Limitations to the superposition principle: Superselection rules in non-relativistic quantum mechanics

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Abstract

The superposition principle is a very basic ingredient of quantum theory. What may come as a surprise to many students and even to some practitioners of the quantum craft, is that superposition has limitations imposed by kinematical or by dynamical requirements of the theory. The discussion of such limitations is the main purpose of this article. For doing this we introduce the central concept of classical superselection variables and of the superselection rules to which they are related. Some of their principal consequences are also discussed. The univalence, mass, and particle number superselection rules, all three resulting from kinematical requirements of non-relativistic quantum mechanics, are next deduced. A brief discussion of dynamically induced superselection rules is next given and they are illustrated with the simple example of the one-dimensional hydrogen atom.
Resumen

El principio de superposición es un ingrediente importantísimo de la mecánica cuántica; por ello, él que esté sujeto a ciertas limitaciones, que le son impuestas por consideraciones cinemáticas o dinámicas, puede ser una sorpresa para muchos estudiantes y aún para algunos practicantes de las artes cuánticas. El propósito de este artículo es la discusión del origen de tales limitaciones. Para ello introducimos la noción fundamental de observable de superselección clásico y de las reglas de superselección a las que aquellos dan siempre lugar. Mencionamos y discutimos las principales consecuencias de la existencia de una regla de superselección. A continuación deducimos las principales reglas de superselección que ocurren en la mecánica cuántica no relativista y que tienen un origen puramente cinemático, a saber, la de univalencia, la asociada con la masa y la de número de partículas. Para terminar, discutimos en forma breve algunas reglas de superselección inducidas por la dinámica e ilustramos todo ello con el ejemplo simple del átomo de hidrógeno unidimensional.

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1. Introduction

Linearity is one of the most basic properties of quantum theory: any set of states of a quantum system, \( |\alpha > \), where \( \alpha \) is just a label for the quantum numbers needed to specify the state, may be linearly superposed to obtain a new state; any combination of the form

\[
|A> = \sum_{\alpha} a_\alpha |\alpha >,
\]

where the \( a_\alpha \) are complex numbers, stands for a new possible state of the system. The physical interpretation of each \( a_\alpha \) is as the probability amplitude for the system to be found in precisely the component state \( |\alpha > \). This requires the linearity of the mathematical structure of the theory, thus, all quantum operators are required to be linear. Linear combinations like (1) are usually referred to as coherent superpositions (Sakurai 1985). A brief discussion of coherent as contrasted to incoherent superpositions is given in section 2.

If the set of states under consideration satisfy a quantum equation of motion (like Dirac’s or Schrödinger’s, for our argument it does not matter which one)

\[
H|\alpha > = i\hbar \frac{\partial}{\partial t}|\alpha >,
\]

the new state (1), a coherent superposition of the original states, also necessarily does:

\[
H|A> = \sum_{\alpha} a_\alpha H|\alpha > = \sum_{\alpha} a_\alpha i\hbar \frac{\partial}{\partial t}|\alpha > = i\hbar \frac{\partial}{\partial t}|A >.
\]

This is the property which explained the puzzling —in the early days of quantum theory— phenomenon of electron diffraction and that put it on the same footing as photon diffraction. The superposition principle is not only basic, it is also responsible, together with the interpretation of the \( a_\alpha \) numbers as probability amplitudes, for some of the strange and counterintuitive aspects of quantum theory, like the so-called Schrödinger’s cat paradox (Sakurai 1985, Sudbery 1986).

An example of the superposition property of quantum systems can be illustrated considering the properties of spin-1/2 particles. The spin state of these particles can be found pointing up \( |+ > \) or pointing down \( |-- > \) along a previously specified \( z \)-direction —specified, for example, by the direction of a given external field. These up and down states are independent from one another and, furthermore, any other spin
states of the particle can be expressed as coherent superpositions in terms of these
two; for example, spin states pointing in the $x$ or the $y$ directions can be respectively
written

$$|x> = \frac{1}{\sqrt{2}} (|-> + |+>) , \quad |y> = \frac{1}{\sqrt{2}} \left( e^{-\frac{i\pi}{4}} |+> + e^{+\frac{i\pi}{4}} |-> \right).$$ (4)

But, despite being fundamental it is easy to realize that the superposition principle
cannot hold unrestricted in every situation. Just think of photons and electrons, then
not mattering what we have just said about linearity and the superposition principle,
no one has ever made sense of any superposition of electronic with photonic states
(Sudbery 1986, Salas-Brito 1990). See the comment on supersymmetry at the end of
section 2.1.

How can we understand this peculiarity within the general structure of quantum
theory? The quantum justification for such fact was found a long time ago (Wick
et al 1952) and involves the curious phenomenon referred to as a superselection rule.
To give a extremely brief description, a superselection rule exist whenever there are
limitations to the superposition principle in quantum theory (Kaempfer 1965, Sudbery
1986), that is, whenever certain superpositions cannot be physically realizable. In fact,
the existence of superselection rules are needed to explain why we can treat certain
observables like the mass or the charge of a particle as parameters rather than as
full-fledged operators.

It is somewhat peculiar that despite the additional insight into the subtleties of
quantum theory that might be offered by superselection rules and of the fundamental
role they have played in explaining puzzling phenomena in atomic, molecular and
quantum field theories, the subject has not yet found a place in many of the excellent
textbooks on quantum mechanics available. Perhaps this lack of attention is due to
the initial belief that superselection rules were only important for quantum field theory
(Streater and Wightman 1964, Lipkin 1973). In this article we want to contribute to
alleviate this situation by first explaining how superselection rules fit in the general
structure of quantum mechanics and then by offering some examples in the context of
non-relativistic quantum mechanics.

2. Superselection rules in quantum theory

2.1 Superposing photons and electrons is forbidden
Let us begin analysing the possibility of coherent superpositions of photon and electron states. Any such superposition could, at least in principle, be written in the form

\[ |\Psi\rangle = a_p |\text{photon}\rangle + a_e |\text{electron}\rangle. \] (5)

where some irrelevant details—irrelevant at least for the present discussion—are left out of the description of the states. Before discussing (5), recall that whereas electrons have a half integer spin, \( s_{\text{electron}} = 1/2 \), a property which makes them fermionic particles, photons have an integer spin, \( s_{\text{photon}} = 1 \), which makes them bosonic particles. Bosons and fermions differ in many ways. Fermions comply with Pauli’s exclusion principle, they cannot ‘stand’ another identical particle in the same state. Bosons, on the contrary, ‘like’ to crowd one over another in the same quantum state (Feynman 1965). Another crucial difference between fermions and bosons is found in their behaviour under rotations. Consider a bosonic state and rotate the system by \( 2\pi \), a rotation which is obviously expected to let the physical content of this and any other state unchanged, the state does indeed not change. On the other hand, a fermionic state has a different behaviour, under that same rotation it changes its phase by \( \pi \), \( i.e. \) the state ends multiplied times \( -1 \). Even so, there is nothing to worry about since the quantum states are determined up to a normalizing factor—or, to describe the situation in other words, the important thing is the modulus of the state since it is related with the actual probability and not just with its amplitude—and thus the final state is also physically indistinguishable from the initial one. However, if we rotate by \( 2\pi \) not a fermionic or a bosonic state separately but the superposition \( |\Psi\rangle \), the rotated state is then

\[ \mathcal{R}(2\pi)|\Psi\rangle = |\Psi'\rangle = a_p |\text{photon}\rangle - a_e |\text{electron}\rangle. \] (6)

Given that \( <\Psi|\Psi> \neq <\Psi'|\Psi'> \), under the rotation the superposition becomes a state essentially different from the original one. This implies that a superposition of the form (5) is devoid of physical meaning unless, of course, \( a_p = 0 \) or \( a_e = 0 \), or, in other words, only if a coherent superposition is not allowed (Lipkin 1973). Thus, to be consistent with the kinematical requirements of rotations, quantum theory is forced to restrict the applicability of the superposition principle and to impose a superposition rule between photons and electrons or, to state the restriction in completely general terms, to impose a superselection rule forbidding superpositions of any fermionic with any bosonic states whatsoever—see Weinberg (1995) for a recent discussion of the
generality and applicability of this restriction. This superselection rule is usually called univalence (Wick et al 1952, Streater and Wightman 1964, Hegerfeld et al 1968).

To avoid possible misunderstandings, it is important to pinpoint that there is a way of making sense of a linear combination like (5) obviously not as a coherent superposition which we have just proved to be impossible but as one of the so-called mixed states or incoherent superpositions. That is, an expression like (5) can have meaning only for describing a statistical ensemble with a fractional population \( a_p \) of photons and a fractional population \( a_e \) of electrons (Landau and Lifshitz 1977, Sakurai 1985, Núnez-Yépez et al 1988, Salas-Brito et al 1990) —as, for example, could be necessary to describe the situation in the interior of a switched-on floodlight.

We should mention also the case of supersymmetric quantum mechanics (Robinet 1997). In supersymmetric systems, for each bosonic state there is an associated fermionic partner required by the formalism. The supersymmetry allows the existence of a sort of coherent superposition of fermionic and bosonic states. That is, we can get bosonic and fermionic states in the same representation multiplet but, as a matter of fact, this behaviour should not be considered a restriction to the univalence superselection rule just discussed. The key to understand the point is the existence of anticommuting Grassmann numbers in the supersymmetric formalism which, under the rotation operator \( R(2\pi) \), change again the sign in Eq (6) and finally recover the plus sign in the rotated system thus behaving as if it were a pure bosonic state (Cooper et al 1995).

2.2 Coherent and incoherent superpositions

Before proceeding any further, it is convenient to review the difference usually made in quantum mechanics between coherent and incoherent superpositions. Take, for example, the spin-1/2 particle states mentioned in the introduction. The most general spin state for one of those particles can be expressed in the form of a coherent superposition

\[
|\phi> = a_+|+> + a_-|->
\]

where the components \( a_+ \) and \( a_- \) are, in general, complex numbers which, if the state is assumed normalized to 1, comply with \( |a_+|^2 + |a_-|^2 = 1 \). The superposition (7) always describes a state with a well-defined spin. States arising as coherent superpositions are sometimes termed pure states. The quantum expectation value of any observable
with respect to the pure state \( |\phi> \) can then be expressed as

\[
< \phi | O | \phi > = |a_+|^2 < |+| > + |a_-|^2 < |-| > + a_+^* a_- < |+| > + a_-^* a_+ < |-| >;
\]  
(8)

the last two terms of equation (8) are called interference terms. These interference terms, basically arising from the information about the relative phase of the states \( |+> \) and \( |-> \) given by the complex numbers \( a_+ \) and \( a_- \), are an unavoidable characteristic of pure states.

However, states of the form (7) are not general enough to describe every possible spin situation. Sometimes the need to describe a so-called mixed spin state, or, as it is also called, an incoherent superposition of spin states, might arise. For example, if the spin of some unfiltered beam of particles is required, a mixed state is necessary to describe the random spin orientations in the beam. It is not possible to do this using coherent superpositions, like in (7), which are only capable of describing particles whose spin is pointing in some definite direction.

To deal with the random spin situation, mixed states of the form

\[
w_+ |+> + w_- |->
\]  
(9)

must be introduced. In the situation described by (9), the numbers \( w_+ \) and \( w_- \) are necessarily real and should be interpreted as just statistical weights describing the fractional population of the state, thence, the usual condition \( w_+ + w_- = 1 \) imposed on them; they should never be interpreted as components in the two-dimensional spin space. Notice thus the complete lack of information on the relative phase between the up and the down states implied by the reality of \( w_+ \) and \( w_- \). In particular this means that interference terms cannot occur if incoherent superpositions are used. In the case of the mixed state (9), the measurement of any property \( O \) is calculated using not the quantum mechanical expectation value but by taking the ensemble average of \( O \) (Sakurai 1985) between the mixed state (9)

\[
[O] \equiv w_+ < |+| > + w_- < |-| > .
\]  
(10)

This expression lacks the interference terms of the quantum expectation value (8) calculated using pure states. We can say that incoherence implies the absence of interference terms. As the next section shows, in systems with superselection rules
there necessarily are states which exist only as mixed, incoherent superpositions and, therefore, that can never interfere with one another.

2.3 Superposition rules in the formalism of quantum theory

In this section we explain how superselection rules are embedded into the structure of quantum theory and discuss some of their consequences.

Let us begin with a definition: If in a quantum system there exist a hermitian operator $G$ which commutes with all the observables $O$, that is, such that

$$[G, O] = 0, \quad \text{for every observable } O,$$

and if $G$ is found not to be a multiple of the identity operator, then, it is said that a superselection rule induced by $G$ is operating in the system. This is in fact closely related to the failure of the superposition principle as is shown in the next subsection.

Since $G$ commutes with every operator, it commutes, in particular, with the Hamiltonian and then, it necessarily corresponds to a conserved quantity. We must distinguish this property of $G$ from the corresponding one of ordinary conserved quantities, like the energy $E$, since physically realizable states exist that are not eigenstates of the Hamiltonian. For example, since we know a superselection rule does not act, nothing prevents us from considering an arbitrary nonstationary state $|\psi\rangle = |n, l, m_s \rangle + \exp(i\delta)|n', l', m'_s \rangle$ (where $n, n', l, l', m_s$ and $m'_s$ are standard hydrogen-like quantum numbers and $\delta$ an arbitrary real number) of an hydrogen atom as physically realizable. On the other hand, when a superselection rule induced by a certain operator $G$ acts on a system, every physically realizable state must be an eigenvalue of $G$ and not every possible superposition can represent a physical state.

The property of commuting with every other operator makes the variable $G$ sharply measurable in every situation being thus rather similar to a $c$-number. It is not surprising then that any such operator is termed by some authors a classical observable of the quantum system. Despite this, we must exercise some care in the use of the term since it is used with a very different meaning in other quantum contexts; to avoid conflicts or misunderstandings, we here have decided to use the term classical superselection variable (CSV for short) for these operators. Thus we can say that every CSV has an associated superselection rule, these two notions are really two manifestations of the same phenomenon.
CSV’s in generic nonrelativistic quantum systems are the mass and the electric charge (Bargman 1954, Foldy 1954); in molecular physics they can be the chirality of a molecule or its tertiary structure (Primas 1981, Pfeifer 1981, 1983, Müller-Herold 1985); in many particle systems, it is the particle number or the total mass (Kaempfffer 1965).

Superselection rules and classical superselection observables can even occur in rather simple systems. For example, in the one-dimensional hydrogen atom with interaction \( V(x) = -e^2/|x| \); the CSV is the one-dimensional Laplace-Runge-Lenz vector, that is, it is just the side of the singularity in the potential in which the particle is confined to move. This was shown by Boya et al (1988) in an interesting and easily readable article. The superselection rule forbidding superpositions among such confined states was shown to exist by Núñez-Yépez et al (1988, 1989), see in particular a contribution to this journal where the problem is discussed from an elementary standpoint (Núñez-Yépez et al 1987).

Let us denote the eigenstates and eigenvalues of \( G \) as

\[
G|g_m; \alpha_m > = g_m|g_m; \alpha_m > , \tag{12}
\]

where \( \alpha_m \) stand for every other quantum number needed to specify the eigenstates. As every operator commutes with \( G \) its eigenstates can always be chosen as eigenstates of the other dynamical variables of the system. Given this, any state \( |S > \) in the Hilbert space of the system can be always expressed as a linear superposition of the eigenstates of \( G \)

\[
|S > = \sum_m a_m|g_m; \alpha_m > ; \tag{13}
\]

thus, the complete Hilbert space of the system \( \mathcal{H} \), is naturally decomposed into a collection of disjoint and mutually orthogonal subspaces \( \mathcal{H}_g \) (this follows since they correspond to different quantum numbers of a Hermitian operator, and, therefore, \( < g_s; \alpha_s|g_m; \alpha_m > = 0 \ (s \neq m) \), each identified by a different eigenvalue of \( G \). The \( \mathcal{H}_g \) subspaces, are called the coherent subspaces or coherent sectors induced by the superselection rule. Recalling the definition of a direct sum between vector spaces, as the existence of a unique expansion like (13) for every element in the original space and where every term in the expansion belongs to only one subspace (Shilov 1977), it is easy to see that the direct sum of all the coherent subspaces
\[ H = H_1 \oplus H_2 \oplus \ldots, \] (14)

with as many terms as different eigenvalues of \( G \) exist, is the complete Hilbert space of the system.

One must be somewhat cautious with the statements (13) and (14), because, although the expansions are allowed in principle, the very existence of the superselection rule forbids any physically meaningful superposition of states in different coherent subspaces. That is, a superselection rule forbids superposing states with different values of the eigenvalue \( g_m \), as is explicitly shown below in subsection 2.4. An additional consequence of the existence of a superselection rule is that not every Hermitian operator, even if related to a supposedly good physical variable, may represent an observable. For this to be true, the operator should not induce transitions between the different coherent sectors of the system. This property is also proved below in subsection 2.5.

2.4 Impossibility of superposing states belonging to different coherent sectors

To show that any superposition of eigenstates of \( G \) with different eigenvalues is not physically admissible, that is, that it cannot be observed in nature, consider first that, on the contrary, the state

\[ |u> = \sum_m u_m |g_m; \alpha_m > \] (15)

is physically reasonable. Since \( G \) commutes with everything, the state (15) should be an eigenstate of every operator in some complete set of commuting observables of the system. Denote this set as \( O_s \), \( s = 1 \ldots, N \), thus

\[ O_s |u> = o_s |u> \] (16)

As a consequence of the superselection rule, the state \( |g_m; \alpha_m > \) is also an eigenstate of \( G \) with eigenvalue \( o_s^m \); then, another way of express state (16), is as

\[ O_s |u> = \sum_m u_m o_s^m |g_m; \alpha_m >, \] (17)

but, all the \( |g_m; \alpha_m > \) being linearly independent from one another, equations (15), (16) and (17) imply that \( o_s^m = o_s \) for every \( m \)-value. Next consider the modified state
\[ |u' > = \sum_m u_m \exp(i\delta_m) |g_m; \alpha_m >, \] (18)

which is just (15) with the phases of each component arbitrarily modified, and apply the operator \( O_s \) to it. In this way the result

\[ O_s |u' > = o_s \sum_m u_m \exp(i\delta_m) |g_m; \alpha_m > = o_s |u' > \] (19)

is easily obtained. On comparing (19) with (16), we can see that both states, \( |u > \) and \( |u' > \), have precisely the same quantum numbers. Therefore, the two states (16) and (19) are completely indistinguishable from one another. No physical process could be able to determine the relative phase between \( |u > \) and \( |u' > \). But they are also completely different vectors in Hilbert space; the only way of avoiding a contradiction is by altogether forbidding superposition of states in different coherent subspaces. This proves the statement and establishes the connection between the existence of operators commuting with everything and the existence of limitations to the superposition principle.

Notice also that, as we have already said, the only way of give physical meaning to superpositions like (15) or (18) is by interpreting them as mixed or impure states, never as coherent superpositions. Thus, in a system with superselection rules, the superposition principle only holds unrestricted within each coherent subspace, superpositions of states from different coherent subspaces can never describe physical states.

2.5 Nonexistence of transitions between different sectors

It is very easy to show the vanishing of any matrix element of an observable \( O \) between states belonging to different coherent sectors \( \mathcal{H}_n \) and \( \mathcal{H}_m, m \neq n \). To accomplish this, keep in mind that the Hermitian operator \( G \) commutes with everything else and that its eigestates can be therefore chosen to have common eigenvalues with any operator, thus

\[ < g_m; \alpha_m | O | g_n; \alpha_n > = o < g_m; \alpha_m | g_n; \alpha_n >; \] (20)

but any states with different quantum numbers (i.e. such that \( n \neq m \)) are necessarily orthogonal, then
\[ \langle g_m; \alpha_m | g_n; \alpha_n \rangle = 0, \quad m \neq m. \] 

This proves that no observable can ever induce transitions between states in different coherent sectors of the Hilbert space. This means, for example, that, according to the univalence superselection rule, it is impossible to produce a single electron from a single photon in any conceivable physical process.

2.6 The most general superselection rules

As it must be clear by now, superselection rules are very important properties of quantum systems; any quantum theory must be formulated taking them into account. The most general superselection rules that, it is believed, must hold in any quantum theory —we are talking here, just in this subsection, of the most general relativistic quantum theories— are associated first with the existence of fermions and bosons, \textit{i.e.} the univalence superselection rule, and then to every absolutely conserved quantum number: there are superselection rules associated to the electric charge, to the barion number, and to the three lepton numbers (Sudbery 1986); this means that all physical states of the basic building blocks of our universe should, according to this point of view, be sharp eigenstates of the operators associated with those properties. But even within the realm of non-relativistic quantum mechanics there are very interesting superselection rules, which, besides, can be obtained by so elementary calculations that they could be part of almost any intermediate course in quantum mechanics. This is exhibited in the next section.

3. Superselection rules in non-relativistic quantum mechanics

The superselection rules acting in non-relativistic quantum mechanics and the CSV associated with them are discussed in this section. We discuss the univalence superselection rule separating the bosonic from the fermionic sectors in the Hilbert space of nonrelativistic quantum mechanics, the superselection rule separating the different mass sectors, which allow the mass to be treated as just a parameter in the Schrödinger equation, and the superselection rule separating states of different particle numbers in Fock space. The electric charge superselection rule also operates in nonrelativistic quantum mechanics but, as its derivation involves field operators and the equations of electrodynamics (Aharonov and Susskind 1967, Strocchi and Wightman 1974), the derivation of this rule is outside of the scope of the article.
In section 2.2, following the arguments of Wick et al (1952), we have already sketched a proof of this superselection rule valid for any rotationally invariant theory. In this section, following the excellent discussion of Müller-Herold (1985), we rederive the univalence superselection rule using arguments closer to the spirit of non-relativistic quantum mechanics. We just have to invoke some consequences of the existence of indistinguishable particles.

Think of a system of any two particles (not necessarily assumed identical at this moment) with equal spin —for the sake of simplicity we describe the argument using just two-particle states but it can be generalized to states of any number of particles (Galindo et al 1962). We can represent the state by the ket $|1, 2>$, where the content of the first slot labels the spatial and spin degrees of freedom of the first particle and the content of the second slot labels the same properties of the second one.

If $P$ stands for the permutation operator for two particle states, we must have

$$P|1, 2> = |2, 1>.$$  \hspace{1cm} (22)

The only eigenvalues of $P$ are $+1$ and $-1$, as it is very easy to prove by applying $P$ twice to any of its eigenstates. The eigenstates of $P$ are the symmetric and antisymmetric states under interchange of particles; these states are called, respectively, bosonic and fermionic states (Feynman 1965). Moreover, every state vector in Hilbert space can be classified as fermionic or bosonic depending only on the spin of the system. This fundamental result, known as the spin-statistics theorem, has a rather difficult proof. The only attempt we know towards simplifying the proof of the spin-statistics theorem is in Feynman’s 1986 Dirac memorial lecture (Feynman and Weinberg 1987).

If the particles being described by (22) happen to be identical in the quantum sense, that is if they are, for example, two electrons or two protons, no property of the system can depend on the order in which their labels appear in (22) (Lipkin 1973). That is, all matrix elements of any observable $O$, between two identical-particle states $|1, 2>$, should be invariant under permutations

$$<1, 2|O|1, 2> = <2, 1|O|2, 1> = <1, 2|P^iOP|1, 2>,$$  \hspace{1cm} (23)

since, being the particles absolutely indistinguishable, no physical property could depend on the order in which we had decided to write their labels. Therefore, as follows
from (23) and the unitarity of the permutation operator (Sakurai 1985), every observable of the system should commute with $P$, i.e. the permutation operator is a CSV and thus induces a superselection rule. Now, consider the projection operators

$$P_\pm = \frac{1}{2} (1 \pm P).$$ \hspace{1cm} (24)

If $|\phi>\$ is an arbitrary state, then $P_+|\phi>\$ is a bosonic and $P_-|\phi>\$ is a fermionic state. These projection operators split the Hilbert space $\mathcal{H}$ in bosonic $\mathcal{H}_+ = P_+ \mathcal{H}$ and fermionic $\mathcal{H}_- = P_- \mathcal{H}$ sectors such that $\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_-$.

If the state vectors $|\phi_\pm>\$ belong to $\mathcal{H}_\pm$ and $O$ is any observable, then its matrix elements between any fermionic and any bosonic states are

$$<\phi_-|O|\phi_+> = <\phi_-|OP|\phi_+>$$
$$= <\phi_-|P^\dagger O|\phi_+>$$
$$= <\phi_-|P|\phi_+>$$
$$= -<\phi_-|O|\phi_+>$$
$$= 0.$$ \hspace{1cm} (25)

since, in the order in which they are used in (25), $P|\phi_+> = +|\phi_+>$, $P$ commutes with any $O$, $P$ is an unitary operator and $P|\phi_-> = -|\phi_->$. Thus, no observable can ever induce transitions from bosonic to fermionic states or viceversa. Furthermore, by applying $P$ to any linear combination of a fermionic with a bosonic state,

$$P(a_+|\phi_+> + a_-|\phi_->) = a_+|\phi_+> - a_-|\phi_->,$$

we get a state linearly independent from the original one. However, as this state ought to indistinguishable from the original one, the only possible conclusion is that a superselection rule is in operation. The operator $P$ is the CSV inducing the univalence superselection rule which, accordingly, has to be regarded as a direct consequence of the existence of indistinguishable particles.

### 3.2 Superselection rule separating different mass sectors

In every elementary application of non-relativistic quantum mechanics, mass is treated as an aprameter. For example, in dealing with the Schrödinger equation, it is always implicitly assumed to be a c-number. Moreover, states superposing different masses are conspicuously absent of every treatment. Mass seems to comply with a superselection
rule. That this statement is true follows from the Galilean invariance of non-relativistic quantum mechanics as was originally noticed by Bargman (1954).

Galilean transformations in quantum mechanics are represented by unitary operators $U_G$ acting upon the states of a system. The change in a state functions when both a displacement of the origin by the constant quantity $r$, and a change between two reference frames moving respect one another at a constant velocity $v$, occur simultaneously; that is, when the Galilean transformation $q' = q + vt + r$ is performed, can be represented by the operator (Kempf 1965)

$$U_G = \exp(i m v \cdot q' - imv^2 t/2).$$  \hspace{1cm} (27)

The action of $U_G$ on a quantum state $|\Psi\rangle$ describing a particle with mass $m$, is

$$|\Psi'\rangle = U_G|\Psi\rangle = \exp \left( i m (v \cdot q' - v^2 t/2) \right) |\Psi\rangle;$$

therefore $<\Psi'|\Psi'> = <\Psi|\Psi>$ and Galilean invariance is manifest in nonrelativistic quantum mechanics. Selecting $r = 0$, specializes (27) to the unitary operator associated to a pure Galileo transformation: $U_V$; whereas selecting $v = 0$, specializes (27) to the operator associated to a pure spatial displacement: $U_r$.

It must be clear that the combined action on $|\Psi\rangle$ of a spatial displacement $U_r$, a pure Galilean transformation $U_V$, the opposite displacement $U_{-r}$ and the inverse Galilean transformation $U_{-v}$, just amounts to an identity transformation. Its net effect, as you may easily check, is again just a change in the phase of the state:

$$U_I|\Psi\rangle = \exp(i m v \cdot r)|\Psi\rangle,$$

where we defined $U_I \equiv U_{-v}U_{-r}U_vU_r$. The important point in (24) is that the phase of the state changes depending on the mass of the particle. The transformation $U_I$ is thus seen to be of trivial consequences as long as we are dealing with definite mass states. This change of phase becomes crucial, however, if we consider states $|\phi_1\rangle$ and $|\phi_2\rangle$, describing particles with different masses $m_1$ and $m_2$. Since if we try any superposition like $|\phi\rangle = a_1|\phi_1\rangle + a_2|\phi_2\rangle$, the new state $|\phi\rangle$ transforms under the identity-equivalent transformation $U_I$ as

$$|\phi'\rangle \equiv U_I|\phi\rangle = a_1 \exp(i m_1 r \cdot v)|m_1\rangle + a_2 \exp(i m_2 r \cdot v)|m_2\rangle.$$  \hspace{1cm} (30)
From (30), you can see that, if \( m_1 \neq m_2 \) and if \( \mathbf{r} \neq 0 \) or \( \mathbf{v} \neq 0 \), the states \( |\phi> \) and \( |\phi'> \) become essentially different. Then, again to avoid inconsistencies, we have to conclude that mass induces a superselection rule in nonrelativistic quantum mechanics. Mass is indeed a CSV in nonrelativistic quantum mechanics, it can always be sharply measured in any situation—but, notice that this is a direct consequence of Galilean invariance, in relativistic quantum theory mass cannot be always sharply measured and there is not a corresponding superselection rule.

The whole Hilbert space associated with nonrelativistic quantum mechanics hence decomposes in a direct sum of sectors corresponding to each possible value the mass can take. This makes nonrelativistic quantum mechanics utterly inadequate for describing states endowed with a mass spectrum or for describing transformations between particles. In ordinary quantum mechanics no operator can exist which produces transitions between states characterized by different masses.

3.3 Particle number superselection rule

Given Bargman’s mass superselection rule, it is rather easy to obtain another, the closely related superselection rule separating states with different particle numbers in Fock space. This superselection rule must be rather intuitive because nobody seems to care about superposing, say, 3-particle states with 11-particle states, it always sounds more reasonable to deal with 14-particle states from the start.

The proof of this rule follows from the following argument (Müller-Herold 1985), given a state \( |n> \) describing \( n \) identical particles, each with the same mass \( m \), this state should transform, under the Galilean transformation \( U(I) \)—which you surely recall, is equivalent to an identity transformation—of section 2.2 as

\[
U(I)|n> = \exp(inm\mathbf{v} \cdot \mathbf{r})|n>.
\]  

From this result and from the fact that the states of \( n \) particles have a total mass \( nm \) different, in general, from the mass \( n'm \) of \( n' \)-particle states, the argument of the previous section unavoidably leads to the conclusion that any superposition of \( n \)-particle with \( n' \)-particles states must describe a mixed state as long as \( n \neq n' \). Therefore, another CSV of nonrelativistic quantum mechanics is the particle number operator \( N \).

4. Dynamically induced superselection rules
In the previous section we have introduced some of the superselection rules acting in non-relativistic quantum theory. The common point of all of them is their kinematical origin. They mainly come about because certain transformation properties must hold for every possible quantum state and to avoid inconsistencies within the theory the superposition of certain states must be avoided. However, not every superselection rule is kinematical in origin there are some that occur when interactions are turned on. Examples of these are the superselection rules induced by the many body couplings in macroscopic bodies. Manifestations of these superselection rules are the CSV’s like the temperature or the chemical potential, that appear as the thermodynamic limit is approached (Primas 1981, Pfeifer 1981, Zureck 1982, Putterman 1983).

4.1 Optical isomers in molecular physics

Another interesting example of dynamically induced superselection rules, is the solution to the so-called paradox of optical isomers proposed in Pfeifer’s dissertation (1980). This superselection rule is basically produced by the coupling of molecular degrees of freedom to the electromagnetic field surrounding the molecule. Under certain conditions this coupling generates the superselection rule that explains, for example, why certain chemical compounds like serine or even a household substance like sugar, can be produced in either right-handed or left-handed forms, but never in a non-chiral pure — in the quantum not the chemical sense— form (Müller-Herold 1985). The only possibility, if we do not want them in their levogirous nor in their dextrogirous forms, is producing them in a mixed quantum state a so-called racemic mixture. The CSV that has been found associated to this superselection rule is chirality (Pfeifer 1980, 1981).

Most superselection rules with a dynamical origin have in common a certain complexity in their causes, since the explanation usually involves systems with an infinite number of degrees of freedom, coming about from either the need of taking the thermodynamic limit as in a macroscopic system, or by the existence of couplings to external fields as in Pfeifer’s rule. It is very interesting, thus, that a rather simple one-dimensional system is capable of showing a dynamically induced superselection rule: the one-dimensional hydrogen atom.

4.2 Superselection rule in the one-dimensional hydrogen atom

The Hamiltonian of the one-dimensional hydrogen atom is
\[ H = \frac{p^2}{2m} - \frac{e^2}{|x|}. \]  

(32)

where \( m \) is the mass and \( e \) electric charge of an electron. The Hamiltonian (32) is parity invariant: the coordinate inversion \( x \rightarrow -x \) leaves (32) unchanged. The normalized eigenstates of the system, in atomic units \( \hbar = m = e = 1 \), and in the coordinate representation, can be written (Núñez-Yépez et al 1987) as right

\[
\psi^+_n(x) = \langle x \mid + n \rangle = \left\{ \begin{array}{ll}
0 & \text{if } x \leq 0, \\
\left( \frac{4}{n^2(n!)^2} \right)^{1/2} (-1)^{n-1} x L^1_{n-1}(2x/n) \exp(-x/n) & \text{if } x > 0,
\end{array} \right.
\]  

(33a)

and left

\[
\psi^-_n(x) = \langle x \mid - n \rangle = \left\{ \begin{array}{ll}
0 & \text{if } x \geq 0, \\
\left( \frac{4}{n^2(n!)^2} \right)^{1/2} (-1)^n x L^1_{n-1}(-2x/n) \exp(+x/n) & \text{if } x < 0,
\end{array} \right.
\]  

(33b)

eigenstates, where the \( L^1_{n-1}(z) \) are generalized Laguerre polynomials and \( n \) is a strictly positive integer—please note that the eigenfunctions in (Núñez-Yépez et al 1987) have certain misprints which have been corrected in (33). The corresponding energy eigenvalues are exactly the same as in the three-dimensional case \( E_n = -1/2n^2 \). As it is rather clear in (33), the attractive potential energy term of the system acts like an impenetrable barrier requiring \( \psi^m_\pm(0) = 0 \) for all \( m \)-values—if this condition is not met then the Hamiltonian fails to be Hermitian. Such properties confine—and explain the names given to the states (33)—the particle to move either to the right or to the left of the origin without any chance of escaping from one region to the other. But, as the one-dimensional hydrogen atom is reflection invariant, we can try to match the eigenfunctions (33) to form parity eigenstates

\[
\phi^+_n(x) = \frac{1}{\sqrt{2}} (\psi^+_n(x) + \psi^-_n(x)),
\]

\[
\phi^-_n(x) = \frac{1}{\sqrt{2}} (\psi^+_n(x) - \psi^-_n(x)),
\]  

(34)

and, in this way, extend the states to the whole \( x \)-axis. The problem is that for the even \( \phi^+_n \) and the odd \( \phi^-_n \) states to be realizable states of the system they must be independent from each other. It is easy to see, however, that the Wronskian determinant \( W(\phi^+_n, \phi^-_n) \) always vanishes and thus that the definite parity states (34) cannot represent independent states of the one-dimensional hydrogen atom. This
means that the relative phase of the states $\psi^n_+$ and $\psi^n_-$ is intrinsically irrelevant in any supposed coherent superposition. Any form of superposing $\psi^n_+$ with $\psi^n_-$ is thus devoid of physical meaning. We must conclude that a superselection rule operates in the system (Núñez-Yépez et al 1988).

The CSV $G$ in the one-dimensional hydrogen atom is the side of the singularity in which the particle is confined to move, or, to be more formal, is the one-dimensional Laplace-Runge-Lenz vector. Notice that $G$ is here also a sort of chiral property as is the case with the CSV in Pfeifer’s superselection rule. Parity, not commuting with $G$, becomes unobservable; we can say that it is spontaneously broken in the one-dimensional hydrogen atom. These properties have been established from a purely classical point of view in (Boya et al 1988) and from a quantum point of view in (Núñez-Yépez et al 1989, Martínez-y-Romero et al 1989a,b). In a way of speaking, the superselection rule in the one-dimensional hydrogen atom is an extremely simplified ‘bare-bones’ version of that of Pfeifer.

We think that this example suffices to exhibit some of the differences between the generally applicable kinematical superselection rules of the previous sections and the superselection rules required by specific features of the interaction, as is the case of the one-dimensional hydrogen atom.

5. Summary

The essential concepts of classical superposition observables and superselection rules have been introduced and its principal consequences have been discussed within the framework of nonrelativistic quantum mechanics. We have exhibited how certain kinematical requirements are capable of constraining the applicability of the superposition principle. Using the basic machinery of nonrelativistic quantum mechanics we have derived three of the basic superselection rules that operate in that theory, namely, univalence, mass and particle number. It is surprising that these very interesting and important ideas are almost never discussed in an intermediate course in quantum mechanics. Our explanation for this curious situation is that, originally, superselection rules were introduced in the context of quantum field theory and they were considered features of infinite dimensional theories. On the other hand, perhaps some theoretical prejudices make people believe that there was impossible to find superselection rules in simple quantum systems. However, we now know that one can find them even in rather large class of one-dimensional quantum systems, those precisely that have
received attention because, it was believed, could furnish counterexamples to the non-degeneracy theorem for bound states in one-dimensional quantum systems; a notable example of this class of systems is the one-dimensional hydrogen atom (Núñez-Yépez et al 1987).

It is also interesting to point out that the so-called environmentally induced superselection rules (Zureck 1982) could perhaps give a way out to one of the main interpretation problems of quantum mechanics, symbolized by the Schrödinger cat paradox. The point is, if it could be established the existence of environmentally induced superselection rules between certain different macroscopic states this would forbid at once the spooky possibility of superposing dead with alive cat states.

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