I. INTRODUCTION

The existence of black holes is a fundamental prediction of general relativity. Isolated individual black holes are stationary solutions to Einstein’s equations, but binary black holes (BBHs) can inspiral and eventually merge. BBH mergers offer a unique opportunity to test general relativity in the strong-field limit, and as such are a primary science target for current and future gravitational-wave (GW) observatories like LIGO, VIRGO, LISA, and the Einstein telescope. BBH mergers are also important for cosmology, as they can serve as standard candles to help determine the geometry and hence energy content of the universe [1, 2]. Astrophysical BBHs are found on at least two very different mass scales. Compact objects believed to be stellar-mass black holes have been observed in binary systems with more luminous companions. These black holes are the remnants of massive main-sequence stars, and binary systems with two such stars may ultimately evolve into BBHs. On larger scales, supermassive black holes (SBHs) with masses \( M > 10^9 \) reside in the centers of most galaxies. They can be observed through their dynamical influence on surrounding gas and stars, and when accreting as active galactic nuclei (AGN). SBHs will form binaries as well, following the merger of two galaxies which each host an SBH at their center.

In order to merge, BBHs must find a way to shed their orbital angular momentum. At large separations, binary SBHs will be escorted inwards by dynamical friction between their host galaxies [3]. The BBHs become gravitationally bound when the sum of their masses \( M = m_1 + m_2 \) exceeds the mass of gas and stars enclosed by their orbit. The binary hardens further by scattering stars on “loss-cone” orbits that pass within a critical radius \( r_\text{crit} \), though this scattering may stall at separations \( r < 0.01 - 1 \) pc unless these orbits are refilled by stellar diffusion [3]. Unlike stars, gas can cool to form a circumbinary disk about the BBHs. A circumbinary disk of mass \( M_d \) and radius \( r_d \) will exert a tidal torque

\[
T_d \sim \frac{q^2 M_d M}{r^3} \left( \frac{r}{r_d - r} \right)^3
\]

on the binary in the limit that the BBH mass ratio \( q \equiv m_2/m_1 \leq 1 \) is small and \( |r_d - r| \ll r \). Throughout this paper we use relativists’ units in which Newton’s constant \( G \) and the speed of light \( c \) are unity. At a sufficiently small separation \( r_{GW} \), the magnitude of this tidal torque will fall below that of the radiation-reaction torque [4]

\[
T_{GW} = \frac{32\eta^2 M_d^{9/2}}{5r_d^{7/2}}
\]

where \( \eta \equiv m_1 m_2/M^2 \) is the symmetric mass ratio. Once \( T_{GW} > T_d \), the inspiral of the BBH is dominated by radiation reaction. The precise value of \( r_{GW} \) depends on...
the properties of the circumbinary disk, but an order-of-
magnitude estimate is given by \[3\]

\[
r_{GW} = (5 \times 10^{16} \text{ cm}) q^{1/4} M_8^{3/4} \left[ \frac{\min(t_h, t_{\text{gas}})}{10^8 \text{ yr}} \right]^{1/4}
\]

\[= (3000 M) \left( \frac{q}{M_8} \right)^{1/4} \left[ \frac{\min(t_h, t_{\text{gas}})}{10^8 \text{ yr}} \right]^{1/4}
\]

where \(M_8\) is the mass of the larger black hole in units of \(10^8 M_\odot\), \(t_h\) is the dynamical friction timescale for a hard binary, and \(t_{\text{gas}}\) is the evolution timescale from gaseous tidal torques.

General relativity completely determines the inspiral of BBH systems from separations less than \(r_{GW}\). These systems are fully specified by 7 parameters: the mass ratio \(q\) and the 3 components of each dimensionless spin \(\chi_{1,2} \equiv S_{1,2}/m_{1,2}^2\). To a good approximation the individual masses and spin magnitudes \(\chi_{1,2} \equiv |\chi_{1,2}|\) remain constant during the inspiral, so only the precession of the two spin directions needs to be calculated. At an initial separation \(r_i = 1000 M \sim r_{GW}\), the binary’s orbital speed \(v/c \ll 1\) and the spin-precession equations can therefore be expanded in this small post-Newtonian (PN) parameter. The PN expansion remains valid until the BBHs reach a final separation \(r_f = 10 M\), after which their evolution can only be described by fully nonlinear numerical relativity (for more precise assessments of the validity of the PN expansion for spinning precessing binaries, see e.g. \[10, 11\]).

Numerical relativists can simulate BBH mergers from separations \(r_{NR} \simeq r_f\) \[12, 13\], but these simulations are too computationally expensive to begin when the binaries are much more widely separated. The GWs produced in the merger and the mass, spin, and recoil velocity of the final black hole depend sensitively on the orientation of the BBH spins at \(r_{NR}\), so it is important to determine what BBH spin orientations are expected at \(r_i\) and whether these orientations are modified by the PN evolution between \(r_i\) and \(r_f\).

The answer to the first of these questions comes from astrophysics, not general relativity. At very large separations, the two black holes are unaffected by each other and one would therefore expect an isotropic distribution of spin directions. However, an isotropic distribution of spins at \(r_f\) would imply that most mergers would result in a gravitational recoil of \(\sim 1000 \text{ km/s}\) for the final black hole \[15, 17\]. Recoll this large would eject SBHs from all but the most massive host galaxies \[18\], in seeming contradiction to the observed tight correlations between SBHs and their hosts \[19, 21\]. This problem can be avoided if Lense-Thirring precession and viscous torques align the spins of the BBHs with the accretion disk responsible for their inwards migration \[22, 24\]. The efficiency of this alignment depends on the properties of the accretion disk, but N-body simulations using smoothed-particle hydrodynamics (SPH) suggest that the residual misalignment of the BBH spins with their accretion disk at \(r_i\) could typically be \(\sim 10^6(30^\circ)\) for cold (hot) accretion disks \[13\].

The second question, does the distribution of spin directions change as the BBHs inspiral from \(r_i\) to \(r_f\), can be answered by evolving this distribution over this interval using the PN spin-precession equations. We will describe these PN equations and our numerical solutions to them in Sec. \[11\]. The precession of a given spin configuration in the PN regime can be understood in terms of the proximity of that configuration to the nearest spin-orbit resonance. Schnittman \[25\] identified a set of equilibrium spin configurations in which both black hole spins and the orbital angular momentum lie in a plane, along with the total angular momentum \(J = L + m_1^2 \chi_1 + m_2^2 \chi_2\). In the absence of radiation reaction, \(J\) is conserved. For these equilibrium configurations, the spins and orbital angular momentum remain coplanar and precess jointly about \(J\) with the angles \(\theta_{1,2}\) between \(L\) and \(\chi_{1,2}\) remaining fixed. The equilibrium configurations can thus be understood as spin-orbit resonances since the precession frequencies of \(L\) and \(\chi_{1,2}\) about \(J\) are all the same. Once radiation reaction is added, the spins and orbital angular momentum remain coplanar as the BBHs inspiral, although \(\theta_1\) and \(\theta_2\) slowly change on the inspiral timescale. Not only do resonant configurations remain resonant, but configurations near resonance can be captured into resonance during the inspiral. The resonances are thus very important for understanding the evolution of generic BBH systems, although the resonances themselves only occupy a small portion of the 7-dimensional parameter space characterizing generic mergers. We shall review these spin-orbit resonances in more detail in Sec. \[11\].

Bogdanović et al. \[23\] briefly considered whether spin-orbit resonances could effectively align SBH spins with the orbital angular momentum following the merger of gas-poor galaxies. They found that for a mass ratio \(q = 9/11\) and maximal spins \(\chi_1 = \chi_2 = 1\), an isotropic distribution of spins at \(r_i = 1000 M\) remains isotropically distributed when evolved to \(r_f = 10 M\). They therefore concluded that an alternative mechanism, such as the accretion torques considered later in their paper, is needed to align the BBH spins with \(L\). This conclusion is supported by a much larger set of PN inspirals presented by Herrmann et al. \[26\] who found that for equal-mass BBHs, an isotropic distribution of spins at \(40 M\) yields a flat distribution in \(\cos \theta_{12} \sim 7.4 M\). Here and in this paper \(\theta_{12}\) is the angle between the two spins \(\chi_1\) and \(\chi_2\). In the final plot of their paper, Herrmann et al. \[26\] revealed their discovery of an anti-correlation between the initial and final values of \(\cos \theta_{12}\) for \(q = 2/3\) BBHs with equal dimensionless spins \(\chi_1 = \chi_2 = 0.05\). Investigation of this anti-correlation was left to future work. Lousto et al. \[27\] also found indications that an initially isotropic distribution of spins can become non-isotropic during the PN stage of the inspiral. For a range of mass ratios \(1/16 \leq q \leq 1\) and equal spins \(\chi_1 = \chi_2 = (0.485, 0.686, 0.97)\), they found that an isotropic spin distribution at \(50 M\) develops a slight but statistically significant tendency towards anti-alignment with the orbital angular momentum \(L\). This amplitude
of anti-alignment scales linearly in the BBH spin magnitudes and appears to decrease as $q \to 0$.

We perform our own study of PN spin evolution from $r_i$ to $r_f$ for several reasons. BBHs get locked into spin-orbit resonances at a separation

$$r_{\text{lock}} \propto \left( \frac{\chi_1 \cos \theta_1 - q^2 \chi_2 \cos \theta_2}{1 - q^2} \right)^2 M,$$

which can become large in the equal-mass ($q \to 1$) limit [22]. This limit is important, as the largest recoil velocities occur for nearly equal-mass mergers. Numerical integration of the PN equations has shown that for a mass ratio $q = 9/11$, spin-orbit resonances affect spin orientations at separations $r \simeq 1000M$. This is a much larger separation than was considered in previous studies [26, 27] of spin alignment, which may therefore have failed to capture the full magnitude of the effect. These studies also focused on whether an initially isotropic distribution of spins becomes anisotropic just prior to merger. However, as discussed above, tidal torques from a circumbinary disk partially align spins with the orbital angular momentum at separations $r \gg r_{\text{GW}}$ before relativistic effects become important. As we will show in Sec. VI, such partially aligned distributions can be strongly affected by spin-orbit resonances despite the fact that isotropic distributions remain nearly isotropic. We will consider how spin precession affects the final spin magnitudes and directions in Sec. VI. The evolution of the distribution of BBH spin directions between $r_i$ and $r_f$ changes the distribution of final spin magnitudes and directions from what it would have been in the absence of precession. In addition, spin precession introduces a fundamental uncertainty in predicting the final spin of a given BBH system. At large separations, a small uncertainty in the separation leads to an uncertainty in the predicted time until merger that exceeds the precession time. In this case, one cannot predict at what phase of the spin precession the merger will occur and thus the resulting final spin. We will explore this uncertainty in Sec. VII. A brief discussion of the chief findings of this paper is given in Sec. VII.

II. POST-NEWTONIAN EVOLUTION

We evolve spinning BBH systems along a sequence of quasi-circular orbits according to the PN equations of motion for precessing binaries first derived by Kidder [28], and later used by Buonanno, Chen and Vallisneri to build matched-filtering template families for GW detection [29]. The adiabatic evolution of the binary’s orbital frequency is described including terms up to 3.5PN order, and spin effects are included up to 2PN order. These evolution equations were chosen for consistency with previous work, in particular with the study by Barausse and Rezzolla [30] of the final spin resulting from the coalescence of BBHs and with the statistical investigation of spinning BBH evolutions using Graphics Processing Units by Herrmann et al. [24]. Lousto et al. [27] evolved a large sample of spinning BBH systems using a non-resummed, PN expanded Hamiltonian. The convergence properties of non-resummed Hamiltonians for spinning BBH systems are somewhat problematic (see e.g. Fig. 1 of Ref. [10]), and it will be interesting to repeat these statistical investigations of precessing BBH systems using the effective-one-body resummations of the PN Hamiltonian recently proposed by Barausse et al. [31, 32].

In our simulations, the spins evolve according to

$$\dot{\hat{S}}_1 = \omega \times \hat{S}_1, \quad \dot{\hat{S}}_2 = \omega \times \hat{S}_2,$$

where

$$\omega = \left( \frac{M}{r^3} \right)^{1/2},$$

is the orbital frequency. In the absence of gravitational radiation, $\mathbf{J}$ and $|\mathbf{L}_N|$ are constant, implying that the direction of the orbital angular momentum evolves according to

$$\dot{\mathbf{L}}_N = -\frac{(M\omega)^{1/3}}{\eta M^2} \frac{d\mathbf{S}}{dt},$$

where $\mathbf{S} = \mathbf{S}_1 + \mathbf{S}_2$. Once radiation reaction is included, the orbital frequency slowly evolves as...
\[
\omega = \omega^2 \left( \frac{96}{5} \eta (M\omega)^{5/3} \left\{ 1 - \frac{743 + 924\eta}{336} (M\omega)^{2/3} + \left( \frac{19}{3} \eta - \frac{113}{12} \right) \chi_s \cdot \hat{L}_N - \frac{1135}{12} \chi_a \cdot \hat{L}_N + 4\pi \right) (M\omega) \right. 
+ \left. \left\{ \frac{34103}{18144} + \frac{13661}{2016} \eta + \frac{59}{18} \eta^2 \right\} - \frac{9\gamma_1\gamma_2}{48} \left( 247\hat{S}_1 \cdot \hat{S}_2 - 721(\hat{L}_N \cdot \hat{S}_1)(\hat{L}_N \cdot \hat{S}_2) \right) \right) 
+ \sum_{i=1}^{2} \left( \frac{m_i \chi_i}{M^2} \right)^2 \left[ \frac{5}{2} \left( 3(\hat{L}_N \cdot \hat{S}_i)^2 - 1 \right) + \frac{1}{96} \left( 7 - (\hat{L}_N \cdot \hat{S}_i)^2 \right) \right] (M\omega)^{1/3} 
- \frac{4159 + 15876\eta}{672} \pi (M\omega)^{5/3} + \left( \frac{1644732263}{139708000} - \frac{1712\gamma E}{105} + \frac{16\pi^2}{3} \right) \left( -\frac{273811877}{1088640} + \frac{451\pi^2}{48} - \frac{88}{3} \frac{\eta}{\Delta \eta} \right) \eta 
+ \frac{541}{896} - \frac{5605}{2592} \eta^2 - \frac{856}{105} \log(16(M\omega)^{2/3}) \right] (M\omega)^2 + \left( \frac{4415}{4032} + \frac{358675}{6048} \eta + \frac{91495}{1512} \right) \pi (M\omega)^{7/3} \right) 
\]

where \( \gamma_E \approx 0.577 \) is Euler’s constant, \( \hat{\theta} \equiv 1039/4620 \), and we have defined

\[
\chi_s \equiv \frac{1}{2} (\chi_1 + \chi_2), \quad (2.7a) \\
\chi_a \equiv \frac{1}{2} (\chi_1 - \chi_2). \quad (2.7b)
\]

The two terms in square parentheses on the third line of Eq. (2.6) are due to the quadrupole-monopole interaction \([34]\) and to the spin-spin self interaction \([33]\), respectively, and they were neglected in the statistical study of Ref. [26]. Their sum agrees with Eq. (5.17) of Ref. [36].

The numerical integration of this system of ordinary differential equations is performed using the adaptive stepsize integrator STEPPEDOPR5 [57]. The evolution of any given BBH system is specified by the following parameters: the initial orbital frequency \( \omega_i \), the binary’s mass ratio \( q \equiv m_2/m_1 \), the dimensionless magnitude of each spin \( \chi_i \), and the relative orientation \( (\theta_i, \phi_i) \) of each spin with respect to the orbital angular momentum at time \( t = 0 \) \( (i = 1, 2) \). To monitor the variables along the whole evolution we output all quantities using a constant logarithmic spacing in the orbital frequency at low frequencies, and the stepsize as used in the integrator at high frequencies. Typically this results in a total of about 64,000 points in the range \( M\omega \in [M\omega_i, M\omega_f] \), where \( M\omega_i = 3.16 \times 10^{-5} \) and \( M\omega_f = 0.1 \). Numerical experimentation indicates that a tolerance parameter \( ATOL = 2 \times 10^{-8} \) in the adaptive stepsize integrator is sufficient for a pointwise accuracy of order 1% or better in the final quantities. Therefore the error induced by the numerical integrations of the PN equations of motion is subdominant with respect to the errors induced by precessional effects and by fits of the numerical simulations, which will be one of the main topics of this paper.

### III. SPIN-ORBIT RESONANCES

In this Section, we review the equilibrium configurations of BBH spins first presented in Schnittman [25] for which the Newtonian orbital angular momentum \( \mathbf{L}_N \) and individual spins \( \mathbf{S}_{1,2} \) all precess at the same resonant frequency. As discussed briefly in the Introduction, at a given binary separation \( r \) fully general quasi-circular BBHs are described by 7 parameters: the mass ratio \( q \) and the 3 components of each black hole spin. In spherical coordinates with \( \mathbf{L}_N \) defining the \( z \)-axis, each spin is given by its magnitude \( S_i = m_i^2 \chi_i \) and direction \( (\theta_i, \phi_i) \) \( (i = 1, 2) \). In the PN limit for which this analysis is valid, a clear hierarchy

\[
t_{\text{orb}} \ll t_p \ll t_{\text{GW}} \quad (3.1)
\]

exists between the orbital time \( t_{\text{orb}} \propto r^{3/2} \), the precession time \( t_p \propto \Omega_{1,2}^{-1} \propto r^{5/2} \), and the radiation time \( t_{\text{GW}} \propto E_{\text{GW}}/E \propto r^4 \). This hierarchy implies that the BBH spins will precess many times before merger leaving only their relative angular separation \( \Delta \phi \equiv \phi_2 - \phi_1 \) in the orbital plane well defined. This reduces the BBH parameter space to 6 dimensions. Since the mass ratio and individual spin magnitudes are preserved during the inspiral, a given BBH evolves through the 3-dimensional parameter space \( (\theta_1, \theta_2, \Delta \phi) \) on the precession timescale \( t_p \). This evolution is governed by the spin precession equations \([24]\).

Schnittman [25] discovered a one-parameter family of equilibrium solutions to these equations for which \( (\theta_1, \theta_2, \Delta \phi) \) remain fixed on the precession timescale \( t_p \). These solutions have \( \Delta \phi = 0^\circ \) or \( 180^\circ \), implying that \( \mathbf{L}_N, \mathbf{S}_1 \) and \( \mathbf{S}_2 \) all lie in a plane and precess at the same resonant frequency about the total angular momentum \( \mathbf{J} \), which remains fixed in the absence of gravitational radiation. The values of \( \theta_1, \theta_2 \) for these resonances can be determined by requiring the first and second time derivatives of \( \mathbf{S}_1 \cdot \mathbf{S}_2 \) to vanish. This is equivalent to satisfying the algebraic constraint

\[
(\Omega_1 \times \mathbf{S}_1) \cdot [\mathbf{S}_2 \times (\mathbf{L}_N + \mathbf{S}_1)] = (\Omega_2 \times \mathbf{S}_2) \cdot [\mathbf{S}_1 \times (\mathbf{L}_N + \mathbf{S}_2)]. \quad (3.2)
\]

Since \( \mathbf{L}_N \) appears in Eq. (3.2) both explicitly and implicitly through \( \Omega_{1,2} \), the resonant values of \( \theta_{1,2} \) vary with
FIG. 1: Spin-orbit resonances for maximally spinning BBHs with a mass ratio of $q = 9/11$. The dotted black diagonal indicates where $\theta_1 = \theta_2$. Solid black curves below (above) this diagonal show $(\theta_1, \theta_2)$ for the one-parameter families of equilibrium spin configurations with $\Delta \phi = 0^\circ$($180^\circ$) at different fixed binary separations. Approaching the diagonal from below, these curves correspond to separations $r = 1000, 500, 250, 100, 50, 10M$. The curves approaching from above correspond to separations $r = 250, 50, 20, 10M$. The long-dashed red curves show how $\theta_{1,2}$ evolve as members of these resonant families inspiral from $r_i = 1000M$ to $r_f = 10M$. The projection $\mathbf{S} \cdot \mathbf{L}_N$ of the total spin $\mathbf{S}$ onto the orbital angular momentum $\mathbf{L}_N$ is constant along the short-dashed blue lines, while the projection $\mathbf{S}_0 \cdot \mathbf{L}_N$ of the EOB spin $\mathbf{S}_0$ is constant along the dot-dashed green lines.

the binary separation. This is crucial, as otherwise these one-parameter families of resonances would affect only a small portion of the 3-dimensional parameter space $(\theta_1, \theta_2, \Delta \phi)$ through which generic BBH configurations evolve. As gravitational radiation slowly extracts angular momentum from the binary on the radiation time $t_{GW}$, the resonances sweep through a significant portion of the $(\theta_1, \theta_2)$ plane. The angular separation $\Delta \phi$ of a generic BBH is varying on the much shorter precession time $t_p$, and thus has a significant chance to closely approach the resonant values $\Delta \phi = 0^\circ$ or $180^\circ$ at some point during the long inspiral. Such generic BBHs will be strongly influenced or even captured by the spin-orbit resonances, as we will see in detail in Sec. IV.

We show the dependence of the spin-orbit resonances on $r$ for maximally spinning BBHs in Figs. 1 and 2. Those resonances with $\Delta \phi = 0^\circ$ (shown in Fig. 2 of [25]) always have $\theta_1 < \theta_2$, and thus appear below the diagonal $\cos \theta_1 = \cos \theta_2$ in our Figs. 1 and 2. Those resonances with $\Delta \phi = 180^\circ$ (shown in Fig. 3 of [25]) have $\theta_1 > \theta_2$ and therefore appear above the diagonal in our Figs. 1 and 2. We plot $(\cos \theta_1, \cos \theta_2)$ rather than $(\theta_1, \theta_2)$ like [25] because isotropically oriented spins should have a flat distribution in these variables.

In the limit $r \to \infty$, so that also $|L_N| \to \infty$, the resonant configurations either have $\mathbf{S}_1$ or $\mathbf{S}_2$ aligned or anti-aligned with $\mathbf{L}_N$ (either $\theta_1$ or $\theta_2$ equals to $0^\circ$ or $180^\circ$). This corresponds to the four edges of the plot in Fig. 1. For smaller fixed values of $|L_N|$, the values $(\theta_1, \theta_2)$ for the one-parameter families of resonant configurations approach the diagonal $\theta_1 = \theta_2$. BBHs in spin-orbit resonances at large values of $|L_N|$ (large $r$) remain resonant as they inspiral. As gravitational radiation carries away angular momentum, $r$ decreases and $\theta_{1,2}$ for individual resonant BBHs evolves towards this diagonal along the red long-dashed curves in Fig. 1. For resonances with $\Delta \phi = 0^\circ$ (those below the diagonal), this evolution aligns the two spins with each other. Symmetry implies that aligning the spins with each other will lead to larger final spins and smaller recoil velocities [38, 39].

The projection

$$\mathbf{S} \cdot \mathbf{L}_N = S_1 \cos \theta_1 + S_2 \cos \theta_2 \quad (3.3)$$

of the total spin $\mathbf{S} = \mathbf{S}_1 + \mathbf{S}_2$ parallel to the orbital an-
angular momentum is constant along the short-dashed blue lines in Figs. 1 and 2. These blue lines have steeper slopes than the red lines along which the resonant binaries inspiral. This implies that the total spin \( S \) becomes anti-aligned (aligned) with the orbital angular momentum for resonant configurations with \( \Delta \phi = 0^\circ (180^\circ) \), leading to smaller (larger) final spins. The families of resonances with \( \Delta \phi = 0^\circ \) (below the diagonal) sweep through a larger area of the \( (\cos \theta_1, \cos \theta_2) \) plane as the BBHs inspiral, and approach the diagonal more closely. This implies that anti-alignment may be more effective than alignment, which might explain the “small but statistically significant bias of the distribution towards counter-alignment” in \( S \cdot \hat{L}_N \) noted in Lousto et al. \[27\]. However, Table IV of \[27\] indicates that both \( S_1 \) and \( S_2 \) individually become anti-aligned with \( \hat{L}_N \), whereas the spin-orbit resonances would align one black hole while anti-aligning the other. All of the PN evolutions in Lousto et al. \[27\] begin at separations of \( r = 50M \), which corresponds to the \( \Delta \phi = 0^\circ \) curve in Fig. 4 that is second closest to the diagonal. The resonances sweep through most of the plane below the diagonal at larger separations, suggesting that these short-duration PN evolutions may have failed to capture the full magnitude of the anti-alignment. We will investigate this possibility in Sec. IV.

Another interesting feature of Figs. 1 and 2 is that the red long-dashed curves along which the BBHs inspiral are nearly parallel to the dot-dashed green lines along which the projection \( S_0 \cdot \hat{L}_N \) of the effective-one-body (EOB) spin \[40\]

\[
S_0 \equiv (1 + q)S_1 + (1 + q^{-1})S_2 \quad (3.4)
\]
is constant. The conservation of this quantity at 2PN order was first noted in Ref. \[33\] and follows directly from Eqs. (2.1), (2.2), and (2.5). The conservation of \( S_0 \cdot \hat{L}_N \) rather than \( S \cdot \hat{L}_N \) itself allows for the possible alignment of the total spin \( S \) discussed in the previous paragraph.

We conclude this Section by briefly discussing how the spin-orbit resonances vary with the mass ratio \( q \), as can be seen by comparing the \( q = 9/11 \) resonances in Fig. 1 with the \( q = 1/3 \) resonances in Fig. 2. The most pronounced differences are that the \( q = 1/3 \) resonances sweep away from the edges of the \( (\cos \theta_1, \cos \theta_2) \) plane at much smaller values of the separation \( r \), and do not approach the diagonal as closely. This is consistent with the decreasing value of \( r_{\text{lock}} \) in Eq. (1.4) as \( q \to 0 \). In this limit both \( t_p \) and \( t_{\text{lock}} \) are proportional to \( q^{-1} \), implying that generic BBHs will be less likely to be affected by the resonances as they sweep through the plane over a smaller range in \( r \). BBHs already in a resonant configuration will also be less affected since the resonant curves do not approach the diagonal as closely. The red long-dashed curves showing the inspiral of resonant configurations have steeper slopes for \( q = 1/3 \), consistent with the larger black hole being immune to its smaller companion in the limit \( q \to 0 \). This seems to contradict the puzzling result presented in Table IV of Lousto et al. \[27\] that it is the smaller companion that remains randomly distributed during the inspiral. We will examine this behavior as well in the next Section.

### IV. SPIN ALIGNMENT

In this Section, we examine the extent to which the spins of generic (i.e. misaligned) BBH configurations become aligned with the orbital angular momentum and each other as the BBHs inspiral from \( r_i = 1000M \) to \( r_f = 10M \). Although we use maximally spinning BBHs to demonstrate this alignment, the magnitude of the alignment is comparable for all BBHs with \( \chi_{1,2} \gtrsim 0.5 \) as shown in Fig. 11 of \[24\]. We first consider initial spin configurations given by a uniform \( 10 \times 10 \times 10 \) grid evenly spaced in \( (\cos \theta_1, \cos \theta_2, \Delta \phi) \). This distribution is isotropic, and would be expected in the absence of an astrophysical mechanism to align the spins. BBHs with isotropically oriented spins might form in gas-poor mergers of SBHs and mergers of stellar-mass black holes in dense clusters.

In Fig. 5 we show how the distribution of...
The precession continues as the BBHs inspiral to the right panels of Fig. 3. By the time they reach the middle θ of the middle region, the spins of all 1000 BBHs precess in a way that conserves the projection of \( \mathbf{S}_i \cdot \mathbf{L}_N \) for this less equal mass ratio. This inhibits their ability to diffuse across the cos \( \theta_1 = \pm 0.4 \) boundaries, again shown by the vertical dotted lines. Even at \( r_f = 10M \) only a few points have trickled between the three regions. Since the spin of the more massive black hole remains aligned with the orbital angular momentum, one would expect a large final spin and an absence of superkicks for such small mass ratios. We will examine in detail how spin alignment affects recoil-velocity distributions in future work.

In Fig. 4 we show how the joint probability distribution function for \( \Delta \phi \) and cos \( \theta_1 \) evolves for our evenly spaced 10 × 10 × 10 grid of initially isotropic BBH spin configurations. As defined in the Introduction, cos \( \theta_1 \) is the cosine of the angle between \( \mathbf{S}_1 \) and \( \mathbf{S}_2 \). It can be expressed in terms of the individual spin angles as

\[
\cos \theta_{12} = \sin \theta_1 \sin \theta_2 \cos \Delta \phi + \cos \theta_1 \cos \theta_2, \tag{4.1}
\]

and has a flat distribution between -1 and 1 for isotropic, uncorrelated spins such as those given by our 10 × 10 × 10 grid. However, as seen in Eq. (4.1), the values of cos \( \theta_{12} \) and cos \( \Delta \phi \) are correlated; for a given value of \( \Delta \phi \) the distribution of cos \( \theta_{12} \) is peaked about cos \( \Delta \phi \) for flat distributions of cos \( \theta_1 \) and cos \( \theta_2 \). This can be seen in Fig. 5 from the clustering of points about the curve cos \( \theta_{12} = \cos \Delta \phi \). Although cos \( \theta_{12} \) and cos \( \Delta \phi \) are correlated even for isotropic spins, geometry implies that both are initially uncorrelated with the value of cos \( \theta_1 \). This is revealed by the identical distributions of the red, green, and blue points in the top left panel of Fig. 4 to within the resolution of our grid. These distributions do not remain identical as the BBHs inspiral from \( r_i = 1000M \) to \( r_f = 10M \). Influenced by the \( \Delta \phi = 0^\circ \) spin-orbit resonances below the diagonal in Fig. 4, the blue points become concentrated about \( \Delta \phi = 0^\circ, \cos \theta_{12} = 1 \) by the time they reach \( r_f \). The red points, similarly influenced by the \( \Delta \phi = \pm 180^\circ \) resonances above the diagonal in Fig. 4, become concentrated about \( \Delta \phi = \pm 180^\circ, \cos \theta_{12} = -1 \). The effect of this spin alignment on the spin of the final black hole will be explored in detail in the next Section, while the effect on recoil velocities will be examined in future work. Qualitatively, alignment of the spins with each other (cos \( \theta_{12} \to 1 \)) increases the final spin and reduces the rate of precession.

![FIG. 4: Distributions of (cos \( \theta_1, \cos \theta_2 \)) at different separations r for 1000 initially isotropic maximally spinning BBHs with a mass ratio \( q = 1/3 \). The different panels, points, and lines are the same as those given for \( q = 9/11 \) in Fig. 3](image-url)
the recoil velocity, while anti-alignment \((\cos \theta_{12} \rightarrow -1)\) does the opposite.

The magnitude of this spin alignment is greatly reduced for smaller mass ratios as seen in Fig. 6 for the case \(q = 1/3\). Although the clustering of all the points about \(\cos \theta_{12} = \cos \Delta \phi\) is again apparent, the distributions of the red, green, and blue points remain similar all the way down to \(r_f = 10 M\) as seen in the lower right panel. The weaker influence of the spin-orbit resonances for \(q = 1/3\) follows from the smaller value of \(r_{lock}\) in Eq. (1.4), and is similarly reflected by the smaller fraction of the \((\cos \theta_1, \cos \theta_2)\) plane occupied by the resonant curves in Fig. 2.

We have provided histograms of \(\cos \theta_{12}\) and \(\Delta \phi\) in Fig. 7 to clarify the differences between Figs. 5 and 6. We see that the distributions of \(\cos \theta_{12}\) and \(\Delta \phi\) are initially flat for both mass ratios, but evolve considerably for \(q = 9/11\) while remaining nearly flat for \(q = 1/3\) within the limits set by Poisson fluctuations. The open blue (red) curves in the left panels of Fig. 7 clearly show distributions peaked at \(\cos \theta_{12} = 1, \Delta \phi = 0^\circ\) \((\cos \theta_{12} = -1, \Delta \phi = \pm 180^\circ)\). Such trends are barely noticeable in the right panels. We will explore the implications of these findings for the final spins in the next Section.

V. FINAL SPIN DISTRIBUTIONS

Several attempts have been made to predict the final dimensionless spin \(\chi_f\) of the black hole resulting from a BBH merger. Initial attempts focused on finding simple phenomenological fitting formulae for the final spin resulting from non-spinning, unequal-mass BBH merger simulations [41–43]. A group at the Albert Einstein Institute (AEI) developed a fitting formula that provides the magnitude and direction of \(\chi_f\) in terms of the initial spins \(\chi_1, \chi_2\) and the mass ratio \(q\) [44–46]. They assumed that the final spin magnitude could be expressed as a polynomial in \(\chi_1, \chi_2\), and the symmetric mass ratio \(\eta\), then made some additional assumptions about the symmetries of this polynomial dependence and how energy and angular momentum are radiated to reduce the number of terms in their expression. The coefficients of the remaining terms were calibrated using numerical-relativity (NR) simulations of BBH mergers in which the initial spins were either aligned or anti-aligned with the orbital angular momentum. We shall refer to this older AEI formula as “AEIo”. A more recent paper [30] by members of this group uses newer NR simulations to recalibrate their coefficients, and replaces earlier assumptions with the con-

FIG. 5: Distributions of \((\Delta \phi, \cos \theta_{12})\) at different separations \(r\) for 1000 initially isotropic maximally spinning BBHs with a mass ratio \(q = 9/11\). The top left panel shows the initial \(10 \times 10 \times 10\) grid of BBH spin configurations, evenly spaced in \((\cos \theta_1, \cos \theta_2, \Delta \phi)\). This distribution is peaked about the curve \(\cos \theta_{12} = \cos \Delta \phi\) shown by the dotted curve. The points are colored according to their initial values of \(\cos \theta_1\) as in Fig. 3. The top right, bottom left, and bottom right panels show the distribution evolves after the BBHs have inspiraled to \(r = 1000, 100\) and \(10\) respectively, also as in Fig. 3.

FIG. 6: Distributions of \((\Delta \phi, \cos \theta_{12})\) at different separations \(r\) for 1000 initially isotropic maximally spinning BBHs with a mass ratio \(q = 1/3\). The different panels, points, and lines are the same as those given for \(q = 9/11\) in Fig. 5.
jecture that the final spin points in the direction of the total angular momentum of the initial BBH at any separation. For consistency, this requires the further assumption that angular momentum is always radiated in the direction of the total angular momentum. We shall refer to this newer AEI formula as “AEIn”. An alternative fitting formula was proposed by a group at Florida Atlantic University (FAU) \cite{FAU}. Following the procedure outlined in \cite{38,39}, the FAU group performed 10 equal-mass misaligned simulations to calibrate the coefficients of fitting formulae for the Cartesian components of $\chi_f$. They then made additional assumptions about the mass-ratio dependence of these formulae, and found good agreement between their predictions and independent NR simulations with mass ratios as small as $q = 1/3$. We shall refer to the formula of this group as “FAU”. The Rochester Institute of Technology (RIT) group proposed yet another fitting formula during the preparation of this paper \cite{48}. This formula includes higher-order terms in the initial spins that may ultimately be needed to describe future high-accuracy NR simulations. However, current simulations are inadequate to calibrate all the terms appearing in the RIT formula, so we will not consider its predictions in this paper.

Other groups have predicted final spins by extrapolating analytical test-particle calculations to finite mass ratios, rather than calibrating fitting formulae with NR simulations. Buonanno, Kidder, and Lehner (BKL) \cite{49} derived a formula for the final spin by assuming, as is true in the test-particle limit, that the angular momentum radiated during the inspiral stage of a BBH merger exceeds that radiated during the plunge and ringdown. Using this assumption, they equated the final spin with the total angular momentum $J = L_{\text{ISCO}} + S_1 + S_2$, where $L_{\text{ISCO}}$ is the orbital angular momentum at the innermost stable circular orbit (ISCO) of a test particle of mass $\eta M$ orbiting a black hole of mass $M$ and dimensionless spin $\chi_f$ equal to that of the final black hole. This counterintuitive but inspired choice correctly provides $\chi_f \to \chi_1$ in the $q \to 0$ limit and respects the symmetry of BBH mergers under exchange of the labels of the two black holes. Though derived only from test-particle calculations, the BKL formula is remarkably successful at predicting final spins even for equal-mass BBH mergers. Kesden \cite{50} slightly modified the BKL spin formula to account for the energy radiated during the inspiral stage of the merger. This change makes the formula accurate to linear order in $q$ in the test-particle limit. It generically increases the magnitude of the predicted dimensionless final spin by reducing the predicted final mass $m_f$ below $M$ in the denominator of the expression $\chi_f = S_f / m_f^2$. This increase improves the agreement with NR simulations of non-spinning BBH mergers, but leads to somewhat larger final spins than the other formulae for mergers of maximally spinning BBHs, such as those considered in this paper. The predictions of this formula are referred to as “Kes” in this paper.

We now present the predictions of the spin formulae summarized above for various distributions of BBH spins that are allowed to inspiral from $r_i = 1000 M$ to $r_f = 10 M$.

A. Spin Magnitudes

In the top panel of Fig. 8 we show the final spin magnitude $\chi_f$ predicted by the AEIn formula for the evenly spaced $10 \times 10 \times 10$ grid of maximally spinning BBHs with $q = 9/11$ described in Sec. IV. The other spin formulae give very similar results; the mean and variance of the final spin distributions predicted by the other formulae for some of the initial distributions described below are provided in Table I. As in Figs. 5-7 the black curves in Fig. 8 refer to all 1000 BBHs, the blue curves to the subset of 300 BBHs with the lowest values of $\theta_1$, and the red curves to the subset of 300 BBHs with the highest values of $\theta_1$. The dotted curves give the final spin distribution predicted for the BBH spin configurations at their initial separation $r_i = 1000 M$, while the solid curves give the final spin distribution predicted when these same BBHs are allowed to inspiral to $r_f = 10 M$. 

FIG. 7: Histograms of $\cos \theta_{12}$ and $\Delta \phi$ for BBHs with initially isotropic spins. The two left panels are for the mass ratio $q = 9/11$, while the two right panels are for $q = 1/3$. The two top panels give the distribution of $\cos \theta_{12}$, while the two bottom panels give the distribution of $\Delta \phi$. The black curves are for all 1000 BBHs in the $10 \times 10 \times 10$ grid discussed in the text, while the blue (red) curves correspond to the blue (red) points in Figs. 3-7 with initial values $\cos \theta_1 > 0.4$ ($\cos \theta_1 < -0.4$). The horizontal dotted lines show the initially flat distributions, while the solid lines show the distributions at $r = 10 M$. 

\[ M \]

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\[ A. \quad \text{Spin Magnitudes} \]
The AEIn formula is unique in that it claims to accurately predict final spins at all separations; separations as large as \( r = 2 \times 10^4 M \) were considered in [30]. The other fitting formulae were intended to apply at \( r_{\text{NR}} \lesssim 10^3 M \), the starting point for the NR simulations with which their coefficients were calibrated. The BKL and Kes formulae were designed for use at the ISCO. Although strictly speaking the formulae other than AEIn cannot be applied to widely separated BBHs, one can imagine that the BBHs inspiral to \( r_f = 10 M \) without spin precession where these formulae are valid. It is in this sense that we consider the predictions of these other formulae when we claim in this Section to apply them to BBH spin configuration at \( r_i = 1000 M \).

According to the PN evolution described in Sec. II, the AEIn formula is applied at the initial distribution at \( r_i = 1000 M \). The dotted and solid black curves in the top panel of Fig. 8 are identical to within the Poisson noise of our limited number of BBH inspirals, confirming the finding of Refs. [23, 26, 27] that isotropic distributions of BBH spins remain nearly isotropic as they inspiral. Even at \( r_i = 1000 M \), the blue (red) subset of spin configurations yields the largest (smallest) predicted final spins, because for these configurations the spin of the more massive black hole is aligned (anti-aligned) with the orbital angular momentum. The spin-orbit resonances further enhance (reduce) the final spins predicted for these subsets by aligning (anti-aligning) the BBH spins with each other during the inspiral for small (large) initial values of \( \theta_1 \). As a result, the solid blue (red) distribution at \( r_f = 10 M \) has a larger (smaller) mean final spin than the initial dotted distribution at \( r_i = 1000 M \). This can be seen in the displacement of predicted final spins for the colored subsets away from \( \chi_f \approx 0.75 \) towards larger and smaller values.

To clarify the magnitude of this effect, we have performed 6 additional sets of BBH inspirals, each of which...
consists of a fixed value of \(\theta_1\) and a \(30 \times 30\) grid evenly spaced in \(\cos \theta_2\) and \(\Delta \phi\). Three of these sets have the spin of the more massive black hole nearly aligned with the orbital angular momentum \((\theta_1 = 10^\circ, 20^\circ, 30^\circ)\), while the other 3 sets have \(\chi_1\) nearly anti-aligned with \(L_N\) \((\theta_1 = 150^\circ, 160^\circ, 170^\circ)\). The choice of aligned distributions was partly motivated by the finding of Ref. 17, that accretion torques will align BBH spins to within 10\(^\circ\) of the orbital angular momentum for a cold (hot) disk. The predicted final spins for these distributions, both at \(r_i = 1000\,M\) and \(r_f = 10\,M\), are shown in the bottom panel of Fig. 3. The final spins for the initially aligned \((\theta_1 \leq 30^\circ)\) BBH distributions are significantly larger when predicted at \(r_f = 10\,M\) than at \(r_i = 1000\,M\), undermining the claim of 30 that the AEIn formula can accurately predict final spins at large separations without the need for PN evolutions. The predicted final spins for the initially anti-aligned \((\theta_1 \geq 150^\circ)\) BBH distributions conversely shift to lower values as the predictions are made later in the inspiral. We provide the mean and standard deviation of the final spins predicted for these 6 new sets of partially aligned BBH distributions for all 5 formulae in Table 1.

To explore the dependence of these effects on the mass ratio, we have provided histograms of the predicted final spins for these same BBH spin distributions with \(q = 1/3\) in Fig. 3. The discrete peaks at low values of \(\chi_f\) in the histograms in the top panel are an artifact of the 10 discrete values of \(\cos \theta_1\) in our \(10 \times 10 \times 10\) grid. Each peak contains 100 points with the same initial value of \(\theta_1\). The decrease in the width of each peak as the BBHs inspiral from \(r_i = 1000\,M\) to \(r_f = 10\,M\) is a consequence of the anti-alignment of the BBH spins for large \(\theta_1\), but the gaps between the peaks would be filled in if we used a finer grid. The shifts in the mean values of the peaks should be robust with respect to the grid spacing. These shifts for the initially aligned BBH distributions are provided in Table II for all 5 formulae for \(q = 1/3\), as well as for the intermediate mass ratio \(q = 2/3\).

### B. Spin Directions

Before providing quantitative results, we need to clarify what is meant by the direction of the spin of the final black hole. In what reference frame is this direction defined? Most of the fitting formulae calibrated with NR simulations attempt to predict the angle

\[
\vartheta_f \equiv \arccos[\hat{L}_N(r_f) \cdot \hat{\chi}_f(r_f)]
\]

between the BBH orbital angular momentum \(L_N\) at the separation \(r_f = r_{\text{NR}}\) where the NR simulations were performed and the final spin \(\chi_f\) predicted from the BBH spin configuration at this same separation. The analytical predictions of BKL and Kes were designed to apply to BBH spin configurations at \(r_f = r_{\text{SCCO}}\). If one assumed that neither the orbital angular momentum nor the BBH spins (upon which the prediction \(\chi_f(r_f)\) depends) precised during the inspiral, one could insert these quantities at any separation into the right-hand side of Eq. 5.1 to predict \(\vartheta_f\). The angle \(\vartheta_f\) is physically interesting because it quantifies the post-merger alignment between \(\chi_f\) and the inner edge of the accretion disk if one assumes that torques have aligned the circumbinary disk with \(L_N\). However, one might also be interested in the alignment between \(\chi_f\) and a feature like the galactic disk that is assumed to be aligned with \(L_N\) at some larger scale \(r_i\). In that case, one would need to compute the angle

\[
\vartheta_i \equiv \arccos[\hat{L}_N(r_i) \cdot \hat{\chi}_f(r_i)]
\]

between \(L_N\) at this larger separation and the final spin \(\chi_f(r_i)\) predicted from the BBH spins at this same separation.

The proper way to predict \(\chi_f\) from the BBH spins at \(r_i\) would be to use PN equations like those specified in Sec. II to propagate those spins and \(L_N\) down to \(r_f\), then insert them into the fitting formula of one’s choice. The AEIn formula is based on the conjecture that \(\chi_f\) points in the direction of the total angular momentum \(J\) at any separation, since angular momentum is always radiated parallel to \(J\), thus preserving its direction. This conjecture is plausible because at large separations, the precession time \(t_p\) is much shorter than the inspiral time \(t_{\text{GW}}\). If the vectors associated with the BBHs precess rapidly enough, all components except those parallel to \(J\) (which varies on the longer timescale \(t_{\text{GW}}\)) will average to zero. The AEIn conjecture is very useful because it allows \(\vartheta_i\) to be computed without solving any PN equations. However, the approximation \(t_p \ll t_{\text{GW}}\) upon which it depends breaks down at small separations. This may lead to incomplete cancellation of the angular momentum radiated perpendicular to \(J\).

We test this possibility by calculating

\[
\theta_f \equiv \arccos[\hat{J}(r_f) \cdot \hat{J}(r_f)]
\]

the angle between the total angular momentum at \(r_i = 1000\,M\) and that after the BBHs have inspiraled to \(r_f = 10\,M\). If the direction of \(J\) really was preserved during the inspiral, \(\vartheta_f\) would vanish. We present histograms of \(\theta_f\) for mass ratio \(q = 9/11\) in Fig. 10. The upper panel shows the \(10 \times 10 \times 10\) grid of BBH spin configurations evenly spaced in \((\cos \theta_1, \cos \theta_2, \Delta \phi)\) that we have discussed previously. The direction of \(J\) changes by \(\theta_J \lesssim 2^\circ\) during most of the inspirals, though a tail extends to larger values for large initial values of \(\theta_1\). This tail can be seen more clearly in the bottom panel for the BBHs with \(\chi_1\) initially anti-aligned with \(L_N\) \((\theta_1 \geq 150^\circ)\). We agree with 30 that these large changes in the direction of \(J\) are likely a consequence of the transitional precession first identified in Ref. 51. This transitional precession occurs to an even greater extent for smaller mass ratios, as can be seen in Fig. 11 for \(q = 1/3\). As in the upper panel of Fig. 9, discrete peaks resulting from the grid spacing in \(\cos \theta_1\) can be seen in the left panel of Fig. 11. The middle panel shows that the direction of \(J\) remains nearly
BBH spins at $r_i = 1000M$ and that at $r_f = 10M$ for our set of 1000 BBHs with $q = 9/11$ and initially isotropic spins. As in previous figures, the blue (red) curve shows the subset of 300 BBHs with the lowest (highest) initial values of $\theta_1$. Bottom panel: Histograms of $\theta_j$ for the 6 sets of 900 BBH mergers with flat distributions in $\cos \theta_2$ and $\Delta \phi$ at $r_i = 1000M$. The red, orange, yellow, green, blue, and purple curves show BBHs that have $\theta_1 = 170^\circ, 160^\circ, 150^\circ, 30^\circ, 20^\circ, \text{and } 10^\circ$ respectively at this initial separation.

FIG. 10: Top panel: Histogram of the angle $\theta_j$ (in degrees) between the total angular momentum $\mathbf{J}$ at $r_i = 1000M$ and that at $r_f = 10M$ for our set of 1000 BBHs with $q = 9/11$ and initially isotropic spins. As in previous figures, the blue (red) curve shows the subset of 300 BBHs with the lowest (highest) initial values of $\theta_1$. Bottom panel: Histograms of $\theta_j$ for the 6 sets of 900 BBH mergers with flat distributions in $\cos \theta_2$ and $\Delta \phi$ at $r_i = 1000M$. The red, orange, yellow, green, blue, and purple curves show BBHs that have $\theta_1 = 170^\circ, 160^\circ, 150^\circ, 30^\circ, 20^\circ, \text{and } 10^\circ$ respectively at this initial separation.

FIG. 11: Left panel: Histogram of the angle $\theta_j$ (in degrees) between the total angular momentum $\mathbf{J}$ at $r_i = 1000M$ and that at $r_f = 10M$ for our set of 1000 initially isotropically spinning BBHs with $q = 1/3$. As in previous figures, the blue (red) curve shows the subset of 300 BBHs with the lowest (highest) initial values of $\theta_1$. Middle panel: Histograms of $\theta_j$ for the 3 sets of 900 BBH mergers initially with $\theta_1 = 10^\circ$ (purple), $20^\circ$ (blue), and $30^\circ$ (green). Right panel: Histograms of $\theta_j$ for the 3 sets of 900 BBH mergers initially with $\theta_1 = 150^\circ$ (yellow), $160^\circ$ (orange), and $170^\circ$ (red).

constant ($\theta_j \lesssim 0.5^\circ$) when $\chi_1$ in closely aligned with $\mathbf{L}_N$ ($\theta_1 \leq 30^\circ$). However, the right panel shows that the assumption of constant $\mathbf{J}$ fails badly for the BBHs with $\theta_1 \geq 150^\circ$, that comprise $\sim 7\%$ of isotropically distributed BBH mergers. The mass ratio $q = 1/3$ is not extreme compared to the majority of astrophysical mergers, so caution should be taken when assuming that $\chi_f$ points in the direction of $\mathbf{J}$ such as in Eq. [5.2].

What about the less ambitious predictions of $\vartheta_f$ from BBH spins at $r_f = 10M$, assuming that NR simulations correctly describe spin precession from this separation until merger? Spin-orbit resonances have significant implications for these predictions as well. We show predictions of $\vartheta_f$ by the AEIn formula for a mass ratio of $q = 9/11$ in Fig. 12. The other formulæ predict very similar results. As in Figs. 8 and 9 the dotted curves show predictions assuming that the initial BBH spin distribution is preserved down to $r_f = 10M$. The solid curves include spin precession from $r_i = 1000M$ to $r_f = 10M$ according to the PN equations of Sec. 11. The difference between the dotted and solid black curves in the top panel is below the Poisson fluctuations, another consequence of the finding of Refs. 23, 26, 27 that isotropically oriented BBH spins remain nearly isotropic as they inspiral. Careful examination of the upper panel reveals that spin precession has shifted the BBHs with $\chi_1$ initially aligned with $\mathbf{L}_N$ (blue distribution) to larger $\vartheta_f$, while the anti-aligned BBHs have conversely shifted to smaller $\vartheta_f$.

This trend is much more pronounced in the middle and bottom panels of Fig. 12. Spin precession actually results in the initially aligned BBHs ($\theta_1 \leq 30^\circ$) having larger values of $\vartheta_f$ at $r_f = 10M$ than the anti-aligned BBHs ($\theta_1 \geq 150^\circ$), a reversal of what would be predicted from the initial spin distributions shown by the dotted curves. The spin-orbit resonances explain this highly counterintuitive result. The BBHs initially with $\theta_1 \leq 30^\circ$ are influenced by the $\Delta \phi = 0^\circ$ resonances which align the BBH spins with each other and anti-align $\mathbf{S} = \mathbf{S}_1 + \mathbf{S}_2$ with $\mathbf{L}_N$. Both effects lead to larger predicted values of $\vartheta_f$. Conversely, the BBHs initially with $\theta_1 \geq 150^\circ$ are influenced by the $\Delta \phi = 180^\circ$ resonances, which greatly decrease the magnitude of $\mathbf{S}$ and align it with $\mathbf{L}_N$. This explains the reduced values of $\vartheta_f$ for these BBHs seen in the bottom panel of Fig. 12. This same effect can
VI. SPIN PRECESSION UNCERTAINTY

So far, we focused on how spin precession between \( r_i \) and \( r_f \) alters the expected distribution of final spins. In this section, we show that spin precession introduces a fundamental uncertainty in predicting the final spin. An uncertainty \( \Delta t_{GW} \) in the BBH separation leads to an uncertainty \( \Delta \theta \) in the time until merger. If this uncertainty is comparable to the precession time \( t_p \), the phase of the spin precession at which the merger occurs will be uncertain as well. This new uncertainty is independent of and may exceed that associated with the NR simulations themselves. Readers only interested in astrophysical distributions of final spins may wish to proceed to the discussion in Sec. VII.

It is often useful to define the final spin direction relative to the orbital angular momentum \( \mathbf{L}_N \) at different separations. We therefore generalize the angles defined in Eqs. (5.1) and (5.2) to the separation-dependent quantities

\[
\phi_f(r) = \arccos[\mathbf{L}_N(r) \cdot \mathbf{\chi}_f(r)], \quad (6.1)
\]

\[
\phi_i(r_i) = \arccos[\mathbf{L}_N(r_i) \cdot \mathbf{\chi}_f(r)], \quad (6.2)
\]

Note that these quantities reduce to the previously defined angles in the appropriate limit: \( \phi_f(r_f) = \phi_f \), \( \phi_i(r_i) = \phi_i \). These definitions address two ambiguities:

(i) the choice of the reference orbital angular momentum and (ii) the separation at which a given fitting formula is evaluated.
The initial parameters of the binary are
arations, correspond to slightly different initial frequencies or sep-
the Kesden formula. The different curves in each panel
of the AEIo, FAU and BKL formulae is quite similar to
and 3

Before we discuss the uncertainties in determining
these angles and the final spin orientation specified by the an-
spins, and initial spin orientation specified by the an-
angles θ₁ = 120°, θ₂ = 60° and Δφ = 288°. The different curves
correspond to initial frequencies Mω₁ = 3.16 × 10⁻⁵ (solid), 3.17 × 10⁻⁵ (long-dashed), 3.18 × 10⁻⁵ (dashed) and 3.19 × 10⁻⁵ (dotted). The envelope determined for Mω₁ = 3.16 × 10⁻⁵ is displayed by thin solid curves. The upper horizontal axis gives the binary separation in units of M; the lower horizontal axis gives the corresponding orbital frequency Mωᵣ.

This figure illustrates two ambiguities in predicting χᵢ: (i) the initial frequency ωᵢ at which the BBH parameters are specified, and (ii) the final separation rᵢ at which the given formula for χᵢ should be applied. Uncertainty in the separation at which the binary decouples from exter-
nal interactions could lead to ambiguity in ωᵢ, in theoretical studies, while uncertainty in the observed distance, projected separation, or line-of-sight velocity could lead to uncertainty in ωᵢ for models of particular systems. Gauge-dependent definitions of rᵢ could lead to uncer-
tainty in the separation at which fitting formulae should be applied. Our task in evaluating the resulting uncer-
tainties for the fitting formulae AEIn, AEIo, FAU, BKL

![Graphs showing the evolution of χᵢ and φᵢ](image-url)
and Kes introduced in Sec. [V] is somewhat simplified because both ambiguities are rooted in the rapid variations of the phase and in the resulting oscillations in the final quantities. These precession-induced oscillations are a clear manifestation of the hierarchy of time scales introduced in Eq. (3.1): $t_p \ll t_{GW}$.

In the upper panels of Fig. 15 we show the angle $\vartheta_i(r)$ for the same binary configuration illustrated in Fig. 14. In the lower panel of Fig. 15 we consider instead, for comparison, a system with lower mass ratio $q = 9/11$ and initial spin orientation $\chi_1 = \chi_2 = 1$, $\theta_1 = 120^\circ$, $\theta_2 = 60^\circ$ and $\Delta \phi = 288^\circ$. The initial frequency is $M\omega_i = 3.16 \times 10^{-5}$ (solid), $3.17 \times 10^{-5}$ (long-dashed), $3.18 \times 10^{-5}$ (dashed) and $3.19 \times 10^{-5}$ (dotted curve). The upper horizontal axis gives the binary separation in units of $M$; the lower horizontal axis gives the corresponding orbital frequency $M\omega$.

In the remainder of this Section, we discuss the uncertainties caused by the rapid spin precession of the following quantities:

1) $\chi_f(r_f)$: The magnitude of the final spin as predicted by applying a given fitting formula at small binary separation $r_f = 10M$, i.e. shortly before merger.

2) $\vartheta_i(r_f)$: The angle between the orbital angular momentum at $r_f = 10M$ and the final spin (as predicted using the binary parameters at $r_f = 10M$).

3) $\theta_i(r_f)$: The angle between the orbital angular momentum of the binary at large separation and the final spin (as predicted using the binary parameters at $r_f = 10M$).

4) $\vartheta_i(r_i)$: The angle between the orbital angular momentum of the binary at large separation and the

FIG. 15: Predicted spin direction $\vartheta_i(r)$ from the AEIn (left panels) and the Kes (right panels) formula. The upper panel shows the evolution of a binary starting with initial parameters $q = 9/11$, $\chi_1 = \chi_2 = 1$, $\theta_1 = 120^\circ$, $\theta_2 = 60^\circ$ and $\Delta \phi = 288^\circ$, as in Fig. 14. For comparison, in the bottom panels we consider a mass ratio $q = 1/3$ and initial spin parameters $\chi_1 = \chi_2 = 1$, $\theta_1 = 154^\circ$, $\theta_2 = 124^\circ$ and $\Delta \phi = 216^\circ$. The initial frequency is $M\omega_i = 3.16 \times 10^{-5}$ (solid), $3.17 \times 10^{-5}$ (long-dashed), $3.18 \times 10^{-5}$ (dashed) and $3.19 \times 10^{-5}$ (dotted curve).
final spin predicted using the binary parameters at this same large separation. We investigate the claim that the AEIn formula, unlike the others, can determine this angle without evolving the BBH parameters down to rf.

These quantities are important for modeling the assembly of supermassive black holes in the context of cosmological structure formation (see e.g. [24, 52, 56]). They are also relevant for electromagnetic counterparts of gravitational-wave sources [57], especially when the invoked mechanism producing the counterparts depends on the recoil velocity of the remnant black hole [58, 60].

We determine the precession-induced uncertainties as follows. Individual evolutions, such as those considered in Fig. [14] suggest that the width of the envelopes or the dispersion induced by varying the initial frequency provide very similar estimates for the uncertainty in χf(rf) and θf(rf). We have verified this conjecture by evolving the evenly spaced 10 × 10 × 10 grid of initially isotropic, maximally spinning BBH configurations introduced in Sec. [IV] for mass ratio q = 2/3 and several slightly different initial frequencies. When we estimate uncertainties by varying Mωi from 3.16 × 10⁻⁵ to 3.22 × 10⁻⁵ in steps of 0.005 × 10⁻⁵ we obtain the red dashed histograms in Fig. [14]. These histograms are in good agreement with the black solid histograms, where the uncertainty was estimated from the width of the envelopes. In order to reduce computational cost, in the remainder of this Section we determine the uncertainties Δχf(rf) and Δθf(rf) by evolving an ensemble of binaries from a single initial frequency (Mωi = 3.16 × 10⁻⁵) and using the envelope method.

Fig. [16] shows that the envelope method does not adequately describe the uncertainty in θi(r). Why does this angle behave so differently from θf(r) as illustrated in Fig. [14]? The direction of ħN(rf) is fixed, while according to the AEIn formula ħf(r) points in the direction of ħ(r). The total angular momentum ħ only varies on the radiation timescale tGW, so according to Eq. [6.2] the AEIn prediction of θi(r) should only vary on this slower timescale as well. The left panels of Fig. [15] at least at small orbital frequencies Mω where tρ < tGW, indeed lack the high-frequency oscillations characteristic of spin precession. In contrast, the Kesden predictions for θi(r) shown in the right panels of Fig. [15] are varying in a more complicated way than the predicted values of θf(r).

Changes in the angle θi(r) between the fixed ħN(rf) and varying ħf(r) reflect the full complexity of spin precession for misaligned, unequal-mass BBHs. The simpler variation in θi(r) occurs because both ħN(r) and ħf(r) are jointly precessing about ħ(r), albeit on the same short timescale tρ.

Since the envelope method fails for θi(r), we somewhat arbitrarily define the uncertainty Δθi(rf) as the maximum deviation of θi(r) from θi(rf = 10rf) in the window rf < r < 2rf. This window covers approximately the range of initial separations within the reach of present and near-future numerical relativity simulations, while smaller separations must be excluded due to the breakdown of the PN expansion. The formulae other than AEIn do not claim to predict ħf from the BBH parameters at large separations. To apply these formulae correctly, one must evolve the BBH parameters inwards to rf according to PN equations such as those in Sec. [II] before applying the formulae. This evolution requires significant additional effort, but if performed properly would only increase Δθf(rf) above Δθi(rf) by the uncertainty in the PN equations themselves. The uncertainty coming from PN evolutions could be quantified by comparing

![Fig. 16: Uncertainties in the final spin magnitude χf(rf) (left) and direction θf(rf) (right) for extremal BBHs with mass ratio q = 2/3. Solid black histograms were obtained by starting the evolutions at Mωi = 3.16 × 10⁻⁵ and using the envelope method of Fig. [14]. Dashed red histograms were obtained by considering the maximum variation in the final quantities as we let Mωi vary from 3.16 × 10⁻⁵ to 3.22 × 10⁻⁵ in steps of 0.005 × 10⁻⁵.](image-url)
different PN orders and pushing the calculation of spin contributions to higher order; such an analysis is beyond the scope of this paper. The AEIn formula is special in that it predicts \( \theta_i(r_i) \) without this additional PN evolution. Since AEIn claims that both \( \dot{\Omega}_N(r_i) \) and \( \dot{\chi}_f(r) \) are independent of \( r \), the uncertainty \( \Delta \theta_i(r_i) \) for this formula is the maximum deviation from \( \theta_i(r_f) = 10 M \) over the entire interval \( r_f < r < r_i \). Since the orbital angular momentum \( L_N \) increasingly dominates over spin contributions in the sum \( J = L_N + S_1 + S_2 \) at large separations, \( L_N \) has little opportunity to precess at large separations and the uncertainty \( \theta_i(r_i) \) asymptotes to a constant value in this limit.

We have evolved the uniform \( 10 \times 10 \times 10 \) grid of maximally spinning binaries introduced in Sec. 11 for three different mass ratios: \( q = 9/11, q = 2/3 \) and \( q = 1/3 \). The average uncertainties (plus or minus their associated standard deviations) are summarized in Table 11.

Errors in the final spin magnitudes due to the rapid spin precession are in the range \( \Delta \theta_i \lesssim 0.03 \) for all mass ratios. The FAU formula performs exceptionally well for nearly equal masses, although it deteriorates to the level of the other predictions for \( q = 1/3 \). We suspect that this is because several of the higher-order terms in \( \eta \) in the FAU formula are symmetric in the dimensionless spins \( \chi_1, \chi_2 \), while physically one would expect the spin of the more massive black hole to be more important in the limit \( q \to 0 \). Overall however, all formulae are able to predict the spin magnitude with rather good accuracy.

The uncertainty \( \Delta \theta_i(r_f) \) in the angle between the final spin and the orbital angular momentum shortly before merger is typically in the range of a few to 20 degrees. Investigation of the angular dependence of the spin uncertainties shows that the AEIn formula tends to behave better for initially aligned spins (small \( \theta_1 \) and \( \theta_2 \)) and worse for anti-aligned cases. This is likely a consequence of anti-aligned binaries being closer to the limit \( L(r) \approx -S(r) \) where transitional precession \( 51 \) occurs, violating assumptions (iii) and (iv) of Ref. 30.

All formulae are able to predict the angle \( \theta_i(r_f) \) between the initial orbital angular momentum and the final spin with decent accuracy. The AEIn predictions are overall more accurate, but investigation of the angular dependence reveals that this accuracy deteriorates (as expected) when \( q = 1/3 \) and the spin of the larger black hole is nearly anti-aligned. In this limit the uncertainties increase up to \( \sim 20^\circ \). This is again a consequence of those configurations approaching the transitional precession regime, where \( L(r) \approx -S(r) \).

The AEIn prediction is unique in that it claims to predict \( \theta_i(r_i) \) using the binary parameters at large separation without PN evolution. Our findings confirm (quite remarkably) that the majority of binaries in an initially isotropic ensemble result in a final spin which is nearly aligned with the orbital angular momentum at large binary separation. The values of \( \theta_f \) shown in Figs. 10 and 11 suggest that this would not be the case for BBHs initially anti-aligned with \( L_N \). The accuracy of the AEIn predictions also decreases for unequal masses (as expected and verified by our results for \( q = 1/3 \)). More extreme mass ratios are expected to play a significant and possibly dominate role in the coalescence of SBH binaries 61, 62, so it will be crucial to test the robustness of the Barausse-Rezzolla predictions for \( q = 1/10 \) and beyond. Accurate PN evolutions are more difficult in this regime, and we plan to investigate more extreme mass ratios in the future.

VII. DISCUSSION

In this paper, we examined how precession affects the distribution of spin orientations as BBHs inspiral from initial separations \( r_i \approx 1000 M \) where gravitational radiation begins to dominate the dynamics, all the way down to separations \( r_f \approx 10 M \) where numerical-relativity simulations typically begin.

We confirmed previous findings that isotropic spin distributions at \( r_i \approx 1000 M \) remain isotropic at \( r_f \approx 10 M \) 23, 24, 27. However, torques exerted by circumbinary disks may partially align BBH spins with the orbital angular momentum at separations \( r > r_i \) before gravitational radiation drives the inspiral 23. Recent simulations suggest that the residual misalignment of the BBH spins with their accretion disk could typically be \( \sim 10^\circ(30^\circ) \) for cold (hot) accretion disks, respectively 17. Partially motivated by these findings, we carried out a more careful analysis of spin distributions that are partially aligned with the orbital angular momentum at \( r = r_i \). We found that spin precession efficiently aligns the BBH spins with each other when the spin of the more massive black hole is initially partially aligned with the orbital angular momentum, increasing the final spin. We found the opposite trend when the spin of the more massive black hole is initially anti-aligned with the orbital angular momentum. Long evolutions are necessary to capture the full magnitude of the spin alignment. This could explain why these trends were not observed in the PN evolutions by Lousto et al. 27, which began at a fiducial binary separation \( r = 50 M \).

Some models of BBH evolution (see e.g. 61, 62) suggest that SBH mergers might have comparable mass ratios \( q \lesssim 1 \) at high redshift and more extreme mass ratios at low redshift. Since spin alignment is stronger for comparable-mass binaries, more alignment might be expected in SBH binaries at high redshifts. Observational arguments (see e.g. 64) and magnetohydrodynamic simulations of accretion disks 65 provide some evidence that black hole spins are related to the radio loudness of quasars. If so, the inefficient alignment (and consequently smaller spins) produced by unequal-mass mergers at low redshift would at least be consistent with recent observational claims that the mean radiative efficiency of quasars decreases at low redshift 66, 67. Stellar-mass black hole binaries should also have comparable mass ratios, so significant spin alignment could occur in such
We also pointed out that predictions of the final spin $\chi_f$ usually suffer from two sources of uncertainty: (i) the uncertainty in the initial frequency $\omega_i$ at which the BBH parameters are specified, and (ii) the uncertainty in the final separation $r_f$ at which the given formula for $\chi_f$ should be applied. Both ambiguities are rooted in the rapid precessional modulation of the orbital parameters, which in turn results from the precessional timescale $t_p$ being much shorter than the radiation timescale $t_{GW}$. Spin precession induces an intrinsic inaccuracy $\Delta \chi_f \lesssim 0.03$ in the dimensionless spin magnitude and $\Delta \theta_f \lesssim 2^\circ$ in the final spin direction.

The spin-orbit resonances studied in this paper should have significant effects on the distribution of gravitational recoil velocities resulting from BBH mergers, because the maximum recoil velocity has a strong dependence on spin alignment $\oplus \ominus \Theta$. We plan to extend this study to investigate the predictions of different formulae for the recoil velocities that have been proposed in the literature.

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TABLE I: Mean and standard deviation of the final spin magnitudes predicted for different sets of maximally spinning BBH mergers. The first column lists the formulae used to predict the final spins, as described in Sec. [V]. The second column gives the mass ratio \( q \). Each set of BBHs begins at \( r_i = 1000M \) with the indicated value of \( \theta_1 \) and flat distributions of \( \cos \theta_2 \) and \( \Delta \phi \). The third, fourth, and fifth columns show the mean and deviation expected if the BBH spins do not precess, thus maintaining their initial distributions at \( r_i = 1000M \) until merger. The sixth, seventh, and eighth columns assume that the spins precess according to the PN equations of Sec. [I] as they inspiral to \( r_f = 10M \), at which separation we apply the spin formulae.

| model | \( q \) | \( \theta_1 = 10^\circ \) | \( \theta_1 = 20^\circ \) | \( \theta_1 = 30^\circ \) | \( \theta_1 = 10^\circ \) | \( \theta_1 = 20^\circ \) | \( \theta_1 = 30^\circ \) |
|-------|--------|----------------|----------------|----------------|----------------|----------------|----------------|
| AEIn  | 9/11   | 0.871 ± 0.046 | 0.863 ± 0.065 | 0.857 ± 0.066 | 0.914 ± 0.034 | 0.905 ± 0.036 | 0.892 ± 0.038 |
| AEIg  | 9/11   | 0.866 ± 0.063 | 0.863 ± 0.064 | 0.856 ± 0.065 | 0.912 ± 0.034 | 0.904 ± 0.036 | 0.891 ± 0.038 |
| FAU   | 9/11   | 0.873 ± 0.059 | 0.868 ± 0.060 | 0.861 ± 0.061 | 0.909 ± 0.035 | 0.901 ± 0.037 | 0.888 ± 0.039 |
| BKL   | 9/11   | 0.862 ± 0.067 | 0.858 ± 0.068 | 0.851 ± 0.070 | 0.905 ± 0.037 | 0.898 ± 0.039 | 0.884 ± 0.042 |
| Kes   | 9/11   | 0.901 ± 0.072 | 0.896 ± 0.073 | 0.889 ± 0.075 | 0.950 ± 0.038 | 0.941 ± 0.041 | 0.927 ± 0.044 |

| model | \( q \) | \( \theta_1 = 150^\circ \) | \( \theta_1 = 160^\circ \) | \( \theta_1 = 170^\circ \) | \( \theta_1 = 150^\circ \) | \( \theta_1 = 160^\circ \) | \( \theta_1 = 170^\circ \) |
|-------|--------|----------------|----------------|----------------|----------------|----------------|----------------|
| AEIn  | 9/11   | 0.551 ± 0.080 | 0.527 ± 0.080 | 0.511 ± 0.079 | 0.535 ± 0.072 | 0.510 ± 0.076 | 0.493 ± 0.080 |
| AEIg  | 9/11   | 0.551 ± 0.080 | 0.527 ± 0.080 | 0.512 ± 0.079 | 0.535 ± 0.072 | 0.510 ± 0.077 | 0.493 ± 0.080 |
| FAU   | 9/11   | 0.542 ± 0.076 | 0.520 ± 0.076 | 0.506 ± 0.076 | 0.530 ± 0.070 | 0.507 ± 0.074 | 0.492 ± 0.076 |
| BKL   | 9/11   | 0.514 ± 0.088 | 0.488 ± 0.087 | 0.471 ± 0.086 | 0.496 ± 0.078 | 0.468 ± 0.083 | 0.449 ± 0.087 |
| Kes   | 9/11   | 0.531 ± 0.091 | 0.504 ± 0.090 | 0.486 ± 0.089 | 0.512 ± 0.081 | 0.483 ± 0.087 | 0.463 ± 0.091 |

| model | \( q \) | \( \Delta \chi_f(r = 10M) \) | \( \Delta \theta_f(r = 10M) \) | \( \Delta \phi_f(r = 10M) \) | \( \Delta \chi_f(r = 1000M) \) | \( \Delta \theta_f(r = 1000M) \) |
|-------|--------|----------------|----------------|----------------|----------------|----------------|
| AEIn  | 9/11   | 0.0159 ± 0.0009 | 8.38 ± 5.30 | 1.47 ± 1.09 | 1.48 ± 1.10 | – |
| AEIg  | 9/11   | 0.0155 ± 0.0098 | 11.38 ± 6.18 | 6.55 ± 2.73 | – | – |
| FAU   | 9/11   | 0.0021 ± 0.0035 | 8.51 ± 4.75 | 3.67 ± 1.68 | – | – |
| BKL   | 9/11   | 0.0153 ± 0.0094 | 11.74 ± 6.39 | 6.89 ± 2.92 | – | – |
| Kes   | 9/11   | 0.0174 ± 0.0105 | 11.99 ± 6.51 | 7.04 ± 2.96 | – | – |

| model | \( q \) | \( \Delta \chi_f(r = 10M) \) | \( \Delta \theta_f(r = 10M) \) | \( \Delta \phi_f(r = 10M) \) | \( \Delta \chi_f(r = 1000M) \) | \( \Delta \theta_f(r = 1000M) \) |
|-------|--------|----------------|----------------|----------------|----------------|----------------|
| AEIn  | 2/3    | 0.0205 ± 0.0127 | 11.96 ± 6.17 | 1.81 ± 1.21 | 1.83 ± 1.24 | – |
| AEIg  | 2/3    | 0.0199 ± 0.0124 | 14.10 ± 6.99 | 7.37 ± 2.80 | – | – |
| FAU   | 2/3    | 0.0034 ± 0.0026 | 10.66 ± 5.61 | 4.41 ± 1.76 | – | – |
| BKL   | 2/3    | 0.0191 ± 0.0108 | 14.52 ± 7.05 | 7.79 ± 2.98 | – | – |
| Kes   | 2/3    | 0.0217 ± 0.0124 | 14.83 ± 7.24 | 8.02 ± 2.99 | – | – |

| model | \( q \) | \( \Delta \chi_f(r = 10M) \) | \( \Delta \theta_f(r = 10M) \) | \( \Delta \phi_f(r = 10M) \) | \( \Delta \chi_f(r = 1000M) \) | \( \Delta \theta_f(r = 1000M) \) |
|-------|--------|----------------|----------------|----------------|----------------|----------------|
| AEIn  | 1/3    | 0.0165 ± 0.0109 | 8.58 ± 4.17 | 3.96 ± 4.46 | 4.25 ± 5.16 | – |
| AEIg  | 1/3    | 0.0156 ± 0.0101 | 9.57 ± 4.62 | 10.45 ± 4.12 | – | – |
| FAU   | 1/3    | 0.0177 ± 0.0090 | 7.80 ± 3.84 | 6.60 ± 2.75 | – | – |
| BKL   | 1/3    | 0.0148 ± 0.0089 | 9.81 ± 4.79 | 11.24 ± 4.45 | – | – |
| Kes   | 1/3    | 0.0167 ± 0.0105 | 10.01 ± 5.06 | 11.49 ± 4.48 | – | – |

TABLE II: Uncertainty distributions in \( \chi_f \) and in the various angles describing the final spin directions, as predicted by the formulae listed in Sec. [V]. The uncertainties and their standard deviations are obtained by evolving uniform 10 × 10 × 10 grids of maximally spinning BBHs with mass ratio \( q = 9/11, 2/3 \) and 1/3, respectively.