On the non-asymptotic and sharp lower tail bounds of random variables

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Funding information
National Institutes of Health, Grant/Award Number: R01 GM131399; National Science Foundation, Grant/Award Number: CAREER-1944904 and DMS-1811868

The non-asymptotic tail bounds of random variables play crucial roles in probability, statistics, and machine learning. Despite much success in developing upper bounds on tail probabilities in literature, the lower bounds on tail probabilities are relatively fewer. In this paper, we introduce systematic and user-friendly schemes for developing non-asymptotic lower bounds of tail probabilities. In addition, we develop sharp lower tail bounds for the sum of independent sub-Gaussian and sub-exponential random variables, which match the classic Hoeffding-type and Bernstein-type concentration inequalities, respectively. We also provide non-asymptotic matching upper and lower tail bounds for a suite of distributions, including gamma, beta, (regular, weighted, and noncentral) chi-square, binomial, Poisson, Irwin–Hall, etc. We apply the result to establish the matching upper and lower bounds for extreme value expectation of the sum of independent sub-Gaussian and sub-exponential random variables. A statistical application of signal identification from sparse heterogeneous mixtures is finally considered.

KEYWORDS
Chernoff–Cramèr bound, concentration inequality, sub-exponential distribution, sub-Gaussian distribution, tail bound

1 | INTRODUCTION

The tail bounds and concentration inequalities, which study how a random variable deviates from some specific value (such as the expectation), are ubiquitous in enormous fields, such as random matrix theory, high-dimensional statistics, and machine learning (Boucheron, Lugosi, & Massart, 2013; Vershynin, 2012, 2018; Wainwright, 2019). Many upper bounds on tail probabilities, such as Markov’s inequality, Chebyshev’s inequality, Hoeffding’s inequality (Hoeffding, 1963), Bernstein’s inequality (Bernstein, 1924, 1937), and Bennett’s inequality (Bennett, 1962), have been well-regarded and extensively studied in literature. The Chernoff–Cramèr bound (Chernoff (1952), Theorem 1; also see, e.g., Boucheron et al. (2013), and Wainwright (2019), with a generic statement given below, has been a basic tool to develop upper bounds of tail probabilities.

Proposition 1 (Chernoff–Cramèr Bound). If $X$ is a real-valued random variable with the moment generating function $\phi_X(t) = \mathbb{E} \exp(tX)$ defined for $t \in T \subseteq \mathbb{R}$. Then for any $x \in \mathbb{R}$,

$$
\mathbb{P}(X \geq x) \leq \inf_{t \in T} \phi_X(t) \exp(-tx); \quad \mathbb{P}(X \leq x) \leq \inf_{t \in T} \phi_X(t) \exp(-tx).
$$

The Chernoff–Cramèr bound has been the beginning step for deriving many probability inequalities, among which Hoeffding’s inequality (Hoeffding, 1963), Bernstein’s inequality (Bernstein, 1924, 1937; Uspensky, 1937), Bennett’s inequality (Bennett, 1962), Azuma’s inequality (Azuma, 1967), and McDiarmid’s inequality (McDiarmid, 1989) are well-regarded and widely used. Motivated by applications in high-dimensional statistics and machine learning, the Chernoff–Cramèr bound has been widely used to develop various concentration inequalities for sub-Gaussian and sub-exponential random variables. These results and applications have been collected in recent textbooks (see, e.g., Boucheron et al., 2013; Lugosi, 2009; Steele, 1997; Vershynin, 2012; Wainwright, 2019) and class notes (see, e.g., Pollard, 2015; Roch, 2015; Sridharan, 2018).

Despite enormous achievements on upper tail bounds in literature, there are relatively fewer results on the corresponding lower bounds: how to find a sharp and appropriate $L > 0$, such that $\mathbb{P}(X \geq x) \leq L$ (or $\mathbb{P}(X \leq x) \leq L$) holds? Among existing literature, the Cramèr–Chernoff theorem...
characterized the asymptotic tail probability for the sum of i.i.d. random variables (Lyons & Pemantle, 1992, and Chernoff, 1952, Theorem 1; also see Van der Vaart, 2000, Proposition 14.23): suppose $Z_1, \ldots, Z_4$ are i.i.d. copies of $Z$. Then,

$$k^{-1} \log \left( \Pr(Z_1 + \ldots + Z_4 \geq ak) \right) \to \log \left( \inf_{c \in \mathbb{C}} \mathbb{E}(e^{Z-\mu}) \right)$$

as $k \to \infty$. The Berry–Esseen central limit theorem (Berry, 1941; Esseen, 1942) provided a non-asymptotic quantification of the normal approximation residual for the sum of independent random variables: let $Z_1, \ldots, Z_4$ be i.i.d. copies of $Z$, where $\mathbb{E}Z = 0$ and $\mathbb{E}|Z|^3 < \infty$; then for all $x \in \mathbb{R}$,

$$\Phi(-x) - \frac{Ce|Z|^3}{\sqrt{k(\text{Var}(Z))^{3/2}}} \leq \Pr\left(Z_1 + \ldots + Z_4 \geq \sqrt{k \text{Var}(Z)}x\right) \leq \Phi(-x) + \frac{Ce|Z|^3}{\sqrt{k(\text{Var}(Z))^{3/2}}}.$$  

(1)

where $\Phi(\cdot)$ is the cumulative distribution function of the standard normal distribution. This lower bound is not universally sharp, as the left-hand side of (1) can be negative for $x \geq C \sqrt{\log(k)}$. Slud (1977), Csiszar (1998), and Cover and Thomas (2012) established upper and lower tail bounds for binomial distribution based on its probability mass function. Kolmogorov introduced the Bernstein-type lower bound for the sum of independent bounded random variables (Ledoux & Talagrand, 2013, Lemma B.1). Gluskin and Kwapień (1995) established tail and moment estimates for the sums of independent random variables with logarithmically concave tails. De la Pena and Giné (2012, Chapter 3) considered decoupling by developing lower tail bounds of hypercontractive variables. Bagdasarov and Ostrovskii (1996) studied a reversion of Chebyshev’s inequality based on the moment generating function and properties of the convex conjugate. Theodosopoulos (2007) proved a tight reversion of the Chernoff bound using the tilting procedure. There is still a lack of easy-to-use and sharp lower bounds on tail probabilities for generic random variables in the finite sample setting.

Departing from existing results, in this paper, we introduce systematic and user-friendly schemes to develop non-asymptotic and sharp lower bounds on tail probabilities for generic random variables. The proofs can be conveniently applied to various settings in statistics and machine learning. We also discuss the following implications of the developed results.

- In Section 3, we establish the lower bounds on tail probabilities for the weighted sums of independent sub-Gaussian and sub-exponential random variables. These new results match the classic Hoeffding-type and Bernstein-type upper bounds in the literature.
- In Section 4, we study the matching upper and lower tail bounds for a suite of commonly used distributions, including gamma, beta, (regular, weighted, and noncentral) chi-square, binomial, Poisson, and Irwin–Hall distributions. To the best of our knowledge, these are the first sharp lower bounds on tail probabilities for these distributions. Especially, we establish a reverse Chernoff–Cramér bound (in the forthcoming Lemma 1) to better cope with binomial and Poisson distributions, which may be of independent interest.
- In Section 5, we consider the applications of the established results. We derive the matching upper and lower bounds for extreme values of the sums of independent sub-Gaussian and sub-exponential random variables. A statistical problem of signal identification from heterogeneous mixtures is finally studied.

## 2 GENERIC LOWER BOUNDS ON TAIL PROBABILITIES

We use uppercase letters, e.g., $X, Z$, to denote random variables and lowercase letters, e.g., $x, t$, to denote deterministic scalars or vectors. $x \wedge y$ and $x \vee y$ respectively represent the maximum and minimum of $x$ and $y$. We say a random variable $X$ is centered if $\mathbb{E}X = 0$. For any random variable $X$, let $\phi_X(t) = \mathbb{E}\exp(tX)$ be the moment generating function. For any vector $u \in \mathbb{R}^d$ and $q \geq 0$, let $\|u\|_q = (\sum |u|^q)^{1/q}$ be the $\ell_q$ norm. In particular, $\|u\|_\infty = \max |u|$. We use $c, C_1, \ldots, c_1, \ldots$ to respectively represent generic large and small constants, whose actual values may differ from time to time. Throughout the paper, we present the previous results as propositions and our new results as theorems/corollaries.

For any centered random variable $X$, if $\phi_X(t) = \mathbb{E}\exp(tX)$ is finite for a range of $t \in T \subseteq \mathbb{R}$, by Taylor’s expansion, we have $\phi_X(t) = 1 + t^2\mathbb{E}X^2 + o(t^2)$ for $t$ in a neighborhood of 0. Thus, there exist constants $c_1, C_1 > 0$, such that

$$\exp \left( c_1(\mathbb{E}X^2)t^2 \right) \leq \phi_X(t) \leq \exp \left( C_1(\mathbb{E}X^2)t^2 \right)$$

(2)

holds for $t$ in a neighborhood of 0. The following Theorem 1 provides the matching upper and lower bounds on tail probabilities for any random variable $X$ satisfying Condition (2) in a certain regime.

**Theorem 1** (Tail probability bound: Small $t$). Suppose $X$ is a centered random variable satisfying

$$c_2 \exp(c_1a^2t^2) \leq \phi_X(t) \leq C_2 \exp(C_1a^2t^2), \forall 0 \leq t \leq M,$$

(3)

where $C_1 \geq c_1 > 0, C_2, c_2 > 0$ are constants. Suppose either of the following statements holds: (1) $aM^2 \geq \frac{16(\log(1/c_2))}{c_1}$; (2) $c_2 = 1$ and $aM^2 \geq c''$ for some constant $c', > 0$. Then there exist constants $c, c', C > 0$ such that

$$c \exp(-Cx^2/a) \leq \Pr(X \geq x) \leq C \exp(-cx^2/a), \forall 0 \leq x \leq c'Mx.$$

(4)
Moreover, if \( a > 0, M = +\infty \), there exist constants \( c, C > 0 \), such that

\[
c \exp \left( -Cx^2/a \right) \leq \mathbb{P}(X \geq x) \leq C \exp \left( -cx^2/a \right), \quad \forall x \geq 0.
\]  

The proof of Theorem 1 relies on the appropriate choice of values \((t, \theta)\) in Paley–Zygmund inequality (Kahane, 1993, Chapter 1.6; Paley & Zygmund, 1932):

\[
\mathbb{P}(\exp(tX) \geq \theta^2 \exp(\mathbb{E}(X))) \geq (1 - \theta^2)^{\frac{(\mathbb{E}(\exp(tX))^2 - \mathbb{E}(\exp(2tX)))}{\mathbb{E}(\exp(tX))}}.
\]

Remark 1. Previously, Gluskin and Kwapien (1995) studied the tail and moment estimates for the sums of independent random variables. Different from their assumptions that \( X \) can be written as the sum of symmetric, independent, and identically distributed random variables with log-concave tails, Theorem 1 studies generic random variables with bounded moment generating function \( \phi_x(t) \) for a range of \( t \). In addition, Gluskin and Kwapien (1995) focused on the tail probability \( \mathbb{P}(X \geq x) \) for large \( x \), while Theorem 1 focuses on bounded \( x \).

Next, we consider the tail probability for large \( x \).

**Theorem 2** (Tail probability bound: Large \( x \)). Suppose \( Y \) and \( Z \) are independent random variables, \( \phi_Z(t) \) is the moment generating function of \( Z \) and \( \phi_Z(t) \leq e^{ct^2} \) for all \(-M \leq t \leq 0\). \( Y \) satisfies the tail inequality \( \mathbb{P}(Y \geq x) \geq T(x) \) for all \( x \geq wa \). Then \( X = Y + Z \) satisfies

\[
\mathbb{P}(X \geq x) \geq (1 - \exp(-\min\{w^2/(16C_1^2), Mw/4\} a)) T(2x), \quad \forall x \geq wa/2.
\]

Theorem 2 immediately yields the lower tail bound for the sum of independent random variables.

**Corollary 1.** Suppose \( Z_1, \ldots, Z_k \) are centered and independent random variables. Assume \( \phi_{Z_i}(t) \leq \exp(C_1 t^2) \) for \(-M \leq t \leq 0\), where \( \phi_{Z_i} \) is the moment generating function of \( Z_i \), \( 1 \leq i \leq k \). \( Z_1 \) satisfies the tail inequality

\[
\mathbb{P}(Z_1 \geq x) \geq T(x), \quad \forall x \geq w_k.
\]

Then, \( X = Z_1 + \ldots + Z_k \) satisfies

\[
\mathbb{P}(X \geq x) \geq (1 - \exp(-\min\{w^2/(16C_1^2), Mw/4\} k)) T(2x), \quad \forall x \geq w_k/2.
\]

### 3 Tail Bounds for the Sums of Independent Random Variables

With the generic lower bounds on tail probabilities developed in the previous section, we are in position to study the tail probability bounds for the sum of independent random variables. We first consider the tail probability bounds for weighted sums of independent sub-Gaussian random variables. The upper tail bound, which is referred to as the Hoeffding-type concentration inequality, has been widely used in high-dimensional statistics and machine learning literature (see, e.g., Vershynin, 2012).

**Proposition 2** (Hoeffding-type inequality for the sum of sub-Gaussians). Suppose \( Z_1, \ldots, Z_k \) are centered and independently sub-Gaussian distributed, in the sense that either of the following holds: (1) \( \phi_{Z_i}(t) \leq \exp(C_1 t^2) \); (2) \( \mathbb{P}(|Z_i| \geq x) \leq C \exp(-cx^2) \). Then \( X = u_1Z_1 + \ldots + u_kZ_k \) satisfies

\[
\mathbb{P}(|X| \geq x) \leq \exp(-c'x^2/\|u\|^2_k), \quad \forall x > 0.
\]

The proof of Proposition 2 can be found in Vershynin (2012, Proposition 5.10). With the additional condition on the lower tail bound of each summand, we can prove the following lower bound on tail probabilities for the sums of independent sub-Gaussians. This result matches the classic Hoeffding-type inequality in Proposition 2.

**Theorem 3** (Hoeffding-type inequality for sub-Gaussians: Matching lower bounds). Suppose \( Z_1, \ldots, Z_k \) are independent, \( \mathbb{E}Z_i = 0 \), and \( \phi_{Z_i} \) is the moment generating function of \( Z_i \). Suppose either of the following statements holds: (1) there exist constants \( c_1, C_1 > 0 \), such that

\[
\exp(c_1 t^2) \leq \phi_{Z_i}(t) \leq \exp(C_1 t^2), \quad \forall t \geq 0, \quad 1 \leq i \leq k;
\]

(2) there exist constants \( c_2, c_3, c_4 > 0 \), such that

\[
\mathbb{P}(Z_i \geq x) \geq c_2 \exp(-c_3 x^2), \quad \mathbb{P}(|Z_i| \geq x) \leq C_2 \exp(-c_3 x^2), \quad \forall x \geq 0, \quad 1 \leq i \leq k.
\]
Then there exist constants \( c, c', C > 0 \), such that for any fixed values \( u_1, \ldots, u_k \geq 0 \), \( X = u_1 Z_1 + \ldots + u_k Z_k \) satisfies

\[
c \exp \left( -C x^2 / \|u\|_2^2 \right) \leq \mathbb{P}(X \geq x) \leq \exp \left( -c' x^2 / \|u\|_2^2 \right), \quad \forall x \geq 0.
\]

(10)

Although Theorem 2 focuses on the right tail bound, i.e., \( \mathbb{P}(X \geq x) \), similar results hold for the left tail, i.e., \( \mathbb{P}(X \leq -x) \), by symmetry. We can further prove the following lower bound on tail probabilities for the sum of random variables with two-sided sub-Gaussian tails.

**Corollary 2.** Suppose \( Z_1, \ldots, Z_k \) are independent, \( \mathbb{E}Z = 0 \), \( \phi_Z \) is the moment generating function of \( Z \). Suppose either of the following statements holds: (1) \( \exp(c_1 t^2) \leq \phi_Z(t) \leq \exp(C_1 t^2) \) for constants \( c_1, C_1 > 0 \), \( t \in \mathbb{R}, 1 \leq i \leq k \); (2) \( c_2 \exp(-C_2 x^2) \leq \mathbb{P}(Z \geq x) \land \mathbb{P}(Z \leq -x) \leq \mathbb{P}(Z_i \geq x) \lor \mathbb{P}(Z_i \leq -x) \leq C_3 \exp(-c_3 x^2) \) for constants \( c_2, C_2, c_3, C_3 > 0 \) and all \( x \geq 0 \), \( 1 \leq i \leq k \). Then there exist constants \( c, c', C > 0 \), for any fixed real values \( u_1, \ldots, u_k \), \( X = u_1 Z_1 + \ldots + u_k Z_k \) satisfies

\[
c \exp \left( -C x^2 / \|u\|_2^2 \right) \leq \mathbb{P}(X \geq x) \leq \exp \left( -c' x^2 / \|u\|_2^2 \right), \quad \forall x \geq 0.
\]  

\[
c \exp \left( -C x^2 / \|u\|_2^2 \right) \leq \mathbb{P}(X \leq -x) \leq \exp \left( -c' x^2 / \|u\|_2^2 \right), \quad \forall x \geq 0.
\]

On the other hand, the class of sub-Gaussian random variables considered in Theorem 3 may fail to cover many useful random variables with heavier tails. To cover broader cases, sub-exponential distributions were introduced and widely used in literature (see the forthcoming Proposition 3 for definition of the sub-exponential distribution). The following Bernstein-type inequality (see, e.g., Vershynin, 2012, Proposition 5.16; Boucheron et al., 2013, Theorem 2.3; and Wainwright, 2019, Proposition 2.2) is a classic result on the tail bound for the sum of sub-exponential random variables.

**Proposition 3** (Bernstein-type inequality for the Sums of independent Sub-exponentials (Vershynin, 2012, Proposition 5.16)). Let \( Z_1, \ldots, Z_k \) be independent centered sub-exponential random variables in the sense that \( \mathbb{E} \exp(tZ_i) \leq \exp(C t^2) \) for all \( |t| \leq c \). Then for every \( u_1, \ldots, u_k \), \( X = u_1 Z_1 + \ldots + u_k Z_k \) satisfies

\[
\mathbb{P}(X \geq x) \leq 2 \exp \left( -\alpha x^2 / \|u\|_2^2 \right), \quad \forall 0 \leq x \leq \|u\|_2^2 / \|u\|_\infty;
\]

\[
\mathbb{P}(X \leq -x) \leq 2 \exp \left( -\alpha x^2 / \|u\|_\infty \right), \quad \forall x \geq \|u\|_2^2 / \|u\|_\infty.
\]

With the additional lower bound on the moment generating function of each summand, we prove the following matching upper and lower tail bounds for the sum of independent sub-exponential random variables.

**Theorem 4** (Bernstein-type inequality for the sums of independent sub-exponential: Matching upper and lower bounds). Suppose \( Z_1, \ldots, Z_k \) are centered independent random variables and \( \phi_Z \) is the moment generating function of \( Z \). Suppose \( Z_i \)'s are sub-exponential in the sense that there exist two constants \( c_1, C_1 > 0 \),

\[
\exp(c_1 t^2) \leq \phi_Z(t) \leq \exp(C_1 t^2), \quad \forall |t| \leq M, 1 \leq i \leq k.
\]

Suppose \( a M^2 \geq c'' \), where \( c'' > 0 \) is a constant. If \( u_1, \ldots, u_k \) are non-negative values, then \( X = u_1 Z_1 + \ldots + u_k Z_k \) satisfies

\[
c \exp \left( -C x^2 / a \|u\|_2^2 \right) \leq \mathbb{P}(X \geq x) \leq \exp \left( -c' x^2 / a \|u\|_2^2 \right)\]

for any \( 0 \leq x \leq \frac{c M \|u\|_\infty^2}{\|u\|_\infty} \). Here \( c, c', C, \alpha > 0 \) are constants.

In addition, if there exists one \( Z_i \) \( (1 \leq i \leq k) \) satisfying

\[
\mathbb{P}(Z_i \geq x) \geq c_0 \exp(-C_0 M x), \quad \forall x \geq 2c' M a,
\]

and \( u_i \geq c_2 \|u\|_\infty \) for constant \( c_2 > 0 \), then we have

\[
c \exp \left( -\hat{c} \frac{M x}{\|u\|_\infty} \right) \leq \mathbb{P}(X \geq x) \leq \exp \left( -\hat{c} \frac{M x}{\|u\|_\infty} \right), \quad \forall x \geq \frac{c M \|u\|_\infty^2}{\|u\|_\infty},
\]

(11)

where \( \hat{c}, \hat{C} > 0 \) are constants.

## 4 SHARP TAIL BOUNDS OF SPECIFIC DISTRIBUTIONS

In this section, we establish matching upper and lower tail bounds for a number of commonly used distributions.
4.1 | Gamma distribution

Suppose $Y$ is gamma distributed with shape parameter $a$, i.e., $Y \sim \text{Gamma}(a)$, where the density is

$$f(y; a) = \frac{1}{\Gamma(a)} y^{a-1} e^{-y}, \quad y > 0.$$  \hfill (12)

Here $\Gamma(a)$ is the gamma function. Although the density of gamma distribution is available, it is highly non-trivial to develop the sharp tail probability bound in a closed form. Previously, Boucheron et al. (2013, pp. 27–29) established an upper bound on the tail probability of gamma distribution: Suppose $Y \sim \text{Gamma}(a)$ and $X = Y - a$. Then

$$P(X \geq \sqrt{2at} + t) \leq e^{-t}, \quad P(X \leq -\sqrt{2at}) \leq e^{-t}, \quad \forall t \geq 0.$$  \hfill (13)

Or equivalently, $P(X \geq x) \leq \exp \left( -\frac{x^2}{x + a + \sqrt{a^2 + 2ax}} \right)$, $P(X \leq -x) \leq \exp \left( -\frac{x^2}{2a} \right)$, $\forall x \geq 0$.

We can prove the following lower bound on the tail probability of gamma distribution that matches the upper bound. Since the density of $\text{Gamma}(a)$ has distinct shapes for $a \geq 1$ and $a < 1$: $\lim_{y \to \infty} f(y; a) = \infty$ if $a < 1$; $\lim_{y \to 0} f(y; a) = 0$ if $a \geq 1$, the tail bound behaves differently in these two cases and we discuss them separately.

**Theorem 5** (Gamma tail bound). Suppose $Y \sim \text{Gamma}(a)$ and $X = Y - a$.

- There exist two uniform constants $c, C > 0$, such that for all $a \geq 1$ and $x \geq 0$,

$$c \exp \left( -Cx \wedge \frac{x^2}{a} \right) \leq P(X \geq x) \leq \exp \left( -cx \wedge \frac{x^2}{a} \right).$$

For any $\beta > 1$, there exists $C_{\beta} > 0$ only relying on $\beta$, such that for all $a \geq 1$ and $0 \leq x \leq \frac{a}{\beta}$, we have

$$c \cdot \exp \left( -C_{\beta} \frac{x^2}{a} \right) \leq P(X \leq -x) \leq \exp \left( -\frac{x^2}{2a} \right).$$

- For all $0 < a < 1$,

$$\frac{1}{e} \cdot \frac{(a + x + 1)^{a} - (a + x)^{a}}{e^{x} \Gamma(a + 1)} \leq P(X \geq x) \leq \frac{e}{e - 1} \cdot \frac{(a + x + 1)^{a} - (a + x)^{a}}{e^{x} \Gamma(a + 1)}, \quad \forall x \geq 0;$$  \hfill (14)

$$\frac{\{(a - x) \vee 0\}^{a}}{e^{a} \Gamma(a + 1)} \leq P(X \leq -x) \leq \frac{\{(a - x) \vee 0\}^{a}}{\Gamma(a + 1)}, \quad \forall x \geq 0.$$  \hfill (15)

The proof for $a \geq 1$ case is based on Theorem 2; the proof of $a < 1$ is via the direct integration and approximation of the gamma density.

4.2 | Chi-square distribution

The chi-square distributions form a special class of gamma distributions and are widely used in practice. Suppose $Y \sim \chi^2_k$, $X = Y - k$. Laurent and Massart (2000, Lemma 1) introduced the following upper tail bound for chi-square distribution: for any $x \geq 0$,

$$P(X \geq x) \leq \frac{x^2}{2(k + x) + 2\sqrt{k^2 + 2kx}}, \quad P(X \leq -x) \leq \frac{x^2}{4k}.$$  

Theorem 5 implies the following lower bound on chi-square distribution tail probability that matches the upper bound.

**Corollary 3** ($\chi^2$ tail bound). Suppose $Y \sim \chi^2_k$ and $X = Y - k$ for integer $k \geq 1$. There exist uniform constants $C, c > 0$ and a constant $C_x > 0$ that only relies on $\epsilon$, such that

$$P(X \geq x) \geq c \exp \left( -Cx \wedge \frac{x^2}{k} \right), \quad \forall x > 0;$$  \hfill (16)

$$P(X \leq -x) \begin{cases} \geq c \exp \left( -C_{\epsilon} \frac{x^2}{k} \right), & \forall 0 < x < (1 - \epsilon)k; \\ = 0, & x \geq k. \end{cases}$$  \hfill (17)

In addition to the regular chi-square distributions, the weighted and noncentral chi-square distributions (definitions are given in the forthcoming Theorems 6 and 7) are two important extensions. We establish the matching upper and lower tail bounds for weighted chi-square distributions in Theorem 6 and noncentral chi-square distributions in Theorem 7, respectively.
Theorem 6 (Tail bounds of weighted $\chi^2$ distribution). Suppose $Y$ is weighted chi-square distributed, in the sense that $Y = \sum_{i=1}^{k} u_i Z_i$, where $u_1, \ldots, u_k$ are fixed non-negative values and $Z_1, \ldots, Z_k \sim N(0,1)$. Then the centralized random variable $X = Y - \sum_{i=1}^{k} u_i$ satisfies

$$c \exp \left(-C x^2 / \|u\|_2^2 \right) \leq \mathbb{P}(X \geq x) \leq \exp(-\hat{C} x^2 / \|u\|_2^2), \quad \forall x \leq \frac{\|u\|_2^2}{\|u\|_\infty};$$

$$\hat{c} \exp \left(-\hat{C} x / \|u\|_\infty \right) \leq \mathbb{P}(X \geq x) \leq \exp(-\hat{C} x / \|u\|_\infty), \quad \forall x > \frac{\|u\|_2^2}{\|u\|_\infty};$$

$$c_1 \exp \left(-C_1 x^2 / \|u\|_2^2 \right) \leq \mathbb{P}(X \leq -x) \leq \exp \left(-\frac{1}{4} x^2 / \|u\|_2^2 \right), \quad \forall 0 \leq x \leq c_1 \frac{\|u\|_2^2}{\|u\|_\infty}. $$

Theorem 7 (Noncentral $\chi^2$ tail bound). Let $Z$ be noncentral $\chi^2$ distributed with $k$ degrees of freedom and noncentrality parameter $\lambda$, in the sense that

$$Z = \sum_{i=1}^{k} Z_i^2, \quad Z_i \sim N(\mu_i, 1) \text{ independently and } \sum_{i=1}^{k} \mu_i^2 = \lambda.$$ 

Then the centralized random variable $X = Z - (k + \lambda)$ satisfies

$$c \exp \left(-C x^2 / k + 2\lambda \right) \leq \mathbb{P}(X \geq x) \leq \exp \left(-\hat{C} x^2 / k + 2\lambda \right), \quad \forall 0 \leq x \leq k + 2\lambda. \quad (18)$$

$$\hat{c} \exp \left(-\hat{C} x \right) \leq \mathbb{P}(X \geq x) \leq \exp(-\hat{C} x), \quad \forall x \geq k + 2\lambda. \quad (19)$$

For all $\beta > 1$, there exists $C_\beta > 0$ that only relies on $\beta$ and a constant $c_1 > 0$,

$$c_1 \exp \left(-C_\beta x^2 / k + 2\lambda \right) \leq \mathbb{P}(X \leq -x) \leq \exp \left(-\frac{1}{4} x^2 / k + 2\lambda \right), \quad \forall 0 < x \leq \frac{k + \lambda}{\beta}. \quad (20)$$

Here, the upper bounds in Theorems 6 and 7 were previously proved by Laurent and Massart (2000, Lemma 1) and Birgé (2001, Lemma 8.1) respectively. We are the first to prove the lower bounds in Theorems 6 and 7 in Sections A.9 and A.10 (Supporting Information), respectively.

4.3 Beta distribution

Beta distribution is a class of continuous distributions that commonly appears in applications. Since the beta distribution is the conjugate prior of Bernoulli, binomial, geometric, and negative binomial, it is often used as the prior distribution for proportions in Bayesian inference. Recently, Marchal and Arbel (2017) proved that the beta distribution is $(\frac{1}{4\alpha + 3})$-sub-Gaussian, in the sense that the moment generating function of $Z \sim \text{Beta}(\alpha, \beta)$ satisfies $\mathbb{E} \left[ \exp \left( t \left( Z - \frac{\alpha}{\alpha + \beta} \right) \right) \right] \leq \exp \left( \frac{c^2}{4\alpha + 3} t^2 \right)$ for all $t \in \mathbb{R}$. They further gave an upper bound on tail probability:

$$\mathbb{P} \left( Z \geq \frac{\alpha}{\alpha + \beta} + x \right) \vee \mathbb{P} \left( Z \leq \frac{\alpha}{\alpha + \beta} - x \right) \leq \exp \left( -2(\alpha + \beta + 1)x^2 \right), \quad \forall x \geq 0. \quad (21)$$

Based on the tail bounds of gamma distribution, we can prove the following matching upper and lower bounds for $\text{Beta}(\alpha, \beta)$.

Theorem 8 (Beta distribution tail bound). Suppose $Z \sim \text{Beta}(\alpha, \beta), \alpha, \beta \geq 1$. There exists a uniform constant $c > 0$ such that

$$\forall 0 < x < \frac{\beta}{\alpha + \beta}, \quad \mathbb{P} \left( Z \geq \frac{\alpha}{\alpha + \beta} + x \right) \leq \begin{cases} \exp \left( -c \left( \frac{\alpha^2}{\beta} \wedge \beta x \right) \right), & \text{if } \beta > \alpha; \\ \exp \left( -c \frac{\alpha^2}{\beta} \right), & \text{if } \alpha \geq \beta. \end{cases}$$

$$\forall 0 < x < \frac{\alpha}{\alpha + \beta}, \quad \mathbb{P} \left( Z \leq \frac{\alpha}{\alpha + \beta} - x \right) \leq \begin{cases} \exp \left( -c \left( \frac{\alpha^2}{\beta} \wedge ax \right) \right), & \text{if } \alpha > \beta; \\ \exp \left( -c \frac{\alpha^2}{\beta} \right), & \text{if } \beta \geq \alpha. \end{cases}$$

In addition, for any $\eta > 1$, there exists $C_\eta > 0$ only depending on $\eta$ and a uniform constant $c > 0$, such that

$$\forall 0 < x \leq \frac{\beta}{\eta(\alpha + \beta)}, \quad \mathbb{P} \left( Z \geq \frac{\alpha}{\alpha + \beta} + x \right) \geq \begin{cases} c \exp \left( -C_\eta \left( \frac{\alpha^2}{\beta} \wedge \beta x \right) \right), & \text{if } \beta > \alpha; \\ c \exp \left( -C_\eta \frac{\alpha^2}{\beta} \right), & \text{if } \alpha \geq \beta. \end{cases} \quad (22)$$
\begin{align*}
\forall 0 < x \leq \frac{a}{\eta(a + \beta)}, \quad & P \left( Z \leq \frac{a}{a + \beta} - x \right) \geq \begin{cases}
    c \exp \left( -C_q \left( \frac{a^2}{x} \right) \right), & \text{if } a > \beta;
    c \exp \left( -C_q \left( \frac{x}{a} \right) \right), & \text{if } \beta \geq a.
\end{cases}
\end{align*}

Moreover,
\begin{align*}
\forall x > \frac{\beta}{a + \beta}, & \quad P \left( Z \geq \frac{a}{a + \beta} + x \right) = 0; \quad \forall x > \frac{a}{a + \beta}, & \quad P \left( Z \leq \frac{a}{a + \beta} - x \right) = 0.
\end{align*}

Theorem (8) implies that (21) may not be sharp when \( a \ll \beta \) or \( a \gg \beta \).

### 4.4 Binomial distribution

The following classic result of upper tail bound for the binomial distribution has been introduced in Arratia and Gordon (1989) and Boucheron et al. (2013, pp. 23–24): Suppose \( Y \sim \text{Bin}(k, p) \) and \( X = Y - kp \) is the centralization. Then for all \( 0 < x < k(1 - p) \),

\begin{equation}
P(X \geq x) \leq \exp \left( -kh_p(x/k + p) \right), \tag{23}
\end{equation}

where \( h_p(v) = v \log \left( \frac{v}{p} \right) + (1 - v) \log \left( \frac{1 - v}{1 - p} \right) \) is the Kullback–Leibler divergence between \( \text{Bernoulli}(u) \) and \( \text{Bernoulli}(v) \) for all \( 0 < u, v < 1 \). Due to the delicate form of the moment generating function of binomial distribution, it may be difficult to select appropriate values of \( (\theta, t) \) in \( P(e^{xt} \geq \theta e^{xt'}) \geq (1 - \theta)^2 \frac{\text{dG}^2(x)}{\text{dG}^2(x')/\theta} \) if we plan to apply Paley–Zygmund inequality to prove the desired lower tail bound. We instead prove the following reverse Chernoff–Cramèr bound as a key technical tool.

**Lemma 1** (A reverse Chernoff–Cramèr bound). Suppose \( X \) is a random variable with moment generating function \( \phi(t) = E \exp(tX) \) defined for \( t \in T \subseteq \mathbb{R} \). Then for any \( x > 0 \), we have

\begin{equation*}
P(X \geq x) \geq \sup_{\theta, \beta \in T} \left\{ \phi(t) \exp(-t\beta x) - \phi(t\theta) \exp(-t\beta \theta x) - \exp(-(t\beta - t')x)\phi(t - t') \right\}.
\end{equation*}

\begin{equation*}
P(X \leq -x) \geq \sup_{\theta, \beta \in T} \left\{ \phi(-t) \exp(-t\beta x) - \phi(-t\theta) \exp(-t\beta \theta x) - \exp(-(t\beta - t')x)\phi(t' - t) \right\}.
\end{equation*}

Then we can prove the following lower bound on tail probability for binomial distribution.

**Theorem 9** (Lower bounds on binomial tail probability). Suppose \( X \) is centralized binomial distributed with parameters \( (k, p) \). For any \( \beta > 1 \), there exist constants \( c_\beta, C_\beta > 0 \) that only rely on \( \beta \), such that

\begin{equation}
P(X \geq x) \begin{cases}
    \geq c_\beta \exp \left( -C_\beta kh_p \left( p + \frac{x}{k} \right) \right), & \text{if } 0 \leq x \leq \frac{k(1 - p)}{\beta} \text{ and } x + kp \geq 1; \\
    = 1 - (1 - p)^k, & \text{if } 0 < kp + x < 1.
\end{cases}
\tag{24}
\end{equation}

\begin{equation}
P(X \leq -x) \begin{cases}
    \geq c_\beta \exp \left( -C_\beta kh_p \left( p - \frac{x}{k} \right) \right), & \text{if } 0 \leq x \leq \frac{k(1 - p)}{\beta} \text{ and } x + kp \geq 1; \\
    = 1 - p^k, & \text{if } 0 < k(1 - p) + x < 1.
\end{cases}
\tag{25}
\end{equation}

**Remark 2** (Comparison to previous results). Previously, Slud (1977) studied the normal approximation for \( X \sim \text{Bin}(k, p) - kp \) and proved that \( P(X \geq x) \geq 1 - \Phi \left( x/\sqrt{kp(1 - p)} \right) \) if either (a) \( p \leq 1/4 \) and \( x \geq 0 \), or (b) \( 0 \leq x \leq k(1 - 2p) \). Here, \( \Phi \) is the cumulative distribution function of the standard normal distribution. However, it is unclear if Slud’s inequality universally provides the sharp lower bound: when \( p \) is close to zero, say \( p = 1/k \), the lower bound provided by Slud’s inequality is approximately \( \exp \left( -\frac{x^2}{2} \right) \) for \( x = \frac{1}{2} \), which does not match the classic upper bound \(( \approx \exp(-k \log k)\)\). By the method of types, Cover and Thomas (2012, Lemma II.2) showed that

\begin{equation*}
P(X \geq x) \geq \frac{1}{k + 1} \exp \left( -kh_p \left( p + \frac{x}{k} \right) \right)
\end{equation*}

when \( 0 \leq x \leq k(1 - p) \) and \( pk + x \) is an integer. This bound is sharp up to a factor of \( \frac{1}{k + 1} \) in comparison to the upper bound (23). Our Theorem 9 yields a sharper rate than the one provided by the method of types for small values of \( x \). In addition, our Theorem 9 allows \( pk + x \) or \( pk - x \) to be more general real values than integers.

The sum of i.i.d. Rademacher random variables is an important instance of binomial distributions that commonly appears in practice. The established binomial tail bounds immediately imply the following result.
Corollary 4 (Sum of i.i.d. Rademacher random variables). Suppose $Z_1, \ldots, Z_k$ are i.i.d. Rademacher random variables, i.e., $P(Z_i = 1) = P(Z_i = -1) = 1/2$. Suppose $X = Z_1 + \cdots + Z_k$. Then, for any $\beta > 1$, there exist constants $c, c_{\beta} > 0$ that only rely on $\beta$, such that

$$c_{\beta} \exp \left(-C \frac{x^2}{k} \right) \leq P(X \geq x) = P(X \leq -x) \leq \exp \left(-\frac{x^2}{4k} \right), \ \forall 0 \leq x \leq \frac{k}{\beta}.$$ 

$$P(X \geq x) = P(X \leq -x) = 0, \ \forall x > k.$$ 

4.5 Poisson distribution

It was shown in literature that the Poisson distribution has the Bennett-type tail upper bound (see, e.g., Boucheron et al., 2013, and Pollard, 2015; also see Bennett, 1962, for Bennett Inequality): Suppose $X \sim \text{Poisson} (\lambda) = Y - \lambda$. Then

$$P(X \geq x) \leq \exp \left(-\frac{x^2}{2\lambda} \psi_{\text{Benn}} \left(\frac{x}{\lambda}\right) \right), \ \forall x \geq 0. \quad (26)$$

$$P(X \leq -x) \leq \exp \left(-\frac{x^2}{2\lambda} \psi_{\text{Benn}} \left(-\frac{x}{\lambda}\right) \right), \ \forall 1 \leq x \leq \lambda. \quad (27)$$

Here, $\psi_{\text{Benn}}$ is the Bennett function defined as

$$\psi_{\text{Benn}}(t) = \begin{cases} \frac{(1+\log(1+t))^2}{t^2}, & \text{if } t > -1, t \neq 0; \\ 1, & \text{if } t = 0. \end{cases} \quad (28)$$

We can prove the following lower bound on the tail probability for Poisson distribution.

Theorem 10 (Lower bound on Poisson tail). Suppose $X$ is centralized Poisson distributed with parameter $\lambda$. Equivalently, $X = Y - \lambda$ and $Y \sim \text{Poisson} (\lambda)$. Then there exist constants $c, C > 0$ such that

$$P(X \geq x) \begin{cases} \geq c \cdot \exp \left(-C \cdot \frac{x^2}{2\lambda} \psi_{\text{Benn}}(x/\lambda) \right), & \text{if } x \geq 0 \text{ and } x + \lambda \geq 1; \\ = 1 - e^{-t}, & \text{if } x + \lambda < 1. \end{cases} \quad (29)$$

For all $\beta > 1$, there exist constants $c_{\beta}, C_{\beta} > 0$ that only rely on $\beta$, such that

$$P(X \leq -x) \geq c_{\beta} \cdot \exp \left(-C_{\beta} \frac{x^2}{2\lambda} \psi_{\text{Benn}}(x/\lambda) \right), \ \text{if } 0 \leq x \leq \frac{1}{\beta}. \quad (30)$$

Remark 3. Since the Poisson distribution is discrete, similarly to the binomial distribution, we need to discuss the boundary case of $x + \lambda < 1$ when analyzing the tail bound $P(X \geq x)$. By comparing to the lower bound in Theorem 10, we can see the classic Bennett-type upper bound (26) is not sharp when $x + \lambda < 1$.

4.6 Irwin–Hall distribution

Suppose $U_1, \ldots, U_k$ are i.i.d. uniformly distributed on $[0, 1]$. Then $Y = \sum_{i=1}^{k} U_i$ satisfies Irwin–Hall distribution. We have the following matching upper and lower bounds on tail probabilities of the Irwin–Hall distribution.

Corollary 5 (Irwin–Hall tail bound). Suppose $Y$ follows the Irwin–Hall distribution with parameter $k$. Denote $X = Y - \frac{k}{2}$. Then for $0 \leq x \leq \frac{k}{2}$,

$$P(X \leq -x) = P(X \geq x) \leq \exp \left(-k \cdot h_{1/2} \left(\frac{1}{2} + \frac{x}{k} \right) \right) \leq \exp \left(-\frac{x^2}{k} \right). \quad (31)$$

There also exist constants $c, c', C > 0$, such that for all $0 \leq x \leq c'k$, we have

$$P(X \leq -x) = P(X \geq x) \geq c \cdot \exp \left(-C \frac{x^2}{k} \right). \quad (32)$$
5 | APPLICATIONS

5.1 | Extreme values of random variables

The extreme value theory plays a crucial role in probability, statistics, and actuarial science (Gumbel, 2012; Smith, 2003; Philippe, 2001). The central goal is to study the distribution of extreme values of a sequence of random variables. The lower bounds on tail probabilities developed in previous sections have a direct implication to extreme values distributions for generic random variables. To be specific, if \( X_1, \ldots, X_n \) are weighted sums of sub-Gaussian or sub-exponential random variables, we can prove the following matching upper and lower bounds for \( \sup_{1 \leq i \leq k} X_i \).

**Theorem 11** (Extreme value of the sums of independent sub-Gaussians). Suppose \( Z_i (1 \leq i \leq k, 1 \leq j \leq n) \) are centered and independent sub-Gaussian random variables satisfying

\[
\exists \text{constants } c_1, c_1 > 0, \quad \exp(c_1 t^2) \leq \exp(C_1 t^2), \quad \forall t \in \mathbb{R}, \quad 1 \leq i \leq k, \quad 1 \leq j \leq n.
\]

If \( u_1, \ldots, u_n \) are non-negative values, \( X_i = u_1 Z_{i1} + \ldots + u_n Z_{in} \) for \( 1 \leq i \leq k \), then there exist constants \( C, c > 0 \),

\[
c \sqrt{a \|u\|_2^2 \log k} \leq E \sup_{1 \leq i \leq k} X_i \leq C \sqrt{a \|u\|_2^2 \log k}.
\]

**Theorem 12** (Extreme value of the sums of independent sub-exponentials). Suppose \( Z_i (1 \leq i \leq k, 1 \leq j \leq n) \) are centered and independent random variables. Suppose \( Z_i \)'s are sub-exponential in the sense that:

\[
\exists \text{constants } c_1, c_1 > 0, \quad \exp(c_1 t^2) \leq \exp(C_1 t^2), \quad \forall |t| < M, \quad 1 \leq i \leq k, \quad 1 \leq j \leq n.
\]

Suppose \( aM^2 \geq c'' \), where \( c'' > 0 \) is a constant. In addition, for any \( j \), there exists one \( Z_j (1 \leq i \leq k) \) further satisfying

\[
P (Z_j \geq x) \geq c_0 \exp(-C_0 Mx), \quad \forall x \geq 2c'Mx,
\]

and \( u_i \geq c_j \|u\|_\infty \), where \( c_j > 0 \) is a constant. If \( u_1, \ldots, u_n \) are non-negative values, \( X_i = u_1 Z_{i1} + \ldots + u_n Z_{in} \) for \( 1 \leq i \leq k \), then there exist constants \( C, c > 0 \),

\[
c \left( \sqrt{a \|u\|_2^2 \log k} \lor \frac{\|u\|_\infty \log k}{M} \right) \leq E \sup_{1 \leq i \leq k} X_i \leq C \left( \sqrt{a \|u\|_2^2 \log k} \lor \frac{\|u\|_\infty \log k}{M} \right).
\]

**Remark 4.** If \( X_1, \ldots, X_k \sim \mathcal{N}_k \), Theorem 12 immediately implies

\[
c \left( \sqrt{n \log k} \lor \log k \right) \leq E \sup_{1 \leq i \leq k} X_i - n \leq C \left( \sqrt{n \log k} \lor \log k \right).
\]

5.2 | Signal identification from sparse heterogeneous mixtures

In this subsection, we consider an application of signal identification from sparse heterogeneous mixtures. Motivated by applications in astrophysical source and genomics signal detections, the sparse heterogeneous mixture model has been proposed and extensively studied in recent high-dimensional statistics literature (see, e.g., Cai et al., 2011, 2007; Donoho & Jin, 2004; Jin, 2008). Suppose one observes \( Y_1, \ldots, Y_k \in \mathbb{R} \). \( Z_1, \ldots, Z_k \in \{0, 1\} \) are hidden labels that indicate whether the observations \( Y_i \) are signal or noise

\[
Y_i = P_0 1_{(Z_i = 0)} + P_1 1_{(Z_i = 1)}.
\]

Here, \( P_0 \) and \( P_1 \) are the noise and signal distributions, respectively. Suppose \( 0 < \varepsilon < 1 \), \( Z_1, \ldots, Z_k \) independently satisfy

\[
P(Z_i = 1) = \varepsilon, \quad P(Z_i = 0) = 1 - \varepsilon.
\]

We aim to identify the set of signals, i.e., \( I = \{ i : Z_i = 1 \} \), based on observations \( Y_1, \ldots, Y_k \). When the observations of \( Y_1, \ldots, Y_k \) are discrete count valued, it is more natural to model the observations as Poisson random variables as opposed to the more commonly studied Gaussian distributions in the literature. In particular, we consider

\[
Y_i \sim \text{Poisson}(\mu 1_{(Z_i = 0)}) + \text{Poisson}(\lambda 1_{(Z_i = 1)}), \quad i = 1, \ldots, k.
\]
where \( \mu \) and \( \lambda \) are the Poisson intensities for noisy and signal observations, respectively. To quantify the performance of any identifier \( \{ \hat{Z}_i \}_{i=1}^k \in \{0, 1\}^k \), we consider the following misidentification rate in the Hamming distance,

\[
M(\hat{Z}) = \frac{1}{k} \sum_{i=1}^k |\hat{Z}_i - Z_i|.
\]

Based on the upper and lower tail bounds of Poisson distributions in Section 4.5, we can establish the following sharp bounds on the misidentification rate.

**Theorem 13.** Suppose \( (Y_1, Z_1), \ldots, (Y_n, Z_n) \) are i.i.d. pairs of observations and hidden labels that satisfy (34) and (35). Assume \( 1 \leq \mu < \lambda, C_1 \mu \leq \lambda \leq C_2 \mu \) for constants \( C_2 \geq C_1 > 1 \), and \( 0 < \epsilon < 1 \). Then for any identification procedure \( \hat{Z} \in \{0, 1\}^k \) based on \( \{Y_i\}_{i=1}^k \), there exist two constants \( C, c > 0 \) such that

\[
EM(\hat{Z}) \geq \begin{cases} 
Ce, & 0 < \epsilon < \epsilon^-; \\
C e^{-C(1+\theta)}(1-\epsilon), & \epsilon^- \leq \epsilon \leq \epsilon^+; \\
C(1-\epsilon), & \epsilon^+ < \epsilon < 1.
\end{cases}
\]

Here,

\[
\epsilon^+ = \frac{1}{\exp(\log(\frac{\lambda}{\mu}) + \mu - \lambda) + 1}, \quad \epsilon^- = \frac{1}{\exp(\log(\frac{\lambda}{\mu}) + \mu - \lambda) + 1}, \quad \theta = \log\left(\frac{1 - \epsilon}{\epsilon}\right) + \lambda - \mu.
\]

\[g(\theta) = -\log\left(1 - \epsilon\right) \exp\left(-\frac{\mu^2}{2\mu - \lambda} - \epsilon\right) + \exp\left(-\frac{(\lambda - \theta)^2}{2\lambda} - \epsilon\right).
\]

\[\psi_{\text{Benn}} \text{ is the Bennett function defined in (28). In particular, the classifier } \hat{Z} = \{\hat{Z}_i\}_{i=1}^k, \hat{Z} = \{0, 1\}^k \text{ achieves the following misclassification rate in the Hamming distance:}
\]

\[
EM(\hat{Z}) \leq \begin{cases}
The performance of this classifier
\end{cases}
\]

**Remark 5.** If the noise and signal distributions in Model (33), i.e., \( P_0 \) and \( P_1 \), are other than Poisson, similar results to Theorem 13 can be established based on the corresponding upper and lower tail bounds in the previous sections.

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**How to cite this article:** Zhang AR, Zhou Y. On the non-asymptotic and sharp lower tail bounds of random variables. Stat. 2020;9:e314. https://doi.org/10.1002/sta4.314