In the present paper we analyze algebraic structures arising in Yang-Mills theory. The paper should be considered as a part of a project started with [15] and devoted to maximally supersymmetric Yang-Mills theories. In this paper we collected those of our results which are correct without assumption of supersymmetry and used them to give rigorous proofs of some results of [15]. We consider two different algebraic interpretations of Yang-Mills theory - in terms of $A_{\infty}$-algebras and in terms of representations of Lie algebras (or associative algebras). We analyse the relations between these two approaches and calculate some Hochschild (co)homology of algebras in question.

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1 Introduction

Suppose \( g \) is a Lie algebra equipped with nondegenerate inner product \(<.,.>\). We consider Yang Mills field \( A \) as \( g \)-valued one-form on complex \( D \)-dimensional vector space \( V \) equipped with symmetric bilinear inner product \((.,.)\). (All vector spaces in this paper are defined over complex numbers.) We are writing this form as \( A = \sum_{i=1}^{D} A_i dx^i \), where \( x^1, \ldots, x^D \) is an orthogonal coordinate system on \( V \) which is fixed for the rest of the paper.

We assume that the field \( A \) interacts with bosonic and fermionic matter fields \( \phi, \psi \) which are functions on vector space \( V \) with values in \( \Phi \otimes g, \Pi S \otimes g \) respectively. The symbol \( \Pi \) stands for the change of parity. In other words matter fields transform according to adjoint representation of \( g \). The linear space \( \Phi \) is equipped with a symmetric inner product \((.,.)\), the linear space \( S \) is equipped with a symmetric bilinear map \( \Gamma : \text{Sym}^2(S) \rightarrow V \). An important example is 10D SUSY Yang-Mills theory where \( D = 10, \Phi = 0, V \) and \( S \) are spaces of vector and spinor representations of \( SO(10) \) and \( \Gamma \) stands for \( SO(10) \)-intertwiner \( \text{Sym}^2S \rightarrow V \).

We will always consider the action functional \( S \) as a holomorphic functional on the space of fields; to quantize one integrates \( \exp(-S) \) over a real slice in this space. (For example, if \( g = \mathfrak{gl}(n) \) one takes \( u(n) \)-valued gauged fields as a real slice in the space of gauge fields). All considerations are local; in other words our fields are polynomials or power series on \( V \). This means that the action functionals are formal expressions (integration over \( V \) is ill-defined). However we work with the equations of motion which are well-defined. It is easy to get rid of this nuisance and make definitions completely rigorous.

Choosing once and for all an orthonormal basis in \( \Phi \) and some basis in \( S \) we can identify \( \phi, \psi \) with \( g \)-valued fields \( (\phi_1, \ldots, \phi_{d'}) \) and \( (\psi^\alpha) \). The Lagrangian in these bases takes the form:

\[
L = \frac{1}{4} \sum_{i,j=1}^{D} <F_{ij}, F_{ij}> + \sum_{i=1}^{D} \sum_{j=1}^{d'} <\nabla_i \phi_j, \nabla_i \phi_j> + \sum_{i=1}^{D} \sum_{\alpha,\beta} <\Gamma_{\alpha\beta}^i, \nabla_i \psi^\alpha, \psi^\beta> - U(\phi, \psi)
\]  

(1)
where $\nabla_i$ stands for covariant derivative built out of $A_i$, $F_{ij} = \partial_i A_j - \partial_j A_i + [A_i, A_j]$ denotes gauge field strength, $\Gamma^i_{\alpha\beta}$ is the matrix of linear map $\Gamma$ in the chosen bases; $U$ is a $g$-invariant potential. Corresponding action functional $S_{cl}$ is gauge invariant and can be extended to solution of BV-master equation in a standard way:

$$S = S_{cl} + \int \left( \sum_{i=1}^D < \nabla_i c, A_i^* > + \sum_{\alpha} < [c, \psi^\alpha], \psi_\alpha^* > + \sum_{j=1}^{d'} < [c, \phi_j], \phi_j^* > + \frac{1}{2} < [c, c^*], c^* > \right) dx^1 \ldots dx^D$$

(2)

Here $c$ stands for Grassmann odd ghost field, $A_i^*, \psi_\alpha^*, \phi_j^*, c^*$ are antifields for $A_i, \psi_\alpha, \phi_j, c$. (The parity of antifields is opposite to the parity of fields.)

The BV action functional $S$ determines a vector field $Q$ on the space of fields, where $Q$ obeys $Q^2 = 0$. The space of solutions of equations of motion in BV-formalism coincides with zero locus of $Q$. Using $Q$ we introduce a structure of $L_\infty$-algebra on the space of fields (see [1] or Appendix C in [15]). (Recall that the Taylor coefficients of vector field $Q$ obeying $Q^2 = 0$ at a point belonging to a zero locus of $Q$ specify an $L_\infty$-algebra. The point in the space of fields where all fields vanish belongs to the zero locus of $Q$; we construct $L_\infty$-algebra using Taylor expansion of $Q$ at this point). Equations of motion can be identified with Maurer-Cartan equations for $L_\infty$-algebra.

The $L_\infty$-algebra $\mathcal{L}$ we constructed depends on the choice of Lie algebra $g$ and other data: potential, inner products on spaces $V$, $\Phi$, bilinear map $\Gamma$ on $S$. When we need to emphasize such dependence we will do it by appropriate subscript: e.g. $\mathcal{L}_{\mathfrak{gl}_n}$ shows that the Lie algebra $\mathfrak{g}$ is $\mathfrak{gl}_n$. We will assume that the potential $U(\phi, \psi)$ in our algebraic approach is a polynomial (or, more generally, formal power series) of $\phi$ and $\psi$. If $\mathfrak{g} = \mathfrak{gl}_n$ it has the form

$$U(\phi, \psi) = tr(P(\phi, \psi))$$

where $P(\phi, \psi)$ is a noncommutative polynomial in matrix fields $\phi_1, \ldots, \phi_{d'}, \psi^\alpha$. In this case we construct $A_\infty$-algebra $\mathcal{A}$ in a such way that $L_\infty$-algebra $\mathcal{L}_{\mathfrak{gl}_n}$ is build in a standard way from $A_\infty$-algebra $\mathcal{A} \otimes Mat_n$. We can say that working with $A_\infty$-algebra $\mathcal{A}$ we are working with all algebras $\mathcal{L}_{\mathfrak{gl}_n}$ at the same time.
(moreover we can say that we are working with gauge theories of all classical
gauge groups at the same time.)

We mentioned already that for a $Q$ manifold $X$ (a supermanifold equipped
with an odd vector field $Q$ obeying $Q^2 = 0$) one can construct an $L_\infty$-algebra
on the vector space $\Pi T^*_x X$ for every point $x_0 \in X$ in zero locus of $Q$. In
finite dimensional case we can identify $L_\infty$-algebra with formal $Q$-manifold. On
the other hand the algebra of functions on formal $Q$-manifold $X$ (= an algebra
of formal power series) is a differential commutative algebra. This algebra by
definition is dual to $L_\infty$-algebra $\mathcal{L}$.

Similar definitions can be given for $A_\infty$-algebras. An algebra of functions on
formal noncommutative manifold $X$ is defined as topological algebra of formal
noncommutative power series. More precisely if $W$ is a $\mathbb{Z}_2$-graded topological
vector space we can consider a tensor algebra $T(W) = \bigoplus_{n \geq 1} W^{\otimes n}$. This algebra
has an additional $\mathbb{Z}$ grading with $n$-th graded component $W^{\otimes n}$ and a descend-
ing filtration $K^n = \bigoplus_{i \geq n} W^{\otimes i}$. The algebra of formal power series $\hat{T}(W)$
is defined as completion of $T(W)$ with respect to this filtration. By definition the
completion $\hat{T}(W)$ consists of infinite series in generators which become finite in
projection to $T(W)/K^n$ for every $n$. The elements of $\hat{T}(W)$ are infinite sums of
monomials formed by elements of a basis of $W$. The algebra $\hat{T}(W)$ is $\mathbb{Z}_2$-graded;
the filtration on $T(W)$ generates a filtration on the completion. $\hat{T}(W)$ can be
considered as inverse limit of spaces $T(W)/K^n$; we equip $\hat{T}(W)$ with the topol-
ogy of inverse limit (the topology on $T(W)/K^n$ is defined as strongest topology,
compatible with linear structure.) A formal noncommutative $Q$-manifold is by
definition a topological algebra $\hat{T}(W)$ equipped with continuous odd differenti-
ation $Q$ obeying $Q^2 = 0$. We say that formal $Q$-manifold $(\hat{T}(W), Q)$ specifies
a structure of $A_\infty$-coalgebra $\mathcal{H}$ on the space $\Pi W$. We are saying that differential
topological algebra $(\hat{T}(W), Q)$ is (bar)-dual to $A_\infty$-coalgebra $\mathcal{H}$. One says
also that differential algebra $(\hat{T}(W), Q)$ is obtained from $A_\infty$-coalgebra $\mathcal{H}$ by
means of bar-construction ; we denote it by Bar$\mathcal{H}$. The homology of $(\hat{T}(W), Q)$
is called Hochschild homology of $\mathcal{H}$. Notice that in this definition Hochschild
homology is $\mathbb{Z}_2$-graded. We obtain also a structure of $A_\infty$-algebra $\mathcal{A} = \mathcal{H}^*$
on \( \Pi W^* \). In finite dimensional case the notion of \( A_\infty \)-algebra on vector space \( V \) is equivalent to the notion of \( A_\infty \)-coalgebra on vector space \( V^* \). However in infinite-dimensional case it is much simpler to use \( A_\infty \)-coalgebras. We will consider the case when the space \( W \) is equipped with descending filtration \( F^n \); then we can extend \( F^n \) to filtration of \( T(W) \) and \( \widehat{T(W)} \) is defined by means of this filtration. (See Appendix, section (3.2) for definitions).

Another way of algebraization of Yang-Mills theory is based on consideration of equations of motion (equations are treated as defining relations in an associative algebra). We analyse relations between two ways of algebraization and study some properties of algebras at hand. In particular we calculate some Hochschild homology.

The paper is organized as follows. In Sec. we formulate our main results. In Sec. we give proofs in the case of Yang-Mills theory reduced to a point and in Sec. we consider more general case of Yang-Mills theory reduced to any dimension. In Sec. we make some homological calculations that allow us to apply general results to the case maximally supersymmetric theories.

All proofs in the paper are rigorous. However, our exposition in Sec. is sometimes sketchy; the exposition of more general results in Sec. is more formal.

**Notations.**

Denote \( \langle a_1, \ldots, a_n \rangle \) a span of vectors \( a_1, \ldots, a_n \) in some linear space.

Denote \( \mathbb{C} \langle a_1, \ldots, a_n \rangle \) a free algebra without a unit on generators \( a_1, \ldots, a_n \).

If \( \langle a_1, \ldots, a_n \rangle = W \) then an alternative notation for \( \mathbb{C} \langle a_1, \ldots, a_n \rangle \) is

\[
T(W) = \bigoplus_{n \geq 1} W^\otimes n
\]

All algebras in this paper are non-unital algebras, unless the opposite is explicitly stated.

Any algebra has a canonical filtration \( F^n = \{ \sum a_1 \ldots a_n \} \).

Suppose \( A \) is an algebra with a unit and augmentation (i.e homomorphism \( \varepsilon : A \to \mathbb{C} \)). Denote \( I(A) = \text{Ker} \varepsilon \).
Suppose $A$ is an algebra (unit is irrelevant). Denote $A = A + \mathbb{C}$-an algebra with the following multiplicative structure: $(a, \alpha)(b, \beta) = (ab + \alpha b + \beta a, \alpha \beta)$. In this construction we formally adjoint a unit to $A$ equal to $(0, 1)$. The algebra $A$ has an augmentation $\varepsilon(a, \alpha) = \alpha$ and $I(A) = A$.

Suppose $\mathbb{Z}_2$-grading of Hochschild homology comes from $\mathbb{Z}$-grading. This happens in the case when algebra $A$ has no differential, or is $\mathbb{Z}$-graded. In $\mathbb{Z}$-graded case by our definition zero Hochschild homology of algebra without unit is equal to zero (sometime it is called reduced homology). Sometimes it will be convenient for us to define $H_0(A) = \mathbb{C}$, so $H_*(A)$ would become an $A_{\infty}$-coalgebra with counit and coaugmentation. Such completion will be denoted as $H_*(A)$. It is easy to see that it is equal to standard unreduced Hochschild homology of the unital algebra $A$ which is denoted as $H_*(A, \mathbb{C})$ and defined in [12]. We will use this notation also in $\mathbb{Z}_2$-graded case.

1.1 Main results

Let us reduce the theory to a point, i.e consider the fields that do not depend on $x^i, i = 1, \ldots, D$ (the case of $x$-dependent fields will be considered later). The BV-action functional becomes a (super)function and takes the form:

$$S = \frac{1}{4} \sum_{i,j=1}^{D} <[A_i, A_j], [A_i, A_j]> + \frac{1}{2} \sum_{i=1}^{D} \sum_{j=1}^{d'} <[A_i, \phi_j], [A_i, \phi_j]> +$$

$$+ \frac{1}{2} \sum_{i,j=1}^{D} \sum_{\alpha\beta} \Gamma_{\alpha\beta}^{i} <[A_i, \psi^\alpha], \psi^\beta> - U(\phi, \psi) +$$

$$+ \sum_{i=1}^{D} <[A_i, c], A^{*i}> + \sum_{j=1}^{d'} <[c, \phi_j], \phi^{*j}> + \sum_{\alpha} <[c, \psi^\alpha], \psi^*_\alpha> + \frac{1}{2} <[c, c], c^*>$$

(4)

Here $A_i, \phi_i, \psi^\alpha, c^*$ are elements of $\mathfrak{g}$, and $A^{*i}, \phi^{*i}, \psi^*_\alpha$ and $c$ are elements of
The vector field $Q$, corresponding to $S$ is given by the formulas:

$$Q(A_i) = [c, A_i]$$
$$Q(\phi_j) = [c, \phi_j]$$
$$Q(\psi^\alpha) = [c, \psi^\alpha]$$
$$Q(c) = \frac{1}{2} [c, c]$$
$$Q(c^*) = \sum_{i=1}^{D} [A_i, A^{*i}] + \sum_{j=1}^{d'} [\phi_j, \phi^{*j}] + \sum_{\alpha} \{\psi^\alpha, \psi^*_\alpha\} + [c, c^*]$$
$$Q(A^{*m}) = -\sum_{i=1}^{D} [A_i, [A_i, A_m]] - \sum_{k=1}^{d'} [\phi_k, [\phi_k, A_m]] + \frac{1}{2} \sum_{\alpha\beta} \Gamma^m_{\alpha\beta} \{\psi^\alpha, \psi^\beta\} - [c, A^{*m}]$$
$$Q(\phi^{*j}) = -\sum_{i=1}^{D} [A_i, [A_i, \phi_j]] - \frac{\partial U}{\partial \phi_j} - [c, \phi^{*j}]$$
$$Q(\psi^*_{\alpha}) = -\sum_{i=1}^{D} \sum_{\beta} \Gamma^i_{\alpha\beta} [A_i, \psi^\beta] - \frac{\partial U}{\partial \psi^\alpha} - [c, \psi^*_{\alpha}]$$

(5)

Let us consider the case $\mathfrak{g} = \mathfrak{gl}(n)$. In this case all fields are matrix valued functions. In order to pass from $L_\infty$ to $A_\infty$ construction we need to assume that the functions $u_i(x)$ are equal to matrix polynomials.

We can construct such an $A_\infty$-algebra $A_0$ that the $L_\infty$-algebra $\mathcal{L}_{\mathfrak{gl}(n)}$ can be obtained as $L_\infty$-algebra corresponding to $A_\infty$-algebra algebra $A_0 \otimes \text{Mat}_n$. The construction is obvious - we consider $A_i, \phi_j, c, A^{*i}, \phi^{*j}, c^*$ in $[\mathfrak{m}]$ not as matrices but as formal generators. Then $Q$ determines derivation $\hat{Q}$ in algebra $\widehat{T(W)}$ of formal power series with respect to free generators (we consider $A_i, \phi_j, \psi^*_{\alpha}, c^*$ as even elements and $A^{*i}, \phi^{*j}, \psi^\alpha, c$ as odd ones). The space $W$ can be considered as a direct sum of spaces $V, \Phi, \Pi S, \Pi C, \Pi V, \Pi \Phi, S^*, C$.

The derivation $\hat{Q}$ obeys $\hat{Q}^2 = 0$, hence it specifies a structure of $A_\infty$-coalgebra on $\Pi W$. The potential $U(\phi, \psi)$ is a linear combination of cyclic words in alphabet $\phi_j, \psi^\alpha$. In other words $U(\varphi, \psi)$ is an element of $\text{Cyc}W = T(W)/[T(W), T(W)]$ or, if we allow infinite sums, an element of completion of this space. The linear space $[T(W), T(W)]$ is spanned by $\mathbb{Z}_2$-graded commuta-
tors. Notice, that for every \( w \in W^* \) one can define derivative \( \partial/\partial w : \text{Cyc} W \to T(W) \); this map can be extended to completions. The derivative \( \partial U/\partial \phi_j \) in the definition of operator \( Q \) should be understood in this way. The derivation \( \hat{Q} \) specifies not only a structure of \( A_\infty \)-coalgebra on \( \Pi W \) but also a structure of \( A_\infty \)-algebra on \( \Pi W^* \). Let us consider for simplicity the case of bosonic theory; writing the potential in the form

\[
U(\phi) = \sum_k c_{j_1 \ldots j_k} \phi_1 \ldots \phi_k
\]

we can represent the operations in the \( A_\infty \)-algebra in the following way:

\[
\begin{align*}
m_k(p^{j_1}, \ldots, p^{j_k}) &= -c_{j_1 \ldots j_{k+1}} p^*_{j_k+1} \quad (k \geq 2) \\
m_2(a^{i_1}_1, a^{i_2}_2) &= m_2(a^*_{i_2}, a^{i_1}_1) = -c^* a^i_{i_2} \\
m_2(p^{j_1}, p^{j_2}) &= m_2(p^*_{j_2}, p^{j_1}) = -c^* \delta_{j_2} \\
m_3(a^{i_1}_1, a^{i_2}_2, a^{i_3}_3) &= - (\delta^{i_1i_2} a^*_{i_3} - 2\delta_{i_1i_3} a^*_{i_2} + \delta^{i_2i_3} a^*_{i_1}) \\
m_3(a^{i_1}_1, a^{i_2}, p^j) &= -\delta_{i_1i_2} p^*_{j} \\
m_3(a^{i_1}_1, p^j, a^{i_2}_2) &= 2\delta_{i_1i_2} p^*_{j} \\
m_3(p^j, a^{i_1}_1, a^{i_2}_2) &= -\delta_{i_1i_2} p^*_{j} \\
m_3(p^{j_1}, p^{j_2}, a^i) &= -\delta_{j_1j_2} a^*_i \\
m_3(p^{j_1}, a^i, p^{j_2}) &= 2\delta_{j_1j_2} a^*_i \\
m_3(a^i, p^{j_1}, p^{j_2}) &= -\delta_{j_1j_2} a^*_i
\end{align*}
\]

(We are using a basis of \( \Pi W^* \) that is dual to the basis of \( W \).) There is an additional set of equations relating \( c \) with the rest of the algebra. They simply assert that \( c \) is a unit.

The ideal \( I(c) \subset T(W) \) generated by element \( c \) is closed under differential \( \hat{Q} \). Denote \( BV_0 \) the quotient differential algebra \( T(W)/I(c) \). To define a filtration on \( BV_0 \) we introduce first of all grading on \( W \) assuming that

\[
deg(c) = 0, \quad \deg(A_i) = \deg(\phi_i) = 2, \quad \deg(\psi^a) = 3, \quad \deg(\psi^*_a) = 5, \quad \deg(A^*_i) = \deg(\phi^{*i}) = 6, \quad \deg(c^*) = 8.
\]

Corresponding multiplicative grading on \( T(W) \) in general does not descends to \( BW_0 \), but the decreasing filtration on \( T(W) \) generated by grading descends to
BV₀ and is compatible with the differential Q in the case when \( \deg(U) \geq 8 \).

(The grading on \( W \) induces a grading on the space \( \text{Cyc}(\Phi + \Pi S) \) of cyclic words and corresponding filtration \( F^k \); the notation \( \deg(U) \geq 8 \) means that \( U \in F^8 \).)

We always impose the condition \( \deg(U) \geq 8 \) considering \( BV₀ \); under this condition we can consider the completion of \( BV₀ \) as filtered differential algebra \( \hat{BV}_0 \) that can be identified with the quotient algebra \( \hat{T}(W)/\hat{I}(c) \). Notice that \( Q \) is a polynomial vector field hence instead of the algebra \( \hat{T}(W) \) of formal power series we can work with tensor algebra \( T(W) = \bigoplus_{k \geq 0} W^\otimes k \), however without additional assumptions on the potential \( U \) the results of our paper are valid only for completed algebra \( \hat{T}(W) \). The differential algebra \( (\hat{T}(W), \hat{Q}) \) is dual to \( A_\infty \)-algebra \( A \). It will be more convenient for us to work with \( A_\infty \)-coalgebras. The motivation is that we would like to avoid dualization in the category of infinite dimensional vector spaces as much as possible. However there is an involutive duality functor on the category of finite-dimensional or graded vector spaces. It implies that a category of finite-dimensional \( A_\infty \)-algebras is dual to the category of \( A_\infty \)-coalgebras. The same statement is true for category of \( A_\infty \)-(co)algebras equipped with additional grading. There is a topological version of such duality which is not auto-equivalence of the appropriate category, rather is equivalence between two different categories.

Another approach to algebraization of Yang-Mills theory is based on consideration of equations of motion. We will illustrate it in the case of Yang-Mills theory, reduced to a point. We consider its equations of motion

\[
\sum_{i=1}^D [A_i, [A_i, A_m]] + \sum_{k=1}^{d'} [\phi_k, [\phi_k, A_m]] = 0 \\
- \frac{1}{2} \sum_{\alpha\beta} \Gamma^m_{\alpha\beta} \{\psi^\alpha, \psi^\beta\} = 0 \quad m = 1, \ldots, D
\]

\[
\sum_{i=1}^D [A_i, \phi_j] + \frac{\partial U}{\partial \phi_j} = 0 \quad j = 1, \ldots, d'
\]

\[
\sum_{\beta} \sum_{i=1}^D \Gamma^i_{\alpha\beta} [A_i, \psi^\beta] + \frac{\partial U}{\partial \psi^\alpha} = 0
\]
as defining relations of associative algebra with generators $A_i, \phi_j, \psi^\alpha$. This algebra will be denoted by $YM_0$. (The algebras $A$ and $YM_0$ depend on the choice of potential $U$, hence more accurate notations should be $A^U_0$ and $YM^U_0$).

One can say that $YM_0$ is a quotient of tensor algebra $T(W_1)$ with generators $A_i, \phi_j, \psi^\alpha$ with respect to some ideal. The grading on $W_1 = V + \Phi + \Pi S$ generates grading on $YM_0$ if $\text{deg}(U) = 8$. It generates a decreasing filtration compatible with algebra structure on $YM_0$ if $\text{deg}(U) \geq 8$; in this case we can introduce an algebra structure on the completion $\widehat{YM}_0$. Graded algebra associated with the filtered algebra $YM_0$ will be denoted by $YM'_0$; it can be described as the algebra $YM_0$ corresponding to component of $U$ having degree 8.

**Theorem 1** If the potential $U$ has degree $\geq 8$, then the differential algebra $(\widehat{BV}_0, \widehat{Q})$ is quasiisomorphic to the algebra $\widehat{YM}_0$. If the potential $U$ has degree 8 we can say also that the differential algebra $(BV_0, Q)$ is quasiisomorphic to the algebra $YM_0$.

**Proof.**

See section (2.1) for the proof. ■

The above constructions can be included into the following general scheme.

Let us consider $\mathbb{Z}_2$-graded vector spaces $W_1 = V$ with basis $e_1, \ldots, e_n$, $W_2 = \Pi V^*$ with dual basis $e^*_{i1}, \ldots, e^*_{in}$, and one-dimensional spaces $W_0$ with odd generator $c$, $W_3 = \Pi W_0^*$ with even generator $c^*$. Take $L \in \text{Cyc}(W_1)$.

Define a differential $Q$ on a free algebra $T(W)$ by the rule:

$$
Q(e_i) = [c, e_i] \\
Q(e^i) = \frac{\partial L}{\partial e^i} + [c, e^*] \\
Q(c) = -\frac{1}{2}[c, c] \\
Q(c^*) = \sum_i [e_i, e^i] + [c, c^*] 
$$

(10)

Denote $W^{red} = W_1 + W_2 + W_3$. Denote the algebra $T(W)$ with differential $Q$ defined by formula (10) as $T(W)^L$. Denote $I(c)$ an ideal generated by $c$. It is easy to see that it is a differential ideal. Denote $BV^L = T(W)^L/I(c)$. 

10
It is easy to see that $BV_0 = BV^L$ for $L$ defined in (11), where one should disregard $<>$ signs. (We consider $A_i, \phi_j, \ldots$ as free variables and $L$ as a $\mathbb{Z}_2$-graded cyclic word.) The algebra $YM^L$ is defined as a quotient of $T(W)$ with respect to the ideal generated by $\partial L/\partial e^i$. There exists a natural homomorphism of $BV^L$ onto $YM^L$ that sends $e_i \rightarrow e_i, e^* \rightarrow 0, \ldots, c^* \rightarrow 0$. If this homomorphism is a quasiisomorphism we say that $L$ is regular. Theorem 1 gives a sufficient condition of regularity.

Theorem 1 can be generalized to unreduced Yang-Mills theory or to theory reduced to $d$ dimensions $0 < d \leq D$. Our consideration will be local; this means that for theory with gauge group $g$ reduced to $d$ dimensions we consider $g$-valued fields $A_i, A^* \i, \phi_j, \phi^* j, c, c^*$ that are formal functions of the first $d$ variables. They span a space $W_d \otimes g$, where $W_d = W \otimes \mathbb{C}[x^1, \ldots, x^d]$. The space $W_d$ is equipped with filtration $F^s$, which induces filtration on $W_d \otimes g$ in a trivial way. The group $F^s$ consists of all power series with coefficients in $W$ with Taylor coefficients vanishing up to degree $s$. The filtration $F^s$ defines a topology in a standard way.

The solutions of the equations of motion in BV-formalism correspond to zeros of vector field $Q$ defining a structure of $A_\infty$-coalgebra on the space $\Pi W_d$; we denote this coalgebra by $bv_d$. We can also work with $A_\infty$-algebra defined on the space $\Pi W^* \otimes \mathbb{C}[x^1, \ldots, x^d]$.

The role of algebra $YM_0$ in the case at hand is played by truncated Yang-Mills algebra $T_dYM$. We consider a set of differentiations $\partial_k : YM_0 \rightarrow YM_0, k = 1, \ldots, d$. The differentiations are defined by the formula:

$$
\begin{align*}
\partial_k(A_i) &= \delta_{ki} \quad 1 \leq i \leq d \\
\partial_k(A_i) &= 0 \quad i > d \\
\partial_k(\phi_j) &= 0 \quad \text{for all } j.
\end{align*}
$$

\textbf{Definition 2} We define $T_dYM$ as $\bigcap_{k=1}^d Ker \partial_k$. We assume that $\partial_k, k = 1, \ldots, d$ is generic and the matrix $\Gamma$ is not degenerate. We define $T_dYM$ in a standard way as $I(T_dYM)$.

The precise meaning of assumptions in this definition can be explained in the
following way.

**Definition 3** We say that the set of differentiations \( \partial_k, k = 1, \ldots, d \) is generic if the restriction of bilinear form from \( V \) to \( T_dYM \cap V \) is non-degenerate. If the set of generators include fermions then we require that there is a subspace \( V' \subset V \) of codimension one such that \( V' \supset T_dYM \cap V \) and the bilinear form \( (s_1, s_2)_v \overset{\text{def}}{=} [\Gamma(s_1, s_2) \rightarrow V/V'] \) is non-degenerate.

**Definition 4** We say that the matrix \( \Gamma \) is non-degenerate if there is at least one generic differentiation from definition [3].

If we do not assume genericity we use an alternative notation \( T_\mu YM \) for \( T_dYM \) which will be adopted throughout the main part of the paper. The algebra \( T_dYM \) is filtered and we can define its completion \( \hat{T}(W_d) \). The ideal \( I(c) \subset \hat{T}(W_d) \) generated by element \( c \) is closed under differential \( \hat{Q} \). Denote \( \hat{BV}_d \) the quotient algebra \( \hat{T}(W_d)/I(c) \). We use notation \( \hat{BV}_\mu \) for a quotient \( \hat{T}_\mu(W)/I(c) \) without assumption of genericity of a family \( \partial_k, k = 1, \ldots, d \).

**Theorem 5** The differential algebra \( (\hat{BV}_d, \hat{Q}) \) is quasiisomorphic to \( \hat{T}_dYM \).

**Proof.**

See proposition [46].

**Corollary 6** The algebra \( \hat{T}_dYM \) is quasiisomorphic to the dual to coalgebra \( bv_d \).

**Proof.**

See proposition [47] where the \( A_\infty \)-coalgebra \( bv_d \) is denoted as \( bv_\mu \).

Let analyze the structure of the algebra \( T_dYM \) for \( d \geq 1 \).

**Definition 7** Define an algebra \( K(q_1, \ldots, q_n|p^1, \ldots, p^n; \psi^1, \ldots, \psi^{n'}) \) as a quotient algebra \( \mathbb{C} < q_1, \ldots, q_n, p^1, \ldots, p^n, \psi^1, \ldots, \psi^{n'} > /I(\omega) \), where the ideal \( I(\omega) \) is generated by an element

\[
\omega = \sum_{i=1}^{n}[q_i, p^j] - \frac{1}{2} \sum_{j=1}^{n'} \{\psi^j, \psi^j\} = 0. \tag{12}
\]
**Theorem 8** The algebra $\hat{T}YM$ is isomorphic to the algebra $\hat{K}$.

**Theorem 9** The algebra $\hat{T_dYM}$ for $d \geq 2$ is isomorphic to the completion of a free algebra $T(\mathcal{H} + \mathcal{S} + \mathcal{G})$, where $\mathcal{H}$ stands for the space of all polynomial harmonic two-forms on $\mathbb{C}^d$ and $\mathcal{S}$ stands for the space of harmonic spinors and $\mathcal{G}$ stands for the space of harmonic polynomials on $\mathbb{R}^d$ with values in $\Phi + \mathbb{C}^{D-d}$.

**Proof.**

The proofs of statements that are more general then Theorems (8, 9) are given in Example 1, Example 2, Example 4, Propositions (50) and (52).

Notice that above theorems have physical interpretation.

Theorems (8) and (9) can be interpreted as a statement that BV formalism is equivalent to the more traditional approach to the theory of the gauge fields. (One can relate this theorem to the calculation of BV-homology in [2]). Theorem (8) is related to Hamiltonian formalism in the gauge theory when we neglect the dependence of fields on all spatial variables. A solution to the equation of motion in such a theory is characterized by a point of a phase space; the degeneracy of Lagrangian leads to constraint (12) on the phase space variables.

Theorems (8) and (9) mean that there exists a one-to-one correspondence between solutions of full Yang-Mills equation of motion and solutions of linearized version of this equation.

The phase space dynamics specifies an action of one-dimensional Lie algebra $a$ on algebra $K$. More precisely, we define an action of exterior derivation $H$, corresponding to generator of $a$ by the rule

$$H(q_i) = p^i$$ (13)

$$H(Q_i) = P^i$$ (14)

$$H(p^i) = -\sum_{i=1}^{D-1} [q_i, [q_i, q_m]] - \sum_{j=1}^{d'} [Q_j, [Q_j, q_m]] + \frac{1}{2} \sum_{\alpha\beta} \Gamma^m_{\alpha\beta} \{\psi'^\alpha, \psi^\beta\}$$ (15)
\[ H(P^j) = - \sum_{i=1}^{D-1} [q_i, [q_i, Q_j]] - \frac{\partial U}{\partial Q_j} \]  

(16)

\[ H(\psi^\alpha) = - \sum_{i=1}^{D-1} \sum_{\beta} \Gamma^i_{\alpha\beta} [q_i, \psi^\beta] - \frac{\partial U}{\partial \psi^\alpha} \]  

(17)

**Definition 10** Suppose \( A \) is an algebra with a unit and \( g \) is a Lie algebra which acts upon \( A \) via derivations. Let

\[ \rho : g \to \text{Der} A \]  

the homomorphism to the Lie algebra of derivations that corresponds to the action. Let \( U(g) \) be the universal enveloping algebra of \( g \). Denote by \( U(g) \ltimes A \) the algebra defined on space \( U(g) \otimes A \) in the following way. It contains \( U(g) \otimes 1 \) and \( 1 \otimes A \) as subalgebras. For \( g \in U(g) \) and \( a \in A \), \( ga = g \otimes a \). If \( g \) is a linear generator of \( g \) then \( ga = ag = \rho(g)a \). We call \( U(g) \ltimes A \) a semi-direct product.

**Theorem 11** Suppose \( \Gamma \)-matrices used in the definition of \( YM_0 \) are not degenerate in a sense of definition (4). Then the algebra \( YM_0 \) is isomorphic to a semidirect product

\[ YM_0 = U(a) \ltimes \hat{K}(q_1, \ldots, q_{D-1}, Q_1, \ldots, Q_d; p_1^1, \ldots, p_1^{D-1}, P_1, \ldots, P_d'; \psi^\alpha). \]  

(19)

In the above formula \( U(a) \) is a universal enveloping algebra of an abelian one-dimensional Lie algebra \( a \) spanned by element \( H \), which acts on \( \hat{K} \) as an outer differentiation.

The formula (19) remains valid for completed algebras \( \hat{YM}_0 \) and \( \hat{K} \) if we replace semidirect product with its completion with respect to multiplicative decreasing filtration \( F^n \) which coincides with intrinsic filtration on \( \hat{K} \) and is determined by condition \( H \in F^2 \).

**Proof.**

See section (2.1) for the proof. ■
Using general results about Hochschild homology (the main reference is [12], see also Appendix) for algebras with one relation and on homology of a cross-product we get the following theorem:

**Theorem 12** The Hochschild homology $\mathcal{H}_*(K(q_1, \ldots, q_n; p^1, \ldots, p^n; \psi^1, \ldots, \psi^n))$ are isomorphic to

$$
\begin{align*}
\mathcal{H}_0(K) &= \mathbb{C} \\
\mathcal{H}_1(K) &= \langle [q_1], \ldots, [q_n], [p_1], \ldots, [p_n], [\psi^1], \ldots, [\psi^n] > \\
\mathcal{H}_2(K) &= \langle r > \\
\mathcal{H}_i(K) &= 0 \text{ for } i \geq 3.
\end{align*}
$$

The symbols $[q_1], \ldots, [q_n], [p_1], \ldots, [p_n], [\psi^1], \ldots, [\psi^n]$ are in one to one correspondence with generators $q_1, \ldots, q_n, p_1, \ldots, p_n, \psi^1, \ldots, \psi^n$. There is a nondegenerate even skew-symmetric pairing on $\mathcal{H}_1(K)$ which depends on a choice of a generator $r$ in $\mathcal{H}_2(K)$. The statement of proposition holds if one replaces algebra $K$ by its completion $\hat{K}$.

**Proof.**

The section (2.1) for the proof.

**Theorem 13** The Hochschild homology $\mathcal{H}_*(\hat{YM}_0)$ are isomorphic to

$$
\begin{align*}
\mathcal{H}_0(\hat{YM}_0) &= W_0 = \langle c > \\
\mathcal{H}_1(\hat{YM}_0) &= W_1 = \Phi + \Pi S \\
\mathcal{H}_2(\hat{YM}_0) &= W_2 = \Pi W_i^* \\
\mathcal{H}_3(\hat{YM}_0) &= W_3 = \langle c^* >.
\end{align*}
$$

There is a graded commutative duality pairing

$$
\mathcal{H}_*(\hat{YM}_0) \otimes \mathcal{H}_{3-i}(\hat{YM}_0) \rightarrow \mathbb{C}
$$

which depends on the choice of a generator $c^*$ in $W_3$.

The algebra $YM_0$ has the same Hochschild homology.
Proof.

See the section (2.1) for the proof. ■

The duality of $A_\infty$-algebras is closely related to Koszul duality of quadratic algebras (see e.g. [11]).

Let $S$ be a spinor representation of $Spin(10)$. Let

$$S = \text{Sym}(S) / \sum_{\alpha\beta} \Gamma_{\alpha\beta} u^{\alpha} u^{\beta}$$

where $\Gamma_{\alpha\beta}$ are spinor $\Gamma$-matrices and $u^1, \ldots, u^{16}$ is a basis of $S$. The algebra $S$ can be considered as an algebra of polynomial functions on the space of pure spinors (spinors in $S^*$ satisfying $\sum_{\alpha\beta} \Gamma_{\alpha\beta} u^{\alpha} u^{\beta} = 0$). Denote

$$B_0 = S \otimes \Lambda(S)$$

with linear generators of $\Lambda(S)$ denoted by $\theta^1, \ldots, \theta^{16}$. Define a differential on the algebra $B_0$ by the rule:

$$Q(\theta^\alpha) = u^\alpha$$

We call the differential algebra $(B_0, Q)$ Berkovits algebra. From now on till the end of the section we assume that $YM_0$ is build from the following data:

$D = 10, d' = 0, S$ is an irreducible spinor representation $s_1$ of $Spin(10), \Gamma_{\alpha\beta}$ are the $\Gamma$-matrices associated with spinor representation $S$. (This means that $YM_0$ is obtained from 10D SUSY YM theory reduced to a point.)

We checked in [15] that $YM_0$ maps to the Koszul dual to Berkovits algebra $(B_0, Q)$ (see [15] and references therein about Koszul duality). In this paper we prove a statement which was formulated in [15] without a proof:

**Theorem 14** Koszul dual to the algebra $(B_0, Q)$ is quasiisomorphic to $YM_0$.

Proof.

To prove this fact we should know the homology of Berkovits algebra. Heuristic calculation of this homology was given in [5]. We present rigorous calculation in Sec. (2.6). ■

**Theorem 15** Berkovits algebra $(B_0, Q)$ is quasiisomorphic to the $A_\infty$-algebra $A$ obtained from $D = 10$ SUSY YM action functional reduced to a point.
This statement was formulated in [15]. It follows from (14) and from relation between Koszul duality and bar-duality.

Let us consider now $d$-dimensional Berkovits algebra $(B_d,Q)$.

**Definition 16** Berkovits algebra $(B_d,Q)$ is defined as an algebra of polynomial functions of pure spinor $u$, odd spinor $\theta = (\theta^1, \ldots, \theta^{16})$ and commuting coordinates $x^1, \ldots, x^d$ $(d \leq 10)$, equipped with differential:

$$Q = \sum_{\alpha=1}^{16} u^\alpha \frac{\partial}{\partial \theta^\alpha} + \sum_{\alpha,\beta=1}^{16} \sum_{i=1}^{d} \Gamma^i_{\alpha\beta} u^\alpha \theta^\beta \frac{\partial}{\partial x^i}$$

The algebra $B_d$ is a quadratic algebra.

**Theorem 17** The Koszul dual to differential algebra $(B_d,Q)$ is quasiisomorphic to truncated Yang-Mills algebra $T_dYM$.

**Proof.**

See propositions [16] and [17].

2 Proofs

2.1 Algebras $\hat{K}$ and $\hat{YM}_0$

**Proof of Proposition 11** We need to rewrite relations (17) in a slightly different form. Introduce notation $q_i = A_{i+1}(i \geq 1)$, $Q_i = \phi_i(i \geq 1)$, $p^i = [A_1, A_{i+1}](i \geq 1)$, $P^j = [A_1, \phi_j](j \geq 1)$. Commutation with $A_1$ preserves the algebra generated by $q_i, Q_i, p^i, P^j, \psi^\alpha$. Denote the operation of commutation with $A_1$ by $H$: $[A_1, x] = H(x)$. Then by definition we have [15] [14].

In the new notations relation (7) when $m = 0$ becomes (12). To prove this we use nondegeneracy of $\Gamma$-matrices. It allows us to make $\Gamma^i_{\alpha\beta} = \delta_{\alpha\beta}$ by appropriate choice of basis in $S$ and in $V$. The relations (7) when $m > 0$ become (15), the relations (8) become (16) and the relations (9) become (17). We can see that in this representation the algebra $\hat{YM}_0$ is a semidirect product of two algebras: a universal enveloping algebra $U(\mathfrak{a})$ of an abelian algebra with one generator $H$ and of the algebra $\hat{K}$. 

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The action of the universal enveloping algebra is given exterior differentiation \( H \) (letter \( H \) stands for the Hamiltonian) by the formulas (13,14,12,15,16,17).

If \( H \) acts on \( K \) via formulas (13,14,12,15,16,17), then the map \( p : YM_0 \to U(a) \ltimes K \), defined by the formulas

\[
p(A_1) = H, \quad p(A_{i+1}) = q_i \quad 1 \leq i \leq D, \quad p(\phi_j) = Q_j \quad 1 \leq j \leq d', \quad p(\psi^\alpha) = \psi^\alpha
\]

is correctly defined and agrees on filtrations. It is because formulas (13,14,12,15,16,17) imply (7,8,9). As a result \( p \) is continuous isomorphism that can be extended to completions.

We need to formulate basic theorems how to compute Hochschild homology of some algebras:

**Proposition 18**

**a.** Suppose \( \mathfrak{g} \) is a Lie algebra. Then \( H_1(\mathfrak{g}, \mathbb{C}) \cong \mathfrak{g}/[\mathfrak{g}, \mathfrak{g}] \), where \([\mathfrak{g}, \mathfrak{g}]\) is an ideal of \( \mathfrak{g} \) consisting of elements of the form \([a, b]\) where \( a, b \in \mathfrak{g} \).

**b.** Suppose \( \mathfrak{g} = F/R \), where \( F \) is a free Lie algebra, and \( R \) is an ideal of relations. Then \( H_2(\mathfrak{g}, \mathbb{C}) \cong R \cap [F, F]/[F, R] \).

**Proof.**

See [12] for the proof.

**Corollary 19** Suppose \( \mathfrak{g} \) is a positively graded Lie algebra.

**a.** Let \( V \subset \mathfrak{g} \) be a minimal generating subspace. Then a canonical map \( V \to \mathfrak{g}/[\mathfrak{g}, \mathfrak{g}] \cong H_1(\mathfrak{g}, \mathbb{C}) \) is an isomorphism.

**b.** Suppose \( \mathfrak{g} = F/R \), where \( F \) is a free algebra, \( R \) is an ideal. Assume also that the minimal linear subspace of relations \( L \) which generates the ideal \( R \) is a subspace of \([F, F]\). Then the canonical map \( L \to R/[F, R] = R \cap [F, F]/[F, R] \cong H_2(\mathfrak{g}, \mathbb{C}) \) is an isomorphism.

**Proposition 20** Let \( A \) be an algebra complete with respect to decreasing multiplicative filtration \( F^s, s \geq 1 \) such that \( \bigcap F^s = 0 \). There is an isomorphism between \( H_1(A) \) and a minimal linear space \( X \subset A \) such that the subalgebra generated by \( X \) is dense (in a sense of topology generated by filtration) in \( A \).
Definition 21 Suppose $A$ is an algebra with a unit, $M$ is a bimodule. If $i \geq 0$ is a minimal number such that Hochschild homology $H_{i+k}(A, M) = 0$ for all $M$, $k > 0$, then one says that homological dimension of $A$ is equal to $i$.

Proposition 22 Suppose a positively graded algebra $A$ is a quotient of a free algebra $T$ by ideal generated by one element $r$. If $r \neq aba$, where $a, b \in T$ and $\deg(a) > 0$, then homological dimension of $A$ is equal to 2 and $H_1(A)$ is isomorphic to a minimal set of generators and $H_2(A) = \langle r \rangle$.

Proposition 23 There is an isomorphism $H_1(T(W)) = W$. All other homology of the free algebra $T(W)$ are trivial.

Proposition 24 Let $A$ be a positively graded algebra and $\hat{A}$ is its completion with respect to filtration associated with grading. Then $H_*(\hat{A}) = \widehat{H}_*(A)$, where $\widehat{H}_*(A)$ stands for the completion of $H_*(A)$ by means of filtration associated with grading.

Proof.
Obvious. ■

Proof of Proposition 23 We can use proposition 22 for computation of homology groups. The algebra has only one relation in degree 2 therefore $H_2(K) = \langle r \rangle$ is one dimensional. There is a comultiplication map

$$\Delta : H_2 \to H_2 \otimes H_0 + H_1 \otimes H_1 + H_0 \otimes H_2$$

(27)

The image of the $r$ in the middle component gives the matrix of the pairing. It is nondegenerate and after the inversion one gets:

$$[q_i] \ast [p^j] = -[p^j] \ast [q_i] = \delta^j_i$$
$$[\psi^i] \ast [\psi^j] = [\psi^j] \ast [\psi^i] = \delta^{ij}$$

(28)

and all other products are equal to zero. The vanishing of $H_i(K) = 0$ for $i \geq 3$ is a corollary of proposition (22). The completion of $K$ is associated with grading, therefore the statement of the proposition for the completed algebra follows from proposition 24. ■
Proof of Proposition 13

By definition we have isomorphisms $H_*(\tilde{Y}M_0) = H_*(YM_0, \C)$ and $H_*(\tilde{K}) = H_*(K, \C)$. The algebra $YM_0$ contains a dense semidirect product $U(a) \rtimes \tilde{K}$ and algebra $YM_0$ is equal to semidirect product.

Let $A$ is one of these algebras. Introduce an increasing filtration $G^n A$ defined as follows. The algebra $U(a) = \C[H]$ has a grading such that $deg(H) = 1$, denote by $G^n U(a)$ the associated increasing filtration. Denote $G^n (A) = G^n U(a) \otimes B$, where $B$ is either $K$ or $\tilde{K}$. This filtration induces a filtration of Bar-complex. It leads to a spectral sequence which is usually attributed to Serre and Hochschild.

The next computations will be carried out for the case of semidirect product $U(a) \rtimes \tilde{K} = YM_0$.

The $E^2$ term of the spectral sequence is $E^2_{ij} = H_i(U(a), H_j(K)) = H_i(a, H_j(K))$ (here $H_i(a, \ldots)$ stands for homology of Lie algebra $a$ with coefficients in some module). We have convergence $E^2_{ij} \Rightarrow H_{i+j}(U(a) \rtimes K, \C)$. The homology with trivial coefficients of algebra $K$ are computed in (12). We have

$$\hat{H}_n(K) = \langle [q_1], \ldots, [q_d], [p^1], \ldots, [p^d], [Q_1], \ldots, [Q_d], [P^1], \ldots, [P^d], [\psi^\alpha] \rangle$$

The action of algebra $a$ or (what is the same) the action of its generator $H$ on homology of $K$ is easy to describe. It is trivial on $H_0(K)$ by obvious reasons. It is trivial on $H_2(K)$ because if one apply a differentiation of a free algebra $\C < q_1, \ldots, q_d-1, p^1, \ldots, P^{d-1}, Q_1, \ldots, Q_d, P^1, \ldots, P^d, \psi^\alpha >$ defined by the formulas (13) to (17) to a LHS of equation (12) one gets zero. The action of $H$ on $[q_1], [Q_j], [\psi^\alpha] \in H_1(K)$ is zero and $H[p^1] = -[q_1], H[\psi^\alpha] = -[Q_j]$ (It follows from the formulas (13) and restriction on degree of potential.

The differential $d_A : C_i(a, H_1(K)) = H_1(K) \overset{H_1(K)}{\longrightarrow} H_1(K) = C_0(a, H_1(K))$ (Here $C_i$ denotes group of $i$-th chains of the abelian Lie algebra $a$). This is the only nontrivial differential in $E_1$ term. The spectral sequence degenerates in $E^2$ term because all linear spaces where higher differentials can hit are equal to $\{0\}$. This implies that the classes $[A_1], \ldots, [A_d], [\phi_1], \ldots, [\phi_d], [\psi^\alpha]$ is a basis of $H_1(U(a) \rtimes K, \C)$. We have a nondegenerate pairing on $E^2$ between $E^2_{ij}$ and $E^2_{1-i,2-j}$. It comes from the diagonal $E^2_{i,2} \rightarrow E^2_{i,j} \otimes E^2_{1-i,2-j}$. This diagonal is a
tensor product of the diagonal from theorem (12) and a diagonal on homology of abelian Lie algebra. It indicates that $H_2(U(a) \ltimes K, \mathbb{C})$ is dual to $H_1(U(a) \ltimes K, \mathbb{C})$ and relations (9) is a minimal set of relations, $H_3(U(a) \ltimes K, \mathbb{C}) = \mathbb{C}$.

All homology groups $H_i(U(a) \ltimes K) i \geq 4$ are equal to zero because all contributors to these groups in the spectral sequence vanish.

This proves the statement for algebra $U(a) \ltimes K = Y M_0$. We could not apply the proof directly to $\hat{Y} M_0$ because it is not a semidirect product. We have the following argument in this case:

The algebra $\hat{Y} M_0$ is filtered with $\text{Gr} \hat{Y} M_0 = \hat{Y} M_0'$. To define $\hat{Y} M_0'$ we need to take the potential $U$ of $\hat{Y} M_0$ and extract its degree 8 homogeneous part. The algebra $\hat{Y} M_0'$ is graded and its completion with respect to decreasing completion associated with the grading is equal to $\hat{Y} M_0'$. By proposition (20) the Hochschild cohomology of $\hat{Y} M_0'$ is equal to completion of homology of $\hat{Y} M_0'$. The later are finite-dimensional. They coincide with homology of $\hat{Y} M_0'$. The spectral sequence associated with filtration $F^a \hat{Y} M_0$ degenerates in $E^1$ term. The proof follows from this fact.

\[ \text{Definition 25} \]
Introduce a multiplicative filtration on $T(W)$ by extending filtration from generating space $W$. After completing algebra $T(W)$ we get $\hat{T}(W)$. Define a continuous differential on the algebra $\hat{T}(W)$ by the formula (14). In assumptions of Theorem (1) differential (13) leaves subalgebra $T(W) \subset \hat{T}(W)$ invariant. Denote the resulting differential algebra by $(T(W), Q)$.

Define a map

\[ p : \hat{B}V_0 \to \hat{Y} M_0 \]

by its values on topological generators:

\[ p(A_i) = A_i \]
\[ p(\phi_j) = \phi_j \]
\[ p(\psi^\alpha) = \psi^\alpha \]
\[ p(A^{*i}) = p(\phi^{*j}) = p(\psi^*_\alpha) = p(\epsilon^*) = 0 \]
then extend it to the entire algebra using properties of homomorphism and continuity. The maps $p : BV_0 \to YM_0$ and $p : BV'_0 \to YM'_0$ are defined by the same formulas.

**Proof of Proposition 1**

We will use Theorems 13 and their corollaries. Hochshild homology of $\hat{YM}_0$ and of $YM_0$ were calculated in Theorem 13. To calculate homology of differential algebra $\hat{BV}_0$ we start with $\hat{BV}_0$ considered as an algebra without differential.

It is easy to check that this is a completed free algebra with free topological generators:

$$A_i, \phi_i, \psi^\alpha, A^*i, \phi^*i, \psi^*_\alpha, c^*.$$

It follows from Propositions (23-24) that $H_1(\hat{BV}_0) = W_1 \oplus W^*_1 \oplus \mathbb{C}$ is the space spanned by free topological generators, $H_i(\hat{BV}_0) = 0$ for $i \neq 1$.

Hochshild homology of differential algebra $(\hat{BV}_0, Q)$ is not $\mathbb{Z}$-graded, but it is $\mathbb{Z}_2$-graded. The calculation of completed Hochshild homology of $(\hat{BV}_0, Q)$ is based on the general lemma (83).

In our case $H$ is the coalgebra that corresponds to the differential $Q$ on $\hat{BV}_0$. This coalgebra is filtered (in a sense described in Appendix) with $F^1(H) = H$, $F^2(H) = 0$. The $A_{\infty}$-coalgebra $H$ has differential equal to zero. We obtain that as $\mathbb{Z}_2$-graded vector space Hochshild homology of $(\hat{BV}_0, Q)$ is isomorphic to Hochshild homology of $\hat{BV}_0$.

More precisely, this homology is spanned by the following cocycles, where
map $\iota$ is identification of generators of $BV_0$ and the homology classes in $\text{Bar}(BV_0, Q)$.

$$
\iota(A^m) = -A^m + \sum_{i=1}^{D} (A_i[A_i, A_m] - [A_i, A_m]A_i) +
\sum_{k=1}^{d'} (\phi_k[A_k, A_m] - [\phi_k, A_m]\phi_k) + \frac{1}{2} \sum_{\alpha\beta} \Gamma^m_{\alpha\beta}\psi^\alpha|\psi^\beta
$$

(31)

$$
\iota(\phi^j) = -\phi^j + \sum_{i=1}^{D} (A_i[A_i, \phi_j] - [A_i, \phi_j]A_i) + \frac{\partial U}{\partial \phi_j}
$$

(32)

$$
\iota(\psi^\alpha) = -\psi^\alpha + \sum_{i=1}^{D} \sum_\beta \left( \Gamma^i_{\alpha\beta}A_i|\psi^\beta - \psi^\beta|A_i \right) + \frac{\partial U}{\partial \psi^\alpha}
$$

(33)

$$
\iota(c^i) = \sum_i \left( A_i((A^* - \iota(A^*))) - (A^* - \iota(A^*))\right|A_i) +
\sum_{j=1}^{d'} (\phi_j((\phi^* - \iota(\phi^*))) - (\phi^* - \iota(\phi^*))|\phi_j) +
\sum_\alpha (\psi^\alpha(|\psi^\alpha - \iota(\psi^\alpha)) + (\psi^\alpha - \iota(\psi^\alpha))|\psi^\alpha)
$$

(34)

The map $\iota$ is the identity map on $A_i, \phi_j, \psi^\alpha$. We need to explain what $\tilde{\cdot}$ means in the formulas (32) and (33). A tensor algebra $T(V)$ generated by linear space $V$ can be considered as a free product of algebra $V$ with zero multiplication and an algebra spanned by $\otimes$ symbol (the multiplication in such algebra is trivial). Consider an algebra spanned by bar symbol $|$ with zero multiplication. Then $\text{Bar}T(V)$ is a subspace of a free product $V \circ \otimes \circ | >$, where $\circ$ denotes a free product. Define a derivation on such algebra by the rule $d(v) = 0, v \in W, d(\otimes) = |, d(|) = 0$. The partial derivatives of $U$ in formulas (32)-(33) are elements of a tensor algebra $T(\phi_1, \ldots, \phi_d, \psi^\alpha) \subset \text{Bar}T(\phi_1, \ldots, \phi_d, \psi^\alpha)$. Then $\frac{\partial U}{\partial \phi_j} = d_1 \frac{\partial U}{\partial \phi_j}$ and $\frac{\partial U}{\partial \psi^\alpha} = d_1 \frac{\partial U}{\partial \psi^\alpha}$, where $d_1$ was extended by continuity to $\text{Bar}T(V)$.

Simple direct calculation using formulas (32)…(34) shows that the homomorphism $(\hat{BV}_0, Q) \to \hat{YM}_0$ induces an isomorphism on Hochshild homology. ■

**Definition 26** Denote a minimal model (see for definition) for coalgebra $\text{Bar}(\hat{BV}_0, d)$ by $\hat{bv}_0$. The linear space of $\hat{bv}_0$ coincides with $W_1 + W_2 + W_3$ and
the structure maps coincide with \( (\text{the variable } c \text{ set to zero}) \). The coalgebras \( bv_0 \) and \( bv'_0 \) are noncomplete and graded version of coalgebra \( \widehat{bv}_0 \).

**Proposition 27** The theorem \( 1 \) together with lemma \( 83 \) can be rephrased as: \( A_\infty \)-coalgebra \( bv_0 \) is bar-dual to algebra \( \widehat{YM}_0 \).

### 2.2 Truncated Yang-Mills algebra

We need to describe notations adopted in this section.

Define a bigraded vector space \( W = \bigoplus_{i=0}^{3} \bigoplus_{j=0}^{8} W_{ij} \)

\[
W_0^0 = <c> \quad W_1^2 = V + \Phi \quad W_3^3 = S
\]

\[
W_3^6 = <c^*> \quad W_2^5 = V^* + \Phi^* \quad W_2^5 = S^*.
\]

We define

\[
W_i = \bigoplus_{j=0}^{8} W_{ij} \quad W^j = \bigoplus_{i=0}^{3} W_{ij}.
\]

We refer to

\[
W = \bigoplus_{i=0}^{3} W_i \text{ as to homological and } W = \bigoplus_{j=0}^{8} W^j \text{ as to additional gradings.}
\]

We also have a filtration.

\[
F^n(W)_i = \bigoplus_{j \geq k} W_{ij}.
\]

The algebra \( \widehat{YM}_0 \) is filtered by multiplicative filtration \( F^n(\widehat{YM}_0) \). On generating space \( W_1 \) it is given by the formula \( 38 \), which determines it uniquely. This filtration was alluded in the end of the proof of proposition \( 13 \).

A continuous differentiation \( \partial_i \) of \( \widehat{YM}_0 \) is defined by the formulas:

\[
\partial_i(A_j) = \delta_{ij} \quad (39)
\]

\[
\partial_i(\phi_k) = \delta_i(\psi'^\alpha) = 0 \quad (40)
\]
We use $\delta_{ij}$ for Kronecker $\delta$ symbol. All such differentiations span $D$-dimensional vector space $V^*$. They can be arranged into one differentiation $\partial : \hat{YM}_0 \to V^* \otimes \hat{YM}_0, \partial(a) = \partial_1(a), \ldots, \partial_d(a)$. A choice of projection

$$\mu : V^* \to V \to 0$$

(41)
on space $V$ specifies a differentiation $\partial_\mu$ of $YM_0$ with values in a bimodule $V \otimes \hat{YM}_0$.

Denote $\hat{YM}_\mu = \text{Ker} \partial_\mu$ and $\hat{YM}_\mu = I(\hat{YM}_\mu)$. The algebra $\hat{YM}_\mu$ is filtered by $\hat{YM}_\mu \cap F^n$ where $F^n = F^n(\hat{YM}_0)$ is a filtration on $\hat{YM}_0$.

Similar constructions hold for algebras $YM_0$ and $YM'_0$. As a result we can define $T_\mu YM$ and $T_\mu YM'$. Define an algebra $\hat{E}_\mu YM = \hat{YM}_0 \otimes \Lambda^\bullet(V)$. The differential is defined by the rule:

$$d(a) = \partial_\mu(a) \in YM_0 \otimes V \subset YM_0 \otimes \Lambda^\bullet(V) \quad a \in YM_0$$

$$d(v) = 0 \quad v \in \Lambda^\bullet(V).$$

(42)

The differential can be extended uniquely to the algebra using the Leibniz rule.

Denote $\hat{E}_\mu YM = I(\hat{E}_\mu YM)$.

A similar construction works for algebras $YM_0$ and $YM'_0$. We can define algebras $E_\mu YM$ and $E_\mu YM'$. Define a multiplicative decreasing filtration $F^s(\hat{E}_\mu YM) = F^s(\hat{E}_\mu YM)$ which extends filtration on $F^s(\hat{YM}_0)$. It is uniquely determined by condition $V \subset F^1$. It defines a filtration on $\hat{E}_\mu YM$ for which we keep the same notation. A similar filtration exist on $E_\mu YM$. The algebra $E_\mu YM'$ is graded, the grading is a multiplicative extension of grading on $YM'_0$, the grading of the space $V \subset E_\mu YM'$ is equal to one. In case of algebras $\hat{E}_\mu YM$, $E_\mu YM$, the differential preserves the filtration. In case of $E_\mu YM'$ the differential preserves the grading.
Lemma 28 Suppose $B$ is an algebra with a unit generated by elements $B_1, \ldots, B_n$. Assume that we given $k$ commuting differentiations $\partial_s$ $s = 1, \ldots, k$ $k \leq n$ of the algebra $B$ such that $\partial_s B_j = \delta_{sj}$. Then there exists an increasing filtration $G^i$

\[i = 0, 1, \ldots,\] such that

a. $\bigcup_i G^i = B$.

b. $G^i G^j \subset G^{i+j}$.

c. $B_i \in G^1$.

Proof.

Define filtration $G^i$ inductively. By definition $G^0 = \bigcap_{s=1}^k \text{Ker} \partial_s$, then

\[G^{i+1} = \{ x | \partial_s(x) \in G^i \text{ for all } s \text{, } 1 \leq s \leq k \} \quad (43)\]

The property b follows from the Leibniz rule. By definition $B_i \in G^1$ hence c follows. Since $B_i$ are generators b and c imply a. ■

Consider an algebra $\text{Gr}_G(B) = \bigoplus_{i=0}^\infty G^{i+1}B/G^iB$. Denote the image of elements $B_1, \ldots, B_k$ in $G^1/G^0$ by $\hat{B}_1, \ldots, \hat{B}_k$.

Lemma 29 The algebra $\text{Gr}_G(B)$ has the following properties:

a. The elements $\hat{B}_1, \ldots, \hat{B}_k$ commute in $\text{Gr}_G(B)$.

b. The elements $\hat{B}_1, \ldots, \hat{B}_k$ commute with $G^0B$.

c. The subalgebra of $\text{Gr}_G(B)$ generated by $G^0B$ and $\hat{B}_1, \ldots, \hat{B}_k$ is isomorphic to $\mathbb{C}[\hat{B}_1, \ldots, \hat{B}_k] \otimes G^0B$.

d. Elements $\hat{B}_1, \ldots, \hat{B}_k$ and $G^0B$ generate $\text{Gr}_G(B)$.

e. We have an isomorphism $\text{Gr}_G(B) = \mathbb{C}[\hat{B}_1, \ldots, \hat{B}_k] \otimes G^0B$.

Proof.

a. $\partial_s[B_i, B_j] = [\delta_{si}, B_j] + [B_i, \delta_{sj}] = 0$, therefore $[B_i, B_j] \in G^0$, hence $[\hat{B}_i, \hat{B}_j] = 0$.

b. Similarly $\partial_s[B_i, m] = [\delta_{si}, m] = 0$ for $m \in G^0 \subset \text{Ker} \partial_s$ for $s = 1, \ldots, k$, Therefore $[B_i, m] \in G^0$ and $[\hat{B}_i, m] = 0$

c. Denote the subalgebra generated by $G^0B$ and $\hat{B}_1, \ldots, \hat{B}_k$ by $C$. There is a surjective map

\[\mathbb{C}[\hat{B}_1, \ldots, \hat{B}_k] \otimes G^0B \rightarrow C\]  \hfill (44)
and an inclusion $G^0 \subset C$. Denote the kernel of the map \((44)\) by $I$. Then

$$I \cap G^0 B = 0. \quad (45)$$

Suppose $0 \neq a = \sum a_{i_1, \ldots, i_k} \hat{B}^{i_1}_{i_1} \cdots \hat{B}^{i_k}_{i_k} \in I$. Since $a$ is a polynomial in $\hat{B}_1, \ldots, \hat{B}_k$, there is $i_1, \ldots, i_k$ such that there is no $a_{i'_1, \ldots, i'_k} \neq 0$ with $i_1 < i'_1, \ldots, i_k < i'_k$. It means that an element $\partial^{i_1}_{i_1} \cdots \partial^{i_k}_{i_k} a \neq 0$ belongs to $I$ and is independent of $\hat{B}_1, \ldots, \hat{B}_k$, which contradicts with \((45)\).

**d.** We are going to prove the statement by induction on the index $i$ in $\text{Gr}^i_G(B)$. If $i = 0$ then there is nothing to prove. Suppose we have an element $\hat{a} \in \text{Gr}^{i+1}_G(B)$. Let $a \in G^{i+1}$ its representative in $B$. Then $\partial_s a = b_s \in G^i$ and by inductive assumption $\hat{b}_s = \hat{b}_s(\hat{B}_1, \ldots, \hat{B}_k) \in C$. The elements $b_s$ satisfy $\partial_i \hat{b}_s = \partial_s \hat{b}_i$. It implies that there is $b \in B$ such that $\partial_s b = \hat{b}_s$. Consider the difference $a - b = c$, $\partial_s c = 0$, hence $\partial_s c \in G^{i-1}$ for every $s$, therefore $c \in G^i$ and $\hat{a} = \hat{b}$.

**Lemma 30** Suppose an algebra $B$ satisfies conditions of lemma \((28)\). Denote by $V$ a linear space with a basis $[\partial_1], \ldots, [\partial_k]$. Define a structure of a complex with differential $\sum_{s=1}^k [\partial_s] \partial_s$ on a linear space $\Lambda^i(V) \otimes B$. Then the cohomology of this complex are concentrated in degree 0 and isomorphic to $G^0$.

**Proof.**

Define a filtration on the complex $H_i = B \otimes \Lambda^i(V)$ by $G^j H_i = G^{i+j} \otimes \Lambda^i(V)$.

The adjoint quotients of this filtration are isomorphic to

$$G^j H_i / G^{j-1} H_i = \Lambda^i(V) \otimes \text{Sym}^{i+j}(V) \otimes G^0. \quad (46)$$

The differential coincides with de Rham differential. It cohomology is isomorphic to $G^0$ in zero degree and 0 in higher degrees. The spectral sequence corresponding to filtration $G^j$ collapses in $E_1$ term and converges to cohomology we are looking for.

**Lemma 31**
a. The embeddings $(TYM_\mu, 0) \to (E_\mu YM, d)$ and $(TYM'_\mu, 0) \to (E_\mu YM', d)$ are quasiisomorphisms.

b. The map

$$(TYM_\mu, 0) \to (E_\mu YM, d)$$

(47)

is a filtered quasiisomorphism of algebras (see Appendix for definition).

Proof.

a. The algebras $YM_0$ and $YM'_0$ satisfy lemma (30), hence the proof follows.

b. The morphism (47) is compatible with filtrations which exist on its range and domain. It induces a map of spectral sequences associated with filtrations. Let us analyse $E^2$-term.

The algebra $A = \text{Gr}_F(\hat{YM}_0)$ is isomorphic to $YM'_0$ where in the relations (8, 9) we drop the potential $U$. The algebra $YM'_0$ is finitely generated, graded, carries no topology. The algebra $A$ satisfies conditions of lemma (30). It implies that there is a quasiisomorphism $(\text{Gr}TYM_\mu, 0) \to (\Lambda(V) \otimes \hat{YM}_0, d)$.

We see that the map (47) induces an isomorphism of $E^1$ term of corresponding spectral sequences. Since the range and domain are complete with respect to filtrations we conclude that the map (47) is quasiisomorphism. The property that it induces quasiisomorphism of adjoint quotients, it is the same as filtered property. ■

Corollary 32 The map $(TYM_\mu, 0) \to (E_\mu YM_0, d)$ is a filtered quasiisomorphism. $(TYM_\mu, 0) \to (E_\mu YM, d)$ and $(TYM'_\mu, 0) \to (E_\mu YM', d)$ are quasiisomorphisms.

Remark 33 The algebra $E_\mu YM$ is topologically finitely generated. It implies that the canonical filtration is comparable with filtration $F^*(E_\mu YM)$. It implies that the algebra $E_\mu YM_0$ is complete with respect to canonical filtration $I^n_{E_\mu YM_0}$ and $TYM_\mu$ with respect to $TYM_\mu \cap I^n_{E_\mu YM_0}$.  

28
2.3 Construction of \( BV_\mu \)

Let \( V \) be a vector space generated by symbols \( \partial_i, i = 1, \ldots, d, \) \( \text{Sym}(V) = \bigoplus_{i=0}^\infty \text{Sym}^i(V) \). Denote a decreasing filtration of \( \text{Sym}(V) \) associated with the grading by \( F^n(\text{Sym}(V)) \). The vector spaces \( W_i, i = 0, \ldots, 3 \) were defined by the formulas (36).

\[
\begin{align*}
\mathbb{W}_i &= W_i \otimes \text{Sym}(V) \\
\mathbb{W} &= \mathbb{W}_0 + \mathbb{W}_1 + \mathbb{W}_2 + \mathbb{W}_3 
\end{align*}
\]  

(48)

The last direct sum decomposition is called homological grading on \( \mathbb{W} \).

Denote two filtrations on \( \mathbb{W} \). The first one is

\[
F^n(\mathbb{W}) = \bigoplus_{i+j \geq n} F^i(W) \otimes F^j(\text{Sym}(V)),
\]  

(49)

where \( F^i(W) \) was defined in (38). The second is

\[
\tilde{F}^n(\mathbb{W}) = W \otimes F^n(\text{Sym}(V)).
\]  

(50)

We have

\[
\tilde{F}^n(\mathbb{W}) \subset F^n(\mathbb{W}) \subset \tilde{F}^{n-\delta}(\mathbb{W})
\]  

(51)

with finite dimensional quotients. Denote the completion of \( \mathbb{W} \) by \( \hat{\mathbb{W}} = \hat{\mathbb{W}}_0 + \hat{\mathbb{W}}_1 + \hat{\mathbb{W}}_2 + \hat{\mathbb{W}}_3 \).

Filtrations \( \tilde{F}^n(\mathbb{W}) \) and \( F^n(\mathbb{W}) \) define two multiplicative filtrations on \( T(\mathbb{W}[1]) \), which we denote by \( \tilde{F}^n(T(\mathbb{W}[1])) \) and \( F^n(T(\mathbb{W}[1])) \). The algebra \( T(\mathbb{W}[1]) \) acquires a homological grading by multiplicative extension of homological grading from \( \mathbb{W} \).

Define an additional grading on \( \mathbb{W} \):

\[
\mathbb{W}^k = \bigoplus_{i+j=k} W^i \otimes \text{Sym}^j(V)
\]  

(52)

The algebra \( T(\mathbb{W}[1]) \) acquires an additional grading by multiplicative extension of additional grading from \( \mathbb{W} \).

**Proposition 34** The completion of \( T(\mathbb{W}[1]) \) with respect filtrations \( \tilde{F}^n(\mathbb{W}) \) and \( F^n(\mathbb{W}) \) coincide and we denote it by \( T(\mathbb{W}[1]) \). Similarly two completions of the space of generators \( \mathbb{W} \) coincide.
Proof.

The filtration satisfy the following inclusions: \( \hat{F}_n(T(\mathbf{W}[1])) \subset F^n(T(\mathbf{W}[1])) \subset \hat{F}^{n-k}(T(\mathbf{W}[1])) \) for some finite \( k \) with finite-dimensional quotients. This is a simple corollary of equation \( 51 \). We see that the filtrations are commensurable. It is an simple exercise to show the completions are equal.

The operators \( \partial_i \) act by multiplication on the set of generators of the algebra \( T(\mathbf{W}[1]) \) (recall that it is a free \( \text{Sym}(V) \)-module). We extend the action of \( \partial_i \) on \( T(\mathbf{W}[1]) \) as a continuous derivation, which we denote by the same symbol \( \partial_i \).

There is a linear map \( \mu : W_1 \to V \). It is extension by zero from \( V \) to \( W_1 = V + \Pi S + \Phi \) of the map \( \mu \) defined in \( 41 \). We used identification \( < A_1, \ldots, A_D > \) and \( < A_1, \ldots, A_D >^* \) provided by the canonical bilinear form . The algebra \( T(\mathbf{W}[1]) \) admits a continuous action of outer derivation \( \nabla_i \) defined by the formula

\[
\nabla_i x = \mu(A_i)x + [A_i, x].
\]

The commutator is defined as:

\[
[\nabla_i, \nabla_j] = \mu(A_i)A_j - \mu(A_j)A_i + [A_i, A_j].
\]

Definition 35 Define a continuous differential in the algebra \( T(\mathbf{W}[1]) \) by the formulas \( 55 \) where \( \text{deg}(U) \geq 8 \). Define a differential on the algebra \( T(\mathbf{W}[1]) \) by the formula \( 55 \) where \( \text{deg}(U) \geq 8 \) and we impose some finitness conditions on the potential. We denote such algebra \( (T(\mathbf{W}[1]), Q^\mu) \). The filtration on algebras \( T(\mathbf{W}[1]) \) and \( T(\mathbf{W}[1]) \) is preserved by the differential.

If one assume that potential \( \text{deg}(U) = 8 \) we denote the algebra \( (T(\mathbf{W}[1]), Q^\mu) \) by \( (T(\mathbf{W}[1])', Q^\mu) \). In this algebra the differential has a degree zero with respect to the additional grading.

One can extend \( Q^\mu \) uniquely to the entire set of generators, using commutation properties with \( \partial_i \).

Proposition 36 The differential \( Q^\mu \) satisfies \( (Q^\mu)^2 = 0 \). The differential has degree \( -1 \) with respect to homological grading.
Observe though the elements $\nabla_i$ do not belong to the algebra $T(\overline{W}[1])$, all RHS expressions in the formulas do.

Define an odd differentiation $\varepsilon$ of the algebra $T(\overline{W}[1])$ by the formula $\varepsilon(c) = 1$, the value of $\varepsilon$ on all other generators is equal to zero. It can be extended by continuity to $\hat{T}(\overline{W}[1])$.

**Proposition 37** The commutator $\{Q^\mu, \varepsilon\}$ is a differentiation $P$ of the algebra $T(\overline{W}[1])$ which on generators is equal to identity transformation. The same holds for $\hat{T}(\overline{W}[1])$.

**Proof.** Direct computation. ■

It implies that $P$ is an invertible linear transformation on $T(\overline{W}[1])$ and on $\hat{T}(\overline{W}[1])$, and $H = \varepsilon/P$ is a contracting homotopy. We are interested in a modification of the algebra $(\hat{T}(\overline{W}[1]), Q^\mu)$, $(T(\overline{W}[1]), Q^\mu)$, $(\hat{T}(\overline{W}[1])', Q^\mu)$. Denote $\hat{I}(c)$ a closure of the ideal generated by $c$. A simple observation is that $\hat{I}(c)$ is a differential ideal. The ideal $\hat{I}(c)$ is not closed under the action of $\partial_i$ however.

**Definition 38** Denote by $\hat{BV}_\mu$ the quotient algebra $T(\overline{W}[1])/\hat{I}(c)$. Similarly define $BV_\mu$ and $BV'_\mu$. The former algebra is filtered the later is graded.

**Remark 39** In contrast with algebras $\hat{BV}_0$, $BV_0$, $BV'_0$ the algebras $\hat{BV}_\mu$, $BV_\mu$, $BV'_\mu$ have nontrivial components with negative homological grading.

**Proposition 40** There is a morphism of differential graded algebras $(\hat{BV}_\mu, Q^\mu) \to (E_\mu YM, d)$ defined by the formulas

$$
p(\partial_i c) = [\partial_i]
\quad p(A_i) = A_i
\quad p(\phi_i) = \phi_i
\quad p(\psi^\alpha) = \psi^\alpha
$$

which preserves the filtrations (so it is continuous). The map is zero on the rest of the generators. The formulas (55) defines a map $(BV_\mu, Q^\mu) \to (E_\mu YM, d)$
which preserves the filtrations and a map $(BV'_\mu, Q^\mu) \to (E_\nu Y M', d)$ of degree zero.

The filtrations $F^n T(\hat{W}[1])$ and $\hat{F}^n T(\hat{W}[1])$ of $T(\hat{W}[1])$ induce similarly named filtrations of $\hat{BV}_\mu$, denoted $F^n \hat{BV}_\mu$ and $\hat{F}^n \hat{BV}_\mu$. Filtration $F^n$ and $\hat{F}^n$ are also defined on $BV_\mu$ and $BV'_\mu$. In the later case filtration $F^n$ coincides with decreasing filtration associated with the grading.

Define $Gr\hat{BV}_\mu = \prod_{n \geq 0} Gr^n \hat{BV}_\mu$ as

$$Gr^n BV_\mu = \hat{F}^n \hat{BV}_\mu / \hat{F}^{n+1} \hat{BV}_\mu.$$ (56)

**Proposition 41** The algebra $Gr\hat{BV}_\mu$ is a completion of a free algebra with the same space of generators as $\hat{BV}_\mu$. The differential $Q$ is defined by the formulas (5), except in the formula (53) one needs to replace

$$\mu(A_i)x + [A_i, x] \Rightarrow [A_i, x].$$ (57)

Similarly $GrBV_\mu$ and $GrBV'_\mu$ coincide with $BV_\mu$ as algebras. In the definition of $Q^\mu$ one has to alter $\nabla_i$ according to the rule (57).

**Proof.**

Obvious. ■

Nonreduced bar-complex of an $A_\infty$-coalgebra $H$ with a counit $\varepsilon$ is by definition bar-complex of $H$ as if it had no counit. A simple theorem asserts that in presence of a counit it is always contractible.

**Definition 42** The algebras $(\hat{T}(\hat{W}[1]), Q^\mu)$ and $(GrT(\hat{W}[1]), GrQ)$ can be thought of as nonreduced bar-complex of $A_\infty$-coalgebras with a counit and coaugmentation which we denote by $(\hat{W}, Q^\mu)$ and $(\hat{W}, Gr(Q^\mu))$, the corresponding coideals are denoted as $bv_\mu$ and $Grbv_\mu$. Similarly we have $A_\infty$-coalgebras $(\hat{W}, Q^\mu), (\hat{W}, Gr(Q^\mu), (\hat{W}', Q^\mu), (\hat{W}', Gr(Q^\mu))$ with coaugmentation coideals $(bv_\mu, Q^\mu), (Grbv_\mu, GrQ^\mu)$ and $(bv'_\mu, Q^\mu), (Grbv'_\mu, GrQ^\mu)$ respectively.
There a general construction of tensor product of $A_\infty$-coalgebras. Its description when one of the tensor factors is an ordinary coalgebra is very simple. Suppose $H$, $G$ are two $A_\infty$-algebras and $G$ was only one nontrivial operation $\Delta_2 = \Delta : G \to G \otimes 2$. Define a mapping $\nu_n \to G \otimes n$ by the formula

$$\nu_n^G = (\Delta \otimes \text{id} \otimes \ldots \text{id}) \circ \cdots \circ \Delta.$$  \hspace{1cm} (58)

The tensor product of $H$ and $G$ has its underlying vector space equal to $H \otimes G$. The operations $\Delta_n^{H \otimes G}$ are defined by the formula:

$$\Delta_n^{H \otimes G} (a \otimes b) = T \Delta_n^H (a) \otimes \nu_n^G (b)$$  \hspace{1cm} (59)

where operator $T$ is a graded permutation which defines isomorphism $H^{\otimes n} \otimes G^{\otimes n} \cong (H \otimes G)^{\otimes n}$. We need to bring readers attention to the fact that though $\Delta_n^H$ is the $n$-th operation in coalgebra $H$, the map $\nu_n^G$ is not such for coalgebra $G$, but rather $n$-th iteration of the binary operation. As you can see this construction is not symmetric.

Observe that on category of $A_\infty$-algebras a similar operation corresponds to extension of the ring of scalars.

It turns out that the algebras $\widehat{\text{Gr}bv_\mu}$, $\text{Gr}bv_\mu$, $\text{Gr}bv'_\mu$ has an alternative description in terms of finite dimensional $A_\infty$-coalgebra $\widehat{bv}_0$, $bv_0$, $bv'_0$ introduced in definition (26).

The Kunneth formula asserts

**Proposition 43** There is a quasiisomorphism of algebras $\widehat{\text{Bar}}(H \otimes G) \to \widehat{\text{Bar}}(H) \hat{\otimes} \widehat{\text{Bar}}(G)$.

**Proof.** See [12] for the proof in the case of algebras. The coalgebra case is similar. ■

**Proposition 44** There is an isomorphism of coalgebras $\text{Gr} bv_\mu = W \otimes \text{Sym}(V)$, $\text{Gr} bv_\mu = W' \otimes \text{Sym}(V)$.

**Proof.**

All nontrivial interactions between $bv_0$ and $\text{Sym}(V)$ parts inside $bv_\mu$ stem from $\mu(A_\pi) x$ part in the formula (33) which we kill passing from $\widehat{bv}_\mu$ to $\text{Gr} \widehat{bv}_\mu$. 

33
In the case at hand $H = \widehat{b_{V_0}}$, $G = \operatorname{Sym}(V)$. The symmetric algebra $\operatorname{Sym}(V)$ has a diagonal $\Delta$ which on $v \in V$ is equal to $\Delta(v) = v \otimes 1 + 1 \otimes v$. The arguments remain to be valid in non-complete and graded case.

**Proposition 45** The map $p : (\widehat{BV}_\mu, Q) \to (E_\mu \widehat{YM}, d)$ defined in equations is a quasi-isomorphism. The map $p : (BV_\mu', Q) \to (E_\mu YM', d)$ is also a quasi-isomorphism.

**Proof.**

Define a filtration $\tilde{F}^n E_\mu YM_0$ by the formula

$$\tilde{F}^n E_\mu YM = \bigoplus_{k \geq n} \Lambda^k(V) \otimes \widehat{YM}_0$$

Filtrations $\tilde{F}^n E_\mu \widehat{YM}$ are compatible with the map $p : \widehat{BV}_\mu \to \widehat{E}_\mu \widehat{YM}$. The map $p$ induces a map of spectral sequences associated with filtrations.

The term $E_0$ of the spectral sequence associated with $\widehat{BV}_\mu$ coincides with $\operatorname{Gr} \widehat{E}_\mu$. The proposition implies a series of quasi-isomorphisms:

$$\operatorname{Gr} BV_\mu \overset{\text{def}}{=} \widehat{\operatorname{Bar}}(\operatorname{Gr} \widehat{b_{V_0}}) = \widehat{\operatorname{Bar}}(b_{V_0} \otimes \widehat{\operatorname{Sym}}(V)) \to \widehat{\operatorname{Bar}}(b_{V_0}) \otimes \widehat{\operatorname{Bar}}(\widehat{\operatorname{Sym}}(V))$$

The map $\operatorname{Gr}(p)$ factors through the map $k : \operatorname{Gr}(p) = p' \circ k$, where $p'$ is

$$\widehat{\operatorname{Bar}}(b_{V_0} \otimes \widehat{\operatorname{Bar}}(\widehat{\operatorname{Sym}}(V))) \overset{p'}{\to} \widehat{YM}_0 \otimes \Lambda(V).$$

The map $p'$ in the formula is a tensor product of two quasi-isomorphisms. The first one is from proposition; the second one is a classical quasi-isomorphism $\widehat{\operatorname{Bar}}(\widehat{\operatorname{Sym}}(V)) \to \Lambda(V)$.

This considerations imply that there is an isomorphism $p : H^* \operatorname{Gr} BV_\mu \to H^* \operatorname{Gr} E_\mu YM$. It means that we have an isomorphism of spectral sequences associated with filtration $\tilde{F}^n$ starting with $E_1$ term.

It implies that the map $p$ induces a quasi-isomorphism of completed complexes: $p : \widehat{BV}_\mu \to \widehat{E}_\mu \widehat{YM}$. The proof goes through in graded case. All the tools which are needed for the proof are collected in the Appendix in the segment devoted homogeneous $A_\infty$-(co)algebras. The obstacle for the proof in
noncomplete case is the absence of quasiisomorphism $\text{Bar}bv_0 = BV_0 \to YM_0$.

\begin{proposition}
$(BV_\mu, Q)$, $T_\mu YM$ and $(BV'_\mu, Q)$, $T_\mu YM'$ are pairs of quasi-isomorphic algebras.
\end{proposition}

\begin{proof}
By proposition (45) the algebra $(BV_\mu, Q)$ is quasiisomorphic to $(E_\mu TYM, Q)$. By lemma (31) the algebra $(E_\mu TYM, Q)$ is quasiisomorphic to $T_\mu YM$. The proof for the second pair is similar.
\end{proof}

\begin{proposition}
Hochschild homology $H_i(T_\mu YM, C)$ as $A_\infty$-coalgebra is isomorphic to $A_\infty$-coalgebra $\text{bv}_\mu$. The same isomorphism holds in graded case.
\end{proposition}

\begin{proof}
There is a series of quasiisomorphisms

$$
\underset{\text{a}}{\hat{T}YM} \overset{a}{\cong} \underset{\text{b}}{E_\mu YM} \overset{b}{\cong} \text{Bar}(bv_\mu) \tag{63}
$$

A theorem (80) assert that if all quasiisomorphisms in equation (63) filtered then we have a quasiisomorphism

$$
\text{Bar}\hat{T}_\mu YM \cong \text{BarBar}(bv_\mu). \quad \tag{64}
$$

Lemma (31) asserts that quasiisomorphism $a$ is filtered. The proof of proposition (45) shows that the quasiisomorphism $b$ is filtered. A proposition (83) claims that for any algebra complete with respect to canonical filtration we have a quasiisomorphism

$$
\hat{bv}_\mu \cong \text{BarBar}(bv_\mu). \quad \tag{65}
$$

By definition homology of algebra $T_\mu YM$ is homology of the bar-complex $\text{Bar}T_\mu YM$. By the result of [13] there is a quasiisomorphism of $A_\infty$-coalgebras $H(T_\mu YM)$ and $\text{Bar}T_\mu YM$. Quasiisomorphisms (64) and (65) finish the proof. The proof in the graded case is similar.
\end{proof}
Proposition 48 The differential $Q_1^\mu$ in $A_\infty$-coalgebra $\hat{w}_\mu$ is defined on $\hat{\text{Sym}}(V)$-generators by the formulas:

\begin{align*}
Q_1^\mu(\phi_k) &= 0 \\
Q_1^\mu(A_i) &= -\mu(A_i)c \\
Q_1^\mu(\psi^\alpha) &= 0 \\
Q_1^\mu(c) &= 0 \\
Q_1^\mu(c^*) &= \sum_{i=1}^D \mu(A_i)A^*_i \\
Q_1^\mu(A^{*m}) &= \sum_{i=1}^D -\mu(A_i)\mu(A_i)A_m + \mu(A_m)\mu(A_i)A_i \\
Q_1^\mu(\phi^{*j}) &= \sum_{i=1}^D -\mu(A_i)\mu(A_i)\phi_j \\
Q_1^\mu(\psi^{*\alpha}) &= \sum_{i=1}^D \sum_{\beta} -\Gamma^i_{\alpha\beta}\mu(A_i), \psi^\beta
\end{align*}

(66)

The same formulas hold in graded case. The homological grading on the complex $\hat{W}$ is defined as follow: the components of homological degree $i$ is equal to $\hat{W}_i$.

Proof. Direct inspection.

2.4 Examples of computations

Example 0

The first trivial example is when $V = 0$ and $\mu = 0$. In this case differential $Q_1$ in (66) are equal to zero and we get that $H_*(\hat{YM}_0) = W$ where graded space $W$ is defined by the formula (36). This is a tautological result.

The second example is when $\text{dim}(V) = 1$. We have to options: restriction of the bilinear form $(.,.)$ on the kernel of the map (41) is a. invertible; b. degenerate.

Example 1
Let us analyse the case a. Below is an explicit description of the complex

\[ L \otimes c^* \xrightarrow{\xi} L \otimes A^1 \]
\[ L \otimes A^2 \xrightarrow{\xi^2} L \otimes A_2 \]
\[ \ldots \]
\[ L \otimes A^D \xrightarrow{\xi^D} L \otimes A_D \]
\[ L \phi^* \xrightarrow{\xi} L \otimes \phi_1 \]
\[ \ldots \]
\[ L \otimes \phi_d^* \xrightarrow{\xi} L \otimes \phi_d' \]
\[ L \otimes \psi_{\alpha} \xrightarrow{\xi} L \otimes \psi_{\alpha} \]
\[ \ldots \]
\[ \begin{array}{ccc}
3 & 2 & 1 \\
L \otimes A_3 & L \otimes A_2 & L \otimes A_1 & L \otimes c \\
\end{array} \]

(67)

where \( L = \mathbb{C}[t] \). The cohomological classes of this complex are

a In dimension 0 it is a space of constants.

b In dimension 1 the space is spanned by \( A_2, \ldots, A_D, tA_2, \ldots, tA_D, \phi_1, \ldots, \phi_d', \phi_1, \ldots, t\phi_d', \psi_{\alpha} \).

c In dimension 2 the space is spanned by \( A^* \).

d In dimension 3 the space of cocycles is zero.

This computation enables us to identify the algebra \( T_\mu YM \) with subalgebra \( K \subset YM_0 \) defined in theorem (13). The connection is \([q_i] = A_{i+1}, [p^j] = tA_{i+1}, [P^j] = \phi_j, [Q^j] = t\phi_j, [\psi_{\alpha}] = \psi_{\alpha} \). The symbol \([a]\) denotes the homology class of a generator \( a \). The cocycle \( A^* \) corresponds to the only relation \( \sum_{i=1}^{D-1} [q_i, p^j] + \sum_{i=1}^{d'} [Q_i, P^j] = \frac{1}{2} \sum_{\alpha} \{ \psi_{\alpha}, \psi_{\alpha} \} \).

The algebra \( K \) has homological dimension 2. There is up to a constant only one homological class which we denote \( f \in H_2(K) \). The algebra \( K \) is a universal enveloping algebra of a Lie algebra \( \mathfrak{k} \) with the same set of generators and relations. We have an isomorphism \( H^2(K) = H^2(\mathfrak{k}, \mathbb{C}) \). It can be used to define a symplectic structure on moduli spaces of representations of \( \mathfrak{k} \) in a semisimple Lie algebra \( \mathfrak{g} \), equipped with an invariant dot-product \( (.,.)_{\mathfrak{g}} \). It is well known what a tangent space to a point \( \rho \) of the moduli of representations of a Lie algebra \( \mathfrak{m} \) is. It is equal to \( H^1(\mathfrak{m}, \mathfrak{g}) \). In our case it is equal to \( H^1(\mathfrak{k}, \mathfrak{g}) \). If we have two elements \( a, b \in H^1(\mathfrak{k}, \mathfrak{g}) \), a cohomological product \( a \cup b \in H^2(\mathfrak{k}, \mathfrak{g} \otimes \mathfrak{g}) \). A composition with \( (.,.)_{\mathfrak{g}} \) gives an element of \( H^2(\mathfrak{k}, \mathbb{C}) \), whose value on homological
class \( \int \) is equal to the value of the symplectic dot-product \( \omega(a, b) \). In more condensed notations we can write:

\[
\omega(a, b) = \int (a, b)_g
\]  

(68)

**Proposition 49** Symplectic form \( \omega(a, b) \) defined on the moduli space \( \text{Mod}_t(g) \) is nondegenerate and closed.

**Proof.**

There is a different description of the space \( \text{Mod}_t(g) \). Consider a linear space \((g + g)^{(D-1)} + (g + g) \otimes \Phi^* + \Pi g \otimes S^*\). We can identify vector space \( g + g \) with \( g + g^* \), by means of invariant bilinear form \((.,.)_g\). The space \( g + g^* \) is a symplectic manifold. The space \( \Pi g \) is an odd-dimensional symplectic manifold with symplectic form equal to \((.,.)_g\). The Lie algebra \( g \) acts on this space by symplectic vector fields. Define a set of functions \( f_i = (e_i, \sum_{k=1}^{D-1} [q_k, p^k] + \sum_{k=1}^{d'} [Q_k, P^k] - \frac{1}{2} \sum_{a} \{\psi^a, \psi^a\}) \). It is easy to see that this set of functions defines a set of Hamiltonians for generators \( e_i \) of the Lie algebra \( g \). A symplectic reduction with respect to the action of \( g \) gives rise precisely the manifold we are studying. The statement of the proposition follows from the general properties of Hamiltonian reduction. ■

**Example 3**

Now we want to discuss the case \( b \) where the restriction of the bilinear form on the kernel of the map \( \mu \) is degenerate. It is easy to see that the null space of the form is one-dimensional. Without loss of generality we may assume that \( \mu(A_0) = t, \mu(A_1) = it \) (\( i \) is the imaginary unit) and the map \( \mu \) on the rest of the generators is equal to zero. It is convenient to make a change of coordinates

\[
\begin{align*}
    v &= \frac{1}{\sqrt{2}} (A_1 + iA_2) \\
    u &= \frac{1}{\sqrt{2}} (A_1 - iA_2) \\
    u^* &= \frac{1}{\sqrt{2}} (A^1 + iA^{*2}) \\
    v^* &= \frac{1}{\sqrt{2}} (A^1 - iA^{*2})
\end{align*}
\]  

(69)
In these notation the differential $Q_1$ looks like:

$$
\begin{array}{cccc}
C[t] \otimes c^* & \sqrt{2}t & C[t] \otimes u^* \\
C[t] \otimes v^* & -2t^2 & C[t] \otimes v \\
C[t] \otimes A^{3} & \frac{2}{3} & C[t] \otimes A_3 \\
\cdots & \cdots & \cdots \\
C[t] \otimes A^{D} & \frac{D}{2} & C[t] \otimes A_D \\
C[t] \phi^* & 0 & C[t] \otimes \phi_1 \\
\cdots & \cdots & \cdots \\
C[t] \otimes \phi^* & 0 & C[t] \otimes \phi_1 \\
C[t] \otimes \psi^* & \frac{D}{2} & C[t] \otimes \psi_1 \\
\cdots & \cdots & \cdots \\
C[t] \otimes u & \sqrt{2}t & C[t] \otimes c \\
\end{array}
$$

(70)

where $G$ is a linear map $S^* \to S$. In degenerate case not much could be said about $G$. If $G$ build from spinorial $\Gamma$-matrices, $G$ has a kernel with dimension equal to $1/2 \dim(S)$. An important observation is that the complex (70) has infinite homology groups in dimensions 1, 2. The homology in dimension 3 is trivial and zero homology is one-dimensional. To simplify formulas for truncated Yang-Mills algebra in this case we get rid of fermions. After change of variable (69) relations (7, 8, 9) become

$$
\begin{align*}
- [v[u, v]] + \sum_{i=3}^{D}[A_i, [A_i, v]] + \sum_{k=1}^{d'}[\phi_k, [\phi_k, v]] &= 0 \\
- [u[u, v]] + \sum_{i=3}^{D}[A_i, [A_i, u]] + \sum_{k=1}^{d'}[\phi_k, [\phi_k, u]] &= 0 \\
[u[v, A_m]] + [v[u, A_m]] + \sum_{i=3}^{D}[A_i, [A_i, A_m]] + \sum_{k=1}^{d'}[\phi_k, [\phi_k, A_m]] &= 0 \\
&= 0 \quad m = 3, \ldots, D \\
[u[v, \phi_j]] + [v[u, \phi_j]] + \sum_{k=3}^{D}[A_k, [A_k, \phi_j]] + \frac{\partial U}{\partial \phi_j} &= 0 \quad j = 1, \ldots, d'
\end{align*}
$$

[39]
The generators of the algebra $T_{\mu}YM$ with $rk(\mu) = 1$ and $ind(\mu) = 0$ are

$$v$$

$$p = [u, v]$$

$$A_{m}^{n} = Ad^{n}(u)A_{m} \quad n \geq 0, m = 3, \ldots D$$

$$\phi_{j}^{n} = Ad^{n}(u)\phi_{j} \quad n \geq 0, j = 1, \ldots d'$$

As in the case of the first example the algebra $YM_{0}$ is a semidirect product of an abelian one-dimensional Lie algebra and algebra $T_{\mu}YM$. The relations in $T_{\mu}YM$ and the action of the generator of the abelian Lie algebra (Hamiltonian) can be read off from equations (71). The action of the Hamiltonian is given by the formulas:

$$H(p) = \sum_{i=3}^{D} [A_{i}^{0}, A_{i}^{1}] + \sum_{k=1}^{d'} [\phi_{k}^{0}, \phi_{k}^{1}]$$

$$H^{m}(A_{i}^{0}) = A_{i}^{m}$$

$$H^{m}(\phi_{i}^{0}) = \phi_{i}^{m}$$

The relations are:

$$[v, p] + \sum_{i=3}^{D} [A_{i}^{0}, [A_{i}^{0}, v]] + \sum_{k=1}^{d'} [\phi_{k}^{0}, \phi_{k}^{0}, v] = 0$$

$$A_{0}^{*m} = [p, A_{m}^{0}] + 2[v, A_{m}^{1}] + \sum_{i=3}^{D} [A_{i}^{0}, [A_{i}^{0}, A_{m}^{0}]] + \sum_{k=1}^{d'} [\phi_{k}^{0}, [\phi_{k}^{0}, A_{m}^{0}]] = 0$$

$$m = 3, \ldots, D$$

$$\phi_{0}^{*j} = [p, \phi_{j}^{0}] + 2[v, \phi_{j}^{1}] + \sum_{k=3}^{D} [A_{k}^{0}, [A_{k}^{0}, \phi_{j}^{0}]] + \frac{\partial U}{\partial \phi_{j}^{0}} = 0 \quad j = 1, \ldots, d'$$

$$A_{n}^{*m} = H^{n}(A_{0}^{*m}) \quad m = 3, \ldots, D \quad n \geq 1$$

$$\phi_{n}^{*j} = H^{n}(\phi_{0}^{*j}) \quad j = 1, \ldots, d' \quad n \geq 1$$

**Proposition 50** There is an isomorphism of $T_{\mu}YM$ and a quotient algebra $\mathbb{C} < v, p, A_{k}^{0}, \phi_{j}^{0}, \psi_{\alpha} > / (I)$ where ideal is generated by relations (72). There is an isomorphism $\mathbb{C}[H] \ltimes T_{\mu}YM \cong YM_{0}$.

**Example 4.**
Suppose $V = V$ and $\mu = id$. We need to compute cohomology of complexes:

\[
\begin{array}{cccc}
S^* \otimes \hat{\text{Sym}}(V) & \overset{D}{\longrightarrow} & S \otimes \hat{\text{Sym}}(V) \\
<c^* > \otimes \hat{\text{Sym}}(V) & d & \Pi V^* \otimes \hat{\text{Sym}}(V) & d \cdot d \\
\Phi^* \otimes \hat{\text{Sym}}(V) & \times ||v||^2 & \Phi \otimes \hat{\text{Sym}}(V) & \\
3 & 2 & 1 & 0
\end{array}
\]

(72)

It is easy to see that homology of the complex (72) coincide with completion of homology of a similar complex with $\hat{\text{Sym}}$ stripped off the completion sign. Therefore we will examine only noncompleted version. It is particularly easy to compute $\Phi^* - \Phi$ part of cohomology. It is equal zero in all dimensions but one where it is $\Phi \otimes \text{Sym}(V)/(||v||^2)$. By $(||v||^2)$ we denote a homogeneous ideal of functions equal to zero on quadric $q$ given by equation $||v||^2 = 0$.

Denote

\[
T = \{(a, b) \in V \times V ||a||^2 = 0, (a, b) = 0\}
\]

\[
X = \{(a, b) \in T ||b| is defined up to addition of multiple of a\}.
\]

(73)

Then $X$ is a quotient bundle of the tangent bundle $T$ to the quadric by one-dimensional subbundle $L$. $L$ consists of all vector fields that are tangent to the projection $q\{0\} \rightarrow \tilde{q} \subset \mathbb{P}^{D-1}$. The space of global sections of $X$ is precisely the first cohomology of the complex (72) in $V \otimes \text{Sym}(V)$-term. The zero cohomology in $< c > \otimes \text{Sym}(V)$ term is one-dimensional by obvious reason. Vanishing of the third and the second cohomology will be proved in proposition (52) under more general assumptions.

There is a standard "adjoint" Dirac operator $D^* : S \otimes \text{Sym}(V) \rightarrow S^* \otimes \text{Sym}(V)$. Together $D$ and $D^*$ satisfy

\[
D^* D = ||.||^2
\]

\[
D D^* = ||.||^2
\]

(74)

where $||.||^2$ is an operator of multiplication on quadric. Equations (74) imply that there is no second cohomology in $S^* \otimes \text{Sym}(V)$-term. There is a similar
geometric interpretation of cohomology in $S \otimes \text{Sym}(V)$-term. Suppose $S$ is a spinor representation of $\text{Spin}(n)$, upon restriction of $S$ onto $\text{Spin}(n-2)$ $S$ splits into two nonisomorphic spinor representation $S_1, S_2$, choses the one from the two which contains the highest vector of $S$ as representation of $\text{Spin}(n)$. The Levi subgroup of stabilizer of a point $l$ of the quadric $q$ is equal to $\text{SO}(n-2)$. One can induce a vector bundle $C$ on $q$ from representation $S_1$ of $\text{Spin}(n-2)$. It is not hard to see that first cohomology of complex (72) in term $S \otimes \text{Sym}(V)$ is isomorphic to direct sum of the space of global section of $C$.

It is useful to use Borel-Weyl theorem to compute the spaces of global section of these bundles.

As an illustration let us carry out such computation in the case $D = 10, d' = 0, N = 1$.

The Dynkin graph of the group $\text{Spin}(10) = \text{Spin}(V)$ is

$$
\begin{array}{cccccc}
& w_1 & w_2 & w_3 & w_4 & w_5 \\
& & & & & \\
\end{array}
$$

We will encode a representation which is labeled by Dynkin diagram above by an array $[w_1, w_2, w_3, w_4, w_5]$. Our convention is that spinor representation $S$ is equal to irreducible representation with highest weight $[0, 0, 0, 1, 0]$, the tautological representation in $\mathbb{C}^{10} = V$ is equal to $[1, 0, 0, 0, 0]$, the exterior square of the later representation is equal to $[0, 1, 0, 0, 0]$. In case of $N = 1, D = 10$ super Yang-Mills theory the cohomology are equal to

$$
\bigoplus_{i \geq 0} [i, 1, 0, 0, 0] \text{ harmonic two-forms}
$$

$$
\bigoplus_{i \geq 0} [i, 0, 0, 1, 0] \text{ harmonic spinors}
$$

An interesting feature of the algebra $T_{d}YM$ is that its second homology vanish. As a result we conclude that the algebra $\hat{T}_{d}YM$ is a completed free algebra. It is useful to exhibit the set of free generators of such algebra. Before
doing this introduce some notations. The space $V$ is a representation of $SO(10)$ and a basis vector $A_D$ can be considered as the highest vector. The element $(A_D)^{\otimes i}$ is a highest vector in the $i$-th symmetric power $\text{Sym}^i(V)$. Let $W$ be an irreducible representation of $Spin(D)$ with highest vector $w$. Then a vector $(A_D)^i \otimes w \in \text{Sym}^i(V) \otimes W$ will be a highest vector and generates an irreducible subrepresentation of $Spin(D)$ in $\text{Sym}^i(V) \otimes W$. Let $W$ be an irreducible representation of $Spin(D)$ with highest vector $w$. Then a vector $(A_D)^i \otimes w \in \text{Sym}^i(V) \otimes W$ will be a highest vector and generates an irreducible subrepresentation of $Spin(D)$ in $\text{Sym}^i(V) \otimes W$. Denote the projection on such representation by $p$. For example representation $[i, 1, 0, 0, 0]$ is isomorphic to the image of $p : \text{Sym}^i(C^{10}) \otimes \Lambda^2(C^{10}) \rightarrow \text{Sym}^i(C^{10}) \otimes \Lambda^2(C^{10})$.

Denote $Ad(x)(y) = [x, y]$, $F_{ij} = [A_i, A_j]$. Introduce an elements $Ad(A_{(i_1)} \ldots Ad(A_{i_{k-1}})F_{ik})$ where () denotes symmetrization. This element belongs to $\text{Sym}^{k-1}(V) \otimes \Lambda^2(V)$.

Similarly elements $Ad(A_{(i_1)} \ldots Ad(A_{i_k})\phi_j$ belong to $\text{Sym}^k(V) \otimes \Phi$ and $Ad(A_{(i_1)} \ldots Ad(A_{i_{k-1}})\psi^\alpha$ belong to $\text{Sym}^k(V) \otimes S$. The elements

$$
p(Ad(A_{(i_1)} \ldots Ad(A_{i_{k-1}})F_{ikij}) \tag{77}
p(Ad(A_{(i_1)} \ldots Ad(A_{i_k})\phi_j)
p(Ad(A_{(i_1)} \ldots Ad(A_{i_{k-1}})\psi^\alpha)
$$

is a topological free set of generators of the algebra $T_dYM$. One can check it by looking at the image of these elements in the first homology.

We would like to elucidate some general features of the complex (66). Its structure depends on the map $\mu$.

**Definition 51** Let $\mu(b)$ be the image of the tensor $q \in \text{Sym}^2(V)$ inverse to the scalar product under the map $\mu$ (see [47]). Denote $\text{ind}(\mu) = rk\mu(b)$ and $rk(\mu) = \text{dim}V$.

There are three classes of maps:

a. $\mu = 0$

b. $\text{ind}(\mu) = 0, \mu \neq 0$

c. $\text{ind}(\mu) = 1, rk(\mu) = 1$.

d. All other cases

The importance of such division is justified by the following Proposition:

**Proposition 52**
a. If condition a. is satisfied the algebra $T_\mu YM$ has homological dimension 3 and coincides with $YM_0$.

b. If condition b. is satisfied the algebra $T_\mu YM$ has homological dimension 2 has infinite number of generators and relations.

c. If condition c. is satisfied then the algebra $T_\mu YM$ has homological dimension 2, has finite number of generators and one relation.

d. If the condition d. is satisfied the algebra $T_\mu YM$ is a completed free algebra, with infinite number of generators.

Proof.

a. Proof is tautology.

If $\mu \neq 0$ the the third homology is equal to zero. Indeed the space of four-chains is equal to zero. On the space of three-chains (equal to $\text{Sym}(V) \otimes c^*$) the differential is injective. It implies that if $\mu \neq 0$ the homological dimension of all algebras in question is less or equal to two.

b. The case when dimension $d$ (the number of generators $A_k$) is less or equal to three and $\text{ind}(\mu) = 0$ was covered by example 2. We may assume that $d \geq 4$. Then the restriction of the map $Q_1: \Pi V^* \otimes \hat{\text{Sym}}(V) \to V \otimes \hat{\text{Sym}}(V)$ contains a free module with at least $[d/2]$ generators ([d/2] is the dimension of maximal isotropic subspace in a space $\Pi V^*$ equipped with the standard bilinear form). It implies that the image $\text{Im}[Q_1: \overline{W}_3 \to \overline{W}_2]$ could not cover $\text{Ker}Q_1 \cap \overline{W}_2$ and the second cohomology are infinite-dimensional.

c. This was worked out as a first nontrivial example of computation of cohomology.

d. If $\text{ind}(\mu) \geq 1$ one can choose a subspace $V' \subset V$ of codimension 1 such that the image of $\mu(b)$ in $\text{Sym}^2(V/V')$ is nonzero. Choose some complement $V''$ to $V'$ such that $V = V' + V''$. Define a linear space $F^1$ as an ideal in $\hat{\text{Sym}}(V)$ generated by $V''$. Introduce a multiplicative descending filtration $F^s$ of $\hat{\text{Sym}}(V)$ generated by $F^1$. Define a filtration of $\hat{W}$ as $F^s(\hat{W}) = W \otimes F^s$. The $E_2$ term of a spectral sequence associated with such filtration is equal to cohomology of $\hat{W}$ where $L = \hat{\mathbb{C}}[t] \otimes \hat{\text{Sym}}(V')$. The element $t$ is a generator of $V''$. The space
of second homology of \( E_2 \)-term is a free \( \hat{\text{Sym}}(V') \)-module generated by \( A^* \) (we can assume that without loss of generality). Similarly the first homology of \( E_2 \) term is a free \( \hat{\text{Sym}}(V') \) module of rank \( D + d' - 1 + l \). The differential in \( E_2 \) is a \( \hat{\text{Sym}}(V') \) homomorphism and it is injective on second homology, if we can prove that it is nonzero on \( A^* \). A simple analysis shows that \( Q_1 A^* \neq 0 \) in \( E_2 \) if the map \( \mu \) satisfies condition \( d \).

2.5 Fano manifolds

Let \( M \) be a smooth manifold of dimension \( n \). Denote \( \Omega^p \) a sheaf of holomorphic \( p \)-forms on \( M \). Let us fix a line bundle \( L \). In most interesting situations \( M \) is a projective manifold and \( L \) is obtained from tautological line bundle \( O(1) \) on projective space by means of restriction to \( M \). We will use the notation \( O(1) \) for \( L \), and \( O(-1) \) for the dual bundle \( L^* \) also in general case. We identify line bundles with invertible sheaves. For any sheaf \( F \) denote \( F(i) \) the sheaf \( F \otimes O(1)^{\otimes i} \) where the tensor product is taken over the structure sheaf \( O \).

Serre algebra \( S \) is defined by the formula

\[
S = \bigoplus_{i=0}^{\infty} S_i = \bigoplus_{i=0}^{\infty} H^0(M, O(i)).
\]  

(78)

It can be embedded in the differential algebra \( B \) (Koszul-Serre algebra) in the following way. As an algebra

\[
B = S \otimes \Lambda(S_1),
\]  

(79)

where \( \Lambda(S_1) \) is the exterior algebra of \( S_1 \). The algebra \( B \) is \( \mathbb{Z} \)-graded: an element \( a \in S_i \otimes \Lambda^j(S_1) \) has degree \( \text{deg}(a) = j \). Let \( v_\alpha, \alpha = 1, \ldots, s \) be a basis of \( S_1 \subset S \) and \( \theta_\alpha \) be the corresponding basis of \( S_1 \subset \Lambda(S_1) \). The algebra \( B \) carries a differential \( d \) of degree \(-1\). If \( a \in S \) then \( d(a) = 0 \), \( d(\theta_\alpha) = v_\alpha \). It can be extended to \( B \) by the Leibniz rule.

There is an additional grading on \( B \) which we denote by \( \text{Deg} \). An element \( a \in S_i \otimes \Lambda^j(S_1) \) has degree \( \text{Deg}(a) = i + j \). The differential has degree zero with respect to additional grading. According to definition in Appendix such

45
algebra is called homogeneous. We can split $\mathcal{B}$ into a sum

$$\mathcal{B} = \bigoplus_{i,j} \mathcal{B}_{j,i}$$  \hspace{1cm} (80)$$

such that $a \in \mathcal{B}_{j,i}$ has the degrees $\text{deg}(a) = j, \text{Deg}(a) = i$. A line bundle $\mathcal{L}$ is called ample if for some positive $n$ the tensor power $\mathcal{L}^\otimes n$ defines an embedding of the manifold $M$ into $\mathbb{P}(H^0(M, \mathcal{L}^\otimes n)^*)$. We assume that

$$M$$ is an algebraic smooth manifold of dimension $n$,
canonical bundle $\Omega^n$ is isomorphic to $\mathcal{O}(-k) \quad k > 0$,  \hspace{1cm} (81)

$\mathcal{O}(1) = \mathcal{L}$ is ample.

The constant $k$ is called index. Manifolds satisfying above assumptions are Fano manifolds (i.e. the anticanonical line bundle is ample). Conversely, every Fano manifold can be equipped with the above structure. (We can always take $k = 1$.)

The goal of this section is to illuminate some properties of cohomology of the following complexes:

$$Q^i_\bullet = (0 \rightarrow \Lambda^i(S_1) \rightarrow \Lambda^{i-1}(S_1) \otimes S_1 \rightarrow \cdots \rightarrow S_i \rightarrow 0) = \bigoplus_{j=0}^i \mathcal{B}_{j,i}$$  \hspace{1cm} (82)$$

Some preliminaries on $H^\bullet(M, \mathcal{O}(i))$

Proposition 53 (Kodaira) Suppose $\mathcal{L}$ is an ample bundle over any complex manifold $N$. Then

$$H^i(N, \Omega^j \otimes \mathcal{L}) = 0$$ if $i + j > n$.  \hspace{1cm} (83)

Corollary 54 In assumptions

$$H^i(M, \mathcal{O}(l)) = 0$$ for $0 < i < n$ and any $l$
$$H^0(M, \mathcal{O}(-l)) = 0$$ for $l > 0$
$$H^0(M, \mathcal{O}(l)) = 0$$ for $l > -k$
$$H^0(M, \mathcal{O}) = \mathbb{C}$

$$H^n(M, \mathcal{O}(-l)) = H^0(M, \mathcal{O}(l-k))^* \quad l \geq k$$
Proof.
The proof is straightforward: use theorem 53 and Serre duality.

**Theorem 55** Suppose \( M \) is an \( n \) dimensional Fano manifold of index \( k \). Let \( \mathcal{B} \) be the differential algebra \([39]\). There exists a nondegenerate pairing

\[
H^{j,i}(\mathcal{B}) \otimes H^{s-n-1-j,s-k-i}(\mathcal{B}) \to H^{s-n-1,s-k}(\mathcal{B})
\]

where \( s = \dim S_1 \).

Proof.
There is a short exact sequence of vector bundles over it (Euler sequence):

\[
0 \to \mathcal{O}(-1) \to S_1^* \to T(-1) \to 0 \tag{84}
\]

Where \( S_1^* \) is a trivial bundle with a fiber \( S_1^* \). The first map is tautological embedding. the vector bundle \( T(-1) \) is a quotient \( S_1^* / \mathcal{O}(-1) \)

The \( i \)-th exterior power of the dual to that sequence gives rise to the complex of vector bundles

\[
N_i = (0 \to E^i \to \Lambda^i(S_1) \to \Lambda^{i-1}(S_1)(1) \to \ldots \to \Lambda^1(S_1)(i-1) \to \mathcal{O}(i) \to 0) \tag{85}
\]

which is acyclic everywhere except in zero degree. The zeroth cohomology is equal to \( \Lambda^i(T(-1)) = E^i \).

Since \( E^s = 0 \) the complex \( N_s \) has a particularly simple form.

After tensoring the resolution \( N_s^\bullet(i) \) on Dolbeault complex \( \Omega^{0,\bullet} \) one can compute diagonal cohomology of corresponding bicomplex (hypercohomology) which we denote by \( \mathbb{H}^\bullet(N_s^\bullet(i)) \). By acyclicity of \( N_s^\bullet(i) \) we have an equality

\[
\mathbb{H}^\bullet(N_s^\bullet(i)) = 0 \tag{86}
\]

There is a spectral sequence of the bicomplex \( N_s^\bullet(i) \otimes \Omega^{0,\bullet} \) whose \( E_1^{p,q} \) term coincides with

\[
E_1^{p,q} = \Lambda^p(S_1) \otimes H^q(\mathcal{O}(i-p)).
\]
According to equation (83) and corollary (54) all nonzero entries of $E_1$ term are concentrated on horizontal segments:

$E_1^{p,0} = \Lambda^p(S_1) \otimes S_{-p}$, $0 \leq p \leq i$  \hspace{1cm} (87)

$E_1^{p,n} = \Lambda^p(S_1) \otimes S^*_{p-i-k}$, $i+k \leq p \leq s$  \hspace{1cm} (88)

All entries $E_1^{p,0}$ not mentioned in the above table are equal to zero.

**Definition 56** Let $K^\bullet = \cdots \rightarrow K^i \rightarrow K^{i+1} \rightarrow \cdots$. The complex $K^\bullet[l]$ is a complex with shifted grading:

$K^i[l] = K^{i+l}$. The differential in the new complex is equal to $(-1)^i d$.

We have an obvious equality of complexes:

$(E^{\bullet,0}_1, d) = Q^*_i$  \hspace{1cm} (89)

$(E^{\bullet,n}_1, d) = Q^*_{s-k-i}[-s]$  \hspace{1cm} (90)

The second isomorphism depends on a choice of a linear functional on the space $\Lambda^i(S_1) \cong \mathbb{C}$. Symbol $*$ means dualisation. The spectral sequence converges to zero in the $n+1$-th term. The differential $d_{n+1}$ on $E^{p,q}_2 = E^{p,q}_{n+1}$ induces an isomorphism

$d_{n+1} : E_2^{p,n} \rightarrow E_2^{p-n-1,0}$  \hspace{1cm} (91)

which is a map of $E_2^{\bullet,0}$ modules, because the spectral sequence is multiplicative.

The isomorphism (91) can be interpreted as nondegenerate pairing:

$(\cdot, \cdot) : H^i(Q_1) \otimes H^{s-n-1-i}(Q_{s-k-i}) \rightarrow \mathbb{C}$  \hspace{1cm} (92)

It satisfies $(ab, c) = (a, bc)$, because $d_{n+1}$ is a map of $E_2^{\bullet,0}$ modules. The pairing can be recovered from the functional $\lambda(a) = (a, 1)$ by the rule $(a, b) = \lambda(ab)$. The functional is not equal to zero only on $H^{s-n-1}(Q_{s-k})$. The proof follows from this.

A direct inspection of a complex $Q_1^\bullet$ shows that $H^i(Q_1^\bullet) = 0$ for $i \neq 0$. It implies by duality (92) that

$H^{i,i}(B) = H^{s-n-1-i,s-k-i}(B) = 0$ for $i \neq 0$.  \hspace{1cm} (93)
Duality also implies that cohomology of $Q^i_r$ are not equal to zero only in the range $0 \leq \text{deg} \leq s - n - 1$, therefore we prove the following Proposition:

**Proposition 57** In the assumptions of Theorem (55)

$$H^{j,i}(B) \neq 0 \text{ only for}$$

$$0 \leq j \leq s - n - 1 \text{ and } j \leq i$$

and by duality $i \leq j + n + 1 - k$.

Notice that an analog of Theorem (55) can be proved for Calabi-Yau manifolds.

**Proposition 58** Suppose an $n$ dimensional smooth algebraic manifold has $\Omega^n = \mathcal{O}$, $H^i(M, \mathcal{O}) = 0$ for $0 < i < n$ and $\mathcal{L} = \mathcal{O}(1)$ is ample. Then there is a non-degenerate pairing

$$H^{j,i}(B) \otimes H^{s-n-1-j,s-i}(B) \to H^{s-n-1,s}(B).$$

for differential graded algebra $B$ defined in (79).

**Proof.**

The proof goes along the same lines as of (55).

### 2.6 Berkovits algebra

In this section we will be dealing with some structures made of 16-dimensional spinor representation $S = s_1$ of group $Spin(10)$ defined over complex numbers. This group is a double cover of a group of all linear transformations of linear space $V$, $\text{dim}(V) = 10$ that preserve a nondegenerate form $(.,.)$ and have determinant equal to one. The Dynkin diagram $D_5$ that corresponds to Lie algebra $SO(V)$ could be found on the picture (79). Our convention is that representation $S$ is equal to irreducible representation with highest weight $[0,0,0,1,0]$. 


Let $S_i$ be $[0,0,0,i,0]$. There is a structure of algebra on

$$\bigoplus_{i \geq 0} S_i = S$$

induced by tensor product of representation and projection on the leading component.

According to Cartan there is a compact Kähler 10-dimensional homogeneous space $OGr(5,10)$ of the group $O(V)$-the Grassmanian of maximal (five-dimensional) isotropic subspaces of $V$. This space is called isotropic Grassmanian. It has two connected components. They are isomorphic as Kähler manifolds but not as homogeneous spaces. An element $e \in O(V)$ with $\det(e) = -1$ swaps the spaces. Let us describe one of the connected components which we denote $Q$ and will call a space of pure spinors. Fix $W_0 \in OGr(5,10)$, then another isotropic subspace $W_1$ belongs to the same component $Q$ if $\dim(W_0 \cap W_1)$ is odd.

The complex group $Spin(V)$ (in fact $SO(V)$) acts transitively on $Q$; corresponding stable subgroup $P$ is a parabolic subgroup. To describe the Lie algebra $\mathfrak{p}$ of $P$ we notice that the Lie algebra $\mathfrak{so}(V)$ of $SO(V)$ can be identified with $\Lambda^2(V)$ (with the space of antisymmetric tensors $\rho_{ab}$ where $a,b = 0, \ldots, 9$). The vector representation $V$ of $SO(V)$ restricted to the group $GL(5,\mathbb{C}) \subset SO(V)$ is equivalent to the direct sum $W \oplus W^*$ of vector and covector representations of $GL(W)$, where $\dim(W) = 5$. The direct sum $W + W^*$ carries a canonical symmetric bilinear form. The Lie algebra of $SO(V)$ as vector space can be decomposed as $\Lambda^2(W) + \mathfrak{p}$ where $\mathfrak{p} = (W \otimes W^*) \oplus \Lambda^2(W^*)$ is the Lie subalgebra of $\mathfrak{p}$. Using the language of generators we can say that the Lie algebra $\mathfrak{so}(10,\mathbb{C})$ is generated by skew-symmetric tensors $m_{ab}$, $n^{ab}$ and by $k^b_a$ where $a,b = 1, \ldots, 5$. The subalgebra $\mathfrak{p}$ is generated by $k^b_a$ and $n^{ab}$. Corresponding
commutation relations are

\[
[m, n] = [n, m'] = 0 \quad (95)
\]

\[
[m, n]_a^b = m_a n_c^b \quad (96)
\]

\[
[m, k]_{ab} = m_a c^b k_c^a + m_c b^a k_c^b \quad (97)
\]

\[
[n, k]_{ab} = n a c^b k_c^a + n c b^a k_c^a \quad (98)
\]

**Proposition 59** (Borel-Weyl-Bott theorem) Suppose \( L \) is an invertible homogeneous line bundle over Kähler compact homogeneous space \( M \) of a semisimple group \( G \). Then \( H^i(M, L) \) could be nonzero only for one value of \( i \). For this value \( H^i(M, L) \) is an irreducible representation.

**Corollary 60** \( H^i(Q, \mathcal{O}) = 0 \) for \( i > 0 \).

Since \( H^1(Q, \mathcal{O}) = 0 \) all holomorphic topologically trivial line bundles are holomorphically trivial. The corollary and Hodge decomposition imply that \( H^2(Q, \mathcal{O}) = H^0(Q, \Omega^2) = 0 \) and \( H^2(Q, \mathbb{Z}) = \text{Pic}(Q) \).

**Proposition 61**

a. The group \( \text{Pic}(Q) = \mathbb{Z} \).

b. The group \( \text{Pic}(Q) \) has an very ample generator which we denote by \( \mathcal{O}(1) = \mathcal{L} \) such that \( H^0(Q, \mathcal{O}(1)) = S \).

c. The canonical class of \( Q \) is isomorphic to \( \mathcal{O}(-8) \).

**Proof.**

We saw that the Levi subgroup of the parabolic group \( P \) contains a center isomorphic to \( \mathbb{C}^\times \), so singular cohomology \( H^1(P, \mathbb{Z}) = \mathbb{Z} \). Transgression arguments imply that \( H^2(Q, \mathbb{Z}) = \mathbb{Z} \). It proves that the Picard group of \( Q \) is equal to \( \mathbb{Z} \).

Denote \( \widetilde{GL}(W) \subset \text{Spin}(V) \) a double cover of \( GL(W) \). A restriction of \( S^* \) on \( \widetilde{GL}(W) \) is isomorphic to

\[
[\mathbb{C} + \Lambda^2(W) + \Lambda^4(W)] \otimes \text{det}^{-1/2}(W). \quad (99)
\]
By Borel-Weyl theorem it implies that the ample generator of $\text{Pic}(\mathcal{Q})$ which we denote by $\mathcal{O}(1)$ has a space of global sections isomorphic to $\mathcal{S}$.

Consider a representation of $\widetilde{GL}(W)$ in $\Lambda^2(W)$. It is easy to see that

$$
det(\Lambda^2(W)) = det^4(W) = (det^{1/2}(W))^8. \tag{100}
$$

We can interpret $\Lambda^2(W)$ as an isotropy representation of parabolic subgroup $P$ in a tangent space of $\mathcal{Q}$ at a point which is fixed by $P$. This implies that canonical class $K$ is isomorphic to $\mathcal{O}(-8)$. ■

By Borel-Weyl-Bott theorem the algebra $\bigoplus_{n \geq 0} H^0(\mathcal{Q}, \mathcal{O}(n))$ is equal to $\mathcal{S}$.

It is possible to write a formula for $\mathcal{S}$ in terms of generators and relations. To do this observe that

$$
\text{Sym}^2(\mathcal{S}) = \mathcal{S}_2 \oplus \mathcal{V}
$$

Denote

$$
\Gamma : \mathcal{V} \to \text{Sym}^2(\mathcal{S})
$$

an inclusion of representations. We use the same letter for projection

$$
\Gamma : \text{Sym}^2(\mathcal{S}) \to \mathcal{V}
$$

To distinguish these two maps we will always specify the arguments. The first map $\Gamma(v)$ has a vector argument $v$. The second map $\Gamma(s_1, s_2)$ has two spinor arguments $s_1, s_2$.

**Proposition 62** (Cartan)\cite{5},\cite{3}

- **a.** Denote $A_1, \ldots, A_{10}$ a basis of $\mathcal{V}$. Then the algebra $\mathcal{S}$ defined in (94) can be described through generators and relations:

$$
\mathcal{S} = \text{Sym}(\mathcal{S})/(\Gamma(A_1), \ldots, \Gamma(A_{10})).
$$

- **b.** The space $\mathcal{Q}$ can be identified with all points $\lambda \in \mathbb{P}(\mathcal{S}^*)$ such that $\Gamma(\lambda, \lambda) = 0$.

Consider a complex

$$
\text{Kos}^\bullet(\mathcal{S})(i) = (0 \to \Lambda^i(\mathcal{S}) \to \Lambda^{i-1}(\mathcal{S}) \otimes \text{Sym}^1(\mathcal{S}) \to \cdots \to \text{Sym}^i(\mathcal{S}) \to 0) \tag{101}
$$

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it is a classical Koszul complex.

The cohomological grading is the degree in the exterior algebra. The sum $Kos^\bullet(S) = \bigoplus_i Kos^\bullet(S)(i)$ is an algebra. It contains $\text{Sym}(S) = \bigoplus_{i \geq 0} \text{Sym}^i(S)$ as a subalgebra. The algebra $S$ is a module over $\text{Sym}(S)$. Then

$$B_0 = Kos^\bullet(S) \otimes_{\text{Sym}(S)} S = \bigoplus_i Q^\bullet(i)$$

(102)

is called (reduced) Berkovits algebra. The complex $Kos^\bullet(S)$ is a free resolution of $C$ over $\text{Sym}(S)$. This implies that we have an identity

$$H^n(Q^\bullet(i)) = \text{Tor}_{\text{Sym}(S)}^{n,i}(C, S)$$

(103)

(the left upper index corresponds to cohomology index, the right index corresponds to homogeneity index). There is a symmetry

$$\text{Tor}_{\text{Sym}(S)}^{n,i}(C, S) = \text{Tor}_{\text{Sym}(S)}^{n,i}(S, C).$$

(104)

One way to compute the groups $H^n(Q^\bullet(i))$ is to construct a minimal resolution of $S$ as a module over $\text{Sym}(S)$ (instead of $C$ as $\text{Sym}(S)$ module). Then the generators of the modules in the resolution will coincide with cohomology classes of the complexes $Q^\bullet(i)$. Such resolution was constructed in [7] (though $Spin(V)$ equivariance in their approach is not apparent). Another option is to compute cohomology of $Q^\bullet(i)$ $i = 0, \ldots, 4$ by brute force using a computer. This has been (partly) done in [8]. In all these approaches due to duality proved in proposition the only nontrivial task is computation of $H^\bullet(Q^\bullet(4))$.

We are going to construct a (partial) free resolution of $\text{Sym}(S)$ module $S$ whose graded components schematically depicted on picture

\[
\begin{align*}
\{0\} & \rightarrow M_3^1 \xrightarrow{\delta_3} M_3^2 \xrightarrow{\delta_2} M_3^3 \xrightarrow{\delta_1} M_3^4 \xrightarrow{\delta_0} S_3 \\
\{0\} & \rightarrow M_2^2 \xrightarrow{\delta_2} M_2^3 \xrightarrow{\delta_1} M_2^4 \xrightarrow{\delta_0} S_2 \\
\{0\} & \rightarrow M_1^1 \xrightarrow{\delta_1} M_1^2 \xrightarrow{\delta_0} M_1^3 \xrightarrow{\delta_0} S_1 \\
\{0\} & \rightarrow M_0^0 \xrightarrow{\delta_0} M_0^1 \xrightarrow{\delta_0} M_0^2 \xrightarrow{\delta_0} M_0^3 \xrightarrow{\delta_0} S_0
\end{align*}
\]

(105)

By definition $M^i = \bigoplus_{j \geq 0} M_j^i$ and $M^0 = \text{Sym}(S)$. 

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Since algebra $S$ is quadratic with ideal of relations $I = \bigoplus_{j \geq 1} I_j$ generated by $V = I_2$ we have $M_2^1 = V, M_1^1 = 0$ and $M^1 = V \otimes \text{Sym}(S)$. Denote $A_i, i = 1, \ldots, 10$ the basis of $V$ and $u^\alpha, \alpha = 1, \ldots, 16$ - the basis of $S$. The map $\delta_1$ is defined by the formula $\delta_1(A_i) = \sum_{\alpha, \beta = 1}^{16} \Gamma_{i\alpha\beta} u^\alpha u^\beta$.

The linear space $M_2^1 = B$ is the kernel of surjection $V \otimes S \to I_3$. In the case of interest the representation content of $I_3$ is $[1, 0, 0, 1, 0]$, representation content of $V \otimes S_1$ is $[0, 0, 0, 0, 1] \oplus [1, 0, 0, 1, 0]$. It implies that $B = [0, 0, 0, 0, 1] = S^*$. Denote a basis of the vector space $S^*$ by $\psi_\alpha, \alpha = 1, \ldots, 16$. The map $\delta_2$ is given on generators as

$$\delta_2 : S^* \to S \otimes V$$

$$\delta_2(\psi_\beta) = \sum_{\alpha = 1}^{16} \sum_{i = 1}^{10} \Gamma_{\alpha\beta i} A_i u^\alpha.$$ (106)

We conclude that module $M^2$ is equal to $S^* \otimes \text{Sym}(S) + \tilde{M}^2$, where $\tilde{M}^2$ is a free module with generators of degree greater than 3. We will see later that $\tilde{M}^2 = 0$. Now need to prove a weaker statement: $\tilde{M}_2^2 = 0$. Indeed it is equal to cohomology of the following complex

$$S^* \otimes S \xrightarrow{\delta_3} V \otimes \text{Sym}^2(S) \xrightarrow{\delta_4} I_4 \to 0$$

in the term $V \otimes \text{Sym}^2(S)$. We have the following representation content:

$$I_4 = [1, 0, 0, 2, 0] \oplus [2, 0, 0, 0, 0]$$

$$V \otimes \text{Sym}^2(S) = [0, 0, 0, 0, 0] \oplus [0, 0, 1, 1] \oplus [1, 0, 0, 0, 0] \oplus [2, 0, 0, 0, 0]$$

$$S^* \otimes S = [0, 0, 0, 0, 0] \oplus [0, 0, 0, 1, 1] \oplus [0, 1, 0, 0, 0].$$ (107)

Since the map $\delta_1$ is surjective we need to check that $S^* \otimes S \xrightarrow{\delta_3} V \otimes \text{Sym}^2(S)$ is inclusion. This can be readily checked by applying map $\delta_2$ to the highest vectors of each representation in decomposition of $S^* \otimes S$. Let us extend partial resolution of $S$ to an arbitrary full resolution. Using this resolution we can
compute $\text{Tor}^{i,j}_{\text{Sym}(S)}(C, S)$. Simple computations give the following answer

\begin{align}
\text{Tor}^{0,i}_{\text{Sym}(S)}(C, S) &= C \text{ if } i = 0 \text{ and } \{0\} \text{ if } i \neq 0 \\
\text{Tor}^{1,i}_{\text{Sym}(S)}(C, S) &= V \text{ if } i = 2 \text{ and } \{0\} \text{ if } i \neq 1 \\
\text{Tor}^{2,i}_{\text{Sym}(S)}(C, S) &= S^* \text{ if } i = 3 \text{ and } \{0\} \text{ if } i < 3 \text{ or } i = 4 \\
\text{Tor}^{3,i}_{\text{Sym}(S)}(C, S) &= \{0\} \text{ if } i < 5
\end{align}

Using the equation (103) and general duality theorem (55) and proposition (57) we prove

**Proposition 63** The cohomology of the algebra $B_0$ is

\begin{align}
H^{0,0} &= C \\
H^{1,2} &= V \\
H^{2,3} &= S^* \\
H^{3,5} &= S \\
H^{4,6} &= V \\
H^{5,8} &= C \\
H^{p,q} &= 0 \text{ for all } p, q \text{ not listed above.}
\end{align}

As we know from [15] the Koszul dual to the algebra $S = F(\hat{Q})$ is the algebra $S'$ with generators $\lambda_\alpha, \alpha = 1, \ldots, 16$ which span a linear space $S^*$ and relations

$$\sum_{\alpha \beta = 1}^{16} \Gamma_{m_1 \ldots m_5}^{\alpha \beta} \{\lambda_\alpha, \lambda_\beta\} = 0. \quad (109)$$

This algebra is a universal enveloping algebra $U(L)$ of a Lie algebra $L$ with the same set of generators and relations.

The Cartan-Eilenberg complex of a positively graded Lie algebra $\mathfrak{g}$ is an exterior algebra $\Lambda(\mathfrak{g}^\dagger) = \Lambda^*(\mathfrak{g}^\dagger)$, the dual complex is denoted by $\Lambda(\mathfrak{g}) = \Lambda^*(\mathfrak{g})$. The sign $\dagger$ denotes dualisation in a category of graded vector spaces (see Appendix).
The differential $\Lambda^\bullet(g^\dagger)$ is given by a formula

$$ (dv)(x) = v([x_1,x_2]) $$

(110)

where $v \in \Lambda^1(g^\dagger)$ is a linear generator. Denote $H^\bullet(g,\mathbb{C})$ the homology of such complex, homology of the dual complex $\Lambda^\bullet(g)$ will be denoted as $H^\bullet(g,\mathbb{C})$ (See [12] for details about (co)homology of Lie algebras). For any positively graded Lie algebra $g$ there is a canonical quasiisomorphism $\Lambda^\bullet(g) \rightarrow \text{Bar}(U(g))$ and the dual quasiisomorphism $\text{Bar}(U(g))^\dagger \rightarrow \Lambda(g^\dagger)$ (see [12] for details).

By one of the properties of Koszul duality transformation $^!$ (see [11]) there is an inclusion $U(g)^! \subset H^\bullet(g,\mathbb{C})$, for any quadratic Lie algebra.

We need the following Proposition:

**Proposition 64** [3] For any compact homogeneous K"ahler manifold $G/P$ of a reductive group $G$ and an ample line bundle $\alpha$ on it the Serre algebra $\bigoplus_{n \geq 0} H^0(G/P,\alpha \otimes n)$ is Koszul.

Since the Koszulity relation is reflexive for the case at hand we have an isomorphism:

$$ U(L)^! = H^\bullet(L,\mathbb{C}) $$

(111)

**Proposition 65** There is a quasiisomorphism $\rho : \Lambda^\bullet(L^\dagger) \rightarrow \mathcal{S}$, which maps linear functional $\lambda^\alpha$ on $L$ into generator $u^\alpha$ of $\mathcal{S}$. This map is zero on subspace $\bigoplus_{i \geq 2} L^\ast_i \subset \Lambda^1(L^\ast)$.

**Proof.**

The only statement which needs to be checked is that $\rho$ commutes with differentials. This is obvious, however.

Our next goal is to relate Berkovits algebra $(B_d,Q)$ with a classical BV approach to YM theory.

Let $\mu : V \rightarrow V$ be a linear surjective map. We assume that the linear space $V$ has an orthonormal basis $A_1,\ldots,A_{10}$ and linear space $V$ has a basis generated by symbols $\frac{\partial}{\partial x^1},\ldots,\frac{\partial}{\partial x^d}$. Let $\text{Sym}(V^\ast)$ be a symmetric algebra on
the space dual to $V$. The space $V^*$ has a basis $x^1, \ldots, x^d$. Then $\mu(A_i)$ defines a linear functional on $V^*$, which can be extended to a derivation of $\text{Sym}(V^*)$. Introduce an algebra

$$B_\mu = B_0 \otimes \text{Sym}(V^*)$$

(112)

and a differential on it by the rule

$$Q = \sum_{\alpha=1}^{16} u^\alpha \frac{\partial}{\partial \theta^\alpha} + \sum_{\alpha\beta=1}^{16} \sum_{i=1}^{10} \Gamma^i_{\alpha\beta} u^\alpha \theta^\beta \mu(A_i).$$

(113)

Recall that differential algebra $(B_d, Q)$ was defined in the introduction in definition (16) and $(B_\mu, Q)$ is a minor generalization of it.

Consider a graded extension $M_\mu$ of the Lie algebra $L$. The linear space $L_1 = S^*$. Then $M_{\mu 0} = S^*$ and this space has a basis $\tau_\alpha, \alpha = 1, \ldots, 16$. A linear space $V$ has a basis $\xi_1, \ldots, \xi_d$ and $M_{\mu 1} = S^* + V$. The linear space $V$ is by definition is dual to the linear space generated by $x^1, \ldots, x^d$ of linear coordinates on $d$-dimensional linear space. For $i \geq 3$ we have $M_{\mu i} = L_i$.

The parity of elements of $M$ is reduction of grading modulo two. The Lie algebra $M$ is equipped with a differential defined by the formulas:

$$
\begin{align*}
d : M_{\mu 1} &\to M_{\mu 0}; S^* + V^0 \overset{id}{\to} S^* \\
d : M_{\mu 2} &\to M_{\mu 1}; V^0 \overset{\mu}{\to} S^* + V \\
d : M_{\mu i} &\to M_{\mu i-1}; i \geq 3, d = 0
\end{align*}
$$

(114)

The commutation relations in the algebra $M$ are of semidirect product $L \ltimes (S^* + V)$, where $S^* \subset M_{\mu 0}, V \subset M_{\mu 1}$. The linear space $S^* + V$ is an abelian ideal. The action of $L$ on $S^* + V$ is given by the rule:

$$
\begin{align*}
[\theta_\alpha, \tau_\beta] &= 2\mu \sum_{i=1}^{10} \Gamma^i_{\alpha\beta} A_i \\
[\theta_\alpha, \xi_i] &= 0
\end{align*}
$$

(115)

One can consider a version of Cartan-Eilenberg complex for differential graded Lie algebra $(M_{\mu}, d)$, in the complex $\Lambda^\bullet(M_{\mu}^*)$ the formula (110) becomes

$$
(D\nu)(x) = \nu([x_1, x_2]) + \nu(d(x_3)).
$$

(116)
There is a map
\[ \chi : (\Lambda(M^\dagger_\mu), D) \to (B_\mu, Q). \]  
(117)

On the generators the map is
\[
\begin{align*}
\chi(\xi^i) &= x^i \\
\chi(\lambda^{*\alpha}) &= u^\alpha \\
\chi(\tau^\alpha) &= \theta^\alpha
\end{align*}
\]  
(118)

\[ \chi \text{ is zero on the rest of the generators.} \]

To make this map a map of complexes one has to modify slightly the grading on \( B_\mu \):
\[
\begin{align*}
\tilde{x}^i &= 2 \\
\tilde{u}^\alpha &= 2 \\
\tilde{\theta}^\alpha &= 1
\end{align*}
\]  
(119)

The grading on \( \Lambda(M^\dagger_\mu) \) is a standard cohomological grading.

**Proposition 66** The map \( \chi \) is correctly defined and is a quasiisomorphism of algebras \( (\Lambda(M^\dagger_\mu), D) \) and \( (B_\mu, Q) \).

**Proof.**

We leave the proof of the first statement as an exercise for the reader. The algebra \( M_\mu \) carries an action of \( \mathbb{C}^\times \) which commutes with differential \( D \). It manifests itself in a grading. In this grading \( \text{DEG}(\lambda_\alpha) = 1 \). This condition allows uniquely spread the grading on the entire algebra. The induced grading on \( \Lambda(M_\mu) \) has the following feature: all graded components become finite-dimensional bounded from both sides complexes. Such grading could be pushed on \( B_\mu \). A simple observation is that the map \( \chi \) is surjective. A filtration of \( \Lambda(M_\mu) \) which leads to Serre-Hochschild spectral sequence, based on the extension
\[
0 \to S^* + V \to M_\mu \to \mathbb{L} \to 0
\]  
(120)
can be pushed to the algebra $B_{\mu}$. The $E_2$-terms of the corresponding spectral sequences are isomorphic to algebra $B_{\mu}$. Therefore the limiting terms of the spectral sequence, which converge strongly, must coincide.

**Proposition 67** The universal enveloping algebra $U(M_{\mu})$ is Koszul dual to $B_{\mu}$.

**Proof.**

The proof is a straightforward application of definitions.

In order to avoid confusion, when we talk about differential Lie algebras by (co)homology we always mean cohomology of Cartan-Eilenberg complex. However the linear space of the algebra itself carries a differential which we call intrinsic differential. Cohomology of such differential will be called intrinsic cohomology.

The algebra $M_{\mu}$ carries an intrinsic differential. It allows to reduce the space of the algebra without affecting its cohomology. Introduce two subalgebras $E_{\mu}M = \bigoplus_{i \geq 1} E_{\mu}M_i$ and $T_{\mu}M = \bigoplus_{i \geq 2} T_{\mu}M_i$ of $M_{\mu}$:

$$E_{\mu}M_1 = V$$
$$E_{\mu}M_i = L_i, i \geq 2$$

the differential is a restriction of the differential on $E_{\mu}M$

$$T_{\mu}M_2 = \ker [\mu : L_2 \to V]$$
$$T_{\mu}M_i = L_i, i \geq 3$$
$$d = 0$$

**Proposition 68** The algebras $E_{\mu}M$, $T_{\mu}M$ quasiisomorphically embed into algebra $M_{\mu}$.

**Proof.**

Obvious.

**Corollary 69** $H^*(T_{\mu}M, \mathbb{C}) = H^*(E_{\mu}M, \mathbb{C}) = H^*(B_{\mu})$, where the first two groups are cohomology of Lie algebras. The last group is intrinsic of differential algebra $B_{\mu}$.  

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We need to identify algebras $T_\mu M$ and $E_\mu M$. Assume that $\mu = 0$. In [15] we indicated that the algebra $L$ contains a homomorphic image of Lie algebra $SYM$. The universal enveloping algebra $U(SYM)$ is isomorphic to $YM_0$ with $D = 10, d' = 0, N = 1$ and potential $U$ equal to zero. The Lie algebra $SYM$ is generated by

$$A_i = \sum_{\alpha \beta} \Gamma_{i}^{\alpha \beta} \{\lambda_\alpha, \lambda_\beta\}$$

$$\psi^\alpha = \sum_{\beta} \sum_{m=1}^{10} \Gamma_{\alpha \beta m} [\lambda_\beta, A_m]$$

with relations (123).

**Proposition 70** The Lie algebra $SYM$ is isomorphic to the algebra $\bigoplus_{i \geq 2} L_i$.

**Proof.**

By construction $SYM$ maps in $\bigoplus_{i \geq 2} L_i$. We need to check that the set of generators of $\bigoplus_{i \geq 2} L_i$ coindex with $A_1, \ldots, A_{10}, \psi^1, \ldots, \psi^{16}$ and there is no relations other then (7,8,9) with $D = 10, d' = 0, N = 1$ and $U = 0$. To do so we take advantage of the proposition (18) and corollary (19). The space $H^*(E_0M, \mathbb{C})$ is quasiisomorphic to $H^*(B_0)$ which were computed in proposition (68). According to this proposition $H^1(E_0M, \mathbb{C}) = V + S^*$ and $H^2(E_0M, \mathbb{C}) = V + S$. It implies th that elements $A_1, \ldots, A_{10}$ which span $V$ and $\psi^1, \ldots, \psi^{16}$ which span $S$ are indeed the generators of the algebra $E_0M$. The relations (7) (if we think of them as of elements of a free algebra) transform as representation $\Pi V^*$, (7) transform as representation $S^*$ are indeed the generators of the ideal of relations of the algebra $E_0M$.

If $\mu = 0$ then $M_\mu$ is quasiisomorphic to $T_0M = \bigoplus_{i \geq 2} L_i$.

**Proposition 71** The Koszul dual to $B_0$ is quasiisomorphic to $U(SYM)$.

**Proof.**

According to proposition (60) $U(SYM)$ is quasiisomorphic to $U(\bigoplus_{i \geq 2} L_i)$. By the remark from the previous paragraph $\bigoplus_{i \geq 2} L_i$ is quasiisomorphic to $T_0M$ which is by proposition (68) is quasiisomorphic to $M_0$. By proposition (67) $M_0$ is Koszul dual to $B_0$. 

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Corollary 72  There is a quasiisomorphism $b_{\nu}^* \cong B_0$.

Proof.

We already established a quasiisomorphism $\Lambda(E_0 M)$ and $B_0$. We have a canonical quasiisomorphism $\text{Bar}(E_0 M)^\dagger \to \Lambda(E_0 M^\dagger)$. On the other hand according to proposition (1) we have a quasiisomorphism $\text{Bar} YM \cong b_{\nu}^0$. Dualisation of the last quasiisomorphism gives the necessary quasiisomorphism.

Proposition 73  There is a quasiisomorphism

$$B_\mu \cong b_{\nu}^*.$$  \hspace{1cm} (124)

Proof.

There is an obvious identification $U(T_\mu M) \cong T_\mu YM$ ($U$ stands for universal enveloping algebra) which comes from identification $T_0 M \cong SYM$. Then by lemma (31) and propositions (45) and (66) we have a quasiisomorphism (124).

Proposition 74  Koszul dual to $B_\mu$ is quasiisomorphic to $U(T_\mu M)$.

Proof.

According to proposition (67) the Koszul dual to $B_\mu$ is equal to $U(M_\mu)$. The result follows from the previous observation and proposition (68).

3 Appendix

Dualisation in the category of linear spaces

Bar duality

3.1 Dual spaces

Suppose we have an inverse system of finite dimensional vector spaces $\to N^{i+1} \overset{i}{\to} N^i \to \cdots \to N^0 (i \geq 0)$, where all maps $i$ are surjective. Let $N = \lim_{\to} N^i$. There
is a canonical map \( N \to N^i \) which in our case is surjective. Denote the kernel if this map by \( J^i \). It is clear that \( J^{i-1} \subset J^i \) and the set of linear spaces \( J^i \) completely determines the inverse system, and for every linear space \( W \) with an decreasing filtration \( J^i \) such that

\[
\bigcap_{i \geq 0} J^i = \{0\}
\]

(125)

\[
dim(J^i/J^{i-1}) < \infty
\]

(126)

we have \( W \subset \lim_i W/L_i = \hat{W} \). Define a direct system of finite dimensional vector spaces \( M^0 \to M^{-1} \to \ldots M^n \to M^{n-1} \to \ldots \) as \( M^n = N^{*n} \). We call such direct system dual to \( N^n \) and denote it by \( N^{*n} \). It should be clear how to define \( N^{*+n} \) and that it is equal to \( N^n \). Observe that \( \text{colim}N^{*n} \) has an increasing filtration by spaces \( F^i = N^{*i} \). Denote \( M = \text{colim}_n N^{*n} \). The filtration satisfies

\[
M = \bigcup_i F^i
\]

(127)

\[
dim(F^{i+1}/F^i) < \infty.
\]

(128)

Skipping all mentionings of limits we can say that there is a dualization invertible functor from category of complete linear spaces with decreasing filtration \( W, J_i \) \((i \leq 0)\), where \( J_i \) satisfies \( \text{(125, 126)} \) and linear spaces equipped with increasing filtration \( M, F^i \) such that \( F^i \) satisfies \( \text{(127)} \) and \( \text{(126)} \). We will refer to such duality as topological.

**Definition 75** Denote \( U = \bigoplus_{i \in \mathbb{Z}} U_i \) a graded vector space with \( \dim(U_i) < \infty \). One can define a dualization functor on the category of such vector spaces. Indeed by definition \( U^\dagger = \bigoplus_{i \in \mathbb{Z}} U_i^* \), the grading of \( U_i^* \) is equal to \(-i\). Observe that the functor \( U^\dagger \) is auto duality in the category of graded linear spaces. A vector space in this since to \( U \) will be called algebraic dual.

Given a graded vector space \( U = \bigoplus_{i \geq 0} U_i \) one can define two linear spaces with filtrations:

a. \( U = \bigoplus_{i \geq 0} U_i \) and a filtration is defined as \( F^n = \bigoplus_{0 \leq i \leq n} U_i \).
b. \( \hat{U} = \prod_{i \geq 0} U_i \) and a filtration is defined as \( J_n = \prod_{i \leq n} U_i \). In future the sign \( \hat{\cdot} \) will always mean completion of the space \( W \) with respect to a decreasing filtration.

In future if we write \( U^* \) for \( U = \bigoplus_{i \geq 0} U_i \) we will always mean the topological dual.

### 3.2 \( A_{\infty} \)-algebras

Let us consider a \( \mathbb{Z}_2 \)-graded vector space \( W \) and corresponding tensor algebra \( T(W) = \bigoplus_{n \geq 1} W^\otimes n \). The tensor algebra \( T(W) \) is \( \mathbb{Z} \)-graded, but it has also \( \mathbb{Z}_2 \)-grading coming from \( \mathbb{Z}_2 \)-grading of \( W \). We say that differential (= an odd derivation having zero square) \( Q \) on \( T(W) \) specifies a structure of \( A_{\infty} \)-coalgebra on \( V = \prod W \).

One can describe the structure of \( A_{\infty} \)-coalgebra on \( V = \Pi W \) as a sequence of linear maps \( \Delta_1 : V \to V, \; \Delta_2 : V \to V \otimes V, \; \Delta_n : V \to V^\otimes n \). Using the Leibniz rule we can extend \( \Delta_1, \Delta_2 \ldots \) to a derivation \( Q \) of \( (T(W); \Delta_1, \Delta_2, \ldots) \). The condition \( Q^2 = 0 \) implies some conditions on \( \Delta_1, \Delta_2, \ldots \). The map \( \Delta_1 \) is a differential \( (\Delta_1^2 = 0) \), the map \( \Delta_2 \) can be interpreted as comultiplication. If \( \Delta_n = 0 \) for \( n \geq 3 \) we obtain a structure of associative coalgebra on \( V \).

One says that differential algebra \( (T(W), Q) \) is (bar-) dual to \( A_{\infty} \)-coalgebra \( (V, m) \) or that \( (T(W), Q) \) is obtained from \( (V, \Delta) \) by means of bar-construction. We will use the notation \( \text{Bar}(V, \Delta) \) for this differential algebra. Hochschild homology of \( (V, \Delta) \) are defined as homology of \( (T(W), Q) \).

An \( A_{\infty} \)-map of coalgebras is defined as a homomorphism of corresponding differential tensor algebras (i.e. as an even homomorphism \( \text{Bar}(V, \Delta) \to \text{Bar}(V', \Delta') \) commuting with differential).

One can describe an \( A_{\infty} \)-map \( \phi(V, \Delta) \to (V', \Delta') \) by means of a sequence of maps \( \varphi_n V \to V'^\otimes n \) where \( V = \Pi W, V' = \Pi W' \).

The map \( \varphi_1 \) commutes with differentials \( (\Delta'_1 \varphi_1 = \varphi_1 \Delta_1) \), hence it induces a homomorphism of homology of \( (V, \Delta_1) \) into homology of \( (V', \Delta'_1) \). If the induced homomorphism is an isomorphism one says that \( A_{\infty} \)-map is a quasiisomorphism.
One can define an $A_\infty$-algebra structure on $\mathbb{Z}_2$-graded vector space $V$ as an odd coderivation $Q$ of tensor coalgebra $T(W) = \bigoplus_{n \geq 1} W \otimes^n$ obeying $Q^2 = 0$. (Here $W = IV$).

Equivalently, $A_\infty$ algebra $(V,m)$ can be defined as $\mathbb{Z}_2$-graded vector space $V$ equipped with a series of operations

$$m_1 : V \to V, m_2 : V^\otimes 2 \to V, \ldots, m_n : V^\otimes n \to V$$

, obeying certain conditions.

One says that the differential coalgebra $(T(W), Q)$ is bar-dual to $A_\infty$ algebra $(V, m)$ or that $(T(W), Q)$ is obtained from $(V, m)$ by means of bar-construction. We will use the notation $\text{Bar}(V, m)$ for differential coalgebra $(T(W), Q)$. Hochschild homology of $(V, m)$ is defined as homology of $(T(W), Q)$.

Usually one considers bar-duality for $\mathbb{Z}$-graded $A_\infty$-algebras and $A_\infty$-coalgebras.

The algebra $\text{Bar}(V, \Delta) = (T(W), Q)$ that is dual to $\mathbb{Z}$-graded $A_\infty$-coalgebra $(V, \Delta)$, will be considered as $\mathbb{Z}$-graded differential algebra. The space $W$ is equal $V[-1]$; in other words $W$ coincides with $V$ with grading shifted by $-1$. Similarly, the $\mathbb{Z}$-graded coalgebra $\text{Bar}(V, m)$ dual to graded $A_\infty$-algebra $(V, m)$ will be considered as graded differential coalgebra.

Let us consider the case of $A_\infty$-(co)algebras $(V, m)$ and $(V', m')$ have an additional positive grading. This is an auxiliary grading which has no correlation with internal (homological) $\mathbb{Z}_2(\mathbb{Z})$-grading. We assume that all structure maps $m_k$ have degree zero with respect do this additional grading. The same applies to $A_\infty$-morphisms. Such $A_\infty$-(co)algebras will be called homogeneous. We will use an abbreviation h.morphism h.quasiisomorphism etc. for homogeneous morphism, homogeneous quasiisomorphism ...

**Theorem 76** Two homogeneous $\text{Bar}(V, \Delta)$ s $A_\infty$-(co)algebras are h.quasiisomorphic iff their dual algebras (coalgebras) are h.quasiisomorphic.

**Theorem 77** If $A_\infty$-morphism $f : (V, m) \to (V', m')$ of homogeneous (co)algebras induces an isomorphism of Hochschild homology then $f$ is h.quasiisomorphism.
For quadratic algebras bar-duality is closely related to Koszul duality. Let $A$ be a quadratic algebra $A = \bigoplus_{n > 0} A^n, \dim A^n < \infty$ and $B = \bigoplus_{n > 0} A^*$ be the dual graded coalgebra.

**Proposition 78** The differential graded algebra bar-dual to the coalgebra $B$ is quasiisomorphic to the Koszul dual $A^!$ if $A$ is a Koszul algebra.

We will consider duality in more general situation when $A_\infty$-coalgebra $(V, \Delta)$ whose descending filtration $F^k$ obeying $F^1 = V, \cap_{k \geq 1} F^k = 0$ and $V$ is complete with respect to filtration. Then we can introduce corresponding filtration $F^k$ on $T(W)$. We define $F^p(T(W))$ by the formula

$$
\sum_{\sum_{r=1}^k n_r \geq p} F^{n_1} \otimes \cdots \otimes F^{n_k}.
$$

We assume that the structure of $A_\infty$-coalgebra is compatible with filtration. This means that

$$
\Delta_k(F^s) \subset \sum_{n_1+\cdots+n_k \geq s} F^{n_1} \otimes \cdots \otimes F^{n_k}, \quad n_k \geq 1.
$$

In the language of tensor algebras we require that $Q(F^k(T(W))) \subset F^k(T(W))$. In particular, for filtered $A_\infty$-coalgebra we have $\Delta_1(F^s) \subset F^s$ hence we can consider homology of $(F^s, \Delta_1)$.

The bar dual to the filtered $A_\infty$-coalgebra $(V, \Delta)$ is defined as topological differential algebra $(\widehat{T(W)}, Q)$ obtained from $(T(W), Q)$ by means of completion with respect to filtration $F^k$.

$A_\infty$-maps of filtered coalgebra should agree with filtrations; they can by considered as continuous homomorphisms of dual topological differential algebras.

Representing $A_\infty$-map $\phi : (V, \Delta_k) \rightarrow (V', \Delta'_k)$ as a series of maps $\varphi_k : V \rightarrow V'^{\otimes k}$ and using that $\Delta'_1 \varphi_1 = \varphi_1 \Delta_1$, $\phi_1$ induces a homomorphism of homology of $(F^k/F^{k+1}, \Delta_1)$ into homology of $(F'^k/F'^{k+1}, \Delta'_1)$, if all of these homomorphisms are isomorphisms we say that $\phi$ is a filtered quasiisomorphism. There are not filtered quasiisomorphisms between filtered objects.
We introduce also a notion of filtered $A_\infty$-algebra $(V, m)$ fixing a decreasing filtration $F^p$ on $V$ $p \geq 1$ that satisfies the following conditions:

$$
\mu_k : F^{s_1} \otimes \cdots \otimes F^{s_k} \to F^{s_1+\cdots+s_k}, \quad k \geq 1 \quad (131)
$$

$$
\bigcap_s F^s = 0 \quad F^1 = V \quad (132)
$$

and $V$ is complete with respect to such filtration. (Notice that the notion of filtered $A_\infty$-algebra is not dual to the notion of filtered $A_\infty$-coalgebra (a filtration that is dual to decreasing filtration is an increasing filtration). The differential coalgebra $\text{Bar}(V, m)$ corresponding to filtered $A_\infty$-algebra can be considered as filtered coalgebra (see formula (130) for filtration). Its completion $\hat{\text{Bar}}(V, m)$ also can be regarded as a filtered differential topological coalgebra.

Let $f$ be an $A_\infty$-morphism of filtered $A_\infty$-algebras $(V, m) \to (V', m')$ that is compatible with filtrations. It induces a map $f_* : \text{Bar}(V, m) \to \text{Bar}(V', m')$ of corresponding dual coalgebras, that can be extended to a map $\hat{f}_* : \hat{\text{Bar}}(V, m) \to \hat{\text{Bar}}(V', m')$.

We need the following statements proved in [13]:

**Theorem 79** If filtered $A_\infty$-coalgebras are quasiisomorphic then dual topological algebras are quasiisomorphic.

**Theorem 80** Let $(V, \Delta)$ and $(V', \Delta')$ be two filtered $A_\infty$-algebras. Then quasiisomorphism of corresponding topological differential coalgebras $(\hat{T}(W), Q)$ and $(\hat{T}(W'), Q')$ implies quasiisomorphism of $A_\infty$-algebras $(V, \Delta)$ and $(V', \Delta')$.

Let $(V, \Delta)$ and $(V', \Delta')$ be two filtered $A_\infty$-algebras. Then a filtered quasiisomorphism of $(V, \Delta)$ and $(V', \Delta')$ implies quasiisomorphism of corresponding topological differential coalgebras $(\hat{T}(W), Q)$ and $(\hat{T}(W'), Q')$.

**Theorem 81** Let $(V, \Delta)$ and $(V', \Delta')$ be two filtered $A_\infty$-coalgebras. Then filtered quasiisomorphism of corresponding topological differential algebras $(\hat{T}(W), Q)$ and $(\hat{T}(W'), Q')$ implies quasiisomorphism of $A_\infty$-coalgebras $(V, \Delta)$ and $(V', \Delta')$.
Theorem 82 If the map $f_*: \widehat{\text{Bar}}(V,m) \to \widehat{\text{Bar}}(V',m)$ is a quasiisomorphism that the original map $f$ is also a quasiisomorphism.

Lemma 83 For any $A_\infty$ filtered coalgebra $H$ there is $A_\infty$-morphism
\[
\widehat{\text{Bar}}\widehat{\text{Bar}}(H) \xrightarrow{\psi} H
\]
(133)
of $A_\infty$-coalgebras. The morphism $\psi$ is a quasiisomorphism.

Similarly for any $A_\infty$ filtered algebra $A$ there is $A_\infty$-i-morphism
\[
A \to \widehat{\text{Bar}}\widehat{\text{Bar}}(A)
\]
(134)
of $A_\infty$-coalgebras. The morphism $\phi$ is a quasiisomorphism.

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