Transport properties in bilayer Quantum Hall systems in the presence of a topological defect

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Abstract. Following a suggestion given in [1], we show how a bilayer Quantum Hall system at fillings \( \nu = \frac{1}{p} + \frac{1}{q} \) can exhibit a point-like topological defect in its edge state structure. Indeed our CFT theory for such a system, the Twisted Model (TM), gives rise in a natural way to such a feature in the twisted sector. Our results are in agreement with recent experimental findings [2] which evidence the presence of a topological defect in the transport properties of the bilayer system.

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INTRODUCTION

Recently bilayer quantum Hall systems have been widely investigated theoretically as well as experimentally [3, 4]. Indeed, when tunneling between the layers is weak, the quantum Hall bilayer state can be viewed as arising from the condensation of an excitonic superfluid in which an electron in one layer is paired with a hole in the other layer. The uncertainty principle makes it impossible to tell which layer either component of this composite boson is in. Equivalently the system may be regarded as a ferromagnet in which all electrons appear in a coherent superposition of the "pseudospin" eigenstates which encode the layer degree of freedom [5][6]. The phase variable of such a superposition fixes the orientation of the pseudospin magnetic moment and its spatial variations govern the low energy excitations in the system. Since Halperin work [7] the concept of edge states was introduced in order to describe transport phenomena in two dimensional electron systems. They arise in a quantized magnetic field at the intersections of the Fermi level with different Landau levels, which are bent up by the edge potential. In particular the formation of a topological defect has been predicted to occur when two edge states with different spins locally switch their positions and thus cross each other at two or more points [8]. More interesting features take place in the transport properties of bilayer systems when also pseudospin (related to the layer index) is involved [6][9]. Recently the presence of edge state crossings and thus of topological defects has been experimentally evidenced in such systems in a quasi-Corbino geometry [10] at filling \( \nu = 3 \) [2] by means of a selective population technique. In particular the application of a suitable gate voltage \( V_g \) and of a magnetic field drives the bilayer in different pseudospin...
states in the gated and ungated regions, so producing a crossing of the edge states which has been detected in the transport properties. The net result is a linear $I - V$ characteristics for the electric transport between two different edges. Because the gate-gap width is smaller than the characteristic equilibration lengths in such a transport between the edge states, it has been argued that a defect must be present, which couples different edge states but only with the same spin in the gate-gap. Such a picture can be destroyed by an in-plane magnetic field component which washes out the above crossing; the $I - V$ curves become then strongly non-linear so signaling the merging of a tunneling process. All the above features in the $I - V$ characteristics appear to be the fingerprints of the presence of a topological defect induced by the different pseudospin configurations in bilayer quantum Hall systems [2].

In this contribution we address theoretically the issue of the presence of topological defects in the Conformal Field Theory (CFT) description of the edge states of bilayer quantum Hall systems in a wide class of filling factors, and in particular the paired states ones, in the framework of our TM approach [1]. In particular we show how such a feature arises in a very natural way in the twisted sector of our theory, as a result of the $m$-reduction technique [11][12]. The transport properties of bilayer systems will be investigated by studying the properties under magnetic translations of the characters of the different sectors, which describe its different non perturbative ground states. The paper is organized as follows. In Section 2 we recall those aspects of our $m$-reduction procedure which turn out to be relevant for the description of bilayer systems with topological defects. In Section 3 we study the transport properties of such systems by means of magnetic translations pointing out how they arise from Laughlin gauge argument. Finally some conclusions and outlooks are given.

**$M$-REDUCTION TECHNIQUE: A DESCRIPTION OF BILAYER SYSTEMS WITH TOPOLOGICAL DEFECTS**

The $m$-reduction technique is based on the simple observation that for any CFT (mother) exists a class of sub-theories parameterized by an integer $m$ with the same symmetry but different representations. The resulting theory (daughter) has the same algebraic structure but a different central charge $c_m = mc$. To obtain the generators of the algebra in the new theory we need to extract the modes which are multiple of the integer $m$. These can be used to reconstruct the primary fields of the daughter CFT. This technique can be generalized and applied to any extended chiral algebra which includes the Virasoro one. Indeed the $m$-reduction preserves the commutation relations between the algebra generators but modifies the central extension (i.e. the level for the WZW models). In particular this implies that the number of primary fields gets modified. Its application to the QHE arises by the incompressibility of the Hall fluid droplet at the plateau, which implies its invariance under the $W_{1,\infty}$ algebra at the different fillings, and by the property of the $m$-reduction procedure to obtain a daughter CFT with the same $W_{1,\infty}$ invariance property of the mother theory. Thus the $m$-reduction furnishes automatically a mapping between different incompressible plateaux of the QHF.

The general characteristics of the daughter theory is the presence of twisted boundary
conditions (TBC) which are induced on the component fields. It is illuminating to give a geometric interpretation of that in terms of the covering on a $m$-sheeted surface or complex curve with branch-cuts, see Fig. 1.

Indeed the fields which are defined on the left boundary have TBC while the fields defined on the right one have periodic boundary conditions (PBC). We point out that fields with TBC describe elegantly the crossing between the layers as a consequence of the presence of a branch-cut. We find different sectors on the torus corresponding to different boundary conditions on the cylinder. Finally we recognize the daughter theory as an orbifold of the usual CFT describing the QHF at a given plateau. The two sheets simulate the two-layers system and the branch cut represents TBC which emerge from the interaction with a localized defect on the edge. This is a key feature of our construction, as we will point out in the following.

In order to see how the $m$-reduction procedure works on the plane [11] and on the torus [12] and how it gives rise to the edge state coupling via a topological defect, let us focus on the paired states fillings in the special $m = 2$ case since we are interested in a system consisting of two parallel layers of 2D electrons gas in a strong perpendicular magnetic field. The filling factor $v^{(a)} = \frac{1}{2p+2}$ is the same for the two $a = 1, 2$ layers while the total filling is $v = v^{(1)} + v^{(2)} = \frac{1}{p+1}$. We point out that our results can be generalized to any bilayer system. The simplest abelian quantum Hall state in the disc topology is written as a generalization of the analytic part of the Laughlin wave function [7]:

$$f \left( z_i^{(a)} \right) = \prod_{a=1,2} \prod_{i<j} \left( z_i^{(a)} - z_j^{(a)} \right)^2 \prod_{i,j} \left( z_i^{(1)} - z_j^{(2)} \right)^p;$$

in particular, for $p = 0$ it describes the bosonic 220 state and, for $p = 1$, the fermionic 331 one. The CFT description for such a system can be given in terms of two compactified chiral bosons $Q^{(a)}$ with central charge $c = 2$. A similar result can be obtained for filling

**FIGURE 1.** The boundaries of the 2-covered cylinder can be viewed as different configurations of the QHF edges described by the 2-reduced CFT.
\( \nu^{(a)} = 1/(2p+1) \) (Jain series).

In order to construct the fields \( Q^{(a)} \) for the TM, let us start from the bosonic “Laughlin” filling \( \nu = 1/2(p+1) \), described by a CFT with \( c = 1 \) in terms of a scalar chiral field \( Q \) compactified on a circle with radius \( R^2 = 1/\nu = 2(p+1) \) (or its dual \( R^2 = 2/(p+1) \)). It is explicitly given by:

\[
Q(z) = q - ip\ln z + i \sum_{n \neq 0} \frac{a_n}{n} z^{-n} \tag{2}
\]

with \( a_n, q \) and \( p \) satisfying the commutation relations \([a_n, a_{n'}] = n\delta_{n,n'}\) and \([q, p] = i\). From such a CFT (mother theory), using the \( m \)-reduction procedure, which consists in considering the subalgebra generated only by the modes in eq. (2) which are a multiple of the integer \( m \), we get a \( c = 2 \) orbifold CFT (daughter theory, i.e. the TM) which describes the LLL dynamics. Then the fields in the mother CFT can be organized into components which have well defined transformation properties under the discrete \( \mathbb{Z}_2 \) (twist) group, which is a symmetry of the TM. Its primary fields content can be expressed in terms of a \( \mathbb{Z}_2 \)-invariant scalar field \( X(z) \), given by

\[
X(z) = \frac{1}{2} \left( Q^{(1)}(z) + Q^{(2)}(-z) \right) \tag{3}
\]

describing the electrically charged sector of the new filling, and a twisted field

\[
\phi(z) = \frac{1}{2} \left( Q^{(1)}(z) - Q^{(2)}(-z) \right) \tag{4}
\]

which satisfies the twisted boundary conditions \( \phi(e^{i\pi} z) = -\phi(z) \) and describes the neutral sector [11]. Such TBC signal the presence of a topological defect which couples, in general, the \( m \) edges in a \( m \)-layers system. In the bilayer system \((m = 2)\) we get a crossing between the two edges as sketched in Fig. 2.

The chiral fields \( Q^{(a)} \), defined on a single layer \( a = 1, 2 \), due to the boundary conditions imposed upon them by the orbifold construction, can be thought of as components of a unique “boson” defined on a double covering of the disc (layer) \((z_1^{(1)} = -z_2^{(2)} = z_t)\).
As a consequence of such a construction the two layers system becomes equivalent to one-layer QHF and the $X$ and $\phi$ fields defined in eqs. (3) and (4) diagonalize the inter-layer interaction. In particular the $X$ field carries the total charge with velocity $v_X$, while $\phi$ carries the charge difference of the two edges with velocity $v_\phi$ i.e. no charge, being the number of electrons the same for each layer (balanced system).

The TM primary fields are composite operators and, on the torus, they are described in terms of the conformal blocks (or characters). Furthermore a topological defect appears in our formalism, being induced by the different isospin configurations on the two layers, which naturally result from our $m$-reduction procedure. The effect of a topological defect in a quantum Hall fluid has been recently evidenced in experimental findings [2], as we will show in the following. In the presence of a localized defect two phenomena can take place. A tunneling of edge quasi-particles at point $x_0$, described by a boundary term Hamiltonian such as:

$$H_P = -t_P \cos \left( Q^{(1)} - Q^{(2)} \right) \delta (x_0).$$

(5)

A second mechanism producing a current flow between the two edges can be addressed to a localized crossing of the edges, which can be represented by a boundary term:

$$H_\beta = \beta \left( Q^{(1)} \partial_t Q^{(2)} - Q^{(2)} \partial_t Q^{(1)} \right) \delta (x_0),$$

(6)

where $\beta = 0 (1/2)$ for PBC (TBC) respectively (see Fig.2). The full Hamiltonian can be written as:

$$H = \frac{1}{2} \sum_{a=1,2} \left[ \left( \Pi^{(a)} \right)^2 + \left( \partial_t Q^{(a)} \right)^2 \right] + H_P + H_\beta + eV \partial_t \left( Q^{(1)} - Q^{(2)} \right),$$

(7)

where $\Pi^{(a)}$ is the momentum conjugate to $Q^{(a)}$. We recognize a kinetic term for the two bosonic fields $Q^{(a)}, a = 1, 2$, a boundary tunneling term which implements the locally applied gate voltage $V_g = t_P \delta (x_0)$, a boundary magnetic term [13] which couples the two fields introducing a topological defect (see ref. [1] for details) and a voltage switching term between the two layers. The last term contains an irrelevant operator, so it doesn’t change the central charge: it behaves as a boundary condition changing operator allowing for the flow from a boundary state to another one. Introducing the charged and neutral fields $X$ and $\phi$ defined in eqs. (3) and (4) we clearly see that the last term in the Hamiltonian is proportional to the neutral current, so it contributes to unbalance the system. Therefore edge-crossing can be described by TBC on the $\phi$ field induced by the boundary magnetic term of eq. (6).

**STUDY OF TRANSPORT PROPERTIES: MAGNETIC TRANSLATIONS AND LAUGHLIN GAUGE ARGUMENT**

The transport properties of the bilayer system under study can be investigated by the application of different chemical potentials between the terminals of Fig. 2, that we
represent by the matrix \( V = \begin{pmatrix} V_{AC} & V_{AD} \\ V_{BC} & V_{BD} \end{pmatrix} \) with entries \( V_{IJ} \), the potentials between the \( I \) and \( J \) terminals. Let us consider the following two cases, the one in which the transport of electrons is on the two independent edges through the points \( A - C - A \), \( B - D - B \) in the non crossed case (PBC see Fig. 2a) and the one in which the transport is through the points \( A - D - B - C - A \) in the crossed edge case (TBC see Fig. 2b). In both cases there is no tunneling (\( t_p = 0 \)) and they correspond respectively to the diagonal (i.e. \( V_{AD} = V_{BC} = 0 \)) and to the anti-diagonal (i.e. \( V_{AC} = V_{BD} = 0 \)) configurations respectively.

In a closed geometry, such as that of a torus, they can be induced by adiabatic magnetic flux insertion through a cycle of the torus (i.e. \( A \) or \( B \) cycle). For example, by inserting a flux quantum \( \frac{hc}{2e} \) through the cycle \( A \), an electromotive force is induced along it with a consequent transport of an electron along the \( B \) cycle.

The foundations of such an issue can be found in the celebrated Laughlin gauge argument [14] which runs as follows. Let us focus on the geometry proposed by Laughlin, that is a ribbon of two-dimensional system bent into a loop of circumference \( L \) and embedded everywhere by a strong magnetic field \( \vec{B} \) normal to its surface (see Fig. 3). Let us also put a small solenoid at the center of the loop, as shown in the figure, and assume that an energy gap separates the ground state from the excited states. In order to force the system to produce Hall current let us also assume that electrons can be fed in at one edge and taken away from the other. Now we switch on the solenoid and adiabatically increase the magnetic flux from zero to \( \Phi_0 = \frac{hc}{e} \). Because of the energy gap, the system remains in a ground state which may be different from the original one. If the ground state is non degenerate, by gauge invariance the system simply returns to the initial state. Because of the phase coherence of the wave function of the system around the loop, the net result of such a process will be the transfer of \( N_0 \) electrons from one edge to the other. The energy increase due to this transfer is [14]

\[
\Delta U = N_0 eV_H
\]

where \( V_H \) is the potential drop from one edge to another. The Hall current is

\[
I_H = \frac{\partial U}{\partial \Phi} = \frac{\Delta U}{\Phi_0} = \frac{V_H N_0 e^2}{h}
\]

and the Hall conductance is

\[
\sigma_H = \frac{I_H}{V_H} = \frac{N_0 e^2}{h}.
\]

In this way quantization of the Hall conductance has been reproduced for integer fillings and the argument has been generalized also to fractional conductance [14]. So in the following we keep in mind this line of reasoning and then produce a potential drop between the four terminals of our bilayer system by adiabatic insertion of a magnetic flux quantum which results in the transport of electrons on each edge and between edges. This allows us to study transport properties.

We focus in particular on the torus topology, where the transport properties can be precisely described in terms of the action of magnetic translations on the conformal blocks of the untwisted and twisted sector respectively. Their explicit description can
be realized by standard calculations on the characters of the TM given in refs [12]. In this letter we just recall that the characters are given in terms of opportune Jacobi theta functions with characteristics 
\[ \theta \left[ \begin{array}{c} \lambda \\ 0 \end{array} \right] (q w(i)|2q\tau), \]
where \( \tau \) is the modular parameter of the torus. \( w^{(i)} = x^{(i)} + y^{(i)} \tau \) is the torus coordinate of the electron and \( q = p + 1 \). Magnetic translations on the \( i \)-layer along the two cycles \( A \) and \( B \) are described by exponential of differential operators acting on the \( w \) dependence of the characters. In the bilayer system the states belong to the \( 1/2 \) representation of the \( su(2) \) pseudospin group. The TM on the torus keeps track of these pseudospin configurations by the \( w \) dependence of the characters, whose charged and neutral components are described in terms of the layers variables \( w^{(1)}, w^{(2)} \) as \( w_c = (w^{(1)} + w^{(2)})/2 \) and \( w_n = (w^{(1)} - w^{(2)})/2 \) respectively. So the two configurations, given above, without tunneling are described on the torus by the following translations on the charged and neutral \( w \) coordinate. In the non crossed case (Fig. 2a) the potential \( V_{AC} \) (\( V_{BD} \)) generates a translation along the first (second) layer, on the variable \( w^{(1)} \) (\( w^{(2)} \)), and it results \( \Delta w_c \propto V_{AC} + V_{BD} \) and \( \Delta w_n \propto V_{AC} - V_{BD} \), while in the crossed case (Fig. 2b) \( \Delta w_c \propto V_{AD} + V_{BC} \) and \( \Delta w_n \propto V_{AD} - V_{BC} \). At this point the study of the transport properties follow by standard analysis [15]. Let us point out that a purely neutral translation \( w_p \) with \( w^{(1)} = -w^{(2)} \) creates the topological defect (and relates the edges switching to the large unbalance phenomenon predicted in [8]). In fact the twisted sector can be realized by a suitable neutral translation starting from the untwisted one and its explicit expression and derivation will be addressed in a forthcoming publication [15]. Finally in the presence of localized tunneling (\( t_p \neq 0 \)) between the layers hybridization takes place. In fact that experimentally corresponds to an equilibration process between the two edge states and results into a breaking of the symmetry of the balanced system described by the TM, due to the breaking of pseudospin symmetry. To take that into account the boundary CFT technology was used in [1], obtaining the characters of the system in the presence of both tunneling and topological defects.
Let us now discuss the transport properties in these unbalanced cases, by describing the tunneling as a small perturbation to the TM, and focus our attention to the terminal $AD$ in the crossed case. The working points are different for the untwisted and the twisted configurations. In the first case the term in eq. (5) for $t_p \ll 1$ is a weak perturbation of the background characterized by $V_{AD} = V_{BC} = 0$ while in the second one it has $V_{AC} = V_{BD} = 0$. The $I - V$ characteristics depend strongly on that. We obtain a different conductance for the two cases. In particular for TBC in the absence of an in plane magnetic field the driving voltage $V_{AD}$ puts the bilayer edges at different chemical potentials and then the ratio of the $AD$ terminal current to $V_{AD}$ is equal to the Hall conductance $\sigma_H = \frac{e^2}{2h}$, of the single layer. Conversely, when the two layers are coupled via the in plane magnetic field, the tunneling of the charge carriers results into a loss in the $AD$ terminal current. The net result is a negative contribution to the current which adds to the previous term, producing a total $AD$ terminal current, which for $p = 0$, can be exactly evaluated in a similar way as in [16], obtaining:

$$ I_{AD}(V_{AD}) = \frac{e^2 V_{AD}}{2h} - \frac{e T_B}{h} \arctan \frac{e V_{AD}}{2 T_B}, $$

(11)

where $T_B = C_1 t_p^{1/(1-\nu)}$ is the analogue of the Kondo temperature, depending on the external parallel magnetic field, $C_1$ is a non-universal constant and $\nu$ is the filling. In this case $\nu = \frac{1}{2}$, for the single layer, but the argument can be generalized to a wide class of fillings.

The non linear behavior of the tunneling characteristics follows by standard analysis (ref. [16]). Indeed for $T_B = 0$ the characteristics has a linear behavior as for the transport in a single layer. Moreover in plane magnetic field removes the twist (topological defect) and re-establishes the non-linear structure characterizing the tunneling phenomenon. Let us notice also that our system is spinless (or fully polarized) while the experimental results in [2] are obtained for spin resolved systems. Therefore we reproduce only the negative branch of the curves given in [2]. No gap is obtained for positive $V_{AD}$.
CONCLUSIONS

In conclusion we point out that the evidence of topological defects, resulting from TBC, is theoretically indispensable for the consistency of our CFT approach to the QHE. It is implied by the $m$-reduction technique.

The presence of topological defects in a double layer induces flux fractionalization described by the special $w_p$ translation and is responsible for linear conduction between different edges with a quantized value of the slope. In [1] the stability of the different ground-states was studied by means of the boundary entropy $g$. It was also related to dissipative quantum mechanics. This is an interesting interpretation of our theoretical results in connection also with the phase-transition between fully polarized/unpolarized pseudospin vacua, in analogy to the observed spin phase-transitions. Our theory for $p \neq 0$ predicts a breaking of the composite fermion picture with a different behavior for the fluxes (vortices), which are not sensible to the topological defect.

We point out that the results of this letter are very general and are relevant for different areas of condensed matter systems at low dimensions. It has been shown that there is a close relation between the existence of topological defects and flux fractionalization in fully frustrated Josephson junction ladders [17]. Furthermore topological defects have been also introduced in the description of dissipation in systems with impurities [1].

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