Determinant Line Bundles Revisited

DANIEL S. FREED

Department of Mathematics
University of Texas at Austin

May 11, 1995

This note is an addendum to joint work with Xianzhe Dai [DF1], [DF2]. In that paper we investigate the geometric theory of η-invariants of Dirac operators on manifolds with boundary. We summarize the main results below. One key geometric observation is that the exponentiated η-invariant naturally takes values in the determinant line of the boundary. As such it is intimately related to the geometry of determinant line bundles for families of Dirac operators. The differential geometry of determinant line bundles was developed first by Quillen [Q] in a special case, and then by Bismut and Freed [BF1], [BF2] in general. (See [F1] for an exposition of these results.) In §5 of [DF1] the results on η-invariants are used to reprove the holonomy formula for determinant line bundles, also known as Witten’s global anomaly formula [W]. However, the argument there is unnecessarily complicated. The main purpose of this note, then, is to reprove both the curvature and holonomy formulas for determinant line bundles using the results of [DF1]. (The argument was sketched in [DF2].)

To avoid repetitious recitation of requirements, we set some conventions here which apply throughout. We work with compact Riemannian manifolds. If the boundary is nonempty we assume that the metric is a product near the boundary. Our results hold for any Dirac operator on a spin\(^c\) manifold coupled to a vector bundle with connection, but for simplicity we state the formulas only for the basic Dirac operator on a spin manifold. Thus all manifolds are assumed spin. We use the \(L^2\) metric on the spinor fields \(S\). A family of Riemannian manifolds is a smooth fiber bundle \(\pi: X \to Z\) together with a metric on the relative (vertical) tangent bundle \(T(X/Z)\) and a distribution of “horizontal” complements to \(T(X/Z)\) in \(TX\). We assume that \(T(X/Z)\) is endowed with a spin structure. Also, when working with families of manifolds with boundary, we assume that the Riemannian metrics on the fibers are products near the boundary. There is an

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To appear in the proceedings of the conference Topological and Geometrical Problems related to Quantum Field Theory, Trieste, Italy, March 13–24, 1995.

The author is supported by NSF grant DMS-9307446, a Presidential Young Investigators award DMS-9057144, and by the O’Donnell Foundation.

\(^1\)In [DF1] the reader will find an extensive discussion of related work and a bibliography.
induced family $\partial \pi : \partial X \to Z$ of closed manifolds. Finally, we will always use ‘$X$’ to denote an odd dimensional manifold and ‘$Y$’ to denote an even dimensional manifold.

As stated earlier this is a continuation of joint work with Xianzhe Dai.

**Eta Invariants on Manifolds with Boundary**

First recall that on a closed odd dimensional manifold $X$ the Dirac operator $D_X$ is self-adjoint and has a discrete spectrum $\text{spec}(D_X)$ extending to $+\infty$ and $-\infty$. The $\eta$-invariant of Atiyah-Patodi-Singer [APS] is defined by meromorphic continuation of the function

$$\eta_X(s) = \sum_{\lambda \neq 0, \lambda \in \text{spec}(D_X)} \frac{\text{sign } \lambda}{|\lambda|^s},$$

which by general estimates converges for Re$(s)$ sufficiently large. In fact, for Dirac operators the meromorphic continuation is analytic for Re$(s) > -2$ [BF2, Theorem 2.6]. In any case $\eta_X$ is regular at $s = 0$, and we set

$$\tau_X = \exp \pi i (\eta_X(0) + \dim \ker D_X) \in \mathbb{C}.$$  

The general theory of $\eta$-invariants shows that $\tau_X$ varies smoothly in families, whereas the $\eta$-invariant $\eta_X(0)$ is discontinuous in general. Note that $|\tau_X| = 1$.

On a manifold with boundary we need to specify elliptic boundary conditions to obtain an operator with discrete spectrum. We use the boundary conditions introduced by Atiyah-Patodi-Singer, but adapted to odd dimensional manifolds $X$. This involves an additional piece of information concerning $\ker D_{\partial X}$. Recall that on an even dimensional manifold $Y$ the spinor fields $S_Y$ split as $S_Y = S_Y^+ \oplus S_Y^-$, and the Dirac operator $D_Y : S_Y^+ \to S_Y^-$ interchanges the positive and negative pieces. (In the sequel we use ‘$D_Y$’ to denote the operator $D_Y : S_Y^+ \to S_Y^-$. If $Y = \partial X$ is the boundary of an odd dimensional manifold $X$, then $\dim \ker^+ D_{\partial X} = \dim \ker^- D_{\partial X}$. The additional piece of information we must choose as part of the boundary condition is an isometry

$$T : \ker^+ D_{\partial X} \to \ker^- D_{\partial X}.$$  

Then the basic analytic properties of $D_X$ with these boundary conditions are the same as those of the Dirac operator on a closed manifold, and so the invariant (1) is defined. Its dependence on $T$ is simple, and factoring this out we observe that

$$\tau_X \in \text{Det}_{\partial X}^{-1}.$$  

where $\text{Det}_{\partial X}$ is the determinant line of the Dirac operator $D_{\partial X}$ on the boundary:

\begin{equation}
\text{Det}_{\partial X} = (\text{Det Ker}^- D_{\partial X}) \otimes (\text{Det Ker}^+ D_{\partial X})^{-1}.
\end{equation}

(Recall that $\text{Det} V = \bigwedge^n V$ for an $n$ dimensional vector space $V$. Also $L^{-1} = L^*$ for a one dimensional vector space $L$.) Properly normalized we have $|\tau_X| = 1$ in the Quillen metric on $\text{Det}_{\partial X}^{-1}$.

Now suppose $X \to Z$ is a family of odd dimensional manifolds with boundary. Then $\partial X \to Z$ is a family of closed even dimensional manifolds. The determinant lines (3) patch together to form a smooth determinant line bundle $\text{Det}_{\partial X/Z} \to Z$. Furthermore, it carries the Quillen metric and a canonical connection $\nabla$, as defined in [BF1]. The exponentiated $\eta$-invariant is now a smooth section

$$\tau_{X/Z} : Z \to \text{Det}_{\partial X/Z}^{-1}.$$ 

There are two basic results about this invariant: a variation formula and a gluing law. The variation formula computes the derivative of $\tau_{X/Z}$ in a family.

**Theorem 4** [DF1,Theorem 1.9]. With respect to the canonical connection $\nabla$ on $\text{Det}_{\partial X/Z}^{-1}$,

$$\nabla \tau_{X/Z} = 2\pi i \left[ \int_{X/Z} \hat{A}(\Omega^{X/Z}) \right] \cdot \tau_{X/Z}.$$ 

Here $\Omega^{X/Z}$ is the Riemannian curvature of $X \to Z$ and $\hat{A}$ is the usual $\hat{A}$-polynomial. (For other Dirac operators substitute the appropriate index polynomial in place of $\hat{A}$.) The ‘(1)’ denotes the 1-form piece of the differential form. For a family of closed manifolds this is a result of Atiyah-Patodi-Singer. The new point here is the relationship of $\tau$ with the canonical connection $\nabla$. This plays a crucial role in the next section.

**Figure 1.** Cutting a closed manifold into two pieces.
The simplest case of the gluing law is for a closed manifold $X$ split into two pieces $X_1, X_2$ along a closed oriented codimension one submanifold $Y \hookrightarrow X$. (See Figure 1.) Then $\tau_{X_i} \in \text{Det}_{-1} Y$ and $\tau_X \in \mathbb{C}$.

**Theorem 5** [DF1,Theorem 2.20]. *In this situation*

$$\tau_X = (\tau_{X_1}, \tau_{X_2})_{\text{Det}_{-1} Y}.$$  

![Figure 2](image.png)  

**Figure 2.** Cutting a manifold along a submanifold.

The more general gluing formula, which we need in the next section, applies when $X$ has boundary. Then for $Y \hookrightarrow X$ a closed oriented codimension one submanifold we cut along $Y$ to obtain a new manifold $X^{\text{cut}}$ with $\partial X^{\text{cut}} = \partial X \sqcup Y \sqcup -Y$. (See Figure 2.) Now

$$\begin{align*}
\tau_X &\in \text{Det}_{-1} \partial X \\
\tau_{X^{\text{cut}}} &\in \text{Det}_{-1} \partial X \otimes \text{Det}_{-1} Y \otimes \text{Det}_{-1} -Y \\
&\cong \text{Det}_{-1} \partial X \otimes \text{L}_{Y} \otimes \text{L}_{-Y},
\end{align*}$$

where $L_Y = \text{Det}_{-1} Y$. There is now a sign which enters the gluing formula, and it is nicely taken care of by the following device. In general we view the determinant line $\text{Det} V$ of a vector space $V$ as a one dimensional graded vector space whose grading is given by $\dim V$. Applied to (3) we see that $\text{Det}_{Y}$ (and so also $\text{Det}_{-Y}$) is graded by the index of the Dirac operator $D_Y$. Notice that in our current situation $Y$ does not necessarily bound a 3-manifold, and so its index may be nonzero. Let

$$\text{Tr}_s : L_Y \otimes L_{-Y} \rightarrow \mathbb{C}$$

be the usual contraction times the grading; i.e., if index $D_Y$ is even it is the usual contraction and if index $D_Y$ is odd it is minus the usual contraction. That understood, we state the general gluing formula.
**Theorem 8** [DF1, Theorem 2.20]. *In this situation*

\[ \tau_X = \text{Tr}_s(\tau_{X,\text{cut}}). \]

One of the novel points of [DF1] is the proof of the gluing law, which we do not discuss here.

**Determinant Line Bundles and Adiabatic Limits**

The application we discuss is to the geometry of the determinant line bundle. Suppose \( \pi : Y \to Z \) is a family of closed even dimensional manifolds. Let \( L = \text{Det}^{-1}_{Y/Z} \) be the inverse determinant line bundle of the family. The results in the last section use the Quillen metric and the construction of the canonical connection \( \nabla \). But they do not depend on the formulas for the curvature and holonomy of \( \nabla \), which were proved in [BF1], [BF2]. Here we derive the curvature and holonomy formulas from Theorem 4 and Theorem 8.\(^2\) The basic idea is to use the \( \tau \)-invariant (2) to define the parallel transport of a new connection \( \nabla' \) on \( L \). Thus suppose \( \gamma : [0,1] \to Z \) is a smooth path\(^3\) in \( Z \). Denote \( I = [0,1] \). Let \( Y_\gamma = \gamma^*(Z) \to I \) be the pullback of the family \( \pi : Y \to Z \) by the path \( \gamma \). Then \( Y_\gamma \) is an odd dimensional manifold with \( \partial Y_\gamma = Z_{\gamma(1)} \sqcup -Z_{\gamma(0)} \). The standard metric \( g_I \) on \( I = [0,1] \) determines a metric on \( Y_\gamma \), since we already have a metric \( g_{Y_\gamma/I} \) on the fibers and a distribution of horizontal planes. (The projection \( \pi : Y_\gamma \to I \) is then a Riemannian submersion.) The \( \tau \)-invariant of \( Y_\gamma \) is a linear map

\[ \tau_{Y_\gamma} : L_{\gamma(0)} \to L_{\gamma(1)}, \]

exactly what we need to define parallel transport. However, (10) does not define parallel transport since it is not independent of the parametrization of the path \( \gamma \). To get a quantity independent of parametrization we introduce the *adiabatic limit* as follows. For each \( \epsilon \neq 0 \) consider the metric

\[ g_\epsilon = g_I \epsilon^2 \oplus g_{Y_\gamma/I} \]

on \( Y_\gamma \) relative to the decomposition \( TY_\gamma \cong \pi^*TI \oplus T(Y_\gamma/I) \). Let \( \tau_{Y_\gamma}(\epsilon) \) be the \( \tau \)-invariant for this metric.

**Lemma 12.** *The adiabatic limit*

\[ \tau_{Y_\gamma} = \text{a-lim} \tau_{Y_\gamma} = \lim_{\epsilon \to 0} \tau_{Y_\gamma}(\epsilon) \]

\(^2\)As was mentioned in the introduction, this was done in [DF1,§5] in an unnecessarily complicated way. Also, there we used the curvature formula instead of proving it. This section should be considered a rewrite of [DF1,§5].

\(^3\)Since we need a cylindrical metric near the boundary of \( Y_\gamma \) defined below, we require that \( \gamma([0,\delta]) \) and \( \gamma([1-\delta,1]) \) be constant for some \( \delta \).
exists and is invariant under reparametrization of $\gamma$.

Notice that the adiabatic limit is introduced for a simple geometrical reason—to scale out the dependence of $\tau$ on the parametrization of $\gamma$.

**Proof.** Here we follow [DF1, §5]. As a preliminary we state without proof a simple result about the Riemannian geometry of adiabatic limits. Let $\nabla_{Y_\gamma}(\epsilon)$ denote the Levi-Civita connection on $Y_\gamma$ of the metric (11) and $\Omega^{Y_\gamma}(\epsilon)$ its curvature. The result we need, which follows from a straightforward computation in local Riemannian geometry, is that \(\operatorname{a-lim} \nabla_{Y_\gamma} = \lim_{\epsilon \to 0} \nabla_{Y_\gamma}(\epsilon)\) exists and is torsionfree.

Furthermore, the curvature of this limiting connection is the limit of the curvatures of $\nabla_{Y_\gamma}(\epsilon)$ and has the form

\[
\operatorname{a-lim} \Omega^{Y_\gamma} = \lim_{\epsilon \to 0} \Omega^{Y_\gamma}(\epsilon) = \left( \begin{array}{cc} 0 & \Omega^{Y_\gamma}/I \\ 0 & \ast \end{array} \right)
\]

relative to a fixed (nonorthonormal) basis. It follows that

\[
\operatorname{a-lim} \hat{A}(\Omega^{Y_\gamma}) = \lim_{\epsilon \to 0} \hat{A}(\Omega^{Y_\gamma}) = \hat{A}(\Omega^{Y_\gamma}/I).
\]

We apply this result to families of adiabatic limits, where it also holds.

To prove that the adiabatic limit exists, consider the family of Riemannian manifolds $Y_\gamma \times \mathbb{R} \neq 0 \to \mathbb{R} \neq 0$, where the metric on the fiber at $\epsilon$ is (11). According to the variation formula Theorem 4 we have

\[
\frac{d}{d\epsilon} \tau_{Y_\gamma}(\epsilon) = 2\pi i \left[ \int_{(Y_\gamma \times \mathbb{R} \neq 0)/\mathbb{R} \neq 0} \hat{A}(\Omega^{(Y_\gamma \times \mathbb{R} \neq 0)/\mathbb{R} \neq 0}) \right]_{(1)}.
\]

Now (14) implies that

\[
\lim_{\epsilon \to 0} \hat{A}(\Omega^{(Y_\gamma \times \mathbb{R} \neq 0)/\mathbb{R} \neq 0}) = \hat{A}(\Omega^{Y_\gamma}/I).
\]

One should understand this as a limit of sections of a bundle on $Y_\gamma$ whose fibers are forms on $Y_\gamma \times \{0\}$. In other words, they are forms on $Y_\gamma$ with a ‘$d\epsilon$’ term as well. Formula (16) implies that there is no $d\epsilon$ term in the limit, and so the integral over the fibers in (15) vanishes. Therefore, \(\lim_{\epsilon \to 0} \frac{d}{d\epsilon} \tau_{Y_\gamma}(\epsilon) = 0\) and so \(\operatorname{a-lim} \tau_{Y_\gamma} = \lim_{\epsilon \to 0} \tau_{Y_\gamma}(\epsilon)\) exists.

A similar argument proves that $\tau_{Y_\gamma}$ is invariant under reparametrization. Let $\mathcal{D}$ denote the space of diffeomorphisms $\phi: [0, 1] \to [0, 1]$ with $\phi(0) = 0$ and $\phi(1) = 1$. We pull back $\pi: Y \to Z$ via the map

\[
[0, 1] \times \mathbb{R} \neq 0 \times \mathcal{D} \to Z, \quad \langle t, \epsilon, \phi \rangle \mapsto \gamma(\phi(t))
\]

4And we correct a mistake in the exposition there.
to construct the family of manifolds
\[ Y \to \mathbb{R}^{\neq 0} \times D, \]
where the metric on the fiber over \((\epsilon, \phi)\) is (11). As in the previous argument we compute the differential of \(\tau_{Y_{\gamma_0}}(\epsilon, \phi)\) in the adiabatic limit:

\[
\lim_{\epsilon \to 0} d\tau_{(\epsilon, \phi)}(\epsilon, \phi) = 2\pi i \sigma^* \left[ \int_{Y/Z} \hat{A}(\Omega_{Y/Z}) \right]_{(2)},
\]

where
\[
\sigma: [0, 1] \times D \to Z \quad \langle t, \phi \rangle \mapsto \gamma(\phi(t))
\]

We conclude that (17) vanishes since the image of \(\sigma\) is one dimensional—the pullback of a 2-form vanishes.

**Lemma 18.** The maps \(\tau_\gamma\) are the parallel transport of a connection \(\nabla'\) on \(L \to Z\).

**Remark.** Since \(\tau_\gamma\) is a unitary transformation (\(|\tau_\gamma| = 1\)), the connection \(\nabla'\) is also unitary.

**Proof.** By a general result [F2, Appendix B] it suffices to show that the fiducial parallel transport \(\tau_\gamma\) is invariant under reparametrization and composes under gluing. The first statement is contained in the previous lemma. For the second, if \(\gamma_1, \gamma_2\) are paths with \(\gamma_2(0) = \gamma_1(1)\), then we can compose to get a path \(\gamma = \gamma_2 \circ \gamma_1\). The gluing law Theorem 8 then implies \(\tau_\gamma = \tau_{\gamma_2} \circ \tau_{\gamma_1}\) as required. (Theorem 8 applies to a fixed metric and then we take the adiabatic limit.)

**Remark.** It is instructive to see in detail how the sign works in this application of the gluing law. Here we cut \(Y_\gamma\) along \(Y = Y_{\gamma_2(0)} = Y_{\gamma_1(1)}\) to obtain \(Y_\gamma^{\text{cut}} = Y_{\gamma_1} \sqcup Y_{\gamma_2}\). So

\[
\tau_{\gamma_1} \in \text{Hom}(L_{\gamma_1(0)}, L_{\gamma_1(1)}) \cong L_Y \otimes L^{-1}_{\gamma_1(0)},
\]
\[
\tau_{\gamma_2} \in \text{Hom}(L_{\gamma_2(0)}, L_{\gamma_2(1)}) \cong L_{\gamma_2(1)} \otimes L^{-1}_Y,
\]

where we write \(L_Y = L_{\gamma_1(1)} = L_{\gamma_2(0)}\). Thus

\[
\tau_{Y_\gamma^{\text{cut}}} = \tau_{\gamma_2} \otimes \tau_{\gamma_1} \in L_{\gamma_2(1)} \otimes L^{-1}_Y \otimes L_Y \otimes L^{-1}_{\gamma_1(0)}.
\]

The key point is that the factors are in a different order than in (6) and (7)—now the factor \(L^{-1}_Y\) precedes the factor \(L_Y\). So the contraction is the usual trace. Put differently, to move (19) to the standard form (6) we introduce a factor of \((-1)^{\text{index } D_Y}\) and this is cancelled by the factor \((-1)^{\text{index } D_Y}\) in the supertrace (9). The upshot is that in this situation the right hand side of (9) is \(\tau_{\gamma_2} \circ \tau_{\gamma_1}\) as desired.

It is quite easy to prove from the variation formula Theorem 4 that this new connection agrees with the canonical connection \(\nabla\).
Proposition 20. $\nabla' = \nabla$.

Proof. We must show that the parallel transports agree. Let $\gamma : [0,1] \to Z$ be a path and fix an element $\ell_0 \in L_{\gamma(0)}$ of unit norm. Then if $\gamma : [0,t] \to Z$, $0 \leq t \leq 1$, is the restriction of $\gamma$, and $\tau_t : L_{\gamma(0)} \to L_{\gamma(t)}$ the parallel transport of $\nabla'$, by definition the path $\ell_t = \tau_t(\ell_0)$ is parallel for $\nabla'$.

It suffices to show that $\frac{D\tau_t}{Dt} = 0$, where $\frac{D}{Dt} = \nabla$ along the path $\gamma$. For then $\frac{D\tau_t(\ell_0)}{Dt} = 0$ as well, since $\ell_0$ is a constant.

Define $T = \{ \langle t, s \rangle \in [0,1] \times [0,1] : s \leq t \}$ with projection

$$\rho : T \to [0,1] = I$$

$$\langle t, s \rangle \mapsto t$$

and a map

$$\Gamma : T \to Z$$

$$\langle t, s \rangle \mapsto \gamma(s).$$

Then the pullback $\pi : \Gamma^*Y \to T$ determines a family of manifolds $\rho \circ \pi : \Gamma^*Y \to [0,1]$ parametrized by $I = [0,1]$. We use the flat metric on $T$ and make $\pi : \Gamma^*Y \to T$ a Riemannian submersion. The variation formula Theorem 4 implies

$$\frac{D\tau_t}{Dt} = 2\pi i \int_{\Gamma^*Y/I} \text{a-lim} \left[ \hat{A}(\Omega^{\Gamma^*Y/I}) \right]_{(1)}.$$  

Even before taking the adiabatic limit, the fact that $\Gamma$ factors through the projection $\langle t, s \rangle \mapsto s$ implies that the right hand side of (21) vanishes.

In view of Proposition 20, to compute the curvature and holonomy of $\nabla$ it suffices to compute the curvature and holonomy of $\nabla'$. Notice that since $L = \text{Det}^{-1}_{Y/Z}$ is the inverse determinant line bundle our formulas here have opposite signs to those for $\text{Det}_{Y/Z}$ computed in [BF1], [BF2]. The holonomy is computed from the parallel transport by a straightforward application of the gluing law. We must only be careful about the spin structure. Recall that $S^1$ has two spin structures. The nonbounding spin structure is the trivial double cover of the circle; the bounding spin structure is the nontrivial double cover.

Theorem 22 [BF2,Theorem 3.18]. Suppose $\gamma : [0,1] \to Z$ is a closed path.\footnote{Recall that we require that $\gamma([0,\delta])$ and $\gamma([1-\delta,1])$ be constant for some $\delta$.} There is an induced manifold $\hat{Y}_\gamma \to S^1$ obtained by gluing the ends of $Y_\gamma$. Then the holonomy of $L$ around $\gamma$ is

$$\text{hol}_L(\gamma) = \begin{cases} (-1)^{\text{index}D_Y} \text{a-lim} \tau_{\hat{Y}_\gamma}, & \text{nonbounding spin structure on } S^1; \\ \text{a-lim} \tau_{\hat{Y}_\gamma}, & \text{bounding spin structure on } S^1. \end{cases}$$
Here the spin structure on $S^1$ combines with the spin structure on $T(\hat{Y}_\gamma/S^1)$ to give a spin structure on $\hat{Y}_\gamma$.

**Proof.** This follows directly from the definition (13) of parallel transport and the gluing law applied to $X = \hat{Y}_\gamma$ and $X_{\text{cut}} = Y_\gamma$. Take first the nonbounding spin structure on $S^1$, lifted to a spin structure on $\hat{Y}_\gamma$. The induced spin structure on the cut manifold $Y_\gamma$ is the standard one, with the ends each identified with $Y_z$, where $z = \gamma(0) = \gamma(1)$. Now for each $\epsilon$ the $\tau$-invariant of $Y_\gamma$ is an element

$$\tau_{Y_\gamma(\epsilon)} \in L_z \otimes L_z^{-1}.$$

Then Theorem 8 implies

$$\tau_{\hat{Y}_\gamma(\epsilon)} = (-1)^{\text{index} D_Y} \tau_{Y_\gamma(\epsilon)},$$

where on the right hand side we identify $L_z \otimes L_z^{-1}$ with $\mathbb{C}$ using the *usual contraction*. Now the first equation in (23) follows from the definition of holonomy in terms of parallel transport. To obtain the second equation, consider the identity map of $Y_z$ lifted to the nontrivial deck transformation on the spin bundle of $Y_z$. It induces multiplication by $(-1)^{\text{index} D_Y}$ on the inverse determinant line $L_z$. Apply this transformation to $Y_\gamma$ before gluing in order to switch spin structures on $\hat{Y}_\gamma$. Then the second equation in (23) follows from the first.

**Theorem 24** [BF2, Theorem 1.21]. *The curvature $\Omega^L$ of the inverse determinant line bundle $L \to Z$ is*

\[
\Omega^L = -2\pi i \left[ \int_{Y/Z} \hat{A}(\Omega^{Y/Z}) \right]_{(2)}.
\]

**Proof.** For any line bundle we can determine the curvature once we know the holonomy as follows. Suppose $\Gamma: D \to Z$ is a map of a disk into $Z$ with boundary map $\gamma$. Let $Y_\Gamma = \Gamma^* Y \to D$ be the pullback manifold; then $\partial Y_\Gamma = \hat{Y}_\gamma$. In the following calculation we use the bounding spin structure on $S^1$ and the induced spin structure on $\hat{Y}_\gamma$.

\[
\int_D \Omega^L = -\log \text{hol}_L(\gamma),
\]

\[
= -\text{a-lim} (-\log \tau_{\hat{Y}_\gamma}),
\]

\[
= -\text{a-lim} \left\{ -2\pi i \int_{Y_\Gamma} \hat{A}(\Omega^{Y_\Gamma}) \right\},
\]

\[
= \int_D (-2\pi i) \int_{Y_\Gamma/D} \hat{A}(\Omega^{Y_\Gamma/D}),
\]

\[
= \int_D \Gamma^* \left\{ -2\pi i \left[ \int_{Y/Z} \hat{A}(\Omega^{Y/Z}) \right]_{(2)} \right\}.
\]
In the fourth line we apply (14). In the third line we apply the index theorem of Atiyah-Patodi-Singer [APS] which asserts that

\[ \int_{\Gamma} \hat{A}(\Omega_{\Gamma}) - \eta_{\Gamma}(0) + \frac{\dim \ker D_{\Gamma}}{2} \]

is a certain index, so in particular is an integer. When \( \Gamma \) shrinks the disk to a point both sides of (26) vanish, so we have chosen the correct logarithm on the right hand side of (26). Since (26) holds for all \( \Gamma : D \to Z \), equation (25) follows.

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