A Short Note on Compressed Sensing with Partially Known Signal Support

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Abstract

This short note studies a variation of the Compressed Sensing paradigm introduced recently by Vaswani et al., i.e. the recovery of sparse signals from a certain number of linear measurements when the signal support is partially known. The reconstruction method is based on a convex minimization program coined innovative Basis Pursuit DeNoise (or iBPDN). Under the common ℓ2-fidelity constraint made on the available measurements, this optimization promotes the (ℓ1) sparsity of the candidate signal over the complement of this known part.

In particular, this paper extends the results of Vaswani et al. to the cases of compressible signals and noisy measurements. Our proof relies on a small adaptation of the results of Candes in 2008 for characterizing the stability of the Basis Pursuit DeNoise (BPDN) program.

We emphasize also an interesting link between our method and the recent work of Davenport et al. on the δ-stable embeddings and the cancel-then-recover strategy applied to our problem. For both approaches, reconstructions are indeed stabilized when the sensing matrix respects the Restricted Isometry Property for the same sparsity order.

We conclude by sketching an easy numerical method relying on monotone operator splitting and proximal methods that iteratively solves iBPDN.

Keywords: Sparse Signal Recovery, Compressed Sensing, Convex Optimization, Instance Optimality.

1 Introduction

The theory of Compressed Sensing (CS) [2, 10] aims at reconstructing sparse or compressible signals from a small number of linear measurements compared to the dimensionality of the signal space. In short, the signal reconstruction is possible if the underlying sensing matrix is well behaved, i.e. if it respects a Restricted Isometry Property (RIP) saying roughly that any small subset of its columns is “close” to an orthogonal basis. The signal recovery is then obtained using non-linear techniques

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based on convex optimization promoting signal sparsity, as the Basis Pursuit DeNoise (BPDN) program \[\text{BPDN}\] [10, 5]. What makes CS more than merely an interesting theoretical concept is that some classes of randomly generated matrices (e.g. Gaussian, Bernoulli, partial Fourier ensemble, etc) satisfy the RIP with overwhelming probability. This happens as soon as their number of rows, i.e. the number of CS measurements, is higher than a few multiples of the assumed signal sparsity.

In this paper we are interested in a variation of the CS paradigm. We assume indeed that the support of the signal to recover is partially known, possibly with a certain error. As explained in [16, 17], this context is indeed well suited to the recovery of (time) sequences of sparse signals when their supports evolves slowly over time. In that case, the support of the recovered signal in a previous (discretized) time can be used to improve the reconstruction of the signal at the next time instance, either by decreasing the required number of measurements for a given quality, or by improving the reconstruction quality for a fixed number of measurements. Recovering a signal with partially known support is also of interest for certain kind of 1-D signals or images. For instance, photographic images, i.e. with positive intensities, have often many non-zero approximation coefficients in their wavelet decomposition [12]; a prior knowledge that can be favorably used in their reconstruction from CS measurements.

By adapting the proof of [1], we show in this short note that the recovery algorithm minimizing the $\ell_1$-norm of the signal candidate over the complement of the known support part, i.e. what we coin innovate Basis Pursuit DeNoising (iBPDN), has a similar stability behavior than the common Basis Pursuit DeNoise program. In particular, this extends the result of [16, 17] to the cases of noisy measurements and of compressible signals, i.e. with non-zero but fast decaying coefficients in a given sparsity basis. We show also that our method shares somehow the conclusion of the cancel-then-recover strategy designed in [9] where Authors propose a recovery algorithm that applies an orthogonal projection to separate the measurements into two components, and then recovers the known support part of the signal separately from the unknown support component.

2 Framework and Notations

Let $x = \Psi \alpha \in \mathbb{R}^n$ be a sparse or a compressible discrete signal in the sparsity basis $\Psi \in \mathbb{R}^{n \times n}$ of $\mathbb{R}^n$, i.e. the vector $\alpha \in \mathbb{R}^n$ has few non-zero or fast decaying components respectively. For the sake of simplicity, we work hereafter with the canonical basis, i.e. $\Psi = \text{Id}$, identifying $\alpha$ with $x$. The present work is however valid for any orthonormal $\Psi$, e.g. the DCT or the Wavelet basis, by integrating $\Psi$ in the sensing model described in Section 3.

We now establish some important notations. We write $\mathcal{N} = \{1, \cdots, n\}$ the index set of the vector components in $\mathbb{R}^n$. For any vector $u \in \mathbb{R}^n$, $u_i$ is the $i^{\text{th}}$ component of $u$ with $i \in \mathcal{N}$, $u_S$ is the vector equal to the components of $u$ on the set $S \subset \mathcal{N}$ and to 0 elsewhere, while $u^l$, with uppercase index $l \in \mathbb{N}$ to avoid confusion, is the vector obtained by zeroing all but the $l$ largest components of $u$ (in amplitude). For non-trivial basis $\Psi$, $u^l$ would be the best $l$-term approximation of $u$ in the $\ell_2$-norm sense. The complement of any set $S \subset \mathcal{N}$ is denoted by $S^c = \mathcal{N} \setminus S$, and the size of $S$ by $\#S$. The $\ell_p$ norm (for $p \geq 1$) of $u \in \mathbb{R}^n$ is $\|u\|_p = \sum_i |u_i|^p$, while its support is written $\text{supp} u \triangleq \{i \in \mathcal{N} : u_i \neq 0\}$. By extension, the $\ell_0$ “norm” is defined as $\|u\|_0 = \#\text{supp } u$.

Let us speak now of the prior knowledge that we have on the signal. In addition to the

\footnote{\text{It is not actually a true norm since for instance it is not positive homogeneous.}}
assumption of sparsity or compressibility, we presume that the support of the signal $x$ is partially known. In the sequel, we denote the known support part by $T \subset N$, while we always refer to its size by the letter $s = \#T$. Notice that in our study nothing prevents $T$ to be corrupted by some “noise”, i.e. a priori $T$ is not fully included to $\mathrm{supp} x$. Moreover, the size of $(\mathrm{supp} x) \setminus T$ is not constrained, what will matter is the values of the components of $x$ on $(\mathrm{supp} x) \setminus T$, i.e. the compressibility of $x$ outside of $T$.

### 3 Sensing Model

Following the common Compressed Sensing model, our vector $x$ is acquired by a sensing matrix $\Phi \in \mathbb{R}^{m \times n}$ subject to an additional white noise $n \in \mathbb{R}^m$, i.e.

$$y = \Phi x + n,$$

where $y \in \mathbb{R}^m$ is the measurement vector. In this model the noise power is assumed bounded by $\epsilon$, $\|n\|_2 \leq \epsilon$.

As shown after, even if a part of the signal support is known, the stability of this sensing model, i.e. our ability to recover or approximate $x$ from $y$, is also linked to the Restricted Isometry Property (RIP) of the sensing matrix [4, 2, 3].

Explicitly, the matrix $\Phi \in \mathbb{R}^{m \times n}$ satisfies the RIP of order $q \in \mathbb{N} (q \leq n)$ and radius $0 \leq \delta_q < 1$, if

$$(1 - \delta_q) \|u\|_2^2 \leq \|\Phi u\|_2^2 \leq (1 + \delta_q) \|u\|_2^2,$$

for all $q$-sparse vectors $u \in \mathbb{R}^n$, i.e. with $\|u\|_0 \leq q$.

### 4 Reconstructing on Innovation

Intuitively, if a part $T \subset N$ of the signal support is known, a possible (non-linear) reconstruction technique of $x$ would simply consist in minimizing the sparsity of a signal candidate $u \in \mathbb{R}^n$ over $T^c$, i.e. the $\ell_0$-norm of $u_{T^c}$, subject to the common $\ell_2$ fidelity constraint $\|\Phi u - y\|_2 \leq \epsilon$ as prescribed by the noise power bound. As underlined many times in the community, such a procedure would result in a combinatorial (NP-hard) problem $[13]$. Here again an $\ell_1$ relaxation must be used, with possibly additional requirements on the RIP-“conditioning” of $\Phi$ [15, 3].

The proposed method is a simple extension of the Modified-CS scheme defined in [16, 17]. We integrate indeed the case of corrupted measurements by defining the following optimization program, coined innovative Basis Pursuit DeNoising (iBPDN),

$$\argmin_u \|u_{T^c}\|_1 \text{ s.t. } \|y - \Phi u\|_2 \leq \epsilon. \quad (i\text{BPDN})$$

The term “innovative” recalls that this program tries to minimize the sparsity of the signal to be reconstructed in the unknown (or innovation) set $(\mathrm{supp} x) \setminus T$ included to $T^c$.  

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2Possibly with high probability.
5 iBPDN and $\ell_2 - \ell_1$ Instance Optimality

The main result of this note provides the conditions under which the solution of iBPDN is close or equal to the initial signal $x$, i.e. the so-called $\ell_2 - \ell_1$ instance optimality [6]. It extends in the same time the conclusion of [16] [17] to the cases of noisy measurements and compressible signals.

**Theorem 1.** Under the condition of the sensing model described above, writing $#T = s$ and given $k \in \mathbb{N}$, let us assume that the matrix $\Phi$ respects the RIP of order $s + 2k$ with radius $\delta_{s+2k} \in (0, 1)$, and that its radius for the smaller order $2k$ is $\delta_{2k} \in (0, 1)$. Then, if $\delta_{2k}^2 + 2\delta_{s+2k} < 1$, iBPDN has the $\ell_2 - \ell_1$ instance optimality meaning that its solution $x^*$ respects

$$
\|x - x^*\|_2 \leq C_{s,k} \epsilon + D_{s,k} e_0(r; k),
$$

where $r$ is the residual $r = x - x_T$, and $e_0(r; k) = k^{-1/2} ||r - r^k||_1$ is the compressibility error[3] at k-term of $r$. The two constants $C_{s,k}$ and $D_{s,k}$, given in the proof, depend on $\Phi$ only. For instance, for small innovation, i.e. when $k \ll s$, if $\delta_{2k} = 0.02$ and if $\delta_{s+2k} = 0.2$, $C_{s,k} < 7.32$ and $D_{s,k} < 3.35$.

**Proof.** We basically adapt the proof of [1] to signal with partially known support.

We define the residual $r = x - x_T$, with $\text{supp } r = (\text{supp } x) \setminus T$. Let us write $x^* = x + h$ with $h \in \mathbb{R}^n$ so that the proof amounts to bound $\|h\|_2$. Let $T_0$ be the support of the $k$ largest coefficients of the residual $r = x - x_T$, i.e. $T_0 = \text{supp } k$ with $T_0 \cap T = \emptyset$.

We define next the sets $T_j$ for $j \geq 1$ as the support of the $k$ largest coefficients of $h_{S_j} = h - h_{S_j}$ with $S_j = T \cup \bigcup_{i=0}^{j-1} T_i$. By construction, we may observe that we got the partition $\bigcup_{i \geq 0} T_i = (\text{supp } x) \setminus T$, with $#T_j = k$ and $T_j \cap T = T_j \cap T_j' = \emptyset$, for $j, j' \geq 0$ and $j \neq j'$.

Let us write $T_{01} = T \cup T_0$ and $T_{01} = T \cup T_0 \cup T_1$, with $#T_0 = s + k$ and $#T_{01} = s + 2k$. The plan of the proof is to first bound $\|h_{T_{01}}\|_2$ and then $\|h_{T_{01}}\|_2$. Using the triangular inequality, we have $\|h_{T_{01}}\|_2 \leq \sum_{j \geq 2} \|h_{T_j}\|_2$. For $j \geq 1$, $\|h_{T_j}\|_1 \geq k\|h_{T_{j+1}}\|_\infty$ by the ordering of the $T_j$’s, and therefore $\|h_{T_{j+1}}\|_2^2 \leq k\|h_{T_{j+1}}\|_\infty^2 \leq \frac{1}{k}\|h_{T_j}\|_1^2$. This leads to

$$
\|h_{T_{01}}\|_2 \leq \frac{1}{k} \sum_{j \geq 1} \|h_{T_j}\|_1 = \frac{1}{k} \|h_{T_0}\|_1. \tag{1}
$$

Since $T^c = T_0 \cup T_{01}^c$ and $\|n\|_2 = \|y - \Phi x\|_2 \leq \epsilon$, and because $x^*$ solves iBPDN, we have

$$
\|x_{T^c}\|_1 \geq \|x_{T^c} + h_{T^c}\|_1 = \|x_{T_0} + h_{T_0}\|_1 + \|x_{T_0} + h_{T_0}\|_1 \geq \|x_{T_0}\|_1 - \|h_{T_0}\|_1 - \|x_{T_0}\|_1 + \|h_{T_0}\|_1,
$$

and therefore,

$$
\|h_{T_0}\|_1 \leq \|x_{T^c}\|_1 + \|x_{T_0}\|_1 + \|h_{T_0}\|_1 - \|x_{T_0}\|_1 = 2\|x_{T_0}\|_1 + \|h_{T_0}\|_1 = 2\|r - r_{T_0}\|_1 + \|h_{T_0}\|_1.
$$

Consequently, using (1) and the equivalence of the norms $\ell_2$ and $\ell_1$, we get

$$
\|h_{T_{01}}\|_2 \leq \sum_{j \geq 2} \|h_{T_j}\|_2 \leq 2e_0(r; k) + \|h_{T_0}\|_2. \tag{2}
$$

[3] It could be called also scaled $\ell_1$-approximation error.
Let us now bound $\|h_{T_{01}}\|^2$. Notice that $h_{T_{01}} = h - \sum_{j \geq 2} h_T_j$, so that, using Cauchy-Schwarz,

$$\|h_{T_{01}}\|^2 = \langle \Phi h_{T_{01}}, \Phi h_{T_{01}} \rangle = \langle \Phi h_{T_{01}}, \Phi h \rangle - \langle \Phi h_{T_{01}}, \sum_{j \geq 2} \Phi h_T_j \rangle \leq \|\Phi h_{T_{01}}\|_2 \|\Phi h\|_2 + \sum_{j \geq 2} \|\langle \Phi h_{T_{01}}, \Phi h_T \rangle\|.$$

By hypothesis, $\Phi$ is RIP of order $q$ and radius $\delta_q$ with $q \in \{2k, s + 2k\}$. It is proved in [1] as a result of the polarization identity, that, for two vectors $u$ and $v$ of disjoint supports and of sparsity $l$ and $l'$ respectively, if $\Phi$ is RIP of order $l + l'$, then $\|\langle \Phi u, \Phi v \rangle\| \leq \delta_{l+l'} \|u\|_2 \|v\|_2$. In addition, since $x^*$ is solution of iBPDN and $x$ is a feasible point of its fidelity constraint, $\|\Phi h\|_2 \leq \|\Phi x^* - y\|_2 + \|y - \Phi x\|_2 \leq 2\epsilon$. Therefore, combining all these considerations,

$$\|h_{T_{01}}\|^2 \leq 2\sqrt{1 + \delta_{s+2k}} \|h_{T_{01}}\|_2 + \sum_{j \geq 2} |\langle \Phi h_{T_{01}}, \Phi h_T \rangle| \leq 2\sqrt{1 + \delta_{s+2k}} \|h_{T_{01}}\|_2 + (\delta_{s+2k} \|h_{T_{01}}\|_2 + \delta_{2k} \|h_{T_{1}}\|_2 + \delta_{2k} \|h_{T_{2}}\|_2) \sum_{j \geq 2} \|h_T\|_2 \leq 2\sqrt{1 + \delta_{s+2k}} \|h_{T_{01}}\|_2 + \mu_{s,k} \|h_{T_{01}}\|_2 \sum_{j \geq 2} \|h_T\|_2,$$

with $\mu_{s,k} = \sqrt{\delta_{s+2k}^2 + \delta_{2k}^2}$.

Since $(1 - \delta_{s+2k}) \|h_{T_{01}}\|_2 \leq \|h_{T_{01}}\|^2$, simplifying the last expression and using [2] lead to

$$(1 - \delta_{s+2k}) \|h_{T_{01}}\|_2 \leq 2\sqrt{1 + \delta_{s+2k}} \epsilon + \mu_{s,k} (2\epsilon_0(r; k) + \|h_{T_0}\|_2),$$

or, since $\|h_{T_0}\|_2 \leq \|h_{T_{01}}\|_2$,

$$\|h_{T_{01}}\|_2 \leq \alpha \epsilon + \beta \epsilon_0(r; k),$$

with $\alpha = 2\sqrt{1 + \delta_{s+2k}} / (1 - \delta_{s+2k} - \mu_{s,k})$ and $\beta = 2\mu_{s,k} / (1 - \delta_{s+2k} - \mu_{s,k})$.

Finally, using again [2],

$$\|h\|_2 \leq \|h_{T_{01}}\|_2 + \|h_{T_{01}}\|_2 \leq \alpha \epsilon + (\beta + 2) \epsilon_0(r; k) + \|h_{T_0}\|_2 \leq C_{s,k} \epsilon + D_{s,k} \epsilon_0(r; k),$$

with

$$C_{s,k} = \frac{4\sqrt{1 + \delta_{s+2k}}}{1 - \delta_{s+2k} - \mu_{s,k}},$$

and

$$D_{s,k} = 2 \frac{1 + \mu_{s,k} - \delta_{s+2k}}{1 - \delta_{s+2k} - \mu_{s,k}}.$$

The denominator of these two constants makes sense only if $1 - \delta_{s+2k} - \mu_{s,k} > 0$, i.e. if $\delta_{2k}^2 + 2\delta_{s+2k} < 1$, which provides the announced reconstruction condition. □
6 Observations

Some observations may be realized from Theorem 1. First, in the case where there is no knowledge about the signal support, i.e. $T = \emptyset$ and $s = 0$, we do find the previous sufficient condition of [1] characterizing when BPDN satisfies the $\ell_2 - \ell_1$ instance optimality, namely $\delta_{2k} < \sqrt{2} - 1$ as involved by $\delta_{2k}^2 + 2\delta_{2k} < 1$.

Second, the condition $\delta_{2k}^2 + 2\delta_{s+2k} < 1$ is satisfied if $\delta_{s+2k} < \sqrt{2} - 1$ since we have always $\delta_{2k} < \delta_{s+2k}$. This seems again a simple generalization of the previous result in [1], i.e. iPBDN is stable if the RIP of $\Phi$ is guaranteed over the sparsity order $s$ characterizing when BPDN satisfies the $\ell_2 - \ell_1$ instance optimality.

Intuitively, the matrix must be sufficiently “well conditionned” to estimate both the unknown $\delta_{2k}$ and $\delta_{s+2k}$ by the pure sensing model $\Phi^* \Phi = \text{Id}$ and the cancelation of the interference (or an equivalent greedy method as CoSaMP [14, 9]. Of course, $x_T = 0$ since this part of $x$ does not

$$\tilde{x} = \arg\min_u \|u\|_1 \text{ s.t. } \tilde{y} = \tilde{\Phi}u,$$

or an equivalent greedy method as CoSaMP [14, 9]. Of course, $x_T = 0$ since this part of $x$ does not
contribute to the fidelity constraint. It is equivalent to say that the reconstruction runs over the space \( P_{14,\perp} \mathbb{R}^n \), where \( P_{14,\perp} u = u_{T^c} \) for any \( u \in \mathbb{R}^n \). Therefore, the estimation error between \( \tilde{x} \) and \( x \) can be bounded over \( T^c \).

For this purpose \( \Phi \) must be characterized in function of \( \Phi \). This can be done by considering a generalization the Restricted Isometry Property: Given \( \delta \in (0,1) \) and two spaces \( U, V \subset \mathbb{R}^n \), a matrix \( \Phi \) realizes a \( \delta \)-stable embedding of \( (U, V) \) if

\[
(1 - \delta) \| u - v \|_2^2 \leq \| \Phi u - \Phi v \|_2^2 \leq (1 + \delta) \| u - v \|_2^2,
\]

for all \( u \in U \) and \( v \in V \). In particular the RIP of order \( q \) and radius \( \delta_q \) is equivalent to a \( \delta_q \)-stable embedding of \((\Sigma_q, \{0\})\), with \( \Sigma_q = \{ u \in \mathbb{R}^n : \|u\|_0 \leq q \} \) the set of \( q \)-sparse signals. The following result provides then the desired characterization.

**Lemma 1** (Corollary 4 in [9]). Suppose that \( \Phi \in \mathbb{R}^{m \times n} \) is a \( \delta \)-stable embedding of \((\Sigma_{2k}, \Sigma_T)\). Then \( \Phi \) is a \( \delta/(1 - \delta) \)-stable embedding of \((P_{14,\perp} \Sigma_{2k}, \{0\})\).

In particular, this Lemma implies that if \( \Phi \) is RIP of order \( s + 2k \) with radius \( \delta_{s+2k} \), it is then a \( \delta_{s+2k} \)-stable embedding of \((\Sigma_{2k}, \Sigma_T)\), and therefore, \( \Phi \) is RIP of order \( 2k \) and radius \( \delta_{s+2k}/(1 - \delta_{s+2k}) \) over the space \( P_{14,\perp} \mathbb{R}^n \simeq \mathbb{R}^{n-s} \). The \( \ell_2 - \ell_1 \) instance optimality of the BP program [1] above holds if \( \delta' = \delta_{s+2k}/(1 - \delta_{s+2k}) < \sqrt{2} - 1 \), i.e. if \( \delta_{s+2k} < (\sqrt{2} - 1)/\sqrt{2} \). In that case,

\[
\| x_{T^c} - \tilde{x}_{T^c} \|_2 \leq \tilde{D}_{\beta'} e_0(x_{T^c}, k) = \tilde{D}_{\beta'} e_0(r, k),
\]

with \( \tilde{D}_{\beta'} = 2 \frac{1+\sqrt{2}-1)\delta'}{1-(\sqrt{2}+1)\delta'} = 2 \frac{1+2-2)\delta_{s+2k}}{1-(\sqrt{2}+2)\delta_{s+2k}} \).

In this paper, we show that iBPDN is optimal when \( \delta_{2k}^2 + 2 \delta_{s+2k} < 1 \). This condition is weaker than the one proposed in [9], i.e. \( \delta_{s+2k} < (\sqrt{2} - 1)/\sqrt{2} \), however it is interesting to notice that both consider also the RIP of order \( 2s + k \) and both are stable for compressible signals. Moreover, iBPDN gives guarantees for the estimation of the whole signal and not only for its behavior over \( T^c \). Of course, if \( x^* \) is the solution of iBPDN (with \( \epsilon = 0 \)), we get similarly

\[
\| x_{T^c} - x^*_{T^c} \|_2 \leq \| x - x^* \|_2 \leq D_{s,k} e_0(r, k),
\]

with \( D_{s,k} < 2 \frac{1+\sqrt{2}-1)\delta_s+2k}{1-(\sqrt{2}+1)\delta_{s+2k}} < \tilde{D}_{\beta'} \).

We can remark also that, conversely to the current cancel-then-recover strategy [1], iBPDN provides stability against noisy measurements. An open question is however that \( \Phi \) in [9] has not to be really RIP of order \( s + 2k \) to valid [3]. As reported in Lemma 1, \( \Phi \) simply needs to provide a \( \delta \)-stable embedding over \((\Sigma_{2k}, \Sigma_T)\) which is weaker than asking the RIP of order \( s + 2k \). Given \( k \) and \( m \), that second requirement holds possibly for a smaller radius \( \delta \) than the RIP radius \( \delta_{s+2k} \).

8 Numerical Method

In this section, we sketch of a simple algorithm for the reader interested in a numerical implementation of iBPDN. This one relies on monotone operator splitting and proximal methods [8] [11].

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4Robustness of this strategy against an additional noise \( n \) could be obtained by bounding the power of \( P_{14,\perp} n \) when \( y = \Phi x + n \).
At the heart of this procedure is the definition of the proximity operator of any convex function \( \varphi : \mathbb{R}^n \to \mathbb{R} \), i.e. the unique solution of \( \text{prox}_\varphi (z) = \arg \min_u \frac{1}{2} \| u - x \|_2^2 + \varphi (z) \).

Both BPDN and iBPDN are special cases of the general minimization problem

\[
\arg \min_{x \in H} f (x) + g (x). \quad (P)
\]

For iBPDN, \( f (u) = \| u_{T^c} \|_1 \) and \( g (u) = \mathbb{I}_{C(\epsilon)} (u) = 0 \) if \( u \in C (\epsilon) \) and \( \infty \) otherwise, i.e. the indicator function of the closed convex set \( C (\epsilon) = \{ v \in \mathbb{R}^n : \| y - \Phi v \|_2 \leq \epsilon \} \).

Of course \( f \) and \( g \) are both non-differentiable, however, since (i) their domain is non-empty, (ii) they are convex and (iii) lower semi-continuous (lsc), i.e. \( \liminf_{u \to u_0} f (u) = f (u_0) \) for all \( u_0 \in \text{dom} f \), iBPDN can be solved by the following Douglas-Rachford iterative method [11]:

\[
u^{(t+1)} = (1 - \alpha_t) u^{(t)} + \alpha_t S_\gamma \circ P_{C(\epsilon)} (u^{(t)}), \quad (4)
\]

where \( A^\circ \triangleq 2A - \text{Id} \) for any operator \( A \), \( \alpha_t \in (0, 2) \) for all \( t \in \mathbb{N} \), \( S_\gamma = \text{prox}_{\gamma f} \) for some \( \gamma > 0 \) and \( P_{C(\epsilon)} = \text{prox}_g \) is the orthogonal projection onto the tube \( C (\epsilon) \). From [7], one can show that the sequence \( (u^{(t)})_{t \in \mathbb{N}} \) converges to some point \( u^* \) and \( x^* = P_{C(\epsilon)} (u^*) \) is the solution of iBPDN.

We may compute that \( S_\gamma z = \text{prox}_{\gamma f} z \) is actually the component-wise soft-thresholding operator of \( z \) on \( T^c \), i.e. \( (S_\gamma z)_i = \text{sign} z_i (| z_i | - \gamma)_+ \) if \( i \in T^c \) and \( z_i \) if \( i \in T \), with, for \( \lambda \in \mathbb{R} \), \( (\lambda)_+ = \lambda \) if \( \lambda \geq 0 \) and \( 0 \) else. Efficient ways to compute \( P_{C(\epsilon)} \) are also given in [11].

9 Conclusion

This short note has studied the modification of Compressed Sensing introduced in [16, 17], i.e. when the signal sparsity assumption is increased by the knowledge of a part of its support. We showed theoretically that a simple generalization of the common Basis Pursuit DeNoise program, i.e. the innovative BPDN, has similar stability guarantees than BPDN with respect to both signal compressibility and noisy measurements. Interestingly, the obtained requirements are related to the conclusion of [9] when the cancel-then-recover strategy is applied to the context of this paper.

In the future, we plan to investigate possible numerical applications of this formalism. In particular, when iBPDN is integrated to the reconstruction of sequences of sparse or compressible signals, we would like to assess the quality of the reconstruction in function of the number of measurements when the amount of innovation, i.e. the ratio between the unknown and the known signal support parts, can be quantified over time.

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