Remotely detecting the signal of a local decohering process in spin chains

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Abstract

We study the dynamics of a one dimensional quantum spin chain evolving from unentangled or entangled initial state. At a given instant of time a quantum dynamical process (ex. quantum measurement operation) is performed on a single spin at one end of the chain, decohering the system. Through the further unitary evolution, a signal propagates in the spin chain, which can be detected from a measurement on a different spin at later times. From the dynamical unitary evolution of the decohered state from the epoch time, it is possible to detect the occurrence of the quantum operation. The propagation of the signal for the dynamical process, and the speed of the signal are investigated for various spin models, viz. using the Ising, Heisenberg and transverse-XY dynamics.

Keywords: spin chains, decoherence, quantum dynamical process, signal propagation

(Some figures may appear in colour only in the online journal)

1. Introduction

Quantum information and communication aspects of quantum spin chains, which can be viewed as multipartite systems of qubits, have been investigated over the last few years, as a spin chain is a possible channel for quantum state transfer [1–3]. Quantum spin chains have been studied extensively as prototype condensed matter systems that exhibit quantum critical behaviour, novel spin states with a variety of spin ordering [4–6]. These systems are generally studied from the quantum dynamics viewpoint, i.e. the evolution of an initial quantum many-body state through the time-dependent Schroedinger equation [7], and from a statistical mechanics viewpoint, i.e. the various thermodynamic phases and transitions [4, 7].

The dynamics of spin chains have been investigated for magnon bound states and scattering [8, 9], spin current dynamics [10], relativistic density wave dynamics [11]. The evolution of quantum correlations after a quantum quench [12], the light-cone in entanglement spreading...
have been studied using the dynamics of model Hamiltonians. All these studies involve the unitary evolution using the Schroedinger equation of initial chosen states or after a quench, and subsequent redistribution of quantum correlations. Now, from the quantum information theory viewpoint, a many-qubit state can undergo various multi-qubit gate operations, both global and local unitaries, and will undergo redistribution of quantum correlations, e.g. entanglement among the various subsystems, but as a whole the multi-qubit system is treated as a closed system. However, a multi-qubit system that is used in any quantum communication protocol will become an open system, as various subparts can undergo non-unitary operations, viz. a general quantum dynamical process (QDP) or a quantum channel action. This is a common source of decoherence, which is a stumbling block for the faithful communication of quantum states.

In this paper, we consider a quantum spin chain that undergoes decoherence, from a QDP that occurs at a given spin. We examine the scenario of a QDP signal propagating through the spin chain, and detecting it from another spin a distance away, whether the QDP occurred or not. As the quantum dynamics is due to a Hamiltonian evolution from local interactions, the efficiency of detection falls as we move away from the site of QDP occurrence. Moreover, it is expected that more distant sites will take longer time to detect the occurrence, as the speed of communication of the fact that QDP occurred will be determined by various details, and similarly for the efficiency of detecting the QDP.

Intuitively we can say that, if the spins interact with their nearest neighbours, if any quantum dynamical process occurred at the boundary spin, its signal can only propagate with a finite speed through the chain. So, the observer will have to wait for some time before detecting the signal. But in the case of a long range interaction we expect a larger speed. The speed of propagation of correlations in spin chains for the case of unitary dynamics, and its dependence on the model parameters has been studied. Similarly in this case of non-unitary dynamics, it is expected that the speed will depend on several factors: the type of the interaction, strength of the interaction, the spatial range of interaction strength, the initial state of the system, the external magnetic field strength. However, the speed of detection may not have the same dependence on the model parameters, as we shall see below.

This paper is arranged as follows. In the next section we describe the general approach and a numerical algorithm for the dynamical evolution of the state, for any type of Hamiltonian that governs the dynamics. We set up a detector function for a simple QDP, specifically a known projective measurement process on a given spin. In section 3, we investigate the simplest form of the evolution—that is, Ising dynamics—and investigate the signal propagation and detection of QDP both analytically and numerically. We consider both entangled and non-entangled initial states. In sections 4 and 5, we analytically calculate the detector function for specific state for Heisenberg and transverse-XY models respectively, along with numerical results. In each of these cases, the dynamics can be investigated analytically for a spin chain, as these models admit of exact solutions for all many-spin eigenstates and eigenvalues, through Bethe ansatz and Jordan–Wigner transformation respectively. In these cases, as we shall see, there is a finite speed of signal propagation giving a definite possibility of detecting the QDP signal. Finally we summarise our results in section 6.

2. Signal from a QDP and its detection in a multi-party system

A multi-partite system can exhibit a variety of correlations among its many various parts. In a general multi-qubit pure state of the system, there can be pairwise quantum correlations between two parts A and B. Let us consider a situation where there is a QDP that
occurs on part A, which leads to decoherence. Now, the question is whether this fact that the QDP occurred on part A can be detected from spatially separated part B. The no communication or no signalling theorem \[19, 26–29\] says that it is impossible, notwithstanding the pre-existing quantum correlations, unless there is a further evolution of the state from the epoch of QDP occurrence to the epoch of detecting it from B.

To see this for a multi-qubit system, let us consider a linear chain of \(N\) spins, with an initial state \(\rho\). The details of the structure of the correlations are not important for arguing the no signalling theorem. Let the first qubit undergo a QDP, a general instantaneous evolution that includes decohering process. This amounts to a quantum channel action of the first qubit. The many-qubit state is instantaneously transformed into a state \(\tilde{\rho}\) through a quantum channel action \(N_1\) on the first qubit. In this evolution, the operation transforms the input state into an output state through the quantum channel action, \(N_1 \times I_2 \times \ldots I_N\). That is, only the first qubit undergoes the QDP, and the rest of the qubits are operated on by the identity operator. This QDP results in an output density matrix, viz.

\[
\rho \rightarrow \tilde{\rho} = \sum_i P_i \rho P_i^\dagger, \tag{1}
\]

where we have used the Kraus operators \(\{P_i\}\) for the quantum channel \[19, 20\], with the constraint that \(\sum_P P_i^\dagger P_i = 1\). Now, we would like to detect this QDP from a different qubit from a measurement on the \(n\)th qubit, which depends only on the reduced density matrix for the desired qubit. However, it is not possible to detect from any other qubit whether or not QDP occurred on the first qubit, due to the no communication theorem. This can easily be seen by comparing the two reduced density matrices for the desired qubit, without and with the QDP, given by

\[
\rho_n = \text{Tr}'\rho, \quad \tilde{\rho}_n = \text{Tr}'\tilde{\rho}, \tag{2}
\]

where the prime indicates a partial trace over all qubits other than the \(n\)th qubit. Any measurement done on the desired qubit will involve an operator \(A_n\) that depends on the spin operators for the \(n\)th qubit. We can see that the two reduced density matrices will give identical results for the expectation value,

\[
\langle A_n \rangle = \text{Tr}A_n\tilde{\rho} = \sum_i \text{Tr}A_nP_i\rho P_i^\dagger = \text{Tr}A_n\rho. \tag{3}
\]

In the last step, we have used the fact that \(P_i\) commutes with \(A_n\) as \(P_i\) depends only on spin operators of the first qubit, and the completeness relation \(\sum P_i^\dagger P_i = 1\). Thus, it is not possible to distinguish between the two reduced density matrices, implying that it is not possible to detect from other qubits whether the QDP occurred or not at the first qubit. This is because we have not considered the evolution of the state after the QDP occurs and before the detection. Below, we will include the further evolution of the state, which will enable the detection of QDP. We will also define a detector function to quantify the efficiency.

Let us consider the scenario of an initial state \(\rho(t = 0) = |\psi(0)\rangle\langle\psi(0)|\), a pure state, evolving through a Hamiltonian evolution, with given Hamiltonian \(H\) that includes pairwise spin interactions. We can write the state using the basis states \(|0\rangle\) and \(|1\rangle\), the eigenstates of \(\sigma^z\) operators of the individual qubits, we have

\[
|\psi(0)\rangle = \sum_{s_1, s_2, \ldots s_N} \psi_{s_1, s_2, \ldots s_N} |s_1, s_2, \ldots s_N\rangle, \tag{4}
\]

where the sum is over qubit states, taking \(s_i = 0, 1\) for the \(n\)th qubit. The different basis states are classified into odd and even magnon states. Even (odd) magnon states have even (odd)
number of qubit flips from the ferromagnetic state $|00..0\rangle$. For simplicity, in the beginning we consider initial states with even-only (or odd-only) magnon basis states with nonzero wave function amplitudes, i.e. only even (or only odd) number of qubits with $s_i = 0$. Later, we will discuss the case of mixing the even and odd sector states. The Hamiltonians for the unitary dynamics that we consider in the later sections, both the Heisenberg and transverse-XY models, conserve the evenness (or oddness) of the number of qubits in the state $|0\rangle$. The Hamiltonian unitary evolution of the initial states involves magnon excitations. We employ the periodic boundary conditions in most of the cases, in finding the excitation spectrum and the eigenfunctions analytically. For numerical calculations, we employ open boundary conditions.

Now, the initial state undergoes transformation through a sequence of operations as shown in figure 1. First, there is a continuous unitary evolution from $t = 0$ to $t = t_0^-$. At $t = t_0$, the state now is given by

$$\rho(t_0^-) = U_{0,0}^\dagger \rho(0) U_{0,0}.$$

In the second step, the system undergoes a QDP, a quantum channel action on the first qubit instantaneously. Let us consider a simple projective measurement done on the first qubit, in the eigenbasis of the operator $\hat{\sigma}_1 \cdot \hat{n}$. The Kraus operators for this QDP are given by, $P_0 = (1 + \hat{\sigma}_1 \cdot \hat{n})/2$, and $P_1 = (1 - \hat{\sigma}_1 \cdot \hat{n})/2$, corresponding to a measurement process (that measures $\hat{\sigma}_1 \cdot \hat{n}$) on the first qubit. Now, the resultant state, immediately after the QDP occurrence, is written as

$$\tilde{\rho}(t_0^+) = P_0 \rho(t_0^-) P_0^\dagger + P_1 \rho(t_0^-) P_1^\dagger.$$

In the third step, the state is further evolved to a time $t > t_0$. Now, the final state $\tilde{\rho}(t)$ is then given by $\tilde{\rho}(t) = U_{1,0} \tilde{\rho}(t_0^+) U_{1,0}^\dagger$, which expands to

$$\tilde{\rho}(t) = \frac{1}{2} \{ |\psi(t)\rangle \langle \psi(t)| + |\tilde{\psi}(t)\rangle \langle \tilde{\psi}(t)| \}.$$

The state is written in terms of two pure states, $|\psi(t)\rangle \equiv e^{-iHt}|\psi(0)\rangle$ (which carries the time evolution without reference to the QDP occurring at $t_0$) and $|\tilde{\psi}(t)\rangle \equiv e^{-iH(t-t_0)} \hat{\sigma}_1 \cdot \hat{n} |\psi(t_0)\rangle$ (which carries the effect of the QDP occurrence). The first term in the above is just $\rho(t)$, the state at time $t$ with no QDP occurring at the first site at time $t = t_0$. This would have been the evolved state—had the decohering QDP not occurred—with a smooth unitary evolution.
The second term has the information regarding the decohering process, through the time evolution from $t = t_0$ onwards. It should be emphasised here that the two states, $\hat{\rho}(t)$ and $\rho(t)$, with and without the local instantaneous QDP intervening in the unitary dynamics, will differ only slightly. This is due to the fact that only one among $N$ spins is affected by the QDP at the epoch time $t_0$, and the state evolves further with the unitary dynamics. Thus, these two states only differ to this extent, and both states are further evolved through the same Hamiltonian dynamics. The spread of correlations would also differ to that extent only. The model parameter dependence of the evolution of correlations is expected to be similar for both these states. Thus, we expect that the efficiency of detecting the QDP from a different qubit will have much weaker dependence on the model parameter. We shall see, in the following sections—using different model Hamiltonian dynamics—the possibility of detecting the QDP signal.

Now, it is possible to detect the QDP from a measurement on $n$th qubit, as the reduced density matrix $\tilde{\rho}_n = \text{Tr}_r \hat{\rho}(t)$ can be differentiated from the reduced density $\rho_n = \text{Tr}_r \rho(t)$. The reduced density matrix for the $n$th site, after tracing out other spins, is given in the diagonal basis of $\sigma^n$ by

$$\tilde{\rho}_n = \begin{bmatrix} (1 + \bar{\sigma}^n_z)/2 & \langle \tilde{\sigma}^n_+ \rangle \\ \langle \tilde{\sigma}^n_- \rangle & (1 - \bar{\sigma}^n_z)/2 \end{bmatrix}. \quad (8)$$

The off-diagonal matrix can be nonzero if the initial state has a mixture of both even- and odd-numbered magnon states and/or if the QDP mixes even and odd states (for example $\bar{n} \neq \bar{z}$ in the Kraus operators above). In all the cases we consider, the Hamiltonian dynamics does not mix even and odd magnon states. We will consider three different situations: (A) Initial state has either even-only or odd-only magnon states, QDP does not mix even and odd states ($\bar{n} = \bar{z}$)—most of the results presented in the following sections will be in this category, where the off-diagonal term above is zero; (B) Initial state has both even and odd magnon states (ex. $|000...00\rangle + |100...00\rangle$), QDP does not mix even and odd states; (C) Initial state has even-only magnon states, but the QDP mixes even and odd sectors (for example $\bar{n} = (\bar{x} + \bar{z})/\sqrt{2}$). In both cases (B) and (C), the reduced density matrix shown above will have nonzero off-diagonal matrix elements, thus the local magnetization $\langle \sigma^z_n \rangle$ alone cannot determine the local von Neumann entropy.

In the situation (A), the Hamiltonian dynamics and the QDP (with $\bar{n} = \bar{z}$) both preserve the evenness of the state, i.e. the operators commute with $(-)^{\sum_i \sigma^i_z}$. Thus, for an initial state having even-only or odd-only magnon states, the reduced density matrix will have only diagonal terms in the computational basis, which are completely characterised by $\langle \sigma^z_n \rangle$. This implies we can construct a QDP detector function using the expectation value of the diagonal spin operator. The local magnetization $\langle \sigma^z_n \rangle$, calculated from the state $\hat{\rho}$ that carries the effect of the QDP occurring at the first spin, will be different from that of $\langle \sigma^z_n \rangle$, calculated from the state $\rho$ which is evolved without the QDP, for times larger than the time required for the signal from the QDP location to propagate to the $n$th spin. Consider the detector function defined as,

$$F_n(t) = \langle \sigma^z_n \rangle_t - \langle \sigma^z_n \rangle_t. \quad (9)$$

The detector function defined above uses the contrast between the reduced density matrices from the two evolved states $\hat{\rho}(t)$ and $\rho(t)$. Since, we are only using the signal information from a single qubit, from the single-qubit reduced density matrices, this is the simplest detector function. There are other detector functions one can construct, for example the relative entropy of the two reduced density matrices, and using two-qubit reduced density matrices, or using even higher multi-qubit information. The above one, the simplest choice, will suffice to detect the QDP from a distant qubit, as we shall see in the next sections.
The detector function can be rewritten using equation (7), we get

\[ F_n(t) = \frac{1}{2} \langle \hat{n} \hat{\sigma}_1(t_0) \sigma_n^z(t) \hat{n} \hat{\sigma}_1(t_0) \rangle - \frac{1}{2} \langle \sigma_n^z(t) \rangle, \]  

(10)

where the time-dependent operator is given by \( \sigma_n^z(t) = e^{iHt} \sigma^z_n(0) e^{-iHt} \), and the expectation value is taken in the initial state \( |\psi(0)\rangle \). Now, the components of the time-evolved operator \( \hat{\sigma}_1(t_0) \) need not commute with \( \sigma_n^z(t) \) in general for \( t \neq t_0 \), unlike in the situation considered in equation (3), due to the dynamics. This implies that the two terms in the above expression need not be equal, making the detector function nonzero, thus making it possible to detect the QDP signal from other qubits. It has become possible to detect the occurrence of QDP, due to the dynamics of evolution of the state from \( t = t_0 \) when the QDP occurs on the first qubit, and at time \( t \) when another qubit is measured. The further the measured qubit from the first qubit, the waiting time (or the evolution time) required for detecting QDP is expected to increase, as we expect the QDP signal will take time to propagate to a far way qubit. As we shall see in the following sections, the detector function \( F_n(t) \) will become nonzero after a waiting time \( t^*_n \). The waiting time increases with \( n \), as further qubits will have to wait longer for the signal, propagating at a finite speed, to arrive. Thus, we can define a signalling speed from the waiting time dependence on the location of the measured qubit. The Schrödinger dynamics, which itself can induce and redistribute quantum correlations between the various qubits of the system, will thus influence the efficiency of the detection of QDP. The signalling speed will depend in general, as we shall see in the following sections, on the nature of interaction, the strength of interaction, the magnetic field, the distribution of quantum correlations present in the initial states. In most of the cases it is difficult to calculate \( F_n(t) \) analytically, and it has to be calculated numerically. We will discuss a few well-known models of spin–spin interactions, and a few simple initial states in the context of the QDP signal propagation and detection. In these cases, the detector function can be calculated analytically.

For the situations (B) and (C), where even and odd magnon sectors are mixed due to either the QDP (for \( \hat{n} \neq \hat{z} \)) or initial state being in a mixed sector, we need to investigate the dynamics of the off-diagonal matrix element of \( \tilde{\rho}_n \). In analogy with the above detector function, we define an off-diagonal detector function, as

\[ O_n(t) = \langle \tilde{\sigma}_n^+(t) \rangle - \langle \sigma_n^+(t) \rangle, \]  

(11)

where the two different terms refer to the two states, with and without the QDP occurrence. Now, we have two different time-dependent detector functions. The information carried by these two can be combined by looking at the von Neumann entropy of the two reduced density matrices. We can use the excess entropy generated to define a single detector function for these situations, we define

\[ D_n(t) = -\text{Tr} \tilde{\rho}_n \log \tilde{\rho}_n + \text{Tr} \rho_n \log \rho_n. \]  

(12)

We shall see in the following sections that all these detector functions show similar behaviour, becoming nonzero after a waiting time \( t^*_n \), after which the QDP signal propagates to \( n \)th qubit.

3. Ising dynamics

The formalism described in the previous section for the QDP signal detection (with \( \hat{n} = \hat{z} \) for simplicity) can be applied to one dimensional spin chains of various kinds of interaction. The simplest of these is obviously the Ising model with nearest neighbour interaction, where the Hamiltonian only depends on one kind of Pauli spin matrix. The Hamiltonian of a spin chain of \( N \) spins is given by
The time evolved operator $\sigma_z^i(t)$ is given by $\sigma_z^i(t) = e^{i\sigma^z_0(\sigma^z_{i-1}+\sigma^z_{i+1})}\sigma_z^i(0)e^{-i\sigma^z_0(\sigma^z_{i-1}+\sigma^z_{i+1})}$.

Hence, the operator $\sigma_z^i(t)$ depends only on the operators corresponding to the site indices $(n-1)$ and $(n+1)$. So the operator $\sigma_z^i(t)$ commutes with the operators $\sigma_z^n(t)$ for all $s$ greater than 2. In this case, from equation (10) we see that the two terms become equal, causing the detector function to vanish, and the dynamics becomes trivial. Figure 2(a) shows the plot of the time evolution of the detector function as a function of time for the second, third and fourth qubits. The detector function is nonzero only for the first neighbour of the site at which the QDP occurs, and displays a periodic behaviour due to the nature of the dynamics. This would imply that it is not possible to detect the signal from any other site than the second one as shown in the figure, and thus that the speed of the signal propagation cannot be defined. This is true for any initial state of the system irrespective of whether the initial state is entangled.
or not. These features can be attributed to the simplest dynamics that we are considering with only nearest neighbour interactions.

To see a non-trivial behaviour in the dynamics, we consider the Ising model with spatially long-ranged interactions among the spins. Here each of the spins interacts with all other spins, and the interaction strength follows an inverse power law determined by the parameter $\delta$. The Hamiltonian is given by

$$ H \equiv \sum_{<ilk>} \frac{J}{|i-k|^\delta} \sigma_i^+ \sigma_k^-, \quad (14) $$

where the sum is over all pairs. Such long-ranged interactions, which can be realized in ion traps by controlling the intensity and polarization of laser fields, can be used to study quantum phase transitions in quantum spins [25]. Now, the time evolved operator $\sigma_n^x(t)$ depends on the operators corresponding to all sites. The expression for $\sigma_n^x(t)$ is given as

$$ \sigma_n^x(t) = \prod_{k \neq n} e^{i\sigma^x_k t} e^{-i\sigma_n^z(0)} e^{-i\sigma_n^z(0) + i\sigma_n^x(0)} (A_n^x - B_n^x) \sigma_n^x(0) - 2A_n B_n \sigma_n^z(0). \quad (15) $$

Here we have introduced the operators $A_n$ and $B_n$, which are given as

$$ A_n = \text{Re} \prod_{k \neq n} \left[ \cos \left( \frac{J}{|n-k|^\delta} \right) + i\sigma^x_k \sin \left( \frac{J}{|n-k|^\delta} \right) \right], $$

$$ B_n = \text{Im} \prod_{k \neq n} \left[ \cos \left( \frac{J}{|n-k|^\delta} \right) + i\sigma^x_k \sin \left( \frac{J}{|n-k|^\delta} \right) \right]. \quad (16) $$

The expectation value of the operator $\sigma_n^x(t)$ can be calculated for the initial state $\rho(0) = |00\ldots0\rangle \langle 00\ldots0|$ as

$$ \langle \sigma_n^x(t) \rangle = \text{Tr} \left[ (A_n^x - B_n^x) \sigma_n^x(0) - 2A_n B_n \sigma_n^z(0) \right] \rho(0), $$

$$ = \prod_{k \neq n} \cos \left( \frac{2J}{|n-k|^\delta} \right). \quad (17) $$

However, in this case the analytical calculation for $F_n(t)$ is difficult, as it contains three time dependent operators. We use a computational method, as discussed below.

For this Hamiltonian the energy eigenstates are the direct product states of $\sigma^z$ eigenstates, $|s_1 s_2 \ldots s_N\rangle$, and the eigenvalues are given by

$$ E(s_1 s_2 \ldots s_N) = \sum_{(i,k)} \frac{J}{|i-k|^\delta} (-1)^{s_i + s_k}, $$

where $s_i$ is the eigenvalue of $\sigma_i^z$. Let $|\Psi(0)\rangle$ be the initial state of the system, which can be expanded in the eigenbasis $|a_i\rangle$ (which happens to be the eigenbasis of $\sigma^z$ in this case) of the Hamiltonian as $|\Psi(0)\rangle = \sum_i c_i |a_i\rangle$. The state at any later time is given by $|\Psi(t)\rangle = \sum_i c_i e^{-iE_i t} |a_i\rangle$. The density matrix at time $t = t_0$ becomes $\rho(t_0) = \sum_{ij} c_j^* c_i e^{i(E_i - E_j)t_0} |a_i\rangle \langle a_j|$. Just after the QDP (with $\tilde{n} = 2$) occurs at the first site the state becomes, at $t = t_0^+,$

$$ \tilde{\rho}(t_0^+) = \frac{1}{2} \sum_{ij} c_j^* c_i e^{i(E_j - E_i)t_0} \{ |a_i\rangle \langle a_j| + \sigma_1^z |a_i\rangle \langle a_j| \sigma_1^z \}. \quad (18) $$
Evolving the state from \( t_0 \) to \( t \), we get the state \( \hat{\rho}(t) \), which has two terms as shown in equation (7).

Now, we can numerically compute the inner product of the initial state \( |\Psi(0)\rangle \) with the eigenstates of the Hamiltonian to find the the state \( |\Psi(t)\rangle \), and hence the detector function \( F_m(t) \). We discuss below the results for specific initial states. For the initial state \( |00...0\rangle \) we have the detector function \( F_m(t) \) nonzero for all sites for any time \( t > t_0 \) as depicted in figure 2(b). So, the observer can detect the signal from any site without any delay. But this is not true for entangled states. Let us consider the state \( (|100..0\rangle + |010..0\rangle)/\sqrt{2} \), where the first spin is maximally entangled with the second one; in this case we see the detector function \( F_m(t) \) is zero only for the second site. The general argument is given as follows. If first site is maximally entangled with the \( m \)th site, the value of \( F_m(t) \) is zero for any time. This can be explained as follows.

For example, let the initial state be \( |\Psi(0)\rangle = (|010_m\rangle + |111_m\rangle) \otimes |00...0\rangle/\sqrt{2} \). Here the first and the \( m \)th site are maximally entangled. The terms like \( \langle a_j |\sigma^+_{i\sigma}|a_i\rangle \) will be nonzero only if the \( m \)th site configuration of the states \( |a_i\rangle \) and \( |a_j\rangle \) (see equation (16)) is different and the rest are the same. However, both the expressions of \( F_m(t) \) and \( \langle \sigma^+_{i\sigma}\rangle \), contain the coefficients \( c_i \) and \( c_j \), which are nothing but the inner product of eigenstates of the \( \sigma^+ \) basis with the initial state of the system. Either of these two states have odd number of \((|S^+, -\rangle)\) spins. Since the initial state has a definite parity, either \( c_i \) or \( c_j \) will be zero. This gives \( F_m(t) = 0 \) for all values of \( t \). A similar argument can also be given for the initial state \( |\Psi(0)\rangle = (|110_m\rangle + |011_m\rangle) \otimes |00...0\rangle/\sqrt{2} \). Figure 2(c) shows that \( F_3(t) \) is zero for the third site when the initial state is \( (|100...0\rangle + |0010...0\rangle)/\sqrt{2} \).

Extending the argument for GHZ like states we can say all the states are maximally entangled with each other. Hence the detector function \( F_m(t) = 0 \) for all values of \( m \) for all time.

We have seen that for nearest-neighbour interactions, only \( F_2(t) \) is nonzero and \( F_4(t) \) vanishes for other sites. In the case of long-ranged interactions, the first site interacts with all other sites with different interaction strengths determined by the Hamiltonian of the system. Now, the detector function \( F_n(t) \) is nonzero for all sites for time \( t > t_0 \). As the distance between the sites increases the interaction strength decreases by power law, as given in equation (12). The slope function \( (df/dt)|_{t=t_0} \), plotted against the site index in figure 2(d), exhibits this same trend. The speed of the signal propagation cannot be defined for the Ising dynamics because the signal does not propagate beyond the second site in the case of nearest-neighbour interactions, and the signal reaches all the sites without any delay for the case of long-ranged interactions.

4. Anisotropic Heisenberg model

One of the first exactly solvable but non-trivial models of quantum mechanics is a one dimensional chain of spins interacting with their nearest neighbour Heisenberg exchange interaction, known as the Heisenberg model [6]. Three kinds of Pauli spin matrices in the Hamiltonian indicates interactions in all three spin dimensions. The anisotropic Heisenberg model Hamiltonian for a spin chain of \( N \) spins is given by

\[
H = \sum_i J_i (\sigma^x_i \sigma^x_{i+1} + \sigma^y_i \sigma^y_{i+1}) + J_z \sigma^z_i \sigma^z_{i+1}.
\]

(19)

For the case of \( J_z = 0 \), this Hamiltonian can be mapped to a free fermion model. For \( J_z > 0 \), the model exhibits antiferromagnetic behaviour, and it is ferromagnetic for \( J_z < 0 \). The model exhibits a Kosterlitz–Thouless-type quantum critical point for the isotropic case of \( J_z = J \). Since the total \( z \)-component of the spin commutes with the Hamiltonian, the eigenstates of this
Hamiltonian have a definite number spins in state $|1\rangle$ (down spin). An eigenstate with $l$ down spins, an $l$-magnon state, can be written as

$$|\psi\rangle = \sum_{x_1, x_2, \ldots} \psi(x_1, x_2, \ldots) |x_1, x_2, \ldots\rangle,$$

where the basis state is labeled by the locations of the $l$ down spins. The eigenfunction $\psi(x_1, x_2, \ldots)$ is given by the Bethe ansatz [22], which will be labeled by the set of momenta $k_i$ of the down spins that are determined by solving $l$ algebraic Bethe ansatz equations, with periodic boundary conditions. The interaction strength $J$ determines the hopping of the down spins to nearby sites, whereas the interaction of the two down spins is determined by $J_z$. The one-magnon eigen energies are independent of $J_z$, as the states carry only one down spin. For $l > 1$, the eigenstates include both scattering states of one magnon, and of the down spins. It is straightforward to see the states $|00\ldots0\rangle$ and $|11\ldots1\rangle$ are eigenstates of the Hamiltonian, where $|0\rangle$, and $|1\rangle$ are eigenstates of $\sigma_z^i$. We have seen that if the initial state is this eigenstate of the Hamiltonian no dynamics will be observed. The dynamics of the magnon bound states using a quench have been studied [8]. We will investigate a linear combination of zero-magnon, one-magnon and two-magnon states that are not eigenstates of the Hamiltonian, using the QDP dynamics below. For the most part we consider $\hat{n} = \hat{z}$, and will consider the case of $n \neq \hat{z}$ towards the end of the section to illustrate the effect of mixing the even and odd sector states.

Let us consider a general initial state, to observe any non-trivial dynamics, given by

$$|\Phi(0)\rangle = \alpha|x\equiv \{x_1,x_2,x_l\} \rangle + \beta|y\equiv \{y_1,y_2,y_m\} \rangle,$$

where $x_i(y_i)$ denotes the location of the $i$th down spin. That is, this state is a superposition of the states with $l$ and $m$ number of down spins; $(x_1 < x_2 < \ldots)$ are the co-ordinates of the sites with down spins in the first part, and similarly for the second part. Such a state can be written as a linear combination of momentum eigenstates $|k\equiv \{k_1,k_2,k_l\} \rangle$, and $|q\equiv \{q_1,q_2,q_m\} \rangle$ of the Hamiltonian, with eigenvalues $\epsilon(k), \epsilon(q)$. Hence,

$$|\Phi(0)\rangle = \alpha \sum_{k_1,k_2} \psi_k^1|k\rangle + \beta \sum_{q_1,q_m} \psi_q^1|q\rangle,$$

where the wave functions denote $\psi_k^1 \equiv \langle k_1,k_2|x_1,x_l\rangle$, and $\psi_q^1 \equiv \langle q_1,q_m|y_1,y_m\rangle$. Now, the time evolution of the state is straightforward, the state after a time $t$ becomes,

$$|\Psi(t)\rangle = \alpha \sum_{k_1,k_2} G^i_k(t)|x_1,x_l\rangle + \beta \sum_{q_1,q_m} G^i_q(t)|y_1,y_m\rangle,$$

where the time-dependent function $G$ is given in terms of the wave functions defined above as

$$G^i_{k_1,k_2}(t) = \sum_{k_1,k_2} \psi_{k_1,k_2}^1 \psi^{*}_{k_1,k_2} e^{-i\epsilon(k_1,k_2)t}.$$

The expectation value of $\sigma_n^z$ for the $n$th site is then given by

$$\langle \sigma_n^z \rangle_t = 1 - 2|\alpha|^2 \sum_{x_1,x_2} |G_{x_1,x_2,x_l}^i|^2 - 2|\beta|^2 \sum_{y_1,y_m} |G_{y_1,y_m,y_m}^i|^2,$$

where the prime over the sums in the above indicates that there is one less free variable to be summed—that is, there are $l - 1$ ($m - 1$) variables $x_i(y_i)$ in the first (second) sum. Just after the system undergoes a QDP (with $n = \hat{z}$) on the first site the state at $t = t_0$ is given by,

$$\tilde{\rho}(t_0+) = \langle \tilde{\Phi}_+(t_0) \rangle \langle \tilde{\Phi}_+ (t_0) | + \langle \tilde{\Phi}_-(t_0) \rangle | \tilde{\Phi}_-(t_0) \rangle,$$
where, $|\tilde{\Phi}_1(t)\rangle \equiv \frac{1+i\epsilon}{2}|\psi(t)\rangle$, we have

$$
|\tilde{\Phi}_+(t)\rangle = \alpha \sum_{x' \neq 1.x', y'} G_{x'}^{x'}(t)|x'\rangle + \beta \sum_{y' \neq 1.y', y''} G_{y'}^{y'}(t)|y'\rangle,
$$

$$
|\tilde{\Phi}_-(t)\rangle = \alpha \sum_{x' = 1.x', y'} G_{x'}^{x'}(t)|x'\rangle + \beta \sum_{y' = 1.y', y''} G_{y'}^{y'}(t)|y'\rangle.
$$

(26)

Further evolution of the system for a time $(t-t_0)$ yields the state at time $t$,

$$
\tilde{\rho}(t) = |\tilde{\Phi}_+(t)\rangle\langle\tilde{\Phi}_+(t)| + |\tilde{\Phi}_-(t)\rangle\langle\tilde{\Phi}_-(t)|,
$$

(27)

where

$$
|\tilde{\Phi}_+(t)\rangle = \alpha \sum_{x', y'} H_{x'}^{x'}(t, t_0)|x'\rangle + \beta \sum_{y'} H_{y'}^{y'}(t, t_0)|y'\rangle,
$$

$$
|\tilde{\Phi}_-(t)\rangle = \alpha \sum_{x', y'} K_{x'}^{x'}(t, t_0)|x'\rangle + \beta \sum_{y'} K_{y'}^{y'}(t, t_0)|y'\rangle.
$$

(28)

Here, the new time-dependent wave functions are given by

$$
H_{x'}^{x'}(t, t_0) = \sum_{x' \neq 1.x', y'} G_{x'}^{x'}(t_0)G_{x'}^{x'}(t-t_0),
$$

$$
K_{x'}^{x'}(t, t_0) = \sum_{x' = 1.x', y'} G_{x'}^{x'}(t_0)G_{x'}^{x'}(t-t_0).
$$

(29)

Using the above functions, a simpler form for the QDP detector function for the $n$th qubit is given as

$$
F_n(t) = 2|\alpha|^2 \sum_{x'} |G_{x_{1.2}}^{x_{1.2}.-n. x_{1..}}|^2 - |H_{x_{1.2}}^{x_{1.2}.-n. x_{1..}}|^2 - |K_{x_{1.2}}^{x_{1.2}.-n. x_{1..}}|^2
$$

$$
+ 2|\beta|^2 \sum_{y'} |G_{y_{1.2}}^{y_{1.2}.-n. y_{1..}}|^2 - |H_{y_{1.2}}^{y_{1.2}.-n. y_{1..}}|^2 - |K_{y_{1.2}}^{y_{1.2}.-n. y_{1..}}|^2.
$$

(30)

In the above, the sums are over the variable sets $x'$ and $y'$ that include the position $n$ as one of the elements, as explicitly shown in the superscripts of the time-dependent functions $G$, $H$, and $K$.

Let us first consider the state $(|100...0\rangle + |010...0\rangle)/\sqrt{2}$. Since this is a one magnon state ($l = m = 1$) equation (19) simplifies to

$$
F_n(t) = \sum_{x=1.2} (|G_{x}^{x}(t)|^2 - |H_{x}^{x}(t, t_0)|^2 - |K_{x}^{x}(t, t_0)|^2).
$$

(31)

For this case, the one-magnon eigenfunction is given by, $\psi(x) = e^{i\epsilon \cdot x}/\sqrt{N}$, and the eigenvalue is given by $\epsilon_k = 4J \cos k$, apart from a constant that depends on $J'$, the diagonal interaction term in the Hamiltonian [22]. We can use the time scaled by $\hbar/4J$, the natural time unit from the spin exchange interaction. In the macroscopic limit, $N \rightarrow \infty$, the sum over the momentum in equation (13) can be converted into an integral; thus, we get

$$
G_{x}^{x}(t) = \frac{1}{2\pi} \int_{0}^{2\pi} e^{-i \epsilon \cdot k \cdot (x-n)} dk = J_{x-n}(t)i^{x-n}.
$$

(32)

Now, using this we can determine the other two time-dependent functions that are needed in $F_n(t)$; we have
\[
H_n^1 = i^{1-n}(J_{1-n}(t) - J_0(t_0)J_{1-n}(t - t_0)), \quad H_n^2 = i^{2-n}J_{2-n}(t - t_0), \\
K_n^1 = i^{1-n}J_0(t_0)J_{1-n}(t - t_0), \quad K_n^2 = i^{2-n}J_1(t_0)J_{1-n}(t - t_0). 
\]

Figure 3(b) shows the detector function \(F_n(t)\) as a function of time for initial state \((|000...0\rangle + |110...0\rangle)/\sqrt{2}\) for the isotropic Heisenberg model with interaction strength \(J = 1\); (b) for the one-magnon initial state \((|010...0\rangle + |100...0\rangle)/\sqrt{2}\) for the isotropic Heisenberg model with interaction strength \(J = 1.0\) with periodic boundary conditions; (c) \(t^*/t_0\) is plotted with \(n\) for different values of \(J\) for the even-magnon initial state \((|000...0\rangle + |110...0\rangle)/\sqrt{2}\), where \(t^*\) is the time when \(|F_n(t)|\) just exceeds a small positive number \(\epsilon\) (taken to be \(10^{-5}\)) for the first time. The curves attain roughly a constant slope for different values of \(J\). The speed of propagation is determined by the inverse of the slope. (d) The speed is plotted against \(J_z/J\) for the anisotropic model, where the reference speed \(v_0 = 10|J|a/\hbar\) in terms of \(a\), the nearest-neighbour separation of the qubits.

\[
H_n^1 = i^{1-n}(J_{1-n}(t) - J_0(t_0)J_{1-n}(t - t_0)), \quad H_n^2 = i^{2-n}J_{2-n}(t - t_0), \\
K_n^1 = i^{1-n}J_0(t_0)J_{1-n}(t - t_0), \quad K_n^2 = i^{2-n}J_1(t_0)J_{1-n}(t - t_0). 
\]
\[ v_0 = a/l_0 = 10|J|a/h \] (as we have used \( J_0/h = 0.1 \) for the numerical plots) in terms of the coupling strength and the separation between the qubits. Using a typical parameter value set pertaining to spin chains, \( J = 10 \text{ meV} \) and \( a = 0.1 \text{ nM} \), we get an estimate of \( v_0 \sim 10^4 \text{ m s}^{-1} \). We can see from figure 3(d) that the speed of propagation \( v \) is of the same order as \( v_0 \).

We now consider the initial state \( |00\ldots0\rangle + |110\ldots0\rangle/\sqrt{2} \), which is a linear combination of the two-magnon eigenstates with \( l = 0, m = 2 \). The first part of the state, with \( l = 0 \), is an eigenstate of the Hamiltonian, which does not change in time till \( t_0 \). The quantum channel action on the first site does not alter the state either. Thus it does not contribute to the dynamics as a whole. The second part of the state, with \( m = 2 \) a two-magnon state, does have very complicated dynamics, with both two one-magnon scattering states and two-magnon bound states contributing to the dynamics. The eigenfunctions and eigenvalues can be obtained from Bethe ansatz [22]—however, the time-dependent functions are quite difficult to calculate. The detector function is given by,

\[
F_n(t) = \sum_{\rho_i} |G^{\rho_i}_n(t)|^2 - |H^{\rho_i}_n(t, 0)|^2 - |K^{\rho_i}_n(t, 0)|^2,
\]

where the sum is over all site indices except \( n \), and the superscript in all the functions is an ordered set. Unlike the previous case of one-magnon states, here both bound and scattering states are possible. In the case of an infinitely long chain it can be shown that the quasi-momenta become a continuous set of numbers from 0 to \( 2\pi \) and \( (-\infty + i/2) \) to \( (-\infty + i/2) \) for scattering and bound states respectively. However, any closed form of \( F_n(t) \) is difficult to find. In this case we have calculated the quantity \( F_n(t) \) by calculating the time-dependent functions \( G, H, \) and \( K \) numerically. It is seen that the result exactly matches with the result from exact numerical diagonalization. Figures 3(a) and (b) show \( F(t) \) for the two initial states \( (|00\ldots0\rangle + |010\ldots0\rangle)/\sqrt{2} \) and \( (|00\ldots0\rangle + |110\ldots0\rangle)/\sqrt{2} \) respectively.

Figure 3(a) shows the detector function \( F(t) \) as a function of time for initial state \( (|00\ldots0\rangle + |110\ldots0\rangle)/\sqrt{2} \). It is clear from the figure that the speed does not depend on the initial states or entanglement of the initial state, though the dynamics varies. We expect that the speed should increase with the interaction strength parameter \( J \) in the case of the isotropic Heisenberg model. Figure 3(c) shows that the time \( t^* \) decreases as the value of \( J \) increases for a fixed site. If we define the speed as \( v = \Delta t^*/\Delta t \) it can be shown that the speed is linearly proportional to \( J \). In the case of the anisotropic Heisenberg model, if \( |J_z| \gg |J| \) the dynamics become trivial, and no speed will be observed. So we expect maximum speed when the parameter \( J_z \) is zero, which is borne out in figure 3(d). It can be seen from the figure that the speed of detection itself has a small variation with the parameter \( J_z/J \), varying in a range of about ten per cent from the maximum value. This is, as argued in section 2, due to the fact that QDP has been affected only on one spin at an epoch time for the state \( \rho(t) \), and the subsequent unitary evolution is the same as the evolution of the state \( \rho(t) \) with no QDP. Both these states would differ only to this extent. We need to study the contrast in the spreading of quantum correlations between the two states, which involves higher marginals, ex. two-qubit reduced density matrices that carry information about pairwise entanglement. The method developed in this section can be used significantly in studying the effect of multi-party correlations and mutual information.

Let us consider the initial state as \( |00..0\rangle \) or \( |11..1\rangle \). These states are the eigenstates of the Hamiltonian with the same eigenvalue. We can see from equation (17) that \( \dot{\rho}(t) = \rho(0) \). This implies no dynamics will be seen for these states. Next, we consider the GHZ like state \( (|00\ldots0\rangle + |11\ldots1\rangle)/\sqrt{2} \), where each of the sites is globally entangled with all other sites.
In this case \( \langle \hat{\sigma}^z_n \rangle = \text{Tr}(\hat{\sigma}^z_n (t) \rho(0)) = 1 \). Also from equation (17) it can be shown that 
\( \langle \hat{\sigma}^x_n \rangle = \text{Tr}(\hat{\sigma}^x_n (t) \rho(0)) = 1 \). Hence the detector function will be zero for this state.

Now, we consider a QDP with \( \tilde{n} = \cos \theta \xi + \sin \theta \tilde{x} \) and an initial state \( |\psi(0)\rangle = |000..0\rangle + |110..0\rangle/\sqrt{2} \). As argued in section 2, the reduced density matrix \( \tilde{\rho}_n \) (see equation (8)) will have a nonzero off-diagonal matrix element. The Hamiltonian unitary dynamics does not mix even and odd magnon states, but the QDP explicitly mixes them. The zero magnon state \( |000..0\rangle \) does not evolve under the unitary evolution, but it becomes a mixture of zero and one magnon states after the QDP occurs. Similarly, the two magnon state will yield a mixture of one, two and three magnon states. Unlike in the previous examples, we have four different magnon sectors contributing for the state \( \tilde{\rho}(t) \), with both even and odd sectors. We need to investigate detector functions \( F_n(t) \) and \( O_n(t) \) corresponding to the diagonal and off-diagonal matrix elements, as discussed in section 2. These detector functions have more complicated expressions in this case, as there are four different magnon sectors contributing to the dynamics for \( t > t_0 \).

The state \( \tilde{\rho}(t_0) \) just after the QDP occurs is composed, as shown in equation (26), of two pure states, but the two pure states in this case are defined as \( |\Phi_{\pm}(t_0)\rangle \equiv 1/2 |\tilde{\rho}_n, |1\rangle + \tilde{\rho}_n, |0\rangle \rangle |\psi(t_0)\rangle \). The operator \( \sigma^x_i \) will create one and three magnon states from zero and two magnon components present in the initial state. Further evolving the state to a time, the two pure states are given (analogously to equation (29)) by

\[
|\Phi_{\pm}(t)\rangle = e^{-iE_\sigma \theta/2} |000..0\rangle \pm \frac{\sin \theta}{2} \sum_{x_i} \left[ e^{-iE_\sigma \theta/2} G_{1,2}^{x_1}(t-t_0) + L_{1,2}^{x_1}(t, t_0) \right] |x_i\rangle \\
\pm \sum_{x_i, x'_i} \left[ \cos \theta/2 H_{1,2}^{x_i, x'_i}(t-t_0) + \frac{1}{2} \pm \sin \theta/2 K_{1,2}^{x_i, x'_i}(t, t_0) \right] |x_i, x'_i\rangle \\
\pm \sin \theta/2 \sum_{x_i, x'_i, x''_i} L_{1,2}^{x_i, x'_i, x''_i}(t, t_0) |x_i, x'_i, x''_i\rangle.
\]

We have defined new propagator functions connecting the even and odd sectors, defined in terms of the previously defined \( G \) functions, as

\[
L_{1,2}^{x_1}(t, t_0) = \sum_{x_i} G_{1,2}^{x_i}(t_0) G_{1,2}^{x_1}(t-t_0),
\]

\[
L_{1,2}^{x_1, x'_1}(t, t_0) = \sum_{x_i} G_{1,2}^{x_i, x'_i}(t, t_0) G_{1,2}^{x_1, x'_1}(t, t_0).
\]

Now, when we calculate the \( \langle \hat{\sigma}^z(t) \rangle \), each term in the pure state will contribute independently as the magnon states are eigenstates for \( \sigma^z_n \), but only the terms with \( L \) functions will contribute to the off-diagonal matrix element \( \langle \hat{\sigma}^z \rangle \). The expressions for these expectation values are very long, but straightforward to evaluate. We have computed them numerically, and the results for \( F_n(t) \) and \( \text{Re}O_n(t) \) are shown in figures 4(a) and (b) respectively, with representative values of the coupling strengths for \( n = 2, 4, 8 \). The excess of the local entropy for the \( n \)th qubit can be calculated from the eigenvalues for the reduced density matrix given by \( \lambda_{\pm} = 1 \pm \sqrt{(\langle \hat{\sigma}^z \rangle)^2 - (\langle \hat{\sigma}^x \rangle)(\langle \hat{\sigma}^y \rangle)} \). Similarly, for the state \( \rho(t) \) with no QDP occurrence, we can calculate the local entropy. We have plotted the local entropy detector function \( D_n(t) \) in figure 4(c) for \( n = 2, 4, 8 \). As can be seen from these figures, in this case all the detector
functions shown have similar behaviour to that of even-only magnon state cases with diagonal reduced density matrix. That is, the detector functions become nonzero after a waiting time, with the waiting time itself increasing with the distance of the measured qubit from the QDP occurrence site. The waiting time \( t^* \) is slightly different for the three different detector functions. But, we have seen that in this case also, the speed of the signal propagation is the same as calculated in the previous case, as the speed depends on the differential increase with \( n \).

5. **XY model with transverse magnetic field**

Another way of getting non-trivial dynamics for the spin chain is to replace the bilinear Ising term with coupling strength \( J_z \) in the last section by a linear transverse magnetic field term. Thus, we will have a quantum spin chain of \( N \) spins interacting with their nearest neighbours in the XY plane, along with an external magnetic field in transverse direction. The Hamiltonian for this case is given by

\[
H = \sum_i (J_x \sigma_i^x \sigma_{i+1}^x + J_y \sigma_i^y \sigma_{i+1}^y) + h \sum_i \sigma_i^z. \tag{38}
\]

This Hamiltonian can be exactly diagonalised and entire eigenvalue spectrum can be found for periodic boundary by employing Jordan–Wigner transformation [5] of spin 1/2 operators to spinless fermionic operators. The ground state exhibits a quantum critical behaviour, for the
isotropic case of $J_x = J_y$ for all values of the magnetic field strength, and for the anisotropic case for $h = J_x + J_y$. In addition to the unitary dynamics of the above Hamiltonian, we will have a QDP ($\hat{n} = 2$ for simplicity) occurring at the first spin at an epoch time $t_0$ like in the previous sections.

It is easy to see that the three different terms in the above Hamiltonian do not commute with each other. The dynamics is similar to the Ising dynamics we have discussed before, when only one of the coupling constants is nonzero. For example, let us take the limit $J_x = J_y = 0$. Then the operator $\sigma^z_i(0)$ does not evolve with time for all sites. This implies $F(t) = 0$ for all sites independent of initial state. So, the detection of QDP is not possible. The case of $J_y = h = 0$, or $J_x = h = 0$, is similar to the discussion of section 2. However, for at least two of three parameters $J_x, J_y$ and $h$ nonzero the two terms in the Hamiltonian do not commute and non-trivial dynamics can be observed. In general, it is expected that the speed should depend on the ratios $J_y/J_x$ and $h/J_x$.

We map spin-1/2 operators in the Hamiltonian to Fermionic creation and annihilation operators by means of Jordan–Wigner transformation [5]. The mapping is given by

$$\sigma^+_i = c_i^\dagger e^{i\sum_{m=1}^{i-1} c_m^c c_m}. \quad (39)$$

The Hamiltonian will have a bilinear form in terms of Fermionic creation and annihilation operators, which can be be brought to a diagonal form by doing a Fourier transformation, followed by a Bogoliubov transformation [4]. Fourier transforming the operators into momentum space, we define

$$c_q = \frac{1}{\sqrt{N}} \sum e^{-iq\ell} c_\ell. \quad (40)$$

Here, the set of allowed momentum values are given by is $q = 2\pi m/N$, with $m = -(N-1)/2..-1/2,1/2..(N-1)/2$ for odd $N$; and $m = -N/2..0..N/2$ for even $N$. In terms of these momentum-space operators the Hamiltonian has a bilinear form with non-diagonal operators $c_q^c c_{-q}^\dagger$ and similar terms.

To diagonalise the Hamiltonian we employ Bogoliubov–Valatin transformation in which new Fermionic creation and annihilation operators are formed as a linear combination of old operators, given as

$$\eta_1 q = u_q c_q - iv_q c_{-q}^\dagger, \quad \eta_2 q = -iv_q c_q + u_q c_{-q}^\dagger. \quad (41)$$

The expansion coefficients and the eigenvalues are given by

$$u_q = \sqrt{\frac{1}{2} + \frac{(J_x + J_y) \cos q + h}{|\omega_q|}}, \quad v_q = \sqrt{1 - u_q^2}, \quad \omega_q = 2\sqrt{|(J_x + J_y) \cos q + h|^2 + [(J_x + J_y) \sin q]^2}. \quad (42)$$

In terms of these new fermion operators, the Hamiltonian is diagonal; we have

$$H = \sum_{0 < q < \pi} |\omega_q| (\eta_1 q \eta_1^\dagger q - \eta_2 q \eta_2^\dagger q). \quad (44)$$

Now, to calculate the QDP detector function for any initial state $|\Psi_0\rangle$, we need to find the time-evolved state for $t > t_0$, or equivalently find the time-evolved operators. The detector function is given by,
\[ F_n(t) = \langle \Psi_0 | \sigma_n^x(t) - \sigma_n^z(t) | \Psi_0 \rangle = -\text{Tr}(\hat{\rho}(t) - \rho(t)). \]

Here, in the first equation we need the difference of the time-evolved operator \( \sigma_n^x \) with the QDP occurring at the first site, and the time-evolved operator \( \sigma_n^z \) without the QDP occurring. 

In the second equation, we need the time-evolved states \( \hat{\rho}(t) \) and \( \rho(t) \) with and without QDP occurring respectively. It is easier to calculate the time-evolved operators first, through the first equation, and then find expectation values in various initial states. However, for some initial states, it is easier to calculate the time-evolved operators first, through the second equation, and then find expectation values in various initial states. 

For some simple initial states, the detector function can be calculated from the time-evolved state. The expectation values of products of fermion operators can be straightforwardly evaluated using analog of Wick’s theorem. However, there will be three sums over the momentum variables that can be carried out numerically.

For some simple initial states, the detector function can be calculated from the time-evolved state, as we outline below. Let us consider a simple initial state \( |00...00\rangle \) with no entanglement. This state can evolve into a complicated state with multipartite entanglement structure [30] through time evolution. We can rewrite the initial state as 

\[ |\Psi(0)\rangle = \prod_{q>0} |0_q\rangle |0_{-q}\rangle = \prod_{q>0} (v_q\eta_q^1 - u_q\eta_q^2)|\text{vac}\rangle, \]

where the vacuum state for the \( \eta \) operators is \(|\text{vac}\rangle = i|0_q\rangle |1_{-q}\rangle\), as a linear combination of the eigenstates of the Hamiltonian. The new basis Hamiltonian being diagonal, each \( q \) mode has independent time evolution. The state after a time \( t \) becomes 

\[ |\Psi(t)\rangle = \prod_{q>0} (v_qe^{i\omega_qt}\eta_q^1 - u_qe^{-i\omega_qt}\eta_q^2)|\text{vac}\rangle \equiv \prod_{q>0} |\phi(t)\rangle_q, \]

where we have defined \(|\phi(t)\rangle_q\), a time-evolved linear combination of eigenstates for each momentum \( q \). Just after the QDP occurs at the first site the state is given by equation (44). Here the Kraus operators \( P_0 \) and \( P_1 \) are written in terms of the new set of fermionic creation and annihilation operators formed via Bogoliubov transformation: 

\[ P_0 = \frac{1}{\sqrt{N}} \sum_{q_1,q_2} c_{q_1}^\dagger c_{q_2} e^{i(q_1-q_2)} = \frac{1}{\sqrt{N}} \sum_{q_1,q_2} (u_{q_1}\eta_{q_1}^1 + v_{q_1}\eta_{q_1}^2)(u_{q_2}\eta_{q_2}^1 + v_{q_2}\eta_{q_2}^2)e^{i(q_1-q_2)} \]

for \( q > 0 \); whereas for \( q < 0 \) it is given by 

\[ P_0 = \frac{1}{\sqrt{N}} \sum_{q_1,q_2} (v_{q_1}\eta_{q_1}^1 - u_{q_1}\eta_{q_1}^2)(v_{q_2}\eta_{q_2}^1 - u_{q_2}\eta_{q_2}^2)e^{i(q_1-q_2)}, \]

and 

\[ P_1 = (1 - P_0). \]

The state \( \hat{\rho}(t_0+) \), just after the QDP, will involve two pure states, and is given by 

\[ \hat{\rho}(t_0+) = |\hat{\Psi}_1(t_0)\rangle \langle \hat{\Psi}_1(t_0)| + |\hat{\Psi}_2(t_0)\rangle \langle \hat{\Psi}_2(t_0)|. \]
Here,

\[ |\tilde{\Psi}_1(t_0)\rangle = \frac{1}{N} \sum_{q_1, q_2} |\phi_+(t_0)\rangle_{q_1} \frac{1}{2^q} |\phi_-(t_0)\rangle_{q_2} \prod_{q \neq q_1, q_2} |\phi(t_0)\rangle_q + \frac{1}{N} \sum_{q_1} |\phi_1(t_0)\rangle_{q_1} \prod_{q \neq q_1} |\phi(t_0)\rangle_q, \]

\[ |\tilde{\Psi}_2(t_0)\rangle = |\tilde{\Psi}(t_0)\rangle - |\Psi_1(t_0)\rangle, \quad (50) \]

where we have used three time-evolved states for a momentum value,

\[ |\phi_+(t_0)\rangle_q = c_1 e^{iqt} |\phi(t_0)\rangle_q, \quad |\phi_-(t_0)\rangle_q = c_q e^{-iqt} |\phi(t_0)\rangle_q, \quad |\phi_1(t_0)\rangle_q = c^{\dagger}_q \phi(t_0)\rangle_q. \quad (51) \]

Evolving further the state for a time \( t > t_0 \) we get,

\[ \tilde{\rho}(t) = |\tilde{\Psi}_1(t)\rangle \langle \tilde{\Psi}_1(t) | + |\tilde{\Psi}_2(t)\rangle \langle \tilde{\Psi}_2(t) |, \quad (52) \]

where the two time-evolved pure states are given as,

\[ |\tilde{\Psi}_1(t)\rangle = \frac{1}{n} \sum_{q_1, q_2} |\phi_+(t)\rangle_{q_1} \frac{1}{2^q} |\phi_-(t)\rangle_{q_2} \prod_{q \neq q_1, q_2} |\phi(t)\rangle_q + \frac{1}{n} \sum_{q_1} |\phi_1(t)\rangle_{q_1} \prod_{q \neq q_1} |\phi(t)\rangle_q, \]

\[ |\tilde{\Psi}_2(t)\rangle = |\tilde{\Psi}(t)\rangle - |\tilde{\Psi}_1(t)\rangle. \quad (53) \]

In the above we have introduced two more time-evolved states for a momentum value, given by

\[ |\tilde{\phi}(t)\rangle_q = (u_q e^{i\omega t} \eta^\dagger_q + v_q e^{-i\omega t} \eta^\dagger_q) |\text{vac}\rangle, \quad |\tilde{\phi}_1(t)\rangle_q = (u_q e^{i\omega t} \eta^\dagger_q + v_q e^{-i\omega t} \eta^\dagger_q) |\text{vac}\rangle. \quad (54) \]

The time evolved state at a time \( t \) are given in equations (49) and (45), with and without the QDP occurring. We can calculate the detector function, using these states, we have

\[ F_n(t) = \frac{4}{N} \text{Re} \left\{ \sum_{q_1, q_2} e^{i(q_1 - q_2) \omega t}|\tilde{\Psi}_1(t)| \langle c_{q_1}^\dagger c_{q_2} |\tilde{\Psi}_1(t)\rangle - |\tilde{\Psi}_1(t)| \langle c_{q_1}^\dagger c_{q_2} |\tilde{\Psi}_2(t)\rangle \right\} + 2|\tilde{\Psi}_1(t)| |\tilde{\Psi}_2(t)\rangle. \quad (55) \]

Now, the expectation value \( \langle \tilde{\Psi}_1(t)|c_{q_1}^\dagger c_{q_2}|\tilde{\Psi}_1(t)\rangle \) is straightforward to evaluate, and similarly the other matrix elements. However, the detector function has a long expression with two sums over the momentum variables, that can be calculated numerically. In this case, we have calculated the quantity \( F_n(t) \) by calculating the sums over \( q_1 \) and \( q_2 \) numerically, and it has been seen that the result matches with the exact diagonalisation results. Figure 5(a) shows \( F(t) \) for the initial state \( |00..0\rangle \). However, for a general entangled state the calculation of the detector function is quite difficult.

We have seen that in the case of the XY model with transverse magnetic field at least two of the three parameters \( J_x, J_y, h \) should be nonzero to obtain non-trivial dynamics. It has been seen that the speed of the signal depends on these three parameters. When the magnitude of the magnetic field is zero we expect maximum speed when \( J_x = J_y \); which is also a quantum anisotropy phase transition point between two ferromagnetic phases. Numerical calculations confirm this. Additionally, when magnetic field is increased the speed also increases, but cannot be increased beyond a certain value of the field. The figure 5(b) shows the dependence of the speed on these three parameters, where the reference speed is \( v_0 = 10|J_x| a / \hbar \), analogously to the earlier discussion; the speed of the signal is of the order \( 10^4 \) m s\(^{-1} \). As compared to the previous case of Heisenberg dynamics that conserves the \( z \)-component of the total spin, in the present case of non-conserving dynamics, we see a stronger dependence of the detection...
The speed on the parameters $h/J_x$ and $J_y/J_x$. This is expected, as we have more parameters and non-conservation. In this case we have studied only a simple initial state analytically, as initial states with entangled pairs of spins are quite difficult to handle for this model Hamiltonian. We take a numerical approach for studying other initial states, as discussed below.

Figure 5(c) shows $F(t)$ as a function of time for the initial state $(|000\ldots0\rangle + |110\ldots0\rangle)/\sqrt{2}$. The actual dynamics is different from that of the direct product state considered in figures 5(b) and (d) show the variation of speed with the magnetic field strength for the two initial states $(|000\ldots0\rangle + |110\ldots0\rangle)/\sqrt{2}$ and $|000\ldots0\rangle$ when $J_y = 0$. Though the two states differ in terms of the initial entanglement (between the first two spins), since the dynamics will generate a variety of entanglement distributions, the speed of the QDP signal is expected to be the same. The signal speed reaches an optimal value as a function of the magnetic field strength, and after that it decreases slowly to zero for large fields. In the limit of a large magnetic field, the interaction terms in the Hamiltonian (equation (38)) are insignificant, and can be ignored. Since, the qubits evolve independently in this case, the signal speed goes to zero, making it impossible to detect the signal.

Figure 5. For the XY dynamics with a transverse magnetic field with open boundary conditions, and the QDP with $\hat{n} = \hat{z}$: (a) $F_d(t)$ is plotted with $t/t_0$ for the initial state $|000\ldots0\rangle$ with interaction strengths $J_x = 0.7$, $J_y = 0.3$ and $h = 1.0$. (b) Density plot of the speed of propagation for various values of $J_x/J_y$ and $h/J_x$, for the initial state $(|000\ldots0\rangle + |110\ldots0\rangle)/\sqrt{2}$. The speed is maximum for the quantum critical case of $J_x = J_y$ independent of $h$. (c) The detector function $F_d(t)$ is plotted with time ($t/t_0$) for the initial state $(|000\ldots0\rangle + |110\ldots0\rangle)/\sqrt{2}$, with interaction strengths $J_x = 0.7$, $J_y = 0.3$ and $h = 1.0$. (d) The speed has been plotted with $h/J_x$ for the two initial states $(|000\ldots0\rangle + |110\ldots0\rangle)/\sqrt{2}$ and $|000\ldots0\rangle$. Here, the reference speed is $v_0 = 10|J_x|a/\hbar$ in terms of $a$, the nearest-neighbour separation between the qubits.
In all the cases studied till now, we have used integrable systems for the Hamiltonian dynamics for the unitary evolution of the state. In the case of non-integrable systems, the behaviour of the wait times could be quite different. To this end, we consider the case of applying a longitudinal field in addition to the transverse field for the XY dynamics that we have studied above. The presence of both the transverse field \( h \) (along the \( z \) direction as shown in equation (38)) and a longitudinal field \( h' \) (along the \( x \) direction that couples to \( \Sigma \sigma^x_i \)), will render the system non-integrable, which has been studied in the context of quantum chaotic behaviour [31]. The system is no longer exactly solvable using the Jordan–Wigner transformation, unlike the case of \( h' = 0 \) we studied above, thus eliminating a analytical tool to analyse the dynamics and getting the equivalent of equations (53) and (55). We can, however, proceed numerically for getting the energy spectrum and the eigenfunctions that are needed to compute the time evolved state, and the detector function. We have plotted the results for the detector function in figure 6, for the initial state \( |000...0\rangle \) with interaction strengths \( J_x = 0.7, J_y = 0.3 \) and the transverse field \( h = 1.0 \), and a longitudinal field \( h' = 1.0 \) for a finite chain with ten qubits. It can be seen that the detector function for different qubits does not show different waiting times, unlike the integrable case shown in figure 5 (with \( h' = 0 \)).

Figure 6. Non-integrable Hamiltonian dynamics: XY model with transverse and longitudinal fields with open boundary conditions. \( F_n(t) \) is plotted with \( t/t_0 \) for the initial state \( |000...0\rangle \) with interaction strengths \( J_x = 0.7, J_y = 0.3 \) and the transverse field \( h = 1.0 \), and a longitudinal field \( h' = 1.0 \) for a finite chain with ten qubits. It can be seen that the detector function for different qubits does not show different waiting times, unlike the integrable case shown in figure 5 (with \( h' = 0 \)).

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6. Conclusions

We investigated a model of decoherence in an interacting spin chain resulting from a Quantum dynamical process on a single site. We have discussed the possibility of detection of the QDP signal from a different site, for different initial states and different Hamiltonian dynamics of the system. In this paper we have presented analytical calculations along with numerical results for specific initial states, viz. polarized direct product state, Hamiltonian eigenstates, pairwise entangled state, globally entangled GHZ state etc.

For the Ising dynamics, we have shown that the effect of QDP cannot propagate through the chain beyond the second site for nearest neighbour interactions. For the case of long-ranged interactions, the effect reaches all the sites instantaneously; which can be intuitively understood. Thus, it is not possible to define a speed of signal propagation for this case. In the case of entangled initial states, the QDP signal cannot be detected from a site which is entangled with the first site initially.

We have shown that it is possible for the QDP signal to reach further sites, within near-neighbour interacting models for the Heisenberg and XY models. In the case of the Heisenberg model the speed is directly proportional to the interaction strength between the spins, which is as expected. Also, for the anisotropic case the speed is maximum when the anisotropy parameter $J_z = 0$. For the anisotropic XY model with the magnetic field in transverse direction, it is seen that the speed depends on both the ratios $J_y/J_x$ and $hJ_x$. Maximum speed is observed for the quantum critical case of $J_y = J_x$ for any value of $h$. However, the speed is not maximum for the quantum critical case of $J_y = 0$ and $h = J_x$. In these cases, any entanglement present in the initial state does not change the QDP signal propagation or the detection.

The dynamics in general depends on the initial state and the nature of interaction between the spins. The speed of the signal depends only the parameters of the Hamiltonian not on the initial state. The speed is similar for an unentangled or entangled state. However, if the initial state is an eigenstate of the Hamiltonian and the measurement operator, it is not possible to detect the signal. We have also seen that for non-integrable systems we cannot define a signal speed. Our method is based on a simplistic model of QDP, where the environment interacts (ex. quantum measurement) with the system instantaneously through only one spin, and that gives rise to decoherence in a many body spin system. However, there can be other complicated models for the same. We have only discussed the case of one dimensional spin chains; the dynamics for higher dimensional cases can be more complicated.

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