TWISTED REPRESENTATIONS OF VERTEX OPERATOR ALGEBRAS

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Abstract. Let $V$ be a vertex operator algebra and $g$ an automorphism of finite order. We construct an associative algebra $A_g(V)$ and a pair of functors between the category of $A_g(V)$-modules and a certain category of admissible $g$-twisted $V$-modules. In particular, these functors exhibit a bijection between the simple modules in each category. We give various applications, including the fact that the complete reducibility of admissible $g$-twisted modules implies both the finite-dimensionality of homogeneous spaces and the finiteness of the number of simple $g$-twisted modules.

1. Introduction

The study of vertex operator algebras has come to play a significant role in disparate areas such as conformal field theory [MS], Moonshine and the Monster [FLM], [B] and elliptic cohomology [T]. Although fairly recent in origin, some of the main problems in the theory of vertex operator algebras are quite classical in nature and concern representation theory.

Let $V$ be a vertex operator algebra and $G$ be a finite automorphism group of $V$. Then the space of $G$-invariants $V^G$ is itself a vertex operator algebra. It is natural to try to understand various module categories for $V^G$. This is so-called orbifold theory in the physical literature [DHVW], [DVVV]. One of the main new features of orbifold theory is the introduction of twisted modules or twisted sectors. Essentially, these are spaces which admit vertex operators indexed by elements of $V$ and satisfying analogues of the Jacobi identity which are “twisted” by elements $g \in G$. Moreover they restrict to “ordinary” modules for $V^G$.

Although, for certain orbifolds, some success has been achieved in the study of these objects in the physics literature [DVVV], [DGM], the mathematical investigation of abstract orbifold models has been hampered by a lack of understanding of the theory of twisted representations of vertex operator algebras. The goals of the present paper are to alleviate this situation.

1991 Mathematics Subject Classification. Primary 17B69; Secondary 17B68, 81T40.

Key words and phrases. vertex operator algebras, twisted modules.

C.D is partially supported by NSF grant DMS-9303374 and a research grant from the Committee on Research, UC Santa Cruz.

G.M. is partially supported by NSF grant DMS-9401272 and a research grant from the Committee on Research, UC Santa Cruz.
Given a vertex operator algebra \( V \) and automorphism \( g \) of finite order \( T \), we will construct an associative algebra \( A_g(V) \) with the property that there is a bijective correspondence between simple \( A_g(V) \)-modules and simple admissible \( g \)-twisted \( V \)-modules. These latter objects are twisted analogues of Zhu’s definition of \( V \)-modules \([Z]\). They carry a grading by \( \frac{1}{T} \mathbb{Z} \), but the homogeneous spaces are neither assumed to be of finite dimension, nor induced from the eigenvalues of the \( L(0) \) operator. If these latter conditions hold, then we have an (ordinary) \( g \)-twisted module as defined in \([FFR]\) and \([D1]\). In fact, the main concern of the paper is the construction of a pair of functors \( L, \Omega \) which we display follows:

\[
A_g(V) \rightarrow \text{Mod} \xrightarrow{\Omega} \text{Adm} \leftarrow g \rightarrow V \rightarrow \text{Mod}
\]

Thus the functor \( L \) constructs a certain admissible \( g \)-twisted \( V \)-module \( L(U) \) from a given \( A_g(V) \)-module \( U \), whilst the functor \( \Omega \) does the opposite. Moreover we have \( \Omega \circ L \cong \text{id} \). Furthermore \( L \) and \( \Omega \) induce bijections on the simple objects of each category. Because of the failure of complete reducibility of appropriate modules one cannot expect \( \Omega \) and \( L \) to be mutually inverse categorical equivalences in general, though we are able to prove this (Theorem 7.2) for the full subcategories of completely reducible objects.

There is an important application of our theory to \( g \)-rational vertex operator algebras; these are vertex operator algebras such that every admissible \( g \)-twisted module is completely reducible. We show (Theorem 8.1) that such vertex operator algebras necessarily have only finitely many inequivalent simple admissible \( g \)-twisted modules, and that every such module is an ordinary \( g \)-twisted module. So for \( g \)-rational vertex operator algebras, \( L \) and \( \Omega \) induce mutually inverse categorical equivalences between the categories of finitely generated \( A_g(V) \)-modules and ordinary \( g \)-twisted \( V \)-modules.

We have already alluded to Zhu’s work \([Z]\). Our theory includes that of Zhu if we take \( g = 1 \), but even in this case our work leads to a strengthening of some of his results as well as a simplification in the proofs. One of our main ideas, which goes back to \([L2]\) if \( g = 1 \), is the introduction of a certain Lie algebra \( V[g] \) into the proceedings. This allows us to replace Zhu’s use of correlation functions, which is quite difficult, with more familiar methods of Lie theory (induced modules, PBW theorem).

We have already made use of our results in several papers \([DLM1]-[DLM2], [DM1]-[DM2]\), and expect that the study of \( g \)-twisted modules will lead to a proof of the generalized Moonshine conjectures when applied to the action of the Monster on the Moonshine Module.

The paper is organized as follows: in Section 2 we introduce the algebra \( A_g(V) \). In Section 3 we discuss the various kinds of \( g \)-twisted \( V \)-modules that we need to deal with. In Section 4 we construct the Lie algebra \( V[g] \) and show that a weak \( g \)-twisted \( V \)-module is a \( V[g] \)-module. Then in Section 5 we construct the functor \( \Omega \); it is obtained essentially as the space of lowest weight vectors for \( V[g] \). Section 6 is technically the most difficult. We construct the functor \( L \), which entails the construction of a certain graded \( V[g] \)-module \( L(U) \) from a given \( A_g(V) \)-module \( U \) and then verifying the twisted Jacobi identity. This
is never easy! We also construct (Theorems 6.2 and 6.3) a certain universal object $\tilde{M}(U)$ in the category of admissible $g$-twisted $V$-modules, and which has $L(U)$ as a quotient. Thus $\tilde{M}(U)$ is a sort of “generalized” Verma module. In Section 7 we prove that $L$ and $\Omega$ are equivalences when restricted to the subcategory of completely reducible objects. Section 8 is concerned with $g$-rational vertex operator algebras and includes the results already mentioned. Section 9 contains some useful applications. It should be emphasized that it remains a conjecture that non-zero $g$-twisted $V$-modules always exist; we prove that this is so if $A_g(V)$ is of finite dimension (Theorem 9.1). We also give some sufficient conditions for the complete reducibility of (admissible and ordinary) $V$-modules.

We expect the reader to be familiar with the elementary theory of vertex operator algebras as found, for example, in [FLM] and [FHL].

2. The associative algebra $A_g(V)$

We fix some notation which will be in force throughout the paper. $(V, Y, 1, \omega)$ denotes, as usual, a vertex operator algebra (cf. [B], [FHL] and [FLM]) and $g$ is an automorphism of $V$ of finite order $T$. Denote the decomposition of $V$ into eigenspaces with respect to the action of $g$ as

$$V = \bigoplus_{r \in \mathbb{Z}/T} V^r$$

where $V^r = \{v \in V | gv = e^{2\pi ir/T}v\}$. (We habitually use $r$ to denote both an integer between 0 and $T - 1$ and its residue class mod $T$ in this situation.)

We are going to construct an associative algebra $A_g(V)$ along the line of Zhu’s construction of his algebra $A(V)$ [Z]. Indeed if $g = 1$ our algebra $A_1(V)$ is precisely $A(V)$. In general one may consider Zhu’s algebra $A(V^0)$ associated with the vertex operator subalgebra $V^0$ of $g$-invariants; our algebra $A_g(V)$ will be a certain quotient of $A(V^0)$.

For homogeneous $u \in V^r$ and $v \in V$ we define

$$u \circ_g v = \text{Res}_z \frac{(1 + z)^{wt_u - 1 + \delta_r + \frac{r}{T}}}{z^{1 + \delta_r}} Y(u, z)v$$

where $\delta_r = 1$ if $r = 0$ and $\delta_r = 0$ if $r \neq 0$.

Let $O_g(V)$ be the linear span of all $u \circ_g v$ and define the linear space $A_g(V)$ to be the quotient $V/O_g(V)$. We will usually write $A(V)$, $O(V)$, $u \circ v$, etc. when $g = 1$.

**Lemma 2.1.** If $r \neq 0$ then $V^r \subseteq O_g(V)$.

**Proof:** It suffices to show that $u \in O_g(V)$ whenever $u \in V^r$ is homogeneous. In this case, take $v = 1$ in (2.2) to see that

$$u \circ_g 1 = \text{Res}_z \frac{(1 + z)^{wt_u - 1 + \frac{r}{T}}}{z} Y(u, z)1 = u \in O_g(V).$$

The lemma follows. □

If we set $I = O_g(V) \cap V^0$, it follows from Lemma 2.1 that $A_g(V) \simeq V^0/I$ (as linear spaces). Notice that $O(V^0) \subset I$, so that $A_g(V)$ is a quotient of $A(V^0)$.
Define a second product $*_{g}$ on $V$ as follows: with $r, u$ and $v$ as above, set

$$u *_{g} v = \begin{cases} \text{Res}_{z}(Y(u, z)\frac{(z+1)^{\text{wt} u}}{z}) & \text{if } r = 0 \\ 0 & \text{if } r > 0 \end{cases} \tag{2.3}$$

Extend linearly to obtain a bilinear product on $V$ which coincides with that of Zhu (loc.cit.) on $V^{0}$. We denote the product (2.3) by $u * v$ in this case. In this way $V$ becomes a (non-associative) algebra with respect to $*_{g}$. Note that if $u \in V^{0}$ then (2.3) may be written in the form

$$u *_{g} v = \sum_{i=0}^{\infty} \left( \frac{\text{wt} u}{i} \right) u_{i-1} v \tag{2.4}$$

Following Lemmas 2.1.2 and 2.1.3 of [Z], we get the following.

**Lemma 2.2.** (i) Assume that $u \in V^{r}$ homogeneous, $v \in V$ and $m \geq n \geq 0$. Then

$$\text{Res}_{z}(1 + z)^{\text{wt} u - 1 + \delta_{r} + \frac{m+n}{z^{m+\delta_{r} + 1}}} Y(u, z)v \in O_{g}(V).$$

(ii) Assume that $u, v \in V^{0}$ are homogeneous. Then

$$u * v - \text{Res}_{z}(1 + z)^{\text{wt} v - 1} Y(v, z)u \in O(V^{0})$$

and

(iii) $u * v - v * u - \text{Res}_{z}(1 + z)^{\text{wt} u - 1} Y(u, z)v \in O(V^{0}).$

**Proposition 2.3.** (i) $O_{g}(V)$ is an two-sided ideal of $V$ with respect to the product $*_{g}$. 

(ii) If $I = O_{g}(V) \cap V^{0}$ then $I/O(V^{0})$ is a two-sided ideal of $A(V^{0})$.

**Proof:** Notice that if $r \neq 0$ then $V^{r} *_{g} V^{0} = 0$ by definition (2.3). Similarly (2.3) shows that $V^{0} *_{g} V^{r} \subset V^{r}$. Since $O_{g}(V) = I \oplus (\oplus_{r=1}^{T-1} V^{r})$ by Lemma 2.1, we see that parts (i) and (ii) are equivalent to each other and to the assertion that $I$ is a 2-sided ideal of $V^{0}$ with respect to $*$. We prove this latter assertion.

Choose $c \in V^{0}$ homogeneous and $u \in I$. We must show that $I$ contains both

$$c * u = \text{Res}_{z}(1 + z)^{\text{wt} c} \frac{Y(c, z)}{z}u \tag{2.5}$$

and

$$u * c \equiv \text{Res}_{z}(1 + z)^{\text{wt} c - 1} \frac{Y(c, z)}{z} u \pmod{I}. \tag{2.6}$$

(For the latter congruence use Lemma 2.2 (ii).) From (2.2) it suffices to take $u = a *_{g} b$ where $a \in V^{r}$ and $b \in V^{T-r}$ are both homogeneous. Set $x_{0} = c * u$, $x_{1} = u * c$ and recall
the Jacobi identity on $V$:

$$z_0^{-1}\delta\left(\frac{z_1 - z_2}{z_0}\right)Y(c, z_1)Y(a, z_2)b - z_0^{-1}\delta\left(\frac{z_2 - z_1}{z_0}\right)Y(a, z_2)Y(c, z_1)b$$

$$= z_2^{-1}\delta\left(\frac{z_1 - z_0}{z_2}\right)Y(Y(c, z_0)v, z_2)b.$$  \hspace{1cm} (2.7)

Using (2.7) we have for $\varepsilon = 0$ or $1$:

$$x_{\varepsilon} = \text{Res}_{z_1}\left(1 + z_1\right)^{\text{wt}_{c - \varepsilon}}\frac{\text{Res}_{z_2}}{z_1}\left(1 + z_2\right)^{\text{wt}_{a-1+\delta_r}+\varepsilon}Y(c, z_1)\text{Res}_{z_2}\frac{\text{Res}_{z_2}}{z_1}\left(1 + z_2\right)^{\text{wt}_{a-1+\delta_r}+\varepsilon}Y(a, z_2)b$$

$$= \text{Res}_{z_1}\text{Res}_{z_2}\left(1 + z_1\right)^{\text{wt}_{c - \varepsilon}}\frac{\text{Res}_{z_2}}{z_1}\left(1 + z_2\right)^{\text{wt}_{a-1+\delta_r}+\varepsilon}Y(c, z_1)\text{Res}_{z_2}\frac{\text{Res}_{z_2}}{z_1}\left(1 + z_2\right)^{\text{wt}_{a-1+\delta_r}+\varepsilon}Y(a, z_2)b$$

$$= \text{Res}_{z_1}\text{Res}_{z_2}\left(1 + z_1\right)^{\text{wt}_{c - \varepsilon}}\frac{\text{Res}_{z_2}}{z_1}\left(1 + z_2\right)^{\text{wt}_{a-1+\delta_r}+\varepsilon}Y(c, z_1)\text{Res}_{z_2}\frac{\text{Res}_{z_2}}{z_1}\left(1 + z_2\right)^{\text{wt}_{a-1+\delta_r}+\varepsilon}Y(a, z_2)b + \text{Res}_{z_1}\text{Res}_{z_2}\text{Res}_{z_0}\left(1 + z_1\right)^{\text{wt}_{c - \varepsilon}}\frac{\text{Res}_{z_2}}{z_1}\left(1 + z_2\right)^{\text{wt}_{a-1+\delta_r}+\varepsilon}Y(c, z_1)b$$

$$= \text{Res}_{z_1}\text{Res}_{z_2}\left(1 + z_1\right)^{\text{wt}_{c - \varepsilon}}\frac{\text{Res}_{z_2}}{z_1}\left(1 + z_2\right)^{\text{wt}_{a-1+\delta_r}+\varepsilon}Y(c, z_1)\text{Res}_{z_2}\frac{\text{Res}_{z_2}}{z_1}\left(1 + z_2\right)^{\text{wt}_{a-1+\delta_r}+\varepsilon}Y(a, z_2)b$$

$$+ \sum_{i,j=0}^{\infty} (-1)^i\left(\frac{\text{wt}_{c - \varepsilon}}{i}\right)\text{Res}_{z_2}\frac{\text{Res}_{z_2}}{z_1}\left(1 + z_2\right)^{\text{wt}_{a-1+\delta_r}+\varepsilon + \text{wt}_{c - \varepsilon}}Y(c_{i+j}a, z_2)b$$

$$= \text{Res}_{z_2}\left(1 + z_2\right)^{\text{wt}_{a-1+\delta_r}+\varepsilon}Y(a, z_2)\text{Res}_{z_2}\frac{\text{Res}_{z_2}}{z_1}\left(1 + z_2\right)^{\text{wt}_{a-1+\delta_r}+\varepsilon}Y(c, z_1)b$$

$$+ \sum_{i,j=0}^{\infty} (-1)^i\left(\frac{\text{wt}_{c - \varepsilon}}{i}\right)\text{Res}_{z_2}\frac{\text{Res}_{z_2}}{z_1}\left(1 + z_2\right)^{\text{wt}(c_{i+j}a)-1+\delta_r+\varepsilon+j+1-\varepsilon}Y(c_{i+j}a, z_2)b.$$ 

The resulting element is in $I$ by the definition of $O_g(V)$ and Lemma 2.2 (i). The proof is complete. □

Our first main result is the following.

**Theorem 2.4.** (i) The product $*_g$ induces the structure of an associative algebra on $A_g(V)$.

(ii) The linear map

$$\phi : a \mapsto e^{L(1)}(-1)^{L(0)}a$$

induces an anti-isomorphism $A_g(V) \rightarrow A_{g-1}(V)$. 
There are identities
\[
1 \ast_g x \equiv x \ast_g 1 \equiv x \pmod{O_g(V)}
\]
\[
\omega \ast_g x \equiv x \ast_g \omega \pmod{O_g(V)}
\]
for \(x \in V\).

Remark 2.5. We have not ruled out the possibility that \(A_g(V)\) is equal to 0, indeed this is a subtle point related to the existence/non-existence of \(g\)-twisted sectors, as we shall see below. Part (iii) says that the vacuum \(1\) maps onto the identity of \(A_g(V)\) as long as \(A_g(V) \neq 0\). Similarly the image of the Virasoro vector \(\omega\) lies in the center of \(A_g(V)\).

Proof of Theorem 2.4: Part (i) follows immediately from Proposition 2.3 and Zhu’s results [Z] that \(A(V^0)\) is an associative algebra with respect to \(\ast\). Similarly part (iii) follows from Theorem 2.1.1 (2), (3) of [Z].

Since part (ii) follows for \(g = 1\) from [Z] (cf. equation (2.1.9) of that paper, though no proof is given) the main point is to show that \(\phi\) maps \(O_g(V) \cap V^0\) into \(O_{g^{-1}}(V) \cap V^0\).

First recall the following conjugation formulas from [FHL]:
\[
z^{L(0)}Y(a, z_0)z^{-L(0)} = Y(z^{L(0)}a, z_0),
\]
\[
e^{zL(1)}Y(a, z_0)e^{-zL(1)} = Y\left(e^{z(1-zz_0)L(1)}(1-zz_0)^{-2L(0)}a, z_0\right).
\]

Then for homogeneous \(a \in V^r\) and \(b \in V^{T-r}\) we have:
\[
\phi(a \circ_g b) = \phi\left(\text{Res}_z \frac{(1+z)^{wt a - 1 + \delta_r + \frac{T}{2}}}{z^{1+\delta_r}} Y(a, z)b\right)
\]
\[
= e^{L(1)(-1)L(0)}\text{Res}_z \frac{(1+z)^{wt a - 1 + \delta_r + \frac{T}{2}}}{z^{1+\delta_r}} Y(a, z)b
\]
\[
= \text{Res}_z \frac{(1+z)^{wt a - 1 + \delta_r + \frac{T}{2}}}{z^{1+\delta_r}} e^{L(1)}Y((-1)L(0)a, -z)(-1)L(0)b
\]
\[
= \text{Res}_z \frac{(1+z)^{wt a - 1 + \delta_r + \frac{T}{2}}}{z^{1+\delta_r}} Y\left(e^{(1+z)L(1)}(1+z)^{-2L(0)}(-1)L(0)a, -z\right) e^{L(1)(-1)L(0)b}.
\]

Replacing \(z\) with \(-\frac{z_0}{1+z_0}\) and using the residue formula for the change of variable (see [Z])
\[
\text{Res}_z g(z) = \text{Res}_{z_0}(g(f(z_0)) \frac{d}{dz_0} f(z_0))
\]
gives
\[ \phi(a \circ b) \]
\[ = -\text{Res}_{z_0}(1 + z_0)^{-w_{t-a}+1-\Delta_r - \frac{1}{z_0}} - \frac{1}{(1 + z_0)^2} \cdot Y(e^{(1+z_0)^{-1}L(1)}(1 + z_0)^2L(0)(-1)L(0)a, z_0) e^{L(1)}(-1)L(0)b \]
\[ = (-1)^w_{t-a} \text{Res}_{z_0} \frac{(1 + z_0)^{w_{t-a}}}{z_0^{1+\Delta_r}} Y(e^{(1+z_0)^{-1}L(1)}a, z_0) e^{L(1)}(-1)L(0)b \]
\[ = \sum_{j=0}^{\infty} \frac{1}{j!} (-1)^w_{t-a} \text{Res}_{z_0} \frac{(1 + z_0)^{w_{t-a-j}}}{z_0^{1+\Delta_r}} Y(L(1)^{j}a, z_0) e^{L(1)}(-1)L(0)b. \]
By considering separately the cases \( r = 0 \) and \( r \neq 0 \), we see that the latter sum lies in \( O_{g^{-1}}(V) \). Thus \( \phi \) maps \( O_g(V) \) to \( O_{g^{-1}}(V) \). Since \( \phi^2 = 1 \) on \( A(V^0) \), it is clear that \( \phi \) maps \( O_g(V) \) onto \( O_{g^{-1}}(V) \). \( \square \)

3. Twisted modules

We discuss the category of weak \( g \)-twisted \( V \)-modules (cf. [D1] and [FFR]). As before \( V \)
is a vertex operator algebra with automorphism \( g \) of order \( T \) and eigenspace decomposition (2.1). We adopt standard notation in using \( W\{z\} \) to denote the space of \( W \)-valued formal series in arbitrary real powers of \( z \) for a vector space \( W \).

**Definition 3.1.** A weak \( g \)-twisted \( V \)-module \( M \) is a vector space equipped with a linear map
\[ V \rightarrow (\text{End } M)\{z\} \]
\[ v \mapsto Y_M(v, z) = \sum_{n \in \mathbb{Q}} v_n z^{-n-1} \quad (v_n \in \text{End } M) \]
which satisfies the following for all \( 0 \leq r \leq T - 1, u \in V^r, v, w \in V, w \in M, \)
\[ Y_M(u, z) = \sum_{n \in \mathbb{Q} + z} u_n z^{-n-1} \quad (3.1) \]
\[ v_l w = 0 \quad \text{for} \quad l >> 0 \quad (3.2) \]
\[ Y_M(1, z) = 1; \quad (3.3) \]
\[ z_0^{-1} \delta \left( \frac{z_1 - z_2}{z_0} \right) Y_M(u, z_1)Y_M(v, z_2) - z_0^{-1} \delta \left( \frac{z_2 - z_1}{z_0} \right) Y_M(v, z_2)Y_M(u, z_1) \]
\[ = z_2^{-1} \delta \left( \frac{z_1 - z_0}{z_2} \right)^{-r/T} \delta \left( \frac{z_1 - z_0}{z_2} \right) Y_M(Y(u, z_0)v, z_2). \quad (3.4) \]
If \( g = 1 \) this reduces to the definition of weak \( V \)-module as given in [DLM1].

One calls (3.4) the twisted Jacobi identity, and again it reduces to the “untwisted” Jacobi identity for \( V \)-modules if \( g = 1 \). Following the arguments in the untwisted case (cf. [DL], [FHL], [FLM]) one can prove that the twisted Jacobi identity is equivalent to the following associativity and commutativity formulas:

\[
(z_0 + z_2)^{k+\frac{r}{T}} Y_M(u, z_0 + z_2)Y_M(v, z_2)w = (z_2 + z_0)^{k+\frac{r}{T}} Y_M(Y(u, z_0)v, z_2)w.
\] (3.5)

where \( w \in M \) and \( k \) is a nonnegative integer such that \( z^{k+\frac{r}{T}} Y_M(u, z)w \) involves only positive powers of \( z \);

\[
[Y_M(u, z_1), Y_M(v, z_2)] = \text{Res}_{z_0} z_0^{-1} \left( \frac{z_1 - z_0}{z_2} \right)^{-r/T} \delta \left( \frac{z_1 - z_0}{z_2} \right) Y_M(Y(u, z_0)v, z_2).
\] (3.6)

Equating the coefficients of \( z_1^{-m-1} z_2^{-n-1} \) in (3.6) yields

\[
[u_m, v_n] = \sum_{i=0}^{\infty} \binom{m}{i} (u_i v)_{m+n-i}.
\] (3.7)

As in the untwisted case [DLM1], we may also deduce from (3.4) the usual Virasoro algebra axioms, namely that if \( Y_M(\omega, z) = \sum_{n \in \mathbb{Z}} L(n) z^{-n-2} \) then

\[
[L(m), L(n)] = (m - n)L(m + n) + \frac{1}{12}(m^3 - m)\delta_{m+n,0} \text{rank } V
\] (3.8)

and

\[
\frac{d}{dz} Y_M(v, z) = Y_M(L(-1)v, z).
\] (3.9)

The category of weak \( g \)-twisted \( V \)-modules has as its objects the weak \( g \)-twisted \( V \)-modules and as morphisms those linear maps \( f : M \to W \) such that

\[
f Y_M(u, z) = Y_W(u, z)f
\] (3.10)

for all \( u \in V \).

**Definition 3.2.** A \( g \)-twisted \( V \)-module is a weak \( g \)-twisted \( V \)-module \( M \) which carries a \( \mathbb{C} \)-grading induced by the spectrum of \( L(0) \). That is, we have

\[
M = \coprod_{\lambda \in \mathbb{C}} M_\lambda
\] (3.11)

where \( M_\lambda = \{ w \in M | L(0)w = \lambda w \} \). Moreover we require that \( \dim M_\lambda \) is finite and for fixed \( \lambda \), \( M_{\frac{r}{T}+\lambda} = 0 \) for all small enough integers \( n \).

In this situation, if \( w \in M_\lambda \) we refer to \( \lambda \) as the weight of \( w \) and write \( \lambda = \text{wt } w \). If \( g = 1 \) then this defines a \( V \)-module as used in [DLM1] and elsewhere. The totality of \( g \)-twisted \( V \)-modules defines a full subcategory of the category of weak \( g \)-twisted \( V \)-modules.

An important and related class of modules are the following.
Definition 3.3. An admissible $g$-twisted $V$-module is a weak $g$-twisted $V$-module $M$ which carries a $\frac{1}{T}\mathbb{Z}_+$-grading

\[ M = \bigoplus_{n \in \frac{1}{T}\mathbb{Z}} M(n) \]  

(3.12)

which satisfies the following

\[ v_m M(n) \subseteq M(n + \text{wt}v - m - 1) \]  

(3.13)

for homogeneous $v \in V$.

The admissible $g$-twisted $V$-modules form a category with morphisms being grade-preserving linear maps satisfying (3.10). Thus a simple object in this category is an admissible $g$-twisted $V$-module $M$ such that $0$ and $M$ are the only graded submodules.

We say that $V$ is $g$-rational if every admissible $g$-twisted $V$-module is completely reducible, i.e., a direct sum of simple admissible $g$-twisted modules.

If $g = 1$, these definitions reduce to the “untwisted” version used in [DLM1].

Lemma 3.4. There is a natural identification of the category of $g$-twisted $V$-modules with a subcategory of the category of admissible $g$-twisted $V$-modules.

Proof: Let $M$ be a $g$-twisted $V$-module with decomposition into $L(0)$-eigenspaces given by (3.11). For each $\lambda \in \mathbb{C}$ for which $M_\lambda \neq 0$, let $\lambda_0$ be the minimal element of the set $\lambda + \frac{1}{T}\mathbb{Z}$ for which $M_{\lambda_0} \neq 0$. Note that $\lambda_0$ exists by definition 3.2. Let $\Lambda$ be the set of all $\lambda_0$ so obtained, and for each $n \in \frac{1}{T}\mathbb{Z}_+$ define

\[ M(n) = \bigsqcup_{\lambda \in \Lambda} M_{n + \lambda}. \]  

(3.14)

It is clear that $M = \bigoplus_n M(n)$, while (3.13) follows from the standard fact that $v_m M_\lambda \subset M_{\lambda + \text{wt}v - m - 1}$.

In this way we have identified $M$ as an admissible $g$-twisted $V$-module. Moreover as a morphism $f$ in the category of $g$-twisted $V$-modules satisfies (3.10) then it preserves $L(0)$-eigenspaces and hence also the grading (3.14). The lemma follows. □

Remark 3.5. We will establish later the less obvious fact that if $V$ is $g$-rational then the two categories of Lemma 3.4 share the same simple objects.

Lemma 3.6. $M$ is a simple weak $g$-twisted $V$-module. Then $M$ has countable dimension.

Proof: One knows (Proposition 2.4 of [DM2] or Lemma 6.1.1 of [L2]) that

\[ M = \text{span}\{a_n u | a \in V, n \in \frac{1}{T}\mathbb{Z}_+\} \]

for any non-zero $u \in M$. Since $V$ has a countable basis, the lemma follows. □

This lemma is useful in the study of contragredient modules, which we now discuss. If $M = \bigoplus_{n \in \frac{1}{T}\mathbb{Z}_+} M(n)$ is an admissible $g$-twisted $V$-module, the contragredient module $M'$ is defined as follows:

\[ M' = \bigoplus_{n \in \frac{1}{T}\mathbb{Z}_+} M(n)^* \]  

(3.15)
where $M(n)^* = \text{Hom}_{\mathbb{C}}(M(n), \mathbb{C})$. The vertex operator $Y_{M'}(a, z)$ is defined for $a \in V$ via

$$
(Y_{M'}(a, z)f, u) = \langle f, Y_M(e^{zL(1)}(-z^{-2})L(0)a, z^{-1})u \rangle
$$

(3.16)

One can prove (cf. [FHL], [X]) the following:

**Lemma 3.7.** $(M', Y_{M'})$ is an admissible $g^{-1}$-twisted $V$-module.

4. **The Lie algebra $V[g]$**

$V$ continues to be a vertex operator algebra with automorphism $g$ of order $T$ and eigenspace decomposition (2.1). Let $t$ be an indeterminate, and consider the tensor product

$$
\mathcal{L}(V) = \mathbb{C}[t^\frac{1}{T}, t^{-\frac{1}{T}}] \otimes V.
$$

(4.1)

Following [B] we give $\mathbb{C}[t^\frac{1}{T}, t^{-\frac{1}{T}}]$ the structure of vertex algebra with vertex operator

$$
Y(f(t), z)g(t) = f(t + z)g(t) = \left( e^{z \frac{d}{dt}} f(t) \right) g(t).
$$

(4.2)

Then $\mathcal{L}(V)$ becomes a tensor product of vertex algebras and hence itself a vertex algebra (cf. [DL], [FHL] and [L2]).

The action of $g$ naturally extends to that of a vertex algebra automorphism

$$
g(t^m \otimes a) = \exp\left( \frac{-2\pi im}{T} \right)(t^m \otimes ga).
$$

(4.3)

Denote the space of $g$-invariants of this action by $\mathcal{L}(V, g)$; it is a vertex subalgebra of $\mathcal{L}(V)$. Of course we have

$$
\mathcal{L}(V, g) = \bigoplus_{r=0}^{T-1} t^r/T \mathbb{C}[t, t^{-1}] \otimes V^r.
$$

(4.4)

The $L(-1)$ operator of $\mathcal{L}(V)$ and $\mathcal{L}(V, g)$ is given by $D = \frac{d}{dt} \otimes 1 + 1 \otimes L(-1)$, and as a consequence one knows [B] that

$$
V[g] = \mathcal{L}(V, g)/D\mathcal{L}(V, g)
$$

(4.5)

carries the structure of Lie algebra with bracket

$$
[u + D\mathcal{L}(V, g), v + D\mathcal{L}(V, g)] = u_0v + D\mathcal{L}(V, g).
$$

(4.6)

As a matter of notation we use $a(q)$ to denote the image of $t^q \otimes a \in \mathcal{L}(V, g)$ in $V[g]$. An easy computation from the definitions yields

**Lemma 4.1.** Let $a \in V^r$, $v \in V^s$ and $m, n \in \mathbb{Z}$. Then

(i) $[\omega(0), a(m + \frac{r}{s})] = -\left( m + \frac{r}{s} \right) a(m - 1 + \frac{r}{s})$,

(ii) $[a(m + \frac{r}{s}), b(n + \frac{s}{r})] = \sum_{i=0}^{\infty} \binom{m+s}{i} a_i b(m+n+s-r-i)$,

(iii) $\omega(0)$ and $1(-1)$ both lie in the center of $V[g]$. 

We can introduce a $\frac{1}{2}\mathbb{Z}$-gradation on $\mathcal{L}(V)$ by defining, for homogeneous $a \in V$,
\[
\deg(t^n \otimes a) = wt a - n - 1.
\] (4.7)

As $D$ increases degree by 1 then $D\mathcal{L}(V,g)$ is a graded subspace of $\mathcal{L}(V,g)$, so that there is a naturally induced $\frac{1}{2}\mathbb{Z}$-gradation on $V[g]$. Let $V[g]_n$ denote the degree $n$ subspace of $V[g]$. After Lemma 4.1 $V[g]$ is in fact a $\frac{1}{2}\mathbb{Z}$-graded Lie algebra and we have a triangular decomposition
\[
V[g] = V[g]_+ \oplus V[g]_0 \oplus V[g]_-,
\] (4.8)
where we have set $V[g]_{\pm} = \sum_{0 < n \in \frac{1}{2}\mathbb{Z}} V[g]_{\pm n}$.

Note that $V[g]_0$ is spanned by elements of the form $a(wt a - 1)$ for homogeneous $a \in V^0$. The bracket is given by
\[
[a(wt a - 1), b(wt b - 1)] = \sum_{j=0}^{\infty} \binom{wt a - 1}{j} a_j b(wt(a_j b) - 1).
\] (4.9)
as we see from Lemma 4.1 (ii).

Following [Z] we set $o(a) = a(wt a - 1)$ for homogeneous $a \in V^0$. So we have a linear map
\[
V^0 \to V[g]_0, \\
a \mapsto o(a) \ (a \text{ homogeneous}).
\] (4.10)
The kernel of the map is precisely $(L(-1) + L(0))V^0$, and (4.10) induces an isomorphism of Lie algebras $V^0/(L(-1) + L(0))V^0 \cong V[g]_0$ where the bracket on the quotient of $V^0$ is as described via
\[
[a, b] = \sum_{j \geq 0} \binom{wt a - 1}{j} a_j b.
\]

**Lemma 4.2.** Let $A_g(V)_{\text{Lie}}$ be the Lie algebra of the associative algebra $A_g(V)$ (cf. Section 2), so that $[u, v] = u \ast_g v - v \ast_g u$. Then the map $o(a) \mapsto a + O_g(V)$ is a Lie algebra epimorphism $V[g]_0 \to A_g(V)_{\text{Lie}}$.

**Proof:** Let $I = O_g(V) \cap V^0$. We have from Lemma 2.1.1 of [Z] that $(L(-1) + L(0))V^0 \subset O(V^0) \subset I$, so we have surjective linear maps
\[
V[g]_0 \cong V^0/(L(-1) + L(0))V^0 \to A(V^0) \to V^0/I \cong A_g(V).
\] (4.11)
Use Lemma 2.2 (ii) to see that if $a, b \in V^0$ are homogeneous then
\[
a \ast b - b \ast a \equiv \text{Res}_z(1 + z)^{wt a - 1} Y(a, z)b = \sum_{i=0}^{\infty} \binom{wt a - 1}{i} a_i b \pmod{O(V^0)}.
\]
In view of the results following (4.10) this says that the map from $V^0/(L(-1) + L(0))V^0$ to $A(V^0)$ in (4.11) is a Lie algebra morphism, and the lemma follows. \(\square\)
Remark 4.3. Observe that $V[g]$ contains $V^0[1] = \mathbb{C}[t, t^{-1}] \otimes V^0/D(\mathbb{C}[t, t^{-1}] \otimes V^0)$ as a graded Lie subalgebra.

5. The functor $\Omega$

In this section we construct a functor $\Omega$ from the category of admissible $g$-twisted $V$-modules to the category of $A_g(V)$-modules. We retain previous notation.

The connection between the Lie algebra $V[g]$ and weak $g$-twisted modules is the following:

Lemma 5.1. Let $M$ be a weak $g$-twisted $V$-module. The map $a(m) \mapsto a_m$ defines a representation of the Lie algebra $V[g]$ on $M$.

Proof: The bracket of elements $a(m), b(n)$ in $V[g]$ $(m, n \in \frac{1}{T} \mathbb{Z})$ is given in Lemma 4.1 (ii), and that of $a_m, b_n$ is given in (3.7). Comparing, we see that it suffices to show that the map $a(m) \mapsto a_m$ is well-defined. Let $t^m \otimes a \in L(V, g)$. Then $D(t^m \otimes a) = mt^{m-1} \otimes a + t^m \otimes L(-1)a \mapsto ma_{m-1} + (L(-1)a)_m = 0$, the latter equality by (3.9). □

Lemma 5.2. Let $M$ be a weak $g$-twisted $V$-module which carries a $\frac{1}{T} \mathbb{Z}_+^+$-grading. Then $M$ is an admissible $g$-twisted $V$-module if, and only if, $M$ is a $\frac{1}{T} \mathbb{Z}_+^+$-graded module for the grade Lie algebra $V[g]$.

Proof: Let $a \in V$ be homogeneous and $a(m) \in V[g]$ with $M(n)$ the $n$-th graded piece of $M$ (cf. (3.12)). The condition that $M$ is graded module for $V[g]$ is this: $a(m)M(n) \subset M(n + \deg a(m)) = M(n + wta - m - 1)$. Using the representation described in Lemma 5.1, this is precisely the condition (ii) of Lemma 4.1 required to make $M$ an admissible $g$-twisted module. □

Recalling the decomposition (4.8) of $V[g]$, consider a module $W$ for the Lie algebra $V[g]$. We let $\Omega(W)$ denote the space of “lowest weight vectors,” that is

$$\Omega(W) = \{u \in W|V[g]_-_u = 0\}. \quad (5.1)$$

Similarly, for a $V^0[1]$-module $W_0$ (cf. Remark 4.3) we set

$$\Omega^0(W_0) = \{u \in W_0|V^0[1]_-_u = 0\}. \quad (5.2)$$

Then $\Omega(W)$ and $\Omega^0(W_0)$ are modules for the Lie algebras $V[g]_0$ and $V^0[1]_0$ respectively. Furthermore it is obvious from Remark 4.3 that we have $\Omega(W) \subset \Omega^0(W)$ for a $V[g]$-module $W$.

Theorem 5.3. Suppose that $M$ is weak $g$-twisted $V$-module. Then there is a representation of the associative algebra $A_g(V)$ on $\Omega(M)$ induced by the map $a \mapsto o(a)$ for homogeneous $a \in V^0$ (cf. (3.10)).

Proof: We start by remarking that if $g = 1$ then this result has been established by Zhu [Z]. Although he works in a less general situation, one easily verifies that his proof goes through in the present situation.
We make use of this as follows: the theorem is correct as applied to the action of $A(V^0)$ on $\Omega^0(M)$. So we are reduced to proving that $\Omega(M)$ is an $A(V^0)$-submodule of $\Omega^0(M)$ on which $O_g(V) \cap V^0$ acts trivially.

To show that $\Omega(M)$ is $A(V^0)$-stable, pick $u \in \Omega(M)$ and homogeneous $a \in V^0$. We must show that $o(a)u$ is annihilated by all $b(n) \in V[g]$ where $b \in V$ is homogeneous and $\deg b(n) < 0$, that is, $b_n o(a)u = 0$.

We have, using (3.7),

$$b_n o(a)u = b_n a_{wt_{a-1}}u = a_{wt_{a-1}}b_n u + \sum_{i=0}^{\infty} \binom{n}{i} (b_n)_{m+n-i-1}u.$$  

Now $b_n u = 0$ since $b(n) \in V[g]_-$ annihilates $\Omega(M)$. And since $b(a(n + wt_a - i - 1)$ has degree equal to $(wtb + wt_a - i - 2) - (n + wt_a - i - 1) = wt_b - n - 1 = \deg b(n) < 0$ then each term $(b_n)_{wt_{a+n-i-1}}u = 0$. So indeed $\Omega(M)$ is an $A(V^0)$-module.

It remains to prove that for any $a \in O_g(V) \cap V^0$, $o(a)$ acts as zero on $\Omega(M)$. If $a \in O(V^0)$ then $o(a) = 0$ since $\Omega(M)$ is an $A(V^0)$-module. Suppose, then, that

$$a = \Res_z (1 + z)^{wt_{c-1} + \frac{r}{T}} Y(u, z)v$$

with $u \in V^r$, $v \in V^{T-r}$, $1 \leq r \leq T - 1$. For any $w \in \Omega(M)$, using a property of the delta-function we can rewrite the Jacobi identity (3.4) as follows:

$$z_1^{-1} \delta \left( \frac{z_0 + z_2}{z_1} \right) Y_M(u, z_1) Y_M(v, z_2) w - z_0^{-1} \delta \left( \frac{z_2 - z_1}{z_0} \right) Y_M(v, z_2) Y_M(u, z_1) w$$

$$= z_1^{-1} \left( \frac{z_2 + z_0}{z_1} \right)^{r/T} \delta \left( \frac{z_2 + z_0}{z_1} \right) Y_M(Y(u, z_0) v, z_2) w.$$  

(5.3)

Apply $\Res_{z_1} z_1^{wt_{u-1} + \frac{r}{T}}$ to (5.3) to obtain the following twisted associativity:

$$\Res_{z_1} z_1^{wt_{u-1} + \frac{r}{T}} z_1^{-1} \delta \left( \frac{z_0 + z_2}{z_1} \right) Y_M(u, z_1) Y_M(v, z_2) w$$

$$= (z_2 + z_0)^{wt_{u-1} + \frac{r}{T}} Y_M(Y(u, z_0) v, z_2) w.$$  

(5.4)

Take $\Res_{z_0} \Res_{z_2} z_0^{-1} z_2^{wt_{v-1} + \frac{r}{T}} (z_2 + z_0)^{wt_{u-1} + \frac{r}{T}} Y_M(Y(u, z_0) v, z_2) w$ of (5.4) to get

$$0 = \Res_{z_0} \Res_{z_2} z_0^{-1} z_2^{wt_{v-1} + \frac{r}{T}} (z_2 + z_0)^{wt_{u-1} + \frac{r}{T}} Y_M(Y(u, z_0) v, z_2) w$$

$$= \sum_{i=0}^{\infty} \binom{wt_{u-1} + \frac{r}{T}}{i} \Res_{z_2} z_2^{wt_{u+v-i} - 1} Y_M(u_{i-1} v, z_2) w$$

$$= \sum_{i=0}^{\infty} \binom{wt_{u-1} + \frac{r}{T}}{i} o(u_{i-1} v) w$$

$$= o \left( \Res_z (1 + z)^{wt_{u-1} + \frac{r}{T}} Y_M(u, z) v \right) w$$

(5.5)
Because our constructions are natural, it is evident that $\Omega$ is a covariant functor from the category of weak $g$-twisted $V$-modules to the category of $A_g(V)$-modules. To be more precise, if $f : M \to N$ is a morphism in the first category (cf. (3.10) we define $\Omega(f)$ to be the restriction of $f$ to $\Omega(M)$. With an obvious notation, (3.10) says that $fa_m^M = a_m^N f$ for $a \in V$ and $m \in \frac{1}{2} \mathbb{Z}$. Then $f$ induces a morphism of $V[g]$-modules $M \to N$ by Lemma 5.1. Moreover $\Omega(f) : \Omega(M) \to \Omega(N)$. Now Theorem 5.3 implies that $\Omega(f)$ is a morphism of $A_g(V)$-modules.

We turn to a consideration of admissible $g$-twisted $V$-modules in this context. Let $M$ be such a module. As long as $M \neq 0$, then some $M(n) \neq 0$ (cf. (3.12), and it is no loss to shift the grading so that in fact $M(0) \neq 0$. If $M = 0$, let $M(0) = 0$. With these conventions we prove

**Proposition 5.4.** Suppose that $M$ is a simple admissible $g$-twisted $V$-module. Then the following hold

(i) $\Omega(M) = M(0)$.

(ii) $\Omega(M)$ is a simple $A_g(V)$-module.

**Proof:** Note that Lemma 5.2 is available in this situation. An easy argument shows that $\Omega(M)$ is a graded subspace of $M$. That is

$$\Omega(M) = \oplus_{n \in \frac{1}{2} \mathbb{Z}} \Omega(M) \cap M(n). \quad (5.6)$$

Set $\Omega(n) = \Omega(M) \cap M(n)$. It is clear that $M(0) \subset \Omega(M)$. In order to prove (i) we must show that $\Omega(n) = 0$ if $n > 0$. We use the PBW theorem to do this.

Let $U(\cdot)$ denote universal enveloping algebra. If $\Omega(n) \neq 0$ then because $M$ is simple we have

$$M = U(V[g]) \Omega(n) = U(V[g]_+) \Omega(n), \quad (5.7)$$

the latter equality thanks to the triangular decomposition of $V[g]$ (1.8). Equation (5.7) tells us that the lowest degree of $M$ is no less than $n$, so we must have $n = 0$ by our convention.

To prove (ii) let $U$ be any nonzero $A_g(V)$-submodule of $M(0)$. Then $U$ is annihilated by $V[g]_-$ and stable under $V[g]_0$ (Lemma 4.2 and Theorem 5.3). So again the PBW theorem yields

$$M = U(V[g])U = U(V[g]_+)U = U \oplus U(V[g]_+)V[g]_+U. \quad (5.8)$$

This implies that $U = M(0)$, and (ii) follows. $\blacksquare$

6. **Generalized Verma modules and the functor $L$**

We consider the possibility of constructing admissible $g$-twisted $V$-modules from a given $A_g(V)$-module $U$, say. We show that there is a universal way to do this. Moreover a certain quotient of the universal object is an admissible $g$-twisted $V$-module $L(U)$ and $L$ defines a functor which is right inverse to the functor $\Omega$. Notation is as before.
Given the $A_g(V)$-module $U$, it is a fortiori a module for $A_g(V)_\text{Lie}$. Thanks to Lemma 4.2 we can lift $U$ to a module for the Lie algebra $V[g]_0$, and then to one for $V[g]_- \oplus V[g]_0$ by letting $V[g]_-$ act trivially. Set $P = V[g]_- \oplus V[g]_0$ and define

$$M(U) = \text{Ind}_{U}^{V[g]}(U) = U(V[g]) \otimes_{U(P)} U.$$ 

If we give $U$ degree 0, the $1/p\mathbb{Z}$-gradation of $V[g]$ lifts to $M(U)$ which thus becomes a $1/p\mathbb{Z}_+$-graded module for $V[g]$. The PBW theorem implies that $M(U)(n) = U(V[g]_+)_nU$ and in particular $M(U)(0) = U$.

Taking our cue from Lemma 5.1, we define for $v \in V^r$,

$$Y_{M(U)}(v,z) = \sum_{m \in \mathbb{P}^2} v(m)z^{-1-m}$$ (6.2)

where we convene that $v(m) = 0$ unless $v(m) = t^m \otimes v$ lies in $V[g]$. Then $Y_{M(U)}(v,z)$ satisfies condition (5.1). Moreover (5.2) and (5.3) are easily confirmed.

Next, thanks to Lemma 4.1 (ii), we see that the identity (5.7) holds. Thus in order to establish the $g$-twisted Jacobi identity for the action (6.2) on $M(U)$ it would be enough to also establish (5.9). In general, however, this is false. Instead we have to divide out by the desired relations.

Precisely, let $W$ be the subspace of $M(U)$ spanned linearly by the coefficients of

$$(z_0 + z_2)^{w_{ta} - 1 + \delta + \frac{r}{2}}Y(a, z_0 + z_2)Y(b, z_2)u - (z_2 + z_0)^{w_{ta} - 1 + \delta + \frac{r}{2}}Y(Y(a, z_0)b, z_2)u$$ (6.3)

for any homogeneous $a \in V^r, b \in V, u \in U$. We set

$$\hat{M}(U) = M(U)/U(V[g])W.$$ (6.4)

In order to prove the first main result of this section we need the following proposition.

**Proposition 6.1.** Let $M$ be a $V[g]$-module such that there is a subspace $U$ of $M$ satisfying the following conditions:

(i) $M = U(V[g])U$;

(ii) For any $a \in V^r$ and $u \in U$ there is a positive integer $k$ such that

$$(z_0 + z_2)^{k + \frac{r}{2}}Y(a, z_0 + z_2)Y(b, z_2)u = (z_0 + z_2)^{k + \frac{r}{2}}Y(Y(a, z_0)b, z_2)u$$ (6.5)

for any $b \in V$. Then $M$ is a weak $V$-module.

**Proof.** It suffices to show that (5.5) holds for all $u \in M$. Let $X$ consist of those $u \in M$ such that for any $a \in V^r$ the associativity (5.3) holds for any $b \in V$. We must show that $X = M$. Since $X$ contains $U$ by (ii) and $M$ is generated by $U$ it is equivalent to show that $V[g]X \subset X$.

Let $u \in X, c \in V$ and $n \in \frac{1}{p\mathbb{Z}}$. Let $k_1$ be a positive integer such that $c_i a = 0$ for $i \geq k_1$. Since $u \in X$, there is a positive integer $k_2$ such that

$$(z_0 + z_2)^{k_2 + \frac{r}{2}}Y(c_i a, z_0 + z_2)Y(b, z_2)u = (z_2 + z_0)^{k_2 + \frac{r}{2}}Y(Y(c_i a, z_0)b, z_2)u,$$ (6.6)

$$(z_0 + z_2)^{k_2 + \frac{r}{2}}Y(a, z_0 + z_2)Y(c_i b, z_2)u = (z_2 + z_0)^{k_2 + \frac{r}{2}}Y(Y(a, z_0)c_i b, z_2)u$$ (6.7)
for any nonnegative integer $i$. Let $k$ be a positive integer such that $k + \frac{r}{t} + n - k_1 > k_2 + \frac{r + s}{t}$. Use (6.6) and (6.7) and the equality (a consequence of (3.7))

$$[a_m, Y(b, z_2)] = \sum_{i=0}^{\infty} \binom{m}{i} z_2^{m-i} Y(a_i, z_2)$$

(6.8)
to obtain

\[
(z_0 + z_2)^{k+\frac{\hat{P}}{2}}Y(a, z_0 + z_2)Y(b, z_2)c_n u \\
= (z_0 + z_2)^{k+\frac{\hat{P}}{2}}c_n Y(a, z_0 + z_2)Y(b, z_2) u \\
- \sum_{i=0}^{\infty} \binom{n}{i} (z_0 + z_2)^{k+\frac{\hat{P}}{2}+n-i}Y(c_i a, z_0 + z_2)Y(b, z_2) u \\
- \sum_{i=0}^{\infty} \binom{n}{i} z_2^{-n-i}(z_0 + z_2)^{k+\frac{\hat{P}}{2}}Y(a, z_0 + z_2)Y(c_i b, z_2) u \\
= (z_0 + z_2)^{k+\frac{\hat{P}}{2}}c_n Y(Y a, z_0)b, z_2) u \\
- \sum_{i=0}^{\infty} \binom{n}{i} (z_2 + z_0)^{k+\frac{\hat{P}}{2}+n-i}Y(Y(c_i a, z_0)b, z_2) u \\
- \sum_{i=0}^{\infty} \binom{n}{i} z_2^{-n-i}(z_2 + z_0)^{k+\frac{\hat{P}}{2}}Y(c_i Y(a, z_0)b, z_2) u \\
+ \sum_{j=0}^{\infty} \sum_{j=0}^{\infty} \binom{n}{j} \binom{n-j}{i-j} z_2^{-n-i}(z_2 + z_0)^{k+\frac{\hat{P}}{2}}z_0^{-j}Y(Y(c_j a, z_0)b, z_2) u \\
= (z_0 + z_2)^{k+\frac{\hat{P}}{2}}c_n Y(Y a, z_0)b, z_2) u \\
- \sum_{i=0}^{\infty} \binom{n}{i} (z_2 + z_0)^{k+\frac{\hat{P}}{2}+n-i}Y(Y(c_i a, z_0)b, z_2) u \\
- \sum_{i=0}^{\infty} \binom{n}{i} z_2^{-n-i}(z_2 + z_0)^{k+\frac{\hat{P}}{2}}Y(c_i Y(a, z_0)b, z_2) u \\
+ \sum_{j=0}^{\infty} \sum_{j=0}^{\infty} \binom{n}{j} \binom{n-j}{i-j} z_2^{-n-i}(z_2 + z_0)^{k+\frac{\hat{P}}{2}}z_0^{-j}Y(Y(c_j a, z_0)b, z_2) u \\
= (z_0 + z_2)^{k+\frac{\hat{P}}{2}}c_n Y(Y a, z_0)b, z_2) u \\
- \sum_{i=0}^{\infty} \binom{n}{i} z_2^{-n-i}(z_2 + z_0)^{k+\frac{\hat{P}}{2}}Y(c_i Y(a, z_0)b, z_2) u \\
= (z_0 + z_2)^{k+\frac{\hat{P}}{2}}c_n Y(Y a, z_0)b, z_2) u \\
- (z_2 + z_0)^{k+\frac{\hat{P}}{2}}c_n Y(Y(a, z_0)b, z_2) u \\
= (z_2 + z_0)^{k+\frac{\hat{P}}{2}}Y(Y(a, z_0)b, z_2)c_n u,
\]
Theorem 6.2. $M(U)$ is an admissible $g$-twisted $V$-module with $M(U)(0) = U$ and with the following universal property: for any weak $g$-twisted $V$-module $M$ and any $A_g(V)$-morphism $\phi : U \to \Omega(M)$, there is a unique morphism $\tilde{\phi} : M(U) \to M$ of weak $g$-twisted $V$-modules which extends $\phi$.

Proof: Clearly $M(U)$ satisfies the conditions placed on $M$ in Proposition 6.1. We see that $M(U)$ is a weak $g$-twisted $V$-module and therefore an admissible $g$-twisted $V$-module. The universal property of $M(U)$ follows from its construction. A proof of that $M(U)(0) = U$ will be given after Proposition 6.10. □

We can now state the second main result of this section.

Theorem 6.3. $M(U)$ has a unique maximal graded $V[g]$-submodule $J$ with the property that $J \cap U = 0$. Then $L(U) = M(U)/J$ is an admissible $g$-twisted $V$-module satisfying $\Omega(L(U)) \cong U$.

$L$ defines a functor from the category of $A_g(V)$-modules to the category of admissible $g$-twisted $V$-modules such that $\Omega \circ L$ is naturally equivalent to the identity.

In the following we let $U^* = \text{Hom}_C(U, \mathbb{C})$ and extend $U^*$ to $M(U)$ by letting $U^*$ annihilate $\oplus_{n>0} M(U)(n)$. Then one easily shows

Lemma 6.4. Let $J$ be the graded $V[g]$-submodule of $M(U)$ maximal subject to $J \cap U = 0$. Then

$$J = \{ v \in M(U) | \langle u', xv \rangle = 0 \text{ for all } u' \in U^*, \text{ all } x \in U(V[g]) \}. $$

The main point in the proof of the Theorems is to show that $U(V[g])W \subset J$. The next three results are devoted to this goal.

Proposition 6.5. The following hold for all homogeneous $a \in V^*, b \in V$, $u' \in U^*, u \in U, j \in \mathbb{Z}_+$,

$$\langle u', (z_0 + z_2)w^{\delta_2+\delta_r} + jy_{M(U)}(a, z_0 + z_2)Y_{M(U)}(b, z_2)u \rangle = \langle u', (z_2 + z_0)w^{\delta_2+\delta_r} + jy_{M(U)}(Y(a, z_0)b, z_2)u \rangle. \quad (6.9)$$

Remark 6.6. In the following we habitually drop subscripts attached to $Y$, which should cause no confusion.

Proof of Proposition 6.5: By linearity we may take $b \in V^*$ homogeneous. Suppose that $v(m) \in V[g]$ and $u \in U$. Notice that $Y(a, z) = \sum_{n \in \mathbb{Z}_+} a(n)z^{-n-1}$. It follows that if $r + s \not\equiv 0 \pmod{T}$ then all coefficients $z_0^{r+s}z_2^j$ of $Y(a, z_0 + z_2)Y(b, z_2)u$ and $Y(a, z_0)b, z_2)$ are either 0 or lie in $M(U)(n)$ for $n > 0$. So in this case both sides of (6.9) are zero since $u'$ annihilates $M(U)(n)$ for $n > 0$.

So we may assume that $r + s \equiv 0 \pmod{T}$. If $r = s = 0$ we may appeal to $[Z]$ or $[L2]$ as this is essentially the case $g = 1$. So from now on we take $1 \leq r \leq T - 1, s = T - r$. Note that $\delta_2 = 0$ in this case. We proceed in several lemmas.
Lemma 6.7. For any \( i, j \in \mathbb{Z}_+ \),
\[
\text{Res}_{\gamma_0} z_0^{-1+i}(z_0 + z_2)^{\text{wt}a-1+i+\psi+j} u', Y(a, z_0 + z_2) Y(b, z_2) u
\]
\[
= \text{Res}_{\gamma_0} z_0^{-1+i}(z_2 + z_0)^{\text{wt}a-1+i+\psi+j} u', Y(Y(a, z_0) b, z_2) u.
\]

Proof: Since \( j \geq 0 \) then \( a(\text{wt}a - 1 + \psi + j) \) lies in \( V[\gamma] \) and hence annihilates \( u \). Then for all \( i \in \mathbb{Z}_+ \) we get
\[
\text{Res}_{z_1}(z_1 - z_2)^i z_1^{\text{wt}a-1+i+\psi+j} Y(b, z_2) Y(a, z_1) u = 0. \tag{6.10}
\]
Note that Lemma 4.2 (ii) is equivalent to
\[
[Y(a, z_1), Y(b, z_2)] = \text{Res}_{z_2} z_2^{-1} \left( \frac{z_2 - z_0}{z_2} \right)^{-r/T} \delta \left( \frac{z_1 - z_0}{z_2} \right) Y(u, z_0) v, z_2 \tag{6.11}
\]
(cf. (3.4)).
Using (6.10) and (6.11) we obtain:
\[
\text{Res}_{z_0} z_0^i (z_0 + z_2)^{\text{wt}a-1+i+\psi} Y(a, z_0 + z_2) Y(b, z_2) u
\]
\[
= \text{Res}_{z_1}(z_1 - z_2)^i z_1^{\text{wt}a-1+i+\psi+j} Y(a, z_1) Y(b, z_2) u
\]
\[
= \text{Res}_{z_1}(z_1 - z_2)^i z_1^{\text{wt}a-1+i+\psi+j} Y(a, z_1) Y(b, z_2) u
\]
\[
- \text{Res}_{z_1}(z_1 - z_2)^i z_1^{\text{wt}a-1+i+\psi+j} Y(b, z_2) Y(a, z_1) u
\]
\[
= \text{Res}_{z_1}(z_1 - z_2)^i z_1^{\text{wt}a-1+i+\psi+j} [Y(a, z_1), Y(b, z_2)] u
\]
\[
= \text{Res}_{z_0} \text{Res}_{z_1}(z_1 - z_2)^i z_1^{\text{wt}a-1+i+\psi+j} \delta \left( \frac{z_1 - z_0}{z_2} \right) \left( \frac{z_1 - z_0}{z_2} \right)^{-r/T} Y(Y(a, z_0) b, z_2) u
\]
\[
= \text{Res}_{z_0} \text{Res}_{z_1} z_0^i z_1^{\text{wt}a-1+i+\psi+j} z_2^{-1} \left( \frac{z_2 + z_0}{z_1} \right)^{\psi} \left( \frac{z_2 + z_0}{z_1} \right) Y(Y(a, z_0) b, z_2) u
\]
\[
= \text{Res}_{z_0} z_0^i (z_2 + z_0)^{\text{wt}a-1+i+\psi+j} Y(Y(a, z_0) b, z_2) u. \tag{6.12}
\]
Thus lemma 6.7 holds if \( i \geq 1 \), and we may now assume \( i = 0 \).
Next use (6.12) to calculate that
\[
\text{Res}_{z_0} z_0^{-1}(z_0 + z_2)^{\text{wt}a-1+i+\psi+j} u', Y(a, z_0 + z_2) Y(b, z_2) u
\]
\[
= \sum_{k=0}^{\infty} \binom{j}{k} \text{Res}_{z_0} z_0^{k-1} z_2^{-k}(z_0 + z_2)^{\text{wt}a-1+i+\psi+j} u', Y(a, z_0 + z_2) Y(b, z_2) u
\]
\[
= \sum_{k=1}^{\infty} \binom{j}{k} \text{Res}_{z_0} z_0^{k-1} z_2^{-k}(z_2 + z_0)^{\text{wt}a-1+i+\psi+j} u', Y(Y(a, z_0) b, z_2) u
\]
\[
+ \text{Res}_{z_0} z_0^{-1} z_2^{j}(z_2 + z_0)^{\text{wt}a-1+i+\psi+j} u', Y(a, z_0 + z_2) Y(b, z_2) u. \tag{6.13}
\]
We claim that we have
\[
\text{Res}_{z_0} z_0^{-1}(z_0 + z_2)^{\text{wt}a-1+i+\psi+j} u', Y(a, z_0 + z_2) Y(b, z_2) u = 0. \tag{6.14}
\]
In the notation of (4.10), (6.17) is equal to

\[ a \]

Since all operators of the form \( k \). Proof: This is true for \( n \geq -1 \) by Lemma 6.7. Let us write \( n = -k + i \) with \( i \in \mathbb{Z}_+ \) and proceed by induction \( k \). Induction yields

\[ \text{Res}_{z_0} z_0^{-1}(z_2 + z_0) \text{wt} a - 1 + \frac{r}{T} \langle u', Y(Y(a, z_0)b, z_2)u \rangle = 0. \]  

(6.15)

If so, then Lemma 6.7 follows from (6.13).

To see (6.14), note that if \( n < 0 \), so that \( b_{\text{wt}b-1-n}u = 0 \) in this case. Then we see that

\[
\begin{align*}
\text{Res}_{z_0} z_0^{-1}(z_2 + z_0)^{\text{wt} a - 1 + \frac{r}{T}}& \langle u', Y(a, z_0 + z_2)Y(b, z_2)u \rangle \\
& = \langle u', \sum_{k \in \mathbb{Z}_+} a(\text{wt}a - 2 - k + \frac{r}{T}) \rangle b(\text{wt}b - 1 - n) z_2^{-\text{wt}b + n + k}u. \\
& = \langle u', \sum_{k \in \mathbb{Z}_+} a(\text{wt}a - 2 - k + \frac{r}{T}) \rangle b(\text{wt}b - 1 - n) z_2^{-\text{wt}b + n + k}u. \\
(6.16)
\end{align*}
\]

Since all operators of the form \( a(q) \) occurring in (6.16) lie in \( V[g] \) then all components lie in \( \oplus_{n \geq 0} M(U)(n) \) and hence are annihilated by \( u' \). So (6.14) follows.

Completely analogous calculation shows that

\[
\begin{align*}
\text{Res}_{z_0} z_0^{-1}(z_2 + z_0)^{\text{wt} a - 1 + \frac{r}{T}}& \langle u', Y(Y(a, z_0)b, z_2)u \rangle = \\
& \langle u', \sum_{k \in \mathbb{Z}_+} a(\text{wt}a - 1 + \frac{r}{T}) \rangle (a_{k-1}b)^{b}(\text{wt}(a_{k-1}b) - 1)u \\
(6.17)
\end{align*}
\]

In the notation of (1.10), (6.17) is equal to

\[
\begin{align*}
\text{Res}_{z_0} z_0^{-1}(z_2 + z_0)^{\text{wt} a - 1 + \frac{r}{T}}& \langle u', Y(Y(a, z_0)b, z_2)u \rangle \\
& = \langle u', \sum_{k \in \mathbb{Z}_+} a(\text{wt}a - 1 + \frac{r}{T}) \rangle o(a_{k-1}b)u \\
& = \langle u', \sum_{k \in \mathbb{Z}_+} a(\text{wt}a - 1 + \frac{r}{T}) \rangle o(a_{k-1}b)u \\
(6.18)
\end{align*}
\]

(note that \( a_{k-1}b \in V^0 \)), and from (2.2) and Lemma 4.2 the sum is precisely the action of \( a \circ b \) on \( u \). Since \( a \circ b \in O_g(V) \) we have \( (a \circ b)u = 0 \) (cf. Theorem 5.3). This completes the proof of (6.15), and with it that of the lemma. □

Proposition 5.5 is a consequence of the next lemma.

**Lemma 6.8.** For all \( n \in \mathbb{Z} \) we have

\[
\begin{align*}
\text{Res}_{z_0} z_0^{-n}(z_2 + z_0)^{\text{wt} a - 1 + \frac{r}{T} + j} & \langle u', Y(Y(a, z_0 + z_2)Y(b, z_2)u) \\
& = \text{Res}_{z_0} z_0^{-n}(z_2 + z_0)^{\text{wt} a - 1 + \frac{r}{T} + j} \langle u', Y(Y(a, z_0)b, z_2)u \rangle.
\end{align*}
\]

**Proof:** This is true for \( n \geq -1 \) by Lemma 6.7. Let us write \( n = -k + i \) with \( i \in \mathbb{Z}_+ \) and proceed by induction \( k \). Induction yields

\[
\begin{align*}
\text{Res}_{z_0} z_0^{-k}(z_2 + z_0)^{\text{wt} a - 1 + \frac{r}{T} + j} & \langle u', Y(Y(a, z_0 + z_2)Y(b, z_2)u) \\
& = \text{Res}_{z_0} z_0^{-k}(z_2 + z_0)^{\text{wt} a - 1 + \frac{r}{T} + j} \langle u', Y(Y(a, z_0)b, z_2)u \rangle.
\end{align*}
\]
Using the residue property \( \text{Res}_z f'(z)g(z) + \text{Res}_z f(z)g'(z) = 0 \) and the \( L(-1) \)-derivation property we have
\[
\begin{align*}
\text{Res}_{z_0} z_0^{-k}(z_0 + z_2)wta + \frac{r}{T} + j | u', Y(L(-1)a, z_0 + z_2)Y(b, z_2)u) \\
= -\text{Res}_{z_0} \left( \frac{\partial}{\partial z_0} z_0^{-k}(z_0 + z_2)wta + \frac{r}{T} + j \right) | u', Y(a, z_0 + z_2)Y(b, z_2)u)
\end{align*}
\]
\[
\begin{align*}
= & \text{Res}_{z_0} z_0^{-k}(z_0 + z_2)wta + \frac{r}{T} + j \langle u', Y(a, z_0 + z_2)Y(b, z_2)u) \\
= & \text{Res}_{z_0} z_0^{-k}(z_0 + z_2)wta + \frac{r}{T} + j \langle u', Y(a, z_0 + z_2)Y(b, z_2)u)
\end{align*}
\]
and
\[
\begin{align*}
\text{Res}_{z_0} z_0^{-k}(z_0 + z_2)wta + \frac{r}{T} + j | u', Y(L(-1)a, z_0)Y(b, z_2)u)
\end{align*}
\]
\[
\begin{align*}
& = -\text{Res}_{z_0} \left( \frac{\partial}{\partial z_0} z_0^{-k}(z_0 + z_2)wta + \frac{r}{T} + j \right) | u', Y(a, z_0)Y(b, z_2)u)
\end{align*}
\]
\[
\begin{align*}
& = \text{Res}_{z_0} z_0^{-k}(z_0 + z_2)wta + \frac{r}{T} + j \langle u', Y(a, z_0)Y(b, z_2)u)
\end{align*}
\]
This yields the identity:
\[
\begin{align*}
\text{Res}_{z_0} z_0^{-k}(z_0 + z_2)wta + \frac{r}{T} + j \langle u', Y(a, z_0 + z_2)Y(b, z_2)u)
\end{align*}
\]
and the lemma is proved. \( \square \)

Let us now introduce an arbitrary \( \frac{1}{2}Z_+ \)-graded \( V[g] \)-module \( M = \bigoplus_{n \in \frac{1}{2}Z_+} M(n) \). As before we extend \( M(0)^\ast \) to \( M \) by letting it annihilate \( M(n) \) for \( n > 0 \). The proof of Proposition of \( [6, 7] \) with \( \langle u', \cdot \rangle \) suitably inserted gives:

**Proposition 6.9.** Assume the following hold:
(i) $M = U(V[g])M(0)$.
(ii) For $a \in V^r$ and $u \in M(0)$ there is $k \in \mathbb{Z}$ such that
\[ \langle u', (z_0 + z_2)^{k+\frac{n}{2}}Y(a, z_0 + z_2)Y(b, z_2)u \rangle = \langle u', (z_2 + z_0)^{k+\frac{n}{2}}Y(Y(a, z_0)b, z_2)u \rangle \] (6.18)
for any $b \in V, u' \in M(0)^*$. Then in fact (6.18) holds for any $u \in M$.

**Proposition 6.10.** Let $M$ be as in Proposition 6.3. Then for any $x \in U(V[g]), a \in V^r, u \in M$, there is an integer $k$ such that
\[ \langle u', (z_0 + z_2)^{k+\frac{n}{2}}x \cdot Y(a, z_0 + z_2)Y(b, z_2)u \rangle = \langle u', (z_2 + z_0)^{k+\frac{n}{2}}x \cdot Y(Y(a, z_0)b, z_2)u \rangle \] (6.19)
for any $b \in V, u' \in M(0)^*$.

**Proof.** Let $L$ be the subspace of $U(V[g])$ consisting of those $x$ for which (6.19) holds. Let $x \in L$, let $c$ be any homogeneous element of $V$, and let $n \in \frac{1}{2}\mathbb{Z}$. Then from (6.8) we have
\[ \langle u', xc_nY(a, z_0 + z_2)Y(b, z_2)u \rangle (z_0 + z_2)^{k+\frac{n}{2}} \]
\[ = \sum_{i=0}^{\infty} \binom{n}{i} (z_0 + z_2)^{k+\frac{n}{2}+n-i} \langle u', xY(cia, z_0 + z_2)Y(b, z_2)u \rangle \]
\[ + \sum_{i=0}^{\infty} \binom{n}{i} z_2^{n-i} (z_0 + z_2)^{k+\frac{n}{2}} \langle u', xY(a, z_0 + z_2)Y(cib, z_2)u \rangle \]
\[ + (z_0 + z_2)^{k+\frac{n}{2}} \langle u', xY(a, z_0 + z_2)Y(b, z_2)c_nu \rangle. \] (6.20)
The same method that was used in the proof of Proposition 6.9 shows that $xc_n \in L$. Since $U(V[g])$ is generated by all such $c_n$’s, and since (6.19) holds for $x = 1$ by Proposition 6.9, we conclude that $L = U(V[g])$, as desired. □

We can now finish the proof of Theorem 6.2. We can take $M = M(U)$ in Proposition 6.10, as we may since $M(U)$ certainly satisfies the conditions placed on $M$ prior to Proposition 6.9 and in Proposition 6.3. Then from the definition of $W$ (6.3), Lemma 6.4 and Propositions 6.3, 6.9 and 6.10 we conclude that $U(V[g])W \subset J$. So $M(U)(0) \cong U$.

Turning to the proof of Theorem 6.3, we have already seen that $U(V[g])W \subset J$. Then from Theorem 6.3 it is clear that $L(U) = M(U)/J$ is a quotient of $M(U)$ and hence an admissible $g$-twisted $V$-module satisfying $L(U)(0) \cong U$. Now clearly $\Omega(L(U)) \supset U$, and if this is not an equality then there is $n > 0$ with $\Omega_n = \Omega(L(U)) \cap L(U)(n) \neq 0$. Then $0 \neq U(V[g])\Omega_n$ intersects $U$ trivially, contradiction. So in fact $\Omega(L(U)) \cong U$. The remainder of Theorem 6.3 is straightforward to prove. □

7. Bijection between simple objects

At this point we have a pair of functors $\Omega, L$ defined on appropriate module categories:
\[ A_g(V) - \text{Mod} \xrightarrow{L} \text{Adm} - g - V - \text{Mod} \] (7.1)
Although $\Omega \circ L$ is equivalent to the identity, one cannot expect that $L \circ \Omega$ is also equivalent to the identity in general. This is essentially because there are examples of vertex operator algebras $V$ for which the category of admissible $V$-modules contains objects which are not completely reducible. For an example see [FZ].

**Lemma 7.1.** Suppose that $U$ is a simple $A_g(V)$-module. Then $L(U)$ is a simple admissible $g$-twisted $V$-module.

**Proof:** If $0 \neq W \subset L(U)$ is an admissible $g$-twisted submodule then, by the definition of $L(U)$, we have $W(0) = W \cap L(U)(0) \neq 0$. As $W(0)$ is an $A_g(V)$-submodule of $U = L(U)(0)$ by Theorem 7.3 then $U = W(0)$, whence $W \supseteq U(V[g])W(0) = U(V[g])U = L(U)$. □

**Theorem 7.2.** $L$ and $\Omega$ are equivalences when restricted to the full subcategories of completely reducible $A_g(V)$-modules and completely reducible admissible $g$-twisted $V$-modules respectively. In particular, $L$ and $\Omega$ induces mutually inverse bijections on the isomorphism classes of simple objects in the category of $A_g(V)$-modules and admissible $g$-twisted $V$-modules respectively.

**Proof:** We have $\Omega(L(U)) \cong U$ for any $A_g(V)$-module by Theorem 7.3.

If $M$ is a completely reducible admissible $g$-twisted $V$-module we must show $L(\Omega(M)) \cong M$. For this we may take $M$ simple, whence $\Omega(M)$ is simple by Proposition 5.4 (ii) and then $L(\Omega(M))$ is simple by Lemma 7.1. Since both $M$ and $L(\Omega(M))$ are simple quotients of the universal object $\bar{M}(\Omega(M))$ then they are isomorphic by Theorems 6.2 and 7.3. □

### 8. $g$-rational vertex operator algebras

The definition of $g$-rational vertex operator algebra prior to Lemma 3.4 says precisely that every object in the category of admissible $g$-twisted $V$-modules is completely reducible. We have the following annulus result.

**Theorem 8.1.** Suppose that $V$ is a $g$-rational vertex operator algebra. Then the following hold:

(a) $A_g(V)$ is a finite-dimensional, semi-simple associative algebra (possibly 0).

(b) $V$ has only finitely many isomorphism classes of simple admissible $g$-twisted modules.

(c) Every simple admissible $g$-twisted $V$-module is an ordinary $g$-twisted $V$-module.

(d) $V$ is $g^{-1}$-rational.

(e) The functors $L, \Omega$ are mutually inverse categorical equivalences between the category of $A_g(V)$-modules and the category of admissible $g$-twisted $V$-modules.

(f) The functors $L, \Omega$ induce mutually inverse categorical equivalences between the category of finite-dimensional $A_g(V)$-modules and the category of ordinary $g$-twisted $V$-modules.

Suppose that (a) holds. Then all objects in the category of $A_g(V)$-modules are completely reducible. Then (e) follows from Theorem 7.3. Moreover as $A_g(V)$ is of finite
dimension it has only finitely many simple modules, whence (b) follows from (e). Similarly (f) follows from (c). So we must prove parts (a), (c), (d).

Proof of (a): It suffices to show that any \( A_g(V) \)-module \( U \) is completely reducible.

Now \( L(U) \) is admissible and hence a direct sum of simple admissible \( g \)-twisted \( V \)-modules. Application of the functor \( \Omega \) shows that \( \Omega(L(U)) \) is also completely reducible, so we are done since \( \Omega(L(U)) \cong U \).

Proof of (c): Let \( M \) be a simple admissible \( g \)-twisted \( V \)-module. Then \( \Omega(M) \) is a simple \( A_g(V) \)-module, call it \( U \), and \( L(U) \cong M \).

Now by Theorem 2.4 (iii), \( \omega + O_g(V) \in A_g(V) \) acts as a scalar \( h \), say, on \( U \). From the construction of \( L(U) \) in Section 6 it follows that the graded subspaces \( M(n) \) of \( M \) are precisely the distinct eigenspaces of \( L(0) \) on \( M \). That is, \( L(0) \) is semi-simple as an operator on \( M \) and for \( n \in \frac{1}{h}\mathbb{Z}_+ \) we have

\[
M(n) = \{ m \in M | L(0) = (n + h)m \}.
\]

Next, if we think of \( U \) as a left \( A_g(V) \)-module then \( U^* \) is naturally a left \( A_g(V) \)\text{-}opp-module (opposite algebra). Theorem 2.4 (ii) tells us that there is a canonical algebra isomorphism \( A_{g^{-1}}(V) \cong A_g(V) \)\text{-}opp, so \( U^* \) is a left \( A_{g^{-1}}(V) \)-module. It is simple by part (a). Now apply Theorem 2.4 (with \( g^{-1} \) in place of \( g \)): we conclude that \( L(U^*) \) is an admissible \( g^{-1} \)-twisted \( V \)-module. Moreover \( L(U^*) \) is simple by Lemma 7.1.

Let \( N = L(U^*)' \). So \( N \) is an admissible \( g \)-twisted \( V \)-module (see Lemma 3.7). We will show that \( N \) is also simple.

Let \( W \) be the admissible \( g \)-twisted submodule of \( N \) generated by \( N(0) = (U^*)^* = U \). As \( V \) is \( g \)-rational then \( N = W \oplus W_0 \) for some admissible \( g \)-twisted submodule \( W_0 \) of \( N \). Obviously \( W_0 \cap U = 0 \), so \( (U^*, W_0) = 0 \). As \( U^* \) generates \( L(U^*) \) we get \( W_0 = 0 \). So \( N = W \) is generated by \( U \).

Now if \( X \) is any non-zero submodule of \( N \) then we have \( X \cap U \neq 0 \) as \( N \) is completely reducible. Since \( U \) is a simple \( A_g(V) \)-module then \( U \subset X \), whence \( U = X \). So indeed \( N \) is simple. Then both \( N \) and \( L(U) \) are simple admissible \( g \)-twisted \( V \)-modules generated by \( U \), so we must have \( N \cong L(U) \) (cf. Theorems 3.2 and 3.3).

Applying Lemma 3.0 shows that \( N \) has countable dimension, so the same is true of each graded subspace. Thus \( (L(U^*)(n))^* \) has countable dimension. This can only happen if \( L(U^*)(n) \) is of finite dimension.

We now deduce that in fact \( L(U^*) \) is an ordinary \( g^{-1} \)-twisted \( V \)-module. Indeed (8.1) applies to \( L(U^*) \) also, so that axiom (3.11) is fulfilled. Then as the contragredient module of \( L(U^*) \), \( L(U) \) is also an ordinary \( g \)-twisted \( V \)-module. This completes the proof of part (c).

Proof of (d): Let \( \{ W^1, \ldots, W^k \} \) be representatives for the equivalence classes of simple \( g \)-twisted \( V \)-modules

Consider any \( \frac{1}{h}\mathbb{Z}_+ \)-graded weak \( g^{-1} \)-twisted \( V \)-module \( M = \oplus_{n \in \frac{1}{h}\mathbb{Z}_+} M(n) \). Then \( M' = \oplus_{n \in \frac{1}{h}\mathbb{Z}_+} M(n)^* \) is an admissible \( g \)-twisted \( V \)-module which is completely reducible. We can write

\[
M' \cong U^1 \otimes W^1 \oplus U^2 \otimes W^2 \oplus \cdots \oplus U^k \otimes W^k
\]
for certain vector spaces $U^i$. Clearly $U^i$ are $\frac{1}{T}\mathbb{Z}_+$-graded:

$$U^i = \bigoplus_{m \in \frac{1}{T}\mathbb{Z}_+} U^i(m)$$

such that

$$M(n)^* = \bigoplus_{i=1}^k \bigoplus_{s,t \in \frac{1}{T}\mathbb{Z}_+, s+t=n} U^i(s) \otimes W^i(t).$$

As $W^i$ is an ordinary $g$-twisted module, each $W^i(t)$ is finite-dimensional. It follows that

$$(M(n)^*)^* = \bigoplus_{i=1}^k \bigoplus_{s,t \in \frac{1}{T}\mathbb{Z}_+, s+t=n} U^i(s)^* \otimes W^i(t)^*$$

and thus

$$(M')^* = \bigoplus_{n \in \frac{1}{T}\mathbb{Z}_+} U^i(n)^* \otimes (W^i)'^*.$$ 

So $(M')'$ is completely reducible, whence so to is $M \subset (M')'$. This completes the proof of Theorem 8.1. □

9. Further Applications

It is a well-known conjecture that $V$ always possess at least one ordinary $g$-twisted module. (Here $V$ is any vertex operator algebra and $g$ an automorphism of order $T$.) Somewhat weaker is the conjecture that $A_g(V)$ is non-zero; this is equivalent to the existence of a simple admissible $g$-twisted $V$-module by Theorem 7.2. We have the following contribution to this problem:

**Theorem 9.1.** Suppose that $A_g(V)$ is of finite dimension. Then there is at least one simple $g$-twisted $V$-module.

We begin the proof with a variation on the theme of Section 8.

**Lemma 9.2.** Let $M = \bigoplus_{n \in \frac{1}{T}\mathbb{Z}_+} M(n)$ be an admissible $g$-twisted $V$-module such that (8.1) holds, i.e., $M(n)$ is the $n+h$-eigenspace for the action of $L(0)$. Assume that the contra-gredient module $M'$ is simple. Then $M$ is a simple (ordinary) $g$-twisted module.

**Proof:** If $W$ is a non-zero submodule of $M$ then $W^\perp = \{ f \in M' | \langle f, W \rangle = 0 \}$ is a weak $g^{-1}$-twisted submodule of $M'$. As $M'$ is simple then $W^\perp = 0$, so $W = M$. This shows that $M$ is simple.

As $M'$ is simple then it has countable dimension by Lemma 3.4, and the remainder of the proof now follows as in the proof of Theorem 8.1 (c). □

Denote by $\mathcal{M}_g(V)$ the (equivalence classes of) simple admissible $g$-twisted $V$-modules. It is a finite set since we are now assuming that $A_g(V)$ is of finite dimension. As before, $L(0)$ is semi-simple as an operator on these modules, and we denote by $S_g(V) \subset \mathbb{C}$ the set of *lowest weights*, i.e., the set of eigenvalues $h$ in the notation of (8.1). Define a partial order $\geq$ on $S_g(V)$ as follows: $h_1 \geq h_2$ if, and only if, $h_1 - h_2 \in \frac{1}{T}\mathbb{Z}_+$. Let $S^*_g(V)$ be the maximal elements in the partial order, and $\mathcal{M}^*_g(V)$ the modules of $\mathcal{M}_g(V)$ whose lowest weights lie in $S^*_g(V)$.

**Lemma 9.3.** Suppose that $M \in \mathcal{M}^*_g(V)$. Then $M$ is a $g$-twisted $V$-module.
Proposition 9.4. Suppose that there are only finitely many simple $g$-twisted $V$-modules (up to equivalence). Then any $g$-twisted $V$-module $M$ has a finite composition series such that each factor is simple.

Proof. Let $\{W^1, \cdots, W^k\}$ be the set of equivalence classes of simple $g$-twisted $V$-modules and let $h_i$ be the lowest weight of $W^i$. Let $M$ be a $g$-twisted $V$-module.

Claim 1: If $h$ is a minimal weight of $M$ in the sense that $M_h \neq 0$ and $M_{h-n} = 0$ for all positive $n \in \mathbb{N}$, then $h \in \{h_1, \cdots, h_k\}$. Suppose $h$ is a minimal weight of $M$. Then $M_h \subseteq \Omega(M)$ and $M_h$ is an $A_g(V)$-submodule of $\Omega(M)$. Let $U$ be an irreducible $A_g(V)$-submodule of $M_h$ and let $W$ be the $g$-twisted $V$-submodule generated by $U$. Then $W$ has a unique simple quotient module, and its lowest weight is $h$. Thus $h = h_i$ for some $i$.

Claim 2: $M$ is generated by $E(M) = \bigoplus_{i=1}^k M_{h_i}$ (note that $E(M)$ is finite-dimensional). Let $W$ be the $g$-twisted $V$-submodule generated by $E(M)$. If $M \neq W$, $M/W$ is a (nonzero) $g$-twisted $V$-module such that no homogeneous subspace has weight $h_i$ for any $i$. So there must be a lowest weight $h$ of $M/W$ such that $h \neq h_i$ for any $1 \leq i \leq k$. Now apply Claim 1 to get a contradiction.

Claim 3: If $W^1$ and $W^2$ are two submodules of $M$ such that $W^1 \cap E(M) = W^2 \cap E(M)$, then $W^1 = W^2$. Since $E(W^1) = W^1 \cap E(M) = W^2 \cap E(M) = E(W^2)$, this follows from Claim 2 immediately.

Let $\mathcal{S}$ be the set of all submodules of $M$ partially ordered by inclusion.

Claim 4: There exists a finite maximal chain in $\mathcal{S}$. It follows from Zorn’s Lemma that there exist maximal chains. Let $\cdots \subseteq M^{-1} \subseteq M^0 \subseteq M^1 \subseteq \cdots$ be an ascending chain in $\mathcal{S}$. Then we have:

$$\cdots \subseteq E(M) \cap M^{-1} \subseteq E(M) \cap M^0 \subseteq E(M) \cap M^1 \subseteq \cdots$$

Since $E(M)$ is finite-dimensional, there are nonnegative integers $m$ and $n$ such that

$$E(M) \cap M^s = E(M) \cap M^{-m}, \quad E(M) \cap M^n = E(M) \cap M^t$$

for any $s \leq -m$ and $t \geq n$. Thus by Claim 3, $M^s = M^{-m}$ and $M^n = M^t$.

It is clear that for any maximal chain, all the factors are simple $g$-twisted $V$-modules. The proof is complete. □
A vertex operator algebra $V$ is said to be holomorphic if $V$ is the only simple $V$-module up to equivalence. The famous moonshine module vertex operator algebra $V^2$ [FLM] is an example [D2]. The following proposition is an application of Proposition [D2].

**Proposition 9.5.** Suppose that $V$ is a holomorphic VOA and that $V$ contains a rational vertex operator subalgebra $U$ (with the same Virasoro element). Then any $V$-module is completely reducible.

We need the following Lemma from [L1].

**Lemma 9.6.** Let $V$ be a vertex operator algebra and $M$ a weak $V$-module.

(i) Let $m \in M$ be a vacuum-like vector, that is, $L(-1)m = 0$. Then the weak submodule of $M$ generated by $m$ is isomorphic to $V$, the isomorphism arising via the map $m \mapsto 1$.

(ii) Conversely, if $V$ is isomorphic to $M$ as weak $V$-modules, then the image of $1$ in $M$ is a vacuum-like vector.

**Proof of Proposition 9.5.** Let $M$ be a $V$-module. By Proposition [D2], there is a composition series $0 \subseteq M^1 \subseteq M^2 \subseteq \cdots \subseteq M^k = M$. Assume that $M^i$ is completely reducible for some $i$. From the assumption, there is a $U$-submodule $B^i$ of $M^{i+1}$ such that $M^{i+1} = B^i \oplus M^i$. Since $M^{i+1}/M^i \cong V$, there is $0 \neq u \in M^{i+1}$ such that $u + M^i$ is a vacuum-like vector in $M^{i+1}/M^i$ by Lemma 9.6. Let $u = a + b$ where $a \in B^i, b \in M^i$. Then $L(-1)u = L(-1)a + L(-1)b \in M^i$. Thus $a \neq 0$ and $L(-1)a = 0$ as $L(-1)$ preserves both $M^i$ and $B^i$. This show that $a$ is a vacuum-like vector in $M^{i+1}$. Again by Lemma 9.6, $a$ generates a $V$-submodule of $M^{i+1}$ isomorphic to $V$. Then $M^{i+1}$ is the sum of $M^i$ and $U(V[1])a$. So $M^{i+1}$ is completely reducible. It follows by induction that each $M^i$ is completely reducible for $i \geq 1$. In particular, $M$ is completely reducible. □

**Remark 9.7.** The moonshine module vertex operator algebra $V^2$ [FLM] has a rational vertex operator subalgebra $U$ which satisfies the conditions of Proposition [D2]. It is isomorphic to the tensor product of 48 irreducible highest weight modules for the Virasoro algebra with central charge $\frac{1}{2}$ [DMZ]. Proposition 9.5 thus gives a proof that any $V^2$-module is completely reducible which is shorter than the original proof [D2].

It is conjectured in [FLM] that any holomorphic vertex operator algebra $V$ of rank 24 with $V_1 = 0$ is isomorphic to the moonshine module vertex operator algebra $V^2$ in [FLM]. The following proposition asserts that any ordinary module for such a vertex operator algebra is completely reducible. When applied to $V^2$ itself, this gives another proof of complete reducibility of any $V^2$-module.

**Proposition 9.8.** Suppose that $V$ is a holomorphic VOA such that $V_1 = 0$. Then any $V$-module is completely reducible.

**Proof.** Let $M$ be any $V$-module and $W \subset M_0$ be the subspace of vacuum-like vectors (cf. Lemma 9.6). Let $M'$ be the submodule of $M$ generated by $W$. Then $M'$ is completely reducible. If $M' \neq M$ consider $M/M'$. Let $u \in M \setminus M'$ such that $u + M'$ is a vacuum-like vector, that is $L(-1)u \in M'$. Note that $u \in M_0$ and $L(-1)u \in M_1$. Since $M$ has a
finite composition series and each factor is isomorphic to $V$ we see that $M_1 = 0$. Thus $L(-1)u = 0$ and $u$ is a vacuum-like vector. This is a contradiction because $u$ is not in $W$. □

**Proposition 9.9.** Suppose that $V^0$ contains a rational vertex operator subalgebra $U$ (with the same Virasoro element) such that the fusion rules among any three irreducible $U$-modules is finite. Then any simple admissible $g$-twisted $V$-module is an ordinary $g$-twisted $V$-module.

**Proof:** Let $M$ be a simple admissible $g$-twisted $V$-module with lowest weight $h$. Since $M$ is a completely reducible $U$-module we can take a simple admissible $U$-submodule $W$ of $M$. Then by Proposition 2.4 of [DM2] or Lemma 6.1.1 of [L2] $M$ is linearly spanned by the coefficients of $Y_M(a,z)u$ for $a \in V$ and fixed $u \in W$. Regarding $V$, $W$ and $M$ as $U$-modules, we have an intertwining operator $Y_M$ of type $(V^M_{VW})$ (see [FHL] for the definition of intertwining operator). It follows from the universal property of the tensor product that there is a $U$-homomorphism $\psi$ from $V \boxtimes W$ onto $M$ (cf. [HL0]-[HL1] and [L2]). From our assumption, $V \boxtimes W$ is a sum of finitely many irreducible $U$-modules, so that any homogeneous subspace is finite-dimensional. Then any homogeneous subspace of $M$ is finite-dimensional. That is, $M$ is an ordinary $g$-twisted $V$-module. □

A similar result has been obtained in [H] in the special case when $g = 1$ and $V$ contains a rational vertex operator subalgebra which is a tensor product of vertex operator algebras associated with the highest weight irreducible representations for the discrete series of the Virasoro algebra.

**Proposition 9.10.** Suppose that $V$ is a holomorphic VOA and that $V$ contains a rational vertex operator subalgebra $U$ (with the same Virasoro element) such that the fusion rules among any three irreducible $U$-modules are finite. Then $V$ is rational.

**Proof.** We need to prove that any admissible $V$-module $M$ is completely reducible. Now both $V$ and $M$ are direct sums of simple $U$-modules as $U$ is rational. Let $W$ be a simple $U$-submodule of $M$ and let $\tilde{W}$ be the weak $V$-submodule generated by $W$. As in the proof of Proposition 9.9, we easily show that any homogeneous subspace of the $U$-module $V \boxtimes W$ is finite-dimensional. Thus being a $U$-homomorphic image of a $U$-module $V \boxtimes W$, $\tilde{W}$ is a $U$-module. By Proposition 9.9, $\tilde{W}$ is a completely reducible $V$-module. Thus $M$ is a completely reducible $V$-module. □

**Remark 9.11.** (i) Recall Remark 9.7. The moonshine module vertex operator algebra $V^\natural$ satisfies the condition of Proposition 9.10. So $V^\natural$ is a rational holomorphic vertex operator algebra.

(ii) A situation in which Proposition 9.9 applies is where $V = V^\natural$ and $g$ any involution in the Monster. This is studied in detail in [DLM2].

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