Elliptic quantum toroidal algebra $U_{q,t,p}(\mathfrak{gl}_{1,\text{tor}})$ and affine quiver gauge theories

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Abstract
We introduce a new elliptic quantum toroidal algebra $U_{q,t,p}(\mathfrak{gl}_{1,\text{tor}})$. Various representations in the quantum toroidal algebra $U_{q,t}(\mathfrak{gl}_{1,\text{tor}})$ are extended to the elliptic case including the level (0,0) representation realized by using the elliptic Ruijsenaars difference operator. Various intertwining operators of $U_{q,t,p}(\mathfrak{gl}_{1,\text{tor}})$-modules w.r.t. the Drinfeld comultiplication are also constructed. We show that $U_{q,t,p}(\mathfrak{gl}_{1,\text{tor}})$ gives a realization of the affine quiver $W$-algebra $W_{q,t}(\Gamma(\hat{A}_0))$ proposed by Kimura–Pestun. This realization turns out to be useful to derive the Nekrasov instanton partition functions, i.e., the $\chi_y$- and elliptic genus, of the 5d and 6d lifts of the 4d $\mathcal{N} = 2^*$ $U(M)$ theories and provide a new Alday–Gaiotto–Tachikawa correspondence.

Keywords Elliptic quantum group · Toroidal algebra · Affine quiver varieties

Mathematics Subject Classification 20C35

1 Introduction

The aim of this paper is to introduce a new elliptic quantum toroidal algebra $U_{q,t,p}(\mathfrak{gl}_{1,\text{tor}})$ associated with the toroidal algebra of type $\mathfrak{gl}_1$ and show its connection...
to the affine quiver $W$-algebra $W_{q,t}(\Gamma(\hat{A}_0))$ and the affine quiver gauge theories in 5d and 6d.

We formulate $U_{q,t,p}(\mathfrak{gl}_{1,tor})$ by generators and relations in the same scheme as the elliptic quantum group $U_{q,p}(\hat{\mathfrak{g}})$ associated with the affine Lie algebra $\hat{\mathfrak{g}}$ [16, 40, 47, 51, 55]. The latter is the Drinfeld realization of the face-type elliptic quantum group [39] in terms of the Drinfeld generators, which are a deformation of the loop generators of the affine Lie algebras [13]. One of the important properties of the elliptic group $[39]$ in terms of the Drinfeld generators, which are a deformation of the loop $47, 51, 55$. The latter is the Drinfeld realization of the face-type elliptic quantum $50$ in the following way. $\hat{\mathfrak{g}}$ the Goddard–Kent–Olive (GKO) coset-type $\hat{\mathfrak{g}}$ and use the same notation $W_{q,t}(\mathfrak{g})$.

Furthermore, for $W_{q,t}(\mathfrak{sl}_N)$ a deformation of the primary fields constructed for example in [8] is realized as certain fusions of intertwiners of the $U_{q,t,p}(\mathfrak{sl}_N)$-modules.

The elliptic quantum toroidal algebra $U_{q,t,p}(\mathfrak{gl}_{1,tor})$ is an elliptic quantum group associated with the toroidal algebra $\mathfrak{gl}_1$. It is an elliptic analogue of the quantum toroidal algebra $U_{q,t}(\mathfrak{gl}_{1,tor})$ introduced by Miki as a deformation of the $W_{1+\infty}$ algebra [62]. Various representations of $U_{q,t}(\mathfrak{gl}_{1,tor})$ have been studied by many papers such as [5, 9, 17, 18, 20–23, 62]. It is also remarkable that $U_{q,t}(\mathfrak{gl}_{1,tor})$ is isomorphic to the elliptic Hall algebra introduced by Schiffmann and Vasserot [17, 18, 78–81]. See also [22, 23, 67] for the shuffle algebra formulation.

Representations of $U_{q,t}(\mathfrak{gl}_{1,tor})$ have many interesting applications to the 5d lifts of the 4d $\mathcal{N} = 2$ SUSY gauge theories, which are the gauge theories associated with the linear quivers, such as a calculation of the Nekrasov instanton partition functions and a study of the 5d analogue of the Alday–Gaiotto–Tachikawa (AGT) correspondence [3]. See for example [5, 6, 10, 11, 63, 86]. The essential properties of $U_{q,t}(\mathfrak{gl}_{1,tor})$ for applications to show AGT correspondence are summarized into the three points:

1) the deformed $W$-algebra $W_{q,t}(\mathfrak{sl}_N)$ is realized on the $N$-fold tensor product of $U_{q,t}(\mathfrak{gl}_{1,tor})$ [6, 9, 23, 62].

2) the intertwiners of $U_{q,t}(\mathfrak{gl}_{1,tor})$-modules w.r.t the Drinfeld comultiplication realize the refined topological vertex [4, 36] as the vertex operators on the bosonic Fock space [5].

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[1] The same was confirmed for the non-simply laced case, say for $U_{q,p}(\mathfrak{sl}_N^{(1)})$ [16, 50], which realizes the deformation of Fateev–Lukyanov’s $\mathcal{W}_l$ algebra [59].
3) a certain block of composition of the intertwiners of \( U_{q,t}(\mathfrak{gl}_{1,\text{tor}}) \)-modules realizes the intertwiner (screened vertex operator) of the elliptic quantum group \( U_{q,p}(\widehat{\mathfrak{sl}}_N) \) \[28, 29, 86]\.

Hence, by 2) taking a composition of the intertwiners of \( U_{q,t}(\mathfrak{gl}_{1,\text{tor}}) \)-modules according to the web diagram \[4, 5, 36\], the expectation value of it gives an instanton partition function of the 5d lift of the 4d \( \mathcal{N} = 2 \) theory and basically by 3) the same quantity is identified with an expectation value of the intertwiners of \( U_{q,p}(\widehat{\mathfrak{sl}}_N) \), i.e., a deformation of the conformal block. This is the 5d analogue of the AGT correspondence. It is also remarkable that thus obtained deformed conformal blocks coincides with the vertex functions introduced by Okounkov \[1, 56, 74\]. See for example (6.4) in \[52\] and take its trigonometric limit \( q^\kappa \rightarrow 0 \). There are some different approaches based on the elliptic Hall algebra \[81\] and the shuffle algebra \[68\]. For the affine Yangian case see \[61\]. See also \[65\].

Instead of taking an expectation value, one can take a trace of the composed intertwining operators of \( U_{q,t}(\mathfrak{gl}_{1,\text{tor}}) \) \[26, 77, 87\]. Then, it gives an instanton partition function of the 6d lift of the theory \[26, 37, 72\] and the same arguments as 1)–3) yield that it is equivalent to the trace of the intertwiners of \( U_{q,p}(\widehat{\mathfrak{sl}}_N) \), i.e., the trace of the vertex operator of \( W_{q,t}(\mathfrak{sl}_N) \) \[29\]. Note that the integral expressions for the latter are given as a part of the integral solution to the elliptic \( q \)-KZ equation \[52, 55\], which has exactly the same structure as discussed as quantum \( q \)-Langlands correspondence in \[1\]. This is the 6d analogue of the AGT correspondence. In this way, the instanton calculus and the AGT correspondence in 5d and 6d lifts of the 4d \( \mathcal{N} = 2 \) SUSY gauge theories associated with the linear quivers can be treated by using the quantum toroidal algebra \( U_{q,t}(\mathfrak{gl}_{1,\text{tor}}) \).

In this paper, we show that the elliptic quantum toroidal algebra \( U_{q,t,p}(\mathfrak{gl}_{1,\text{tor}}) \) provides a relevant quantum group structure to treat the 5d and 6d lifts of the 4d \( \mathcal{N} = 2^* \) SUSY gauge theories, which are the \( \mathcal{N} = 2 \) SUSY gauge theories coupled with the adjoint matter \[69, 70\]. These are gauge theories associated with the Jordan quiver. We also formulate the AGT correspondence between these 5d and 6d lifts and the affine quiver \( W \) algebra \( W_{q,t}(\Gamma(\widehat{\mathbb{A}}_0)) \). For this purpose, we construct some useful representations of \( U_{q,t,p}(\mathfrak{gl}_{1,\text{tor}}) \) and give a realization of \( W_{q,t}(\Gamma(\widehat{\mathbb{A}}_0)) \) \[41\] in the same way as the above b) and c) for \( U_{q,p}(\widehat{\mathfrak{g}}) \). Namely the level \((1, N)\) elliptic currents \( x^\pm(z) \) of \( U_{q,t,p}(\mathfrak{gl}_{1,\text{tor}}) \) give the screening currents \( S^\pm(z) \) of \( W_{q,t}(\Gamma(\widehat{\mathbb{A}}_0)) \), and a certain composition of the “intertwiners” of \( U_{q,t,p}(\mathfrak{gl}_{1,\text{tor}}) \)\footnote{A composition of the type I intertwiner \( \Phi(u) \) of the \( U_{q,t,p}(\mathfrak{gl}_{1,\text{tor}}) \)-module and its shifted inverse \( \Phi^*(u) \). See Sect. 5} gives a realization of the generating function \( T(u) \). There one of the key properties of \( U_{q,t,p}(\mathfrak{gl}_{1,\text{tor}}) \) is that the level \((1, N)\) representation possesses the four parameters \( q, t, p, p^* \) satisfying \( p/p^* = t/q \). They play the role of the \( SU(4) \) \( \Omega \) deformation parameters introduced by Nekrasov \[70\], i.e., two of three independent parameters are the 5d analogue the \( \Omega \) deformation parameters \( \epsilon_1, \epsilon_2 \) \[69\] and the remaining one is a deformation of the adjoint mass parameter in the 4d \( \mathcal{N} = 2^* \) SUSY gauge theories.

Realizing the generating function \( T(u) \) of \( W_{q,t}(\Gamma(\widehat{\mathbb{A}}_0)) \), it becomes trivial that the vacuum expectation value of \( T(u) \) gives the Nekrasov instanton partition function of the 5d lift of the 4d \( \mathcal{N} = 2^* U(1) \) theory, i.e., the generating function of the \( \chi_y \) genus.
of the Hilbert scheme $\text{Hilb}_n(\mathbb{C}^2)$ of $n$ points on $\mathbb{C}^2$ [41, 42], where we take $y = p$. However, this fact and our realization of $T(u)$ as a composition of the “intertwiners” lead us to identify $T(u)$ with a basic refined topological vertex operator corresponding to the diagram Fig. 1 studied in [32, 34, 36]. Then, such realization makes calculation of compositions of $T(u)$ easy and allows us to apply them to various instanton calculus in the 5d and 6d lifts of the 4d $\mathcal{N} = 2^*$ theories. This includes $\mathcal{N} = 2^*$ $U(M)$ theories, whose instanton partition function is the generating function of the $\chi_y$ genus of the moduli space of the rank $M$ instantons with charge $n$. Their 6d lifts give the generating functions of the elliptic genus of the same moduli space. Hence, we establish a new AGT correspondence between the 5d and 6d lifts of the 4d $\mathcal{N} = 2^*$ SUSY gauge theories and the affine quiver $W$ algebra $W_{q,t}(\widehat{A})$.

This paper is organized as follows. In Sect. 2, after reviewing basic facts in the quantum toroidal algebra $U_{q,t}(\mathfrak{gl}_{1,\text{tor}})$, we give a definition of the elliptic quantum toroidal algebra $U_{q,t,p}(\mathfrak{gl}_{1,\text{tor}})$. The $\mathbb{Z}$-algebra structure, an elliptic analogue of Miki’s automorphism and Hopf algebroid structure are also given. In Sect. 3, we construct the level $(1, N, 0, 1)$ and $(0, 0)$ representations. In particular, the level $(0, 0)$ representation is given by using the elliptic Ruijsenaars difference operator on $C[[x_1, \ldots, x_N]]$. In Sect. 4, we construct the type I and the type II dual vertex operator as intertwining operators of the $U_{q,t,p}(\mathfrak{gl}_{1,\text{tor}})$-modules constructed in Sect. 3. We also give a definition of their shifted inverse operators. In Sect. 5, we give a realization of the affine quiver $W$ algebra $W_{q,t}(\Gamma(\widehat{A}))$ by using the level $(1, N)$ representation of $U_{q,t,p}(\mathfrak{gl}_{1,\text{tor}})$. In Sect. 6, several calculations of the instanton partition functions of the 5d and 6d lifts of the 4d $\mathcal{N} = 2^*$ SUSY gauge theories are given. The results indicate a new AGT correspondence between those gauge theories and $W_{q,t}(\Gamma(\widehat{A}))$. In Sect. 7, we summarize calculations of basic correlation functions of our intertwining operators, which should provide the $\mathcal{N} = 2^*$ analogues of the instanton partition functions of the 5d and 6d lifts of the 4d $\mathcal{N} = 2$ pure $SU(N)$ theory and $SU(N)$ theory coupled with $2N$ fundamental matters obtained, for example, in [4, 5]. Appendix A is a list of formulas, which is used to discuss the $\mathbb{Z}$-algebra structure in Sect. 1. Appendix B is a direct check of the statement on the level $(0, 1)$ representation. In Appendices C and D, proofs of Theorem 4.2 and 4.7 are given. In Appendix F, some useful formulas of the Nekrasov function are collected.

Partial results in Sects. 2, 3, 4, 5 have been presented by H.K. at several workshops [54].

2 Elliptic quantum toroidal algebra $U_{q,t,p}(\mathfrak{gl}_{1,\text{tor}})$

After reviewing the quantum toroidal algebra $U_{q,t}(\mathfrak{gl}_{1,\text{tor}})$, we introduce the elliptic quantum toroidal algebra $U_{q,t,p}(\mathfrak{gl}_{1,\text{tor}})$. The $\mathbb{Z}$-algebra structure, an elliptic analogue of Miki’s automorphism and Hopf algebroid structure are also given.

2.1 The quantum toroidal algebra $U_{q,t}(\mathfrak{gl}_{1,\text{tor}})$

This section is a review of Miki’s results [62] in our notation.
2.1.1 Definition

Definition 2.1 Let $q$, $t$ be nonzero complex numbers such that $q$, $t$, $q/t$ are not roots of unity, and set

$$
\kappa_m = (1 - q^m)(1 - t^{-m})(1 - (t/q)^m),
$$
$$
G^{\pm}(z) = (1 - q^{\pm 1}z)(1 - t^{\mp 1}z)(1 - (q/t)^{\mp 1}z).
$$

The quantum toroidal algebra $U_{q,t} = U_{q,t}(\mathfrak{gl}_{1,\text{tor}})$ is a $\mathbb{C}$-algebra generated by

$$
a_m, \quad X^\pm_n, \quad \gamma^{\pm 1/2}, \quad (\psi_0^\pm)^{\pm 1} \quad (m \in \mathbb{Z}\setminus\{0\}, \ n \in \mathbb{Z}).
$$

By using the generating functions

$$
X^\pm(z) = \sum_{n \in \mathbb{Z}} X^\pm_n z^{-n},
$$
$$
\phi(z) = \psi_0^+ \exp \left\{ \sum_{m > 0} a_m \gamma^{m/2} z^{-m} \right\},
$$
$$
\psi(z) = \psi_0^- \exp \left\{ - \sum_{m > 0} a_m \gamma^{m/2} z^m \right\},
$$

the defining relations are

$$
\gamma^{\pm 1/2} : \text{central},
$$
$$
[a_m, a_n] = -\delta_{m+n,0} \frac{\kappa_m}{m} (\gamma^m - \gamma^{-m}) \gamma^{-|m|},
$$
$$
[a_m, X^+(w)] = -\frac{\kappa_m}{m} \gamma^{-|m|-m/2} w^m X^+(w),
$$
$$
[a_m, X^-(w)] = \frac{\kappa_m}{m} \gamma^{-m/2} w^m X^-(w),
$$
$$
[X^+(z), X^-(w)] = \frac{(1 - q)(1 - 1/t)}{(1 - q/t)} \left( \delta(\gamma^{-1}z/w) \phi(w) - \delta(\gamma z/w) \psi(\gamma^{-1}w) \right),
$$
$$
z^3 G^{\pm}(w/z) X^\pm(z) X^\pm(w) = -w^3 G^{\pm}(z/w) X^\pm(w) X^\pm(z),
$$
$$
\text{Sym}_{z_1, z_2, z_3, z_2 z_3^{-1}} [X^\pm(z_1), [X^\pm(z_2), X^\pm(z_3)]] = 0,
$$

where $\delta(z) = \sum_{n \in \mathbb{Z}} z^n$.

Remark By rescaling the generators $X^\pm_m$ as

$$
X^{'\pm}(z) = (1 - t/q) X^\pm(z), \quad X^{'-}(z) = (1 - q/t) X^-(z),
$$
the relation (2.5) can be rewritten as

\[ [X^+(z), X^-(w)] = \kappa_1(\delta(y^{-1}z/w)\phi(w) - \delta(yz/w)\psi(y^{-1}w)) \]

so that the whole relations of \( \mathcal{U}_{q,t} \) are symmetric under any permutations among \( q_1 = q, q_2 = t^{-1}, q_3 = t/q \). Note also that (2.2)–(2.4) are equivalent to

\[
\begin{align*}
\phi(z)\phi(w) &= \phi(w)\phi(z), \\
\frac{G^+(y^{-1}w/z)}{G^+(y/w/z)}\phi(z)\psi(w) &= \frac{G^+(yz/w)}{G^+(y^{-1}z/w)}\psi(w)\phi(z), \\
z^3G^+(y^{-1}w/z)\phi(z)X^+(w) &= -\gamma^3w^3G^+(yz/w)^{-1}X^+(w)\phi(z), \\
z^3G^+(w/z)\psi(z)X^+(w) &= -w^3G^+(z/w)X^+(w)\psi(z), \\
z^{-3}G^+(w/z)^{-1}\phi(z)X^-(w) &= -w^{-3}G^+(z/w)^{-1}X^-(w)\phi(z), \\
z^{-3}G^+(y^{-1}w/z)^{-1}\psi(z)X^-(w) &= -\gamma^3w^{-3}G^+(yz/w)^{-1}X^-(w)\psi(z).
\end{align*}
\]

### 2.1.2 Automorphism of \( \mathcal{U}_{q,t} \)

Let us set \( \omega = (1 - q)(1 - t^{-1}) \) and define \( Y_l^\pm \in \mathcal{U}_{q,t} \) (\( l \in \mathbb{Z} \)) by

\[
\begin{align*}
Y_l^+ &= \begin{cases} 
(\psi_0^+)^i(\text{ad } X_0^+)^{-1}X_1^+/(\omega x)^{l-1} & (l > 0) \\
\gamma/2a_1 & (l = 0) \\
-(q/t)^{|l|}\gamma(\text{ad } X_0^-)^{-1}X_1^+/(\omega x)^{|l|-1} & (l < 0)
\end{cases} \\
Y_l^- &= \begin{cases} 
-(t/q)\gamma^{-1}(\text{ad } X_0^-)^{-1}X_1^+/(\omega x)^{l-1} & (l > 0) \\
\gamma/2a_1 & (l = 0) \\
(q/t)^{|l|-1}(\psi_0^-)^i(\text{ad } X_0^-)^{-1}X_1^+/(\omega x)^{|l|-1} & (l < 0)
\end{cases}
\end{align*}
\]

where \( (\text{ad } x)y = [x, y] \) for \( x, y \in \mathcal{U}_{q,t} \). In addition, for \( k \in \mathbb{Z} \setminus \{0\} \) we set

\[
h_k = \begin{cases} 
[X_{-1}^+, X_0^+, \ldots, X_0^+, X_1^+] & (k \geq 2) \\
X_0^+ & (k = 1) \\
-(q/t)X_0^- & (k = -1) \\
-(q/t)^{|k|}[X_1^-, X_0^-, \ldots, X_0^-, X_{-1}^-] & (k \leq -2)
\end{cases}
\]

where for \( x_1, x_2, \ldots, x_n \in \mathcal{U}_{q,t} \)

\[ [x_n, \ldots, x_1] = [x_n, [x_{n-1}, \ldots, [x_2, x_1] \cdots]]. \]
In terms of $h_k$, we define $b_k \in \mathcal{U}_{q,t}$ \((k \in \mathbb{Z} \backslash \{0\})\) by the relations

\[
\exp \left( \sum_{k>0} b_k (\psi_0^+)^{k/2} z^{-k} \right) = 1 + \kappa_1 \sum_{k \geq 1} \frac{h^k}{\omega^k} z^{-k},
\]

\[
\exp \left( - \sum_{k>0} b_{-k} (\psi_0^-)^{k/2} z^k \right) = 1 + \kappa_1 \sum_{k \geq 1} \frac{h^{-k}}{\omega^k} z^k,
\]

where we set $\omega = (1-q)(1-t^{-1})$. Then, we have

**Theorem 2.2** (Miki) *The map*

\[
\Psi : a_1 \mapsto (1-t/q)^{1/2} X_0^+ , \quad a_{-1} \mapsto -(1-q/t)^{1/2} X_0^-,
\]

\[
X_0^+ \mapsto \frac{\gamma^{1/2}}{1-t/q} a_1, \quad X_0^- \mapsto \frac{\gamma^{1/2}}{1-q/t} a_1,
\]

\[
\gamma \mapsto \psi_0^+, \quad \psi_0^- \mapsto \gamma^{-1}
\]

gives an automorphism of $\mathcal{U}_{q,t}$ satisfying $\Psi^4 = 1$. Moreover, $\Psi$ maps

\[
a_k \mapsto b_k , \quad X_i^\pm \mapsto Y_i^\pm , \quad b_k \mapsto a_{-k} , \quad Y_i^\pm \mapsto X_i^\mp
\]

For readers’ convenience, we listed below the correspondence between our notations and those used in [62]. The symbols with $M$ denote the ones in [62].

\[
\gamma_M = q^{1/2}, \quad q_M = (t/q)^{1/2}, \quad C_M^\pm = \gamma^\pm, \quad C_M^\pm = \psi_0^\pm,
\]

\[
(q_M - q_M^{-1})(\gamma_M^k - \gamma_M^{-k}) a_k^M = \gamma^{k/2} a_k , \quad (q_M - q_M^{-1})(\gamma_M^k - \gamma_M^{-k}) b_k^M = (\psi_0^+)^{|k|/2} b_k ,
\]

\[
h_k^M = \left( \frac{t^{1/2}}{1-q} \right)^{|k|} h_k ,
\]

\[
X_i^{M^+} = \frac{t^{1/2}}{1-q} X_i^+, \quad X_i^{M^-} = -\frac{t^{1/2}}{1-q} X_i^-, \quad Y_i^{M^+} = \frac{t^{1/2}}{1-q} Y_i^+, \quad Y_i^{M^-} = -\frac{t^{1/2}}{1-q} Y_i^-.
\]

### 2.2 The elliptic quantum toroidal algebra $\mathcal{U}_{q,t,p}(\mathfrak{gl}_{1,\text{tor}})$

Let $p$ be an indeterminate and set

\[(z; p)_n = (1 - z)(1 - zp) \cdots (1 - zp^{n-1}), \quad (z; p)_0 = 1,
\]

for $n \in \mathbb{N}$, $z \in \mathbb{C}$. We define $\mathcal{U}_{q,t,p}(\mathfrak{gl}_{1,\text{tor}})$ by generators and relations as follows.

**Definition 2.3** The elliptic quantum toroidal algebra $\mathcal{U}_{q,t,p} = \mathcal{U}_{q,t,p}(\mathfrak{gl}_{1,\text{tor}})$ is a $\mathbb{C}[[p]]$-algebra generated by $a_m, x_i^\pm$, \((m \in \mathbb{Z} \backslash \{0\}, n \in \mathbb{Z})\) and invertible elements $\psi_0^\pm, \gamma^{1/2}$. The defining relations can be conveniently expressed in terms of the generating functions, which we call the elliptic currents,
The defining relations are

\[
\psi_0^\pm, \gamma^{1/2} : \text{central,} \quad (2.14)
\]

\[
[\alpha_m, \alpha_n] = -\frac{k_m}{m} (\gamma^m - \gamma^{-m}) \gamma^{-m} \frac{1 - p^m}{1 - p^s m} \delta_{m+n,0}, \quad (2.15)
\]

\[
[\alpha_m, x^+(z)] = -\frac{k_m}{m} \frac{1 - p^m}{1 - p^s m} \gamma^{-3m/2} z^m x^+(z) \quad (m \neq 0). \quad (2.16)
\]

\[
[\alpha_m, x^-(z)] = \frac{k_m}{m} \gamma^{-m/2} z^m x^-(z) \quad (m \neq 0), \quad (2.17)
\]

\[
[x^+(z), x^-(w)] = \frac{(1 - q)(1 - 1/t)}{(1 - q/t)} \left( \delta(\gamma^{-1} z/w) \psi^+(w) - \delta(\gamma z/w) \psi^-(\gamma^{-1} w) \right),
\]

\[
z^3 G^+(w/z) g(w/z; p^*) x^+(z) x^+(w) = -w^3 G^+(z/w) g(z/w; p^*) x^+(w) x^+(z), \quad (2.18)
\]

\[
z^3 G^-(w/z) g(w/z; p) x^-(-z) x^-(w) = -w^3 G^-(z/w) g(z/w; p) x^-(w) x^-(z), \quad (2.19)
\]

\[
g(w/z; p^*) g(u/w; p^*) g(u/z; p^*) \left( \frac{w}{u} + \frac{w}{z} - \frac{z}{w} - \frac{u}{w} \right) x^+(z) x^+(w) x^+(u) + \text{permutations in } z, w, u = 0, \quad (2.20)
\]

\[
g(w/z; p) x^-(-z) x^-(w) x^-(-u) + \text{permutations in } z, w, u = 0, \quad (2.21)
\]

\[
g(w/z; p) x^-(-z) x^-(w) x^-(-u) + \text{permutations in } z, w, u = 0, \quad (2.22)
\]

where we set \( p^* = p \gamma^{1/2} \) and

\[
g(z; s) = \exp \left( \sum_{m>0} \frac{k_m}{m} \frac{s^m}{1 - s^m z^m} \right) \in \mathbb{C}[z] \quad (2.23)
\]

for \( s = p, p^* \).

We treat these relations as formal Laurent series in \( z, w \) and \( u \). All the coefficients in \( z, w, u \) are well defined in the \( p \)-adic topology \([24, 25, 51]\).
It is sometimes convenient to set

$$\alpha'_m = \frac{1 - p^{sm}}{1 - p^m} \gamma^m \alpha_m \quad (m \in \mathbb{Z}_{\neq 0})$$

which satisfy

$$[\alpha'_m, \alpha'_n] = -\frac{k_m}{m} (\gamma^m - \gamma^{-m}) \gamma^m \frac{1 - p^{sm}}{1 - p^m} \delta_{m+n,0}. \quad (2.24)$$

**Remark** In the same way as $U_{q,t}$, by rescaling the generators $x^\pm_m$ as

$$x'^+(z) = (1 - t/q)x^+(z), \quad x'^-(z) = (1 - q/t)x^-(z),$$

the relation (2.18) can be rewritten as

$$[x'^+(z), x'^-(w)] = \kappa_1 \left( \delta(\gamma^{-1}z/w) \psi^+(w) - \delta(\gamma z/w) \psi^-(\gamma^{-1}w) \right)$$

so that the whole relations of $U_{q,t,p}$ are symmetric under any permutations among $q_1 = q$, $q_2 = t^{-1}$, $q_3 = t/q$.

**Remark** On $U_{q,t,p}$-modules, where the central element $\gamma^{1/2}$ takes a complex value, we regard $p, p^* = py^{-2}$ as a generic complex number with $|p| < 1, |p^*| < 1$, and have

$$g(z; p) = \frac{(pqz; p)_\infty}{(pq^{-1}z; p)_\infty} \frac{(pt^{-1}z; p)_\infty}{(ptz; p)_\infty} \frac{(p(t/q)z; p)_\infty}{(p(t/q)^{-1}z; p)_\infty}.$$

Here, we set

$$(z; p)_\infty = \prod_{n=0}^{\infty} (1 - zp^n) \quad |z| < 1.$$

From this, one can rewrite (2.15–2.17), (2.19) and (2.20) in the sense of analytic continuation as follows:

$$\psi^\pm(z)\psi^\pm(w) = \frac{g_\theta(w/z; p^*)}{g_\theta(w/z; p)} \psi^\pm(w)\psi^\pm(z),$$

$$\psi^+(z)\psi^-(w) = \frac{g_\theta(\gamma^{-1}w/z; p^*)}{g_\theta(\gamma w/z; p)} \psi^-(w)\psi^+(z),$$

$$\psi^+(z)x^+(w) = g_\theta(\gamma^{-1}w/z; p^*)x^+(w)\psi^+(z),$$

$$\psi^-(z)x^+(w) = g_\theta(w/z; p^*)x^+(w)\psi^-(z),$$

$$\psi^+(z)x^-(w) = g_\theta(z/w; p)x^-(w)\psi^+(z),$$

$$\psi^-(z)x^-(w) = g_\theta(\gamma z/w; p)x^-(w)\psi^-(z),$$

$$x^+(z)x^+(w) = g_\theta(w/z; p^*)x^+(w)x^+(z),$$

$$x^-(z)x^-(w) = g_\theta(z/w; p)x^-(w)x^-(z),$$
where

\[ g_\theta(z; p) = \frac{\theta_p(q^{-1}z)\theta_p((q/t)z)\theta_p(tz)}{\theta_p(qz)\theta_p((q/t)^{-1}z)\theta_p(t^{-1}z)}, \]

\[ \theta_p(z) = (z; p)_\infty (p/z; p)_\infty. \]

\[ g_\theta(z; p^*) = g_\theta(z; p)\big|_{p \mapsto p^*}. \]

Note that

\[ g_\theta(z^{-1}; p) = g_\theta(z; p)^{-1}, \quad g_\theta(pz; p) = g_\theta(z; p). \]

### 2.3 The \( Z \) Algebra Structure

We next show that the \( Z \) algebra structure of \( U_{q,t,p} \), i.e., it is not elliptically deformed. This is a common feature in the elliptic algebras \( U_{q,t,p}(\hat{\mathfrak{g}}) \) [16].

In this subsection, we assume \( \gamma \neq 1 \). Set

\[ E^\pm(\alpha, z) = \exp\left\{ \pm \sum_{n>0} \frac{1}{\gamma^n - \gamma^{-n}} \alpha_{\pm n}(\gamma^{-1/2}z)^\mp n \right\}, \]

\[ E^\pm(\alpha', z) = \exp\left\{ \mp \sum_{n>0} \frac{1}{\gamma^n - \gamma^{-n}} \alpha'_{\pm n}(\gamma^{-1/2}z)^\pm n \right\}, \]

and define \( Z^\pm(z) \) by

\[ Z^+(z) = E^-(\alpha, z)x^+(z)E^+(\alpha, z), \]

\[ Z^-(z) = E^-(\alpha', z)x^-(z)E^+(\alpha', z). \]

Then, from (A.1)–(A.4) in Appendix A, we have

\[ [\alpha_m, Z^\pm(z)] = 0, \quad (m \in \mathbb{Z} \neq 0). \]

Furthermore, the relations in Definition 2.3 and Lemma A.1 lead to the following theorem.

**Theorem 2.4**

\[ z^3 G^-(w/z)g(w/z; \gamma^2)Z^\pm(z)Z^\pm(w) = -w^3 G^-(z/w)g(z/w; \gamma^2)^{-1}Z^\pm(w)Z^\pm(z), \]

\[ g(\gamma w/z)g(\gamma w/z; \gamma^2)Z^+(z)Z^-(w) - g(\gamma z/w)g(\gamma z/w; \gamma^2)Z^-(w)Z^+(z) \]

\[ = \frac{(1-q)(1-1/t)}{1-q/t} \left\{ \delta(\gamma^{-1}z/w)\psi_0^+ - \delta(\gamma z/w)\psi_0^- \right\}, \]

\[ g(w/z)^{-1}g(u/w)^{-1}g(u/z)^{-1}g(w/z; \gamma^2)^{-1}g(u/w; \gamma^2)^{-1}g(u/z; \gamma^2)^{-1} \]
\[
\times \left( \frac{w}{u} + \frac{w}{z} - \frac{z}{w} - \frac{u}{w} \right) Z^+(z)Z^+(w)Z^+(u) + \text{permutations in } z, w, u = 0, \quad (2.34)
\]
\[
g(w/z; \gamma^2)^{-1} g(u/w; \gamma^2)^{-1} g(u/z; \gamma^2)^{-1} \left( \frac{w}{u} + \frac{w}{z} - \frac{z}{w} - \frac{u}{w} \right) Z^-(z)Z^-(w)Z^-(u) + \text{permutations in } z, w, u = 0. \quad (2.35)
\]

Here, \( g(z; \gamma^2) \) is given in (2.23) with \( s = \gamma^2 \) and we also set
\[
g(z) = \exp \left( \sum_{m>0} \frac{\kappa_m}{m} z^m \right) \in \mathbb{C}[[z]]. \quad (2.36)
\]

We call the algebra generated by the coefficients of \( Z^\pm(z) = \sum_{\nu \in \mathbb{Z}} Z^{\pm}_\nu z^{-\nu} \) in \( z \) satisfying the relations in Theorem 2.4 the \( Z \) algebra associated with \( \mathcal{U}_{q,t,p} \).

**Remark.** Noting that the whole relations in this theorem are independent of the elliptic nome \( p \), one finds that these relations coincides with those in Proposition 4.2 in [62] under the identification
\[
\mathcal{K}_M = \gamma, \quad K_M^{-1} = \psi_0, \\
T^\pm(z)_M = \pm Z^\pm(z)/(1 - q^\pm)(1 - t^\pm)
\]
and the correspondence listed below Theorem 2.2.

Hence, the elliptic quantum toroidal algebra \( \mathcal{U}_{q,t,p} \) possesses the same deformed Virasoro and \( W_3 \) algebra structures through the same \( Z \) algebra structure as \( \mathcal{U}_{q,t} \) discussed in Theorem 4.1 and Remark 4.7 of [62].

### 2.4 Algebra homomorphism from \( \mathcal{U}_{q,t,p}(\mathfrak{gl}_{1,\text{tor}}) \) to \( \mathcal{U}_{q,t}(\mathfrak{gl}_{1,\text{tor}}) \)

In [40], an homomorphism from the elliptic quantum algebra \( \mathcal{U}_{q,p}(\widehat{\mathfrak{g}}) \) to the quantum affine algebra \( \mathcal{U}_q(\widehat{\mathfrak{g}}) \) with \( \widehat{\mathfrak{g}} \) being an untwisted affine Lie algebra was given. This allows us to extend any representations of \( \mathcal{U}_q(\widehat{\mathfrak{g}}) \) with generic \( q \) to those of \( \mathcal{U}_{q,p}(\widehat{\mathfrak{g}}) \) with generic \( p, q \). Such homomorphism can be generalized to the cases with \( \widehat{\mathfrak{g}} \) being any twisted affine Lie algebras, for example see [46], as well as any toroidal Lie algebras \( \mathfrak{g}_{\text{tor}} \) [23, 54]. They include the cases with \( \widehat{\mathfrak{g}} \) being (affine or toroidal) Lie super algebras. See for example [44].

**Proposition 2.5** Let us define \( u^\pm(z, p) \) by
\[
u^+(\gamma^{-1/2}z, p) = \exp \left( - \sum_{m>0} \frac{p^m}{1 - p^m a_m z^m} \right),
\]
\[
u^-(\gamma^{1/2}z, p) = \exp \left( \sum_{m>0} \frac{p^m}{1 - p^m a_m z^{-m}} \right).
\]
where \( p^* = p\gamma^{-2} \) as above. Then, the following gives a homomorphism from \( \mathcal{U}_{q,t,p} \) to \( \mathcal{U}_{q,t} \otimes \mathbb{C}[[p]] \).

\[
\begin{align*}
x^+(z) &\mapsto u^+(z, p)X^+(z), \\
x^-(z) &\mapsto X^-(z)u^-(z, p), \\
\psi^+(z) &\mapsto u^+(\gamma z, p)\phi(z)u^-(z, p), \\
\psi^-(z) &\mapsto u^+(z, p)\psi(z)u^-(\gamma z, p).
\end{align*}
\tag{2.37-2.40}
\]

In particular, (2.39) (or (2.40)) is equivalent to

\[
\alpha_m \mapsto a_m, \quad \alpha_{-m} \mapsto \frac{1 - p^m}{1 - p^{m+1}}a_m \quad (m > 0).
\]

The statement is proved by using the following lemma.

**Lemma 2.6**

\[
\begin{align*}
[a_n, u^+(z, p)] &= \frac{k_n}{n} \gamma^n - \gamma^{-n} \left( p^* \gamma^{-1/2}z \right)^n u^+(z, p) \quad (n > 0), \\
[a_{-n}, u^-(z, p)] &= \frac{k_n}{n} \gamma^{-n} - \gamma^n \left( p^{-1} \gamma^{1/2}z \right)^{-n} u^-(z, p) \quad (n > 0), \\
u^+(z, p)X^+(w) &= g(z/w; p^*)X^+(w)u^+(z, p), \\
u^+(z, p)X^-(w) &= g(\gamma z/w; p^*)X^+(w)u^+(z, p), \\
u^-(z, p)X^+(w) &= g(\gamma^{-1}w/z; p)X^+(w)u^-(z, p), \\
u^-(z, p)X^-(w) &= g(w/z; p)X^-(w)u^-(z, p), \\
\phi(z)u^+(w, p) &= g(\gamma w/z; p^*)g(\gamma^{-1}w/z; p^*)^{-1} u^+(w, p)\phi(z), \\
u^-(z, p)\psi(w) &= g(\gamma w/z; p)g(\gamma^{-1}w/z; p)\psi(w)u^-(z, p), \\
\phi(z)u^-(w, p) &= u^-(w, p)\phi(z), \\
u^+(z, p)\psi(w) &= \psi(w)u^+(z, p), \\
u^+(z, p)u^-(w, p) &= g(p^* \gamma w/z; p^*)g(p^* \gamma^{-1}w/z; p)\psi(w)u^-(w, p)u^+(z, p).
\end{align*}
\tag{2.41-2.51}
\]

Here, \( g(z; s) \), \( s = p \), \( p^* \) is given in (2.23).

It is obvious that the homomorphism in Proposition 2.5 is invertible. Hence, one has the following isomorphism.

**Theorem 2.7** For generic \( q, t, p \),

\[
\mathcal{U}_{q,t,p} \cong \mathcal{U}_{q,t} \otimes \mathbb{C}[[p]]
\]

as a topological algebra.
2.5 Elliptic analogue of Miki’s automorphism

By using a set of new generators $b_m, Y^\pm_l$ of $U_{q,t}(\mathfrak{gl}_{1,\text{tor}})$ defined in Sect. 2.1.2, let us consider their generating functions.

$$Y^\pm(z) = \sum_{l \in \mathbb{Z}} Y^\pm_l z^{-l},$$

$$\phi_b(z) = \gamma^{-1} \exp \left\{ \sum_{m > 0} b_m (\psi_0^+)^{m/2} z^{-m} \right\},$$

$$\psi_b(z) = \gamma \exp \left\{ - \sum_{m > 0} b_{-m} (\psi_0^+)^{m/2} z^m \right\}.$$

Applying the homomorphism in Sect. 2.4, one obtains a new elliptic quantum toroidal algebra $U_{q,t,p}^b(\mathfrak{gl}_{1,\text{tor}})$ generated by $\beta_m, y^\pm_l, \psi_0^+, \gamma^{-1}$, defined by

$$y^\pm(z) = \sum_{l \in \mathbb{Z}} y^\pm_l z^{-l},$$

$$\psi_b^+((\psi_0^+)^{1/2}z) = \gamma^{-1} \exp \left( - \sum_{m > 0} \frac{p^m}{1 - p^m} \beta_{m-2} z^m \right) \exp \left( \sum_{m > 0} \frac{1}{1 - p^m} \beta_{m-2} z^{-m} \right),$$

where

$$y^+(z) := u_b^+(z, p) Y^+(z),$$

$$y^-(z) := Y^-(z) u_b^-(z, p),$$

$$\psi_b^+(z) := u_b^+(\psi_0^+ z, p) \phi_b(z) u_b^-(z, p),$$

$$\psi_b^-(z) := u_b^+(z, p) \psi_b(z) u_b^-(\psi_0^+ z, p)$$

with

$$u_b^+((\psi_0^+)^{-1/2} z, p) := \exp \left( - \sum_{m > 0} \frac{p^m}{1 - p^m} b_{m-2} z^m \right),$$

$$u_b^-((\psi_0^+)^{1/2} z, p) := \exp \left( \sum_{m > 0} \frac{p^m}{1 - p^m} b_{m-2} z^{-m} \right),$$

and $p_b^* = p(\psi_0^+)^{-2}$.

Then, we obtain the following isomorphism.

**Corollary 2.8**

$$U_{q,t,p}(\mathfrak{gl}_{1,\text{tor}}) \cong U_{q,t,p}^b(\mathfrak{gl}_{1,\text{tor}}),$$
by 

\[ \alpha_m \mapsto \beta_m, \quad x_i^\pm \mapsto y_i^\pm, \quad \gamma \mapsto \psi_0^+, \quad \psi_0^+ \mapsto \gamma^{-1}. \]

**Proof** The statement follows from Theorems 2.2 and 2.7. \(\square\)

### 2.6 Hopf algebroid structure

In this section, we introduce a Hopf algebroid structure into \( \mathcal{U}_{q,t,p} \). This structure was introduced by Etingof and Varchenko [14, 15] and developed in [43, 49].

For \( F(z, p) \in \mathbb{C}[[z, z^{-1}]][[p]] \), let \( \tilde{\otimes} \) denote the usual tensor product with the following extra condition

\[ F(z, p^*)a \tilde{\otimes} b = a \tilde{\otimes} F(z, p)b \quad (2.56) \]

Here, \( F(z, p^*) \) denotes the same \( F \) with replacing \( p \) by \( p^* = p\gamma^{-2} \).

Define two moment maps \( \mu_l, \mu_r : \mathbb{C}[[z, z^{-1}]][[p]] \to \mathcal{U}_{q,t,p}[[z, z^{-1}]] \) by

\[ \mu_l(F(z, p)) = F(z, p), \quad \mu_r(F(z, p)) = F(z, p^*). \]

Let \( \gamma(1) = \gamma \tilde{\otimes} 1, \gamma(2) = 1 \tilde{\otimes} \gamma \) and \( p_i^* = p_{i^*}^{-2} \) \((i = 1, 2)\). Let us define two algebra homomorphisms \( \Delta : \mathcal{U}_{q,t,p} \to \mathcal{U}_{q,t,p} \otimes \mathcal{U}_{q,t,p} \) and \( \varepsilon : \mathcal{U}_{q,t,p} \to \mathbb{C} \) by

\[
\begin{align*}
\Delta(\gamma^{\pm 1/2}) &= \gamma^{\pm 1/2} \tilde{\otimes} \gamma^{\pm 1/2}, \quad (2.57) \\
\Delta(\psi^\pm(z)) &= \psi^\pm(\gamma(2)_{1/2} z) \tilde{\otimes} \psi^\pm(\gamma(1)_{-1/2} z), \quad (2.58) \\
\Delta(x^+(z)) &= 1 \tilde{\otimes} x^+(\gamma(1)_{-1/2} z) + x^+(\gamma(2)_{1/2} z) \tilde{\otimes} \psi^-(\gamma(1)_{-1/2} z), \quad (2.59) \\
\Delta(x^-(z)) &= x^-(\gamma(2)_{1/2} z) \tilde{\otimes} 1 + \psi^+(\gamma(1)_{-1/2} z) \tilde{\otimes} x^-(\gamma(2)_{1/2} z), \quad (2.60) \\
\Delta(\mu_l(F(z, p))) &= \mu_l(F(z, p)) \tilde{\otimes} 1, \quad \Delta(\mu_r(F(z, p))) = 1 \tilde{\otimes} \mu_r(F(z, p)), \quad (2.61) \\
\varepsilon(\gamma^{1/2}) &= \varepsilon(\psi_0^+) = 1, \quad \varepsilon(\psi^\pm(z)) = 1, \quad \varepsilon(x^\pm(z)) = 0, \quad (2.62) \\
\varepsilon(\mu_l(F(z, p))) &= \varepsilon(\mu_r(F(z, p))) = F(z, p). \quad (2.63)
\end{align*}
\]

Note that \( \Delta \) is the so-called Drinfeld comultiplication. Then, we have

**Proposition 2.9** The maps \( \varepsilon \) and \( \Delta \) satisfy

\[
\begin{align*}
(\Delta \otimes \text{id}) \circ \Delta &= (\text{id} \otimes \Delta) \circ \Delta, \quad (2.64) \\
(\varepsilon \otimes \text{id}) \circ \Delta &= \text{id} = (\text{id} \otimes \varepsilon) \circ \Delta. \quad (2.65)
\end{align*}
\]

We also define an algebra anti-homomorphism \( S : \mathcal{U}_{q,t,p} \to \mathcal{U}_{q,t,p} \) by

\[
\begin{align*}
S(\gamma^{1/2}) &= \gamma^{-1/2}, \\
S(\psi^\pm(z)) &= \psi^\pm(z)^{-1}, \\
S(x^+(z)) &= -x^+(z)\psi^-(z)^{-1}, \\
S(x^-(z)) &= \psi^+(z)^{-1}x^-(z), \\
S(\mu_l(F(z, p))) &= \mu_l(F(z, p)), \\
S(\mu_r(F(z, p))) &= \mu_r(F(z, p)).
\end{align*}
\]
\[ S(x^-(z)) = -\psi^+(z)x^-(z), \]
\[ S(\mu_l(F(z, p))) = \mu_r(F(z, p)), \quad S(\mu_r(F(z, p))) = \mu_l(F(z, p)). \]

Then, one can check the following.

**Proposition 2.10**

\[ m \circ (\text{id} \otimes S) \circ \Delta(a) = \mu_l(\varepsilon(a))1 \quad \forall a \in U_{q, t, p}, \]
\[ m \circ (S \otimes \text{id}) \circ \Delta(a) = \mu_r(\varepsilon(a))1. \]

These Propositions indicate that \((U_{q, t, p}, \Delta, \varepsilon, \mu_l, \mu_r, S)\) is a Hopf algebroid \([14, 15, 43, 48, 49]\).

**Remark** For representations on which \(\gamma^{1/2} = 1\) hence \(p = p^*\), the following gives an opposite Drinfeld comultiplication.

\[ \Delta^{\text{op}}(\psi^\pm(z)) = \psi^\pm(z) \tilde{\otimes} \psi^{\pm}(z), \quad \text{(2.66)} \]
\[ \Delta^{\text{op}}(x^+(z)) = x^+(z) \tilde{\otimes} 1 + \psi^-(z) \tilde{\otimes} x^+(z), \quad \text{(2.67)} \]
\[ \Delta^{\text{op}}(x^-(z)) = 1 \tilde{\otimes} x^-(z) + x^-(z) \tilde{\otimes} \psi^+(z). \quad \text{(2.68)} \]

### 3 Representations of \(U_{q, t, p}(\mathfrak{gl}_{1, \text{tor}})\)

Let \(\mathcal{V}\) be a \(U_{q, t, p}\) module. For \((k, l) \in \mathbb{C}^2\), we say that \(\mathcal{V}\) has level \((k, l)\), if the central elements \(\gamma^{1/2}\) and \(\psi_0^+\) act as

\[ \gamma^{1/2}v = (t/q)^{k/4}v, \quad \psi_0^+v = (t/q)^{-l/2}v \quad \forall v \in \mathcal{V}. \]

In the rest of this paper, we regard \(p, p^* = p\gamma^{-2}\) as a generic complex number with \(|p| < 1, |p^*| < 1\).

More generally, for \(p_1, \ldots, p_r \in \mathbb{C}\) with \(|p_i| < 1 (1 \leq i \leq r, r = 1, 2, \ldots)\), we set

\[ (z; p_1, \ldots, p_r)_\infty = \prod_{n_1, \ldots, n_r = 0}^{\infty} (1 - z p_1^{n_1} \cdots p_r^{n_r}). \]

We use the following multiple elliptic Gamma functions defined by \([66, 73]\)

\[ \Gamma_r(z; p_1, \ldots, p_r) = (z; p_1, \ldots, p_r)^{(-1)^{r-1}} (p_1 \cdots p_r/z; p_1, \ldots, p_r)_\infty. \]

Note

\[ \Gamma_1(z; p_1) = \theta_{p_1}(z), \]
\[ \Gamma_2(z; p_1, p_2) \equiv \Gamma(z; p_1, p_2) = \frac{(p_1 p_2/z; p_1, p_2)_\infty}{(z; p_1, p_2)_\infty}. \]
are Jacobi’s odd theta function and the elliptic Gamma function \([76]\), respectively. We have

**Proposition 3.1**

\[
\Gamma_r(p_j z; p_1, \ldots, p_r) = \Gamma_{r-1}(z; p_1, \ldots, \tilde{p}_j, \ldots, p_r) \Gamma_r(x; p_1, \ldots, p_r)
\]

for \(r \geq 2\), where \(\tilde{p}_j\) means the excluding of \(p_j\).

### 3.1 Level \((1, N)\) representation of \(U_{q,t,p}(\mathfrak{sl}_1, \text{tor})\)

For \(u \in \mathbb{C}^*\), let \(\mathcal{F}_u^{(1,N)} = \mathbb{C}[\alpha_m (m > 0)]1_u^{(N)}\) be a Fock space on which the Heisenberg algebra \((\alpha_m (m \in \mathbb{Z}_{\neq 0})\) and the central elements \(\gamma^{1/2}, \psi_0^+\) act as

\[
\gamma^{1/2} \cdot 1_u^{(N)} = (t/q)^{1/4} 1_u^{(N)}, \quad \psi_0^+ \cdot 1_u^{(N)} = (t/q)^{-N/2} 1_u^{(N)}, \quad \alpha_m \cdot 1_u^{(N)} = 0,
\]

\[
\alpha_m \cdot \xi = \frac{\kappa_m}{m} (1 - (q/t)^m) \frac{1 - p^m}{1 - p^m} \frac{\partial}{\partial \alpha_m} \xi
\]

for \(m > 0\), \(\xi \in \mathcal{F}_u^{(1,N)}\). Note that \(p^+ = pq/t\) on \(\mathcal{F}_u^{(1,N)}\).

**Theorem 3.2** The following assignment gives a level \((1, N)\) representation of \(U_{q,t,p}\) on \(\mathcal{F}_u^{(1,N)}\).

\[
x^+(z) = uz^{-N} (t/q)^{3N/4} \exp \left\{ - \sum_{n>0} \frac{(t/q)^{n/4}}{1 - (t/q)^n} \alpha_{-n} z^n \right\} \exp \left\{ \sum_{n>0} \frac{(t/q)^{3n/4}}{1 - (t/q)^n} \alpha_{-n} z^n \right\},
\]

\[
x^-(z) = u^{-1} z^N (t/q)^{-3N/4} \exp \left\{ \sum_{n>0} \frac{(t/q)^{n/4}}{1 - (t/q)^n} \alpha'_{-n} z^n \right\} \exp \left\{ - \sum_{n>0} \frac{(t/q)^{3n/4}}{1 - (t/q)^n} \alpha'_{-n} z^n \right\},
\]

\[
\psi^+(z) = (t/q)^{-N/2} \exp \left\{ - \sum_{n>0} \frac{p^n (t/q)^{-n/4}}{1 - p^n} \alpha_{-n} z^n \right\} \exp \left\{ \frac{(t/q)^{n/4}}{1 - p^n} \alpha_{-n} z^n \right\},
\]

\[
\psi^-(z) = (t/q)^{N/2} \exp \left\{ - \sum_{n>0} \frac{(t/q)^{n/4}}{1 - p^n} \alpha_{-n} z^n \right\} \exp \left\{ \frac{p^n (t/q)^{-n/4}}{1 - p^n} \alpha_{-n} z^n \right\}.
\]

**Proof** Let us set \(x^\pm(z) = u^{\pm} z^\mp N (t/q)^{\pm 3N/4} \psi^\pm(z)\), \(\psi^\pm(z) = (t/q)^{\mp N/2} \psi^\pm(z)\). The statement follows from the operator product expansion (OPE) formulas listed below.

\[
\psi^+(z)x^+(t/q)^{1/2}w = f(w/z; p^+), \quad \psi^+(z)x^+(t/q)^{1/2}w : |w/z| < 1,
\]

\[
x^+(t/q)^{1/2}w)\psi^+(z) = f(p^+ w/z; p^+), \quad \psi^+(z)x^+(t/q)^{1/2}w : |z/w| < 1,
\]

\[
\psi^-(z)x^+(t/q)^{1/2}w = f(p^+ (t/q)^{1/2}w/z; p^+), \quad \psi^-(z)x^+(t/q)^{1/2}w : |w/z| < 1,
\]

\[
x^+(t/q)^{1/2}w)\psi^-(z) = f((t/q)^{-1/2}w/z; p^+), \quad \psi^-(z)x^+(t/q)^{1/2}w : |z/w| < 1,
\]
the
\[ p(w)x(w) = \psi(pwq^j u/w)[w] \]
and the formulas
\[ \psi^+(z)x^-(w) = \tilde{\psi}^+(z), \]  
\[ \psi^+(z)x^-(w) = \tilde{\psi}^-(\gamma^{-1} z). \]  

Here, \( f(z; s) \) is given by
\[ f(z; s) = \frac{(q^{-1} z; s)_{\infty}(t z; s)_{\infty}(qt^{-1} z; s)_{\infty}}{(qz; s)_{\infty}(t^{-1} z; s)_{\infty}(tq^{-1} z; s)_{\infty}} \]
for \( s = p, p^* \).

A similar representation is given in Lemma A.16 in [23].

### 3.2 Vector representation and the \( q \)-Fock space representation

We next consider the elliptic analogue of a representation given in [17, 18, 22], i.e., the \( q \)-Fock space representation of \( \U_{q,t,p} \).

For \( u \in \mathbb{C}^* \), let \( V(u) \) be a vector space spanned by \([u]_j \) \((j \in \mathbb{Z})\).

**Proposition 3.3** By the following action, \( V(u) \) is a level-(0, 0) \( \U_{q,t,p} \)-module. We call \( V(u) \) the vector representation.

\[
\begin{align*}
x^+(z)[u]_j &= a^+(p)\delta(q^j u/z)[u]_{j+1}, \\
x^-(z)[u]_j &= a^-(p)\delta(q^{-1} u/z)[u]_{j-1}, \\
\psi^\pm(z)[u]_j &= \frac{\theta_p(q^j u/z)\theta_p(q^{-1} u/z)}{\theta_p(q^j u/z)\theta_p(q^{-1} u/z)} |u\rangle_j, \\
\psi^+_0[u]_j &= [u]_j,
\end{align*}
\]
where we set
\[
\begin{align*}
a^+(p) &= (1 - t)\frac{(pt/q; p)_{\infty}(p/t; p)_{\infty}}{(p; p)_{\infty}(pq; p)_{\infty}}, \\
a^-(p) &= (1 - t^{-1})\frac{(pq/t; p)_{\infty}(pt; p)_{\infty}}{(p; p)_{\infty}(pq; p)_{\infty}}.
\end{align*}
\]

**Proof** Use a formula in Lemma 3.11.  

Let \( \mathcal{F}_u \) be a vector space spanned by \([\lambda]_u \) \((\lambda \in \mathcal{P}^+)\), where
\[ \mathcal{P}^+ = \{ \lambda = (\lambda_1, \lambda_2, \ldots) \mid \lambda_i \geq \lambda_{i+1}, \lambda_i \in \mathbb{Z}, \lambda_l = 0 \text{ for sufficiently large } l \}. \]  

(3.9)

We denote by \( \ell(\lambda) \) the length of \( \lambda \in \mathcal{P}^+ \), i.e., \( \lambda \ell(\lambda) > 0 \) and \( \lambda \ell(\lambda) + 1 = 0 \). We also set \( |\lambda| = \sum_{i \geq 1} \lambda_i \) and denote by \( \lambda' \) the conjugate of \( \lambda \).

**Theorem 3.4** The following action gives a level \((0,1)\) representation of \( \mathcal{U}_{q,t,p} \) on \( \mathcal{F}_u \). We denote this by \( \mathcal{F}^{(0,1)}_u \).

\[ \gamma^{1/2}|\lambda\rangle_u = |\lambda\rangle_u, \]

(3.10)

\[ x^+(z)|\lambda\rangle_u = a^+(p) \sum_{i=1}^{\ell(\lambda)} A^+_{\lambda,i}(p) \delta(u_i/z)|\lambda + 1_i\rangle_u, \]

(3.11)

\[ x^-(z)|\lambda\rangle_u = (qt)^{1/2}a^-(p) \sum_{i=1}^{\ell(\lambda)} A^-_{\lambda,i}(p) \delta(q^{-1}u_i/z)|\lambda - 1_i\rangle_u, \]

(3.12)

\[ \psi^+(z)|\lambda\rangle_u = (qt)^{1/2}B^+_{\lambda}(u/z; p)|\lambda\rangle_u, \]

(3.13)

\[ \psi^-(z)|\lambda\rangle_u = (qt)^{-1/2}B^-_{\lambda}(z/u; p)|\lambda\rangle_u, \]

(3.14)

where we set

\[ A^+_{\lambda,i}(p) = \prod_{j=1}^{i-1} \frac{\theta_p(tu_i/u_j)\theta_p(qt^{-1}u_i/u_j)}{\theta_p(qu_i/u_j)\theta_p(u_i/u_j)}, \]

(3.15)

\[ A^-_{\lambda,i}(p) = \prod_{j=i+1}^{\ell(\lambda)} \frac{\theta_p(qt^{-1}u_j/u_i)}{\theta_p(u_j/u_i)} \prod_{j=i+1}^{\ell(\lambda)+1} \frac{\theta_p(tu_j/u_i)}{\theta_p(qu_j/u_i)}, \]

(3.16)

\[ B^+_{\lambda}(u/z; p) = \prod_{i=1}^{\ell(\lambda)} \frac{\theta_p(u_i/tz)}{\theta_p(u_i/qz)} \prod_{i=1}^{\ell(\lambda)+1} \frac{\theta_p(tu_i/qz)}{\theta_p(u_i/z)}, \]

(3.17)

\[ B^-_{\lambda}(z/u; p) = \prod_{i=1}^{\ell(\lambda)} \frac{\theta_p(tz/u_i)}{\theta_p(qz/u_i)} \prod_{i=1}^{\ell(\lambda)+1} \frac{\theta_p(qz/tu_i)}{\theta_p(z/u_i)}. \]

(3.18)

A direct proof showing these actions satisfy the defining relation of \( \mathcal{U}_{q,t,p} \) is given in Appendix B. In Appendix C, we also give an inductive derivation of Theorem 3.4.

**Remark 1** In the trigonometric case obtained by taking \( p \to 0 \), the level \((0,1)\) representation in Theorem 3.4 is identified with the geometric representation of \( \mathcal{U}_{q,t} \) on \( \bigoplus_N K_T(\text{Hilb}_N(\mathbb{C}^2)) \), the direct sum of the \( T = \mathbb{C}^\times \times \mathbb{C}^\times \text{equivariant } K\text{-theory of the Hilbert scheme of } N \text{ points on } \mathbb{C}^2 \) [17, 18, 22]. There the basis \( \{ |\lambda\rangle_u \} \) in \( \mathcal{F}_u \) is identified with the fixed point basis \( \{ [\lambda] \} \) in \( K_T(\text{Hilb}_N(\mathbb{C}^2)) \). We conjecture [54] that the same is true in the elliptic case. Namely, if one could properly formulate a geometric action of \( \mathcal{U}_{q,t,p} \) on the direct sum of the equivariant elliptic cohomology of the Hilbert scheme \( \bigoplus_N E_T(\text{Hilb}_N(\mathbb{C}^2)) \), it should be identified with the level \((0,1)\).
representation in Theorem 3.4 by identifying $|\lambda_u\rangle$ with the fixed point class $[\lambda]$ in $\bigoplus_N E_T(\text{Hilb}_N(\mathbb{C}^2))$, where the latter bases can be realized in terms of the elliptic stable envelopes [2] on $E_T(\text{Hilb}_N(\mathbb{C}^2))$ constructed in [84] by

$$[\lambda] = \sum_{\mu \in \mathcal{P}^+} \text{Stab}_e^{-1}(\mu)|_{\text{Stab}}(\mu)$$

in the similar way as the case of the equivariant elliptic cohomology of the partial flag variety [53]. See also [82, 83].

**Remark 2** Again from the trigonometric limit of (C.4), one finds the eigenvalue of $\alpha_1(= a_1)$ on $|\lambda_u^{(N)}\rangle$ is given by

$$-(1 - t/q)(1 - t) t^{-N} \sum_{j=1}^{N} q^{\lambda_j} t^{N-j}.$$  

Under the automorphism in Theorem 2.2, $a_1$ is identified with $(1 - t/q)X_0^+$. Hence, we obtain the eigenvalue of $X_0^+$ as $-(1 - t)t^{-N} \sum_{j=1}^{N} q^{\lambda_j} t^{N-j}$. Then, under the identification of $q^{N|\lambda_u^{(N)}\rangle}$ with the specialization of the Macdonald symmetric polynomial $P_{\lambda}(t^0 q^\lambda)$, where the variables $x = (x_1, \ldots, x_N)$ are specialized by $t^0 q^\lambda = (t^{-N} q^{\lambda_1}, t^{-N-2} q^{\lambda_2}, \ldots, q^{\lambda_N})$. This is consistent with the identification that the trigonometric limit of (C.2) is the Pieri formula [60]:

$$e_1(x) P_{\lambda}(x) = \sum_{i=1}^{N} \psi_{\lambda+1_i/\lambda} P_{\lambda+1_i}(x)$$

where $e_1(x) = \sum_{j=1}^{N} x_j$ and

$$\psi_{\lambda+1_i/\lambda} = \prod_{j=1}^{i-1} \frac{(1 - tu_i/u_j)(1 - qu_i/tu_j)}{(1 - u_i/u_j)(1 - qu_i/u_j)}.$$  

See also [17, 18] for such identification and its extension to the case $N \to \infty$ in [22]. Hence, it is natural to expect that the action of the elliptic generator $X_0^+ = \oint_0 \frac{dz}{2\pi i z} x^+(z)$ on $|\lambda_u^{(N)}\rangle$ in (C.2) gives an elliptic analogue of the Pieri formula [54] and the eigenvalue of $x_0^+$ should be given in terms of an elliptic analogue of $e_1(t^0 q^\lambda)$, which is unknown yet, times $t^{-N} a^+(p)$.

These two remarks suggest that $x_0^+$ behaves as the elliptic Ruijsenaars difference operator on $W^{(N)}(u)$ and the bases vector $|\lambda_u^{(N)}\rangle$ as an elliptic analogue of the Macdonald symmetric polynomials with the eigenvalue given by an elliptic analogue of $e_1(t^0 q^\lambda)$, which should satisfy the elliptic analogue of the Pieri formula (C.2). See also [58, 64], where the fixed point classes in the equivariant homology groups of Hilbert schemes of points on $\mathbb{C}^2$ are identified with the Jack symmetric functions.
Sect. 3.3, we give more explicit relation of $x_0^+$ to the elliptic Ruijsenaars difference operator.

### 3.2.1 Some elliptic formulas

Let us consider the combinatorial factors $c_\lambda, c'_\lambda$ appearing in the inner product of the Macdonald symmetric functions as

\[
\langle P_\lambda, P_\lambda \rangle_{q,t} = \frac{c'_\lambda}{c_\lambda},
\]

\[
c_\lambda = \prod_{\square \in \lambda} (1 - q^{a(\square)} t^{\ell(\square)+1}) = \prod_{1 \leq i \leq j \leq \ell(\lambda)} \frac{(q^{\lambda_i - \lambda_j} t^{j-i+1}; q)_\infty}{(q^{\lambda_i - \lambda_j+1} t^{j-i+1}; q)_\infty},
\]

\[
c'_\lambda = \prod_{\square \in \lambda} (1 - q^{a(\square)}+1 t^{\ell(\square)}) = \prod_{1 \leq i \leq j \leq \ell(\lambda)} \frac{(q^{\lambda_i - \lambda_j+1} t^{j-i}; q)_\infty}{(q^{\lambda_i - \lambda_j+1+t^{j-i}}; q)_\infty}.
\]

Here, $a(\square) \equiv a_\lambda(\square) = \lambda_i - j$, $\ell(\square) \equiv \ell_\lambda(\square) = \lambda'_j - i$ for $\square = (i, j) \in \lambda$. The second expressions for $c_\lambda, c'_\lambda$ follow from the formula due to Macdonald \[60\]

\[
(1 - q) \sum_{(i,j)\in\lambda} q^{\lambda_i-j} t^{\lambda'_j-i+1} = t \sum_{1 \leq i \leq \ell(\lambda)} q^{\lambda_i-\lambda_j} t^{j-i} - \sum_{1 \leq i < \ell(\lambda)+1} q^{\lambda_i-\lambda_j} t^{j-i}.
\]

We introduce elliptic analogues of $c_\lambda, c'_\lambda$ as follows.

\[
c_\lambda(p) = \prod_{\square \in \lambda} \theta_p(q^{a(\square)} t^{\ell(\square)+1}) = \prod_{1 \leq i \leq j \leq \ell(\lambda)} \frac{\Gamma(q^{\lambda_i - \lambda_j+1} t^{j-i+1}; q, p)}{\Gamma(q^{\lambda_i - \lambda_j+1} t^{j-i+1}; q, p)},
\]

\[
c'_\lambda(p) = \prod_{\square \in \lambda} \theta_p(q^{a(\square)}+1 t^{\ell(\square)}) = \prod_{1 \leq i \leq j \leq \ell(\lambda)} \frac{\Gamma(q^{\lambda_i - \lambda_j+1} t^{j-i}; q, p)}{\Gamma(q^{\lambda_i - \lambda_j+1+t^{j-i}}; q, p)}.
\]

One can verify the following.

**Proposition 3.5**

\[
\frac{c_{\lambda+1k}(p)}{c_\lambda(p)} = \prod_{i=1}^{k-1} \frac{\theta_p(t q^{-1} u_i / u_k)}{\theta_p(q^{-1} u_i / u_k)} \prod_{i=k+1}^{\ell(\lambda)+1} \frac{\theta_p(u_k / u_i)}{\theta_p(q t u_k / u_i)},
\]

\[
\frac{c'_{\lambda+1k}(p)}{c'_\lambda(p)} = \prod_{i=1}^{k-1} \frac{\theta_p(u_i / u_k)}{\theta_p(t^{-1} u_i / u_k)} \prod_{i=k+1}^{\ell(\lambda)+1} \frac{\theta_p(q t u_k / u_i)}{\theta_p(q u_k / u_i)}.
\]

The following formulas are useful in Sect. 4.
Proposition 3.6  Let us consider
\[ A^+_{\lambda, i}(p) = \prod_{j=i+1}^{\ell(\lambda)} \frac{\theta_p(tu_i/u_j)}{\theta_p(qu_i/u_j)} \prod_{j=i+1}^{\ell(\lambda)+1} \frac{\theta_p(qu_i/tu_j)}{\theta_p(u_i/u_j)}, \]
\[ A^-_{\lambda, i}(p) = \prod_{j=1}^{i-1} \frac{\theta_p(tu_j/u_i)\theta_p(qu_j/tu_i)}{\theta_p(qu_j/u_i)\theta_p(u_j/u_i)}. \]

We have
\[ A^{+'}_{\lambda, i}(p) = \frac{c_{\lambda}(p)c'_{\lambda+1_i}(p)}{c'_{\lambda}(p)c_{\lambda+1_i}(p)} A^+_{\lambda, i}(p) \]
\[ A^{-''}_{\lambda, i}(p) = \frac{c_{\lambda}(p)c'_{\lambda-1_i}(p)}{c'_{\lambda}(p)c_{\lambda-1_i}(p)} A^-_{\lambda, i}(p). \]

By a direct calculation, one can also verify the following formulas.

Proposition 3.7
\[ A^{+'}_{\lambda, i}(p) = (q/t)A^-_{\lambda+1, i}(p), \] \[ A^{-''}_{\lambda, i}(p) = (t/q)A^+_{\lambda-1, i}(p). \]

Hence, we have

Proposition 3.8
\[ \frac{c_{\lambda}(p)}{c_{\lambda+1}(p)} \frac{c'_{\lambda+1_i}(p)}{c'_{\lambda}(p)} A^+_{\lambda, i}(p) = (q/t)A^-_{\lambda+1, i}(p), \]
\[ \frac{c_{\lambda}(p)}{c_{\lambda-1}(p)} \frac{c'_{\lambda-1_i}(p)}{c'_{\lambda}(p)} A^-_{\lambda, i}(p) = (t/q)A^+_{\lambda-1, i}(p). \]

We also need to introduce \( N_{\lambda}(p) \) and \( N_{\lambda}'(p) \) by
\[ \frac{N_{\lambda}(p)}{N_{\lambda+1}(p)} = \prod_{j=1}^{i-1} \frac{(pu_j/tu_i; p)_{\infty}(ptu_j/qu_i; p)_{\infty}}{(pu_j/qu_i; p)_{\infty}(pu_j/u_i; p)_{\infty}} \]
\[ \times \prod_{j=i+1}^{\ell(\lambda)} \frac{(pu_i/u_j; p)_{\infty}}{(ptu_i/u_j; p)_{\infty}} \prod_{j=i+1}^{\ell(\lambda)+1} \frac{(ptu_i/tu_j; p)_{\infty}}{(pu_i/tu_j; p)_{\infty}}. \]
\[ \times \prod_{j=i+1}^{\ell(\lambda)} \frac{(pu_j/tu_i; p)_\infty}{(pu_j/qu_i; p)_\infty} \prod_{j=i+1}^{\ell(\lambda)+1} \frac{(ptu_j/qu_i; p)_\infty}{(pu_j/u_i; p)_\infty}. \] (3.35)

One can derive the following expressions.

\[
N_\lambda(p) = \prod_{1 \leq i < j \leq \ell(\lambda)} \frac{(pu_j/u_i; q, p)_\infty}{(ptu_i/u_j; q, p)_\infty} \prod_{1 \leq i \leq j \leq \ell(\lambda)+1} \frac{(pu_i/u_j; q, p)_\infty}{(pqu_i/tu_j; q, p)_\infty},
\] (3.36)

\[
N'_\lambda(p) = \prod_{1 \leq i < j \leq \ell(\lambda)+1} \frac{(pu_j/u_i; q, p)_\infty}{(pqu_j/tu_i; q, p)_\infty} \prod_{1 \leq i \leq j \leq \ell(\lambda)+1} \frac{(pu_i/u_j; q, p)_\infty}{(pu_i/qu_j; q, p)_\infty},
\] (3.37)

Then, one finds the following property.

**Proposition 3.9**

\[
\frac{c'_\lambda N_\lambda(p)}{c'_\lambda N'_\lambda(p)} = \frac{c'_\lambda(p)}{c_\lambda(p)}. \] (3.38)

### 3.3 Level (0,0) representation and elliptic Ruijsenaars operators

The two remarks after Proposition 3.9 suggest a connection of the level (0,1) or (0,0) representations of $U_{q,t,p}$ to a possible elliptic analogue of the Macdonald symmetric functions, which are expected to be eigenfunctions of the elliptic Ruijsenaars difference operator. In this subsection, we show a direct relation between the level (0,0) representation and the elliptic Ruijsenaars difference operator

\[
D = \sum_{i=1}^{N} \prod_{j \neq i} \frac{\theta_p(x_i/x_j)}{\theta_p(x_i/x_j)} T_{q,x_i},
\]

which acts on $\mathbb{C}[[x_1^{\pm 1}, \ldots, x_N^{\pm 1}]]$. Here, $T_{q,x_i}$ denotes the shift operator

\[ T_{q,x_i} f(x_1, \ldots, x_i, \ldots x_n) = f(x_1, \ldots, qx_i, \ldots x_n). \]

The following theorem is an elliptic analogue of Proposition 3.3 in [62].
Theorem 3.10 \textit{The following assignment gives a level (0, 0) representation of } U_{q,t,p} \textit{on } \mathbb{C}[[x_1^{\pm 1}, \ldots, x_N^{\pm 1}]].

\begin{equation}
\alpha_m = \frac{(1 - t^{-m})(1 - (q/t)^{-m})}{m^N m} \sum_{j=1}^{N} x_j^m \quad (m \in \mathbb{Z} \setminus \{0\}), \tag{3.41}
\end{equation}

or

\begin{equation}
\psi^{+}(z) = \prod_{j=1}^{N} \frac{\theta_p(t^{-1}x_j/z) \theta_p(q^{-1}tx_j/z)}{\theta_p(x_j/z) \theta_p(q^{-1}x_j/z)}, \tag{3.42}
\end{equation}

\begin{equation}
\psi^{-}(z) = \prod_{j=1}^{N} \frac{\theta_p(tz/x_j) \theta_p(q^{-1}z/x_j)}{\theta_p(z/x_j) \theta_p(qz/x_j)}. \tag{3.43}
\end{equation}

In particular, the zero-mode \( x_0^{+} = \oint_{|z|=0} \frac{dz}{2\pi i z} x^{+}(z) \) acts as the elliptic Ruijsenaars difference operator

\[ x_0^{+} = a^{+}(p) \sum_{i=1}^{N} \prod_{j \neq i} \frac{\theta_p(tx_i/x_j)}{\theta_p(x_i/x_j)} T_{q,x_i} \]

\textbf{Proof} Let us check the relation (2.18).

\[ [x^{+}(z), x^{-}(w)] = -a^{+}(p)a^{-}(p)\delta(z/w) \sum_{i=1}^{N} \left( \prod_{j \neq i} \frac{\theta_p(tx_i/x_j)}{\theta_p(x_i/x_j)} \prod_{k \neq i} \frac{\theta_p(qx_i/tx_k)}{\theta_p(x_i/tx_k)} \delta(x_i/z) \right. \]

\[ -\prod_{j \neq i} \frac{\theta_p(tx_i/qx_j)}{\theta_p(x_i/qx_j)} \prod_{k \neq i} \frac{\theta_p(qx_i/x_k)}{\theta_p(x_i/x_k)} \delta(q^{-1}x_i/z) \].

Then, the relation (2.18) holds by the following lemma.
Lemma 3.11
\[
\prod_{j=1}^{N} \frac{\theta_p(qz/tx_j)\theta_p(tz/x_j)}{\theta_p(z/x_j)\theta_p(qz/x_j)} - \prod_{j=1}^{N} \frac{\theta_p(qz/tx_j)\theta_p(tz/x_j)}{\theta_p(z/x_j)\theta_p(qz/x_j)} = -\frac{\theta_p(q/t)\theta_p(t)}{(p; p)_{\infty}^2 \theta_p(q)} \left( \sum_{i=1}^{N} \prod_{j \neq i} \frac{\theta_p(tx_i/x_j) \theta_p(qx_i/tx_k) \theta_p(x_i/tx_k)}{\theta_p(x_i/qx_j) \theta_p(x_i/qx_k)} \delta(x_i/z) \right).
\]

Proof Note the partial fraction expansion formula, see for example [75],
\[
\theta_p(s/b_{m+1}) \prod_{j=1}^{m} \frac{\theta_p(s/b_j)}{\theta_p(s/a_j)} = -\sum_{k=1}^{m} \frac{1}{\theta_p(a_k/s)} \prod_{j \neq k}^{m+1} \frac{\theta_p(a_k/b_j)}{\theta_p(a_k/a_j)} \quad (3.44)
\]
with the balancing condition \( b_1 \cdots b_{m+1} = a_1 \cdots a_m s \). Let us consider the case \( m = 2N \) and take
\[
s = z, \quad a_j = x_j, \quad a_{N+j} = q^{-1}x_j, \quad b_j = (t/q)x_j, \quad b_{N+j} = \beta^{-1}x_j \quad (j = 1, \ldots, N), \quad b_{2N+1} = (\beta/t)^N z,
\]
for some constant \( \beta \). Then, one has
\[
\prod_{j=1}^{N} \frac{\theta_p(qz/tx_j)}{\theta_p(z/x_j)} \prod_{j=1}^{N} \frac{\theta_p(\beta z/x_j)}{\theta_p(qz/x_j)} = -\sum_{i=1}^{N} \frac{\theta_p((t/\beta)^N x_i/z) \theta_p(q/t)}{\theta_p((t/\beta)^N x_i/x_j) \theta_p(q/tx_j)} \prod_{j \neq i}^{N} \frac{\theta_p(qx_i/tx_k) \theta_p(\beta x_i/x_k)}{\theta_p(qx_i/x_j) \theta_p(x_i/qx_k)} \times \frac{\theta_p(\beta)}{\theta_p(q)}
\]
\[
-\sum_{i=1}^{N} \frac{\theta_p((t/\beta)^N x_i/z) \theta_p(\beta/q)}{\theta_p((t/\beta)^N x_i/qz) \theta_p(x_i/qz)} \prod_{j \neq i}^{N} \frac{\theta_p(qx_i/tx_k) \theta_p(\beta x_i/x_k)}{\theta_p(qx_i/x_j) \theta_p(x_i/qx_k)} \prod_{k \neq i}^{N} \frac{\theta_p(\beta x_i/qx_k)}{\theta_p(x_i/x_k)}.
\]

Hence, we obtain
\[
\prod_{j=1}^{N} \frac{\theta_p(qz/tx_j)}{\theta_p(z/x_j)} \prod_{j=1}^{N} \frac{\theta_p(\beta z/x_j)}{\theta_p(qz/x_j)} = -\frac{\theta_p(\beta) \theta_p(q/t)}{(p; p)_{\infty}^2 \theta_p(q)} \sum_{i=1}^{N} \theta_p((x_i/z) \prod_{j \neq i}^{N} \frac{\theta_p(qx_i/tx_j)}{\theta_p(x_i/x_j)} \prod_{k \neq i}^{N} \frac{\theta_p(\beta x_i/x_k)}{\theta_p(qx_i/x_k)}.
\]
\[ -\theta_p(t^{-1})\theta_p(\beta/q) \sum_{i=1}^{N} \delta(q^{-1}x_i/z) \cdot \prod_{j=1 \atop j \neq i}^{N} \theta_p(x_i/x_j) \cdot \prod_{k=1 \atop k \neq i}^{N} \theta_p(\beta x_i/q x_k) \]

Then, taking the limit \( \beta \to t \), one obtains the desired formula.

4 The vertex operators

We construct the two vertex operators \( \Phi(u) \) and \( \Psi^*(u) \) of \( U_{q,t,p} \) called the type I and the type II dual vertex operators [38] as intertwining operators of \( U_{q,t,p} \)-modules. These two vertex operators are the elliptic analogues of those constructed in [5], whose matrix elements reproduce the refined topological vertex in [4, 36, 85]. We also construct a shifted inverse of them denoted by \( \Phi^*(u) \) and \( \Psi(u) \), respectively. These vertex operators turn out to be useful to realize the affine quiver \( W \) algebra and instanton calculus in the affine quiver gauge theories. See Sects. 5 and 6.

4.1 The type I vertex operator

The type I vertex operator is the intertwining operator

\[ \Phi(u) : \mathcal{F}_{-uv}^{(1,N+1)} \to \mathcal{F}_u^{(0,1)} \otimes \mathcal{F}_v^{(1,N)} \]

w.r.t. the comultiplication \( \Delta \) satisfying

\[ \Delta(x)\Phi(u) = \Phi(u)x \quad (\forall x \in U_{q,t,p}). \quad (4.1) \]

We define the components of \( \Phi(u) \) by

\[ \Phi(u)|\xi\rangle = \sum_{\lambda \in \mathcal{P}^+} \frac{c_{\lambda}(p)}{c_{\lambda}'(p)} \Phi_{\lambda}(u)|\xi\rangle \quad \forall |\xi\rangle \in \mathcal{F}_{-uv}^{(1,N+1)}, \quad (4.2) \]

where we set

\[ |\lambda\rangle_u = \frac{c_{\lambda}(p)}{c_{\lambda}'(p)} |\lambda\rangle_u. \quad (4.3) \]

Lemma 4.1 The intertwining relation (4.1) reads

\[ \Phi_{\lambda}(u)\psi^+((t/q)^{1/4}z) = (q/t)^{1/2}B^+_{\lambda}(u/z; p)\psi^+((t/q)^{1/4}z)\Phi_{\lambda}(u), \quad (4.4) \]

\[ \Phi_{\lambda}(u)\psi^-((t/q)^{-1/4}z) = (q/t)^{-1/2}B^-_{\lambda}(z/u; p)\psi^-((t/q)^{-1/4}z)\Phi_{\lambda}(u), \quad (4.5) \]

\[ \Phi_{\lambda}(u)x^+((t/q)^{-1/4}z) = x^+((t/q)^{-1/4}z)\Phi_{\lambda}(u) + qf(1; p)^{-1}\psi^-((t/q)^{-1/4}z) \]
\[ \times \sum_{i=1}^{\ell(\lambda)+1} a^-_{\lambda,i}(p)A^-_{\lambda,i}(p)\delta(q^{-1}u_i/z)\Phi_{\lambda-1,i}(u), \quad (4.6) \]
\[
\Phi_\lambda(u)x^-(t/q)^{1/4}z = (q/t)^{1/2}B^+_\lambda(u/z; p)x^-(t/q)^{1/4}z\Phi_\lambda(u) + q^{-1}f(1; p) \\
\times(q/t)^{1/2}\sum_{i=1}^{\ell(\lambda)}a^+(p)\Lambda^+_{\lambda,i}(p)\delta(u_i/z)\Phi_{\lambda+1}(u).
\] (4.7)

**Proof** For example, let us consider

\[
\Delta(x^+(z))\Phi(u)|\xi\rangle = \Phi(u)x^+(z)|\xi\rangle \quad \forall|\xi\rangle \in F^{(1,N+1)}.
\]

\[
\text{RHS} = \sum_\lambda |\lambda\rangle_u^\dagger \tilde{\om}_{\lambda}(u)x^+(z)|\xi\rangle,
\]

\[
\text{LHS} = \sum_\lambda |\lambda\rangle_u^\dagger \tilde{\phi} x^+(z)\Phi_\lambda(u)|\xi\rangle + \sum_\lambda x^+(t/q)^{1/4}z|\lambda\rangle_u^\dagger \tilde{\om}^-(z)\Phi_\lambda(u)|\xi\rangle
\]

\[
= \sum_\lambda |\lambda\rangle_u^\dagger \tilde{\phi} x^+(z)\Phi_\lambda(u)|\xi\rangle
\]

\[
+ \sum_{\lambda} \sum_{i=1}^{\ell(\lambda)+1} a^+(p)\Lambda^+_{\lambda,i}(p)\delta((t/q)^{-1/4}u_i/z)|\lambda + 1\rangle_u^\dagger \tilde{\om}^-(z)\Phi_\lambda(u)|\xi\rangle
\]

\[
= \sum_\lambda |\lambda\rangle_u^\dagger \left\{ x^+(z)\Phi_\lambda(u)
\right. \]

\[
+ qf(1, p)^{-1}\sum_{\lambda} \sum_{i=1}^{\ell(\lambda)+1} a^-(p)\Lambda^-_{\lambda,i}(p)\delta((t/q)^{-1/4}u_i/z)|\lambda + 1\rangle_u^\dagger \tilde{\om}^-(z)\Phi_\lambda(u)|\xi\rangle
\]

The third equality follows from (3.30). \hfill \Box

By using the representations in Theorems 3.2 and 3.4, one can solve these intertwining relations and obtain the following result.

**Theorem 4.2**

\[
\Phi_\lambda(u) = \frac{q^n(\lambda') N_\lambda(p)t^*(\lambda', u, v, N)}{c_\lambda} \tilde{\Phi}_\lambda(u),
\]

\[
\tilde{\Phi}_\lambda(u) := \Phi_\theta(u) \prod_{i=1}^{\ell(\lambda)} \prod_{j=1}^{\lambda_j} \bar{x}^-((t/q)^{1/4}q^{j-1}i^{-i+1}u),
\]

\[
\Phi_\theta(u) = \exp \left\{ -\sum_{m>0} \frac{1}{\kappa_m} \alpha'_m((t/q)^{1/2}u)^m \right\} \exp \left\{ \sum_{m>0} \frac{1}{\kappa_m} \alpha'_m((t/q)^{1/2}u)^{-m} \right\}
\]
where \( x^-(z) = u^{-1} z^{N (t/q)^{-3 N/4}} \tilde{x}^-(z) \) on \( \mathcal{F}^{(1,N)}_\mu \)

\[
n(\lambda) = \sum_{i \geq 1} (i - 1) \lambda_i, \quad n(\lambda') = \sum_{i \geq 1} (i - 1) \lambda'_i = \frac{\sum_{i \geq 1} \lambda_i (\lambda_i - 1)}{2},
\]

\[
t^*(\lambda, u, v, N) = (q^{-1} v)^{-|\lambda|} (-u)^{|\lambda|} f_{\lambda}(q, t)^N,
\]

\[
f_{\lambda}(q, t) = (-1)^{|\lambda|} q^{n(\lambda')} + (-1)^{|\lambda|/2} t^{-n(\lambda) - |\lambda|/2}.
\]

The factor \( t^*(\lambda, u, v, N) \) was introduced in [5] and in particular \( f_{\lambda}(q, t) \) is called the framing factor [4, 85]. Our vertex operator \( \Phi(u) \) is the elliptic analogue of \( \Phi^{*}(u) \) in [5]. One should note that our comultiplication is opposite from the one in [5]. A proof of the statement is given in Appendix D.

In later sections, the following formula is useful.

**Proposition 4.3**

\[
\tilde{\Phi}_{\lambda}(u) = \exp \left( \sum_{m \neq 0} \frac{1}{k_m} \mathcal{E}_{\lambda,m} \alpha'_m ((t/q)^{1/2} u)^{-m} \right),
\]

where we set

\[
\mathcal{E}_{\lambda,m} = \frac{1}{1 - t^m} + \sum_{j=1}^{\ell(\lambda)} (q^{-m \lambda_j} - 1) t^{m(j-1)} \quad (m \in \mathbb{Z}_{\neq 0}).
\]

### 4.2 The type II dual vertex operator

The type II dual vertex operator is the intertwining operator \( \Psi^*(v) \) satisfying

\[
\lambda^-(\lambda, u, v, N) = (q^{-1} v)^{-|\lambda|} (-u)^{|\lambda|} f_{\lambda}(q, t)^N.
\]

We call \( \Psi^*(v) \) the type II dual vertex operator. We define its components by

\[
\Psi^*_{\lambda}(v) |\xi\rangle = \Psi^*(v) (|\xi\rangle \otimes |\lambda\rangle) \quad \forall |\xi\rangle \in \mathcal{F}^{(1,N)}_\mu.
\]

**Lemma 4.4** The intertwining relation (4.13) is equivalent to

\[
\psi^+((q/t)^{1/4} z) \Psi^*_{\lambda}(v) = (q/t)^{1/2} B^+_\lambda (v/z; p^*) \Psi^*_{\lambda}(v) \psi^+((q/t)^{1/4}),
\]

\[
\psi^-((q/t)^{-1/4} z) \Psi^*_{\lambda}(v) = (q/t)^{-1/2} B^-_\lambda (z/v; p^*) \Psi^*_{\lambda}(v) \psi^-((q/t)^{-1/4}).
\]
\[ x^-( (q/t)^{1/4} z) \Psi_\lambda^*(v) = \Psi_\lambda^*(v) x^- ( (q/t)^{1/4} z) + (q/t)^{1/2} a^- (p^*) \]
\[ \times \sum_{i=1}^{\ell(\lambda)} A_{\lambda,i}^- (p^*) \delta (q^{-1} v_i / z) \tilde{\Psi}_{\lambda^{-}1_i}^*(v) \psi^+ ( (q/t)^{1/4} z), \]
\[ (4.17) \]
\[ x^+ ( (q/t)^{-1/4} z) \Psi_\lambda^*(v) = (q/t)^{-1/2} B_\lambda^-(z/v; p^*) \Psi_\lambda^*(v) x^+ ( (q/t)^{-1/4} z) \]
\[ + a^+ (p^*) \sum_{i=1}^{\ell(\lambda)+1} A_{\lambda,i}^+ (p^*) \delta (v_i / z) \tilde{\Psi}_{\lambda^{-}1_i}^*(v). \]
\[ (4.18) \]

Here, \( p^* = pq/t \) associated with \( F_u^{(1,N)} \).

One can prove this in the similar way to Lemma 4.1.

By using the representations in Theorem 3.2, 3.4 and (4.3), one can solve these intertwining relations and obtain the following result.

**Theorem 4.5**

\[ \Psi_\lambda^*(v) = \frac{q^{n(\lambda')} t(\lambda, u, v, N)}{c_\lambda N'_\lambda (p^*)} \tilde{\Psi}_\lambda^*(v), \]
\[ \tilde{\Psi}_\lambda^*(v) = \Psi_{\theta}^* (v) \prod_{i=1}^{\ell(\lambda)} \prod_{j=1}^{\lambda_i} x^+ ( (t/q)^{1/4} q^{-1} t^{-i+1} v), \]
\[ (4.19) \]
\[ \Psi_{\theta}^*(v) = \exp \left\{ \sum_{m > 0} \frac{1}{\kappa_m} \alpha_{-m} ((t/q)^{1/2} v)^m \right\} \exp \left\{ - \sum_{m > 0} \frac{1}{\kappa_m} \alpha_{m} ((t/q)^{1/2} v)^{-m} \right\}. \]
\[ (4.20) \]

where \( x^+ (z) = uz^{-N} (t/q)^{3N/4} x^+ (z) \) on \( F_u^{(1,N)} \), and

\[ t(\lambda, u, v, N) = (-uv)^{\lambda^0} (-v)^{-(N+1)^0} f_{\lambda}(q, t)^{-N-1} \]

with \( f_{\lambda}(q, t) \) given in (4.10).

Our vertex operator \( \Psi^*(u) \) is the elliptic analogue of \( \Phi(u) \) in [5]. A proof of the statement is similar to the one in Appendix D.

We have a similar formula to Proposition 4.3.

**Proposition 4.6**

\[ \tilde{\Psi}_\lambda^*(v) = : \exp \left\{ - \sum_{m \neq 0} \frac{1}{\kappa_m} E_{\lambda,m} \alpha_{m} ((t/q)^{1/2} v)^{-m} \right\} :. \]
\[ (4.21) \]
4.3 The shifted inverse of $\Phi(u)$ and $\Psi^*(v)$

We next introduce the shifted inverse of $\Phi(u)$ and $\Psi^*(v)$ denoted by $\Phi^*(u)$ and $\Psi(v)$, respectively. In Sects. 5 and 6, we show that the vertex operators here and in the previous subsections are useful to derive physical and mathematical quantities associated with the Jordan quivers.

4.3.1 The vertex operator $\Phi^*(u)$

Let us consider the linear map

$$
\Phi^*(u) : \mathcal{F}^{(0,1)}_{u} \otimes \mathcal{F}^{(1,N)}_{v} \rightarrow \mathcal{F}^{(1,N+1)}_{-u}.
$$

whose components are defined by

$$
\Phi^*(u) \left( (\lambda)_{u} | \xi \rangle \right) = \Phi^+_\lambda (u) | \xi \rangle, \quad \forall | \xi \rangle \in \mathcal{F}^{(1,N)}_{v},
$$

$$
\Phi^+_\lambda (u) = \frac{q^{n(\lambda)} N^1_{\lambda}(p) t(\lambda, v, u p^{-1}, N)}{c_{\lambda}^{'}} : \tilde{\Phi}_\lambda (p^{-1} u)^{-1} :.
$$

Note that from Proposition 4.3 we have

$$
: \tilde{\Phi}_\lambda (p^{-1} u)^{-1} := \exp \left( - \sum_{m \neq 0} \frac{1 - t^m}{\kappa_m} p^m \epsilon_{\lambda,m} q_m (t/q)^{1/2} u^{-m} \right) :
$$

**Proposition 4.7** The vertex operator $\Phi^+_\lambda (u)$ satisfies the following relations.

$$
\psi^+((t/q)^{1/4} z) \Phi^+_\lambda (u) = (t/q)^{-1/2} B^+_{\lambda}(p^{-1} u/z; p) \Phi^+_\lambda (u) \psi^+((t/q)^{1/4} z),
$$

$$
\psi^-((t/q)^{-1/4} z) \Phi^+_\lambda (u) = (t/q)^{1/2} B^-_{\lambda}(p z/u; p) \Phi^+_\lambda (u) \psi^-((t/q)^{-1/4} z),
$$

$$
x^+((t/q)^{-1/4} z) \Phi^+_\lambda (u) = \Phi^+_\lambda (u) x^+((t/q)^{-1/4} z)
$$

$$
+ (t/q)^{-1/2} a^+(p) \sum_{i=1}^{\ell(\lambda)+1} A^+_\lambda i(p) \delta(p^{-1} i u / q z) \Phi^+_{\lambda+1, i} (u) \psi^+((t/q)^{1/4} q z / t),
$$

$$
x^-((t/q)^{1/4} z) \Phi^+_\lambda (u) = (t/q)^{-1/2} B^+_{\lambda}(p^{-1} u/z; p) \Phi^+_\lambda (u) x^-((t/q)^{1/4} z)
$$

$$
+ (t/q) a^- (p) \sum_{i=1}^{\ell(\lambda)} A^-\lambda i(p) \delta(q^{-1} p^{-1} i u / z) \Phi^+_{\lambda-1, i} (u).
$$

A proof of this statement is given in Appendix E.

These relations allow us to expect that $\Phi^*(u)$ is the intertwiner satisfying

$$
\Phi^*(u) \Delta(x) = x \Phi^*(u) \quad \forall x \in \mathcal{U}_{q,t, p}.
$$
Proposition 4.8 The intertwining relation (4.29) reads

\[
\psi^+ (t/q)^{1/4} z \Phi^*_\lambda(u) = (t/q)^{-1/2} B^+_{\lambda}(u/z; p) \Phi^*_\lambda(u) \psi^+ (t/q)^{1/4} z, \tag{4.30}
\]

\[
\psi^- (t/q)^{-1/4} z \Phi^*_\lambda(u) = (t/q)^{1/2} B^-_{\lambda}(z/u; p) \Phi^*_\lambda(u) \psi^- (t/q)^{-1/4} z, \tag{4.31}
\]

\[
x^+ (t/q)^{-1/4} z \Phi^*_\lambda(u) = \Phi^*_\lambda(u) x^+ ((t/q)^{-1/4} z)
+ a^+ (p) \sum_{i=1}^{\ell(\lambda)+1} A^+_{\lambda,i}(p) \delta(u_i/z) \Phi^*_{\lambda+1}(u) \psi^- ((t/q)^{-1/4} z), \tag{4.32}
\]

\[
x^- ((t/q)^{1/4} z) \Phi^*_\lambda(u) = (t/q)^{-1/4} z B^+_{\lambda}(u/z; p) \Phi^*_\lambda(u) x^- ((t/q)^{1/4} z)
+ (t/q)^{-1/2} a^-(p) \sum_{i=1}^{\ell(\lambda)} A^-_{\lambda,i}(p) \delta(q^{-1} u_i/z) \Phi^*_{\lambda-1}(u). \tag{4.33}
\]

This discrepancy is probably due to a lack of understanding the dual representation to \( F^{(0,1)}_u \). It is hence an open problem to find a representation theoretical meaning of \( \Phi^*(u) \).

4.3.2 The vertex operator \( \Psi(v) \)

Similar to \( \Phi^*(u) \), we consider the linear map

\[
\Psi(v) : \mathcal{F}^{(1,N+1)}_{-uv} \to \mathcal{F}^{(1,N)}_u \otimes \mathcal{F}^{(0,1)}_v
\]

and define its components by

\[
\Psi(v)|\xi\rangle = \sum_{\lambda \in \mathcal{P}^+} \Psi^v(\lambda)|\xi\rangle \otimes |\lambda\rangle_v \quad \forall |\xi\rangle \in \mathcal{F}^{(1,N+1)}_{-uv},
\]

\[
\Psi^v(\lambda) = \frac{q^{n(\lambda')}}{c^v_\lambda N^v(\lambda')} t^v(\lambda, p^*v, u, N) : \Psi^*_\lambda(p^*v)^{-1} :. \tag{4.35}
\]

Note that from Proposition 4.6 we have

\[
: \Psi^*_\lambda(p^*v)^{-1} := \exp \left(- \sum_{m \neq 0} \frac{1 - t^m}{\kappa_m} p_*^{-m} E_m \alpha_m ((t/q)^{1/2} v)^{-m} \right) :.
\]

One finds the following relations satisfied.

Proposition 4.9

\[
\Psi^v(\lambda) \psi^+ ((t/q)^{-1/4} z) = (t/q)^{-1/2} B^+_{\lambda}(p^*v/z; p^*) \psi^+ ((t/q)^{-1/4} z) \Psi^v(\lambda), \tag{4.36}
\]
\begin{equation}
\Psi_\lambda(v)\psi^-(t/q \frac{1}{4}z) = (t/q)^{1/2} B^-_\lambda(p^*z/v; p^*)\psi^-((t/q)\frac{1}{4}z)\Psi_\lambda(v),
\end{equation}

\begin{equation}
\Psi_\lambda(v)\psi^-((t/q)\frac{1}{4}z) = x^-((t/q)\frac{1}{4}z)\Psi_\lambda(v)
+ (t/q)a^-(p^*) \sum_{i=1}^{\ell(\lambda)+1} A^+_{\lambda,i}(p^*)\delta(p^*tv_i/qz)\psi^-((t/q)\frac{1}{4}z)\Psi_{\lambda+1_i}(v),
\end{equation}

\begin{equation}
\Psi_\lambda(v)\psi^+((t/q)\frac{1}{4}z) = (t/q)^{1/2} B^+_\lambda(z/v; p^*)\psi^+((t/q)\frac{1}{4}z)\Psi_\lambda(v)
+ (t/q)^{-1/2}a^+(p^*) \sum_{i=1}^{\ell(\lambda)} A^-_{\lambda,i}(p^*)\delta(p^*v_i/qz)\Psi_{\lambda-1_i}(v).
\end{equation}

A proof of this is similar to the one in Appendix E.
Again, these relations do not coincide with the intertwining relation

\begin{equation}
\Psi(v)x = \Delta(x)\Psi(v) \quad \forall x \in \mathcal{U}_{q,t,p},
\end{equation}

which is equivalent to the following relations:

**Proposition 4.10**

\begin{equation}
\Psi_\lambda(v)\psi^+((t/q)\frac{1}{4}z) = (t/q)^{-1/2} B^+_\lambda(v/z; p^*)\psi^+((t/q)\frac{1}{4}z)\Psi_\lambda(v),
\end{equation}

\begin{equation}
\Psi_\lambda(v)\psi^-((t/q)\frac{1}{4}z) = (t/q)^{1/2} B^-_\lambda(z/v; p^*)\psi^-((t/q)\frac{1}{4}z)\Psi_\lambda(v),
\end{equation}

\begin{equation}
\Psi_\lambda(v)x^-((t/q)\frac{1}{4}z) = x^-((t/q)\frac{1}{4}z)
+ (t/q)^{1/2}a^-(p^*) \sum_{i=1}^{\ell(\lambda+1_i)} A^+_{\lambda,i}(p^*)\delta(q^\lambda t^{-i+1}v/z)\psi^+((t/q)\frac{1}{4}z)\Psi_{\lambda+1_i}(v),
\end{equation}

\begin{equation}
\Psi_\lambda(v)x^+((t/q)\frac{1}{4}z) = (t/q)^{1/2} B^-_\lambda(z/v; p^*)x^+(((t/q)\frac{1}{4}z)\Psi_\lambda(v)
+ a^+(p^*) \sum_{i=1}^{\ell(\lambda-1)_i+1} A^-_{\lambda,i}(p^*)\delta(q^\lambda t^{-i+1}v/z)\Psi_{\lambda-1_i}(v).
\end{equation}

Hence, again it is an open problem to find a representation theoretical meaning of \(\Psi(u)\).

### 5 Affine quiver W-algebra \(W_{p,p^*}(\Gamma(\widehat{A}_0))\)

One of the importance to consider the elliptic quantum group is that it gives a realization of the deformed W algebras such as \(W_{p,p^*}(g)\) [27] and provides an algebraic structure, i.e., a co-algebra structure, which enables us to define the intertwining operators (the vertex operators) as deformation of the primary fields in CFT. In this section, we realize the deformed W algebra \(W_{p,p^*}(\Gamma(\widehat{A}_0))\) associated with the Jordan quiver \(\widehat{A}_0\).
by using the level \((1, N)\) representation of \(\mathcal{U}_{q,t,p}\) given in Sect. 3.1, where in particular \(\gamma = (t/q)^{1/2}\), in the same scheme as it was done for \(W_{p,p^*}(\hat{g})\) in terms of the level 1 representation of the elliptic quantum group \(U_{q,p}(\hat{g})\) [16, 40, 45, 47, 50].

5.1 Screening currents

Let us set

\[
s^+_m = \frac{(t/q)^{m/2}}{1 - (t/q)^m}{\alpha}_m, \quad s^-_m = \frac{(t/q)^{m/2}}{1 - (t/q)^m}{\alpha}'_m.
\]

Then from (2.15) and (2.24) with \(\gamma = (t/q)^{1/2}\), hence \(p^* = pq/t\), one can show the following commutation relations

\[
[s^+_m, s^+_n] = -\frac{1}{m} \frac{1 - p^m}{1 - p^{*m}}(1 - q^m)(1 - t^{-m})\delta_{m+n,0},
\]

\[
[s^-_m, s^-_n] = -\frac{1}{m} \frac{1 - p^{*m}}{1 - p^{-m}}(1 - q^m)(1 - t^{-m})\delta_{m+n,0}.
\]

Moreover, one can rewrite the elliptic currents \(x^\pm(z)\) in Theorem 3.2 as

\[
x^\pm((t/q)^{1/4}z) = ((t/q)^{N/2}u/z^N)^\pm1 \exp\left\{ \pm \sum_{m \neq 0} s^\pm_m z^{-m} \right\}.
\]

Hence, one of \(x^\pm((t/q)^{1/4}z)\) coincides with the screening currents of \(W_{p,p^*}(\Gamma(\hat{A}_0))\) [41, 42] with the \(SU(4)\ \Omega\)-deformation parameters \(p, p^*, q, t\) [70] satisfying

\[
p/p^* = t/q.
\]

In our knowledge the elliptic quantum toroidal algebra \(\mathcal{U}_{q,t,p}\) is the first quantum group structure which possesses the \(SU(4)\ \Omega\)-deformation parameters.

One should also note that in [41, 42] Kimura and Pestun constructed only one type of screening current. However, it is natural for the (deformed) \(W\) algebras that there are two types of screening currents [7, 12, 19, 27]. In this sense, our realization completes their construction.

5.2 Generating function

To obtain the generating function of \(W_{p,p^*}(\Gamma(\hat{A}_0))\), we apply the same scheme as used in [50]. Namely consider the composition of \(\Phi(u)\) and \(\Phi^*(u)\), which are given in Theorem 4.2 and Sect. 4.3.1, respectively.

\[
T(u) = \Phi^*(u)\Phi(u) = \sum_{\lambda \in \mathcal{P}^+} \Phi^*_\lambda(u)\Phi_\lambda(u) : \mathcal{F}^{(1,N+1)}_{-uv} \to \mathcal{F}^{(1,N+1)}_{-uv},
\]

\(\square\) Springer
Note that one can choose \( v \in \mathbb{C}^* \), \( N \in \mathbb{Z} \) arbitrarily. Taking the normal ordering one obtains
\[
\Phi^*_\lambda(u) \Phi_\lambda(u) = C_\lambda(q, t, p) : \tilde{\Phi}_\lambda(u) \tilde{\Phi}^*_\lambda(u) :.
\]

Then, one finds that the operator part is given by
\[
: \tilde{\Phi}^*_\lambda(u) \tilde{\Phi}_\lambda(u) := \prod_{\square \in A(\lambda)} Y(u/q^{\square}) \prod_{\blacksquare \in R(\lambda)} Y\left((q/t)u/q^{\blacksquare}\right)^{-1}:
\tag{5.1}
\]
with
\[
Y(u) =: \exp \left\{ \sum_{m \neq 0} y_m u^{-m} \right\} :
\tag{5.2}
\]
Here, we set \( q^{\square} \equiv t^{-i-1}q^{-j+1} \) for \( \square = (i, j) \in \lambda \), etc., and \( y_m = \frac{1 - t^m}{\kappa_m} (t/q)^{-m/2} \alpha'_m \).

The symbols \( R(\lambda) \) and \( A(\lambda) \) denote the set of removable and addable boxes in the Young’s diagram \( \lambda \), respectively. The main structure of (5.1) is due to Proposition 4.3, which yields
\[
: \tilde{\Phi}^*_\lambda(u) \tilde{\Phi}_\lambda(u) := \exp \left( \sum_{m \neq 0} \frac{(1 - t^m)(1 - p^m)}{\kappa_m} E_{\lambda, m} \alpha'_m ((t/q)^{1/2}u)^{-m} \right) :,
\tag{5.3}
\]
and the following combinatorial formula.

**Proposition 5.1**
\[
E_{\lambda, m} = \frac{1}{1 - t^m} \left( \sum_{\square \in A(\lambda)} q^m \square - (t/q)^m \sum_{\blacksquare \in R(\lambda)} q^m \blacksquare \right)
\tag{5.4}
\]

**Proof** The statement follows from
\[
E_{\lambda, m} = \frac{1}{1 - t^m} \left( 1 - (1 - q^{-m})(1 - t^m) \sum_{\square \in \lambda} q^m \square \right)
\]
and
\[
\bullet (i, \lambda_i) \in R(\lambda) \iff \lambda_i > \lambda_{i+1}
\]
\[
\bullet \text{if } (i, \lambda_i) \in R(\lambda) \iff (i+1, \lambda_{i+1} + 1) \in A(\lambda)
\]
for \( \lambda \in \mathcal{P}^+ \).
Moreover, from (2.24) with \( \gamma = (t/q)^{1/2} \), one finds the following commutation relation.

\[
[y_m, y_n] = -\frac{1}{m} \frac{(1 - p^{*m})(1 - p^{-m})}{(1 - q^m)(1 - t^{-m})} \delta_{m+n,0}.
\]

This agrees with the one in [41].

The coefficient part in \( \Phi_\lambda^*(u) \Phi_\lambda(u) \) can be calculated by combining the normalization factors of the vertex operators and the OPE coefficient. The calculation of the latter coefficient is essentially due to the following formula [4] obtained by considering a \( q \)-analogue of (3.6) in [71].

**Proposition 5.2**

\[
-\frac{1 - t^m}{1 - q^m} E_{\lambda, -m} E_{\mu, m} = \frac{t^m}{(1 - q^m)(1 - t^m)} + \sum_{\square \in \mu} q^{m a_\lambda(\square)} t^m (\ell_\mu(\square) + 1) + \sum_{\square \in \lambda} q^{-m (a_\mu(\square) + 1)} t^{-m (\ell_\lambda(\square))}.
\]

(5.5)

Then, one finds

\[
\mathcal{C}_\lambda(q, t, p) = C q^{|\lambda|} \hat{\mathcal{A}}_\lambda^0(t, q^{-1}, p),
\]

where

\[
q = p^{*q-1} p^{N-1} (t/q)^{1/2},
\]

(5.6)

\[
\hat{\mathcal{A}}_\lambda^0(t, q^{-1}, p) = \prod_{\square \in \lambda} \frac{(1 - p q^{a(\square) + 1} t^{\ell(\square)}) (1 - p q^{-a(\square)} t^{-\ell(\square)} - 1)}{(1 - q^{a(\square) + 1} t^{\ell(\square)}) (1 - q^{-a(\square)} t^{-\ell(\square)} - 1)},
\]

(5.7)

\[
C = \frac{(p^{-1} t; q, t, p)_\infty}{(q; q, t, p)_\infty}.
\]

(5.8)

Note that the sum \( \sum_{\lambda, |\lambda|=n} \hat{\mathcal{A}}_\lambda^0(t, q^{-1}, p) \) coincides with the equivariant \( \chi_y \)-genus of the Hilbert scheme of \( n \) points on \( \mathbb{C}^2 \), \( \text{Hilb}_n(\mathbb{C}^2) \), [36, 57] with \( y = p \). Note also that one can rewrite (5.7) as

\[
\hat{\mathcal{A}}_\lambda^0(t, q^{-1}, p) = \frac{N_{\lambda, \mu}(pq/t)}{N_{\lambda, \mu}(q/t)}
\]

(5.9)

in terms of the 5d analogue of the Nekrasov function \( N_{\lambda, \mu}(x) \) given by

\[
N_{\lambda, \mu}(x) = \prod_{\square \in \lambda} (1 - x q^{a(\square) - 1} t^{-\ell(\square)}) \prod_{\square \in \mu} (1 - x q^{a(\square)} t^{\ell(\square) + 1}).
\]

(5.10)
Fig. 1 Graphical expression of \( \sum_{\lambda} \Phi^*_\lambda(u) \Phi_\lambda(u) \). The two horizontal lines with || are glued together.

Hence, the whole operator

\[
T(u) = C \sum_{\lambda} q^{\lambda|} Z_{\lambda}^{\hat{A}_0}(t, q^{-1}, p) : \prod_{\square \in A(\lambda)} Y(u/q^{\square}) \prod_{\blacksquare \in R(\lambda)} Y((q/t)u/q^{\blacksquare})^{-1} : \tag{5.11}
\]

agrees with the generating function of \( W_{p, p^*}(\Gamma(\hat{A}_0)) \) in \([41, 42]\) up to an over all constant factor. Note that we have the symmetry

\[
Z_{\lambda}^{\hat{A}_0}(t, q^{-1}, p) = (pp^*)^{\lambda|} Z_{\lambda}^{\hat{A}_0}(t, q^{-1}, p^{*-1}) = p^{2\lambda|} Z_{\lambda}^{\hat{A}_0}(t^{-1}, q, p^{-1}) = (t/q)^{\lambda|} Z_{\lambda}^{\hat{A}_0}(t^{-1}, q, p^*). 
\]

From (5.11), it is immediate to obtain the rank 1 instanton partition function of the 5d lift of the 4d \( N = 2^+ \) theory \([4, 32–34]\) by taking the vacuum expectation value:

\[
\langle 0 | T(u) | 0 \rangle = C \sum_{\lambda} q^{\lambda|} Z_{\lambda}^{\hat{A}_0}(t, q^{-1}, p). \tag{5.12}
\]

It is then important to recognize that this result and our realization \( T(u) = \sum_{\lambda} \Phi^*_\lambda(u) \Phi_\lambda(u) \) lead to the identification of \( T(u) \) with the basic refined topological vertex depicted in Fig. 1, which was introduced in \([32, 34, 36]\). Once obtaining such basic operator, one can apply it to various calculations presented in the subsequent sections.

### 5.3 The higher rank extension

To extend \( W_{p, p^*}(\Gamma(\hat{A}_0)) \) to the one associated with the higher rank instantons, one need to take a composition of \( T(u) \)’s. By using (5.3) and Propositions 5.1 and 5.2, one
obtains the following expression.

\[ T(u_1) \cdots T(u_M) = C_M \sum_{k=0}^{\infty} q \sum_{\lambda^{(1)}, \ldots, \lambda^{(M)}} \prod_{i,j=1}^{M} \frac{N_{\lambda^{(i)} \lambda^{(j)}}(p q u_{i,j} / t)}{N_{\lambda^{(i)} \lambda^{(j)}}(q u_{i,j} / t)} \]

\[ \times : \prod_{l=1}^{M} \prod_{\Box \in A(\lambda^{(i)})} Y(u_l / q^{\Box}) \prod_{\blacksquare \in R(\lambda^{(i)})} Y((q/t) u_l / q^{\blacksquare})^{-1} : \]

(5.13)

where we set \( u_{j,i} = u_j / u_i \), and

\[ q_M = q p_{-(M-1)} = p^{s-1} p^{M+N} (t/q)^{1/2}, \]

\[ C_M = \left( \frac{p^{-1} t; q, t, p}{p^{-1} t; q, t, p} \right)^{M} \prod_{1 \leq i < j \leq M} \frac{p^{-1} t u_{j,i} ; q, t, p}{p^{-1} t u_{j,i} ; q, t, p} \frac{p q u_{j,i} ; q, t, p}{p q u_{j,i} ; q, t, p} \]  

(5.15)

In fact, the sum

\[ \chi_p(M_k, M) = \sum_{\lambda^{(1)}, \ldots, \lambda^{(M)}} \prod_{i,j=1}^{M} \frac{N_{\lambda^{(i)} \lambda^{(j)}}(p q u_{i,j} / t)}{N_{\lambda^{(i)} \lambda^{(j)}}(q u_{i,j} / t)} \]

(5.16)

coincides with the equivariant \( \chi_y (y = p) \) genus of the moduli space of rank \( M \) instantons with charge \( k \) [34].

Note that \( T(u) \) gives a non-commutative 5d-analogue of Nekrasov’s \( qq \)-character of the \( \mathcal{N} = 2^* U(1) \) theory [41]. We expect that \( T(u_1) \cdots T(u_M) \) gives a non-commutative 5d-analogue of Nekrasov’s \( qq \)-character of the \( \mathcal{N} = 2^* U(M) \) theory.

### 6 Instanton calculus in the Jordan quiver gauge theories

By using the generating function \( T(u) \) of \( W_{p, p^*}(\Gamma(A_0)) \) constructed in the previous section, we show some demonstrations of deriving the instanton partition functions of the 5d and 6d lifts of the 4d \( \mathcal{N} = 2^* \) SUSY gauge theories. Hence, the elliptic quantum toroidal algebra \( \mathcal{U}_{q,t,p} \) is a relevant quantum group structure of dealing with such theories.
6.1 Instanton partition functions for the 5d and 6d lifts of the $\mathcal{N} = 2^*$ $U(1)$ theory

As mentioned in Sect. 5.2, the vacuum expectation value of $T(u)$ gives the instanton partition functions of the 5d lift of the 4d $\mathcal{N} = 2^* U(1)$ gauge theory.

$$\langle 0 | T(u) | 0 \rangle = C \sum_{\lambda} q^{\|\lambda\|} Z_{\lambda}^{\hat{A}_0}(t, q^{-1}, p). \quad (6.1)$$

We then identify $T(u) = \sum_{\lambda} \Phi^\ast_{\lambda}(u) \Phi_{\lambda}(u)$ with the basic topological vertex depicted in Fig. 1.

An immediate next application is to take a trace of $T(u)$. Let us introduce the degree counting operator

$$d = - \sum_{m > 0} m^2 \kappa_m \frac{1 - p^m}{1 - p^{-m} \alpha' m \alpha'}$$

such that

$$[d, \alpha'_m] = m \alpha'_m \quad m \in \mathbb{Z} \neq 0.$$

Then, the following trace yields the 6d version of the partition function of the rank 1 instantons [4, 32–34].

$$\text{tr}_{\mathfrak{g}(1,N+1)} Q^d T(u) = C_Q \sum_{\lambda} q^{\|\lambda\|} Z_{\lambda}^{\hat{A}_0}(t, q^{-1}, p; Q), \quad (6.2)$$

where

$$C_Q = \frac{1}{(Q; Q)_{\infty} (q; q, t, p)_{\infty} (pq Q; q, t, Q)_{\infty} (p q Q; q, t, Q)_{\infty} (t Q; q, t, Q)_{\infty}}, \quad (6.3)$$

$$Z_{\lambda}^{\hat{A}_0}(t, q^{-1}, p; Q) = \frac{N^{\theta}_{\lambda,\mu}(pq/t; Q)}{N^{\theta}_{\lambda,\mu}(q/t; Q)}. \quad (6.4)$$

Here $N^{\theta}_{\lambda,\mu}(x; Q)$ denotes the theta function analogue of the Nekrasov function given by

$$N^{\theta}_{\lambda,\mu}(x; Q) = \prod_{\square \in \lambda} \theta_Q(x q^{-a_{\mu}(\square) - 1} t^{-\ell_{\lambda}(\square)}) \prod_{\boxdot \in \mu} \theta_Q(x q a_{\lambda}(\boxdot) t^{\ell_{\mu}(\boxdot) + 1}). \quad (6.5)$$

In fact the sum $\sum_{\lambda, \|\lambda\|=n} Z_{\lambda}^{\hat{A}_0}(t, q^{-1}, p; Q)$ gives the equivariant elliptic genus of $\text{Hilb}_n(\mathbb{C}^2)$ [57].
6.2 The 5d and 6d lifts of the $\mathcal{N} = 2^* U(M)$ theory

The higher rank instanton partition functions can be obtained from the composition $T(u_1) \cdots T(u_M)$ in (5.13). The vacuum expectation value gives the instanton partition function of the 5d lift of the 4d $\mathcal{N} = 2^* U(M)$ theory [34].

$$\langle 0 | T(u_1) \cdots T(u_M) | 0 \rangle = C_M \sum_{k=0}^{\infty} q_M^k \chi_p(\mathcal{M}_{k,M}),$$

(6.6)

where $\chi_p(\mathcal{M}_{k,M})$ is given by (5.16).

Furthermore, taking the trace of (5.13), one obtains

$$\text{tr}_{\mathcal{F}(1,N+1)} T(u_1) \cdots T(u_M) = C_{Q,M} \sum_{k=0}^{\infty} q_M^k \mathcal{E}_{p,Q}(\mathcal{M}_{k,M}),$$

(6.7)

where $u_1 v_1 = u_2 v_2 = \cdots = u_M v_M$ with arbitrary $v_1, \cdots, v_M \in \mathbb{C}^*$. We here also set

$$\mathcal{E}_{p,Q}(\mathcal{M}_{k,M}) = \sum_{\chi^{(1)}, \ldots, \chi^{(M)} \atop \sum_j |\chi^{(j)}| = k} N^\theta_{\chi^{(1)}, \chi^{(j)}}(pqu_{i,j}/t; Q) / N^\theta_{\chi^{(j)}, \chi^{(j)}}(qu_{i,j}/t; Q),$$

(6.8)

$$C_{Q,M} = \frac{1}{(Q; Q)^\infty} \left( (t; q, t)^\infty \Gamma_3(p^{-1}t; q, t, Q) \right)^M \prod_{1 \leq i < j \leq M} \frac{\Gamma_3(p^{-1}tu_{j,i}; q, t, Q) \Gamma_3(pqu_{j,i}; q, t, Q)}{\Gamma_3(tu_{j,i}; q, t, Q) \Gamma_3(qu_{j,i}; q, t, Q)},$$

(6.9)

The sum $\mathcal{E}_{p,Q}(\mathcal{M}_{k,M})$ gives the equivariant elliptic genus of the moduli space of rank $M$ instantons with charge $k$ [33, 35]. Hence, (6.7) gives the instanton partition function of the 6d lift of the $\mathcal{N} = 2^* U(M)$ theory.

7 Correlation functions of $\Phi(u)$ and $\Psi^*(v)$

We here give some $U_{q,t,p}$-analogues of the formulas of those obtained in Propositions 5.1 and 5.2 in [5]. We expect that they give the $\mathcal{N} = 2^*$ theory (Jordan quiver gauge theory) analogue of the partition function of the 5d lift of the pure $SU(N)$ theory and the 5d and 6d lifts of the $SU(N)$ theory with $2N$ fundamental matters, respectively. The latter theories are the $A_1$ linear quiver gauge theories.

7.1 OPE formulas

From Propositions 4.3, 4.6 and 5.2, we have the following OPE formulas for the operator parts of the type I $\Phi(u)$ and the type II dual $\Psi^*(v)$ intertwining operators.
Proposition 7.1

\[ \tilde{\Phi}_\lambda(u)\tilde{\Phi}_\mu(v) = \frac{G(v/u, p)}{N_{\mu\lambda}(v/u; p)} : \tilde{\Phi}_\lambda(u)\tilde{\Phi}_\mu(v) : , \]

\[ \tilde{\Psi}_\lambda^*(u)\tilde{\Psi}_\mu^*(v) = \frac{G^*(qv/tu, p)}{N_{\mu\lambda}^*(qv/tu; p)} : \tilde{\Psi}_\lambda^*(u)\tilde{\Psi}_\mu^*(v) : , \]

\[ \Phi_\lambda(u)\tilde{\Psi}_\mu^*(v) = \frac{N_{\mu\lambda}(t/q)^{-1/2}v/u}{G(v/u)} : \Phi_\lambda(u)\tilde{\Psi}_\mu^*(v) : , \]

\[ \tilde{\Psi}_\lambda^*(u)\Phi_\mu(v) = \frac{N_{\mu\lambda}(t/q)^{-1/2}v/u}{G(v/u)} : \tilde{\Psi}_\lambda^*(u)\Phi_\mu(v) : . \]

Here \( N_{\lambda\mu}(u; p) \) and \( N_{\lambda\mu}^*(u; p) \) denote \( p \)-deformations of \( N_{\lambda\mu}(u) \) in (5.10) given by

\[ N_{\lambda\mu}(u; p) = \prod_{\square \in \lambda} \frac{(uq^{-a_\mu(\square)}-1)_{t-\ell_\lambda(\square); p}}{(puq^{-a_\mu(\square)}-1)_{t-\ell_\lambda(\square); p}} \prod_{\square \in \mu} \frac{(uq^{a_\mu(\square)}t^{\ell_\mu(\square)+1}; p)_\infty}{(puq^{a_\mu(\square)}t^{\ell_\mu(\square)+1}; p)_\infty}, \]

\[ N_{\lambda\mu}^*(u; p) = N_{\lambda\mu}(u; p) \Big|_{p \leftrightarrow p^*}. \]

We also set

\[ G(u) = \exp \left\{ \sum_{n>0} \frac{1}{n} \frac{t^n}{(1-q^n)(1-t^n)} u^n \right\} = \frac{1}{(tu; q, t)_\infty}, \]

\[ G(u, p) = \exp \left\{ \sum_{n>0} \frac{1}{n} \frac{(1-p^n)t^n}{(1-q^n)(1-t^n)(1-p^n)} u^n \right\} = \frac{(p^*tu; q, t, p)_\infty}{(tu; q, t, p)_\infty}, \]

\[ G^*(u, p) = G(u, p) \Big|_{p \leftrightarrow p^*}. \]

7.2 The four points operator

Remember

\[ \Phi(-x) \mathcal{F}_u^{(1,N+1)} \to \mathcal{F}_{-x}^{(0,1)} \otimes \mathcal{F}_u^{(1,N)} , \]

\[ \Psi^*(-x) \mathcal{F}_u^{(1,N)} \otimes \mathcal{F}_{-x}^{(0,1)} \to \mathcal{F}_u^{(1,N+1)} . \]

Let us consider the following composition.

\[ \phi(w) := (\Psi^*(-w) \otimes \text{id}) \circ (\text{id} \otimes \Phi(-w)) \mathcal{F}_v^{(1,M)} \otimes \mathcal{F}_u^{(1,L)} \to \mathcal{F}_{vw}^{(1,M+1)} \otimes \mathcal{F}_{uw}^{(1,L-1)} . \]
This is an intertwining operator satisfying
\[ \Delta(x) \phi(w) = \phi(w) \Delta(x) \quad \forall x \in \mathcal{U}_{q,t,p}. \]

We call \( \phi(w) \) the basic four points operator.

**Proposition 7.2** The action of \( \phi(w) \) on \( |\xi^M_u \rangle \otimes |\eta^L_u \rangle \in \mathcal{F}_v^{(1,M)} \otimes \mathcal{F}_u^{(1,L)} \) is given by
\[ \phi(w) |\xi^M_u \rangle \otimes |\eta^L_u \rangle = \sum_{\lambda} \frac{c_\lambda(p^*)}{c_\lambda(p^*)} \Psi^*_\lambda(-w) |\xi^M_u \rangle \otimes \Phi^*_\lambda(-w) |\eta^L_u \rangle. \]

**Proof** Use (4.2), (4.3), (4.14) and (2.56). \( \square \)

**Proposition 7.3** The composition of \( \phi(w) \)'s gives the following generalized four points intertwining operator. (Fig. 2)
\[ \phi(w_N) \circ \cdots \circ \phi(w_1) \mathcal{F}_v^{(1,M)} \otimes \mathcal{F}_u^{(1,L)} \rightarrow \mathcal{F}_v^{(1,M+N)} \otimes \mathcal{F}_u^{(1,L-N)}. \]

The action on \( |\xi^M_u \rangle \otimes |\eta^L_u \rangle \in \mathcal{F}_v^{(1,M)} \otimes \mathcal{F}_u^{(1,L)} \) is given by
\[ \phi(w_N) \circ \cdots \circ \phi(w_1) |\xi^M_u \rangle \otimes |\eta^L_u \rangle = \sum_{\lambda(1), \ldots, \lambda(N)} \prod_{a=1}^{N} c_{\lambda(1)}(p^*) \psi^*_\lambda(N)(-w_N) \cdots \psi^*_\lambda(1)(-w_1) |\xi^M_u \rangle \otimes \phi^*_\lambda(N)(-w_N) \cdots \phi^*_\lambda(1)(-w_1) |\eta^L_u \rangle. \]

**Corollary 7.4** For a vector \( |\omega^{M+N}_{u_1 \cdots u_N} \rangle \otimes |\xi^{L-N}_{u/u_1 \cdots u_N} \rangle \in \mathcal{F}_v^{(1,M+N)} \otimes \mathcal{F}_u^{(1,L-N)} \) one has the following expectation value.
\[ \langle \omega^{M+N}_{u_1 \cdots u_N} | \phi(w_N) \circ \cdots \circ \phi(w_1) |\xi^M_u \rangle \otimes |\eta^L_u \rangle \]
\[ = \sum_{\lambda(1), \ldots, \lambda(N)} \prod_{a=1}^{N} c_{\lambda(1)}(p^*) \langle \omega^{M+N}_{u_1 \cdots u_N} | \psi^*_\lambda(N)(-w_N) \cdots \psi^*_\lambda(1)(-w_1) |\xi^M_u \rangle \]
\[ \langle \xi^{L-N}_{u/u_1 \cdots u_N} | \phi^*_\lambda(N)(-w_N) \cdots \phi^*_\lambda(1)(-w_1) |\eta^L_u \rangle. \]

Note that this gives the \( \mathcal{N} = 2^* \) theory analogue of the partition function of the 5d pure \( SU(N) \) theory obtained, for example, in [4, 5].

### 7.3 The six points and higher operators

Next let us consider the operator
\[ \phi \left( w', \frac{y}{x} \right) = (\phi(-y) \otimes \text{id}) \circ (\psi^*(-w) \otimes \psi^*(x)) \circ (\text{id} \otimes \Phi(-w) \otimes \text{id}) \]
This is an intertwining operator satisfying

$$\phi(w; y_x) (\Delta \otimes \text{id}) \Delta(a) = (\Delta \otimes \text{id}) \Delta(a) \phi(w; y_x) \quad \forall a \in U_{q,t,p}.$$ 

We call $\phi(w; y_x)$ the six points operator.

**Proposition 7.5** For $\mu \in P^+$, the action of $\phi(w; y_x)$ on $|\xi^0_v, \eta^0_u, \mu\rangle_{-x}$ in $F_v^{(1,0)} \otimes F_u^{(1,0)} \otimes F^{(0,1)}_{-x}$ is given by

$$\phi(w; y_x) |\xi^0_v, \eta^0_u, \mu\rangle_{-x} = \sum_{\lambda, \sigma \in P^+} c^\lambda(p^\sigma) c^\lambda(p^\sigma) |\sigma\rangle_{-y} \otimes \Phi_{\sigma}(-y) \Psi^*_\lambda(-w) |\xi^0_v, \Psi^*_\lambda(-x) \Phi_{\lambda}(-w) |\eta^0_u\rangle.$$ 

Let us set

$$\phi^{(j)}(w; y_x) := \text{id} \otimes \cdots \otimes \text{id} \phi(w; y_x) \otimes \text{id} \otimes \cdots \otimes \text{id}.$$
Proposition 7.6 By composing $\phi^{(j)} \left( w_j; x_j \right)$ ($j = 1, \ldots, N$), we obtain the following $2(N+2)$ points intertwining operator (Fig. 3).

$$
\phi^{(N)} \left( w_N; x_N \right) \circ \cdots \circ \phi^{(1)} \left( w_1; x_1 \right)
: \mathcal{F}^{(1)}_{w_1} \otimes \cdots \otimes \mathcal{F}^{(1)}_{-x_1} \rightarrow \mathcal{F}^{(0,1)}_{y_1} \otimes \cdots \otimes \mathcal{F}^{(0,1)}_{-y_1} \otimes \mathcal{F}^{(1)}_{w_{N+1}} \otimes \mathcal{F}^{(1)}_{-w_{N+1}}.
$$

For $\mu = (\mu^{(1)}, \ldots, \mu^{(N)}) \in (\mathcal{P}^+)^N$, the action on $|\xi^0_{\nu} \otimes |\eta^0_{\nu} \otimes |\mu^{(1)}\rangle \rangle_{x_1} \otimes \cdots \otimes |\mu^{(N)}\rangle \rangle_{x_N}$ is given by

$$
\phi^{(N)} \left( w_N; x_N \right) \circ \cdots \circ \phi^{(1)} \left( w_1; x_1 \right) |\xi^0_{\nu} \otimes |\eta^0_{\nu} \otimes |\mu^{(1)}\rangle \rangle_{-x_1} \otimes \cdots \otimes |\mu^{(N)}\rangle \rangle_{-x_N}
= \sum_{\lambda^{(1)}, \ldots, \lambda^{(N)} \in (\mathcal{P}^+)^N} \prod_{\sigma^{(1)}, \ldots, \sigma^{(N)} \in (\mathcal{P}^+)} \mathcal{F}^{(1)}_{\lambda^{(a)} \nu} \otimes \mathcal{F}^{(1)}_{\lambda^{(a)} \nu} \otimes \mathcal{F}^{(0,1)}_{\mu^{(a)} \nu} \otimes \mathcal{F}^{(0,1)}_{\mu^{(a)} \nu},
$$

where $\lambda = (\lambda^{(1)}, \ldots, \lambda^{(N)})$, $\sigma = (\sigma^{(1)}, \ldots, \sigma^{(N)}) \in (\mathcal{P}^+)^N$.

Corollary 7.7 For a vector

$$
|v^{(1)}\rangle' \otimes \cdots \otimes |v^{(N)}\rangle' \otimes |\theta_{w_1 \cdots w_N}^0 \rangle \otimes |\xi^0_{\nu} \rangle \otimes |\mu^{(1)}\rangle \rangle \in \mathcal{F}^{(0,1)}_{-y_1} \otimes \cdots \otimes \mathcal{F}^{(0,1)}_{-y_N},
$$

we have the expectation value

$$
-|v^{(1)}\rangle \otimes \cdots \otimes |v^{(N)}\rangle \otimes |\theta_{w_1 \cdots w_N}^0 \rangle \otimes |\xi^0_{\nu} \rangle \otimes |\mu^{(1)}\rangle \rangle_{-x_1} \otimes \cdots \otimes |\mu^{(N)}\rangle \rangle_{-x_N}
= \sum_{\lambda^{(1)}, \ldots, \lambda^{(N)} \in (\mathcal{P}^+)^N} \prod_{\sigma^{(1)}, \ldots, \sigma^{(N)} \in (\mathcal{P}^+)} \mathcal{F}^{(1)}_{\lambda^{(a)} \nu} \otimes \mathcal{F}^{(1)}_{\lambda^{(a)} \nu} \otimes \mathcal{F}^{(0,1)}_{\mu^{(a)} \nu} \otimes \mathcal{F}^{(0,1)}_{\mu^{(a)} \nu},
$$

where $y' = (y^{(1)}, \ldots, y^{(N)})$ is defined by

$$
y' (\mu | v) = \delta_{\mu, v} \quad v \in \mathcal{P}^+.
$$
Corollary 7.8 When $x_1 \cdots x_N = y_1 \cdots y_N = w_1 \cdots w_N$, one can take the following trace

\[
\begin{align*}
&\text{tr}_{F(1,0)} Q^d \otimes \text{tr}_{F(1,0)} Q^d \\
&\quad \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \·
Here, we set \( p^{**} = p^*(q/t) = p(q/t)^2 \) and

\[
C_N(q, t, p, Q) = \frac{1}{(Q; Q)_\infty^2} \left( \frac{(p^*tQ; q, t, p, Q)_\infty}{(tQ; q, t, p, Q)_\infty} \left( \frac{pqQ; q, t, p^*, Q}_\infty \right) \left( \frac{p^*tQ; q, t, p^*, Q}_\infty \right) \right)^N.
\]

\[
N^N_\mu(u, x, w; q, t, p, Q) = \prod_{a=1}^N \left( -t^{-1}u \prod_{k=1}^a \frac{x_k}{w_k} \right) t^{-n(\mu^{(a)})} c^{(a)}_\mu(p^{**})
\times \prod_{1 \leq a < b \leq N} \Gamma(\sqrt{qt}x_a/x_b; q, t) \Gamma_4(qx_a/x_b; q, t, p^{**}, Q) N^\theta_{\mu^{(a)}\mu^{(b)}}(x_a/x_b; p^{**})
\times \prod_{a, b=1}^N N^{\Gamma}_{\mu^{(a)}\mu^{(b)}}(qx_a/tx_b; p^{**}, Q) \Gamma_3(\sqrt{qt}x_a/x_b; q, t, Q),
\]

\[
N^N_\nu(v, y, w; q, t, p, Q) = \prod_{a=1}^N \left( -t^{-1}u \prod_{k=1}^a \frac{y_k}{w_k} \right) t^{-n(\nu^{(a)})} c^{(a)}_\nu(p)
\times \prod_{1 \leq a < b \leq N} \Gamma(\sqrt{qt}y_a/y_b; q, t) \Gamma_4(qy_a/y_b; q, t, p, Q) N^\theta_{\nu^{(a)}\nu^{(b)}}(qy_a/ty_b; p)
\times \prod_{a, b=1}^N N^{\Gamma}_{\nu^{(a)}\nu^{(b)}}(qy_a/ty_b; p, Q) \Gamma_3(\sqrt{qt}y_a/y_b; q, t, Q).
\]
Here, \( N_{\lambda,\mu}^\Gamma (x; p, Q) \) denotes the elliptic Gamma-analogues of the Nekrasov function defined by

\[
N_{\lambda,\mu}^\Gamma (x; p, Q) = \prod_{\square \in \lambda} \Gamma(xq^{-a_{\mu}(\square)} - 1^{-\ell(\square)}; p, Q) \prod_{\square \in \mu} \Gamma(xq^{a_{\lambda}(\square)} + 1; p, Q).
\] (7.3)

**Proof** The statement follows from Proposition 7.1 and the trace formula given, for example, in Appendix E of [55]. \(\square\)

As for the functions \( N_{\lambda,\mu}^\theta (x; p) \) and \( N_{\lambda,\mu}^\Gamma (x; p, Q) \), the following properties are useful.

**Proposition 7.10**

\[
N_{\lambda,\mu}^\theta (x; p) = \left( x \sqrt{\frac{t}{q}} \right)^{|\lambda|+|\mu|} f_{\lambda}(q,t) f_{\mu}(q,t) N_{\lambda,\mu}^\theta (q/tx; p),
\]

\[
N_{\lambda,\mu}^\Gamma (x; p, Q) = \frac{1}{N_{\mu,\lambda}^\Gamma (pq/qtx; p, Q)} = \left( x \sqrt{\frac{t}{q}} \right)^{|\lambda|+|\mu|} f_{\lambda}(q,t) f_{\mu}(q,t) \frac{1}{N_{\lambda,\mu}^\Gamma (x; p) N_{\lambda,\mu}^\theta (q/tx; p, Q)}.
\]

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**Declarations**

**Conflict of interest** The authors declare that they have no conflict of interest.

### A Formulas for \( E^\pm (\alpha, z) \) and \( E^\pm (\alpha', z) \)

**Lemma A.1** The operators \( E^\pm (\alpha, z) \) and \( E^\pm (\alpha', z) \) defined in (2.28) and (2.29) satisfy the following relations.

\[
[\alpha_{-n}, E^+ (\alpha, z)] = \frac{\kappa_n}{n} \frac{1 - p^n}{1 - p^{*-n}} y^{-n/2} z^{-n} E^+ (\alpha, z) \quad (n > 0),
\] (A.1)

\[
[\alpha_n, E^- (\alpha, z)] = \frac{\kappa_n}{n} \frac{1 - p^n}{1 - p^{*-n}} y^{-3n/2} z^n E^- (\alpha, z) \quad (n > 0),
\] (A.2)

\[
[\alpha_{-n}, E^+ (\alpha', z)] = -\frac{\kappa_n}{n} y^{n/2} z^{-n} E^+ (\alpha', z) \quad (n > 0),
\] (A.3)

\[
[\alpha_n, E^- (\alpha', z)] = -\frac{\kappa_n}{n} y^{-n/2} z^n E^- (\alpha', z) \quad (n > 0),
\] (A.4)
\[ E^+(\alpha, z)E^-(\alpha, w) = g(w/z)^{-1}g(w/z; \gamma^2)^{-1}g(w/z; p^*)^{-1}E^-(\alpha, w)E^+(\alpha, z), \]

(A.5)

\[ E^+(\alpha', z)E^-(\alpha', w) = g(w/z; \gamma^2)^{-1}g(w/z; p)E^-(\alpha', w)E^+(\alpha', z), \]

(A.6)

\[ E^+(\alpha, z)E^-(\alpha, w) = g(\gamma w/z)g(\gamma w/z; \gamma^2)E^- (\alpha', w)E^+(\alpha, z), \]

(A.7)

\[ E^+(\alpha', z)E^-(\alpha, w) = g(\gamma w/z)g(\gamma w/z; \gamma^2)E^- (\alpha', w)E^+(\alpha', z), \]

(A.8)

\[ E^+(\alpha, z)x^+(w) = g(w/z)g(w/z; p^*)x^+(w)E^+(\alpha, z), \]

(A.9)

\[ E^+(\alpha, z)x^-(w) = g(\gamma w/z)^{-1}g(\gamma w/z; \gamma^2)x^- (w)E^+(\alpha, z). \]

(A.10)

\[ E^-(\alpha, z)x^+(w) = g(z/w)^{-1}g(z/w; \gamma^2)^{-1}g(z/w; p^*)^{-1}x^+(w)E^-(\alpha, z), \]

(A.11)

\[ E^-(\alpha, z)x^-(w) = g(\gamma z/w)g(\gamma z/w; \gamma^2)x^- (w)E^-(\alpha, z), \]

(A.12)

\[ E^+(\alpha', z)x^+(w) = g(\gamma w/z)^{-1}g(\gamma w/z; \gamma^2)^{-1}x^+(w)E^+(\alpha', z), \]

(A.13)

\[ E^+(\alpha', z)x^-(w) = g(w/z; \gamma^2)g(w/z; p)^{-1}f(pw/z; p)x^- (w)E^+(\alpha', z), \]

(A.14)

\[ E^-(\alpha', z)x^+(w) = g(\gamma z/w)g(\gamma z/w; \gamma^2)x^+(w)E^-(\alpha', z), \]

(A.15)

\[ E^-(\alpha', z)x^-(w) = g(z/w; \gamma^2)^{-1}g(z/w; p)x^- (w)E^-(\alpha', z). \]

(A.16)

Here, \( g(z; s) \), \( s = p, p^*, \gamma^2 \) and \( g(z) \) are given in (2.23) and (2.36), respectively.

Proof The statement follows from direct calculations using (2.15)–(2.17) and (2.24).

\[ \square \]

**B Direct Check of the Level (0,1) Representation**

Let us check the relation (2.18).

\[ x^+(z)x^-(w)|\lambda\rangle_u \]

\[ = a^+(p)a^-(p)(t/q)^{-1/2} \sum_{i=1}^{\ell} \sum_{j=1}^{\ell+1} \delta(q^{-1}u_i/w)\delta(u_j/z)A^+_{\lambda, i}(p)A^-_{\lambda-1, j}(p)|\lambda - 1_i + 1_j\rangle_u \]

\[ + a^+(p)a^-(p)(t/q)^{-1/2} \sum_{i=1}^{\ell} \sum_{j=1}^{\ell+1} \delta(q^{-1}u_i/w)\delta(z/w)A^-_{\lambda, i}(p)A^+_{\lambda-1, j}(p)|\lambda\rangle_u. \]

(B.1)

\[ x^-(w)x^+(z)|\lambda\rangle_u \]

\[ = a^+(p)a^-(p)(t/q)^{-1/2} \sum_{i=1}^{\ell} \sum_{j=1}^{\ell+1} \delta(q^{-1}u_i/w)\delta(u_j/z)A^+_{\lambda, j}(p)A^-_{\lambda+1, i}(p)|\lambda + 1_j - 1_i\rangle_u \]

\[ + a^+(p)a^-(p)(t/q)^{-1/2} \sum_{i=1}^{\ell+1} \sum_{j=1}^{\ell+1} \delta(u_i/w)\delta(z/w)A^+_{\lambda, i}(p)A^-_{\lambda+1, j}(p)|\lambda\rangle_u. \]

(B.2)
One finds that the first terms in (B.1) and (B.2) coincide. Hence,

\[
[x^+(z), x^-(w)]|\lambda\rangle_u = a^+(p)a^-(p)(t/q)^{-1/2}\delta(z/w) \\
\times \left( \sum_{i=1}^{\ell} \delta(q^{-1}u_i/w)\delta(z/w)A^-_{\lambda,i}(p)A^+_{\lambda-1,i}(p) - \sum_{i=1}^{\ell+1} \delta(u_i/w)A^+_\lambda(p)A^-_{\lambda+1,i}(p) \right) |\lambda\rangle_u.
\]

One also shows

\[
a^+(p)a^-(p) = \frac{(1 - q)(1 - t^{-1})}{1 - q/t} \frac{\theta_p(q/t)\theta_p(t)}{(p; p)^2_{\infty}} \theta_p(q)
\]

\[
A^-_{\lambda,i}(p)A^+_{\lambda-1,i}(p) = \prod_{j=1, \neq i}^{\ell} \frac{\theta_p(qu_j/tu_i)}{\theta_p(u_j/u_i)} \prod_{j=1, \neq i}^{\ell+1} \frac{\theta_p(tu_j/u_i)}{\theta_p(qu_j/u_i)}
\]

\[
= (t/q) \prod_{j=1, \neq i}^{\ell} \frac{\theta_p(tu_j/u_i)}{\theta_p(u_j/u_i)} \prod_{j=1, \neq i}^{\ell+1} \frac{\theta_p(u_j/tu_j)}{\theta_p(qu_j/u_i)}
\]

\[
= -(t/q) \frac{(p; p)^2\theta_p(q)}{\theta_p(q/t)\theta_p(t)} \text{Res}_{z=q^{-1}u} B^-(z/u; p) \frac{dz}{z},
\]

\[
A^+_{\lambda,i}(p)A^-_{\lambda+1,i}(p) = \prod_{j=1, \neq i}^{\ell} \frac{\theta_p(u_j/tu_i)}{\theta_p(u_j/qui)} \prod_{j=1, \neq i}^{\ell+1} \frac{\theta_p(tu_j/qui)}{\theta_p(u_j/qui)}
\]

\[
= (t/q) \prod_{j=1, \neq i}^{\ell} \frac{\theta_p(tu_j/u_i)}{\theta_p(qu_j/u_i)} \prod_{j=1, \neq i}^{\ell+1} \frac{\theta_p(qui/tu_j)}{\theta_p(qui/u_i)}
\]

\[
= (t/q) \frac{(p; p)^2\theta_p(q)}{\theta_p(q/t)\theta_p(t)} \text{Res}_{z=u} B^-(z/u; p) \frac{dz}{z}.
\]

Hence,

\[
[x^+(z), x^-(w)]|\lambda\rangle_u = (1 - q)(1 - t^{-1}) \delta(z/w) \left( - \sum_{i=1}^{\ell} \delta(q^{-1}u_i/w)\text{Res}_{z=q^{-1}u} (t/q)^{1/2} B^-(z/u; p) \frac{dz}{z} \\
- \sum_{i=1}^{\ell+1} \delta(u_i/w)\text{Res}_{z=u} (t/q)^{1/2} B^-(z/u; p) \frac{dz}{z} \right) |\lambda\rangle_u.
\]

Using the partial fraction expansion formula (3.44), one can show
$B^-_\lambda(z/u; p)_{|u/z| < 1} - B^-_\lambda(z/u; p)_{|u/z| > 1}$

$$= - \sum_{k=1}^\ell \delta(q^{-1}u_k/z)\text{Res}_{z=q^{-1}u_k} B^-_\lambda(z/u; p) \frac{dz}{z} - \sum_{k=1}^{\ell+1} \delta(u_k/z)\text{Res}_{z=q^{-1}u_k} B^-_\lambda(z/u; p) \frac{dz}{z}$$

Therefore, we have

$$[x^+(z), x^-(w)]|\lambda\rangle_u$$

$$= \frac{(1-q)(1-t^{-1})}{1-q/t} \delta(z/w)$$

$$\times \left( (t/q)^{1/2} B^-_\lambda(z/u; p)_{|u/z| < 1} - (t/q)^{1/2} B^-_\lambda(z/u; p)_{|u/z| > 1} \right) |\lambda\rangle_u$$

$$= \frac{(1-q)(1-t^{-1})}{1-q/t} \delta(z/w) \left( (t/q)^{1/2} (t/q)^{-1} B^+_\lambda(u/z; p)_{|u/z| < 1} - (t/q)^{1/2} B^-_\lambda(z/u; p)_{|u/z| > 1} \right) |\lambda\rangle_u$$

$$= \frac{(1-q)(1-t^{-1})}{1-q/t} \delta(z/w) \left( \psi^+(z) - \psi^-(z) \right) |\lambda\rangle_u .$$

\[ \square \]

**C Inductive derivation of Theorem 3.4**

We consider a tensor product of the vector representations in Proposition 3.3. Define

$$V^{(N)}(u) = V(u) \tilde{\otimes} V(u(t/q)^{-1}) \tilde{\otimes} V(u(t/q)^{-2}) \tilde{\otimes} \cdots \tilde{\otimes} V(u(t/q)^{-N+1}).$$

Set

$$\mathcal{P}^{(N)} = \{ \lambda = (\lambda_1, \lambda_2, \ldots, \lambda_N) \in \mathbb{Z}^N \mid \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_N \},$$

$$|\lambda\rangle_u^{(N)} = [u]_{\lambda_1} \tilde{\otimes} [u(t/q)^{-1}]_{\lambda_2-1} \tilde{\otimes} [u(t/q)^{-2}]_{\lambda_3-2} \cdots \tilde{\otimes} [u(t/q)^{-N+1}]_{\lambda_N-N+1}$$

and define $W^{(N)}(u)$ to be a subspace of $V^{(N)}(u)$ spanned by $\{|\lambda\rangle_u^{(N)} \mid \lambda \in \mathcal{P}^{(N)}\}$. An action of $U_{q,t,p}$ on the tensor product space can be constructed by using the comultiplication $\Delta^{op}$ in (2.66)–(2.68) repeatedly.

One can verify the following two propositions.

**Proposition C.1** By applying $\Delta^{op}$ repeatedly, the following gives a level-$(0,0)$ $U_{q,t,p}$-module structure on $W^{(N)}(u)$.

$$\gamma^{1/2}|\lambda\rangle_u^{(N)} = |\lambda\rangle_u^{(N)} , \quad (C.1)$$

$$x^+(z)|\lambda\rangle_u^{(N)} = a^+(p) \sum_{i=1}^N A^{(N)+}_{\lambda,i} (p) \delta(u_i/z) |\lambda + 1_i\rangle_u^{(N)} , \quad (C.2)$$
Elliptic quantum toroidal algebra $U_{q,t,p}(gl_{1,tor})$ and affine...

Let us define $\tau$ to be the subspace of $F_{u}$ to be the subspace of $u = q^{N}t^{-i+1}u$, $\lambda \pm 1 = (\lambda_{1}, \ldots, \lambda_{i} \pm 1, \ldots, \lambda_{N})$ and we set

$$A_{\lambda,i}^{(N)+}(p) = \prod_{j=1}^{i-1} \frac{\theta_{p}(tu_{i}/u_{j})\theta_{p}(qt^{-1}u_{i}/u_{j})}{\theta_{p}(u_{i}/u_{j})}\theta_{p}(u_{i}/u_{j})$$

$$A_{\lambda,i}^{(N)-}(p) = \prod_{j=i+1}^{N} \frac{\theta_{p}(tu_{j}/u_{i})\theta_{p}(qt^{-1}u_{j}/u_{i})}{\theta_{p}(u_{j}/u_{i})}\theta_{p}(u_{j}/u_{i})$$

$$B_{\lambda}^{(N)+}(u/z; p) = \prod_{j=1}^{N} \frac{\theta_{p}(t^{-1}u_{j}/z)\theta_{p}(q^{-1}t_{u_{j}}/z)}{\theta_{p}(u_{j}/z)\theta_{p}(q^{-1}u_{j}/z)}$$

$$B_{\lambda}^{(N)-}(z/u; p) = \prod_{j=1}^{N} \frac{\theta_{p}(zu_{j}/z)\theta_{p}(qt^{-1}u_{j}/z)}{\theta_{p}(z/u)\theta_{p}(qz/u)}.$$

In particular, we have

$$\alpha_{m}|\lambda\rangle_{u}^{(N)} = \frac{(1-t^{-m})(1-(q/t)^{-m})}{m} \sum_{j=1}^{N} u_{j}^{m}|\lambda\rangle_{u}^{(N)} \quad (m \in \mathbb{Z}\setminus\{0\}). \quad (C.6)$$

An inductive limit $N \to \infty$ can be taken in the same way as in the trigonometric case [17, 18]. Let

$$\mathcal{P}^{(N)+} = \{\lambda \in \mathcal{P}^{(N)} | \lambda_{N} \geq 0 \}.$$

and define $W^{(N)+}(u)$ to be the subspace of $W^{(N)}(u)$ spanned by $|\lambda\rangle_{u}^{(N)}, \lambda \in \mathcal{P}^{(N)+}$. Let us define $\tau_{N} : \mathcal{P}^{(N)+} \to \mathcal{P}^{(N+1)+}$ by

$$\tau_{N}(\lambda) = (\lambda_{1}, \lambda_{2}, \ldots, \lambda_{N}, 0).$$

This induces the embedding $W^{(N)+}(u) \hookrightarrow W^{(N+1)+}$. We then define a semi-infinite tensor product space $F_{u}$ by the inductive limit

$$F_{u} = \lim_{N \to \infty} W^{(N)+}(u).$$

The action of $U_{q,t,p}$ on $F_{u}$ via $\Delta^{op}$ is defined inductively as follows. Define

$$x^{+}[N](z) = x^{+}(z), \quad x^{-}[N](z) = (q/t)^{1/2}\frac{\theta_{p}(q^{-1}t^{-N+1}u/z)}{\theta_{p}(t^{-N}u/z)}x^{-}(z),$$
\[
\psi^+[N](z) = (q/t)^{1/2} \frac{\theta_p(q^{-1}t^{-N+1}u/z)}{\theta_p(t^N u/z)} \psi^+(z),
\]
\[
\psi^-[N](z) = (q/t)^{-1/2} \frac{\theta_p(qt^{-1}z/u)}{\theta_p(t^{-N}z/u)} \psi^+(z).
\]

Then, we have

**Lemma C.2** For \( x = x^\pm, \psi^\pm \), we have

\[
\tau_N(x^{[N]}(z)|\lambda)_u = x^{[N+1]}(z)\tau_N(|\lambda)_u.
\]

Thanks to this lemma, one can define the action of \( U_{q,t,p} \) on \( \mathcal{F}_u \) by

\[
x(z)|\lambda)_u = \lim_{N \to \infty} x^{[N]}(z)|(\lambda_1, \lambda_2, \ldots, \lambda_N).
\]

This gives the level (0, 1) representation in Theorem 3.4.

**D Proof of Theorem 4.2**

Proof of (4.4): From Theorem 3.2 one has

\[
\Phi_\theta(u)\psi^+((t/q)^{1/4}z)|_{\mathcal{F}(1,N+1)} = (t/q)^{-(N+1)/2} \frac{(pqz/tu; p)_\infty}{(pz/u; p)_\infty} \Phi_\theta(u)\widetilde{\psi}^+((t/q)^{1/4}z) :.
\]

\[
\psi^+((t/q)^{1/4}z)|_{\mathcal{F}(1,N)} \Phi_\theta(u) = (t/q)^{-N/2} \frac{(u/z; p)_\infty}{(tu/qz; p)_\infty} : \widetilde{\psi}^+((t/q)^{1/4}z)\Phi_\theta(u) :.
\]

Hence, we have

\[
\Phi_\theta(u)\psi^+((t/q)^{1/4}z) = (t/q)^{-1/2} \frac{\theta_p(tu/qz)}{\theta_p(u/z)} \psi^+((t/q)^{1/4}z)\Phi_\theta(u).
\]

Then, (4.4) follows from

\[
: \widetilde{\psi}^+((t/q)^{1/4}z)^{-1} \Phi_\lambda(u)\widetilde{\psi}^+((t/q)^{1/4}z) : = : \widetilde{\psi}^+((t/q)^{1/4}z)^{-1} \Phi_\theta(u)\widetilde{\psi}^+((t/q)^{1/4}z)^{-1} \prod_{(i,j) \in \lambda} \widetilde{\psi}^+((t/q)^{1/4}z)^{-1} \chi^-((t/q)^{1/4}u_{i,j}) \widetilde{\psi}^+((t/q)^{1/4}z) :.
\]

where \( u_{i,j} = q^{j-1}t^{-i+1}u \), and the following proposition.
Proposition D.1

\[
\prod_{i=1}^{\ell(\lambda)} \prod_{j=1}^{\lambda_i} g(q^{j-1}t^{-i+1}; p) = \frac{\theta_p(z)}{\theta_p((t/q)z)} B^\pm_\lambda(z; p).
\]

Proof of (4.6): From Theorem 3.2 one has

\[
\Phi_\theta(u)\tilde{x}^+(t/q)^{-1/4}z = (1 - qz/tu) : \Phi_\theta(u)\tilde{x}^+(t/q)^{-1/4}z :
\]

\[
\tilde{x}^+(t/q)^{-1/4}z \Phi_\theta(u) = (1 - tu/qz) : \Phi_\theta(u)\tilde{x}^+(t/q)^{-1/4}z :
\]

\[
\tilde{x}^+(t/q)^{-1/4}z\tilde{x}^-(t/q)^{1/4}w = \frac{1 - tw/z(1 - w/qz)}{1 - tw/z(1 - w/z)} : \tilde{x}^+(t/q)^{-1/4}z\tilde{x}^-(t/q)^{1/4}w :
\]

Hence, we have

\[
\tilde{\Phi}_\lambda(u)\tilde{x}^+(t/q)^{-1/4}z = (1 - t^{\ell(\lambda)} qz/tu) \prod_{j=1}^{\ell(\lambda)} \frac{1 - qz/tu_j}{1 - qz/u_j} : \tilde{\Phi}_\lambda(u)\tilde{x}^+(t/q)^{-1/4}z :
\]

for \(|z/u_j| < 1\),

\[
\tilde{x}^+(t/q)^{-1/4}z\tilde{\Phi}_\lambda(u) = (1 - t^{-\ell(\lambda)} tu/qz) \prod_{j=1}^{\ell(\lambda)} \frac{1 - tu_j/qz}{1 - u_j/qz} : \tilde{\Phi}_\lambda(u)\tilde{x}^+(t/q)^{-1/4}z :
\]

for \(|u_j/z| < 1\)

and

\[
: \tilde{\Phi}_\lambda(u)\tilde{x}^+(t/q)^{-1/4}q^{-1}u_k :
\]

\[
= : \Phi_\theta(u) \prod_{(i, j) \in \lambda} \tilde{x}^-((t/q)^{1/4}u_{i,j})\tilde{x}^+(t/q)^{-1/4}q^{-1}u_k :
\]

\[
= : \Phi_\theta(u) \prod_{(i, j) \in \lambda - 1_k} \tilde{x}^-((t/q)^{1/4}u_{i,j}) \cdot \tilde{x}^-((t/q)^{1/4}u_{k,\lambda_k})\tilde{x}^+(t/q)^{-1/4}q^{-1}u_k :
\]

\[
=: \tilde{\Phi}_{\lambda-1_k}(u)\tilde{\psi}^-((t/q)^{-1/4}q^{-1}u_k) . \quad (D.1)
\]

The last equality follows from

\[
: \tilde{x}^-((t/q)^{1/4}z)\tilde{x}^+(t/q)^{-1/4}z := \tilde{\psi}^-((t/q)^{-1/4}z).
\]

Therefore, noting

\[
x^+(z)|_{\mathcal{F}_{+}(N)} = vz^{-N}(t/q)^{3N/4}\tilde{x}^+(z),
\]

\[
x^+(z)|_{\mathcal{F}_{+}(N+1)} = -uvz^{-(N+1)}(t/q)^{3(N+1)/4}\tilde{x}^+(z),
\]
one obtains

\[
(-uvz^{-(N+1)}(t/q)^{N+1})^{-1} \left( \tilde{\Phi}_\lambda(u)x^+((t/q)^{-1/4}z) \bigg|_{x^{(1,N+1)}} - x^+((t/q)^{-1/4}z) \bigg|_{x^{(1,N)}} \tilde{\Phi}_\lambda(u) \right)
\]

\[
= \left( 1 - t^{\ell(\lambda)}qz/tu \right) \prod_{j=1}^{\ell(\lambda)} \left. \frac{1 - qz/tu_j}{1 - qz/u_j} \right|_{|z/u_j| < 1}
\]

\[
+ (t/q)^{-1}z/u(1 - t^{-\ell(\lambda)}tu/qz) \prod_{j=1}^{\ell(\lambda)} \left. \frac{1 - tu_j/qz}{1 - u_j/qz} \right|_{|u_j/z| < 1}
\]

\[
\times: \tilde{\Phi}_\lambda(u)\tilde{x}^+((t/q)^{1/4}(t/q)^{-1/2}z) :
\]

\[
= \left( 1 - t^{\ell(\lambda)}qz/tu \right) \prod_{j=1}^{\ell(\lambda)} \left. \frac{1 - qz/tu_j}{1 - qz/u_j} \right|_{|z/u_j| < 1}
\]

\[
- (1 - t^{\ell(\lambda)}qz/tu) \prod_{j=1}^{\ell(\lambda)} \left. \frac{1 - qz/tu_j}{1 - qz/u_j} \right|_{|u_j/z| < 1}
\]

\[
\times: \tilde{\Phi}_\lambda(u)\tilde{x}^+((t/q)^{1/4}(t/q)^{-1/2}z) :
\]

\[
= \sum_{k=1}^{\ell(\lambda)} \delta(q^{-1}u_k/z) \prod_{j=1}^{k-1} \left. \frac{1 - u_k/tu_j}{1 - u_k/u_j} \right|_{|t/u_j| < 1}
\]

\[
\times \prod_{j=k+1}^{\ell(\lambda)} \left. \frac{1 - u_k/tu_j}{1 - u_k/u_j} \right|_{|u_j/tu_k| < 1}\tilde{\Phi}_\lambda(u)\tilde{x}^+((t/q)^{1/4}(t/q)^{-1/2}z) :
\]

\[
= \sum_{k=1}^{\ell(\lambda)} \delta(q^{-1}u_k/z)(1 - t^{-1})(q/t)t^{-k+1}c_{\lambda}A_{\lambda-1k}^{-}\tilde{\Phi}_\lambda-1_k(u)\tilde{\psi}^-((t/q)^{-1/4}u_k) :
\]

(D.2)

In the last equality we used (D.1). The third equality follows from the formula

\[
(1 - s/b_{n+1}) \prod_{j=1}^{n} \left. \frac{1 - s/b_j}{1 - s/a_j} \right|_{|s/a_j| < 1} - (1 - s/b_{n+1}) \prod_{j=1}^{n} \left. \frac{1 - s/b_j}{1 - s/a_j} \right|_{|a_j/s| < 1}
\]

\[
= \sum_{k=1}^{n} \delta(s/a_k) (1 - a_k/b_{n+1}) \frac{\prod_{j=1}^{n} (1 - a_k/b_j)}{\prod_{j=1}^{n} (1 - a_k/a_j)}. \quad \text{(D.3)}
\]

This is obtained from the partial fraction formula for \( \prod_{j=1}^{n} \left. \frac{1 - s/b_j}{1 - s/a_j} \right|_{|z| < 1} \) and

\[
\frac{1}{1 - z} \bigg|_{|z| < 1} - \frac{1}{1 - z} \bigg|_{|z| > 1} = \delta(z).
\]
On the other hand, we have

$$
\tilde{\psi}^-((t/q)^{-1/4}z) \Phi_{\lambda}(u) = \frac{(pu/z; p)_{\infty}}{(ptu/qz; p)_{\infty}} : \tilde{\psi}^-((t/q)^{-1/4}z) \Phi_{\lambda}(u) :,
$$

$$
\tilde{\psi}^-((t/q)^{-1/4}z) x^-((t/q)^{1/4}w) = h(w/z) : \tilde{\psi}^-((t/q)^{-1/4}z) x^-((t/q)^{1/4}w) :,
$$

where

$$
h(w) = \frac{(pqw; p)_{\infty}(ptw/q; p)_{\infty}(pw/t; p)_{\infty}}{(pw/q; p)_{\infty}(pqw/t; p)_{\infty}(ptw; p)_{\infty}}.
$$

We have

$$
\prod_{(i, j) \in \lambda - 1_k} h(u_i, j / z) \bigg|_{z = q^{-1} u_k} = - t \frac{a^- (p)}{a^+ (p)} \frac{(pu/u_k; p)_{\infty}}{(pu_j/u_k; p)_{\infty}} \prod_{j=1}^{\ell(\lambda)} \frac{(pu_j/u_k; p)_{\infty}}{(pu_j/tu_k; p)_{\infty}} \prod_{j=1}^{\ell(\lambda) + 1} \frac{(pqu_j/u_k; p)_{\infty}}{(pqu_j/tu_k; p)_{\infty}}.
$$

Hence,

$$
\tilde{\psi}^-((t/q)^{-1/4}z) \Phi_{\lambda - 1_k}(u) \bigg|_{z = q^{-1} u_k} = - t \frac{a^- (p)}{a^+ (p)} \prod_{j=1}^{\ell(\lambda)} \frac{(pu_j/u_k; p)_{\infty}}{(pu_j/tu_k; p)_{\infty}} \prod_{j=1}^{\ell(\lambda) + 1} \frac{(pqu_j/u_k; p)_{\infty}}{(pqu_j/tu_k; p)_{\infty}}.
$$

Substituting this into (D.2), one obtains

$$
\tilde{\Phi}_{\lambda}(u) x^+((t/q)^{-1/4}z) \bigg|_{x^+((t/q)^{-1/4}z)} - x^+((t/q)^{-1/4}z) \bigg|_{x^+((t/q)^{-1/4}z)} \tilde{\Phi}_{\lambda}(u)
$$

$$
= u v z^{-(N+1)} (t/q)^{N+1} \sum_{k=1}^{\ell(\lambda)} \delta(q^{-1}u_k/z) t^{-k} a^+ (p) c_{N_k - 1_k} \Phi_{\lambda - 1_k}(p) (t/q)^{-N/2}
$$

$$
\times \psi^-((t/q)^{-1/4}z) \tilde{\Phi}_{\lambda - 1_k}(u)
$$

$$
= \frac{c_{\lambda}}{q^{n(\lambda)}} N_{\lambda}(p) r^*(\lambda, u, v, N) \sum_{k=1}^{\ell(\lambda)} \delta(q^{-1}u_k/z) t^{-1} a^+ (p) A_{\lambda - 1_k}^- (p) \psi^-((t/q)^{-1/4}z)
$$

$$
\times \frac{q^{n(\lambda - 1_k)}}{c_{\lambda - 1_k}} N_{\lambda - 1_k}^- (p) r^*(\lambda - 1_k, u, v, N) \tilde{\Phi}_{\lambda - 1_k}(u).
$$

Here, we used
\[
\frac{t^*(\lambda - 1_k, u, v, N)}{t^*(\lambda, u, v, N)} = q^{-1}v(q^{-1}u_k)^{-N(t/q)^{N/2}}.
\]

Proof of (4.7): From Theorem 3.2 we have

\[
\tilde{\Phi}_\lambda(u)\tilde{x}^-((t/q)^{1/4}z) = \prod_{i=1}^{\ell(\lambda)} \frac{(tz/nu_i; p)_\infty}{(pz/nu_i; p)_\infty}
\]

\[
\times \prod_{i=1}^{\ell(\lambda)+1} \frac{(pz/nu_i; p)_\infty}{(zu_i; p)_\infty} : \tilde{x}^-((t/q)^{1/4}z)\tilde{\Phi}_\lambda(u) : \text{ for } |z/u| < 1
\]

\[
\tilde{x}^-((t/q)^{1/4}z)\tilde{\Phi}_\lambda(u) = \prod_{i=1}^{\ell(\lambda)} \frac{(q^{-1}u_i/z; p)_\infty}{(pt^{-1}u_i/z; p)_\infty}
\]

\[
\times \prod_{i=1}^{\ell(\lambda)+1} \frac{(pu_i/z; p)_\infty}{(tu_i/qz; p)_\infty} : \tilde{x}^-((t/q)^{1/4}z)\tilde{\Phi}_\lambda(u) : \text{ for } |u/z| < 1
\]

and

\[
B^+_{\lambda}(u/z, p)\tilde{x}^-((t/q)^{1/4}z)\tilde{\Phi}_\lambda(u) = -z/u \prod_{i=1}^{\ell(\lambda)} \frac{(tz/nu_i; p)_\infty}{(pz/nu_i; p)_\infty}
\]

\[
\times \prod_{i=1}^{\ell(\lambda)+1} \frac{(pz/nu_i; p)_\infty}{(zu_i; p)_\infty} : \tilde{x}^-((t/q)^{1/4}z)\tilde{\Phi}_\lambda(u) : .
\]

Hence, noting

\[
x^-((z))\bigg|_{\mathcal{F}^{(1,N+1)}_{-uv}} = (-uv)^{-1}z^{N+1}(t/q)^{-3(N+1)/4}\tilde{x}^-((z),
\]

\[
x^-((z))\bigg|_{\mathcal{F}^{(1,N)}_{x}} = v^{-1}z^N(t/q)^{-3N/4}\tilde{x}^-((z),
\]

one gets

\[
\tilde{\Phi}_\lambda(u)x^-((t/q)^{1/4}z) - (t/q)^{-1/2}B^+_{\lambda}(z, p)x^-((t/q)^{1/4}z)\tilde{\Phi}_\lambda(u)
\]

\[
= -(uv)^{-1}z^{N+1}(t/q)^{-(N+1)/2}
\]

\[
\times \prod_{i=1}^{\ell(\lambda)} \frac{(tz/nu_i; p)_\infty}{(pz/nu_i; p)_\infty} \prod_{i=1}^{\ell(\lambda)+1} \frac{(pz/nu_i; p)_\infty}{(zu_i; p)_\infty} |_{|z/u|<1} - \prod_{i=1}^{\ell(\lambda)} \frac{(tz/nu_i; p)_\infty}{(pz/nu_i; p)_\infty}
\]

\[
\times \prod_{i=1}^{\ell(\lambda)+1} \frac{(pz/nu_i; p)_\infty}{(zu_i; p)_\infty} |_{|u/z|<1}
\]

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\begin{align*}
&(uv)^{-1}z^{N+1}(t/q)^{-(N+1)/2} \left( \prod_{i=1}^{\ell(\lambda)+1} \frac{1-iz/\ell_i}{1-z/\ell_i} \bigg|_{z/\ell_i<1} \right) \\
&\quad - \prod_{i=1}^{\ell(\lambda)+1} \frac{1-iz/\ell_i}{1-z/\ell_i} \bigg|_{z/\ell_i<1} \prod_{i=1}^{\ell(\lambda)+1} \frac{(pqz/u_i; p)_{\infty}}{(pz/u_i; p)_{\infty}} \prod_{i=1}^{\ell(\lambda)+1} \frac{(ptz/u_i; p)_{\infty}}{(pqz/u_i; p)_{\infty}} \\
&\quad \times : \Phi_\lambda(u) \widehat{x}^- ((t/q)^{1/4}z) : .
\end{align*}

\text{(D.4)}

Applying the formula

\begin{align*}
\prod_{j=1}^{n-1} \frac{(1-s/b_j)}{(1-s/a_j)} &\bigg|_{|s/a|<1} - \prod_{j=1}^{n-1} \frac{(1-s/b_j)}{(1-s/a_j)} \bigg|_{|s/a|<1} = \sum_{k=1}^{n} \frac{\delta(a_k/s)}{\prod_{j=1}^{n} (1-a_k/b_j)} \\
&\text{and using}
\end{align*}

\begin{align*}
: \Phi_\lambda(u) \widehat{x}^- ((t/q)^{1/4}z) : \bigg|_{z=a_k} = \widetilde{\Phi}_{\lambda+1_k}(u),
\end{align*}

the RHS of (D.4) is

\begin{align*}
&(uv)^{-1}z^{N+1}(t/q)^{-(N+1)/2} \sum_{k=1}^{\ell(\lambda)+1} \delta(u_k/z) \\
&\quad \prod_{i=1}^{\ell(\lambda)} \frac{(tu_k/u_i; p)_{\infty}}{(pu_k/u_i; p)_{\infty}} \frac{\prod_{j=1}^{\ell(\lambda)+1} (pu_k/tu_i; p)_{\infty}}{\prod_{i=1}^{\ell(\lambda)+1} (u_k/u_i; p)_{\infty}} \frac{1}{(p; p)_{\infty}} \widetilde{\Phi}_{\lambda+1_k}(u) \\
&= (uv)^{-1}z^{N+1}(t/q)^{-(N+1)/2} \sum_{k=1}^{\ell(\lambda)+1} \delta(u_k/z) t^k a^{-}(p) \frac{c_\lambda}{c_{\lambda+1_k}} A_{\lambda,k}^+ (p) \frac{N_{\lambda+1_k}(p)}{N_\lambda(p)} \widetilde{\Phi}_{\lambda+1_k}(u) \\
&= q^{n(\lambda^+)N_\lambda(p)t^*(\lambda, u, v, N)} q^{-1} f(1; p)a^+(p)(t/q)^{-1/2} \\
&\quad \times \sum_{k=1}^{\ell(\lambda)+1} \delta(u_k/z) A_{\lambda,k}^+ (p) q^{n((\lambda+1)_k^*) N_{\lambda+1_k}(p)t^*(\lambda+1_k, u, v, N)} \frac{N_{\lambda+1_k}(p)}{N_{\lambda+1_k}(p)} \widetilde{\Phi}_{\lambda+1_k}(u).
\end{align*}

\[ \square \]

\section*{E Proof of Proposition 4.7}

Proof of (4.30): From (4.4), we have

\[ \psi^+((t/q)^{1/4}z) \Phi_{\lambda}(p^{-1}u)^{-1} = (q/t)^{1/2} B_{\lambda}^+(p^{-1}u/z; p) \Phi_{\lambda}(p^{-1}u)^{-1} \psi^+((t/q)^{1/4}z). \]

\[ \square \]
Proof of (4.32): Similar to the calculations in Appendix D, we have
\[ \tilde{\Phi}_\lambda(p^{-1}u)^{-1}\tilde{\beta}^+((t/q)^{-1/4}z) = \frac{1}{1 - pt^l qz/tu} \]
\[ \times \prod_{j=1}^{l} \frac{1 - pqz/u_j}{1 - pqz/tu_j} : \tilde{\Phi}_\lambda(p^{-1}u)^{-1}\tilde{\beta}^+((t/q)^{-1/4}z) : \]
for \(|z/u_j| < 1\),
\[ \tilde{\beta}^+((t/q)^{-1/4}z)\tilde{\Phi}_\lambda(p^{-1}u)^{-1} = \frac{1}{1 - p^{-1}t^{-l}tu/qz} \]
\[ \times \prod_{j=1}^{l} \frac{1 - p^{-1}u_j/qz}{1 - p^{-1}tu_j/qz} : \tilde{\Phi}_\lambda(p^{-1}u)^{-1}\tilde{\beta}^+((t/q)^{-1/4}z) : \]
for \(|u_j/z| < 1\)

and
\[ : \tilde{\Phi}_\lambda(p^{-1}u)^{-1}\tilde{\beta}^+((t/q)^{-1/4}p^{-1}tu_k/q) : \]
\[ = : \Phi_\varphi(p^{-1}u)^{-1} \prod_{(i,j) \in \lambda} \tilde{\beta}^-((t/q)^{1/4}p^{-1}u_{i,j})^{-1}\tilde{\beta}^+((t/q)^{-1/4}tu_k/q) : \]
\[ = : \Phi_\varphi(p^{-1}u)^{-1} \prod_{(i,j) \in \lambda + 1_k} \tilde{\beta}^-((t/q)^{1/4}p^{-1}u_{i,j})^{-1} \]
\[ \times \tilde{\beta}^-((t/q)^{1/4}u_k,\lambda_k + 1)\tilde{\beta}^+((t/q)^{-1/4}p^{-1}tu_k/q) : \]
\[ = : \tilde{\Phi}_{\lambda + 1_k}(p^{-1}u)^{-1}\tilde{\beta}^+((t/q)^{1/4}p^{-1}u_k) : . \quad (E.1) \]

Therefore, noting
\[ x^+(z) \big|_{\mathcal{F}_v^{(1, N)}} = vz^{-N}(t/q)^{3N/4}x^+(z), \]
\[ x^+(z) \big|_{\mathcal{F}_v^{(1, N+1)}} = -p^{-1}uvz^{-(N+1)}(t/q)^{3(N+1)/4}x^+(z), \]
one obtains
\[ x^+((t/q)^{-1/4}z) \big|_{\mathcal{F}_v^{(1, N+1)}} \tilde{\Phi}_\lambda(p^{-1}u)^{-1} - \tilde{\Phi}_\lambda(p^{-1}u)^{-1}x^+((t/q)^{-1/4}z) \big|_{\mathcal{F}_v^{(1, N)}} \]
\[ = -p^{-1}uvz^{-(N+1)}(t/q)^{N+1} \]
\[ \times \left( \frac{1}{1 - p^{-1}t^{-l}tu/qz} \prod_{j=1}^{\ell} \frac{1 - p^{-1}u_j/qz}{1 - p^{-1}tu_j/qz} \big|_{u_j/z < 1} \right) \]
\[ + p(t/q)^{-1}z|_u \frac{1}{1 - pt^l qz/tu} \prod_{j=1}^{\ell} \frac{1 - pqz/u_j}{1 - pqz/tu_j} \big|_{|z/u_j| < 1} \]
\[ \times : \tilde{\Phi}_\lambda(p^{-1}u)^{-1}\tilde{\beta}^+((t/q)^{-1/4}z) : \]
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\[
\begin{aligned}
&= - p^{-1} uvz^{-(N+1)}(t/q)^{N+1}pqz/tu \\
&\times \left( - \frac{1}{1-tqz/tu} \prod_{j=1}^{\ell} \frac{1 - pqz/u_j}{1 - pqz/tu_j} \bigg|_{u_j/z < 1} + \frac{1}{1-tqz/tu} \prod_{j=1}^{\ell} \frac{1 - pqz/u_j}{1 - pqz/tu_j} \bigg|_{z/u_j < 1} \right) \\
&\times : \Phi_\lambda(p^{-1}u)^{-1}x^+((t/q)^{-1/4}z) : \\
&= - p^{-1} uvz^{-(N+1)}(t/q)^{N+1}pqz/tu \\
&\times \sum_{k=1}^{\ell+1} \delta(p^{-1}tu_k/zq) \prod_{\substack{j=1 \atop j \neq k}}^{\ell+1} \frac{1 - tu_k/u_j}{1 - u_k/u_j} : \Phi_{\lambda+1_k}(p^{-1}u)^{-1}\tilde{\psi}^+((t/q)^{-1/4}u_k) : .
\end{aligned}
\]

(E.2)

The last equality follows from (D.5).

On the other hand, we have

\[
\begin{aligned}
\Phi_\emptyset(u)^{-1}\tilde{\psi}^-((t/q)^{-1/4}z) &= \frac{(z/u; p)_\infty}{(qz/tu; p)_\infty} : \tilde{\psi}^-((t/q)^{-1/4}z)\Phi_\emptyset(u)^{-1} : , \\
\tilde{x}^-((t/q)^{1/4}u)^{-1}\tilde{\psi}^-((t/q)^{-1/4}z) &= h(p^{-1}z/u)^{-1} : \tilde{\psi}^-((t/q)^{-1/4}z)\tilde{x}^-((t/q)^{1/4}u)^{-1} : ,
\end{aligned}
\]

where

\[
h(w) = \frac{(pqw; p)_\infty(pqw/tu; p)_\infty(pqw/t; p)_\infty(pqw/tu; p)_\infty}{(pw/t; p)_\infty(pqw/t; p)_\infty(pqw/tu; p)_\infty(pqw/tu; p)_\infty}. \]

We have

\[
\Phi_{\lambda+1_k}(p^{-1}u)^{-1}\tilde{\psi}^-((t/q)^{-1/4}u_k) \\
= \frac{(pu_k/u; p)_\infty}{(pqu_k/tu; p)_\infty} \prod_{(i,j) \in \lambda+1_k} h(u_k/u_{i,j})^{-1} : \Phi_{\lambda+1_k}(p^{-1}u)^{-1}\tilde{\psi}^-((t/q)^{-1/4}u_k) :
\]

We have $u_k, (\lambda+1_k)_k = u_k$,

\[
\prod_{(i,j) \in \lambda+1_k} h(u_k/u_{i,j})^{-1} = \prod_{(i,j) \in \lambda} h(u_k/u_{i,j})^{-1} \times h(u_k/u_{k,(\lambda+1)_k})^{-1} \\
= - \frac{a^-(p)}{a^+(p)} \frac{(pqu_k/tu; p)_\infty}{(pu_k/u; p)_\infty} \prod_{j=1}^{l} \frac{(pqu_k/u_j; p)_\infty}{(ptu_k/u_j; p)_\infty} \prod_{j=1}^{l+1} \frac{(pu_k/u_j; p)_\infty}{(pqu_k/tu_j; p)_\infty}. \]

Hence,

\[
\Phi_{\lambda+1_k}(p^{-1}u)^{-1}\tilde{\psi}^-((t/q)^{-1/4}u_k) \\
= - \frac{a^-(p)}{a^+(p)} \prod_{j=1}^{l} \frac{(pqu_k/u_j; p)_\infty}{(ptu_k/u_j; p)_\infty} \prod_{j=1}^{l+1} \frac{(pu_k/u_j; p)_\infty}{(pqu_k/tu_j; p)_\infty} : \\
\times \Phi_{\lambda+1_k}(p^{-1}u)^{-1}\tilde{\psi}^-((t/q)^{-1/4}u_k) : .
\]
Therefore,

\[
\prod_{j=1}^{\ell}(1 - t u_k / u_j) \prod_{j=1}^{\ell+1} (1 - u_k / u_j) \neq k = \widetilde{\Phi}_{\lambda + 1 k}(p^{-1} u) -1 \tilde{\psi}^-((t / q)^{-1/4} u_k) :
\]

\[
= t^{k-1} a^+(p) \frac{c'_{\lambda, N'_{\lambda, k}(p)}}{c'_{\lambda + 1 k, N'_{\lambda, k}(p)}} A_{\lambda,k}^+(p) \tilde{\Phi}_{\lambda + 1 k}(p^{-1} u) -1 \tilde{\psi}^-((t / q)^{-1/4} u_k).
\]

Substituting this into (D.2), one obtains

\[
x^+((t / q)^{-1/4} z) \bigg|_{\mathcal{F}(1, N + 1)_{-p^{-1} u v}} \tilde{\Phi}_\lambda(p^{-1} u) -1 - \tilde{\Phi}_\lambda(p^{-1} u) -1 x^+((t / q)^{-1/4} z) \bigg|_{\mathcal{F}(1, N)}
\]

\[
= \frac{c'_{\lambda}}{q^{n(\lambda')'} t(\lambda + 1 k, v, p^{-1} u, N) N'_{\lambda, k}(p)} (t / q)^{-1/2} a^+(p)
\]

\[
\times \sum_{k=1}^{\ell+1} \delta(p^{-1} tu_k / q z) A_{\lambda,k}^+(p) \frac{q^{n(\lambda + 1 k)'} t(\lambda + 1 k, v, p^{-1} u, N) N'_{\lambda + 1 k}(p)}{c'_{\lambda + 1 k}}
\]

\[
\times \tilde{\Phi}_{\lambda + 1 k}(p^{-1} u) -1 \tilde{\psi}^+((t / q)^{1/4} q z / t).
\]

Here, we used

\[
t(\lambda + 1 k, v, p^{-1} u, N)
\]

\[
= - p^{-1} u v z^{-(N + 1)} (t / q)^{3(N + 1)/2},
\]

\[
pq z / tu \cdot t^{k-1} = \lambda_{\lambda,k} = \frac{q^{n(\lambda + 1 k)'}}{q^{n(\lambda')}}.
\]

\[\square\]

Proof of (4.28): We have

\[
\tilde{\Phi}_\lambda(u)^{-1} \tilde{\chi}^-((t / q)^{1/4} z) = \prod_{i=1}^{l(\lambda)} (p q z / u_i ; p)^{\infty} (t z / u_i ; p)^{\infty}
\]

\[
\times \prod_{i=1}^{l(\lambda) + 1} \frac{(z / u_i ; p)^{\infty}}{(p q z / t u_i ; p)^{\infty}} : \tilde{\chi}^-((t / q)^{1/4} z) \tilde{\Phi}_\lambda(u)^{-1} : \text{ for } |z / u| < 1,
\]

\[
\tilde{\chi}^-((t / q)^{1/4} z) \tilde{\Phi}_\lambda(u)^{-1} = \prod_{i=1}^{l(\lambda)} (p u_i / t z ; p)^{\infty} (u_i / q z ; p)^{\infty}
\]

\[
\times \prod_{i=1}^{l(\lambda) + 1} \frac{(t u_i / q z ; p)^{\infty}}{(p u_i / z ; p)^{\infty}} : \tilde{\chi}^-((t / q)^{1/4} z) \tilde{\Phi}_\lambda(u)^{-1} : \text{ for } |u / z| < 1
\]

and

\[\square\]
\[ B_\lambda^+(p^{-1}u/z, p) \tilde{\Phi}_\lambda(p^{-1}u)^{-1} \bar{x}^-((t/q)^{1/4}z) \]

\[ = -pz/u \prod_{i=1}^{l(\lambda)} \frac{(u_i/tz; p)_{\infty}}{(p^{-1}u_i/qz; p)_{\infty}} \prod_{i=1}^{l(\lambda)+1} \frac{(p^{-1}tu_i/qz; p)_{\infty}}{(u_i/z; p)_{\infty}} \tilde{\Phi}_\lambda(p^{-1}u)^{-1} \bar{x}^-((t/q)^{1/4}z) : . \]

Noting

\[ x^-((t/q)^{1/4}z) \bigg|_{x^-(1,N+1)} = (-p^{-1}uv)^{-1} z^{N+1} (t/q)^-(N+1)/2 \bar{x}^-((t/q)^{1/4}z), \]
\[ x^-((t/q)^{1/4}z) \bigg|_{x^-(1,N)} = v^{-1} z^N (t/q)^{-N/2} \bar{x}^-((t/q)^{1/4}z), \]

one gets

\[ x^-((t/q)^{1/4}z) \bigg|_{x^-(1,N+1)} = -(pv)^{-1} z^{N+1} (t/q)^-(N+1)/2 \]
\[ \times \left( \prod_{i=1}^{l(\lambda)} \frac{(u_i/tz; p)_{\infty}}{(p^{-1}u_i/qz; p)_{\infty}} \prod_{i=1}^{l(\lambda)+1} \frac{(p^{-1}tu_i/qz; p)_{\infty}}{(u_i/z; p)_{\infty}} \right) \bigg|_{|u/z|<1} \]
\[ \times : \tilde{\Phi}_\lambda(p^{-1}u)^{-1} \bar{x}^-((t/q)^{1/4}z) : \]
\[ = -(uv)^{-1} p^{N+1} (t/q)^-(N+1)/2 \]
\[ \times \left( \prod_{i=1}^{l(\lambda)} \frac{(1-p^{-1}tu_i/qz)}{(1-p^{-1}u_i/qz)} \bigg|_{|u_i|<1} \right) \]
\[ \times \left( \prod_{i=1}^{l(\lambda)} \frac{(1-p^{-1}tu_{i+1}/qz)}{(1-p^{-1}u_{i+1}/qz)} \bigg|_{|z/u_i|<1} \right) \]
\[ \times \prod_{i=1}^{l(\lambda)} \frac{(u_i/tz; p)_{\infty}}{(u_i/qz; p)_{\infty}} \prod_{i=1}^{l(\lambda)+1} \frac{(tu_i/qz; p)_{\infty}}{(u_i/z; p)_{\infty}} : \tilde{\Phi}_\lambda(p^{-1}u)^{-1} \bar{x}^-((t/q)^{1/4}z) : . \] (E.3)

Applying the formula (D.3) and using

\[ : \tilde{\Phi}_\lambda(p^{-1}u)^{-1} \bar{x}^-((t/q)^{1/4}z) : \bigg|_{z=p^{-1}uk, \lambda_k} = \tilde{\Phi}_{\lambda_k-1k}(p^{-1}u)^{-1}, \]

the RHS of (E.3) is

\[ -(uv)^{-1} p^{N+1} (t/q)^-(N+1)/2 ta^- (p) \]
\[ \times \sum_{k=1}^{l(\lambda)} \delta(pqz/u_k) \prod_{i=1}^{l(\lambda)} \frac{(pqu_i/tu_k; p)_{\infty}}{(u_i/u_k; p)_{\infty}} \prod_{i=1}^{l(\lambda)+1} \frac{(tu_i/u_k; p)_{\infty}}{(pqu_i/u_k; p)_{\infty}} \times \tilde{\Phi}_{\lambda_k-1k}(p^{-1}u)^{-1} \]
\[
= -(uv)^{-1} \left( t^{N+1} \right) \left( (t/q)^{-N+1} \right) / 2 \\
\times \sum_{k=1}^{\ell(\lambda)} \delta(pqz/uk) q^{-\lambda_k a^- (p)} c'_{\lambda} \frac{N'_{\lambda-1_k}(p)}{\lambda'_{\lambda-1_k}(p)} A_{\lambda,k}(p) \Phi_{\lambda-1_k}(p^{-1}u)^{-1} \\
= \frac{c'_{\lambda}}{q^{n(\lambda)} t(\lambda, v, p^{-1}u, N)N_{\lambda}(p)} (t/q) a^- (p) \\
\times \sum_{k=1}^{\ell(\lambda)} \delta(pqz/uk) A_{\lambda,k}(p) \frac{q^{n(\lambda-1_k)} t(\lambda+1_k, v, p^{-1}u, N)N'_{\lambda-1_k}(p)}{c'_{\lambda-1_k}} \Phi_{\lambda-1_k}(p^{-1}u)^{-1}.
\]

\[\square\]

**F Useful formulas for the Nekrasov function**

For readers’ convenience, we list some useful formulas for the Nekrasov function \( N_{\lambda, \mu}(x) \). Let \( \lambda, \mu \in \mathcal{P}^+ \). Then for any integer \( \ell \geq \ell(\lambda), \ell(\mu) \), one has the following formulas \([4, 5]\):

\[
N_{\lambda, \mu}(x) = \prod_{\square \in \lambda} (1 - x q^{-a_{\square}(\square) - 1} t^{\ell, (\square)}) \prod_{\square \in \mu} (1 - x q^{a_{\square}(\square) + \ell(\square) + 1}) \\
= \prod_{1 \leq i \leq j \leq \ell} (x q^{\mu_i + \lambda_j t^{-i-j}; q}) \prod_{1 \leq r \leq s \leq \ell} (x q^{\lambda_r - \mu_s t^{-r-s+1}; q}) \\
= \prod_{1 \leq i \leq j \leq \ell} (x q^{\mu_i + \lambda_j t^{-i-j}; q}) (x q^{\lambda_r - \mu_s t^{-r-s+1}; q}) \\
= \prod_{i, j = 1}^{\ell} (x q^{\mu_i - \lambda_j t^{-i-j+1}; q}) \prod_{r = 1}^{\ell} (x q^{\mu_i t^{-\ell + r}; q}) \\
= \prod_{r = 1}^{\ell} (x q^{\mu_i t^{-\ell + r}; q}) \\
\]

where

\[
(x; q)_n = (1 - x)(1 - x q) \cdots (1 - x q^{n-1}).
\]

The second equality follows from the formula

\[
(1 - q) \sum_{(i, j) \in \lambda} q^{\mu_i - j t^{\lambda_j - i+1}} = t \sum_{1 \leq i \leq \ell(\lambda)} q^{\mu_i - \lambda_j t^{-i}} - \sum_{1 \leq i \leq \ell(\lambda) + 1} q^{\mu_i - \lambda_j t^{-i}}.
\]

\[\text{(F.1)}\]

which is derived by using \([60]\)

\[
(1 - q) \sum_{j=1}^{\lambda_i} q^{-j t^{\lambda_j}} = \sum_{j=1}^{\ell(\lambda)} q^{\lambda_j} (1 - q^{\lambda_j - 1} t).
\]

\[\text{(F.2)}\]
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References

1. Aganagic, M., Frenkel, E., Okounkov, A.: Quantum $q$-Langlands correspondence. Trans. Moscow Math. Soc. 79, 1–83 (2018)
2. Aganagic, M., Okounkov, A.: Elliptic Stable Envelopes (2016). arXiv:1604.00423
3. Alday, L.F., Gaiotto, D., Tachikawa, Y.: Liouville correlation functions from four-dimensional gauge theories. Lett. Math. Phys. 91, 167–197 (2010)
4. Awata, H., Kanno, H.: Refined BPS state counting from Nekrasov’s formula and Macdonald functions. Int. J. Mod. Phys. A 24, 2253–2306 (2009)
5. Awata, H., Feigin, B., Shiraishi, J.: Quantum algebraic approach to refined topological vertex. J. High Energy Phys. 041, 34 (2012)
6. Awata, H., Kanno, H., Matsumoto, T., Mironov, A., Morozov, A., Morozov, A., Ohkubo, Y., Zenkevich, Y.: Explicit examples of DIM constraints for network matrix models. J. High Energy Phys. 07, 103 (2016)
7. Awata, H., Kubo, H., Odake, S., Shiraishi, J.: Quantum $\mathfrak{W}_N$ algebras and Macdonald polynomials. Commun. Math. Phys. 179, 401–416 (1996)
8. Bourgine, J.-E., Fukuda, M., Matsuo, Y., Zhang, H., Zhu, R.-D.: Coherent states in quantum, $W_{1+\infty}$ algebra and $qq$-character for 5d super Yang–Mills. PTEP 2016(12), 123B05 (2016)
9. Bourgine, J.-E., Fukuda, M., Harada, K., Matsuo, Y., Zhu, R.-D.: $(p, q)$-webs of DIM representations, 5d $\mathcal{N}=1$ instanton partition functions and $qq$-characters. arXiv:1703.10759
10. Bourgine, J.-E., Fukuda, M., Harada, K., Matsuo, Y., Zhu, R.-D.: Exchange dynamical quantum groups. Commun. Math. Phys. 196, 591–640 (1998)
11. Farghly, R.M., Konno, H., Oshima, K.: Elliptic algebra $U_{q,p}(\hat{\mathfrak{g}})$ and quantum $Z$-algebras. Algebras Represent Theory (2014). https://doi.org/10.1007/s10468-014-9483-x
12. Feigin, B., Feigin, E., Jimbo, M., Miwa, T., Mukhin, E.: Quantum continuous $\mathfrak{gl}_{\infty}$: semi-infinite construction of representations. Kyoto J. Math. 51, 337–364 (2011)
13. Feigin, B., Frenkel, E.: Quantum $W$-algebras and elliptic algebras. Commun. Math. Phys. 178, 653–678 (1996)
14. Foda, O., Iohara, K., Jimbo, M., Kedem, R., Miwa, T., Yan, H.: Notes on Highest Weight Modules of the Elliptic Algebra $A_{q,p}(\mathfrak{sl}_2)$. Quantum field theory, integrable models and beyond (Kyoto, 1994). Progr. Theoret. Phys. Suppl. 118, 1–34 (1995)
15. Foda, O., Zhu, R.-D.: An Elliptic Topological Vertex. arXiv:1805.12073
16. Frenkel, E., Reshetikhin, N.: Deformation of $\mathcal{W}$-algebras associated to simple Lie algebras. Commun. Math. Phys. 197, 1–31 (1998). arxiv:q-alg/9708006
17. Fukuda, M., Ohkubo, Y., Shiraishi, J.: Generalized Macdonald functions on Fock tensor spaces and duality formula for changing preferred direction. arXiv:1903.05905
29. Fukuda, M., Ohkubo, Y., Shiraishi, J.: Non-stationary Ruijseenaars functions for $\kappa = r^{-1/N}$ and intertwining operators of Ding–Iohara–Miki algebra. arXiv:2002.00243
30. Goddard, P., Kent, A., Olive, D.: Virasoro algebras and coset space models. Phys. Lett. B 152, 88 (1985)
31. Goddard, P., Kent, A., Olive, D.: Unitary representations of the Virasoro and super-Virasoro algebras. Commun. Math. Phys. 103, 105–119 (1986)
32. Hollowood, T., Iqbal, A., Vafa, C.: Matrix models, geometric engineering and elliptic genera. J. High Energy Phys. 2008, 069 (2008)
33. Hohenegger, S., Iqbal, A.: M-strings, elliptic genera and $\mathcal{N} = 4$ string amplitudes. Fortsch. Phys. 62, 155–206 (2014)
34. Haghighat, B., Iqbal, A., Kozcaz, C., Lockhart, G., Vafa, C.: Matrix models, geometric engineering and elliptic genera. J. High Energy Phys. 2008, 069 (2008)
35. Kimura, T.: Double quantization of Seiberg–Witten geometry and $\mathcal{W}$-algebras. Lett. Math. Phys. 108, 1351–1381 (2018)
36. Kimura, T.: Double quantization of Seiberg–Witten geometry and $\mathcal{W}$-algebras. In: Proceedings of Symposia in Pure Mathematics, pp. 405–431. AMS (2018). https://doi.org/10.1090/pspum/100/01762
37. Kojima, T., Konno, H.: The elliptic algebra $\mathbb{U}_{q,\lambda}$. Commun. Math. Phys. 199, 605–647 (1999)
38. Kojima, T., Pestun, V.: Quiver $\mathcal{W}$-algebras. Adv. Math. 334, 779–842 (2015)
39. Kojima, T., Konno, H.: The elliptic algebra $\mathbb{U}_{q,\lambda}$. Commun. Math. Phys. 334, 779–842 (2015)
40. Kojima, T., Konno, H.: The elliptic algebra $\mathbb{U}_{q,\lambda}$. Commun. Math. Phys. 334, 779–842 (2015)
41. Kimura, T., Pestun, V.: Quiver $\mathcal{W}$-algebras. Adv. Math. 334, 779–842 (2015)
42. Kimura, T.: Double quantization of Seiberg–Witten geometry and $\mathcal{W}$-algebras. Lett. Math. Phys. 108, 1351–1381 (2018)
43. Kimura, T.: Double quantization of Seiberg–Witten geometry and $\mathcal{W}$-algebras. In: Proceedings of Symposia in Pure Mathematics, pp. 405–431. AMS (2018). https://doi.org/10.1090/pspum/100/01762
44. Kojima, T.: Elliptic deformed superalgebra $\mathbb{U}_{q,\lambda}\widehat{\mathfrak{sl}}_2$. Commun. Math. Phys. 239, 405–447 (2003)
45. Kojima, T., Konno, H.: The elliptic algebra $\mathbb{U}_{q,\lambda}\widehat{\mathfrak{sl}}_2$ and the Drinfeld realization of the elliptic quantum group $\mathbb{B}_{q,\lambda}\widehat{\mathfrak{sl}}_2$. Commun. Math. Phys. 239, 405–447 (2003)
46. Kojima, T.: The Drinfeld realization of the elliptic quantum group $\mathbb{B}_{q,\lambda}\widehat{\mathfrak{sl}}_2$. J. Math. Phys. 45, 3146–3179 (2004)
47. Konno, H.: An elliptic algebra $\mathbb{U}_{q,\lambda}\widehat{\mathfrak{sl}}_2$ and the fusion RSOS models. Commun. Math. Phys. 195, 373–403 (1998)
48. Konno, H.: Elliptic quantum group $\mathbb{U}_{q,\lambda}\widehat{\mathfrak{sl}}_2$ and vertex operators. J. Phys. A 41, 194012 (12pp) (2008)
49. Konno, H.: Elliptic quantum group $\mathbb{U}_{q,\lambda}\widehat{\mathfrak{sl}}_2$, Hopf algebroid structure and elliptic hypergeometric series. J. Geom. Phys. 59, 1485–1511 (2009)
50. Konno, H.: “Elliptic Quantum Group, Drinfeld Coproduct and Deformed $\mathcal{W}$-Algebras”, Talk Given at the Workshop Recent Advances in Quantum Integrable Systems 2014, Dijon; “Elliptic Quantum Group $\mathbb{U}_{q,\lambda}\widehat{\mathfrak{sl}}_2$ and the Drinfeld Coproduct and Deformed $\mathcal{W}$-Algebras”, Talk Delivered at RIMS Seminar on Representation Theory, July 11, 2014, Kyoto
51. Konno, H.: Elliptic quantum groups $\mathbb{U}_{q,\lambda}\widehat{\mathfrak{gl}}_N$ and $\mathbb{E}_{q,\lambda}(\widehat{\mathfrak{gl}}_N)$. Adv. Stud. Pure Math. 76, 347–417 (2018)
52. Konno, H.: Elliptic weight functions and elliptic $q$–KZ equation. J. Integr. Syst. 2, 1–43 (2017). https://doi.org/10.1093/integr/xyx011
53. Konno, H.: Elliptic stable envelopes and finite-dimensional representations of elliptic quantum group. J. Integr. Syst. 3, 1–43 (2018). https://doi.org/10.1093/integr/xyy012
54. Konno, H.: “Elliptic Quantum Toroidal Algebras”, Talk Given at the Workshop Representation Theory and Special Functions, 18–20, February, 2019, Univ. of the Ryukyus, Okinawa; “Elliptic Quantum Groups, Representations and Geometry”, Talk Given at the Workshop Elliptic Cohomology Days, 10–15, June, 2019, Univ. of Illinois, Urbana-Champaign; “Elliptic Quantum (Toroidal) Algebras and Their Representations”, Talk Given at the Workshop Elliptic Integrable Systems, Special Functions and Quantum Field Theory, 16–20, June, 2019 NORDITA, Stockholm; “Elliptic Quantum Groups and Deformed $\mathcal{W}$-Algebras”, Talk Given at the Workshop Representation Theory and Integrable Systems, 12–16, August, 2019, ETH Zurich
Elliptic quantum toroidal algebra $U_{q,t,p}(\mathfrak{gl}_{1,\text{tor}})$ and affine… Page 63 of 64

55. Konno, H.: Elliptic Quantum Groups, Representations and Related Geometry, Springer Briefs in Mathematical Physics, vol. 37. Springer, Berlin (2020)

56. Koroteev, P., Pushkar, P., Smirnov, A., Zeitlin, A.: Quantum K-Theory of Quiver Varieties and Many-Body Systems. arXiv:1705.10419

57. Li, J., Liu, K., Zhou, J.: Topological string partition functions as equivariant indices. Asian J. Math. 10, 081–114 (2006)

58. Li, W.-P., Qin, Z., Wang, W.: Rings the cohomology, of Hilbert schemes via Jack polynomials, algebraic structures and moduli spaces. In: CRM Proceedings of Lecture Notes, vol. 38. AMS, Providence, pp. 249–258 (2004)

59. Lukyanov, S.L., Fateev, V.A.: Additional symmetries and exactly-soluble models in two-dimensional conformal field theory. Sov. Sci. Rev. A Phys. 15, 1–117 (1990)

60. Macdonald, I.: Symmetric Functions and Hall Polynomials, 2nd edn. Oxford Univ. Press, New York (1995)

61. Miki, K.: A $(q,\gamma)$ analogue of the $W_{1+\infty}$ algebra. J. Math. Phys. 48, 123520, 35 (2007)

62. Morozov, A., Mironov, A., Zenkevich, Y.: Ding–Iohara–Miki Symmetry of Network Matrix Models. arXiv:1603.05467

63. Nakajima, H.: More Lectures on Hilbert Schemes of Points on Surfaces (2014). arXiv:1401.6782

64. Nakajima, H.: Lectures on Perverse Sheaves on Instanton Moduli Spaces (2016). arXiv:1604.06316

65. Nekrasov, N.: Seiberg–Witten prepotential from instanton counting. Adv. Theor. Math. Phys. 7, 831–864 (2004)

66. Nekrasov, N.: BPS/CFT correspondence: non-perturbative Dyson-Schwinger equations and $qq$-characters. J. High Energy Phys. 03, 181 (2016)

67. Schiffmann, O.: Drinfeld Realization of the Elliptic Hall Algebra. arXiv:1004.2575

68. Schiffmann, O., Vasserot, E.: The elliptic Hall algebra and the equivariant cohomology of the moduli space of instantons on $\mathbb{A}^2$. Publ. Math. Inst. Hautes Etudes Sci. 118, 213–342 (2013). arXiv:1202.2756

69. Shenfeld, D.: Abelianization of Stable Envelopes in Symplectic Resolutions, PhD Thesis, Princeton University (2013)

70. Smirnov, A.: Polynomials Associated with Fixed Points on the Instanton Moduli Space (2014). arXiv:1404.5304

71. Zhu, R.-D.: An Elliptic Vertex of Awata–Feigin–Shiraishi-Type for M-Strings. arXiv:1712.10255
