EVEN ORDER PERIODIC OPERATORS ON THE REAL LINE

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Abstract. We consider \( 2p \geq 4 \) order differential operator on the real line with a periodic coefficients. The spectrum of this operator is absolutely continuous and is a union of spectral bands separated by gaps. We define the Lyapunov function, which is analytic on a \( p \)-sheeted Riemann surface. The Lyapunov function has real or complex branch points. We prove the following results: (1) The spectrum at high energy has multiplicity two. (2) Endpoints of all gaps are periodic (or anti-periodic) eigenvalues or real branch points. (3) The spectrum of operator has an infinite number of open gaps and there exists only a finite number of non-real branch points for some specific coefficients (the generic case). (4) The asymptotics of the periodic, anti-periodic spectrum and branch points are determined at high energy.

1. Introduction and main results

Consider the self-adjoint periodic operator \( H \) acting in \( L^2(\mathbb{R}) \) and given by

\[
H = H_0 + q, \quad H_0 = (-1)^{p} \frac{d^{2p}}{dt^{2p}}, \quad q = \sum_{j=0}^{p-1} \frac{d^{j}}{dt^{j}} q_{j+1} \frac{d^{j}}{dt^{j}}, \quad p \geq 2, \quad (1.1)
\]

where \( \mathbb{Z} \) is the set of all integers. Let \( W^2_j(\mathbb{R}), j \in \mathbb{N} = \mathbb{Z} \cap [1, \infty) \), be the Sobolev space of functions \( f, f^{(j)} \in L^2(\mathbb{R}) \). Here and below we use the notation \( f' = \frac{df}{dt}, f^{(j)} = \frac{d^j f}{dt^j} \). We define the self-adjoint operator \( H \) using the quadratic form with the form domain \( \text{Dom}_{fd}(H) = W^2_j(\mathbb{R}) \) (see Proposition 3.1).

It is well known (see \[\text{[DS]}, \text{Ch. XIII.7.64}\]) that the spectrum \( \sigma(H) \) of \( H \) for the sufficiently smooth coefficients \( q_j, j \in \mathbb{N}_p = \{1, 2, \ldots, p\}, \quad T = \mathbb{R}/\mathbb{Z}, \quad (1.2) \)

where \( \mathbb{Z} \) is the set of all integers. Let \( W^2_j(\mathbb{R}), j \in \mathbb{N} = \mathbb{Z} \cap [1, \infty) \), be the Sobolev space of functions \( f, f^{(j)} \in L^2(\mathbb{R}) \). Here and below we use the notation \( f' = \frac{df}{dt}, f^{(j)} = \frac{d^j f}{dt^j} \). We define the self-adjoint operator \( H \) using the quadratic form with the form domain \( \text{Dom}_{fd}(H) = W^2_j(\mathbb{R}) \) (see Proposition 3.1).

It is well known (see \[\text{[DS]}, \text{Ch. XIII.7.64}\]) that the spectrum \( \sigma(H) \) of \( H \) for the sufficiently smooth coefficients \( q_j, j \in \mathbb{N}_p, \) is absolutely continuous and consists of non-degenerated intervals \( \mathcal{S}_n, n = 1, \ldots, N_G < \infty \). These intervals \( \mathcal{S}_n \) and \( \mathcal{S}_{n+1} \) are separated by the gap \( g_n \) with length \( |g_n| > 0 \) and \( N_G - 1 \) is a number of the gaps. Theorem 1.1 extends this result to the larger case \( q_j \in L^1(\mathbb{R}) \).

The typical applications of our operator \( H \) are the vibrations of beams, plates and shells:

(1) The standard Kirchhoff-Love model of the bend of beams and plates provides the Euler-Bernoulli equation \( y''' = \lambda ay \) (see \[\text{[TYW]}, \text{Ch. 5.9}\]).

(2) The Vlasov model of the bend of cylinder shells (see \[\text{[NCM]}, \text{Ch. I.1.14}\]) gives the equations of vibration having the form \( y^{(8)} + b_1 y = \lambda by \).

Here \( y \) is the normal displacement of the plate (or shell), the functions \( a \) (or \( b, b_1 \)) are defined by the parameters of the plate (or shell): Young’s modulus, Poisson’s modulus, rigidity and thickness.

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The high order differential operators arise in the inverse problem method of integration of non-linear evolution equations. There exist the Lax pairs, where the self-adjoint operator is a high order operator and the corresponding non-linear Lax equation is integrable by the inverse problem method, see [DKN]. Many physically interesting equations have this form, see [AC].

Recall that the spectral theory for the high order operators with decreasing coefficients is well developed, see [BDT], [Su] and the references therein. The results for high order periodic operators are still modest.

We describe our goal. In the case of spectral bands, but all endpoints of the bands are 2-periodic eigenvalues of the equation \(-y'' + q_1 y = \lambda y\). In the case \(p = 2\) the spectrum of the operator \(H\) is also a union of spectral bands, but all endpoints of the bands are 2-periodic eigenvalues of the equation \(y'''' + qy = \lambda y\) or the branch points of the Lyapunov function [BK2]. Until now there are no any results about the multiplicity of the spectrum at high energy, number of gaps (is it finite or infinite ?), asymptotics and type of endpoints of the gaps at high energy etc for the operators \(H, p > 2\). Our main goal is to answer some of these questions.

In order to describe our results we consider the equation

\[-(1)^p y'''' + qy = \lambda y, \quad y = \sum_{j=0}^{p-1} \frac{d^j}{dt^j} q_{j+1} \frac{d^j}{dt^j} \quad (t, \lambda) \in \mathbb{R} \times \mathbb{C}, \quad (1.3)\]

where \(\mathbb{C}\) is the complex plane. If all coefficients \(q_j, q_j^{(j-1)} \in L^1(\mathbb{T})\), then the standard monodromy matrix is well defined (see [DS], Ch. XIII.7). If some coefficient \(q_j \in L^1(\mathbb{T}), q'_j \not\in L^1(\mathbb{T})\), then the standard monodromy matrix is not well-defined, since, in general, the derivative of \(y^{(2p-1)}\) is not continuous. In this case we will introduce the modified symplectic monodromy matrix, see (1.10). We think that it will be convenient even for smooth coefficients \(q_j\). We rewrite the equation (1.3) in the vector form by

\[Y' - \mathcal{P}(\lambda) Y = \mathcal{Q} Y, \quad (t, \lambda) \in \mathbb{R} \times \mathbb{C}, \quad (1.4)\]

see [Na], Ch. II.4, where the vector-valued function \(Y\) is given by

\[
Y = \begin{pmatrix}
y_1 \\
y_2 \\
\vdots \\
y_{p+1} \\
y_{p+2} \\
y_{p+3} \\
\vdots \\
y_{2p}
\end{pmatrix} = \begin{pmatrix}
y \\
y' \\
\vdots \\
y'_p \\
y'_{p+1} + (-1)^p q_p y_p \\
y'_{p+2} + (-1)^p q_{p-1} y_{p-1} \\
\vdots \\
y'_{2p-1} + (-1)^p q_{2p} y_2
\end{pmatrix}, \quad (1.5)
\]

and the \(2p \times 2p\) matrices-valued functions \(\mathcal{P}, \mathcal{Q}\) are given by

\[
\mathcal{P} = \begin{pmatrix}
\mathbb{1}_{2p-1,1} & \mathbb{1}_{2p-1} \\
(-1)^p \lambda & \mathbb{1}_{1,2p-1}
\end{pmatrix}, \quad \mathcal{Q} = (-1)^p + 1 \begin{pmatrix}
\mathbb{0}_{p,p} & \mathbb{0}_{p,p} \\
0 & \mathbb{0}_{p,p} \\
0 & \mathbb{0}_{p,p} \\
\vdots & \vdots & \mathbb{0}_{p,p} \\
q_1 & \mathbb{0}_{p,p} & \mathbb{0}_{p,p}
\end{pmatrix}, \quad (1.6)
\]
The monodromy matrix $M = M(1, \cdot)$ is symplectic, i.e. it satisfies the identity

$$M^\top J M = J,$$

where $J = \left( \begin{array}{cc} \mathbb{0}_{p,p} & J_p \\ (-1)^p J_p & \mathbb{0}_{p,p} \end{array} \right)$, \quad $J_p = \left( \begin{array}{cccc} 0 & \cdots & 0 & 1 \\ 0 & \cdots & 0 & -1 \\ 0 & \cdots & 1 & 0 \\ (-1)^{p+1} & \cdots & \cdots & 0 \end{array} \right)$ \hspace{1cm} (1.10)

and $J^\top = -J$. Furthermore, there exists an analytic function $\Delta$ on the connected $p$-sheeted Riemann surface $\mathbb{R}$ having the following properties:

i) All branches of $\Delta$ have the form $\Delta_j = \frac{1}{2}(\tau_j + \tau_j^{-1})$, \quad $j \in \mathbb{N}_p$, and satisfy:

$$D(\tau, \lambda) = \Phi(\nu, \lambda) = \prod_{j=1}^p (\nu - \Delta_j(\lambda)), \quad \nu = \frac{\tau + \tau^{-1}}{2}, \quad (\tau, \lambda) \in \mathbb{C}^2, \quad \tau \neq 0 \hspace{1cm} (1.11)$$

$$\Delta_j(\lambda) = \cosh z\omega_j + O\left(e^{\left|\text{Re} z\omega_j\right|}\right) \text{ as } |\lambda| \to \infty, \quad \lambda \in \mathbb{C}_+. \hspace{1cm} (1.12)$$

ii) If $\Delta_j(\lambda) \in (-1, 1)$ for some $(j, \lambda) \in \mathbb{N}_p \times \mathbb{R}$, and $\lambda$ is not a branch point of $\Delta_j$, then $\Delta_j(\lambda) \neq 0$.

iii) The spectrum $\sigma(H)$ of the operator $H$ satisfies

$$\sigma(H) = \sigma_{ac}(H) = \{\lambda \in \mathbb{R} : \Delta_j(\lambda) \in [-1, 1] \text{ for some } j \in \mathbb{N}_p\}. \hspace{1cm} (1.13)$$
Figure 1. The Riemann surface of the Lyapunov function for $p = 3$ for the case of small coefficients. Identities (1.22) give $\Delta_2(r_{1,n}^+) = \Delta_3(r_{1,n}^+)$ and $\Delta_1(r_{2,n}^+) = \Delta_2(r_{2,n}^+)$. Then the second and third sheets of the surface are attached along the cuts $(r_{1,n-1}, r_{1,n})$, $n \in \mathbb{N}$. We attach the upper (lower) edge of each cut on the second sheet to the lower (upper) edge of the same cut on the third sheet. Similarly, first and second sheets of the surface are attached along the cuts $(r_{2,n-1}, r_{2,n})$, $n \in \mathbb{N}$. Thus, whenever we cross the cut, we pass from one sheet to another.

Remark. 1) If $\lambda \in \sigma(H)$, then some branch $\Delta_j(\lambda)$ is real and the corresponding multiplier $\tau_j(\lambda)$ is complex and $|\tau_j(\lambda)| = 1$. It is more convenient to study the real function $\Delta_j(\lambda)$ on the spectrum $\sigma(H)$, than the complex multiplier $\tau_j(\lambda)$ on $\sigma(H)$.

2) The surface $\mathcal{R}$ is connected. For the first and second order operators with the matrix-valued potentials the corresponding surface may be disconnected (see [CK], [K1], [K2]).

3) The proof of i), ii) repeats essentially the argument from [CK], [K1], [K2].

4) The monodromy matrix for the second order operators has asymptotics in terms of cos and sin bounded on the real line. The monodromy matrix for high order operators has asymptotics in terms of cosh and sinh, see (3.11), unbounded on the real line.

The zeros of $D(1, \cdot)$ (or $D(-1, \cdot)$) are periodic (or antiperiodic) eigenvalues of the equation $(-1)^py^{(2p)} + qy = \lambda y$, where $y$ are 1-periodic (or 1-antiperiodic) functions. Denote by $\lambda_0^+, \lambda_2^+, n \geq 1$, the periodic eigenvalues and by $\lambda_{2n-1}^+, n \geq 1$, the antiperiodic eigenvalues labeling by (counted with multiplicity)

$$\lambda_0^+ \leq \lambda_2^- \leq \lambda_4^- \leq \lambda_6^- \leq \ldots, \quad \lambda_1^- \leq \lambda_3^- \leq \lambda_5^- \leq \lambda_7^- \leq \ldots.$$

For the polynomial $\Phi$ given by (1.11) we introduce the discriminant $\rho(\lambda), \lambda \in \mathbb{C}$, by

$$\rho = \prod_{1 \leq j < k \leq p} (\Delta_j - \Delta_k)^2. \quad (1.14)$$

A zero of $\rho$ is a ramification point (or simply a ramification) of the Lyapunov function $\Delta$.

Remark. 1) Ramification is a geometric term used for 'branching out', in the way that the square root function, for complex numbers, can be seen to have two branches differing in sign. We also use it from the opposite perspective (branches coming together) as when a
covering map degenerates at a point of a space, with some collapsing together of the fibers of the mapping.

2) Recall that all endpoints of gaps of the spectrum of the Hill operator (i.e., \( p = 1 \)) are periodic or anti-periodic eigenvalues. The situation is more complicated for the high order periodic operators and the periodic operators with the matrix potentials. In these cases the endpoints of gaps are periodic or anti-periodic eigenvalues, or ramifications (zeros of the function \( \rho \)). The numerical analysis for the fourth order operators and the second order periodic operators with the \( 2 \times 2 \) matrix potential shows that ramifications can be non-real for some values of the coefficients and can become real and to create the gap for some other values of the coefficients. This behavior is similar to the behavior of the resonances in the scattering problem for the Schrödinger operator (see, e.g., [K5], [Z]). In fact, this was a reason for us to use the term resonance for the zero of the function \( \rho \) for the periodic operators in our previous papers [BK1], [BK2], [BBK], [CK], ... But now we will use the term ramification for such points because the term resonance is overloaded and is used in different other senses.

We shortly describe the unperturbed operator \( H^0 = (-1)^p \frac{d^{2p}}{d\tau^{2p}} \), see more in Section 2. The unperturbed multipliers \( \tau_j^0 \), the Lyapunov function \( \Delta^0 \) with all branches \( \Delta_j^0 \) are given by

\[
\tau_j^0(\lambda) = e^{\omega_j z}, \quad \Delta^0(\lambda) = \cos \lambda \frac{1}{\pi}, \quad \Delta_j^0(\lambda) = \cosh \omega_j, \quad (j, \lambda) \in \mathbb{N}_{2p} \times \mathbb{C}_+.
\]

The unperturbed discriminant \( \rho^0 \) has the form

\[
\rho^0 = \prod_{1 \leq j < \ell \leq p} (\cosh \omega_j - \cosh \omega_k)^2. \quad (1.15)
\]

The 2-periodic eigenvalues \( \lambda_n^{0,\pm} = (\pi n)^{2p}, n \geq 1 \), have multiplicity 2 and the periodic eigenvalue \( \lambda_0^{0,+} = 0 \) has multiplicity 1. The function \( \rho^0 \) is entire and has the zeros \( r_{k,n}^0, k \in \mathbb{N}_{p-1}, n \geq 0 \), given by

\[
r_{k,n}^0 = (-1)^k \left( \frac{\pi n}{c_k} \right)^{2p}, \quad c_k = \cos \frac{\pi k}{2p}, \quad 1 > c_1 > c_2 > ... > c_{p-1} > 0. \quad (1.16)
\]

The zero \( \lambda = 0 \) of the function \( \rho^0 \) has the multiplicity \( p - 1 \) and each another zero has the multiplicity 2. The spectrum \( \sigma(H^0) \) has the multiplicity 2. The Riemann surface \( \mathcal{R}^0 \) of the Lyapunov function \( \Delta^0 \) for the operator \( H^0 \) coincides with the Riemann surface of the function \( \lambda^2 \) with the unique branch point at \( \lambda = 0 \).

We determine the sharp asymptotics of the ramifications.

**Theorem 1.2.** i) The function \( \rho \) is entire, real on \( \mathbb{R} \) and satisfies

\[
\rho(\lambda) = \rho^0(\lambda)(1 + O(|z|^{-1})) \quad \text{as} \quad |\lambda| \to \infty, \quad |\lambda - r_{k,n}^0| > 1, \quad \forall (k, n) \in \mathbb{N}_{p-1} \times \mathbb{N}. \quad (1.17)
\]

ii) The function \( \rho \) has the zeros \( r_{k,n}^+, r_{k,n}^-, (k, n) \in \mathbb{N}_{p-1} \times \mathbb{N}, \) which satisfy

\[
r_{k,n}^\pm = (-1)^k \left( \frac{\pi n}{c_k} \right)^{2p} \left( 1 + \frac{c_k^2}{(\pi n)^2} \left( -1 \right)^{p+1} \hat{q}_{p,0} \pm c_k |\hat{q}_{p,n}| + O\left( \frac{1}{n} \right) \right) \quad (1.18)
\]

as \( n \to \infty \), where

\[
\hat{q}_{p,n} = \int_0^1 q_p(t)e^{-i2\pi nt}dt, \quad n \geq 0. \quad (1.19)
\]
Remark. 1) Asymptotics (1.17) show that $\rho \neq 0$, since $\rho^0 \neq 0$. Note that for the second (and first) order operators with the $p \times p$ matrix-valued potential the corresponding function may be equal to 0 (see [CK], [K1], [K2]).

2) The Riemann surface $\mathcal{R}$, roughly speaking, is close to $\mathcal{R}^0$ at high energy. In general, the points $r_{k,n}^\pm$ are simple branch points of $\Delta$ (square root type) for $n$ large enough. The surface $\mathcal{R}$ for $p = 3$ is shown by Fig. 1.

We describe the structure of the bands and the gaps at high energy.

Theorem 1.3. i) The branch $\Delta_\rho$ is real analytic function on the interval $(\lambda_{n_0}^+, \infty)$ for some $n_0 \in \mathbb{N}$ and $\lambda_{n_0}^+ < \lambda_{n_0+1}^- \leq \lambda_{n_0+2}^+ < \lambda_{n_0+2}^- < \ldots$. Moreover, if $n \geq n_0$, then each interval $[\lambda_n^+, \lambda_{n+1}^-]$ is a spectral band with the spectrum of the multiplicity 2, and each interval $(\lambda_n^-, \lambda_n^+)$ is a gap and $\Delta_\rho^2(\lambda_n^+) = 1$. There are no other bands of $H$ to the right of $\lambda_{n_0}^+$.

ii) The periodic and antiperiodic eigenvalues $\lambda_n^\pm$ satisfy:

$$
\lambda_n^\pm = (\pi n)^{2p} \left( 1 + \frac{1}{(\pi n)^2} \left[ (-1)^{p+1} \hat{q}_{p,0} \pm |\hat{q}_{p,n}| + \frac{O(1)}{n} \right] \right) \text{ as } n \to \infty.
$$

(1.20)

Remark. 1) The spectrum of $H$ has multiplicity 2 at high energy. The spectrum of the Schrödinger operator with the $p \times p$ matrix-valued potential and the first order operator with the $2p \times 2p$ matrix-valued potential has multiplicity 2p at high energy (see [CK], [K1]).

2) The periodic and antiperiodic eigenvalues accumulate at $+\infty$. The ramifications accumulate at $\pm\infty$. But for second (or first) order systems the periodic and antiperiodic eigenvalues and the ramifications accumulate at $+\infty$ (or $\pm\infty$), see [CK], [K2] (or [K1]).

3) The spectrum of $H$ at high energy is described by the branch $\Delta_\rho$ of the Lyapunov function. The structure of the spectrum at high energy is similar to the structure of the spectrum of the Hill operator:

(a) the spectra of $H$ and of the Hill operator are similar as the sets, including multiplicities;
(b) endpoints of gaps are periodic and antiperiodic eigenvalues only;
(c) the sharp asymptotics of these eigenvalues are expressed in terms of the Fourier coefficients of the potential.

Recall that there exists an infinite number of open gaps in the spectrum of the first and the second order operators for some specific potentials [CK], [K1], [K2]. Now we describe this situation for our case.

Corollary 1.4. Let the coefficient $q_p$ satisfy $| \int_0^1 q_p(t)e^{-it\pi n_j}\,dt | \geq \frac{1}{n_j^\alpha}$ for some infinite sequence of indices $n_j \to \infty$ and some $0 < \alpha < 1$. Then

i) All gaps $\gamma_n = (\lambda_n^-, \lambda_n^+)$ are real and the gap-length $|\gamma_n| \to \infty$ as $j \to \infty$.

ii) All ramifications $r_{k,n_j}^\pm$, $k \in \mathbb{N}_{p-1}$ are real and there exist an infinite number of the non-empty intervals $(r_{k,n_j}^-, r_{k,n_j}^+)$ in $\mathbb{R}$ with the length $|r_{k,n_j}^+ - r_{k,n_j}^-| \to \infty$ as $j \to \infty$.

Note that if $| \int_0^1 q_p(t)e^{-it\pi n_j}\,dt | \geq \frac{1}{n_j^\alpha}$ as $n \to \infty$ and for some $0 < \alpha < 1$ (generic periodic coefficients). Then there exists only a finite number of non-real ramifications (branch points of the Lyapunov function) and all high energy gaps are open.

A great number of papers is devoted to the inverse spectral theory for the Hill operator: Dubrovin [D], Garnett and Trubowitz [GT], Its and Matveev [IM], Kappeler [Kap], Kargaev and Korotyaev [KK], Korotyaev [K3], Marchenko and Ostrovski [MO], Novikov [No], etc. Note that Korotyaev [K4] extended the results of [MO], [GT], [Kap], [KK], [K3] for the case $-y'' + qy$ to the case of periodic distributions, i.e. $-y'' + qy$ on $L^2(\mathbb{R})$, where periodic $q \in L^2_{\text{loc}}(\mathbb{R})$. 


We describe now the results for vector differential equations. Recently the inverse problem for vector-valued Sturm-Liouville operators on the unit interval with Dirichlet boundary conditions, including characterization, was solved by Chelkak, Korotyaev [CK1], [CK2]. The periodic case is more complicated and a lot of papers are devoted only to the direct problem of periodic systems: Carlson [Ca1], [Ca2], Gelfand and Lidskii [GL], Gesztesy and coauthors [CL], Korotyaev and coauthors [CK], [BBK], [K1], [K2], etc. We describe results for first and second order operators with the periodic $p \times p$ matrix-valued potential from [CK], [K1], [K2]:

1) the properties of the Lyapunov function, defined on the Riemann surface, are described,

2) the conformal mapping with real part given by the integrated density of states and imaginary part given by the Lyapunov exponent is constructed and the main properties are obtained,

3) trace formulas (similar to the case of the Hill operators) are determined,

4) an estimate of gap lengths in terms of potentials is obtained,

5) sharp asymptotics of periodic eigenvalues and ramifications are determined.

Note that the discrete periodic systems were studied in [KKu1], [KKu2]. The results for first and second order operators are important for us, since we plan to repeat one for even order periodic operators. In fact this is the motivation of our paper. Note that the case of even order periodic operators is more complicated than the case of first and second order operators, since in the first case only one fundamental solution is bounded on the real line and all other fundamental solutions are unbounded on the real line.

We describe the fourth order operators $H = \partial^4 + \partial q_2 \partial^2 + q_1$. The results for decreasing coefficients are more developed, see [AP], [GM], [HLO], [LO]. We mention the paper [CPS], [McL] about the inverse problem for fourth order operators on the unit interval. Now we describe the periodic case. The authors [BK2] obtained the following results for the operator $H = \partial^4 + \partial q_2 \partial^2 + q_1$ (the case $q_2 = 0$ see in [BK1]):

1) The properties of the Lyapunov function on the 2-sheeted Riemann surface are described. The asymptotics of the spectral gaps and ramifications are determined at high energy.

2) If $q_2 = 0$, $q_1 \to 0$ or $q_1 = 0$, $q_2 \to 0$, then there exists a small non-empty spectral band with the spectrum of multiplicity 4. The beginner of this band is the ramification, which coincides with the top of the spectrum. The spectrum in all other bands has multiplicity 2.

3) There exist both real and non-real ramifications for some specific potentials.

The spectral properties of the periodic Euler-Bernoulli equation $(a y''')'' = \lambda y$ were studied by Papanicolaou [P1], [P2], [PK] (jointly with Kravarritis). It was shown that the spectrum is a union of non-overlapping bands of multiplicity 2, similar to the case of the scalar Hill operator. The beginning of the spectrum is both a simple periodic eigenvalue and a branch point of the Lyapunov function. All other ramifications are negative.

Consider the operator $H$ with $p \geq 2$. The old well known results see in the book [Na]. Tkachenko [Tk] obtained the eigenfunction expansion formula for the operator $H$. Mikhailets and Molyboga [MM1], [MM2] determined asymptotics of eigenvalues for the operator $(-1)^p \partial^{2p} + q$ on the circle $T = \mathbb{R}/\mathbb{Z}$, where $q$ is a distribution. Galunov and Oleinik [GO] considered the operator $(-1)^p \partial^{2p} + \delta_{\text{per}}$ on the real line, where $\delta_{\text{per}}$ is a periodic $\delta$-function.

It is important that for $p = 1$ the spectral analysis of the operator on the circle (the periodic and antiperiodic spectrum) is equivalent to one of the operator $H$ on the real line. The main tool is the analysis of the entire Lyapunov function. The situation for $p \geq 2$ is much more complicated (see [BK1], [BK2]). In this case the Lyapunov function $\Delta$ has the complicated $p$ sheeted Riemann surface.
In the present paper we extend some of results from [BK1], [BK2] about the case $p = 2$ to the case $p \geq 2$. We construct the Riemann surface for the Lyapunov function of $H$ and describe this surface for large $|\lambda|$. Moreover, we determine asymptotics of the ramifications and periodic and antiperiodic eigenvalues at high energy.

The plan of the paper is as follows. In Sect. 2 we describe the multipliers for the unperturbed operator. In Sect. 3 we describe the basic properties of the monodromy matrix $M$. In order to determine the asymptotics of the monodromy matrix $M$ at high energy we use so-called Jost type solutions with ”good” asymptotics at high energy. In Sect. 4 we obtain the main properties of the multipliers, the Lyapunov function and the function $\rho$ and prove Theorem 1.1. Moreover, we consider some simple examples. In Sect. 5 we prove our main Theorems 1.2 and 1.3. In the proof using the mix of arguments both for the fourth order operator [BK1], [BK2] and for the systems [CK], [K1], [K2], we determine the asymptotics of the ramifications and periodic eigenvalues analyzing directly the determinant $D$ of the monodromy matrix in the neighborhoods of ramifications (see Lemma 5.2 and the proof of Theorem 1.2, 1.3). In the end of Sect. 5 we prove the simple Corollary 1.4 from Theorem 1.3. Some technical proofs are placed in Appendix.

2. Properties of the unperturbed operator

The numbers $\omega_j$, given by (1.9), satisfy

$p$ odd: \[ \text{Re} \, \omega_{2p} = \text{Re} \, \omega_{2p-1} < \ldots < \text{Re} \, \omega_4 = \text{Re} \, \omega_3 < \text{Re} \, \omega_2 = \text{Re} \, \omega_1, \]
\[ \text{Im} \, \omega_{2j-1} < 0, \quad \omega_{2j} = \overline{\omega}_{2j-1}, \quad \text{all} \quad j \in \mathbb{N}_p, \quad (2.1) \]

$p$ even: \[ -1 = \omega_{2p} < \text{Re} \, \omega_{2p-1} = \text{Re} \, \omega_{2p-2} < \ldots < \text{Re} \, \omega_5 = \text{Re} \, \omega_4 < \text{Re} \, \omega_3 = \text{Re} \, \omega_2 < \omega_1 = 1, \]
\[ \text{Im} \, \omega_{2j} < 0, \quad \omega_{2j+1} = \overline{\omega}_{2j}, \quad \text{all} \quad j \in \mathbb{N}_{p-1}, \quad (2.2) \]

see Fig.2. Moreover,
\[ \omega_{p+j+1} = \varepsilon_j \pi_j, \quad \omega_{p+j} = \overline{\varepsilon}_j \overline{\pi}_j, \quad \omega_{p+j} - \omega_{p+j+1} = (1)^{j+1} 2i c_j \pi_j, \quad (2.3) \]
for all \( j = -p + 1, -p + 2, \ldots, p - 1 \), where
\[
c_j = \cos \frac{\pi j}{2p}, \quad \varepsilon_j = \begin{cases} \frac{i e^{i \pi j / 2p}}{2p}, & j \text{ even} \\ -\frac{i e^{-i \pi j / 2p}}{2p}, & j \text{ odd} \end{cases}, \quad \eta_j = \begin{cases} 1, & j \text{ even} \\ e^{i \pi j}, & j \text{ odd} \end{cases}.
\]

We introduce the step functions \( \Omega_j(\lambda), j, \lambda \in \mathbb{N}_{2p} \times \mathbb{C} \), which are constant in each half-plane \( \mathbb{C}_\pm = \{ \lambda \in \mathbb{C} : \pm \text{Im} \lambda > 0 \} \) and given by
\[
\Omega_j(\lambda) = \begin{cases} \omega_j, & \text{Im} \lambda > 0 \\ \overline{\omega_j} = \omega_\ell, & \text{Im} \lambda < 0 \end{cases},
\]
where
\[
\text{odd } p : \quad \ell = \begin{cases} j + 1, & j \text{ odd} \\ j - 1, & j \text{ even} \end{cases}, \quad \text{even } p : \quad \ell = \begin{cases} j + 1, & j \text{ even}, j \neq 2p \\ j - 1, & j \text{ odd}, j \neq 1 \\ j, & j = 1, 2p \end{cases}.
\]

**Lemma 2.1.** The functions \( \Omega_j(\lambda), \lambda \in \mathbb{C}, j \in \mathbb{N}_{2p} \), satisfy
\[
\text{Re}(z\Omega_2(\lambda)) \leq \cdots < \text{Re}(z\Omega_1(\lambda)) \leq \text{Re}(\Omega_2(\lambda)) \leq \text{Re}(\Omega_1(\lambda)),
\]
\[
\text{Re}(z\Omega_j(\lambda) - z\Omega_{j+2}(\lambda)) > a|z|, \quad \lambda \neq 0, \quad \text{all } j \in \mathbb{N}_{2p-2},
\]
where
\[
a = 2c_{p-1} \sin \frac{\pi}{4p} > 0, \quad z = \lambda^{1/2p} \in S = \left\{ z \in \mathbb{C} : \text{arg } z \in \left( -\frac{\pi}{2p}, \frac{\pi}{2p} \right) \right\}.
\]

**Proof.** Assume that (2.7), (2.8) hold for \( \text{Im} \lambda > 0 \). Then identities (2.5) give these estimates for \( \text{Im} \lambda < 0 \).

We will prove (2.7), (2.8) for \( \text{Im} \lambda \geq 0 \), i.e. \( 0 \leq \text{arg } z \leq \frac{\pi}{2p} \). Identities (2.3) yield
\[
\text{Re}(z\Omega_{p+j}(\lambda) - z\Omega_{p+j+1}(\lambda)) = \text{Re}(\omega_{p+j} - \omega_{p+j+1}) = 2c_j(-1)^j \text{Im}(z\overline{\eta}_j)
\]
for all \( j = -p + 1, -p + 2, \ldots, p - 1 \). Identities
\[
(-1)^j \text{Im}(z\overline{\eta}_j) = |z| \begin{cases} \text{sin } \text{arg } z, & \text{if } j \text{ is even} \\ \text{sin } \left( \frac{\pi}{2p} - \text{arg } z \right), & \text{if } j \text{ is odd} \end{cases} \geq 0
\]
yield estimates (2.7). Furthermore, identities (2.10) and estimates (1.10) imply
\[
\text{Re}(z\Omega_{p+j}(\lambda) - z\Omega_{p+j+2}(\lambda)) = \text{Re}(\omega_{p+j} - \omega_{p+j+1}) + \text{Re}(\omega_{p+j+1} - \omega_{p+j+2})
\]
\[
= (-1)^j 2c_j \text{Im}(z\overline{\eta}_j) + (-1)^{j+1} 2c_{j+1} \text{Im}(z\overline{\eta}_{j+1}) \geq 2c_{j+1}((-1)^j \text{Im}(z\overline{\eta}_j) + (-1)^{j+1} \text{Im}(z\overline{\eta}_{j+1})).
\]
Identity (2.11) and estimates (1.10) imply
\[
\text{Re}(z\Omega_j(\lambda) - z\Omega_{j+2}(\lambda)) > 2c_{p-1} \max\left\{ \text{sin } \text{arg } z, \text{sin } \left( \frac{\pi}{2p} - \text{arg } z \right) \right\} |z|,
\]
which yields (2.8). \( \blacksquare \)

We define the branches \( \tau_j^0, j \in \mathbb{N}_{2p} \), of the multiplier function \( \tau^0 = e^{i \lambda^{1/2p}} \) for the unperturbed operator \( H^0 \) in the upper half-plane by the identities
\[
\tau_j^0(\lambda) = e^{i \omega_j}, \quad \text{all } \lambda \in \mathbb{C}_+.
\]
For each $j \in \mathbb{N}_p$ the functions $\tau_{p-j}^0, \tau_{p-j+1}^0$ are single-valued analytic functions in the domain $\mathcal{D}_{p+j}^0$, where

$$\mathcal{D}_{p+j}^0 = \mathbb{C} \setminus \mathbb{R}, \quad \text{all } j \in \mathbb{N}_{p-1}, \quad \mathcal{D}_{2p}^0 = \begin{cases} \mathbb{C} \setminus \mathbb{R}_- & \text{for even } p \\ \mathbb{C} \setminus \mathbb{R}_+ & \text{for odd } p \end{cases}. \quad (2.12)$$

**Lemma 2.2.** i) The unperturbed multipliers satisfy the identities

$$\tau_j^0(\lambda) = \overline{\tau_j^0(\lambda)}, \quad \tau_j^0(\lambda) = e^{\omega_j^0(\lambda)}, \quad \text{all } (j, \lambda) \in \mathbb{N}_{2p} \times (\mathbb{C} \setminus \mathbb{R}). \quad (2.13)$$

$$\tau_{p-j+1}^0(\lambda) = (\tau_{p-j}^0(\lambda))^{-1}, \quad \text{all } (j, \lambda) \in \mathbb{N}_p \times \mathcal{D}_{p+j}^0. \quad (2.14)$$

ii) Let $\tau_{p+k}^0(\lambda) = \tau_{p+j}^0(\lambda)$ for some $\lambda \in \mathbb{C} \setminus \mathbb{R}, 0 \leq k < j \leq p$. Then $j = k + 1$ and $\lambda = r_{k,n}^0 + i0$ or $\lambda = r_{k,n}^0 - i0$ for some $n \in \mathbb{N}$. Moreover,

$$\tau_{p+k}^0(r_{k,n}^0 + i0) = \tau_{p+k}^0(r_{k,n}^0 - i0) = \tau_{p+k+1}^0(r_{k,n}^0 + i0) = \tau_{p+k+1}^0(r_{k,n}^0 - i0), \quad k \in \mathbb{N}_{p-1} = \{0, 1, ..., p-1\}. \quad (2.15)$$

**Proof.** i) The Riemann surface $\mathcal{L}^0$ of the function $\tau^0$ coincides with the Riemann surface of the function $\lambda^{1/2p}$ with the unique branch point at $\lambda = 0$. This surface has $2p$ sheets $\mathcal{L}_j^0, j \in \mathbb{N}_{2p}$, corresponding to the branches $\tau_j^0$, that is $\tau_j^0(\lambda) = \tau_j^0(w)|_{w \in \mathcal{L}_j^0}$, where $\lambda \in \mathbb{C}$ is the projection of the point $w \in \mathcal{L}^0$. For each $j \in \mathbb{N}_p$ the projection of the sheet $\mathcal{L}_{p+j}^0$ and $\mathcal{L}_{p+j+1}^0$ is the domain $\mathcal{D}_{p+j}$ and has the cut along the real axis or semi-axis, see (2.12). The upper (lower) edge of each cut on the sheet $\mathcal{L}_{p+j}^0, j \in \mathbb{N}_p$, is attached to the lower (upper) edge of the corresponding cut on the sheet $\mathcal{L}_{p+j+1}^0$. Similarly the upper (lower) edge of each cut on the sheet $\mathcal{L}_{p-j}^0, j \in \mathbb{N}_p$, is attached to the lower (upper) edge of the corresponding cut on the sheet $\mathcal{L}_{p-j+1}^0$. 

**Figure 3.** The $\zeta$-plane for the cases $p = 2$ and $p = 3$, where $\zeta = iw^{1/2p}, w \in \mathcal{L}^0$. Each sector $S_j^0$ is the image of the $\mathbb{C}_+$ half-plane on the $j$-th sheet of the surface $\mathcal{L}^0$. The values of the function $\tau_0^0(w) = e^\zeta$ at the points, connected by the lines with arrows, are equal one with other.
The simple parametrization of the surface $\mathcal{L}^0$ is given by the analytical mapping $W$ having the form $w \rightarrow \zeta = iw^{-\frac{1}{2p}}$, where $w \in \mathcal{L}^0$. We have

$$W(\mathcal{L}^0) = \mathbb{C}.$$  

Describe this parametrization in more details. Introduce the sectors (see Fig. 3)

$$S^+ = \left\{ z \in \mathbb{C} : 0 < \arg z < \frac{\pi}{2p} \right\}, \quad S_{j}^+ = \omega_j S^+, \quad S_{j}^- = \left\{ \zeta \in \mathbb{C} : \bar{\zeta} \in S_{j}^+ \right\}, \quad j \in \mathbb{N}_{2p}. \tag{2.16}$$

Let $\mathcal{L}_{j}^{0,\pm}$, $j \in \mathbb{N}_{2p}$, be the half-plane $\mathbb{C}_\pm$ on the sheet $\mathcal{L}_{j}^{0}$ of the surface $\mathcal{L}^0$. Then

$$W(\mathcal{L}_{j}^{0,\pm}) = S_{j}^\pm, \quad W(\mathcal{L}_{j}^{0}) = S_{j}, \quad \text{all } j \in \mathbb{N}_{2p}, \quad \text{where } S_{j} = S_{j}^+ \cup S_{j}^- \tag{2.17}$$

Each function $\tau_j^0, j \in \mathbb{N}_{2p}$, satisfies the identity $\tau_j^0(\lambda) = \tau_j^0(w)|_{w \in \mathcal{L}_{j}^{0}}$, where $\lambda \in \mathbb{C}$ is the projection of the point $w \in \mathcal{L}^0$. Let $\zeta = W(w) \in \mathbb{C}, w \in \mathcal{L}^0$. Then $\tau_j^0(w) = e^\zeta$. Using identities (2.17) we obtain $(2.16)$. Then $\bar{\lambda} = (\lambda - \zeta_j^0)$ (see (2.16)) and the conditions $\lambda \in \mathbb{C}_{\pm} \cap \mathcal{L}^0$. The second identities in (2.18) imply the identities $\tau_j^0(\lambda) = e^\zeta_j^0 = \lambda$, $\zeta_{p+k} = \zeta_{p+j}$, where $\zeta_{p+k} = 0, \zeta_{p+j} = 0$. Moreover, $\text{Im} \lambda = 0$. Identities (2.16) and the condition $k < j$ (see also Fig. 3) give: a) $j = k + 1$; b) $\zeta_{p+k} = \frac{2\pi}{\eta_k} e^{\pm \frac{2\pi}{k}}$, i.e. $\lambda = \zeta_{p+k} = (-1)^p(\frac{2\pi}{k})^{2p} = \bar{r}_{k,n}$; and c) identities (2.15).

Consider the operator $H^\mu = (-1)^p \frac{d^{2p}}{dt^{2p}} + \mu \frac{d^{2p-2}}{dt^{2p-2}}, \mu \in \mathbb{R}$. The equation

$$(2.19)$$

has the solutions $e^{\omega_j^\mu(t)}$, $j \in \mathbb{N}_{2p}$, where $\omega_j^\mu(t)$, $j \in \mathbb{N}_{2p}$, are the solutions of the equation

$$(2.20)$$

The functions $\omega_j^\mu, j \in \mathbb{N}_{2p}$, constitute branches of the analytic function $\omega^\mu$ having only algebraic singularities. For $\lambda \in \Lambda_R$ for some $R > 0$ large enough, we define these branches by the asymptotics $\omega_j^\mu(\lambda) = \omega_j + o(1)$ as $|\lambda| \rightarrow \infty$, here and below

$$\Lambda_R = \{ \lambda \in \mathbb{C} : |\lambda| > R^{2p} \}, \quad R > 0, \quad \Lambda_R^\pm = \Lambda_R \cap \mathbb{C}_\pm. \tag{2.15}$$

The multipliers have the form $\tau_j^\mu(\lambda) = e^{\omega_j^\mu(\lambda)}$ for all $(j, \lambda) \in \mathbb{N}_{2p} \times \Lambda_R$ for some $R > 0$ large enough. Then

$$\tau_j^\mu(\lambda) = e^{\omega_j^\mu(\lambda)}, \quad \text{all } (j, \lambda) \in \mathbb{N}_{2p} \times \Lambda_R, \quad \text{where } \Omega_j^\mu(\lambda) = \begin{cases} \omega_j^\mu(\lambda), \quad \text{Im } \lambda \geq 0, \\ \omega_j^\mu(\lambda), \quad \text{Im } \lambda < 0. \end{cases} \tag{2.21}$$

where $\ell$ is given by identities (2.6). Each function $\Omega_j^\mu(\lambda), j \in \mathbb{N}_{2p}, \mu \in \mathbb{R}$, is analytic in $\lambda \in \Lambda_R^\pm$ and piecewise-continuous in $\lambda \in \mathbb{C}$, but the set $\{\Omega_j^\mu(\lambda), j \in \mathbb{N}_{2p}, \mu \in \mathbb{R}\}$ is continuous in $\lambda \in \mathbb{C}$. The branches of the Lyapunov function are given by $\Delta_j^\mu(\lambda) = \cosh z \Omega_j^\mu(\lambda), (j, \lambda) \in \mathbb{N}_p \times \Lambda_R$. 


Lemma 2.3. i) Each function $\omega^\mu_j$, $j \in \mathbb{N}_{2p}$, satisfies the asymptotics

$$\omega^\mu_j(\lambda) = \omega_j - \frac{(-1)^p \mu}{2p\omega_j z^2} + O(|z|^{-4}) \quad \text{as} \quad |\lambda| \to \infty,$$

$$\omega^\mu_j(\lambda + \varepsilon) = \omega^\mu_j(\lambda) + O(|z|^{-4}) \quad \text{as} \quad |\lambda| \to \infty, \quad \varepsilon = O(|z|^{2p-2}).$$

(2.22)

(2.23)

ii) The periodic and antiperiodic eigenvalues for equation $(-1)^p y^{(2p)} + \mu y^{(2p-2)} = \lambda y$ satisfy:

$$\lambda_{0}^{\mu,+} = 0, \quad \lambda_{n}^{\mu,-} = \lambda_{n}^{\mu,+} = (\pi n)^{2p} - (-1)^p \mu(n)^{2p-2}, \quad \text{all} \quad n \geq 1.$$

(2.24)

iii) Each function $\Omega^\mu_j$, $j \in \mathbb{N}_{2p}$, satisfies the asymptotics

$$\Omega^\mu_j(\lambda) = \Omega_j(\lambda) + O(|z|^{-2}) \quad \text{as} \quad |\lambda| \to \infty,$$

(2.25)

Proof. i) Substituting $\omega^\mu_j = \omega_j + \delta, \delta = o(1)$, into the identity $(\omega^\mu_j)^{2p} + (-1)^p (\omega^\mu_j)^{2p-2} \mu z^{-2} - (-1)^p = 0$ we obtain

$$(\omega_j + \delta)^{2p} + (-1)^p (\omega_j + \delta)^{2p-2} \mu z^{-2} = (-1)^p.$$

(2.26)

This identity gives $2p\omega_j^{2p-1}\delta + O(\delta^2) + O(|z|^{-2}) = 0$, which yields $\delta = O(|z|^{-2})$. Using (2.26) again we obtain $2p\omega_j^{2p-1}\delta + (-1)^p \omega_j^{2p-2} \mu z^{-2} = O(|z|^{-4})$. This asymptotics gives $\delta = (-1)^p (2p\omega_j)^{-1} \mu z^{-2} + O(|z|^{-4})$, which yields (2.23). Asymptotics (2.22) yield

$$\omega^\mu_j(\lambda + \varepsilon) - \omega^\mu_j(\lambda) = \frac{(-1)^p \mu}{2p\omega_j} \left( \frac{1}{z^2} - \frac{1}{\zeta^2} \right) + O(|z|^{-4}) \quad \text{as} \quad |\lambda| \to \infty,$$

(2.27)

where $\zeta = (\lambda + \varepsilon)^{1/2}$. Using the asymptotics $\zeta = z + O(|z|^{-1})$, we obtain (2.23).

ii) We will prove (2.24) for the periodic eigenvalues. The proof for the antiperiodic eigenvalues is similar. The periodic eigenvalues $\lambda_{0}^{\mu,+}, \lambda_{n}^{\mu,\pm}, n \geq 1$, are zeros of the entire function $D_+^\mu = \prod_{j=1}^p (\Delta^\mu_j - 1)$. We have

$$D^\mu_+ = \prod_{j=1}^p (cosh \omega^\mu_j z - 1) = 2^p \prod_{j=1}^p \sinh^2 \frac{\omega^\mu_j z}{2} = \frac{\lambda}{2^p} \prod_{j=1}^p (\omega^\mu_j)^2 \prod_{n=1}^{\infty} \left( 1 + \frac{(\omega^\mu_j)^2 z^2}{2(n\pi)^2} \right)^2.$$  

(2.28)

Using the simple identity

$$\omega^{2p} + (-1)^p \omega^{2p-2} \mu z^{-2} - (-1)^p = \prod_{j=1}^p (\omega^2 - (\omega^\mu_j)^2), \quad \text{all} \quad \omega \in \mathbb{C},$$

we obtain $\prod_{j=1}^p (\omega^\mu_j)^2 = -1$ (put $\omega = 0$) and

$$\prod_{j=1}^p \left( 1 + \frac{(\omega^\mu_j)^2 z^2}{2(n\pi)^2} \right) = \frac{(-1)^p \mu}{(2n\pi)^2} \prod_{j=1}^p \left( \frac{2n\pi}{z} \right)^2 - (\omega^\mu_j)^2 \right) = 1 - \frac{(-1)^p \mu}{(2n\pi)^2} - \frac{\lambda}{(2n\pi)^{2p}}.$$

Substituting these identities into (2.28) we obtain

$$D^\mu_+ = -\frac{\lambda}{2^p} \prod_{n=1}^{\infty} \left( 1 - \frac{(-1)^p \mu}{(2n\pi)^2} - \frac{\lambda}{(2n\pi)^{2p}} \right)^2,$$

which yields (2.24) for the periodic eigenvalues.

iii) Asymptotics (2.22) and definition (2.21) of $\Omega^\mu_j$ yield (2.25). □

Remark. The periodic eigenvalue $\lambda = 0$ for equation (2.19) is simple and other periodic and antiperiodic eigenvalues have multiplicities 2.
3. Fundamental solutions

In this section we consider the operator $H = H_0 + q$, $H_0 = (-1)^p \frac{d^{2p}}{dt^{2p}}$ and recall that

$$q = \sum_{j=0}^{p-1} \frac{d^j}{dt^j} q_{j+1} \frac{d^j}{dt^j}, \quad q_j \in L^1_{\text{real}}(\mathbb{T}), \quad j \in \mathbb{N}_p, \quad j = \{1, 2, \ldots, p\}, \quad p \geq 2,$$

The form domain of the self-adjoint operator $H_0$ is the set $\text{Dom}_{f,d}(H_0) = W^2_p(\mathbb{R})$. The quadratic form $(qy, y)$ is defined by $(qy, y) = \sum_{j=0}^{p-1} (-1)^j (q_{j+1} y^{(j)}, y^{(j)})$, $y \in W^2_p(\mathbb{R})$. Let $q_j = \hat{q}_{j,0} + a_j$, where $\hat{q}_{j,0} = \int_0^1 q_j(t)dt$ and $a_j \in W^1_1(\mathbb{T}), \int_0^1 a_j(t)dt = 0, j \in \mathbb{N}_p$. The integration by parts gives the form

$$(qy, y) = \sum_{j=0}^{p-1} (-1)^j \left( \hat{q}_{j+1,0} y^{(j)} y^{(j)} - 2 \text{Re}(a_{j+1} y^{(j+1)}, y^{(j)}) \right), \quad (3.1)$$

correctly defined on the form domain $y \in W^2_p(\mathbb{R})$.

For each $q_j, j \in \mathbb{N}_p$, we introduce the sequence $(q_{j,n})_{n=1}^{\infty}$ of the smooth functions $q_{j,n} \in W^1_{j-1}(\mathbb{T})$, such that

$$\int_0^1 (q_j - q_{j,n}) dt = 0 \quad \text{all} \quad n \in \mathbb{N}, \quad \beta_n = \sup_{j \in \mathbb{N}_p} \int_0^1 |q_j - q_{j,n}| dt \to 0 \quad \text{as} \quad n \to \infty. \quad (3.2)$$

Let

$$H_n = H_0 + q(n), \quad n \in \mathbb{N}, \quad \text{where} \quad q(n) = \sum_{j=0}^{p-1} \frac{d^j}{dt^j} q_{j+1,n} \frac{d^j}{dt^j}. \quad (3.3)$$

**Proposition 3.1.**

i) The quadratic form $(qy, y)$ satisfies

$$|(qy, y)| \leq \frac{1}{2} \|y^{(p)}\|^2 + C \|y\|^2, \quad \text{all} \quad y \in W^2_p(\mathbb{R}) \quad (3.4)$$

for some constant $C > 0$, where $\|y\|^2 = (y, y)$ is the scalar product in $L^2(\mathbb{R})$.

ii) There exists a unique self-adjoint operator $H = H_0 + q$ with the form domain $\text{Dom}_{f,d}(H) = W^2_p(\mathbb{R})$ and

$$(Hy, y_1) = (H_0 y, y_1) + (qy, y_1), \quad \text{all} \quad y, y_1 \in W^2_p(\mathbb{R}). \quad (3.5)$$

iii) Let $y \in W^2_p(\mathbb{R})$. Then there exists the sequence $(\epsilon_n)_{n=1}^{\infty}$ such that $\epsilon_n > 0$ for all $n \in \mathbb{N}$, $\epsilon_n \to 0$ as $n \to \infty$, and

$$|(H - H_n)y, y| \leq \epsilon_n (|(Hy, y)| + \|y\|^2), \quad \text{all} \quad n \in \mathbb{N}. \quad (3.6)$$

**Proof.**

i) We get

$$2|a_j y^{(j)}(y^{(j-1)})| \leq 2b \|y^{(j)}\| \|y^{(j-1)}\| \leq \epsilon \|y^{(j)}\|^2 + \frac{b^2}{\epsilon} \|y^{(j-1)}\|^2, \quad b = \sup_{(t, j) \in \mathbb{T} \times \mathbb{N}_p} |a_j(t)| \quad (3.7)$$

for any $\epsilon > 0$, where $(y, y) = \|y\|^2 = \int_{\mathbb{R}} |y|^2 dt$. Thus (3.1) yields

$$|(qy, y)| \leq \epsilon \|y^{(p)}\|^2 + \tilde{C} \sum_{0}^{p-1} \|y^{(j)}\|^2, \quad \tilde{C} = q_0 + \epsilon + \frac{b^2}{\epsilon}, \quad \hat{q}_0 = \sum_{0}^{p-1} |\hat{q}_{j,0}|. \quad (3.8)$$
Using the simple estimate \( \sum_{0}^{p-1} \|y^{(j)}\|^2 \leq \varepsilon^2 \|y^{(p)}\|^2 + C_\varepsilon \|y\|^2 \) for some \( C_\varepsilon > 0 \), we obtain \[
|(qy, y)| \leq \varepsilon (1 + \varepsilon \tilde{C}) \|y^{(p)}\|^2 + \tilde{C} C_\varepsilon \|y\|^2, \] (3.9) which yields (3.4) with \( \frac{1}{2} = \varepsilon (1 + \varepsilon \tilde{C}) \) and \( C = \tilde{C} C_\varepsilon \) for \( \varepsilon \) small enough.

ii) The operator \( q \) on the domain \( \text{Dom}_{\mathcal{H}}(H_0) \) is given by (3.1) and satisfies the estimate (3.4). Using the KLMN theorem (see [RS1]) we obtain that there exists a unique self-adjoint operator \( H = H_0 + q \) with the form domain \( W_p^2(\mathbb{R}) \) and identity (3.5) holds true.

iii) Repeating the previous arguments and using \( \|y^{(j)}\|^2 \leq \|y^{(p)}\|^2 + \|y\|^2 \) we obtain \[
|((q_j - q_j,n)y^{(j-1)}, y^{(j-1)})| \leq 2\beta_n \|y^{(j)}\| \|y^{(j-1)}\| \leq \beta_n (\|y^{(j)}\|^2 + \|y^{(j-1)}\|^2) 
\leq 2\beta_n (\|y^{(p)}\|^2 + \|y\|^2) = 2\beta_n ((H_0y, y) + \|y\|^2),
\] where \( \beta_n \) is given by (3.2). This yields \[
|((H - H_n)y, y)| \leq \sum_{0}^{p-1} |((q_j+1 - q_j,n)y^{(j)}, y^{(j)})| \leq 2p\beta_n ((H_0y, y) + \|y\|^2). \] (3.10)

Using estimates (3.4) we obtain \[
(H_0y, y) \leq |(H, y)| + |(qy, y)| \leq |(H, y)| + \frac{1}{2} (H_0y, y) + C \|y\|^2,
\] which yields \( |(H_0y, y)| \leq 2 |(H, y)| + 2C \|y\|^2 \). Substituting this estimate into (3.10) we obtain (3.6), where \( C_\varepsilon = 4p\beta_n (1 + C) \).

Below a vector \( h = (h_n)^N_1 \in \mathbb{C}^N \) has the norm \( \|h\| = \sum_{1}^{N} |h_n| \), while an \( N \times N \) matrix \( A = (A_{ij})_{i,j=1}^{N} \) has the operator norm given by \( |A| = \sup_{\|h\|=1} |Ah| = \max_{1 \leq j \leq N} \sum_{i=1}^{N} |A_{ij}| \).

Always below we denote \( n \times n \) diagonal matrices by \( \text{diag}(a_{j,k})_{j,k=1}^{n} = (a_j \delta_{jk})_{j,k=1}^{n} \). The following Lemma (proof in Appendix) describes the basic properties of the monodromy matrix \( \mathcal{M}(1, \lambda) \).

**Lemma 3.2.** The matrix-valued function \( \mathcal{M}(1, \lambda) \) is entire and satisfies:
\[
|\mathcal{M}(1, \lambda)| \leq 2pe^{\sigma_0 + \kappa}, \quad \text{all } \lambda \in \mathbb{C}, \quad |Z^{-1}(\lambda)\mathcal{M}(1, \lambda)Z(\lambda)| \leq 2pe^{\sigma_0 + \kappa}, \quad \text{all } |\lambda| \geq 1, \tag{3.11}
\]

where
\[
Z = \text{diag}(z^{j-1})_{j=1}^{2p}, \quad \kappa = \max_{j \in \mathbb{N}_p} \int_{0}^{1} |q_j(t)| dt, \quad \sigma_0 = \max_{j \in \mathbb{N}_p} |\text{Re}(z \omega_j)|. 
\]
Moreover, \( \mathcal{M}(1, \lambda) \) is a continuous function of \((q_j)^{p}_{j=1} \in L^1(\mathbb{T})^p \).

**Lemma 3.3.** Let each \( q_j \in W_{j-1}^1(\mathbb{T}), \ j \in \mathbb{N}_p \) and let \( \lambda \in \mathbb{C} \). Then the spectrum of the \( 2p \times 2p \) matrix \( (\varphi^{(k-1)}_{j}(1, \lambda))_{k,j=1}^{2p} \) coincides with the spectrum of the matrix \( \mathcal{M}(1, \lambda) \), counted with multiplicity. Moreover, in this case
\[
\sigma(H) = \{ \lambda \in \mathbb{R} : |\tau_j(\lambda)| = 1 \text{ for some } j \in \mathbb{N}_p \}. \tag{3.12}
\]

**Proof.** Introduce the vector-valued function \( \tilde{Y} = (y^{(j-1)})_{j=1}^{2p} \). Identity (1.3) shows that
\[
Y = S\tilde{Y}, \quad \text{where } S = \begin{pmatrix} I_{p+1} & 0 & 0 \\ \tilde{S} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \tilde{S} = \begin{pmatrix} 0 & 0 & \ldots & 0 & q_p & 0 \\ 0 & 0 & \ldots & q_{p-1} & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & q_2 & \ldots & 0 & 0 \end{pmatrix}. \tag{3.13}
\]
Let $\tilde{M}(t, \lambda) = (\varphi_j^{(k-1)}(t, \lambda))_{k,j=1}^{2p}$, $(t, \lambda) \in \mathbb{R} \times \mathbb{C}$. Identity (3.13) shows that each matrix-valued function $S\tilde{M}(\cdot, \lambda)$, $\lambda \in \mathbb{C}$, satisfies equation (1.2), which yields $M(t, \lambda) = S(t)\tilde{M}(t, \lambda)S^{-1}(0)$. Using $S(1) = S(0)$ we obtain $M(1, \lambda) = S(0)\tilde{M}(1, \lambda)S^{-1}(0)$. Thus the matrices $M(1, \lambda)$ and $\tilde{M}(1, \lambda)$ are similar and the spectra of these matrices coincide one with other, counted with multiplicity. Identity (3.12) follows (see, e.g., [DS], Th. XIII.7.64).

We will introduce the Jost type fundamental matrix solution $T(t, \lambda)$ of equation (3.17). This solution will be described below. Recall that the monodromy matrix has the form $M(1, \lambda)$, where the matrix-valued function $M(t, \lambda)$ satisfies the matrix equation (1.4). Rewrite equation (1.4) in the form
\[
M' - P^\mu(\lambda)M = Q^\mu(t)M, \quad (t, \lambda) \in \mathbb{R} \times \mathbb{C},
\]
where the $2p \times 2p$ matrices $P^\mu$ and $Q^\mu$ are given by
\[
P^\mu = P - (-1)^p \mu E, \quad Q^\mu = Q + (-1)^p \mu E, \quad \mu = \int_0^1 q_p(t) dt \in \mathbb{R},
\]
the matrices $P$, $Q$ are defined by (1.6), and the matrix $E$ is given by
\[
E = (E_{jk})_{j,k=1}^{2p}, \quad E_{p+1,p} = 1, \quad E_{jk} = 0, \quad \text{all } (j, k) \neq (p+1, p).
\]
Each matrix $P^\mu(\lambda)$, $\lambda \in \Lambda_R = \{ \lambda \in \mathbb{C} : |\lambda| > R^{2p} \}$ for some $R > 0$ large enough, has eigenvalues $z\Omega_j^\mu(\lambda)$, $j \in \mathbb{N}_{2p}$, all these eigenvalues are simple and the corresponding eigenvectors are given by
\[
U_j = \begin{pmatrix}
1 \\
(\Omega_j^0)^2 \\
(\Omega_j^p)^p \\
(\Omega_j^{p+1})^2 + (-1)^p \mu (\Omega_j^p)^{p-1} \\
\vdots \\
(\Omega_j^{2p})^2 + (-1)^p \mu (\Omega_j^p)^{2p-3}
\end{pmatrix}, \quad j \in \mathbb{N}_{2p}.
\]
Then the matrix $P^\mu$ is similar to the diagonal matrix
\[
zB^\mu = U^{-1}P^\mu U, \quad \text{where} \quad B^\mu = \text{diag}(\Omega_j^p)_{j=1}^{2p},
\]
and the $2p \times 2p$ matrix $U$ has the form
\[
U = (U_1 \ U_2 \ \ldots \ U_{2p}).
\]
We rewrite equation (3.14) in the form
\[
\tilde{M}' - zB^\mu(\lambda)\tilde{M} = \tilde{Q}(t, \lambda)\tilde{M}, \quad (t, \lambda) \in \mathbb{R} \times \Lambda_R,
\]
where
\[
\tilde{M} = U^{-1}MU, \quad \tilde{Q} = (\tilde{Q}_{ij})_{i,j=1}^{2p} = U^{-1}Q^\mu U.
\]
The following Lemma, proved in Appendix, shows that the matrix $\tilde{Q}$ is decreasing at large $|\lambda|$.
Lemma 3.4. The matrix-valued function \( \tilde{Q} \) satisfies the following asymptotics:
\[
\tilde{Q}(t, \lambda) = \frac{(-1)^{p+1}}{z}(q_p(t) - \mu)L(\lambda) + b(t)O(|\lambda|^{-1}), \quad L = (L_{jk})_{j,k=1}^{2p}, \quad L_{jk} = \frac{\partial^p G_{j,k}}{\partial \lambda^p},
\] (3.18)
uniformly on \( t \in [0, 1] \) as \( |\lambda| \to \infty \), where \( b(t) = \max_{t \neq \lambda} |g_j(t)| \).

Consider equation (3.17). Assume that this equation has the \( 2p \times 2p \) matrix-valued solution \( T(t, \lambda) \) for all \( t \in [0, 1] \) and some \( \lambda \in \Lambda_R \). Then \( \tilde{T}(t, \lambda) = T(t, \lambda)T(0, \lambda)^{-1} \) and
\[
\tilde{M}(t, \lambda) = \mathcal{U}(\lambda) \tilde{T}(t, \lambda) \mathcal{U}^{-1}(\lambda) = \mathcal{U}(\lambda)T(t, \lambda)T(0, \lambda)^{-1}U^{-1}(\lambda), \quad (t, \lambda) \in [0, 1] \times \Lambda_R. \] (3.19)

In order to analyze equation (3.17) by the Birkhoff method (see [Na]), we write the solution \( T \) in the form
\[
T = \mathcal{G}e^{zB^\mu t},
\] (3.20)
where \( \mathcal{G}(t, \lambda), (t, \lambda) \in [0, 1] \times \Lambda_R \), is some matrix-valued function. Substituting (3.20) into (3.17) we obtain
\[
\mathcal{G}' + z(\mathcal{G}B^\mu - B^\mu \mathcal{G}) = \tilde{Q}\mathcal{G}.
\] (3.21)
This equation is equivalent to the integral equation
\[
\mathcal{G} = \mathbb{1}_{2p} + K\tilde{Q}\mathcal{G},
\] (3.22)
where \( K \) is an integral operator given by
\[
(K\mathcal{A})_{ij}(t, \lambda) = \left\{ \begin{array}{ll}
\int_0^1 e^{z(t-s)(\Omega_{ij}(\lambda)\lambda - \Omega_{ij}(\lambda))}A_{ij}(s)ds, & \text{if } i > j \\
\int_{-1}^0 e^{z(t-s)(\Omega_{ij}(\lambda)\lambda - \Omega_{ij}(\lambda))}A_{ij}(s)ds, & \text{if } i \leq j
\end{array} \right. = \int_0^1 e_{ij}(t-s)\lambda A_{ij}(s)ds
\] (3.23)
for the matrix-valued function \( \mathcal{A} \), where \( (i, j, t, \lambda) \in \mathbb{N}_2 \times [0, 1] \times \Lambda_R \) and
\[
e_{ij}(t, \lambda) = \left\{ \begin{array}{ll}
e^{z(t)(\Omega_{ij}(\lambda)\lambda - \Omega_{ij}(\lambda))}\chi(t), & \text{if } i > j \\
-e^{-z(t)(\Omega_{ij}(\lambda)\lambda - \Omega_{ij}(\lambda))}\chi(-t), & \text{if } i \leq j
\end{array} \right., \quad \chi(t) = \left\{ \begin{array}{ll}
0, & \text{if } t < 0 \\
1, & \text{if } t \geq 0
\end{array} \right.
\] (3.24)
In fact, differentiating (3.22) we obtain
\[
\mathcal{G}'_{ij}(t, \lambda) = (K\tilde{Q}\mathcal{G})'_{ij}(t, \lambda) = \frac{d}{dt} \left\{ \int_0^1 e^{z(t-s)(\Omega_{ij}(\lambda)\lambda - \Omega_{ij}(\lambda))}(\tilde{Q}\mathcal{G})_{ij}(s)ds, \text{ if } i > j \right. \\
\left. - \int_{-1}^0 e^{z(t-s)(\Omega_{ij}(\lambda)\lambda - \Omega_{ij}(\lambda))}(\tilde{Q}\mathcal{G})_{ij}(s)ds, \text{ if } i \leq j \right.
\]
\[
= (\tilde{Q}\mathcal{G})_{ij}(t) + z(\Omega^\mu_{ij}(\lambda) - \Omega^\mu_{ij}(\lambda))(K\tilde{Q}\mathcal{G})_{ij}(t, \lambda), \quad \text{all } (i, j, t, \lambda) \in \mathbb{N}_2 \times [0, 1] \times \Lambda_R.
\] (3.25)
Using the identities
\[
(\Omega^\mu_{ij} - \Omega^\mu_{ij})(K\tilde{Q}\mathcal{G})_{ij} = (B^\mu(K\tilde{Q}\mathcal{G}) - (K\tilde{Q}\mathcal{G})B^\mu)_{ij} = (B^\mu \mathcal{G} - \mathcal{G}B^\mu)_{ij}
\]
we obtain \( \mathcal{G}' = \tilde{Q}\mathcal{G} + z(\mathcal{G}B^\mu - \mathcal{G}B^\mu) \). Thus \( \mathcal{G} \) satisfies (3.21), which yields the equivalence of equations (3.21) and (3.22).

In Lemma 3.3, proved in Appendix, we describe the Jost type fundamental matrix solution \( \mathcal{T} \) of equation (3.17). In fact we will prove that equation (3.22) has a unique solution for \( |\lambda| \) large enough. For the \( 2p \times 2p \) matrix-valued function \( \mathcal{A} \in L^\infty(0, 1) \) we introduce the norm
\[
\|\mathcal{A}\|_\infty = \sup_{t \in [0, 1]} |\mathcal{A}(t)|.
\]
Lemma 3.5. i) Let $A \in L^\infty(0, 1)$ be a $2p \times 2p$ matrix-valued function. Then for each $\lambda \in \Lambda_R$ for some $R > 0$ large enough the operator $K$ satisfies the estimate
\[
\|(K \bar{Q}A)(\cdot, \lambda)\|_\infty \leq \frac{\xi}{|z|}\|A\|_\infty, \quad \text{where} \quad \xi = \max\left\{4 \int_0^1 |q_p(t) - \mu| dt; 1\right\}.
\] (3.26)

ii) For each $\lambda \in \Lambda_R$ the integral equation (3.22) has the unique solution $G(t, \lambda)$. Each matrix-valued function $G(t, \cdot), t \in [0, 1]$, is analytic in $\Lambda_R^+$ and satisfies the estimates
\[
\|G(\cdot, \lambda)\|_\infty \leq 2, \quad \|G(\cdot, \lambda) - \mathbb{I}_{2p}\|_\infty \leq \frac{2\xi}{|z|}, \quad \|G(\cdot, \lambda) - \mathbb{I}_{2p} - G_1(t, \lambda)\|_\infty \leq \frac{2\xi^2}{|z|^2},
\] (3.27)
for all $\lambda \in \Lambda_{2R}$, where $G_1 = K\bar{Q}$,
\[
(G_1(t, \lambda))_{ij} = \frac{(-1)^{p+1}\Omega^p_{ij}}{2pz} \int_0^1 e_{ij}(t - s, \lambda)(q_p(s) - \mu)ds + O(|z|^{-2}), \quad i, j \in \mathbb{N}_{2p},
\] (3.28)
as $|\lambda| \to \infty$, uniformly on $t \in [0, 1]$.

iii) For each $t \in [0, 1]$ the function $T(t, \cdot)$ is analytic in $\Lambda_R^+$ and satisfies
\[
T(t, \lambda) = e^{\varepsilon B_r(\lambda)}(\mathbb{I}_{2p} + O(|z|^{-1})) \quad \text{as} \quad |\lambda| \to \infty,
\] (3.29)
uniformly on $t \in [0, 1]$.

Now we will prove the main result of this Section. In this Lemma 3.6 we obtain the representation and asymptotics of the monodromy matrix $M$, see (3.30)-(3.34).

Lemma 3.6. i) The monodromy matrix $M(1, \cdot)$ satisfies the identity
\[
M(1, \cdot) = UG(0, \cdot)F e^{\varepsilon B_r'(UG(0, \cdot))^{-1}}, \quad \text{where} \quad F = G^{-1}(0, \cdot)G(1, \cdot).
\] (3.30)

ii) The matrix-valued function $F = (F_{ij})_{i,j=1}^{2p}$ is analytic in $\Lambda_R^+$ and satisfies the asymptotics
\[
F_{ij}(\lambda) = \delta_{ij} + \frac{(-1)^p+1}{2pz} \Omega^p_{ij} \int_0^1 \xi_{ij}(t, \lambda)(q_p(t) - \mu)dt + O(|z|^{-2}),
\] (3.31)
as $|\lambda| \to \infty$, where
\[
\xi_{ij}(t, \lambda) = \begin{cases} e^{\varepsilon(1-t)\Omega(t, \lambda) - \Omega(t, \lambda)} & \text{if } i > j, \\ e^{-\varepsilon t\Omega(t, \lambda) - \Omega(t, \lambda)} & \text{if } i \leq j. \end{cases}
\] (3.33)

iii) The functions $F_{p+k,p+k+1}$ and $F_{p+k+1,p+k}$ for all $k \in \mathbb{N}_{p-1} = \{0, 1, \ldots, p-1\}$ satisfy
\[
F_{p+k,p+k+1}(\lambda) = f_{k,n} + O(n^{-2}), \quad F_{p+k+1,p+k}(\lambda) = \tilde{f}_{k,n} + O(n^{-2}),
\] (3.34)
as $n \to \infty$, $\lambda = (-1)^k \frac{(n \rightarrow odd)}{2\pi \mu} 2p + O(n^{-2p}), \text{Im } \lambda \geq 0$, where
\[
f_{k,n} = \frac{i(-1)^{k+1}c_k}{2\pi \mu n} \begin{cases} e^{\frac{i\pi}{2p\mu}} q_{p,n}, & \text{if } k \text{ odd,} \\ -e^{-i\frac{\pi}{2p\mu}} q_{p,n}, & \text{if } k \text{ even.} \end{cases}
\] (3.35)

Proof. i) Identities (3.19), (3.20) yield $M(t, \lambda) = U(\lambda)G(t, \lambda)e^{\varepsilon B_r'(\mathbb{I}^{-1}(0, \lambda)U^{-1}(\lambda))}$, which implies (3.30).

ii) Estimates (3.27) yield $G(t, \lambda) = \mathbb{I}_{2p} + G_1(t, \lambda) + O(|z|^{-2})$ as $|\lambda| \to \infty$, uniformly on $t \in [0, 1]$, and
\[
F(\lambda) = G^{-1}(0, \lambda)G(1, \lambda) = \mathbb{I}_{2p} + G_1(1, \lambda) - G_1(0, \lambda) + O(|z|^{-2}) \quad \text{as} \quad |\lambda| \to \infty.
\]
Substituting (3.28) into this asymptotics we obtain
\[ F_{ij}(\lambda) = \delta_{ij} + \frac{(-1)^{p+1}}{2p} \eta_j \int_0^1 (e_{ij}(1-t, \lambda) - e_{ij}(-t, \lambda))(q_p(t) - \mu)dt + O(|z|^{-2}) \]
as \( |\lambda| \to \infty \). Substituting (3.24) into the last asymptotics and using (2.25) we obtain (3.31), which yields the first asymptotics in (3.32). The second asymptotics in (3.32) follows from the identities (3.35) and the identity \( \int_0^1 q_p(t)dt = \mu \), see (3.15).

iii) Identity (2.3) gives \( \Omega_j = \omega_j \) for all \( j \in \mathbb{N}_{2p} \). Identities (2.3), \( \varepsilon_k^{-1} = \xi_k \) and \( \varepsilon_k^{2p} = (-1)^{p+k} \) (see (2.4)) yield for \( k \in \mathbb{N}_{p-1}^0 \):
\[ \bar{\omega}^p \omega_{s+1}^p = \varepsilon_k^{2p-1} \eta_k = (-1)^{s} \xi_k \eta_k, \quad \bar{\omega}_{s+1}^p = \xi_k^{2p-1} \eta_k = (-1)^{s} \varepsilon_k \eta_k, \quad s = p + k. \] (3.36)

Substituting identities (3.36) into asymptotics (3.31) and using (3.33) we obtain
\[ F_{s,s+1}(\lambda) = \frac{(-1)^{k+1} \xi_k \varepsilon_k}{2p} \int_0^1 e^{\varepsilon_k (t)(\omega_s - \omega_{s+1})} (q_p(t) - \mu)dt + O(|z|^{-2}), \]
\[ F_{s+1,s}(\lambda) = \frac{(-1)^{k+1} \xi_k \varepsilon_k}{2p} \int_0^1 e^{\varepsilon_k (t)(\omega_{s+1} - \omega_s)} (q_p(t) - \mu)dt + O(|z|^{-2}) \] (3.37)
as \( |\lambda| \to \infty \), \( k \in \mathbb{N}_{p-1}^0 \). Let \( \lambda = (-1)^{k} \frac{2p}{c_k} + O(n^{2p-2}) \) as \( n \to \infty \). Then \( z = \lambda \frac{2p}{c_k} + O(n^{-1}) \) and using (2.3) we have \( z(\bar{\omega}_s - \omega_{s+1}) = i(-1)^{k+1}2\pi n + O(n^{-1}) \). Substituting this asymptotics into (3.37) we obtain
\[ F_{s,s+1}(\lambda) = \frac{(-1)^{k+1} \xi_k c_k}{2p} \int_0^1 e^{(-1)^{k}2\pi n t} (q_p(t) - \mu)dt + O(n^{-2}), \]
\[ F_{s+1,s}(\lambda) = \frac{(-1)^{k+1} \xi_k c_k}{2p} \int_0^1 e^{(-1)^{k+1}2\pi n t} (q_p(t) - \mu)dt + O(n^{-2}) \] (3.38)
as \( n \to \infty \). Substituting \( \varepsilon_k \) from (2.4) into (3.38), we get (3.33).

4. Properties of the multipliers

Define the single-valued branches of the multiplier \( \tau \) at high energy. Here we use the results of Lemma 4.3 which will be proved later. The zeros of the function \( \rho \) for high energy are close to the real axis. Then the functions \( \tau_j, j \in \mathbb{N}_{2p} \), are analytic in the domain \( \Lambda_R \cap \{ \lambda \in \mathbb{C} : \delta < \arg \lambda < \pi - \delta \} \) for some \( R > 0 \) large enough and for any \( \delta > 0 \) small enough. Asymptotics (1.8) define the branches \( \tau_j, j \in \mathbb{N}_{2p} \), of the function \( \tau \) in this domain.

Moreover, the points \( r_{k,n}^+ \in \Lambda_R, k \in \{ j - 1, j \}, j \in \mathbb{N}_{p-1}, n \geq n_0 \) for some (large) \( n_0 \geq 1 \), are ramification points of the functions \( \tau_{p+j} \) and \( \tau_{p-j+1} \) and these functions have no any other singularities in \( \Lambda_R \) (see discussion after the proof of Lemma 4.4). Here \( r_{k,n}^+ \) satisfy:

1) \( |r_{k,n}^+ - r_{k,n}^-| < 1 \) for all \( k \in \mathbb{N}_{p-1}^0 = \{ 0, ..., p - 1 \}, n \geq n_0 \),
2) all \( r_{0,n}^+ \), \( n \geq n_0 \), are positive numbers (anti-periodic and periodic eigenvalues),
3) all \( r_{k,n}^+ \), \( k \in \mathbb{N}_p, n \geq n_0 \), are real or non-real numbers (ramification points of the Lyapunov function), \( \pm \text{Im} r_{k,n}^+ \geq 0 \) and if \( \text{Im} r_{k,n}^+ > 0 \) for some \( k \in \mathbb{N}_p, n \geq n_0 \), then \( r_{k,n}^- = \pi_{k,n}^+ \).

Then each function \( \tau_{p+j}, \tau_{p-j+1}, j \in \mathbb{N}_p \), is a single-valued analytic function in the domain \( \mathcal{D}_{p+j} \), where
\[ \mathcal{D}_{p+j} = \Lambda_R \setminus \cup_{n \geq n_0}(\Gamma_{j-1,n} \cup \Gamma_{j,n}) \quad \text{all} \quad j \in \mathbb{N}_{p-1}, \quad \mathcal{D}_{2p} = \Lambda_R \setminus \cup_{n \geq n_0} \Gamma_{p-1,n}. \]
Lemma 4.1. i) Let $k$ monodromy matrix $M_R \in \tau$ and the matrix-valued function $W_{j,n}$ multiplier function $i)$. Recall the following Gershgorin’s result from the matrix theory, see [HJ]:

point $\lambda$ where we used the estimate $|Gershgorin’s theorem, the disc $z$ every eigenvalue of $A$ $K$ $| \lambda | < R$ then $k = j + 1$. No more than two multipliers $\tau_j, j \in \mathbb{N}_p$, can coincide one with other at the point $\lambda$.

ii) The multipliers satisfy the identities

$$\tau_{p-j+1}(\lambda) = \tau_{p-j}^{-1}(\lambda), \quad \text{all } (j, \lambda) \in \mathbb{N}_p \times \mathbb{D}_{p+j}.$$ (4.2)

Proof. i) Recall the following Gershgorin’s result from the matrix theory, see [HJ]:

Let $A = (A_{ij})_{i,j=1}^m$ be a complex $m \times m$ matrix and let $R_i = \sum_{j=1,j \neq i}^m |A_{ij}|$ for $i \in \mathbb{N}_m$. Then every eigenvalue of $A$ lies within at least one of the discs \( \{ \tau \in \mathbb{C} : |\tau - A_{ii}| < R_i \}, i \in \mathbb{N}_m \). If the union of $k$ discs is disjoint from the union of the other $n-k$ discs then the former union contains exactly $k$ and the latter $n-k$ eigenvalues of $A$.

Identity (3.30) implies that the matrices $\mathcal{M}(1, \cdot)$ and $\mathcal{X} = e^{zB^0} \mathcal{F}$ have the same eigenvalues (the multipliers). Asymptotics (2.25), (3.32) give

$$\mathcal{X} = e^{zB^0} \left( I_{2p} + W(z) \right) \left( \frac{1}{|z|} \right), \quad \text{where } B^0 = \text{diag}(\Omega_j)^{2p},$$

and the matrix-valued function $W(z) = (w_{ij}(z))_{i,j=1}^{2p}$ is uniformly bounded on $|z| > R$ for some $R > 0$. By Gershgorin’s theorem every multiplier $\tau$ lies within at least one of the discs

$$\mathcal{K}_j = \left\{ \tau \in \mathbb{C} : |\tau - e^{iz}\Omega_j| < |e^{iz}\Omega_j| w_j \right\}, \quad j \in \mathbb{N}_{2p}, \quad \text{where } w_j = \max_{|z| > R} \sum_{k=1}^{2p} |w_{jk}(z)|.$$

Firstly, let the disc $\mathcal{K}_j$ for some $j \in \mathbb{N}_{2p}$ be disjoint from the other discs $\mathcal{K}_k, k \neq j$. Then, by Gershgorin’s theorem, the disc $\mathcal{K}_j$ contains exactly one multiplier $\tau_j(z)$, which satisfies the estimate $|\tau_j(z) - e^{iz}\Omega_j| < |e^{iz}\Omega_j| w_j$. This estimate gives (4.1) for this case.

Secondly, consider all $k, j, 1 \leq j < k \leq 2p$, such that $\mathcal{K}_j \cap \mathcal{K}_k \neq \emptyset$. Then the distance between the centers of these discs is less than the sum of their radii: $|e^{iz}\Omega_j - e^{iz}\Omega_k| < \frac{1}{|z|} (|e^{iz}\Omega_j| w_j + |e^{iz}\Omega_k| w_k)$. Then

$$|e^{iz}(\Omega_k - \Omega_j) - 1| < \frac{1}{|z|} (w_j + w_k),$$ (4.3)

where we used the estimate $|e^{iz}(\Omega_k - \Omega_j)| = e^{Re(z)(\Omega_k - \Omega_j)} \leq 1$ (see (2.7)). If $k \geq j + 2$, then estimates (2.8) together with (2.7) yields $Re(z(\Omega_k - \Omega_j) < -a|z|, a > 0$, and then $e^{iz(\Omega_k - \Omega_j)} \to 0$ as $|z| \to \infty$. For $|z|$ large enough we have a contradiction with (4.3). Hence $k = j + 1$.

Moreover, the similar arguments show that only two domains $\mathcal{K}_j, \mathcal{K}_{j+1}$ can intersect each
Then the matrix $M$ and $K$ and $\tau$ imply
\[ |e^{z\Omega_j} - e^{z\Omega_{j+1}}| < \frac{1}{|z|}|e^{z\Omega_j}w_j + |e^{z\Omega_{j+1}}w_{j+1}|, \]
\[ |e^{z\Omega_{j+1}} - e^{z\Omega_{j+2}}| < \frac{1}{|z|}(e^{z\Omega_{j+1}}w_{j+1} + |e^{z\Omega_{j+2}}w_{j+2}|), \]
which yield
\[ |e^{z\Omega_j} - e^{z\Omega_{j+2}}| \leq |e^{z\Omega_j} - e^{z\Omega_{j+1}}| + |e^{z\Omega_{j+1}} - e^{z\Omega_{j+2}}| < \frac{1}{|z|}(e^{z\Omega_j}w_j + 2|e^{z\Omega_{j+1}}w_{j+1}| + |e^{z\Omega_{j+2}}w_{j+2}|). \]
Then
\[ |1 - e^{z(\Omega_{j+2} - \Omega_j)}| < \frac{1}{|z|}(w_j + 2w_{j+1} + w_{j+2}), \]
which is in contradiction with the estimate $Re z(\Omega_{j+2} - \Omega_j) < -a|z|, a > 0$. Thus only two domains $\mathcal{K}_j, \mathcal{K}_{j+1}$ can intersect each other and they are disjoint from other domains $\mathcal{K}_m, m \neq j, j + 1$.

Let $\mathcal{K}_j \cap \mathcal{K}_{j+1} \neq \emptyset$ for some $j \in \mathbb{N}_{2p-1}$. By Gershgorin’s theorem, the domain $\mathcal{K}_j \cup \mathcal{K}_{j+1}$ contains exactly two multipliers $\tau_j(z), \tau_{j+1}(z)$ and $|\tau_{j+1} - e^{z\Omega_{j+1}}| < \frac{1}{|z|}(2|e^{z\Omega_j}|w_j + |e^{z\Omega_{j+1}}w_{j+1}|)$, which implies
\[ |\tau_{j+1}e^{-z\Omega_{j+1}} - 1| < \frac{1}{|z|}(2w_j + |e^{z(\Omega_{j+1} - \Omega_j)}w_{j+1}|). \]
Estimate (4.3) yields that $e^{z(\Omega_{j+1} - \Omega_j)} = 1 + O(|z|^{-1})$ and then $\tau_{j+1}$ satisfies asymptotics (4.1).

The similar arguments show that $\tau_j$ also satisfies (4.1). In fact, $|\tau_j - e^{z\Omega_j}| < \frac{1}{|z|}(e^{z\Omega_j}w_j + 2|e^{z\Omega_{j+1}}w_{j+1}|)$, which implies
\[ |\tau_j e^{-z\Omega_j} - 1| < \frac{1}{|z|}(2w_{j+1} + |e^{z(\Omega_{j+1} - \Omega_j)}w_{j+1}|). \]
Since $e^{Re z(\Omega_{j+1} - \Omega_j)} \leq 1$, we obtain asymptotics (4.1) for $\tau_j$.

ii) Asymptotics (4.1) and identities (2.14) provide (4.2).

**Proof of Theorem 1.1.** The proof of identity (1.10) is standard. We rewrite equation (1.4) in the form $JM' = H M$, where $H = J(P + Q)$. $J Q$ is a diagonal $2p \times 2p$ matrix and
\[ J P = \begin{pmatrix} (-1)^p \lambda & O_{2p-1,p} \\ O_{1,2p-1} & J \end{pmatrix}, \quad J = \begin{pmatrix} O_{p-1,p} & O_{p-1,1} & -J_{p-1} \\ O_{1,p-1} & 1 & O_{1,1} \\ -J_{p-1} & O_{p-1,1} & O_{p-1,p} \end{pmatrix}. \]

Then the matrix $H$ is symmetric: $H^T = H$. Using the identity $J^T = -J$ we obtain $-(M^T)' J = M H M'$, which yields
\[ (M^T J M)' = (M^T)' J M + M^T J M' = -M^T H M + M^T H M = 0. \]

Then $M^T J M = \text{const}$ and using $M(0) = I_{2p}$ we obtain (1.10).

i) We will use the arguments from [CK]. Identity (1.10) yields
\[ D(\tau, \cdot) = \tau^{2p} D(\tau^{-1}, \cdot), \quad \tau \neq 0, \quad (4.4) \]
and $D(\tau, \lambda) = \sum_{k=0}^{2p} x_k(\lambda) \tau^{2p-k}$, where the functions $x_k$ are given by
\[ x_0 = 1, \quad x_1 = -2p T_1, \quad x_2 = -\frac{2p}{2}(T_2 + T_1 x_1), \quad ..., x_k = -\frac{2p}{k} \sum_{j=0}^{k-1} T_{k-j} x_j, ..., \quad T_k = \frac{\text{Tr} M^k(1, \cdot)}{2p}, \]
(see [RS2], p.331-333). By Lemma 3.2 the coefficients $\zeta_{k}(\lambda)$ are entire in $\lambda \in \mathbb{C}$. Using the identity (4.4) we obtain $x_{2p-j} = \zeta_{j}$, $j \in \mathbb{N}_{p}$, which yields $D(\tau, \cdot) = (\tau^{2p} + 1) + \zeta_{1}(\tau^{2p-1} + \tau) + \ldots + \zeta_{p-1}(\tau^{p+1} + \tau^{p-1}) + \zeta_{p} \tau^{p}$. Then
\[
D(\tau, \lambda) = \nu^{p} + f_{1}(\lambda) \nu^{p-1} + \ldots + f_{p}(\lambda), \quad \nu = \frac{\tau + \tau^{-1}}{2},
\] (4.5)
where $f_{1}, \ldots, f_{p}$ are some linear combinations of $x_{0}, \ldots, x_{p}$. In particular, all coefficients $f_{1}(\lambda), \ldots, f_{p}(\lambda)$ are entire functions. The function $\Phi(\nu, \lambda) = \frac{D(\nu, \lambda)}{(2\tau)^{p}}$ is a polynomial of $\nu$ degree $p$. Each zero $\Delta_{j}$, $j \in \mathbb{N}_{p}$, of this function satisfies $\Delta_{j} = \frac{1}{2}(\tau_{j} + \tau_{j}^{-1}), j \in \mathbb{N}_{p}$, where $\tau_{j}, \tau_{j}^{-1}$ are multipliers, and identity (1.11) holds. Asymptotics (1.8) yields (1.12). Recall that $\tau_{j}$ are branches of the function $\tau$ analytic on the $2p$ sheeted Riemann surface (see Sect.1). Then $\Delta_{j}(\lambda), j \in \mathbb{N}_{p}$, constitute $p$ branches of one analytic function $\Delta(\lambda)$ on the connected $p$-sheeted Riemann surface $\mathcal{R}$.

ii) Proof repeats the standard arguments (see [CK]).

iii) Let $\mathcal{M}_{n}, n \geq 1$, be the monodromy matrix for the operator $H_{n}$ given by (3.3). Let $\tau_{j,n}, j \in \mathbb{N}_{p}$, be the multipliers of $H_{n}$. Identity (3.12) gives
\[
\sigma(H_{n}) = \{ \lambda \in \mathbb{R} : |\tau_{j,n}(\lambda)| = 1 \text{ for some } j \in \mathbb{N}_{p} \}. \tag{4.6}
\]
Lemma 3.2 provides $\mathcal{M}_{n} \to \mathcal{M}(1, \cdot)$ as $n \to \infty$ uniformly on any compact in $\mathbb{C}$. Then $\tau_{j,n} \to \tau_{j}$ as $n \to \infty$ uniformly on any compact in $\mathbb{C}$ for all $j \in \mathbb{N}_{p}$. Identity (4.6) implies
\[
\sigma(H_{n}) \to \{ \lambda \in \mathbb{R} : |\tau_{j}(\lambda)| = 1 \text{ for some } j \in \mathbb{N}_{p} \} \quad \text{as } n \to \infty. \tag{4.7}
\]
Assume that
\[
\sigma(H_{n}) \to \sigma(H) \quad \text{as } n \to \infty. \tag{4.8}
\]
Then using (4.7) we obtain $\sigma(H) = \{ \lambda \in \mathbb{R} : |\tau_{j}(\lambda)| = 1 \text{ for some } j \in \mathbb{N}_{p} \}$, which yields (4.13).

Now we will prove (4.8). We need the following result (see [Ka], Th. VI.5.13, Cor. V.4.2):

Let $A \geq 0$ be an operator in a Hilbert space $\mathcal{H}$ and let $Q_{n}$ be a symmetric operator in $\mathcal{H}$ with the form domain $\text{Dom}_{fd} Q_{n} \subset \text{Dom}_{fd} A$. Assume that
\[
\| (Q_{n}v, u) \| \leq \varepsilon_{n}(\| u \|^{2} + (Au, u)), \quad u \in \text{Dom}_{fd} Q_{n}, \quad \text{where } \varepsilon_{n} > 0, \ \varepsilon_{n} \to 0.
\]
Then the the Friedrichs extension $A_{n}$ of the operator $A + Q_{n}$ is selfadjoint for sufficiently large $n$ and $A_{n} \to A$ in the uniform resolvent sense. If $\sigma(A)$ has a gap at $\alpha$, then $\sigma(A_{n})$ has a gap at $\alpha$ for sufficiently large $n$.

Estimate (3.4) shows that $H_{n} \to H$ in the uniform resolvent sense (and then in the strong resolvent sense) as $n \to \infty$, and $\lim_{n \to \infty} \sigma(H_{n}) \subset \sigma(H)$. Then relation (4.8) is obtained from the following result (see [Ka], Th. VIII.1.14):

Let $H, H_{n}, n \geq 1$, be selfadjoint operators in a Hilbert space $\mathcal{H}$ and let $H_{n} \to H$ in the strong resolvent sense. Then every open set containing a point of $\sigma(H)$ contains at least a point of $\sigma(H_{n})$ for sufficiently large $n$. ■

Consider two simple examples.

**Example 1.** Consider the operator $H = (-1)^{p} \frac{d^{2p}}{dx^{2p}} + \sum_{j=0}^{p-1} q_{j+1} \frac{d^{j}}{dx^{j}}$ with the constant coefficients $q_{j}$. Equation (1.3) has the solutions $e^{\pm \zeta_{j}(\lambda)t}$, where $\zeta_{j}(\lambda) = \sqrt{w_{j}(\lambda)}, j \in \mathbb{N}_{p}$, and $w_{j}$ are values of the algebraic function $w(\lambda)$ which is a solution of the equations $P(w) - \lambda = 0, P(w) = w^{p} + \sum_{j=0}^{p-1} (-1)^{j} q_{j+1} w^{j}$. Then the multipliers have the form $e^{\pm \zeta_{j}(\lambda)}$ and the Lyapunov function is given by $\Delta_{j}(\lambda) = \cos \zeta_{j}(\lambda)$. If the polynomial $P(w) - \lambda$ for some $\lambda \in \mathbb{R}$ has $n$ $(1 \leq n \leq p)$ positive simple zeros, then the spectrum $\sigma(H)$ in some interval $(\lambda - \varepsilon, \lambda + \varepsilon), \varepsilon > 0$,
has multiplicity $2n$. Let $P(w) = T_p(w - 1)$, where $T_p(w) = \sum_{i=0}^{[p/2]} \binom{p}{2i} (w^2 - 1)^i w^{p-2i}$ is the Chebyshev polynomial. The properties of these polynomials (see [AS]) provide that the spectrum is given by $\sigma(H) = [-1, +\infty)$ and has multiplicity $2p$ (maximal multiplicity) on the interval $(-1, 1)$ and the multiplicity 2 (minimal multiplicity) on $(1, +\infty)$.

**Example 2.** Consider the operator $H = \tilde{H}^p$, where $\tilde{H} = -\frac{d^2}{dz^2} + q$ is the Hill operator with the 1-periodic function $q \in L^2(\mathbb{T})$. The branches of the Lyapunov function are given by $\Delta_j(\lambda) = \tilde{\Delta}(-z^2\omega_j^2)$, $j \in \mathbb{N}_p$, where $\tilde{\Delta}$ is the (entire) Lyapunov function of $\tilde{H}$. Recall that the spectrum $\sigma(H)$ is semi-bounded below and consists of bands separated by gaps. Let $\sigma_j = \{ \lambda \in \mathbb{R} : \tilde{\Delta}(-z^2\omega_j^2) \in [-1, 1], j \in \mathbb{N}_p, z = \lambda^\frac{1}{2} \in S \}$. The spectrum of $H$ satisfies the identity $\sigma(H) = \cup_p \sigma_j = \sigma_1 \cup \sigma_p$. We have

$$\sigma_1 = \begin{cases} 
\{ \lambda \in [0, \infty) : -z^2 \in \sigma_{-} \}, & p \text{ even} \\
\{ \lambda \in (-\infty, 0) : e^{-i\pi z^2} < 0 \} \cap \sigma_{-} \end{cases}, \quad \sigma_p = \{ \lambda \in [0, \infty) : z^2 \in \sigma(\tilde{H}) \cap [0, \infty) \},$$

where $\sigma_\ast = \sigma(\tilde{H}) \cap (-\infty, 0]$. If $p$ is odd, then the spectrum $\sigma(H)$ has multiplicity 2. If $p$ is even, then the spectrum has multiplicity 4 in the set $\sigma_1 \cap \sigma_p$ and multiplicity 2 in the other intervals.

Recall that all functions $\rho, D_\pm = 2^{-n}D(\pm1, \cdot) = \prod_{j=1}^{p} (\Delta_j \mp 1)$ are entire. The zeros of $\rho$ are ramifications of the Lyapunov function, the zeros of $D_\pm$ are periodic and antiperiodic eigenvalues. We introduce the contours $C_n(r) = \{ \lambda : |\lambda - \pi n| = \pi r \}$, $r > 0, n \geq 0$.

**Lemma 4.2.** Let $N \in \mathbb{N}$ be large enough. Then the function $D_+$ (and $D_-$) has exactly $2N + 1$ (and $2N$) zeros in the domain $\{|z| < 2\pi(N + \frac{1}{2})\}$ (and in $\{|z| < 2\pi N\}$), counted with multiplicity. Moreover, for each $n > N$ the function $D_+$ (and $D_-$) has exactly two zeros in the disk $\{|z - 2\pi n| < \frac{\pi}{2}\}$ (and in $\{|z - \pi(2n + 1)| < \frac{\pi}{2}\}$), counted with multiplicity. There are no other zeros.

**Proof.** We consider the function $D_+$. The proof for $D_-$ is similar. The function $D_+$ for the operator $H^0$ is given by

$$D_+^0 = \prod_{j=1}^{p} (\cosh z\omega_j - 1) = -\frac{\lambda}{2^p} \prod_{n=1}^{\infty} \left( 1 - \frac{\lambda}{(2n\pi)^2} \right)^2.$$ 

Assume that for each $j \in \mathbb{N}_p$ and for some $R > 0$ large enough

$$|\cosh z\omega_j - 1| = 2\left| \sinh \frac{z\omega_j}{2} \right| > \frac{1}{8}e^{\text{Re} z\omega_j}, \quad \text{all } \lambda \in \Lambda_R \setminus \cup_{n \in \mathbb{N}} \{ |z - 2\pi n| \leq \frac{\pi}{2} \}. \quad (4.9)$$

Then asymptotics (4.12) and estimates (4.9) yield

$$\frac{D_+(\lambda)}{D_+^0(\lambda)} = \prod_{j=1}^{p} \frac{\Delta_j(\lambda) - 1}{\cosh z\omega_j - 1} = \prod_{j=1}^{p} \cosh z\omega_j - 1 + O(|z|^{-1}e^{\text{Re} z\omega_j}) \cosh z\omega_j - 1 = 1 + O(|z|^{-1}) \quad (4.10)$$

as $\lambda \in \mathbb{C} \setminus \cup_{n \in \mathbb{N}} \{ |z - 2\pi n| \leq \frac{\pi}{2} \}, |\lambda| \to \infty$. Let $N' > N$ be another integer. Let $\lambda$ belong to the contours $C_0(2N + \frac{1}{2}), C_0(2N' + \frac{1}{2}), C_2n(\frac{1}{2}), |n| > N$. Asymptotics (4.10) yields

$$|D_+(\lambda) - D_+^0(\lambda)| = |D_+^0(\lambda)| \left| \frac{D_+(\lambda)}{D_+^0(\lambda)} - 1 \right| = |D_+^0(\lambda)||O(|z|^{-1}) < |D_+^0(\lambda)|.$$
on all contours. Hence, by Rouché’s theorem, $D_+$ has as many zeros, as $D_0^+$ in each of the bounded domains and the remaining unbounded domain. Since $D_0^+$ has exactly one simple zero at $\lambda = 0$ and exactly one zero of multiplicity two at $(2\pi n)^2$, $n \geq 1$, and since $N' > N$ can be chosen arbitrarily large, the statement for $D_+$ follows.

We will prove (4.9). Using the simple estimate $\eta \left| \frac{1}{\sin z} \right| < 4 |\sin z|$ as $|z - \pi n| > \frac{\pi}{2}$ for all $n \in \mathbb{Z}$ (see [PT], Lemma 2.1) we obtain that estimates (4.9) hold in the domain $\mathbb{R} \setminus \bigcup_{n \in \mathbb{Z}} \{|z\omega_j + i2\pi n| \leq \frac{\pi}{2}\}$. For each $j \neq p$ the estimates $|z\omega_j + i2\pi n| > \frac{\pi}{2}$ hold for all $n \in \mathbb{Z}$ and $|\lambda|$ large enough. Moreover, the estimates $|z\omega_p + i2\pi n| = |z - 2\pi n| > \frac{\pi}{2}$ hold for all $n \leq 0$ and $|\lambda|$ large enough. Thus, estimates (4.9) hold in $\Lambda_R \setminus \bigcup_{n \in \mathbb{N}} \{|z - 2\pi n| \leq \frac{\pi}{2}\}$. □

Now we will describe the zeros of the function $\rho$. Identifying the sides of the sector $S = \{z \in \mathbb{C} : \arg z \in (-\frac{\pi}{2p}, \frac{\pi}{2p})\}$ (i.e. we identify each point $xe^{i\pi\frac{m}{p}}, x \in \mathbb{R}_+$, on the $z$-plane with the point $xe^{-i\pi\frac{m}{p}}$) we obtain the cone $S_{con}$. For each $(k, n) \in \mathbb{N}_{p-1} \times \mathbb{N}_0, \mathbb{N}_0 = \mathbb{N} \cup \{0\}$, we introduce the domain $U_{k,n}$ given by

$$U_{k,n} = \left\{ \lambda = z^{2p}, z \in S_{con} : \left| z - \pi n \frac{\eta_k}{c_k} \right| < \beta \right\}, \quad \beta > 0 \quad \text{is small enough.} \quad (4.11)$$

Each domain $U_{k,n}$ is a neighborhood of the zero $r^0_{k,n} = (\pi n_c)^{2p}$ of the function $\rho^0$, see (1.16), where $\eta_k$ is given by (2.3).

We have $U_{1,0} = \cdots = U_{p-1,0}$ and $U_{1,0} \cap U_{k,n} = \emptyset$ for all $(k, n) \in \mathbb{N}_{p-1} \times \mathbb{N}$. If $p = 2$ or 3, then the domains $U_{k,n}, U_{k',n'}; (k, n), (k', n') \in \mathbb{N}_{p-1} \times \mathbb{N}, (k', n') \neq (k, n)$, are separated, that is $U_{k,n} \cap U_{k',n'} = \emptyset$. The situation is more complicated for $p \geq 4$ (see Fig. 4). In this case we have $U_{2k,n} \cap U_{2k'-1,n'} = \emptyset$ for all $2k, 2k' - 1 \in \mathbb{N}_{p-1}, n, n' \in \mathbb{N}$. However, the domains $U_{2k,n}, U_{2k'-1,n'}$ can have non-empty intersection for some $2k, 2k' \in \mathbb{N}_{p-1}, n, n' \in \mathbb{N}$ such that $(k', n') \neq (k, n)$. The similar statement for the domains $U_{2k-1,n}, U_{2k'-1,n'}$ holds.

We introduce the cluster decomposition of the set of indices $(k, n) \in \mathbb{N}_{p-1} \times \mathbb{N}_0$, having the form $\bigcup_{j=-\infty}^\infty \mathcal{C}_j = \mathbb{N}_{p-1} \times \mathbb{N}_0$, where

i) The indices $(k, n), (l, m)$ belong to the cluster $\mathcal{C}_j$ iff the domains $U_{k,n}$ and $U_{l,m}$ are connected one with another by a chain of the pairwise intersecting domains $U_{k,n,z}$.\footnote{\textit{Figure 4.} The domains $U_{k,n}$ in the $z$-plane for $p = 6$.}

ii) $\mathcal{C}_0 = \{(k, 0), k \in \mathbb{N}_{p-1}\}$.\footnote{\textit{Figure 4.} The domains $U_{k,n}$ in the $z$-plane for $p = 6$.}

iii) If $r^0_{k,n} < r^0_{k',n'}$ and $(k, n) \in \mathcal{C}_j, (k', n') \in \mathcal{C}_{j'}$, then $j \leq j'$.\footnote{\textit{Figure 4.} The domains $U_{k,n}$ in the $z$-plane for $p = 6$.}
These domains satisfy the following relations:
\[ \nu_j \subset \{ \lambda \in \mathbb{C} : \Im z < \beta \}, \quad \nu_j' \subset \{ \lambda \in \mathbb{C} : \Im z e^{-\frac{\beta}{\sqrt{p}}} < \beta \}, \]
for all \( j \geq 0 \). Moreover, if \( \lambda_j \in \nu_j \), \( \lambda_j' \in \nu_j' \), and \( j < j' \), then \( \Re \lambda_j < \Re \lambda_j' \), and each domain \( \nu_{j-1} \) is separated from \( \nu_j, \lambda_j \in \mathbb{Z} \), by the line \( \{ \lambda : \Re z = R_j \} \) for some \( R_j \in \mathbb{R} \).

**Lemma 4.3.** The function \( \rho \) has as many zeros, counted with multiplicity, as the function \( \rho^0 \), in each domain \( \{ z : \Re z < R_N, \Im z \notin N \} \) and in each domain \( \nu_j \), \( |j| > N \) for \( N \in \mathbb{N} \) large enough. There are no other zeros.

**Proof.** Recall that
\[ \rho^0 = \prod_{1 \leq j < \ell \leq p} (\cosh z \omega_j - \cosh z \omega_\ell)^2 = -\frac{(-1)^{\frac{p(p+1)}{2}} p^p \lambda^{p-1}}{2(p-1)p} \prod_{n=1}^{p-1} \prod_{k=1}^{\infty} \left( 1 - \frac{\lambda}{\ell_{k,n}} \right)^2. \]
Assume that for each \( 1 \leq j < \ell \leq p \), and for some \( c > 0 \)
\[ |\cosh z \omega_j - \cosh z \omega_\ell| > c e^{\max(|\Re z \omega_j|, |\Re z \omega_\ell|)} \quad \text{as} \quad \left| z - \pi n \frac{\eta_k}{c_k} \right| > \beta, \quad \text{all} \quad (k, n) \in \mathbb{N}_{p-1} \times \mathbb{N}. \]

Asymptotics (4.12) and estimates (4.15) yield
\[ \frac{\rho(\lambda)}{\rho^0(\lambda)} = \prod_{1 \leq j < \ell \leq p} \left( \frac{\Delta_j(\lambda) - \Delta_\ell(\lambda)}{\cosh z \omega_j - \cosh z \omega_\ell} \right) = \prod_{1 \leq j < \ell \leq p} \left( \frac{\cosh z \omega_j - \cosh z \omega_\ell + O(|z|^{-1} e^{\Re z \omega_j}) + O(|z|^{-1} e^{\Re z \omega_\ell})}{\cosh z \omega_j - \cosh z \omega_\ell} \right)^2 = 1 + O(|z|^{-1}) \]
as \( |\lambda| \to \infty \). Let \( N \in \mathbb{N} \) be large enough and let \( N' > N \) be another integer. Let \( \lambda \) belong to the contours \( C_0(R_N), C_0(R_N'), \partial Y_n, |n| > N \). Asymptotics (4.16) on all contours yields
\[ |\rho(\lambda) - \rho^0(\lambda)| = |\rho^0(\lambda)| O(|z|^{-1}) < \rho^0(\lambda). \]
Hence, by Rouché’s theorem, \( \rho \) has as many zeros, as \( \rho^0 \) in each of the bounded domains and the remaining unbounded domain. Since \( N_1 > N \) can be chosen arbitrarily large, the statement follows.

We have to prove estimates (4.15). Let \( \left| z - \pi n \frac{\eta_k}{c_k} \right| > \beta \) for all \( (k, n) \in \mathbb{N}_{p-1} \times \mathbb{N} \). Then
\[ 2e^{\pi n c_k} - 2\pi n > 2\beta c_k. \]
Identities (2.23) give \( |z(\omega_{p+k} - \omega_{p+k+1}) + \pi 2n| > \beta c_k \) and a fortiori
\[ |z(\omega_j + \omega_\ell)| > \beta c_k \]
for all \( 1 \leq j < \ell \leq p \). Using the standard estimates we obtain
\[ |\cosh z \omega_j - \cosh z \omega_\ell| = \left| \sinh \frac{z(\omega_j - \omega_\ell)}{2} \right| \left| \sinh \frac{z(\omega_j + \omega_\ell)}{2} \right| > c e^{|\Re z (\omega_j - \omega_\ell)| + |\Re z (\omega_j + \omega_\ell)|}, \]
for some $c > 0$, which yields (4.15). ■

**Lemma 4.4.** i) Let $\tau_{p+k}(\lambda) = \tau_{p+j}(\lambda)$ for some $0 \leq k < j \leq p, \lambda \in \Lambda_R$, where $R > 0$ is large enough. Then $j = k + 1$ and $\lambda \in \mathcal{U}_{k,n}$ for some (large) $n \in \mathbb{N}$. Moreover, in this case $\lambda \in \mathcal{U}_{k,n}$ and $\lambda$ is also the zero of the function $\tau_{p+k} - \tau_{p+k+1}$.

ii) Let $r_{k,n}^{\mu,\pm}, k \in \mathbb{N}_{p-1}, n \geq 0$, be ramifications of the Lyapunov function for the operator $H^\mu$.

Then

$$r_{k,n}^{\mu,\pm} = (-1)^k \left( \frac{\pi n}{c_k} \right)^{2p} \left[ 1 + (-1)^{p+1} \frac{\mu c_k^2}{(\pi n)^2} \left( 1 + O(n^{-2}) \right) \right] \quad \text{as} \quad n \to \infty, \quad \text{all} \quad k \in \mathbb{N}_{p-1}. \quad (4.17)$$

Moreover, all ramifications $r_{k,n}^{\mu,\pm}$ are real for $n$ large enough.

**Proof.** i) Let $\tau_{p+k}(\lambda) = \tau_{p+j}(\lambda)$ for some $0 \leq k < j \leq p, \lambda \in \Lambda_R$. Then, due to Lemma 4.1 $j = k + 1$. Asymptotics (1.8) gives

$$e^{z(\Omega_{p+k}-\Omega_{p+k+1})} = 1 + O(|z|^{-1}) \quad \text{as} \quad |\lambda| \to \infty, \quad \text{where} \quad z = \lambda^{1/2p}. \quad (4.18)$$

Substituting identity (2.3) into (4.18) we obtain $z = \pi n \frac{2\mu}{c_k} + O(n^{-1})$ as $n \to \infty$. Then $\lambda \in \mathcal{U}_{k,n}$ for $n \in \mathbb{N}$ large enough.

The domain $\mathcal{U}_{k,n}$ is symmetric with respect to the real axis, then $\lambda \in \mathcal{U}_{k,n}$. The function $\rho$ is real on $\mathbb{R}$ (see Theorem 1.2 i), then $\lambda$ is the ramification. Note that $\lambda$ is a zero of the function $\tau_{p+k} - \tau_{p+k+1}$. Then asymptotics (4.1) and the first identity in (2.13) show that $\lambda$ is also a zero of the function $\tau_{p+k} - \tau_{p+k+1}$.

ii) Let $\lambda = r_{k,2n}^{\mu,\pm}$ for some $(k, n) \in \mathbb{N}_{p-1} \times \mathbb{N}$. By Lemma 4.2 i), $z = (r_{k,2n}^{\mu,\pm})^{1/2} = z_0 + \varepsilon$, where $z_0 = (r_{k,2n}^{\mu,\pm})^{1/2}$ and $|\varepsilon| < p\beta$ for all $n \in \mathbb{N}$ large enough. Then $1 = \tau_{k,n}^\mu(\lambda)(\tau_{k,n}^\mu)^{-1}(\lambda) = e^{z(\Omega_0(\lambda)-\Omega_{k,n+1}(\lambda))}$, where $s = p + k$ and we used identities (2.21). Consider the case $\text{Im} \lambda \geq 0$, the proof for the other case is similar. Then $z$ satisfies the identity

$$z = \frac{2\pi ni}{\omega_s^\mu(\lambda) - \omega_{s+1}^\mu(\lambda)}. \quad (4.19)$$

Asymptotics (2.22) yields

$$z = \frac{2\pi ni}{\omega_s - \omega_{s+1}} + O(n^{-2}) = \frac{(-1)^{k+1} \pi n \eta_k}{c_k} + O(n^{-2}) \quad \text{as} \quad n \to \infty, \quad (4.20)$$

where we used (2.3). Asymptotics (2.22) and (4.20) give

$$\omega_s^\mu(\lambda) - \omega_{s+1}^\mu(\lambda) = \omega_s - \omega_{s+1} - \frac{(-1)^p \mu}{2p\varepsilon^2} \left( \frac{1}{\omega_s} - \frac{1}{\omega_{s+1}} \right) + O(n^{-4})$$

$$= (-1)^{k+1} \frac{2i c_k}{\eta_k} \left( 1 + \frac{(-1)^p \mu c_k^2}{2p(\pi n)^2} \right) + O(n^{-4}), \quad (4.21)$$

where we used (2.3) and the simple identity $\omega_s \omega_{s+1} = \eta_k^{-2}$. Substituting asymptotics (4.21) into identity (4.19) we obtain

$$z = \frac{(-1)^{k+1} \pi n \eta_k}{c_k} \left( 1 - \frac{(-1)^p \mu c_k^2}{2p(\pi n)^2} \right) + O(n^{-3}),$$

which yields (4.17).
Assume that \( r_{k,n}^{\mu,\pm} \) are non-real for some \( n \in \mathbb{N} \) large enough. Then \( r_{k,n}^{\mu,\pm} = \overline{r_{k,n}^{\mu,\pm}} \), which is in contradiction with asymptotics (4.17). Hence \( r_{k,n}^{\mu,\pm} \in \mathbb{R} \). ■

We introduce the labeling of the ramifications at high energy:

For each \( k \in \mathbb{N}_{p-1}, n \geq n_0 \) for some \( n_0 \in \mathbb{N} \) large enough, \( r_{k,n}^{\pm} \) are zeros of the function \( \tau_{p+k} - \tau_{p+k+1} \) and \( r_{k,n}^{\pm} \in \mathcal{U}_{j,n} \). Moreover, we assume that \( \text{Im} r_{k,n}^{\pm} \geq 0 \) and

- if \( \text{Im} r_{k,n}^{+} > 0 \), then \( r_{k,n}^{-} = \overline{r_{k,n}^{+}} \),
- if \( \text{Im} r_{k,n}^{-} = 0 \), then \( (-1)^k r_{k,n}^{-} < (-1)^k r_{k,n}^{+} \).

**Corollary 4.5.** The following identities hold true

\[
\tau_{p+k}(r_{k,n}^{\pm}) = \tau_{p+k+1}(r_{k,n}^{\pm}), \quad \tau_{p-k}(r_{k,n}^{\pm}) = \tau_{p-k+1}(r_{k,n}^{\pm}), \quad \Delta_{p-k}(r_{k,n}^{\pm}) = \Delta_{p-k+1}(r_{k,n}^{\pm}),
\]

for all \( k \in \mathbb{N}_{p-1}, n \geq n_0 \) for some \( n_0 \in \mathbb{N} \) large enough.

**Proof.** The results of Lemmas 4.3 and 4.4(i) yield the first identities in (4.22). Identities (4.22) give the second identities in (4.22). The definition of the functions \( \Delta_{j} \), see Theorem 1.1(i), implies the third identities in (4.22). ■

**Remark.** Identities (4.22) define the order of attachment of the sheets of the Riemann surface \( \mathcal{A} \) at high energy (see Fig. 1).

5. Asymptotics

Now we will determine the rough asymptotics of the periodic and antiperiodic eigenvalues and ramifications of the Lyapunov function.

**Lemma 5.1.** The periodic and antiperiodic eigenvalues \( \lambda_{n}^{\pm} \) and the ramifications \( r_{k,n}^{\pm} \) satisfy:

\[
\lambda_{n}^{\pm} = (\pi n)^{2p} + O(n^{2p-2}) \quad \text{as} \quad n \to \infty, \quad (5.1)
\]

\[
r_{k,n}^{\pm} = r_{k,n}^{0} + O(n^{2p-2}) \quad \text{as} \quad n \to \infty, \quad k \in \mathbb{N}_{p-1}, \quad r_{k,n}^{0} = (-1)^{k} \left( \frac{\pi n}{c_k} \right)^{2p}. \quad (5.2)
\]

**Proof.** Let \( \lambda = \lambda_{n}^{\pm} \) for some \( n \in \mathbb{N} \). Lemma 4.2 gives \( z = (\lambda_{n}^{\pm})^{1/2p} = \pi n + \delta \), where \( |\delta| < \frac{\pi}{2} \) for all \( n \geq 1 \) large enough. The periodic and antiperiodic eigenvalues are real zeros of the functions \( \tau_{j}^{2} - 1, j \in \mathbb{N}_{p} \). Asymptotics (4.1) show that these functions, with only exception \( \tau_{p}^{2} - 1 \), have no any large real zeros. Then \( 1 = \tau_{p}^{2}(\lambda) = e^{2i\delta}(1 + O(n^{-1})) \), where we used (4.1).

Substituting \( z = \pi n + \delta \) into this identity we obtain \( e^{2i\delta} = 1 + O(n^{-1}) \). Then \( \delta = O(n^{-1}) \) and \( z = \pi n + O(n^{-1}) \), which yields (5.1).

We will prove (5.2). Let \( \lambda = r_{k,n}^{\pm} \) for some \( (k,n) \in \mathbb{N}_{p-1} \times \mathbb{N} \). By Lemma 4.2 \( z = (r_{k,n}^{\pm})^{1/2p} = z_{0} + \varepsilon \), where \( z_{0} = (r_{k,n}^{0})^{1/2p} \) and \( |\varepsilon| < p\beta \) for all \( n \in \mathbb{N} \) large enough. Identities (4.22) show that \( 1 = \tau_{s}(\lambda) \tau_{s+1}^{-1}(\lambda) = e^{\varepsilon}(1 + O(n^{-1})) \), where \( s = p + k \) and we used asymptotics (4.1). Substituting \( z = z_{0} + \varepsilon \) into this identity and using \( e^{\varepsilon}(1 + O(n^{-1})) = 1 \) we obtain \( e^{\varepsilon} = 1 + O(n^{-1}) \). Then \( \varepsilon = O(n^{-1}) \) and \( z = z_{0} + O(n^{-1}) \), which yields (5.2). ■

In order to improve asymptotics (5.1), (5.2) we determine the asymptotics of the function \( D(\tau, \lambda) = \det(\mathcal{M}(1, \lambda) - \tau \mathbb{I}_{2p}) \) in the neighborhoods of the unperturbed ramifications at high energy.
Lemma 5.2. Let \( k \in \mathbb{N}_{0}^{p-1} \) and let \( \lambda \in \mathbb{C}_{+} \) and \( \tau \in \mathbb{C} \) satisfy

\[
\lambda = \nu_{k,n}^{0} + O(n^{2p-2}), \quad \tau = \tau_{p+k}^{0}(\lambda) \left( 1 + O(n^{-1}) \right) \quad \text{as} \quad n \to \infty.
\]

Then

\[
\tau = \tau_{p+k+1}^{0}(\lambda) \left( 1 + O(n^{-1}) \right) \quad \text{as} \quad n \to \infty.
\]

Moreover, the determinant \( D(\tau, \lambda) \), given by (1.7), satisfies the asymptotics

\[
D(\tau, \lambda) = \alpha(\tau, \lambda) \det \left( \begin{pmatrix}
1 - \left( \tau_{p+k}^{\mu}(\lambda) \right)^{-1} \tau & f_{k,n} \\
\frac{1}{f_{k,n}} - \left( \tau_{p+k+1}^{\mu}(\lambda) \right)^{-1} \tau & O(n^{-2})
\end{pmatrix} \right)
\]

as \( n \to \infty \), where \( \alpha = \tau^{-p-k-1} \prod_{j=p+k+2}^{2p} (\tau_{j}^{\mu}(\lambda))^{-1} \neq 0 \) and \( f_{k,n} \) are given by (3.33).

Proof. Identities (2.21) give \( \Omega_{f}^{\mu}(\lambda) = \omega_{f}^{\mu} \) for all \( j \in \mathbb{N}_{2p} \). We have \( z = \lambda^{\frac{1}{p}} = z^{0} + O(n^{-1}) \), where \( z^{0} = (r_{k,n}^{0})_{2p} = \pi n^{\frac{1}{p}} c_{k} \). Asymptotics (5.3) and identities (2.3) give

\[
\tau e^{-2\omega_{s+1}} = e^{z(\omega_{s+1})} (1 + O(n^{-1})) = e^{z_{0}(\omega_{s} - \omega_{s+1} + O(n^{-1}))} (1 + O(n^{-1})) = 1 + O(n^{-1}), \quad s = p+k
\]

as \( n \to \infty \). Asymptotics (5.6) yields (5.4).

We will prove asymptotics (5.5). Identity (3.30) yields

\[
D(\tau, \cdot) = \det(\mathcal{F}e^{z \mathcal{B}_{s}} - \tau \mathbb{I}_{2p}) = \det(\mathcal{F} - \tau e^{-z \mathcal{B}_{s}}) = \det \begin{pmatrix}
A_{1} - \tau e^{-z B_{1}} & A_{2} \\
A_{3} & A_{4} - \tau e^{-z B_{2}}
\end{pmatrix},
\]

where the matrices \( B_{1} = \text{diag}(\omega_{1}^{\mu}, \ldots, \omega_{s+1}^{\mu}) \), \( B_{2} = \text{diag}(\omega_{s+2}^{\mu}, \ldots, \omega_{2p}^{\mu}) \),

\[
A_{1} = \begin{pmatrix}
\mathcal{F}_{11} & \ldots & \mathcal{F}_{1,s+1} \\
\vdots & \ddots & \vdots \\
\mathcal{F}_{s+1,1} & \ldots & \mathcal{F}_{s+1,s+1}
\end{pmatrix}, \quad A_{2} = \begin{pmatrix}
\mathcal{F}_{1,s+2} & \ldots & \mathcal{F}_{1,2p} \\
\vdots & \ddots & \vdots \\
\mathcal{F}_{s+1,s+2} & \ldots & \mathcal{F}_{s+1,2p}
\end{pmatrix},
\]

\[
A_{3} = \begin{pmatrix}
\mathcal{F}_{s+2,1} & \ldots & \mathcal{F}_{s+2,s+1} \\
\vdots & \ddots & \vdots \\
\mathcal{F}_{2p,1} & \ldots & \mathcal{F}_{2p,2p}
\end{pmatrix}, \quad A_{4} = \begin{pmatrix}
\mathcal{F}_{s+2,s+2} & \ldots & \mathcal{F}_{s+2,2p} \\
\vdots & \ddots & \vdots \\
\mathcal{F}_{2p,2p} & \ldots & \mathcal{F}_{2p,2p}
\end{pmatrix}.
\]

Due to (3.32), the matrices \( A_{1}, \ldots, A_{4} \) are bounded for \( |\lambda| > 0 \) large enough. Identity (5.7) yields

\[
D(\tau, \cdot) = \alpha \det \begin{pmatrix}
A_{1} - \tau e^{-z B_{1}} & A_{2} \\
\tau^{-1} e^{-z B_{2}} & A_{4} - \mathbb{I}_{p-k-1}
\end{pmatrix}, \quad \alpha = \alpha(\tau, \lambda).
\]

Estimates (2.7), asymptotics (2.25) and (5.3) yield \( |\tau|^{-1} e^{z B_{2}(\lambda)} = e^{-z_{0} \omega_{s+2}} e^{z \omega_{s+2}} (1 + O(n^{-1})) \) as \( n \to \infty \). Relations (2.8) show that \( \text{Re} z(\omega_{s} - \omega_{s+2}) > a |z|, a > 0 \). Then \( |\tau|^{-1} e^{z B_{2}(\lambda)} = O(e^{-a n}) \).

These asymptotics show that the matrix \( \tau^{-1} e^{z B_{2}} A_{4} - \mathbb{I}_{p-k-1} \) is invertible for \( n \) large enough. Using the standard formula (see [Ga], Ch.2.5) we obtain

\[
D(\tau, \cdot) = \alpha \det(A_{1} - \tau e^{-z B_{1}} - A_{2}(\tau^{-1} e^{z B_{2}} A_{4} - \mathbb{I}_{p-k-1})^{-1} \tau^{-1} e^{z B_{2}} A_{3}) \det(\tau^{-1} e^{z B_{2}} A_{4} - \mathbb{I}_{p-k-1}).
\]

Substituting the asymptotics \( |\tau|^{-1} e^{z B_{2}(\lambda)} = O(e^{-a n}) \) into identity (5.8) we obtain

\[
D(\tau, \lambda) = \alpha \left( \det(A_{1}(\lambda) - \tau e^{-z B_{1}(\lambda)}) + O(e^{-a n}) \right) \quad \text{as} \quad n \to \infty, \quad a > 0.
\]

Furthermore, we have

\[
A_{1} - \tau e^{-z B_{1}} = \begin{pmatrix}
A_{5} - \tau e^{-z B_{5}} & A_{6} \\
A_{7} & A_{0} - \tau e^{-z B_{0}}
\end{pmatrix}.
\]
where $B_3 = \text{diag}(e^{-zw_1^\mu}, ..., e^{-zw_{s-1}^\mu}), B_0 = \text{diag}(e^{-zw_1^\mu}, e^{-zw_{s+1}^\mu}),$

$$A_5 = \left( \begin{array}{ccc} F_{1,1} & \cdots & F_{1,s-1} \\ \cdots & \cdots & \cdots \\ F_{s-1,1} & \cdots & F_{s-1,s-1} \end{array} \right), \quad A_6 = \left( \begin{array}{ccc} F_{1,1} & \cdots & F_{1,s+1} \\ \cdots & \cdots & \cdots \\ F_{s-1,1} & \cdots & F_{s-1,s+1} \end{array} \right),$$

$$A_7 = \left( \begin{array}{ccc} F_{s,1} & \cdots & F_{s,s-1} \\ \cdots & \cdots & \cdots \\ F_{s+1,1} & \cdots & F_{s+1,s-1} \end{array} \right), \quad A_0 = \left( \begin{array}{ccc} F_{s,1} & \cdots & F_{s,s+1} \\ \cdots & \cdots & \cdots \\ F_{s+1,1} & \cdots & F_{s+1,s+1} \end{array} \right).$$

Estimates (2.27) imply $\tau e^{-B_3(\lambda)} = \tau e^{-z_{s-1}}(1 + o(1)) = e^{zw_{s+1}}e^{-zw_{s-1}}(1 + o(1))$, where we used (5.4). Relations (2.8) show that $\text{Re}(\omega_{s-1} - \omega_{s+1}) > a|z|$. Then $\tau e^{-B_3(\lambda)} = O(e^{-an})$. Asymptotics (3.32) show that $F_{jj}(\lambda) = 1 + O(n^{-2}), j \in \mathbb{N}_{2p}$, which yields

$$A_5(\lambda) = \tau e^{-B_3(\lambda)} = \mathbb{I}_{s-1} + O(n^{-2}) \quad \text{as} \quad n \to \infty. \quad (5.11)$$

Thus the matrix $A_5(\lambda) - \tau e^{-B_3(\lambda)}$ is invertible for large $n$ and (5.10) gives

$$\det(A_1 - \tau e^{-B_1}) = \det(A_5 - \tau e^{-B_5(\lambda)}) \det(A_0 - \tau e^{-B_0(\lambda)} - A_7(A_5 - \tau e^{-B_3(\lambda)})^{-1}A_6). \quad (5.12)$$

Substituting asymptotics (5.11) into identity (5.12) we get

$$\det(A_1(\lambda) - \tau e^{-B_1(\lambda)}) = \det(A_0(\lambda) - \tau e^{-B_0(\lambda)} + O(n^{-2})) \quad \text{as} \quad n \to \infty.$$ 

Substituting this asymptotics into (5.9) we have

$$D(\tau, \lambda) = \alpha \det \left( \left( \begin{array}{ccc} F_{s,s}(\lambda) & -\tau e^{-zw_{s}^\mu(\lambda)} \\ F_{s+1,s}(\lambda) & F_{s+1,s+1}(\lambda) - \tau e^{-zw_{s+1}^\mu(\lambda)} \end{array} \right) + O(n^{-2}) \right).$$

Substituting (3.32), (3.34) into the last asymptotics we obtain (5.5). □

**Proof of Theorem 1.2.** i) Repeating the arguments from [CK] we obtain that $\rho$ is entire and real on $\mathbb{R}$. Asymptotics (1.12) yields (1.17).

ii) Let $\lambda = \tau_{k,n}^{-\ast}$ for some $(k, n) \in \mathbb{N}_{p-1} \times \mathbb{N}$. We assume that $\text{Im} \lambda \geq 0$. Then $\Omega_j = \omega_j$ and $\Omega_j^\mu = \omega_j^\mu$ for all $j \in \mathbb{N}_{2p}$. Using the identity $\tau_{k,n}^{-\ast} = \tau_{k,n}^{-\ast}$ we obtain asymptotics for $\tau_{k,n}^{-\ast} \in \mathbb{C}_-$. Let $\lambda^\mu = \tau_{k,n}^{-\ast}$, where $\mu = \hat{q}_{p,0}$, be the unperturbed ramification. Asymptotics (1.17), (5.2) yield

$$z = \lambda^\mu + \xi^\delta, \quad \text{where} \quad z^\mu = (\lambda^\mu)^{\hat{q}_{p,0}}, \quad \xi = \frac{\eta_k}{c_k}, \quad \delta = O(n^{-1}) \quad \text{as} \quad n \to \infty. \quad (5.13)$$

Since $\lambda = \tau_{k,n}^{-\ast}$ is the ramification, the monodromy matrix $\mathcal{M}(1, \lambda)$ has the eigenvalue $\tau$ of multiplicity 2, i.e. its characteristic polynomial $D(\cdot, \lambda)$ has the zero $\tau$ of multiplicity 2. If $n \in \mathbb{N}$ is large enough, then identities (1.22) show that $\tau = \tau_{p+k}^0(\lambda) = \tau_{p+k+1}(\lambda)$. Using asymptotics (1.11) we obtain $\tau = \tau_{p+k}^0(\lambda)(1 + O(n^{-1}))$ as $n \to \infty$. Then we can apply asymptotics (5.5).

Note that the function $A(\tau) = \det \left( \begin{array}{ccc} a_1 - \tau a_2 & a_3 \\ a_4 & a_5 - \tau a_6 \end{array} \right)$, where $a_j \in \mathbb{C}$ for all $j \in \mathbb{N}_6$, has the zero of multiplicity 2 iff $(a_2a_5 - a_1a_6)^2 + 4a_2a_6a_3a_4 = 0$. Using asymptotics (5.3) we deduce that

$$\left( (\tau^\mu_{s})(\lambda))^{-1}(1 + O(n^{-2})) - (\tau^\mu_{s+1}(\lambda))^{-1}(1 + O(n^{-2})) \right)^2$$

$$+ 4(\tau^\mu_{s}(\lambda)\tau^\mu_{s+1}(\lambda))^{-1}(\tilde{f}_{k,n} + O(n^{-2}))(\tilde{f}_{k,n} + O(n^{-2})) = 0 \quad \text{as} \quad n \to \infty. \quad (5.14)$$
where \( s = p + k \) and \( \tau^\mu_j \) are given by (2.21). Identity (5.14) yields

\[
\left((\tau^\mu_s(\lambda))^{-\frac{1}{2}}(\tau^\mu_{s+1}(\lambda))\right)^2 (1 + O(n^{-2})) - (\tau^\mu_s(\lambda))^{-\frac{1}{2}}(\tau^\mu_{s+1}(\lambda))^{-\frac{1}{2}} (1 + O(n^{-2})) \right)^2 + 4|f_{k,n}|^2 = n^{-3}\ell^2(n)
\]

as \( n \to \infty \), where we used \( f_{k,n} = \ell^2(n)O(n^{-1}) \), see (3.35).

Identities (4.22), applied to the operator \( H^\mu \), yields \( \tau^\mu_{s+1}(\lambda^\nu) = \tau^\mu_{s+1}(\lambda^\nu) \). Asymptotics (5.17) imply

\[
\tau^\mu_s(\lambda)(\tau^\mu_{s+1}(\lambda))^{-1} = \tau^\mu_s(\lambda)(\tau^\mu_{s+1}(\lambda))^{-1} \tau^\mu_{s+1}(\lambda^\nu) = \epsilon(\lambda^\nu)\left(\omega^\nu(\lambda) - \omega^\nu_{s+1}(\lambda)\right)\epsilon^{-1}(\lambda^\nu)\left(\omega^\nu(\lambda) - \omega^\nu_{s+1}(\lambda)\right)
\]

Asymptotics (2.23) gives \( \omega^\nu_j(\lambda) = \omega_j(\lambda) + O(n^{-2}) \). Asymptotics (2.23) yields \( \omega^\nu_j(\lambda) - \omega^\nu_j(\lambda^\nu) = O(n^{-4}) \) as \( n \to \infty, j \in \mathbb{N}_2p \). Then (5.16) implies

\[
(r^\pm_{k,n})^\mu = z^\mu \pm \frac{(-1)^k\epsilon_k|\hat{q}_{p,n}|}{2p\pi n} + O(n^{-2}) \quad \text{as} \quad n \to \infty,
\]

where we used (3.35). Then

\[
(z^\mu)^{2p-1}(-1)^k\epsilon_k|\hat{q}_{p,n}| + O(n^{2p-3}) = r^\mu_{k,n} \pm \left(\frac{\pi n}{c_k}\right)^{2p-2} c_k|\hat{q}_{p,n}| + O(n^{2p-3}),
\]

where we used \( z^\mu = \xi \pi n + O(n^{-1}) \), see (4.17), and \( \eta_k^{2p} = (-1)^k \), see (2.4). Substituting asymptotics (4.17) into (5.17) we obtain (1.18).  

**Proof of Theorem 1.3**: i) Asymptotics (1.12) shows that the branches of the Lyapunov function \( \Delta \) on the interval \( (K, +\infty) \) for some \( K \in \mathbb{R} \) satisfy:

- if \( p \) is odd, then there is exactly one real branch \( \Delta_j \) and the other branches are non-real;
- if \( p \) is even, then there are two real branches \( \Delta_1 \) and \( \Delta_2 \) and the other branches are non-real.

Moreover, \( \Delta_j(\lambda) \neq \Delta_j(\lambda^\nu) \) for all \( j = 2, \ldots, p, \lambda \in (K, +\infty) \), hence \( \Delta_1 \) is analytic on the interval \( (K, +\infty) \). Asymptotics (1.12) for \( \Delta_1 \) and Theorem (1.1) ii) show that the function \( \Delta_1 \) oscillates on \( (K, +\infty) \) similar to the Lyapunov function for the Hill operator. Then using identity (1.13) and the standard arguments (see [BK1]) we obtain the needed statement.

ii) Let \( \lambda = \lambda^\mu_n \) for some \( n \in \mathbb{N} \). Recall that \( \lambda \in \mathbb{R} \) and satisfies

\[
D_n = D((-1)^n, \lambda) = \det(\mathcal{M}(1, \lambda) - (-1)^n \mathbb{I}_{2p}) = 0.
\]

Let \( \lambda^\mu = \lambda^\mu_n,^\pm \) be the unperturbed 2-periodic eigenvalues. Asymptotics (2.24), (5.1) give \( z = z^\mu + \delta \), where \( z^\mu = (\lambda^\mu)^\frac{1}{p} \), \( \delta \in \mathbb{R}, \delta = O(n^{-1}) \) as \( n \to \infty \), \( \lambda^\mu \) satisfies (2.24).
Asymptotics (5.5) for $k = 0$ give

$$
D_n = \alpha_n \det \left( \begin{array}{c}
1 - (-1)^n (\tau_j^p(\lambda))^{-1} \\
\frac{f_{0,n}}{j_{0,n}} \\
1 - (-1)^n (\tau_{p+1}^p(\lambda))^{-1}
\end{array} \right) + O(n^{-2}) \quad \text{as} \quad n \to \infty,
$$

(5.19)

where $\alpha_n = \alpha((-1)^n, \lambda_n^\pm) \neq 0$, $f_{0,n} = -i\frac{\delta_{0,n}}{2\pi n}$. Each $\lambda^\pm$ is a periodic or antiperiodic eigenvalue of the operator $H^\mu$ and the corresponding multipliers satisfy the identities $\tau_j^p(\lambda^\pm) = \tau_{p+1}^p(\lambda^\pm) = (-1)^n$. Then

$$
(1 - (-1)^n (\tau_j^p(\lambda))^{-1} (\tau_j^p(\lambda))^{-1} = (\tau_j^p(\lambda))^{-1} = e^{-(z^\mu + \delta)\omega_j^\mu(\lambda)} e^{z^\mu\omega_j^\mu(\lambda)} = e^{-\delta\omega_j^\mu} e^{-z^\mu(\omega_j^\mu(\lambda) - \omega_j^\mu(\lambda))}
$$

(5.20)

for $j = p, p + 1$. Asymptotics (2.23) yield $e^{-\delta\omega_j^\mu(\lambda)} = e^{-\delta\omega_j(1 + O(n^{-3}))}$ as $n \to \infty$. Asymptotics (5.5) gives

$$
(\tau_j^p(\lambda))^{-1} = (1 - (-1)^n (1 - \omega_j^p + O(n^{-2})) = (1 - n^2(1 - \omega_j^p) + O(n^{-2}) \quad \text{as} \quad n \to \infty, \quad j = p, p + 1.
$$

Using the identities $\omega_p = -\omega_{p+1} = -i$ we obtain

$$
(\tau_p^p(\lambda))^{-1} = (1 + i\delta) + O(n^{-2}), \quad (\tau_{p+1}^p(\lambda))^{-1} = (1 - i\delta) + O(n^{-2}) \quad \text{as} \quad n \to \infty.
$$

Substituting these asymptotics into (5.19) we obtain

$$
D_n = \alpha_n \det \left( \begin{array}{c}
-\delta i \\
\frac{f_{0,n}}{j_{0,n}} \\
\delta i
\end{array} \right) + O(n^{-2}) = \alpha_n \det (N + \delta \mathbb{I}_2 + O(n^{-2})), \quad N = \left( \begin{array}{cc}
0 & i f_{0,n} \\
-i f_{0,n} & 0
\end{array} \right).
$$

Using identity (5.18) we conclude that $\delta$ is an eigenvalue of the matrix $N + O(n^{-2})$. Since the eigenvalues of the matrix $N$ have the form $\pm \frac{|f_{0,n}|}{2\pi n}$, we obtain $\delta = \pm \frac{|f_{0,n}|}{2\pi n} + O(n^{-2})$. Using the identity $|f_{0,n}| = \frac{|f_{0,n}|}{2\pi n} + O(n^{-2})$. Then (2.24) gives (1.20). $
$

**Proof of Corollary 1.4**

i) Asymptotics (1.20) and the estimates $|\hat{q}_{p,n}| \geq \frac{1}{n^{\kappa}}$ give the asymptotics

$$
|\gamma| = \lambda_n^+ - \lambda_n^- = 2(\pi n)2^{p-2}|\hat{q}_{p,n}| + O(n^{2p-3}) \quad \text{as} \quad n \to \infty,
$$

and the estimates $|\gamma_n| \geq 2\pi e^{-2\alpha} (1 + O(n^{-\alpha}))$ as $k \to \infty$, which yields the statement.

ii) Assume that $r_{k,nj}^\pm$ are non-real. Then $r_{k,nj}^\pm = r_{k,nj}^\mp$, which is in contradiction with asymptotics (1.18). Hence $r_{k,nj}^\pm \in \mathbb{R}$. Moreover, asymptotics (1.18) gives

$$
|r_{k,nj}^+ - r_{k,nj}^-| = 2\left( \frac{\pi n}{C_k} \right)^{2p-2} \left( |\hat{q}_{p,n}| + O(n^{2p-3}) \quad \text{as} \quad n \to \infty,
$$

which yields $|r_{k,nj}^+ - r_{k,nj}^-| \to \infty$ as $k \to \infty$. $
$

6. Appendix

**Proof of Lemma 3.2.** The standard arguments yield that the fundamental solution $\mathcal{M}(t, \lambda)$ of equation (1.5), with the initial condition $\mathcal{M}(0, \lambda) = \mathbb{I}_{2p}$, satisfies the integral equation

$$
\mathcal{M}(t, \lambda) = \mathcal{M}_0(t, \lambda) + \int_0^t \mathcal{M}_0(t - s, \lambda) Q(s) \mathcal{M}(s, \lambda) ds,
$$

(6.1)

where $\mathcal{M}_0(t, \lambda) = e^{t\mathcal{P}(\lambda)}$ is a solution at $Q = 0$.

Describe the properties of the matrix-valued function $\mathcal{M}_0$. Each function $\mathcal{M}_0(t, \cdot)$, $t \in \mathbb{R}$, is entire. Moreover, $\mathcal{M}_0 = ((\varphi_j^0)^{(k-1)})_{k,j=1}$, where $\varphi_j^0, j \in \mathbb{N}_{2p}$, are the solutions of the equation
$y^{(2p)} = \lambda y$, satisfying the conditions $(\varphi_j^0)^{(k-1)}(0, \lambda) = \delta_{jk}$ for all $k \in \mathbb{N}_{2p}$. These solutions are given by the identities

$$
\varphi_1^0 = \frac{1}{2p} \sum_{t=1}^{2p} e^{z\omega_t}, \quad \varphi_{j+1}^0(t, \lambda) = \int_0^t \varphi_j^0(s, \lambda) ds, \quad \text{all } j \in \mathbb{N}_{2p-1}.
$$

Then $|(\varphi_j^0)^{(k-1)}(t, \lambda)| \leq e^{z|t|}$ for all $j, k \in \mathbb{N}_{2p}, (t, \lambda) \in \mathbb{R} \times \mathbb{C}$, and

$$
|\mathcal{M}_0(t, \lambda)| \leq 2pe^{z|t|}, \quad \text{all } (t, \lambda) \in \mathbb{R} \times \mathbb{C}. \quad (6.2)
$$

Estimate (6.2) will be useful for obtaining the first estimate in (3.11). Now we will prove the other estimate of $\mathcal{M}_0$ (see below (6.3)), which will be effective to obtain the second estimate in (3.11). In fact, direct calculations show that $\mathcal{P} = z\mathcal{ZC}\mathcal{B}(\mathcal{ZC})^{-1}$, where the diagonal matrix $\mathcal{B}$ is given by $\mathcal{B} = \text{diag}(\omega_j)^{2p}$, and the matrix $\mathcal{C}$ has the form $\mathcal{C} = (\omega_j^{-1})^{2p}_{k,j=1}$. Then $\mathcal{C}^{-1} = \frac{1}{2p}\mathcal{C}^*$, $\mathcal{M}_0 = \mathcal{ZC}e^{z\mathcal{B}(\mathcal{ZC})^{-1}}$ and

$$
|\mathcal{Z}^{-1}(\lambda)\mathcal{M}_0(t, \lambda)\mathcal{Z}(\lambda)| \leq 2pe^{z|t|}, \quad \text{all } (t, \lambda) \in \mathbb{R} \times (\mathbb{C} \setminus \{0\}). \quad (6.3)
$$

The standard iterations in (6.1) yield

$$
\mathcal{M}(t, \lambda) = \sum_{n=0}^{\infty} \mathcal{M}_n(t, \lambda), \quad \mathcal{M}_n(t, \lambda) = \int_0^t \mathcal{M}_0(t-s, \lambda)Q(s)\mathcal{M}_{n-1}(s, \lambda) ds. \quad (6.4)
$$

and $\mathcal{M}_n(t, \lambda)$ is given by

$$
\mathcal{M}_n(t, \lambda) = \int_0^t \prod_{k=1}^n \left( \mathcal{M}_0(t_{k+1} - t_k, \lambda)Q(t_k) \right) \mathcal{M}_0(t_1, \lambda) dt_1 dt_2 \ldots dt_n \quad (6.5)
$$

the factors are ordering from right to left, $T = \{0 < t_1 < \ldots < t_n < t_{n+1} = t\}$. Substituting estimates (6.2) into identities (6.5) we obtain

$$
|\mathcal{M}_n(t, \lambda)| \leq \frac{2p}{n!}e^{z|t|} \left( 2p \int_0^t |Q(s)| ds \right)^n, \quad \text{all } (n, t, \lambda) \in \mathbb{N} \times \mathbb{R} \times \mathbb{C}. \quad (6.6)
$$

These estimates show that for each fixed $t \in \mathbb{R}$ the formal series (6.4) converges absolutely and uniformly on bounded subset of $\mathcal{C}$. Each term of this series is an entire function. Hence the sum is an entire function. Summing the majorants we get

$$
|\mathcal{M}(t, \lambda)| \leq 2pe^{z|t| + \alpha^*} \int_0^t |Q(s)| ds, \quad \text{all } (t, \lambda) \in \mathbb{R} \times \mathbb{C}, \quad (6.7)
$$

which yields the first estimate in (3.11). Moreover, we deduce that the monodromy matrix $\mathcal{M}(1, \lambda)$ is a continuous function of all $q_j \in L^1(\mathbb{T}), j \in \mathbb{N}_p$.

Substituting estimates (6.3) into identities (6.5) we obtain

$$
|\mathcal{Z}^{-1}(\lambda)\mathcal{M}_n(t, \lambda)\mathcal{Z}(\lambda)| \leq \frac{2p}{n!}e^{z|t|} \left( 2p \int_0^t |\mathcal{Z}^{-1}(\lambda)Q(s)\mathcal{Z}(\lambda)| ds \right)^n
$$

for all $(n, t, \lambda) \in \mathbb{N} \times \mathbb{R} \times (\mathbb{C} \setminus \{0\})$. Substituting these estimates into the series (6.4) we obtain

$$
|\mathcal{Z}^{-1}(\lambda)\mathcal{M}(t, \lambda)\mathcal{Z}(\lambda)| \leq 2pe^{z|t| + \alpha^*} |\mathcal{Z}^{-1}(\lambda)Q(s)\mathcal{Z}(\lambda)| ds, \quad \text{all } (t, \lambda) \in \mathbb{R} \times (\mathbb{C} \setminus \{0\}),
$$

which yields the second estimate in (3.11). 

**Proof of Lemma 3.4**. Asymptotics (2.25) implies

$$
\mathcal{U}(\lambda) = \mathcal{Z}(\lambda)\mathcal{U}_0(\mathbb{1}_{2p} + O(|z|^{-2})), \quad \text{where } \mathcal{Z} = \text{diag}(z^j)^{2p}_{j=1}, \quad \mathcal{U}_0 = (\Omega_j^{2p})^{j=1}_{j,k=1}.
$$
\[
Q(t, \lambda) = U^{-1}(\lambda) Q^u(t, \lambda) U(\lambda) = U_0^{-1} Z^{-1}(\lambda) Q^u(t, \lambda) Z(\lambda) U_0(1_{2p} + O(|z|^{-2})) \quad \text{as } |\lambda| \to \infty
\]
uniformly on \( t \in [0, 1] \). We have
\[
Z^{-1}(\lambda) Q^u(t, \lambda) Z(\lambda) = \frac{(-1)^{p+1}}{z} \left((q_p(t) - \mu)E + b(t)O(|z|^{-1})\right) \quad \text{as } |\lambda| \to \infty,
\]
uniformly on \( t \in [0, 1] \), where \( E \) is given by (3.16). Then we obtain (3.18) with \( L = U_0^{-1} E U_0 \).

Using the identity \( U_L \) uniformly on \( t \)

Then substituting estimates (6.9) into (3.23) we obtain

\[
\mathcal{L}_{jk} = \frac{1}{2p} \sum_{l,n=1}^{2p} \tilde{\Omega}_{ji}^{-1} E_{ln} n^{-1} = \frac{\tilde{\Omega}_{jk} \Omega_{lk}^{n-1}}{2p},
\]
which yields the identity for \(\mathcal{L}_{jk}\) in (3.18).

**Proof of Lemma 3.5.**

i) Asymptotics (3.18) show that the estimate

\[
\max_{k,j \in \mathbb{N}_{2p}} |\tilde{Q}_{kj}(t, \lambda)| \leq \frac{\max\{q_p(t) - \mu|, \frac{1}{p}\}}{p|z|}, \quad (t, \lambda) \in [0, 1] \times \Lambda_{R_1}
\]

for some \( R_1 > 0 \). Assume that

\[
\max_{k,j \in \mathbb{N}_{2p}} |e_{kj}(t, \lambda)| \leq 2, \quad (t, \lambda) \in \mathbb{R} \times \Lambda_{R_2}, \quad \text{for some } R_2 > 0.
\]

Then substituting estimates (6.9) into (3.23) we obtain

\[
\|(K \tilde{Q} A)(\cdot, \lambda)\|_\infty = \max_{t \in [0, 1]} \|(K \tilde{Q} A)(t, \lambda)\| = \max_{(t,j) \in [0,1] \times \mathbb{N}_{2p}} \sum_{i=1}^{2p} \left| \int_0^1 e_{ij}(t-s, \lambda) \sum_{n=1}^{2p} \tilde{Q}_{in}(s, \lambda) A_{nj}(s) ds \right|
\]

\[
\leq 2 \max_{1 \leq j \leq 2p} \sum_{i,n=1}^{2p} \left| \tilde{Q}_{in}(s, \lambda) \right| |A_{nj}(s)| ds \leq 2 \max_{1 \leq j \leq 2p} \sum_{i,n=1}^{2p} \left| A_{nj}(t) \right| \int_0^1 |\tilde{Q}_{in}(s, \lambda)| ds.
\]

for all \( \lambda \in \Lambda_{\max\{R_1, R_2\}} \). Using estimates (6.8) we obtain

\[
\|(K \tilde{Q} A)(\cdot, \lambda)\|_\infty \leq \frac{\xi}{|z|} \max_{t \in [0,1]} \sum_{1 \leq j \leq 2p} \sum_{n=1}^{2p} |A_{nj}(t)|,
\]

which yields (3.20).

We will prove (6.9). We will consider the case \( i > j \). The proof for \( i \leq j \) is similar. Identities (3.23) shows that \( e_{ij}(t, \lambda) = 0 \) for \( t < 0 \). Asymptotics (2.25) shows that

\[
\Omega_j^u = \Omega_j + \frac{\Omega_j}{z}, \quad j \in \mathbb{N}_{2p}, \quad \text{where } \max_{(j,\lambda) \in \mathbb{N}_{2p} \times \{|z| > R_2\}} |\Omega_j(\lambda)| < \frac{1}{2} \log 2
\]

for some \( R_2 > 0 \). Estimates (2.7) give \( \text{Re}(\Omega_i - \Omega_j) \leq 0 \) for \( i > j \). Then

\[
|e_{ij}(t, \lambda)| = e^{t \text{Re}(\Omega_j^u(\lambda) - \Omega_j(\lambda))} \leq e^{t \text{Re}(\Omega_j(\lambda) - \Omega_i(\lambda) + t \text{Re}(\Omega_j(\lambda) - \Omega_i(\lambda)))} \leq e^{t|\Omega_i(\lambda) - \Omega_j(\lambda)|} \leq 2,
\]

for \( t > 0, i > j, |z| > R_2 \), which yields estimates (6.9) for \( i > j \).

ii) The standard iterations in (3.22) give

\[
\mathcal{G} = \sum_{n=0}^{\infty} \mathcal{G}_n, \quad \mathcal{G}_0 = 1_{2p}, \quad \mathcal{G}_n = K \tilde{Q} \mathcal{G}_{n-1} = (K \tilde{Q})^n, \quad \text{all } n \in \mathbb{N}.
\]

\[
\]
Estimate (3.26) yields
\[ \|G_n(\cdot, \lambda)\|_\infty = \|(K\tilde{Q})^n(\cdot, \lambda)\|_\infty \leq \left( \frac{\xi}{|z|} \right)^n, \quad \text{all} \ (n, \lambda) \in \mathbb{N} \times \Lambda_{\max\{R_1, R_2\}}. \] (6.13)
These estimates show that the formal series (6.12) converges absolutely and uniformly on any bounded subset of \( \Lambda_R^\pm \), \( R = \max\{R_1, R_2, \xi^{2p}\} \). Hence it gives the unique solution of equation (3.22). Each term of this series is analytic in \( \Lambda_R^\pm \). Hence the matrix-valued function \( G \) is analytic in \( \Lambda_R^\pm \). If \( \lambda \in \Lambda_{2R} \), then substituting estimates (6.13) into the series (6.12) we obtain
\[ \|G(\cdot, \lambda)\|_\infty \leq \sum_{n=0}^{\infty} \left( \frac{\xi}{|z|} \right)^n = \frac{1}{1 - \frac{\xi}{|z|}} \leq 2, \quad \|G(\cdot, \lambda) - \mathbb{I}_{2p}\|_\infty \leq \sum_{n=1}^{\infty} \left( \frac{\xi}{|z|} \right)^n = \frac{\xi}{1 - \frac{\xi}{|z|}} \leq \frac{2\xi}{|z|}; \]
which yields (6.27). Substituting (3.18) into the identity \( G_1 = K\tilde{Q} \) we obtain
\[ (G_1(t, \lambda))_{ij} = \frac{(-1)^{p+1}}{z} \int_0^1 e_{ij}(t-s, \lambda)(q_p(s) - \mu)L_{ij}(\lambda)ds + a_{ij}(t, \lambda)O(|z|^{-2}), \quad i, j \in \mathbb{N}_{2p}, \]
where \( a_{ij}(t, \lambda) = (-1)^{p+1} \int_0^1 e_{ij}(t-s, \lambda)b(s)ds, i, j \in \mathbb{N}_{2p} \). Estimate (6.11) gives \( |a_{ij}(t, \lambda)| \leq 2||b|| \), which yields (3.28).
iii) The matrix-valued function \( G(t, \cdot) \), \( t \in [0, 1] \), is analytic in \( \Lambda_{2R} \), then \( T(t, \cdot) \) is also analytic. The second estimate in (3.27) yields asymptotics (3.29).

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