Freely Falling 2-Surfaces and the Quasi-Local Energy

Keita Ikumi
Department of Physics, The University of Tokyo, Tokyo 113, Japan

Tetsuya Shiromizu
Department of Physics, The University of Tokyo, Tokyo 113, Japan

and
Research Center for the Early Universe (RESCEU),
The University of Tokyo, Tokyo 113, Japan

We derive an expression for effective gravitational mass for any closed spacelike 2-surface. This effective gravitational energy is defined directly through the geometrical quantity of the freely falling 2-surface and thus is well adapted to intuitive expectation that the gravitational mass should be determined by the motion of test body moving freely in gravitational field. We find that this effective gravitational mass has reasonable positive value for a small sphere in the non-vacuum space-times and can be negative for vacuum case. Further, this effective gravitational energy is compared with the quasi-local energy based on the $(2 + 2)$ formalism of the General Relativity. Although some gauge freedoms exist, analytic expressions of the quasi-local energy for vacuum cases are same as the effective gravitational mass. Especially, we see that the contribution from the cosmological constant is the same in general cases.

I. INTRODUCTION

One frequently wants to define the local energy in order to investigate the local structure of the dynamical space-times. However, it is well known that, due to the equivalence principle, the gravitational field does not have local (point-wise) energy density in General Relativity. Hence, it might be impossible to construct the combined energy density of gravity and matter in a purely local manner.

Fortunately, for asymptotically flat spacetimes, it was shown that the notion of the total energy for the whole 3-space exists and that one can consistently define the total energy at spatial and null infinity: the ADM energy $E_{\text{ADM}}$ and the Bondi-Sachs energy $E_{\text{BS}}$, respectively. They have several nice features. They are defined in entirely covariant ways. The positivity of these energy related to the stability of spacetimes has been proven by [5,6]. The relation between $E_{\text{ADM}}$ and $E_{\text{BS}}$ is revealed in Ref. [7].

In the case of asymptotically de Sitter spacetimes, one can define the Abbott-Deser energy [8]. This is an integral of the conserved charge and has the nature such as the total energy of the whole 3-space. In spherical cases, Nakao, Shiromizu and Maeda [9] showed that the Abbott-Deser energy picks up correctly the gravitational mass, which determines the tidal force. Further, they found an example in which the Abbott-Deser energy is negative. One of the present authors showed that it is positive in cases corresponding with ‘static’ like spacetimes [10].

The above success of these total energies in spacetimes with different asymptotic structure urges again people to construct the local notion of the gravitational energy, that is, one want to define the useful total energy of compact regions which does not depend on the asymptotic structure of spacetimes [11].

In the present paper, we consider the motion of a freely falling 2-surface and then define the effective gravitational mass for that surface. Then we evaluate it for small spheres in non-vacuum space-times and in Schwarzschild space-time. Furthermore, we show that it resembles the quasi-local energy derived from the total Hamiltonian of the $(2 + 2)$ formalism.

*Present Address: DAMTP, Univ. of Cambridge, Silver Street, Cambridge 3DN 9EW, UK
II. GRAVITATIONAL MASS AND FREELY FALLING 2-SURFACE

In the Newtonian theory of gravity, the motion of a test particle with vanishing angular momentum is determined by
\[ \ddot{r} = -\frac{\mathcal{M}}{r^2}. \] (2.1)
This equation relates the gravitational mass \( \mathcal{M} \) with the time evolution of a geometrical quantity \( \ddot{r} \). We seek for a similar relation in the context of General Relativity. In the following, we first establish the mass-geometry relation in terms of the quantities associated with freely falling 2-surface in Newtonian theory and then extend that relation to the case of General Relativity.

A. Effective mass in Newtonian theory

We consider an arbitrary 2-surface \( S \) in the Newtonian absolute spacetime. Imagine that \( S \) is entirely covered with freely infalling test particles. Denote the tangent vector of the world line of each test particle by \( t^a \) and define the 3-velocity \( v^a \) by \( t^a = (\partial/\partial t)^a + v^a \), where \( t \) is the absolute time. Write the flat metric of the absolute space as \( g_{ab} \) and the natural derivative operator of the spacetime as \( \nabla_a \). Then the equation of motion takes the form
\[ t^a \nabla_a v^b = -g^{ab} \nabla_a \Phi, \]
where \( \Phi \) is the gravitational potential which satisfies the Poisson equation
\[ g^{ab} \nabla_a \nabla_b \Phi = 4\pi \rho, \] (2.2)
where \( \rho \) is the matter density.

Denote the projection tensor onto \( S \) by \( h^a_b \), where \( h_{ab} \equiv h^c_a h_{cb} \) is the metric of \( S \). These metrics satisfy \( g^{ab} = h^{ab} + n^a n^b \) where \( n^a \) is the normal vector of \( S \) pointing outwardly and \( h^{ab} \equiv g^{ac} h^c_b \). Define \( \theta \) by \( h^c_a h^d_b \nabla_c \mu_{cd} = \theta^{\mu_{ab}} \) where \( \mu_{ab} \) is the area 2-form of \( S \). Then \( \dot{\theta} = h^a_b \nabla_a t^b = h^a_b \nabla_b \mu_{ab} \) gives the evolution rate of the infinitesimal area element \( \delta A \) which consists of some fixed members of the test particles: \( \dot{\theta} = t^a \nabla_a (\delta A)/\delta A \). \( \dot{\theta} \) is also the expansion of the congruence of the test particles. Next, introduce the shear \( \sigma_{ab} \equiv 2 h^c_a h^d_b \nabla_c (\mu_{cd}) - \dot{\theta} h_{ab} \) and the rotation \( \omega_{ab} \equiv 2 h^c_a h^d_b \nabla_c (\mu_{cd}) \) of the congruence. Then we can introduce time evolution of \( t^b \nabla_b \mu_{ab} \) as follows.

Since \( t^b \nabla_b \mu_{ab} \) gives zero when contracted with \( (\partial/\partial t)^a \) and \( n^a \), it has only components parallel to \( S \). Take an arbitrary vector field \( X^a \) tangent \( S \). Then we have
\[ X^a t^b \nabla_b \mu_{ab} = -(h^b_c \nabla_b X^a) \mu_{ab} \]
\[ = -(\nabla_c X^a + X^b \nabla_b t^a) \mu_{ab} \]
\[ = -X^b \nabla_b t^a. \] (2.3)
The last equality holds because \( S \) is Lie-propagated along \( t^a \), which is assumed implicitly in the above construction. Since \( X^a \) is arbitrary other than \( X^a \) should be tangent to \( S \), we have the relation \( t^b \nabla_b \mu_{ab} = -h^b_a \nabla_a v^b \). Now we can obtain the time evolution of \( \dot{\theta} \) as follows.
\[ t^c \nabla_c \dot{\theta} = t^c \nabla_c (h^b_c \nabla_a v^b) \]
\[ = (t^c \nabla_c h^b_c) \nabla_a v^b + h^b_c t^c \nabla_c \nabla_a v^b \]
\[ = t^c \nabla_c (g^b_c - n_b n^a) \nabla_a v^b + h^b_c \left( \nabla_a (t^c \nabla_c v^b) - (\nabla_a t^c)(\nabla_c v^b) \right) \]
\[ n_d(h_b^a \nabla_c v^d)n^a \nabla_a v^b + n_b n_d(h^{ac} \nabla_c v^d) \nabla_a v^b \\
+ b^a_b(-\nabla_a (g^{bc} \nabla_c \Phi) - (\nabla_a v^c)(\nabla_c v^b)) \\
= -h_b^a h_d^c(\nabla_a v^d)(\nabla_c v^b) + h^{ab}(n_c \nabla_a v^c)(n_d \nabla_b v^d) \\
- h_b^a \nabla_a (h^{bc} \nabla_c \Phi + n^b n^c \nabla_c \Phi) \\
= -\frac{1}{4} \left( \dot{\theta} h_b^a + \dot{\sigma}_b^a + \ddot{\omega}_b^a \right) (\ddot{h}_b^a + \ddot{\sigma}_b^a + \ddot{\omega}_b^a) \\
+ h^{ab}(n_c \nabla_a v^c)(n_d \nabla_b v^d) - D_a(h^{ab} D_b \Phi) - h_b^a(\nabla_a n^b) n^c \nabla_c \Phi. \] (2.4)

Thus we have an equation

\[ t^a \nabla_a \dot{\theta} + \frac{1}{2} \dot{\theta}^2 + \frac{1}{4} \dot{\sigma}_{ab} \dot{\sigma}^{ab} - \frac{1}{4} \ddot{\omega}_{ab} \ddot{\omega}^{ab} = -\frac{\theta n^a \nabla_a \Phi - D_a(h^{ab} D_b \Phi) + h^{ab}(n_c \nabla_a v^c)(n_d \nabla_b v^d)}, \] (2.5)

where \( \theta \equiv h^{ab} \nabla_a n_b \) is the trace of the extrinsic curvature of \( S \) in the Euclid space. This equation resembles the Raychaudhuri’s equation. Integrating the above equation over the closed 2-surface \( S \), we have

\[ \int_S \mu \left( t^a \nabla_a \dot{\theta} + \frac{1}{2} \dot{\theta}^2 + \frac{1}{4} \dot{\sigma}_{ab} \dot{\sigma}^{ab} - \frac{1}{4} \ddot{\omega}_{ab} \ddot{\omega}^{ab} \right) = \int_S \mu \left[ -\frac{\theta n^a \nabla_a \Phi + h^{ab}(n_c \nabla_a v^c)(n_d \nabla_b v^d) \right]. \] (2.6)

The first term on the right hand side basically gives the material mass inside \( S \) and the second term corresponds to the contribution of centrifugal force. This is most easily seen for the initial configuration in which \( S \) is a sphere of constant radius \( r \) and \( v^a \) is tangent to \( S \). For such surface, \( \theta = 2/r \) and \( n_c h^{ab} \nabla_b v^c = -v^c h^{ab} \nabla_b n_c = -v^a/r \). Thus the value of the right hand side of (2.4) is, considering eq. (2.2),

\[ \int_S \mu \left( -\frac{2}{r} n^a \nabla_a \Phi + \frac{v^2}{r^2} \right) = 8\pi \left( -\frac{M}{r} + \frac{l^2}{2r^2} \right), \]

where \( M \equiv \int_V dV \rho \) with \( V \) being the region inside \( S \) and \( l \equiv (\int_S \mu v^2/4\pi)^{1/2} \) is the averaged angular momentum of the test particles of unit mass. This is identical to (up to a numerical factor 8\( \pi \)) an effective potential of the test particles with the angular momentum \( l \).

From these considerations, in general cases, we can define the effective mass for general closed surface \( S \) by

\[ M_{\text{eff}}(S) \equiv -\left( \frac{A}{4\pi} \right)^{\frac{1}{2}} \int_S \frac{\mu}{8\pi} \left( t^a \nabla_a \dot{\theta} + \frac{1}{2} \dot{\theta}^2 + \frac{1}{4} \dot{\sigma}_{ab} \dot{\sigma}^{ab} - \frac{1}{4} \ddot{\omega}_{ab} \ddot{\omega}^{ab} \right), \]

\[ = -\left( \frac{A}{4\pi} \right)^{\frac{1}{2}} \int_S \frac{\mu}{8\pi} \left[ -\frac{\theta n^a \nabla_a \Phi + h^{ab}(n_c \nabla_a v^c)(n_d \nabla_b v^d) \right]. \] (2.7)

**B. Effective mass in General Relativity**

Now consider a freely falling 2-surface in General Relativity. Let \( t^a \) be a unit timelike vector field orthogonal to \( S \). Extend each \( t^a \) to be geodesic along its direction. As a result we have a two-dimensional geodesic congruence which starts from \( S \). Denote the metric of the 2-surface which is Lie-propagated along \( t^a \) by \( h_{ab} \). Then \( t^a \) is always orthogonal to \( S \). We define the expansion of the congruence as \( \dot{\theta} \equiv h^{ab} \nabla_a t_b \). The expansion \( \dot{\theta} \) again satisfies \( h_a^c h_b^d \mathcal{L}_{t^c} \epsilon_{ac} = \dot{\theta} \epsilon_{ac} \) and equals to the evolution rate of the infinitesimal area element which is spanned by some fixed members of the congruence: \( \dot{\theta} = t^a \nabla_a (\delta A)/\delta A \). In this case the rotation \( \dot{\omega}_{ab} = 2 h_a^c h_b^d \nabla_c t^d - \dot{\theta} h_{ab} \), without symmetrization over the indices \( c \) and \( d \).

We can show, by the similar argument to the one in the Newtonian case, that the evolution of the unit spacelike vector \( n^a \) orthogonal to both \( h_{ab} \) and \( t^a \) is given by the equation \( t^c \nabla_c n^a = -n_b h^{ab} \nabla_c t^b \). Then the evolution of \( \theta \) is given by

\[
\dot{t}^c \nabla_c \dot{\theta} = t^c \nabla_c (h_a^b \nabla_a t^b) \\
= (t^c \nabla_c h_b^a) \nabla_a t^b + h_a^b t^c \nabla_c \nabla_a t^b \\
= t^c \nabla_c (g_b^a + h_b^a t^a - n_b n^a) \nabla_a t^b + h_a^b t^c (\nabla_c \nabla_a t^b + \nabla_a \nabla_c t^b)
\]
\[ n_a \left( h_b^a \nabla_c t^d \right) n^a \nabla_a t^b + n_b n_d \left( h^{ac} \nabla_c t^d \right) \nabla_a t^b + h_b^a \left( t^e R_{dca}^b t^d - \left( \nabla_a t^e \right) \left( \nabla_c t^b \right) \right) = -h_b^a h_d^c \left( \nabla_c t^d \right) \left( \nabla_a t^b \right) + h^{ab} \left( n_c \nabla_a t^c \right) \left( n_d \nabla_b t^d \right) - h^{ab} R_{abcd} t^c t^d. \quad (2.8) \]

The curvature term \( h^{ab} R_{abcd} t^c t^d \) is related to the parallel component of Weyl tensor as

\[ R_{abcd} h^{ab} t^c t^d = \frac{1}{2} C_{abcd} h^{ab} + R_{ab} t^a t^b - \frac{1}{2} R_{ab} h^{ab} + \frac{3}{2} R, \quad (2.10) \]

so we obtain

\[ t^c \nabla_c \dot{\theta} + \frac{1}{2} \dot{\theta}^2 + \frac{1}{4} \dot{\sigma}_{ab} \dot{\sigma}^{ab} = -\frac{1}{2} C_{abcd} h^{ac} h^{bd} - R_{ab} t^a t^b + \frac{1}{2} R_{ab} h^{ab} - \frac{R}{3} + \omega^a \omega_a, \]

where \( \omega_a = n_a h_b^a \nabla_b t^c \) and it expresses the “centrifugal force component.” This equation should be compared to (2.3). Thus, in the case of General Relativity, from the Newtonian analogy of eq.(2.7), we can define the effective mass for the freely falling 2-surface \( S \) as

\[ M_{\text{eff}}(S) = -\left( \frac{A}{4\pi} \right)^{\frac{1}{2}} \int_S \frac{\mu}{8\pi} \left( t^a \nabla_a \dot{\theta} + \frac{1}{2} \dot{\theta}^2 + \frac{1}{4} \dot{\sigma}_{ab} \dot{\sigma}^{ab} \right) \]

\[ = \left( \frac{A}{4\pi} \right)^{\frac{1}{2}} \int_S \mu \left[ \frac{1}{16\pi} \left( C_{abcd} h^{ac} h^{bd} - 2 \omega_a \omega^a - \frac{4}{3} \Lambda \right) + T_a t^a t^b - \frac{1}{2} T_{ab} h^{ab} + \frac{1}{3} T_a \right], \quad (2.11) \]

where we have used the Einstein equations \( R_{ab} - g_{ab} R/2 = \pi T_{ab} - \Lambda g_{ab} \). We propose the effective mass as the quasi-local energy. We note that the effective mass \( M_{\text{eff}}(S) \) can be defined for any 2-surface \( S \) although the definition is based on the freely falling test particles, because, given any 2-surface \( S \), the above argument can be applied to the sequence of 2-surfaces generated by the motion of freely falling test particles which start off from that particular 2-surface.

One can show that the effective mass \( M_{\text{eff}}(S) \) coincides exactly with the ADM energy and the Bondi-Sachs energy with appropriate limits in asymptotically flat spacetime. Both of the ADM energy and the Bondi-Sachs energy can be expressed as an asymptotic limit of the integral \((\frac{A}{4\pi})^{1/2} \int_S \frac{\mu}{8\pi} C_{abcd} h^{ac} h^{bd}. \) Taking into consideration the asymptotic behaviour of the twist \( \omega_a \omega^a \sim O(r^{-6}) \) \([12]\), the standard falloff conditions of the energy-momentum tensor in the asymptotic region tell us that \( E_{\text{ADM}} \) and \( E_{\text{BS}} \) are given as the appropriate limits of the effective mass:

\[ E_{\text{ADM}} = \lim_{S \to \infty} M_{\text{eff}}(S), \quad E_{\text{BS}} = \lim_{S \to J} M_{\text{eff}}(S). \]

We give the explicit values of \( M_{\text{eff}}(S) \) for various exact solutions with spherical symmetry here. In all cases \( S \) is taken to be a sphere of symmetry in \( t = \) constant surface:

\[ M_{\text{eff}}(S) = 0 \quad (\text{Minkowski}) \]

\[ M \quad (\text{Schwarzschild}) \]

\[ M = -\frac{e^2}{r} \quad (\text{Reissner-Nordström}) \]

\[ M = -\frac{\Lambda}{3} r^3 \quad (\text{Schwarzschild-de Sitter}) \]

\[ M = \frac{4\pi}{3} r^3 (\rho + 3P) - \frac{\Lambda}{3} r^3 \quad (\text{Friedmann-Robertson-Walker}). \quad (2.12) \]

All these values satisfy eq.(2.11) for radial timelike geodesic regarding \( M_{\text{eff}}(S) \) as gravitational mass, as is expected from the above derivation, since \( M_{\text{eff}}(S) \) is defined directly through the behaviour of freely falling 2-surface. Actually, one can see that \( M_{\text{eff}}(S) \) always gives the correct gravitational mass in the above sense in spherically symmetric spacetimes when \( S \) is a symmetric sphere. Proof: In such spacetimes, the area of \( S \) is expressed as \( A = 4\pi r^2 \) and the shear \( \dot{\sigma}_{ab} \) vanishes. In addition, \( \theta \) is constant over \( S \) and equals to \( \dot{\theta}/A = 2\dot{r}/r \). Thus the effective energy is easy to compute and gives \( M_{\text{eff}}(S) = -r^2 \dot{r}. \)

Here we mention two features of \( M_{\text{eff}}(S) \). First, in the Reissner-Nordström (RN) spacetime, it does not coincide with the Misner-Sharp energy \([14]\), which has been widely accepted as the correct quasi-local energy in the spherically
C. The evaluation on small spheres

A few years ago Bergqvist [13] studied the energy of small spheres and showed that Hayward’s energy [12] becomes negative for a small sphere in vacuum case. The effective mass $M_{\text{eff}}$ agrees with the Hayward energy in vacuum spacetimes as shown in the next section, so we investigate the properties of $M_{\text{eff}}$ for small spheres in this subsection.

First, we consider non-vacuum case, that is, $T_{ab} \neq 0$. In this case, as the same way of Bergqvist’s estimation, one can estimate the present effective gravitational mass easily. The leading term is given by

$$M_{\text{eff}}(S) \sim \frac{4\pi}{3} r^3 (T_{ab}t^a t^b + T_{ab}q^{ab}) - \frac{\Lambda}{3} r^3$$

and

$$= \frac{r^3}{3} R_{ab} n^a n^b,$$ (2.14)

where $q_{ab}$ is the metric of the hypersurface orthogonal to $t^a$. If one defines the effective local energy density and pressure by

$$\rho_{\text{eff}} := T_{ab} t^a t^b \quad \text{and} \quad P_{\text{eff}} := \frac{1}{3} T_{ab} q^{ab},$$ (2.15)

the expression of the leading term becomes

$$M_{\text{eff}}(S) \sim \frac{4\pi}{3} r^3 (\rho_{\text{eff}} + 3P_{\text{eff}}) - \frac{\Lambda}{3} r^3.$$ (2.16)

Here note that the pressure term exists. Such term does not exist for Hayward’s and Hawking’s energies which have only the local energy density term [13] [14]. As the pressure can be source of gravity also in general relativity, our result is more reasonable than that for Hayward’s and Hawking’s energies.

Next, we consider a small sphere in the Schwarzschild spacetime for an example of the vacuum case. Here we adopt the isotropic coordinate for Schwarzschild space-time:

$$ds^2 = -\left(1 - \frac{M}{2r}\right)^2 dt^2 + \left(1 + \frac{M}{2r}\right)^4 d\mathbf{x}'^2.$$ (2.17)

Let us consider a small sphere outside the black hole whose center is located at $\mathbf{x}' = \mathbf{a}$. We assume that the sphere has coordinate radius $r = r_0$ in the transformed coordinate $\mathbf{x} = \mathbf{x}' - \mathbf{a}$. In this coordinate, the metric becomes

$$ds^2 = -\left(1 - \frac{M}{2r}\right)^2 dt^2 + \left(1 + \frac{M}{2r}\right)^4 (dr^2 + r^2 d\Omega_2^2)$$ (2.18)

where $r^2 = r^2 + a^2 + 2ar \cos \theta$ and $\cos \theta := a \cdot r / |a||r|$. Here we assume that the initial velocity of the surface is zero, so the “centrifugal force component” $\omega_a = n_c h^k_a \nabla_k u^c = 0$. Thus the effective mass can be expressed as

$$M_{\text{eff}} = \left(\frac{A}{4\pi}\right)^\frac{1}{2} \int_S \frac{dS}{16\pi} \left(C_{abcd} h^{ac} h^{bd} - 2\omega_a \omega^a\right)$$

and

$$= \left(\frac{A}{4\pi}\right)^\frac{1}{2} \int_S \frac{dS}{16\pi} R_{abcd} h^{ac} h^{bd}.$$ (2.19)

Using the extended Gauss-Codacci relation, we have
\[ R_{abcd} h^{ac} h^{bd} = (2) R + \sum_{i=t,r} g^{ii} (K_{iA} B K_{iB} - (K_{iA})^2) \]  

(2.21)

where \( A, B \) run on \( \theta, \phi \) and \( K_{iAB} \) is the second fundamental form of the normal vectors \( \partial_i \) and \( \partial_r \).

By virtue of the Gauss theorem \( \int_S (2) R = 8\pi \), we can arrive at the form

\[ \int \frac{dS}{16\pi} R_{abcd} h^{ac} h^{bd} = \frac{1}{2} - \frac{1}{4} \int_{-1}^1 d(\cos \theta) \left[ 1 - \left( 1 + \frac{M}{2r} \right)^{-1} \frac{M}{r^3} (r + a \cos \theta) \right]^2 \]  

(2.22)

without much effort.

This integral can be evaluated analytically by converting the integral variable from \( \cos \theta \) to \( r' \). In the case \( r = r_0 < a \),

\[ \int \frac{dS}{16\pi} R_{abcd} h^{ac} h^{bd} = \frac{1}{am} (a^2 - r_0^2 + m^2) + \frac{3}{4am^2r_0} (F(m) - F(0)) - \frac{1}{2} + \frac{1}{2am} \frac{(a^2 - r_0^2 - m^2)^2}{(a + m)^2 - r_0^2} \]  

(2.23)

\[ = -\frac{4}{5} \left( \frac{m}{a} \right)^2 \left( 1 + \frac{m}{a} \right)^{-4} \left( \frac{r_0}{a} \right)^4 \left[ 1 + O \left( \left( \frac{r_0}{a} \right)^2 \right) \right] \]  

(2.24)

where \( m = M/2, F(m) := (a^2 - r_0^2 - m^2)(a^2 - r_0^2 + m^2) \log(a + r_0 + m)/(a - r_0 + m) \).

The area \( A \) can be estimated by similar method and

\[ A = 2\pi \int_{-1}^1 d(\cos \theta) \left( 1 + \frac{M}{2r} \right)^{4/3} \]  

(2.25)

\[ = 4\pi r_0^2 \left( 1 + \frac{4m}{a} + \frac{3m^2 \log a + r_0}{a - r_0} + \frac{4m^3}{a(a^2 - r_0^2)} + \frac{m^4}{(a^2 - r_0^2)^2} \right) \]  

(2.26)

\[ = 4\pi r_0^2 \left( 1 + \frac{m}{a} \right)^4 \left[ 1 + O \left( \left( \frac{r_0}{a} \right)^2 \right) \right] \]  

(2.27)

Hence the effective mass is

\[ M_{\text{eff}} = -\frac{1}{5} \left( \frac{M}{a} \right)^2 \left( 1 + \frac{M}{2a} \right)^{-2} \left( \frac{r_0}{a} \right)^4 r_0 \left[ 1 + O \left( \left( \frac{r_0}{a} \right)^2 \right) \right] \]  

(2.28)

\( r_0 < a \)

The negativity means that the effect of the tidal force along the direction of \( a \) which prolongs the sphere dominates over the effect of the tidal force normal to \( a \) which squeezes the sphere. That is, on the whole, a small sphere which does not enclose the central black hole must expand due to the tidal force at the first moment. When one considers a sufficient small sphere in vacuum spacetimes, the gravity is too weak and then the small sphere cannot collapse gravitationally at the first moment. The negativity of the effective mass on the small sphere reflect certainly such kind of reasonable feature.

On the other hand, on the whole, a large sphere which encloses black hole should shrink at the first moment, so the effective mass is expected to positive. Actually we can obtain the exact expression for \( M_{\text{eff}} \) for this case, too. The above integral \( 2.22 \) is valid for large sphere \( r_0 > a \) as long as the sphere does not intersect with the horizon and gives

\[ M_{\text{eff}} = M - \left( 1 + \frac{2M}{r_0} \right) a^2M^2 r_0^{-4} \left[ 1 + O \left( \left( \frac{a}{r_0} \right)^2 \right) \right] \]  

(2.29)

\( r_0 > a \),

which confirms the above expectation.

### III. QUASI-LOCAL ENERGY BASED ON THE (2 + 2) FORMALISM

Since the construction of the effective mass comes from the dynamics of the test 2-surfaces, one can expect that the similar form can be obtained by another procedure in which 2-surfaces is basic tool. In this section, we will show that our quasi-local energy is in fact similar to the quasi-local energy derived from the total Hamiltonian of the (2 + 2) formalism under the special choice of the gauge. The formalism is just the double null foliation of 2-surfaces and then this is a good example for the demonstration.

\[ 6 \]
A. Brief review on the \((2+2)\) formalism

In this sub-section, we review the \((2+2)\) formalism and introduce the Hamiltonian. Broadly, we follow Ref. \[17\]. We take two commutable vector fields \(u^a\) and \(v^a\), and regard them as the evolution vectors. One can take the evolitional direction to be null in the neighbourhood of regular region without loss of generality. This “double null foliation” is assumed hereafter. We take the parameters of \(u^a\) and \(v^a\) to be \(\xi\) and \(\eta\). Then the 2-surfaces \(\{S_{\xi,\eta}\}\), generated by Lie-propagating a fixed two-dimensional spacelike surface \(S\) along \(u^a\) and \(v^a\), serve as a foliation of the spacetime in the neighbourhood of \(S\). We take the origins of the parameters \(\xi\) and \(\eta\) so that \(S_{0,0}\) coincides with \(S\). Since we assumed the double null foliation, we restrict ourselves to the cases such that \(u^a\) and \(v^a\) give null three-surfaces. Define \(h_{ab}\), the induced metric on \(S_{\xi,\eta}\), and \(r_a \equiv h_{ab}u^b\), \(s_a \equiv h_{ab}v^b\) and \(m \equiv -\log(-(u-r)_a(v-s)^a)\). It is easy to see that \(u-r\) and \(v-s\) are the null normal vectors to the foliation. Thus, using these quantities, the metric can be written as

\[ g_{ab} = h_{ab} - e^m((u-r)_a(v-s)_b + (v-s)_a(u-r)_b). \]

The dynamical equations for the system are derived from the variational principle. The Lagrangian \((4)\) is the sum of the Einstein-Hilbert Lagrangian \((4)\)\(\epsilon R/16\pi\) and the matter Lagrangian \((4)\)\(L_m\) and \((4)\)\(\epsilon\) is four-dimensional volume form and \(R\) is the Ricci scalar of the spacetime metric.

It is easily verified that the dynamical equations obtained by extremizing the action integral \((3.1)\) are written as the Euler-Lagrange equations

\[ \frac{\delta L}{\delta u^q} + \frac{\delta L}{\delta v^q} - \frac{\delta L}{\delta q} = 0, \]

where \(q\) denotes the dynamical degrees of freedom such as \(h_{ab}, r^a, s^a, m\) and matter fields.

It is also easy to verify that the Euler-Lagrange equations \((3.2)\) are equivalent to a Hamiltonian system. The Hamiltonian \(H\) is given by

\[ H = pL_u + \hat{p}L_v - L, \]

where \(p\) and \(\hat{p}\) are the canonical momenta defined by

\[ p \equiv \frac{\delta L}{\delta u^q}, \quad \hat{p} \equiv \frac{\delta L}{\delta v^q}. \]

The gravitational part of the Lagrangian which follows from the Einstein-Hilbert Lagrangian with the cosmological constant, after removing the total divergence, becomes

\[ L_G = \mu \frac{e^{-m}}{16\pi} \left[ \mathcal{R} + \frac{e^m}{2} \left( \text{Tr}(L_{u-r}hL_{v-s}h) - \text{Tr}L_{u-r}h\text{Tr}L_{v-s}h ight) \right. \]

\[ \left. + \text{Tr}L_{u-r}hL_{v-s}m + \text{Tr}L_{v-s}hL_{u-r}m \right] + \frac{1}{2}D_aD^a m + 2\omega a \omega^a - 2\Lambda, \]

where \(\mu, \mathcal{R}\) and \(D_a\) are the area two-form, the scalar curvature and the covariant derivative of the 2-surface \(S_{\xi,\eta}\), respectively. We also define the following quantities:

\[ \text{Tr} L_u h \equiv h_{ab}L_u h_{ab} \]

\[ \text{Tr} L_v h \equiv h_{ab}L_v h_{ab} \]

\[ \text{Tr} (L_u h L_v h) \equiv h_{ab}h_{cd}L_u h_{ac}L_v h_{bd} \]

\[ \omega_a \equiv h_{ab} [u-r, v-s]^b \]

\[ 2 \cdot (u-r)(v-s)_c. \]

We note that the “centrifugal force component” \(\omega_a\), introduced in the previous section, coincides with the twist \(\omega_a\) defined here for a particular choice of the foliation. See Appendix A for proof.
B. Quasi-local energies derived from \((2 + 2)\) formalism

In the standard \((3 + 1)\) formalism, the total energy associated with an asymptotically flat spacelike hypersurface is defined as the integral of the Hamiltonian over that hypersurface \[\int_S \mathcal{H}.\] The analogous quantity for the \((2 + 2)\) formalism is the integral of the Hamiltonian \(\mathcal{H}\) over the surface \(S\). We would like to relate the integral \(\int_S \mathcal{H}\) to the quasi-local energy associated with \(S\).

Since the integral \(\int_S \mathcal{H}\) does not have the correct dimension of energy and is dimensionless (in geometrical units \(c = G_N = 1\)), we multiply it with the area radius \(r \equiv (A/4\pi)^{1/2}\) where \(A\) is the area of \(S\). However, the quantity \(r \int \mathcal{H}\) still cannot be viewed as the quasi-local energy of \(S\) without restriction since its value is not a geometrically invariant quantity. Rather, it depends on the choice of the foliation surfaces around \(S\). So we try to fix this ambiguity.

Of the geometrical quantities introduced in section \(\mathrm{III}A\), \(r^a\), \(s^a\) and \(m\) actually represents the coordinate freedom and can be set to zero on any particular surface \(S\). More precisely, for any given spacelike 2-surface, one can always find a double null foliation around \(S\) such that \(r^a = s^a = 0\), \(m = 0\), \(\nabla_a m = 0\) on \(S\) (See Appendix B). Note that they cannot be set to zero throughout the foliation. Under this gauge, for example, the gravitational part of the Hamiltonian \(\mathcal{H}_G\), which is derived from the gravitational part of the Lagrangian \(G\), becomes simple:

\[
\mathcal{H}_G = \frac{\mu}{16\pi} \left[ -R - \frac{1}{2}(\text{Tr} \mathcal{L}_a h \text{Tr} \mathcal{L}_a h - \text{Tr} (\mathcal{L}_a h \mathcal{L}_a h)) + 2\omega_a \omega^a + 2\Lambda \right]
\]

\[
= \frac{\mu}{16\pi} \left( -C_{abcd} h^{ac} h^{bd} - R_{ab} h^{ab} + \frac{R}{3} + 2\omega_a \omega^a + 2\Lambda \right),
\]

where \(C_{abcd}\), \(R_{ab}\) and \(R\) are the four-dimensional Weyl tensor, Ricci tensor and Ricci scalar, respectively.

Now having fixed some gauge, we define the quasi-local energy by the total Hamiltonian as follows:

\[
E(S) \equiv -\left(\frac{A}{4\pi}\right)^{\frac{1}{2}} \int_S \mathcal{H}.
\]

This is an analogue of the ADM energy, which is constructed from the total Hamiltonian in \((3 + 1)\) formalism. Recently Hayward \([12]\) defined his quasi-local energy as:

\[
E_{\text{Hay}}(S) \equiv -\left(\frac{A}{4\pi}\right)^{\frac{1}{2}} \int_S \mathcal{H}_G \bigg|_{\Lambda=0}
\]

\[
= \left(\frac{A}{4\pi}\right)^{\frac{1}{2}} \int_S \frac{\mu}{16\pi}\left(C_{abcd} h^{ac} h^{bd} + R_{ab} h^{ab} - \frac{R}{3} - 2\omega_a \omega^a \right).
\]

The difference with \(E(S)\) is that Hayward has used only the gravitational part of the Hamiltonian without the cosmological constant, not the total one. When there is no matter field and the cosmological constant vanishes, \(E(S)\) reduce to \(E_{\text{Hay}}(S)\) if we impose Ricci flat condition. As the calculation is rather complicated for the cases with matters

\[\text{---}\]

\[\text{---}\]

\[\text{---}\]
in the double null formalism, we concentrate on the vacuum case here. That is, if there is only cosmological constant \( \Lambda \) and no matter fields exist, the quasi-local energy \( E(S) \) is

\[
E(S) = \left( \frac{A}{4\pi} \right)^{\frac{1}{2}} \int_S \frac{\mu}{16\pi} \left( C_{abcd} h^{ac} h^{bd} - 2 \omega_a \omega^a - \frac{4}{3} \Lambda \right).
\]

If one chooses another gauge fixing, one will obtain another form. However, the fact that we could obtain the same form with the effective mass defined in the section 2 is important. As we guessed, one can see that the effective gravitational mass derived from the physical argument of freely falling 2-surface really has a relation with the Hamiltonian energy derived from the \((2 + 2)\) formalism.

The similarity of the expression between \( M_{\text{eff}}(S) \) and \( E(S) \) should be remarked and we have the interpretation such that one can give support \( M_{\text{eff}}(S) \) from the theoretical point of view.

**IV. SUMMARY**

We have defined the quasi-local energy from the concept of the effective gravitational mass for freely falling 2-surface \( S \). Its expression is given by

\[
M_{\text{eff}}(S) = \left( \frac{A}{4\pi} \right)^{\frac{1}{2}} \int_S \mu \left[ \frac{1}{16\pi} \left( C_{abcd} h^{ac} h^{bd} - 2 \omega_a \omega^a - \frac{4}{3} \Lambda \right) + T_{ab} t^a t^b - \frac{1}{2} T_{ab} h^{ab} + \frac{2}{3} T^a_a \right].
\]

It has the advantage that the gravitational mass is related directly with the motion of a body under free fall, so is well adapted to the intuitive physical expectation. It is not obscured by mathematical complication which sometimes covers over the quasi-local energies proposed so far. We have also found that it reduces to the ADM energy and Bondi-Sachs energy at the infinity in the asymptotically flat spacetimes. In spherically symmetric spacetimes, it gives the appropriate gravitational mass for radially infalling test particles. We also found the similarity of the effective mass with the quasi-local energy derived from the total Hamiltonian of the \((2 + 2)\) formalism in the vacuum cases.

Furthermore, we evaluated the effective mass for small spheres. In the non-vacuum case, we obtain the leading term

\[
M_{\text{eff}}(S) \sim \frac{1}{3} R_{ab} t^a t^b
\]

and, in vacuum space-times without \( \Lambda \) term, we observe the effective mass for small sphere outside the black hole is negative in the Schwarzschild spacetime. We discussed that the negativity is reasonable from the view point of the tidal force. Hence, for our effective energy, the negativity is no problem in spite of Bergqvist’s claim. Rather one should prove that the effective mass must have the negative lower bound.

From the construction we expect that the effective mass is useful to investigate the dynamics of the space-time. The application will be considered in the future work.

**ACKNOWLEDGEMENT**

We would like to thank Sean A. Hayward for his important suggestion and discussion. We would also like to thank Katsuhiro Sato, Yasushi Suto, Takahiro T. Nakamura and Gen Uchida for useful comments and discussions.

**APPENDIX A: EQUIVALENCE OF TWO DEFINITIONS OF \( \omega_a \)**

In this appendix we show that two definitions of \( \omega_a \), namely, \( n_c h^b_a \nabla_b t^c \) and eq.(3.3), are equivalent for a particular choice of gauge.

Consider a 1-parameter family of 2-surfaces generated by Lie-propagating the initial 2-surface \( S \) along the free-fall vector \( t^a \). We would like to construct a double null foliation around \( S \) some surfaces of which coincide with the members of this 1-parameter family.

If such foliation exists, each 2-surface of the 1-parameter family is a cross section of two null hypersurfaces generated by null geodesics normal to 2-surfaces on \( \mathcal{L}_u S \) and \( \mathcal{L}_v S \). Conversely, if we generate two null hypersurfaces by null geodesics normal to a member of the 1-parameter family and repeat this procedure for each member of the family, we have the desired double null foliation around \( S \).
Choose two future directed null vector fields \( n^a_+ \) on \( \mathcal{L}_S \) such that \( n^a_+ n_{-a} = -1, (n^a_+ + n^a_-)/\sqrt{2} = t^a \). Then we have \( n^a = (n^a_+ - n^a_-)/\sqrt{2} \). Extend \( n^a_+ \) along its direction by parallel transport: \( n^a_+ \nabla_a n^b_+ = 0 \). Similarly, extend \( n^a_- \) in the same way: \( n^a_- \nabla_a n^b_- = 0 \). Demanding the normalization condition \( n^a_+ n_{-a} = -1 \), we have normalized null normal fields to the double null foliation around \( S \). Define \( t^a \equiv (n^a_+ + n^a_-)/\sqrt{2} \) and \( n^a \equiv (n^a_+ - n^a_-)/\sqrt{2} \) on the entire foliation.

Now we examine the \( \omega_{ij} \) for this foliation. Since \( u - r \) and \( v - s \) in eq.(3.3) are the null normals to the foliation and proportional to \( t^a - n^a \) and \( t^a + n^a \), respectively, it is easy to see that

\[
\omega_{ij} = \frac{h_{ab}}{2} \frac{[t - n, t + n]}{(t - n)^a(t + n)_b} = -\frac{h_{ab}}{2} (t'^c n^c, n^b) = n_a h^b_c n^c t^a t^b + h^b_c n^c t^a t^b.
\]

Thus our aim is to show \( X^a n^b \nabla_{[t,n]} = 0 \) on \( S \) for an arbitrary vector field \( X^a \) tangent to \( S \). Now we have

\[
2X^a n^b \nabla_{[t,n]} = t_a [X, n]^a = (n^a_+ + n^a_-)[X, n_+ - n_-]^a. \tag{A1}
\]

Since \( [X, n_+] = \) is tangent to the null hypersurface \( \mathcal{L}_a S \), it is orthogonal to its normal vector \( n^a_+ \). A similar relation holds for \( n^a_- \), too:

\[
n_{+a}[X, n^+_a] = n_{-a}[X, n^-_a] = 0. \tag{A2}
\]

On \( \mathcal{L}_S \), we also have

\[
[t, X]^a = [n_+ + n_-, X]^a \parallel S.
\]

So, contracting \( n_{+a} \) and \( n_{-a} \) with this expression and using eq.(A2), we have

\[
n_{+a}[X, n^-_a] = 0, n_{-a}[X, n^+_a] = 0. \tag{A3}
\]

Eqs.(A2,A3) tell us the expression (A1) vanishes. This completes the proof.

### APPENDIX B: ACHIEVING THE GAUGE \( R^A = S^A = 0, M = 0, \nabla_A M = 0 \) ON \( S \)

Here we show the existence of the double null foliation satisfying the gauge condition \( n^a = s^a = 0, m = 0, \nabla_a m = 0 \) on \( S \), which was briefly stated in [12]. First we show that there always exists a double null foliation such that \( r^a = s^a = 0 \) on \( S \). Introduce internal coordinates \( (\vartheta, \varphi) \) on \( \mathcal{S}, \eta \) so that the evolution vectors are expressed as partial derivatives \( u^a = (\partial_\vartheta)^a, v^a = (\partial_\varphi)^a \). They lie on the intersection of the 2-surface \( \vartheta, \varphi = \) const and the null hypersurfaces \( \eta = \) const, \( \xi = \) const, respectively. Thus to show the existence of the foliation with \( r^a|S = s^a|S = 0 \), it suffices to show the existence of a coordinate chart \( (\vartheta, \varphi) \) such that each 2-surface \( \vartheta, \varphi = \) const is normal to \( S \), which is obvious.

Modifying this foliation, it is possible to achieve \( m = 0, \nabla_a m = 0 \) on \( S \). (Intuitively, this is obvious since \( m \) represents the ‘density’ of the foliation surfaces.) In general, any two foliations are related by a coordinate transformation \( \xi \rightarrow \xi' = \xi f, \eta \rightarrow \eta' = \eta g \) where \( f, g \) are some smooth functions. The reason why \( \xi \) and \( \eta \) can be factored out in \( \xi' \) and \( \eta' \) is that the null hypersurfaces \( \xi = 0 \) and \( \eta = 0 \) are uniquely determined from \( S \) and do not depend on the choice of the foliation. Since the inverse of the spacetime metric is expressed in the original coordinate as

\[
g^{ab} = -e^m((u - r)^a(v - s)^b + (v - s)^a(u - r)^b) + h^{ij}(\partial_i)^a(\partial_j)^b
\]

where the indexes \( i, j \) run on \{\( \vartheta, \varphi \),

\[
e^m|S = -g^{-1}(d\xi', dh') = fge^m|S
\]

\[
e^m dm|S = e^m(fgdm|S + df|S + dg|S + f\partial_\xi df|S + g\partial_\eta df|S).
\]

The condition \( m'|S = 0 \) is thus equivalent to \( fg|S = e^{-m} \). With this condition satisfied, the condition \( \nabla_a m'|S = 0 \) is equivalent to \( \partial_dm'|S = \partial_\eta m'|S = 0 \) which is further equivalent to \( e^{-m}\partial_\xi m|S + f\partial_\xi g|S + 2g\partial_\xi f|S = 0 \) and \( e^{-m}\partial_\eta m|S + g\partial_\eta f|S + 2f\partial_\eta g|S = 0 \). These conditions only specify the behaviour of \( f, g \) and their first order derivatives on \( S \) and it is not difficult to see that they are compatible with the double null conditions \( g^{-1}(d\xi', d\xi') = g^{-1}(dh', dh') = 0 \). This establishes the existence of the desired foliation.
[1] Arnowitt R, Deser S and Misner C W 1962 \textit{Gravitation: An introduction to Current Research} Witten L (New York: Wiley) p 227
[2] Bondi H, van der Burg M G J and Metzner A W K 1962 \textit{Proc. Roy. Soc. Lond.} \textbf{A269} 21
[3] Sachs R K 1962 \textit{Proc. Roy. Soc. Lond.} \textbf{A270} 103
[4] Ashtekar A and Hansen R O 1978 \textit{J. Math. Phys.} \textbf{19} 1542
[5] Schoen R and Yau S T 1981 \textit{Comm. Math. Phys.} \textbf{97} 231
[6] Witten E 1981 \textit{Comm. Math. Phys.} \textbf{80} 381
[7] Ashtekar A and Magnon-Ashtekar A 1979 \textit{J. Math. Phys.} \textbf{20} 793
[8] Abott L and Deser S 1982 \textit{Nucl. Phys.} \textbf{B195} 76
[9] Nakao K, Shiromizu T and Maeda K 1994 \textit{Class. Quantum Grav.} \textbf{11} 2059
[10] Shiromizu T 1994 \textit{Phys. Rev.} \textbf{D49} 5026
[11] Misner C W and Sharp D H 1964 \textit{Phys. Rev.} \textbf{136} B571; Hawking S W 1968 \textit{J. Math. Phys.} \textbf{9} 598; Bergqvist G 1992 \textit{Class. Quantum Grav.} \textbf{9} 1753; and reference therein.
[12] Hayward S A 1994 \textit{Phys. Rev.} \textbf{D49} 831
[13] Misner C W and Sharp D H 1964 \textit{Phys. Rev.} \textbf{136} B571
[14] Ashtekar A 1980 \textit{General Relativity and Gravitation} vol 2 Held A (New York: Plenum) p 37
[15] Bergqvist G 1994 \textit{Class. Quantum Grav.} \textbf{11} 3013
[16] Horowitz G T and Schmidt B G 1982 \textit{Proc. R. Soc. Lond.} \textbf{A381} 215
[17] Hayward S A 1993 \textit{Class. Quantum Grav.} \textbf{10} 779
[18] Regge T and Teitelboim C 1974 \textit{Ann. Phys.} \textbf{88} 286