Lifting of Multicuts

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Abstract
For every simple, undirected graph $G = (V, E)$, a one-to-one relation exists between the decompositions and the multicuts of $G$. A decomposition of $G$ is a partition $\Pi$ of $V$ such that, for every $U \in \Pi$, the subgraph $(U, E \cap (U \cup (V \setminus U)))$ of $G$ is connected. A multicut of $G$ is a subset of edges, $M \subseteq E$, such that, for every (chordless) cycle $C \subseteq E$ of $G$, $|M \cap C| \neq 1$. The characteristic function $x \in \{0, 1\}^E$ of a multicut $M = x^{-1}(1)$ of $G$ makes explicit, for every $\{v, w\} \in E$, whether $v$ and $w$ are in distinct components. In order to make explicit, for every $\{v, w\} \in E'$ with $E \subseteq E' \subseteq \binom{V}{2}$, whether $v$ and $w$ are in distinct components, we define a lifting of the multicuts of $G$ to multicuts of $G' = (V, E')$. We show that, if $G$ is connected, the convex hull, in $R^{E'}$, of the characteristic functions of those multicuts of $G'$ that are lifted from $G$ is a full, $|E'|$-dimensional 01-polytope.

Figure 1 A decomposition of a graph (Def. 1) is a partition of the node set into connected subsets (depicted in green). Any decomposition is characterized by the set of those edges (depicted as dotted lines) that straddle distinct components. Such sets of edges are precisely the multicuts of the graph (Def. 2), by Lemma 1.

1 Decompositions and Multicuts

This work is concerned with decompositions of graphs.

Definition 1 For any graph $G = (V, E)$, a partition $\Pi$ of $V$ is called a decomposition of $G$ iff, for every $U \in \Pi$, the graph $(U, E \cap (U \cup (V \setminus U)))$ is connected.

The term decomposition is meaningful because every element of any decomposition of a graph $G$ is a component (i.e., a connected subgraph) of $G$. Useful in the study of decompositions are the multicuts of a graph:

Definition 2 [1] For any graph $G = (V, E)$, an $M \subseteq E$ is called a multicut of $G$ iff, for every cycle $C \subseteq E$ of $G$, $|C \cap M| \neq 1$.

One reason why multicuts are useful in the study of decompositions is that, for any graph $G$, a one-to-one relation exists between the decompositions and the multicuts of $G$ (Lemma 1). See also Fig. 1. Hence, the study of decompositions is the study of multicuts.

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Figure 2 For any connected graph $G$ (left), the characteristic functions of all multicuts of $G$ (middle) span, as their convex hull in $R^E$, the multicut polytope of $G$ (right), a 01-polytope that is $|E|$-dimensional [1].

Lemma 1 For any graph $G = (V, E)$, the map $\phi_G : 2^V \rightarrow 2^E$ such that, for any $\Pi \subseteq 2^V$ and any $\{v, w\} \in E$,

$$\{v, w\} \in \phi_G(\Pi) \iff \forall U \in \Pi(v \notin U \lor w \notin U) \quad (1)$$

is a bijection between the decompositions and the multicuts of $G$.

A second reason why multicuts are useful in the study of decompositions is that, for every graph, the characteristic functions of its multicuts offer a 01-encoding of fixed length $|E|$ of the decompositions of the graph:

Lemma 2 [1] For any graph $G = (V, E)$ and any $x \in \{0, 1\}^E$, the set $x^{-1}(1)$ is a multicut of $G$ iff

$$\forall C \in \text{cycles}(G) \forall e \in C : \ x_e \leq \sum_{e' \in C \setminus e} x_{e'} \quad (2)$$

It is sufficient in (2) to consider only chordless cycles.

For any graph $G = (V, E)$, we denote by $X_G$ the set of all $x \in \{0, 1\}^E$ that satisfy (2).

1.1 Multicut Polytope

Definition 3 [1, 2] For any graph $G = (V, E)$, $\Xi_G := \text{conv} \ X_G$ is called the multicut polytope of the graph $G$.

An example is depicted in Fig. 2.
graph decompositions $\equiv$ multicuts
set partitions $\equiv$ equivalence relations

Table 1 The definition of a graph generalizes the definition of a set if we identify any set $V$ with the complete graph $K_V$ on itself. The definition of a decomposition of a graph (Def. 1) generalizes the definition of a partition of a set, specializing to the latter for complete graphs. The definition of a multicut of a graph (Def. 2) generalizes the definition of an equivalence relation on a set in the sense of Lemma 3. The bijection between the decompositions and the multicut of a graph (Lemma 1) specializes, for complete graphs, to the bijection between partitions and equivalence relations (by Lemma 3).

1.2 Complete Graphs
The decompositions of a complete graph $K_V := (V, (\binom{V}{2}))$ are precisely the partitions of the node set $V$ (by Def. 1).

The multicut of a complete graph $K_V$ are related one-to-one to the equivalence relations on $V$:

**Lemma 3** For any set $V$, the complete graph $K_V = (V, (\binom{V}{2}))$ and $E := (\binom{V}{2})$, the map $\psi : 2^E \rightarrow V \times V$ such that, for all $M \subseteq E$ and all $v, w \in V$,

$$(v, w) \in \psi(M) :\iff \{v, w\} \notin M$$

is a bijection between the multicut of $K_V$ and the equivalence relations on $V$.

Hence, the bijection between the decompositions and the multicut of a graph (Lemma 1) specializes, for complete graphs, to the bijection between partitions and equivalence relations (by Lemma 3). In this sense, decompositions and multicut of graphs generalize partitions of and equivalence relations on sets. See Tab. 1.

2 Lifting of Multicut

For any graph $G = (V, E)$, the characteristic function $x \in X_G \subseteq \{0, 1\}^E$ of a multicut $x^{-1}(1)$ of $G$ makes explicit, for every $\{v, w\} \in E$, whether $v$ and $w$ are in distinct components. In order to make explicit, for every $\{v, w\} \in E'$ with $E \subseteq E'$, whether $v$ and $w$ are in distinct components, we define a lifting of the multicut of $G$ to multicut of $G' = (V, E')$:

**Definition 4** For any graphs $G = (V, E)$ and $G' = (V, E')$ with $E \subseteq E'$, the map $\lambda_G : \phi_G \circ \phi_G^{-1}$ is called the lifting of multicut from $G$ to $G'$.

For any graphs $G = (V, E)$ and $G' = (V, E')$ with $E \subseteq E'$, we define $F_G' := E' \setminus E$.

**Lemma 4** For any graphs $G = (V, E)$ and $G' = (V, E')$ such that $E \subseteq E'$ and for any $x \in \{0, 1\}^{E'}$, the set $x^{-1}(1)$ is a multicut of $G'$ lifted from $G$ iff

$$\forall C \in \text{cycles}(G) \forall e \in C : x_e \leq \sum_{e' \in C \setminus \{e\}} x_{e'}$$

$$\forall uv \in F_{G'} \forall P \in uv\text{-paths}(G) : x_{uv} \leq \sum_{e \in P} x_e$$

$$\forall uv \in F_{G'} \forall C \in uv\text{-cuts}(G) : x_{uv} \leq \sum_{e \in C} (1 - x_e)$$

For any graphs $G = (V, E)$ and $G' = (V, E')$ such that $E \subseteq E'$ we denote by $X_{G'}$ the set of all $x \in \{0, 1\}^{E'}$ that satisfy (4)–(6).

2.1 Lifted Multicut Polytope

**Definition 5** For any graphs $G = (V, E)$ and $G' = (V, E')$, $\Xi_{G'} := \text{conv} X_{G'}$ is called the lifted multicut polytope with respect to $G$ and $G'$.

An example is depicted in Fig. 3. In this example, the lifted multicut polytope with respect to graphs $G$ and $G'$ (Fig. 3) is a proper subset of the multicut polytope of the graph $G'$ (Fig. 2). If only $G$ is connected, the lifted multicut polytope is full-dimensional:

**Theorem 1** For any connected graph $G = (V, E)$ and any graph $G' = (V, E')$ such that $E \subseteq E'$, $\dim \Xi_{G'} = |E'|$.

We prove Theorem 1 by constructing $|E'| + 1$ multicuts of $G'$ lifted from $G$ whose characteristic 01-vectors are affine independent. The strategy is to consider the edges $e \in E'$ in order and, for each edge, set $x_e = 0$ and as many other edge labels as possible to 1. The challenge is that edge labels are not independent. In particular, for $f \in F_{G'}$, $x_f = 0$ can imply, for certain $f' \in F_{G'} \setminus \{f\}$, that $x_{f'} = 0$, as illustrated in Fig. 4. This structure is made explicit below.

**Definition 6** For any connected graph $G = (V, E)$ and any graph $G' = (V, E')$ such that $E \subseteq E'$, the sequence $(F_n)_{n \in \mathbb{N}}$ of subsets of $F_{G'}$ defined below is called the hierarchy of $F_{G'}$ with respect to $G$:

(i) $F_0 = \emptyset$

(ii) For any $n \in \mathbb{N}$ and any $\{v, w\} = f \in F_{G'}$, $\{v, w\} \in F_n$ iff there exists a $vw$-path in $G$ such that, for any distinct nodes $v'$ and $w'$ in the path such that $\{v', w'\} \neq \{v, w\}$, either $\{v', w'\} \notin F_{G'}$ or there exists a natural number $j < n$ such that $\{v', w'\} \in F_j$.
For any connected graph \( G = (V,E) \), any graph \( G' = (V,E') \) with \( E \subseteq E' \) and any \( f \in F_{GG'} \), there exists an \( n \in \mathbb{N} \) such that \( f \in F_n \).

**Lemma 5** For any connected graph \( G = (V,E) \), any graph \( G' = (V,E') \) with \( E \subseteq E' \) and any \( f \in F_{GG'} \), there exists an \( n \in \mathbb{N} \) such that \( f \in F_n \).

**Definition 7** For any connected graph \( G = (V,E) \), any graph \( G' = (V,E') \) such that \( E \subseteq E' \), the map \( l : F_{GG'} \rightarrow \mathbb{N} \) such that \( \forall f \in F_{GG'} \forall n \in \mathbb{N} : l(f) = n \Leftrightarrow f \in F_n \wedge f \notin F_{n-1} \) is called the level function of \( F_{GG'} \).

**Lemma 6** For any connected graph \( G = (V,E) \), any graph \( G' = (V,E') \) with \( E \subseteq E' \) and for any \( f \in F_{GG'} \), there exists an \( x \in X_{GG'} \), called \( f \)-feasible, such that

(i) \( x_f = 0 \)

(ii) \( x_f = 1 \) for all \( f' \in F_{GG'} \setminus \{ f \} \) with \( l(f') \geq l(f) \).

3 Conclusion

For any simple, undirected graph \( G = (V,E) \), the decompositions of \( G \) are related one-to-one to the multicut of \( G \). For any \( G' = (V,E') \) with \( E \subseteq E' \), a multicut of \( G' \) lifted from \( G \) makes explicit, for all \( \{v,w\} \in E' \), whether \( v \) and \( w \) are in distinct components. If \( G \) is connected, the convex hull in \( E' \), of the characteristic functions of those multicut of \( G' \) that are lifted from \( G \) is a full, \( |E'| \)-dimensional 01-polytope.

**References**

[1] Sunil Chopra and M.R. Rao. The partition problem. Mathematical Programming, 59(1-3):87–115, 1993.

[2] Michel Marie Deza and Monique Laurent. Geometry of Cuts and Metrics. Springer, 1997.

**A Proofs**

**Proof of Lemma 5** Let \( \{v,w\} = f \in F_{GG'} \) and let \( d(v,w) \) be the length of a shortest \( vw \)-path in \( G \). Then, \( d(v,w) > 0 \) because \( F_{GG'} \cap E = \emptyset \).

If \( d(v,w) = 2 \), there exists a \( u \in V \) such that \( \{v,u\} \in E \) and \( \{u,w\} \in E \). Moreover, \( \{v,u\} \notin F_{GG'} \) and \( \{u,w\} \notin F_{GG'} \) as \( F_{GG'} \cap E = \emptyset \). Thus, \( f \in F_1 \).

If \( d(v,w) > 2 \), consider any shortest \( vw \)-path \( P \) in \( G \). Moreover, let \( F' \subseteq F_{GG'} \) such that, for any \( \{v',w'\} = f' \in F_{GG'} \), \( f' \in F' \) iff \( v' \in P \) and \( w' \in P \) and \( f' \neq f \). If \( F' = \emptyset \) then \( f \in F_1 \). Otherwise, \( \forall \{v',w'\} \in F' : d(v',w') < m \)

and thus,

\[ \forall f' \in F' \exists n_{f'} \in \mathbb{N} : f' \in F_{n_{f'}} \]

by induction (over \( m \)). Let

\[ n = \max_{f' \in F'} n_{f'} \ . \]

Then, \( f \in F_{n+1} \).

**Proof of Lemma 6** For any \( \{v,w\} = f \in F_{GG'} \), let \( P \) be a shortest \( vw \)-path in \( G \) and let

\[ F_{GG'}' := \{(v',w') \in F_{GG'} : v' \in P \wedge w' \in P \} \quad (7) \]

\[ F_{GG'} := F_{GG'} \setminus F_{GG'}' \quad (8) \]

Moreover, let \( x \in \{0,1\}^E \) with \( x_P = 0 \) and \( x_{E \setminus P} = 1 \) and \( x_{E_{GG'}} = 0 \) and \( x_{E_{GG'}'} = 1 \). \( P \) has no chord in \( E \) (because it is a shortest path). Thus, \( x \in X_{GG'} \).

**Proof of Theorem 1** \( x^* \in \{0,1\}^E \) such that \( x^* = 1 \) holds \( x^* \in X_{GG'} \).

For any \( e \in E \), \( x^e \in \{0,1\}^E \) such that \( x^e = 0 \) and \( x^e_{E \setminus \{e\}} = 1 \) and \( x^e_{E_{GG'}} = 1 \) holds \( x^e \in X_{GG'} \).

For any \( f \in F_{GG'} \), any \( f \)-feasible \( x^f \in \{0,1\}^E \) holds \( x^f \in X_{GG'} \). Moreover, \( x^f \) can be chosen such that one shortest path connecting the two nodes in \( f \) is the only component containing more than one node.

For any \( e \in E \), let \( y^e \in \mathbb{R}^E \) such that

\[ y^e = x^* - x^e \ . \]

For any \( f \in F_1 \), choose an \( f \)-feasible \( x^f \) and let \( y^f \in \mathbb{R}^E \) such that

\[ y^f = x^* - x^f - \sum_{\{e \in E|y^e = 0\}} y^e \ . \]

For any \( n \in \mathbb{N} \), such that \( n > 1 \) and any \( f \in F_n \), choose an \( f \)-feasible \( x^f \) and let \( y^f \in \mathbb{R}^E \) such that

\[ y^f = x^* - x^f - \sum_{\{f' \in F_{GG'}|f' \neq f \wedge x^f_{f'} = 0\}} y^f - \sum_{\{e \in E|y^e = 0\}} y^e \ . \]

Here, \( l(f') < l(f) \leq n \), by definition of \( f \)-feasibility. Thus, all \( y^f \) are well-defined by induction (over \( n \)).

Observe that \( \{y^e \mid e \in E' \} \) is the unit basis in \( \mathbb{R}^{E'} \).

Moreover, each of its elements is a linear combination of \( \{x^* - x^e \mid e \in E' \} \) which is therefore linearly independent.

Thus, \( \{x^*\} \cup \{x^e \mid e \in E' \} \) is affine independent. It is also a subset of \( X_{GG'} \) and, therefore, a subset of \( \Xi_{GG} \). Thus, \( \dim \Xi_{GG} = |E'| \).