Topology and existence of 3D anisotropic filamentary kinematic dynamos

by

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Abstract

Curvature and helicity topological bounds for the magnetic energy of the streamlines magnetic structures of a kinematic dynamo flow are computed. The existence of the filament dynamos are determined by solving the magnetohydrodynamic equations for 3D flows and the solution is used to determine these bounds. It is shown that in the limit of zero resistivity filamentary dynamos always exists in the isotropic case, however when one takes into account that the Frenet frame does not depend only of the filament length parameter $s$, (anisotropic case) the existence of the filamentary dynamo structure depends on the curvature in the case of screwed dynamos. Frenet curvature is associated with forld and torsion to twist which allows us to have a stretch, twist, and fold method to build fast filament dynamos. Arnold theorem for the helicity bounds of energy of a divergence-free vector field is satisfied for these streamlines and the constant which depends on the size of the compact domain $MCR^2$, where the vector field is defined is determined in terms of the dimensions of the constant cross-section filament. It is shown that when the Arnold theorem is violated by the filament amplification of the magnetic field structure appears, the magnetic field decays in space. PACS numbers: 02.40.Hw:Riemannian geometries
I Introduction

The topology and geometry of hydrodynamical and MHD dynamos have been mainly developed by Arnold and Khesin [1] and by Childress and Gilbert [2] using non chaotic flows and the twist, stretch and fold technique developed by Moffatt [3] to investigate fast dynamos. Dynamos in chaotic flows have been developed lately by Thiffeault and Boozer [4]. On the other hand twisted filamentary magnetic structures have been important in plasma and solar physics [5] in the investigation of electric carrying-current loops. Though we know [2] that planar dynamos do not exist they amplify the magnetic fields. It is also important to recognize when we have a dynamo or anti-dynamo from the magnetic profile. Since we only have planar dynamos in very specific circumstances, when we have folds of filaments for example, we here consider three dimensional 3D dynamos where the twist can be associated with torsion of the filament and curvature to fold processes. In this paper we consider the topology and geometrical bound obtained from the solution of magnetic MHD equations with a dissipative term in the absence of electric potential. The scalar MHD equations to be solved are obtained by expressing the MHD equations in the Frenet frame anisotropic basis where the basis depend upon not only the position of the frame along the filament but also of the normal and binormal direction of the frame as well as the time. The filament are considered to be planar with constant curvature. The existence of anti-dynamos where the magnetic field is amplifies, as in the case of galactic magnetic fields, depends upon the relation between the right (on the left) handness of curvature of the filament and the normal direction to the filament which is assumed to move along the binormal direction such as some vortex filaments. Actually this is not the first time Frenet curvature is used to investigate dynamos, in 2001 Schekochihin et al [6] used the statistical Frenet curvature in the small-scale magnetic fields in kinematic dynamos. This paper is organized as follows: In section 2 we compute the filamentary solution of the magnetic filament. In section 3 we compute the energy bounds from helicity and curvature and examine the relation with the existence of the filamentary dynamos. The conclusions are presented in section 4.
II MHD scalar equations for kinematical dynamos

Let us now start by considering the MHD field equations

\[ \nabla \cdot \vec{B} = 0 \]  
\[ \frac{\partial}{\partial t} \vec{B} - \nabla \times [\vec{u} \times \vec{B}] - \epsilon \nabla^2 \vec{B} = 0 \]  

where \( \vec{u} \) is a solenoidal field while \( \epsilon \) is the diffusion coefficient. Equation (II.2) represents the induction equation. The magnetic field \( \vec{B} \) is chosen to lie along the filament and is defined by the expression \( \vec{B} = B(s, n)\vec{t} \) and \( \vec{u} = u\vec{b} \), chosen by the similarity with vortex filaments, is the speed of the flow. The remaining coordinate \( n \) is orthogonal to the filament all along its extension, and the arc length \( s \) measures distances along the the filament itself. The vectors \( \vec{t} \) and \( \vec{n} \) along with binormal vector \( \vec{b} \) together form the Frenet frame which obeys the Frenet-Serret equations

\[ \vec{t}' = \kappa \vec{n} \]  
\[ \vec{n}' = -\kappa \vec{t} + \tau \vec{b} \]  
\[ \vec{b}' = -\tau \vec{n} \]  

the dash represents the ordinary derivation with respect to coordinate \( s \), and \( \kappa(s, t) \) is the curvature of the curve where \( \kappa = R^{-1} \). Here \( \tau \) represents the Frenet torsion. We follow the assumption that the Frenet frame \([7]\) may depend on other degrees of freedom such as that the gradient operator becomes

\[ \nabla = \vec{t} \frac{\partial}{\partial s} + \vec{n} \frac{\partial}{\partial n} + \vec{b} \frac{\partial}{\partial b} \]  

The other equations for the other legs of the Frenet frame are

\[ \frac{\partial}{\partial n} \vec{t} = \theta_{ns} \vec{n} + [\Omega_b + \tau] \vec{b} \]  
\[ \frac{\partial}{\partial n} \vec{n} = -\theta_{ns} \vec{t} - (\text{div}\vec{b}) \vec{b} \]  
\[ \frac{\partial}{\partial n} \vec{b} = -[\Omega_b + \tau] \vec{t} - (\text{div}\vec{b}) \vec{n} \]  
\[ \frac{\partial}{\partial b} \vec{t} = \theta_{bt} \vec{b} - [\Omega_n + \tau] \vec{n} \]
\[ \frac{\partial}{\partial b} \vec{n} = \left[ \Omega_n + \tau \right] \vec{t} - \kappa + (\text{div} \vec{n}) \vec{b} \] (II.11)

\[ \frac{\partial}{\partial b} \vec{b} = -\theta_{bs} \vec{t} - [\kappa + (\text{div} \vec{n})] \vec{n} \] (II.12)

Another set of equations which we shall need here is the time derivative of the Frenet frame given by

\[ \dot{\vec{t}} = [\kappa' \vec{b} - \kappa \tau \vec{n}] \] (II.13)

\[ \dot{\vec{n}} = \kappa \tau \vec{t} \] (II.14)

\[ \dot{\vec{b}} = -\kappa' \vec{t} \] (II.15)

A long and straightforward computation, specially due to the computation of \( \nabla^2 A \), where the vector potential \( \vec{A} = A(t, s, n) \) in principle. Substituting these equations for the dynamics of the Frenet frame leads to the scalar MHD expressions

\[ \partial_t A = -\partial_s \phi + \left[ \partial^2_n A - A(\theta_{ns}^2 - \kappa_0^2) \right] \] (II.16)

\[ -\kappa \tau A = -uB + \epsilon \left[ 2 \partial_n A + (\Omega_s + \tau) \theta_{ns} A \right] \] (II.17)

\[ -\theta_{bs} A = \epsilon \left[ 2 \partial_n A \Omega_s + \Omega^2 A \right] \] (II.18)

where \( \kappa_0 \) is the Frenet curvature of the streamlines. These equations have already been simplified by using the relations

\[ \nabla \times \vec{A} = \vec{B} \] (II.19)

which yields the following differential scalar equations

\[ B = -A[\Omega_b + \tau] \] (II.20)

\[ \partial_n A + \kappa A = 0 \] (II.21)

\[ A(\Omega_n + \tau) = 0 \] (II.22)

Where the \( \Omega' \)’s represent the abnormalities of the streamlines of the flow. Here

\[ \theta_{ns} = \vec{n} \cdot \frac{\partial}{\partial n} \vec{t} \] (II.23)
When the $\Omega_s$ vanishes we note the geodesic streamlines are obtained. As we shall see below here we are not consider geodesic flows dynamos. By considering planar flows where torsion vanishes and the gauge condition

$$\nabla \vec{A} + \frac{\partial \phi}{\partial t} = 0$$  \hspace{1cm} (II.24)

This equation can be expressed as

$$\partial_s A + [\theta_{ns} + \theta_{bs}]A = 0$$  \hspace{1cm} (II.25)

Now by considering that A does not depend on the coordinate s this expression reduces to

$$[\theta_{ns} + \theta_{bs}]A = 0$$  \hspace{1cm} (II.26)

which reduces to $\theta_{ns} = -\theta_{bs}$. By making use of this expression and the assumption that $\phi = 0$ one simplifies the MHD scalar equations to

$$\partial_t A = \left[\partial^2_n A - A(\theta_{ns}^2 - \kappa_0^2)\right]$$  \hspace{1cm} (II.27)

$$-\kappa_0 \tau_0 A = -uA[\Omega_b + \tau_0] + \epsilon[2\kappa_0 + (\Omega_s + \tau_0)\theta_{ns}]A$$  \hspace{1cm} (II.28)

Where we have used the hypothesis of helical or screwed dynamos where torsion and curvature coincides and are constants and equal to $\tau_0$ and $\kappa_0$. Simple algebraic manipulation of these equations, in the limit of zero resistivity, or $\epsilon = 0$ reduce them to

$$\kappa_0^2 + u\kappa_0 + \Omega_b u = 0$$  \hspace{1cm} (II.29)

which is a second order algebraic equation, and

$$\partial_t A = \left[2\kappa_0^2 - \theta_{ns}^2\right]A$$  \hspace{1cm} (II.30)

$$\partial_n A = \kappa_0 A$$  \hspace{1cm} (II.31)

The discriminant of the algebraic equation is assumed to vanish or $\Delta = u^2 - 4\Omega_b u = 0$ which yields the solution $u = -4\Omega_b$. A simple solution of the system can now be obtained by separation of variables $A(t, n) = H(n)T(t)$, substitution of this product in the equations yields

$$A = A_0 e^{[\lambda - \kappa_0 n]}$$  \hspace{1cm} (II.32)
which by the relation between the fields B and A yields

$$B = A_0 (\Omega_b + \tau_0) e^{[\lambda - \kappa_0 n]}$$  \hspace{1cm} (II.33)

where $\lambda := [2\kappa_0^2 - \theta_{ns}^2]$. Both results shows that the filament magnetic field can only be maintained in time when $\lambda > 0$ or $2\kappa_0^2 > \theta_{ns}^2$ otherwise the magnetic field decay very fast in time and the filament is actually an antidynamo. Note that when the filament is isotropic in the sense the frame only depends upon the arc length s then the filaments are always dynamos since then $\theta_{ns} = 0$.

III  Topology bounds the energy from magnetic helicity

The magnetic energy of a divergence vector field on a compact planar domain M reads

$$E_B = \frac{1}{8\pi} \int B^2 dV$$  \hspace{1cm} (III.34)

substitution of the formulas obtained in the last section yields

$$E_B = \frac{a^2}{8} \int A_0^2 [\Omega_b + \tau_0^2] ds$$  \hspace{1cm} (III.35)

where we consider that the magnetic filaments leads constant cross-section area as $S = \pi a^2$. In terms of the vector potential component $A_n$ expression (III.35) reduces to

$$E_B = \frac{La^2}{8} [B^2]$$  \hspace{1cm} (III.36)

where $L = \int ds$ is the length of the filament. Now the Arnold theorem states that a divergence-free vector field $\alpha$ defined a compact manifold $M$, the helicity have an upper limit exactly given by the modulus of the magnetic helity

$$H = \int \bar{\alpha} \cdot \nabla \times \bar{\alpha} dV$$  \hspace{1cm} (III.37)

where the topological bound is given by $[?]

$$E_B > C|H|$$  \hspace{1cm} (III.38)
where C is a positive constant which depends upon the size and form of the compact manifold. Let us now apply the Arnold’s theorem to our solution and analyze the implications for the anti-dynamo problem. First of all we must compute the magnetic helicity in the example given here. This yields

$$|H| = \pi a^2 L[\Omega_b + \tau_0]A^2 e^{[\lambda t - \kappa_0]}$$

(III.39)

Thus to examine the comparison with the Arnold’s theorem we have

$$C = \frac{E_B}{|H|} = \frac{\pi}{8}[\Omega_b + \tau_0]$$

(III.40)

which is the Arnold’s constant C which clearly fulfill the geometrical requirements of Arnold’s theorem. One must notice that when the dynamo conditions discussed in the previous section is fulfilled the Arnolds theorem is also obeyed, however when dynamos existence is not possible Arnold’s theorem seems to be violated and an lower bound for the energy is not obtained anymore.

**IV Conclusions**

While the existence of planar MHD solutions does not warrant the existence of dynamos (with the possible exception of the presence of fold of filaments) it is enough to warrant the existence of the amplification of magnetic fields. Unfortunately a simple way, though not the only, of getting fold of the filaments is by introducing Frenet torsion in the filaments but this would violate the condition of planarity. In conclusion, the investigation of kinematical anti-dynamos and dynamos filamentary MHD in generalised Frenet frame shows that is possible to test the Arnold’s theorem against the dynamo conditions on Frenet curvature and topology comparing it with the magnetic helicity of streamlines. We pretend in near future to investigate this theorem for a more general class of fluids where the topological numbers of twist and writhe can be computed for other more complex filamentary geometries.

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