Quantum vertex algebras and their $\phi$-coordinated quasi modules

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Abstract
We develop a theory of $\phi$-coordinated (quasi) modules for a nonlocal vertex algebra and we establish a conceptual construction of nonlocal vertex algebras and their $\phi$-coordinated (quasi) modules, where $\phi$ is what we call an associate of the one-dimensional additive formal group. By specializing $\phi$ to a particular associate, we obtain a new construction of weak quantum vertex algebras in the sense of $\textsc{Li3}$. As an application, we associate weak quantum vertex algebras to quantum affine algebras, and we also associate quantum vertex algebras and $\phi$-coordinated modules to a certain quantum $\beta\gamma$-system.

1 Introduction
In the general field of vertex algebras, arguably a central problem (see $\textsc{FJ}$, $\textsc{EFK}$) is to develop a suitable theory of quantum vertex algebras so that quantum vertex algebras can be associated to quantum affine algebras in the same way that vertex algebras were associated to affine Lie algebras. With solving this problem as one of the main goals, in the past we have conducted a series of studies. In $\textsc{Li3}$, we formulated and studied a notion of (weak) quantum vertex algebra, inspired by Etingof-Kazhdan’s notion of quantum vertex operator algebra (see $\textsc{EK}$), and we established a conceptual construction of nonlocal vertex algebras and weak quantum vertex algebras together with their modules. Nonlocal vertex algebras (which are weak $G_1$-vertex algebras in the sense of $\textsc{Li2}$ and are essentially field algebras in the sense of $\textsc{BK}$) are analogs of noncommutative associative algebras, in contrast with that vertex algebras are analogs of commutative associative algebras. Furthermore, weak quantum vertex algebras are nonlocal vertex algebras that satisfy a certain braided locality (commutativity). As an application, we had associated nonlocal vertex algebras to quantum affine algebras. Unfortunately, the associated nonlocal vertex algebras are not weak quantum vertex algebras. A crucial question is whether this theory of (weak) quantum vertex algebras is the (or a) right one for solving the aforementioned problem.

The main goal of this paper is to answer this very question. In this paper, we develop a theory of what we call $\phi$-coordinated (quasi) modules for nonlocal vertex algebras (including vertex algebras and weak quantum vertex algebras) and we establish a conceptual construction of nonlocal vertex algebras and their $\phi$-coordinated (quasi) modules. In this new theory, the parameter $\phi$, which is an

$^1$Partially supported by NSF grant DMS-0600189
element of \( \mathbb{C}((x))[[z]] \) satisfying certain conditions, is what we call an associate of the one-dimensional additive formal group \( F_\alpha(x, y) = x + y \). When \( \phi(x, z) = x + z \) (the additive formal group itself), this construction of nonlocal vertex algebras reduces to the construction of \([13]\) while the notion of \( \phi \)-coordinated (quasi) module reduces to the notion of (quasi) module. Specializing \( \phi \) to another particular associate of \( F_\alpha(x, y) \), we obtain a new construction of weak quantum vertex algebras, which enables us to associate weak quantum vertex algebras to quantum affine algebras through \( \phi \)-coordinated quasi modules.

We now go into some technical details to describe the contents of this paper. Let \( W \) be a general vector space and set \( \mathcal{E}(W) = \text{Hom}(W, W(\langle x \rangle)) \). In \([13]\), we studied the vertex algebra-like structures generated by various types of subsets of \( \mathcal{E}(W) \), where the most general type consists of what were called quasi compatible subsets. On \( \mathcal{E}(W) \), we considered partial operations \((a(x), b(x)) \mapsto a(x)_n b(x)\) for any quasi compatible pair \((a(x), b(x))\) and for \( n \in \mathbb{Z} \). Roughly speaking, they were defined in terms of the generating function \( Y_\mathcal{E}(a(x), z) b(x) = \sum_{n \in \mathbb{Z}} a(x)_n b(x) z^{-n-1} \) by

\[
"Y_\mathcal{E}(a(x), z) b(x) = [a(x_1) b(x)]|_{x_1 = x + z},"
\]

which essentially uses what physicists call the operator product expansion. It was proved therein that any quasi compatible subset of \( \mathcal{E}(W) \) generates a nonlocal vertex algebra with \( W \) as a quasi module in a certain sense. This generalizes the corresponding result of \([12]\). Furthermore, it was proved that every what we called (resp. quasi) \( \mathcal{S} \)-local subset of \( \mathcal{E}(W) \) generates a weak quantum vertex algebra with \( W \) as a (resp. quasi) module, which generalizes the corresponding results of \([11]\) and \([14]\).

The essence of this present paper is a family generalization of the vertex operator operation \( Y_\mathcal{E} \), parameterized by a formal series \( \phi(x, z) \in \mathbb{C}((x))[[z]] \), satisfying

\[
\phi(x, 0) = x, \quad \phi(\phi(x, y), z) = \phi(x, y + z).
\]

We call such a formal series an associate of the one-dimensional additive formal group \( F_\alpha(x, y) = x + y \), where a formal group to an associate is like a group \( G \) to a \( G \)-set. It is proved that for any \( p(x) \in \mathbb{C}((x)) \), \( e^{zp(x) (d/dx)} x \) is an associate and that every associate is of this form. In particular, we have \( \phi(x, z) = x + z = F_\alpha(x, z) \) for \( p(x) = 1 \) and \( \phi(x, z) = xe^z \) for \( p(x) = x \). For a quasi compatible pair \((a(x), b(x))\) in \( \mathcal{E}(W) \), we define \( a(x)_n^\phi b(x) \) for \( n \in \mathbb{Z} \) in terms of the generating function

\[
Y_\mathcal{E}^\phi(a(x), z) b(x) = \sum_{n \in \mathbb{Z}} a(x)_n^\phi b(x) z^{-n-1}
\]

roughly by

\[
"Y_\mathcal{E}^\phi(a(x), z) b(x) = [a(x_1) b(x)]|_{x_1 = \phi(x, z)},"
\]

(see Section 2 for the precise definition). We prove that any quasi compatible subset \( U \) of \( \mathcal{E}(W) \) generates a nonlocal vertex algebra under the operation \( Y_\mathcal{E}^\phi \). To describe
the relationship between such nonlocal vertex algebras and the space $W$, we introduce a notion of $\phi$-coordinated (quasi) module. In terms of this notion, the space $W$ becomes a $\phi$-coordinated (quasi) module for those nonlocal vertex algebras.

To deal with quantum affine algebras, we formulate notions of quasi $S_{\text{trig}}$-local subset and $S_{\text{trig}}$-local subset of $\mathcal{E}(W)$ with $W$ a general vector space. We prove that every quasi $S_{\text{trig}}$-local subset generates under $Y_{\phi}$ with $\phi(x, z) = xe^z$ a weak quantum vertex algebra in the sense of [Li3]. If $W$ is taken to be a highest weight module for a quantum affine algebra, the generating functions of the generators in the Drinfeld realization form a quasi $S_{\text{trig}}$-local subset and hence they generate a weak quantum vertex algebra with $W$ as a $\phi$-coordinated quasi module by our conceptual result. In this way, we obtain a canonical association of quantum affine algebras with weak quantum vertex algebras. In a sequel, we shall study this association in a deeper level, to determine the structure of the associated weak quantum vertex algebras and prove that they are indeed quantum vertex algebras.

On the other hand, as a toy example we apply this general machinery to a certain quantum $\beta\gamma$-system. To this system we associate quantum vertex algebras and $\phi$-coordinated modules explicitly. This particular quantum $\beta\gamma$-system is in fact a one-dimensional trigonometric type Zamolodchikov-Faddeev algebra (see [Fad], [ZZ]). Previously, rational type Zamolodchikov-Faddeev algebras have been associated with quantum vertex algebras and modules (see [Li3], [Li4], [KL]). The quantum vertex algebras associated to the trigonometric type quantum $\beta\gamma$-system in this paper are described by a certain rational type quantum $\beta\gamma$-system.

While this paper has set up a basic foundation for the theory of $\phi$-coordinated (quasi) modules for nonlocal vertex algebras, including vertex (operator) algebras in particular, there are many aspects to be explored further. We note that though this paper is among a series of papers in a long program, it is pretty much self contained and can be read independently.

This paper is organized as follows: In Section 2, we define the notion of associate of the additive formal group and we construct and classify all the associates. In Section 3, we introduce a notion of $\phi$-coordinated (quasi) module for a nonlocal vertex algebra. In Section 4, we give a conceptual construction of nonlocal vertex algebras and their $\phi$-coordinated (quasi) modules. In Section 5, we study $\phi$-coordinated modules for (weak) quantum vertex algebras with $\phi(x, z) = xe^z$. In Section 6, we study two quantum $\beta\gamma$-systems in terms of quantum vertex algebras and their $\phi$-coordinated modules.

2 Associates of the one-dimensional additive formal group

In this section, we formulate and study a notion of associate for a one-dimensional formal group. For the one-dimensional additive formal group we construct and classify its associates.
Throughout this paper, we use the usual symbols \( \mathbb{C} \) for the complex numbers, \( \mathbb{Z} \) for the integers, and \( \mathbb{N} \) for the nonnegative integers. For this paper, we shall be working on \( \mathbb{C} \) and we use the fairly standard formal variable notations and conventions (see [FLM], [FHL]; cf. [LL]).

We first recall the notion of formal group (see [Bo]).

**Definition 2.1.** A one-dimensional formal group over \( \mathbb{C} \) is a formal power series \( F(x, y) \in \mathbb{C}[[x, y]] \) such that

\[
F(x, y) = x + y + \text{terms of high degree},
\]

\[
F(x, F(y, z)) = F(F(x, y), z).
\]

The simplest example is the one-dimensional additive formal group

\[
F_a(x, y) = x + y.
\] (2.1)

We formulate the following notion, which is an analog of the notion of \( G \)-set for a group \( G \) to a certain extent:

**Definition 2.2.** Let \( F(x, y) \) be a one-dimensional formal group over \( \mathbb{C} \). An associate of \( F(x, y) \) is a formal series \( \phi(x, z) \in \mathbb{C}((x))[[z]] \), satisfying the condition that

\[
\phi(x, 0) = x,
\]

\[
\phi(\phi(x, x_2), x_0) = \phi(x, F(x_0, x_2)).
\] (2.2)

**Remark 2.3.** We here verify the well-definedness of the two expressions on both sides of (2.2). Note that as \( \phi(x, z) \in \mathbb{C}((x))[[z]] \) with \( \phi(x, 0) = x \), \( \phi(x, z) \) is a unit in the algebra \( \mathbb{C}((x))[[z]] \), so that it is well understood that

\[
\phi(x, z)^m \in \mathbb{C}((x))[[z]] \quad \text{for } m \in \mathbb{Z}.
\] (2.3)

Write \( \phi(x, z) = x + zA \) with \( A \in \mathbb{C}((x))[[z]] \). For \( f(x) = \sum_{m \geq k} a_m x^m \in \mathbb{C}((x)) \) with \( k \in \mathbb{Z}, \ a_m \in \mathbb{C} \), we have

\[
f(\phi(x, x_2)) = \sum_{m \geq k} a_m \phi(x, x_2)^m = \sum_{m \geq k} \sum_{i \geq 0} \binom{m}{i} a_m x_2^i A^i \in \mathbb{C}((x))[[x_2]].
\]

Set \( \phi(x, z) = \sum_{n \geq 0} f_n(x) z^n \) with \( f_n(x) \in \mathbb{C}((x)) \). By definition we have

\[
\phi(\phi(x, x_2), x_0) = \sum_{n \geq 0} f_n(\phi(x, x_2)) x_0^n \in \mathbb{C}((x))[[x_0, x_2]].
\]

On the other hand, as \( F(0, 0) = 0, \ \phi(x, F(x_0, x_2)) \) also exists in \( \mathbb{C}((x))[[x_0, x_2]] \).

For this paper, our interest is on the additive formal group \( F_a(x, y) \). The following is an explicit construction of associates for \( F_a(x, y) \):
Proposition 2.4. Let \( p(x) \in \mathbb{C}((x)) \). Set

\[
\phi(x, z) = e^{zp(x) \frac{d}{dx}} = \sum_{n \geq 0} \frac{z^n}{n!} \left( p(x) \frac{d}{dx} \right)^n x \in \mathbb{C}((x))[z].
\]

Then \( \phi(x, z) \) is an associate of \( F_a(x, y) \). Furthermore, every associate \( \phi(x, z) \) of \( F_a(x, y) \) is of this form with \( p(x) \) uniquely determined.

Proof. For the first assertion, clearly, \( \phi(x, 0) = x \). Since \( e^{zp(x)(d/dx)} \) is an automorphism of the algebra \( \mathbb{C}((x))[[x_0, x_2]] \), we have

\[
e^{zp(x)(d/dx)} f(x_0, x_2) = f(e^{zp(x)(d/dx)} x, x_0, x_2)
\]

for \( f(x_0, x_2) \in \mathbb{C}((x))[[x_0, x_2]] \). Then

\[
\phi(x_0 + x_2) = e^{(x_0 + x_2)p(x)(d/dx)} x = e^{zp(x)(d/dx)} e^{x_0p(x)(d/dx)} x
\]

\[
= e^{zp(x)(d/dx)} \phi(x_0) = \phi(e^{zp(x)(d/dx)} x, x_0) = \phi(\phi(x, x_2), x_0).
\]

This proves that \( \phi(x, z) \) is an associate of \( F_a(x, y) \).

On the other hand, let \( \phi(x, z) \) be any associate. Denote the formal partial derivatives of \( \phi(x, z) \) by \( \phi_x(x, z) \) and \( \phi_z(x, z) \). Set \( p(x) = \phi_z(x, 0) \in \mathbb{C}((x)) \). We claim that \( \phi(x, z) = e^{zp(x)(d/dx)} x \). By definition we have

\[
\phi(\phi(x, y), z) = \phi(x, y + z).
\]

Extracting the coefficients of \( z \), we get

\[
\phi_z(\phi(x, y), 0) = \phi_y(x, y).
\]

Noticing that \( \phi_z(\phi(x, y), 0) = p(\phi(x, y)) \), we see that \( \phi(x, z) \) satisfies the differential equation

\[
\frac{\partial}{\partial z} \phi(x, z) = p(\phi(x, z))
\]

with initial condition \( \phi(x, 0) = x \). For any \( k \in \mathbb{N} \), we have

\[
\left( \frac{\partial}{\partial z} \right)^{k+1} \phi(x, z) = \left( \frac{\partial}{\partial z} \right)^k p(\phi(x, z)),
\]

which gives a recursion of the coefficients of \( z^n \) in \( \phi(x, z) \) for \( n \in \mathbb{N} \). It follows that the initial value problem has a unique solution. Thus \( \phi(x, z) = e^{zp(x)(d/dx)} x \). We also see that \( p(x) \) is uniquely determined by \( p(x) = \phi_z(x, 0) \). \( \square \)

Example 2.5. Here we work out some special examples by using Proposition 2.4. We have \( \phi(x, z) = x \) for \( p(x) = 0 \), \( \phi(x, z) = e^{z(d/dx)} x = x + z = F_a(x, z) \) for \( p(x) = 1 \), \( \phi(x, z) = e^{zx(d/dx)} x = xe^z \) for \( p(x) = x \). We also have \( \phi(x, z) = e^{xx(d/dx)} x = \frac{x}{1-xx} \) for \( p(x) = x^2 \) and \( \phi(x, z) = x + \log(1 + z/x) \) for \( p(x) = x^{-1} \).
Remark 2.6. We here discuss a certain formal substitution slightly different from those discussed in Remark 2.3. Let \( \phi(x, z) \in \mathbb{C}((x))[z] \) with \( \phi(x, 0) = x \) and let \( f(x_1, x) = \sum_{m,n \geq k} a(m, n)x_1^m x^n \in \mathbb{C}((x_1, x)) \) with \( k \in \mathbb{Z} \). Set

\[
f(\phi(x, z), x) = \sum_{m,n \geq k} a(m, n)\phi(x, z)^m x^n = \sum_{m,n \geq k} \sum_{i \geq 0} \binom{m}{i} a(m, n)z^i A^i x^{m+n-i},
\]

which exists in \( \mathbb{C}((x))[z] \), where \( \phi(x, z) = x + zA \) with \( A \in \mathbb{C}((x))[z] \). We have

\[
f(\phi(x, z), x) = \text{Res}_{x_1} x_1^{-1} \delta \left( \frac{\phi(x, z)}{x_1} \right) f(x_1, x).
\]

Furthermore, for \( A(x_1, x_2) \in \text{Hom}(W, W((x_1, x_2))) \) with \( W \) a vector space, we have

\[
A(\phi(x_2, z), x_2) \in \text{Hom}(W, W((x_2))[z]) = (\text{Hom}(W, W((x_2))))[[z]].
\]

The following technical result plays an important role later:

Lemma 2.7. Let \( \phi(x, z) \) be an associate of \( F_a(x, y) \) with \( \phi(x, z) \neq x \). Then

\[
f(\phi(x, z), x) \neq 0 \quad \text{for any nonzero } f(x_1, x) \in \mathbb{C}((x_1, x)).
\]

Proof. It suffices to prove the assertion for \( f(x_1, x) \in \mathbb{C}[[x_1, x]] \). Now assume that \( f(\phi(x, z), x) = 0 \) for some

\[
f(x_1, x) = \sum_{m,n \geq 0} a(m, n)x_1^m x^n \in \mathbb{C}[[x_1, x]].
\]

We claim

\[
\sum_{m,n \geq 0} m^r a(m, n)x^{m+n} = 0 \quad \text{for all } r \geq 0,
\]

where \( m^r = 1 \) for \( m = r = 0 \) as a local convention. By Proposition 2.4, we have \( \phi(x, z) = e^{zp(x)(d/dx)}x \) for some nonzero \( p(x) \in \mathbb{C}((x)) \). As \( f(\phi(x, z), x) = 0 \), we have

\[
\sum_{m,n \geq 0} a(m, n)x^n e^{zp(x)(d/dx)} x^m = \sum_{m,n \geq 0} a(m, n)x^n (e^{zp(x)(d/dx)} x)^m = 0.
\]

Extracting the constant term with respect to \( z \) (equivalently setting \( z = 0 \)), we get

\[
\sum_{m,n \geq 0} a(m, n)x^{m+n} = 0,
\]

proving the base case with \( r = 0 \). Extracting the coefficient of \( z \) we get

\[
\left( \sum_{m,n \geq 0} ma(m, n)x^{m+n} \right) p(x)x^{-1} = 0,
\]

which completes the proof.
which implies
\[ \sum_{m,n \geq 0} ma(m,n)x^{m+n} = 0, \]
confirming the case with \( r = 1 \). Assume it is true for \( 0 \leq r \leq k \) with \( k \geq 1 \). Then
\[ \sum_{m,n \geq 0} a(m,n)x^ng(x) \left( \frac{d}{dx} \right)^r x^m = g(x) \sum_{m,n \geq 0} m^r a(m,n)x^{m+n} = 0 \] (2.8)
for \( 0 \leq r \leq k \), \( g(x) \in \mathbb{C}(\!(x)\!) \). Extracting the coefficient of \( z^{k+1} \) from (2.7) we have
\[ \sum_{m,n \geq 0} a(m,n)x^n \left( p(x) \frac{d}{dx} \right)^{k+1} x^m = 0. \] (2.9)
Noticing that
\[ \left( p(x) \frac{d}{dx} \right)^{k+1} - (x^{-1}p(x))^{k+1} \left( x \frac{d}{dx} \right)^{k+1} \in \sum_{j=1}^{k} \mathbb{C}(\!(x)\!) \left( x \frac{d}{dx} \right)^j, \]
using (2.9) and (2.8) we get
\[ \sum_{m,n \geq 0} a(m,n)x^n(x^{-1}p(x))^{k+1} \left( x \frac{d}{dx} \right)^{k+1} x^m = 0. \]
Then we obtain
\[ \sum_{m,n \geq 0} m^{k+1}a(m,n)x^{m+n} = 0, \]
completing the induction and proving (2.6).

Now, from (2.6) we get
\[ \sum_{m=0}^{l} m^r a(m, l - m) = 0 \]
for all \( \ell, r \geq 0 \). It follows that \( a(m, l - m) = 0 \) for all \( 0 \leq m \leq l \). Therefore, we have \( f(x_1, x) = 0 \) proving the assertion. \(\square\)

**Remark 2.8.** Here, we collect some simple facts which we need later. Let \( \phi(x, z) \) be an associate of \( F_a(x, y) \). Let \( h(x_1, x_0) \in \mathbb{C}(\!(x_1)\!)((x_0)) \). We see that
\[ h(\phi(x_2, x_0), x_0) \text{ exists in } \mathbb{C}(\!(x_2)\!)((x_0)). \]
Furthermore, we have
\[ (h(x_1, x_0)|_{x_1=\phi(x_2,x_0)})|_{x_2=\phi(x_1,-x_0)} = h(x_1, x_0), \] (2.10)
noticing that \( \phi(\phi(x, -z), z) = \phi(x, 0) = x \). For \( A(x_1, x_2) \in \mathbb{C}((x_1, x_2)) \), we have
\[
A(x_1, x_2)|_{x_1=\phi(x_2,x_0)} = (A(x_1, x_2)|_{x_2=\phi(x_1,-x_0)})|_{x_1=\phi(x_2,x_0)}. \tag{2.11}
\]
Assume \( \phi(x, z) \neq x \). By Lemma 2.7, for \( A(x_1, x_2), B(x_1, x_2) \in \mathbb{C}((x_1, x_2)) \), the relation
\[
A(x_1, x_2)|_{x_1=\phi(x_2,x_0)} = B(x_1, x_2)|_{x_1=\phi(x_2,x_0)}
\]
implies \( A(x_1, x_2) = B(x_1, x_2) \).

3 \( \phi \)-coordinated quasi modules for nonlocal vertex algebras

This is a short preliminary section. In this section, we recall the definitions of a nonlocal vertex algebra and a (quasi) module, and we define the notion of \( \phi \)-coordinated (quasi) module for a nonlocal vertex algebra. We also give a construction of \( \phi \)-coordinated modules through Borcherds’s construction of nonlocal vertex algebras.

We begin by recalling the notion of nonlocal vertex algebra, which plays a central role in this paper. A nonlocal vertex algebra is a vector space \( V \) equipped with a linear map
\[
Y(\cdot, x) : V \rightarrow \text{Hom}(V, V((x))) \subset (\text{End}V)[[x, x^{-1}]]
\]
and equipped with a distinguished vector \( 1 \in V \), satisfying the conditions that
\[
Y(1, x)v = v, \quad Y(v, x)1 \in V[[x]] \quad \text{and} \quad \lim_{x \to 0} Y(v, x)1 \ (= v_{-1}1) = v \quad \text{for} \ v \in V,
\]
and that for \( u, v, w \in V \), there exists \( l \in \mathbb{N} \) such that
\[
(x_0 + x_2)^lY(u, x_0 + x_2)Y(v, x_2)w = (x_0 + x_2)^lY(Y(u, x_0)v, x_2)w \tag{3.1}
\]
(the weak associativity).

Let \( V \) be a nonlocal vertex algebra, which is fixed throughout this section. Let \( D \) be the linear operator on \( V \) defined by \( Dv = v_{-2}1 \) for \( v \in V \). Then (see [Li2])
\[
Y(v, x)1 = e^{xD}v, \quad [D, Y(v, x)] = Y(Dv, x) = \frac{d}{dx}Y(v, x). \tag{3.2}
\]

Among general nonlocal vertex algebras, what we called weak quantum vertex algebras in [Li3] form a distinguished family. A weak quantum vertex algebra is defined by using the same set of axioms except replacing the weak associativity axiom with the condition that for any \( u, v \in V \), there exist
\[
u^{(i)} \in V, \quad f_i(x) \in \mathbb{C}((x)) \quad (i = 1, \ldots, r)
\]
such that
\[ x_0^{-1} \delta \left( \frac{x_1 - x_2}{x_0} \right) Y(u, x_1)Y(v, x_2) = -x_0^{-1} \delta \left( \frac{x_2 - x_1}{x_0} \right) \sum_{i=1}^{r} f_i(-x_0)Y(v^{(i)}, x_2)Y(u^{(i)}, x_1) \]
\[ = x_1^{-1} \delta \left( \frac{x_2 + x_0}{x_1} \right) Y(Y(u, x_0)v, x_2). \] (3.3)

A weak quantum vertex algebra can also be defined to be a nonlocal vertex algebra \( V \) that satisfies \( S \)-locality: For any \( u, v \in V \), there exist \( u^{(i)}, v^{(i)} \in V, f_i(x) \in \mathbb{C}(x) \) \((i = 1, \ldots, r)\) such that
\[ (x_1 - x_2)^k Y(u, x_1)Y(v, x_2) = (x_1 - x_2)^k \sum_{i=1}^{r} f_i(x_2 - x_1)Y(v^{(i)}, x_2)Y(u^{(i)}, x_1) \] (3.4)
for some \( k \in \mathbb{N} \).

The following notion was introduced in [Li2]:

**Definition 3.1.** A \( V \)-module is a vector space \( W \) equipped with a linear map
\[ Y_W(\cdot, x) : V \to \text{Hom}(W, W((x))) \subset (\text{End}W)[[x, x^{-1}]] \]
\[ v \mapsto Y_W(v, x), \]
satisfying the conditions that \( Y_W(1, x) = 1_W \) (the identity operator on \( W \)) and that for \( u, v \in V, w \in W \), there exists \( l \in \mathbb{N} \) such that
\[ (x_0 + x_2)^l Y_W(u, x_0 + x_2)Y_W(v, x_2)w = (x_0 + x_2)^l Y_W(Y(u, x_0)v, x_2)w. \] (3.5)

**Remark 3.2.** We note that from [LTW] (Lemma 2.9), the weak associativity axiom in the definition of a \( V \)-module can be equivalently replaced by the condition that for any \( u, v \in V \), there exists \( k \in \mathbb{N} \) such that
\[ (x_1 - x_2)^k Y_W(u, x_1)Y_W(v, x_2) \in \text{Hom}(W, W((x_1, x_2)))) \] (3.6)
and
\[ ((x_1 - x_2)^k Y_W(u, x_1)Y_W(v, x_2)) \big|_{x_1=x_2+x_0} = x_0^k Y_W(Y(u, x_0)v, x_2). \] (3.7)

Here, by \( ((x_1 - x_2)^k Y_W(u, x_1)Y_W(v, x_2)) \big|_{x_1=x_2+x_0} \) we mean
\[ t_{x_2, x_0} \left( ((x_1 - x_2)^k Y_W(u, x_1)Y_W(v, x_2)) \big|_{x_1=x_2+x_0} \right), \]
the expansion in the nonnegative powers of the second variable \( x_0 \). Note that \( (Y_W(u, x_1)Y_W(v, x_2)) \big|_{x_1=x_2+x_0} \) does not exist in general. (On the other hand, the substitution \( (Y_W(u, x_1)Y_W(v, x_2)) \big|_{x_1=x_0+x_2} \) always exists.) Thus (3.6) is a precondition for (3.7) to make sense. The same principle also applies to Definitions 3.3 and 3.4 below.
The following is a modification of the same named notion defined in [Li3]:

**Definition 3.3.** A *quasi V-module* is defined as in Definition 3.1 except replacing the weak associativity axiom with the condition that for $u, v \in V$, there exists a nonzero power series $p(x, y) \in \mathbb{C}[x, y]$ such that

$$p(x_1, x_2)Y_W(u, x_1)Y_W(v, x_2) \in \text{Hom}(W, W((x_1, x_2))),$$

$$p(x_0 + x_2, x_2)Y_W(Y(u, x_0)v, x_2) = (p(x_1, x_2)Y_W(u, x_1)Y_W(v, x_2))|_{x_1=x_2+x_0}. \quad (3.8)$$

Now, let $\phi(x, z)$ be an associate of the additive formal group $F_n(x, y)$.

**Definition 3.4.** A *$\phi$-coordinated quasi V-module* is defined as in Definition 3.1 except replacing the weak associativity axiom with the condition that for $u, v \in V$, there exists a (nonzero) power series $p(x, y) \in \mathbb{C}[x, y]$ such that $p(\phi(x, z), x) \neq 0$,

$$p(x_1, x_2)Y_W(u, x_1)Y_W(v, x_2) \in \text{Hom}(W, W((x_1, x_2))), \quad (3.9)$$

$$p(\phi(x_2, x_0), x_2)Y_W(Y(u, x_0)v, x_2) = (p(x_1, x_2)Y_W(u, x_1)Y_W(v, x_2))|_{x_1=\phi(x_2, x_0)}. \quad (3.10)$$

A *$\phi$-coordinated V-module* is defined as above except that $p(x_1, x_2)$ is assumed to be a polynomial of the form $(x_1 - x_2)^k$ with $k \in \mathbb{N}$.

**Example 3.5.** Let $W$ be a vector space. The space $(\text{End}W)((x))$ is naturally an associative algebra with identity. Let $p(x) \in \mathbb{C}((x))$. Then $p(x)\frac{d}{dx}$ is a derivation of $(\text{End}W)((x))$. By Borcherds’ construction, $(\text{End}W)((x))$ becomes a nonlocal vertex algebra with

$$Y(a(x), z)b(x) = \left(e^{zp(x)\frac{d}{dx}}a(x)\right)b(x) = a\left(e^{zp(x)\frac{d}{dx}}x\right)b(x)$$

for $a(x), b(x) \in (\text{End}W)((x))$. Define $Y_W(a(x), z) = a(z)$ for $a(x) \in (\text{End}W)((x))$. We have $Y_W(1, z) = 1_W$, and for $a(x), b(x) \in (\text{End}W)((x))$,

$$Y_W(a(x), x_1)Y_W(b(x), x_2) = a(x_1)b(x_2) \in \text{Hom}(W, W((x_1, x_2)))$$

and

$$Y_W(Y(a(x), x_0)b(x), x_2) = Y(a(x_2), x_0)b(x_2) = (a(x_1)b(x_2))|_{x_1=e^{zp(x_2)(d/dx_2)x_2}} = (Y_W(a(x), x_1)Y_W(b(x), x_2))|_{x_1=e^{zp(x_2)(d/dx_2)x_2}}.$$

Thus $W$ is a $\phi$-coordinated module for the nonlocal vertex algebra $(\text{End}W)((x))$ with $\phi(x, z) = e^{zp(x)(d/dx)}x$ and with $Y_W(a(x), x_0) = a(x_0)$ for $a(x) \in (\text{End}W)((x))$. In the next section we shall give a much more sophisticated construction of nonlocal vertex algebras and their $\phi$-coordinated (quasi) modules.
The following is a convenient technical result:

**Lemma 3.6.** Let $V$ be a nonlocal vertex algebra and let $(W,Y_W)$ be a $\phi$-coordinated quasi $V$-module. Let $u, v \in V$ and suppose that $q(x_1, x_2) \in \mathbb{C}[[x_1, x_2]]$ satisfies

$$q(x_1, x_2)Y_W(u, x_1)Y_W(v, x_2) \in \text{Hom}(W, W((x_1, x_2))).$$

Then

$$q(\phi(x_2, x_0), x_2)Y_W(Y(u, x_0)v, x_2) = (q(x_1, x_2)Y_W(u, x_1)Y_W(v, x_2))|_{x_1=\phi(x_2, x_0)}.$$

**Proof.** By definition there exists $p(x_1, x_2) \in \mathbb{C}[[x_1, x_2]]$ with $p(\phi(x, z), x) \neq 0$, satisfying the condition in Definition 3.4. Then

$$p(\phi(x_2, x_0), x_2)q(\phi(x_2, x_0), x_2)Y_W(Y(u, x_0)v, x_2)$$

$$= q(\phi(x_2, x_0), x_2) (p(x_1, x_2)Y_W(u, x_1)Y_W(v, x_2))|_{x_1=\phi(x_2, x_0)}$$

$$= (p(x_1, x_2)q(x_1, x_2)Y_W(u, x_1)Y_W(v, x_2))|_{x_1=\phi(x_2, x_0)}$$

$$= p(\phi(x_2, x_0), x_2) (q(x_1, x_2)Y_W(u, x_1)Y_W(v, x_2))|_{x_1=\phi(x_2, x_0)}.$$

Noticing that the powers of $x_0$ in both

$$q(\phi(x_2, x_0), x_2)Y_W(Y(u, x_0)v, x_2)$$

and

$$(q(x_1, x_2)Y_W(u, x_1)Y_W(v, x_2))|_{x_1=\phi(x_2, x_0)}$$

are truncated from below and that $p(\phi(x_2, x_0), x_2) \in \mathbb{C}((x_2))(x_0)$ is nonzero, we obtain the desired relation by cancellation. 

We also have the following result:

**Lemma 3.7.** Let $V$ be a nonlocal vertex algebra and let $(W,Y_W)$ be a $\phi$-coordinated quasi $V$-module. Then

$$Y_W(e^{x_0p} v, x) = Y_W(v, \phi(x, x_0)) \text{ for } v \in V. \tag{3.11}$$

**Proof.** For $v \in V$, as $Y_W(1, x) = 1_W$, we have

$$Y_W(v, x_1)Y_W(1, x_2) = Y_W(v, x_1) \in \text{Hom}(W, W((x_1, x_2))).$$

By Lemma 3.6 we have

$$Y_W(Y(v, x_0)1, x_2) = Y_W(v, x_1)|_{x_1=\phi(x_2, x_0)} = Y_W(v, \phi(x_2, x_0)),$$

which gives (3.11) as $Y(v, x_0)1 = e^{x_0p} v.$

**Remark 3.8.** Recall from Example 2.5 that $\phi(x, z) = x$ is a particular associate. Let $V$ be a nonlocal vertex algebra and suppose that $V$ admits a faithful $\phi$-coordinated quasi module $(W,Y_W)$ with $\phi(x, z) = x$. Let $u, v \in V$. By definition, there exists $p(x_1, x_2) \in \mathbb{C}[[x_1, x_2]]$ with $p(x_2, x_2) \neq 0$ such that

$$p(x_1, x_2)Y_W(u, x_1)Y_W(v, x_2) \in \text{Hom}(W, W((x_1, x_2))).$$
\[ p(x_2, x_2) Y_W(Y(u, x_0)v, x_2) = (p(x_1, x_2) Y_W(u, x_1) Y_W(v, x_2)) |_{x_1=x_2}. \]

Notice that the right-hand side is independent of \( x_0 \). As the map \( Y_W \) is assumed to be injective, it follows that \( u_n v = 0 \) whenever \( n \neq -1 \). Consequently, \( V \) is merely an ordinary associative algebra. Now, let \((W, Y_W)\) be a general \( \phi \)-coordinated quasi \( V \)-module with \( \phi(x, z) = x \). It can be readily seen that \( \ker Y_W \) is a two-sided ideal of \( V \). We have that \( V/(\ker Y_W) \) is an ordinary associative algebra.

4 A construction of nonlocal vertex algebras and their \( \phi \)-coordinated modules

In this section we present the conceptual construction of nonlocal vertex algebras and their \( \phi \)-coordinated (quasi) modules, by using (quasi) compatible subsets of formal vertex operators on a vector space. This generalizes significantly the construction of nonlocal vertex algebras and their (quasi) modules in [Li2] and [Li3].

We begin with certain generalized iota maps (cf. [FHL]). Denote by \( F(\mathbb{C}[[x_1, x_2]]) \) the fraction field of the ring \( \mathbb{C}[[x_1, x_2]] \). Since \( \mathbb{C}[[x_1, x_2]] \) is also a subring of the field \( \mathbb{C}((x_1))((x_2)) \), there exists a field embedding

\[ \iota_{x_1, x_2} : F(\mathbb{C}[[x_1, x_2]]) \to \mathbb{C}((x_1))((x_2)), \tag{4.1} \]

which is uniquely determined by the condition \( \iota_{x_1, x_2} |_{\mathbb{C}[[x_1, x_2]]} = 1 \). In fact, we have

\[ \iota_{x_1, x_2} |_{\mathbb{C}((x_1, x_2))} = 1, \]

noticing that \( \mathbb{C}((x_1, x_2)) \) is a subalgebra of both \( F(\mathbb{C}[[x_1, x_2]]) \) and \( \mathbb{C}((x_1))((x_2)) \). This map \( \iota_{x_1, x_2} \) naturally extends the algebra embedding \( \iota_{x_1, x_2} \) of \( \mathbb{C}_*(x_1, x_2) \) into \( \mathbb{C}((x_1))((x_2)) \), which was used in [L3] (cf. [L5]), where \( \mathbb{C}_*(x_1, x_2) \) is the algebra extension of \( \mathbb{C}[[x_1, x_2]] \) by inverting all the nonzero polynomials.

**Remark 4.1.** We here discuss certain cancellation rules which shall use extensively in this work. Let \( W \) be a vector space. The space \( \text{Hom}(W, W((x))) \) is naturally a vector space over the field \( \mathbb{C}((x)) \). Furthermore, \( \text{Hom}(W, W((x_1))((x_2))) \) is a vector space over the field \( \mathbb{C}((x_1))((x_2)) \), while \( \text{Hom}(W, W((x_2))((x_1))) \) is a vector space over \( \mathbb{C}((x_2))((x_1)) \). In view of this, for any

\[ A(x_1, x_2), B(x_1, x_2) \in \text{Hom}(W, W((x_1))((x_2))), \]

if

\[ f(x_1, x_2) A(x_1, x_2) = f(x_1, x_2) B(x_1, x_2) \]

for some nonzero \( f(x_1, x_2) \in \mathbb{C}[[x_1, x_2]] \subset \mathbb{C}((x_1))((x_2)) \), then \( A(x_1, x_2) = B(x_1, x_2) \). On the other hand, we have

\[ \mathbb{C}((x_1))((x_2)) \cap \mathbb{C}((x_2))((x_1)) = \mathbb{C}((x_1, x_2)), \]

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and both $\text{Hom}(W,W((x_1))((x_2)))$ and $\text{Hom}(W,W((x_2))((x_1)))$ are $\mathbb{C}((x_1, x_2))$-modules. (Note that $\mathbb{C}((x_1, x_2))$ is an algebra but not a field.) In view of this, for any

$$A(x_1, x_2), B(x_1, x_2) \in \text{Hom}(W,W((x_1))((x_2))) + \text{Hom}(W,W((x_2))((x_1)))$$

($\subset (\text{End} W)[[x_1^{\pm 1}, x_2^{\pm 1}]]$), if

$$f(x_1, x_2)A(x_1, x_2) = f(x_1, x_2)B(x_1, x_2)$$

for some invertible element $f(x_1, x_2)$ of $\mathbb{C}((x_1, x_2))$, then $A(x_1, x_2) = B(x_1, x_2)$.

Let $W$ be a vector space (over $\mathbb{C}$), which is fixed throughout this section. Set

$$\mathcal{E}(W) = \text{Hom}(W,W((x))) \subset (\text{End} W)[[x, x^{-1}]].$$

The identity operator on $W$ is denoted by $1_W$, which is a typical element of $\mathcal{E}(W)$. Recall the notion of compatibility from [Li2]: A finite sequence $a_1(x), \ldots, a_r(x)$ in $\mathcal{E}(W)$ is said to be compatible if there exists a nonnegative integer $k$ such that

$$\left( \prod_{1 \leq i < j \leq r} (x_i - x_j)^k \right) a_1(x_1) \cdots a_r(x_r) \in \text{Hom}(W,W((x_1, \ldots, x_r))). \tag{4.2}$$

Furthermore, a subset $U$ of $\mathcal{E}(W)$ is said to be compatible if every finite sequence in $U$ is compatible.

We formulate the following notion of quasi compatibility:

**Definition 4.2.** A finite sequence $a_1(x), \ldots, a_r(x)$ in $\mathcal{E}(W)$ is said to be quasi compatible if there exists a nonzero power series $p(x_1, x_2) \in \mathbb{C}[[x_1, x_2]]$ such that

$$\left( \prod_{1 \leq i < j \leq r} p(x_i, x_j) \right) a_1(x_1) \cdots a_r(x_r) \in \text{Hom}(W,W((x_1, \ldots, x_r))). \tag{4.3}$$

Furthermore, a subset $U$ of $\mathcal{E}(W)$ is said to be quasi compatible if every finite sequence in $U$ is quasi compatible.

Note that this notion of quasi compatibility slightly generalizes the same named notion defined in [Li3] in the way that the modifier of $p(x_1, x_2)$ is changed from a nonzero polynomial to a nonzero power series.

**Definition 4.3.** Let $\phi$ be an associate of $F_a$. A finite sequence $a_1(x), \ldots, a_r(x)$ in $\mathcal{E}(W)$ is said to be $\phi$-quasi compatible if there exists a power series $p(x, y) \in \mathbb{C}[[x, y]]$ with $p(\phi(x, z), x) \neq 0$ such that (4.3) holds. A subset $U$ of $\mathcal{E}(W)$ is said to be $\phi$-quasi compatible if every finite sequence in $U$ is $\phi$-quasi compatible.

From definition, $\phi$-quasi compatibility implies quasi compatibility. In fact, for almost all $\phi$, $\phi$-quasi compatibility is the same as quasi compatibility. We have the following lemma, where the first assertion is straightforward while the second immediately follows from Lemma 2.7.
Lemma 4.4. Let $\phi$ be an associate of $F_\alpha$ with $\phi(x, z) \neq x$. Every compatible subset of $E(W)$ is $\phi$-quasi compatible and every quasi compatible subset is $\phi$-quasi compatible.

From now on, we fix an associate $\phi(x, z)$ of $F_\alpha(x, y)$.

Definition 4.5. Let $(a(x), b(x))$ be a $\phi$-quasi compatible pair in $E(W)$. We define

$$a(x)_n^\phi b(x) \in E(W) \quad \text{for } n \in \mathbb{Z}$$

in terms of the generating function

$$Y_\phi^\phi(a(x), z)b(x) = \sum_{n \in \mathbb{Z}} a(x)_n^\phi b(x)z^{-n-1} \quad (4.4)$$

by

$$Y_\phi^\phi(a(x), z)b(x) = p(\phi(x, z), x)^{-1}(p(x_1, x)a(x_1)b(x))|_{x_1=\phi(x, z)}, \quad (4.5)$$

which lies in $(\text{Hom}(W, \mathbb{W})))((z)) = E(W)((z))$, where $p(x_1, x_2)$ is any power series with $p(\phi(x, z), x) \neq 0$ such that $(4.3)$ holds and where $p(\phi(x, z), x)^{-1}$ stands for the inverse of $p(\phi(x, z), x)$ in $\mathbb{C}((x))(\text{C})$.

Just as with $Y_\phi$ (see [Li5]), it is straightforward to show that $Y_\phi^\phi$ is well defined, i.e., the expression on the right-hand side does not depend on the choice of $p(x_1, x_2)$. From definition we have

$$p(\phi(x, z), x)Y_\phi^\phi(a(x), z)b(x) = (p(x_1, x)a(x_1)b(x))|_{x_1=\phi(x, z)} \quad (4.6)$$

for any power series $p(x_1, x_2)$ with $p(\phi(x, z), x) \neq 0$ such that $(4.3)$ holds.

We shall need the following technical result:

Lemma 4.6. Let $(a_i(x), b_i(x))$ $(i = 1, \ldots, n)$ be $\phi$-quasi compatible ordered pairs in $E(W)$. Suppose that

$$\sum_{i=1}^n g_i(x_1, x_2)a_i(x_1)b_i(x_2) \in \text{Hom}(W, W((x_1, x_2))) \quad (4.7)$$

with $g_1(x_1, x_2), \ldots, g_n(x_1, x_2) \in \mathbb{C}((x_1, x_2))$. Then

$$\sum_{i=1}^n g_i(\phi(x, z), x)Y_\phi^\phi(a_i(x), z)b_i(x) = \left(\sum_{i=1}^n g_i(x_1, x)a_i(x_1)b_i(x)\right)|_{x_1=\phi(x, z)}. \quad (4.8)$$

Proof. There exists $g(x_1, x_2) \in \mathbb{C}[[x_1, x_2]]$ with $g(\phi(x, z), x) \neq 0$ such that $g(x_1, x_2)a_i(x_1)b_i(x_2) \in \text{Hom}(W, W((x_1, x_2)))$ for $i = 1, \ldots, n$.

Then

$$g(\phi(x, z), x)Y_\phi^\phi(a_i(x), z)b_i(x) = (g(x_1, x)a_i(x_1)b_i(x))|_{x_1=\phi(x, z)}$$

for $i = 1, \ldots, n$.
for $i = 1, \ldots, n$. Using (4.7) we get

$$g(\phi(x, z), x) \sum_{i=1}^{n} g_i(\phi(x, z), x) Y_{\xi}^\phi(a_i(x), z) b_i(x)$$

$$= \sum_{i=1}^{n} g_i(\phi(x, z), x) \left( g(x_1, x) a_i(x_1) b_i(x) \right) |_{x_1 = \phi(x, z)}$$

$$= \left( g(x_1, x) \sum_{i=1}^{n} g_i(x_1, x) a_i(x_1) b_i(x) \right) |_{x_1 = \phi(x, z)}$$

$$= g(\phi(x, z), x) \left( \sum_{i=1}^{n} g_i(x_1, x) a_i(x_1) b_i(x) \right) |_{x_1 = \phi(x, z)}.$$

Notice that both $\sum_{i=1}^{n} g_i(\phi(x, z), x) Y_{\xi}^\phi(a_i(x), z) b_i(x)$ and

$$\left( \sum_{i=1}^{n} g_i(x_1, x) a_i(x_1) b_i(x) \right) |_{x_1 = \phi(x, z)}$$

lie in $(\text{Hom}(W,W)((x)))(|z|)$. Now, it follows immediately from cancellation as $g(\phi(x, z), x) \in \mathbb{C}((x))[|z|]$ is nonzero (recall Remark 4.1).

**Definition 4.7.** Let $U$ be a subspace of $\mathcal{E}(W)$ such that every ordered pair in $U$ is $\phi$-quasi compatible. We say that $U$ is $Y_{\xi}^\phi$-closed if

$$a(x)_n b(x) \in U \quad \text{for } a(x), b(x) \in U, \ n \in \mathbb{Z}. \quad (4.9)$$

We are going to prove that every $Y_{\xi}^\phi$-closed $\phi$-quasi compatible subspace of $\mathcal{E}(W)$, which contains $1_W$, is a nonlocal vertex algebra. First we have:

**Lemma 4.8.** Assume that $V$ is a subspace of $\mathcal{E}(W)$ such that every sequence of length 2 or 3 in $V$ is $\phi$-quasi compatible and such that $V$ is $Y_{\xi}^\phi$-closed. Let $a(x), b(x), c(x) \in V$ and let $f(x, y)$ be a nonzero power series such that

$$f(y, z) b(y) c(z) \in \text{Hom}(W, W((y, z))), \quad (4.10)$$

$$f(x, y) f(x, z) f(y, z) a(x) b(y) c(z) \in \text{Hom}(W, W((x, y, z))). \quad (4.11)$$

Then

$$f(\phi(x, x_1), x_1) f(\phi(x, x_2), x_2) f(\phi(x, x_1), \phi(x, x_2)) Y_{\xi}^\phi(a(x), x_1) Y_{\xi}^\phi(b(x), x_2) c(x)$$

$$= (f(y, x) f(z, x) f(y, z) a(y) b(z) c(x) ) |_{y = \phi(x, x_1), z = \phi(x, x_2)}. \quad (4.12)$$

**Proof.** With (4.10), by Lemma 4.6 we have

$$f(\phi(x, x_2), x) Y_{\xi}^\phi(b(x), x_2) c(x) = (f(z, x) b(z) c(x)) |_{z = \phi(x, x_2)}, \quad (4.13)$$
which gives
\[
    f(y, x)f(y, \phi(x, x_2))f(\phi(x, x_2), x)a(y)Y^{\phi}_{\mathcal{E}}(b(x), x_2)c(x)
\]
\[
    = (f(y, x)f(y, z)f(z, x)a(y)b(z)c(x))|_{z=\phi(x, x_2)}. \quad (4.14)
\]
From (4.11) we see that the right-hand side of (4.14) lies in \((\text{Hom}(W, W((x, y)))[[x_2]]\), so does the left-hand side. That is,
\[
f(y, x)f(y, \phi(x, x_2))f(\phi(x, x_2), x)a(y)Y^{\phi}_{\mathcal{E}}(b(x), x_2)c(x) \in (\text{Hom}(W, W((y, x)))[[x_2]]\).
\]
Notice that because \(b(x)^n_0c(x) = 0\) for \(m\) sufficiently large, for every \(n \in \mathbb{Z}\), the coefficient of \(x_2^n\) is of the form
\[
    \sum_{j=r}^{s} g_j(y, x)a(y)(b(x)^j_0c(x))
\]
with \(r, s \in \mathbb{Z}\) and \(g_j(y, x) \in \mathbb{C}((x, y))\). By considering the coefficient of each power of \(x_2\) and then using Lemma 4.6 we have
\[
f(\phi(x, x_1), x)f(\phi(x, x_1), \phi(x, x_2))f(\phi(x, x_2), x)Y^{\phi}_{\mathcal{E}}(a(x), x_1)Y^{\phi}_{\mathcal{E}}(b(x), x_2)c(x)
\]
\[
    = \left( f(y, x)f(y, \phi(x, x_2))f(\phi(x, x_2), x)a(y)Y^{\phi}_{\mathcal{E}}(b(x), x_2)c(x) \right)|_{y=\phi(x, x_1)}. \quad (4.15)
\]
Using this and (4.13) we obtain
\[
f(\phi(x, x_1), x)f(\phi(x, x_1), \phi(x, x_2))f(\phi(x, x_2), x)Y^{\phi}_{\mathcal{E}}(a(x), x_1)Y^{\phi}_{\mathcal{E}}(b(x), x_2)c(x)
\]
\[
    = \left( f(y, x)f(y, \phi(x, x_2))f(y, x)a(y)Y^{\phi}_{\mathcal{E}}(b(x), x_2)c(x) \right)|_{y=\phi(x, x_1)}
\]
\[
    = (f(z, x)f(y, x)f(y, z)a(y)b(z)c(x))|_{y=\phi(x, x_1), z=\phi(x, x_2)},
\]
completing the proof. \(\square\)

Now we are in a position to prove our first key result:

**Theorem 4.9.** Let \(V\) be a subspace of \(\mathcal{E}(W)\), that contains \(1_W\), such that every sequence of length 2 or 3 in \(V\) is (resp. \(\phi\)-quasi) compatible and \(V\) is \(Y^{\phi}_{\mathcal{E}}\)-closed. Then \((V, Y^{\phi}_{\mathcal{E}}, 1_W)\) carries the structure of a nonlocal vertex algebra and \(W\) is a faithful \(\phi\)-coordinated (resp. quasi) \(V\)-module with \(Y_W(\alpha(x), x_0) = \alpha(x_0)\) for \(\alpha(x) \in V\).

**Proof.** For any \(a(x), b(x) \in V\), from definition we have \(a(x)^n_0b(x) = 0\) for \(n\) sufficiently large and \(a(x)^n_0b(x) \in V\) for any \(n \in \mathbb{Z}\) by assumption. We also have
\[
    Y^{\phi}_{\mathcal{E}}(1_W, z)b(x) = (1_W(x_1)b(x))|_{x_1=\phi(x, z)} = b(x)
\]
and
\[
    Y^{\phi}_{\mathcal{E}}(a(x), z)1_W = (a(x_1)1_W)|_{x_1=\phi(x, z)} = a(\phi(x, z)).
\]
Since $\phi(x, z) \in \mathbb{C}((x))[[z]]$ and $\phi(x, 0) = x$, we have
\[ Y_{\mathcal{E}}^\phi(a(x), z)1_w \in \mathcal{E}(W)[[z]] \quad \text{and} \quad \lim_{z \to 0} Y_{\mathcal{E}}^\phi(a(x), z)1_w = a(x). \]

Now, for the assertion on the nonlocal vertex algebra structure, it remains to prove weak associativity, i.e., for $a(x), b(x), c(x) \in V$, there exists a nonnegative integer $k$ such that
\[ (x_0 + x_2)^k Y_{\mathcal{E}}^\phi(a(x), x_0 + x_2)Y_{\mathcal{E}}^\phi(b(x), x_2)c(x) = (x_0 + x_2)^k Y_{\mathcal{E}}^\phi(a(x), x_0)b(x), x_2)c(x). \]

Let $f(x, y)$ be a power series such that $f(\phi(x, z), x) \neq 0$, and
\[ f(x, y)a(x)b(y) \in \text{Hom}(W, W((x, y))), \]
\[ f(x, y)b(x)c(y) \in \text{Hom}(W, W((x, y))), \]
\[ f(x, y)f(x, z)f(y, z)a(x)b(y)c(z) \in \text{Hom}(W, W((x, y, z))). \]

By Lemma 4.18 we have
\[ f(\phi(x, x_2), x)f(\phi(x, x_0 + x_2), x)f(\phi(x, x_0), x)Y_{\mathcal{E}}^\phi(a(x), x_0 + x_2)Y_{\mathcal{E}}^\phi(b(x), x_2)c(x) = (f(z, x)f(y, x)f(y, z)a(y)b(z)c(x)) |_{y = \phi(x, x_0 + x_2), z = \phi(x, x_2)}. \quad (4.16) \]

On the other hand, let $n \in \mathbb{Z}$ be arbitrarily fixed. Since $a(x)^\phi_m b(x) = 0$ for $m$ sufficiently large, there exists a power series $p(x, y)$, depending on $n$, such that $p(\phi(x, z), x) \neq 0$, and
\[ p(\phi(x, x_2), x)(Y_{\mathcal{E}}^\phi(a(x)^\phi_m b(x), x_2)c(x) = (p(z, x)(a(z)^\phi_m b(z))c(x)) |_{z = \phi(x, x_2)} \quad (4.17) \]
for all $m \geq n$. With $f(x, y)a(x)b(y) \in \text{Hom}(W, W((x, y)))$, we have
\[ f(\phi(x, x_0), x)(Y_{\mathcal{E}}^\phi(a(x^2), x_0)b(x_2))c(x) = (f(y, x_2)a(y)b(x_2)c(x)) |_{y = \phi(x_2, x_0)}. \quad (4.18) \]

Set
\[ Y_{\mathcal{E}}^\phi(a(x), x_0)^{\geq n}b(x) = \sum_{m \geq n} a(x)^\phi_m b(x). \]

Then for any $q(x) \in \mathbb{C}[[x]]$ we have
\[ \text{Res}_x x^n q(x)Y_{\mathcal{E}}^\phi(a(x), x_0)b(x) = \text{Res}_x x^n q(x)Y_{\mathcal{E}}^\phi(a(x), x_0)^{\geq n}b(x). \quad (4.19) \]

Using (4.19), (4.17) and (4.18) we get
\[ \text{Res}_x x^n f(\phi(x, x_0 + x_2), x)f(\phi(x, x_0), x)p(\phi(x, x_2), x)Y_{\mathcal{E}}^\phi(a(x), x_0)b(x), x_2)c(x) \]
\[ = \text{Res}_x x^n f(\phi(x, x_0 + x_2), x)f(\phi(x, x_0), x)p(\phi(x, x_2), x)Y_{\mathcal{E}}^\phi(a(x), x_0)^{\geq n}b(x), x_2)c(x) \]
\[ = \text{Res}_x x^n f(\phi(x, x_0 + x_2), x)f(\phi(x, x_0), x)\left(p(z, x)Y_{\mathcal{E}}^\phi(a(z), x_0)^{\geq n}b(z))c(x) \right) |_{z = \phi(x, x_2)} \]
\[ = \text{Res}_x x^n f(\phi(x, x_0 + x_2), x)f(\phi(x, x_0), x)\left(p(z, x)Y_{\mathcal{E}}^\phi(a(z), x_0)b(z))c(x) \right) |_{z = \phi(x, x_2)} \]
\[ = \text{Res}_x x^n (f(y, x)f(y, z)p(z, x)a(y)b(z)c(x)) |_{y = \phi(z, x), z = \phi(x, x_2)} \]
\[ = \text{Res}_x x^n (f(y, x)f(y, z)p(z, x)a(y)b(z)c(x)) |_{y = \phi(x, x_2), z = \phi(x, x_2)}. \quad (4.20) \]
As \( \phi(\phi(x, y), z) = \phi(x, y + z) \), combining (4.20) with (4.10) we get

\[
\text{Res}_{x_0} x_0^nf(\phi(x, x_2), x)f(\phi(x, x_0 + x_2), x)f(\phi(x, x_0), x)p(\phi(x, x_2), x) \\
\cdot Y^\phi_x(a(x), x_0 + x_2)Y^\phi_x(b(x), x_2)c(x) \\
= \text{Res}_{x_0} x_0^n f(\phi(x, x_2), x)f(\phi(x, x_0 + x_2), x)f(\phi(x, x_0), x)p(\phi(x, x_2), x) \\
\cdot Y^\phi_x(Y^\phi_x(a(x), x_0)b(x), x_2)c(x). \tag{4.21}
\]

Notice that both sides of (4.21) involve only finitely many negative powers of \( x_2 \). Multiplying both sides by \( p(\phi(x, x_2), x)^{-1} f(\phi(x, x_2), x)^{-1} (\in \mathbb{C}((x))((x_2))) \) we get

\[
\text{Res}_{x_0} x_0^n f(\phi(x, x_0 + x_2), x)f(\phi(x, x_0), x)Y^\phi_x(a(x), x_0 + x_2)Y^\phi_x(b(x), x_2)c(x) \\
= \text{Res}_{x_0} x_0^n f(\phi(x, x_0 + x_2), x)f(\phi(x, x_0), x)Y^\phi_x(Y^\phi_x(a(x), x_0)b(x), x_2)c(x).
\]

Since \( f(x, y) \) does not depend on \( n \) and since \( n \) is arbitrary, we have

\[
f(\phi(x, x_0 + x_2), x)f(\phi(x, x_0), x)Y^\phi_x(a(x), x_0 + x_2)Y^\phi_x(b(x), x_2)c(x) \\
= f(\phi(x, x_0 + x_2), x)f(\phi(x, x_0), x)Y^\phi_x(Y^\phi_x(a(x), x_0)b(x), x_2)c(x).
\]

In view of Remark 4.1 we can multiply both sides by \( f(\phi(x, x_0), x)^{-1} \) to get

\[
f(\phi(x, x_0 + x_2), x)Y^\phi_x(a(x), x_0 + x_2)Y^\phi_x(b(x), x_2)c(x) \\
= f(\phi(x, x_0 + x_2), x)Y^\phi_x(Y^\phi_x(a(x), x_0)b(x), x_2)c(x). \tag{4.22}
\]

Write \( f(\phi(x, z), x) = z^k g(x, z) \) for some \( k \in \mathbb{N} \), \( g(x, z) \in \mathbb{C}((x))[[z]] \) with \( g(x, 0) \neq 0 \). Then

\[
f(\phi(x, x_0 + x_2), x) = (x_0 + x_2)^k g(x, x_0 + x_2)
\]

and \( g(x, x_0 + x_2) \) is a unit in \( \mathbb{C}((x))[[x_0, x_2]] \). By cancellation we obtain

\[
(x_0 + x_2)^k(Y^\phi_x(a(x), x_0 + x_2)Y^\phi_x(b(x), x_2)c(x) \\
= (x_0 + x_2)^kY^\phi_x(Y^\phi_x(a(x), x_0)b(x), x_2)c(x),
\]

as desired.

With \( Y_W(a(x), z) = a(z) \) for \( a(x) \in V \), we have \( Y_W(1_W, z) = 1_W \). Furthermore, for \( a(x), b(x) \in V \), there exists \( h(x, y) \in \mathbb{C}[[x, y]] \) with \( h(\phi(x, z), x) \neq 0 \) such that

\[
h(x_1, x_2)a(x_1)b(x_2) \in \text{Hom}(W, W((x_1, x_2))).
\]

Then

\[
h(x_1, x_2)Y_W(a(x), x_1)Y_W(b(x), x_2) = h(x_1, x_2)a(x_1)b(x_2) \in \text{Hom}(W, W((x_1, x_2)))
\]

and

\[
h(\phi(x_2, x_0), x_2)(Y^\phi_x(a(x), x_0)b(x))|_{x=x_2} = (h(x_1, x_2)a(x_1)b(x_2)) |_{x_1=\phi(x_2, x_0)}.
\]
That is,
\[
\begin{align*}
  h(\phi(x_2, x_0), x_2)Y_W(Y_{\phi}^\phi(a(x), x_0)b(x), x_2) \\
  = (h(x_1, x_2)Y_W(a(x), x_1)Y_W(b(x), x_2))|_{x_1=\phi(x_2, x_0)}.
\end{align*}
\]

Therefore, \( W \) is a \( \phi \)-coordinated quasi \( V \)-module. The furthermore assertion is clear from the proof. \( \square \)

Next, we are going to prove that every \( \phi \)-quasi compatible subset of \( \mathcal{E}(W) \) generates a nonlocal vertex algebra. To achieve this goal, we first establish the following key result:

**Proposition 4.10.** Let \( \psi_1(x), \ldots, \psi_r(x), a(x), b(x), \phi_1(x), \ldots, \phi_s(x) \in \mathcal{E}(W) \). Assume that the ordered sequences \((a(x), b(x))\) and
\[
(\psi_1(x), \ldots, \psi_r(x), a(x), b(x), \phi_1(x), \ldots, \phi_s(x))
\]
are \( \phi \)-quasi compatible. Then for any \( n \in \mathbb{Z} \), the ordered sequence
\[
(\psi_1(x), \ldots, \psi_r(x), a(x)^n b(x), \phi_1(x), \ldots, \phi_s(x))
\]
is \( \phi \)-quasi compatible. The same assertion holds without the prefix “quasi.”

**Proof.** Let \( f(x, y) \in \mathbb{C}[x, y] \) be such that \( f(\phi(x, z), x) \neq 0 \),
\[
f(x_1, x_2)a(x_1)b(x_2) \in \text{Hom}(W, W((x_1, x_2)))
\]
and
\[
\begin{align*}
  &\left( \prod_{1 \leq i < j \leq r} f(y_i, y_j) \right) \left( \prod_{1 \leq i \leq r, 1 \leq j \leq s} f(y_i, z_j) \right) \left( \prod_{1 \leq i < j \leq s} f(z_i, z_j) \right) \\
  &\cdot f(x_1, x_2) \left( \prod_{i=1}^r f(y_i, x_1)f(x_2, y_i) \right) \left( \prod_{i=1}^s f(x_1, z_i)f(x_2, z_i) \right) \\
  &\cdot \psi_1(y_1) \cdots \psi_r(y_r)a(x_1)b(x_2)\phi_1(z_1) \cdots \phi_s(z_s) \\
  &\in \text{Hom}(W, W(((y_1, \ldots, y_r, x_1, x_2, z_1, \ldots, z_s)))) \tag{4.23}
\end{align*}
\]

Set
\[
P = \prod_{1 \leq i < j \leq r} f(y_i, y_j), \quad Q = \prod_{1 \leq i < j \leq s} f(z_i, z_j), \quad R = \prod_{1 \leq i \leq r, 1 \leq j \leq s} f(y_i, z_j).
\]

From Proposition 2.4 we have
\[
\phi(x, z) = e^{zp(x)(d/dx)} x
\]
for some \( p(x) \in \mathbb{C}((x)) \). Let \( n \in \mathbb{Z} \) be arbitrarily fixed. There exists a nonnegative integer \( k \) such that

\[
x_0^{k+n} f(\phi(x, x_0), x)^{-1} \in \mathbb{C}((x))[x_0].
\] (4.24)

Using this and the fact

\[
\phi(\phi(x, z), -z) = \phi(x, 0) = x = \phi(\phi(x, -z), z),
\] (4.25)

we obtain

\[
\prod_{i=1}^{r} f(x_2, y_i)^k \prod_{j=1}^{s} f(x_2, z_j)^k \psi_1(y_1) \cdots \psi_r(y_r)(a(x)^{\phi}(x_0)b(x_2)\phi_1(z_1) \cdots \phi_s(z_s))
\]

\[
= \operatorname{Res}_{x_0} x_0^n \prod_{i=1}^{r} f(x_2, y_i)^k \prod_{j=1}^{s} f(x_2, z_j)^k \\
\cdot \psi_1(y_1) \cdots \psi_r(y_r)(Y_1^0(a(x_2), x_0)b(x_2)\phi_1(z_1) \cdots \phi_s(z_s))
\]

\[
= \operatorname{Res}_{x_1, \operatorname{Res}_{x_0}} x_0^n \prod_{i=1}^{r} f(\phi(x_1, -x_0), y_i)^k \prod_{j=1}^{s} f(\phi(x_1, -x_0), z_j)^k f(\phi(x_2, x_0), x_2)^{-1}
\]

\[
\cdot \psi_1(y_1) \cdots \psi_r(y_r)(Y_1^0(\phi_1, x_0)b(x_2)\phi_1(z_1) \cdots \phi_s(z_s))
\]

\[
= \operatorname{Res}_{x_1, \operatorname{Res}_{x_0}} x_0^n e^{-x_0 p(x_1)} \frac{\partial}{\partial x_1} \left( \prod_{i=1}^{r} f(x_1, y_i) \prod_{j=1}^{s} f(x_1, z_j) \right)^k f(\phi(x_2, x_0), x_2)^{-1}
\]

\[
\cdot \psi_1(y_1) \cdots \psi_r(y_r)(\phi_1, x_0)b(x_2)\phi_1(z_1) \cdots \phi_s(z_s))
\]

\[
= \operatorname{Res}_{x_1, \operatorname{Res}_{x_0}} \sum_{t=0}^{k-1} \frac{(-1)^t}{t!} x_0^{n+t} \left( p(x_1) \frac{\partial}{\partial x_1} \right)^t \left( \prod_{i=1}^{r} f(x_1, y_i) \prod_{j=1}^{s} f(x_1, z_j) \right)^k
\]

\[
\cdot f(\phi(x_2, x_0), x_2)^{-1} \frac{\partial}{\partial x_1} \left( \frac{\phi(x_2, x_0)}{x_1} \right)
\]

\[
\cdot (f(x_1, x_2)\psi_1(y_1) \cdots \psi_r(y_r)a(x_1)b(x_2)\phi_1(z_1) \cdots \phi_s(z_s)).
\]

Notice that for any power series \( B \) and for \( 0 \leq t \leq k - 1 \), \( \left( p(x_1) \frac{\partial}{\partial x_1} \right)^t B^k \) is a
multiple of $B$. Using (4.13) we have

$$PQR \prod_{i=1}^{r} f(x_2, y_i) \prod_{j=1}^{s} f(x_2, z_j)$$

$$\cdot \sum_{t=0}^{k-1} \frac{(-1)^t}{t!} x_0^{n+t} \left( \frac{p(x_1)}{\partial x_1} \right)^t \left( \prod_{i=1}^{r} f(y_i, x_1) \prod_{j=1}^{s} f(x_1, z_j) \right) f(\phi(x_2, x_0), x_2)^{-1}$$

$$\cdot x_1^{-1} \left( \frac{\phi(x_2, x_0)}{x_1} \right) (f(x_1, x_2) \psi_1(y_1) \cdots \psi_r(y_r) a(x_1) b(x_2) \phi_1(z_1) \cdots \phi_s(z_s))$$

$$\in \text{Hom}(W, W((y_1, \ldots, y_r, x_2, z_1, \ldots, z_s))) ([x_1, x_1^{-1}]).$$

Then

$$PQR \prod_{i=1}^{r} f(x_2, y_i)^{k+1} \prod_{j=1}^{s} f(x_2, z_j)^{k+1}$$

$$\cdot \psi_1(y_1) \cdots \psi_r(y_r) (a(x)^{\phi(b(x))}(x_2) \phi_1(z_1) \cdots \phi_s(z_s))$$

$$\in \text{Hom}(W, W((y_1, \ldots, y_r, x_2, z_1, \ldots, z_s))). \quad (4.26)$$

This proves that the sequence $(\psi_1(x), \ldots, \psi_r(x), a(x)^{\phi(b(x))}, \phi_1(x), \ldots, \phi_s(x))$ is $\phi$-quasi compatible. The last assertion also follows from the proof.

The following is the main result of this section:

**Theorem 4.11.** Let $W$ be a vector space, $\phi(x, z)$ an associate of the additive formal group $F_3(x, y)$, and $U$ a (resp. $\phi$-quasi) compatible subset of $\mathcal{E}(W)$. There exists a $Y^\phi_\varepsilon$-closed (resp. $\phi$-quasi) compatible subspace of $\mathcal{E}(W)$, that contains $U$ and $1_W$. Denote by $\langle U \rangle_\phi$ the smallest such subspace. Then $(\langle U \rangle_\phi, Y^\phi_\varepsilon, 1_W)$ carries the structure of a nonlocal vertex algebra and $W$ is a $\phi$-coordinated (resp. quasi) $\langle U \rangle_\phi$-module with $Y_W(\alpha(x), z) = \alpha(z)$ for $\alpha(x) \in \langle U \rangle_\phi$.

**Proof.** By Zorn’s lemma, there exists a maximal quasi compatible subspace $V$ of $\mathcal{E}(W)$, containing both $U$ and $1_W$. It follows from Proposition 4.10 that $V$ is $Y^\phi_\varepsilon$-closed. This proves the first assertion. Furthermore, by Theorem 4.9 $(V, Y^\phi_\varepsilon, 1_W)$ carries the structure of a nonlocal vertex algebra with $W$ as a $\phi$-coordinated quasi module. By definition, $\langle U \rangle_\phi$ is the intersection of all $Y^\phi_\varepsilon$-closed (resp. $\phi$-quasi) compatible subspaces of $\mathcal{E}(W)$, containing both $U$ and $1_W$. The rest follows from Theorem 4.9.

Just as with usual quasi modules for a nonlocal vertex algebra, the state-field correspondence for $\phi$-coordinated quasi modules is also a homomorphism.

**Proposition 4.12.** Let $V$ be a nonlocal vertex algebra and let $(W, Y_W)$ be a $\phi$-coordinated quasi $V$-module. Then

$$Y_W(Y(u, x_0)v, x) = Y^\phi_\varepsilon(u(x), x_0)v(x) \quad (4.27)$$

for $u, v \in V$, where $u(x) = Y_W(u, x)$, $v(x) = Y_W(v, x) \in \mathcal{E}(W)$. 21
Proof. For $u, v \in V$, there exists $p(x, y) \in \mathbb{C}[x, y]$ such that $p(\phi(x, z), x) \neq 0$, $p(x_1, x_2)u(x_1)v(x_2) \in \text{Hom}(W, W((x_1, x_2)))$ (4.28) and $p(\phi(x, x_0), x) Y_W(Y(u, x_0)v, x) = (p(x_1, x)u(x_1)v(x))|_{x_1=\phi(x, x_0)}.$

With (4.28), we also have $p(\phi(x, x_0), x) Y^\phi_W(u(x), x_0)v(x) = (p(x_1, x)u(x_1)v(x))|_{x_1=\phi(x, x_0)}.$

Thus $p(\phi(x, x_0), x) Y_W(Y(u, x_0)v, x) = p(\phi(x, x_0), x) Y^\phi_W(u(x), x_0)v(x).$

As the powers of $x_0$ in both $Y_W(Y(u, x_0)v, x)$ and $Y^\phi_W(u(x), x_0)v(x)$ are lower truncated, with $p(\phi(x, x_0), x) \in \mathbb{C}((x))[[x_0]]$ nonzero we obtain the desired relation by cancellation.

\[\text{Remark 4.13.} \text{ Consider the case with } \phi(x, z) = x + z. \text{ From Lemma 4.4, } \phi\text{-quasi compatibility is the same as quasi compatibility. Furthermore, for a quasi compatible pair } (a(x), b(x)) \text{ in } \mathcal{E}(W), \text{ we have } Y^\phi_W(a(x), z)b(x) = Y_\phi(a(x), z)b(x), \text{ which was defined in [L3] by }\]

\[Y_\phi(a(x), z)b(x) = t_{x, z}(1/p(x + z, x))(p(x_1, x)a(x_1)b(x))|_{x_1=x+z},\]

where $p(x, y)$ is any nonzero element of $\mathbb{C}[[x, y]]$ such that $p(x_1, x_2)a(x_1)b(x_2) \in \text{Hom}(W, W((x_1, x_2))).$

On the other hand, with $\phi(x, z) = x + z$, a $\phi$-coordinated quasi module a nonlocal vertex algebra is simply a quasi module. In view of these, Theorem 4.11 generalizes the corresponding results of [L3].

\[\text{Remark 4.14.} \text{ Consider the extreme case with } \phi(x, z) = x. \text{ A pair } (a(x), b(x)) \text{ in } \mathcal{E}(W) \text{ is } \phi\text{-quasi compatible if and only if there exists } p(x_1, x_2) \in \mathbb{C}[[x_1, x_2]] \text{ with } p(x_2, x_2) \neq 0 \text{ such that } p(x_1, x_2)a(x_1)b(x_2) \in \text{Hom}(W, W((x_1, x_2))).\]

Assuming that $(a(x), b(x))$ is $\phi$-quasi compatible with $p(x_1, x_2) \in \mathbb{C}[[x_1, x_2]]$ satisfying the above condition, we have $Y^\phi_W(a(x), z)b(x) = p(x, x)^{-1}(p(x_1, x)a(x_1)b(x))|_{x_1=x},$

which is independent of $z$, where $p(x, x)^{-1}$ stands for the inverse of $p(x, x)$ in $\mathbb{C}((x))$. Then the nonlocal vertex algebra $\langle U \rangle_\phi$, associated to a $\phi$-quasi compatible subset $U$ of $\mathcal{E}(W)$ by Theorem 4.11, is in fact an ordinary associative algebra. From Theorem 4.11, the vector space $W$ is a $\phi$-coordinated quasi module for $\langle U \rangle_\phi$ viewed as a nonlocal vertex algebra, but $W$ is not a module in the usual sense for $\langle U \rangle_\phi$ viewed as either an associative algebra or a nonlocal vertex algebra in general.
5\ φ-coordinated modules for weak quantum vertex algebras with \(\phi(x, z) = xe^z\)

In this section, we study \(\phi\)-coordinated quasi modules for weak quantum vertex algebras with \(\phi\) specialized to \(\phi(x, z) = xe^z\). We first continue with Section 4 to formulate notions of (quasi) \(S_{\text{trig}}\)-local subset and prove that the nonlocal vertex algebra generated by any quasi \(S_{\text{trig}}\)-local subset is a weak quantum vertex algebra. We then present certain axiomatic results on \(\phi\)-coordinated modules for weak quantum vertex algebras. In particular, we establish a Jacobi-type identity.

Let \(W\) be a vector space as in Section 4. Throughout this section, we assume \(\phi(x, z) = xe^z\) and we denote \(Y_\phi\) by \(Y_e\). That is,

\[
Y_e(a(x), z)b(x) = p(xe^z, x)^{-1}(p(x_1, x)a(x_1)b(x)) |_{x_1=xe^z} \quad (5.1)
\]

for any quasi compatible pair \((a(x), b(x))\) in \(E(W)\) with nonzero \(p(x_1, x_2) \in \mathbb{C}[x_1, x_2]\) such that (4.3) holds.

First we formulate the following notions:

**Definition 5.1.** A subset \(U\) of \(E(W)\) is said to be \(S_{\text{trig}}\)-local if for any \(a(x), b(x) \in U\), there exist \(u_i(x), v_i(x) \in U, q_i(x) \in \mathbb{C}(x) \ (i = 1, \ldots, r)\), where \(\mathbb{C}(x)\) denotes the field of rational functions, such that

\[
(x_1 - x_2)^k a(x_1)b(x_2) = (x_1 - x_2)^k \sum_{i=1}^{r} t_{x_2, x_1}(q_i(x_1/x_2))u_i(x_2)v_i(x_1) \quad (5.2)
\]

for some \(k \in \mathbb{N}\). The notion of quasi \(S_{\text{trig}}\)-local subset is defined by weakening the above condition as

\[
p(x_1/x_2)a(x_1)b(x_2) = p(x_1/x_2) \sum_{i=1}^{r} t_{x_2, x_1}(q_i(x_1/x_2))u_i(x_2)v_i(x_1) \quad (5.3)
\]

for some nonzero polynomial \(p(x) \in \mathbb{C}[x_1, x_2]\).

These notions single out a family of compatible subsets and a family of quasi compatible subsets as we show next.

**Lemma 5.2.** Every (resp. quasi) \(S_{\text{trig}}\)-local subset of \(E(W)\) is (resp. quasi) compatible.

**Proof.** Let us first consider the quasi case. Let \(U\) be a quasi \(S_{\text{trig}}\)-local subset. We must prove that every finite sequence in \(U\) is quasi compatible. To prove this we use induction on the length \(n\) of sequences. Let \((a(x), b(x))\) be an ordered pair in
By assumption, there exist \(0 \neq p(x) \in \mathbb{C}[x], a^{(i)}(x), b^{(i)}(x) \in U \) and \(q_i(x) \in \mathbb{C}(x)\) for \(i = 1, \ldots, r\) such that

\[
p(x_1/x_2)a(x_1)b(x_2) = p(x_1/x_2) \sum_{i=1}^r \ell_{x_2,x_1}(q_i(x_1/x_2))b^{(i)}(x_2)a^{(i)}(x_1). \tag{5.4}
\]

The expression on the left-hand side lies in \(\text{Hom}(W,W((x_1))(x_2)))\) while the expression on the right-hand side lies in \(\text{Hom}(W,W((x_2))((x_1)))\). This forces the expressions on both sides to lie in \(\text{Hom}(W,W((x_1, x_2)))\). Thus \((a(x), b(x))\) is quasi compatible, proving the case for \(n = 2\).

Now assume that \(n \geq 2\) and that any sequence in \(U\) of length \(n\) is quasi compatible. Let \(\psi^{(1)}(x), \ldots, \psi^{(n+1)}(x) \in U\). From the inductive hypothesis, there exists \(0 \neq f(x) \in \mathbb{C}[x]\) such that

\[
\left( \prod_{2 \leq i < j \leq n+1} f(x_i/x_j) \right) \psi^{(2)}(x_2) \cdots \psi^{(n+1)}(x_{n+1}) \in \text{Hom}(W,W((x_2, \ldots, x_{n+2}))). \tag{5.5}
\]

By assumption there exist \(0 \neq p(x) \in \mathbb{C}[x], a^{(i)}(x), b^{(i)}(x) \in U \) and \(q_i(x) \in \mathbb{C}(x)\) for \(i = 1, \ldots, r\) such that

\[
p(x_1/x_2)\psi^{(1)}(x_1)\psi^{(2)}(x_2) = p(x_1/x_2) \sum_{i=1}^r \ell_{x_2,x_1}(q_i(x_1/x_2))b^{(i)}(x_2)a^{(i)}(x_1). \tag{5.6}
\]

From the inductive hypothesis again, there exists \(0 \neq g(x) \in \mathbb{C}[x]\) such that

\[
\left( \prod_{1 \leq i < j \leq n+1, i,j \neq 2} g(x_i/x_j) \right) a^{(s)}(x_1)\psi^{(3)}(x_3) \cdots \psi^{(n+1)}(x_{n+1})
\in \text{Hom}(W,W((x_1, x_3, x_4, \ldots, x_{n+1}))) \tag{5.7}
\]

for \(s = 1, \ldots, r\). Using (5.6) we have

\[
\left( \prod_{2 \leq i < j \leq n+1} f(x_i/x_j) \prod_{1 \leq i < j \leq n+1, i,j \neq 2} g(x_i/x_j) \right) p(x_1/x_2)\psi^{(1)}(x_1) \cdots \psi^{(n+1)}(x_{n+1})
\]

\[
= \left( \prod_{2 \leq i < j \leq n+1} f(x_i/x_j) \prod_{1 \leq i < j \leq n+1, i,j \neq 2} g(x_i/x_j) \right) p(x_1/x_2)
\]

\[
\cdot \sum_{s=1}^r \ell_{x_2,x_1}(q_i(x_1/x_2))b^{(s)}(x_2)a^{(s)}(x_1)\psi^{(3)}(x_3) \cdots \psi^{(n+1)}(x_{n+1}). \tag{5.8}
\]

From (5.5), the expression on the left-hand side of (5.8) lies in \(\text{Hom}(W,W((x_1))((x_2, x_3, x_4, \ldots, x_{n+1})))\).
and by (5.7), the expression on the right-hand side of (5.8) lies in
\[ \text{Hom}(W, W((x_1, x_3, x_4, \ldots, x_{n+1}))). \]
This forces the expressions on both sides to lie in the space
\[ \text{Hom}(W, W((x_1, x_2, x_3, x_4, \ldots, x_{n+1}))). \]
In particular, the expression on the left-hand side (5.8) lies in
\[ \text{Hom}(W, W((x_1, x_2, x_3, x_4, \ldots, x_{n+1}))). \]
This proves that the sequence \((\psi^{(1)}(x), \ldots, \psi^{(n+1)}(x))\) is quasi compatible, completing the induction. From the proof, it is clear that if \(U\) is \(S_{\text{trig}}\)-local, then \(U\) is compatible. \(\square\)

In view of Lemma 5.2 and Theorem 4.11, for any quasi \(S_{\text{trig}}\)-local subset \(U\) of \(E(W)\), we have a nonlocal vertex algebra \(\langle U \rangle_\phi\) generated by \(U\) with \(\phi(x, z) = xe^z\).

In the following we are going to prove that \(\langle U \rangle_\phi\) is a weak quantum vertex algebra.

**Proposition 5.3.** Let \(V\) be a \(Y^e_{\text{E}}\)-closed quasi compatible subspace of \(E(W)\). Suppose
\[ a(x), b(x), u_i(x), v_i(x) \in V, \ 0 \neq p(x) \in \mathbb{C}[x], \ q_i(x) \in \mathbb{C}(x) \quad (i = 1, \ldots, r) \]
satisfy
\[ p(x_1/x_2)a(x_1)b(x_2) = \sum_{i=1}^{r} p(x_1/x_2)t_{x_2,x_1}(q_i(x_1/x_2))u_i(x_2)v_i(x_1). \quad (5.9) \]

Then
\[ p(e^{x_1-x_2})Y^e_{\text{E}}(a(x), x_1)Y^e_{\text{E}}(b(x), x_2) \]
\[ = p(e^{x_1-x_2}) \sum_{i=1}^{r} t_{x_2,x_1}(q_i(e^{x_1-x_2}))Y^e_{\text{E}}(u_i(x), x_2)Y^e_{\text{E}}(v_i(x), x_1). \quad (5.10) \]
Furthermore, we have
\[ (x_1 - x_2)^kY^e_{\text{E}}(a(x), x_1)Y^e_{\text{E}}(b(x), x_2) \]
\[ = (x_1 - x_2)^k \sum_{i=1}^{r} t_{x_2,x_1}(q_i(e^{x_1-x_2}))Y^e_{\text{E}}(u_i(x), x_2)Y^e_{\text{E}}(v_i(x), x_1), \quad (5.11) \]
where \(k\) is the multiplicity of the zero of \(p(x)\) at \(x = 1\).
Proof. Let \( \theta(x) \in V \) be arbitrarily fixed. There exists \( 0 \neq f(x_1, x_2) \in \mathbb{C}[[x_1, x_2]] \) such that

\[
\begin{align*}
&f(z, x)b(z)\theta(x) \in \text{Hom}(W, W((x, z))), \\
&f(y, z)f(y, x)f(z, x)a(y)b(z)\theta(x) \in \text{Hom}(W, W((x, y, z))).
\end{align*}
\]

By Lemma 4.8 we have

\[
\begin{align*}
f(xe^{x_1}, xe^{x_2})f(xe^{x_1}, x)f(xe^{x_2}, x)Y^e_\xi(a(x), x_1)Y^e_\xi(b(x), x_2)\theta(x) &= (f(y, z)f(y, x)f(z, x)a(y)b(z)\theta(x))|_{y=xe^{x_1}, z=xe^{x_2}}.
\end{align*}
\]

Replacing \( f(x_1, x_2) \) with a multiple of \( f(x_1, x_2) \) if necessary, we also have

\[
\begin{align*}
f(xe^{x_1}, xe^{x_2})f(xe^{x_1}, x)f(xe^{x_2}, x)Y^e_\xi(u_i(x), x_2)Y^e_\xi(v_i(x), x_1)\theta(x) &= (f(y, z)f(y, x)f(z, x)u_i(z)v_i(y)\theta(x))|_{z=xe^{x_2}, y=xe^{x_1}}
\end{align*}
\]

for \( i = 1, \ldots, r \). Therefore,

\[
\begin{align*}
f(xe^{x_1}, xe^{x_2})f(xe^{x_1}, x)f(xe^{x_2}, x)p(e^{x_1-x_2})Y^e_\xi(a(x), x_1)Y^e_\xi(b(x), x_2)\theta(x) &= (f(y, z)f(y, x)f(z, x)p(y/z)a(y)b(z)\theta(x))|_{y=xe^{x_1}, z=xe^{x_2}} \\
&= \sum_{i=1}^{r} (f(y, z)f(y, x)f(z, x)p(y/z)q_i(y/z)u_i(z)v_i(y)\theta(x))|_{z=xe^{x_2}, y=xe^{x_1}} \\
&= f(xe^{x_1}, xe^{x_2})f(xe^{x_1}, x)f(xe^{x_2}, x)p(e^{x_1-x_2}) \\
&\quad \cdot \sum_{i=1}^{r} l_{x_2, x_1}(q_i(e^{x_1-x_2}))Y^e_\xi(u_i(x), x_2)Y^e_\xi(v_i(x), x_1)\theta(x).
\end{align*}
\]

In view of Remark 4.1 (by cancellation) we have

\[
\begin{align*}
f(xe^{x_1}, xe^{x_2})p(e^{x_1-x_2})Y^e_\xi(a(x), x_1)Y^e_\xi(b(x), x_2)\theta(x) &= f(xe^{x_1}, xe^{x_2})p(e^{x_1-x_2})\sum_{i=1}^{r} l_{x_2, x_1}(q_i(e^{x_1-x_2}))Y^e_\xi(u_i(x), x_2)Y^e_\xi(v_i(x), x_1)\theta(x).
\end{align*}
\]

Write \( f(z_1, z_2)p(z_1/z_2) = (z_1 - z_2)^s z_2^{s'} g(z_1, z_2) \) with \( s, s' \in \mathbb{N}, g(z_1, z_2) \in \mathbb{C}[[z_1, z_2]] \) such that \( g(z, z) \neq 0 \). Then

\[
\begin{align*}
f(xe^{x_1}, xe^{x_2})p(e^{x_1-x_2}) &= (x_1 - x_2)^s x^s E(x_1, x_2)^s (xe^{x_2})^{-s'} g(xe^{x_1}, xe^{x_2}),
\end{align*}
\]

where \( E(x_1, x_2) = \sum_{n \geq 1} \frac{1}{n!} (x_1^{n-1} - x_2^{n-1}) \) is a unit in \( \mathbb{C}[[x_1, x_2]] \). Noticing that \( g(xe^{x_1}, xe^{x_2}) \) is a unit in \( \mathbb{C}(x) [[x_1, x_2]] \), by cancellation we get

\[
\begin{align*}
(x_1 - x_2)^s Y^e_\xi(a(x), x_1)Y^e_\xi(b(x), x_2) &= \sum_{i=1}^{r} (x_1 - x_2)^s l_{x_2, x_1}(q_i(e^{x_1-x_2}))Y^e_\xi(u_i(x), x_2)Y^e_\xi(v_i(x), x_1).
\end{align*}
\]
Combining this with weak associativity (Theorem 4.9) we obtain
\[ x_0^{-1} \delta \left( \frac{x_1 - x_2}{x_0} \right) Y_\mathcal{E}^e(a(x), x_1) Y_\mathcal{E}^e(b(x), x_2) \]
\[ -x_0^{-1} \delta \left( \frac{x_2 - x_1}{x_0} \right) \sum_{i=1}^{r} l_{x_2,x_1}(q_i(e^{x_1-x_2})) Y_\mathcal{E}^e(u_i(x), x_2) Y_\mathcal{E}^e(v_i(x), x_1) \]
\[ = x_2^{-1} \delta \left( \frac{x_1 - x_0}{x_2} \right) Y_\mathcal{E}^e(Y_\mathcal{E}^e(a(x), x_0)b(x), x_2). \] (5.12)

From (5.9) we have
\[ p(x_1/x_2)a(x_1)b(x_2) \in \text{Hom}(W,W((x_1, x_2))), \]
so that
\[ p(e^{x_0}) Y_\mathcal{E}^e(a(x), x_0)b(x) = (p(x_1/x)a(x_1)b(x)) |_{x_1 = x_0}. \]
which involves only nonnegative integer powers of $x_0$. Multiplying the both sides of (5.12) by $p(e^{x_0})$ and then taking Res$_{x_0}$ we get
\[ p(e^{x_1-x_2}) Y_\mathcal{E}^e(a(x), x_1) Y_\mathcal{E}^e(b(x), x_2) \]
\[ = p(e^{x_1-x_2}) \sum_{i=1}^{r} l_{x_2,x_1}(q_i(e^{x_1-x_2})) Y_\mathcal{E}^e(u_i(x), x_2) Y_\mathcal{E}^e(v_i(x), x_1). \]

Let $k$ be the multiplicity of the zero of $p(x)$ at $x = 1$. Then $p(x) = z^k h(z)$ where $h(z) \in \mathbb{C}[[z]]$ with $h(0) \neq 0$. By cancellation we obtain
\[ (x_1 - x_2)^k Y_\mathcal{E}^e(a(x), x_1) Y_\mathcal{E}^e(b(x), x_2) \]
\[ = \sum_{i=1}^{r} (x_1 - x_2)^k l_{x_2,x_1}(q_i(e^{x_1-x_2})) Y_\mathcal{E}^e(u_i(x), x_2) Y_\mathcal{E}^e(v_i(x), x_1), \]
as desired. \hfill \Box

As the main result of this section we have:

**Theorem 5.4.** Let $W$ be a vector space and let $U$ be any (resp. quasi) $\mathcal{S}_{\text{trig}}$-local subset of $\mathcal{E}(W)$. Then $\langle U \rangle_\phi$ is a weak quantum vertex algebra and $W$ is a $\phi$-coordinated (resp. quasi) module with $\phi(x, z) = xe^z$.

**Proof.** We only need to prove that $\langle U \rangle_\phi$ is a weak quantum vertex algebra. As $\langle U \rangle_\phi$ is the smallest $Y_\mathcal{E}^e$-closed quasi compatible subspace containing $U$ and $1_W$, we see that $\langle U \rangle_\phi$ as a nonlocal vertex algebra is generated by $U$. Given that $U$ is quasi $\mathcal{S}_{\text{trig}}$-local, from Proposition 5.3 we have that
\[ \{Y_\mathcal{E}^e(a(x), z) \mid a(x) \in U \} \]
is an $\mathcal{S}$-local subset of $\mathcal{E}(\langle U \rangle_\phi)$ in the sense of [Li3]. Then by [LTW] (Proposition 2.6), $\langle U \rangle_\phi$ is a weak quantum vertex algebra. \hfill \Box

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Remark 5.5. Let $W$ be a highest weight module for a quantum affine algebra $U_q(\hat{\mathfrak{g}})$ with $q$ a complex number (see [Dr], [FJ]). It is straightforward to see that the generating functions of the generators in the Drinfeld realization form a quasi $\mathcal{S}_{rig}$-local subset $U_W$ of $E(W)$. By Theorem 5.4, $U_W$ generates a weak quantum vertex algebra with $W$ as a $\phi$-coordinated quasi module where $\phi(x, z) = xe^z$. In a sequel, we shall study the associated weak quantum vertex algebras in detail.

We next study $\phi$-coordinated quasi modules for a general weak quantum vertex algebra.

**Proposition 5.6.** Let $V$ be a nonlocal vertex algebra and let $(W, Y_W)$ be a $\phi$-coordinated quasi $V$-module. Assume that $u, v, u^{(i)}, v^{(i)} \in V$, $f_i(x) \in \mathbb{C}(x)$ $(i = 1, \ldots, r)$ satisfy the relation

\[
(x_1 - x_2)^k Y(u, x_1)Y(v, x_2) = (x_1 - x_2)^k \sum_{i=1}^{r} t_{x_2, x_1}(f_i(e^{x_1-x_2}))Y(v^{(i)}, x_2)Y(u^{(i)}, x_1) \tag{5.13}
\]

for some $k \in \mathbb{N}$. Suppose that $p(x_1, x_2) \in \mathbb{C}[[x_1, x_2]]$ is nonzero such that

\[
p(x_1, x_2)Y_W(u, x_1)Y_W(v, x_2) \in \text{Hom}(W, W((x_1, x_2))).
\]

Then

\[
p(x_1, x_2)Y_W(u, x_1)Y_W(v, x_2) = p(x_1, x_2) \sum_{i=1}^{r} t_{x_2, x_1}(f_i(x_1/x_2))Y_W(v^{(i)}, x_2)Y_W(u^{(i)}, x_1). \tag{5.14}
\]

**Proof.** With (5.13), by Corollary 5.3 of [Li3] we have

\[
Y(u, x)v = \sum_{i=1}^{r} t_{x, 0}(f_i(e^x))e^{xD}Y(v^{(i)}, -x)u^{(i)}.
\]

From definition, there exists a nonzero polynomial $q(x_1, x_2)$ such that

\[
q(x_1, x_2)Y_W(u, x_1)Y_W(v, x_2) \in \text{Hom}(W, W((x_1, x_2)))
\]

and such that $q(x_1, x_2)f_i(x_1/x_2) \in \mathbb{C}[x_1, x_2]$,

\[
q(x_1, x_2)f_i(x_1/x_2)Y_W(v^{(i)}, x_2)Y_W(u^{(i)}, x_1) \in \text{Hom}(W, W((x_1, x_2))).
\]
for \( i = 1, \ldots, r \). Then, using Lemma 3.7, we get
\[
(q(x_1, x_2)Y_W(u, x_1)Y_W(v, x_2))|_{x_1 = x_2 e^{x_0}} \\
= q(x_2 e^{x_0}, x_2)Y_W(Y(u, x_0)v, x_2) \\
= \sum_{i=1}^{r} (q(x_2 e^{x_0}, x_2)f_i(e^{x_0}))Y_W(e^{x_0}D Y(v^{(i)}, -x_0)u^{(i)}, x_2) \\
= \sum_{i=1}^{r} (q(x_2 e^{x_0}, x_2)f_i(e^{x_0}))Y_W(Y(v^{(i)}, -x_0)u^{(i)}, x_2 e^{x_0}).
\]

We also have
\[
\left(q(x_1, x_2) \sum_{i=1}^{r} f_i(x_1/x_2)Y_W(v^{(i)}, x_2)Y_W(u^{(i)}, x_1)\right)|_{x_2 = x_1 e^{-x_0}} \\
= \sum_{i=1}^{r} (q(x_1, x_1 e^{-x_0})f_i(e^{x_0}))Y_W(Y(v^{(i)}, -x_0)u^{(i)}, x_1).
\]

Then using Remark 2.8 we have
\[
(q(x_1, x_2)Y_W(u, x_1)Y_W(v, x_2))|_{x_1 = x_2 e^{x_0}} \\
= \left(\left(q(x_1, x_2) \sum_{i=1}^{r} f_i(x_1/x_2)Y_W(v^{(i)}, x_2)Y_W(u^{(i)}, x_1)\right)|_{x_2 = x_1 e^{-x_0}}\right)|_{x_1 = x_2 e^{x_0}} \\
= \left(q(x_1, x_2) \sum_{i=1}^{r} f_i(x_1/x_2)Y_W(v^{(i)}, x_2)Y_W(u^{(i)}, x_1)\right)|_{x_1 = x_2 e^{x_0}}.
\]

Using Remark 2.8 again we get
\[
q(x_1, x_2)Y_W(u, x_1)Y_W(v, x_2) = \sum_{i=1}^{r} q(x_1, x_2)f_i(x_1/x_2)Y_W(v^{(i)}, x_2)Y_W(u^{(i)}, x_1).
\]

Then
\[
q(x_1, x_2)(p(x_1, x_2)Y_W(u, x_1)Y_W(v, x_2)) \\
= q(x_1, x_2) \left(p(x_1, x_2) \sum_{i=1}^{r} t_{x_2, x_1}(f_i(x_1/x_2))Y_W(v^{(i)}, x_2)Y_W(u^{(i)}, x_1)\right).
\]

Multiplying both sides by the inverse of \( q(x_1, x_2) \) in \( \mathbb{C}((x_2))((x_1)) \) we obtain the desired relation. \( \square \)

**Remark 5.7.** Note that for any \( f(x) \in x\mathbb{C}[[x]] \), \( g(x) \in \mathbb{C}((x)) \), the composition \( g(f(x)) \) exists in \( \mathbb{C}((x)) \). Set
\[
\log(1 + x) = \sum_{n \geq 1} (-1)^{n-1} \frac{1}{n} x^n \in x\mathbb{C}[[x]]. \quad (5.15)
\]
For any \( f(x) \in xC[[x]] \), we have
\[
\log(1 + f(x)) = \sum_{n \geq 1} (-1)^{n-1} \frac{1}{n} f(x)^n \in xC[[x]],
\]
\[
e^{f(x)} = \sum_{n \geq 0} \frac{1}{n!} f(x)^n \in C[[x]].
\]

Using formal calculus one can show
\[
e^{\log(1+z)} = 1 + z \quad \text{and} \quad \log(1 + (e^x - 1)) = x.
\]

Let \( E(x_1, x_2) \in C((x_1, x_2)) \). Set
\[
F(x_0, x_2) = E(x_2e^{x_0}, x_2) \in C((x_2))[x_0].
\]

Then
\[
F(\log(1+z), x_2) = E(x_2(1+z), x_2) \in C((x_2))[[z]].
\]

**Lemma 5.8.** Let \( W \) be any vector space and let
\[
A(x_1, x_2) \in \text{Hom}(W, W((x_1))((x_2))), \quad B(x_1, x_2) \in \text{Hom}(W, W((x_2))((x_1))),
\]
\[
C(x_0, x_2) \in (\text{Hom}(W, W((x_2))))((x_0)).
\]

If there exists a nonnegative integer \( k \) such that
\[
(x_1 - x_2)^k A(x_1, x_2) = (x_1 - x_2)^k B(x_1, x_2),
\]
\[
((x_1 - x_2)^k A(x_1, x_2)) |_{x_1 = x_2 e^{x_0}} = x_2^k (e^{x_0} - 1)^k C(x_0, x_2),
\]
then
\[
(x_2 z)^{-1} \delta \left( \frac{x_1 - x_2}{x_2 z} \right) A(x_1, x_2) - (x_2 z)^{-1} \delta \left( \frac{x_2 - x_1}{-x_2 z} \right) B(x_2, x_1)
\]
\[
= x_1^{-1} \delta \left( \frac{x_2(1+z)}{x_1} \right) C(\log(1+z), x_2).
\]

Furthermore, the converse is also true.

**Proof.** We have the standard delta-function identity
\[
x_0^{-1} \delta \left( \frac{x_1 - x_2}{x_0} \right) - x_0^{-1} \delta \left( \frac{x_2 - x_1}{-x_0} \right) = x_1^{-1} \delta \left( \frac{x_2 + x_0}{x_1} \right)
\]
(see [FLM]). Substituting \( x_0 = x_2 z \) with \( z \) a new formal variable, we have
\[
(x_2 z)^{-1} \delta \left( \frac{x_1 - x_2}{x_2 z} \right) - (x_2 z)^{-1} \delta \left( \frac{x_2 - x_1}{-x_2 z} \right) = x_1^{-1} \delta \left( \frac{x_2(1+z)}{x_1} \right), \quad (5.18)
\]
where it is understood that for \( n \in \mathbb{Z} \),
\[
(1 + z)^n = \sum_{j \geq 0} \binom{n}{j} z^j \in \mathbb{C}[[z]].
\]

Then using Remark 5.7 we obtain
\[
(x_2 z)^{-1} \delta \left( \frac{x_1 - x_2}{x_2 z} \right) (x_2 z^k A(x_1, x_2) - (x_2 z)^{-1} \delta \left( \frac{x_2 - x_1}{-x_2 z} \right) (x_2 z)^k B(x_2, x_1)
\]
\[
= (x_2 z)^{-1} \delta \left( \frac{x_1 - x_2}{x_2 z} \right) (x_1 - x_2)^k A(x_1, x_2)
\]
\[
- (x_2 z)^{-1} \delta \left( \frac{x_2 - x_1}{-x_2 z} \right) (x_1 - x_2)^k B(x_2, x_1)
\]
\[
= x_1^{-1} \delta \left( \frac{x_2(1 + z)}{x_1} \right) ((x_1 - x_2)^k A(x_1, x_2))
\]
\[
= x_1^{-1} \delta \left( \frac{x_2(1 + z)}{x_1} \right) ((x_1 - x_2)^k A(x_1, x_2)) \bigg|_{x_1 = x_2(1 + z)}
\]
\[
= x_1^{-1} \delta \left( \frac{x_2(1 + z)}{x_1} \right) (((x_1 - x_2)^k A(x_1, x_2)) \bigg|_{x_1 = x_2 e^{x_0}}) \bigg|_{x_0 = \log(1 + z)}
\]
\[
= x_1^{-1} \delta \left( \frac{x_2(1 + z)}{x_1} \right) (x_2 e^{x_0} - 1)^k C(x_0, x_2) \bigg|_{x_0 = \log(1 + z)}
\]
\[
= x_1^{-1} \delta \left( \frac{x_2(1 + z)}{x_1} \right) (x_2 z)^k C(\log(1 + z), x_2),
\]
which implies (5.17).

Conversely, assume (5.17). Let \( k \in \mathbb{N} \) be such that
\[
x_0^k C(x_0, x_2) \in (\text{Hom}(W, W((x_2))))[[x_0]].
\]

Then
\[
z^k C(\log(1 + z), x_2) \in (\text{Hom}(W, W((x_2))))[[z]],
\]
as \( \log(1 + z) = zg(z) \) with \( g(z) \in \mathbb{C}[[z]] \) invertible. Applying Res_{z}z^{k} to (5.17) we get
\[
(x_1 - x_2)^k A(x_1, x_2) = (x_1 - x_2)^k B(x_1, x_2).
\]

Using this and (5.18) we get
\[
x_1^{-1} \delta \left( \frac{x_2(1 + z)}{x_1} \right) ((x_1 - x_2)^k A(x_1, x_2))
\]
\[
= x_1^{-1} \delta \left( \frac{x_2(1 + z)}{x_1} \right) (x_2 z)^k C(\log(1 + z), x_2).
\]

Substituting \( z = e^{x_0} - 1 \), we get
\[
x_1^{-1} \delta \left( \frac{x_2 e^{x_0}}{x_1} \right) ((x_1 - x_2)^k A(x_1, x_2)) = x_1^{-1} \delta \left( \frac{x_2 e^{x_0}}{x_1} \right) (x_2 z)^k C(x_0, x_2),
\]
which implies
\[
((x_1 - x_2)^k A(x_1, x_2)) |_{x_1 = x_2 e^{x_0}} = x_2^k (e^{x_0} - 1)^k C(x_0, x_2).
\]
This completes the proof. \qed

Now we are ready to present our second main result of this section.

**Proposition 5.9.** Let \( V \) be a weak quantum vertex algebra and let \((W,Y_W)\) be a \(\phi\)-coordinated module for \( V \) viewed as a nonlocal vertex algebra. Let \( u, v \in V \) and assume that
\[
(x_1 - x_2)^k Y(u, x_1) Y(v, x_2) = (x_1 - x_2)^k \sum_{i=1}^r t_{x_2, x_1} (f_i(e^{x_1 - x_2})) Y(u^{(i)}, x_2) Y(v^{(i)}, x_1)
\]
with \( k \in \mathbb{N} \), \( f_i(x) \in \mathbb{C}(x) \), \( u^{(i)}, v^{(i)} \in V \) for \( 1 \leq i \leq r \). Then
\[
(x_2 z)^{-1} \frac{1}{x_2 z} Y_W(u, x_1) Y_W(v, x_2) = Y_W(Y(u, \log(1 + z)) v, x_2).
\]
Furthermore, we have
\[
Y_W(u, x_1) Y_W(v, x_2) - \sum_{i=1}^r t_{x_2, x_1} (f_i(x_1 / x_2)) Y_W(v^{(i)}, x_2) Y_W(u^{(i)}, x_1) = \Res_{x_0} x_1^{-1} \delta \left( \frac{x_2 e^{x_0}}{x_1} \right) x_2 e^{x_0} Y_W(Y(u, x_0) v, x_2).
\]

**Proof.** From definition, there exists a nonnegative integer \( l \) such that
\[
(x_1 - x_2)^l Y_W(u, x_1) Y_W(v, x_2) \in \text{Hom}(W, W((x_1, x_2)))
\]
and
\[
x_2^l (e^{x_0} - 1)^l Y_W(Y(u, x_0) v, x_2) = ((x_1 - x_2)^l Y_W(u, x_1) Y_W(v, x_2)) |_{x_1 = x_2 e^{x_0}}.
\]
On the other hand, by Proposition 5.6 we also have
\[
(x_1 - x_2)^l Y_W(u, x_1) Y_W(v, x_2) = (x_1 - x_2)^l \sum_{i=1}^r t_{x_2, x_1} (f_i(x_1 / x_2)) Y_W(v^{(i)}, x_2) Y_W(u^{(i)}, x_1).
\]

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Then the first assertion follows immediately from Lemma 5.8. Furthermore, applying \( \text{Res}_z x_2 \) we get

\[
Y_W(u, x_1)Y_W(v, x_2) - \sum_{i=1}^r t_{x_2, x_1}(f_i(x_1/x_2))Y_W(v(i), x_2)Y_W(u(i), x_1)
= \text{Res}_z x_1^{-1} \delta \left( \frac{x_2(1 + z)}{x_1} \right) x_2 Y_W(Y(u, \log(1 + z))v, x_2)
= \text{Res}_x x_1^{-1} \delta \left( \frac{x_2e^{x_0}}{x_1} \right) x_2 e^{x_0} Y_W(Y(u_0)Y(v, x_2))
\]

proving the second assertion.

As a consequence we have:

**Corollary 5.10.** Let \( W \) be a vector space and let \( U \) be an \( S_{\text{trig}} \)-local subset of \( E(W) \). Assume that \( a(x), b(x), a^{(i)}(x), b^{(i)}(x) \in U, f_i(x) \in \mathbb{C}(x) \ (i = 1, \ldots, r) \) satisfy

\[
(x_1 - x_2)^k a(x_1)b(x_2) = (x_1 - x_2)^k \sum_{i=1}^r t_{x_2, x_1}(f_i(x_1/x_2))b^{(i)}(x_2)a^{(i)}(x_1)
\]

for some nonnegative integer \( k \). Then

\[
(xz)^{-1} \delta \left( \frac{x_1 - x}{xz} \right) a(x_1)b(x) - (xz)^{-1} \delta \left( \frac{x - x_1}{-xz} \right) \sum_{i=1}^r f_i(x_1/x)b^{(i)}(x)a^{(i)}(x_1)
= x_1^{-1} \delta \left( \frac{x(1 + z)}{x_1} \right) Y_\mathcal{E}_c(a(x), \log(1 + z))b(x).
\]

In particular,

\[
Y_\mathcal{E}_c(a(x), \log(1 + z))b(x) = \text{Res}_{x_1} (xz)^{-1} \delta \left( \frac{x_1 - x}{xz} \right) a(x_1)b(x)
- \text{Res}_{x_1} (xz)^{-1} \delta \left( \frac{x - x_1}{-xz} \right) \sum_{i=1}^r f_i(x_1/x)b^{(i)}(x)a^{(i)}(x_1).
\]

**Proof.** As \( U \) is \( S_{\text{trig}} \)-local, by Theorem 5.4 \( \langle U \rangle_\mathcal{E} \) is a weak quantum vertex algebra with \( W \) as a \( \phi \)-coordinated module. By Proposition 5.9 we have (5.21), which immediately implies (5.22).
6 Quantum vertex algebras associated with quantum $\beta\gamma$-system

In this section we associate quantum vertex algebras to certain quantum $\beta\gamma$-systems, using the general machinery developed in previous sections. We first introduce a quantum $\beta\gamma$-system of trigonometric type, which is a modified version of the quantum $\beta\gamma$-system in \cite{EFK}. Then we introduce a quantum $\beta\gamma$-system of rational type, to describe the quantum vertex algebras constructed from the trigonometric type quantum $\beta\gamma$-system.

We start with the following quantum $\beta\gamma$-system:

**Definition 6.1.** Let $q$ be a nonzero complex number. Define $A_{\text{trig}}^q(\beta\gamma)$ to be the associative algebra with identity over $\mathbb{C}$ with generators $\tilde{\beta}_n, \tilde{\gamma}_n$ ($n \in \mathbb{Z}$), which are considered as the coefficients of the currents

$$\tilde{\beta}(x) = \sum_{n \in \mathbb{Z}} \tilde{\beta}_n x^{-n-1}, \quad \tilde{\gamma}(x) = \sum_{n \in \mathbb{Z}} \tilde{\gamma}_n x^{-n-1},$$

subject to relations

$$\tilde{\beta}(x)\tilde{\beta}(z) = t_{z,x} \left( \frac{x - qz}{qx - z} \right) \tilde{\beta}(z)\tilde{\beta}(x),$$

$$\tilde{\gamma}(x)\tilde{\gamma}(z) = t_{z,x} \left( \frac{x - qz}{qx - z} \right) \tilde{\gamma}(z)\tilde{\gamma}(x),$$

$$\tilde{\beta}(x)\tilde{\gamma}(z) - t_{z,x} \left( \frac{qx - z}{x - qz} \right) \tilde{\gamma}(z)\tilde{\beta}(x) = \delta \left( \frac{x}{z} \right). \quad (6.1)$$

When $q = 1$, it can be readily seen that $\tilde{\beta}(x)$ and $x^{-1}\tilde{\gamma}(x)$ form the standard $\beta\gamma$-system where $A_{\text{trig}}^q(\beta\gamma)$ is isomorphic to the universal enveloping algebra of an infinite-dimensional Heisenberg Lie algebra.

**Remark 6.2.** Here we give some details about the definition of $A_{\text{trig}}^q(\beta\gamma)$. Set

$$t_{z,x} \left( \frac{x - qz}{qx - z} \right) = \sum_{k \geq 0} \lambda_k (x/z)^k, \quad t_{z,x} \left( \frac{qx - z}{x - qz} \right) = \sum_{k \geq 0} \lambda'_k (x/z)^k,$$

with $\lambda_k, \lambda'_k \in \mathbb{C}$ for $k \geq 0$. The defining relations (6.1) amount to

$$\tilde{\beta}_m \tilde{\beta}_n = \sum_{k \geq 0} \lambda_k \tilde{\beta}_{n-k} \tilde{\beta}_{m+k}, \quad \tilde{\gamma}_m \tilde{\gamma}_n = \sum_{k \geq 0} \lambda_k \tilde{\gamma}_{n-k} \tilde{\gamma}_{m+k},$$

$$\tilde{\beta}_m \tilde{\gamma}_n - \sum_{k \geq 0} \lambda'_k \tilde{\gamma}_{n-k} \tilde{\beta}_{m+k} = \delta_{m+n+2,0} \quad (6.2)$$

for $m, n \in \mathbb{Z}$. Let $T$ be the free associative algebra over $\mathbb{C}$ with generators $\beta_n, \gamma_n$ ($n \in \mathbb{Z}$). Define

$$\text{deg } \beta_n = \text{deg } \gamma_n = n + 1 \quad \text{for } n \in \mathbb{Z},$$

$$\text{deg } \tilde{\beta}_n = \text{deg } \tilde{\gamma}_n = n + 2 \quad \text{for } n \in \mathbb{Z}.$$
to make $T$ a $\mathbb{Z}$-graded algebra whose homogeneous subspace of degree-$n$ is denoted by $T(n)$. Following [FZ], for $n \in \mathbb{Z}$, $k \geq 1$, set

$$T(n, k) = \sum_{r \geq k} T(n - r)T(r) \subset T(n).$$

We have $\cap_{k \geq 1} T(n, k) = 0$. Equip $T(n)$ with the topology with $a + T(n, k)$ for $a \in T(n)$, $k \geq 0$ as a basis of open sets. Let $\overline{T(n)}$ be the formal completion of $T(n)$. Set $\overline{T} = \bigoplus_{n \in \mathbb{Z}} \overline{T(n)}$. Then the algebra $A_{q}^{\text{trig}}(\beta \gamma)$ can be defined as the quotient algebra of $\overline{T}$ modulo the relations (6.2). Since all the relations are homogeneous, $A_{q}^{\text{trig}}(\beta \gamma)$ is a $\mathbb{Z}$-graded algebra.

As $(x - z)\delta \left( \frac{z}{x} \right) = 0$, from the third relation in (6.1) we get

$$(x - z)\tilde{\beta}(x)\tilde{\gamma}(z) = (x - z)\iota_{z,x} \left( \frac{qx - z}{x - qz} \right) \tilde{\gamma}(z)\tilde{\beta}(x). \quad (6.3)$$

By a restricted $A_{q}^{\text{trig}}(\beta \gamma)$-module we mean an $A_{q}^{\text{trig}}(\beta \gamma)$-module $W$ which equipped with the discrete topology is a continuous module. Then a restricted $A_{q}^{\text{trig}}(\beta \gamma)$-module amounts to a $T$-module $W$ such that for every $w \in W$, $\beta_n w = \gamma_n w = 0$ for $n$ sufficiently large and the relations corresponding to (6.2) applied to $w$ hold. Let $W$ be a restricted $A_{q}^{\text{trig}}(\beta \gamma)$-module. With the relations (6.1) and (6.3), we see that $\{\tilde{\beta}(x), \tilde{\gamma}(x)\}$ is an $S_{\text{trig}}$-local subset of $\mathcal{E}(W)$. In view of Theorem 5.4, $\{\tilde{\beta}(x), \tilde{\gamma}(x)\}$ generates a weak quantum vertex algebra $V_{W}$ inside $\mathcal{E}(W)$. To describe the structure of $V_{W}$ we need another algebra.

**Definition 6.3.** Let $q$ be a complex number. Define $A_{q}^{\text{rat}}(\beta \gamma)$ to be the associative algebra with identity over $\mathbb{C}$ with generators $\hat{\beta}_n$, $\hat{\gamma}_n$ ($n \in \mathbb{Z}$), subject to relations

$$\hat{\beta}(x)\hat{\beta}(z) = \iota_{z,x} \left( \frac{e^{x-z} - q}{qe^{x-z} - 1} \right) \hat{\beta}(z)\hat{\beta}(x),$$

$$\hat{\gamma}(x)\hat{\gamma}(z) = \iota_{z,x} \left( \frac{e^{x-z} - q}{qe^{x-z} - 1} \right) \hat{\gamma}(z)\hat{\gamma}(x),$$

$$\hat{\beta}(x)\hat{\gamma}(z) - \iota_{z,x} \left( \frac{qe^{x-z} - 1}{e^{x-z} - q} \right) \hat{\gamma}(z)\hat{\beta}(x) = z^{-1} \delta \left( \frac{x}{z} \right). \quad (6.4)$$

**Remark 6.4.** Here is a precise definition of the algebra $A_{q}^{\text{rat}}(\beta \gamma)$. Set

$$\frac{e^{x-z} - q}{qe^{x-z} - 1} = \sum_{k \geq 0} \mu_k (x - z)^k, \quad \frac{qe^{x-z} - 1}{e^{x-z} - q} = \sum_{k \geq 0} \mu'_k (x - z)^k$$

with $\mu_k, \mu'_k \in \mathbb{C}$ for $k \geq 0$. The defining relations (6.4) read as

$$\hat{\beta}_m \hat{\beta}_n = \sum_{k,i \geq 0} \binom{k}{i} (-1)^i \mu_k \hat{\beta}_{n+i} \hat{\beta}_{m+k-i}, \quad \hat{\gamma}_m \hat{\gamma}_n = \sum_{k,i \geq 0} \binom{k}{i} (-1)^i \mu'_k \hat{\gamma}_{n+i} \hat{\gamma}_{m+k-i},$$

$$\hat{\beta}_m \hat{\gamma}_n - \sum_{k,i \geq 0} \binom{k}{i} (-1)^i \mu'_k \hat{\gamma}_{n+i} \hat{\beta}_{m+k-i} = \delta_{m+n+1,0} \quad (6.5)$$
for $m, n \in \mathbb{Z}$. Let $T$ be the free associative algebra as in Remark 6.2, generated by $\beta_n, \gamma_n$ for $n \in \mathbb{Z}$, and equip $T$ with the same $\mathbb{Z}$-grading. For $k \geq 0$, set

$$T[k] = \sum_{n \geq k} T(n) \subset T.$$ 

Equip $T$ with the topology with $a + T(k)$ for $a \in T$, $k \geq 0$ as a basis of open sets. Let $\tilde{T}$ be the formal completion of $T$. The algebra $A^{rat}_q(\beta \gamma)$ can be defined as the quotient algebra of $\tilde{T}$ modulo the relations (6.5). Since the defining relations are inhomogeneous, the algebra $A^{rat}_q(\beta \gamma)$ is not $\mathbb{Z}$-graded in the obvious way.

**Remark 6.5.** Notice that when $q = 1$, the quantum $\beta \gamma$-system defined in Definition 6.3 is exactly the standard $\beta \gamma$-system. If $q = -1$, the defining relations become

$$\hat{\beta}(x) \hat{\gamma}(z) = -\hat{\gamma}(z) \hat{\beta}(x),$$

$$\hat{\gamma}(x) \hat{\gamma}(z) = -\hat{\gamma}(z) \hat{\gamma}(x),$$

$$\hat{\beta}(x) \hat{\gamma}(z) + \hat{\gamma}(z) \hat{\beta}(x) = z^{-1} \delta \left( \frac{x}{z} \right). \quad (6.6)$$

In this case, $A^{rat}_q(\beta \gamma)$ is an (infinite-dimensional) Clifford algebra.

Just as with $A^{trig}_q(\beta \gamma)$, by a restricted $A^{rat}_q(\beta \gamma)$-module we mean an $A^{rat}_q(\beta \gamma)$-module $W$ which equipped with the discrete topology is a continuous module. A restricted $A^{rat}_q(\beta \gamma)$-module simply amounts to a module for the free algebra $T$ such that for any $w \in W$, $\beta_n w = 0 = \gamma_n w$ for $n$ sufficiently large and such that the relations corresponding to (6.5) after applied to each vector $w \in W$ hold.

**Definition 6.6.** A vacuum $A^{rat}_q(\beta \gamma)$-module is a restricted $A^{rat}_q(\beta \gamma)$-module $W$ equipped with a vector $w_0 \in W$, satisfying the condition that $W = A^{rat}_q(\beta \gamma) w_0$,

$$\hat{\beta}_n w_0 = \hat{\gamma}_n w_0 = 0 \quad \text{for } n \geq 0. \quad (6.7)$$

We sometimes denote a vacuum module by a pair $(W, w_0)$.

We are going to prove that the weak quantum vertex algebra $V_W$ associated to a restricted $A^{trig}_q(\beta \gamma)$-module $W$ is naturally a vacuum $A^{rat}_q(\beta \gamma)$-module. To achieve this goal, we shall need the following technical result:

**Lemma 6.7.** Let $W$ be a vector space and let $a(x), b(x) \in \mathcal{E}(W)$. Assume that there exist

$$0 \neq p(x) \in \mathbb{C}[x], \quad q_i(x) \in \mathbb{C}((x)), \quad u^{(i)}(x), v^{(i)}(x) \in \mathcal{E}(W) \quad (1 \leq i \leq r)$$

such that

$$p(x_1/x_2) a(x_1) b(x_2) = \sum_{i=1}^r q_i(x_1/x_2) u^{(i)}(x_2) v^{(i)}(x_1). \quad (6.8)$$
Then \((a(x), b(x))\) is quasi compatible and

\[
p(e^{x_0})Y^e_x(a(x), x_0)b(x)
= \text{Res}_{x_1} \left( \frac{1}{x_1-xe^{x_0}} p(x_1/x)a(x_1)b(x) - \frac{1}{-xe^{x_0}+x_1} \sum_{i=1}^{r} q_i(x_1/x)u^{(i)}(x)v^{(i)}(x_1) \right).
\]

Furthermore, if \(k\) is the order of zero of \(p(x)\) at 1, then \(a(x)^n b(x) = 0\) for \(n \geq k\) and

\[
\frac{1}{k!} p^{(k)}(1) a(x)_{k-1}^e b(x)
= \text{Res}_{x_1} \left( \frac{1}{x_1-x} p(x_1/x)a(x_1)b(x) - \frac{1}{-x+x_1} q(x_1/x)u^{(i)}(x)v^{(i)}(x_1) \right). 
\]

Proof. By observing both sides of (6.8) we see that

\[
p(x_1/x_2)a(x_1)b(x_2) \in \text{Hom}(W,W((x_1, x_2))),
\]

which implies that \((a(x), b(x))\) is quasi compatible. Furthermore, we have

\[
p(e^{x_0})Y^e_x(a(x), x_0)b(x)
= (p(x_1/x)a(x_1)b(x)) |_{x_1=xe^{x_0}}
= \text{Res}_{x_1} x_1^{-1} \delta \left( \frac{xe^{x_0}}{x_1} \right) \left( p(x_1/x)a(x_1)b(x) \right)
= \text{Res}_{x_1} \left( \frac{1}{x_1-xe^{x_0}} p(x_1/x)a(x_1)b(x) - \frac{1}{-xe^{x_0}+x_1} (p(x_1/x)a(x_1)b(x)) \right)
= \text{Res}_{x_1} \left( \frac{1}{x_1-xe^{x_0}} p(x_1/x)a(x_1)b(x) - \frac{1}{-xe^{x_0}+x_1} \sum_{i=1}^{r} q_i(x_1/x)u^{(i)}(x)v^{(i)}(x_1) \right)
\]

as \(x_1^{-1} \delta \left( \frac{xe^{x_0}}{x_1} \right) = \frac{1}{x_1-xe^{x_0}} - \frac{1}{-xe^{x_0}+x_1} \). This proves the first part of the lemma.

Note that \(p(e^{x_0})Y^e_x(a(x), x_0)b(x)\) involves only nonnegative powers of \(x_0\). As \(k\) is the order of zero of \(p(x)\) at 1, we have \(p(e^{x_0}) = x_0^k g(x_0)\) for some \(g(x) \in \mathbb{C}[[x]]\) with \(g(0) \neq 0\). Since \(g(x_0)\) is a unit in \(\mathbb{C}[[x_0]]\), we have that \(x_0^k Y^e_x(a(x), x_0)b(x)\) involves only nonnegative powers of \(x_0\). That is, \(a(x)^n b(x) = 0\) for \(n \geq k\). Then, applying \(\text{Res}_{x_0} x_0^{-1}\), (or setting \(x_0 = 0\)), we obtain (6.9). 

Now we have:

**Proposition 6.8.** Let \(W\) be a restricted \(A^{\text{trig}}(\beta\gamma)\)-module and let \(V_W\) be the weak quantum vertex algebra generated by the \(S_{\text{trig}}\)-local subset \(\{\tilde{\beta}(x), \tilde{\gamma}(x)\}\) of \(E(W)\). Then \(V_W\) is an \(A^{\text{rat}}(\beta\gamma)\)-module with \(\tilde{\beta}(z)\) and \(\tilde{\gamma}(z)\) acting as \(Y^e_x(\tilde{\beta}(x), z)\) and \(Y^e_x(\tilde{\gamma}(x), z)\), respectively. Furthermore, \((V_W, 1_W)\) is a vacuum \(A^{\text{rat}}(\beta\gamma)\)-module.
Proof. With the relations (6.1) and (6.3), in view of Proposition 5.3 we have

\[ Y^e_\xi(\tilde{\beta}(x), x_1)Y^e_\xi(\tilde{\beta}(x), x_2) = \left( \frac{e^{x_1-x_2} - q}{qe^{x_1-x_2} - 1} \right) Y^e_\xi(\tilde{\beta}(x), x_2)Y^e_\xi(\tilde{\beta}(x), x_1), \]

\[ Y^e_\xi(\tilde{\gamma}(x), x_1)Y^e_\xi(\tilde{\gamma}(x), x_2) = \left( \frac{e^{x_1-x_2} - q}{qe^{x_1-x_2} - 1} \right) Y^e_\xi(\tilde{\gamma}(x), x_2)Y^e_\xi(\tilde{\gamma}(x), x_1), \]

\[ (x_1 - x_2)Y^e_\xi(\tilde{\beta}(x), x_1)Y^e_\xi(\tilde{\gamma}(x), x_2) \]

\[ = (x_1 - x_2) \left( \frac{qe^{x_1-x_2} - 1}{e^{x_1-x_2} - q} \right) Y^e_\xi(\tilde{\gamma}(x), x_2)Y^e_\xi(\tilde{\beta}(x), x_1). \]

Furthermore, due to the last relation, we have

\[ x_0^{-1} \delta \left( \frac{x_1 - x_2}{x_0} \right) Y^e_\xi(\tilde{\beta}(x), x_1)Y^e_\xi(\tilde{\gamma}(x), x_2) \]

\[ - x_0^{-1} \delta \left( \frac{x_2 - x_1}{-x_0} \right) \left( \frac{qe^{x_1-x_2} - 1}{e^{x_1-x_2} - q} \right) Y^e_\xi(\tilde{\gamma}(x), x_2)Y^e_\xi(\tilde{\beta}(x), x_1) \]

\[ = x_2^{-1} \delta \left( \frac{x_1 - x_0}{x_2} \right) Y^e_\xi(Y^e_\xi(\tilde{\beta}(x), x_0)\tilde{\gamma}(x), x_2). \quad (6.10) \]

Combining (6.3) with Lemma 6.7 we get \( \tilde{\beta}(x)_n^e\tilde{\gamma}(x) = 0 \) for \( n \geq 1 \) and we have

\[ \tilde{\beta}(x)_0^e\tilde{\gamma}(x) = \text{Res}_{x_1} \left( x^{-1} \tilde{\beta}(x_1) \tilde{\gamma}(x) - x^{-1} \left( \frac{qx_1/x - 1}{x_1/x - q} \right) \tilde{\gamma}(x)\tilde{\beta}(x_1) \right) \]

\[ = \text{Res}_{x_1} x^{-1} \delta \left( \frac{x_1}{x} \right) \]

\[ = 1. \]

Then applying Res\(_{x_0}\) to (6.10) we obtain

\[ Y^e_\xi(\tilde{\beta}(x), x_1)Y^e_\xi(\tilde{\gamma}(x), x_2) - \left( \frac{qe^{x_1-x_2} - 1}{e^{x_1-x_2} - q} \right) Y^e_\xi(\tilde{\gamma}(x), x_2)Y^e_\xi(\tilde{\beta}(x), x_1) \]

\[ = x_2^{-1} \delta \left( \frac{x_1}{x_2} \right) Y^e_\xi(\tilde{\beta}(x)_0^e\tilde{\gamma}(x), x_2) \]

\[ = x_2^{-1} \delta \left( \frac{x_1}{x_2} \right). \]

Now, we see that with \( \tilde{\beta}(z) \) and \( \tilde{\gamma}(z) \) acting as \( Y^e_\xi(\tilde{\beta}(x), z) \) and \( Y^e_\xi(\tilde{\gamma}(x), z) \), respectively, \( V_W \) becomes an \( A_q^{rat}(\beta\gamma) \)-module. Since \( V_W \) as a nonlocal vertex algebra is generated by \( \tilde{\beta}(x) \) and \( \tilde{\gamma}(x) \), it follows that \( V_W \) as an \( A_q^{rat}(\beta\gamma) \)-module is generated by \( 1_W \). We have \( \tilde{\beta}(x)_n 1_W = 0 \) and \( \tilde{\gamma}(x)_n 1_W = 0 \) for \( n \geq 0 \). Therefore \( (V_W, 1_W) \) is a vacuum \( A_q^{rat}(\beta\gamma) \)-module.

Next, we construct a universal vacuum \( A_q^{rat}(\beta\gamma) \)-module, following [Li4] (Section 4). Let \( T_+ \) denote the subspace of \( T \), linearly spanned by the vectors

\[ a_{n_1}^{(1)} \cdots a_{n_r}^{(r)} \]
for \( r \geq 1 \), \( a^{(i)} \in \{ \beta, \gamma \} \), \( n_i \in \mathbb{Z} \) with \( n_1 + \cdots + n_r \geq 0 \). As \( TT_+ \) is a left ideal of \( T \), \( T/TT_+ \) is naturally a \( T \)-module, which we denote by \( \tilde{V}(\beta\gamma) \). From definition, for any \( w \in \tilde{V}(\beta\gamma) \), \( T(\eta)w = 0 \) for \( n \) sufficiently large. Then we let \( J_q(\beta\gamma) \) be the submodule of \( \tilde{V}(\beta\gamma) \), generated by the following vectors:

\[
\begin{align*}
\beta_m\beta_n w &= - \sum_{k,i \geq 0} \binom{k}{i} (-1)^i \mu_k \beta_{n+i} \beta_{m+k-i} w, \\
\gamma_m\gamma_n w &= - \sum_{k,i \geq 0} \binom{k}{i} (-1)^i \mu_k \gamma_{n+i} \gamma_{m+k-i} w, \\
\beta_m\gamma_n w &= - \sum_{k,i \geq 0} \binom{k}{i} (-1)^i \mu_k \gamma_{n+i} \beta_{m+k-i} w - \delta_{m+n+1,0} w
\end{align*}
\]

for \( m, n \in \mathbb{Z} \), \( w \in \tilde{V}(\beta\gamma) \) (recall Remark 6.4). Set

\[
V_q(\beta\gamma) = \tilde{V}(\beta\gamma)/J_q(\beta\gamma)
\]

and set \( 1 = 1 + J_q(\beta\gamma) \in V_q(\beta\gamma) \). From the construction, \( (V_q(\beta\gamma), 1) \) is naturally a vacuum \( A_q^{rat}(\beta\gamma) \)-module. Set

\[
\hat{\beta} = \hat{\beta}_1 1, \quad \hat{\gamma} = \hat{\gamma}_1 1 \in V_q(\beta\gamma).
\]

**Theorem 6.9.** Let \( q \) be any nonzero complex number. The vacuum \( A_q^{rat}(\beta\gamma) \)-module \( (V_q(\beta\gamma), 1) \) is universal in the obvious sense and there exists a weak quantum vertex algebra structure on \( V_q(\beta\gamma) \), which is uniquely determined by the condition that \( 1 \) is the vacuum vector and

\[
Y(\hat{\beta}, x) = \hat{\beta}(x), \quad Y(\hat{\gamma}, x) = \hat{\gamma}(x).
\]

Furthermore, \( V_q(\beta\gamma) \) is an irreducible quantum vertex algebra. On the other hand, for every restricted \( A_q^{rat}(\beta\gamma) \)-module \( W \), there exists a \( V_q(\beta\gamma) \)-module structure \( Y_W \) on \( W \), which is uniquely determined by the condition that

\[
Y_W(\hat{\beta}, x) = \hat{\beta}(x), \quad Y_W(\hat{\gamma}, x) = \hat{\gamma}(x).
\]

**Proof.** Let \( H \) be a vector space with \( \{ \beta, \gamma \} \) as a basis and define a linear map \( S(x) : H \otimes H \to H \otimes H \otimes \mathbb{C}[[x]] \) by

\[
S(x)(\beta \otimes \beta) = (\beta \otimes \beta)f(x), \quad S(x)(\gamma \otimes \gamma) = (\gamma \otimes \gamma)f(x),
\]

\[
S(x)(\beta \otimes \gamma) = (\beta \otimes \gamma)f(x), \quad S(x)(\gamma \otimes \beta) = (\gamma \otimes \beta)g(x),
\]

where \( f(x) \) and \( g(x) \) are the formal Taylor series expansions at 0 of \( \frac{e^{-x} - q}{q e^{-x} - 1} \) and \( \frac{q e^{-x} - 1}{e^{-x} - q} \), respectively. Then \( V_q(\beta\gamma) \) is simply the \((H, S)\)-module \( V(H, S) \) in [Li4]. In view of this, universality follows from Proposition 4.3 of [Li4]. The assertion
on weak quantum vertex algebra structure and the assertion on module structure follow immediately from Proposition 4.2 of [KL] (cf. [Li3], Propositions 2.18 and 4.3). As for the irreducibility assertion we shall use a result of [KL]. A special case of Theorem 4.9 of [KL] states that for any $p(x) \in \mathbb{C}[[x]]$ with $p(0) = 1$, there exists a (nonzero) weak quantum vertex algebra $V$ which is generated by two linearly independent vectors $u$ and $v$ such that

\[
Y(u, x_1)Y(u, x_2) = -(p(x_1 - x_2)/p(x_2 - x_1))Y(u, x_2)Y(u, x_1),
\]
\[
Y(v, x_1)Y(v, x_2) = -(p(x_1 - x_2)/p(x_2 - x_1))Y(v, x_2)Y(v, x_1),
\]
\[
Y(u, x_1)Y(v, x_2) + (p(x_2 - x_1)/p(x_1 - x_2))Y(v, x_2)Y(u, x_1) = x_2^{-1}\delta\left(\frac{x_1}{x_2}\right),
\]

and furthermore, all such $V$ are irreducible quantum vertex algebras and isomorphic to each other. For $q = 1$, it is known that $V_q(\beta\gamma)$ is a simple (equivalently irreducible) vertex algebra. Assume $q \neq 1$. Set

\[
p(x) = (e^{-x/2} - qe^{x/2})/(1 - q) \in \mathbb{C}[[x]].
\]

We have $p(0) = 1$ and

\[
p(-x)/p(x) = \frac{e^x - q}{1 - qe^x} = -\frac{e^x - q}{qe^x - 1}.
\]

Then by Theorem 4.9 of [KL], $V_q(\beta\gamma)$ is an irreducible quantum vertex algebra. □

**Remark 6.10.** As $V_q(\beta\gamma)$ is an irreducible quantum vertex algebra, $V_q(\beta\gamma)$ is an irreducible $A_q^{rat}(\beta\gamma)$-module. It follows that every nonzero vacuum $A_q^{rat}(\beta\gamma)$-module is irreducible and isomorphic to $V_q(\beta\gamma)$.

The following is a connection between quantum vertex algebra $V_q(\beta\gamma)$ and restricted $A_q^{trig}(\beta\gamma)$-modules:

**Theorem 6.11.** Let $q$ be a nonzero complex number and let $W$ be a restricted $A_q^{trig}(\beta\gamma)$-module. Then there exists a $\phi$-coordinated $V_q(\beta\gamma)$-module structure $Y_W$ on $W$ with $\phi(x, z) = xe^z$, which is uniquely determined by the condition that

\[
Y_W(\hat{\beta}, x) = \hat{\beta}(x), \quad Y_W(\hat{\gamma}, x) = \hat{\gamma}(x).
\]

On the other hand, for any $\phi$-coordinated $V_q(\beta\gamma)$-module $(W, Y_W)$, $W$ is a restricted $A_q^{trig}(\beta\gamma)$-module with $\hat{\beta}(x)$ and $\hat{\gamma}(x)$ acting as $Y_W(\hat{\beta}, x)$ and $Y_W(\hat{\gamma}, x)$, respectively.

**Proof.** It is similar to the proof of Theorem 6.9. First, by Proposition 6.8 the weak quantum vertex algebra $V_W$ with $1_W$ is a vacuum $A_q^{rat}(\beta\gamma)$-module. As the vacuum $A_q^{rat}(\beta\gamma)$-module $V_q(\beta\gamma)$ is universal, there exists an $A_q^{rat}(\beta\gamma)$-module homomorphism $\theta$ from $V_q(\beta\gamma)$ to $V_W$, sending $1$ to $1_W$. It follows that $\theta$ is a homomorphism of weak quantum vertex algebras. As $W$ is a canonical $\phi$-coordinated $V_W$-module, $W$ is a $\phi$-coordinated $V_q(\beta\gamma)$-module.

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On the other hand, assume that \((W, Y_W)\) is a \(\phi\)-coordinated \(V_q(\beta\gamma)\)-module. In view of Propositions 5.6 and 5.9, we have

\[
Y_W(\hat{\beta}, x_1)Y_W(\hat{\beta}, x_2) = \iota_{x_2, x_1}\left(\frac{x_1 - qx_2}{qx_1 - x_2}\right) Y_W(\hat{\beta}, x_2)Y_W(\hat{\beta}, x_1),
\]

\[
Y_W(\hat{\gamma}, x_1)Y_W(\hat{\gamma}, x_2) = \iota_{x_2, x_1}\left(\frac{x_1 - qx_2}{qx_1 - x_2}\right) Y_W(\hat{\gamma}, x_2)Y_W(\hat{\gamma}, x_1),
\]

\[
Y_W(\hat{\beta}, x_1)Y_W(\hat{\beta}, x_2) - \iota_{x_2, x_1}\left(\frac{qx_1 - x_2}{x_1 - qx_2}\right) Y_W(\hat{\gamma}, x_2)Y_W(\hat{\beta}, x_1)
= \text{Res}_{x_0} x_1^{-1} \delta \left(\frac{x_2 e^{x_0}}{x_1}\right) x_2 e^{x_0} Y_W(Y(\hat{\beta}, x_0)\hat{\gamma}, x_2)
= \delta \left(\frac{x_2}{x_1}\right),
\]

where we are using the relation

\[\hat{\beta}_n \hat{\gamma} = \delta_{n,0} 1 \quad \text{for } n \geq 0.\]

Thus, \(W\) is a restricted \(A^\text{rig}_q(\beta\gamma)\)-module with \(\hat{\beta}(z)\) and \(\hat{\gamma}(x)\) acting as \(Y_W(\hat{\beta}, x)\) and \(Y_W(\hat{\gamma}, x)\), respectively. \(\square\)

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