Simulation of the electrostatic fields in devices with complex geometric shapes

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Abstract. The problems of physical and mathematical simulation of electrostatic field were considered. The antetype of experimental assembly of field investigation was created. The calculation algorithm for numerical solution of the problem based on the mathematical model of the field and Green theorem was created. The algorithm was implemented as computer program. The results of the numerical and live experiments were obtained and compared.

1. Introduction
Simulation of the electromagnetic field using weakly conducting materials today is widely used in the educational process. Acquaintance of students of many universities with the concepts of charge and electric field begins with the construction of equipotential and force lines of the model field during the laboratory class. To reduce the costs and simplify the laboratory bench, water is often used as a weakly conducting operating environment, and the charges are simulated using electrodes plunge in it [1]. On the one hand, using such assembling, it is possible to simulate the electromagnetic field of a system of charges of arbitrary shape (and in the three-dimensional case), on the other hand, problems related to the accuracy and speed of experiments arise (for example, electrodes are oxidized during measurements, which leads to potential drop at the contact between the electrode and water).

In the student experimental laboratory of physics (SELPH) of the Physics Department, Bauman Moscow State Technical University a new cycle of laboratory work to demonstrate and study the phenomena of electrodynamics is developed. In particular, the issue of improving the laboratory bench for simulating an electrostatic field is developed. In [2], an analysis of weakly conducting solid materials was carried out, and it was found that the graphite coating is optimal from the point of view of a balance between such characteristics as the possibility of uniform coating of the matter, the wear resistance of the target coating, the ease of providing electrical contact and costs.

This paper is devoted to the analysis of electrostatic fields in devices with complex geometric shapes by mathematical and physical simulation. Section 2 presents a mathematical justification for the correctness of physical simulation; special attention was paid to the simulation the boundary conditions. Section 3 describes the design features of the laboratory bench. Section 4 describes numerical methods for solving the problem. Section 5 presents the results of numerical and live experiments.
2. Mathematical model

The complexity of electrostatic measurements led to the development of a special method for studying electrostatic fields by simulating it, i.e. artificial reproduction of the structure of fields in conducting environments through which a stationary (direct) current is passed.

The simulation method is based on the similarity of equipotential surfaces in a uniform conductor and in vacuum while maintaining the similarity of the shape of the electrodes and their potentials. This similarity is based on the fact that currents in conductors obey Ohm’s law and the continuity equation [1, 3, 4]. If the conducting matter is homogeneous and has low conductivity, the equations for the electric field strength in the model will coincide with the Maxwell equations for the electrostatic field in the absence of charges [4]

\[ \text{div} \mathbf{E} = 0, \]

\[ \text{rot} \mathbf{E} = 0, \]  

where \( \mathbf{E} \) — electric field vector.

Electrostatic field is potential, after the introduction of potential \( \phi \): \( \mathbf{E} = -\nabla \phi \) equation (2) becomes an identity, and equation (1) transforms to:

\[ \Delta \phi = 0. \]  

Consider in detail the problem of setting boundary conditions.

First, there are conductors in the study area. According to the Gauss theorem, the relation [5] should be satisfied at the boundary between the conductor and the dielectric:

\[ -\frac{\partial \phi}{\partial \mathbf{n}} = E_n = \gamma, \]

where \( \mathbf{n} \) — external to conductor normal, \( \gamma \) — surface charge density, the index \( n \) denotes the normal component of the vector \( (E_n = (E; \mathbf{n})) \). For the physical model the equation (4) is correct, and surface charge density \( \gamma_m = \frac{j}{\sigma} \), where \( j \) — current density, \( \sigma \) — environment conductivity.

Thus, using a physical model, the problem of finding the charge distribution over the conductor surface if the location of the conductors and their potentials are known can be solved.

Secondly, as a rule, in electrostatics, a system of charges in an unlimited space is considered. The physical model has finite dimensions. In physical modeling, it is necessary to artificially limit the area by the \( \Gamma \) boundary. A current does not flow through such a boundary, therefore the boundary condition for the model can be written as:

\[ \frac{\partial \phi_m}{\partial \mathbf{n}} = 0. \]  

As the size of the physical model increases (if the size of the conductors is maintained), the influence of the boundary on the field in the vicinity of the conductors will decrease. This effect can be estimated using the third Green formula for the Laplace operator [6]

\[ \phi(P) = \int_D \Phi^L P f \, dV + \int_{\partial D} \left( \Phi^L P \frac{\partial \phi}{\partial \mathbf{n}} - \phi \frac{\partial \Phi^L P}{\partial \mathbf{n}} \right) dS, \]  

where \( \Phi^L P \) — elementary solution of the Laplace operator, \( f \) — right side of the Poisson equation, \( D \) — considered area with boundary \( \partial D \). Due to the simplistic right-hand side of (3) and condition (5), equation (6) is converted to:

\[ \phi_m(P) = \sum_i \int_{S_i} \left( \Phi^L P \frac{\partial \phi_m}{\partial \mathbf{n}} - \phi_m \frac{\partial \Phi^L P}{\partial \mathbf{n}} \right) dS - \int_{S_m} \frac{\partial \Phi^L P}{\partial \mathbf{n}} dS, \]  

where \( S_i \) — the boundary of \( i \) conductor.
Thus, the influence of the $\Gamma$ boundary is fully described by the last component in expression (7). By calculating the correction value for each point of the studied area

$$w(P) = -\int_{\Gamma} \phi_m \frac{\partial \Phi_p}{\partial n} dS,$$

simulation in an unbounded region using a physical model can be carried out.

3. Laboratory bench

Creating a laboratory bench requires solving a number of engineering problems. Based on the analysis of solid conductive coatings [2], graphite varnish was chosen as a weakly conducting environment. It is applied on a textolite blade with electrodes got out of the desired shape. For ease of use of the blades in the SELPH, a body with an integrated voltmeter (Figure 1) has been developed.

For quick and accurate determination of coordinates, an electronic stylus is used as a feeler gage [7]. To automate the process of building equipotential and field lines, a driver for an electronic stylus and a program for processing results have been developed [8].

This laboratory bench can be used not only to plot a picture of the model field, but also to study the main theorems of electrostatics: Gauss theorem and circulation theorem. Reformulate the statements of the theorems in terms of measurable quantities as shown in [9]. In this case, write the main theorems of electrostatics in the integral form [4, 5]:

$$\oint_S \mathbf{E} \cdot d\mathbf{S} = \frac{q}{\varepsilon_0},$$

$$\int_I \mathbf{E} \cdot d\mathbf{r} = 0,$$

where $q$ — the charge inside the volume covered by the closed surface $S$, $\varepsilon_0$ is the electric constant.

Replace the vector analysis operators in (9) and (10) with their difference analogues, in result:

$$\sum_{i=1}^{N} \frac{U_i}{l} \Delta r_i \approx \frac{q/h}{\varepsilon_0},$$

$$\sum_{i=1}^{N} \frac{U_i}{l} \Delta r_i \approx 0,$$

where $U_i$ is the voltage differential on the segment $\Delta r_i$, $l$ is the distance between the feeler gages, $q/h$ is the specific density of the charge. In the case when the system of two conductors is analysed, the charge $q$ linearly depends on the voltage between the plates [4]:

$$q = CU,$$

where $U$ is the voltage differential between the conductors, $C$ is the electrical capacity.

The capacity of a system of two charges depends only on the geometrical parameters of the conductors and in some cases can be easily calculated analytically. Thus, the basic electrostatic theorems are written in terms of measurable quantities.

4. Numerical simulation is the electrostatic field

With the development of computer machines, live experiments began to be replaced by numerical ones. Construct an algorithm for the numerical simulation of an electrostatic field. Due to the complex
geometry of the studied configurations of charges, the finite element method (FEM) on a triangular grid will be used [10].

Before performing the calculations, the field simulation problems were divided into two classes: problems in a bounded region (for example, field modeling in a cylindrical capacitor) and problems in an unbounded region (modeling a plate capacitor of finite size). The solution of the first class of problems is not complex and is described in the classical literature, for example, in [10].

To solve the problem in an unbounded region, the two-step method described in [11] will be used. At the first stage, the problem in a certain finite region will be solved (Figure 2):

\[
\begin{align*}
\Delta \phi &= 0, \\
\phi_{L,S} &= \phi_1, \\
\partial \phi_{L,S} / \partial n_{L,S} &= 0,
\end{align*}
\]

where \( \partial S \) is the boundaries of conducting subregions. After that in certain region \( D' \subseteq D \) the corrections using the third Green formula will be carried out:

\[
\phi_{L'} = \phi + \int_{D'} \left( \phi(r) \frac{\partial \Phi_{L}(r)}{\partial n} \right) d\mathbf{r},
\]

where \( \Phi_{L}(r) = -\frac{1}{2\pi} \ln \left( \frac{1}{|r - r_s|} \right) \) is the fundamentation solution of the Laplace operator, \( r_s \) is the radius – vector of point \( P \).

5. Results

5.1. Simulation of the atmospheric electricity

The patterns of different physical phenomena are often described by the same differential equations and boundary conditions. This identical mathematical description makes it possible to replace a complex study of one phenomenon with a simple experiment in another field. In particular, using the developed laboratory bench, it is possible to simulate atmospheric electricity, to solve problems of hydrodynamics [3].

Consider in more detail the problem of simulation of the electric field in the vicinity of the launch complex of the spaceport (Figure 3).

Even on a usual day, over the desert plain or above the sea, the electric potential increases with each meter by about 100 volts, i.e. the air has a vertical electric field \( \mathbf{E} \) of 100 V / m. The sign of the field corresponds to the negative charge of the earth's surface. Objects on earth form an equipotential surface with it. Usually equipotential surfaces are parallel to the ground, but in the presence of high objects (for example, rockets on the launch pad) they are shifted [13].

To simulate the electric field in the vicinity of the spaceport, a special board was created and an experiment was performed. In figure 4 the result of the construction of equipotential and force lines using the developed equipment. Due to the fact that far from the lightning rods the rockets and the
equipotential lines are parallel to the ground, the condition (5) is satisfied at the side boundaries, i.e. the physical model is correct. In the result, picture of the field (Figure 4) shows that the lightning rods standing next to the rocket protect it from the lighting discharge from the thunderstorm front.

In electrical engineering, capacitors of various forms and types are widely used. Consider a plate capacitor of finite size (Figure 5). Physical and mathematical simulation of the electromagnetic field in the vicinity of the plates of such a capacitor will be carried out.

In figure 6, the result of physical simulation obtained using the developed assembly and special software is presented. In theory [13], the electric field strength at the edges of the capacitor plates is higher than inside it. This means that a charge with a higher surface density accumulates on the edges of a charged capacitor than on its inner side surfaces.

It is known that in the case when the dimensions of the capacitor plates are much larger than the distance between them to calculate the capacity, the formula [5] can be used:

\[ C = \frac{\varepsilon_0 S}{d}, \]  

where \( S = ah \) is lateral area of the capacitor plate, \( d \) is the distance between the plates.

\[ (14) \]
In this case (Figure 5), the relations between the geometric dimensions of the device are such that formula (14) is inapplicable. According to (11) and (13), using the physical model, the dimensionless volumetric efficiency of a capacitor can be calculated by the formula:

$$\tilde{C}_2 = \frac{1}{U} \sum_{i=1}^{N} \Delta r_i.$$  \hspace{1cm} (15)

Table 1. Calculation of the plate capacitor volumetric efficiency using the physical model

| Number of points of integration $N$ | Integration step $\Delta r_i$ | $\tilde{C}_2$ | $|\tilde{C}_2 - \tilde{C}_2^{N=64}|/\tilde{C}_2^{N=64}$ |
|------------------------------------|-------------------------------|--------------|------------------------------------------|
| 8                                 | 43.15                         | 2,343        | 0.070                                    |
| 16                                | 21.58                         | 2,389        | 0.023                                    |
| 32                                | 10.79                         | 2,419        | 0.007                                    |
| 64                                | 5.39                          | 2,412        | —                                        |

To calculate the capacity using the formula (15), a template, fixing sixty-four positions for a measuring tool on the border of a certain ellipse 345.2 mm long, was created. According to the measurement results (Table 1), it can be concluded that the method (15) has the second order of accuracy according to the integration step $\Delta r$, and $\tilde{C}_2 \approx 2.4$.

Unfortunately, the value $\tilde{C}_2$ obviously includes the error associated with the formulation of the boundary conditions (5) in the physical model. To calculate the capacity, provided that the capacitor is located in an unbounded region, a numerical simulation will be conducted. The results of the calculation of the electromagnetic field are presented in Figure 7 and figure 8. It is possible to notice a good similarity of the results presented in figure 6 and figure 7.

In order to verify the convergence of the numerical method, a range of computational experiments were carried out, the results are shown in Table 2 ($\tilde{C}_3$ is the dimensionless volumetric efficiency of the capacitor when setting the boundary conditions (5), $\tilde{C}_4$ is the dimensionless volumetric efficiency of the capacitor in case it is in unbounded space).

Table 2. Calculation of the volumetric efficiency of the plate capacitor using the computational experiment

| Step | $\tilde{C}_3$ | $|\tilde{C}_3 - \tilde{C}_3^{h/8}|/\tilde{C}_3^{h/8}$ | $\tilde{C}_4$ | $|\tilde{C}_4 - \tilde{C}_4^{h/8}|/\tilde{C}_4^{h/8}$ |
|------|--------------|---------------------------------|--------------|---------------------------------|
| $h$  | 2.489        | 0.041                           | 2.904        | 0.042                           |
| $h/2$ | 2.513        | 0.032                           | 2.931        | 0.033                           |
| $h/4$ | 2.568        | 0.010                           | 2.997        | 0.011                           |
| $h/8$ | 2.595        | —                               | 3.030        | —                               |
For clarity, the results obtained using the formula (14) ($\hat{C}_1$), physical simulation ($\hat{C}_2$) and mathematical simulation ($\hat{C}_3, \hat{C}_4$) are presented in Table 3. Note that the results of physical and computational experiments in the final domain differ by only 7%, which confirms the quality of the developed equipment. The discrepancy between the physical experiment and the calculation in an unbounded domain is 20%, which indicates the importance of properly taking into account the boundary conditions. The error of the formula (14) for the capacitor shown in figure 5 is 42%, which confirms the inapplicability of formula (14) for calculating the capacitance of a capacitor with a ratio $S/d \ll 1$.

Table 3. Comparison of the physical and computational experiments

|     | $\hat{C}_1$ | $\hat{C}_2$ | $\hat{C}_3$ | $\hat{C}_4$ |
|-----|-------------|-------------|-------------|-------------|
|     | 1.75        | 2.416       | 2.595       | 3.030       |

6. Conclusion
Simulation using a stationary electric field is widely used in the study of various physical fields, so the creation of reliable laboratory equipment and software with an intuitive interface is still an urgent task. An antetype of such a software and hardware complex was created in SELPH.

Using numerical methods, the statements of the main theorems of electrostatics are written through measured values, and control measurements are carried out. The results of solving practical problems, such as simulation of an electric field at the launch site of the spaceport, calculating the charge distribution on the plate of a plate capacitor are presented.

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