Nonideal quantum detectors in Bayesian formalism

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The Bayesian formalism for a continuous measurement of solid-state qubits is derived for a model which takes into account several factors of the detector nonideality. In particular, we consider additional classical output and backaction noises (with finite correlation), together with quantum-limited output and backaction noises, and take into account possible asymmetry of the detector coupling. The formalism is first derived for a single qubit and then generalized to the measurement of entangled qubits.

I. INTRODUCTION

The problem of continuous qubit measurement is of a significant importance for solid-state quantum computing because the measurement of a solid-state qubit typically requires no significant time and thus can interplay nontrivially with the intrinsic evolution of the qubit system. The evolution of a single solid-state qubit (without ensemble averaging) due to continuous measurement can be described by the Bayesian formalism (for review see Ref.) which takes into account the noisy measurement output of the detector. The Bayesian formalism practically coincides with the version of the quantum trajectory formalism adapted to solid-state setups from the theory developed for quantum optics.

One of the main predictions of the Bayesian formalism is the absence of the single qubit decoherence during the measurement by a good (ideal) detector. In contrast to decoherence of an ensemble of qubits. Moreover, the state of a solid-state qubit can be gradually purified due to continuous measurement. In particular, this makes possible to monitor the phase of quantum coherent (Rabi) oscillations of the qubit. Such monitoring can be naturally used in the quantum feedback control of the Rabi oscillations which suppresses the qubit decoherence due to environment (for quantum feedback in quantum optics see, e.g., Ref.). Another potentially useful application of the Bayesian formalism is a recent prediction that two qubits can be made fully entangled by their continuous measurement by an equally coupled detector.

The efficiency of the quantum feedback loop operation crucially depends on the ideality (quantum efficiency) of the detector. For example, 100% synchronization between the qubit Rabi oscillations and desired pure oscillations is possible only for 100% ideal detector. Many other effects related to continuous measurement of solid-state qubits, which have been predicted using the Bayesian formalism (see, e.g., Ref.) also depend significantly on the detector ideality. The ideality $\eta$ of a continuously operating solid-state detector can be generally defined as a ratio between the detector performance and the performance of a quantum-limited detector, in which the output and backaction noises are strictly related by the lower bound of an inequality similar to the Heisenberg uncertainty relation. More exact definition will be discussed later.

A Quantum Point Contact (QPC) at low temperature is theoretically an ideal quantum detector, that follows from the results of Ref.. A nearly ideal operation of the QPC has been demonstrated experimentally. The fact that a SQUID can theoretically reach the limit of an ideal detector follows from the results of Ref. A normal state single-electron transistor (SET) is not a good quantum detector at usual operating points above the Coulomb blockade threshold. However, its quantum efficiency improves when we go closer to the threshold and becomes much better when the operating point is in the cotunneling range (below the threshold), in which case the limit of an ideal detector can be achieved. Superconducting SET is generally better than normal SET as a quantum-limited detector and can approach 100% ideality in the supercurrent regime as well as in the double Josephson-plus-quasiparticle regime. Finally, the resonant-tunneling SET can reach complete ideality in the small-bias limit.

In the simplest version of the Bayesian formalism, a nonideal solid-state detector is modeled as an ideal symmetrically coupled detector and a “pure dephasor” in parallel (environment or just extra backaction noise). In this case the nonideality leads to an extra term in the Bayesian equations, which introduces the gradual decay of the nondiagonal elements of the density matrix of the measured qubit. It was implied that such backaction dephasing is also equivalent to the extra noise at the detector output. However, the equivalence has never been proven explicitly, and this is one of the goals of the present paper.

In a more advanced version of the Bayesian formalism a possible correlation between the output noise of a nonideal detector and the backaction noise is taken into account. However, the formulas for the evolution of the qubit density matrix in this case have been presented without any derivation, just from physical intuition. Moreover, comparison of these formulas with the results of Ref. for an ideal but asymmetrically coupled detector (which shifts the energy levels of the measured qubit) reveals some difference. Even though the
difference is minor (second order in the detector response, which is assumed to be small), it points to some incorrectness of the initial formulas of Ref. 23 (corrected formulas can be found in Ref. 24). The main goal of this paper is to present a mathematical derivation of the Bayesian formalism for a nonideal detector with correlated output and backaction noises, using the phenomenological model which adds correlated classical noise to the quantum noise of an asymmetric ideal detector. We start with the measurement of one qubit and then generalize the formalism to the continuous measurement of an arbitrary number of entangled qubits.

Notice that the issue of the asymmetric detector coupling to qubit has been recently discussed for a QPC in terms of the tunneling phase control by the qubit state. 25 For a small-transparency QPC the formalism is significantly simplified and is a direct generalization of the model of Ref. 24. We will use the results of Ref. 24 to model an ideal asymmetric detector.

While we model the detector nonideality by an additional classical noise, let us mention a different approach to the nonideality in Ref. 24, in which a random fraction of electrons tunneled through the detector is assumed to be missing. In our opinion, such model is not well applicable to solid-state detectors, even though it perfectly describes the inefficiency of a photodetector in a similar problem in quantum optics.

II. MODEL

We will use the phenomenological model of a nonideal solid-state detector of a qubit state shown in Fig. 1. It consists of an ideal detector and three sources of additional classical noise. We assume that the detector output is the noisy current $I(t)$ (we have in mind a QPC or a SET as a detector). The ideal detector is characterized by the output noise spectral density $S_0$ [we assume flat ("white") noise spectrum] and its backaction onto the measured qubit which will be called "quantum noise". (Actually, because of the quantum relation between the output noise and the backaction, the output noise could also be called quantum; however, we will avoid such terminology, emphasizing the assumption that the quantum behavior does not propagate beyond the ideal detector.)

The first source of an additional classical noise adds the noisy component $\xi_1(t)$ with the white spectral density $S_1$ to the output $I_d(t)$ of the ideal detector, so that the total output is $I(t) = I_d(t) + \xi_1(t)$. The second noise source is the classical noise $\xi_2(t)$ which is 100% correlated with (proportional to) the noise $\xi_1(t)$ and affects the qubit energy asymmetry $\varepsilon$. The qubit Hamiltonian is

$$H_{qh} = \frac{\varepsilon}{2} (c^\dagger_0 c_2 - c^\dagger_1 c_1) + H (c^\dagger_0 c_2 + c^\dagger_1 c_1),$$

where the tunneling strength $H$ is assumed to be real without loss of generality. The relative magnitude of the noise $\xi_2(t) = A \xi_1(t)$ is characterized by the parameter $A$. Finally, the third classical noise source is the white noise $\xi_3(t)$ which also affects the qubit energy asymmetry $\varepsilon$ [so that $\varepsilon \rightarrow \varepsilon + \xi_2(t) + \xi_3(t)$]. The second and third noise sources together are obviously equivalent to one white noise source, partially correlated with $\xi_1(t)$. However, we prefer to split it into the fully correlated and uncorrelated parts for clarity. Obviously, the qubit parameter $H$ can also be affected by the detector noise; however, we do not take this effect into account, because the qubit dephasing is more naturally caused by the noise of its energy asymmetry $\varepsilon$ (which corresponds to the measured degree of freedom) and also because the induced noise of $H$ is negligible, for example, for a single-Cooper-pair qubit measured by an SET.

Let us start with the symmetric ideal detector, neglect all classical noises $\xi_{1,2,3}(t)$ and use the basic Bayesian formalism to describe the measurement process (i.e., the result of quantum backaction onto qubit), then the evolution of the qubit density matrix $\rho_{ij}(t)$ is

$$\dot{\rho}_{11} = -\dot{\rho}_{22} = -\frac{H}{\hbar} \text{Im} \rho_{12} + \rho_{11} \rho_{22} \frac{2\Delta I}{S_0} \left[I_d(t) - I_0\right], \quad (2)$$

$$\dot{\rho}_{12} = \frac{\varepsilon}{\hbar} \rho_{12} + \frac{H}{\hbar} \left(\rho_{11} - \rho_{22}\right)$$

$$-\left(\rho_{11} - \rho_{22}\right) \frac{\Delta I}{S_0} \left[I_d(t) - I_0\right] \rho_{12}. \quad (3)$$

Here $\Delta I \equiv I_1 - I_2$ is the detector response, $I_1$ is the average detector current for the qubit state $|1\rangle$, $I_2$ is the average current for the state $|2\rangle$, and $I_0 \equiv (I_1 + I_2)/2$. For the validity of Eqs. (2)–(3) we have to assume the weakly responding detector, $|\Delta I| \ll I_0$, sufficiently large detector voltage (much larger than the qubit energies), and assume that the passage of individual electrons in the detector is much faster than the qubit evolution, $I_0/\varepsilon \gg (4H^2 + \varepsilon^2)/\hbar$, so that the current can be considered as

![FIG. 1. Schematic of a nonideal solid state detector measuring a qubit state. The detector is modeled as an ideal quantum detector and three sources of additional classical noise: output noise $\xi_1(t)$ with the white spectral density $S_1$, backaction noise $\xi_2(t)$ fully correlated with $\xi_1(t)$, and uncorrelated backaction noise $\xi_3(t)$. The total noise density $S_0 + S_1$ of the output detector current $I(t)$ includes the contribution $S_0$ from the noise of ideal detector current $I_d(t)$.](image-url)
continuous.

For simulations, Eqs. (3)–(5) should be complemented by the equation

\[ I_d(t) - I_0 = \frac{\Delta I}{2} (\rho_{11} - \rho_{22}) + \xi_0(t), \]  

where \( \xi_0(t) \) is the pure output noise of the ideal detector with flat spectral density \( S_0 \).

Equations (3)–(5) are written in the so-called Stratonovich form, which assumes symmetric definition of the derivative, \( \dot{\rho}(t) = \lim_{\tau \to 0} [\rho(t + \tau/2) - \rho(t - \tau/2)]/\tau \). For the forward definition of derivative, \( \dot{\rho}(t) = \lim_{\tau \to 0} [\rho(t + \tau) - \rho(t)]/\tau \) (Itô form), the Eqs. (3)–(5) transform into

\[
\dot{\rho}_{11} = -\dot{\rho}_{22} = -2 \frac{H}{\hbar} \text{Im} \rho_{12} + \rho_{11} \rho_{22} \frac{2\Delta I}{S_0} \xi_0(t),
\]

\[
\dot{\rho}_{12} = i \frac{\varepsilon}{\hbar} \rho_{12} + i \frac{H}{\hbar} (\rho_{11} - \rho_{22})
- (\rho_{11} - \rho_{22}) \frac{\Delta I}{S_0} \rho_{12} \xi_0(t) - \frac{(\Delta I)^2}{4S_0} \rho_{12},
\]

while the relation (4) remains unchanged. The general rule of transformation is the following: for an arbitrary system of equations \( \dot{x}_i(t) = G_i(x, t) + F_i(x, t) \xi(t) \) in the Stratonovich form, the corresponding equations in the Itô form are \( \dot{x}_i(t) = G_i(x, t) + F_i(x, t) \xi(t) + (S_0/4) \sum_k F_k(x, t) \partial F_i(x, t)/\partial x_k \) \. The advantage of the Itô form is that the averaging over the noise \( \xi_0(t) \) is straightforward (we just need to eliminate terms with \( \xi_0 \)), so it is easy to obtain the ensemble averaged evolution:

\[
\dot{\rho}_{11} = -\dot{\rho}_{22} = -2 \frac{H}{\hbar} \text{Im} \rho_{12},
\]

\[
\dot{\rho}_{12} = i \frac{\varepsilon}{\hbar} \rho_{12} + i \frac{H}{\hbar} (\rho_{11} - \rho_{22}) - \frac{(\Delta I)^2}{4S_0} \rho_{12}.
\]

On the other hand, the advantage of the Stratonovich form is the validity of usual calculus rules (which do not work in the Itô form) and therefore easier physical interpretation of equations.

Let us emphasize that the single qubit in this model (which assumes ideal detector) does not decohere during the measurement process, as easier to see from Eqs. (3)–(5). However, because of the probabilistic nature of quantum measurement, the ensemble of qubits does decohere (different qubits will go along different “trajectories”). The ensemble decoherence rate \( (\Delta I)^2/4S_0 \) is determined by this quantum randomness and therefore can be naturally called “quantum-limited” decoherence rate. (It can also be called “information-limited” ensemble decoherence, since its origin is the tendency of qubit state to evolve either into state \( |1 \rangle \) or \( |2 \rangle \), corresponding to the information acquired from the measurement.)

While it is not trivial to take into account additional classical noises \( \xi_1 \) and \( \xi_2 \) (this will be done in the following Sections), the account of the noise \( \xi_3 \) is very simple. It leads to the additional dephasing term \( -\gamma_3 \rho_{12} \) in Eqs. (8), (9), and (10), where \( \gamma_3 = S_3/4\hbar^2 \) is proportional to the spectral density \( S_3 \) of \( \xi_3(t) \). We will characterize this noise by the dephasing rate \( \gamma_3 \) instead of characterizing it by \( S_3 \). (The relation \( \gamma_3 = S_3/4\hbar^2 \) can be easily derived adding \( \xi_3(t) \) to \( \varepsilon \) in the Stratonovich form, then translating the equation into Itô form, and averaging over \( \xi_3 \).)

The natural definition of the detector ideality factor \( \eta \) in this case is

\[
\eta \equiv \frac{\Gamma_0}{\Gamma_S},
\]

where \( \Gamma_0 \equiv (\Delta I)^2/4S_0 \) is the quantum-limited contribution and \( \Gamma_S = \Gamma_0 + \gamma_3 \) is the total ensemble dephasing rate. Simply speaking, this definition of ideality is the ratio between quantum contribution and total backaction noise.

III. IDEAL SYMMETRIC DETECTOR AND ADDITIONAL OUTPUT NOISE

Let us now take into account additional output noise \( \xi_1(t) \), while \( \xi_2(t) \) is still zero. We also switch off \( \xi_3(t) \), since it is trivial to add its effect later. In order to derive Bayesian equations in this case, let us also assume \( H = \varepsilon = 0 \) (“frozen” qubit) and add the effects of \( H \) and \( \varepsilon \) later. For \( H = \varepsilon = 0 \), Eqs. (3)–(5) have a simple solution (11), which can be interpreted as a consequence of the “Quantum Bayes theorem” (12):

\[
\rho_{11}(\tau) = \left[ 1 + \frac{\rho_{22}(0)}{\rho_{11}(0)} \right]^{-1}
\]

\[
\rho_{22}(\tau) = 1 - \rho_{11}(\tau),
\]

\[
\rho_{12}(\tau) = \rho_{12}(0) \left[ \frac{\rho_{11}(0) \rho_{22}(0)}{\rho_{11}(0) \rho_{22}(0)} \right]^{1/2}
\]

\[
= \rho_{12}(0) \exp\left[ -\frac{(\Delta I)^2}{4S_0} \right] + \rho_{22}(0) \exp\left[ -\frac{(\Delta I)^2}{4S_0} \right],
\]

where \( T_\text{d} \) is the average of the detector current during the time interval between 0 and \( \tau \):

\[
T_\text{d} = \frac{1}{\tau} \int_0^\tau I_d(t) dt.
\]

Here Eq. (11) is the consequence of the classical Bayes theorem (12) and Eq. (13) says that the degree of the qubit purity is conserved. It is easy to include the effect of finite \( \varepsilon \), which just leads to an extra factor \( \exp(i\varepsilon \tau/\hbar) \) in Eqs. (12) and (13); however, we will not do that in order to keep the formulas shorter.

Since the detector output is now \( I(t) = I_d(t) + \xi_1(t) \), we have to express \( \rho_{ij}(\tau) \) in terms of \( I(t) \) and average it.
over the noise \(\xi_1(t)\). Naively thinking, we have to use the substitution

\[
\overline{T}_d = T - x, \quad T = \frac{1}{\tau} \int_0^T I(t)dt, \quad x = \frac{1}{\tau} \int_0^T \xi_1(t)dt, \quad (15)
\]

and average Eqs. (10) and (13) over the noise contribution \(x\) using the weight factor \(p(x) = (2\pi D_1)^{-1/2} \exp(-x^2/2D_1)\) where \(D_1 = S_1/2\tau\) is the variance of \(x\). However, this is not a correct procedure because the probability distribution of \(x\) is correlated with \(T\) (though it is not correlated with \(\overline{T}_d\)). So instead, we have to use the conditional distribution of \(x\) for a given \(T\):

\[
p(x) = P(x)/ \int P(x')dx', \quad (16)
\]

\[
P(x) = \frac{\exp(-x^2/2D_1)}{\sqrt{2\pi D_1}} \left[ \frac{\rho_{11}(0)}{\rho_{11}(0)} \exp(-T-x-I_1^2/2D_0) \sqrt{2\pi D_0} 
+ \frac{\rho_{22}(0)}{\rho_{11}(0)} \exp(-T-x-I_2^2/2D_0) \sqrt{2\pi D_0} \right], \quad (17)
\]

where \(D_0 = S_0/2\tau\). (Let us stress again that both \(\overline{T}_d\) and \(x\) are assumed to be classical quantities, so the derivation is not applicable to the case when the detector output \(I_d\) is involved in further quantum interactions.) Introducing the weight factor \(p(x)\) into Eq. (10), substituting \(\overline{T}_d = T - x\), and integrating over \(x\), we get the averaged equation

\[
\rho_{11}(\tau) = \left[ 1 + \frac{\rho_{22}(0)}{\rho_{11}(0)} \exp(-T-I_2^2/2S_2) \right]^{-1}, \quad (18)
\]

where \(S_2 \equiv S_0 + S_1\) is the total output noise. The only difference compared with Eq. (10) is the change of \(\overline{T}_d\) into \(T\) and change of \(S_0\) into \(S_2\) (this is quite expected since \(\rho_{11}\) behaves as a classical probability and the classical Bayes formula still works).

To calculate \(\rho_{12}(\tau)\) averaged over the noise \(\xi_1(t)\), we have to do a similar procedure. We multiply Eq. (13) by the weight factor \(p(x)\), use substitution \(\overline{T}_d = T - x\), and integrate over \(x\). In this way we obtain

\[
\rho_{12}(\tau) = \frac{\rho_{12}(0) \exp[-(A/T)^2/4S_0] \exp[-(T-I_2)^2/2S_2]}{\rho_{11}(0) \exp[-(T-I_2)^2/2S_2] + \rho_{22}(0) \exp[-(T-I_2)^2/2S_2]}, \quad (19)
\]

Comparing Eqs. (13) and (19), we see that \(\overline{T}_d\) changes into \(T\) and \(S_0\) changes into \(S_2\), except in the second factor of the numerator, where \(S_0\) remains unchanged. Let us represent this factor as \(\exp[-(A/T)^2/4S_0]\exp(-\gamma_1\tau)\), where

\[
\gamma_1 = \frac{(A/T)^2 S_1}{4S_0 S_2}. \quad (20)
\]

So, the effect of additional output noise \(\xi_1\) on the Eqs. (11)–(13) is the following: the output current \(I_d\) from the ideal part of the detector changes into the output current \(I\), the spectral density \(S_0\) corresponding to the ideal part of the detector changes into the total output noise \(S_2\), and the nondiagonal matrix element acquires the dephasing factor \(\exp(-\gamma_1\tau)\). Differentiating the new equations over time (if we do it in a simple first-order way, we automatically get equations in the Stratonovich form) and adding terms due to \(H\), \(\epsilon\), and noise \(\xi_3\), we obtain

\[
\dot{\rho}_{11} = -\dot{\rho}_{22} = -\frac{2}{\hbar} \text{Im} \rho_{12} + \rho_{11}\rho_{22} \frac{2\Delta I}{S_2} [I(t) - I_0], \quad (21)
\]

\[
\dot{\rho}_{12} = i \frac{\epsilon}{\hbar} \rho_{12} + \frac{1}{\hbar} (\rho_{11} - \rho_{22}) \frac{\Delta I}{S_2} [I(t) - I_0] \rho_{12} - (\gamma_1 + \gamma_3) \rho_{12}. \quad (22)
\]

We see that the effect of the extra output noise \(\xi_1(t)\) is similar to the effect of the extra backaction noise \(\xi_3(t)\) and both lead to the qubit dephasing.

From physical reasoning, the way of separation of the detector into the ideal part and additional noise sources \(\xi_1(t)\) and \(\xi_3(t)\) is arbitrary, as long as the total output noise \(S_2\) and total ensemble qubit dephasing rate \(\Gamma_2\) are fixed (in other words, \(S_2\) and \(\Gamma_2\) are the only physically relevant quantities). It is easy to check that Eqs. (21)–(22) satisfy this requirement because

\[
\gamma_1 + \gamma_3 = \Gamma_2 - \frac{(\Delta I)^2}{4S_2}; \quad \Gamma_2 = \frac{(\Delta I)^2}{4S_0} + \gamma_3. \quad (23)
\]

[The total ensemble dephasing rate \(\Gamma_2\) can be formally found from Eqs. (21)–(22) by translating them into II form that adds ensemble dephasing rate \((\Delta I)^2/4S_2).\]

Comparing Eqs. (21)–(22) with Eqs. (2)–(3), we naturally introduce a more general definition of the detector ideal part:

\[
\eta = \frac{(\Delta I)^2/4S_2}{\Gamma_2}, \quad (24)
\]

which is again the ratio of the quantum-limited part of the ensemble qubit dephasing and its total dephasing rate. (Notice that the numerator is not the “real” quantum backaction determined by \(S_0\), but the information-limited backaction determined by \(S_2\).) In particular, for our model in the case \(\xi_3(t) = 0\) (no classical backaction) we obtain \(\eta = S_0/(S_0 + S_1)\).

### IV. Correlated Output and Backaction Noises

Now let us add the classical backaction noise \(\xi_2(t)\) which affects the qubit energy \(\epsilon\) so that \(\epsilon \rightarrow \epsilon + \xi_2(t)\), and which is 100% correlated with the output noise source, \(\xi_2(t) = A\xi_1(t)\). We again start with Eqs. (11)–(13) for the “frozen” qubit. Addition of \(\xi_2(t)\) does not
affect diagonal matrix elements, so Eq. (18) remains unchanged. In order to calculate $\rho_{12}(\tau)$ averaged over the noises $\xi_1$ and $\xi_2$, we multiply Eq. (19) by the factor $\exp[ih^{-1} \int_0^\tau \xi_2(t)dt] = \exp(iA\xi\tau/h)$ and average it over $x$ with the weight $p(x)$ given by Eqs. (40)–(47). This leads to the equation

$$\rho_{12}(\tau) = \frac{\rho_{12}(0)\exp[-(\Delta I)^2/4S_\Sigma]}{\rho_{11}(0)\exp[-(\Delta I)^2/4S_\Sigma]} \rho_{22}(0)\exp(-i(\xi_2)^2/4S_\Sigma) \times \exp[ih(\bar{I} - I_0)A S_1 S_2/4h^2],$$

(25)
in which only the second line is different from Eq. (19).

Differentiating this equation over time (again, we automatically obtain the Stratonovich form) and adding terms due to $H$, $\varepsilon$ and $\xi_3$, we get

$$\dot{\rho}_{12} = i\varepsilon \rho_{12} + \frac{H}{h}(\rho_{11} - \rho_{22}) + \frac{i}{h}[I(t) - I_0]A S_1 S_2 / 4h^2 \rho_{12} - (\rho_{11} - \rho_{22}) \Delta I / S_\Sigma [I(t) - I_0] \rho_{12} - (\gamma_1 + \gamma_2 + \gamma_3) \rho_{12},$$

(26)

where $\gamma_1$ is given by Eq. (20) and $\gamma_2$ is

$$\gamma_2 = \frac{S_0 S_1 A^2}{4S_\Sigma h^2}.$$

(27)

Physically relevant parameters of the detector are the total output noise $S_\Sigma = S_0 + S_1$, total ensemble dephasing rate $\Gamma_\Sigma$, and the correlation between output and backaction noises. Since three sources of the backaction noise in Fig. 3 are uncorrelated, the ensemble dephasing rate is

$$\Gamma_\Sigma = (\Delta I)^2 / 4S_0 + A^2 S_1 / 4h^2 + \gamma_3$$

(28)

(the same result can be obtained by translating Eq. (20) into Itô form and performing ensemble averaging). Following Refs. (23)–(26), let us characterize the noise correlation by the magnitude (real number)

$$K = \frac{S_{\bar{I}}}{hS_\Sigma},$$

(29)

where $S_{\bar{I}}$ is the mutual spectral density of the detector output noise and induced fluctuations of $\varepsilon$ (strictly speaking, $S_{\bar{I}}$ in our notation is only the real part of the mutual spectral density, while the imaginary part can formally describe the detector response). In our case $K = AS_1/hS_\Sigma$. Expressing qubit evolution only in terms of physically relevant quantities, we obtain

$$\dot{\rho}_{11} = -\dot{\rho}_{22} = -2 \frac{H}{h} \text{Im} \rho_{12} + \rho_{11}\rho_{22} \frac{2\Delta I}{S_\Sigma} [I(t) - I_0],$$

(30)

$$\dot{\rho}_{12} = i\varepsilon \rho_{12} + \frac{H}{h}(\rho_{11} - \rho_{22}) + iK[I(t) - I_0] \rho_{12} - (\rho_{11} - \rho_{22}) \frac{\Delta I}{S_\Sigma} [I(t) - I_0] \rho_{12} = \hat{\gamma} \rho_{12},$$

(31)

where

$$\hat{\gamma} = \Gamma_\Sigma - (\Delta I)^2 / 4S_\Sigma - K^2 S_\Sigma / 4.$$ 

(32)

Eqs. (30)–(32) is the main result of this Section. Comparing them with similar equations presented (but not derived) in Refs. (23)–(26), we notice a difference: the term $iK[I(t) - I_0] \rho_{12}$ was erroneously replaced in Refs. (23)–(26) by the term $iK[I(t) - (\rho_{11} + \rho_{22})] \rho_{12}$. Notice though that the effect of their difference $iK(\Delta I/2)(\rho_{11} - \rho_{22}) \rho_{12}$ is minor since $\rho_{11} - \rho_{22}$ is the oscillating magnitude and averages to zero. As will be mentioned later, Eqs. (30)–(32) in the case $\hat{\gamma} = 0$ (this is possible for asymmetric ideal detector) coincide with the corresponding equations of Ref. (23) if $\varepsilon$ includes the detector-induced shift.

To translate Eqs. (30)–(32) from Stratonovich to Itô form, notice that

$$I(t) - I_0 = \frac{\Delta I}{2} (\rho_{11} - \rho_{22}) + \xi_0(t) + \xi_1(t)$$

(33)

and the sum of two output white noises $\xi_{0+1}(t) \equiv \xi_0(t) + \xi_1(t)$ is the white noise with the spectral density $S_\Sigma$. Then using the standard rule of translation, we obtain Itô equations

$$\dot{\rho}_{11} = -\dot{\rho}_{22} = -2 \frac{H}{h} \text{Im} \rho_{12} + \rho_{11}\rho_{22} \frac{2\Delta I}{S_\Sigma} \xi_{0+1}(t),$$

(34)

$$\dot{\rho}_{12} = i\varepsilon \rho_{12} + \frac{H}{h}(\rho_{11} - \rho_{22}) + iK\xi_{0+1}(t) \rho_{12} - (\rho_{11} - \rho_{22}) \frac{\Delta I}{S_\Sigma} \xi_{0+1}(t) \rho_{12} - \left(\hat{\gamma} + \frac{(\Delta I)^2}{4S_\Sigma} + \frac{K^2 S_\Sigma}{4}\right) \rho_{12},$$

(35)

while the relation between output current $I(t)$ and pure noise $\xi_{0+1}(t)$ is still given by Eq. (33). (Notice that the above mentioned incorrect term is correct in the Itô form of the equation, so the mistake was due to mixing of the Stratonovich and Itô forms.) The corresponding ensemble-averaged equations can be obtained by erasing terms containing $\xi_{0+1}(t)$ in Eqs. (24)–(26): 

$$\dot{\rho}_{11} = -\dot{\rho}_{22} = -2 \frac{H}{h} \text{Im} \rho_{12},$$

(36)

$$\dot{\rho}_{12} = i\varepsilon \rho_{12} + \frac{H}{h}(\rho_{11} - \rho_{22}) - \Gamma_\Sigma \rho_{12},$$

(37)

where the total ensemble decoherence rate $\Gamma_\Sigma$ is given by Eqs. (23) and/or (22).

Since $\hat{\gamma} > 0$ [otherwise solution of Eqs. (24)–(26) would violate inequality $|\rho_{12}|^2 \leq |\rho_{11}|^2 + |\rho_{22}|^2$], we obtain the fundamental limitation for the ensemble dephasing:

$$\Gamma_\Sigma \geq \frac{(\Delta I)^2}{4S_\Sigma} + \frac{K^2 S_\Sigma}{4}.$$ 

(38)

Besides the definition of the detector ideality $\eta$ given by Eq. (24) which would give $\eta = 1 - (\hat{\gamma} + K^2 S_\Sigma/4)/\Gamma_\Sigma$, it is natural to introduce another definition of the ideality [3].
\[ \tilde{\eta} = 1 - \frac{\tilde{\gamma}}{\Gamma_\Sigma} = \frac{(\Delta T)^2/4S_\Sigma + K^2S_\Sigma/4}{\Gamma_\Sigma}, \]  
(39)

since the term \( K^2S_\Sigma/4 \) does not correspond to the dephasing of a single qubit. One more possible definition of ideality (which also gives 100% if \( \tilde{\gamma} = 0 \)) is

\[ \tilde{\eta}_2 = \frac{1}{1 + \tilde{\gamma}/[(\Delta T)^2/4S_\Sigma]} = \frac{(\Delta T)^2/4S_\Sigma}{\Gamma_\Sigma - K^2S_\Sigma/4}, \]  
(40)

so that \((\tilde{\eta}_2)^{-1/2}\) directly corresponds to the total energy sensitivity of the detector in units of \( \hbar/2 \). In the case \( K = 0 \) all the definitions coincide, \( \eta = \tilde{\eta} = \tilde{\eta}_2 \).

V. ACCOUNT OF ASYMMETRIC IDEAL DETECTOR

So far we have assumed that the ideal part of the detector (in Fig. 1) does not induce the shift of the qubit energy asymmetry \( \varepsilon \) (i.e. in our terminology assumed symmetrically coupled detector). However, in general the coupling with detector changes \( \varepsilon \), so it should be treated self-consistently. As an example, the operating point of an SET as a detector is slightly shifted by different charge states of a measured qubit. This generally leads to the change of the average potential \( v \) of the SET island, which affects back the qubit energy asymmetry \( \varepsilon \). Notice that \( v \) can also be temporarily shifted by a fluctuation of the current through SET, leading to the correlation between the output and backaction noises. So, in this example the shift of \( \varepsilon \) and noise correlation are closely related. Similar situation occurs when as a detector we use a QPC, which location relative to the qubit is geometrically asymmetric. Then the qubit state affects the phase of the QPC current (of course, this should happen before the current becomes a classical quantity), and in return each electron passing through the QPC affects the phase difference between qubit states \( |1\rangle \) and \( |2\rangle \), thus leading to effective shift of \( \varepsilon \). Correspondingly, the noise of the QPC current causes the correlated noise of \( \varepsilon \), so again these effects are closely related.

The asymmetric coupling can be relatively easy taken into account for a small transparency QPC using the model analyzed in Ref. 23. The detector and its interaction with the qubit are described by Hamiltonians

\[ \mathcal{H}_{\text{det}} = \sum_i E_i a_i^\dagger a_i + \sum_r E_r a_r^\dagger a_r + \sum_{l,r} (T a_l^\dagger a_l + T^* a_l^\dagger a_r), \]

\[ \mathcal{H}_{\text{int}} = \sum_{l,r} (c_l^\dagger c_1 - c_2^\dagger c_2) \left( \frac{\Delta T}{2} a_l^\dagger a_l + \frac{\Delta T^*}{2} a_l^\dagger a_r \right), \]  
(41)

in which we neglect the dependence of tunnel matrix elements \( T \) and \( \Delta T \) on the electron states \((l, r)\) in electrodes. The only difference of this model from the model of Ref. 23 [which leads to Eqs. (3)–(8)] is the possibility of complex \( T \) and \( \Delta T \) (actually, \( T \) can be assumed real without loss of generality). Following the procedure developed in Ref. 23 it is possible to show that in the corresponding Bloch equation for \( \rho_{ij} \) (where \( \rho_{ii} \) is the density matrix with account of the number \( n \) of electrons passed through the detector) the term \( \sqrt{\Gamma_\Sigma} \rho_{ij} n^{-1} \) should be replaced by \( \exp(i\phi)\sqrt{\Gamma_\Sigma} \rho_{ij} n^{-1}, \) where \( \phi = \text{arg}[(T + \Delta T/2)(T^* - \Delta T^*/2)]. \) Therefore, each electron tunneling through the detector adds the phase \( \phi \) to \( \rho_{ij}, \) thus affecting the qubit energy asymmetry \( \varepsilon. \)

The assumption of weak detector response implies \( |\Delta T| \ll |T|, \) so that \( |\phi| \ll 1. \) The extra phase leads to the extra term \( i(\theta/\hbar)I_0 \rho_{12} \) in Eq. (8) where \( \theta \equiv \hbar \varepsilon/c. \) Separating it into the average and fluctuating parts, we obtain the following equations for the asymmetric ideal detector in the Stratonovich form:

\[ \dot{\rho}_{11} = -\frac{\tilde{\varepsilon}}{\hbar} \rho_{11} + \frac{H}{\hbar} \text{Im} \rho_{12} + \frac{2\Delta I}{S_0} [I(t) - I_0], \]  
(42)

\[ \dot{\rho}_{12} = i \left( \frac{\tilde{\varepsilon}}{\hbar} \rho_{21} + \frac{H}{\hbar} \rho_{11} - \frac{2\Delta I}{S_0} \right) [I(t) - I_0] \rho_{12} - (\rho_{11} - \rho_{22}) \frac{2\Delta I}{S_0} [I(t) - I_0] \rho_{12}, \]  
(43)

where \( \tilde{\varepsilon} = \varepsilon + \Delta \varepsilon \) and \( \Delta \varepsilon = \theta I_0 = \hbar \varepsilon I_0/c. \) Equations (42)–(43) in Itô form have been obtained in Ref. 23 (notice a different definition of \( \theta. \)) Let us mention that even though Eqs. (12)–(13) have been derived for a particular model of the detector [Eq. (4)], it is expected that they are applicable to a significantly broader class of asymmetric ideal detectors (i.e. finite-transparency QPCs, quantum-limited dc SQUIDs, etc.). Notice that Eqs. (12)–(13) are formally similar to Eqs. (10)–(11) except \( \tilde{\gamma} = 0 \) (ideal detector) and \( \varepsilon \) is replaced by the self-consistent value \( \tilde{\varepsilon}. \)

The solution of Eqs. (12)–(13) in the simple case \( H = 0 \) is still given by Eqs. (10)–(11) for \( \rho_{11} \) and \( \rho_{22}, \) while Eqs. (12)–(13) for \( \rho_{12} \) should be replaced by

\[ \rho_{12}(\tau) = \rho_{12}(0) \left[ \frac{\rho_{11}(\tau)\rho_{22}(\tau)}{\rho_{11}(0)\rho_{22}(0)} \right]^{1/2} \exp[i(\quad \theta I_0 - \tilde{T} - T_0)\tau/\hbar] \]  
(44)

\[ = \rho_{12}(0) \exp[- \frac{(\quad \Delta T)\tau}{4S_0} - \frac{\tilde{T} - T_0}{8T_0}] \exp[i(\quad \theta I_0 - \tilde{T} - T_0)\tau/\hbar], \]  
(45)

Let us now add the classical noises \( \xi_1, \xi_2 \) and \( \xi_3 \) (see Fig. 1) to our model of asymmetric ideal detector. Using the procedure explained in two previous Sections, we multiply Eq. (42) by the factor \( \exp(i4\pi x/\hbar) \) where \( x = \tau^{-1} \int_0^\tau \xi_1(\tau) d\tau, \) average the resulting equation and Eq. (10) over the distribution \( p(x) \) given by Eqs. (14)–(15), differentiate the result over time, and add the terms due to \( H \) and \( \xi_3 \). In this way we obtain the following Stratonovich equations:

\[ \dot{\rho}_{11} = -\frac{\tilde{\varepsilon}}{\hbar} \rho_{11} - \frac{2\Delta I}{S_\Sigma} [I(t) - I_0], \]  
(46)

\[ \dot{\rho}_{12} = i \left( \frac{\tilde{\varepsilon}}{\hbar} \rho_{21} + \frac{H}{\hbar} \rho_{11} - \frac{2\Delta I}{S_\Sigma} \right) [I(t) - I_0] \rho_{12} - (\rho_{11} - \rho_{22}) \frac{2\Delta I}{S_\Sigma} [I(t) - I_0] \rho_{12}. \]  
(47)
\[ + \frac{i}{\hbar} \frac{AS_1 + \theta S_0}{S_\Sigma} [I(t) - I_0] \rho_{12} \]
\[- (\rho_{11} - \rho_{22}) \frac{\Delta I}{S_\Sigma} [I(t) - I_0] \rho_{12} - \gamma \rho_{12}, \]
\[ \gamma = \frac{(\Delta I)^2 S_1}{4S_0 S_\Sigma} + \frac{S_0 S_1 (A - \theta)^2}{S_\Sigma^2} \frac{\Delta I}{4\hbar^2} + \gamma_3. \]
\[ \Gamma_\Sigma = \frac{(\Delta I)^2}{4S_0 + \theta^2 S_0/4\hbar^2 + A^2 S_1/4\hbar^2 + \gamma_3}. \]

It is interesting to notice that the single-qubit decoherence rate \( \gamma \) can be decreased by adding the backaction noise \( \xi_2(t) = A\xi_1(t) \) with \( A = \theta \) (i.e., having the same correlation with the output noise as the quantum backaction), even though it increases the ensemble dephasing rate
\[ \Gamma_\Sigma = \frac{(\Delta I)^2}{4S_0 + \theta^2 S_0/4\hbar^2 + A^2 S_1/4\hbar^2 + \gamma_3}. \]

Introducing the total correlation between the output noise and the backaction (including quantum noise),
\[ K = \frac{AS_1 + \theta S_0}{hS_\Sigma}, \]
we reduce Eqs. (44)–(47) to Eqs. (24)–(27) with the only substitution of \( \varepsilon \) by the self-consistent value \( \bar{\varepsilon} \). Therefore, the corresponding Itô equations are still given by Eqs. (23)–(25) and the ensemble-averaged equations are still given by Eqs. (24)–(27) (with \( \varepsilon \to \bar{\varepsilon} \)), while the relation between the single-qubit decoherence rate \( \gamma \) and ensemble decoherence \( \Gamma_\Sigma \) still satisfies Eq. (22), we conclude that the Bayesian description of the measurement process given by Eqs. (24)–(27), which is expressed in terms of the physically relevant quantities \( K, S_\Sigma, \) and \( \Gamma_\Sigma \), remains valid in the case when the ideal part of the detector is assumed asymmetrically coupled to the qubit. Similarly, the limitation (48) for \( \Gamma_\Sigma \) remains valid and the definitions (49)–(50) of the detector ideality can still be used.

Let us emphasize that for the Bayesian description of the continuous measurement of a single qubit, the detector in our model is completely described by six quantities: dc output current (“operating point”) \( I_0 \), response \( \Delta I \), output noise \( S_\Sigma \), ensemble dephasing rate \( \Gamma_\Sigma \), correlation magnitude \( K \), and the induced qubit energy shift \( \Delta \varepsilon \) (the shift \( \Delta \varepsilon \) can also have a classical contribution due to shift of the detector operating point). The quantities \( S_\Sigma, \Gamma_\Sigma, \) and \( K \) are analogous to the output, input, and cross-correlation noise terms, which are usually used for the description of a classical amplifier (and similarly for the description of a quantum amplifier – see, e.g., Ref. [4] and references therein). The induced energy shift \( \Delta \varepsilon \) is somewhat similar to the effect of a finite amplifier input impedance onto the previous stage parameters.

VI. DETECTOR MEASURING ENTANGLED QUBITS

Finally, let us consider the case when the detector is coupled to \( N \) arbitrarily entangled and arbitrary inter-
acting qubits. Following Ref. [4], we introduce the measurement basis consisting of \( 2^N \) states \( |i\rangle \) and up to \( 2^N \) different levels \( I_i \) of the detector average current (some average currents can coincide). For an ideal symmetric detector and “frozen” qubits, \( H_{\text{obs}} = 0 \), where \( H_{\text{obs}} \) is the Hamiltonian of intrinsic evolution of the qubits, the evolution of the density matrix \( \rho \) of qubits due to measurement is described by simple “Quantum Bayes” equation [4,11]
\[ \rho_{ij}(\tau) = \rho_{ij}(0) \sqrt{P_i(\tau)P_j(\tau)} \sum_k \rho_{ikk}(0)P_k(\tau). \]

where \( P_i(\tau) \) is the probability of obtaining particular measurement result \( I_d \) in this case) assuming state \( |i\rangle \) of the measured system:
\[ P_i(\tau) = \frac{1}{\sqrt{2\pi D_0}} \exp\left[-\frac{(I_d - I_i)^2}{2D_0}\right], \quad D_0 = \frac{S_0}{2\tau} \]

(we again assume that the currents \( I_i \) do not differ much, and therefore the detector noise \( S_0 \) is state-independent).

To take into account additional output noise \( \xi_1(t) \) with spectral density \( S_1 \), we have to perform the averaging of the density matrix \( \rho_{ij} \) over \( x = I - I_d \) with the weight factor
\[ p(x) = P(x)/\int P(x')dx', \]
\[ P(x) = \frac{\exp(-x^2/2D_I)}{\sqrt{2\pi D_I}} \sum_k \rho_{kk}(0) \exp\left[-\frac{(I_d - I_k)^2}{2D_0}\right] \]

This procedure (without account of other noise sources) will lead to the equation presented in Ref. [4] and corresponds to the detector ideality \( \eta = S_0/(S_0 + S_1) \), similar to the one-qubit case.

For the classical backaction noise which is 100% correlated with \( \xi_1(t) \), we should take into account that it can be coupled differently to different qubits. Let us assume that the energy of each state \( |i\rangle \) is affected by the classical backaction noise, proportional to \( \xi_1 \), so that \( \varepsilon_i \to \varepsilon_i + A_i \xi_1(t) \), where \( A_i \) are arbitrary constants. Then Eq. (31) should be multiplied by the factor \( \exp[i(A_j - A_i)\tau/\hbar] \) (in the one-qubit case the previously defined \( A \) would correspond to \( A_2 - A_1 \)).

Similarly, to take into account the possible asymmetry of the quantum backaction noise, let us assume that each electron tunneling through the ideal part of the detector shifts the phases corresponding to states \( |i\rangle \) (differently for different states), that leads to the extra factor \( \exp[i(\theta_i - \theta_j)\tau/\hbar] \) in Eq. (31).

The uncorrelated classical noise \( \xi_2(t) \) is also assumed to be coupled differently to the states \( |i\rangle \), so that \( \varepsilon_i \to \varepsilon_i + g_i \xi_2(t) \), where \( g_i \) are some constants. The averaging over noise \( \xi_2 \) is simple and leads to the extra factor \( \exp[-(g_i - g_j)^2S_3\tau/4\hbar^2] \) in Eq. (31).

Taking into account the effect of \( \theta_i \), averaging over the noise \( \xi_1(t) \) (and fully correlated backaction noise) and
\( \xi(t) \), differentiating equations over time, and adding the intrinsic evolution of qubits, we finally obtain the following equation in the Stratonovich form:

\[
\dot{\rho}_{ij} = -\frac{i}{\hbar}[H_{qbs}, \rho_{ij}] + \frac{i}{\hbar} \Delta \xi_{ij} \rho_{ij} + iK_{ij}[I(t) - \frac{I_i + I_j}{2}] \rho_{ij} \\
+ \rho_{ij} \frac{1}{S_{\Sigma}} \sum_k \rho_{kk} \left[ |I(t) - \frac{I_i + I_k}{2}||I_j - I_k| \right] - \tilde{\gamma}_{ij} \rho_{ij},
\]

where the first term describes the intrinsic evolution of qubits due to \( H_{qbs} \), \( \Delta \xi_{ij} = (\theta_i - \theta_j)(I_i + I_j)/2 \) is the effective energy shift due to detector asymmetry, \( S_{\Sigma} = S_0 + S_1 \) is the total output noise,

\[
K_{ij} = \frac{(A_j - A_i)S_1 + (\theta_i - \theta_j)S_0}{\hbar S_{\Sigma}}
\]

is the correlation factor between output and backaction noises, and the dephasing rate is

\[
\tilde{\gamma}_{ij} = \frac{(I_i - I_j)^2 S_1}{4S_{\Sigma}^2} + \frac{(g_i - g_j)^2 S_1}{4\hbar^2} + \frac{S_0 S_1}{4\hbar^2 S_{\Sigma}} [(A_j - A_i) - (\theta_i - \theta_j))^2].
\]

Notice that there are obviously no dephasing terms for diagonal matrix elements. Also notice that Eq. (55) is applicable to both linear and nonlinear detectors, including purely quadratic detectors, as long as the condition of weak response is satisfied.

Translating Eq. (55) into Itô form, we obtain

\[
\dot{\rho}_{ij} = -\frac{i}{\hbar}[H_{qbs}, \rho_{ij}] + \frac{i}{\hbar} \Delta \xi_{ij} \rho_{ij} + iK_{ij}[I(t) - \sum_k \rho_{kk}I_k] \rho_{ij} \\
+ \rho_{ij} \frac{1}{S_{\Sigma}} [I(t) - \sum_k \rho_{kk}I_k] (I_i + I_j - 2\sum_k \rho_{kk}I_k) \\
- \tilde{\gamma}_{ij} \rho_{ij},
\]

where the ensemble dephasing \( \Gamma_{ij} \) is related to the single system dephasing \( \tilde{\gamma}_{ij} \) as

\[
\Gamma_{ij} = \frac{(I_i - I_j)^2}{4S_{\Sigma}} + \frac{K_{ij}^2}{4} + \tilde{\gamma}_{ij}
\]

and in our particular case is equal to

\[
\Gamma_{ij} = \frac{(I_i - I_j)^2}{4S_0} + \frac{(A_j - A_i)^2 S_1}{4\hbar^2} + \frac{(\theta_i - \theta_j)^2 S_0}{4\hbar^2} + \frac{(g_i - g_j)^2 S_3}{4\hbar^2}.
\]

Since the combination \( I(t) - \sum_k \rho_{kk}I_k \) in Eq. (58) is a pure noise because of the relation

\[
I(t) = \sum_k \rho_{kk}I_k + \xi_0(t) + \xi_1(t), \tag{61}
\]

the ensemble averaged evolution is described by the reduced equation

\[
\dot{\rho}_{ij} = -\frac{i}{\hbar}[H_{qbs}, \rho_{ij}] + \frac{i}{\hbar} \Delta \xi_{ij} \rho_{ij} - \Gamma_{ij} \rho_{ij}. \tag{62}
\]

Because of the reciprocity, it is natural to assume that the backaction couplings \( A_j - A_i, \theta_i - \theta_j, \) and \( g_i - g_j \) are proportional to the signal coupling \( I_i - I_j \), so that \( A_j - A_i = (I_i - I_j)a, \theta_i - \theta_j = (I_i - I_j)\Theta, \) and \( g_i - g_j = (I_i - I_j)g. \) (Actually, this assumption implies detector linearity and also that all interactions with qubits occur via one “port of entry”. It is not valid, for example, when several geometrical parts of the detector interact with qubits in different ways.) With this assumption the parameters \( \Delta \xi_{ij}, K_{ij}, \tilde{\gamma}_{ij} \), and \( \Gamma_{ij} \) used in evolution equations (55) and (58) become

\[
\Delta \xi_{ij} = \Theta(I_i^2 - I_j^2)/2, \tag{63}
\]

\[
K_{ij} = \frac{aS_1 + \Theta S_0}{\hbar S_{\Sigma}} (I_i - I_j), \tag{64}
\]

\[
\tilde{\gamma}_{ij} = (I_i - I_j)^2 \left[ \frac{S_1}{4S_{\Sigma}} + \frac{(a - \Theta)S_2S_1}{4\hbar^2 S_{\Sigma}} + \gamma_{3,n}, \right] \tag{65}
\]

\[
\Gamma_{ij} = (I_i - I_j)^2 \left( \frac{1}{4S_0} + \frac{a^2S_1}{4\hbar^2} + \frac{\Theta^2 S_0}{4\hbar^2} + \gamma_{3,n} \right), \tag{66}
\]

where \( \gamma_{3,n} = gS_3/4\hbar^2 \). Notice that there will be no dephasing between states \( |i\rangle \) and \( |j\rangle \) if the detector is equally coupled to these states, \( I_i = I_j \).

The detector ideality in this case can be characterized by a single number (or few numbers for different definitions), which does not depend on the state of the measured system. Extending the definitions (24), (39), and (40) discussed in previous Sections, the detector ideality can be characterized by the parameter combinations

\[
\eta = \frac{1}{4S_{\Sigma}} + \frac{K_{\Sigma}^2 S_\Sigma}{4}, \tag{67}
\]

\[
\tilde{\eta} = \frac{1}{4S_{\Sigma}} - \frac{K_{\Sigma}^2 S_\Sigma}{4},
\]

where \( K_{\Sigma} \equiv (aS_1 + \Theta S_0)/\hbar S_{\Sigma} \) and \( S_{\Sigma} = 1/4S_0 + a^2S_1/4\hbar^2 + \Theta^2 S_0/4\hbar^2 + \gamma_{3,n} \). In the case \( a = \Theta = 0 \) all definitions of ideality coincide and the evolution equation (55) reduces to the equation derived in Ref. (4). In the case of finite \( a \) and/or \( \Theta \), more natural definitions are \( \eta \) and \( \tilde{\eta} \) (again, \( \tilde{\eta}^{-1/2} \) is the total energy sensitivity in units of \( \hbar/2 \)). However, ideality \( \eta \) can also be a useful parameter, for example, if there is no way to control the degree of freedom affected by the backaction noise \( \Theta \xi_0 + a \xi_1 \), and therefore the corresponding dephasing cannot be reduced by a feedback procedure.

VII. CONCLUSION

In this paper we have analyzed the process of continuous measurement of a solid state qubit by a nonideal solid
state detector. We have considered the phenomenological model of the detector (Fig. 1) consisting of an ideal (quantum-limited) part and classical noise sources which contribute to the output ($\xi_1$) and backaction ($\xi_2 + \xi_3$) noises. The possible correlation between classical output and backaction noise sources is taken into account by separating the backaction noise into a contribution $\xi_2(t)$ fully correlated with output noise $\xi_1(t)$ and the uncorrelated contribution $\xi_3(t)$. For the description of the ideal part we have started with the Bayesian equations of Refs. 34 and 36 and then used the model of an asymmetrically coupled ideal detector developed in Ref. 37. The asymmetric coupling changes the self-consistent energy difference between two qubit states. Also, this change fluctuates in time and the fluctuations are correlated with the output noise, thus producing an effect similar to the correlation of classical noises.

The main result of the paper for the one-qubit case is the derivation of evolution equations (35) and (38) in Stratonovich and Itô form, respectively (the qubit correlation of classical noises should be treated self-consistently, as discussed in section V). In these equations the detector is characterized by the total output noise $S_{\Sigma}$, induced ensemble qubit decoherence rate $\Gamma_{\Sigma}$, and the total correlation $K$ [see Eq. (34)] between output and backaction noises, so that the detector separation into the quantum part and extra noises is irrelevant. (Notice that these three quantities are the counterparts of output, input, and cross-correlation noise terms used for the description of a classical amplifier; the induced qubit energy shift $\Delta \epsilon$ is somewhat analogous to a backaction due to finite input impedance.) The relation between ensemble and single qubit decoherence rates is given by Eq. (35), which leads to the fundamental limitation (36) for the ensemble decoherence rate. The discussed definitions of the detector ideality [see Eqs. (24), (27), and (29)] are various combinations of the single qubit decoherence rate, ensemble decoherence, and the “information acquisition” rate $(\Delta I)^2/4S_{\Sigma}$. A 100% ideal detector corresponds to the absence of single qubit decoherence.

The theory developed for a single qubit measurement is generalized to the case of entangled qubits in Section VI. The evolution equation is given by Eqs. (55) and (58), while the relation between ensemble and single system decoherence rates is given by Eq. (59).

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In Ref. 20 it is stated that the effect of finite $\theta$ is always small in the “quantum diffusion” (“weakly responding”) case, $|\Delta I| \ll I_0$. We think that actually its effect can be significant.

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