c=1 Liouville Theory Perturbed by the Black-Hole Mass Operator

TOHRU EGUCHI

Department of Physics, Faculty of Science
University of Tokyo
Tokyo, Japan 113

ABSTRACT

We discuss the properties of the Liouville theory coupled to the $c = 1$ matter when perturbed by an operator, the screening operator of the $SL(2; R)$ current algebra, which is supposed to generate the mass of the two-dimensional black hole. Mimicking the standard KPZ scaling theory of the Liouville system perturbed by the cosmological constant operator, we develop a scaling theory of correlation functions as functions of the mass of the black hole. Contrary to the case of KPZ, the present theory does not have the $c = 1$ barrier and seems somewhat insensitive to the details of the matter content of the theory; the string susceptibility equals 1 independent of the matter central charge. It turns out that our scaling exponents agree with those of the deformed matrix model recently proposed by Jevicki and Yoneya.
In this article we would like to discuss the behavior of the Liouville theory coupled to the $c = 1$ matter when it is perturbed by an operator $V$ which generates a mass for the two-dimensional black hole. The black-hole mass operator is essentially the screening operator of the $SL(2; R)$ current algebra [1] and has appeared in the study of the two-dimensional black hole based on the $SL(2; R)/U(1)$ gauged WZW model [2]. If one uses the operator correspondence between the $c = 1$ Liouville and $SL(2; R)/U(1)$ coset theories [3], the operator $V$ is expressed as

$$V = (\partial X - i \sqrt{\frac{k'}{k}} \partial \phi) \exp(-\sqrt{\frac{2}{k'}} \phi)$$

(1)

where $\phi$ and $X$ are the Liouville and matter field, respectively and the $k = 9/4$ is the level of $SL(2; R)$ current algebra ($k' = k - 2 = 1/4$). Up to a total derivative the operator $V$ coincides with the $j = 1, m = 0$ member of the negative (anti-Seiberg) discrete state of the $c = 1$ Liouville theory, $W_{1,0}^- = \partial X \exp(-2\sqrt{2} \phi)$ (we use the same notations as in ref.[4]). It is easy to see that the action of the Liouville theory perturbed by the black-hole mass term

$$S_M = \frac{1}{8\pi} \int d^2 x \sqrt{\hat{g}} \{ g^{ab} (\partial_a \phi \partial_b \phi + \partial_a X \partial_b X) - Q \hat{R}^{(2)} \hat{g} \}
- \sqrt{\frac{k'}{2}} M g^{ab} (\partial_a X - i \sqrt{\frac{k'}{k}} \partial_a \phi) (\partial_b X - i \sqrt{\frac{k'}{k}} \partial_b \phi) \exp(-\sqrt{\frac{2}{k'}} \phi) \}, \; Q = \sqrt{\frac{2}{k'}}$$

(2)

has the familiar form of the two-dimensional black hole [5,2]

$$S = \frac{k}{4\pi} \int d^2 x \sqrt{g} g^{ab} (\partial_a r \partial_b r + \tanh^2 r \partial_a \theta \partial_b \theta) - \frac{1}{4\pi} \int d^2 x \sqrt{g} \hat{R}^{(2)} \log \cosh r$$

(3)

after a change of variables

$$\phi = \sqrt{2k'} \log \cosh r + \phi^*, \; X = \sqrt{2k} (\theta - i \log \tanh r)$$

(4)

in the leading order of $1/k$. In eq.(2) $M$ denotes the mass of the black hole and is
related to the value of the dilaton field $\Phi = \sqrt{\frac{2}{k'}}\phi$ at the horizon $r = 0$ \cite{2},

$$M = \sqrt{\frac{2}{k'}} \exp(\Phi(r = 0)) = \sqrt{\frac{2}{k'}} \exp\left(\sqrt{\frac{2}{k'}}\phi^*\right). \quad (5)$$

It turns out that the mass-perturbed Liouville theory eq.(2) has properties which are very different from those of the Liouville system perturbed by the cosmological constant operator. The action of the latter theory is given by

$$S_M = \frac{1}{8\pi} \int d^2x \sqrt{g} \left\{ \hat{g}^{ab} \left( \partial_a \phi \partial_b \phi + \partial_a X \partial_b X \right) - 2i\alpha_0 \hat{R}^{(2)} X - Q \hat{R}^{(2)} \phi \right. + \left. \mu W_{0,0}^- \bar{W}_{0,0}^- \right\}, \quad Q = \sqrt{\frac{25 - c}{3}}. \quad (6)$$

In (6) the Liouville field is coupled to the matter with a central charge $c = 1 - 12\alpha_0^2$ and the cosmological constant operator is given by

$$W_{0,0}^- = W_{0,0}^+ = \exp \alpha \phi, \quad \alpha = \frac{-\sqrt{25 - c} + \sqrt{1 - c}}{\sqrt{12}}. \quad (7)$$

As is well-known, the two-dimensional gravity coupled to matter fields (6) exists only for $c \leq 1$: the exponent $\alpha$ becomes complex for $c > 1$ and the theory does not make sense above the $c = 1$ barrier.

On the other hand, our theory (2) does not seem to have the barrier at $c = 1$ and exists for all values of $c < 25$. In the case of general values of the central charge eq.(2) is replaced by

$$S_M = \frac{1}{8\pi} \int d^2x \sqrt{\hat{g}} \left\{ \hat{g}^{ab} \left( \partial_a \phi \partial_b \phi + \partial_a X \partial_b X \right) - 2i\alpha_0 \hat{R}^{(2)} X - Q \hat{R}^{(2)} \phi \right. - \left. \sqrt{\frac{k'}{2}} M \hat{g}^{ab} \left( \partial_a X - i \sqrt{\frac{k'}{k}} \partial_a \phi \right) (\partial_b X - i \sqrt{\frac{k'}{k}} \partial_b \phi) \exp \left( - \sqrt{\frac{2}{k'}} \phi^* \right) \right\}, \quad Q = \sqrt{\frac{2}{k'}} \quad (8)$$

where $k = 2(28 - c)/(25 - c), k' = k - 2$ and $c = 1 - 12\alpha_0^2$. As we see in eq.(8) the exponent of our perturbation operator is equal to (minus) the background charge $Q$ for all values of $c$ and is well-defined up to $c < 25$. 

3
We can in fact mimick the KPZ scaling theory [6] of the two-dimensional gravity (6) and develop a similar scaling analysis of our system (8). As one can check easily, critical exponents of the theory eq.(8) are all well-defined for \( c < 25 \) and do not show pathologies at \( c > 1 \).

Let us first consider the partition function of (8)

\[
Z(M) = \int \mathcal{D}\phi \mathcal{D}X \exp(-S_M(X, \phi))
\]  

(9)

(we suppress contributions from the ghost fields) and assume its power-behavior in the parameter \( M \)

\[
Z(M) \approx M^{2-\gamma_M}.
\]  

(10)

Eq.(10) defines the string susceptibility \( \gamma_M \) of our system. In order to compute \( \gamma_M \) we use the heuristic method of derivation of the KPZ scaling [7]: one shifts the Liouville field by a constant \( \phi \to \phi + \rho \) in eq.(9) and uses the Gauss-Bonnet formula

\[
\frac{1}{4\pi} \int d^2x \sqrt{\hat{g}} \hat{R}^{(2)} = 2 - 2g
\]  

(\( g \) is the genus of the world-sheet) to obtain,

\[
Z(M) = e^{Q\rho(2-2g)/2} Z(Me^{-Q\rho})
\]  

(11)

We put \( g = 0 \) hereafter. Then the string susceptibility of our model equals

\[
\gamma_M = 1
\]  

(12)

and is independent of the matter central charge.

Let us recall that in case of the two-dimensional gravity

\[
Z(\mu) = \int \mathcal{D}\phi \mathcal{D}X \exp(-S_\mu(X, \phi))
\]  

\[
\approx \mu^{2-\gamma_\mu}
\]  

(13)

the string susceptibility is given by

\[
\gamma_\mu = -\frac{1}{p}
\]  

(14)

for the Liouville theory coupled to the \((p, p + 1)\) unitary minimal matter with \( c = 1 - 6/p(p + 1) \). \( \gamma_\mu \) of (14) is negative for \( c < 1 \) and vanishes at \( c = 1 \).
By comparing (12) and (14) we see that our system and two-dimensional gravity have very different characteristics. (At $c = 1$ there exists a logarithmic scaling violation and $Z(\mu)$ behaves as $Z(\mu) \approx \mu^2 \log \mu$. We also expect a logarithmic scaling violation at $c = 1$ in the mass-perturbed theory and $Z(M) \approx M \log M$. Logarithmic terms are not detected by the simple method of shifting the Liouville field).

Let us next turn to the discussion on correlation functions and their scaling relations. We first consider the case $c < 1$ and recall the KPZ scaling exponents for the gravitationally-dressed primary fields of the $(p, p + 1)$ minimal unitary series [6],

$$
\langle \Phi_{r,s}\rangle_\mu \equiv Z(\mu)^{-1} \int D\phi DX \int d^2x \sqrt{g} \Phi_{r,s}(x) \exp(\beta_{r,s}\phi(x)) \exp(-S_\mu(X, \phi)) \\
\approx \mu^{-1+\Delta_{r,s}}, \\
\Delta_{r,s} = 1 - \frac{\beta_{r,s}}{\alpha} = \frac{r(p + 1) - sp - 1}{2p}, \quad 1 \leq s \leq r \leq p - 1.
$$

(15)

On the other hand, our theory predicts

$$
\langle \Phi_{r,s}\rangle_M \equiv Z(M)^{-1} \int D\phi DX \int d^2x \sqrt{g} \Phi_{r,s}(x) \exp(\beta_{r,s}\phi(x)) \exp(-S_M(X, \phi)) \\
\approx M^{-1+\Delta_{r,s}}, \\
\Delta_{r,s} = 1 + \frac{\beta_{r,s}}{Q} = \frac{1}{2} + \frac{r(p + 1) - sp}{2(2p + 1)}, \quad 1 \leq s \leq r \leq p - 1.
$$

(16)

Thus the cosmological constant operator ($r = s = 1$) now has a scaling exponent $\Delta_{1,1} = (p + 1)/(2p + 1)$.

Let us next turn to the case of $c = 1$. BRST invariant observables of the $c = 1$ Liouville theory are well-known [8,9,4,10]. There exist tachyon states given by

$$
T_p^+(z) = \exp(i\sqrt{2}pX(z)) \exp(\sqrt{2}(-1 + |p|)\phi(z)), \\
T_p^-(z) = \exp(i\sqrt{2}pX(z)) \exp(\sqrt{2}(-1 - |p|)\phi(z))
$$

(17)

where the Euclidean momentum $p$ takes arbitrary real values. There also exists discrete states which occur at special values of the momenta. $W_\infty$ currents, for
instance, are given by

\[ W_{j,j}^\pm (z) = T_j^\pm (z) , \quad j = 0, 1/2, 1, \cdots \]

\[ W_{j,m}^\pm (z) = \left( \oint \exp (-i\sqrt{2}X(w))dw \right)^{j-m} W_{j,j}^\pm (z) , \quad -j \leq m \leq j . \tag{18} \]

Let us first consider the tachyon 2-point function,

\[ G(M) = \langle T_+^- T_+^p \rangle_M \equiv \int \mathcal{D} \phi \mathcal{D} X T_+^- T_+^p \exp (-S_M(X, \phi)) \tag{19} \]

where

\[ T_+^p = \int d^2 x \sqrt{g(x)} T_+^p (x) . \tag{20} \]

Again by shifting the Liouville field \( \phi \to \phi + \rho \) we obtain

\[ G(M) = e^{2\sqrt{2}(1-|p|)\rho} e^{2\sqrt{2}\rho} G(M e^{-2\sqrt{2}\rho}) \tag{21} \]

Hence

\[ G(M) = c_M(p) M^{|p|} . \tag{22} \]

\( c_M(p) \) is some function depending only on the momentum \( p \). \( \tag{22} \) is to be contrasted with the case of the \( c = 1 \) two-dimensional gravity [11,12,13]

\[ G(\mu) = \langle T_+^- T_+^\mu \rangle_\mu \equiv \int \mathcal{D} \phi \mathcal{D} X T_+^- T_+^\mu \exp (-S_\mu(X, \phi)) \]

\[ = c_\mu(p) \mu^2 |p| . \tag{23} \]

The normalization factor \( c_\mu(p) \) in the right-hand-side of (23) is known to have (double) poles at \( |p| = \text{half-integers} \) [12]

\[ c_\mu(p) \sim (\Gamma(1-2|p|))^2 \tag{24} \]

In the present case it is somewhat difficult to compute \( c_M(p) \) in a closed form due to the presence of the derivative terms \( \partial \phi, \partial X \) in the perturbing operator. It is,
however, possible to check that \( c_M(p) \) has a double pole at \(|p| = 1\) but it is regular at \(|p| = 1/2\). It appears quite likely that \( c_M(p) \) has the following pole structure

\[
c_M(p) \sim (\Gamma(1 - |p|))^2.
\]  

(25)

Appearance of poles in the 2-point function at special values of the momenta indicates some resonance phenomenon. We would like to point out that the behavior (25) suggests the existence of the ringing or quasi-normal modes in two-dimensional black hole which are quite analoguous to those known in the four-dimensional black holes. In order to make further analysis let us now switch to the target-space description of the tachyon field and consider its propagation equation in the background of the two-dimensional black hole

\[
\nabla^2 T - 2\nabla \Phi \nabla T + 2T = 0.
\]  

(26)

Under a suitable choice of coordinates eq.(26) is reduced to a hypergeometric equation and one can easily construct its solutions [2,14,15,16]. One considers a solution of (26) which is regular at the horizon and propagates towards the center of the black hole and decomposes it into a sum of ougoing and incoming waves at asymptotic null-infinities. After simple calculations one finds

\[
T_p = \frac{1}{\pi} \Gamma(1 + |p|) \Gamma(-|p|) e^{i \sqrt{2} pt} e^{-\sqrt{2}(1+|p|)r} + \frac{\Gamma(1 + |p|) \Gamma(|p|) e^{i \sqrt{2} pt} e^{-\sqrt{2}(1-|p|)r}}{\Gamma(1/2 + |p|)^2}.
\]  

(27)

The first (second) term in the right-hand-side of eq.(27) represents an outgoing (incoming) wave at infinity. If we renormalize the tachyon wave-function by the factor \( \Gamma(1 - |p|) \) as suggested by eq.(25), the coefficient of the incoming wave of (27) vanishes at (Euclidean momoenta) \(|p|\)=positive integers. Eq.(27) is then interpreted as describing the decay of a resonance into waves which either fall into the black hole or escape to asymptotic infinity. This phenomenon is quite analogous to the quasi-normal modes or ringing modes known in the four-dimensional black hole
theory (see for instance, [17]). Quasi-normal modes are defined by the expansion coefficients of a solution of the wave equation propagating towards the interior of the black hole at the horizon into a sum of waves at asymptotic infinity as in eq.(27),

$$T(\omega) = a(\omega)e^{i\omega t-i\omega r} + b(\omega)e^{i\omega t+i\omega r}. \quad (28)$$

The zero of the coefficient function $b(\omega)$ in the upper-half complex $\omega$ plane gives the characteristic frequency of a decaying resonant state. It is known that in a large class of perturbations and black hole geometries there exist an infinite number of quasi-normal modes [17]. The peculiarity of our two-dimensional case is that the quasi-normal frequencies occur all at pure imaginary values. It will be very interesting to check if other discrete states \{\text{W}^\pm_{j,m}\} besides the discrete tachyons \{\text{W}^+_{j,j}, j = 1, 2, \cdots\} are interpreted as ringing modes of the two-dimensional black hole.

It is well-known that the two-dimensional gravity coupled to matter fields (6) has an alternative, discrete formulation by means of the large-$N$ matrix models. In particular the $c = 1$ case may be described by the matrix quantum mechanics

$$H = \frac{1}{2} Tr\left(\frac{d\Phi(t)}{dt}\right)^2 - \frac{1}{2} Tr\Phi(t)^2. \quad (29)$$

where $\Phi(t)$ is an $N \times N$ hermitian matrix and the theory (29) is solved by converting it into that of free fermions. We wonder if there exists a similar discrete formulation for our mass-perturbed $c = 1$ Liouville system.

Recently a new type of matrix model, the deformed matrix model, has been introduced by Jevicki and Yoneya [18] in order to discuss black holes within the framework of the matrix model. The deformed model is defined by the Hamiltonian with an inverse-squared potential

$$H = \frac{1}{2} Tr\left(\frac{d\Phi(t)}{dt}\right)^2 - \frac{1}{2} Tr\Phi(t)^2 + \frac{M}{N^2} Tr\Phi(t)^{-2}, \quad (30)$$

where the parameter $M$ is to identified as the mass of the black hole. Contrary to the treatment of the model (29) where the system approaches criticality as the fermi
energy reaches its critical value, $\mu_{cr} - \mu_F = \mu \to 0$, one puts $\mu = 0$ in the deformed model (30) and drives the system to a critical point by taking the strength of the repulsive potential go to zero, $M \to 0$. Unfortunately, it is not straightforward to interpret the model (30) as describing the geometry of the black hole. One has to introduce a certain assumption on the relation between the field variables of the model and the tachyon wave functions in the black hole background. This relation (somewhat modified from the formulas proposed in [19,20]) does not follow from the model itself.

It turns out, however, the deformed model predicts the same scaling exponents as ours and appears closely related to our continuum discussions. First of all the free-energy of the model has been computed [18] and behaves as $Z(M) \equiv M \log M$. Thus the exponent $\gamma$ equals 1 up to a logarithmic correction. Also the characteristic features of our tachyon correlation functions (22), (25) agree with those of the deformed model (for further discussions on the deformed model see, [21]).

In ref.[4] Witten suggested a characterization of the integral perturbations of the $c = 1$ Liouville theory as the deformation of the quadratic relation $a_1a_2 - a_3a_4 = 0$ into a form

$$a_1a_2 - a_3a_4 = \sum_n \varepsilon_n (a_1a_2)^n ,$$

where $a_1 = x\bar{x}, a_2 = y\bar{y}, a_3 = x\bar{y}$ and $a_4 = y\bar{x}$ and $x, y$ are the generators of the chiral ground ring. Correspondence to the matrix model is given by $a_1 = p + x, a_2 = p - x$ where $x$ is the matrix eigenvalue and $p$ is its conjugate momentum. In the case of the perturbation by the cosmological constant operator, it is known [22] that the $n = 0$ term contributes in the right-hand-side of (31) and the quadratic relation is deformed to $p^2 - x^2 = \mu$ which is exactly the equation for the fermi surface of the matrix quantum mechanics. This result may be understood as the consequence of the Liouville-momentum conservation: the ground ring elements $x, y$ have the form $(bc, \partial \phi, \partial X) \exp(\pm i \frac{1}{\sqrt{2}} X) \exp(\frac{1}{\sqrt{2}} \phi)$ ($b, c$ are the ghost fields) and hence the $a_1a_2$ carries a Liouville-momentum $\sqrt{2}$. On the other hand, the cosmological constant operator carries a momentum $-\sqrt{2}$. Then the momentum
of the operator $a_1a_2$ is cancelled after perturbation and we obtain the $n = 0$ term in the right-hand-side of (31).

If we apply the same analysis in the case of the mass perturbation, we obtain the $n = -1$ term in (31): the mass operator carries a momentum $-2\sqrt{2}$ which converts the momentum of $a_1a_2$ to $-\sqrt{2}$ under perturbation. Thus eq.(31) is expected to behave as

$$p^2 - x^2 = \frac{M}{p^2 - x^2}.$$  \hspace{1cm} (32)

If we ignore the $p^2$ piece in the denominator of the right-hand-side, we obtain the Hamiltonian of the deformed model. Thus there is a good chance that the deformed matrix model is equivalent to the mass-perturbed Liouville theory.

As we have remarked before, the black-hole mass term coincides with the discrete state operator $W_{1,0}^- W_{1,0}^- \bar{W}_{1,0}^- \bar{W}_{1,0}^-$ up to a total derivative. Since the cosmological constant operator $W_{0,0}^- W_{0,0}^- \bar{W}_{0,0}^- \bar{W}_{0,0}^-$ has also a similar structure, we may consider a general class of perturbations of the Liouville theory due to negative discrete states $W_{j,0}^- W_{j,0}^- \bar{W}_{j,0}^- \bar{W}_{j,0}^-$, $j = 1, 2, \cdots$. There exists some indication that the operator $W_{j,0}^- W_{j,0}^- \bar{W}_{j,0}^- \bar{W}_{j,0}^-$ corresponds to a potential $Tr\Phi^{-2j}$ in the matrix model: if we again use the Liouville-momentum analysis, we find that the $n = -j$ term is generated in the right-hand-side of (31) under the perturbation by $W_{j,0}^- W_{j,0}^- \bar{W}_{j,0}^- \bar{W}_{j,0}^-$. The $n = -j$ term corresponds to the potential $Tr\Phi^{-2j}$. It will be extremely interesting to see if we can establish a definite relation between the perturbed Liouville theories and the deformed matrix models.

Materials of this paper have been presented at the workshop ”Quantum Aspects of the Black Hole” held at ITP, Santa Barbara, June 21-26. I would like to thank participants of the meeting for their discussions. In particular I am grateful to Prof. T. Yoneya for detailed discussions on the deformed matrix model. I am also grateful to Prof. C. Vafa for explaining a somewhat different view on the two-dimensional black hole. Research of T. Eguchi is partly supported by the Grant-in-Aid for Scientific Research on Priority Area ”Infinite Analysis”.

10
REFERENCES

1. M. Bershadsky and D. Kutasov, *Phys. Lett.* **B266** (1991) 345.

2. E. Witten, *Phys. Rev.* **D44** (1991) 314.

3. T. Eguchi, H. Kanno and S.K. Yang, *Phys. Lett.* **B298** (1993) 73.

4. E. Witten, *Nucl. Phys.* **B373** (1992) 187.

5. G. Mandal, A. Sengupta and S. Wadia, *Mod. Phys. Lett.* **A6** (1991) 1685; I. Bars and D. Nemeschansky, *Nucl. Phys.* **B348** (1991) 89; S. Elizur, A. Forge and E. Rabinovici, *Nucl. Phys.* **B359** (1991) 581.

6. V. Knizhnik, A. Polyakov and A. Zamolodchikov, *Mod. Phys. Lett.* **A3** (1988) 819.

7. F. David, *Mod. Phys. Lett.* **A3** (1988) 1651; J. Distler and H. Kawai, *Nucl. Phys.* **B321** (1989) 509.

8. B. Lian and G. Zuckerman, *Phys. Lett.* **B266** (1991) 21.

9. P. Bouwknegt, J. McCarthy and K. Pilch, *Comm. Math. Phys.* **145** (1992) 541.

10. I. Klebanov and A. Polyakov, *Mod. Phys. Lett.* **A6** (1991) 3237.

11. I. Kostov, *Phys. Lett.* **B215** (1988) 499.

12. D. Gross, I. Klebanov and M. Newman, *Nucl. Phys.* **B350** (1991) 621.

13. Y. Kitazawa, *Phys. Lett.* **B265** (1991) 262; VI. S. Dotsenko, *Mod. Phys. Lett.* **A6** (1991) 3601; N. Sakai and Y. Tanii, "Correlation Functions of c=1 Matter coupled to Two-Dimensional Gravity" , Tokyo Institute of Technology preprint TIT/HEP-168, March 1991.

14. R. Dijkgraaf, H. Verlinde and E. Verlinde, *Nucl. Phys.* **B371** (1992) 269.
15. S. Chaudhuri and D. Minic, "On the Black Hole Background of Two-Dimensional String Theory", Univ. of Texas preprint, UTTG-31-92, December 1992; S.K. Rama, "Tachyon Back Reaction on d=2 Black Hole", Trinity College Dublin preprint, TCD-3-93, March 1993.

16. N. Marcus and Y. Oz, "The Spectrum of the 2D Black Hole or Does the 2D Black Hole has Tachyonic or W Hair?", Tel-Aviv University preprint, TAUP-2046-93, May 1993.

17. S. Chandrasekhar, *The Mathematical Theory of Black Holes*, Clarendon Press, Oxford, England 1983.

18. A. Jevicki and T. Yoneya, "A Deformed Matrix Model and the Black Hole Background in Two-Dimensional String Theory", Brown preprint BROWN-HEP-904, May 1993.

19. E. Martinec and S. Shatashvili, *Nucl. Phys.* **B368** (1992) 338.

20. S. Das, *Mod. Phys. Lett.* **A8** (1993) 69; A. Dhar, G. Mandal and S. Wadia, *Mod. Phys. Lett.* **A7** (1992) 3703.

21. Ulf Danielsson, "A Matrix-Model Black Hole", CERN preprint, CERN-TH 6916/93, June 1993; K. Demeterfi and J. Rodrigues, "States and Quantum Effects in the Collective Field Theory of a Deformed Matrix Model", Princeton preprint PUPT-1407, June 1993.

22. S. Kachru, *Mod. Phys. Lett.* **A7** (1992) 1419; J. Barbon, "Perturbing the Ground Ring of 2-D String Theory", CERN preprint, CERN-TH 6379/92, January 1992.