Beyond Fourier transform: break the axial resolution limit in optical coherence tomography

Yuye Ling\textsuperscript{1,\,*}, Mengyuan Wang\textsuperscript{1}, Yu Gan\textsuperscript{2}, Xinwen Yao\textsuperscript{3,4,5}, Leopold Schmetterer\textsuperscript{3,4,5,6}, Chuanqing Zhou\textsuperscript{7,\,*}, Yikai Su\textsuperscript{8}

\textsuperscript{1} John Hopcroft Center for Computer Science, Shanghai Jiao Tong University, Shanghai, China
\textsuperscript{2} Department of Electrical and Computer Engineering, The University of Alabama, AL, USA
\textsuperscript{3} SERI-NTU Advanced Ocular Engineering (STANCE), Singapore
\textsuperscript{4} Institute for Health Technologies, Nanyang Technological University, Singapore
\textsuperscript{5} Singapore Eye Research Institute, Singapore National Eye Centre, Singapore
\textsuperscript{6} School of Chemical and Biomedical Engineering, Nanyang Technological University, Singapore
\textsuperscript{7} Institute of Biomedical Engineering, Shenzhen Bay Lab, Shenzhen, China
\textsuperscript{8} State Key Lab of Advanced Optical Communication Systems and Networks, Department of Electronic Engineering, Shanghai Jiao Tong University, Shanghai, China
\textsuperscript{\,*} yuye.ling@sjtu.edu.cn and zhoucq@sjtu.edu.cn

January 10, 2020

Abstract

Optical coherence tomography (OCT) is a volumetric imaging modality that empowers the clinicians and scientists to noninvasively visualize and examine the microscopic cross-sectional architectures of biological samples. By taking advantages of the coherence gating, OCT could achieve micrometer-level axial sectioning with high sensitivity. In the past decade, substantial efforts have been made to push the axial resolution of Fourier-domain OCT (FD-OCT) further into the sub-micron regime via physically extending the system’s spectral bandwidth. Here, we would like to offer a new perspective on this problem. We argue that the coupling between the axial resolution and the spectral bandwidth in FD-OCT is induced by the conventional Fourier transform-based reconstruction algorithm. To surpass this limitation, we present a novel optimization-based reconstruction algorithm to recover the image from ordinary FD-OCT measurement with proper assumptions. We retrieved images at a resolution higher than the theoretical prediction in both numerical simulations and proof-of-concept experiments.

The everlasting technological advancement in optical imaging is one of the driving forces that reshape the landscape of modern healthcare and medical research [1, 2, 3, 4]. The success of OCT in ophthalmology is a prominent example: the volumetric imaging technique pivots from bench to bedside and evolves into
one of the gold standards in less than 30 years. It is estimated that more than 30 million OCT exams are performed annually across the globe [5]. There is therefore a continuing interest in elevating the performance of OCT, especially for its axial resolution.

Fourier-domain OCT (FD-OCT), the latest generation of OCT, typically possesses an axial resolutions ranging from a few to tens of micrometers. Unlike its time-domain counterpart, FD-OCT utilizes a simple inverse discrete Fourier transform (IDFT) to recover the axial profile of the object from its spectral interferogram, which have enabled numerous exciting applications including real-time imaging [6] and functional imaging [7, 8]. Based on the current FD-OCT theory, it is believed that the axial resolution of the reconstruction is theoretically limited by the full width half maximum (FWHM) of the light source’s coherence function, which is inversely proportional to the emission bandwidth of the light source [9, 10].

Signal processing techniques including deconvolution [11], spectral shaping [12], and numerical dispersion compensation [13, 14] are widely deployed in FD-OCT to improve the image quality. Novel techniques such as sparse representation [15] and deep learning [16, 17] are also gaining traction in retrieving high-resolution images from low-resolution ones, despite requiring a learning or training process. Nonetheless, the axial resolution of the processed image would still be lower than the aforementioned theoretical limit.

As a result, “increasing emission bandwidth” essentially becomes a synonym for “developing a high-resolution FD-OCT system” [18] in the community. To date, multiple groups have showcased FD-OCT with about 1 μm axial resolution. Most of which are obtained by an 800 nm system with a bandwidth about 300 nm [19, 20, 21]. It is worth noting that a further expansion in the bandwidth would be contingent on the technological progression in laser sources. Moreover, the excessive dispersion stems from the broader bandwidth might even counter the benefits.

One intriguing question could thus be raised: is increasing the bandwidth the sole path to higher axial resolution, or is there a way to circumvent?

Here, we propose an optimization-based reconstruction framework to super-resolve the OCT images. The basic concept can be understood as follows. The image reconstruction is first modeled as a mixed-signal system and formulated into an inverse problem: to recover the object function from its sampled spectral interferogram. In contrast to the IDFT-based reconstruction, we then integrate apparent physical insights of the object, including sparsity, to the solution in the proposed scheme. This type of prior knowledge could help regularize the originally ill-posed inverse problem and provides a superior axial resolving capability. Our claim is supported by both numerical simulation and proof-of-concept experiments on biological samples.

Since the proposed method is developed upon ordinary FD-OCT measurement, it could be retrospectively applied onto any existing datasets to improve the axial resolution as long as the raw spectral measurements are accessible. To facilitate the process, we also make our source codes fully available online.

2
Results

Super-resolving OCT: an *ill-posed* inverse problem

Figure 1: Schematic illustration of the fringe formation and the image reconstruction in FD-OCT. After FD-OCT imaging, the analog object \( R_S(z) \) forms an analog spectral interferogram pattern \( I(k) \), which will later be digitized by measurement devices. Conventionally, an IDFT is performed directly on the resultant digital signal \( I[m] \) to reconstruct the object. However, the reconstruction resolution \( \delta_z' \) and image depth \( \Delta z' \) are thus implicitly defined by the spectral sampling interval \( \delta k \) and bandwidth \( \Delta k \). Here we proposed an optimization approach to this process: both the grid size \( \delta z \) and image depth \( \Delta z \) could be arbitrarily defined by the user to achieve best reconstruction performance.

The FD-OCT is modeled as a mixed-signal system as illustrated in Fig 1: the physical formation of the spectral interferogram \( I(k) \) is in the analog domain, while the image reconstruction is in the digital domain. Two domains are interfaced by measurement devices, which perform the analog-to-digital conversion (ADC). The measured spectral interferogram \( I[m] \) could be given by (See Supplementary Note 1),

\[
I(k) = S(k) \cdot \int_{m}^{m+\Delta z} \sqrt{R_S(z)} \cos(2kz) \, dz
\]

\[
I[m] = S[m] \cdot \int_{m}^{m+\Delta z} \sqrt{R_S(z)} \cos(2kz) \, dz
\]
\[ I[m] = 2S(k_m) \int_{z_0}^{z_0 + \Delta z} r_s(z) \cos(2k_m z) \, dz + N(k_m), \quad m = 0, \ldots, M - 1, \quad (1) \]

where \( S(k_m) \) is the sampled power emission spectrum of the light source, \( N(k_m) \) is the noise presented, \( k_m \) is the \( m \)th sample in the wavenumber domain, \( r_s(z) \) is the reflectivity profile (object function) of the sample, and \([z_0, z_0 + \Delta z]\) defines its support.

In order to numerically solve Eq. 1, we discretize its RHS uniformly by \( N \) times, and rewrite the equation in short form by using matrix-vector notation \[22\],

\[ \mathbf{i} = \delta_z \cdot (\mathbf{s} \odot \mathbf{C} \mathbf{r}) + \mathbf{n} \quad (2) \]

where \( \mathbf{i} = \{I[0], I[1], \ldots, I[M - 1]\}^\top, \mathbf{r} = \{r_s(z_0), r_s(z_1), \ldots, r_s(z_{N-1})\}^\top, \mathbf{s} = \{S(k_0), S(k_1), \ldots, S(k_{M-1})\}^\top, \delta_z = \Delta z/(N - 1), \mathbf{C} \) is an \( M \times N \) matrix with \( C_{mn} = \cos(2k_m z_n) \), and \( \odot \) represents for Hadamard product or element-wise multiplication. It is worth noting that the discretization error \( \mathbf{n}_{\text{disc}} \) is absorbed into the new noise term \( \mathbf{n} \),

\[ \mathbf{n} = \{N(k_0), N(k_1), \ldots, N(k_{M-1})\}^\top + \mathbf{n}_{\text{disc}} \quad (3) \]

and \( \mathbf{n}_{\text{disc}} \) would asymptotically converge to zero when the sampling number \( N \) approaching infinity (See Supplementary Note 1). Therefore, we will mostly end up with \( N \gg M \).

It is now apparent that the image reconstruction in FD-OCT could be categorized as an ill-posed inverse problem \[23\]: the aim is to restore an \( N \)-dimensional object \( \mathbf{r} \) from an \( M \)-dimensional measurement \( \mathbf{i} \) by solving a set of linear equations (Eq. 2). Naturally, this problem could be solved via \( \ell_2 \)-norm minimization,

\[ \arg \min_{\hat{\mathbf{r}}} \| \mathbf{i} - \frac{\Delta z}{N} \cdot \mathbf{A} \hat{\mathbf{r}} \|_2^2 \quad (4) \]

where \( \hat{\mathbf{r}} \) is the reconstructed object discretized on a grid that is arbitrarily defined by the user.

**Reconstruction via constrained optimization**

Before diving into the vast literature that discusses how to solve ill-posed inverse problems \[24, 25\], we would like to first offer an analysis on the current paradigm adopted in FD-OCT community. Conventionally, Eq. 2 is solved by directly performing an IDFT on the spectral measurements \[26\], which is equivalent to left-multiplying \( \mathbf{i} \) by an IDFT matrix \( \mathbf{F}^{-1} \),

\[ \hat{\mathbf{r}}_{\text{IDFT}} = \mathbf{F}^{-1} \mathbf{i} \quad (5) \]

This procedure automatically defines the basis functions \( \{\phi_m = \delta(z - m\delta_z) | m = 0, 1, \ldots, M - 1\} \) for the reconstruction \( \hat{\mathbf{r}}_{\text{IDFT}} \) \[27\], and the number of the bases equals to the number of the spectral measurements \( M \); the grid spacing \( \delta_z = \frac{\Delta z}{N - 1} \).
1/\Delta k and the grid length (imaging depth) \Delta z' = 1/\delta k are also related to their spectral counterparts via reciprocity relations [27].

It is interesting to observe that \( \hat{r}_{\text{IDFT}} \) actually solves Eq. 4, if we ignore the modulation of the emission spectrum and if the discretization grid in the object domain is identical to that of the reconstruction domain. In other words,

\[
\hat{r}_{\text{IDFT}} = \mathcal{F}^{-1}i = \arg \min_{\hat{r}} \|i - \frac{\Delta z}{N} \cdot C\hat{r}\|_2^2
\]

(6)

However, this rarely happens. Considering that Eq. 4 is often heavily under-determined, the reconstructed \( \hat{r}_{\text{IDFT}} \) would thus suffer from the spectral leakage and the entire system would become shift-variant (See Supplementary Note 2).

Fortunately, the OCT image reconstruction is a physical problem: the solution that we are seeking should not only fit the mathematical equations described above, but also satisfy some additional constraints coming from the physical picture. We can therefore apply regularization to Equation 4 to acquire a feasible solution.

One such prior information available is the predefined support for the object \( r \) [28]. For example, a rough estimation of the support could be obtained from the low-resolution \( \hat{r}_{\text{IDFT}} \).

Another a priori we could utilize is the sparsity of the object \( r \). Considering that \( N \) is a huge number and \( \delta_z \) be a fraction of the incident wavelength, it is plausible for us to assume the refractive index distribution of the object to be slow varying and piece-wise constant at this scale. This will lead to a sparse reflectivity profile \( r \), and we could use \( \ell_1 \)-norm minimization to promote the sparsity in the solution [29, 30]. It is worth noting that the sparsity presented here is independent of the structure of the object, but relies on the sampling density.

We would like to focus on using the later constraint in the current study, although there are infinitely many possible assumptions we could reasonably make towards the structure of the solution [31]. The OCT reconstruction is thus formatted as a constrained optimization problem,

\[
\begin{align*}
\text{minimize} & \quad \|\hat{r}\|_1 \\
\text{subject to} & \quad \|i - \frac{\Delta z}{N} \cdot A\hat{r}\|_2^2 \leq \epsilon
\end{align*}
\]

(7)

Axial super-resolution demonstration

To prove the concept, numerical simulation was first conducted. We used two closely located Kronecker delta functions as the target and numerically synthesized their spectral interferograms. Six different techniques including plain IDFT, plain IDFT followed by Lucy-Richardson deconvolution [32], \( \ell_1 \)- and \( \ell_2 \)-norm minimization with both coarse [33] and fine grid were used to reconstruct the object. Here, the coarse grid is same as that defined by the Fourier relationship, while the fine grid has eight times smaller grid step. The reconstructed results were plotted in Fig. 2 for different separations. Considering that the
theoretical axial resolution of the system is $\sim 19 \, \mu m$ (See Methods), it is expected that the conventional IDFT-based reconstruction could only resolve the spikes when they are separated by $20 \, \mu m$ as shown in Fig. 2a. The resolving capability of the system is improved when deconvolution algorithm is applied afterwards: the $15 \, \mu m$’s separation could be discriminated as illustrated in Fig. 2b. Neither $\ell_1$- nor $\ell_2$-norm minimization with coarse grid obtains better performance than that of the deconvolution. However, using $\ell_2$-norm minimization in a finer grid succeeds in resolving spikes that are only $10 \, \mu m$ apart, and using $\ell_1$-norm minimization in a finer grid further push the limit to $5 \, \mu m$. Resolution beyond theoretical prediction is achieved in both cases, and they are depicted in Fig. 2c and 2d, respectively.

Figure 2: Simulation of six different reconstruction techniques for FD-OCT. Two Kronecker delta functions, the object, are separated by (a) $20 \, \mu m$, (b) $15 \, \mu m$, (c) $10 \, \mu m$, and (d) $5 \, \mu m$, respectively. The theoretical axial resolution of the simulated system is $\sim 19 \mu m$. The proposed $\ell_1$-norm minimization with fine grid shows the best performance.

To further validate our idea, we applied our method on onion images acquired from a commercial FD-OCT system. We first reconstructed images from full-spectrum measurements, and use them as references as shown in Fig. 3(b)-(e). We then repeated the procedures for reduced-bandwidth measurements as illustrated in Fig. 3(a), in which we are mimicking the scenarios of having lower-
Figure 3: Single-frame comparisons of the reconstructed onion obtained by IDFT and the proposed technique. (a) Exemplary input spectrum for different spectral bandwidth. Different bandwidths are obtained via numerical truncation. (b)-(e), Reconstructed images obtained by using conventional IDFT from full bandwidth, half bandwidth, quarter bandwidth, and eighth bandwidth, respectively. (f)-(i), Reconstructed images obtained by using optimization-based technique from the same input as in (b)-(e). (j)-(m), Magnified views of the regions highlighted in (b)-(e). (n)-(q), Magnified views of the regions highlighted in (f)-(i). The red arrows point out the double-layer structure of the onion skin that could be visually distinguished.
Figure 4: The super-resolving capability of the proposed technique. A lateral position within the region pointed out by red arrows in Fig. 3 is selected. The A-line profile excerpts at the selected location reconstructed by using both IDFT and proposed techniques from (a) full bandwidth, (b) half bandwidth, (c) quarter bandwidth, and (d) eighth bandwidth are plotted, respectively. The corresponding theoretically predicted point spread functions and their FWHM are also given for reference. The separation between the two layers of the onion skin is measured 7.8 µm. The proposed method managed to resolve this structure even when the theoretical resolution is larger (c.f. (c)).
resolution systems. We compared the images obtained from the latter (Fig. 3(f)-(q)) with the references. Specifically, we visually examined whether the double-layer skin structure pointed out by the red arrows is visible in various reduced-bandwidth cases. By using the proposed method, we could even distinguish the double-layer structure when the bandwidth is quartered (Fig. 3(m)). Finally, we plotted the A-line profiles of the onion skin obtained by both methods for different bandwidth cases in Fig. 4. The theoretically predicted point spread functions for each case are also provided for references. It is confirmed that the proposed technique does enable higher resolution in experimental situations and outperforms its IDFT-based counterpart as illustrated in Fig. 4(c). In fact, the proposed method achieved a resolution higher than the theoretical value just as predicted in the numerical simulation.

We also applied our technique on existing dataset to retrospectively super-resolve the images. Specifically, the original data was obtained from a custom-built ultra-high-resolution SD-OCT [34] on \textit{in vivo} monkey cornea [35]. The conventional IDFT-based reconstruction is presented in Fig. 5(a), which is the same one rendered as Fig. 5(a) in [35], and its super-resolved counterpart is given in Fig. 5(b). The super-resolved image shows better axial resolution with slightly reduced SNR. Moreover, the Descemet’s membrane, which could barely be resolved, is clearly observed in Fig. 5(b). Magnified views of the same region are illustrated in Fig. 5(c) and (d) for better comparison.
Discussion

For such a long period of time, the usage of IDFT to reconstruct the FD-OCT is simply taken for granted. Although there are tons of literature discussing how to use advanced signal processing techniques to perform the post-processing, there are few that touches the reconstruction procedure. We deem this as abnormal: in other branches of biomedical imaging such as magnetic resonance imaging, computer tomography and even optical diffraction tomography, inverse problem based reconstruction techniques have been widely explored and deployed. This pilot research shows the great potential of using such a framework on FD-OCT.

In conclusion, a novel image reconstruction algorithm for FD-OCT based on optimization is presented. By exploiting the physical priors of the object, the proposed technique can outperform its IDFT-based counterpart, and the result is validated both numerically and experimentally. This work shall offer a new perspective to the FD-OCT community as to the image reconstruction problem: IDFT-based construction might not be the only solution not to mention its optimality. In the future, more sophisticated physical priors could be integrated to the proposed framework to further improve the image quality. Theoretical analysis on the convergence of the algorithm, the lower bound of the resolution, as well as the robustness of the algorithm would also be the future direction of this study.

Acknowledgement

The authors would like to thank Dr. Bo Jiang and Dr. Wenxuan Liang for their enlightening discussions. This project is supported by National Natural Science Foundation of China (NSFC) (61905141, 61875123), Shanghai Science and Technology Committee (1744190202900), and Shanghai Jiao Tong University Interdisciplinary Research Fund of Biomedical Engineering (YG2016ZD07). We acknowledge computing resources from Shanghai Jiao Tong University high-performance computing facilities π2.0-cluster.

References

[1] Huang, D. et al. Optical coherence tomography. *Science* **254**, 1178–1181 (1991).

[2] Hell, S. W. & Wichmann, J. Breaking the diffraction resolution limit by stimulated emission: stimulated-emission-depletion fluorescence microscopy. *Opt. Lett.*** 19**, 780–782 (1994).

[3] Greenbaum, A. et al. Imaging without lenses: achievements and remaining challenges of wide-field on-chip microscopy. *Nat. Methods*** 9**, 889 (2012).
[18] Cense, B. et al. Ultrahigh-resolution high-speed retinal imaging using spectral-domain optical coherence tomography. *Opt. Express* **12**, 2435–2447 (2004).

[19] Liu, L. et al. Imaging the subcellular structure of human coronary atherosclerosis using micro-optical coherence tomography. *Nat. Med* **17**, 1010 (2011).

[20] Yadav, R. et al. Micrometer axial resolution oCT for corneal imaging. *Biomed. Opt. Express* **2**, 3037–3046 (2011).

[21] Bizheva, K. et al. Sub-micrometer axial resolution oCT for in-vivo imaging of the cellular structure of healthy and keratoconic human corneas. *Biomed. Opt. Express* **8**, 800–812 (2017).

[22] Stoer, J. & Bulirsch, R. *Introduction to numerical analysis*, vol. 12 (Springer, 2013).

[23] Bertero, M. & Boccacci, P. *Introduction to inverse problems in imaging* (CRC press, 1998).

[24] Vogel, C. R. *Computational Methods for Inverse Problems* (Society for Industrial and Applied Mathematics, 2002).

[25] Hansen, P. C. *Discrete inverse problems: insight and algorithms*, vol. 7 (Society for Industrial and Applied Mathematics, 2010).

[26] Drexler, W. & Fujimoto, J. G. *Optical Coherence Tomography: Technology and Applications* (Springer), second edition edn.

[27] Briggs, W. L. & Henson, V. E. *The DFT: An Owner’s Manual for the Discrete Fourier Transform* (Society for Industrial and Applied Mathematics, 1995).

[28] Vaswani, N. & Lu, W. Modified-cs: Modifying compressive sensing for problems with partially known support. *IEEE Trans. Signal Process* **58**, 4595–4607 (2010).

[29] Donoho, D. L. Compressed sensing. *IEEE Trans. Inf. Theory* **52**, 1289–1306 (2006).

[30] Candes, E. J., Romberg, J. & Tao, T. Robust uncertainty principles: Exact signal reconstruction from highly incomplete frequency information. *IEEE Trans. Inf. Theory* **52**, 489–509 (2006).

[31] Lubk, A. *Chapter Three - Tomography*, vol. 206, 59 – 104 (Elsevier, 2018).

[32] Liu, Y., Liang, Y., Mu, G. & Zhu, X. Deconvolution methods for image deblurring in optical coherence tomography. *J. Opt. Soc. Am. A* **26**, 72–77 (2009).
Methods

Proposed optimization-based reconstruction algorithm

Although the proposed technique can incorporate various physical constraints into the reconstructed image, we focus on the use of sparsity properties of the sample here as an example. This case was referred as ℓ1-norm minimization in the main text, and was elaborated in Equation 4.

We employed the Alternating Direction Method of Multipliers (ADMM) [36] to solve Equation 4, and the derived algorithm is summarized in Algorithm 1.

The algorithm is implemented in GNU Octave (version 4.4.1) on a high-performance computing facility (Shanghai Jiao Tong University π2.0-cluster, China). It takes about 4s to reconstruct an A-line with 2,048 sampling points. The maximum iteration number is set to 1,000 to balance the reconstruction speed and the quality.

Numerical simulation

We assume the power spectral density $S(k)$ have a Gaussian shape, the central wavelength $\bar{\lambda}$ is 1310 nm, and the FWHM is 40 nm. The spectral interferogram is sampled by $M = 1,024$ times within a range of 100 nm (from 1260 nm to 1360 nm).

Two Kronecker delta functions are used as the object. The discretization interval of the object was set to be an extremely small value (100 nm here) to minimize the error. The support size of the object was set to 1 mm. We numerically computed the spectral measurement $I[m]$ as the input.

For reconstruction, the size of the coarse grid was equated to the reconstruction definition in the IDFT case ($\sim 8.4 \mu$m), while the size of the fine grid to be much smaller ($\sim 1 \mu$m). We used the built-in function `IFFT` and `deconvlucy` in Octave to implement the IDFT and Lucy-Richardson algorithm, respectively. The ℓ1- and ℓ2-norm minimization are both implemented by the aforementioned
Algorithm 1: ADMM algorithm for solving Equation 4

Problem: \( \arg \min_{\hat{r}} \frac{1}{2} ||i - \frac{\Delta z}{N} \cdot A \hat{r}||^2_2 + \lambda ||s||_1 \) s.t. \( \hat{r} - s = 0 \)

Input: \( A \) in \( \mathbb{R}^{m \times n} \), \( i \) and \( \rho, \lambda, \alpha \)

Initialize: \( \hat{r}^{(0)}, s^{(0)}, u^{(0)} \) where \( u^{(0)} = \frac{1}{\rho} y^{(0)} \)

While: \( \hat{r}^{(k)}, s^{(k)}, u^{(k)} \) do not converge (or reach maximum iteration)

\( \hat{r}^{(k+1)} \leftarrow \arg \min_{\hat{r}} L(\hat{r}^{(k)}, s^{(k)}, u^{(k)}) \)
\( = \arg \min f(\hat{r}^{(k)}) + \frac{\rho}{2} ||\hat{r}^{(k+1)} - v^{(k)}||^2_2 \)
\( = (AA^T + \rho)^{-1} (A^{-1}b + \rho v^{(k)}) \)

\( s^{(k+1)} \leftarrow \arg \min_s L(\hat{r}^{(k+1)}, s^{(k)}, u^{(k)}) \)
\( = \arg \min f(\hat{r}^{(k+1)}) + \frac{\rho}{2} ||\hat{r}^{(k+1)} - v^{(k)}||^2_2 \)

\( u^{(k+1)} \leftarrow u^{(k)} + \hat{r}^{(k+1)} - s^{(k+1)} \)

end While

Output: \( \hat{r}^* \leftarrow \hat{r} \)

ADMM-based algorithm; \( \lambda \) was set to 100 for \( \ell_1 \)-norm minimization and 0 for \( \ell_2 \).

Experimental set-up

The onion imaging was performed on a commercially available high-resolution spectral-domain optical coherence tomography (SD-OCT) system (GAN620C1, Thorlabs, USA). The system is centered at 892.8 nm, and the spectrometer measures from 791.6 nm to 994.0 nm with 2,048 pixels. The reconstruction grid step and imaging depth, based on IDFT method, are 1.94 \( \mu \)m and 1.9 mm, respectively. The grid size and support size for the optimization-based method was set to 1 \( \mu \)m and 1 mm, respectively.

We extracted the \( k \)-linearized spectra directly from the SD-OCT system. No preprocessing was performed. To obtain the reduced-bandwidth measurement, we numerically truncated the original measurement as illustrated in Fig. 3(a). The corresponding theoretical PSF is obtained by directly performing IDFT on the resultant emission spectra. During the reconstruction, the Lagrange multiplier \( \lambda \) for full, half, quarter, and eighth bandwidth is 1,000, 1,000, 100, and 10, respectively.
1 An inverse problem model for FD-OCT

First conceived by Fercher et al in 1995 under the name of “backscattering spectral interferometry” [1], FD-OCT offers a one-dimensional approximate solution to the optical inverse scattering problem [2]. The authors derived that the backscattered field amplitude $A_S(k)$ is proportional to the Fourier transform of the scattering potential $F_S(z)$, when the measurement is taken in the far field from a weakly scattering object [1]. This important finding laid the theoretical foundation of FD-OCT: the depth profile of an unknown object could be simply retrieved by performing an inverse Fourier transform on the spectrum of the backscattered light. However, in most real-world implementations, backscattered light intensity $I(k) = |A_S(k)|^2$ is only available in its truncated and sampled form. Handily, most researchers either explicitly or implicitly use the inverse discrete Fourier transform (IDFT) in place of the inverse Fourier transform to recover the object function without further justifications [3, 4, 5].

Unfortunately, this signal processing procedure might be erroneous; the Fourier transform relationship existed between $F_S(z)$ and $A_S(k)$ does not guarantee the discrete Fourier transform relationship between their sampled counterparts in the discrete domain. For example, Dirac delta function $f(z) = \delta(z - z_0)$ and $F(k) = \exp(-jkz_0)$ forms a Fourier transform pair in the continuous domain, but their truncated and sampled copies do not necessarily form a “discrete Fourier transform pair”. Generally speaking, performing an IDFT on the truncated and sampled $F(k)$ would not bring back the expected Kronecker delta function, unless the truncated interval is an integer multiple of $1/z_0$. Instead, issues including sampling errors, alias, and leakage could arise [6] and compromise the elegant physical picture depicted in the continuous domain.

In the following pages, we would like to offer a mathematical model to rigorously describe the mixed-signal nature of the FD-OCT signal processing.

1.1 The effect of digitization

In the following derivations, we will use power reflectivity and scattering potential interchangeably.

Assume the weakly scattering object under investigation has a reflectivity profile $r_S(z)$, which is a real-valued function defined on a finite support of...
\[ z_0, z_0 + \Delta z \] as illustrated in Figure 1. The scattered electric field \( E_S(k) \) in the sample arm and reflected electric field \( E_R(k) \) in the reference arm are given by

\[
E_S(k) = E_i(k) \cdot \int_{z_0}^{z_0+\Delta z} r_S(z)e^{-j2kz} \, dz
\]

and

\[
E_R(k) = E_i(k) \cdot r_R
\]

where \( k \) is the wavenumber, \( z \) is the displacement between the reference mirror and sample, \( E_i(k) \) is the incident electric field, and \( r_R \) is the reflectivity of the reference mirror. Here, we implicitly place the reference mirror at \( z = 0 \).

The interferometric pattern \( I(k) \) formed by a typical FD-OCT (before being measured) could be expressed as,

\[
I(k) = |E_S(k) + E_R(k)|^2
\]

\[
= |E_S(k)|^2 + |E_R(k)|^2 + E_S^*(k)E_R(k) + E_S(k)E_R^*(k)
\]

\[
= \text{autocorrelation} + \text{DC} + S(k) \cdot r_R \cdot \int_{-z_0-\Delta z}^{z_0+\Delta z} f(z)e^{-j2kz} \, dz
\]

where \( S(k) = E_i^*(k)E_i(k) \) is the power emission spectrum of the light source. Generally speaking, the autocorrelation term could be omitted when the FD-OCT system is operated in shot-noise limited regime and the DC term could be filtered out. Therefore, Equation could be rewritten as,

\[
I(k) = S(k) \cdot r_R \cdot \int_{-z_0-\Delta z}^{z_0+\Delta z} f(z)e^{-j2kz} \, dz
\]

where

\[
f(z) = \begin{cases} 
  r_S(z), & z \in [z_0, z_0 + \Delta z] \\
  r_S(-z), & z \in [-z_0 - \Delta z, z_0] \\
  0, & \text{otherwise}
\end{cases}
\]

The newly constructed object function \( f(z) \) is just the original sample profile plus its mirror. Clearly, the spectral interferogram \( I(k) \) and object function \( f(z) \) forms a Fourier transform pair.

Once the interferogram is measured, \( I(k) \) will be digitized, i.e. both truncated and sampled. Without losing generality, we assume the signal is uniformly sampled in the \( k \) domain by \( M \) times. The resultant discrete signal could then be written as,

\[
I[m] = I(k_m)
\]

\[
\sim S(k_m) \cdot \int_{-z_0-\Delta z}^{z_0+\Delta z} f(z)e^{-j2k_m z} \, dz, \quad m = 0, \ldots, M-1,
\]

\[
= 2S(k_m) \cdot \int_{z_0}^{z_0+\Delta z} r_S(z) \cos(2k_m z) \, dz, \quad m = 0, \ldots, M-1,
\]
where \( k_m = k_0 + m\delta_k \), \( k_0 \) is the starting wavenumber for the spectral measurement, \( \delta_k \) is the sampling interval in the \( k \) domain, and \( m \) is the corresponding index. And constant scalars, \( \rho \) and \( r_R \), are ignored in our analysis.

It is now intriguing for us to quickly evaluate the effect of digitization based on Equation 6: the analog signal \( I(k) \) is not only sampled at an interval of \( \delta_k \) but also truncated to a shorter range \([k_0, k_{M-1}]\). According to the property of Fourier transform, the first operation, spectral sampling, is equivalent to create a periodic summation of the original function in the object domain and the periodicity is equal to \( 1/2\delta_k \). However, the second operation, truncation, would convolve the object function with a complex-valued function, which is surprisingly shift-variant in its general form. We will come back to this point in Note 2.

1.2 Image retrieval in OCT as an inverse problem

As stated at the beginning of the section, the fundamental problem in FD-OCT is to recover the original object profile \( r_S(z) \) from its spectral measurement \( I[m] \). This could be addressed by solving Equation 6 as an inverse problem, which gives the forward relationship between \( I[m] \) and \( r_S(z) \).

In order to numerically solve Equation 6, which is an integral equation, we could discretize the integral on its RHS,

\[
\sum_{n=0}^{N-1} r_S(z_n) \cos (2k_m z_n) \cdot \delta_z, \quad n = 0, \ldots, N - 1,
\]

where \( z_n = z_0 + n\delta_z \), \( \delta_z = \Delta z/N \) is the digitization definition, and \( n \) is the corresponding index. It is worth noting that the left Riemann sum presented in Equation 7 only equals to the original integral when the digitization definition \( \delta_z \) approaches zero,

\[
\int_{z_0}^{z_0+\Delta z} r_S(z) \cos (2k_m z) \, dz
\]

\[
= \lim_{\delta_z \to 0} \sum_{n=0}^{N-1} \sqrt{R_S(z_n)} \cos (2k_m z_n) \cdot \delta_z, \quad n = 0, \ldots, N - 1
\]

Unfortunately, it is impossible for us to obtain \( \delta_z \to 0 \) (or equivalently \( N \to \infty \)); a discretization error \( n_{\text{disc}} \) would inevitably occur during the process. Therefore, a feasible alternative is to reduce \( \delta_z \) to a value that is small enough so that \( n_{\text{disc}} \) falls below a predefined threshold \( \epsilon_0 \).

Under this condition, Equation 6 can now be transformed into a set of \( M \)
summation equations, where $r_S(z_n)$ are the $N$ unknowns we want to solve,

\[
\begin{align*}
I[0] & \approx 2S(k_0) \cdot \sum_{n=0}^{N-1} r_S(z_n) \cos(2k_0z_n) \cdot \delta_z \\
I[1] & \approx 2S(k_1) \cdot \sum_{n=0}^{N-1} r_S(z_n) \cos(2k_1z_n) \cdot \delta_z \\
& \vdots \\
I[M-1] & \approx 2S(k_{M-1}) \cdot \sum_{n=0}^{N-1} r_S(z_n) \cos(2k_{M-1}z_n) \cdot \delta_z
\end{align*}
\]

(9)

Equation 9 could further be re-arranged into a matrix form as given in Equation 2 in the main text,

\[
i \simeq \frac{\Delta z}{N} \cdot (s \odot Cr)
\]

(10)

where $i = \{I[0], I[1], \ldots, I[M-1]\}^T$, $r = \{r_S(z_0), r_S(z_1), \ldots, r_S(z_{N-1})\}^T$, $s = \{S(k_0), S(k_1), \ldots, S(k_M)\}^T$, $C$ is an $M \times N$ matrix with $C_{mn} = \cos(2k_m z_n)$, and $\odot$ represents for Hadamard product or element-wise multiplication.

The OCT imaging problem is now converted to an inverse problem defined by Equation 10, whose aim is to recover the spatial object $r$ from spectral measurement $i$.

2 The essence of IDFT-based reconstruction

Conventionally, the reconstruction is performed by directly applying an IDFT on the measured spectrum $I[m]$. Mathematically, it is equivalent to left multiplying Equation 10 by an IDFT matrix

\[
\mathcal{F}^{-1} = \begin{bmatrix}
    \exp(j2z'_0 k_0) & \exp(j2z'_0 k_1) & \cdots & \exp(j2z'_0 k_{M-1}) \\
    \exp(j2z'_1 k_0) & \exp(j2z'_1 k_1) & \cdots & \exp(j2z'_1 k_{M-1}) \\
    \vdots & \vdots & \ddots & \vdots \\
    \exp(j2z'_{M-1} k_0) & \exp(j2z'_{M-1} k_1) & \cdots & \exp(j2z'_{M-1} k_{M-1})
\end{bmatrix}
\]

where $z'_m = z_0 + m\delta_{z'}$ defines the grid in the reconstruction domain, $\delta_{z'} = 1/\Delta k$ is the grid spacing, and $\Delta z' = M\delta_{z'}$ is the grid length (imaging depth). The obtained reconstruction $\mathbf{r}_{\text{IDFT}}$ could be given by,

\[
\mathbf{r}_{\text{IDFT}} = \mathcal{F}^{-1} i \\
= \frac{\Delta z}{N} \cdot \mathcal{F}^{-1}(s \odot Cr) \\
= \frac{\Delta z}{N} \cdot (\mathcal{F}^{-1} s \odot \mathcal{F}^{-1} Cr)
\]

(11)
where we applied convolution theorem in the last step.

For an ideal reconstruction, \( \hat{r}_{\text{IDFT}} \) should be identical to the object \( r \) to enable a perfect imaging. However, the direct IDFT reconstruction would inevitably cause some distortions as shown in Equation 11: the object \( r \) is first transformed by \( \mathcal{F}^{-1}C \) and then convolved with \( \mathcal{F}^{-1}s \).

### 2.1 The impact of optical transfer function

The vector \( s \) functions effectively as an optical transfer function (OTF). Its IDFT, \( \mathcal{F}^{-1}s \), gives the axial point spread function (PSF) of the system; the full-width-half-maximum (FWHM) of the magnitude of the PSF \( h(z_m) = |\mathcal{F}^{-1}s| \) is defined as the axial resolution of the FD-OCT system based on Rayleigh criterion.

The impact of the OTF \( s \) on the reconstructed object \( \hat{r}_{\text{IDFT}} \) could be understood from two perspectives. First of all, it is band-pass in the \( k \)-domain, since all the spectral components of the original object \( r \) residing outside the support \([S(k_0), S(k_{M-1})]\) will be implicitly eliminated. This process is believed to be irreversible and physically limiting the axial resolution of FD-OCT in conventional theory[7]. In other words, we could achieve a higher axial resolution only by increasing the bandwidth of \( s \). Secondly, the band-passed portion of \( r \) is further modulated by the envelope of \( s \). This effect, on the other hand, is often regarded as reversible. Techniques including spectral shaping [8] or deconvolution are proposed to alleviate it.

### 2.2 The impact of grid discretization

As to \( \mathcal{F}^{-1}C \), there is no prior literature discussing its impact to our best knowledge. To analytically study its behaviour, we can explicitly write out its expression [9],

\[
G_{m,n} = g_{zn}(z'_m) = \begin{cases} 
M, & z'_m = \pm z_n, \\
\exp[-j2(k_0 + \frac{M - 1}{2}\delta_k)zn - j\frac{m\pi}{M}] \sin(M\delta_k zn) \\
+ \exp[j2(k_0 + \frac{M - 1}{2}\delta_k)zn - j\frac{m\pi}{M}] \sin(M\delta_k zn) 
\end{cases} 
\text{otherwise.} 
\]

(12)

which essentially defines a shift-variant mapping from \( \mathbb{R}^{N \times 1} \) (object domain) to \( \mathbb{C}^{M \times 1} \) (reconstruction domain): a Kronecker delta in the object domain that is fed into the system \( G \) would lead to different responses in the reconstruction domain depending on the impulse location \( z_n \). It is also interesting to notice that \( g_{zn}(z'_m) \) consists of two mirrored parts: just as we often observed in OCT reconstructions. In this note, we will confine our discussion on the one in the positive half plane of \( z \),

5
The first term, \( \exp \{ -j2(k_0 + (M - 1)/2\delta_k)z_n - j(m\pi)/M \} \), is a complex exponential function, which determines the phase of the reconstructed object \( \hat{r}_{\text{IDFT}} \). Since its phase is linearly modulated by the \( z_n \), one can localize the original object at a precision even finer than a fraction of the incident wavelength. This essentially manifests the foundation of the so-called phase-sensitive or phase-resolved OCT techniques [10]. However, this type of techniques are only suitable for isolated objects, and a phenomenon known as signal competition would emerge if multiple samples are placed closely [11].

The second term, \( \sin (M\delta_k z_n)/\sin (\delta_k z_n - m\pi/M) \), is the Dirichlet kernel, which is the sampling (or interpolation) kernel for the IDFT. Both terms are shift-variant in their general forms.

### 2.2.1 Case I: matched grids

Let’s now consider the first case that the sampling grid in the object domain exactly matches that in the reconstruction domain,

\[
\begin{align*}
\delta_z &= \delta_{z'} \\
M &= N
\end{align*}
\]

Therefore, the \( g_{z_n,+}(z'_m) \) will become a sinc function for a given \( z_n \), and matrix \( G \) would degenerate to an Identity matrix. In this case, the reconstructed \( \hat{r}_{\text{IDFT}} \) will only be degraded by the axial PSF.

### 2.2.2 Case II: mismatched grids cross the domains

Unfortunately, it is extremely unlikely, if not possible, to encounter the scenario depicted in case I: the grids will hardly be matched cross the domains in real-world scenarios. Therefore, we need to tackle the general form of \( g_{z_n,+}(z'_m) \) as that in Equation 13.

Recall the Dirichlet kernel, \( \sin (M\delta_k z_n)/\sin (\delta_k z_n - m\pi/M) \), is a shift-variant function: it would change its shape upon different \( z_n \). To illustrate this point, we set \( z_n \) to be 4.6 \( \mu \)m, 4.7 \( \mu \)m, 4.8 \( \mu \)m, 4.9 \( \mu \)m, and 5.0 \( \mu \)m, and plot the corresponding Dirichlet kernel in Fig. 1(a). We also plotted their absolute value in Fig. 1(b). It is obvious that the system is shift-variant as we expected. Finally, we plot the reconstruction against different input location as a color-coded image as shown in Fig. 1(c).

It is quite clear now that convolving the object \( r \) with \( g_{z_n,+}(z'_m) \) would impair the imaging quality of the system in a special way: since \( g_{z_n,+}(z'_m) \) is shift-variant, we could not use conventional tools such as deconvolution to resolve it. This also partially explained why the deconvolution is not that popular in OCT communities.
Figure 1: The reconstruction is shift-variant. (a) The Dirichlet kernel manifests different shape for different input locations. (b) Its absolute value is also shift-variant. (c) $g_{z_n,+(z_m')}$ is plotted.
2.3 Summary

To summarize, the IDFT-based technique provides us with a simple and satisfactory reconstruction result. However, the axial resolution of the reconstruction is theoretically limited by the emission bandwidth. Moreover, the accuracy of the reconstruction is degraded by a shift-variant impulse response.

On the other hand, it is rather common to have an imaging depth $\Delta z'$ to be much larger than the object’s support $\Delta z$ considering the shallow penetration depth of OCT. In other words, part of the equations in Equation 7 are “wasted” only to solve unknowns $\{z_n|z_n \in (\Delta z, \Delta z']\}$ that are known to be zeros.

It is now quite clear that the conventional IDFT-based reconstruction only provides a very special solution to the inverse problem as stated in Equation 7 even without considering the effect of PSF $s(\cdot)$: it offers an $l_2$-regularized solution provided the digitization of the original object coincides that defined by the Fourier-transform relationship. In addition, this process is purely mathematical with no prior knowledge imposed. Therefore, the proposed optimization-based technique should should in theory provide better reconstruction performance than the plain IDFT.

References

[1] Fercher, A. F., Hitzenberger, C. K., Kamp, G. & El-Zaiat, S. Y. Measurement of intraocular distances by backscattering spectral interferometry. Opt. Commun 117, 43–48 (1995). URL http://www.sciencedirect.com/science/article/pii/003040189500119S.

[2] Wolf, E. Three-dimensional structure determination of semi-transparent objects from holographic data. Opt. Commun 1, 153–156 (1969). URL http://www.sciencedirect.com/science/article/pii/0030401869900522.

[3] Leitgeb, R., Hitzenberger, C. & Fercher, A. Performance of fourier domain vs. time domain optical coherence tomography. Opt. Express 11, 889–894 (2003). URL http://www.opticsexpress.org/abstract.cfm?URI=oe-11-8-889.

[4] Choma, M., Sarunic, M., Yang, C. & Izatt, J. Sensitivity advantage of swept source and fourier domain optical coherence tomography. Opt. Express 11, 2183–2189 (2003). URL http://www.opticsexpress.org/abstract.cfm?URI=oe-11-18-2183.

[5] de Boer, J. F. et al. Improved signal-to-noise ratio in spectral-domain compared with time-domain optical coherence tomography. Opt. Lett 28, 2067–2069 (2003). URL http://ol.osa.org/abstract.cfm?URI=ol-28-21-2067.
[6] Briggs, W. L. & Henson, V. E. *The DFT: An Owner’s Manual for the Discrete Fourier Transform* (Society for Industrial and Applied Mathematics, 1995).

[7] Drexler, W. & Fujimoto, J. G. *Optical Coherence Tomography: Technology and Applications* (Springer, 2015), second edition edn.

[8] Tripathi, R., Nassif, N., Nelson, J. S., Park, B. H. & de Boer, J. F. Spectral shaping for non-gaussian source spectra in optical coherence tomography. *Opt. Lett.* **27**, 406–408 (2002). URL http://ol.osa.org/abstract.cfm?URI=ol-27-6-406.

[9] Ling, Y., Gan, Y., Yao, X. & Hendon, C. P. Phase-noise analysis of swept-source optical coherence tomography systems. *Opt. Lett.* **42**, 1333–1336 (2017).

[10] Choma, M. A., Ellerbee, A. K., Yang, C., Creazzo, T. L. & Izatt, J. A. Spectral-domain phase microscopy. *Opt. Lett.* **30**, 1162–1164 (2005).

[11] Lin, N. C., Hendon, C. P. & Olson, E. S. Signal competition in optical coherence tomography and its relevance for cochlear vibrometry. *J. Acoust. Soc. Am* **141**, 395–405 (2017).