NONNOETHERIAN LORENTZIAN MANIFOLDS

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ABSTRACT. A nonnoetherian spacetime is a Lorentzian manifold that contains a set of causal curves with no distinct interior points, called ‘pointal curves’. This new geometry recently arose in the study of nonnoetherian coordinate rings in algebraic geometry. We investigate properties of metrics on nonnoetherian spacetimes, and use the Hodge star operator to show that free dust particles have spin $\frac{1}{2}$. We also reproduce the Kochen-Specker $\psi$-epistemic model of spin using the nonnoetherian metric, and show similarities between our model and spin entanglement for Bell states and four-photon entanglement swapping. Finally, we determine the stress-energy tensor of dust on such spacetimes, and find that it is only nonzero at points where dust is created or annihilated.

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1. INTRODUCTION: SPACETIME GEOMETRY FROM INDISTINGUISHABILITY

We introduce two modifications to Lorentzian manifolds that are based on indistinguishability, and investigate the fundamental properties of dust and the tangent bundle on such spacetimes. One of these modifications is that the worldline of a dust particle is replaced by a curve that contains no distinct interior points. In particular, we consider spacetimes with positive dimensional ‘smeared-out’ points. This new type of geometry was recently introduced in the context of algebraic geometry.
to describe affine varieties with nonnoetherian coordinate rings \( [B1] \). Our aim in applying this nonnoetherian structure to general relativity is to model certain quantum phenomena, such as path superposition, entanglement, and spin, using the geometry of spacetime alone.

The modifications we impose are based on indistinguishability, and are thus similar in spirit to the equivalence principle. Indeed, the equivalence principle states that an accelerated frame is locally indistinguishable from a gravitational field, and therefore they are the same. This identification led Einstein to propose that spacetime is a manifold whose curvature is gravitation. The equivalence principle is a particular application of a general principle called the identity of indiscernibles. This general principle, due to Leibniz, asserts that if two things are indistinguishable, then they are the same. We will refer to the identity of indiscernibles and its converse as the \textit{indistinguishable principle}. It is well-known that the indistinguishable principle is fundamental in quantum statistics, as well as a resource for quantum entanglement (e.g., \( [KHLZ, LHLBZ] \)). The principle also plays a fundamental role in Smolin’s real ensemble interpretation\(^1\) \( [S] \), Penrose’s twistor theory, and shape dynamics \( [ABFKO, GGK] \).

We briefly describe the applications of the indistinguishable principle that we introduce in this article. Let \( \tilde{M} \) be an orientable Lorentzian manifold, and consider a collection of free dust particles on \( \tilde{M} \). We assume that the geodesic hypothesis holds; the worldlines of the free dust particles are then causal geodesics on \( \tilde{M} \). We modify \( \tilde{M} \) by incorporating two further applications of the indistinguishable principle, thus obtaining a new spacetime \( M \):

- Since dust particles are non-interacting, they do not detect change, and therefore time does not pass along their worldlines. \( \text{In Sections 2 and 7, we show that this feature yields electric charge, the Kochen-Specker } \psi \text{-epistemic model of spin, and an approximate model of quantum nonlocality. In particular, we find that dust particles have spin } \frac{1}{2}. \text{ In the companion article } [B3], \text{ we show that this feature also gives rise to quark color charge.} \)

- If the precise direction that a dust particle at \( p \in \tilde{M} \) propagates is undetectable, then each of the indistinguishable directions in the tangent space \( T_p \tilde{M} \) are identified as the same direction in \( M \). \( \text{In Section 3 we show that this feature produces path superposition, the wave-like propagation of dust particles, and destructive interference.} \)

We call the manifold \( \tilde{M} \) a \textit{depiction} of the nonnoetherian spacetime \( M \), following the terminology used in algebraic geometry. Based on the geodesic hypothesis and the indistinguishable principle, we propose the following approximate model of quantum superposition:

\(^1\)In the Smolin interpretation, spacetime does not fundamentally exist, in contrast to our model.
Nature takes a set of all indistinguishable stationary paths. Thus, by the indistinguishable principle, Nature takes a single stationary path.

Path superposition then results whenever two distinct paths in $\tilde{M}$ are indistinguishable to all pointal particles, and are thus identified as the same path in $M$. Conversely, state reduction occurs when these paths become distinguishable.

In Sections 3 and 7, we show that our framework is compatible with first quantization, at least to a naive approximation, but is incompatible with second quantization from the outset. However, the worldlines of dust particles with the same orientation and spin are not allowed to intersect by the Pauli exclusion principle [B3], and this imposes a strong constraint on what ‘pointal’ Feynman diagrams are allowed to exist in superposition. This constraint then suggests that the discrepancy between our model and quantum field theory will provide a new regularization scheme that renders quantum field theory finite in the ultraviolet. In this framework, then, we expect that quantum field theory is an effective description of particle physics.

Finally, we note that the original motivation for the development of nonnoetherian geometry arose from string theory: the vacuum moduli spaces of certain non-superconformal quiver gauge theories, called brane tilings, initially had no concrete geometric description, other than to say they were nonnoetherian schemes. Nonnoetherian geometry, in terms of depictions and geometric height, was then developed to provide such a description [B2]. However, in this article we apply nonnoetherian geometry directly to general relativity.

**Notation:** Tensors labeled with upper and lower indices $a, b, \ldots$ represent covector and vector slots respectively in Penrose’s abstract index notation (so $v^a \in V$ and $v_a \in V^*$), and tensors labeled with indices $\mu, \nu, \ldots$ denote components with respect to a coordinate basis. Given a curve $\beta : I \to \tilde{M}$, we often denote its image $\beta(I)$ also by $\beta$. We use natural units $\hbar = c = G = 1$, and the signature $(-, +, +, +)$ unless stated otherwise.

2. **Indistinguishability applied to time: electric charge and spin**

2.1. **Pointal particles.** The indistinguishable principle implies that time passes if and only if something changes.\(^2\) To incorporate this operational notion of time into general relativity, consider an isolated free particle of dust. By the geodesic hypothesis, the particle’s worldline is a causal (i.e., timelike or null) geodesic $\beta$. Since the particle is free, it does not detect any change. Therefore, by the indistinguishable principle, *time does not advance for the particle*. Consequently, each point $p$ along the worldline $\beta$ is the same point, that is, all the points along $\beta$ are identified. We are thus motivated to make the following definition.

\(^2\)Aristotle proposed this notion of time in the fourth century BCE.
Definition 2.1. Let $(\tilde{M}, g)$ be a $3 + 1$-dimensional orientable Lorentzian manifold. Let $B$ be a collection of piecewise causal geodesic curves with no endpoints, called pointal curves. Consider the multivalued map

$$
\pi : \tilde{M} \to M := \{ \{p\} \mid p \in \tilde{M} \setminus \cup_{\beta \in B} \beta \} \cup B,
$$

$$
p \mapsto q \text{ if } q \ni p.
$$

Note that if $p \in \beta \cap \beta'$, then $\pi(p) \supseteq \{\beta, \beta'\}$. We call $M$ a (nonnoetherian) spacetime, and $\tilde{M}$ a depiction of $M$.

We define fundamental particles in terms of the geometry of spacetime alone:

Definition 2.2. A pointal particle is a particle whose worldline is a segment of a pointal curve. A pointal geodesic is a geodesic segment of a pointal curve.

Consequently, the worldline $\beta$ of a pointal particle $\beta(t)$ is continuum of distinct 0-dimensional points in $\tilde{M}$, and a single 1-dimensional point in $M$.

If two pointal particles meet at a point $p \in \tilde{M}$ and interact, then they would each detect change. Thus, by the indistinguishable principle, their worldlines in $M$ would necessarily end at $p$. However, since pointal curves do not have endpoints, the worldlines of the two particles must be segments of the same pointal curve. We therefore conclude

Pointal particles only interact by creation and annihilation in pairs.

Definition 2.3. We call a point in $\tilde{M}$ where two pointal particles are created or annihilated an apex.

Consequently, in order to model more complex interactions, composite bound states of pointal particles are needed. We call such bound states atoms. Atoms may interact with each other by exchanging their constituent pointal particles. We introduce a model of quarks, leptons, and vector bosons using this framework in [B3].

2.2. Pointal metrics. As is well known, for any Riemannian manifold $M$, there is a geodesic ball $B_p$ about each point $p \in M$ for which the exponential map $\exp_p : T_p M \to M$ is injective. We want to define a metric that preserves this property on nonnoetherian spacetimes.

Let $M$ be a nonnoetherian spacetime depicted by a Lorentzian manifold $(\tilde{M}, g)$. Fix $p \in \tilde{M}$, suppose $\pi(p) = \beta$. Further suppose that $\beta$ is locally a timelike geodesic at $p$ with tangent 4-vector $v = v^a \in T_p \tilde{M}$. Then for all $\lambda, \lambda'$ in a sufficiently small interval $(-\varepsilon, \varepsilon)$, we have

$$
\pi(\exp_p(\lambda v)) = \beta = \pi(\exp_p(\lambda' v)).
$$

The notion of a ‘positive-dimensional point’ is made mathematically precise using ideals and over-rings in [B1].
Thus, a ‘pointal metric’ \( h \) at \( p \) should identify \( \lambda v \) and \( \lambda'v \), and therefore satisfy
\[
h(\lambda v, w) = h(\lambda'v, w)
\]
for all \( w \in T_p\tilde{M} \). But then
\[
\frac{\varepsilon}{2}h(v, w) = h\left(\frac{\varepsilon}{2}v, w\right) = h\left(-\frac{\varepsilon}{2}v, w\right) = -\frac{\varepsilon}{2}h(v, w).
\]
Consequently,
\[
h_{ab}v^a = h(v, -) \equiv 0.
\]

**Definition 2.4.** The pointal metric of \( g \) at \( p \in \tilde{M} \) is the symmetric rank-2 tensor
\[
h_{ab} = h : T_p\tilde{M} \otimes T_p\tilde{M} \to \mathbb{R}
\]
defined by the following:

(i) If \( v \in T_p\tilde{M} \) is a timelike tangent 4-vector to some \( \beta \in \pi(p) \), then
\[
h(v, -) = 0.
\]
(ii) If \( w \in T_p\tilde{M} \) satisfies \( g(w, v) = 0 \) for each vector \( v \) tangent to some \( \beta \in \pi(p) \) at \( p \), then
\[
h(w, -) = g(w, -).
\]

The following lemma is clear.

**Lemma 2.5.** Fix \( p \in \tilde{M} \). If \( \pi(p) = \beta \) and \( p \) is a smooth point of \( \beta \), then \( h \) is the projection of \( T_p\tilde{M} \) onto the subspace orthogonal to the tangent 4-vector \( v = v^a \) to \( \beta \) at \( p \),
\[
h_{\alpha b} = v^a_{\perp b} := g_{\alpha b} + v^a v_b.
\]
In particular, if \( v \) is null, then \( h = g \).

More generally, suppose \( \pi(p) = \{\beta_1, \ldots, \beta_\ell\} \), with \( p \) not necessarily a smooth point of any \( \beta_j \). Then
\[
h_{\alpha b} = v^1_{\perp a} v^1_{\perp b} \cdot \cdots \cdot v^\ell_{\perp b},
\]
where \( v_1, \ldots, v_\ell \) and \( v_{\ell+1}, \ldots, v_{2\ell} \) are the incoming and outgoing tangent 4-vectors to \( \beta_1, \ldots, \beta_\ell \) at \( p \).

The pointal metric \( h_{ab} \) defines the projection
\[
h : T_p\tilde{M} \to T_p\tilde{M}, \quad w^a \mapsto h_{ab}w^b.
\]
We will use the induced map,
\[
h : T_p\tilde{M} \to T_p\tilde{M} / \ker h \cong \text{im} h,
\]
to model the state reduction of spin in Section 2.5.

**Definition 2.6.** We call the kernel of \( h \),
\[
\ker h := \{ v^a \in T_p\tilde{M} | h_{ab}v^b = 0 \},
\]
and its image \( \text{im} h \) the *indistinguishable* and *distinguishable* subspaces of \( T_p\tilde{M} \), respectively.
2.3. Electric charge from Hodge duality. We first briefly recall the following definitions to establish notation. Let \((M^n, g)\) be an orientable pseudo-Riemannian manifold. The coordinate expression for the volume form \(\text{vol}^4 \in \Lambda^n(T^*_p M)\) at \(p \in M\) is
\[
\text{vol}^4 = o\sqrt{|g|} dx^1 \wedge \cdots \wedge dx^n,
\]
where \(g = \det(g_{ij}(p))\), and \(o := o(\partial_{x_1}, \ldots, \partial_{x_n}) \in \{\pm 1\}\) is a chosen orientation for \(T_p M\). For \(k\)-forms \(\alpha, \beta \in \Lambda^k(T^*_p M)\), the Hodge dual \(*\beta\) of \(\beta\) is defined by
\[
\alpha \wedge *\beta = \langle \alpha, \beta \rangle \text{vol}^4,
\]
where
\[
\langle \alpha, \beta \rangle := \sum_{i_1 < \cdots < i_k} g^{i_1 j_1} \cdots g^{i_k j_k} \alpha_{i_1 \cdots i_k} \beta_{j_1 \cdots j_k}
\]
is an inner product on \(k\)-forms (e.g., \([Fr]\)). The Hodge star operator \(*\) then acts on the exterior algebra of the cotangent bundle \(T^* M\), taking \(k\)-forms to \(pseudo\) \((n - k)\)-forms. Furthermore, the Hodge dual of a vector \(v \in T_p M\) is given by the interior product \(*v = i_v \text{vol}^4\), where \(i_v : \Lambda^k(T^* M) \to \Lambda^{k-1}(T^* M)\) is defined by
\[
i_v \alpha(w_2, \ldots, w_k) = \alpha(v, w_2, \ldots, w_k).
\]

We now return to our nonnoetherian spacetime \(M\). Fix \(p \in \tilde{M}\), suppose \(\pi(p) = \beta\), where \(\beta\) is a timelike geodesic near \(p\) with tangent 4-vector \(v^a \in T_p M\).

By \([1]\), a timelike tangent 4-vector \(v\) to a pointal curve \(\beta\) vanishes with respect to the pointal metric \(h\):
\[
h_{ab} v^b = 0.
\]
In contrast, the Hodge dual pseudo 3-form \(*v_a = i_{v^a} \text{vol}^4\) of \(v_a = g_{ab} v^b\) is nonvanishing with respect to \(h\), because it is a wedge product of 1-forms that live in the orthogonal complement to \(v_a\) in \(T^*_p M\). We may thus consider \(*v\) in place of the tangent 4-vector \(v\).

However, \(*v\) is not uniquely determined by \(\beta\):
\[(1)\] To specify \(*v\), an orientation \(o \in \{\pm 1\}\) of \(\text{vol}^4\) must be chosen (that is, \(*v\) is a pseudo tensor).
\[(2)\] Since \(\beta\) consists of only a single point in \(M\), the length \(|v| = \sqrt{-v^a v_a}\) of \(v\) is not defined by \(\beta\) in \(M\). Thus, to specify \(*v\), a length \(|v|\) must also be chosen.

Therefore three pieces of data are required in order to specify a timelike pointal particle: its worldline \(\beta\), a choice of orientation \(o_\beta := o\), and a length \(|v|\). We consider the length \(|v|\) and its physical interpretation in Section [5].

Let \(V = \mathbb{R}^3\) denote the spatial subspace of \(T_p \tilde{M}\) in the inertial frame of \(\beta\) at a point \(p \in \beta(I)\), and parallel transport \(V\) along \(\beta\). With a fixed orientation of a basis for \(V\), the specification of \(o_\beta\) is equivalent to the specification of an orientation of \(\beta\) in \(\tilde{M}\), that is, a choice of whether the tangent vector \(v \in T_p \tilde{M}\) of \(\beta\) is future directed or past directed. Indeed, the timelike orientation of \(v\) is not determined by
the corresponding pointal particles’ worldline \( \beta \subset \tilde{M} \), since \( \pi(\beta) \) is a single point of spacetime \( M \).

This choice of time orientation is similar to the Stückelberg-Feynman interpretation of antimatter in quantum field theory, where matter particles travel forward in time and antimatter particles travel backward in time \([5,6]\). Since electrons and positrons are distinguished by their electric charge, we are led to identify \( o \) with the electric charge of the pointal particle \( \beta(t) \). We note, however, that in contrast to the Stückelberg-Feynman interpretation, time does not flow along the worldline of a pointal particle: time does not flow backwards along \( \beta \) just as it does not flow forwards.

**Definition 2.7.** The tangent form to a pointal geodesic \( \beta \) is the Hodge dual pseudo 3-form \( *v = i_v vol^4 \) of the tangent 4-vector \( v \in T_p \tilde{M} \) of \( \beta \) at any point \( p \) along \( \beta \). Since \( *v \) is a pseudo-form, an orientation \( o_\beta \in \{ \pm 1 \} \) of \( T_p \tilde{M} \) must be chosen; we call this orientation the electric charge of \( \beta \).

**2.4. A model of electrons, positrons, and photons.** We model electrons and positrons as a future-directed and past-directed pointal particles, respectively. Two distinct pointal curves may intersect at a point \( p \in \tilde{M} \) if they are distinguishable at \( p \). Thus we may consider two pointal curves \( \beta, \beta' \) that have coincident worldlines \( \beta(I) = \beta'(I) \) but opposite orientations

\[ o_\beta = -o_{\beta'} , \]

that is, opposite electric charges. Such a bound state would have zero electric charge, and so we model the photon by such a configuration. The electron, positron, and photon are shown in Figure 1 for clarity we draw the two pointal curves in a photon as separated.

We call a bound state of pointal particles, such as the photon, an atom. Electrons and positrons are regarded as atoms consisting of single pointal particles. We call a point \( p \in \tilde{M} \) where atoms exchange pointal particles a splitting. The electron-photon splittings are shown in Figure 1. A scattering where an electron emits two photons is shown in Figure 2. In [B3], we present a similar model of atoms for all leptons, quarks, and gauge bosons.

**2.5. Spin from Hodge duality.** Let \( p \in \tilde{M} \) be a trivalent splitting, such as one of the electron-photon splittings shown in Figure 1. Let \( \beta, \beta' \) be (the) pointal curves that intersect at \( p = \beta(0) = \beta'(0) \), such that for sufficiently small \( \varepsilon > 0 \), \( \beta((-,\varepsilon]) \) and \( \beta'([0,\varepsilon)) \) are timelike geodesics.

**Definition 2.8.** If the tangent 4-vectors \( v_1, v_2, v_3 \in T_p \tilde{M} \) to the three legs of the splitting are generic, then we call the one-dimensional subspace

\[ L := \text{im} \, h = \mathbb{R} \cdot * (v_1 \wedge v_2 \wedge v_3) \subset T_p \tilde{M} \]
Figure 1. From left to right: an electron; a positron; a photon; an electron and positron join to form a photon (two vertices, no apices); a photon splits into an electron and positron (two vertices, no apices); an electron absorbs a photon (one vertex, one apex); and an electron emits a photon (one vertex, one apex). In this model, an electron is a future-directed pointal particle, a positron is a past-directed pointal particle, and a photon is a bound state of an electron and a positron.

Figure 2. Two entangled photons emitted from an electron, as in a cascade. The single lines are electron worldlines, and the double lines (which have opposite orientation) are photon worldlines. The two photons are entangled because they share a common pointal curve (drawn in red).

the magnetic line at \( p \). We call the two unit vectors \( s^a, -s^a \) in \( L \) spin vectors, and label them spin up and spin down.

Suppose a spin vector \( s^a \) has been parallel transported along the past worldline of \( \beta \) to \( \beta(0) = p \). At \( p \), we apply the pointal metric \( h_{ab} \) to \( s^a \), which projects \( s^a \) onto the magnetic line \( L \) at \( p \):

\[
h(s) := h^a_b s^b \in L \cong T_p \bar{M} / \ker h.
\]

This projection corresponds precisely to the quantum state reduction of spin at \( p \).

By the indistinguishable principle, the physical state of the spin vector is the image of \( h(s) \) in the isomorphic quotient \( T_p \bar{M} / \ker h \), namely, the class

\[
\bar{h}(s) := h(s) + \ker h.
\]
We would now like to parallel transport $\bar{h}(s)$ along the future worldline of $\beta'$. However, in order to do this, $\bar{h}(s)$ must leave the quotient $T_p\tilde{M}/\ker h$. Let $v_j$ be the tangent 4-vector to $\beta'([0,\varepsilon])$ at $p$. Then $\langle v_j \rangle : = \mathbb{R} v_j \subset \ker h$. Thus there is a projection

$$T_p\tilde{M}/\langle v_j \rangle \longrightarrow T_p\tilde{M}/\ker h.$$  

Choose a section of this projection

(2) $$\sigma : T_p\tilde{M}/\ker h \longrightarrow T_p\tilde{M}/\langle v_j \rangle,$$

and apply it to $\bar{h}(s)$. We may then parallel transport the normalized vector

$$s' = \frac{\sigma \bar{h}(s)}{\sigma \bar{h}(s)}$$

along the future worldline of $\beta'$. The choice of section corresponds precisely to the randomness in the outcome of a quantum measurement.

Let $V$ and $V'$ be the spatial subspaces of $T_p\tilde{M}$ in the inertial frames of the timelike geodesics $\beta((-\varepsilon,0])$ and $\beta'([0,\varepsilon])$. We identify the unit sphere $S^2$ in $V$ and $V'$ as the Bloch sphere, and parallel transport $S^2$ along $\beta((-\varepsilon,0])$ and $\beta'([0,\varepsilon])$. The spin vectors $s$ and $s'$ then live in the Bloch sphere, and thus may be identified with a vector in the spin Hilbert space $H = \mathbb{C}|\uparrow\rangle + \mathbb{C}|\downarrow\rangle$ by the standard correspondence

(3) $$s = (\sin \theta \cos \phi, \sin \theta \sin \phi, \cos \theta) \longleftrightarrow |s\rangle = \cos(\theta/2)|\uparrow\rangle + e^{i\phi} \sin(\theta/2)|\downarrow\rangle,$$

where $0 \leq \theta \leq \pi$ and $0 \leq \phi < 2\pi$.

The model of spin we have presented is a geometric realization of the Kochen-Specker $\psi$-epistemic model [KS]. Indeed, in the Kochen-Specker model, each spin state $\psi \in H$ represents all ontic states $\lambda^\alpha \in \Lambda := S^2$ that lie in the hemisphere ‘above’ the plane orthogonal to $\psi^\alpha$, that is, all (spatial) 3-vectors $\lambda^\alpha$ satisfying

$$\psi^\alpha \lambda^\alpha > 0.$$  

Thus, for example, if a physical ontic state has quantum state $|0\rangle$ (resp. $|1\rangle$), then the ontic state is specified by some vector in the northern hemisphere (resp. southern hemisphere) of $\Lambda$.

The 3-vector $\psi^\alpha$ corresponds to the spin 4-vector $s^a \in T_p\tilde{M}$ in our model by

$$(s^a) = (0, \psi^\alpha),$$

where we have used Fermi normal coordinates of $\beta'([0,\varepsilon])$.

In the Kochen-Specker model, the probability that a preparation procedure $P_\psi$ produces an ontic state $\lambda^\alpha$ represented by $\psi$ is given by

(4) $$p(\lambda^\alpha | P_\psi) = \frac{1}{\pi} H(\psi^\alpha \lambda^\alpha) \psi^\beta \lambda^\beta,$$

Note that $p(\lambda^\alpha | \psi)$ is correctly normalized: it suffices to suppose $$(\lambda^\alpha) = (\sin \theta \cos \varphi, \sin \theta \sin \varphi, \cos \theta)$$ and $$(\psi^\alpha) = (0, 0, 1).$$ Whence $\int_\Lambda d\lambda^\alpha H(\psi^\alpha \lambda^\alpha) \psi^\beta \lambda^\beta = \int_{-\pi}^{\pi} \int_{0}^{\pi/2} \cos \theta \sin \theta \, d\theta d\varphi = \pi$. 


where $H$ is the heaviside function

$$H(x) = \begin{cases} 
1 & \text{if } x > 0 \\
0 & \text{if } x \leq 0
\end{cases}.$$ 

In our model, the choice of $\lambda^\alpha$ corresponds to the choice of section $\sigma$ in (2),

$$(\sigma \tilde{h}(s)) = (0, \lambda^\alpha).$$

This follows since in the inertial frame of $\beta'$, $\nu^\beta_0$ has Fermi normal coordinates $(\nu^\beta_0) = (1, 0, 0, 0)$, and thus $\sigma$ is a choice of some vector $\lambda^\alpha$ in $V'$ for which $s^\alpha \lambda^\alpha > 0$.

Since $p(\lambda^\alpha | P_\psi)$ is independent of the preparation $P$ in this model, we write it as $p(\lambda^\alpha | \psi)$. Furthermore, the probability that the projection measurement $|\phi \rangle \langle \phi|$ of $\lambda^\alpha$ yields $\phi$ is given by

$$p(\phi | \lambda^\alpha) = H(\phi^\alpha \lambda^\alpha).$$

The measurement $|\phi \rangle \langle \phi|$ corresponds to the orthogonal projection of the pointal metric $h^{ab}: T_p \tilde{M} \to L$, since $L$ is a one-dimensional subspace.

It is well known that the conditional probabilities (4) and (5) reproduce quantum statistics:

$$p(\phi | \psi) = \int_{\mathbb{R}} d\lambda^\alpha p(\phi | \lambda^\alpha) p(\lambda^\alpha | \psi) = \int_{\mathbb{R}} d\lambda^\alpha H(\phi^\alpha \lambda^\alpha) \frac{1}{\pi} H(\psi^\beta \lambda^\beta) \psi^\gamma \lambda^\gamma = |\langle \psi | \phi \rangle|^2.$$

Furthermore, this model is $\psi$-epistemic since the product

$$p(\lambda^\alpha | \psi)p(\lambda^\alpha | \phi) = \frac{1}{\pi^2} H(\psi^\alpha \lambda^\alpha) H(\phi^\beta \lambda^\beta) \psi^\gamma \lambda^\gamma \phi^\delta \lambda^\delta$$

is nonzero for some $\lambda^\alpha$ whenever $\psi^\alpha$ and $\phi^\alpha$ are not orthogonal (see Appendix A).

3. Indistinguishability Applied to Direction: Path Superposition

3.1. Path superposition. By ‘path’ we often mean the worldline of a particle. In our framework we maintain the geodesic hypothesis, but allow distinct paths in $\tilde{M}$ to be indistinguishable. We therefore propose

\textit{Nature takes a set of all indistinguishable stationary paths.}

\textit{Thus, by the indistinguishable principle, Nature takes a single stationary path.}

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5For completeness, we show (6) following the derivation of (5) Appendix B. It suffices to suppose $(\psi^\alpha) = (1, 0, 0)$ and $(\phi^\alpha) = (\cos \varphi, \sin \varphi, 0)$. The Heaviside functions in (6) imply the restrictions $-\frac{\pi}{2} < \varphi < \frac{\pi}{2}$ and $-\frac{\pi}{2} + \varphi < \varphi < \frac{\pi}{2} + \varphi$. Thus, if $\varphi$ is positive, we have

$$\int_{-\pi}^{\pi} \int_{0}^{\pi} H(\phi^\alpha \lambda^\alpha) \frac{1}{\pi} H(\psi^\beta \lambda^\beta) \psi^\gamma \lambda^\gamma \sin \theta \sin \varphi d\theta d\varphi = \frac{1}{\pi} \int_{-\frac{\pi}{2} + \varphi}^{\frac{\pi}{2}} \int_{0}^{\pi} \sin \theta \cos \varphi \sin \theta d\theta d\varphi$$

$$= \frac{1}{\pi} \int_{-\frac{\pi}{2} + \varphi}^{\frac{\pi}{2}} \sin \varphi d\varphi = \frac{1}{2} \left(1 + \cos \varphi\right) = \left|\cos \frac{\pi}{4} \cos \frac{\pi}{4} (0|0) + e^{i\varphi} \sin \frac{\pi}{4} \sin \frac{\pi}{4} (1|1)\right|^2 = |\langle \phi | \psi \rangle|^2,$$

where (1) holds by (3). A similar computation holds if $\varphi$ is negative.
Classical physics and our hypothesis share the same underlying assumption, namely, that Nature takes a single stationary path determined by a Lagrangian. However, in contrast to classical physics, our hypothesis yields an approximate model of quantum superposition. Indeed, path superposition occurs whenever two distinct worldlines in $\tilde{M}$ are indistinguishable to all pointal particles, and are thus identified as the same worldline in $M$. Conversely, state reduction occurs when the worldlines become distinguishable; see Section 4 below. Our model of path superposition is therefore ontic and $\psi$-complete, with the caveat that, by the indistinguishable principle, path superposition does not exist in spacetime $M$.

**Definition 3.1.** Consider segments of pointal curves $\beta, \beta'$ with affine parameterizations

$$\beta : [a, b] \to \tilde{M} \quad \text{and} \quad \beta' : [a, b] \to \tilde{M},$$

such that for some $c \in (a, b)$, we have

$$\beta(t) = \beta'(t) \quad \forall t \in [a, c], \quad \text{resp.} \quad \forall t \in [c, b].$$

We call the point $\beta(c) \in \tilde{M}$ with $c$ maximum resp. minimum a **branching point** of $\beta$.

**Example 3.2.** Suppose a photon decays into its constituent pointal particles $e^+$ and $e^-$ at $p \in \tilde{M}$. Fix a spatial hypersurface $V = \mathbb{R}^3$ of $T_p\tilde{M}$, and let

$$\Phi \subseteq S^2 \subset V$$

be the set of directions for which the electron $e^-$ may propagate. Further suppose the specific direction chosen $v \in \Phi$ is undetectable to all pointal particles for some open interval of time. Then, by the indistinguishable principle, all of the directions in $\Phi$ are identified as the same direction in $M$. Therefore the electron travels in a superposition of paths in $M$,

$$\exp_p(tv) \in \tilde{M}, \quad v \in \Phi \subset T_p\tilde{M},$$

but only along a single path in $M$. The point $p$ is thus a branching point of the electron.

Note that if $\Phi = S^2$, then the electron propagates as an expanding sphere; see Appendix B. **Pointal particles can therefore propagate as spherical waves.**

**Example 3.3.** Now consider a photon meeting a beam splitter such that no pointal particle in its environment can detect whether the photon is transmitted or reflected. Then the constituent pointal particles of the photon branch at some point within the beam splitter; see Figure 3.i. Observe that the photon remains undetected by the electron in the beam splitter, since neither pointal curve of the photon joins with the electron at an apex.

Now consider the same scattering, but as a standard Feynman diagram; see Figure 3.ii. In this case, the incoming photon is detected by the electron at a photon-electron
Figure 3. (i) A photon meeting a beam splitter, represented here by a single electron. The branching points are labeled with red dots. (ii) The reflected photon and electron in (i), drawn as a standard Feynman diagram.

vertex. Consequently, the transmitted and reflected paths cannot exist in superposition by the indistinguishable principle. This example shows that our hypothesis of path superposition is incompatible with standard Feynman diagrams.

3.2. Destructive interference from branching points. We assume that the branching of pointal curves is symmetric under time reversal (as is the case for other fundamental laws of Nature). In particular, the time reversal of Figure 3.i is allowed to exist. We thus obtain a geometric model of destructive interference. Indeed, the time reversal of Figure 3.i would occur at the final beam splitter in a Mach-Zehnder interferometer with a single initial photon, and we therefore obtain the destructive interference between the two path eigenstates of the photon. Conversely, if two branches of a photon meet in a beam splitter but they are out of phase (defined in (12) below), then we would not have the time reversal of a branching point, and therefore the two photon branches would not destructively interfere.

We note that we have only modelled the case of complete destructive interference; we leave the case of arbitrary wavefunction phase interference for future work.

3.3. Relation to first and second quantization. Our geometric model appears to be approximately compatible with first quantization, that is, quantum mechanics. We give evidence of this claim in specific examples of spin entanglement in Section 7.

In contrast, our model is fundamentally incompatible with second quantization, that is, quantum field theory. Unlike first quantization, second (i.e., canonical) quantization does not follow from our application of the indistinguishable principle: based on Dirac’s path integral (together with the Huygen-Fresnel principle), Feynman concluded that ‘Nature takes every possible path’ [F]. Of course, this is in stark contrast
to the geodesic hypothesis. In our model, second quantization becomes an approximate description that overcounts the eigenstates which contribute to a superposition. In particular, second quantization **overcounts the Feynman diagrams involved in a scattering event.** It is therefore possible that our model may yield a novel regularization scheme for quantum field theory.

Recall that in first quantization, particles cannot be created or annihilated. In second quantization, however, particles are replaced by fields, and at various points of \( \tilde{M} \) these fields can be created or annihilated. However, our model also allows pointal particles to be created or annihilated in pairs (in a spatial hypersurface of \( \tilde{M} \)), since a pointal geodesic \( \beta \) terminates at any point \( p \) of \( \tilde{M} \) where change is detected, and thus at any point where the pointal particle interacts with another pointal particle.

### 3.4. An example: off-shell particles that do not violate relativity.

Consider the Dirac Lagrangian \( L_0 = \bar{\psi} (i\partial - m) \psi \). If the specific value of the rest energy \( E_0 \) of a particle is indistinguishable, then we may modify the relation \( E_0 = m \) by introducing an ‘indistinguishable parameter’ \( \lambda \in \mathbb{R} \):

\[
E_0 = \lambda m.
\]

This in turn modifies the Dirac Lagrangian to

\[
L = \bar{\psi} (i\partial - \lambda m) \psi.
\]

The classical equations of motion of \( L \), namely

\[
i\partial \psi = \lambda m \psi,
\]

coincide with certain quantum perturbations determined by the path integral \( Z = \int D\psi e^{i \int d^4x L_0} \) of the Dirac Lagrangian \( L_0 \), namely

\[
i\partial \psi \neq m \psi.
\]

Recall that a particle is off-shell if its four-momentum \( p^a \) satisfies \( p^2 \neq m^2 \). Our modification \( L \) then admits a model of off-shell particles **for which relativity is never violated.** Indeed, in our model the relation

\[
p^2 = \lambda^2 m^2 = E_0^2
\]

always holds, and describes an off-shell particle whenever \( \lambda \neq 1 \).

A significant difference remains, however, between the classical dynamics determined by \( L \) and quantum perturbations of \( L_0 \): vacuum fluctuations arise from the path integral of \( L_0 \), but not from the modified Lagrangian \( L \) alone. In our model,

---

\[6\] We note that this is similar to the algebro-geometric process of blowing up a point of a variety.
then, apexes would only arise so that pointal curves that are off-shell can become
on-shell, and would not spontaneously occur without cause.

4. A MODEL OF STATE REDUCTION

Recall our assumption from Section 3 that path state reduction occurs whenever
two or more paths which were indistinguishable become distinguishable. In the fol-
lowing, we address the question of what constitutes distinguishability in the context
of pointal geodesics.

Since a pointal geodesic $\beta$ is a single point of spacetime $\pi(\beta) \in M$, time does not
flow along $\beta$ in $M$. Thus, by the indistinguishable principle, a pointal particle cannot
detect change, whence undergo any fundamental interaction, without simultaneously
being annihilated. Therefore distinguishability occurs at a point $p \in \tilde{M}$ if and only
if $p$ is the endpoint of a pointal geodesic. Consequently,

*State reduction occurs at $p \in \tilde{M}$ if and only if $p$ is the endpoint of a pointal geodesic.*

Consider an electron absorbing and then emitting a photon as shown in Figure 4.i. The electron first meets the positive pointal particle of the photon, and the two
particles annihilate at an apex. At this precise point, the quantum state of the system
is reduced. This may constitute, for example, the transfer of which-way information
from the photon to the surviving electron, or a measurement of the initial electron’s
position. After annihilation, the surviving negative pointal particle propagates as a
free electron.

In Figure 4.ii, a similar electron-photon scattering takes place, but no pointal
geodesic terminates. This configuration shows how an electron and photon are able
to influence each other without reducing the quantum state of the system. The
configuration could describe, for example, the passage of a photon through a polarizer
or beam splitter, where the photon’s quantum state does not collapse during the
scattering.

5. THE 4-MOMENTUM AND STRESS-ENERGY TENSOR ALONG A POINTAL CURVE

5.1. The 4-momentum of a pointal particle. Let $\beta$ be a pointal curve, and let
$p \in \beta(I)$ be a smooth point of $\beta$ contained in an open interval $I = (-\varepsilon, \varepsilon)$ for
which $\beta$ is timelike. Consider the spin vector $s^a = s$ of $\beta$ in the spatial hypersurface
$V \subset T_{\beta(I)}\tilde{M}$ of the inertial frame of $\beta$ near $p$, parallel transported along $\beta(I)$.

In order to specify the tangent form $\ast \nu$ of $\beta$ near $p$, a length of $\nu$ must be chosen:

$$\nu = \omega \hat{v},$$

Footnote: Vacuum fluctuations account for the Casimir force and the positive cosmological constant, as
well as provide a physical mechanism that corrects bare masses to renormalized masses. Further
development of our framework is therefore required to address these issues.
where $\omega \in \mathbb{R}_{>0}$, and $\hat{v}$ is a unit vector, $\hat{v}^a \hat{v}_a = -1$. Since $\beta$ is a pointal curve, this length is independent of the worldline $\beta(I) \subset \tilde{M}$ itself. Indeed, for sufficiently small $\lambda, \lambda' \in \mathbb{R}$, we have

$$\pi(\exp_p(\lambda v)) = \beta = \pi(\exp_p(\lambda' v)).$$

Thus, just like the choice of orientation $o$ of the volume form $\text{vol}^4$, the choice of $\omega$ is additional data that is required to specify a pointal curve, and is independent of the trace of the curve in $\tilde{M}$.

In Section 2.3 we showed that the orientation $o$ should be identified with the electric charge of the pointal particle, and now we would like to find a similar physical interpretation for $\omega$.

Fix an orthonormal basis $\{dx^1, dx^2, dx^3\}$ of $V^*$ for which $dx^3 = s_a$. Recall that $o_\beta \in \{\pm 1\}$ is the orientation of the volume 4-form $\text{vol}^4$. Consider the contraction of

**Figure 4.** How which-way information can arise in the interaction of a photon and an electron. The scattering (i) contains two apexes, and the scattering (ii) contains no apexes. These two scatterings cannot exist in superposition because they do not share identical apexes in $\tilde{M}$. 
the tangent form $\ast v$ with the (unit) spin vector $s^a = s$:

$$(\ast v)_{abc} s^c = o_\beta \sqrt{|g|} \omega(dx^1 \wedge dx^2 \wedge dx^3)(s)$$

$$= o_\beta \sqrt{|g|} \omega(s_a s^a)dx^1 \wedge dx^2$$

(i) $$= o_\beta \sqrt{|g|} \omega \hat{r}_a \wedge \hat{u}_b$$

(ii) $$= \ast o_\beta o_V \omega(\hat{r} \times \hat{u})_d$$

(iii) $$= \ast o_t \frac{u}{r}(\hat{r} \times \hat{u})_d$$

(iv) $$= \ast o_t \frac{1}{r^2}(r \times u)_d$$

(v) $$= \ast o_t \omega s_{ab}.$$ 

Indeed, in (i) we rename the unit spatial 1-forms at $t = 0$, $dx^1 = \hat{r}_a(0)$ and $dx^2 = \hat{u}_a(0)$.

In (ii), the spatial Hodge star operator $\ast$ in $V^*$ is applied. To define $\ast$, an orientation $o_V \in \{\pm 1\}$ of the volume 3-form $\text{vol}^3$ of $V$ must be chosen.

In (iii), we set

$$(8) \quad o_t := o_\beta o_V^{-1} = o_\beta o_V \in \{\pm 1\},$$

which is the orientation of the time coordinate. Moreover, we use the relation $u = \omega r$ for a particle in $\tilde{M}$ that lifts to a circular trajectory with tangential velocity $u > 0$, angular velocity $\omega > 0$, and radius $r > 0$, to the plane $P \subset V$ orthogonal to the spin vector $s$:

$$(9) \quad \alpha(t) = \exp_{\beta(t)}(r \cos(\omega \cdot o_t t) \partial_{x_1} + r \sin(\omega \cdot o_t t) o_V \partial_{x_2})$$

$$= \exp_{\beta(t)}(r \cos(\omega t) \partial_{x_1} + o_\beta r \sin(\omega t) \partial_{x_2}).$$

The plane $P$ is then spanned by $\partial_{x_1} = \hat{r}^a(0)$ and $\partial_{x_2} = \hat{u}^a(0)$. Observe that the electric charge $o_\beta$ of the pointal particle near $p$ determines the direction of rotation of $\alpha(t)$ about the spin vector $s$.

In (iv), we set

$$r := r \hat{r} \quad \text{and} \quad u := u \hat{u}.$$

Finally, (v) is a standard identity.

We make the following observations:

- If the spatial orientation $o_V$ is fixed, then flipping the electric charge $o_\beta$ is equivalent to flipping the time orientation $o_t = o_V o_\beta$ by (8), and thus the spin direction, up or down, by the relation $\ast v(s) = \ast o_t \omega s$:

$$\ast v(-s) = \ast o_t \omega(-s) = \ast(-o_t) \omega s.$$
• If the time orientation $o_t$ is fixed, then flipping the spatial orientation $o_V$, say from right-handed to left-handed, is equivalent to flipping the electric charge $o_B$ by $[8]$. Consequently, we obtain the elementary fact that electrons, subject to a Lorentz force, follow the right-hand rule iff positrons follow the left-hand rule.

**Definition 5.1.** We define the rest energy $E_0$ of the pointal particle to be the ‘length’ $\omega$ of its tangent form $*v$,

$$ E_0 = \hbar \omega = \omega, $$

and its mass $m$ to be\(^8\)

$$ m = \frac{\hbar}{cr} = \frac{1}{r}. $$

We call $\alpha(t) \in \hat{M}$ in [9] the dual particle to the pointal particle $\beta(t) \in \hat{M}$.

It follows from the identification $E_0 = \omega$ that the tangent 4-vector of a pointal particle in $\hat{M}$ is its wave 4-vector $\kappa^a$:

$$ v^a = \omega \hat{v}^a = \kappa^a. $$

Furthermore, from (10), (11), and the relation $u = \omega r$, we obtain

$$ \frac{E_0}{u} = \frac{\hbar \omega}{u} = \frac{\hbar}{r} = mc. $$

Whence,

$$ E_0 = mcu = mu. $$

Einstein’s relation $E_0 = m$ is then recovered if $u = 1$, that is, if the tangential speed of the particle $\alpha(t)$ equals the speed of light.

The relation [13] modifies the Dirac Lagrangian so that it is precisely of the form given in Section 3.4 with $\lambda = u$. Thus, a pointal particle is on-shell if $u = 1$, and off-shell otherwise; nevertheless, $p^2 = \kappa^a \kappa_a = E_0^2$ always holds. We therefore obtain a geometric model of off-shell pointal particles for which relativity is never violated.

Finally, we note that we do not have a way to construct the dual particle to a null pointal particle. Nevertheless, we may define the 4-momentum $\kappa^a$ of $\beta$ to be its tangent 4-vector $v^a$, independent of whether $v^a$ is null or timelike.

**5.2. The stress-energy tensor of pointal particle.** In this section we use the timelike signature $(+, -, -, -)$.

Let $\beta$ be a pointal curve, and let $p$ be a smooth point of $\beta \subset \hat{M}$. To determine the stress-energy tensor of the corresponding pointal particle near $p$, we use a Lagrangian

\(^8\)Note that $r = h(cm)^{-1}$ is the reduced Compton wavelength.
that describes the particle’s worldline. Near $p$, the worldline is causal geodesic, and
is thus described by the equations of motion of the Lagrangian
\[ \tilde{L} = \frac{1}{2} g_{ab} \dot{\beta}^a \dot{\beta}^b, \]
where $\dot{\beta}^a(s)$ is the tangent 4-vector of $\beta$ with affine parameter $s$. In particular,
if $\beta$ is timelike, resp. lightlike, then
\[ \tilde{L} = \frac{1}{2} \text{ resp. } \tilde{L} = 0. \]
However, we found in Section 5.1 that the tangent form of $\beta$ on $M$ is the Hodge dual
of the tangent 4-vector $v^a = \omega \dot{v}^a$, for some fixed choice of $\omega > 0$. Thus, to determine
the stress-energy tensor of the pointal particle we use the Lagrangian
\[ L = \frac{1}{2} g_{ab} v^a v^b = \omega^2 \tilde{L}. \]
Note that this (constant) rescaling does not affect the geodesic equation. The stress-
energy tensor of the pointal particle on $\tilde{M}$ is therefore
\[ \tilde{T}^{ab} := 2 \frac{\partial L}{\partial g_{ab}} = v^a v^b \left( \frac{(i)}{r^2} \dot{\beta}^a \dot{\beta}^b = \frac{u^2 m}{r} \dot{v}^a \dot{v}^b, \right. \]
where $(i)$ assumes that $v^a$ is timelike. Recall that $u = 1$ if and only if the pointal
particle is on-shell. Furthermore, the $r^{-2}$ factor looks like a sectional curvature, which
is suggestive of the Einstein tensor; we will return to this observation below.

In the following definition, we specify how to transport the stress-energy tensor
$\tilde{T}^{ab}$ from $\tilde{M}$ to nonnoetherian spacetime $M$:

**Definition 5.2.** We define the pointal stress-energy tensor $T_{ab}$ at a point $p \in \tilde{M}$ to be
\[ (14) \quad T_{ab} := h_a^c h_b^d \tilde{T}_{cd}, \]
where $\tilde{T}_{ab}$ is the stress-energy tensor of $\tilde{M}$ at $p$, and $h_{ab}$ is the pointal metric at $p$.
Einstein’s field equation on nonnoetherian spacetime $M$ is then
\[ G_{ab} = 8\pi T_{ab} = 8\pi h_a^c h_b^d \tilde{T}_{cd}, \]
where $G_{ab} := R_{ab} + \frac{1}{2} R g_{ab}$ is the Einstein tensor on $\tilde{M}$.

The definition of $T^{ab}$ in (14) yields a model where state reduction and gravitation
are intimately related. To show this, we determine the pointal stress-energy tensor
along $\beta$.

First, suppose that the only pointal curve that passes through $p$ is $\beta$. In particular,$p$ is not a point on the worldline of a photon, and so $\beta$ is a timelike geodesic near $p$.
Since $\beta$ has no interior points in $M$, there is no flow of momentum at $p$, that is, there
is no momentum flux across any surface at $p$. Thus, since $\tilde{T}^{\mu\nu}$ is the flux of the $\mu$-th
component of the 4-momentum $\kappa^a$ across the $x^\nu$ surface, the pointal stress-energy
tensor at $p$ should vanish. Indeed, since $h_a^b v_b = 0$ we find
\[ T_{ab} = h_a^c h_b^d \tilde{T}_{cd} = h_a^c h_b^d v_c v_d = 0. \]
Now suppose multiple pointal curves intersect at \( p \). The 4-momentum of an atom of pointal particles is the signed sum of the individual 4-momenta,

\[
k^a = \sum_i o_\beta_i \kappa^a_i,
\]

where the orientations \( o_\beta_i \) must be included so that each vector in the sum, \( o_\beta_i \kappa^a_i \), has the same orientation with respect to the chosen orientation of the underlying vector space \( T_pM \). Furthermore, similar to Feynman vertices in quantum field theory, we assume that 4-momentum is conserved at a splitting,

\[
\sum_{\text{incoming}} k^a_i = \sum_{\text{outgoing}} k^a_j =: K^a.
\]

Set

\[
\mu := \sqrt{K^a K_a}.
\]

In the single particle on-shell case we have

\[
\tilde{T}^{ab} = v^a v^b = \kappa^c \kappa^d \hat{v}^a \hat{v}^b = m^2 \hat{v}^a \hat{v}^b,
\]

where the second equality holds by (12). To generalize this to multiple pointal particles at \( p \), we propose that the stress-energy tensor should be

\[
(15) \quad \tilde{T}^{ab} = \frac{1}{2} \mu^2 \sum_i \hat{v}_i^a \hat{v}_i^b,
\]

where \( i \) runs over all incoming and outgoing tangent 4-vectors at \( p \) (whence the factor of \( \frac{1}{2} \)). At an electron-photon splitting, as given in Figure 1, (15) simplifies to

\[
(16) \quad \tilde{T}^{ab} = \mu^2 (\hat{v}_1^a \hat{v}_1^b + \hat{v}_2^a \hat{v}_2^b + \hat{v}_3^a \hat{v}_3^b),
\]

where \( v_1, v_2 \) are tangent 4-vectors to the electron/positron legs, and \( v_3 \) is a tangent 4-vector to the photon leg.

In the following theorem, we restrict our attention to electrons, positrons, photons, and the splittings given in Figure 1.

**Theorem 5.3.** The pointal stress-energy tensor is only nonvanishing at a splitting point of \( \tilde{M} \) which contains an apex and an on-shell photon leg. Thus, by Einstein’s field equation, gravitation is only sourced at apexes with an on-shell photon leg.

**Proof.** First suppose \( p \) is a smooth point on the worldline of a photon; then

\[
K^a = k^a = 2 v^a.
\]

There are two cases:

(a.i) If the photon is on-shell, then

\[
\mu^2 = K^a K_a = 4 v^a v_a = 0,
\]

whence \( \tilde{T}^{ab} = 0 \).
(a.ii) If the photon is off-shell, then \( v^{ab}_\perp \) is not the identity map. Thus,

\[
T^{ab} = h^a_c h^b_d \bar{T}^{cd} = h^a_c v^{\perp c e} v^{\perp d e} \mu^2 \hat{v}^c \hat{v}^d = 0.
\]

Therefore, in either case, \( T^{ab} = 0 \).

Now suppose \( p \) is a splitting point. There are three cases:

(b.i) If the photon leg is off-shell, then

\[
h^a_b = v^{\perp a c} v^{\perp d b}.
\]

Whence \( T^{ab} = h^a_b h^b_d \bar{T}^{cd} = 0 \) by (16).

(b.ii) If the splitting contains no apices (the first two splittings of Figure 1), then

\[
\mu^2 = K^{a c} K^{b d} = k^a_3 k^a_3 = 4 \nu^a_3 \nu^a_3 = 0.
\]

Thus again \( T^{ab} = 0 \).

(b.iii) Finally, if the splitting contains an apex (the second two splittings of Figure 1), then \( \mu \neq 0 \). Furthermore, if the photon is on-shell, then \( v^{\perp a}_3 \) is the identity map, and consequently

\[
h^a_b = v^{\perp a}_1 v^{\perp c}_2 b.
\]

It follows that

\[
T^{ab} = \mu^2 \pi(v^a_3) \pi(v^b_3),
\]

where \( \pi(v_3) \) is the projection of \( \hat{v}_3 \) onto the spatial hypersurface of \( T_p \bar{M} \) orthogonal to \( \hat{v}_1 \) and \( \hat{v}_2 \). Therefore, in this case, \( T^{ab} \neq 0 \).

□

Recall that the mass \( m \) of a pointal particle is defined as the inverse radius of its dual particle, \( m = r^{-1} \). We can then ask whether the total incoming (or outgoing) rest energy \( \mu \) at an apex \( p \) is also naturally given by some inverse radii. Indeed, consider the only nonvanishing case of \( T^{ab} \), namely (b.iii). Let \( \{e_0, e_1, e_2, e_3\} \) be an orthonormal basis for \( T_p \bar{M} \) for which

\[
e_3 = \frac{\pi(v_3)}{|\pi(v_3)|}.
\]

Then, using (17) and Einstein’s field equation, the curvature at \( p \) on \( \bar{M} \) is the sum of the sectional curvatures [11.5]

\[
8\pi \mu^2 = G(e_3, e_3) = K(e_0 \wedge e_1) + K(e_0 \wedge e_2) + K(e_1 \wedge e_2).
\]

Einstein’s field equation thus yields our desired relationship between the total incoming rest energy \( \mu \) and inverse radii: \( \mu^2 \) is a sum of sectional curvatures.
6. THE SPINOR CHIRALITY OF A POINTAL PARTICLE

Consider a pointal particle with worldline $\beta \subset \tilde{M}$, and let $p$ be a point of $\beta$ for which the tangent 4-vector to $\beta$ at $p$ is timelike. In Section 2.5, we found that the particle has spin $\frac{1}{2}$ near $p$, and now we determine its spinor chirality.

We briefly recall the chiral decomposition of the Lorentz algebra. The generators of the Lorentz algebra $\mathfrak{so}(1,3)$, namely the three rotations $J_i$ and three boosts $K_i$ admit linear combinations

$$A_i = \frac{1}{2}(J_i + iK_i)$$

and

$$B_i = \frac{1}{2}(J_i - iK_i)$$

that satisfy

$$[A_i, A_j] = i\epsilon_{ijk}A_k, \quad [B_i, B_j] = i\epsilon_{ijk}B_k, \quad [A_i, B_j] = 0.$$

The complexification $\mathfrak{so}(1,3)_\mathbb{C} := \mathfrak{so}(1,3) \otimes \mathbb{C}$ therefore decomposes as a direct sum

$$\mathfrak{so}(1,3)_\mathbb{C} \cong \mathfrak{su}(2)_\mathbb{C} \oplus \mathfrak{su}(2)_\mathbb{C}.$$

Since there is a bijection between real representations of a real Lie algebra and complex representations of its complexification, (18) implies that $\mathfrak{so}(1,3)$ and $\mathfrak{su}(2) \oplus \mathfrak{su}(2)$ have the same irreducible representations.

Let $o_{12} = o(\partial_1, \partial_2)$ and $o_{03} = o(\partial_0, \partial_3)$ be orientations of the respective planes $dx^1 \wedge dx^2$ and $dx^0 \wedge dx^3$ in $T_p\tilde{M}$ in the inertial frame of $\beta$. These orientations determine which $\mathfrak{su}(2)_\mathbb{C}$ subgroup of the complexified Lorentz algebra $\mathfrak{so}(1,3)_\mathbb{C}$ the pointal particle belongs: Set

$$C_\beta := \frac{1}{2}(o_{12}J_3 + o_{03}iK_3).$$

Then

$$C_\beta \in \mathfrak{su}(2)_\mathbb{C} \oplus 0 \iff o_\beta = o_{12}o_{03} = +1,$$

$$C_\beta \in 0 \oplus \mathfrak{su}(2)_\mathbb{C} \iff o_\beta = o_{12}o_{03} = -1.$$

Therefore, the pointal particle is represented by the chiral spinor

$$\begin{cases}
\psi_- \in \mathbb{C}^2 \oplus 0 = (\frac{1}{2}, 0) & \text{if } o_\beta = +1 \\
\psi_+ \in 0 \oplus \mathbb{C}^2 = (0, \frac{1}{2}) & \text{if } o_\beta = -1
\end{cases}$$

In particular, the chirality of a pointal particle is its electric charge $o_\beta$. The chiral decomposition of a (4-component) Dirac spinor $\psi$, namely

$$\psi_+ := \frac{1}{2}(1 + \gamma^5)\psi \quad \text{or} \quad \psi_- := \frac{1}{2}(1 - \gamma^5)\psi,$$

thus decomposes $\psi$ into two 2-component spinors of opposite charge. The role of $\gamma^5$ in the context of charge conjugation is studied in [B4].

In Section 2.1, we found that pointal particles are only allowed to interact by creation and annihilation at apices. Since a timelike pointal particle is represented

$${[K_i, K_j] = -i\epsilon_{ijk}J_k, \quad [J_i, J_j] = i\epsilon_{ijk}J_k, \quad [J_i, K_j] = i\epsilon_{ijk}K_k.}$$
by a free chiral spinor, and has rest energy $E_0 = \mu$ by (13), its dynamics are determined by the chiral decomposition of the free Dirac Lagrangian,

$$\mathcal{L} := i\bar{\psi}(i\partial - \mu)\psi,$$

$$= i\bar{\psi}_+ \partial \psi_+ - \mu \bar{\psi}_+ \psi_+ + (\leftrightarrow -).$$

The mass terms $m\bar{\psi}_+ \psi_-$ and $m\bar{\psi}_- \psi_+$ then imply that pointal particles may only interact at apices, where two pointal particles of opposite charge are created or annihilated. Furthermore, the mass $m$ of a pointal particle is precisely the coupling constant of this interaction. In particular, since the mass terms $m\bar{\psi}_+ \psi_+$ and $m\bar{\psi}_- \psi_-$ are absent from $\mathcal{L}$, a pointal particle is (gravitationally) massless as it propagates. We therefore reproduce the general conclusion of Theorem 5.3 using the Dirac Lagrangian.

7. Similarities between entanglement and pointal curves

In the framework of nonnoetherian spacetime, quantum nonlocality arises from the assumption that time does not flow along the worldline of a pointal particle. Using our model of electrons and photons from Section 2.4, we find similarities between certain cases of quantum entanglement and pointal curves. The similarities are obtained by making the following assumptions:

- Two pointal atoms are entangled if they share a common pointal curve. In particular:
  - Two photons are entangled if they share a common pointal curve; see Figures 2 and 5.
  - Two electrons are entangled if they are segments of the same pointal curve; see Figure 6.
- Entanglement can also arise from path superposition; see Figure 7.

Recall the Bell states

$$\Psi := \cos(\theta/2) |\uparrow \downarrow\rangle + e^{i\phi} \sin(\theta/2) |\downarrow \uparrow\rangle,$$

$$\Phi := \cos(\theta/2) |\uparrow \uparrow\rangle + e^{i\phi} \sin(\theta/2) |\downarrow \downarrow\rangle.$$

Example 7.1. In (delayed choice) entanglement swapping, introduced in [HBGSSZ], two pairs of photons, 1, 2 and 3, 4, are each produced in the state $\Psi^-$. Photons 2 and 3 then travel to a beam splitter, interfere, and a Bell state measurement is made. This process is given by the tensor product decomposition

$$(19) \quad \Psi_{12} \otimes \Psi_{34} = \frac{1}{2} \left( \Psi_{14}^+ \otimes \Psi_{23}^+ - \Psi_{24}^- \otimes \Psi_{31}^- - \Phi_{14}^+ \otimes \Phi_{23}^+ + \Phi_{14}^- \otimes \Phi_{23}^- \right),$$

where the four-photon state is represented in the initial physical basis on the left, and the final physical basis on the right. Remarkably, (19) shows that the measurement causes photons 1 and 4 to become entangled, and disentangles the pairs 1, 2 and 3, 4. In Figure 5 we introduce a model of this process using pointal curves. This
Figure 5. A model of four-photon entanglement swapping using pointal particles. The single lines are electron worldlines, and the double lines are photon worldlines. Photons 2 and 3 become entangled at the beam splitter (drawn as a dashed line), causing photons 1 and 4 to be entangled, and disentangling the pairs 1, 2 and 3, 4.

Figure 6. A model of two neighboring entangled electrons, such as a cooper pair. A photon bounces back and forth between the two electrons.

This model reproduces the decomposition (19) since any two photons in the diagram are entangled if and only if they share a common pointal curve.

Example 7.2. In Figure 6 we model two neighboring entangled electrons, such as a cooper pair. The two electrons are entangled because they each consist of segments of the same pointal curve (drawn in red).

Example 7.3. In Figure 7 we model entangled electrons in a Bell state using branching points. Observe that diagrams (i) and (ii) are indistinguishable scenarios in $M$, and diagram (iii) shows their superposition in $\tilde{M}$. Furthermore, diagrams (i) and (ii) necessarily have identical apexes, labeled by purple dots; otherwise the two scenarios would be distinguishable.
Figure 7. A model of two electrons in the Bell state $\frac{1}{\sqrt{2}} (|↑↓⟩ + |↓↑⟩)$ using path superposition of pointal curves.

Appendix A. The Harrigan-Spekkens classification of ontological models

We briefly review the classification of ontological (hidden variable) models given by Harrigan and Spekkens [HS]. Consider a quantum system with Hilbert space $\mathcal{H}$, and suppose the system possesses an underlying ontic state space $\Lambda$. Let $p(\lambda|P)$ be the probability distribution that an ontic state $\lambda \in \Lambda$ results from the preparation procedure $P$; and let $p(k|M, \lambda)$ be the probability distribution that the outcome $k$ results from the measurement $M$ of $\lambda$. Let $\rho$ be the density operator associated to $P$, and let $E_k$ be the POVM associated to the outcome $k$ of $M$. In order for the model to reproduce quantum statistics, it must satisfy Born’s rule [HS Definition 1]:

$$p(k|M, P) := \int_{\Lambda} d\lambda p(k|M, \lambda)p(\lambda|P) = \text{tr}(\rho E_k).$$

The relationship between $\Lambda$ and $\mathcal{H}$ specifies the following classes of ontological models [HS Definitions 4 and 5].

(a) $\psi$-ontic: Each ontic state $\lambda \in \Lambda$ is represented by a unique quantum state $\psi \in \mathcal{H}$. Thus, if $\psi \neq \phi$, then

$$p(\lambda|P_\psi)p(\lambda|P_\phi) = 0.$$

There are two subclasses of $\psi$-ontic models:

(i) $\psi$-complete: Each quantum state $\psi$ represents a unique ontic state $\lambda$.

(ii) $\psi$-supplemented: There is a quantum state $\psi$ that represents more than one ontic state in $\Lambda$.

(b) $\psi$-epistemic: There is an ontic state $\lambda$ that is represented by more than one quantum state in $\mathcal{H}$; thus [A] does not hold for $\lambda$.

For our purposes, $P$ and $M$ produce the pure states $\psi$ and $\phi$ in $\mathcal{H}$ respectively; whence $\rho = |\psi\rangle \langle \psi|$, $E_k = |\phi\rangle \langle \phi|$, and $\text{tr}(\rho E_k) = |\langle \phi|\psi\rangle|^2$. 

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A quantum state $\psi$ is therefore $\psi$-ontic if and only if a variation of $\psi$ implies a variation of reality, and $\psi$-epistemic if and only if a variation of $\psi$ does not imply a variation of reality. In terms of mappings, a model is $\psi$-complete if the correspondence $\Lambda \rightarrow \mathbb{PH}$ is bijective; $\psi$-supplemented if the correspondence $\Lambda \rightarrow \mathbb{PH}$ is surjective but not injective; and $\psi$-epistemic if the opposite correspondence $\mathbb{PH} \rightarrow \Lambda$ is surjective but not injective.

**Appendix B. The wave propagation of a particle: a thought experiment**

Imagine waking up in a large, completely black, empty room, so that there are no visible features that distinguish the different directions before you. If you stand up and walk in a straight line, any direction you choose to walk is indistinguishable from any other direction. In particular, walking five steps in one direction is indistinguishable from walking five steps in another direction. Thus, by the indistinguishable principle, regardless of what direction you choose to walk, you will arrive at the same location after five steps. Therefore the circle of radius five steps, centered at your initial location $O$, becomes identified as a single 1-dimensional point.

In your simple act of walking, you have unwittingly changed the topology of the flat floor you are walking on. Suppose you walk a distance $r > 0$. Then the indistinguishable principle implies that the circle $C$ of radius $r$, centered at $O$, becomes a single point. Consequently, the closed disc $D$ (on the floor) with boundary $\partial D = C$ becomes topologically a 2-sphere. On this sphere, the point $O$ and the boundary circle $C$ are antipodal points; if we take these points to be the north and south poles, then the circle of radius $\frac{r}{2}$ centered at $O$ may be mapped to the equator.

Now suppose there are small tables placed throughout the room, each with a unique bell on it. You are not able to see these tables, however, since the room is completely black.

If, in your forward walking, you happen to stumble into a table, then its bell will ring. The bell will thus specify your location in the room. (This is analogous to wavefunction collapse.) At such a moment, the 1-dimensional circular point you are standing on will vanish, and you will once again be standing on a 0-dimensional point. Furthermore, the topology of the floor will resume its initial state of being perfectly flat.

If instead you happen to miss the table (unknowst to you), then the arc along the circle $C$ that encounters the table will vanish from the 1-dimensional point you are standing on. If you miss $n \geq 2$ tables, say, then $C$ will be reduced to a disjoint union of $n$ circular arcs, all of which will remain identified as a one single point on the floor of the room. (This is analogous to partial wavefunction collapse, due to partial which-way information.) Of course, this thought experiment would not actually hold on our macroscopic scale, because the atoms in the floor would detect your footsteps, and thus distinguish the direction you had chosen to walk.
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