Integrable Models of Horizons and Cosmologies

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Abstract

A new class of integrable theories of 0+1 and 1+1 dimensional dilaton gravity coupled to any number of scalar fields is introduced. These models are reducible to systems of independent Liouville equations whose solutions must satisfy the energy and momentum constraints. The constraints are solved thus giving the explicit analytic solution of the theory in terms of arbitrary chiral fields. In particular, these integrable theories describe spherically symmetric black holes and branes of higher dimensional supergravity theories as well as superstring motivated cosmological models.

1 Introduction

In the last decade 1+1 and 0+1 dimensional dilaton gravity coupled to scalar matter fields proved to be a reliable model for higher dimensional black holes and string inspired cosmologies. The connection between high and low dimensions has been demonstrated in different contexts of gravity and string theory - symmetry reduction, compactification, holographic principle, AdS/CFT correspondence, duality. For spherically symmetric configurations the description of static black holes, branes, and of cosmological solutions even simplifies to 0+1 dimensional dilaton gravity - matter models, which in many interesting cases are explicitly analytically integrable (see e.g. [1] - [12] and references therein).

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However, generally they are not integrable. For example, spherical black holes coupled to Abelian gauge fields are usually described by integrable 0+1 dimensional models, while the addition of the cosmological constant term destroys integrability. In 1+1 dimension, pure dilaton gravity is integrable but the coupling to scalar matter fields usually destroys integrability. The one very well studied exception is the CGHS model. In [6] a more general integrable model of dilaton gravity coupled to matter, which incorporates as limiting cases the CGHS and other known integrable models was proposed. It reduces to two Liouville equations, whose solutions should satisfy two constraints. Because the general analytic solution of the constraints had been not found at that time, the model of Ref.[6] received little attention and was not studied in detail\(^1\).

Recently, the author has proposed a class of more general integrable dilaton gravity models in dimension 1+1, which are reducible to \(N\) Liouville equations (a brief summary is published in Ref.[14]). For these models the general analytic solution of the constraints has been found. We demonstrated that the \(N\)-Liouville models are closely related to physically interesting solutions of higher dimensional supergravity theories describing the low energy limit of superstring theories. These 1+1 dimensional \(N\)-Liouville theories describe the solutions of higher dimensional theories in some approximation. On the other hand, their reduction to dimensions 1+0 (cosmological) and 0+1 (static black holes) give the exact solution of higher dimensional theories.

Static black holes and cosmological models are described by one dimensional solutions of the 1+1 dimensional theories. In the standard approach the deep connection between black holes and cosmologies is not transparent and is usually ignored (even the precise relation between the dimensional reductions used by ‘cosmologists’ and by ‘black holes investigators’ is not quite obvious). We thus start from the 1+1 dimensional formulation to get a unified description of these two objects. A characteristic feature of the static solutions of the models derived from string theory is the existence of horizons with nontrivial scalar field distributions (what must be characteristic features of string cosmologies is as yet a much discussed problem).

It is well known that in the Einstein - Maxwell theories minimally coupled

\(^1\)Some applications of this model were discussed in [13]. Note also that some recent cosmological models use potentials similar to those introduced in [6]. Such applications will be considered in a separate paper.
to scalar fields the spherical static horizons disappear if the scalar fields have a nontrivial space distribution (this is the so-called ‘no-hair theorem’). In Ref. [6], a local version of the no-hair theorem (we call it the ‘no horizon’ theorem) was formulated and proved. It states that, under certain conditions, there exists no static solution with a horizon in a class of 0+1 dimensional dilaton gravity theories coupled to scalar matter (the important requirement is that the scalar fields vary in space and are finite on the horizon). The theorem is local, and does not require any boundary conditions at infinity.

However, the ‘no horizon’ theorem is not true (as is known in several examples) for Einstein - Yang-Mills theories [15] as well as for solutions of higher dimensional supergravities, see e.g. [9]. In all these cases the static solutions of higher dimensional theories may be constructed by using the 1+1 or 0+1 dimensional dilaton gravity coupled to matter. In the integrable models we discuss here the solutions with horizons are completely identified and described in very simple terms. One may also consider the global properties of the solutions with or without horizons but we will not discuss this subject here.

In Section 2 we briefly demonstrate that spherically symmetric black holes and branes of higher dimensional supergravity theories, as well as superstring motivated cosmological models, may be described in terms of 0+1 and 1+1 dimensional dilaton gravity theories. In Sections 3, 4 a new class of integrable theories of 0+1 and 1+1 dimensional dilaton gravity coupled to any number of scalar fields is introduced. In Section 5 we discuss possible applications of the integrable models and some unsolved problems.

2 High dimensional dilaton gravity

We first write the higher dimensional theories which, under dimensional reductions, produce special examples of integrable theories introduced in [6] and [14]. They all come from the low energy limit of the superstring theories, which is described by 10 dimensional supergravities. The bosonic part of the 10 dimensional supergravities of type II (corresponding to the type II

\footnote{One may also start with the 11 dimensional supergravity, which is believed to be related to the so called ‘M - theory’, and reduce to 10 dimension by compactifying one dimension. Note also that here we are not attempting to consider compactifications of the most general supergravities. We only demonstrate the main features of the connection between low dimensional and high dimensional theories}
superstrings) may be written as
\[ \mathcal{L}^{(10)} = \mathcal{L}_{NS-NS}^{(10)} + \mathcal{L}_{RR}^{(10)}, \] (1)

In this brief discussion it is sufficient to consider the first Lagrangian\(^3\):
\[ \mathcal{L}_{NS-NS}^{(10)} = \sqrt{-g^{(10)}} e^{-2\phi_s} \left[ R^{(10)} + 4(\nabla \phi_s)^2 - \frac{1}{12} H_3^2 \right], \] (2)

Here \(\phi_s\) is the dilaton, related to the string coupling constant; \(H_3 = dB_2\) is a 3-form; \(g^{(10)}\) and \(R^{(10)}\) are the 10 dimensional metric and scalar curvature respectively.

There are many ways to reduce high dimensional theories to low dimensions 1+1 and 0+1. We only mention here those that may lead to integrable theories. First, one may compactify the \(d\) dimensional theory on a \(p\) dimensional torus \(T^p\) (or on several tori, including the circle \(S^1\)) using the Kaluza - Mandel - Fock - Klein (KMFK) mechanism\(^4\). This introduces \(p\) Abelian gauge fields and at least \(p\) scalar fields. Antisymmetric tensor fields (\(n\)-forms), which may be present in the high dimensional theory, will produce lower-rank forms and, eventually, other scalar fields. Thus we get a theory of gravity coupled to matter fields (scalars, Abelian gauge fields and, possibly, higher-rank forms) in a \(d\) dimensional space, \(d = D - p\). The next step is to reduce further its dimension using some symmetry of the \(d\) dimensional theory. The most typical one is the spherical symmetry (the axial symmetry leads to much more complex low dimensional theories and is not considered here). This step produces a 1+1 dimensional dilaton gravity theory coupled to scalar and gauge fields. The simplest example is the spherical reduction of the \(d\) dimensional Einstein - Maxwell theory - the resulting 1+1 dimensional dilaton gravity is actually equivalent to a 0+1 dimensional theory.

The 1+1 dimensional dilaton gravity theories so derived may describe static black holes (static solutions), spherically symmetric evolution of the

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\(^3\)The second one gives similar 1+1 dimensional theories. Eventually, all bosonic and fermionic parts of the higher dimensional Lagrangians give in 1+1 dimension dilaton gravity coupled to scalar matter fields.

\(^4\)It is usually called the Kaluza - Klein (KK) mechanism but this is not justified historically. Actually, the Russian theorists George Mandel and Vladimir Fock have written their papers, in which they generalized the Kaluza theory, even somewhat earlier than Oscar Klein and published them in the same journal. We hope to redress this historical injustice in a separate publication.
black holes (collapse of matter) and of the universe (expansion of the universe). In this sense, the flat space cosmological models and static black holes may be regarded as the 1+0 and 0+1 dimensional reductions of the 1+1 dimensional theory and they can be connected in the frame of the 1+1 dimensional model.

Note that the final step in the chain of dimensional reductions in cosmology is usually somewhat different from that in black hole physics. Cosmological models are normally obtained by reducing the d dimensional theory directly to dimension 1+0. Indeed, isotropy and homogeneity of the universe require a higher symmetry than the spherical one - the whole space should have constant curvature $k$, which may be equal to zero or ±1. These cosmologies can be selected from the set of the 1+0 dimensional solutions of the 1+1 dimensional theory by choosing a proper dimensional reduction of the metric and of the dilaton. We will not go into a detailed description of dimensional reductions, referring the reader to an instructive example in [3], [4], [9] and to reviews [10] - [12]. Instead we give a simplified typical chain of dimensional reductions leading to simple and important two dimensional and one dimensional dilaton gravity models.

Reducing to $d$ dimensions by different sorts of dimensional reduction (KMFK, compactification on tori, etc.) we obtain an effective Lagrangian $\mathcal{L}^{(d)}$. For our purposes it is sufficient to consider the following expression

$$\mathcal{L}^{(d)} = \sqrt{-g^{(d)}} e^{-2\phi_d} \left( R^{(d)} + 4(\nabla \phi_d)^2 - \frac{1}{2} (\nabla \psi)^2 - X_0 - X_1 (\nabla \sigma)^2 - X_2 F_2^2 \right).$$

Here $\phi_d$ is a new dilaton, $F_2$ is a 2-form (an Abelian gauge field); $X_a$ are functions of $\phi_d$ and $\psi$. Actually, the Lagrangian should depend on several $F_2$-fields, several $\psi$-fields, and may depend on several $\sigma$-fields as well as on higher - rank forms. However, after further reduction to two dimension only 2-forms and scalar fields will survive (in fact, the 2-forms can also be excluded by writing an effective potential depending on electric or magnetic charges).

We further reduce the $d$ dimensional theory to dimension 1+1 by spherical symmetry. Before and after doing so one may transform this Lagrangian by the Weyl conformal transformation, $g_{\mu\nu} \Rightarrow \tilde{g}_{\mu\nu} \equiv \Omega^2 g_{\mu\nu}$, where $\Omega$ depends on the corresponding dilaton. Expressing $R$ in terms of the new metric,

$$R = \Omega^2 \left[ \hat{R} + 2(d - 1) \nabla^2 \ln \Omega - (d - 1)(d - 2)(\nabla \ln \Omega)^2 \right],$$

one can easily find the new expression for the Lagrangian. For $d > 2$ one can cancel the multiplier $e^{-2\phi_d}$ by choosing an appropriate function $\Omega(\phi_d)$ and
thus write the Lagrangian in the so called Einstein frame (as distinct from the string frame expressions above). In dimension $d = 2$ it is impossible to remove the dilaton multiplier but, instead, one can remove the dilaton gradient term.

Now consider the spherically symmetric solutions of the $d$ dimensional theory (3). Usually, it is more convenient to remove the dilaton factor by a Weyl transformation and rewrite the action (3) in the Einstein frame,

$$L^{(d)}_E = \sqrt{-g^{(d)}} \left[ R^{(d)} - \frac{1}{2} (\nabla \chi)^2 - \frac{1}{2} (\nabla \psi)^2 - X_0 e^{a_0 \chi} - X_1 e^{a_1 \chi} (\nabla \sigma)^2 - X_2 e^{a_2 \chi} F^2 \right],$$

(5)

where $\phi_d \equiv \chi$ and $a_k$ are known constants depending on $d$. Then we parameterize the spherically symmetric metric by the general 1+1 dimensional metric $g_{ij}$ and the dilaton $\varphi$ ($\nu \equiv 1/n$, $n \equiv d - 2$),

$$ds^2 = g_{ij} dx^i dx^j + \varphi^{2\nu} d\Omega^2_{(d-2)},$$

(6)

introduce appropriate spherical symmetry conditions for the fields, which from now on will be functions of the variables $x_0$ and $x_1$ (or, $t$ and $r$), and integrate out the other (angular) variables from the $d$ dimensional action.

Applying, in addition, the Weyl transformation that removes the dilaton gradient term we thus obtain the effective 1+1 dimensional action

$$L = \sqrt{-g} \left[ \varphi R + n(n - 1) \varphi^{-\nu} - X_0 e^{a_0 \chi} \varphi^\nu - X_2 e^{a_2 \chi} \varphi^{2 - \nu} F^2 - \frac{1}{2} \varphi \left( (\nabla \chi)^2 + (\nabla \psi)^2 + 2X_1 e^{a_1 \chi} (\nabla \sigma)^2 \right) \right].$$

(7)

Here $\varphi$ is the 2-dilaton field that is often denoted by $e^{-2\varphi}$; the scalar fields $\psi$ may have different origins – they may be former dilaton fields, KMK scalar fields, reduced $p$-forms, etc. The functions $X_k$ (we call them potentials) depend on the scalar fields $\chi$ and $\psi$, which from now on will be called scalar matter fields. Also the field $\sigma$ may be regarded as a matter field but it plays a special role that will be discussed later. Notice that the potentials are positive and that $n(n - 1)$ is positive or zero.$^5$

$^5$This term is the curvature of the $n$ dimensional sphere whose metric is given by the second term in (5). If, instead of the spherical symmetry, we used a pseudo-spherical one, the sign would be negative. If the $n$ dimensional subspace is flat this term will be absent.
For dimensionally reduced supergravity theories one can often find a parameterization of the fields in which the potentials are exponentials of the matter fields while the kinetic (gradient) terms have the above simple structure. These 1+1 dimensional theories may have integrable one dimensional sector describing static (0+1) or cosmological (1+0) solutions of the higher dimensional theories. The 1+1 dimensional theories obtained by dimensional reductions are usually not integrable but may be approximated by explicitly analytically integrable 1+1 dimensional theories.

As it was mentioned above, the cosmological models are usually obtained from higher dimensional theories by a different dimensional reduction. To describe the homogeneous and isotropic universe one supposes that the metric may be written in the form

\[ ds^2 = -e^{2\nu(t)} dt^2 + e^{2\mu(t)} d\Omega^2_{(d-1)}(k), \]  

(8)

where \( k = 0 \) for the flat space and \( k = \pm 1 \) for the space of constant positive (negative) curvature. Now, in cosmological models somewhat different reductions of the fields are of interest because the terms generated by the higher rank forms (characteristic of string theories) are believed to be of interest. Anyway, after reducing to one dimension also the higher rank forms give scalar fields of either \( \psi \) or \( \sigma \) type. For example, in a typical reduction of the type IIA 10 dimensional supergravity to dimension 4 (compactification on an isotropic six dimensional torus \( T^6 \)) and then to 1+0 dimensional dilaton gravity (see e.g. [16]), the 3-form produces in one dimensional theory a \( \sigma \) term while the 4-form generates an \( X_0 \)-type potential. The cosmologies so obtained are in general not integrable.

3 1+1 dimensional dilaton gravity

Now let us consider a general 1+1 dimensional dilaton gravity coupled to Abelian gauge fields \( F^{(a)}_{ij} \) and to scalar fields \( \psi_n \). The general Lagrangian can be written as

\[
\mathcal{L} = \sqrt{-g} \left[ U(\varphi) R(g) + V(\varphi) + W(\varphi)(\nabla \varphi)^2 + X(\varphi, \psi, F^2_{(1)}, ..., F^2_{(4)}) + Y(\varphi, \psi) + \sum_n Z_n(\varphi, \psi)(\nabla \psi_n)^2 \right].
\]  

(9)

Here \( g_{ij} \) is a generic 1+1 dimensional metric with signature (-1,1) and \( R \) is the Ricci curvature; \( U(\varphi), V(\varphi), W(\varphi) \) are arbitrary functions of the
dilaton field; \( X, Y \) and \( Z_n \) are arbitrary functions of the dilaton field and of \((N-2)\) scalar fields \( \psi_n \) \((Z_n < 0)\); \( X \) also depends on \( A \) Abelian gauge fields \( F_{(a)ij} \equiv F_{ij}^{(a)} \), \( F_{(a)}^2 \equiv g^{i\prime}g^{j\prime}F_{ij}^{(a)}F_{i'j'}^{(a)} \). Notice that in dimensionally reduced theories (see (7)) both the scalar fields and the Abelian gauge fields are non-minimally coupled to the dilaton.

The equations of the theory (9) can be solved for arbitrary potentials \( U, V, W \) and \( X \) if \( \partial\psi X \equiv 0 \) (for the simplest explicit solution in case of \( X \) linear in \( F^2 \) see e.g [6] and references therein as well as the recent review [18]). Actually, only \( V(\phi) \) and \( X \) are really arbitrary functions. Moreover, for general potentials \( X(\phi, \psi, F^2) \) one may solve the equations for \( F_{ij}^{(a)} \) and then construct the effective action

\[
\mathcal{L}_{\text{eff}} = \sqrt{-g} \left[ \varphi R(g) + V_{\text{eff}}(\varphi, \psi) + \sum_n Z_n(\varphi, \psi) g^{ij}\partial_i\psi_n\partial_j\psi_n \right].
\]

(10)

Here the effective potential \( V_{\text{eff}} \) (below we omit the subscript) depends also on charges introduced by solving the equations for the Abelian fields. Note also that we use a Weyl transformation to exclude the kinetic term for the dilaton and also choose the simplest, linear parameterization for \( U(\psi) \).

If the effective potential does not depend on \( \psi \), one can find the general solution for the matter vacuum when all \( \psi \) are constant. In this case the equation of motion actually reduce to those of the pure dilaton gravity not coupled to scalar matter. Few 1+1 dimensional models are integrable. The best studied ones are the CGHS and JT models. They were essentially generalized in [6]. In all these models \( Z \)-potentials are constant (so called minimal coupling). The only integrable class of models with non minimal coupling to scalar fields (with some special functions \( Z_n(\phi) \)) was proposed in [17].

For the sake of completeness we write here the equations of motion in the light cone \((u,v)\) coordinates (see e.g. [4]; to simplify the formulas we keep only one scalar field and omit the superscript of the effective potential):

\[
\partial_u\partial_v\varphi + f V(\varphi,\psi) = 0,
\]

(11)

\[
\partial_v(Z\partial_u\psi) + \partial_u(Z\partial_v\psi) + fV_\psi(\varphi,\psi) = Z_\psi \partial_u\psi \partial_v\psi,
\]

(12)

\[
f\partial_i\left( \frac{\partial_i\varphi}{f} \right) = Z(\partial_i\psi)^2, \quad i = u, v,
\]

(13)

\[
\partial_u\partial_v \ln|f| + fV_\varphi(\varphi,\psi) = Z_\varphi \partial_u\psi \partial_v\psi,
\]

(14)

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*If \( U'(\varphi) \) has zeroes, this parameterization, as well as the more popular exponential one, \( U = e^{\lambda\varphi} \), is valid only between two consecutive zeroes.*
where $V_\varphi = \partial_\varphi V$, $V_\psi = \partial_\psi V$, $Z_\varphi = \partial_\varphi Z$, $Z_\psi = \partial_\psi Z$. Equations (11) – (14) are not independent. Actually, (14) follows from equations (11) – (13). Alternatively, if (11), (13), (14) are satisfied, (12) is satisfied.

Now we introduce a more general class of integrable 1+1 dimensional dilaton gravity models with minimal coupling to scalar fields. They are defined by the Lagrangian (10) with the following potentials:

$$Z_n = -1; \quad |f|V = \sum_{n=1}^{N} 2g_n e^{\eta_n}.$$  \hspace{1cm} (15)

Here $f$ is the light cone metric, $ds^2 = -4f(u,v) du dv$, and

$$q_n \equiv F + a_n \phi + \sum_{m=3}^{N} \psi_m a_{mn} \equiv \sum_{m=1}^{N} \psi_m a_{mn},$$  \hspace{1cm} (16)

where $\psi_1 + \psi_2 \equiv \ln |f| \equiv F$ ($f \equiv \varepsilon e^F$) and $\psi_1 - \psi_2 \equiv \phi$.

Now, varying the Lagrangian (10) in $(N-2)$ scalar fields, dilaton and in $g_{ij}$ and then passing to the light cone metric we find $N$ equations of motion for $N$ functions $\psi_n$,

$$\epsilon_n \partial_u \partial_v \psi_n = \sum_{m=1}^{N} \varepsilon g_m e^{q_n} a_{mn}; \quad \epsilon_1 = -1, \quad \epsilon_n = +1, \text{ if } n \geq 2,$$  \hspace{1cm} (17)

and two constraints,

$$C_i \equiv f \partial_i (\partial_i \phi / f) + \sum_{n=3}^{N} (\partial_i \psi_n)^2 = 0, \quad i = (u,v).$$  \hspace{1cm} (18)

For arbitrary coefficients $a_{mn}$ the equation of motion are not integrable. However, if the $N$-component vectors $v_n \equiv (a_{mn})$ are pseudo-orthogonal, the equations of motion can be reduced to $N$ Liouville equations for $q_n$,

$$\partial_u \partial_v q_n - \tilde{g}_n e^{\eta_n} = 0,$$  \hspace{1cm} (19)

where $\tilde{g}_n = \varepsilon \lambda_n g_n$, $\lambda_n = \sum \epsilon_m a_{mn}^2$, and $\varepsilon \equiv |f| / f$ (note that the equations for $q_n$ depend on $\epsilon_n$ only implicitly, through the normalization factor $\lambda_n$).

The most important fact is that the constraints can be explicitly solved. By writing the solutions of the Liouville equations in the form suggested by the conformal field theory,

$$e^{-q_n / 2} = a_n(u)b_n(v) + \tilde{a}_n(u)\tilde{b}_n(v),$$  \hspace{1cm} (20)
where \(a\) and \(b\) can be expressed in terms of \(a\) and \(b\), i.e.

\[
e^{-q_n/2} = a_n(u)b_n(v)\left[1 - \frac{1}{2}\tilde{g}_n \int \frac{du}{a_n^2(u)} \int \frac{du}{b_n^2(v)}\right],
\]

we may rewrite the constraints in the form

\[
C_u = \sum_{n=1}^{N} a_n''(u)[\lambda_n a_n(u)]^{-1}, \quad C_v = \sum_{n=1}^{N} b_n''(v)[\lambda_n b_n(v)]^{-1}.
\]

Using the fact that the norms \(\lambda_n\) satisfy the constraint \(\sum_{n=1}^{N} \lambda_n^{-1} = 0\) (this is a consequence of the pseudo-orthogonality conditions) we can solve these constraints. The solution has the following form:

\[
a_n'(u) = a_n(u) - \frac{\sum_{n=2}^{N} \lambda_n^{-1}(\alpha_n' + \alpha_n^2)}{2 \sum_{n=2}^{N} \lambda_n^{-1} \alpha_n},
\]

where \(\alpha_1 = 0\) and the other \(\alpha_n\) are arbitrary functions of \(u\). The ratios \(b_n'(v)/b_n(v)\) are expressed by the same equation in terms of functions \(\beta_n(v)\).

By integrating the first order differential equations for \(a_n(u)\) and \(b_n(v)\) we thus find the general solution of the \(N\)-Liouville dilaton gravity in terms of \((2N-2)\) arbitrary chiral fields \(\alpha_n(u)\) and \(\beta_n(v)\). With proper conditions of convergence one may use this solution also for \(N = \infty\).

4 Integrable 0+1 dimensional dilaton gravity coupled to matter

The dimensional reduction from 1+1 to 0+1 dimensions in the light cone coordinates \((u,v)\) is very simple. If we suppose that \(\varphi = \varphi(\tau), \psi_n = \psi_n(\tau)\) where \(\tau = a(u) + b(v)\), we find from the 1+1 dimensional equations of motion that

\[
f(u,v) = \mp h(\tau) a'(u)b'(v); \quad ds^2 = -4f du dv = \pm 4h(\tau)dad\tau.
\]

Defining two dimensional space and time coordinates, \(r = a \pm b\) and \(t = a \mp b\) we find that

\[
ds^2 = h(\tau)(dt^2 - dr^2), \quad \text{where } \tau = r \text{ or } t,
\]

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and thus the reduced solution may be either static or cosmological one\(^7\).

However, this is not the most general way for obtaining 0+1 or 1+0 dimensional theories from higher dimensional ones. Not all possible reductions can be derived by this simple dimensional reduction of the 1+1 dimensional gravity. For example, to derive the cosmological solutions corresponding to the reductions \(^8\) one should use a more complex dimensional reduction of the 1+1 dimensional dilaton gravity, which will be discussed elsewhere.

It is not difficult to show that the 0+1 dimensional equations are described by the Lagrangian \((F \equiv \ln |h|, \varepsilon \equiv \pm)\) \(^9\):

\[
\mathcal{L} = -\frac{1}{l} \left( \dot{\phi} F + \sum_n Z_n(\varphi, \psi) \dot{\psi}_n^2 \right) + \varepsilon e^F V(\varphi, \psi),
\]

(26)

where \(l(t)\) is the Lagrange multiplier (related to the general metric \(g_{ij}\)).

Now, the two-dimensionally integrable \(N\)-Liouville theories presented above are also integrable in 0+1 dimension. Moreover, as we can solve the Cauchy problem in dimension 1+1 we can study the evolution of the initial configurations to stable static solutions, e.g., black holes, which are special solutions of the 0+1 dimensional reduction. However, the reduced theories can be explicitly solved for much more general potentials \(Z_n\) and \(V\).

Suppose that for \((N-2)\) scalar fields \(\psi_n\) \((n = 3, \ldots, N)\) the ratios of the \(Z\)-potentials are constant so that we can write \(Z_n = -\gamma_n/\phi'(\varphi)\) (in eq. \(^7\) these are the fields \(\chi\) and \(\psi\) and \(\phi = \ln \varphi\)). Suppose that all the potentials \(Z_n\) and \(V\) be independent of the scalar fields \(\psi_{N+k}\) with \(k = 1, \ldots, K\) (in eq. \(^7\) this is the field \(\sigma\)). Then, we first remove the factor \(\phi'(\varphi)\) by defining the new Lagrange multiplier \(\bar{l} = l(\tau)\phi'(\varphi)\) and absorb the factors \(\gamma_n > 0\) in the corresponding scalar fields. In this way we may introduce the new dynamical variable \(\phi\) instead of \(\varphi\). Now we can solve the equations for the \(\sigma\)-fields and construct the effective Lagrangian \(^8\). We thus may arrive at the effective Lagrangian

\[
\mathcal{L}_{\text{eff}} = -\frac{1}{\bar{l}} \left[ \dot{\phi} \bar{F} - \sum_{n=3}^N \dot{\psi}_n^2 \right] + \varepsilon e^F V_{\text{eff}}(\phi, \psi) + V_{\sigma}(\phi, \psi).
\]

(27)

\(^7\)Of course, in 2d theories this distinction is not very important. However, when we know the higher dimensional theory from which our 2d dilaton gravity originated, we can reconstruct the higher dimensional metric and thus find the higher dimensional interpretation of our solutions. In the remaining part of this paper we do not use \(r\) and \(t\), take in \(^24\) the upper sign and usually call all one dimensional solutions static.

\(^8\)It is better to do this in the Hamiltonian formalism but space limitations force us to omit details of our derivations.
Here \( V_{\text{eff}} = V/\phi' (\phi) \) and \( V_\sigma = \sum_k C_k^2/Z_{N+k} \phi' (\phi) \) where \( \phi \) must be expressed in terms of \( \phi \). If the original potentials in eq. (26) are such that \((Z_{N+k} \phi' (\phi))^{-1}\) and \( V/\phi' \) can be expressed in terms of sums of exponentials of linear combinations of the fields \( \phi \) and \( \psi \), then there is a chance that the 0+1 dimensional theory can be reduced to Liouville or Toda equations (the Toda case is possible only if \( V_\sigma \neq 0 \)).

The pure Liouville case was introduced in [14] and is described by the Lagrangian (in notation of eq. (16))

\[
\mathcal{L} = \frac{1}{l} (\dot{\psi}_1^2 + \sum_{n=2}^{N} \dot{\psi}_n^2) + l \sum_{n=1}^{N} 2g_n e^{q_n}.
\]  

(28)

If the \( a_{mn} \) satisfy our pseudo-orthogonality conditions, the equations of motion are reduced to \( N \) independent one dimensional Liouville equations whose solutions have to satisfy the energy constraint. The solutions are expressed in terms of elementary exponentials (for simplicity, we write the solution in the gauge \( l(\tau) \equiv 1 \) but all the results are actually gauge invariant):

\[
e^{-q_n} = \frac{|\tilde{g}_n|}{2\mu_n^2} \left[ e^{\mu_n (\tau - \tau_n)} + e^{-\mu_n (\tau - \tau_n)} + 2\varepsilon_n \right],
\]

(29)

where \( \varepsilon_n = -|\tilde{g}_n|/\tilde{g}_n \), \( \mu_n \) and \( \tau_n \) are the integration constants (\( \mu_n^2 \) and \( \tau_n \) are real). The constraint can be shown to be \( \sum_n \mu_n^2/\lambda_n = 0 \), and its solution is trivial. The space of the solutions is thus defined by the \((2N-2)\) dimensional module space (one of the \( \tau_n \) may be fixed). One can show that the solutions have at most 2 horizons\(^9\) and the space of the solutions with horizons has dimension \((N-1)\). There exist integrable models having solutions with horizons and no singularities but their relation to the high dimensional world is at the moment not clear.

Note that the solution (29) is written in a rather unusual coordinate.

One may write a more standard representation remembering that the dilaton \( \phi \) is related to the coordinate \( r \) (see [13]). This may be useful for a geometric analysis of some simple solutions (e.g. Schwarzschild or Reissner - Nordstrøm) but in general the standard representation is very inconvenient for analyzing solutions of the \( N \)-Liouville theory.

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\(^9\)To prove this one should analyze the behavior of \( q_n \) for \(|\tau| \to \infty \) and for \(|\tau - \tau_n| \to 0 \) (if \( \varepsilon_n < 0 \)). The horizons appear when \( F \to -\infty \) while \( \phi \) and \( \psi_n \) for \( n \geq 2 \) tend to finite limits. This is possible for \(|\tau| \to \infty \) if and only if \( \mu_n = \mu \). When \( F \to F_0 \) and \( \phi \to \infty \) we have the flat space limit, e.g. exterior of the black hole. The singularities in general appear for \(|\tau - \tau_n| \to 0 \) if \( \varepsilon_n < 0 \).
5 Discussion and outlook

The explicitly analytically integrable models presented here may be of interest for different applications. The most obvious is to use them to construct first approximations to generally non integrable theories describing black holes and cosmologies. Realistic theories of these objects are usually not integrable (even in dimension 0+1). Having explicit general solutions of the zeroth approximation in terms of elementary functions it is not difficult to construct different sorts of (classical) perturbation theory.

For example, spherically symmetric static black holes non minimally coupled to scalar fields are described by the integrable 0+1 dimensional $N$-Liouville model. However, the corresponding 1+1 dimensional theory is not integrable because the scalar coupling potentials $Z_n$ are not constant (see eq.\((7)\)). To obtain approximate analytic solutions of the 1+1 dimensional theory one may try to approximate $Z_n$ by properly chosen constants.

It may be useful to combine this approach with the recently proposed analytic perturbation theory allowing to find solutions near horizons for most general non integrable 0+1 dilaton gravity theories \[19\]. The detailed description of the $N$-Liouville (and of the Toda type) theory as well as applications to black holes and cosmology will be given elsewhere \[20\]. The Toda type theories were earlier introduced by direct reductions of higher dimensional theories to cosmological models (see e.g. \[12\], \[8\] and references therein).

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6 Appendix

To help the reader in keeping trace of relations between dimensions 1+1, 1+0 and 0+1 we write here a simple expression for the curvature in dimension

\[^{10}\text{This paper is an extended version of the report that will be published in the Proceedings of the third Sakharov Conference (sc3). Neither the report nor earlier papers \[14\] in which I very briefly described the integrable models discussed here were sent to hep-th. A more detailed presentation of the models and their applications will be given \[20\].}^\]
1+1. We take the diagonal metric

\[ ds^2 = -e^{2\nu} dt^2 + e^{2\mu} dr^2. \]

The Ricci scalar \( R \) for this metric is

\[ R = 2e^{-2\nu}(\ddot{\mu} + \dot{\mu}^2 - \dot{\mu}\dot{\nu}) - 2e^{-2\mu}(\ddot{\nu} + \dot{\nu}^2 - \dot{\nu}\dot{\mu}). \]

For this metric, one may also need the expression for \( \nabla^2 \phi \), where \( \phi \) is an arbitrary scalar field:

\[ \nabla^2 \equiv \nabla^m \nabla_m \phi = -e^{-2\nu}(\dddot{\phi} + (\dot{\mu} - \dot{\nu})\dot{\phi}) + e^{-2\mu}(\dddot{\phi} + (\dot{\nu} - \mu')\dot{\phi}). \]

In the \((u, v)\) coordinates the curvature \( R \) is simply

\[ R = \frac{1}{f} \partial_u \partial_v \ln |f|. \]

Note that the equations of motion (but not the constraints) may be derived from the gauge fixed Lagrangian

\[ \mathcal{L} = \varphi \partial_u \partial_v F + fV - Z \partial_u \psi \partial_v \psi. \]

It is easy to derive from it the dimensionally reduced one dimensional equations of motion and the Lagrangian (26) restoring the constraint, if one passes to the Hamiltonian formalism and recalls that the constraint in one dimension requires that the Hamiltonian should vanish (see [6]).

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