On the Maximum Number of Vertices of Minimal Embedded Graphs

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Abstract
Consider graphs embedded in a Riemannian manifold, attached at \( n \) fixed points. Say such a graph is minimal if it is a critical point of the length functional. In this paper, we give a sharp upper bound for the maximal number of vertices of planar 3-regular minimal graphs.

1 Minimal graphs

Definition 1.1 Let \( M \) be a Riemannian manifold and let \( A \) be a finite subset of \( M \). A minimal graph with attaching points \( A \) is a finite embedded graph \( G \) in \( M \) such that the following conditions are satisfied:

I. Each edge of the graph is a geodesic segment

II. Every \( a \in A \) is a vertex of degree 1.

III. The sum of unit vectors of edges outcoming from each vertex of degree greater than 1 is equal to zero.

Minimal graphs are critical points for the length functional on the space of embedded graphs with fixed attaching points. In this paper we are not interested in the length of graphs. The purpose of our study here is the combinatorial structure of minimal graphs.

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There exist minimal graphs with empty attaching set. For example on the canonical two sphere we can have a 3-regular minimal graph with only two vertices. This cannot exist in the plane. For more details about minimal graphs see [7].

In this paper, we will consider only the classes of 3 or 4-regular minimal graphs. We are concerned with estimates on the number of vertices of such graphs with $n$ attaching points for a given number $n$. Using the term ”regular” graph is abusive as all our graphs will have some vertices of degree 1 (attaching points), so in our use, a regular graph is everywhere regular except at the attaching points.

**Notation 1** Let $n \geq 2$, we denote by $f_3(n)$ (resp. $f_4(n)$) the supremum of the numbers of vertices of 3 (resp. 4)-regular minimal graphs on the plane attached to $n$ points.

Here are the two main theorems of this paper.

**Theorem 1** The maximal number of vertices of a 3-regular minimal graph on the plane with $n$ attaching points, $f_3(n)$ satisfies the following equalities

- if $n = 6k$, $f_3(n) = 6k^2 + 6k.$ \hspace{1cm} (1)
- if $n = 6k + 1$, $f_3(n) = 6k^2 + 8k.$ \hspace{1cm} (2)
- if $n = 6k + 2$, $f_3(n) = 6k^2 + 10k + 2.$ \hspace{1cm} (3)
- if $n = 6k + 3$, $f_3(n) = 6k^2 + 12k + 4.$ \hspace{1cm} (4)
- if $n = 6k + 4$, $f_3(n) = 6k^2 + 14k + 6.$ \hspace{1cm} (5)
- if $n = 6k + 5$, $f_3(n) = 6k^2 + 16k + 8.$ \hspace{1cm} (6)

Theorem 1 is sharp. Furthermore, the combinatorial structure of minimal graphs which maximize the number of vertices is unique and will be described in Definition 3.3.
Theorem 2 Let $G$ be a 4-regular minimal graph on the plane with $n$ attaching points. Then $G$ has at most $\binom{n/2}{2} + n$ vertices. In other words

$$f_4(n) = \begin{cases} \binom{n/2}{2} + n & \text{if } n \text{ is even} \\ 0 & \text{otherwise} \end{cases}.$$ 

This is sharp. For each $n$, there is a minimal 4-regular graph which achieves this bound.

Theorem 1 (resp. 2) is proven in sections 2 and 3 (resp. 4). Both proofs are elementary. As paradoxal as it can be, for a variant of this problem, where graphs are allowed to bear densities, there is an opposite result: no bound on the number of vertices. We will give more details in Section 5, together with some open questions.

2 3-regular minimal graphs

In this section we prove Theorem 1, i.e. an upper bound on the number of vertices of 3-regular minimal graphs with $n$ attaching points.

2.1 Preliminaries

Let us give a simple example which illustrates definitions.

Example 2.1 The minimum number of attaching points of such a graph is 2 and for this case we have $f_3(2) = 2$ (the graph consists of only one edge connecting the two attaching points). Things get more interesting for $n = 3$ because in this case we know that there exists some configuration of 3 points on the plane and a 3-regular minimal tree with only one vertex of degree 3 attached to these points. And so $f_3(3) \geq 4$, but can we actually find a minimal graph attached to 3 points with more than 4 vertices?

The answer of this question turns out to be negative but we will have to wait a little before proving this statement.

Let $G$ be a 3 regular minimal graph with $n$ attaching points. By the definition 1.1 we know that the angle between the edges directed from every vertex is equal to 120 degree. This is actually the only geometric restriction put on the graph which will make the estimation of the function $f_3(n)$ easy.

Notation 2 For every graph $G$ we denote by $\#G$ the number of vertices of $G$. 

Lemma 2.1 $f_3(n)$ is an increasing function of $n$.

Proof of Lemma 2.1. Let $G$ be a 3-regular graph attached to $n$ points. We choose one attaching point $x$. As we supposed that all attaching points have degree equal to 1 then there is only one edge $e$ directed from $x$. We add two edges directed from $x$ in a way that the angles between each of them and $e$ are equal to 120 degrees. The new graph $G'$ is 3-regular and minimal with $n + 1$ attaching points, and $\#G' = \#G + 1$. This completes the proof of the lemma.

Definition 2.2 (Cycle, Interior) A cycle $C$ in $G$ is a 2-regular subgraph of $G$. As a cycle is a simple closed curve in the plane, it separates the plane into two components. The bounded one is called the interior of $C$, and denoted by $\text{Int}(C)$.

Definition 2.3 (Ingoing and Outgoing vertices and edges) Let $C$ be a cycle in $G$.

- A vertex $v$ of $C$ is called ingoing if the one edge $e$ at $v$ which does not belong to $C$ is contained in the interior of $C$. $e$ is called an ingoing edge.
- Otherwise we call the vertex $v$ and the edge $e$ outgoing.

We denote by $V_{\text{out}}^C$ the number of outgoing vertices of the cycle $C$, and by $V_{\text{in}}^C$ the number of ingoing vertices of the cycle $C$.

We define a partial order on the set of cycles.

Definition 2.4 (Maximal cycles) Let $C$ and $C'$ be two cycles. We define $C \leq C'$ if $\text{Int}(C) \subseteq \text{Int}(C')$. We call a cycle $C'$ of $G$ maximal if $C'$ is maximal for this partial order.

Lemma 2.2 Let $C$ be a maximal cycle. For every outgoing vertex $v$ of $C$ the outgoing edge attached to $v$ does not belong to any cycle.

Proof of Lemma 2.2. By contradiction. Assume there is an outgoing edge $e$ of $C$ which belongs to a cycle $C'$. The cycles $C$ and $C'$ will have some edges in common. The outer boundary component of $\text{Int}(C) \cup \text{Int}(C')$ is a cycle. This new cycle contradicts the maximality of $C$.

Lemma 2.3 If a graph does not have any maximal cycle then this graph is a tree.

Proof of the Lemma. If the graph does not have any cycle then there is nothing to prove, so we assume that the graph has some cycles. In this case the set of cycles is non empty so it must have a maximal element for the partial order of definition 2.4. And the proof follows.
Lemma 2.4

\[ V^C_{out} - V^C_{in} = 6. \] (7)

Proof of Lemma 2.4

This is a simple consequence of the Gauss-Bonnet theorem. Let us walk along \( C \), keeping the interior of \( C \) on our left hand side. The cycle \( C \) consist of finitely many line segments with exterior angles equal to +60 degrees at outgoing vertices and −60 degrees at ingoing vertices. So

\[ 60V^C_{in} - 60V^C_{out} = 360. \]

Corollary 2.5 Let \( G \) be a minimal graph which is not a tree. Then \( G \) has at least 6 attaching points.

Proof of the Corollary.

By assumption, \( G \) contains a maximal cycle \( C \). Let \( G' = G \) with the edges of \( C \) removed. Then no two outgoing vertices of \( C \) can be connected in \( G' \). Otherwise, let \( v_1, v_2 \) be outgoing vertices of \( C \) connected in \( G' \) by an arc \( \sigma \) of minimal length (over all paths and all pairs \( v_1, v_2 \)). Then \( \sigma \cap C = \{v_1, v_2\} \). Let \( \delta \) be one of the arcs of \( C \) from \( v_1 \) to \( v_2 \). Then \( \sigma \cup \delta \) is a cycle, contradicting maximality of \( C \).

Each connected component of \( G' \) sitting outside \( C \) is minimal with attaching points consisting of a subset of attaching points of \( G \) and exactly one outgoing vertex of \( C \). It must have at least 2 attaching points. Therefore

\[ \sharp \text{att}(G) \geq V^C_{out} = 6 + V^C_{in} \geq 6. \]

where \( \sharp \text{att}(G) \) denotes the number of attaching points of \( G \).

Example 2.5 In example 2.1, we showed that \( f_3(3) \geq 4 \). It is only here, at this stage, that we can give the exact value of \( f_3(3) = 4 \).

Indeed, we saw in the previous two lemmas that a graph with 3 attaching points must be a tree, and then it will have 4 vertices.

Beside this example, for minimal graphs having 4 and 5 attaching points, the optimal minimal graphs maximising the number of vertices have to be trees, hence \( f_3(4) = 6 \) and \( f_3(5) = 8 \). Moreover, the isomorphism classes of such trees are unique. Hence for \( n \leq 5 \) we have a classification of 3-regular minimal graphs maximising the number of vertices.

Lemma 2.6 Let \( F \) be a disjoint union of \( k \) 3-regular trees attached at a total of \( n \) points (i.e. \( F \) has \( n \) vertices of degree 1, all others have degree 3). Then \( \sharp F = 2n - 2k \).

Proof.

For each component \( T \) of \( F \) with \( n_T \) attaching points, \( \sharp T = 2n_T - 2 \). Summing over all components yields \( \sharp F = 2n - 2k \).
2.2 Three types of graphs

Now, let us come back to the estimation of the function $f_3$. There are 3 possibilities for a graph $G$:

I. $G$ is a tree.

II. $G$ has one and only one maximal cycle.

III. $G$ has more than one maximal cycle.

Accordingly the proof of Theorem 1 splits into 3 cases, covered in the following 3 lemmas.

**Lemma 2.7** Let $G$ be a minimal graph attached on $n$ points. If $G$ is a tree, then $\sharp G \leq 2n - 2$.

**Proof.**

The number of vertices of the binary tree attached to $n$ points is equal to $2n - 2$ ($n$ attaching points plus $n - 2$ vertices of degree equal 3). Therefore $\sharp G \leq 2n - 2$.

**Lemma 2.8** Let $G$ be a minimal graph attached on $n$ points. If $G$ has only one maximal cycle then $\sharp G \leq f_3(n - 6) + 2n$. If equality holds, then either $n = 6$ and $G$ is the union of a 6-cycle and its 6 outgoing edges, or $n > 6$ and $G$ is obtained from a minimal graph $G''$ attached at $n - 6$ points by completing its $n - 6$ attaching points into a $2n - 6$-cycle and adding the $n$ outgoing edges of this cycle.

**Proof.**

$G$ has exactly 1 maximal cycle $C$. Let $m'$ denote the number of outgoing vertices of the cycle. From Lemma 7, the number of ingoing vertices is equal to $m' - 6$. Let $n'$ be the number of attaching points outside the cycle. Of course the number of attaching points inside the cycle is equal to $n - n'$.

If outside the cycle $C$, there exists some (non maximal) cycle then the set of cycles outside $C$ must have a maximal element, hence a maximal cycle which is disjoint from the cycle $C$. This contradicts the assumption. Hence outside $C$ the graph is a forest (a disjoint union of trees).

Now let us remove all the edges of $C$, and consider vertices of $C$ as attaching points for the remaining graph $G'$. $G'$ consists of a graph $G''$ whose edges were inside $C$ and of a collection $F$ of trees whose edges were outside $C$. $F$ has $m' + n'$ attaching points.

Each outgoing vertex of $C$ is the root of one of the trees of the forest $F$. Thus the number of components of $F$ is $m'$. From Lemma 2.6 $\sharp F = 2(m' + n') - 2m' = 2n'$.

Each of these trees has at least one attaching point, apart from its root, thus $n' \geq m'$. 

6
$G''$ has at most $n - n' + m' - 6$ attaching points. By the definition of the function $f_3(n)$ we know that the number of the vertices of $G''$ is at most $f_3(n + m' - n' - 6)$. Then

$$
\sharp G \leq f_3(n + m' - n' - 6) + 2n'.
$$

Thus there exists a $k \leq n - 6$ such that

$$
\sharp G \leq f_3(k) + 2n'.
$$

On the other hand, we showed that $f_3(n)$ is nondecreasing. We conclude that

$$
\sharp G \leq f_3(n - 6) + 2n. \quad (8)
$$

Equality implies that $n = n' = m'$ and that $F$ is a disjoint union of $n$ edges. Thus $F$ consists of the outgoing edges of $C$. If $G''$ is nonempty, the attaching points of $G''$ are the ingoing vertices of $C$. Otherwise, $C$ has no ingoing vertices, this means that $C$ is a 6-cycle. This completes the proof of Lemma 2.8.

**Lemma 2.9** Let $G$ be a minimal graph attached on $n$ points. If $G$ has more than one maximal cycle, then $\exists n' \; 6 \leq n' \leq n - 4$ such that

$$
\sharp G \leq f_3(n') + f_3(n - n' + 2). \quad (9)
$$

Proof. We shall use the following terminology.

**Definition 2.6** Let $C$ and $D$ be two maximal cycles in a minimal 3-regular graph. A connecting set for $C$ and $D$ is a triple $(v_C, v_D, \sigma)$ such that $v_C$ (resp. $v_D$) is an outgoing vertex of $C$ (resp. $D$) and $\sigma$ a path which joins these vertices outside $C \cup D$.

Let $C$ and $D$ be two maximal cycles. Let $\sigma$ be a path connecting $C$ to $D$ with a minimum number of edges. Let $e$ be the first edge traversed by $\sigma$. Then $e$ disconnects $C$ from $D$. Indeed, otherwise, there would exist a path connecting $C$ to $D$ away from $e$. Let $\gamma$ be the shortest path in $G \setminus \{e\}$ joining the endpoints of $e$. Then $\gamma \cup e$ is a cycle touching $C$ and thus contradicting maximality of $C$. So cutting the edge $e$ will disconnect $C$ from $D$.

Let $e' \subseteq e$ be a proper interval. Let $G'(\text{resp. } G'')$ be the connected component of $G \setminus \{e\}$ containing $C$ (resp. $D$). Let $n' = \sharp att(G')$. Then $G''$ has $n - n' + 2$ attaching points (the cut through $e'$ produces two extra attaching points). By Lemma 7 we conclude that $n' \geq 6$ and $n - n' + 2 \geq 6$.

And so in final for an $n'$ such that $6 \leq n' \leq n - 4$, we have

$$
\sharp G \leq f_3(n') + f_3(n - n' + 2). \quad (9)
$$

This completes the proof of Lemma 2.9.
2.3 Combining three recursion inequations

To sum up, for every minimal graph $G$ with $n$ attaching points, the number of vertices of $G$ satisfies

either $\#G \leq 2n - 2,$

or $\#G \leq f_3(n - 6) + 2n,$

or $(\exists \ 6 \leq k \leq n - 4) \ \#G \leq f_3(k) + f_3(n - k + 2).$

So $f_3$ satisfies at least one of three recursion inequations.

**Lemma 2.10** Let $f$ be a function on integers. Assume that $f(2) = 2$ and, for every $n \geq 3,$

$$f(n) \leq \max \begin{cases} 2n - 2, \\ f(n - 6) + 2n \\ \max \{ f(k) + f(n - k + 2); \ 6 \leq k \leq n - 4 \} \\ \end{cases} \quad \text{if } n \geq 6,$$

Then $f_3(n) \leq \frac{1}{6}n^2 + n.$ If equality holds for some $n,$ then $f(n) = f(n - 6) + 2n.$

Proof:

We prove this lemma by induction on $n.$

For $n = 2$ the inequality is verified as we assumed $f(2) = 2.$

Let’s suppose that the inequality is verified for every $k \leq n - 1$ and we want to prove it for $k = n.$

If $f(n)$ satisfies the first inequation:

$$f(n) \leq 2n - 2 < \frac{1}{6}n^2 + n.$$

If $f(n)$ satisfies the second inequation:

$$f(n) \leq f(n - 6) + 2n$$

$$= \frac{1}{6}(n - 6)^2 + (n - 6) + 2n$$

$$= \frac{1}{6}n^2 + n.$$

If $f(n)$ satisfies the last inequation, there exists $6 \leq k \leq n - 4$ such that

$$f(n) \leq f(k) + f_3(n - k + 2) \leq \frac{1}{6}k^2 + \frac{1}{6}(n - k + 2)^2 + n + 2$$

$$= \left( \frac{1}{6}n^2 + n \right) + \frac{1}{3}(k^2 - kn + 2n - 2k + 8).$$

8
But as $n \geq 10$ and $6 \leq k \leq (n - 4)$, we have

$$\frac{1}{3}((k - n)(k - 2) + 8) \leq \frac{1}{3}(8 - n) < 0.$$ 

And so

$$f(n) \leq \frac{1}{6}n^2 + n.$$ 

Note that equality can hold only in the second case. This completes the proof of Lemma 2.10, and the

Corollary 2.11

$$f_3(n) \leq \frac{1}{6}n^2 + n$$

And of Theorem 1 except for the equality case which will be discussed in the next section.

3 3-regular minimal graphs which maximise the number of vertices

Theorem 1.1 gives an upper bound for the maximal number of vertices of 3-regular minimal graphs. A natural question to ask is how good is this estimate.

In this section we will study the equality case of the inequations of the last section. For each $n$, we will actually find the combinatorial class of graphs which maximise the number of vertices.

Before characterising these graphs, we need a construction which applies to a class of minimal graphs.

Definition 3.1 (Simple minimal graphs) Let $G$ be a minimal embedded graph in the plane. Say that $G$ is simple if

- Either $G$ is a tree with the following property: it does not contain paths consisting of 5 edges and turning on the same side (like 5 consecutive edges of a convex hexagon).
Forbidden configuration in a simple minimal tree.

- Or $G$ has a unique maximal cycle which surrounds all vertices except attaching points. Furthermore, no two consecutive vertices in the maximal cycle are both ingoing vertices.

Forbidden configuration in the maximal cycle of a simple minimal graph

The pictures of section 6 all feature simple minimal graphs. Note that, on the set of attaching points of a simple minimal graph, there is a natural circular order.

**Definition 3.2 (Padding of a simple minimal graph)** Let $G$ be a simple minimal embedded graph in the plane. The padding of $G$, denoted by $P(G)$ is the minimal graph obtained as follows.

We number the attaching points of $G$ in circular order $a_1, \ldots, a_n$ where $a_{n+1} = a_1$. From each attaching point $a_i$ we draw two half-lines, $\alpha_i^+$ and $\alpha_i^-$ obtained by turning the edge which connects $a_i$ to $G$ by respectively 120 and 240 degrees. Next, one considers the
portion of the cycle between $a_i$ and $a_{i+1}$, and completes this set of edges into an hexagon having two edges carried by $\alpha_i^+$ and $\alpha_{i+1}^-$. For this, one is led to place between 1 and 4 new vertices, depending on the configuration.

- If the angle $\angle(\alpha_i^+, \alpha_{i+1}^-) = \frac{-2\pi}{3}$, we cut the half-lines $\alpha_i^+$ and $\alpha_{i+1}^-$ on their intersection point $b_i$. We obtain two edges making an angle equal to 120 degrees at $b_i$.

![](image1)

Adding one vertex

- If the angle $\angle(\alpha_i^+, \alpha_{i+1}^-) = \frac{-\pi}{3}$, we place on $\alpha_i^+$ (resp $\alpha_{i+1}^-$), two points $b_i^-$ (resp $b_i^+$) such that the vector $b_i^+b_i^-$ makes an angle equal to $\frac{\pi}{3}$ with $\alpha_i^+$ (i.e $\angle(\alpha_i^+, b_i^+b_i^-) = \frac{\pi}{3}$ and $\angle(b_i^+b_i^-, \alpha_{i+1}^-) = \frac{\pi}{3}$).

![](image2)

Adding two vertices

- If the angle $\angle(\alpha_i^+, \alpha_{i+1}^-) = 0$, we add a point $b_i^-$ on $\alpha_i^+$, a point $b_i^+$ on $\alpha_{i+1}^-$ and a point $i$ such that $a_i b_i^- c_i b_i^+ a_{i+1} \gamma_i$ is a hexagon with interior angles all equal to 120 degree and where $\gamma_i$ is the vertex connected to $a_i$ and $a_{i+1}$ by two edges of $G$. 

11
Adding three vertices

• If the angle \( \angle(\alpha_i^+, \alpha_{i+1}^-) = \pi/3 \) (this happens only if \( G \) is the one-edge graph), we place a point \( b_i^- \) on \( \alpha_i^+ \), a point \( b_{i+1}^+ \) on \( \alpha_{i+1}^- \) and points \( c_i^- \), \( c_i^+ \) such that \( a_i b_i^- c_i^- c_i^+ b_{i+1}^+ a_{i+1} \) is a hexagon with interior angles all equal to 120 degree.

Adding four vertices

The edges \( \alpha_i^+ (b_i^+, \gamma_i, b_{i+1}^-) \alpha_{i+1}^- \) form a cycle for which the vertices \( a_i \) are ingoing vertices and the \( b_i, b_i^+, b_{i+1}^-, \gamma_i \) are outgoing vertices. We add to \( G \) these edges with segments (and vertices) attached to each outgoing vertices which will form the attaching points of the new minimal graph \( P(G) \).

The padded graph \( P(G) \) is a simple minimal graph. Indeed, by construction, it has a cycle which surrounds all vertices except attaching points. In this cycle, the \( a_i \)'s are ingoing vertices, and between two consecutive \( a_i \)'s, outgoing vertices (\( b_i^\pm \)'s and \( c_i^\pm \)'s) are inserted. Therefore, the padding operation can be iterated.

3.1 The Graphs \( H_n \)

For each \( n \), we define a graph \( H_n \) which is a subgraph of the standard tiling of the plane by regular hexagons. This definition is by induction on \( n \). We begin by defining these graphs for \( 2 \leq n \leq 7 \). As we saw in the previous section, for \( n \leq 5 \), the minimal graphs
are trees and it is not hard to know that for each \( n \leq 5 \), there exists only one class of isomorphism of a tree which maximises the number of vertices. We denote an element of this class which is a subset of the standard hexagonal tiling by \( H_n \). \( H_2 \) consists of two vertices joined by a single edge.

By convention, \( H_6 \) is a minimal graph consisting of a maximal cycle of length 6 (a hexagon) and the outgoing edges and vertices attached to the hexagon. \( H_7 \) is the minimal graph which consists of adding a minimal tree of length 3 to a vertex of the hexagon of \( H_6 \) (see the first 7 pictures of section 6).

We are now ready to define the family of graphs \( H_n \).

**Definition 3.3** For each \( n \geq 8 \), define \( H_n \) inductively as follows. \( H_n \) is the graph obtained by padding \( H_{n-6} \), i.e.

\[
H_n = P(H_{n-6}).
\]

Section 6 shows the first 19 \( H_n \).

**Remark** For every \( n \geq 6 \), \( H_n \) has \( n \) attaching points and only one maximal cycle of length \( 2n - 6 \). The \( n \) attaching points correspond to the outgoing vertices of the cycle.

Next we enumerate for each \( n \), the number of vertices of \( H_n \).

**Lemma 3.1** Denote by \( N_n \) the number of vertices of \( H_n \). If \( n = 6k + i \), \( 0 \leq i \leq 5 \) and \( n \geq 2 \), \( N_n = \frac{1}{6}(6k)^2 + 6k + 2ik + \epsilon(i) \) where

\[
\epsilon(0) = 0, \quad \epsilon(1) = 0, \quad \epsilon(2) = 2, \quad \epsilon(3) = 4, \quad \epsilon(4) = 6, \quad \epsilon(5) = 8.
\]

**Proof of the Lemma.**
By the recursive definition of \( H_n \), we can easily conclude that

\[
N(n) = N(n-6) + 2n.
\]

This gives the fact that \( \epsilon \) is periodic.

We will show now that the graphs \( H_n \) maximize the number of vertices.

**Lemma 3.2** For every \( n \), \( H_n \) has the maximum number of vertices among all the 3-regular minimal graphs attached to \( n \) points.

**Proof of the Lemma.**
For \( n = 6k, 6k + 2, 6k + 3, 6k + 4 \), by the obvious following calculation

\[
\left( \frac{1}{6}(6k + i)^2 + (6k + i) \right) - \left( \frac{1}{6}(6k)^2 + (6k) \right) - 2ik = i + \frac{i^2}{6},
\]
we deduce $\frac{1}{6}n^2 + n - N(n) < 1$ hence $f_3(n) \leq \lfloor \frac{1}{6}n^2 + n \rfloor \leq N(n)$. Then the Lemma follows from Corollary 2.1.

The difficulty is for the two cases $n = 6k + 1$ and $n = 6k + 5$, when the above calculation shows the possibility of existence of a minimal graph having one vertex more than $H_n$ (see Lemma 3.1). We prove the Lemma for $n = 6k + 1$, for the other case, the argument is the same.

Let suppose that there exists a graph $G$, attached to $n = 6k + 1$ vertices, and having $(6k^2 + 6k) + (2k + 1)$ vertices. It is obvious that $G$ is not a tree. If $G$ has at least two maximal cycles then by Lemma 2.10 we know that there exists $k$ with $6 \leq k \leq n - 4$ such that $f(n) \leq f(k) + f(n - k + 2)$. Following the calculation in the proof of Lemma 2.10 we have

\[
\begin{align*}
f(n) - \left(\frac{1}{6}n^2 + n\right) &\leq \frac{1}{3}((k - n)(k - 2) + 8) \\
&\leq \frac{4}{3}(4 - k) \\
&\leq \frac{8}{3} \\
&\leq -2
\end{align*}
\]

and hence if $G$ has at least two maximal cycles

\[
f(n) \leq \frac{1}{6}n^2 + n - 2.
\]

with $n = 6k + 1$ and $f(n) = 6k^2 + 8k + 1$ we find a contradiction.

Then $G$ has one maximal cycle. To every outgoing vertex of the maximal cycle, there is a tree which is attached. If the tree is not a segment, then we can eliminate two neighbouring edges of the tree and obtain a new graph $G'$ with $6k$ attaching points. The number of vertices of $G'$ is equal to $6k^2 + 8k - 1$. We know that the maximum number of vertices of a minimal graph attached to $6k$ points is equal to $6k^2 + 6k$. but

\[6k^2 + 8k - 1 \geq 6k^2 + 6k\]

and this is not possible. So we deduce that to every outgoing vertex of the maximal cycle of the graph $G$ only a segment can be attached. This is important for us because we can now use induction. Eliminating the outgoing vertices of the maximal cycle of $G$ with the attaching segments (and points) attached to them, we obtain a new graph $G'$ with $6(k - 1) + 1$ attaching points (by Lemma 7) and $f(n) - 12k - 2$ vertices. As

\[f(n) - 12k - 2 = 6k^2 + 8k + 1 - 12k - 2 = 6k^2 - 4k - 1 = \frac{1}{6}(6(k-1)^2) + 6(k-1) + 2(k-1) + 1\]
we can apply the same operation as we did on the graph $G$ to $G'$. Carrying the induction we arrive to the case where $k = 1$, which means a minimal graph attached to 7 points which has 15 vertices. The next Lemma provides the desired contradiction.

**Lemma 3.3** The maximum number of vertices of a 3-regular minimal graph $H_7$ attached to 7 points is equal to 14. Furthermore, every 3-regular minimal graph with 7 attaching points and 14 vertices is isomorphic to $H_7$.

*Proof of the Lemma*

$H_7$ can be presented in the tiled plane by a hexagon with 5 segments attached to 5 vertices of the hexagon and a tree with 4 vertices having one attaching point in the 6th vertex of the hexagon.

We need to prove that this graph has the maximum number of vertices among all 3-regular minimal graphs with 7 attaching points.

Let suppose that there exists a graph $H$ having more vertices than $H_7$. By Corollary 2.5, $H$ has only one maximal cycle and the length of the maximal cycle is equal to 6. Then each vertex of the maximal cycle can be considered as an attaching point for a minimal tree (attached to the vertex). If there exist two vertices of the maximal cycle such that the two minimal trees attached to them are not segments, then the number of attaching points of $H$ will be more than 7 and this is not possible. So to 5 vertices of the maximal cycle are attached 5 segments, and the proof of the Lemma follows.

By Lemma 3.3 we can conclude that a 3-regular minimal graph attached to $n = 6k + 1$ points can’t have $f(n) = 6k^2 + 8k + 1$ vertices and the proof of the lemma follows.

The conclusion of Lemma 3.2 is that for every $n$, $H_n$ maximises the number of vertices for $n$ attaching points. In fact, we can prove more.

**Lemma 3.4** For every $n$, a graph $G$ which maximises the number of vertices among 3-regular minimal graphs attached to $n$ points is combinatorially isomorphic to $H_n$.

*Proof of the Lemma*

As seen before, Lemma 3.5 holds for all $n \leq 7$. For $n \geq 8$ we prove it by induction on $n$ (the proof repeats some arguments in Lemma 3.3). Let $n \geq 8$. Let $G$ be a 3-regular minimal graph attached to $n$ points with $N(n)$ vertices. Copying the proof of Lemma 3.2 we know that $G$ has one maximal cycle. We also know that a segment is attached to every outgoing vertex of the maximal cycle (otherwise by eliminating one edge from a (non trivial) tree attached to an outgoing vertex of the maximal cycle we obtain a graph $G'$ with $n - 1$ attaching points having more than $\frac{1}{6}(n - 1)^2 + (n - 1)$ vertices, which is impossible). Eliminating the outgoing vertices of the maximal cycle and the segments attached to them, we obtain a graph $G'$ with $n - 6$ attaching points. The number of
vertices of $G'$ is equal $N(n - 6)$. Then $G'$ is a minimal graph attached to $n - 6$ points and maximising the number of vertices among all 3-regular minimal graphs attached to $n - 6$ points. By induction hypothesis, $G'$ is isomorphic to $H_{n-6}$. To reconstruct $G$ from $G'$, one must first glue in a cycle whose ingoing vertices are the attaching points of $G'$. This operation is exactly the padding operation of Definition 3.1, hence $P(G')$ is isomorphic to $P(H_{n-6})$ and the proof of the Lemma follows.

4 4-regular minimal graphs

Here we prove Theorem 2. Let $G$ be a 4-regular minimal graph with $n$ attaching points. Then $G$ is made up of line segments intersecting each other in the way that when any two segments intersect at a point (vertex of $G$) there are no other segments passing through the intersecting point. Thus every intersection points will be a vertex of $G$ and the minimality condition is verified. Every line segment joins two of the attaching points. Then the problem of estimating $f_4(n)$ is equivalent to finding the maximum number of intersection points of $n/2$ line segments in the plane such that only two lines pass through the intersecting points.

For $n$ odd it is impossible to attach a minimal graph of degree 4 to $n$ points, and $f_4(n) = 0$ in this case. For $n$ even, the number of intersecting points will not exceed $\binom{n/2}{2}$.

To complete the proof of the theorem, we show that, for every even $n$, there exists a collection of $n/2$ line segments intersecting at exactly $\binom{n/2}{2}$ points. We prove the existence of such a collection by induction on $n$. For $n = 1$ there is nothing to prove and I guess for $n = 2$ my non-born baby could find two lines which intersect at one point in the plane. We suppose that such a collection is constructed for $n$ and we need to add a single line $L$ to this collection such that $L$ does not pass throw the intersection points and such that $L$ intersects all the lines of the collection. As the number of lines and their intersection is finite, it is always possible to add such a line $L$ with required property. From the existence of such a collection the prove of Theorem 2 follows.

5 Remarks and open questions

The problem of estimating the maximum number of vertices of a minimal graph attached to some points in the plane for the case of 3 and 4-regular graphs turned out to be very elementary. We saw that without any difficulties we could even classify maximizing graphs. But the same question for a non necessarily regular graph is violently more
complicated. Indeed for vertices of degree 3, the angle between any two outgoing edges is equal to 120 degree, this simple fact let us have a Gauss-Bonnet type lemma and made our estimates possible. But for degrees greater than 4, there exist infinitely many possible geometric configurations of outgoing edges.

**Notation 3** Let \( \delta \) be a natural number, we denote by \( g_\delta(n) \) the supremum of the number of vertices of minimal graphs with \( n \) attaching points and degree bounded by \( \delta \).

The general questions are

- for \( \delta = 4 \), find a sharper upper bound for \( g_4(n) \).
- for \( \delta > 4 \), is \( g_\delta(n) \) finite?

A first guess is \( g_\delta(n) = f_3(n) \) and that the class of 3-regular graphs have the largest \( g_\delta(n) \) for all the value of \( n \) and among all the minimal graphs with bounded degrees. Indeed, one can imagine that locally every minimal graph with degree greater than 3 can be mapped by a homotopy to a 3-regular minimal graph such that the number of vertices of the image by the homotopy increase. The non-obvious part is that these local homotopies will move the position of the attaching points and that we can’t glue back the local part of the graph correctly together and get a new 3-regular minimal graph. Thus the initial guess may be misleading.

Let us modify the problem by introducing *weights*. We consider finite planar graphs equipped with a positive weight for each edge. We replace the total length functional by the weighted length, i.e. the sum of lengths of edges multiplied with weights. This changes the minimality condition slightly. Allard and Almgren gave an example of a family of 3-regular weighted minimal graphs with 16 attaching points and with arbitrarily large numbers of vertices. These examples are known as the spider web-like varifolds, see [1], [2] and [3]. They motivate the following conjecture.

**Conjecture 5.1** There exist minimal graphs attached to some fixed finite set of points in the plane with arbitrary large number of vertices.

If this conjecture is true, it will be interesting to study infinite minimal graphs. This can motivate also the study of Morse theory in the space of 1-cycles with infinitely many edges.

However (paradoxically), the author conjectures

**Conjecture 5.2** For all \( d \geq 3 \), the number of vertices of a \( d \)-regular minimal graph attached to \( n \) points is \( \leq f_3(n) \).
One can begin with the case where all the angles of the $d$-regular minimal graphs are equal to $\frac{2\pi}{d}$ and try to obtain an intermediate result like Lemma 2.4.

The problem of estimating the maximum number of vertices in a minimal graph with some fixed conditions can also be asked in a more general context. One can ask the same questions about the graphs embedded in compact Riemannian manifold such as spheres, tori, etc.

The known results concern mostly the two sphere with a non necessarily canonical metric (see [5] and [6]).
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References

[1] W.K. Allard and Jr. Almgren F.J. An introduction to regularity theory for parametric elliptic variational problems. *Proc.symp.PureMath.*, XXIII,Amer.Math.Soc.(7):231–260, March 17 1973.

[2] W.K. Allard and Jr. Almgren F.J. The structure of stationary one dimensional varifolds with positive density. *Inventiones mathematicae*, 34:83–97, 1976.

[3] Jr. Almgren F.J. *Plateau’s Problem An invitation to Varifold Geometry*. 1966.

[4] M. Gromov. Singularities, expanders and topology of maps. part 1: Homology versus volume in the spaces of cycles. *preprint*.

[5] J. Hass and F. Morgan. Geodesic nets on the 2-sphere. *Proceedings of the American Mathematical Society*, 124(12):3843–3850, December 1996.

[6] A. Heppes. On the partition of the 2-sphere by geodesic nets. *Proceedings of the American Mathematical Society*, 127(7):2163–2165, March 17 1999.

[7] S. Markvorsen. Minimal webs in Riemannian manifolds. *Geom.Dedicata*, 133:7–34, 2008.