The family $G_T$ of graded Artinian quotients of $k[x, y]$ of given Hilbert function

Anthony Iarrobino

Department of Mathematics, Northeastern University, Boston, MA 02115, USA.

Joachim Yaméogo

Lab. J.-A. Dieudonné, UMR CNRS 6621, Univ. de Nice-Sophia Antipolis, F-06108 Nice 06034 cedex 02, France

August 26, 2002

Abstract

Let $R = k[x, y]$ be the polynomial ring over an algebraically closed field $k$ of characteristic zero of char $k > j$. Let $T = (1, 2, \ldots, \mu, \mu, \mu + 1, \ldots, t_j, 0)$ with $\mu \geq t_\mu \geq t_{\mu+1} \geq \ldots \geq t_j \geq 0$ be a sequence of nonnegative integers. The nonsingular projective variety $G_T$ parametrizes all graded ideals $I$ of $R = k[x, y]$ for which the Hilbert function $H(R/I) = T$. Let $E$ be a monomial ideal of $R$, satisfying $H(R/E) = T$: the cell $V(E)$ is the family of graded ideals having initial monomials $E$ in a suitable partial order. The graded ideal $I$ belongs to the cell $V(E)$ if the divisibility of a standard basis of each degree-$i$ homogeneous piece $I_i$ by powers of $x$ matches that of $E$; this specifies the ramification $\text{QRAM}(I_i, x)$ partition of the linear system $L(I_i)$ determined by $I_i$ at the point $p_x : x = 0$ of the projective line $\mathbb{P}^1$. Likewise, letting $p$ be a point of $\mathbb{P}^1$ defined by $ax + by = 0$, we define cells $V(E, p)$, specifying the ramification at the point $p : ax + by = 0$. When $j = \mu$, $V(E, p)$ is a Schubert cell of a Grassman variety $\text{Grass}(d, R_j)$, $d = j + 1 - t_j$. In this case the intersection of ramification conditions at different points is given by the Schubert calculus (see [I2, EH1, EH2]). Our goal is to generalize this calculus to determine the homology ring $H^*(G_T)$, in terms of the classes of the cells $V(E)$ giving the natural cellular decomposition of $G_T$, a problem that remains open in general.

We show that $G_T$ is birational to a certain product $S\text{Grass}(T)$ of small Grassman varieties (Proposition 3.15). We show (Theorem 3.28)

Theorem. Let $k = \mathbb{C}$. The birational map from $G_T$ to $S\text{Grass}(T)$ induces an additive $\mathbb{Z}$ isomorphism $\tau : H^*(G_T) \rightarrow H^*(S\text{Grass}(T))$ of homology groups.

When $\mu < j$ this isomorphism is not usually an isomorphism of rings. We determine the ring $H^*(G_T)$ when $T = T(\mu, j) = (1, 2, \ldots, \mu - 1, \mu, \ldots, \mu, t_j = 1)$, where $G_T \subset \mathbb{P}^\mu \times \mathbb{P}^j$ (Theorem 4.14). In this case $G_T$ is a desingularisation of the $\mu$-secant bundle $\text{Sec}(\mu, j)$ of the degree-$j$ rational normal curve, or, equivalently, of the determinantal variety of rank-$\mu$ Hankel matrices. We use this ring $H^*(G_T)$ to determine the number of ideals satisfying an intersection of ramification conditions at different points (Example 4.15). We also determine the classes in $H^*(G_T)$ of the pullback of the singular locus of $\text{Sec}(\mu, j)$ and of the pullbacks of the higher singular loci – the rank-$i$, with $i < \mu$ determinantal varieties of the Hankel matrix (Theorem 4.11).

A main tool we use from Part I [LY2] is the hook code or pruning code, a map $D$ taking the partition $P(E)$ determined by the monomials of $E^*$, the complement of the monomial ideal $E$, to a sequence $D(P(E))$ of smaller partitions, each corresponding to a cell in a small Grassman variety of size determined by $T$. 

1
1 Introduction

The Hilbert function $H(A)$ of an Artin algebra $A$ over a field $k$ specifies the dimension of each degree-$i$ piece $A_i$ of $A$. Let $T$ be a sequence of nonnegative integers possible for a Hilbert function of an Artin quotient $A = R/I$ of the polynomial ring $R = k[x, y]$. Then

$$T = (1, 2, \ldots, \mu, t_\mu, \ldots, t_j, 0), \text{ where } \mu \geq t_\mu \geq \cdots \geq t_j > 0. \quad (1.1)$$

Here $\mu$ is the order of $T$, the order or initial degree of the ideal $I$. Letting $M = (x, y)$ be the maximal ideal of $R$, we have $M^\mu \supset I \supset M^{j+1}$. We denote by $G_T$ the variety that parametrizes the graded ideals $I$ of $R$ defining a quotient algebra $A = R/I$ of Hilbert function $T$. The first author had shown that $G_T$ is a smooth projective variety [H1]. We consider these first a natural class of varieties worthy of study in their own right. The classical Grassmann varieties $Grass(d, R)$ are the special case $j = \mu$. G. Gotzmann has used the properties of $G_T$ in his proof of the simple connectivity of the Hilbert scheme of curves [Gm1]; he has also used the varieties $G_T$ to determine the dimension of the postulation strata $Hilb^H(P^2)$ of the punctual Hilbert scheme $Hilb^nP^2$ (see [Gm1] [Gm3], [IK] Theorem 5.51). We will note several other applications shortly.

J. Briançon and the second author had studied a certain “vertical stratification” of the family $Hilb^n R$ of all colength $n$ ideals in the local ring $k[x, y]$ (see [BS] [Y1] [Y2]). L. Götsche then used the methods of G. Ellingsrud and S. A. Strømme ([ES1] [ES2]) and of A. Białynicki-Birula [B] to give a cellular decomposition of the variety $Z_T$, parametrizing all ideals $I \subset R$ with Hilbert function $H(R/I) = T$, and a similar decomposition of $G_T$ (graded ideals). Each cell $V(E)$ corresponds to a monomial ideal $E$ of $R$ defining a quotient algebra of Hilbert function $T$; the cell consists of the ideals collapsing to $E$ under a $\mathbb{C}^*$-action. The second author showed that the Ellingsrud-Strømme-Göttche cells $Z(E)$ on $Z_T$ and also $V(E)$ on $G_T$ are in fact the vertical cells of Briançon [BS]. In the special case of graded ideals the cell $V(E)$ is just the family of graded ideals having initial monomial ideal $E$ (see [Y4]). J. Yaméogo has studied the closure of these cells of $Z_T$ and $G_T$ in [Y3] [Y4] [Y5]. L. Götsche used the cellular decomposition to determine the Betti numbers of $Z_T$ and of $G_T$, when $k = \mathbb{C}$ [G02] [G04]. G. Gotzmann independently obtained a simple formula for the Betti numbers in codimension one [Gm3]. Here we give a new description of the homology groups $H^*(G_T)$, generalizing Gotzmann’s. Letting $\Delta = \Delta(T)$ denote the difference function of $T$, so $\delta_i = t_{i-1} - t_i$, we denote by

$$SGrass(T) = \prod_{\mu \leq i \leq j} Grass(\delta_{i+1}, 1 + \delta_i + \delta_{i+1}) \quad (1.2)$$

a product of “small” Grassmann varieties. We first show that $G_T$ is birational to $SGrass(T)$ (Proposition 3.15). We denote by $H^*(G_T)$ the homology group in codimension $i$. We then show (Theorem 3.28).

**Theorem.** Suppose that $k = \mathbb{C}$. The birational map from $G_T$ to $SGrass(T)$, the product of small Grassmannians induces an additive isomorphism $\tau : H^*(G_T) \rightarrow H^*(SGrass(T))$ of homology groups:

$$H^*(G_T) \cong_{add} \prod_{\mu(T) \leq i \leq j(T)} H^*(Grass(\delta_{i+1}, \delta_i + \delta_{i+1} + 1)) \quad (1.3)$$

The simple nature of the isomorphism $\tau$ in terms of a “hook code” for partitions is striking. If $E$ is a monomial ideal of $R$, the quotient $R/E$ has a basis of monomials in the shape of a partition $P(E)$, having “diagonal lengths” $T$ (Definition 3.7). A hook in the shape $P(E)$ consists of a corner monomial $\omega$, an arm $(\omega, x\omega, \ldots, \mu = x^{n-1}\omega)$, and a foot $(\omega, y\omega, \ldots, \mu = y^{n-1}\omega)$, with $x\mu$ and $y\mu \in E$, but $\mu, \mu \notin E$. Such a hook has hand $\mu = x^{n-1}\omega$ and arm minus leg difference $a - b$. We denote by $\mathcal{H}(P)$ the set of “difference-one” hooks of $P$, those for which $a - b = 1$. We show (Theorem 3.12).
Theorem. The dimension of the cell \( V(E) \) satisfies \( \dim(V(E)) = \# \mathcal{H}(P(E)) \), the total number of difference-one hooks of the partition \( P(E) \).

We then define a partition \( Q_i(P) \) with \( \delta_{i+1} = t_i - t_{i+1} \) parts. Each part of \( Q_i(P) \) is the number of difference-one hooks having a given degree-\( i \) hand monomial \( \mu \). The hook code \( P \to D(P) \) is the sequence \( D(P) = (Q_0(P), \ldots, Q_j(P)) \). The partition \( Q_i(P) \) determines in the usual way a Schubert variety and homology class in \( \text{Grass}(\delta_{i+1}, 1 + \delta_i + \delta_{i+1}) \). By Theorem 3.26, this class is the degree-\( i \)-component of the isomorphism \( \tau \) shown in Theorem 3.28. The hook code is studied in detail in Part I (see [Y2]), and we give the results we need in Section 3-D.

A second reason for studying \( G_T \) is that it parametrizes ideals of linear systems on the projective line \( \mathbb{P}^1 \). A linear system on \( \mathbb{P}^1 \) is determined by a \( d \)-dimensional vector subspace \( V \) of the space \( R_j \) of degree-\( j \) forms, and consists of the zero-sets \( p \in \mathbb{P}^1 \) of the elements \( f \in V \) (Definition 2.1). The degree sequence \( N_p(V) \) at a point \( p : ax + by = 0 \) of \( \mathbb{P}^1 \) is an increasing sequence of integers specifying the order of vanishing at \( p \) of a “good” basis of \( V \) at \( p \). If \( V_u(p) \) is the \( u \)-dimensional subspace of \( V \) maximally divisible by \( ax + by \), and \( n_u(p, V) \) is the power of \( ax + by \) dividing \( V_{d+u-1}(p) \), then \( N_p(V) = (n_1(p, V), \ldots, n_d(p, V)) \). The ramification of a linear system at \( p \) is a partition \( QRAM_p(V) \) whose parts are the sequence \( N_p(V) \) decreased by \( (0, 1, \ldots, d - 1) \) (Definition 2.6). The problem of characterizing the possible ramification sequences at different points for linear systems on a curve was a classical topic of study. When the linear system is the canonical linear system, this was the classical study of Weierstrass points. The first author had used Schubert calculus to study the problem of ramification for linear systems over \( \mathbb{P}^1 \) (see the last appendix of [P2]). D. Eisenbud and J. Harris elaborated this viewpoint to study the ramification of linear systems on curves in a series of papers (see [E1, E2, E3]). Our goal is to extend these results to ideals of linear systems.

A third application of the varieties \( G_T \) is as a desingularization of the secant bundles to a degree-\( j \) rational curve. When \( T = T(\mu, j) = (1, 2, \ldots, \mu, \mu, \ldots, \mu, 1) \) and \( 2\mu < j + 1 \), then \( G_T \) is a natural desingularisation of the determinantal variety \( V(\mu, j) \) of \( (\mu + 1) \times (\mu + 1) \) minors of the generic \( (\mu + 1) \times (j + 1 - \mu) \) Hankel matrix (see section 4-C). These determinantal varieties are also the secant bundles of the degree-\( j \) normal rational curve, and as well have applications to control theory. The homology of these determinantal varieties has been studied in non-projective context by A. L. Gorodentsov and B. Z. Shapiro in [Sh]. If \( V(\mu, j) \) is given its natural stratification by rank, then \( G_T \) is a semismall resolution in the sense of intersection theory, and the homology ring structure of \( G_T \) in this case is known (Theorem 4.4). As a first step to determining the homology of \( V(\mu, j) \) we determine the class of the pullback of the rank \( i \) \( \mu \) locus in Theorem 4.11.

For some other choices of \( T \) the varieties \( G_T \) are also natural desingularizations of interesting varieties \( Y_T \): for example, take \( Y_T \) to be the closure of a Hilbert function stratum for any one of the three families of algebras associated to the Grassmanian \( \text{Grass}(d, R_j) \) parametrizing \( d \)-dimensional vector space of forms of degree \( j \). These are the family \( \text{Grass}(d, j) \) of ancestor algebras, the family \( LA(d, j) \) of level algebras, or the family \( GA(d, j) \) of algebras \( A = R/(V) \) determined by the ideal of \( V \) [R3, Theorem 2.32]. The Hilbert function strata in the last example \( GA(d, j) \) are in fact the strata giving the decomposition of the restricted tangent bundle to a rational curve in \( \mathbb{P}^j \) as a direct sum of line bundles \( GA(n) \).

A fourth reason for studying \( G_T \) is that the variety \( Z_T \) parametrizing all ideals of Hilbert function \( T \) in \( k[[x, y]] \) is fibred over \( G_T \) by an affine space of known dimension; and the union of the \( Z_T \) for all \( T \) of fixed length \( n \) form the punctual Hilbert scheme \( \text{Hilb}^n_k[[x, y]] \), the fibre of \( \text{Hilb}^n(\mathbb{P}^2) \) over a point \( npt_0 \) of the symmetric product \( \text{Sym}^n(\mathbb{P}^2) \). The punctual Hilbert scheme has shown itself important lately in several ways. M. Haiman has used \( \text{Hilb}^n \mathbb{C}[[x, y]] \) in solving an important combinatorial problem, an \( n! \)-conjecture involving representation theory [Ha]. J. Cheah has recently determined the homology groups of nonsingular varieties \( \text{Hilb}^{n,n-1}_k[[x, y]] \) parametrizing pairs \( (I_n \subset I_{n-1}) \) where \( I_n \) is a colength-\( n \) ideal and \( I_{n-1} \) is a colength-\( n - 1 \) ideal in \( k[[x, y]] \). She also uses the difference-\( a \) hooks in her description of these groups [Ch]. Her study and also her compact review of the known homology results for \( \text{Hilb}^n_k[x, y] \) due to G. Gotzmann, L. Göttsche,
et al. is a complement to our study of the homology of the fine strata $G_T$.

Motivated by physics, I. Grojnowski, H. Nakajimina, and many others have found deep connections among the homology for different $n$ of the Hilbert scheme $\text{Hilb}^n(X)$ for surfaces $X$ (see, for example, [Na1, Na2, LQW]). On the one hand, the projective varieties $G_T$ and the bundles $Z_T$ over them may be regarded as subschemes of the local punctual Hilbert scheme, the fibre over the point $n \cdot p_0$ of the symmetric product. On the other hand, the varieties $G_T$ appear to be rather more complicated, since the Picard group $\text{Pic}(\text{Hilb}^n\mathbb{P}^2) \otimes \mathbb{Q} = \mathbb{Q} \oplus \mathbb{Q}$, whereas the Picard group of $G_T$ is by Theorem 3.28 in general of higher rank.

This article is a revision of the preprint [IY1]; we have changed the organization, title and exposition throughout. It is the geometric portion, originally entitled “Part II” of a work whose first avatar was a 1991/1992 preprint. We refer here to the preprint [IY2] on the combinatorial aspects of the hook code for partitions as “Part I”, even though this article is no longer titled “Part II”, and we plan to split “Part I” into two shorter articles.

We now summarize the sections. In Section 2 we study the special case $\mu = j$ where there is a single linear system, rather than an ideal. We define the ramification of a linear system at a point of $\mathbb{P}^1$ and we connect the ramification – a partition – with the cells $\mathcal{V}(E)$ parametrizing vector spaces having a given initial monomial vector space $E$. We then use the Wronskian of a vector space of forms $V$ in $R_j$ to determine the total ramification number of $V$ at all points of $\mathbb{P}^1$. In Section 2-C we apply the Schubert calculus to this case, and in Section 2-C we summarize the known results on the intersection of ramification conditions on linear systems over $\mathbb{P}^1$.

In Section 3 we extend our study of ramification to an ideal of linear systems, our main theme. Using our results from Section 2 we show that the ramification loci are the same as the cells $\mathcal{V}(E)$ (Proposition 3.6).

Proposition. Let $E$ be a monomial ideal with $H(R/E) = T$, and $p \in \mathbb{P}^1$, and suppose that $I$ is an ideal with $H(R/I) = T$. The following are equivalent:

i. $I \in \mathcal{V}(E,p)$

ii. $\text{QRAM}(I,p) = \text{QRAM}(E)$.

Thus, if $p : ax + by = 0$ is a point of $\mathbb{P}^1$, the condition $I \in \mathcal{V}(E,p)$, specifies both the initial form ideal of $I$ in a certain basis $(ax + by, C)$ for $k[x, y]$, and also the ramification of $I$ at $p$. We also define the partition $P(E)$ determined by the monomial ideal $E$: its Ferrers graphs is that of a complementary basis $E^c$ to $E$ in $R$ (Definition 3.7). In Section 3-B we show we show that the cell $\mathcal{V}(E)$ is an affine space, with parameters given by certain coefficients of the generators of $I \in \mathcal{V}(E)$, corresponding to the difference-one hooks of the partition $P(E)$ (Theorem 3.12). We also show that there is a birational map: $G_T \to \text{SGrass}(T)$, the product of small Grassmanians (Proposition 3.13). In Section 3-C after giving a formula for the dimension of the affine space fibre $F_{\mathcal{V}/\mathcal{V}}$ of the cell $Z(E)$ (all ideals) over $\mathcal{V}(E)$ (graded ideals) in Proposition 3.15, we reconcile our dimension formulas for $Z(E)$ and $\mathcal{V}(E)$ with formulas of L. Göttshe. In Section 3-D we define the hook code of a partition $P$ of diagonal lengths $T$, and we show that the hook code gives an isomorphism between two distributive lattices, the second related to the product of Schubert cells on the small Grassmanians studied earlier (Theorem 3.20). In Section 3-E we show that there is an additive isomorphism given by the hook code between the homology $H^*(G_T)$, and $H^*(\text{SGrass}(T))$ (Theorem 3.28), and we determine the Poincaré polynomial of $G_T$ (Theorem 3.29). In Section 3-F we summarize what we know about the ramification loci for ideals $I \in G_T$. The main limitation in comparison with the similar theory for Grass($d, R_j$) is that the intersection of ramification conditions $Z = \overline{\mathcal{V}(E,p) \cap \mathcal{V}(E',p')}$ at different points is not necessarily dimensionally proper!. This was shown by the second author for $T = (1, 2, 3, 2, 1)$ [Y1]. When $Z$ has the right codimension, then its homology class $[Z]$ can be read from the homology ring $H^*(G_T)$ (Theorem 3.11). In particular if $Z$ is a zero-dimensional set, the number of vector spaces satisfying given ramification conditions can be calculated, if we know the homology ring $H^*(G_T)$ in terms of the classes of the cells $\mathcal{V}(E)$. 


In Section 1-A we review what is known about the homology ring structure \( H^\ast(G_T) \). There is a natural immersion \( i : G_T \to B\text{Grass}(T) \) into a variety \( B\text{Grass}(T) \) that is a product of big Grassmannians \( \text{Grass}(i) = \text{Grass}(i+1-t_i, i+1) \); we take the ideal \( I \) into its degree-\( i \) pieces, for each \( i \). Recently, A. King and C. Walter have shown that \( i^* : H^\ast(B\text{Grass}(T)) \to H^\ast(G_T) \) is a surjection [KW]. A result of the second author exhibits \( G \) in finding the homology ring \( H^\ast(G_T) \) at all points \( T \). Ideals satisfying a certain intersection of ramification conditions on \( G \) (Theorem 1.1). We illustrate our approach by determining the ideals satisfying a certain intersection of ramification conditions on \( G_T, T = T(\mu,j) \) (Example 1.3). In Section 1-C we study the \( \mu \)-secant variety \( \text{Sec}(\mu,j) \) to the degree-\( j \) rational normal curve, of which \( G_T, T = T(\mu,j) \) is a desingularisation. We determine the classes in \( H^\ast(G_T) \) of the pullbacks of the higher singular loci of \( \text{Sec}(\mu,j) \) (Theorem 1.11).

As a result of Theorem 3.20 and Proposition 3.6 we have coded the ramification conditions on ideals of \( G_T \) at a fixed point \( p \) by the hook code. It remains to read off the ramification \( \text{QRAM}(I_t,p) \) of each piece \( I_t \), given the code. This problem is studied and solved in Algorithm 2.29 of Part I [IY2]. There we defined a “strand map” and with it we constructed a partition \( Q_t(E,p) \) that determines \( E_t \) directly from \( D(E) \). Here we show that \( Q_t(E,p) \) is the dual of the complement \( \text{QRAM}(E_t,p) \) (Lemma 2.10).

The sum of the parts of \( \text{QRAM}(E_t,p) \) is the codimension of the condition on \( \text{Grass}(i) \) that a vector space \( V \) in \( R_t \) satisfies \( \text{QRAM}(V,p) = \text{QRAM}(E_t,p) \). Concerning the ramification of an ideal \( I \) at different points, we show, denoting by \( \ell(P) \) the sum of the parts of \( P \) (Lemma 2.11 and Proposition 2.10),

**Theorem.** If \( I \in G_T \), then for each \( i \),

\[
\sum_{p \in \mathbb{P}^1} \ell(\text{QRAM}(I_t,p)) = t_i(i+1-t_i).
\] (1.4)

Summing over \( i \), we find that the sum of the lengths of the ramification conditions that \( I \) satisfies at all points \( p \) is

\[
\sum_{p \in \mathbb{P}^1} \ell(\text{QRAM}(I,p)) = \dim(B\text{Grass}(T)).
\]

This result is a consequence of a stronger result that for each \( i \) the set of ramification partitions \( \text{QRAM}(I_t,p), p \in \mathbb{P}^1 \), or, equivalently, the set of Schubert classes \( \{ Q(I_t,p), p \in \mathbb{P}^1 \} \) must be “complementary”: their intersection can be calculated using the Littlewood-Richardson rule: the intersection must be nonempty and nonzero provided the codimensions are not too large, and the calculated homology class is nonzero. (Proposition 2.17). There is a Wronskian morphism

\[
W : G_T \to \mathbb{P}, \quad \text{where } \mathbb{P} = \prod_{i=\mu}^{\ell} \mathbb{P}^{N_i}, \quad n_i = t_i(i+1-t_i).
\]

The morphism \( W \) is a product of finite covering maps \( w_i : \text{Grass}(i) \to \mathbb{P}^{N_i} \) studied in Section 2 (Proposition 2.15). The map \( W \) is a finite cover of its image (Proposition 1.8), but in general its image has large codimension in \( \mathbb{P} \). What are the equations describing the image of \( W \)? These equations constitute mysterious, hidden relations among the ramification of \( I \) at different points of \( \mathbb{P}^1 \).

This article and Part I [IY2] replace the preprint [I-Y1]. Part I is the combinatorial portion; this is the algebraic-geometric portion. Our intent in following a referee’s suggestion to split the article is to make the results more accessible to specialists in each area.

**Acknowledgment.** It was a question of G. Gotzmann [Gm-1] about the relation of his calculation of a simple formula for the rank of \( \text{Pic}(G_T) \) (see Theorem 3.20) with the results of L. G"ottsche.
that led to our work. We thank J. Briançon, S. Diesel, D. Eisenbud, G. Ellingsrud, J. En- salem, A. Geramita, G. Gottzmann, A. Hirschowitz, P. LeBarz, K. Ohara, M. Merle, R. Stanley, A. Suciu, S. Xambó-Descamps, J. Weyman, B. Zaslov and A. Zelevinsky for their comments. We thank H. Matsumura, organizer of Commutative Algebra and Combinatorics in Nagoya in 1990, A. Galligo and other organizers of the MEGAS seminar at Nice in April 1992, E. Kunz, H.J. Nastold, and L. Szpiro, organizers of a Commutative Algebra meeting at Oberwohlbach in 1992, and B. Sturmfels and D. Cox, organizers of a portion of the 1992 Regional Geometry Institute on Computational Algebraic Geometry. Revisions and the addition of Section 4-C were made during a visit of the first author to the Laboratoire J.-A. Dieudonné, UMR CNRS 6621, in 1997, Further revisions were made soon after S. Kleiman’s 60th birthday conference. We thank Steve Kleiman for informative discussions and encouragement.

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2 Vector spaces of forms, linear systems on $\mathbb{P}^1$, and Wronskians

We let $R$ denote the polynomial ring $k[x, y]$ in two variables, where $k$ is an algebraically closed field. We let $R_j = \langle x^j, x^{j-1}y, \ldots, y^j \rangle$ be the subspace of degree-$j$ homogeneous polynomials. In Section 2-A we first describe how a $d$-dimensional vector space $V$ of $R_j$ is a linear system on the projective line $\mathbb{P}^1$. We then define the ramified points $p \in \mathbb{P}^1$ of $V$, and a ramification partition $\text{QRAM}_p(V)$ of the vector space $V$ at each point $p \in \mathbb{P}^1$ (Definition 2.3). Recall that Grass($d, R_j$) is the Grassmanian parametrizing $d$-dimensional subspaces of $R_j$; it has dimension $N = d(j+1-d)$. We define a Wronskian morphism from Grass($d, R_j$) to $\mathbb{P}^N$, and use it to show that the total lengths of the ramification partitions for a fixed $V$ is $N = \dim(V) \cdot \text{cod}(V)$ (Definition 2.3, Lemma 2.7).

We will also show that $\text{QRAM}_p(V)$ is determined by the initial monomials for $V$ in a particular basis for $R$ (Lemma 2.10). Given $E$ a set of monomials of $R_j$ we define the “cell” $V(E)$ as the
subscheme of the Grassmanian having initial monomials $E$, and $\forall(E)$ an analogal cell at the point $p \in \mathbb{P}^1$ (Definition 2.12).

In Section 2-B we show that if $E$ is a monomial vector space, the subscheme $\forall(E,p) \subset \text{Grass}(d,R_j)$ is a Schubert cell of the Grassmannian (Lemmas 2.13, 2.14). We also show that the Wronskian morphism $w : \text{Grass}(d,R_j) \rightarrow \mathbb{P}^N$ is a finite cover (Proposition 2.15). The intersection of ramification loci at $s$ different points of $\mathbb{P}^1$ is nonempty and has a class given by the Schubert calculus, provided their codimensions sum to less than $N$ and that class is nonzero (Proposition 2.17). In Section 2-C we summarize in this context what is known classically for the intersection of ramification loci of linear systems at distinct points of $\mathbb{P}^1$. We will give a similar summary in Section 3.3 for the varieties $G_T$.

### 2-A Ramification and the Wronskian

**Definition 2.1. Linear System of a Vector Space $V \subset R_j$.** If $p = (\alpha_p, \beta_p)$ is a point of the projective space $\mathbb{P}^1 = \mathbb{P}(R_1)$, we let $L_p$ be any linear form $ax + by \in R_1$ vanishing at $p : a\alpha_p + b\beta_p = 0$; the vector space $\langle L_p \rangle$ and the class $\{L_p \text{ up to } k^* - \text{ multiple} \}$ in $\mathbb{P}^1 = \mathbb{P}(R_1)$ are uniquely determined by $p$. Conversely, given $\lambda \in R_1$ we let $p\lambda \in \mathbb{P}^1 = \mathbb{P}(R_1)$ be the point where it vanishes. The vector space $\langle f \rangle$ spanned by a single degree-$j$ homogeneous polynomial $f$ in $R = \mathbb{K}[x, y]$, $f = L_1 \cdots L_i$ corresponds to a zero cycle $p_f = p_{L_1} + \cdots + p_{L_i}$ of $j$ points - counting multiplicities - on the projective line $\mathbb{P}^1$. A $d$-dimensional vector subspace $V$ of $R_j$ corresponds to a unique linear system on $\mathbb{P}^1$:

$$\mathcal{L}(V) = \{p_f \mid f \in V \} \subset \mathbb{P}^1 = \text{Sym}^d(\mathbb{P}^1).$$

(2.1)

$\mathcal{L}(V)$ is a linear subspace of $\mathbb{P}^1$ and has projective dimension $(d - 1)$.

**Example 2.2.** Let

$$V_a = \langle y^2x - a^2x^3, \ yx^2 + ax \rangle = \langle y(xy + ax^2), \ x(xy + ax^2) \rangle.$$

The general element of the one-dimensional family $\mathcal{L}(V_a)$ is the zero set $p_{F_i}$ of

$$F_i = t(y^2x - a^2x^3) + (1 - t)(yx^2 + ax).$$

Here, $\mathcal{L}(V_a)$ consists of all sets of three points in $\mathbb{P}^1$ including both $P : x = 0$ and $P' : y + ax = 0$, the base points of $\mathcal{L}(V_a)$. In particular, $\mathcal{L}(V_a)$ contains $2P + P'$ and $2P' + P$. (See Figure 7.)

We now define the ramification $\text{QRAM}_p(V)$ of $V$ at a point $p \in \mathbb{P}^1$. It will be a subpartition of the $\dim V \times \text{cod} V$ rectangular partition $B(d,j + 1 - d)$ with $d$ parts of size $j + 1 - d$. We will consider $\text{QRAM}_p(V)$ to have $d$ parts, although some of the parts may be zero.

**Definition 2.3 (Ramification of $V \subset R_j$ at $p$).** Suppose that $L$ is a linear form $L = ax + by$ corresponding to the point $p : ax + by = 0$ of $\mathbb{P}^1$, that $V = \langle f_1, \ldots, f_d \rangle$ is a $d$-dimensional subspace of $R_j$, and that $F = (f_1, \ldots, f_d)$ satisfies

$$f_i = L^n_i(p,V)g_i(p,V), \quad n_1(p,V) < n_2(p,V) < \cdots < n_d(p,V).$$

(2.2)

The degree sequence $N_p(V)$ of $V$ at $p$ is

$$N_p(V) = (n_1(p,V), \ldots, n_d(p,V)).$$

(2.3)

It is easy to see that there is a basis $F$ of $V$ satisfying (2.2) and that $N_p(V)$ is independent of the choice of $F$. The ramification sequence $\text{QRAM}_p(V) \subset B(d,j + 1 - d)$ of $V$ at the point $p \in \mathbb{P}^1$ is the partition constructed in the usual way from $N_p(V)$:

$$\text{QRAM}_p(V) = (r_1(p,V), \ldots, r_d(p,V)), \quad r_i(p,V) = n_i(p,V) - (i - 1).$$

(2.4)
We say that $V$ is unramified at $p$ if $N_p(V) = (0, 1, \ldots, d - 1)$, the minimum possible sequence, so $\text{QRAM}_p(V) = (0, \ldots, 0)$. Otherwise, $V$ is ramified at $p$. The total ramification $r(p, V)$ of $V$ at $p$ satisfies

$$r(p, V) = \sum_i r_i(p, V) = \ell(\text{QRAM}_p(V)), \quad (2.5)$$

the length of $\text{QRAM}_p(V)$. Thus, $V$ is ramified at $p$ iff the total ramification $r(p, V) > 0$.

**Example 2.4.** Let $p$ be $x = 0$, and let $E$ be the monomial vector space

$$E = \{x^{n_1}y^{-n_1}, \ldots, x^{n_d}y^{-n_d}\}, \quad n_1 < \ldots < n_d$$

then we denote by $\text{QRAM}(E)$ the sequence $\text{QRAM}_x(E) = (n_1 - 0, \ldots, n_d - (d - 1))$, and by $r(E)$ the total ramification

$$r(E) = \ell(\text{QRAM}(E)) = (\sum_{1 \leq i \leq d} n_i) - d(d - 1)/2 \quad (2.6)$$

of $E$ at $p$. (Recall that $\ell(P)$ for a partition $P$ is the sum of its parts.)

**Remark.** For most vector spaces $V$ of dimension two in $R_3$, there are four distinct ramification points $P_i$, $i = 1, \ldots, 4$ such that $2P_i + P'_i$ is an element of the linear system $L(V)$ for some $P'_i$. For the space $V_a$ of Example 2.2 there are only two ramification points $P_1$ and $P_2$, at each of which $\text{QRAM}_{P_1}(V_a) = \text{QRAM}_{P_2}(V_a) = (1, 1)$; thus, $V_a$ has total ramification two at each of $P_1$, $P_2$. In either case the sum of the total ramification of $V$ over all points of $\mathbb{P}^1$ is $4 = \text{cod}(V) \cdot \text{dim}(V)$. We now introduce a form, the Wronskian determinant $W(V)$ of $V$. We will show in Lemma 2.7 that the multiplicity of its roots at $p$ is the total ramification of $V$ at $p$, and that its degree is $N = \text{cod}(V) \cdot \text{dim}(V)$.

**Definition 2.5 (Wronskian).** Suppose that $V$ is a $d$-dimensional subspace of $R_j$. We will define up to nonzero constant multiple a degree-$N$ Wronskian form $R$, where $N = d(j + 1 - d)$. The Wronskian form determines a unique element, the Wronskian determinant $W(V) \in \mathbb{P}(R_N)$, the projective space. We let $R' = k[x, y, dx, dy]$, the polynomial ring and let $R = k[dx, dy]$. We define a derivation $D : R' \rightarrow R'$ by

$$D = 0 \text{ on } R,$$

$$D : R \rightarrow R' \text{ satisfies } Df = f_xdx + f_ydy.$$
Thus if \( f_i \in R \) and \( g_i \in \mathbb{R} \) we have \( D(\sum f_i g_i) = \sum g_i (f_i x dx + f_i y dy) \). If \( V = \langle f_1, \ldots, f_d \rangle \) is a \( d \)-dimensional vector subspace of \( R_j \), we let \( N = d(j+1-d) = (\dim(V)) \cdot (\text{cod}(V)) \), and we define the following degree-\( N \) homogeneous polynomial \( W(f_1, \ldots, f_d) \):

\[
W(f_1, \ldots, f_d) = \det \left( \begin{array}{ccc}
f_1 & \cdots & f_d \\
Df_1 & \cdots & Df_d \\
\vdots & \ddots & \vdots \\
D^{d-1}f_1 & \cdots & D^{d-1}f_d \\
\end{array} \right) / (x dy - y dx)^{d(d-1)/2} \in R_N.
\] (2.7)

The polynomial \( W(f_1, \ldots, f_d) \) is a degree-\( N \) homogeneous form so is an element of \( R_N \). Its class \( W(V) \mod k^*\)-multiple is independent of the basis chosen for \( V \), and we define

\[
W(V) = W(f_1, \ldots, f_d) \mod k^* - \text{multiple in } \mathbb{P}(R_N).
\] (2.8)

We denote by \( \text{Grass}(d, R_j) \), the Grassmannian parametrizing \( d \)-dimensional vector subspaces of \( R_j \). The map \( V \mapsto W(V) \) defines a Wronskian morphism

\[
w : \text{Grass}(d, R_j) \longrightarrow \mathbb{P}^N, \quad N = d(j+1-d)
\] (2.9)

**Example 2.6.** When \( V_a = \langle y^2x - a^2x^3, yx^2 + ax^3 \rangle \) the Wronskian \( W(V_a) = x^2(y + ax)^2 \) up to \( k^*\)-multiple.

**Lemma 2.7.** If \( \text{char } k > j \) or is 0, and \( V \subset R_j \), then

\[
W(V) = \prod_{p \in \mathbb{P}^1} L_r(p,V) \text{ up to } k^* - \text{multiple.}
\] (2.10)

The product is over points \( p \) of \( \mathbb{P}^1 \). Any vector space \( V \) is unramified at all but a finite number of points of \( \mathbb{P}^1 \). The linear form \( L_r \) divides \( W(V) \) iff \( \exists f \in V \) such that \( (L_r)^d | f \). We have

\[
\sum_{p \in \mathbb{P}^1} r(p,V) = N.
\] (2.11)

**Proof.** For (2.10), it suffices, considering the \( \text{PGL}(1) \) action on \( W(V) \), to take \( L = x \). By substituting \( y = 1 \), then \( W(V)(x,1) \) is a usual Wronskian determinant, and has the value \( cx^{r(x,V)} \) as a consequence of a Van der Monde style calculation. From Definition 2.6 degree \( W(V) = d(j+1-d) \), provided \( (x dy - y dx)^{d(d-1)/2} \) divides the numerator of (2.7), which is easily verified. The formula (2.11) then follows from (2.10). That \( L_p W(V) \leftrightarrow (L_p)^d \) divides some \( f \in V \) follows from (2.10).

□

We now define a partition \( Q(V,p) \subset B(\text{cod}(V), \dim V) \), determined by a standard basis of \( V \) in the direction \( p \). If \( p \in \mathbb{P}^1 \) is the point \( L = 0 \), and \( C \in R_1 \) satisfies \( \langle C \rangle \neq \langle L \rangle \) we use the reverse alphabetic order on the set of degree-\( j \) monomials of \( R \), \( C^j < C^{j-1}L < \ldots < L^j \). If \( f \in R_j \), we let \( \text{In}_p(f) \) = the initial monomial of \( f \) in this basis. If \( V \subset R_j \), we let \( \text{In}_p(V) = \langle \text{In}_p(f), f \in V \rangle \), a space spanned by monomials in \( C, L \). If \( E_L \) is spanned by monomials in \( C, L \) we let \( E^c_L \) be the complementary set of monomials in \( C, L \).

**Definition 2.8 (Partition \( Q(V,p) \)).** If \( V \subset R_j \), if \( p \in \mathbb{P}^1 \) is the point \( L = 0 \), and if \( V \) has dimension \( d \) and codimension \( t = j+1-d \) in \( R_j \), we let \( E_L = \text{In}_p(V) \), and

\[
E^c_L = \{ L^{a_1}C^{j-a_1}, \ldots, L^{a_t}C^{j-a_t} \}, \quad a_1 < \ldots < a_t.
\]

We define the partition \( Q(V,p) \subset B(t,d), t = j+1-d \) by

\[
Q(V,p) = \langle a_1, a_2 - 1, \ldots, a_t - (i-1), \ldots, a_t - (t-1) \rangle.
\] (2.12)
The partition \( Q(V,p) \) has \( t = \text{cod}(V) \) parts, some of which may be zero. Each part is no greater than \( j + 1 - t = \dim(V) \) (Lemma 2.9).

We now give an equivalent definition of this partition. If \( E_L \) is a set of \( d \) degree–\( j \) monomials in \((L,C)\), and \( E'_L \) is its complement, we let \( S(E_L) \) denote the following set of ordered pairs of degree–\( j \) monomials in \( L,C \):

\[
S(E_L) = \{(\mu,\nu) : \mu \in E_L, \nu \in E'_L, \text{ and } \mu < \nu\}.
\]

Given a monomial \( \mu \in E'_L \), we define a subset of degree–\( j \) monomials

\[
S_{E_L}(\nu) = \{\mu \in E_L \mid \mu < \nu\}.
\]

If \( P \) is a partition, we let \( \ell(P) \) be the number it partitions.

**Lemma 2.9.** Let \( \text{In}_p(V) = E_L \). There is one part \( q_\nu \) of the partition \( Q(V,p) \) corresponding to each of the \( t \) cobasis monomials \( \nu \in E'_L \). The part \( q_\nu \) is \( \#S(E_L(\nu)) \) and satisfies \( q_\nu \leq j + 1 - t \), thus we have \( Q(V,p) \subseteq B(\text{cod}V, \dim V) \). Each partition \( Q \subseteq B(t,j+1-t) \) occurs as the partition \( Q(E_L,p) \) for a unique monomial vector space \( E_L \).

**Proof.** If \( \nu = L^{a_1}C^{j-a_1} \in E'_L \), then \( i - 1 \leq a_i \leq j - (t-1) \), to leave room for the other monomials \( \nu' \in E'_L \) before and after \( \nu \). Also,

\[
S_{E_L}(\nu) = \{C^j, LC^{j-1}, \ldots, L^{a_i}C^{j-a_i}\} - \{L^{a_1}C^{j-a_1}, \ldots, L^{a_i}C^{j-a_i}\},
\]

has \((a_i) + (i) = a_i - (i-1)\) elements. This is the \( i \)-th part \( q_i \) of \( Q(E,p) \), and \( q_i \) thus satisfies \( 0 \leq q_i \leq j + 1 - t \). Conversely, given \( Q \) let \( a_i = q_i + (i+1) \). Choosing monomials \( \nu_i = L^{a_i}C^{j-a_i} \), we obtain a set \( E_L(Q) \) of \( t \) distinct degree–\( j \) monomials such that \( Q(E_L,p) = Q \). The space \( E_L \) is easily seen to be independent of the choice of \( C \), so is unique.

**Notation** Let \( P \) be a subpartition of the rectangular partition \( B(t,j+1-t) \) with \( t \) parts of size \( j+1-t \), so \( P \) has \( t \) parts (some possibly zero), each of size no greater than \( (j+1-t) \). We denote by \( P^c \) its complement in the rectangular partition \( B(t,j+1-t) \), and we denote by \( P^n \subseteq B(j+1-t,t) \) its dual partition (switch rows and columns in the Ferrers diagram).

**Lemma 2.10 (Ramification partition QRAM\(_p\)(V) and QRAM\(_p\)(V,p)).** If \( p \) is any point of \( \mathbb{P}^1 \), the partition \( Q(V,p) \subseteq B(t,j+1-t) \) is related to \( \text{QRAM}_p(V) \subseteq B(j+1-t,t) \) by

\[
\text{QRAM}_p(V) = (Q(V,p)^c)^e.
\]

Each partition having \( \dim(V) \) parts, each no greater than \( \text{cod}(V) \) occurs as \( \text{QRAM}_p(E_L) \) for a unique monomial vector space \( E_L \).

**Proof.** If \( \text{In}_p(V) = E_L \) (\( \ell^aC^{j-n_1}, \ldots, L^{a_i}C^{j-n_i}, \ldots, L^{a_d}C^{j-n_d} \)), then by Definition 2.8 \( Q(V) = (n_1, \ldots, n_i - (i-1), \ldots, n_d - (d-1)) \). The partition \( Q(V,p) \) in Definition 2.8 is constructed similarly from the complementary set of monomials \( E'_L \). The formula (2.13) thus reduces to an easily shown combinatorial identity. The last statement is a consequence of Lemma 2.9.

**Example 2.11.** Let \( V = (x^4, x^3y, x(x+y)^3) \) and the point \( p : x = 0 \); then \( n_p(V) = (4,3,1) \) so \( \text{QRAM}_p(V) = (4,3,1) - (2,1,0) = (2,2,1) \). We have \( E = \text{In}(V) = (xy^3, x^5y, x^4) \) so \( E^c = (y^4, y^2x^2) \) and \( Q(V,p) = (1,0) \subseteq B(3,3) \). The complementary partition \( Q(V,p)^c \) in \( B(3,3) \) is \( (3,2) \), whose dual partition is \( (2,2,1) \); thus \( \text{QRAM}(V,p) = (Q(V,p)^c)^e \) (Lemma 2.10).

**Definition 2.12 (The cells \( \mathbb{V}(E,p) \) in Grass(d,R\(_j\))).** If \( E \) is a \( d \)-dimensional vector space spanned by monomials, and \( p \in \mathbb{P}^1 \) is the point \( p : x = 0 \), then we denote by \( \mathbb{V}(E) \) or \( \mathbb{V}(E,p) \) the subfamily of the Grassmannian \( \text{Grass}(d,R_j) \)

\[
\mathbb{V}(E) = \{V \mid \text{In}(V) = \text{In}(E)\}.
\]
Let $p \in \mathbb{P}^1$ be arbitrary, and let $E = \{x^{n_1}y^{j-n_1}, \ldots, x^{n_d}y^{j-n_d}\}$, $n_1 < \ldots < n_d$ be a set of $d$ monomials in $x,y$. We denote by $E_L = \{L^{n_1}C^{j-n_1}, \ldots, L^{n_d}C^{j-n_d}\}$, the corresponding set of monomials in any basis $(L,C)$ for $R$, $L = L_p$, $C \in R_1$ with $\langle C \rangle \neq \langle L \rangle$. We denote by $\mathcal{V}(E,p)$ the monomials in any basis $(L,C)$ for $R$ : $\mathcal{V}(E,p)$ the set of vector spaces whose initial forms in the basis $C^j < C^{j-1}L < \ldots < L^j$ for $R_j$ is $E_L$. We have

$$\mathcal{V}(E,p) = \{V \subseteq R_j \mid \dim_k V = d, \text{ and } \operatorname{QRAM}(V,p) = \operatorname{QRAM}(E)\}. \quad (2.15)$$

We give $\mathcal{V}(E,p)$ the reduced subscheme structure inherited from $\operatorname{Grass}(d, R_j)$. We will show that the closure $\overline{\mathcal{V}(E,p)}$ is a Schubert cell of the Grassmannian $\operatorname{Grass}(d, R_j)$ below in Lemma 2.14.

### 2-B Ramification and Schubert cells of the Grassmanian $\operatorname{Grass}(d, R_j)$

Let $E$ be a subspace of $R_j$ spanned by monomials in $x,y$; we let $E^c$ denote the degree—$j$ monomials not in $E$, which we term *cobasis* monomials. For $p \in \mathbb{P}^1$ we denote by $\mathcal{F}_j(p)$ the flag

$$\mathcal{F}_j(p) : \emptyset = F_0(p) \subset F_1(p) \subset \cdots \subset F_i(p) \subset \cdots \subset F_{j+1}(p) = R_j$$

of subspaces of $R_j$, where $C \in R_1$ satisfies $\langle C \rangle \neq \langle L_p \rangle$ and where

$$F_i(p) = \langle L_p^i, L_p^{i-1}C, \ldots, L_p^0, L_p^{j-i}C^{j-i-1} \rangle. \quad (2.17)$$

#### Lemma 2.13

Let $E = \{x^{n_1}y^{j-n_1}, \ldots, x^{n_d}y^{j-n_d}\}$, $n_1 < n_2 < \ldots < n_d$, and consider a point $p$ of $\mathbb{P}^1$. The following are equivalent:

1. $V \subseteq \mathcal{V}(E,p)$, or, equivalently, $Q(V,p) = Q(E)$.
2. $\operatorname{QRAM}(V,p) = \operatorname{QRAM}(E)$.
3. $\dim_k (V \cap F_{j+1-n_i}(p)) = d + 1 - i, \quad i = 1, \ldots, d$. \quad (2.18)

#### Lemma 2.14

For any $p \in \mathbb{P}^1$, the subvariety $\mathcal{V}(E,p)$ is an open dense subset of a Schubert cell $\overline{\mathcal{V}(E,p)}$ on $\operatorname{Grass}(d, R_j)$ and it is an affine space. The dimension of $\mathcal{V}(E,p)$ is

$$\dim(\mathcal{V}(E,p)) = \# \mathcal{S}(E) = N - r(E) = \ell(Q(E)). \quad (2.19)$$

where $r(E) = \ell(\operatorname{QRAM}(E))$ of (2.10) and it has codimension in $\operatorname{Grass}(d, R_j)$

$$\operatorname{cod}(\mathcal{V}(E,p)) = r(E) = \ell(\operatorname{QRAM}(E)) = \left( \sum_{1 \leq i \leq d} n_i \right) - d(d-1)/2. \quad (2.20)$$

#### Proof of Lemma 2.13

It suffices to consider $p: x = 0$. We have $V \subseteq \mathcal{V}(E)$ iff $\operatorname{In}(V) = E$. Let $(f_1, \ldots, f_d)$ be a basis of $V$ such that $\operatorname{In}(f_i) = x^{n_i}y^{j-n_i} \in E$, for each $i$. Then $n_1 < n_2 < \ldots < n_d$, $x^{n_i}f_i$, and it is easy to see that there is no basis $F'$ of $V$ with larger $x$-powers than $n_i$. By Definition 2.23, $\operatorname{QRAM}(V,p) = \operatorname{QRAM}(E)$. From $\operatorname{In}(V) = E$ we see that (2.18) is satisfied. Thus $i. \implies ii. \implies iv.$, and the converse implications are an easy consequence. Lemma 2.10 shows $ii. \iff iii. \quad \Box$

#### Proof of Lemma 2.14

The identification of $\mathcal{V}(E)$ with a Schubert cell is immediate from (2.18). That $\mathcal{V}(E)$ is an affine space, and $\operatorname{cod}(\mathcal{V}(E)) = r(E)$ in $\operatorname{Grass}(d, R_j)$ is a standard result (see §1.5 of [GrH], or see the proof below of Theorem 3.12 in the special case $T = (1, 2, \ldots, j, t, 0)$). The formula (2.19) follows from (2.20) and Lemma 2.10 \quad \Box

If $G$ is a complex algebraic variety, we grade the homology ring $H^*(G, \mathbb{Z})$ by codimension, and consider the intersection product $H^*(G, \mathbb{Z}) \times H^*(G, \mathbb{Z}) \rightarrow H^{*+*}(G, \mathbb{Z})$. Since our varieties are connected, if $N = \dim(G) = d(j + 1 - d)$, $H^N(G) \cong \mathbb{Z} \cdot \zeta_0$ and has basis the class $\zeta_0$ of a point. If $a\zeta_0 \in H^N(G)$ we will denote by $[a\zeta_0]$, the integer $a$.  

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Proposition 2.15. Assume $k = \mathbb{C}$. The Wronskian morphism $w : \text{Grass}(d, R_j) \to \mathbb{P}^N$ is a finite cover of degree $[c_1^N] = N!/(j^d)$ where $c_1$ is the class of a hyperplane in the homology $H^*(\text{Grass}(d, R_j))$. The degree of $w$ is the same as that of Grass($d, R_j$) in the Plücker embedding.

Proof. Clearly $w$ is a proper algebraic morphism. We let $q \in \mathbb{P}^N = \text{Sym}^N(\mathbb{P}^1)$ correspond to a set $\{g\} = p_1, \ldots, p_N$ of $N$ distinct points of $\mathbb{P}^1$. For each $i$, the condition that $\text{QRAM}(V, p_i) = (0, \ldots, 0, 1), 1 \leq i \leq N$ is by Lemma 2.10 a simplest Schubert condition on Grass($d, R_j$), whose class is $c_i$ in the homology ring $H^*(G_T)$. Since the points $p_1, \ldots, p_N$ are distinct the Schubert conditions are distinct. The intersection of all $N$ conditions will consist of $m = [c_1^N]$ points of Grass($d, R_j$), provided that the intersection is proper. Should the intersection not be proper, and there is a one dimensional family of vector spaces in the intersection, then we would impose an additional condition $\text{QRAM}(V, p_{N+1}) = 0$ at any further point $p_{N+1} \in \mathbb{P}^1$, and find a vector space having $N + 1$ ramification points, contradicting Lemma 2.7. Thus, there will be $[c_1^N] = N!/(j^d)$ vector spaces $V$ having total ramification one at each of the points $p_i$. By (2.21) any such vector space $V$ can have no further ramification, and $w(V) = q$. We have shown that $w : w^{-1}(\mathbb{P}^N \setminus \Delta) \to \mathbb{P}^N \setminus \Delta$ is an $m$-to-one cover, where $\Delta$ is the large diagonal; since Grass($d, R_j$) is irreducible, $w$ is a finite cover as claimed. That the integer $[c_1^N]$ is the the degree of Grass($d, R_j$) in its Plücker embedding is well known. \ensuremath{\square}

Definition 2.16. Recall that the homology class of the intersection $\mathbb{V}_{Q_1} \cap \mathbb{V}_{Q_2}$ of two Schubert subvarieties of Grass($d, R_j$) contains a term corresponding to $Q_a$ iff there are permutations $\tau$ and $\sigma$ of $(1, \ldots, j + 1 - d)$ such that we can write each part $q_a$ of $Q_a$ as $q_a = q_{a_1}(\tau(i)) + q_{a_2}(\tau(i))$, the sum of parts in $Q_1$ and $Q_2$. We say $Q_a \subset Q_1 + Q_2$, in this case, and extend this definition to sums of $s$ partitions.

Proposition 2.17. Let $p_1, \ldots, p_s$ be distinct points of $\mathbb{P}^1$, and let $E_1, \ldots, E_s$ be monomial vector spaces of dimension $d$ in $R_j$. Assume that $\sum \ell(Q(E_i)) \leq N = d(j + 1 - d)$. Then the homology class of the intersection $\mathbb{V}(E_1, p_1) \cap \ldots \cap \mathbb{V}(E_s, p_s)$ of cells satisfies

$$[\mathbb{V}(E_1, p_1) \cap \ldots \cap \mathbb{V}(E_s, p_s)] = \sum n_a[Q_a], \enspace Q_a \subset Q(E_1) + \ldots + Q(E_s). \tag{2.21}$$

Here the coefficients $n_a$ are given by the Schubert calculus. If the above class in (2.21) is nonzero, the codimension of the intersection is

$$\sum r(E_i) = N - \sum \ell(Q(E_i)). \tag{2.22}$$

Proof. Immediate from Lemmas 2.10 and 2.14 and the Schubert calculus, provided that the intersection $Y = \mathbb{V}(E_1, p_1) \cap \ldots \cap \mathbb{V}(E_s, p_s)$ is proper and nonempty. The condition (2.21) for nonemptiness is a consequence of the Schubert calculus. It remains to show that the intersection is proper if nonempty. WLOG we may add codimension 1 conditions $\text{QRAM}(V, p_i) = (0, \ldots, 0, 1)$ at a finite number of points $p_{s+1}, \ldots, p_{s'}$ so that $s' - s = N - \sum r(E_i)$, obtaining $Y' = Y \cap H_{s+1} \cap \ldots \cap H_{s'}$. Then the Wronskian of $V \in Y'$ satisfies: $W(V) = \prod_{i \leq i \leq s} L_{p_i}(E_i) \prod_{s < i \leq s'} L_{p_i}$. Since $w : \text{Grass}(d, R_j) \to \mathbb{P}^N$ is a finite cover, there are only a finite number of vector spaces $V$ having a given Wronskian, so $Y'$ is a finite set of points. It follows that $Y$ is proper if nonempty. \ensuremath{\square}

Table 2 gives the six cells of Grass($2, R_3$), for the $p : x = 0$ direction. At left the cobasis monomials for the cell $\mathbb{V}(E)$ are shaded; the initial monomials of generators of $V$ in the standard basis are unshaded. We give at right the partition $Q(E)$ (see Definition 2.8). The codimension of a cell $\mathbb{V}(E)$ is its depth below the top of the table. Thus, cells $\mathbb{V}(E)$ with higher powers of $x$ dividing the cobasis $E^c$ have a higher shaded portion $E^c$ in Table 2 these cells correspond to codings $Q(E)$ with larger area $\ell(Q(E))$ giving the dimension of the cell, or equivalently, to vector spaces $V \in \mathbb{V}$ with smaller ramification at $p$. 

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Table 2: The cells $\mathcal{V}(E)$ of Grass(2, 4) and their codings $Q(E)$. 

$\mathcal{V}(E)$ Cell | $Q(E)$ Code
--- | ---

A | $y^2 + a_1yx^2 + b_1x^3$

B | $y^3 + a_1y^2x + b_2x^3$

C | $y^3 + a_1y^2x + b_2x^3$

D | $y^2x + a_1x^3$

E | $y^7x + a_1yx^2$

F | $y^8x + a_1yx^2$

G | $y^2x^2$

$\emptyset$
Example 2.18 (Coding of a cell). The partition $Q(B)$ of Table 2 for the second cell $B$ has two rows. The length, two, of the first row $Q_{i_3}$, $v_1 = x^3$ counts the two pairs of monomials $(y^2, x^3)$ and $(y^3, x^3)$; equivalently, the length counts the two coefficients $b_1, b_2$ of the basis of $V$ on $v = x^3$. The length, one, of the second row $Q_{i_2}$, $v_2 = y^2x$ counts the single pair $(y^3, y^2x)$; equivalently, it counts the single coefficient $a_2$ of $y^2 + a_1y^2x + b_2$ on $y^2x$.

A given vector space $V$ has a coding $Q(V, p)$ for each point $p \in \mathbb{P}^1$. By Lemmas 2.7 and 2.13 for all but a finite number of points $p$ the partition $Q(V, p)$ is $B(j + 1 - d, d)$ and $QRAM(V, p) = 0$: we say that $V$ is in the generic cell for such points $p$ in which the partition $Q(V, p)$ is $B(j + 1 - d, d)$. We set cod $(Q(V, p)) = N - \ell(Q(V, p))$. By Lemma 2.10 cod $(Q(V, p)) = \ell(QRAM(V, p))$. Equations 2.23 and 2.11 imply

**Lemma 2.19.** We have

$$\sum_{p \in \mathbb{P}^1} \text{cod} (Q(V, p)) = N = \text{cod} (V) \cdot \dim(V).$$  \hspace{2cm} (2.23)

Example 2.20. In Table 2 the vector space $V_1 = (yx^2, x^3)$ is the smallest cell in the $p : x = 0$ direction and has codimension four. By the sum formula 2.23, $V_1$ lies in the generic cell in any other direction. The vector space $V_2 = (y^3x + x^3y^2)$ lies in the cell of codimension two at the right of the table, so must lie either in one cell of codimension two in some other direction, or in two cells of codimension one in two different directions (here, the latter, in directions $y - x$, $y + x$).

2-C Classical results for ramification conditions in Grass($d, R_j$)

We state known results for the Schubert cells $V(E, p)$ of the Grassmannian variety $\text{Grass}(d, R_j)$. Each cell consist of all vector spaces $V$ having given ramification $QRAM_p(V) = QRAM(E)$ at a point $p$ of $\mathbb{P}^1$; the cell also consists of all the vector spaces $V$ having given initial space $In(V) = E_L$ in a basis $(L_p, C)$ of $R$ (Lemma 2.10). This approach to ramification of linear systems over $\mathbb{P}^1$ by Schubert calculus has been extended to families of linear systems on curves in [E-H1, E-H2]. Our approach here emphasizes the connections with combinatorics, in preparation for considering ideals of linear systems in Section 3.

We recall our notation. We fix $d = \dim(V)$, $j = \deg(V)$, so cod $(V) = j + 1 - d$, and we let $N = d(j + 1 - d)$, the dimension of $\text{Grass}(d, R_j)$. We let cod $(V(E)) = N - \dim(V(E))$ be the codimension of the cell $V(E)$ in the Grassmannian $\text{Grass}(d, R_j)$. The partitions $Q$ we consider will have $j + 1 - d$ parts $0 \leq q_1 \leq \cdots \leq q_{j+1-d} \leq d$. We partially order the partitions by $Q \leq Q'$ iff $q_i \leq q'_i$ for each i. We denote by $Q^\land$ the dual partition obtained by exchanging rows and columns in the shape of $Q$: it has $d$ parts of sizes between zero and $j + 1 - d$. The shape of $Q$ is included in a cod $(V) \times \dim(V)$ rectangle $B$; we denote by $Q^c$ its complement in $B$. If $E = \{x^{n_1}y^{j-n_1}, \ldots, x^{n_d}y^{j-n_d}\}$ we denote by $E^\land$ the space of monomials $E^\land = \{y^{n_1}x^{j-n_1}, \ldots, y^{n_d}x^{j-n_d}\}$. The “length” $\ell(Q)$ of a partition $Q$ is $\ell(Q) = \sum q_i$, the “area” of its Ferrers graph, or shape.

The result Cii. below follows from D. Eisenbud and J. Harris in [E-H2]; the rest are consequences of Lemmas 2.13, 2.14, Propositions 2.15 and 2.17 and the Schubert calculus (see Appendix 3); or [Ste, E-H1].

A. The cells $V(E)$ are affine spaces.

B. The set of Schubert cells $V(E)$ correspond 1-1 with the set of ordinary partitions $Q(E)$ of integers into no more than cod $(V)$ parts, each of size no greater than $\dim(V)$.
The partition \( Q(E) \) is related to the ramification \( \text{QRAM}(E) \) of \( E \) at \( x \) by

\[
Q(E) = \text{QRAM}(E)^c.
\]  

If \( V \in \mathcal{V}(E) \), then the ramification of \( V \) at \( p \) satisfies \( \text{QRAM}(V, p) = \text{QRAM}(E) \) by Lemma 2.10. The total ramification of \( V \) at \( p \) is the length of \( \text{QRAM}(V, p) \) (area of its Ferrers graph) and satisfies

\[
r(V, p) = N - \ell(Q(E)).
\]  

C. The dimension of \( \mathcal{V}(E) = \ell(Q(E)) \). And \( Q(E^\wedge) = Q(E)^c \).

Ci. The cells have dimensionally proper intersections. The codimension of the intersection of ramification cells at two different points \( x, L \) is the sum of their codimensions,

\[
\text{cod}(\mathcal{V}(E, p)) \cap \mathcal{V}(E', p') = \text{cod}(\mathcal{V}(E, p)) + \text{cod}(\mathcal{V}(E', p')),
\]  

provided that the intersection of the Schubert cycles corresponding to \( E, E' \) are nonzero. Otherwise, the intersection is empty.

Cii. Ramification behaves well under specialization: If \( V_t \mid t \neq 0 \) and \( W_t \mid t \neq 0, t \in \mathbb{A}^1 \) are algebraic families of vector spaces satisfying \( V_t \in \mathcal{V}(E, p_t), W_t \in \mathcal{V}(E', p'_t) \) tending to a common limit \( V_0 = W_0 \), if \( \mathcal{V}(E, p_t) \) and \( \mathcal{V}(E', p'_t) \) have dimensionally proper intersection, and \( p_t = p'_t \), then \( V_0 \) belongs to some cell \( \mathcal{V}(E'', p_0) \) where the class of \( E'' \) occurs with positive coefficient in the intersection of Schubert cycles corresponding to \( \mathcal{V}(E, p) \cap \mathcal{V}(E', p') \). (See [EH2].)

D. Exact duality: if \( \mathcal{V}(E) \) and \( \mathcal{V}(E') \) have complementary dimension, then the intersection \( \mathcal{V}(E) \cap \mathcal{V}(E') = V(E_0) \) is the class of a point iff \( E' = E^\wedge \); the intersection is empty if \( E' \neq E^\wedge \).

E. Frontier property: The closure \( \overline{\mathcal{V}(E)} \) is the union of those cells \( \mathcal{V}(E') \) such that \( Q(E) \geq Q(E') \) in the sense of inclusion of Ferrers diagrams.

Our goal in this article is to see which of these remarkably good properties for the ramification/Schubert cells of \( \text{Grass}(d, R) \) parametrizing \( d \)-dimensional vector spaces \( V \subset R_j, R = k[x,y] \) extends to the analogous ramification cells for the variety \( G_T \) parametrizing ideals \( I \) of \( R \) having Hilbert function \( H(R/I) = T \). This will lead us to study the homology ring of \( G_T \), and to determine it in some special cases.

# 3 Ramification cells in the family of graded ideals of \( R \)

We now show our main results, determining the homology groups of \( G_T \), by connecting the dimension of the cells \( \mathcal{V}(E) \) with the hook code of the partition \( P(E) \) having diagonal lengths \( T \). In Section 3-A we extend our definitions from Section 2 to ideals, and we show the equivalence between suitable ramification \( \text{QRAM}(I, p) = E \) of the graded ideal \( I \) at \( p \in \mathbb{P}^1 \), and the ideal having initial monomial ideal \( E \) (Proposition 3.6). We also define the partition \( P(E) \) (Definition 3.7). In Section 3-B we show that the cell \( \mathcal{V}(E) \) is an affine space, with parameters given by certain coefficients of generators, determined by the difference-one hooks of \( P(E) \) (Theorem 3.12). We also show that there is a birational map \( G : G_T \to S\text{Grass}(T) \), the product of small Grassmanians (Proposition 3.13). In Section 3-C we determine the fibre dimension of the cell \( \mathcal{Z}(E) \) (all ideals) over \( \mathcal{V}(E) \) (graded ideals) in Proposition 3.15 and we reconcile our formulas with those of
L. Götsche, using properties of the difference-zero hooks of \( P(E) \) (Lemmas 3.21, 3.22). Remark 3.23. In Section 3.1 we define the hook code of a partition \( P \) of diagonal lengths \( T \), and we show that the hook code gives an isomorphism between the distributive lattice \( \mathcal{P}(T) \) of all partitions of diagonal lengths \( T \), and the lattice corresponding to the direct product of the lattices of partitions whose Ferrer’s graphs are enclosed in boxes \( B_i(T), \mu \leq i \leq j \) (Theorem 3.24). In Section 3.2 we show that there is an additive isomorphism over \( \mathbb{Z} \) given by the hook code between the homology \( H^*(G_T) \), and \( H^*(S\text{Grass}(T)) \), that respects the \( \mathbb{Z}_2 \) action arising from complementation or duality (Theorem 3.28), and we determine the Poincaré polynomial of \( G_T \) (Theorem 3.29).

3-A Graded ideals of \( R = k[x, y] \), and linear systems on \( \mathbb{P}^1 \)

Let \( R = k[x, y] \) and \( A \) be an \( R \)-module, we let \( A_i \) denote \( M^iA/M^{i+1}A \), where \( M \) is the maximal ideal \( (x, y) \) of \( R \), and we let \( H(A) = (t_0, t_1, \ldots) \), \( t_i = \dim_k A_i \) denote the Hilbert function of \( A \). If \( I \) is an ideal of \( R \), we let \( d(I) = \min\{i \mid I_i \neq 0\} \) be the order of \( I \), and if \( I \) has finite colength, \( j(I) = \max\{i \mid I_i \neq R_i\} \) is the socle degree of \( I \). The Hilbert function \( T = H(A) = R/I \) of an Artin quotient of \( R \) satisfies

\[
T = (1, 2, \ldots, \mu, t_{\mu}, \ldots, t_j, 0), \quad \mu \geq t_{\mu} \geq \cdots \geq t_j > 0.
\]

If \( T \) satisfies \( (3.1) \), we define the set \( \{G_T\} = \{ \text{graded ideals } I \subset R \mid H(R/I) = T \} \). There is a natural inclusion \( \iota : \{G_T\} \to \{B\text{Grass}(T)\} \), of the set \( \{G_T\} \) into the set of closed points of \( B\text{Grass}(T) \), a product of “Big” Grassmannians: \( B\text{Grass}(T) = \prod_{\mu \leq i < j} \text{Grass}(i + 1 - t_i, R_i) \).

\[
\iota : \{G_T\} \subset \{B\text{Grass}(T)\} = \prod_{\mu \leq i < j} \{\text{Grass}(i + 1 - t_i, R_i)\}
\]

\[
I \mapsto (I_\mu, I_{\mu+1}, \ldots, I_j)
\]

We give \( G_T \) the reduced subscheme structure induced from this inclusion \( \iota \). Then \( G_T \) is a closed, projective, nonsingular variety having a cover by affine spaces of the same dimension ([11], Theorem 2.9).

Example 3.1. Let \( T = (1, 2, 3, 2, 1) \), then \( I_2 = 0, I_3 = \langle f, g \rangle \) is two-dimensional, \( I_4 \) has dimension four as vector subspace of \( R_4 \), while \( I_5 = R_5 \). The inclusion \( \iota : G_T \hookrightarrow B\text{Grass}(T) \), is

\[
\iota : G_T \subset \text{Grass}(2, R_3) \times \text{Grass}(4, R_4) = \text{Grass}(2, 4) \times \mathbb{P}^4, \quad I \mapsto (I_3, I_4).
\]

In this example, \( G_T \) is the locus of pairs of vector spaces \( (V_3, V_4) \in \text{Grass}(2, R_3) \times \text{Grass}(4, R_4) \) such that \( (xV_3, yV_3) \subset V_4 \).

Given an ideal \( I \) of \( G_T \) we let \( \iota(I) \) denote the sequence of vector spaces \( (I_\mu, \ldots, I_j) \) and we let \( \mathcal{L}(I) = \mathcal{L}(I_\mu), \ldots, \mathcal{L}(I_j) \) denote the corresponding sequence of linear systems on \( \mathbb{P}^1 \) (see Definition 2.1).

Lemma 3.2. Suppose that \( T \) satisfies \( (3.1) \). The maps \( \iota : I \mapsto \iota(I) \) and \( \mathcal{L} : I \mapsto \mathcal{L}(I) \) give isomorphisms among

i. The set of all ideals \( I \in G_T \).

ii. The set of all collections of vector spaces \( (V_{\mu}, \ldots, V_j) \) such that for each \( i, \mu \leq i \leq j \), \( V_i \subset R_i, \dim_k V_i = i + 1 - t_i \), and such that

\[
f \in V_i, \quad L \in R_i \implies Lf \in V_{i+1} \quad \text{for } \mu \leq i < j.
\]

iii. The set of sequences \( \mathcal{L} = (\mathcal{L}_\mu, \ldots, \mathcal{L}_j) \) of linear systems on \( \mathbb{P}^1 \) such that for \( \mu \leq i < j \),

\[
\deg(\mathcal{L}_i) = i, \quad \dim_k(\mathcal{L}_i) = i + 1 - t_i, \quad \text{and satisfying, for } \mu \leq i < j:
\]
If \( p_f = (\sum_u n_u p_u) \), \( p_u \in \mathbb{P}^1 \), satisfies \( p_f \in \mathcal{L}_i \) and if \( p \in \mathbb{P}^1 \), then \((p + p_f) \in \mathcal{L}_{i+1} \). (3.4)

**Proof.** Immediate from the definitions. \(\square\)

**Definition 3.3.** The ramification QRAM\((I, p)\) at \( p \) of an ideal \( I \in \text{Grass}(T) \) in the direction \( p \in \mathbb{P}^1 \), given by \( p : L = 0, L \in \mathcal{R}_i \) is the sequence

\[
\text{QRAM}(I, p) = (\ldots, \text{QRAM}(I, p), \ldots), \ \mu \leq i \leq j,
\]

where QRAM\((I, p)\) is from Definition 1.2. We denote the ramification in the direction \( p : x = 0 \) by QRAM\((I)\).

Suppose that \( \wp = (\wp_i, \mu \leq i \leq j) \) is a sequence of partitions, and that each \( \wp_i \) is included in \( B_i(T) \), a \( \dim(I_i) \times \text{cod}(I_i) \) rectangle. This is the condition for each \( \wp_i \) to be possible as a ramification partition by Lemma 1.9. Suppose also that \( p \in \mathbb{P}^1 \) is a point.

**Question 4.3.1.** What is the structure of the subfamily \( \mathcal{V}_{\wp, p} \subset \text{Grass}(T) \) of graded ideals \( I \) such that QRAM\((I, p) = \wp? \) What is its dimension?

For each \( i \), the condition QRAM\((I_i, p) = \wp_i \) is by Lemma 2.1B an open dense subset of a Schubert condition on the big Grassmannian Grass\((i + 1 - t_i, i + 1) \). But these conditions for different \( i \) do not intersect properly in the product BGrass\((T) \). First, in order to be compatible, there must be a monomial ideal \( E \) in \( \text{Grass}(T) \) such that QRAM\((E_i) = \wp_i \) for each \( i \). Second, we need to study the cell \( \mathcal{V}(E, p) \). We answer Question 4.3.1 in Lemmas 3.4, Proposition 3.6, and Theorem 3.12 below.

**Lemma 3.4.** The family \( \mathcal{V}_{\wp, p} \subset \text{Grass}(T) \) of graded ideals satisfying QRAM\((I, p) = \wp \) is nonempty if and only if there is a monomial ideal \( E \) such that QRAM\((E) = \wp \). Then \( \mathcal{V}_{\wp, p} = \mathcal{V}(E, p) \), the family of graded ideals with initial ideal \( E_L \) in the basis \((L_p, C)\) for \( R \).

**Proof.** By Lemma 2.10, the family \( \mathcal{V}_{\wp, p} \) is the family of all graded ideals \( I \) of Hilbert function \( H(R/I) = T \), such that for each \( i \), \( \mu \leq i \leq j \), \( Q(I_i, p) = (B(i + 1 - t_i, t_i) - \wp_i)^\wedge \). By Lemma 2.13 each \( I_i \) has the unique initial monomial vector space \( E_L(i) \) in the basis \((L, C), L = L_p \) for \( R \), such that \( Q(E_L(i), p) = Q(I_i, p) \). The initial monomial vector spaces \( E_L(i) \) form an ideal \( E_L \). The family of graded ideals \( \mathcal{V}_{\wp, p} \) is just \( \mathcal{V}(E, p) \), the family having initial ideal \( E_L \). The corresponding monomial ideal \( E \) in the variables \((x, y)\) satisfies QRAM\((E) = \wp \). \(\square\)

**Definition 3.5 (The cell \( \mathcal{V}(E, p) \) of \( \text{Grass}(T) \)).** If \( E \in \text{Grass}(T) \) is an ideal generated by monomials in \((L, C)\) then \( \mathcal{V}(E, p) \) is the family of all graded ideals \( I \in \text{Grass}(T) \) such that \( \text{In}_p(I) = E \).

**Proposition 3.6.** Let \( E \) be a monomial ideal with \( H(R/E) = T \), and \( p \in \mathbb{P}^1 \), and suppose that \( I \) is an ideal with \( H(R/I) = T \). The following are equivalent:

i. \( V \in \mathcal{V}(E, p) \)

ii. QRAM\((I_i, p) = \text{QRAM}(E_i) \), i.e. QRAM\((I_i, p) = Q(E_i) \) for each \( i, \mu \leq i \leq j \). (3.6)

iii. \( Q(I_i, p) = Q(E_i) \) for each \( i, \mu \leq i \leq j \).

For each \( i \),

\[
\text{QRAM}(I_i, p) = (Q(E_i)^\wedge)^{\wedge},
\]

the dual \((Q(E_i)^\wedge)^{\wedge}\) of the complement \( Q(E_i)^\wedge \) to \( Q(E_i) \) in the \((t_i) \times (i + 1 - t_i) \) rectangle \( B_i(T) \).

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Proof. The first formula is a consequence of (2.18), applied to each degree-$i$ piece $I_i$ and $E_i$, $\mu \leq i \leq j$. The second follows from Lemma (2.10).

**Definition 3.7 (Partition $P(E)$ of a monomial ideal $E$).** A monomial ideal $E \in G_T$ has a complementary basis $E'$ of monomials, that we arrange in standard array form, with the monomials $(1, x, \ldots, x^{p_0(E)-1})$ in the cobasis forming the top row, and the monomials $(y, yx, \ldots, y^{p_1(E)-1})$ forming the $i$-th row from the top, $i = (0,1,\ldots,s)$. Then $E'$ has the shape of a partition $P(E) = (p_0, p_1, \ldots, p_s)$ whose diagonal lengths are given by $T$. Conversely, the partition $P$ determines a unique monomial ideal $E = E(P)$ satisfying $P = P(E(P))$.

![Figure 3: Partition $P(T)$ of diagonal lengths $T = (1, 2, 3, 2, 1)$, $T = (x^5, x^2y, xy^2, y^4)$. See Example 3.8.](image)

**Example 3.8.** Let $T$ be the monomial ideal $T = (x^5, x^2y, xy^2, y^4)$; then $P(T) = (5, 2, 1, 1)$. The diagonals of the shape $P(C)$ are those of $T'$: namely, $(1), (x, y), (x^2, xy, y^2), (x^3, y^3), (x^4)$, whose lengths are counted by the Hilbert function $T(R/T) = (1, 2, 3, 2, 1)$. (See Figure 3.)

3-B Parameters for the cell $V(E)$ of $G_T$

In this section, we will further study the family $V(E)$ of graded ideals having given monomial initial ideal $E$. We give $V(E)$ the reduced subscheme structure coming from $G_T$. We show that the cell is an affine space of known dimension. Recall that we order the monomials of $R$, $1 < y < x < y^2 < xy < x^2 < y^3 < \cdots$ by total degree, then $y$-degree.

We now define parameters for ideals $I$ in the cell $V(E)$.

**Definition 3.9.** Given a monomial ideal $E$ or partition $P = P(E)$ we let $S_i(E)$ or $S_i(P)$ denote the set of ordered pairs of monomials

$$S_i(E) = \{(\mu, \nu) \mid \mu \in E, (\mu : y) \notin E, \mu \notin E, x\mu \in E$$

\text{degree } \mu = \text{degree } \nu = i, \text{ and } \mu < \nu \}.$$  

We let $S(E)$ or $S(P)$ denote the union

$$S(E) = \bigcup_{\mu \leq \nu \leq j} S_i(E).$$  

If $\mu$ is a monomial of $E$ and $I$ is an ideal with initial ideal $\text{In}(I) = E$ we let

$$f(\mu, I) = \mu - \sum_{\nu > \mu} \alpha_{\mu\nu}(I) \cdot \nu, \quad \alpha_{\mu\nu} \in k, \quad \nu \notin E, \quad \deg(\nu) = \deg(\mu)$$

denote the unique element of $I$ with leading term $\mu$. The parameters for the graded ideals $I$ in the cell $V(E)$ will be the set of coefficients $\{\alpha_{\mu\nu}(I) \mid (\mu, \nu) \in S(E)\}$. (See Example 3.10 and Figure 4.)
Example 3.10. If $T = (1,2,3,4,1)$ and the initial ideal is $E = (x^6, yx^4, y^2x^3, y^3x, y^5)$ then $P(E) = (6,4,3,1,1)$. If $\mu = y^3x$, and $I \in \mathbb{V}(E)$, then $f(\mu, I) = \mu - \alpha_{\mu_1}\nu_1 - \alpha_{\mu_2}\nu_2 - \alpha_{\mu_3}\nu_3 = \mu - \alpha_{\mu_1}y^2x^2 - \alpha_{\mu_2}y^3x^2 - \alpha_{\mu_3}y^4x \in I$, and $(\alpha_{\mu_1}, \alpha_{\mu_2})$ are among the parameters for $I$. Each pair $(\mu, \nu) \in S(E)$ corresponds to a difference-one hook of $P$ with hand $\nu$ and foot $\mu : y$. (See Figure 4).

Figure 4: Here $P = (6,4,3,1,1)$. $E = (x^6, yx^4, y^2x^3, y^3x, y^5)$, $\mu = y^3x$, $f(\mu, I) = \mu - \alpha_{\mu_1}\nu_1 - \alpha_{\mu_2}\nu_2 - \alpha_{\mu_3}\nu_3 \in I$, and $(\alpha_{\mu_1}, \alpha_{\mu_2})$ are parameters for $I$. The pair $(\mu, \nu) \in S_i(I)$ corresponds to the shaded hook of $P$. See Example 3.10 and Remark 3.11.

Remark 3.11. Note that $S_i(I)$ consists of the pairs $(\mu, \nu)$ of monomials of degree $i$, where $\nu$ is the right endpoint of a row of $E^c$, where $\mu$ in $E$ is vertically just below a column of $E^c$, and $\mu$ is diagonally below $\nu$, when we view the monomials of $E^c$ as filling the shape of the partition $P(E)$. The degree-$i$ difference-one hooks of $P$ is the set $\mathcal{H}(P)_i = \{(\mu, \nu) \mid (\mu, \nu, \nu') \in S_i(P)\}$. (See Figure 4 and Definition 3.17).

Theorem 3.12. The cell $\mathcal{V}(E)$ of $G_T$ is an affine space $A^{s(E)}$ of dimension $s(E) = \#S(E) = \#\mathcal{H}(P)$. The parameters of $\mathcal{V}(E)$ are $\{\alpha_{\mu_1}(I) \mid (\mu, \nu) \in S(E)\}$.

Historical remarks. Analogous results were well known to the Nice Hilbert scheme group, in particular J. Briançon, as early as 1972, for the family $Z_T$ of all ideals (not just graded) defining quotient algebras $A = R/I$ of Hilbert function $T$. J. Briançon studied “vertical strata” $Z(E)$ of the punctual Hilbert scheme Hilb$^n R$, which when restricted to graded ideals are identical with the cells $\mathcal{V}(E)$; his vertical strata involve ideals with different Hilbert functions, and are a key tool in his proof of the irreducibility of the local punctual Hilbert scheme Hilb$^n \mathbb{C}\{x,y\}$ [53]. J. Yaméogo gives a proof of Theorem 3.12 in [Y3] and [Y4] using the map from $I$ to $(I : x)$ and an induction, following the approach of Briançon. Our proof here generalizes that given in [11] for the special case of the “generic” or big cell $E(0)$ of maximum dimension, where $E(0)_i = \langle y^i, \ldots, y^i x^i \rangle$.

J. Yaméogo showed that the “vertical strata” subvarieties $Z(E)$ and $\mathcal{V}(E)$ are identical to the families of ideals collapsing to $E$ under a $C^*$-action (see [Y3] [Y4]). L. Göttsche proved that these latter are affine spaces, and he calculated the dimension of $Z(E)$, and then of $\mathcal{V}(E) \subset G_T$ from the dimension of the fibre of $Z_T$ over $G_T$ (G52). Given this identification, L. Göttsche’s formula for $\dim \mathcal{V}(E)$ (see Proposition 3.20 below) was the first covering explicitly all the cells $\mathcal{V}(E)$ in the graded case. But his dimension formula is different than ours in Theorem 3.12. We reconcile the two formulas using a combinatorial result from Part I [1Y2] (see Remark 3.23). In our proof of Theorem 3.12 the parameters for $\mathcal{V}(E)$ give the geometric meaning of the “hook code” of 3.10.

Proof of Theorem 3.12. The key to the proof is that since we are working in only two variables, the minimal relations between standard generators of ideals in $\mathcal{V}(E)$ have an echelon or upper triangular form. List the monomials $\beta_0, \ldots, \beta_p$, $p = p_0(E)$, in $E$ just below the pattern: $\beta_i = x^i y^{q(i)} \in E$ and $(\beta_i : y) \in E^c$. 

The sequence \((q(0), q(1), \ldots, q(p_0) = 0)\) is the dual partition \(P^\wedge\) to \(P(E)\). Since \((x^i) = (\beta_1, \ldots, \beta_p) \oplus E^c \cap (x^i)\), and \(\beta_i\) is the leading form of \(f(\beta_i, I)\), it follows that for ideals \(I \in \mathcal{V}(E)\),

\[
(x^i) \cap I = (f(\beta_1, I), f(\beta_{i+1}, I), \ldots, f(\beta_p, I)).
\] (3.11)

**Induction step : comparison to \((I : x^i)\)**. We will let \((E : x^i)\) denote \((E \cap (x^i) : (x^i))\), the monomial ideal with cobasis \((E^c : x^i) = (E^c \cap (x^i) : x^i)\). The shape \(P(E : x^i)\) of the cobasis \((E^c : x^i)\) is that of \(P\) with the first \(i\)-columns omitted. Thus \(P(E : x^i)\) is the partition dual to \((q(i), q(i+1), \ldots, q(p_0))\).

We regard the monomials \(\mu, \nu\) of \((E : x^i)\) and \((E^c : x^i)\) as those of \(E^c\) and \(E\) shifted left by \(i\), and will thus for convenience set \(\mu = (\mu : x^i)\) and \(\nu = (\nu : x^i)\) in Claim B below. The cell \(\mathcal{V}(E : x^i)\) of Claim B below lies on \(G_{T(i)}\) where \(T(i)\) is the Hilbert function of \(R/(E : x^i)\).

**Claim A.** If \(I \in \mathcal{V}(E)\), then for each \(i \leq p\) the polynomial \(f(\beta_i, I)\) is uniquely determined by the set of coefficients

\[
\{ \alpha_{\mu, \nu}(I) \mid \mu \geq \beta_i \text{ and } (\mu, \nu) \in \mathcal{S}(E) \}.
\] (3.12)

**Claim B.** For each \(i \leq p\), given any set of constants \(\{ \psi_{\mu, \nu} \in k \mid (\mu, \nu) \in \mathcal{S}(E) \text{ and } \mu \geq \beta_i \}\) there is a unique ideal \(I^{(i)}_\psi \in \mathcal{V}(E : x^i)\) such that \(\alpha_{\mu, \nu}(I^{(i)}_\psi) = \psi_{\mu, \nu}\). The ideal \(x^iI^{(i)}_\psi\) has generators \(f(\beta_1), \ldots, f(\beta_p)\) with initial terms \(\beta_1, \ldots, \beta_p\), respectively, and satisfies

\[
x^iI^{(i)}_\psi \oplus E^c \cap (x^i) = (x^i).
\] (3.13)

![Figure 5: Partition \(P(E) = (6, 4, 3, 1, 1)\), and, shaded, \(P(E : x^2) = (4, 2, 1)\). Here \(xf(\beta_1, I) \subset (x^2) \cap I = (f(\beta_2, I), \ldots, f(\beta_5, I), \beta_6)\). See Example 3.13](image)

**Proof of Claims.** We will show these by descending induction on \(i\), beginning with \(i = p\). Since \(I \supset x^p = \beta_p\), and the sets of A, B are vacuous when \(i = p\), the claims are satisfied for \(i = p\). Suppose they are satisfied for \(i+1\), and that \(I \in \mathcal{V}(E)\). Then \(xf(\beta_i, I) \in (x^{i+1}) \cap I\).

The coefficients \(\{ \alpha_{\mu, \nu}(I) \mid \mu = \beta_i \text{ and } (\mu, \nu) \in \mathcal{S}(E) \}\) are precisely those of \(f(\beta_i, I)\) on monomials \(\nu\) such that \(x\nu\) lands \textit{outside} \(\langle E^c \rangle\) in \(xf(\beta_i, I)\). These can be reduced in a standard way by elements of \((f(\beta_{i+1}, I), \ldots, f(\beta_p, I))\) to a remainder \(g_i(I)\) in \(\langle E^c \rangle_{i+1}\). The remaining coefficients of \(f(\beta_i, I)\) are on monomials \(\nu\) such that \(x\nu\) lands \textit{inside} \(\langle E^c \rangle_{i+1}\): these must be \(-g_i(I)\), since their sum with the remainder \(g_i(I)\) is in \(I \cap \langle E^c \rangle\) so must be 0. This proves Claim A.

Suppose Claim B is satisfied for \(i+1\), and that the set \(\{ \psi_{\mu, \nu} \in k \mid (\mu, \nu) \in \mathcal{S}(E) \text{ and } \mu \geq \beta_i \}\) is specified, and let \(I^{(i+1)}\) denote the unique ideal in \(\mathcal{V}(E : x^{i+1})\) satisfying Claim B. We have thus determined an ideal \(x^{i+1}I^{(i+1)} = (f(\beta_{i+1}), \ldots, f(\beta_p))\) such that

\[
x^{i+1}I^{(i+1)} \oplus E^c \cap (x^{i+1}) = (x^{i+1}).
\] (3.14)

By the proof of Claim A, there is a unique homogenous polynomial \(f(\beta_i)\) having the designated coefficients \(\{ \alpha_{\mu, \nu} = \psi_{\mu, \nu} \in k \mid (\mu, \nu) \in \mathcal{S}(E) \text{ and } \mu = \beta_i \}\), and satisfying \(xf(\beta_i) \in (f(\beta_{i+1}), \ldots, f(\beta_p))\), so we have a relation \(r_i\),

\[
r_i : xf(\beta_i) = h_{i+1}f(\beta_{i+1}) + \ldots + h_pf(\beta_p).
\] (3.15)

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Since $\beta_i$ has leading term of $x$-degree $i$, any other relation $r$ between $f(\beta_i)$ and $f(\beta_{i+1}), \ldots, f(\beta_p)$ involves a multiple of $xf(\beta_i)$, so can be reduced by the relation $r_i$ to one involving only $f(\beta_{i+1}), \ldots, f(\beta_p)$. But (3.14) together with the form of $f(\beta_i)$ now show that

$$f(\beta_1), \ldots, f(\beta_p) \oplus E^2 \cap (x^i) = (x^i).$$  

We set $I^{(i)} = (f(\beta_1) : x^i, \ldots, f(\beta_p) : x^i)$. By (3.16) and the form of $f(\beta_i)$, the ideal $I^{(i)}$ is in $\mathfrak{V}(E : x^i)$. This completes the proof of Claim B. Taking $i = 0$, and $I = I^0$ we obtain Theorem 3.12 \hfill \Box

**Example 3.13 (Induction step).** Consider $P(E) = (6,4,3,1,1), \beta_2 = xy^3$, and $f(\beta_2, I) = \mu - \alpha_{\mu \nu_1} y^2 x^2 - \alpha_{\mu \nu_2} y x^3 - \alpha_{\mu} x^4$. If $(x^2 \cap I)$ has been chosen, the new information in $f(\beta_2, I)$ is the coefficients $(\alpha_{\mu \nu_1}, \alpha_{\mu \nu_2})$ on those monomials $xv_1, xv_2$ such that $xv_1, xv_2$ lie outside of $P(E)$; these may be chosen arbitrarily (Claim B). In the usual standard basis way, $xf(\beta_2, I)$ may be reduced using $(x^2 \cap I = (f(\beta_2, I), \ldots, \beta_6)$ to a linear combination of cobasis monomials in $P(E) \cap (M^5)$, which must be zero. The coefficient $\alpha_{\mu \nu}$ of $f$ on a monomial $\nu = x^4$ such that $xv$ lies inside of $P(E)$ is thus determined by $(x^2) \cap I$ and by $(\alpha_{\mu \nu_1}, \alpha_{\mu \nu_2})$ (Claim A). (See Figure 7).

**Definition 3.14 (Small Grassmannians and the big cell of $G_T$).** Let $T$ satisfy (3.21). Then we let $\delta_i = \delta_i(T) = t_{i-1} - t_i$. We let $\text{SG}_{T}$ denote the product of small Grassmannians,

$$\text{SG}_{T}(T) = \prod_{\mu(T) \leq i \leq j(T)} \text{Grass}(\delta_{i+1}, 1 + \delta_i + \delta_{i+1}).$$

We let $E = E_0$ denote the big cell of $G_T$ determined by the unique partition $P_0(E) = (p_1, p_2, \ldots, p_r)$ of diagonal lengths $T$ having distinct parts: $P_0(T)$ is maximal in the partial order on $P(T)$ defined by the inclusion of the partitions $Q(E(P))$ (See also Part I, [LYX Definition 2.18B]). For $i \geq d$ and $E = E_0$ we define the vector spaces $U_i = (E^e_{i+1} : x)$, $V_i = (x(E^e_{i-1} + y(E^e_{i-1})) / U_i$, and $W_i = (E^e_{i}) / U_i$. Then $\dim_k W_i = \delta_{i+1}$, and $\dim_k V_i = t_{i-1} + 1 - t_{i+1} = \delta_i + \delta_{i+1} + 1$. Given $I \in \mathfrak{V}(E)$ we define

$$G_i(I) = (I_i \cap (x(E^e_{i-1} + y(E^e_{i-1})) + U_i) / U_i, \quad (3.18)$$

a complement to $W_i$ in $V_i$. We define $G : \mathfrak{V}(E_0) \rightarrow \text{SG}_{T}$ by

$$G(I) = (G_{\mu(T)}(I), \ldots, G_{j(T)}(I)).$$

**Proposition 3.15.** There is a birational map $G : G_T \rightarrow \text{SG}_{T}$, that is defined on the big cell $\mathfrak{V}(E_0)$.

**Proof.** The proof of Theorem 3.12 shows that $\mathfrak{V}(E_0)$ is an open dense affine cell in $G_T$, and gives its parameters; these parameters can be identified with the usual parameters for the corresponding open dense cell $G(\mathfrak{V}(E_0))$ of $\text{SG}_{T}$.

3-C Dimension of the cell $V(E)$ and the hooks of $P(E)$

In this section we obtain a new formula for the dimension of the cells $\mathfrak{V}(E)$, and compare with a previous dimension result of L. Göttsche. Since Göttsche’s result is based on his dimension formulas for cells of nongraded ideals, we introduce the family $Z_T$ of all algebra quotients $A = R/I$ of $R$ with Hilbert function $H(A) = T$.

**Definition 3.16.** Let $Z(E) \subset Z_T$ parametrize the family of all ideals of $R$ having initial monomial ideal $E$. We let

$$W^+(E) = \{(\mu, \nu) \mid \mu \in E, (\mu : y) \notin E, (\nu \notin E, xv \in E, \text{ and degree } \mu < \text{ degree } \nu\}, \quad (3.20)$$

$$W^-(E) = \{(\mu, \nu) \mid (\mu : x) \notin E, (\nu \notin E, yv \in E, \text{ and degree } \mu < \text{ degree } \nu\}. \quad (3.21)$$

$$W(E) = W^+(E) \cup W^-(E)$$

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and we let \( w(E) \) denote \( \# W(E) \).

We let \( h : Z_T \to G_T \) denote the morphism defined on closed points by \( h(A) = Gr_m(A) \), the associated graded algebra \( A^* \) of \( A \) with respect to the maximal ideal \( M = (x, y) \) of \( R \). Recall that \( n(T) = \sum t_i \), that the initial degree of ideals in \( G_T \) is \( \mu(T) = \min \{ i \mid t_i \leq i \} \), the socle degree of \( R/I \) is \( j(T) = \max \{ i \mid t_i \neq i \} \), and we let \( \delta_i = t_{i-1} - t_i \). In [11], Theorems 2.11 and 3.14, the first author showed the first of the following two Propositions. The second readily follows from similar arguments.

**Proposition 3.17 (A. Iarrobino, [I1]).** If \( \text{char}(k) = 0 \) or is greater than \( j(T) \), then the morphism \( h : Z_T \to G_T \) is a locally trivial fibration whose fiber is an affine space \( A^{\mu(T)} \), of dimension

\[
f(T) = n(T) - \sum_{i \geq \mu(T)} \delta_i (\delta_i + 1)/2 - \sum_{i \geq \mu(T)} (\delta_i + 1)\delta_{i+1}.
\]

(3.22)

Here \( \dim G_T = \sum_{i \geq \mu(T)} (\delta_i + 1)\delta_{i+1} \), and \( \dim Z_T = n(T) - \sum_{i \geq \mu(T)} \delta_i (\delta_i + 1)/2 \).

**Proposition 3.18.** The fiber \( F_{2/\nu}(E) \) of the cell \( Z(E) \) over \( \nu(E) \) satisfies

\[
F_{2/\nu}(E) \cong A^{w(E)}, \text{ an affine space}. \tag{3.23}
\]

**Comment on proof.** The proofs of Propositions 3.17 and 3.18 are analogous to that of Theorem 3.12. However Proposition 3.17 requires in addition a proof that in a generic basis \( X, Y \) for \( R_I \), a given ideal \( I \) lies in the maximal cell \( E_0 \): this requires the assumption on \( \text{char}(k) \). A simple reconciliation of the dimension formulas in Proposition 3.17 and Proposition 3.18 follows from our combinatorial results in Part I [12]; see Remark 3.12 below, which gives an alternative proof of Proposition 3.18.

L. Göttzsche gives a different formula for the dimension of the cell \( Z(E) \), from which he derives a formula for \( \dim(\nu(E)) \) by using Proposition 3.18. In order to show the Göttzsche formula from ours, we need to define the set \( \mathcal{H}^a(P)_i \) of degree-\( a \) difference-a hooks of \( E^c \).

**Definition 3.19 (Difference-a hooks).** Suppose \( P = P(E) \) and \( a \geq 0 \); we denote by \( \mathcal{H}^a(P) \) or \( \mathcal{H}^a(E) \) the set \( \mathcal{H}^a(P) = \bigcup \mathcal{H}^a(P)_i \) where

\[
\mathcal{H}^a(P)_i = \{ (\mu, \nu) \mid \mu \neq \nu \in E, y\mu \in E, x\nu \in E, \text{degree } x^a \mu = \text{degree } \nu, = i, \text{ and } x^a \mu \leq \nu \}. \tag{3.24}
\]

These pairs of monomials are the endpoints of difference-a hooks of the shape \( P(E) \) whose arm length is \( a \) units greater than the leg length. When \( a = 0 \), we call such hooks balanced. When \( a = 1 \), we write \( \mathcal{H}(E)_i, \mathcal{H}(E), \mathcal{H}^1(E)_i, \mathcal{H}^1(E) \).

**Figure 6.** A hook in the shape of \( P \) is a subshape as shown in Figure 6 consisting of an arm whose hand or right endpoint \( \nu \) is the right endpoint of a row of \( P \), and a leg whose foot \( \mu \) or lowermost entry is the endpoint of a column of \( P \). The hook shown has an arm of length four, and a leg of length three, so has arm-leg difference equal one.

We let \( E^\wedge \) denote the monomial ideal \( \sigma(E) \) obtained by switching the variables \( x, y \) in \( E \). The associated partition \( P(E^\wedge) \) satisfies \( P(E^\wedge) = (P(E))^\wedge \), the dual of \( P(E) \), obtained by switching rows and columns in the shape of \( P(E) \). L. Göttzsche showed, using somewhat different language,

**Proposition 3.20 (L. Göttzsche [Gö2, Gö4]).** The dimension \( z(E) \) of the cell \( Z(E) \) of \( Z_T \) satisfies

\[
z(E) = n(T) - \# S(E^\wedge) - \# \mathcal{H}^0(E). \tag{3.25}
\]
The dimension \( v(E) \) of the cell \( V(E) \) of \( G_T \) satisfies
\[
v(E) = n(T) - f(T) - \#S(E^\wedge) - \#H^0(E). \tag{3.26}
\]
We will now show Proposition 3.20 as a consequence of Theorem 3.12. We need two results from Part I [LY2]. Let \( \#H^0(E)_i \), \( S_i(E) \), respectively, denote the subsets of monomial pairs in \( H^0(E) \), \( S(E) \) having degree \( i \).

**Lemma 3.21.** If \( i \geq \mu(T) - 1 \), the number of difference-zero degree-i hooks \( \#(H^0(E)_i) \) satisfies
\[
\#H^0(E)_i = \binom{\delta_i + 1}{2} \tag{3.27}
\]

**Lemma 3.22.** The sizes of \( S_i(E) \) and of \( S_i(E^\wedge) \) are related by
\[
\#S_i(E^\wedge) = (\delta_i + 1)\delta_{i+1} - \#S_i(E) \tag{3.28}
\]

**Proof.** These are Theorem 1.17, and a consequence of Theorem 1.8, respectively of Part I [LY2]. \( \square \)

**Proof of Proposition 4.15.** Since the morphism \( h : Z_T \to G_T \) satisfies \( h^{-1}(V(E)) = Z(E) \), Theorem 3.12, Propositions 3.17 and 3.18 imply that
\[
z(E) = \#S(E) + f(T)
= \dim G_T - \#S(E^\wedge) + f(T)
= \dim G_T - \#S(E^\wedge) + (n(T) - \#H^0(E) - \dim G_T)
= n(T) - \#S(E^\wedge) - \#H^0(E),
\]
which is Göttsche’s formula 3.25. \( \square \)

**Remark 3.23.** We can now reconcile the formulas 3.28 and 3.22, for the fibre dimension of the cell \( Z(E) \) over \( V(E) \), as a direct consequence of Lemmas 3.21 and 3.22. The size \( \#W(E) \) of the fibre counts the size of \( \{ H^a(E) \mid a \neq 0, 1, -1 \} \), the number of hooks of \( E \) having hook-difference any integer but \( 0, 1 \), and \( -1 \). There is exactly one hook having corner \( \mu \) for each monomial \( \mu \) of \( E^\wedge \), so there are \( n(T) \) hooks in all. Thus \( \#W(E) \) satisfies
\[
\#W(E) = n(T) - \#H^0(E) - \#H^1(E) - \#H^{-1}(E)
= n(T) - \sum_{i \geq \mu(T)} \delta_i(\delta_i + 1)/2 - \sum_{i \geq \mu(T)} (\delta_i + 1)\delta_{i+1}, \tag{3.29}
\]
by Lemmas 3.21, 3.22, since \( \#H^1(E) = \#S(E) \), and \( \#H^{-1}(E) = \#S(E^\wedge) \).

**3-D The cells of \( G_T \) and the hook code**

We recall from Part I [LY2] the “hook code” for the cells \( V(E) \) of \( G_T \). The hook code is important, because it gives a simple way of understanding the Betti numbers of \( G_T \), and because the homology ring structure on \( H^*(G_T) \) in the cases it is known, is most simply stated in terms of the hook code. We show here that the hook code arises in natural way form the local parameters of the cell \( V(E) \) of \( G_T \).
Recall that when $T = (1, 2, \ldots, \mu, t_\mu, \ldots, t_j, 0)$, satisfies the condition $\mu \geq t_\mu \geq \ldots \geq t_j > 0$ of §3.26.1, we let $\mu(T) = \mu$ and $j(T) = j$. Let $P$ be a partition whose shape has diagonal lengths $T$. Then $\mu(T)$ is the index of the first diagonal of the shape of $P$ - counting from zero - in which there are spaces, and $j(T)$ is the index of the last nonempty diagonal.

**Definition 3.24 (Partially ordered sets $\mathcal{P}(T)$ and $\mathcal{Q}(T)$).** Recall that $\mathcal{P}(T)$ is the set of partitions $P$ having diagonal lengths $T$. We denote by $\mathcal{B}(T)$ the sequence $\mathcal{B}(T) = (\mathcal{B}_\mu(T), \ldots, \mathcal{B}_j(T))$ where

$$
\mathcal{B}_i(T) = \text{the partition of rectangular shape having } t_i - t_{i+1}
$$

parts each of length $(1 + t_{i-1} - t_i)$ for $\mu \leq i \leq j$. We denote by $\mathcal{Q}(T)$ the set of sequences of partitions $Q_i$, with $Q_i$ included in $\mathcal{B}_i(T)$.

$$
\mathcal{Q}(T) = \{ Q_\mu, \ldots, Q_i, \ldots, Q_j \, | \, Q_i \subset \mathcal{B}_i(T) \}. \tag{3.30}
$$

If $Q = (Q_\mu, \ldots, Q_j) \in \mathcal{Q}(T)$, then $(Q_i)^c = \mathcal{B}_i(T) - Q_i$ is the complement of $Q_i$ in $\mathcal{B}_i(T)$. We also let $\#Q = \sum_i \#(Q_i)$ be the total length of the partitions in the sequence $Q$.

We partially order $\mathcal{Q}(T)$ by inclusion of the component partitions, and we partially order $\mathcal{P}(T)$ by inclusion of each of the diagonal partitions $\mathcal{Q}(E_i)$, $\mu(T) \leq i \leq j(T)$, $E = E(P)$. Both $\mathcal{Q}(T)$ and $\mathcal{P}(T)$ are distributive lattices (see Part I, [IY2]).

**Definition 3.25 (Hook code).** Suppose $P = P(E) \in \mathcal{P}(T)$ is a partition of diagonal lengths $T$, and $\mu \leq i \leq j$. We let $Q_i(P)$ be the partition of the integer $\#\mathcal{H}^i(E)_i$, determined by the distribution of degree $i$ difference-one hooks $(\mu, \nu)$ according to the hand monomial $\nu$. We order the $\delta_{i+1} = t_i - t_{i+1}$ monomials of $(E^i)_i$ such that $xv \in E_{i+1}$ in the reverse alphabetic order. Then

$$
Q_i(P) = (q_{i1}, \ldots, q_{i\delta_{i+1}}), q_{ik} = \#\{ (\mu, \nu) \in \mathcal{H}^i(E)_i \, | \, \nu = \nu_k \}. \tag{3.32}
$$

The partition $Q_i(P)$ is considered to be inside the box $B_i(T)$, so it has $\delta_{i+1}$ parts, some possibly zero. The hook code $\mathcal{D}(P)$ of $P$ is the sequence

$$
\mathcal{D}(P) = (Q_\mu(P), \ldots, Q_j(P)) \subset \mathcal{B}(T). \tag{3.33}
$$

The length $\ell(\mathcal{D}(P)) = \ell(Q_\mu(P)) + \ldots + \ell(Q_j(P))$, and is $\#\mathcal{H}^1(E)$.

**Theorem 3.26.** The coding map $\mathcal{P} \to \mathcal{D}(P)$ gives an isomorphism of distributive lattices $\mathcal{D} : \mathcal{P}(T) \to \mathcal{Q}(T)$, such that

i. The dimension and codimension of the cell $V(E)$ satisfies

$$
dim V(E) = \ell(\mathcal{D}(P(E))) \tag{3.34}
$$

$$
cod V(E) = \ell(\mathcal{D}(P(E)^c)) = \#\mathcal{H}^{-1}(E). \tag{3.35}
$$

ii. $Q(P^c) = (Q(P))^c$.

**Proof of Theorem 3.26.** The dimension formula in Part i. of the Theorem is immediate from Theorem 3.24 and the definition of $\mathcal{D}$. Part ii. of the Theorem is the case $a = 1$ of Theorem 1.8 of Part I [IY2]. That $\mathcal{D}$ is an isomorphism of distributive lattices is Theorem 2.28 of [IY2]. \qed

**Remark.** J. Yaméogo in [Y3] outlines a different proof of Theorem 3.26. He uses a parametrized family of linkings to reduce from the cell $V(E)$ to the cell $V((x^a, y^b) : E)$ whose partition $P'$ is the complement of $P(E)$ in a $b \times a$ rectangle.
Warning. The hook partition \( Q_i(E) \) is quite different from the partition \( Q(E_i) \) that we defined above. Neither determines the other, except in special cases such as \( i = \mu(T) = j(T) \).\(^1\) However, the sequence of partitions \( (Q_\mu(E), \ldots, Q_j(E)) \) determines \( (Q(E_\mu), \ldots, Q(E_j)) \) and vice versa. The connection between the sequences is explained in §2C of Part I: see Theorem 2.23f and Example 2.31 of \( \text{[Y2]} \).

![Diagram](image)

The forms \( p_1, p_2 \) in \( I \) determine
\[
\begin{align*}
f &= x^2y + bx^3, \quad yf = x^2y^3 + byx^3 \\
g &= x^2y^2 - b^2x^3, \quad yg = x^2y^3 - b^2yx^3 \text{ in } I
\end{align*}
\]

Cell \( \mathcal{V}(C) \), \( C = (x^5, x^2y, xy^2, y^4) \)

![Diagram](image)

The coefficients \( a_1, a_2, b \)
determine \( I \) in the cell \( \mathcal{V}(D) \)

Cell \( \mathcal{V}(D) \), \( D = (x^5, x^2y, y^3) \)

Table 7: Two cells of \( G_T, T = (1, 2, 3, 2, 1) \) and their codes. See Example 3.27.

Example 3.27. Table 4 depicts several cells in \( G_T, T = (1, 2, 3, 2, 1) \) with their parameters and corresponding pruning code.

The first cell, labeled \( C \), corresponds to the partition \( P(C) = (5, 2, 1, 1) \), and is important for a counterexample of J. Yaméogo (see (Ci) in §2A above). In Table 4, the shape or Ferrer’s graph of \( P \) is shaded, and corresponds to the cobasis \((1, x, x^2, x^3, x^4, y, yx; y^2); y^3)\). The cell \( \mathcal{V}(C) \) consists of all ideals \( I \) in \( R \) having the monomial initial ideal \( C = (x^5, x^2y, xy^2, y^4) \). If \( I \) is in the cell, the two key elements \( p_1 = x^3y + bx^4 \) and \( p_2 = y^4 + ax^4 \) of \( I \) determine the \( I \) and are listed at right. They have the form \((\mu + cv), \) where \( c \in k, \) where \( v \) darkly shaded is a monomial at the right endpoint of a row, and \( \mu \) indicated by a space with “O” inside is a monomial just below the shape, and lying on the same diagonal as \( v \) but lower than \( v \). There are two such pairs \((\mu, v)\) with \( v = x^4; \) the two pairs count as the unique part of the code partition \( q_4 = (2) \). There are no such pairs with \( v = y^3, \) and we have written at right \( q_3 = (\emptyset) \), the empty partition.

The second cell in Table 4 corresponds to the partition \( P = (5, 2, 2) \). The ideal \( I \) is in the cell if its monomial initial ideal is \((x^5, yx^2, y^3); \) if so, the ideal is completely determined by the

---

\(^1\)Here are two such special cases, where \( Q_i \) determines \( Q_j \). When \( T = (1, 2, \ldots, \mu(T), t, 0) \) then \( i = j(T) = \mu(T), G_T = \text{Grass}(i + 1 - t, i + 1) \) as in Section 2 and \( Q_i = Q_t \). When \( T = (1, 2, \ldots, \mu(T), a, \ldots, a) \) then \( Q_{j(T)} \) determines \( Q_{\mu}, Q_{\mu+1}, \ldots, Q_j \) and any \( Q_i \) determines \( Q_{j(T)} \).
coefficients \( a_1, a_2, b \) of the forms \( p_1, p_2, p_3 \). At right we have written the code partition \( q_4 = (2) \) corresponding to the pairs of monomials
\[
(\mu, \nu) = (\text{initial monomial, cobasis monomial})
\]
\[
= (x^3y, x^4), (xy^3, x^4) \text{ of the coefficients } a_1, a_2.
\]
The code partition \( q_3 = (1) \) of the second cell corresponds to the pair \( (y^3, y^2x) \) of the coefficient \( b \) in \( p_3 \).

3-E The homology groups \( H^*(G_T) \)

We apply Theorem 3.26 to determining the Betti numbers of the variety \( G_T \). First, recall that the generating function \( B(a, b; q) = \sum_n p(a, b; n)q^n \) for the number \( p(a, b; n) \) of partitions of \( n \) into at most \( b \) parts each less or equal to \( a \) is the \( a \)-binomial coefficient,
\[
\sum_n p(a, b, n)q^n = \left[ \frac{a+b}{a} \right] \frac{(q)_{a+b}}{(q)_a(q)_b}, \text{ where } (q)_a = (1-q)(1-q^2)\cdots(1-q^a) . \tag{3.36}
\]
The polynomial \( B(a, b; q^2) \) is also the Poincaré polynomial \( B(\text{Grass}(a + b)) \) of the Grassmannian of \( a \)-subspaces of a complex vector space of dimension \( a + b \).

A partition \( P \) of diagonal lengths \( T \) corresponds to a unique monomial ideal \( E(P) \), defining a quotient \( R/E(P) \) of Hilbert function \( T \); and conversely, a monomial ideal \( E \) determines a unique partition \( P = P(E) \) (see Definition 3.14).

We let \( b(T; h) \) denote the number of partitions \( P \) of \( n = \sum t_i \) having diagonal lengths \( T \) and for which the corresponding cell \( V(E(P)) \) in \( G_T \) has dimension \( h \), or, equivalently, for which \( P \) has \( h \) difference-one hooks (Theorem 3.12).

In the following result, we consider \( G_T \) as complex variety, because of our use of the theorem of Białynicki-Birula. The results concerning cellular decomposition of \( G_T \) are valid in characteristic zero, or char \( k = p > j \), and the results below will count the cells of given dimension for such fields \( k \). We denote by \( H^*(G_T) \) the homology group in codimension \( i \). Note that \( \text{Grass}(a, \delta_i + \delta_i + \delta_i + 1) = \text{Grass}(t_i - t_{i+1}, 1 + t_{i-1} - t_i) \). Over the complexes the cells occur in even (real) dimensions, and we denote by \( B(T, q^2) \) the Poincaré polynomial of \( G_T \), \( B(T, q^2) = \sum \dim H^i(G_T)q^i \).

**Theorem 3.28 (G. Gotzmann in codimension one \( \text{Grass}(T) \)).** Suppose that \( k = \mathbb{C} \). The isomorphism \( D \) of Theorem 3.26 induces an additive isomorphism of homology groups \( \tau : H^*(G_T) \to H^*(\text{Grass}(T)) \), over \( \mathbb{Z} \), where \( \text{Grass}(T) \) is the product of small Grassmannians (Definition 3.14):
\[
\tau : H^*(G_T) \cong_{\add} \prod_{\mu(T) \leq i \leq j(T)} H^i(\text{Grass}(\delta_i + 1 + \delta_i + 1, \delta_i + 1)) \tag{3.37}
\]
The homomorphism \( \tau \) respects the \( \mathbb{Z}_2 \) duality action.

**Theorem 3.29.** When \( k = \mathbb{C} \) the Poincaré polynomial \( B(G_T) = B(T; q^2) \) satisfies
\[
B(T; q^2) = \prod_{\mu \leq i \leq j} B(\text{Grass}(\delta_i, 1 + \delta_i + \delta_i + 1)) = \prod_{\mu \leq i \leq j} B(1 + \delta_i, \delta_i + 1, q^2). \tag{3.38}
\]
When char \( k = 0 \), or char \( k > j(T) \) the coefficient of \( q^{2u} \) in \( B(G_T) \) counts the number of cells \( V(E) \) having \( k \)-dimension \( 2u \); and the coefficient counts also the number of partitions of diagonal lengths \( T \) having \( u \) hooks of difference one.

The total number of cells \( V(E) \subset G_T \) is \( b(T) = B(T; 1) \), which is the following product of binomial coefficients
\[
b(T) = \prod_{\mu(T) \leq i \leq j(T)} \left( \frac{1 + t_{i-1} - t_{i+1}}{t_i - t_{i+1}} \right) . \tag{3.39}
\]
The product \( b(T) \) also counts the number of monomial ideals \( E \) in \( k[x, y] \) defining a quotient algebra of Hilbert function \( T \); and when \( k = \mathbb{C} \) it is also the Euler characteristic \( \chi(G_T) \).

**Proof.** By Theorem 3.12 the cells \( \{V(E) \mid E \text{ is a monomial ideal with } H(R/E) = T \} \) form a cellular decomposition of \( G_T \), in any characteristic \( p > j \), or in characteristic 0. When \( k = \mathbb{C} \), it follows from [B] that the classes of the cells form a \( \mathbb{Z} \)-basis of the homology \( H^*(G_T) \). Theorems 3.28 and 3.29 now follow from Theorem 3.26 and the well-known homology of \( \text{Grass}(a, b) \). □

### 3-F Do the classical results for the Grassmannian extend to \( G_T? \)

In Section 2-C we listed the classical results for the Grassmannian \( \text{Grass}(d, R_j) \). Here we state which extend to \( G_T \).

A. \( G_T \) has a cellular decomposition whose cells \( V(E) \) correspond 1 − 1 with initial monomial ideals \( E \), such that the Hilbert function \( H(R/E) = T \), and \( I \in V(E) \implies \text{In}(I) = E \). (Theorem 3.12). The cells \( V(E) \) correspond 1 − 1 with the partitions \( P(E) \) of \( n = \sum t_i \) having diagonal lengths \( T \). The shape of the partition \( P(E) \) can be identified with the standard cobasis \( E^\circ \) for the ideal \( E \). (See Proposition 3.6, Theorem 3.12)

B. The set of cells \( V(E) \) of \( G_T \) correspond 1 − 1 to the set of all sequences \( Q = (Q_i, \ldots, Q_j) \) where each \( Q_i \) is an ordinary partition having no more than \( (t_i - t_{i+1}) \) nonzero parts, each of size no greater than \( (1 + t_{i-1} - t_i) \). Thus, each \( Q_i(P) \) is a subpartition of a \( (t_i - t_{i+1}) \times (1 + t_{i-1} - t_i) \) rectangle \( B_i(T) \). (See Theorem 3.20)

C. The dimension of \( V(E) \) satisfies \( \dim V(E) = (\ell Q(P(E))) = \# H(P(E)) \), the total length of the partitions comprising the code \( Q(P(E)) \), or the total number of difference one hooks of \( P(E) \). The code of the dual partition \( P^\circ \) is the complement of \( Q(P) \): \( Q(P^\circ) = (Q(E)^c) \) (Theorem 3.20).

Ci. Dimensional properness fails when \( T = (1, 2, 3, 2, 1) \) and the cell is \( C \) of codimension 2 in \( G_T \) from Table 7. The family of ideals \( I_a = (y^4 + ax^4, yx^2, y^2 x, a \neq 0) \) has initial ideal \( (y^4, yx^2, y^2 x) \) corresponding to the cell \( C \) in the basis \( (x, y) \) where \( y < x \), and has the corresponding initial ideal \( (x^4, xy^2, x^2 y) \) in the basis \( (y, x) \), \( y > x \). Thus, the 1 dimensional family \( I_a \) satisfies

\[ I_a \subset \overline{V}(C, x) \cap \overline{V}(C, y) \]

so the intersection is not proper (Example of J. Yaméogo [Y1]). When the intersection is proper, the homology class of intersection \( \overline{V}(E, p) \cap \overline{V}(E^\circ, p') \) of several ramification conditions at distinct points can be calculated from the still unknown homology ring of \( G_T \). See Theorem 4.1 Example 4.3 and the Problem at the end of Section 4.1. In Section 4.2A we list the cases where the ring structure \( H^*(G_T) \) is known.

Cii. The analog of the Eisenbud-Harris result concerning specialization of intersections on \( \text{Grass}(d, R_j) \) (see [2-C] (Cii)) is still open.

D. The Poincaré duality is not exact, when expressed in terms of the cells. J. Yaméogo shows that the intersection \( \overline{C} \cdot \overline{C} = [\text{point}] = \overline{C} \cdot \overline{C^\circ} \) when \( T = (1, 2, 3, 2, 1) \). However, he has also shown that the duality is always exact in codimension one (See [Y6]).

E. J. Yaméogo has shown that the closure \( \overline{C} \) is not a union of cells. He proves in [Y-5] the following result:

**Theorem.** If \( U_+ \) is a cell in \( G_T \) of dimension \( c \) then there are cells \( \overline{U_+} \supset \overline{U} \supset \overline{U_-} \), where \( \dim U_+ = c + 1 \) and \( \dim U_- = c - 1 \).
His method is to use a parametrized family of linkings to reduce the question to cells of lower dimension.

**Conjecture 1 (Chain conjecture).** The cells $U_c, U_e$ of dimensions $c, e$ in $G_T$ satisfy $U_c \supset U_e$ iff there is a chain of cells $U_c \supset U_{c+1} \supset \cdots \supset U_e$ such that $U_{i+1}$ has codimension 1 in $U_i$.

### 4 Homology ring of $G_T$ and the intersection of cells

In Section 4-A we note the relevance of the homology ring $H^*(G_T)$ in computing the intersection of ramification loci at different points $p \in \tilde{\mathbb{P}}^1$ (Theorem 4.1), and we summarize what is known about the homology ring. In Section 4-B we determine the homology ring for $G_T$ when $T = T(\mu, j) = (1, 2, \ldots, \mu, \mu, 1 = \mu)$ (Theorem 4.4). In Section 4-C we show that this $G_T, T = T(\mu, j)$ is a desingularisation of the $\mu$-secant variety Sec($\mu, j$) to the degree-$j$ rational normal curve, and we identify the homology classes of the pullbacks to $H^*(G_T)$ of the higher singular loci of Sec($\mu, j$) (Theorem 4.11).

#### 4-A Intersection of ramification conditions

Knowledge of the homology ring $H^*(G_T)$ would allow us to understand the homology class of the intersection of ramification conditions $V(E; x_0) \cap V(E', x_1)$ on ideals of linear systems at different points $x_0, x_1$ of the curve $\mathbb{P}^1$. We give an example of this in 4-B (Example 4.5). Although by Theorem 3.28 the homology of $G_T$ is additively isomorphic to that of a product $S\text{Grass}(T)$ of small Grassmanians, the ring structure is not in general isomorphic to that of the product. Nevertheless there are some simplifications. We first formulate the basic result, motivating our study of the homology ring of $G_T$. We suppose that $k = \mathbb{C}$ in this section.

**Theorem 4.1.** Let $p_1, p_2, \ldots, p_s$ be $s$ points of $\tilde{\mathbb{P}}^1$, let $V(E_1, p_1), \ldots, V(E_s, p_s) \subset G_T$ be the cells associated to $\text{QRAM}(V, p_1) = \text{QRAM}(E_1), \ldots, \text{QRAM}(V, p_s) = \text{QRAM}(E_s)$, and suppose that the intersection $Z = V(E, p) \cap V(E', p')$ is proper. Then the homology class of $Z$ is the intersection product of the classes of $V(E_1, p_1), \ldots, V(E_s, p_s)$. The homology class of the closure $\overline{V(E, p)}$ is independent of $p$.

**Proof.** The independence of the homology class of $\overline{V(E, p)}$ is result of the PGL(1) action on $G_T$; if $g(p) = p'$ then $g$ takes $\overline{V(E, p)}$ to $\overline{V(E, p)}$. The rest is a consequence of homology theory on the nonsingular projective algebraic variety $G_T$. □

**Remark:** Despite the counterexample of Yaméogo (see (Ci) of 3-F), the intersection $Z$ of two such conditions often is proper, and there is an actual ramification calculus in those cases.

There are two major simplifications of the problem of determining the homology ring structure $H^*(G_T)$.

1. **Decomposition into a product of elementary $G_{T_k}$.** We may assume that $T$ has no consecutive constant values less than $\mu(T)$. We say that $T$ is *elementary* if

$$t_i \neq t_{i+1} \text{ for } t_i < \mu(T).$$

**Lemma 4.2.** There is a decomposition of $G_T$ as a product

$$G_T = \prod_k G_{T_k} \text{ for } T_k \text{ elementary.}$$

(4.1)
For details see [Y3, IY1].

ii. Surjectivity of \( \iota^* \). A result of A. King and C. Walter [KW] shows that the inclusion \( \iota : G_T \to BGrass(T) \) of [KW] into a product of big Grassmannian varieties satisfies

**Theorem.** [KW] The homomorphism \( \iota^* : H^*(BGrass(T)) \to H^*(G_T) \) of homology rings is a surjection (and \( \iota_* : H^*(G_T) \to H^*(BGrass(T)) \) is an inclusion).

**Remark.** Status of problem of determining \( H^*(G_T) \).

**Cases where the ring structure** \( H^*(G_T) \) **is known:**

A. For \( T = (1, 2, 3, 2, 1) \) (see J. Yaméogo, cite Y3, Y6).

B. For \( T = T(\mu, j) = (1, 2, \ldots, \mu, \mu, \ldots, \mu, 1) \) of socle degree \( j \) (see (4.3) below, and a sequel article).

C. We have determined the class \( \iota_* (G_T) \) in \( BGrass(T) \) for \( T = (1, 2, \ldots, \mu, a, b, 0) \), using a vector bundle argument suggested by G. Ellingsrud (See [IY1], and a sequel article under preparation).

By (4.3) these results determine \( H^*(G_T) \) for any \( T \) composed of the elementary Hilbert functions in A. and B.

**Sections of a vector bundle.** J. Yaméogo has shown that if we set \( T' = (t_0, \ldots, t_{j-1}) \), then \( G_T \) is the zeroes of a section of a known vector bundle \( E_T \) on \( G_{T'} \times Grass(j + 1 - t_j, j + 1) \), where the codimension of \( G_T \) in the product is equal to the rank of \( E_T \). [Y6]. This allows computation of the class \( \iota_* (G_T) \) in \( BGrass(T) \) for \( T = (1, 2, \ldots, \mu, a, b, 0) \), using a vector bundle argument suggested by G. Ellingsrud (See [IY1], and a sequel article under preparation).

By (4.3) these results determine \( H^*(G_T) \) for any \( T \) composed of the elementary Hilbert functions in A. and B.

**4-B The ring** \( H^*(G_T) \) **when** \( T = T(\mu, j) \)

In this section we determine the ring \( H^*(G_T) \) for the special case \( T = T(\mu, j) \):

\[
T(\mu, j) = (1, 2, l \ldots \mu, \ldots, \mu, 1 = T_j) \quad (4.2)
\]

We then use the calculation to illustrate Theorem (4.3) determining the number of ideals of linear systems over \( \mathbb{P}^1 \) satisfying an intersection of ramification conditions (Example (15)). The ring structure of \( H^*(G_T) \), \( T = T(\mu, j) \) involves binomial coefficients, which are related to the degrees of ideals of determinantal minors of circulant matrices

\[
\begin{pmatrix}
a_0 & \cdots & a_\mu \\
a_1 & \cdots & a_{\mu+1} \\
\vdots & \vdots & \vdots \\
a_{j-\mu} & \cdots & a_j
\end{pmatrix}
\]

The variety \( G_T(\mu, j) \) has dimension \( 2\mu - 1 \), and has the structure of a projective \( \mathbb{P}^{\mu-1} \) bundle over \( \mathbb{P}^\mu \). An ideal \( I \) having Hilbert function \( T \) is determined by \( I_\mu = \langle f \rangle \) and by a codimension one subspace \( I_j \) of \( R_j \), since \( I_i = \langle f \rangle \cap R_i \) for \( \mu < i < j \). Thus, we have an inclusion of projective
By (3.30) the block corresponding to the cell $a$ of codimension varieties:

$$t : G_T \subset \prod_{i=\mu,j} (\text{Large Grassmanians}) = \text{Grass}(1, \mu + 1) \times \text{Grass}(j, j + 1)$$

$$I \rightarrow (I_{\mu}, I_j) \subset \mathbb{P}^\mu \times \mathbb{P}^j. \quad (4.3)$$

The projection $pr_1 : G_T \rightarrow \mathbb{P}^\mu$ has fibre over $(f) \in R_\mu$ the projective space $\mathbb{P}(R_j/(R_{j-\mu}f)) \cong \mathbb{P}^{\mu-1}$. By this fact, or by Theorem 3.28, the additive structure of $H^*(G_T)$ is

$$H^*(G_T) \cong \text{add } H^*(\text{Grass}(\mu - 1, \mu)) \otimes H^*(\text{Grass}(1, \mu))$$

$$\cong \text{add } H^*(\mathbb{P}^{\mu-1}) \otimes H^*(\mathbb{P}^\mu).$$

By (3.30) the block $\mathcal{B}_{j-1}(T)$ is $(\mu - 1) \times 1$, and $\mathcal{B}_j(T)$ is $(1 \times \mu)$. We let $[a, b]$ denote the cohomology class corresponding to the cell

$$E[a, b] = D^{-1}((\mu - 1 - a)\wedge, (\mu - b), 0 \leq a \leq \mu - 1, 0 \leq b \leq \mu.$$ 

of codimension $a + b$ in $G_T$. We let $[a, b] = 0$ when $a > \mu - 1$ or $b > \mu$. Recall that $H^*(G_T)$ denotes the homology ring of $G_T$ graded by codimension of the homology class. If $Y$ is subvariety we denote by $[Y]$ its homology class in the relevant homology ring.

**Lemma 4.3.** Suppose $T = T(\mu, j)$. Let $\zeta$ be the cohomology class of a hyperplane section of $\mathbb{P}^\mu$, and $\eta$ the class of a hyperplane section of $\mathbb{P}^j$. Then the class of $\iota_* (G_T)$ satisfies

$$[\iota_* (G_T)] = (\zeta + \eta)^{j+1-\mu}. \quad (4.4)$$

Furthermore, $\iota_*$ is injective, and $\iota^*$ is surjective on homology or cohomology and they satisfy

$$\iota^*(\zeta^i \eta^j) = \begin{cases} [u, v] \text{ for } u + v < \mu, \\ \sum_{i=0}^{j+1-\mu} [u + i - 1, v - i + 1] \text{ for } u + v \geq \mu. \end{cases} \quad (4.5)$$

$$\iota_* [a, b] = \begin{cases} [\zeta^{a+1} \eta^{b+1-\mu}] \text{ for } a + b \geq \mu, \\ \zeta^a \eta^b [\iota_* (G_T)] \text{ for } a + b < \mu. \end{cases} \quad (4.6)$$

**Comment.** The $\zeta^i$ term in (4.4) is zero unless $i \leq \mu$, as $\zeta^{\mu+1} = 0$. We originally proved Lemma 4.3 and Theorem 1.3 by intersecting $G_T$ with cycles of $H^*(\mathbb{P}^\mu \times \mathbb{P}^j)$ having complementary dimension, obtaining the degree of certain determinantal minors of circulant matrices, which we then calculated. We are grateful to G. Ellingsrud, who suggested the approach here. A similar method applies to Hilbert functions of the types $T = (1, 2, \ldots, \mu, \ldots, \mu, a, 0)$ or $T = (1, 2, \ldots, \mu, a, b, 0)$. That the formula (4.3) depends on $j$ shows that the homology ring structure is not that of $S\text{Grass}(T)$ which here depends only on $\mu$.

**Proof of Lemma 4.3.** On $\mathbb{P}^\mu \cong \text{Grass}(1, R_\mu)$ we let $-\zeta = \mathcal{O}(-\zeta) \cong \mathcal{O}(1)$ denote the universal rank one subbundle, and on $\mathbb{P}^j$ we let $\mathcal{O}(\eta) \cong \mathcal{O}(-1)$ denote the universal rank one quotient. Let $p_1$, $p_2$ denote the projections from $\mathbb{P} = \mathbb{P}^\mu \times \mathbb{P}^j \rightarrow \mathbb{P}^\mu$ or $\mathbb{P}^j$, respectively. Then we have on $\mathbb{P}^\mu \times \mathbb{P}^j$ a composite map

$$\alpha : p_1^*(R^j - \mu \otimes \mathcal{O}(-\zeta)) \rightarrow p_2^*(R^j) \rightarrow p_2^*(\mathcal{O}(\eta))$$

whose vanishing at $p \in \mathbb{P}^\mu \times \mathbb{P}^j$ is a condition for $p \in G_T$. The vanishing of $\alpha$ is equivalent to that of $\alpha^\vee$

$$\alpha^\vee : p_2^*(\mathcal{O}(\eta)) \rightarrow p_1^*(R^j - \mu \otimes \mathcal{O}(\zeta)).$$
Finally the monomial ideal \( E + x, y, x \)

by the the powers of \( x \) which is nonzero only when \((\zeta + \eta)^{j+1-\mu}\), proving (4.4).

We omit the proofs of the remaining formulas of Lemma 4.3 which are now straightforward.

**Theorem 4.4.** When \( T = T(\mu, j) \) the product \([a, b] \cdot [c, e]\) satisfies

\[
[a, b] \cdot [c, e] = \begin{cases} 
[a + c, b + e] & \text{if } a + b + c + e < \mu, a + b < \mu \text{ and } c + e < \mu, \\
\sum_{i=0}^{i=j+1-\mu} (j+1-\mu)[a + c + i - 1, b + e - i + 1] & \text{or vice-versa,} \\
0 & \text{if } a + b \geq \mu \text{ and } c + e \geq \mu.
\end{cases}
\]

**Proof.** The proof of Theorem 4.4 is a straightforward calculation, given Lemma 4.3.

**Remark.** When the classes \([a, b]\) and \([c, e]\) are in complementary dimensions, so \((a + b + c + e) = 2\mu - 1\), then \(a + b < \mu\) implies \(c + e \geq \mu\), and vice-versa. Thus the product

\([a, b] \cdot [c, e] = [a + c, b + e] = [a + c, 2\mu - 1 - (a + c)],\)

which is nonzero only when \(a + c = \mu - 1\), and \(b + e = \mu\), namely when \(E[a, b] = (E[c, e])^\vee\), the class of the dual partition. It follows that when \(T = T(\mu, j)\) the classes of cells do in fact give the exact duality in complementary dimensions for \(H^\ast(G_T)\), unlike what is true in general.

**Example 4.5 (Determining ideals with given ramification).** Let \(\mu = 3\) and \(j = 6\), so that \(T = (1, 2, 3, 3, 3, 3, 1)\). We wish to find the number of ideals having the ramification \([1, 1]\) at \(x\), \([0, 2]\) at \(y\), and \([1, 0]\) at \((x + y)\) classes of total codimension \(2 + 2 + 1 = 5 = \dim(G_T)\). By Theorem 4.4 the product \([1, 1] \cdot [0, 2]\) satisfies

\([1, 1] \cdot [0, 2] = \sum_{0 \leq i \leq 4} \binom{4}{i} \cdot [i, 4 - i].\)

By the exact duality, the intersection of this product with \([1, 0]\) picks out the coefficient for \(i = 1\), namely four. There are thus four ideals - counting multiplicities, having this ramification at \(x, y, x + y\). What are these four ideals? Each of the four ideals has ramification at \(x = 0\) given by the the powers of \(x\) in the monomial ideal \(E_1 = (y^{2}x, y^{3}, x^{6})\) with cobasis given by the partition \((6, 6, 1, 1, 1, 1)\) of code \(D(E_1) = (Q_5 = (1, 0); Q_6 = (2))\). The monomial ideal \(E_2\) in coordinates \((y, C)\) with code \(D(E_2) = (Q_5 = (1, 1); Q_6 = 1)\) at \(y\) is, \(E_2 = (y^{6}, C y^{5}, C^{3})\), of partition \((6, 5, 5)\). Finally the monomial ideal \(E_3\) in coordinates \((x + y, C)\) for the diagram \(D(E_3) = (Q_5 = 1; Q_6 = 3)\) at \(x + y\) is that of the ideal \((x + y)^{7}, C (x + y)^{5}, C^{2}(x + y), C^{6})\), with cobasis given by the partition \((7, 5, 1, 1, 1, 1)\). It follows that the generator \(f = f_{a}\) satisfies

\(f = x(x + y)(x + ay)\) where \(a\) is not zero.
We know that $I_6$ contains $y^6, xy^5,$ and $x^6$, as well as $R_3 f,$ but has dimension only 6. It follows that there is a dependency in the columns of the $4 \times 4$ matrix $M$ obtained by writing as rows $x^3 f, x^2 y f, xy^2 f, y^3 f$ of $(R_6 f/\langle y^6, xy^5, x^6 \rangle)$ in the basis $(x^5 y, x^4 y^2, x^3 y^3, x^2 y^4)$ for $(R_6/\langle y^6, xy^5, x^6 \rangle)$:

$$M = \begin{pmatrix}
1 + a & a & 0 & 0 \\
1 & 1 + a & a & 0 \\
0 & 1 & 1 + a & a \\
0 & 0 & 1 & 1 + a
\end{pmatrix}.$$ 

The four roots $a_1, \ldots, a_4$ of $\det M = 0$ give the four ideals

$$I_u = (f_a, y^6, xy^5, x^6), \; a = a_u, \; u = 1, \ldots, 4.$$  \hspace{1cm} (4.8)

having the given ramification at $x, y,$ and $x + y$.

**Proposition 4.6.** If $I \in G_T$, the sum of the codimensions or lengths of the ramification conditions it satisfies at all points of $\mathbb{P}^1$ is

$$\sum_{p \in \mathbb{P}^1} \ell(QRAM(I, p)) = \dim(B\text{Grass}(T)) = \sum_{p(T) \leq a(j(T))} (t_i (i + 1 - t_i))$$  \hspace{1cm} (4.9)

**Proof.** Immediate from Lemma 2.7 applied to each $I_i$. $\square$

**Example 4.7 (Overdetermination of ideals by ramification).** Each of the four solution ideals $I$ for Example 4.5 has non-generic ramification at $(x + a_u y)$ because $x + a_u y$ is a factor of $f$. Each solution ideal $I(u), u = 1, \ldots, 4$ has ramification also at three other points $L_{uv} = 0, v = 1, 2, 3$ satisfying $L_{uv} \in I(u)_6$ (see Lemma 2.7).

We define a Wronskian morphism $W : G_T \to \mathbb{P}$, $\mathbb{P} = \prod \mathbb{P}^N$, $N_i = t_i (i + 1 - t_i)$ as the composition of $\iota : G_T \to B\text{Grass}(T)$ and of the product $w$ of the Wronskian maps $w_i : \text{Grass}(i + 1 - t_i, i + 1) \to \mathbb{P}^N$ (Definition 2.6). Each $w_i$ is a finite cover of degree given by Proposition 2.15. Hence we have

**Proposition 4.8.** The Wronskian morphism $W : G_T \to \mathbb{P}$ is a finite cover of its image, of degree $\prod_{i=0}^t \deg w_i$.

**Remark.** RELATIONS AMONG THE RAMIFICATION CONDITIONS. It follows from Propositions 4.6 and 4.8 that in general there are relations between the set of ramifications $\{QRAM(I, L_p) \mid p \in \mathbb{P}^1\}$ in different directions $p$ of an ideal $I \in G_T$. This contrasts with the case of a single linear system, where the total codimension of the ramification conditions adds up to the dimension $N_i$ of Grass($d_i, R_t$) (Lemma 2.7). For ideals the total codimension of the ramification conditions adds up to the dimension of the product $B\text{Grass}(T)$ of big Grassmannians, to which $G_T$ is embedded; this is much larger than the dimension of $G_T$. We expect that there is a kind of algebraic variety of relations among the ramification conditions for ideals in $G_T$. This can be seen in the special case of Example 4.7, where evidently the “extra” ramification points $x + a_u y = 0,$ and at $L_{uv} = 0,$ for $I(u)$ satisfy algebraic relations. What are the equations defining the image $W(\mathbb{G}_T)$ in $\mathbb{P}$?

**Problem A GENERALIZED SCHUBERT CALCULUS?** The intersection of the classes of cells in $H^*(\mathbb{G}_T), \; T = T(\mu, j)$ can be expressed simply, using the codes $[a, b]$ of cells; the intersection numbers involve binomial coefficients. When $d = j$, then $T = (1, 2, \ldots, \mu, a, 0)$, and $G_T$ is a Grassmann variety Grass($\mu + 1 - a, R_{\mu}$). In this case the cells are the Schubert cells, and the intersection numbers are given by the Littlewood-Richardson rule. When $k = \mathbb{C}$, $H^*(\mathbb{G}_T)$ is additively a Z-module with basis the classes of the cells $\mathbb{V}(E(P))$, as $E(P)$ runs through the
monomial ideals $E(P)$ attached to partitions $P$ of diagonal lengths $T$. The question of finding the ring structure on $H^*(G_T)$ in terms of this basis asks for a generalization of (at least) two rules. First, the Schubert calculus and its Littlewood-Richardson rule when $\mu = j$; second, the intersection numbers given above for $T = T(d,j)$ involving binomial coefficients (Theorem 4.4). The intersection numbers there are most simply expressed in terms of the hook code of the cells.

The problem of determining the homology ring of $G_T$, generalizing the special case $\mu = j$ which is Schubert calculus, is, given two partitions $P, Q$ of diagonal lengths $T$, to find the intersection numbers $\alpha(P, Q : S)$, such that the homology classes $[P], [Q]$ of the cells $\mathcal{V}(P)$ and $\mathcal{V}(Q)$ satisfy

$$[P] \cdot [Q] = \sum_S \alpha(P, Q : S)[S],$$

(4.10)

Here $S$ runs through the partitions of diagonal lengths $T$, having codimension $\text{cod } \mathcal{V}(S) = \text{cod } \mathcal{V}(P) + \text{cod } \mathcal{V}(Q)$. The related geometric problem is to connect these numbers naturally to the geometry of the rational normal curve.

**4-C Desingularization of the secant bundle $V(\mu, j)$ to the rational normal curve**

Let $k[A]$ be the polynomial ring $k[A] = k[a_0, \ldots, a_j]$, suppose $2\mu < j + 1$ and denote by $V(\mu, j)$ the projective subvariety of $\mathbb{P}^j = \text{Proj}(k[A])$ defined by $I(\mu, j - \mu, A)$, the ideal of $(\mu + 1) \times (\mu + 1)$ minors of the Hankel matrix

$$\text{HANKEL}(\mu, j - \mu, A) = \begin{pmatrix} a_0 & a_1 & \ldots & a_{j-\mu} \\ a_1 & a_2 & \ldots & a_{j+1-\mu} \\ \vdots & \vdots & \ddots & \vdots \\ a_\mu & a_{\mu+1} & \ldots & a_j \end{pmatrix}.\quad (4.11)$$

It is well known that $V(\mu, j)$ is irreducible, and that $I(\mu, j - \mu, A)$ is also the ideal of $(\mu + 1) \times (\mu + 1)$ minors of $\text{HANKEL}(\mu', j - \mu', A)$ provided $\mu \leq \min(\mu', j - \mu')$ (see [GP, Wa, E, Ge], [IK, §1.3]). It is also well known that if $\text{char } k = 0$ or $\text{char } k > j$, then $V(\mu, j)$ is the $\mu$-secant locus $\text{Sec}(\mu, j)$ to the degree-$j$ rational normal curve $X(j)$ parametrizing projective $\mathbb{P}^{\mu-1}$-secants to $X(j)$. We explain this connection briefly. Suppose that

$$F_a = a_0 x^j + \binom{j}{1} a_1 x^{j-1} y + \ldots + \binom{j}{i} a_i x^{j-i} y^i + \ldots + a_j y^j \quad (a_i \in k)$$

is an element of $R_j, R = k[x,y]$. $\text{Sec}(\mu, j)$ is the closure of the family of forms $F$ that can be written as the sum of $F = L_1^\mu + \ldots + L_\mu^\mu$ of $\mu$ $j$-th powers of linear forms $L_i \in R_1$. Then

**Lemma 4.9.** $F_a \in \text{Sec}(\mu, j)$ iff $a \in V(\mu, j)$.

We now explain the connection with the variety $G_T, T = T(\mu, j)$. Following the classical theory of apolarity, let $R$ act on $R$ as higher order partial differential operators: $h \circ f = h(\partial/\partial x, \partial/\partial y) \circ f$. Then if $F \in R_j$, the ideal $I_F = \text{Ann } (f)$ is a Gorenstein ideal - hence complete intersection of $R$, and we have $I_F = (g,h)$, with $\mu = \mu_F = \deg(g) \leq \deg(h) = j+2-\mu$ (see [IK, Theorem 1.54]). We have

**Lemma 4.10.** $\mu_F \leq \mu$ iff $F \in \text{Sec}(\mu, j)$.

Let $T = T(\mu, j)$. We define a morphism $\alpha : G_T \to \text{Sec}(\mu, j)$ by $\alpha(I) = F_I, F_I = (I_j)^+ \in R_j$. Thus, $\alpha$ is identical to (the dual of) $pr_2 : G_T \to \mathbb{P}^j$, in the notation of [LE]. Since, as is easy to show, $I_\mu = g$ satisfies $g \circ F_I = 0$, we have $\mu_F \leq \mu$, implying by the Lemma that $F_I \in \text{Sec}(\mu, j)$.
We stratify $V(\mu, j)$ by rank: $V(\mu, j) \supset V(\mu - 1, j) \supset \ldots \supset V(1, j)$. It is easy to see that the strata satisfy $\dim(V(i, j)) = 2i - 1$ if $i \leq (j + 1)/2$, or $\dim(V(i, j)) = 2i - 2$ if $i = (j + 2)/2$. If $2\mu < j + 1$ and $i > 0$ the singular locus of $V(i, j)$ is $V(i - 1, j)$; if $i = \mu$ and $2\mu = j + 1$, or $2\mu = j + 2$ then $V(\mu, j) = \mathbb{P}^j$ or Grass$(2, R_j)$, respectively, and is non singular.

Recall that a desingularization is semismall iff the fiber over a codimension $c$ stratum has dimension at most $2c$ (see §6 of [Na2] or [BoM]). Below $\iota: G_T \to \mathbb{P}^\mu \times \mathbb{P}^j$ is the inclusion of $\mathbb{P}^\mu$.

We have

**Theorem 4.11.** If $2\mu < j + 1$, the morphism $\alpha$ makes $G_T$ a semismall desingularization of $\text{Sec}(\mu, j)$, whose fiber over a point $p \in V(\mu, i) - V(\mu - 1, j)$, for $1 \leq i \leq \mu$ is a projective space $\mathbb{P}^{\mu - i}$. The class in the homology ring $H^*(G_T)$ of $\alpha^{-1}(V(\mu, j))$ satisfies

$$|\alpha^{-1}(V(\mu, j))| = \iota^*(\mu \cdot \eta - (j + 2 - 2\mu)\zeta) = \mu[0, 1] - (j + 2 - 2\mu)[1, 0],$$

(4.12)

and

$$|\alpha^{-1}(V(\mu, j))| = \text{coeff of } t^{\mu - i} \text{ in } \iota^*\{(1 - \zeta t)^{\mu - i + 1}(1 + \eta t)^{i + 1}\}$$

(4.13)

$$= \sum_{u + v = \mu - i} \binom{j - \mu - i + 1}{u} \binom{i + 1}{v} [u, v].$$

(4.14)

**Proof.** First, if the point $p \in V(\mu, j) - V(\mu - 1, j)$ corresponds to $F$, then $I_F = (g', h')$, with $\deg(g') = i$, and the fibre of $G_T$ over $p$ is all pairs $(I_\mu, I_j)$ with $I_j = (I_F)$ and $\eta|\mu$; thus, $I_\mu = g'g'' >, g'' \in R_{\mu - i}$ and the fiber over $p$ is $\mathbb{P}(R_{\mu - i}) = \mathbb{P}^{\mu - i}$.

Consider $G_T \subset \mathbb{P}^\mu \times \mathbb{P}^j$, let $pr_1$ be the projection of $G_T$ on $\mathbb{P}^\mu$ and $pr_2$ its projection on $\mathbb{P}^j$. Let $S_\mu$ denote the pull-back on $G_T$ of the tautological sub-bundle of $\mathbb{P}^\mu$ and $Q_j$ the pull-back on $G_T$ of the tautological quotient-bundle of $\mathbb{P}^j$. For $1 \leq i \leq \mu$, we have an injection $S_\mu \otimes R_{j-\mu-i} \hookrightarrow R_{j-i}$ of vector bundles on $G_T$ that on fibres maps $f \otimes h$ to the product $fh$. We identify $S_\mu \otimes R_{j-\mu-i}$ to a sub-bundle of $R_{j-i}$, and let $E_i = R_{j-i}/(S_\mu \otimes R_{j-\mu-i})$. Now consider the homomorphism of vector bundles on $G_T$, $\phi_i: E_i \to Q_j^{\oplus i+1}$, that on fibres maps the class $[T]$ to $[x^1 \cdot T, \ldots, x^{i-1} \cdot T, \ldots, y^j \cdot T]$. It is easy to see that $\alpha^{-1}(V(i, j))$ is the locus where $\phi_i$ has rank less or equal $i$. Since $\alpha^{-1}(V(i, j))$ has the right codimension $(\mu - i)$ in $G_T$, we have $|\alpha^{-1}(V(i, j))| = c_{\mu - i}(Q_j^{\oplus i+1} - E)$, where

$$c_t (Q_j^{\oplus i+1} - E) = c_t(Q_j^{\oplus i+1})/c_t(E) = c_t(S_\mu)^{\mu - i + 1}c_t(Q_j)^{i + 1} = (1 - \zeta t)^{\mu - i + 1}(1 + \eta t)^{i + 1}.$$

This proves the equation (4.13). The first equation (4.12) is just a particular case of (4.13) when $i = \mu - 1$. \[\square\]

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Anthony A. Iarrobino,
Mathematics Department, 567 LA
Northeastern University Boston, MA 02115, USA
e-mail: iarrobin@neu.edu

Joachim Yaméogo,
Laboratoire J.-A. Dieudonné, UMR CNRS 6621
Université de Nice-Sophia Antipolis,
F-06108 Nice cedex 02, France
e-mail: yameogo@math.unice.fr