Envelopes of equivalent martingale measures and a generalized no-arbitrage principle in a finite setting

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Abstract
We consider a one-period market model composed by a risk-free asset and a risky asset with \( n \) possible future values (namely, a \( n \)-nomial market model). We characterize the lower envelope of the class of equivalent martingale measures in such market model, showing that it is a belief function. Then, we reformulate a general one-period pricing problem in the framework of belief functions: this allows to model frictions in the market and can be justified in terms of partially resolving uncertainty according to Jaffray. We provide a generalized no-arbitrage condition for a generic one-period market model under partially resolving uncertainty and show that the “risk-neutral” belief function arising in the one-period \( n \)-nomial market model does not allow to satisfy such condition. Finally, we derive a generalized arbitrage-free lower pricing rule through an inner approximation of the “risk-neutral” belief function arising in the one-period \( n \)-nomial market model.

Keywords Equivalent martingale measures · Belief functions · Generalized no-arbitrage principle · Lower pricing rule

1 Introduction
As is well-known, the one-period binomial model is the most simple example of financial market showing all features of no-arbitrage pricing [see, e.g., Cox et al. (1979), Pliska (1997)]. Such model is based on two assets: a risk-free asset (bond) and a risky asset (stock), the latter having only two possible future values. Assuming no-arbitrage condition to hold, which happens under a suitable choice of parameters, this model is proved to be complete, that
is there exists a unique equivalent martingale measure that allows to compute no-arbitrage prices as discounted expectations.

As soon as we allow more than two possible values for the risky asset, obtaining a \textit{n-nominal model}, completeness is lost. In this case, the no-arbitrage condition is equivalent to the existence of an infinite class \( Q \) of equivalent martingale measures [see, e.g., Černý (2009), Pliska (1997)].

One of the basic underlying assumptions of no-arbitrage pricing models is the absence of frictions in the market [see, e.g., Dybvig and Ross (1989)], which materializes in the linearity of the price functional. Hence, in case of an incomplete market we need to get rid of non-uniqueness of the equivalent martingale measure for reaching linearity either by completing the market with extra securities or by choosing one of the equivalent martingale measures by some suitable criterion of choice.

Decision models involving sets of probability measures have been extensively studied in the decision theory literature connoting situations of \textit{ambiguity} [see, e.g., Etner et al. (2012), Gilboa and Marinacci (2016)]. The incompleteness of the market in a \( n \)-nominal market model generates a form of “objective” ambiguity as one needs to deal with the class of equivalent martingale measures.

In this paper, referring to the one-period \( n \)-nominal market model, we characterize the corresponding set of equivalent martingale measures for every \( n > 2 \). We provide a closed form expression for the lower envelope \( Q \) of the class of equivalent martingale measures, further showing that it is a \textit{belief function} in the Dempster–Shafer theory of evidence (Dempster, 1967; Shafer, 1976). In spite of the very particular form of the lower envelope, the closure of the set of equivalent martingale measures \( Q \) is shown not to coincide with \( \text{core}(Q) \) in general, that is with the set of all probability measures dominating \( Q \) [see, e.g., Grabisch (2016)].

As discussed in Amihud and Mendelson (1986, 1991), real markets show the presence of frictions mainly in the form of bid-ask spreads and this amounts in giving up on the linearity of the price functional. Since we have a set of equivalent martingale measures \( Q \), we could think to use a suitable closed subset \( Q' \subseteq Q \) to define a \textit{lower pricing rule} as a discounted lower expectation, in a way to allow frictions in the market. The approach to pricing through lower/upper expectation functionals has been investigated in several papers [see, e.g., Bensaid et al. (1992), El Karoui and Quenez (1995), Jouini and Kallal (1995)]. The choice of \( Q' \) is not free of issues since a reasonable criterion should be provided. The most natural way to get \( Q' \) is to consider a finite set of random payoffs \( \mathcal{G} \subseteq \mathbb{R}^\Omega \), and a \textit{lower price assessment} \( \pi : \mathcal{G} \rightarrow \mathbb{R} \). The problem is to look for a closed \( Q' \subseteq Q \) such that \( \pi(X) = \min_{Q \in Q'} (1 + r)^{-1} \mathbb{E}_Q(X) \), for every \( X \in \mathcal{G} \), where \( 1 + r \) is the risk-free return. Unfortunately, we show that in general this problem could not have a solution.

On the other hand, the fact that \( Q \) is a belief function suggests to derive a lower pricing rule from it as a discounted Choquet expectation. We stress that, working directly in the framework of belief functions allows to incorporate naturally frictions in the market, nevertheless, for such a lower pricing rule to be acceptable the classical notion of arbitrage must be generalized.

For that, we reformulate a general one-period pricing problem over a finite state space in the framework of belief functions. We provide a generalized avoiding Dutch book condition and a generalized no-arbitrage condition for a lower price assessment based on the \textit{partially resolving uncertainty} principle proposed by Jaffray (1989). Adopting such principle, we allow that an agent may only acquire the information that an event \( B \neq \emptyset \) occurs, without knowing which is the true state of the world \( \omega \in B \). In turn, this translates in considering payoffs of portfolios on every event \( B \neq \emptyset \) adopting a systematically pessimistic behavior, that is always considering the minimum of random payoffs. This is in contrast to the usual
completely resolving uncertainty assumption according to which the agent will always acquire which is the true state of the world in $\Omega$.

First, we show that the generalized avoiding Dutch book condition is necessary and sufficient for the existence of a belief function whose corresponding discounted Choquet expectation functional agrees with the lower price assessment, even though the positivity of the belief function cannot be guaranteed. The lack of positivity is an issue in the context of pricing since assets with non-negative and non-null payoff should have positive lower price.

Second, we prove that the proposed generalized no-arbitrage condition is equivalent to the existence of a strictly positive belief function whose corresponding discounted Choquet expectation functional agrees with the lower price assessment. The theorem we prove is the analogue of the first fundamental theorem of asset pricing, formulated in the context of belief functions. In particular, our result specializes results given in Cerreia-Vioglio et al. (2015) and Chateauneuf et al. (1996), where the authors characterize an upper pricing rule that can be expressed as a discounted Choquet expectation with respect to a concave (or 2-alternating) capacity. Working with belief functions in place of 2-monotone capacities (that are dual of 2-alternating capacities), allows to introduce non-linearity departing the less from the classical no-arbitrage setting.

Concerning the original problem of deriving a lower pricing rule from the “risk-neutral” belief function $Q$ arising in the $n$-nomial market model, we show that the discounted Choquet expectation with respect to $Q$ is not consistent with the lower price $\pi(S^1_1) = S^1_0$ of the stock. Indeed, we would get a lower price assessment that does not satisfy the generalized avoiding Dutch book condition. Hence, we propose a procedure for determining a belief function inner approximating $Q$ and giving rise to a generalized arbitrage-free lower pricing rule. Such procedure relies on the choice of a reference equivalent martingale measure $Q_0 \in \mathcal{Q}$, and on the determination of an inner approximating martingale belief function $\hat{\text{Bel}}$ for $Q$ complying only with the stock lower price assessment $\pi(S^1_1) = S^1_0$. The latter task is achieved by minimizing a suitable distance, subject to a system of linear constraints, similarly to Miranda et al. (2021) and Montes et al. (2018, 2019). In this way we get an equivalent inner approximating martingale belief function $\hat{\text{Bel}}_e$ for $Q$, as the $\epsilon$-contamination [see, e.g., Huber (1981)] of $Q_0$ with respect to $\text{core}(\hat{\text{Bel}})$. Finally, we show that if we further require $\hat{\text{Bel}}$ to comply also with the upper price assessment $\overline{\pi}(S^1_1) = S^1_0$ (arriving to an inner approximating strong martingale belief function), both $\hat{\text{Bel}}$ and $\hat{\text{Bel}}_e$ reduce to probability measures.

The paper is structured as follows. In Sect. 2 we collect some preliminaries. Section 3 introduces the one-period $n$-nomial market model and provides the characterization of the lower envelope $Q$ of the set $\mathcal{Q}$ of equivalent martingale measures. In Sect. 4 we formulate a general one-period pricing problem in the context of belief functions and introduce the generalized avoiding Dutch book condition and the generalized no-arbitrage condition. Then, in Sect. 5 we cope with the problem of inner approximating the “risk-neutral” belief function $Q$ arising in the $n$-nomial market model. Finally, Sect. 6 gathers conclusions and future perspectives.

### 2 Preliminaries

Let $\Omega$ be a finite non-empty set and $\mathcal{F} = \mathcal{P}(\Omega)$, where $\mathcal{P}(\Omega)$ denotes the power set of $\Omega$. To avoid cumbersome notation, in the rest of the paper we assume that $\Omega = \{1, \ldots, n\}$ with $n \in \mathbb{N}$. 

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2.1 Uncertainty measures

Definition 1 A function \( \varphi : \mathcal{F} \to [0, 1] \) is called a (normalized) capacity if:

(i) \( \varphi(\emptyset) = 0 \) and \( \varphi(\Omega) = 1 \);
(ii) \( \varphi(A) \leq \varphi(B) \) when \( A \subseteq B \), for every \( A, B \in \mathcal{F} \).

Further, a capacity \( \varphi \) is called a:

- probability measure if \( \varphi(A \cup B) = \varphi(A) + \varphi(B) \), for every \( A, B \in \mathcal{F} \) with \( A \cap B = \emptyset \);
- belief function if it is completely monotone, i.e., for every \( k \geq 2 \) and every \( A_1, \ldots, A_k \in \mathcal{F} \),

\[
\varphi \left( \bigcup_{i=1}^{k} A_i \right) \geq \sum_{\emptyset \neq I \subseteq \{1, \ldots, k\}} (-1)^{|I|+1} \varphi \left( \bigcap_{i \in I} A_i \right);
\]

- \( 2 \)-monotone (or convex) capacity if \( \varphi(A \cup B) \geq \varphi(A) + \varphi(B) - \varphi(A \cap B) \), for every \( A, B \in \mathcal{F} \);
- (coherent) lower probability if there exists a class \( Q \) of probability measures on \( \mathcal{F} \) such that, for every \( A \in \mathcal{F} \),

\[
\varphi(A) = \inf_{Q \in Q} Q(A).
\]

Probability measures are particular belief functions, belief functions are particular \( 2 \)-monotone capacities and the latter are particular lower probabilities [see, e.g., Grabisch (2016)].

As shown in Walley (1982), in case of a lower probability we can always consider the closure \( \operatorname{cl}(Q) \) of \( Q \) in the product topology, for which it holds that, for every \( A \in \mathcal{F} \),

\[
\varphi(A) = \min_{Q \in \operatorname{cl}(Q)} Q(A).
\]

Given a capacity \( \varphi : \mathcal{F} \to [0, 1] \), we can define its dual capacity \( \psi : \mathcal{F} \to [0, 1] \) by setting, for every \( A \in \mathcal{F} \),

\[
\psi(A) = 1 - \varphi(A^c).
\]

The dual of a belief function is called plausibility function, the dual of a \( 2 \)-monotone capacity is called \( 2 \)-alternating capacity, and the dual of a lower probability is called upper probability.

As proved in Chateauneuf and Jaffray (1989), every capacity \( \varphi \) is completely characterized by its Möbius inverse \( \mu : \mathcal{F} \to \mathbb{R} \) through the relations,

\[
\mu(A) = \sum_{B \subseteq A} (-1)^{|A \setminus B|} \varphi(B) \quad \text{and} \quad \varphi(A) = \sum_{B \subseteq A} \mu(B),
\]

where \( A \in \mathcal{F} \), that imply \( \mu(\emptyset) = 0 \).

In this paper we will be mainly concerned with belief and plausibility functions that have been introduced by Dempster (1967) and Shafer (1976) in their theory of evidence. Following the usual custom, a belief function is denoted by \( \text{Bel} \) and its dual plausibility function by \( \text{Pl} \).

In the rest of the paper \( \mathbf{B}(\Omega, \mathcal{F}) \) stands for the set of all belief functions on \( (\Omega, \mathcal{F}) \) and \( \mathbf{P}(\Omega, \mathcal{F}) \) for the subset of all probability measures on \( (\Omega, \mathcal{F}) \).
As already pointed out, belief functions are particular lower probabilities as they induce the closed (in the product topology) convex set of probability measures on $\mathcal{F}$ called core, defined as

$$\text{core}(\text{Bel}) = \{Q \in \mathcal{P}(\Omega, \mathcal{F}) : Q \geq \text{Bel}\}. \quad (3)$$

that satisfies, for every $A \in \mathcal{F},$

$$\text{Bel}(A) = \min_{Q \in \text{core}(\text{Bel})} Q(A). \quad (4)$$

Furthermore, the Möbius inverse of a belief function is such that $\mu(A) \geq 0$, for every $A \in \mathcal{F}$, and $\sum_{A \in \mathcal{F}} \mu(A) = 1$. In particular, the Möbius inverse of a probability measure can be positive only on singletons [see, e.g., Grabisch (2016), Shafer (1976)]

Denote by $R_\Omega$ the set of all real-valued random variables defined on $\Omega$. For $A \in \mathcal{F}$, we denote by $1_A : \Omega \to \{0, 1\}$ the indicator of $A$, defined as $1_A(i) = 1$ if $i \in A$ and 0 otherwise. In order to avoid cumbersome notation, for every $a \in \mathbb{R}$, in what follows we identify $a$ with the constant random variable $a\,1_\Omega$.

Given a capacity $\phi$ on $\mathcal{F}$ and $X \in R_\Omega$, we can introduce the Choquet expectation of $X$ with respect to $\phi$ defined [see, e.g., Denneberg (1994), Grabisch (2016)] through the Choquet integral

$$C_\phi(X) = \sum_{i=1}^{n} (X(\sigma(i)) - X(\sigma(i+1)))\phi(E^\sigma_i), \quad (5)$$

where $\sigma$ is a permutation of $\Omega$ such that $X(\sigma(1)) \geq \ldots \geq X(\sigma(n))$, $E^\sigma_i = \{\sigma(1), \ldots, \sigma(i)\}$ for $i = 1, \ldots, n$, and $X(\sigma(n+1)) = 0$. In particular, if $\phi$ reduces to a probability measure $Q$, then

$$C_Q(X) = \mathbb{E}_Q(X),$$

where $\mathbb{E}_Q$ denotes the usual expectation operator with respect to $Q$. For a general capacity $\phi$, $C_\phi$ is not linear, but if $X, Y \in R_\Omega$ are comonotone, that is $(X(i) - X(j))(Y(i) - Y(j)) \geq 0$ for every $i, j \in \Omega$, then

$$C_\phi(X + Y) = C_\phi(X) + C_\phi(Y).$$

Further, $C_\phi$ is always monotone, that is, for every $X, Y \in R_\Omega$ with $X \leq Y$, we have that $C_\phi(X) \leq C_\phi(Y)$.

Finally, if $\psi$ is the dual capacity of $\phi$, then, for every $X \in R_\Omega$,

$$C_\psi(X) = -C_\phi(-X). \quad (6)$$

As proved in Schmeidler (1989), if $\phi$ reduces to a belief function $\text{Bel}$, then

$$C_{\text{Bel}}(X) = \min_{Q \in \text{core}(\text{Bel})} \mathbb{E}_Q(X), \quad (7)$$

thus $C_{\text{Bel}}(X)$ can be interpreted as a lower expectation. Furthermore, in this case it is also possible to provide an expression of $C_{\text{Bel}}(X)$ relying on the Möbius inverse of $\text{Bel}$. At this aim, let $\mathcal{U} = \mathcal{F}\setminus\{\emptyset\}$. For every random variable $X \in R_\Omega$ define the function $X^L : \mathcal{U} \to \mathbb{R}$ setting, for every $B \in \mathcal{U},$

$$X^L(B) = \min_{i \in B} X(i). \quad (8)$$
If $\mu$ is the Möbius inverse of $Bel$, then [see, e.g., Gilboa and Schmeidler (1994), Grabisch (2016)]

$$
C_{Bel}(X) = \sum_{B \in \mathcal{D}} X^1(B) \mu(B).
$$  \quad (9)

We further have that $C_{Bel}$ is completely monotone [see e.g., Troffaes and Cooman (2014)] that is, for every $k \geq 2$ and every $X_1, \ldots, X_k \in \mathbb{R}^\Omega$, it holds that

$$
C_{Bel} \left( \bigvee_{i=1}^k X_i \right) \geq \sum_{\emptyset \neq I \subseteq \{1, \ldots, k\}} (-1)^{|I|+1} C_{Bel} \left( \bigwedge_{i \in I} X_i \right),
$$  \quad (10)

where $\vee$ and $\wedge$ denote the pointwise minimum and maximum.

Equations (3), (4), (7), (8) and (9) continue to hold even if we take a 2-monotone capacity $\varphi$ in place of a belief function and we consider the Choquet integral with respect to $\varphi$. On the other hand, condition (10) is guaranteed to hold only for $k = 2$ when a 2-monotone capacity is considered and $C_{\varphi}$ is said to be 2-monotone, in this case. Moreover, as proved in Troffaes and Cooman (2014), $C_{\varphi}$ is superadditive, meaning that, for every $X, Y \in \mathbb{R}^\Omega$,

$$
C_{\varphi}(X + Y) \geq C_{\varphi}(X) + C_{\varphi}(Y).
$$  \quad (11)

**Remark 1** When a family of probabilities $Q$ is available and $\varphi = \min \text{cl}(Q)$ is a 2-monotone (completely monotone) capacity, but $\text{cl}(Q)$ is strictly contained in $\text{core}(\varphi)$, then the equality (7) may fail to hold. Indeed, computing the Choquet integral with respect to $\varphi$ we are actually computing the lower expectation functional on $\mathbb{R}^\Omega$ determined by $\text{core}(\varphi)$, which is dominated by the lower expectation functional on $\mathbb{R}^\Omega$ determined by $Q$ [see de Cooman et al. (2008)]. This latter functional is 2-monotone (completely monotone) on indicators of events but may fail 2-monotonicity (complete monotonicity) on the whole $\mathbb{R}^\Omega$. This means that for some $X \in \mathbb{R}^\Omega$ we can have $C_{\varphi}(X) < \min_{Q \in \text{cl}(Q)} E_Q(X)$.

### 2.2 Classical one-period no-arbitrage pricing theory

The classical one-period no-arbitrage pricing theory [see, e.g., Černý (2009), Delbaen and Schachermayer (2006), Pliska (1997)] refers to a financial market which is open at times $t = 0$ and $t = 1$. In such market we consider a risk-free asset (bond) and a fixed number $m$ of risky assets (stocks). Such assets are associated with the price processes $\{S^0_k, S^1_k\}$ for the bond and $\{S^0_k, S^1_k\}$, for $k = 1, \ldots, m$, for the stocks. All the processes are assumed to be adapted to the filtration $\{\mathcal{F}_0, \mathcal{F}_1\}$ with $\mathcal{F}_0 = \{\emptyset, \Omega\}$ and $\mathcal{F}_1 = \mathcal{F}$.

It is usually assumed that $S^0_0 = 1$ and $S^0_1 = 1 + r > 0$, where $r$ is the risk-free interest rate on the market. Moreover $S^0_k$, for $k = 1, \ldots, m$, is a scalar value expressing the price of the $k$th risky asset at time $t = 0$, while $S^1_k$, for $k = 1, \ldots, m$, is a random variable expressing the payoff at time $t = 1$. If we consider the set of random variables $G = \{S^1_1, \ldots, S^m_1\}$ then we can define a function $\pi : G \rightarrow \mathbb{R}$ setting, for $k = 1, \ldots, m$, $\pi(S^1_k) = S^k_0$ that corresponds to a price assessment on the market at time $t = 0$.

The filtered measurable space $(\Omega, \{\mathcal{F}_0, \mathcal{F}_1\}, \mathcal{F})$ is endowed with a probability measure $P$ such that $P((i)) = p_i > 0$ for all $i \in \Omega$, which is dubbed “real-world” probability measure. All market agents are assumed to share the same beliefs that are encoded in the probability measure $P$, though they may have different utility functions resulting in different preferences.

The classical theory assumes that the market is frictionless, i.e., there are no transaction costs nor taxes, securities are infinitely divisible and short selling is allowed. The market is also assumed to be competitive, i.e., market agents are profit maximizers and price takers.
The risk-free asset plays a special role as it is used as a numéraire: this allows to associate every risky asset to the discounted process \( \{ \tilde{S}_0^k, \tilde{S}_1^k \} \), for \( k = 1, \ldots, m \), where \( \tilde{S}_0^k = S_0^k \) and \( \tilde{S}_1^k = (1 + r)^{-1} S_1^k \). Switching to discounted processes permits to disregard the risk-free asset, as \( \tilde{S}_0^0 = 1 \) and \( \tilde{S}_1^0 = 1_\Omega \) [see, e.g., Delbaen and Schachermayer (2006), Pliska (1997)].

A portfolio is a vector \( \lambda = (\lambda_0, \ldots, \lambda_m)^T \in \mathbb{R}^{m+1} \) where \( \lambda_0 \) is the number of units of the risk-free asset while, for \( k = 1, \ldots, m \), \( \lambda_k \) is the number of units of the \( k \)th risky asset, to buy or short sell. Each portfolio \( \lambda \) is associated with a random variable expressing the final (discounted) payoff of the portfolio \( Z_\lambda : \Omega \to \mathbb{R} \) defined, for every \( i \in \Omega \), as

\[
Z_\lambda(i) = \lambda_0 + \sum_{k=1}^m \lambda_k \tilde{S}_1^k(i).
\] (12)

Moreover, \( \lambda \) is also associated with the scalar value

\[
\pi_\lambda = \lambda_0 + \sum_{k=1}^m \lambda_k \pi(S_1^k)
\] (13)

expressing the price at time \( t = 0 \) of the portfolio.

Under the above assumptions, the market is said to satisfy the no-arbitrage principle [see, e.g., Černý (2009)] if for every portfolio \( \lambda \in \mathbb{R}^{m+1} \) none of the following conditions holds:

(a) \( Z_\lambda(i) = 0 \), for \( i = 1, \ldots, n \), and \( \pi_\lambda < 0 \);
(b) \( Z_\lambda(i) \geq 0 \), for \( i = 1, \ldots, n \), with at least a strict inequality, and \( \pi_\lambda \leq 0 \).

Descriptively, condition (a) encodes a situation in which we are paid at time \( t = 0 \) to hold a portfolio that pays nothing at time \( t = 1 \). On the other hand, condition (b) refers to a situation in which we pay nothing or we are paid at time \( t = 0 \) to hold a portfolio that guarantees a non-negative payoff at time \( t = 1 \) which is strictly positive in at least one state.

In turn, the first fundamental theorem of asset pricing [see, e.g., Černý (2009), Pliska (1997)] states that, under the above conditions, the no-arbitrage principle is equivalent to the existence of a probability measures \( Q \in \mathcal{P}(\Omega, \mathcal{F}) \) with the following properties:

(i) \( Q \sim P \), meaning that \( Q \) is equivalent to \( P \) that is \( P(A) = 0 \iff Q(A) = 0 \), for every \( A \in \mathcal{F} \);
(ii) \( (1 + r)^{-1} \mathbb{E}_Q(S_1^k) = \pi(S_1^k) \), for \( k = 1, \ldots, m \).

Since \( P \) is assumed to be positive on \( \mathcal{F}\setminus\{\emptyset\} \), property (i) translates in the positivity of \( Q \) on \( \mathcal{F}\setminus\{\emptyset\} \). We stress that positivity of \( Q \) is a desideratum in the context of pricing since it assures that a risky asset with a non-negative and non-null payoff has a positive price. On the other hand, property (ii) can be expressed, for \( k = 1, \ldots, m \), as

\[
\mathbb{E}_Q(\tilde{S}_1^k) = \tilde{S}_0^k,
\] (14)

that is the discounted process of every risky asset is a martingale under \( Q \). For these reasons the probability measure \( Q \) is called an equivalent martingale measure. We also notice that, if \( S_0^k \neq 0 \), then

\[
\mathbb{E}_Q\left(\frac{S_1^k}{S_0^k}\right) = 1 + r,
\] (15)

that is the expected return of the \( k \)th risky asset under \( Q \) is equal to the risk-free return. This fact justifies the name risk-neutral probability for \( Q \).

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We notice that the first fundamental theorem of asset pricing does not guarantee the uniqueness of $Q$. Hence, we generally have a set $Q$ of equivalent martingale measures and the market is said to be complete if $Q$ reduces to a singleton.

An equivalent martingale measure $Q \in Q$ induces a pricing rule [see, e.g., Cerreia-Vioglio et al. (2015)] through the corresponding discounted expectation that extends the price assessment $\pi$ from $G$ to the entire $\mathbb{R}^\Omega$. The no-arbitrage principle only assures that such an extension exists but it will be unique only in a complete market. In particular, the pricing rule induced by a $Q \in Q$ is completely determined by a family of prices $\{\pi(1_{[i]}) : i \in \Omega\}$ called Arrow-Debreu state prices corresponding to the Arrow-Debreu securities $\{1_{[i]} : i \in \Omega\}$. Indeed, we have that

$$\pi(1_{[i]}) = (1 + r)^{-1} Q([i]).$$

The pricing rule induced by an equivalent martingale measure does not take into account market agents’ preferences and is only determined by the no-arbitrage principle, which is intended as a normative principle.

### 3 Envelopes of equivalent martingale measures in the $n$-nomial market model

In this section we consider a particular market model, according to Sect. 2.2. For $n \geq 2$, a one-period $n$-nomial market model related to times $t = 0$ and $t = 1$, is composed by a risk-free asset (bond) and a risky asset (stock). The prices of the two securities are modeled by the processes $\{S_0^0, S_0^1\}$ and $\{S_1^0, S_1^1\}$ defined on the filtered probability space $(\Omega, \{\mathcal{F}_0, \mathcal{F}_1\}, \mathcal{F}, P)$, with $\Omega = \{1, \ldots, n\}$, $P(i) = p_i > 0$ for all $i \in \Omega$, $\mathcal{F}_0 = \{\emptyset, \Omega\}$ and $\mathcal{F}_1 = \mathcal{F} = \mathcal{P}(\Omega)$.

Assume $S_0^0 = 1$ and $S_0^1 = s > 0$, while the prices at the end of the period satisfy

$$\frac{S_1^0}{S_0^0} = 1 + r, \quad \text{and} \quad \frac{S_1^1}{S_0^1} = \begin{cases} m_1, & \text{with probability } p_1, \\ m_2, & \text{with probability } p_2, \\ \vdots \\ m_n, & \text{with probability } p_n, \end{cases}$$

where $m_1 > m_2 > \cdots > m_n > 0$ and $1 + r > 0$.

To avoid cumbersome notation, in what follows, every element $Q \in P(\Omega, \mathcal{F})$ is identified with the vector $Q \equiv (q_1, \ldots, q_n)^T \in [0, 1]^n$, where $Q(i) = q_i$, for all $i \in \Omega$.

In this model, the set of equivalent martingale measures is defined as

$$Q = \{ Q \in P(\Omega, \mathcal{F}) : (1 + r)^{-1} E_Q(S_1^1) = S_1^1, \ Q \sim P \}. \tag{18}$$

As is well-known [see, e.g., Pliska (1997)], this set is not empty if $m_1 > 1 + r > m_n$, moreover, $Q$ is convex but generally not closed. In the particular case $n = 2$ this set reduces to the singleton $Q = \{Q\}$ where

$$Q \equiv \left( \begin{array}{c} \frac{(1 + r) - m_2}{m_1 - m_2}, \frac{m_1 - (1 + r)}{m_1 - m_2} \end{array} \right)^T.$$

To avoid triviality, in what follows we assume $n > 2$, moreover, we denote by $\text{cl}(Q)$ and $\text{ext}(\text{cl}(Q))$ the closure of $Q$ and the set of extreme points of the closure.

We characterize the properties of the set of equivalent martingale measures $Q$ because, despite each $Q \in Q$ complies with the no-arbitrage assumption, we cannot account the whole
set. Hence our aim is to work with the lower envelope $Q$ of $\text{cl}(Q)$ defined, for every $A \in \mathcal{F}$, as

$$Q(A) = \min_{Q \in \text{cl}(Q)} Q(A).$$

(19)

We first provide a characterization of $\text{ext}(\text{cl}(Q))$.

**Lemma 1** For $n > 2$, if $m_{s-1} > 1 + r \geq m_s$ for some $s \in \{2, \ldots, n\}$ and $1 + r \neq m_n$, let $I = \{1, \ldots, s-1\}$ and $J = \{s, \ldots, n\}$, then

$$\text{ext}(\text{cl}(Q)) = \{Q_{i,j} \in \mathbf{P}(\Omega, \mathcal{F}) : (i, j) \in I \times J\},$$

where $Q_{i,j} \equiv (0, \ldots, 0, q_i, 0, \ldots, 0, q_j, 0, \ldots, 0)^T$, with

$$q_i = \frac{(1+r) - m_j}{m_i - m_j} \quad \text{and} \quad q_j = \frac{m_i - (1+r)}{m_i - m_j}.$$  

**Proof** We have that $Q \equiv (q_1, \ldots, q_n)^T$ is an element of $\text{cl}(Q)$ if and only if it solves the system

$$\begin{cases}
\sum_{k=1}^{n} q_k = 1, \\
\sum_{k=1}^{n} m_k q_k = 1 + r, \\
q_k \geq 0, & \text{for } k = 1, \ldots, n.
\end{cases}$$

It is immediate to see that the coefficient matrix associated with the first two equations has full rank, so it admits infinite solutions depending on $n - 2$ real parameters. The set of such solutions is a closed subset of $\mathbb{R}^n$, while $\text{cl}(Q)$ is the intersection of such set with the non-negative orthant.

As the rank is 2 [see, e.g., Faigle et al. (2002)], the set $\text{ext}(\text{cl}(Q))$ can be generated by selecting all the possible pairs of distinct indices $i, j \in \{1, \ldots, n\}$, and verifying if the vector $Q_{i,j}$, defined as in the statement of the theorem, is a solution of the above system. In turn, since $Q_{i,j}$ is a solution of the above system if and only if $i \in I$ and $j \in J$, the claim follows.

In particular, if $1 + r > m_s$, we have $q_i, q_j \in (0, 1)$ for every $i \in I$ and $j \in J$. On the other hand, if $1 + r = m_s$, we have $q_i, q_j \in (0, 1)$ for every $i \in I$ and $j \in J \backslash \{s\}$, while for $j = s$

$$q_i = \frac{(1+r) - m_s}{m_i - m_s} = \frac{m_s - m_s}{m_i - m_s} = 0 \quad \text{and} \quad q_j = q_s = \frac{m_i - m_s}{m_i - m_s} = 1,$$

thus $Q_{i,j}$ reduces to

$$Q_{i,j} = Q_s \equiv (0, \ldots, 0, q_s = 1, 0, \ldots, 0)^T.$$  

Now we provide a characterization of the lower envelope $\underline{Q}$.

**Lemma 2** For $n > 2$, if $m_{s-1} > 1 + r \geq m_s$ for some $s \in \{2, \ldots, n\}$ and $1 + r \neq m_n$, let $I = \{1, \ldots, s-1\}$ and $J = \{s, \ldots, n\}$, then, for every $A \in \mathcal{F}$,

$$\underline{Q}(A) = \left\{ \begin{array}{ll}
1 & \text{if } A = \Omega, \\
\frac{(1+r) - m_j}{m_i - m_j} & \text{if } 1 + r \neq m_s \text{ and } I \subseteq A \neq \Omega, \\
\frac{m_i - m_j}{m_i - m_s} & \text{or } 1 + r = m_s \text{ and } I \cup \{s\} \subseteq A \neq \Omega, \\
\frac{m_i - (1+r)}{m_i - m_s} & \text{if } J \subseteq A \neq \Omega, \\
0 & \text{otherwise},
\end{array} \right.$$
where \( \underline{j} = \min\{j \in J : j \notin A\} \) and \( \bar{t} = \max\{i \in I : i \notin A\} \).

**Proof** We first prove the case \( 1 + r \neq m_s \). We have that \( Q(A) = 1 \) if and only if for all \( (i, j) \in I \times J, (i, j) \subseteq A \), and this happens if and only if \( A = \Omega \). Moreover, \( Q(A) = 0 \) if and only if there exists \( (i, j) \in I \times J, A \subseteq \{i, j\}^c \), and this happens if and only if \( I \notin A \) and \( J \notin A \).

For the remaining \( A \)'s, two situations can occur: either \( (a) I \subseteq A \neq \Omega \) or \( (b) J \subseteq A \neq \Omega \).

(a) If \( I \subseteq A \neq \Omega \), then

\[
Q(A) = \min_{(i,j) \in I \times J} \left[ 1_A(i) \frac{(1+r) - m_j}{m_i - m_j} + 1_A(j) \frac{m_i - (1+r)}{m_i - m_j} \right]
\]

\[
= \min_{(i,j) \in I \times J} \frac{(1+r) - m_j}{m_i - m_j}
\]

Suppose \( i \in I \) and let \( j \in J \) be such that \( j \notin A \), with \( m_1 > m_i > 1 + r > m_j \). Since

\[
\frac{(1+r) - m_j}{m_1 - m_j} - \frac{(1+r) - m_j}{m_i - m_j} = \frac{((1+r) - m_j)(m_i - m_1)}{(m_1 - m_j)(m_i - m_j)} < 0
\]

we have \( \frac{(1+r) - m_j}{m_1 - m_j} < \frac{(1+r) - m_j}{m_i - m_j} \). Suppose \( j, j' \in J \) are such that \( j, j' \notin A \) with \( m_1 > 1 + r > m_j > m_{j'} \). Since

\[
\frac{(1+r) - m_j}{m_1 - m_j} - \frac{(1+r) - m_{j'}}{m_i - m_{j'}} = \frac{((1+r) - m_j)(m_{j'} - m_1)}{(m_1 - m_j)(m_{j'} - m_1)} < 0
\]

we have \( \frac{(1+r) - m_j}{m_1 - m_j} < \frac{(1+r) - m_{j'}}{m_1 - m_{j'}} \). Hence, if \( j \) is the minimum element of \( J \) such that \( j \notin A \) we have that

\[
Q(A) = \min_{(i,j) \in I \times J} \frac{(1+r) - m_j}{m_i - m_j} = \frac{(1+r) - m_\bar{t}}{m_\bar{t} - m_j}.
\]

(b) If \( J \subseteq A \neq \Omega \), then

\[
Q(A) = \min_{(i,j) \notin A} \left[ 1_A(i) \frac{(1+r) - m_j}{m_i - m_j} + 1_A(j) \frac{m_i - (1+r)}{m_i - m_j} \right]
\]

\[
= \min_{(i,j) \notin A} \frac{m_i - (1+r)}{m_i - m_j}
\]

Suppose \( j \in J \) and let \( i \in I \) be such that \( i \notin A \) with \( m_1 > 1 + r > m_j > m_n \). Since

\[
\frac{m_i - (1+r)}{m_i - m_j} - \frac{m_i - (1+r)}{m_i - m_n} = \frac{(m_i - (1+r))(m_j - m_n)}{(m_i - m_j)(m_i - m_n)} > 0
\]

we have \( \frac{m_i - (1+r)}{m_i - m_j} > \frac{m_i - (1+r)}{m_i - m_n} \). Suppose \( i, i' \in I \) are such that \( i, i' \notin A \) with \( m_i > m_{i'} > 1 + r > m_n \). Since

\[
\frac{m_i - (1+r)}{m_i - m_n} - \frac{m_{i'} - (1+r)}{m_{i'} - m_n} = \frac{((1+r) - m_n)(m_i - m_{i'})}{(m_i - m_n)(m_{i'} - m_n)} > 0
\]
we have \( \frac{m_i-(1+r)}{m_i-m_n} > \frac{m_i-(1+r)}{m_j-m_n} \). Hence, if \( \tilde{I} \) is the maximum element of \( I \) such that \( \tilde{I} \notin A \) we have that
\[
Q(A) = \min_{(i,j) \in I \times J : i \notin A, j \in A} \frac{m_i-(1+r)}{m_i-m_j} = \frac{m_{\tilde{I}}-(1+r)}{m_{\tilde{I}}-m_n}.
\]

Finally, we prove the case \( 1 + r = m_s \). As before, we have that \( Q(A) = 1 \) if and only if \( A = \Omega \). Moreover, \( Q(A) = 0 \) if and only if \( I \cup \{s\} \not\subseteq A \) and \( J \not\subseteq A \).

For the remaining \( A \)'s, two situations can occur: either \( (a') \) \( I \cup \{s\} \subseteq A \neq \Omega \) or \( (b') \) \( J \subseteq A \neq \Omega \). Situation \( (b') \) coincides with \( (b) \), thus it is proved in the same way.

\( (a') \) If \( I \cup \{s\} \subseteq A \neq \Omega \), then proceeding as in the proof of \( (a) \) we have
\[
Q(A) = \min_{(i,j) \in (I \cup \{s\}) \times J} \left[ 1_A(i) \frac{(1+r)-m_j}{m_i-m_j} + 1_A(j) \frac{m_i-(1+r)}{m_i-m_j} \right] = \min_{(i,j) \in (I \cup \{s\}) \times J} \frac{(1+r)-m_j}{m_i-m_j} = \frac{(1+r)-m_j}{m_1-m_j}.
\]

\( \square \)

In the next lemma we finally characterize the Möbius inverse of \( \underline{\mu} \).

**Lemma 3** For \( n > 2 \), if \( m_{s-1} > 1 + r \geq m_s \) for some \( s \in \{2, \ldots, n\} \) and \( 1 + r \neq m_n \), let \( I = \{1, \ldots, s-1\} \) and \( J = \{s, \ldots, n\} \). Let \( \mu : \mathcal{F} \to \mathbb{R} \) be the Möbius inverse of \( \underline{\mu} \). Then, for every \( A \in \mathcal{F} \),

\[
\mu(A) = \begin{cases} 
\frac{(1+r)-m_s}{m_1-m_s} & \text{if } 1 + r \neq m_s \text{ and } A = I, \\
\frac{(1+r)-m_s}{m_1-m_s} - \frac{(1+r)-m_k}{m_{s-1}-(1+r)} & \text{if } A = \{1, \ldots, k\} \text{ and } I \subseteq A \neq \Omega, \\
\frac{m_k-(1+r)}{m_k-m_{s-1}} & \text{if } A = \{k, \ldots, n\} \text{ and } J \subseteq A \neq \Omega, \\
0 & \text{otherwise.}
\end{cases}
\]

**Proof** We first prove the case \( 1 + r \neq m_s \), by considering all the possibilities for \( A \in \mathcal{F} \).

\( a \) If \( J \not\subseteq A \) and \( I \not\subseteq A \), then \( \mu(A) = 0 \). Indeed, by Theorem 2 we have that \( \underline{Q}(B) = 0 \) for every \( B \subseteq A \) and this implies \( \mu(B) = 0 \) for every \( B \subseteq A \).

\( b \) If \( A = I \), then by Theorem 2 the only \( B \subseteq A \) with \( \underline{Q}(B) \neq 0 \) is \( B = A = I \). Hence,

\[
\mu(I) = \underline{Q}(I) = \frac{(1+r)-m_s}{m_1-m_s},
\]

as \( s \) is the minimum element of \( J \) not in \( A = I \).

\( c \) If \( A = J \), then by Theorem 2 the only \( B \subseteq A \) with \( \underline{Q}(B) \neq 0 \) is \( B = A = J \). Hence,

\[
\mu(J) = \underline{Q}(J) = \frac{m_{s-1}-(1+r)}{m_{s-1}-m_n},
\]

as \( s-1 \) is the maximum element of \( I \) not in \( A = J \).

\( d \) If \( A \neq \{1, \ldots, k\} \) and \( I \subseteq A \neq \Omega \), then \( \mu(A) = 0 \). To see this, let \( A = \{1, \ldots, k\} \cup B \) with \( I \subseteq \{1, \ldots, k\} \neq \Omega, B \neq \emptyset \), and \( B \cap \{1, \ldots, k+1\} = \emptyset \). Since for all \( E \subseteq A \) not containing \( I \) we have \( \underline{Q}(E) = 0 \), we can write

\[
\mu(A) = \sum_{E \subseteq A \setminus I} (-1)^{|A \setminus E|} \underline{Q}(E).
\]
For every $s - 1 \leq t \leq k$, if $E$ contains $\{1, \ldots, t\}$ but not $\{1, \ldots, t + 1\}$, it follows that $\frac{Q(E)}{m_1 - m_r} = (1 + r) - m_{s-1}$ and all of such sets are of the form $\{1, \ldots, t\} \cup C$ with $C \subseteq F$, where $F = \{t + 2, \ldots, k\} \cup B$ if $t + 2 \leq k$ and $F = B$ otherwise. Moreover, we have

$$Q((1, \ldots, t) \cup F) - \sum_{|D| = |F| - 1} D \subseteq F Q((1, \ldots, t) \cup D) + \sum_{|D| = |F| - 2} D \subseteq F Q((1, \ldots, t) \cup D) + \cdots + (-1)^{|F|} \frac{Q((1, \ldots, t))}{|F|} = 0,$$

since all terms are equal in absolute value and the number of positive terms is equal to that of negative terms. In turn, this implies that $\mu(A) = 0$.

(e) If $A \neq \{k, \ldots, n\}$ and $J \subset A \neq \varnothing$, then $\mu(A) = 0$. The proof of this claim is analogous to point (d).

(f) If $A = \{1, \ldots, k\}$ and $I \subset A \neq \varnothing$, i.e., $s \leq k \leq n - 1$, then, taking into account points (a)–(e),

$$Q(A) = \sum_{t=s-1}^{k} \frac{(1 + r) - m_{k+1}}{m_1 - m_{k+1}} \mu([1, \ldots, t]).$$

Hence, we have that

$$\mu(A) = Q(A) - \sum_{t=s-1}^{k-1} \mu([1, \ldots, t]) = \frac{(1 + r) - m_{k+1}}{m_1 - m_{k+1}} - \frac{(1 + r) - m_k}{m_1 - m_k}.$$

(g) If $A = \{k, \ldots, n\}$ and $J \subset A \neq \varnothing$, i.e., with $2 \leq k \leq s - 1$, then proceeding as in point (f) we get $\mu(A) = \frac{m_{k-1} - (1 + r)}{m_1 - m_k} - \frac{m_{k-1} - (1 + r)}{m_1 - m_{k-1}}$.

(h) If $A = \varnothing$, then $\mu(A) = 0$. Indeed, by points (a)–(g), for every $A \in \mathcal{F}\{\varnothing\}$, $\mu(A) \geq 0$ and, in particular, $\mu$ is strictly positive on the families

$$C_1 = \{[1, \ldots, s-1], [1, \ldots, s], \ldots, [1, \ldots, n-1]\},
\text{ and } C_2 = \{[s, \ldots, n], [s-1, \ldots, n], \ldots, [2, \ldots, n]\},$$

while it is 0 otherwise. By the properties of the Möbius inverse, it must be $\sum_{A \in \mathcal{F}} \mu(A) = 1$, and since

$$\sum_{A \in \mathcal{F}\{\varnothing\}} \mu(A) = \sum_{A \in C_1} \mu(A) + \sum_{A \in C_2} \mu(A) = \frac{(1 + r) - m_n}{m_1 - m_n} - \frac{m_1 - (1 + r)}{m_1 - m_n} = 1,$$

it follows that $\mu(\varnothing) = 0$.

Finally, we prove the case $1 + r = m_s$. Proceeding as in points (a)–(g) by taking $I \cup \{s\}$ in place of $I$, it is possible to show that $\mu$ is strictly positive on the families $C_2$ and $C_1' = \{[1, \ldots, s], \ldots, [1, \ldots, n-1]\}$, while it is 0 on $\mathcal{F}\{\varnothing\} \cup C_1' \cup C_2$. Thus, in analogy to point (h), since

$$\sum_{A \in \mathcal{F}\{\varnothing\}} \mu(A) = \sum_{A \in C_1'} \mu(A) + \sum_{A \in C_2} \mu(A) = 1,$$

it follows that $\mu(\varnothing) = 0$. \hfill \Box

The following theorem shows that $Q$ is a completely monotone lower probability, therefore it is a belief function.
Theorem 1 The lower probability $Q$ is completely monotone, that is, for every $k \geq 2$ and every $A_1, \ldots, A_k \in \mathcal{F}$, it holds that

$$Q \left( \bigcup_{i=1}^{k} A_i \right) \geq \sum_{\varnothing \neq J \subseteq \{1, \ldots, k\}} (-1)^{|J|+1} Q \left( \bigcap_{i \in J} A_i \right).$$

Proof The proof is an immediate consequence of the previous Lemmas 1–3 showing that the Möbius inverse $\mu$ of $Q$ is such that $\mu(\varnothing) = 0$, $\mu(A) \geq 0$, for every $A \in \mathcal{F}$, and $\sum_{A \in \mathcal{F}} \mu(A) = 1$. In turn, this is equivalent to the complete monotonicity of $Q$ [see Chateauneuf and Jaffray (1989), Grabisch (2016)].

The following example shows the computation of the lower envelope $Q$.

Example 1 Let $\Omega = \{1, 2, 3, 4\}$ and $m_1 = 4$, $m_2 = 2$, $m_3 = \frac{1}{2}$, $m_4 = \frac{1}{4}$, and $1 + r = 1$. To avoid cumbersome notation, we denote events omitting braces and commas. In this case we have $I = \{1, 2\}$, $J = \{3, 4\}$ and $\text{ext}(\text{cl}(Q)) = \{Q_{1,3}, Q_{1,4}, Q_{2,3}, Q_{2,4}\}$ inducing the $Q$ reported below

| $\mathcal{F}$ | \(\varnothing\) | 1 | 2 | 3 | 4 | 12 | 13 | 14 | 23 | 24 | 34 | 123 | 124 | 134 | 234 | $\Omega$ |
|----------------|---------|---|---|---|---|----|----|----|----|----|----|------|------|------|------|------|
| $Q_{1,3}$      | 0       | 15/105 | 0 | 90/105 | 90/105 | 15/105 | 0 | 15/105 | 90/105 | 0 | 15/105 | 1 | 15/105 | 0 | 15/105 | 0 | 15/105 | 0 | 15/105 | 90/105 | 90/105 | 1 |
| $Q_{1,4}$      | 0       | 21/105 | 0 | 21/105 | 70/105 | 0 | 21/105 | 70/105 | 0 | 21/105 | 1 | 0 | 84/105 | 21/105 | 1 | 1 | 0 | 84/105 | 21/105 | 1 | 1 | 84/105 | 1 |
| $Q_{2,3}$      | 0       | 0 | 35/105 | 0 | 70/105 | 35/105 | 0 | 70/105 | 35/105 | 0 | 70/105 | 1 | 35/105 | 70/105 | 1 | 0 | 35/105 | 70/105 | 1 | 1 | 70/105 | 1 |
| $Q_{2,4}$      | 0       | 0 | 45/105 | 0 | 60/105 | 45/105 | 0 | 60/105 | 45/105 | 0 | 60/105 | 1 | 60/105 | 45/105 | 1 | 0 | 60/105 | 45/105 | 1 | 1 | 45/105 | 1 |
| $Q$            | 0       | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 24/105 | 0 |

Since $\mu(A) \geq 0$ for every $A \in \mathcal{F}$, then $Q$ is a belief function.

Even though $Q$ is a belief function, we have that, for $n > 2$, $\text{cl}(Q) \neq \text{core}(Q)$ in general, as shown in the following example.

Example 2 Consider $\Omega = \{1, 2, 3, 4\}$, $m_1 = 4$, $m_2 = 2$, $m_3 = \frac{1}{2}$, $m_4 = \frac{1}{4}$, and $1 + r = 1$, $Q$ and $Q$ of Example 1.

A straightforward computation shows that $\text{cl}(Q) \neq \text{core}(Q)$, since [see, e.g., Grabisch (2016), Schmeidler (1986)], assuming $S_0^1 = s > 0$,

$$C_Q \left( \frac{S_1^1}{S_0^1} \right) = \frac{54}{105} < 1 = \min_{Q \in \text{cl}(Q)} E_Q \left( \frac{S_1^1}{S_0^1} \right).$$

Recall that $\text{ext}(\text{core}(Q)) = \{Q^\sigma : \sigma \in \Sigma\}$, where $\Sigma$ is the set of permutations of $\Omega$ [see, e.g., Grabisch (2016)]. In particular, taking the permutation $\sigma = (1, 2, 3, 4)$ and defining the probability measure

$$Q^\sigma = \left( Q(1), Q(12) - Q(1), Q(123) - Q(12), Q(1234) - Q(123) \right)^T$$

$$= \left( 0, \frac{15}{105}, \frac{6}{105}, \frac{84}{105} \right)^T.$$
we have that $Q^\sigma / \not\in \cl(Q)$ which further proves that $\cl(Q) \neq \text{core}(Q)$.

As is well-known [see, e.g., Delbaen and Schachermayer (2006)], given a random variable $X \in \mathbb{R}^\Omega$ expressing the payoff at time $t = 1$ of a contract, its no-arbitrage price at time $t = 0$ can be computed relying on the set of equivalent martingale measures $Q$, by computing

$$
\pi_s(X) = \min_{Q \in \cl(Q)} (1 + r)^{-1} \mathbb{E}_Q(X) \quad \text{and} \quad \pi^*(X) = \max_{Q \in \cl(Q)} (1 + r)^{-1} \mathbb{E}_Q(X).
$$

It holds that [see, e.g., Černý (2009), Pliska (1997)]:

- if $\pi_s(X) = \pi^*(X)$, then their common value $\pi(X)$ is the no-arbitrage price at time $t = 0$ of $X$;
- if $\pi_s(X) < \pi^*(X)$, then the no-arbitrage price $\pi(X)$ at time $t = 0$ of $X$ belongs to the open interval $(\pi_s(X), \pi^*(X))$.

If we have two contracts with payoffs $X, Y \in \mathbb{R}^\Omega$ at time $t = 1$, then their no-arbitrage price intervals are

$$(\pi_s(X), \pi^*(X)) \quad \text{and} \quad (\pi_s(Y), \pi^*(Y)),$$  

nevertheless, we are not free to choose a value in one interval independently of the other, as shown in the following example.

**Example 3** Let $\Omega = \{1, 2, 3\}$, $m_1 = 4$, $m_2 = 2$, $m_3 = \frac{1}{2}$, $1 + r = 1$ and $S_0^1 = 20$. In this case we have $I = \{1, 2\}$, $J = \{3\}$ and $\text{ext}(\cl(Q)) = \{Q_{1,3}, Q_{2,3}\}$ inducing the $Q$ reported below

| $\mathcal{F}$ | $\emptyset$ | 1 | 2 | 3 | 12 | 13 | 23 | $\Omega$ |
|---------------|-------------|----|----|----|-----|-----|-----|--------|
| $Q_{1,3}$     | 0           | 21 | 0  | 84 | 21  | 1   | 84  | 1      |
| $Q_{2,3}$     | 0           | 0  | 45 | 60 | 45  | 60  | 1   | 1      |
| $Q$           | 0           | 0  | 0  | 60 | 21  | 60  | 84  | 1      |
| $\mu$         | 0           | 0  | 0  | 60 | 21  | 0   | 24  | 0      |

Consider the following payoffs at time $t = 1$

| $\Omega$ | 1 | 2 | 3 |
|----------|---|---|---|
| $X$      | 20| 10| 10|
| $Y$      | 10| 10| 20|

We have that

$$
\pi_s(X) = \min\{\mathbb{E}_{Q_{1,3}}(X), \mathbb{E}_{Q_{2,3}}(X)\} = 10,
$$  

$$
\pi^*(X) = \max\{\mathbb{E}_{Q_{1,3}}(X), \mathbb{E}_{Q_{2,3}}(X)\} = 12,
$$  

$$
\pi_s(Y) = \min\{\mathbb{E}_{Q_{1,3}}(Y), \mathbb{E}_{Q_{2,3}}(Y)\} = \frac{110}{7} \approx 15.7.
$$
\[ \pi^*(Y) = \max\{E_{Q_{1.3}}(Y), E_{Q_{2.3}}(Y)\} = 18, \]

so, we can consider the price assessment \( \pi(S_1^1) = 20, \pi(X) = 11, \pi(Y) = 17. \) It holds that the partial price assessments \( \{\pi(S_1^1), \pi(X)\} \) and \( \{\pi(S_1^1), \pi(Y)\} \) are arbitrage-free, while the global price assessment \( \{\pi(S_1^1), \pi(X), \pi(Y)\} \) is not, as there is no \( Q \in Q \) such that \( \pi(S_1^1) = E_Q(S_1^1), \pi(X) = E_Q(X), \pi(Y) = E_Q(Y). \)

One of the main hypotheses underlying the one-period \( n \)-nomial market model is the absence of frictions that, together with the no-arbitrage principle, imply the linearity of the price functional. Nevertheless, as largely acknowledged in the literature [see, e.g., Amihud and Mendelson (1986, 1991)] real markets show frictions, mainly in the form of bid-ask spreads, that translate in the non-linearity of the price functional.

Since we have a set of equivalent martingale measures \( Q \), we could look for a suitable closed subset \( Q' \subseteq Q \) to define a lower pricing rule as a discounted lower expectation, in a way to allow frictions in the market. In the literature, several papers investigated the problem of pricing using lower/upper expectation functionals [see, e.g., Bensaid et al. (1992), El Karoui and Quenez (1995), Jouini and Kallal (1995)]. The choice of \( Q' \) is not free of issues since a reasonable criterion should be provided. The most natural way to get \( Q' \) is to consider a finite \( G \subseteq \mathbb{R}^Q \), and a lower price assessment \( \pi : G \to \mathbb{R} \). Here, the problem is to look for a closed \( Q' \subseteq Q \) such that

\[ \pi(X) = \min_{Q \in Q'} (1 + r)^{-1} E_Q(X), \text{ for every } X \in G. \]

A first (trivial) constraint for \( \pi \) is, for every \( X \in G \),

\[ \pi_s(X) < \pi(X) < \pi^*(X), \text{ if } \pi_s(X) < \pi^*(X), \]

and \( \pi(X) = \pi_s(X) = \pi^*(X) \) otherwise, which, however, does not assure the existence of such a \( Q' \), as shown in the following example.

**Example 4** Let \( \Omega, m_1, m_2, m_3, 1 + r, S_0^1, X \) and \( Y \) as in Example 3. Consider the lower price assessment \( \pi(S_1^1) = 20, \pi(X) = 11 \) and \( \pi(Y) = 17. \) We have that there is no closed subset \( Q' \subseteq Q \) such that the corresponding discounted lower expectation functional agrees with \( \pi \), in fact the following system

\[
\begin{align*}
q_1 + q_2 + q_3 &= 1, \\
4q_1 + 2q_2 + \frac{q_3}{4} &= 1, \\
20q_1 + 10q_2 + 10q_3 &= 11, \\
10q_1 + 10q_2 + 20q_3 &\geq 17, \\
q_k &\geq 0, \\
k &= 1, 2, 3,
\end{align*}
\]

is not compatible. Notice that the constraint related to \( \pi(S_1^1) = 20 \) is not reported since it is implied by the second equation.

We stress that, more generally, for the above assessment there is no closed subset \( Q'' \subseteq \mathcal{P}(\Omega, \mathcal{F}) \) whose corresponding discounted lower expectation functional agrees with \( \pi \). To see this, it is sufficient to consider the above system and relax the second constraint in a greater than or equal to constraint, as this result in an incompatible system.

If we take the lower prices of securities in \( G \) as fixed, then the non-existence of a suitable closed \( Q' \subseteq Q \) forces us to depart from the set \( Q \), in a way to derive a consistent discounted lower expectation. Thus, we should face a problem of correction of the set \( Q \) that necessarily introduces some imprecision with respect to \( Q \).
On the other hand, instead of looking for a closed $Q' \subseteq Q$, we could try to derive a lower pricing rule from the lower envelope $\overline{Q}$, which has been proved to be a belief function. The most natural way to get a lower pricing rule is to consider a discounted Choquet expectation derived from the “risk-neutral” belief function $\overline{Q}$. We stress that, working directly in the framework of belief functions allows to incorporate “naturally” frictions in the market, nevertheless, for such a lower pricing rule to be acceptable the classical notion of arbitrage must be generalized. This will be the objective of the next section. By considering only the lower envelope $\overline{Q}$ we forget of the set $Q$ and actually work with $\text{core}(\overline{Q})$, thus also in this case we introduce some imprecision with respect to $Q$.

4 A generalized no-arbitrage principle

In this section we still refer to a finite measurable space $(\Omega, \mathcal{F})$, with $\Omega = \{1, \ldots, n\}$ and $\mathcal{F} = \mathcal{P}(\Omega)$, endowed with a belief function $\text{Bel}$ encoding the market beliefs. Throughout this section we assume $\text{Bel}(A) > 0$, for every $A \in \mathcal{F}\{\emptyset\}$. Such a belief function $\text{Bel}$ plays the same role of the “real-world” probability measure $P$ in the classical formulation of a one-period market model [see, e.g., Delbaen and Schachermayer (2006)]. For this, $\text{Bel}$ can be dubbed as “real-world” belief function.

**Definition 2** Given two belief functions $\text{Bel}$, $\widehat{\text{Bel}}$ on $\mathcal{F}$, we say that $\widehat{\text{Bel}}$ is equivalent to $\text{Bel}$, in symbol $\widehat{\text{Bel}} \sim \text{Bel}$, if $\text{Bel}(A) = 0 \iff \widehat{\text{Bel}}(A) = 0$, for every $A \in \mathcal{F}$.

Let us stress that, since $\text{Bel}$ is positive on $\mathcal{F}\{\emptyset\}$, $\widehat{\text{Bel}} \sim \text{Bel}$ if and only if the same holds for $\widehat{\text{Bel}}$ and this happens if and only if its Möbius inverse $\widehat{\mu}$ is positive over the singletons.

Also in this case, we refer to the filtration $\{\mathcal{F}_0, \mathcal{F}_1\}$ with $\mathcal{F}_0 = \{\emptyset, \Omega\}$ and $\mathcal{F}_1 = \mathcal{F}$.

**Remark 2** In classical finite-state no-arbitrage pricing models [see, e.g., Pliska (1997)], the positivity of the “real-world” probability $P$ is motivated by the fact that only “realistic” states of nature are taken into account, i.e., states with null measure are discarded. Furthermore, a common assumption in finance is that all market agents share the same probabilistic opinions, i.e., they have the same $P$. The positivity requirement for $\text{Bel}$ could follow from a probabilistic interpretation with the same rationale. Indeed, the belief function $\text{Bel}$ gives rise to $\text{ext}(\text{core}(\text{Bel}))$, which is a finite set of “extreme” probabilistic opinions on the market, that can be associated with some reference agents. Such probabilistic opinions, though possibly different, are assumed to agree on states that are perceived as “unrealistic”. Hence, the positivity of $\text{Bel}$ generalizes the classical assumption on market beliefs as reference agents can have different probabilistic opinions, though it still requires some agreement among them on “realistic” states. A more general formulation can be given referring to an arbitrary belief function $\text{Bel}$, still adopting the notion of equivalent belief function given in Definition 2. In turn this requires to keep track of events with null belief, that will play a role in the formation of prices: this is a non-trivial generalization that could be the subject of future research.

We still consider a one-period market model related to times $t = 0$ and $t = 1$ where there is a risk-free bond assuring the return $1 + r > 0$. Such a bond has price $S^0_0 = 1$ at time $t = 0$ and payoff $S^0_1 = 1 + r$ at time $t = 1$. Here, the goal is to allow frictions in the market by considering, for a random variable $X \in \mathbb{R}^\Omega$, a lower price $\pi(X)$ at time $t = 0$ and, if available, a corresponding upper price $\overline{\pi}(X)$, with $\pi(X) \leq \overline{\pi}(X)$, to be interpreted as bid-ask prices. In the case of the risk-free bond we assume absence of frictions, meaning that the lower price coincides with the upper price $\pi(S^0_0) = \overline{\pi}(S^0_1) = S^0_0$, thus we simply call it price.
We consider a finite non-empty collection of random variables
\[ G = \{S_1, \ldots, S_m\} \subset \mathbb{R}^\Omega, \tag{21} \]
expressing random payoffs at time \( t = 1 \) and a lower price assessment \( \pi : G \to \mathbb{R} \) related to time \( t = 0 \). In analogy with the classical formulation of no-arbitrage pricing (Delbaen & Schachermayer, 2006), we do not require the risk-free bond to be part of \( G \) as it possesses a special role being used as numéraire.

Our aim is to determine a necessary and sufficient condition for the existence of a belief function \( \hat{\text{Bel}} \sim \text{Bel} \) such that, for \( k = 1, \ldots, m \), it holds that
\[ (1 + r)^{-1} C_{\hat{\text{Bel}}}(S_1^k) = \pi(S_1^k). \]
By the positive homogeneity property of the Choquet integral [see, e.g., Grabisch (2016)],
\[ (1 + r)^{-1} C_{\hat{\text{Bel}}}(S_1^k) = C_{\hat{\text{Bel}}}(1 + r)^{-1} S_1^k), \]
thus we can consider the discounted payoff \( \tilde{S}_1^k = (1 + r)^{-1} S_1^k \), for \( k = 1, \ldots, m \), and write
\[ C_{\hat{\text{Bel}}}(\tilde{S}_1^k) = \pi(S_1^k). \tag{22} \]

Also in this case we have that \( \tilde{S}_1^0 = 1_\Omega \).

Let us point out that, like in classical no-arbitrage theory, the request of positivity of \( \hat{\text{Bel}} \) is a desideratum in the context of pricing since it assures that a security with a non-negative and non-null payoff at time \( t = 1 \) will have a positive lower price at time \( t = 0 \).

Here we assume to know only the lower price of every \( S_1^k \), for \( k = 1, \ldots, m \). This is not restrictive since, if also the upper price assessment \( \bar{\pi} : G \to \mathbb{R} \) is available, then the problem can be reformulated by considering
\[ G' = \{S_1^1, \ldots, S_m^1, -S_1^1, \ldots, -S_m^1\} \tag{23} \]

\[ \pi'(S_1^k) = \pi(S_1^k) \quad \text{and} \quad \pi'(-S_1^k) = -\pi(S_1^k). \tag{24} \]

As usual, a portfolio is a vector \( \lambda = (\lambda_0, \ldots, \lambda_m)^T \in \mathbb{R}^{m+1} \), where \( \lambda_0 \) is the number of units of the risk-free asset while, for \( k = 1, \ldots, m \), \( \lambda_k \) is the number of units of every contract related to the random payoffs in \( G \), to buy or short sell.

Here, we assume the partially resolving uncertainty principle proposed by Jaffray (1989) according to which the agent may only acquire the information that an event \( B \neq \emptyset \) occurs, without knowing which is the true state of the world \( \omega \in B \). Further, we assume that the agent is systematically pessimistic in his/her quantitative evaluations. As such, both in computing his/her (discounted) payoff related to a portfolio of securities and in the corresponding gain, the agent considers all non-impossible events in \( \mathcal{U} = \mathcal{F}\backslash\{\emptyset\} \) further, for every \( X \in \mathbb{R}^\Omega \), he/she considers the corresponding \( X^L \in \mathbb{R}^{\mathcal{U}} \) built taking minima of \( X \) as in (8). This is in contrast with the principle of completely resolving uncertainty which is usually tacitly adopted and amounts in assuming that the agent will always acquire the information on the true state of the world \( \omega \in \Omega \).

Working under partially resolving uncertainty, the final (discounted) payoff of the portfolio is the function \( Z_\lambda : \mathcal{U} \to \mathbb{R} \) defined, for every \( B \in \mathcal{U} \), as
\[ Z_\lambda(B) = \lambda_0 + \sum_{k=1}^m \lambda_k (\tilde{S}_1^k)^L(B), \tag{25} \]
while we interpret the quantity $\pi_k = \lambda_0 + \sum_{k=1}^{m} \lambda_k \pi_k(S_1^k)$ as the hypothetical price at time $t = 0$ of the portfolio that we would have if we were in a situation of completely resolving uncertainty. Hence, we can define the function $G_\lambda : \mathcal{U} \rightarrow \mathbb{R}$ setting, for every $B \in \mathcal{U}$,

$$G_\lambda(B) = Z_\lambda(B) - \pi_\lambda = \sum_{k=1}^{m} \lambda_k \left( (S_1^k)^L(B) - \pi_k(S_1^k) \right),$$

(26)

that can be interpreted as a random gain under partially resolving uncertainty.

**Theorem 2** The following conditions are equivalent:

(i) there exists a belief function $\hat{\mathcal{B}}$ such that $C_{\hat{\mathcal{B}},\Omega}(\tilde{S}_1^k) = \pi(S_1^k)$, for $k = 1, \ldots, m$;

(ii) for every $\lambda = (\lambda_0, \ldots, \lambda_m)^T \in \mathbb{R}^{m+1}$ it holds that

$$\min_{B \in \mathcal{U}} G_\lambda(B) \leq 0 \leq \max_{B \in \mathcal{U}} G_\lambda(B).$$

**Proof** The proof can be obtained applying Theorem 4.1 in Coletti et al. (2020), working with the dual capacity of $\hat{\mathcal{B}}$, which is a plausibility function. Here we provide a direct proof for the sake of completeness.

Fix an enumeration of $\mathcal{U} = \{B_1, \ldots, B_{2^n-1}\}$. Condition (i) is equivalent to the solvability of the following system

$$\begin{cases}
Ax = b, \\
x \geq 0,
\end{cases}$$

where $x = (\hat{\mu}(B_1), \ldots, \hat{\mu}(B_{2^n-1}))^T \in \mathbb{R}^{2^n-1}$ is an unknown column vector, $A \in \mathbb{R}^{(m+1) \times (2^n-1)}$ is the coefficient matrix with

$$A = \begin{pmatrix}
1\Omega^L(B_1) & \cdots & 1\Omega^L(B_{2^n-1}) \\
(S_1^1)^L(B_1) & \cdots & (S_1^m)^L(B_{2^n-1}) \\
\vdots & \vdots & \vdots \\
(S_1^1)^L(B_1) & \cdots & (S_1^m)^L(B_{2^n-1})
\end{pmatrix},$$

and $b = (1, \pi(S_1^1), \ldots, \pi(S_m^1))^T \in \mathbb{R}^{m+1}$.

By Farkas’ lemma (Mangasarian, 1994), the system above is compatible if and only if the following system is not compatible

$$\begin{cases}
A^Ty \leq 0, \\
b^Ty > 0,
\end{cases}$$

where $y = (\lambda_0, \ldots, \lambda_m)^T \in \mathbb{R}^{m+1}$ is an unknown column vector. It holds that $A^Ty \in \mathbb{R}^{2^n-1}$ and, for $i = 1, \ldots, 2^n - 1$, the $i$th component of constraint $A^Ty \leq 0$ is

$$\lambda_0 + \sum_{k=1}^{m} \lambda_k (S_1^k)^L(B_i) \leq 0,$$

moreover, subtracting the positive quantity $b^Ty$ we get

$$\sum_{k=1}^{m} \lambda_k \left( (S_1^k)^L(B_i) - \pi_k(S_1^k) \right) < 0.$$

Thus, condition (i) is equivalent to the existence of $i \in \{1, \ldots, 2^n - 1\}$ such that the above inequality does not hold, which, in turn, is equivalent to (ii). \qed
The above theorem says that, working under partially resolving uncertainty, in order to have a discounted totally monotone Choquet expectation representation of the lower price assessment \( \pi \) independent of the “real-world” belief function of the market, it is necessary and sufficient that every portfolio \( \lambda \) does not give rise to a sure loss or a sure gain over \( \mathcal{U} \). In other terms, the above condition can be considered a \textit{generalized avoiding Dutch book condition}, working under partially resolving uncertainty. Nevertheless, the condition (ii) of Theorem 2 does not assure that \( \hat{\text{Bel}} \sim \text{Bel} \), that is we do not have any guarantee that \( \hat{\text{Bel}}(A) > 0 \), for every \( A \in \mathcal{F} \setminus \{\emptyset\} \).

We stress again that, following the classical financial pricing models (Černý, 2009; Pliska, Pliska (1997)), we look for a lower pricing rule that assures a positive lower price for non-negative and non-null payoffs. Hence, we provide a necessary and sufficient condition for the existence of an equivalent belief function positive on the entire \( \mathcal{F} \setminus \{\emptyset\} \) that can be dubbed “risk-neutral” belief function. Besides equivalence between \( \hat{\text{Bel}} \) and \( \text{Bel} \) no other relation is asked to hold.

Such theorem is the analogue of the \textit{first fundamental theorem of asset pricing}, formulated in the Dempster–Shafer theory of evidence.

**Theorem 3** The following conditions are equivalent:

(i) there exists a belief function \( \hat{\text{Bel}} \sim \text{Bel} \), i.e., \( \hat{\text{Bel}}(A) > 0 \), for every \( A \in \mathcal{F} \setminus \{\emptyset\} \), such that \( \mathbb{C}_{\hat{\text{Bel}}} (\hat{\mathcal{S}}^k) = \hat{\pi}(\mathcal{S}^k) \), for \( k = 1, \ldots, m \);

(ii) for every \( \lambda = (\lambda_0, \ldots, \lambda_m)^T \in \mathbb{R}^{n+1} \) none of the following conditions holds:

(a) \( Z_\lambda([i]) = 0 \), for \( i = 1, \ldots, n \), \( Z_\lambda(B) \geq 0 \), for all \( B \in \mathcal{U} \setminus \{i : i \in \Omega\} \) and \( \pi_\lambda < 0 \);

(b) \( Z_\lambda([i]) \geq 0 \), for \( i = 1, \ldots, n \), with at least a strict inequality, \( Z_\lambda(B) \geq 0 \), for all \( B \in \mathcal{U} \setminus \{i : i \in \Omega\} \), and \( \pi_\lambda \leq 0 \).

**Proof** Since every belief function is completely characterized by its Möbius inverse, statement (i) is equivalent to the existence of a non-negative function \( \hat{\mu} : \mathcal{F} \to \mathbb{R} \) such that

\[
\hat{\mu}(\emptyset) = 0, \quad \sum_{A \in \mathcal{F}} \hat{\mu}(A) = 1, \quad \text{and} \quad \hat{\text{Bel}}(A) = \sum_{B \subseteq A} \hat{\mu}(B), \quad \text{for every} \ A \in \mathcal{F},
\]

further satisfying \( \hat{\mu}((i)) > 0 \), for all \( i \in \Omega \), and

\[
\mathbb{C}_{\hat{\text{Bel}}} (\hat{\mathcal{S}}^k) = \sum_{B \in \mathcal{U}} (\hat{\mathcal{S}}^k_B L(B)) \hat{\mu}(B) = \hat{\pi}(\mathcal{S}^k_B), \quad \text{for} \ k = 1, \ldots, m.
\]

Fix an enumeration of \( \mathcal{U} = \{B_1, \ldots, B_{2^n-1}\} \) such that \( B_i = [i] \), for \( i = 1, \ldots, n \), and consider the matrices \( \mathbf{A} \in \mathbb{R}^{(2(m+1)+2^n-(n+1)) \times (2^n-1)} \) and \( \mathbf{B} \in \mathbb{R}^{n \times (2^n-1)} \) defined as

\[
\mathbf{A} = \begin{pmatrix} \mathbf{C} \\ \mathbf{O}_1 | -\mathbf{I}_{(2^n-(n+1))} \end{pmatrix} \quad \text{and} \quad \mathbf{B} = (\mathbf{-I}_n | \mathbf{O}_2),
\]

where \( \mathbf{C} \in \mathbb{R}^{2(m+1) \times (2^n-1)} \) is defined as

\[
\mathbf{C} = \begin{pmatrix} \mathbf{1}_L^1(B_1) & \cdots & \mathbf{1}_L^1(B_{2^n-1}) \\
-\mathbf{I}_L^2(B_1) & \cdots & -\mathbf{I}_L^2(B_{2^n-1}) \\
\mathbf{\hat{S}}_1^1L(B_1) & \cdots & \mathbf{\hat{S}}_1^1L(B_{2^n-1}) \\
-(\mathbf{\hat{S}}_1^1)^L(B_1) & \cdots & -(\mathbf{\hat{S}}_1^1)^L(B_{2^n-1}) \\
\vdots & \vdots & \vdots \\
\mathbf{\hat{S}}_m^1L(B_1) & \cdots & \mathbf{\hat{S}}_m^1L(B_{2^n-1}) \\
-(\mathbf{\hat{S}}_1^m)^L(B_1) & \cdots & -(\mathbf{\hat{S}}_1^m)^L(B_{2^n-1}) \end{pmatrix},
\]
in which \( \mathbf{1}_{(2^n-(n+1))} \in \mathbb{R}^{(2^n-(n+1)) \times (2^n-(n+1))} \) and \( \mathbf{I}_n \in \mathbb{R}^{n \times n} \) are identity matrices, and \( \mathbf{O}_1 \in \mathbb{R}^{(2^n-(n+1)) \times n} \) and \( \mathbf{O}_2 \in \mathbb{R}^{n \times (2^n-(n+1))} \) are null matrices. Take the vector

\[
\mathbf{b} = (1, -1, \pi(S^1_1), -\pi(S^m_1), \ldots, \pi(S^m_1), -\pi(S^m_1), 0, \ldots, 0)^T
\]

with \( \mathbf{b} \in \mathbb{R}^{(2(n+1)+2^n-(n+1))} \) and consider the unknown vector

\[
\mathbf{x} = (\tilde{\mu}(B_1), \ldots, \tilde{\mu}(B_{2^n-1}))^T
\]

with \( \mathbf{x} \in \mathbb{R}^{2^n-1} \). Condition (i) turns out to be equivalent to the solvability of the following system

\[
\begin{align*}
\mathbf{A} \mathbf{x} & \leq \mathbf{b}, \\
\mathbf{B} \mathbf{x} & < \mathbf{0}.
\end{align*}
\]

By a well-known version of Motzkin’s theorem of the alternative [see, e.g., Theorem 1 in Ben-Israel (2001)] the above system is solvable if and only if, for every \( \mathbf{y} = (y_0, y'_0, y_1, y'_1, \ldots, y_m, y'_m, \alpha_{n+1}, \ldots, \alpha_{2^n-1})^T \in \mathbb{R}^{(2(n+1)+2^n-(n+1))} \) and \( \mathbf{z} = (z_1, \ldots, z_n)^T \in \mathbb{R}^n \) with \( \mathbf{y} \geq \mathbf{0} \) and \( \mathbf{z} \geq \mathbf{0} \), none of the following conditions holds:

- \( \mathbf{A}^T \mathbf{y} + \mathbf{B}^T \mathbf{z} = \mathbf{0}, \mathbf{z} = \mathbf{0} \) and \( \mathbf{b}^T \mathbf{y} < \mathbf{0} \);
- \( \mathbf{A}^T \mathbf{y} + \mathbf{B}^T \mathbf{z} = \mathbf{0}, \mathbf{z} \neq \mathbf{0} \) and \( \mathbf{b}^T \mathbf{y} \leq \mathbf{0} \).

In turn, setting \( \lambda_k = y_k - y'_k \), for \( k = 0, \ldots, m \), and considering \( \tilde{\mathbf{y}} = (\lambda_0, \ldots, \lambda_m, \alpha_{n+1}, \ldots, \alpha_{2^n-1})^T \) such that \( \alpha_{n+1}, \ldots, \alpha_{2^n-1} \geq 0 \),

\[
\tilde{\mathbf{A}} = \left( \frac{\tilde{\mathbf{C}}}{\mathbf{O}_1 - \mathbf{I}_{(2^n-(n+1))}} \right)
\]

and \( \tilde{\mathbf{b}} = (1, \pi(S^1_1), \ldots, \pi(S^m_1), 0, \ldots, 0)^T \),

where \( \tilde{\mathbf{C}} \in \mathbb{R}^{(m+1) \times (2^n-1)} \) is defined as

\[
\tilde{\mathbf{C}} = \begin{pmatrix}
\mathbf{1}_{L_1}^L(B_1) & \cdots & \mathbf{1}_{L_1}^L(B_{2^n-1}) \\
(S^1_1)^{L_1} \mathbf{1}^L(B_1) & \cdots & (S^1_1)^{L_1} \mathbf{1}^L(B_{2^n-1}) \\
\vdots & & \vdots \\
(S^m_1)^{L_1} \mathbf{1}^L(B_1) & \cdots & (S^m_1)^{L_1} \mathbf{1}^L(B_{2^n-1})
\end{pmatrix},
\]

the above conditions can be rewritten as:

- \( \tilde{\mathbf{A}}^T \tilde{\mathbf{y}} + \tilde{\mathbf{b}}^T \tilde{\mathbf{z}} = \mathbf{0}, \mathbf{z} = \mathbf{0} \) and \( \tilde{\mathbf{b}}^T \tilde{\mathbf{y}} < \mathbf{0} \);
- \( \tilde{\mathbf{A}}^T \tilde{\mathbf{y}} + \tilde{\mathbf{b}}^T \tilde{\mathbf{z}} = \mathbf{0}, \mathbf{z} \neq \mathbf{0} \) and \( \tilde{\mathbf{b}}^T \tilde{\mathbf{y}} \leq \mathbf{0} \).

Denoting \( \lambda = (\lambda_0, \ldots, \lambda_m)^T \in \mathbb{R}^{m+1} \), we have that

\[
(\tilde{\mathbf{A}}^T \tilde{\mathbf{y}} + \tilde{\mathbf{b}}^T \tilde{\mathbf{z}})_i = \begin{cases}
Z_\lambda(B_i) - z_i, & \text{for } i = 1, \ldots, n, \\
Z_\lambda(B_i) - \alpha_i, & \text{for } i = n + 1, \ldots, 2^n - 1,
\end{cases}
\]

and further \( \tilde{\mathbf{b}}^T \tilde{\mathbf{y}} = \pi_\lambda \).

Hence, for every \( \lambda = (\lambda_0, \ldots, \lambda_m)^T \in \mathbb{R}^{m+1} \), the above conditions can be rewritten as

(a) \( Z_\lambda([i]) = 0, \) for \( i = 1, \ldots, n, \) \( Z_\lambda(B) \geq 0, \) for all \( B \in \mathcal{U}\setminus\{[i]: i \in \Omega\} \) and \( \pi_\lambda < 0; \)
(b) \( Z_\lambda([i]) \geq 0, \) for \( i = 1, \ldots, n, \) with at least a strict inequality, \( Z_\lambda(B) \geq 0, \) for all \( B \in \mathcal{U}\setminus\{[i]: i \in \Omega\}, \) and \( \pi_\lambda \leq 0. \)

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Recall that we interpret $\pi_\lambda$ as the hypothetical price of the portfolio $\lambda$ as if we were in a situation of completely resolving uncertainty. In this light, conditions (ii.a) and (ii.b) of previous theorem can be interpreted as two generalized forms of arbitrage, working under partially resolving uncertainty. Avoiding condition (ii.a) assures that we cannot find a portfolio $\lambda$ whose hypothetical price $\pi_\lambda$ is negative (that is we are paid for it), resulting in a uniformly non-negative payoff $Z_\lambda$ in all the possible events in $\mathcal{U}$, with null value on the singletons (i.e., on those events where we have completely resolving uncertainty). Avoiding condition (ii.b) assures that we cannot find a portfolio $\lambda$ whose hypothetical price $\pi_\lambda$ is negative or null (that is we are paid or we do not pay anything for it), resulting in a uniformly non-negative payoff $Z_\lambda$ in all the possible events in $\mathcal{U}$, with at least a strictly positive value on the singletons (i.e., on those events where we have completely resolving uncertainty).

It is immediate to see that the generalized no-arbitrage principle expressed by statement (ii) of Theorem 3 implies the generalized avoiding Dutch book condition in statement (ii) of Theorem 2. In particular, if a portfolio $\lambda$ satisfies condition (ii.a) of Theorem 3 then the corresponding gain $G_\lambda$ is such that $\min_{B \in \mathcal{U}} G_\lambda(B) > 0$, thus violating the generalized avoiding Dutch book condition.

Let us stress that the generalized no-arbitrage principle of Theorem 3 is actually weaker than the classical no-arbitrage principle. This is due to the fact that if a portfolio $\lambda$ gives rise to a generalized arbitrage of the form (ii.a) or (ii.b) then it also gives rise to a classical arbitrage, while a portfolio $\lambda$ giving rise to a classical arbitrage does not generally give rise to a generalized arbitrage.

Let us stress that the price functional determined by the discounted Choquet expectation with respect to a $\widehat{\text{Bel}}$ with Möbius inverse $\widehat{\mu}$ as in Theorem 3 is generally not linear. In particular, we have that

$$
\sum_{B \in \mathcal{U}} Z_\lambda(B) \widehat{\mu}(B) = \sum_{B \in \mathcal{U}} \left( \lambda_0 + \sum_{k=1}^{m} \lambda_k (\widehat{S}^k_1)^{L}(B) \right) \widehat{\mu}(B) = \lambda_0 + \sum_{k=1}^{m} \lambda_k \left( \sum_{B \in \mathcal{U}} (\widehat{S}^k_1)^{L}(B) \widehat{\mu}(B) \right) = \lambda_0 + \sum_{k=1}^{m} \lambda_k \mathbb{C}_{\widehat{\text{Bel}}} (\widehat{S}^k_1) = \lambda_0 + \sum_{k=1}^{m} \lambda_k \mathbb{I}_{\pi_{\lambda}} (S^k_1) = \pi_{\lambda}. \quad (27)
$$

Nevertheless, considering the random variable $\lambda_0 + \sum_{k=1}^{m} \lambda_k \hat{S}^k_1 \in \mathbb{R}^\mathcal{U}$, though $\mathbb{C}_{\widehat{\text{Bel}}} \left( \lambda_0 + \sum_{k=1}^{m} \lambda_k \hat{S}^k_1 \right) = \lambda_0 + \sum_{k=1}^{m} \lambda_k \mathbb{C}_{\widehat{\text{Bel}}} (\hat{S}^k_1)$, in general we have that

$$
\mathbb{C}_{\widehat{\text{Bel}}} \left( \sum_{k=1}^{m} \lambda_k \hat{S}^k_1 \right) \neq \sum_{k=1}^{m} \mathbb{C}_{\widehat{\text{Bel}}} (\lambda_k \hat{S}^k_1) \quad \text{and} \quad \sum_{k=1}^{m} \mathbb{C}_{\widehat{\text{Bel}}} (\lambda_k \hat{S}^k_1) \neq \sum_{k=1}^{m} \lambda_k \mathbb{C}_{\widehat{\text{Bel}}} (\hat{S}^k_1).
$$

Clearly, in the above formulas we have equalities in case $\widehat{\text{Bel}}$ reduces to a probability measure. On the other hand, in the particular case $\hat{S}^h_1, \hat{S}^f_1$ are comonotone and $\lambda_1, \lambda_2 \geq 0$, it holds that

$$
\mathbb{C}_{\widehat{\text{Bel}}} \left( \lambda_1 \hat{S}^h_1 + \lambda_2 \hat{S}^f_1 \right) = \lambda_1 \mathbb{C}_{\widehat{\text{Bel}}} (\hat{S}^h_1) + \lambda_2 \mathbb{C}_{\widehat{\text{Bel}}} (\hat{S}^f_1) = \lambda_1 \mathbb{I}_{\pi_{\lambda}} (\hat{S}^h_1) + \lambda_2 \mathbb{I}_{\pi_{\lambda}} (\hat{S}^f_1).
$$

Furthermore, denoting by $\widehat{\mathcal{P}}_I$ the dual plausibility function of $\widehat{\text{Bel}}$, we have that for a generic random variable $X \in \mathbb{R}^{\mathcal{U}}$, it holds that

$$
(1 + r)^{-1} \mathbb{C}_{\widehat{\text{Bel}}} (X) \leq (1 + r)^{-1} \mathbb{C}_{\widehat{\mathcal{P}}_I} (X), \quad (28)
$$

$\odot$ Springer
i.e., the two values above should be interpreted as lower and upper prices.

The following example shows a lower price assessment violating the generalized no-arbitrage principle expressed in Theorem 3.

**Example 5** Let \( \Omega = \{1, 2, 3, 4\} \), \( \mathcal{F} = \mathcal{P}(\Omega) \) and consider three contracts whose payoffs in euros at time \( t = 1 \) are

| \( \Omega \) |
|---|
| 1 |
| 2 |
| 3 |
| 4 |

| \( S_1^1 \) |
|---|
| 10 |
| 10 |
| 20 |
| 20 |

| \( S_2^1 \) |
|---|
| 0 |
| 10 |
| 0 |
| 10 |

| \( S_3^1 \) |
|---|
| 10 |
| 30 |
| 20 |
| 40 |

Assume that the lower prices at time \( t = 0 \) are fixed to \( \pi(S_1^1) = 15 \), \( \pi(S_2^1) = 5 \) and \( \pi(S_3^1) = 20 \) and that the risk-free interest rate is \( r = 0 \), so we have \( S_k^1 = S_k^1 \), for \( k = 1, 2, 3 \).

This lower price assessment violates the generalized no-arbitrage principle of Theorem 3 as, in particular, it violates the generalized avoiding Dutch book condition expressed in Theorem 2. Indeed, every belief function \( \hat{\text{Bel}} \) on \( \mathcal{F} \) induces a Choquet expectation functional on \( \mathbb{R}^\Omega \) which is positively homogeneous and superadditive, therefore, assuming \( \mathbb{C}_{\hat{\text{Bel}}}(S_k^1) = \pi(S_k^1) \), for \( k = 1, 2, 3 \), it should be, as \( S_3^1 = S_1^1 + 2S_2^1 \),

\[
\mathbb{C}_{\hat{\text{Bel}}}(S_3^1) = \mathbb{C}_{\hat{\text{Bel}}}(S_1^1 + 2S_2^1) \geq \mathbb{C}_{\hat{\text{Bel}}}(S_1^1) + 2\mathbb{C}_{\hat{\text{Bel}}}(S_2^1) = 25.
\]

Denoting \( \mathcal{U} = \mathcal{F} \setminus \{\emptyset\} \) and omitting braces and commas to have a lighter set notation, if we consider the portfolio \( \lambda = (0, -1, -2, 1)^T \) we have that \( \pi_\lambda = -5 \) and

| \( \mathcal{U} \) |
|---|
| 1 |
| 2 |
| 3 |
| 4 |
| 12 |
| 13 |
| 14 |
| 23 |
| 24 |
| 34 |
| 123 |
| 124 |
| 134 |
| 234 |
| 1234 |

| \( (S_1^1)_L \) |
|---|
| 10 |
| 10 |
| 20 |
| 20 |
| 10 |
| 10 |
| 10 |
| 10 |
| 20 |
| 20 |
| 20 |
| 10 |
| 10 |
| 10 |
| 10 |

| \( (S_2^1)_L \) |
|---|
| 0 |
| 10 |
| 0 |
| 0 |
| 0 |
| 0 |
| 0 |
| 0 |
| 0 |
| 0 |
| 0 |
| 0 |
| 0 |
| 0 |
| 0 |

| \( (S_3^1)_L \) |
|---|
| 10 |
| 30 |
| 20 |
| 40 |
| 10 |
| 10 |
| 20 |
| 30 |
| 20 |
| 10 |
| 10 |
| 20 |
| 10 |
| 0 |
| 0 |

| \( Z_\lambda \) |
|---|
| 5 |
| 5 |
| 5 |
| 5 |
| 5 |
| 5 |
| 5 |
| 5 |
| 5 |
| 5 |
| 15 |
| 5 |
| 5 |
| 5 |

Hence, since \( \min_{B \in \mathcal{U}} G_\lambda(B) > 0 \), the generalized avoiding Dutch book condition is not satisfied, therefore there is no belief function \( \hat{\text{Bel}} \) such that \( \mathbb{C}_{\hat{\text{Bel}}} \) agrees with the assessed lower prices. Moreover, the same \( \lambda \) shows that we have a generalized arbitrage in the form of (ii.a) of Theorem 3, since \( Z_\lambda(i) = 0 \), for \( i = 1, \ldots, 4 \), \( Z_\lambda(B) \geq 0 \), for all \( B \in \mathcal{U} \setminus \{i : i \in \Omega\} \) and \( \pi_\lambda < 0 \).

On the other hand, taking the portfolio \( \lambda' = (0, -2, -10, 4)^T \) we have that \( \pi_{\lambda'} = 0 \) and

| \( \mathcal{U} \) |
|---|
| 1 |
| 2 |
| 3 |
| 4 |
| 12 |
| 13 |
| 14 |
| 23 |
| 24 |
| 34 |
| 123 |
| 124 |
| 134 |
| 234 |
| 1234 |

| \( Z_{\lambda'} \) |
|---|
| 20 |
| 0 |
| 40 |
| 20 |
| 20 |
| 20 |
| 60 |
| 0 |
| 40 |
| 20 |
| 20 |
| 20 |
| 60 |
| 20 |
therefore, we have a generalized arbitrage in the form of (ii.b) of Theorem 3, since $Z_{\lambda'}(\{i\}) \geq 0$, for $i = 1, \ldots, 4$, with at least a strict inequality, $Z_{\lambda'}(B) \geq 0$, for all $B \in \mathcal{U}\backslash\{\{i\} : i \in \Omega\}$ and $\pi_{\lambda'} \leq 0$.

The following example shows a lower price assessment violating the classical no-arbitrage principle but not the generalized no-arbitrage principle.

**Example 6** Let $\Omega, \mathcal{F}, r$, and $S_1^1, S_2^1, S_3^1$ as in Example 5. Consider the lower price assessment $\pi(S_1^1) = 15, \pi(S_2^1) = 5$ and $\pi(S_3^1) = 26$. Such an assessment violates the classical no-arbitrage principle, indeed, every probability measure $Q$ on $\mathcal{F}$ gives rise to a positive, linear and normalized functional $E_Q$ on $R^\Omega$. Hence, assuming $E_Q(S_k^1) = \pi(S_k^1)$, for $k = 1, 2, 3$, it should be, as $S_3^1 = S_1^1 + 2S_2^1$,

$$E_Q(S_3^1) = E_Q(S_1^1 + 2S_2^1) = E_Q(S_1^1) + 2E_Q(S_2^1) = 25.$$

On the other hand, there exists a belief function $\widehat{Bel}$ on $\mathcal{F}$ which is strictly positive on $\mathcal{F}\backslash\{\emptyset\}$, whose corresponding Choquet expectation functional $C_{\widehat{Bel}}$ agrees with the given lower price assessment. For instance, we can take the $\widehat{Bel}$ whose Möbius inverse $\hat{\mu}$ is such that

\[
\begin{array}{cccccccccccccc}
\mathcal{U} & 1 & 2 & 3 & 4 & 12 & 13 & 14 & 23 & 24 & 34 & 123 & 124 & 134 & 234 & 1234 \\
(S_1^1)_L & 10 & 10 & 20 & 20 & 10 & 10 & 10 & 10 & 10 & 10 & 10 & 10 & 10 & 10 & 10 \\
(S_2^1)_L & 0 & 10 & 0 & 10 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
(S_3^1)_L & 10 & 30 & 20 & 40 & 10 & 10 & 10 & 20 & 30 & 10 & 10 & 10 & 20 & 10 & 10 \\
\hat{\mu} & \frac{2}{10} & \frac{1}{10} & \frac{1}{10} & \frac{4}{10} & \frac{1}{10} & 0 & 0 & \frac{1}{10} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{array}
\]

For such a $\widehat{Bel}$ we have that

$$C_{\widehat{Bel}}(S_1^1) = 10 \cdot \frac{2}{10} + (10 + 20 + 10 + 10) \cdot \frac{1}{10} + 20 \cdot \frac{4}{10} = 15,$$

$$C_{\widehat{Bel}}(S_2^1) = 10 \cdot \frac{1}{10} + 10 \cdot \frac{4}{10} = 5,$$

$$C_{\widehat{Bel}}(S_3^1) = 10 \cdot \frac{2}{10} + (30 + 20 + 10 + 20) \cdot \frac{1}{10} + 40 \cdot \frac{4}{10} = 26.$$

Hence, by Theorem 3 we cannot find a portfolio $\lambda$ giving rise to a generalized arbitrage in the form of (ii.a) or (ii.b). On the other hand, the portfolio $\lambda = (0, 1, 2, -1)^T$ gives rise to a classical arbitrage since $\pi_\lambda = -1 < 0$ and

\[
\Omega \quad 1 \quad 2 \quad 3 \quad 4 \\
S_1^1 \quad 10 \quad 10 \quad 20 \quad 20 \\
S_2^1 \quad 0 \quad 10 \quad 0 \quad 10 \\
S_3^1 \quad 10 \quad 30 \quad 20 \quad 40 \\
\lambda_0 + \sum_{k=1}^{3} \lambda_k S_k^1 \quad 0 \quad 0 \quad 0 \quad 0
\]
A non-linear pricing rule defined as a discounted Choquet expectation with respect to a concave capacity and satisfying a form of put-call parity has been axiomatically characterized, respectively, in Chateauneuf et al. (1996) and Cerreia-Vioglio et al. (2015). Such a functional, that can be interpreted as an upper pricing rule, allows to model frictions in the market. Our generalized no-arbitrage condition is equivalent to the existence of a (possibly not unique) completely monotone discounted Choquet expectation that can be interpreted as a lower pricing rule still allowing for frictions in the market.

Let us stress that, if \( \{1_B : B \in \mathcal{U}\} \subseteq \mathcal{G} \) and \( \pi : \mathcal{G} \to \mathbb{R} \) satisfies the generalized no-arbitrage principle, then there exists a unique \( \widehat{\text{Bel}} \), positive on \( \mathcal{F} \setminus \{\emptyset\} \), such that the corresponding discounted Choquet expectation functional on \( \mathbb{R}^\Omega \) agrees with \( \pi \). The payoffs in \( \{1_B : B \in \mathcal{U}\} \) can be considered as generalized Arrow-Debreu securities [see, e.g., Černý (2009), Dybvig and Ross (1989), Pliska (1997)], working under partially resolving uncertainty. Indeed, the family of lower prices \( \{\pi(1_B) : B \in \mathcal{U}\} \) uniquely determines \( \widehat{\text{Bel}} \) since, for every \( B \in \mathcal{U} \), we have that
\[
\pi(1_B) = (1 + r)^{-1}\widehat{\text{Bel}}(B),
\]
therefore such lower prices can be dubbed generalized Arrow-Debreu lower prices.

We notice that the wider framework of 2-monotone capacities could be considered instead of Dempster–Shafer theory. The reason why we stick to Dempster–Shafer theory is that the generalized no-arbitrage principle has a normative purpose: respecting it we derive a (non-necessarily unique) lower pricing rule that allows us to price securities taking care of bid-ask spreads. Besides equivalence, \( \widehat{\text{Bel}} \) has no other relation with \( \text{Bel} \) nor with market agents’ utility functions as it is only determined by the generalized no-arbitrage principle. Switching to the 2-monotone setting, though mathematically possible, makes the no-arbitrage condition much more involved and its interpretation is difficult to justify from a normative point of view.

5 Equivalent inner approximating martingale belief functions

We turn back to the lower envelope \( Q \) of the class \( \mathcal{Q} \) of equivalent martingale measures induced by the n-nomial market model characterized in Sect. 3. Recall that in this context we have only one risky asset whose price process is \( \{S_0^1, S_1^1\} \). In what follows we assume that the lower price of the risky asset is \( \pi(S_1^1) = S_0^1 \), which is justified by the fact that \( \min_{Q \in \mathcal{Q}} (1 + r)^{-1} \mathbb{E}_Q(S_1^1) = S_0^1 \). Moreover, here the “real-world” probability \( P \) can play the role of the “real-world” belief function introduced in Sect. 4.

As already pointed out in Corollary 1, \( Q \) is a belief function that we know is not positive over \( \mathcal{F} \setminus \{\emptyset\} \), for \( n > 2 \). Moreover, we have that assessing \( \pi(S_1^1) = S_0^1 \) is generally not consistent with the lower pricing rule obtained as the discounted Choquet expectation with respect to \( Q \). Indeed, if we define \( \pi : \mathcal{G} \to \mathbb{R} \) on \( \mathcal{G} = \{S_1^1\} \cup \{1_B : B \in \mathcal{U}\} \) such that
\[
\pi(S_1^1) = S_0^1 \quad \text{and} \quad \pi(1_B) = (1 + r)^{-1}Q(B), \quad \text{for all } B \in \mathcal{U},
\]
we get that \( \pi \) violates both the generalized avoiding Dutch book condition and the generalized no-arbitrage principle, as Example 7 shows.

**Example 7** Consider \( \Omega = \{1, 2, 3, 4\} \), \( m_1 = 4 \), \( m_2 = 2 \), \( m_3 = \frac{1}{2} \), \( m_4 = \frac{1}{4} \), \( 1 + r = 1 \), \( Q \) and \( Q \) of Example 1. Let \( S_0^1 = 100 \) and take the lower price assessment \( \pi \) on \( \mathcal{G} = \{S_1^1\} \cup \{1_B : B \in \mathcal{U}\} \) defined as in (30).
Denote by $\lambda_{S^0}, \lambda_{S^1}$ the numbers of units of the risk-free and risky assets, and by $\lambda_i, \lambda_{ij}, \lambda_{ijk}, \lambda_{1234}$ those associated with $\mathbf{1}_{[i]}, \mathbf{1}_{[i,j]}, \mathbf{1}_{[i,j,k]}, \mathbf{1}_\Omega$, respectively, in a portfolio $\lambda$ on $\mathcal{G}$. If we take $\lambda_{S^0} = 0, \lambda_{S^1} = -\frac{4}{23}, \lambda_1 = 7, \lambda_2 = \lambda_{13} = \lambda_{124} = \lambda_{1234} = 10, \lambda_3 = \lambda_4 = \lambda_{34} = -10, \lambda_{12} = 8, \lambda_{14} = \lambda_{123} = 5, \lambda_{23} = -1, \lambda_{24} = \lambda_{234} = -5, \lambda_{134} = 9$ we get that $\pi_\lambda = -7$ and further

| $\mathcal{U}$ | 1 | 2 | 3 | 4 | 12 | 13 | 14 | 23 | 24 | 34 | 123 | 124 | 134 | 234 | 1234 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $Z_\lambda$ | 0 | 0 | 0 | 0 | 1 | 26 | 30 | 1 | 6 | 0 | 7 | 16 | 15 | 1 | 6 |

In turn, since $\pi_\lambda < 0$ and $G_\lambda$ is defined by (26), it follows that $\min_{B \in \mathcal{U}} G_\lambda (B) > 0$, thus $\lambda$ violates the avoiding Dutch book condition. We also notice that this portfolio gives rise to a generalized arbitrage of type (ii.a) in Theorem 3, since $Z_\lambda ([i]) = 0$, for $i = 1, \ldots, 4$, $Z_\lambda (B) \geq 0$, for all $B \in \mathcal{U}\setminus\{(i) : i \in \Omega\}$ and $\pi_\lambda < 0$.

We stress that $\pi$ fails the generalized avoiding Dutch book condition (and, therefore, the generalized no-arbitrage principle) since $S^1$ is mispriced as it should be $\pi(S^1) = \mathbb{C}_Q(S^1) = \frac{5400}{105}$ to be consistent with the generalized Arrow–Debreu lower prices.

Furthermore, if we take the portfolio $\lambda'$ on $\mathcal{G}$ with entries $\lambda_{S^0}' = 0, \lambda_{S^1}' = -\frac{1}{3}, \lambda_1' = \lambda_2' = \lambda_3' = \lambda_4' = \lambda_{12} = \lambda_{14} = \lambda_{123} = \lambda_{124} = \lambda_{134} = \lambda_{1234} = 10, \lambda_{234}' = 6, \lambda_{234}' = -5$ we get that $\pi_{\lambda'} = 0$ and $\pi_{\lambda'} < 0$.

| $\mathcal{U}$ | 1 | 2 | 3 | 4 | 12 | 13 | 14 | 23 | 24 | 34 | 123 | 124 | 134 | 234 | 1234 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $Z_{\lambda'}$ | 0 | 25 | 51 | 56 | 0 | 30 | 35 | 15 | 20 | 16 | 10 | 15 | 14 | 15 | 14 |

It is easily seen that the portfolio $\lambda'$ gives rise to a generalized arbitrage of type (ii.b) in Theorem 3, since $Z_{\lambda'} ([i]) \geq 0$, for $i = 1, \ldots, 4$, with at least a strict inequality, $Z_{\lambda'} (B) \geq 0$, for all $B \in \mathcal{U}\setminus\{(i) : i \in \Omega\}$, and $\pi_{\lambda'} < 0$.

Let us stress that another problem related to $Q$ is its non-positivity on $\mathcal{F}\setminus\{\emptyset\}$. To see this, let $\mathcal{G}' = \{\mathbf{1}_B : B \in \mathcal{U}\}$ and consider the restriction of the lower price assessment (30) defined as $\pi' = \pi_{|\mathcal{G}'}$. In this case, since the lower price assessment is consistent with the discounted Choquet expectation with respect to $Q$, no generalized arbitrage in the form of (ii.a) in Theorem 3 can be built, as it would imply the violation of the generalized avoiding Dutch book condition. On the other hand, a generalized arbitrage in the form of (ii.b) in Theorem 3 can be found as the following Example 8 shows.

**Example 8** Let $\Omega, m_1, m_2, m_3, m_4, 1 + r, Q, Q'$ and $S^1_0$ be defined as in Example 7. Denote by $\lambda''$ a portfolio on $\mathcal{G}'$, whose components are $\lambda''_{S^0}, \lambda''_i, \lambda''_{ij}, \lambda''_{ijk}, \lambda''_{1234}$, referring to $\mathbf{1}_{[i]}, \mathbf{1}_{[i,j]}, \mathbf{1}_{[i,j,k]}, \mathbf{1}_\Omega$, respectively. If we take $\lambda''_{S^0} = 0, \lambda''_1 = \lambda''_2 = \lambda''_3 = \lambda''_4 = \lambda''_{13} = \lambda''_{14} = \lambda''_{23} = \lambda''_{24} = \lambda''_{124} = \lambda''_{134} = \lambda''_{1234} = 10, \lambda_{12} = \lambda_{13} = \lambda_{123} = \lambda_{1234} = -10$ we get that $\pi_{\lambda''} = 0$ and also
One immediately observes that the portfolio $\lambda''$ gives rise to a generalized arbitrage of type (ii.b) in Theorem 3.

The previous examples show that if we seek a positive belief function whose discounted Choquet expectation is consistent with the generalized no-arbitrage principle and the assessment $\pi(S_1^1) = S_0^1$, then we must depart from $Q$.

It is easily seen that every equivalent martingale measure $Q \in \mathcal{Q}$ is a belief function, positive on $\mathcal{F}\{\emptyset\}$, nevertheless, the choice of a particular $Q_0$ in the class $\mathcal{Q}$ is a problematic task, as one needs to provide a reasonable choice criterion. For instance, a possible choice is $Q_0 = \frac{1}{|\text{ext}(\text{cl}(Q))|} \sum_{Q \in \text{ext}(\text{cl}(Q))} Q$, which belongs to $\mathcal{Q}$ since it is a strict convex combination of elements of $\text{ext}(\text{cl}(Q))$.

Once an equivalent martingale measure $Q_0$ has been chosen, some of the information contained in the class $\mathcal{Q}$ can be preserved if we consider [see, e.g., Huber (1981), Walley (1991)] the $\epsilon$-contamination of $Q_0$ with respect to $\text{cl}(Q)$, where $\epsilon \in (0, 1)$. This amounts to consider the closed subset of $\mathcal{Q}$ given by

$$Q_\epsilon = \{Q \in \mathcal{Q} : Q = (1 - \epsilon)Q_0 + \epsilon Q', Q' \in \text{cl}(Q)\},$$

whose lower envelope $\underline{Q}_\epsilon$ is defined, for every $A \in \mathcal{F}$, as

$$\underline{Q}_\epsilon(A) = (1 - \epsilon)Q_0(A) + \epsilon Q(A).$$

In particular, since $\underline{Q}_\epsilon$ is the strict convex combination of the two belief functions $Q_0$ and $Q$, we have that $\underline{Q}_\epsilon$ is a belief function which, in turn, is strictly positive over $\mathcal{F}\{\emptyset\}$, as $Q_0$ is.

The idea is to directly use $\underline{Q}_\epsilon$ in order to derive a lower pricing rule through a discounted Choquet expectation. In the light of previous section, the obtained lower pricing rule would be acceptable if it satisfied the generalized no-arbitrage condition expressed by Theorem 3.

The following Example 9 shows that, even if $\underline{Q}_\epsilon$ is a positive belief function, the corresponding lower pricing rule obtained as discounted Choquet expectation is not consistent with $\pi(S_1^1) = S_0^1$. In particular, this means that we can build portfolios resulting in generalized arbitrage opportunities and generalized Dutch books. Indeed, the lower pricing rule with respect to $\underline{Q}_\epsilon$ can be expressed as a convex combination of the discounted Choquet expectations with respect to $Q$ and $Q_0$. The problem with $\underline{Q}_\epsilon$ is that $\text{cl}(Q)_\epsilon$ is strictly contained in $\text{core}(Q)_\epsilon$, as discussed in Remark 1.

**Example 9** Consider $\Omega = \{1, 2, 3, 4\}$, $m_1 = 4$, $m_2 = 2$, $m_3 = \frac{1}{2}$, $m_4 = \frac{1}{4}$, $1 + r = 1$, $\mathcal{Q}$ and $Q$ of Example 1, and let $Q_0$ be an arbitrary element of $\mathcal{Q}$. Take $\epsilon \in (0, 1)$ and consider the $\epsilon$-contamination class of $Q_0$ with respect to $\text{cl}(Q)$, whose lower envelope is $Q_\epsilon = (1 - \epsilon)Q_0 + \epsilon Q$.

The lower pricing rule obtained as the discounted Choquet expectation with respect $Q_\epsilon$ is not consistent with the assessment $\pi(S_1^1) = S_0^1$ as it should be

$$(1 + r)^{-1}C_{\underline{Q}_\epsilon}(S_1^1) = S_0^1.$$
which is equivalent, since $S_0^1 = s > 0$, to

$$\mathbb{C}_Q \left( \frac{S_1^1}{S_0^1} \right) = 1 + r.$$ 

Nevertheless, due to linearity of the Choquet integral with respect to the integrating capacity [see, e.g., Denneberg (1994), Grabisch (2016)], we have that

$$\mathbb{C}_Q \left( \frac{S_1^1}{S_0^1} \right) = (1 - \epsilon) \mathbb{C}_0 \left( \frac{S_1^1}{S_0^1} \right) + \epsilon \mathbb{C}_Q \left( \frac{S_1^1}{S_0^1} \right) = (1 - \epsilon) + \epsilon \frac{54}{105} < 1 + r,$$

since $\mathbb{C}_Q \left( \frac{S_1^1}{S_0^1} \right) = \frac{54}{105}$.

We stress that the result in Example 9 does not depend on $Q_0$. Indeed, for every choice of $Q_0 \in \mathbb{Q}$, then $Q_\epsilon$ gives rise to a lower pricing rule that assigns positive lower price to every security with non-negative and non-null payoff but is not consistent with $\pi(S_1^1) = S_0^1$. Indeed, the issue of $\epsilon$-contamination in this case is due to the fact that $\text{cl}(Q)$ is strictly contained in $\text{core}(Q)$, so we have a situation like that described in Remark 1 for the corresponding Choquet expectation $\mathbb{C}_Q$.

A possible way to keep the assessment $\pi(S_1^1) = S_0^1$ and fulfill the generalized no-arbitrage principle is to look for an inner approximation $\widehat{\text{Bel}}$ of $Q$ satisfying the generalized avoiding Dutch book condition and then define, for $\epsilon \in (0, 1)$,

$$\widehat{\text{Bel}}_\epsilon = (1 - \epsilon) Q_0 + \epsilon \widehat{\text{Bel}}.$$ 

(33)

The belief functions $\widehat{\text{Bel}}$ and $\widehat{\text{Bel}}_\epsilon$ will be referred to as inner approximating martingale belief function and equivalent inner approximating martingale belief function, according to the following definition.

**Definition 3** A belief function $\widehat{\text{Bel}}$ on $\mathcal{F}$ is called:

- an **inner approximation** for $Q$ if, for every $A \in \mathcal{F}$, it holds that $Q(A) \leq \widehat{\text{Bel}}(A)$;

- a **martingale belief function** if $$(1 + r)^{-1} \mathbb{C}_{\widehat{\text{Bel}}}(S_0^1) = S_0^1;$$

- an **inner approximating martingale belief function** for $Q$ if it is both an inner approximation for $Q$ and a martingale belief function;

- an **equivalent inner approximating martingale belief function** for $Q$ if it is an inner approximating martingale belief function for $Q$ and $\widehat{\text{Bel}}_\sim P$.

Our goal is to select a belief function $\widehat{\text{Bel}} \in \mathcal{B}(\Omega, \mathcal{F})$ which is an inner approximating martingale belief function for $Q$ and is closest to $Q$ with respect to a suitable distance $d$ defined on the set $\mathcal{B}(\Omega, \mathcal{F})$.

A different approach would be to look for an outer approximation as done in Miranda et al. (2021) and Montes et al. (2018, 2019), i.e., a belief function such that $\widehat{\text{Bel}} \leq Q$, where comparisons are pointwise on $\mathcal{F}$. Our choice of an inner approximation, rather than an outer
approximation, is due to the fact that the latter would imply a greater dilation in the price interval with respect to the inner approximation. Moreover, an outer approximation $\hat{Bel}$ would induce $\text{core}(\hat{Bel})$ that contains $\text{core}(Q)$, and we already know that $\text{cl}(Q) \subseteq \text{core}(Q)$, so the martingale property for the stock cannot be enforced.

The entire focus of the paper is on Dempster–Shafer theory since $Q$ is a belief function, thus the aim is to arrive to a consistent lower pricing rule inside of the same framework. However, the same process could be carried out in the more general framework of 2-monotone capacities, as shown in Montes et al. (2018, 2019).

It is easy to see that every $Q \in \text{cl}(Q)$ is an inner approximating martingale belief function for $Q$, while every $Q \in Q$ is an equivalent inner approximating martingale belief function for $Q$. Thus, a possible procedure, which is that normally carried out in finance, would be to choose a probability measure $Q \in Q$ that is a trivial equivalent inner approximating martingale belief function for $Q$. Nevertheless, this procedure suffers from the well-known problem of providing a criterion to choose a $Q \in Q$ and further it does not allow to model frictions in the market as bid-ask spreads. Hence, we avoid this trivial case and look for a non-additive equivalent inner approximating martingale belief function.

In order to choose an approximation, following Montes et al. (2018, 2019), two possible choices for the distance $d$ are

$$d_1(Bel_1, Bel_2) = \sum_{A \in \mathcal{F}} |Bel_1(A) - Bel_2(A)|,$$

$$d_2(Bel_1, Bel_2) = \sum_{A \in \mathcal{F}} (Bel_1(A) - Bel_2(A))^2,$$

that can be seen as distances induced by the $L^1$ and $L^2$ norms on $[0, 1]^{\mathcal{F}}$. In particular, $d_2$ is the squared Euclidean distance.

Thus, for a fixed distance $d$, an optimal inner approximating martingale belief function $\hat{Bel}$ for $Q$ can be found by solving the following optimization problem:

minimize $d(\hat{Bel}, Q)$

subject to:

$$\hat{Bel}(A) \geq Q(A), \quad \text{for every } A \in \mathcal{F},$$

$$(1 + r)^{-1} \mathbb{C}_{\hat{Bel}}(S_1^1) = S_0^1,$$

$$\hat{Bel} \in \mathcal{B}(\Omega, \mathcal{F}).$$

Denote as before $\mathcal{U} = \mathcal{F}\setminus\{\emptyset\}$. The searched $\hat{Bel}$ is completely characterized by its Möbius inverse $\hat{\mu}$ that must satisfy $\hat{\mu}(\emptyset) = \hat{Bel}(\emptyset) = Q(\emptyset) = 0.$ Moreover, since $S_0^1 = s > 0$, it holds that

$$(1 + r)^{-1} \mathbb{C}_{\hat{Bel}}(S_1^1) = S_0^1$$

is equivalent to

$$\mathbb{C}_{\hat{Bel}} \left( \begin{array}{c} S_1^1 \\ S_0^1 \end{array} \right) = 1 + r,$$

where

$$\mathbb{C}_{\hat{Bel}} \left( \begin{array}{c} S_1^1 \\ S_0^1 \end{array} \right) = \sum_{B \in \mathcal{U}} \left( \begin{array}{c} S_1^1 \\ S_0^1 \end{array} \right)^L (B) \hat{\mu}(B) = \sum_{i=1}^{n} m_i \left( \sum_{(i) \subseteq B \subseteq \{1, \ldots, i\}} \hat{\mu}(B) \right).$$
Hence, the above problem (36) is equivalent to the following optimization problem with linear constraints, whose unknowns are the values of $\hat{\mu}$ on $U$:

$$\begin{align*}
\text{minimize} & \quad d(\hat{\text{Bel}}, Q) \\
\text{subject to:} & \\
& \sum_{\emptyset \neq B \subseteq A} \hat{\mu}(B) \geq Q(A), \quad \text{for every } A \in U, \\
& \sum_{i=1}^{n} m_i \left( \sum_{\emptyset \neq B \subseteq \{1,\ldots,i\}} \hat{\mu}(B) \right) = 1 + r, \\
& \sum_{B \in U} \hat{\mu}(B) = 1, \\
& \hat{\mu}(B) \geq 0, \quad \text{for every } B \in U.
\end{align*}$$

(37)

It is easily seen that every $Q \in \text{cl}(Q)$ gives rise to a Möbius inverse that satisfies all the constraints in problem (37). Hence, the feasible region of problem (37) is a non-empty convex compact subset of $\mathbb{R}^{2^n-1}$, endowed with the product topology.

We notice that, as already pointed out in Montes et al. (2018, 2019), if we consider the distance $d_1$ and take into account that $\hat{\text{Bel}} \geq Q$, we have that

$$d_1(\hat{\text{Bel}}, Q) = \sum_{A \in U} \left[ \left( \sum_{\emptyset \neq B \subseteq A} \hat{\mu}(B) \right) - Q(A) \right]$$

(38)

where $\sum_{A \in U} Q(A)$ is a constant, since $Q$ is given. Therefore, problem (37) reduces to a linear programming problem.

The following example shows the computation of an equivalent inner approximating martingale belief function, relying on the distance $d_1$.

**Example 10** Consider $\Omega = \{1, 2, 3, 4\}$, $m_1 = 4$, $m_2 = 2$, $m_3 = \frac{1}{2}$, $m_4 = \frac{1}{4}$, $1 + r = 1$, and $Q$ and $\overline{Q}$ of Example 1.

An inner approximating martingale belief function $\hat{\text{Bel}}$ minimizing the $d_1$ distance is reported below

| $\mathcal{F}$ | $\emptyset$ | 1 | 2 | 3 | 4 | 12 | 13 | 14 | 23 | 24 | 34 | 123 | 124 | 134 | 234 | $\Omega$ |
|--------------|-------|---|---|---|---|-----|-----|-----|----|----|----|------|------|------|-----|-------|
| $Q$          | 0     | 0 | 0 | 0 | 0 | 15  | 0   | 0   | 0  | 0  | 0  | 60   | 21   | 15  | 60  | 84   | 1     |
| $\mu$        | 0     | 0 | 0 | 0 | 0 | 15  | 0   | 0   | 0  | 0  | 0  | 60   | 21   | 15  | 60  | 105  | 6     |
| $\hat{\mu}$  | 0     | 21 | 0 | 0 | 0 | 0   | 0   | 0   | 0  | 0  | 0  | 60   | 0    | 0   | 0   | 105  | 1    |
| $\hat{\text{Bel}}$ | 0    | 21 | 0 | 0 | 0 | 21  | 21  | 21  | 0  | 0  | 0  | 105  | 21   | 21  | 21  | 81   | 84   | 1    |

for which we have that $d_1(\hat{\text{Bel}}, Q) = \frac{96}{105}$.

Define $Q_0 = \frac{1}{|\text{ext}(\text{cl}(Q))|} \sum_{Q \in \text{ext}(\text{cl}(Q))} Q$, whose values are reported below

$\hat{\text{Bel}}$
Finally, for $\epsilon = \frac{1}{2}$, define $\hat{B}el_\epsilon = \frac{1}{2} Q_0 + \frac{1}{2} \hat{B}el$, whose values are reported below

| $\mathcal{F}$ | $\emptyset$ | 1 | 2 | 3 | 4 | 12 | 13 | 14 | 23 | 24 | 34 | 123 | 124 | 134 | 234 | $\Omega$ |
|----------------|----------|----|----|----|----|-----|----|----|-----|----|----|-------|------|------|-------|----|
| $Q_{1,3}$     | 0        | 15 | 0  | 90 | 0   | 15  | 1   | 15 | 0   | 90  | 0   | 1    | 15   | 0    | 15    | 0   |
| $Q_{1,4}$     | 0        | 0   | 1  | 105| 84  | 21  | 1   | 84 | 21  | 1   | 0   | 105  | 0    | 1    | 105   | 0   |
| $Q_{2,3}$     | 0        | 0   | 25 | 70 | 0   | 25  | 0   | 25 | 0   | 70  | 0   | 1    | 35   | 0    | 35    | 1   |
| $Q_{2,4}$     | 0        | 45  | 0  | 60 | 45  | 0   | 60  | 45 | 0   | 60  | 45  | 1    | 60   | 1    | 60    | 1   |
| $Q_0$         | 0        | 56  | 160| 112| 116 | 196 | 190 | 224| 307 | 276 | 260 | 340  | 384  | 1    | 384   | 1   |

| $\mathcal{F}$ | $\emptyset$ | 1 | 2 | 3 | 4 | 12 | 13 | 14 | 23 | 24 | 34 | 123 | 124 | 134 | 234 | $\Omega$ |
|----------------|----------|----|----|----|----|-----|----|----|-----|----|----|-------|------|------|-------|----|
| $\hat{B}el_\epsilon$ | 0 | 60 | 40 | 80 | 72 | 100 | 140 | 132 | 120 | 112 | 180 | 172 | 332 | 360 | 1   |

We have that $\hat{B}el_\epsilon$ is an equivalent inner approximating martingale belief function for $Q$, furthermore, it is the lower envelope of the class of probability measures on $\mathcal{F}$

$$\hat{Q}_\epsilon = \{ Q \in P(\Omega, \mathcal{F}) : Q = (1 - \epsilon) Q_0 + \epsilon Q' \text{, } Q' \in \text{core}(\hat{B}el) \}.$$

A direct computation shows that

$$\mathbb{C}_{\hat{B}el_\epsilon} \left( \frac{S_1}{S_0} \right) = \min_{Q \in \hat{Q}_\epsilon} \mathbb{E}_Q \left( \frac{S_1}{S_0} \right) = 1 + r.$$

The use of $d_1$ has been justified in Miranda et al. (2021) and Montes et al. (2018, 2019) since this distance is the most intuitive as it measures the imprecision added when we replace $Q$ with $\hat{B}el$. Despite using $d_1$ we get a linear programming problem the main disadvantage is that the optimal solution is generally not unique, as shown in the following example.

**Example 11** Consider $\Omega = \{ 1, 2, 3, 4 \}$, $m_1 = 5$, $m_2 = 3$, $m_3 = 2$, $m_4 = \frac{1}{2}$ and $1 + r = 4$. According to Theorem 1, we have $I = \{ 1 \}$, $J = \{ 2, 3, 4 \}$ and $\text{ext}(\text{cl}(Q)) = \{ Q_{1,2}, Q_{1,3}, Q_{1,4} \}$ inducing $\underline{Q}$ and $\mu$ reported below

| $\mathcal{F}$ | $\emptyset$ | 1 | 2 | 3 | 4 | 12 | 13 | 14 | 23 | 24 | 34 | 123 | 124 | 134 | 234 | $\Omega$ |
|----------------|----------|----|----|----|----|-----|----|----|-----|----|----|-------|------|------|-------|----|
| $Q_{1,2}$     | 0        | 9  | 9  | 0  | 0  | 1   | 9   | 9   | 9   | 0  | 1   | 9    | 9   | 9    | 1    |
| $Q_{1,3}$     | 0        | 1  | 6  | 0  | 12 | 1   | 18  | 17  | 17  | 6  | 12  | 18   | 17  | 17   | 1    |
| $Q_{1,4}$     | 0        | 0  | 4  | 14 | 14 | 0   | 4   | 14  | 14  | 1   | 4   | 14   | 14  | 1    | 1    |
| $Q$           | 0        | 9  | 0  | 0  | 0  | 0   | 9   | 0   | 0   | 0  | 0   | 9    | 0   | 0    | 0    |
| $\mu$         | 0        | 9   | 0  | 0  | 0  | 0   | 9   | 0   | 0   | 0  | 0   | 9    | 0   | 0    | 0    |
The following two belief functions have Möbius inverse minimizing the distance $d_1$ and it holds that $d_1(\hat{Bel}_1, Q) = d_1(\hat{Bel}_2, Q) = \frac{20}{18}$. ♦

The problem of not uniqueness of the solution could be dealt with some further procedures adopting different criteria as, for example, taking into account measures of specificity (Yager, 1983) or other distances (Miranda et al., 2021) in order to select a unique solution among those that are optimal according to $d_1$.

The main feature of distance $d_2$ is that problem (37) admits a unique optimal solution as the objective function turns out to be strictly convex [see, e.g., Montes et al. (2018, 2019)]. In other terms, the choice of $d_2$ amounts in computing the orthogonal projection of $Q$ onto the set of inner approximating martingale belief functions for $Q$.

Example 12 Consider $\Omega, m_1, m_2, m_3, m_4$, and $1+r$ as in Example 11. In this case, the unique optimal solution $\hat{Bel}$ minimizing $d_2$ has Möbius inverse $\hat{\mu}$ such that $\hat{\mu}(1) = 0.628655$, $\hat{\mu}(2) = 0.0087719$, $\hat{\mu}(12) = 0.149123$, $\hat{\mu}(23) = 0.18421$, $\hat{\mu}(234) = 0.0292399$, and 0 otherwise. In this case we have $d_2(\hat{Bel}, Q) = 0.169591$. ♦

Distances $d_1$ and $d_2$ face the problem of approximation from a metric point of view. Another possibility is to take as $d$ the Bregman divergence induced by a bounded (strictly) proper scoring rule [see, e.g., Censor and Zenios (1997), Predd et al. (2009)]. Indeed, as shown in Petturiti and Vantaggi (2023), proper scoring rules allow to introduce a notion of coherence for belief functions through a penalty criterion that generalizes classical results for probabilities [see, e.g., Predd et al. (2009)]. In particular, as shown in Gilio and Sanfilippo (2011), every bounded proper scoring rule gives rise to a Bregman divergence that can be used in the approximation. It actually turns out that $d_2$ is the Bregman divergence induced by the Brier quadratic scoring rule, so it has a justification in terms of the penalty criterion for belief functions [see Petturiti and Vantaggi (2023)]. We stress that if we take a Bregman divergence for $d$, then (37) is generally a non-linear problem with linear constraints.

Furthermore, besides minimizing a distance or a divergence, other approaches are available, like minimizing a measure of nonspecificity (or imprecision) as done in Denœux (2006).

Up to now, we have considered only the lower price assessment $\pi(S^1_1) = S^0_1$. Nevertheless, we also know that $\max_{Q \in \text{cl}(Q)} (1+r)^{-1}E_{Q}(S^1_1) = S^0_1$, thus we could consider an upper price assessment for the stock. If we further impose to respect the upper price assessment $\overline{\pi}(S^1_1) = S^0_1$, then the notion of martingale belief function given in Definition 3 can be strengthened as follows.

Definition 4 A belief function $\hat{Bel}$ on $\mathcal{F}$ is called:

- a strong martingale belief function if

  $$(1+r)^{-1}C_{\hat{Bel}}(S^1_1) = S^0_1 \quad \text{and} \quad (1+r)^{-1}C_{\hat{Bel}}(-S^1_1) = -S^0_1;$$
– an inner approximating strong martingale belief function for $Q$ if it is both an inner approximation for $Q$ and a strong martingale belief function;
– an equivalent inner approximating strong martingale belief function for $Q$ if it is an inner approximating strong martingale belief function for $Q$ and $\hat{\text{Bel}} \sim P$.

Still referring to a distance $d$ defined on $B(\Omega, \mathcal{F})$, an optimal inner approximating strong martingale belief function $\hat{\text{Bel}}$ for $Q$ can be found by solving the following optimization problem:

$$\begin{align*}
\text{minimize} & \quad d(\hat{\text{Bel}}, Q) \\
\text{subject to:} & \quad \begin{cases} 
\hat{\text{Bel}}(A) \geq Q(A), & \text{for every } A \in \mathcal{F}, \\
(1 + r)^{-1} C_{\hat{\text{Bel}}}(S_1^1) = S_0^1, \\
(1 + r)^{-1} C_{\hat{\text{Bel}}}(-S_1^1) = -S_0^1, \\
\hat{\text{Bel}} \in B(\Omega, \mathcal{F}). 
\end{cases}
\end{align*}$$

(39)

Also in this case, problem (39) can be reformulated as follows

$$\begin{align*}
\text{minimize} & \quad d(\hat{\text{Bel}}, Q) \\
\text{subject to:} & \quad \begin{cases} 
\sum_{B \subseteq A, \emptyset \neq B} \hat{\mu}(B) \geq Q(A), & \text{for every } A \in \mathcal{U}, \\
\sum_{i=1}^{n} m_i \left( \sum_{B \subseteq \{i, \ldots, j\}} \hat{\mu}(B) \right) = 1 + r, \\
\sum_{i=1}^{n} m_i \left( \sum_{B \subseteq \{i, \ldots, n\}} \hat{\mu}(B) \right) = 1 + r, \\
\sum_{B \in \mathcal{U}} \hat{\mu}(B) = 1, \\
\hat{\mu}(B) \geq 0, & \text{for every } B \in \mathcal{U}. 
\end{cases}
\end{align*}$$

(40)

The following theorem states that any inner approximating strong martingale belief function $\hat{\text{Bel}}$ for $Q$ is actually a probability measure belonging to $\text{cl}(Q)$.

**Theorem 4** For every distance $d$ defined on $B(\Omega, \mathcal{F})$, the set of feasible solutions of problem (39) is $\text{cl}(Q)$. Further, if $d = d_1$ then the set of optimal solutions of problem (39) coincides with $\text{cl}(Q)$, while if $d = d_2$ then there is a unique optimal solution.

**Proof** First notice that inner approximating strong martingale belief functions for $Q$, that is feasible solutions of problem (39), are in one-to-one correspondence with feasible solutions of problem (40). Define $\mathcal{V} = \mathcal{U} \setminus \{\iota : i \in \Omega\}$. Subtracting memberwise the second equation to the third equation of problem (40) we get

$$\sum_{B \in \mathcal{V}} \left( \max_{i \in B} m_i - \min_{i \in B} m_i \right) \hat{\mu}(B) = 0.$$ 

Hence, since for every $B \in \mathcal{V}$, $\left( \max_{i \in B} m_i - \min_{i \in B} m_i \right) > 0$ and $\hat{\mu}(B) \geq 0$, any feasible solution of problem (40) is such that $\hat{\mu}(B) = 0$, for every $B \in \mathcal{V}$. In turn, this implies that any feasible solution of problem (40) is the Möbius inverse of a probability measure which is a feasible solution of problem (39). Thus, if $\hat{\text{Bel}}$ is a feasible solution of problem (39)
we have $C\hat{Bel}\left(\frac{s^1}{x_0}\right) = E\hat{Bel}\left(\frac{s^1}{x_0}\right) = 1 + r$, implying that $\hat{Bel} \in \text{cl}(Q)$. Vice versa, every element of $\text{cl}(Q)$ is easily seen to be a feasible solution of problem (39).

If $d = d_1$, since every feasible solution $\hat{Bel}$ of problem (39) is a probability measure with Möbius inverse $\hat{\mu}$, by (38) we get that

$$d_1(\hat{Bel}, Q) = \sum_{\{i\} \in U} 2^{\Omega \setminus \{i\}} \hat{\mu}(\{i\}) - \sum_{A \in E} Q(A) = 2^{n-1} - \sum_{A \in E} Q(A),$$

that does not depend on $\hat{Bel}$. Hence, all the elements of $\text{cl}(Q)$ are optimal according to $d_1$.

If $d = d_2$, then the uniqueness of the optimal solution immediately follows since the objective function of (39) is strictly convex.

Hence, using $d_1$ any element of $\text{cl}(Q)$ turns out to be optimal, while using $d_2$ we get the orthogonal projection of $Q$ onto the set of inner approximating strong martingale belief functions for $Q$, which is by Theorem 4 the set $\text{cl}(Q)$.

Example 13 Consider $\Omega, m_1, m_2, m_3, m_4,$ and $1 + r$ as in Example 11. Using $d_1$, the set of optimal inner approximating strong martingale belief functions for $Q$ is $\text{cl}(Q)$ and for every $\hat{Bel} \in \text{cl}(Q)$ we have $d_1(\hat{Bel}, Q) = \frac{8}{3}$.

On the other hand, if we use the distance $d_2$ we have a unique optimal solution which is the following inner approximating strong martingale belief function (probability measure)

$$\hat{Bel} = 0.638889 0.188956 0.102339 0.0701755$$

for which we have $d_2(\hat{Bel}, Q) = 0.572124$.

Let us stress that, for a fixed $Q_0 \in Q$, if $\hat{Bel}$ is an inner approximating strong martingale belief function (probability measure), then $\text{core}(\hat{Bel}) = \{\hat{Bel}\}$. In this case $\hat{Q}_e$ reduces to the singleton $\hat{Q}_e = \{(1-\epsilon)Q_0 + \epsilon \hat{Bel}\}$, thus the lower envelope $Q_\epsilon$ is an equivalent martingale measure, that is $Q_\epsilon = (1-\epsilon)Q_0 + \epsilon \hat{Bel} \in Q$.

The following proposition states that, both for $d_1$ and $d_2$, an optimal inner approximating martingale (strong martingale) belief function does not dominate any other inner approximating martingale (strong martingale) belief function. This is inline with results proved in Montes et al. (2018, 2019), where the problem of finding an outer approximating belief function for a lower probability is studied.

Proposition 1 Let $d = d_1$ or $d = d_2$. If $\hat{Bel}$ is an optimal solution of problem (36) [problem (39)] then there is no feasible solution $\hat{Bel}'$ of problem (36) [problem (39)] such that $\hat{Bel}' \neq \hat{Bel}$ and $Q \leq \hat{Bel}' \leq \hat{Bel}$.

Proof The proof can be obtained by a straightforward adaptation of Lemma 14 in Montes et al. (2018).

6 Conclusions

In this paper we characterize the lower envelope of the set of equivalent martingale measures arising in a one-period $n$-nomial market model, showing that it is a belief function. This
suggests to use such lower envelope to derive a lower pricing rule accommodating frictions in the market, in the form of bid-ask spreads.

For that we formulate a general one-period pricing problem and prove a version of the first fundamental theorem of asset pricing in the context of belief functions. The theorem relies on a generalized definition of arbitrage assuming partially resolving uncertainty, according to Jaffray.

Finally, we cope with the derivation of a generalized arbitrage-free lower pricing rule stemming from the “risk-neutral” belief function $Q$ arising in the one-period $n$-nomial market model. This amounts in choosing an equivalent martingale measure and in producing an $\epsilon$-contamination relying on a suitable inner approximation of $Q$.

As a topic of future research, we aim at extending the introduced notion of arbitrage to the multi-period case. For this to be possible, the issue of dynamic consistency needs to be taken into account (Asano & Kojima, 2019; Kaste et al., 2014).

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