Probing non-locality of interactions in a Bose–Einstein condensate using solitons

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Abstract

We consider a Bose–Einstein condensate (BEC) with non-local inter-particle interactions. The local Gross–Pitaevskii (GP) equation is valid for the gas parameter $\nu = \frac{a^3 n_0}{\lambda^2} \ll 1$, but for $\nu \to 1$, the BEC is described by a modified GP equation (MGPE). We study the exact solutions of the MGPE describing bright and dark solitons. It turns out that the width of these non-local solitons has qualitatively similar behaviour as the modified healing length due to the non-local interactions of the MGPE. We also study the effect of the non-locality and gas parameter ($\nu$) on the stability of the solitons using the Vakhitov–Kolokolov (VK) stability criterion. We show that these soliton solutions are stable according to the VK criterion. Further, the stability of these soliton solutions gets enhanced due to the non-locality of interactions.

Keywords: Bose–Einstein condensates, solitons, modified Gross–Pitaevskii equation, non-local interactions

(Some figures may appear in colour only in the online journal)

1. Introduction

Solitons are solitary, shape preserving travelling waveforms. These solutions are peculiar to non-linear differential equations and arise as a result of the balance between dispersion and non-linearity [1]. Atomic Bose–Einstein condensates (BECs) are described by the non-linear Gross–Pitaevskii (GP) equation which exhibits such soliton solutions [2]. The sign of the non-linearity in a BEC depends on the attractive or repulsive nature of the inter-particle interactions in the BEC. As a result, one obtains bright soliton solution in an attractive BEC and dark soliton solution in a repulsive BEC [3, 4]. Such solitons have been realized in trapped atomic condensates [5, 6]. They are of interest mainly due to the tunability by the control which atomic BEC systems provide in experiments. These control parameters include wide tunability of the s-wave scattering length in a BEC, the availability of trapping techniques in order to modify the external potential and the achievement of multi-component BEC [7–10].

The GP equation used to describe BECs considers s-wave scattering between bosons. The low energy scattering characteristics can be obtained by an integration over the actual interaction potential [11]. Hence, the actual interaction potential between bosons is replaced by a δ-function pseudo-potential to obtain the local GP equation. This shape-independent interaction approximation assumes the gas parameter $\nu = \frac{a^3 n_0}{\lambda^2} \ll 1$, where $a$ is the s-wave scattering length and $n_0$ is the average density of the BEC. However, making use of the tunability of the s-wave interactions, there have been many experiments which achieve $\nu \sim 0.05$ or higher [12, 13]. In this regime, the shape dependence of the s-wave interactions is expected to come into play. There have been various proposals to account for the shape dependence of the inter-boson interactions using a modified GP equation (MGPE) [14, 15]. The MGPE is obtained by introducing an extra nonlinear term to the local GP equation which takes care of non-local interactions. The local GP equation admits a soliton solution, so the natural question would be to ask whether one can obtain a soliton solution to the MGPE as well and as to how it compares with the former [16].

An equation similar to the MGPE is used to describe a weakly non-local Kerr medium in optics [17]. Although the form of the non-locality is similar, the origin of the non-locality is quite different in optics and BEC. Exact soliton solutions to the non-local Kerr medium have been obtained and studied, and are found to be stable [18]. However, to our knowledge,
the implications of these solutions in a non-local BEC along with the stability considerations have not been studied. Also, in light of the stability quantifier proposed [19], the effect of the non-local term to the stability of soliton solution in a BEC is of interest. In this paper we present a detailed analysis of the soliton solution of the MGPE in order to systematically manifest the role of microscopic interaction on such typical nonlinear states.

The spectrum of elementary excitations for the local GP equation shows a phonon like behaviour \( \omega \propto k \) for small wave-numbers \( k \) and particle like behaviour \( \omega \propto k^2 \) for large \( k \). The healing length is the length scale where there is a transition of this spectrum from phonon-like behaviour to particle-like behaviour. In the local GP equation the width of the soliton solution scales as \( \xi_0 = 1/\sqrt{8\pi m n_0} \). For the MGPE, there is a change in the healing length and this modified healing length \( \xi \) shrinks with increasing non-locality [20] for a BEC with repulsive interactions. Whereas, it is shown in this paper that for an attractive BEC, the modified healing length \( \xi \) is bigger than the \( \xi_0 \). Solitons are the structures whose width scales as the healing length of the system and therefore solitons should capture this modification in the healing length resulting from the non-locality of inter-particle interactions. We show that the width of the soliton shows qualitatively similar behaviour as \( \xi \).

1D solitons in local GP equation are known to be unstable to transverse small amplitude oscillations and as such they have to be strongly confined along the radial directions [21]. Thus, soliton solutions are obtained only for effective 1D BEC. The Vakhitov–Kolokolov (VK) stability condition ensures the stability of solitons for the 1D local GP equation [22]. This stability check is also of interest for the soliton solutions in the MGPE. The VK condition determines the stability of the solitons based on the sign of derivative of the momentum versus speed graph for dark solitons and particle number versus chemical potential graph for bright solitons. There has also been a proposal for a quantifier of soliton stability which depends on the magnitude of the aforementioned derivatives and not just the sign [19]. In this paper we show that according to these criteria, not only are the solitons in the MGPE stable, but their stability is enhanced by the non-locality of interactions.

We start by briefly describing the well known soliton solution in a local GP equation in section 2. Then, in section 3, the MGPE is introduced. In section 4 we use the soliton solutions previously obtained by Krolikowski et al [18] and study the bright soliton solution for the case of non-local BEC. In section 5 the same is done for a dark soliton in a non-local BEC. We end with a discussion on the scope of the soliton solutions in a non-local BEC.

2. Soliton solution in local GP equation

Let us briefly review the well-known soliton solution in the local GP equation (LGPE) [3, 4]. The GP equation with contact interactions \( (\nu \ll 1) \) in the absence of an external potential is given by

\[
i\hbar \frac{\partial \psi(\mathbf{r}, t)}{\partial t} = -\frac{\hbar^2}{2m} \nabla^2 \psi(\mathbf{r}, t) + g|\psi(\mathbf{r}, t)|^2 \psi(\mathbf{r}, t),
\]

where the + sign in the final term indicates a BEC with repulsive interactions and − sign indicates one with attractive interactions. Here, ‘\( m \)’ is the mass of the boson, \( g = 4\pi \hbar^2 a/m \) is the inter-boson interaction strength and ‘\( a \)’ is the s-wave scattering length. In the absence of the non-linear term, one would get a dispersive system such that any local disturbance in the system would disperse. The non-linearity counters this dispersion and leads to creation of localized structures like soliton [1]. We want to look at 1D soliton solutions of this equation. Therefore, we assume that \( \psi(\mathbf{r}, t) \) varies only along one direction. Let that direction be the \( z \)-direction. This can be achieved in experiment, by strong harmonic confinement in the radial direction. Consider the form of the solution as

\[
\psi(z,t) = \sqrt{n_0} \text{sech} \left( \frac{z - vt}{\xi_0} \right) e^{i \left( \omega t - k_z z \right)},
\]

where \( v \) is the soliton speed and \( \xi_0 = \hbar/\sqrt{2m n_0} \) is the healing length. One can see that the bright soliton density far away from the soliton centre tends to zero, the density in a dark soliton centre. Unlike the bright soliton where the density far from the soliton centre falls off to 0. The bright soliton moves along the \( z \)-direction like a particle.

For a BEC with repulsive inter-boson interactions, one can obtain an exact dark soliton solution given by

\[
\psi(z,t) = \sqrt{n_0} \text{sech} \left( \frac{z - vt}{\xi_0} \right) e^{i \left( \omega t - k_z z \right)},
\]

where \( c = \sqrt{gn_0/m} \) is the speed of sound in BEC, \( v \) is the soliton speed and \( n_0 = \sqrt{2} \) is the uniform density far away from the soliton centre. Unlike the bright soliton where the density far from the soliton centre tends to zero, the density in a dark soliton heals to a uniform density far away from the soliton centre. Also, the dark soliton has a phase which depends both on the co-moving coordinate \( z - vt/\xi_0 \) and the speed \( v \) of the soliton. Note that, soliton width in both cases scales as \( \xi_0 \).

3. The MGPE

In this section, we consider a model to account for the non-locality of the s-wave interactions. This model considers an additional term to the local GP equation and is referred to as the MGPE in literature [14, 15]. The MGPE, in the absence of an external potential, is given by

\[
i\hbar \frac{\partial \psi(\mathbf{r}, t)}{\partial t} = -\frac{\hbar^2}{2m} \nabla^2 \psi(\mathbf{r}, t) + g|\psi(\mathbf{r}, t)|^2 \psi(\mathbf{r}, t) + g |\nabla \psi(\mathbf{r}, t)|^2 |\psi(\mathbf{r}, t)|^2,
\]

where \( g \) is the inter-boson interaction strength.
where $g_2 = \left(\frac{4}{\hbar^2} - \frac{m}{2\pi^2}\right)$ and the $+$ sign implies a BEC with repulsive interactions and $-$ sign implies a BEC with attractive interactions. The effective range of the interactions is captured by $r_e$. This equation is valid only for $g_2 > 0$. This means that $r_e < 2a/3$, since $r_e = 2a/3$ would correspond to scattering by a hard sphere [15]. The name ‘effective range’ does not imply that $r_e$ is the width of the interaction potential, but that $r_e$ is approximately determined by the width and the strength of the effective potential. The limit $r_e \leq 2a/3$ can be understood as follows. Consider the solution to the s-wave scattering problem given by $\psi(k, r)$, where $k$ is the wave momentum and is small since we are considering low energy scattering. Let the solution far away from the scattering region be given by $\phi(k, r)$. In other words for large $r$, $\psi(k, r) = \phi(k, r)$. Then the value of effective range is given by $r_e = 2\int dr [\phi^2(0, r) - \psi^2(0, r)]$ [23]. By definition, in a region of space where the scattering potential vanishes, $\psi(k, r) = \phi(k, r)$. Now, let the scattering potential have a range $r_0$ such that the scattering potential is zero for $r > r_0$. In such a case, the contribution to $r_e$ would come from the integral from 0 to $r_0$ since for $r > r_0$ the integrand vanishes. Further for $k = 0$, we have $\psi(0, r) = 1 - (r/a)$, where $a$ is the s-wave scattering length [23]. Now, looking at the expression of $r_e$, it may be easily inferred that the maximum value of the integrand is when $\psi(0, r) = 0$, which vanishes in the region $r \leq r_0$. This situation occurs when the scattering potential is a hard sphere potential in which case the potential is infinite for $r \leq r_0$ and zero for $r > r_0$. A simple calculation with $\psi(0, r) = 1 - (r/a)$ shows that for hard sphere scattering $r_e = 2a/3$. For any other potential, the value of integrand is smaller than this limiting value and hence $r_e \leq 2a/3$. The details of this calculation may be found in the book by Newton [23].

Another way to obtain corrections to the LGPE is by considering the pseudopotential to have a finite width rather than assuming it to be a delta function. The LGPE given by equation (1) is obtained from the following equation by considering an effective interaction potential given by a delta function

$$\hbar \frac{\partial \psi(r, t)}{\partial t} = -\frac{\hbar^2}{2m} \nabla^2 \psi(r, t)$$

$$\pm \psi(r, t) \int dr' \phi^{*}(r', t) V_{\text{eff}}(r - r') \psi(r', t),$$

where $V_{\text{eff}}(r - r')$ is the effective potential. The effective potential is constrained by the condition $\int dr' V_{\text{eff}}(r - r') = g$. To obtain corrections to the LGPE, we may consider the simplest modification to the effective potential of the form $V_{\text{eff}}(r - r') = 3g/4\pi r_0^3$ for $|r - r'| \leq r_0$ and $V_{\text{eff}}(|r - r'|) = 0$ for $|r - r'| > r_0$. In short, instead of considering the effective potential to have zero width, it is considered to have a finite width given by $r_0$. We can write equation (3) in the form

$$\hbar \frac{\partial \psi(r, t)}{\partial t} = -\frac{\hbar^2}{2m} \nabla^2 \psi(r, t)$$

$$\pm \psi(r, t) \int dr' |\psi(r', t)|^2 V_{\text{eff}}(r - r'),$$

Considering the effective potential to have finite width as mentioned above, we may then Taylor expand $|\psi(r', t)|^2$ around $r$ to get the series of corrections to the LGPE of the form

$$\hbar \frac{\partial \psi(r, t)}{\partial t} = -\frac{\hbar^2}{2m} \nabla^2 \psi(r, t) + g|\psi(r, t)|^2 \psi(r, t)$$

$$\pm g \frac{3\hbar^2}{10} \psi(r, t) \nabla^2 |\psi(r, t)|^2$$

$$\pm g \frac{3\hbar^2}{72} \psi(r, t) \nabla^4 |\psi(r, t)|^2 \pm ...$$

To note is the important fact that the expansion above has the natural limit in that the corrections to the LGPE vanish as $r_0 \to 0$. This expansion makes use of the range of the potential rather than employing an effective range. A comparative study of the corrections obtained using the effective range expansion and by employing an effective potential with finite width is worth exploring and we shall do so in future works. For now, we shall use the existing model given by equation (2).

As stated in the introduction, the non-local correction would usually become important at a higher value of the gas parameter $\nu$. The energy functional of such a MGPE is given by [24]

$$F = \int dx \frac{\hbar^2}{2m} \nabla^2 |\psi(r, t)|^2 + \frac{g^2}{2} |\psi(r, t)|^4 + \frac{8g^4}{2} |\psi(r, t)|^2 \nabla^2 |\psi(r, t)|^2.$$  (6)

The energy functional above differs from the energy functional of the LGPE by the term $\frac{g^2}{2} |\psi(r, t)|^4 \nabla^2 |\psi(r, t)|^2$. In general for addition of a term proportional to $|\psi(r, t)|^4 \nabla^2 |\psi(r, t)|^2$ to the dynamics of the LGPE, the addition to the energy functional is proportional to $\frac{1}{l} (|\psi(r, t)|^2 \nabla^2 |\psi(r, t)|^2)$, only for even $l$, such a correspondence does not exist. A rigorous proof of this general statement has been published by one of us earlier [24]. However for quick reference of the reader, in appendix A we present a proof of equation (6) being the correct energy functional of the MGPE given by equation (2).

As done for the local GP equation, we now want to look at 1D soliton solutions for the MGPE. For this, we consider an elongated BEC with tight trapping along the radial direction and loose trapping along the axial direction. Such a trapping would give us an effective 1D BEC. To obtain an effective 1D MGPE, we use the 3D MGPE given by equation (2) and follow the approach used by Salasnich et al [25] to reduce the 3D equation. Equation (2) in the presence of an external potential becomes

$$\hbar \frac{\partial \Psi(r, t)}{\partial t} = -\frac{\hbar^2}{2m} \nabla^2 \Psi(r, t) + g|\Psi(r, t)|^2 \Psi(r, t)$$

$$+ V_{\text{ext}}(r) \Psi(r, t) + g g_2 |\Psi(r, t)|^2 \nabla^2 |\Psi(r, t)|^2.$$  (7)

Let us consider harmonic trapping along the radial direction using the form of the external potential as $V_{\text{ext}}(r, t) = \frac{1}{2} \mu a_r^2 (x^2 + y^2) + V(z)$, where $V(z)$ is the potential along the axial direction. Considering tight trapping along
the radial direction, we use a variable separated trial wave function of the form \( \Psi(r, t) = \Phi(x, y; \sigma(z, t)) \psi(z, t) \), where \( \Phi \) is represented by a Gaussian function
\[
\Phi(x, y; \sigma(z, t)) = e^{-[(x^2 + y^2)/2\sigma^2(z, t)]} / \sigma(z, t) \sqrt{\pi},
\]
where \( \sigma(z, t) \) defines the standard deviation of the Gaussian. Plugging in the above given form of \( \psi \) in equation (7) and integrating out the solution along the \( x \) and \( y \) direction, we get
\[
\frac{\partial \psi(t, z)}{\partial t} = -\frac{\hbar^2}{2m} \frac{\partial^2 \psi(t, z)}{\partial z^2} + \frac{\hbar^2}{2m \sigma^2} \psi(t, z) + V(z) \psi(t, z) + \frac{gN}{2\pi \sigma^2} |\psi(t, z)|^2 \psi(t, z) + \frac{gg_2 N}{2\pi \sigma^2} |\psi(t, z)|^2 \frac{\partial^2 |\psi(t, z)|^2}{\partial z^2},
\]
where \( N \) is the total number of atoms in the condensate. The variational parameter \( \sigma(z, t) \) in the above expression can be obtained by minimizing the energy functional for the MGPE in the presence of an external potential given by
\[
F = \int \frac{\hbar^2}{2m} |\nabla \Psi(r, t)|^2 + V_{\text{ext}}(r, t)|\Psi(r, t)|^2 + \frac{g}{2} |\Psi(r, t)|^4 + \frac{gg_2 N}{2\pi \sigma^2} |\psi(t, z)|^2 |\nabla \psi(t, z)|^2.
\]
Defining the the oscillator length along the radial direction as \( a_r = \sqrt{\frac{\hbar m}{2\sigma_0^2}} \), we obtain the following expression by minimizing the energy functional
\[
\frac{\sigma^2}{a_r^2} \left( 1 + \frac{2Na_l |\psi(z, t)|^2}{\sigma^2} + \frac{3ag_2 N}{2} \frac{\partial^2 |\psi(z, t)|^2}{\partial z^2} \right) + 12g_2 a_l N |\psi(z, t)|^2 a_r^2 = 0.
\]

Firstly, by definition of \( g_2 \), we have \( g_2 \propto a_r^2 \). The effective 1D BEC obtained experimentally satisfies the condition \( a_r a_c < 1 \) \( [5] \). Therefore, \( g_2 a_r^2 \ll 1 \). Secondly, considering a smooth solution along the \( z \) direction such that \( \partial^2 |\psi(z, t)|^2 / \partial z^2 \) is finite, we can neglect the terms dependent on \( \psi(z, t) \) in equation (10) as long as \( Na_l |\psi(z, t)|^2 \ll 1 \). This condition may be rewritten as \( \pi a_r^2 \frac{Na_l |\psi(z)|^2}{\sigma^2} \ll 1 \), where we have multiplied and divided by the cross sectional area of the condensate given by \( \pi a_r^2 \). The healing length of the condensate is given by \( \xi_0 = \frac{1}{\sqrt{\pi a_n n}} \) where \( n \) is the average density of the condensate. This gives the condition for neglecting the \( \psi \) dependent terms as \( a_r^2 / \xi_0^2 \ll 1 \), meaning that the trapping along the radial direction is so tight that the healing length is much larger than the oscillator length. As we have already considered \( g_2 / a_r^2 \ll 1 \), we can safely neglect all the \( \psi(z, t) \) dependent terms in equation (10) to obtain \( \sigma \sim a_r \). Thus we obtain the effective 1D MGPE as
\[
\frac{\hbar^2}{2m} \frac{\partial^2 \psi(t, z)}{\partial z^2} + \frac{\hbar^2}{2m \sigma^2} \psi(t, z) + \frac{gN}{2\pi \sigma^2} |\psi(t, z)|^2 \psi(t, z) + \frac{gg_2 N}{2\pi \sigma^2} |\psi(t, z)|^2 \frac{\partial^2 |\psi(t, z)|^2}{\partial z^2},
\]
where the constant terms which do not affect the dynamics have been neglected. The term \( \frac{3g_2}{\sigma^2} \) in the equation above can be neglected as \( g_2 / a_r^2 \ll 1 \). Thus, the equation above assumes the form
\[
\frac{\hbar^2}{2m} \frac{\partial^2 \psi(t, z)}{\partial z^2} + \frac{gN}{2\pi \sigma^2} |\psi(t, z)|^2 \psi(t, z) + \frac{gg_2 N}{2\pi \sigma^2} |\psi(t, z)|^2 \frac{\partial^2 |\psi(t, z)|^2}{\partial z^2}.
\]

The factor scaling the nonlinear terms can be eliminated by considering the average density of the BEC ‘\( n \)’ in place of the particle number \( N \). Also, we may absorb the \( n \) into the wave-function by scaling \( \psi(z, t) \rightarrow \psi(z, t) / \sqrt{n} \). Lastly, we shall consider a weak trapping along the \( z \) – axis and hence neglect the trapping potential along the \( z \) – axis \( V(z) \) to give the effective 1D MGPE as
\[
\frac{\hbar^2}{2m} \frac{\partial^2 \psi(t, z)}{\partial z^2} + \frac{gN}{2\pi \sigma^2} |\psi(t, z)|^2 \psi(t, z) + \frac{gg_2 N}{2\pi \sigma^2} |\psi(t, z)|^2 \frac{\partial^2 |\psi(t, z)|^2}{\partial z^2}.
\]

We shall use this effective 1D MGPE to study the 1D soliton solution.

In what follows, we use \( \gamma = 2g_2 / \xi_0^2 \) as a non-locality parameter. Also, instead of using \( \mu = gn_0 \), as is the case for ground state condensate, we use \( \mu = \alpha gn_0 \) where ‘\( \alpha \)’ is a factor that takes into account the change in the chemical potential due to the presence of soliton in the BEC.

4. Bright soliton in an attractive MGPE

A bright soliton solution can be obtained in an attractive BEC. This corresponds to a self-focussing nonlinearity with non-locality for an optical medium, for which exact implicit solutions were obtained by Krolikowski et al \( [18] \). The solutions obtained shall be used to see the behaviour of bright solitons in a BEC in what follows. Using equation (12), the 1D MGPE for attractive BECs can be written as
\[
\frac{\hbar^2}{2m} \frac{\partial^2 \psi(t, z)}{\partial z^2} - g\psi(t, z)^2 \psi(t, z) - g g_2 \psi(t, z)^2 \frac{\partial^2 |\psi(t, z)|^2}{\partial z^2}.
\]
where \( g \) is taken to be positive and the attractive nature of BEC is expressed by putting a minus sign in the equation itself.
Notice that since \( g \) is positive, so is \( a \) in the above equation. This is to say that \( a \) physically is negative, but the negative sign is taken out of \( a \) and written explicitly in the equation.

Let us consider here a system with finite size which is very large as compared to the soliton width and the healing length. Let \( N \) be the number of bosons in the system and \( V \) be its volume. For the state with uniform density, the density for the major part of the volume would be \( n_0 = N/V \), barring the healing at the boundaries. This healing is neglected in the analysis that follows, since the system size is taken to be large as compared to the healing length. To obtain a bright soliton solution, moving with a speed \( v \), the \( z \) coordinate is scaled as \( \rho = \left( z - vt \right) / \left( \zeta_0 \sqrt{2} \right) \), where \( \zeta_0 = h / \sqrt{2m \hbar} \). Further, we write \( \psi(z,t) = \sqrt{m_0} f(\rho) e^{i \left( \alpha_0 q_0 - 4\pi \rho^2 \right)} \exp(2i \nu z / h) \) where \( n_0 \) is the uniform density in the absence of soliton and \( \alpha \) is the factor which accounts for the presence of soliton in the system. This turns the above equation to

\[
\frac{\partial f}{\partial \rho} + 2(f f^* - \frac{\alpha}{2}) f + \frac{g_2}{\zeta_0^2} |f|^2 \frac{\partial f}{\partial \rho^2} = 0. \tag{14}
\]

This equation can be integrated once, to give

\[
\left( \frac{\partial f}{\partial \rho} \right)^2 = \frac{(\alpha - f^2 f^*)^2}{1 + 2(\frac{\alpha}{2}) f^2}. \tag{15}
\]

The central peak density of the bright soliton can be evaluated from the above equation. This can be done by setting \( \frac{\partial f}{\partial \rho} = 0 \) for \( \rho = 0 \). This gives the peak density as \( f(0) = \alpha \). The factor \( \alpha \) is evaluated by evaluating the energy functional \( \langle F \rangle \) and then from \( \frac{d \langle F \rangle}{d N} \). This means that the central density of the soliton would depend on number of particles \( (N) \) and volume \( (V) \) of the system.

A further integral of the above equation leads to the implicit soliton solution \([18]\)

\[
\pm \rho = \tanh^{-1} \left( \frac{\sigma}{\sqrt{\alpha}} \right) + \sqrt{\gamma} \tan^{-1} \left( \frac{\sigma}{\sqrt{\gamma}} \right),
\]

where \( \sigma = 2g_2 / \xi_0^2 \) and \( \gamma = \frac{\sigma - f^2 f^*}{1 + \sigma f^2 f^*} \). In this implicit solution, square roots of \( \alpha \) and \( \gamma \) appear which may have a \( + \) or \( - \) sign. However, for the solution to exist, it is essential that only the \( + \) or the \( - \) square root of \( \gamma \) are considered consistently. For the root of \( \alpha \), only the \( + \) square root can be taken for the solution to exist, otherwise the sign preceding the \( \tanh^{-1} \) term would become negative. Since there are two instances when \( \gamma \) appears on the RHS, taking different sign for two \( \gamma \)'s would imply a change of sign between the \( \tanh^{-1} \) and tan \( -1 \) terms. This implicit expression would no longer be a solution of equation (14) as can be verified by differentiating it twice.

We can obtain the density profile of the bright soliton from the implicit equation (16). For simplicity, we consider the density profile for \( \alpha = 1 \). By definition of \( \psi(z,t) \), we have \( |\psi(z,t)|^2 = n_0 f^2(\rho) \) and hence \( f(\rho) \) goes from 1 at \( \rho = 0 \), to 0 as \( \rho \rightarrow \pm \infty \). Therefore, varying the value of \( f(\rho) \) from \( 0, 1 \) in equation (16) gives us the density profile of the bright soliton. Figure 1 shows this profile for \( U = 0 \) and different values of \( r_e \), where \( U = v / (\sqrt{2}) \). The inset shows variation of width with respect to \( r_e \) for \( U = 0 \). From this we can see the effect of non-locality on the width of the bright soliton. For \( U = 0 \), the width of the bright soliton decreases if we increase \( r_e \).

Using the bright soliton solution in equation (16), the energy for a bright soliton on top of the uniform density state can be evaluated using the expression for energy functional in equation (6). Here too, \( \alpha = 1 \) for simplicity. Since the integrand involves second derivative of \( |f|^2 \), it is convenient to change the integration variable from \( \rho \rightarrow f \). This can be done using equation (15) giving \( d\rho = df \sqrt{1 - f^2} / f^2 \). Also, using equation (15), one can find \( d^2 |f|^2 / df^2 = 2 \times [(df/d\rho)^2 + f(df^2 / df^2)] \). Further, subtracting the energy of the uniform density state from this integral gives the soliton energy. This gives the expression for energy integral as

\[
E = \frac{gn^2}{2} - \sqrt{2} g_0 \sqrt{g_2} \int_0^1 \left[ f^2 \left( \frac{1 - f^2}{1 + \gamma f^2} \right) + 2f^2 - 1 - f^4 \right]
\]

\[
\times \left( \frac{2f^4 - 3f^6 - \gamma f^8}{1 + \gamma f^2} \right) \times \sqrt{\frac{1 + \gamma f^2}{f^2 \left(1 - f^2 \right)}} \, df.
\]

Figure 2 shows the variation of energy with respect to \( \gamma \) for certain values of \( U \). This figure shows that the effect of the non-locality in an attractive BEC is to decrease the magnitude of energy of the soliton as compared to the uniform density state for a fixed speed.

We define the width of the bright soliton to be the \( \rho \) where the soliton density falls to half of its peak density. Substituting \( f^2 = 1/2 \) in equation (16), we take \( r_e = \beta a \) giving \( \gamma = 2g_2 / \xi_0^2 = 16\pi \nu (1 - a^2) / 2 \). Here, \( \beta \) is a constant proportionality factor. Thus, one obtains, in figure 3, a plot of the width of the bright soliton as a function of \( \nu \) for different values of \( r_e \). Note that, the width is independent of the speed \( v \) of the soliton. By definition of \( g_2 \), \( r_e \sim 0.67 \) corresponds to \( g_2 = 0 \). In figure 3, one can see that \( r_e = 0.67 \) corresponds to a line parallel to the \( \nu \) axis. This implies that \( w / \xi_0 \) is constant, or \( w \propto \xi_0 \), i.e. the width of the soliton scales as \( \xi_0 \). However for \( \gamma_e \neq 0.67 (g_2 \neq 0) \), the plot of variation of \( w / \xi_0 \) with respect
to $\nu$ is no longer a line parallel to the $\nu$ axis. This implies that for $g_2 \neq 0$, the width no longer scales as $\xi_0$.

The width can be analytically found by putting $f^2 = 1/2$ in equation (16). This gives us the width to be

$$w = \tanh^{-1}\left(\frac{2\alpha - 1}{\alpha(2 + \gamma)}\right) + \sqrt{\gamma} \tanh^{-1}\left(\frac{\gamma(2\alpha - 1)}{2 + \gamma}\right).$$

Here, $w/\xi_0$ is written so as $\rho$ contains a scaling factor of $\xi_0$. In the absence of the non-local correction, $\gamma = 0$ and $w/\xi_0 = \tanh^{-1}(\sqrt{\gamma/2\alpha})$, meaning $w/\xi_0$ is a constant. This implies that the length scale of width of the soliton is $\xi_0$. However, in the presence of the non-local correction, the width gets modified and the modified length scale is given by $B\xi_0$, where $B = \tanh^{-1}\left(\frac{2\alpha - 1}{\alpha(2 + \gamma)}\right) + \sqrt{\gamma} \tanh^{-1}\left(\frac{\gamma(2\alpha - 1)}{2 + \gamma}\right)$.

By considering a small amplitude oscillatory disturbance around the uniform density ground state of the form, $\psi(r, t) = (\sqrt{N} + \nu e^{ikr} e^{-i\omega t} + \nu e^{-ikr} e^{i\omega t}) e^{-im^{2}/\hbar}$, one can obtain the dispersion relation for small amplitude oscillations as $\omega = \pm k \sqrt{-\frac{\alpha m}{\mu} + \frac{\hbar k^2 s}{2m}}$ for the local GP equation. The value of $k$ for which the dispersion relation turns from phonon-like ($\omega \sim k$) to particle like ($\omega \sim k^2$) is $\sim \xi_0$. Since an attractive BEC is unstable, the dispersion relation becomes imaginary for small $k$. Nevertheless, one can define the healing length to be the length scale when magnitude of the phonon-like ($\omega \sim k$) term becomes comparable to the magnitude of the particle-like term ($\omega \sim k^2$). A similar analysis can be done for the MGPE to obtain the spectrum of elementary excitations which comes out to be $\omega = \pm k \sqrt{-\frac{\alpha m}{\mu} + \frac{\hbar k^2 s}{2m} + \frac{\hbar k^2 g_2 s}{m}}$ [20]. The length scale of phonon–particle transition of elementary excitation spectrum can be found by equating the magnitude of terms which go as $k$ and the term which goes as $k^2$. This length scale turns out to be $\xi = \xi_0 \sqrt{1 + \gamma/\sqrt{2}}$. This change in the healing length indicates that there has to be a broadening of the width of the bright soliton in the presence of non-local interactions and, here, such a change can be seen. So, in the presence of non-local interactions the width of the bright soliton somewhat follows the modified healing length. In other words, the widening of the width of a bright soliton from the healing length is a proof of existence of non-local interactions and corresponding change in the healing length.

Having seen the behaviour of soliton width and energy in the presence of the non-local correction, the next natural step is to check the stability of the bright soliton solution. This stability condition is given by the sign of $dN/d\mu$ [26], where $N = \int |\psi(z, t)|^2$ is the total number of particles in the BEC and $\mu = \alpha n_0/2$. The bright soliton is stable if $dN/d\mu > 0$. The bright soliton solution above can be used to obtain

$$N = \hbar \sqrt{n_0} \left(\sqrt{\alpha} + \frac{1 + \gamma\alpha}{\sqrt{\gamma}} \tan^{-1}\left(\frac{\sqrt{\gamma}}{\alpha}\right)\right).$$

This gives

$$dN/d\mu \propto \frac{1}{\sqrt{\alpha} + \sqrt{\gamma} \tan^{-1}\left(\frac{\sqrt{\gamma}}{\alpha}\right)}. \quad (17)$$

As explained before, $\sqrt{\gamma}$ and $\sqrt{\alpha}$ are positive and hence $\tan^{-1}\left(\sqrt{\gamma}/\sqrt{\alpha}\right)$ is also positive, which ensures that $dN/d\mu > 0$. This shows that bright solitons are stable. Figure 4 shows the variation $dN/d\mu$ as a function of $\gamma$. This graph shows that the bright soliton solution is indeed stable. There has been a proposal of quantifying the stability based on $dN/d\mu$ [19]. It says that the soliton stability increases with increase in the magnitude of $dN/d\mu$. In this regard, it can be seen from figure 4 that the soliton stability is enhanced by increasing the non-locality of interactions.

5. Dark soliton in a repulsive MGPE

Let us now look at the bright soliton solutions which can be obtained in a repulsive BEC. This corresponds to a self-defocussing non-linearity with non-locality in an optical medium. Using equation (12), the 1D MGPE for a repulsive BEC can be written as

$$\hbar \frac{\partial \psi(z, t)}{\partial t} = -\frac{\hbar^2}{2m} \frac{\partial^2 \psi(z, t)}{\partial z^2} + g|\psi(z, t)|^2 \psi(z, t) + g\xi^2 |\psi(z, t)|^4 \psi(z, t). \quad (18)$$

As for the attractive BEC case in the previous section, let us start by considering $N$ bosons occupying a volume $V$. The uniform density state has $n_0 = N/V$. The healing at the boundaries may be neglected by assuming that the system is large compared to the healing length. To find moving solutions, it is convenient to consider a moving frame of reference $s = (z - vt)/\sqrt{2}\xi_0$ and write $\psi = \psi(s)e^{-ims/\hbar}$. The form of $\psi(s)$ is taken as $\psi(s) = \sqrt{n_0} \sqrt{\lambda(s)} e^{i\phi(s)}$. Using such a form of $\psi$, equation (18) gives two equations for real and imaginary parts as

\begin{align*}
\hbar \frac{\partial \phi(s)}{\partial s} &= -\frac{\hbar^2}{2m} \frac{\partial^2 \phi(s)}{\partial s^2} + g|\psi(s)|^2 \psi(s) + g\xi^2 |\psi(s)|^4 \psi(s), \\
\hbar \frac{\partial \lambda(s)}{\partial s} &= -\frac{\hbar^2}{2m} \frac{\partial^2 \lambda(s)}{\partial s^2} + g|\psi(s)|^2 \psi(s) + g\xi^2 |\psi(s)|^4 \psi(s).
\end{align*}
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\[ \lambda (1 - \gamma \lambda) \frac{d^2 \lambda}{ds^2} - 8\lambda^2 (\lambda - \alpha) \]

\[ - 4\lambda^2 \left( \frac{d\phi}{ds} \right)^2 + 8U\lambda^2 \frac{d\phi}{ds} = 0; \]

\[ \frac{d}{ds} \left[ \lambda \left( \frac{d\phi}{ds} - U \right) \right] = 0, \]

\[ \lambda (s) = \left( \frac{4\alpha}{\sqrt{1 - \gamma \lambda}} \right)^{1/2}, \]

where \( \gamma = \frac{2g_2/\xi_0^2}{U} \) and \( U = \sqrt{2m}\nu\xi_0/h \). Since dark soliton solutions are localized with an intensity minima at \( s = 0 \), we integrate the equations above. Putting the boundary conditions of \( d\lambda/ds = 0 \) for \( s = 0 \) and \( s \to \pm \infty \), one gets from the above equation the central dip \( \lambda(0) = U^2 \) and the background density \( \lambda(s \to \pm \infty) = \alpha \). The lower of the two equations gives, \( d\phi/ds = U \) when integrated. This can be used to integrate the upper equation giving

\[ \left( \frac{d\lambda}{ds} \right)^2 = \frac{4(\lambda - U^2)(\alpha - \lambda)^2}{1 - \gamma \lambda}. \]

The above equation can then be integrated to give the following implicit solution for a dark soliton [18]

\[ \pm s = \frac{1}{\delta_0} \tanh^{-1} \left( \frac{\delta}{\delta_0} + \sqrt{\gamma} \tan^{-1} (\delta/\sqrt{\gamma}) \right) \]

\[ \pm \phi(s) = \tan^{-1} \left( \frac{\delta}{U} \right) - U \sqrt{\gamma} \tan^{-1} (\delta/\sqrt{\gamma}) \]

where \( \delta^2 = \frac{\lambda(0) - U^2}{1 - \gamma \lambda} \) and \( \delta_0^2 = \frac{\lambda(0) - U^2}{1 - \gamma \lambda} \). Here, \( n_0 \) is the background density over which one observes the dark soliton solution. As for the bright soliton solution, there appears a square root of \( \gamma \) which may have a + or − sign. As before, only take the + or the − square root of \( \gamma \) can be taken consistently. Taking different signs for two \( \gamma \)’s would imply a change of sign between the \( \tanh^{-1} \) and \( \tan^{-1} \) terms which would no longer be a solution of equation (19).

Figure 5 shows the density profile of the dark soliton for different values of \( r_e \) and \( \alpha = 1 \) for \( U = 0 \). The inset shows the variation of the width of dark soliton with variation in \( r_e \) (in units of \( a \)) for \( \nu = 0.1 \) for \( U = 0 \).

Figure 3. Figures show the variation of the width of the bright soliton with change in the gas parameter \( \nu \) with \( \alpha = 1 \). In (a), the width is scaled with the healing length \( \xi_0 \), whereas in (b), the width is not scaled. (a) \( U = 0 \), graph scaled by \( \xi_0 \). (b) \( U = 0 \), graph unscaled.

Figure 4. The figure shows the variation of \( dN/d\mu \) as we change \( r_e \) (in scale of \( a \)). Here, \( \alpha = 1 \) and \( \nu = 0.1 \).

Figure 5. The figure shows the density profile of the dark soliton for different values of \( r_e \) and \( \alpha = 1 \) for \( U = 0 \). The inset shows the variation of the width of dark soliton with variation in \( r_e \) (in units of \( a \)) for \( \nu = 0.1 \) for \( U = 0 \).
is a constant, meaning that $\xi_0$ is the length scale for solitons, in the case of local GP dynamics. However, when $\gamma \neq 0$, it can be seen that the scale is different from $\xi_0$ and depends on $\gamma$. Let us say that due to a finite $\gamma$, $w/(A \times \xi_0)$ is a constant now instead of $w/\xi_0$. Then the new length scale for the width is $A\xi_0$. To find this new length scale, we can use equation (21) and demand that $w/(A \times \xi_0)$ be a constant. This demand gives us the value of $A$ as $A = \sqrt{1 - \gamma} \tan^{-1} \left( \frac{1 - \gamma}{2} \right) + \sqrt{\gamma} \tan^{-1} \left( \frac{\gamma}{2 - \gamma} \right)$. This gives us the new length scale $A\xi_0$.

As before doing a small amplitude oscillation analysis around the ground state of the form $\psi(r,t) = (\sqrt{\mathcal{N}_0} + \text{we}^{ikr} e^{-i\omega t} + \text{ve}^{-ikr} e^{-i\omega t}) e^{-i\omega t}$, one can obtain the dispersion relation for small amplitude oscillations as $\omega = \pm k \sqrt{\frac{m}{\mathcal{N}}} + \frac{\mathcal{N} \lambda}{\frac{\mathcal{N}}{m}}$ for the local GP equation. Note here that since the repulsive is stable, $\omega$ is real. The healing length as before is $\sim \xi_0$. Doing such an analysis for the MGPE with repulsive interactions gives the modified healing length as $\xi = \xi_0 \sqrt{1 - \gamma}/\sqrt{2}$. This change in the healing length indicates that there has to be a shrinking in the width of the dark soliton in the presence of non-local interactions which can be observed here, as for the bright soliton. Hence, for dark solitons too, the width of the dark soliton somewhat follows the modified healing length in the presence of non-local interactions.

However, the width of the soliton in the presence of non-local interactions although qualitatively follows the modified healing length $\xi$, the width is not exactly scaled by $\xi$. It scales with $c\xi$, where $c$ is a $\gamma$-dependent function.

One of the popular methods of obtaining dark solitons experimentally is the phase imprinting method [6, 27]. From equation (20), it is evident that along with the spatial density variation, dark solitons are accompanied by a spatial variation of phase of the wave function, given by $\phi(s)$. The phase imprinting method, as the name suggests, uses a laser which is phase-imprinted, as the name suggests, uses a laser which is phase-imprinted. For these which have to be imprinted. For dark solitons too, the phase variation of the wave function, given by $\phi(s)$, does not play a role. However, as $U$ increases, the variation of the phase with respect to $s$ is significantly altered by $\gamma$ and hence the phase to be imprinted for soliton generation changes with $\gamma$. Figure 8 shows the phase variation of the dark soliton as a function of $s$ for few different values of $r_e$, with $U = 0.8$.

As for the bright soliton, the stability of the dark soliton solution of the MGPE is of interest. The stability of dark solitons is provided by the Vakhitov–Kolokolov (VK) conditions which determine the stability based on the sign of $dQ/du$, where $Q$ is the winding number.

$$Q = \frac{i}{2} \int_{-\infty}^{\infty} \left[ \left( \frac{\partial}{\partial z} \psi^*(z,t) \right) \left( \frac{\partial}{\partial z} \psi(z,t) - \psi^*(z,t) \frac{\partial}{\partial z} \psi(z,t) \right) - \frac{n_0}{|\psi(z,t)|^2} \right] dz.$$
parameter is fixed at $\nu = 0.1$, $\alpha = 1$ and $U = 0.8$.

Dark solitons are stable if $dQ/dU > 0$ according to the VK condition. Further, as for the bright soliton, there is a proposal for a stability quantifier using the magnitude of $dQ/dU$. It states that the stability is enhanced as the value of $dQ/dU$ increases beyond 0 [19].

Using the dark soliton solution, the expression for $Q$ is given by

$$Q = -2\alpha \tan^{-1} \left( \frac{\delta_0}{U} \right) + (\alpha - U^2) \frac{U}{\delta_0} + \frac{U}{\sqrt{\gamma}} \left\{ 1 + \gamma(2\alpha - U^2) \right\} \tan^{-1}(\delta_0\sqrt{\gamma}).$$

The integration is done by using the change of coordinates from $z \rightarrow \lambda$ as before.

Using this expression for $Q$, one can evaluate $dQ/dU$, which is given by

$$\frac{dQ}{dU} = \frac{2U^4\gamma - 2U^2\alpha - \alpha U^2\gamma + 3\alpha^2}{\delta_0(\alpha - U^2\gamma)} + \frac{U^2(1 - U^2\gamma + 2\alpha\gamma)}{\delta_0[-1 + (1 + U^2 - \alpha)\gamma]} + \frac{\left( 1 - 3U^2\gamma + 2\alpha\gamma \right)}{\sqrt{\gamma}} \tan^{-1}(\delta_0 \sqrt{\gamma}).$$

To bear in mind is the fact that $\delta_0$ is a function of $U$.

Figure 9 shows the variation of $dQ/dU$ with respect to $U$ for different values of the effective range $r_e$. These graphs are plotted for $\nu = 0.5$ and $\alpha = 1$. This graph shows that the solitons are stable even for such large values of $\nu$. Note that even for higher and lower value of $\nu$, one can use equation (22) to draw plots similar to figure 9 to show that solitons are stable for a range of $\nu$ values. However, at larger values of $\nu$, the higher order corrections would come into picture. For lower values of $\nu$, only the lower order correction used in MGPE is sufficient and as such the solitons are stable using the VK conditions.

Figure 9 shows that the VK stability condition is satisfied for all $U$ for $\alpha = 1$. A simple check can be done to show that $dQ/dU > 0$ for all $\alpha$ for $U = 0$. This can be done by putting $U = 0$ in equation (22), which gives $dQ/dU = (3\alpha/\delta_0) + [(1 + 2\alpha \sqrt{\gamma} \tan^{-1}(\delta_0 \sqrt{\gamma}))/\sqrt{\gamma}]$, which is >0.

As for the bright solitons, $|dQ/dU|$ increases for a fixed value of gas parameter as we increase $\gamma$ (i.e. decrease $r_e$) for $U = 0$. However, from figure 9 one can say with certainty that the increased nonlocality provides additional stability to the dark soliton [19] for $U$ close to $U = 0$. 

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{fig7.png}
\caption{Figures show the variation of the width of the black soliton with change in the gas parameter $\nu$ for $\alpha = 1$. In (a), the width is scaled with the healing length $\xi_0$, whereas in (b), the width is not scaled. (a) $U = 0$, graph scaled by $\xi_0$. (b) $U = 0$, graph unscaled.}
\end{figure}

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{fig8.png}
\caption{The figure shows the variation of phase of the dark soliton $\phi(s)$ as a function of $s$ for a few values of $r_e$. The gas parameter is fixed at $\nu = 0.1$, $\alpha = 1$ and $U = 0.8$.}
\end{figure}

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{fig9.png}
\caption{The figure shows the value of $dQ/dU$ as a function of $U$ for a few values of $r_e$. The gas parameter is fixed at $\nu = 0.5$ and $\alpha = 1$.}
\end{figure}
Bright and dark soliton generation in BEC has been achieved by several groups [28, 29]. Bright solitons are generated by trapping a BEC with repulsive interactions and then using Feshbach resonance to modify the scattering length from positive values (repulsive BEC) to negative values (attractive BEC). On the other hand, dark solitons are generated in a repulsive BEC by using methods like phase-imprinting and density engineering. Both these solitons require strong anisotropic confinement in order to freeze out the instability modes perpendicular to the 1D soliton. As shown in our analysis, the non-locality of the interactions manifest upon increasing the gas parameter $\nu(= a^2 n_0)$. Experimentally, $\nu$ can be increased by tuning the s-wave scattering length, using the Feshbach resonance. As mentioned in the introduction, high values of $\nu$ have been achieved in trapped condensates [12, 13]. However, its effect on solitons in a BEC has not been experimentally studied to the best of our knowledge. Thus, after generation of a bright or dark soliton in a BEC, the s-wave scattering length can be tuned to increase $\nu$ and its effect on the width of the soliton can be observed by using imaging techniques. By studying the variation of the soliton versus $\nu$, the parameter $r_e$ can be evaluated. Thus, studying the effect of non-local interactions on the soliton width provides for a measure of the effective range parameter $r_e$.

In our work, considering the exact solutions for the non-local case, there are two free parameters, one is $\alpha$ and the other is the speed of the frame $v$. We have taken here the chemical potential to be $\mu gn$ instead of $gn$ to make allowance for the fact that we cannot consider $\mu = gn$ as for the uniform density BEC. If it is possible to fix the chemical potential for the soliton and hence fix $\alpha$, then there is only one free parameter which is the speed $v$. This $v$ can then be fixed by minimization of the free energy. One would then get an exact selection for the speed of the soliton which would no longer be arbitrary as allowed by the Galilean invariance. But in the present work we are interested in a broader picture as a function of the soliton speed and that is why we have kept the solution as a one parameter family.

6. Discussion

In this paper, we have studied the 1D soliton solutions for a modified Gross–Pitaevskii equation (MGPE). The MGPE takes into account the non-locality of s-wave interactions in a Bose–Einstein condensate (BEC). Using the exact solutions obtained by Krolikowski et al, we have studied the behaviour of these solutions for a BEC. As the soliton solution is obtained due to a balance between dispersion and non-linearity, the introduction of correction term to the GP equation changes the length scale of the solitons. Consequently, the width of the solitons change as well. This change in the length scale of soliton width shows a similar behaviour to the modified healing length for the MGPE. The change in the width can be experimentally verified as a confirmation of the existence of non-locality of interactions in a BEC. An important consequence of it is that the value of effective range of interactions $r_e$ can be experimentally determined. We have also studied the energetics of the soliton solution and further explored their stability.

The effective range of interactions $r_e$ depends on the details of the interaction potential which is system specific. However, the strength of the non-local interactions depends not only on $r_e$, but also on the s-wave scattering length ($\alpha$). The scattering length can be tuned using Feshbach resonance to amplify the effects of change in soliton width. This would lead to a considerable shift of the width from that of the healing length $\xi_0$. The width of the soliton can then be compared with $\xi_0$ to determine the value of $r_e$.

Do note that the change in the width of the dark and bright solitons are opposite in nature and that readily provides one with a qualitative verification of the existence of change in healing length due to non-locality of interactions. Moreover, in this paper we have also given exact quantitative measure for the change in width as a function of $r_e$ for the dark and bright solitons. In an experiment, if one uses similar condensates for positive as well as negative scattering lengths, obtained using Feshbach resonance, one can actually measure $r_e$ of the system in two different ways, i.e. from the shrinking and from the broadening of the respective solitons.

An exact solution provides with the opportunity of analyzing the energetics of the structure and we have fully utilized this opportunity to explore the stability of solitons in the presence of non-local interactions. The soliton solutions of the MGPE are stable even as one increases the gas parameter($\nu$) to small but finite values. The non-locality of interactions in a BEC imparts additional stability to the solitons, considering the proposal of quantifying soliton stability [19]. However, as seen from figure 9, beyond a certain value of $U$, the stability of solitons with stronger non-local interactions decreases. It would be interesting to analyze such a behaviour with respect to the proposal quantifying soliton stability. Such a behaviour might arise due to the modified behaviour of dispersive and focussing effects in a MGPE. However, a detailed study of such a behaviour might throw more light on the stability of solitons in 1D.

It would also be interesting to see the effect of transverse modes which make the soliton unstable, on the soliton solutions of the MGPE. The non-locality of interactions may impart extra stability in the transverse direction as well, allowing for weaker transverse trapping as compared to the local GP case. We shall investigate this feature in a future study.

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Appendix. Proof of energy functional

The energy functional $F$ of a system is defined by the stationarity condition of the action integral yielding the equation for dynamics as
$$\frac{\delta F}{\delta \psi^*(r, t)} = i\hbar \frac{\partial \psi(r, t)}{\partial t}.$$ 

Let us denote the energy functional of the LGPE by $E$ which is well known and is given by [3]

$$E = \int dr \left[ \frac{\hbar^2}{2m} |\nabla \psi^*(r, t)|^2 \pm \frac{g}{2} |\psi(r, t)|^4 \right],$$

where the + and − signs denote BEC with repulsive and attractive interactions respectively. The term added on top of the energy functional above in equation (6) is $\pm \frac{g_2}{2} |\psi(r, t)|^2 |\nabla^2 \psi(r, t)|^2$. Let us denote this additional term by $F_1$. Therefore, it would suffice to prove that

$$\frac{\delta F_1}{\delta \psi^*} = \pm g_2 |\psi(r, t)||\nabla^2 \psi(r, t)|^2.$$ 

The argument of $\psi$ would be dropped for this derivation for the purpose of brevity. The functional derivative is defined as [30]

$$\frac{\delta F_1}{\delta \psi^*} = \left( \frac{\partial}{\partial \psi^*} - \nabla \cdot \frac{\partial}{\partial (\nabla \psi^*)} + \nabla^2 \cdot \frac{\partial}{\partial (\nabla^2 \psi^*)} - \ldots \right) F_1.$$ 

The term $\nabla^2 |\psi|^2$ in $F_1$ may be expanded to give

$$F_1 = \frac{g_2}{2} \psi^* \psi (\nabla^2 \psi^* + 2(\nabla \psi^*) \cdot (\nabla \psi^*)) + \psi^* \nabla^2 \psi.$$ 

The functional derivative of this form of $F_1$ gives

$$\frac{\delta F_1}{\delta \psi^*} = \left[ g_2 \left( \frac{\psi^* \nabla^2 \psi^*}{2} + \psi \nabla (\psi^*) \cdot (\nabla \psi^*) + \psi^* \nabla^2 \psi^* \right) \right]$$

$$- \left[ g_2 \left( 2 \psi^* (\nabla \psi^*)^2 + \psi \nabla (\psi^*) \cdot (\nabla \psi^*) + \psi^* \nabla^2 \psi^* \right) \right]$$

$$+ \left[ g_2 \left( 2 \psi (\nabla \psi^*) \cdot (\nabla \psi^*) + \psi^* (\nabla \psi^*)^2 \right) \right]$$

$$+ \frac{\psi^* \nabla^2 \psi^*}{2} + \psi^* \nabla^2 \psi.$$ \hspace{1cm} (A.1)

In the equation above, the first line corresponds to the term \(\frac{\partial F_1}{\partial \psi^*}\), second line corresponds to the term \(\nabla \cdot \frac{\partial F_1}{\partial (\nabla \psi^*)}\) and the third and fourth lines line corresponds to \(\nabla^2 \cdot \frac{\partial F_1}{\partial (\nabla^2 \psi^*)}\). Due to the absence of higher derivatives of $\psi^*$ in $F_1$ further terms are not considered. Adding terms in equation (A.1) gives us

$$\frac{\delta F_1}{\delta \psi^*} = g_2 \psi \left( \psi \nabla^2 \psi^* + 2(\nabla \psi^*) \cdot (\nabla \psi^*) + \psi^* \nabla^2 \psi\right)$$

$$= g_2 \psi \nabla \psi^* (\nabla^2 \psi^* |\psi|^2).$$

Therefore, the energy functional given by equation (6) corresponds to the MGPE given by equation (2).

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