Quark-quark interaction and quark matter in neutron stars

Y. Yamamoto$^1$, N. Yasutake$^2$, and Th. A. Rijken$^3$

$^1$RIKEN Nishina Center, 2-1 Hirosawa, Wako, Saitama 351-0198, Japan
$^2$Department of Physics, Chiba Institute of Technology, 2-1-1 Shibazono Narashino, Chiba 275-0025, Japan
$^3$IMAPP, Radboud University, Nijmegen, The Netherlands

Hyperon (Y) mixing in neutron-star matter brings about a remarkable softening of the equation of state (EoS) for dense nuclear matter. The observed masses of neutron stars J1614−2230 [1], J0348+0432 [2] and J0740+6620 [3] are given as (1.97 ± 0.04)M⊙, (2.01 ± 0.04)M⊙ and (2.17 ± 0.11)M⊙, respectively, being important conditions for the stiffness of the EoS of neutron-star matter. In non-relativistic approaches, the stiff EoS giving the maximum mass of 2M⊙ can be derived from the existence of strongly repulsive effects such as three-nucleon repulsions in the high-density region [4].

The hyperon (Y) mixing in neutron-star matter brings about a remarkable softening of the EoS and a maximum mass is reduced to a value far less than 2M⊙. The mechanism of EoS softening is understood as follows: With increasing of baryon density toward centers of neutron stars, chemical potentials of neutrons become high so that neutrons at Fermi surfaces are changed to hyperons (Y) via strangeness non-conserving weak interactions overcoming rest masses of hyperons. Then, it should be noted that naively such a mechanism of EoS softening works also for mixing of any exotic particles such as quarks into neutron matter.

One of the ideas to avoid this “hyperon puzzle in neutron stars” is to assume that the many-body repulsions work universally for every kind of baryons [5]. In Refs [6][7][8], the multi-pomeron exchange potential (MPP) was introduced as a model of universal repulsions among three and four baryons on the basis of the extended soft core (ESC) baryon-baryon interaction model developed by two of the authors (T.R. and Y.Y.) and M.M. Nagels [9].

Another solution for the hyperon puzzle has been suggested by taking into account quark deconfinement phase transitions from a hadronic-matter EoS (H-EoS) to a sufficiently stiff quark-matter EoS (Q-EoS) in the neutron-star interiors, namely by studying hybrid stars having quark matter in their cores [10][11][12][13][14][15][16][17][18][19]. It is well known that repulsive effects in quark phases are needed to result in massive neutron stars of 2M⊙. In the Nambu-Jona-Lasinio (NJL) model, for instance, repulsions to stiffen EoSs are given by vector interactions [20], strengths of which are treated as phenomenological parameters to stiffen the EoSs. Note that NJL models, including extended ones, are mainly based on mean-field approximations, in which two-body quark-quark interactions are not used explicitly. In spite of many works for hadron-quark phase transitions in neutron-star matter, there is not yet a unified theory of both the hadronic and quark phases.

In this work, our approach to hadron-quark phase transitions is different from the usual methods in which deconfined quark phases are treated in mean-field approximations. We handle here the quark matter with the two-body quark-quark (QQ) potentials derived as follows: The meson-exchange quark-quark potentials are derived from the ESC baryon-baryon (BB) potentials in the framework of the constituent quark model (CQM). The quark-quark-meson (QQM) vertices are defined such that, upon folding with the Gaussian ground-state baryonic quark wave functions, the BB potentials are reproduced [21]. In this process the QQM couplings are related to the BBM couplings, and the extra interactions at the quark level necessary to achieve this connection are completely determined. (Like in the ESC16 BB-potentials, relativistic effects are included in the QQ-
potentials via the small components of the Dirac-spinors and a $1/M_Q$-expansion.) The quark-quark instanton-exchange potential is derived from tuning the baryon masses $(N, \Lambda, \Sigma, \Xi)$, and $\Delta Q$ in the CQM. Here, also the one-gluon-exchange (OGE) and the confining potential are included.

With use of these $QQ$ potentials together with the ESC $BB$-potentials, baryonic matter and quark matter are treated in the common framework of the Brueckner-Bethe-Goldstone (BBG) theory, where the transitions between them are described in a reasonable way. It should be emphasized here that our $QQ$ potentials are determined on the basis of the terrestrial data and do not include parameters only for the purpose of stiffening the quark-matter EoS.

Recently, the radius measurement has been performed for the most massive neutron star PSR J0740+6620: The two analyses have been done independently for the X-ray data taken by the Neutron Star Interior Composition Explorer (NICER) and the X-ray Multi-Mirror (XMM-Newton) observatory. The radius and mass are $12.39^{+1.30}_{-0.98}$ km and $2.072^{+0.067}_{-0.064} M_\odot$ or $13.7^{+2.6}_{-1.5}$ km (68%) and $2.08 \pm 0.07 M_\odot$ [23]. The radius of a typical $1.4 M_\odot$ neutron star $R_{1.4 M_\odot}$ has been estimated by combining the NICER measurements and the other multimesseger data [24-25]. In Ref.[24], the two values of $R_{1.4 M_\odot} = 12.33^{+0.76}_{-0.81}$ km and $R_{1.4 M_\odot} = 12.18^{+0.56}_{-0.79}$ km are obtained for the two different high-density EoSs of a piecewise-polytropic (PP) model and a model based on the speed of sound, respectively. In Ref. [25], the estimated value is $R_{1.4 M_\odot} = 11.94^{+0.78}_{-0.87}$ km at 90% confidence. These values of radii are rather similar to each other. On the other hand, when the implication of PREX-II for the neutron skin thickness of heavy nuclei are taken into account on the neutron-star EoS, they obtain $13.33$ km < $R_{1.4 M_\odot}$ < $14.26$ km [26]. Our obtained EoSs in this work are investigated in the light of these new data.

This paper is organized as follows: In Sect.II, the hadronic-matter EoS (H-EoS) is recapitulated on the basis of our previous works. In Sect.III, on the basis of realistic $QQ$ interaction models, the BBG theory is applied to quark matter: In III-A, the G-matrix framework is outlined for quark matter. In III-B, our $QQ$ potentials are explained, which are composed of the extended meson-exchange potential, the multi-pomeron potential, the instanton potential and the one-gluon exchange potential. In III-C, the $QQ$ G-matrix interactions in coordinate space are parameterized as density-dependent interactions. In Sect.IV, there are obtained the quark-matter EoSs (Q-EoS) and $MR$ diagrams of hybrid stars: In IV-A, Q-EoSs are derived. In IV-B, hadron-quark phase transitions in hybrid stars are investigated on the basis of the obtained EoSs. In IV-C, the $MR$ relations of hybrid stars are obtained by solving the TOV equation. The conclusion of this paper is given in Sect.V.

II. HADRONIC-MATTER EOS

Here, the hadronic matter is defined exactly as $\beta$-stable baryonic matter including leptons. On the basis of the BBG theory, the hadronic-matter EoS (H-EoS) is derived with use of the ESC baryon-baryon interaction model [27-28]. Then, the EoS is stiff enough to assure the neutron-star masses of $2M_\odot$, if the strong three-nucleon repulsion is taken into account. However, the hyperon (Y) mixing results in remarkable softening of the EoS canceling this repulsive effect. In order to avoid this “hyperon puzzle”, it is assumed that the repulsions work universally for $YNN$, $YYN$ $YYY$ as well as for $NNN$. In [27-28], such universal repulsions are modeled as the multi-pomeron exchange potential (MPP). In Ref.[8] they proposed three versions of MPP: MPa, MPa$^+$, MPb. MPa and MPa$^+$ (MPb) include the three-and four-body (only three-body) MPPs, where mixings of $\Lambda$ and $\Sigma^-$ hyperons are taken into account. The three-body part of MPa (MPa$^+$) is less repulsive than (equal to) that of MPb, and the four-body parts of MPa and MPa$^+$ are equal to each other. The EoSs for MPa and MPa$^+$ are stiffer than that for MPb, because of which radii of neutron stars obtained from the formers are larger than those from the latter. Our ESC $BB$ interactions including MPb, MPa and MPa$^+$ are named as H1, H2, H3, for simplicity. In addition, we introduce two versions $H0$ and $H1'$ for comparative studies: H0 is the nucleon-nucleon part of H1, being used in nuclear-matter EoSs with no hyperons. H1' is the $BB$ interaction H1 in which MPP works only among nucleons. In the case of H1', the remarkable softening of the EoS is brought about by hyperon mixing.

As shown later, neutron-star radii $R$ for masses lower than about $1.5 M_\odot$ are determined by H-EoSs even in our $MR$ diagrams including hadron-quark transitions. For the H-EoSs derived from the above $BB$ interactions, the obtained values of radii at $1.4 M_\odot$ ($R_{1.4 M_\odot}$) are $12.4$ km (H1), $13.3$ km (H2) and $13.6$ km (H3).

III. QUARK-QUARK INTERACTION AND QUARK MATTER

A. G-matrix framework

The BBG theory is adopted for studies of quark matter on the basis of two-body $QQ$ potentials given in Ref.[21]. Here, correlations induced by $QQ$ potentials are renormalized into coordinate-space G-matrix interactions, being considered as effective $QQ$ interactions to derive the Q-EoS. In this stage to construct G-matrix interactions in quark matter, color quantum numbers are not taken into account.

We start from the G-matrix equation for the quark pair $f_1f_2$ in quark matter, where $f_1$ and $f_2$ denote flavor
quantum numbers \((u, d, s)\):

\[
G_{cc_0} = v_{cc_0} + \sum_{c'} v_{cc'} \frac{Q_{y'} - \epsilon_{f_1} - \epsilon_{f_2}}{\omega} \cdot G'_{c'c_0} \quad (3.1)
\]

where \(c\) denotes a relative state \((y, T, L, S, J)\) with \(y = f_1 f_2\), \(S\) and \(T\) being spin and isospin quantum numbers, respectively. Orbital and total angular momenta are denoted by \(L\) and \(J\), respectively, with \(J = L + S\). A two-quark state is specified by \(2S+1\) \(L\). In Eq. (3.1), \(\omega\) gives the starting energy in the starting channel \(c_0\). The Pauli operator \(Q_{y'}\) acts on intermediate quark states with \(y = f_1 f_2\). We adopt for simplicity the gap choice for the intermediate states in the G-matrix equation, meaning that an intermediate energy \(\epsilon_{f_1}\) is replaced by a kinetic energy operator. The G-matrix equation (3.1) is represented in the coordinate space, whose solutions give rise to G-matrix elements.

The quark single particle (s.p.) energy \(\epsilon_{f}(k_f)\) in quark matter is given by

\[
\epsilon_{f}(k_f) = \frac{\hbar^2 k_f^2}{2m_f} + U_f(k_f) \quad (3.2)
\]

where \(k_f\) is a \(f\)-quark momentum \((f = u, d, s)\). The potential energy \(U_f\) is obtained self-consistently in terms of the G-matrix as

\[
U_f(k_f) = \sum_{|k_f'|} (k_f k_{f'} | G_{f'f}(\omega = \epsilon_{f}(k_f) + \epsilon_{f'}(k_{f'})) | k_f k_{f'}) \quad (3.3)
\]

where \((T L S J)\) quantum numbers are implicit. Then, the potential energy per particle \(\langle U \rangle = \sum_f \langle U_f \rangle\) is obtained by averaging \(U_f(k_f)\) over \(f\):

\[
\langle U \rangle = \frac{3}{2} \sum_f \omega_f \int_0^{k_F^2} \frac{d^3 k_f}{(2\pi)^3} \cdot U_f(k_f) \quad (3.4)
\]

where \(\omega_f = \rho_f / (\sum_f \rho_f)\) with a \(f\)-quark density \(\rho_f\). Making a partial wave reduction of Eq. (3.3) with explicit use of \(T L S J\) quantum numbers, \(U_f(k_f)\) is represented as a sum of \(G_{f'f}^{T L S J}(k_f)\) obtained from G-matrix elements.

\[
G_{f'f}^{T L S J}(k_f) \quad (3.5)
\]

B. Quark-Quark interactions

Our QQ interaction is given by

\[
V_{QQ} = V_{EME} + V_{INS} + V_{OGE} + V_{MPP} \quad (3.6)
\]

where \(V_{EME}, V_{INS}, V_{OGE}\) and \(V_{MPP}\) are the extended meson-exchange potential, the instanton exchange potential, the one-gluon exchange potential and the multi-pomeron potential, respectively [21]. The included parameters in our QQ potential are chosen so as to be consistent with physical observables as much as possible. The contributions of the confining potential \(V_{conf}\) to \(V_{QQ}\) are minor in quark matter, being omitted in this work.

The \(V_{EME}\) QQ potential is derived from the ESC16 BB potential [4] so that the QQM couplings are related to the BBM couplings through folding procedures with Gaussian baryonic quark wave functions. Then, the \(V_{EME}\) QQ potential is basically of the same functional expression as the ESC16 BB potential. The explicit expressions for \(V_{EME}\) QQ potentials are given in Ref. [21]. In the ESC modeling, the strongly repulsive components in \(BB\) potentials are described mainly by vector-meson and pomeron exchanges between baryons. It should be noted that this feature persists in the \(V_{EME}\) QQ potential, which includes the strongly repulsive components originated from vector-meson and pomeron exchanges between quarks.

Multi-pomeron exchanges are expected to work not only among baryons but also among quarks, in which the baryon mass \(M_B\) is replaced by the quark mass \(M_Q = M_B/3\) and the pomeron-baryon-baryon coupling constant \(g_{ppB}\) is replaced by the pomeron-quark-quark coupling constant \(g_{pQQ}\). In this work, the QQ multi-pomeron potential \(V_{MPP}\) is derived from the version MPa for the MPP among baryons.

The included parameters included in \(V_{INS}\) and \(V_{OGE}\) are chosen so as to reproduce basic features of baryon mass spectra. The form of the one-gluon exchange potential is given as

\[
V_{OGE}(r) = \frac{1}{4} (\lambda_1^C \cdot \lambda_2^C) \sigma S V_{vector}(m_G; r) \quad (3.7)
\]

where \(\lambda_a^C\), \(a = 1,..,8\) are the Gell-Mann matrices in color SU(3) space and \(V_{vector}(m_G; r)\) is the vector-type one boson exchange potential. Its explicit form is given by Eq.(E9a) in Ref. [21]. The strength of \(V_{OGE}\) is determined by the quark-gluon coupling constant \(\sigma S\), being fixed as \(\sigma S = 0.25\) in this work. The gluon mass \(m_G\) is taken as 420 MeV [21]. In quark matter, \((\lambda_1^C \cdot \lambda_2^C) = -8/3, +4/3, +4/3, -8/3\) in states of \((S, T) = (0,0), (0,1), (1,0), (1,1)\), respectively.

The instanton potential \(V_{INS}\) is based on the SU(3) generalization of the ’t Hooft interaction for \((u, d, s)\) quarks. In the configuration space, with the addition of the Gaussian cut-off \(\exp(-k^2 / \Lambda_1^2)\), the local instanton potential is given as [21]

\[
V_{INS}(r) = -4(3 - \lambda_1^F \cdot \lambda_2^F) G_I \left( \frac{\Lambda_I}{2\sqrt{\pi}} \right)^3 \times \left[ 1 + \frac{\Lambda_1^2}{2\Lambda_Q^2} \left( 3 - \frac{1}{2} \Lambda_2^2 \right)^2 \right] \exp\left( -\frac{1}{4} \Lambda_3^2 \right) \quad (3.8)
\]

where \(\lambda_a^F\), \(a = 1,..,8\) are the Gell-Mann matrices in flavor SU(3) space and \(m_Q\) is the quark mass. In two-quark states, \(X_F\) operators are reduced to \(\tau\) operators
of isospin. The strength of $V_{INS}$ is determined by coupling constant $G_I$ and cut-off mass $\Lambda_I$. They are taken as $G_I = 2.5$ GeV$^{-2}$ and $\Lambda_I = 0.55$ GeV, being estimated from the $\pi - \rho$ mass splitting.

### FIG. 1: Averaged single particle potentials \( \langle U \rangle \) in quark matter as a function of the baryon number density \( \rho_B \) in the case of \( \rho_u = \rho_d = \rho_s \). The solid, short-dashed, long-dashed and dot-dashed curves are the contributions to \( \langle U \rangle \) from $V_{EME}$, $V_{MPP}$, $V_{OGE}$ and $V_{INS}$, respectively. The bold-solid curve is obtained from $V_{EME}$, $V_{MPP}$, $V_{OGE}$ and $V_{INS}$, respectively. The bold-solid curve is obtained by the sum of $V_{EME}$, $V_{MPP}$, $V_{INS}$, and $V_{OGE}$. The strongly-repulsive nature of \( \langle U(\rho_B) \rangle \) is the key point in this work, which leads to the quark-matter EoS stiff enough to reproduce neutron-star masses over 2$M_\odot$. In the figure, the repulsive contribution of $V_{EME}$ is found to be essential the repulsive nature of \( \langle U(\rho_B) \rangle \), where the repulsive contributions of $V_{OGE}$ and $V_{MPP}$ are considerably canceled by the attractive contribution of $V_{INS}$. It is worthwhile to say that the repulsive components in $V_{EME}$ are from the vector-meson and pomeran exchange. This feature persists from the ESC BB interaction model.

In neutron matter, \( \langle U \rangle \) includes repulsive contributions from the multi-pomeron potential MPP, being quite large in high density regions. The strengths of three- and four-body parts of MPP are proportional to $(g_{PB})^3$ and $(g_{PB})^4$, respectively, $g_{PB}$ being the pomeran-baryon-baryon coupling constant. In $Q\bar{Q}$ potentials, $g_{BB}$ is replaced by the pomeran-quark-quark coupling constant $g_{QPQ}$. Because of the relation $g_{QQ} = \frac{1}{2}g_{PB}$, the strengths of three- and four-body parts of MPP among quarks are far smaller than those among baryons. Therefore, MPPs among quarks are not so remarkable in comparison with those among baryons.

The even- and odd-state contributions to \( \langle U \rangle \) are denoted as \( \langle U_{\text{even}} \rangle \) and \( \langle U_{\text{odd}} \rangle \), respectively. In Fig. 2, the solid curve shows \( \langle U(\rho_B) \rangle \) obtained from $V_{EME}$, and \( \langle U_{\text{even}}(\rho_B) \rangle \) and \( \langle U_{\text{odd}}(\rho_B) \rangle \) are given by the dashed and short-dashed curves. The dotted curves are the even- and odd-state contributions of averaged neutron potentials \( \langle U_{\text{even}} \rangle \) and \( \langle U_{\text{odd}} \rangle \) in neutron matter. The remarkable feature of \( \langle U_{\text{even}} \rangle \) and \( \langle U_{\text{odd}} \rangle \) is that they are attractive and repulsive, respectively.

When our $Q\bar{Q}$ potentials are used in quark-matter calculations, it is reasonable to assume the constituent quark masses originated from the chiral symmetry breaking as the QCD non-perturbative effect. Then, it is probable that the constituent quark masses in quark matter become smaller than those in vacuum and move to current masses in the high-density limit. At the mean-field (MF) level usually, the density-dependent quark masses in matter have been derived from the MF-Lagrangian such as that of the NJL model. In the present approach, we introduce phenomenologically the density-dependent quark mass

$$M_Q(\rho_B) = M_0/\left[1 + \exp\{\gamma(\rho_Q - \rho_c)\}\right] + m_0 + C \quad (3.8)$$

with $C = M_0 - M_0[1 + \exp(-\gamma\rho_c)]$ assuring $M_Q(0) = M_0 + m_0$, where $\rho_Q$ is number density of quark matter, and $M_0$ and $m_0$ are taken as 300 (360) MeV and 5 (140) MeV for $u$ and $d$ (s) quarks. Then, we have $M_Q(0) = 305$ (500) MeV for $u$ and $d$ (s) quarks. The adjustable parameters $\rho_c$ and $\gamma$ are used to control mainly the onset densities of quark phases into hadronic phases.
Furthermore, because the quark mass reduction has to bring about an increase of the vacuum energy $B$, we assume simply

$$B(\rho_Q) = M_Q^2(0) - M_Q(\rho_Q). \quad (3.9)$$

It is well known that there are three schemes for the density-dependent quark mass \[^{28}\]: (i) a constant quark mass, (ii) a linear density dependence (Brown-Rho scaling \[^{29}\]), (iii) a density-dependence within a higher-order NJL model \[^{30} \textit{et al.}\]. Eq. (3.9) includes these schemes, representing (i) for $\gamma = 0$, (ii) for small values of $\gamma$ and (iii) for large values of $\gamma$. The parameter $\rho_B$ is chosen as $6\rho_0$ by referring to forms of (iii) derived from the higher-order NJL models.

We define the following five sets with different values of $\gamma$ of $QQ$ interactions for deriving $Q$-EoSs.

- **Q0**: $V_{EME}$ with $\gamma=1.2$
- **Q1** ($Q1e$): $V_{EME} + V_{INS} + V_{OGE}$ with $\gamma=1.0 \ (\gamma=2.6)$
- **Q2** ($Q2e$): $V_{EME} + V_{MPP} + V_{INS} + V_{OGE}$ with $\gamma=1.6 \ (\gamma=2.2)$

In the cases of Q0, Q1 and Q2, the values of $\gamma$ are chosen so that the critical chemical potentials and densities for phase transitions are as small as possible. In the cases of Q1e and Q2e, they are chosen so that critical densities are near crossing points of hadronic and quark energy densities.

In Fig. 3 the quark mass $M^2_Q(\rho_Q)$ $(Q = u, d)$ as a function of the baryon number density $\rho_B = \rho_Q/3$ is plotted in the cases of (a) Q1, (b) Q1e, (c) Q2 and (d) Q2e. The density-dependent quark masses in these cases (especially Q1e and Q2e) are found to be rather close to (ii) with the Brown-Rho scaling.

![FIG. 3: Quark mass as a function of the baryon number density $\rho_B$ for the (a) Q1, (b) Q1e, (c) Q2 and (d) Q2e models. As a reference, also the Brown-Rho scaling is shown by the dashed line.](image)

It is quite important to use the density-dependent quark masses together with our $QQ$ potentials. When constant quark masses are used, hadron-quark transitions derived from our $QQ$ potentials occur in density regions over $5\rho_0$ in hadronic matter giving $2M_\odot$ masses. In such a case, quark phases have no effect on masses and radii of neutron stars, even if they exist in inner cores.

For instance, when the baryon-baryon interaction H2 is used together with quark-quark interaction Q2, the combined set is denoted as H2+Q2. Hereafter, combinations of BB and QQ interactions are expressed like this.

### C. Effective Quark-Quark interactions

For applications to quark-matter calculations, we construct density-dependent effective local interactions $G_{QQ}(\rho_Q; r)$ simulating G-matrices in coordinate space, where $\rho_Q$ is number density of quark matter. We use here the method given in Ref.\[^{32}\].

The effective interactions are written as $G_{QQ} = G_{EME} + G_{MPP} + G_{INS} + G_{OGE}$ approximately corresponding to $V_{QQ} = V_{EME} + V_{MPP} + V_{INS} + V_{OGE}$. Though they can be obtained for each $(j f^\prime, T, L, S, J)$ state, for simplicity, the dependence on $L$ is approximated by that on parity $P$ and the dependence on $J$ is averaged: Quantum numbers $T L S J$ are reduced to $TSP$. The respective interactions are represented in two- or one-range Gaussian forms, and coefficients are adjusted so that s.p. potentials $U^{TSP}_f$ obtained from $G^{TSP}_f$ simulate the original G-matrix results. It is far easier to derive quark-matter EoSs with use of these density-dependent interactions $G_{QQ}$ than derivations by G-matrix calculations with $V_{QQ}$.

The density-dependent effective interactions $G_{EME}$ and $G_{OGE}$ derived from $V_{EME}$ and $V_{OGE}$, respectively, are parameterized in a two-range Gaussian form as

$$G_{EME, OGE}(\rho, r) = \left( a \rho^2 + b \rho^3 \right) \cdot \exp\left( -\frac{r}{0.8} \right) + c \cdot \exp\left( -\frac{r}{1.6} \right). \quad (3.10)$$

The parameter set $(a, b, c)$ in Eq. (3.10) is given for each $(y, T, S, P)$ state with $y = qq, qs, ss \ (q = u, d)$. In Tables II and III the values of parameters are tabulated for $G_{EME}$ and $G_{OGE}$, respectively.

$G_{INS}$ derived from $V_{INS}$ is parameterized in an one-range Gaussian form as

$$G_{INS}(\rho, r) = \left( a \rho^2 + b \rho^3 \right) \cdot \exp\left( -\frac{r}{0.6} \right). \quad (3.11)$$

The parameter set $(a, b, c)$ in Eq. (3.11) is given for each $(y, T, S, P)$ state with $y = qq, qs, ss \ (q = u, d)$. The values of them are given in Table IV.

$G_{MPP}$ derived from $V_{MPP}$ is parameterized in an one-range Gaussian form as

$$G_{MPP}(\rho, r) = \left( a + b \rho \right) \cdot \exp\left( -\frac{r}{1.3} \right)^2 \quad (3.12)$$

being independent of $(y, T, S)$ and given only for $P$. The values of parameters $(a, b, c)$ are given in Table IV.
TABLE I: $G_{EME}(\rho, r) = (a \rho^2 + b \rho^3) \cdot \exp(-r/0.8^2) + c \cdot \exp(-r/1.0^2)$. $y = qq,_qs, ss$ ($q = u, d$).

| y  | T   | S   | P   | a   | α   | b   | β   | c   |
|----|-----|-----|-----|-----|-----|-----|-----|-----|
| qq | 1/0 | +   | 1   | -3.520 | -1 | -17.94 | 0 | -0.9978 |
|   | 0/1 | +   | 1   | -2.871 | -1 | -30.59 | 0 | -0.8389 |
|   | 0/0 | -   | 1   | 43.34 | -1 | 192.8 | 0 | 3.896 |
|   | 1/1 | -   | 1   | 6.621 | -1 | 102.5 | 0 | 1.595 |
| qs | 1/2 | 0   | 1   | -0.5716 | -1 | -28.27 | 0 | -0.4530 |
|   | 1/2 | 1   | 1   | -0.6959 | -1 | -24.58 | 0 | -0.1939 |
|   | 1/2 | 0   | 1   | -1.597 | -1 | 149.0 | 0 | 1.568 |
|   | 1/2 | 1   | 1   | 1.183 | -1 | 75.98 | 0 | 1.217 |
| ss | 0/0 | +   | 1   | -2.755 | -1 | -26.37 | 0 | -0.1212 |
|   | 0/1 | -   | 1   | -1.651 | -1 | 51.06 | 0 | 0.3558 |

TABLE II: $G_{OGE}(\rho, r) = (a \rho^2 + b \rho^3) \cdot \exp(-r/0.8^2) + c \cdot \exp(-r/1.0^2)$. $y = qq, qs, ss$ ($q = u, d$).

| y  | T   | S   | P   | a   | α   | b   | β   | c   |
|----|-----|-----|-----|-----|-----|-----|-----|-----|
| qq | 1/0 | +   | 1   | 8.565 | 1   | 3.892 | 0.3742 | 0.5185 |
|   | 0/1 | +   | 1   | 7.543 | 1   | 1.977 | 0.6431 | 1.142 |
|   | 0/0 | -   | 1   | -0.8959 | 1   | 9.982 | 0.2741 | 0.5027 |
|   | 1/1 | -   | 1   | 8.094 | 1   | 11.64 | 0.2881 | 1.147 |
| qs | 1/2 | 0   | 1   | -4.733 | -1 | -1.359 | 0.4161 | -0.2593 |
|   | 1/2 | 1   | 1   | -3.658 | -1 | -0.7316 | 0.6347 | -0.5709 |
|   | 1/2 | 0   | 1   | -1.282 | -1 | -3.141 | 0.3675 | -0.2514 |
|   | 1/2 | 1   | 1   | -5.645 | -1 | -3.831 | 0.3727 | -0.5736 |
| ss | 0/0 | +   | 1   | 10.48 | 1   | 1.478 | 0.5930 | 0.5185 |
|   | 0/1 | -   | 1   | 11.23 | 1   | 7.743 | 0.3250 | 1.147 |

TABLE III: $G_{INS}(\rho, r) = (a \rho^2 + b \rho^3) \cdot \exp(-r/0.6^2)$. $y = qq, qs$ ($q = u, d$).

| y  | T   | S   | P   | a   | α   | b   | β   |
|----|-----|-----|-----|-----|-----|-----|-----|
| qq | 0/1 | +   | 1   | 0.2132 | -1 | -130.0 | 0 |
|   | 0/0 | -   | 1   | -124.8 | 0   | -38.82 | 0.4227 |
| qs | 1/2 | 0   | 1   | -1.504 | -1 | -61.35 | 0.0705 |
|   | 1/2 | 1   | 1   | -0.1638 | -1 | -63.47 | 0 |
|   | 1/2 | 0   | 1   | 59.03 | 0   | -17.06 | 0.4966 |
|   | 1/2 | 1   | 1   | -36.95 | 0   | -5.749 | 0.3192 |

TABLE IV: $G_{MPP}(\rho, r) = (a + b \rho^3) \cdot \exp(-r/1.3^2)$.

| P   | a   | b   | β   |
|-----|-----|-----|-----|
| +   | 0.3597 | 1.600 | 1.490 |
| −   | -0.4338 | 2.618 | 1.384 |

IV. EOS AND MR DIAGRAM OF HYBRID STAR

A. Derivation of quark-matter EoS

Let us derive the EoS of quark matter composed of quarks with flavor $f = u, d, s$. In this derivation, we use the density-dependent $QQ$ interactions Eq. (3.10), Eq. (3.11), Eq. (3.12) based on the non-relativistic formalism. Relativistic expressions are used only for kinetic energies.

A single $f$ quark potential in quark matter composed of $f'$ quarks is given by

$$U_f(k) = \sum_{f'} U^{(f')}_f(k) = \sum_{k' < k_{f'}} \langle kk' | G_{f'f} | kk' \rangle$$

with $f, f' = u, d, s$, where spin and isospin quantum numbers are implicit. The quark energy density is given by

$$\varepsilon_f = 2N_c \int_0^{k_f} \frac{d^3k}{(2\pi)^3} \left\{ \sqrt{\hbar^2 k^2 + M_f^2 + 1/2 U_f(k)} \right\} + B(\rho_Q)$$

where $N_c = 3$ is the number of quark colors. The quark number density is given as $\rho_Q = \sum_f \rho_f$ with $\rho_f = N_c \frac{k_f^3}{(2\pi)^3}$. The chemical potential $\mu_f$ and pressure $P_Q$ are expressed as

$$\mu_f = \frac{\partial \varepsilon_f}{\partial \rho_f},$$

$$P_Q = \rho_Q^2 \frac{\partial (\varepsilon_f/\rho_Q)}{\partial \rho_Q} = \sum_f \mu_f \rho_f - \varepsilon_Q.$$  

Here, we consider the EoS of $\beta$-stable quark matter composed of $u, d, s, e^-$. The equilibrium conditions are summarized as follows:

1. chemical equilibrium conditions,

$$\mu_d = \mu_s = \mu_u + \mu_e$$

2. charge neutrality,

$$0 = \frac{1}{3} (2\rho_u - \rho_d - \rho_s - \rho_e)$$

3. baryon number conservation,

$$\rho_B = \frac{1}{3} (\rho_u + \rho_d + \rho_s) = \frac{1}{3} \rho_Q$$

In the parabolic approximation, the following relation can be derived:

$$\mu_e = \mu_d - \mu_u = 4\beta E_{sym}$$

where $x = \rho_u/(\rho_u + \rho_d)$ and $\beta = 1 - 2x$. $E_{sym}$ is the symmetric energy of ud part.
When the chemical potentials are substituted into (4.5), the chemical equilibrium conditions are represented as equations for densities $\rho_u$, $\rho_d$, $\rho_s$ and $\rho_e$. Then, equations (4.5) – (4.7) are solved iteratively, and densities and chemical potentials in equilibrium are obtained. Finally, energy densities (4.2) and pressures (4.4) can be calculated.

An example of solution is demonstrated in Fig. 4. The number fractions of quarks and electrons in $\beta$-stable quark matter are plotted as a function of the baryon density $\rho_B$ in the case of using Q2, where solid (dashed) curves are for $u$, $d$ and $s$ quarks (electrons). In the figure, the electron fractions are not visible below the $s$-quark onset. The reason of such small values are because the symmetry energies $E_{\text{sym}}$ in Eq.(4.8) are not so large in the case of our QQ interactions.

![FIG. 4: The number fractions of quarks and electrons in $\beta$-stable quark matter as a function of the baryon density $\rho_B$. The fractions of $u$, $d$ and $s$ quarks are given by solid curves, and that of electrons $e^-$ are by dashed curve.](image)

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**B. Phase transition from hadronic matter to quark matter**

The EoSs are shown in Fig. 5 where pressures of quark matter are given as a function of the energy density $\epsilon$ and compared to those of hadronic matter. Steeper slopes of curves correspond to stiffer EoSs: The Q-EoSs are stiffer than the H-EoSs, and the EoSs for (b) Q1 and (c) Q2 are stiffer than that for (a) Q0 owing to the repulsive contributions of $V_{\text{OGE}}$ and $V_{\text{MPP}}$. As shown later, these features are clearly reflected in the $MR$ curves of hybrid stars.

In order to construct the hybrid EoS including a transition from hadronic phase and quark phase, we use the replacement interpolation method \[33\] \[14\], being a simple modification of the Maxwell and the Glendenning (Gibbs) constructions \[34\]. In our actual calculations, we follow the interpolation formula given in Ref.\[14\]. Then, interpolated regions can be considered as mixed phases. Both of H-EoSs and Q-EoSs are assumed to fulfill separately the charge-neutrality and $\beta$-equilibrium conditions. The EoSs of hadronic and quark phases and that of mixed phase are described with the relations between pressures and chemical potentials $P_H(\mu)$, $P_Q(\mu)$ and $P_M(\mu)$, respectively. The critical chemical potential $\mu_c$ for the transition from the hadronic phase to the quark phase is obtained from the Maxwell condition

$$P_Q(\mu_c) = P_H(\mu_c) = P_c.$$  \hspace{1cm} (4.9)

The pressure of the mixed phase is represented by a polynomial ansatz

$$P_M(\mu) = \sum_{q=1}^{N} \alpha_q (\mu - \mu_c)^q + P_c + \Delta P$$  \hspace{1cm} (4.10)

where the pressure shift $\Delta P$ at $\mu_c$ is treated as a free parameter. The pressure of the mixed phase at $\mu_c$ is determined by $P_M(\mu_c) = P_c + \Delta P = (1 + \Delta P)P_c$, with $\Delta P = \Delta P/P_c$. Then, the matching chemical potential $\mu_Q$ of $P_M(\mu)$ to $P_H(\mu)$ ($P_Q(\mu)$) can be obtained from the continuity condition. The corresponding matching densities $\rho_H$ and $\rho_Q$ are obtained with use of $\rho(\mu) = dP(\mu)/d\mu$. The finite values of $\Delta P = 0.05 - 0.07$ corresponds to the Glendenning construction \[14\]. We choose $\Delta P = 0.07$ in this work.
FIG. 6: Pressures as a function of the chemical potential $\mu_B$. Short-dashed, long-dashed and dot-dashed curves are pressures of hadronic matter for H1, H2 and H3, respectively. In the left panel, solid curves are pressures of quark matter obtained from (a) Q0, (b) Q1 and (c) Q2. In the right panel, they are obtained by using constant quark masses without density dependences Eq. (3.8) and vacuum energies Eq. (3.9).

In Fig. 6 pressures are drawn as a function of the chemical potential $\mu_B$, where short-dashed, long-dashed and dot-dashed curves are pressures of hadronic matter for H1, H2 and H3, respectively. In the left panel, solid curves are pressures of quark matter obtained from (a) Q0, (b) Q1 and (c) Q2. The crossing of the hadronic and the quark-matter curves is considered to be a condition for phase transition to occur. The values of $P$ at crossing points give the critical pressures $P_c$ for phase transitions. The hadronic and quark-matter curves are connected smoothly by Eq. (4.10). Then, the effective-mass parameter $\gamma$ in Eq. (3.8) is adjusted so that cross points appear at similar values of $\mu_B \sim 1200$ MeV. In the right panel, on the other hand, solid curves are obtained from Q0, Q1 and Q2 by using constant quark masses $M_Q^0(\rho_Q = 0)$ without density dependences Eq. (3.8) and vacuum energies Eq. (3.9). It is found that there is no crossing point in this region of $\mu_B$. Thus, the density-dependent quark mass plays a decisive role in the occurrence of phase transition.

In Fig. 7 pressures are given as a function of the chemical potential $\mu_B$ in the transition region. The short-dashed curve is obtained by the H-EoS for H2 and the solid curve is by the Q-EoS for Q2. The bold-solid curve is the interpolated one.

Our phase transition is specified by the pressures $P_c$ at critical chemical potentials $\mu_c$ and boundary values of chemical potentials and densities for mixed phases. They are shown in the cases of phase transitions from the H-EoS for H1, H2 and H3 to the Q-EoSs for Q0, Q1, Q2, Q1e and Q2e. In Table V the chemical potentials at matching points are given by values of $\mu_H$ and $\mu_Q$. In Table VI $\rho_H$ and $\rho_Q$ are densities at matching points in phase transitions, and $\rho_{cH}$ and $\rho_{cQ}$ are critical densities defined by the $P_H(\rho_{cH}) = P_Q(\rho_{cQ}) = P_c$ in the case of $\Delta P = 0$. It is reasonable that that the values of $\rho_{cH}$ and $\rho_{cQ}$ are between the values of $\rho_H$ and $\rho_Q$. The Maxwell construction is conditioned by $\rho_{cH} < \rho_{cQ}$. As found in Ta
TABLE V: Pressures $P_c$ at critical chemical potentials $\mu_c$ in phase transitions from the hadronic phases for H1, H2 and H3 to the quark-matter phases for Q0, Q1 and Q2. Values of $\mu_H$ ($\mu_Q$) are chemical potentials at matching points between mixed phases and hadron (quark) phases.

|        | $P_c$ (MeV/fm$^3$) | $\mu_c$ (MeV) | $\mu_H$ (MeV) | $\mu_Q$ (MeV) |
|--------|-------------------|--------------|--------------|--------------|
| H1+Q0  | 125.7             | 1241         | 1186         | 1373         |
| H1+Q1  | 92.55             | 1183         | 1141         | 1277         |
| H1+Q2  | 110.4             | 1215         | 1095         | 1368         |
| H2+Q0  | 139.6             | 1254         | 1199         | 1386         |
| H2+Q1  | 102.3             | 1193         | 1149         | 1282         |
| H2+Q2  | 138.2             | 1252         | 1141         | 1415         |
| H2+Q1e | 132.9             | 1189         | 1243         | 1446         |
| H2+Q2e | 209.6             | 1360         | 1261         | 1600         |
| H3+Q0  | 136.0             | 1251         | 1198         | 1382         |
| H3+Q1  | 101.2             | 1191         | 1148         | 1279         |
| H3+Q2  | 131.6             | 1243         | 1142         | 1412         |

TABLE VI: Critical densities (fm$^{-3}$) of phase transitions: $\rho_H$ and $\rho_Q$ are densities at matching points in phase transitions from the hadronic phases for H1, H2 and H3 to the quark-matter phases for Q0, Q1, Q2, Q1e and Q2e. $\rho_H$ and $\rho_Q$ are critical densities for the Maxwell construction defined by $P_H(\rho_H) = P_Q(\rho_Q) = P_c$. Values of $\rho_E$ are densities at crossing points of energy densities $\epsilon_H(\rho)$ and $\epsilon_Q(\rho)$. There is no crossing point in the case of H1+Q2.

|        | $\rho_H$ | $\rho_Q$ | $\rho_{H}^{\prime}$ | $\rho_{Q}^{\prime}$ | $\rho_E$ |
|--------|----------|----------|----------------------|----------------------|----------|
| H1+Q0  | 0.566    | 0.661    | 0.673                | 0.784                |
| H1+Q1  | 0.490    | 0.574    | 0.544                | 0.918                |
| H1+Q2  | 0.407    | 0.623    | 0.584                | --                   |
| H2+Q0  | 0.521    | 0.664    | 0.694                | 0.702                |
| H2+Q1  | 0.446    | 0.573    | 0.561                | 0.753                |
| H2+Q2  | 0.433    | 0.661    | 0.620                | 0.716                |
| H2+Q1e | 0.506    | 0.650    | 0.707                | 0.643                |
| H2+Q2e | 0.608    | 0.790    | 0.722                | 0.695                |
| H3+Q0  | 0.482    | 0.616    | 0.689                | 0.659                |
| H3+Q1  | 0.416    | 0.568    | 0.559                | 0.692                |
| H3+Q2  | 0.407    | 0.608    | 0.612                | 0.660                |

by the Q-EoS for Q2. The bold-solid curve is pressure in the interpolated region.

![Figure 8: Pressures as a function of density](image)

It is worthwhile to point out that our hybrid-EoSs are consistent with the picture of hadron-quark continuity [17][18]. In these references, the interpolated pressures are given in the density region of $2 < \rho_B/\rho_0 < (4 - 7)$, where quark degrees of freedom gradually emerge. Correspondingly, our mixed phases are given in the region of $(2.4 - 3.3) < \rho_B/\rho_0 < (4.1 - 6.0)$, as found in Table VI.

Let us demonstrate that the Maxwell construction appears in the the $\Delta_P = 0$ limit. In Fig. 9 we show pressure $P$ as a function of density $\rho_B$ in the case of using H2+Q1e, where the dashed (short-dashed) curves are for the hadronic (quark) matter. The horizontal solid lines show the range of the hadron-quark mixed phase. It is well known that the Maxwell construction is specified by the horizontal line in the $P-\rho$ diagram, where the density values at ends of horizontal and vertical lines are given by $\rho_H^{\prime}$ and $\rho_Q^{\prime}$. The difference of the curve for $\Delta_P = 0$ from the dot-dashed curve for $\Delta_P = 0.07$ is found to be small, the appearance of which is seen in the corresponding MR curves as shown later. Not only in the case of He+Q1e, there appear the similar curves specifying the Maxwell construction in the cases of $\rho_H^{\prime} < \rho_Q^{\prime}$ in Table VI.

Our hybrid-star EoS is composed of H-EoS and Q-EoS, being combined by the interpolation formula including the parameter $\Delta_P$. The $MR$ relations of hybrid stars can be obtained by solving the Tolmann-Oppenheimer-Volkoff (TOV) equation, where our hybrid EoSs are connected smoothly to the crust EoS [25][30] in the low-density side.
FIG. 9: Pressure $P$ as a function of density $\rho_B$ in the case of $H_2+Q_1e$, where the dashed (short-dashed) curves are for hadronic (quark) matter. The horizontal solid line shows the range of the hadron-quark mixed phases in case of $\Delta P = 0$. The dot-dashed curve is in the case of $\Delta P = 0.07$.

FIG. 10: Hybrid-star masses as a function of radius $R$, where the short-dashed curves are obtained by the H-EoS for $H_2$. In the left panel, the solid and dashed curves are obtained by $H_2+Q_1e$ in cases of $\Delta P = 0.07$ and $\Delta P = 0$, respectively. In the right panel, the upper (lower) solid curves are obtained by $H_2+Q_2$ ($H_2+Q_1$), and the upper (lower) dashed curves are by $H_2+Q_2e$ ($H_2+Q_1e$).

In Fig. 9, hybrid-star masses are given as a function of radius $R$ (left panel) and central baryon density $\rho_{Bc}$ (right panel). The curves obtained by the Q-EoSs for (a) $Q_0$, (b) $Q_1$ and (c) $Q_2$ are given by solid curves, and those by the H-EoS for $H_2$ are given by the short-dashed curves. The former curves are connected from the latter curves by the hadron-quark phase transitions. The maximum masses for (b) $Q_1$ and (c) $Q_2$ are over $2M_\odot$ substantially. It is noted that the Q-EoS for (a) $Q_0$ derived from $V_{EME}$ is still stiff enough to reach $2M_\odot$ without help of the repulsive contributions of $V_{OGE}$ and $V_{IMPP}$. This repulsive components in $V_{EME}$ come from vector-meson and pomeran exchanges between quarks. The rectangle in the left panel indicates the region of mass $2.072^{+0.067}_{-0.066}M_\odot$ and radius $12.39^{+1.30}_{-0.98}$ km for the most massive neutron star PSR J0740+6620. The $MR$ curves for $Q_1$ and $Q_2$ are found to pass through this rectangle.

In the left panel of Fig. 10, hybrid-star masses are drawn as a function of radius $R$, where the Q-EoSs for $Q_2$ and H-EoSs for (a) $H_1$, (b) $H_2$ and (c) $H_3$ are used. Short-dashed, long-dashed and dot-dashed curves are obtained with H-EoSs for $H_1$, $H_2$ and $H_3$, respectively. Solid curves show deviations by transitions from hadronic-matter to quark-matter phases. The maximum masses in the figures are as follows: In the cases of H-EoSs, they are $1.82M_\odot$ ($H_1$), $1.94M_\odot$ ($H_2$) and $2.07M_\odot$ ($H_3$). In the cases of including hadron-quark transitions, they are $2.25M_\odot$ ($H_1+Q_2$), $2.25M_\odot$ ($H_2+Q_2$) and $2.28M_\odot$ ($H_3+Q_2$). The maximum masses are noted to be determined by the Q-EoSs, being larger than those given by the H-EoSs. The rectangle in the left panel is the same as that in Fig. 11, indicating the mass-radius region obtained from the observation [22]. The $MR$ curves for the Q-EoSs pass through the rectangle, though those

In our approach, there is no clear criteria to decide which of $Q_2$ ($Q_1$) and $Q_2e$ ($Q_1e$) is more appropriate. In the following section, we use $Q_1$ and $Q_2$ because they seem to be more suitable than $Q_1e$ and $Q_2e$ in the light of the recent observations for the maximum masses.

C. $MR$ diagrams of hybrid stars

In Fig. 11 hybrid-star masses are given as a function of radius $R$ (left panel) and central baryon density $\rho_{Bc}$ (right panel). The curves obtained by the Q-EoSs for (a) $Q_0$, (b) $Q_1$ and (c) $Q_2$ are given by solid curves, and those by the H-EoS for $H_2$ are given by the short-dashed curves. The former curves are connected from the latter curves by the hadron-quark phase transitions. The maximum masses for (b) $Q_1$ and (c) $Q_2$ are over $2M_\odot$ substantially. It is noted that the Q-EoS for (a) $Q_0$ derived from $V_{EME}$ is still stiff enough to reach $2M_\odot$ without help of the repulsive contributions of $V_{OGE}$ and $V_{IMPP}$. This repulsive components in $V_{EME}$ come from vector-meson and pomeran exchanges between quarks. The rectangle in the left panel indicates the region of mass $2.072^{+0.067}_{-0.066}M_\odot$ and radius $12.39^{+1.30}_{-0.98}$ km for the most massive neutron star PSR J0740+6620. The $MR$ curves for $Q_1$ and $Q_2$ are found to pass through this rectangle.

In the left panel of Fig. 12, hybrid-star masses are drawn as a function of radius $R$, where the Q-EoSs for $Q_2$ and H-EoSs for (a) $H_1$, (b) $H_2$ and (c) $H_3$ are used. Short-dashed, long-dashed and dot-dashed curves are obtained with H-EoSs for $H_1$, $H_2$ and $H_3$, respectively. Solid curves show deviations by transitions from hadronic-matter to quark-matter phases. The maximum masses in the figures are as follows: In the cases of H-EoSs, they are $1.82M_\odot$ ($H_1$), $1.94M_\odot$ ($H_2$) and $2.07M_\odot$ ($H_3$). In the cases of including hadron-quark transitions, they are $2.25M_\odot$ ($H_1+Q_2$), $2.25M_\odot$ ($H_2+Q_2$) and $2.28M_\odot$ ($H_3+Q_2$). The maximum masses are noted to be determined by the Q-EoSs, being larger than those given by the H-EoSs. The rectangle in the left panel is the same as that in Fig. 11, indicating the mass-radius region obtained from the observation [22]. The $MR$ curves for the Q-EoSs pass through the rectangle, though those...
for the H-EoSs (dashed curves) are below this rectangle.

The radii $R$ at $1.4 M_\odot$ ($R_{1.4 M_\odot}$) are given as follows: The values of $R_{1.4 M_\odot}$ are 12.5 km (H1+Q2), 13.3 km (H2+Q2) and 13.6 km (H3+Q2), being obtained in the cases of including hadron-quark transitions. The similar values are obtained by using H-EoSs only, which means that the values of $R_{1.4 M_\odot}$ are determined by H-EoSs. In the figure, dotted and solid line segments indicate $R_{1.4 M_\odot} = 12.33^{+0.76}_{-0.81}$ km (PP model) and $R_{1.4 M_\odot} = 12.18^{+0.56}_{-0.79}$ km (CS model) \cite{24}, and dashed and dot-dashed ones do $R_{1.4 M_\odot} = 11.94^{+0.76}_{-0.87}$ km \cite{25} and $R_{1.4 M_\odot} = 13.80 \pm 0.47$ km \cite{20}, respectively. The former three line segments (dotted, solid and dashed lines) are similar with each other, and the $MR$ curve for H1 intersects them. On the other hand, the $MR$ curves for H2 and H3 intersect the dot-dashed line, but do not intersect the other three lines. In the present stage of the observations for radii of neutron stars, it is difficult to determine which one of H1, H2 and H3 lead to the most reasonable EoS.

In the right panel of Fig.13 hybrid-star masses are drawn as a function of central baryon density $\rho_{Bc}$, where the Q-EoS for Q2 is used. Short-dashed, long-dashed and dot-dashed curves are obtained with H-EoSs for H1, H2 and H3, respectively. Solid curves show the deviations by transitions from hadronic-matter to quark-matter phases. In the cases of including hadron-quark transitions, the onset values of $\rho_{Bc}$ for quark phases are $0.40$ fm$^{-3}$ (H1+Q2), $0.46$ fm$^{-3}$ (H2+Q2) and $0.43$ fm$^{-3}$ (H3+Q2).

It is useful to compare our results for $MR$ diagrams with those in Ref.\cite{14}, because we employ the method in this reference for the hadron-quark phase transitions. Though their quark-matter EoS is based on the nonlocal Nambu-Jona-Lasinio (nlNJL) model differently from ours, it is found that the quark-phase regions of the $MR$ curves in Fig.4 of \cite{14} are similar to ours qualitatively. Especially, maximum masses of $2M_\odot$ are reproduced well, namely the Q-EoSs are stiff similarly in both cases of ours and \cite{14}. However, the hadronic-matter regions are different from each other, since softer H-EoSs are used in \cite{14} than ours.

As stated before, the hyperon mixing results in remarkable softening of the EoS. In order to avoid this “hyperon puzzle”, the universal repulsions modeled as MPP are included in our derivations of our H-EoSs. Here, let us try to use the H-EoS for H1’ in which the MPP repulsions work only among nucleons. The BB interaction used in \cite{14} is of this type. In Fig.13 hybrid-star masses are given as a function of radius $R$ (left panel) and of central density $\rho_{Bc}$ (right panel). The top dashed curve (a) is obtained from the H-EoS for H0 without hyperons. The middle dashed curve (b) is from the H-EoS for H1. The bottom dashed curve (c) is from the H-EoS for H1’ including hyperons, in which the MPP repulsions work only among nucleons. The solid curves show the deviations by transitions from hadronic phase to quark-matter phase for Q1. It should be noted that the large difference from the top dashed curve to the bottom dashed curve demonstrates the softening of the EoS by hyperon mixing. Then, the solid curves show the deviations by transitions from hadronic phases to quark-matter phases for Q1. The lowering of the maximum mass by the EoS
FIG. 12: Hybrid-star masses as a function of radius $R$ (left panel) and central density $\rho_{Bc}$ (right panel), where the Q-EoS for Q2 is used. Short-dashed, long-dashed and dot-dashed curves are obtained with H-EoSs for (a) H1, (b) H2 and (c) H3, respectively. Solid curves show the deviations by transitions from hadronic-matter to quark-matter phases. In the left panel, the rectangle indicates the region of mass $2.072^{+0.067}_{-0.066} M_\odot$ and radius $12.39^{+1.30}_{-0.98}$ km [22]. Dotted and solid line segments indicate $R_{1.4M_\odot} = 12.33^{+0.76}_{-0.81}$ km (PP model) and $R_{1.4M_\odot} = 12.15^{+0.56}_{-0.78}$ km (CS model) [24], and dashed and dot-dashed ones do $R_{1.4M_\odot} = 11.94^{+0.76}_{-0.87}$ km [25] and $R_{1.4M_\odot} = 13.80 \pm 0.47$ km [26], respectively.

Our $MR$ diagrams of hybrid stars are derived from H-EoSs for $BB$ interactions (H1, H2, H3) and Q-EoSs for QQ interactions (Q0, Q1, Q2). There are nine combinations of H-EoSs and Q-EoSs, among which some combinations are used in the above results. In Table VII, features of the obtained $MR$ diagrams in all combinations of (H1, H2, H3) and (Q0, Q1, Q2) are demonstrated by showing the calculated values of maximum masses $M_{max}$ and radii $R_{M_{max}}$, and radii at $1.4M_\odot$ ($R_{1.4M_\odot}$). For comparison, those for H1' and H1'+Q1 are added. Here, the important features are as follows: (1) In all cases, the Q-EoSs combined with the H-EoSs are stiff enough to reproduce maximum masses over $2M_\odot$. (2) The values of $R_{1.4M_\odot}$ are specified by the H-EoSs.

Another constraint for the EoS is given by the tidal deformability, being the induced quadrupole polarizability. The dimensionless tidal deformability $\Lambda$ is defined as $\Lambda = (2/3)k_2(c^2 R/GM)^5$ [37], where $c$ is the speed of light, $R$ and $M$ are radius and mass of a neutron star and $G$ is the gravitational constant. $k_2$ is the tidal Love number describing the response of each star to the external disturbance. The binary neutron star merger GW170817 give the upper limit on the tidal deformability of a neutron star with mass $1.4M_\odot$: $\Lambda_{1.4M_\odot} \leq 800$ [38]. In Table VII are given the calculated values of $\Lambda_{1.4M_\odot}$ for our EoSs, where all values are less than the upper limit of 800. It should be noted that the values of $\Lambda_{1.4M_\odot}$ are determined by the H-EoSs, even if the Q-EoSs are combined with them.

### Table VII: Maximum masses $M_{max}$ and radii $R_{M_{max}}$, radii at $1.4M_\odot$ $R_{1.4M_\odot}$, dimensionless tidal deformability at $1.4M_\odot$ $\Lambda_{1.4M_\odot}$

|          | $M_{max}/M_\odot$ | $R_{M_{max}}$ (km) | $R_{1.4M_\odot}$ (km) | $\Lambda_{1.4M_\odot}$ |
|----------|-------------------|---------------------|----------------------|---------------------|
| H1       | 1.82              | 10.4                | 12.4                 | 422                 |
| H1+Q0    | 1.99              | 10.0                | 12.4                 | 422                 |
| H1+Q1    | 2.14              | 10.3                | 12.4                 | 422                 |
| H1+Q2    | 2.25              | 10.7                | 12.5                 | 422                 |
| H1'      | 1.52              | 10.4                | 12.1                 | 334                 |
| H1'+Q1   | 2.10              | 10.0                | 12.2                 | 337                 |
| H2       | 1.94              | 10.3                | 13.3                 | 671                 |
| H2+Q0    | 2.01              | 10.4                | 13.3                 | 671                 |
| H2+Q1    | 2.16              | 10.6                | 13.3                 | 671                 |
| H2+Q2    | 2.25              | 10.9                | 13.3                 | 671                 |
| H3       | 2.07              | 10.7                | 13.6                 | 771                 |
| H3+Q0    | 2.04              | 10.7                | 13.6                 | 771                 |
| H3+Q1    | 2.18              | 10.8                | 13.6                 | 771                 |
| H3+Q2    | 2.28              | 11.2                | 13.6                 | 771                 |

Softening turns out to be recovered by the transition to the quark-matter phase given by the stiff EoS. It is interesting that the curves for H0+Q1 are similar to those for H0. The basic feature of the $MR$ curve for H1'+Q1 is similar to those of the curves in [14].
FIG. 13: hybrid-star masses are given as a function of radius \( R \) (left panel), and as a function of central density \( \rho_{\text{Bc}} \) (right panel). The top dashed curve (a) is obtained from the H-EoS for H0 without hyperons. The middle dashed curve (b) is from the H-EoS for H1. The bottom dashed curve (c) is from the H-EoS for H1' including hyperons, in which the MPP repulsions work only among nucleons. The solid curves show the deviations by transitions from hadronic phase to quark-matter phase for Q1.

V. CONCLUSION

The EoSs and \( MR \) diagrams of hybrid stars are obtained on the basis of our \( QQ \) interaction model composed of the extended meson exchange potential (\( V_{\text{EME}} \)), the multi-pomeron exchange potential (\( V_{\text{MPP}} \)), the instanton exchange potential (\( V_{\text{INS}} \)) and the one gluon exchange potential (\( V_{\text{OGE}} \)), whose strengths are determined on the basis of terrestrial data with no adhoc parameter to stiffen EoSs. The repulsive nature of our \( QQ \) interaction in high density region are basically given by \( V_{\text{EME}} \) including strongly repulsive components owing to vector-meson and pomeron exchanges. Additional repulsions (attractions) are given by \( V_{\text{MPP}} \) and \( V_{\text{OGE}} \). The resultant repulsions included in our \( QQ \) interaction are so strong that the quark-matter EoSs become stiff enough to give maximum masses of hybrid stars over \( 2M_\odot \).

Hadronic-matter EoSs (H-EoS) and quark-matter EoSs (Q-EoS) are derived in the same framework based on the BBG theory. In quark matter, density-dependent quark masses are introduced phenomenologically, playing a decisive role in the occurrence of phase transition. Parameters of density dependences are taken so that hadron-quark phase transitions are able to occur at reasonable density region owing to the reduction of the quark masses and chemical potentials in quark matter. Our resulting density dependence of effective quark mass is similar to the Brown-Rho scaling.

Our H-EoSs are still not stiff enough to give maximum masses of neutron stars over \( 2M_\odot \) due to the softening by hyperon mixing, although the stiffness is recovered substantially by universal many-body repulsions. In the case of using our \( QQ \) interaction model, the Q-EoSs are stiffer than the H-EoSs and \( MR \) curves of hybrid stars shift above those of stars with hadronic phases only. The maximum masses of the formers including quark phases become larger than those of the latters, and \( MR \) curves are characterized by Q-EoSs in the mass region higher than about \( 1.5M_\odot \). Our Q-EoSs for Q1 and Q2 are stiff enough to give a maximum mass over \( 2M_\odot \). The derived mass and radius are consistent with the recent measurement for the most massive neutron star PSR J0740+6620, obtained by the combining analysis for the NICER and the other multimessenger data.

In our approach, star radii \( R_{1.4M_\odot} \) given by hadronic-matter EoSs do not changed by the hadron-quark phase transitions, namely they are determined by H-EoSs regardless of Q-EoSs. There are three estimates of \( R_{1.4M_\odot} \) based on the NICER measurements and the other multimessenger data. Two of them give \( R_{1.4M_\odot} = 11.1 - 13.1 \) km, and the other \( R_{1.4M_\odot} = 13.1 - 14.4 \) km. Our H-EoS for H1 (H2 or H3) is consistent with the former (latter). Our H-EoSs and Q-EoSs lead to \( MR \) diagrams of hybrid stars consistent with the recent observations for masses and radii.
Acknowledgments

The authors would like to thank D. Blaschke for valuable comments and fruitful discussions. This work was supported by JSPS KAKENHI (No.20K03951 and No.20H04742).

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