AN ISOMORPHISM THEOREM FOR PARABOLIC PROBLEMS
IN HÖRMANDER SPACES AND ITS APPLICATIONS

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Abstract. We investigate a general parabolic initial-boundary value problem with zero Cauchy data in some anisotropic Hörmander inner product spaces. We prove that the operators corresponding to this problem are isomorphisms between appropriate Hörmander spaces. As an application of this result, we establish a theorem on the local increase in regularity of solutions to the problem. We also obtain new sufficient conditions under which the generalized derivatives, of a given order, of the solutions should be continuous.

1. Introduction. General linear parabolic initial-boundary value problems have been investigated completely enough on the classical scales of Hölder–Zygmund spaces and Sobolev spaces [1, 8–10, 12, 18, 20]. The central result of the theory of these problems states that they are well posed by Hadamard in appropriate pairs of the function spaces belonging to these scales; in other words, the operators corresponding to the parabolic problems are isomorphisms on these pairs. This fact has important applications to the investigation of the regularity of solutions to parabolic problems, to the study of properties of Green functions of these problems, to the optimal control problems and others.

However, for some applications to differential equations, the Hölder–Zygmund and Sobolev scales are not calibrated finely enough by number parameters [13, 14, 34]. In this connection, L. Hörmander [13, Section 2.2] introduced and investigated normed function spaces for which a function parameter, not a number, serves as an

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index of regularity of functions or distributions. L. Hörmander [13,14] gave applications of these spaces to the investigation of regularity of solutions to hypoelliptic partial differential equations, among which are the parabolic equations.

Of course, the inner product spaces are of the most important among Hörmander spaces. Lately Mikhailets and Murach [25–29,31,32,34,36] built the theory of general elliptic differential operators on manifolds and elliptic boundary-value problems in Hilbert scales formed by the Hörmander spaces

$$H^{s,\varphi} := \mathcal{B}_{2,\mu} := \{ w \in \mathcal{S}'(\mathbb{R}^n) : \mu Fw \in L^2(\mathbb{R}^n) \} \quad (1)$$

and their analogs for Euclidean domains and smooth manifolds. Here, the function

$$\mu(\xi) := (1 + |\xi|^2)^{s/2} \varphi \left( (1 + |\xi|^2)^{1/2} \right) \quad \text{of} \quad \xi \in \mathbb{R}^n$$

serves as the index of regularity, with $s$ being a real number and with $\varphi$ being a slowly varying function at infinity in the sense of J. Karamata. (As usual, $F$ denotes the Fourier transform.) The class of the spaces (1) contains the Sobolev scale $\{H^s\} = \{H^{s,1}\}$ and is attached to it by means of $s$ but is calibrated more finely than the Sobolev scale.

This theory is based on the method of the interpolation with a function parameter between Hilbert spaces, specifically, between Sobolev spaces. Using this interpolation systematically, Mikhailets and Murach transferred the classical theory of elliptic equations from Sobolev spaces to wide classes of Hörmander inner product spaces. Their results were supplemented in [2, 3, 37, 46] for the class of all Hilbert spaces that are interpolation spaces between Sobolev inner product spaces. (This class admits a description in terms of Hörmander spaces and is closed with respect to the interpolation with a function parameter [33,35].)

Note that the methods of interpolation with a number parameter are fruitful in the theory of partial differential equations [4,19,20,44]. However, to pass from the classical spaces parametrized by numbers parameters to more general classes of spaces parametrized by function parameters, we need to use the interpolation with a function parameter [7,24,30].

The purpose of this paper is to prove a theorem about an isomorphism generated by the general initial-boundary value parabolic problem in anisotropic analogs of the Hörmander spaces (1). We consider the problem in a bounded many-dimensional cylinder and suppose that the initial conditions are zero Cauchy data. (The case of inhomogeneous Cauchy data can be reduced to the homogeneous ones.) We will deduce this theorem from the classical result by M. S. Agranovich and M. I. Vishik [1, Theorem 12.1] by means of the interpolation with a function parameter between anisotropic Sobolev spaces. To this end, we investigate necessary interpolation properties of the used anisotropic spaces. We also give some applications of this isomorphism theorem to investigation of local regularity of solutions to the parabolic problem.

Some results of this paper are announced (without proofs) in [22]. The case of a bounded two-dimensional cylinder was investigated in [21,23]. In this case, the Hörmander spaces over the lateral area of the cylinder are isotropic, which essentially simplifies the reasoning.

The paper consists of eight sections and Appendix. Section 1 is Introduction. In Section 2, we state an initial-boundary value problem for a general parabolic equation with zero Cauchy data. The necessary anisotropic Hörmander spaces are introduced in Section 3. The next Section 4 contains the main results of the paper. They are the isomorphism Theorem 4.1 and its applications, Theorems 4.2...
and 4.3. Theorem 4.2 deals with the local regularity of generalized solutions to the parabolic problem in the Hörmander spaces. Theorem 4.3 gives sufficient conditions under which the generalized derivatives (of a prescribed order) of the solutions are continuous on a given subset of the cylinder. These conditions are formulated in terms of Hörmander spaces. In Section 5, we discuss the isomorphism theorem in the case of anisotropic Sobolev spaces and compare it with the known result by M. S. Agranovich and M. I. Vishik. Section 6 is devoted to the method of the interpolation with a function parameter between Hilbert spaces. We recall its definition and some of its properties. In Section 7, we prove formulas that connect anisotropic Sobolev spaces with Hörmander spaces by means of this interpolation. Section 8 contains the proofs of the main results of this paper. In Appendix, we show that the function parameters used in the paper as the index of regularity for Hörmander spaces satisfies Hörmander’s condition, so the spaces introduced in Section 3 are well defined.

2. Statement of the problem. Let an integer \( n \geq 2 \) and a real number \( \tau > 0 \) be arbitrarily chosen. Suppose that \( G \) is a bounded domain in \( \mathbb{R}^n \) and that its boundary \( \Gamma := \partial G \) is an infinitely smooth closed manifold (of dimension \( n - 1 \)), with the \( C^\infty \)-structure on \( \Gamma \) being induced by \( \mathbb{R}^n \). Let \( \Omega := G \times (0, \tau) \) and \( S := \Gamma \times (0, \tau) \). Thus, \( \Omega \) is an open cylinder in \( \mathbb{R}^{n+1} \), and \( S \) is its lateral area, with their closures \( \overline{\Omega} = \overline{G} \times [0, \tau] \) and \( \overline{S} = \overline{\Gamma} \times [0, \tau] \).

We consider the following parabolic initial-boundary value problem in \( \Omega \):

\[
A(x, t, D_x, \partial_t)u(x, t) = \sum_{|\alpha|+2\beta \leq 2m} a^{\alpha,\beta}(x, t) D_x^\alpha \partial_t^\beta u(x, t) = f(x, t)
\tag{2}
\]

for all \( x \in G \) and \( t \in (0, \tau) \);

\[
B_j(x, t, D_x, \partial_t)u(x, t) = \sum_{|\alpha|+2\beta \leq m_j} b_j^{\alpha,\beta}(x, t) D_x^\alpha \partial_t^\beta u(x, t) = g_j(x, t)
\tag{3}
\]

for all \( x \in \Gamma \), \( t \in (0, \tau) \), and \( j \in \{1, \ldots, m\} \);

\[
\partial_t^k u(x, t) \big|_{t=0} = 0 \quad \text{for all} \quad x \in G \quad \text{and} \quad k \in \{0, \ldots, \tau - 1\}.
\tag{4}
\]

Note that the initial data (4) are assumed to be zero. Here, \( b \), \( m \), and all \( m_j \) are arbitrarily given integers such that \( m \geq b \geq 1 \), \( \kappa := m/b \in \mathbb{Z} \), and \( m_j \geq 0 \).

All coefficients of the linear partial differential expressions \( A := A(x, t, D_x, \partial_t) \) and \( B_j := B_j(x, t, D_x, \partial_t) \), with \( j \in \{1, \ldots, m\} \), are supposed to be infinitely smooth complex-valued functions given on \( \overline{\Omega} \) and \( \overline{S} \) respectively; i.e., each

\[
a^{\alpha,\beta} \in C^\infty(\overline{\Omega}) := \{ w \mid \overline{w}: w \in C^\infty(\mathbb{R}^{n+1}) \} \]

and each

\[
b_j^{\alpha,\beta} \in C^\infty(\overline{S}) := \{ v \mid \overline{v}: v \in C^\infty(\Gamma \times \mathbb{R}) \}.
\]

(Naturally, we consider \( \Gamma \times \mathbb{R} \) as an infinitely smooth manifold with the \( C^\infty \)-structure induced by \( \mathbb{R}^{n+1} \).

We use the notation \( D^\alpha_x := D_1^{\alpha_1} \cdots D_n^{\alpha_n} \), with \( D_k := i \partial/\partial x_k \), and \( \partial_t := \partial/\partial t \) for partial derivatives of functions depending on \( x = (x_1, \ldots, x_n) \in \mathbb{R}^n \) and \( t \in \mathbb{R} \). Here, \( i \) is imaginary unit, and \( \alpha = (\alpha_1, \ldots, \alpha_n) \) is a multi-index, with \( |\alpha| := \alpha_1 + \cdots + \alpha_n \).

In formulas (2) and (3) and their analogs, we take summation over the integer-valued nonnegative indices \( \alpha_1, \ldots, \alpha_n \) and \( \beta \) that satisfy the condition written under the integral sign. As usual, \( \xi^\alpha := \xi_1^{\alpha_1} \cdots \xi_n^{\alpha_n} \) for \( \xi := (\xi_1, \ldots, \xi_n) \in \mathbb{C}^n \).
We recall [1, Section 9, Subsection 1] that the initial-boundary value problem (2)–(4) is said to be parabolic in \( \Omega \) if the following Conditions 2.1 and 2.2 are fulfilled.

**Condition 2.1.** For arbitrary \( x \in \overline{\Omega}, t \in [0, \tau], \xi \in \mathbb{R}^n, \) and \( p \in \mathbb{C} \) with \( \text{Re} \, p \geq 0, \) we have
\[
A^\alpha(x, t, \xi, p) := \sum_{|\alpha|+2b\beta=2m} a^{\alpha,\beta}(x, t) \xi^\alpha p^\beta \neq 0 \quad \text{whenever} \quad |\xi| + |p| \neq 0.
\]

To formulate Condition 2.2, we arbitrarily choose a point \( x \in \Gamma, \) real number \( t \in [0, \tau], \) vector \( \xi \in \mathbb{R}^n \) tangent to the boundary \( \Gamma \) at \( x, \) and number \( p \in \mathbb{C} \) with \( \text{Re} \, p \geq 0 \) such that \( |\xi| + |p| \neq 0. \) Let \( \nu(x) \) is the unit vector of the inward normal to \( \Gamma \) at \( x. \) It follows from Condition 2.1 and the inequality \( n \geq 2 \) that the polynomial \( A^\alpha(x, t, \xi + \zeta \nu(x), p) \) in \( \zeta \in \mathbb{C} \) has \( m \) roots \( \zeta_j^+(x, t, \xi, p), j = 1, \ldots, m, \) with positive imaginary part and \( m \) roots with negative imaginary part provided that each root is taken the number of times equal to its multiplicity.

**Condition 2.2.** For each choice indicated above, the polynomials
\[
B_j^\alpha(x, t, \xi + \zeta \nu(x), p) := \sum_{|\alpha|+2b\beta=m_j} b_{\alpha,\beta}(x, t) (\zeta + \zeta \nu(x))^\alpha p^\beta, \quad j = 1, \ldots, m,
\]
in \( \zeta \in \mathbb{C} \) are linearly independent modulo
\[
\prod_{j=1}^m (\zeta - \zeta_j^+(x, t, \xi, p)).
\]

Note that Condition 2.1 is the condition for the partial differential equation \( Au = f \) to be \( 2b \)-parabolic in \( \overline{\Omega} \) in the sense of I. G. Petrovskii [40], whereas Condition 2.2 claims that the system of boundary partial differential expressions \( \{B_1, \ldots, B_m\} \) covers \( A \) on \( \overline{S}. \)

We associate the linear mapping
\[
C^\infty_+(\overline{\Omega}) \ni u \mapsto (Au, Bu) := (Au, B_1 u, \ldots, B_m u) \in C^\infty_+(\overline{\Omega}) \times (C^\infty_+(\overline{S}))^m \quad (5)
\]
with the parabolic problem (2)–(4). Here and below,
\[
C^\infty_+(\overline{\Omega}) := \{ w \mid \overline{\Omega} : w \in C^\infty(\mathbb{R}^{n+1}), \supp \, w \subseteq \mathbb{R}^n \times [0, \infty) \},
\]
\[
C^\infty_+(\overline{S}) := \{ h \mid \overline{S} : h \in C^\infty(\Gamma \times \mathbb{R}), \supp \, h \subseteq \Gamma \times [0, \infty) \}.
\]

In the paper, all functions and distributions are supposed to be complex-valued.

The mapping (5) is a bijection of \( C^\infty_+(\overline{\Omega}) \) onto \( C^\infty_+(\overline{\Omega}) \times (C^\infty_+(\overline{S}))^m. \) This follows specifically from [1, Theorem 12.1] (see the reasoning at the end of Section 5).

The main purpose of the paper is to prove that the mapping (5) extends uniquely (by continuity) to an isomorphism between appropriate Hörmander inner product spaces.

### 3. Hörmander spaces

Here, we will define the Hörmander inner product spaces being used in the paper. They are built on the base of the anisotropic function spaces \( H^{s, s/(2b)}(\mathbb{R}^{k+1}) \) given over \( \mathbb{R}^{k+1}, \) with \( k \geq 1. \) These spaces are parametrized with the pair of the real numbers \( s \) and \( s/(2b) \) and with the function \( \varphi \in \mathcal{M}. \)

By definition, the class \( \mathcal{M} \) consists of all Borel measurable functions \( \varphi : [1, \infty) \to (0, \infty) \) that satisfy the following two conditions:

(i) both the functions \( \varphi \) and \( 1/\varphi \) are bounded on each compact interval \([1, b] \) with \( 1 < b < \infty; \)
Here and below, we use the notation $R$ with $k$ of $C$ polynomial space complete (i.e., Hilbert) and separable and is continuously embedded in the linear to-
that $B$ Section 2.1, and therefore the space $\mu$ will show in Appendix that the function $p$ [14, Section 10.1] and in the
The spaces $B$ embeds space and is denoted by $H$ and the function parameter $L$. Hörmander [13, Section 2.2]. Namely, $\phi$ $[6,42]$. An important example of a function $\phi \in M$ is given by a continuous function $\phi : [1, \infty) \to (0, \infty)$ such that
$$
\phi(r) = (\log r)^{q_1} (\log \log r)^{q_2} \cdots (\log \log \cdots \log r)^{q_k} \quad \text{for} \quad r \gg 1,
$$
with $k \in \mathbb{Z}$, $k \geq 1$, and $q_1, q_2, \ldots, q_k \in \mathbb{R}$.
Let $s \in \mathbb{R}$, $\phi \in M$, and real $\gamma > 0$. Although we need the space $H^{s,\gamma;\varphi}(\mathbb{R}^{k+1})$ only in the case where $\gamma = 1/(2b)$, it is naturally to introduce this space for arbitrary $\gamma > 0$.
By definition, the complex linear space $H^{s,\gamma;\varphi}(\mathbb{R}^{k+1})$ consists of all tempered distributions $w$ on $\mathbb{R}^{k+1}$ whose (complete) Fourier transform $\hat{w}$ is locally Lebesgue integrable over $\mathbb{R}^{k+1}$ and satisfies the condition
$$
\int_{\mathbb{R}^k} \int_{\mathbb{R}^k} r_\gamma^2(\xi, \eta) \varphi^2(r_\gamma(\xi, \eta)) |\hat{w}(\xi, \eta)|^2 d\xi d\eta < \infty.
$$
Here and below, we use the notation
$$
r_\gamma(\xi, \eta) := (1 + |\xi|^2 + |\eta|^{2\gamma})^{1/2}, \quad \text{with} \quad \xi \in \mathbb{R}^k \quad \text{and} \quad \eta \in \mathbb{R}.
$$
This space is equipped with the inner product
$$
(w_1, w_2)_{H^{s,\gamma;\varphi}(\mathbb{R}^{k+1})} := \int_{\mathbb{R}^k} \int_{\mathbb{R}^k} r_\gamma^2(\xi, \eta) \varphi^2(r_\gamma(\xi, \eta)) \hat{w}_1(\xi, \eta) \overline{\hat{w}_2(\xi, \eta)} d\xi d\eta
$$
of $w_1, w_2 \in H^{s,\gamma;\varphi}(\mathbb{R}^{k+1})$. The inner product naturally induces the norm
$$
\|w\|_{H^{s,\gamma;\varphi}(\mathbb{R}^{k+1})} := (w, w)^{1/2}_{H^{s,\gamma;\varphi}(\mathbb{R}^{k+1})}.
$$
We note that $H^{s,\gamma;\varphi}(\mathbb{R}^{k+1})$ is a special case of the spaces $B_{p,\mu}$ introduced by L. Hörmander [13, Section 2.2]. Namely, $H^{s,\gamma;\varphi}(\mathbb{R}^{k+1}) = B_{p,\mu}$ provided that $p = 2$ and the function parameter
$$
\mu(\xi, \eta) = r_\gamma^s(\xi, \eta) \varphi(r_\gamma(\xi, \eta)) \quad \text{for all} \quad \xi \in \mathbb{R}^k \quad \text{and} \quad \eta \in \mathbb{R}.
$$
The spaces $B_{p,\mu}$ were systematically investigated by L. Hörmander [13, Section 2.2], [14, Section 10.1] and in the $p = 2$ case by L. R. Volevich and B. P. Paneah [45]. We will show in Appendix that the function $\mu(\xi, \eta)$ satisfies Hörmander’s condition [13, Section 2.1], and therefore the space $B_{p,\mu}$ is well defined. If $\gamma = 1/(2b)$, then we say that $H^{s,\gamma;\varphi}(\mathbb{R}^{k+1})$ is a $2b$-anisotropic Hörmander space.
According to [13, Theorem 2.2.1], the Hörmander space $H^{s,\gamma;\varphi}(\mathbb{R}^{k+1})$ is complete (i.e., Hilbert) and separable and is continuously embedded in the linear topological space $S'(\mathbb{R}^{k+1})$ of tempered distributions on $\mathbb{R}^{k+1}$. Furthermore, the set $C_0^\infty(\mathbb{R}^{k+1})$ of test functions on $\mathbb{R}^{k+1}$ is dense in $H^{s,\gamma;\varphi}(\mathbb{R}^{k+1})$.
In the $\varphi(r) \equiv 1$ case, the space $H^{s,\gamma;\varphi}(\mathbb{R}^{k+1})$ becomes an anisotropic Sobolev space and is denoted by $H^{s,\gamma}(\mathbb{R}^{k+1})$. Generally, we have the continuous and dense embeddings
$$
H^{s_1,\gamma_1}(\mathbb{R}^{k+1}) \hookrightarrow H^{s,\gamma;\varphi}(\mathbb{R}^{k+1}) \hookrightarrow H^{s_0,\gamma_0}(\mathbb{R}^{k+1})
$$
whenever $s_0 < s < s_1$. (6)
They result directly from the following property of $\varphi \in \mathcal{M}$: for every $\varepsilon > 0$ there exists a number $c = c(\varepsilon) \geq 1$ such that $c^{-1}r^{-\varepsilon} \leq \varphi(r) \leq cr^{-\varepsilon}$ for all $r \geq 1$ (see [42, Section 1.5, Subsection 1]).

The embeddings (6) clarify the role of the function parameter $\varphi \in \mathcal{M}$ in the class of Hilbert function spaces

$$\{H^{s,s\gamma;\varphi}(\mathbb{R}^{k+1}) : s \in \mathbb{R}, \varphi \in \mathcal{M}\}. \quad (7)$$

We see that $\varphi$ defines a supplementary (subpower) regularity of distributions with respect to the basic (power) regularity given by the pair of numbers $(s, s\gamma)$. Specifically, if $\varphi(r) \to \infty$ [or $\varphi(r) \to 0$] as $r \to \infty$, then $\varphi$ defines a positive [or negative] supplementary regularity. So, we can briefly say that $\varphi$ refines the power anisotropic regularity $(s, s\gamma)$.

Note that in the $\gamma = 1$ case the space $H^{s,s\gamma;\varphi}(\mathbb{R}^{k+1})$ becomes the isotropic Hörmander space denoted by $H^{s;\varphi}(\mathbb{R}^{k+1})$. The spaces $H^{s;\varphi}(\mathbb{R}^{k+1})$, with $s \in \mathbb{R}$ and $\varphi \in \mathcal{M}$, form the refined Sobolev scale introduced and investigated by Mikhailets and Murach [25, 27]. This scale has various applications in the theory of elliptic partial differential equations [32, 34].

Using the scale (7), we now introduce the function spaces relating to the problem (2)–(4). Let $V$ be an open nonempty set in $\mathbb{R}^{k+1}$. (Specifically, $V = \Omega$, with $k = n$.) We put

$$H^{s,s\gamma;\varphi}_{+}(V) := \{w \mid V : w \in H^{s,s\gamma;\varphi}(\mathbb{R}^{k+1}), \text{supp} \ w \subseteq \mathbb{R}^{k} \times [0, \infty)\}. \quad (8)$$

The norm in the linear space (8) is defined by the formula

$$\|u\|_{H^{s,s\gamma;\varphi}_{+}(V)} := \inf \left\{ \|w\|_{H^{s,s\gamma;\varphi}(\mathbb{R}^{k+1})} : \right\}
\begin{align*}
&w \in H^{s,s\gamma;\varphi}(\mathbb{R}^{k+1}), \text{supp} \ w \subseteq \mathbb{R}^{k} \times [0, \infty), u = w \mid V, \\
&\text{with } u \in H^{s,s\gamma;\varphi}_{+}(V).
\end{align*}
\quad (9)$$

Considering the $V = \mathbb{R}^{k+1}$ case, we note that $H^{s,s\gamma;\varphi}_{+}(\mathbb{R}^{k+1})$ consists of all $w \in H^{s,s\gamma;\varphi}(\mathbb{R}^{k+1})$ with $\text{supp} \ w \subseteq \mathbb{R}^{k} \times [0, \infty)$ and is a (closed) subspace of the Hilbert space $H^{s,s\gamma;\varphi}(\mathbb{R}^{k+1})$. According to [45, Lemma 3.3], the set

$$C^{\infty}_{0}(\mathbb{R}^{k} \times (0, \infty)) := \{w \in C^{\infty}(\mathbb{R}^{k+1}) : \text{supp} \ w \subseteq \mathbb{R}^{k} \times (0, \infty)\}$$

is dense in the space $H^{s,s\gamma;\varphi}(\mathbb{R}^{k+1})$.

Generally, $H^{s,s\gamma;\varphi}_{+}(V)$ is a Hilbert space because formulas (8) and (9) mean that $H^{s,s\gamma;\varphi}_{+}(V)$ is the factor space of the Hilbert space $H^{s,s\gamma;\varphi}_{+}(\mathbb{R}^{k+1})$ by its subspace

$$H^{s,s\gamma;\varphi}_{+}(\mathbb{R}^{k+1}, V) := \{w \in H^{s,s\gamma;\varphi}_{+}(\mathbb{R}^{k+1}) : w = 0 \text{ in } V\}. \quad (10)$$

The norm (9) is induced by the inner product

$$(u_{1}, u_{2})_{H^{s,s\gamma;\varphi}_{+}(V)} := (w_{1} - \Upsilon w_{1}, w_{2} - \Upsilon w_{2})_{H^{s,s\gamma;\varphi}_{+}(\mathbb{R}^{k+1})},$$

with $u_{1}, u_{2} \in H^{s,s\gamma;\varphi}_{+}(V)$. Here, $w_{j} \in H^{s,s\gamma;\varphi}_{+}(\mathbb{R}^{k+1})$, $w_j = u_j$ in $V$ for every $j \in \{1, 2\}$, and $\Upsilon$ is the orthogonal projector of the space $H^{s,s\gamma;\varphi}_{+}(\mathbb{R}^{k+1})$ onto its subspace (10).

In the Sobolev case of $\varphi(r) \equiv 1$, we will omit the index $\varphi$ in the designations of $H^{s,s\gamma;\varphi}_{+}(V)$ and similar spaces. We note the continuous and dense embeddings

$$H^{s,s\gamma}_{+}(V) \hookrightarrow H^{s,s\gamma;\varphi}_{+}(V) \hookrightarrow H^{s_{0},s_{0}\gamma}_{+}(V) \text{ whenever } s_{0} < s < s_{1}. \quad (11)$$

They result from (6) and the density of the set

$$\{w \mid V : w \in C^{\infty}_{0}(\mathbb{R}^{k} \times (0, \infty))\}$$
in the spaces appearing in (11).

As to the problem (2)–(4), we need the space $H^s_{\ast,\gamma,v}(V)$ in the case where $k = n$ and $V = \Omega$. Note that the set $C^\infty_\ast(\Omega)$ is dense in $H^s_{\ast,\gamma,v}(\Omega)$.

We also need to introduce an analog of the space $H^s_{\ast,\gamma,v}(V)$ for the lateral area $S$ of the cylinder $\Omega$. It is sufficient for our purposes to restrict ourselves to the case where $k = n - 1$ and $V = \Pi$. We will define the space $H^s_{\ast,\gamma,v}(S)$ on the base of the space $H^s_{\ast,\gamma,v}(\Pi)$ with the help of special local charts on $\mathfrak{S}$.

We arbitrarily choose a finite atlas from the $C^\infty$-structure on the closed manifold $\Gamma$. Let this atlas be formed by the local charts $\theta_j : \mathbb{R}^{n-1} \leftrightarrow \Gamma_j$, with $j = 1, \ldots, \lambda$. Here, each $\theta_j$ is a $C^\infty$-diffeomorphism of the whole Euclidean space $\mathbb{R}^{n-1}$ onto an open subset $\Gamma_j$ of $\Gamma$. Moreover, $\Gamma := \Gamma_1 \cup \cdots \cup \Gamma_\lambda$, i.e. the open sets $\Gamma_1, \ldots, \Gamma_\lambda$ make up a covering of $\Gamma$. Besides, we arbitrarily choose functions $\chi_j \in C^\infty(\Gamma)$, with $j = 1, \ldots, \lambda$, such that $\text{supp} \chi_j \subset \Gamma_j$ and $\chi_1 + \cdots + \chi_\lambda = 1$ on $\Gamma$. So, these functions form a $C^\infty$-partition of unity on $\Gamma$ which is subordinate to the covering.

This atlas of $\Gamma$ induces the collection of the special local charts

$$\theta_j^* : \mathbb{R}^{n-1} \times [0, \tau] \leftrightarrow \Gamma_j \times [0, \tau], \quad j = 1, \ldots, \lambda,$$

of $\mathfrak{S} = \Gamma \times [0, \tau]$ by the formula $\theta_j^*(x,t) := (\theta_j(x),t)$ for all $x \in \mathbb{R}^{n-1}$ and $t \in [0, \tau]$. Consider the functions $\chi_j^*(x,t) := \chi_j(x)$ of $x \in \Gamma$ and $t \in [0, \tau]$, with $j = 1, \ldots, \lambda$. They form a $C^\infty$-partition of unity on $\mathfrak{S}$ which is subordinate to the covering $\{\Gamma_j \times [0, \tau] : j = 1, \ldots, \lambda\}$ of $\mathfrak{S}$.

Now, we put

$$H^s_{\ast,\gamma,v}(S) := \{ v \in L_2(S) : (\chi_j^* v) \circ \theta_j^* \in H^s_{\ast,\gamma,v}(\Pi) \text{ for all } j \in \{1, \ldots, \lambda\} \}. \quad (12)$$

Here, recall, $s > 0$, and, $L_2(S)$ is the Hilbert space of all square integrable functions $v : S \to \mathbb{C}$ with respect to the Lebesgue measure on the smooth surface $S$. As usual, $\circ$ is the sign of composition of functions or mappings, so that

$$((\chi_j^* v) \circ \theta_j^*)(x,t) = (\chi_j v)(\theta_j(x),t) = \chi_j(\theta_j(x)) v((\theta_j(x),t))$$

for all $x \in \mathbb{R}^{n-1}$ and $t \in (0, \tau)$. The inner product in the linear space (12) is defined by the formula

$$(v_1, v_2)_{H^s_{\ast,\gamma,v}(S)} := \sum_{j=1}^\lambda ((\chi_j^* v_1) \circ \theta_j^*, (\chi_j^* v_2) \circ \theta_j^*)_{H^s_{\ast,\gamma,v}(\Pi)}, \quad (13)$$

with $v_1, v_2 \in H^s_{\ast,\gamma,v}(S)$. This inner product naturally induces the norm

$$\|v\|_{H^s_{\ast,\gamma,v}(S)} := (v, v)^{1/2}_{H^s_{\ast,\gamma,v}(S)}.$$

**Lemma 3.1.** Let $s > 0$, $\gamma > 0$, and $\varphi \in \mathcal{M}$. Then we state the following:

(i) The space $H^s_{\ast,\gamma,v}(S)$ is complete (i.e., Hilbert) and separable and does not depend up to equivalence of norms on the indicated choice of an atlas and partition of unity on $\Gamma$.

(ii) The set $C^\infty_\ast(\mathfrak{S})$ is dense in this space.

We will prove this lemma at the end of Section 7.

Ending the present section, we note that statement (11) about continuous and dense embeddings also holds true in the case where $V = S$ and $s_0 > 0$. This follows
directly from the validity of this statement for the open set \( V = \Pi \subset \mathbb{R}^n \) and from Lemma 3.1(ii).

4. Main results. Here, we formulate an isomorphism theorem for the parabolic problem (2)–(4) in Hörmander spaces introduced above and then consider applications of this theorem to the investigation of the regularity of the generalized solutions to the problem.

Let \( \sigma_0 \) denote the smallest integer such that
\[
\sigma_0 \geq 2m, \quad \sigma_0 \geq m_j + 1 \quad \text{for each } j \in \{1, \ldots, m\}, \quad \text{and} \quad \frac{\sigma_0}{2b} \in \mathbb{Z}.
\]

Note, if \( m_j \leq 2m - 1 \) for each \( j \in \{1, \ldots, m\} \), then \( \sigma_0 = 2m \).

Isomorphism Theorem is formulated as follows.

**Theorem 4.1.** For every real number \( \sigma > \sigma_0 \) and every function parameter \( \varphi \in \mathcal{M} \), the mapping (5) extends uniquely (by continuity) to an isomorphism
\[
(A, B) : H_{+}^{\sigma, \sigma/(2b); \varphi}(\Omega) \leftrightarrow \mathcal{H}_{+}^{\sigma - 2m, (\sigma - 2m)/(2b); \varphi}(\Omega, S),
\]
(14)

where
\[
\mathcal{H}_{+}^{\sigma - 2m, (\sigma - 2m)/(2b); \varphi}(\Omega, S)
:= H_{+}^{\sigma - 2m, (\sigma - 2m)/(2b); \varphi}(\Omega) \oplus \bigoplus_{j=1}^{m} H_{+}^{\sigma - m - 1/2, (\sigma - m - 1/2)/(2b); \varphi}(S).
\]
(15)

In the Sobolev case of \( \varphi(r) \equiv 1 \) and \( \sigma/(2b) \in \mathbb{Z} \), this theorem follows from the result by M. S. Agranovich and M. I. Vishik [1, Theorem 12.1], the limiting case of \( \sigma = \sigma_0 \) being included. This will be demonstrated in Section 5. In the general situation, we will deduce Theorem 4.1 from the Sobolev case with the help of the interpolation with a function parameter between Hilbert spaces. This will be done in Section 8 after we investigate the necessary interpolation properties of the Hörmander spaces appearing in (14) and (15).

As has just been mentioned, the mapping (5) extends by continuity to an isomorphism
\[
(A, B) : H_{+}^{\sigma_0, \sigma_0/(2b); \varphi}(\Omega) \leftrightarrow \mathcal{H}_{+}^{\sigma_0 - 2m, (\sigma_0 - 2m)/(2b); \varphi}(\Omega, S),
\]
(16)

which acts between some anisotropic Sobolev spaces. All the isomorphisms (14), with \( \sigma > \sigma_0 \) and \( \varphi \in \mathcal{M} \), are restrictions of (16). This results from the embeddings (11) being valid for \( V = \Omega \) and \( V = S \).

Every vector
\[
(f, g_1, ..., g_m) \in \mathcal{H}_{+}^{\sigma_0 - 2m, (\sigma_0 - 2m)/(2b); \varphi}(\Omega, S)
\]
(17)
has a unique preimage \( u \in H_{+}^{\sigma_0, \sigma_0/(2b); \varphi}(\Omega) \) relative to the one-to-one mapping (16). The function \( u \) is said to be a (strong) generalized solution to the parabolic problem (2)–(4) with the right-hand sides (17).

Let us now discuss the regularity properties of this solution in Hörmander spaces. The next result follows from Theorem 4.1.

**Corollary 1.** Assume that \( u \in H_{+}^{\sigma_0, \sigma_0/(2b); \varphi}(\Omega) \) is a generalized solution to the problem (2)–(4) whose right-hand sides satisfy the condition
\[
(f, g_1, ..., g_m) \in \mathcal{H}_{+}^{\sigma - 2m, (\sigma - 2m)/(2b); \varphi}(\Omega, S)
\]
for some \( \sigma > \sigma_0 \) and \( \varphi \in \mathcal{M} \). Then \( u \in H_{+}^{\sigma, \sigma/(2b); \varphi}(\Omega) \).
Indeed, assume that the condition of this corollary is satisfied. Then, by Theorem 4.1, there is a function \( w \in H_{+}^{\sigma,\sigma/(2b)}/\varphi(\Omega) \) such that \( (A, B)w = (f, g_{1}, \ldots, g_{m}) \). Hence, \( (A, B)(u - w) = (0, 0, \ldots, 0) \), where \( u - w \in H_{+}^{\sigma,\sigma/(2b)}(\Omega) \) due to the right-hand embedding in (11). Therefore \( u - w = 0 \) according to the isomorphism (16).

Now we conclude that \( u = w \in H_{+}^{\sigma,\sigma/(2b)}/\varphi(\Omega) \).

We observe that the supplementary regularity \( \varphi \) of the right-hand sides is inherited by the solution.

Let us formulate a local version of this result. Let \( U \) be an open set in \( \mathbb{R}^{n+1} \), and let \( \omega := U \cap \Omega \neq \emptyset \), \( \pi_{1} := U \cap \partial \Omega \), and \( \pi_{2} := U \cap S \). We need to introduce local analogs of the spaces \( H_{+}^{s,\gamma}(\Omega) \) and \( H_{+}^{s,\gamma}(S) \) with \( s > 0, \gamma = 1/(2b) \), and \( \varphi \in \mathcal{M} \).

We let \( H_{+}^{s,\gamma}(\omega, \pi_{1}) \) denote the linear space of all distributions \( u \) on \( \Omega \) such that \( \chi u \in H_{+}^{s,\gamma}(\Omega) \) for each function \( \chi \in C_{0}^{\infty}(\Omega) \) with \( \text{supp} \chi \subset \omega \cup \pi_{1} \). The topology in this space is given by the seminorms

\[
u \mapsto \|\chi u\|_{H_{+}^{s,\gamma}(\omega, \pi_{1})},
\]

where \( \chi \) is an arbitrary above-mentioned function. Analogously, we let \( H_{+}^{s,\gamma}(\pi_{2}) \) denote the linear space of all distributions \( v \) on \( S \) such that \( \chi v \in H_{+}^{s,\gamma}(S) \) for each function \( \chi \in C_{0}^{\infty}(S) \) with \( \text{supp} \chi \subset \pi_{2} \). The topology in this space is given by the seminorms

\[
u \mapsto \|\chi v\|_{H_{+}^{s,\gamma}(\pi_{2})},
\]

where \( \chi \) is an arbitrary function mentioned just now.

**Theorem 4.2.** Let \( u \in H_{+}^{\sigma_{0},\sigma_{0}/(2b)}(\Omega) \) be a generalized solution to the parabolic problem (2)–(4) with the right-hand sides (17). Assume that

\[
f \in H_{+}^{\sigma-2m,\sigma/(2b)}/\varphi(\omega, \pi_{1}),
\]

\[
g_{j} \in H_{+}^{\sigma-m_{j}-1/2,\sigma/(2b)}/\varphi(\pi_{2}), \quad \text{with} \quad j = 1, \ldots, m,
\]

for some \( \sigma > \sigma_{0} \) and \( \varphi \in \mathcal{M} \). Then \( u \in H_{+}^{\sigma,\sigma/(2b)}/\varphi(\omega, \pi_{1}) \).

In the special case where \( \omega = \Omega \) and \( \pi_{1} = \partial \Omega \) (then \( \pi_{2} = S \)), Theorem 4.2 is just a repetition of Corollary 1. If \( \pi_{1} = \emptyset \), then this theorem asserts that the regularity increases in neighbourhoods of interior points of the closed domain \( \Omega \).

Using Hörmander spaces, we can obtain fine sufficient conditions under which the generalized solution \( u \) and its generalized derivatives of a prescribed order are continuous on \( \omega \cup \pi_{1} \).

**Theorem 4.3.** Let an integer \( p \geq 0 \) be such that \( p + b + n/2 > \sigma_{0} \), and let \( u \in H_{+}^{\sigma_{0},\sigma_{0}/(2b)}(\Omega) \) be a generalized solution to the parabolic problem (2)–(4) with the right-hand sides (17). Suppose that they satisfy conditions (18), (19) for \( \sigma := p + b + n/2 \) and some function parameter \( \varphi \in \mathcal{M} \) subject to

\[
\int_{1}^{\infty} \frac{dr}{r \varphi^{2}(r)} < \infty.
\]

Then the solution \( u(x, t) \) and all its generalized derivatives \( D_{x}^{\alpha} \partial_{t}^{\beta} u(x, t) \) with \( |\alpha| + 2b \beta \leq p \) are continuous on \( \omega \cup \pi_{1} \).
Remark 1. Condition (20) in Theorem 4.3 is sharp. Namely, let \( \sigma := p + b + n/2 \) and \( \varphi \in \mathcal{M} \) and assume that for every function \( u \in H^{\sigma_0,\sigma_0/(2b)}(\Omega) \) the following implication holds:

\[
( u \text{ is a solution to problem (2)–(4) for some right-hand sides (18), (19) }) \implies ( u \text{ satisfies the conclusion of Theorem 4.3 } )
\]

Then \( \varphi \) satisfies condition (20).

Remark 2. If we formulate an analog of Theorem 4.3 for the Sobolev case of \( \varphi \equiv 1 \), we have to change the condition of this theorem for a stronger one because (20) is not valid in this case. Namely, we have to claim that the right-hand sides of the problem (2)–(4) satisfy conditions (18), (19) for certain \( \sigma > p + b + n/2 \). This claim is stronger than the condition of Theorem 4.3 due to the left-hand embedding in (11).

In Section 8, we will deduce Theorem 4.2 from Isomorphism Theorem 4.1 and will show that Theorem 4.3 is a consequence of Theorem 4.2 and a version of Hörmander’s embedding theorem [13, Theorem 2.2.7]. We will also justify Remark 1.

5. Isomorphism Theorem in the Sobolev case. The goal of this section is to show that Theorem 4.1 follows from the above-mentioned result by M. S. Agranovich and M. I. Vishik [1, Theorem 12.1] in the Sobolev case where \( \varphi(r) \equiv 1 \) and \( \sigma/(2b) \in \mathbb{Z} \). Beforehand, we will prove a lemma about a description of the spaces \( H^{s,s\gamma}(\Omega) \) and \( H^{s,s\gamma}(S) \) in terms of the Hilbert spaces \( H^{s,s\gamma}(\Omega) \) and \( H^{s,s\gamma}(S) \) used in this result. The latter two are defined quite similarly to the first two, with no restrictions on support of distributions being imposed. For the reader’s convenience, we give the relevant definitions.

Let real \( s > 0 \) and \( \gamma > 0 \). Suppose that \( V \) is an open nonempty set in \( \mathbb{R}^{k+1} \), with \( k \geq 1 \). We put

\[
H^{s,s\gamma}(V) := \{ w \upharpoonright V : w \in H^{s,s\gamma}(\mathbb{R}^{k+1}) \}.
\]

The norm in the linear space (21) is defined by the formula

\[
\| u \|_{H^{s,s\gamma}(V)} := \inf \{ \| w \|_{H^{s,s\gamma}(\mathbb{R}^{k+1})} : w \in H^{s,s\gamma}(\mathbb{R}^{k+1}), \ u = w \upharpoonright V \},
\]

with \( u \in H^{s,s\gamma}(V) \). This space is Hilbert with respect to the norm (22). We are interested in the case where \( V = \Omega \), with \( k = n \), and also in the case where \( V = \Pi \), with \( \Pi := \mathbb{R}^{n-1} \times (0, \tau) \) and \( k = n - 1 \). Using \( H^{s,s\gamma}(\Pi) \), we now define the Hilbert space \( H^{s,s\gamma}(S) \) by formulas (12), (13) in which we omit the subscript +. The anisotropic Sobolev spaces just defined are well known in the theory of parabolic equations [1,10,20,43].

We have the continuous embeddings

\[
H^{s,s\gamma}_+(\Omega) \hookrightarrow H^{s,s\gamma}(\Omega) \quad \text{and} \quad H^{s,s\gamma}_+(S) \hookrightarrow H^{s,s\gamma}(S).
\]

They follow immediately from the definitions of the spaces appearing in (23).

Besides, we let \( H^{\theta}(U) \) denote the isotropic Sobolev inner product space of order \( \theta \) over a Euclidean domain \( U \); specifically, \( H^{0}(U) = L_2(U) \).

We need the following version of the result by M. S. Agranovich and M. I. Vishik [1, Proposition 8.1].
Lemma 5.1. Let $s > 0$, $\gamma > 0$, and $s\gamma - 1/2 \notin \mathbb{Z}$. Then the space $H_s^{s,\gamma}(\Omega)$ consists of all functions $u \in H^{s,\gamma}(\Omega)$ such that
\[
\partial_t^k u(x,t) \big|_{t=0} = 0 \quad \text{for almost all} \quad x \in G \\
\text{whenever} \quad k \in \mathbb{Z} \quad \text{and} \quad 0 \leq k < s\gamma - 1/2.
\] (24)
Moreover, the norm in $H_s^{s,\gamma}(\Omega)$ is equivalent to the norm in $H^{s,\gamma}(\Omega)$. This lemma remains valid if we replace $\Omega$ by $S$ and $G$ by $\Gamma$.

Lemma 5.1 is close to the result by M. S. Agranovich and M. I. Vishik [1, Proposition 8.1]. They found necessary and sufficient conditions under which the extension of every function $u \in H^{s,\gamma}(G \times (0,\theta))$ by zero belongs to $H^{s,\gamma}(G \times (-\infty,\theta))$ with $0 < \theta \leq \infty$, they restricting themselves to the case where $s \in \mathbb{Z}$ and $\gamma = 1/(2b)$. These conditions are tantamount to (24) and imply equivalence of the norms of $u$ and its extension by zero. M. S. Agranovich and M. I. Vishik also considered the case of functions given on $\Gamma \times (0,\theta)$.

Proof of Lemma 5.1. We note first that condition (24) is well posed by virtue of the trace theorem for anisotropic Sobolev spaces (see, e.g., [43, Part II, Theorem 4]). Let $\mathcal{Y}^{s,\gamma}(\Omega)$ denote the linear manifold of all functions $u \in H^{s,\gamma}(\Omega)$ that satisfy (24). According to this trace theorem, we may and will consider $\mathcal{Y}^{s,\gamma}(\Omega)$ as a (closed) subspace of $H^{s,\gamma}(\Omega)$. It follows directly from (23) that we have the continuous embedding $H_s^{s,\gamma}(\Omega) \hookrightarrow \mathcal{Y}^{s,\gamma}(\Omega)$. Therefore (in view of the Banach theorem on inverse operator) it remains to prove the converse inclusion $\mathcal{Y}^{s,\gamma}(\Omega) \subseteq H_s^{s,\gamma}(\Omega)$.

Let $u \in \mathcal{Y}^{s,\gamma}(\Omega)$. We must prove that $u = w$ on $\Omega$ for a certain function $w \in H_s^{s,\gamma}(\mathbb{R}^{n+1})$. To this end, we use the following three extension operators $O$, $T_\tau$, and $T_G$ acting between some isotropic Sobolev spaces.

Given a function $v \in L_2((0,\infty))$, we define the function $Ov \in L_2(\mathbb{R})$ by the formulas $(Ov)(t) := v(t)$ for $t > 0$ and $(Ov)(t) := 0$ for $t \leq 0$. Thus, we introduce a bounded linear operator
\[
O : L_2((0,\infty)) \rightarrow L_2(\mathbb{R}).
\] (25)
Let $\mathcal{Y}^{s,\gamma}((0,\infty))$ denote the linear manifold of all functions $v \in H^{s,\gamma}((0,\infty))$ such that $v^{(k)}(0) = 0$ whenever $k \in \mathbb{Z}$ satisfies $0 \leq k < s\gamma - 1/2$. By the trace theorem, $\mathcal{Y}^{s,\gamma}((0,\infty))$ is a subspace of $H^{s,\gamma}((0,\infty))$. It follows directly from [44, Theorems 2.9.3(a) and 2.10.3(b)] and the condition $s\gamma - 1/2 \notin \mathbb{Z}$ that the restriction of (25) to $H^{s,\gamma}((0,\infty))$ defines an isomorphism
\[
O : \mathcal{Y}^{s,\gamma}((0,\infty)) \leftrightarrow H_s^{s,\gamma}(\mathbb{R}).
\] (26)
Here, $H_s^{s,\gamma}(\mathbb{R})$ consists, by definition, of all functions $v \in H^{s,\gamma}(\mathbb{R})$ with supp $v \subseteq [0,\infty)$ and is regarded as a subspace of $H^{s,\gamma}(\mathbb{R})$.

We consider a bounded linear operator
\[
T_\tau : L_2((0,\tau)) \rightarrow L_2((0,\infty))
\] (27)
such that $T_\tau v = v$ on $(0,\tau)$ for every function $v \in L_2((0,\tau))$ and that the restriction of the mapping $T_\tau$ to the space $H^{s,\gamma}((0,\tau))$ is a bounded operator
\[
T_\tau : H^{s,\gamma}((0,\tau)) \rightarrow H^{s,\gamma}((0,\infty)).
\] (28)
We also consider a bounded linear operator
\[
T_G : L_2(G) \rightarrow L_2(\mathbb{R}^n)
\] (29)
such that $T_G h = h$ on $G$ for every function $h \in L_2(G)$ and that the restriction of the mapping $T_G$ to the space $H^s(G)$ is a bounded operator

$$T_G : H^s(G) \to H^s(\mathbb{R}^n).$$

(30)

Operators of this kind exist [44, Theorems 4.2.2 and 4.2.3].

It is known that

$$H^{s,s\gamma}(\Omega) = H^s(G) \otimes L_2((0,\tau)) \cap L_2(G) \otimes H^{s\gamma}((0,\tau))$$

(31)

and

$$H^{s,s\gamma}(\mathbb{R}^{n+1}) = H^s(\mathbb{R}^n) \otimes L_2(\mathbb{R}) \cap L_2(\mathbb{R}^n) \otimes H^{s\gamma}(\mathbb{R})$$

(32)

up to equivalence of norms; see, e.g., [1, § 8, Subsection 1]. (As usual, $E \otimes F$ denotes the tensor product of arbitrary Hilbert spaces $E$ and $F$. Besides, their intersection $E \cap F$ is considered as a Hilbert space endowed with the inner product $(v_1, v_2)_{E \cap F} := (v_1, v_2)_E + (v_1, v_2)_F$ of vectors $v_1, v_2 \in E \cap F$.)

It follows directly from (27), (28), (31) and the inclusion $u \in \mathcal{Y}^{s,s\gamma}(\Omega)$ that

$$(I \otimes T_r) u \in H^s(G) \otimes L_2((0,\infty)) \cap L_2(G) \otimes \mathcal{Y}^{s\gamma}((0,\infty)).$$

Here, $I$ is the identity operator on $L_2(G)$. Then

$$w := (T_G \otimes (OT_r)) u = (T_G \otimes O)(I \otimes T_r) w$$

$$\in H^s(\mathbb{R}^n) \otimes L_2(\mathbb{R}) \cap L_2(\mathbb{R}^n) \otimes H^{s\gamma}(\mathbb{R}) = H^{s,s\gamma}(\mathbb{R}^{n+1})$$

(33)

in view of formulas (25), (26), (29), (30), and (32). Besides, $w = u$ on $\Omega$. Thus, $u \in H^{s,s\gamma}(\Omega)$.

The same reasoning shows that Lemma 5.1 remains valid if we replace $\Omega$ by $\Pi := \mathbb{R}^{n-1} \times (0,\tau)$ and $G$ by $\mathbb{R}^{n-1}$. (Of course, we take $\mathbb{R}^n$ instead of $\mathbb{R}^{n+1}$ and need not use the extension operator $T_G$ in this case.) It follows directly from this fact and the definitions of $H^{s,s\gamma}_+(S)$ and $H^{s,s\gamma}(S)$ that Lemma 5.1 also remains valid if we replace $\Omega$ by $S$ and $G$ by $\Gamma$.

Considering Theorem 4.1, in this section we restrict ourselves to the Sobolev case where $\varphi(\tau) \equiv 1$ and $\sigma/(2b) \in \mathbb{Z}$ and suppose that $\sigma \geq \sigma_0$. Let us show that Theorem 4.1 in this case follows from M. S. Agranovich and M. I. Vishik’s result [1, Theorem 12.1].

We consider the parabolic initial-boundary value problem (2)–(4) for arbitrarily chosen right-hand sides

$$(f, g_1, \ldots, g_m) \in \mathcal{H}_{+}^{\sigma - 2m, (\sigma - 2m)/(2b)}(\Omega, S).$$

(34)

Of course, the equalities and partial derivatives appearing in this problem are interpreted in the sense of the theory of distributions. The vector (34) satisfies the compatibility condition [1, § 11] in the case of zero initial data (4). M. S. Agranovich and M. I. Vishik’s theorem [1, Theorem 12.1] asserts in view of (23) that the problem (2)–(4) has a unique solution $u \in H^{\sigma,\sigma/(2b)}(\Omega)$ and that this solution satisfies the two-sided estimate

$$\|u\|_{H^{\sigma,\sigma/(2b)}(\Omega)} \leq c_1 \| (f, g_1, \ldots, g_m) \| \mathcal{H}_{+}^{\sigma - 2m, (\sigma - 2m)/(2b)}(\Omega, S)$$

$$\leq c_2 \| u \|_{H^{\sigma,\sigma/(2b)}(\Omega)}$$

(35)
with some positive numbers $c_1$ and $c_2$ being independent of (34) and $u$. Here, we put

$$
\mathcal{H}^{-2m, (\sigma - 2m)/(2b)}(\Omega, S) := H^{-2m, (\sigma - 2m)/(2b)}(\Omega) \oplus \bigoplus_{j=1}^{m} H^{-m_j - 1/2, (\sigma - m_j - 1/2)/(2b)}(S).
$$

Let us show that $u \in \mathcal{H}^{-\sigma, \sigma/(2b)}(\Omega)$. To this end, we use Lemma 5.1 with $s := \sigma$ and $\gamma := 1/(2b)$. According to (4), the function $u$ satisfies condition (24) provided that $0 \leq k \leq \infty - 1$. Here,

$$
\infty - 1 = \frac{2m}{2b} - 1 < \frac{\sigma}{2b} - \frac{1}{2}.
$$

The fulfilment of (24) for the rest values of the integer $k$ (when $\infty - 1 < k < \sigma/(2b) - 1/2$) is proved (if these values exist) in the following way.

Let the number of these values is $l \geq 1$; then

$$
\infty + l - 1 < \frac{\sigma}{2b} - \frac{1}{2} < \infty + l.
$$

The parabolicity Condition 2.1 in the case of $\xi = 0$ and $p = 1$ means that the coefficient $a^{(0, \ldots, \infty)}(x, t) \neq 0$ for all $x \in \Omega$ and $t \in [0, \tau]$. Therefore we can resolve the parabolic equation (2) with respect to $\partial^{\infty} u(x, t)$; namely, we can write

$$
\partial^{\infty} u(x, t) = \sum_{|\alpha| + 2|\beta| \leq 2m,} a^{\alpha, \beta}_0(x, t) D_x^\alpha \partial^\beta_t u(x, t) + (a^{(0, \ldots, (\alpha, \infty)}(x, t))^{-1} f(x, t) (37)
$$

for some functions $a^{\alpha, \beta}_0 \in \mathcal{C}^\infty(\overline{\Omega})$. If $l \geq 2$, then we differentiate the equality (37) $l - 1$ times with respect to $t$ and obtain $l - 1$ equalities

$$
\partial^{\infty+j} u(x, t) = \sum_{|\alpha| + 2|\beta| \leq 2m + 2b_j, \atop |\alpha| \leq 2m, \beta \leq \infty + j - 1} a^{\alpha, \beta}_j(x, t) D_x^\alpha \partial^\beta_t u(x, t) + \partial^\beta_t ((a^{(0, \ldots, (\alpha, \infty)}(x, t))^{-1} f(x, t),
$$

with $j = 1, \ldots, l - 1$. (38)

Here, each $a^{\alpha, \beta}_j(x, t)$ is a certain function from $\mathcal{C}^\infty(\overline{\Omega})$. The equalities (37) and (38) are considered on $\Omega = \{(x, t) : x \in G, 0 < t < \tau\}$. Since $f \in \mathcal{H}^{-2m, (\sigma - 2m)/(2b)}(\Omega)$, then $\partial^\beta_t f(x, t)|_{t=0} = 0$ for almost all $x \in G$ and for each integer $j \in \{0, \ldots, l - 1\}$ by virtue of Lemma 5.1 and (36). Using these equalities, we deduce successively from (37) and (38) that $\partial^{\infty+j} u(x, t)|_{t=0} = 0$ for the same $x$ and $j$.

Thus, the function $u \in \mathcal{H}^{-\sigma, \sigma/(2b)}(\Omega)$ due to Lemma 5.1. Moreover, according to this lemma and formulas (34) and (35), we can write

$$
\|u\|_{\mathcal{H}^{-\sigma, \sigma/(2b)}(\Omega)} \leq c_3 \| (f, g_1, \ldots, g_m) \|_{\mathcal{H}^{-2m, (\sigma - 2m)/(2b)}(\Omega, S)} \leq c_4 \| u \|_{\mathcal{H}^{-\sigma, \sigma/(2b)}(\Omega)}.
$$

Here, $c_3$ and $c_4$ are some positive numbers that do not depend on (34) and $u$.

Thus, we conclude that for an arbitrary vector (34) there exists a unique solution $u \in \mathcal{H}^{-\sigma, \sigma/(2b)}(\Omega)$ of the parabolic problem (2)–(4) and that this solution
there exists a unique function for this interpolation (see also [30, Section 2]).

Hilbert spaces. We follow the monograph [34, Section 1.1], which systematically sets

sufficient for our purposes to restrict the discussion to the case of separable complex

a function parameter will play a key role in the proof of Isomorphism Theorem. It is

Section 1, Subsection 10] and [19, Chapter 1, Sections 2 and 5]). Interpolation with

is a natural generalization of the classical interpolation method by S. G. Krein

was introduced by C. Foiaş and J.-L. Lions [11, p. 278]. This interpolation

method of the interpolation with a function parameter between Hilbert spaces,

We have justified the bijection (40).

lev case under consideration. Therefore Theorem 4.1 in this case is a consequence

satisfies (39). Evidently, this conclusion is equivalent to Theorem 4.1 in the Sobolev case under consideration. Therefore Theorem 4.1 in this case is a consequence of M. S. Agranovich and M. I. Vishik’s result [1, Theorem 12.1].

Completing this section, we will show that the mapping (5) is a bijection

\[(A, B) : C^\infty_+(\Omega) \leftrightarrow C^\infty_+(\Omega) \times \left(C^\infty_+(\mathbb{S})\right)^m,\]  

(40)
as we asserted at the end of Section 2. Let us use Theorem 4.1 in the Sobolev case just considered, namely, where \(\sigma = 2bl > \sigma_0\) with \(l \in \mathbb{Z}\). The mapping (5) is injective by this theorem. It remains to show that (5) is surjective. Let a vector \((f, g_1, \ldots, g_m)\) belong to the space \(C^\infty_+(\Omega) \times \left(C^\infty_+(\mathbb{S})\right)^m\). According to this theorem, there exists a unique function

\[u \in \bigcap_{l \in \mathbb{Z}, 2bl > \sigma_0} H^{2bl,l}_+(\Omega)\]  

(41)such that \((A, B)u = (f, g_1, \ldots, g_m)\). To deduce the desired inclusion \(u \in C^\infty_+(\Omega)\) from formula (41), we use the extension operators \(O, T_\tau,\) and \(T_G\) from the proof of Lemma 5.1. We can suppose that the mappings \(T_\tau\) and \(T_G\) are independent of \(s > 0\). The extension operators of this kind are constructed, e.g., in [41]. Then, according to (33) (with \(s = 2bl\) and \(\gamma = 1/(2b)\)) and (41), we can write

\[w := (T_G \otimes (OT_\tau))u \in \bigcap_{l \in \mathbb{Z}, 2bl > \sigma_0} H^{2bl,l}_+(\mathbb{R}^{n+1}) \subset C^\infty_+(\mathbb{R}^{n+1}).\]  

(42)

Here, we note that \(u \in \mathcal{Y}^{2bl,l}(\Omega)\) by Lemma 5.1 for \(l\) indicated in (42), the space \(\mathcal{Y}^{2bl,l}(\Omega)\) being defined in the proof of this lemma. We also remark that the latter inclusion in (42) holds true by the Sobolev embedding theorem (see also [43, Part II, Theorem 13]). Hence, \(u = w \mid \Omega \in C^\infty_+(\Omega)\). Thus, the mapping (5) is surjective. We have justified the bijection (40).

6. Interpolation with a function parameter. In this section we discuss the

method of the interpolation with a function parameter between Hilbert spaces, which was introduced by C. Foiaş and J.-L. Lions [11, p. 278]. This interpolation is a natural generalization of the classical interpolation method by S. G. Krein and J.-L. Lions to the case where a sufficiently general function is used instead of a number as an interpolation parameter (see, e.g., monographs [17, Chapter IV, Section 1, Subsection 10] and [19, Chapter 1, Sections 2 and 5]). Interpolation with a function parameter will play a key role in the proof of Isomorphism Theorem. It is sufficient for our purposes to restrict the discussion to the case of separable complex Hilbert spaces. We follow the monograph [34, Section 1.1], which systematically sets forth this interpolation (see also [30, Section 2]).

Let \(X := [X_0, X_1]\) be an ordered pair of separable complex Hilbert spaces such that \(X_1 \subseteq X_0\) and this embedding is continuous and dense. This pair is said to be admissible. There is a positive-definite self-adjoint operator \(J\) on \(X_0\) with the domain \(X_1\) such that \(\|Jv\|_{X_0} = \|v\|_{X_1}\) for every \(v \in X_1\). This operator is uniquely determined by the pair \(X\) and is called a generating operator for \(X\) (see, e.g., [17, Chapter IV, Theorem 1.12]). It defines an isometric isomorphism \(J : X_1 \leftrightarrow X_0\).

Let \(\mathcal{B}\) denote the set of all Borel measurable functions \(\psi : (0, \infty) \to (0, \infty)\) such that \(\psi\) is bounded on each compact interval \([a, b]\), with \(0 < a < b < \infty\), and that \(1/\psi\) is bounded on every semiaxis \([a, \infty)\), with \(a > 0\).

Given a function \(\psi \in \mathcal{B}\), we consider the (generally, unbounded) operator \(\psi(J)\), which is defined on \(X_0\) as the Borel function \(\psi\) of \(J\). This operator is built with the
help of Spectral Theorem applied to the self-adjoint operator $J$. Let $[X_0, X_1]_\psi$ or, simply, $X_\psi$ denote the domain of $\psi(J)$ endowed with the inner product

$$(v_1, v_2)_{X_\psi} := (\psi(J)v_1, \psi(J)v_2)_{X_0}.$$  

The linear space $X_\psi$ is Hilbert and separable with respect to this inner product. The latter induces the norm $\|v\|_{X_\psi} := \|\psi(J)v\|_{X_0}$.

A function $\psi \in \mathcal{B}$ is called an interpolation parameter if the following condition is fulfilled for all admissible pairs $X = [X_0, X_1]$ and $Y = [Y_0, Y_1]$ of Hilbert spaces and for an arbitrary linear mapping $T$ given on $X_0$: if the restriction of $T$ to $X_j$ is a bounded operator $T : X_j \to Y_j$ for each $j \in \{0, 1\}$, then the restriction of $T$ to $X_\psi$ is also a bounded operator $T : X_\psi \to Y_\psi$.

If $\psi$ is an interpolation parameter, then we say that the Hilbert space $X_\psi$ is obtained by the interpolation with the function parameter $\psi$ of the pair $X = [X_0, X_1]$ (or, otherwise speaking, between the spaces $X_0$ and $X_1$). In this case, we have the dense and continuous embeddings $X_1 \hookrightarrow X_\psi \hookrightarrow X_0$.

It is known that a function $\psi \in \mathcal{B}$ is an interpolation parameter if and only if $\psi$ is pseudoconcave in a neighbourhood of infinity, i.e. there exists a concave positive function $\psi_1(r)$ of $r \gg 1$ such that both the functions $\psi/\psi_1$ and $\psi_1/\psi$ are bounded in some neighbourhood of infinity. This criterion follows from J. Peetre’s [38, 39] description of all interpolation functions for the weighted $L_p(\mathbb{R}^n)$-type spaces (this result of J. Peetre is also set forth in the monograph [5, Theorem 5.4.4]). The proof of the criterion is given in [34, Section 1.1.9].

We will use the next consequence of this criterion [34, Theorem 1.11].

**Proposition 1.** Suppose that a function $\psi \in \mathcal{B}$ varies regularly of index $\theta$ at infinity, with $0 < \theta < 1$, i.e.

$$\lim_{r \to \infty} \frac{\psi(\lambda r)}{\psi(r)} = \lambda^\theta \text{ for every } \lambda > 0.$$  

Then $\psi$ is an interpolation parameter.

The notion of a regularly varying function belongs to J. Karamata [16]. It is evident that a function $\psi : (r_0, \infty) \to (0, \infty)$, with $r_0 \in \mathbb{R}$, varies regularly of index $\theta \in \mathbb{R}$ at infinity if and only if $\psi(r) \equiv r^\theta \psi_0(r)$ for a certain function $\psi_0 : (r_0, \infty) \to (0, \infty)$ that varies slowly at infinity, both functions being assumed to be Borel measurable.

Note that, in the case of power functions, Proposition 1 leads us to the above-mentioned classical result by J.-L. Lions and S. G. Krein. Namely, they proved that the function $\psi(r) \equiv r^\theta$ is an interpolation parameter whenever $0 < \theta < 1$. In this case, the exponent $\theta$ is regarded as a number parameter of the interpolation.

We end this section with two properties of the interpolation, which will be used in our proofs. The first of them enables us to reduce the interpolation of subspaces or factor spaces to the interpolation of initial spaces (see [34, Section 1.1.6] or [44, Section 1.17]). Note that subspaces (of Hilbert spaces) are assumed to be closed and that we generally consider nonorthogonal projectors onto subspaces.

**Proposition 2.** Let $X = [X_0, X_1]$ be an admissible pair of Hilbert spaces, and let $Y_0$ be a subspace of $X_0$. Then $Y_1 := X_1 \cap Y_0$ is a subspace of $X_1$. Suppose that there exists a linear mapping $P : X_0 \to X_0$ such that $P$ is a projector of the space $X_j$ onto its subspace $Y_j$ for every $j \in \{0, 1\}$. Then the pairs $[Y_0, Y_1]$ and $[X_0/Y_0, X_1/Y_1]$ are
admissible, and
\[ [Y_0, Y_1]_\psi = X_\psi \cap Y_0, \]  
\[ [X_0/Y_0, X_1/Y_1]_\psi = X_\psi / (X_\psi \cap Y_0) \] (43) (44)
with equivalence of norms for an arbitrary interpolation parameter \( \psi \in B \). Here, \( X_\psi \cap Y_0 \) is a subspace of \( X_\psi \).

The second property reduces the interpolation of orthogonal sums of Hilbert spaces to the interpolation of their summands.

**Proposition 3.** Let \( [X_0^{(j)}, X_1^{(j)}] \), with \( j = 1, \ldots, q \), be a finite collection of admissible pairs of Hilbert spaces. Then
\[ \bigoplus_{j=1}^q X_0^{(j)}, \bigoplus_{j=1}^q X_1^{(j)} \big|_\psi = \bigoplus_{j=1}^q [X_0^{(j)}, X_1^{(j)}]_\psi \] with equality of norms for every function \( \psi \in B \).

7. **The interpolation between anisotropic Sobolev spaces.** The purpose of this section is to prove that the Hörmander spaces appearing in Theorem 4.1 can be obtained by means of the interpolation with a function parameter between their Sobolev analogs.

We assume that \( s, s_0, s_1, \gamma \in \mathbb{R}, \ s_0 < s < s_1, \ \gamma > 0, \) and \( \varphi \in M \). (45)

Consider the function
\[ \psi(r) := \begin{cases} r^{(s-s_0)/(s_1-s_0)} \varphi(r^{1/(s_1-s_0)}) & \text{for } r \geq 1, \\ \varphi(1) & \text{for } 0 < r < 1. \end{cases} \] (46)

By Proposition 1, this function is an interpolation parameter because \( \psi \) varies regularly of index \( \theta := (s-s_0)/(s_1-s_0) \) at infinity, with \( 0 < \theta < 1 \). We will interpolate pairs of Sobolev spaces with the function parameter \( \psi \).

Beforehand, we will derive the necessary interpolation formulas for the basic Hörmander spaces \( \dot{H}^{s, s_0 \gamma} \varphi (\mathbb{R}^{k+1}) \) and \( \dot{H}^{s_1, s_1 \gamma} \varphi (\mathbb{R}^{k+1}) \), with the integer \( k \geq 1 \). All the results of this section are formulated as lemmas.

**Lemma 7.1.** On the assumption (45) we have
\[ H^{s, s_0 \gamma} \varphi (\mathbb{R}^{k+1}) = [H^{s_0, s_0 \gamma} (\mathbb{R}^{k+1}), H^{s_1, s_1 \gamma} (\mathbb{R}^{k+1})]_\psi \] (47)
with equality of norms.

The proof of this lemma is analogous to the proof of [21, Lemma 5.1], where the \( k = 1 \) case was examined. Nevertheless, we give this proof for the reader’s convenience.

**Proof.** According to (6), the pair
\[ X := [H^{s_0, s_0 \gamma} (\mathbb{R}^{k+1}), H^{s_1, s_1 \gamma} (\mathbb{R}^{k+1})] \]
of anisotropic Sobolev spaces is admissible. It follows immediately from their definition that the generating operator for \( X \) is given by the formula
\[ J : w \mapsto \mathcal{F}^{-1}[r^{s_1-s_0} \mathcal{F} w], \] with \( w \in H^{s_1, s_1 \gamma} (\mathbb{R}^{k+1}) \).

Here, \( \mathcal{F} [\mathcal{F}^{-1} \) respectively] denotes the operator of the direct [inverse] Fourier transform in all variables of tempered distributions given in \( \mathbb{R}^{k+1} \).
The generating operator $J$ is reduced to the operator of multiplication by $r_{\gamma}^{s_{1}-s_{0}}$ by means of the Fourier transform which sets an isometric isomorphism

$$F : H^{s_0-s_0,\gamma}(\mathbb{R}^{k+1}) \leftrightarrow L_2(\mathbb{R}^{k+1}, r_{\gamma}^{s_{0}}(\xi, \eta)d\xi d\eta).$$

Here, of course, the second Hilbert space consists of all functions (of $\xi \in \mathbb{R}^{k}$ and $\eta \in \mathbb{R}$) that are square integrable over $\mathbb{R}^{k+1}$ with respect to the Radon measure $r_{\gamma}^{s_{0}}(\xi, \eta)d\xi d\eta$. Hence, $F$ reduces $\psi(J)$ to the operator of multiplication by the function

$$\psi(r_{\gamma}^{s_{1}-s_{0}}(\xi, \eta)) \equiv r_{\gamma}^{s_{1}-s_{0}}(\xi, \eta) \varphi(r_{\gamma}(\xi, \eta)),$$

the identity being due to (46).

Therefore, given $w \in C_0^\infty(\mathbb{R}^{k+1})$, we can write

$$\|w\|^2_{X_\psi} = \|\psi(J)w\|^2_{H^{s_0-s_0,\gamma}(\mathbb{R}^{k+1})} = \int_{\mathbb{R}^{k}} \int_{\mathbb{R}} |\psi(r_{\gamma}^{s_{1}-s_{0}}(\xi, \eta)) (Fw)(\xi, \eta)|^2 d\xi d\eta$$

$$= \|w\|^2_{H^{s_1-s_0,\gamma}(\mathbb{R}^{k+1})}. $$

This implies the equality of spaces (47) as the set $C_0^\infty(\mathbb{R}^{k+1})$ is dense in both of them. Here, we remark that this set is dense in the second space denoted by $X_\psi$ because $C_0^\infty(\mathbb{R}^{k+1})$ is dense in the space $H^{s_1-s_0,\gamma}(\mathbb{R}^{k+1})$ embedded continuously and densely in $X_\psi$. \qed

Lemma 7.2. Assume in addition to (45) that $s_0 \geq 0$. Then

$$H^{s_1-s_0,\gamma}(\mathbb{R}^{k+1}) = [H^{s_0-s_0,\gamma}(\mathbb{R}^{k+1}), H^{s_1-s_0,\gamma}(\mathbb{R}^{k+1})]_\psi$$

with equivalence of norms.

Proof. We will deduce this lemma from Lemma 7.1 with the help of Proposition 2. To this end, we need to present a linear mapping $P$ on $L_2(\mathbb{R}^{k+1})$ that the restriction of $P$ to each space $H^{s_j,s_j,\gamma}(\mathbb{R}^{k+1})$, with $j \in \{0,1\}$, is a projector of $H^{s_j,s_j,\gamma}(\mathbb{R}^{k+1})$ onto $H^{s_j,s_j,\gamma}(\mathbb{R}^{k+1})$.

Let us consider a bounded linear operator

$$T : L_2((-\infty,0)) \rightarrow L_2(\mathbb{R})$$

such that $Th = h$ on $(-\infty,0)$ for every function $h \in L_2((-\infty,0))$ and that the restriction of the mapping $T$ to the Sobolev space $H^{s_j,\gamma}((-\infty,0))$ is a bounded operator

$$T : H^{s_j,\gamma}((-\infty,0)) \rightarrow H^{s_j,\gamma}(\mathbb{R}) \text{ for each } j \in \{0,1\}. $$

The operator $T$ exists [44, Lemma 2.9.3]. Using the tensor product of bounded operators on Hilbert spaces, we obtain the bounded linear operator

$$I \otimes T : L_2(\mathbb{R}^{k} \times (-\infty,0)) \rightarrow L_2(\mathbb{R}^{k+1}) $$

such that $(I \otimes T)v = v$ on $\mathbb{R}^{k} \times (-\infty,0)$ for every function $v \in L_2(\mathbb{R}^{k} \times (-\infty,0))$. Here, $I$ is the identity operator on $L_2(\mathbb{R}^{k})$.

Given $j \in \{0,1\}$, we can write

$$H^{s_j,s_j,\gamma}(\mathbb{R}^{k+1}) = H^{s_j}(\mathbb{R}^{k}) \otimes L_2((-\infty,0)) \cap L_2(\mathbb{R}^{k}) \otimes H^{s_j,\gamma}((-\infty,0)) $$

and

$$H^{s_j,s_j,\gamma}(\mathbb{R}^{k+1}) = H^{s_j}(\mathbb{R}^{k}) \otimes L_2(\mathbb{R}) \cap L_2(\mathbb{R}^{k}) \otimes H^{s_j,\gamma}(\mathbb{R}).$$
These equalities of spaces hold true up to equivalence of norms. Being based on (49), (50), (52), and (53), we conclude that the restriction of the operator (51) to the space $H^{s_j,s_j\gamma}(\mathbb{R}^k \times (-\infty, 0))$ is a bounded operator

$$I \otimes T : H^{s_j,s_j\gamma}(\mathbb{R}^k \times (-\infty, 0)) \to H^{s_j,s_j\gamma}(\mathbb{R}^{k+1}).$$

(54)

Consider the linear mapping

$$P : w \mapsto w - (I \otimes T)(w | (\mathbb{R}^k \times (-\infty, 0))), \quad w \in L_2(\mathbb{R}^{k+1}).$$

It is easy to see that supp $Pw \subseteq \mathbb{R}^k \times [0, \infty)$ and that the inclusion supp $w \subseteq \mathbb{R}^k \times [0, \infty)$ implies the equality $Pw = w$ on $\mathbb{R}^{k+1}$. Using these properties of $P$ and the boundedness of the operator (54), we conclude that the mapping $P$ is required.

Now, by virtue of Proposition 2 (formula (43)) and Lemma 7.1, we can write

$$\begin{align*}
[&H_{+}^{s_0,s_0\gamma}(\mathbb{R}^{k+1}), H_{+}^{s_1,s_1\gamma}(\mathbb{R}^{k+1})]_\psi \\
= &H_{+}^{s_0,s_0\gamma}(\mathbb{R}^{k+1}), H_{+}^{s_1,s_1\gamma}(\mathbb{R}^{k+1})]_\psi \cap H_{+}^{s_0,s_0\gamma}(\mathbb{R}^{k+1}) \\
= &H_{+}^{s_1,s_1\gamma}(\mathbb{R}^{k+1}) \\
= &H_{+}^{s_1,s_1\gamma}(\mathbb{R}^{k+1}).
\end{align*}$$

These equalities of spaces hold true up to equivalence of norms. (Note that the first pair is admissible by Proposition 2.) Thus, we have proved (48). □

**Lemma 7.3.** Assume in addition to (45) that $s_0 \geq 0$ and

$$s_j\gamma - 1/2 \notin \mathbb{Z} \quad \text{for each} \quad j \in \{0, 1\}. \tag{55}$$

Then

$$H_{+}^{s_j,s_j\gamma}(\Omega) = [H_{+}^{s_0,s_0\gamma}(\Omega), H_{+}^{s_1,s_1\gamma}(\Omega)]_\psi, \tag{56}$$

and

$$H_{+}^{s_j,s_j\gamma}(S) = [H_{+}^{s_0,s_0\gamma}(S), H_{+}^{s_1,s_1\gamma}(S)]_\psi, \tag{57}$$

with equivalence of norms.

**Proof.** We will first prove (56). Recall that, by definition,

$$H_{+}^{s_j,s_j\gamma}(\Omega) = H_{+}^{s_j,s_j\gamma}(\mathbb{R}^{n+1})/H_{+}^{s_j,s_j\gamma}(\mathbb{R}^{n+1}, \Omega) \tag{58}$$

and

$$H_{+}^{s_j,s_j\gamma}(\Omega) = H_{+}^{s_j,s_j\gamma}(\mathbb{R}^{n+1})/H_{+}^{s_j,s_j\gamma}(\mathbb{R}^{n+1}, \Omega) \tag{59}$$

for each $j \in \{0, 1\}$. Here, the denominators are defined by (10). We will deduce formula (56) from Lemma 7.2 with the help of Proposition 2, the interpolation of factor spaces being used. For this purpose, we need to present a linear mapping $P$ on $H_{+}^{s_0,s_0\gamma}(\mathbb{R}^{n+1})$ that the restriction of $P$ to each $H_{+}^{s_j,s_j\gamma}(\mathbb{R}^{n+1})$, with $j \in \{0, 1\}$, is a projector of the space $H_{+}^{s_j,s_j\gamma}(\mathbb{R}^{n+1})$ onto its subspace $H_{+}^{s_j,s_j\gamma}(\mathbb{R}^{n+1}, \Omega)$.

Let us make use of the reasoning and notation given in the proof of Lemma 5.1. The justification of (33) presented in this proof shows also that we have the bounded operator

$$T_{+} := T_G \otimes (OT_{+}) : Y_{+}^{s_j,s_j\gamma}(\Omega) \to H_{+}^{s_j,s_j\gamma}(\mathbb{R}^{n+1}) \tag{60}$$

for each $j \in \{0, 1\}$. Here, the operators $T_G$ and $T_{+}$ respectively are restrictions of the mappings (27) and (29), which do not depend on $j$ (see (44, Theorems 4.2.2 and 4.2.3)). Therefore the operator (60) with $j = 1$ is a restriction of its counterpart with $j = 0$. Besides, as we have mentioned just after (33), the equality $T_{+}u = u$ holds on $\Omega$ for every $u \in Y_{+}^{s_j,s_j\gamma}(\Omega)$. Note also that $Y_{+}^{s_j,s_j\gamma}(\Omega) = H_{+}^{s_j,s_j\gamma}(\Omega)$ due to Lemma 5.1 and the condition (55).
Let us consider the linear mapping

$$P : w \mapsto w - T_+(w | \Omega), \quad w \in H^{s_0, s_0 \gamma}_+(\mathbb{R}^{n+1}).$$

Note that $Pw = 0$ on $\Omega$ and that the condition $w = 0$ on $\Omega$ implies the equality $Pw = w$ on $\mathbb{R}^{n+1}$. Taking into account these properties of $P$ and the boundedness of the operator (60), we conclude that the mapping $P$ is required.

Now, using (59), Proposition 2 (formula (44)), Lemma 7.2, and (58) successively, we write the following:

$$[H^{s_0, s_0 \gamma}_+(\Omega), H^{s_1, s_1 \gamma}_+(\Omega)] \psi = [H^{s_0, s_0 \gamma}_+(\mathbb{R}^{n+1}), H^{s_1, s_1 \gamma}_+(\mathbb{R}^{n+1})] \psi$$

$$= X_\psi / (X_\psi \cap H^{s_0, s_0 \gamma}_+(\mathbb{R}^{n+1}))$$

$$= H^{s, s_0 \gamma}_+(\mathbb{R}^{n+1}) / (H^{s, s_0 \gamma}_+(\mathbb{R}^{n+1}) \cap H^{s_0, s_0 \gamma}_+(\mathbb{R}^{n+1}))$$

$$= H^{s, s_0 \gamma}_+(\mathbb{R}^{n+1}) / H^{s, s_0 \gamma}_+(\mathbb{R}^{n+1}), \quad \Omega)$$

$$= H^{s, s_0 \gamma}_+(\mathbb{R}^{n+1}).$$

Here,

$$X_\psi := [H^{s_0, s_0 \gamma}_+(\mathbb{R}^{n+1}), H^{s_1, s_1 \gamma}_+(\mathbb{R}^{n+1})] \psi = H^{s, s_0 \gamma}_+(\mathbb{R}^{n+1}).$$

These equalities of spaces hold true up to equivalence of norms. (Remark also that formula (57) from its analog (56) is admissible due to Lemma 3.1. (Its proof given at the end of this section does not use Lemma 7.3 in the case where $\nu(r) \equiv 1$ and $s_0 - 1/2 \not\in \mathbb{Z}$.)) We will deduce formula (57) from its analog

$$H^{s, s_0 \gamma}_+(\mathbb{R}^{n+1}, H^{s_1, s_1 \gamma}_+(\mathbb{R}^{n+1})) \psi = (H^{s, s_0 \gamma}_+(\Pi))^\lambda \quad (61)$$

for $\Pi := \mathbb{R}^{n-1} \times (0, \tau)$. The proof of (61) is the same as that of (56), but with $\Pi$, $\mathbb{R}^n$, and the identity operator instead of $\Omega$, $\mathbb{R}^{n+1}$, and $T_G$ respectively.

Using the definition of anisotropic Hörmander spaces over $S$ given in (12) and (13), we will deduce (57) from (61) with the help of certain operators of flattening and sewing of the manifold $S$. We define the flattening operator by the formula

$$L : v \mapsto ((\chi_1^1 v) \circ \theta_1^1, \ldots, (\chi_2^1 v) \circ \theta_2^1), \quad v \in L_2(S).$$

Its restrictions to the spaces $H^{s_0, s_0 \gamma}_+(S)$ and $H^{s_1, s_1 \gamma}_+(S)$ are isometric linear operators

$$L : H^{s_0, s_0 \gamma}_+(S) \rightarrow (H^{s, s_0 \gamma}_+(\Pi))^\lambda \quad (62)$$

and

$$L : H^{s_1, s_1 \gamma}_+(S) \rightarrow (H^{s, s_0 \gamma}_+(\Pi))^\lambda \quad \text{for each} \quad j \in \{0, 1\}. \quad (63)$$

This follows immediately from the definition of these spaces. Interpolating with the function parameter $\psi$ between the spaces in (63), we obtain a bounded operator

$$L : [H^{s_0, s_0 \gamma}_+(S), H^{s_1, s_1 \gamma}_+(S)] \psi \rightarrow [(H^{s_0, s_0 \gamma}_+(\Pi))^\lambda, (H^{s_1, s_1 \gamma}_+(\Pi))^\lambda] \psi. \quad (64)$$

The latter interpolation space is equal to $(H^{s_0, s_0 \gamma}_+(\Pi))^\lambda$ up to equivalence of norms by virtue of Proposition 3 and formula (61). Hence, (64) is a bounded operator

$$L : [H^{s_0, s_0 \gamma}_+(S), H^{s_1, s_1 \gamma}_+(S)] \psi \rightarrow (H^{s_0, s_0 \gamma}_+(\Pi))^\lambda. \quad (65)$$
We define the sewing operator by the formula

\[ K : (h_1, \ldots, h_\lambda) \mapsto \sum_{k=1}^{\lambda} O_k((\eta_k^* h_k) \circ \theta_k^{-1}) \text{, with } h_1, \ldots, h_\lambda \in L_2(\Pi). \]

Here, each function \( \eta_k \in C_0^\infty(\mathbb{R}^{n-1}) \) is chosen so that \( \eta_k = 1 \) on the set \( \theta_k^{-1}(\text{supp } \chi_k) \); next \( \eta_k^*(x, t) := \eta_k(x) \) for all \( x \in \mathbb{R}^{n-1} \) and \( t \in (0, \tau) \). Besides, \( O_k \) denotes the operator of the extension (of functions) by zero from \( \Gamma_k \times (0, \tau) \) to \( S \); thus, for every \( y \in \Gamma \) and \( t \in (0, \tau) \), we have

\[ (O_k((\eta_k^* h_k) \circ \theta_k^{-1}))(y, t) = \begin{cases} \eta_k(x) h_k(x, t) & \text{if } y = \theta_k(x) \in \Gamma_k \text{ with } x \in \mathbb{R}^{n-1}; \\ 0 & \text{otherwise.} \end{cases} \]

The mapping \( K \) is left-inverse to \( L \). Indeed,

\[ KLv = \sum_{k=1}^{\lambda} O_k((\eta_k^* ((\chi_k^* v) \circ \theta_k^*)) \circ \theta_k^{-1}) \]

\[ = \sum_{k=1}^{\lambda} O_k((\chi_k^* v) \circ \theta_k \circ \theta_k^{-1}) = \sum_{k=1}^{\lambda} \chi_k^* v = v, \]

that is

\[ KLv = v \text{ for every } v \in L_2(S). \] (66)

Let us show that the restriction of the linear mapping \( K \) to \( (H_{+}^{s, \gamma; \varphi}(\Pi))^\lambda \) is a bounded operator

\[ K : (H_{+}^{s, \gamma; \varphi}(\Pi))^\lambda \to H_{+}^{s, \gamma; \varphi}(S). \] (67)

Given a vector-valued function

\[ h := (h_1, \ldots, h_\lambda) \in (H_{+}^{s, \gamma; \varphi}(\Pi))^\lambda, \]

we can write

\[ \|K h\|_{H_{+}^{s, \gamma; \varphi}(S)}^2 = \sum_{l=1}^{\lambda} \|(\chi_l^* K h) \circ \theta_l^*)\|_{H_{+}^{s, \gamma; \varphi}(\Pi)}^2 \]

\[ = \sum_{l=1}^{\lambda} \left\|\left(\chi_l^* \sum_{k=1}^{\lambda} O_k((\eta_k^* h_k) \circ \theta_k^{-1})\right) \circ \theta_l^*\right\|_{H_{+}^{s, \gamma; \varphi}(\Pi)}^2 \]

\[ = \sum_{l=1}^{\lambda} \left\|\sum_{k=1}^{\lambda} Q_{k, l} h_k\right\|_{H_{+}^{s, \gamma; \varphi}(\Pi)}^2. \] (68)

Here, we define

\[ (Q_{k, l} w)(x, t) := \eta_{k, l}(\beta_{k, l}(x)) w(\beta_{k, l}(x), t) \] (69)

for all \( w \in L_2(\Pi) \), \( x \in \mathbb{R}^{n-1} \), and \( t \in (0, \tau) \), where

\[ \eta_{k, l} := (\chi_l \circ \theta_k) \eta_k \in C_0^\infty(\mathbb{R}^{n-1}) \]

and, moreover, \( \beta_{k, l} : \mathbb{R}^{n-1} \mapsto \mathbb{R}^{n-1} \) is an infinitely smooth diffeomorphism such that \( \beta_{k, l} = \theta_k^{-1} \circ \theta_l \) in a neighbourhood of \( \text{supp } \eta_{k, l} \). As is known [15, Theorem B.1.8], the operator \( \omega \mapsto (\eta_{k, l} \omega) \circ \beta_{k, l} \) is bounded on every Sobolev space \( H^s(\mathbb{R}^{n-1}) \) with \( s \in \mathbb{R} \). Therefore the operator \( w \mapsto Q_{k, l} w \) defined by formula (69) for all \( w \in L_2(\mathbb{R}^n) \), \( x \in \mathbb{R}^{n-1} \), and \( t \in \mathbb{R} \) is bounded on each space

\[ H^{s_{\gamma}, s_{\gamma} \gamma}(\mathbb{R}^n) = H^{s_{\gamma}}(\mathbb{R}^{n-1}) \otimes L_2(\mathbb{R}) \cap L_2(\mathbb{R}^{n-1}) \otimes H^{s_{\gamma}}(\mathbb{R}) \].
with \( j \in \{0, 1\} \). Hence, the restriction of the mapping \( w \mapsto Q_{k,l} w \), with \( w \in L^2(\Pi) \),
to each space \( H^{s,j\gamma} + (\Pi) \) is a bounded operator on this space. Then, owing to the
terpolation formula (61), the restriction of this mapping to the space \( H^{s,j\gamma,\gamma}(\Pi) \)
is a bounded operator on this space. Combining the latter conclusion with (68), we can write
\[
\| K \|_{H^{s,j\gamma,\gamma}(\Pi)}^2 = \sum_{l=1}^\Lambda \left( \sum_{k=1}^\lambda Q_{k,l} \| h_k \|_{H^{s,j\gamma,\gamma}(\Pi)}^2 \right) \leq \epsilon \sum_{k=1}^\Lambda \| h_k \|_{H^{s,j\gamma,\gamma}(\Pi)}^2
\]
for some number \( c > 0 \) that does not depend on \( h \). Thus, we have proved the
boundedness of the operator (67).

The same reasoning also proves that the restriction of the mapping \( K \) to the
space \( (H^{s,j\gamma,\gamma}(\Pi))^\Lambda \) is a bounded operator
\[
K : (H^{s,j\gamma,\gamma}(\Pi))^\Lambda \to H^{s,j\gamma,\gamma}(S)
\]
Interpolating between the spaces in (70) with the function parameter \( \psi \) and using
Proposition 3 and formula (61), we obtain a bounded operator
\[
K : (H^{s,j\gamma,\gamma}(\Pi))^\Lambda \to [H^{s_0,s_0,\gamma}(S), H^{s_1,s_1,\gamma}(S)]_{\psi}.
\]
Now it follows directly from (62), (71), and (66) that the identity operator \( KL \)
realizes a continuous embedding of the space \( H^{s,j\gamma,\gamma}(S) \) in the interpolation space
\[
[H^{s_0,s_0,\gamma}(S), H^{s_1,s_1,\gamma}(S)]_{\psi}.
\]
Moreover, it follows immediately from (65) and (67) that the identity operator \( KL \) establishes the inverse continuous embedding. Thus, we have proved that the
equality (57) is true up to equivalence of norms.

**Remark 3.** Lemma 7.3 remains valid without the assumption (55). Indeed, if
\( s_j \gamma - 1/2 \in \mathbb{Z} \) for some \( j \in \{0, 1\} \), then the interpolation formulas (56) and (57)
can be deduced from this lemma with the help of the reiteration property of the
interpolation [34, Theorem 1.3].

At the end of this section, we will prove Lemma 3.1.

**Proof of Lemma 3.1.** We first examine the case where \( \varphi(r) \equiv 1 \) and \( s_\gamma - 1/2 \notin \mathbb{Z} \).
Lemma 5.1 implies that \( H^{s,\gamma}(S) \) is equal up to equivalence of norms to a certain
subspace of \( H^{s,\gamma}(S) \). Therefore, assertion (i) follows directly from its known analog
for the anisotropic Sobolev space \( H^{s,\gamma}(S) \); see [43, Chapter I, § 5].

Let us prove assertion (ii) with the help of the flattening operator \( L \) and sewing
operator \( K \) used in the proof of Lemma 7.3. As has been stated in Section 3, the
set
\[
\Upsilon_0^\infty(\Pi) := \{ w | \Pi : w \in C_0^\infty(\mathbb{R}^{n-1} \times (0, \infty)) \}
\]
is dense in \( H^{s,\gamma}(\Pi) \). Given a function \( v \in H^{s,\gamma}(S) \), we approximate the vector \( Lv \)
by the sequence of vectors \( h^{(j)} \in (\Upsilon_0^\infty(\Pi))^\Lambda \) in the norm of the space \( (H^{s,\gamma}(\Pi))^\Lambda \).
Then the sequence of functions \( K h^{(j)} \in C_0^\infty(S) \) approximates the function \( KL v = v \)
in the norm of the space \( H^{s,\gamma}(S) \). Assertion (ii) is proved in the case examined.

In the general situation, Lemma 3.1 follows from this case with the help of
Lemma 7.3. Indeed, the space \( H^{s,\gamma,\gamma}(S) \) is complete and separable due to properties
of the interpolation used in formula (57). Moreover, let \( \mathcal{A}_1 \) and \( \mathcal{A}_2 \) be two pairs
each of which consists of an atlas on \( \Gamma \) and a partition of unity used in the definitions (12) and (13). Let \( H^{s,\gamma,\gamma}(\mathcal{A}_1) \) and \( H^{s,\gamma,\gamma}(\mathcal{A}_2) \), with \( j \in \{0, 1\} \), denote
the spaces $H^{s_1,s_2}_+(S)$ and $H^{s_j,s_j}_+(S)$ corresponding to the pair $A_k$ with $k \in \{0,1\}$. Consider the identity mapping $I : v \mapsto v$, with $v \in L^2(S)$. As has been stated in the previous paragraph, a restriction of $I$ is an isomorphism between the spaces $H^{s_1,s_1}_+(S,A_0)$ and $H^{s_j,s_j}_+(S,A_1)$ for each $j \in \{0,1\}$. Therefore, by virtue of the interpolation formula (57), a restriction of $I$ is an isomorphism between $H^{s_1,s_1}_+(S,A_0)$ and $H^{s_j,s_j}_+(S,A_1)$. This means that the space $H^{s_1,s_1}_+(S)$ does not depend on the choice of an atlas on $\Gamma$ and a partition of unity. Finally, assertion (ii) in the general situation results from the density of $H^{s_1,s_1}_+(S)$ in $H^{s_1,s_1}_+(S)$. (However, this assertion can be proved in the same way as that in the previous paragraph.)

8. **Proofs of the main results.** In this section, we will prove Theorems 4.1, 4.2, and 4.3. We will also justify Remark 1.

**Proof of Theorem 4.1.** Let $\sigma > \sigma_0$ and $\varphi \in M$. We choose an integer $\sigma_1 > \sigma$ so that $\sigma_1/(2b) \in \mathbb{Z}$. According to M. S. Agranovich and M. I. Vishik’s result [1, Theorem 12.1], the mapping (5) extends uniquely (by continuity) to isomorphisms

$$(A,B) : H^{\sigma_1\varphi,\sigma_k}(\Omega) \leftrightarrow H^{\varphi_m,\sigma_k}(\Omega, S) \quad \text{with} \quad k \in \{0,1\}. \quad (72)$$

(Here, recall, the second space is defined by formula (15) with $\sigma := \sigma_k$ and $\varphi \equiv 1$.)

Let us define the interpolation parameter $\psi$ by formula (46) in which $s := \sigma$, $s_0 := \sigma_0$, and $s_1 := \sigma_1$. Interpolating with the function parameter $\psi$ between the spaces in (72), we obtain an isomorphism

$$(A,B) : [H^{\sigma_0\varphi_0/(2b)}(\Omega), H^{\sigma_1\varphi_1/(2b)}(\Omega)]_\psi \leftrightarrow [H^{\varphi_m,\sigma_0/(2b)}(\Omega, S), H^{\varphi_m,\sigma_1/(2b)}(\Omega, S)]_\psi. \quad (73)$$

It is a restriction of the operator (72) with $k = 0$.

According to Lemma 7.3 and Proposition 3 we can write

$$[H^{\sigma_0\varphi_0/(2b)}(\Omega), H^{\sigma_1\varphi_1/(2b)}(\Omega)]_\psi = H^{\sigma,\varphi/(2b)}(\Omega)$$

and

$$[H^{\varphi_0,\sigma_0/(2b)}(\Omega, S), H^{\varphi_1,\sigma_1/(2b)}(\Omega, S)]_\psi = H^{\varphi_m,\sigma/(2b)}(\Omega, S).$$

These equalities of spaces hold true up to equivalence of norms. Thus, the isomorphism (73) becomes (14). This isomorphism is an extension by continuity of the mapping (5) because the set $C^\infty_0(\Omega)$ is dense in the space $H^{\sigma/(2b);\varphi}(\Omega). \quad \square$

**Proof of Theorem 4.2.** We will first prove that, under the conditions (18) and (19) of this theorem, the implication

$$u \in H^{\lambda-\lambda_1,\varphi/(2b)}(\omega, \pi_1) \Rightarrow u \in H^{\lambda-\lambda_1,\varphi/(2b)}(\omega, \pi_1) \quad (74)$$

holds for each integer $\lambda \geq 1$ subject to $\sigma - \lambda + 1 > \sigma_0$. 

We arbitrarily choose a function \( \chi \in C^\infty(\Omega) \) with supp \( \chi \subseteq \omega \cup \pi_1 \). For \( \chi \) there exists a function \( \eta \in C^\infty(\Omega) \) such that supp \( \eta \subseteq \omega \cup \pi_1 \) and \( \eta = 1 \) in a neighbourhood of supp \( \chi \). Interchanging each of the differential operators \( A \) and \( B_j \) with the operator of the multiplication by \( \chi \), we can write
\[
(A, B)(\chi u) = (A, B)(\chi \eta u) = \chi (A, B)(\eta u) + (A', B')(\eta u) = \chi (A, B)u + (A', B')(\eta u) = \chi (f, g_1, \ldots, g_m) + (A', B')(\eta u).
\] (75)
Here, \( (A', B') := (A', B'_1, \ldots, B'_m) \) is a differential operator with components
\[
A'(x, t, D_x, \partial_t) = \sum_{|\alpha| + 2\beta \leq 2m - 1} a_1^{\alpha, \beta}(x, t) D_x^\alpha \partial_t^\beta \quad (76)
\]
and
\[
B'_j(x, t, D_x, \partial_t) = \sum_{|\alpha| + 2\beta \leq m_j - 1} b_1^{\alpha, \beta}(x, t) D_x^\alpha \partial_t^\beta, \quad j = 1, \ldots, m, \quad (77)
\]
where all \( a_1^{\alpha, \beta} \in C^\infty(\Omega) \) and \( b_1^{\alpha, \beta} \in C^\infty(S) \). This operator acts continuously between the spaces
\[
(A', B') : H_{+}^s(B; \varphi/(2b)) = 1 \to H_{+}^{s+1-2m,.(s+1-2m)/(2b)}(\Omega, S) \quad (78)
\]
for every \( s > \sigma_0 - 1 \). In the case where \( \varphi \equiv 1 \) and where the second superscripts are not half-integers, this follows directly from (76), (77), Lemma 5.1 and the known properties of the anisotropic Sobolev space \( H_{+}^{s,s/(2b)}(\Omega) \) (see, e.g., [43, Chapter I, Lemma 4, and Chapter II, Theorems 3 and 7]). The boundedness of the operator (78) in the general situation is plainly deduced from this case with the help of the interpolation Lemma 7.3.

By the conditions (18) and (19), we obtain the inclusion
\[
\chi (f, g_1, \ldots, g_m) \in H_{-}^{\sigma - 2m,.(\sigma - 2m)/(2b)}(\Omega, S).
\]
Besides, according to (78) with \( s := \sigma - \lambda \), we have the implication
\[
\begin{align*}
u & \in H_{+}^{\sigma - \lambda,.(\sigma - \lambda)/(2b)}(\omega, \pi_1) \\
\Rightarrow (A', B')(\eta u) & \in H_{+}^{\sigma - \lambda + 1 - 2m,.(\sigma - \lambda + 1 - 2m)/(2b)}(\Omega, S).
\end{align*}
\]
Hence, using (75) and Corollary 1, we can write
\[
\begin{align*}
u & \in H_{+}^{\sigma - \lambda,.(\sigma - \lambda)/(2b)}(\omega, \pi_1) \\
\Rightarrow (A, B)(\chi u) & \in H_{+}^{\sigma - \lambda + 1 - 2m,.(\sigma - \lambda + 1 - 2m)/(2b)}(\Omega, S) \\
\Rightarrow \chi u & \in H_{+}^{\sigma - \lambda + 1,.(\sigma - \lambda + 1)/(2b)}(\Omega).
\end{align*}
\]
Note that Corollary 1 is applicable here because \( \chi u \in H_{+}^{\sigma_0,.(\sigma_0)/(2b)}(\Omega) \) by the condition of the theorem and because \( \sigma - \lambda + 1 > \sigma_0 \). Thus, owing to our choice of \( \chi \), we have proved the implication (74).

Let us use this implication in our proof of the inclusion \( u \in H_{+}^{\sigma,.(\sigma)/(2b)}(\omega, \pi_1) \). We will separately examine the case of \( \sigma \notin \mathbb{Z} \) and the case of \( \sigma \in \mathbb{Z} \).

Consider first the case of \( \sigma \notin \mathbb{Z} \). In this case, there exists an integer \( \lambda_0 \geq 1 \) such that
\[
\sigma - \lambda_0 < \sigma_0 < \sigma - \lambda_0 + 1. \quad (79)
\]
Using the implication (74) successively with \( \lambda := \lambda_0, \lambda := \lambda_0 - 1, \ldots, \) and \( \lambda := 1, \) we deduce the desired inclusion in the following way:

\[
\begin{align*}
\forall \lambda \geq 1, \quad u & \in H^\sigma_{+; +, \varphi}(\Omega) \\
\Rightarrow u & \in H^\sigma_{+; +, \varphi}(\Omega) \\
\Rightarrow \ldots \Rightarrow u & \in H^\sigma_{+; +, \varphi}(\Omega).
\end{align*}
\]

Note that \( u \in H^\sigma_{+; +, \varphi}(\Omega) \) by the condition of the theorem.

Consider now the case of \( \sigma \in \mathbb{Z}. \) In this case, there is no integer \( \lambda_0 \) that satisfies (79). Nevertheless, since \( \sigma - 1/2 \notin \mathbb{Z} \) and \( \sigma - 1/2 > \sigma_0, \) the inclusion

\[
u \in H^\sigma_{+; +, \varphi}(\Omega)
\]

holds true as we have proved in the previous paragraph. Hence, using the implication (74) with \( \lambda := 1, \) we deduce the desired inclusion; namely:

\[
\begin{align*}
\forall \lambda \geq 1, \quad u & \in H^\sigma_{+; +, \varphi}(\Omega) \\
\Rightarrow u & \in H^\sigma_{+; +, \varphi}(\Omega).
\end{align*}
\]

To deduce Theorem 4.3 from Theorem 4.2 and to justify Remark 1, we use a version of Hörmander’s embedding theorem [13, Theorem 2.2.7].

**Lemma 8.1.** Let \( p \in \mathbb{Z}, \) \( p \geq 0, \) \( s := p + b + n/2, \) and \( \varphi \in \mathcal{M}. \) The following two assertions are true:

(i) If \( \varphi \) satisfies (20), then every function \( u \in H^{s, s/(2b); \varphi}(\mathbb{R}^{n+1}) \) has the following property: all its generalized partial derivatives \( D^j \varphi \) with \( 0 \leq |\alpha| + 2b\beta \leq p \) are continuous on \( \mathbb{R}^{n+1}. \)

(ii) Let \( V \) be a nonempty open subset of \( \mathbb{R}^{n+1}, \) and let an integer \( k \) be such that \( 1 \leq k \leq n. \) If every function \( u \in H^{s, s/(2b); \varphi}(\mathbb{R}^{n+1}) \) with \( \supp \subset V \) satisfies the condition \( \varphi \) of (20) for each \( j \in \mathbb{Z} \) with \( 0 \leq j \leq p, \) then \( \varphi \) satisfies (20). Here, \( \partial^j \varphi \) denotes the generalized partial derivative \( \partial^j \varphi \) of the function \( \varphi = |\xi + \eta|^2 \).

**Proof.** (i) Given a function \( u \in H^{s, s/(2b); \varphi}(\mathbb{R}^{n+1}), \) we consider its arbitrary partial derivative \( \partial^j \varphi \) of \( u \) with \( 0 \leq |\alpha| + 2b\beta \leq p. \) The condition

\[
\int \int \frac{[\xi_n^2 |\eta|^2 \varphi^{(2b)}(r_{\gamma}(\xi, \eta))]}{r_{\gamma}^{2\gamma}(\xi, \eta)} \, d\xi \, d\eta < \infty
\]

implies the inclusion \( \partial^j \varphi \in C(\mathbb{R}^{n+1}); \) here, \( \gamma := 1/(2b). \) Indeed, by the Schwarz inequality, we can write

\[
\int \int \frac{[\xi_n^2 |\eta|^2 \varphi^{(2b)}(r_{\gamma}(\xi, \eta))]}{r_{\gamma}^{2\gamma}(\xi, \eta)} \, d\xi \, d\eta \leq \|w\|_{H^{s, s/(2b); \varphi}(\mathbb{R}^{n+1})} \int \int \frac{|\xi_n^2 |\eta|^2 \varphi^{(2b)}(r_{\gamma}(\xi, \eta))]}{r_{\gamma}^{2\gamma}(\xi, \eta)} \, d\xi \, d\eta,
\]

Hence, if the condition (80) is fulfilled, then the function \( \partial^j \varphi \) is integrable over \( \mathbb{R}^{n+1} \) and therefore its inverse Fourier transform \( \partial^j \varphi \) is continuous on \( C(\mathbb{R}^{n+1}). \)

Let us show that (20) implies this condition. In the multiple integral written in (80), we change the variable \( \eta = \eta_{2b}, \) then pass to the spherical coordinates...
(θ_1, \ldots, θ_n) with \( \varrho = (|ξ|^2 + η^2)^{1/2} \), and finally change the variable \( r = (1 + \varrho^2)^{1/2} \).

So, we write the following:

\[
\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|ξ|^2 |η|^{2b} dξ dη}{r^{2s}(ξ, η) \varphi^2(r(ξ, η))} = 2^{n+1} \int_0^∞ \int_0^∞ \frac{|ξ|^2 |η|^{2b} dξ dη}{r^{2s}(ξ, η) \varphi^2(r(ξ, η))}
\]

\[
= 2^{n+1} \int_0^∞ \int_0^∞ \frac{2b |ξ|^2 η^{4b+2b-1} dξ dη}{(1 + |ξ|^2 + η^2)^s \varphi^2(\sqrt{1 + |ξ|^2 + η^2})}
\]

\[
= c_{α, β} \int_0^∞ \frac{β^{2|α|+4bβ+2b-1} dβ}{(1 + β^2)^s \varphi^2(r^2 - 1)^{1/2}} = c_{α, β} \int_1^∞ \frac{(r^2 - 1)^{α-1 - δ(α, β)} r dr}{r^{2s-1} \varphi^2(r)}
\]

here, \( c_{α, β} \) is a certain positive number, and

\[ δ(α, β) := p - |α| - 2bβ \in [0, p]. \]

Thus,

\[
\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|ξ|^2 |η|^{2b} dξ dη}{r^{2s}(ξ, η) \varphi^2(r(ξ, η))} = c_{α, β} \int_1^∞ \frac{(r^2 - 1)^{α-1 - δ(α, β)} r dr}{r^{2s-1} \varphi^2(r)}. \quad (81)
\]

Note that

\[ (20) \Leftrightarrow \int_1^∞ \frac{(r^2 - 1)^{s-1}}{r^{2s-1} \varphi^2(r)} dr < ∞ \quad (82) \]

\[ \Rightarrow \int_1^∞ \frac{(r^2 - 1)^{s-1 - δ(α, β)}}{r^{2s-1} \varphi^2(r)} dr < ∞. \]

The latter implication holds true because \( δ(α, β) \geq 0 \) and

\[ s - 1 - δ(α, β) \geq p + b + n/2 - 1 - p \geq 0. \]

Therefore (20) implies (80) in view of (81). Assertion (i) is proved.

(ii) Let the assumption made in assertion (ii) be fulfilled. Following Hörmander’s proof of the necessity part of his embedding theorem [13, Theorem 2.2.7], we conclude that

\[
\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|ξ|^2 |η|^{2b} dξ dη}{r^{2s}(ξ, η) \varphi^2(r(ξ, η))} < ∞.
\]

This property together with (81) gives

\[ \int_1^∞ \frac{(r^2 - 1)^{s-1}}{r^{2s-1} \varphi^2(r)} dr < ∞. \quad (83) \]

Here, we use (81) in the case where \( α_k = p, \ α_q = 0 \) whenever \( q \neq k \), and \( β = 0 \), then \( δ(α, β) = 0 \). Now (20) follows from (82) and (83). Assertion (ii) is proved. \( \square \)
Proof of Theorem 4.3. We arbitrarily choose a point \( M \in \omega \cup \pi_1 \). Let a function \( \chi \in C^\infty (\Omega) \) satisfy the following conditions: \( \text{supp} \chi \subseteq \omega \cup \pi_1 \), and \( \chi = 1 \) in a certain neighbourhood \( V(M) \subseteq \Omega \) of \( M \). According to Theorem 4.2 we have the inclusion \( \chi \in \Omega \subseteq H^{n/2} (\mathbb{R}^n) \). Hence, there exists a function \( w \in H^{n/2} (\mathbb{R}^n) \) such that \( w = \chi u = u \) on the set \( V(M) \). According to Lemma 8.1(i), each generalized partial derivative \( D^s_x \partial^j_t w(x, t) \), with \( 0 \leq |\alpha| + 2b \beta \leq p \), is continuous on \( \mathbb{R}^n \). Thus, the derivative \( D^s_x \partial^j_t u(x, t) \) is continuous in the neighbourhood \( V(M) \) of \( M \). Since the point \( M \in \omega \cup \pi_1 \) is arbitrarily chosen, this derivative is continuous on \( \omega \cup \pi_1 \). 

At the end of this section, we will justify Remark 1. Let \( \varphi \in M \), and let an integer \( p \geq 0 \) be subject to \( \sigma := p + b + n/2 > \sigma_0 \). Assume that every function \( u \in H^{\sigma_0, \sigma_0/(2b)} (\Omega) \) satisfies the implication given in Remark 1. Hence if for an arbitrary function \( u \in H^{\sigma_0, \sigma_0/(2b)} (\Omega) \), we put 

\[
(f, g_1, \ldots, g_m) := (A, B)u \in H^{\sigma_0, \sigma_0/(2b)} (\Omega),
\]

then we infer that \( u \) satisfies the conclusion of Theorem 4.3. Thus, if the function \( u \) belongs to \( H^{\sigma_0, \sigma_0/(2b)} (\Omega) \), then all its generalized derivatives \( D^s_x \partial^j_t u(x, t) \) with \( |\alpha| + 2b \beta \leq p \) are continuous on \( \omega \cup \pi_1 \). Specifically, each derivative \( \partial^j_t u \) with \( 0 \leq j \leq p \) is continuous on \( \omega \cup \pi_1 \).

Let \( V \) be a nonempty open subset of \( \mathbb{R}^{n+1} \) such that \( V \subset \omega \). We arbitrarily choose a function \( w \in H^{\sigma_0, \sigma_0/(2b)} (\mathbb{R}^{n+1}) \) such that \( \text{supp} w \subset V \), and we put 

\[
(\varphi := w (x, t) \in H^{\sigma_0, \sigma_0/(2b)} (\Omega)).
\]

The function \( w \) and its generalized derivatives \( \partial^j_t w \) with \( 1 \leq j \leq p \) are continuous on \( \mathbb{R}^{n+1} \) due to the property of \( u \) deduced in the previous paragraph under the assumption made. Hence, \( \varphi \) satisfies (20) due to Lemma 8.1(ii). Thus, we have justified Remark 1.

Appendix. Let \( \varphi \in M \), \( \gamma > 0 \), and \( s \in \mathbb{R} \). In Section 3, we introduced the function 

\[
\mu (\xi, \eta) = r^s (\xi, \eta) \varphi (r \xi, \eta) \text{ of } \xi \in \mathbb{R}^k \text{ and } \eta \in \mathbb{R}
\]

and referred to the Hörmander space \( B_{p, \mu} \) over \( \mathbb{R}^k \), with \( 1 \leq k \in \mathbb{Z} \) and \( p \geq 1 \). The space \( B_{p, \mu} \) is well defined if the function \( \mu \) satisfies the following Hörmander’s condition: there exist numbers \( c \geq 1 \) and \( l > 0 \) such that

\[
\frac{\mu (\xi', \eta_1)}{\mu (\xi''', \eta_2)} \leq c (1 + |\xi' - \xi'''| + |\eta_1 - \eta_2|)^l
\]

for arbitrary \( \xi', \xi''' \in \mathbb{R}^k \) and \( \eta_1, \eta_2 \in \mathbb{R} \). This condition is given in [13, Section 2.1, Remark] (see also [45, Chapter I, Section 1, Subsection 1]).

Let us prove that \( \mu \) satisfies Hörmander’s condition. Since \( \varphi \in M \), there exists a number \( c_\varphi \geq 1 \) such that

\[
\frac{\varphi (\lambda r)}{\varphi (r)} \leq c_\varphi \lambda \quad \text{and} \quad \frac{\varphi (r)}{\varphi (\lambda r)} \leq c_\varphi \lambda
\]

for arbitrary \( r \geq 1 \) and \( \lambda \geq 1 \); see [34, Section 2.4.1, formula (2.91)]. Besides, according to [45, Chapter I, Section 2, Subsection 2] there exist numbers \( c_\gamma \geq 1 \)
and \( l, \gamma > 0 \) such that
\[ r_{\gamma}(\xi', \eta_1) \leq c_\gamma (1 + |\xi' - \xi''| + |\eta_1 - \eta_2|)^{l_\gamma} \tag{86} \]
for arbitrary \( \xi', \xi'' \in \mathbb{R}^k \) and \( \eta_1, \eta_2 \in \mathbb{R} \).

Let \( \xi', \xi'' \in \mathbb{R}^k \) and \( \eta_1, \eta_2 \in \mathbb{R} \). In the case of \( r_{\gamma}(\xi', \eta_1) \geq r_{\gamma}(\xi'', \eta_2) \), we obtain the following bound:
\[
\frac{\mu(\xi', \eta_1)}{\mu(\xi'', \eta_2)} = \frac{r^{s}_{\gamma}(\xi', \eta_1) \varphi(r_{\gamma}(\xi', \eta_1))}{r^{s}_{\gamma}(\xi'', \eta_2) \varphi(r_{\gamma}(\xi'', \eta_2))} \leq c_\varphi \frac{r^{s+1}_{\gamma}(\xi', \eta_1)}{r^{s+1}_{\gamma}(\xi'', \eta_2)}
\leq c_\varphi c_\gamma^{\max(s+1,0)} (1 + |\xi' - \xi''| + |\eta_1 - \eta_2|)^{l_\gamma \max(s+1,0)}.
\]

Here, we use the first inequality in (85) and then apply (86). Similarly, in the opposite case where \( r_{\gamma}(\xi', \eta_1) < r_{\gamma}(\xi'', \eta_2) \), we get
\[
\frac{\mu(\xi', \eta_1)}{\mu(\xi'', \eta_2)} \leq c_\varphi \frac{r^{1-s}_{\gamma}(\xi', \eta_1)}{r^{1-s}_{\gamma}(\xi'', \eta_2)} = c_\varphi \frac{r^{1-s}_{\gamma}(\xi', \eta_1)}{r^{1-s}_{\gamma}(\xi'', \eta_2)}
\leq c_\varphi c_\gamma^{\max(1-s,0)} (1 + |\xi' - \xi''| + |\eta_1 - \eta_2|)^{l_\gamma \max(1-s,0)}.
\]

Here, we use the second inequality in (85) and then apply (86) again. Thus, in any case we obtain the required inequality (84) with
\[ c := c_\varphi c_\gamma^{\max(s+1,1-s,0)} \]
and \( l := l_\gamma \max(s+1,1-s,0) \).

Note finally that L. Hörmander [13, Definition 2.1.1] initially assumes the function \( \mu \) to satisfy some stronger condition than (84). But he remarks below [13, Section 2.1, Remark] that the set of the functions \( \mu \) satisfying (13, Definition 2.1.1) and the set of the continuous functions \( \mu \) satisfying (84) give the same collection of the spaces \( B_{p,\mu} \). As L. R. Volevich and B. P. Paneah [45, Chapter I, Section 1, Subsection 1] note, the condition of continuity of \( \mu \) is not essential and can be replaced with the weaker condition of Borel measurability. Specifically, since \( \varphi \in \mathcal{M} \), there exists a function \( \varphi_1 \in \mathcal{M} \cap C^\infty([1, \infty)) \) that both the functions \( \varphi/\varphi_1 \) and \( \varphi_1/\varphi \) are bounded on \( [1, \infty) \) (this follows from [42, Section 1.4, property 1\(^{\circ} \)])). Therefore the spaces \( B_{p,\mu} \) and \( B_{p,\mu_1} \) coincide up to equivalence of norms; here, \( \mu_1(\xi, \eta) := r^{s}_{\gamma}(\xi, \eta) \varphi_1(r_{\gamma}(\xi, \eta)) \) is a continuous function on \( \mathbb{R}^{k+1} \). Moreover, given the infinitely smooth function
\[ \mu_2(\xi, \eta) := r^{s}_{\gamma}(\xi, \sqrt{1 + \eta^2}) \varphi_1(r_{\gamma}(\xi, \sqrt{1 + \eta^2})) \]
on \( \mathbb{R}^{k+1} \), we conclude that the spaces \( B_{p,\mu} \) and \( B_{p,\mu_2} \) coincide up to equivalence of norms as well.

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**REFERENCES**

[1] M. S. Agranovich and M. I. Vishik, Elliptic problems with parameter and parabolic problems of general form, (Russian), Uspehi Mat. Nauk, 19 (1964), 53–161 [English translation in Russian Math. Surveys, 19 (1964), 53–157].

[2] A. V. Anop and A. A. Murach, Parameter-elliptic problems and interpolation with a function parameter, Methods Funct. Anal. Topology, 20 (2014), 103–116.

[3] A. V. Anop and A. A. Murach, Regular elliptic boundary-value problems in the extended Sobolev scale, Ukrainian Math. J., 66 (2014), 969–985.

[4] Yu. M. Berezansky, Expansions in Eigenfunctions of Selfadjoint Operators, Transl. Math. Monogr., vol. 17, American Mathematical Society, Providence, R.I., 1968.

[5] J. Bergh and J. Löfström, Interpolation Spaces, Grundlehren Math. Wiss., band 223, Springer-Verlag, Berlin-New York, 1976.
[6] N. H. Bingham, C. M. Goldie and J. L. Teugels, *Regular Variation*, Encyclopedia Math. Appl., vol. 27, Cambridge University Press, Cambridge, 1989.
[7] F. Cobos and D. L. Fernandez, *Hardy-Sobolev spaces and Besov spaces with a function parameter*, in *Function Spaces and Applications* (eds. M. Cwikel, J. Peetre, Y. Sagher and H. Wallin), Lecture Notes in Math., vol. 1302, Springer, Berlin, (1988), 158–170.
[8] S. D. Eidel’man, *Parabolic Systems*, North-Holland Publishing Co., Amsterdam-London; Wolters-Noordhoff Publishing, Groningen, 1969.
[9] S. D. Eidel’man, Parabolic equations, in *Encyclopaedia Math. Sci.*, vol. 63 (Partial Differential Equations, VI. Elliptic and Parabolic Operators) (eds. Yu.V. Egorov and M.A. Shubin), Springer, Berlin, (1994), 205–316.
[10] S. D. Eidel’man and N. V. Zhitarashu, *Parabolic Boundary Value Problems*, Oper. Theory Adv. Appl., vol. 101, Birkhäuser Verlag, Basel, 1998.
[11] C. Foiaş and J.-L. Lions, Sur certains théorèmes d’interpolation, *Acta Scient. Math. Szeged*, 22 (1961), 209–282.
[12] A. Friedman, *Partial Differential Equations of Parabolic Type*, Prentice-Hall Inc., Englewood Cliffs, N.J., 1964.
[13] L. Hörmander, *Linear Partial Differential Operators*, Grundlehren Math. Wiss., band 116, Academic Press, Inc., Publishers, New York; Springer-Verlag, Berlin-Göttingen-Heidelberg, 1963.
[14] L. Hörmander, *The Analysis of Linear Partial Differential Operators, vol. II*, Differential Operators with Constant Coefficients, Grundlehren Math. Wiss., band 257, Springer-Verlag, Berlin, 1983.
[15] L. Hörmander, *The Analysis of Linear Partial Differential Operators, Vol. III*, Pseudo-Differential Operators, Grundlehren Math. Wiss., band 274, Springer-Verlag, Berlin, 1985.
[16] J. Karamata, Sur certains Tauberian theorems de M. M. Hardy et Littlewood, *Mathematica (Cluj)*, 3 (1930), 33–48.
[17] S. G. Krein, Yu. L. Petunin and E. M. Semënov, *Interpolation of Linear Operators*, Transl. Math. Monogr., vol. 54, American Mathematical Society, Providence, R.I., 1982.
[18] O. A. Ladyženskaja, V. A. Solonnikov and N. N. Ural’tzeva, *Linear and Quasilinear Equations of Parabolic Type*, Transl. Math. Monogr., vol. 23, American Mathematical Society, Providence, R.I., 1968.
[19] J.-L. Lions and E. Magenes, *Non-Homogeneous Boundary-Value Problems and Applications*, vol. I, Grundlehren Math. Wiss., band 181, Springer-Verlag, New York-Heidelberg, 1972.
[20] J.-L. Lions and E. Magenes, *Non-Homogeneous Boundary-Value Problems and Applications*, vol. II, Grundlehren Math. Wiss., band 182, Springer-Verlag, New York-Heidelberg, 1972.
[21] V. Los and A. A. Murach, Parabolic problems and interpolation with a function parameter, *Methods Funct. Anal. Topology*, 19 (2013), 146–160.
[22] V. Los and A. A. Murach, Parabolic mixed problems in spaces of generalized smoothness, (Russian), *Dopov. Nats. Acad. Nauk. Ukr. Mat. Prirodozn. Tehn. Nauki*, 6 (2014), 23–31.
[23] V. Los, Mixed problems for the two-dimensional heat-conduction equation in anisotropic Hörmander spaces, *Ukrainian Math. J.*, 67 (2015), 735–747.
[24] C. Merucci, Application of interpolation with a function parameter to Lorentz, Sobolev and Besov spaces, in *Interpolation Spaces and Allied Topics in Analysis* (eds. M. Cwikel and J. Peetre), Lecture Notes in Math., vol. 1070, Springer, Berlin, (1984), 183–201.
[25] V. A. Mikhailets and A. A. Murach, Elliptic operators in a refined scale of function spaces, *Ukrainian Math. J.*, 57 (2005), 817–825.
[26] V. A. Mikhailets and A. A. Murach, Refined scales of spaces, and elliptic boundary-value problems. I, *Ukrainian Math. J.*, 58 (2006), 244–262.
[27] V. A. Mikhailets and A. A. Murach, Refined scales of spaces, and elliptic boundary value problems. II, *Ukrainian Math. J.*, 58 (2006), 398–417.
[28] V. A. Mikhailets and A. A. Murach, Refined scales of spaces, and elliptic boundary-value problems. III, *Ukrainian Math. J.*, 59 (2007), 744–765.
[29] V. A. Mikhailets and A. A. Murach, A regular elliptic boundary-value problem for a homogeneous equation in a two-sided refined scale of spaces, *Ukrainian Math. J.*, 58 (2006), 1748–1767.
[30] V. A. Mikhailets and A. A. Murach, Interpolation with a function parameter and refined scale of spaces, *Methods Funct. Anal. Topology*, 14 (2008), 81–100.
[31] V. A. Mikhailets and A. A. Murach, An elliptic boundary-value problem in a two-sided refined scale of spaces, *Ukrainian Math. J.*, 60 (2008), 574–597.
[32] V. A. Mikhailets and A. A. Murach, The refined Sobolev scale, interpolation, and elliptic problems, *Banach J. Math. Anal.*, 6 (2012), 211–281.

[33] V. A. Mikhailets and A. A. Murach, Extended Sobolev scale and elliptic operators, *Ukrainian Math. J.*, 65 (2013), 435–447.

[34] V. A. Mikhailets and A. A. Murach, *Hörmander Spaces, Interpolation, and Elliptic Problems*, De Gruyter Studies in Math., vol. 60, De Gruyter, Berlin, 2014.

[35] V. A. Mikhailets and A. A. Murach, Interpolation Hilbert spaces between Sobolev spaces, *Results Math.*, 67 (2015), 135–152.

[36] A. A. Murach, Elliptic pseudo-differential operators in a refined scale of spaces on a closed manifold, *Ukrainian Math. J.*, 59 (2007), 874–893.

[37] A. A. Murach and T. Zinchenko, Parameter-elliptic operators on the extended Sobolev scale, *Methods Funct. Anal. Topology*, 19 (2013), 29–39.

[38] J. Peetre, On interpolation functions, *Acta Sci. Math. (Szeged)*, 27 (1966), 167–171.

[39] J. Peetre, On interpolation functions II, *Acta Sci. Math. (Szeged)*, 29 (1968), 91–92.

[40] I. G. Petrovskii, On the Cauchy problem for systems of partial differential equations in the domain of non-analytic functions, (Russian) *Bull. Mosk. Univ., Mat. Mekh.*, 1 (1938), 1–72.

[41] V. S. Rychkov, On restrictions and extensions of the Besov and Triebel–Lizorkin spaces with respect to Lipschitz domain, *J. London Math. Soc.*, 60 (1999), 237–257.

[42] E. Seneta, *Regularly Varying Functions*, Lecture Notes in Math., vol. 508, Springer, Berlin, 1976.

[43] L. N. Slobodeckii, Generalized Sobolev spaces and their application to boundary problems for partial differential equations, (Russian), *Leningrad. Gos. Ped. Inst. Uchen. Zap.*, 197 (1958), 54–112 [English translation in *Amer. Math. Soc. Transl. (2)*, 57 (1966), 207–275].

[44] H. Triebel, *Interpolation Theory, Function Spaces, Differential, Operators*, 2nd edition, Johann Ambrosius Barth, Heidelberg, 1995.

[45] L. R. Volevich and B. P. Paneah, Certain spaces of generalized functions and embedding theorems, (Russian), *Uspekhi Mat. Nauk*, 20 (1965), 3–74 [English translation in *Russian Math. Surveys*, 20 (1965), 1–73].

[46] T. N. Zinchenko and A. A. Murach, Douglis–Nirenberg elliptic systems in Hörmander spaces, *Ukrainian Math. J.*, 64 (2013), 1672–1687.

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