Perturbation Analysis of Deformed Q-Ball and Primordial Magnetic Field

Tetsuya Shiromizu

DAMTP, University of Cambridge
Silver Street, Cambridge CB3 9EW, UK

Department of Physics, The University of Tokyo, Tokyo 113-0033, Japan
and
Research Center for the Early Universe (RESCEU),
The University of Tokyo, Tokyo 113-0033, Japan

Tomoko Uesugi

Institute for Cosmic Ray Research, The University of Tokyo, Tokyo 188-8502, Japan

Mayumi Aoki

Graduate School of Humanities and Sciences, Ochanomizu University, Tokyo 112-8610, Japan

to be published in Phys. Rev. D

We study the excited states of the Q-balls by performing stationary perturbation on the spherical Q-balls. We find the exact solution of the stationary perturbation of the global Q-ball with thin wall approximation. For local Q-balls we solve the equations of motion for the perturbative part approximately by using expansion about the coupling constant. Furthermore we comment on the magnetic field generated by the excited Q-balls during the phase transition precipitated by soliton synthesis and give an implication into cosmology.

I. INTRODUCTION

The existence of the coherent magnetic field over various astrophysical scales has been established up to the present [1]. To explain the observed magnetic field for galaxies, clusters and inter-clusters, we need a generation mechanism of the seed of primordial magnetic fields [2]. This is because magnetohydrodynamics shows that the magnetic fields cannot exist unless there is non-zero field strength as a initial condition.

It is known that if the seed of magnetic fields with $10^{-19}$ Gauss exists over the present comoving scale of the proto galaxy, $\sim 100$ kpc, these fields might be amplified to the observed galactic magnetic fields, $10^{-6}$ Gauss, by the dynamo mechanism [3].

Recently, several generation mechanism of the magnetic field in the electroweak phase transition have been proposed. One of them is based on the thermal fluctuations [6]. However, this mechanism cannot provide enough coherent scale of the magnetic field, and the net magnetic field will be too small to explain the present observation. Another mechanism is founded on the bubble nucleation [7]. This mechanism is able to supply the large coherent length of

*JSPS Postdoctoral Fellowship for Research Abroad
†Research Fellow of the Japan Society for the Promotion of Science.
‡Kulsrud and Anderson pointed out that the kinetic dynamo theory breaks down in the interstellar mediums [8]. Whether the mechanism can work sufficiently or not is in the debate.
the magnetic field and it can provide the enough field strength for the onset condition of the dynamo mechanism. In this case, however, one needs to assume that the phase transition is strongly first order one and completed by the expansion of the critical bubbles. Although several attempts to clarify the detail of the phase transition have been performed, the order of the phase transition has not been clear at this stage.

In this paper, we considered the magnetogenesis due to excited states of Q-balls during the phase transition, suggested by one of the present authors. Q-ball is a non-topological soliton solution of the complex scalar field arising in the theory with an unbroken continuous symmetry. The generation of Q-ball and the possibility of the phase transition precipitated by solitonsynthesis have been actively investigated. These studies are based on the ground state of Q-ball. If we consider the excited states of Q-ball, however, it is conceivable that Q-ball has angular momentum. Then, it could also have the magnetic moment if it is the gauged Q-ball.

We re-analyze the excited state of the Q-ball by the stationary perturbation analysis, not by the surface wave investigated by Coleman. We should note that the stationary perturbation is not general. However, it is still worth to investigate them because one may expect that the thermal distribution is decided by stationary states. We found that the $\ell = 1$ mode which does not exist as in the case of surface wave. The result might have some important effects on cosmology, since the contribution of the $\ell = 1$ mode is larger than the model $\ell = 2$ in general. Furthermore, taking account of the above fact and assuming copious production of gauged Q-balls, we estimate the magnetic field of the excited Q-balls.

The rest of the present paper is organized as follows. In Sec. II, we analyze the stationary perturbation on the global Q-balls and local Q-balls of the ground states and estimate the conserved quantities such as the angular momentum and the magnetic field etc. In the subsections of the above two sections, we review the feature of the Q-balls in ground state, and we also give a sketch of the phase transition due to gauged Q-balls. Finally, we give an implication into cosmology in Sec. IV.

II. GLOBAL Q-BALLS

A. Review of Q-Balls in the Ground State

In this subsection we review briefly global Q-balls in the ground state. For simplicity, we consider only the complex scalar field with a global U(1) symmetry. According to the Coleman’s study, the Q-ball has stationary and spherical configuration,

$$\phi_0 = \varphi_0(r) e^{i\omega t}. \quad (2.1)$$

For the existence of the Q-ball solution, we consider here the potential which has two minima, that is to say, true vacuum and false vacuum. In the thin wall approximation, the spatial configuration of the scalar field, $\varphi_0(r)$, is characterized by the radius of Q-ball, $R$, the wall width, $\delta$, and the field values at the center of configuration, $\sigma_\pm$. Here we set the field value at the minima: $0, \sigma_+$, and the top of the potential barrier: $\sigma_-$. Then the total energy can be written by

$$E_0 = \frac{Q_0^2}{2 \int \varphi_0^2 d^3x} + \frac{1}{2} \int (\nabla \varphi_0)^2 d^3x + \int U(\varphi_0) d^3x$$

$$= \frac{3}{8\pi} \frac{Q_0^2}{R^3 \sigma_+^2} + 2\pi R^2 \delta \left( \frac{\sigma_+}{\delta} \right)^2 + 4\pi R^2 \delta U_- + \frac{4\pi}{3} U_+ R^3, \quad (2.2)$$

where $U_\pm := U(\sigma_\pm)$ and $Q_0$ is the total conserved charge: $Q_0 = \int d^3x \psi_0^2$.

The most favourable configuration is determined by the variational principle to minimize the total energy. From $\partial E_0 / \partial \delta |_{\delta = \delta_*} = 0$, the wall width has the expression

$$\delta_* = \frac{1}{\sqrt{2}} \frac{\sigma_+}{\sqrt{U_-}}. \quad (2.3)$$

Thus, the total energy becomes

$$E_{0*} = E_0(\delta_*) = \frac{3}{8\pi} \frac{Q_0^2}{R^3 \sigma_+^2} + 4\sqrt{2\pi} \sigma_+ \sqrt{U_-} R^2 + \frac{4\pi}{3} U_+ R^3. \quad (2.4)$$

If the potential has two degenerate minima, $U_+ = U(0) = 0$, the total energy becomes
\[ E_0 = \frac{3}{8\pi} \frac{Q_0^2}{R^3 \sigma^2} + 4\sqrt{2\pi \sigma + \sqrt{\pi R^2}}. \]  

(2.5)

From \( \partial E_0 / \partial R |_{R=R_*} = 0 \), one obtains the radius at which the energy takes minimum value,

\[ R_* = \left( \frac{9}{64\sqrt{2\pi^2}} \right)^{1/5} \frac{Q_0^{2/5}}{\sigma_+^{1/5} U_*^{1/10}}. \]  

(2.6)

Then, the total energy becomes

\[ E_* = E_0(R_*) = \frac{15}{16\pi} \left( \frac{64\sqrt{2\pi^2} \sigma_+}{9} \right)^{3/5} Q_0^{4/5} U_0^{3/10} - \sigma_+^{1/5}. \]  

(2.7)

For simplicity, we write \( R_0^* \) as \( R \) hereafter.

**B. Stationary Perturbation on Q-Balls**

Let us consider the stationary perturbation on the Q-balls. The perturbation was analyzed in Ref. [10] from the viewpoint of the stability. In this paper, we give explicit solutions of the stationary perturbation. Although we will not give any implication to cosmology in this calculation on the global Q-balls, the analysis give a educational examination for the extension into the local Q-ball in which we can give an implication into cosmology.

The Lagrangian of the complex scalar fields is

\[ L = \int d^3x \left[ \frac{1}{2} |\phi|^2 - \frac{1}{2} |\nabla \phi|^2 - U(|\phi|) \right], \]  

(2.8)

where \( U(|\phi|) \) is the potential. The energy momentum tensor becomes

\[ T_{\mu\nu} = \frac{1}{2} \nabla_\mu \phi^* \nabla_\nu \phi + \frac{1}{2} \nabla_\mu \phi \nabla_\nu \phi^* - g_{\mu\nu} \left[ \frac{1}{2} \nabla^\rho \phi^* \nabla_\rho \phi + U(|\phi|) \right] \]  

(2.9)

We add the stationary perturbation on the Q-ball in the ground state as follows,

\[ \phi = \phi_0 + \phi_1 = \left( \varphi_0(r) + \sqrt{4\pi \varphi_{\ell m}(r)} Y_{\ell m}(\theta, \varphi) \right) e^{i\omega t}, \]  

(2.10)

where \( \ell \geq 1 \) and \( \ell \geq |m| \). Then the Lagrangian becomes

\[ L = L_0 + L_1 \]

\[ = \int d^3x \left[ \frac{1}{2} \omega^2 \varphi_0^2 - \frac{1}{2} |\nabla \varphi_0|^2 - U(\varphi_0) \right] + \frac{1}{2} \int d^3x \left[ -\left( \frac{d\varphi_{\ell m}}{dr} \right)^2 - V_\ell(r)(\varphi_{\ell m})^2 \right], \]  

(2.11)

where

\[ V_\ell(r) = U''(\varphi_0) - \omega^2 + \frac{\ell(\ell + 1)}{r^2}. \]  

(2.12)

One can see easily from the eq. (2.11) that the perturbation \( \varphi_{\ell m} \) satisfies

\[ -\frac{d^2\varphi_{\ell m}}{dr^2} - \frac{2}{r} \frac{d\varphi_{\ell m}}{dr} + V_\ell(r)\varphi_{\ell m} = 0. \]  

(2.13)

The above equation is similar to the Schrödinger equation with the potential \( V_\ell(r) \) in the ground state. In the thin wall approximation, one can set

\[ U''(\psi_0(r)) = \mu_0^2 \theta(r - R) + \mu^2 \theta(R - r), \]  

(2.14)

where \( \mu_0^2 = U''(0) \) and \( \mu^2 = U''(\sigma_+) \), and then the eq. (2.13) can be solved exactly. For \( r < R \), the solution is
\[ \varphi_{\ell m}(r) = \varphi_\ell^m(r) = A_{\ell m} \partial_{xj}(x) \]
\[ = A_{\ell m} \left( \frac{1}{r} \frac{d}{dr} \right)^{\ell} \left( \frac{\sin(\lambda r)}{r} \right), \]  
(2.15)

where \( x = \lambda r \) and \( \lambda = \sqrt{\omega^2 - \mu^2} \). For \( r > R \),
\[ \varphi_{\ell m}(r) = \varphi_\ell^m(r) = B_{\ell m} h_\ell^{(1)}(y) \]
\[ = B_{\ell m} \frac{d}{d(r^2)} \left( \frac{e^{-\gamma r}}{r} \right), \]  
(2.16)

where \( y = i \gamma r \) and \( \gamma = \sqrt{\mu_0^2 - \omega^2} \). Here remember that the condition \( \mu_0 > \omega > \mu \) is necessary for the existence of the stable Q-ball \[9\] \[10\].

First, we consider the \( \ell = 1 \) case. In this case the solution becomes
\[ \varphi_1^1(r) = \varphi_{1m}(r) = \frac{A_1}{r^2} \left( -\sin(\lambda r) + \lambda \cos(\lambda r) \right) \]  
(2.17)

and
\[ \varphi_{1m}(r) = \varphi_1^m(r) = \frac{B_1}{2r^2} \left( -\gamma r - 1 \right) e^{-\gamma r}. \]  
(2.18)

The relation between the prefactors, \( A_1 \) and \( B_1 \), is determined by the matching conditions of the field on the wall. As the potential \( V_\ell \) is continuous, one should require the \( \varphi_1^1(R) = \varphi_1^1(R) \) and \( \varphi_1^m(R) = \varphi_1^m(R) \). Each requirements imply
\[ (1 + \gamma R) e^{-\gamma R} B_1 = 2 \left[ \sin(\lambda R) - \lambda R \cos(\lambda R) \right] A_1, \]  
(2.19)

and
\[ A_1 \left( -2 \lambda R \cos(\lambda R) + 2 \sin(\lambda R) - (\lambda R)^2 \sin(\lambda R) \right) = B_1 \left( \frac{1}{2} (\gamma R)^2 + \gamma R + 1 \right) e^{-\gamma R}. \]  
(2.20)

Thus, one finds
\[ \frac{-2 \alpha \cos \alpha + (2 - \alpha^2) \sin \alpha}{\sin \alpha - \alpha \cos \alpha} = \frac{\beta^2 + 2 \beta + 2}{1 + \beta}, \]  
(2.21)

where \( \alpha = \lambda R \) and \( \beta = \gamma R \). For \( R_\mu = 1 \) and \( R_\mu = 4 \), the eigenvalue \( \omega_1 \) is \( \omega_1 \approx 3.6 R^{-1} \). For \( R_\mu = 1 \) and \( R_\mu = 5 \), the eigenvalue \( \omega_1 \) is \( \omega_1 \approx 3.8 R^{-1} \). The perturbation is permitted only in the case of the discretized frequency. Note that \( \ell = 1 \) modes of the surface wave do not exist in the Coleman’s study. As \( \ell = 1 \) modes can be easily created than \( \ell = 2 \) modes in general, the existence of the \( \ell = 1 \) modes is important in the context of cosmology.

Next, we give some conserved quantities which specify the feature of the perturbation. The charge of the perturbation is
\[ \delta Q_m = \int d^3 x j_0 = \omega_1 \int d^3 x \varphi_1^2 \]
\[ = A_1 \frac{\omega_1}{R} \left[ \left( \frac{1}{2} - \frac{1}{2} \alpha^2 + \frac{1}{2} \cos(2\alpha) + \frac{1}{4} \sin(2\alpha) + \left( \frac{\sin(\alpha) - \alpha \cos(\alpha)}{1 + \beta} \right) \right)^2 (1 + \frac{1}{2} \beta) \right]. \]  
(2.22)

The angular momentum \( J_{1m} \) is
\[ J_{1m} = \int d^3 x T_{0\varphi} = m \omega_1 \int d^3 x \varphi_1^2 = m \delta Q. \]  
(2.23)

Finally, we consider the \( \ell = 2 \) case. From the eqs. (2.15) and (2.16) the solutions are
\[ \varphi_2^m(r) = \varphi_{2m}(r) = \frac{A_2}{r^3} \left[ 3 \sin(\lambda r) - 3 \lambda \cos(\lambda r) - (\lambda r)^2 \sin(\lambda r) \right] \]  
(2.24)

and
\[ \varphi_2^\pm (r) = \varphi^{2m} (r) = \frac{B_2}{4r^3} (3 + 3\gamma r + \gamma^2 r^2) e^{-\gamma r}. \] (2.25)

The matching conditions, \( \varphi_2^+ (R) = \varphi_2^- (R) \) and \( \varphi_2^+ (R) = \varphi_2^- (R) \) at \( r = R \), give the relation

\[
\frac{(3 - \alpha^2) \sin \alpha - 3 \alpha \cos \alpha}{(9 + 4\alpha^2) \sin \alpha + \alpha (9 - \alpha^2) \cos \alpha} = -\frac{3 + 3\beta + \beta^2}{9 + 9\beta + 4\beta^2 + \beta^3}.
\] (2.26)

For \( \mu R = 1 \) and \( \mu_0 R = 4 \), the solution of \( \omega \) does not exist. This comes from the fact that the bottom of the potential \( V_\ell \) raises as \( \ell \) increase and then decrease the number of the bound state. For \( \mu R = 1 \) and \( \mu_0 R = 5 \), one has the solution \( \omega_2 \simeq 4.66R^{-1} \).

### III. EXCITED STATE OF LOCAL Q-BALLS

So far, we have analyzed the perturbation on the global Q-balls. In this section, we perform the perturbation analysis on the local Q-balls coupled to a U(1) gauge field. This analysis will be significant to cosmology because such object might be able to supply a seed of astrophysical magnetic field.

#### A. Local Q-balls in the Ground State and Phase Transition

The basic features of the local Q-ball was studied by Lee et al in Ref. [14]. We consider the theory with a complex scalar field \( \phi \) coupled to a U(1) gauge field \( A_\mu \). The Lagrangian density is

\[ \mathcal{L} = \frac{1}{2} [\partial_\mu - ieA_\mu] \phi |^2 - U(|\phi|) - \frac{1}{4} F_{\mu\nu} F^{\mu\nu}, \] (3.1)

where \( F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \). The conserved current and the energy momentum tensor which induce the conserved quantities are

\[ j_\mu = \frac{i}{2} [\phi (\partial_\mu + ieA_\mu) \phi^* - \phi^* (\partial_\mu - ieA_\mu) \phi] \] (3.2)

and

\[ T_{\mu\nu} = \frac{1}{2} \left[ (\partial_\mu - ieA_\mu) \phi (\partial_\nu + ieA_\nu) \phi^* + (\partial_\nu - ieA_\nu) \phi (\partial_\mu + ieA_\mu) \phi^* \right] - F_{\mu\sigma} F^{\sigma\nu} - \mathcal{L}_{\eta_{\mu\nu}}, \] (3.3)

respectively.

To find the Q-balls in the ground state, we seek for the solution such that

\[ \phi = \varphi(r) e^{i\omega t} \quad \text{and} \quad A_\mu = A_0(r) \delta_{\mu0} \] (3.4)

Using the above configurations, the Lagrangian becomes

\[ L = 4\pi \int dr r^2 \left[ -\frac{1}{2} \varphi'^2 + \frac{1}{2eg} g^2 + \frac{1}{2} g^2 \varphi^2 - U(\varphi) \right], \] (3.5)

where \( g = \omega - eA_0(r) \). The conserved charge associated with the U(1) symmetry becomes

\[ Q = \int d^3x j_0 = \int d^3x (\omega - eA_0) \varphi^2 = \int d^3x g \varphi^2. \] (3.6)

Assumed the thin wall approximation, as well as the case of global Q-balls, the configuration of the scalar field, \( \varphi \), is roughly expressed by the radius, \( R \), and the field value at the minimum of the potential, \( \sigma_+ \).\[8\]

\[ ^8\text{Rigourously speaking, the value of the scalar field } \varphi \text{ for } r < R \text{ should be determined by the variational principle with respect to the expression for the total energy } E \text{ [14]. One can approximate it as } \sigma_+, \text{ however, if only the leading-order term is considered. The radius } R \text{ is decided so that the energy of the non-trivial configuration takes minimum value.} \]
It is given by
\[ \varphi(r) = \sigma + \theta(R - r). \] (3.7)
In this case, we can solve the equation of motion for \( g \) [14]. The solution becomes
\[
\begin{align*}
g(r) &= \left( \frac{\omega - \frac{e^2 Q}{4\pi R}}{\sinh(e\sigma)} \right) \frac{R \sinh(e\sigma_r)}{r} \quad \text{for} \quad r \leq R, \\
g(r) &= \omega - \frac{e^2 Q}{4\pi r} \quad \text{for} \quad r > R.
\end{align*}
\] (3.8)
(3.9)

Inserted the above solution into the eq. (3.6), the relation between \( Q \) and \( \omega \) is determined by
\[ \omega = \frac{e^2 Q}{4\pi R} \left( 1 - \frac{1}{1 - \tanh x} \right), \] (3.10)
where \( x = e\sigma + R \). The total energy of the Q-ball becomes\[ E = \frac{e^2 Q^2}{8\pi R} + 4\sqrt{2\pi \sigma} \sqrt{U(R)} + \frac{4\pi}{3} U(R^3), \] (3.11)
If \( e\sigma + R > 1 \), as will be discussed later, the energy can be approximately written by
\[ E \approx \frac{e^2 Q^2}{8\pi R} + 4\sqrt{2\pi \sigma} \sqrt{U(R)} + \frac{4\pi}{3} U(R^3). \] (3.12)

Let us consider the phase transition. We note that our study is different from the previous one by Ellis et al [15]. In Ref. [15], they discussed the phase transition which occurs above the critical temperature, and took no account of the existence of the critical charge. We discuss the phase transition which takes place below the critical temperature and above the temperature at which the standard nucleation [16] starts.

As the universe expands, the temperature of the universe cools down. At the critical temperature \( T_c \), the potential has the two degenerate minima at \( \varphi = \sigma_+ \) and \( \varphi = 0 \); \( U_+ = U(0) = 0 \). In this case, the total energy becomes
\[ E = \frac{e^2 Q^2}{8\pi R} + 4\sqrt{2\pi \sigma} \sqrt{U(R)}^2. \] (3.13)
There exists the upper limit on charge and radius of Q-ball [14]. From \( \partial E/\partial R|_{R=R_*} = 0 \), the radius of Q-ball is given by
\[ R_* = \left( \frac{e^2 Q^2}{64\sqrt{2\pi} \sigma} \right)^{1/3}. \] (3.14)

Inserted this equation into the eq. (3.13), the minimum energy becomes
\[ E_* = E(R_*) = \frac{3}{2 \cdot 2^{5/6} \pi^{1/3}} (\sigma_+ \sqrt{U(R)})^{1/3}(eQ)^{1/3}. \] (3.15)

Here note that \( E_* \) is the monotonically increasing function of \( Q \). To construct stable Q-balls, however, the total energy \( E_* \) cannot exceed the energy for \( Q \)'s free particles with the mass \( \mu \), otherwise the Q-balls will decay into \( Q \)'s free particles. Thus, from the condition \( \mu = \partial E_*/\partial Q|_{Q=Q_{\text{max}}} \), one can obtain the upper bound on the total charge \( Q_{\text{max}} \) as
\[ ** Although the configuration of the scalar field is approximately expressed by the step function, we considered the surface term with the finite width here. Readers might feel a confusion. However, after all, if we assume \( e\sigma + R > 1 \), one see that the inconsistency of the approximation can vanish.
The upper bound on the radius $R_{\text{max}}$ of Q-ball is also given by
\[ R_{\text{max}} = \frac{1}{4\sqrt{2}e^2} \frac{\mu^2}{\sigma_+ \sqrt{U_-}}. \] (3.17)

We now consider the temperature $T_q (< T_c)$. We assumed that $T_q$ is higher than the temperature at which the bubble nucleation starts. In Ref. [13], it was pointed out that the Q-ball has the critical charge $Q_c$. If the charge exceeds this critical value, Q-ball can expand. If the charge is smaller than the critical one, there exists the range of the total energy in which the Q-ball is bounded. At $T = T_q$, if Q-ball is able to have the critical charge, it is possible that the phase transition is precipitated by the Q-ball growing up into the macroscopic size. For the Q-ball with critical charge, various quantities are decided by the variational principle so as to satisfy the conditions
\[ \frac{\partial E}{\partial R}\big|_{R=R_c,Q=Q_c} = \frac{\partial^2 E}{\partial R^2}\big|_{R=R_c,Q=Q_c} = 0. \] (3.18)

Resultant expressions for $Q_c, R_c,$ and $E_c$ are respectively obtained as
\[ Q_c = 6\sqrt{6}e \frac{\sigma_+^2 U_-}{e(U_+)^{3/2}}, \] (3.19)
\[ R_c = \frac{3}{\sqrt{2}} \frac{\sigma_+ \sqrt{U_-}}{|U_+|}, \] (3.20)
and
\[ E_c = 18\sqrt{2}e \frac{\sigma_+^3 U_-^{3/2}}{|U_+|^2}. \] (3.21)

As in the case of $T = T_c$, the energy must satisfy
\[ E_c/Q_c = \sqrt{3}e\sigma_+ \left(\frac{U_-}{U_+}\right)^{1/2} < \mu. \] (3.22)

According to Kusenko [13], the large charge can be attained soon and then the almost of Q-balls has the maximum charge $Q_{\text{max}}$. If $Q_{\text{max}} > Q_c$, Q-ball can expand as soon as the temperature becomes lower than the critical one because the growth rate of the charge is $\sim \eta n_r (4\pi R_c^2) \sqrt{T/m_c} \gg H \sim 10^{-15}\text{GeV}$ for TeV scale. Thus, the expansion of the Q-ball will start before the nucleation of the critical bubble. To compare $Q_c$ with $Q_{\text{max}}$, let us consider the ratio
\[ \frac{Q_c}{Q_{\text{max}}} = 12\sqrt{3}e \sigma_+ \left(\frac{U_-}{U_+}\right)^{3/2}, \] (3.23)
where we used the approximation, $U_-(T_q) \sim U_-(T_c)$. The condition $Q_c/Q_{\text{max}} < 1$ implies
\[ 2^{2/3}\sqrt{3}e\sigma_+ \left(\frac{U_-}{U_+}\right)^{1/2} < \mu. \] (3.24)

Comparing eq. (3.22) with eq. (3.24), one can see that the most strict constraint is the latter one (eq. (3.24)).

Here we must remember that $e\sigma_+ R > 1$ was assumed in the derivation of the eq. (3.12). The conditions $e\sigma_+ R_{\text{max}} > 1$ and $e\sigma_+ R_c > 1$, therefore, must be satisfied, so that the coupling constant $e$ is given by
\[ \frac{\sqrt{2}}{3} \frac{|U_+|}{\sigma^2 \sqrt{U_-}} < e < \frac{1}{4\sqrt{2}} \frac{\mu^2}{\sqrt{U_-}}. \] (3.25)

To see the details, we consider a potential
\[ U(\varphi) = \frac{1}{2} \mu^2 \varphi^2 - \frac{1}{6} M \varphi^3 + \frac{\lambda}{24} \varphi^4, \] (3.26)
where $\mu$, $M$, and $\lambda$ depend on the temperature of the universe. At the near critical temperature, one can take $y = \lambda^{1/2}(\mu/M) = y_c + \epsilon$, where $\epsilon$ is a small non-dimensional quantity, and $y_c = 1/\sqrt{3}$ is the value of $y$ at the critical temperature, decided by $U_+ = 0$. The leading order of these quantities become $U_+ \simeq \frac{\sigma_+^2 M^2 y_c}{\lambda} \epsilon$, $U_- \simeq \frac{\sigma_-^2 M^2}{\lambda}$, $\sigma_+ \simeq \frac{2\lambda}{M}$, $\sigma_- \simeq \frac{\lambda}{M}$, and $\mu \simeq y_c \frac{M}{\sqrt{\lambda}}$. Substituting the above equations into eq. (3.24), one can obtain the constraint on the coupling constant $\epsilon$:

$$e < \frac{2^{5/6}}{3^{3/4}} \sqrt{\lambda |\epsilon|}. \quad (3.27)$$

Furthermore, from the eq. (3.24), the range where the present approximation is valid becomes

$$\frac{4}{3} \lambda^{1/2} |\epsilon| < e < \frac{1}{2\sqrt{3}} \lambda^{1/2}. \quad (3.28)$$

We can see that the constraint (eq. (3.27)) can work for a sufficiently small value of $\epsilon$.

We investigated the phase transition caused by not global Q-ball but local Q-ball. To accomplish the phase transition caused by local Q-ball, we showed that the coupling constant $\epsilon$ must be too small. In other words, if the model satisfies

$$\frac{3^{3/2} \epsilon^2}{2^{5/3}} < \lambda |\epsilon|, \quad (3.29)$$

the charge of Q-ball can exceed the critical value as soon as the universe cools down under the critical temperature. In this situation, the phase transition will go as follows. By solitonsynthesis, Q-balls with maximum charge $\tilde{Q}_{\text{max}}$ is created above the critical temperature. If the coupling constant satisfies the above constraint (eq.(3.29)), the charge of Q-ball exceeds the critical value, $Q_c$. Then all the Q-balls must expand to macroscopic size at the temperature lower than the critical one. Such phenomena always happen before bubble nucleation starts. Thus, the new picture of the phase transition due to gauged Q-balls could become real one.

We note that the present scenario of the phase transition is slightly different from the Kusenko’s one. In Ref. [13], the accretion of charge which happens only below the critical temperature was considered. However, it is conceivable that such mechanism will take place above the critical temperature too, and we have taken account of it.

### B. Stationary Perturbation on Local Q-balls

Let us consider the stationary perturbation on the local Q-balls. The perturbed quantities are written by

$$\phi = (\varphi_0 + \sqrt{4\pi} \varphi_{\ell m} Y_{\ell m}) e^{i \omega t} \quad (3.30)$$

and

$$A_\mu = (0) A_\mu + (1) A_\mu \quad (3.31)$$

We assume the following ansatz on perturbation of the vector potential: $(1) A_0 = (1) A_r = 0$, and $(1) A_I = \sqrt{\frac{4\pi}{\ell I}} g_{\ell m}(r) \partial_I (Y_{\ell m} - Y_{\ell m}^*)$, where suffix $I$ runs over $\theta, \varphi$.

The Lagrangian for the perturbations becomes

$$\delta L = \int d^3x \left[ \frac{1}{2} \left( \varphi'_{\ell m} \right)^2 + \frac{1}{2} g_0^2 \varphi^2_{\ell m} - \frac{1}{2} U''(\varphi_0) \varphi^2_{\ell m} - \frac{\ell (\ell + 1)}{2 r^2} \varphi^2_{\ell m} \right]$$

$$+ \int d^3x \left[ - (g'_{\ell m})^2 - e^2 \varphi_0^2 g_{\ell m} + e g_{\ell m} \varphi_0 \varphi_{\ell m} \right] \frac{\ell (\ell + 1)}{r^2}. \quad (3.32)$$

Thus, the equations of the perturbation become

$$- \varphi''_{\ell m} + e^2 \varphi_0^2 \varphi_{\ell m} = \frac{1}{2} e \varphi_0 \varphi_{\ell m} \quad (3.33)$$

$\dagger$Note that all other types of the expression cannot couple with $\varphi_{\ell, m}$.
\[-\frac{d^2}{dr^2} - \frac{2}{r} \frac{d}{dr} - g_0^2 + U''(\varphi_0) + \frac{\ell(\ell + 1)}{r^2} \] \varphi_{\ell m} = \frac{e\ell(\ell + 1)}{r^2} \varphi_0 g_{\ell m}, \tag{3.34}

respectively.

The total charge, the angular momentum and the total energy are given by

\[ Q = \int d^3 x j_0 = \int d^3 x g_0 (\varphi_0^2 + \varphi_{\ell m}^2) = Q_0 + \delta Q_m, \tag{3.35} \]
\[ J_m = \int d^3 x T_{0\varphi} = m \int d^3 x g_0 \left[ \varphi_{\ell m}^2 - e g_0 \varphi_0 g_{\ell m} \varphi_{\ell m} \right], \tag{3.36} \]
\[ E = \frac{1}{2} \omega Q_0 + \int d^3 x \left[ \frac{1}{2} \varphi_0^2 + U(\varphi_0) \right] + \frac{1}{2} \omega \delta Q_m + \frac{1}{2} \int d^3 x g_0^2 \varphi_{\ell m}^2 \]
\[ =: E_0 + \delta E, \tag{3.37} \]

where \( \delta E = (1/2)\omega \delta Q_m + (1/2) \int d^3 x g_0^2 \varphi_{\ell m}^2 \).

As we stated in Introduction, the perturbed Q-ball has the magnetic field. The averaged magnetic field \( B^i = \epsilon^{ijk} \partial_j A_k \) over the angular direction is

\[ B^2(r) := \frac{1}{4\pi} \int d\Omega B^2 = 2\ell(\ell + 1) \frac{g_{\ell m}^2}{r^2}. \tag{3.38} \]

We consider only the \( \ell = 1 \) mode of excited Q-balls, since it is unrealistic that the Q-balls with much higher excitations actually affect. For the order estimation, we assume \( e^2 \sigma R \ll 1 \). In this case \( g_0 \sim \omega - e^2 Q_0 / 4\pi R \). Furthermore, we also suppose the small coupling constant, \( e \ll 1 \), and expand the part of perturbation in powers of \( e \),

\[ \varphi_1 = \varphi_{1 m} = (0) \varphi_1 + e(1) \varphi_1 \quad \text{and} \quad g_1 = g_{\ell m} = (0) g_1 + e(1) g_1. \tag{3.39} \]

The equations in the order \( O(e^0) \) become

\[ \left[ -\frac{d^2}{dr^2} - \frac{2}{r} \frac{d}{dr} + U''(\varphi_0) - \omega^2 + \frac{2}{r^2} \right] (0) \varphi_1 = 0 \quad \text{and} \quad \frac{d^2}{dr^2} (0) g_1 = 0. \tag{3.40} \]

The solutions of these equations become

\[ (0) \varphi_1^{-} = A_1 \frac{d}{dr} \frac{\sin(\lambda r)}{r} \quad \text{and} \quad (0) \varphi_1^{+} = B_1 r \frac{d}{dr^2} \frac{e^{-\gamma r}}{r} \tag{3.41} \]

and

\[ (0) g_1 = 0 \tag{3.42} \]

Assumed that \( \alpha \ll 1 \) and \( \beta \ll 1 \), the expression of \( \delta Q_1 \) becomes

\[ \delta Q_1 = \frac{2}{15} \pi A_1^2 \frac{\omega}{R} \alpha^6. \tag{3.43} \]

Up to the order of \( O(e) \), one can obtain the relation

\[ (1) g_1 \simeq \varphi_0 R^2 (0) \varphi_1. \tag{3.44} \]

The energy induced by the magnetic field becomes

\[ E_M \simeq 4\pi \int_0^R drr^2 B^2(r) \]
\[ \simeq \frac{16\pi}{3} e^2 \sigma R^3 (0) \varphi_1(R)^2 \]
\[ \simeq \frac{16\pi}{27} e^2 A_1^2 \sigma R^5 \lambda^6 \]
\[ \simeq \frac{40\pi}{9} e^2 Q_0 \frac{\sigma^2}{\omega}, \tag{3.45} \]

9
where we used $\delta Q \sim Q_0$. The mean magnetic field becomes

$$B \simeq \sqrt{\frac{2E_M}{3}} \frac{4\pi}{R^3} \simeq \epsilon Q_0^{1/2} \left( \frac{\sigma^2}{\omega R^2} \right)^{1/2},$$

(3.46)

where $\rho = \omega R$. For the potential (3.26), the order of the magnitude becomes

$$B \simeq 24(2\pi)^{1/2} e \mu^2 \rho^{-1/2}.$$  

(3.47)

**IV. AN IMPLICATION INTO COSMOLOGY – PRIMORDIAL MAGNETIC FIELD –**

In this paper, we investigated the perturbation on the ground state of Q-balls. Under the thin-wall approximation, we solved the equation for the part of perturbation analytically in the global Q-balls. For the local Q-balls, we solved them by expanding in powers of coupling constant $e$. Furthermore, we estimated the mean magnetic field for each Q-balls. In the context of cosmology, we need to average it. We assume that gauged Q-balls can be copiously produced. Since it is natural to suppose that the excited states of the Q-ball obeys the thermal distribution in such situation, the cosmological mean magnetic field generated by the excitation of the local Q-ball can be evaluated by multiplying the factor $e^{-(1/2)\delta E}$, where $\delta E$ is the energy of the excitation and is given by

$$\delta E \sim 2 \times \frac{1}{2} \omega Q_1 \sim \omega Q_0 \sim \frac{4\pi}{e^2} \mu.$$  

(4.1)

Thus, the mean value becomes

$$\langle B \rangle_{R_{\text{max}}} \sim 50 \frac{e}{\lambda \rho^{1/2} \mu^2} e^{-\xi}.$$  

(4.2)

where $\xi := (1/2)^{\delta E} = \frac{2\pi \rho \mu}{e^2}$. At the endpoint of the phase transition, the magnetic field becomes

$$\langle B \rangle_{R_f} \sim \left( \frac{R_{\text{max}}}{R_f} \right)^2 \langle B \rangle_{R_{\text{max}}} \simeq 0.3 \frac{g^* \lambda}{f_B^{1/2} \rho^{1/2} m_p^2} e^{-\xi} \sim 10 \times \left( \frac{g^*}{100} \right) \left( \frac{\lambda}{1.0} \right) \left( \frac{f_b}{10^{-3}} \right) e^{-2} \left( \frac{e}{0.1} \right) \left( \frac{\rho}{4.0} \right)^{-1/2} \left( \frac{T_f}{100 \text{GeV}} \right)^4 \exp \left[ -800 \pi \left( \frac{\rho}{4.0} \right) \left( \frac{e}{0.1} \right)^{-2} \mu \right] \mu \text{ Gauss.}$$  

(4.3)

where $R_f$ denotes the size of Q-ball at that time and $f_b = R_f H_f$. Since the Reynolds number becomes $\text{Re} \sim 10^{12}$ at the endpoint of the phase transition, turbulence for the electroweak plasma occurs over the scale of the radius of domain and then the energy of the magnetic field is equipartitioned with the energy of the fluid. Thus the final field strength becomes

$$B(R_f) \sim \rho^{1/2} v \sim g^*^{1/2} \frac{v}{T_f^2} \sim 10^{24} \left( \frac{T_f}{100 \text{GeV}} \right)^2 \text{ Gauss},$$  

(4.4)

where $\rho$ and $v$ are the density and the velocity of the fluid, respectively.

If the coherent scale is comoving, the number of magnetic domains inside the galaxy scale is

$$N \sim 10^{10}(10^{-3}/f_b)(T_f/100 \text{GeV}).$$

Thus, averaging over the galaxy scale, one can obtain the present mean value

$$\langle B \rangle_{\text{now}} \sim \frac{1}{N^{3/2}} \left( \frac{a_f}{a_{\text{now}}} \right)^2 B(R_f) \sim 10^{-21} \left( \frac{f_b}{10^{-3}} \right)^{3/2} \left( \frac{T_f}{100 \text{GeV}} \right)^{-3/2} \text{ Gauss.}$$  

(4.5)

††Although the amplification due to the turbulence is not clear at present, it has been used in the generation mechanism of the primordial magnetic field during the first order phase transition.
The above value satisfies the onset condition of the dynamo mechanism.

In the above derivation, we assumed for simplicity that the evolution of the magnetic field goes only by the red-shift effect. Although the definite scenario of the evolution has not been composed yet, some studies related with this problem have been done. Finally, we will refer to the studies which deal with the details of the evolution of the magnetic field. In Ref. [19], the damping effect due to photon diffusion around the recombination era were studied. In Ref. [20], Olesen et al showed that an inverse cascade happens if the power of the initial spectrum is larger than $-3$, and this means that the coherent scale of the magnetic field will be able to extend.

**Acknowledgment**

TS is grateful to Gary Gibbons and DAMTP relativity group for their hospitality. This work is supported by JSPS fellow.

[1] P.P. Kronberg, Pep. Prog. Phys. 57, 325(1994)
[2] For example, A. V. Olinto, ‘Cosmological Magnetic Field’ in the Proceeding of 3rd RESCEU Symposium ‘Particle Cosmology’, eds. K. Sato, T. Yanagida and T. Shiromizu, Univ. Acad. Press, p151(1998)
[3] For example, E. N. Parker, Cosmological Magnetic Field (Oxford Univ. Press, Oxford, 1979)
[4] R. M. Kulsrud and S. W. Anderson, Astrophys. J. 396 (1992) 606
[5] T. Shiromizu, Phys. Rev. D58 107301 (1998)
[6] T. Vachaspati, Phys. Lett. B265 (1991), 258.
[7] G. Baym, D. Bödeker and L. McLerran, Phys. Rev. D53 (1996), 662
[8] For example, T. Uesugi, M. Morikawa and T. Shiromizu, Prog. Theor. Phys. 96, 377 (1996);
K. Kajantie, M. Laine, K. Rummukainen and M. Shaposhnikov, Nucl.Phys. B493, 413(1997) and reference therein
[9] S. Coleman, Nucl. Phys. B262 (1985), 263
[10] T. D. Lee and Y. Pang, Phys. Rep. 221, 251(1992)
[11] K. Griest and E. W. Kolb, Phys. Rev. 40 (1989), 3231;
J. A. Frieman, A. V. Olinto, M. Gleiser and C. Alcock, Phys. Rev. D40, (1989), 3241;
K. Griest, E. W. Kolb and A. Massarotti, Phys. Rev. 40 (1989), 3529
[12] A. Kusenko, Phys. Lett. bf 405, 108(1997);
A. Kusenko and M. Shapiro, Phys. Rev. B418, 46(1998);
A. Kusenko, V. Kuzmin, M. Shaposhnikov and P. G. Tinyakov, Phys. Rev. Lett. 80, 3185(1998);
A. Kusenko, M. Shaposhnikov, P. G. Tinyakov and I. I. Tkachev, Phys. Lett. B423, 104(1998);
G. Dvali, A. Kusenko and M. Shaposhnikov, Phys. Lett. B417, 99(1998)
[13] A. Kusenko, Phys. Lett. B406, 26(1997)
[14] K. Lee, J. A. Stein-Schabes, R. Watkins and L. M. Widrow, Phys. Rev. D39, 1665(1989)
[15] J. Ellis, K. Enqvist, D. V. Nanopoulos and K. A. Olive, Phys. Lett. B225 (1989), 313
[16] A. Linde, Particle Physics and Inflationary Cosmology, Harwood Acad. Pub. (1990)
[17] G. Sigl, A. Olinto and K. Jedamik, Phys. Rev. D55 (1997), 4582
[18] K. Enqvist and P. Olesen, Phys. Lett. B319 (1993), 178
[19] K. Jedamik, V. Katalinic and A. V. Olinto, Phys. Rev. D57 (1998), 3264
[20] P. Olesen, Phys. Lett. B398 (1997), 321;
A. Brandenburg, K. Enqvist and P. Olesen, Phys. Lett. B391 (1997), 395;
T. Shiromizu, astro-ph/9810339, to be published in Phys. Lett. B