Boundedness of Classical Solutions to a Degenerate Keller–Segel Type Model with Signal-Dependent Motilities

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Abstract
In this paper, we consider the initial Neumann boundary value problem for a degenerate kinetic model of Keller–Segel type. The system features a signal-dependent decreasing motility function that vanishes asymptotically, i.e., degeneracies may take place as the concentration of signals tends to infinity. In the present work, we are interested in the boundedness of classical solutions when the motility function satisfies certain decay rate assumptions. Roughly speaking, in the two-dimensional setting, we prove that classical solution is globally bounded if the motility function decreases slower than an exponential speed at high signal concentrations. In higher dimensions, boundedness is obtained when the motility decreases at certain algebraical speed. The proof is based on the comparison methods developed in our previous work (Fujie and Jiang in J. Differ. Equ. 269:5338–5778, 2020; Fujie and Jiang in Calc. Var. Partial Differ. Equ. 60:92, 2021) together with a modified Alikakos–Moser type iteration. Besides, new estimations involving certain weighted energies are also constructed to establish the boundedness.

Keywords Classical solutions · Boundedness · Degeneracy · Chemotaxis · Keller–Segel models

Mathematics Subject Classification 35B60 · 35K20 · 35K65 · 35M33 · 35Q92

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1 Introduction

This paper is a continuation of our previous work [10, 11] on the following initial boundary value problem:

\[
\begin{align*}
  u_t &= \Delta (\gamma(v)u) \quad &x \in \Omega, \ t > 0, \\
  \varepsilon v_t - \Delta v + v &= u \quad &x \in \Omega, \ t > 0, \\
  \partial_n u &= \partial_n v = 0, \quad &x \in \partial \Omega, \ t > 0, \\
  u(x, 0) &= u_0(x), \quad \varepsilon v(x, 0) = \varepsilon v_0(x), \quad x \in \Omega.
\end{align*}
\]

Here, \(u\) and \(v\) stand for the density of cells and the concentration of signals, respectively, and \(\gamma\) represents cell motility. System (1.1) is a special version of Keller–Segel type model with signal-dependent diffusion rates and chemo-sensitivities introduced by Keller and Segel in their seminal works [18–20]. In fact, in [19], the evolution of cell density was described by the following equation:

\[
  u_t = \nabla \cdot (\mu(v) \nabla u - u \chi(v) \nabla v),
\]

where the cell diffusion rate \(\mu\) and chemo-sensitivity \(\chi\) are linked via

\[
  \chi(v) = (\sigma - 1)\mu'(v).
\]

Note that a direct decomposition of the right-hand side of the first equation in (1.1) yields the following variant form

\[
\begin{align*}
  u_t &= \nabla \cdot (\gamma(v) \nabla u) + \nabla \cdot (u \gamma'(v) \nabla v), \\
  \varepsilon v_t - \Delta v + v &= u,
\end{align*}
\]

which corresponds to a special case of (1.2) with \(\sigma = 0\) in (1.3). Recall that the parameter \(\sigma\) is proportional to the distance between chemical receptors in the cells. In the case \(\sigma = 0\), the distance between receptors is zero. In other words, chemotaxis occurs because of an undirected effect on activity due to the presence of a chemical sensed by a single receptor (local sensing), which is distinct from the directed chemotactic movement when \(\sigma > 0\) induced by comparing the chemical concentrations at different spots (gradient sensing). It is worth mentioning that in the local sensing case, \(\gamma' < 0\) corresponds to a chemo-attraction phenomenon while \(\gamma' > 0\) accounts for a chemo-repulsion movement. Simulations carried out in [7] indicated that pattern formation may take place in (1.1) with some non-increasing motility function.

Recently system (1.1) with an additional logistic source term was also proposed in some Biophysical work [9, 23] (see also [35]) to describe the process of pattern formations via the so-called “self-trapping” mechanism. There, the cellular motility \(\gamma(\cdot)\) was assumed to be suppressed by the concentration of signals, which characterizes a repressive effect of the signal concentration on the cellular motility. In other words, \(\gamma(\cdot)\) is a signal-dependent decreasing function, i.e., \(\gamma'(v) \leq 0\).

Theoretical analysis for problem (1.1) has attracted a lot interest in some recent studies. When \(\gamma' \leq 0\), the system features a possible vanishing macroscopic motility as \(v\) becomes unbounded. Thus the major difficulty in analysis comes from a possible degeneracy, which was tackled in some previous work [1, 6, 7, 17, 22, 24, 30, 33, 36] basically by energy method. In fact, standard elliptic/parabolic regularity theory tells that \(L^\infty_t L^p_x\)-boundedness
of \( u \) with any \( p > \frac{n}{2} \) will yield to an upper bound of \( v \). Thus, an indirect way to prevent degeneracy is to establish higher integrability of \( u \). However, this idea seems only efficient for some specific cases, where several additional assumptions are needed to achieve the \( L_t^\infty L_x^p \)-boundedness of \( u \). For example, smallness of some coefficients [36], particular choices of the motility functions [1, 36], or a presence of logistic source term in the first equation [17, 22, 24, 33], etc.

In contrast, a new comparison method based on a careful observation of the delicate nonlinear structure was developed to establish the upper bound of \( v \) directly in our previous work [10, 11]. We proved that in any spatial dimensions and with any motility function satisfying (A0) when \( \varepsilon = 0 \), or additionally (A1) when \( \varepsilon > 0 \) below, the upper bound of \( v \) grows at most exponentially in time and thus degeneracy cannot take place in finite time. Then we showed that classical solution always exists globally in dimension two. Moreover, under certain polynomial growth condition on \( 1/\gamma \), we discussed uniform-in-time boundedness when \( n = 2, 3 \). More importantly, occurrence of exploding solutions was examined for the first time. In the case \( \gamma(v) = e^{-v} \), a novel critical-mass phenomenon in the two-dimensional setting was observed that with any sub-critical mass, the global solution is uniformly-in-time bounded while with certain super-critical mass, the global solution will blow up at time infinity. We mention that in the special case \( \gamma(v) = e^{-v} \), global boundedness with sub-critical mass and possible blowup at unspecified blow-up time with super-critical mass was also proved in [16] by energy method. In [6], the authors also verified that blowup of classical solution must occur at time infinity by duality method and moreover, weak solutions was obtained in any dimensions when \( \gamma(v) = e^{-v} \).

In the present work, we aim to continue our discussion on uniform-in-time boundedness of classical solutions with generic motility functions and arbitrarily large initial data. Throughout this paper, we assume \( \Omega \subset \mathbb{R}^n \) with \( n \geq 2 \) being a smooth bounded domain and

\[
(u_0, v_0) \in C^0(\Omega) \times W^{1,\infty}(\Omega), \quad u_0 \geq 0, \quad v_0 > 0 \quad \text{in} \quad \overline{\Omega}, \quad u_0 \not\equiv 0.
\] (1.5)

For \( \gamma \), we first require in general that

\[
(A0) : \gamma(v) \in C^3[0, +\infty), \quad \gamma(v) > 0, \quad \gamma'(v) \leq 0 \quad \text{on} \quad (0, +\infty).
\] (1.6)

Additionally if \( \varepsilon > 0 \), we need the following asymptotic property:

\[
(A1) : \lim_{s \to +\infty} \gamma(s) = 0.
\] (1.7)

Moreover, in order to study the uniform-in-time boundedness, we will propose certain decreasing assumptions on \( \gamma \). In particular we shall consider the form \( \gamma(s) = s^{-k} \) with some \( k > 0 \) as a toy model. In this case, the variant form reads

\[
\begin{cases}
    u_t = \nabla \cdot [v^{-k}(\nabla u - ku \nabla \log v)], \\
    \varepsilon v_t - \Delta v + v = u,
\end{cases}
\] (1.8)

which resembles the classical Keller–Segel model with a logarithmic chemo-sensitivity:

\[
\begin{cases}
    u_t = \nabla \cdot (\nabla u - ku \nabla \log v), \\
    \varepsilon v_t = \Delta v - v + u.
\end{cases}
\] (1.9)

Up to now, theoretical results on (1.9) are far from being satisfactory. Roughly speaking, existence of global solutions or blowups seems to be determined by the size of \( k \). Blowup
solution was constructed only in the radial symmetric case when $\varepsilon = 0, n \geq 3$ and $k > \frac{2n}{n-2}$ [27]. On the other hand, there are several attempts on enlarging the admissible range of $k$ for global existence and however, the threshold number is still unclear. Since a complete description of related results can be found in [12, 13, 21], we omit a detailed review here.

For the degenerate model under consideration, the result in our work [10, 11] in the two-dimensional setting asserts that any polynomial decreasing motility would give rise to globally boundedness whereas the exponentially decaying type will result in a critical mass phenomenon, which strongly indicates that the dynamic of solutions is closely related to the decay rate of $\gamma(\cdot)$. In this regard, one motivation of the current work is to understand the connection between decay rate and global existence or boundedness. In the context of particular choice $\gamma(s) = s^{-k}$, it suffices to find out an admissible range of $k$ as well. Note that systems (1.8) and (1.9) share the same set of equilibria. Thus a study on our system with $\gamma(s) = s^{-k}$ may also lead us to a better understanding of the mechanism of the logarithmic Keller–Segel model.

Now we recall some related results on the toy model case (1.8). If $\varepsilon = 0$, global classical solution with uniform-in-time bounds was obtained by delicate energy estimates in [1] when $n \leq 2$ for any $k > 0$ or $n \geq 3$ for $k < \frac{2}{n-2}$ (see [32] for an alternative proof; see also [36] for global existence under certain smallness assumptions). In our previous work [10, 11], we generalized above boundedness results in dimension two for any $\varepsilon \geq 0$ under the following weaker assumption:

\[ (A_2) : \quad \text{there is } k > 0 \text{ such that } \lim_{s \to +\infty} s^k \gamma(s) = +\infty. \quad (1.10) \]

More precisely, we proved that if $n = 2$, then problem (1.1) with any $\varepsilon \geq 0$ has a uniformly-in-time bounded classical solution provided that $\gamma$ satisfies $(A_0) - (A_1)$ as well as $(A_2)$. Note that $(A_2)$ allows $\gamma$ to take any decreasing form within a finite region and moreover, other algebraically decreasing functions are permitted as well, for example, $\gamma(v) = \frac{1}{v^k \log(1+v)}$ with any $k > 0$. Furthermore, if $n = 3$ and $\varepsilon > 0$, we also obtained globally bounded solution provided additionally that

\[ (A_3) : \quad 2|\gamma'(s)|^2 \leq \gamma(s) \gamma''(s), \quad \forall s > 0. \quad (1.11) \]

Under assumptions $(A_0)$, $(A_1)$ and $(A_3)$, $1/\gamma(s)$ can grow at most linearly in $s$. Correspondingly, if we take $\gamma(s) = s^{-k}$, then $(A_3)$ will yield to a constraint $k \leq 1$.

The present contribution focuses on a study of the homogeneous problem (1.1). From an analysis point of view, the presence of a logistic source term alters significantly the mathematical properties of (1.1) and its dissipative effect fosters boundedness of classical solutions. However, it was indicated in [17] that the logistic source is an important factor to explain the numerical phenomenon observed in [9, 23]. We refer the readers to [17, 22, 24, 33] for studies in this direction and a brief review can be found in our previous work [10, 11]. We also refer to [25, 26, 34] for studies on the stationary problem with/without a source term.

Now we are in a position to state the main results of the current work. First, we give uniform-in-time boundedness for the two dimensional case. Note in previous work [1, 10, 11], $\gamma$ can decrease at most algebraically in $v$.

**Theorem 1.1** Assume $n = 2$. Suppose that $\gamma$ satisfies $(A_0)$ if $\varepsilon = 0$ and additionally $(A_1)$ if $\varepsilon > 0$. Moreover, suppose that

\[ (A_2') : \quad \lim_{s \to +\infty} e^{\alpha s} \gamma(s) = +\infty, \quad \text{for all } \alpha > 0. \quad (1.12) \]
Then problem (1.1) has a unique global classical solution which is uniformly-in-time bounded.

On the other hand, if there is $\chi > 0$ such that

\[(A2'') : \lim_{s \to +\infty} e^{\chi s} \gamma(s) = +\infty, \quad (1.13)\]

then the solution of (1.1) is uniformly-in-time bounded provided that $\|u_0\|_{L^1(\Omega)} < \frac{4\pi}{\chi}$.

**Remark 1.1** Recall that in [10, 11], we have proved that in the two dimensional setting, classical solution always exists globally provided that $\gamma$ satisfies (A0) if $\varepsilon = 0$ and additionally (A1) if $\varepsilon > 0$. In addition, it was proved in [10, 11, 17] that when $\gamma(s) = e^{-\xi s}$ for some fixed $\xi > 0$, the solution is uniformly-in-time bounded with any initial condition provided that $\|u_0\|_{L^1(\Omega)} < 4\pi/\xi$. Note in this case, (A2'') is satisfied with all $\chi > \xi$. Since for any given initial condition with $\|u_0\|_{L^1(\Omega)} < 4\pi/\xi$ one can always find some $\chi > \xi$, such that $\|u_0\|_{L^1(\Omega)} < 4\pi/\chi$ as well, due to the second part of Theorem 1.1, the solution must be uniformly bounded. Thus, Theorem 1.1 yields the same assertion as the boundedness results in [10, 11, 17] when $\gamma(s) = e^{-\xi s}$.

**Remark 1.2** When $\gamma(s) = e^{-\chi s}$ with some $\chi > 0$, a decomposition of the Laplacian part in the first equation yields that

\[u_t = \nabla \cdot (e^{-\chi v} (\nabla u - \chi u \nabla v)). \quad (1.14)\]

Under the circumstances, system (1.1) looks very like the minimal/classical Keller–Segel model but with an extra weight $e^{-\chi v}$. Moreover, as pointed out in our previous work [10, 11], they also share the same energy functional $E(u, v) = \int_\Omega (u \log u - uv + \frac{1}{2}(|\nabla v|^2 + v^2)) \, dx$ and similar dissipation terms (also with an extra weight $e^{-\chi v}$). Such an energy-dissipation relation plays a key role in studying the dynamic behavior of the solutions for both our system and the classical Keller–Segel model. For bounded domains, an applications of Moser-Trudinger’s inequality is crucial to get a lower bound for the energy functional with sub-critical mass [14, 28], while for the whole space this is achieved via an application of the logarithmic Hardy-Littlewood-Sobolev inequality [4, 8].

In the second part of Theorem 1.1, we only assume that $\gamma$ decays at an exponential rate at infinity and thus remove the requirement of the specific form $\gamma(s) = e^{-\xi s}$ in our previous work. Above mentioned techniques fail now since we do not have the energy-dissipation relation any more. We develop a new method relying on the duality estimate together with an application of the celebrated Brezis-Merle inequality to establish the boundedness with sub-critical mass. However, it is not clear whether there is infinite blowup under condition (A2'') when $\gamma(s)$ takes more general form other than $e^{-\chi s}$.

**Remark 1.3** Note that (A2') is weaker than (A2). For example, $\gamma(s) = e^{-\chi s^\beta}$ with any $\chi > 0$ and $0 < \beta < 1$ is excluded from (A2), but satisfies (A2'). In this regard, Theorem 1.1 partially indicates that in 2D the exponentially decay rate of $\gamma$ is critical for global boundedness of the classical solutions with large mass.

Next, we consider higher dimensional cases. For the parabolic-elliptic case $\varepsilon = 0$, we obtain that
Theorem 1.2 Assume \( n \geq 3 \) and \( \epsilon = 0 \). Suppose that \( \gamma \) satisfies (A0) and the following condition:

\[
(A3a) : \quad \sqrt{\frac{n}{2}} |\gamma'(s)|^2 < \gamma(s)\gamma''(s), \quad \forall s > 0. \tag{1.15}
\]

Then problem (1.1) has a unique global classical solution.

In addition, if \( \gamma \) satisfies (A1) and

\[
(A3u) : \quad l_0|\gamma'(s)|^2 \leq \gamma(s)\gamma''(s), \quad \text{with some } l_0 > \frac{n}{2} \text{ for all } s > 0, \tag{1.16}
\]

then the global solution is uniformly-in-time bounded.

Remark 1.4 We point out that conditions like (A3a) or (A3u) on \( \gamma \) tacitly requires that \( \gamma \) is convex. According to Lemma 3.9 below, if \( \gamma(\cdot) \) satisfies (A0), (A1) and (A3u), then it must fulfill assumption (A2) with some \( k < \frac{2}{n-2} \). In this regard, our uniform-in-time boundedness result covers those in [1] established for the special case \( \gamma(s) = s^{-k} \) with any \( 0 < k < \frac{2}{n-2} \).

We remark that a similar condition as (A3u) has been proposed in [32] and our result on uniform-in-time boundedness recovers those in [32] for the specific case \( \gamma(s) = s^{-k} \) by a different method.

On the other hand when \( \epsilon > 0 \), we prove the following boundedness result.

Theorem 1.3 Assume \( n \geq 3 \) and \( \epsilon > 0 \). Suppose that \( \gamma \) satisfies (A0) – (A1) and the following condition:

\[
(A3b) : \quad \left(1 + \left[\frac{n}{2}\right]\right)|\gamma'(s)|^2 \leq \gamma(s)\gamma''(s), \quad \forall s > 0, \tag{1.17}
\]

where \( \left[\frac{n}{2}\right] \) denotes the maximal integer less or equal to \( \frac{n}{2} \). Then problem (1.1) has a unique global classical solution, which is uniformly-in-time bounded.

Remark 1.5 For the sake of simplicity, we normalize all physical parameters except \( \epsilon \) and in the proof we take \( \epsilon = 1 \) for the case \( \epsilon > 0 \). But the statements of our results and assumptions (A3a), (A3u) and (A3b) are independent of the choice of parameters.

Remark 1.6 Since \( v_0 > 0 \) in \( \overline{\Omega} \), thanks to the strictly positive time-independent lower bound \( v_\ast \) of \( v \) for \( (x, t) \in \Omega \times [0, \infty) \) given in Lemma 2.4 and Lemma 2.5 in the next section, our existence and boundedness results also hold true if \( \gamma(s) \) has singularities at \( s = 0 \), for example \( \gamma(s) = s^{-k} \) with \( k > 0 \). In such cases, we can simply replace \( \gamma(s) \) by a new motility function \( \tilde{\gamma}(s) \) which satisfies (A0) and coincides with \( \gamma(s) \) for \( s \geq \frac{v_\ast}{2} \).

In particular, for the typical case \( \gamma(v) = v^{-k} \), we have

Corollary 1.1 Suppose that \( \gamma(v) = v^{-k} \) and \( n \geq 3 \). Then,

- when \( \epsilon = 0 \), problem (1.1) has a unique global classical solution provided that \( k < \frac{\sqrt{2n+2}}{n-2} \).
  In addition, the global solution is uniformly-in-time bounded if \( k < \frac{2}{n-2} \);
- when \( \epsilon > 0 \), problem (1.1) has a uniformly-in-time bounded global solution provided that \( k \leq 1/\left[\frac{n}{2}\right] \).
Now, let us sketch the main idea of our proof for boundedness in higher dimensions. First, it is necessary to briefly recall some related results in our work [10, 11]. Denote $w(x, t)$ the unique non-negative solution of the following Helmholtz equation:

$$\begin{cases}
-\Delta w + w = u, & x \in \Omega, \ t > 0, \\
\partial_n w = 0, & x \in \partial\Omega, \ t > 0.
\end{cases}$$

Then we found that ([10, Lemma 3.1] or [11, Lemma 4.1])

$$w_t + \gamma(v)u = (I - \Delta)^{-1}[\gamma(v)u],$$

(1.18)

which unveils the intrinsic mechanism of the nonlinear structure. Here, $(I - \Delta)^{-1}$ denotes the inverse operator of $I - \Delta$ and $\Delta$ is the Laplacian operator with homogeneous Neumann boundary condition. Using comparison principle of the elliptic equations together with Gronwall’s inequality, we proved from the above key identity that ([10, Lemma 3.2] or [11, Lemma 4.1])

$$w(x, t) \leq w_0(x)e^{Ct},$$

for all $x \in \Omega$ and $t \geq 0$,

where $w_0 \triangleq (I - \Delta)^{-1}u_0$ and $C > 0$ depends only on $\gamma, \Omega$ and the initial data. Note that in the parabolic-elliptic case, i.e., $\varepsilon = 0$ in (1.1), $w$ is identical to $v$. On the other hand when $\varepsilon > 0$, thanks to the above identity again, upon an application of the comparison principle for parabolic equations, we proved that ([11, Lemma 4.3])

$$v(x, t) \leq C(w(x, t) + 1)$$

for all $x \in \Omega$ and $t \geq 0$.

(1.19)

with $C > 0$ depending only on $\gamma, \Omega$ and the initial data. In a word, $v(x, t)$ can grow pointwisely at most exponentially in time in both cases.

In addition, under certain decay assumptions for example, $(A2')$ when $n = 2$, or $(A2)$ with some $k < \frac{2}{n-2}$ when $n \geq 3$, the above upper bound estimate can be further improved. Take $\varepsilon = 0$ and $n \geq 3$ for example and recall that $w = v$ in this case. In [10, 11], time-independent upper bounds of $v$ were proved directly when $n \leq 3$ by simple arguments based on an application of the uniform Gronwall inequalities. However, since we made use of the Sobolev embedding $H^2 \hookrightarrow L^\infty$ there, the technique fails in higher dimensions. In this work, observing that the key identity also reads ($\varepsilon = 0$)

$$v_t - \gamma(v)\Delta v + v\gamma(v) = (I - \Delta)^{-1}[\gamma(v)u],$$

(1.20)

we develop an alternative approach based on a delicate Alikakos–Moser type iteration to achieve the same goal in higher dimensions. More precisely, for any $n \geq 3$ we are able to prove that under the assumptions $(A0)$ and $(A2)$ with any $k < \frac{2}{n-2}$ when $\varepsilon > 0$, or additional $(A1)$ when $\varepsilon > 0$, $v$ has a time-independent upper bound; see Proposition 3.1. Note here the uniform-in-time upper bound of $v$ is obtained independently of $u$ under a much weaker decay rate assumption than that in [1].

In order to establish the global existence or time-independent boundedness, it remains to derive $L^\infty_t L^p_x$ (time-independent) boundedness of $u$ with some $p > \frac{n}{2}$ due to standard bootstrap argument. Here, the key idea is to construct an estimate for a weighted energy $\int_\Omega u^p v^q$ with some $p > \frac{n}{2}$ and $q > 0$. Since $v$ is bounded from above now, $\gamma(v)$ is bounded from below thanks to its decreasing property. Then $L^\infty_t L^p_x$ boundedness of $u$ follows from the boundedness of the above weighted energy. Adjusting the parameters $p, q$
carefully and using the key identity again, we are able to construct a new estimation involving the weighted energy which gives rise to the desired boundedness.

We remark that at the present stage, we cannot obtain boundedness results for the case $\varepsilon > 0$ under the same condition (A3u) as for the case $\varepsilon = 0$. The main obstacle comes from the different equations for $v_t$, where an additional diffusion coefficient $\gamma(v)$ in (1.20) helps to weaken the constraint when $\varepsilon = 0$. Besides, in the fully parabolic case $\varepsilon > 0$, we cannot simply adjust $p, q$ in a single estimation involving $\int_{\Omega} u^n \gamma^q(v)$ to get the desired result as done for the case $\varepsilon = 0$. A different strategy used here is to list out a system of estimations involving the weighted energies with $p = 2, 3, \ldots, 1 + \left\lceil \frac{n}{2} \right\rceil$ and $q = 0, 1, \ldots, p - 1$. Then by a careful recombination of such estimations and an iteration argument together with an application of the uniform Gronwall inequality, we prove the time-independent boundedness of the weighted energies.

Before concluding this part, we would like to stress some new features of the present work. Firstly, we improve the boundedness result with arbitrarily large initial data in dimension two, which partially indicates that the exponential decay case is critical for boundedness with large mass. We remark that it is still unknown whether the 2-D global classical solution would be bounded or blow up at time infinity with large initial data if $\gamma$ decays at a speed faster than exponential rate. Secondly, uniform boundedness for $v$ is independently proved provided that $\gamma$ satisfies (A2) with some $k < \frac{2}{n-2}$ when $n \geq 3$ by delicate iterations. For the case $\varepsilon = 0$, boundedness of $u$ is achieved under a slightly stronger assumption (A3u). Here, our work also provides an alternative proof for the result in [1] concerning the particularly chosen motility $\gamma(v) = v - k$ with any $k < \frac{2}{n-2}$. Alt is still challenging whether one can prove boundedness of $u$ under the same decay condition as $v$, or the slightly stronger one (A3u).

The rest of the paper is organized as follows. In Sect. 2, we provide some preliminary results and recall some useful lemmas. Then in Sect. 3 we use modified Alikakos–Moser iteration to derive the uniform-in-time upper bounds of $v$. In Sect. 4, we study the parabolic-elliptic case $\varepsilon = 0$ and establish the boundedness of weighted energy. In Sect. 5, we prove boundedness of weighted energy for the fully parabolic case.

2 Preliminaries

In this section, we recall some useful lemmas. First, local existence and uniqueness of classical solutions to system (1.1) can be established by the standard fixed point argument and regularity theory for elliptic/parabolic equations. Similar proof can be found in [1, Lemma 3.1] or [17, Lemma 2.1] and hence here we omit the detail here.

**Theorem 2.1** Let $\Omega$ be a smooth bounded domain of $\mathbb{R}^n$. Suppose that $\gamma(\cdot)$ satisfies (A0) and $(u_0, v_0)$ satisfies (1.5). Then there exists $T_{\text{max}} \in (0, \infty)$ such that problem (1.1) permits a unique non-negative classical solution $(u, v) \in (C^0(\bar{\Omega} \times [0, T_{\text{max}}]) \cap C^{2,1}(\bar{\Omega} \times (0, T_{\text{max}})))^2$.

Moreover, the following mass conservation holds

$$\int_{\Omega} u(\cdot, t) dx = \int_{\Omega} u_0 dx \quad \text{for all } t \in (0, T_{\text{max}}).$$

If $T_{\text{max}} < \infty$, then

$$\limsup_{t \searrow T_{\text{max}}} \|u(\cdot, t)\|_{L^\infty(\Omega)} = \infty.$$
In the same manner as to classical Keller–Segel systems, we can prove the following criterion (see e.g., [1, Lemma 4.3]).

**Lemma 2.1** For any \( p > \frac{n}{2} \), if the solution of (1.1) satisfies that
\[
\|u(\cdot, t)\|_{L^p(\Omega)} \leq C, \quad \text{for all } t \in (0, T_{\text{max}})
\] (2.1)
with some \( C > 0 \), then \( T_{\text{max}} = \infty \) and there holds
\[
\sup_{t > 0} \left( \|u(\cdot, t)\|_{L^\infty(\Omega)} + \|v(\cdot, t)\|_{W^{1, \infty}(\Omega)} \right) \leq C'
\] (2.2)
with some \( C' > 0 \). Moreover, if the above constant \( C > 0 \) is time-independent, then the global solution has a uniform-in-time bound as well.

Next, as done in our previous work [10, 11], we introduce an auxiliary variable \( w(x, t) \), which is the unique non-negative solution of the following Helmholtz equation:
\[
\begin{aligned}
-\Delta w + w &= u, \quad x \in \Omega, \quad t > 0, \\
\frac{\partial w}{\partial n} &= 0, \quad x \in \partial \Omega, \quad t > 0.
\end{aligned}
\] (2.3)

Now, we recall the following lemma given in [1] about estimates for the solution of Helmholtz equations. Let \( a_+ = \max\{a, 0\} \). Then we have

**Lemma 2.2** Let \( \Omega \) be a smooth bounded domain in \( \mathbb{R}^n, n \geq 1 \) and let \( f \in C(\bar{\Omega}) \) be a non-negative function such that \( \int_\Omega f \, dx > 0 \). If \( z \) is a \( C^2(\Omega) \) solution to
\[
\begin{aligned}
-\Delta z + z &= f, \quad x \in \Omega, \\
\frac{\partial z}{\partial n} &= 0, \quad x \in \partial \Omega,
\end{aligned}
\] (2.3)
then if \( 1 \leq q < \frac{n}{(n-2)_+} \), there exists a positive constant \( C = C(n, q, \Omega) \) such that
\[
\|z\|_{L^q(\Omega)} \leq C \|f\|_{L^1(\Omega)}. \] (2.4)

When \( n = 2 \), we need the following result given in [32, Lemma 3.3], which is similar to the celebrated Brezis–Merle inequality [5, Theorem 1], see also [29, Lemma A.3].

**Lemma 2.3** Let \( \Omega \subset \mathbb{R}^2 \) be a smooth bounded domain. For any \( f \in L^1(\Omega) \) such that
\[
\|f\|_{L^1(\Omega)} = \Lambda, \] (2.5)
with some \( \Lambda > 0 \), there is \( C > 0 \) such that the solution of (2.3) satisfies
\[
\int_\Omega e^{Az} \, dx \leq C \] (2.6)
for any \( 0 < A < \frac{4\pi}{\Lambda} \).

Besides, a strictly positive uniform-in-time lower bound for \( w = (I - \Delta)^{-1}[u](x, t) \) is given by the positivity of the Green function to the Helmholtz equation ([15]) and the mass conservation. See also [3, Lemma 3.3].
Lemma 2.4 Suppose \((u, v)\) is the classical solution of (1.1) up to the maximal time of existence \(T_{\text{max}} \in (0, \infty]\). Then, there exists a strictly positive constant \(w_\ast = w_\ast(n, \Omega, \|u_0\|_{L^1(\Omega)})\) such that for all \(t \in (0, T_{\text{max}})\), there holds

\[
\inf_{x \in \Omega} w(x, t) \geq w_\ast.
\]

Similarly, a strictly positive uniform-in-time lower bound for \(v\) was given in [12, Lemma 2.1] provided that \(v_0\) is strictly positive in \(\Omega\).

Lemma 2.5 Assume that \((u_0, v_0)\) satisfies (1.5). If \((u, v)\) is the solution of (1.1) in \(\Omega \times (0, T)\), then there exists some \(v_\ast > 0\) such that

\[
\inf_{x \in \Omega} v(x, t) \geq v_\ast > 0 \quad \text{for all } t \in (0, T).
\]

Here the constant \(v_\ast\) is independent of \(T > 0\).

By the comparison method developed in our previous work, we proved the following upper bounds for \(w\) and \(v\) (see [10, Lemma 3.1] and [11, Lemma 4.1, Lemma 4.3 & Remark 4.1]).

Lemma 2.6 Assume \(n \geq 1\) and suppose that \(\gamma\) satisfies (A0). For any \(0 < t < T_{\text{max}}\), there holds

\[
w_t + \gamma(v) u = (I - \Delta)^{-1}[\gamma(v) u].
\]

Moreover, for any \(x \in \Omega\) and \(t \in [0, T_{\text{max}}]\), we have

\[
w(x, t) \leq w_0(x)e^{\gamma(v_\ast)t}.
\]

Lemma 2.7 Assume that \(\varepsilon > 0\). Suppose \(\gamma\) satisfies (A0) and the following asymptotic property:

\[
(A1') : \lim_{s \to +\infty} \gamma(s) < 1/\varepsilon.
\]

Then there exist \(K > 0\) depending on \(\gamma, \varepsilon\) and the initial data and a generic constant \(\tilde{C} > 0\) independent of \(\gamma\) such that for all \((x, t) \in \Omega \times [0, T_{\text{max}}]\),

\[
v(x, t) \leq \tilde{C}\left(w(x, t) + K\right).
\]

Furthermore, if \(\gamma\) satisfies (A1) instead of (A1'), then \(\tilde{C}\) can be chosen as an arbitrary constant larger than 1.

Finally, we need the following uniform Gronwall inequality [31, Chapter III, Lemma 1.1] to deduce uniform-in-time estimates for the solutions.

Lemma 2.8 Let \(g, h, y\) be three positive locally integrable functions on \((t_0, \infty)\) such that \(y'\) is locally integrable on \((t_0, \infty)\) and the following inequalities are satisfied:

\[
y'(t) \leq g(t)y(t) + h(t) \quad \forall \ t \geq t_0,
\]
where \( r, a_i, (i = 1, 2, 3) \) are positive constants. Then

\[
y(t + r) \leq \left( \frac{a_3}{r} + a_2 \right) e^{a_1}, \quad \forall t \geq t_0.
\]

3 Time-Independent Upper Bounds of \( v \)

This section is devoted to the following uniform-in-time boundedness result of \( v \). In the two-dimensional case, the proof is based on a simple application of the 2D Sobolev embeddings together with the uniform Gronwall inequality while in higher dimensions, the boundedness is achieved via a modified Alikakos–Moser type iteration argument.

3.1 The Two-Dimensional Case

In two dimensions, it was proved in [10, 11] that global classical solution always exists provided that \( \gamma \) satisfies (A0) if \( \varepsilon = 0 \) and additionally (A1) if \( \varepsilon > 0 \). In order to establish the boundedness, we first prove the following result.

**Lemma 3.1** Under the same assumptions of Theorem 1.1, there is \( C > 0 \) depending only on the initial data, \( \gamma, \varepsilon \) and \( \Omega \) such that

\[
\sup_{t \geq 0} \left( \| \nabla w \|_{L^2(\Omega)}^2 + \| w \|_{L^2(\Omega)}^2 + \int_0^{t+1} \int_\Omega \gamma(v) u^2 dx ds \right) \leq C. \tag{3.1}
\]

**Proof** Multiplying the key identity (2.7) by \( u \) and recalling that \( w = (I - \Delta)^{-1}[u] \), we obtain

\[
\frac{1}{2} \frac{d}{dt} \left( \| \nabla w \|_{L^2(\Omega)}^2 + \| w \|_{L^2(\Omega)}^2 \right) + \int_\Omega \gamma(v) u^2 dx = \int_\Omega (I - \Delta)^{-1} [\gamma(v) u] u dx
\]

\[
= \int_\Omega \gamma(v) w dx
\]

\[
\leq \gamma(v^*) \int_\Omega w dx.
\]

On the other hand, by integrating by parts it follows

\[
\| \nabla w \|_{L^2(\Omega)}^2 + \| w \|_{L^2(\Omega)}^2 = \int_\Omega w udx.
\]

Combining the above inequalities, we have

\[
\frac{d}{dt} \left( \| \nabla w \|_{L^2(\Omega)}^2 + \| w \|_{L^2(\Omega)}^2 \right) + \| \nabla w \|_{L^2(\Omega)}^2 + \| w \|_{L^2(\Omega)}^2 + 2 \int_\Omega \gamma(v) u^2 dx
\]

\[
= (2\gamma(v^*) + 1) \int_\Omega w dx
\]

\[
\leq \int_\Omega \gamma(v) u^2 dx + \frac{(2\gamma(v^*) + 1)^2}{4} \int_\Omega \gamma^{-1}(v) w^2 dx.
\]
Thus we obtain that
\[
\frac{d}{dt} \left( \| \nabla w \|^2_{L^2(\Omega)} + \| w \|^2_{L^2(\Omega)} \right) + \| \nabla w \|^2_{L^2(\Omega)} + \| w \|^2_{L^2(\Omega)} + \int_{\Omega} \gamma(v) u^2 \, dx \\
\leq C \int_{\Omega} \gamma^{-1}(v) w^2 \, dx 
\]
with some \( C > 0 \). In view of our assumption (A2'), we may infer that for any \( b > 0, \alpha > 0 \), there exists \( s_b > v_a \) depending on \( \alpha \) and \( b \) such that for all \( s \geq s_b \)
\[ \gamma^{-1}(s) \leq be^{\alpha s} \]
and on the other hand, since \( \gamma(\cdot) \) is decreasing,
\[ \gamma^{-1}(s) \leq \gamma^{-1}(s_b) \]
for all \( 0 \leq s < s_b \). Therefore, for all \( s \geq 0 \), there holds
\[ \gamma^{-1}(s) \leq be^{\alpha s} + \gamma^{-1}(s_b). \]  
(3.3)

Thus, we deduce from above and Lemma 2.7 that
\[
\int_{\Omega} \gamma^{-1}(v) w^2 \, dx \leq \int_{\Omega} (be^{\alpha v} + \gamma^{-1}(s_b)) w^2 \, dx \\
\leq \int_{\Omega} \left( be^{\tilde{C} \alpha (w + K)} + \gamma^{-1}(s_b) \right) w^2 \, dx. 
\]  
(3.4)

Invoking Young’s inequality, we observe that
\[
\int_{\Omega} e^{\tilde{C} \alpha (w + K)} w^2 \, dx \leq e^{\tilde{C} K \alpha} \left( \int_{\Omega} e^{2 \tilde{C} \alpha w} \, dx \right)^{1/2} \left( \int_{\Omega} w^4 \, dx \right)^{1/4}, 
\]
and thus
\[
\int_{\Omega} \gamma^{-1}(v) w^2 \, dx \leq e^{\tilde{C} K \alpha} \left( \int_{\Omega} e^{2 \tilde{C} \alpha w} \, dx \right)^{1/2} \left( \int_{\Omega} w^4 \, dx \right)^{1/4} + \gamma^{-1}(s_b) \int_{\Omega} w^2 \, dx. 
\]

Here we apply Lemma 2.3 by taking \( \| u_0 \|_{L^1(\Omega)} = \Lambda \) and sufficiently small \( \alpha > 0 \) such that \( 2\tilde{C}\alpha < A \) (recall that \( \tilde{C} \) is independent of \( \gamma \)), and also invoke Lemma 2.2 to have
\[
\int_{\Omega} \gamma^{-1}(v) w^2 \, dx \leq C. 
\]
Combining (3.2) with the above estimate completes the proof by solving the above differential inequality. \( \square \)

**Remark 3.1** If \( \varepsilon = 0 \), we have
\[
\sup_{t \geq 0} \left( \| \nabla v \|^2_{L^2(\Omega)} + \| v \|^2_{L^2(\Omega)} + \int_t^{t+1} \int_{\Omega} \gamma(v) u^2 \, dx \, ds \right) \leq C. 
\]  
(3.5)
Remark 3.2 If there is \( 0 < \chi < \frac{4\pi}{\Lambda} \) with \( \|u_0\|_{L^1(\Omega)} = \Lambda \) such that

\[
\text{(A2') : } \lim_{s \to +\infty} e^{\chi s} \gamma(s) = +\infty,
\]

we can argue in the same manner as before to deduce that

\[
\int_\Omega \gamma^{-1}(v) w^2 \leq b \int_\Omega e^{C_\chi(w+K)} w^2 + \int_\Omega (s_b) w^2.
\]

Since \( \lim_{s \to 0} \gamma(s) = 0 \), by Lemma 2.7, \( \tilde{C} > 1 \) above can be chosen arbitrarily close to 1 such that

\[
\tilde{C}_\chi < \frac{4\pi}{\Lambda}.
\]

Moreover, we may fix some \( p > 1 \) such that

\[
\tilde{C}_p \chi < \frac{4\pi}{\Lambda}.
\]

Thus, thanks to Young’s inequality, Lemma 2.2 and Lemma 2.3, we infer that

\[
\int_\Omega e^{\tilde{C}_\chi(w+K)} w^2 \leq e^{\tilde{C}_K \chi} \left( \int_\Omega w^2 \right)^{1/p} \left( \int_\Omega \gamma(v) \right)^{1-2p/p} \leq C
\]

where \( 1/p + 1/p' = 1 \). Thus, if \( \Lambda < \frac{4\pi}{\chi} \), there also holds

\[
\sup_{t \geq 0} \left( \|\nabla w\|_{L^2(\Omega)} + \|w\|_{L^2(\Omega)} + \int_t^{t+1} \int_\Omega \gamma(v) u^2 dx ds \right) \leq C. \tag{3.6}
\]

**Lemma 3.2** **Under the assumption of Theorem 1.1, we have**

\[
\sup_{t \geq 0} \left( \|w\|_{L^\infty(\Omega)} + \|v\|_{L^\infty(\Omega)} \right) \leq C. \tag{3.7}
\]

**Proof** Recall that \( w - \Delta w = u \). For any fixed \( 1 < p < 2 \), we infer by the Sobolev embedding theorem and (3.3) that

\[
\|w\|_{L^\infty(\Omega)} \leq C \|u\|_{L^p(\Omega)}
\]

\[
\leq \left( \int \Omega u^2 \gamma(v) dx \right)^{1/2} \left( \int \Omega (\gamma(v))^{-\frac{p}{2-p}} dx \right)^{\frac{2-p}{2p}}
\]

\[
\leq C \left( \int \Omega u^2 \gamma(v) dx \right)^{1/2} \left( \int \Omega \left( be^{\tilde{C}_\alpha (w+K)} \gamma^{-1}(s_b) \right)^{\frac{p}{2-p}} dx \right)^{\frac{2-p}{2p}}.
\]

Picking \( \alpha > 0 \) small such that \( \frac{\tilde{C}_p \alpha}{2-p} < A \), we deduce by Lemma 2.3 that

\[
\|w\|_{L^\infty(\Omega)} \leq C \left( \int \Omega u^2 \gamma(v) dx \right)^{1/2}. \tag{3.8}
\]

\[\text{Springer} \]
Then by Lemma 3.1, for any \( t > 0 \) we obtain that

\[
\int_t^{t+1} \| w \|_{L^\infty(\Omega)} ds \leq \int_t^{t+1} \int_\Omega u^2 \gamma(v) dx + C \leq C
\]

and thus for any fixed \( x \in \Omega \),

\[
\sup_{t > 0} \int_t^{t+1} w(s, x) ds \leq C
\]

with \( C > 0 \) depending only on the initial data, \( \gamma \) and \( \Omega \). Finally, observing that

\[
w_t + u \gamma(v) = (I - \Delta)^{-1}[u \gamma(v)] \leq \gamma(v_*)(I - \Delta)^{-1}[u] = \gamma(v_*)w
\]

we may apply the uniform Gronwall inequality Lemma 2.8 to obtain that for any \( x \in \Omega \)

\[
w(x, t) \leq C \quad \text{for } t \geq 1,
\]

with some \( C > 0 \) independent of \( x \in \Omega \), which together with Lemma 2.6 for \( t \leq 1 \) gives rise to the following estimate

\[
w(x, t) \leq C \quad \text{for } t \geq 0.
\]

Finally, recall Lemma 2.7, we also have

\[
v(x, t) \leq C(w(x, t) + 1) \leq C \quad \text{for } t \geq 0,
\]

which concludes the proof. \( \square \)

**Remark 3.3** Under the same assumption of Remark 3.2, we can choose some \( 1 < p < 2 \) such that \( \frac{p}{2-p} \) larger than 1 but sufficiently close to 1 such that

\[
\tilde{C} \frac{p \chi}{2 - p} < \frac{4\pi}{\Lambda}.
\]

Then, in the same manner as before, we can still deduce that

\[
\| w \|_{L^\infty(\Omega)} \leq C \| u \|_{L^p(\Omega)}
\]

\[
\leq \left( \int_\Omega u^2 \gamma(v) dx \right)^{\frac{1}{2}} \left( \int_\Omega (\gamma(v))^{-\frac{p}{2-p}} dx \right)^{\frac{2-p}{2}}
\]

\[
\leq C \left( \int_\Omega u^2 \gamma(v) dx \right)^{\frac{1}{2}} \left( \int_\Omega (be \tilde{\chi}(w + K) + \gamma^{-1}(s_\theta))^{\frac{p}{2-p}} dx \right)^{\frac{2-p}{2}}
\]

\[
\leq C \left( \int_\Omega u^2 \gamma(v) dx \right)^{\frac{1}{2}}.
\]

Then, we can similarly prove that

\[
\sup_{t \geq 0} \left( \| w \|_{L^\infty(\Omega)} + \| v \|_{L^\infty(\Omega)} \right) \leq C.
\]
Once \( v \) is uniformly-in-time bounded from above, we can prove in the same manner as in [10, 11] to get the uniform boundedness of the classical solutions and thus Theorem 1.1 is proved. We omit the detail here.

### 3.2 Higher-Dimensional Cases

In this part, we aim to establish uniform-in-time upper bound for \( v \) in higher dimensions when \( \gamma \) decreases algebraically at large concentrations.

**Proposition 3.1** Assume \( n \geq 3 \). Suppose \( \gamma \) satisfies (A0) and (A2) with some \( 0 < k < \frac{2}{n - 2} \) when \( \varepsilon = 0 \), and \( \gamma \) satisfies (A1') additionally when \( \varepsilon > 0 \). Then there is \( C > 0 \) depending only on the initial data, \( \gamma \), \( \varepsilon \) and \( \Omega \) such that

\[
\sup_{0 \leq t < T_{\text{max}}} \| v(\cdot, t) \|_{L^\infty(\Omega)} \leq C.
\]  

(3.9)

The proof of the above result consists of several steps. To begin with, we prove the following time-independent estimates.

**Lemma 3.3** Under the same assumptions of Proposition 3.1, there is \( C > 0 \) depending only on the initial data, \( \gamma \), \( \varepsilon \) and \( \Omega \) such that

\[
\sup_{0 \leq t < T_{\text{max}}} \left( \| \nabla w \|_{L^2(\Omega)}^2 + \| w \|_{L^2(\Omega)}^2 \right) \leq C,
\]  

(3.10)

and for any \( t \in (0, T_{\text{max}} - \tau) \) with \( \tau = \min\{1, \frac{1}{2} T_{\text{max}}\} \),

\[
\int_t^{t+\tau} \int_\Omega \gamma(v) u^2 dx ds \leq C.
\]  

(3.11)

**Proof** Proceeding the same lines as in Lemma 3.1, we arrive at

\[
\frac{d}{dt} \left( \| \nabla w \|_{L^2(\Omega)}^2 + \| w \|_{L^2(\Omega)}^2 \right) + \| \nabla w \|_{L^2(\Omega)}^2 + \| w \|_{L^2(\Omega)}^2 + \int_\Omega \gamma(v) u^2 dx
\]

\[
\leq C \int_\Omega \gamma^{-1}(v) w^2 dx,
\]  

(3.12)

and we will estimate the right-hand side. Under the assumption of Proposition 3.1, we may infer that there exist \( k \in (0, \frac{2}{n - 2}) \), \( b > 0 \) and \( s_b > v_* \) such that for all \( s \geq s_b \)

\[
\gamma^{-1}(s) \leq bs^k
\]

and on the other hand, since \( \gamma(\cdot) \) is decreasing,

\[
\gamma^{-1}(s) \leq \gamma^{-1}(s_b)
\]

for all \( 0 \leq s < s_b \). Therefore, for all \( s \geq 0 \), there holds

\[
\gamma^{-1}(s) \leq bs^k + \gamma^{-1}(s_b).
\]  

(3.13)
Thus, we deduce from above and Lemma 2.7 that
\[ \int_{\Omega} \gamma^{-1}(v) w^2 \, dx \leq \int_{\Omega} (b v^k + \gamma^{-1}(s_b)) w^2 \, dx \]
\[ \leq \int_{\Omega} \left( b (C(w + 1))^k + \gamma^{-1}(s_b) \right) w^2 \, dx \]
\[ \leq C \int_{\Omega} w^{k+2} \, dx + C \]
with \( C > 0 \) depending only on the initial data, \( \gamma \), \( \varepsilon \) and \( \Omega \).

Recall that \( \|w\|_{L^q(\Omega)} \) with any \( q \in \left[ 1, \frac{n}{n-2} \right) \) is bounded due to Lemma 2.2. Thus if \( k + 2 < \frac{n}{n-2} \), which only occurs when \( n = 3 \) and \( 0 < k < 1 \), there holds
\[ \int_{\Omega} w^{k+2} \, dx \leq C. \]

On the other hand, if \( 1 \leq k < 2 \) when \( n = 3 \) or \( 0 < k < \frac{2}{n-2} \) when \( n \geq 4 \), we can check
\[ \frac{n}{n-2} \leq k + 2 \leq q_*, \quad \frac{nk}{2} < \frac{n}{n-2}, \]
with \( q_* \triangleq \frac{2n}{n-2} \). Here we can pick up \( q \geq 1 \) satisfying
\[ \frac{nk}{2} < q < \frac{n}{n-2}, \]
and make use of the interpolation inequality and the Sobolev embedding \( H^1 \hookrightarrow L^{q_*} \) to have
\[ \int_{\Omega} w^{k+2} \, dx \leq \|w\|_{L^{q_*}(\Omega)}^{\beta_1(k+2)} \|w\|_{L^q(\Omega)}^{(1-\beta_1)(k+2)} \leq C \|w\|_{H^1(\Omega)}^{\beta_1(k+2)} \]
with some \( C > 0 \) depending only on \( n, \Omega \) and the initial data, and
\[ \beta_1 = \left( \frac{1}{q} - \frac{1}{k+2} \right)/(\frac{1}{q} - \frac{1}{q_*}). \]

Moreover, we can easily confirm that
\[ 0 \leq \beta_1(k+2) < 2 \]
due to \( \frac{nk}{2} < q \). By invoking Young’s inequality we arrive at
\[ \int_{\Omega} \gamma^{-1}(v) w^2 \, dx \leq C \int_{\Omega} w^{k+2} \, dx + C \leq \frac{1}{2} \|w\|_{H^1(\Omega)}^2 + C. \quad (3.14) \]

In summary, for \( n \geq 3 \) and \( 0 < k < \frac{2}{n-2} \), (3.12) and (3.14) implies
\[ \frac{d}{dt} \left( \|\nabla w\|_{L^2(\Omega)}^2 + \|w\|_{L^2(\Omega)}^2 \right) + \frac{1}{2} \left( \|\nabla w\|_{L^2(\Omega)}^2 + \|w\|_{L^2(\Omega)}^2 \right) + \frac{1}{2} \int_{\Omega} \gamma(v) u^2 \, dx \leq C, \quad (3.15) \]
where \( C > 0 \) depends only on the initial data, \( \gamma \), \( \varepsilon \) and \( \Omega \). Then a direct ODE analysis will finally yield to our assertion. This completes the proof. \( \square \)
With the above result, we can establish the uniform-in-time upper bounds of \( v \). For comparison, we first provide a simple proof in the same spirit as given in Sect. 3.1 which relies on an application of the uniform Gronwall inequality and the three-dimensional embeddings.

**Lemma 3.4** Assume \( n = 3 \) and suppose \( \gamma \) satisfies (A0) and (A2) with some \( 0 < k < 2 \). Moreover, \( \gamma \) satisfies (A1’) additionally when \( \varepsilon > 0 \). There is \( C > 0 \) depending only on \( \Omega, k, \varepsilon \) and the initial data such that

\[
\sup_{0 \leq t < T_{\max}} \left( \| w \|_{L^\infty(\Omega)} + \| v \|_{L^\infty(\Omega)} \right) \leq C. \tag{3.16}
\]

**Proof** For any \( \frac{3}{2} < p < 2 \), due to the three-dimensional Sobolev embedding theorem and Hölder’s inequality, we have

\[
\| w \|_{L^\infty(\Omega)} \leq C \| u \|_{L^p(\Omega)}
\leq C \left( \int_{\Omega} \gamma(v) u^2 \, dx \right)^{1/2} \left( \int_{\Omega} (\gamma(v))^{-\frac{p}{2-p}} \, dx \right)^{\frac{2-p}{2}}.
\]

In the same manner as before, we infer that

\[
\int_{\Omega} (\gamma(v))^{-\frac{p}{2-p}} \, dx \leq \int_{\Omega} \left( b v^k + \gamma^{-1}(s_b) \right)^{\frac{p}{2-p}} \, dx
\leq \int_{\Omega} \left( b (C(1 + w))^{k} + \gamma^{-1}(s_b) \right)^{\frac{p}{2-p}} \, dx \tag{3.17}
\leq C \int_{\Omega} w^{\frac{pk}{2-p}} \, dx + C,
\]

where \( C > 0 \) depends only on the initial data, \( \gamma, \varepsilon \) and \( \Omega \).

Since \( 0 < k < 2 \), we can always pick \( \frac{3}{2} < p < 2 \) such that \( \frac{pk}{2-p} \leq 6 \) and hence by the three-dimensional Sobolev embeddings and Lemma 3.3,

\[
\left( \int_{\Omega} (\gamma(v))^{-\frac{p}{2-p}} \, dx \right)^{\frac{2-p}{2}} \leq C \left( \int_{\Omega} w^{\frac{pk}{2-p}} \, dx \right)^{\frac{2-p}{2}} + C \leq C\| w \|_{H^1(\Omega)}^{\frac{1}{2}} + C \leq C.
\]

As a result, invoking Lemma 3.3 again, for any \( t \in (0, T_{\max} - \tau) \) with \( \tau = \min\{1, \frac{1}{2} T_{\max}\} \),

\[
\int_{t}^{t+\tau} \| w \|_{L^\infty(\Omega)} \leq C \int_{t}^{t+\tau} \int_{\Omega} \gamma(v) u^2 \, dx \, ds + C \leq C. \tag{3.18}
\]

It follows that for any fixed \( x \in \Omega \) and any \( t \in (0, T_{\max} - \tau) \) with \( \tau = \min\{1, \frac{1}{2} T_{\max}\},

\[
\int_{t}^{t+\tau} w(x, s) \, ds \leq \int_{t}^{t+\tau} \| w \|_{L^\infty(\Omega)} \leq C. \tag{3.19}
\]

Then, we recall the key identity (2.7) and deduce by the comparison principle of elliptic equations that

\[
w_t + \gamma(v) u = (I - \Delta)^{-1} [\gamma(v) u] \leq (I - \Delta)^{-1} [\gamma(v) u] = \gamma(v) w.
\]
Since \( u\gamma(v) \geq 0 \), with the aid of the uniform Gronwall inequality (Lemma 2.8), we infer for any \( x \in \Omega \) and \( t \in (\tau, T_{max}) \) that

\[
w(x, t) \leq C
\]

with some \( C > 0 \) independent of \( x, t \) and \( T_{max} \) which together with Lemma 2.6 for \( t \leq \tau \) gives rise to the uniform-in-time boundedness of \( w \) such that for all \( (x, t) \in \Omega \times [0, T_{max}) \),

\[
w(x, t) \leq C.
\]

This completes the proof in view of Lemma 2.7. \( \Box \)

For higher dimensions, the preceding argument fails. We provide the following alternative proof which is based on a modified Alikakos–Moser iteration \cite{2}. First, we begin with the case \( \varepsilon = 0 \) and keep in mind that in such case \( w \) is identical to \( v \).

\textbf{Lemma 3.5} Assume that \( n \geq 3 \) and \( \varepsilon = 0 \). Suppose \( \gamma \) satisfies (A0) and (A2) with some \( 0 < k < \frac{2}{n-2} \). There is \( C > 0 \) depending only on \( \Omega \), \( k \) and the initial data such that

\[
\sup_{0 \leq t < T_{max}} \|v\|_{L^\infty(\Omega)} \leq C. \tag{3.20}
\]

We prepare the following auxiliary lemmas.

\textbf{Lemma 3.6} Assume that \( n \geq 3 \) and \( \varepsilon = 0 \). Suppose \( \gamma \) satisfies (A0) and (A2) with some \( 0 < k < \frac{2}{n-2} \). There exist some \( \lambda_1 > 0 \) and \( \lambda_2 > 0 \) such that for any \( p > \frac{n}{n-2} \),

\[
\frac{d}{dt} \int_{\Omega} v^p + \lambda_2 p \int_{\Omega} v^p + \lambda_1 p (p - k - 1) \int_{\Omega} |\nabla v|^{p-k} \int_{\Omega} v^p \leq 2 \lambda_2 p \int_{\Omega} v^p. \tag{3.21}
\]

\textbf{Proof} Let \( p > \frac{q^*}{2} \) with \( q^* = \frac{2n}{n-2} \). Multiplying the key identity (2.7) by \( v^{p-1} \), we obtain that

\[
\frac{1}{p} \frac{d}{dt} \int_{\Omega} v^p + \int_{\Omega} u\gamma(v) v^{p-1} = \int_{\Omega} (I - \Delta)^{-1} [u\gamma(v)] v^{p-1} \leq \gamma(v^*_u) \int_{\Omega} v^p. \tag{3.22}
\]

Thanks to (3.13), it follows that

\[
\int_{\Omega} u\gamma(v) v^{p-1} dx \geq \int_{\Omega} u \left( b v^k + \gamma^{-1}(s_h) \right)^{-1} v^{p-1} dx \geq C \int_{\Omega} (v^k + 1)^{-1} v^{p-1} u dx
\]

with \( C > 0 \) independent of \( p \) and time. Since \( v^k \geq v^*_u \) by Lemma 2.5, there holds

\[
(v^k + 1)^{-1} v^{p-1} \geq (v^k + v^*_u^{-1} v^k)^{-1} v^{p-1} = \frac{v^{p-k-1}}{1 + v^*_u^{-k}} \tag{3.23}
\]

from which we deduce that

\[
\int_{\Omega} u\gamma(v) v^{p-1} dx \geq C \int_{\Omega} v^{p-k-1} u dx \tag{3.24}
\]
where $C > 0$ depends on the initial data, $\Omega$ and $\gamma$, but is independent of $p$ and time.

Next, recalling that $v - \Delta v = u$, we observe that

$$
\int_\Omega v^{p-k-1}udx = \int_\Omega v^{p-k-1}(v - \Delta v)dx
= \int_\Omega v^{p-k}dx + (p - k - 1) \int_\Omega |\nabla v|^{2}v^{p-k-2}
= \int_\Omega v^{p-k}dx + \frac{4(p - k - 1)}{(p-k)^2} \int_\Omega |\nabla v^{p-k}|^2.
$$

Therefore, we arrive at

$$
\frac{d}{dt} \int_\Omega v^p + \frac{\lambda_1 p(p - k - 1)}{(p-k)^2} \int_\Omega |\nabla v^{p-k}|^2 + \lambda_1 p \int_\Omega v^{p-k} \leq \lambda_2 p \int_\Omega v^p
$$

(3.25)

with some $\lambda_1, \lambda_2 > 0$ independent of $p$ and time. Adding $\lambda_2 p \int_\Omega v^p$, we complete the proof. □

**Lemma 3.7** Assume that $n \geq 3$ and $\varepsilon = 0$. Suppose $\gamma$ satisfies (A0) and (A2) with some $0 < k < \frac{2}{n-2}$. Let $L > 1$. There exists $C_0 > 0$ depending only on the initial data, $\Omega$, $k$ and $n$ such that for any $p > q \geq \frac{n}{n-2}$ satisfying

$$
q < p = 2q - \frac{nk}{2},
$$

there holds

$$
\frac{d}{dt} \int_\Omega v^p + \lambda_2 p \int_\Omega v^p \leq C_0 L^{\frac{n}{2}} p^{\frac{n+2}{2}} \left( \int_\Omega v^q \right)^2.
$$

**Proof** Let $p > q \geq \frac{q_*}{2}$ satisfying

$$
q < p = 2q - \frac{nk}{2} = 2q - \frac{kq_*}{q_* - 2}.
$$

Denote $\eta = v^{\frac{p-k}{2}}$ and define

$$
\alpha = \frac{(p-k)(p-q)}{p(p-k-2q/q_*)}.
$$

(3.26)

One easily checks that $\alpha \in (0, 1)$. Indeed,

$$
p - k - \frac{2q}{q_*} = q - \frac{2q}{q_*} - k = \frac{q_* - 2}{q_*} q - k \\
\geq \frac{q_* - 2}{q_*} \frac{q_*}{2} - k = \frac{2}{n-2} - k > 0
$$

and on the other hand, solving $\alpha < 1$ yields $p > \frac{kq_*}{q_* - 2}$, which is guaranteed by $p > q_*/2$ since $\frac{q_*}{2} > \frac{kq_*}{q_* - 2}$ and $k < \frac{2}{n-2}$. Moreover, since $k < \frac{2}{n-2}$, there holds $\frac{2pq}{p-k} < 2$ provided that...
Recall the Sobolev embedding inequality
\[
\| \eta \|_{L^{q_*}(\Omega)} \leq \lambda_* \| \eta \|_{H^1(\Omega)}
\]
where \( q_* = \frac{2n}{n-2} \) and \( \lambda_* > 0 \) depends only on \( n \) and \( \Omega \). In view of the fact \( \frac{2pa}{p-k} < 2 \), invoking Young’s inequality, we obtain that

\[
\lambda_2 p \int_{\Omega} v^p \, dx \leq \lambda_2 p \| \eta \|_{L^{q_*}(\Omega)}^{\frac{2pa}{p-k}} \left( \int_{\Omega} v^q \right)^{\frac{2pa}{p-k} \frac{(p-k)q_* - 2pa}{(p-k)q_*}}
\]

\[
\leq \lambda_2 p \left( \lambda_* \| \eta \|_{H^1(\Omega)} \right)^{\frac{2pa}{p-k}} \left( \int_{\Omega} v^q \right)^{\frac{2pa}{p-k} \frac{(p-k)q_* - 2pa}{(p-k)q_*}}
\]

\[
\leq \frac{p\alpha \delta}{p-k} \| \eta \|_{H^1(\Omega)}^2 + \frac{p-k - p\alpha}{p-k} \lambda_*^{\frac{2pa}{p-k} \frac{(p-k)q_* - 2pa}{(p-k)q_*}} \left( \frac{\lambda_2 p}{p-k} \right)^{\frac{2pa}{p-k} \frac{(p-k)q_* - 2pa}{(p-k)q_*}} \left( \int_{\Omega} v^q \right)^{\frac{2pa}{p-k} \frac{(p-k)q_* - 2pa}{(p-k)q_*}},
\]

where \( \delta > 0 \) satisfies

\[
\frac{p\alpha \delta}{p-k} = \frac{\lambda_1 p(p-k-1)}{2L(p-k)^2}.
\]

It follows from above and (3.26) that

\[
\frac{p-k - p\alpha}{p-k} \lambda_*^{\frac{2pa}{p-k} \frac{(p-k)q_* - 2pa}{(p-k)q_*}} \left( \frac{\lambda_2 p}{p-k} \right)^{\frac{2pa}{p-k} \frac{(p-k)q_* - 2pa}{(p-k)q_*}} \left( \int_{\Omega} v^q \right)^{\frac{2pa}{p-k} \frac{(p-k)q_* - 2pa}{(p-k)q_*}}
\]

\[
= \frac{p-k - p\alpha}{p-k} \left( \frac{2L(p-k)\lambda_*}{\lambda_1(p-k-1)} \right)^{\frac{2pa}{p-k} \frac{(p-k)q_* - 2pa}{(p-k)q_*}} \left( \frac{\lambda_2 p}{p-k} \right)^{\frac{2pa}{p-k} \frac{(p-k)q_* - 2pa}{(p-k)q_*}} \left( \int_{\Omega} v^q \right)^{\frac{2pa}{p-k} \frac{(p-k)q_* - 2pa}{(p-k)q_*}}
\]

\[
= \frac{(q_* - 2)q - kq_*}{q_* (p-k) - 2q} \left( \frac{2L(p-k)^2(p-q)\lambda_*^2}{\lambda_1(p-k-1)(p-k-2q/q_*)} \right)^{\frac{2pa}{p-k} \frac{(p-k)q_* - 2pa}{(p-k)q_*}} \left( \frac{\lambda_2 p}{q_* (p-k) - 2q} \right)^{\frac{2pa}{p-k} \frac{(p-k)q_* - 2pa}{(p-k)q_*}} \times \left( \int_{\Omega} v^q \right)^{\frac{2pa}{p-k} \frac{(p-k)q_* - 2pa}{(p-k)q_*}}.
\]
Since \( p > q \geq \frac{n}{2} \) satisfying
\[
q < p = 2q - \frac{nk}{2} = 2q - \frac{2kq_\ast}{q_\ast - 2},
\]
one easily checks that
\[
\frac{q_\ast(p - k) - 2p}{q_\ast(q - k) - 2q} = 2,
\]
\[
\frac{(q_\ast - 2)q - kq_\ast}{q_\ast(p - k) - 2q} = \frac{2}{n + 2},
\]
\[
\frac{(p - q)q_\ast}{q(q_\ast - 2) - kq_\ast} = \frac{n}{2},
\]
\[
\frac{q_\ast(p - k) - 2q}{(q_\ast - 2)q - kq_\ast} = \frac{n + 2}{2},
\]
and
\[
\frac{p - q}{p - k - 2q/q_\ast} = \frac{n}{n + 2}.
\]
Moreover, since \( p > \frac{n}{2} > 1 \),
\[
\frac{(p - k)^2}{p(p - k - 1)} = \frac{(p - k - 1)^2 + 2(p - k - 1) + 1}{p(p - k - 1)}
= \frac{p - k - 1}{p} + \frac{2}{p} + \frac{1}{p(p - k - 1)}
< 3 + \frac{1}{p - k - 1}
< 3 + \frac{1}{n - 2 - k},
\]
and
\[
\frac{(p - k)^2}{p(p - k - 1)} > \frac{p - k}{p} = 1 - \frac{k}{p} > 1 - \frac{2k}{q_\ast} = 1 - \frac{(n - 2)k}{n} > 0. \tag{3.29}
\]
Therefore by the above calculations, it follows
\[
\frac{2L(p - k)^2(p - q)\lambda^2_{\ast\ast}}{\lambda_1 p(p - k - 1)(p - k - 2q/q_\ast)} \leq \frac{2L\lambda^2_{\ast\ast}}{\lambda_1} \cdot \frac{n}{n + 2} \cdot \frac{(p - k)^2}{p(p - k - 1)}
< \frac{2Ln\lambda^2_{\ast\ast}}{\lambda_1(n + 2)} \left(3 + \frac{1}{n - 2 - k}\right).
\]
Hence one can find \( C_0 > 0 \) being a constant depending only on the initial data, \( \Omega, k \) and \( n \) such that
\[
\frac{(q_\ast - 2)q - kq_\ast}{q_\ast(p - k) - 2q} \left(\frac{2L(p - k)^2(p - q)\lambda^2_{\ast\ast}}{\lambda_1 p(p - k - 1)(p - k - 2q/q_\ast)}\right)^{\frac{(p - q)q_\ast}{q(q_\ast - 2) - kq_\ast}} \lambda^2_{\ast\ast} \left(\frac{q_\ast(p - k) - 2q}{(p - q)q_\ast - kq_\ast}\right)^{\frac{(p - q)q_\ast}{q(q_\ast - 2) - kq_\ast}}.
\]
\[
\begin{align*}
&\leq \frac{2}{n+2} \cdot \left\{ \frac{2L \ln \lambda_{\infty}^2}{\lambda_1(n+2)} \left( 3 + \frac{1}{\frac{n}{n-2} - k} \right) \right\} \left( \lambda_2 p \right)^{\frac{n+2}{2}} \\
&\leq \frac{C_0}{2} L^2 p^{\frac{n+2}{2}}.
\end{align*}
\]
Therefore by the above and (3.27) we have
\[
2\lambda_2 p \int_{\Omega} v^{p} dx \leq \frac{\lambda_1 p(p-k-1)}{L(p-k)^2} \| v \|_{H^1(\Omega)}^2 + C_0 L^\frac{n}{2} p^{\frac{n+2}{2}} \left( \int_{\Omega} v^q \right)^2.
\]
Combining Lemma 3.6 and recalling \( L > 1 \), we obtain the following inequality
\[
\frac{d}{dt} \int_{\Omega} v^{p} + \lambda_2 p \int_{\Omega} v^{p} \leq C_0 L^\frac{n}{2} p^{\frac{n+2}{2}} \left( \int_{\Omega} v^q \right)^2.
\]
Now we are in a position to give a proof of Lemma 3.5.

**Proof** For all \( r \in \mathbb{N} \) we define
\[
p_r \triangleq 2^{r-1} (q_* - nk) + \frac{nk}{2}, \quad p_0 = q_*/2.
\]
Then \( p_r > \frac{n}{n-2} \) and \( p_r = 2p_{r-1} - \frac{nk}{2} \). We apply Lemma 3.7 with \((p, q) = (p_r, p_{r-1})\) to have
\[
\frac{d}{dt} \int_{\Omega} v^{p_r} + \lambda_2 p_r \int_{\Omega} v^{p_r} \leq \lambda_2 p_r A_r (M_{r-1})^2,
\]
where
\[
M_r \triangleq \sup_{0 \leq t < T_{\max}} \int_{\Omega} v^{p_r} \quad \text{and} \quad A_r \triangleq \frac{C_0 L^\frac{n}{2} p_r^{\frac{q}{2}}}{\lambda_2}.
\]
By solving the above ODE, it follows that for all \( r \in \mathbb{N} \)
\[
M_r = \sup_{0 \leq t < T_{\max}} \int_{\Omega} v^{p_r} \leq \max \{ A_r M_{r-1}^2, \| v_0 \|_{L^\infty(\Omega)}^{p_r} \}.
\]
Since \( p_r \geq q_* / 2 \) for all \( r \geq 1 \), one can choose \( L > 1 \) sufficiently large depending only on the initial data, \( \Omega, n \) and \( k \) such that \( A_r > 1 \) for all \( r \geq 1 \). Moreover, adjusting \( C_0 \) by a proper larger number, we have
\[
A_r \leq C_0 a^r
\]
with some \( a > 0 \) depending only on the initial data, \( \Omega, k \) and \( n \). In addition, due to Lemma 3.3, we may find some large constant \( K_0 > 1 \) that dominates \( \| v_0 \|_{L^\infty} \) and \( \int_{\Omega} v^{q_*/2} \) for all time.
Iteratively, we deduce that
\[
\int_{\Omega} v^{p_r} \leq \max\{\mathcal{A}_r\mathcal{A}_r^2, \mathcal{A}_r K_0^{2p_r-1}, K_0^{p_r}\} \\
= \max\{\mathcal{A}_r\mathcal{A}_r^2, \mathcal{A}_r K_0^{2p_r-1}\} \\
\leq \cdots \\
\leq \max\{\mathcal{A}_r\mathcal{A}_r^2, \mathcal{A}_r^2\mathcal{A}_r^2, \mathcal{A}_r^2 \mathcal{A}_r^2 \cdots \mathcal{A}_r^2 \mathcal{A}_r^2 \mathcal{A}_r^2 \mathcal{A}_r^2 K_0^{2p_r-1} p_1\} \\
\leq \max\{\mathcal{A}_r\mathcal{A}_r^2, \mathcal{A}_r^2\mathcal{A}_r^2, \mathcal{A}_r^2 \mathcal{A}_r^2 \cdots \mathcal{A}_r^2 \mathcal{A}_r^2 \mathcal{A}_r^2 \mathcal{A}_r^2 K_0^{2p_r-1} p_1\} \\
\leq C_0^{2^0+1+\cdots+2^r-1} \times a^{1r+2(r-1)+2^2(r-2)+\cdots+2^{r-1}(r-(r-1))} \times \tilde{K}_r^{2^r} \\
=C_0^{2^r-1} a^{2^{1+r-r-2}} \tilde{K}_r^{2^r}
\]

where \(\tilde{K} = \max\{K_0, K_0^{p_1}\}\). Finally, recalling that \(p_r = 2^{r-1}(q - n) + \frac{n^2}{2}\), we deduce that
\[
\|v\|_{L^\infty(\Omega)} \leq \lim_{r, r' \to \infty} \left( C_0^{2^r-1} a^{2^{1+r-r-2}} \tilde{K}_r^{2^r} \right)^{1/p_r} = \left( C_0 a^2 \tilde{K}_0 \right)^{\frac{2}{n^2-nk}},
\]
which concludes the proof. \(\square\)

Next, we turn to consider the fully parabolic case \(\varepsilon > 0\). Without loss of generality, we assume \(\varepsilon = 1\). For any \(\varepsilon > 0\) we can proceed the same lines to obtain the following lemma.

**Lemma 3.8** Assume that \(n \geq 3\), \(\varepsilon = 1\). Suppose that \(\gamma\) satisfies (A0), (A1’) and (A2) with some \(k < \frac{2}{n-2}\). There is \(C > 0\) depending only on \(\Omega\), \(k\), \(\varepsilon\) and the initial data such that
\[
\sup_{0 \leq t < T_{\max}} \left( \|w\|_{L^\infty(\Omega)} + \|v\|_{L^\infty(\Omega)} \right) \leq C.
\]

**Proof** First of all, we note that since \(w \geq w_* > 0\) by Lemma 2.4. It follows from Lemma 2.7 that
\[
v \leq C(w + 1) \leq C(w + \frac{w}{w_*}) = C(1 + \frac{1}{w_*}) w
\]
with some \(C > 0\) depending only on \(\Omega\), \(\gamma\), \(\varepsilon\) and the initial data. Hence by the non-increasing property of \(\gamma\),
\[
\gamma(v) \geq \gamma(C(w + 1)) \geq \gamma(C'w)
\]
with \(C' = C(1 + \frac{1}{w_*})\). Now denoting \(\tilde{w} = C'w\), it follows from (2.7) that
\[
\tilde{w}_t + C'u\gamma(\tilde{w}) \leq C'(I - \Delta)^{-1}[u\gamma(v)] \leq \gamma(v_*) \tilde{w}. \tag{3.30}
\]

Now, multiplying (3.30) by \(\tilde{w}^{p-1}\), we get
\[
\frac{1}{p} \frac{d}{dt} \int_{\Omega} \tilde{w}^p dx + C' \int_{\Omega} u\gamma(\tilde{w}) \tilde{w}^{p-1} dx \leq \gamma(v_*) \int_{\Omega} \tilde{w}^p dx.
\]
Here, we note that $\tilde{w} - \Delta \tilde{w} = C'u$. Then in the same manner as done in proof of Lemma 3.5, one proves that there is $C > 0$ depending only on the initial data, $\Omega, \varepsilon$ and $k$ such that

$$\sup_{0 \leq t < T_{\max}} \|\tilde{w}\|_{L^\infty(\Omega)} \leq C.$$ 

This completes the proof since $v \leq \tilde{w}$ point-wisely. 

Before concluding this section, we show the relationship between $(A2)$ and $(A3)$-type conditions provided that assumptions $(A0)$ and $(A1)$ are satisfied.

**Lemma 3.9** A function satisfying $(A0)$, $(A1)$ and

$$(A3c): \quad l|\gamma'(s)|^2 \leq \gamma(s)\gamma''(s), \quad \forall s > 0 \quad (3.31)$$

with some $l > 1$ must fulfill assumption $(A2)$ with any $k > \frac{1}{l-1}$.

**Proof** First, we point out that under the assumptions $(A0)$, $(A1)$ and $(A3c)$, $\gamma'(s) < 0$ on $[0, \infty)$. In fact, due to $(A0)$ and $(A3c)$, we have $\gamma''(s) \geq 0$ for all $s > 0$. Then if there is $s_1 \geq 0$ such that $\gamma'(s_1) = 0$, it must hold that $0 = \gamma'(s_1) \leq \gamma'(s) \leq 0$ for all $s \geq s_1$, which contradicts to the positivity of $\gamma$ in assumptions $(A0)$ and the asymptotically vanishing assumption $(A1)$.

Now, we may divide $(3.31)$ by $-\gamma(s)\gamma'(s)$ to obtain that

$$-\frac{l\gamma'(s)}{\gamma(s)} \leq -\frac{\gamma''(s)}{\gamma'(s)}, \quad \forall s > 0,$$

which indicates that

$$\left(\log(-\gamma^{-l}\gamma')\right)' \leq 0.$$ 

An integration of above ODI from $v_*$ to $s$ yields that

$$-\gamma^{-l}(s)\gamma'(s) \leq -\gamma^{-l}(v_*)\gamma'(v_*) \triangleq d > 0,$$

which further implies that

$$\left(\frac{1}{(l-1)\gamma'^{l-1}(s)}\right)' \leq d.$$ 

Thus for any $s \geq v_*$, there holds

$$\frac{1}{\gamma'^{l-1}(s)} \leq d(l-1)(s - v_*) + \frac{1}{\gamma'^{l-1}(v_*)}.$$ 

As a result, for any $k > \frac{1}{l-1}$, we have

$$\frac{1}{[s^k\gamma(s)]^{l-1}} \leq \frac{d(l-1)(s - v_*)}{s^{k(l-1)}} + \frac{1}{s^{k(l-1)}\gamma'^{l-1}(v_*)} \to 0, \quad \text{as} \ s \to +\infty.$$

This completes the proof. \qed
Corollary 3.1 Assume that $n \geq 3$, $\varepsilon \geq 0$ and $\gamma(\cdot)$ satisfies (A0), (A1) and (A3u). Then $v$ has a uniform-in-time upper bound in $\Omega \times [0, T_{\text{max}})$.

Proof Note that $\frac{1}{n-1} < \frac{2}{n-2}$ when $l_0 > \frac{n}{2}$. Thus $\gamma$ satisfies (A2) with some $k < \frac{2}{n-2}$ and due to Lemma 3.5 and Lemma 3.8, $v$ has a uniform-in-time upper bound. \hfill \Box

4 The Parabolic-Elliptic Case

This section is devoted to the proof of Theorem 1.2. With the upper bound of $v$ at hand, in view of Lemma 2.1, it suffices to establish an estimate for the weighted energy $\int_{\Omega} u^p \gamma^q(v)$ for some $p > \frac{n}{2}$ and $q > 0$.

4.1 Global Existence

First, we prove existence of global classical solutions which is given by the following lemma.

Lemma 4.1 Assume that $\varepsilon = 0$ and $\gamma(\cdot)$ satisfies (A0) and (A3a). Then for any given $0 < T < T_{\text{max}}$, there exist $p > \frac{n}{2} - 1$ and $C_T > 0$ such that

$$\sup_{0 \leq t \leq T} \int_{\Omega} u^{1+p} \leq C_T,$$

where $p$ may depend on $n$, $\Omega$, $\gamma$ and $T$.

Proof Recall that $v = w$ when $\varepsilon = 0$. Multiplying the key identity (2.7) by $qu^{p+1} \gamma^{q-1}(v) \times \gamma'(v)$ with $p, q > 0$ to be specified later and integrating with respect to $x$ yields

$$\frac{d}{dt} \int_{\Omega} u^{p+1} \gamma^q(v)dx - (p + 1) \int_{\Omega} \gamma^q(v)u^p u_t dx - q \int_{\Omega} u^{p+1} \gamma^q(v)\gamma'(v) \Delta v dx$$

$$- q \int_{\Omega} (I - \Delta)^{-1}[u\gamma'(v)]u^{p+1} \gamma^{q-1}(v)\gamma'(v) v dx = - q \int_{\Omega} u^{p+1} \gamma^q(v)\gamma'(v)v dx,$$

where we used the fact that $-\Delta v + v = u$.

By the first equation of (1.1) and integration by parts, we infer that

$$- (p + 1) \int_{\Omega} \gamma^q(v)u^p u_t dx$$

$$= - (p + 1) \int_{\Omega} \gamma^q(v)u^p \Delta(\gamma(v)u) dx$$

$$=(p + 1) \int_{\Omega} (\gamma'(v)u_{\nabla} + \gamma'(v)u\nabla v) \left( pu^{p-1} \gamma^q(v)\nabla u + qu^p \gamma^{q-1}(v)\gamma'(v)\nabla v \right) dx$$

$$= p(p + 1) \int_{\Omega} u^{p+1} \gamma^{q+1}(v)|\nabla u|^2 dx + q(p + 1) \int_{\Omega} u^{p+1} \gamma^{q+1}(v)\gamma'(v)^2 |\nabla v|^2 dx$$

$$+ (p + 1)(p + q) \int_{\Omega} u^p \gamma^q(v)\gamma'(v) \nabla u \cdot \nabla v dx,$$
and by integration by parts again,

\[-q \int_\Omega u^{p+1} \gamma^q(v) \gamma'(v) \Delta v dx\]

\[= q^2 \int_\Omega u^{p+1} \gamma^{q-1}(v) |\gamma'(v)|^2 |\nabla v|^2 dx + q \int_\Omega u^{p+1} \gamma^q(v) |\nabla v|^2 dx\]

\[+ q(p + 1) \int_\Omega u^p \gamma^q(v) \gamma'(v) \nabla u \cdot \nabla v dx.\]  

(4.3)

Then we arrive at

\[
\frac{d}{dt} \int_\Omega u^{p+1} \gamma^q(v) dx + (p + 1) p \int_\Omega \nabla u^{p+1} |\nabla u|^2 dx \\
+ q \int_\Omega \left( (p + q + 1) |\gamma'(v)|^2 + \gamma \gamma'' \right) u^{p+1} \gamma^{q-1}|\nabla v|^2 dx \\
- q \int_\Omega (I - \Delta)^{-1}[u \gamma'(v)] u^{p+1} \gamma^{q-1}(v) \gamma'(v) dx \\
= - (p + 1)(p + 2q) \int_\Omega u^p \gamma^q(v) \gamma'(v) \nabla u \cdot \nabla v dx - q \int_\Omega u^{p+1} \gamma^q(v) \gamma'(v) v dx.\]

(4.4)

Now applying Young’s inequality, we infer that

\[-(p + 1)(p + 2q) \int_\Omega u^p \gamma^q(v) \gamma'(v) \nabla u \cdot \nabla v dx \\
\leq (p + 1) p \int_\Omega \nabla u^{p+1} |\nabla u|^2 dx + \frac{(p + 1)(p + 2q)^2}{4p} \int_\Omega u^{p+1} \gamma^{q-1} |\gamma'|^2 |\nabla v|^2 dx.\]

We further require that

\[
\frac{(p + 1)(p + 2q)^2}{4p} \int_\Omega u^{p+1} \gamma^{q-1} |\gamma'|^2 |\nabla v|^2 dx \\
\leq q \int_\Omega \left( (p + q + 1) |\gamma'(v)|^2 + \gamma \gamma'' \right) u^{p+1} \gamma^{q-1} |\nabla v|^2 dx,\]

(4.5)

which is satisfied provided that

\[(p^2 + p^3 + 4q^2)|\gamma'|^2 \leq 4pq \gamma \gamma'' \quad \text{a.e.} \]

(4.6)

Next, letting \(q = \lambda p\) with some \(\lambda > 0\), then (4.6) is equivalent to the following

\[(1 + p + 4\lambda^2)|\gamma'|^2 \leq 4\lambda \gamma \gamma'' \quad \text{a.e.} \]

(4.7)

Note that \(\frac{1+p+4\lambda^2}{4\lambda}\) attains its minimum value \(\sqrt{1+p}\) when \(\lambda = \frac{\sqrt{1+p}}{2}\).

For any given \(0 < T < T_{\max}\), due to Lemma 2.4 and Lemma 2.6, \(v\) is bounded on \([0, T] \times \Omega\) from above and below by some strictly positive constants depending only on the initial data, \(\gamma\), \(T\) and \(\Omega\), which is also true for \(|\gamma'(v)|^2\) and \(\gamma(v) \gamma''(v)\) on \([0, T] \times \Omega\) due to our
assumption on \( \gamma \). Then under the assumption (A3a), one can always find \( p > \frac{n}{2} - 1 \) and \( \lambda = \frac{\sqrt{1+p}}{2} \) such that (4.7) holds on \([0, T] \times \overline{\Omega}\). As a result, one obtains that

\[
\frac{d}{dt} \int_{\Omega} u^{p+1} \gamma^q(v) dx - q \int_{\Omega} (I - \Delta)^{-1}[u \gamma(v)]u^{p+1} \gamma^{q-1}(v)\gamma'(v)dx \\
\leq -q \int_{\Omega} u^{p+1} \gamma^q(v)\gamma'(v)v dx.
\]

(4.8)

Then by Gronwall’s inequality, we get

\[
\int_{\Omega} u^{p+1} \gamma^q(v) dx \leq C_T.
\]

This concludes the proof since \( \gamma^q(v) \) is bounded from below.

**Corollary 4.1** Assume \( \epsilon = 0 \), \( \gamma(v) = v^{-k} \) and \( n \geq 3 \). Then there exists a unique global classical solution provided that \( k < \frac{\sqrt{2n+2}}{n-2} \).

### 4.2 Uniform-in-Time Boundedness

In this part we prove the uniform-in-time boundedness in Theorem 1.2. To this aim, we establish time-independent bounds for the weighted energy.

**Lemma 4.2** Assume that \( \epsilon = 0 \), \( \gamma(\cdot) \) satisfies (A0), (A1) and (A3u). The there holds

\[
\sup_{t \geq 0} \int_{\Omega} u^p dx \leq C \tag{4.9}
\]

with \( p > \frac{n}{2} \) and \( C > 0 \) depending only on the initial data, \( \Omega \) and \( \gamma \).

**Proof** Under our assumption, condition (4.7) holds for any \( p > 1 \) such that

\[
\frac{1 + p + 4\lambda^2}{4\lambda} \leq l_0
\]

holds with some \( \lambda > 0 \). Define

\[
f(\lambda) = 4\lambda l_0 - 4\lambda^2
\]

for all \( \lambda > 0 \). We observe that \( f(\lambda) \) attains its maximum value \( l_0^2 \) at \( \lambda_0 = l_0/2 \). Since \( l_0 > \frac{n}{2} \), there holds

\[
l_0^2 > \frac{n^2}{4}.
\]

Thus, for any \( 1 + p \in (1, \frac{n^2}{4}] \) and \( \lambda = \lambda_0 \), there holds

\[
1 + p \leq \frac{n^2}{4} < l_0^2 = f(\lambda_0). \tag{4.10}
\]
In other words, there holds
\[ 1 + p + 4\lambda_0^2 \left| \gamma'(s) \right|^2 < l_0 \left| \gamma'(s) \right|^2 \leq \gamma \gamma'', \quad \forall s > 0 \]
for any \( 1 + p \in (1, \frac{n^2}{4}] \). In particular, recalling the time-independent lower and upper bounds for \( v \) given by Corollary 3.1,
\[ v_* \leq v(x, t) \leq v^* \quad \text{on } \overline{\Omega} \times [0, \infty) \]
with \( v_*, v^* > 0 \), we infer that
\[ 1 + p + 4\lambda_0^2 \left| \gamma'(v(x, t)) \right|^2 < \gamma(v(x, t)) \gamma''(v(x, t)), \quad \text{on } \overline{\Omega} \times [0, \infty) \]
for any \( 1 + p \in (1, \frac{n^2}{4}] \). In addition, for any \( 1 + p \in (1, \frac{n^2}{4}] \), we can further find time-independent \( \delta_0 = \delta_0(p, \lambda_0) > 0 \) such that
\[ 1 + p + 4\lambda_0^2 + 4\lambda_0 \delta_0 (1 + p + \lambda_0 p) \frac{1}{4\lambda_0 (1 - \delta_0)} \left| \gamma'(v(x, t)) \right|^2 < \gamma(v(x, t)) \gamma''(v(x, t)), \quad \text{on } \overline{\Omega} \times [0, \infty). \]
As a result, based on a similar argument as from (4.5) to (4.7), we have
\[ \frac{(p + 1)(p + 2q)^2}{4p(1 - \delta_0)} \int_\Omega u^{p + 1} \gamma^{q - 1} |\gamma'|^2 |\nabla v|^2 \]
\[ \leq q \int_\Omega \left( (p + q + 1) |\gamma'(v)|^2 + \gamma \gamma'' \right) u^{p + 1} \gamma^{q - 1} |\nabla v|^2 \]
with \( q = \lambda_0 p \). Thus by Young’s inequality,
\[ - (p + 1)(p + 2q) \int_\Omega u^p \gamma^q(v) \gamma'(v) \nabla u \cdot \nabla v dx \]
\[ \leq (p + 1) p(1 - \delta_0) \int_\Omega u^{p - 1} \gamma^{q + 1} |\nabla u|^2 dx \]
\[ + \frac{(p + 1)(p + 2q)^2}{4p(1 - \delta_0)} \int_\Omega u^{1 + p} \gamma^{q - 1} |\gamma'|^2 |\nabla v|^2 dx, \quad (4.11) \]
we obtains an improved version of (4.8) as follows
\[ \frac{d}{dt} \int_\Omega u^{p + 1} \gamma^q(v) dx + \delta_0 (p + 1) p \int_\Omega u^{p - 1} \gamma^{q + 1} |\nabla u|^2 dx \]
\[ - q \int_\Omega (I - \Delta)^{-1}[u \gamma(v)] u^{p + 1} \gamma^{q - 1}(v) \gamma'(v) dx \]
\[ \leq - q \int_\Omega u^{p + 1} \gamma^q(v) \gamma'(v) v dx \quad (4.12) \]
with any \( 1 + p \in (1, \frac{n^2}{4}] \), \( q = \frac{pl_0}{2} \) and some \( \delta_0 = \delta_0(p, l_0) > 0 \).
Now, recalling Lemma 3.3 and the time-independent boundedness of \( v \), there holds
\[
\sup_{t \geq 0} \int_t^{t+1} \int_{\Omega} u^2 dx \ dt \leq C \tag{4.13}
\]
with \( C > 0 \) depending only on the initial data, \( \Omega \) and \( \gamma \).

Next, we take \( p = 1 \) such that \( 1 + p = 2 < \frac{n^2}{4} \) and \( q = \frac{b_0}{2} \) in (4.12). Since now \( v \) is bounded from above and below, we obtain that
\[
\frac{d}{dt} \int_{\Omega} u^2 v^{-\frac{b_0}{2}} dx + C \int_{\Omega} |\nabla u|^2 dx \leq C \int_{\Omega} u^2 dx \tag{4.14}
\]
with \( C > 0 \) independent of time.

In view of (4.13), an application of the uniform Gronwall inequality together with the local boundedness yields that
\[
\sup_{t \geq 0} \int_{\Omega} u^2 dx \leq C.
\]

Besides, an integration of (4.14) from \( t \) to \( t + 1 \) further gives rise to
\[
\sup_{t \geq 0} \int_t^{t+1} \int_{\Omega} |\nabla u|^2 dx \ dt \leq C.
\]

Thus, by the Sobolev embedding
\[
\|\xi\|_{L^{r^*_s}(\Omega)} \leq C \|\nabla \xi\|_{L^2(\Omega)}^2 \|\xi\|_{L^2(\Omega)}^{r^*_s-2} + C \|\xi\|_{L^1(\Omega)}^{r^*_s} \tag{4.15}
\]
with \( r^*_s = 2 + \frac{4}{n} \), we infer that
\[
\sup_{t \geq 0} \int_t^{t+1} \int_{\Omega} u^{r^*_s} dx \leq C.
\]

Then, we divide the discussion into several cases regarding the spatial dimensions. First, when \( n = 3 \), one notes that \( r^*_s = 2 + \frac{4}{n} = \frac{10}{3} > \frac{9}{4} = \frac{n^2}{4} \) and we may take \( 1 + p = \frac{9}{4} \) and \( q = \frac{5b_0}{8} \) in (4.12). In the same manner as before, by the uniform Gronwall inequality, we deduce that
\[
\sup_{t \geq 0} \int_{\Omega} u^{\frac{9}{4}} dx \leq C.
\]

When \( n \geq 4 \), there holds \( r^*_s = 2 + \frac{4}{n} < \frac{n^2}{4} \). We can take \( 1 + p = r^*_s \) and \( q = \frac{b_0}{2} (r^*_s - 1) \) in (4.12) to obtain in the same manner as before that
\[
\sup_{t \geq 0} \left( \int_{\Omega} u^{r^*_s} + \int_t^{t+1} \int_{\Omega} |\nabla u|^{\frac{2}{r^*_s}} dx \right) \leq C. \tag{4.16}
\]

Note that when \( n = 4, 5 \), we have \( r^*_s = 2 + \frac{4}{n} > \frac{6}{7} \).
It remains to consider the case \( n \geq 6 \). First, using the embedding (4.15) with \( \xi = u^{2r} \), we infer from (4.16) that

\[
\sup_{t \geq 0} \int_t^{t+1} \int_\Omega u^{2r_2} \, dx \, ds \leq C.
\]

On the other hand when \( n \geq 6 \), one can always find \( m \in \mathbb{N} \) such that \( \frac{n^2}{2} < r_m^2 \leq \frac{n^2}{4} \). Indeed, let \( m \) be the integer such that \( r_m^2 \leq \frac{n^2}{4} \) and \( r_{m+1}^2 > \frac{n^2}{4} \). Then we observe that

\[
\frac{r_m^2}{2} > \frac{n^2}{4r_*} = \frac{n}{2} \times \frac{n}{2r_*}
\]

where \( \frac{n}{2r_*} > 1 \) if \( n \geq 6 \).

Using the embedding (4.15) with \( \xi = u^{2r_l} \) with \( l = 1, 2, \ldots, m - 1 \), repeating the above steps, we can finally prove that

\[
\sup_{t \geq 0} \int_\Omega u^{2r_m} \, dx \leq C.
\]

This completes the proof. \( \square \)

5 The Fully Parabolic Case

In this section, we consider the fully parabolic case and give a proof for Theorem 1.3. The idea is basically a generalization of [11, Lemma 5.5] to higher dimensions. Indeed, we list out a system of estimations involving the weighted energies \( \int u^{1+p} \gamma^q(v) \) with the varying parameters \( p, q \). Luckily, by a careful recombination we are able to obtain the uniform-in-time boundedness.

**Lemma 5.1** Assume \( n \geq 3 \). Suppose that \( \gamma(\cdot) \) satisfies (A0), (A1), and (A3b). Then there is \( C > 0 \) depending only on the initial data and \( \Omega \) such that

\[
\sup_{0 \leq t < T_{\max}} \int_\Omega u^{1+\frac{n}{2}} \, dx \leq C.
\]

**Proof** In the same manner as before, we first compute by integration by parts to obtain that

\[
\frac{d}{dt} \int_\Omega u^{1+p} \gamma^q(v) \, dx = (1 + p) \int_\Omega u^p \gamma^q(v) u_t + q \int_\Omega u^{1+p} \gamma^{q-1}(v) \gamma'(v) v_t
\]

\[
= (1 + p) \int_\Omega u^p \gamma^q(v) \Delta(u \gamma(v)) + q \int_\Omega u^{1+p} \gamma^{q-1}(v) \gamma'(v)(u - v + \Delta v)
\]

\[
= - (1 + p) \int_\Omega \nabla(u^p \gamma^q(v)) \cdot \nabla(u \gamma(v)) + q \int_\Omega u^{2+p} \gamma^{q-1}(v) \gamma'(v)
\]

\[
- q \int_\Omega u^{1+p} \gamma^{q-1}(v) \gamma'(v) v - q \int_\Omega \nabla(u^{1+p} \gamma^{q-1}(v) \gamma'(v)) \cdot \nabla v.
\]
The main difference here is that we need to use the second equation in (1.1) to replace $v_t$.

Recalling (4.2),

\[
(1 + p) \int_\Omega \nabla(u^p \gamma^q(v)) \cdot \nabla(u \gamma(v))
\]

\[
= p(1 + p) \int_\Omega u^{p-1} \gamma^{1+q} |\nabla u|^2 + q(1 + p) \int_\Omega u^{1+p} \gamma^{q-1} |\gamma'|^2 |\nabla v|^2
\]

\[
+ (1 + p)(p + q) \int_\Omega u^p \gamma^q \gamma' \nabla u \cdot \nabla v,
\]

and by integration by parts again,

\[
q \int_\Omega \nabla(u^{1+p} \gamma^{q-1}(v) \gamma'(v)) \cdot \nabla v
\]

\[
= q(1 + p) \int_\Omega u^p \gamma^{q-1} \gamma' \nabla u \cdot \nabla v + q \int_\Omega u^{1+p} \gamma^{q-2}(1 + q |\gamma'|^2 + \gamma \gamma'') |\nabla v|^2.
\]

As a result, we obtain that

\[
\frac{d}{dt} \int_\Omega u^{1+p} \gamma^p(v) dx + p(1 + p) \int_\Omega u^{p-1} \gamma^{1+q} |\nabla u|^2 + q(1 + p) \int_\Omega u^{1+p} \gamma^{q-1} |\gamma'|^2 |\nabla v|^2
\]

\[
+ q \int_\Omega u^{1+p} \gamma^{q-2}(1 + q |\gamma'|^2 + \gamma \gamma'') |\nabla v|^2 - q \int_\Omega u^{2+p} \gamma^{q-1}(v) \gamma'(v)
\]

\[
= - (1 + p)(p + q) \int_\Omega u^p \gamma^q \gamma' \nabla u \cdot \nabla v - q(1 + p) \int_\Omega u^p \gamma^{q-1} \gamma' \nabla u \cdot \nabla v
\]

\[
- q \int_\Omega u^{1+p} \gamma^{q-1}(v) \gamma'(v)v.
\]

(5.1)

In particular, if $p = q$, since

\[
(1 + p) \int_\Omega \nabla(u^p \gamma^p(v)) \cdot \nabla(u \gamma(v))
\]

\[
= p(1 + p) \int_\Omega (u \gamma)^{p-1} |\nabla(u \gamma)|^2,
\]

one obtains the following estimate

\[
\frac{d}{dt} \int_\Omega u^{1+p} \gamma^p(v) dx + p(1 + p) \int_\Omega (u \gamma)^{p-1} |\nabla(u \gamma)|^2
\]

\[
+ p \int_\Omega u^{1+p} \gamma^{p-2}(1 + q |\gamma'|^2 + \gamma \gamma'') |\nabla v|^2 - p \int_\Omega u^{2+p} \gamma^{p-1}(v) \gamma'(v)
\]

\[
= - p(1 + p) \int_\Omega u^p \gamma^{p-1} \gamma' \nabla u \cdot \nabla v - p \int_\Omega u^{1+p} \gamma^{p-1}(v) \gamma'(v)v.
\]

Now, let $1 \leq p \leq \left\lfloor \frac{n}{2} \right\rfloor$ be any integer fixed. In addition, assume $q = j$ in (5.1) with $j = 0, 1, 2, \ldots, p$ and multiply the $j$th formula by $\lambda_{p,j} > 0$ to be specify later. Then a summation
yields that
\[
\frac{d}{dt} \sum_{j=0}^{p} \lambda_{p,j} \int_{\Omega} u^{1+p} \gamma^j(v) dx + \lambda_{p,p} p(1 + p) \int_{\Omega} (u\gamma)^{p-1} |\nabla(u\gamma)|^2 \\
\]
\[
+ \left\{ \int_{\Omega} u^{1+p} \gamma^{p-2} \left( (p-1)(p\lambda_{p,p} + (p+1)\lambda_{p-1,j}) |\gamma'|^2 + p\lambda_{p,p} \gamma^{p''} \right) |\nabla v|^2 \\
+ \lambda_{p,p-1} p(p+1) \int_{\Omega} u^{p-1} \gamma^p |\nabla u|^2 + (1 + p)(p\lambda_{p,p} + (2p-1)\lambda_{p,p-1}) \\
\times \int_{\Omega} u^p \gamma^{p-1} \gamma' \nabla u \cdot \nabla v \right\} \\
+ \left\{ \int_{\Omega} u^{1+p} \gamma^{p-3} \left( (p-2)((p-1)\lambda_{p,p-1} + (p+1)\lambda_{p-2,j}) |\gamma'|^2 + (p-1)\lambda_{p,p-1} \gamma^{p''} \right) \\
\times |\nabla v|^2 \\
+ \lambda_{p,p-2} p(p+1) \int_{\Omega} u^{p-1} \gamma^{p-1} |\nabla u|^2 + (1 + p)((p-1)\lambda_{p,p-1} + (2p-2)\lambda_{p,p-2}) \\
\times \int_{\Omega} u^p \gamma^{p-2} \gamma' \nabla u \cdot \nabla v \right\} \\
+ \cdots \\
\right. \\
\left. + \left\{ \lambda_{p,1} \int_{\Omega} u^{1+p} \gamma'' |\nabla v|^2 + \lambda_{p,0} p(p+1) \int_{\Omega} u^{p-1} \gamma |\nabla u|^2 \\
\right. \\
\left. +(1 + p)(\lambda_{p,1} + p\lambda_{p,0}) \int_{\Omega} u^p \gamma' \nabla u \cdot \nabla v \right\} \\
- \sum_{j=1}^{p} j \lambda_{p,j} \int_{\Omega} u^{2+p} \gamma'(v) \gamma^{j-1} \\
= - \sum_{j=1}^{p} j \lambda_{p,j} \int_{\Omega} u^{1+p} \gamma'(v) \gamma^{j-1} v.
\]
For \(1 \leq j \leq p\), we define
\[
\Lambda_{p,j} = \int_{\Omega} u^{1+p} \gamma^{j-2} \left( (j-1)(j\lambda_{p,j} + (p+1)\lambda_{p,j-1}) |\gamma'|^2 + j \lambda_{p,j} \gamma^{p''} \right) |\nabla v|^2 \\
+ \lambda_{p,j-1} p(p+1) \int_{\Omega} u^{p-1} \gamma^j |\nabla u|^2 + (1 + p)(j\lambda_{p,j} + (p+j-1)\lambda_{p,j-1}) \\
\times \int_{\Omega} u^p \gamma^{j-1} \gamma' \nabla u \cdot \nabla v.
\]
Then we obtain that
\[
\frac{d}{dt} \sum_{j=0}^{p} \lambda_{p,j} \int_{\Omega} u^{1+p} \gamma^j(v) dx + \lambda_{p,p} p(1 + p) \int_{\Omega} (u\gamma)^{p-1} |\nabla(u\gamma)|^2
\]
\[
+i=1 \sum \int_{\Omega} u^{2+p} \gamma'(v) \gamma^{j-1} \\
= -i=1 \sum \int_{\Omega} u^{1+p} \gamma'(v) \gamma^{j-1} v.
\]

Invoking Young’s inequality, we infer that
\[
(1 + p)(j \lambda_{p,j} + (p + j - 1) \lambda_{p,j-1}) \int_{\Omega} u^p \gamma^{j-1} \gamma' \nabla u \cdot \nabla v
\leq \lambda_{p,j-1} p(p + 1) \int_{\Omega} u^{p-1} \gamma' |\nabla u|^2 + \frac{(1 + p) \left[ j \lambda_{p,j} + (p + j - 1) \lambda_{p,j-1} \right]^2}{4p \lambda_{p,j-1}}
\times \int_{\Omega} u^{1+p} \gamma^{j-2} |\gamma'|^2 |\nabla v|^2.
\]

Hence, in the same spirit as before, we have \( \Lambda_{p,j} \geq 0 \) provided that for all \( s > 0 \)
\[
\frac{(1 + p) \left[ j \lambda_{p,j} + (p + j - 1) \lambda_{p,j-1} \right]^2}{4p \lambda_{p,j-1}} |\gamma'(s)|^2
\leq (j - 1)(j \lambda_{p,j} + (p + 1) \lambda_{p,j-1}) |\gamma'(s)|^2 + j \lambda_{p,j} \gamma(s) \gamma''(s),
\]
which by simple computations is equivalent to
\[
(1 + p) j^2 \lambda_{p,j}^2 + (1 + p)(p + 1 - j)^2 \lambda_{p,j-1}^2 + 2j \lambda_{p,j-1} \lambda_{p,j}(p^2 - pj + 2p + j - 1)
\leq \gamma \gamma''.
\]

Observe that by Young’s inequality again,
\[
(1 + p) j^2 \lambda_{p,j}^2 + (1 + p)(p + 1 - j)^2 \lambda_{p,j-1}^2 + 2j \lambda_{p,j-1} \lambda_{p,j}(p^2 - pj + 2p + j - 1)
\geq 4pj \lambda_{p,j-1} \lambda_{p,j} (p + 2 - j)
\]
where the minimum is attained provided that
\[
j \lambda_{p,j} = (p + 1 - j) \lambda_{p,j-1}.
\]
Thus, if we take \( \lambda_{p,0} = 1 \) and \( \lambda_{p,j} = (p + 1 - j) \lambda_{j-1} / j \) for \( 1 \leq j \leq p \), condition (5.3) reads
\[
(p + 2 - j)|\gamma'|^2 \leq \gamma \gamma'', \quad \forall s > 0 \text{ and for all } 1 \leq j \leq p.
\]
Thus, under the assumption (A3b), one can always find $\lambda_{p,j} > 0$ such that $\Lambda_{pj} \geq 0$ for any fixed $1 \leq p \leq \lceil \frac{n}{2} \rceil$ with all $1 \leq j \leq p$. As a result, there holds

$$
\frac{d}{dt} \sum_{j=0}^{j=p} \lambda_{p,j} \int_{\Omega} u^{1+p} \gamma^j(v) dx + \lambda_{p,p} (1 + p) \int_{\Omega} (u \gamma)^{p-1} |\nabla (u \gamma)|^2
$$

$$
- \sum_{j=1}^{j=p} \lambda_{p,j} \int_{\Omega} u^{2+p} \gamma^j(v) \gamma^{-j} \gamma^j
$$

$$
\leq - \sum_{j=1}^{j=p} \lambda_{p,j} \int_{\Omega} u^{1+p} \gamma^j(v) \gamma^{-j} v.
$$

(5.5)

Now, we recall that $v$ is uniformly-in-time bounded from above and below under the assumption (A3b). Moreover by Lemma 3.3,

$$
\int_t^{t+\tau} \int_{\Omega} u^2 \gamma^j(v) dx ds \leq C
$$

with $C > 0$ independent of time. We can first take $p = 1$ in (5.5) and use the uniform Gronwall inequality together with the above estimates to derive that

$$
\sup_{0 \leq t < T_{\max}} \int_{\Omega} u^2 dx \leq C.
$$

Moreover thanks to the third term on the left-hand side of (5.5), there holds

$$
\int_t^{t+\tau} \int_{\Omega} u^3 dx ds \leq C.
$$

Subsequently, in the same manner as above, we can deduce by iterations that

$$
\sup_{0 \leq t < T_{\max}} \int_{\Omega} u^{1+\lceil \frac{n}{2} \rceil} dx \leq C.
$$

Remark 5.1 Our assumption (A3b) is independent of the coefficients of the system. Indeed, if we replace the second equation of system (1.1) by $u_t - \alpha \Delta v + \beta v = \theta u$ with some $\alpha, \beta, \theta > 0$, one easily checks that condition (5.3) becomes

$$
\frac{(1 + p)\alpha^2 j^2 \lambda_j^2 + (1 + p) (p + 1 - j)^2 \lambda_{j-1}^2 + 2\alpha j \lambda_{j-1} \lambda_j (p^2 - pj + 2p + j - 1)}{4\alpha p j \lambda_{j-1} \lambda_j} |\gamma'|^2
$$

$$
\leq \gamma \gamma'”,
$$

which still yields to

$$
(p + 2 - j)|\gamma'|^2 \leq \gamma \gamma’”, \ \forall \ s \geq 0 \ \text{and} \ \forall \ 1 \leq j \leq p,
$$

if we take $\lambda_0 = 1$ and $\lambda_j = \frac{(p+1-j)\lambda_{j-1}}{ja}$ for $1 \leq j \leq p$.

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