A Group-theoretic Approach to Fast Matrix Multiplication

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Abstract

We develop a new, group-theoretic approach to bounding the exponent of matrix multiplication. There are two components to this approach: (1) identifying groups $G$ that admit a certain type of embedding of matrix multiplication into the group algebra $\mathbb{C}[G]$, and (2) controlling the dimensions of the irreducible representations of such groups. We present machinery and examples to support (1), including a proof that certain families of groups of order $n^{2+o(1)}$ support $n \times n$ matrix multiplication, a necessary condition for the approach to yield exponent 2. Although we cannot yet completely achieve both (1) and (2), we hope that it may be possible, and we suggest potential routes to that result using the constructions in this paper.

1. Introduction

Strassen [14] made the startling discovery that one can multiply two $n \times n$ matrices in only $O(n^{2.81})$ field operations, compared with $2n^3$ for the standard algorithm. This immediately raises the question of the exponent of matrix multiplication: what is the smallest number $\omega$ such that for each $\varepsilon > 0$, matrix multiplication can be carried out in at most $O(n^{\omega+\varepsilon})$ operations? Clearly $\omega \geq 2$. It is widely believed that $\omega = 2$, but the best bound known is $\omega < 2.38$, due to Coppersmith and Winograd [6], following a sequence of improvements to Strassen’s original algorithm (see [5] p. 420 for the history). It is known that all the standard linear algebra problems (for example, computing determinants, solving systems of equations, inverting matrices, computing LUP decompositions—see Chapter 16 of [4]) have the same exponent as matrix multiplication, which makes $\omega$ a fundamental number for understanding algorithmic linear algebra. In addition, there are non-algebraic algorithms whose complexity is expressed in terms of $\omega$ (see, e.g., Section 16.9 in [4]).

Several fairly elaborate techniques for bounding $\omega$ are known, but since 1990 nobody has been able to improve on them. In this paper:

- We develop a new approach to bounding $\omega$ that imports the problem into the domain of group theory and representation theory. The approach is relatively simple and almost entirely separate from the existing machinery built up since Strassen’s original algorithm.

- We demonstrate the feasibility of the group theory aspect of the approach by identifying a family of groups for which a parameter that mirrors $\omega$ approaches 2. We also exhibit techniques for bounding this critical parameter and prove non-trivial bounds for a number of diverse groups and group families.

- We pose a question in representation theory (Question 4.1 below) that represents a potential barrier to directly obtaining non-trivial bounds on $\omega$ using this approach. We do not know the answer to this question. A positive answer would illuminate a path that might lead to $\omega = 2$ using the techniques that we present in this paper.

Our approach is reminiscent of a question asked by Coppersmith and Winograd (in Section 11 of [4]) about avoiding “three disjoint equivoluminous subsets” in abelian groups, which would lead to $\omega = 2$ if it has a positive answer. However, our technique is completely different, and our framework seems to have more algebraic structure to make use of (whereas theirs is more combinatorial).

1.1. Analogy with fast polynomial multiplication

There is a close analogy between the framework we propose in this paper and the well-known algorithm for multiplying two degree $n$ polynomials in $O(n \log n)$ operations using the Fast Fourier Transform (FFT). In this section we elucidate this analogy to give a high-level description of our technique.
Suppose we wish to multiply the polynomials \( A(x) = \sum_{i=0}^{n-1} a_i x^i \) and \( B(x) = \sum_{i=0}^{n-1} b_i x^i \). The naïve way to do this is to compute \( n^2 \) products of the form \( a_i b_j \), and from these the \( 2n-1 \) coefficients of the product polynomial \( A(x) \cdot B(x) \). Of course a far better algorithm is possible; we describe it below in language that easily translates into our framework for matrix multiplication.

Let \( G \) be a group and let \( \mathbb{C}[G] \) be the group algebra—that is, every element of \( \mathbb{C}[G] \) is a formal sum \( \sum_{g \in G} a_g g \) with \( a_g \in \mathbb{C} \), and the product of two such elements is
\[
\left( \sum_{g \in G} a_g g \right) \cdot \left( \sum_{h \in G} b_h h \right) = \sum_{f \in G} \left( \sum_{gh=f} a_g b_h \right) f.
\]

We often identify the element \( \sum_{g \in G} a_g g \) with the vector of its coefficients. If \( G \) is the cyclic group of order \( m \), then the product of two elements \( a = (a_g)_{g \in G} \) and \( b = (b_g)_{g \in G} \) is a cyclic convolution of the vectors \( a \) and \( b \). The important observation is that a cyclic convolution is almost what is needed to compute the coefficients of the product polynomial \( A(x) \cdot B(x) \)—the only problem is that it wraps around. To avoid this problem, we embed \( A(x) \) and \( B(x) \) as elements \( \tilde{A}, \tilde{B} \in \mathbb{C}[G] \) as follows: Let \( z \) be a generator of \( G \), which we assume to be a cyclic group of order \( m > 2n-1 \), and define
\[
\tilde{A} = \sum_{i=0}^{n-1} a_i z^i \quad \text{and} \quad \tilde{B} = \sum_{i=0}^{n-1} b_i z^i.
\]

Since the group size \( m \) is large enough to avoid wrapping around, we can read off the coefficients of the product polynomial from the element \( \tilde{A} \tilde{B} \in \mathbb{C}[G] \): the coefficient of \( x^i \) in \( A(x)B(x) \) is the coefficient of the group element \( z^i \) in \( \tilde{A} \tilde{B} \). This is a wordy account of a so-far simple correspondence, but the payoff is near. The Discrete Fourier Transform (DFT) for \( \mathbb{C}[G] \) is an invertible linear transformation \( D : \mathbb{C}[G] \to \mathbb{C}[G] \), which turns multiplications in \( \mathbb{C}[G] \) into pointwise multiplication of vectors in \( \mathbb{C}[G] \). We can therefore compute the product \( \tilde{A} \tilde{B} \) by first computing \( D(\tilde{A}) \) and \( D(\tilde{B}) \) and then computing the inverse DFT of their pointwise product. Thus, using the \( O(n \log m) \) Fast Fourier Transform algorithm, we can perform multiplication in \( \mathbb{C}[G] \) (and therefore polynomial multiplication, via the embedding above) in \( O(n \log m) \) operations.

One of the main results of the present paper is that matrix multiplication can be embedded into group algebra multiplication in an analogous way. The embedding is not as simple as the embedding of polynomial multiplication, but it has a natural and clean description in terms of a property of subsets of \( G \) (which we often take to be subgroups). In particular, if \( S, T, \) and \( U \) are subsets of \( G \) and \( A = (a_{s,t})_{s \in S, t \in T} \) and \( B = (b_{t,u})_{t \in T, u \in U} \) are \( |S| \times |T| \) and \( |T| \times |U| \) matrices, respectively, then we define
\[
\tilde{A} = \sum_{s,t} a_{s,t} s^{-1} t \quad \text{and} \quad \tilde{B} = \sum_{t,u} b_{t,u} t^{-1} u.
\]

If \( S, T, U \) satisfy the triple product property (see Definition 2.1), then we can read off the entries of the product matrix \( AB \) from \( \tilde{A} \tilde{B} \in \mathbb{C}[G] \): entry \( (AB)_{s,u} \) is simply the coefficient of the group element \( s^{-1} u \).

In the case of polynomial multiplication, the simplicity of the embedding obscures the fact that if \( G \) is too large (e.g., if \( |G| = n^2 \) rather than \( O(n) \)), then the benefit of the entire scheme is destroyed. Avoiding this pitfall turns out to be the main challenge in the new setting. We wish to embed matrix multiplication into a group algebra over a small group \( G \), as the size of \( G \) is a lower bound on the complexity of multiplication in \( \mathbb{C}[G] \). It is not surprising, for example, that \( n \times n \) matrix multiplication can be embedded into the group algebra of a group of order \( n^3 \). We show that abelian groups cannot beat \( n^3 \) and we identify families of non-abelian groups of size \( n^{2+o(1)} \) that admit such an embedding.

It might seem that this result together with the above trick for performing group algebra multiplication (i.e., taking the DFT, multiplying in the Fourier domain, and transforming back) would imply that \( \omega = 2 \). There are, however, two complications introduced by the fact that we are forced to work with non-abelian groups. The first is that we know of fast algorithms to compute the DFT only for limited classes of non-abelian groups (see Section 13.5 in [4]). However, the DFT is linear, and because of the recursive structure of divide and conquer matrix multiplication algorithms, linear transformations applied before and after the recursive step are “free.” For example, in Strassen’s original matrix multiplication algorithm, the number of matrix additions and scalar multiplications in the recursive step does not affect the bound on \( \omega \). So this potential complication is in fact no problem at all.

The second complication is that for \( \mathbb{C}[G] \) when \( G \) is non-abelian, multiplication in the Fourier domain is not simply pointwise multiplication of vectors in \( \mathbb{C}[G] \). Instead it is block-diagonal matrix multiplication, where the dimensions of the blocks are the dimensions of the irreducible representations of \( G \). We thus obtain a reduction of \( n \times n \) matrix multiplication to a number of smaller matrix multiplications of varying sizes, which gives rise to an inequality involving the exponent \( \omega \) of matrix multiplication. If the size of \( G \) were exactly \( n^2 \), then this inequality would imply that \( \omega = 2 \). However, the smallest one can make \( |G| \) is \( n^{2+o(1)} \), and then the question of whether the inequality implies \( \omega = 2 \) turns on the representation theory of \( G \). We show that when \( |G| = n^{2+o(1)} \), even slight control over the dimension of the largest irreducible representation is sufficient to achieve \( \omega = 2 \). Some control is necessary to avoid trivialities such as reducing to an even larger matrix multi-
plication problem. We can achieve that much control; the issue of whether it is possible to achieve more control is the subject of Question 4.1.

1.2. Outline

Following some preliminaries below, Sections 2 through 5 are devoted to outlining our approach. In Sections 6 and 7 we show that a variety of different types of groups support matrix multiplication within our framework, and in the process demonstrate a number of useful proof techniques. Section 5 highlights linear groups, whose representation theory makes them especially attractive for our purposes. Section 6 describes a parallel with the combinatorial notion of Sperner theory: the group algebra of a group may support matrix multiplication even when the group itself fails to support it.

In Section 7.2 we consider wreath product constructions, and linear groups may indeed be a fruitful line of inquiry. In Section 7.3 we use the combinatorial notion of Sperner capacity to demonstrate the surprising fact that the triple product property. If we wish to emphasize the specific subsets, we say that G realizes ω through 4.1.

In most of our examples, matrix multiplication will be realized through subgroups H1, H2, H3 of G, rather than arbitrary subsets. In that case, the triple product property is especially simple, because Q(H1) = H1: it states that if h1h2h3 = 1 with h1 ∈ H1, then h1 = h2 = h3 = 1. An equivalent formulation replaces h1h2h3 = 1 with h1h2 = h3.

Perhaps the simplest example comes from the product Cn × Cm × Cp of cyclic groups, which clearly realizes ⟨n, m, p⟩ through Cn × {1} × {1}, {1} × Cm × {1}, and {1} × {1} × Cp. We will see a number of less trivial examples shortly.

Lemma 2.1. If G realizes ⟨n1, n2, n3⟩, then it does so for every permutation of n1, n2, n3.

Proof. Suppose G realizes ⟨n1, n2, n3⟩ through S1, S2, S3, and suppose s1, s′ 1 ∈ S1. We need to show that the order in which 1, 2, and 3 appear in the equation

s′ 1s−1 1s′ 2s−1 2s′ 3s−1 3 = 1

is irrelevant. Conjugating by s′ 1s−1 1 shows that it is equivalent to

s′ 2s−1 2s′ 3s−1 3s′ 1s−1 1 = 1,

so we can perform a cyclic shift. To get a transposition, we take the inverse of the initial equation, which yields

s′ 3s′ 1 1 s′ 2s−1 2s′ 3s−1 3 = 1,

i.e., a transposition of 1 with 3 (the roles of s and s′ have been reversed, but that is irrelevant). These two permutations generate all permutations of {1, 2, 3}.

Lemma 2.2. If N is a normal subgroup of G that realizes ⟨n1, n2, n3⟩ and G/N realizes ⟨m1, m2, m3⟩, then G realizes ⟨n1m1, n2m2, n3m3⟩.

Proof. Suppose N realizes ⟨n1, n2, n3⟩ through S1, S2, S3, and suppose T1, T2, T3 are lifts to G of the three subsets of G/N that realize ⟨m1, n2, m3⟩. Then we claim that G realizes ⟨n1m1, n2m2, n3m3⟩ through the pointwise products

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$S_1T_1, S_2T_2, S_3T_3$. We need to check that for $s_i, s'_i \in S_i$ and $t_i, t'_i \in T_i$,

$$(s'_i t'_i)(s_i t_i)^{-1}(s_2 t_2)(s'_2 t'_2)^{-1}(s_3 t_3)(s'_3 t'_3)^{-1} = 1$$

iff $s_i = s'_i$ and $t_i = t'_i$ for all $i$. If we reduce this equation modulo $N$, we find that $t_i = t'_i \mod N$, and hence also in $G$. The equation in $G$ then becomes

$$s'_1 s_1^{-1} s_2 s_2^{-1} s_3 s_3^{-1} = 1,$$

from which we deduce $s_i = s'_i$, as desired. \qed

One useful special case of Lemma 2.2 is that if $G_1$ realizes $\langle n_1, m_1, p_1 \rangle$ and $G_2$ realizes $\langle n_2, m_2, p_2 \rangle$, then $G_1 \times G_2$ realizes $\langle n_1 n_2, m_1 m_2, p_1 p_2 \rangle$.

Our first theorem describes the embedding of matrix multiplication into group algebra multiplication:

**Theorem 2.3.** Let $F$ be any field. If $G$ realizes $\langle n, m, p \rangle$, then the number of field operations required to multiply $n \times m$ with $m \times p$ matrices over $F$ is at most the number of operations required to multiply two elements of $F[G]$. Furthermore, $\langle n, m, p \rangle_F \leq F[G]$.

For the definition of the restriction relation $\leq$ in the last sentence, see Section 14.3 of [4].

**Proof.** Let $G$ realize $\langle n, m, p \rangle$ through subsets $S, T, U$. Suppose $A$ is an $n \times m$ matrix, and $B$ is an $m \times p$ matrix. We will index the rows and columns of $A$ with the sets $S$ and $T$, respectively, those of $B$ with $T$ and $U$, and those of $AB$ with $S$ and $U$.

Consider the product

$$\left( \sum_{s \in S} A_{st} s^{-1} t \right) \left( \sum_{t' \in T, u \in U} B_{t' u} t'^{-1} u \right)$$

in the group algebra. We have

$$(s^{-1} t)(t'^{-1} u) = s'^{-1} u'$$

iff $s = s'$, $t = t'$, and $u = u'$, so the coefficient of $s'^{-1} u$ in the product is

$$\sum_{t' \in T} A_{st} B_{t'u} = (AB)_{su}.$$ 

Thus, one can simply read off the matrix product from the group algebra product by looking at the coefficients of $s'^{-1} u$ with $s \in S, u \in U$, and the assertions in the theorem statement follow. \qed

### 3. The pseudo-exponent

The pseudo-exponent of a group measures the quality of the embedding afforded by Theorem 2.3 in a single, well-behaved parameter, which in some ways mirrors the exponent $\omega$ of matrix multiplication.

**Definition 3.1.** The pseudo-exponent $\alpha(G)$ of a non-trivial finite group $G$ is the minimum of

$$\frac{3 \log |G|}{\log nmp}$$

over all $n, m, p$ (not all 1) such that $G$ realizes $\langle n, m, p \rangle$. The pseudo-exponent of the trivial group is 3.

When it is clear from the context which group is intended, we often write $\alpha$ instead of $\alpha(G)$. Note that in the special case that $G$ realizes $\langle n, n, n \rangle$, its pseudo-exponent satisfies $\alpha \leq \log n |G|$. In general, if $G$ realizes $\langle n, m, p \rangle$, then

$$\alpha \leq \log \frac{nmp |G|}{|G|^3}.$$ 

**Lemma 3.1.** The pseudo-exponent of a finite group $G$ is always greater than 2 and at most 3. If $G$ is abelian, then it is exactly 3.

**Proof.** The upper bound of 3 is trivial: use the subgroups $H_1 = H_2 = \{1\}$ and $H_3 = G$.

For the lower bounds, suppose $G$ realizes $\langle n_1, n_2, n_3 \rangle$ (with $n_1 n_2 n_3 > 1$) through subsets $S_1, S_2, S_3$. It follows from the definition of realization that the map $(x, y) \mapsto x^{-1} y$ is injective on $S_1 \times S_2$ and its image intersects the quotient set $Q(S_3)$ only in the identity. Thus, $|G| \geq n_1 n_2$, and $|G| > n_1 n_2$ unless $n_3 = 1$. Similarly, $|G| \geq n_2 n_3$ with equality only if $n_1 = 1$, and $|G| \geq n_1 n_3$ with equality only if $n_2 = 1$. Thus, $|G|^3 > (n_1 n_2 n_3)^2$, so $\alpha(G) > 2$.

If $G$ is abelian, then the product map $S_1 \times S_2 \times S_3 \to G$ must be injective, so $|G| \geq n_1 n_2 n_3$ and $\alpha(G) \geq 3$. \qed

The pseudo-exponent is well-behaved with respect to group extensions:

**Lemma 3.2.** If $N$ is a normal subgroup of $G$, then $\alpha(G) \leq \max(\alpha(N), \alpha(G/N))$.

**Proof.** Suppose $N$ realizes $\langle n_1, n_2, n_3 \rangle$ and $G/N$ realizes $\langle m_1, m_2, m_3 \rangle$. Then Lemma 2.2 implies that the pseudo-exponent of $G$ is at most

$$\frac{3 \log |G|}{\log n_1 m_1 n_2 m_2 n_3 m_3} = \frac{3 \log |N| + 3 \log |G/N|}{\log n_1 n_2 n_3 + \log m_1 m_2 m_3},$$

which is bounded above by the larger of

$$\frac{3 \log |N|}{\log n_1 n_2 n_3} \quad \text{and} \quad \frac{3 \log |G/N|}{\log m_1 m_2 m_3},$$

as desired. \qed
Non-abelian groups can have pseudo-exponent less than 3. The smallest example is the symmetric group $S_3$ on 3 elements. It realizes $(2, 2, 2)$ through its three subgroups of order 2, so it has pseudo-exponent at most $\log_2 6$ (and one can check that it is exactly $\log_2 6$). Next, we generalize this construction to show that it is possible to come arbitrarily close to pseudo-exponent 2, as follows.

Given a triangular array of points in the plane, as in Figure 1, we consider the group of permutations of the points, together with three subgroups, one for each side of the triangle. Each subgroup permutes the set of points on each line parallel to its side of the triangle. The proof of Theorem 3.3, while not phrased in geometric terms, shows that these subgroups satisfy the triple product property.

**Theorem 3.3.** The pseudo-exponent of $S_{n(n+1)/2}$ is at most 2 + $\frac{2 - \log 2}{\log n} + O \left( \frac{1}{\log n} \right)$.

**Proof.** There are $n(n+1)/2$ triples $(a, b, c)$ with $a, b, c \geq 0$ and $a + b + c = n - 1$. We view $S_{n(n+1)/2}$ as the group of permutations of these triples. Let $H_i$ be the subgroup that fixes the $i$-th coordinate. The size of this subgroup is $1!2! \ldots n!$, so the pseudo-exponent bound is

$$ \frac{\log(n(n+1)/2)!}{\log 1!2! \ldots n!} = 2 + \frac{2 - \log 2}{\log n} + O \left( \frac{1}{\log n} \right), $$

assuming these subgroups satisfy the triple product property. For that, we need to prove that if $h_1 h_2 h_3 = 1$ with $h_i \in H_i$, then $h_1 = h_2 = h_3 = 1$.

Suppose $h_1 h_2 h_3 = 1$ with $h_i \in H_i$. We will order the triples lexicographically, so that $(0, 0, n-1)$ is the smallest triple and $(n-1, 0, 0)$ is the largest, and prove by induction using this ordering that $h_1, h_2, h_3$ fix every triple.

Suppose all triples smaller than $(a, b, c)$ are fixed by each of $h_1, h_2, h_3$ (in the base case, the set of such triples is empty). The permutation $h_3$ cannot send $(a, b, c)$ to a smaller triple, since all smaller triples are fixed points, so $h_3$ must send it to $(a + i, b - i, c + j)$ with $i \geq 0$. Then $h_2$ sends that to $(a + i + j, b - i, c - j)$ for some $j$. The only way $h_1$ can return to $(a, b, c)$ is if $i + j = 0$, so that must be the case. However, $h_1$ fixes $(a, b - i, c + i)$ for $i > 0$ (since such a triple is smaller than $(a, b, c)$), so we must have $i = 0$. It follows that $(a, b, c)$ is fixed by each of $h_1, h_2, h_3$, so by induction all triples are fixed and hence $h_1 = h_2 = h_3 = 1$.

The same holds for all symmetric groups, since one can look at the largest subgroup of the form $S_{n(n+1)/2}$.

**4. Relating the pseudo-exponent to $\omega$**

In this section we relate the pseudo-exponent $\alpha$ to the exponent of matrix multiplication $\omega$. As with many of the results since Strassen’s algorithm, our main theorems are stated as bounds on $\omega$, rather than explicit algorithms, but of course algorithms are implicit in the proofs.

**Theorem 4.1.** Suppose $G$ has pseudo-exponent $\alpha$, and the character degrees of $G$ are $\{d_i\}$. Then

$$ |G|^\omega / \alpha \leq \sum_i d_i^\omega. $$

The intuition is simple: the problem of multiplying matrices of size $|G|^{1/\alpha}$ reduces to multiplication in $C[G]$, which is equivalent to multiplying a collection of matrices of sizes $d_i$. These multiplications should take about $d_i^\omega$ operations, so $\sum_i d_i^\omega$ should be an approximate upper bound for the number of operations required to multiply matrices of size $|G|^{1/\alpha}$, i.e., roughly $|G|^\omega / \alpha$. It is convenient that when one makes this idea precise, these crude approximations become exact bounds.

**Proof.** Suppose $G$ realizes $(n, m, p)$ with $nmp = |G|^{3/\alpha}$ (it follows from the definition of the pseudo-exponent that $G$ realizes such a tensor). By Theorem 3.3

$$ \langle n, m, p \rangle \leq C[G] \simeq \bigoplus_i \langle d_i, d_i, d_i \rangle. \quad (1) $$

We will need two facts about the rank of matrix multiplication: for all $n', m', p'$,

$$ (n' m' p')^{\omega/3} \leq R(\langle n', m', p' \rangle) $$

(Proposition 15.5 in [4]), and for each $\varepsilon > 0$ there exists $C > 0$ such that for all $k$,

$$ R(\langle k, k, k \rangle) \leq Ck^{\omega + \varepsilon} $$

(Proposition 15.1 in [4]).

The $\ell$-th tensor power of $\langle n, m, p \rangle$ is

$$ \langle n', m', p' \rangle \leq \bigoplus_{i_1, \ldots, i_\ell} \langle d_{i_1}, \ldots, d_{i_\ell}, d_{i_1}, \ldots, d_{i_\ell} \rangle, $$

Figure 1. A triangular array of points.
if we use
\[ \langle n_1, m_1, p_1 \rangle \otimes \langle n_2, m_2, p_2 \rangle \simeq \langle n_1 n_2, m_1 m_2, p_1 p_2 \rangle. \]
It follows from taking the rank of both sides that
\[ |G|^{\omega/\alpha} \leq C \left( \sum_i d_i^{\epsilon + \epsilon} \right)^{\ell}, \]
and if we take the \( \ell \)-th root and let \( \ell \) go to infinity, then we deduce that
\[ |G|^{\omega/\alpha} \leq \sum_i d_i^{\omega + \epsilon}. \]
Finally, because this inequality holds for all \( \epsilon > 0 \), it must hold for \( \epsilon = 0 \) as well, by continuity.

Notice that if \( \alpha(G) \) were 2, then this theorem would imply that \( \omega = 2 \) (using \( \sum_i d_i^2 = |G| \), the Cauchy-Schwarz inequality, and the fact that every non-trivial group has at least two irreducible representations). In general, though, we need to control the character degrees of \( G \). The maximum possible character degree for any non-trivial group is \( (|G| - 1)^{1/2} \); we show below that an upper bound of \( |G|^{1/2 - \epsilon} \) for fixed \( \epsilon > 0 \) would be sufficient to obtain \( \omega = 2 \) from a family of groups with pseudo-exponent approaching 2 (and that even a much weaker bound suffices).

We define \( \gamma(G) \), or simply \( \gamma \) when \( G \) is clear from the context, so that \( |G|^{1/\gamma} \) is the maximum character degree of \( G \) \( (\gamma(G) = \infty \) if \( G \) is abelian). Ideally, we’d like the exponent of matrix multiplication \( \omega \) to be bounded above by the pseudo-exponent \( \alpha \). The following corollary shows that in the region near 2, this actually happens, with a correction factor that depends on \( \gamma \).

**Corollary 4.2.** Let \( G \) be a finite group. If \( \alpha(G) < \gamma(G) \), then
\[ \omega \leq \alpha \left( \frac{\gamma - 2}{\gamma - \alpha} \right). \]

*Proof.* Let \( \{d_i\} \) denote the character degrees. Then by Theorem 4.1,
\[ |G|^{\omega/\alpha} \leq \sum_i d_i^{\omega - 2} d_i^2 \]
\[ \leq |G|^{(\omega - 2)/\gamma} \sum_i d_i^2 \]
\[ = |G|^{1 + (\omega - 2)/\gamma}, \]
which implies \( \omega(1/\alpha - 1/\gamma) \leq 1 - 2/\gamma \). Dividing by \( 1/\alpha - 1/\gamma \) (which is positive by assumption) yields the stated result.

Like \( \alpha(G) \), we have \( \gamma(G) > 2 \) for all \( G \), and Corollary 4.2 shows that our approach amounts to a race between \( \alpha(G) \) and \( \gamma(G) \) to see which approaches 2 faster. The most attractive form of this corollary is the following special case:

**Corollary 4.3.** Suppose there exists a family \( G_1, G_2, \ldots \) of finite groups such that \( \alpha(G_i) = 2 + o(1) \) as \( i \to \infty \), and furthermore \( \alpha(G_i) - 2 = o(\gamma(G_i) - 2) \). Then the exponent of matrix multiplication is 2.

These corollaries are weakenings of Theorem 4.1, the advantage being that they only require knowledge of \( \gamma(G) \), which is typically easier to work with than the complete set of character degrees that is required for Theorem 4.1.

It is reasonable to ask whether the requirement \( \alpha < \gamma \) which occurs in Corollary 4.2 is necessary. It turns out that it is, because if \( \alpha \geq \gamma \), then for all \( \omega > 0 \),
\[ |G|^{\omega/\alpha} \leq |G|^{\omega/\gamma} \leq \sum_i d_i^{\omega}, \]
where the second inequality holds because \( |G|^{1/\gamma} = d_i \) for some \( i \). Then the inequality in Theorem 4.1 holds even for \( \omega = 3 \). The necessity of \( \alpha < \gamma \) makes perfect sense, because when it fails to hold, the approach amounts to a reductio of matrix multiplication to several instances, one of which is as large as the original instance. In fact, the construction in the proof of Theorem 5.3 succumbs to this problem: there we proved that \( \alpha(S_{n(n+1)/2}) \leq 2 + O(1/\log n) \), but it turns out that \( \gamma(S_{n(n+1)/2}) = 2 + \Theta(1/(n \log n)) \) (see [15]). Moreover, there exist non-abelian groups for which \( \alpha < \gamma \) and \( \alpha < 3 \); one example is the group in Proposition 4.4 below.

If we do have access to the complete set of character degrees then there is a relatively simple condition to check to determine whether the inequality in Theorem 4.1 yields a non-trivial bound on \( \omega \). The condition is that \( |G|^{3/\alpha} > \sum_i d_i^3 \). To see this observe that the inequality in Theorem 4.1 is equivalent to
\[ \frac{\omega}{\alpha} \log |G| \leq \log \sum_i d_i^\alpha. \]
The right-hand side is convex as a function of \( \omega \), and the left-hand side is linear. Furthermore, as \( \omega \to \infty \), the right-hand side is asymptotic to
\[ \frac{\omega}{\gamma} \log |G|, \]
which is smaller than the left-hand side when \( \alpha < \gamma \) (which is the non-trivial case). Therefore [2] gives no information about \( \omega \) in the interval \([2, 3] \) unless it rules out \( \omega = 3 \), which is equivalent to the above stated condition. We do not have examples of groups meeting this condition.

We are thus led to pose the following question in representation theory:

**Question 4.1.** Does there exist a finite group that realizes \( \langle n, m, p \rangle \) and has character degrees \( \{d_i\} \) such that
\[ nmp > \sum_i d_i^3. \]
5. Linear groups

Matrix groups over finite fields are an important class of finite groups. They are especially attractive for our purposes because their representation sizes, as measured by \( \gamma \), are well behaved. We will focus on the case of \( SL_n(F_q) \) for simplicity, although we see no reason why it should perform better than other linear groups. If \( n > 1 \) is held fixed, \( \gamma(SL_n(F_q)) \) approaches \( 2 + 2/n \) as \( q \) tends to infinity (which can be deduced from [2], according to a private communication from G. Lusztig). Thus, if one could prove that \( \alpha(SL_n(F_q)) = 2 + \alpha(1) \) for some fixed \( n \), then Corollary 4.2 would imply \( \omega = 2 \). Even if one lets \( n \) grow, one might still hope that \( \omega \) would tend to \( 2 \) faster than \( \gamma \). We cannot prove that \( \alpha \) even approaches \( 2 \) at all as \( n, q \to \infty \), but comparison with Theorem 6.1 below suggests that it does. In this section we concentrate on the case of \( SL_2(F_q) \).

For later reference, we collect here the character degrees of \( SL_2(F_q) \):

| Degree       | Multiplicity (\( q \) odd) | Multiplicity (\( q \) even) |
|--------------|----------------------------|----------------------------|
| \( q + 1 \)  | \( (q - 3)/2 \)            | \( (q - 2)/2 \)            |
| \( q \)      | 1                          | 1                          |
| \( (q + 1)/2 \) | \( (q - 1)/2 \)            | \( q/2 \)                  |
| \( (q - 1)/2 \) | 2                          | 0                          |
| \( 1 \)      | 1                          | 1                          |

(See Exercise 28.2 and its solution in [11] for \( q \) even, and [13] for \( q \) odd, but note that [13] has a typo in the multiplicity for degree \( q + 1 \) at the bottom of the first column on page 122.)

Proposition 5.1. The group \( SL_2(F_q) \) of order \( q^6 - q^2 \) realizes \( (q, q, q) \).

Unfortunately, this pseudo-exponent bound tends to \( 3 \) as \( q \to \infty \), but at least it is always strictly better than \( 3 \). (We can also prove similarly that \( \alpha(SL_n(F_q)) < 3 \).)

Proof. Consider the three parabolic subgroups

\[
H_1 = \left\{ \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} : x \in F_q \right\},
\]

and

\[
H_2 = \left\{ \begin{pmatrix} 1 & 0 \\ y & 1 \end{pmatrix} : y \in F_q \right\},
\]

and

\[
H_3 = \left\{ \begin{pmatrix} 1 + z & z \\ -z & 1 - z \end{pmatrix} : z \in F_q \right\}.
\]

We need to check that for \( h_1 \in H_1 \), if \( h_1h_2 = h_3 \), then \( h_1 = h_2 = h_3 = 1 \). To check that, we multiply to get

\[
\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ y & 1 \end{pmatrix} = \begin{pmatrix} 1 + xy & x \\ y & 1 \end{pmatrix}.
\]

That can be of the form

\[
\begin{pmatrix} 1 + z & z \\ -z & 1 - z \end{pmatrix}
\]

only if \( x = y = z = 0 \), as desired.

One might hope that \( SL_n(F_q) \) realizes

\[
(q^{(n-1)/2}, q^{n(n-1)/2}, q^{n(n-1)/2})
\]

through three conjugates of the group of upper-triangular matrices with 1’s on the diagonal. However, that fails for \( q = 2 \) and \( n = 3 \), according to calculations using the computer program GAP (see [2]); furthermore, no subgroups of these orders work for \( q = 2 \) and \( n = 3 \).

Proposition 5.2. The group \( SL_2(F_q^2) \) of order \( q^6 - q^2 \) realizes \( (q^2, q^2, q^2) \).

Proof. Let \( x \mapsto \bar{x} \) denote the Frobenius automorphism of \( F_q^2 \) over \( F_q \). The three subgroups we will use are

\[
H_1 = \left\{ \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} : x \in F_q^2 \right\};
\]

\[
H_2 = \left\{ \begin{pmatrix} 1 & 0 \\ y & 1 \end{pmatrix} : y \in F_q^2 \right\};
\]

and

\[
H_3 = SU_2(F_q) = \left\{ \begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix} : a, b \in F_q^2, a\bar{a} + b\bar{b} = 1 \right\}.
\]

Note that to check that \( |H_3| = q^3 - q \), one just needs to count solutions to \( a\bar{a} + b\bar{b} = 1 \). For a fixed \( b \) with \( b\bar{b} \neq 1 \), there are \( q + 1 \) corresponding choices of \( a \) that work; if \( b\bar{b} = 1 \), then \( a = 0 \). There are \( (q^2 - 1) - (q + 1) \) non-zero choices of \( b \) with \( b\bar{b} \neq 1 \) (to which we must add \( b = 0 \)) and \( q + 1 \) with \( b\bar{b} = 1 \). Thus, there are \( (q^2 - 1)(q + 1) + (q + 1) = q^3 - q \) elements of \( H_3 \).

As in the previous proof, checking the triple product property amounts to checking that

\[
\begin{pmatrix} 1 + xy & x \\ y & 1 \end{pmatrix} = \begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix}
\]

implies \( x = y = b = 0 \) and \( a = 1 \), which is a trivial calculation.
Proposition \ref{proposition:lie-pseudo-exponent} proves that
\[
\liminf_{q \to \infty} \alpha(SL_2(\mathbb{F}_q)) \leq 18/7,
\]
which is substantially better than 3 but still not near 2. Using Theorem \ref{theorem:lie-pseudo-exponent} and the character degrees of $SL_2(\mathbb{F}_q)$, one can show that if
\[
\liminf_{q \to \infty} \alpha(SL_2(\mathbb{F}_q)) < 9/4,
\]
than Question \ref{question:lie-pseudo-exponent} has a positive answer.

6. Lie groups

In the category of Lie groups, one can set up a theory parallel to that of the previous sections. We do not know how to use it to bound the exponent of matrix multiplication (because of course Lie groups of positive dimension are infinite). However, we have had more luck constructing examples using Lie groups than with finite linear groups, and this success seems a good reason to be optimistic about matrix groups over finite fields. All examples involving Lie groups can be skipped by a reader who cares only about finite groups and matrix multiplication.

Recall that $Q(S)$ denotes the right quotient set of $S$.

**Definition 6.1.** Let $G$ be a Lie group, with submanifolds $M_1, M_2, M_3$ such that for $q_i \in Q(M_i)$, if $q_1 q_2 q_3 = 1$ then $q_1 = q_2 = q_3 = 1$. We say that $G$ has Lie pseudo-exponent at most
\[
\frac{\dim(G)}{\dim(M_1) + \dim(M_2) + \dim(M_3)/3}.
\]

We usually take the submanifolds to be Lie subgroups. If $G$ and the three subgroups are algebraic groups defined over a number field, then it is natural to ask what pseudo-exponent may be achieved when one reduces modulo a prime ideal, to get a finite quotient group. If the triple product property still holds, then as the finite field size tends to infinity, the pseudo-exponent bound of this finite group approaches the Lie pseudo-exponent. However, the triple product property may not be preserved, as we will show after the following theorem.

**Theorem 6.1.** The group $SL_n(\mathbb{R})$ has Lie pseudo-exponent at most $2 + 2/n$.

**Proof.** The three subgroups are the group $U$ of upper-triangular matrices with 1’s on the diagonal, the group $L$ of lower-triangular matrices with 1’s on the diagonal, and $SO_n(\mathbb{R})$. Each subgroup has dimension $n(n - 1)/2$, and $SL_n(\mathbb{R})$ has dimension $n^2 - 1$, so assuming the triple product property holds, the Lie pseudo-exponent is at most
\[
\frac{n^2 - 1}{n(n - 1)/2} = 2 + \frac{2}{n}.
\]

Let $M \in SO_n(\mathbb{R})$, $A \in U$, and $B \in L$. We wish to prove that if $MA = B$, then $M = A = B = I$. Let $e_1, \ldots, e_n$ be the standard basis of $\mathbb{R}^n$. We will prove by induction on $i$ that $Me_i = e_i$. Once we know that $M = I$, it follows that $A = B$, and thus $A = B = I$ because $U$ and $L$ are disjoint except for the identity. ($A = B = I$ will also follow directly from the proof that $M = I$.)

Let $A_i$ and $B_i$ denote the $i$-th columns of $A$ and $B$, and denote their $j$-th entries by $A_{ij}$ and $B_{ij}$. Note that this indexing of rows and columns is opposite to the standard convention, but it will be more convenient in this proof. Because $MA = B$, we have
\[
MA_i = B_i.
\]

We start with the base case $i = 1$. Since $A$ is in $U$, we have $A_1 = e_1$. Thus, $|B_1| = |MA_1| = |Me_1| = |e_1| = 1$, since $M$ is an orthogonal matrix. Because $B_{11} = 1$, the only way $|B_1|$ can be 1 is if $B_1 = e_1$. Thus, $Me_1 = e_1$.

Now suppose that $Me_j = e_j$ for all $j < i$. Because $A$ is in $U$,
\[
A_i = e_i + \sum_{j < i} A_{ij} e_j,
\]
and because $B$ is in $L$,
\[
B_i = e_i + \sum_{j > i} B_{ij} e_j.
\]

Now the induction hypothesis implies that
\[
B_i = MA_i = Me_i + \sum_{j < i} A_{ij} e_j,
\]
so
\[
Me_i = e_i + \sum_{j > i} B_{ij} e_j - \sum_{j < i} A_{ij} e_j.
\]

Since $M$ is orthogonal, $|Me_i| = |e_i| = 1$. The coefficient of $e_i$ in $Me_i$ is already 1, so the other coefficients must be zero and thus $Me_i = e_i$, as desired.

The same holds for $SL_n(\mathbb{C})$ with $SO_n(\mathbb{R})$ replaced by $SU_n$, but not by $SO_n(\mathbb{C})$: the orthogonal matrix
\[
\begin{pmatrix}
1 & -1+i & 1+i \\
1 & 1+i & 1-i \\
-i & 1 & 1
\end{pmatrix}
\]
equals
\[
\begin{pmatrix}
1 & 0 & 0 \\
1 & 1 & 0 \\
-i & 1 & 1
\end{pmatrix} \begin{pmatrix}
1 & -1+i & 1+i \\
0 & 1 & 2 \\
0 & 0 & -1
\end{pmatrix}.
\]

Of course the same obstacle arises over finite fields (a sum of non-zero squares may vanish).
7. Additional examples

In this section we explore a variety of different types of groups, and prove non-trivial pseudo-exponent bounds for them. We hope that these examples (together with the ones we have already seen) will serve as something of a tool kit for constructing a group that might answer Question 4.1 and possibly even a family of groups that prove $\omega = 2$.

7.1. Solvable groups

Non-abelian simple (or almost simple) groups appear to be a fruitful source of groups with small pseudo-exponents. However, solvable groups also do quite well. In this section, we will construct solvable groups that have Lie pseudo-exponent tending to 2, and finite solvable groups with pseudo-exponent bounds of 2.5 and 2.4811 . . . (which, GAP tells us, is the best pseudo-exponent attained using three subgroups in any group of order up to 100).

Let $F$ be a field, and $\langle , \rangle$ a symmetric bilinear form on $F^n$. Define multiplication in

$$G = \{(x, y, \alpha) : x, y \in F^n, \alpha \in F\}$$

via

$$(x, y, \alpha)(u, v, \beta) = (x + u, y + v, \alpha + \beta + 2\langle u, y \rangle),$$

and define the three subgroups

$$H_1 = \{(x, 0, 0) : x \in F^n\},$$

$$H_2 = \{(0, y, 0) : y \in F^n\},$$

and

$$H_3 = \{(z, z, \langle z, z \rangle) : z \in F^n\}.$$ 

**Proposition 7.1.** If the only element $z \in F^n$ satisfying $\langle z, z \rangle = 0$ is $z = 0$, then $H_1$, $H_2$, and $H_3$ satisfy the triple product property.

**Proof.** We simply need to check that $H_3$ avoids all elements of the form $(x, 0, 0)(0, y, 0) = (x, y, 0)$, except when $x = y = 0$. The only way such an element can be in $H_3$, i.e., of the form $(z, z, \langle z, z \rangle)$, is if $x = y = z$ and $\langle z, z \rangle = 0$. That means $z = 0$ and thus $x = y = 0$, as desired.

When $F = \mathbb{R}$, the group described above is a Heisenberg group, and we obtain the following bound:

**Corollary 7.2.** In the above framework, with $F = \mathbb{R}$, and $\langle , \rangle$ the standard inner product, the Lie group $G$ has Lie pseudo-exponent at most $2 + 1/n$.

**Proof.** It is clear that Proposition 7.1 is satisfied; the group dimension is $2n + 1$, and the three subgroups each have dimension $n$.

When $F$ is a finite field, the group described above is an extraspecial group, and we obtain the following bound:

**Corollary 7.3.** In the above framework, with $F = \mathbb{F}_q$ of odd characteristic, $n = 2$, and $(x, y) = x_1y_1 - wx_2y_2$ for some $w \in F$ that is not a square, the finite group $G$ has pseudo-exponent at most 2.5.

Here, $x_i$ denotes the $i$-th coordinate of the vector $x$.

**Proof.** Note that $\langle z, z \rangle = 0$ implies $z_1^2 = wz_2^2$, which by our choice of $w$ can only happen when $z = 0$. Thus Proposition 7.1 is satisfied. The group has order $q^5$, and the three subgroups have size $q^2$, leading to a pseudo-exponent bound of 2.5 as claimed.

A slight variant of this construction works for even $q$ as well, but the pseudo-exponent bound is identical so we omit the details.

One quite different example is the following Frobenius group of order 80. We found the group by a brute force search using GAP, and Michael Aschbacher supplied the following humanly understandable proof that it works.

Let $C_5 \subset \mathbb{F}_{16}^*$ be the unique subgroup of order 5. Consider its semidirect product $G = C_5 \ltimes \mathbb{F}_{16}$ with the additive group of $\mathbb{F}_{16}$, where multiplication is defined by

$$(\alpha, x)(\beta, y) = (\alpha\beta, \beta x + y).$$

**Proposition 7.4.** The group $G = C_5 \ltimes \mathbb{F}_{16}$ realizes $\langle 5, 5, 8 \rangle$, and thus $\omega(G) \leq 3 \log_{200} 80 = 2.4811 . . . .

**Proof.** Let

$$H_1 = \{(\alpha, 0) : \alpha \in C_5\}$$

and

$$H_2 = \{(\alpha, \alpha - 1) : \alpha \in C_5\}$$

(i.e., $H_2$ is $H_1$ conjugated by $(1, 1)$). Let

$$H_3 = \{(1, x) : x \in \mathbb{F}_{16}, \text{Tr} x = 0\},$$

where Tr denotes the trace from $\mathbb{F}_{16}$ to $\mathbb{F}_2$. These groups satisfy $|H_1| = |H_2| = 5$ and $|H_3| = 8$. All we need to check is the triple product property.

We must verify that unless $\alpha$ and $\beta$ are both 1, the product

$$(\alpha, 0)(\beta, \beta - 1) = (\alpha\beta, \beta - 1)$$

is not in $H_3$. For it to be in $H_3$, we must have $\alpha = \beta^{-1}$ and $\text{Tr} (\beta - 1) = 0$. However,

$$\text{Tr} (\beta - 1) = \text{Tr} \beta - \text{Tr} 1 = \text{Tr} \beta,$$

and $\text{Tr} \beta = 1$ for $\beta \in C \setminus \{1\}$ because the minimal polynomial over $\mathbb{F}_2$ of such a $\beta$ is $1 + \beta + \beta^2 + \beta^3 + \beta^4$. 

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This proposition generalizes as follows (see [3] for background on cohomology): Let $G$ be a group that acts on an abelian group $A$, $\theta : G \to A$ a 1 cocycle, and $B \subseteq A$ a subgroup. If $\theta(g) \in B$ implies $g = 1$ for all $g \in G$, then the semidirect product $G \ltimes A$ realizes $[G],[G],[B]$ via the subgroups $G \times \{0\}, \{(g,\theta(g)) : g \in G\}$, and $\{1\} \times B$. (In Proposition 7.4 the 1 cocycle is a coboundary.) Unfortunately, we do not know any other good examples.

Unlike the cases of extraspecial groups and matrix groups, we do not know how to generalize Proposition 7.4 to achieve Lie pseudo-exponent arbitrarily near 2. The best we know how to do is the following. Let $\mathbb{H}$ be the quaternions, and $U \subset \mathbb{H}^\times$ be the group of unit quaternions (which is isomorphic to $SU(2)$). Then within the semidirect product $U \ltimes \mathbb{H}$, the three subgroups $U \times \{0\}$, $\{(u,u-1) : u \in U\}$, and $\{(0,x) : \text{Tr}x = 0\}$ satisfy the triple product property and prove that the Lie pseudo-exponent of $U \ltimes \mathbb{H}$ is at most 7/3.

### 7.2. Wreath products

In this section we present another family of groups that achieves pseudo-exponent $2+\alpha(1)$. This family is described in terms of the wreath product: if $A$ is a group, then the wreath product $A \wr S_n$ is the semidirect product $S_n \ltimes A^n$, where $S_n$ acts on $A^n$ by permuting the coordinates (and the multiplication is of course via $(\pi,u)(\pi',v) = (\pi\pi',\pi'u+v)$).

**Theorem 7.5.** Let $A$ be the cyclic group of order $2n$, and let $G_n = A \wr S_n$. Then

$$\alpha(G_n) \leq \gamma(G_n) = 2 + \frac{1 + \log 2}{\log n} + O\left(\frac{1}{(\log n)^2}\right).$$

**Proof.** We view $G_n$ as the semidirect product $S_n \ltimes A^n$, and will use the three subgroups

$$
H_1 = \{(\pi,0) : \pi \in S_n\},
H_2 = \{(\pi,\pi u - u) : \pi \in S_n\}, 	ext{ and }
H_3 = \{(\pi,\pi v - v) : \pi \in S_n\},
$$

where $u = (1,2,\ldots,n)$, and $v = (n,n-1,\ldots,1)$. As each subgroup has size $n!$ in a group of size $n!(2n)^n$,

$$\alpha \leq \frac{\log(n!(2n)^n)}{\log n},$$

assuming the triple product property holds. The largest character degree of $G_n$ is $|S_n| = n!$ (see Theorem 25.6 in [10]) and so $|G|^1/\gamma = n!$, which implies

$$\gamma = \frac{\log(n!(2n)^n)}{\log n}.$$ 

By Stirling’s formula,

$$\frac{\log(n!(2n)^n)}{\log n} = 2 + \frac{1 + \log 2}{\log n} + O\left(\frac{1}{(\log n)^2}\right),$$

so all that remains is to verify the triple product property.

Suppose $h_1 = (\pi',0) \in H_1$ and $h_2 = (\pi,\pi u - u) \in H_2$. Their product is $(\pi'\pi,\pi u - u)$, and if it equals $h_3 = (\sigma,\sigma v - v) \in H_3$, then $\pi u - u = \sigma v - v$. The $i$-th coordinate of $\pi u - u$ is $\pi(i) - i$, and that of $\sigma v - v$ is $(n+1 - \sigma(i)) - (n+1 - i) = i - \sigma(i)$. Thus, $h_1h_2 = h_3$ implies $\pi(i) + \sigma(i) = 2i$ for all $i$. This is an equation in $A$, and hence holds only modulo $2n$. However, $\pi(i),\sigma(i)$, and $i$ are all in $\{1,\ldots,n\}$, so the equation holds in the integers as well. Because $\pi(1)$ and $\sigma(1)$ are both at least 1, we conclude from $\pi(1) + \sigma(1) = 2$ that $\pi(1) = \sigma(1) = 1$. Then $\pi(2)$ and $\sigma(2)$ must be at least 2, and $\pi(2) + \sigma(2) = 4$, so $\pi(2) = \sigma(2) = 2$, etc. We conclude that $\pi$ and $\sigma$ are both trivial, as is $\pi'$ because $\pi'\pi = \sigma$. Thus, $h_1 = h_2 = h_3 = 1$, as desired.

This construction is an improvement over Theorem 3.3 because it achieves essentially the same pseudo-exponent bound, while at the same time $\alpha \leq \gamma$. A more complicated variant of this construction achieves a comparable pseudo-exponent and has $\alpha < \gamma$.

### 7.3. Direct products and the Sperner capacity

It is natural to attempt to improve the pseudo-exponent of a finite group $G$ by forming some group derived from it, such as a power $G^k$. We know that $\gamma(G^k) = \gamma(G)$, so that parameter becomes no smaller. Lemma 5.2 implies that $\alpha(G^k) \leq \alpha(G)$, and in this section we show that it is possible to achieve $\alpha(G^k) < \alpha(G)$.

We will be led for the first time since Lemma 2.2 to realize matrix multiplication through quotient sets that are not subgroups. Proposition 7.6 below proves that this complication is necessary to determine the pseudo-exponents of certain groups.

Let $D_m$ be the dihedral group generated by $x$ and $y$, with the relations $y^2 = x^m = 1$ and $yxy = x^{-1}$.

**Proposition 7.6.** For every $m$, $D_m$ realizes $(2,2,2|m/3|)$, and hence $\alpha(D_m) < 3$ for $m \geq 9$. If $m$ is a prime greater than 3, then no three subgroups prove $\alpha(D_m) < 3$.

**Proof.** Let $S_1 = \langle y \rangle$ be the subgroup generated by $y$, $S_2 = \langle yx^2 \rangle$, and $S_3 = \langle x^k, yx^{3k+1} : 0 \leq k < (m-2)/3 \rangle$. Then one can check by simple case analysis that $D_m$ realizes $(2,2,2|m/3|)$ through $S_1, S_2, S_3$. Note that $S_3$ is a subgroup iff $m$ is a multiple of 3.

When $m$ is prime, all subgroups of $D_m$ have order 1, 2, $m$, or $2m$, and it is easy to rule out each case (except when $m = 3$, in which case three subgroups of order 2 prove $\alpha(D_3) < 3$).
Proposition 7.7. If $S \subseteq (\mathbb{Z}/m\mathbb{Z})^k$ is a subset in which no two distinct vectors differ by an element of $\{0, 1\}^k$, then $D_m^k$ realizes $(2^k, 2^k, |S|)$. 

Proof. We identify $\mathbb{Z}/m\mathbb{Z}$ with the subgroup $\langle x \rangle \subseteq D_m$ (via $i \leftrightarrow x^i$), so that $S \subseteq \langle x \rangle \subseteq D_m^k$. The subgroups $\langle y \rangle$ and $\langle xy \rangle$ of $D_m$ have pointwise product $\langle y \rangle \langle xy \rangle = \{1, y, xy, x\}$. Therefore the condition on differences of elements in $S$ implies that $\langle y \rangle^k \langle xy \rangle^k$, and $S$ satisfy the triple product property, since $(\langle y \rangle^k \langle xy \rangle^k) \cap \langle x \rangle^k = \{1, x\}^k$, and $Q(S) \subseteq \langle x \rangle^k$ avoids $\{1, x\}$.

The problem of making $S$ as large as possible has been studied before; a generalization of this problem is known as the Sperner capacity of a directed graph [8, 12]. It is known that $S$ can be achieved by the following construction: 

\[ \text{Proposition 7.8. We have } \alpha(D_5^k) \leq (3 + o(1)) \log_{12} 8, \text{ which approaches } 3 \log_{12} 8 = 2.51 \ldots \text{ as } \alpha \to \infty. \]

This pseudo-exponent bound comes tantalizingly close to settling Question 4.1 if

\[ \liminf_{k \to \infty} \alpha(D_5^k) < 3 \log_{12} 8, \]

then the answer to Question 4.1 is “yes.”

Also, note that Lemma 3.2 implies that for all $G$, 

\[ \liminf_{k \to \infty} \alpha(G^k) = \inf_{k \geq 1} \alpha(G^k). \]

Thus, even if the answer to Question 4.1 is “no,” there are combinatorial consequences. For example, knowing that $\alpha(D_{2n}^k) \geq 3 \log_{8n-4} 4n$ for all $n$ and $k$ would give a new proof of the Sperner capacity bound $|S| \leq (m - 1)^k$ above, in the case of even $m$.

8. Concluding comments

8.1. Open questions

The most pressing question arising in this paper is Question 4.1 which represents a potential barrier to obtaining non-trivial bounds on $\omega$ using our techniques. However, there are numerous other open questions that are relevant to Question 5.1 and the ultimate goal of proving $\omega = 2$.

Matrix groups. As pointed out in Section 5, matrix groups seem to be one of the most promising families of examples, but we still know very little about them. Can our bounds for $\alpha(SL_2(\mathbb{F}_q))$ be improved? We see no reason why they should be optimal. Recall that beating 9/4 asymptotically would settle Question 4.1. We know even less about $SL_n(\mathbb{F}_q)$ (only that $\alpha(SL_n(\mathbb{F}_q)) < 3$), so any non-trivial construction would be of interest. The only other finite matrix groups that we have studied are those closely connected to $SL_n$ (such as $PSL_n$ or $GL_n$), but there are a number of other families. What can one say about the pseudo-exponents of the groups in these families?

Quotient sets. The examples in Subsection 7.4 show that quotient sets sometimes outperform subgroups. For which groups does this occur? Are there general constructions of useful quotient sets other than via Sperner capacity? Can they be used to improve our constructions for $S_n$ or the wreath product? What about matrix groups?

Lie groups. Can one use Lie groups to prove anything about $\omega$ directly? Do results on the Lie pseudo-exponent imply anything about the pseudo-exponents of related finite groups? Compact Lie groups seem more closely analogous to finite groups than non-compact Lie groups are, so studying them might be illuminating. (All of the Lie groups in this paper are non-compact.)

Group extensions. Extensions of groups with pseudo-exponent 3 can have substantially smaller pseudo-exponents, as demonstrated by the solvable groups in Subsection 7.4. (Recall that solvable groups are formed from
abelian groups by taking repeated extensions.) Is there a general way to lower \( \alpha \) or raise \( \gamma \) by taking extensions? As a first step, can one find a family of solvable groups with pseudo-exponents tending to 2?

**Powers of groups.** The simplest case of group extensions is taking powers of a group. Given \( G \), what can one say about the asymptotic pseudo-exponent \( \inf_{k \geq 1} \alpha (G^k) \) of \( G \)? As noted in Subsection 7.2, \( \gamma (G^k) = \gamma _{(G)} \), so if there exists a group such that \( \inf_{k \geq 1} \alpha (G^k) = 2 \), then \( \omega = 2 \) by Corollary 4.3.

### 8.2. Extensions

It is natural to attempt to extend our methods in various ways. For example, one might try to obtain bounds on border ranks of tensors, perhaps by using deformations of group algebras. It is also reasonable to ask whether our approach (given its reliance on representation theory) works in finite characteristic, as well as over \( \mathbb{C} \). As Theorem 2.3 indicates, one can just as easily embed matrix multiplication into \( F[G] \) rather than \( \mathbb{C}[G] \), where \( F \) has characteristic \( p \).

As long as \( p \) does not divide \( |G| \), the representation theory of \( G \), and all other aspects of our approach, work out identically, assuming \( F \) is algebraically closed. Schönhage has shown that the exponent of matrix multiplication over arbitrary fields depends only on the characteristic (see Corollary 15.18 in [4]), so we lose nothing by requiring that \( F \) be algebraically closed.

We conclude by mentioning a particular variant of our approach that does not require any control of the character degrees, and thus may still be viable even if there is a negative answer to Question 4.1. We have found less structure to the exponent of matrix multiplication with arbitrary fields depends only on the characteristic (see Corollary 15.18 in [4]), so we lose nothing by requiring that \( F \) be algebraically closed.

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