/|[g]|\).

### C Induced representations

In this section we will argue that the irreducible components of restrictions of representations of \(G\) to a normal subgroup \(K\) span all irreducible representations of \(K\). The argument is a short exercise in induced representations, which we review.

Let \(K\) be a normal subgroup of a finite group \(G\), and let \(\rho\) be a representation of \(K\), acting on a vector space \(V\). Briefly, \(\rho\) induces a representation \(\tilde{\rho}\) of \(G\), which can be described as the vector bundle over \(G/K\) associated to the principal \(K\) bundle \(G \to G/K\). (Note that as a vector space, this is \(|G/K|\) copies of \(V\).) This has a natural \(G\)-equivariant structure, and so gives a representation of \(G\).
In more detail, let \( \{g_i\} \) be a set of representatives of \( G/K \), then the vector space \( \tilde{V} \) on which the induced representation is the sum

\[
\tilde{V} = \bigoplus_i g_i V. \tag{C.1}
\]

Any element \( g \in G \) acts as follows. For each \( i \), write

\[
 gg_i = g_j(k) \tag{C.2}
\]

for some \( k \in K \), whose action is given by \( \rho \).

It will be helpful to consider an example. Let \( G = D_4 \), the eight-element finite group, and let \( K = \mathbb{Z}_2 \), its center. We present \( D_4 \) as

\[
D_4 = \{1, a, b, z, az, bz, ab, ba = abz\}, \tag{C.3}
\]

where \( a^2 = 1 = z^2, b^2 = z, \) and \( z \) generates the center. Let the representatives of the coset \( G/K \) be \( \{1, a, b, ab\} \), so that, for example,

\[
aa = 1, \ ab = ab, \ aab = b, \tag{C.4}
\]

\[
ba = abz, \ bb = z, \ bab = a, \tag{C.5}
\]

\[
aba = bz, \ abb = az, \ abab = 1. \tag{C.6}
\]

Let \( \rho \) be the nontrivial one-dimensional representation of \( K = \mathbb{Z}_2 \). The induced representation \( \tilde{\rho} \) has vector space

\[
\tilde{V} = \bigoplus_4 V = V \oplus aV \oplus bV \oplus abV. \tag{C.7}
\]

Let \( E \equiv (x_1, x_a, x_b, x_{ab}) \in \tilde{V} \), then from the multiplication rules above, we have

\[
z \cdot E = (-x_1, -x_a, -x_b, -x_{ab}), \tag{C.8}
\]

\[
a \cdot E = (x_a, x_1, x_{ab}, x_b), \tag{C.9}
\]

\[
b \cdot E = (-x_b, x_{ab}, x_1, -x_a), \tag{C.10}
\]

\[
(ab) \cdot E = (x_{ab}, -x_b, -x_a, x_1), \tag{C.11}
\]

so that

\[
a \cdot (b \cdot E) = (x_{ab}, -x_b, -x_a, x_1) = (ab) \cdot E, \tag{C.12}
\]

\[
b \cdot (a \cdot E) = (-x_{ab}, x_b, x_a, -x_1) = (ba = abz) \cdot E, \tag{C.13}
\]

as expected.

If we restrict \( \tilde{\rho} \) to \( K \), we get four copies of \( \rho \):

\[
\tilde{\rho}|_K = \bigoplus_4 \rho. \tag{C.14}
\]
For another example, suppose \( G = \mathbb{H} \), the eight-element group of unit quaternions, and \( K = \langle i \rangle \cong \mathbb{Z}_4 \). Let \( \rho \) be an irreducible representation of \( \mathbb{Z}_4 \) that maps the generator of \( \mathbb{Z}_4 \) to multiplication by \( \xi \), for \( \xi \) some fourth root of unity. Let \( \{1, j\} \) represent cosets in \( G/K = \mathbb{Z}_2 \), so that

\[
\tilde{V} = V \oplus jV
\]

for \( V \cong \mathbb{C} \). Then, from the multiplications

\[
ij = k = j(-i), \quad jj = -1, \quad kj = -i,
\]

for \( E = (x_1, x_j) \in \tilde{V} \), we have

\[
i \cdot E = (\xi x_1, \xi^3 x_j), \quad j \cdot E = (\xi^2 x_j, x_1), \quad k \cdot E = (\xi^3 x_j, \xi^3 x_1),
\]

so that, for example,

\[
i \cdot (j \cdot E) = (\xi^3 x_j, \xi^3 x_1) = (k = ij) \cdot E,
\]

as expected.

In this case, if we restrict the induced representation \( \tilde{\rho} \) to \( K \), we find

\[
\tilde{\rho}|_K = \rho \oplus \rho^3.
\]

In general, given any irreducible representation \( \rho \) of \( K \), the restriction of the induced representation will be a sum of irreducible representations of \( K \), as many as elements of \( G/K \), with at least one copy (over the identity coset) equal to \( \rho \). If \( K \) is central, then all the irreducible representations appearing in the restriction will be copies of \( \rho \).

In particular, every irreducible representation of \( K \) appears as a summand in the decomposition of the restriction of representations of \( G \) to \( K \): for any representation \( \rho \) of \( K \), at least one summand in the restriction of the induced representation will be a copy of \( \rho \).

## D Miscellaneous group cohomology results

In this section we collect a handful of pertinent statements in group cohomology that will be used elsewhere.

First, we often use the fact that for a group 2-cochain \( \omega : G \times G \to U(1) \) (with trivial action on the coefficients),

\[
(d\omega)(a, b, c) = \frac{\omega(b, c)\omega(a, bc)}{\omega(ab, c)\omega(a, b)}.
\]
We will typically use normalized cocycles, by which we mean that
\[ \omega(1, g) = 1 = \omega(g, 1) \]  
(D.2)
for all \( g \in G \). In addition, we can impose two more constraints:

1. \( \omega(g, g^{-1}) = 1 = \omega(g^{-1}, g) \),

2. for \( \omega \)-regular elements \( g \in G \), for \( h \) commuting with \( g \),
\[ \omega(h, g)\omega(hg, h^{-1}) = 1. \]  
(D.3)

The first of these two conditions can be demonstrated as follows. From \( (d\omega)g, g^{-1}, g \) = 1 and the normalization condition, we have
\[ \omega(g, g^{-1}) = \omega(g^{-1}, g), \]  
(D.4)
and by picking a coboundary \( \mu \) such that \( \mu(1) = 1, \mu(g)\mu(g^{-1}) = \omega(g, g^{-1})^{-1} \), we can replace \( \omega \) by \( \omega' \) such that \( \omega'(g, g^{-1}) = 1 \).

The second condition can be demonstrated as follows. First, recall that for an element \( g \in G \) to be \( \omega \)-regular means that for all \( h \) that commute with \( g \), \( \omega(g, h) = \omega(h, g) \). Then, from the cocycle condition \( (d\omega)(g, h, h^{-1}) = 1 \), we have that
\[ \omega(h, g)\omega(hg, h^{-1}) = \omega(h, h^{-1})\frac{\omega(h, g)}{\omega(g, h)}. \]  
(D.5)
From \( \omega \)-regularity, the right-hand side reduces to \( \omega(h, h^{-1}) \), which as already demonstrated can be chosen to equal 1.

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