Existence and stability of a positive solution for nonlinear hybrid fractional differential equations with singularity

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ABSTRACT
In this paper, we will study solution existence and its stability for hybrid fractional DE with fractional integral, fractional differential derivative and $\Phi$-operator in Caputo sense. Our problem deals with two basic types of fractional order derivatives, that is, Riemann-Liouville derivative of order $\delta$ and Caputo fractional derivative of order $\lambda$, where $n-1<\lambda,\sigma\leq n$, and $n\geq 3$. We will transform the problem into an integral equation by using Green function and examine whether it is increasing or decreasing and positive or negative function. Some fixed point theorems (Krasnoselski Theorem) are utilized for the existence of a positive solution (EPS). Addition to studying HU-stability technique for our suggested problem. An example is included to apply the results.

1. Introduction

Fractional calculus is a wonderful tool of mathematics, a more generic for the applied mathematics and have caught consideration of researchers in several branches of science and engineering in the previous two decades and applying this wonderful tool in the various fields. The fractional calculus has opened its wings to absorb the dynamics of the complex real world and new thoughts have begun to be implemented and tested on real data. Where it focused on the role of fractional DE in modelling the problems like set theory, in processing image, control theory, biochemistry, computer networking, medicine (the modelling of human tissue), in mechanics (theory of viscoelasticity), in electrical engineering (transmission of ultrasound waves) and many others. For detail, see related literature to the topics [1–12]. We are interested to study the hybrid fractional (DE) with non-integer order, because most of the Mathematical models in applied fields in the time include the fractional order integrals and fractional order derivatives. For further information, we recommend reading these papers [13–15]. Therefore, a great number of scientists studied various side of the arbitrary order hybrid fractional (DE). In this literature, we present some contribution of researchers for investigating of existence and stability the solution (EUS) for hybrid fractional differential equations the different (HFDE). Ahmad B and Ntouyas SK [16] studied the (ES) for a nonlinear hybrid fractional differential of a boundary value problem by using a fixed point, given by

$$D^\lambda\left(\frac{z(t)}{f(t,z(t))}\right) \in F(t,z(t)), \quad 1< t < e, 1< \lambda \leq 2,$$

$$z(1) = z(e) = 0, \quad (1.1)$$

where $D^\lambda$ is the Hadamard fractional derivative, $f \in C([1,e] \times \mathbb{R}, \mathbb{R}\setminus\{0\})$. Further examples we refer the readers to see these papers [17–22]. Herzallah and Baleanu [23] considered the EUS for the following first type and second type hybrid FDEs

$$D^\delta\left(\frac{z(t)}{f(t,z(t))}\right) = g(t), \quad z(0) = z_0 \in \mathbb{R},$$

$$D^\lambda(z(t) - f(t)) = g(t), \quad z(0) = z_0 \in \mathbb{R}, \quad (1.2)$$

where $t \in [0,T], D^\alpha$ is Caputo fractional derivative of order $0<\lambda<1$. Further examples, we indicate readers to see these papers [24–28].

Bashir Ahmad et al. [29] studied the existence of solutions for a system of coupled hybrid fractional differential equations with Dirichlet boundary conditions

$$D^\lambda\left(\frac{z(t)}{f_1(t,z(t),w(t))}\right) = g_1(t,z(t),w(t)), \quad 0< t < 1, 1< \lambda \leq 2,$$

$$D^\sigma\left(\frac{z(t)}{f_2(t,z(t),w(t))}\right) = g_2(t,z(t),w(t)), \quad 0< t < 1, 1< \sigma \leq 2,$$

$$z(0) = z(1), \quad w(0) = w(1), \quad (1.3)$$
where $D^{\lambda}, D^{\sigma}$ are Caputo fractional derivatives of order $\lambda, \sigma$, $f_i \in C([0,1] \times R \times R, R)$ and $g_i \in C([0,1] \times R \times R, R), i = 1, 2$. Further examples, we indicate readers to see these papers [30–32].

Bashiri et al. [33] investigated the existence of solutions for the system of fractional hybrid differential equations given by

\[
D^{\lambda}(\mu(t) - z(t, \mu(t))) = x(t, v(t)), \quad t \in \mathbb{R},
\]

\[
D^{\sigma}(v(t) - z(t, v(t))) = x(t, \mu(t)), \quad t \in \mathbb{R},
\]

where $D^{\lambda}$ is Riemann-Liouville fractional derivative $j = [0, 1]$, the function $y : j \times R \rightarrow R \setminus \{0\}, z : j \times R \rightarrow R$, $z(0) = 0$, and $x : j \times R \rightarrow R$. The main goal form of this paper investigates the (EUS), addition to studying Hyers-Ulam stability theorem for the following (SHFDE):

\[
D^{\lambda}(\phi_p D^{\lambda} (u(t) - Q^{\lambda}_j (u(t)))) + K(t) Q^{\lambda}_j (u(t)) = 0,
\]

\[
(\phi_p D^{\lambda} (u(t) - Q^{\lambda}_j (u(t))))^{(i)} \big|_{t=0} = 0, \quad i = 0, 1, 2, 3, \ldots, n - 1,
\]

\[
i^{\lambda-1} (u(t) - Q^{\lambda}_j (u(t))) \big|_{t=0} = 0, \quad k = 2, 3, \ldots, n,
\]

\[
D^{\lambda-1} (u(t) - Q^{\lambda}_j (u(t))) \big|_{t=1} = 0,
\]

where $D^{\lambda}$ is the Caputo derivative and $D^{\sigma}$ is Riemann-Liouville derivative $n - 1 < \lambda, \sigma \leq n, n \in \{3, 4, 5, \ldots\}$, and $1 < \beta \leq 2$. The $\phi_p$ denotes p-Laplacian, $\phi_p(\theta) = \theta |\theta|^{p-2}, \phi_p(0) = 0, \phi_q = \phi_p^{-1}$, such that $1/p + 1/q = 1$. The $Q^{\lambda}_j, Q^{\sigma}_j, K$ are continuous and increasing for, $t \in (0, 1)$, while singular at some points in $[0, 1]$, $Q^{\lambda}_j$ satisfies that $t^{\lambda-1} ||Q^{\lambda}_{j\tau}|| \leq Q^{\lambda}_{j\omega}, \forall t \in (0, 1)$. Our supposed problem will more complicated and general than the problems studied before and mentioned above.

Our paper consists of 6 parts. In the first part, we offer an introduction, which includes the paper reviews of hybrid fractional DE for the (EPS). In the second part, we narrative several important definitions, theorems and lemmas, which we will be in need of them to prove a solution to our problem. In the third part, we test the increasing or decreasing on period $(0,1)$ for the Green function. In the fourth part, we use fixed point theorems (Krasnoselskii theorem) to prove the existence of a positive solution for our proposed problem (3.1). In the fifth part, we use Hyers-Ulam technique to prove stability to our proposed problem (3.1). In the sixth part, we present an example to emphasize our results. In the seventh part the conclusion.

\section{Axillary results}

\textbf{Definition 2.1:} [7] For $Q(\xi) : (0, +\infty) \rightarrow \mathbb{R}$, fractional derivative for Riemann-Liouville of order $\delta > 0$, is defined by

\[
D^\delta Q(\xi) = \frac{1}{\Gamma(n - \delta)} \frac{d^n}{d\gamma^n} \int_0^\xi (\xi - \gamma)^{n-\delta-1} Q(\gamma) d\gamma,
\]

where $n - 1 < \delta < n$, in which the integral on the right side is pointwise defined on $(0, +\infty), Q(\xi)$ continuous function.

\textbf{Definition 2.2:} [7] For $Q(\xi) : (0, +\infty) \rightarrow \mathbb{R}$, Fractional integral for Riemann–Liouville of order $\delta > 0$, is defined by

\[
\int^\xi \frac{d^n}{d\gamma^n} (\xi - \gamma)^{n-\delta-1} Q(\gamma) d\gamma,
\]

where $\delta > 0$, in which the integral on the right side is pointwise defined on $(0, +\infty)$ and, where $\Gamma(\delta)$ denoted to Gamma function of $\delta$ and $Q(\xi)$ continuous function, given by

\[
\Gamma(\delta) = \int_{0}^{\infty} e^{-\gamma} \gamma^{\delta-1} d\gamma.
\]

\textbf{Definition 2.3:} [2,4] Caputo fractional derivative of order $\lambda > 0$, of a continuous function $Q(\xi) : (0, +\infty) \rightarrow \mathbb{R}$, is given by

\[
D^\lambda Q(\xi) = \frac{1}{\Gamma(n - \lambda)} \frac{d^n}{d\xi^n} \int_0^\xi (\xi - \gamma)^{n-\lambda-1} Q^{(n)}(\gamma) d\gamma,
\]

where $-1 < \lambda < n$, in which the integral on the right side is pointwise defined on $(0, +\infty)$.

\textbf{Lemma 2.1:} [2,4] For a fractional order $\delta \in (n - 1, n], Q \in C^{n-1}$, and $D^{\lambda}$ is Caputo fractional derivative, and then

\[
\int_0^\xi D^\lambda Q(\xi) = Q(\xi) + b_1 + b_2 \xi + b_3 \xi^2 + \cdots + b_n \xi^{n-1},
\]

for the $b_i \in \mathbb{R}$ for $i = 1, 2, 3, \ldots, n$.

\textbf{Lemma 2.2:} [2,4] Consider $\delta \in (n - 1, n], Q \in C^{n-1}$, and $D^{\sigma}$ is fractional derivative for Riemann-Liouville, and then

\[
\int_0^\xi D^\sigma Q(\xi) = Q(\xi) + c_1 \xi^{\delta-1} + c_2 \xi^{\delta-2} + c_3 \xi^{\delta-3} + \cdots + c_n \xi^{\delta-n},
\]

for the $c_i \in \mathbb{R}$ for $i = 1, 2, 3, \ldots, n$.

\textbf{Lemma 2.3:} [2,4] For $\delta, \lambda > 0$, the following regulations satisfying:

\[
D^\lambda t^\delta = \frac{\Gamma(\delta + 1)}{\Gamma(1 + \delta - \lambda)} t^{\delta-\lambda},
\]

\[
\int_0^\xi D^\lambda t^\delta = \frac{\Gamma(\delta + 1)}{\Gamma(1 + \delta + \lambda)} t^{\delta+\lambda}.
\]

\textbf{Definition 2.4:} [34] An operator $M^*: P \cap (W_2 \setminus W_1) \rightarrow P$ is called $(\mathcal{N}_1)$ uniformly bounded, if there exists a constant $S$, such that $|M^*(t)| \leq S$ for all $t \in P \cap (W_2 \setminus W_1); (\mathcal{N}_2)$ equicontinuous, if for every $\varepsilon > 0,$
there exists $\eta(\epsilon) > 0$, such that $|M^*(t_1) - M^*(t_2)| < \epsilon$ for all $t_1, t_2 \in P \cap (\mathcal{W}_2 \setminus \mathcal{W}_1)$ with $|t_1 - t_2| < \eta$.

**Definition 2.5:** [35] Consider $(\Omega, \cdot)$ be a Banach space. A nonempty, cambered, closed set $P \subset \Omega$ is called a cone only if it satisfies the following: $(\mathbb{T}_1)$ if $\psi \in P$ and $\psi \geq 0$, then $\mu \psi \in P$; $(\mathbb{T}_2)$ if $\psi \in P$ and $\psi \geq 0$, then $\psi = 0$.

Assume that $W_1, W_2$ are two bounded subsets of $\Omega$, such that $0 \in W_1, W_1 \subset W_2$, and $M^* : P \cap (\mathcal{W}_2 \setminus \mathcal{W}_1) \rightarrow P$ is an operator.

**Theorem 2.1:** (Krasnoselskii Theorem) [36] If $M^* : P \cap (\mathcal{W}_2 \setminus \mathcal{W}_1) \rightarrow P$ is a completely continuous operator such that $(\mathbb{U}_1)$ $|M^*(u)|| \leq ||u||$ if $u \in P \cap \partial W_1$ and $|M^*(u)|| \geq ||u||$ if $u \in P \cap \partial W_2$; or $(\mathbb{U}_2)$ $|M^*(u)|| \geq ||u||$ if $u \in P \cap \partial W_1$ and $|M^*(u)|| \leq ||u||$ if $u \in P \cap \partial W_2$ is satisfying, and then $M^*$ has a fixed point $p \in P \cap (\mathcal{W}_2 \setminus \mathcal{W}_1)$.

**Theorem 2.2:** [Arzelà–Ascoli’s Theorem] [37] Let $M^* : P \cap (\mathcal{W}_2 \setminus \mathcal{W}_1) \rightarrow P$; we say $M^*$ is compact operator iff it is uniformly bounded and equicontinuous.

**Lemma 2.4:** [38] Let $\phi_p : R \rightarrow R$ be a nonliner $p$-Laplacian operator, $\phi_p(\rho) = |\rho|^{p-2} \rho, \rho \in R$. Then $\frac{d}{d\rho} \phi_p(\rho) = (p-1) |\rho|^{p-2}$. The basic properties of $\phi_p$ operator are the following:

$(\sigma_1)$ If $0 < p \leq 2, \theta_1, \theta_2 > 0$ and $|\theta_1|, |\theta_2| \geq \rho > 0$, then $|\phi_p(\theta_1) - \phi_p(\theta_2)| \leq (p-1) \rho^{p-2} |\theta_1 - \theta_2|.$ \hspace{1cm} (2.9)

$(\sigma_2)$ If $p > 2$ and $|\theta_1|, |\theta_2| \leq \rho^*, \rho > 0$, then $|\phi_p(\theta_1) - \phi_p(\theta_2)| \leq (p-1) \rho^{p-2} |\theta_1 - \theta_2|.$ \hspace{1cm} (2.10)

### 3. Main results

**Theorem 3.1:** Presume an integrable function $Q^*_t, K \in C[0,1]$ satisfying (3.1). Then, for $\lambda, \delta \in (3,n)$, and positive integer $n \geq 4$, the positive solution of the following (SHFDE) with nonlinear $p$-Laplacian operator

\[
^cD^\delta[\phi_p D^\delta(u(t) - Q^*_t(u(t)))] + K(t)Q^*_t(u(t)) = 0,
\]

\[
(\phi_p D^\delta(u(t) - Q^*_t(u(t))))|_{t=0} = 0,
\]

where \(i = 0, 1, 2, 3, \ldots, n - 1, \)

\(\phi_p D^\delta(u(t) - Q^*_t(u(t)))|_{t=0} = 0, \quad k = 2, 3, \ldots, n, \)

\(D^\delta[\phi_p D^\delta(u(t) - Q^*_t(u(t)))]|_{t=1} = 0, \quad (3.1)\)

is

\[
u(t) = Q^*_t(u(t)) + \int_0^1 G^\delta(s, t)\phi_q(\frac{1}{\Gamma(\lambda)}) ds,
\]

\[
\times \int_0^s (s - \tau)^{\lambda - 1}[K(t)Q^*_t(u(t))] d\tau dt, \quad (3.2)
\]

where $G^\delta(t, s)$ is a Green’s function given by

\[
G^\delta(t, s) = \begin{cases} \frac{-(t-s)^{\delta-1}}{\Gamma(\delta)}, & s \leq t \leq 1, \\ \frac{t^{\delta-1}(s-1)^{\delta-\beta}}{\Gamma(\delta)}, & t \leq s \leq 1, \end{cases}
\]

(3.3)

**Proof:** By using Lemma (2.1) and applying integral operator $I^\delta$ on (3.1), we get an alternate form of problem (3.1) as below

\[
\phi_p [D^\delta(u(t) - Q^*_t(u(t)))] = -I^\delta [K(t)Q^*_t(u(t))]
\]

\[
\times + b_1 t + b_2 t^2 + b_3 t^3 + \cdots + c_n t^{n-1}. \quad (3.4)
\]

For the values $i = 0, 1, 2, \ldots, n - 1$, by the conditions $(\phi_p D^\delta(u(t) - Q^*_t(u(t))))|_{t=0} = 0$, the coefficients $b_1 = b_2 = b_3 = b_4 = \cdots = b_n = 0$. Substituting the values $b_i$ for $i = 1, 2, 3, \ldots, n$, and (3.4), we get

\[
\phi_p [D^\delta(u(t) - Q^*_t(u(t)))] = -I^\delta [K(t)Q^*_t(u(t))]. \quad (3.5)
\]

Applying $\phi_p^{-1} = \phi_q$ on both sides of (3.5), we get

\[
D^\delta(u(t) - Q^*_t(u(t))) = -I^\delta [K(t)Q^*_t(u(t))]. \quad (3.6)
\]

By using Lemma 2.2 and applying integral operator $I^\delta$ on both sides of (3.6), we get

\[
(u(t) - Q^*_t(u(t))) = -I^\delta [K(t)Q^*_t(u(t))] + c_1 t^{\delta-1} + c_2 t^{\delta-2} + c_3 t^{\delta-3} + \cdots + c_n t^{n-\delta}. \quad (3.7)
\]

Using the conditions $\phi_p D^\delta(u(t) - Q^*_t(u(t)))|_{t=0} = 0$, for $k = 2, 3, \ldots, n$ in (3.7), we obtain $c_2 = c_3 = c_4 = \cdots = c_n = 0$, and then we get

\[
(u(t) - Q^*_t(u(t))) = -I^\delta [K(t)Q^*_t(u(t))], \quad (3.8)
\]

Using condition $D^\delta[u(t) - Q^*_t(u(t))]|_{t=1} = 0$, in (3.8), we get

\[
\frac{c_1}{\Gamma(\delta - \beta + 1)} + (\phi_q I^\delta [K(t)Q^*_t(u(t))]|_{t=1}. \quad (3.9)
\]

Using the value of $c_1$ in (3.8), we get

\[
u(t) = Q^*_t(u(t)) + \frac{b_1 t}{\Gamma(\delta - \beta + 1)} (\phi_q I^\delta [K(t)Q^*_t(u(t))]|_{t=1})
\]

\[
\times - b_1 t - b_2 t^2 - b_3 t^3 - \cdots - b_n t^{n-1} \quad \text{in (3.8), we get}
\]

\[
u(t) = Q^*_t(u(t)) + \frac{b_1 t}{\Gamma(\delta - \beta + 1)} (\phi_q I^\delta [K(t)Q^*_t(u(t))]|_{t=1})
\]

\[
\times - b_1 t - b_2 t^2 - b_3 t^3 - \cdots - b_n t^{n-1} \quad \text{in (3.8), we get}
\]
This implies
\[ G(t, t) = \frac{1}{\Gamma(\delta)} \int_0^t (t - s)^{\delta - 1} \phi_q(s) \frac{1}{\Gamma(\lambda)} ds \]
where \( G(t, s) \) is a Green function such that defined by (3.3).

**Lemma 3.1:** The Green function \( G(t, s) \) defined by (3.3), satisfies the properties:

1. \( G(t, s) < 0 \) for all \( 0 < s, t < 1 \);
2. \( G(t, s) \) is increasing function and \( \max_{t \in [0,1]} G(t, s) = G(1, s) \);
3. \( G(t, s) \geq t^{\delta - 1} \max_{t \in [0,1]} G(t, s) \) for \( 0 < s, t < 1 \).

**Proof:** For the prove of (A1), we consider:

**Case 1:** When \( s \leq t \), we have
\[
G(t, s) = \frac{-(t - s)^{\delta - 1}}{\Gamma(\delta)} + \frac{t^{\delta - 1}(1 - s)^{\delta - \beta}}{\Gamma(\delta)}
\]
\[
= \left[ \frac{-t^{\delta - 1}(1 - s)^{\delta - 1}}{\Gamma(\delta)} + \frac{t^{\delta - 1}(1 - s)^{\delta - \beta}}{\Gamma(\delta)} \right]
\[
\geq \frac{-t^{\delta - 1}(1 - s)^{\delta - 1}}{\Gamma(\delta)} + \frac{t^{\delta - 1}(1 - s)^{\delta - \beta}}{\Gamma(\delta)}
\]
\[
= t^{\delta - 1} \left[ \frac{(1 - s)^{\delta - 1} - (1 - s)^{\delta - \beta}}{\Gamma(\delta)} \right] = 0
\]
This implies
\[ G(t, s) \geq 0, \text{ for } s \leq t < 1. \]  
(3.11)

**Case 2:** If \( t \leq s < 1 \), then \( t^{\delta - 1} - (1 - s)^{\delta - \beta} \geq 0 \), therefore
\[
G(t, s) = \frac{t^{\delta - 1}(1 - s)^{\delta - \beta}}{\Gamma(\delta)} \geq 0.
\]
This implies
\[ G(t, s) \geq 0, \text{ for } t \leq s < 1. \]  
(3.12)

With (3.11), (3.12), it is evaluated that for all
\[ G(t, s) > 0, \text{ for } s, t \in (0, 1). \]  
(3.13)

Next, for prove (A2), we consider:

**Case 1:** When \( s \leq t \), we proceed
\[
\frac{\partial G(t, s)}{\partial t} = \frac{-(t - s)^{\beta - 2}}{\Gamma(\delta - 1)} + \frac{t^{\beta - 2}(1 - s)^{\delta - \beta}}{\Gamma(\delta - 1)}
\]
\[
= -t^{\beta - 2} \left( \frac{1 - s)^{\delta - 2}}{\Gamma(\delta - 1)} + \frac{(1 - s)^{\delta - \beta}}{\Gamma(\delta - 1)} \right) \geq 0.
\]
This implies
\[ \frac{\partial G(t, s)}{\partial t} > 0, \text{ for } s \leq t < 1. \]  
(3.14)

**Case 2:** For \( t \leq s < 1 \), \( t^{\delta - 2} - (1 - s)^{\delta - \beta} > 0 \), we get
\[
\frac{\partial G(t, s)}{\partial t} = \frac{t^{\beta - 2}(1 - s)^{\delta - \beta}}{\Gamma(\delta - 1)} > 0.
\]
(3.15)

From (3.14) and (3.15), we conclude
\[
\frac{\partial G(t, s)}{\partial t} > 0, \text{ for } s, t \in (1, 0). \]  
(3.16)

Consequently, we can see that \( \frac{\partial G(t, s)}{dt} > 0, \text{ for } s, t \in (1, 0) \). This means that \( G(t, s) \) is non-decreasing function versus \( t \). Therefore, we get
\[
\max_{t \in [0,1]} G(t, s) = \frac{(1 - s)^{\delta - 1}}{\Gamma(\delta)} + \frac{(1 - s)^{\delta - \beta}}{\Gamma(\delta)} = G(1, s). \]  
(3.17)

For prove (A3), we assume:

**Case 1:** If \( s \leq t \), then
\[
G(t, s) = \frac{-(t - s)^{\delta - 1}}{\Gamma(\delta)} + \frac{t^{\delta - 1}(1 - s)^{\delta - \beta}}{\Gamma(\delta)}
\]
\[
= -t^{\delta - 1} \left( \frac{1 - s)^{\delta - 1}}{\Gamma(\delta)} + \frac{(1 - s)^{\delta - \beta}}{\Gamma(\delta)} \right)
\]
\[
= t^{\delta - 1} \max_{t \in [0,1]} G(t, s) = t^{\delta - 1} G(1, s).
\]
This implies
\[ G(t, s) \geq t^{\delta - 1} \max_{t \in [0,1]} G(t, s) \text{ for } s \leq t < 1. \]  
(3.18)
Case 2: For \( t \leq s < 1, t^\frac{1}{\lambda}(1 - s)^{\frac{1}{\lambda} - \beta} > 0 \), then

\[
G^s(t, s) = \frac{t^\frac{1}{\lambda}(1 - s)^{\frac{1}{\lambda} - \beta}}{\Gamma(\delta)} \\
\geq t^\frac{1}{\lambda}(1 - s)^{\frac{1}{\lambda} - \beta} \frac{1}{\Gamma(\delta)} \\
= t^\frac{1}{\lambda}\max_{s \in [0, 1]} G^s(t, s) \\
= t^\frac{1}{\lambda}G^s(1, s).
\]  

(3.19)

This means

\[
G^s(t, s) = t^\frac{1}{\lambda}\max_{s \in [0, 1]} G^s(t, s) \\
= t^\frac{1}{\lambda}G^s(1, s), \text{ for } t \leq s < 1.
\]  

(3.20)

By (3.18) and (3.20), we conclude

\[
G^s(t, s) = t^\frac{1}{\lambda}\max_{s \in [0, 1]} G^s(t, s) \\
= t^\frac{1}{\lambda}G^s(1, s), \text{ for } s, t \in (0, 1).
\]  

(3.21)

4. Existence results

Consider a Banach space \( \Omega = C[0, 1] \) with a norm \( \| u \| = \max_{s \in [0, 1]} \| u(t) \| : u \in \Omega \) and \( P \) be a cone containing positive functions in the space \( \Omega \), where \( P = \{ u \in \Omega : u(t) \geq t^\frac{1}{\lambda} \| u \|, t \in [0, 1] \} \).

Let \( W(r) = \{ u \in P : \| u \| < r, \partial W(r) = \{ u \in P : \| u \| = r \} \). By Theorem (3.1), an alternate form of (3.1) is

\[
u(t) = Q^s_2(t, u(t)) + \int_0^1 G^s(t, s)\phi(t) \left( \frac{1}{\Gamma(\lambda)} \right) \\
\times \int_0^s (s - \tau)^{\lambda - 1}[K(\tau)Q^s_1(\tau, u(\tau))]d\tau ds.
\]  

Define \( M^* : P \setminus [0] \rightarrow \Omega \) by

\[
M^* u(t) = Q^s_2(t, u(t)) + \int_0^1 G^s(t, s)\phi(t) \left( \frac{1}{\Gamma(\lambda)} \right) \\
\times \int_0^s (s - \tau)^{\lambda - 1}[K(\tau)Q^s_1(\tau, u(\tau))]d\tau ds
\]  

(4.2)

Now, we will transform Equation (4.2) to fixed point function \( u(t) \) of the operator \( M^* \) defined as

\[
u(t) = M^* u(t).
\]  

(4.3)

We need to presumptions the following conditions to complete our results.

- \((\mathcal{H}_1)\) \( Q^s_1 : ((0, 1) \times (0, +\infty)) \rightarrow [0, +\infty) \) is continuous.
- \((\mathcal{H}_2)\) \( K : (0, 1) \rightarrow [0, +\infty) \) is non-vanishing and continuous on \((0, 1)\) with

\[
K = \max_{t \in [0, 1]} |K(t)| < +\infty
\]

- \((\mathcal{H}_3)\) For \( a_1, a_2, U^1_1, U^2_1 \) non-negative constants and \( h_1 \in [0, 1] \), non-decreasing function such that \( Q^s_1, Q^s_2 \) are continuous on \((0, 1)\) with

\[
|Q^s_1(t, u(t))| \leq \phi(t)(a_1 u(t)^{h_1} + U^1_1), \\
|Q^s_2(t, u(t))| \leq \phi(t)(a_2 u(t)^{h_2} + U^2_2).
\]

- \((\mathcal{H}_4)\) For a constant value \( \gamma, \gamma > 0 \) and \( z, y \in \Omega \), the functions \( Q^s_1, Q^s_2 \) satisfy

\[
|Q^s_1(t, u(t)) - Q^s_1(t, y(t))| \leq \gamma, \\
|Q^s_2(t, u(t)) - Q^s_2(t, y(t))| \leq \gamma.
\]

Theorem 4.1: Assume that conditions \((\mathcal{H}_1) - (\mathcal{H}_3)\) satisfying. Then \( M^* \) is a completely continuous operator.

Proof: For any \( u \in \overline{W(r_2)} \setminus W(r_1) \), from Lemma 3.1 and Equation (4.2), we have

\[
M^* u(t) = Q^s_2(t, u(t)) + \int_0^s G^s(t, s)\phi(t) \left( \frac{1}{\Gamma(\lambda)} \right) \\
\times \int_0^s (s - \tau)^{\lambda - 1}[K(\tau)Q^s_1(\tau, u(\tau))]d\tau ds \leq Q^s_2(t, u(t)) + \int_0^s G^s(t, s)\phi(t) \left( \frac{1}{\Gamma(\lambda)} \right) \\
\times \int_0^s (s - \tau)^{\lambda - 1}[K(\tau)Q^s_1(\tau, u(\tau))]d\tau ds
\]

(4.4)

and

\[
M^* u(t) = Q^s_2(t, u(t)) + \int_0^s G^s(t, s)\phi(t) \left( \frac{1}{\Gamma(\lambda)} \right) \\
\times \int_0^s (s - \tau)^{\lambda - 1}[K(\tau)Q^s_1(\tau, u(\tau))]d\tau ds \geq Q^s_2(t, u(t)) + t^\frac{1}{\lambda}\int_0^s G^s(t, s)\phi(t) \left( \frac{1}{\Gamma(\lambda)} \right) \\
\times \int_0^s (s - \tau)^{\lambda - 1}[K(\tau)Q^s_1(\tau, u(\tau))]d\tau ds
\]

(4.5)

With help of (4.4) and (4.5), we get

\[
M^* u(t) \geq t^\frac{1}{\lambda} M^* u(t), 0 \leq t \leq 1.
\]  

(4.6)

This implies \( M^* : \overline{W(r_2)} \setminus W(r_1) \rightarrow P \) is closed.
Now, in order to show that $M^*$ is continuous, we prove $||M^*(u_n) - M^*(u)|| \to 0$ as $n \to \infty$ as follows:

$$
|M^*(u_n(t)) - M^*(u(t))| = |Q_2^*(t, u_n(t)) - Q_2^*(t, u(t))| \\
\quad + \int_0^1 G^1(t, s) \phi_q \left( \frac{1}{\Gamma(\delta)} \right) ds \\
\quad \times \left( s - t \right)^{\delta - 1} [K(\tau)] |Q_1^*(\tau, u_n(\tau))] d\tau \\
\quad - \phi_q \left( \frac{1}{\Gamma(\delta)} \right) \int_0^1 (s - \tau)^{\delta - 1} [K(\tau)] |Q_1^*(\tau, u(\tau))] d\tau \\
\quad \times \left( s - t \right)^{\delta - 1} |Q_1^*(\tau, u_n(\tau))] d\tau \\
\quad \times \left( s - t \right)^{\delta - 1} [K(\tau)] |Q_1^*(\tau, u(\tau))] d\tau \\
\leq |Q_2^*(t, u_n(t)) - Q_2^*(t, u(t))| + \int_0^1 G^1(t, s) \phi_q \left( \frac{1}{\Gamma(\delta)} \right) ds \\
\quad \times \left( s - t \right)^{\delta - 1} [K(\tau)] |Q_1^*(\tau, u_n(\tau))] d\tau \\
\quad - \phi_q \left( \frac{1}{\Gamma(\delta)} \right) \int_0^1 (s - \tau)^{\delta - 1} [K(\tau)] |Q_1^*(\tau, u(\tau))] d\tau \\
\quad \times \left( s - t \right)^{\delta - 1} [K(\tau)] |Q_1^*(\tau, u_n(\tau))] d\tau \\
\quad \times \left( s - t \right)^{\delta - 1} [K(\tau)] |Q_1^*(\tau, u(\tau))] d\tau
$$

(4.7)

With the help of (4.7), and continuity of $Q_1^*, Q_2^*$ we have $|M^*(u_n) - M^*(u)| \to 0$ as $n \to \infty$. This proves that $M^*$ is continuous, for prove the uniformly boundedness of $M^*$, by Equation (4.2) and presumption ($\mathcal{H}_1$) we get

$$
|M^*(u(t_1)) - M^*(u(t_2))| = |Q_2^*(t_1, u(t_1)) - Q_2^*(t_2, u(t_2))| \\
\quad + \int_0^1 G^1(t_1, s) \phi_q \left( \frac{1}{\Gamma(\delta)} \right) ds \\
\quad \times \left( s - t \right)^{\delta - 1} [K(\tau)] |Q_1^*(\tau, u(t_1))] d\tau \\
\quad - \phi_q \left( \frac{1}{\Gamma(\delta)} \right) \int_0^1 (s - \tau)^{\delta - 1} [K(\tau)] |Q_1^*(\tau, u(\tau))] d\tau \\
\quad \times \left( s - t \right)^{\delta - 1} [K(\tau)] |Q_1^*(\tau, u(t_1))] d\tau \\
\quad \times \left( s - t \right)^{\delta - 1} [K(\tau)] |Q_1^*(\tau, u(\tau))] d\tau
$$

(4.9)

From (4.9), when $t_1 \to t_2$, we have that (4.9) approaches to zero. So $M^*$ is equicontinuous by the Theorem 2.2, $M^*(P \cap \overline{W(t_2)} \cap W(r_1))$ is compact in $P \cap \overline{W(t_2)} \cap W(r_1))$. Consequently, $M^*: \overline{W(t_2)} \cap W(r_1) \to P$ is completely continuous.

To complete the proof of our results of hybrid function $Q^*(t, x(t))$ for $x > 0$, we need to define the
Theorem 4.2: Assume that supposition (\(\mathcal{H}_1\)) to (\(\mathcal{H}_3\)) hold and there exist \(h, r \in \mathbb{R}^+\), such that

\[
(B_1) \quad h \leq ||Q_2^*(t, u(t))|| + \int_0^1 G^1(1, s)\phi_q \\left( \frac{1}{\Gamma(\lambda)} \right) \int_0^s (s - \tau)^{\lambda - 1} [K(\tau)\phi_{\min}(\tau, h)] d\tau ds
\]

\[< +\infty \quad \text{and} \]

\[
||Q_2^*(t, u(t))|| + \int_0^1 G^1(1, s)\phi_q \\left( \frac{1}{\Gamma(\lambda)} \right) \int_0^s (s - \tau)^{\lambda - 1} [K(\tau)\phi_{\max}(\tau, r)] d\tau ds < r \quad \text{or;}
\]

\[
(B_2) \quad ||Q_2^*(t, u(t))|| + \int_0^1 G^1(1, s)\phi_q \\left( \frac{1}{\Gamma(\lambda)} \right) \int_0^s (s - \tau)^{\lambda - 1} [K(\tau)\phi_{\max}(\tau, h)] d\tau ds < h \quad \text{and}
\]

\[r \leq ||Q_2^*(t, u(t))|| + \int_0^1 G^1(1, s)\phi_q \\left( \frac{1}{\Gamma(\lambda)} \right) \int_0^s (s - \tau)^{\lambda - 1} [K(\tau)\phi_{\max}(\tau, r)] d\tau ds < +\infty.
\]

is satisfied. Then the hybrid fractional DE of the problem (3.1) has a positive solution \(u^* \in P \) and \(h \leq ||u|| \leq r\).

Proof: Without loss of generality, consider the case \((B_1)\). If \(x \in \partial W(h)\), then we have \(||u|| = h\) and \(t^{\lambda - 1} h \leq u(t) \leq h, t \in [0, 1]\). With the help of Equation (4.10), we get

\[
\frac{\partial}{\partial t} ||Q_2^*(t, u(t))|| + \int_0^1 G^1(1, s)\phi_q \\left( \frac{1}{\Gamma(\lambda)} \right) \int_0^s (s - \tau)^{\lambda - 1} [K(\tau)\phi_{\min}(\tau, h)] d\tau ds
\]

\[\geq t^{\lambda - 1} \int_0^1 G^1(1, s)\phi_q \\left( \frac{1}{\Gamma(\lambda)} \right) \int_0^s (s - \tau)^{\lambda - 1} [K(\tau)\phi_{\max}(\tau, h)] d\tau ds
\]

\[\geq \int_0^1 G^1(1, s)\phi_q \\left( \frac{1}{\Gamma(\lambda)} \right) \int_0^s (s - \tau)^{\lambda - 1} [K(\tau)\phi_{\min}(\tau, r)] d\tau ds \geq h = ||u||.
\]

This implies

\[
||M^*(u(t))|| \geq h \geq ||u||.
\]

If \(u(t) \in \partial W(h)\), then \(||u|| = r\) and \(t^{\lambda - 1} r \leq u \leq r, t \in [0, 1]\). Using (4.10), we have \(\phi_{\max}(t, u) \geq Q^*(t, u)\) for \(t \in (0, 1)\). Thus, we have

\[
||M^*(u(t))|| = \max\{ ||Q_2^*(t, u(t))|| + \int_0^1 G^1(1, s)\phi_q \\left( \frac{1}{\Gamma(\lambda)} \right) \int_0^s (s - \tau)^{\lambda - 1} [K(\tau)\phi_{\min}(\tau, u(t))] d\tau ds
\]

\[\leq \int_0^1 ||Q_2^*(t, u(t))|| + G^1(1, s)\phi_q \\left( \frac{1}{\Gamma(\lambda)} \right) \int_0^s (s - \tau)^{\lambda - 1} [K(\tau)\phi_{\max}(\tau, r)] d\tau ds
\]

\[\leq ||Q_2^*(t, u(t))|| + \int_0^1 G^1(1, s)\phi_q \\left( \frac{1}{\Gamma(\lambda)} \right) \int_0^s (s - \tau)^{\lambda - 1} [K(\tau)\phi_{\max}(\tau, r)] d\tau ds \leq r = ||u||.
\]

This implies

\[
||M^*(u(t))|| \leq r = ||u||.
\]

By Equation (4.11), (4.12), and Theorem (2.1), we get that has a fixed point in \([h, r]\), say \(u^* \in (W(r) \cap W(h))\) such that \(h \leq ||u^*|| \leq r\). By Lemma 3.1 and Theorem 3.1 implies \(u^*(t) \leq t^{\lambda - 1} ||u^*|| \leq h t^{\lambda - 1} > 0\), for \(t \in (0, 1)\) and

\[
\frac{\partial}{\partial t} u^*(t) = \frac{\partial}{\partial t} M^*(u(t)) = \frac{\partial}{\partial t} Q_2^*(t, u(t))
\]

\[+ \int_0^1 \frac{\partial}{\partial t} G^1(1, s)\phi_q \\left( \frac{1}{\Gamma(\lambda)} \right) \int_0^s (s - \tau)^{\lambda - 1} [K(\tau)\phi_{\max}(\tau, r)] d\tau ds > 0.
\]

Thus \(u^*\) is a positive solution for \(t \in (0, 1)\).

5. Hyers-Ulam stability

Here, we offer Hyers-Ulam stability for the hybrid FDE with nonlinear \(\phi_{\nu}\)-Laplacian operator in suggested problem (3.1).

Definition 5.1: We say that integral equation (4.1) is HU-stability if there exists positive constant value \(\nu_2\), satisfying: For every \(\alpha > 0\), if

\[
|u(t) - Q_2^*(t, u(t))| + \int_0^1 G^1(1, s)\phi_q \\left( \frac{1}{\Gamma(\lambda)} \right) \int_0^s (s - \tau)^{\lambda - 1} [K(\tau)\phi_{\min}(\tau, u(t))] d\tau ds \leq \alpha,
\]

there exist \(v(t)\) satisfying that

\[
|v(t) - Q_2^*(t, v(t))| + \int_0^1 G^1(1, s)\phi_q \\left( \frac{1}{\Gamma(\lambda)} \right) \int_0^s (s - \tau)^{\lambda - 1} [K(\tau)\phi_{\max}(\tau, v(t))] d\tau ds \leq \alpha,
\]

there exist \(\nu_2\) such that the pair \((u(t), v(t))\) is satisfy in

\[
|u(t) - v(t)| \leq \alpha \nu_2.
\]
Theorem 5.1: The singular hybrid FDE with \( \phi_p \)-operator of suggested problem (3.1) is Hyers-Ulam stable provided that (H1), (H2) and (H4) hold true.

Proof: By using Definition 5.1 and Theorem 3.3, suppose that \( u(t) \) be the real solution of the hybrid fractional DE of Equation (4.1) and \( v(t) \) be an approximate solution satisfying (5.2). Then, we have

\[
|u(t) - v(t)| = \left| \int_0^t G^i(t,s) \phi_q \left( \frac{1}{\Gamma(\lambda)} \int_0^s (s - \tau)^{\lambda - 1} \times [K(\tau)Q^1_i(\tau, u(\tau))] d\tau \right) ds \right| \\
\leq \int_0^t \left| G^i(t,s) \phi_q \left( \frac{1}{\Gamma(\lambda)} \int_0^s (s - \tau)^{\lambda - 1} \times [K(\tau)Q^1_i(\tau, u(\tau))] d\tau \right) ds \right| \\
\leq \frac{1}{\Gamma(\lambda)} \int_0^t \left| G^i(t,s) \phi_q \left[ \frac{1}{\Gamma(\lambda)} \int_0^s (s - \tau)^{\lambda - 1} \times [K(\tau)Q^1_i(\tau, u(\tau))] d\tau \right] ds \right| \\
\leq \frac{1}{\Gamma(\lambda)} \int_0^t \left| G^i(t,s) \phi_q \left[ \frac{1}{\Gamma(\lambda)} \int_0^s (s - \tau)^{\lambda - 1} \times [K(\tau)Q^1_i(\tau, u(\tau))] d\tau \right] ds \right| \\
\leq \frac{1}{\Gamma(\lambda)} \int_0^t \left| G^i(t,s) \phi_q \left[ \frac{1}{\Gamma(\lambda)} \int_0^s (s - \tau)^{\lambda - 1} \times [K(\tau)Q^1_i(\tau, u(\tau))] d\tau \right] ds \right|
\]

where \( \lambda = \delta - 1 + 1 \). Hence, by (5.4), the integral Equation (4.1) is HU-stable. Consequently, the SHFDE with nonlinear \( p \)-Laplacian operator (3.1) is Hyers-Ulam stable.

6. Illustrative example

In this section, we give application for the characterization of the results proved in Sections 4 and 5, for the EPS of HFDE of the type (3.1).

Example 1. For

\[
t \in [0,1], Q^1_i(t, u(t)) = u^{2/3}(t) + 7(u(t))^{-1/3}, q = \frac{5}{3}, \delta = 4, \lambda = \frac{5}{2}, \beta = \frac{3}{2}, K(t) = t(3 - 3t)^{-1/3}, \]

we consider the following singular fractional DE with \( \phi_p \)-operator:

\[
\frac{d^p}{ds^p} \left[ \phi_p D^p (u(t) - Q^1_i(t, u(t))) \right] + K(t)Q^1_i(t, u(t)) = 0, \]

where \( i = 0, 1, 2, \ldots, n, \)

\[
\Delta^k \left[ u(t) - Q^1_i(t, u(t)) \right] |_{t=0} = 0, k = 2, 3, \ldots, n, \]

\[
D^{\alpha - 1}[u(t) - Q^1_i(t, u(t))] |_{t=1} = 0, \]

The functions clearly \( K \in C((0,1), [0, +\infty)), Q^* \in C((0,1) \times (0, +\infty), [0, +\infty)).\) Height functions

\[
\varphi_{\text{max}}(t,x) = \max[u^{2/3} + 7(u(t))^{-1/3} : t^2 x \leq u \leq x] \leq t^2 x + 7/8, \]

\[
\varphi_{\text{min}}(t,x) = \min[u^{2/3} + 7(u(t))^{-1/3} : t^2 x \leq u \leq x] \geq t^2 x + 7/8.
\]

Then, we have

\[
(1 + t) \frac{1}{10000} + \int_0^1 G^i(1,s) \phi_q \left( \frac{1}{\Gamma(\lambda)} \int_0^s (s - \sigma)^{\lambda - 1} \times [K(\sigma)\varphi_{\text{min}}(\sigma, h)] d\sigma \right) ds \\
= (1 + t) \frac{1}{10000} + \int_0^1 G^i(1,s) \phi_q \left( \frac{1}{\Gamma(\lambda)} \int_0^s (s - \sigma)^{\lambda - 1} \times [K(\sigma)\varphi_{\text{min}}(\sigma, 10^{-4})] d\sigma \right) ds \\
\geq 0.0001 + \int_0^1 G^i(1,s) \phi_q \left( \frac{1}{\Gamma(\lambda)} \int_0^s (s - \sigma)^{\lambda - 1} \times [\varphi_{\text{min}}(3 - 3\sigma)^{-1/2} (\sigma^2 10^{-8/3} + 7(10)^{4/3})] d\sigma \right) ds \\
\geq 0.000049645 \geq 10^{-4},
\]

By Theorem 3.3, the problem (3.1) has a solution \( u^* \) and \( 10^{-4} \leq |u^*| \leq 1. \)
7. Conclusion

By the help of fixed point theorems of Krasnoselskii and function analysis on Banach space, we have proved existence, uniqueness, and Hyers-Ulam stability of solutions for hybrid fractional DEs. For these aims, we transformed the proposed problem (3.1) into an integral equation by Green function. After that, Green function was tested in the period (0,1) for being increasing or decreasing and positive or negative. For an application of our results, we included an example using Mathematica.

Acknowledgements

The authors are thankful to the unknown reviewers and the editorial council for their value propositions that have to ameliorate the quality of the literature. The first author is thankful to the unknown reviewers and the editorial council for their value propositions that have to ameliorate the quality of the literature. The first author is thankful for Nanjing of science and technology University in order to obtain a research grant under the China Government Excellent Young Talents Program through postdoctoral studies.

Disclosure statement

No potential conflict of interest was reported by the authors.

Funding

This work was supported by Nanjing of Science and Technology University. [Grant Number 717113010063].

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