The Post-Newtonian Hill Problem

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Abstract. A relativistic approach of the Hill Problem is presented by using an approximate binary system metric obtained from the first post-Newtonian expansion (1PN). We employ Poincaré maps and Lyapunov exponents to study the stability of bounded orbits of the system for different mass arrangements, and compare with the classical problem based on Newtonian dynamics. We find that for larger masses the system become totally stable, a striking behaviour that is not predicted by the Newtonian dynamics.

1. Introduction
The Hill Problem was formulated in the nineteenth century by Hill [1, 2], in order to study the Moon-Earth-Sun system. The Hill Problem is still applied in solar system models where bodies in nearly circular orbits are perturbed by other far away massive bodies, and it is very useful in the study of stellar dynamics. In many systems the Hill Problem can be taken as a first approximation and can easily accommodate necessary modifications (see for instance Heggie [3]). The interaction of a Keplerian binary system with a normally incident circularly polarized gravitational wave can be represented by a Hill system, as shown by Chicone et al. [4]. The classical Hill Problem was proved to be non-integrable by Meletlidou et al. [5], and it is chaotic, as shown by Simó and Stuchi [6].

2. The Post-Newtonian Hill Problem
Before we study the post-Newtonian Hill problem we must situate the restricted three-body problem in the context of general relativity. We consider the spacetime under influence of the binary system composed by BH1 and BH2. The motion of the third, small mass body, is given by a timelike geodesic of this spacetime.

The exact metric for a binary system is not known. Due to the lack of symmetries of a such mass distribution the search for an exact solution in this case it does look unworkable. There are several approximate metrics that can represent different regions of the spacetime with limited accuracy. The approximate metric proposed by Alvi [7] is a composition of several different metrics, each one valid for a different region of the spacetime. The delimiters \( r_{1n} \), \( r_{2n} \) and \( r_{out} \) are given by \( r_{1n} = \sqrt{Gm_1 R / c^2} \), \( r_{2n} = \sqrt{Gm_2 R / c^2} \) and \( r_{out} = \lambda_c / 2\pi \), where \( R \) is the separation...
distance between BH1 and BH2 and \( \lambda_c \) is the characteristic wavelength of gravitational radiation emitted by the binary system.

In this work we consider the motion of the small mass body only in the Region III (the zone between BH1 and BH2). We could have considered as well the Region II (close to BH2 but outside the apparent horizon). The choice of Region III is justifiable as \( r^3_{\mu} \) is very small for typical systems, and the orbits we are considering are located in the Buffer Zone between the two regions. The analysis of the other regions shall be considered in future works. In Region II we have \( r_1 > r^3_{\mu}, r_2 > r^3_{\mu} \) and \( r < r^{\text{out}} \), where \( r_1 = \sqrt{(x-x_1)^2 + (y-y_1)^2 + (z-z_1)^2} \), \( r_2 = \sqrt{(x-x_2)^2 + (y-y_2)^2 + (z-z_2)^2} \) and \( r = \sqrt{x^2 + y^2 + z^2} \) are the distances of the small mass body from BH1, BH2 and the center of mass, respectively. This region can be described using post-Newtonian expansions [8]. We carry out the post-Newtonian expansion of the binary system only up to first order (1PN), so we consider the motion of the two massive bodies BH1 and BH2 in Newtonian orbits (0PN). According to the statements of the Hill Problem, we make the additional assumption that these bodies are in circular orbits around their center of mass with angular velocity \( \omega \), given by \( \omega = \sqrt{G(m_1 + m_2)/R^3} \).

We use this metric to obtain the equations of motion from the geodesic equations. We use units such that \( R = 1, \omega = 1 \), and we change the origin and the scale of the problem with the transformation \( x \rightarrow \mu^{1/3} x + 1, \ y \rightarrow \mu^{1/3} y \), where \( \mu = m_2/(m_1 + m_2) \) is the mass ratio. The 1PN Hill Problem is then obtained by considering \( m_1 \gg m_2 \) such that \( \mu \approx m_2/m_1 \) is small and keeping terms up to the \( \mu^{2/3} \) order of approximation. We have,

\[
\frac{d^2x}{dt^2} = 2 \frac{dy}{dt} + (3 - \frac{1}{r^3}) x + \frac{1}{c^2 r^2} \left[ -\frac{5}{\mu^{1/3}} + \left( -12 + \frac{3}{r} \right) x + 2 \frac{dy}{dt} \right],
\]

\[
\frac{d^2y}{dt^2} = -2 \frac{dx}{dt} + \frac{y}{r^3} + \frac{1}{c^2 r^2} \left( \left( 5 + \frac{3}{r} \right) y - 2 \frac{dy}{dt} \right),
\]

where \( c^* \) is given by \( c/\mu^{1/3} \), in the new units of the problem. These are the Hill equations, in the first post-Newtonian approximation. The terms on the right-hand side of each equation with no dependency on \( c^* \) correspond to the Newtonian Hill problem [1, 9].

3. Stability of Orbits

In this work we consider the variation of the masses such that the mass ratio \( \mu \) is fixed. We use \( r^3_S = 2Gm_1/c^2\mu^{1/3} \) as a parameter. A variation of \( r^3_S \) is associated with the variation of \( m_1 \), and the dynamics of the system is changed. We label the different systems with the corresponding value of \( r^3_S \), as in our previous work [10]. For values less than \( r^3_S = 10^{-4} \) the 1PN expansion is still valid, but for larger values more terms in our series expansions ought to be considered.

3.1. Poincaré Sections

We study surfaces of section (Poincaré sections) by evaluating the orbits for different values of the Jacobi constant and registering the crossings of the hypersurface \( y = 0 \) with \( \dot{y} > 0 \).

The results for \( C_f = -2.17 \) (typical value for a bounded system) are shown on the Figure 1 for several values of \( r^3_S \). The first figure, in upper left, shows a section for \( r^3_S = 5.10^{-12} \). The other figures, in upper right, lower left and lower right, show the sections obtained for \( r^3_S = 5.10^{-6} \), \( r^3_S = 10^{-5} \) and \( r^3_S = 5.10^{-5} \), respectively. We see that, as \( r^3_S \) increases, less Kolmogorov-Arnold-Moser (KAM) tori are destroyed, indicating an increasing stability. In particular, the system with \( r^3_S = 5.10^{-3} \) seems to be completely stable. Besides, the dependence on initial conditions, the main characteristic of chaotic systems, still must be analyzed by appropriate tools, like Lyapunov exponents. This analysis can decide if a system is more unstable than other [11].
Figure 1. Poincaré sections for different values of the parameter $r^*_S$. From left to right and top to bottom: $r^*_S = 5 \cdot 10^{-12}$, $r^*_S = 5 \cdot 10^{-6}$, $r^*_S = 10^{-5}$ and $r^*_S = 5 \cdot 10^{-5}$. The rate of destruction of KAM tori decreases and the system becomes more stable.

3.2. Lyapunov Exponents

To analyze quantitatively the orbits stability we study the Lyapunov exponents for the systems described above. We get the largest $\lambda$ by applying the technique suggested by Benettin et al. [12] and the algorithm of Wolf et al. [13].

Figure 2. Lyapunov exponents for different values of $r^*_S$ for the 1PN Hill Problem.

The Lyapunov exponents are not absolute, but dependent on the choice of the time scale. We recall that we have fixed the time scale for each system by requiring $\omega = 1$, so the different
Lyapunov exponents must be normalized to a fixed time unit in order to be comparable. Each coefficient was computed until convergence is reached. To achieve this precision the system of equations was integrated for at least one hundred thousand periods. The calculations are performed with the aid of the Burlisch-Stoer method with step control, that works well for non-stiff systems, and the error due to the integration is proportional to the tolerance imposed ($10^{-10}$). For the Newtonian system we have $\lambda = 0.141$ that is independent of the mass of the bodies.

The Lyapunov exponents obtained for the relativistic systems are shown in Figure 2. We can see in this figure, that the value of the Lyapunov exponents increases until it reaches a maximum, approximately at $r_S^* = 8.10^{-6}$ and begins to decrease, reaching approximately zero for values of $r_S^*$ larger than $5.10^{-5}$. These results confirm the analysis of the Poincaré sections shown in the last section.

4. Conclusions

The dynamics of the relativistic system is also different from the dynamics of systems based on pseudo-Newtonian potentials, in particular the Paczyński-Wiita potential [10, 14]. The Paczyński-Wiita potential is obtained from a slight modification of the Newtonian potential by changing $1/r$ for $1/(r - r_S)$. It correctly reproduces the locations of innermost stable circular orbit and the marginally bound orbit, and it is widely used in numerical simulations of black hole accretion. However, it does not correspond to the actual post-Newtonian approximation. The singularity present on the Paczyński-Wiita potential is different from the general relativistic system, so each system present a different dynamics when the masses are increased. Namely the relativistic system becomes more unstable in a rate greater than in the case of the pseudo-Newtonian system. Moreover, the pseudo-Newtonian the systems becomes more unstable as $r_S^*$ increases. On the other hand, the relativistic system achieves a maximum and then starts to decrease, until it reaches a almost stable state.

The behaviour of the relativistic system must be analysed more carefully. The decrease in the stability can be due to the fact that we carried the post-Newtonian expansion only up to first order. By considering more terms we can confirm if our results indicate the true behaviour of the relativistic system. However, this is a non trivial task, as these expansions need long calculations to be performed.

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