RELATIVE GARSIDE ELEMENTS OF ARTIN MONOIDS

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ABSTRACT. We introduce a relative Garside element, the quotient of the corresponding Garside elements \( \Delta(\Gamma_{n-1}) \) and \( \Delta(\Gamma_n) \), for a pair of Artin monoids associated to Coxeter graphs \( \Gamma_{n-1} \subset \Gamma_n \), the second graph containing a new vertex. These relative elements give a recurrence relation between Garside elements. As an application, we compute explicitly the Garside elements of Artin monoids corresponding to spherical Coxeter graphs or the longest elements of the associated finite Coxeter groups.

Key words: Coxeter groups, Artin monoids, Garside elements.

1. INTRODUCTION

The Garside element for the braid group (and for the braid monoid) corresponding to Artin’s presentation (see [1], [13])

\[
B_n = \langle x_1, x_2, \ldots, x_{n-1} \mid x_i x_j = x_j x_i \text{ if } |i - j| \geq 2, x_{i+1} x_i x_{i+1} = x_i x_{i+1} x_i \text{ if } 1 \leq i \leq n-2 \rangle
\]
is given by (see [10])

\[
\Delta_n = x_1(x_2 x_1)(x_3 x_2 x_1) \cdots (x_{n-1} x_{n-2} \cdots x_1)
\]
(to represent \( \Delta_n \), or, more general, to represent an element of a monoid as a product of generators, we chose the smallest word in the length-lexicographic order induced by the order between generators \( x_1 < x_2 < \ldots < x_{n-1} \)). For other representations of \( \Delta_n \), including the most used formula, see [3]. The right quotient \( \Delta_n^{-1} \Delta_{n+1} = x_n x_{n-1} \ldots x_1 \) will be called the relative Garside element corresponding to the embedding of the Coxeter graphs \( A_{n-1} \subset A_n \):

\[
A_{n-1} : x_1 \quad x_2 \quad x_3 \quad \ldots \quad x_{n-1} \subset A_n : x_1 \quad x_2 \quad x_3 \quad \ldots \quad x_{n-1} \quad x_n
\]

This element and its left divisors play a central role in the construction of the Gröbner basis for the classical braid monoid (see [2], [5]). In this paper we give few characterizations of the

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relative Garside element corresponding to an extension of Coxeter graphs by one new vertex, see Propositions 1-4, and we will use these to compute inductively the Garside elements of the Artin monoids of spherical type, see Corollaries 1-7.

We start to recollect some facts about Coxeter graphs, Coxeter and Artin groups, Artin monoids, and Garside elements. Let $S$ be a set. A Coxeter matrix over $S$ is a square matrix $M = (m_{st})_{s,t \in S}$ indexed by the elements of $S$ such that

$m_{ss} = 1$ for all $s \in S$ and $m_{st} = m_{ts} \in \{2, 3, 4, \ldots, \infty\}$ for all $s, t \in S, s \neq t$.

The associated Coxeter graph $\Gamma = \Gamma(M)$ is a labeled graph defined by the following data:

- $S$ is a set of vertices of $\Gamma$;
- two vertices $s, t \in S$ are joined by an edge if $m_{st} \geq 3$, with label $m_{st}$ if $m_{st} \geq 4$.

A Coxeter matrix $M = (m_{st})_{s,t \in S}$ is usually represented by its Coxeter graph $\Gamma(M)$.

**Definition 1.** Let $M = (m_{st})_{s,t \in S}$ be the Coxeter matrix of the Coxeter graph $\Gamma$. Then the group defined by

$$W(\Gamma) = \langle s \in S \mid (st)^{m_{st}} = 1 \text{ for all } s, t \in S \text{ satisfying } m_{st} \neq \infty \rangle$$

is called the Coxeter group of type $\Gamma$.

In an equivalent way we can write $W(\Gamma) = \langle s \in S \mid s^2 = 1, \underbrace{sts\ldots}_{m_{st} \text{ factors}} = \underbrace{tst\ldots}_{m_{st} \text{ factors}} \rangle$.

We call $\Gamma$ to be of spherical type if $W(\Gamma)$ is a finite group. A graph is of spherical type if and only if it has finitely many connected components, any of them from the next list (see [6]):

- $(A_n)_{n \geq 1}$:
  - $(B_n)_{n \geq 2}$:
  - $(D_n)_{n \geq 4}$:
  - $(E_n)_{n=6,7,8}$:
  - $F_4$:
  - $G_2$:
Definition 2. If $\Gamma$ is a Coxeter graph, its associated Artin group is defined by

$$A(\Gamma) = \langle s \in S \mid \underbrace{sts \ldots}_{m_{st} \text{ factors}} = \underbrace{tst \ldots}_{m_{st} \text{ factors}} \text{ for all } s, t \in S \text{ satisfying } m_{st} \neq \infty \rangle.$$

The set of positive elements (i.e. the elements which are product of generators with positive exponents) in an Artin group $A(\Gamma)$ is called the associated Artin monoid and it can be also defined by the monoid presentation (see [14])

$$M(\Gamma) = \langle s \in S \mid \underbrace{sts \ldots}_{m_{st} \text{ factors}} = \underbrace{tst \ldots}_{m_{st} \text{ factors}} \text{ for all } s, t \in S \text{ satisfying } m_{st} \neq \infty \rangle.$$

There are two obvious surjective morphisms:

$$M(\Gamma) \twoheadrightarrow A(\Gamma) \rightarrow W(\Gamma).$$

Now we recall some basic properties of the (absolute) Garside element of an Artin spherical monoid $M(\Gamma)$ (see [10], [7], [9], [12], [14], and also Section 2 for notation). This element is the least common left-multiple of the set of generators $x_1, \ldots, x_n$: we have, for any $i = 1, \ldots, n$, $x_i \mid_L \Delta$ and if $x_i \mid_L \omega$ for all $i$, then $\Delta \mid_L \omega$. The element $\Delta(\Gamma)$ is square free (there is no generator $x_i$ such that $x_i^2 \mid \Delta$). In some cases $\Delta(\Gamma)$ itself is a square: for example

$$\Delta(A_3) = x_1(x_2x_1)(x_3x_2x_1) = (x_1x_3x_2)^2;$$
$$\Delta(I_2(4k)) = (x_1x_2 \ldots x_2)^{2k \text{ times}}.$$

Also, there is a bijection $\sigma : \{1, \ldots, n\} \rightarrow \{1, \ldots, n\}$ such that $x_i\Delta = \Delta x_{\sigma(i)}$. The image of $\Delta(\Gamma)$ in the corresponding Coxeter group $W(\Gamma)$ is the (unique) longest element of this group and it has order two (see [6] and [8]). The length $l(\Gamma)$ of the Garside element $\Delta(\Gamma)$ is equal to the number of the reflections (i.e. the conjugates of the generators in the Coxeter group) and it is given by the following table (see [11] and [4]):

| $\Gamma$ | $A_n$ | $B_n$ | $D_n$ | $E_6$ | $E_7$ | $E_8$ | $F_4$ | $G_2$ | $H_3$ | $H_4$ | $I_2(p)$ |
|-----------|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|---------|
| $l(\Gamma)$ | $\binom{n+1}{2}$ | $n^2$ | $n^2 - n$ | 36 | 63 | 120 | 24 | 6 | 15 | 60 | $p$ |
2. Relative Garside Elements

Let us fix the notation. We already used the divisibility relation between two elements of a monoid $\mathcal{M}$, $\alpha \mid \beta$: this is equivalent to $\beta = \lambda \alpha \rho$ for some elements $\lambda, \rho \in \mathcal{M}$. If $\lambda = 1$, we write $\alpha \mid_L \beta$ and similarly, if $\rho = 1$, we write $\alpha \mid_R \beta$ and we say that $\alpha$ is a left and right divisor of $\beta$ respectively. The element $\alpha$ in a monoid $\mathcal{M}$ generated by $x_1, x_2, \ldots$ is said to be rigid if $\alpha$ can be represented in a unique way as a word in $x_1, x_2, \ldots$.

Suppose we have an inclusion of Coxeter graphs $\Gamma_{n-1} \subset \Gamma_n$ with vertices $\{x_1, \ldots, x_{n-1}\}$ and $\{x_1, \ldots, x_n\}$, respectively. Using the definition of Garside elements we have $\Delta(\Gamma_{n-1}) \mid_L \Delta(\Gamma_n)$.

**Definition 3.** The relative Garside element $\Delta(\Gamma_n, \Gamma_{n-1})$ is defined as a right quotient:

$$\Delta(\Gamma_n) = \Delta(\Gamma_{n-1}) \Delta(\Gamma_n, \Gamma_{n-1}).$$

If there is no ambiguity concerning the inclusion $\Gamma_{n-1} \subset \Gamma_n$, we will use the simple notation $\Delta_n = \Delta_{n-1}R_n$. Now we present some properties of the relative Garside element $R_n$ which characterize this element.

**Proposition 1.** The relative Garside element $R_n = \Delta(\Gamma_n, \Gamma_{n-1})$ satisfies the properties:

a) $R_n$ is square free;

b) $x_i \mid_L R_n$ if and only if $i = n$;

c) there is a bijection $\sigma : \{1, \ldots, n-1\} \to \{1, \ldots, n\} \setminus \{\sigma(n)\}$ such that $x_i R_n = R_n x_{\sigma(i)}$;

d) $x_j \mid_R R_n$ if and only if $j = \sigma(n)$.

**Proof.** Let $\Delta_n = \Delta_{n-1}R_n$. Then we have:

a) $R_n$ is square free because $\Delta_n$ is square free;

b) $x_i (1 \leq i \leq n-1)$ cannot be a left divisor of $R_n$: otherwise, $R_n = x_i U$. But $x_i \mid_R \Delta_{n-1}$ implies $\Delta_{n-1} = V x_i$, therefore we have $\Delta_n = \Delta_{n-1} R_n = V x_i^2 U$, a contradiction.

c) Let $\sigma_{n-1} : \{1, \ldots, n-1\} \to \{1, \ldots, n-1\}$ and $\sigma_n : \{1, \ldots, n\} \to \{1, \ldots, n\}$ be the bijections defined by the conjugation with $\Delta_{n-1}$ and $\Delta_n$, respectively: $x_i \Delta_{n-1} = \Delta_{n-1} x_{\sigma_{n-1}(i)}$, $x_i \Delta_n = \Delta_n x_{\sigma_n(i)}$. Now, for $i \in \{1, \ldots, n-1\}$, we have:

$$\Delta_{n-1} x_i R_n = x_{\sigma_{n-1}(i)} \Delta_{n-1} R_n = x_{\sigma_{n-1}(i)} \Delta_n = \Delta_n x_{\sigma_n \circ \sigma_{n-1}(i)} = \Delta_{n-1} R_n x_{\sigma_n \circ \sigma_{n-1}(i)}.$$
therefore \( x_i R_n = R_n x_{\sigma(i)} \), where \( \sigma = \sigma_n \circ \sigma_{n-1}^{-1} \). The image of \( \sigma \) contains all the elements \( 1, \ldots, n \), but not \( \sigma_n(n) = m \).

d) If \( x_j \neq x_m \), then there exists \( i \in \{1, \ldots, n-1\} \) such that \( j = \sigma(i) \) and \( x_i R_n = R_n x_j \). Suppose that \( x_j \mid R_n \), i.e. \( R_n = Sx_j \). We obtain a contradiction: \( x_i R_n = R_n x_j = Sx_j^2 \), because \( x_i R_n \) is a right divisor of \( \Delta_n \).

**Remark 1.** From this proposition, the first and the last factors of \( R_n \) are uniquely defined. In some cases \( R_n \) is completely rigid, for instance \( R_3(A_3, A_2) = x_3x_2x_1 \), in other cases only the interior factors can be changed, for instance \( R_3(A_3, A_1 \times A_1) = x_2x_1x_3x_2 = x_2x_3x_1x_2 \).

\[
\begin{align*}
A_2 : & \quad \bullet \quad \subset A_3 : \quad \bullet \quad \bullet \quad \bullet \quad x_1 \quad x_2 \quad x_3 \\
A_1 \times A_1 : & \quad \bullet \quad \subset A_3 : \quad \bullet \quad \bullet \quad \bullet \quad x_1 \quad x_2 \quad x_3
\end{align*}
\]

We will use the notation \( m = \sigma_n(n) \).

**Proposition 2.** If \( U_n \in \mathcal{M}(\Gamma_n) \) satisfies the next three conditions:

a) \( x_m^2 \nmid U_n \),

b) \( x_i \mid L U_n \) if and only if \( i = n \), and

c) there is a bijection \( \tau : \{1, \ldots, n-1\} \to \{1, \ldots, \hat{k}, \ldots, n\} \) such that \( x_i U_n = U_n x_{\tau(i)} \),

then \( U_n = R_n \) (and also \( k = m, \tau = \sigma \)).

**Proof.** Let us define \( D = \Delta_{n-1} U_n \). Because of the relations \( x_i \mid L \Delta_{n-1}, i = 1, \ldots, n-1 \) and also from \( x_n \mid L D \), a consequence of b) and c), we have \( x_i \mid L \Delta_{n-1} U_n \) for any \( i = 1, \ldots, n \), therefore \( \Delta_n \mid L D \). This implies that \( \Delta_{n-1} R_n \mid L \Delta_{n-1} U_n \), hence \( R_n \mid L U_n \). If \( R_n \neq U_n \), then \( U_n = R_n x_j W \) for some \( j \). If \( j \neq m \), then \( j = \sigma_n(i) \) for some \( i \in \{1, \ldots, n-1\} \), hence \( U_n = R_n x_{\sigma(i)} W = x_i R_n W \) and this contradicts b). If \( j = m \), then \( U_n = (Sx_m)x_m W \) which contradicts a).

**Proposition 3.** If \( U_n \in \mathcal{M}(\Gamma_n) \) satisfies the next three conditions:

a) \( x_n \mid L U_n \),

b) there is a bijection \( \tau : \{1, \ldots, n-1\} \to \{1, \ldots, \hat{k}, \ldots, n\} \) such that \( x_i U_n = U_n x_{\tau(i)} \), and

c) \( U_n \) has minimal length among the words satisfying a) and b),

then \( U_n = R_n \) (and also \( k = m, \tau = \sigma \)).
Proof. Let us define $D = \Delta_{n-1} U_n$. As in the previous proof, a) and b) implies $x_i \mid_L D$ for all $i = 1, \ldots, n$. Therefore $\Delta \mid_L D$, hence $R_n \mid_L U_n$. We have $|U_n| \geq |R_n|$ and $R_n$ satisfies a) and b), therefore condition c) implies $R_n = U_n$. □

With the same proof, we have another version of the previous proposition:

**Proposition 4.** If $U_n \in \mathcal{M}(\Gamma_n)$ satisfies the next three conditions:

a) $x_n \mid_L U_n$,

b) there is a bijection $\tau : \{1, \ldots, n-1\} \to \{1, \ldots, \hat{k}, \ldots, n\}$ such that $x_i U_n = U_n x_{\tau(i)}$, and

c) the length of $U_n$ is the expected length $l(\Gamma_n) - l(\Gamma_{n-1})$ (from the table in Section 1),

then $U_n = R_n$ (and also $k = m, \tau = \sigma$).

3. The Relative Garside and Garside Elements for the Infinite Series

After the computation of various relative Garside elements, we will describe the Garside elements associated to the classical list of the connected Coxeter diagrams: the infinite series in this section, the exceptional cases in the next section. Using the following obvious result, these will give formulae for the Garside elements of all Artin monoids of spherical type.

**Lemma 1.** If the graph $\Gamma$ is the disjoint union of $\Gamma_1$ and $\Gamma_2$, then $\mathcal{M}(\Gamma) \cong \mathcal{M}(\Gamma_1) \times \mathcal{M}(\Gamma_2)$ and

$\Delta(\Gamma) = \Delta(\Gamma_1)\Delta(\Gamma_2) = \Delta(\Gamma_2)\Delta(\Gamma_1)$.

**A\textsubscript{n} series.** We start with $A_1$: $\bullet x_1$ and its Garside element $\Delta(A_1) = \Delta_2 = x_1$.

Construction of $\Delta(A_n, A_{n-1})$: The element $M_n = x_n x_{n-1} \ldots x_1$ satisfies the conditions of Proposition[2]

a) $M_n$ is square free: obvious because $M_n$ is rigid;

b) $x_i \mid_L M_n$ if and only if $i = n$ for the same reason;

c) $x_i M_n = M_n x_{i+1}$, therefore $\sigma(i) = i + 1$, where $i = 1, \ldots, n - 1$, and $m = 1$.

**Corollary 1.** (classical Garside element)

$\Delta(A_n) = \Delta_{n+1} = x_1(\ldots)(x_{n-1} x_1)(x_n \ldots x_1)$.

**B\textsubscript{n} series.** We start with $B_2$: $\bullet 4 \xrightarrow{1} x_2$ and its Garside element $\Delta(B_2) = x_1(2x_1x_2)$, see the construction $\Delta(I_2(p), A_1)$. 


Construction of $\Delta(B_n, B_{n-1})$: The element $N_n = x_n x_{n-1} \ldots x_2 x_1 x_n$ satisfies the conditions of Proposition 2:

a) $N_n$ is square free: obvious, because $N_n$ is rigid in $\mathcal{M}(B_n)$;

b) $x_i \mid_L N_n$ if and only if $i = n$ for the same reason;

c) $x_i N_n = N_n x_i$, therefore $\sigma(i) = i$, where $i = 1, \ldots, n - 1$, and $m = n$.

**Corollary 2.**

\[
\Delta(B_n) = x_1 (x_2 x_1 x_2) (x_3 x_2 x_1 x_2 x_3) \ldots (x_n x_{n-1} \ldots x_2 x_1 x_n).
\]

**$D_n$ series.** We start with $D_3$: and its Garside element $\Delta(D_3) = x_1 (x_2 x_1) (x_3 x_1 x_2)$ (this is $\Delta(A_3, A_2) = x_3 x_2 x_1$ with a change of notation).

Construction of $\Delta(D_n, A_{n-1})$: Consider the product

\[
P_n = (x_n x_{n-2} x_{n-3} \ldots x_1)(x_{n-1} x_{n-2} \ldots x_2)(x_n x_{n-2} x_{n-3} \ldots x_3)(x_{n-1} x_{n-2} \ldots x_4) \ldots.
\]

We prove by induction that $P_n = R_n$. The initial step is given by the previous formula: $\Delta(D_3, A_2) = (x_3 x_1) x_2$. We check the conditions in Proposition 2 in the case of $n$ even, when the product ends with the generator $x_n$:

\[
P_n = (x_n x_{n-2} x_{n-3} \ldots x_1)(x_{n-1} x_{n-2} x_{n-3} \ldots x_2)(x_n x_{n-2} x_{n-3} \ldots x_3) \ldots (x_{n-1} x_{n-2}) x_n.
\]

a) First we prove that $P_n$ is square free and also that $x_n x_{n-1}$ is not a divisor of $P_n$.

\[
\begin{align*}
D_n : & \quad x_1 \quad x_2 \quad \ldots \quad x_{n-2} \quad x_n \quad x_{n-1} \\
A_{n-1} : & \quad y_1 \quad y_2 \quad \ldots \quad y_{n-2} \quad y_{n-1} \\
\downarrow & \quad \downarrow \\
P_n & \quad \Delta_{n-1}
\end{align*}
\]

Define a projection $pr : \mathcal{M}(D_n) \to \mathcal{M}(A_{n-1})$ given by the diagram: $pr(x_i) = y_{\min(i, n-1)}$.

The image of $P_n$ under the map $pr$ is $\Delta_{n-1}$ which is square free: if $x_j^2 \mid P_n$, then we have $pr(x_j^2) = y_h^2 \mid \Delta_{n-1}$ ($h = j$ if $j \leq n - 1$ and $h = n - 1$ if $j = n$), a contradiction. Therefore $P_n$ is square free. In the same way, if $x_n x_{n-1} \mid P_n$, then $pr(x_n x_{n-1}) = y_{n-1}^2 \mid \Delta_{n-1}$, a contradiction.

b) Obviously $x_n \mid_L P_n$. Let us suppose that $x_i \mid_L P_n$ for some $i \leq n - 1$. By induction we know that $R_{n-1}$ is given by the next formula and we will analyze the following three cases:
1) \( i = n - 1 \), 2) \( i = n - 2 \), and 3) \( i \leq n - 3 \):

\[
R_{n-1} = (x_{n-1}x_{n-3}x_{n-4} \ldots x_1)(x_{n-2}x_{n-3} \ldots x_2)(x_{n-1}x_{n-3}x_{n-4} \ldots x_3) \ldots x_{n-2}.
\]

Using the inclusion morphism \( in : \mathcal{M}(D_{n-1}) \to \mathcal{M}(D_n) \), \( z_1 \mapsto x_2 \), \( z_2 \mapsto x_3 \), \ldots , \( z_{n-3} \mapsto x_{n-2} \), \( z_{n-2} \mapsto x_n \), \( z_{n-1} \mapsto x_{n-1} \), given by the inclusion of graphs:

\[
\begin{align*}
D_{n-1} & : \quad z_1 \quad z_2 \ldots z_{n-3} \quad z_{n-1} \\
D_n & : \quad x_1 \quad x_2 \quad x_3 \ldots x_{n-2} \quad x_{n-1}
\end{align*}
\]

the image of

\[
R_{n-1} = (z_{n-1}z_{n-3}z_{n-4} \ldots z_1)(z_{n-2}z_{n-3} \ldots z_2)(z_{n-1}z_{n-3} \ldots z_3) \ldots (z_{n-1}z_{n-3})z_{n-2}
\]

is \( in(R_{n-1}) = R'_{n-1} \), given by

\[
R'_{n-1} = (x_{n-1}x_{n-2}x_{n-3} \ldots x_2)(x_{n}x_{n-2}x_{n-3} \ldots x_3)(x_{n-1}x_{n-2} \ldots x_4) \ldots (x_{n-1}x_{n-2})x_n
\]

and we have \( P_n = (x_nx_{n-2}x_{n-3} \ldots x_1)R'_{n-1} \). If \( \alpha_{i} \mid_{L} R'_{n-1} \), then \( i = n - 1 \) (by induction this is true for \( i \in \{2, 3, \ldots , n, n - 1\} \) and also \( \alpha_1 \mid_{L} R'_{n-1} \) is impossible because \( R'_{n-1} \) does not contain \( x_1 \)). In the case 1), \( \alpha_{n-1} \mid_{L} P_n \) implies \( x_{n-1} \alpha_1 = P_n \) (by Garside Lemma, see Section 5) and \( pr(P_n) \) contains \( y_{n-1}^2 \), a contradiction. In the case 2), \( \alpha_{n-2} \mid_{L} P_n \), and Garside Lemma implies that \( x_n x_{n-2} x_{n-3} = x_{n-3} x_{n-4} \ldots x_1 R_{n-1} \), hence \( \alpha_2 = x_{n-3} x_{n-4} \ldots x_1 R_{n-1} \) and \( x_n x_{n-3} \ldots x_1 \alpha_3 = x_{n-3} \ldots x_1 R_{n-1} \), and this gives a contradiction: \( x_\alpha_3 = R_{n-1} \). In the last case, 3), if \( \alpha_i \mid_{L} P_n (i = 1, \ldots , n - 3) \), then \( \alpha_{n} = x_{n} x_{n-2} \ldots x_{i+1}x_i \ldots x_1 R_{n-1} \), and also \( x_i x_{n-2} \ldots x_{i+2} \alpha_2 = x_{n-2} \ldots i+2 x_{i+1} x_i \ldots x_1 R_{n-1} \). We obtain \( x_i \beta_2 = x_{i+1}x_i \ldots x_1 R_{n-1} \), next \( x_{i+1} x_i \beta_3 = x_{i+1} x_i \beta_1 \beta_4 = x_{i} \ldots x_1 R_{n-1} \), and also \( x_{i+1} x_i \beta_1 = x_{n} x_{n-2} \ldots x_{i+1} x_i \ldots x_1 R_{n-1} \); this gives another contradiction: \( x_{i+1} \mid_{L} R_{n-1} \).

c) We have \( x_{i} P_n = P_n x_{n-i} \), hence \( \sigma(i) = n - i \). Therefore \( P_n = R_n \).

Similarly one can check the conditions a), b), c) of Proposition 2 for \( n \) odd:

\[
R_n = (x_n x_{n-2} x_{n-3} \ldots x_1)(x_{n-1} x_{n-2} \ldots x_2)(x_n x_{n-2} x_{n-3} \ldots x_3) \ldots (x_n x_{n-2})x_{n-1}.
\]
Corollary 3.
\[ \Delta(D_n) = x_1(x_2x_1) \cdots (x_{n-1} \cdots x_1)(x_nx_{n-2}x_{n-3} \cdots x_1)(x_{n-1}x_{n-2} \cdots x_2)(x_nx_{n-2} \cdots x_3) \cdots \]

I\(_2(p)\) series, B\(_2\) and G\(_2\). Construction of \(\Delta(I_2(p), A_1)\): Let us define \(Q_2(p) = x_2x_1x_2x_1x_2 \cdots\), \((p - 1)\text{ factors}\). This element satisfies the conditions in Proposition[3]

- a) clearly \(x_2 \mid L Q_2(p)\);
- b) we have \(x_1Q_2(2p + 1) = Q_2(2p + 1)x_2 (m = 1)\) and \(x_1Q_2(2p) = Q_2(2p)x_1 (m = 2)\);
- c) any relation \(x_1V = Vx_{\sigma(1)}\) should involve the unique defining relation \(x_2^2x_1x_2 \cdots (p \text{ factors}) = x_1x_2x_1 \cdots (p \text{ factors})\), so the length of \(V\) is greater than or equal to \(p - 1\) and \(Q_2(p)\) has minimal length among the words satisfying a) and b).

Corollary 4.
\[
\Delta(G_2) = x_1x_2x_1x_2x_1x_2 , \quad \Delta(I_2(p)) = x_1x_2x_1x_2 \cdots (p \text{ factors}).
\]

4. The relative Garside elements for the exceptional series

As a consequence of the results in Section 3 we obtained a new proof for the lengths of the Garside elements corresponding to the infinite series \(A_\infty, B_\infty, D_\infty\) and \(I(\ast)\) (including \(G_2\)). In this section we will use the lengths of the Garside elements to find the relative Garside elements and also the Garside elements of the monoids corresponding to the exceptional Coxeter graphs \(E_6, E_7, E_8, F_4, H_3\) and \(H_4\).

**F\(_4\) case.** We consider the inclusion:

\[
B_3: \begin{array}{ccc}
\cdot & 4 & \cdot \\
x_1 & x_2 & x_3 \\
\end{array} \subset \begin{array}{cccc}
\cdot & 4 & \cdot & \cdot \\
x_1 & x_2 & x_3 & x_4 \\
\end{array} \quad (F_4:)
\]

Construction of \(\Delta(F_4, B_3)\): Let us define \(T_3 = x_3x_2x_1x_3x_2x_3\); the Garside element of \(B_3\) (with the change of marking \(x_1 \leftrightarrow x_3\)) is \(\Delta(B_3) = x_1x_2x_1T_3\). Defining \(R_4 = x_4T_3x_4T_3x_4\), we have \(x_iR_4 = R_4x_i, i = 1, 2, 3,\) and \(l(R_4) = 15\); from the table in Section 1 we have \(l(F_4) - l(B_3) = 24 - 9 = 15\), hence, using Proposition[4] we obtain
\[
\Delta(F_4, B_3) = x_4T_3x_4T_3x_4.
\]

Corollary 5.
\[
\Delta(F_4) = x_1x_2x_1(x_3x_2x_1x_3x_2x_3)x_4(x_3x_2x_1x_3x_2x_3)x_4(x_3x_2x_1x_3x_2x_3)x_4.
\]
**H\textsubscript{n=3,4} series.** We consider the inclusions:

- \( I_2(5) : \begin{array}{ccc} x_1 & x_2 & H_3 : \end{array} \begin{array}{ccc} x_1 & x_2 & x_3 \end{array} \subset H_4 : \begin{array}{ccc} x_1 & x_2 & x_3 & x_4 \end{array} \)

Construction of \( \Delta(H_3, I_2(5)) \) and \( \Delta(H_4, H_3) \): The element

\[
S_3 = (x_3x_2x_1x_2x_1)(x_3x_2x_1x_2)x_3
\]

satisfies the commutation rules \( x_1S_3 = S_3x_2 \), \( x_2S_3 = S_3x_1 \), and its length is 10. From the length table we find that \( l(\Delta(H_3, I_2(5))) = 15 - 5 = 10 \), so

\[ \Delta(H_3, I_2(5)) = S_3. \]

Similarly, the element

\[
S_4 = x_4x_3x_4x_3x_4x_3x_4x_3x_4
\]

verifies \( x_iS_4 = S_4x_i, i = 1, 2, 3 \), and it has the expected length: \( l(S_4) = 45 = 60 - 15 = l(H_4) - l(H_3) \), therefore

\[ \Delta(H_4, H_3) = S_4. \]

**Corollary 6.**

- \( \Delta(H_3) = x_1x_2x_1x_2x_1S_3 = x_1x_2x_1x_2x_1 \cdot (x_3x_2x_1x_2x_1)(x_3x_2x_1x_2)x_3 \)
- \( \Delta(H_4) = x_1x_2x_1x_2x_1S_4S_4 = x_1x_2x_1x_2x_1S_3x_4S_3x_4S_3x_4S_3x_4S_3x_4 \)

**E\textsubscript{n=6,7,8} series.** We consider the inclusions:

- \( D_5 : \begin{array}{ccc} x_1 & x_2 & x_3 & x_5 \end{array} \subset \bigcap \begin{array}{ccc} x_4 \end{array} \)
- \( E_8 : \begin{array}{ccc} x_1 & x_2 & x_3 & x_5 \end{array} \subset \bigcup \begin{array}{ccc} x_4 \end{array} \)
- \( E_6 : \begin{array}{ccc} x_1 & x_2 & x_3 & x_5 \end{array} \subset \bigcap \begin{array}{ccc} x_4 \end{array} \)
- \( E_7 : \begin{array}{ccc} x_1 & x_2 & x_3 \end{array} \subset \bigcup \begin{array}{ccc} x_4 \end{array} \)

Construction of \( \Delta(E_6, D_5), \Delta(E_7, E_6), \) and \( \Delta(E_8, E_7) \): Let us define the element

\[
V_6 = x_6\Delta(D_5, D_4)x_6x_5x_3x_2x_1 = (x_6x_5x_3x_2x_1x_4x_3x_2x_5x_3x_4)(x_6x_5x_3x_2x_1).
\]
This verifies the commutation relations:

\[ x_1V_6 = V_6x_6, x_2V_6 = V_6x_5, x_3V_6 = V_6x_3, x_4V_6 = V_6x_2 \text{ and } x_5V_6 = V_6x_4. \]

Define also the elements

\[ V_7 = x_7V_6(x_7x_6x_5x_3x_2x_4x_3x_5x_6)x_7 \text{ and } V_8 = x_8V_7x_8V_7x_8. \]

These elements verify the commutation relations:

\[ x_iV_7 = \begin{cases} V_7x_{7-i}, & i = 1, 2, 5, 6, \\ V_7x_i, & i = 3, 4, \end{cases} \quad \text{and} \quad x_iV_8 = V_8x_i, \quad i = 1, \ldots, 7. \]

Counting the lengths we obtain

\[ l(V_6) = 16 = 36 - 20 = l(E_6) - l(D_5), \]
\[ l(V_7) = 27 = 63 - 36 = l(E_7) - l(E_6), \]
\[ l(V_8) = 57 = 120 - 63 = l(E_8) - l(E_7). \]

From Proposition 4 we obtain the relative Garside elements:

\[ \Delta(E_6, D_5) = V_6, \Delta(E_7, E_6) = V_7, \Delta(E_8, E_7) = V_8. \]

**Corollary 7.**

\[ \Delta(E_6) = \Delta(D_5)V_6 = \Delta(A_4)\Delta(D_5, A_4)V_6, \]
\[ \Delta(E_7) = \Delta(E_6)V_7 = \Delta(A_4)\Delta(D_5, A_4)V_6V_7, \]
\[ \Delta(E_8) = \Delta(E_7)V_8 = \Delta(A_4)\Delta(D_5, A_4)V_6V_7V_8. \]

### 5. Garside Lemma and Few Computations

The next lemma was proved by Garside for the braid monoid (or \( A_n \) series), see [10], and generalized for an arbitrary Artin monoid by Brieskorn and Saito, see [7]:

**Lemma 2. (Garside Lemma)** Let \( W \) be an element in the Artin monoid \( M \) such that \( x_i \mid_L W \) and \( x_j \mid_L W \) (i ≠ j). Then there is an element \( Z \in M \) such that

\[ W = (x_i x_j x_i x_j \ldots)_{m_{ij} \text{ times}}Z = (x_j x_i x_j x_i \ldots)_{m_{ij} \text{ times}}Z. \]

Now we give the details for the proof of two commutation relations described in Section 4. First a short computation:

**Lemma 3.** In \( F_4 \) we have \( x_1R_4 = R_4x_1 \).
Proof. The factors which are transformed under Coxeter relations are written in bold characters:
\[
\begin{align*}
\mathbf{x}_1 \cdot x_4(x_3x_2x_1x_3x_2x_3)x_4(x_3x_2x_1x_3x_2x_3)x_4 &= \\
&= x_4(x_3\mathbf{x}_1x_2x_1x_3x_2x_3)x_4(x_3x_2x_1x_3x_2x_3)x_4 = \\
&= x_4(x_3x_2x_1\mathbf{x}_2x_3x_2x_3)x_4(x_3x_2x_1x_3x_2x_3)x_4 = \\
&= x_4(x_3x_2x_1x_3x_2x_3)x_4(\mathbf{x}_2x_3x_2x_3x_1x_3x_2)x_4 = \\
&= x_4(x_3x_2x_1x_3x_2x_3)x_4(x_3x_2x_3\mathbf{x}_2x_1x_3x_2)x_4 = \\
&= x_4(x_3x_2x_1x_3x_2x_3)x_4(x_3x_2x_3x_1x_2\mathbf{x}_1x_3)x_4 = \\
&= x_4(x_3x_2x_1x_3x_2x_3)x_4(x_3x_2x_1x_3x_2x_3)x_4 \cdot \mathbf{x}_1.
\end{align*}
\]
\[
\square
\]

And now a long computation:

Lemma 4. In $E_8$ we have $x_7V_8 = V_8x_7$.

Proof. In $E_8$ we have the following sequence of equalities:
\[
\begin{align*}
\alpha & \equiv x_3x_2x_4x_3x_2x_4 = x_3x_2\mathbf{x}_3x_4x_3x_2 = \mathbf{x}_2x_3x_2x_4x_3x_2 \equiv \\
&\equiv x_2x_3\mathbf{x}_2x_4x_3x_2 = x_2x_3x_4x_3x_2\mathbf{x}_3 = x_2\mathbf{x}_3x_4x_3x_2x_3 = \\
&= x_2x_4x_3x_2x_4x_3 \equiv \beta,
\end{align*}
\]
and from the equality $\alpha = \beta$ we get
\[
\begin{align*}
\gamma & \equiv x_3(x_3x_2x_4x_3x_5)(\mathbf{x}_3x_2x_1x_4x_3x_2) = x_3(x_3\mathbf{x}_5x_2x_4x_3x_5)(x_2x_1x_4x_3x_2) = \\
&= \mathbf{x}_3x_5(x_3x_2x_4x_3\mathbf{x}_5)(x_2x_1x_4x_3x_2) = x_3x_5(x_3x_2x_4x_3)(x_2x_1x_4\mathbf{x}_5x_3x_2) \equiv \\
&\equiv x_3x_5(x_3x_2x_4x_3)(\mathbf{x}_2x_1x_4x_3x_2) = x_3x_5(x_3x_2x_4x_3)(\mathbf{x}_2x_4\mathbf{x}_1x_5x_3x_2) \equiv \\
&\equiv x_3x_5\alpha\mathbf{x}_1x_5x_3x_2 = x_3x_5\beta\mathbf{x}_1x_5x_3x_2 \equiv x_3x_5(x_2x_4x_3x_2x_4x_3\mathbf{x}_1x_5x_3x_2) = \\
&= x_3\mathbf{x}_5(x_2x_4x_3x_2\mathbf{x}_1x_4x_3x_5x_3x_2) = x_3\mathbf{x}_5(x_2x_4x_3\mathbf{x}_5)(x_2x_1x_4x_3x_5x_2) = \\
&= x_3(x_2x_4\mathbf{x}_5x_3x_5)(x_2x_1x_4x_3x_5\mathbf{x}_2) = (x_3x_2x_4x_3x_5)(\mathbf{x}_3x_2x_1x_4x_3\mathbf{x}_2)x_5 \equiv \delta,
\end{align*}
\]
and also, from $\gamma = \delta$, we obtain
\( \eta \equiv x_6(x_5 x_3 x_2 x_4 x_3 x_5 x_6)(x_5 x_3 x_2 x_1 x_4 x_3 x_2 x_5 x_3 x_4) = \\
\equiv x_6(x_5 x_6)(x_3 x_2 x_4 x_3 x_5 x_6)(x_3 x_2 x_1 x_4 x_3 x_2 x_5 x_3 x_4) = \\
\equiv x_5 x_6(x_3 x_2 x_4 x_3 x_5)(x_3 x_2 x_1 x_4 x_3 x_2) x_6 x_5 x_3 x_4 = \\
\equiv x_5 x_6 x_6(x_3 x_2 x_4 x_3 x_5 x_6)(x_3 x_2 x_1 x_4 x_3 x_2 x_5 x_6 x_3 x_4) = \\
\equiv x_5 x_6 x_6(x_3 x_2 x_4 x_3 x_5 x_6)(x_3 x_2 x_1 x_4 x_3 x_2 x_5 x_6 x_3 x_4) = \\
\equiv (x_5 x_3 x_2 x_4 x_3 x_5 x_6)(x_5 x_3 x_2 x_1 x_4 x_3 x_2 x_5 x_3 x_4) x_6 \equiv \theta.

For the final step we use the next equality

\[
\lambda \equiv x_7(x_6 x_5 x_3 x_2 x_4 x_3 x_5 x_6 x_7) V_6 \equiv \\
\equiv x_7(x_6 x_5 x_3 x_2 x_4 x_3 x_5 x_6 x_7)(x_6 x_5 x_3 x_2 x_1 x_4 x_3 x_2 x_5 x_3 x_4)(x_6 x_5 x_3 x_2 x_1) = \\
\equiv x_7(x_6 x_5 x_3 x_2 x_4 x_3 x_5 x_6 x_7)(x_6 x_5 x_3 x_2 x_1 x_4 x_3 x_2 x_5 x_3 x_4)(x_6 x_5 x_3 x_2 x_1) = \\
\equiv x_6 x_7 x_6 x_5 x_3 x_2 x_1 = x_6 x_7 x_6 x_5 x_3 x_2 x_1 \equiv \\
\equiv x_6 x_7(x_5 x_3 x_2 x_4 x_3 x_5 x_6)(x_5 x_3 x_2 x_1 x_4 x_3 x_2 x_5 x_3 x_4)(x_6 x_7 x_6 x_5 x_3 x_2 x_1) = \\
\equiv (x_6 x_5 x_7 x_3 x_2 x_4 x_3 x_5 x_6 x_7)(x_5 x_3 x_2 x_1 x_4 x_3 x_2 x_5 x_3 x_4)(x_6 x_7 x_5 x_3 x_2 x_1) = \\
\equiv (x_6 x_5 x_3 x_2 x_4 x_3 x_5 x_6 x_7)(x_6 x_5 x_3 x_2 x_1 x_4 x_3 x_2 x_5 x_3 x_4)(x_6 x_5 x_3 x_2 x_1) x_7 \equiv \\
\equiv (x_6 x_5 x_3 x_2 x_4 x_3 x_5 x_6 x_7) V_6 x_7 \equiv \mu,
\]

and we find

\( x_7 V_8 \equiv x_7(x_8 V_7 x_8 V_7 x_8) \equiv \\
\equiv x_7(x_8 V_7 x_6 x_5 x_3 x_2 x_4 x_3 x_5 x_6)(x_7 x_8 V_6)(x_7 x_6 x_5 x_3 x_2 x_4 x_3 x_5 x_6) x_7 x_8 = \\
\equiv x_8 x_7 V_6(x_8 x_7 x_6 x_5 x_3 x_2 x_4 x_3 x_5 x_6)(x_8 x_7 V_6)(x_8 x_7 x_6 x_5 x_3 x_2 x_4 x_3 x_5 x_6) x_7 x_8 = \\
\equiv x_8 x_7 V_6 x_7 x_8(x_7 x_6 x_5 x_3 x_2 x_4 x_3 x_5 x_6)(x_7 V_6)(x_8 x_7 x_6 x_5 x_3 x_2 x_4 x_3 x_5 x_6) x_7 x_8 = \\
\equiv x_8 x_7 V_6 x_7 x_8 \lambda(x_8 x_7 x_6 x_5 x_3 x_2 x_4 x_3 x_5 x_6) x_7 x_8 = \\
\equiv x_8 x_7 V_6 x_7 x_8 \mu(x_8 x_7 x_6 x_5 x_3 x_2 x_4 x_3 x_5 x_6) x_7 x_8 \equiv \\
\equiv x_8 x_7 V_6 x_7 x_8(x_8 x_7 x_6 x_5 x_3 x_2 x_4 x_3 x_5 x_6) x_7 x_8 = \\
\equiv x_8 x_7 V_6(x_7 x_8 x_6 x_5 x_3 x_2 x_4 x_3 x_5 x_6) x_7 x_8 V_6(x_7 x_8 x_6 x_5 x_3 x_2 x_4 x_3 x_5 x_6) x_7 x_8 = \\
\equiv [x_8 x_7 V_6(x_7 x_8 x_6 x_5 x_3 x_2 x_4 x_3 x_5 x_6) x_7 x_8 V_6(x_7 x_8 x_6 x_5 x_3 x_2 x_4 x_3 x_5 x_6) x_7 x_8] x_7 \equiv \\
\equiv (x_8 V_7 x_8 V_7 x_8) x_7 \equiv V_8 x_7.\]

□
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