Beyond sets with atoms
Definability in first order logic

Michal R. Przybylek
Faculty of Mathematics, Informatics and Mechanics
University of Warsaw
Warsaw, Poland
mrp@mimuw.edu.pl

Abstract
Sets with atoms serve as an alternative to ZFC foundations for mathematics, where some infinite, though highly symmetric sets, behave in a finitistic way. Therefore, one can try to carry over analysis of the classical algorithms from finite structures to symmetric infinite structures. Recent results show that this is indeed possible and leads to many practical applications: automata over infinite alphabets [6], model checking [23], constraint satisfaction solving [34], [22], programming languages [25], [27] and [13], to name a few. In this paper we shall take another route to finite analysis of infinite sets, which extends and sheds more light on sets with atoms. As an application of our theory we give a characterisation of languages recognized by automata definable in fragments of first order logic.

CCS Concepts • Theory of computation → Computability; Constructive mathematics; Automata over infinite objects; Regular languages;

Keywords Zermelo-Fraenkel set theory with atoms, nominal sets, automata theory, model theory, classifying topos

1 Introduction
In the late ’70s Stephen Schanuel working on the theory of combinatorial functions studied the topos of pullback preserving functors from the category of finite sets and injections to the category of sets, which is nowadays known as the Schanuel topos. Shortly afterwards, when the theory of classifying toposes emerged, it has been discovered that the Schanuel topos is the classifying topos for the first order theory of infinite decidable objects

[42].

The Schanuel topos was then rediscovered by James Gabbay and Andrew Pitts [16] as an elegant formalism for reasoning about name bindings in formal languages. This idea was further pursued [35] and the Schanuel topos earned a new name — the topos of nominal sets — starting a completely new life in theoretical computer science. A decade later, the connection between nominal sets and the theory of classifying toposes was forgotten and some of the classical results were discovered again by the Warsaw Logical Group [7], [9] and again in [5]. Nonetheless, many well-known classical results are still unknown.

This paper presents nominal sets, and their older cousins: sets with atoms, as a part of a bigger picture (see Figure 1, which will be explained throughout the paper) — the theory of classifying toposes for the positive existential fragment of intuitionistic first order logic. According to this picture, generalised nominal sets are precisely the classifying toposes for \( \omega \)-categorical structures, whereas set theories with atoms are precisely the filtered colimits of some canonical diagrams of generalised nominal sets. We shall focus on the aspects of computability in positive existential logic — which algorithms can be effectively executed, when the domains of the variables are interpreted as "potentially infinite" definable sets. This goes beyond theories of oligomorphic structures (Example 1.1 and Example 1.2). Our framework is suitable for \( \omega \)-categorical structures, which are not

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1 Classically, this is just the theory of pure sets. See Example 2.6.
oligomorphic (Example 1.3), structures build from \( \omega \)-categorical structures by adding infinitely many constants (Example 1.4), classical non-complete theories (Example 1.5), intuitionistic propositional theories (Example 1.6 and Example 1.7), and many more.

**Example 1.1 (Pure sets).** Let \( N = \{0, 1, 2, \ldots\} \) be a countably infinite set over empty signature \( \Xi \). Then the first order theory of \( N \) is \( \omega \)-categorical, i.e. there is exactly one countable model of the theory up to an isomorphism. This theory is called the theory of "pure sets".

**Example 1.2 (Rational numbers with ordering).** Let \( Q = \langle Q, \leq \rangle \) be the structure whose universe is interpreted as the set of rational numbers \( Q \) with a single binary relation \( \leq \subseteq Q \times Q \) interpreted as the natural ordering of rational numbers. Then the first order theory of \( Q \) is \( \omega \)-categorical.

**Example 1.3 (Multi-sorted \( \omega \)-categorical theory).** Let \( S \) be a structure consisting of countably many countable sorts identified with natural numbers \( N \) and such that the \( i \)-th sort interprets constants \( \{0, 1, \ldots, i - 1\} \). Then the theory \( Th(S) \) is \( \omega \)-categorical. However, the group of automorphisms \( Aut(S) \) of \( S \) in not oligomorphic — since the automorphisms act independently on each sort, the group has infinitely many orbits.

**Example 1.4 (Pure sets with constants).** Let \( N \sqcup N \) be the structure from Example 1.1 over an extended signature consisting of all constants \( n \in N \). Then the first order theory of \( N \sqcup N \) has countably many non-isomorphic countable models.

**Example 1.5 (Dense linear order).** Let \( T \) be the first order theory of dense linear orders, i.e.: it is a theory over signature consisting of a single binary predicate \(<\), with the following axioms (written as first order sequents):

\[
\begin{align*}
 a < a & \vdash \bot \\
 a < b \land b < c & \vdash a < c \\
 & \vdash a < b \lor b < a \lor a = b \\
 a < c & \vdash \exists b \ a < b \land b < c
\end{align*}
\]

This theory is not complete, as it does not specify whether a given linear order has the smallest and the largest element, and if so, whether or not they coincide.

**Example 1.6 (Propositional theory with one variable).** By propositional theory with one variable we shall mean the empty positive existential theory over zero-sorted signature \( \Xi_1 \) with a single nullary relation \( p \subseteq |\Xi_1|^0 = 1 \). A model of this theory in any topos is an internal truth value (i.e. subobject of the terminal object). For example, in Set there are exactly two models: one in which \( p \) is false, and another in which \( p \) is true.

**Example 1.7 (Seemingly impossible theory).** By seemingly impossible theory we shall mean the positive existential theory over zero-sorted signature \( \Xi_N \) with countably many nullary relations \( \{n\}_{n \in N} \) with following axioms: \( \neg(n + 1) \vdash \neg n \).

In [8] a concept of a while-program with semantics in definable sets with atoms \( \mathcal{A} \) has been defined. The authors examine conditions on \( \mathcal{A} \) that ensure that certain while-programs terminate. As an illustrative example, consider the reachability problem on directed graphs. A while-program for this problem is presented as Algorithm 1. This algorithm can be actually implemented in a natural way in a programming language that supports computation on sets with atoms, for instance: LOIS or \( N \lambda^2 \) (see [26] and [24], also [25], [27] and [13] for more details). We will see in Section 2.1 that by transfer principle (Theorem 2.2), the program can be actually executed in the category \( Cont(Aut(\mathcal{A} \sqcup A_0)) \) of continuous actions of the topological group of automorphisms of structure \( \mathcal{A} \sqcup A_0 \) for some finite \( A_0 \subset A \). Moreover, the conditions the authors examine imply that \( \mathcal{A} \) is oligomorphic and \( Cont(Aut(\mathcal{A} \sqcup A_0)) \) is the classifying topos for the theory of \( \mathcal{A} \sqcup A_0 \). Therefore, (see Section 2.4) their framework restricts to sets definable in the first order theory of oligomorphic structure \( \mathcal{A} \sqcup A_0 \). We will see that Algorithm 1 can be effectively executed on sets definable in theories from all our Examples 1.1, 1.2, 1.3, 1.4, 1.5, 1.6 and 1.7.

In parallel with Algorithm 1, we shall study the languages that can be recognised by a generalisation of finite memory machines in the sense of Kaminiski and Francez [21]. An example of such a machine is presented on Figure 2. The machine has a single register \( R \) and can test for equality and inequality only. It starts in state "SET PASSW", where it awaits for the user to provide a password \( x \). This password is then stored in register \( R \), and the machine enters state "START". Inside the top rectangle the machine can perform actions that do not require authentication, whereas the actions that require authentication are presented inside the bottom rectangle. The bottom rectangle can be entered by state "GRANT AUTH", which can be accessed from one of three authentication states. In order to authorise, the machine moves to state "AUTH TRY 1", where it gets input \( x \) from the user. If the input is the same as the value previously stored in register \( R \), then the machine enters state "GRANT".

| Algorithm 1 Reachability algorithm |
|-------------------------------------|
| **procedure** REACHABLE(E, a, b) |
| \( R' \leftarrow \emptyset \) |
| \( R \leftarrow \{b\} \) |
| **while** \( R' \neq R \) **do** |
| **if** \( a \in R \) **then** **return** \( \top \) |
| \( R' \leftarrow R \) |
| **for** \((x, y) \in E \) **do** |
| **if** \( y \in R' \) **then** |
| \( R \leftarrow R \cup \{x\} \) |
| **return** \( \bot \) |

A working implementation of \( N \lambda \), a functional programming language capable of processing infinite structures with atoms, is available through the web-site: https://www.mimuw.edu.pl/~szynwelski/nlambda/.
by a definable automata is almost never definable. This problem can be overtaken by describing a language $L$ as a collection of languages $(L^k)_{k \in \mathbb{N}}$, where $L^k$ is definable and consists of these words of $L$ whose length is at most $k$. Nonetheless, the full justification of such a definition is difficult without the theory of classifying toposes, and the explicit calculations are messy. Therefore, we review some basic facts about classifying toposes in Subsection 2.4 and perform all of the necessary computations in Section 3 inside the classifying topos of the theory, where the concept of a language can live naturally. For more information about automata in categories we refer to [2], [1], [15] and [14].

Section 2 is devoted to explaining Figure 1. Subsection 2.1 explains the left side of part 1 on the picture: how Zermelo-Frankel set theory with atoms can be constructed from toposes of continuous actions of topological groups. We state here a meta-theorem (Theorem 2.2) allowing us to delegate computations from ZFA to toposes of continuous actions of topological groups. The right side of 1 on the picture together with 2 is explained in Subsection 2.2. We investigate there possible extensions to definability in sets with atoms and prove Theorem 2.5 indicating why such attempts might be futile in general. The right square of 3 is explained in Subsection 2.3, where we study definability in positive existential theories. Finally, the outer square of 3 is roughly explained in Subsection 2.4; for more information about Grothendieck toposes we refer the reader to [29], [19] and [11].

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2 The roadmap

In this section we shall explain Figure 1 in detail and discuss how the computations with atoms can be carried over to more general framework of classifying toposes for positive existential theories.

2.1 Set with atoms

Let $\mathcal{A}$ be an algebraic structure (both operations and relations are allowed) with universum $A$. We shall think of elements of $\mathcal{A}$ as “atoms”. A von Neumann-like hierarchy $V_\alpha(\mathcal{A})$ of sets with atoms $\mathcal{A}$ can be defined by transfinite induction [33], [17]:

- $V_0(\mathcal{A}) = A$
- $V_{\alpha+1}(\mathcal{A}) = \mathcal{P}(V_\alpha(\mathcal{A})) \cup V_\alpha(\mathcal{A})$
- $V_\lambda(\mathcal{A}) = \bigcup_{\alpha < \lambda} V_\alpha(\mathcal{A})$ if $\lambda$ is a limit ordinal

Then the cumulative hierarchy of sets with atoms $\mathcal{A}$ is just $V(\mathcal{A}) = \bigcup_{\alpha: \text{ord}} V_\alpha(\mathcal{A})$. Observe, that the universe $V(\mathcal{A})$ carries a natural action $\times$: $\text{Aut}(\mathcal{A}) \times V(\mathcal{A}) \rightarrow V(\mathcal{A})$ of the automorphism group $\text{Aut}(\mathcal{A})$ of structure $\mathcal{A}$ — it is just applied pointwise to the atoms of a set. If $X \in V(\mathcal{A})$ is a set with atoms then by its set-wise stabiliser we shall mean the

![Figure 2. A register machine that models access control to the dashed part of the system.](image-url)
set: $\Aut(\mathcal{A})_X = \{ \pi \in \Aut(\mathcal{A}) : \pi \cdot X = X \}$; and by its pointwise stabiliser the set: $\Aut(\mathcal{A})_{X(x)} = \{ \pi \in \Aut(\mathcal{A}) : \forall x \in X \pi \cdot x = x \}$. Moreover, for every $X$, these sets inherit a group structure from $\Aut(\mathcal{A})$.

There is an important sub-hierarchy of the cumulative hierarchy of sets with atoms $\mathcal{A}$, which consists of "symmetric sets" only. To define this hierarchy, we have to equip $\Aut(\mathcal{A})$ with the structure of a topological group. A set $X \in V(\mathcal{A})$ is symmetric if the set-wise stabilisers of all of its descendants $Y$ is an open set (an open subgroup of $\Aut(\mathcal{A})$), i.e. for every $Y \subseteq X$ we have that: $\Aut(\mathcal{A})_Y$ is open in $\Aut(\mathcal{A})$, where $\subseteq^*$ is the reflexive-transitive closure of the membership relation $\in$. A function between symmetric sets is called symmetric if its graph is a symmetric set. Of a special interest is the topology on $\Aut(\mathcal{A})$ inherited from the product topology on $\prod_{\mathcal{A}} = \mathcal{A}^\mathcal{A}$. We shall call this topology the canonical topology on $\Aut(\mathcal{A})$. In this topology, a subgroup $\mathcal{H}$ of $\Aut(\mathcal{A})$ is open if there is a finite $A_0 \subseteq A$ such that: $\Aut(\mathcal{A})_{A_0} \subseteq \mathcal{H}$, i.e.: group $\mathcal{H}$ contains a pointwise stabiliser of some finite set of atoms. The sub-hierarchy of $V(\mathcal{A})$ that consists of symmetric sets according to the canonical topology on $\Aut(\mathcal{A})$ will be denoted by $\ZFA(\mathcal{A})$ (it is a model of Zermelo-Fraenkel set theory with atoms).

**Remark 2.1.** The above definition of hierarchy of symmetric sets is equivalent to another one used in model theory, namely, the definition involving a normal filter of subgroups.

**Example 2.1** (The basic Fraenkel-Mostowski model). Let $N$ be the structure from Example 1.1. We call $\ZFA(N)$ the basic Fraenkel-Mostowski model of set theory with atoms. Observe that $\Aut(\mathcal{N})$ is the group of all bijections (permutations) on $N$. The following are examples of sets in $\ZFA(N)$:

- all sets without atoms, e.g. $\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}, \ldots, \ldots$
- all finite subsets of $N$, e.g. $\{\emptyset\}, \{\emptyset, 1, 2, 3\}, \ldots$
- all cofinite subsets of $N$, e.g. $\{1, 2, 3, \ldots\}, \{4, 5, 6, \ldots\}, \ldots$
- $N \times N$
- $\{\langle a, b \rangle \in N^2 : a \neq b \}$
- $N^* = \bigcup_{k \in \mathbb{N}} N^k$
- $\mathcal{K}(N) = \{N_0 : N_0 \subseteq N, N_0$ is finite\}$
- $\mathcal{P}_3(N) = \{N_0 : N_0 \subseteq N, N_0$ is symmetric\}$

Here are examples of sets in $V(N)$ which are not symmetric:

- $\{0, 2, 4, 6, \ldots\}$
- $\{\langle n, m \rangle \in N^2 : n \leq m \}$
- the set of all functions from $N$ to $N$
- $\mathcal{P}(N) = \{N_0 : N_0 \subseteq N$}

**Example 2.2** (The ordered Fraenkel-Mostowski model). Let $Q$ be the structure from Example 1.2. We call $\ZFA(Q)$ the ordered Fraenkel-Mostowski model of set theory with atoms. Observe that $\Aut(Q)$ is the group of all order-preserving bijections on $Q$. All symmetric sets from Example 2.1 are symmetric sets in $\ZFA(Q)$ when $N$ is replaced by $Q$. Here are some further symmetric sets:

- $\{\langle p, q \rangle \in Q^2 : p \leq q \}$

**Example 2.3** (The second Fraenkel-Mostowski model). Let $S = (\mathbb{Z}^*, -, (\lfloor a \rfloor)_{a \in \mathbb{N}})$ be the structure of non-zero integer numbers, with unary "minus" operation ($-$): $\mathbb{Z}^* \rightarrow \mathbb{Z}^*$ and with unary relations $\lfloor a \rfloor \subseteq \mathbb{Z}^*$ defined in the following way: $\lfloor z \rfloor_n \Leftrightarrow |z| = n$. We call $\ZFA^1(\mathbb{Z}^*)$ the second Fraenkel-Mostowski model of set theory with atoms. Observe that $\Aut(\mathbb{Z}^*) \simeq \mathbb{Z}_2^*$, therefore the following sets are symmetric in $\ZFA^1(\mathbb{Z}^*)$:

- $\{\ldots, -6, -4, -2, 2, 4, 6, \ldots\}$
- $\{\langle x, y \rangle \in \mathbb{Z}^* \times \mathbb{Z}^* : x = 3y\}$

Observe that the group $\Aut(\mathcal{A}(a_0))$ is actually the group of automorphism of structure $\mathcal{A}$ extended with constants $a_0$, i.e.: $\Aut(\mathcal{A}(a_0)) = \Aut(\mathcal{A} \cup a_0)$. Then a set $X \in V(\mathcal{A})$ is symmetric if and only if there is a finite $A_0 \subseteq A$ such that: $\Aut(\mathcal{A} \cup a_0) \subseteq \mathcal{H}$ and the canonical action of topological group $\Aut(\mathcal{A} \cup a_0)$ on discrete set $X$ is continuous. A symmetric set is called $A_0$-equivariant (or equivariant in case $A_0 = \emptyset$) if $\Aut(\mathcal{A} \cup a_0) \subseteq \mathcal{H}$. Therefore, the (non-full) subcategory of $\ZFA(\mathcal{A})$ on $A_0$-equivariant sets and $A_0$-equivariant functions (i.e. functions whose graphs are $A_0$-equivariant) is equivalent to the category $\Con(\Aut(\mathcal{A} \cup a_0))$ of continuous actions of the topological group $\Aut(\mathcal{A} \cup a_0)$ on discrete sets.

**Example 2.4** (Equivariant sets). In the basic Fraenkel-Mostowski model:

- all sets without atoms are equivariant
- all finite subsets $N_0 \subseteq N$ are $N_0$-equivariant
- all finite subsets $N_0 \subseteq N$ are $N \setminus N_0$-equivariant
- $N \times N, N^{|N|}, \mathcal{K}(N), \mathcal{P}_3(N)$ are equivariant

In most works on computations in sets with atoms, the authors focus on equivariant sets and equivariant functions (i.e. the category $\Con(\Aut(\mathcal{A}))$) and claim that the results carry over to $\ZFA(\mathcal{A})$. We shall now give a formal argument why such claims are valid.

**Lemma 2.1** (Presentation of $\ZFA(\mathcal{A})$). Let $\mathcal{A}$ be an algebraic structure. Then there is a functor $\Theta: K(\mathcal{A}) \rightarrow \Log$ from the poset $K(\mathcal{A})$ of finite subsets of $\mathcal{A}$ seen as a posetal category to the 2-category $\Log$ of elementary toposes and logical functors. This functor maps $A_0$ to $\Con(\Aut(\mathcal{A} \cup a_0))$ and $A_0 \subseteq A_1$ to the logical embedding: $\Con(\Aut(\mathcal{A} \cup a_0)) \rightarrow \Con(\Aut(\mathcal{A} \cup a_1))$. Moreover, $\ZFA(\mathcal{A})$ is the colimit of $\Theta$ in $\Log$ — the canonical embeddings $\Con(\Aut(\mathcal{A} \cup a_0)) \rightarrow \ZFA(\mathcal{A})$ for finite $A_0 \subseteq A$ are logical embeddings (i.e. preserve elementary topos structure).

Because the forgetful functor from $\Log$ to the category of locally small categories $\Cat$ preserves filtered colimits, $\ZFA(\mathcal{A})$ is also the filtered colimit of logical embeddings in $\Cat$. Therefore, every diagram $D$ in $\ZFA(\mathcal{A})$ of the shape of
C for a finite category C (i.e. a functor C \rightarrow ZFA(\mathcal{A})) factors via some embedding Cont(Aut(A \sqcup A_0)) \rightarrow ZFA(\mathcal{A}): 

\[
\begin{array}{c}
\text{C} \\
\text{ZFA(\mathcal{A})} \\
\text{Cont(Aut(A \sqcup A_0))}
\end{array}
\]

Theorem 2.2 (Transfer principle). Every categorical reasoning concerning all elementary topos operations, such as: finite limits and colimits, exponentials, power objects, quotients, internal quantifiers, etc. can be studied in Cont(Aut(A \sqcup A_0)) for some finite A_0 \subseteq A and then the results can be transferred back to ZFA(\mathcal{A}).

Corollary 2.3. If \mathcal{A} is \omega-categorical (resp. extremely amenable) then for every finite A_0 \subseteq A, the structure \mathcal{A} \sqcup A_0 is \omega-categorical (resp. extremely amenable) as well. Therefore, every theorem involving elementary topos construction that holds for every \omega-categorical (resp. extremely amenable) \mathcal{A} in Cont(Aut(A \sqcup A_0)) also holds in ZFA(\mathcal{A}).

2.2 First order structures

We shall say that an A_0-equivariant set X ∈ ZFA(\mathcal{A}) is of finitary type if its canonical action has only finitely many orbits, i.e. if the relation x \equiv y ⇔ \exists x ∈ Aut(\mathcal{A} \sqcup A_0) x = x \cdot y has finitely many equivalence classes. The reason behind this terminology is that X is of a finitary type if it is compact when treated as an object of category Cont(Aut(\mathcal{A} \sqcup A_0)).

Let us recall the formal definition of a compact object in a general category with filtered colimits.

Definition 2.1 (Compact object). An object X of a category \mathcal{C} is called compact if its co-representation hom_{\mathcal{C}}(X, -) : \mathcal{C} \rightarrow Set preserves filtered colimits of monomorphisms.

Example 2.5. Here are some examples of compact objects in toposes:

- a set X in classical Set is compact iff it is finite
- a continuous G-set X in Cont(\mathcal{G}) is compact iff it has finitely many orbits
- a function X : A → B thought of as an object in Set^{\ast \to \ast} is compact if its graph is finite — i.e. if A and B are finite sets
- a chain of functions (X_i : A_i → A_{i+1})_{i \in \mathbb{N}} thought of as an object in Set^{\ast \to \ast \to \ast \to \ast \to \ast} is compact if all A_i are finite and the chain is eventually bijective

Observe that we cannot speak about compact objects in ZFA(\mathcal{A}), because ZFA(\mathcal{A}) does not have filtered colimits of monomorphisms. For a counterexample consider the chain: 

\[
\{\} \subset \{a_1\} \subset \{a_1, a_2\} \subset \{a_1, a_2, a_3\} \subset \cdots
\]

where a_i \in A. This chain cannot have a colimit, since not every function f with domain A is symmetric, but every restriction of f to a finite set is symmetric. Therefore, by Theorem 2.2, the notion of a set of finitary type in ZFA(\mathcal{A}) is a reflection of the notion of compactness in Cont(Aut(\mathcal{A} \sqcup A_0)).

Every set of finitary type is isomorphic to a set that is hereditarily of finitary type, therefore without loss of generality we can assume that all finitary sets are of this form. We call a set in ZFA(\mathcal{A}) "definable with atoms" if it is hereditarily of finitary type. The category of definable sets and functions with atoms will be denoted by Def(\mathcal{A}), and its subcategory of A_0-equivariant sets by Def_{A_0}(\mathcal{A}).

Theorem 2.4 (Presentation of Def(\mathcal{A})). Category Def(\mathcal{A}) is the filtered colimit of categories Def_{A_0}(\mathcal{A}) and natural embeddings for finite subsets A_0 ⊆ A.

Blass and Scedrov in [4] proved that Cont(Aut(\mathcal{A})) is a coherent topos if and only if \mathcal{A} is \omega-categorical. Moreover, in such a case first order definable subsets\(^3\) of \mathcal{A} coincide (up to isomorphisms) with compact objects in Cont(Aut(\mathcal{A})).

The last statement is a very special case of the characterisation theorem for coherent toposes by Alexander Grothendieck and we return to it in Section 2.4. We point out, that one direction of this theorem for oligomorphic structures was recently rediscovered in [7].

Let us recall that by Ryll-Nardzewski theorem [38], a structure \mathcal{A} (in a countable language) is \omega-categorical if and only if for every k, there are only finitely many non-equivalent formulas with k free variables. By the above considerations, this can be equivalently expressed by the following property of Cont(Aut(\mathcal{A})): every compact object has only finitely many subobjects; or by the property of ZFA(\mathcal{A}): every set of finitary type has only finitely many A_0-equivariant subsets (for every finite A_0 ⊆ A). This property allows for effective algorithms in ZFA(\mathcal{A}) on sets of finitary type. This has been observed in [7]. If one is careful to use only elementary topos operations in algorithms then, because every algorithm is finite, it can be executed in Cont(Aut(\mathcal{A})) and by transfer principle its outcome transfers to ZFA(\mathcal{A}). Notice however, that a power set of a set of finitary type needs not be of a finitary type. Therefore, one should further restrict to the operations that are stable under definability, i.e. one may use: finite limits, finite coproducts and coequalisers of kernel pairs (i.e. quotient sets), Boolean operations on definable subsets, images, inverse images and dual images of definable sets under definable functions.

Let us assume that \mathcal{A} is \omega-categorical with a deciding theory. Algorithm 1 can be run on a definable relation E and two elements a, b ∈ Def(\mathcal{A}). Moreover, it always terminates on such inputs. Its run can be seen as a computation of a partial transitive-reflexive closure of a relation E. By transfer principle, we can assume that the inputs are equivariant. Then there is a formula \phi that defines E, and all relations: id, E, E^2, \ldots have the same context as \phi. Thus there are only finitely many different E^k and the process

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\(^3\)Definability means here: "definable in the theory of \mathcal{A} extended with elimination of imaginaries". For more details see the next subsection.
terminates after finitely many steps. We are generally interested in the structures $\mathcal{A}$ with the property that algorithms like Algorithm 1 can be effectively realized. Unfortunately, if $\mathcal{A}$ is not $\omega$-categorical, then such a problem is not even well-defined, because the correspondence between sets definable in the theory of $\mathcal{A}$ and sets of finitary types fails badly; there is no longer a correspondence between complete types over $\mathcal{A}$ and orbits of $\text{Aut}(\mathcal{A})$; moreover, finite sets of types does not correspond to formulas (for more details consult [18] Chapter 10). We shall see later in Section 2.4 that equivariant definable sets $\text{Def}_0(\mathcal{A})$ can be recovered from the classifying topos of the first order theory of $\mathcal{A}$ as the full subcategory on a special type of compact objects called coherent. Example 2.5 shows that compact objects in $\text{Cont}(\text{Aut}(\mathcal{A}))$ can have only finitely many orbits, therefore there cannot be a one-to-one correspondence between compact (nor coherent) objects in $\text{Aut}(\mathcal{A})$ and equivariant definable sets $\text{Def}_0(\mathcal{A})$ for non-$\omega$-categorical $\mathcal{A}$. In fact, Blass and Scedrov (see Section 2.4) showed that the classifying topos for the first order theory of $\mathcal{A}$ cannot be even Boolean unless $\mathcal{A}$ is $\omega$-categorical.

Instead of diving into classifying toposes, which for general structures may be difficult to describe, we may like to reverse our thinking and treat definable algorithms as formulas themselves: in the sense of dynamic logic. Then the question about effective realisation of algorithms turns into the question of decidability of the first order theory extended with dynamic logic of structure $\mathcal{A}$. Let us focus on the following well-understood fragment of dynamic logic: $\mu$-calculus — i.e. extension of first order logic with the least fixed-point operator. That is, together with the usual first order formulas, we also have formulas of the form: $\mu X[\gamma], \phi(X, \gamma)$, where $X$ (must occur positively in $\phi$) is a "predicate" variable of arity equal to the length of sequence of "parameters" $\gamma$. The semantics of this formula (in a given algebraic structure) is the least set $X^*$ such that: $X^*(\gamma) \iff \phi(X^*, \gamma)$. For example, if $E$ is a formula representing a binary relation, then:

$$\mu X[y_1, y_2]. (y_1 = y_2) \lor (\exists z E(y_1, z) \land X(z, y_2))$$

defines the transitive-reflexive closure of $E$. The least fixed point is the union of the following sequence defined by transfinite induction:

- $X_0(\gamma) = \bot$
- $X_{\alpha+1}(\gamma) = \phi(X_\alpha, \gamma)$
- $X_\lambda(\gamma) = \bigvee_{\alpha < \lambda} \phi(X_\alpha, \gamma)$ if $\lambda$ is a limit ordinal

Because the above sets (called the stages of computation) are bounded by the context of $\gamma$, the transfinite sequence must stabilize at some ordinal $\alpha$, in which case we put $X^* = X_\alpha$. In particular, if $\mathcal{A}$ is countable, which is the case of computations in set with atoms (see [7], [6], [9]), $\alpha$ is bounded by $\omega$, and the computation of the least fixed-point can be realized by a standard while-program. Therefore, the least fixed-points are computable by while-programs if all stages of computations stabilize after finitely many steps. This is the case of an $\omega$-categorical structure $\mathcal{A}$ (since then, in any context $\gamma$ there are only finitely many definable sets). Unfortunately, this is a general phenomenon in case of decidability.

**Theorem 2.5 ($\mu$-elimination in decidable theories).** Let $\mathcal{A}$ be a first order structure. If the first order theory extended by the least fixed-point operator of $\mathcal{A}$ is decidable, then every fixed point formula in $\mathcal{A}$ is first order definable.

**Proof.** Let $\mu \gamma[\gamma], \phi(\gamma, \gamma)$ be a formula such that $\phi$ is first order. Moschovakis's stage comparison theorem [32] (see also [31] and [28]) says that the relation $\gamma \leq \gamma' \iff (X_\mu(\gamma') \rightarrow X_\mu(\gamma))$ is also definable in $\mu$-calculus. Let us write $\gamma \equiv \gamma'$ for $\gamma \leq \gamma' \land \gamma' \leq \gamma$ and $\gamma < \gamma'$ for $\gamma \leq \gamma' \land \gamma' \not< \gamma$. Then we can define natural numbers up to $\equiv$-equivalence in the usual way: i.e. $\text{zero}(\gamma) : \forall \gamma'. \gamma' \equiv \gamma' \rightarrow \text{succ}(\gamma, \gamma') \equiv \gamma < \gamma' \land \gamma < \gamma'$. Then $\gamma' \equiv \gamma$ relation as the least fixed point $\mu \gamma[\gamma, \gamma'].(\text{zero}(\gamma) \land \gamma \equiv \gamma' \lor \exists \gamma'. \text{succ}(\gamma, \gamma') \land \text{succ}(\gamma, \gamma') \land \gamma \equiv \gamma')$, and the multiplication in a similar fashion. Therefore, if the theory is decidable then there cannot be any formula $\mu \gamma[\gamma], \phi(\gamma, \gamma)$ such that there is an infinite ascending chain $X_0 < X_1 < X_2 < \cdots$. But in such a case, $X^* = X_k$ for some finite $k$, thus $X^*$ is first order definable. \hfill \Box

The above theorem says that a decidable theory has to eliminate least fixed-points operators. As mentioned in the above, a practical consequence of this fact is that if $\mathcal{A}$ is a general countable structure, then if an algorithm over $\mathcal{A}$ can be effectively realized, then its result must be already present in the first order theory of $\mathcal{A}$. We claim that it is the consequence of Theorem 2.5 that is crucial for effective realisation of the algorithms, rather than a much stronger property of $\omega$-categoricity. This claim leads us to the following definition.

**Definition 2.2 (Locally $\omega$-categorical theory).** Let $T$ be a (not necessarily complete) first order theory. Then we say that $T$ is locally $\omega$-categorical if every finite set of formulas in $T$ closed under logical connectives of first order logic yields a finite set of formulas up to equivalence in $T$.

Of course, $\omega$-categorical theories are locally $\omega$-categorical. Theories from Example 1.4 and Example 1.5 are locally $\omega$-categorical, but not $\omega$-categorical. We have the following theorem.

**Theorem 2.6.** Let $T$ be a decidable locally $\omega$-categorical theory. Then $T$ eliminates least fixed-points.

In fact, if $T$ is a decidable locally $\omega$-categorical theory, then every while-program terminates on $T$-definable sets. We will elaborate more on $T$-definable sets in non-complete theories in the next subsection.

In the reminder we shall consider positive existential theories that satisfy some weaker versions of the conclusion...
of Theorem 2.5. The pay-off for such generality is that we lose correspondence between sets with atoms and definable sets — the role of \( \text{Cont}(\text{Aut}(\mathcal{A})) \) will be played by the classifying topos \( \text{Set}[T] \) of a positive existential theory \( T \). Note, however, that because of Theorem 2.2 this lost is not that severe.

### 2.3 Positive existential theories

If \( \mathcal{A} \) is a single-sorted algebraic structure, then there is a one-to-one correspondence between first order definable subsets of \( \mathcal{A}^k \) and first order formulas in the language of \( \mathcal{A} \) up to equivalence modulo the theory \( Th(\mathcal{A}) \) of \( \mathcal{A} \). The reason for this is that in a complete theory (and \( Th(\mathcal{A}) \) is clearly complete) two formulas are equivalent iff they have the same interpretation in any model of the theory. Of course, the same is true if move to multi-sorted structures \( \mathcal{A} \), with the obvious correction that we have to consider definable subsets of \( \prod_{i \in \mathcal{I}} A_i \), where \( \mathcal{I} \) is a finite set of indices of sorts of \( \mathcal{A} \). It is tempting then to extend the notion of definability from structures \( \mathcal{A} \) to non necessarily complete theories \( T \) in the following way: we say that the class of formulas \( \phi \) up to equivalence modulo \( T \) is a \( T \)-definable set. With one caveat: we are not interested in all first order formulas, but in the formulas from a restricted fragment of intuitionistic first order logic, called positive existential logic, which we shall formally define now. Let \( (X_i)_{i \in I} \) be a set of variables. Positive existential formulas in variables \( (X_i)_{i \in I} \) over signature \( \Sigma \) with sorts \( (A_i)_{i \in I} \) are defined inductively according to the following rules:

- \( T, \bot \)
- \( R(t_1, t_2, \ldots, t_k) \) for a relation symbol \( R \subseteq A_{i_1} \times \cdots \times A_{i_k} \) in \( \Sigma \) and terms \( t_i : A_{i_1}, \ldots, t_k : A_{i_k} \) in \( \Sigma \)
- \( t = q \) for terms \( t, q \) over the same sort in \( \Sigma \)
- \( \phi \land \psi \land \psi \) for formulas \( \phi, \psi \)
- \( \exists x \in X \phi \) for a formula \( \phi \) and a variable \( x \in X \)

The reason for this restriction is that it gives us much more flexibility in deciding which sets are definable, and which are not. Let us also recall that on the syntactic level one may substitute a classical first order theory with a positive existential theory having the same definable sets. The idea is to introduce two new relational symbol \( P_\phi \) and \( N_\phi \) for every first order formula \( \phi \), then force \( P_\phi \) and \( N_\phi \) to be equivalent to \( \phi \) and \( \neg \phi \) to its negation. This process is known under the name “atomisation”, or “Morleyisation” (see [18] Chapter 2 or [19] Chapter D1.5). Therefore, one may recover the full power of classical first order logic in positive existential logic.

### Example 2.6 (Infinite decidable objects)

The first order theory of pure sets from Example 1.1 is equivalent to the following positive existential theory, called the theory of infinite decidable objects. The theory is over signature with a single sort \( N \) and one binary relation \( \# \subseteq N \times N \) and consists of the following axioms:

- \( \vdash a \neq b \lor a = b \)
- \( \exists x_1 \exists x_2 \cdots \exists x_n x_1 \neq x_2 \land \cdots \land x_1 \neq x_1 \land x_{n-1} \neq x_n \)

The first two axioms say that relation \( \# \) is complemented by the equality relation \( = \). The last axiom scheme describes an infinite sequence of axioms, whose \( n \)-th axiom says that there are at least \( n \) different elements.

If \( \phi \) is a definable set in the context \( \prod_{i \in \mathcal{I}} A_i \) and \( \psi \) is a definable set in the context \( \prod_{i \in \mathcal{J}} B_i \), then a definable function \( f : \phi \rightarrow \psi \) from \( \phi \) to \( \psi \) is a definable set in the context \( \prod_{i \in \mathcal{I}} A_i \times \prod_{i \in \mathcal{J}} B_i \) satisfying the following (positive existential) axioms:

\[
\begin{align*}
\forall \overline{x}, \overline{y} (f(\overline{x}, \overline{y}) & \rightarrow \phi(\overline{x}) \land \psi(\overline{y})) \\
\forall \overline{x} (\phi(\overline{x}) & \rightarrow \exists \overline{y} f(\overline{x}, \overline{y})) \\
f(\overline{x}, \overline{y}) & \land f(\overline{x}, \overline{z}) \rightarrow \overline{y} = \overline{z}
\end{align*}
\]

**Definition 2.3.** Let \( T \) be a positive existential theory. By \( \text{Def}(T) \) we shall denote the category of \( T \)-definable sets and \( T \)-definable functions with natural identities and compositions.

Category \( \text{Def}(T) \) is a coherent category, i.e. it has finite limits, stable existential quantifiers and stable unions of subobjects. Moreover, in case \( T \) is a classical first order theory, \( \text{Def}(T) \) also has stable universal quantifiers and is Boolean (i.e. it is Boolean Heyting category).

Unfortunately, \( \text{Def}(T) \) may lack finite disjoint coproducts in general. It has been observed by Makkai and Reyes [30] that any positive existential theory \( T \) can be extended to a positive existential theory \( T^{li} \) in such a way that \( T \) and \( T^{li} \) have essentially the same models and \( T^{li} \)-definable sets admit disjoint coproducts. Their construction follows an intuitive idea of “encoding” disjoint coproducts directly in the language of the theory. Theory \( T^{li} \) is obtained from \( T \) by extending the signature of \( T \) with a new sort \( \prod_{i \in \lambda} A_i \) for every finite cardinal \( \lambda \), together with new functional symbols \( i_j : A_j \rightarrow \prod_{i \in \lambda} A_i \) for every \( j \in \lambda \) and introducing axioms expressing that \( \prod_{i \in \lambda} A_i \) are disjoint coproducts with injections \( i_j \):

\[
\begin{align*}
i_j(\overline{x}) &= i_j(\overline{y}) \rightarrow \overline{x} = \overline{y} \\
i_j(\overline{x}) &= i_j(\overline{y}) \rightarrow \bot \\
\bigvee_{i \in \lambda} \exists \overline{x} \in A_i i_j(\overline{x}) &= \overline{z} \text{ for all } i \neq j;
\end{align*}
\]

It is routine to check that category \( \text{Def}(T^{li}) \) has disjoint coproducts.

There is one more important set theoretic construction that may be missing in \( \text{Def}(T) \). Let \( R \) be an equivalence relation on a set \( X \). Then, one may form the quotient set \( X/R \) of \( X \) by \( R \):

\[
X/R = \{ \langle x, \{ y : R(x, y) \} \rangle : x \in X \}
\]

Moreover, there is also a canonical surjection \( [-] : X \rightarrow X/R \) sending an element of \( X \) to its abstraction class:

\[
[x] = \langle x, \{ y : R(x, y) \} \rangle
\]
Conversely, if \( \phi(\overline{x}, \overline{y}) \) is any formula, then one may form an equivalence formula:

\[
\hat{\phi}(\overline{x}, \overline{x'}) = \forall_{\overline{y}} \phi(\overline{x}, \overline{y}) \leftrightarrow \phi(\overline{x'}, \overline{y})
\]

and represent \( \{\overline{y} : \phi(\overline{x}, \overline{y})\} \) by an imaginary element of \( \hat{\phi}(\overline{x}, \overline{x'}) \) of \( T^* \). Then by induction over structure of nested-sets one can show that every nested-definable set is \( T^+ \)-definable.

The above is also a consequence of the characterisation theorem for Boolean coherent toposes discussed in the next subsection.

### 2.4 Classifying topos

One important connection between Grothendieck toposes and logic is through the concept of classification. The general statement says that, for every logical theory \( T \) formalized in a positive existential fragment of infinitary first order logic there is a Grothendieck topos \( \text{Set}(T) \), and a generic model \( M_T \) of \( T \) inside \( \text{Set}(T) \). The term generic means that every model of the theory in any Grothendieck topos can be obtained from \( M_T \) as an application of the inverse image part of some geometric morphism into \( \text{Set}(T) \). Roughly speaking, a generic model of a theory is a well-behaved model that contains all of the information about the theory. The topos \( \text{Set}(T) \) is called the classifying topos for \( T \). Moreover, the above general statement is definitive, because every Grothendieck topos arises as the classifying topos for some positive existential fragment of infinitary first-order theory.

In this paper we are interested in theories \( T \) defined in positive existential fragment of finitary first order logic. In such a case, the classifying topos \( \text{Set}(T) \) is called coherent topos and can be obtained as the topos of sheaves on \( \text{Def}(T) \) with the usual coherent coverage (i.e. coverage generated by finite jointly regular-epimorphic families of morphisms; for more information consult [19] Chapter D3, especially Section D3.3, or Volume 3 of [11]) Moreover, the categories of sheaves on \( \text{Def}(T) \) and on \( \text{Def}(T^*) \) are equivalent, i.e. \( \text{Set}(T) \approx \text{Set}(T^*) \).

#### Example 2.8 (Sierpiński topos). The classifying topos of the propositional theory from Example 1.6 is the Sierpiński topos \( \text{Set}^{\rightarrow\rightarrow} \) — i.e. the topos of sheaves on the Sierpiński space \( \Sigma \). To see this from the perspective of the definable sets, observe that there are exactly three definable sets in this theory: corresponding to the false formula \( \bot \), to nullary relation \( p \) itself and to the true formula \( T \). Moreover, the coherent topology on the definable sets is the same as the topology of the Sierpiński space \( \{0, 1\} \), whose open sets are \( \bot = \emptyset, p = \{0\} \) and \( T = \{0, 1\} \).

#### Example 2.9 (Impossible topos). The classifying topos of the seemingly impossible theory of Example 1.7 is the presheaf topos \( \text{Set}^{\rightarrow\rightarrow\rightarrow\rightarrow} \). Like in Example 2.8 this topos can be presented as a topos of sheaves on a suitable topological space.

Let us recall the definition of a coherent object.
**Definition 2.4** (Coherent object). An object $A$ in a category with filtered colimits and kernel pairs is coherent if it is compact and for every morphisms $f : B \to A$ from a compact object $B$ the kernel $\ker(f)$ of $f$ is compact.

It is a classical result of Alexander Grothendieck that $\text{Def}(T^+)$ can be recovered from the classifying topos $\text{Set}(T)$ as the full subcategory spanned on coherent objects (see Corollary 3.3.8 in Chapter D of [19]).

**Example 2.10.** Continuing Example 2.5: all compact objects in $\text{Set}$, $\text{Set}^{\omega\omega}$ and $\text{Set}^{\omega\omega\cdots}$ are coherent. More generally, in a coherent topos, compact objects coincide with coherent objects iff every sub-compact object is compact. Therefore, they coincide in $\text{Cont}(\mathbb{G})$ iff $\mathbb{G}$ is (equivalent to) a group of automorphism of an $\omega$-categorical structure. For a counterexample, consider $\text{Set}^{\mathbb{G}} = \text{Cont}(\mathbb{G})$ for a discrete group $\mathbb{G}$. Coherent objects in $\text{Set}^{\mathbb{G}}$ are these sets $X$ with finitely many orbits whose stabilisers $\{\pi \in \mathbb{G} : \pi \bullet x = x\}$ are finite at every $x \in X$.

During 1974-1975 Walter Roelcke in a course on topology at the University of Munich introduced and systematically developed the theory of four natural uniform structures (or uniformities) on topological groups [37]. The lower (infinium) uniformity plays a crucial role in model theory and is nowadays known as Roelcke uniformity. Specifically, a topological group whose Roelcke uniformity is compact is called Roelcke precompact. An important characterisation theorem of Roelcke precompact groups is given in [41] as Theorem 2.4.: a topological subgroup $\mathbb{G} \leq \mathbb{S}_\omega$ (i.e. a non-Archimedean group) is Roelcke precompact iff for every continuous action $\mathbb{G}$ on a countable, discrete set $X$ with finitely many orbits, the induced action on $X^n$ has finitely many orbits for each natural $n$. This theorem says that Roelcke precompact groups are generalizations of oligomorphic groups, capturing their most important properties. In fact, Roelcke precompact groups are multi-sorted metric analogue of oligomorphic groups form the classical model theory [43] [3].

In the early '80s Andreas Blass and Andre Scedrov [4] tuning the representation theorem of André Joyal and Myles Tierney [20] (see also [12]) to Boolean toposes introduced the notion of a coherent group. A topological group $\mathbb{G}$ is coherent if the topos of its continuous actions $\text{Cont}(\mathbb{G})$ is a coherent topos. From this definition, they obtained the following characterisation: a topological group $\mathbb{G}$ is coherent if its every open subgroup $\mathbb{H} \subseteq \mathbb{G}$ has only finitely many double cosets: $\mathbb{H}x\mathbb{H} = \{h \bullet x \bullet k : h \in \mathbb{H}, k \in \mathbb{H}\}$ and $x \in \mathbb{G}$. Independently, this property, has been also shown to characterise Roelcke precompact groups with small open subgroups. Therefore, Roelcke precompact groups and coherent groups coincide (with small open subgroups).

The abovementioned theorem states that Boolean coherent Grothendieck toposes are precisely the finite products of categories of actions of topological groups of automorphisms of multi-sorted $\omega$-categorical structures [4] (this justifies the correspondence between nominal sets and classifying toposes from Figure 1). Moreover, positive existential theories $T$ that are classified by Boolean Grothendieck toposes are characterised by the following properties: a) in every context there are only finitely many formulas up to equivalence modulo $T$, b) every first order formula is classically equivalent to a coherent formula modulo the theory $[4]$.

Such theories $T$ have only finitely many completions, all of which are $\omega$-categorical. In fact, for a complete theory, the first property is equivalent to $\omega$-categoricity of the theory by (generalised) Ryll-Nardzewski theorem. The classifying topos may be constructed as the product of categories of the form $\text{Cont}(\text{Aut}(M_i))$, where $\text{Aut}(M_i)$ is the topological group of automorphism of the unique countable model $M_i$ corresponding to the $i$-th completion of theory $T$. The theory of dense linear orders from Example 1.5 satisfies these properties. Moreover, all theories that satisfy these properties are locally $\omega$-categorical [4].

As mentioned in the introduction, we are interested in theories $T$ such that Algorithm 1 is effective on $T$-definable sets. One property of such theories is local $\omega$-categoricity introduced in Section 2.2 for classical first order theories. We could define a suitable version of local $\omega$-categoricity for positive existential theories, but for the purpose of this paper (termination of Algorithm 1) it suffices to require a weaker property. Let us recall that by Theorem 2.6, every locally $\omega$-categorical theory eliminates least fixed points. We shall focus on elimination of transitive closures only.

**Definition 2.5** (Elimination of transitive closures). Let $T$ be a positive existential theory. We say that $T$ eliminates transitive closures if for every $T$-definable binary relation $R$ there is a $T$-definable relation $R^+$ such that: $R^+ = \bigvee_i R_i$, where: $R_i = R$ and $R_{i+1} = R \circ R_{i+1}$.

Observe that the above definition is not first order, and in fact cannot be axiomatised by first order formulas.

**Example 2.11.** All working examples of theories in this paper eliminate transitive closures:

- if $T$ is $\omega$-categorical then for every $T$-definable binary relation $R$, there are only finitely many $T$-definable relations in the same context; therefore $\bigvee_i R_i$ may be reduced to a finite disjunction; this includes Example 1.1, Example 1.2 and Example 1.3
- more generally, if $T$ is locally $\omega$-categorical then since $T$ eliminates least fixed points, it also eliminates transitive closures; this includes Example 1.4 and Example 1.5

---

4To the best knowledge of the authors, this coincidence has not been observed before and the connection between metric model theory and classifying toposes has never been exploited.
• if $T$ is the propositional theory from Example 1.6 or the seemingly impossible theory from Example 1.7, then every infinite ascending chain of $T^+$-definable sets $\phi_0 \subseteq \phi_1 \subseteq \phi_2 \subseteq \cdots$ stabilizes, therefore $\bigvee_i R^i = \bigvee_{i \leq k} R^i$ for some finite $k$

**Theorem 2.7.** Let $T$ be a decidable positive existential theory that eliminates transitive closures. Then Algorithm 1 is effective on $T$-definable inputs.

### 3 Automata

For this section we fix a single positive existential theory $T$ that has disjoint coproducts and eliminates imaginaries. Moreover, if not stated otherwise, all objects and morphisms live in $\text{Set}(T)$ — the classifying topos of $T$. The subobject classifier will be denoted by $\Omega$, and the characteristic function of a subobject $s : A_0 \rightarrow A$ by $\chi_s : A \rightarrow \Omega$.

#### 3.1 Preliminaries

An object of words in an alphabet $\Sigma$ is the free monoid $(\Sigma^*, \text{concat}, e)$ over $\Sigma$ generators. It consists of the concatenation morphism $\text{concat} : \Sigma^* \times \Sigma^* \rightarrow \Sigma^*$ and an empty word $e : 1 \rightarrow \Sigma^*$. By a language $L$ over alphabet $\Sigma$, we shall mean a subobject of $\Sigma^*$, or equivalently a morphism $L : \Sigma^* \rightarrow \Omega$.

**Definition 3.1 (Automaton).** A non-deterministic automaton $A$ is a quadruple $A = \langle s_0 : I \rightarrow S, s_f : F \rightarrow S, \sigma : \Sigma \times S \rightarrow S \rangle$, where:

- $s_0 : I \rightarrow S$ is a monomorphism of initial states
- $s_f : F \rightarrow S$ is a monomorphism of final states
- $\sigma : \Sigma \times S \rightarrow S$ is the transition relation

Automaton $A$ is called deterministic if $I$ is the terminal object $1$ and the transition relation $\sigma$ is functional.

An automaton $A$ is called a $T$-automaton if all the data from its definition are $T$-definable. Observe that for any object $S$, we have the canonical monoidal structure on $\Omega^{S \times S}$, given by the internal composition of binary relations. Therefore, the adjoint transposition $\sigma^! : \Sigma \rightarrow \Omega^{S \times S}$ induces a unique homomorphisms of monoids: $\sigma^! : \Sigma^* \rightarrow \Omega^{S \times S}$, and by transposition a relation: $\overline{\sigma} : \Sigma^* \times S \rightarrow S$.

**Definition 3.2 (Language of an automaton).** Let $A = \langle s_0 : I \rightarrow S, s_f : F \rightarrow S, \sigma : \Sigma \times S \rightarrow S \rangle$ be a non-deterministic automaton. By the language $L(A)$ recognized by $A$ we shall mean the subobject of $\Sigma^*$ that corresponds to the following relation:

$$\Sigma^* \xrightarrow{id_{\Sigma^*} \times s_0} \Sigma^* \times S \xrightarrow{\Sigma^* \times \sigma} S \xrightarrow{s_f} 1$$

In case automaton $A$ is deterministic, the monoid of relations $\Omega^{S \times S}$ can be substituted by the monoid of functions $S^S$ with internal composition, and $L(A)$ can be constructed as the pullback of $s_f$ along $\overline{\sigma} \circ (id_{\Sigma^*} \times s_0) : \Sigma^* \rightarrow S$.

Before moving to more advanced theory, let us prove a simple theorem.

**Theorem 3.1.** Let $T$ be a decidable theory that eliminates transitive closures of binary relations. Then the problem of emptiness of a $T$-definable automaton is decidable.

**Proof.** The problem of emptiness of an automaton is equivalent to the problem of reachability of a final state from one of the initial states. Therefore it suffices to compute the transitive-reflexive closure $\phi$ of its underlying transition relation by Algorithm 1 and then check if the formula $\exists s \in s_0 \exists f : s_f, \phi(s, f)$ is provable in $T$. \[\square\]

Kaminski and Francez studied, so called, finite memory automata [21]: i.e. automata augmented with a finite set of registers, each of which can hold a natural number, and the automata can test for equality between registers and alphabets. Here is a suitable generalisation of this definition to a general structure $\mathcal{A}$.

**Definition 3.3 (Register automata).** An $\mathcal{A}$-automata with $k$ registers over alphabet $\Sigma$ is a quadruple $(S, s_0, s_f, F)$ such that:

- $S$ is a finite set of states
- $I \subseteq S$ is a set of initial states, and $\phi_I \subseteq A^k$ is a set of possible initial configurations of registers
- $F \subseteq S$ is a set of final states, and $\phi_F \subseteq A^k$ is a set of possible final configurations of registers
- $\delta \subseteq (\Sigma \times S^k \times (S \times A^k))$ is a transition relation such that for every $s, s' \in S$ the relation $\delta(s, s') \subseteq (\Sigma \times S^k) \times A^k$ is $\mathcal{A}$-definable.

We could state an even more general definition suitable for any theory $T$, but we refrain from doing this for the following reason: every register $\mathcal{A}$-automata with states $S$ and $k$-registers is equivalent to:

1. A register $\mathcal{A}$-automata with a single state.
2. An $\mathcal{A}$-automata (without registers).

Because $\text{Def}(\mathcal{A})$ has disjoint coproducts, it can interpret finite cardinals. Moreover, every function between finite cardinals is definable. Let us assume that the context of $S$ is $A^m$. Therefore, $S' = S \times A^k$ can be thought of as either the object of states of a definable automaton, or as the $\mathcal{A}$-automata with $k + m$ registers and a single state 1.

#### 3.2 Myhill-Nerode theorem

Consider the following morphism:

$$\Sigma^* \xrightarrow{\text{concat}} \Sigma^* \xrightarrow{L} \Omega$$

Its transposition $(L \circ \text{concat})^! : \Sigma^* \rightarrow \Omega^{\Sigma^*}$ maps a word $w$ to the predicate: $\lambda x. w x \in L$.

**Definition 3.4 (Myhill-Nerode relation).** Let $L : \Sigma^* \rightarrow \Omega$ be a language. By the Myhill-Nerode relation $\text{MN}(L)$ of $L$, we shall mean the kernel relation of $(L \circ \text{concat})^!$, and by the the $\text{Myhill-Nerode quotient of } L$ we shall mean the coequaliser of this kernel pair:

$$\text{MN}(L) \leadsto \Sigma^* \xrightarrow{(L \circ \text{concat})^!} \Omega^{\Sigma^*}$$
Intuitively, two words \( w, v \in \Sigma^* \) are related by Myhill-Nerode relation \( MN(L) \) iff for every \( x \in \Sigma^* \) we have that: \( wx \in L \iff vx \in L \).

**Lemma 3.2.** Let \( A = (S_0: 1 \to S, F; K \to S, \sigma: \Sigma \times S \to S) \) be a deterministic automaton. The Myhill-Nerode quotient of \( L(A) \) is a sub-quotient of \( S \).

**Proof.** We have the following morphism:

\[
\begin{array}{ccc}
\Sigma^* \times \Sigma^* & \xrightarrow{\text{concat}} & \Sigma^* \\
\tau \times \sigma & \downarrow \tau \times \sigma & \downarrow \tau \times \sigma \\
S^S \times S^S & \xrightarrow{\circ} & S^S \times S \rightarrow S \\
\sigma_{id_S \times S_0} & \downarrow & \sigma_{id_S \times S_0} \\
\end{array}
\]

which by transposition corresponds to the morphism:

\[ k: \Sigma^* \rightarrow \Omega^S \]

The kernel pair \( \text{Ker}(k) \Rightarrow \Sigma^* \) of this morphism \( k \), is the Myhill-Nerode relation of the language \( L(A) \). Such an equivalence relation induces a quotient object \( \Sigma^*/k \) as the coequaliser of the kernel pair:

\[ \text{Ker}(k) \Rightarrow \Sigma^* \xrightarrow{[\text{-}k]} \Sigma^*/k \]

On the other hand, the morphism \( \tau \circ (id_{\Sigma^*} \times S_0): \Sigma^* \rightarrow S \), which will be denoted by \( s \), has its own kernel pair:

\[ \text{Ker}(s) \Rightarrow \Sigma^* \xrightarrow{\pi_1} S \]

We want to show that \( \Sigma^*/k \) is a quotient of \( \Sigma^*/s \), or equivalently that the relation \( \text{Ker}(s) \) is coarser than \( \text{Ker}(k) \). We shall prove it on generalised elements: \( x, y: X \rightarrow \Sigma^* \). That is, we want to show that if \( \exists s = x \circ y \) then \( \exists k = x \circ y \).

But, by the triangle equality for exponent: \( k \circ x = k \circ y \) iff \( k^0 \circ (x \times id_{\Sigma^*}) = k^1 \circ (y \times id_{\Sigma^*}) \). Moreover, because \( \epsilon \) is the unit for \( \text{concat} \), the following diagram commutes:

\[
\begin{array}{ccc}
X \times \Sigma^* & \xrightarrow{(id_X, \epsilon)} & X \\
\xrightarrow{x \times id_{\Sigma^*}} & \downarrow & \\
\Sigma^* \times \Sigma^* & \xrightarrow{\text{concat}} & \Sigma^* \\
\end{array}
\]

with the top row being mono. Therefore, \( k^0 \circ (x \times id_{\Sigma^*}) = k^1 \circ (y \times id_{\Sigma^*}) \) iff \( (x \times id_{\Sigma^*}) \circ (id_X, \epsilon) = (y \times id_{\Sigma^*}) \circ (id_X, \epsilon) \) iff \( F \circ s \circ x = F \circ s \circ y \), which completes the proof of the claim. Now, because \( s \circ \pi_1^* = s \circ \pi_2^* \), we have that: \( k \circ \pi_1^* = k \circ \pi_2^* \) and by the definition of the kernel of \( k \) there is a unique monomorphism of relations \( j: \text{Ker}(s) \rightarrow \text{Ker}(k) \), i.e.: \( j \circ \pi_1^* = \pi_1^* \) and \( j \circ \pi_2^* = \pi_2^* \). Therefore, by the universal property of the coequaliser \( \Sigma^*/s \), there is a unique (necessarily epi) morphism: \( \Sigma^*/s \rightarrow \Sigma^*/k \). \( \Box \)

**Lemma 3.3.** Let \( L: \Sigma^* \rightarrow \Omega \) be a language. The Myhill-Nerode quotient of \( L \) can be equipped with the structure of a deterministic automaton that recognizes \( L \).

**Theorem 3.4 (Subcompact rational languages).** Sub-compact deterministic automata recognize the languages whose Myhill-Nerode quotients are sub-compact.

**Proof.** If \( X \) is a sub-quotient of \( A \) via \( A_0 \), then we may form the pushout:

\[
\begin{array}{ccc}
A & \rightarrow & P \\
\downarrow & & \downarrow \\
A_0 & \rightarrow & X
\end{array}
\]

A pushout of an epimorphism \( e: A_0 \rightarrow X \) is an epimorphism, thus \( P \) is a quotient of \( A \). But a quotient of a compact object is compact, so \( P \) is compact if \( A \) is. Moreover, in a topos a pushout of a monomorphism is again monomorphism. Therefore, if \( X \) is a sub-quotient of a compact object, then it is actually a quotient of a compact object. \( \Box \)

From the above theorem we can instantly get the generalisation of Myhill-Nerode Theorem for nominal sets.

**Corollary 3.5.** In a topos of continuous actions of a topological group, deterministic automata with finitely many orbits recognize exactly the languages whose Myhill-Nerode relations have finitely many orbits.

**Theorem 3.6.** Let \( T \) be a theory that eliminates transitive closures of binary relations. Then \( T \)-definable deterministic automata recognize exactly the languages whose Myhill-Nerode quotients are \( T \)-definable.

**Proof.** A definable morphism \( \sigma: \Sigma \times S \rightarrow S \) induces a binary relation on \( S: R_\sigma(a, b) \leftrightarrow \exists x, y: \sigma(x, a) = b \). Since \( T \) eliminates transitive closures, the transitive closure \( R_\sigma^n \) of \( R_\sigma \) factors as: \( R_\sigma \cup R_\sigma^2 \cup \ldots \cup R_\sigma^n \) for some finite \( n \). Unwinding the definition of \( R_\sigma^n \), this yields: \( R_\sigma^n(a, b) \leftrightarrow \exists y: \sigma(x, a) = b \). Therefore, \( R_\sigma^n(a, b) \leftrightarrow \exists y: \sigma(x, a) = y \), which means that the image of \( \sigma: \Sigma \times S \rightarrow S \) factors through \( \Sigma^n \rightarrow \Sigma^1 \). This means that \( \Sigma^n \) is coherent. On the other hand, \( \Sigma^1/k \) can be described as the filtered colimit of \( \Sigma^1/k \circ j \), where \( j: \Sigma^1 \rightarrow \Sigma^* \) is the natural injection of coproducts. Therefore, the epimorphism \( \Sigma^1/k \rightarrow \Sigma^j/k \circ j \) followed by a monomorphism \( \Sigma^j/k \circ j \rightarrow \Sigma^*/k \). By the uniqueness of epi-mono factorisation, \( \Sigma^j/k \circ j \approx \Sigma^*/k \), and \( \Sigma^*/k \) is coherent. \( \Box \)

**Example 3.1.** In all of the theories from Examples 1.1, 1.2, 1.3, 1.4, 1.5, 1.6 and 1.7 definable deterministic automata recognize exactly the languages whose Myhill-Nerode quotients are definable.

Definable non-deterministic automata are generally more expressive than definable deterministic automata. The reason is that, unlike finite sets, definable sets are not stable under the power-set construction.

**Example 3.2 (Definable deterministic vs. non-deterministic automata).** Consider the following language in \( \text{Set}(N) \):

- the alphabet is the set of all atoms, i.e.: \( \Sigma = N \)
• the language consists of all words over alphabet $\Sigma$, such that in each word there is a letter that appears at least twice, i.e.: $L = \{w''nw'w' : n \in \Sigma \land w, w', w'' \in \Sigma^*\}$

One may check that the Myhill-Nerode quotient of $L$ has infinitely many orbits, therefore $L$ cannot be recognised by a deterministic automaton. On the other hand, the non-deterministic automaton from Figure 3 recognizes it: the automaton loops in state $s_0$ for a number of times, non-deterministically moves to the state "n" after seeing a letter $n \in \Sigma$, and then loops in that state until another letter $n$ appears in the word, in which case the automaton moves to the final state $sf$.

### 3.3 Recognition by monoids

We say that a language $L$ over alphabet $\Sigma$ is recognized by a monoid $M$ if there is a subobject $F$ of $M$ and a homomorphism $h: \Sigma^* \to M$ such that: $\chi_F \circ h: \Sigma^* \to \Omega$ is the characteristic morphism of $L$. It is well-known that classical regular languages (i.e. languages recognised by finite automata in Set) are precisely the languages recognised by finite monoids. The correspondence does not carry over to definable regular languages and definable monoids — in general the notion of a language recognised by a coherent deterministic automaton is much stronger than the notion of a language recognised by a coherent monoid.

**Example 3.3** (Definable deterministic automata vs. definable monoids). Consider the following language in $\text{Set}(N)$:

- the alphabet is the set of all atoms, i.e.: $\Sigma = N$
- the language consists of all words over alphabet $\Sigma$, such that in each word the first appears at least twice, i.e.: $L = \{nw'nw' : n \in \Sigma \land w, w' \in \Sigma^*\}$

One may check that $L$ cannot be recognised by a monoid that has only finitely many orbits. On the other hand, the deterministic part (without the dashed transition) of the automaton from Figure 3 clearly recognizes it: the automaton moves to the state $n$ after seeing a letter $n \in \Sigma$, and then loops in that state until another letter $n$ appears in the word, in which case the automaton moves to the final state $sf$.

Therefore, to hope for such a correspondence, we need a more general notion of a monoid, or a more restrictive notion of an automaton. Languages recognized by finitary monoids in $\text{ZFA}(N)$ are the subject of the thesis of Rafał Stefanski [40], [10]. The author developed a model of restricted deterministic automata whose languages are recognizable by finitary monoids. We shall take another path and generalise the concept of a monoid. If $\text{Set}(T)$ is the topos under consideration, then by $\text{Rel}(T)$ we shall denote the category of binary relations in $\text{Set}(T)$. Category $\text{Rel}(T)$ equipped with the cartesian product $\times$ and the terminal object 1 from $\text{Set}(T)$ forms a monoidal category. By a monoid in $\text{Set}(T)$ we shall mean a monoid object in $\text{Rel}(T)$. Explicitly, a monoid $M$ consists of an object $M$ together with the multiplication relation $\mu: M \times M \to M$ and the unital monomorphism $\eta: M_0 \to M$ subject to the usual monoid laws. The category of monoids and their homomorphisms will be denoted by $\text{ProMon}(T)$. Because $\text{Rel}(T)$ has small coproducts inherited from $\text{Set}(T)$, for every $\Sigma$ there is a free monoid $\Sigma^*$, which coincides with the free monoid in $\text{Set}(T)$.

We should also observe that every promonoid has a representation as a monoid, i.e.: every promonoid $M$ gives rise to the power monoid $\mathcal{P}(M)$ by convolution: the unit $1_\mathcal{P}$ is just the characteristic map of $\eta$, and the multiplication $\Omega^M \times \Omega^M \to \Omega^M$ is given as the free cocontinous extension of $M \times M \to \Omega^M$ on each coordinate.

To our surprise, the concept of recognisability by promonoids has not been studied before. Therefore, the next theorem and the following Corollary 3.8 has been unknown even in case of the usual nominal sets.

**Theorem 3.7** (Characterisation of non-deterministic regular languages). Let $\mathcal{K}$ be a class of objects closed under binary products. The languages recognized by non-deterministic $\mathcal{K}$-automata coincide with the languages recognized by $\mathcal{K}$-promonoids.

**Proof.** Let us observe that for every object $M$ the object $\Omega^{M \times M}$ carries a canonical monoidal structure of binary relations under composition. Because the composition is cocontinuous in both variables, $\Omega^{M \times M}$ is freely generated by its restriction to the singletons, i.e. by a promonoid $\mathcal{R}_M = \langle M \times M, \circ, \cdot, \rangle$, which we shall call the promonoid of binary relations on $M$. Every promonoid $M = \langle M, \mu, \eta \rangle$ has a relational representation as a submonoid of the promonoid of $\mathcal{R}_M$ given by the transposition of its multiplication $\mu$, i.e. $\mu^\dagger: M \to M \times M$ is a homomorphism in the category of promonoids$^5$.

We claim that if a language $L$ is recognized by a monoid $M$ then it is recognized by promonoid $\mathcal{R}_M$. Let $F: M \to 1$ be a characteristic function of a subobject of $M$. Then $\eta^\dagger \times F: M \times M \to 1$ is a characteristic function of a subobject of $M \times M$. Moreover, $\mu^\dagger \circ(\eta^\dagger \times F) = F$ by the definition of the transposition and neutrality of $\eta$ under $\mu$. Therefore, if there is a relational homomorphism $h: \Sigma^* \to M$ such that: $L = F \circ h$, then $L$ is recognised by homomorphism $\mu^\dagger \circ h: \Sigma^* \to M\mathcal{R}_M$ with subobject $\eta^\dagger \times F$.

---

$^5$One may treat this fact as the generalisation of the Cayley representation for a monoid $M$ as a submonoid of the endo-monoid $M^\dagger$ under functional composition.
Now, if we define a non-deterministic automaton $\mathcal{A}$ as:
- its object of states is $M$
- its transition relation is $\mu \circ (h \times id_M) : \Sigma \times M \rightarrow M$
- its initial states are $\eta$
- its final states are $F$
then, $L(\mathcal{A})$ is given by:
\[
\Sigma^* \xrightarrow{id_{\Sigma} \times \eta} \Sigma^* \times M \xrightarrow{\Sigma^* \times id_M} M \times M \xrightarrow{\mu} M \xrightarrow{\eta} 1
\]
because $\mu^* \circ h$ is a homomorphism as has been shown in the above. But $F \circ \mu \circ (h \times id_M) \circ (id_{\Sigma} \times \eta) = F \circ \mu \circ (h \times \eta) = F \circ h$ which completes this part of the proof.

In the other direction, let us assume that $L$ is recognized by an automaton $A = \langle s_0 : I \rightarrow S, s_f : F \rightarrow S, \sigma : \Sigma \times S \rightarrow S \rangle$. Then $L$ is given as the left path on the following diagram:
\[
\Sigma^* \xrightarrow{id_{\Sigma} \times \chi(s_0)} S \times S \xrightarrow{\chi^{\top} \times id_S} \chi(s_0 \times id_S) \xrightarrow{\chi(s_0 \times id_S)} 1
\]
The square commutes by the definition of relational composition. Therefore, if we equip $R_S$ with $\chi(s_0 \times \chi(s_f))$, then $\sigma^{\top} : \Sigma^* \rightarrow R_S$ will recognize $L$. □

Because $T$-definable objects are closed under binary products, from Theorem 3.7 we can get the following characterisation of definable non-deterministic languages.

**Corollary 3.8.** A language can be recognised by a $T$-definable promonoid if and only if it can be recognised by a $T$-definable non-deterministic automaton.

### 4 Conclusions and further work

This paper makes the following contributions. First of all, we show that mathematics can be transferred back and forth between sets with atoms and categories of continuous actions of topological groups (Theorem 2.2). Because the topos of continuous actions is much better behaved than the topos of sets with atoms, this allows for a simplification of the mathematical reasoning. For our second contribution, we showed the limits of the classical approach to computability in sets with atoms. It may be inferred from the analysis in [7] that effectiveness of the naive algorithms to the reachability-like problems defined in a decidable complete first order theory is equivalent to $\omega$-categoricity of the theory. Our Theorem 2.5 shows that $\omega$-categoricity of the theory is actually equivalent to the existence of any effective algorithm for reachability-like problems. This leads to our third contribution. We showed how to move the concept of algorithms and automata beyond complete first order theories. This requires replacing toposes of continuous actions of topological groups by general classifying toposes for positive existential theories. We have coined a new property of a theory: “elimination of transitive closures” and showed that in some aspects it behaves like $\omega$-categoricity for complete first order theories. This includes Theorem 2.7 for the reachability problem, Theorem 3.1 for the emptiness problem of an automaton, and Nihil-Nerode like Theorem 3.6, which is central for studying behaviours of deterministic automata. For our forth contribution, we established a general correspondence between languages of non-deterministic automata and relational monoids in Theorem 3.7. This correspondence has not been known before even for very restricted cases (like nominal sets). The meta-contribution of this paper is in showing that many concepts incarnate in different areas of mathematics and by linking these incarnations together we can simplify our thoughts and proofs. For example, the connection between coherent groups (defined by Blass and Scedrov in ’80s to characterise coherent toposes of continuous actions of topological gorups) and Roelcke precompact groups (defined by Roelcke in ’70s to characterise topological dynamics) has not been observed before. Similarly, most of the theorems from [5] with advanced proofs are easy consequences of the facts from the theory of classifying toposes and the connection established in this paper (compare the proof of Theorem 5.1 in [5] with our Theorem 2.7).

For further work we shall study other concepts and algorithms definable in positive existential theories, e.g.: constraint satisfaction problems with definable sets of constraints, definable pushdown automata, definable Turing machines, etc. Our recent paper [36] shows that carrying over some of these results to non-Boolean classifying toposes is a challenging task.

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A Categories and sheaves

**Definition A.1** (Kernel pair). Let $f : A \to B$ be a morphism.

By a kernel pair $\text{Ker}(f) \triangleright\triangleleft A$ we shall mean the following pullback provided it exists:

$$
\begin{array}{c}
\text{Ker}(f) \\
\parallel \\
\downarrow \pi_1 \\
A \\
\pi_2 \\
\downarrow \\
B
\end{array}
$$

**Definition A.2** (Regular category). A category $\mathcal{C}$ is regular if it has finite limits, every kernel pair $\text{Ker}(f) \triangleright\triangleleft A$ has a coequaliser, and regular epimorphisms are stable under pullbacks.

Every kernel pair $\text{Ker}(f) \triangleright\triangleleft A$ is an equivalence relation on $A$, but the converse need not be true.

**Definition A.3** (Effective regular category). A regular category $\mathcal{C}$ is effective if every equivalence relation $\mathcal{R} \triangleright\triangleleft A$ is the kernel pair of some morphism.
the full subcategory of the slice \( \mathcal{C}/A \) spanned on monomorphisms \( X \rightarrow A \) for some \( X \in \mathcal{C} \). Because monomorphisms are stable under pullbacks, every morphism \( f : A \rightarrow B \) induces a functor \( f^* : \text{Sub}_{\mathcal{C}}(B) \rightarrow \text{Sub}_{\mathcal{C}}(A) \), i.e.: the inverse image functor.

**Definition A.4** (Coherent category). A regular category \( \mathcal{C} \) is coherent if for every \( A \in \mathcal{C} \) the category \( \text{Sub}_{\mathcal{C}}(A) \) has finite coproducts and for every morphism \( f : A \rightarrow B \) the inverse image functor \( f^* : \text{Sub}_{\mathcal{C}}(B) \rightarrow \text{Sub}_{\mathcal{C}}(A) \) preserves finite coproducts.

If \( \mathcal{C} \) is regular then \( f^* \) has the left adjoint \( \exists_{f} : \text{Sub}_{\mathcal{C}}(A) \rightarrow \text{Sub}_{\mathcal{C}}(B) \), but may lack the right adjoint in general.

**Definition A.5** (Heyting category). A coherent category \( \mathcal{C} \) is Heyting if for every morphism \( f : A \rightarrow B \) the inverse image functor \( f^* : \text{Sub}_{\mathcal{C}}(B) \rightarrow \text{Sub}_{\mathcal{C}}(A) \) has the right adjoint \( \forall_{f} : \text{Sub}_{\mathcal{C}}(A) \rightarrow \text{Sub}_{\mathcal{C}}(B) \).

**Definition A.6** (Boolean category). A category \( \mathcal{C} \) is Boolean if for every object \( A \in \mathcal{C} \) the category of subobjects \( \text{Sub}_{\mathcal{C}}(A) \) is a Boolean algebra.

**Definition A.7** (Pretopos). An coherent category is a pretopos if it is effective regular and finite coproducts are disjoint.

**Definition A.8** (Topos). A topos is a finitely complete cartesian closed category with a subobject classifier.

A functor between toposes that preserves the topos-theoretic structure is called a logical functor.

**Definition A.9** (Logical functor). A functor between toposes is called logical if it preserves finite limits, exponents and the subobject classifier.

### A.1 Grothendieck toposes

Let \( \mathcal{C} \) be a category.

**Definition A.10** (Covering family). A covering family of an object \( U \in \mathcal{C} \) is a collection of morphisms \( i_{ij} : U_{ij} \rightarrow U \). For convenience, we shall index the morphisms \( i_{ij} \) by some set \( I \) and write \( (i_{ij})_{i \in I} \).

\[
\begin{array}{c}
U_{ij} \\
\ldots
\end{array}
\xymatrix{
\ar[rr]^{i_{ij}} & & \ar[r] & U
\end{array}
\]

If \( F : \mathcal{C}^{op} \rightarrow \text{Set} \) is a presheaf and \( (i_{ij} : U_{ij} \rightarrow U)_{i \in I} \) a covering family on \( U \in \mathcal{C} \) then the idea of a sheaf is that elements of \( F(U) \) should be uniquely determinable by the elements of \( F(U_{ij}) \). Not every family \( (x_{i} \in F(U_{ij}))_{i \in I} \) can describe an element of \( F(U) \) — to do this, elements \( (x_{i})_{i \in I} \) have to be compatible with each other. That is, for every pair of covers \( i_{ij} : U_{ij} \rightarrow U, i_{ji} : U_{ji} \rightarrow U \) restricting \( x_{i} \in F(U_{ij}) \) and \( x_{j} \in F(U_{ji}) \) to their common domain, i.e. restricting to the intersection of \( U_{ij} \) with \( U_{ji} \), should give the same element. If \( \mathcal{C} \) has enough pullbacks then we may formalize this condition as follows — a family \( (x_{i} \in F(U_{ij}))_{i \in I} \) describe “the same element” if for every pair of indices \( i, j \in I \) the restrictions of \( x_{i} \) and \( x_{j} \) along the pullback projections \( \pi_{i} : U_{ij} \times_{U} U_{j} \rightarrow U_{i} \) and \( \pi_{j} : U_{i} \times_{U} U_{j} \rightarrow U_{j} \) respectively:

\[
\begin{array}{ccc}
U_{ij} \times_{U} U_{j} & \xrightarrow{\pi_{i}} & U_{i} \\
\downarrow & & \downarrow \\
U_{j} & \xrightarrow{\pi_{j}} & U_{j}
\end{array}
\]

are equal: \( F(\pi_{i})(x_{i}) = F(\pi_{j})(x_{j}) \). Moreover, this condition is equivalent to the condition that for any object \( V \in \mathcal{C} \) and any morphism \( h : V \rightarrow U_{i} \times_{U} U_{j} \) we have that \( F(\pi_{i} \circ h)(x_{i}) = F(\pi_{j} \circ h)(x_{j}) \). By the universal property of the pullback this, in turn, is equivalent to: for any object \( V \) and any morphisms \( f : V \rightarrow U_{i} \) and \( g : V \rightarrow U_{j} \) that commute with \( i_{i} \) and \( i_{j} \):

\[
\begin{array}{ccc}
V & \xrightarrow{f} & U_{i} \\
\downarrow & & \downarrow \\
V & \xrightarrow{g} & U_{j}
\end{array}
\]

i.e. \( i_{i} \circ f = i_{j} \circ g \), we have that \( F(f)(x_{i}) = F(g)(x_{j}) \). Notice that the last condition is pullback-free and can be stated in any category \( \mathcal{C} \). Therefore, we take it for the definition of a family that describes “the same element”. Such families are called “matching families”.

**Definition A.11** (Matching family). Let \( F : \mathcal{C}^{op} \rightarrow \text{Set} \) be a presheaf on a category \( \mathcal{C} \). We say that a collection of elements \( (x_{i} \in F(U_{ij}))_{i \in I} \) constitutes a matching family with respect to a covering family \( (i_{ij} : U_{ij} \rightarrow U)_{i \in I} \) if for any morphisms \( f : V \rightarrow U_{i}, g : V \rightarrow U_{j} \) such that \( i_{i} \circ f = i_{j} \circ g \) we have \( F(f)(x_{i}) = F(g)(x_{j}) \).

**Definition A.12** (Sheaf condition). A presheaf \( F : \mathcal{C}^{op} \rightarrow \text{Set} \) satisfies the sheaf condition for a covering family \( (i_{ij} : U_{ij} \rightarrow U)_{i \in I} \) if for every family \( (x_{i} \in F(U_{ij}))_{i \in I} \) that describes "the same element" (i.e. for any matching family) this abstract element has a materialization in an actual element \( x \) — i.e. there is a unique element \( x \in F(U) \) such that \( F(i_{ij})(x) = x_{i} \) for all \( i \in I \):
Definition A.15 (Trivial coverage). Let $s$ be a pre-coverage on $C$ such that the only covering families are singletons consisting of identities — i.e. for every $U \in C$ we put $s(U) = \{(id_U): U \to U\}$. We call pre-coverage $s$ the trivial (pre-)coverage on $C$.

Every presheaf is a sheaf for the trivial pre-coverage and so $Sh(C,s) = Set^{op}$.

Example A.1 (Atomic pre-coverage). Let $s$ be a pre-coverage on $C$ such that every morphism is covering — i.e. for every $U \in C$ we put $s(U) = \{\{f\} : f : X \to U \in U\}$. We call pre-coverage $s$ the atomic (pre-)coverage on $C$.

Example A.2 (Coherent pre-coverage). Let $C$ be a coherent category. Let $s$ be a pre-coverage on $C$ consisting of finite jointly regular-epimorphic families. We call pre-coverage $s$ the coherent (pre-)coverage on $C$.

Definition A.16 (Coverage). A pre-coverage $s$ is called coverage if it is stable under weak pullbacks:

- for every morphism $f : V \to U$ and every covering family $(i_t : U_t \to U)_{t \in I} \in s(U)$, there is a covering family $(i'_t : V \to V')_{t \in I} \in s(V)$ and a collection of morphisms $(h_{i,t,j} : V_j \to U_{i_t})_{t \in I,j \in J}$ such that $f \circ i'_t = i_t \circ h_{i_t,j}$.

Remark A.1. The stability under weak-pullbacks tries to express stability of coverages under pullbacks in case the category does not have pullbacks. Indeed, if $C$ has pullbacks then the above property can be reformulated as follows:

- for every morphism $f : V \to U$ and every covering $(i_t : U_t \to U)_{t \in I} \in s(U)$, the pullback family $(f^*(i_t) : f^*(U_t) \to V)_{t \in I}$ is a covering family on $V$, that is $(f^*(i_t))_{t \in I} \in s(V)$.

without changing the notion of sheaf (i.e. for any pre-coverage, there is a coverage satisfying the above property with the same category of sheaves).

Let $C$ be a small category with a coverage $s$. Then the category of sheaves $Sh(C,s)$ is a cocomplete topos with a small generating family. The converse is true as well. Such toposes are called Grothendieck toposes.

Definition A.17 (Classifying topos for positive existential theory). Let $T$ be a positive existential theory. By the classifying topos $Set(T)$ for $T$ we shall mean the topos of sheaves on $Def(T)$ with the coherent (pre-)coverage.

B Topological dynamic

Definition B.1 (Topological space). A topological space on a set $A$ is a collection $\tau$ of subsets $U \subseteq A$ containing $\emptyset$ and $A$ and closed under arbitrary unions and finite intersections.

Elements of $\tau$ are called open sets. A completion of an open set is called a closed set.

Definition B.2 (Continuous function). Let $A$ and $B$ be two topological spaces. A function $f : A \to B$ is called continuous if for every open set $U$ of $B$, the inverse image $f^{-1}[U]$ is open in $A$.

Topological sets together with continuous functions for a category $\text{Top}$. This category has all small limits and colimits. There is a forgetful functor $U : \text{Top} \to \text{Set}$ from the category of topological spaces to the category of sets. This functor has a fully faithful left adjoint that assigns to a set $A$ the discrete topological space on $A$, i.e. the topology whose every set is open. Therefore, we may treat sets as topological spaces with the discrete topology. In particular, for a set $A$ thought of as a discrete topological space, the set $A^A = \prod_A A$ may be equipped with the product topology (the Tychonoff topology).

Definition B.3 (Hausdorff space). A topological space $A$ is Hausdorff if the diagonal set $\{(a,a) : a \in A\}$ is a closed subset of the product space $A \times A$.

We say that a family $O$ of open sets in $A$ is a covering family of a set $A_0 \subseteq A$ if $A_0 \subseteq \bigcup O$. A closure of a set $A_0 \subseteq A$ is the smallest closed subset $\overline{A_0}$ such that $A_0 \subseteq \overline{A_0}$.

Definition B.4 (Compact set). Let $A$ be a topological space. A set $A_0 \subseteq A$ is compact if for every covering family $O$ of $A_0$ there exists a finite subset $F \subseteq O$ that is also a covering family of $A_0$.

A topological space $A$ is called compact if the set $A$ is compact.

A topological group is a group object in the category of topological spaces $\text{Top}$. Explicitly, we have the following definition.

Definition B.5 (Topological group). A group $\mathbb{G} = (G, \cdot, (\cdot)^{-1})$ is a topological group if $G$ is equipped with a topology and both the multiplication $\cdot : G \times G \to G$ and the inverse operation $(\cdot)^{-1} : G \to G$ are continuous.

Definition B.6 (Continuous actions of a topological group). Let $\mathbb{G}$ be a topological group and $A$ a topological space. A continuous action $\mathbb{G} \times A \to A$ of $\mathbb{G}$ on $A$ is a group action that is continuous as a function.

Definition B.7 (Extremely amenable group). A topological group $\mathbb{G}$ is called extremely amenable if its every action $\mathbb{G} \times A \to A$ on a non-empty compact Hausdorff space $A$ has a fixed point.

If a topological group $\mathbb{G}$ acts continuously on a space $A$ via $h : G \times A \to A$, then it also acts continuously on space $A^k$ in a canonical way, i.e.:

$$h^k(g, (a_1, a_2, \ldots, a_k)) = (h(g, a_1), h(g, a_2), \ldots, h(g, a_k))$$

Definition B.8 (Oligomorphic action). Let a topological group $\mathbb{G}$ act continuously on a topological space $A$ via $h : G \times A \to A$. We say that action $h$ is oligomorphic if for every finite $k$ the canonical action $h^k$ of $\mathbb{G}$ on $A^k$ has only finitely many orbits.
Let $\mathcal{A}$ be an algebraic structure. Then the group of automorphism $\text{Aut}(\mathcal{A})$ treated as a subspace of $A^\mathcal{A}$ with the Tychonoff topology is a topological group.

**Definition B.9** (Oligomorphic group). Let $\mathcal{A}$ be an algebraic structure. Then $\text{Aut}(\mathcal{A})$ (or just $\mathcal{A}$) is oligomorphic if the canonical action of $\text{Aut}(\mathcal{A})$ on $A$ given by $\pi \circ a = \pi(a)$ is oligomorphic.

To define a uniformly continuous function we need a stronger notion than the notion of a topological space.

**Definition B.10** (Uniform structure). A uniform structure (or a uniformity) on a set $A$ consists of a filter $F$ of reflexive binary relations on $A$, such that:

- for every $R \in F$ there exists $S \in F$ such that $S \circ S \subseteq R$
- for every $R \in F$ there exists $S \in F$ such that $S^{op} \subseteq R$

Elements of $F$ are called entourages. Every uniform structure $A$ induces a canonical topology on $A$, called the uniform topology. This topology consists open sets $U$ such that for every $x \in U$ there exists an entourage $R$, such that $\{ y \in A : R(x, y) \} \subseteq U$.

**Definition B.11** (Uniformly continuous function). Let $A$ and $B$ be two uniform structures. A function $f : A \to B$ is called uniformly continuous if for every entourage $R$ of $B$, the inverse image $(f \times f)^{-1}[R]$ is an entourage of $A$.

If $G$ is a topological group then we may define four natural uniformities on it. Of a special interest is the uniformity generated by entourages of the form $\{ \langle g, \alpha \cdot g \cdot \beta \rangle : g \in G, \alpha, \beta \in U \}$ for some open $U$ containing the neutral element of $G$ (i.e. the neighbourhood of the identity) such that $\{ g^{-1} : g \in U \} = U$. This uniformity is called the Roelcke uniformity on $G$.

**Definition B.12**. A topological group $G$ is called Roelcke precompact if for every entourage $R$ in its Roelcke uniformity there exists a finite set $G_0 \subseteq G$ such that $R[G_0] = G$. 

