On Grothendieck–Serre conjecture in mixed
class characteristic for $SL_{1,D}$

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Abstract

Let $R$ be an unramified regular local ring of mixed characteristic, $D$ an Azumaya
$R$-algebra, $K$ the fraction field of $R$, $Nrd : D^\times \to R^\times$ the reduced norm homomor-
phism. Let $a \in R^\times$ be a unit. Suppose the equation $Nrd = a$ has a solution over
$K$, then it has a solution over $R$.

Particularly, we prove the following. Let $R$ be as above and $a, b, c$ be units in
$R$. Consider the equation $T_1^2 - aT_2^2 - bT_3^2 + abT_4^2 = c$. If it has a solution over $K$, then
it has a solution over $R$.

Similar results are proved for regular local rings, which are geometrically regular
over a discrete valuation ring. These results extend result proven in [PS] to the
mixed characteristic case.

1 Introduction

A well-known conjecture due to J.-P. Serre and A. Grothendieck [Sc] Remarque, p.31],
[Gr1] Remarque 3, p.26-27], and [Gr2] Remarque 1.10.a] asserts that, for any regular local
ring $R$ and any reductive group scheme $G$ over $R$ rationally trivial $G$-homogeneous spaces
are trivial. Our results correspond to the case when $R$ an unramified regular local ring
of mixed characteristic and $G$ is the group $SL_1(D)$ of norm one elements of an Azumaya
$R$-algebra $D$. Our results extend to the mixed characteristic case the ones proven in
[PS] by A.Suslin and the author. Note that our Theorem 3.2 is much stronger than [P1,
Theorem 8.1]. Also, details of the proof are given better in this preprint.

Our approach is this. Using the D.Popescu theorem we reduce the question to the case
when $R$ is essentially smooth over $\mathbb{Z}(p)$ and $D$ is defined over $R$. Then, using a geometric
presentation lemma due to Cesnavicius [C Proposition 4.1] and the method as in [PS]
we prove a purity result for the functor $K_1(-, D)$. Finally, a diagram chasing shows the
result mentioned above.

A rather good survey of the topic is given in [Pan1]. Point out the conjecture is solved
in the case, when $R$ contains a field. More precisely, it is solved when $R$ contains an

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infinite field, by R. Fedorov and the author in [FP]. It is solved by the author in the case, when \( R \) contains a finite field in [Pan3] (see also [P]).

The case of mixed characteristic is widely open. Here are several references.

- The case when the group scheme is \( \text{PGL}_n \) and the ring \( R \) is an arbitrary regular local ring is done by A. Grothendieck in 1968 in [Gr2].
- The case of an arbitrary reductive group scheme over a discrete valuation ring or over a henselian ring is solved by Y. Nisnevich in 1984 in [N].
- The case, where \( G \) is an arbitrary torus over a regular local ring, was settled by J.-L. Colliot-Thélène and J.-J. Sansuc in 1987 in [C-T/S].
- The case, where \( G \) is quasi-split reductive group scheme over an arbitrary two-dimensional local rings, is solved by Y. Nisnevich in 1989 in [?].
- The case when \( G \) is the unitary group scheme \( U_{A,\sigma} \) is solved by S. Gille and the author recently in [GP]; here \( R \) is an unramified regular local ring of characteristic \((0, p)\) with \( p \neq 2 \) and \((A, \sigma)\) is an Azumaya \( R \)-algebra with involution;
- In [PSU] the conjecture is solved for any semi-local Dedekind domain providing that \( G \) is simple simply-connected and \( G \) contains a torus \( \mathbb{G}_{m,R} \);
- The latter result is extended in [NG] to arbitrary reductive group schemes \( G \) over any semi-local Dedekind domain;
- There are as two very interesting recent publications [Fe1], [Fe2] by R. Fedorov;
- The case, when \( G \) is quasi-split reductive group scheme over an unramified regular local ring is solved recently by K. Cesnavicius in [G].

2 Agreements

Through the paper
\( A \) is a d.v.r., \( m_A \subseteq A \) is its maximal ideal;
\( \pi \in m_A \) is a generator of the maximal ideal;
\( k(v) \) is the residue field \( A/m_A \);
\( p > 0 \) is the characteristic of the field \( k(v) \);
it is supposed in this preprint that the fraction field of \( A \) has characteristic zero;
\( d \geq 1 \) is an integer;
\( X \) is an irreducible \( A \)-smooth affine \( A \)-scheme of relative dimension \( d \);
So, all open subschemes of \( X \) are regular and all its local rings are regular.
If \( x_1, x_2, \ldots, x_n \) are closed points in the scheme \( X \), then write
\( O \) for the semi-local ring \( O_{X,\{x_1,x_2,\ldots,x_n\}} \), \( K \) for the fraction field of \( O \),
\( U \) for \( \text{Spec}(O) \), \( \eta \) for \( \text{Spec}(K) \).
Recall that a regular local ring \( R \) of mixed characteristic \((0, p)\) is called unramified if the ring \( R/pR \) is regular;
recall from [SP, 0382] that a Noetherian algebra over a field \( k \) is geometrically regular if its base change to every finite purely inseparable (equivalently, to every finitely generated) field extension of \( k \) is regular;
one says that a regular local \( A \)-algebra \( R \) is geometrically regular if the \( k(v) \)-algebra \( R/mR \) is geometrically regular.

### 3 Main results

**Theorem 3.1.** Let \( R \) be an unramified regular semi-local domain of mixed characteristic \((0,p)\), \( D \) an Azumaya \( R \)-algebra, \( K \) the fraction field of \( R \), \( Nrd : D^\times \to R^\times \) the reduced norm homomorphism. Let \( a \in R^\times \) be a unit. Suppose the equation \( Nrd = a \) has a solution over \( K \), then it has a solution over \( R \).

The following result is an extension of Theorem 3.1.

**Theorem 3.2.** Let \( R \) be a geometrically regular semi-local integral \( A \)-algebra. Let \( D \) be an Azumaya \( R \)-algebra, \( K \) the fraction field of \( R \), \( Nrd : D^\times \to R^\times \) the reduced norm homomorphism. Let \( a \in R^\times \) be a unit. Suppose the equation \( Nrd = a \) has a solution over \( K \), then it has a solution over \( R \).

**Corollary 3.3.** Let \( R \) be a geometrically regular semi-local integral \( A \)-algebra, \( K \) the fraction field of \( R \). Let \( a, b, c \) be units in \( R \). Consider the equation

\[
T_1^2 - aT_2^2 - bT_3^2 + abT_4^2 = c. \tag{1}
\]

If it has a solution over \( K \), then it has a solution over \( R \).

**Remark 3.4.** Let \( a, b, c \) be units in \( R \) as in the Corollary. Let \( D \) be the generalised quaternion \( R \)-algebra given by generators \( u, w \) and relations \( u^2 = a, w^2 = b, uw = -wu \). Then the reduced norm \( Nrd : D \to R \) takes a quaternion \( \alpha + \beta u + \gamma w + \delta uw \) to the element \( \alpha^2 - a\beta^2 - b\gamma^2 + ab\delta^2 \). This is why the Corollary is a consequence of Theorem 3.2.

### 4 Cesnavicius geometric presentation proposition

Let \( A \) be a d.v.r. and \( X \) be an \( A \)-scheme as in the section 2. The following result is due to Cesnavicius [C, Proposition 4.1].

**Theorem 4.1.** (Proposition 4.1, [C]) Let \( x_1, x_2, \ldots, x_n \) be closed points in the scheme \( X \). Let \( Z \) be a closed subset in \( X \) of codimension at least 2 in \( X \). Then there are an affine neighborhood \( X^\circ \) of points \( x_1, x_2, \ldots, x_n \), an open affine subscheme \( S \subseteq A^{d-1}_A \) and a smooth \( A \)-morphism

\[
q : X^\circ \to S
\]

of pure relative dimension \( d - 1 \) such that \( Z^\circ \) is \( S \)-finite, where \( Z^\circ = Z \cap X^\circ \).
5 Proof of the geometric case of Theorem 3.2

Let $A, p > 0, d \geq 1, X, x_1, x_2, \ldots, x_n \in X, \mathcal{O}$ and $U$ be as in Section 2. Write $\mathcal{K}$ for the fraction field of the ring $\mathcal{O}$.

Theorem 5.1. Let $D$ be an Azumaya $\mathcal{O}$-algebra and $\text{Nrd}_D : D^\times \to \mathcal{O}^\times$ be the reduced norm homomorphism. Let $a \in \mathcal{O}^\times$. If $a$ is a reduced norm for the central simple $\mathcal{K}$-algebra $D \otimes_\mathcal{O} \mathcal{K}$, then $a$ is a reduced norm for the algebra $D$.

Proposition 5.2. Let $D$ be an Azumaya $\mathcal{O}$-algebra. Let $K_s$ be the Quillen $K$-functor. Then for each integer $n \geq 0$ the sequence is exact

$$K_n(D) \to K_n(D \otimes_\mathcal{O} \mathcal{K}) \xrightarrow{\eta^*} \bigoplus_{y \in U^{(1)}} K_{n-1}(D \otimes_\mathcal{O} k(y))$$

(2)

Particularly, it is exact for $n = 1$.

Reducing Theorem 5.1 to Proposition 5.2. Consider the commutative diagram of groups

$$
\begin{array}{ccc}
K_1(D) & \xrightarrow{\eta^*} & K_1(D \otimes_\mathcal{O} \mathcal{K}) \\
\text{Nrd} & & \text{Nrd} \\
\mathcal{O}^\times & \xrightarrow{\eta^*} & \mathcal{K}^\times \\
\end{array}
\begin{array}{c}
\bigoplus_{y \in U^{(1)}} K_0(D \otimes_\mathcal{O} k(y)) \\
\bigoplus_{y \in U^{(1)}} K_0(k(y))
\end{array}
$$

(3)

By Proposition 5.2 the complex on the top is exact. The bottom map $\eta^*$ is injective. The right hand side vertical map $\text{Nrd}$ is injective. Thus, the map

$$
\mathcal{O}^\times/\text{Nrd}(K_1(D)) \to \mathcal{K}^\times/\text{Nrd}(K_1(D \otimes_\mathcal{O} \mathcal{K}))
$$

is injective. The image of the left vertical map coincides with $\text{Nrd}(D^\times)$ and the image of the middle vertical map coincides with $\text{Nrd}((D \otimes_\mathcal{O} \mathcal{K})^\times)$. Thus, the map

$$
\mathcal{O}^\times/\text{Nrd}(D^\times) \to \mathcal{K}^\times/\text{Nrd}((D \otimes_\mathcal{O} \mathcal{K})^\times)
$$

is injective. The derivation of Theorem 5.1 from Proposition 5.2 is completed.

Proof of Proposition 5.2. To prove this proposition it sufficient to prove vanishing of the support extension map $\text{ext}_{2,1} : K'_n(U; D)_{\geq 2} \to K'_n(U; D)_{\geq 1}$. Prove that $\text{ext}_{2,1} = 0$. Take an $a \in K'_n(U; D)_{\geq 2}$. We may assume that $a \in K'_n(Z; D)$ for a closed $Z$ in $U$ with $\text{codim}_U(Z) \geq 2$. Enlarging $Z$ we may assume that each its irreducible component contains at least one of the point $x_i$’s and still $\text{codim}_U(Z) \geq 2$. Our aim is to find a closed subset $Z_{\text{ext}}$ in $U$ containing $Z$ such that the element $a$ vanishes in $K'_n(Z_{\text{ext}}; D)$ and $\text{codim}_U(Z_{\text{ext}}) \geq 1$. We will follow the method of [PS].

Let $\bar{Z}$ be the closure of $Z$ in $X$. Shrinking $X$ and $\bar{Z}$ accordingly we may and will suppose that $D$ is an Azumaya algebra over $X$ and there is an element $\bar{a} \in K'_n(\bar{Z}; D)$ such that $\bar{a}|_Z = a$. By Theorem 4.1 there are an affine neighborhood $X^\circ$ of points $x_1, x_2, \ldots, x_n$, an...
open affine subscheme \( S \subseteq A^{d-1}_A \), a smooth \( A \)-morphism

\[ q : X^o \to S \]

of relative dimension one such that \( Z^o/S \) is finite, where \( Z^o = \overline{Z \cap X^o} \). Put \( a^o = \tilde{a}|_{Z^o} \).

Put \( s_i = q(x_i) \). Consider the semi-local ring \( \mathcal{O}_{S,s_1,\ldots,s_n} \), put \( B = \text{Spec} \ \mathcal{O}_{S,s_1,\ldots,s_n} \) and \( X_B = q^{-1}(B) \subset X^o \). Put \( Z_B = Z^o \cap X_B \). Write \( q_B \) for \( q|_{X_B} : X_B \to B \). Note that \( Z_B \) is finite over \( B \). Since \( B \) is semi-local, hence so is \( Z_B \).

Let \( W \subseteq X_B \) be an open containing \( Z_B \). Write \( Z_W \) for \( Z_B \times_B W \). Let \( \Pi : Z_W \to W \) be the projection to \( W \) and \( q_Z : Z_W \to Z_B \) be the projection to \( Z_B \). Since \( Z_B/B \) is finite the morphism \( \Pi \) is finite. Since \( \Pi \) is finite the subset of \( B \) finite over \( Z \) in \( \text{inj} \) homomorphisms contains \( B \). Clearly, \( Z_B \) contains all the points \( x_1, \ldots, x_n \). This shows that \( W \) contains \( U \) and \( Z_B \) contains \( Z \). One can check that \( Z = Z_B \cap U \). Write \( j : Z \to Z_B \) for the inclusion.

Put \( Z_{\text{ext}} = U \cap Z_{\text{new}} \). Since \( Z \) is in \( U \cap Z_{\text{new}} = Z_{\text{ext}} \), hence we have an inclusion \( \text{in} : Z \hookrightarrow Z_{\text{ext}} \) of closed subsets in \( U \). The inclusion \( U \hookrightarrow W \) is a flat morphism. Hence the inclusions \( \text{inj} : Z_{\text{ext}} \to Z_{\text{new}} \) and \( j : Z \to Z_B \) are also flat morphisms. Thus the homomorphisms \( \text{inj}^* : K'_n(Z_{\text{new}}; D) \to K'_n(Z_{\text{ext}}; D) \) and \( j^* : K'_n(Z_B; D) \to K'_n(Z; D) \) are well-defined. Moreover \( \text{inj}^* \circ i_* = i_* \circ j^* \).

As explained just above the generic point of \( W \) is not in \( Z_{\text{new}} \). Thus, the generic point of \( U \) is not in \( Z_{\text{ext}} \) and \( Z_{\text{ext}} \) has codimension at least one in \( W \). Recall the Azumaya algebra \( D \) is an Azumaya algebra over \( X \). We still will write \( D \) for \( D|_W \). Also we are given with an element \( a_B := a^o|_{Z_B} \in K'_n(Z_B; D) \) such that \( j^*(a_B) = a \in K'_n(Z; D) \).

Claim. There is an open \( W \) in \( X_B \) containing \( Z_B \) such that for the closed inclusion \( i : Z_B \hookrightarrow Z_{\text{new}} \) the map \( i_* : K'_n(Z_B; D) \to K'_n(Z_{\text{new}}; D) \) vanishes.

Given this Claim complete the proof of the proposition as follows: \( \text{in}_*(a) = \text{in}_*(j^*(a_B)) = \text{inj}^*(i_*(a_B)) = \text{inj}^*(0) = 0 \) in \( K'_n(Z_{\text{ext}}; D) \) and \( Z_{\text{ext}} \subseteq U \).

In the rest of the proof we prove the Claim. So, let \( W \subseteq X_B \) be as in the Claim. Let \( Z_W = Z_B \times_B W \) and the projections \( \Pi : Z_W \to W \), \( q_Z : Z_W \to Z_B \) be as above in this proof. The closed embedding \( \text{in} : Z_B \hookrightarrow W \) defines a section \( \Delta = (\text{id} \times \text{in}) : Z_B \hookrightarrow Z_W \) of the projection \( q_Z \). Also one has an equality \( \Delta = \Pi \circ \Delta \). Put \( D_Z = D|_{Z_B} \) and write \( z D_W \) for the Azumaya algebra \( \Pi^*(D) \otimes q^*_Z(D^p_{Z}) \) over \( Z_W \). The \( O_{Z_W} \)-module \( \Delta_* (D_Z) \) has an obvious left \( z D_W \)-module structure. And it is equipped with an obvious epimorphism \( \pi : z D_X \to \Delta_* (D_Z) \) of the left \( z D_W \)-modules. Following [PS] one can see that \( I := \text{Ker}(\pi) \) is a left projective \( z D_W \)-module. Hence the left \( z D_W \)-module \( \Delta_* (D_Z) \) defines an element \( [\Delta_* (D_Z)] = [z D_W] - [I] \) in \( K_0(Z_W; z D_W) \). This element has rank zero. Hence by [DeM] it vanishes semi-locally on \( Z_W \). Thus, there is a neighborhood \( W \) of \( Z_B \times_B Z_B \) in \( Z_W \) such that \( \Delta_* (D_Z) \) vanishes in \( K_0(W; z D_W) \). It is easy to see that \( W \) contains a neighborhood of \( Z_B \times_B Z_B \) of the form \( Z_{W'} \), where \( W' \subset X_B \) is an open containing \( Z_B \).
Thus, replacing notation we may suppose that $W \subset X_B$ is as in the Claim and the element $[\Delta_*(D_Z)] = [\mathbb{Z}_D] - [I]$ vanishes in $K_0(\mathbb{Z}_W; \mathbb{Z}_D)$. It remains to check that for this specific $W$ the map $i_* : K'_n(Z_B; D) = K'_n(Z_B; D_Z) \to K'_n(Z_{new}; D)$ vanishes.

The functor $(P, M) \mapsto P \otimes q^*_{\mathbb{Z}_D}(D_Z) M$ induces a bilinear pairing

$$\cup_{q^*_{\mathbb{Z}_D}(D_Z)} : K_0(\mathbb{Z}_W; \mathbb{Z}_D) \times K'_n(\mathbb{Z}_W; q^*_{\mathbb{Z}_D}(D_Z)) \to K'_n(\mathbb{Z}_W; \Pi^*(D))$$

Each element $\alpha \in K_0(\mathbb{Z}_W; \mathbb{Z}_D)$ defines a group homomorphism

$$\alpha_* = \Pi_* \circ (\alpha \cup_{q^*_{\mathbb{Z}_D}(D_Z)} \iota^* q^*_{\mathbb{Z}_D}) : K'_n(Z_B; D) = K'_n(\mathbb{Z}_B; \mathbb{Z}_D) \to K'_n(Z_{new}; D)$$

which takes an element $b \in K'_n(\mathbb{Z}_B; \mathbb{Z}_D)$ to the one $\Pi_*(\alpha \cup_{q^*_{\mathbb{Z}_D}(D_Z)} q^*_{\mathbb{Z}_D}(b))$ in $K'_n(Z_{new}; D)$.

Following [PS] we see that the map $[\Delta_*(D_Z)]_*$ coincides with the map $i_* : K'_n(Z_B; D) = K'_n(\mathbb{Z}_B; \mathbb{Z}_D) \to K'_n(Z_{new}; D)$. The equality $0 = [\Delta_*(D_Z)] \in K_0(\mathbb{Z}_W; \mathbb{Z}_D)$ proven just above shows that the map $i_*$ vanishes. The Claim is proved. The proposition is proved. □

Let $A$, $p > 0$, $d \geq 1$, $X$ be as in Section 2. Let $y_1, y_2, \ldots, y_n \in \mathbb{X}$ be points not necessary closed. Write $\mathcal{O}_y$ for the semi-local ring $\mathcal{O}_{X, \{y_1, y_2, \ldots, y_n\}}$ and $U_y$ for $\text{Spec} \mathcal{O}_y$. Write $K$ for the fraction field of the ring $\mathcal{O}_y$. The following result can be derived from Theorem 5.1 in a standard way.

**Theorem 5.3.** Let $D$ be an Azumaya $\mathcal{O}_y$-algebra and $\text{Nrd}_D : D^\times \to \mathcal{O}_y^\times$ be the reduced norm homomorphism. Let $a \in \mathcal{O}_y^\times$. If $a$ is a reduced norm for the central simple $\mathcal{K}$-algebra $D \otimes_{\mathcal{O}_y} K$, then $a$ is a reduced norm for the algebra $D$.

**Proof of Theorem 5.3** One can choose closed points $x_1, x_2, \ldots, x_n$ in $X$ such that $x_i$ is in the closure of $y_i$, $D = D \otimes_{\mathcal{O}_y} \mathcal{O}_y$ for an Azumaya $\mathcal{O}$-algebra $\mathcal{D}$, where $\mathcal{O} := \mathcal{O}_{X, \{x_1, x_2, \ldots, x_n\}}$, $a \in \mathcal{O}^\times$. In particular, $\mathcal{O} \subseteq \mathcal{O}_y$. Now by Theorem 5.1 the element $a$ is a reduced norm from $\mathcal{D}$. Thus, $a$ is a reduced norm from $\mathcal{D}$. □

**Proof of Theorem 5.3** By Popescu theorem [Po], [Sw], the ring $R$ is a filtered direct limit of smooth $A$-algebras. Thus, a limit argument allows us to assume that $R$ is the semilocalization of a smooth $A$-algebra at finitely many primes. Theorem 5.3 completes the proof. □

**References**

[C] Cesnavicius K. Grothendieck–Serre in the quasi-split unramified case, preprint (2021). Available at https://arxiv.org/abs/2009.05299v2.

[C-T/S] Colliot-Thélène J.-L., Sansuc J.-J. Principal Homogeneous Spaces under Flasque Tori: Applications. Joural of Algebra, (1987), 106, 148–205.
[C-T/O] Colliot-Thélène J.-L., Ojanguren M. Espaces Principaux Homogènes Localement Triviaux, Publ. Math. IHÉS 75 (1992), no. 2, 97–122.

[DeM] DeMeyer F.R. Projective modules over central separable algebras, Canad. J. Math. 21 (1969), 39–43.

[FP] Fedorov R., Panin I. A proof of Grothendieck–Serre conjecture on principal bundles over a semilocal regular ring containing an infinite field, Publ. Math. Inst. Hautes Etudes Sci., Vol. 122, 2015, pp. 169–193.

[Fe1] Fedorov R. On the Grothendieck–Serre conjecture on principal bundles in mixed characteristic, Trans. Amer. Math. Soc., to appear (2021), arXiv:1501.04224v3

[Fe2] Fedorov R. On the Grothendieck–Serre Conjecture about principal bundles and its generalizations, Algebra Number Theory, to appear (2021), arXiv:1810.11844v2

[GiP] Gille S., Panin I. On the Gersten conjecture for hermitian Witt groups, arXiv:2201.10715v1

[GL] Gillet H., Levine M. The relative form of Gersten’s conjecture over a discrete valuation ring: the smooth case, J. Pure Appl. Algebra, 1987, (1), 46, 59–71.

[Gr1] Grothendieck, A. Torsion homologique et section rationnelles, in Anneaux de Chow et applications, Séminaire Chevalley, 2-e année, Secrétariat mathématique, Paris, 1958.

[Gr2] Grothendieck, A. Le group de Brauer II, in Dix exposés sur la cohomologique de schémas, Amsterdam, North-Holland, 1968.

[N] Nisnevich Ye. Espaces homogènes principaux rationnellement triviaux et arithmétique des schémas en groupes réductifs sur les anneaux de Dedekind”. C. R. Acad. Sci. Paris Sér. I Math. 299.1 (1984), pp. 5–8.

[NG] Guo, N. The Grothendieck–Serre conjecture over semi-local Dedekind rings Transformation rings, 2020, to appear. arXiv:1902.02315v2

[Oj1] Ojanguren M. A splitting theorem for quadratic forms. Comment. Math. Helv. 57.1 (1982), pp. 145–157.

[Oj2] Ojanguren M. Unités représentées par des formes quadratiques ou par des normes réduites. Algebraic K-theory, Part II (Oberwolfach, 1980). Vol. 967. Lecture Notes in Math. Springer, Berlin-New York, 1982, 291–299.

[OP2] Ojanguren M., Panin I. Rationally trivial hermitian spaces are locally trivial, Math. Z. 237 (2001), 181–198.
Panin, I.; Stavrova, A.; Vavilov, N. On Grothendieck—Serre’s conjecture concerning principal $G$-bundles over reductive group schemes: I, Compositio Math. 151 (2015), 535–567.

Panin, I. Proof of Grothendieck–Serre conjecture on principal bundles over regular local rings containing a finite field, preprint (2015). https://www.math.uni-bielefeld.de/lag/man/559.pdf.

Panin, I. Proof of the Grothendieck–Serre conjecture on principal bundles over regular local rings containing a field, Izvestiya: Mathematics, 2020, 84:4, 780–795.

Panin, I. On Grothendieck–Serre conjecture concerning principal bundles, Proceedings of the International Congress of Mathematicians, vol. 2 (Rio de Janeiro 2018), World Sci. Publ., Hackensack, NJ 2018, pp. 201–221.

Panin, I. Moving lemmas in mixed characteristic and applications, arXiv: 2202.00896v1

I. A. Panin and A. K. Stavrova On the Grothendieck–Serre conjecture concerning principal $G$-bundles over semi-local Dedekind domains. Zap. Nauchn. Sem. S.-Peterburg. Otdel. Mat. Inst. Steklov. (POMI) Vol.443, 2016, 133–146. arXiv: 1512.00354.

Panin, I.; Suslin, A. On a conjecture of Grothendieck concerning Azumaya algebras, St.Petersburg Math. J., 9, no.4, 851–858 (1998)

Popescu, D. General Néron desingularization and approximation, Nagoya Math. J. 104 (1986), 85–115.

Quillen, D. Higher K-theory-I, Lect. Notes Math. 341 (1973), 85–147.

Serre, J.-P. Espaces fibrés algébriques, in Anneaux de Chow et applications, Séminaire Chevalley, 2-e année, Secrétariat mathématique, Paris, 1958.

Swan, R. G. Néron—Popescu desingularization, Algebra and Geometry (Taipei, 1995), Lect. Algebra Geom. 2, Internat. Press, Cambridge, MA, 1998, 135–192.