Cauchy-Carlitz numbers

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Abstract
In 1935 Carlitz introduced Bernoulli-Carlitz numbers as analogues of Bernoulli numbers for the rational function field \( \mathbb{F}_r(T) \). In this paper, we introduce Cauchy-Carlitz numbers as analogues of Cauchy numbers. By using Stirling-Carlitz numbers, we give their arithmetical and combinatorial properties and relations with Bernoulli-Carlitz numbers for \( \mathbb{F}_r(T) \). Several new identities are also obtained by using Hasse-Teichmüller derivatives.

1 Introduction
In 1935, L. Carlitz ([2]) introduced analogues of Bernoulli numbers for the rational function field \( K = \mathbb{F}_r(T) \), which are called Bernoulli-Carlitz numbers now. He proved an analogue of the von Staudt-Clausen theorem ([3],[4]). Some identities for Bernoulli-Carlitz numbers were found in [6]. In [11], explicit formulae of Bernoulli-Carlitz numbers were given by using the basic properties of the Hasse-Teichmüller derivatives. In [17], it was shown a necessary and sufficient condition for a nonzero prime ideal of the rational function.
field divides the $n$-th Bernoulli-Carlitz number. A recent exposition of Bernoulli-Carlitz numbers can be seen in [15]. We refer to [7] for an exposition and the modern notation. The Carlitz exponential $e_C(x)$ is defined by

$$ e_C(x) = \sum_{i=0}^{\infty} \frac{x^r^i}{D_i}, \quad (1) $$

where $D_i = [i][i-1] \cdots [1]^i \cdot \cdots \cdot [1] (i \geq 1)$ with $D_0 = 1$, and $[i] = T^r^i - T$. The Carlitz exponential $e_C(x)$ is defined by

$$ e_C(x) = \sum_{i=0}^{\infty} (-1)^i \frac{x^r^i}{L_i}, \quad (2) $$

where $L_i = [i][i-1] \cdots [1] (i \geq 1)$ with $L_0 = 1$. Notice that

$$ e_C(\log_C(x)) = \log_C(e_C(x)) = x. \quad (3) $$

eC(x) and logC(x) have the functional equations

$$ e_C(Tx) = Te_C(x) + e_C(x)^r \quad (4) $$

and

$$ T\log_C(x) = \log_C(Tx) + \log_C(x^r), \quad (5) $$

respectively.

The Carlitz factorial $\Pi(i)$ is defined by

$$ \Pi(i) = \prod_{j=0}^{m} D_j^{c_j} \quad (6) $$

for a non-negative integer $i$ with $r$-ary expansion:

$$ i = \sum_{j=0}^{m} c_j r^j \quad (0 \leq c_j < r). \quad (7) $$

Therefore,

$$ \Pi(i) = \prod_{j=0}^{m} \left( \prod_{k=1}^{j} (T^r^k - T^r^j-k) \right)^{c_j} $$

$$ = \prod_{k=1}^{m} (T^r^k - T)^{c_k+c_{k+1}r+\cdots} = \prod_{k=1}^{m} (T^r^k - T)[i/r^k] \quad ([\cdot] \text{ denotes the floor function.}) $$

$$ = \prod_{k=1}^{m} (T^r^k - T)^{c_k+c_{k+1}r+\cdots+c_m r^{m-k}} \quad (8) $$

and

$$ \Pi(r^d - 1) = (D_0 \cdots D_{d-1})^{r^{-1}} = \frac{D_d}{L_d} \quad (d \geq 0). \quad (9) $$
2 Cauchy-Carlitz numbers

The Bernoulli-Carlitz numbers $BC_n$ are defined by

$$\frac{x}{e^C(x)} = \sum_{n=0}^{\infty} \frac{BC_n}{\Pi(n)} x^n$$

as analogues of the classical Bernoulli numbers $B_n$, defined by

$$\frac{x}{e^x - 1} = \sum_{n=0}^{\infty} \frac{B_n}{n!} x^n.$$  \hspace{1cm} (10)

As analogues of the classical Cauchy numbers $c_n$, defined by

$$\frac{x}{\log(1 + x)} = \sum_{n=0}^{\infty} \frac{c_n}{n!} x^n$$

we define the Cauchy-Carlitz numbers $CC_n$ by

$$\frac{x}{\log_C(x)} = \sum_{n=0}^{\infty} \frac{CC_n}{\Pi(n)} x^n.$$  \hspace{1cm} (11)

In addition, as analogues of the Stirling numbers of the first kind $[^n_k]$, defined by

$$\frac{(-\log(1-t))^k}{k!} = \sum_{n=0}^{\infty} [^n_k] t^n$$

we define the Stirling-Carlitz numbers of the first kind $[^n_k]_C$ by

$$\frac{(\log_C(z))^k}{\Pi(k)} = \sum_{n=0}^{\infty} [^n_k]_C z^n$$

As analogues of the Stirling numbers of the second kind $\{^n_k\}$, defined by

$$\frac{(e^t - 1)^k}{k!} = \sum_{n=0}^{\infty} \{^n_k\} t^n$$

we define the Stirling-Carlitz numbers of the second kind $\{^n_k\}_C$ by

$$\frac{(e^C(z))^k}{\Pi(k)} = \sum_{n=0}^{\infty} \{^n_k\}_C z^n$$

By the definition (14), we have

$$[^n_0]_C = 0 \ (n \geq 1), \ \ [^n_m]_C = 0 \ (n < m) \ \text{and} \ \ [^n_n]_C = 1 \ (n \geq 0).$$  \hspace{1cm} (16)
Similarly, we see
\[
\begin{align*}
\binom{n}{0}_C &= 0 \quad (n \geq 1), \\
\binom{n}{m}_C &= 0 \quad (n < m) \quad \text{and} \\
\binom{n}{n}_C &= 1 \quad (n \geq 0).
\end{align*}
\] (17)

It is known that poly-Cauchy numbers \( c^{(k)}_n \) are expressed in terms of the Stirling numbers of the first kind:
\[
c^{(k)}_n = \sum_{m=0}^{n} \left\lfloor \frac{n}{m} \right\rfloor \frac{(-1)^{n-m}}{(m+1)^k}
\] (18)

([12], Theorem 1). If \( k = 1 \), this is an explicit expression of the classical Cauchy numbers \( c_n \) ([5, Ch. VII], [14, p.1908]).

As analogues, we express Cauchy-Carlitz numbers as certain finite sums of Stirling-Carlitz numbers of the first kind. Thus, Cauchy-Carlitz numbers are calculated by this expression because Stirling-Carlitz numbers of the first kind are computed by (14).

**Theorem 1.**
\[
CC_n = \sum_{j=0}^{\infty} \frac{1}{L_j} \left[ \binom{n}{r^j - 1} \right]_C.
\] (19)

**Proof.** Note that the right-hand side of (19) is a finite sum by the second relation of (16).

Observe that
\[
\frac{z}{\log_C(z)} = \frac{e_C(\log_C(z))}{\log_C(z)} = \frac{e_C(t)}{t} \bigg|_{t = \log_C(z)} = \sum_{j=0}^{\infty} \frac{(\log_C(z))^{r^j - 1}}{D_j}
\]
\[
= \sum_{j=0}^{\infty} \frac{1}{D_j} \prod (r^j - 1) \sum_{n=0}^{\infty} \left[ \binom{n}{r^j - 1} \right]_C \frac{z^n}{\Pi(n)}
\]
\[
= \sum_{j=0}^{\infty} \frac{1}{L_j} \sum_{n=0}^{\infty} \left[ \binom{n}{r^j - 1} \right]_C \frac{z^n}{\Pi(n)}
\]
\[
= \sum_{n=0}^{\infty} \left( \sum_{j=0}^{\infty} \frac{1}{L_j} \left[ \binom{n}{r^j - 1} \right]_C \right) \frac{z^n}{\Pi(n)}.
\]

By the definition (12), we get (19).

Similarly, as an analogous expression of the classical Bernoulli numbers
\[
B_n = \sum_{m=0}^{n} \left\lfloor \frac{n}{m} \right\rfloor \frac{(-1)^{n-m}m!}{m + 1},
\]

\(^1\)In this paper we use the notation \( c^{(k)}_n \) in order to distinguish from the Cauchy numbers of high order in later section.
we get that Bernoulli-Carlitz numbers are equal to certain finite sums of Stirling-Carlitz numbers of the second kind. In particular, Bernoulli-Carlitz numbers are calculated by (15).

**Theorem 2.**

\[ BC_n = \sum_{j=0}^{\infty} \frac{(-1)^j D_j}{L_j^2} \begin{bmatrix} n \\ rj - 1 \end{bmatrix}_C. \] (20)

Let \( f(z) \) be a formal power series of the form \( f(z) = \sum_{i=0}^{\infty} f_i z^i \in \mathbb{F}_r[[z]]. \) If \( f_0 \neq 0, \) then let \( g(z) = \sum_{i=0}^{\infty} g_i z^i \) be the inverse function of \( f(z). \) Set

\[ h(z) = \sum_{n=0}^{\infty} h_n z^n := \frac{zf'(z)}{f(z)}. \] (21)

Then (10) and (12) are special examples of (21). Carlitz [2] studied the coefficients \( h_n \) in the case where \( n \) satisfies certain assumptions. For instance, he showed that if \( f_0 = 1, \) then \( h_{r^k-1} = g_k. \) Similarly, it is seen that if \( f_0 \neq 0, \) then \( h_{r^k-1} = f_0^{r^k} g_k. \)

J. A. Lara Rodríguez and D. S. Thakur [16] gave other relations as follows: Let \( l \) be a integer with \( 1 \leq l \leq r \) and \( k, k_1, \ldots, k_l \) integers with \( 0 \leq k_j \leq k \) for any \( 1 \leq j \leq l. \) Then

\[ \prod_{j=1}^{l} h_{r^k - r^{k_j}} = h_{\sum_{(r^k - r^{k_j})}}. \]

### 3 Examples

In this section we give examples of Stirling-Carlitz numbers of the first and second kind. Moreover, by using Theorem 1 and Theorem 2, we calculate examples of Cauchy-Carlitz numbers and Bernoulli-Carlitz numbers. In the rest of this section, we assume that \( r = 3. \)

Observe that

\[ \sum_{n=0}^{\infty} \begin{bmatrix} n \\ 2 \end{bmatrix}_C \frac{z^n}{\Pi(n)} = \frac{(\log_C(z))^2}{\Pi(2)} = \left(z - \frac{1}{L_1} z^3 + \frac{1}{L_2} z^9 - + \cdots \right)^2 = z^2 - \frac{2}{[1]} z^4 + \frac{1}{[1]^2} z^6 + 0 \cdot z^8 + \cdots. \]

Hence, we get

\[ \begin{bmatrix} 4 \\ 2 \end{bmatrix}_C = -\frac{2}{[1]} \Pi(4) = -2 = 1, \quad \begin{bmatrix} 6 \\ 2 \end{bmatrix}_C = \frac{1}{[1]^2} \Pi(6) = 1 \quad \text{and} \quad \begin{bmatrix} 8 \\ 2 \end{bmatrix}_C = 0. \]
By combining Theorem 1, (16), and the equality above, we obtain

\[ CC_2 = \frac{1}{L_0} \begin{bmatrix} 2 \\ 0 \end{bmatrix}_C + \frac{1}{L_1} \begin{bmatrix} 2 \\ 2 \end{bmatrix}_C = \frac{1}{T^3 - T} = \frac{1}{T^3 + 2T}, \]
\[ CC_4 = \frac{1}{L_1} \begin{bmatrix} 4 \\ 2 \end{bmatrix}_C = \frac{1}{T^3 + 2T}, \quad CC_6 = \frac{1}{L_1} \begin{bmatrix} 6 \\ 2 \end{bmatrix}_C = \frac{1}{T^3 + 2T}, \]

and

\[ CC_8 = \frac{1}{L_1} \begin{bmatrix} 8 \\ 2 \end{bmatrix}_C + \frac{1}{L_2} \begin{bmatrix} 8 \\ 8 \end{bmatrix}_C = \frac{1}{L_2} = \frac{1}{(T^3 - T)(T^9 - T)} = \frac{1}{(T^3 + 2T)(T^9 + 2T)}. \]

In the same way, by using

\[ \sum_{n=0}^{\infty} \left\{ \begin{array}{c} n \\ 2 \end{array} \right\}_C \frac{z^n}{\Pi(n)} = \left( \frac{e_C(z)}{\Pi(n)} \right)^2 = z^2 + \frac{2}{1}z^4 + \frac{1}{[1]^2}z^6 + 0 \cdot z^8 + \cdots, \]

we get

\[ \left\{ \begin{array}{c} 4 \\ 2 \end{array} \right\}_C = 2, \quad \left\{ \begin{array}{c} 6 \\ 2 \end{array} \right\}_C = 1 \quad \text{and} \quad \left\{ \begin{array}{c} 8 \\ 2 \end{array} \right\}_C = 0. \]

By combining Theorem 2, (17), and the equality above, we obtain the following:

\[ BC_2 = -\frac{D_1}{L_1^2} \left\{ \begin{array}{c} 2 \\ 2 \end{array} \right\}_C = \frac{-1}{T^3 - T} = \frac{2}{T^3 + 2T}, \]
\[ BC_4 = -\frac{D_1}{L_1} \left\{ \begin{array}{c} 4 \\ 2 \end{array} \right\}_C = \frac{1}{T^3 + 2T}, \quad BC_6 = -\frac{D_1}{L_2} \left\{ \begin{array}{c} 6 \\ 2 \end{array} \right\}_C = \frac{2}{T^3 + 2T}, \]
\[ BC_8 = -\frac{D_1}{L_1} \left\{ \begin{array}{c} 8 \\ 2 \end{array} \right\}_C + \frac{D_2}{L_2} \left\{ \begin{array}{c} 8 \\ 8 \end{array} \right\}_C = \frac{T^3 - T}{T^6 + T^4 + T^2 + 1} = \frac{1}{T^6 + T^4 + T^2 + 1}. \]

4 Hasse-Teichmüller derivatives

Let \( \mathbb{F} \) be a field of any characteristic, \( \mathbb{F}[[z]] \) the ring of formal power series in one variable \( z \), and \( \mathbb{F}((z)) \) the field of Laurent series in \( z \). Let \( n \) be a nonnegative integer. We define the Hasse-Teichmüller derivative \( H^{(n)} \) of order \( n \) by

\[ H^{(n)} \left( \sum_{m=R}^{\infty} c_m z^m \right) = \sum_{m=R}^{\infty} c_m \binom{m}{n} z^{m-n} \]

for \( \sum_{m=R}^{\infty} c_m z^m \in \mathbb{F}((z)) \), where \( R \) is an integer and \( c_m \in \mathbb{F} \) for any \( m \geq R \).

The Hasse-Teichmüller derivatives satisfy the product rule \[18\], the quotient rule \[8\] and the chain rule \[10\]. One of the product rules can be described as follows.
Lemma 1. For $f_i \in \mathbb{F}[z]$ ($i = 1, \ldots, k$) with $k \geq 2$ and for $n \geq 1$, we have

$$H^{(n)}(f_1 \cdots f_k) = \sum_{i_1, \ldots, i_k \geq 0 \atop i_1 + \cdots + i_k = n} H^{(i_1)}(f_1) \cdots H^{(i_k)}(f_k).$$

The quotient rules can be described as follows.

Lemma 2. For $f \in \mathbb{F}[z]\{0\}$ and $n \geq 1$, we have

$$H^{(n)} \left( \frac{1}{f} \right) = \sum_{k=1}^{n} \frac{(-1)^k}{f^{k+1}} \sum_{i_1, \ldots, i_k \geq 1 \atop i_1 + \cdots + i_k = n} H^{(i_1)}(f) \cdots H^{(i_k)}(f)$$

$$= \sum_{k=1}^{n} \left( \frac{n+1}{k+1} \right) \frac{(-1)^k}{f^{k+1}} \sum_{i_1, \ldots, i_k \geq 0 \atop i_1 + \cdots + i_k = n} H^{(i_1)}(f) \cdots H^{(i_k)}(f).$$

By using the Hasse-Teichmüller derivative of order $n$, some explicit expressions of Bernoulli-Carlitz numbers are obtained in [11]. In this section we study explicit expressions of Cauchy-Carlitz numbers and Cauchy numbers, by using the Hasse-Teichmüller derivatives. First, we express Cauchy-Carlitz numbers in terms of $L_i$ without using Stirling-Carlitz numbers of the first kind, which gives a new method to calculate Cauchy-Carlitz numbers.

Theorem 3. For $n \geq 1$,

$$CC_n = \Pi(n) \sum_{k=1}^{n} (-1)^k \sum_{i_1, \ldots, i_k \geq 1 \atop r^{i_1} + \cdots + r^{i_k} = n+k} \frac{(-1)^{i_1 + \cdots + i_k}}{L_{i_1} \cdots L_{i_k}}.$$

Proof. Put

$$g := \log C(z) = \sum_{j=0}^{\infty} (-1)^j \frac{z^{r^j-1}}{L_j}.$$ 

Note that

$$H^{(e)}(g)_{z=0} = \sum_{j=0}^{\infty} \frac{(-1)^j}{L_j} \left( \frac{r^j-1}{e} \right) z^{r^j-1-e} \bigg|_{z=0}$$

$$= \begin{cases} \frac{(-1)^j}{L_j} & \text{if } e = r^j - 1, \\ 0 & \text{otherwise}. \end{cases} \quad (24)$$
Thus, by using Lemma 2 (22) and (24), we have

\[
\frac{CC_n}{\Pi(n)} = \left. H^{(n)} \left( \frac{z}{\log_C(z)} \right) \right|_{z=0} = \left. H^{(n)} \left( \frac{1}{g} \right) \right|_{z=0}
\]

\[
= \sum_{k=1}^{n} (-1)^k \sum_{\epsilon_1 + \cdots + \epsilon_k \geq 1 \atop \epsilon_1, \ldots, \epsilon_k} H^{(\epsilon_1)}(g) \cdots H^{(\epsilon_k)}(g) \bigg|_{z=0}
\]

\[
= \sum_{k=1}^{n} (-1)^k \sum_{\epsilon_1, \ldots, \epsilon_k \geq 1 \atop r_1 + \cdots + r_k \geq n + k} \frac{(-1)^i_1 + \cdots + i_k}{L_1 \cdots L_{i_k}}.
\]

Example. Let \( r = 3 \) and \( n = 8 \). For \( 1 \leq k \leq n \), put

\[
S_k = \{(i_1, \ldots, i_k) \mid i_1, \ldots, i_k \geq 1, \ 3^{i_1} + \cdots + 3^{i_k} = 8 + k\}.
\]

Then \( S_k \) is empty except the cases of \( k = 1, S_1 = \{(2)\} \) and \( k = 4, S_4 = \{(1, 1, 1, 1)\} \). By Theorem 3 we get

\[
CC_8 = \Pi(8) \left( \frac{(-1)^2}{L_2} + \frac{(-1)^4}{L_4} \right)
\]

\[
= (T^3 - T)^2 \left( -\frac{1}{(T^3 - T)(T^9 - T)} + \frac{1}{(T^3 - T)^4} \right)
\]

\[
= \frac{1}{(T^3 - T)(T^9 - T)}.
\]

Recall that if \( k = 1 \), then (18) gives an explicit formula for the classical Cauchy numbers in terms of the Stirling numbers of the first kind. By applying the method for the proof of Theorem 3, we get an explicit formula for the classical Cauchy numbers without using the Stirling numbers of the first kind.

Theorem 4. For \( n \geq 1 \),

\[
c_n = (-1)^n n! \sum_{k=1}^{n} (-1)^k \sum_{i_1, \ldots, i_k \geq 2 \atop i_1 + \cdots + i_k = n + k} \frac{1}{i_1 \cdots i_k}.
\]

Proof. Put

\[
h := \log(1 + z) = \frac{z}{\log(1 + z)} = \sum_{j=0}^{\infty} (-1)^j \frac{z^j}{j+1}.
\]

Note that

\[
H^{(i)}(h) \bigg|_{z=0} = \left. \sum_{j=0}^{\infty} \frac{(-1)^j}{j+1} \binom{j}{i} z^{j-i} \right|_{z=0} = \frac{(-1)^i}{i+1}.
\]
Hence, by using Lemma 2 (22), we have

\[
\frac{c_n}{n!} = H^{(n)} \left( \frac{z}{\log(1 + z)} \right) \bigg|_{z=0} = H^{(n)} \left( \frac{1}{h} \right) \bigg|_{z=0} \\
= \sum_{k=1}^{n} (-1)^k \sum_{i_1, \ldots, i_k \geq 1 \atop i_1 + \cdots + i_k = n} H^{(i_1)}(h)|_{z=0} \cdots H^{(i_k)}(h)|_{z=0} \\
= \sum_{k=1}^{n} (-1)^k \sum_{i_1, \ldots, i_k \geq 1 \atop i_1 + \cdots + i_k = n} \frac{(-1)^{i_1+\cdots+i_k}}{(i_1+1) \cdots (i_k+1)} \\
= (-1)^n \sum_{k=1}^{n} \frac{(-1)^k}{i_1 \cdots i_k} \\
\]

\(\square\)

We can express the Cauchy numbers also in terms of the binomial coefficients. In fact, by using Lemma 2 (23) instead of Lemma 2 (22) in the proof of Theorem 4, we obtain the following:

**Proposition 1.** For \(n \geq 1\),

\[
c_n = (-1)^n n! \sum_{k=1}^{n} (-1)^k \binom{n+1}{k+1} \sum_{i_1, \ldots, i_k \geq 1 \atop i_1 + \cdots + i_k = n+k} \frac{1}{i_1 \cdots i_k}.
\]

From (13) we have

\[
\left( -\log(1 - z) \right)^k = \sum_{n=k}^{\infty} k! \binom{n}{k} \frac{z^{n-k}}{n!} = \sum_{n=0}^{\infty} \frac{k!}{(n+k)!} \binom{n+k}{k} z^n.
\]  

(25)

Applying Lemma 11 with

\[
f_1(z) = \cdots = f_k(z) = \frac{-\log(1 - z)}{z},
\]

we get

\[
\sum_{i_1, \ldots, i_k \geq 0 \atop i_1 + \cdots + i_k = n} \frac{1}{(i_1+1) \cdots (i_k+1)} = \frac{k!}{(n+k)!} \binom{n+k}{k}.
\]

(26)

Together with Proposition 11, we deduce a simple expression for Cauchy numbers in terms of the binomial coefficients and Stirling numbers of the first kind.
Proposition 2. For \( n \geq 1 \)

\[
c_n = (-1)^n \sum_{k=1}^{n} (-1)^k \binom{n+1}{k+1} \binom{n+k}{k}.
\]

By using Proposition 2 immediately we get some initial values of Cauchy numbers:

\[
c_1 = \frac{1}{2}, \quad c_2 = -\frac{1}{6}, \quad c_3 = \frac{1}{4}, \quad c_4 = -\frac{19}{30}, \quad c_5 = \frac{9}{4}, \quad c_6 = -\frac{863}{84}, \quad c_7 = \frac{1375}{24}.
\]

Define the Cauchy numbers \( c_n^{(m)} \) of order \( m \) by

\[
\left( \frac{z}{\log(1+z)} \right)^m = \sum_{n=0}^{\infty} \frac{c_n^{(m)} z^n}{n!}.
\]  \hspace{1cm} (27)

Notice that the concept of Cauchy numbers of higher order is different from that of poly-Cauchy numbers ([12]), though we use the similar notation here.

We now introduce formulae for Cauchy numbers of higher order, by using Hasse-Teichmüller derivatives. The first two formulae (Proposition 3 and Proposition 4) give expressions for Cauchy numbers in terms of multinomial coefficients, binomial coefficients, and the Stirling numbers of the first kind. The last formula (Proposition 5) gives a simple expression without using multinomial coefficients, which is useful to calculate Cauchy numbers of higher order.

Let again \( h(z) = (\log(1 + z))/z \). By applying Lemma 2 (22) with \( f(z) = h(z)^m \), we see, by (27),

\[
\frac{c_n^{(m)}}{n!} = \sum_{k=1}^{n} (-1)^k \sum_{i_1, \ldots, i_k \geq 1 \atop \sum_{i_1 \ldots i_k = n}} H^{(i_1)} (h^m) \big|_{z=0} \cdots H^{(i_k)} (h^m) \big|_{z=0}.
\]

Using Lemma 1 and the identity (26), we get an explicit formula for Cauchy numbers of higher order.

Proposition 3. For \( n \geq 1 \)

\[
c_n^{(m)} = (-1)^n \sum_{i_1, \ldots, i_k \geq 1 \atop \sum_{i_1 \ldots i_k = n}} \binom{n}{i_1 \ldots i_k} \frac{(i_1 + m) \ldots (i_k + m)}{m} \left[ \begin{array}{c} i_1 + m \\ m \end{array} \right] \cdots \left[ \begin{array}{c} i_k + m \\ m \end{array} \right].
\]

If we use Lemma 2 (23) instead of Lemma 2 (22), we obtain the following.
Proposition 4. For \( n \geq 1 \)

\[
c_n^{(m)} = (-1)^n \sum_{k=1}^{n} (-1)^k \binom{n+1}{k+1} \sum_{\substack{i_1, \ldots, i_k \geq 0 \\ i_1 + \cdots + i_k = n}} \frac{\binom{n}{i_1, \ldots, i_k}}{(n+m)^{i_1+i_2+\cdots+i_k+m}} \left[ \begin{array}{c} i_1+m \\ m \end{array} \right] \cdots \left[ \begin{array}{c} i_k+m \\ m \end{array} \right].
\]

Applying Lemma 1 with

\[
f_1(z) = \cdots = f_k(z) = \left( -\frac{\log(1-z)}{z} \right)^m,
\]

we get, by (25),

\[
\frac{(mk)!}{(n+mk)!} \left[ \begin{array}{c} n+mk \\ mk \end{array} \right] = \sum_{\substack{i_1, \ldots, i_k \geq 0 \\ i_1 + \cdots + i_k = n}} \frac{m!}{(i_1+m)!} \cdots \frac{m!}{(i_k+m)!} \left[ \begin{array}{c} i_1+m \\ m \end{array} \right] \cdots \left[ \begin{array}{c} i_k+m \\ m \end{array} \right].
\]

Multiplying the both sides of the equality above by \( n! \), we deduce a different explicit expression of \( c_n^{(m)} \) by Proposition 4.

Proposition 5. For \( n \geq 1 \)

\[
c_n^{(m)} = (-1)^n \sum_{k=1}^{n} (-1)^k \frac{\binom{n+1}{k+1}}{(n+mk)_n} \left[ \begin{array}{c} n+mk \\ mk \end{array} \right].
\]

For example, when \( m = 3 \), we have

\[
c_1^{(3)} = \frac{3}{2}, \ c_2^{(3)} = 1, \ c_3^{(3)} = 0, \ c_4^{(3)} = \frac{1}{10}, \ c_5^{(3)} = -\frac{1}{4}, \ c_6^{(3)} = \frac{16}{21}, \ c_7^{(3)} = -\frac{11}{4}, \ c_8^{(3)} = \frac{329}{30}.
\]

Define the Cauchy-Carlitz numbers \( CC_n^{(m)} \) of order \( m \) by

\[
\left( \frac{x}{\log C(x)} \right)^m = \sum_{n=0}^{\infty} \frac{CC_n^{(m)}}{\Pi(n)} x^n.
\]

In the rest of this section, we show that Cauchy-Carlitz numbers of higher order are also expressed only in terms of \( L_i \) in the same way as Theorem 3.

Let again \( g(z) = (\log C(z))/z \). Applying Lemma 2 with \( f(z) = (g(z))^m \), we get, by (28),

\[
\frac{CC_n^{(m)}}{\Pi(n)} = \sum_{k=1}^{n} (-1)^k \sum_{\substack{i_1, \ldots, i_k \geq 1 \\ i_1 + \cdots + i_k = n}} H^{(i_1)}(g^m) \big|_{z=0} \cdots H^{(i_k)}(g^m) \big|_{z=0}.
\]

By applying Lemma 1 to the right-hand side of the equality above, and using the identity (24), we get the following:
Proposition 6. For \( n \geq 1 \),
\[
CC_n = \Pi(n) \sum_{k=1}^{n} (-1)^k \sum_{i_1, \ldots, i_k \geq 1 \atop i_1 + \cdots + i_k = n} M^{(m)}(i_1) \cdots M^{(m)}(i_k),
\]
where
\[
M^{(m)}(i) = \sum_{j_1, \ldots, j_m \geq 0 \atop r_1 + \cdots + r_m = i} (-1)^{j_1 + \cdots + j_m} \frac{L_{j_1} \cdots L_{j_m}}{L_{i+m}}.
\]

5 Stirling-Carlitz numbers

One of the most useful identities of Stirling numbers is the pair of inversion properties:
\[
\sum_{m=k}^{n} (-1)^{n-m} \left[ \begin{array}{c} n \\ m \end{array} \right] \left\{ \begin{array}{c} m \\ k \end{array} \right\} = \delta_{n,k},
\]
\[
\sum_{m=k}^{n} (-1)^{m-k} \left\{ \begin{array}{c} m \\ k \end{array} \right\} \left[ \begin{array}{c} n \\ m \end{array} \right] = \delta_{n,k}.
\]

Stirling-Carlitz numbers also satisfy the similar orthogonal identities.

Theorem 5. Let \( n, k \) be nonnegative integers with \( n \geq k \). Then
\[
\sum_{m=k}^{n} \left[ \begin{array}{c} n \\ m \end{array} \right] C_{k+m}^{\left\{ \begin{array}{c} m \\ k \end{array} \right\}} = \delta_{n,k},
\]
\[
\sum_{m=k}^{n} \left\{ \begin{array}{c} m \\ k \end{array} \right\} C_{n+m}^{\left[ \begin{array}{c} n \\ m \end{array} \right]} = \delta_{n,k}.
\]

Proof. We may assume that \( k \geq 1 \) because if \( k = 0 \), then (29) and (30) are easily checked by (16) and (17). We see that
\[
z^k = \left( e_C (\log_C(z)) \right)^k
\]
\[
= \sum_{m=k}^{\infty} \left\{ \begin{array}{c} m \\ k \end{array} \right\} C \frac{\Pi(k)}{\Pi(m)} (\log_C(z))^m
\]
\[
= \sum_{m=k}^{\infty} \left\{ \begin{array}{c} m \\ k \end{array} \right\} C \frac{\Pi(k)}{\Pi(m)} \sum_{n=m}^{\infty} \left[ \begin{array}{c} n \\ m \end{array} \right] C \frac{\Pi(m)}{\Pi(n)} z^n
\]
\[
= \sum_{n=k}^{\infty} \frac{\Pi(k)}{\Pi(n)} z^n \sum_{m=k}^{n} \left[ \begin{array}{c} n \\ m \end{array} \right] C \left\{ \begin{array}{c} m \\ k \end{array} \right\} C,
\]
which implies (29). In the same way, we deduce (30). \( \square \)
In the rest of this section we introduce more properties on Stirling-Carlitz numbers of the first and second kind \( [n \atop m]_C \) and \( \{ n \atop m \}_C \) in the case where \( n \) and \( m \) satisfy certain conditions on the sum of digits. When a nonnegative integer \( i \) is expressed as \( r \)-ary expansion (7), we write the sum of digits of \( i \) by

\[
\lambda(i) := \sum_{j=0}^{m} c_j \in \mathbb{Z}.
\]

First we show that if \( \lambda(n) = \lambda(m) = 1 \), then the Stirling-Carlitz numbers are expressed as certain products of the terms of \( D_i \) and \( L_j \).

**Proposition 7.** Let \( a \) and \( b \) be nonnegative integers with \( a \geq b \). Then we have the following:

\[
\begin{align*}
[r^a \atop r^b]_C &= \frac{D_a}{D_b} \cdot \frac{(-1)^{a-b}}{L_{a-b}^r}, \quad (31) \\
\{r^a \atop r^b\}_C &= \frac{D_a}{D_b} \cdot \frac{1}{D_{a-b}^r}. \quad (32)
\end{align*}
\]

**Proof.** Note for any \( i \geq 0 \) that \((-1)^{r^i} = -1\). In fact, if \( r \) is even, then \(-1 = 1\) because the characteristic of \( \mathbb{F}_r \) is 2. Since

\[
\left( \log C(z) \right)^r = \sum_{i=0}^{\infty} \left[ r^i \atop r^b \right]_C \frac{D_b}{D_i} z^{r^i}
\]

\[
= \sum_{i=0}^{\infty} \frac{(-1)^i}{L_i^r} z^{r^{i+b}} = \sum_{i=b}^{\infty} \frac{(-1)^{i-b}}{L_{i-b}^r} z^{r^i},
\]

we obtain (31). Similarly, calculating the coefficients of \( (e_C(z))^{r^b} \), we see (32). \( \square \)

Next, we show that if \( \lambda(n) > \lambda(m) \), then the Stirling-Carlitz numbers vanish.

**Proposition 8.** Let \( n \) and \( m \) be positive integers with \( \lambda(n) > \lambda(m) \). Then

\[
\begin{align*}
[n \atop m]_C &= \{ n \atop m \}_C = 0.
\end{align*}
\]

**Proof.** Let \( s := \lambda(m) \). Then \( (\log C(z))^m \) is written as

\[
\left( \log C(z) \right)^m = \sum_{j=1}^{\infty} \left[ j \atop m \right]_C \prod_{j}^{m} \frac{\Pi(m)}{\Pi(j)} z^{j} = \prod_{i=1}^{s} \left( \log C(z) \right)^{a(i)},
\]

where \( a(1), \ldots, a(s) \) are nonnegative integers. For example, \( a(1) = \cdots = a(c_0) = 0 \), \( a(c_0 + 1) = \cdots = a(c_1) = 1 \), \ldots, \( a(c_{m-1} + 1) = \cdots = a(c_m) = m \). For any \( j \geq 1 \), the coefficient of \( z^j \) is not zero only if \( \lambda(j) \leq \lambda(m) \). Thus, we get

\[
[n \atop m]_C = 0.
\]
Similarly, we see
\[
\left\{ \binom{n}{m} \right\}_{C} = 0.
\]
\[\square\]

For example, consider the case of \( r = 3, n = 8 \) and \( m = 2 \). Using \( \lambda(8) = 4, \lambda(2) = 2 \), we see
\[
\left[ \begin{array}{c}
8 \\
2 
\end{array} \right]_{C} = \left\{ \binom{8}{2} \right\}_{C} = 0
\]
by Proposition 8.

6 Some properties of Cauchy-Carlitz numbers

It is known that poly-Cauchy numbers \( c^{(k)}_{n} \) satisfy
\[
\sum_{m=0}^{n} \left\{ \binom{n}{m} \right\}^{(k)} c^{(k)}_{m} = \frac{1}{(n+1)^{k}}
\]
([12, Theorem 3]). If \( k = 1 \), this identity is the same as that in [14, Theorem 2.3]. For Cauchy-Carlitz numbers, we obtain an analogous identity.

**Theorem 6.** For a nonnegative integer \( n \), we have
\[
\sum_{m=0}^{n} \left\{ \binom{n}{m} \right\} CC_{m} = \left\{ \frac{1}{L_{j}} \right\} \text{ if } n = r^{j} - 1, \quad 0 \text{ otherwise}.
\]

**Proof.** By Theorem 1 and Theorem 5, we have
\[
\sum_{m=0}^{n} \left\{ \binom{n}{m} \right\} CC_{m} = \sum_{m=0}^{n} \left\{ \binom{n}{m} \right\} C \sum_{j=0}^{\infty} \frac{1}{L_{j}} \left[ \binom{m}{r^{j} - 1} \right]_{C}
\]
\[
= \sum_{j=0}^{\infty} \frac{1}{L_{j}} \sum_{m=0}^{n} \left\{ \binom{n}{m} \right\} C \left[ \binom{m}{r^{j} - 1} \right]_{C}
\]
\[
= \sum_{j=0}^{\infty} \frac{1}{L_{j}} \delta_{n,r^{j}-1}
\]
\[
= \left\{ \frac{1}{L_{j}} \right\} \text{ if } n = r^{j} - 1, \quad 0 \text{ otherwise}.
\]
\[\square\]
It is known that
\[
\frac{1}{n!} \sum_{m=0}^{n} (-1)^m \binom{n+1}{m+1} B_m = \frac{1}{n+1}.
\]

Similarly, we have the following.

**Theorem 7.** For a nonnegative integer \( n \), we have
\[
\sum_{m=0}^{n} \binom{n}{m} BC_m = \begin{cases} (-1)^j D_j L_j & \text{if } n = r^j - 1, \\ 0 & \text{otherwise.} \end{cases}
\]

There are alternating expressions between poly-Bernoulli numbers and poly-Cauchy numbers ([12, 13]). When \( k = 1 \), they are reduced to the relations between classical Bernoulli numbers and classical Cauchy numbers.

\[
B_n^{(k)} = \sum_{l=0}^{n} \sum_{m=0}^{n} (-1)^{n-m} m! \binom{n}{m} \binom{m}{l} c_l^{(k)},
\]
\[
c_n^{(k)} = \sum_{l=0}^{n} \sum_{m=0}^{n} \frac{(-1)^{n-m}}{m!} \binom{n}{m} \binom{m}{l} B_l^{(k)}.
\]

As analogues, we have the following.

**Theorem 8.**
\[
BC_n = \sum_{l \geq 0} \sum_{m=r^j-1 \geq 0} (-1)^j \Pi(m) \binom{n}{m} \binom{m}{l} CC_l,
\]
\[
CC_n = \sum_{l \geq 0} \sum_{m=r^j-1 \geq 0} \frac{(-1)^j}{\Pi(m)} \binom{n}{m} \binom{m}{l} BC_l.
\]

**Proof.** By Theorem [6] and Theorem [2] we have
\[
\sum_{l \geq 0} \sum_{m=r^j-1 \geq 0} (-1)^j \Pi(m) \binom{n}{m} \binom{m}{l} CC_l
\]
\[
= \sum_{j=0}^{\infty} (-1)^j \Pi(r^j - 1) \binom{n}{r^j - 1} \frac{1}{L_j} C L_j
\]
\[
= \sum_{j=0}^{\infty} \frac{(-1)^j D_j}{L_j^2} \binom{n}{r^j - 1} C
\]
\[
= BC_n.
\]
By Theorem \[7\] and Theorem \[1\] we have

\[
\sum_{l \geq 0} \sum_{m=r^j-1 \geq 0} \frac{(-1)^j}{\Pi(m)} \left\lbrack \frac{n}{m} \right\rbrack_C \left\lbrack \frac{m}{l} \right\rbrack_C BC_l
\]
\[
= \sum_{j=0}^{\infty} \Pi(r^j - 1) \left\lbrack \frac{n}{r^j - 1} \right\rbrack_C \frac{(-1)^j D_j}{L_j^2} = \sum_{j=0}^{\infty} \frac{1}{L_j} \left\lbrack \frac{n}{r^j - 1} \right\rbrack_C
\]
\[
= CC_n.
\]

\[\square\]

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