The Norton algebra of a $Q$-polynomial distance-regular graph

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Abstract

We consider the Norton algebra associated with a $Q$-polynomial primitive idempotent of the adjacency matrix for a distance-regular graph. We obtain a formula for the Norton algebra product that we find attractive.

Keywords. Bose-Mesner algebra; Krein parameter; $Q$-polynomial; Leonard system.

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1 Introduction

There is a family of highly regular graphs said to be distance-regular [1,2,4]. Examples include the Johnson graphs [2, Section 9.1], the Hamming graphs [2, Section 9.2], the Grassmann graphs [2, Section 9.3], and the dual polar graphs [2, Section 9.4]. The graphs in these four families are particularly attractive for several reasons: (i) they have a $Q$-polynomial structure, according to which their Krein parameters vanish in a certain attractive pattern; (ii) these graphs come with a ranked partially ordered set that can be used to analyze the graph.

In the analysis of any distance-regular graph $\Gamma$, one often considers the eigenspaces for the adjacency matrix $A$ of $\Gamma$. By [3] these eigenspaces possess an algebra structure, called the Norton algebra, that is commutative but not necessarily associative. The Norton product $\star$ is described as follows. Let $X$ denote the vertex set of $\Gamma$. The rows and columns of $A$ are indexed by $X$. The matrix $A$ acts on a vector space $V$ over $\mathbb{R}$, consisting of column vectors whose coordinates are indexed by $X$. For $x \in X$ let $\hat{x}$ denote the vector in $V$ that has $x$-coordinate 1 and all other coordinates zero. So $\{\hat{x} \mid x \in X\}$ is a basis for $V$. The entry-wise product $\circ : V \times V \to V$ satisfies $\hat{x} \circ \hat{y} = \delta_{x,y}\hat{x}$ for all $x, y \in X$. The matrix $A$ is diagonalizable since it is symmetric, so $V$ is a direct sum of the eigenspaces of $A$. For an eigenspace of $A$, the corresponding primitive idempotent $E$ acts as the identity on the eigenspace, and zero on the other eigenspaces of $A$. Thus $E$ is the projection from $V$ onto the eigenspace. The eigenspace is $EV$. For $u, v \in EV$ we have $u \star v = E(u \circ v)$.

Earlier we mentioned $Q$-polynomial structures. For a given $Q$-polynomial structure on $\Gamma$ the adjacency matrix $A$ has a distinguished primitive idempotent, said to be $Q$-polynomial. Recently, several authors have considered the Norton algebra $EV$ for a $Q$-polynomial primitive idempotent $E$ of $A$. This was done by C. Maldonado and D. Penazzi in [8], under the
assumption that Γ is a Johnson graph, Hamming graph, or Grassmann graph. It was done by F. Levstein, C. Maldonado, and D. Penazzi in [7], under the assumption that Γ is a dual polar graph. In both articles the authors compute \( u \star \bar{v} \) for all \( u, v \in L_1 \), where \( \{u|u \in L_1\} \) is a certain spanning set for EV indexed by the set \( L_1 \) of rank 1 elements in the associated poset. The results of [7][8] are used by J. Huang in [5] to investigate the extent to which the Norton product is nonassociative.

In the present paper we consider the Norton algebra \( EV \), where \( E \) is a \( Q \)-polynomial primitive idempotent of the adjacency matrix \( A \) for any distance-regular graph \( \Gamma \) with diameter \( d \geq 2 \). For all vertices \( x, y \) of \( \Gamma \) we give an explicit formula for the Norton product \( E\hat{x} \star E\hat{y} \), in terms of a few eigenvalues \( \theta_0, \theta_1, \theta_2 \) of \( A \) and a sequence of scalars \( \{\theta_i^*\}_{i=0}^d \) called the dual eigenvalues of \( \Gamma \) associated with \( E \). We give two versions of our formula. The first version is more straightforward. To obtain the second version, we use the balanced set condition [9][10] to make the symmetry \( E\hat{x} \star E\hat{y} = E\hat{y} \star E\hat{x} \) explicit. Our main results are Theorems 3.7 and 4.4.

The paper is organized as follows. In Section 2 we review some basic concepts concerning distance-regular graphs. In Section 3 we recall the Norton algebra and obtain the first version of our Norton product formula. In Section 4 we use the balanced set condition to obtain the second version of our Norton product formula. In Section 5 we remark how certain equations in Sections 3, 4 can be obtained using the theory of Leonard systems.

## 2 Preliminaries

In this section we review some basic concepts concerning distance-regular graphs. For more background information we refer the reader to [1][2][4].

Let \( \mathbb{R} \) denote the real number field. Let \( X \) denote a nonempty finite set. Let \( \text{Mat}_X(\mathbb{R}) \) denote the \( \mathbb{R} \)-algebra consisting of the matrices that have rows and columns indexed by \( X \) and all entries in \( \mathbb{R} \). Let \( I \) denote the identity matrix in \( \text{Mat}_X(\mathbb{R}) \), and let \( J \) denote the matrix in \( \text{Mat}_X(\mathbb{R}) \) that has all entries 1. Let \( V = \mathbb{R}X \) denote the vector space over \( \mathbb{R} \) consisting of the column vectors that have coordinates indexed by \( X \) and all entries in \( \mathbb{R} \).

The algebra \( \text{Mat}_X(\mathbb{R}) \) acts on \( V \) by left multiplication. For \( x \in X \) let \( \hat{x} \) denote the vector in \( V \) that has \( x \)-coordinate 1 and all other coordinates 0. The vectors \( \{\hat{x}|x \in X\} \) form a basis for \( V \). Let \( 1 \) denote the vector in \( V \) that has all entries 1. So \( 1 = \sum_{x \in X} \hat{x} \). Note that \( J\hat{x} = 1 \) for all \( x \in X \).

Let \( \Gamma = (X, R) \) denote an undirected connected graph, without loops or multiple edges, with vertex set \( X \), edge set \( R \), and path-length distance function \( \partial \). Recall the diameter \( d = \max\{\partial(x, y)|x, y \in X\} \). For \( x \in X \) and an integer \( i \geq 0 \) define \( \Gamma_i(x) = \{y \in X|\partial(x, y) = i\} \).

We abbreviate \( \Gamma(x) = \Gamma_1(x) \). For an integer \( k \geq 0 \), \( \Gamma \) is said to be regular with valency \( k \) whenever \( k = |\Gamma(x)| \) for \( x \in X \). The graph \( \Gamma \) is said to be distance-regular whenever for all integers \( h, i, j \) \( (0 \leq h, i, j \leq d) \) and all \( x, y \in X \) at distance \( \partial(x, y) = h \), the scalar \( p_{ij}^h = |\Gamma_i(x) \cap \Gamma_j(y)| \) is independent of \( x \) and \( y \). The scalars \( p_{ij}^h \) are called the intersection numbers of \( \Gamma \). For the rest of this paper, we assume that \( \Gamma \) is distance-regular with diameter \( d \geq 2 \). By construction \( p_{ij}^h = p_{ji}^h \) for \( 0 \leq h, i, j \leq d \). By the triangle inequality we find that for \( 0 \leq h, i, j \leq d \),
(i) \(p_{i,j}^h = 0\) if one of \(h, i, j\) is greater than the sum of the other two;

(ii) \(p_{i,j}^h \neq 0\) if one of \(h, i, j\) is equal to the sum of the other two.

We abbreviate \(c_i = p_{1,i-1}^1 (1 \leq i \leq d), a_i = p_{1,i}^1 (0 \leq i \leq d), b_i = p_{1,i+1}^1 (0 \leq i \leq d - 1)\). By construction \(c_i \neq 0\) for \(1 \leq i \leq d\) and \(b_i \neq 0\) for \(0 \leq i \leq d - 1\). The graph \(\Gamma\) is regular with valency \(k = b_0\). Moreover \(k = c_i + a_i + b_i\) for \(0 \leq i \leq d\), where \(c_0 = 0\) and \(b_d = 0\).

Next we recall the Bose-Mesner algebra of \(\Gamma\). For \(0 \leq i \leq d\) let \(A_i\) denote the matrix in \(\text{Mat}(\mathbb{R})\) that has \((x,y)\)-entry

\[
(A_i)_{x,y} = \begin{cases} 
1, & \text{if } \partial(x,y) = i; \\
0, & \text{if } \partial(x,y) \neq i 
\end{cases} \quad (x,y \in X).
\]

We call \(A_i\) the \(i\)-th distance-matrix of \(\Gamma\). Note that \(A_0 = I\). We abbreviate \(A = A_1\) and call this the adjacency matrix of \(\Gamma\). By the construction,

\[
A_i A_j = \sum_{h=0}^{d} p_{i,j}^h A_h \quad (0 \leq i, j \leq d).
\]

By these comments the matrices \(\{A_i\}_{i=0}^{d}\) form a basis for a commutative subalgebra of \(\text{Mat}_X(\mathbb{R})\). This algebra is denoted by \(M\) and called the Bose-Mesner algebra of \(\Gamma\). The algebra \(M\) is generated by \(A\) \([\text{II} \ p. 190]\).

Next we recall the primitive idempotents and eigenvalues of \(\Gamma\). By [2] p. 45] the vector space \(M\) has a basis \(\{E_i\}_{i=0}^{d}\) such that (i) \(E_0 = |X|^{-1} J\); (ii) \(I = \sum_{i=0}^{d} E_i\); (iii) \(E_i E_j = \delta_{i,j} E_i (0 \leq i, j \leq d)\). This basis is unique up to permutation of \(\{E_i\}_{i=0}^{d}\). We call \(\{E_i\}_{i=0}^{d}\) the primitive idempotents of \(M\) (or \(\Gamma\)). The primitive idempotent \(E_0\) is called trivial. By construction, there exist real numbers \(\{\theta_i\}_{i=0}^{d}\) such that \(A = \sum_{i=0}^{d} \theta_i E_i\). The \(\{\theta_i\}_{i=0}^{d}\) are mutually distinct since \(A\) generates \(M\). Using \(E_0 = |X|^{-1} J\) we obtain \(\theta_0 = k\). The scalars \(\{\theta_i\}_{i=0}^{d}\) are called the eigenvalues of \(A\) (or \(\Gamma\)).

Next we recall the Krein parameters of \(\Gamma\). For \(0 \leq i, j \leq d\) we have \(A_i \cdot A_j = \delta_{i,j} A_i\), where \(\cdot\) denotes the entry-wise product for \(\text{Mat}_X(\mathbb{R})\). Therefore \(M\) is closed under the \(\cdot\) product. Consequently there exist \(q_{i,j}^h \in \mathbb{R} (0 \leq h, i, j \leq d)\) such that

\[
E_i \cdot E_j = |X|^{-1} \sum_{h=0}^{d} q_{i,j}^h E_h \quad (0 \leq i, j \leq d).
\]

By construction \(q_{i,j}^h = q_{j,i}^h\) for \(0 \leq h, i, j \leq d\). By [2] Proposition 4.1.5 we have \(q_{i,j}^h \geq 0\) for \(0 \leq h, i, j \leq d\). The scalars \(q_{i,j}^h\) are called the Krein parameters of \(\Gamma\).

We describe one significance of the Krein parameters. In this description, we will use the following notation. For \(u \in V\) and \(x \in X\) let \(u_x\) denote the \(x\)-coordinate of \(u\). So \(u = \sum_{x \in X} u_x \hat{x}\). For \(u, v \in V\) their entry-wise product \(u \circ v\) is the vector in \(V\) that has \(x\)-coordinate \(u_x v_x\) for all \(x \in X\). So \(u \circ v = \sum_{x \in X} u_x v_x \hat{x}\). For \(x, y \in X\) we have

\[
\hat{x} \circ \hat{y} = \begin{cases} 
\hat{x}, & \text{if } x = y; \\
0, & \text{if } x \neq y.
\end{cases}
\]
For \( v \in V \) we have \( 1 \circ v = v \). For subspaces \( Y, Z \) of \( V \) define \( Y \circ Z = \text{Span}\{y \circ z|y \in Y, z \in Z\} \).

By [3, Proposition 5.1] we have

\[
E_i V \circ E_j V = \sum_{0 \leq h \leq d}^{q_{ij}^h \neq 0} E_h V \quad (0 \leq i, j \leq d).
\] (3)

Next we recall the \( Q \)-polynomial property. The given ordering \( \{E_i\}_{i=1}^d \) of the nontrivial primitive idempotents of \( \Gamma \) is said to be \( Q \)-polynomial whenever for \( 0 \leq h, i, j \leq d \),

(i) \( q_{i,j}^h = 0 \) if one of \( h, i, j \) is greater than the sum of the other two;

(ii) \( q_{i,j}^h \neq 0 \) if one of \( h, i, j \) is equal to the sum of the other two.

Let \( E \) denote a nontrivial primitive idempotent of \( \Gamma \). We say that \( E \) is \( Q \)-polynomial whenever there exists a \( Q \)-polynomial ordering \( \{E_i\}_{i=1}^d \) of the nontrivial primitive idempotents of \( \Gamma \) such that \( E = E_1 \). For the rest of this paper we assume that \( E \) is \( Q \)-polynomial. By construction, there exist real numbers \( \{\theta_i^*\}_{i=0}^d \) such that

\[
E = |X|^{-1} \sum_{i=0}^d \theta_i^* A_i. \quad (4)
\]

By [1, p. 260] the scalars \( \{\theta_i^*\}_{i=0}^d \) are mutually distinct. The scalars \( \{\theta_i^*\}_{i=0}^d \) are called the dual eigenvalues of \( \Gamma \) associated with \( E \). For notational convenience let \( \theta_{-1}^* \) and \( \theta_{d+1}^* \) denote indeterminates. Taking the trace of each side of (4) yields \( \theta_0^* = \text{rank}(E) \). Also, multiplying both sides of (4) by \( A \) and evaluating the result yields

\[
c_i \theta_{i-1}^* + a_i \theta_i^* + b_i \theta_{i+1}^* = \theta_i \theta_i^* \quad (0 \leq i \leq d). \quad (5)
\]

### 3 The Norton algebra

We continue to discuss the distance-regular graph \( \Gamma \) and its \( Q \)-polynomial primitive idempotent \( E \). In this section we turn the vector space \( EV \) into a commutative nonassociative algebra called the Norton algebra.

**Definition 3.1.** (See [3, Proposition 5.2].) The **Norton algebra** of \( \Gamma \) consists of the vector space \( EV \), together with the product

\[
u * v = E(\circ v) \quad (u, v \in EV).
\]

The Norton algebra is commutative, but not necessarily associative.

The vector space \( EV \) is spanned by the vectors \( \{E \hat{x}|x \in X\} \). These vectors are nonzero, mutually distinct, and linearly dependent [3, Theorem 1.1]. As we investigate the Norton product \( * \) it is natural to consider \( E \hat{x} * E \hat{y} \) for all \( x, y \in X \). In the next two lemmas we discuss some extremal cases.

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3 The Norton algebra...
Lemma 3.2. For $x \in X$,

$$E\hat{x} \star E\hat{x} = |X|^{-1}q_{1,1}^1 E\hat{x}. \quad (6)$$

Proof. By (11) we have $E \cdot E = |X|^{-1}\sum_{h=0}^d q_{1,1}^h E_h$. For this equation, multiply each side by $E$ to obtain $E(E \cdot E) = |X|^{-1}q_{1,1}^1 E$. For this equation, compare column $x$ of each side to obtain (6). \hfill \square

Lemma 3.3. The following are equivalent:

(i) $E\hat{x} \star E\hat{y} = 0$ for all $x, y \in X$;

(ii) $u \star v = 0$ for all $u, v \in EV$;

(iii) The Krein parameter $q_{1,1}^1 = 0$.

Proof. By (3) and the construction. \hfill \square

We have been discussing some extremal cases. Before we proceed to the general case, we bring in some notation. To motivate things, observe that for $x \in X$ and $0 \leq i \leq d$,

$$A_i\hat{x} = \sum_{z \in \Gamma_i(x)} \hat{z}. \quad (7)$$

Lemma 3.4. For $x, y \in X$ and $0 \leq i, j \leq d$ we have

$$A_i\hat{x} \circ A_j\hat{y} = \sum_{z \in \Gamma_i(x) \cap \Gamma_j(y)} \hat{z}. \quad (8)$$

Proof. Use (2) and (7). \hfill \square

Definition 3.5. Pick $x, y \in X$ and write $i = \partial(x, y)$. Define

$$x^+_y = A\hat{x} \circ A_{i+1}\hat{y} = \sum_{z \in \Gamma(x) \cap \Gamma_{i+1}(y)} \hat{z}, \quad (8)$$

$$x^0_y = A\hat{x} \circ A_{i}\hat{y} = \sum_{z \in \Gamma(x) \cap \Gamma_i(y)} \hat{z}, \quad (9)$$

$$x^-_y = A\hat{x} \circ A_{i-1}\hat{y} = \sum_{z \in \Gamma(x) \cap \Gamma_{i-1}(y)} \hat{z}, \quad (10)$$

where we understand $A_{-1} = 0$, $\Gamma_{-1}(x) = \emptyset$ and $A_{d+1} = 0$, $\Gamma_{d+1}(x) = \emptyset$.

We clarify the notation (8)–(10). Pick $x, y \in X$. If $\partial(x, y) = d$ then $x^+_y = 0$. If $\partial(x, y) = 1$ then $x^-_y = \hat{y}$. If $x = y$ then $x^0_y = 0$ and $x^-_y = 0$.

Lemma 3.6. For $x, y \in X$ we have

$$x^+_y + x^0_y + x^-_y = A\hat{x}, \quad (11)$$

$$Ex^+_y + Ex^0_y + Ex^-_y = \theta_1 E\hat{x}. \quad (12)$$
Proof. To verify (11), note that each side is equal to $\sum_{x \in \Gamma(x)} \hat{z}$. To get (12), apply $E$ to each side of (11), and use $EA = \theta_1 E$.

The following is our first main result.

**Theorem 3.7.** Assume that $\Gamma$ is $Q$-polynomial with respect to $E$. Then for all $x, y \in X$ we have

$$E\hat{x} \ast E\hat{y} = \frac{(\theta_{i-1} - \theta_i)Ex^- + (\theta_{i+1} - \theta_i)Ex^+ + (\theta_1 - \theta_2)\theta_i E\hat{x} + (\theta_2 - \theta_0)E\hat{y}}{|X|(|\theta_1 - \theta_2|)}$$

(13)

where $i = \partial(x, y)$. Here $\theta_{i-1}$ and $\theta_{i+1}$ denote indeterminates.

**Proof.** We consider the vector

$$E(A\hat{x} \circ E\hat{y}) - \theta_2 E(\hat{x} \circ E\hat{y}).$$

We evaluate this vector in two ways. For the first evaluation, use $A - \theta_2 I = \sum_{h=0}^{d}(\theta_h - \theta_2)E_h$ to obtain

$$E(A\hat{x} \circ E\hat{y}) - \theta_2 E(\hat{x} \circ E\hat{y}) = \sum_{h=0}^{d}(\theta_h - \theta_2)E(E_h\hat{x} \circ E\hat{y}).$$

For the above sum, we examine the $h$-summand for $0 \leq h \leq d$. For $h = 0$ the summand is $(\theta_0 - \theta_2)|X|^{-1}E\hat{y}$ because

$$E_0 \hat{x} \circ E\hat{y} = |X|^{-1}J\hat{x} \circ E\hat{y} = |X|^{-1}1 \circ E\hat{y} = |X|^{-1}E\hat{y}.$$  

For $h = 1$ the summand is $(\theta_1 - \theta_2)E\hat{x} \ast E\hat{y}$ by Definition 3.1. For $h = 2$ the summand is zero by construction. For $3 \leq h \leq d$ the summand is zero by (3) and the definition of $Q$-polynomial. By these comments,

$$E(A\hat{x} \circ E\hat{y}) - \theta_2 E(\hat{x} \circ E\hat{y}) = (\theta_0 - \theta_2)|X|^{-1}E\hat{y} + (\theta_1 - \theta_2)E\hat{x} \ast E\hat{y}.$$  

(14)

For the second evaluation, use $E = |X|^{-1} \sum_{\ell=0}^{d} \theta_\ell A_\ell$ to obtain

$$(A - \theta_2 I)\hat{x} \circ E\hat{y} = |X|^{-1} \sum_{\ell=0}^{d} \theta_\ell (A - \theta_2 I)\hat{x} \circ A_\ell \hat{y}.$$  

For the above sum, we examine the $\ell$-summand for $0 \leq \ell \leq d$. The term $A\hat{x} \circ A_\ell \hat{y}$ is equal to $x^-_\ell$ (if $\ell = i-1$) and $x^0_\ell$ (if $\ell = i$) and $x^+_\ell$ (if $\ell = i+1$) and zero (if $|\ell - i| > 1$). The term $\hat{x} \circ A_\ell \hat{y}$ is equal to $\hat{x}$ (if $\ell = i$) and zero (if $\ell \neq i$). By these comments,

$$(A - \theta_2 I)\hat{x} \circ E\hat{y} = |X|^{-1}(\theta_{i-1} x^-_i + \theta_i x^0_i + \theta_{i+1} x^+_i - \theta_2 \theta_i \hat{x}).$$

Therefore

$$E(A\hat{x} \circ E\hat{y}) - \theta_2 E(\hat{x} \circ E\hat{y}) = |X|^{-1}(\theta_{i-1}Ex^-_i + \theta_i Ex^0_i + \theta_{i+1}Ex^+_i - \theta_2 \theta_i E\hat{x}).$$  

(15)

Comparing (14), (15) we obtain

$$|X|(\theta_1 - \theta_2)E\hat{x} \ast E\hat{y} = \theta_{i-1}Ex^-_i + \theta_i Ex^0_i + \theta_{i+1}Ex^+_i - \theta_2 \theta_i E\hat{x} + (\theta_2 - \theta_0)E\hat{y}.$$  

(16)

In (16) eliminate the term $Ex^0_i$ using (12). In the resulting equation, solve for $E\hat{x} \ast E\hat{y}$ and we are done.  

\end{proof}
Referring to Theorem 3.7, the formula for $E\hat{x} \ast E\hat{y}$ can be simplified if $i \in \{0, 1, d\}$. This simplification is discussed next.

**Corollary 3.8.** Assume that $\Gamma$ is $Q$-polynomial with respect to $E$. Then (i)--(iii) hold below.

(i) For $x \in X$,

$$E\hat{x} \ast E\hat{x} = \frac{\theta_1\theta_1^* - \theta_2\theta_0^* + \theta_2 - \theta_0}{|X|}(\theta_1 - \theta_2)E\hat{x}.$$ 

(ii) For $x, y \in X$ at distance $\partial(x, y) = 1$,

$$E\hat{x} \ast E\hat{y} = \frac{(\theta_2^* - \theta_1^*)Ex_y^+ + (\theta_1 - \theta_2)\theta_1^*Ex_x + (\theta_2 - \theta_0 + \theta_0^* - \theta_1^*)Ey_y}{|X|}(\theta_1 - \theta_2).$$

(iii) For $x, y \in X$ at distance $\partial(x, y) = d$,

$$E\hat{x} \ast E\hat{y} = \frac{(\theta_d - \theta)^*Ex_y^+ + (\theta_1 - \theta_2)\theta_d^*Ex_x + (\theta_2 - \theta_0)Ey_y}{|X|}(\theta_1 - \theta_2).$$

**Proof.** (i) We evaluate (13) with $y = x$ and $i = 0$. We have $x_y^- = 0$ and $x_0^y = 0$, so $Ex_y^+ = \theta_1Ex_x$ by Lemma 3.6.

(ii) Set $i = 1$ in (13) and use $x_y^- = \hat{y}$.

(iii) Set $i = d$ in (13) and use $x_y^+ = 0$. \qed

**Corollary 3.9.** Assume that $\Gamma$ is $Q$-polynomial with respect to $E$. Then the Krein parameter $q_{1,1}^1$ satisfies

$$q_{1,1}^1 = \frac{\theta_1\theta_1^* - \theta_2\theta_0^* + \theta_2 - \theta_0}{\theta_1 - \theta_2}.$$ 

**Proof.** Compare Lemma 3.2 and Corollary 3.8(i). \qed

The eigenvalue $\theta_2$ appears in the above results. By [12] Lemma 19.21 we find that $1 + \theta_1$, $1 + \theta_1^*$ are nonzero and

$$\frac{1 + \theta_1}{\theta_0 - \theta_2} = \frac{1 + \theta_1^*}{\theta_0^* - \theta_2^*}. \tag{17}$$

**4 The Norton product in symmetric form**

We continue to discuss the distance-regular graph $\Gamma$ and its $Q$-polynomial primitive idempotent $E$. Pick distinct $x, y \in X$. In the formula (13) we computed $E\hat{x} \ast E\hat{y}$. We have $E\hat{x} \ast E\hat{y} = E\hat{y} \ast E\hat{x}$, so the right-hand side of (13) must be invariant if we interchange $x, y$. In this section, we express the right-hand side of (13) in a form that makes this invariance explicit. We will use a result known as the balanced set condition.
Lemma 4.1. (See [9] Theorem 1.1, [10] Theorem 3.3.) For distinct \( x, y \in X \) we have
\[
Ex_y^+ - Ey_x^+ = b_i \frac{\theta_i^* - \theta_{i+1}^*}{\theta_0^* - \theta_i^*} (E\hat{x} - E\hat{y}),
\tag{19}
\]
\[
Ex_y^- - Ey_x^- = c_i \frac{\theta_i^* - \theta_{i-1}^*}{\theta_0^* - \theta_i^*} (E\hat{x} - E\hat{y}),
\tag{18}
\]
where \( i = \partial(x, y) \).

Corollary 4.2. For distinct \( x, y \in X \) we have
\[
C(x, y) = C(y, x), \quad B(x, y) = B(y, x)
\]
where
\[
C(x, y) = Ex_y^- - c_i \frac{\theta_i^* - \theta_{i-1}^*}{\theta_0^* - \theta_i^*} E\hat{x},
\tag{20}
\]
\[
B(x, y) = Ex_y^+ - b_i \frac{\theta_1^* - \theta_i^*}{\theta_0^* - \theta_i^*} E\hat{x}
\tag{21}
\]
and \( i = \partial(x, y) \).

Proof. Rearrange the terms in (18), (19).

We clarify the meaning of (20) and (21). For \( i = 1 \) we have \( C(x, y) = E\hat{x} + E\hat{y} \). For \( i = d \) we have \( B(x, y) = 0 \).

For distinct \( x, y \in X \) we are going to express \( E\hat{x} \star E\hat{y} \) in terms of \( C(x, y) \) and \( B(x, y) \). The following equation will be useful.

Lemma 4.3. We have
\[
c_i \frac{(\theta_i^* - \theta_{i-1}^*)(\theta_{i-1}^* - \theta_i^*)}{\theta_0^* - \theta_i^*} + b_i \frac{(\theta_1^* - \theta_i^*)(\theta_{i+1}^* - \theta_i^*)}{\theta_0^* - \theta_i^*} = (\theta_2 - \theta_1)\theta_i^* + \theta_2 - \theta_0
\]
for \( 1 \leq i \leq d - 1 \) and
\[
c_d \frac{(\theta_1^* - \theta_d^*)(\theta_{d-1}^* - \theta_d^*)}{\theta_0^* - \theta_d^*} = (\theta_2 - \theta_1)\theta_d^* + \theta_2 - \theta_0.
\]

Proof. In the equation \( 0 = E\hat{x} \star E\hat{y} - E\hat{y} \star E\hat{x} \), expand the right-hand side using Theorem 3.7 and evaluate the result using Lemma 4.1. Examine the outcome using the fact that \( E\hat{x} \neq E\hat{y} \).

The following is our second main result.

Theorem 4.4. Assume that \( \Gamma \) is \( Q \)-polynomial with respect to \( E \). Then for distinct \( x, y \in X \) we have
\[
E\hat{x} \star E\hat{y} = \frac{(\theta_{i-1}^* - \theta_i^*)C(x, y) + (\theta_{i+1}^* - \theta_i^*)B(x, y) + (\theta_2 - \theta_0)(E\hat{x} + E\hat{y})}{|X| (\theta_1 - \theta_2)}
\tag{22}
\]
where \( i = \partial(x, y) \). Here \( \theta_{d+1}^* \) denotes an indeterminate.
Proof. To verify (22), expand the right-hand side using (20), (21) and evaluate the result using Theorem 3.7 along with Lemma 4.3.

Referring to Theorem 4.4, the formula for \( E^\hat{x} \ast E^\hat{y} \) can be simplified if \( i \in \{1, d\} \). This simplification is discussed next.

**Corollary 4.5.** Assume that \( \Gamma \) is \( Q \)-polynomial with respect to \( E \). Then (i), (ii) hold below.

(i) For \( x, y \in X \) at distance \( \partial(x, y) = 1 \),

\[
E^\hat{x} \ast E^\hat{y} = \frac{(\theta_2^* - \theta_1^*)B(x, y) + (\theta_2 - \theta_0 + \theta_0^* - \theta_1^*)(E^\hat{x} + E^\hat{y})}{|X|(\theta_1 - \theta_2)}.
\]

(ii) For \( x, y \in X \) at distance \( \partial(x, y) = d \),

\[
E^\hat{x} \ast E^\hat{y} = \frac{(\theta_d^* - \theta_d^*)(C(x, y) + (\theta_2 - \theta_0)(E^\hat{x} + E^\hat{y})}{|X|(\theta_1 - \theta_2)}.
\]

Proof. (i) Set \( i = 1 \) in (22) and use \( C(x, y) = E^\hat{x} + E^\hat{y} \).

(iii) Set \( i = d \) in (22) and use \( B(x, y) = 0 \).

---

5 Remarks

The algebraic structure of a \( Q \)-polynomial distance-regular graph can be described using the concept of a Leonard system [11, Definition 1.4]; this concept was motivated by a theorem of D. A. Leonard [1, p. 260], [6]. We refer the reader to [11, 12] for the standard notation and basic results about Leonard systems. The equations below are routinely obtained using the formulas in [12, Sections 19, 20]. Let \( \Phi \) denote any Leonard system with diameter \( d \geq 2 \).

For \( \Phi \) we have

\[
a_i^* = \frac{(\theta_1 - a_0)\theta_1^* - (\theta_2 - a_0)\theta_0^* + (\theta_2 - \theta_0)c_i^*}{\theta_1 - \theta_2}.
\]

If we set \( a_i^* = q_{i,1}^1 \) and \( a_0 = 0 \) and \( c_1^* = 1 \) then we recover the formula in Corollary 3.9.

For \( \Phi \) we also have

\[
c_1 - a_0 + \theta_1 \quad \frac{c_1^* - a_0^* + \theta_1^*}{\theta_0^* - \theta_2^*}.
\]

If we set \( c_1 = 1, \ a_0 = 0 \) and \( c_1^* = 1, \ a_0^* = 0 \) then we recover (17).

For \( \Phi \) we also have

\[
c_i \frac{(\theta_1 - \theta_1^*)(\theta_i^* - \theta_i^*)}{\theta_0^* - \theta_i^*} + b_i \frac{(\theta_1 - \theta_i^*)(\theta_i^* - \theta_i^*)}{\theta_0^* - \theta_i^*} = (\theta_2 - \theta_1)(\theta_i^* - a_0^*) + (\theta_2 - \theta_0)c_i^*
\]

for \( 1 \leq i \leq d - 1 \) and

\[
c_d \frac{(\theta_1^* - \theta_d^*)(\theta_d^* - \theta_d^*)}{\theta_0^* - \theta_d^*} = (\theta_2 - \theta_1)(\theta_d^* - a_0^*) + (\theta_2 - \theta_0)c_1^*.
\]

If we set \( a_0^* = 0 \) and \( c_1^* = 1 \) then we recover the formulas in Lemma 4.3.
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