THE SPACE OF HERMITIAN TRIPLES AND THE ASHTEKAR — ISHAM QUANTIZATION

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Abstract. It was pointed out ([5]) that the space of hermitian triples is an analogy of the hermitian connection space. Generalizing the Ashtekar — Isham procedure one can quantize the space of hermitian triples as well as the original one. Here we add an example how this similarity can be exploited in a quantum theory of riemannian geometry.

§0. Introduction

Let $X$ be an orientable compact smooth riemannian 4-dimensional manifold. Consider the space of all triples

$$\mathcal{M}_X = \{(g, J, \omega)\}$$

where $g$ is a riemannian metric, $J$ is a compatible almost complex structure and $\omega$ is the corresponding almost kaehler form. This space naturally splits with respect to a discrete parametrization such that every connected component can be described as follows

$$\mathcal{M}^K = \{(g, J, \omega) \in \mathcal{M}_X \quad | \quad K_J = K \in H^2(X, \mathbb{Z})\}$$

where $K_J$ is the canonical class of $J$. All details of the local geometry can be found in [5], [6].

For the hermitian connection space it was constructed a quantization programme (see f.e. [2]), dealing with based loops on 3-dimensional based manifold $M$. One can generalize this procedure for the space of hermitian triples over the product $X = M \times S^1$ where for every loop $\alpha$ on $M$ we have the corresponding 2-dimensional torus $\Sigma_\alpha$ on $X$ (or if one uses graphs for the definition as in [1] then one gets a Riemann surface). Thus one can arrange a complex bounded function on the space $\mathcal{M}^K$ associated with each loop $\alpha$ and then repeat the construction of the corresponding $C^*$-algebra. As well one can introduce the notion of cylindrical functions and generate a regular measure on the compactified space $\overline{\mathcal{M}^K}$.

Extending this approach to the framework of quantum theory of riemannian geometry (see [1]) one can exploit ”duality” between two spaces in some special abelian case. In this case we take a 3-dimensional orientable compact smooth riemannian manifold as the based manifold and a complex line bundle $L \to M$. The configuration space is the compactified space of abelian hermitian connection $\overline{\mathcal{A}}$ (in notations of [1]) but one takes the product

$$\overline{\mathcal{A}} \times \overline{\mathcal{M}^K}_M$$
as the phase space of the system. Here $\mathcal{M}_M^K$ is a subspace of the space $\mathcal{M}^K$ of all hermitian triples over the product manifold $X = M \times S^1$ consists of all triples invariant under all $S^1$-rotation. The key point is that the product space

$$\mathcal{A} \times \mathcal{M}_M^K$$

is a Poisson manifold thus one can say that in some weak sense the space $\mathcal{M}_M^K$ is dual to $\mathcal{A}$. Using this Poisson structure one could come in the same way as it is proposed for the cotangent bundle $T^*\mathcal{A}$ (see section 3A in [1]). Moreover since the quantizations of both spaces depend on the same loops $\alpha$ in $M$ one can introduce the notions of cylindrical function simultaneously on both spaces and then gets well defined Poisson brackets on the space of cylindrical functions on the product space.

§1. The Ashtekar — Isham Quantization

In this section we follow [2].

Let $M$ be a real smooth connected orientable 3-manifold. On the principal $SU(2)$-bundle one has the space of hermitian connections denoted as $\mathcal{A}$ with natural gauge group $\mathcal{G}$ action. Fixing a point $x_0 \in M$ consider the space of based loops denoted as $L_{x_0}$ which admits a group structure. A natural equivalence relation on $L_{x_0}$ is given by the requirement

$$H(\alpha_1, A) = H(\alpha_2, A)$$

for every connection $A$ where $H(\alpha_i, A)$ is the corresponding holonomy. The collection of the equivalence classes is denoted as $\mathcal{H}\mathcal{G}$.

Every element $\tilde{\alpha} \in \mathcal{H}\mathcal{G}$ defines a function on the quotient space $\mathcal{A}/\mathcal{G}$ by

$$T_{\tilde{\alpha}}([A]) = \frac{1}{2} Tr H(\alpha, A)$$

where $\alpha$ represents class $\tilde{\alpha}$ and $A$ represents class $[A]$. Due to $SU(2)$ trace identities one establishes that the set of such functions is closed under multiplication thus one gets a $C^*$- algebra $\mathcal{H}\mathcal{A}$, consisting of all finite linear combination of $T_{\tilde{\alpha}_i}$ with complex coefficients. This one is a subalgebra in $C^*$-algebra of all complex valued bounded continuous functions on $\mathcal{A}/\mathcal{G}$ and one can take the completion $\overline{\mathcal{H}\mathcal{A}}$ of $\mathcal{H}\mathcal{A}$ under the supremum norm in the ambient $C^*$-algebra. This one is called the Ashtekar — Isham $C^*$-algebra.

Briefly speaking the Gelfand spectrum of $\overline{\mathcal{H}\mathcal{A}}$ is a compactification of the quotient space $\mathcal{A}/\mathcal{G}$; the Hilbert space of the quantized theory is represented by $L^2(\mathcal{A}/\mathcal{G}, d\mu)$ where $d\mu$ is a regular diffeomorphism invariant measure on the compactified quotient space. The construction of such a measure is a crucial step in the programme. The definition uses the notion of cylindrical functions — special functions which form a dense subset in $\mathcal{H}\mathcal{A}$. Namely for a number of loops $\alpha_1, ... \alpha_n$ consider the following projection

$$p_{\alpha_1, ..., \alpha_n} : \mathcal{A}/\mathcal{G} \to (SU(2))^n,$$

defined by the holonomies around the loops. Then for every regular function on the target space one takes the corresponding lifting to the quotient space — it gives a cylindrical function on the last one. As well one lifts the product Haar measure to $\mathcal{A}/\mathcal{G}$; taking the limit one gets a regular $DiffM$-invariant measure on the compactified space. Moreover this measure corresponds to an invariant of framed knots in $M$ (see [2], [3]).
§2. THE SAME CONSTRUCTION FOR HERMITIAN TRIPLES

Let us adopt this construction to the case of hermitian triples. Instead of sufficiently wide generalization made in [5] now we’d like to consider only a special case. Namely let \( M \) be as above; consider the product manifold

\[ X = M \times S^1, \]

and let an orientation on \( S^1 \) is fixed as well as on \( M \). Then we can consider the space \( \mathcal{M}_X^K \) with a fixed topological datum \( K \in H^2(X, \mathbb{Z}) \). For every loop \( \alpha \in \mathcal{L}_{x_0} \) one has the function

\[ P_\alpha : \mathcal{M}_X^K \rightarrow \mathbb{C} \]

defined as follows. Let \( \Sigma_\alpha \) be the torus in \( X \) corresponds to \( \alpha \):

\[ \Sigma_\alpha = \alpha \times S^1. \]

This torus inherits its own orientation from a parametrization of \( \tilde{\alpha} \). We take

\[ P_\alpha((g, J, \omega)) = e^{i (\text{sgn} \Sigma_\alpha \text{Vol}_g \Sigma_\alpha + \int_{\Sigma_\alpha} \omega)} \]

where \( \text{sgn} \) equals to +1 if the own orientation of \( \Sigma_\alpha \) is compatible to the fixed orientation on \( X \) and −1 otherwise. It’s clear that \( P_\alpha \) is a bounded continuous complex function on \( \mathcal{M}_X^K \). Thus again we can make all steps getting the corresponding \( C^* \) - algebra \( \overline{\mathcal{M}}_X^K \) with Gelfand spectrum denoted as \( \overline{\mathcal{M}}_X^K \). But in what follows we need just a subspace in \( \mathcal{M}_X^K \) corresponds to such triples \( (g, J, \omega) \in \mathcal{M}_X^K \) which are invariant under all \( S^1 \) - rotations defined on \( X \) due to its product structure. This subspace we denote as \( \mathcal{M}_M^K \). It depends only on the based manifold \( M \) and can be described without any references to the product manifold \( X \) (see [7]). Again using the quantization procedure we get the corresponding compactification \( \overline{\mathcal{M}}_M^K \).

As above we can define the notion of cylindrical functions on \( \overline{\mathcal{M}}_M^K \). The definition will be the same: for each set of loops \( \alpha_1, ..., \alpha_n \) one has a projection

\[ p_{\alpha_1, ..., \alpha_n} : \mathcal{M}_M^K \rightarrow (U(1))^n. \]

In terms of this projection we again define the cylindrical functions and the corresponding regular measure. The limit gives what we need. All details can be found in [7].

§3. "DUALITY"

In a classical setup the hermitian connection space is the configuration space. At the same time one takes the cotangent bundle \( T^*A \) as the phase space. It is endowed with natural Poisson brackets but this brackets can not be extended to a sufficiently wide quantization procedure in the Ashtekar — Isham framework (see section 3 A in [1]). We’d like to propose an appropriate way avoiding the difficulties in the abelian case. We claim that in this case the space of hermitian triples is almost dual to the hermitian connection space over a 3 - dimensional based manifold.

Let \( M \) be as above 3 - dimensional based manifold. Consider a complex line bundle \( L \) over \( M \) with a fixed hermitian structure. Let us fix an appropriate element \( K \in H^2(X, \mathbb{Z}) \) corresponding to an almost complex structure on \( M \times S^1 \). We take the induced component \( \mathcal{M}_M^K \) and consider the direct product

\[ Y = A(L) \times \mathcal{M}_M^K. \]
Claim. The space $Y$ is a Poisson manifold.

(see [8] for the definition and properties of Poisson manifolds and [7] for detailed explanations).

The key point is that in each point $(a, (g, J, \omega)) \in Y$ the tangent space is isomorphic to

$$T_{pt} Y = \Omega^1(i\mathbb{R})_M \oplus \Omega^2_M \oplus \Omega^1_M$$

and this representation is locally trivial. Thus for any point of $Y$ the tangent space is represented as

$$T_{pt} Y = V \oplus V^* \oplus V$$

(we rescale all vectors from the first summand by $\iota$). For a pair of smooth functions $F, G \in C^\infty(Y \to \mathbb{R})$ let $dF_i, dG_i$ be components belong to $i$ - summands in the dual decomposition

$$T^*_{pt} Y = V^* \oplus V \oplus V^*.$$

The Poisson structure is given by formula

$$\{F, G\} |_{pt} = dF_1(dG_2) - dG_1(dF_2) \in \mathbb{R}.$$ 

In local coordinate $(p, q, s)$ where $dp$ belongs to the first summand in the decomposition above $dq$ to the second and $ds$ to the third the Poisson structure has absolutely classical form

$$\{F, G\} = \sum_{i<j} \left( \frac{\partial F}{\partial p_i} \frac{\partial G}{\partial q_j} - \frac{\partial F}{\partial q_i} \frac{\partial G}{\partial p_j} \right),$$

while the functions depend only on $s$ form the center of the corresponding Poisson algebra.

Now we can quantize the picture using the same loops for both components in $Y$. We introduce the spaces of cylindrical functions for both $\mathcal{A}$ and $\mathcal{M}_M^K$ denoting the first one as $Cyl_p$ and the second one as $Cyl_q$. Then one has the big space $Cyl_{tot} = Cyl_p \otimes Cyl_q$ endowed with the product regular measure. The corresponding limit gives us an universal space $\mathcal{C}$ on $\overline{Y}$ which is the product of quantized spaces

$$\overline{Y} = \overline{\mathcal{A}} \times \overline{\mathcal{M}_M^K}$$

and we have the corresponding $DiffM$ - invariant measure on the product space. The point is that on this space we have the extended Poisson brackets

$$\{;\} : \mathcal{C} \times \mathcal{C} \to \mathcal{C},$$

satisfying obviously the distinguishing properties namely

$$\{f_1, f_2\} \equiv 0,$$

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where $f_i$ doesn’t depend on the second ”variable” $\overline{\mathcal{M}_M^K}$ being a pure function on $\overline{\mathcal{A}}$ and $F_i$ depends only on the second ”variable”.
Consider the mixed situation when one takes a cylindrical function \( f \in Cyl_p \) and a cylindrical function \( F \in Cyl_q \) pairing these functions. Let \( f \) is defined by a set of loops \( \alpha_1, ..., \alpha_k \) and \( F \) uses a set \( \beta_1, ..., \beta_l \). In the total space \( Cyl_{tot} \) we have a component corresponds to the union set \( \alpha_1, ..., \alpha_k, \beta_1, ..., \beta_l \). Since the Poisson structure defined above is a constant Poisson structure then the Poisson brackets \( \{ f, F \}_\Pi \) should belong to the component in \( Cyl_{tot} \).

Moreover one can consider the second type functions defined on \( \overline{M^K_M} \) as derivations of the original space \( L^2(\overline{A}, d\mu) \). Really if we denote the corresponding \( DiffM \)-invariant measure on the compactified \( \overline{M^K_M} \) as \( d\mu' \) then the action

\[
N_F(f) = \int_{\overline{M^K_M}} \{ f, F \}_{\Pi} d\mu'
\]

is correctly defined and satisfies the Leibnitz rule thus it can be used in quantization constructions. This action on cylindrical functions gives cylindrical functions again but changes a “grading” because \( N_F(f) \) is defined via the extended loop set \( \alpha_1, ..., \alpha_k, \beta_1, ..., \beta_l \) rather then the original function \( f \) with the loop set \( \alpha_1, ..., \alpha_k \). As in [1] the construction easily extends to the situation with graphs instead of loop sets used above. The details are contained in [7].

**Conclusion**

As it was pointed out in [5] one has a twisted version of the construction above. In the case above our ingredients in the direct product

\[
Y = A \times M^K_M
\]

are independent ”variables”. But one can twist the picture imposing for example the following condition. Every hermitian triple \((g, J, \omega)\) from \( M^K_M \) defines a hermitian structure on a real rank 2 subbundle of \( TM \) (see [5], [7]). So each point \( pt \in M^K_M \) gives a complex line bundle with a fixed hermitian structure. Let us take the corresponding hermitian connection space \( A \) as the fiber getting globally topologically nontrivial bundle

\[
Y \to M^K_M
\]

with \( A \)-fibers. One could try to exploit a Poisson structure on \( Y \) familiar with the original one described above. But there exists an alternative way. The point is that \( Y \) looks like an even super symplectic manifold (the definition and properties can be found in [4]). The based space \( M^K_M \) has a symplectic form defined as follows. Let us recall from above that the tangent space in each point is isomorphic to the direct sum

\[
T_{pt}M^K_M \cong \Omega^1_M \oplus \Omega^2_M.
\]

Thus one has a constant symplectic form defined by

\[
\Omega((u_1 \oplus w_1); (u_2 \oplus w_2)) = \int_M (u_1 \wedge w_2 - u_2 \wedge w_1).
\]
The fiber of \( Y \) is an affine space over a real vector space endowed with inner product (over point \((g, J, \omega)\) the riemannian metric \(g\) should be exploited for the definition). For the even super symplectic structure it remains to take an appropriate affine connection on \( Y \). Thus in the construction one can consider over \( Y \) the corresponding super brackets instead of the classical one.

On the other hand the example has been discussed in the present article looks like too special. The construction hasn’t been extended to a nonabelian case. The crucial point for a virtual extension is that the case of hermitian triples is ”abelian” in some sense. One could expect that a nontrivial generalization of the example lies in the framework of lorentzian geometry when one considers triples consist of lorentzian metrics and ”compatible” almost complex structures. Anyway the formula of the Poisson structure \( \{ \cdot, \cdot \}_n \) on the product space \( A \times M^K_M \) can be easily generalized to the case when \( A \) is the connection space on \( SU(2) \)- bundle over \( M \) but this generalization doesn’t look like a gauge invariant one being the result of pure formal approach so we finish this example hoping that more geometric approach will be found in a future.

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