Strong excitations of a Bose–Einstein condensate: Barrier resonances

Juan J. G. Ripoll and Víctor M. Pérez-García

Departamento de Matemáticas, Escuela Técnica Superior de Ingenieros Industriales
Universidad de Castilla-La Mancha, 13071 Ciudad Real, Spain

We study the dynamics of a forced condensed atom cloud and relate the behavior to a classical Mathieu oscillator in a singular potential. It is found that there are wide resonances which can strongly affect the dynamics even when dissipation is present. The behavior is characteristic of condensed clouds of any shape and has experimental relevance.

The recent experimental realization of Bose–Einstein condensation (BEC) in ultra-cold atomic gases [1] has triggered the theoretical exploration of the properties of Bose gases. The current model used to describe a system with a fixed mean number of weakly interacting bosons, trapped in a parabolic potential \( V \) given by the trap imposes the condition anisotropies of the trap [2]. In real experiments with stations with a fixed mean number of Bose gases. The current model used to describe a system with a fixed mean number of weakly interacting bosons, trapped in a parabolic potential \( V \) when the particle density and temperature of the condensate are small enough is the so called Gross–Pitaevskii equation (GPE)

\[
i\hbar \partial_t \psi = -\frac{\hbar^2}{2m} \nabla^2 \psi + V(r,t) \psi + U_0 |\psi|^2 \psi,
\]

where \( U_0 = 4\pi \hbar^2 a/m \) is defined in terms of the ground state scattering length \( a \). The normalization for \( \psi \) is \( N = \int |\psi|^2 \, d^3r \), and the trapping potential is given by \( V(r,t) = \frac{1}{2} m \nu^2 (\lambda_x^2(t)x^2 + \lambda_y^2(t)y^2 + \lambda_z^2(t)z^2) \). \( \lambda_n \), \( (n = x, y, z) \) are, as usual, functions that describe the anisotropies of the trap [3]. In real experiments with stationary systems they are constants and the geometry of the trap imposes the condition \( \lambda_x = \lambda_y = 1 \). \( \lambda_z = \nu_z / \nu \) is the quotient between the trap frequency along the \( z \)-direction \( \nu_z \) and the radial one \( \nu_r = \nu \).

The problem of the excitations of the condensate under a periodic driving was studied experimentally in Ref. [3]. On the theoretical side an analysis in the Thomas–Fermi limit which providing an approximation to the low-energy excitation spectra were done in Ref. [4] and numerical simulations done in Ref. [3]. More accurate theoretical predictions were found using time dependent variational methods [2]. Other results have been found using different approaches to the problem for the single [3] and double condensate [2] cases. Recent research on this area has focused also in the effects of damping on the spectrum of low energy excitations [2][4].

Most of the theoretical work done up to now is devoted to the analysis of the condensate properties for weak perturbations. It is our intention here to extend the analysis to the case when the perturbation is stronger or is maintained for a longer time than in the first experiments [3].

An unexpected result is the existence of a parametric resonance which has important experimental implications.

Let us start by deriving our model equations. It can be proved that every solution of Eq. (1) is a stationary point of an action corresponding, up to a divergence, to the Lagrangian density \( \mathcal{L} \):

\[
\mathcal{L} = \frac{i\hbar}{2} (\psi \partial_t \psi^* - \psi^* \partial_t \psi)
+ \frac{\hbar^2}{2m} |\nabla \psi|^2 + V(r) |\psi|^2 + U_0 |\psi|^4,
\]

where the asterisk denotes complex conjugation. That is, instead of working with the GPE we can treat the recent experimental realization of Bose–Einstein condensate: Barrier resonances. We use this trial function to obtain an averaged Lagrangian density and study the time evolution of the parameters that define that family. A natural choice, as discussed in Ref. [3], is a three dimensional Gaussian-like function

\[
\psi(r,t) = A \prod_{\eta = x, y, z} e^{-\frac{\eta^2}{2\nu^2} - i\eta \alpha - i\eta^2 \beta},
\]

We use this trial function to obtain an averaged Lagrangian and then the Lagrange equations to find the evolution equations for all the parameters. Of special relevance are the equations for the widths, which after defining the constants \( P = \sqrt{2/\pi N a / a_0} \) and \( a_0 = \sqrt{\hbar / (m \nu)} \), as well as a set of rescaled variables for time, \( \tau = \nu t \), and the widths, \( \nu_i = a_0 \nu_i, (\nu = x, y, z) \), are found to be

\[
\begin{align*}
\ddot{\nu}_x + \lambda_x^2(t) \nu_x &= \frac{1}{\nu_x^3} + \frac{P}{\nu_x^2 \nu_y \nu_z}, \\
\ddot{\nu}_y + \lambda_y^2(t) \nu_y &= \frac{1}{\nu_y^3} + \frac{P}{\nu_x \nu_y^2 \nu_z}, \\
\ddot{\nu}_z + \lambda_z^2(t) \nu_z &= \frac{1}{\nu_z^3} + \frac{P}{\nu_x \nu_y \nu_z^2}.
\end{align*}
\]

As can be seen, the variational equations remain the same as those of Ref. [3] with the only change of time-dependent \( \lambda \) values. Although a lot of different dynamics are possible in Eqs. (4a)(4c), we will restrict ourselves to the parameter ranges relevant for current Bose–Einstein condensation experiments.
To gain insight on the problem let us first analyze the radially symmetric version of Eq. (4), which is

\[ \ddot{v} + \lambda^2(t)v = \frac{1}{v^3} + \frac{P}{v^4}, \tag{5} \]

This equation is a Hill’s equation with a singular potential given by the right hand side of (3). Approaching the experimental setups, we will treat the case where the potential strength is harmonic, i.e., \( \lambda^2(t) = 1 + \epsilon \cos(\omega t) \).

First, as it can be seen in Fig. 1 if one reproduces the evolution of the Gaussian condensate with a far from equilibrium width in a stationary trap (\( \epsilon = 0 \)), what comes out is a periodic orbit with a frequency that is slightly different from the natural one, and with fast bounces near the origin. In other words, for large amplitude motion, the origin acts as an elastics barrier and the harmonic frequencies of the trap gain importance over the details of the potential well and its linearization.

Let us now concentrate on the time dependent problem (\( \epsilon \neq 0 \)). We have solved it numerically, as stated in Eq. (3). Scanning the parameter space \((\omega, \epsilon)\) one finds at least two wide resonance regions where the radial width \(v\) grows exponentially as far as the perturbation is maintained (Fig. 2(a)). Both regions have the form of wedges, with a starting point, \((\omega_{\text{min}}, \epsilon_{\text{min}})\), and a growing width as \(\epsilon\) is increased (Fig. 2(b)).

Exhaustive simulations have also been made for several values of the parameter \(P\) going from \(P = 9.2\) – the JILA \cite{9} experiment – to 20 times this value, and replacing the singular potential with others of a similar shape \((1/v^4, 1/v^3, \text{etc.})\). The description is the same as the one stated in the preceding paragraphs, the base frequencies \(\omega_{\text{min}} = 2\) and \(\omega_{\text{min}} = 1\) – remaining the same up to a 0.5% precision.

A great care is needed when treating Eq. (3) numerically because of the singularity at \(v = 0\). We have used a Dormand–Prince pair \cite{13}, the ODE Suite of MATLAB, and Vazquez’s conservative scheme \cite{14}. When one approaches a resonance condition all those methods fail after a sufficiently long run and we have employed several stiff methods: the BDF formulas \cite{13}, the LSODE Fortran library and the stiff integrators included in the MATLAB ODE suite.

Since the origin acts as an elastic wall it is intuitively appealing to replace the singular potential in (3) with a bounce condition on the origin, i.e., an impact oscillator \cite{11}. We replace Eq. (3) with the following one

\[ \lim_{t \to t_c^-} (v, \dot{v}) = (0^+, V_c) \quad \iff \quad \lim_{t \to t_c^+} (v, \dot{v}) = (0^+, -V_c), \tag{6} \]

where \(t_c\) denotes any isolated instance when the system bounces against the \(v = 0\) singularity. Let us show that this equation is in turn equivalent to an elastic oscillator without barrier conditions. We introduce the change of variables \(v = |u|\), where \(u\) is an unrestricted real function which satisfies the following one–dimensional harmonic oscillator equation

\[ \ddot{u} + \lambda^2(t)u = 0, \tag{7} \]

This equation is thus a Mathieu equation which arises in the analysis of parametrically forced oscillators. It is a well know problem where one can analytically obtain \cite{12} a lot of information. It is now easy to prove that every solution of Eq. (3) provides a solution of Eq. (4).

And vice versa, from every solution of Eq. (4) it is possible to construct a solution of Eq. (3), unique up to a sign. Among the properties of the Mathieu equation the one that we are concerned most about is the existence of instability regions in the parameter space. The limits of these zones can be found by means of Floquet’s theory, and have the shape of wedges that start on the points \((\omega_{\text{min}}, \epsilon_{\text{min}}) = (2, 0), (1, 0), (2/3, 0), \ldots, \text{and widen as } \epsilon \text{ is increased from zero, in close similarity to our numerical results from Eq. (3)}.\]

Either with an asymptotic method, or by making use of the singular perturbation theory, it is possible to study the evolution of the condensate around the resonances. For a perturbation frequency close enough to the first resonance, that is for \(|\omega - 2| = 2\delta = o(1)\), an asymptotic method \cite{12} yields, up to first order,

\[ q = \pm \sqrt{\frac{\epsilon^2}{4\omega^2} - \delta^2}, \tag{8a} \]

\[ u(t) \simeq ce^{qt} \cos(\omega t + \theta_0). \tag{8b} \]

Here we see that for some values of \(\delta\) and \(\epsilon\) the exponent \(q\) is a positive real number and the amplitude of the oscillations grows unlimitedly. Also, the strength of the resonance is maximum for a value of \(\epsilon = 0\). We have observed at least two subharmonic resonances in our numerical simulations.

The resonances here found resist the presence of dissipation. Adding a viscous damping term to Eq. (3) it becomes

\[ \ddot{u} + (1 + \epsilon \cos(\omega t))u + \gamma \dot{u} = 0. \tag{9} \]

As before, it is also possible to find the approximate form of the dominant contribution on resonance

\[ u(t) \simeq ce^{(\alpha - \gamma)t} \cos(\omega t + \theta_0), \tag{10} \]

where \(q\) is given by an equation similar to (3a). Due to \(\gamma\), the resonance regions in the parameter space are
constrained to values of \((\omega, \epsilon)\) for which the strength of the resonance, \(q\), is greater that of the dissipative term.

For instance, taking the data from the JILA experiment \([9]\), we can estimate a condensate lifetime of about 110 ms and a value for \(\gamma\) of 0.15 in natural units of the condensate. Such a damping makes the \(\epsilon_{\text{min}}\) value rise from 0.09 to 0.18 for the \(P = 9.2\) case. Thus, the instability should not be appreciated unless the perturbation amplitude exceeds the 20%.

An interesting effect of damping is that the evolution of a continuously perturbed condensate outside the instability regions develops a simple limit cycle perfectly synchronized to the frequency of the parametric perturbation, and with a size that depends only on the perturbation parameters, \((\omega, \epsilon)\). The optimal frequency for the creation of the limit cycle depends on the amplitude of the perturbation. For very small perturbations the frequency is close to that of the small amplitude oscillations. However, as the perturbation is increased, the frequency rapidly approaches the Mathieu resonance. The appearance of a limit cycle opens the door to a wide family of phenomena, from chaotic motion to bifurcation theory. This limit cycle would exist under a great variety of dissipative terms, and is not exclusive of linear damping.

Apart from studying the variational reduction of the radially symmetric problem, we have also studied numerically the exact radially symmetric version of Eq. \(1\). In a rather complete inspection of the parameter space, we have found that, up to the point in which the numerical simulations could be continued, the resonances exist and behave like the variational model predicted. This is a somewhat surprising result since the partial differential equation has a lot of degree of freedom available and one could think that the resonant gaussian–like mode would decay into a combination of nonresonant modes. Although some contribution of higher order modes is generated by the perturbation the energy pumping mechanism provided by the resonance is quite efficient and seemingly active as far as the perturbation is maintained. More details of the analysis will be provided elsewhere.

Finally let us comment on the nonsymmetric case ruled by the full set of Eqs. \([1-12]\). Following the experimental setups \([1,2]\), we can once more take a sinusoidal time dependence for every \(\lambda_\eta\) (t) coefficient:

\[
\lambda_\eta^2(t) = \lambda_\\eta_0^2(1 + \epsilon_\eta \cos(\omega t)), \eta = x, y, z. \tag{11}
\]

This choice accounts both for the \(m = 0\) \((\epsilon_x = \epsilon_y, \epsilon_z = 0)\) and the \(m = 2\) \((\epsilon_x = -\epsilon_y, \epsilon_z = 0)\) perturbations from the JILA experiment \([3]\). In the latter case the potential is a parabolic one, with fixed frequencies on a rotating frame.

Substituting our effective perturbation frequencies into Eq. \(4\) we get a set of three coupled Mathieu equations with a potential that is singular on the \(v_x = 0\), \(v_y = 0\) and \(v_z = 0\) planes. The singularities are at least as strong as \(1/v^3\), and the numerical simulations again confirm that they act as elastical walls, so we now proceed with the change of variables \(v_\eta = |u_\eta|\) to find

\[
\ddot{u}_\eta + \lambda_\eta^2(t) u_\eta = 0 \tag{12}
\]

for \(\eta = x, y, z\). Now the situation is a bit more complex. The first new feature we find is the existence of several sets of instability regions. Due to having three \(a \text{ priori}\) different constants \(\lambda_\eta\), the three oscillators in Eq. \(12\) are not equivalent and we may get three sets of resonances in the \((\epsilon_\eta, \omega)\) space. Further details will be given in a future publication \([13]\).

A very interesting result can be found by analyzing exactly the center of mass movement. Defining \(\langle \eta \rangle = \int \eta|\psi|^2 \, dr\), and computing its time derivatives using Eq. \(1\) we arrive again to a Mathieu equation for the center of mass \([14]\)

\[
\frac{d^2}{dt^2} <\eta > + \frac{1}{2} m \omega^2 \lambda_\eta^2(t) <\eta > = 0 \tag{13}
\]

with \(\eta = x, y, z\). This has a serious consequence which is that feeding the condensate will result in the exponential amplification of any initial displacement of the center of mass. This is an exact prediction based only on the GPE. On the other hand, we can imagine two ways in which this effect could have experimental relevance. First, the study of the strength of the resonance should account for possible dissipation effects due to collisions. And secondly, it would be interesting to study how the condensed and the noncondensed centers of mass response to the perturbation, because their different dissipation regimes may lead to an effective separation of both clouds. It also imposes restrictions on the time the perturbation can be applied if the condensate is to remain stable.

Summarizing, we have found that medium to large amplitude oscillations of a condensate approach the harmonic trap frequencies, not the ones resulting from the linearization of the variational equations. Even when damping is added only frequencies close to the Mathieu resonance regions do excite the condensate as a whole in an efficient way, causing the appearance of a stable limit cycle.

We have also found that for this kind of parametrical drive the resonances are naturally wide in all perturbation regimes for this kind of parametrical drive. This width grows with the strength of the interaction, a fact that can be checked in the experiments by forcing the system for a longer time than what it is currently done.

The variational method showed that the kinetic terms in the evolution equations guarantee a \(1/v^3\) singularity as far as we impose a repulsive interaction between the atoms in the cloud. This singularity is enough to cause the appearance of the instability regions. Thus, we have a wide family of systems that behave much the same. Also the response of the noncondensed atoms under the parametrical perturbation will be qualitatively similar to
that of the condensed ones, with the only difference that the former are subject to a more intense dissipation. This dissipation can be enough to distinguish both kind of fluids: while the condensed part might suffer an exponential growth, the uncondensed part might develop bounded oscillations.

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FIG. 1. Orbits in the phase space evolution for the variational approximation of a spherically symmetric condensate with $P = 9.2$, $\epsilon = 0$, and several different initial conditions.

FIG. 2. Example of resonance (a) Time dependent behavior of $v$ for $P = 9.2$, $\omega = 2.04$ and $\epsilon = 0.15$ and initial conditions $(v, \dot{v}) = (1.6, 0)$ close to the equilibrium point (b) One of the regions of instability (shaded) in the $(\omega, \epsilon)$ plane corresponding to the resonance $\omega = 2.04$.

FIG. 3. Plot of the evolution of a cylindrically symmetric condensate for $P = 9.2$ under a sinusoidal perturbation $(\omega, \epsilon) = (2.04, 0.1)$ of the radial strength of the trap. Both (a) the axial $v_z$ and (b) the radial $v_r$ widths are plotted.

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