MOMENTUM OPERATORS IN TWO INTERVALS:
SPECTRA AND PHASE TRANSITION

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Abstract. We study the momentum operator defined on the disjoint union of two intervals. Even in one dimension, the question of two non-empty open and non-overlapping intervals has not been worked out in a way that extends the cases of a single interval and gives a list of the selfadjoint extensions. Starting with zero boundary conditions at the four endpoints, we characterize the selfadjoint extensions and undertake a systematic and complete study of the spectral theory of the selfadjoint extensions. In an application of our extension theory to harmonic analysis, we offer a new family of spectral pairs. Compared to earlier studies, it yields a more direct link between spectrum and geometry.

CONTENTS

1. Introduction 2
1.1. Two Intervals 3
1.2. Unbounded Operators 4
1.3. Summary 4
1.4. Prior Literature 5
2. Momentum Operators 5
2.1. Applications 5
2.2. The boundary form, spectrum, and the group $U(2)$ 6
3. Spectral Theory 8
3.1. Boundary Cases 10
3.2. Generic Cases 11
3.3. Irrational $\beta - \alpha$ 16
4. Unitary One-Parameter Groups 19
5. Spectral Pairs 20
5.1. Two Theorems 21
5.2. Applications 26
5.3. Examples 26
6. Questions 32
Acknowledgments 33
References 33

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1. Introduction

An early paper of John von Neumann [vN32] deals with unbounded linear operators in Hilbert space, and their spectral theory. By now the theory is part of functional analysis, and the problems entail the study of “deficiency indices;” now important tools in mathematical physics; see e.g., [Kre49].

Indeed, von Neumann’s initial motivation came directly from quantum mechanics: measurements of observables, prepared in states. These indices have now become the accepted framework in spectral theory, as well as in quantum mechanics.

In quantum theory, states are realized by vectors (of norm one) in Hilbert space $\mathcal{H}$, and the quantum observables correspond to selfadjoint operators in $\mathcal{H}$ (the state space.)

Our system of operators in two intervals has a number of applications not contained in other studies. For example, when modeling quantum states (Section 4), we show how a two-component quantum system offer a binary model for the dynamics of wave functions. Indeed the parameters for our family of selfadjoint operators allow a direct computation of transition probabilities.

Here we study an instance of the index/spectral theory question, and in computations we use the spectral theorem as it is applied to selfadjoint operators. However the deficiency indices are precisely the obstruction to selfadjointness for unbounded Hermitian operators.

This is illustrated with examples, naturally occurring, and quite simple, from quantum theory: There one studies wave-functions with confinement to a finite region (particles in a box), e.g., confinement in a finite interval. In these cases, the physical operators are not selfadjoint, but only Hermitian.

But the unbounded linear operators will still have dense domain in the physical Hilbert space (of states). Therefore, an assignment of boundary conditions will amount to a subtle extension of the initial dense operator domain.

In several variables, one is then looking for commuting selfadjoint extensions; see [Fug74, Jør82, PW01, Ped04, JP00, JP99, JP98, DJ07] for a study of this problem, in several applications.

For these problems, and for applications in the theory of linear partial differential operators, one is led to a more axiomatic consideration of boundary conditions. Indeed, J. von Neumann, in his paper [vN32], identified such conditions in the Hilbert space theoretic framework. For a single Hermitian operator, they take the form of deficiency spaces, and deficiency indices, see details below. In one dimension, and for a finite interval, say $I$, a popular model example has its deficiency indices come out $(1, 1)$. Hence, by von Neumann’s theorem, this yields a corresponding one-parameter family of selfadjoint extensions. In the case of one interval, they can readily be computed, and it is possible to read off the spectrum of each of the selfadjoint extensions. Each one is the generator of a unitary one-parameter group $U(t)$ acting in $\mathcal{H}$. And each of these unitary groups of operators in turn translates the wave-functions in the interior of the interval, and then assigns a phase-factor at the endpoints of $I$. The corresponding dynamics of the wave-functions turns into a fixed phase-shift, i.e., a matching with a phase-factor of incoming and outgoing waves.

Moreover, for each selfadjoint extension, the corresponding spectrum is a Fourier spectrum for the Hilbert space $\mathcal{H} = L^2(I)$ where $I$ is the given interval. By “Fourier spectrum” we mean that the spectrum (of the selfadjoint extension) is discrete of
uniform multiplicity one, and that the eigenfunctions are complex exponentials. We say that the pair, set and spectrum, is a spectral pair. See [JP98] for the theory of spectral pairs.

Indeed, for more general models with wave equations, and in higher dimensions, there is a scattering theory model which accounts for scattering of waves at obstacles, see for example [LP68]. Yet, still staying with the translation model, but going from one dimension to \( \mathbb{R}^d \), Fuglede [Fug74] asked for an analogue of the translation model in one dimension, but now for bounded open domains in \( \mathbb{R}^d \).

Nonetheless, even for one dimension, the question of two non-empty open and non-overlapping intervals has not been worked out in a way that extends the cases of a single interval and gives a list of the selfadjoint extensions.

### 1.1. Two Intervals.

This paper is devoted to the two-interval case. Now, starting with zero boundary conditions at the four endpoints, we show that the deficiency indices are then \((2, 2)\), see [Kre49]. Moreover, we offer a characterization and parameterization of all the selfadjoint extensions. For each one, we compute the spectrum, and we write down the corresponding unitary one-parameter groups.

We show that, by contrast to the one-interval case, that some of the selfadjoint extension may have points in the spectrum of multiplicity two.

Our analysis lies at the crossroads of operator theory, spectral geometry, and harmonic analysis. For the two-interval case, we find the selfadjoint extension operators \( P_B \), and their spectra. Each operator \( P_B \), indexed by the group \( U(2) \) of unitary 2 by 2 matrices, has pure eigenvalue spectrum \( \{ \lambda_n(B) \} \); the multiplicity can be 1 or 2. In fact, we have the eigenvalues represented as functions of both \( B \) and of the choice of endpoints in \( I_2 \).

We study all possible configurations of the two intervals. It turn out that only the ratio \( r \) of their lengths is important. The nature of this number has subtle spectral theoretic implications; e.g., the distinction between the cases when \( r \) is rational vs irrational is important. When it is irrational, the points in the spectrum of \( P_B \) form a generator for pseudo-random numbers [BAAS11]; see Section 3 for details.

For the general case, we display the points \( \{ \lambda_n \} \) in the spectrum of \( P_B \) as functions of 5 variables, 4 for \( B \) in \( U(2) \), and one for \( r \).

In Section 5, we determine completely the cases when we the two intervals, together with spectrum of \( P_B \), yield spectral pairs (SP). It turns out to be a small subset of the total parameter space. In the SP-case, we show that the set \( \Lambda \) of points \( \{ \lambda_n \} \) fall in two simple cases: Modulo the angular part of \( B \), \( \Lambda \) is either the union of one or two lattice co-sets. For simplicity suppose the intervals are \([0, 1] \cup [\alpha, \beta] \) where \( 1 < \alpha < \beta \). In Theorems 5.3 and 5.8, we give a complete analysis of the SP cases. Modulo angular variables, this is merely a “small” discrete subfamily: Firstly, SP forces either \( \beta/(1 + \beta - \alpha) \) to be an integer or \( \beta - \alpha = 1 \); and secondly, we get the eigenvalue lists as follows: Let \( w \) be the radial variable in \( B \). If \( \beta/(1 + \beta - \alpha) \) is an integer then the points \( \lambda_n \) form a lattice, and if \( \beta - \alpha = 1 \) the points \( \lambda_n \) (as a function of \( w \)) contains the branches of the arc-cos function. In summary, the SP cases form a singular family, representing a broken symmetry, or phase transition; a discrete family within a much bigger continuum. For comparison, note that in [Rob71], dealing with second order operators in a single interval, the author finds an analogous spectral analysis of phase transitions.
1.2. Unbounded Operators. We recall the following fundamental result of von Neumann on extensions of Hermitian operators.

**Lemma 1.1** (see e.g. [DS88]). Let \( L \) be a closed Hermitian operator with dense domain \( \mathcal{D}_0 \) in a Hilbert space. Set

\[
\mathcal{D}_\pm = \{ \psi_\pm \in \text{dom}(L^*) \mid L^* \psi_\pm = \pm i \psi_\pm \},
\]

\[
\mathcal{E}(L) = \{ U : \mathcal{D}_+ \to \mathcal{D}_- \mid U^* U = P_{\mathcal{D}_+}, UU^* = P_{\mathcal{D}_-} \} \tag{1.1}
\]

where \( P_{\mathcal{D}_\pm} \) denote the respective projections. Set

\[
\mathcal{E}(L) = \{ S \mid L \subseteq S, S^* = S \}.
\]

Then there is a bijective correspondence between \( \mathcal{C}(L) \) and \( \mathcal{E}(L) \), given as follows:

If \( U \in \mathcal{C}(L) \), and let \( L_U \) be the restriction of \( L^* \) to

\[
\{ \varphi_0 + f_+ + U f_+ \mid \varphi_0 \in \mathcal{D}_0, f_+ \in \mathcal{D}_+ \}. \tag{1.2}
\]

Then \( L_U \in \mathcal{E}(L) \), and conversely every \( S \in \mathcal{E}(L) \) has the form \( L_U \) for some \( U \in \mathcal{C}(L) \). With \( S \in \mathcal{E}(L) \), take

\[
U := (S - iI)(S + iI)^{-1} \big|_{\mathcal{D}_+} \tag{1.3}
\]

and note that

1. \( U \in \mathcal{C}(L) \), and
2. \( S = L_U \).

Vectors \( f \) in \( \text{dom}(L^*) \) admit a unique decomposition \( f = \varphi_0 + f_+ + f_- \) where \( \varphi_0 \in \text{dom}(L) \), and \( f_\pm \in \mathcal{D}_\pm \). For the boundary-form \( B(\cdot, \cdot) \), we have

\[
iB(f, f) = \langle L^* f, f \rangle - \langle f, L^* f \rangle = \|f_+\|^2 - \|f_-\|^2.
\]

1.3. Summary. We undertake a systematic study of interconnections between geometry and spectrum for a family of selfadjoint operator extensions indexed by two thing: by (i) the prescribed configuration of the two intervals, and by (ii) the von Neumann parameters. This turns out to be subtle, and we show in detail how variations in both (i) and (ii) translate into explicit spectral properties for the extension operators. Indeed, for each choice in (i), i.e., relative length of the two intervals, we have a Hermitian operator with deficiency indices \((2, 2)\). Our main theme is spectral theory of the corresponding family of \((2, 2)\)-selfadjoint extension operators. Accidentally, this more general problem has implications for a question from [Łab01] on spectral pairs. But our study of the spectral theory dictated by (i) and (ii) goes far beyond its implications for spectral pairs.

Another advantage of our approach via a definite 6-parameter family of selfadjoint operators is that the verification for completeness of the associated function-systems becomes easy. We first prove that all these operators have pure point-spectrum, see Theorem 2.4. In each verification, one may then take advantage of the spectral theorem applied to the particular selfadjoint operators under study. So this applies whether or not one has a spectral pair. Otherwise the completeness property is typically subtle in spectral geometry and in physics applications; see e.g., [DS88, Gil72, KM06, Lab01, Lab02, LP68, Rob71].

One of our applications will be to the characterization of spectral pairs in the two-interval case(s). While this question was addressed in the paper [Lab01], it was attacked there with the use of geometric tools; specifically by listing the ways
two intervals can tile the real line under translations. We obtain the same results regarding spectral pairs, but now with the use of operator/spectral theory. Even though most of our computations are for interval examples, we believe they are of general interest, and that they offer intriguing extensions to more structured models.

1.4. Prior Literature. There are related investigations in the literature on spectrum and deficiency indices. For the case of indices (1, 1), see for example [ST10]. For a study of odd-order operators, see [BH08]. Operators of even order in a single interval are studied in [Oro05]. The paper [BV05] studies matching interface conditions in connection with deficiency indices \((m, m)\). Dirac operators are studied in [Sak97]. For the theory of selfadjoint extensions operators, and their spectra, see [Šmu74, Gil72], for the theory; and [Naz08, VGT08, Vas07, Sad06, Mik04, Min04] for recent papers with applications. For applications to other problems in physics, see e.g., [PR76, Bar49].

2. Momentum Operators

By momentum operator \(P\) we mean the generator for the group of translations in \(L^2(−\infty, \infty)\). There are several reasons for taking a closer look at restrictions of the operator \(P\). In our analysis, we study spectral theory determined by a choice of two intervals (details below.) Our motivation derives from quantum theory, and from the study of spectral pairs in geometric analysis; see e.g., [DJ07], [Fug74], [JP99], [Łab01], and [PW01]. In our model, we examine how the spectral theory depends on both variations in the choice of the two intervals, as well as on variations in the von Neumann parameters.

Granted that in many applications, one is faced with vastly more complicated data and operators; nonetheless, it is often the case that the more subtle situations will be unitarily equivalent to a suitable model involving \(P\). This is reflected for example in the conclusion of the Stone-von Neumann uniqueness theorem: The Weyl relations for quantum systems with a finite number of degree of freedom are unitarily equivalent to the standard model with momentum and position operators \(P\) and \(Q\).

2.1. Applications. In Lax-Phillips scattering [LP68] theory for the acoustic wave equation, one obtains the solution to a particular wave equation represented by a unitary one-parameter group acting on Hilbert space. Wave equations preserve energy; hence the Hilbert space.

With scattering against a finite obstacle in some number \(d\) of dimensions, one then gets two closed subspaces corresponding to incoming and outgoing waves, respectively, here referring to the whole space, in this case \(\mathbb{R}^d\). The Lax-Phillips scattering operator is derived from the two. Again, in the two subspaces, the respective unitary one-parameter groups are unitary equivalent to a \(P\)-model. By taking into account uniform multiplicity, one arrives at the operator \(P\).

For other uses of \(P\) and its restrictions, see [Jor81] and [Rob71]. The application in [Rob71] is to quantum statistical mechanics.

In geometric analysis, one studies second order Hermitian PDOs on a finite region in \(\mathbb{R}^d\), and selfadjoint boundary condition, for example Dirichlet conditions, or Neumann conditions, see also [LP68], [DS88] and [Rob71].
In quantum systems, the initial interest is in the Schrödinger equation for the dynamics of quantum states. And once again an essential part of the problem is turned into the study of a unitary one-parameter group in Hilbert space governing the solution to the Schrödinger equation: We are faced with determining the spectrum of the selfadjoint generator. Even if there may not be a direct unitary equivalence, there will often be a part of the dynamics, e.g., bound-states, localization, and quantum tunneling, where the two-interval model offers insight. In [Rob71] for example, the operator is second order, and it is used in the study of confinement quantum states (wave functions) in a single interval. There is a connection between this, and the details below. Below we have a first-order operator, and two intervals. But in both instances, the models are represented through the study of deficiency indices $(2,2)$. In both cases, one is able to determine an eigenvalue spectrum (i.e., bound states) from a system of curves with intersection, but the systems vary from one application to the other. For details, in [Rob71], see Figures 1 through 8. The analogous system of curve intersections of our present two-interval models are given in Examples 3.21 and 5.15 below.

In studies of quantum scales, one is faced with quantum tunneling, see e.g., [Rob71], a process that cannot be directly perceived at a macroscopic scale. Quantum particles (or rather waves) travel between potential barriers. With a small probability, wave functions will crossing barriers. See Corollary 4.1 for a discussion. The reason is duality for quantum states; i.e., states having simultaneously properties of waves and particles. This leads to computation of probability density for particle’s position, thus describing the probability that the particle is at any given place. In the limit of large barriers, the probability of tunneling decreases for taller and wider barriers. The simplest and best understood tunneling-barriers are rectangular barriers, offering approximate solutions.

### 2.2. The boundary form, spectrum, and the group $U(2)$. Since the problem is essentially invariant under affine transformations we may assume the two intervals are $I_1 := [0,1]$ and $I_2 := [\alpha, \beta]$. $L^2(I_1 \cup I_2)$ is a Hilbert space with respect to the inner product

\[ \langle f \mid g \rangle := \int_{I_1} f \overline{g} + \int_{I_2} f \overline{g}. \] (2.1)

The maximal momentum operator is

\[ P := \frac{1}{i2\pi} \frac{d}{dt} \] (2.2)

with domain $\mathcal{D}(P)$ equal to the set of absolutely continuous functions on $I_1 \cup I_2$.

The boundary form associated with $P$ is the form

\[ B(f,g) := \langle Pf \mid g \rangle - \langle f \mid Pg \rangle \] (2.3)

on $\mathcal{D}(P)$. Clearly,

\[ B(f,g) = f(1)\overline{g(1)} - f(0)\overline{g(0)} + f(\beta)\overline{g(\beta)} - f(\alpha)\overline{g(\alpha)}. \] (2.4)

For $f \in \mathcal{D}(P)$, let $\rho_1(f) := (f(1), f(\beta))$ and $\rho_2(f) := (f(0), f(\alpha))$. Then

\[ B(f,g) = \langle \rho_1(f) \mid \rho_1(g) \rangle - \langle \rho_2(f) \mid \rho_2(g) \rangle . \] (2.5)

Hence $(\mathbb{C}^2, \rho_1, \rho_2)$ is a boundary triple for $P$. The set of selfadjoint restrictions of $P$ is parametrized by the set of unitary $2 \times 2$ matrices, see e.g., [dO09]. Explicitly,
any unitary $2 \times 2$ matrix $B$ determines a selfadjoint restriction $P_B$ of $P$ by setting
\[ \mathcal{D}(P_B) := \{ f \in \mathcal{D}(P) \mid B\rho_1(f) = \rho_2(f) \}. \quad (2.6) \]
Conversely, every selfadjoint restriction of $P$ is obtained in this manner. For completeness, we begin by working out the details connecting the boundary form formulation to the von Neumann deficiency space approach in our setting.

**Definition 2.1.** For $I_1$ and $I_2$, consider the following two copies of the two-dimensional Hilbert space $\mathbb{C}^2$:
\[
\rho_2(\mathcal{D}(P)) = \mathcal{B}_L = \left\{ \left( \begin{array}{c} f(0) \\ f(\alpha) \end{array} \right) \mid f \in \mathcal{D}(P) \right\},
\]
and
\[
\rho_1(\mathcal{D}(P)) = \mathcal{B}_R = \left\{ \left( \begin{array}{c} f(1) \\ f(\beta) \end{array} \right) \mid f \in \mathcal{D}(P) \right\},
\]
where $\rho_1$ and $\rho_2$ are the respective boundary restrictions.

For $f \in \mathcal{D}(P)$, let $f = \varphi_0 + f_+ + f_-$ be the decomposition in (1.2), i.e., $\varphi_0 \in \mathcal{D}_0$, $f_\pm \in \mathcal{D}_\pm$. Note that, for $f \in \mathcal{D}(P)$, we have
\[
B(f,f) = \|f_R\|^2 - \|f_L\|^2 = \|f_+\|^2 - \|f_-\|^2. \quad (2.7)
\]
Set
\[
\mathcal{C}_b = \{ B : \mathcal{B}_L \to \mathcal{B}_R, \text{ isometric} \},
\]
i.e., a copy of the matrix-group of all unitary $2 \times 2$ matrices.

**Proposition 2.2.** In the two-interval example, there is a natural isomorphism $\mathcal{C}_b \cong \mathcal{C}(L)$ where $\mathcal{C}(L)$ is defined as in Lemma 1.1.

**Proof.** Note that by Lemma 1.1, each of the selfadjoint extensions from the minimal domain $\mathcal{D}_0$ (zero-boundary conditions) is determined by some $U \in \mathcal{C}(L)$ via $f_- = Uf_+ + f_+ \in \mathcal{D}_+$. Specializing (2.7) to the domain $\mathcal{D}_U$ in (1.2), i.e., $f = \varphi_0 + f_+ + Uf_+$, $\varphi_0 \in \mathcal{D}_0$, $f_+ \in \mathcal{D}_+$, we get
\[
B(f,f) = \|f_R\|^2 - \|f_L\|^2 = \|f_+\|^2 - \|Uf_+\|^2 = 0
\]
for all $f \in \mathcal{D}_U$. It follows that
\[
f_L \mapsto f_R, \quad (2.8)
\]
defined for $f \in \mathcal{D}_U$, defines a unique element $B \in \mathcal{C}_b$, and that
\[
\mathcal{C}(L) \ni U \mapsto B \in \mathcal{C}_b \quad (2.9)
\]
is an isomorphism. ■

Since the two deficiency spaces $\mathcal{D}_\pm$ are easy to compute in each of the two-interval examples, we can write down two isomorphisms $\mathcal{D}_+ \xrightarrow{b_+} \mathcal{B}_L$, and $\mathcal{D}_- \xrightarrow{b_-} \mathcal{B}_R$; note that all four spaces are two-dimensional. With this, we get the following representation of the isomorphism in Proposition 2.2:

**Corollary 2.3** (Matrix realization). Let $U$, $b_{\pm}$, and $B$ be the operators in Proposition 2.2, then $B : \mathcal{B}_L \to \mathcal{B}_R$ is $B = b_-Ub_+^{-1}$; see Figure 2.1.
With (2.9) in Proposition 2.2, we now parametrize the collection of unitary $2 \times 2$ matrices by

$$B = \begin{pmatrix}
    w e(\phi) & -\sqrt{1-w^2} e(\theta - \psi) \\
    \sqrt{1-w^2} e(\psi) & w e(\theta - \phi)
\end{pmatrix},$$

(2.10)

where $0 \leq w \leq 1$, $\phi, \psi, \theta \in \mathbb{R}$, and

$$e(x) := e^{i2\pi x}.$$  
(2.11)

Some of our results about the spectrum are summarized in

**Theorem 2.4.** If $P_B = P_{w,\phi,\psi,\theta}$ is the selfadjoint momentum operator on $[0,1] \cup [\alpha,\beta]$ associated with the unitary matrix (2.10) via (2.6), then $P_B$ has pure point spectrum. In fact, the spectrum of $P_B$ is the set of all real solutions $\lambda$ to the equation

$$(e(\phi + \lambda) - w) e(\theta - \phi + (\beta - \alpha) \lambda) = w e(\phi + \lambda) - 1.$$  
(2.12)

In particular, the spectrum is independent of the parameter $\psi$ and does not depend on the location of the interval $[\alpha,\beta]$ only on its length.

**Remark 2.5.** Our analysis also works when $\alpha = 1$. In this case one has to be careful with the interpretation of the function values at the point of intersection of $[0,1]$ and $[\alpha,\beta]$. One way is the interpretations

$$f(1) = f(1-) = \lim_{t \nearrow 1} f(t), \text{ and } f(\alpha) = f(1+) = \lim_{t \searrow 1} f(t).$$

We leave the details to the interested reader.

**Remark 2.6.** For applications, the role of the four parameters in (2.10) is taken up in Section 4 below. Here we merely add a note about the geometry entailed by the two extreme cases (i) $w = 0$, and (ii) $w = 1$. In the first case, the matrix $B$ is off-diagonal, with zeros in the two diagonal slots, while for (ii) it is diagonal. As a result, for (ii), we are merely dealing with the orthogonal direct sum of separate boundary value problems for the two intervals $I_1$ and $I_2$ treated in isolation; so two separate index (1,1) problems. By contrast, for (i), the two intervals get exchanged each time a local translation reaches a boundary point. As noted in Section 4 below, the dynamics in the intermediate case $0 < w < 1$, may be computed with the use of a binary random-walk model.

### 3. Spectral Theory

This section is about a family of selfadjoint operators $P_B$ arising as restrictions to the union of two intervals of the maximal momentum operator. We explore how these selfadjoint operators $P_B$ depend both on the prescribed boundary matrix $B$ from eq. (2.10), as well as the two intervals, i.e., on the choice of the two numbers $\alpha$ and $\beta$, see eq. (2.1). So the operators and their spectra depend on six parameters in all, and this dependency is charted in complete detail below.
Note that the difference $\beta - \alpha$ is the length of the second interval. Further recall that the unitary 2 by 2 matrix $B$ has four parameters, see (2.10); and that its radial parameter $w$ lies in the closed interval from 0 to 1. We say that the endpoints constitute the extreme cases, and the open interval $0 < w < 1$, the generic case. The role of $w$ is analyzed further in Section 4 which deals with scattering theory for the unitary one-parameter groups generated by each of the operators $P_B$.

In the present section, we show that each operator $P_B$ has pure eigenvalue spectrum $\{\lambda_n(B)\}$ with the eigenvalues depending on all parameters, i.e., on both $\alpha$ and $\beta$, and on the matrix $B$. This dependence is detailed below: In Section 3.1 we deal with two extreme cases for $B$; Section 3.2 the generic case; and in Section 3.3 we deal with the dichotomy for the value of the length of the second interval, rational vs irrational.

We saw in Corollary 2.3 that every selfadjoint restriction of the maximal momentum operator $P$ has the form $P_B$ for some $2 \times 2$ matrix $B$ as in (2.10). We begin our investigation of how the spectrum of $P_B$ depends on the parameters $1 < \alpha < \beta$, $\phi$, $\psi$, $\theta$, and $0 \leq w \leq 1$ by stating two simple lemmas.

Fix $1 < \alpha < \beta$. Let $I_1 := [0, 1]$, and $I_2 := [\alpha, \beta]$. Let $e_x(y) := e(xy)$. Any complex number $\lambda$ is an eigenvalue of the maximal momentum operator $P$ in (2.2).

For each complex number $\lambda$ the formal solutions to $Pf = \lambda f$ are

$$e_{(a,b)}(x) := (a\chi_{I_1}(x) + b\chi_{I_2}(x))e_\lambda(x), \quad a, b \in \mathbb{C}. \quad (3.1)$$

**Lemma 3.1.** The maximal operator $P$ has spectrum equal to the complex plane and each point in the spectrum is an eigenvalue with multiplicity two.

*Proof.* Since $e_{(a,b)}^\lambda$ is in $L^2(I_1 \cup I_2)$, it follows from $Pe_{(a,b)}^\lambda = \lambda e_{(a,b)}^\lambda$ and (3.1) that each $\lambda \in \mathbb{C}$ is an eigenvalue of $P$ with multiplicity two. $\blacksquare$

**Lemma 3.2.** The spectrum of any selfadjoint restriction $\tilde{P}$ of $P$ equals the set of eigenvalues of $\tilde{P}$ and each eigenvalue has multiplicity one or two.

*Proof.* By Lemma 3.1 the spectrum of $\tilde{P}$ is the $\lambda$ for which $e_{(a,b)}^\lambda$ is in the domain of $\tilde{P}$ for some choice of constants $a, b$. The multiplicity is determined by the dimension of this set of constants.

Suppose $P_B$ is the selfadjoint restriction of $P$ determined by the matrix (2.10) via (2.6). When the parameters for $B$ are fixed, the expression on the right-hand side in eq. (3.1) is a function of the $x$-variable. The constants $a$ and $b$ in front of the indicator functions depend on $B$, and, in turn, they determine the eigenfunctions, and therefore the spectrum, of the associated selfadjoint operator $P_B$.

Suppose $P_B$ is the selfadjoint restriction of $P$ determined by the matrix (2.10) via (2.6). Then the boundary condition $B\rho_1(f) = \rho_2(f)$ states

$$\begin{pmatrix} w e(\phi) & -\sqrt{1-w^2} e(\theta - \psi) \\ \sqrt{1-w^2} e(\psi) & w e(\theta - \phi) \end{pmatrix} \begin{pmatrix} f(1) \\ f(\beta) \end{pmatrix} = \begin{pmatrix} f(0) \\ f(\alpha) \end{pmatrix}. \quad (3.2)$$

Consequently, $\lambda$ is an eigenvalue for $P_B$ iff there is are complex numbers $a, b$ such that $f = e_{(a,b)}^\lambda$ satisfies the boundary condition (3.2), i.e.,

$$\begin{pmatrix} w e(\phi) & -\sqrt{1-w^2} e(\theta - \psi) \\ \sqrt{1-w^2} e(\psi) & w e(\theta - \phi) \end{pmatrix} \begin{pmatrix} a e(\lambda) \\ b e(\lambda \beta) \end{pmatrix} = \begin{pmatrix} a \\ b e(\lambda \alpha) \end{pmatrix}. \quad (3.3)$$

Since $P_B$ is selfadjoint any eigenvalue $\lambda$ must be a real number. $\blacksquare$
Below we show that the real solutions \( \lambda \) to (3.3) is a discrete set having no other accumulation points than plus/minus infinity.

3.1. **Boundary Cases.** Our analysis depends on the parameter \( w \). We begin by considering the extreme cases \( w = 1 \) and \( w = 0 \).

**Theorem 3.3** \((w = 1)\). Fix \( 1 < \alpha < \beta \). Let \( I_1 = [0, 1] \), and \( I_2 := [\alpha, \beta] \). Suppose \( P_B \) be the selfadjoint restriction associated with (2.10) via (2.6) and \( w = 1 \). Then \( P_B \) has pure point spectrum \( \Lambda_1 \cup \Lambda_2 \), where

\[
\Lambda_1 = -\phi + Z \quad \text{(3.4)}
\]

\[
\Lambda_2 = \frac{\phi - \theta}{\beta - \alpha} + \frac{1}{\beta - \alpha} Z \quad \text{(3.5)}
\]

and corresponding eigenfunctions are

\[
e_{\phi+n}^{(1,0)} = \chi_{[0,1]} e^{\phi+n}, \quad \text{and} \quad e_{\phi+n}^{(0,1)} = \chi_{[\alpha,\beta]} e^{\phi+n}. \quad \text{(3.6)}
\]

Consequently, each eigenvalue has multiplicity one, except the eigenvalues, if any, in \( \Lambda_1 \cap \Lambda_2 \) have multiplicity two.

**Proof.** In this case the boundary condition (3.3) takes the form

\[
\begin{pmatrix}
 e(\phi) & 0 \\
 0 & e(\theta - \phi)
\end{pmatrix}
\begin{pmatrix}
 a e(\lambda) \\
 b e(\lambda\beta)
\end{pmatrix}
= 
\begin{pmatrix}
 a \\
 b e(\lambda\alpha)
\end{pmatrix}. \quad \text{(3.7)}
\]

That is

\[
a e (\phi + \lambda) = a \\
b e (\theta - \phi + \beta\lambda) = b e (\alpha\lambda).
\]

The stated conclusions are now immediate. \( \blacksquare \)

**Remark 3.4.** It is easy to see that (3.4) and (3.5) are the solutions to (2.12) when \( w = 1 \).

We present a few examples in order to illustrate some of the possibilities.

**Example 3.5.** If \( \beta - \alpha = 1 \) and \( \theta = 2\phi \) the spectrum equals \(-\phi + Z\) and has uniform multiplicity equal to two.

**Example 3.6.** If \( \phi = 0 \) is rational and \( \beta - \alpha \) is irrational, then the spectrum has uniform multiplicity equal to one.

**Example 3.7.** If \( \phi = \theta = 0 \) and \( \beta - \alpha = 2 \), then the spectrum is \( \frac{1}{2}Z \), the eigenvalues in \( Z \) have multiplicity two and the non-integer eigenvalues have multiplicity one.

**Example 3.8.** If \( \phi = \theta = 0 \) and \( \beta - \alpha \) is irrational, then 0 has multiplicity two and the other eigenvalues have multiplicity one.

In the other extreme case we have

**Theorem 3.9** \((w = 0)\). Fix \( 1 < \alpha < \beta \). Let \( I_1 = [0, 1] \), and \( I_2 := [\alpha, \beta] \). Suppose \( P_B \) be the selfadjoint restriction associated with (2.10) via (2.6) and \( w = 0 \). Then \( P_B \) has pure point spectrum

\[
\frac{1}{2} - \theta \\
\frac{1}{1 + \beta - \alpha} + \frac{1}{1 + \beta - \alpha} Z
\]

(3.8)
with uniform multiplicity equal to one and an eigenfunction corresponding to the eigenvalue $\frac{1}{2} - \theta + n$, $n \in \mathbb{Z}$, is
\[
\left( e\left(\frac{1}{2} - \psi + \frac{(1 + n)\beta - (1 - \alpha)\theta}{1 + \beta - \alpha}\right) \chi_{I_1} + \chi_{I_2}\right) e^{\frac{1}{2} - \theta + n}. 
\] (3.9)

**Proof.** In this case the boundary condition (3.3) takes the form
\[
\begin{pmatrix}
0 & -e(\theta - \psi) \\
e(\psi) & 0
\end{pmatrix}
\begin{pmatrix}
a e(\lambda) \\
b e(\lambda\beta)
\end{pmatrix} =
\begin{pmatrix}
a \\
b e(\lambda\alpha)
\end{pmatrix}. 
\] (3.10)

Writing (3.10) as two equations we get
\[
-b e(\theta - \psi + \lambda\beta) = a \tag{3.11}
\]
\[
a e(\psi + \lambda) = b e(\lambda\alpha). \tag{3.12}
\]

The first equation shows that $|a| = |b|$. In particular, we may set $b = 1$ and $a = -e(\theta - \psi + \lambda\beta)$. Then the eigenvalues are determined by the second equation which now states
\[-e(\theta + \lambda + \beta\lambda) = e(\lambda\alpha). \]

Hence the eigenvalues $\lambda$ are determined by
\[
\theta + \lambda(1 + \beta - \alpha) \in \frac{1}{2} + \mathbb{Z}. 
\]

The stated conclusions are now immediate. $\blacksquare$

**Remark 3.10.** It is easy to see that (3.8) are the solutions to (2.12) when $w = 0$.

3.2. **Generic Cases.** We now consider the non-extreme cases $0 < w < 1$. We begin by showing that the eigenvalues are the solutions to (2.12), thereby completing the proof of Theorem 2.4.

**Lemma 3.11.** Fix $1 < \alpha < \beta$. Let $I_1 = [0,1]$, and $I_2 := [\alpha,\beta]$. Suppose $P_B$ is the selfadjoint restriction associated with (2.10) via (2.6) and $0 < w < 1$. Then a point $\lambda$ in $\mathbb{R}$ is an eigenvalue of $P_B$ if and only it is a solution to the equation
\[
e(\theta - \phi + (\beta - \alpha)\lambda) = \frac{w e(\phi + \lambda) - 1}{e(\phi + \lambda) - w}. \tag{3.13}
\]
in terms of the argument this is
\[
\theta - \phi + (\beta - \alpha)\lambda = \frac{1}{2\pi} \arctan\left(\frac{(1 - w^2)\sin(2\pi(\phi + \lambda))}{2w - (1 + w^2)\cos(2\pi(\phi + \lambda))}\right) + \mathbb{Z}. \tag{3.14}
\]

When $\lambda$ is a real solution to (3.13) then any corresponding eigenfunction is a multiple of
\[
e^{(\alpha,1)}(x) = (a\chi_{I_1}(x) + \chi_{I_2}(x)) e_\lambda(x), \tag{3.15}
\]
where $a$ is determined by (3.18) below. In particular, the spectrum has uniform multiplicity equal to one.

**Proof.** The eigenvalue equation (3.3) is
\[
a w e(\phi + \lambda) - b \sqrt{1 - w^2} e(\theta - \psi + \beta\lambda) = a \tag{3.16}
\]
\[
a \sqrt{1 - w^2} e(\psi + \lambda) + b w e(\theta - \phi + \beta\lambda) = b e(\lambda\alpha). \tag{3.17}
\]

Equation (3.16) shows $a = 0 \iff b = 0$. So, we may set $b = 1$, which gives (3.15).
Solving both (3.16) and (3.17) for \( a \), shows \( a \) equals
\[
\frac{\sqrt{1 - w^2} e(\theta - \psi + \beta \lambda)}{w e(\phi + \lambda) - 1} = \frac{1 - w e(\theta - \phi + (\beta - \alpha)\lambda)}{\sqrt{1 - w^2} e(\psi + \lambda - \alpha \lambda)}.
\] (3.18)

Cross-multiplying gives
\[
(1 - w^2) e(\theta + \lambda + (\beta - \alpha)\lambda) = (1 - w e(\theta - \phi + (\beta - \alpha)\lambda))(w e(\phi + \lambda) - 1).
\]

Expanding the right hand side we see
\[
(1 - w^2) e(\theta + \lambda + (\beta - \alpha)\lambda) = -1 - w^2 e(\theta + \lambda + (\beta - \alpha)\lambda) + w(e(\theta - \phi + (\beta - \alpha)\lambda) + e(\phi + \lambda)).
\]

Adding \( w^2 e(\theta + \lambda + (\beta - \alpha)\lambda) \) to both sides gives
\[
e(\theta + \lambda + (\beta - \alpha)\lambda) = -1 + w(e(\theta - \phi + (\beta - \alpha)\lambda) + e(\phi + \lambda)).
\]

Subtracting \( w e(\theta - \phi + (\beta - \alpha)\lambda) \) yields
\[
e(\theta + \lambda + (\beta - \alpha)\lambda) - w e(\theta - \phi + (\beta - \alpha)\lambda) = -1 + w e(\phi + \lambda).
\]

Factoring the left hand side
\[
(e(\lambda) - w e(-\phi)) e(\theta + (\beta - \alpha)\lambda) = -1 + w e(\phi + \lambda)
\]
or
\[
(e(\phi + \lambda) - w) e(\theta - \phi + (\beta - \alpha)\lambda) = -1 + w e(\phi + \lambda). \quad (3.19)
\]

Rearranging (3.19) gives (3.13).

The following lemma established some simple facts about the right hand side of (3.13).

**Lemma 3.12.** Fix \( 0 < w < 1 \). Consider the Möbius transformation
\[
Mz := \frac{wz - 1}{z - w}, \quad z \in \mathbb{C}. \quad (3.20)
\]

Then \( M = M_w \) is a bijection of the unit circle \( T \) onto itself and there is a decreasing continuous bijection \( g \) of the real line \( \mathbb{R} \) onto itself such that \( g(0) = -1/2 \) and \( Me(t) = e(g(t)) \) for all \( t \in \mathbb{R} \).

**Proof.** Recall any Möbius transformation is a continuous bijection of the Riemann sphere onto itself. Note \( M \) is its own inverse. Taking the modulus we see
\[
|w e(t) - 1| = |e(t) - w|, \quad t \in \mathbb{R}.
\]

Hence \( M \) maps \( T \) into itself and consequently, \( M \) is a bijection of \( T \) onto itself.

In particular, \( \gamma(t) := Me(t), t \in \mathbb{R} \) has minimal period equal to one and maps any interval of non-zero integer length onto \( T \). Since \( M \) has positive coefficients \( M \) maps the extended real line onto itself. Since \( Mi \) is in the first quadrant \( M \) maps upper half-plane onto itself, in particular, \( \gamma([0, 1/2]) \) is the part of \( T \) in the upper half-plane. Note \( \gamma(0) = M1 = -1 = e(-1/2) \).

By the path lifting theorem there is a unique continuous function \( f : [0, 1] \to \mathbb{R} \) such that \( f(0) = -1/2 \) and \( e(f(t)) = \gamma(t) \) for \( t \in [0, 1] \). Since \( e(f(1/2)) = \gamma(1/2) = M(-1) = 1 \) and \( e(f ([0, 1/2])) = \gamma ([0, 1/2]) \) is the part of \( T \) in the upper half-plane, it follows that \( f(1/2) = -1 \). Consequently, \( f \) is a continuous bijection of \( [0, 1] \) onto \([ -1/2, -1/2] \).
The desired function is determined by

\[ g(t + n) := f(t) - n \]

for \( 0 \leq t < 1 \) and integers \( n \).

Remark 3.13. Since \( g(1/4) \) is in the first quadrant and \( g(1/2) = 1 \) then \( e(g[0,1/4]) \) is the part of \( T \) in the fourth quadrant and part of the first quadrant and \( e(g[1/4,1/2]) \) is the remaining part of \( T \) in the first quadrant. This “explains” the general shape of the curve in our pictures below.

We are now ready to finish our analysis of the spectrum of \( P_B \).

**Theorem 3.14** \((0 < w < 1)\). Let the two intervals \( I_1 \) and \( I_2 \) be as described, with \( \alpha \) and \( \beta \) fixed and denoting the endpoints of \( I_2 \), i.e., \( 1 < \alpha < \beta \). Pick parameters \( \phi, \psi, \) and \( w \), \( 0 < w < 1 \), describing one of the selfadjoint extensions \( P_B \). Then the spectrum of \( P_B \) is pure point, with the eigenvalue list equal to the sequence \( \{ \lambda_n \} \) of real numbers solving equation (3.13). Every number \( \lambda_n \) in the spectrum of \( P_B \) is isolated, and, for \( n \) fixed, the corresponding eigenspace is spanned by the associated eigenfunction from (3.15).

**Proof.** By Lemma 3.11 the eigenvalues of \( P_B \) are the real solution to

\[ e(\theta - \phi + (\beta - \alpha) \lambda) = M e(\phi + \lambda) \]

where \( M \) is the Möbius transformation (3.20). Hence, if \( g \) is the function from the conclusion of Lemma 3.12, then the eigenvalues are the solution to

\[ e(\theta - \phi + (\beta - \alpha) \lambda - g(\phi + \lambda)) = 1. \]

Hence the eigenvalues are the \( \lambda \) for which

\[ \theta - \phi + (\beta - \alpha) \lambda - g(\phi + \lambda) \in \mathbb{Z}. \]  

(3.21)

Let \( h : \mathbb{R} \to \mathbb{R} \) be determined by

\[ h(t) := \theta - \phi + (\beta - \alpha) t - g(\phi + t). \]  

(3.22)

If \( t \) is an integer and \( n \) is an integer, then \( g(t + n) = g(t) - n \), consequently,

\[ h([t, t + n)) = [h(t), h(t) + (1 + (\beta - \alpha)) n]. \]  

(3.23)

is an interval of length \( n(1 + \beta - \alpha) \). In particular, it contains at most \( 1 + n(1 + \beta - \alpha) \) integers, hence \( n \) points from the spectrum of \( P_B \).

Figure 3.1 shows three plots of the left hand side of (3.21) as a function of \( t \), the plots illustrate that the curve is less linear for larger values of \( w \).
Figure 3.1. $y = \theta - \phi + (\beta - \alpha)t - g(\phi + t)$ for $\theta, \psi, \phi = 0$, $\beta - \alpha = 1$, and $w = 0.1, 0.5, 0.9$.

The spectrum of $P_B$ is continuous in $B$ in the sense that

**Corollary 3.15.** For integers $n$, let $\lambda_n$ denote the solution to $\theta - \phi + (\beta - \alpha)t - g(\phi + t) = n$. The spectrum of $P_B$ is the set $\{\lambda_n \mid n \in \mathbb{Z}\}$, $\lambda_n < \lambda_{n+1}$, and for fixed $n$, $(w, \phi, \psi, \theta, \alpha, \beta) \to \lambda_n$ is continuous. The dependence of the eigenvalues on the parameter $w$ in $B$ is harmonic. Specifically in the formula (3.24) the circle is given by the parameter $t$, and when viewed as a function on the circle, we therefore get an extension to the inside disk, represented in polar coordinates $(w, u)$ where $w$ is radius and $u$ is angle; and with continuity up to $w = 1$.

**Proof.** Let $h$ be as in (3.22). Then $\lambda_n$ is the first coordinate of the point of intersection of the curve $y = h(t)$ and the horizontal line $y = n$. Clearly the curve is continuous in $\phi, \psi, \theta, \alpha,$ and $\beta$. By construction, $g$ is the argument of a continuous logarithm. Consequently,

$$g(t) = -\frac{1}{2} + \frac{1}{2\pi} \text{Im} \int_0^t \frac{\gamma'}{\gamma} = -\frac{1}{2} - \int_0^t \frac{1 - w^2}{1 - 2w \cos(2\pi u) + w^2} du,$$

(3.24)

where $\gamma(t) = Me(t)$ as in Lemma 3.12. To get the harmonic extension, note that the Poisson kernel in (3.24) offers a harmonic extension from the circle to the inside disk. We use the parameter $t$ for the circle, and the extension is to the inside disk in polar coordinates $(w, u)$ with $w$ as radius and $u$ angle. So the statement about continuity up to $w = 1$ is just continuity up to the boundary circle for the harmonic extension. The latter is a known property of Poisson kernel. ■

**Remark 3.16.** The spectrum of $P_B$ is uniform in the sense that for any integer $n$, we have

$$h([\lambda_n, \lambda_n + 1]) = [n, n + 1 + \beta - \alpha].$$

Where $\lambda_n$ is as in Corollary 3.15. Consequently, the interval $[\lambda_n, \lambda_n + 1)$ contains exactly $\lfloor 1 + \beta - \alpha \rfloor$ points from the spectrum. Here, $\lfloor 1 + \beta - \alpha \rfloor$ is an integer such that $\beta - \alpha < 1 + \beta - \alpha \leq 1 + \beta - \alpha$.

Points in the spectrum are separated in the sense that

**Corollary 3.17.** There exists $\delta > 0$, so that if $\lambda \neq \lambda'$ are eigenvalues of $P_B$, then $|\lambda - \lambda'| > \delta$. 

Proof. The function \( t \to \langle e_t \mid 1 \rangle = \int_0^1 e_t + \int_0^\beta e_t \) is continuous and equals 1 at \( t = 0 \). Hence there is a \( \delta > 0 \), such that \( |t| \leq \delta \) implies \( |\langle e_t \mid 1 \rangle| > \frac{1}{2} \). Since \( \lambda - \lambda' \) is a root of \( t \to \langle e_t \mid 1 \rangle \), we conclude \( |\lambda - \lambda'| > \delta \).

Let \( \lambda_n \) be the enumeration of the spectrum of \( P \) from Corollary 3.15. The following figures shows plots of \( \lambda_n \) as a function of \( w \) for several values of \( n \). Here both the selfadjoint operator \( P_B \), and the numbers \( \lambda_n \) in its eigenvalue list, depend on the choice of the matrix \( B \), but for brevity, in some formulas, we have suppressed the \( B \)-dependence. Now \( \lambda_n = \lambda_n(B) \), but in Figure 3.2-3.4 we zoom in on the dependence of the \( w \) parameter from \( B \). Figure 3.2–3.4 show examples where \( \delta_w \to 0 \) as \( w \to 1 \) and where \( \delta_w \geq \delta > 0 \) for all \( w \). Here \( \delta_w = \delta \) is determined in Corollary 3.17 above.

\[ \text{Figure 3.2. } \Omega = (0, 1) \cup (2, 3), \text{ and } \theta, \psi, \phi = 0; \quad \Lambda_{w=0} = \frac{1}{4} + \frac{1}{2} \mathbb{Z}, \]
\[ \text{and } \Lambda_{w=1} = \mathbb{Z}. \]

\[ \text{Figure 3.3. } \Omega = (0, 1) \cup (2, 4), \text{ and } \theta, \psi, \phi = 0; \quad \Lambda_{w=0} = \frac{1}{6} + \frac{1}{3} \mathbb{Z}, \]
\[ \text{and } \Lambda_{w=1} = \frac{1}{2} \mathbb{Z}. \]

\[ \text{Figure 3.4. } \Omega = (0, 1) \cup (2, 3), \psi, \phi = 0, \text{ and } \theta = 1/2; \quad \Lambda_w = \frac{1}{2} \mathbb{Z}, \]
\[ \text{for all } 0 \leq w \leq 1. \]
Corollary 3.18. Suppose $\beta - \alpha$ is rational and $p,q > 0$ are integers such that $\beta - \alpha = p/q$. Then there is a finite set $L$ such that the spectrum of $P_B$ is $L + qZ$.

Proof. Suppose $p > 0$. Let $\lambda_n$ be as in Corollary 3.15. In particular, $h(\lambda_0) = 0$ and $h(\lambda_0 + q) = h(\lambda_0) + (1 + \beta - \alpha)q = q + p$.

Hence, setting $t = \lambda_0$ and $n = q$ in (3.23) gives

$$h([\lambda_0, \lambda_0 + q)) = [0, q + p)$$

which contains $q + p$ integers. Let $L := \{\lambda_0, \lambda_1, \cdots, \lambda_{q+p-1}\}$ be the corresponding eigenvalues of $P_B$. By construction $L \subset [\lambda_0, \lambda_0 + q)$ and the spectrum of $P_B$ is $L + qZ$. \[\blacksquare\]

3.3. Irrational $\beta - \alpha$. In the remainder of this section we discuss the spectrum of $P_B$ when $\beta - \alpha$ is irrational.

Corollary 3.19 (Asymptotics). Let $\lambda_n$ be the enumeration of the spectrum of $P$ from Corollary 3.15. If $\beta - \alpha$ is irrational, then

$$\lambda_0 + \frac{k - 1}{1 + \beta - \alpha} - 1 \leq \lambda_k < \lambda_0 + \frac{k - 1}{1 + \beta - \alpha} + 1$$

for all $k \in \mathbb{Z}$.

Proof. For any real number $x$, let $[x]$ be the real number satisfying $0 < [x] \leq 1$ and $x - [x] \in \mathbb{Z}$. Well, for any integer $q > 0$, the set $h([\lambda_0, \lambda_0 + q))$ is the interval $[0, (1 + \beta - \alpha)q)$. By irrationality of $\beta - \alpha$, the interval $[\lambda_0, \lambda_0 + q)$ contains $[(1 + \beta - \alpha)q]$ points from the spectrum. Similarly, $[\lambda_0, \lambda_0 + q + 1)$ contains $[(1 + \beta - \alpha)(q + 1)]$ points from the spectrum. Consequently,

$$\lambda_0 + q \leq \lambda_{[(1+\beta-\alpha)q]+1} < \lambda_{[(1+\beta-\alpha)q]+2} < \cdots < \lambda_{[(1+\beta-\alpha)(q+1)]} < \lambda_0 + q + 1.$$

Restating we see

$$\lambda_0 + q \leq \lambda_k < \lambda_0 + q + 1,$$

when $[(1 + \beta - \alpha)q] + 1 \leq k \leq [(1 + \beta - \alpha)(q + 1)]$. Now

$$[(1 + \beta - \alpha)q] + 1 \leq k \leq [(1 + \beta - \alpha)(q + 1)]$$

implies

$$(1 + \beta - \alpha)q + 1 \leq k \leq (1 + \beta - \alpha)(q + 1) + 1.$$

Solving for $q$ gives

$$q \leq \frac{k - 1}{1 + \beta - \alpha} \leq q + 1.$$

The desired conclusions now follow from (3.25).

We leave the discussion of the case $q < 0$ for the reader. \[\blacksquare\]

In the general case (3.14) we study the intervals in the $\Lambda$ axis between neighboring asymptotes: a sequence of equal-length intervals. The difference $\beta - \alpha$: rational vs irrational. We now study the points on the axis corresponding to the intersections in the above examples. By Lemma 3.11 these points constitute the spectrum $\Lambda$.

Remark 3.20. If $\beta - \alpha$ is irrational, the spectrum is purely aperiodic, see details below in Example 3.21.
The examples below serve to illustrate a number of spectral theoretic issues accounted for in our theorems.

They are motivated by several considerations:

(i) The examples, both in this section, and in Section 5, show that the possibilities, outlined in the abstract, may indeed arise.

(ii) A vector bundle over $U(2)$: This is illustrated below. Recall, for each $B$ in $U(2)$, the eigenspaces for $P_B$ form a line bundle over the group $U(2)$ as base-space, with degeneracies at places when the multiplicity jumps from 1 to 2;

(iii) The examples further serve to make concrete the subtle interplay between geometry and spectrum;

(iv) They illustrate some additional issues dealing with the way a continuum of parameters has symmetry breaking, and thus produce cases with properties of special interest; for example illustrating how the spectral pairs (SP) form a small subfamily; and

(v) The various families of two-interval cases serve as models for scattering theory in physics (see Section 2.12.1), so they are models for systems with a much richer structure, models for potential barriers, for quantum jumps, . . . . By model, we mean, up to unitary equivalence, component-by component.

Example 3.21. Choose $w = \frac{1}{\sqrt{3}}$, $\theta = -1/4$, $\phi = -1/8$, $\alpha = 3$, $\beta = 3 + \sqrt{2}$.

Let $J_s = [s + 1/8, s + 1 + 1/8]$, $s \in \mathbb{Z}$, i.e., $J_s$ is the $s^{th}$ interval between two neighboring branch cuts. Let $L_s$ be the eigenvalues in $J_s$, then $\Lambda = \bigcup_{s \in \mathbb{Z}} L_s$, and the set $\bigcup_{s \in \mathbb{Z}} (L_s - s)$ is dense in $J_0$. See Figures 3.5-3.7.

This conclusion holds whenever $I_2$ is chosen with irrational length $\beta - \alpha$. The reason is that this number then induces an irrational rotation which is known to have dense orbits; a basic example in symbolic dynamics.

Figures 3.5-3.7 are based on (3.13), the argument function from the proof of Lemma 3.12, reduction modulo one gives the branch curves. More precisely, the figures are obtained in a number of steps. Step 1. Fix $B$ as in (2.10), and let $P_B$ be the corresponding selfadjoint operator. Consider the curves for argument function from the right hand side of (3.11) and the lines with slope $\beta - \alpha$ from the left hand side of (3.11). We are assuming here that this slope is irrational.

Step 2. Identify the asymptotes, and fix an interval between two branch cuts; for example the interval $J_0$ on the axis closest to 0. Note that all the intervals between neighboring branch cuts have the same fixed unit-length. They extend both to the left and to the right of $J_0$. Step 3. Since the spectrum of $P_B$ is discrete and infinite, it intersects all these intervals between branch cuts. Now, translate all of these finite intersections down to $J_0$. Conclusion. Since the line-slope is irrational, the union of all these sets (inside $J_0$) is dense in $J_0$.

Remark 3.22. When the difference $\beta - \alpha$ is irrational, then the distribution of points in $\Lambda$ changes as you move around on the axis between the cut-intervals. See Remark 3.20, and Example 3.21 above for a detailed discussion.

Corollary 3.23. Let $\lambda_n$ be the enumeration of the spectrum of $P$ from Corollary 3.15. For a real number $x$, let $[x]$ be the real number satisfying $0 \leq [x] < 1$. If $\beta - \alpha$ is irrational, then $[\lambda_n] \neq [\lambda_m]$ for all $n \neq m$ and the set $\{[\lambda_n] | n \in \mathbb{Z}\}$ is dense in the interval $[0, 1]$. 
Proof. \( \lambda_n \) is a solution, for \( t \), to
\[
\theta - \phi + (\beta - \alpha)t = n + g(\phi + t)
\]
if \( \phi + \lambda_n \) is a solution, for \( t \), to
\[
\theta - (1 + \beta - \alpha)\phi + (\beta - \alpha)t = n + g(t).
\]
Consequently, replacing \( \phi \) by 0 corresponds to translating the spectrum and replacing \( \theta \) by \( \theta - (1 + \beta - \alpha)\phi \). We will therefore assume \( \phi = 0 \) in (3.26).

Suppose \([\lambda_m] = [\lambda_n]\). Then \( \lambda_m = \lambda_n + k \) for some integer \( k \). Hence \( M e(\lambda_m) = M e(\lambda_n) \) and therefore Lemma 3.11 implies \( e(\theta + (\beta - \alpha)\lambda_m) = e(\theta + (\beta - \alpha)\lambda_n) \). Consequently,
\[
\theta + (\beta - \alpha)(\lambda_n + k) = \theta + (\beta - \alpha)\lambda_n + j
\]
for some integer \( j \). It follows that \( \beta - \alpha = \frac{q}{r} \). Contradicting that \( \beta - \alpha \) is irrational.

We proceed the proof of the density claim by an alternative description of the set \( \{[\lambda_n] \mid n \in \mathbb{Z}\} \). For real \( x \) let \( -\frac{1}{2} \leq x < \frac{1}{2} \) be chosen such that \( x - \{x\} \) is an
integer. Let \( f \) be the path lifting function from the proof of Lemma 3.12. Replacing \( t \) in (3.26) by \( t + m \) where \( 0 \leq t < 1 \), we get
\[
\{ \theta + (\beta - \alpha)(t + m) \} = 1 + f(t)
\]
since \( \{ n + g(t + m) \} = \{ n + f(t) - m \} = 1 + f(t) \).
Let \( \ell_m(t) := \{ \theta + (\beta - \alpha)(t + m) \} \).
We will say \( \ell_m \) is a line. Then, the set \( \{ [\lambda_n] \mid n \in \mathbb{Z} \} \) is the set of solutions to the equations
\[
\ell_m(t) = 1 + f(t), \quad m \in \mathbb{Z}
\]
in the interval \([0, 1]\).

To show that \( \{ [\lambda_n] \mid n \in \mathbb{Z} \} \) is dense in the interval \([0, 1]\) we must show that any subinterval \([a, b] \subseteq [0, 1]\) contain at least one point of the form \( [\lambda_n] \). Consider the rectangle \( R = [a, b] \times [1 + f(b), 1 + f(a)] \). Note the graph of \( 1 + f(t) \), \( a \leq t \leq b \) is a monotonically decreasing function passing through the upper left hand corner of \( R \) and the lower right hand corner of \( R \). Since \( \{ [m(\beta - \alpha)] \mid m \in \mathbb{Z} \} \) is dense in \([0, 1]\), there are lines \( \ell_m \) passing through a dense set of point of the bottom \([a, b] \times [1 + f(b), 1 + f(b)] \) of \( R \). Since the lines \( \ell_m \) all have fixed positive slope \( \beta - \alpha \) we can ensure that they pass through the right hand edge \([b, b] \times [1 + f(b), 1 + f(a)] \) of \( R \). These lines must intersect the graph of \( 1 + f(t) \), the first coordinates of these intersection points are the points in \( \{ [\lambda_n] \mid n \in \mathbb{Z} \} \).

**Proposition 3.24.** Let \( I_1, I_2 \) be the two intervals as before, and \( P_B \) one of the selfadjoint extensions. Let \( \Lambda \) be the spectrum of \( P_B \). Then \( \Lambda = F + \mathbb{Z} \), where \( F \) is some finite subset in \( \mathbb{R} \), if and only if \( \beta - \alpha \) is rational.

### 4. Unitary One-Parameter Groups

Consider the Hilbert space \( \mathcal{H} = L^2(\Omega) \), \( \Omega = I_1 \cup I_2 \), and with the two intervals \( I_1 \) and \( I_2 \) as described. As before let
\[
B = \begin{pmatrix}
w e(\phi) & -\sqrt{1 - w^2} e(\theta - \psi) \\
\sqrt{1 - w^2} e(\psi) & w e(\theta - \phi)
\end{pmatrix}
\]
be a unitary matrix with the parameters \( w, \varphi, \theta, \psi \) chosen.

Consider the Hermitian operator \( L = \frac{1}{i\pi} \frac{d}{dx} \) on its minimal domain \( D_0 \) corresponding to function \( f \in \mathcal{H} \) such that \( f' \in \mathcal{H} \) and \( f(0) = f(1) = f(\alpha) = f(\beta) = 0 \). Then the selfadjoint extensions \( P_B \) are prescribed by the choice in (4.1). Specifically,
\[
\text{dom}(P_B) = \{ f \in \text{dom}(P) \mid f_R = B f_L \}
\]
where \( P \) is the maximal operator and where
\[
\begin{align*}
f_R &= \begin{pmatrix} f(1) \\ f(\beta) \end{pmatrix}, \quad \text{and} \\
f_L &= \begin{pmatrix} f(0) \\ f(\alpha) \end{pmatrix}.
\end{align*}
\]

We get a unitary one-parameter group \( U_B(t) : \mathcal{H} \to \mathcal{H}, t \in \mathbb{R} \) such that for \( f \in \text{dom}(P_M) \)
\[
\lim_{t \to 0} \frac{1}{t} (U_B(t)f - f) = iP_B f.
\]

The following is a consequence of the results in Sections 2 - 3.

**Corollary 4.1.**

1. Let \( f \) be some wave-function localized in \( I_1 \), then if \( x \) and \( x - t \) are in \( I_1 \), then
\[
(U_B(t)f)(x) = f(x - t).
\]
(2) As the support of $U_B(t)f$ hits $x = 1$ (the right-hand side boundary point), then it transfers to the two parts $0$ (left-hand side point in $I_1$), and to $\alpha$ (left-hand side part in $I_2$) with probabilities $w^2$ and $1 - w^2$, respectively.

(3) In the transfer from $x = 1$ to $x = 0$, the phase is shifted by $e(\varphi)$; and from $x = 1$ to $x = \alpha$, it is phase-shifted by $e(\psi)$.

(4) The boundary conditions (4.2) are preserved by $U_B(t)$ for all $t \in \mathbb{R}$; i.e., we have

$$\begin{pmatrix} (U_B(t)f)(1) \\ (U_B(t)f)(\beta) \end{pmatrix} = B \begin{pmatrix} (U_B(t)f)(0) \\ (U_B(t)f)(\alpha) \end{pmatrix}$$ (4.5)

for all $f \in \text{dom}(P_B)$; see also eq. (4.3).

**Proof.** It follows from (4.4) that $\text{dom}(P_B)$ is invariant under $U_B(t)$ for all $t \in \mathbb{R}$, and (4.5) follows from this and Corollary 2.3. ■

![Figure 4.1. Quantum Binary Model](image)

5. Spectral Pairs

In the study of the spectral pair-problem for subsets in $\mathbb{R}^d$ ([Fug74, Joer81, JP98, JP99]) one considers subsets $\Omega$ in $\mathbb{R}^d$ of finite positive measure. If the Hilbert space $L^2(\Omega)$ has a Fourier basis, we say that $\Omega$ is a **spectral set**. This means that there is a discrete subset $\Lambda$ in $\mathbb{R}^d$ such that $E(\Lambda) := \{e_\lambda | \lambda \in \Lambda\}$ is an orthogonal basis in $L^2(\Omega)$. We then say that $(\Omega, \Lambda)$ is a **spectral pair**, and the set $\Lambda$ is called a **Fourier spectrum**. The spectrum is never unique. We say $\Omega$ is a **tile**, or that $\Omega$ tiles by translations, if there is a set $A$ such that $\sum_{a \in A} \chi_{\Omega}(t - a) = 1$ for a.e. $t \in \mathbb{R}^d$. The set $A$ is called a **tiling set** for $\Omega$. 
An old question from [Fug74] asks if $\Omega$ is spectral if and only if it tiles $\mathbb{R}^d$ by translations. The question is known to have negative answers for $d \geq 3$. But it is open for $d = 2, 1$. Another reason for our focus here on $d = 1$ and $\Omega = \text{union of two intervals}$, is that, in this case, we prove in Corollary 5.13 below that the two properties are then equivalent. In other words, Fuglede is affirmative in 1D for these sets.

While the equivalence of the two notions, “tile by translations” and “spectral set” is open for general measurable subsets $\Omega$ in $\mathbb{R}$, of finite positive measure, we show below that, in case when $\Omega$ is a union of two intervals, then the equivalence is affirmative. For a discussion of the question for $d \geq 3$, see e.g., [KM06, IKT03, Łab02, IP98, Ped96, LW97]. The first result proving non-equivalence was for $d = 5$, the paper [Tao04]. Others followed shortly after, proving non-equivalence for $d = 4$, and $d = 3$.

Let $1 < \alpha < \beta$ and suppose $P_B$ is the selfadjoint restriction of the maximal operator $P$ corresponding to the matrix $B = B(w, \phi, \psi, \theta)$ in (2.10). Let $\Lambda_B$ be the spectrum of $P_B$. By (3.1) any eigenfunction of $P_B$ is of the form
\[ (a_\lambda \chi_{[0,1]} + b_\lambda \chi_{[\alpha,\beta]}) e^\lambda \]
for some complex numbers $a_\lambda$ and $b_\lambda$. We say $P_B$ is spectral if $a_\lambda = b_\lambda$ for all $\lambda \in \Lambda_B$.

**Lemma 5.1.** If $P_B$ is spectral, then $([0,1] \cup [\alpha,\beta], \Lambda_B)$ is a spectral pair. Conversely, if $([0,1] \cup [\alpha,\beta], \Lambda)$ is a spectral pair, then there is a spectral $P_B$ such that $\Lambda = \Lambda_B$.

**Proof.** Since the eigenfunctions of $P_B$ form an orthogonal basis the first claim is obvious.

Conversely, suppose $([0,1] \cup [\alpha,\beta], \Lambda)$ is a spectral pair. Then the closure of
\[ \sum c_\lambda e_\lambda \rightarrow \sum \lambda c_\lambda e_\lambda \]
determines a selfadjoint restriction $\tilde{P}$ of $P$. The spectrum of this selfadjoint restriction is $\Lambda$ and the eigenfunctions are (the multiples of) the $e_\lambda$’s with $\lambda \in \Lambda$. Since any selfadjoint restriction of $P$ is of the form $P_B$ for some $B$ the proof is complete. ■

When $([0,1] \cup [\alpha,\beta], \Lambda)$ is a spectral pair eq. (3.3) can be solved to find a corresponding matrix $B$.

**Theorem 5.2.** Let $\Lambda_B$ be the spectrum of the selfadjoint restriction $P_B$ corresponding to the matrix $B = B(w, \phi, \psi, \theta)$ in (2.10). If $w = 1$, then $P_B$ is not spectral and consequently $([0,1] \cup [\alpha,\beta], \Lambda_B)$ is not a spectral pair for any choice of the remaining parameters $\alpha, \beta, \theta, \psi, \phi, w$.

**Proof.** By Theorem 3.3 every eigenfunction has $a_\lambda = 0$ or $b_\lambda = 0$. Hence this is a direct consequence of Lemma 5.1. ■

5.1. Two Theorems. With a choice of two intervals, specified by the endpoints $\alpha$ and $\beta$ for the second interval, we proceed to characterize the four parameters in the boundary matrix $B$ from (2.10) in Corollary 2.3 which yield a spectral pair. In other words, we determine when there is an orthogonal Fourier basis in $L^2$ of the two intervals. Our results are Theorems 5.3 and 5.8 below. Together, they exhaust the possibilities for spectral pairs formed from the union of two intervals.
In particular, we show that all the two-interval spectral pairs fall in two classes, (i) lattice tilings (Theorems 5.3), and (ii) non-lattice tilings (Theorems 5.8). In both cases, we find the particular instances of spectrum($P_B$) occurring as the second part in a spectral pair. Since we are in one dimension, (i) represents the cases when the two intervals tile $\mathbb{R}$ by a fixed multiple of the integers $\mathbb{Z}$. In this case, a Fourier spectrum may be taken to be the dual lattice in $\mathbb{R}$. We show that, for (ii), the possible Fourier spectra arising as spectrum($P_B$) must be the union of two lattice co-sets.

**Theorem 5.3.** Let $\Lambda_B$ be the spectrum of the selfadjoint restriction $P_B$ corresponding to the matrix $B = B(w, \phi, \psi, \theta)$ in (2.10). If $w = 0$, then $P_B$ is spectral iff

$$\frac{\beta}{1 + \beta - \alpha} \in \mathbb{N} \quad (5.1)$$

and the parameters $\phi, \theta$ satisfy

$$-\psi + \frac{(\theta - \frac{1}{2})(1 - \alpha)}{1 + \beta - \alpha} \in \mathbb{Z}. \quad (5.2)$$

In the affirmative case the spectrum is $\Lambda_B = \frac{\frac{3}{2} - \theta}{1 + \beta - \alpha} + \frac{1}{1 + \beta - \alpha} \mathbb{Z}$.  

**Proof.** By Theorem 3.9 and Lemma 5.1 $P_B$ is spectral iff

$$-e\left(\theta - \psi + \beta \frac{\frac{1}{2} - \theta + n}{1 + \beta - \alpha}\right) = 1$$

for all $n \in \mathbb{Z}$. That is iff

$$-\psi + \frac{(\theta - \frac{1}{2})(1 - \alpha) + \beta n}{1 + \beta - \alpha} \in \mathbb{Z}$$

for all $n \in \mathbb{Z}$. Considering $n = 0$ and $n = 1$ it follows that $P_B$ is spectral iff the conditions (5.2) are satisfied.

The formula for the spectrum is from Theorem 3.9. □

**Corollary 5.4.** If (5.1) and (5.2) then \([0, 1] \cup [\alpha, \beta], \frac{1}{1 + \beta - \alpha} \mathbb{Z})\) is a spectral pair and \([0, 1] \cup [\alpha, \beta]\) is a tile with tiling set \((1 + \beta - \alpha)\mathbb{Z}\).

**Proof.** If $k = \frac{\beta}{1 + \beta - \alpha}$, then

$$\lfloor \alpha - (1 + \beta - \alpha)(k - 1), \beta - (1 + \beta - \alpha)(k - 1)\rfloor = [1, 1 + \beta - \alpha],$$

consequently, \([0, 1] \cup [\alpha, \beta]\) is a tile with tiling set \((1 + \beta - \alpha)\mathbb{Z}\), when $k$ is an integer. □

**Remark 5.5.** In Theorem 5.3 the condition (5.1) restricts the geometry of the intervals. Specific examples of sets satisfying this condition includes \([0, 1] \cup [\frac{5}{12}, 3]\) and \([0, 1] \cup [7\pi + 1, 14\pi]\). The other condition in (5.2) is a restriction on the set of selfadjoint restrictions, that is on the values of $\theta$ and $\psi$, that determine $B$ and the spectrum.

For every $\beta - \alpha \in \mathbb{Z}_+$, we may construct tiling spectral pairs. In fact, fix $p = \beta - \alpha$, and set $\beta = 2(1 + p)$, $\alpha = 2 + p$. Then

$$\Omega = [0, 1] \cup [2 + p, 2p + 2]$$
tiles with $(1 + p)\mathbb{Z}$. 

Example 5.6. Suppose $\beta - \alpha = 2$. The condition $\frac{\beta}{1+\beta-\alpha} = \frac{\beta}{3} \in \mathbb{Z}$, see (5.2), implies $\beta \in 3\mathbb{Z}$. For $\beta = 6$, the set $[0,1] \cup [4,6]$ tiles $\mathbb{R}$ with $3\mathbb{Z}$; see Figure 5.1. By contrast, $[0,1] \cup [2,4]$ does not tile, and note that (5.2) does not hold.

Example 5.7. Suppose $\beta - \alpha = 3$. Then $\frac{\beta}{1+\beta-\alpha} = \frac{\beta}{4} \in \mathbb{Z}$ implies $\beta \in 4\mathbb{Z}$. For $\beta = 8$, the set $[0,1] \cup [5,8]$ tiles with $4\mathbb{Z}$, by Theorem 5.3.

Figure 5.1. $\Omega \cup (\Omega + 3) \cup (\Omega - 3) \cup (\Omega - 6)$, where $\Omega = [0,1] \cup [4,6]$. 

Figure 5.2. $\Omega \cup (\Omega + 4) \cup (\Omega - 4)$, where $\Omega = [0,1] \cup [5,8]$. 

Theorem 5.8. Let $\Lambda_B$ be the spectrum of the selfadjoint restriction $P_B$ corresponding to the matrix $B = B(w, \phi, \psi, \theta)$. If $0 < w < 1$, then $P_B$ is spectral iff

$$1 < \alpha \text{ is an integer, } \beta = \alpha + 1,$$

and the parameters $\theta, \phi, \psi, \text{ and } w$ satisfies

$$\theta - 2\phi \in \mathbb{Z}, \psi + (\alpha - 1) \phi \in \frac{1}{2} \mathbb{Z}, \text{ and } w \in \left\{ \cos \left( \frac{2\pi}{1+2k} \right) \right\}_{k \in \mathbb{Z}}. \quad (5.4)$$

In the affirmative case $\Lambda_B = \left\{ -\phi \pm c^{-1}(w) \right\} + \mathbb{Z}$, hence $\Lambda_B = \left\{ -\phi \pm \frac{1+2k}{4\alpha} \right\} + \mathbb{Z}$, for some integer $k$.

Proof. We saw earlier that $\lambda$ is a eigenvalue for $P_B$ iff

$$e (\theta - \phi + (\beta - \alpha) \lambda) = \frac{e (\phi + \lambda) - 1}{e (\phi + \lambda) - w} \quad (5.5)$$

and if $\lambda$ is an eigenvalue, then $a_\lambda = b_\lambda = 1$ iff

$$\sqrt{1 - w^2} e (\psi + \lambda - \alpha \lambda) = 1 - w e (\theta - \phi + (\beta - \alpha) \lambda). \quad (5.6)$$

Taking the square of the modulus to both sides of (5.6) we get

$$0 = 2w \left( w - c (\theta - \phi + (\beta - \alpha) \lambda) \right).$$

Where $c(t) := \cos(2\pi t)$. Since $w \neq 0$, we conclude

$$w = c (\theta - \phi + (\beta - \alpha) \lambda). \quad (5.7)$$

Replacing $w$ in (5.5) by $c (\theta - \phi + (\beta - \alpha) \lambda)$ and simplifying using $s(t) := \sin(2\pi t)$ and $c(t) = c(t) + is(t)$ we arrive at

$$i (c (\phi + \lambda) - c (\theta - \phi + (\beta - \alpha) \lambda)) s (\theta - \phi + (\beta - \alpha) \lambda)$$

$$= -1 + c^2 (\theta - \phi + (\beta - \alpha) \lambda) + s (\phi + \lambda) s (\theta - \phi + (\beta - \alpha) \lambda).$$
Consequently,

\[ c(\phi + \lambda) - c(\theta - \phi + (\beta - \alpha)\lambda) s(\theta - \phi + (\beta - \alpha)\lambda) = 0 \]  \hspace{1cm} (5.8)

\[ c^2(\theta - \phi + (\beta - \alpha)\lambda) + s(\phi + \lambda) s(\theta - \phi + (\beta - \alpha)\lambda) = 1. \]  \hspace{1cm} (5.9)

Since \(0 < w < 1\), if follows from (5.7) that \(s(\theta - \phi + (\beta - \alpha)\lambda) \neq 0\), hence (5.8) implies

\[ c(\phi + \lambda) = c(\theta - \phi + (\beta - \alpha)\lambda) = w. \]

It follows from (5.7) and (5.9) that

\[ s(\phi + \lambda) = s(\theta - \phi + (\beta - \alpha)\lambda) = \pm \sqrt{1 - w^2}. \]

Consequently,

\[ e(\theta - \phi + (\beta - \alpha)\lambda) = e(\phi + \lambda) = w \pm i\sqrt{1 - w^2}. \]  \hspace{1cm} (5.10)

is one of the two fixed points \(w \pm i\sqrt{1 - w^2}\) of the Möbius transformation \(M = M_w\) from Lemma 3.12. Note (5.10) implies (5.5). Hence, if \(\lambda\) satisfies (5.10), then \(\lambda\) is in \(\Lambda_B\). Plugging (5.10) into (5.6) we get

\[ e(\psi + \alpha\lambda) = \sqrt{1 - w^2} \pm iw = \pm i\left(w \mp i\sqrt{1 - w^2}\right). \]  \hspace{1cm} (5.11)

We have seen that \(\lambda\) satisfies (5.5) and (5.6) iff \(\lambda\) satisfies (5.10) and (5.11).

Let \(\lambda := \cos^{-1}(w)/2\pi\). Then

\[ e\left(\pm \lambda\right) = w \pm i\sqrt{1 - w^2} \]

by (5.10). And \(c(\phi + \lambda) = w\), implies \(\lambda \in \left\{ -\phi \pm \lambda \right\} + \mathbb{Z}\), hence

\[ \Lambda_B \subseteq \left\{ -\phi \pm \lambda \right\} + \mathbb{Z}. \]  \hspace{1cm} (5.12)

By [Lan67] \(\Lambda_B\) has density \(1 + \beta - \alpha > 1\). Hence one of

\[ \Lambda_B \cap \left\{ -\phi + \lambda \right\} + \mathbb{Z} \quad \text{and} \quad \Lambda_B \cap \left\{ -\phi - \lambda \right\} + \mathbb{Z} \]

has density \(> 1/2\). Assume it is \(\Lambda_B \cap \left\{ -\phi + \lambda \right\} + \mathbb{Z}\).

Let \(Z_+\) be the integers \(k\) such that \(\lambda_k^+ := -\phi + \lambda + k \in \Lambda_B\) and similarly, let \(Z_-\) be the integers \(k\) such that \(\lambda_k^- := -\phi - \lambda + k \in \Lambda_B\). Plugging \(\lambda = \lambda_k^+ = -\phi + \lambda + k\) into (5.11) gives

\[ e\left(\psi - (\phi + \lambda)(1 - \alpha) + (1 - \alpha)k\right) = ie\left(-\lambda\right) = e\left(1 + \lambda\right). \]

for \(k \in Z_+\). Rewriting we get

\[ e((1 - \alpha)k) = e\left(1 + \alpha\lambda\right) e(\phi(1 - \alpha) - \psi) \]

for \(k \in Z_+\). Since the right hand side is independent of \(k \in Z_+\) and \(Z_+\) has density \(> 1/2\), \(1 - \alpha\) is an integer. Similarly, it follows from (5.10) that \(\beta - \alpha\) is an integer. The set \(\left\{ -\phi \pm \lambda \right\} + \mathbb{Z}\) has density equal to two, hence the subset \(\Lambda_B\) has density at most two, so by [Lan67] \(1 + \beta - \alpha \leq 2\). It follows that \(\beta = 1 + \alpha\) and \(\Lambda_B = \left\{ -\phi \pm \lambda \right\} + \mathbb{Z}\). In particular, \(Z_+ = Z_- = \mathbb{Z}\), \(\alpha\) is an integer > 1 and \(\beta = 1 + \alpha\).
Using $\beta - \alpha = 1$ and $\lambda = -\phi + \bar{\lambda} + k$ for $k \in \mathbb{Z}$ we can write (5.10) as
\[ e \left( \theta - 2\phi \pm \bar{\lambda} \right) = e \left( \pm \bar{\lambda} \right), \]
thus $\theta - 2\phi$ is an integer. Similarly, (5.11) becomes
\[ e (\psi + (\alpha - 1) \phi) = e \left( \pm \frac{1}{4} \pm \alpha \bar{\lambda} \right). \]
Consequently,
\[ \pm \frac{1}{4} \pm \alpha \bar{\lambda} \in \psi + (\alpha - 1) \phi + \mathbb{Z}. \]
Adding these equations shows $2 (\psi + (\alpha - 1) \phi) \in \mathbb{Z}$. Subtracting them gives $2\alpha \bar{\lambda} \in -\frac{1}{2} + \mathbb{Z}$. But this is equivalent to
\[ \alpha \bar{\lambda} \in = \left\{ -\frac{1}{2} + \frac{2k}{4} \mid k \in \mathbb{Z} \right\}. \]
This completes the proof.

Corollary 5.9. If (5.3) and (5.4), then
\[ (0, 1] \cup [\alpha, \beta], \left\{ \pm \cos^{-1}(\phi) \right\} + \mathbb{Z} \]
is a spectral pair and $[0, 1] \cup [\alpha, \beta]$ is a tile with tiling set $\{0, 1, 2, \ldots, \alpha - 1\} + 2\alpha \mathbb{Z}$.

Remark 5.10. In Theorem 5.8 the condition (5.3) restricts the geometry of the intervals. The set $[0, 1] \cup [2, 3]$ satisfies (5.3) but not (5.1). The set $[0, 1] \cup [7\pi + 1, 14\pi]$ satisfies (5.1) but not (5.3). Finally, $[0, 1] \cup [3, 4]$ satisfies both (5.1) and (5.3). The other condition in Theorem 5.8 (5.4) is a restriction on the set of selfadjoint restrictions, that is on the values of $w, \theta, \phi$ and $\psi$, that determine $B$ and the spectrum.

Remark 5.11. (i) Let $\Omega := [0, 1] \cup [\alpha, \beta]$. Suppose (5.1) or (5.3) is satisfied, then $\Omega$ is a spectral set. Any choice of a unitary matrix gives us a spectrum $\Lambda_B$. Most of these spectra will not satisfy (5.2) or (5.4). Hence for most $B$ the pair $(\Omega, \Lambda)$ is not a spectral pair, even when $\Omega$ is a spectral set.

(ii) Suppose $(\Omega, \Lambda)$ is a spectral pair. By Lemma 5.1 there is a unitary matrix $B$, such that $P_B$ is spectral and $\Lambda_B = \Lambda$. Construct a new unitary matrix $B_0$ by replacing $\psi$ by a $\psi_0$, (keeping $w, \phi$, and $\theta$ fixed) such that the appropriate one of (5.2) or (5.4) fail. Then $P_{B_0}$ is not spectral, yet, since the spectrum is independent of $\psi$ we have $\Lambda_{B_0} = \Lambda_B = \Lambda$.

(iii) If $(\Omega, \Lambda)$ is a spectral pair we can choose selfadjoint restrictions $P_1$ and $P_2$ of the maximal operator $P$ such that $\Lambda = \Lambda_{P_1} = \Lambda_{P_2}$ and the functions $e_\lambda, \lambda \in \Lambda$ are eigenfunctions for $P_1$ but not for $P_2$.

Corollary 5.12. The set $[0, 1] \cup [\alpha, \beta]$ is a spectral set iff either $\frac{\beta}{1+\beta-\alpha}$ is an integer, or $\alpha > 1$ is an integer and $\beta = \alpha + 1$.

Corollary 5.13. If the set $[0, 1] \cup [\alpha, \beta]$ is a spectral set, then it tiles the real line by translations.

Corollary 5.14. If $[0, 1] \cup [\alpha, \beta]$ is a spectral set and $\beta - \alpha > 1$, then $[0, 1] \cup [\alpha, \beta]$ is a lattice tile.
5.2. Applications. While our results for spectral pairs built with two intervals offer the list of possibilities in terms of the two parameters $\alpha, \beta$, and the boundary matrix (2.10), it is helpful to visualize some of the possibilities, and discuss geometric implications. For example, even though the sets spectrum($P_B$) is determined in all cases, its properties are reflected very concretely by curve intersections, see Figures 5.6 through 5.12 below. And our brief discussions of the examples below serve to highlight main points. For example, one arrives at a visual for the fact that the difference $\beta - \alpha$ represents a slope. Also when it is irrational the restriction to spectrum($P_B$) of natural homomorphism from $\mathbb{R}$ to the quotient $\mathbb{R}/\mathbb{Z}$ (the circle group) has dense range. See Figures 5.4 and 3.5.

In the discussion below, we further illustrate how the possibility for spectral type for our 6-parameter family of selfadjoint operators depend the roots of modulus 1 in certain characteristic polynomials; see Example 5.18, Figure 5.8; Example 5.19, Figure 5.9; and Example 5.21, Figure 5.11. In detail; depending on the relative position of two intervals with integer endpoints, we get a complex polynomial $P(z)$. The question of how many roots are on the circle $|z| = 1$ decide a number of properties of the spectrum of an associated operator $P_B$.

Example 5.15. Let $\Omega = [0, 1] \cup [\alpha, \beta]$, and fix the matrix $B = B(w, \phi, \psi, \theta)$. The spectrum $\Lambda_B$ of $P_B$ is determined by

$$e(\theta - \phi + (\beta - \alpha) \lambda) = \frac{w e(\phi + \lambda) - 1}{e(\phi + \lambda) - w} = e(g(t)).$$

See Lemma 3.11 and 3.12 for details. That is, $\lambda \in \Lambda_B$ iff it is at the intersection of the linear function $\theta - \phi + (\beta - \alpha) \lambda \pmod{1}$, and the monotone decreasing function $g(t) \pmod{1}$. Density of $\Lambda_B$ is reflected in the slope $(\beta - \alpha)$. Choose $w = \frac{1}{\sqrt{3}}$, $\theta = -1/4$, $\phi = -1/8$, $\alpha = 3$, and let $\beta = 4, 5, 6, 7$. Since $\beta - \alpha$ is rational, the spectrum is periodic; see Figure 5.3. On the other hand, let $\beta = 3 + \{1, 2, 3, 4\} \sqrt{2}$, so that $\beta - \alpha$ is irrational, and the spectrum is purely aperiodic; see Figure 5.4.

5.3. Examples. What follows are examples 5.16 - 5.22; and a summary of selected conclusions from these examples.

In Examples 5.16 and 5.17, we compute the boundary matrix $B$ for the case when $\alpha = 2, \beta = 3$. The union of the corresponding two intervals offers a spectral pair which is not a lattice tile.

Continuing with cases when the second interval has the form $[\alpha, \beta]$ with integer endpoints, we follow with a family of examples $\beta - \alpha = 1$; i.e., same length. They again are spectral pairs, but when $\alpha$ is odd, then they are also in the more restrictive class, the lattice tiles. Indeed, for $\alpha$ odd, it turns out that there is then both a lattice spectrum, and a non-lattice spectrum. See Example 5.20 which also shows that different choices of boundary matrix $B$, while leading to the same spectrum, yet can result in different systems of eigenfunctions. In Figures 5.7 - 5.10, we illustrate determination of spectra from curve crossings. Examples 5.21 - 5.22 illustrate computation of spectrum($P_B$) for sets that are unions of pairs of intervals, different length, and producing non-spectral pairs, i.e., not spectral pairs for any choice of $\Lambda$. 
Figure 5.3. $\beta - \alpha$ is rational, and the spectrum is periodic.

Figure 5.4. $\beta - \alpha$ is rational, and the spectrum is purely aperiodic.
**Example 5.16.** We know that \([0, 1] \cup [2, 3]\) is a spectral set with spectrum \(\mathbb{Z} \cup \left(\frac{1}{4} + \mathbb{Z}\right)\). In this case \(B = \left(\begin{array}{ccc} w & x \\ y & z \end{array}\right)\) must satisfy the equations

\[
\left(\begin{array}{ccc} w & x \\ y & z \end{array}\right) \left(\begin{array}{c} 1 \\ 1 \end{array}\right) = \left(\begin{array}{c} 1 \\ 1 \end{array}\right), \lambda \text{ integer}
\]

\[
\left(\begin{array}{ccc} w & x \\ y & z \end{array}\right) \left(\begin{array}{c} i \\ -i \end{array}\right) = \left(\begin{array}{c} 1 \\ -1 \end{array}\right), \lambda \text{ integer plus } 1/4
\]

Hence

\[
B = \frac{1}{\sqrt{2}} \left(\begin{array}{ccc} 1 - i & 1 + i \\ 1 + i & 1 - i \end{array}\right).
\] (5.13)

The boundary matrix \(B\) in (5.13) has parameters \(w = \sqrt{2}/2\), \(\phi = -\frac{1}{8}\), \(\psi = \frac{1}{8}\), and \(\theta = -\frac{1}{4}\). By (3.13), \(\lambda\) is in the spectrum if and only if

\[
e(\lambda - \frac{1}{8}) = \frac{\sqrt{2}}{2} \left(\frac{\lambda - \frac{1}{8}}{\lambda - \frac{1}{8}} - \sqrt{2}\right).
\]

Since both sides of the above equation have modulus one, comparing arguments shows that

\[
\lambda - \frac{1}{8} = \pm \frac{1}{8} + \mathbb{Z}
\]

i.e., \(\lambda = \{0, \frac{1}{8}\} + \mathbb{Z} = \mathbb{Z} \cup \left(\frac{1}{4} + \mathbb{Z}\right)\) confirming the claim at the beginning. We revisit this example several times below.

In the following examples we apply Corollary 5.12 in order to decide when \(L^2(I_1 \cup I_2)\) has a Fourier basis, i.e., when we get spectral pairs. In the examples below this amounts to counting the number of root on the unit circle of a certain polynomial, see also [Ped96].

We illustrate the results with two classes of examples, see Figures 5.5, 5.6. Fix the parameters by \(w = \frac{1}{\sqrt{2}}\), \(\theta = -1/4\), \(\phi = -1/8\), \(\psi = 1/8\), and consider the two intervals \(I_1, I_2\) given by \([0, 1] \cup [2, 2 + k]\) in Class A, and \([0, 1] \cup [3, 3 + k]\) in Class B, where \(k = 1, 2, 3, \ldots\). Note that

1. The spectrum \(\Lambda\) depends on \(\beta - \alpha\) but not the location of the interval \([\alpha, \beta]\);
2. The coefficient \(a(\lambda)\) depends on all 6 parameters.
Example 5.17. Spectral pair: $[0, 1] \cup [2, 3]$, $w = \frac{1}{\sqrt{2}}$, $\theta = -1/4$, $\phi = -1/8$, $\psi = 1/8$.
Spectrum

$$\Lambda = \{0, 1/4\} + \mathbb{Z} = \mathbb{Z} \cup (1/4 + \mathbb{Z})$$

Coefficients

$$a_\lambda = 1, \forall \lambda \in \Lambda$$

Example 5.18. Not spectral pair: $[0, 1] \cup [2, 4]$, $w = \frac{1}{\sqrt{2}}$, $\theta = -1/4$, $\phi = -1/8$, $\psi = 1/8$.
Spectrum

$$\Lambda \approx \{0, 0.1825, 0.5675\} + \mathbb{Z}$$

Coefficients

$$a_0 = 1$$
$$a_{0, 0.1825} \approx 1.61716 - 0.455719i$$
$$a_{0, 0.5675} \approx -0.367157 + 0.205719i$$
Example 5.19. Not spectral pair: $[0, 1] \cup [2, 5]$ , $w = \frac{1}{\sqrt{2}}$, $\theta = -1/4$, $\phi = -1/8$, $\psi = 1/8$.

Spectrum

$$\Lambda \approx \{0, 0.1550, 0.3913, 0.7037\} + \mathbb{Z}$$

Coefficients

$$a_0 = 1$$

$$a_{0.1550} \approx 2.07306 - 0.464978i$$

$$a_{0.3913} \approx 0.253715 + 0.489548i$$

$$a_{0.7037} \approx -0.326779 - 0.27457i$$

Example 5.20. Spectral pair: $[0, 1] \cup [3, 4]$ , $w = \frac{1}{\sqrt{2}}$, $\theta = -1/4$, $\phi = -1/8$, $\psi = 1/8$.

Spectrum

$$\Lambda = \{0, 1/4\} + \mathbb{Z} = \mathbb{Z} \cup (1/4 + \mathbb{Z})$$

But the coefficients are

$$a_0 = 1, \ a_{1/4} = i$$

so the parameters above lead to non-spectral eigenfunctions. Illustrating that different parameters $B = B(\theta, \psi, \phi, w)$ can lead to the same spectrum yet to different eigenfunctions.
Example 5.21. Not spectral pair: $[0, 1] \cup [3, 5]$, $w = \frac{1}{\sqrt{2}}$, $\theta = -1/4$, $\phi = -1/8$, $\psi = 1/8$.
Spectrum

$$\Lambda \approx \{0, 0.1825, 0.5675\} + \mathbb{Z}$$

Coefficients

\[
a_0 &= 1 \\
\tilde{a}_{0.1825} &\approx 1.08072 + 1.28644i \\
\tilde{a}_{0.5675} &\approx 0.419281 - 0.0364378i
\]

Example 5.22. Not spectral pair: $[0, 1] \cup [3, 6]$, $w = \frac{1}{\sqrt{2}}$, $\theta = -1/4$, $\phi = -1/8$, $\psi = 1/8$, $p(z) = (z - 1)(1 + z^3(1 + z))$.
Spectrum

$$\Lambda \approx \{0, 0.1550, 0.3913, 0.7037\} + \mathbb{Z}$$

Coefficients

\[
a_0 &= 1 \\
\tilde{a}_{0.1550} &\approx 1.55016 + 1.45286i \\
\tilde{a}_{0.3913} &\approx -0.505763 - 0.219617i \\
\tilde{a}_{0.7037} &\approx -0.1694 + 0.391761i
\]
Motivated by connections to problems from mathematical physics, we include a brief list of questions arising naturally in connection with properties of spectrum \(P_B\) for the selfadjoint operators \(P_B\) considered here. Even though [Rob71] deals with second order operators, there are a number of parallels, for example both in [Rob71] and here, we are dealing with deficiency indices \((2, 2)\). And in both cases, we get that, when the boundary points are fixed, then the individual eigenvalues \(\lambda_n(B)\) from the list spectrum \((P_B)\), for each \(n\), depend continuously on \(B\).

1. Robinson [Rob71]: one interval, but a second order operator, indices \((2, 2)\), and a formula where the spectrum is a sequence of points in \(\lambda\) (on the real line) arising by graph intersections. Our case: two intervals, and a first order operator, and the same conclusion as in Robinson, only with graphs of a different pair of functions. Open question: What can be said about “similar” of extensions of Hermitian operators with indices \((2, 2)\)?

2. Which self-adjoint restrictions of \(d^2/dx^2\) are not \(P^2\) for some self-adjoint restriction \(P\) of \(id/dx\)?

3. Obtain a formula for the dynamics, i.e., description of \(t \to e^{itP_B}f(x)\)?

4. Corollary 3.17 and plots of \(\lambda_n\) as a function of \(n\), suggests it might be interesting to obtain a quantitative estimate, depending on the other parameters, on \(\delta = \delta_w\)?

5. More than two intervals. Some results in the paper do carry over to sets formed as union of \(n\) intervals, but not our detailed classifications. With \(n\) intervals, the preliminaries in Sections 1 - 2 still hold: the deficiency indices will be \((n, n)\). The boundary form in Lemma 1.1 will also have a matrix representation. Corollary 2.3 will still hold; but instead there will be \(n\) left interval endpoints, and a corresponding set of right-hand side interval endpoints. As a result, the boundary matrix \(B\) will now be an \(n\) by \(n\) complex unitary matrix; every \(B\) in the group \(U(n)\) allows an Iwasawa decomposition, and we will get an analogue of \(w\) in (2.10), only now it will be an positive selfadjoint matrix \(W\), satisfying \(0 \leq W \leq I\), referring to the usual order of selfadjoint matrices. One will also have a matrix version of the Möbius transform in (3.20) in Lemma 3.12. There is a theory of matrix operations, making use of fractional linear matrix operations. There is even a matrix version of the Poisson kernel (3.24). But our detailed classification in Theorem 3.14 will
not carry over. Nonetheless, each selfadjoint operator $P_B$ will still have a pure point spectrum, but now with a much more subtle multiplicity configuration. And, for the general multiple-interval case, it would be difficult to get a classification of all the spectral pairs anything close to what we have in our theorems in Section 5 above.

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