INDEX REDUCTION FOR DEGENERATED DIFFERENTIAL-ALGEBRAIC EQUATIONS BY EMBEDDING AND REAL NUMERICAL ALGEBRAIC GEOMETRY *

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Abstract. To find consistent initial data points (witness points) for a system of differential-algebraic equations, requires the identification of its missing (hidden) constraints arising from differentiation of the system. An efficient class of so-called structural methods exploiting a dependency graph for this task was initiated by Pantileles. The choice of graph is related to the choice of ordering that minimize the solution dimension of the problem. However, this method may fail. More complete methods rely on differential-algebraic geometry but suffer from other issues (e.g. high complexity and instability on approximate data). In this paper we give a new class of efficient structural methods combined with new tools from numerical real algebraic geometry that has much improved completeness properties. Existing structural methods may fail for a system of differential-algebraic equations if its Jacobian matrix after differentiation is still singular due to symbolic cancellation or numerical degeneration. Existing structural methods can only handle degenerated cases caused by symbolic cancellation. However, if a system has parameters, then its parametric Jacobian matrix may be still singular after application of the structural method for certain values of the parameters. This case is called numerical degeneration.

In this paper, for polynomially nonlinear systems of differential-algebraic equations, numerical methods are given to solve both degenerated cases using numerical real algebraic geometry. First, we introduce a witness point method, which produces at least one witness point on every constraint component (manifold). This can help to ensure constant rank and detection of degeneration on all components of such systems. Secondly, we present a Constant Rank Embedding Lemma, then based on this lemma we propose an Index Reduction by Embedding (IRE) method which can construct an equivalent system with a full rank Jacobian matrix. Thirdly, IRE leads to a global structural differentiation method, to solve degenerated differential-algebraic equations on all components numerically. Application examples of models from circuits, mechanics, and motion are used to demonstrate our method and its advantages.

Key words. real algebraic geometry, constant rank, witness points, differential-algebraic equations, structural methods.

AMS subject classifications. 68Q25, 68R10, 68U05

1. Introduction. Systems of differential-algebraic equations are widely used to model and simulate dynamical systems such as mechanical systems, electrical circuits, and chemical reaction plants [21]. We will often use the abbreviation DAE1 for a system of differential-algebraic equations. The name arose since such systems usually contain differential equations with derivatives and algebraic equations without derivatives. It was initially believed that any such DAE could be easily converted by coordinate changes and eliminations to a traditional explicit ODE - the so-called underlying ODE2. However, they are now recognized as common and natural in applications, to the

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1A system of differential-algebraic equations will be denoted by DAE while DAEs will denote several such systems.
2A system of explicit ordinary differential equations, in solved form for their highest derivatives, will be denoted by ODE.
extent that several modeling environments have them as their central object. Indeed, the explicit underlying ODE may be too expensive to uncover, and modern codes for numerical solution of a DAE have made this unnecessary in most cases. Even the idea there is a unique underlying ODE is misleading since the DAE may yield several component manifolds with different behavior and different underlying ODE on each component.

DAEs are a subset of the set of general systems of partial differential equations. The great geometer Cartan conjectured but was not able to prove that after a finite number of prolongations (differentiations) of such systems, they would become involutive, and a local existence and uniqueness theorem could be stated for their solutions. Another greater geometer Kuranishi eventually proved this result, albeit under conditions that are difficult to render explicitly [26]. The number of prolongations (differentiations) to uncover the underlying ODE (i.e. the differential index) corresponds to this result for DAEs; and it is equivalent to obtaining all the constraints on initial data for existence and uniqueness of solutions. We note that the differential index and underlying ODE may be different on different components of the DAE.

One may try to numerically solve a DAE directly without reducing its index by prolongation. However, properly posing initial values for a DAE requires that they satisfy the missing constraints, and hence implicitly requires knowledge of the prolonged form of the DAE. This direct approach is prone to order reduction, instability, inaccuracy and the tendency for the approximate solution to drift off the constraints that increases with the index. It is usually only feasible for low index problems [30, 48, 31, 1, 42]. For references related to differential and perturbation index see Hairer and Wanner [11], Campbell and Gear [9], and Reid, Lin and Wittkopf [45].

Note that the name DAE misleadingly suggests that a DAE can be partitioned into differential equations and non-differential equations (algebraic equations) where the latter are regarded as constraints. Consider the DAE

\[ u''u' + uu' + x = 0, ((u')^2 + u^2 + x^2 - 1)((u')^2 + u^2 + x^2 - 4) = 0 \]

where \( u \) is a unknown function of \( x \). Then \((u')^2 + u^2 + x^2 - 1)((u')^2 + u^2 + x^2 - 4) = 0\) is a constraint even though it contains derivatives. Geometrically there are 2 constraint components (spheres of radius 1 and 2).

In contrast to the above direct approach, indirect and widely used approaches first use index reduction only by differentiation [43, 42, 41, 13, 12] followed by consistent initial point determination [3, 39, 41].

In this paper we make contributions to such indirect approaches. In particular for polynomially nonlinear DAEs we apply a new efficient prolongation method to reduce their index, which implicitly gives the hidden constraint components of initial data, then determine consistent initial points using new methods from real numerical algebraic geometry.

### 1.1. Previous Work.

#### 1.1.1. Consistent Point.
Finding at least one consistent point on each constraint component of a DAE, is an important problem as it determines the initial point for a numerical solution [39, 41]. Commonly used methods to obtain such consistent initial points are the approximation method [29] and the transformation method [52, 4]. The approximation method starts with a guess for an initial point and
then iteratively refines it [47]. A good guess is critical for convergence of the iterative method [47].

We note that most treatments assume there is just one constraint component, and indeed that equations whose set of solutions correspond to this constraint component (constraint equations) can be explicitly found. Geometrically constraint components are projections from the higher dimensional space with the derivatives regarded as indeterminates (the so-called Jet space of the DAE). For linear DAEs and polynomially nonlinear DAEs with rational coefficients elimination algorithms are known for explicitly rendering equations for their constraints. But no algorithms are known for the general case of analytic DAEs, though there are some known for subclasses of analytic DAEs.

1.1.2. Index Reduction. Indices are used to measure how far a DAE is from a DAE which includes its missing constraints, or is implicitly equivalent to an ODE. The Kronecker index [14, 27] is applicable to constant coefficient linear DAEs. The tractability index [27, 32, 18] and strangeness index [24] applies to linear variable coefficient DAEs. Further, the tractability index can be extended to a topological index [53] in some applications, and the strangeness index also can be extended to non-square DAE [25]. The perturbation index [9] is defined in terms of perturbations of nonlinear autonomous DAEs. The differential index [9, 8] is the minimum number of differentiation times required to transform a DAE system into an explicit ODE system, and is used in our paper.

In order to solve a DAE accurately, index reduction is an essential and important operation. The accurate numerical solution of a high index ($\geq 2$) DAE is difficult to obtained directly [47]. Thus, we emphasize the need to convert a higher index DAE to a low ($\leq 1$) index DAE. After sufficient differentiation, all the time derivatives of the existing differential variables can be replaced by new variables to realize the index reduction [41]. Gear [13] proposed a method by repeatedly finding algebraic equations and dealing with them by differential processing until the system becomes an ODE. However, these methods are notoriously hard for large and non-linear systems. The arguments in [41] and [13] depend on liberal use of the implicit function theorem for analytic functions under tacitly assumed unstated rank conditions. The general finite termination of prolongation of analytic systems of partial differential equations yielding in finite steps involutive systems for which an existence and uniqueness theorem can be given, was conjectured by Cartan. Kuranishi [26] eventually proved this famous and difficult result that had eluded Cartan.

For polynomially nonlinear DAEs with rational coefficients, there are symbolic differential-elimination algorithms that reduce index of DAE, but these algorithms are often unstable when applied to approximate DAE and also have high worst case complexity. Fortunately there are some efficient methods based on bipartite graph preprocessing that can sometimes reduce the differential index. These methods have been implemented in DAE simulation packages such as Dymola, Open-Modelica, MapleSim [12], Simulink and IsamDAE [6]. Such methods originated with work by Pantelides [41] who presented a graph-based preprocessing method that can sometimes by prolongation reduce a DAE to involutive (index 0 or 1) form containing the underlying ODE that decides consistent initial data for numerical solutions. Crucially Pantelide’s method and its later developments have proven to be successful often enough in applications that they have become a standard part of the software environments mentioned above. Such developments include Mattsson-Söderlind’s (MS) Method [34] which employs an amending method to introduce new variables.
to replace dummy derivatives [36]. Pryce [43] further generalized it to a more direct and widely applicable method by solving an assignment problem. Zolfaghari, Taylor and Spiteri [60] further extended Pryce method to the application of integro-differential–algebraic equations.

1.1.3. Improved Structural Methods. Despite the success of structural analysis by index reduction, the methods may fail for a DAE if its Jacobian after differentiation is singular, and it is essential to develop improved structure methods.

Campbell [7] proposed a direct method, which can regularize a DAE in theory by sufficiently differentiating the DAE and simplifying it with an elimination method. But the symbolic elimination process can be very complex and inefficient for nonlinear DAE.

Linear DAE with constant coefficients can be transformed into the canonical form of Weierstrass, the Kronecker index determined and then the DAE can be solved directly [16]. This transformation is neither a strict equivalence transformation [22] nor a unimodular transformation [37]. Murota [37] proposed a general framework “combinatorial relaxation” algorithm to compute the degree of a certain determinant based on its upper bound, which is defined in terms of the maximum weight of a perfect matching in an associated graph. Iwata [22] improved the combinatorial relaxation algorithm by an equivalence transformation with constant matrices, reducing computational complexity. X. Wu [59] applied the modified combinatorial relaxation algorithm to analyze the resulting error behavior. In particular he gave an algorithm to detect and correct the error from structural index reduction by matrix pencils.

Compared with the method of X. Wu et al. [59], the LC-method of Tan et al. [51] also considers equations and their derivatives, with better results for some nonlinear DAEs. Unfortunately, although this method may guarantee a global equivalence transformation, it can only be used by its norm space. The ES-method [51] uses new variables to seek the solution in a projection of a higher dimensional space, and it can be considered as a supplement for the LC-method. The LC-method replaces equations, while the ES-method replaces variables. If the global equivalence transformation of both methods or neither of them is guaranteed, then the LC-method can be used, otherwise, the ES-method can be used. Both of the above two methods can also be extended and applied to some integro-differential–algebraic equations [60]. The substitution method [39] aims to modify non-linear DAE, and it is a local equivalence method. Like the method of Campbell, it avoids excessive elimination through targeted variable selection. For DAE with high non-linearity, this method is usually too complex to be applied. In order to avoid the complexity of elimination calculations, the augmentation method [39] adopts the principle similar to the ES-method, and is also a local equivalence method.

1.2. Problem Statement. Mathematical models of circular motion in kinematics, mechanical structures and chemical processes etc., often provide polynomially nonlinear DAEs. Consequently they may have more than one solution component (see Example 1.2). For global information about solutions, we need at least one consistent initial point on each component. Such consistent initial points are hard to obtain for a polynomial system with many variables by using symbolic computation, e.g. by using Groebner Bases [15] or Triangular Decomposition [5]. Further, Newton iterative solvers usually require a starting point sufficiently close to a solution. To obtain such global information is one of the two main goals in this paper.
Moreover, the success of structural analysis methods for DAE depends on the regular Jacobian assumption after index reduction. In many cases, this assumption is valid. However, we will present counterexamples from real applications.

Such cases are called "degeneration" cases, which means the Jacobian matrix is singular on a whole component, including two types: symbolic cancellation (see Example 1.1) and numerical degeneration (see Example 1.2).

Remark 1.1. In fact, singularities can occur only at special points along a solution. For example in Equation (1.1), when $u$ is increasing, $u'$ will gradually change to 0, leading to a singular Jacobian for the equation. We will not consider this kind of problem in this paper.

Example 1.1. Symbolic Cancellation: Consider the following DAE [39]:

$$\begin{align*}
\dot{x}_1 + \dot{x}_2 + x_3 &= 0 \\
\dot{x}_1 + \dot{x}_2 &= 0 \\
x_2 + \dot{x}_3 &= 0
\end{align*}$$

Symbolic cancellation occurs when the determinant of the Jacobian matrix of the DAE is identically zero. This case can be regularized by a number of methods: a combinatorial relaxation method, a linear combination (LC) method, and an expression substitution (ES) method [51], a substitution method and an augmentation method [39].

Unfortunately, there is little research on failure caused by numerical degeneration. This could happen for a parametric DAE model with a non-zero determinant, where parameters take some specific values, and the determinant equals zero after substituting any initial value on a component defined by the constraints.

Example 1.2. Numerical Degeneration:

In the bending deformation of a beam described in [44], the deformation of any moment acting at a point $x$ satisfies $\frac{d^2}{dx^2} y(x) = \frac{M(x)}{E(x)I(x)}$. When two collinear bending moments act on the beam at the same time, the deformation is the superposition of the effects of these moments. Suppose that the elastic deformation energy of the bending moment at each point on the beam is linear in $\lambda$, then the deformation variables $y_1(x)$,
\[ y_2(x) \text{ satisfy:} \]
\[ \left\{ \begin{array}{c}
\frac{d^2}{dx^2} y_1(x) + \frac{d^2}{dx^2} y_2(x) + \frac{1}{5} \cdot (1 - \sin(x)) + y_1(x) = 0 \\
\lambda \cdot y_1(x)^2 - y_2(x)^2 = 0
\end{array} \right. \]

In this example, the determinant of the Jacobian matrix is \(-2(y_2 + \lambda y_1)\). When the parameter \(\lambda\) is equal to 1, the constraint becomes \(y_1^2 - y_2^2 = (y_1 + y_2)(y_1 - y_2) = 0\).

In the view of physics, this means that the elastic deformation energy of each point on the beam is the same. Obviously, two consistent initial points can be selected from two different components, respectively. If the point is on the component \(y_1 - y_2 = 0\), then Pryce’s structural method works well. But for any initial point on the component \(y_1 + y_2 = 0\), we always encounter a singular Jacobian. Note that, this “bad” initial value usually can only be obtained approximately, since there is no explicit expression for the roots of general degree 5 polynomials. Furthermore, such choices of the parameter values if exist, often satisfy a nonlinear system and are only obtained numerically. Therefore, we call this case numerical degeneration.

Such a degeneration is of potentially great importance in designing control parameters in architecture, aviation and biochemistry. So a key question deserving further study both theoretically and computationally which is addressed in our paper is how to identify and solve such degenerated systems.

In summary, we aim to solve the following problems in this paper:
- To detect at least one initial point on each real component of a polynomially nonlinear DAE.
- To propose a global structural method to restore full rank Jacobian matrices without algebraic elimination.

2. Preliminaries. In what follows we will use algorithmic aspects of the formal (Jet) theory of differential equations \([45, 46]\). Jet theory enables two inter-related views of differential equations to be algorithmically and correctly manipulated. One view is in terms of local solutions of the differential equations.

Let \(\mathbb{I}\) be a nonempty sub-interval of \(\mathbb{R}\). Let \(t \in \mathbb{I} \subset \mathbb{R}\) and suppose \(x, x^{(1)}, \ldots, x^{(\ell)}\) are vectors in \(\mathbb{R}^n\), where \(\ell\) is a fixed positive integer. Here we consider maps \(F : \mathbb{I} \times \mathbb{R}^{n+n} \to \mathbb{R}^m\) which are polynomially nonlinear in \(x, x^{(1)}, \ldots, x^{(\ell)}\) and real analytic in \(t\), where possibly \(m \neq n\).

An algebraic solution of this system is a point \((t, x, x^{(1)}, \ldots, x^{(\ell)}) \in \mathbb{I} \times \mathbb{R}^{n+n}\) that satisfies \(F(t, x, x^{(1)}, \ldots, x^{(\ell)}) = 0\). A differential solution \(x(t)\) is a solution for \(t\) in some subinterval of \(\mathbb{I}\) that satisfies the differential equations \(F(t, x(t), x^{(1)}(t), \ldots, x^{(\ell)}(t)) = 0\) where now \(x^{(k)}(t)\) denotes the \(k\)-order derivative of \(x(t)\). Sometimes we will consider the system as an algebraic system and sometimes a differential system, which should be clear from the context.

In particular, we consider systems which are not solved for their highest derivatives, and regard such a system as a DAE. The differential-algebraic systems we consider have Jet form

\begin{equation}
(2.1) \quad F(t, x, x^{(1)}, \ldots, x^{(\ell)}) = 0
\end{equation}

Let \(D\) be the formal total derivative operator with respect to independent variable \(t\):

\begin{equation}
(2.2) \quad D = \frac{\partial}{\partial t} + \sum_{k=0}^{\infty} x^{(k+1)} \frac{\partial}{\partial x^{(k)}}
\end{equation}
Regarding $F$ in its algebraic (jet) form a single prolongation of $F$ is the differentiation of each $F_i$ with respect to $t$, in which $F_i$ is the $i$-th equation of $F$, and it is denoted by

\begin{equation}
F^{(1)} = DF \cup D^0F = \{DF_1,\ldots,DF_n\} \cup F
\end{equation}

It easily follows that the prolongation of $F$ is a linear system with respect to the “new” dependent variable $x^{(\ell+1)}$. Thus, we can rewrite

\begin{equation}
DF = S(t,x,x^{(1)},\ldots,x^{(\ell)}) \cdot x^{(\ell+1)} + G(t,x,x^{(1)},\ldots,x^{(\ell)})
\end{equation}

where $S$ is an $n \times n$ matrix called the “symbol matrix” and $x^{(\ell+1)}$ is a column vector and $G$ contains all the remaining terms. Note that $S$ is also the Jacobian matrix of $F$ with respect to its highest order derivative $x^{(\ell+1)}$.

If we specify the prolongation order for $F_i$ to be $c_i$, then $c_i \geq 0$, for $i = 1,\ldots,n$. For notational brevity, we will write $(c_1,\ldots,c_n) = c \geq 0$. Then the prolongation of $F$ up to the order $c$ is

\begin{equation}
F^{(c)} = \{F_1,DF_1,\ldots,D^nF_1\} \cup \cdots \cup \{F_n,DF_n,\ldots,D^nF_n\} = D^cF
\end{equation}

If $c > 0$, then $F^{(c)}$ also has linear structure similar to (2.4). The number of equations of $F^{(c)}$ is $n + \sum_{i=1}^n c_i$.

2.1. Structural Prolongation Methods for DAE. In [43], Pryce reinterpreted Pantelides' algorithm [41] as an assignment problem that reveals structural information about DAEs. This structural method for square DAEs is a special case with roots in the work of Jacobi [40] and yields a local existence and uniqueness result. The most important feature of Pryce's method is that it finds all the local constraints for a large class of square DAEs only using prolongation. A generalization of this construction to partial differential-algebraic equations can be found in [58].

Suppose that the $k$-th order of derivative of $x_j$ occurs in $F_i$, then the total derivative $\partial F_i/\partial x_j^{(k)}$ is not identically zero. The leading derivative of an equation or a system $F$ with respect to $x_j$ is denoted by $LD(F,x_j)$ and is the highest order of derivative such that some $F_i \in F$ depends on $x_j^{(k)}$ for some $k \in \mathbb{Z}^+$. We define the leading class derivatives of a system $F$ by

\[ \text{LCD}(F) := \{LD(F,x_j) : 1 \leq j \leq n\} \]

Then we obtain an $n \times n$ matrix $\sigma = (\sigma_{i,j})_{1 \leq i \leq n,1 \leq j \leq n}$ which is called the signature matrix of $F$ by Pryce [43]:

\begin{equation}
(\sigma_{i,j})(F) := \begin{cases} 
\text{the order of } LD(F_i,x_j); \\
-\infty, \text{ otherwise.}
\end{cases}
\end{equation}

Suppose that the highest order derivative of $x_j$ appearing in $F^{(c)}$, defined in Equation (2.5), is $d_j$. From the definition of $\sigma_{i,j}$, clearly $d_j$ is the largest of $c_i + \sigma_{i,j}$ for $i = 1,\ldots,n$, which implies that

\begin{equation}
d_j - c_i \geq \sigma_{i,j}, \text{ for all } i, j.
\end{equation}

Obviously, there are at most $n + \sum j d_j$ derivatives and $n + \sum c_i$ equations in $F^{(c)}$. The dimension of $F^{(c)}$ usually is $\sum j d_j - \sum c_i$. Roughly speaking, finding all the
The triangular block structure of \( F^{(c_p)} \) for the case of \( c_p = c_{p+1} + 1 \): For \( 0 \leq p < k_c \), \( B_i \) has fewer jet variables than \( B_{p+1} \).

| \( B_0 \) | \( B_1 \) | \( \cdots \) | \( B_{k_c-1} \) | \( B_{k_c} \) |
|---|---|---|---|---|
| \( F_1^{(0)} \) | \( F_1^{(1)} \) | \( \cdots \) | \( F_1^{(c_1-1)} \) | \( F_1^{(c_1)} \) |
| \( F_2^{(0)} \) | \( \cdots \) | \( F_2^{(c_2-1)} \) | \( F_2^{(c_2)} \) |
| \( \vdots \) | \( \vdots \) | \( \vdots \) |
| \( F_n^{(0)} \) | \( \cdots \) | \( F_n^{(c_n)} \) |

Table 1

The constraints is equivalent to minimizing the dimension of \( F^{(c)} \). This can be formulated as an integer linear programming (ILP) problem in the variables \( c = (c_1, \ldots, c_n) \) and \( d = (d_1, \ldots, d_n) \):

\[
\begin{align*}
\delta(F) & \quad \text{Minimize } \delta = \sum d_j - \sum c_i, \\
& \quad \text{where } d_j - c_i \geq \sigma_{ij}, \\
& \quad c_i \geq 0
\end{align*}
\]

Let \( \delta(F) \) be the optimal value of the problem (2.8).

**Remark 2.1.** When a DAE has no redundant equations, the optimal value \( \delta(F) \) can be regarded as degree of freedom (DOF) measure for the DAE, and it also equals the number of variables of \( F \) minus the number of equations of \( F \). In this paper, we usually only consider cases without redundant equations in theoretical derivation. Some cases with redundant equations will be addressed in Section 7.2. We will also show the computational performance of our approach in our experiments.

After we obtain the number of prolongation steps \( c_i \) for each equation \( F_i \) by applying an ILP solver to Equation (2.8), we can construct the partially prolonged system \( F^{(c)} \) using \( c \). We note that \( F^{(c)} \) has a favorable block triangular structure enabling us to compute consistent initial values more efficiently.

Without loss of generality, we assume \( c_1 \geq c_2 \geq \cdots \geq c_n \), and let \( k_c = c_1 \), which is closely related to the index of system \( F \) (see [43]). The \( r \)-th order derivative of \( F_j \) with respect to \( t \) is denoted by \( F_j^{(r)} \). Then we can partition \( F^{(c)} \) into \( k_c + 1 \) parts (see Table 1), for \( 0 \leq p \in \mathbb{Z} \leq k_c \) given by

\[
B_p := \{ F_j^{(p+c_j-k_c)} : 1 \leq j \leq n, p + c_j - k_c \geq 0 \}.
\]

Here, we call \( B_{k_c} \) the top block of \( F^{(c)} \) and \( F^{(c-1)} = \{ B_0, \ldots, B_{k_c-1} \} \) the constraints.

Similarly, let \( k_d = \max(d_j) \) and we can partition all the variables into \( k_d + 1 \) parts:

\[
X^{(q)} := \{ x_j^{(q+d_j-k_d)} : 1 \leq j \leq n, q + d_j - k_d \geq 0 \}.
\]

For each \( B_i, 0 \leq i \leq k_c \), we define the Jacobian Matrix

\[
\mathbf{J}_i := \left( \frac{\partial B_i}{\partial X^{(i+k_d-k_c)}} \right).
\]

So \( \mathbf{J}_{k_c} \) is the Jacobian Matrix of the top block in the table, and it is a square matrix.
Proposition 2.1. Let \( \{J_i\} \) be the set of Jacobian matrices of \( \{B_i\} \). For any \( 0 \leq i < j \leq k_c \), \( J_i \) is a sub-matrix of \( J_j \). Moreover, if \( J_{k_c} \) has full rank, then any \( J_i \) also has full rank.

See [58] for the proof.

Suppose \((t^*, X^*)\) is a point satisfying the constraints \( \{B_0, \ldots, B_{k_c-1}\} \) and \( J_{k_c} \) has full rank at this point. Then Pryce’s structural method has successfully finished the index reduction. However, it fails if \( J_{k_c} \) is still singular, i.e. \( J_{k_c} \) is degenerated.

Obviously, the definition of optimal value \( \delta(F) \) is limited to square systems, and we need to extend the definition for non-square systems \( F^{(c)} \).

**Definition 2.1.** Let a dae \( F \) consist of two blocks \( A \) and \( B \), where \( F \) contains \( p \) equations and \( n \) dependent variables, and the signature matrix of \( A \) be an \( n \times n \) square matrix. So \( B \) contains the remaining \( (p-n) \) equations. Let \( \delta(A) \) be the optimal value of the \( \text{ILP} \) of \( A \)'s signature matrix. We define \( \delta(F) = \delta(A) - \#eqns(B) \), where \#eqns(B) is the number of equations in \( B \). Meanwhile, \( \delta(F) \) also equals the DOF [51] of \( F \), which equals the number of dependent variables minus the number of equations in the prolongation of \( F \).

In the case of a square signature matrix of a dae \( F \), we have \#eqns = 0, and the extended definition of \( \delta(F) \) is equivalent to the original definition.

**Proposition 2.2.** Let \((c, d)\) be the optimal solution of Problem (2.8) for a given dae \( F \). Then \( \delta(F) = \delta(F^{(c)}) = \sum d_j - \sum c_i \).

**Proof.** For a prolonged dae system \( F^{(c)} = \{B_{k_c}, F^{(c-1)}\} \), the signature matrix of the top block \( B_{k_c} \) is square.

We construct a pair \((\hat{c}, d)\), for \( i = 1, \cdots, n \) and \( j = 1, \cdots, n \), \( \hat{c}_i = 0 \) and \( \hat{d}_j = d_j \). Since \((c, d)\) is the optimal solution for \( F \), and \( B_{k_c} \) is the top block of \( F^{(c)} \), it follows that \((\hat{c}, d)\) is the optimal solution of \( B_{k_c} \), \( \delta(B_{k_c}) = \sum d_j \).

By Definition 2.1, and using \#eqns\( (F^{(c-1)}) = \sum c_i \), we obtain

\[
\delta(F^{(c)}) = \delta(B_{k_c}) - \#eqns(F^{(c-1)}) = \sum d_j - \sum c_i = \delta(F).
\]

\[\Box\]

2.2. Framework for Improved Structural Methods. Many improved structural methods have been proposed to regularize the Jacobian matrices of daes. See [16, 22, 37, 7, 59] for methods for linear dae. For non-linear daes, improved structural methods are based on a combinatorial relaxation framework [39] with the following steps:

**Phase 1.** Compute the solution \((c, d)\) of ILP problem \( \delta(F) \). If there is no solution, the dae do not admit perfect matching, and the algorithm ends with failure.

**Phase 2.** Determine whether \( J_{k_c} \) is identically singular or not. If not, the method returns \( F^{(c)} \) and halts.

**Phase 3.** Construct a new dae \( \hat{F} \), such that its solution space in \( x \) dimension is the same as dae \( F \) and \( 0 \leq \delta(\hat{F}) < \delta(F) \). Then go to Phase 1.

**Remark 2.2.** The key part of an improved structural method is to exploit different regularization method for \( F \) in Phase 3. In this paper, our global structural differentiation method mainly focuses on this phase.

Phase 2 above is only to check for symbolic cancellation. As pointed out in Example 1.2, \( \det(J_{k_c}) \) may not be identically zero, but \( \det(J_{k_c}) = 0 \) at any consistent initial point of \( Z(F^{(c)}) \) — the zero set of \( F^{(c)} \). Since \( F \) is a polynomial system
in \{x, x^{(1)}, ..., x^{(t)}\}, \mathbf{F}^{(e)} can be considered as a polynomial system in the variables \{X^{(0)}_, X^{(k_e)}\}. In the language of algebraic geometry, it means that det \mathbf{J}_{k_e} \in \sqrt{\langle \mathbf{F}^{(e)} \rangle}$ or equivalently $Z_{\mathbb{R}}(\mathbf{F}^{(e)}) \subseteq Z_{\mathbb{R}}(\mathbf{J}_{k_e})$.

In the rest of the paper, we usually suppress the subscript in \mathbf{J}_{k_e} so it becomes \mathbf{J} unless the subscript is needed.

**Example 2.1.** Consider the following dae with dependent variables $x(t)$ and $y(t)$:

\begin{equation}
\mathbf{F} = \{2y \frac{d^2 x}{dt^2} - x \frac{d^2 y}{dt^2} + 2x \left( \frac{dx}{dt} \right)^2 - \frac{dx}{dt} + \sin(t), y - x^2 \}.
\end{equation}

Applying the structural method yields $c = (0, 2)$ and $d = (2, 2)$. Then

$\mathbf{F}^{(e)} = \{(2yx_{tt} - yx_{tt} + 2xx_t^2 - x_t + \sin(t), y_{tt} - 2x_t^2 - 2xx_{tt}, \{-2xx_t + y_t}, \{-x^2 + y\}]$

and the Jacobian matrix of the top block is $\mathbf{J} = \begin{pmatrix} 2y & -x \\ -2x & 1 \end{pmatrix}$.

Although the determinant of the Jacobian $2y - 2x^2$ is not identically zero, it must equal zero at any initial point, since the determinant belongs to the polynomial ideal generated by the constraints, i.e. $2y - 2x^2 \in (-x^2 + y)$.

Checking if a polynomial belongs to an ideal can be done by a standard ideal membership test using a Gröbner basis of the ideal. In general, it is challenging to compute the Jacobian determinant and the associated Gröbner basis if the system is quite large. See the text [10] for more details about polynomial ideals, varieties and Gröbner bases. Algorithmic algebraic geometry exploits Gröbner bases and related techniques to compute features of solutions of general polynomial systems with exact (e.g. rational) coefficients. Numerical versions of these algorithms, where exact numbers are replaced with approximate numbers have largely been expensive and often unstable.

In this paper, we propose a numerical approach based on real algebraic geometry to detect such degenerated cases without using determinants or Gröbner bases. It exploits a new generation of algorithms using a fundamentally different and more thoroughly numerical approach, centered around the concept of witness points on solution components and is discussed in the next section.

**2.3. Numerical Real Algebraic Geometry.** Numerical algebraic geometry [50, 20] was pioneered by Sommese, Wampler, Verschelde and others (see [2, 49] for references and background). The approach is built on witness points which arise by slicing the complex variety with appropriate random planes of complementary dimension. These complex witness points can be efficiently computed by homotopy continuation solvers [28], and are theoretically guaranteed to compute at least one such point on each solution component.

For the real case, the methods in [56, 57] yield real witness points as critical points of the distance from a random hyperplane to the real variety. Alternatively, the real witness points can be considered as critical points of the distance from a random point to the real variety [19].

More precisely, to solve a polynomial system $\mathbf{f} = \{f_1, ..., f_k\} \subset \mathbb{R}[x_1, ..., x_n]$, we first choose a random point $a \in \mathbb{R}^n$, then there is at least one point on each connected component of $V_{\mathbb{R}}(\mathbf{f})$ with minimal distance to $a$ satisfying the following problem:
The optimization problem can be formulated as a square system by using Lagrange multipliers, i.e.

\begin{equation}
\mathbf{g} = \{ \mathbf{f}, \sum_{i=1}^{k} \lambda_{i} \nabla f_{i} + x_{i} - a_{i} \} = 0
\end{equation}

When \( \mathbf{f} \) satisfies the regularity assumptions in [56], all the real solutions of \( \mathbf{g} = 0 \) can be obtained by the homotopy continuation method. These points are called real witness points of \( V_{R}(\mathbf{f}) \), where \( V_{R}(\mathbf{f}) = \{ x \in \mathbb{R}^{n} : \mathbf{f}(x) = 0 \} \). These real solutions of the constraint equations provide initial points for every component of a non-linear DAE.

DEFINITION 2.2. For a polynomial system \( \mathbf{f} \), let \( \mathcal{S} \) be the set of singular points of \( V_{R}(\mathbf{f}) \). If a finite set \( \mathcal{W} \subseteq \mathbb{R}^{n} \) contains at least one point on each connected component of \( V_{R}(\mathbf{f}) \setminus \mathcal{S} \), then this set is called the real witness set of \( V_{R}(\mathbf{f}) \) and these points are called \textbf{real witness points}.

However, if \( \mathbf{f} \) does not satisfy the regularity assumptions due to high multiplicity or a non-real radical ideal, then we apply a critical point approach [55] based on a penalty factor.

For example if \( \mathbf{f} = (x^{2} + y^{2} - 1)^{2} \), then we choose a random point \( a = (a, b) \) and a large penalty factor \( \beta \). The corresponding regular system is

\begin{equation}
\begin{pmatrix}
x \\
y
\end{pmatrix} + \beta \cdot \mathbf{f} \cdot 
\begin{pmatrix}
4x(x^{2} + y^{2} - 1) \\
4y(x^{2} + y^{2} - 1)
\end{pmatrix} = 
\begin{pmatrix}
a \\
b
\end{pmatrix}
\end{equation}

If the random point is \( (a, b) = (1, 1) \) and a large penalty factor \( \beta = 100000 \) is chosen, then 4 approximate solutions can be calculated by the homotopy continuation method. Here, two of these solutions are far away from components, which can be verified by substituting them into the equation \( \mathbf{f} \). The verified solutions \( [x = 0.985220, y = 0.172402], [x = -0.652031, y = -0.758442] \) are close to the real variety which is the unit circle. For more details, please see [55].

3. Detecting Degeneration by Witness Points. To build a solid foundation of our theory, we need some results from the theory of real analytic functions of several variables [23].

DEFINITION 3.1. A function \( \mathbf{f} \), with domain an open subset \( U \subset \mathbb{R}^{n} \) and range \( \mathbb{R} \), is called \textbf{real analytic on} \( U \), if for each \( \mathbf{p} \in U \) the function \( \mathbf{f} \) can be represented by a convergent power series in some neighbourhood of \( \mathbf{p} \).

PROPOSITION 3.1 (Proposition 2.2.8 of [23]). If \( f_{1}, ..., f_{m} \) are real analytic in some neighbourhood of the point \( \mathbf{p} \in \mathbb{R}^{n} \) and \( g \) is real analytic in some neighbourhood of the point \( (f_{1}(\mathbf{p}), ..., f_{m}(\mathbf{p})) \in \mathbb{R}^{m} \), then the composition of functions \( g(f_{1}(\mathbf{x}), ..., f_{m}(\mathbf{x})) \) is real analytic in a neighborhood of \( \mathbf{p} \).

THEOREM 3.1 (Real Analytic Implicit Function Theorem [23]). Given a set of equations \( f_{i}(x_{1}, ..., x_{m}; y_{1}, ..., y_{n}) = 0, \quad i = 1, 2, ..., n, \) where each \( f_{i} \) is real analytic,
suppose that \((p, q) = (p_1, \ldots, p_m; q_1, \ldots, q_n)\) is a solution with nonsingular Jacobian \(\frac{\partial f}{\partial q}(p, q)\).

Then there exists a neighborhood \(U \subset \mathbb{R}^m\) of \(p\) and a set of real analytic functions \(\phi_j : U \to \mathbb{R}, j = 1, 2, \ldots, n\), such that \(\phi_j(p) = q_j, j = 1, 2, \ldots, n\), and

\[f_i(x; \phi_1(x), \ldots, \phi_n(x)) = 0, i = l, 2, \ldots, n,\]

hold for \(x \in U\).

**Theorem 3.2** (Identity Theorem for Real Analytic Functions). Given two real analytic functions \(f\) and \(g\) on an open and connected set \(U \subset \mathbb{R}^n\), if \(f = g\) on a nonempty open subset \(S \subseteq U\), then \(f = g\) on the whole set \(U\).

**Proof.** Define a set where \(f\) and \(g\) have the same power series:

\[D = \left\{ c \in U : \frac{\partial | \mu |}{\partial x^\mu} f(c) = \frac{\partial | \mu |}{\partial x^\mu} g(c) \text{ for all } \mu \right\}\]

where \(\mu\) is a multi-index in \((\mathbb{Z}^+)^n\). Firstly, since \(f = g\) on a nonempty open subset \(S\), we have \(D \supseteq S\) and consequently \(D \neq \emptyset\). Secondly, \(f\) and \(g\) are real analytic on \(U\), then for any \(c \in D\) the power series have a non-zero radius of convergence. It implies that \(D\) is open. Meanwhile,

\[D = \bigcap_{\mu} \left\{ c \in U : \frac{\partial | \mu |}{\partial x^\mu} f(c) = \frac{\partial | \mu |}{\partial x^\mu} g(c) \right\}\]

which is an intersection of closed sets, so it is closed.

Therefore, by the connectedness of \(U\), \(D\) must be equal to the whole set \(U\). \(\square\)

**Lemma 3.3.** Let \(C\) be a connected real analytic manifold in \(\mathbb{R}^{m+n}\) of dimension \(m\) and let \(f\) be a real analytic function on \(\mathbb{R}^{m+n}\). Then the intersection \(C \cap Z_\mathbb{R}(f)\) is equal to \(C\) or has measure zero over \(C\).

**Proof.** Since \(C\) is a smooth manifold described by an analytic system implicitly, by the Implicit Function Theorem 3.1, locally \(C\) can be parameterized by \(m\) free coordinates. To be rigorous, we need an atlas over \(C\) which is a collection of charts depending on the free coordinates. Without loss of generality, we assume \(y_j = \phi_j(x_1, \ldots, x_m), j = 1, \ldots, n\) where \(\phi_j\) is real analytic.

Suppose the intersection \(C \cap Z_\mathbb{R}(f)\) has non-zero measure. Then there is a nonempty open subset \(S\) of \(C\) where \(g(x) = f(x; \phi_1(x), \ldots, \phi_n(x)) = 0\). By Proposition 3.1, \(g\) is real analytic. Due to the Identity Theorem 3.2, \(g = 0\) on the whole component, and thus \(C \subseteq Z_\mathbb{R}(f)\). \(\square\)

The real zero set of an analytic system \(f\) is denoted by \(Z_\mathbb{R}(f)\), whereas the real zero set of a polynomial system \(f\) is often denoted by \(V_\mathbb{R}(f)\).

**Theorem 3.4.** Let \(g\) be a polynomial system and \(W\) be a real witness set of \(V_\mathbb{R}(g)\). If another polynomial system \(f(p) = 0\) for any \(p \in W\), then \(V_\mathbb{R}(g) \subseteq V_\mathbb{R}(f)\) with probability one.

**Proof.** First \(V_\mathbb{R}(g) \setminus \mathcal{G}\) consists of finitely many smooth connected components and isolated real points. By Lemma 3.3, for each smooth connected component \(C\), the intersect \(C \cap V_\mathbb{R}(f)\) has measure zero over \(C\), unless \(C \subseteq V_\mathbb{R}(f)\). Since \(p\) is a random
point on $C$, it belongs to a measure zero set with probability zero. For the isolated points, $f(p) = 0 \Rightarrow p \in V_R(f)$. Therefore, $V_R(g) \setminus \mathcal{S} \subseteq V_R(f)$ with probability one.

Since $V_R(f)$ is a closed set, the closure of $V_R(g) \setminus \mathcal{S}$, which is $V_R(g)$, must be contained in $V_R(f)$.

This theorem actually gives a probabilistic method to check if $f \in \sqrt{\langle g \rangle}$ without using Gröbner bases.

**Remark 3.1.** After we obtain a witness set $W$ of the constraints, it is unnecessary to compute the determinant of $\mathcal{J}$ during the detection of degenerated cases by Theorem 3.4. We can simply substitute a real witness point into the Jacobian matrix and compute its smallest singular value by numerical methods. If the smallest singular values at all witness points are close to zero, then the Jacobian is degenerated with probability close to one. If some of them are almost zero and the remaining singular values are non-zero, then the determinant vanishes on some components of the constraints, and further work is needed for the DAE on these components.

### 4. Index Reduction by Embedding for Degenerated Systems

Consider a smooth connected component $C$ of $Z_R(F(c))$ with a real point $p \in \mathbb{R}^n$. Suppose $\text{rank} \mathcal{J}(p) = r < n$. Without loss of generality, we assume that the sub-matrix $\mathcal{J}(p)[1: r, 1: r]$ has full rank. In this section we will show that the rank is almost a constant over the whole component.

**Lemma 4.1.** Let $C$ be a smooth connected component. If $\mathcal{J}[1: r, 1: r]$ has full rank at a random point $p$ on $C$. Then it is non-singular over the whole component except some set with measure zero. Moreover, if a minor of $\mathcal{J}$ at this point is singular, then it is singular over the whole component with probability one.

**Proof.** Let $f$ be the determinant of $\mathcal{J}[1: r, 1: r]$. If $f(p) = 0$, then by Lemma 3.3 we have $C \subseteq Z_R(f)$ with probability one, implying that the Jacobian is singular over the whole component.

If $f(p) \neq 0$, then $C \nsubseteq Z_R(f)$ and Lemma 3.3 implies that $C \cap Z_R(f)$ has measure zero over $C$. \qed

Jacobians with constant rank enable us to embed the zero set into a higher dimensional space.

**Lemma 4.2** (Constant Rank Embedding). Let

$$f = \{f_1(x, y, z), ..., f_r(x, y, z)\} \quad \text{and} \quad g = \{g_1(x, y, z), ..., g_{n-r}(x, y, z)\}$$

be two sets of analytic functions, where $x = (x_1, ..., x_r)$, $y = (y_1, ..., y_{n-r})$ and $z = (z_1, ..., z_n)$. Let $C$ be a smooth connected component in $\mathbb{R}^{m+n}$. If the Jacobian matrices $\frac{\partial f}{\partial (x, y)}$ and $\frac{\partial f}{\partial z}$ have constant rank $r$ on $C$. Then

$$Z_R(f, g) \cap C = \pi Z_R(f(x, y, z), f(u, \xi, z), g(u, \xi, z)) \cap C$$

where $u = (u_1, ..., u_r)$ and $\xi$ is a constant vector and $\pi$ is the projection from $(x, y, z)$-space to $(x, y, z)$-space.

**Proof.** Since $\frac{\partial f}{\partial z}$ has constant rank $r$ on $C$, by the Implicit Function Theorem 3.1 and the Identity Theorem 3.2, there exist a set of real analytic functions $\phi = \{\phi_1, ..., \phi_r\}$ such that $f(\phi(y, z), y, z) = 0$ for any $(y, z) \in \pi_{yz}(C)$. Thus,

$$\frac{\partial f}{\partial x} \frac{\partial \phi}{\partial y} + \frac{\partial f}{\partial y} = 0.$$
Since $\frac{\partial (f,g)}{\partial (x,y)}$ also has constant rank $r$, $(\frac{\partial g}{\partial y}, I)^t$ is in the null-space of $\frac{\partial (f,g)}{\partial (x,y)}$. So

$$\frac{\partial g}{\partial x} \frac{\partial \phi}{\partial y} + \frac{\partial g}{\partial y} = 0.$$

Let $G(y,z) = g(\phi(y,z), y,z)$. We have $\frac{\partial G}{\partial y} = \frac{\partial g}{\partial y} + \frac{\partial g}{\partial y} = 0$, which implies that $G(y,z) = G(\xi, z)$ for any constant $\xi$ on $C$. If $p = (p_x, p_y, p_z) \in Z_R(f, g) \cap C$, then $p_x = \phi(p_y, p_z)$. Let $p_u = \phi(\xi, p_z)$ for some constant vector $\xi$, and let $\hat{p} = (p_x, p_y, p_z, p_u)$. It is straightforward to verify that $f(p_u, \xi, p_z) = 0$ and $g(p_u, \xi, p_z) = G(\xi, p_z) = G(p_y, p_z) = g(p_x, p_y, p_z) = 0$. Therefore, $\hat{p} \in Z_R(f(x, y, z), f(u, \xi, z), g(u, \xi, z)) \cap C$. Thus $p \in \pi Z_R(f(x, y, z), f(u, \xi, z), g(u, \xi, z)) \cap C$.

For any $p = (p_x, p_y, p_z) \in \pi Z_R(f(x, y, z), f(u, \xi, z), g(u, \xi, z)) \cap C$, we have $p_x = \phi(p_y, p_z)$ and $u = \phi(\xi, p_z)$. Also $g(u, \xi, p_z) = 0 \Rightarrow 0 = G(\xi, p_z) = G(p_y, p_z) = g(p_x, p_y, p_z)$. So $p \in Z_R(f, g) \cap C$.

If we have the witness set, then according to Lemma 4.1, then the rank of Jacobian matrix of the DAE on whole component can be calculated by singular value decomposition (SVD) given by Algorithm 3.

Suppose a prolonged system $Z_R(F^{(c)})$ has constant rank i.e.

$$\text{rank} J = r = \text{rank} J[1 : r, 1 : r] < n$$

over a smooth component $C$ of $Z_R(F^{(c-1)})$. To simplify our description, we specify the full rank submatrix to be $J[1 : r, 1 : r]$, which always can be done by proper permutations of variables and equations given by Algorithm 4.

**Definition 4.1. Index Reduction by Embedding (IRE):** Suppose $(c, d)$ is the optimal solution of Problem (2.8) for a given DAE $F$, and then prolonged DAE $F^{(c)} = \{B_{k_c}, F^{(c-1)}\}$ has constant rank $\text{rank} J = r < n$. Let $s = (x_1^{d_1}, ..., x_r^{d_r})$, $y = (x_{r+1}^{d_{r+1}}, ..., x_n^{d_n})$ and $z = (t, X^{(1)}, ..., X^{(k_d-1)})$, then $B_{k_c} = \{f(s, y, z), g(s, y, z)\}$, where $f(s, y, z) = F_{c_1}^{(1)}, ..., F_{c-1}^{(c-1)}$ and $g(s, y, z) = F_{c-1}^{(c+1)}$, ..., $F_{c}^{(c+n)}$. We can construct $G = \{F^{(aug)}_{c-1}\}$ in which $F^{(aug)}_{c-1} = \{f(s, y, z), f(u, \xi, z), g(u, \xi, z)\}$. Then $F^{(aug)}$ is constructed by the following steps:

1. Introduce $n$ new equations $\hat{F} = \{f(u, \xi, z), g(u, \xi, z)\}$: to replace $s$ in the top block $B_{k_c}$ by $r$ new dependent variables $u = (u_1, ..., u_r)$ respectively, and simultaneously replace $y$ in the top block $B_{k_c}$ by $n - r$ random constants $\xi = (\xi_1, ..., \xi_{n-r})$ respectively.

2. Construct a new square subsystem

$$F^{(aug)} = \{f(s, y, z), \hat{F}\},$$

where $F^{(aug)}$ has $n + r$ equations with $n + r$ leading variables $\{X^{(k_d)}, u\}$ and $X^{(k_d)} = \{s, y\}$.

Since this reduction step introduces a new variable $u$, the corresponding lifting of the consistent initial values must be addressed. One approach to this problem is to solve the new system $F^{(aug)}$ to obtain lifted consistent initial values. But this approach is unnecessary and expensive. According to Definition 4.1, the consistent initial values of the new variables $u$ can simply be taken as the initial values of their replaced variables $s$. Then $\xi$ takes the same initial value as was assigned to $y$. 
where $G - \delta Z$ random constants involved can be arbitrarily ascribed, we easily get By the Constant Rank Embedding Lemma 4.2 and Definition 4.1, since the polynomial system and we lack of such a global solver for analytic systems.

Because the homotopy continuation methods can provide all solutions of a square polynomial system and we lack of such a global solver for analytic systems.

\[ \text{Let } (c, d) = \text{the optimal solution of Problem (2.8) for a given DAE } F. \text{ Let } F^{(c)} = \{B_k, F^{(c-1)}\} \text{ as defined in Equation (2.9). If } F^{(c)} \text{ satisfies (4.1), and } C \text{ is a smooth connected component in } \mathbb{R}^{\sum n + n}, \text{ then} \]

\[ Z_R(F^{(c)}) \cap C = \pi Z_R(G) \cap C \]

where $G = \{F^{aug}, F^{(c-1)}\}$ as defined in Definition 4.1. Moreover, we have $\delta(G) \leq \delta(F) - (n - r)$.

**Proof.** By the Constant Rank Embedding Lemma 4.2 and Definition 4.1, since the random constants involved can be arbitrarily ascribed, we easily get $Z_R(B_k) \cap C = \pi Z_R(F^{aug}) \cap C$. Further, since $F^{(c-1)}$ is common to both $F^{(c)}$ and $G$ we have $Z_R(F^{(c)}) \cap C = \pi Z_R(G) \cap C$.

According to Table 2, we construct a pair $(\tilde{c}, \tilde{d})$:

\[ \tilde{c}_i = \begin{cases} 0, & i = 1, \ldots, r \\ 1, & i = (r + 1), \ldots, (n + r) \end{cases}, \tilde{d}_j = \begin{cases} d_j, & j = 1, \ldots, n \\ 1, & j = (n + 1), \ldots, (n + r) \end{cases} \]

For $1 \leq i \leq r$ and $1 \leq j \leq n$, the signature matrix of $F^{aug}$ is the same as $B_k[1 : r, 1 : n]$, implying that $\sigma_{i,j}(f) \leq d_j - 0 = d_j - \tilde{c}_i$.

For $1 \leq i \leq r$ and $(n + 1) \leq j \leq (n + r)$, $\sigma_{i,j}(f) = -\infty < 1 - 0 = d_j - \tilde{c}_i$.

For $(r + 1) \leq i \leq (n + r)$ and $1 \leq j \leq n$, since $s$ and $y$ in $\tilde{F}$ have been replaced with dummy variables and constants, we have:

\[ \sigma_{i,j}(\tilde{F}) \leq \delta_{i,j}(B_k) - 1 - d_j - 1 = \tilde{d}_j - \tilde{c}_i \]

For $(r + 1) \leq i \leq (n + r)$ and $(n + 1) \leq j \leq (n + r)$, $\sigma_{i,j}(\tilde{F}) \leq 0 = \tilde{d}_j - \tilde{c}_i$.

To sum up, $(\tilde{c}, \tilde{d})$ is a pair of feasible solutions of the ILP (2.8) for $F^{aug}$. Thus,

\[ \delta(F^{aug}) \leq \sum_{j=1}^{n+r} \tilde{d}_j - \sum_{i=1}^{n+r} \tilde{c}_i = \sum_{j=1}^{n} d_j - (n - r) = \delta(B_k) - (n - r). \]

Obviously, since both $F^{(c)}$ and $G$ have the same block of constraints $F^{(c-1)}$, according to Definition 2.1, it follows that $\delta(G) - \delta(F^{(c)}) = \delta(F^{aug}) - \delta(B_k) \leq -(n - r)$. Finally, $\delta(G) \leq \delta(F^{(c)}) - (n - r) = \delta(F) - (n - r)$, since $\delta(F) = \delta(F^{(c)})$ by Proposition 2.2.

Since rank $\mathcal{F} = r$, restoring regularity is equivalent is some sense to finding $n - r$ hidden constraints by elimination.

**Remark 4.1.** Actually, most of the results in the paper can be generalized to real analytic functions. We only consider polynomially nonlinear DAEs in this paper, because the homotopy continuation methods can provide all solutions of a square polynomial system and we lack of such a global solver for analytic systems.

Although there are more dependent variables in $G$, the computational cost is much lower than explicit symbolic elimination, since $G$ and the corresponding lifted
witness points can be easily constructed. Moreover, in the IRE method, the feasible solution \((\bar{c}, \bar{d})\) given in Equation (4.3) without ILP solving is an optimal solution in all examples in Section 6. Theoretically, Lemma 4.4 below shows that the feasible solution \((\bar{c}, \bar{d})\) is optimal under some reasonable assumptions.

**Lemma 4.4.** Suppose each equation \(F_j\) in the top block \(B_k\) of a DAE \(F\) contains at least one variable \(x_j \in X^{(k_j)}\). If \(F\) is also a perfect match, then \((\bar{c}, \bar{d})\) in Equation (4.3) is an optimal solution and \(\delta(G) = \delta(F) - (n - r)\).

**Proof.** According to the Table 2, since \(f(s, y, z)\) is a part of DAE \(F\), its corresponding \((\bar{c}[1 : r], \bar{d}[1 : n])\) is optimal. If \((\bar{c}, \bar{d})\) is not an optimal solution, then there must be a feasible solution \((c, d)\) satisfies one of the following four cases, such that \((\sum \bar{d} - \sum \bar{c}) \leq \sum (\bar{d} - \sum \bar{c})\). The Lemma is now proved by contradiction.

1. \(e = \bar{c}\) and at least one element in \(d[(n + 1) : (n + r)]\) is 0. It is easy to prove it does not satisfy \(d_j - c_i \geq \sigma_{ij}\).
2. \(d = \bar{d}\) and at least one element in \(c[(r + 1) : (n + r)]\) is more than 1. It also does not satisfy \(d_j - c_i \geq \sigma_{ij}\).
3. Some elements in \(c[(r + 1) : (n + r)]\) are zeros and more elements in \(d[(n + 1) : (n + r)]\) are also zeros. That implies, at least 2 of the highest derivatives of variables in \(u\) only occur in 1 equation. This contradicts the perfect match condition.
4. Some elements in \(c[(r + 1) : (n + r)]\) are > 1, and some elements in \(d[(n + 1) : (n + r)]\) are also > 1, such that \(\sum \bar{d}[(n + 1) : (n + r)] - \sum \bar{c}[(r + 1) : (n + r)] < (r - n)\). Since at least one of \(X^{(k_d)}\) occurs in every equation of \(F(z, \xi, u)\), all elements in \(c[(r + 1) : (n + r)]\) must be \(\leq 1\), contrary to our assumption.

Since \((\bar{c}, \bar{d})\) is an optimal solution of ILP (2.8) for \(F^{aug}\) it follows that, \(\delta(F^{aug}) = \delta(B_k) - (n - r) \Rightarrow \delta(G) = \delta(F) - (n - r)\).

\(\square\)

5. Algorithms. This section provides a global structural differentiation method (Algorithm 7) for solving a polynomially non-linear DAE, based on the IRE method (Algorithm 6) — the key algorithm to restore the regularity and to reduce index. Also, we need to recall some existing subroutines given in Algorithms 1–5.

Algorithm 1 is used to find an optimal solution \((c, d)\) of ILP (2.8) of a DAE \(F\) with variables \(x\), which helps to prolong DAE in a special pattern to reduce its differential index.

Algorithm 2 is used to find a real witness set \(W = \{p_i | i = 1, ..., m\}\) by the homotopy continuation method. Here, the input \(f\) is considered as a polynomial system by taking all derivatives of \(x\) as new variables. For constraints of a DAE, the obtained real witness points can be considered as candidate initial points. Crucially, this algorithm can find all constraint components of a DAE.

Algorithm 3 is the Singular Value Decomposition (SVD). The purpose of this algorithm is to find the numerical rank of Jacobian matrix \(J\) at a real witness point \(p\) with absolute tolerance \(AbsTol\).

Algorithm 4 is a sorting method to find a sub-matrix with constant rank by swapping the equations of the top block \(B_k\), and the highest derivative variables \(X^{k_d}\). The output is a new \(B_k\) whose Jacobian matrix at a given real witness point \(p\) has a full rank sub-matrix \(J(p)[1 : r, 1 : r]\), where \(r\) is determined by Algorithm 3. Firstly, calculate permutation vectors of rows and columns for \(J(p)\) respectively by householder QR (HQR). Then, swap equations and variables according to permutation vectors respectively. Before returning the sorted matrix, we will verify the rank of
Algorithm 5 is a low index DAE solver implemented by one-step projection and one-step prediction. Obviously a low index DAE $F^{(c)}$ can be divided into two parts — constraints $F^{(c-1)}$ and a square ODE $B_k$. Firstly, since an initial value may be not a consistent initial value of the ODE, the initial value point needs to be projected back onto the constraints by Newton iteration to find a nearby consistent initial value point satisfying the constraints. Secondly, an ODE solver, such as the Runge-Kutta method or the Euler method, is used to make a one-step prediction from the previous consistent initial value point. Through step-by-step iteration, the DAE can be solved numerically, where the tolerance can be set as needed.

Algorithm 5.1
1: $(c, d) = \text{Structure}(F, x)$, such as Pryce method see [38].

Algorithm 5.2
1: $W = \{p_i | i = 1, ..., m\} = \text{witness}(f)$, see [57]. // $m$ is the number of real witness points.

Algorithm 5.3
1: $r = \text{Rank}(J(p), \text{AbsTol})$, see Section 2.5 [17].

Algorithm 5.4
Input: the top block equations $B_k$, Jacobian matrix $J$ with witness point $p$, the constant rank $r$, absolute tolerance $AbsTol$
Output: recombination of the top block equations $B_k$
1: function SORT($B_k, J(p), r, AbsTol$)
2: $\text{piv}_{row} = \text{HQR}(J(p), AbsTol)$, see Section 5.2 [17];
3: $\text{piv}_{col} = \text{HQR}(J^T(p), AbsTol)$; // $\text{piv}_{row}$ and $\text{piv}_{col}$ are the permutation vector of rows and columns, respectively;
4: $B_k = B_k[\text{piv}_{row}]$, // swap equations;
5: $B_k = B_k[X_{k_{\text{piv}_{col}}}]$, // swap the highest derivative variables;
6: verify the rank of $J(p)[1 : r, 1 : r]$ by SVD.
7: end function

Algorithm 5.5
Input: low index DAE equations $F^{(c)}$ and dependent variables $x$ with independent variable $t \in [t_0, t_{\text{end}}]$, initial point $p$, absolute tolerance $AbsTol$ and relative tolerance $RelTol$
Output: numerical solutions of DAE $x(t)$
1: function DAESOLVER($F^{(c)}, x, p, [t_0, t_{\text{end}}], AbsTol, RelTol$) //
2: $j = 0$, $x(t_0) = p$, set step $h$ and the maximum number of iterations $N$;
3: while $t_j < t_{\text{end}}$ do
4: $x(t_j) = \text{Newton}(F^{(c-1)}, x(t_j), AbsTol, N)$ // Refinement, see [33];
5: $x(t_{j+1}) = \text{OdeSolver}(B_k, x(t_j), AbsTol, RelTol, h)$ // such as ode45, Euler method, ode15i etc.;
6: $j = j + 1$;
7: end while
8: end function
**Algorithm 6** Index Reduction by Embedding

**Input:** DAE equations $F$ and dependent variables $x$ with independent variable $t \in [t_0, t_{end}]$, real witness point $p$, absolute tolerance $AbsTol$

**Output:** modified DAE new equations $F^{(e)}$ and new real witness point $p$

```plaintext
1: function IRE($F, x, p_i, AbsTol$)
2:  while true do
3:    Structural Analysis: $(c, d) = Structure(F, x)$
4:    $n = \text{length}(c)$, $k_d = \text{max}(d_j)$, $k_c = \text{max} c_i$, $\delta = \sum d_j - \sum c_i$
5:    Construct: $F^{(e)}$, $B_{k_c}$, $J$ by Equation (2.5, 2.9, 2.11)
6:    $r = \text{Rank}(J(p), AbsTol)$
7:    if $r = n$ then
8:      return $F^{(e)}$, $p$
9:    else if $\delta - (n - r) \leq 0$ then
10:       return Error // this DAE does not have a solution.
11:  end if
12:  \{ $f(s, y, z), g(s, y, z)$ \} = SORT($B_{k_c}, J(p), r, AbsTol$)
13:  Introduce $n$ new equations $\tilde{F} = \{ f(s, y, z), g(s, y, z) \}$
14:  Replace $s$ by $u$ in $\tilde{F}$ // refer to Definition 4.1
15:  Replace $y$ by random constants $\xi$ in $\tilde{F}$
16:  Substitute $\{ t_0, p, \xi \}$ into $f(s, y, z)$ to calculate $s$, note as $\hat{u}$
17:  $p \leftarrow (p, \hat{u})$ // corresponding lifting of consistent initial value
18:  $F^{aug} = \{ f(s, y, z), \tilde{F} \}$
19:  $F \leftarrow \{ F^{(e-1)}, F^{aug} \}$, $x \leftarrow (x, u)$ //extend equations and variables
20:  if $F^{aug}$ satisfies Lemma 4.4 then
21:     $c = [0, 1_n]$, $\bar{d} = [\bar{d}, 1_n]$ by Equation (4.3)
22:     $\bar{c} = [0, \sum c_j], \bar{d} = \bar{d}$
23:     Goto 4
24:  end if
25: end while
26: end function
```

**Algorithm 7** Global Structural Differentiation Method

**Input:** DAE equations $F$ and dependent variables $x$ with independent variable $t \in [t_0, t_{end}]$, absolute tolerance $AbsTol$ and relative tolerance $RelTol$

**Output:** numerical solutions of DAE $x^*(t)$

```plaintext
1: Initialization: check the number of equations #eqns and dependent variables #dvars
2: if #eqns $\neq$ #dvars then
3:     return False
4: end if
5: Set $x^*(t) = \{ \}$
6: Structural Analysis: $(c, d) = Structure(F, x)$
7: Construct: prolonged system $F^{(e)}$, Jacobian matrix $J$
8: Find real witness points: $W = \{ p_i = \text{witness}(F^{(e-1)}(t_0)) | i = 1, ..., m \} // m$ is number of real witness points.
9: for $p_i \in P$ do
10:   $\{ F, \tilde{p}_i \} := IRE(F, x, p_i, AbsTol)$
11:   $\hat{x}(t) = \text{DAESOLVER}(F, x, \tilde{p}_i, [t_0, t_{end}], AbsTol, RelTol)$
12:   $x^*(t) = \{ x^*(t), \hat{x}(t)[1, ..., n] \}$
```
6. Examples. In this section, we use five examples. These included three symbolic cancellation examples: transistor amplifier, modified pendulum and ring modulator. Also included are two numerical degeneration examples: Example 2.1 and the bending deformation of a beam.

In a similar manner to that described in [39], we compare several methods on the DAE for the above examples. In particular, we apply the following four methods to the above 5 DAEs: (a) Pryce method, (b) the substitution method, (c) the augmentation method, (d) the IRE method. We use Matlab R2021a for the numerical computations with the error settings AbsTol $= 10^{-6}$ and RelTol $= 10^{-3}$.

6.1. Transistor Amplifier (index-1). First, we discuss a transistor amplifier example existing in electrical network [35]. It’s a linear DAE for the above examples. In particular, we apply the following four methods to the modulator. Also included are two numerical degeneration examples: Example 2.1 with the error settings AbsTol $= 10^{-6}$ method, (d) the IRE method. We use Matlab R2021a for the numerical computations above 5.

Obviously, we still cannot solve the system directly after the Pryce method. Fortunately, as it is a linear DAE, almost all existing improved structural methods can be used to regularize it.

It is easy to get $F = F(c)$ since $c$ is a zero vector. By the IRE method, according to Definition 4.1, we have $s = \{\dot{x}_3, \dot{x}_6, \dot{x}_3, \dot{x}_1, \dot{x}_7\}$, $y = \{\dot{x}_2, \dot{x}_5, \dot{x}_8\}$, $f(s, y, z) = \{F_4, F_5, F_3, F_1, F_7\}$ and $g(s, y, z) = \{F_2, F_5, F_8\}$. Thus, $\hat{F} = \{f(u, \xi, z), g(u, \xi, z)\}$, where $s$ and $y$ are replaced by $(u_1, u_2, u_3, u_4, u_5)$ and some random constants $(\xi_1, \xi_2, \xi_3)$ respectively. Finally, we construct a new top block $F^{aug} = \{f(s, y, z), \hat{F}\}$ of the prolonged DAE where $\hat{F}$ is given below.

$$\hat{F} = \left\{ \begin{array}{l}
C_1 \cdot (u_4 - \xi_1) + (x_1 - U_e) / R_0 = 0 \\
C_2 \cdot u_3 + x_3 / R_0 - f(x_2 - x_3) = 0 \\
C_3 \cdot (u_1 - \xi_2) + x_1 / R_0 - U_b / R_4 + \alpha \cdot f(x_2 - x_3) = 0 \\
C_4 \cdot (u_4 - \nu_2) - x_5 / R_6 + \beta \cdot f(x_5 - x_6) = 0 \\
C_5 \cdot (u_5 - \xi_3) + x_7 / R_8 - U_b / R_8 + \alpha \cdot f(x_5 - x_6) = 0 \\
C_6 \cdot (u_5 - \xi_3) - x_8 / R_9 = 0
\end{array} \right\}$$

After the IRE method, we can directly construct an optimal solution of ilp with $\hat{c} = (0_{1 \times 5}, 1_{1 \times 8})$ and $\hat{d} = (1_{1 \times 8}, 1_{1 \times 5})$ by Lemma 4.4. Actually it is equivalent to the optimal solution $\hat{c} = (0_{1 \times 5}, 1, 1, 0, 1, 0, 1, 0, 1, 0, 1, 1)$ and $\hat{d} = (1_{1 \times 8}, 1, 0, 1, 0, 1, 1)$ calculated by ilp and both give the same optimal value of the new system $\tilde{\delta} = \delta - n + r = 8 - 8 + 5$.

Then we can verify that the determinant of the new Jacobian matrix is a non-zero constant. Furthermore, the IRE method in this example finish the index reduction just by one step, rather than 3 steps repeatedly by the substitution method or the augmentation method shown in section 6.2 [39]. In other words, it shows the IRE method is more efficient for this example.

Specifically, for numerical solution, the initial value of $u(0)$ and $\xi$ are set corresponding to $\tilde{x}(0)$ in Section A.2 of [39], respectively.

6.2. Non-linearly Modified Pendulum (index-3). This nonlinear DAE system consisting of 4 differential equations and 1 algebraic equation, is obtained
After structural analysis, we get the dual optimal solution is $c = (0, 0, 1, 0, 0)$ and $d = (1, \cdots, 1)$, with $\delta = 4$ and $n = 5$. Moreover, the rank of jacobian matrix is $\text{rank} J = r = \text{rank} J[(3, 2, 1, 4), (3, 1, 4, 5)] = 4$. Thus, the constraint is $F(c-1) = \{x_2^2 + x_2^2 \cdot \sin(x_3)^2 - 1 = 0\}$.

From Section 6.1, by the IRE method, let $s = \{\dot{x}_3, \dot{x}_1, \dot{x}_4, \dot{x}_5\}$, $y = \{\dot{x}_2\}$, $f(s, y, z) = \{F_3, F_2, F_1, F_3\}$ and $g(s, y, z) = \{F_5\}$. Then we need to replace $s$ by $\{u_1, u_2, u_3, u_4\}$ and $y$ by a random constant $\xi$ in $\hat{F}$, respectively. Finally, we can get a modified dae $\{F^{(c-1)}, F^{\text{aug}}\}$, in which $F^{\text{aug}} = \{f(s, y, z), \hat{F}\}$.

\[
\begin{aligned}
\hat{F} &= \begin{cases}
  u_3 - x_1 \cdot x_2 \cdot \cos(x_3) &= 0 \\
  u_4 - x_2^2 \cdot \cos(x_3) \cdot \sin(x_3) + g &= 0 \\
  2 \cdot x_1 \cdot u_1 + 2 \cdot x_2 \cdot \xi \cdot \sin(x_3)^2 + 2 \cdot x_2^2 \cdot \sin(x_3) \cdot \cos(x_3) \cdot u_2 &= 0 \\
  \tanh((u_1 - x_4)) &= 0 \\
  \xi \cdot \sin(x_3) + x_2 \cdot u_2 \cdot \cos(x_3) - x_5 &= 0
\end{cases}
\end{aligned}
\]

We can construct a new optimal solution $(\tilde{c}, \tilde{d})$ by ILP for $F^{\text{aug}}$ by Lemma 4.4 directly, which yields $c = (0_1x_4, 0, 0_1x_3)$ and $d = (1_{1x_5}, 1, 1, 0, 0)$ with the same optimal value $\tilde{\delta} = \sum \tilde{d}_j - \sum \tilde{c}_i - \#eqns(F^{(c-1)}) = 9 - 5 - 1 = \delta - n + r = 4 - 5 + 4$.

Unfortunately, the Jacobian matrix of the new top block $F^{\text{aug}}$ is also singular, with $\text{rank} J(F^{\text{aug}}) = \text{rank} J[(1 : 6, 8 : 9), (1 : 3 : 9)] = 8$. Similarly, we need another modification of $F^{\text{aug}}$ by the IRE method. Finally, this dae system has been regularized. The final optimal value is $2 = \delta - 9 + 8$. The numerical results are shown in Figure 3.
INDEX REDUCTION FOR DEGENERATED DAEs

Compared with the one additional equation of the augmentation method, the IRE method in this example will introduce more equations which will affect the efficiency of the numerical solution, although both methods can be successful after two steps of regularization. However, this adverse effect only exists when the Jacobian matrix is very close to being full rank, i.e. \( r = n - 1 \).

6.3. Ring Modulator (index-2). This DAE of index 2, consists of 11 differential equations and 4 algebraic equations and originates from electrical circuit analysis, describing the behavior of a ring modulator. For more details, see [39].

In this DAE, the prolongation order \( c \) is a zero vector, and the highest derivative of variables \( \mathbf{d} = (1, 1, \mathbf{0}_{1 \times 4}, 1_{1 \times 9}) \), which means \( \mathbf{F} = \mathbf{F}^{(c)} \) and \( \delta = 11, n = 15 \). The system’s Jacobian matrix, has constant rank \( \text{rank} J = r = \text{rank} J[(1 : 2, 4 : 15), (1 : 10, 12 : 15)] = 14 \). That means \( f(s, y, z) = \{F_1, F_2, F_4, \ldots, F_{15}\} \), and \( \tilde{F} \) consists of 15 new equations in which the highest derivative of variables \( \{x_1, \ldots, x_{10}, x_{12}, \ldots, x_{15}\} \) are replaced by \( \{u_1, \ldots, u_{14}\} \) respectively, and \( \dot{x}_{11} \) is replaced by a random constant \( \xi \). Further, we can get \( \mathbf{F}^{aug} = \{f(s, y, z), \tilde{F}\} \) with 29 variables and a non-singular Jacobian matrix. By Lemma 4.4, the optimal value of the new DAE is \( \bar{\delta} = \delta - n + r = 11 - 15 + 14 \).

Note that although some highest derivatives of variables are zero, such as the highest derivatives of \( x_3, x_4, x_5, x_6 \), they also need to be replaced. Numerical results are shown in Figure 4.

6.4. Example 2.1 (index-2). As shown in Example 2.1, this DAE is an example of numerical degeneration. The exact solution of this DAE is \( x(t) = C - \cos(t) \)
and \( y(t) = x(t)^2 \). The IRE method is essential to address the difficulties posed by numerical degeneration for this example.

\[
\mathbf{F}_{aug} = \begin{cases}
2 \cdot u_1 \cdot y - \xi \cdot x + 2x \left( \frac{dx}{dt} \right)^2 - \frac{dx}{dt} + \sin(t) = 0 \\
\xi - 2 \cdot u_1 \cdot x - 2 \cdot \left( \frac{dx}{dt} \right)^2 = 0 \\
2y \frac{d^2x}{dt^2} - x \frac{d^2y}{dt^2} + 2x \left( \frac{dx}{dt} \right)^2 - \frac{dx}{dt} + \sin(t) = 0
\end{cases}
\]

After the IRE method, the new Jacobian matrix is

\[
\mathbf{J} = \begin{pmatrix}
4x \cdot x_t - 1 & 0 & 2y \\
-4x_t & 0 & -2x \\
2y & -1 & 0
\end{pmatrix}
\]

It is obvious that the determinant of the new Jacobian matrix will not degenerate to a singular matrix by virtue of the constraints. Numerical results for \( C = 2 \) are shown in Figure 5.

6.5. Analysis of Bending Deformation of Beam (index-2). The specific description is given in Example 1.2. In this example, when the elastic deformation energies of forces are the same, we can set \( \lambda = 1 \). By structural analysis, the optimal solutions is \( c = (0, 2) \) and \( d = (2, 2) \).

This non-linear DAE has two components resulting from its constraints: one component results from \( y_1 = y_2 \), the other component results from \( y_1 = -y_2 \). In detail, two witness points are computed by the Homotopy continuation method [54]
INDEX REDUCTION FOR DEGENERATED DAE

(a) Pryce method  
(b) substitution  
(c) augmentation  
(d) IRE

Fig. 5. Numerical Solution of Example 2.1

where each point has coordinates \((y_1, y_2, \dot{y}_1, \dot{y}_2)\):

\[
\begin{pmatrix}
-0.43092053722 & -0.43092060160 & -0.27565041470 & -0.27565030340 \\
-0.19993949748 & +0.19993723792 & +0.64332968577 & -0.64333747822
\end{pmatrix}
\]

Because the Jacobian matrix of the polynomial constraints is singular here, a large penalty factor should be introduced in order to improve convergence. These witness points are approximate points near the consistent initial value points, which need to be refined by Newton iteration.

By symbolic computation, we can get two exact solutions of above DAE as

\[
y_1(x) = +y_2(x) = C_1 \cdot \sin\left(\frac{\sqrt{2}x}{2}\right) + C_2 \cdot \cos\left(\frac{\sqrt{2}x}{2}\right) - \frac{1}{5} \cdot \sin(x) - \frac{1}{5}
\]

\[
y_1(x) = -y_2(x) = -\frac{1}{5} \cdot (1 - \sin(x))
\]

Here \(C_1\) and \(C_2\) are constants depending on consistent initial conditions. These exact solutions can be used to check the correctness of our numerical solution of the global structural differentiation method.

Obviously, the Jacobian matrix is non-singular for any witness point from the component with \(y_1 = y_2\). This case can be solved directly after applying the Pryce method as shown in Figure 6. On the contrary, for any witness point on the component with \(y_1 = -y_2\), the Jacobian matrix will degenerate to a singular matrix. For this case, we have to construct its equivalent DAE, and its numerical results are shown in Figure 7.
Fig. 6. Numerical Solution of Beam (Nonsingular Component)

Fig. 7. Numerical Solution of Beam (Singular Component)
6.6. Result Analysis. We used MATLAB’s ode15i as the ode solver of Algorithm 5 for numerical solution of the examples in our paper.

By comparing Figures 2, 3, 4, it can be seen that the substitution method and the augmentation method are effective in dealing with symbolic cancellation DAEs, as well as the IRE method. Without index reduction, ode15i only works well when the index is low and its solutions of high index DAEs become unstable.

In Figure 5, although this DAE is low-index, the former three methods all fail at time $t = 3.952847 \times 10^{-4}$, as they cannot detect that the Jacobian matrix has been constrained to be singular. Furthermore, the Homotopy continuation method used in the IRE method helps to detect numerical degeneration by computing a constant rank of Jacobian matrix at witness points.

By the Homotopy continuation method, all possible consistent initial value paths can be tracked, for the two components in the bending deformation of beam (see Figures 6, 7, respectively). Thus, by the IRE method, structural information of each path can be obtained separately, and all real solutions of the DAE can be approximated.

7. Two Types of Challenge DAEs for Structural s. Beside the degradation of the Jacobian matrix in the previous sections, the Pryce method will also fail in dealing with unreduced models. Such models may be due to unreduced descriptions of DAEs in the modeling process, e.g. DAEs with mixed signature matrix or high multiplicity. Next, we will discuss how to apply the IRE method to solve such DAEs.

7.1. Linear Recombination. Here mixed signature matrix means that all rows of the signature matrix are exactly same. To produce this case, the original DAE is multiplied by a non-singular constant matrix. Its structural information is hidden and it causes trouble for the Pryce method. This type of DAE belongs to the case of symbolic cancellation. Obviously, in theory, the IRE method can deal with this kind of case well. However, due to the missing structural information, it is necessary to call the IRE method several times.

Consider example 2.1. Suppose there is a matrix $A = (1,0;1,1)$, and the new DAE is $A \cdot F$. Its structural information by the Pryce method is $c = (0,0)$, $d = (2,2)$. Compared with Section 6.1 where the DAE is missing 1 hidden constraint equation, the new DAE is missing 2 additional hidden constraints caused by structural method failure.

Then we need to make additional calls of the IRE method to find missing hidden constraints. In the first call, the size $n = 2$ and rank $r = 1$, so we can only find $n-r = 1$ additional hidden constraint equations. Then we need a second call, which yields size $n = 3$ and rank $r = 2$. So the remaining $n-r = 1$ additional hidden constraint equation has also been found. However, because the original DAE is missing 1 hidden constraint equation, as in Section 6.1, a third call is necessary. That yields size $n = 5$ and rank $r = 4$, and $n-r = 1$ hidden constraint equation is found, yielding a full rank Jacobian.

Finally, the numerical solution in Figure 8 shows that the structure information of the new DAE by the IRE method is correct and reliable. Furthermore, it is easy to deduce that linear recombination cases can be handled well by IRE method.

7.2. DAEs with High Multiplicity. A simple way to produce this case is that some equations of a DAE are squared. It will also lead to the singularity of Jacobian matrix. The determinant of the Jacobian is nonzero with a factor which is the constraint. This is the case of numerical degeneration, and common factors lead to redundancy.
Actually, due to the existence of redundant equations, Problem (2.8) and Definition 2.1 must be reformulated, which will lead to the invalidation of $\delta(G) \leq \delta(F) - (n - r)$ in Theorem 4.3. But this is beyond the scope of this paper. Instead we will present some interesting observations concerning the IRE method.

Note that for high multiplicity case, Theorem 4.3 can still guarantee the equivalence of DAE before and after application of the IRE method.

Consider Example 2.1, and suppose the constraint equation is replaced by $(y(t) - x^2(t))^2$, whose structural information is $c = (0, 2), D = (2, 2)$. Apparently the number of equations is 4, the number of variables is 6, and the optimal value is $6 - 4 = 2$. But in fact, the rank of its equations is 2. Compared with Section 6.1 where 2.1 is missing 1 hidden constraint equation, the new DAE is missing $4 - 2 = 2$ additional hidden constraint equations.

In a similar manner to Section 7.1, we make multiple calls of IRE method to find hidden constraints of the new DAE. The first call of IRE yields 1 hidden constraint equation of the new DAE, with size $n = 2$ and rank $r = 1$. In the second call, we also found 1 hidden constraint equation of the new DAE, with size $n = 3$ and rank $r = 2$. However, because the original DAE is missing 1 hidden constraint equation, as in Section (6.1), a third call is necessary for the new DAE, and yields size $n = 5$ and rank $r = 4$. Finally, the last 1 hidden constraint equation is found, and the final Jacobian is non-singular. After the structural method is applied, the numerical solution is shown in Figure 9.

In this example, the IRE method can also find $n - r$ hidden constraint equations at once. In other words, IRE method has the possibility of regularizing the structure method, but further research is needed. From the perspective of hidden constraints, we can redefine the optimal value as $\#\text{var} - \text{rank}(F)$, where $\#\text{var}$ is the number of variables. In Theorem 4.3, we conjecture that a more general theorem can be obtained by replacing $\delta(G) \leq \delta(F) - (n - r)$ by finding at least $n - r$ hidden constraints.

8. Conclusions. In this paper, we first gave a framework for improved structural methods in Section 2.2. In Section 4 we proposed an improved structural method — the IRE method — based on witness point techniques described in Section 3.

The IRE method avoids the direct elimination of non-linear DAEs in other
improved structural methods by introducing new variables and equations to increase the dimensions of space in which the DAE resides. The IRE method is efficient and intuitive, and enables the simultaneous regularization of all the equations of a DAE, rather than one specific equation at a time. The more rank deficiency, the higher efficiency, but the scale of the equation will also increase. A strong feature of our approach, is that Homotopy continuation methods can be naturally and efficiently combined with the IRE method, which can help to deal with almost all degeneration issues for DAEs. Unlike the local equivalence methods, such as the substitution method and the augmentation method, the IRE method is proved to be a global equivalence method in Lemma 4.2.

To better demonstrate our methods, we describe specific algorithms in Section 5, and give 5 numerical examples in Section 6. The experimental results are summarized in Table 3, which show that global structural differentiation method based on the IRE
method can deal with symbolic cancellation DAEs and numerically degenerated DAEs whether these DAEs are high-index or not.

In Section 7 two challenging special cases are discussed whose structural information is wrong after application of structural methods. In particular Section 7.1 considers the problem of linear recombinations and a modification of the IRE method is given which addresses this case. Section 7.2 considers the problem of DAEs with high multiplicity. For this case, the IRE method still works on examples, but the approach lacks a theoretical justification, which is a problem for future research.

Although the IRE method performs well, it may fail when dealing with DAEs with transcendental equations or strong non-linearity in applications. This is due to the limitation of the Homotopy method in solving the constraints of these DAEs in which we cannot find witness points on every component. Indeed such problems may have infinitely many components, unlike polynomially nonlinear DAE. Here, if a consistent initial point can be obtained by numerical iteration, the global structural differentiation method based on the IRE method can still give some solutions of the DAE.

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REFERENCES

[1] F. Awawdeh, H. Jaradat, and O. Alsayyed, Solving System of DAEs by Homotopy Analysis Method, Chaos, Solitons and Fractals, 42 (2009), pp. 1422–1427.
[2] D. J. Bates, A. J. Sommese, J. D. Hauenstein, and C. W. Wampler, Numerically Solving Polynomial Systems with Bertini, Society for Industrial and Applied Mathematics, Philadelphia, PA, 2013.
[3] K. E. Brenan, S. L. Campbell, and L. R. Petzold, Numerical Solution of Initial-Value Problems in Differential-Algebraic Equations, Society for Industrial and Applied Mathematics, 1995.
[4] P. N. Brown, A. C. Hindmarsh, and L. R. Petzold, Consistent initial condition calculation for differential-algebraic systems, SIAM Journal on Scientific Computing, 19 (1998), pp. 1495–1512.
[5] G. E. Buchberger, B. Collins and R. Loos, Computer Algebra: Symbolic and Algebraic Computation, Springer-Verlag Vienna, Vienna, 1982.
[6] B. Caillaud, M. Malandain, and J. Thibault, Demo: IsamDAE, an Implicit Structural Analysis Tool for Multimode DAE Systems. HSCC 2020 - 23rd ACM International Conference on Hybrid Systems: Computation and Control, Apr 2020.
[7] S. L. Campbell, Least squares completions for nonlinear differential algebraic equations, Numer. Math., 65 (1993), p. 77–94.
[8] S. L. Campbell, High-Index Differential Algebraic Equations, Mechanics of Structures and Machines, 23 (1995), pp. 199–222.
[9] S. L. Campbell and C. W. Gear, The Index of General Nonlinear DAEs, Numerische Mathematik, 72 (1995), pp. 173–196.
[10] D. O. David A. Cox, John Little, Ideals, Varieties, and Algorithms, Springer-Verlag Vienna, 2007.
[11] G. W. Ernst Hairer, Solving Ordinary Differential Equations II, vol. 14, Springer-Verlag, Berlin Heidelberg, 1991.
[12] P. Fritzson, Principles of Object Oriented Modeling and Simulation with Modelica 3.3 (A Cyber-Physical Approach), Wiley-IEEE Press, Hoboken, 2014, ch. 17, pp. 977–991.
[13] C. W. Gear, Differential-Algebraic Equation Index Transformations, SIAM Journal on Scientific and Statistical Computing, 9 (1988), pp. 39–47.
[14] C. W. Gear and L. R. Petzold, Differential/Algebraic Systems and Matrix Pencils, in Matrix Pencils, B. Kågström and A. Ruhe, eds., Berlin, Heidelberg, 1983, Springer Berlin Heidelberg, pp. 75–89.
[15] K. O. Geddes, S. R. Czapor, and G. Labahn, Algorithms for Computer Algebra, Springer US, Boston, MA, 1992, ch. Gröbner Bases for Polynomial Ideals, pp. 429–471.
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[16] M. Gerdts, Optimal Control of ODEs and DAEs, De Gruyter, 2011.
[17] G. H. Golub and C. F. Van Loan, Matrix Computations (4rd Ed.), Johns Hopkins University Press, USA, 2013.
[18] N. Guzel and M. Bayram, On the Numerical Solution of Differential-Algebraic Equations with Index-3, Applied Mathematics and Computation, 175 (2006), pp. 1320–1331.
[19] J. D. Hauenstein, Numerically Computing Real Points on Algebraic Sets, Acta Applicandae Mathematicae, 125 (2012), pp. 105–119.
[20] J. D. Hauenstein and A. J. Sommese, What is Numerical Algebraic Geometry?, Journal of Symbolic Computation, 79 (2017), pp. 499–507. Numerical Algebraic Geometry.
[21] A. Ilchmann and T. Reis, Surveys in Differential-Algebraic Equations I, Springer, Berlin, Heidelberg, 2013.
[22] S. Iwata, Computing the Maximum Degree of Minors in Matrix Pencils via Combinatorial Relaxation, Algorithmica, 36 (2003), pp. 331–341.
[23] S. G. Krantz and H. R. Parks, Some Questions of Hard Analysis, Birkhäuser Boston, Boston, MA, 2002.
[24] P. Kunkel and V. Mehrmann, Canonical Forms for Linear Differential-Algebraic Equations with Variable Coefficients, Journal of Computational and Applied Mathematics, 56 (1994), pp. 225–251.
[25] P. Kunkel and V. Mehrmann, Differential-Algebraic Equations. Analysis and Numerical Solution, European Mathematical Society, 01 2006.
[26] M. Kuranishi, On É Cartan’s Prolongation Theorem of Exterior Differential Systems, Amer. J. Math, 79 (1957), pp. 1–47.
[27] R. Lamour, R. März, and C. Tischendorf, Differential-Algebraic Equations: A Projector Based Analysis, Springer, Berlin, Heidelberg, 1 ed., 01 2013.
[28] T.-L. Lee, T. Li, and C. Tsai, Hom4ps-2.0: A software package for solving polynomial systems by the polyhedral homotopy continuation method, Computing, 83 (2008), pp. 109–133.
[29] B. Leimkuhler, L. Petzoldt, and C. W. Gear, Approximation Methods for the Consistent Initialization of Differential-Algebraic Equations, SIAM Journal on Numerical Analysis, 28 (1991), pp. 205–226.
[30] C.-S. Liu, Elastoplastic Models and Oscillators Solved by a Lie-group Differential Algebraic Equations Method, International Journal of Non-Linear Mechanics, 69 (2015), pp. 93–108.
[31] H. Liu and Y. Song, Differential Transform Method Applied to High Index Differential-Algebraic Equations, Applied Mathematics and Computation, 184 (2007), pp. 748–753.
[32] R. März, The index of linear differential algebraic equations with properly stated leading terms, Results in Mathematics, 42 (2002), pp. 308–338.
[33] J. H. Mathews and K. K. Fink, Numerical Methods Using Matlab (4th Edition), Pearson, 4 ed., jan 2004.
[34] S. E. Mattsson and G. Söderlind, Index Reduction in Differential-Algebraic Equations Using Dummy Derivatives, SIAM Journal on Scientific Computing, 14 (1993), pp. 677–692.
[35] M. C. K. Mazza F., Test Set for Initial Value Problem Solvers. Department of Mathematics, 2008.
[36] R. McKenzie and J. Pryce, Structural Analysis Based Dummy Derivative Selection for Differential Algebraic Equations, BIT Numerical Mathematics, 57 (2017), pp. 433–462.
[37] K. Muñoz, Computing the Degree of Determinants via Combinatorial Relaxation, SIAM J. Comput., 24 (1995), pp. 765–796.
[38] N. Nedialkov and J. Pryce, Solving Differential Algebraic Equations by Taylor Series (III): the DAETS Code, European Society of Computational Methods in Sciences and Engineering (ESCMSE) Journal of Numerical Analysis, Industrial and Applied Mathematics, 3 (2008), pp. 61–80.
[39] T. Ollivier, Improved structural methods for nonlinear differential-algebraic equations via combinatorial relaxation, CoRR, abs/1907.04511 (2019).
[40] F. Ollivier, Jacobi’s Bound and Normal Forms Computations. A Historicial Survey, 2009.
[41] C. C. Pantelides, The consistent initialization of differential-algebraic systems, SIAM Journal on Scientific and Statistical Computing, 9 (1988), pp. 213–231.
[42] J. D. Pryce, Solving High-index DAEs by Taylor Series, Numerical Algorithms, 19 (1998), pp. 195–211.
[43] J. D. Pryce, A simple structural analysis method for daes, BIT Numerical Mathematics, 41 (2001), pp. 364–394.
[44] C. Rans and S. T. D. Freitas, Bending Deflection - Differential Equation Method. Aerospace Structures and Materials, 2016.
[45] G. J. Reid, P. Lin, and A. D. Wittkopf, Differential Elimination - Completion Algorithms
for DAE and PDAE, Studies in Applied Mathematics, 106 (2001).

[46] W. Seiler, *Involution - The Formal Theory of Differential Equations and its Applications in Computer Algebra*, vol. 24 of Algorithms and Computation in Mathematics, Springer, Berlin, Heidelberg, 01 2010.

[47] L. Shampine, *Solving $y'=f(t,y(t),y'(t))$ in Matlab*, Journal of Numerical Mathematics, 10 (2002), pp. 291–310.

[48] L. M. Skvortsov, *Runge-Kutta Collocation Methods for Differential-Algebraic Equations of Indices 2 and 3*, Computational Mathematics and Mathematical Physics, 52 (2012), pp. 1373–1383.

[49] A. Sommese and C. Wampler, *The Numerical Solution of Systems of Polynomials Arising in Engineering and Science*, World Scientific Pub Co Inc, 03 2005.

[50] A. J. Sommese, J. Verschelde, and C. W. Wampler, *Solving Polynomial Equations: Foundations, Algorithms, and Applications*, Springer Berlin Heidelberg, Berlin, Heidelberg, 2005, ch. Introduction to Numerical Algebraic Geometry, pp. 301–337.

[51] G. Tan, N. Nedialkov, and J. Pryce, *Conversion Methods for Improving Structural Analysis of Differential-Algebraic Equation Systems*, BIT Numerical Mathematics, 57 (2017), pp. 845–865.

[52] R. Vieira and E. Biscailza, *Direct Methods for Consistent Initialization of DAE Systems*, Computers and Chemical Engineering, 25 (2001), pp. 1299–1311.

[53] C. von Tischendorf, *Topological Index Calculation of DAEs in Circuit Simulation*, ZAMM - Journal of Applied Mathematics and Mechanics / Zeitschrift für Angewandte Mathematik und Mechanik, 78 (1998).

[54] Y. Wang, W. Wu, and B. Xia, *A Special Homotopy Continuation Method for a Class of Polynomial Systems*, in Computer Algebra in Scientific Computing, V. P. Gerdt, W. Koepf, W. M. Seiler, and E. V. Vorozhtsov, eds., Cham, 2017, Springer International Publishing, pp. 362–376.

[55] W. Wu, C. Chen, and G. Reid, *Penalty Function Based Critical Point Approach to Compute Real Witness Solution Points of Polynomial Systems*, in Computer Algebra in Scientific Computing, V. P. Gerdt, W. Koepf, W. M. Seiler, and E. V. Vorozhtsov, eds., Cham, 2017, Springer International Publishing, pp. 377–391.

[56] W. Wu and G. Reid, *Finding Points on Real Solution Components and Applications to Differential Polynomial Systems*, in Proceedings of the 38th International Symposium on Symbolic and Algebraic Computation, ISSAC ’13, New York, NY, USA, 2013, Association for Computing Machinery, pp. 339–346.

[57] W. Wu, G. Reid, and Y. Feng, *Computing Real Witness Points of Positive Dimensional Polynomial Systems*, Theoretical Computer Science, 681 (2017), pp. 217–231. Symbolic Numeric Computation.

[58] W. Wu, G. Reid, and S. Ilie, *Implicit Riquier Bases for PDAE and Their Semi-discretizations*, Journal of Symbolic Computation, 44 (2009), pp. 923–941. International Symposium on Symbolic and Algebraic Computation.

[59] X. Wu, Y. Zeng, and J. Cao, *The Application of the Combinatorial Relaxation Theory on the Structural Index Reduction of DAE*, in 2013 12th International Symposium on Distributed Computing and Applications to Business, Engineering Science, 2013, pp. 162–166.

[60] R. Zolfaghari, J. Taylor, and R. J. Spiteri, *Structural analysis of integro-differential-algebraic equations*, Journal of Computational and Applied Mathematics, 394 (2021), p. 113568.