A CHARACTERIZATION OF THE RATIONAL NORMAL CURVE

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Abstract. We give a characterization of the rational normal curve in terms of the rank function associated to a curve.

1. Introduction

Let $C \subset \mathbb{P}^n$ be a nondegenerate and nonsingular curve over $\mathbb{C}$. Let $x \in \mathbb{P}^n$ be a point. We define the $C$-rank of $x$ as the smallest number $r$ such that $x$ lies in the linear span of $r$ points of $C$. In [CS02] we describe the strata of points having constant rank in the case in which $C$ is a rational normal curve, that is, a degree $n$ curve. It is shown there that all points belonging to a tangent line to $C$ have rank $n$, excepting the point of tangency, which has rank one. In [Com07], it is shown that if $C$ has positive genus, and the immersion is given by a complete linear system of degree $d$ divisors (with $d \geq 10g - 1$), then all points in $\mathbb{P}^n$ have rank less or equal than $n - g$, and all points in tangent lines have rank $n - g$ (excepting the points of tangency).

In this article we show that the only curve $C \subset \mathbb{P}^n$ such that all points belonging to a tangent line to $C$ have rank $n$ (excepting the point of tangency), is the rational normal curve.

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2. Main result.

Definition 1. Let $C \subset \mathbb{P}^n$ be a nondegenerate and nonsingular curve. Let $x \in \mathbb{P}^n$ be a point. We define the $C$-rank of $x$ (and note it $\text{rk}_x$) as the smallest number $r$ such that $x$ lies in the linear span of $r$ points of $C$.

We will prove the following theorem

Theorem 2. The rational normal curve is the only nondegenerate and nonsingular curve in $\mathbb{P}^n$ such that all the points in its tangent lines (excepting the point of tangency) have rank $n$.

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First we show that the rational normal curve has the desired property.

**Theorem 3.** Let $C \subset \mathbb{P}^n$ be the rational normal curve, let $p \in C$ and let $x \in T_p(C)$ a point such that $x \neq p$. Then $\text{rk}(x) = n$.

**Proof.** Assume that $r = \text{rk}(x) < n$. Then $x \in \langle p_1, \ldots, p_r \rangle$ for $p_1, \ldots, p_r \in C$. Let $L$ be the tangent line to $C$ at $p$, $M$ be the linear span of $p_1, \ldots, p_r$ and $N$ be the linear span of $L$ and $M$. We have that $x \in L \cap M$, therefore $N$ has dimension less than or equal to $r$.

If $L \cap M = \{x\}$, that is, if $p \neq p_i \forall i$, then $\text{dim} \ N = r$.

If, on the other hand, $p = p_i$ for some $i$, then $L \cap M$ contains the line spanned by $p$ and $x$, and therefore $L \cap M = L$. Then $\text{dim} \ N = r - 1$.

But the intersection of a linear variety of dimension $k$ cuts the rational normal curve in a divisor of degree less than or equal to $k + 1$. In both cases the intersection of $N$ with $C$ is a divisor of degree $\text{dim} \ N + 2$.

Therefore $\text{rk}(x) \geq n$.

Now we show that every point in $\mathbb{P}^n$ has rank less than or equal to $n$. Let us consider the linear system $D$ of hyperplanes that contain $x$. It is clear that the intersection of all the members in $D$ is the set which has $x$ as its only element.

We want to show that one of the members in $D$ has multiple points. If every member has multiple points, then by Bertini’s Theorem there exists a point $p \in C$ that is a multiple point of every member in $D$. But then the intersection of all the hyperplanes contains the tangent line to $C$ at $p$, which is a contradiction. Therefore one of the members in $D$ has no multiple points, and the rank of $x$ is less than or equal to $n$. $\square$

**Proof of Theorem 2.** We already proved that the rational normal curve has the desired property. Now we will show that if $C$ is not the rational normal curve, then there are points on its tangent lines which have neither rank 1 nor rank $n$. Since $C$ is not the rational normal curve, the degree of $C$ is greater than $n$.

If $n = 2$, a generic tangent line to $C$ cuts $C$ in another point. Therefore the generic tangent line to $C$ contains a point that is not the point of tangency, which has rank different from 2.

From now on $n \geq 3$. Let $p \in C$ be a point. If the tangent line to $C$ at $p$ cuts $C$ in another point $q$ we are done, since $q$ has rank one. Since there are only finite points $p$ such the tangent line to $C$ at $p$ cuts $C$ with multiplicity greater than 2, we assume that the intersection of the tangent line to $C$ at $p$ and $C$ is the divisor $2p$.

Let us consider first the case $n \geq 4$. Let $L$ be the tangent line to $C$ at $p$ and let $\pi_L : \mathbb{P}^n \setminus L \to \mathbb{P}^{n-2}$ be the projection with center $L$. The restriction of $\pi_L$ to $C \setminus \{p\}$ extends to a function on all $C$. Let $C'$ be the image $\pi_L(C)$, which is a nondegenerated degree $d - 2$ curve $(d - 2 \geq 3)$. By the general position theorem ([ACGH85], pag. 109), the generic hyperplane section to $C'$ cuts $C'$ in $d - 2$ disctint
points such that any \( n - 2 \) are in general position. Furthermore, we can assume that the generic hyperplane section does not contain any of the singular points of \( C' \). We can also assume that the generic hyperplane section does not contain the image of \( p \) by \( \pi_L \). Let \( q_1, \ldots, q_{d-2} \in C' \) be the points of intersection of \( C' \) with a general hyperplane. By the assumptions we made we can choose \( p_1, \ldots, p_{d-2} \in C \) (with \( p_i \neq p_j \)) such that \( \pi_L(p_i) = q_i \).

We now use the fact that there is a bijection between hyperplanes section of \( C' \) and hyperplanes sections to \( C \) that contain \( L \). By this bijection the hyperplane \( \langle q_1, \ldots, q_{d-2} \rangle \) corresponds to the hyperplane \( \langle L, p_1, \ldots, p_{d-2} \rangle \). Let \( \Lambda \subset \mathbb{P}^n \) be the linear variety \( \Lambda = \langle p_1, \ldots, p_{d-2} \rangle \). As \( \pi_L(\Lambda) \) is a variety of dimension \( n - 3 \), the possible values of \( \dim \Lambda \) are \( n - 3, n - 2 \) or \( n - 1 \). If \( \dim \Lambda = n - 2 \), then the intersection \( L \cap \Lambda \) is a point \( x \notin C \). We can choose \( n - 1 \) of the \( p_i \)'s such that they generate \( \Lambda \). Then we have \( \text{rk} x \leq n - 1 \). If \( \dim \Lambda = n - 1 \) then \( L \subset \Lambda \). Let us choose \( n \) of the \( p_i \)'s that generate \( \Lambda \). The linear variety generated by any \( n - 1 \) of these points cuts \( L \) in a point \( x \) not in \( C \). Then \( \text{rk} x \leq n - 1 \).

To finish the demonstration we have to prove that for the generic hyperplane section \( H' \) of \( C' \) the corresponding \( \Lambda \) has dimension \( n - 1 \) or \( n - 2 \). So let us assume that \( \dim \Lambda = n - 3 \). As \( d - 2 > n - 2 \), we have that \( \Lambda \) is a \( (n - 3) \)-plane that cuts \( C \) in a divisor of degree greater than \( n - 2 \). Now we compare dimensions. There are \( \infty^{n-2} \) generic hyperplane sections of \( C' \). On the other hand, a nonsingular and nondegenerate curve can have at most \( \infty^{n-3} \) \( (n-1) \)-secant \( (n-3) \)-planes ([ACGH85], pag. 152). Therefore, for the generic hyperplane section \( H' \) we must have \( \dim \Lambda = n - 1 \) or \( n - 2 \).

If \( n = 3 \) we study the family of planes containing \( L \). If a plane \( H \) cuts \( C \) in less than \( d - 2 \) points, then there are double points in this intersection. That means that \( H \) contains another tangent line \( L' \) that must cut \( L \). Since two generic tangent lines to \( C \) do no meet, for general \( H \) containing \( L \) the intersection of \( H \) and \( C \) is the divisor \( 2p + p_1 + \cdots + p_{d-2} \) with \( p_i \neq p \forall i \). We consider the linear variety \( \Lambda = \langle p_1 + \cdots + p_{d-2} \rangle \). If \( \dim \Lambda = 1 \), then the intersection of \( L \) and \( \Lambda \) is a point that must have rank 2 since \( L \) does not cut \( C \) away from \( p \). If on the other hand \( \dim \Lambda = 2 \), we can choose 3 of the \( p_i \)'s that generate \( \Lambda \). The intersection of \( L \) with one of the lines generated by two of the chosen \( p_i \)'s is a 2 rank point.

\[ \square \]

References

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