Abstract. We prove a formula for a characteristic polynomial of an operator expressed as a polynomial of rank 1 operators. The formula uses a discrete analog of path integration and implies a generalization of the Forman–Kenyon’s formula for a determinant of the graph Laplacian \[9, 7\] (which, in its turn, implies the famous matrix-tree theorem by Kirchhoff) as well as its level 2 analog, where the summation is performed over triangulated nodal surfaces with boundary.

Introduction

The primary impulse to write this article was the famous Matrix-tree theorem (MTT), first discovered by Kirchhoff \[10\] in 1847 and re-proved more than a dozen times since then. The theorem in its simplest form expresses the number of spanning trees in a graph as a determinant of a suitably chosen matrix \(M\). See \[6\] for a review of existing proofs and generalizations. One of the proofs, given in \[3\], makes use of the fact that the matrix in question is a weighted sum of special rank 1 matrices (identity minus reflection). This structure of \(M\) explains why the MTT appears sometimes in quite unexpected contexts (see \[4\] for just an example).

Let \(M\) be an operator expressed explicitly as a noncommutative polynomial of arbitrary operators \(M_i\) of rank 1. The article contains a formula (Theorem 1.1) for the characteristic polynomial of \(M\) in terms of \(M_i\). Corollaries of the formula include the MTT (cf, Corollary 1.3), the \(D\)-analog of the MTT (\[3\], cf. Corollary 1.4), a formula for the determinant of the Laplacian of a line bundle on a graph \[7, 9\] (Section 1.2.2 actually, we compute the whole characteristic polynomial), and a level 2 analog of the formula from \[7\] proved in Sections 1.2.3 and 2.3.

The right-hand side of the main formula (1.4) involve summation over the graphs consisting of several cycles and/or several chains. The summand (the weight \(W_P\), see (1.2)) is a function on edges of the graph obtained also by symmation over the set of paths joining the endpoints of the edge. Consequently, corollaries of the main theorem involve summation over various objects, including trees (the MTT), hypertrees (the Massbaum–Vaintrob theorem), cycle-rooted trees (the Forman–Kenyon formula) and nodal surfaces with boundary (the level 2 analog from Section 1.2.3). The standard expression of a determinant via summation over the permutation group also follows from Theorem 1.1 — see Section 1.2.1.

Structure of the article. Section 1.1 contains the formulation and the proof of the main theorem (Theorem 1.1). Section 1.2 lists several corollaries of the theorem. Some corollaries are proved immediately, proofs of some others require not some additional reasoning, which is given in Section 2. Section 2.1 contains technical lemmas (including a duality lemma 2.5) for the angle between two subspaces of a
Euclidean space). Section 2.2 contains the proof of the generalization of Forman’s formula (Theorem 1.2), and Section 2.3 is devoted to the proof of Theorem 1.9 about the level 2 analog of the graph Laplacian.

Open questions and future research. Generalizations of the matrix-tree theorem are plentiful and versatile (see e.g. [6] and the references therein), and Theorem 1.1 covers surprisingly many of them. Nevertheless, there are numerous results in the field whose relations with the discrete path integration technique are yet to be clarified. One of such results is the Hyperpfaffian-cactus theorem by A. Abdessalam [1]; it is a generalization of the Pfaffian Hypertree theorem of [11]. The latter theorem is not included into this paper but was proved in [3] by a method close to Theorem 1.1 (the original proof due to G. Massbaum and A. Vaintrob was quite different); we were unable, though, to extend this proof to hyperpfaffians. Also, some results of the MTT type appear in the theory of determinantal point processes, see [5] and [2]. It would be interesting to know whether these results are covered by Theorem 1.1 or by its suitable generalization.

The summation in Theorem 1.1 is performed over the set of graphs that deserve to be called “discrete 1-manifolds” (oriented, possibly with boundary). The summand is obtained by a procedure which is quite natural to understand as a “discrete path integration”. A tempting direction of the future research here is to send dimension to infinity, getting “real” path integration and summation over 2-varieties. The author plans to write a special paper about this.

1. Sums of operators of rank 1

1.1. The main theorem. Let \( V \) be a vector space of dimension \( n \) with a scalar product \( \langle \cdot, \cdot \rangle \). Choose an integer \( N \) and fix two sequences of vectors, \( e_1, \ldots, e_N \in V \) and \( \alpha_1, \ldots, \alpha_N \in V \). For any \( i \) denote by \( M_i \) the operator given by \( M_i(v) = \langle \alpha_i, v \rangle e_i \); one has \( \text{rk} M_i = 1 \) or \( M_i = 0 \). Consider an operator

\[
M = P(M_1, \ldots, M_N)
\]

where

\[
P(x_1, \ldots, x_N) = \sum_{s=1}^{k} \sum_{1 \leq i_1, \ldots, i_s \leq N} p_{i_1, \ldots, i_s} x_{i_1} \cdots x_{i_s}
\]

is a noncommutative polynomial of degree \( k \). This section contains a description of the characteristic polynomial \( \text{char}_M(t) \) of the operator \( M \).

Let \( G \) be a finite graph with the vertices 1, 2, \ldots, \( N \); let \( a \) and \( b \) be its vertices. Define the weight \( W_P(a, b) \) by

\[
W_P(a, b) = \sum_{s=1}^{k} \sum_{i_1, \ldots, i_s} p_{i_1, \ldots, i_s} \langle \alpha_{i_2}, e_{i_1} \rangle \langle \alpha_{i_3}, e_{i_2} \rangle \cdots \langle \alpha_{i_s}, e_{i_{s-1}} \rangle,
\]

(the internal summation is taken over the set of paths \( i_1, \ldots, i_s \) of length \( s \) joining the vertices \( a = i_1 \) and \( b = i_s \)). Also, denote

\[
W_P(G) \overset{\text{def}}{=} \prod_{(a, b) \text{ is an edge of } G} W_P(a, b)
\]

A directed graph \( G \) with the vertices 1, 2, \ldots, \( N \) is called a discrete oriented one-dimensional manifold with (possibly empty) boundary (abbreviated as DOOMB) if every its connected component is either an oriented chain (a graph with \( \ell \) distinct
vertices $i_1, \ldots, i_\ell$ and the edges are $(i_1, i_2), (i_2, i_3), \ldots, (i_{\ell-1}, i_\ell)$ or an oriented cycle (the same edges but the vertices are $i_1, \ldots, i_{\ell-1}$ and $i_\ell = i_1$).

**Theorem 1.1.** \( \text{char}_M(t) = \sum_{k=0}^n (-1)^k \mu_k t^{n+1-k} \) where

(1.4) \[
\mu_k = \sum_{G \in D_{n,k}} WP(G) \det(\langle \alpha_{d^-}, e_{d^+} \rangle).
\]

Here \( D_{n,k} \) is the set of DOOMBs with the vertices 1, 2, \ldots, \( n \) and \( k \) edges; \( d^- \) and \( d^+ \) run through the set of all edges of the graph \( G \); \( d^- \) and \( d^+ \) are the initial and the terminal vertex of the directed edge \( d \).

**Remark.** The main theorem of \[3\] is a particular case of Theorem 1.1.

**Proof.** Consider an orthonormal basis \( u_1, \ldots, u_n \in V \) and fix a sequence \( j_1, \ldots, j_k, 1 \leq j_1 < \cdots < j_k \leq N \). Then

\[
M(u_{j_1} \wedge \cdots \wedge u_{j_k}) = \sum_{s_1, \ldots, s_k} \sum_{1 \leq i_m(0) \leq N} \prod_{q=1}^k \left( p_{i_1^{(q)}}, \ldots, i_k^{(q)} \prod_{r=2}^{s_q} (\alpha_{i_r^{(q)}}, e_{i_{r-1}^{(q)}}) \right)
\]

\[
\times \prod_{q=1}^k (\alpha_{i_1^{(q)}}, u_{j_q}) \times e_{i_1^{(1)}} \wedge \cdots \wedge e_{i_k^{(k)}}
\]

\[
= \sum_{s_1, \ldots, s_k} \sum_{1 \leq i_m(0) \leq N} \prod_{q=1}^k \left( p_{i_1^{(q)}}, \ldots, i_k^{(q)} \prod_{r=2}^{s_q} (\alpha_{i_r^{(q)}}, e_{i_{r-1}^{(q)}}) \right)
\]

\[
\times \prod_{q=1}^k (\alpha_{i_1^{(q)}}, u_{j_q}) \times e_{i_1^{(1)}} \wedge \cdots \wedge e_{i_k^{(k)}}
\]

The coefficient at \( u_{j_1} \wedge \cdots \wedge u_{j_k} \) in \( M(u_{j_1} \wedge \cdots \wedge u_{j_k}) \) is then equal to

\[
\sum_{s_1, \ldots, s_k} \sum_{1 \leq i_m(0) \leq N} \prod_{q=1}^k \left( p_{i_1^{(q)}}, \ldots, i_k^{(q)} \prod_{r=2}^{s_q} (\alpha_{i_r^{(q)}}, e_{i_{r-1}^{(q)}}) \right)
\]

\[
\times \det(\langle \alpha_{i_r^{(q)}}, u_{j_r} \rangle)_{1 \leq q, r \leq k} \times \det(\langle u_{j_r}, e_{i_{r-1}^{(q)}} \rangle)_{1 \leq q, r \leq k}.
\]

Hence

(1.5) \[
\mu_k = \text{Tr } M^\wedge k = \sum_{s_1, \ldots, s_k} \sum_{1 \leq i_m(0) \leq N} \prod_{q=1}^k \left( p_{i_1^{(q)}}, \ldots, i_k^{(q)} \prod_{r=2}^{s_q} (\alpha_{i_r^{(q)}}, e_{i_{r-1}^{(q)}}) \right)
\]

\[
\times \det(\langle \alpha_{i_r^{(q)}}, e_{i_{r-1}^{(q)}} \rangle)_{1 \leq q, r \leq k}
\]

A multi-index \( i_1^{(q)}, \ldots, i_k^{(q)}, 1 \leq q \leq k \), can be interpreted as a directed graph \( G \) with the edges \( (i_1^{(q)}, i_k^{(q)}), 1 \leq q \leq k \), and a path \( i_1^{(q)}, \ldots, i_k^{(q)} \) joining endpoints

\[\text{(1.5)}\]
$d_-=i_1^{(q)}$ and $d_+=i_{kq}$ of every edge. Conversely, a graph plus collection of paths is just the multi-index. So one can rearrange summation in (1.5) to sum over the graphs $G$ first, and then over the set of all paths for a given graph, getting $W(G)$ times the determinant. The determinant may be nonzero only if no two $i_1^{(q)}$ and no two $i_{kq}$ are equal. Thus, in a graph $G$ entering the sum every vertex has at most one outgoing edge and at most one incoming edge. This means that $G$ is a DOOMB. □

1.2. Graph Laplacians and other examples. Here are some corollaries of Theorem 1.1.

1.2.1. Determinants. It’s a funny result demonstrating the nature of Theorem 1.1.

Let $u_1,\ldots,u_n$ be an orthonormal basis in $\mathbb{R}^n$. Take $e_{ij} \overset{\text{def}}{=} u_i$ and $\alpha_{ij} \overset{\text{def}}{=} u_j$ for all $1 \leq i,j \leq n$. Take

$$P(x_{11},\ldots,x_{nn}) \overset{\text{def}}{=} \sum_{i,j=1}^{n} a_{ij} x_{ij}$$

and apply Theorem 1.1. The matrix of the operator $M$ in the basis $u_1,\ldots,u_n$ is $(\alpha_{ij})$. The polynomial $P$ is linear, so the paths entering equation (1.2) all have length 1. Consequently, the DOOMB $G$ of (1.4) must be a union of $k$ loops attached to the vertices $(i_1,j_1),\ldots,(i_k,j_k)$. So the summation in (1.4) is over the set of unordered $k$-tuples $(i_1,j_1),\ldots,(i_k,j_k)$ with $1 \leq i_s,j_s \leq N$ for all $s$. In other words, the summation is over the set of graphs $F$ with the vertices $1,2,\ldots,n$ and $k$ unnumbered directed edges (loops are allowed).

One has $\langle \alpha_{ij},e_{kl} \rangle = \delta_{jk}$, so the contribution of a graph $F$ into (1.4) is equal to $a_{i_1,j_1}\cdots a_{i_k,j_k} \det(\delta_{i_q,j_p})_{1 \leq p \leq k}$. It is easy to see that the determinant is nonzero only if all the $i_p$ and all the $j_q$ are distinct (else the matrix has identical rows or columns), and for every $q$ there is $p = \sigma(q)$ such that $j_q = i_p$ (else a matrix has a zero row). If these conditions are satisfied, the determinant is obviously $(-1)^{\text{sgn}(\sigma)}$ where $\text{sgn}(\sigma)$ is the parity of the permutation $\sigma$. Hence, Theorem 1.1 in this case is reduced to the usual formula expressing coefficients of the characteristic polynomial of the operator via its matrix elements.

1.2.2. Graph Laplacians. Let $F$ be a directed graph without loops. Following [9], give the following definition:

**Definition.** A line bundle with connection on $F$ is a function attaching a number $\varphi_e \neq 0$ to every directed edge $e$ of the graph. By definition, also take $\varphi_{-e} = \varphi_e^{-1}$ where $-e$ is the edge $e$ with the direction reversed.

To explain the name, attach a one-dimensional space $\mathbb{R}$ (a fiber of the bundle) to every vertex of $F$ and interpret the number $\varphi_e$ as the $1 \times 1$-matrix of the operator of parallel transport along the edge $e$. For a path $\Lambda = (e_1,\ldots,e_k)$ denote $\varphi_{\Lambda} \overset{\text{def}}{=} \varphi_{e_1}\cdots\varphi_{e_k}$ (the operator of parallel transport along $\Lambda$). If the path $\Lambda$ is a cycle then $\varphi_{\Lambda}$ is called a holonomy of the cycle.

Suppose now that $F$ has the vertices $1,2,\ldots,n$ and no multiple edges. Supply also every edge $(i,j)$ of $H$ with a weight $c_{ij} = c_{ji}$. Take $N = n(n-1)/2$.

$$P(x_{12},\ldots,x_{n-1,n}) = \sum_{1 \leq i<j \leq n} c_{ij} x_{ij},$$

(1.6)
and \( c_{ij} \equiv u_i - \varphi_{ij}u_j \) and \( \alpha_{ij} \equiv u_i - \varphi_{ji}u_j \), and consider the operator \( M \) like in (1.4).

If \( v = \sum_{i=1}^{n} v_i u_i \) then

\[
M(v) = \sum_{1 \leq i < j \leq n} c_{ij}(v_i - \varphi_{ji}v_j)(u_i - \varphi_{ij}u_j) = \sum_{i=1}^{n} u_i \sum_{j \neq i} c_{ij}(v_i - \varphi_{ji}v_j).
\]

The operator \( M \) is called in [9] a Laplacian of the bundle.

Call a graph \( F \) a mixed forest if every its connected component is either a tree or a graph with one cycle (a connected graph with the number of vertices equal to the number of edges). The graphs where each component contains one cycle are called cycle-rooted spanning forests (CRSF) in [9]; hence the name “mixed forest” here.

The following corollary of Theorem 1.1 generalizes the Matrix-CRSF theorem of [7] and [9]:

**Theorem 1.2.** The characteristic polynomial of the Laplacian (1.7) of a line bundle on a graph is equal to \( \sum_{k=0}^{n} (-1)^k \mu_k t^k \) where

\[
(1.8) \quad \mu_k = \sum_{F \in \mathcal{M}_F} \prod_{(pq) \text{ is an edge of } F} c_{pq} \prod_{i=1}^{n-k} (m_i + 1) \prod_{j=1}^{\ell} (1 - w_j)(1 - 1/w_j).
\]

Here \( \mathcal{M}_F \) is the set of mixed forests containing \( n \) vertices, \( k \) edges and split into \( n-k \) tree components and \( \ell \) one-cycle components; \( m_i \) is the number of edges in the \( i \)-th component, and \( w_j \) is the holonomy of the cycle in the \( j \)-th component.

A special case of Theorem 1.2 arises if \( \varphi_{ij} = 1 \) for all \( i, j \). Then (1.4) implies \( M = \sum_{1 \leq p < q \leq n} c_{pq}(1 - \sigma_{pq}) \) where \( \sigma_{pq} \) is a reflection exchanging the \( p \)-th and the \( q \)-th coordinate: \( \sigma_{p}(u_p) = u_q \), \( \sigma_{pq}(u_q) = u_p \) and \( \sigma_{pq}(u_i) = u_i \) for \( i \neq p, q \). The reflections \( \sigma_{pq} \) generate the Coxeter group \( A_n \). The holonomies here are all equal to 1, so a mixed forest entering a summation in Theorem 1.2 cannot have cycles and should be a “real” forest:

**Corollary 1.3.** The characteristic polynomial of the operator \( M = \sum_{1 \leq p < q \leq n} c_{pq}(1 - \sigma_{pq}) \) is equal to \( \sum_{k=0}^{n} (-1)^k \mu_k t^k \) where

\[
\mu_k = \sum_{F \in \mathcal{F}} \prod_{(pq) \text{ is an edge of } F} c_{pq} \prod_{i=1}^{n-k} (m_i + 1).
\]

Here \( \mathcal{F} \) is the set of forests with \( n \) vertices, \( k \) edges and \( n-k \) components; \( m_i \) is the number of edges in the \( i \)-th component.

Apparently, \( \det M = 0 \) (there are no forests with \( n \) vertices and \( n \) edges), so the summation is indeed up to \( k = n - 1 \). This corollary follows also from the classical Principal Minors Matrix-Tree Theorem, see e.g. [6] for proofs and related results.

Another possibility is to join every pair of vertices \((i, j)\) with two edges: \((i, j)_-\) with \( \varphi_{ij} = 1 \) (“a -edge”), because \( c_{ij}^- = u_i - u_j \) and \((i, j)_+\) with \( \varphi_{ij}^+ = -1 \) (“a +edge” because \( c_{ij}^+ = u_i + u_j \)); the weights are \( c_{ij}^- \) and \( c_{ij}^+ \), respectively. The holonomy of a cycle is \( w = (-1)^d \) where \( d \) is the number of +edges in the cycle. By (1.7), \( M = \sum_{1 \leq p < q \leq n} c_{pq}^-(1 - \sigma_{pq}) + c_{pq}^+(1 - \tau_{pq}) \) where \( \sigma_{pq} \) is as before and \( \tau_{pq} \)
Corollary 1.4. The characteristic polynomial of the operator

\[ M = \sum_{1 \leq p < q \leq n} c_{pq}^-(1 - \sigma_{pq}) + c_{pq}^+(1 - \tau_{pq}) \]

is equal to \( \sum_{k=0}^{n} (-1)^k \mu_k t^k \) where

\[ \mu_k = \sum_{F \in \mathcal{MFD}_{n,k,\ell}} \prod_{(pq) \text{ is an edge of } F} c_{pq}^{n-k} \prod_{i=1}^{n-k} (m_i + 1). \]

Here \( \mathcal{MFD}_{n,k,\ell} \) is the set of mixed forests with \( k \) edges \( (pq)_s \), \( n-k \) tree components and \( \ell \) one-cycle components such that the number of + -edges in every cycle is odd; \( m_i \) is the number of edges in the \( i \)-th components.

This corollary generalizes [3, Theorem 3.2].

1.2.3. The level 2 Laplacian. Take up the same setting as in Section 1.2.2 (a line bundle with a connection over a graph \( F \)), and take

\[ P(x_{12}, \ldots, x_{n-1,n}) = \sum_{1 \leq i \leq n, 1 \leq j, k \leq n} c_{ijk} (x_{ij}x_{ik} - x_{ik}x_{ij}) = \sum_{1 \leq i, j, k \leq n} c_{ijk} x_{ij} x_{ik}; \]  

here the constants \( c_{ijk} \) are defined for all \( 1 \leq i, j, k \leq n \) and possess the property \( c_{ijk} = -c_{ikj} \) for all \( i, j, k \) (in particular, \( c_{ijj} = 0 \) for all \( i, j \)). Define \( M \) by (1.1):

\[ M \overset{\text{def}}{=} P(M_{12}, \ldots, M_{n-1,n}) \]  

where \( M_{ij}(v) = \langle \alpha_{ij}, v \rangle \epsilon_{ij}. \)

Explicitly, if \( v = \sum_{p=1}^{n} v_p u_p \) then

\[ M_{ij}(v) = (v_i - \varphi_{ji} v_j)(u_i - \varphi_{ji} u_j) = (v_i - \varphi_{ji} v_j)u_i + (v_j - \varphi_{ji} v_i)u_j. \]

and

\[ M(v) = \sum_{i \neq j} u_i v_j \left( \varphi_{ij} \sum_{k \neq i, j} (c_{ijk} + c_{kij}) + \sum_{k \neq i, j} c_{kij} \varphi_{ik} \varphi_{kj} \right). \]

Remark 1.5. Note that \( M_{ij} = M_{ji} \) because \( \alpha_{ji} = -\varphi_{ji} \alpha_{ij} \) and \( \epsilon_{ji} = -\varphi_{ij} \epsilon_{ij} \). Nevertheless, since \( \alpha \) and \( \epsilon \) enter the equation (1.4) separately, one has to choose the ordering of indices in every pair \( (i, j) \) used; the final result, of course, does not depend on the choice. We will write \( \alpha_{[i,j]} \) meaning \( \alpha_{ij} \) or \( \alpha_{ji} \) depending on the choice; the same for \( \epsilon \).

Remark 1.6. In particular, if \( \varphi_{ij} = 1 \) for all \( i, j \), then \( M(v) = \sum_{1 \leq i < j < k \leq n} w_{ijk} v_i u_j \) where \( w_{ijk} \overset{\text{def}}{=} c_{ijk} + c_{kji} + c_{kij} \). Note that \( w_{ijk} \) is totally skew-symmetric in all the three indices, and therefore the operator \( M \) is skew-symmetric. In [11] a beautiful formula for the Pfaffian of \( M \) was proved; see also [3] for a proof of the same formula using a technique close to Theorem 1.1.

We call \( M \) the level 2 Laplacian of the bundle, by an apparent analogy with the operator defined by (1.7). Note that \( M_{ij} \) and \( M_{kl} \) commute if \( \{i, j\} \cap \{k, l\} = \emptyset \) or \( \{i, j\} = \{k, l\} \), that’s why \( P \) contains no terms like \( x_{ij} x_{kl} - x_{kl} x_{ij}. \)
Application of Theorem 1.1 to the level 2 Laplacian gives the formula similar to (1.8), where the summation is done over the set of triangulated polyhedra of special kind.

A nodal surface is obtained from a smooth surface (a 2-manifold, not necessarily connected, possibly with boundary), by gluing a finite number of points. If the surface has boundary then boundary points also can be glued. A boundary of the nodal surface is still well-defined, but unlike the boundary of a smooth surface, it can be any graph, not just collection of circles. Nodal surfaces with boundary attracted much attention in recent years due to their connection with the geometry of moduli spaces of complex structures, see [12].

Depending on the surface, we will speak about nodal disks, annuli, Moebius bands, etc.

A compact 2-dimensional polyhedron $H$ is called reducible if it can be split into a union $H = H_1 \cup \cdots \cup H_\ell$ where the polyhedra $H_i$ are compact and edge-disjoint (but not necessarily vertex-disjoint!). Every reducible polyhedron is a union of irreducible components.

A 2-dimensional polyhedron is called a cycle polyhedron if it is irreducible, homeomorphic to a nodal surface, and every its face is a triangle with exactly one side on the boundary. A 2-dimensional polyhedron is called a chain polyhedron, if the last condition is satisfied for all the faces except two. Each of these two faces has two sides on the boundary. One face is called an initial face; one of its boundary sides is marked and called an initial side. The other exceptional face is called a terminal one; one of its boundary sides is marked and called a terminal side.

Choosing an orientation of a face of a cycle polyhedron is equivalent to ordering its sides: the first internal side, the second, the boundary side. For a chain polyhedron the rule is the same except for the initial and the terminal face. For the initial face the ordering is: the initial side, the internal side, the second boundary side; for the terminal face: the internal side, the terminal side, the second boundary side. Say that an orientation of two adjacent faces sharing a side $a$ is compatible if $a$ is the first internal side in one face and the second internal side in the other.

**Lemma 1.7.** A cycle polyhedron is one of the following:

1. a nodal annulus where all vertices lie in the boundary;
2. a nodal Moebius band with the same property;
3. a disk (smooth) with a vertex in the interior joined by edges with all the other vertices, which lie on the boundary.

A chain polyhedron is a nodal disk.

For every cycle polyhedron there are exactly two ways to orient all its faces compatibly. For every chain polyhedron (where the initial and terminal sides are chosen) there is exactly one way to orient all its faces compatibly.

See Section 2.3 for proof.

**Example 1.8.** See Fig. 11. The left-hand side is a cycle polyhedron (a nodal Moebius band with the node $c$), the right-hand side is a chain polyhedron (a nodal disk with the node $c$). Solid lines represent internal edges and exceptional edges ($ab$ and $ef$ at the nodal disk); dotted lines represent boundary edges. The graph $G$ is drawn below; the cycle is directed clockwise and a line is directed left to right. The corresponding ordering of edges inside every face of $H$ is shown by the numbers $1, 2, 3$. 
Figure 1. A cycle polyhedron and a chain polyhedron

Denote by $\mathcal{CP}_{n,k}$ the set of polyhedra with the vertices 1, 2, ..., $n$, having $k$ faces, such that every its irreducible component is either a cycle polyhedron or a chain polyhedron with compatibly oriented faces. By $\text{ess}_1(H)$ denote the graph formed by the internal edges of $H$ and initial and terminal edges of its chain components. The initial and the terminal edges themselves denote by $I_{1}(H), \ldots, I_{s}(H)$, and $T_{1}(H), \ldots, T_{s}(H)$, respectively. Let $H \in \mathcal{CP}_{n,k}$ be such that the graphs $H_{I} \, \text{def} := \text{ess}_1(H) \setminus \{I_{1}(H), \ldots, I_{s}(H)\}$ and $H_{T} \, \text{def} := \text{ess}_1(H) \setminus \{T_{1}(H), \ldots, T_{s}(H)\}$ are mixed forests. Let $H_{I}^{1}, \ldots, H_{I}^{t}$ and $H_{T}^{1}, \ldots, H_{T}^{t}$ be connected components of $H_{I}$ and $H_{T}$, respectively, that are trees (the number of such components $t = m - n - s$, where $m$ is the number of edges in $H$ is the same for both graphs). Choose in every tree a root $r_{i}^{\alpha} \in H_{I}^{i}$ and $r_{j}^{e} \in H_{T}^{j}$ and consider a matrix $M(H)$ such that

$$M(H)_{i}^{j} = \sum_{\Lambda \in L_{i,j}} \varphi_{\Lambda}^{2} n_{\Lambda}$$

where $L_{i,j}$ is the set of paths joining $r_{j}^{e}$ with $r_{i}^{\alpha}$, and $n_{\Lambda}$ is the number of vertices along the path $\Lambda$ that belong both to $H_{I}^{i}$ and $H_{T}^{j}$.

For an oriented face $F$ of a cycle or a chain polyhedron denote by $s_{i}(F)$ its $i$-th side ($i = 1, 2, 3$); by $v_{i}(F)$ denote the vertex opposite to the side number $i$. The internal sides are directed away from their common point; direction of the boundary side is not important.

**Theorem 1.9.** Let $M$ be a level 2 Laplacian of a line bundle on a graph $F$. Then $\text{char}(M(t)) = \sum_{k=0}^{n} (-1)^{k} \mu_{k} t^{k}$ where

$$\mu_{k} = \sum_{H \in \mathcal{CP}_{n,k}} \prod_{\Phi \in \text{a face of } H} c_{\Phi}(\alpha_{s_{1}(\Phi)}, \alpha_{s_{2}(\Phi)}, v_{1}(\Phi), v_{2}(\Phi)) \varphi_{pq}$$

$$\times \det(M(H) \times \prod_{i,j=1}^{s} \prod_{(pq) \in H \text{ lies in a path joining } r_{i}^{e} \text{ with } r_{j}^{e}} \varphi_{pq})$$

(1.11)

See Remark 1.5 for the notation $\alpha_{[s_{1}]}, c_{[s_{2}]}$.

2. Proofs
2.1. Technical lemmas. Let $H$ be a directed graph without loops or multiple edges.

**Definition.** A line bundle with connection on $H$ is a function a number $\varphi_{ij} \neq 0$ to every directed edge $(i, j)$ of the graph. By definition, also take $\varphi_{ji} = \varphi_{ij}^{-1}$.

To explain the name, attach a one-dimensional space $\mathbb{R}$ (a fiber of the bundle) to every vertex and interpret the number $\varphi$ to every directed edge $(i,j)$. Consider the space $\mathbb{R}^n$ (which can be interpreted as a total space of the bundle) with the standard basis $u_1, \ldots, u_n$. For every $i, j$ consider the vectors $\alpha_{ij} \overset{\text{def}}{=} u_i - \varphi_{ij} u_j$ and $e_{ij} \overset{\text{def}}{=} u_i - \varphi_{ji} u_j$. Denote by $A_H$ and $E_H$ the sets of vectors $\alpha_{ij}$ or $e_{ij}$, respectively, where $(i, j)$ runs through the edges of $H$.

Introduce in $\mathbb{R}^n$ a standard scalar product making $u_1, \ldots, u_n$ an orthonormal basis. For two sequences of vectors $M = (\mu_1, \ldots, \mu_k)$ and $X = (\xi_1, \ldots, \xi_k)$ of the same size $k$ denote by $G(M, X)$ the $k \times k$-matrix where the $(i,j)$-th entry is equal to the scalar product $\langle \mu_i, \xi_j \rangle$. Call two sequences, $M$ and $M'$, elementarily equivalent, if $M'$ can be obtained from $M$ by a finite number of substitutions $\mu_i \mapsto \mu_i + t \mu_j$ where $t$ is any number. The matrices $G(M, X)$ and $G(M', X)$ are also connected by elementary equivalence (a row is replaced by the sum of itself with a multiple of another row), and therefore $\det G(M, X) = \det G(M', X)$; the same applies to $X$.

Number the edges of $H$ arbitrarily and consider the matrices $P_H \overset{\text{def}}{=} G(A_H, A_H)$, $Q_H \overset{\text{def}}{=} G(A_H, E_H)$, $R_H \overset{\text{def}}{=} G(E_H, E_H)$. The matrix element of $P_H$, $Q_H$ and $R_H$ corresponding to the pair of edges $s$ and $t$ depends on their mutual position as follows:

- If $s$ and $t$ do not intersect then the matrix element is zero.
- If $s$ and $t$ have one common vertex $b$ then the matrix element is equal to the product $\psi_{s,a} \psi_{t,a}$ where

$$\psi_{r,a} = \begin{cases} 1, & \text{if the edge } r \text{ points away from the vertex } a, \\ -\varphi_r, & \text{if the edge } r \text{ points towards the vertex } a \\ -\varphi_r^{-1}, & \text{if the edge } r \text{ points towards the vertex } a \end{cases}$$

and corresponds to the vector $\alpha$.

- If the edges have two common vertices $a$ and $b$ then the product is equal to the sum $\psi_{s,a} \psi_{t,a} + \psi_{s,b} \psi_{t,b}$.

The most important cases later will be when $H$ is a tree or a graph with one cycle. Describe the determinants of $P_H$, $Q_H$ and $R_H$ for such $H$. For a directed path $\Lambda = (\lambda_0, \ldots, \lambda_k)$ in the graph denote by $\varphi_\Lambda$ the product $\varphi_{\lambda_0, \lambda_1} \cdots \varphi_{\lambda_{k-1}, \lambda_k}$.

**Lemma 2.1.** Let $H$ be a rooted tree. Then

$$\det P_H = \prod_{i,j} \varphi_{ij}^2 \sum_\Lambda \varphi_\Lambda$$

where the product is taken over all the edges of $H$ directed away from the root, and the sum, over all chains $\Lambda$ joining vertices of the tree with the root. Similarly,

$$\det R_H = \prod_{i,j} \varphi_{ji}^2 \sum_\Lambda \varphi_\Lambda^{-2}$$
in the same notation.

Proof. Clearly, the statements for \( P_H \) and \( R_H \) are equivalent, so consider the \( P_H \) case. Since \( \alpha_{ji} = \varphi_{ji} \alpha_{ij} \), changing the direction of an edge \( (i, j) \rightarrow (j, i) \) multiplies both sides of (2.12) by \( \varphi_{ji}^2 \). So, it is enough to prove the lemma for trees where all the edges are directed away from the root.

Let \( p \) be a hanging edge of \( H \) and \( q \), its parent. Then \( (\alpha_p, \alpha_p) = 1 + \varphi_p^2 \), \( (\alpha_p, \alpha_q) = -\varphi_q \) (the edge \( q \) is directed towards \( p \), and \( p \), away from \( q \)), and \( (\alpha_p, \alpha_e) = 0 \) for all the other edges \( e \). Develop the determinant \( \det P_H \) by the row \( p \) and then by the row \( q \); this gives a relation

\[
\det P_H = (1 + 1/\varphi_p^2) \det P_H' - 1/\varphi_q^2 \cdot \det P_H''
\]

where \( H' \) and \( H'' \) are \( H \) with \( p \) deleted and \( p \) and \( q \) deleted, respectively.

Suppose by induction that the theorem is proved for \( H' \) and \( H'' \). Multiply both sides of (2.13) by \( \prod_{e} \varphi_e^2 \) where \( e \) runs through the set of edges of \( \Gamma \). It gives

\[
\prod_{e} \varphi_e^2 \det P_H = \sum_{\Lambda'} \varphi_{\Lambda'}^2 + 1/\varphi_p^2 \cdot \sum_{\Lambda'} \varphi_{\Lambda'}^2 - 1/\varphi_p^2 \sum_{\Lambda''} \varphi_{\Lambda''}^2
\]

where \( \Lambda', \Lambda'' \) are chains in \( H' \) and \( H'' \), respectively, joining vertices with the root. The first sum is over all the chains in \( \Gamma \) not containing \( p \). The second sum is over all the chains containing \( p \), plus the union of \( p \) with a chain in \( \Gamma'' \) (because \( p \) is hanging). The third sum cancels the second term.

**Lemma 2.2.** If \( H \) is a tree with \( m \) edges (and \( m+1 \) vertices) then \( \det Q_H = m+1 \).

Proof. Let \((i, j)\) be an edge of \( H \). The graph \( H \setminus (i, j) \) is a union of two trees \( H_1 \) and \( H_2 \) containing vertices \( i \) and \( j \), respectively. The matrix \( Q_H \) looks like

\[
Q_H = \begin{pmatrix}
2 & 1 & \ldots & 1 & -\varphi_{ji} & \ldots & -\varphi_{ji} \\
1 & Q_{H_1} & 0 & 0 & 0 & 0 & 0 & 0 \\
\vdots & & & & & & & \\
1 & 0 & Q_{H_2} & 0 & 0 & 0 & 0 & 0 \\
-\varphi_{ij} & 0 & 0 & Q_{H_1} & 0 & 0 & 0 & 0 \\
\vdots & & & & & & & \\
-\varphi_{ij} & 0 & 0 & 0 & Q_{H_2} & 0 & 0 & 0 \\
\end{pmatrix}
\]

(we suppose that the edge \((i, j)\) corresponds to the first row and the first column).

Denote the sizes of the first and the second block (i.e. the numbers of edges in \( H_1 \) and \( H_2 \)) by \( m_1 \) and \( m_2 \) respectively (thus, \( m = m_1 + m_2 + 1 \)). Let \( p > m_1 > q \). Then the last \( m_2 \) rows of the minor \( [Q_H]_{1,p+1}^{1,q+1} \) (deleted columns 1 and \( p+1 \) and rows 1 and \( q+1 \)) have zeros at the \( m_1 \) initial positions; only the last \( m-2-m_1 = m_2-1 \) positions may be nonzero. Hence the rows are linearly dependent, so that \( \det [Q_H]_{1,p+1}^{1,q+1} = 0 \). The same is true if \( p < m_1 < q \). Having \( \det Q_H \) decomposed by the first row and then by the first column, one is left only with the terms \((Q_H)_{1,p+1}(Q_H)_{q+1,1} \det [Q_H]_{1,p+1}^{1,q+1} \) where either \( p, q \leq m_1 \) or \( p, q \geq m_1 + 1 \). In both cases \((Q_H)_{1,p+1}(Q_H)_{q+1,1} = 1 \), and therefore \( \det Q_H \) does not depend on \( \varphi_{ij} \) (recall that \( Q_{H_1} \) and \( Q_{H_2} \) contain no \( \varphi_{ij} \) because neither \( H_1 \) nor \( H_2 \) have an edge \((i, j)\)). Since \((i, j)\) is just an arbitrary edge of \( H \), it proves that \( \det Q_H \) does not depend on any \( \varphi_{pq} \) and is a constant.

Suppose now that \( \varphi_{pq} = 1 \) for all \( p \) and \( q \). Then \( e_{pr} = u_p - u_r = (u_p - u_q) + (u_q - u_r) = e_{pq} + e_{qr} \), and the same is true for \( \alpha \). Suppose the tree \( H \) contains edges \( pq \)
and \( qr \). Consider a tree \( H' \) where the edge \( qr \) is replaced by \( pr \); then the systems of vectors \( A_H \) and \( A_{H'} \), as well as \( E_H \) and \( E_{H'} \), differ by an elementary transformation, and therefore \( \det Q_H = \det Q_{H'} \). By such operations one can convert \( H \) into a line, i.e. a tree with the edges \((p_0, p_1), (p_1, p_2), \ldots, (p_{m-1}, p_m)\). The matrix \( Q_H \) for such tree is: \((Q_H)_{ii} = 2, (Q_H)_{i,i+1} = (Q_H)_{i,i-1} = 1\) for all \( i \). An easy induction shows that \( \det Q_H = m + 1 \). \( \square \)

Let now \( H \) be a graph with one cycle, i.e. a connected graph with \( n \) vertices and \( n \) directed edges. It consists of a cycle \( p_1, \ldots, p_s \) and, possibly, some trees (“antlers”) attached to the vertices \( p_i \). The direction of edges in the cycle and in the antlers can be arbitrary. Following [9], call the holonomy of the cycle the product \( w \defeq \varphi_{p_1}^\pm \cdots \varphi_{p_s}^\pm \) where the \( i \)-th exponent is +1 if the corresponding edge is directed along the cycle (from \( p_i \) to \( p_{i+1} \)) and −1 otherwise.

**Lemma 2.3.** Let \( H \) be a graph with one cycle. Then

\[
\det P_H = (1 - w)^2 \prod_{i,j} \varphi_{ij}^2,
\]
\[
\det R_H = (1 - w^{-1})^2 \prod_{i,j} \varphi_{ij}^{-2},
\]
\[
\det Q_H = -(1 - w)(1 - w^{-1}),
\]

where the product is taken over the set of all the edges in the antlers directed away from the cycle.

**Proof.** The proofs of all the three statements are similar; here is the proof of the statement about \( Q_H \).

Consider a system of vectors \( E_H^{(1)} \) elementarily equivalent to \( E_H \). To obtain \( E_H^{(1)} \) from \( E_H \) replace every vector \( \varepsilon_i \equiv e_{p_ip_{i+1}} = u_{p_i} - \varphi_{p_{i+1}p_i} u_{p_{i+1}}, i = 1, \ldots, s \), with \( \varepsilon_i^{(1)} \) where

\[
\varepsilon_s^{(1)} \defeq \varepsilon_s + \varphi_{p_s} \varepsilon_{s-1} = u_{p_s} - \varphi_{p_s} u_{p_s} + \varphi_{p_s} u_{p_s} u_1,
\]
\[
\varepsilon_{s-2}^{(1)} = \varepsilon_{s-2} + \varphi_{p_{s-1}p_s} \varepsilon_{s-1} = u_{s-2} - \varphi_{p_{s-1}p_s} u_{s-2} + \varphi_{p_{s-1}p_s} u_{p_s} u_1,
\]
\[
\vdots
\]
\[
\varepsilon_1^{(1)} = u_1 (1 - \varphi_{p_{i+1}} \cdots \varphi_{p_1}) = (1 - 1/w) u_1.
\]

Consider also a system \( A_H^{(1)} \) obtained from \( A_H \) in a similar way. One has \( G(E^{(1)}, A_H^{(1)}) = Q_H \) by elementary equivalence.

If \( w = 1 \) then \( \varepsilon_1^{(1)} = 0 \), so \( \det Q_H = \det G(E^{(1)}, A_H^{(1)}) = 0 \), and the lemma is proved; we suppose from now on that \( w \neq 1 \).

Consider now the sequence \( E_H^{(2)} \), which is \( E_H^{(1)} \) with every \( \varepsilon_i^{(1)} \) replaced with

\[
\varepsilon_i^{(2)} = \varepsilon_i^{(1)} - \varepsilon_i^{(1)}/(1 - 1/w) = -\varphi_{p_i} u_{p_{i+1}} u_i.
\]

Consider also \( A_H^{(2)} \) obtained from \( A_H^{(1)} \) in a similar way. The sequences \( E_H^{(2)}, A_H^{(2)} \) are elementarily equivalent to \( E_H^{(1)}, A_H^{(1)} \), so \( \det G(E_H^{(2)}, A_H^{(2)}) = Q_H \).

Denote by \( H_1 \) a subtree of \( H \) attached to the vertex \( p_i \), \( 1 \leq i \leq s \), and let \( \beta_j = e_{q_jr_j}, 1 \leq j \leq m_i \) be vectors in \( E_H \) (and \( E_H^{(2)} \)) corresponding to its edges. For
all edges attached immediately to $p_i$ (so that $q_i = p_i$) replace $\beta_j \mapsto \beta_j^{(1)} = \beta_j + \varepsilon_i \varphi_{p_i+1, p_i} = -\varphi_{p_i+1, p_i}$. Then do the same for all the edges attached to endpoints of $\beta_j^{(1)}$, etc. Having done this for all $i$, $1 \leq i \leq s$, one obtains the system $E_H' = ((1-w)u_1, -\varphi_{12}u_2, \ldots, -\varphi_{1m}u_m)$ elementarily equivalent to $E_H$. Similarly, $A_H$ is elementarily equivalent to $A_H' = ((1-w)u_1, -\varphi_{12}u_2, \ldots, -\varphi_{1m}u_m)$. Now det $Q_H = \det G(E_H', A_H')$: the matrix $G(E_H', A_H')$ is a diagonal matrix with $(1-w)(1-1/w)$ in the corner and 1 in all the other positions on the main diagonal.

Below we will need two more statements from the general linear algebra.

**Lemma 2.4.** Let $e_1, \ldots, e_n$ be an orthonormal basis in $\mathbb{R}^n$, and $\mu_i = \sum_{j=1}^n a_{ij}e_j$. Then $\det G(M, M)$ is equal to the sum of squares of all the $k \times k$-minors of the matrix $A = (a_{ij})_{1 \leq i \leq n}$. The lemma is classical; see e.g. [8, IX§5] for proof.

Let now $M, X \subset \mathbb{R}^n$ be two linear subspaces of the same dimension $k$, and $M = (\mu_1, \ldots, \mu_k)$ and $X = (\xi_1, \ldots, \xi_k)$ be bases in them. Denote

$$\angle(M, X) \overset{\text{def}}{=} \det G(M, X)^2 / (\det G(M, M) \cdot \det G(X, X)).$$

**Lemma 2.5.**

1. $\angle(M, X)$ depends only on the subspaces and not on the choice of the bases $M$ and $X$ in them.
2. $\angle(M^\perp, X^\perp) = \angle(M, X)$.

**Proof.** Let $N = (v_1, \ldots, v_k)$ be another basis in $M$; denote by $A = (a_{ij})$ the transfer matrix: $v_i = \sum_{j=1}^k a_{ij}v_j$. Then $G(N, X) = AG(M, X)$, and $G(N, N) = AG(M, M)A^*$; so $G(N, X) = (\det G(N, N) \cdot \det G(X, X)) = A^2 \det G(M, M) \cdot \det G(X, X)$. The same is for $X$.

Let now $M_1, M_2 \subset \mathbb{R}^n$ be two spaces of equal dimension, and let $M \overset{\text{def}}{=} \langle M_1 \cup M_2 \rangle$ be their linear hull. Then $M_i^\perp = M_i^\perp, M_i \oplus M_i^\perp, i = 1, 2$; here $\perp$ means an orthogonal complement in $\mathbb{R}^n$, and $(\perp, M)$ means an orthogonal complement in $M$. Choose an orthonormal basis $T = (\tau_1, \ldots, \tau_q)$ in $M^\perp$, and bases $\Lambda_i = (\lambda^{(i)}_1, \ldots, \lambda^{(i)}_s)$ in $M_i^\perp, M_i$, $i = 1, 2$, normal to $M^\perp$. Denote $Y_1 = (\lambda^{(1)}_1, \ldots, \lambda^{(1)}_s, \tau_1, \ldots, \tau_q)$ and $Y_2 = (\lambda^{(2)}_1, \ldots, \lambda^{(2)}_s, \tau_1, \ldots, \tau_q)$ are bases in $M_1^\perp$ and $M_2^\perp$, respectively. By the first statement of the theorem, $\angle(M_1^\perp, M_2^\perp) = \det G(Y_1, Y_2)^2 / (\det G(Y_1, Y_1) \cdot \det G(Y_2, Y_2))$. The matrix $G(Y_1, Y_2)$ is block diagonal; its first block is the matrix $G(\Lambda_1, A_2)$, and the second block is the unit matrix $G(T, T)$; thus $\det G(Y_1, Y_2) = \det G(\Lambda_1, A_2)$. Similarly, $\det G(Y_1, Y_1) = \det G(\Lambda_1, \Lambda_1)$ and $\det G(Y_2, Y_2) = \det G(\Lambda_2, A_2)$. Hence, $\angle(M_1^\perp, M_2^\perp) = \angle(M_1^{\perp, M_1}, M_2^{\perp, M_2})$.

Let now $X = M_1 \cap M_2 \neq \emptyset$. A similar choice of a basis shows that $\angle(M_1, M_2) = \angle(X, Y)$ where $X = \langle X, X \rangle$ and $X = \langle X, X \rangle$. So, it is enough to prove the second statement of the theorem for the $k$-dimensional subspaces $M_1, M_2 \subset \mathbb{R}^n$ such that $n = 2k$ and $M_1 \cap M_2 = 0$, so that $\mathbb{R}^n = \langle M_1 \cap M_2 \rangle$. To do this take an orthonormal basis $e_1, \ldots, e_{2k}$ in $\mathbb{R}^n$ such that $X_1 \overset{\text{def}}{=} \langle e_1, \ldots, e_k \rangle$ is a basis in $M_1$, and $X_2 \overset{\text{def}}{=} \langle e_{k+1}, \ldots, e_{2k} \rangle$ is a basis in $M_2$. Let $F = (f_1, \ldots, f_k)$ be a basis in $\mathbb{R}^k$. Then $G(F, X_1) = B$, and therefore $\det G(F, X_1) = \Delta \overset{\text{def}}{=} \det B$. By Lemma 2.4, the
determinant \( \det G(F, F) \) is equal to the sum of squares of all \( k \times k \)-minors of the \( k \times (2k) \)-matrix composed of two blocks, \( B \) and the identity \( k \times k \)-matrix. It is easy to see that the latter sum is the sum of squares of all the minors of \( B \) (of all sizes), including \( \Delta^2 \) and 1 (the square of the empty minor).

Consider now the vectors \( h_i = -e_i + \sum_{j=1}^k b_{ji}e_{j+k}, \ i = 1, \ldots, k \). Apparently, \( H = (h_1, \ldots, h_k) \) is a basis in \( \mathcal{M}_G \); also \( G(H, X_2) = B^T = G(F, X_1)^T \), and also \( \det G(H, H) = \det G(F, F) \) by Lemma 2.4. This proves the lemma. \( \Box \)

2.2. **Proof of Theorem 1.2** To apply Theorem 1.1 note that the polynomial \( P \) of (1.8) has degree 1, so every path involved should contain one vertex only. Therefore, the DOOMB \( G \) must be a union of \( k \) loops attached to the vertices \( (i_1, j_1), \ldots, (i_k, j_k) \). So the summation is over the set of unordered \( k \)-tuples \( (i_1, j_1), \ldots, (i_k, j_k) \) with \( 1 \leq i_s, j_s \leq N \) for all \( s \). In other words, the summation is over the set of graphs \( F \) with the vertices \( 1, 2, \ldots, n \) and \( k \) unnumbered directed edges.

Let \( F_1, \ldots, F_k \) be connected components of \( F \). If the edges \( d_1 \) and \( d_2 \) belong to different components then \( (a_{d_1}, e_{d_2}) = 0 \). So the matrix \( Q_F \) is block diagonal, and \( \det Q_F = \det Q_{F_1} \cdots \det Q_{F_k} \). If \( F \) is a connected graph with \( m \) edges \( (i_1, j_1), \ldots, (i_m, j_m) \) and \( \mu < m \) vertices then the vectors \( e_{i_1,j_1}, \ldots, e_{i_m,j_m} \) belong to a vector space of dimension \( \mu \) spanned by the corresponding basis elements \( u_\mu \). Therefore they are linearly dependent, so that \( \det Q_F = 0 \). Thus, if \( \det Q_F \neq 0 \) then every connected component \( F_i \) of \( F \) should be either a tree (with \( \mu = m + 1 \)) or a graph with one cycle (\( \mu = m \)). So, \( F \) is a mixed forest.

Theorem 1.2 now follows from Theorem 1.1 Lemma 2.2 and Lemma 2.3.

2.3. **Proof of Theorem 1.9** Apply Theorem 1.1 to the operator \( M \) of (1.10). Like in Section 2.2 vertices of the graph \( G \) are pairs \( (i, j) \), \( 1 \leq i < j \leq n \), that is, edges of a directed graph with the vertices \( 1, \ldots, n \). The polynomial \( P \) contains only terms \( x_{ij}x_{jk} \); therefore if \( (a, b) \) is an edge of \( G \) then the pairs \( a = \{i, j\} \) and \( b = \{i, k\} \) have exactly one common element. Represent such edge by a triangle \( ijk \). Color its sides \( ij \) and \( ik \) (corresponding to the vertices of \( G \)) black, and the third side \( jk \), red (they are shown as sold and the dashed lines, respectively, in Fig. 1). Thus a graph \( G \) is represented by 2-dimensional polyhedron (call it \( H \)) with triangular faces and edges colored red and black so that every face has two black sides and one red side. The black edges of \( H \) form a graph denoted by \( \text{ess}_1(H) \) and called an essential 1-skeleton of \( H \).

If \( G \) consists of connected components \( G_1, \ldots, G_s \) then the corresponding sub-polyhedra \( H_1, \ldots, H_s \) of \( H \) are edge-disjoint but not necessarily vertex-disjoint; thus, \( H \) is reducible, and \( H_1, \ldots, H_s \) are its irreducible components. By Theorem 1.1 every \( G_i \) is either an oriented cycle or an oriented chain.

Let first \( G_i \) be an oriented cycle. In the corresponding \( H_i \) every black edge belongs to two faces and every red edge, to one face. Hence \( H_i \) is a nodal surface with boundary where every face has two internal sides and one boundary side. An orientation of an edge of \( G_i \) gives rise to an orientation of the corresponding face of \( H_i \); since the orientations of the edges in a cycle \( G_i \) are compatible, so are orientations of the faces in \( H_i \). The graph \( G_i \) is connected, so the polyhedron \( H_i \) is irreducible. Hence, \( H_i \) is a cycle polyhedron. Conversely, if \( H_i \) is a cycle polyhedron, then \( G_i \) is connected and every its vertex has valency 2 — hence, \( G_i \) is a cycle.
In a similar manner one proves that $G_i$ is an oriented chain if and only if $H_i$ is a chain polyhedron.

Proof of Lemma 1.7. Let $H$ be a cycle polyhedron with the vertices $1, 2, \ldots, n$. Build the graph $G$ such that the vertices of $G$ are internal edges of $H$ (elements of $\text{ess}_1(H)$), and two vertices are joined by an edge if the corresponding edges belong to a face.

As proved before, the graph $G$ is an oriented cycle where every vertex is marked by two indices $(i, j)$, and the neighboring vertices have exactly one common index. For every index $i$ denote by $K_i$ the set of vertices in $G$ containing $i$ as one of the indices. If for some $i$ the corresponding $K_i$ contains all the vertices, then the second indices $j$ in all the vertices are pairwise distinct, and we have a disk described in clause 3 of the lemma. From now on suppose that for every index $i$ there is at least one vertex of $G$ not containing it.

Every $K_i$ is a union of several “solid arcs” $K_{i,1}, \ldots, K_{i,s_i}$ (a solid arc is one vertex or a sequence of several successive vertices in a cycle). Consider an auxiliary graph $G'$ where for all $i$ and $p = 1, \ldots, s_i$ the index $i$ in the vertices in $K_{i,p}$ is renamed into a new index $i_p$. Thus, the polyhedron $H$ is obtained from the polyhedron $H'$ corresponding to $G'$ by identification of some vertices. Thus it is enough to prove that if every $K_i$ is one solid arc (but not the whole cycle) then the polyhedron $H$ is an annulus or a Moebius band.

Let $G$ contain $m$ vertices. Consider an auxiliary circle $S^1$ with $m$ equidistant points $a_1, \ldots, a_m$ on it. Let $K_i$ be a solid arc covering the vertices $p, p+1, \ldots, p+q$. Define a subset $L_i \subset S^1$ as an arc from $a_p$ to $a_{p+q}$, including both endpoints. Every two neighboring vertices in $G$ have a common index $i$, so every point of every arc $[a_p, a_{p+1}]$ belongs to some $L_i$, hence $\bigcup L_i = S^1$. Triple intersections of different $L_i$s are all empty; the intersections $L_i \cap L_j$ are either empty or one point. There are exactly $m$ pairs with nonempty intersection, because $L_i$ are arcs. Thus, the Euler characteristics is

$$0 = \chi(S^1) = \chi(\bigcup L_i) = \sum_i \chi(L_i) - \sum_{i,j} \chi(L_i \cap L_j).$$

One has $\chi(L_i) = 1$ for all $i$ and $\chi(L_i \cap L_j) = 1$ for $m$ pairs $i, j$ where the intersection is nonempty, and $\chi(L_i \cap L_j) = 0$ otherwise. So, the number of indices $i$, that is, the number of vertices of the polyhedron $H$, is equal to $m$. The total number of its faces is the number of edges in $G$, that is, $m$. Also $H$ contains $m$ red edges (one per face) and $m$ black edges (corresponding to the vertices of $G$). Thus $\chi(H) = m - 2m + m = 0$, and $H$ is either an annulus or a Moebius band.

The proof in the chain case is similar.

So, summation in the right-hand side of Theorem 1.1 for the $\text{char}_M(t)$ is performed over the set of nodal sufraces with boundary $H = H_1 \cup \cdots \cup H_m$ where the irreducible components $H_1, \ldots, H_s$ are either cycle polyhedra or chain polyhedra.

The weight $W_P(H)$ of a polyhedron is equal to the product of the weights of the faces. The weight of a triangular face $pqr$ where $pq$ is the first internal edge, $pr$, the second, and $qr$, a boundary edge, is equal to $e_{pq} \cdot e_{pr} \cdot e_{qr}$. So, the second factor is equal to $1, -\varphi_{pq}, -\varphi_{pr}$ or $\varphi_{pq} \varphi_{pr}$ depending on how the edges $pq$ and $pr$ are directed. The weight of $H$ is $W_P(H) = W_P(H_1) \cdots W_P(H_m)$.

Let $H_i, \ldots, H_s$ be chain polyhedra with the initial edges $I_1(H), \ldots, I_s(H)$ and the terminal edges $T_1(H), \ldots, T_s(H)$, and $H_{s+1}, \ldots, H_m$ be cycle polyhedra. The
The determinant in (1.4) is equal to $\det G(A_{H^I}, E_{H^T})$ where $H^I \defeq \ess I(H) \setminus \{I_1(H), \ldots, I_n(H)\}$ and $H^T \defeq \ess H \setminus \{T_1(H), \ldots, T_s(H)\}$. In particular, if the determinant is nonzero, then $e_{i, i} \neq I_1(H), \ldots, I_n(H)$, as well as $\alpha_j, j \neq T_1(H), \ldots, T_s(H)$, are linearly independent. According to Section 2.2, this implies that the graphs $H^I$ and $H^T$ are mixed forests.

Thus, equation (1.4) for the case considered looks as follows: if $M$ is the level 2 Laplacian then $\text{char}_M(t) = \sum_{k=0}^n (-1)^k \mu_k t^k$ where

$$\mu_k = \sum_{H \in \text{CP}_n \cap \text{mixed forests}} H^I \text{ and } H^T \text{ are two trees} \prod_{\Phi \text{ is a face of } H,} c_{v_1(\Phi)v_2(\Phi)} \langle \alpha_{[s_1(\Phi)], e_{s_2(\Phi)}} \rangle \times \det G(A_{H^I}, E_{H^T})$$

(2.14)

The determinantal term in (2.14) can be simplified using Lemma 2.5.

$$\left(\det G(A_{H^I}, E_{H^T})\right)^2 = \langle (A_{H^I}), (E_{H^T}) \rangle \cdot \det G(A_{H^I}, A_{H^T}) \det (E_{H^T}, E_{H^T}) \langle (A_{H^I})^\perp, (E_{H^T})^\perp \rangle \cdot \det G(A_{H^I}, A_{H^T}) \det (E_{H^T}, E_{H^T})$$

$$= \frac{\det G(A_{H^I}, A_{H^T})^2}{\det G(A_{H^I}, A_{H^T}) \det G(E_{H^T}, E_{H^T})} \times \det G(A_{H^I}, A_{H^T}) \det (E_{H^T}, E_{H^T})$$

(2.15)

where $\langle X \rangle$ mean the subspace in $\mathbb{R}^n$ spanned by $X$, $\perp$ means an orthogonal complement, and $A_{H^I}^\perp, E_{H^T}^\perp$ are bases in $\langle A_{H^I} \rangle^\perp$ and $\langle E_{H^T} \rangle^\perp$, respectively.

By Lemma 2.5, the formula above is true for any choice of the bases $A_{H^I}^\perp, E_{H^T}^\perp$; below we describe orthogonal bases the most convenient for our purposes. For any graph $G$ denote by $\mathcal{C}(G)$ the set of its connected components. Let $\mathcal{C}(H^I) = \{H_1^I, \ldots, H_i^I\}$ and $\mathcal{C}(H^T) = \{H_1^T, \ldots, H_i^T\}$; denote by $V_i^I \subset \mathbb{R}^n$ and $V_i^T \subset \mathbb{R}^n$ the subspaces spanned by the vertices of $H_i^I$ and $H_i^T$, respectively. Then

$$\langle A_{H^I} \rangle^\perp = \bigoplus_{i=1}^s \langle A_{H_i^I} \rangle^\perp, V_i^I,$$

where the summands are pairwise orthogonal; the same is true for $E_{H^T}$.

Every $H_i^I$ and $H_i^T$ is either a tree or a graph with one cycle; choose a root in every tree component and denote it by $r_i^I$ and $r_i^T$, respectively.

**Lemma 2.6.** If $H$ is a tree with a root $r$, then the spaces $\langle A_H \rangle^\perp$ and $\langle E_H \rangle^\perp$ have dimension 1 and are spanned by vectors $b_H \defeq \sum u \varphi_{ur}$ and $f_H \defeq \sum u \varphi_{ru}$, respectively, where the summation is over the set of vertices of $H$, and $ur, ru$ are paths joining $u$ with $r$ and $r$ with $u$.

If $H$ is a graph with one cycle with monodromy $w \neq 1$ then $\langle A_H \rangle = \langle E_H \rangle \equiv \mathbb{R}^n$.

**Proof.** Let $H$ be a tree. The spaces $\langle A_H \rangle$ and $\langle E_H \rangle$ are spanned by $n - 1$ vectors in $\mathbb{R}^n$; the vectors are linearly independent by Lemma 2.2. Hence, $\dim \langle A_H \rangle^\perp = \dim \langle E_H \rangle^\perp = 1$. Apparently, $(b_H, \alpha_{ij}) = 0$ and $(f_H, e_{ij}) = 0$ for any edge $ij$ of $H$, and the first statement follows.

The second statement follows from Lemma 2.6.

The number of rows and columns of the matrix $G(A_{H^I}, E_{H^T})$ is equal to the number of tree components of the graphs $H^I$ and $H^T$. If $H_i^I$ is a tree component of $H^I$ and $H_i^T$ is a tree component of $H^T$, then, obviously, $G(A_{H^I}, E_{H^T})_{ii} = $
\((b_{H^I}, I_{H^T}) = \sum_{\Lambda \in L_{ij}} \varphi^2_{\Lambda} n_{\Lambda} = M(H)^{ij}_i\) (recall that \(L_{ij}\) is the set of paths joining \(r^e_i\) with \(r^e_j\)) and \(n_{\Lambda}\) is the number of vertices along the path \(\Lambda\) that belong both to \(H^I_i\) and \(H^T_j\).

Lemmas 2.6 and 2.1 imply now that

\[
\det G(A_{H^I}, E_{H^T}) = \det M(H) \times \prod_{i=1}^{s} \prod_{\langle pq \rangle \in H^I_i \text{ looks away from } r^e_i} \varphi_{pq} \times \prod_{j=1}^{\infty} \prod_{\langle pq \rangle \in H^T_j \text{ looks away from } r^e_j} \varphi_{qp}
\]

\[
= \det M(H) \times \prod_{i,j=1}^{s} \prod_{\langle pq \rangle \in H \text{ lies in a path joining } r^e_i \text{ with } r^e_j} \varphi_{pq},
\]

and Theorem 1.9 is proved.

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