Long-time asymptotics of solutions
of the heat equation

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Abstract. The long-time asymptotics of solutions of the Cauchy problem for the heat equation are constructed in the case when the initial function at infinity has power asymptotics.

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1 One-dimensional heat equation

Consider the Cauchy problem for the one-dimensional heat equation with a bounded and continuous initial function $\Lambda : \mathbb{R} \to \mathbb{R}$:

\[ \frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}, \quad t \geq 0, \quad u(x,0) = \Lambda(x), \quad x \in \mathbb{R}. \]

The solution of this problem is written in the form of convolution

\[ u(x,t) = \frac{1}{2\sqrt{\pi t}} \int_{-\infty}^{+\infty} \Lambda(s) \exp \left\{ -\frac{(s-x)^2}{4t} \right\} \, ds. \]

Investigation of the asymptotic behavior of integral (1.3) in addition to direct applications to physical processes of heat conduction and diffusion is of independent interest for asymptotic analysis, because the necessity to solve such problems arises under applying the matching method [1].

We assume that there hold the following asymptotic relations:

\[ \Lambda(x) = \sum_{n=0}^{\infty} \frac{\Lambda_{\pm n}}{x^n}, \quad x \to \pm \infty. \]

Let us represent function (1.3) in the form

\[ u(x,t) = U^-(x,t) + U^-(x,t) + U^+(x,t) + U^+(x,t), \]

where

\[ U^-(x,t) = \int_{-\infty}^{-\sigma} \ldots \, ds, \quad U^+(x,t) = \int_{-\sigma}^{0} \ldots \, ds, \]

[1]
\[ U_0^+(x, t) = \int_0^\sigma \ldots ds, \quad U_1^+(x, t) = \int_0^{+\infty} \ldots ds, \]

\[ \sigma = (x^2 + t)^{p/2}, \quad 0 < p < 1. \]

In the integral \( U_1^+(x, t) \) we make the change \( s = 2z\sqrt{t} \). Setting

\[ \mu = \frac{\sigma}{2\sqrt{t}}, \quad \eta = \frac{x}{2\sqrt{t}} \]

and using condition (1.3), we obtain

\[ U_1^+(x, t) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{+\infty} \Lambda(2z\sqrt{t}) e^{-\left(\frac{z-\eta}{\sqrt{\mu}}\right)^2} dz = \]

\[ = \sum_{n=0}^{N-1} \frac{\Lambda_n}{\sqrt{\pi^{2n}}} t^{-n/2} \int_{-\infty}^{+\infty} z^{-n} e^{-\left(\frac{z-\eta}{\sqrt{\mu}}\right)^2} dz + \int_{-\infty}^{+\infty} R_N(z\sqrt{t}) e^{-\left(\frac{z-\eta}{\sqrt{\mu}}\right)^2} dz, \quad (1.5) \]

where \(|R_N(s)| \leq K_N s^{-N}\).

From the last inequality we obtain the estimate

\[ \left| \int_{-\infty}^{+\infty} R_N(z\sqrt{t}) e^{-\left(\frac{z-\eta}{\sqrt{\mu}}\right)^2} dz \right| \leq k_N t^{-N/2} \int_{-\infty}^{+\infty} z^{-N} dz = k_N \mu^{-N}. \quad (1.6) \]

Let us extract the dependence on the parameter \( \mu \) in the integral

\[ \int_{-\infty}^{+\infty} z^{-n} e^{-\left(\frac{z-\eta}{\sqrt{\mu}}\right)^2} dz. \]

For \( n = 0 \) and \( t \geq |x|^\alpha, \quad 1 + p < \alpha < 2 \), we have

\[ \int_{-\infty}^{+\infty} e^{-\left(\frac{z-\eta}{\sqrt{\mu}}\right)^2} dz = \int_{-\infty}^{0} e^{-\left(\frac{z-\eta}{\sqrt{\mu}}\right)^2} dz - \int_{0}^{+\infty} e^{-\left(\frac{z-\eta}{\sqrt{\mu}}\right)^2} dz = \]

\[ = \sqrt{\pi} \text{erfc}(-\eta) + \sum_{r=1}^{N-1} \mu^r e^{-\eta^2} P_{r-1}(\eta) + O(\sigma^{-\gamma N}), \quad \sigma \to \infty, \quad (1.7) \]

where \( P_m(\eta) \) denotes polynomials of the \( m \)th degree in \( \eta \),

\[ \gamma = \frac{\alpha}{2p} - 1 > 0, \quad \text{erfc}(x) = \frac{1}{\sqrt{\pi}} \int_{x}^{+\infty} e^{-s^2} ds. \]

To obtain (1.7) we make use of estimates

\[ \mu = O\left(\frac{t^{2p-\alpha}}{2^{\alpha}}\right), \quad \sigma = O\left(t^{p/\alpha}\right), \quad \mu = O\left(\sigma^{-\gamma}\right) \quad (1.8) \]
as \( \sigma \to \infty \) on the set
\[
T_\alpha = \{(x, t) : x \in \mathbb{R}, \; t \geq |x|^\alpha \}.
\]

However, it is seen that (1.7) remains valid on the set
\[
X_\alpha = \{(x, t) : x \in \mathbb{R}, \; 0 < t < |x|^\alpha \}
\]
as well, where
\[
\mu = o(\eta), \quad |\eta| \geq \frac{|x|^{1-\frac{\alpha}{2}}}{2} \to \infty \quad \text{for} \quad \sigma \to \infty.
\]

For the calculation of \( U_1^- (x, t) \) it is useful an analogous relation
\[
\int_{-\infty}^{-\mu} e^{-(z-\eta)^2} dz = \sqrt{\pi} \text{erfc}(\eta) + \sum_{r=1}^{N-1} \mu^r e^{-\eta^2} P_{r-1}(\eta) + O(\sigma^{-N}).
\]

For \( n \geq 1 \) we have
\[
\int_{\mu}^{+\infty} z^{-n} e^{-(z-\eta)^2} dz = \int_{1}^{+\infty} z^{-n} e^{-(z-\eta)^2} dz + \int_{\mu}^{1} \Psi_n(z, \eta) dz + e^{-\eta^2} \sum_{r=0}^{n-1} P_r(\eta) \int_{\mu}^{1} \frac{e^{-z^2} P_r(z)}{z^{n-r}} dz,
\]
where
\[
\Psi_n(z, \eta) = z^{-n} \left[ e^{-(z-\eta)^2} - e^{-\eta^2} \sum_{r=0}^{n-1} z^r P_r(\eta) \right],
\]
and the sum in \( r \) is a partial sum of the Taylor series for the function \( \exp(2z\eta - z^2) \) in the variable \( z \). Thus,
\[
\int_{\mu}^{+\infty} z^{-n} e^{-(z-\eta)^2} dz = \int_{1}^{+\infty} z^{-n} e^{-(z-\eta)^2} dz + e^{-\eta^2} P_{n-1}(\eta) \ln \mu + \sum_{r=0}^{n-2} \mu^{r-n+1} e^{-\eta^2} P_r(\eta) + \int_{0}^{1} \Psi_n(z, \eta) dz - \int_{0}^{\mu} \Psi_n(z, \eta) dz.
\]

From formula (1.9) we conclude that \( \Psi_n(z, \eta) \) has no singularities; from the same formula we obtain
\[
\int_{0}^{\mu} \Psi_n(z, \eta) dz = \sum_{r=1}^{N-1} \mu^r e^{-\eta^2} P_{r+1+n}(\eta) + O(\sigma^{-N}).
\]

Substituting relation (1.7) and (1.10) in formula (1.5) and using the estimate (1.6), we obtain
\[
U_1^+ (x, t) = \Lambda_0^- \text{erfc}(-\eta) + \sum_{n=1}^{N-1} \frac{\Lambda_n^+}{\sqrt{\pi} 2^n} t^{-n/2} \left[ \int_{1}^{+\infty} z^{-n} e^{-(z-\eta)^2} dz + \int_{0}^{1} \Psi_n(z, \eta) dz \right] + O(\sigma^{-\gamma N}).
\]
Here and further, by \( V_i(\mu, \eta, t) \) we denote expressions of the form
\[
e^{-\eta^2} \sum_{r^2 + q^2 \neq 0} b_r \eta^{m+1} \mu^r s \ln^q \mu.
\]

Since \(|x| \leq \sigma^{1/p}\), and \(t \geq \sigma^{\alpha/p}\) on the set \(T_\alpha\), we conclude that for \(0 \leq s \leq \sigma\) there hold the estimates
\[
x \frac{s}{t} = O(\sigma^{-\rho}), \quad \frac{s^2}{t} = O(\sigma^{-2\rho}), \quad \rho = \frac{\alpha - 1}{p} - 1 > 0.
\]

Using these estimates, we represent the integral \( U_0^+(x, t) \) in the following form:
\[
U_0^+(x, t) = \frac{1}{2 \sqrt{\pi t}} \int_0^\sigma \Lambda(s) \exp \left\{ -\frac{(s-x)^2}{4t} \right\} ds =
\]
\[
= \frac{e^{-\eta^2}}{2 \sqrt{\pi t}} \int_0^\sigma \Lambda(s) \sum_{m=0}^{N-1} \frac{1}{m!} \left( \frac{\eta s}{\sqrt{t}} - \frac{s^2}{4t} \right)^m ds + O(\sigma^{-\rho N}).
\]

Because of the factor \(\exp(-\eta^2)\) the estimate of the remainder is valid on the set \(X_\alpha\) as well. Expanding the power and changing the order of summation, we obtain
\[
U_0^+(x, t) = \frac{e^{-\eta^2}}{2 \sqrt{\pi t}} \sum_{n=1}^{N-1} t^{-n/2} \sum_{k=0}^{[n/2]} \frac{\eta^{n-2k-1} e^{-\eta^2}}{2^{k} \pi^{k!} k!(n-2k-1)!} \int_0^\sigma s^{n+k} \Lambda(s) ds + O(\sigma^{-\rho N}) =
\]
\[
= \sum_{n=1}^{N-1} t^{-n/2} \sum_{k=0}^{[n/2]} \frac{\eta^{n-2k-1} e^{-\eta^2}}{2^{k} \pi^{k!} k!(n-2k-1)!} \int_0^\sigma s^{n-1} \Lambda(s) ds + O(\sigma^{-\rho N}).
\]

Let us transform the integral as follows:
\[
\int_0^\sigma s^{n-1} \Lambda(s) ds = \int_0^1 s^{n-1} \Lambda(s) ds + \int_1^\sigma \left( \Lambda_0^+ s^{n-1} + \cdots + \Lambda_{n-1}^+ s^{n-1} + \frac{\Lambda_n^+}{s} \right) ds +
\]
\[
+ \int_1^\sigma s^{n-1} \left[ \Lambda(s) - \Lambda_0^+ - \cdots - \frac{\Lambda_{n-1}^+}{s^{n-1}} - \frac{\Lambda_n^+}{s^n} \right] ds =
\]
\[
= \int_0^1 s^{n-1} \Lambda(s) ds + \int_1^{+\infty} \Phi_n^+(s) ds - \sum_{m=1}^n \frac{\Lambda_{n-m}^+}{m} +
\]
\[
+ \Lambda_n^+ \ln \sigma + \sum_{m=1}^n \frac{\Lambda_{n-m}^+}{m} \sigma^m - \int_\sigma^{+\infty} \Phi_n^+(s) ds,
\]
(1.12)
where
\[ \Phi_n^+(s) = s^{n-1} \left[ \Lambda(s) - \sum_{m=0}^{n} \frac{\Lambda_m^+}{s^m} \right]. \]

From condition (1.4) we obtain the estimate
\[ |\Phi_n^+(s)| \leq C_n s^{-2}, \quad s \geq 1; \]
hence,
\[ \int_{\sigma}^{+\infty} \Phi_n^+(s) ds = \sum_{m=1}^{N-1} \psi_{n,m} \sigma^{-m} + O(\sigma^{-N}), \quad \sigma \to \infty. \]

Taking into account this relation and substituting \( \sigma = 2\mu \sqrt{t} \) in (1.12), we obtain
\[ \int_{0}^{\sigma} s^{n-1} \Lambda(s) ds = I_n + \frac{\Lambda_n^+}{2} \ln t + \Lambda_n^+ \ln \mu + \sum_{r \neq 0} a_n, \mu, \mu^r r^{r/2} + O(\sigma^{-N}), \]
where \( I_n \) are constants. Then
\[ U_0^+ (x, t) = \sum_{n=1}^{N-1} t^{-n/2} e^{-\eta^2} \left[ P_{n-1}(\eta) + \tilde{P}_{n-1}(\eta) \ln t \right] + V_0(\mu, \eta, t) + O(\sigma^{-\rho N}). \quad (1.13) \]

Expressions \( V_0(\mu, \eta, t) \) and \( V_1(\mu, \eta, t) \) in formulas (1.13) and (1.11) asymptotically diminish. Thus, we arrive at the following statement (details of the proof see in [2]).

**Theorem 1.** As \( |x| + t \to \infty \) the asymptotics of the solution to problem (1.1)–(1.2) with an initial function satisfying relations (1.4) has the form
\[ u(x, t) = \Lambda_0^- \text{erfc}(\eta) + \Lambda_0^+ \text{erfc}(-\eta) + \sum_{n=1}^{\infty} t^{-n/2} (H_{n,0}(\eta) + H_{n,1}(\eta) \ln t), \quad (1.14) \]
where \( H_{n,m} \) are \( C^\infty \)-smooth functions of the self-similar variable
\[ \eta = \frac{x}{2\sqrt{t}}. \]

Notice that \( H_{n,1} \) are the Hermite functions:
\[ H_{n,1}(\eta) = \frac{\Lambda_n^+ - \Lambda_n^-}{4\sqrt{\pi}} \sum_{k=0}^{[(n-1)/2]} \frac{\eta^{n-2k-1} e^{-\eta^2}}{4^k k!(n-2k-1)!}. \]

The above reasoning without essential changes can be applied to the case when \( u(x, 0) \) grows power-like as \( x \to \infty \). For a function \( \Lambda \), satisfying the asymptotic relations
\[ \Lambda(x) = x^p \sum_{n=0}^{\infty} \frac{\Lambda_n^\pm}{x^n}, \quad x \to \pm \infty, \]

where $p$ is a natural number, the expansion of the solution has the following form:

$$u(x, t) = \sum_{m=0}^{p} t^{m/2} \left[ \Lambda_{p-m}(\eta) \text{erfc}(\eta) + \Lambda_{p-m}^+(\eta) \text{erfc}(-\eta) + P_{m-1}(\eta) \exp(-\eta^2) \right] + \sum_{n=1}^{\infty} t^{-n/2} \left[ G_{n,0}(\eta) + G_{n,1}(\eta) \ln t \right],$$

where $\Pi_m^\pm(\eta)$ are polynomials of degree $m$, whose coefficients are constants, and $P_{m-1}(\eta)$ are polynomials of degree $m - 1$ ($P_{-1}(\eta) \equiv 0$), $G_{n,k}$ are smooth functions.

## 2 Heat equation on a plane

Consider the Cauchy problem for the heat equation on a plane with a locally Lebesgue-integrable initial function $\Lambda : \mathbb{R}^2 \to \mathbb{R}$ of slow growth:

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2}, \quad t > 0, \quad (2.1)$$

$$u(x_1, x_2, 0) = \Lambda(x_1, x_2), \quad (x_1, x_2) \in \mathbb{R}^2. \quad (2.2)$$

The solution of this problem has the form

$$u(x_1, x_2, t) = \frac{1}{4\pi t} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \Lambda(s_1, s_2) \exp \left\{ -\frac{(s_1 - x_1)^2 + (s_2 - x_2)^2}{4t} \right\} ds_1 ds_2. \quad (2.3)$$

Let us construct a uniform asymptotic expansion as $|x_1| + |x_2| + t \to \infty$ under the assumption that

$$\Lambda(x_1, x_2) = 0, \quad x_1 < 0, \quad (2.4)$$

$$\Lambda(x_1, x_2) = x_1^p \sum_{n=0}^{\infty} \frac{\Lambda_n(x_2)}{x_1^n}, \quad x_1 \to +\infty, \quad (2.5)$$

where $p$ is a nonnegative integer, $\Lambda_n$ are continuous functions. In addition, we assume that

$$\text{supp } \Lambda \subset \{(x_1, x_2) : x_1 > 0, \ |x_2| < |x_1|^\nu, \ \nu > 0\},$$

$$\text{supp } \Lambda_n \subset [-R_n, R_n], \quad R_n > 0. \quad (2.6)$$

Taking into account condition (2.4), we represent function (2.3) in the form

$$u(x_1, x_2, t) = U_0(x_1, x_2, t) + U_1(x_1, x_2, t), \quad (2.7)$$

where

$$U_0(x_1, x_2, t) = \int_{0}^{+\infty} ds_1 \int_{-\infty}^{+\infty} ds_2 \ldots, \quad U_1(x_1, x_2, t) = \int_{-\infty}^{+\sigma} ds_1 \int_{-\infty}^{+\sigma} ds_2 \ldots,$$

$$\sigma = (x_1^2 + x_2^2 + t)^{\beta/2}, \quad 0 < \beta < 1, \quad (2.8)$$
and dots denote the integrand from formula (2.3) together with the factor $(4\pi t)^{-1}$. In the integral $U_1(x_1, x_2, t)$ we make the change $s_1 = 2z\sqrt{t}$. Setting

$$
\mu = \frac{\sigma}{2\sqrt{t}}, \quad \eta_1 = \frac{x_1}{2\sqrt{t}}
$$

and using by condition (2.5), we obtain

$$
U_1(x_1, x_2, t) = \frac{1}{2\sqrt{\pi t}} \int_{-\infty}^{+\infty} e^{-(\mu - z_1)^2} \int_{-\infty}^{+\infty} \Lambda(2z\sqrt{t}, s_2) \exp \left\{ -\frac{(s_2 - x_2)^2}{4t} \right\} ds_2dz =
$$

$$
= \frac{t^{p/2}}{\sqrt{\pi}} \sum_{n=0}^{N-1} 2^{p-n} t^{-n/2} \int_{-\infty}^{+\infty} z^{p-n} e^{-(z-n)^2} dz \times
$$

$$
\times \frac{1}{2\sqrt{\pi t}} \int_{-\infty}^{+\infty} \Lambda_n(s_2) \exp \left\{ -\frac{(s_2 - x_2)^2}{4t} \right\} ds_2 + O(\sigma^{-\rho_1 N}), \quad \rho_1 > 0.
$$

According to Theorem 1, from condition (2.6) we have

$$
\frac{1}{2\sqrt{\pi t}} \int_{-\infty}^{+\infty} \Lambda_n(s_2) \exp \left\{ -\frac{(s_2 - x_2)^2}{4t} \right\} ds_2 =
$$

$$
= \exp(-\eta_2^2) \sum_{m=1}^{N} t^{-m/2} Q_{n,m-1}(\eta_2) + O((x_2^2 + t)^{-\rho_3 N}), \quad \rho_3 > 0,
$$

where $Q_{n,m-1}(\eta_2)$ are polynomials of degree $m - 1$ in $\eta_2 = \frac{x_2}{2\sqrt{t}}$, whose coefficients depend on $n$. Then

$$
U_1(x_1, x_2, t) = t^{p/2} \sum_{n=1}^{N} t^{-n/2} \tilde{S}_n(\eta_1, \eta_2) + V_1(\mu, \eta_1, \eta_2, t) + O(\sigma^{-\rho_4 N}), \quad \rho_4 > 0,
$$

$$
V_1(\mu, \eta_1, \eta_2, t) = \exp \left\{ -(\eta_1^2 + \eta_2^2) \right\} \sum_{r_1^2 + q_2^2 \neq 0} a'_s \eta_1^{r_1} \eta_2^{q_2} t^{r_1} \mu^{q_2} \ln^{q_2} \mu,
$$

where $a'_s$ are constants. Coefficients $\tilde{S}_n(\eta_1, \eta_2)$ are smooth functions; in particular,

$$
\tilde{S}_1(\eta_1, \eta_2) = \exp \left\{ -(\eta_1^2 + \eta_2^2) \right\} \left[ \Pi_{p}^{(1)}(\eta_1) \exp(\eta_1^2) \text{erfc}(-\eta_1) + \Pi_{p-1}^{(2)}(\eta_1) \right],
$$

where $\Pi_{p}^{(1)}(\eta_1)$ and $\Pi_{p-1}^{(2)}(\eta_1)$ are polynomials of degree $p$ and $p - 1$, respectively.

Let us represent the integral $U_0(x_1, x_2, t)$ in the following form:

$$
U_0(x_1, x_2, t) = \frac{\exp \left\{ -(\eta_1^2 + \eta_2^2) \right\}}{4\pi t} \times
$$

$$
\times \int_{0}^{\sigma} \int_{-\infty}^{\infty} \Lambda(s_1, s_2) \sum_{m=0}^{N} \frac{1}{m!} \left( \frac{\eta_1 s_1 + \eta_2 s_2}{\sqrt{t}} - \frac{s_1^2 + s_2^2}{4t} \right)^m ds_2 ds_1 + O(\sigma^{-\rho_5 N}), \quad \rho_5 > 0.
$$
From this formula we obtain

\[ U_0(x_1, x_2, t) = \exp \left\{ -(\eta_1^2 + \eta_2^2) \right\} \sum_{n=2}^{N} t^{-n/2} \sum_{0 \leq m_1, m_2 \leq n-2} a_{m_1, m_2, l_1, l_2} \eta_1^{m_1} \eta_2^{m_2} \times \]

\[ \times \int_{0}^{\sigma} \int_{-\infty}^{\infty} s_1^{l_1} s_2^{l_2} \Lambda(s_1, s_2) ds_1 ds_2 + O(\sigma^{-\rho_6 N}), \]

where \( a_{m_1, m_2, l_1, l_2} \) are some constants. Let us transform the integral as follows:

\[ \int_{0}^{\sigma} \int_{-\infty}^{\infty} s_1^{l_1} s_2^{l_2} \Lambda(s_1, s_2) ds_1 ds_2 = \int_{0}^{1} \int_{-\infty}^{\infty} s_1^{l_1} s_2^{l_2} \Lambda(s_1, s_2) ds_1 ds_2 + \]

\[ + \int_{1}^{\sigma} \int_{-\infty}^{\infty} s_1^{l_1} s_2^{l_2} \left[ \Lambda(s_1, s_2) - s_1^p \Lambda_0(s_2) - \ldots - s_1^{-l_1-1} \Lambda_{p+l_1+1}(s_2) \right] ds_1 ds_2 = \]

\[ = A_{l_1, l_2} + \ln \sigma \int_{-\infty}^{\infty} s_2^{l_2} \Lambda_{p+l_1+1}(s_2) ds_2 + \sum_{k=1}^{N-1} (c_k \sigma^k + c_{-k} \sigma^{-k}) + O(\sigma^{-N}) = \]

\[ = A_{l_1, l_2} + B_{l_1, l_2} \ln t + 2B_{l_1, l_2} \ln(2\mu) + \sum_{k=1}^{N-1} (c'_k \mu^{k/2} + c'_{-k} \mu^{-k/2}) + O(\sigma^{-N}), \]

where \( A_{l_1, l_2}, B_{l_1, l_2}, c'_k \) and \( c'_{-k} \) are constants. Then

\[ U_0(x_1, x_2, t) = \exp \left\{ -(\eta_1^2 + \eta_2^2) \right\} \sum_{n=2}^{N} t^{-n/2} \left[ \Pi_{n-2}(\eta_1, \eta_2) + \Pi^*_{n-2}(\eta_1, \eta_2) \ln t \right] + \]

\[ + V_0(\mu, \eta_1, \eta_2, t) + O(\sigma^{-\rho_6 N}), \quad \rho_6 > 0, \quad (2.14) \]

where \( \Pi_{n-2}(\eta_1, \eta_2) \) and \( \Pi^*_{n-2}(\eta_1, \eta_2) \) are polynomials of degree \( n - 2 \), and the expression

\[ V_0(\mu, \eta_1, \eta_2, t) = \exp(-\eta_1^2 - \eta_2^2) \sum_{r^2 + q^2 \neq 0} a'' \eta_1^{m_1} \eta_2^{m_2} t^{l_1} \mu^{l_2} \ln^6 \mu, \]

where \( a'' \) are some constants, is obtained similarly to expression (2.13). Expressions \( V_0(\mu, \eta, t) \) and \( V_1(\mu, \eta, t) \) asymptotically diminish

\[ V_0(\mu, \eta_1, \eta_2, t) + V_1(\mu, \eta_1, \eta_2, t) = O(\sigma^{-\rho N}), \quad (x_1, x_2, t) \in T_\alpha, \]

where \( \rho = \min\{\rho_4, \rho_6\} \). Then we arrive at the following statement (details of the proof see in [3]).
Theorem 2. As $|x_1| + |x_2| + t \to \infty$ the asymptotics of the solution of equation (2.1) with conditions (2.2), (2.4), (2.5) and (2.6) has the form

$$u(x_1, x_2, t) = \sum_{n=1}^{\infty} t^{-n/2} \left[ t^{p/2} S_n(\eta_1, \eta_2) + t^{-1/2} \ln t \Pi_n(\eta_1, \eta_2) \exp(-\eta_1^2 - \eta_2^2) \right],$$

where $S_n(\eta_1, \eta_2)$ are $C^\infty$-smooth functions of slow growth, $\Pi_n(\eta_1, \eta_2)$ are polynomials of degree $n$ in the self-similar variables

$$\eta_1 = \frac{x_1}{2\sqrt{t}}, \quad \eta_2 = \frac{x_2}{2\sqrt{t}}.$$

3 Multidimensional equation

For a multidimensional problem, there holds the following result.

Theorem 3. Let $u(x_1, \ldots, x_m, t)$ be the solution of the Cauchy problem

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x_1^2} + \ldots + \frac{\partial^2 u}{\partial x_m^2}, \quad t > 0,$$

$$u(x_1, \ldots, x_m, 0) = \Lambda(x_1, \ldots, x_m), \quad (x_1, \ldots, x_m) \in \mathbb{R}^m,$$

with a locally integrable initial function $\Lambda$ of slow growth. If the conditions

$$\Lambda(x_1, \ldots, x_m) = 0, \quad x_1 < 0,$$

$$\Lambda(x_1, \ldots, x_m) = x_1^p \sum_{n=0}^{\infty} \frac{\Lambda_n(x_2, \ldots, x_m)}{x_1^n}, \quad x_1 \to +\infty,$$

where

$$\text{supp } \Lambda \subset \{(x_1, \ldots, x_m) : x_1 > 0, \ |x_2| + \ldots + |x_m| < |x_1|^\nu, \ \nu > 0\},$$

$$\text{supp } \Lambda_n \subset [-R_n, R_n]^{m-1}, \quad R_n > 0,$$

are fulfilled, then there holds the asymptotic formula

$$u(x_1, \ldots, x_m, t) = t^{-m/2} \sum_{n=0}^{\infty} t^{-n/2} \left[ t^{(p+1)/2} S_n(\eta_1, \ldots, \eta_m) + \ln t \Pi_n(\eta_1, \ldots, \eta_m) \exp(-\eta_1^2 - \ldots - \eta_m^2) \right], \quad |x_1| + \ldots + |x_m| + t \to \infty,$$

where $S_n(\eta_1, \ldots, \eta_m)$ are smooth functions of slow growth, $\Pi_n(\eta_1, \ldots, \eta_m)$ are polynomials of degree $n$ in the self-similar variables

$$\eta_1 = \frac{x_1}{2\sqrt{t}}, \ldots, \quad \eta_m = \frac{x_m}{2\sqrt{t}}.$$
References

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