Results on fuzzy soft topological spaces

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May 5, 2014

Abstract

B. Tanay et. al. [4] introduced and studied fuzzy soft topological spaces. Here we introduce fuzzy soft point and study the concept of neighborhood of a fuzzy soft point in a fuzzy soft topological space. We also study fuzzy soft closure and fuzzy soft interior. Separation axioms and connectedness are introduced and investigated for fuzzy soft topological spaces.

Fuzzy soft topological space, fuzzy soft closure, fuzzy soft connectedness, fuzzy soft separation.
MSC: 06D72.

1 Introduction and Preliminaries

Molodtsov [2] introduced the concept of soft sets in the year 1999. Fuzzy soft set was introduced by Maji et. al. [1]. Since then many researchers have been working for theoretical and practical development of this topic. B. Tanay et. al. introduced topological structure of fuzzy soft set in [4] and gave a introductory theoretical base to carry further study on this topic. S. Roy and T. K. Samanta also studied fuzzy soft topological space in [3].

This paper continues the study of Tanay et. al. to strengthen the theoretical pedestal of fuzzy soft topological spaces.

Here are some definitions and results required in the sequel.

Let $U$ be an initial universe, $E$ be the set of parameters, $\mathcal{P}(U)$ be the set of all subsets of $U$ and $\mathcal{FS}(U; E)$ be the family of all fuzzy soft sets over $U$ via parameters in $E$.

**Definition 1.1.** [7] Let $A \subseteq E$ and $\mathcal{F}(U)$ be the set of all fuzzy sets in $U$. Then the pair $(f, A)$ is called a fuzzy soft set over $U$, denoted by $f_A$, where $f : A \to \mathcal{F}(U)$ is a function.
Definition 1.2. Two fuzzy soft sets \( f_A \) and \( g_B \) are said to be disjoint if \( f(a) \cap g(b) = \Phi, \forall a \in A, b \in B \).

Definition 1.3. [4] Let \( f_A \) be a fuzzy soft set, \( FS(f_A) \) be the set of all fuzzy soft subsets of \( f_A \) and \( \tau \) be a subfamily of \( FS(f_A) \). Then \( \tau \) is called a fuzzy soft topology on \( f_A \) if the following conditions are satisfied.

i. \( \Phi \sim f_A \) belongs to \( \tau \);

ii. \( h_A, g_B \in \tau \Rightarrow h_A \sim g_B \in \tau \);

iii. \( \{ (h_A)_\lambda \mid \lambda \in \Lambda \} \subset \tau \Rightarrow \sim \bigcap_{\lambda \in \Lambda} (h_A)_\lambda \in \tau \).

Then \( (f_A, \tau) \) is called a fuzzy soft topological space. Members of \( \tau \) are called fuzzy soft open sets and their complements are called fuzzy soft closed sets.

Definition 1.4. [4] Let \( (f_A, \tau) \) be a fuzzy soft topological space and \( g_A \in FS(f_A) \). Then the fuzzy soft topology \( \tau_{g_A} = \{ g_A \sim h_A \mid h_A \in \tau \} \) is called fuzzy soft subspace topology and \( (g_A, \tau_{g_A}) \) is called fuzzy soft subspace of \( (f_A, \tau) \).

Definition 1.5. [4] Let \( (f_A, \tau) \) be a fuzzy soft topological space and \( h_A, g_B \) be fuzzy soft sets in \( FS(f_A) \) such that \( g_B \sim h_A \). Then \( g_B \) is called an interior fuzzy soft set of \( h_A \) iff \( h_A \) is a neighborhood of \( g_B \).

The union of all interior fuzzy soft sets of \( g_A \) is called the interior of \( g_A \) and is denoted by \( g_A^0 \).

Definition 1.6. [4] Let \( (f_A, \tau_1) \) and \( (f_A, \tau_2) \) be two fuzzy soft topological spaces. If each \( g_A \in \tau_1 \) is in \( \tau_2 \), then \( \tau_2 \) is called fuzzy soft finer than \( \tau_1 \), or \( \tau_1 \) is called fuzzy soft coarser than \( \tau_2 \).

2 Fuzzy soft neighborhood, fuzzy soft closure and fuzzy soft interior

In [4], authors defined neighborhood of a fuzzy soft set but not for a point. Here we introduce and study fuzzy soft point and its fuzzy soft neighborhood. Further fuzzy soft interior and fuzzy soft closure of a fuzzy soft set in a fuzzy soft topological space are investigated.

Definition 2.1. A fuzzy soft set \( g_A \) is said to be a fuzzy soft point, denoted by \( e_{g_A} \), if for the element \( e \in A, g(e) \neq \Phi \) and \( g(e') = \Phi, \forall e' \in A - \{ e \} \).

Definition 2.2. The complement of a fuzzy soft point \( e_{g_A} \) is a fuzzy soft point \( (e_{g_A})^c \) such that \( g_{e_{g_A}}^c(e) = 1 - g(e) \) and \( g_{e_{g_A}}^c(e') = \Phi, \forall e' \in A - \{ e \} \).

Example 2.3. Let \( U = \{ h^1, h^2, h^3, h^4 \} \), \( A = \{ e_1, e_2, e_3, e_4, e_5 \} \subset E \), the set of parameters. Then \( e_{g_A} = \{ e_1 \} = \{ h^1_{0.1}, h^2_{0.9}, h^3_{0.4} \} \) is a fuzzy soft point whose complement is \( (e_{g_A})^c = \{ e_1 \} = \{ h^1_{0.9}, h^2_{0.1}, h^3_{0.6} \} \).
Definition 2.4. A fuzzy soft point $e_{g_A}$ is said to be in a fuzzy soft set $h_A$, denoted by $e_{g_A} \subseteq h_A$ if for the element $e \in A, g(e) \leq h(e)$.

Theorem 2.5. Fuzzy soft points satisfy the following properties.

i. If a fuzzy soft point $e_{g_A} \sim g_A$ then $e_{g_A} \notin g_A$;  

ii. $e_{g_A} \sim g_A \Rightarrow e_{g_A} \sim g_A'$;  

iii. Union of all the fuzzy soft points of a fuzzy soft set is equal to the fuzzy soft set;  

iv. $e_{g_A} \in e_{h_B} \Leftrightarrow g(e) \leq h(e)$ and $A \subseteq B$;  

v. $e_{g_A} \in \bigcup \{h_{\lambda A} \mid \lambda \in \Lambda \} \Leftrightarrow \exists \lambda \in \Lambda$ such that $e_{g_A} \in h_{\lambda A}$;  

vi. $e_{g_A} \in \bigcap \{h_{\lambda A} \mid \lambda \in \Lambda \} \Leftrightarrow \forall \lambda \in \Lambda$ $e_{g_A} \in h_{\lambda A}$.

Following is an example in favor of theorem 2.5.(ii).

Example 2.6. Let $U = \{h_1, h_2\}, E = \{e_1, e_2\}$. Consider the fuzzy soft point $e_{g_A} = \{e_1 = (h_{0.1}, h_{0.2}), e_2 = (h_{0.2}, h_{0.3})\}$, which is contained in the fuzzy soft set $h_A = \{e_1 = (h_{0.1}, h_{0.2}), e_2 = (h_{0.1}, h_{0.1})\}$. Then $e_{g_A} = \{e_1 = (h_{0.9}, h_{0.1}), e_2 = (h_{0.8}, h_{0.7})\}$ does not contain $e_{g_A} = \{e_1 = (h_{0.9}, h_{0.8})\}$.

Definition 2.7. A fuzzy soft set $g_A$ in a fuzzy soft topological space $(f_A, \tau)$ is said to be a fuzzy soft neighborhood of a fuzzy soft point $e_{g_A}$ if $\exists$ a fuzzy soft open set $h_A$ such that $e_{g_A} \subseteq h_A \subseteq g_A$.

Example 2.8. Consider fuzzy soft topological space $(f_A, \sim)$ as defined in Example 2.2 of [2]. Here $\{e_3 = (h_{0.2}, h_{0.3}, h_{0.4}, h_{0.5}, h_{0.6})\}$ is a fuzzy soft neighborhood of the fuzzy soft point $\{e_3 = (h_{0.1}, h_{0.2}, h_{0.3}, h_{0.4}, h_{0.6})\}$.

The family of all neighborhoods of $e_{g_A}$ is called its neighborhood system and is denoted by $N_{\tau}(e_{g_A})$.

Theorem 2.9. A fuzzy soft set in a fuzzy soft topological space is fuzzy soft open if and only if it is a fuzzy soft neighborhood of each of its fuzzy soft points.

Proof. Let $(f_A, \tau)$ be a fuzzy soft topological space and $e_{g_A}$ be a fuzzy soft point in a fuzzy soft open set $g_A$. Then by definition, $g_A$ is a fuzzy soft neighborhood of $e_{g_A}$.

Conversely, let $g_A$ be a fuzzy soft set such that it is fuzzy soft neighborhood of each of its fuzzy soft points, say $e_{g_A}$. Then for each $\lambda \in \Lambda$, $\exists$ a fuzzy soft open set $h_{\lambda A}$ such that $e_{\lambda A} \subseteq h_{\lambda A} \subseteq g_A$. Now $g_A = \bigcup_{\lambda \in \Lambda} e_{\lambda A} \Rightarrow g_A = \bigcup_{\lambda \in \Lambda} h_{\lambda A} \Rightarrow g_A$ is fuzzy soft open being the union of arbitrary family of fuzzy soft open sets.

\[\square\]
Theorem 2.10. The neighborhood system of $\mathcal{N}_\tau(e_{gA})$ in a soft topological space $(f_A, \tau)$ satisfies the following properties:

i. If $g_A \in \mathcal{N}_\tau(e_{gA})$, then $e_{gA} \sim g_A$;

ii. A fuzzy soft superset of a fuzzy soft neighborhood of a fuzzy soft point is also a fuzzy soft neighborhood of the point;

iii. Intersection of two fuzzy soft neighborhoods of a fuzzy soft point is again a fuzzy soft neighborhood;

iv. $k_A \in \mathcal{N}_\tau(e_{gA}) \Rightarrow \exists h_A \in \mathcal{N}_\tau(e_{gA})$ such that $h_A \sim k_A$ and $h_A \in \mathcal{N}_\tau(e_{hA})$.

Proof. i. If $g_A \in \mathcal{N}_\tau(e_{gA})$, then $\exists$ a fuzzy soft open set $h_A$ such that $e_{gA} \sim h_A \subseteq g_A$.

ii. Let $g_A \in \mathcal{N}_\tau(e_{gA}) \Rightarrow \exists$ a fuzzy soft open set $h_A$ such that $e_{gA} \sim h_A \subseteq g_A$ and $h_A \subseteq k_A \Rightarrow e_{gA} \sim h_A \subseteq k_A \Rightarrow k_A \in \mathcal{N}_\tau(e_{gA})$.

iii. Let $g_A, k_A \in \mathcal{N}_\tau(e_{gA})$ then there exists fuzzy soft open sets $h_A$ and $s_A$ such that $e_{gA} \sim h_A \subseteq g_A$ and $e_{gA} \sim s_A \subseteq k_A \Rightarrow e_{gA} \sim h_A \cap s_A \subseteq g_A \cap k_A$.

Now $h_A \cap s_A$ is fuzzy soft open and hence $g_A \cap k_A \in \mathcal{N}_\tau(e_{gA})$.

iv. $k_A \in \mathcal{N}_\tau(e_{gA}) \Rightarrow \exists$ a fuzzy soft open set $s_A$ such that $e_{gA} \sim s_A \subseteq k_A$.

By definition $s_A$ is a fuzzy soft neighborhood of each of its points, so $s_A \in \mathcal{N}_\tau(e_{sA})$.

Definition 2.11. Let $(f_A, \tau)$ be a fuzzy soft topological space and $g_A$ be a fuzzy soft set.

i. The fuzzy soft closure of $g_A$ is a fuzzy soft set

$$fsclg_A = \bigcap \{h_B \mid g_A \subseteq h_B \text{ and } h_B \text{ is fuzzy soft closed set}\};$$

ii. The fuzzy soft interior of $g_A$ is a fuzzy soft set

$$fsintg_A = \bigcup \{h_B \mid h_B \subseteq g_A \text{ and } h_B \text{ is fuzzy soft open set}\}.$$

In [4], authors defined fuzzy soft interior of a fuzzy soft set. But it is clear that both the definitions are equivalent.

Theorem 2.12. A fuzzy soft set $g_A$ is fuzzy soft closed iff $fsclg_A = g_A$.

Theorem 2.13. Let $(f_A, \tau)$ be a fuzzy soft topological space and $g_A, h_A$ be fuzzy soft sets. Then

i. $(fsclg_A)^c = fsintg_A^c$;

ii. $(fsintg_A)^c = fsclg_A^c$;
iii. \( g_A \subseteq h_A \Rightarrow f\text{scl}g_A \subseteq f\text{scl}h_A \);
iv. \( g_A \sim h_A \Rightarrow f\text{sint}g_A \subseteq f\text{sint}h_A \);
v. \( f\text{scl}(f\text{scl}g_A) = f\text{scl}g_A \);
vi. \( f\text{sint}(f\text{sint}g_A) = f\text{sint}g_A \);
vii. \( f\text{scl}f_A = f \) and \( f\text{sclf}A = f \);
viii. \( f\text{sint}f_A = f \) and \( f\text{sint}f_A = f \);
ix. \( f\text{scl}(g_A \cup h_A) = f\text{scl}g_A \cup f\text{scl}h_A \);
x. \( f\text{sint}(g_A \cap h_A) = f\text{sint}f_A \cap f\text{sint}h_A \);
xi. \( f\text{scl}(g_A \cap h_A) \subset f\text{scl}g_A \cap f\text{scl}h_A \);
xii. \( f\text{sint}(g_A \cup h_A) \subset f\text{sint}g_A \cup f\text{sint}h_A \);

Proof. Straightforward.

**Theorem 2.14.** The fuzzy soft set \( h_A \) is fuzzy soft closed in a subspace \((g_A, \tau_{g_A})\) of \((f_A, \tau_f)\) iff \( h_A = k_A \cap g_A \) for some fuzzy soft closed set \( k_A \) in \( f_A \).

**Theorem 2.15.** The fuzzy soft closure of a fuzzy soft set \( h_A \) in a subspace \((g_A, \tau_{g_A})\) of \((f_A, \tau_f)\) equals \( f\text{scl}(h_A) \cap g_A \).

**Proof.** We know \( f\text{scl}h_A \) is a fuzzy soft closed set in \( f_A \Rightarrow f\text{scl}h_A \cap g_A \) is fuzzy soft closed set in \( g_A \). Now \( h_A \cap f\text{scl}h_A \cap g_A \) and fuzzy soft closure of \( h_A \) in \( g_A \) is the smallest fuzzy closed set containing \( h_A \), so fuzzy soft closure of \( h_A \) in \( g_A \) is contained in \( f\text{scl}h_A \cap g_A \).

On the other hand, if \( w_A \) denotes the fuzzy soft closure of \( h_A \) in \( g_A \), then \( w_A \) is a fuzzy soft closed set in \( g_A \Rightarrow w_A = k_A \cap g_A \) where \( k_A \) is a fuzzy soft closed set in \( f_A \) (by theorem 2.14). Then \( k_A \) is fuzzy soft closed containing \( h_A \Rightarrow f\text{scl}h_A \subset k_A \Rightarrow f\text{scl}h_A \cap g_A \subset k_A \cap g_A = w_A \).

\[ \square \]

### 3 Fuzzy Soft separation axioms

Here, we introduce and study various separation axioms for a fuzzy soft topological space.

**Definition 3.1.** A fuzzy soft topological space \((f_A, \tau_f)\) is said to be a fuzzy soft \( T_0 \)-space if for every pair of disjoint fuzzy soft points \( e_{h_A}, e_{g_B} \), \( \exists \) a fuzzy soft open set containing one but not the other.
Example 3.2. A discrete fuzzy soft topological space is a fuzzy soft \( T_0 \)– space since every \( e_{h_A} \) is a fuzzy soft open set in the discrete space.

Theorem 3.3. A fuzzy soft subspace of a fuzzy soft \( T_0 \)– space is fuzzy soft \( T_0 \).

Proof. Let \( (g_A, \tau_{g_A}) \) be a fuzzy soft subspace of a fuzzy soft \( T_0 \)– space \( (f_A, \tau) \) and let \( e_{k_1}, e_{k_2} \) be two distinct fuzzy soft points of \( g_A \). Then these fuzzy soft points are also in \( f_A \Rightarrow \exists \) a fuzzy soft open set \( h_A \) containing one fuzzy soft point but not the other \( \Rightarrow g_A \cap h_A \), where \( h_A \in \tau \) is a fuzzy soft open set in \( \tau_{g_A} \) containing one fuzzy soft point but not the other. \qed 

Definition 3.4. A fuzzy soft topological space \( (f_A, \tau) \) is said to be a fuzzy soft \( T_1 \)– space if for distinct pair of fuzzy soft points \( e_{g_A}, e_{k_A} \) of \( f_A \), \( \exists \) fuzzy soft open sets \( s_A \) and \( h_A \) such that
\[
e_{g_A} \not\in s_A \text{ and } e_{g_A} \not\in h_A;
\]
\[
e_{k_A} \not\in h_A \text{ and } e_{k_A} \not\in s_A.
\]

Theorem 3.5. If every fuzzy soft point of a fuzzy soft topological space \( (f_A, \tau) \) is fuzzy soft closed then \( (f_A, \tau) \) is fuzzy soft \( T_1 \).

Proof. Let \( e_{h_A} = \{ e_j = \{ h_{\alpha_i}^i \mid i = 1, 2, \ldots, n \} \} \), \( e_{k_A} = \{ e_m = \{ h_{\beta_i}^i \mid i = 1, 2, \ldots, n \} \} \), where \( e_j, e_m \) are distinct parameters be distinct fuzzy soft point of \( f_A \).

i. \( \alpha_i, \beta_i \leq 0.5 \).
Then we can always find some \( \gamma_i \) and \( \delta_i \) such that \( \alpha_i \leq \gamma_i, \beta_i \leq \delta_i \Rightarrow \alpha_i \leq 1 - \gamma_i, \beta_i \leq 1 - \delta_i \Rightarrow \) the fuzzy soft sets \( e_{l_A} = \{ e_j = \{ h_{\delta_i}^i \mid i = 1, 2, \ldots, n \} \} \) and \( e_{l_A} = \{ e_m = \{ h_{\gamma_i}^i \mid i = 1, 2, \ldots, n \} \} \) are such that their complements are disjoint fuzzy soft open sets containing \( e_{h_A} \) and \( e_{k_A} \) respectively.

ii. \( \alpha_i, \beta_i > 0.5 \).
Then we can always find some \( \gamma_i \) and \( \delta_i \) such that \( \gamma_i \leq \alpha_i, \delta_i \leq \beta_i \Rightarrow \alpha_i \leq 1 - \gamma_i, \beta_i \leq 1 - \delta_i \Rightarrow \) the fuzzy soft sets \( e_{l_A} = \{ e_j = \{ h_{\delta_i}^i \mid i = 1, 2, \ldots, n \} \} \) and \( e_{l_A} = \{ e_m = \{ h_{\gamma_i}^i \mid i = 1, 2, \ldots, n \} \} \) are such that their complements are disjoint fuzzy soft open sets containing \( e_{h_A} \) and \( e_{k_A} \) respectively. \qed 

Theorem 3.6. A fuzzy soft subspace of a fuzzy soft \( T_1 \)– space is fuzzy soft \( T_1 \).

Definition 3.7. A fuzzy soft topological space \( (f_A, \tau) \) is said to be a fuzzy soft \( T_2 \)– space if and only if for distinct fuzzy soft points \( e_{g_A}, e_{k_A} \) of \( f_A \), \( \exists \) disjoint fuzzy soft open sets \( h_A \) and \( s_A \) such that \( e_{g_A} \not\in h_A \) and \( e_{k_A} \not\in s_A \).

Theorem 3.8. If every fuzzy soft point of a fuzzy soft topological space \( (f_A, \tau) \) is fuzzy soft closed then \( (f_A, \tau) \) is fuzzy soft \( T_2 \).

Theorem 3.9. A fuzzy soft subspace of a fuzzy soft \( T_2 \)– space is fuzzy soft \( T_2 \).
Theorem 3.10. A fuzzy soft topological space \((f_A, \tau)\) is fuzzy soft T2 if and only if for distinct fuzzy soft points \(e_{g_A}, e_{k_A}\) of \(f_A, \exists\) a fuzzy soft open set \(s_A\) containing \(e_{g_A}\) but not \(e_{k_A}\) such that \(e_{k_A} \notin \text{scls}_A\).

Proof. \((\Rightarrow)\) Let \((f_A, \tau)\) be fuzzy soft T2 and \(e_{g_A}, e_{k_A}\) be distinct fuzzy soft points. So \(\exists\) distinct fuzzy soft open sets \(h_A\) and \(b_A\) such that \(e_{k_A} \in h_A, e_{g_A} \in b_A \Rightarrow e_{g_A} \in h'_A\). So \(h'_A\) is a fuzzy soft open set containing \(e_{g_A}\) but not \(e_{k_A}\) and \(fscl h'_A = h'_A\). \((\Leftarrow)\) Take a pair of distinct fuzzy soft points \(e_{g_A}\) and \(e_{k_A}\) of \(f_A, \exists\) a fuzzy soft open set \(s_A\) containing \(e_{g_A}\) but not \(e_{k_A}\) such that \(e_{k_A} \notin \text{scls}_A \Rightarrow e_{k_A} \in (\text{scls}_A)^c \Rightarrow s_A\) and \((\text{scls}_A)^c\) are disjoint fuzzy soft open set containing \(e_{g_A}\) and \(e_{k_A}\) respectively.

\(\square\)

Definition 3.11. A fuzzy soft topological space \((f_A, \tau)\) is said to be a fuzzy soft regular space if for every fuzzy soft point \(e_{h_A}\) and fuzzy soft closed set \(k_A\) not containing \(e_{h_A}\), \(\exists\) disjoint fuzzy soft open sets \(g_{1A}, g_{2A}\) such that \(e_{h_A} \notin g_{1A}\) and \(k_A \subseteq g_{2A}\).

A fuzzy soft regular \(T_1\) space is called a fuzzy soft \(T_2\) space,

Remark 3.12. It can be shown that the property of being fuzzy soft \(T_3\) is hereditary.

Remark 3.13. Every fuzzy soft \(T_3\) space is fuzzy soft \(T_2\) space, every fuzzy soft \(T_2\) space is fuzzy soft \(T_1\) space and every fuzzy soft \(T_1\) space is fuzzy soft \(T_0\) space.

Theorem 3.14. A fuzzy soft topological space \((f_A, \tau)\) in which every fuzzy soft point is fuzzy soft closed, is fuzzy soft regular iff for a fuzzy soft open set \(g_A\) containing a fuzzy soft point \(e_{h_A}\), there exists a fuzzy soft open set \(s_A\) containing \(e_{h_A}\) such that \(fscl s_A \subset g_A\).

Proof. Take a fuzzy soft open set \(g_A\) containing \(e_{h_A}\) in a regular fuzzy soft topological space \((f_A, \tau\) Then \(g_A^c\) is fuzzy soft closed. By hypothesis, \(\exists\) disjoint fuzzy soft open sets \(s_A\) and \(w_A\) such that \(e_{h_A} \subset s_A\) and \(g_A^c \subset w_A\). Now, \(s_A\) and \(w_A\) are disjoint, so \(s_A \subset w_A^c \Rightarrow fscl s_A \subset w_A^c \Rightarrow fscl s_A \subset g_A\).

Conversely, assume the hypothesis. Take a fuzzy soft closed set \(k_A\) not containing a fuzzy soft point \(e_{h_A} \notin k_A\). Then \(k_A^c\) is a fuzzy soft open set containing the fuzzy \(e_{h_A}\) \(\Rightarrow\) a fuzzy soft open set \(s_A\) containing \(e_{h_A}\) such that \(fscl s_A \subset k_A^c \Rightarrow k_A \subset (fscl s_A)^c \Rightarrow (fscl s_A)^c\) is a fuzzy soft open set containing \(k_A\) and \(s_A \cap (fscl s_A)^c = \emptyset\) \(\square\)

Definition 3.15. A fuzzy soft topological space \((f_A, \tau)\) is said to be a fuzzy soft normal space if for every pair of disjoint fuzzy soft closed sets \(h_A\) and \(k_A, \exists\) disjoint fuzzy soft open sets \(g_{1A}, g_{2A}\) such that \(h_A \subseteq g_{1A}\) and \(k_A \subseteq g_{2A}\).
A fuzzy soft normal $T_1$-space is called a fuzzy soft $T_4$-space.

**Remark 3.16.** Every fuzzy soft $T_4$-space is fuzzy soft $T_3$.

**Theorem 3.17.** A fuzzy soft topological space $(f_A, \tau)$ is fuzzy soft normal iff for any fuzzy soft closed set $h_A$ and fuzzy soft open set $g_A$ containing $h_A$, there exists a fuzzy soft open set $s_A$ such that $h_A \subseteq s_A$ and $f_{scl} s_A \subseteq g_A$.

**Proof.** Let $(f_A, \tau)$ be fuzzy soft normal space and $h_A$ be a fuzzy soft closed set and $g_A$ be a fuzzy soft open set containing $h_A \Rightarrow h_A$ and $g_A^c$ are disjoint fuzzy soft closed sets $\Rightarrow \exists$ disjoint fuzzy soft open sets $g_{1A}, g_{2A}$ such that $h_A \subseteq g_{1A}$ and $g_A^c \subseteq g_{2A}$. Now $g_{1A} \subseteq g_{2A} \Rightarrow f_{scl} g_{1A} \subseteq f_{scl} g_{2A}^c = g^c_{2A}$ Also, $g_A^c \subseteq g_{2A} \Rightarrow g_{2A}^c \subseteq g_A \Rightarrow f_{scl} g_{1A} \subseteq g_A$.

Conversely, let $l_A$ and $k_A$ be any disjoint pair fuzzy soft closed sets $\Rightarrow l_A \subseteq k_A^c$, then by hypothesis there exists a fuzzy soft open set $s_A$ such that $l_A \subseteq s_A$ and $f_{scl} s_A \subseteq k_A = k_A \subseteq (f_{scl} s_A)^c \Rightarrow s_A$ and $(f_{scl} s_A)^c$ are disjoint fuzzy soft open sets such that $l_A \subseteq s_A$ and $k_A \subseteq (f_{scl} s_A)^c$. 

**Theorem 3.18.** A fuzzy soft closed subspace of a fuzzy soft normal space is fuzzy soft normal.

## 4 Fuzzy soft connectedness

In this section, we introduce and study fuzzy soft connectedness of fuzzy soft topological spaces.

**Definition 4.1.** A fuzzy soft separation of a fuzzy soft topological space $(f_A, \tau)$ is a pair $h_A, k_A$ of disjoint non empty fuzzy soft open sets whose union is $f_A$.

If there does not exist a fuzzy soft separation of $f_A$, then the fuzzy soft topological space is said to be fuzzy soft connected, otherwise fuzzy soft disconnected.

**Example 4.2.**

i. The discrete fuzzy soft topological space with more than one member is always disconnected;

ii. The indiscrete fuzzy soft topological space is always connected.

**Theorem 4.3.** A fuzzy soft topological space $(f_A, \tau)$ is fuzzy soft disconnected $\iff \exists$ a non empty proper fuzzy soft subset of $f_A$ which is both fuzzy soft open and fuzzy soft closed.

**Proof.** Let $k_A$ be a non empty proper subset of $f_A$ which is both fuzzy soft open and fuzzy soft closed. Now $h_A = (k_A)^c$ is non empty proper subset of $f_A$ which is also both fuzzy soft open and fuzzy soft closed $\Rightarrow f_{scl} k_A = k_A$ and $f_{scl} h_A = h_A \Rightarrow f_A$ can be expressed as the union of two separated fuzzy soft sets $k_A, h_A$ and so, is fuzzy soft disconnected.

Conversely, let $f_A$ be fuzzy soft disconnected $\Rightarrow \exists$ non empty fuzzy soft subsets $k_A$ and $h_A$ such that $f_{scl} k_A \cap h_A = \emptyset, k_A \cap f_{scl} h_A = \emptyset$ and $k_A \cup h_A = f_A$. 

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Now \( k_A \subseteq f\text{scl}k_A \) and \( f\text{scl}k_A \cap h_A = \Phi_A \Rightarrow k_A \cap h_A = \Phi_A \Rightarrow h_A = (k_A)^c \).

Then \( k_A \cap f\text{scl}h_A = f_A \) and \( k_A \cap f\text{scl}h_A = \Phi_A \Rightarrow k_A = (f\text{scl}h_A)^c \) and similarly \( h_A = (f\text{scl}k_A)^c \Rightarrow k_A, h_A \) are fuzzy soft open sets being the complements of fuzzy soft closed sets. Also \( h_A = (k_A)^c \Rightarrow \) they are also fuzzy soft closed.

**Theorem 4.4.** If the fuzzy soft sets \( h_A \) and \( k_A \) form a fuzzy soft separation of \( f_A \), and if \((g_A, \tau_{g_A})\) is a fuzzy soft connected subspace of \( f_A \), then \( g_A \subseteq h_A \) or \( g_A \subseteq k_A \).

**Proof.** Since \( h_A \) and \( k_A \) are disjoint fuzzy soft open sets, so are \( h_A \cap g_A \) and \( k_A \cap g_A \) and their union gives \( g_A \), i.e. they would constitute a fuzzy soft separation of \( g_A \), a contradiction. Hence, one of \( h_A \cap g_A \) and \( k_A \cap g_A \) is empty and so \( g_A \) is entirely contained in one of them. \( \square \)

**Theorem 4.5.** If \( g_A \) is a fuzzy soft subspace of \( f_A \), a separation of \( g_A \) is a pair of disjoint non empty fuzzy soft sets \( k_A \) and \( h_A \) whose union is \( g_A \), such that \( k_A \cap f\text{scl}h_A = \Phi \) and \( h_A \cap f\text{scl}k_A = \Phi \).

**Proof.** Suppose \( k_A \) and \( h_A \) forms a separation of \( g_A \). Then \( k_A \) is both fuzzy soft open and fuzzy soft closed in \( g_A \). The fuzzy soft closure of \( k_A \) in \( g_A \) is \( f\text{scl}k_A \cap g_A \). Since \( k_A \) is fuzzy soft closed in \( g_A \), \( k_A = f\text{scl}k_A \cap g_A \Rightarrow f\text{scl}k_A \cap h_A = \Phi \). By similar argument \( f\text{scl}h_A \cap k_A = \Phi \).

Conversely, let \( k_A \) and \( h_A \) are disjoint non empty fuzzy soft sets whose union is \( g_A \) such that \( k_A \cap f\text{scl}h_A = \Phi \) and \( h_A \cap f\text{scl}k_A = \Phi \Rightarrow g_A \cap f\text{scl}h_A = \Phi \) and \( g_A \cap f\text{scl}k_A = \Phi \Rightarrow h_A \) and \( k_A \) are fuzzy soft closed in \( g_A \). Also \( h_A = k_A^c \) implies both \( k_A \) and \( h_A \) are fuzzy soft open in \( g_A \). \( \square \)

**Theorem 4.6.** Let \( g_A \) be a fuzzy soft connected subspace of \( f_A \). If \( g_A \subseteq k_A \subseteq f\text{scl}g_A \), then \( k_A \) is also fuzzy soft connected.

**Proof.** Let the soft set \( k_A \) satisfies the hypothesis. If possible, let \( h_A \) and \( s_A \) form a fuzzy soft separation of \( k_A \). Then by theorem 4.2, \( g_A \subset h_A \) or \( g_A \subset s_A \). Let \( g_A \subset h_A \Rightarrow f\text{scl}g_A \subset f\text{scl}h_A \); since \( f\text{scl}h_A \) and \( s_A \) are disjoint, \( f\text{scl}g_A \) cannot intersect \( s_A \). This contradicts the fact that \( s_A \) is a nonempty. \( \square \)

**Remark 4.7.** In particular \( f\text{scl}g_A \) is fuzzy soft connected if \( g_A \) is fuzzy soft connected.

**Remark 4.8.** A fuzzy soft topological space is fuzzy soft connected iff \( \Phi \) and \( f_A \) are the only sets which are both fuzzy soft open and fuzzy soft closed.

**Theorem 4.9.** Arbitrary union of fuzzy soft connected subsets of \((f_A, \tau)\) that have non empty intersection is fuzzy soft connected.
Proof. Let \( \{(g_{\lambda}, \tau_{g_{\lambda}}) \mid \lambda \in \Lambda \} \) be a collection of fuzzy soft connected subspaces of \((f_A, \tau)\) with non empty intersection. If possible, take a fuzzy soft separation \(h_A, k_A\) of \(g_A = \bigcup_{\lambda \in \Lambda} g_{\lambda}\). Now for each \(\lambda\), \(h_A \cap g_{\lambda}\) and \(k_A \cap g_{\lambda}\) are disjoint fuzzy soft open sets in the subspace such that their union gives \(g_{\lambda}\). As \(g_{\lambda}\) is connected for each \(\lambda\), one of \(h_A \cap g_{\lambda}\) and \(k_A \cap g_{\lambda}\) must be empty (by theorem 4.2). Suppose, \(h_A \cap g_{\lambda} = \not\emptyset\) \(\Rightarrow\) \(k_A \cap g_{\lambda} = g_{\lambda} \subset k_A \forall \lambda \in \Lambda \Rightarrow \bigcup_{\lambda \in \Lambda} g_{\lambda} \subset k_A \Rightarrow h_A \not\subset k_A \Rightarrow\) \(h_A\) is empty, a contradiction.

**Theorem 4.10.** Arbitrary union of a family of fuzzy soft connected subsets of \((f_A, \tau)\) such that one of the members of the family has non empty intersection with every member of the family, is fuzzy soft connected.

Proof. Let \(\{(g_{\lambda}, \tau_{g_{\lambda}}) \mid \lambda \in \Lambda \}\) be a collection of fuzzy soft connected subspaces of \((f_A, \tau)\) and \(g_{\lambda_0}\) be a fixed member such that \(g_{\lambda_0} \not\subset \not\emptyset\) for each \(\lambda \in \Lambda\). Then by theorem 4.7, \(h_{\lambda} = g_{\lambda_0} \bigcup g_{\lambda}\) is a fuzzy soft connected for each \(\lambda \in \Lambda\). Now,

\[
\bigcup_{\lambda \in \Lambda} h_{\lambda} = \bigcup_{\lambda \in \Lambda} (g_{\lambda} \bigcup g_{\lambda}) = \bigcup_{\lambda \in \Lambda} g_{\lambda} \text{ and }
\]

\[
\bigcap_{\lambda \in \Lambda} h_{\lambda} = \bigcap_{\lambda \in \Lambda} (g_{\lambda} \bigcup g_{\lambda}) = g_{\lambda_0} \bigcap_{\lambda \in \Lambda} (\bigcup g_{\lambda}) \not\subset \not\emptyset.
\]

Therefore, by theorem 4.7 \(\bigcup_{\lambda \in \Lambda} h_{\lambda} = \bigcup_{\lambda \in \Lambda} g_{\lambda}\) is fuzzy soft connected.

**Theorem 4.11.** If \((f_A, \tau_2)\) is a fuzzy soft connected space and \(\tau_1\) is fuzzy soft coarser than \(\tau_2\), then \((f_A, \tau_1)\) is also fuzzy soft connected.

Proof. Assume that \(k_A, h_A\) form a fuzzy soft separation of \((f_A, \tau_1)\). Now \(k_A, h_A \in \tau_1 \Rightarrow k_A, h_A \in \tau_2 \Rightarrow k_A, h_A\) form a fuzzy soft separation of \((f_A, \tau_2)\), a contradiction.

\[
\square
\]

5 Conclusion

This paper investigates properties of separation axioms and connectedness of fuzzy soft topological spaces. Several properties of neighborhood system of a fuzzy soft point are discussed. Other concepts like, compactness and continuity for a fuzzy soft topological space etc can be studied further.

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