On Pole Placement and Invariant Subspaces

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Abstract—The classical eigenvalue assignment problem is revisited in this note. We derive an analytic expression for pole placement which represents a slight generalization of the celebrated Bass-Gura and Ackermann formulae, and also is closely related to the modal procedure of Simon and Mitter.

I. INTRODUCTION

For a single-input linear time-invariant system \( \dot{x} = Ax + bu \) with \( x \in \mathbb{R}^n \), \( A \in \mathbb{R}^{n \times n} \), \( b \in \mathbb{R}^{n \times 1} \), the solution to the pole placement problem provides the feedback gain \( k \in \mathbb{R}^n \) in \( u = k^T x \), such that the open-loop eigenvalues \( \lambda(A) \) are shifted to some prespecified values \( \lambda(\tilde{A}) \) where \( \tilde{A} := A + bk^T \). We also stress its similar matrix of \( \omega \) into a product for special cases. Indeed, it will be shown later in the paper that it includes both the Ackermann and Bass-Gura formulae as well as the multiset of self-conjugate eigenvalues \( \{\lambda_i\}_{i=1}^n \in \mathbb{C} \), there exists always a unique state feedback gain \( k \in \mathbb{R}^n \) which solves the pole assignment problem [5]. In the sequel, we provide an original method for computation of \( k \).

Let \( \omega_{n-1} \in \mathbb{C}^n \) be a left eigenvector of the closed-loop system matrix \( \tilde{A} := A + bk^T \) corresponding to an arbitrary eigenvalue \( \lambda_1 \in \mathbb{C} \). Then, with \( \omega_{n-1}^H (A + bk^T) = \lambda_1 \omega_{n-1}^H \), we claim:

\[
k^T = \omega_{n-1}^H (\lambda_1 I - A) - \omega_{n-1}^H b = 1,
\]

whereby in light of implementation, care has to be taken in selecting a pair \( \omega_{n-1} \) and \( \lambda_1 \) that guarantee a real outcome \( k \in \mathbb{R}^n \). Observe, that the right-hand side statement in (2) results from the fact that \( \omega_{n-1}^H b \) is a left eigenvector of \( \tilde{A} \), as well, and the condition \( \omega_{n-1}^H b \neq 0 \) which is guaranteed by the controllability of the pair \((A, b)\). Indeed, if the opposite would hold true, i.e. if \( \omega_{n-1}^H b = 0 \), we would have: \( \omega_{n-1}^H (A + bk^T) = \omega_{n-1}^H A = \lambda_1 \omega_{n-1}^H \) for all \( k \), indicating that \( \lambda_1 \) is an eigenvalue of \( A \) and \( \tilde{A} \) simultaneously, i.e. it cannot be shifted by any \( k \), which contradicts the controllability of \((A, b)\).

Furthermore, equation (2) reveals that the remainder eigenvalues in the multiset \( \{\lambda_i\}_{i=2}^n \) are uniquely specified by the left eigenvector \( \omega_{n-1} \). Hence, it is natural to pose the spectrum assignment in terms of computing the eigenvector \( \omega_{n-1} \) such that a prespecified multiset of self-conjugate (not necessarily distinct) eigenvalues \( \{\lambda_j\}_{j=2}^n \) are assigned to

\[
\tilde{A} = (I - \omega_{n-1}^H b) A + \lambda_1 \omega_{n-1}^H.
\]

To this end, we start with the characteristic polynomial of the closed loop matrix \( \tilde{A} \), which (with a little of technical effort) is shown to be given by:

\[
\det(\lambda I - \tilde{A}) = (\lambda - \lambda_1) \omega_{n-1}^H \text{adj}(\lambda I - A^T) b.
\]

Next, consider the controller canonical form \( \xi = Ae\xi + b\xi u \), with \( TA_c = AT, Tb_c = b \), and

\[
A_c = \begin{pmatrix}
0 & 1 & \cdots & 0 \\
\vdots & \ddots & \ddots & \vdots \\
0 & 0 & \cdots & 1 \\
-a_n & -a_{n-1} & \cdots & -a_1
\end{pmatrix},
\]

the shorthand: \( r := \{1, \ldots, r\} \) for \( r \in \mathbb{N} \) in \( i \in r \) to indicate \( i \in \{1, \ldots, r\} \). If \( r \in \mathbb{R}_0 \) allows \( i \) to take also the value 0.
Here, $T := CC_c^{-1}$ indicates the transformation $x = T\xi$, where, for convenience, we denote by $C := \text{Con}(\alpha, b)$ and $C_c := \text{Con}(A_c, b_c)$ the open-loop and closed-loop controllability matrix [5], respectively. The characteristic polynomial of $A$ then reads:

$$p(\lambda) = \det(\lambda I - A) = \lambda^n + a_1\lambda^{n-1} + \ldots + a_n.$$  (5)

Following the discussion related to equation (2), if we let

$$\gamma_{n-1}^H := [\gamma_{n-1,n-1}, \ldots, \gamma_{n-1,1}, 1]$$  (6)

represent the desired left eigenvector, and $\lambda_1$ the corresponding eigenvalue of the closed loop $A_c := A_c + b_c k_c^T = T^{-1} A T$ in the $\xi$-coordinates, then from (4) we get

$$\det(\lambda I - A_c) = (\lambda - \lambda_1) \gamma_{n-1}^H \Upsilon(\lambda),$$  (7)

where we introduce: $\Upsilon(\lambda) := [1 \lambda \ldots \lambda^{n-1}]^T = \text{adj}(\lambda I - A_c^T)_{\ast}$. From (7) it is obvious that the eigenvalues $\{\lambda_j\}_{j=2}^n$ of the closed-loop matrix $A_c$, that is, of $\tilde{A}$, as well are independent of the parameters $a_1, \ldots, a_n$, and they are entirely determined by the left eigenvector $\gamma_{n-1}$. On the other hand, let (4) be specified by a desired closed-loop characteristic polynomial of the form:

$$q_n(\lambda) = \lambda^n + a_1\lambda^{n-1} + \ldots + a_n.$$  (8)

Equation (7) then hosts the parameters of the polynomial $q_{n-1}(\lambda)$, where $q_n(\lambda) = (\lambda - \lambda_1) q_{n-1}(\lambda)$. Explicitly, it can be checked that $\gamma_{n-1}$, as defined in (9), is given by the recursive algorithm: $\gamma_{n-1,i} = \alpha_i + \gamma_{n-1,i-1}\lambda_1$ for $i \in n - 1$, where, in accordance with our adoption in (4): $\gamma_{n-1,0} = 1$. Moreover, with $\tilde{A}$ and $A_c$, being similar, we have

$$\omega_{n-1}^H = \gamma_{n-1}^H C_c^{-1}, \quad k^T = \omega_{n-1}^H (\lambda_1 I - A).$$  (9)

This represents our initial pole assignment formula. Next, we generalize it and demonstrate its relationship to the Bass-Gura and Ackermann formulas. First, it is readily verified that

$$\gamma_{n-1}^H (\lambda_1 I - A_c) = [\alpha_1 - a_1, \ldots, \alpha_n - a_n] := \gamma_n^H,$$  (10)

indicating that all the closed-loop eigenvalues in $\{\lambda_j\}_{j \in n}$ are “encoded” in the (real) vector $\gamma_n$, whereas $\gamma_{n-1}$ carries the information about $\{\lambda_j\}_{j=2,n}$. Then, the Bass-Gura formula:

$$k^T = \gamma_n^H C_c^{-1}$$  (11)

results immediately, if we rewrite (2) as: $k^T = \gamma_{n-1}^H (\lambda_1 I - A_c) C_c^{-1}$, with the term $T^{-1} = C_c C_c^{-1}$ shifted right most.

Equation (10) can be interpreted as “pulling out” or “carrying over” the eigenvalue $\lambda_1$ from $\gamma_n$ via the factor $\lambda_1 I - A_c$, this necessarily introducing $\gamma_{n-1}$. By proceeding in the same way, one can pullout the eigenvalue $\lambda_2$ from $\gamma_{n-1}$ by means of $\lambda_2 I - A_c$, $\lambda_3$ from $\gamma_{n-2}$ via $\lambda_3 I - A_c$, and so on. Hence, we can introduce

$$\gamma_{n-r}^H := [\gamma_{n-r,n-1}, \ldots, \gamma_{n-r,r}, 1, 0, \ldots, 0], \quad r \in n$$  (12)

using:

$$\gamma_n^H = \gamma_{n-r}^H \prod_{i=1}^{r} (\lambda_i I - A_c),$$  (13)

where the $(r-1)$ zeros (for $r \geq 2$) result due to the “absence” of the eigenvalues $\lambda_2, \ldots, \lambda_r$ in $\gamma_{n-r}$, while the $n - r$ non-zero terms carry the information about $\lambda_{r+1}, \ldots, \lambda_n$. In this sense, by substituting (13) into (11), our spectrum assignment formula (9) can be set in the general form:

$$k^T = \gamma_{n-r}^H C_c^{-1} \prod_{i=1}^{r} (\lambda_i I - A), \quad r \in n,$$  (14)

which can be slightly generalized to

$$k^T = \omega_{n-r}^H q_r(A), \quad r \in n_0,$$  (15)

with $q_0(A) := I_n$ and otherwise:

$$\omega_{n-r}^H := \gamma_{n-r}^H C_c^{-1}, \quad q_r(A) := \prod_{i=1}^{r} (\lambda_i I - A).$$  (16)

Clearly, equation (15) represents the generalized form of our initial expression in (9). For $r \geq 1$ the vector $\gamma_{n-r}$ is simply defined by the coefficients of the polynomial $q_r(\lambda)$, where

$$q_n(\lambda) = q_{n-r}(\lambda) q_r(\lambda).$$  (17)

The definition of $\gamma_n$ (i.e. reflecting the Bass-Gura formula with $r = 0$, c.f. (10) represents an exception to this rule.

Now, consider the special case with $r = n$ and let $q_n(A)$ denote the real matrix polynomial corresponding to the desired characteristic polynomial $q_n(\lambda)$ from (3). Then, using $\gamma_0^H = [1, 0, \ldots, 0]$ from (12), and: $[1, 0, \ldots, 0] C_c = [0, \ldots, 0, 1]$, we obtain the Ackermann formula directly from (14):

$$k^T = [0, \ldots, 0, 1] C_c^{-1} q_n(A).$$  (18)

A. Comments

(i) Expressions (14) and (15) provide a direct link of the Bass-Gura and Ackermann formulae. Moreover, it represents a generalization thereof: the former one results with $r = 0$ (leading to the definition (10) for $\gamma_n$), while the latter one for $r = n$ in (14). Notice that from (18) we immediately obtain

$$\omega_0^T = [0, \ldots, 0, 1] C_c^{-1}.$$ (19)

(ii) The desired conjugate eigenpairs should be “encoded” jointly in (14), either in the real vector $\omega_{n-r}$, or in the real matrix polynomial $q_r(A)$ to benefit from the numerical computation with real numbers. Therefore, without loss of generality we may consider

$$k^T = \omega_{n-r}^T q_r(A), \quad r \in n_0,$$  (19)

as the general form of our spectrum assignment formula. In this sense, it is also convenient to use a real $\lambda_1$ in (2).

(iii) If $\omega_{n-1}$ in (2) is selected to be the left eigenvector of the open-loop matrix $A$ corresponding to a real eigenvalue, say $\mu_1$, then from (3) we have $\tilde{A} = A + \Delta b \omega_{n-1}^T$, with $\Delta := \lambda_1 - \mu_1$ referring to a real shift. The remainder open-loop eigenvalues $\{\mu_i\}_{i=2}^n$ are thereby unaltered, as for any right eigenvector $v_{n-i}$ of $A$ corresponding to the eigenvalue $\mu_i$, we have $Av_{n-i} = \mu_i v_{n-i}$, $i \in \{2, \ldots, n\}$ (as a consequence of $\omega_{n-1}^T v_{n-i} = 0$). In this case we retain:

$$k^T = \Delta_1 \omega_{n-1}^T,$$
which represents the well-known result of Simon and Mitter [2] (cf. pp.338). It is important to observe in this case the geometric interpretation of the vector term $\omega_{n-1}$ in [2]: it is orthogonal to the invariant subspace corresponding to the eigenvalues that remain unchanged. We discuss this more generally in the next section.

(iv) Finally, due to the presence of the factor $C^{-1}$, which for large $n$ is typically ill-conditioned, related well-known numerical robustness problems are inherent in the expression (14). In the sequel, we discuss the avoidance of such difficulties.

B. Partial spectrum assignment

Next, we consider the usability of the vector $\omega_{n-r} \in \mathbb{R}^n$ in the context of the partial spectrum assignment and a sequential spectrum assignment based thereon, which consists in shifting a multiset of open-loop self-conjugate eigenpairs, say $M_r = \{\mu_i\}_{i=1}^r$, to some prescribed self-conjugate $L_r = \{\lambda_i\}_{i=1}^r$, while keeping the remainder $(n-r)$-ones of $M_{n-r} = \{\mu_i\}_{i=r+1}^n$ unaltered ($r \in \mathbb{N}$).

To this end, consider the operator description of $A$:

$$A = \begin{bmatrix} X & 0 \\ * & Y \end{bmatrix} U \odot \odot V, \tag{20}$$

corresponding to the real Schur decomposition:

$$A(U,V) = (U,V) \left( \begin{bmatrix} X & 0 \\ * & Y \end{bmatrix} : \oplus \oplus \right), \tag{21}$$

where $U \oplus V = \mathbb{R}^n$ (i.e. $U$ and $V$ are complementary subspaces), $U = \text{Range}(U) \subseteq \mathbb{R}^n$, $V = \text{Range}(V) \subseteq \mathbb{R}^{n-n-r}$ is the $A$-invariant subspace (i.e. $AV = VY$) corresponding to the eigenvalues in $M_{n-r}$, and $[U,V] \in \mathbb{R}^{n \times n}$ is orthogonal (i.e., $U$ and $V$ are mutually orthogonal subspaces). Next, introducing

$$\omega_{n-r} = U\eta \tag{22}$$

in terms of $\eta \in \mathbb{R}^r$ in (19), it can be readily checked that the block-triangular form is preserved under feedback [6]:

$$\begin{bmatrix} U^T \\ V^T \end{bmatrix} A(U,V) = \begin{bmatrix} X + U^T b\eta T q_r(X)^{-1} & 0 \\ * & Y \end{bmatrix}. \tag{23}$$

Note that due to the re-appearance of $Y$ in the diagonal, the eigenvalues in $M_{n-r}$ remain unaltered in $A$, while those from $M_r$ change subject to the parameter $\eta$ in the term $X + U^T b\eta T q_r(X)$. The latter expression suggests using the Ackermann formula for computation of $\eta$ in shifting the eigenvalues $M_r$ of $X$ to $L_r$:

$$\omega_{n-r}^T = \eta^T U^T, \quad \eta^T = [0, \ldots, 1] C^{-1}(X, U^T b). \tag{24}$$

In words, if $\omega_{n-r}$ is fixed perpendicularly to the invariant subspace $V$, then the corresponding open-loop eigenvalues remain unchanged if we apply the feedback of the form (19) with (22) and (24). This fact provides a geometric interpretation for the term $\omega_{n-r}$ in the expression (19).

With reference to (27), it is easily seen that the invertibility of the controllability matrix $C^{-1}(X, U^T b)$ in (24) requires

$$\text{rank}(U^T [b, Ab, \ldots, A^{r-1}b]) = r,$$

which refers to the projected subsystem $(U^T A U, U^T b)$ onto the subspace $U \subseteq \mathbb{R}^r$ [6].

C. Sequential spectrum assignment

Comment (iv) indicates the difficulties with the invertibility of the underlying controllability matrix, while in the previous section we saw that the latter is reduced due to the projection of the system matrix onto a subspace of a lower dimension. This idea can now be utilized sequentially as suggested by the following algorithm. Let

$$A(A) = \bigcup_{\ell=1}^m M_\ell, \quad \Lambda(A) = \bigcup_{\ell=1}^m L_\ell, \tag{25}$$

where $M_\ell$ includes a multiset of self-conjugate open-loop eigenvalues, and $L_\ell$ the corresponding desired self-conjugate closed-loop eigenvalues. In other words, the eigenvalues in $M_\ell$ are to be shifted to $L_\ell$ for all $\ell \in m$. Then, introduce:

$$u_\ell = \omega_\ell^T q_r(\bar{A}_\ell) x + u_{\ell+1}, \quad \ell \in m \tag{26}$$

with $u = u_1, u_{m+1} = 0, \bar{A}_1 = A, \bar{A}_{\ell+1} = \bar{A}_\ell + b\omega_\ell^T q_r(\bar{A}_\ell)$,

$$\bar{A}_\ell = \begin{bmatrix} X_\ell & 0 \\ * & Y_\ell \end{bmatrix} : \oplus \oplus \odot \odot V_\ell, \tag{27}$$

where $V_\ell = \text{Range}(V_\ell)$ represents the $\bar{A}_\ell$-invariant subspace corresponding to the eigenvalues $\Lambda(\bar{A}_\ell) \setminus M_\ell, U_\ell = \text{Range}(U_\ell)$ is orthogonal to $V_\ell$ in $\mathbb{R}^n$,

$$\omega_\ell^T = \eta_\ell^T U_\ell^T, \quad \eta_\ell^T = [0, \ldots, 1] C^{-1}(X_\ell, U_\ell^T b) \tag{28}$$

and $q_r(\cdot)$ is the characteristic polynomial corresponding to the desired eigenvalues in $L_\ell$. Effectively, we obtain:

$$k^T = \sum_{\ell=1}^m \omega_\ell^T q_r(\bar{A}_\ell). \tag{29}$$

In words, the vector $\omega_\ell$ is set perpendicularly to the invariant subspaces $V_\ell$ and the $A_\ell$-invariant subspace corresponding to the eigenvalues $\Lambda(\bar{A}_\ell) \setminus M_\ell$, $U_\ell = \text{Range}(U_\ell)$ is orthogonal to $V_\ell$ in $\mathbb{R}^n$.

III. Conclusion

This short note introduces a slightly generalized version of the pole placement formulae and discusses its relationships to Ackermann, Bass-Gura and Simon & Mitter algorithms. It extends and completes initial ideas of [3]. The author thanks Dietrich Flockerzi for useful discussions.

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