Group-theoretical search for rows or columns of the lepton mixing matrix

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March 8, 2018

Abstract

We have used the \texttt{SmallGroups} library of groups, together with the computer algebra systems \texttt{GAP} and \texttt{Mathematica}, to search for groups with a three-dimensional irreducible representation in which one of the group generators has a twice-degenerate eigenvalue while another generator has non-degenerate eigenvalues. By assuming one of these group generators to commute with the charged-lepton mass matrix and the other one to commute with the neutrino (Dirac) mass matrix, one derives group-theoretical predictions for the moduli of the matrix elements of either a row or a column of the lepton mixing matrix. Our search has produced several realistic predictions for either the second row, or the third row, or for any of the columns of that matrix.

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1 Introduction

Over the past few years—starting with ref. [1]—a paradigm has been developed in which the moduli of the entries of the lepton mixing (Pontecorvo–Maki–Nakagawa–Sakata or PMNS) matrix $U$ are fixed by some discrete, finite, non-Abelian group $G$. In this paradigm, the (unspecified) family theory of the leptons is symmetric under $G$. That group breaks, through some unspecified mechanism, into two different subgroups $G_\ell \subset G$ and $G_\nu \subset G$, which only intersect at the identity element of $G$. The mass matrices $M_\ell$ and $M_\nu$ of the charged leptons and of the neutrinos, respectively, are separately invariant under $G_\ell$ and $G_\nu$, respectively. As a consequence, the unitary matrices that diagonalize $M_\ell$ and $M_\nu$, named $U_\ell$ and $U_\nu$, respectively, also diagonalize the matrices of the restrictions to $G_\ell$ and $G_\nu$, respectively, of a three-dimensional representation of $G$. The diagonalization of $M_\ell$ and of $M_\nu$ is thus replaced by the diagonalization of the matrices representing the two subgroups $G_\ell$ and $G_\nu$ in some representation of $G$. The numerical values of the moduli of the matrix elements of $U = U_\ell^* U_\nu$ are traced back in this way to a three-dimensional representation of a finite group $G$. The irreducible representations of finite groups are finite in number and may be systematically studied. That study may be carried out exclusively through theoretical means [2], but is nowadays greatly facilitated by the free availability of the computer software GAP [3], which manipulates groups and their representations, and of the complete library SmallGroups of all the discrete groups of order less than 2000 [4].

The paradigm mentioned in the previous paragraph has firstly been developed under the assumption that the neutrinos are Majorana fields. In that case, the group $G_\nu$ must be (isomorphic to) the Klein group $\mathbb{Z}_2 \times \mathbb{Z}_2$ because the neutrino Majorana masses must remain real and positive under a rephasing of the neutrino fields and therefore this rephasing can at most be a change of sign. Systematic searches using GAP were produced under this assumption [5, 6] and a thorough classification of the PMNS matrices achievable in this way has been derived [7]. The paradigm has been extended to the cases of Dirac neutrinos [2] and of quarks [8]; then, $G_\nu$ may be a general $\mathbb{Z}_n$ group with $n > 2$. An extensive theoretical as well as SmallGroups investigation of those cases has been presented in ref. [9].

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1 We have assumed in this statement that $G_\nu$ is a subgroup of $SU(3)$. If $G_\nu$ is a subgroup of $U(3)$ but not of $SU(3)$, then it should be of the form $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$. 

2
The papers mentioned in the previous paragraph aimed at fixing the whole matrix \( |U|^2 \), defined as \( (|U|^2)_{ij} \equiv |U_{ij}|^2 \), through group theory. In this paper, following ref. [10], we have the more modest aim of only fixing either one row or one column of \( |U|^2 \). This allows the prediction of two out of the four parameters of \( |U|^2 \). This happens when either \( G_\ell \) or \( G_\nu \), respectively, is represented by twice-degenerate matrices, i.e. by \( 3 \times 3 \) matrices which have two equal eigenvalues while the third eigenvalue is different. For instance, if the matrices representing \( G_\ell \) have two of their eigenvalues equal, then only one of columns of their diagonalizing matrix \( U_\ell \) is well-defined, i.e. defined but for an arbitrary overall phase. Choosing that column to be the third one, the third row of \( U = U_\ell^\dagger U_\nu \) is well-defined while the first and second rows may be mixed and cannot, therefore, be predicted. In ref. [10] this job of predicting either one row or one column of \( |U|^2 \) was undertaken under the assumption that the neutrinos are Majorana fields; it was found that most rows/columns thus found have some zero entries and are therefore of no practical interest, since the phenomenology indicates the absence of zeros from the PMNS matrix.

In this paper we shall assume instead that neutrinos are Dirac fields\(^2\), this allows for a larger variety of groups \( G_\ell \) and \( G_\nu \)—namely, they will either be cyclic groups of order larger than 2 or possibly groups \( \mathbb{Z}_m \times \mathbb{Z}_n \)—and consequently to a much larger variety of predictions for rows/columns of \( |U|^2 \). In section 2 we expose the theory behind our group search. In section 3 we explain how we have used GAP to perform the search. Section 4 is devoted to the presentation of the rows/columns that resulted from the search. Section 5 compares our presumptive rows/columns to the actual phenomenological values of \( |U|^2 \), checking which rows/columns are realistic. In section 6 we make a short summary of our work. Appendix A is devoted to some definitions in group theory and may be skipped by an uninterested reader.

We clarify that this paper reports on a pure computational search made by using GAP. We have made neither any attempt at studying analytically a particular group or set of groups, nor at explaining analytically the results.

\(^2\)Those four parameters are usually taken to be three mixing (‘Euler’) angles and one phase, but they may alternatively be chosen to be four of the entries of \( |U|^2 \) [11].

\(^3\)We thus treat the lepton sector in exactly the same way as the quark sector, just as was done in ref. [9]. We drop any attempt to explain the smallness of the neutrino masses through the usual see-saw mechanism. Other versions to that mechanism, like for instance a see-saw mechanism for the vacuum expectation values of Higgs doublets [12], or a see-saw mechanism with extra vector-like neutrinos [13], may possibly be employed.
found through our computational search. For comparison, ref. [14] gives other papers relying strongly on the power of the GAP software.

2 Theory

We work in the context of the three-generation Standard Model with the addition of three right-handed neutrinos. The neutrinos are assumed to be standard Dirac fields; they have no Majorana mass terms. The charged-lepton mass matrix $M_\ell$ and the neutrino mass matrix $M_\nu$ are defined through the mass terms

$$L_{\text{lepton mass}} = -\overline{\ell_L}M_\ell\ell_R - \overline{\nu_L}M_\nu\nu_R + \text{H.c.}$$

(1)

The matrices $H_\ell \equiv M_\ell M_\ell^\dagger$ and $H_\nu \equiv M_\nu M_\nu^\dagger$ are diagonalized as

$$U_\ell^\dagger H_\ell U_\ell = \text{diag} (m_\ell^2, m_\mu^2, m_\tau^2),$$

(2a)

$$U_\nu^\dagger H_\nu U_\nu = \text{diag} (m_1^2, m_2^2, m_3^2),$$

(2b)

where $U_\ell$ and $U_\nu$ are $3 \times 3$ unitary matrices. The PMNS matrix is $U = U_\ell^\dagger U_\nu$.

We assume that the matrices $H_\ell$ and $H_\nu$ are invariant under the action of two invertible matrices $T_\ell$ and $T_\nu$, respectively. This invariance is defined through

$$T_\ell H_\ell T_\ell^{-1} = H_\ell,$$

(3a)

$$T_\nu H_\nu T_\nu^{-1} = H_\nu.$$  

(3b)

Equation (3a) states that $T_\ell$ and $H_\ell$ commute, therefore they are simultaneously diagonalizable. Similarly, equation (3b) implies that $T_\nu$ and $H_\nu$ are simultaneously diagonalizable. Therefore, $U_\ell$ diagonalizes $T_\ell$ and $U_\nu$ diagonalizes $T_\nu$:

$$U_\ell^\dagger T_\ell U_\ell = \hat{T}_\ell \equiv \text{diag} (l_1, l_2, l_3),$$

(4a)

$$U_\nu^\dagger T_\nu U_\nu = \hat{T}_\nu \equiv \text{diag} (n_1, n_2, n_3).$$

(4b)

We now make the crucial assumption that $T_\ell$ and $T_\nu$ together generate a (three-dimensional representation of a) group which is finite. This assumption allows one to restrict the PMNS matrix, which in general belongs to the continuous set of unitary matrices, to a discrete set of matrices defined by
the theory of finite groups, thereby generating some predictive power. This predictive power is further enhanced if we assume that the group generated by \( T_\ell \) and \( T_\nu \) is small, i.e. that its order is smaller than some arbitrary number. In our practical search we have assumed that the order of \( G \) is smaller than 2000, since this is the present reach of the SmallGroups library.

### 2.1 Main search

In our main search we have assumed that two, and only two, of the three eigenvalues \( l_{1,2,3} \) of \( T_\ell \) are equal, while the eigenvalues \( n_{1,2,3} \) of \( T_\nu \) are all distinct. Suppose for instance that \( l_1 = l_2 \neq l_3 \). Then, the third column of \( U_\ell \), which according to equation (4a) is the normalized eigenvector of \( T_\ell \) corresponding to the eigenvalue \( l_3 \), is well-defined—only its overall phase is arbitrary—but the first two columns of \( U_\ell \) are not, because they are eigenvectors corresponding to the same eigenvalue \( l_1 = l_2 \) and may therefore be arbitrarily mixed between themselves. As a consequence, the third row of \( U = U_\ell \dagger U_\nu \) will be fixed except for its phase, while the first two rows will remain arbitrary (they will only be restricted to being orthogonal to the third row and to each other). Our assumption thus allows us to ‘predict’ the moduli of the matrix elements of the third row of \( U \). In the same way, if \( \ell_1 = \ell_3 \neq \ell_2 \), then the second row of \( U \) is predicted; if \( \ell_2 = \ell_3 \neq \ell_1 \), then the first row of \( U \) is predicted.

In practice, we compute those moduli in the following way. Let \( p \) and \( q \) be two integers, then, from equations (4) \(^4\)

\[
\begin{align*}
\text{tr}(T_\ell^p T_\nu^q) &= \text{tr}(U_\ell^\dagger T_\ell^p U_\ell \dagger U_\nu T_\nu^q) = \text{tr}(U_\ell^\dagger T_\ell^p U_\ell \dagger U_\nu T_\nu^q) = \sum_{i=1}^{3} \sum_{j=1}^{3} l_i^p n_j^q |U_{ij}|^2 \\
&= l_1^p n_1^q (1 - |U_{21}|^2 - |U_{31}|^2) + l_2^p n_2^q (1 - |U_{22}|^2 - |U_{32}|^2) \\
&\phantom{=} + l_3^p n_3^q (|U_{21}|^2 + |U_{31}|^2 + |U_{22}|^2 + |U_{32}|^2 - 1) + l_2^p n_1^q |U_{21}|^2 \\
&\phantom{=} + l_3^p n_2^q |U_{22}|^2 + l_2^p n_3^q (1 - |U_{21}|^2 - |U_{22}|^2) + l_3^p n_1^q |U_{31}|^2 \\
&\phantom{=} + l_3^p n_2^q |U_{32}|^2 + l_3^p n_3^q (1 - |U_{31}|^2 - |U_{32}|^2) \\
&= (l_2^p - l_1^p) (n_1^q - n_3^q) |U_{21}|^2 + (l_3^p - l_1^p) (n_2^q - n_3^q) |U_{22}|^2 \\
&\phantom{=} + (l_3^p - l_2^p) (n_1^q - n_2^q) |U_{31}|^2 + (l_2^p - l_3^p) (n_2^q - n_3^q) |U_{32}|^2 \\
&\phantom{=} + l_1^p n_1^q + l_2^p n_2^q - l_1^p n_3^q + l_3^p n_3^q + l_1^p n_3^q + l_2^p n_2^q.
\end{align*}
\]

\(^4\)The use of traces like those in equation (5) has been first advocated in ref. [11].
Next, using our extra assumption that \( l_1 = l_2 \neq l_3 \),

\[
\text{tr} (T^p \ell T^q \nu) = \left( p^2 - p^2_1 \right) (n^q_1 - n^q_3) |U_{31}|^2 + \left( p^2 - p^2_1 \right) (n^q_2 - n^q_3) |U_{32}|^2 \\
+ p^2_1 n^q_1 + p^2_1 n^q_2 + p^2_1 n^q_3.
\]  

(6)

Writing both \( \text{tr} (T^p \ell T^q \nu) \) and \( \text{tr} (T^p \ell T^q \nu) \) as in equation (6) (i.e. with \( p = 1 \) and \( q = 1 \) and 2, respectively), one obtains two equations for \( |U_{31}|^2 \) and \( |U_{32}|^2 \).

They are solved to yield

\[
|U_{31}|^2 = \frac{\text{tr} (T^p \ell T^q \nu) + l_1 \chi - (n_2 + n_3) \text{tr} (T^p \ell T^q \nu) - l_1 n^q_1 + l_3 n_2 n_3}{(l_3 - l_1) (n_1 - n_2) (n_1 - n_3)}, \]  

(7a)

\[
|U_{32}|^2 = \frac{\text{tr} (T^p \ell T^q \nu) + l_1 \chi - (n_1 + n_3) \text{tr} (T^p \ell T^q \nu) - l_1 n^q_2 + l_3 n_1 n_3}{(l_3 - l_1) (n_2 - n_1) (n_2 - n_3)}, \]  

(7b)

\[
|U_{33}|^2 = \frac{\text{tr} (T^p \ell T^q \nu) + l_1 \chi - (n_1 + n_2) \text{tr} (T^p \ell T^q \nu) - l_1 n^q_3 + l_3 n_1 n_2}{(l_3 - l_1) (n_3 - n_1) (n_3 - n_2)}, \]  

(7c)

where \( \chi \equiv n_1 n_2 + n_1 n_3 + n_2 n_3 \). (We have used \( |U_{33}|^2 = 1 - |U_{31}|^2 - |U_{32}|^2 \) to derive equation (7c) from equations (7a) and (7b).) Equations (7) allow us to compute the third row of \( |U| \) from the matrices \( T^p \ell \) and \( T^q \nu \) and from their eigenvalues, without having to explicitly diagonalize those matrices.

We emphasize that our use of traces \( \text{tr} (T^p \ell T^q \nu) \) constitutes an important technical progress over other searches using \textit{GAP}, because \textit{GAP} frequently gives the matrices of a group representation in non-unitary form; by using the traces one may directly use those matrices in that form, without having firstly to unitarize them and then to compute their eigenvectors; one thus saves a considerable amount of computer time.

### 2.2 Secondary search

We have also considered the possibility that there are two matrices \( T^p \nu_1 \) and \( T^p \nu_2 \) which commute both with each other and with the Hermitian matrix \( H^p \nu \); therefore, they are all diagonalized by the same unitary matrix \( U^p \nu \):

\[
U^p \nu_1 T^p \nu_1 U^p \nu = \text{diag} (n_1, n_2, n_3), \quad \text{(8a)}
\]

\[
U^p \nu_2 T^p \nu_2 U^p \nu = \text{diag} (\bar{n}_1, \bar{n}_2, \bar{n}_3). \quad \text{(8b)}
\]

We have considered the situation in which both \( T^p \nu_1 \) and \( T^p \nu_2 \) have two identical eigenvalues, but together they act as if all their eigenvalues are different:
\[n_1 = n_2 \neq n_3 \text{ and } \bar{n}_1 = \bar{n}_3 \neq \bar{n}_2.\] We identify this property by computing the invariant quantity

\[q \equiv 3 [\text{tr} (T_{\nu_1} T_{\nu_2})]^2 - 2 (\text{tr} T_{\nu_1}) (\text{tr} T_{\nu_2}) \text{tr} (T_{\nu_1} T_{\nu_2}) + (\text{tr} T_{\nu_1})^2 \text{tr} (T_{\nu_2})^2 + \text{tr} (T_{\nu_1})^2 (\text{tr} T_{\nu_2})^2 - 3 \text{tr} (T_{\nu_1}^2) \text{tr} (T_{\nu_2}^2). \quad (9)\]

It is easy to check that \[q = -(n_1 - n_3)^2 (\bar{n}_1 - \bar{n}_2)^2 \neq 0\] if \(n_1 = n_2\) and \(\bar{n}_1 = \bar{n}_3\), while \(q = 0\) if \(n_1 = n_2\) and \(\bar{n}_1 = \bar{n}_2\). So, we select \(q \neq 0\).

Now, with

\[U_{\ell}^\dagger T_{\ell} U_{\ell} = \text{diag} (l_1, l_1, l_3), \quad (10a)\]
\[U_{\nu_1}^\dagger T_{\nu_1} U_{\nu} = \text{diag} (n_1, n_1, n_3), \quad (10b)\]
\[U_{\nu_2}^\dagger T_{\nu_2} U_{\nu} = \text{diag} (\bar{n}_1, \bar{n}_2, \bar{n}_1), \quad (10c)\]

and \(U = U_{\ell}^\dagger U_{\nu}\) as before, one easily finds that

\[|U_{33}|^2 = \frac{\text{tr} (T_{\ell} T_{\nu_1}) - l_1 n_1 - l_1 n_3 - l_3 n_1}{(l_1 - l_3) (n_1 - n_3)}, \quad (11a)\]
\[|U_{32}|^2 = \frac{\text{tr} (T_{\ell} T_{\nu_2}) - l_1 \bar{n}_1 - l_1 \bar{n}_2 - l_3 \bar{n}_1}{(l_1 - l_3) (\bar{n}_1 - \bar{n}_2)}. \quad (11b)\]

So, in this case one can once again derive the entries in the third row of \(|U|^2\) by using invariant traces.

### 3 The search

We have scanned all the finite groups with order less than 2000 by making use of the library SmallGroups. We have identified which of them have three-dimensional faithful irreducible representations (‘irreps’) and, moreover, are not the direct product of a group \(\mathbb{Z}_n\) with \(n \geq 2\) by some other group.\(^5\) We only need to scan for faithful representations; if a representation is unfaithful (\(i.e.\) if it represents several elements of the group \(G\) by the same matrix), then it is the faithful representation of a subgroup \(G' \subset G\) and we will find it when we scan for the faithful representations of \(G'\). Moreover, we only need

\(^5\)The cyclic group \(\mathbb{Z}_n\) is formed by the \(n\)-roots of unity under the standard multiplication of complex numbers. It is of course an Abelian group, because the multiplication of complex numbers is commutative.
to scan for irreducible representations: if a representation is reducible, then all its matrices may simultaneously be rotated to a basis in which they are all block-diagonal; hence, \( T_\ell \) and \( T_\nu \) will be block-diagonal in some basis, and therefore \( U_\ell \), \( U_\nu \), and the PMNS matrix will also be block-diagonal; but this is in contradiction with the phenomenology, which indicates that \( U \) has no zero matrix elements. Finally, we shed groups \( G \) of the form \( G = G' \times \mathbb{Z}_n \) with \( n \geq 2 \), because a faithful representation of \( G' \times \mathbb{Z}_n \) is necessarily also a faithful representation of the smaller group \( G' \) and we will find it when we study \( G' \).

Since the number of groups of order 1536 is much too large for all of them to be scanned within a reasonable time, we have used the conjecture in ref. \[9\] that both nilpotent groups\[6\] and groups with a normal Sylow 3-subgroup\[7\] never have three-dimensional faithful irreps. Since 99.97% of the groups of order 1536 are in one of those two categories, this conjecture has allowed us to outright disconsider most of the groups of that order.

Tables 1, 2, and 3 present all the groups \( G \) that we have found not to be of type \( G = \mathbb{Z}_n \times G' \) and to possess at least one faithful three-dimensional irrep. Tables 1 and 2 reproduce equation (42) of ref. \[16\], while table 3 is new.

We have explicitly constructed all the three-dimensional irreps of the groups in tables 1, 2, and 3. From each of those irreps we have discarded matrices proportional to the unit matrix. We have divided the remaining matrices (of each irrep of each group) into two sets: the ones that have non-degenerate eigenvalues (set \( N \)) and the ones that have twice-degenerate eigenvalues (set \( T \)). We recall that a matrix \( M \) has degenerate eigenvalues when

\[
4 [I_2 (M)]^3 - (\text{tr } M)^2 [I_2 (M)]^2 + 27 (\det M)^2 + 4 (\text{tr } M)^3 \det M - 18 (\text{tr } M) [I_2 (M)] \det M = 0, \tag{12}
\]

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6One can check whether the group with SmallGroups identifier \([m,n]\) (\(m\) and \(n\) are integers; \(m\) is the order of the group) is nilpotent by first typing the GAP command \( G := \text{SmallGroup}([m,n]) \) and then the GAP command \( \text{IsNilpotentGroup}(G) \). The latter command produces the answer \text{True} if the group \([m,n]\) is nilpotent.

7The Sylow 3-subgroups of a group \( G \) may be found by typing the GAP command \( \text{SylowSubgroup}(G,3) \). The GAP command \( \text{IsNormal}(G,U) \) returns \text{True} if \( U \) is a normal subgroup of \( G \).

8In appendix A we attempt to explain in simple terms what are nilpotent groups and groups with a normal Sylow 3-subgroup [15].

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Table 1: The SmallGroups identifiers of the groups $G$ with three-dimensional irreducible representations. Part 1: groups with $\text{order}(G) < 768$. The identifiers in boldface denote the groups which have three-dimensional irreducible representations in which some of the matrices have twice-degenerate eigenvalues; only those groups are relevant for this paper.
Table 2: The SmallGroups identifiers of the groups with three-dimensional irreducible representations. Part 2: groups \( G \) with \( 768 \leq \text{order}(G) < 1536. \)

The identifiers in boldface stand for groups with three-dimensional irreducible representations in which some of the matrices have twice-degenerate eigenvalues.

where \( I_2(M) \equiv M_{11}M_{22} + M_{11}M_{33} + M_{22}M_{33} - M_{12}M_{21} - M_{13}M_{31} - M_{23}M_{32} \) is the second-order invariant of \( M \). We have found that there are many groups for which set \( T \) is empty; we have discarded those groups. The remaining
Table 3: The SmallGroups identifiers of the groups with three-dimensional irreducible representations. Part 3: groups with $1536 \leq \text{order}(G) < 2000$.

The identifiers in boldface denote groups with three-dimensional irreducible representations with matrices having twice-degenerate eigenvalues.

| 1536, 408544632 | 1536, 408544641 | 1536, 408544678 | 1536, 408544687 | 1539, 16 | 1539, 25 | 1539, 26 | 1539, 27 | 1539, 29 | 1539, 32 | 1539, 35 | 1539, 37 | 1539, 47 | 1548, 11 | 1569, 1 | 1575, 7 | 1587, 2 | 1596, 55 | 1596, 56 | 1623, 1 | 1629, 1 | 1641, 1 | 1641, 6 | 1647, 9 | 1647, 10 | 1647, 11 | 1647, 12 | 1659, 3 | 1659, 4 | 1668, 11 | 1677, 3 | 1677, 4 | 1701, 68 | 1701, 102 | 1701, 112 | 1701, 115 | 1701, 126 | 1701, 127 | 1701, 128 | 1701, 130 | 1701, 131 | 1701, 135 | 1701, 138 | 1701, 240 | 1701, 261 | 1713, 1 | 1728, 3 | 1728, 185 | 1728, 953 | 1728, 1286 | 1728, 1290 | 1728, 1291 | 1728, 2785 | 1728, 2847 | 1728, 2855 | 1728, 2929 | 1731, 1 | 1734, 5 | 1737, 1 | 1764, 11 | 1764, 91 | 1767, 3 | 1767, 4 | 1776, 60 | 1791, 1 | 1803, 1 | 1809, 6 | 1809, 9 | 1809, 10 | 1809, 11 | 1809, 12 | 1812, 11 | 1821, 1 | 1839, 1 | 1857, 1 | 1872, 60 | 1875, 16 | 1884, 14 | 1893, 1 | 1899, 1 | 1911, 3 | 1911, 4 | 1911, 14 | 1929, 1 | 1944, 35 | 1944, 37 | 1944, 70 | 1944, 707 | 1944, 746 | 1944, 832 | 1944, 833 | 1944, 849 | 1944, 1123 | 1944, 2293 | 1944, 2294 | 1944, 2333 | 1944, 2363 | 1944, 2415 | 1944, 3448 | 1953, 3 | 1953, 4 | 1956, 11 | 1971, 6 | 1971, 9 | 1971, 10 | 1971, 11 | 1971, 12 | 1983, 1 |

We have then explicitly computed the eigenvalues of all the matrices in both sets $N$ and $T$.

In our main search, we have considered all possible pairs of one matrix $T_\ell$ from set $T$ and one matrix $T_\nu$ from set $N$. For each of those pairs we have computed the three $|U_{3j}|^2$ ($j = 1, 2, 3$) by using equations (7); in those equations, $l_1$ is the degenerate eigenvalue of $T_\ell$, $l_3$ is the non-degenerate eigenvalue of $T_\ell$, and $n_{1,2,3}$ are the three eigenvalues of $T_\nu$. We have discarded the set of the three $|U_{3j}|^2$ whenever any one of them turned out to vanish; we have only collected the sets for which all three numbers $|U_{31}|^2$, $|U_{32}|^2$, and $|U_{33}|^2$ were non-zero.
For our secondary search we have picked all possible couples of two matrices \( T_{\nu 1} \) and \( T_{\nu 2} \) from the set \( T \) and selected those couples that commute and that moreover have a non-zero quantity \( q \) defined in equation (9). We have then picked one third matrix \( T_\ell \), and have computed the three \( |U_{3j}|^2 \) by using equations (11) together with \( |U_{31}|^2 = 1 - |U_{32}|^2 - |U_{33}|^2 \). Once again, we have discarded all the sets of three \( |U_{3j}|^2 \) in which any one of them happened to vanish.

Since all the finite-dimensional representations of finite groups are unitary, all the representations that we have dealt with are in principle equivalent to representations through unitary matrices. GAP usually gives the representations in non-unitary form, but we never have had to bring the representations to unitary form, because all our computations were performed in terms of basis-invariant quantities.

At the end of our search we have used GAP to find out the form of the group generated by the matrices \( T_\ell \) and \( T_\nu \) (in the main search) alone. Let \( \langle T_\ell, T_\nu \rangle \) denote that group. It coincides in most cases with the initial group \( G \), but sometimes it is just a subgroup of it.

### 4 Results

The searches described in the previous section have produced a total of sixty sets of three non-zero numbers \( |U_{31}|^2 \), \( |U_{32}|^2 \), and \( |U_{33}|^2 \) in each set, \( |U_{31}|^2 + |U_{32}|^2 + |U_{33}|^2 = 1 \). From now on, we let \( V_j \equiv |U_{3j}|^2 \) \( (j=1,2,3) \) denote the three numbers in each set and we assume that they have been ordered as \( V_1 \leq V_2 \leq V_3 \). We christen each such set \( \{V_1, V_2, V_3\} \) a ‘structure’. We have plotted the sixty structures that we have found as sixty—blue, green, and red—points in figure 1.

We note that all the sixty structures that we have found would also have been found if we had restricted our search to subgroups of \( SU(3) \). Namely, for all the structures, there is always at least one case in which both matrices \( T_\ell \) and \( T_\nu \) have determinant 1. We do not know why this happens but, like Ludl has pointed out [17], it is possible that every finite subgroup of \( U(3) \) is equivalent in physical terms to some finite subgroup of \( SU(3) \), because they can only produce the same Lagrangians.

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\(^9\)Actually, all the sets except one have been produced by the main search. All but one of the sets produced by the secondary search merely reproduce sets that had already been obtained in the main search.
Figure 1: A depiction of the sixty structures \(\{V_1, V_2, V_3\}\) that have been produced by our searches. The horizontal line gives \(V_1\) and the vertical line gives \(V_2\), while \(V_3 = 1 - V_1 - V_2\). We have discarded structures in which any of the three \(V_j\) \((j = 1, 2, 3)\) is zero. The structures are ordered as \(V_1 \leq V_2 \leq V_3\). All the points are within the triangle bounded by the vertical axis (equivalent to \(V_1 = 0\)) and by the green line. The latter is composed of two segments with equations \(V_1 = V_2\) and \(V_1 + 2V_2 = 1 \Leftrightarrow V_2 = V_3\). The two segments meet at the point \(V_1 = V_2 = V_3 = 1/3\). The blue curve unites points with similar features (see text).

We divide all the structures into three types. Each of these types is described in one of the following subsections.
4.1 Structures on the blue curve

The first type encompasses a total of 44 structures. They are described by the analytical formula

$$V_2 = \frac{1}{2} \left( 1 - V_1 - \sqrt{2V_1 - 3V_1^2} \right).$$

(13)

This is depicted as a blue curve in figure 1. The relevance of that curve had already been noticed in refs. [6, 18].

All but two of the 44 structures on the blue curve may be written

$$V_1 = \frac{1}{3} \left( 1 - \cos \frac{2\pi k}{3n} \right),$$

(14a)

$$V_2 = \frac{1}{3} \left[ 1 - \cos \frac{2\pi (k - n)}{3n} \right],$$

(14b)

$$V_3 = \frac{1}{3} \left[ 1 - \cos \frac{2\pi (k + n)}{3n} \right],$$

(14c)

for positive integers $2 \leq n \leq 17$ and $k < n/2$ given in tables 4 and 5.

| $n$ | 17 | 16 | 14 | 13 | 11 |
|-----|----|----|----|----|----|
| $k$ | 1, 2, 3, 4, 5, 6, 7, 8 | 1, 3, 5, 7 | 1, 3, 5 | 1, 2, 3, 4, 5, 6 | 1, 2, 3, 4, 5 |

Table 4: The values of $n$ and $k$ to be used in equations (14).

| $n$ | 10 | 9 | 8 | 7 | 6 | 5 | 4 | 3 | 2 |
|-----|----|---|---|---|---|---|---|---|---|
| $k$ | 1, 3 | 1, 2, 4 | 1, 3 | 1, 2, 3 | 1 | 1, 2 | 1 | 1 | 1 |

Table 5: Continuation of table 4.

There are two more structures that are also described by equation (13) but cannot be described through equations (14) for any small integers $n$ and $k$. They are

$$\left\{ \frac{1}{6 + \csc \frac{3\pi}{14}}, \frac{1}{6 - \sec \frac{\pi}{7}}, \frac{1}{6 - \csc \frac{\pi}{11}} \right\} \approx \{0.131512, 0.204495, 0.663993\}$$

(15)
and

\[ \left\{ \frac{3 - \sqrt{5}}{8}, \frac{3 + \sqrt{5}}{8} \right\} \approx \{0.0954915, 0.25, 0.654508\}. \quad (16) \]

The structures given by equations (14) are produced by groups \( \langle T_\ell, T_\nu \rangle \) which are either \( \Delta(6n^2) \) or \( D(9n, 1; 2, 1, 1) \), where \( n \) is a positive integer [19]. We recall that those are subgroups of \( SU(3) \). The group \( \Delta(6n^2) \) [20] has \( 6n^2 \) elements and is generated by

\[
E = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix}, \quad \text{and} \quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & \eta & 0 \\ 0 & 0 & \eta^{-1} \end{pmatrix},
\]

where \( \eta = \exp(2i\pi/n) \). The group \( D(9n, 1; 2, 1, 1) \) has \( 162n^2 \) elements and is generated by \( E, B, \) and \( \text{diag}(\epsilon, \epsilon, \epsilon^{-2}) \), where \( \epsilon = \exp[2i\pi/(9n)] \) [21]. In tables 6 and 7 we give the SmallGroups identifiers of these groups, which

| \( \Delta(6 \times 2^2) \) | \( \Delta(6 \times 3^2) \) | \( \Delta(6 \times 4^2) \) | \( \Delta(6 \times 5^2) \) | \( \Delta(6 \times 6^2) \) |
|----------------|----------------|----------------|----------------|----------------|
| [24, 12] | [54, 8] | [96, 64] | [150, 5] | [216, 95] |
| \( \Delta(6 \times 7^2) \) | \( \Delta(6 \times 8^2) \) | \( \Delta(6 \times 9^2) \) | \( \Delta(6 \times 10^2) \) | \( \Delta(6 \times 11^2) \) |
| [294, 7] | [384, 568] | [486, 61] | [600, 179] | [726, 5] |
| \( \Delta(6 \times 12^2) \) | \( \Delta(6 \times 13^2) \) | \( \Delta(6 \times 14^2) \) | \( \Delta(6 \times 15^2) \) | \( \Delta(6 \times 16^2) \) |
| [864, 701] | [1014, 7] | [1176, 243] | [1350, 46] | [1536, 408544632] |
| \( \Delta(6 \times 17^2) \) | \( \Delta(6 \times 18^2) \) | | | |
| [1734, 5] | [1944, 849] | | | |

Table 6: The SmallGroups identifiers of the groups \( \Delta(6n^2) \) with order smaller than 2000.

| \( n = 1 \) | \( n = 2 \) | \( n = 3 \) |
|-------------|-------------|-------------|
| [162, 14]   | [648, 259]  | [1458, 659] |

Table 7: The SmallGroups identifiers of the groups \( D(9n, 1; 2, 1, 1) \) \( \equiv D_{9n,3n}^{(1)} \) [21] with order smaller than 2000.
have structures

\[(\mathbb{Z}_n \times \mathbb{Z}_n) \rtimes S_3 \quad \text{and} \quad (\mathbb{Z}_m \times \mathbb{Z}_3n) \rtimes S_3,\]  

respectively, where \(S_3\) is the permutation group of three objects.

The structure (15) is produced in the main search by the group \(\langle T_l, T_\nu \rangle = [168, 42] = \Sigma (168)\), which is an ‘exceptional’ finite subgroup of \(SU(3)\). The structure (16) is produced by another exceptional subgroup of \(SU(3)\), the group \([60, 5] = \Sigma (60) \cong A_5\) (the symmetry group of the icosahedron) in the secondary search [22]. That structure is the only one produced in the secondary search that was not also a result of the main search; it had already been found in refs. [22] [10].

It is convenient to number all the structures belonging to the blue curve of figure 1 according to increasing values of \(V_1\). We thus construct the following list of the 44 structures:

structure 1 \((n = 17, k = 1)\) : \(\{0.0025265, 0.463262, 0.534212\}\);
structure 2 \((n = 16, k = 1)\) : \(\{0.00285171, 0.460894, 0.536254\}\);
structure 3 \((n = 14, k = 1)\) : \(\{0.00372306, 0.455114, 0.541163\}\);
structure 4 \((n = 13, k = 1)\) : \(\{0.00431658, 0.451535, 0.544148\}\);
structure 5 \((n = 11, k = 1)\) : \(\{0.00602377, 0.442356, 0.55162\}\);
structure 6 \((n = 10, k = 1)\) : \(\{0.00728413, 0.436339, 0.556377\}\);
structure 7 \((n = 9, k = 1)\) : \(\{0.00898504, 0.428934, 0.562081\}\);
structure 8 \((n = 17, k = 2)\) : \(\{0.0100677, 0.424554, 0.565378\}\);
structure 9 \((n = 8, k = 1)\) : \(\{0.0113581, 0.419606, 0.569036\}\);
structure 10 \((n = 7, k = 1)\) : \(\{0.0148091, 0.407507, 0.577684\}\);
structure 11 \((n = 13, k = 2)\) : \(\{0.0171545, 0.400009, 0.582837\}\);
structure 12 \((n = 6, k = 1)\) : \(\{0.0201025, 0.391216, 0.588681\}\);
structure 13 \((n = 17, k = 3)\) : \(\{0.0225093, 0.384464, 0.593027\}\);
structure 14 \((n = 11, k = 2)\) : \(\{0.0238774, 0.380772, 0.595351\}\);
structure 15 \((n = 16, k = 3)\) : \(\{0.0253735, 0.376842, 0.597784\}\);
structure 16 \((n = 5, k = 1)\) : \(\{0.0288182, 0.368176, 0.603006\}\);
structure 17 \((n = 14, k = 3)\) : \(\{0.0330104, 0.358243, 0.608746\}\);
structure 18 \((n = 9, k = 2)\) : \(\{0.0354558, 0.352715, 0.611829\}\);
structure 19 \((n = 13, k = 3)\) : \(\{0.0381813, 0.346755, 0.615063\}\);
structure 20 \((n = 17, k = 4)\): \{0.0396626, 0.343598, 0.616739\};
structure 21 \((n = 4, k = 1)\): \{0.0446582, 0.333333, 0.622008\};
structure 22 \((n = 11, k = 3)\): \{0.0529155, 0.317473, 0.629612\};
structure 23 \((n = 7, k = 2)\): \{0.0579204, 0.308423, 0.633656\};
structure 24 \((n = 17, k = 5)\): \{0.0612677, 0.302577, 0.636155\};
structure 25 \((n = 10, k = 3)\): \{0.063661, 0.298491, 0.637848\};
structure 26 \((n = 13, k = 4)\): \{0.0668524, 0.293154, 0.639993\};
structure 27 \((n = 16, k = 5)\): \{0.068822, 0.289825, 0.641293\};
structure 28 \((n = 3, k = 1)\): \{0.0779852, 0.275451, 0.646564\};
structure 29 \((n = 17, k = 6)\): \{0.086997, 0.262022, 0.650981\};
structure 30 \((n = 14, k = 5)\): \{0.0889827, 0.25916, 0.651858\};
structure 31 \((n = 11, k = 4)\): \{0.0920887, 0.254747, 0.653164\};
structure 32 : \{0.0954915, 0.25, 0.654508\};
structure 33 \((n = 8, k = 3)\): \{0.0976311, 0.24706, 0.655309\};
structure 34 \((n = 13, k = 5)\): \{0.102425, 0.240594, 0.656981\};
structure 35 \((n = 5, k = 2)\): \{0.11029, 0.230328, 0.659383\};
structure 36 \((n = 17, k = 7)\): \{0.116461, 0.222548, 0.660991\};
structure 37 \((n = 7, k = 3)\): \{0.125503, 0.211553, 0.662944\};
structure 38 \((n = 16, k = 7)\): \{0.130413, 0.205772, 0.663815\};
structure 39 : \{0.131512, 0.204495, 0.663993\};
structure 40 \((n = 9, k = 4)\): \{0.13428, 0.201307, 0.664413\};
structure 41 \((n = 11, k = 5)\): \{0.139981, 0.194862, 0.665157\};
structure 42 \((n = 13, k = 6)\): \{0.143978, 0.190436, 0.665586\};
structure 43 \((n = 17, k = 8)\): \{0.149212, 0.184754, 0.666034\};
structure 44 \((n = 2, k = 1)\): \{1/6, 1/6, 2/3\}.

4.2 Structures on the green line

The second type of structures features two equal \(V_j\), \textit{i.e.} either \(V_1 = V_2\) or \(V_2 = V_3\). These structures straddle the green line in figure\textsuperscript{11} \(\text{One of them is of course structure 44, which corresponds to the point where the green line meets the blue curve. There are six more such structures on the green line.}

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They are

structure 45 \( V_1 = \frac{3 - \sqrt{5}}{12} \): \( \{0.063661, 0.063661, 0.872678\} \);  
structure 46 \( V_1 = \frac{5 - \sqrt{5}}{20} \): \( \{0.138197, 0.138197, 0.723607\} \);  
structure 47 \( V_1 = \frac{1}{4} \): \( \{0.25, 0.25, 0.5\} \);  
structure 48 \( V_1 = \frac{1}{3} \): \( \{0.333333, 0.333333, 0.333333\} \);  
structure 49 \( V_1 = \frac{5 - \sqrt{5}}{10} \): \( \{0.276393, 0.361803, 0.361803\} \);  
structure 50 \( V_1 = \frac{3 - \sqrt{5}}{6} \): \( \{0.127322, 0.436339, 0.436339\} \).

The structures 45, 46, 49, and 50 all originate in \( \langle T_e, T_\nu \rangle = A_5 \). Structure 47 originates in the permutation group \( S_4 \cong \Delta(6 \times 2^2) \). Structure 48 originates in the alternating group \( A_4 \cong \Delta(3 \times 2^2) \), with SmallGroups identifier [12, 3].

Both structures 46 and 49 justify the Ansatz \([23]\)

\[
\cot \theta_{12} = \varphi \equiv \frac{1 + \sqrt{5}}{2},
\]

relating the lepton mixing angle \( \theta_{12} \) to the ‘golden ratio’ \( \varphi \). Indeed, \( 2V_1 = 1/(1 + \varphi^2) \) in structure 46 and \( V_1 = 1/(1 + \varphi^2) \) in structure 49.

### 4.3 Isolated structures

Besides the points on the blue curve and the points on the green line, there are ten isolated points marked red in figure\([\text{11}]\). The corresponding structures
are

structure 51 : \{0.0347854, 0.166667, 0.798548\}

\[
= \left\{ \frac{5 - \sqrt{21}}{12}, \frac{1}{6}, \frac{5 + \sqrt{21}}{12} \right\};
\]

structure 52 : \{0.0389375, 0.306554, 0.654508\}

\[
= \left\{ \frac{5 + \sqrt{3} - \sqrt{5} - \sqrt{15}}{16}, \frac{5 - \sqrt{3} - \sqrt{5} + \sqrt{15}}{16}, \frac{3 + \sqrt{5}}{8} \right\};
\]

structure 53 : \{0.0442811, 0.25, 0.705719\} = \left\{ \frac{3 - \sqrt{7}}{8}, \frac{1}{4}, \frac{3 + \sqrt{7}}{8} \right\};

structure 54 : \{0.0625591, 0.138197, 0.799244\}

\[
= \left\{ \frac{15 + \sqrt{5} - \sqrt{150} + 30 \sqrt{5}}{40}, \frac{5 - \sqrt{5}}{20}, \frac{15 + \sqrt{5} + \sqrt{150} + 30 \sqrt{5}}{40} \right\};
\]

structure 55 : \{0.08592426701, 0.201689718788, 0.712386014201\}

\[
= \left\{ \frac{2}{3 (2 + \csc \frac{\pi}{18})}, \frac{2}{3 (2 + \sec \frac{2 \pi}{9})}, \frac{2}{3 (2 - \sec \frac{\pi}{9})} \right\};
\]

structure 56 : \{0.0914501, 0.361803, 0.546747\}

\[
= \left\{ \frac{15 - \sqrt{5} - \sqrt{150} - 30 \sqrt{5}}{40}, \frac{5 + \sqrt{5}}{20}, \frac{15 - \sqrt{5} + \sqrt{150} - 30 \sqrt{5}}{40} \right\};
\]

structure 57 : \{0.0954915, 0.101940, 0.802569\}

\[
= \left\{ \frac{3 - \sqrt{5}}{8}, \frac{5 - \sqrt{3} + \sqrt{5} - \sqrt{15}}{16}, \frac{5 + \sqrt{3} + \sqrt{5} + \sqrt{15}}{16} \right\};
\]

structure 58 : \{0.105662, 0.394338, 0.5\} = \left\{ \frac{3 - \sqrt{3}}{12}, \frac{3 + \sqrt{3}}{12}, \frac{1}{2} \right\};

structure 59 : \{0.158494, 0.25, 0.591506\} = \left\{ \frac{3 - \sqrt{3}}{8}, \frac{1}{4}, \frac{3 + \sqrt{3}}{8} \right\};

structure 60 : \{1/6, 1/3, 1/2\}.
All structures 51–60 but structures 55 and 58 are either rows or columns of matrices found in ref. [7]; we have taken their analytic expressions from that paper.

Structures 51–60 originate in the following groups \( \langle T_\ell, T_\nu \rangle \):

- The exceptional \( SU(3) \) subgroup \( \Sigma (168) \), which has \texttt{SmallGroups} identifier [168, 42], for structures 51 and 53;
- The exceptional \( SU(3) \) subgroup \( \Sigma (360 \times 3) \), which has \texttt{SmallGroups} identifier [1080, 260], for structures 52, 54, 56, and 57;
- any of three groups of order 648—with \texttt{SmallGroups} identifiers [648, 531], [648, 532], and [648, 533]—for structures 55, 58, and 60. Those three groups have similar structure \{[(\Z_3 \times \Z_3) \rtimes \Z_3] \rtimes Q_8 \} \rtimes \Z_3, where \( Q_8 \) is the quaternion group, a subgroup of \( SU(2) \). The group [648, 532] is the \( SU(3) \) exceptional subgroup \( \Sigma (216 \times 3) \).

- for structure 59, \( \langle T_\ell, T_\nu \rangle \) may be either the exceptional \( SU(3) \) subgroup \( \Sigma (36 \times 3) \), with \texttt{SmallGroups} identifier [108, 15], or some other groups with analogous structures and orders which are multiple of 108, like [216, 25], [324, 111], [432, 57], and so on.

Table 8 gives the \texttt{SmallGroups} identifiers of all six exceptional subgroups of \( SU(3) \).

| \( \Sigma (60) \) | \( \Sigma (168) \) | \( \Sigma (36 \times 3) \) | \( \Sigma (72 \times 3) \) | \( \Sigma (216 \times 3) \) | \( \Sigma (360 \times 3) \) |
|---|---|---|---|---|---|
| [60, 5] | [168, 42] | [108, 15] | [216, 88] | [648, 532] | [1080, 260] |

Table 8: The \texttt{SmallGroups} identifiers of the exceptional finite subgroups of \( SU(3) \).

5 Comparison with the data

The matrix \( |U|^2 \) is parameterized as

\[
|U|^2 = \begin{pmatrix}
  s_{12}^2 c_{23}^2 + c_{12}^2 c_{23}^2 & \frac{c_{12}^2 c_{13}^2}{2} & s_{12}^2 c_{13}^2 & s_{12}^2 c_{13}^2 & s_{13}^2 \\
  s_{12}^2 s_{23}^2 s_{13}^2 & c_{12}^2 c_{23}^2 & \frac{c_{12}^2 c_{13}^2}{2} & s_{12}^2 s_{23}^2 s_{13}^2 & s_{13}^2 \\
  s_{12}^2 s_{23}^2 s_{13}^2 & c_{12}^2 c_{23}^2 & \frac{c_{12}^2 c_{13}^2}{2} & s_{12}^2 s_{23}^2 s_{13}^2 & s_{13}^2 \\
  s_{12}^2 s_{23}^2 s_{13}^2 & c_{12}^2 c_{23}^2 & \frac{c_{12}^2 c_{13}^2}{2} & s_{12}^2 s_{23}^2 s_{13}^2 & s_{13}^2 \\
  s_{12}^2 s_{23}^2 s_{13}^2 & c_{12}^2 c_{23}^2 & \frac{c_{12}^2 c_{13}^2}{2} & s_{12}^2 s_{23}^2 s_{13}^2 & s_{13}^2 \\
\end{pmatrix}, \quad (22)
\]
where $c_{ij} \equiv \cos \theta_{ij}$ and $s_{ij} \equiv \sin \theta_{ij}$ for $(ij) = (12), (23), (13)$. The quantity $Y \equiv 2c_{12}s_{12}c_{23}s_{23}s_{13}\cos \delta$.

There are in the literature three global phenomenological fits to the parameters $\theta_{12}, \theta_{23}, \theta_{13},$ and $\delta$. The specific bounds on each parameter depend on which of refs. [24], [25], or [26] one uses and also on whether a ‘normal’ or ‘inverted’ ordering is assumed for the neutrino masses. For definiteness, we shall use the values of ref. [24] for an inverted ordering. They are

$$s_{12}^2 \in [0.278, 0.375], \quad s_{23}^2 \in [0.403, 0.640], \quad s_{13}^2 \in [0.0183, 0.0297]$$  \hspace{1cm} (23)

at 3$\sigma$ level,

$$s_{12}^2 \in [0.292, 0.357], \quad s_{23}^2 \in [0.432, 0.621], \quad s_{13}^2 \in [0.0202, 0.0278]$$  \hspace{1cm} (24)

at 2$\sigma$ level, and

$$s_{12}^2 \in [0.307, 0.339], \quad s_{23}^2 \in [0.530, 0.598], \quad s_{13}^2 \in [0.0221, 0.0259], \quad \delta \in [1.16\pi, 1.82\pi]$$  \hspace{1cm} (25)

at 1$\sigma$ level. Note that $\cos \delta$ is free at both 3$\sigma$ and 2$\sigma$ levels.

Up to now, we have considered that the structures correspond to predictions for the third row of $|U|^2$. This is because $T_\ell$ and $H_\ell$ are simultaneously diagonalizable and we have assumed that the eigenvalues of $T_\ell$ obey $l_1 = l_2 \neq l_3$. However,

- we might instead have assumed either $l_1 = l_3 \neq l_2$ or $l_2 = l_3 \neq l_1$, and then we would have predicted, through exactly the same mathematics, either the second or the first row, respectively, of $|U|^2$;
- equations (7) are invariant under simultaneous permutations of

$$\left( |U_{31}|^2, |U_{32}|^2, |U_{33}|^2 \right) \quad \text{and} \quad (n_1, n_2, n_3),$$

hence, by altering the ordering of the eigenvalues of $T_\nu$ one may alter the ordering of the matrix elements in the row of $|U|^2$ that one is predicting;
- $(H_\ell, T_\ell)$ and $(H_\nu, T_\nu)$ may be interchanged in the reasoning of section 2 and then we would be predicting a column instead of a row of $|U|^2$. 

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Thus, the structures in the previous section must be interpreted as predictions for either any row or any column of $|U|^2$; moreover, for each such column or row of $|U|^2$ the three numbers of the structure may be taken in any order.

We may now state the results of the confrontation of our structures with the data:

If one tries to fit the first row of $|U|^2$ through one of our structures, one finds that none of them is able to do it. Indeed, though structures 12–16 have $V_1$ adequate to fit $|U_{13}|^2$, their $V_2$ is much too large to agree with the upper bound on $s_{12}^2$.

If one tries to fit the third column of $|U|^2$ through one of our structures, one finds that structures 12 and 16 do it at $3\sigma$ level and structures 13–15 manage it at $2\sigma$ level.

Many structures on the blue curve of figure 1 fit the first column of $|U|^2$: structures 29–44 do it at $1\sigma$ level, structures 26–28 at $2\sigma$ level, and structures 24 and 25 at $3\sigma$ level. The structures on the green line in figure 1 (with the exception of structure 44) and the isolated structures in that figure cannot fit the first column of $|U|^2$, even though some of the corresponding points appear close to the blue line in figure 1. Notice that structure 32 incorporates one of the ‘golden ratio’ Ansätze for the mixing angle $\theta_{12}$, namely $\cos\theta_{12} = \varphi/2 \leftrightarrow \cos^2\theta_{12} = \frac{(3 + \sqrt{5})}{8}$ [27]; it was already pointed out in ref. [22] that that particular structure gives an excellent fit to the first column of $|U|^2$.

The second column of $|U|^2$ is fitted at $2\sigma$ level by structure 48 and at $3\sigma$ level by structures 49 and 60. Structure 48 corresponds to the well-known ‘trimaximal mixing’ [28] or ‘TM$_2$’ [29] Ansatz for that column of $|U|^2$. Structure 49 corresponds to one of the two [30] ‘golden ratio’ Ansätze for $|U_{12}|^2$.

The second row of $|U|^2$ may be fitted at $1\sigma$ level by structure 56, at $2\sigma$ level by structures 47, 50, and 58–60, and at $3\sigma$ level by structure 21.

The third row of $|U|^2$ may be fitted at $1\sigma$ level by structure 50, at $2\sigma$ level by structures 47, 56, 58, and 60, and at $3\sigma$ level by structure 49. The adequateness of structure 47 had already been pointed out in ref. [10].
It should be stressed that the fits in this section are rather sensitive to the precise phenomenological bounds that one utilizes for the three $s_{ij}^2$ and for $\cos \delta$. We have used here the bounds in ref. [24] with an inverted neutrino mass ordering but, if we had instead used the bounds in either ref. [25] or ref. [26], or the bounds for a normal ordering, then our results would have differed somewhat.

6 Summary

In this paper we have assumed that neutrinos are Dirac particles, i.e. that the lepton sector is similar to the quark sector. We have assumed that mixing in the lepton sector originates from a discrete horizontal symmetry group $G$ that breaks into two distinct subgroups $G_\ell$ and $G_\nu$ under which the mass matrices of the charged leptons and of the neutrinos are separately invariant. We have considered the special situation in which one of the subgroups $G_{\ell,\nu}$ is generated by a matrix with two equal eigenvalues while the other subgroup is generated by a matrix with non-degenerate eigenvalues. Under these assumptions, by making a GAP/SmallGroups search of all possible groups $G$ of order smaller than 2000, we have found 60 possible structures for either a row or a column of the lepton mixing matrix. Several of those structures constitute realistic predictions for either any of the columnsor for the second row or for the third row of the PMNS matrix.

A Nilpotent groups and groups with a normal Sylow 3-subgroup

A.1 Nilpotent groups

The commutator $[f, g]$ of two group elements $f$ and $g$ is the group element $f^{-1}g^{-1}fg$. Clearly, if $f$ commutes with $g$, i.e. if $fg = gf$, then $[f, g]$ is the identity element $e$ of the group; the converse is also true: if $[f, g] = e$ then $fg = gf$.

Let $G$ be a finite group. Let $F$ be one of its subgroups. Then, we define $[F, G]$ as the subgroup of $G$ generated by all the elements $[f, g]$ of $G$ which are the commutators of some $f \in F$ and some $g \in G$. (Notice that
the commutators do not in general close under the group multiplication; therefore, one must use them to generate a subgroup of $G$.)

We next define the *descending central series* of a group $G$. This is the series $G, G_1, G_2, G_3, \ldots$ of subgroups of $G$ defined through the procedure $G_1 = [G, G]$, $G_2 = [G_1, G]$, $G_3 = [G_2, G]$, and so on.

We finally define a *nilpotent* group: it is a group whose descending central series ends up in the trivial group—the one consisting solely of the identity element.

Let us give an example of a nilpotent group: the $D_4$ group generated by the two matrices

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$  \hfill (A1)

Clearly,

$$[A, A] = [B, B] = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad [A, B] = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}. \hfill (A2)$$

Therefore, $[D_4, D_4] = \mathbb{Z}_2$ is formed by the two matrices in (A2). Since those two matrices commute with all the matrices of $D_4$ (indeed, they commute with any $2 \times 2$ matrix), $[[D_4, D_4], D_4]$ is just the unit $2 \times 2$ matrix, *i.e.* it is the trivial group. Therefore, $D_4$ is nilpotent.

We next give an example of a non-nilpotent group: the $D_3$ group generated by the two matrices

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad C = \begin{pmatrix} \omega & 0 \\ 0 & \omega^2 \end{pmatrix}, \hfill (A3)$$

where $\omega = \exp(2i\pi/3)$. Clearly,

$$[A, A] = [C, C] = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad [A, C] = \begin{pmatrix} \omega^2 & 0 \\ 0 & \omega \end{pmatrix} = C^{-1}. \hfill (A4)$$

Therefore, $[D_3, D_3] = \mathbb{Z}_3$ is the group generated by $C^{-1}$. We compute $[C^{-1}, A] = C^{-1}$ to conclude that $[[D_3, D_3], D_3]$ is again $\mathbb{Z}_3$. Therefore, the descending central series of $D_3$ is $D_3, \mathbb{Z}_3, \mathbb{Z}_3, \mathbb{Z}_3, \ldots$; this central series never ends up in the trivial group, hence $D_3$ is not nilpotent.
A.2 Groups with a normal Sylow 3-subgroup

Let $G$ be a finite group. Let $F$ be one of its subgroups. Let $g \in G$ be an element of $G$. Then, the left coset of $F$ with respect to $g$ is the set of all elements $h \in G$ that may be written $h = gf$ for some $f \in F$. Similarly, the right coset of $F$ with respect to $g$ is defined to be the set of all the $h' \in G$ that may be written $h' = fg$ for some $f \in F$. The subgroup $F$ of $G$ is said to be normal if its left coset with respect to any $g \in G$ coincides with the right coset with respect to $g$.

Take for instance the group $D_3$ generated by the matrices in (A3). It has a $\mathbb{Z}_3$ subgroup formed by $C$, $C^2$, and the $2 \times 2$ unit matrix. It is easily seen that $AC = C^2A$ and $AC^2 = CA$. Therefore, the left and right cosets of $\mathbb{Z}_3$ relative to $A$ are identical; hence, $\mathbb{Z}_3$ is a normal subgroup of $D_3$.

The order of an element $g$ of a finite group $G$ is the smallest positive integer $o$ such that $g^o = e$ is the identity of $G$.

Let $p$ be a prime number, then a $p$-group is a group where all the elements have order $o$ which is a power of $p$ (different elements may have different orders, but all the orders are powers of $p$). Thus, a 3-group is a group where all the elements either have order one, or order three, or order nine, etc. An obvious example is the well-known group $\Delta(27)$, generated by

$$
\begin{pmatrix}
0 & 1 & 0 \\
0 & 0 & 1 \\
1 & 0 & 0
\end{pmatrix}
\quad \text{and} \quad
\begin{pmatrix}
1 & 0 & 0 \\
0 & \omega & 0 \\
0 & 0 & \omega^2
\end{pmatrix}
$$

(A5)

all 27 elements of $\Delta(27)$ except the identity have order three. Similarly, the group $D_4$ generated by the matrices in (A1) is a 2-group—all its elements either have order one, or order two, or order four. In general, a finite group is a $p$-group if and only if the number of elements of the group is a power of $p$; thus, the 3-groups are the groups with $3^n$ elements, for some integer $n$.

A subgroup $S$ of a group $G$ is called a Sylow $p$-subgroup if it is a $p$-group and if there is no larger $p$-subgroup of $G$ that contains $S$ as a proper subgroup.

Thus, a normal Sylow 3-subgroup $S$ of a finite group $G$ is a normal subgroup of $G$ with $3^n$ elements such that there is no subgroup of $G$ with $3^m$ elements, $m > n$, that contains $S$.

For instance, the $\mathbb{Z}_3$ subgroup of $D_3$ generated by the matrix $C^{-1}$ in (A3) is a Sylow 3-subgroup of $D_3$; indeed, the group $D_3$ has six elements and therefore any three-element subgroup of it is necessarily a Sylow 3-subgroup. Since we already know that $\mathbb{Z}_3$ is a normal subgroup of $D_3$, we conclude
that $\mathbb{Z}_3$ is a normal Sylow 3-subgroup of $D_3$. The conjecture mentioned in section 3 then informs us that $D_3$ does not have a faithful three-dimensional irrep; this is indeed true.

In the same way, a normal Sylow 3-subgroup of a group $G$ of order $1536 = 3 \times 2^9$ is just a normal $\mathbb{Z}_3$ subgroup of $G$.

A different example is the group $A_4$ generated by

$$D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \quad \text{and} \quad E = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}.$$  \hspace{1cm} (A6)

The group $A_4$ has $12 = 3 \times 2^2$ elements and therefore its $\mathbb{Z}_3$ subgroup formed by $E$, $E^2$, and the unit $3 \times 3$ matrix is automatically a Sylow 3-subgroup. However, it is not a normal subgroup, because the set $\{DE, DE^2\}$ does not coincide with the set $\{ED, E^2D\}$. Thus, $A_4$ does not have a normal Sylow-3 subgroup and it is allowed by the conjecture of section 3 to have a faithful three-dimensional irrep; this is indeed the representation generated by the matrices in (A6).

Acknowledgements: We gratefully acknowledge the collaboration of Patrick Otto Ludl in the beginning of this work; we also thank him for valuable discussions all along. The work of D.J. was supported by the Lithuanian Academy of Sciences through project DaFi2016. The work of L.L. was supported by the Portuguese Fundação para a Ciência e a Tecnologia through the projects CERN/FIS-NUC/0010/2015 and UID/FIS/00777/2013, which are partially funded by POCTI (FEDER), COMPETE, QREN, and the European Union.

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