LOWER BOUNDS FOR PROJECTIVE DESIGNS, CUBATURE FORMULAS AND RELATED ISOMETRIC EMBEDDINGS

YU. I. LYUBICH

Abstract. Yudin’s lower bound [21] for the spherical designs is generalized to the cubature formulas on the projective spaces over a field $K \subset \{\mathbb{R}, \mathbb{C}, \mathbb{H}\}$ and thus to isometric embeddings $l^m_{2,K} \rightarrow l^n_{p,K}$ with $p \in 2\mathbb{N}$. For large $p$ and in some other situations this is essentially better than those known before.

AMS Classification: 46B04, 05B30

1. Introduction

In the theory of spherical and projective designs some important lower bounds were obtained [2, 6] by maximization of the functional

$$D(f) = \frac{f(1)}{c_0[f]}, \quad f \in K_l,$$

where $K_l$ is the set of nonnegative on $(-1, 1)$ nonzero polynomials $f$, $\deg f \leq l$, and

$$c_0[f] = \int_{-1}^{1} f(t) \omega_{\alpha,\beta}(t) dt, \quad \omega_{\alpha,\beta}(t) = (1 - t)^\alpha (1 + t)^\beta.$$  \hspace{1cm} (1.1)

Here the numbers $l \in \mathbb{N}$ and $\alpha, \beta > -1$ depend on the design.

Obviously, $\sup \{D(f) : f \in K_l\} = \sup \{f(1) : f \in K_l, c_0[f] = 1\}$. The solution to the latter linear programming problem is classical, the extremal polynomial $f_{\text{max}}$ is unique and can be expressed in terms of the Jacobi polynomials, see [19], Section 7.7.1. For the designs of cardinality $n$ this yields

$$n \geq \tau_{\alpha,\beta} f_{\text{max}}(1) = \tau_{\alpha,\beta} \max \{D(f) : f \in K_l\},$$  \hspace{1cm} (1.2)

where

$$\tau_{\alpha,\beta} = \int_{-1}^{1} \omega_{\alpha,\beta}(t) dt.$$  \hspace{1cm} (1.3)

We denote by $L^2_{2,\alpha,\beta}(-1, 1)$ the space of complex-valued measurable functions $f$ on $(-1, 1)$ such that

$$\|f\|^2 \equiv \int_{-1}^{1} |f(t)|^2 \omega_{\alpha,\beta}(t) dt < \infty.$$

The corresponding Jacobi polynomials $P_k(t)$ constitute an orthogonal basis in $L^2_{2,\alpha,\beta}$, so that

$$f(t) = \sum_{k=0}^{\infty} \nu_k c_k[f] P_k(t), \quad f \in L^2_{2,\alpha,\beta},$$  \hspace{1cm} (1.4)

where

$$c_k[f] = \int_{-1}^{1} f(t) P_k(t) \omega_{\alpha,\beta}(t) dt, \quad \nu_k = 1/\|P_k\|^2.$$  \hspace{1cm} (1.5)
The Jacobi-Fourier series (1.4) converges to $f$ in $L^2_2(-1, 1)$. The coefficient $c_0[f]$ in (1.5) coincides with that of (1.1) since $P_0(t) \equiv 1$, according to the usual standardization
\[ \deg P_k = k, \quad P_k(1) = \left( \frac{\alpha + k}{k} \right) \quad (k = 0, 1, 2, \ldots). \]
For the same reason $\nu_0 = 1/\tau_{\alpha, \beta}$.

The linear programming bound (1.2) can be extended to the set $K_{l,l'}$, $l' > l$, of the polynomials $f \neq 0$, $\deg f \leq l'$, such that $f(t) \geq 0$ for $|t| \leq 1$ and $c_k[f] \leq 0$ for $l+1 \leq k \leq l'$. In this way Boyvalenkov and Nikova [3, 4, 5] obtained a series of new concrete lower bounds for the projective designs. For the spherical designs Yudin [21] considered the limit case $l' = \infty$. Its class $K_{l,\infty}$ consists of all nonnegative nonzero continuous functions $f(t)$, $|t| \leq 1$, such that $c_k[f] \geq 0$ for all $k \geq l + 1$. A suitable choice of a function $f \in K_{l,\infty}$ yields a lower bound asymptotically better than classical one that comes from (1.2).

In the present paper we generalize Yudin’s result on the projective designs and even the cubature formulas on the projective spaces $\mathbb{K}P^{m-1}$, where $\mathbb{K} \subset \{ \mathbb{R}, \mathbb{C}, \mathbb{H} \}$. (Recall that $\mathbb{H}$ is the standard notation for the quaternion field.) The extension of the linear programming bound from the projective designs to the general cubature formulas is technically simple but important since the latter are equivalent to the isometric embeddings $l^m_{2;\mathbb{K}} \to l^p_{\mathbb{K}}$, $p \in 2\mathbb{N}$. (See [14] and references therein.) Note that with the standard inner product $(x,y)$ the space $l^m_{2;\mathbb{K}}$ is Euclidean, its unit sphere is $S = S^{2m-1}$, where $\delta = \delta(\mathbb{K})$ such that $\delta(\mathbb{R}) = 1, \delta(\mathbb{C}) = 2, \delta(\mathbb{H}) = 4$.

From now on we assume $m \geq 2$, $p \in 2\mathbb{N}$, and denote by $\Phi_\mathbb{K}(m,p)$ the space of complex-valued functions $\phi(x)$, $x \in S$, satisfying the following conditions, see [14, 15].

1. $\phi = \psi|S$, where $\psi$ is a homogeneous polynomial of degree $p$ (“$p$-forms”) on the space $\mathbb{K}^m \equiv \mathbb{R}^{2m}$;
2. $\psi$ is invariant in the sense that
\[ \psi(w\alpha) = \psi(w) \quad (w \in \mathbb{K}^m, \alpha \in \mathbb{K}, \ |\alpha| = 1). \]

A fortiori, $\phi(x\alpha) = \phi(x)$ that allows us to naturally transfer $\phi$ to the projective space $\mathbb{K}P^{m-1}$. However, we will consider $\phi$ on $S$ which is equivalent but more elementary. In this setting a projective cubature formula of index $p$ on $S$ is
\[ \int \phi \, d\sigma = \sum_{i=1}^n \phi(x_i)\rho_i, \quad \phi \in \Phi_\mathbb{K}(m,p), \quad (1.6) \]
where $\sigma$ is the normalized Lebesgue measure on $S$, the nodes $x_i \in S$ are projectively distinct and the weights $\rho_i$ are positive. (Note that $\sum \rho_i = 1$ automatically by the restriction of $(x,x)^{p/2}$ to $S$.) In the case of equal $\rho_i$ the set $\{x_i\}_{1}^{n}$ is nothing but a projective $p/2$-design, c.f. [9].

2. Basic theory

First of all, we have the decomposition
\[ \Phi_\mathbb{K}(m,p) = \sum_{k=0}^{p/2} \text{Harm}_\mathbb{K}(m,2k) \quad (2.1) \]
where the space \( \text{Harm}_K(m, 2k) \) consists of restrictions to \( S \) of the invariant harmonic \( 2k \)-forms. Regarding to the inner product \( (\psi_1, \psi_2) = \int \bar{\psi}_1 \psi_2 \, d\sigma \) the decomposition (2.1) is orthogonal.

For any orthonormal basis \( \{\phi_{ks}\}_{s=1}^{d_{m,2k}} \) of \( \text{Harm}_K(m, 2k) \) the addition formula
\[
\sum_{s=1}^{d_{m,2k}} \phi_{ks}(x) \phi_{ks}(y) = b_{m,k} P_k(xy) \quad (x, y \in S)
\]
holds with
\[
b_{m,k} = \tau_{\alpha,\beta} \nu_k P_k(1), \quad \alpha = \frac{\delta(m-1) - 2}{2}, \quad \beta = \frac{\delta - 2}{2},
\]
and
\[
xy = 2|(x,y)|^2 - 1,
\]
see [8, 12, 14, 17]. Later on we operate only with \( \alpha, \beta \) given by (2.3).

Now let \( X \) be a finite nonempty subset of \( S \), and let \( A(X) \) be its angle set, i.e.
\[
A(X) = \{xy : x, y \in X, x \neq y\}.
\]
The addition formula easily implies the following

**Lemma 2.1.** Let the series
\[
\sum_{k=0}^{\infty} a_k P_k(t)
\]
converge to a function \( f(t) \) for every \( t \in A(X) \) and for \( t = 1 \). Then
\[
\sum_{x,y \in X} f(xy) \bar{\lambda}(x) \lambda(y) = \sum_{k=0}^{\infty} a_k b_{m,k}^{-1} \sum_{s=1}^{d_{m,2k}} \left| \sum_{x \in X} \phi_{ks}(x) \lambda(x) \right|^2,
\]
where \( \lambda \) is an arbitrary function \( X \to \mathbb{C} \).

With \( \lambda(x) \equiv 1 \) formula (2.5) plays a fundamental role in the design theory [6, 10, 14]. In the context of cubature formulas we need (2.4) with arbitrary \( \lambda(x) > 0 \), \( \sum \lambda(x) = 1 \), c.f. [14], §5. Also note that, in contrast to those which are quoted above, now we have to apply (2.5) to the non-polynomial functions \( f \in K_{l,\infty} \). It is possible because of

**Lemma 2.2.** The Jacobi-Fourier series of any function \( f \in K_{l,\infty} \) converges to \( f(t) \) for all \( t \in [-1, 1] \).

**Proof.** Since \( f(t) \) is continuous, its Jacobi-Fourier series at \( t = 1 \) is summable to \( f(1) \) by a Cesaro method, see [19], Theorem 9.13. Therefore, it is summable to \( f(1) \) by the Abel method, see [7], Theorem 5.5. Hence, this series converges to \( f(1) \) since \( c_k f \leq 0 \) for \( k \geq l + 1 \). It remains to refer to Theorem 7.32.1 from [19] which states that
\[
\max_{|t| \leq 1} |P_k(t)| = P_k(1)
\]
if \( \max(\alpha, \beta) \geq -1/2 \). The latter is fulfilled because of (2.3) and \( m \geq 2 \). \( \square \)

**Corollary 2.3.** Formula (2.5) is true for every \( f \in K_{l,\infty} \) with \( a_k = \nu_k c_k[f], k \geq 0 \).
Remark 2.4. In [21] the absolute convergence of the corresponding series is mentioned without proof. The proof of Lemma 2.2 shows that in our situation the convergence is absolute and uniform.

Now we can prove the following linear programming bound.

**Proposition 2.5.** The inequality

\[ n \geq \tau_{\alpha,\beta} \sup \{ D(f) : f \in K_{p/2,\infty} \} \]  \hspace{1cm} (2.7)

holds for any projective cubature formula of shape \((1.6)\).

**Proof.** We have

\[ \sum_{i=1}^{n} \phi(x_i) e_i = 0, \quad \phi \in \text{Harm}(m, 2k), \quad 1 \leq k \leq p/2. \]  \hspace{1cm} (2.8)

Applying Corollary 2.3 and Lemma 2.1 to \(f \in K_{p/2,\infty}\), \(X = \{x_i\}_{n}^{p}\) and \(\lambda(x) = \rho(x)\), \(x \in X\), we obtain

\[ f(1) \sum_{x \in X} \rho^2(x) \leq \sum_{x,y \in X} f(xy) \rho(x) \rho(y) \leq a_0 b_{m,0}^{-1} \left( \sum_{x} \rho(x) \right)^2. \]

Indeed, on the left side of (2.5) all summands are \(\geq 0\). On the right side the summands are \(\leq 0\) for \(k \geq p/2 + 1\) and vanish for \(1 \leq k \leq p/2\) by (2.5). It remains to recall that \(\phi(x) = 1\), therefore, \(\sum \rho^2(x) \geq n^{-1}\); on the other hand, \(a_0 b_{m,0}^{-1} = c_0 / \tau_{\alpha,\beta}\) since \(b_{m,0} = 1\). \(\square\)

**Remark 2.6.** The inequality (2.7) implies

\[ n \geq \tau_{\alpha,\beta} \sup \{ D(f) : f \in K_{p/2,l'} \}, \quad l' \geq p/2, \]

since \(K_{p/2,l'} \subset K_{p/2,\infty}\). (For \(l' = l\) we set \(K_{l,l} = K_{l}\).)

Since with any given \(m, p\) a projective cubature formula exists (or, equivalently, there exists an isometric embedding \(l_m^{2,k} \rightarrow l_n^{p,k}\)), we have

**Corollary 2.7.** \(\sup \{ D(f) : f \in K_{p/2,\infty} \} < \infty\).

The supremum in question is unknown but a “good” test function can be constructed using the “convolution”

\[ \int g(xu) h(uy) d\sigma(u) \quad (x, y \in S) \]  \hspace{1cm} (2.9)

of two suitable functions \(g(t)\) and \(h(t)\), \(-1 \leq t \leq 1\), c.f. [21].

**Lemma 2.8.** For any \(e \in L_{1}^{(\alpha,\beta)}(-1,1)\) the function \(u \mapsto e(xu)\), \(u \in S\), belongs to \(L_1(S, \sigma)\) for every \(x \in S\) and

\[ \int e(xu) d\sigma(u) = \frac{1}{\tau_{\alpha,\beta}} \int_{-1}^{1} e(t) \omega_{\alpha,\beta}(t) dt \]  \hspace{1cm} (2.10)

**Proof.** This follows by calculation in spherical coordinates. \(\square\)

**Corollary 2.9.** With \(g, h \in L_{2}^{(\alpha,\beta)}(-1,1)\) the integral (2.9) exists for all \(x, y \in S\).
Since any ordered pair \( x', y' \in S \) with \( x'y' = xy \) can be obtained from \( x, y \) by an isometry of \( L^2_{\mathbb{R}^2} \), the integral (2.9) depends on \( xy \) only. Thus, we have a function 
\[
(g \ast h)(t), \quad -1 \leq t \leq 1,
\]
such that
\[
(g \ast h)(xy) = \int g(xu)h(uy) \, d\sigma(u).
\]
(2.11)

In particular, for \( x = y \) (2.11) yields
\[
(g \ast h)(1) = \int g(xu)h(xu) \, d\sigma(u) = \frac{1}{\tau_{\alpha,\beta}} \int_{-1}^{1} g(t)h(t)\omega_{\alpha,\beta}(t) \, dt,
\]
(2.12)

by (2.10). Moreover, applying the Schwartz inequality to (2.11) and using (2.10) by this inequality and bilinearity, the convolution \( g \ast h \) determines a continuous mapping \((L^2_{(\alpha,\beta)})^2 \to L_{\infty}\).

Lemma 2.10. With \( g, h \in L^2_{(\alpha,\beta)} \) the function \((g \ast h)(t)\) is continuous, and the series
\[
(g \ast h)(t) = \sum_{k=0}^{\infty} (\nu_k^2/b_{m,k})c_k[g]c_k[h]P_k(t)
\]
converges uniformly.

Proof. Let
\[
g_N = \sum_{j=0}^{N} \nu_j c_j[g]P_j, \quad h_N(t) = \sum_{k=0}^{N} \nu_k c_k[h]P_k.
\]
Then
\[
g_N \ast h_N = \sum_{k=0}^{N} (\nu_k^2/b_{m,k})c_k[g]c_k[h]P_k
\]
since
\[
P_j \ast P_k = b_{m,k}^{-1} P_k \delta_{jk}
\]
by the addition formula. Since \( g_N \to g \) and \( h_N \to h \) \((N \to \infty)\) in \( L^2_{(\alpha,\beta)}\), we obtain \( g_N \ast h_N \to g \ast h \) uniformly. Thus, the limit function is continuous.

Corollary 2.11. \( c_k[g \ast h] = \nu_k c_k[g]c_k[h]/b_{m,k} = c_k[g]c_k[h]/\tau_{\alpha,\beta} P_k(1) \).

3. A function \( f_l \in K_{l,\infty} \)

Recall that all roots of every \( P_k \) are simple and lie on \((-1,1)\). The roots of the derivative \( P'_k \) alternate them, so they are also simple and lie on \((-1,1)\). Now we introduce a function \( f_l \) by setting
\[
f_l = g \ast h, \quad g(t) = \begin{cases} P_r(t) - P_r(\xi), & t \geq \xi, \\ 0, & t < \xi \end{cases}, \quad h(t) = \begin{cases} 1, & t \geq \xi, \\ 0, & t < \xi \end{cases}
\]
(3.1)

where \( r = l + 1 \) and \( \xi \) is the largest root of \( P'_r \). We have to verify that \( f_l \in K_{l,\infty} \).

By Lemma 2.10, \( f_l \) is continuous. The inequality \( f_l \geq 0 \) follows from (2.11) since \( \xi < 1 \). Moreover, \( f_l(1) > 0 \) by (2.12), thus, \( f_l \neq 0 \). It remains to prove that \( c_k[f_l] \leq 0 \) for \( k \geq r \). In (21) a rather complicated vector analysis on
$\mathbb{R}^m$ was used at this point. We manage without a generalization of this technique
to $\mathbb{C}^m$ and $\mathbb{H}^m$ by dealing with the corresponding Jacobi polynomials.

Our starting point is the differential equation

$$\Delta t \equiv (\omega_{\alpha+1,\beta+1} P_t')' + i(i + \lambda)\omega_{\alpha,\beta} P_t = 0, \quad i \geq 0, \quad (3.2)$$

where $\lambda = \alpha + \beta + 1$, see [19], formula(4.2.1). Note that $\lambda \geq 0$ by (2.3). From (3.2) it follows that

$$0 = \int_\xi^1 (P_t \Delta k - P_k \Delta r) dt =$$

$$= (k - r)(k + r + \lambda) \int_\xi^1 \omega_{\alpha,\beta} P_t P_k dt + \int_\xi^1 \{\omega_{\alpha+1,\beta+1} (P_t P_k')' - P_r P_t'\}' dt =$$

$$= (k - r)(k + r + \lambda) \int_\xi^1 \omega_{\alpha,\beta} P_t P_k dt - (\omega_{\alpha+1,\beta+1} P_t') (r)$$

since $\omega_{\alpha+1,\beta+1}(1) = 0$, $P_t'(r) = 0$. For $k \neq r$ we obtain

$$\int_\xi^1 P_t P_k \omega_{\alpha,\beta} dt = \frac{(\omega_{\alpha+1,\beta+1} P_t P_k') (r)}{(k - r)(k + r + \lambda)}.$$

This formula extends to $r = 0$ since $P_0(t) \equiv 1$, so $P_0'(\xi) = 0$. Thus,

$$\int_\xi^1 P_k \omega_{\alpha, \beta} dt = \frac{(\omega_{\alpha+1,\beta+1} P_t')(r)}{k(k + \lambda)},$$

and then

$$\int_\xi^1 P_t P_k \omega_{\alpha, \beta} dt = \frac{k(k + \lambda)P_t (\xi)}{(k - r)(k + r + \lambda) \int_\xi^1 P_k \omega_{\alpha, \beta} dt}.$$

As a result,

$$c_k [g] = \int_\xi^1 g P_k \omega_{\alpha, \beta} dt = \frac{r(r + \lambda)P_t (\xi)}{(k - r)(k + r + \lambda)} \int_\xi^1 P_k \omega_{\alpha, \beta} dt = \frac{r(r + \lambda)P_t (\xi)}{(k - r)(k + r + \lambda)} c_k [h],$$

and, by Corollary 2.11

$$c_k [f_i] = \frac{r(r + \lambda)P_t (\xi)}{(k - r)(k + r + \lambda) P_k (1) \tau_{\alpha, \beta}} (c_k [h])^2 \quad (k \neq r). \quad (3.3)$$

Since $P_k(1) > 0$, formula (3.3) yields $c_k [f_i] = \text{sign}(P_t (\xi))$, $k > r$. But $\text{sign} P_t (\xi) = -1$ since $\xi$ lies in between two largest roots of $P_r(t)$ and $P_t(1) > 0$. Thus, $c_k [f_i] < 0$ for $k > r$. In addition, $c_r [f_i] = 0$ since $c_r [h] = 0$. The latter follows from (3.2) with $i = r$ by integration over $[\xi, 1]$.

In conclusion we note that $\xi$ in (3.1) is actually the largest root of $P_t^{(\alpha+1,\beta+1)}(t)$, see [19], formula (4.21.7).

4. Main Theorem

Now we are in position to prove the following

**Theorem 4.1.** The number $n$ of nodes of every projective cubature formula of
index $p$ on $S^{km-1}$ satisfies the inequality

$$n \geq \frac{\Gamma(\alpha + 2) \Gamma(\beta + 1)}{\Gamma(\alpha + \beta + 2) F(-\beta, \alpha + 1, \alpha + 2, \varepsilon)} \left( \frac{1}{\omega} \right)^{\delta(m-1)/2}, \quad (4.1)$$
where $F$ is the hypergeometric function, the numbers $\alpha$ and $\beta$ are given by (2.3), $\varepsilon = (1 - \xi)/2$, $\xi$ is the largest root of the Jacobi polynomial $P_{\xi}^{(\alpha+1,\beta+1)}(t)$.

Proof. Using $f_{p/2}(t)$ as a test function in (2.7) we get

$$n \geq \frac{\tau_{\alpha,\beta} f_{p/2}(1)}{c_0[f_{p/2}]}.$$

By (2.12) and (3.1) we have

$$\tau_{\alpha,\beta} f_{p/2}(1) = \int_1^{\xi} g h \omega_{\alpha,\beta} dt = \int_0^1 g \omega_{\alpha,\beta} dt = c_0[h].$$

On the other hand, $c_0[f_{p/2}] = c_0[g]c_0[h]/\tau_{\alpha,\beta}$ by Corollary 2.11. Hence,

$$n \geq \frac{\tau_{\alpha,\beta}}{c_0[h]} = \frac{\int_0^1 (1 - t)^{\alpha + t} dt}{\int_0^1 (1 - t)^{\alpha + t} dt} \int_0^1 (1 - s)^{\alpha + \beta + 1} ds.$$

Now we substitute $t = 1 - 2s$ into the numerator and $t = 1 - 2\varepsilon s$ into the denominator. This yields (4.1) since

$$F(-\beta, \alpha + 1, \alpha + 2, \varepsilon) = (\alpha + 1) \int_0^1 s^\alpha (1 - \varepsilon s)^\beta ds,$$

(c.f. [1], formula (15.3.1)) and

$$\int_0^1 s^\alpha (1 - s)^\beta ds = \frac{\Gamma(\alpha + 1)\Gamma(\beta + 1)}{\Gamma(\alpha + \beta + 2)}.$$

(Also note that $\alpha + 1 = (\delta m - \delta)/2$ by (2.3).) □

Remark 4.2. By substitution $t = 2s^2 - 1$ in both integrals (4.2) we obtain

$$n \geq \frac{\int_0^1 (1 - s)^{2\alpha + 1} ds}{\int_0^\eta (1 - s)^{2\alpha + 1} ds}, \quad \eta = \sqrt{(1 + \xi)/2}.$$

In particular, for $K = \mathbb{R}$ we have $\alpha = (m - 3)/2, \beta = -1/2$, see (2.3). Hence,

$$n \geq \frac{\int_0^1 (1 - s)^{(m-3)/2} ds}{\int_0^\eta (1 - s)^{(m-3)/2} ds} \quad (K = \mathbb{R}),$$

(4.3)

where $\eta$ is the largest root of the polynomial

$$P_{p/2}^{(m-1)/2, 1/2}(2s^2 - 1) = \text{const} \cdot P_{p+1}^{(m-1)/2, (m-1)/2}(s)/s,$$

(4.4)

or, equivalently, of the Gegenbauer polynomial $C_{p+1}^{m/2}(s)$ (see (19), formulas (4.1.5) and (4.7.1)). In the case of antipodal spherical $(p+1)$-design the lower bound turns into (3) of (21) up to the additional factor 2 in the latter. Note that the factor 2 is just the degree of the natural mapping $S^{m-1} \to \mathbb{R}P^{m-1}$.

Remark 4.3. By (2.3) we have $\alpha = m - 2, \beta = 0$ for $K = \mathbb{C}$, and $\alpha = 2m - 3, \beta = 1$ for $K = \mathbb{H}$. Accordingly, (4.1) reduces to

$$n \geq \left(\frac{1}{\varepsilon}\right)^{m-1} \quad (K = \mathbb{C}),$$

(4.5)
and to
\[ n \geq \frac{1}{(2m - 1) - (2m - 2)\varepsilon} \left( \frac{1}{\varepsilon} \right)^{2m - 2} (\mathbb{K} = \mathbb{R}). \] (4.6)

In the real case the hypergeometric function in (4.1) is not a polynomial of \( \varepsilon \).

Now we denote by \( N_K(m, p) \) the minimal number \( n \) of nodes in the cubature formula (1.4) or, equivalently, the minimal \( n \) such that there is isometric embedding \( l_{2,\mathbb{K}}^m \rightarrow l_{p,\mathbb{K}}^n \). In this notation Theorem 4.1 states that
\[ N_K(m, p) \geq \frac{\Gamma(\alpha + 2)\Gamma(\beta + 1)}{\Gamma(\alpha + \beta + 2)F(-\beta, \alpha + 1, \alpha + 2, \varepsilon)} \left( \frac{1}{\varepsilon} \right)^{\delta(m - 1)/2}. \] (4.7)

We will compare this result to the linear programming bound (1.2) with \( l = p/2 \).

An explicit form of the latter is
\[ N_K(m, p) \geq \Lambda_K(m, q), \quad q = p/2, \] (4.8)

where
\[ \Lambda_K(m, q) = \begin{cases} \frac{(m + q - 1)^{m - 1}}{(m + [q/2] - 1)^{m + [q/2] - 2} - 1}, & (\mathbb{K} = \mathbb{R}); \\ \frac{1}{2m - 1} \left( \frac{2m + [q/2] - 2}{2m - 2} \right)^{m + [q/2] - 1}, & (\mathbb{K} = \mathbb{C}); \\ \frac{1}{2m - 1} \left( \frac{2m + q - 1}{2m - 2} \right)^{m + [q/2] - 1}, & (\mathbb{K} = \mathbb{H}), \end{cases} \] (4.9)

while
\[ N_K(m, p) \leq \Lambda_K(m, p), \] (4.10)

(See [14] and the references therein.)

5. Asymptotic Analysis

From (4.8) and (4.9) it follows that
\[ N_K(m, p) \gtrsim \frac{p^{\delta(m - 1)}}{\lambda_K(m)}, \quad p \to \infty, \] (5.1)

where
\[ \lambda_K(m) = \begin{cases} 2^{m - 1}(m - 1)!, & (\mathbb{K} = \mathbb{R}); \\ 2^{4(m - 1)}(m - 1)!, & (\mathbb{K} = \mathbb{C}); \\ 2^{8(m - 1)}(2m - 1)!(2m - 2)!, & (\mathbb{K} = \mathbb{H}). \end{cases} \] (5.2)

or, in an unified form,
\[ \lambda_K(m) = \frac{\Gamma(\delta m/2)\Gamma(\delta (m - 1)/2 + 1)}{\Gamma(\delta/2)} \cdot 2^{2\delta(m - 1)} = \frac{\Gamma(\alpha + \beta + 2)\Gamma(\alpha + 2)}{\Gamma(\beta + 1)} \cdot 2^{2\delta(m - 1)}. \] (5.3)

As to (5.1), \( \varepsilon \) is the only parameter depending on \( p \). (Of course, \( \varepsilon \) also depends on \( m \).) By definition, \( \varepsilon = (1 - \xi)/2 = \sin^2(\theta/2) \) where \( \theta = \arccos \xi \). This \( \theta \) is the smallest root of the polynomial \( P^{(\alpha + 1, \beta + 1)}(\cos \theta) \). By Theorem 8.1.2 from [19] we have \( \theta \sim 2j_{\alpha + 1, 1}/p \) where \( j_{\alpha + 1, 1} \) is the smallest positive root of the Bessel’s function \( J_{\alpha + 1}(z) \). Therefore, \( \varepsilon \sim j_{\alpha + 1, 1}^2/p^2 \), and (4.7) yields
\[ N_K(m, p) \gtrsim \frac{\Gamma(\alpha + 2)\Gamma(\beta + 1)}{\Gamma(\alpha + \beta + 2)} \cdot \frac{p^{\delta(m - 1)}}{j_{\alpha + 1, 1}^{\delta(m - 1)}}, \quad p \to \infty, \] (5.4)

since \( \varepsilon \to 0, F(\cdot, \cdot, \cdot, 0) = 1. This estimate is better than (5.1) because of
Proposition 5.1. The inequality
\[ j_{\delta(m-1)}^{\alpha+1} < \frac{\Gamma(\alpha + 2)\Gamma(\beta + 1)}{\Gamma(\alpha + \beta + 2)} \lambda_{K}(m) \] (5.5)
holds for all \( m \geq 2 \), except for the case \( m = 2, \delta = 1 \), when (5.5) changes for an equality.

Proof. By (5.3) the inequality (5.5) is equivalent to
\[ j_{\delta(m-1)}^{\alpha+1} < \frac{\Gamma(\alpha + 2)^2 \cdot 2^{\delta(m-1)}}{\Gamma(\alpha + 1)^2 \cdot 16^\nu} \] (5.6)
We set \( \alpha + 1 = \nu \), so that \( \delta(m-1) = 2\nu \), and (5.6) takes the form
\[ j_{\nu}^{2\nu} < \frac{\Gamma(\nu + 1)^2 \cdot 16^\nu}{\Gamma(\nu + 1)^2 \cdot 16^\nu}. \] (5.7)
The number \( \nu \) is positive integer or half-integer, \( \nu \geq 1/2 \), and \( \nu = 1/2 \) if and only if \( m = 2, \delta = 1 \). In this case \( j_{1/2} \) is proportional to \( \sin \frac{z}{\sqrt{z}} \). On the other hand, \( \Gamma(3/2)^2 \cdot 16^{1/2} = \pi \) as well. Thus, (5.7) changes for an equality.

Now let \( \nu \geq 1 \). By the inequality \( j_{\nu}^{\nu} < \sqrt{2(\nu + 1)(\nu + 3)} \) (see [20], Section 15.3) it suffices to prove that
\[ (\nu + 1)^{\nu} (\nu + 3)^{\nu} \leq \Gamma(\nu + 1)^2 \cdot 8^\nu. \] (5.8)
By Stirling’s lower bound the inequality (5.8) follows from
\[ \left(1 + \frac{1}{\nu}\right)^{\nu} \left(1 + \frac{3}{\nu}\right)^{\nu} < 2\pi\nu \left(\frac{8}{e^2}\right)^{\nu}. \]
A fortiori, (5.8) follows from
\[ 2\pi\nu \left(\frac{8}{e^2}\right)^{\nu} > e^4. \]
But the latter is indeed true if \( \nu \geq \nu_0 \) where \( \nu_0 \) is a unique root of the equation \( 2\pi\nu(8/e^2)^\nu = e^4 \). It is easy to see that \( \nu_0 < 6 \), so (5.8) is valid for \( \nu \geq 6 \). For \( \nu < 6 \), i.e. \( \nu = 1, 3/2, 2, \ldots, 5, 11/2 \), the inequality (5.8) can be checked numerically. □

The inequalities (5.1) and (5.4) can be rewritten as
\[ \liminf_{p \to \infty} p^{-\delta(m-1)} N_{K}(m, p) \geq 1/\lambda_{K}(m) \] (5.9)
and
\[ \liminf_{p \to \infty} p^{-\delta(m-1)} N_{K}(m, p) \geq \frac{\Gamma(\alpha + 2)\Gamma(\beta + 1)}{\Gamma(\alpha + \beta + 2)} J_{\alpha+1,1}^{\delta(m-1)} \] (5.10)
respectively. By Proposition 5.1 the quotient \( \kappa_{K}(m) \) of the lower bounds (5.9) and (5.10) is less than 1, except for the case \( m = 2, \delta = 1 \). Moreover, \( \kappa_{K}(m) \) exponentially decays as \( m \to \infty \), c.f. [21] for the spherical designs.

Proposition 5.2. Asymptotically,
\[ \kappa_{K}(m) \sim \frac{1}{\pi \delta m} \left(\frac{e}{4}\right)^{\delta(m-1)}, \quad m \to \infty. \] (5.11)

Proof. In the same notation as before we have
\[ \kappa_{K}(m) = \frac{j_{\nu}^{2\nu}}{\Gamma(\nu + 1)^2 \cdot 16^\nu}. \]
Using the relation \( j_{\nu,1} = \nu + O(\sqrt[3]{\nu}) \), \( \nu \to \infty \), (see [20], Section 15.83) and Stirling’s asymptotic formula we obtain
\[
\kappa_K(m) \sim \frac{1}{2\pi \nu} \left( \frac{e}{4} \right)^{2\nu}
\]
that is equivalent to (5.11) since \( 2^{\nu} \sim \delta m \).

\[\square\]

**Remark 5.3.** From (4.10) the asymptotic upper bound
\[
\operatorname{lim sup}_{p \to \infty} p^{-\delta(m-1)} N_K(m,p) \leq 2^{\delta(m-1)} / \lambda_k(m)
\]
follows. We see that there is an exponential gap between (5.12) and (5.10) as \( m \to \infty \). Indeed, the quotient of these bounds is
\[
2^{\delta(m-1)} \kappa_K(m) \sim \frac{1}{\pi \delta m} \left( \frac{e}{2} \right)^{\delta(m-1)}, \quad m \to \infty.
\]

(5.13)

6. The case \( m = 2 \)

In this case we discuss the real, complex and quaternion situation separately.

6.1. \( \mathbb{K} = \mathbb{R} \). Then the inequalities (4.7) and (4.8) are both the equalities, so they coincide. Indeed, \( N_R(2,p) = p/2 + 1 \), according to [16, 18], and, on the other hand, \( \Lambda_R(2,q) = q + 1 = p/2 + 1 \) by (4.9). Furthermore, in the real case (4.7) is equivalent to (4.3). For \( m = 2 \) this yields \( N_R(2,p) \geq \pi/2 \arccos \eta = p/2 + 1 \).

Indeed, in this context \( \eta \) is the largest root of the Gegenbauer polynomial \( C_{p+1}^1(s) = \sin(p+2) \theta / \sin \theta \) where \( \theta = \arccos s \).

6.2. \( \mathbb{K} = \mathbb{C} \). By (4.9)
\[
N_C(2,p) \geq \left[ \left( \frac{p}{4} + 1 \right)^2 \right],
\]
c.f. [11]. On the other hand, our bound (4.5) for \( m = 2 \) is
\[
N_C(2,p) \geq \left[ \frac{2}{1 - \xi_p} \right]
\]
where \( \xi_p \) is the largest root of \( P_{p/2}^{(1,1)}(t) \) and \( |\zeta| \) means the smallest integer \( \geq \zeta \), \( \zeta \in \mathbb{R} \). A numerical evaluation shows that (6.2) coincides with (6.1) for \( p \leq 16 \), but exceeds it for \( 18 \leq p \leq 90 \). Moreover, the difference \( \Delta_C(p) \) between the lower bounds (6.2) and (6.1) is nondecreasing in this range, as we see from the table

\[
\begin{array}{cccccccccccccccc}
| p | & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 & 16 & 18 & 24 & 26 & 28 & 30 & 32 & 34, 36 & 38 & 40 & 42 & 44 & 46 & 48 & 50 & 52 & 54 & 56 & 58 & 60 & 62 & 64 & 66 & 68 & 70 & 72 & 74 & 76 & 78 & 80 & 82 & 84 & 86 & 88 & 90 & 34 & 36 & 38
\end{array}
\]

The table also shows that the “derivative” \( \Delta_C^\prime(p) = \Delta_C(p) - \Delta_C(p-2) \) is nondecreasing (rather slowly).
6.3. \( \mathbb{H} = \mathbb{H} \). We have

\[
N_{\mathbb{H}}(2, p) \geq \frac{1}{3} \left( \left\lceil \frac{p}{2} \right\rceil + 2 \right) \left( \left\lceil \frac{(p+2)/2}{2} \right\rceil + 3 \right)
\]  

(6.3)

from (4.8) and (4.9), but (4.6) yields

\[
N_{\mathbb{H}}(2, p) \geq \left\lceil \frac{4}{(2 + \eta_p)(1 - \eta_p)^2} \right\rceil
\]  

(6.4)

where \( \eta_p \) is the largest root of \( P_{2^{-1, 2}}(t) \).

Comparing (6.4) to (6.3) one can see a small advantage of (6.3) when \( 4 \leq p \leq 20 \), \( p \neq 18 \). Namely, for the difference \( \Delta_H(p) \) between the lower bounds (6.4) and (6.3) we have

| \( p \) | 2 | 4 | 8 | 8 | 10 | 12 | 14 | 16 | 18 | 20 |
|---|---|---|---|---|---|---|---|---|---|---|
| \( \Delta_H(p) \) | 0 | -1 | -1 | -4 | -6 | -6 | -6 | -1 | -1 |

Table 1.

However, for \( p \geq 22 \) this difference increases rather rapidly:

| \( p \) | 22 | 24 | 26 | 28 | 30 | 32 | 34 | 36 | 38 | 40 | 42 | 44 | 46 | 48 | 50 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| \( \Delta_H(p) \) | 12 | 14 | 35 | 42 | 75 | 90 | 138 | 165 | 231 | 274 | 364 | 426 | 544 | 631 | 782 |

Table 2.

Also, an interesting observable phenomenon is a regular oscillation of \( \Delta'_H(p) \) in contrast to the monotonicity of \( \Delta'_C(p) \). Indeed, in both tables 1 and 2 we have

\[
\text{sign } \Delta'_H(p) = (-1)^{p/2+1}.
\]  

(6.5)

for the second difference. This can be conjectured for all \( p \) as well as the results of observations above.

REFERENCES

[1] M. Abramowitz and I. A. Stegun. *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables*, volume 55 of Nat. Bureau of Stand. Appl. Math. Ser. U.S. Gov. Print. Off., Washington, D.C., 1964.

[2] E. Bannai and S. G. Hoggar. On tight \( \ell \)-designs in compact symmetric spaces of rank one. *Proc. Japan Acad. Ser. A Math. Sci.*, 61(3):78–82, 1985.

[3] P. Boyvalenkov. Extremal polynomials for obtaining bounds for spherical codes and designs. *Discrete Comput. Geom.*, 14(2):167–183, 1995.

[4] P. Boyvalenkov and S. Nikova. New lower bounds for some spherical designs. In *Algebraic coding*, volume 781 of Lecture Notes in Comput. Sci., pages 207–216. Springer, Berlin, 1994.

[5] P. Boyvalenkov and S. Nikova. On lower bounds on the size of designs in compact symmetric spaces of rank 1. *Arch. Math.*, 68(1):81–88, 1997.

[6] P. Delsarte, J. M. Goethals, and J. J. Seidel. Spherical codes and designs. *Geom. Dedicate*, 6(3):363–388, 1977.

[7] G. H. Hardy. *Divergent Series*. Clarendon Press, Oxford, 1949.

[8] S. G. Hoggar. Zonal functions and the symplectic group. Preprint, Mat. Inst. Aarhus Univ. 1–22, 1977.

[9] S. G. Hoggar. \( t \)-designs in projective spaces. *Eur. J. Comb.*, 3(3):233–254, 1982.

[10] S. G. Hoggar. \( t \)-designs in Delsarte spaces. In *Coding theory and design theory, Part II*, volume 21 of *IMA Vol. Math. Appl.*, pages 144–165. Springer, New York, 1990.
[11] H. König. Isometric imbeddings of Euclidean spaces into finite-dimensional $l_p$-spaces. volume 34 of *Banach Center Publ.*, pages 79–87. Polish Acad. Sci., Warsaw, 1995.
[12] T. Koornwinder. The addition formula for Jacobi polynomials and spherical harmonics. *SIAM J. Appl. Math.*, 25:236–246, 1973.
[13] V. I. Levenshtein. Designs as maximum codes in polynomial metric spaces. *Acta Appl. Math.*, 29(1-2):1–82, 1992.
[14] Yu. I. Lyubich and O. A. Shatalova. Isometric embeddings of finite-dimensional $l_p$-spaces over the quaternions. *St. Petersburg Math. J.*, 16(1):9–24, 2005.
[15] Yu. I. Lyubich and O. A. Shatalova. Polynomial functions on the classical projective spaces. *Studia Math.*, 170(1):77–87, 2005.
[16] Yu. I. Lyubich and L. N. Vaserstein. Isometric embeddings between classical Banach spaces, cubature formulas, and spherical designs. *Geom. Dedicata*, 47(3):327–362, 1993.
[17] C. Müller. *Spherical Harmonics*, volume 17 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 1966.
[18] B. Reznick. Sums of even powers of real linear forms. *Mem. Amer. Math. Soc.*, 96(463), 1992.
[19] G. Szegö. *Orthogonal Polynomials*. AMS, Providence, R.I., 1975.
[20] G. N. Watson. *A Treatise on the Theory of Bessel Functions*. Cambridge Univ. Press, Cambridge, 1944.
[21] V. A. Yudin. Lower bounds for spherical designs. *Izv. Math.*, 61(3):673–683, 1997.