\[ N = 4 \text{ SYM STRUCTURE CONSTANTS AS DETERMINANTS} \]

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Abstract. We obtain a determinant expression for the tree-level structure constant of three non-extremal single-trace operators in the \( SU(2) \) sector of planar \( N = 4 \) supersymmetric Yang-Mills theory.

0. Overview

This note is based on [1], where a computationally tractable expression for a class of structure constants in \( N = 4 \) supersymmetric Yang-Mills theory, SYM\(_4\), is obtained, and on [2, 3], where a restricted version of Slavnov’s scalar product in XXZ spin-\( \frac{1}{2} \) chains, of which the XXX spin-\( \frac{1}{2} \) chain discussed in this note is a special case, is discussed.

To put the result of this note in context, we start the rest of this section with a brief overview of some of the highlights of integrability in SYM\(_4\), together with references to original works as well competent reviews. Following that, we recall basic definitions from the theory of quantum integrable models that are needed to explain our result. We refer the reader to the literature for technical details. Finally, we outline our result and the contents of the rest of the sections.

0.1. Integrability in planar SYM\(_4\). The discovery of integrable structures, on both sides of Maldacena’s AdS/CFT correspondence [4], is undoubtedly one of the major developments in mathematical physics in the past ten years [5]. This is not only because of the obvious intrinsic importance of building bridges between subjects that would otherwise remain unrelated, but also because integrability may be the right approach to put the correspondence on a rigorous footing.

In this note, we restrict our attention to integrability in planar SYM\(_4\) on the CFT side of AdS/CFT. The planar limit (the number of colours \( N_c \to \infty \), the gauge coupling \( g_{YM} \to 0 \), while the ’t Hooft coupling \( \lambda = g_{YM}^2 N_c \) remains finite) allows SYM\(_4\) to be integrable. It is possible that integrability persists beyond the planar limit, but at this stage, this is a wide open question.

0.2. SYM\(_4\) and spin chains. 1-loop results. SYM\(_4\) contains an SO(6) invariant scalar sector, that consists of six real scalars \( \phi_i, i \in \{1, \cdots , 6\} \). In [6], Minahan and Zarembo showed that the action of the 1-loop dilatation operator \( D \) on single-trace operators \( \{O\} \) with 1-loop conformal dimensions \( \{\Delta_O\} \), in the scalar sector map to the action of the Hamiltonian on states in an integrable periodic SO(6) spin-chain with nearest-neighbour interactions.

The single-trace operators \( \{O\} \) map to eigenstates of the spin-chain Hamiltonian. Their conformal dimensions \( \{\Delta_O\} \) map to the corresponding eigenvalues. In [7], Beisert extended the result of [6] to all fundamental fields in SYM\(_4\).

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0.3. SYM$_4$ and spin chains. Higher loop results. The six scalar fields $\phi_i$, $i \in \{1, \cdots, 6\}$, can be combined into three charged scalars $\{X,Y,Z\}$ and their charge conjugates $\{\bar{X}, \bar{Y}, \bar{Z}\}$. Any two non-conjugate fields, such as $\{X,Y\}$, form a closed $SU(2)$ subsector.

In [8], Beisert, Kristjansen and Staudacher established integrability in the $SU(2)$ scalar sector, up to 3-loops. However, beyond 1-loop order, the action of the dilatation operator on gauge-invariant states can no longer be represented in terms of a nearest-neighbour spin-chain Hamiltonian.

In [9], Serban and Staudacher matched the dilatation operator in the $SU(2)$ sector with higher Hamiltonians in the Inozemtsev model, which is a spin-chain with long range interactions, up to 3-loop level, and an asymptotic Bethe Ansatz was proposed to obtain the Bethe eigenstates and eigenvalues in the long chain-length limit, $L \to \infty$. These results can also be obtained using the Hubbard model [10] [11]. But these models do not match the dilatation operator beyond 3-loop level and the final word on the integrable model that describes SYM$_4$ to all loop order remains to be written.

0.4. All-sector, all-loop asymptotic Bethe Ansatz equations. In [12], Beisert, Dippel and Staudacher proposed asymptotic (valid with no corrections only in the long chain-length limit, $L \to \infty$) all-loop Bethe Ansatz equations in the $SU(2)$ sector. These equations require a dressing factor to match predictions made in the strong coupling limit. In [13], Janik proposed an equation that the dressing factor must satisfy. In [14], Beisert, Hernandez and Lopez solved Janik’s equation. In [15] Beisert, Eden and Staudacher showed that this solution has the right properties in the weak coupling limit. In [16], Beisert and Staudacher proposed asymptotic Bethe Ansatz equations that hold for all sectors to all loops, in the $L \to \infty$ limit. This proposal was confirmed in [17].

0.5. Finite-size corrections. The asymptotic Bethe Ansatz equations are valid without corrections only in the $L \to \infty$ limit. For long but finite length chains, we need to compute the finite size corrections.

One approach to computing finite-size corrections is Lüscher’s method, introduced in the context of weak coupling integrability by Janik and Lukowsky [18] and applied by Bajnok and Janik [19]. For an introduction to this method in AdS/CFT, see [20]. Another approach to finite-size corrections is the thermodynamic Bethe Ansatz, TBA, first considered in the AdS/CFT framework by Ambjorn, Janik and Kristjansen [21]. It relies on the equivalence of a finite-size, zero-temperature theory to an infinite-size, finite-temperature mirror theory. The ground state energy is then computed by solving sets of coupled nonlinear integral TBA equations [22].

TBA equations can be put in an elegant, universal form called Y-systems, which are systems of difference equations that appear in diverse topics in classical as well as quantum integrability. For a comprehensive review of Y-systems in AdS/CFT, see [23]. For a review of applications of Y-systems in AdS/CFT, see [24].

0.6. Weakly-coupled, planar SYM$_4$. The $SU(2)$ scalar sector. In this note, we restrict ourselves to weakly-coupled planar SYM$_4$, where perturbation theory in ’t Hooft’s coupling constant $\lambda$ is valid and we can consistently work up to 1-loop order. When this is the case, we can make use of mappings to integrable spin chains with nearest neighbour interactions, and conventional tools, such as the algebraic Bethe Ansatz apply.
Furthermore, we deal only with sectors with two complex scalars, so that the mapping is to $SU(2)$ spin-$\frac{1}{2}$ chains. It is only in the case of spin chains based on rank-1 Lie algebras that we have a determinant expression for the inner product of a Bethe eigenstate and a generic state [23], which will be the main tool that we will use to obtain determinant expressions for structure constants.

0.7. Conformal invariance and 2-point functions. Because SYM$_4$ is conformally-invariant at the quantum level, it contains a basis of local gauge-invariant composite operators $\{\mathcal{O}\}$ such that each $\mathcal{O}_i \in \{\mathcal{O}\}$ is an eigenstate of the dilatation operator $D$, with a corresponding eigenvalue $\Delta_{\mathcal{O}_i}$, equal to the conformal dimension of $\mathcal{O}$. The 2-point function of $\mathcal{O}_i$ and $\mathcal{O}_j$ can be written as

$$\langle \mathcal{O}_i(x) \bar{\mathcal{O}}_j(y) \rangle = \left( \mathcal{N}_i \mathcal{N}_j \right)^{1/2} \frac{\delta_{ij}}{|x-y|^{2\Delta_i}},$$

where $\bar{\mathcal{O}}_j$ is the Wick conjugate of $\mathcal{O}_i$, $\Delta_i$ is again the conformal dimension of $\mathcal{O}_i$, and $\mathcal{N}_i$ is a normalization factor. The 2-point functions of $\{\mathcal{O}\}$ and their conformal dimensions $\{\Delta_{\mathcal{O}}\}$ are by now well-understood [5], and the next logical step is to study 3-point functions of $\{\mathcal{O}\}$ and their structure constants [26, 1, 27, 28].

0.8. 3-point functions and structure constants. A 3-point function of basis local operators in SYM$_4$, is restricted by conformal symmetry to be of the form

$$\langle \mathcal{O}_i(x_i) \mathcal{O}_j(x_j) \mathcal{O}_k(x_k) \rangle = \left( \mathcal{N}_i \mathcal{N}_j \mathcal{N}_k \right)^{1/2} \frac{C_{ijk}}{|x_{ij}|^{\Delta_i+\Delta_j-\Delta_k}|x_{jk}|^{\Delta_j+\Delta_k-\Delta_i}|x_{ki}|^{\Delta_k+\Delta_i-\Delta_j}},$$

where $x_{ij} = x_i - x_j$, and $C_{ijk}$ is the structure constant.

In this work, we restrict our attention to the weak-coupling limit where perturbation theory in the ‘t Hooft coupling constant $\lambda$ makes sense, and we can further restrict our analysis to 1-loop level perturbation theory. In this limit, we can describe the integrability of SYM$_4$ in terms of spin-chains with nearest-neighbour interactions where conventional tools such as the algebraic Bethe Ansatz are most effective.

In [1], Escobedo, Gromov, Sever and Vieira (EGSV) obtained an expression for the structure constants of non-extremal single-trace operators in the scalar sector of SYM$_4$ that contains two charged scalars $\{Z, X\}$ and their conjugates $\{\bar{Z}, \bar{X}\}$. The EGSV expression is in terms of a sum over partitions of a set of rapidities into two distinct subsets. In this paper, the sum expression of EGSV is evaluated in determinant form. This determinant turns out to be (a restriction of) the well-known Slavnov determinant in exact solutions in statistical mechanics. It is equal to the inner product of a Bethe eigenstate and a generic state in Heisenberg XXZ spin-$\frac{1}{2}$ chains.

0.9. Rapidity variables, generic Bethe states and Bethe eigenstates. States in a closed length-$L$ XXX spin-$\frac{1}{2}$ chain depend on two sets of rapidity variables, auxiliary space rapidity variables, ‘auxiliary rapidities’, and quantum space rapidity variables, ‘quantum rapidities’. When all quantum rapidities are set equal to the same constant value, the spin chain is ‘homogeneous’. At each

1Later, we will choose $\mathcal{N}_i$ to be (the square root of) the Gaudin norm of the corresponding spin-chain state.

2We restrict our attention to this spin chain, and use ‘spin chain’ to refer to that.
of the $L$ sites, there is a state variable, or equivalently, a spin variable, that is represented by an arrow that can be either up or down. A state with all spins up is a ‘reference state$^3$. Initial and final generic Bethe states, $|O\rangle$ and $\langle O|$ are created by the action of algebraic Bethe Ansatz (BA) operators on initial and final spin-chain reference states. They are characterized by auxiliary rapidities that are free variables, and they are not eigenstates of the spin-chain transfer matrix. Initial and final Bethe eigenstates, $|O\rangle_\beta$ and $\langle O|_\beta$, are also created by the action of BA operators on reference states. However, their auxiliary rapidities satisfy Bethe equations, and consequently, they are eigenstates of the spin-chain transfer matrix. We use the subscript $\beta$ to distinguish between these two types of states, and refer to them as ‘generic states’ and ‘eigenstates’, respectively.

0.10. An expression for the structure constant. In [1], Escobedo, Gromov, Sever and Vieira (EGSV) obtained a computationally tractable expression for the tree-level structure constant $c_{ijk}^{(0)}$ of three operators, $O_i$, $i \in \{1, 2, 3\}$ with definite 1-loop anomalous conformal dimensions, in the $SU(2)$ sector in planar SYM$_4$. We restrict our attention to these operators in this sector of this theory, and use ‘operators’ and ‘structure constants’ in the sense of this restriction. We use $c_{ijk}^{(0)}$ rather than $c_{ijk}$ when this simplifies the notation with no loss of generality.

In [1], EGSV make use of the connection of weakly coupled SYM$_4$ to integrable spin chains to map the operators $O_i$, to eigenstates $|O_i\rangle_\beta$. Following that, they 1. Split each initial eigenstate $|O_i\rangle_\beta$ into two initial generic states, $|O_i\rangle_r$ and $|O_i\rangle_l$. 2. Map the three initial generic states $|O_i\rangle_r$, to the corresponding three final generic states $r\langle O_i|$ and finally 3. Compute the structure constants by taking scalar products of specific pairs of initial and final generic states.

From the above outline one expects two complications. A. From step 1, one expects a sum over many possible ways of splitting each eigenstate into two generic states, and B. From step 3, one expects that there are three non-trivial scalar products to evaluate. Both of these expectations turn out to be incorrect.

0.11. A constraint that leads to two simplifications. In formulating $c_{ijk}^{(0)}$ in BA terms, EGSV start with three initial eigenstates, $|O_1\rangle_\beta$, $|O_2\rangle_\beta$, and $|O_3\rangle_\beta$, characterized by sets of auxiliary rapidities $\{u\}_{\beta N_1}$, $\{v\}_{\beta N_2}$ and $\{w\}_{\beta N_3}$ with cardinalities $N_1$, $N_2$ and $N_3$, respectively, that satisfy Bethe equations$^5$. The set $\{N\} \equiv \{N_1, N_2, N_3\}$ will appear frequently in the sequel. Remarkably, it turns out that $N_1 = N_2 + N_3$. This constraint distinguishes the eigenstate $|O_1\rangle_\beta$, and admits one and only one way to split each eigenstate into left and right generic states. This removes complication A. It also reduces the number of scalar products that one expects to evaluate. One scalar product is constrained to be between two reference states and therefore trivial. A second scalar product is constrained to be between two dual reference states (states with all spins down) and therefore straightforward to compute. Only one scalar product remains to be evaluated and this removes complication B.

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$^3$In this note, ‘state variables’, ‘spin variables’, and ‘arrows’ can be used interchangeably.

$^4$For an introduction to integrability in gauge and string theory, see [2].

$^5$In this note, the set $\{u\}$ will always have cardinality $N_1$ and satisfy Bethe equations, hence the subscript $\beta$. The sets $\{v\}$ and $\{w\}$ will have cardinalities $N_2$ and $N_3$. They satisfy Bethe equations, but this fact is not used, and their Bethe equations will play no role. The quantum rapidities $\{z\}$ will have cardinality $L$, and do not satisfy Bethe equations.
0.12. A generic scalar product that is a weighted sum. The remaining scalar product is generic in the sense that it involves two generic states with rapidities that do not satisfy Bethe equations, and neither is a reference or a dual reference state. There is no simple expression (such as a determinant) for a generic scalar product, but using the commutation relations of the BA operators, one can express it as a manageable sum [29]. EGSV use this sum form of the generic scalar product to obtain a computationally tractable weighted sum over all partitions of the set \( \{u\}_{N_1} \) of cardinality \( N_1 \) into two sets \( \alpha \) and \( \bar{\alpha} \) of cardinality \( N_2 \) and \( N_3 \), respectively.

0.13. Bethe equations, Slavnov’s scalar product and the result in this note. This note is based on the observation that \( c^{(0)}_{123} \) as defined in [1] is (up to a factor) a restricted version of a Slavnov scalar product of a generic state and an eigenstate. This restricted version is discussed in [2] and was used in [3] to obtain a recursive proof of the determinant expression of Slavnov’s scalar product [3].

This observation allows us to implicitly use the Bethe equations satisfied by \( \{u\}_{\beta N_1} \) to evaluate the EGSV weighted sum over partitions of \( \{u\}_{\beta N_1} \), and to write \( c^{(0)}_{123} \) as a determinant of an \( (N_1 \times N_1) \)-matrix [3].

0.14. Outline of contents. The subject of this note is at the intersection of supersymmetric Yang-Mills theory and integrable statistical mechanical models. We cannot review either of these topics in any technical detail. Overall, we can only recall the very basics that are needed to obtain our result and refer the reader to [1, 2, 3] for a more complete discussion and references to the original literature.

On the other hand, our presentation is elementary. In particular, we rephrase the operator language of spin chains in terms of the diagrammatic language of the six vertex model, in the hope that this will make our arguments more accessible to readers with minimal background in quantum integrable models.

In Section 1, we review standard facts related to the rational six-vertex model, which is basically another way to consider XXX spin-\( \frac{1}{2} \) chains, but as mentioned above, we find that the diagrams that represent the vertex model lattice configurations better suit our purposes. Following [2, 3], we introduce the \([L, N_1, N_2]\)-configurations that will be central to our result. In Section 2, we review standard facts related to XXX spin-\( \frac{1}{2} \) chains, and rephrase various ingredients of the BA solution in terms of vertex model lattice configurations. Following [2, 3], we introduce restricted versions \( S[L, N_1, N_2] \) of Slavnov’s scalar product, that can be evaluated in determinant form. \( S[L, N_1, N_2] \) will turn out to be the partition function of the \([L, N_1, N_2]\)-configurations introduced in Section 1. In Section 3, we recall the EGSV expression of the structure constants, and express it in terms of vertex model lattice configurations. In Section 4, we identify the weighted sum in the EGSV expression with the restricted Slavnov scalar product \( S[L, N_1, N_2] \) introduced earlier, thereby showing that, up to a multiplicative factor, \( c^{(0)}_{123} \) can be written as a determinant. In Section 5, we collect a number of comments and remarks.

In [3], one can also find a representation of this restricted scalar product in terms of six-vertex model diagrams. We will use this representation in this note.

The auxiliary rapidities \( \{v\} \) and \( \{w\} \) also satisfy Bethe equations, but this fact is not used in this note.
In this section, we recall the 2-dimensional rational six-vertex model in the absence of external fields. From now on, ‘six-vertex model’ will refer to that. It is equivalent to the XXX spin-1/2 chain that appears in [1], but affords a diagrammatic representation that suits our purposes. We introduce quite a few terms to make this corresponds clear and the presentation precise, but any reader with basic familiarity with quantum integrable models can skip all these.

1.1. Lattice lines, orientations, and rapidity variables. Consider a square lattice with \( L_h \) horizontal lines and \( L_v \) vertical lines that intersect at \( L_h \times L_v \) points. There is no restriction, at this stage, on \( L_h \) or \( L_v \). We order the horizontal lines from top to bottom and assign the \( i \)-th line an orientation from left to right and a rapidity variable \( u_i \). We order the vertical lines from left to right and assign the \( j \)-th line an orientation from top to bottom and a rapidity variable \( z_j \). See Figure 1. The orientations that we assign to the lattice lines are matters of convention and are only meant to make the vertices of the six-vertex model, that we will introduce shortly, unambiguous. We orient the vertical lines from top to bottom to agree with the direction of the ‘spin set evolution’ that we will introduce shortly.

1.2. Bulk and boundary line segments, arrows, and vertices. Each lattice line is split into segments by all other lines that are perpendicular to it. Bulk segments that are attached to two intersection points, and boundary segments that are attached to one intersection point only. Assign each segment an arrow that can point in either direction, and define the vertex \( v_{ij} \) as a set of the three elements. 1. The intersection point of the \( i \)-th horizontal line and the \( j \)-th vertical line, 2. The four line segments attached to this intersection point, and 3. The arrows on these segments (regardless of their orientations). Assign \( v_{ij} \) a weight that depends on the specific orientations of its arrows, and the rapidities \( u_i \) and \( z_j \) that flow through it.

1.3. Six vertices that conserve ‘arrow flow’. Since every arrow can point in either direction, there are \( 2^4 = 16 \) possible types of vertices. In this note, we are interested in a model such that only those vertices that conserves ‘arrow flow’ (that is, the number of arrows that point toward the intersection point is equal to the number of arrows that point away from it) have non-zero weights. There are six such vertices. They are shown in Figure 2. We assign these vertices

Figure 1: A square lattice with oriented lines and rapidity variables. Lattice lines are assigned the orientations indicated by the white arrows.
non-vanishing weights. We assign the rest of the 16 possible vertices zero weights \[30\].

In the rational six-vertex model, and in the absence of external fields, the six vertices with non-zero weights form three equal-weight pairs of vertices, as in Figure 2. Two vertices that form a pair are related by reversing all arrows, thus the vertex weights are invariant under reversing all arrows. In the notation of Figure 2, the weights of the rational six-vertex model, in the absence of external fields, are

\[(3) \quad a[u_i, z_j] = \frac{(u_i - z_j + \eta)}{(u_i - z_j)}, \quad b[u_i, z_j] = 1 \quad c[u_i, z_j] = \frac{\eta}{(u_i - z_j)}\]

The assignment of weights in Equation (3) satisfies unitarity, crossing symmetry, and most importantly the Yang-Baxter equations \[30\]. It is not unique since one can multiply all weights by the same factor without changing the final physical results \[8\].

1.4. Remarks. 1. The spin chain that is relevant to SYM\(_4\) is homogeneous since all quantum rapidities are set equal to the same constant value \(z\). In our conventions, \(z = \frac{1}{2}\sqrt{-1}\). 2. The rational six-vertex model that corresponds to the homogeneous XXX spin-\(\frac{1}{2}\) chain used in \[1\] will have, in our conventions, all vertical rapidity variables equal to \(\frac{1}{2}\sqrt{-1}\). In this note, we start with inhomogeneous vertical rapidities, then take the homogeneous limit at the end. 3. In a 2-dimensional vertex model with no external fields, the horizontal lines are on equal footing with the vertical lines. To make contact with spin chains, we will treat these two sets of differently. 4. In figures in this note, a line segment with an arrow on it obviously indicates a definite arrow assignment. A line segment with no arrow on it implies a sum over both arrow assignments.

\[8\] The normalization of the vertex weights in Equation (3) is different from that in \[2\]. The latter is such that \(b[u_i, z_j] = 1\). We will comment on this again in Section \[4\].
1.5. **Weighted configurations and partition functions.** By assigning every vertex $v_{ij}$ a weight $w_{ij}$, a vertex model lattice configuration with a definite assignment of arrows is assigned a weight equal to the product of the weights of its vertices. The partition function of a lattice configuration is the sum of the weights of all possible configurations that the vertices can take and that respect the boundary conditions. Since the vertex weights are invariant under reversal of all arrows, the partition functions is also invariant under reversal of all arrows.

1.6. **Rows of segments, spin systems, spin system states and net spin.** ‘A row of segments’ is a set of vertical line segments that start and/or end on the same horizontal line(s). An $(L_v \times L_h)$ six-vertex lattice configuration has $(L_v + 1)$ rows of segments. On every length-$L_h$ row of segments, one can assign a definite spin configuration, whereby each segment carries a spin variable (an arrow) that can point either up or down. A spin system on a specific row of segments is a set of all possible definite spin configurations that one can assign to that row. ‘A spin system state’ is a one definite such configuration. Two neighbouring spin systems (or spin system states) are separated by a horizontal lattice line. The spin systems on the top and the bottom rows of segments are initial and final spin systems, respectively. Consider a specific spin system state. Assign each up-spin the value $+1$ and each down-spin the value $-1$. The sum of these values is the net spin of this spin system state. In this note, we only consider six-vertex model configurations such that all elements in a spin system will the same net spin.

1.7. **Four types of horizontal lines.** Each horizontal line has two boundary segments. Each boundary segment has as an arrow that can point into the configuration or away from it. Accordingly, we can distinguish four types of horizontal lines, as in Figure 3. We will refer to them as $A$-, $B$-, $C$- and $D$-lines.

An important property of a horizontal line is how the net spin changes as one moves across it from top to bottom. Given that all vertices conserve ‘arrow flow’, one can easily show that, scanning a configuration from top to bottom, $B$-lines change the net spin by $-1$, $C$-lines increase it by $+1$, while $A$- and $D$-lines...
preserve the net spin. This can be easily understood by working out a few simple examples.

1.8. Remarks. 1. There is of course no ‘time variable’ in the six-vertex model, but one can think of a spin system as a dynamical system that evolves in discrete steps as one scans a lattice configuration from top to bottom. Starting from an initial spin set and scanning the configuration from top to bottom, one can think of the intermediate spin sets as consecutive states in the history of a dynamical system, ending with the final spin set. One can think of this evolution as caused by the action of the horizontal line elements.

2. In this note, all elements in a spin system, that live on a certain row of segments, have the same net spin. The reason is that vertically adjacent spin systems are separated by horizontal lines of a fixed type that change the net spin by the same amount (±1) or keep it unchanged. Since we consider only lattice configurations with given horizontal lines (and do not sum over different types), the net spin of all elements in a spin system change by the same amount.

1.9. Initial and final reference states, dual reference states, and a variation. An initial (respectively, final) reference state \(|[L^\wedge]|\) (respectively, \(<[L^\wedge]|\)) is a spin system set on a top (respectively, bottom) row of segments with \(L\) arrows that are all up. An initial (final) dual reference state \([|L^\vee]\) (\(<|L^\vee|\)) is a spin system set on a top (bottom) row of segments with \(L\) arrows that are all down. The state \(<|N_3^\vee, (L - N_3)^\wedge|\) is a spin system state on a bottom row of segments with \(L\) arrows such that the first \(N_3\) arrows from the left are down, while the right \((L - N_3)\) arrows are up. We will not need the initial version of this state or their duals.

1.10. Four types of configurations. 1. A \(B\)-configuration is a lattice configuration with \(L\) vertical lines and \(N\) horizontal lines, \(N \leq L\), such that \(A\). The initial spin system is an initial reference state \([|L^\wedge]|\), and \(B\). All horizontal lines \(B\)-lines. An example is on the left hand side of Figure 4.

2. A \(C\)-configuration is a lattice configuration with \(L\) vertical lines and \(N\) horizontal lines, \(N \leq L\), such that \(A\). All horizontal lines are \(C\)-lines, and \(B\). The final spin system is a final reference state \(<|L^\wedge|\). An example is on the right hand side of Figure 4.
Figure 5: A six-vertex model $BC$-configuration. $L = 12$, and $N_1 = 5$, or equivalently $L_h = 2 \times 5 = 10$ and $L_v = 12$. The top $N$ horizontal lines represent $B$-operators. The bottom $N$ horizontal lines represent $C$-operators. The initial (top) as well as the final (bottom) boundary spin systems are reference states.

3. A $BC$-configuration is a lattice configuration with $L$ vertical lines and $2N_1$ horizontal lines, $0 \leq N_1 \leq L$, such that A. The initial spin system is an initial reference state $|\{L\}^\wedge\rangle$, B. The first $N_1$ horizontal lines from top to bottom are $B$-lines, C. The following $N_1$ horizontal lines are $C$-lines, D. The final spin system is a final reference state $\langle\{L\}^\wedge|$. See Figure 5.

4. An $[L,N_1,N_2]$-configuration, $0 \leq N_2 \leq N_1$, is identical to a $BC$-configuration except that it has $N_1$ $B$-lines, and $N_2$ $C$-lines. When $N_3 = N_1 - N_2 = 0$, we evidently recover a $BC$-configuration. The case $N_2 = 0$ will be discussed below. For intermediate values of $N_2$, we obtain restricted $BC$-configurations whose partition functions will turn out to be essentially the structure constants.

1.11. $[L,N_1,N_2]$-configurations as restrictions of $BC$-configurations. Consider a $BC$-configuration with no restrictions. To be specific, let us consider the configuration in Figure 5 where $N_1 = 5$ and $L = 12$. Consider the vertex at the bottom-left corner. For convenience, we label the $\{v\}$ rapidities from bottom to top. The $\{v\}$ rapidities are labeled from top to bottom as before.

From Figure 2 it is easy to see that this can be either a $b$- or a $c$-vertex. Since the $\{v\}$ variables are free, set $v_1 = z_1$, thereby setting the weight of all configurations with a $b$-vertex at this corner to zero, and forcing the vertex at this corner to be $c$-vertex.

Referring to Figure 2 again, one can see that not only is the corner vertex forced to be type-$c$, but the orientations of all arrows on the horizontal lattice line with rapidity $v_1$, as well all arrows on the vertical line with rapidity $z_1$ but below the horizontal line with rapidity $u_1$ are also frozen to fixed values.

*For visual clarity, we have allowed for a gap between the $B$-lines and the $C$-lines in Figure 5. There is also a gap between the $N_3$-th and $(N_3 + 1)$-st vertical lines, where $N_3 = 3$ in the example shown, that indicates separate portions of the lattice that will be relevant shortly. The reader should ignore this at this stage.*
Figure 6: The effect of forcing the three vertices at the intersection of the \{v_1, z_1\}, \{v_2, z_2\} and \{v_3, z_3\} rapidity lines to be a \(c\)-vertices. We used the notation \(N_3 = N_1 - N_2\).

The above exercise in ‘freezing’ vertices and arrows can be repeated and to produce a non-trivial example, we do it two more times. Setting \(v_2 = z_2\) forces the vertex at the intersection of the lines carrying the rapidities \(v_2\) and \(z_2\) to be a \(c\)-vertex and freezes all arrows to the right as well as all arrows above that vertex and along \(C\)-lines. Setting \(v_3 = z_3\), we end up with the lattice configuration in Figure 6.

From Figure 6 one can see that 1. All arrows on the lower \(N_3\) horizontal lines, where \(N_3 = 3\) in the specific example shown, are frozen, and 2. All lines on the \(N_3\) left most vertical lines in the lower half of the diagram, where they intersect with \(C\)-lines. Removing the lower \(N_3\) \(C\)-lines we obtain the configuration in Figure 7. This configuration has a subset (rectangular shape on lower left corner) that is also completely frozen. All vertices in this part are \(a\)-vertices, hence from Equation 3 their contribution to the partition function of this configuration is trivial.

An \([L, N_1, N_2]\)-configuration, as in Figure 7 interpolates between an initial reference state \([L^\Lambda]\) and a final \([\langle N_3^\vee, (L - N_3)^\Lambda \rangle]\) state, using \(N_1\) \(B\)-lines followed by \(N_2\) \(C\)-lines.

Setting \(v_i = z_i\) for \(i = 1, \cdots, N_1\), we freeze all arrows that are on \(C\)-lines or on segments that end on \(C\)-lines. Discarding these we obtain the lattice configuration in Figure 8.

Removing all frozen vertices (as well as the extra space between two sets of vertical lines, that is no longer necessary), one obtains the domain wall configuration in Figure 9 which is characterized as follows. All arrows on the left and right boundaries point inwards, and all arrows on the upper and lower boundaries point outwards. The internal arrows remain free, and the configurations that are consistent with the boundary conditions are summed over. Reversing the orientation of all arrows on all boundaries is a dual a domain wall configuration.
Figure 7: A restricted $[L, N_1, N_2]$-configuration. In this example, $N_1 = 5$, $N_2 = 2$, and as always $N_3 = N_1 - N_2$.

Figure 8: A restricted $[L, N_1, N_2]$-configuration. In this example, $N_1 = 5$ and $N_2 = 5$. Equivalently, the left half is an $(N_1 \times N_1)$ domain wall configuration, where $N_1 = 5$, with an additional totally frozen lattice configuration to its right.

Figure 9: The left hand side is an $(N \times N)$ domain wall configuration, where $N = 5$. The right hand side is the corresponding dual configuration.

1.12. Remarks on domain wall configurations. 1. One can generate a domain wall configuration directly starting from a length-$N$ initial reference state followed by $N$ $B$-lines. 2. One can generate a dual domain wall configuration.
Izergin’s determinant expression for the domain wall partition function is used by Wheeler in [3] to provide a recursive proof of Slavnov’s determinant corresponding spin chain.

Izergin’s expression for the domain wall partition function.

Z_N \left( \{w\}_N, \{z\}_N \right) = \frac{\prod_{i,j=1}^{N} (w_i - z_j + \eta)}{\prod_{1 \leq i < j \leq N} (w_i - w_j)(z_j - z_i)} \det \left( \frac{\eta}{(w_i - z_j + \eta)(w_i - z_j)} \right)_{1 \leq i,j \leq N}

Dual domain wall configurations have the same partition functions due to invariance under reversing all arrows. For the result of this note, we need the homogeneous limit of the above expression. Taking the limit \( z_i \to z, \{i = 1, \cdots, L\} \), we obtain

\[ Z_N^\text{hom} \left( \{w\}_N, z \right) = \frac{\prod_{1 \leq i \leq N} (w_i - z + \eta)^N}{\prod_{1 \leq i < j \leq N} (w_i - w_j)} \det \left( \phi^{(j-1)}(w_i, z) \right)_{1 \leq i,j \leq N}
\]

\[ \phi^{(j)}(w_i, z) = \frac{1}{j!} \frac{\eta}{(w_i - z + \eta)(w_i - z)} \]

2. The XXX spin-\( \frac{1}{2} \) chain

In this section, we recall the XXX spin-\( \frac{1}{2} \) chain that we need to discuss the EGSV expression of the structure constants in II. Our aim is to motivate the connection with the rational six-vertex model discussed in Section I.

2.1. Closed spin chains, open lattice segments, and spin variables. Consider a length-\( L \) closed spin chain. Label the sites sequentially using \( i \in \{1, 2, \cdots, L\} \) and represent the closed spin chain as a length-\( L \) segment of a 1-dimensional open lattice. Assign site \( i \) a spin variable \( \sigma_i, \sigma_1 \equiv \sigma_{L+1} \), that takes one of two possible values in a 2-dimensional space \( h_i \) with a basis \( \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \), which we refer to as ‘up’ and ‘down’. The space of states \( \mathcal{H} \) is the tensor product \( \mathcal{H} = h_1 \otimes \cdots \otimes h_L \). Every state in \( \mathcal{H} \) is an assignment of \( L \) definite-value (either up or down) spin variables to the sites of the spin chain. In computing scalar products, we wish to think of states in \( \mathcal{H} \) as initial states.

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The following result does not require that any set of rapidities satisfy Bethe equations.
2.2. Initial reference and dual reference states. $\mathcal{H}$ contains two distinguished states,

\begin{equation}
[L^\wedge] = \bigotimes_{i=1}^{L} \left( \begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array} \right)_i, \quad [L^\vee] = \bigotimes_{i=1}^{L} \left( \begin{array}{cc} 0 & 1 \\ 1 & 0 \end{array} \right)_i
\end{equation}

where $[L^\wedge]$ indicates $L$ spin states all of which are up, and $[L^\vee]$ indicates $L$ spin states all of which are down. These are the reference state and the dual reference state.

2.3. Final reference and dual reference states, and a variation. Consider a length-$L$ spin chain, and assign each site $i$ the space $h_i^*$ with the basis $\left( \begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array} \right)_i$. We construct a final space of states as the tensor product $\mathcal{H}^* = h_1^* \otimes \cdots \otimes h_L^*$. $\mathcal{H}^*$ contains two distinguished states

\begin{equation}
\langle [L^\wedge] \rangle = \bigotimes_{i=1}^{L} \left( \begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array} \right)_i, \quad \langle [L^\vee] \rangle = \bigotimes_{i=1}^{L} \left( \begin{array}{cc} 0 & 1 \\ 1 & 0 \end{array} \right)_i
\end{equation}

where all spins are up, and all spins are down. Finally, we consider the state

\begin{equation}
\langle [N_3^\wedge, (L - N_3)^\vee] \rangle = \bigotimes_{1 \leq i \leq N_3} \left( \begin{array}{cc} 0 & 1 \\ 1 & 0 \end{array} \right)_i \bigotimes_{(N_3+1) \leq i \leq L} \left( \begin{array}{cc} 0 & 1 \\ 1 & 0 \end{array} \right)_i
\end{equation}

where first $N_3$ spins from the left are down, and all remaining spins up.

2.4. Remark. The connection to the six-vertex model is clear. Every state of the periodic spin chain is analogous to a spin set in the six-vertex model. Periodicity is not manifest in the latter representation for the same reason that it is not manifest once we choose a labeling system. The initial and final reference and dual reference states are the spin-chain analogues of those discussed in Section 1.

2.5. The $R$-matrix. From an initial reference state, we can generate all other states in $\mathcal{H}$ using operators that flip the spin variables, one spin at a time. Defining these operators requires defining a sequence of objects. 1. The $R$-matrix, 2. The $L$-matrix, and finally, 3. The monodromy or $M$-matrix.

The $R$-matrix assigns a weight to the transition from a pair of initial spin states (for example the definite spin states on the left and lower segments that meet at a certain vertex) to a pair of final spin states (the definite spin states on the right and upper segments that meet at the same vertex as the initial ones). In the case of the rational XXX spin-$\frac{1}{2}$, this a transition between four possible initial spin states and four final spin states and the $R$-matrix is the $(4 \times 4)$-matrix

\begin{equation}
R_{ab}(u_a, u_b) = \begin{pmatrix}
a[u_a, u_b] & 0 & 0 & 0 \\
0 & b[u_a, u_b] & c[u_a, u_b] & 0 \\
0 & c[u_a, u_b] & b[u_a, u_b] & 0 \\
0 & 0 & 0 & a[u_a, u_b]
\end{pmatrix}_{ab}
\end{equation}

More formally, the $R$-matrix is an element of $\text{End}(h_a \otimes h_b)$, where $h_a$ is an auxiliary space and $h_b$ is another auxiliary space of the spin chain. The variables $u_a, u_b$ are the corresponding rapidity variables. The $R$-matrix intertwines these spaces.

The elements of the $R$-matrix in Equation 9 are the weights of the vertices of the rational six-vertex model. This is the origin of the connection of the two models. One can graphically represent the elements of 9 to obtain the
six vertices of the rational six-vertex model in Figure 2. Naturally, they satisfy the same properties, namely unitarity, crossing symmetry and the crucial Yang-Baxter equations that are required for integrability.

2.6. The $L$-matrix. The $L$-matrix of the XXX spin chain is a local operator that acts non-trivially on one site of the spin chain only. It acts non-trivially on the auxiliary space $h_a$ and on the $i$-th quantum space, and acts trivially all other quantum spaces. The mechanics of the construction and the precise action of the $L$-matrix require more space than we can afford in this note. We refer the reader to [29] for a detailed exposition.

2.7. The Monodromy matrix. The monodromy matrix is a global operator that acts on all sites in the spin chain. It is constructed as an ordered direct product of the $L$-matrices that act on single sites. It is typically written in $(2 \times 2)$ block form as

$$M_a(x, \{z\}_L) = \begin{pmatrix} A(x) & B(x) \\ C(x) & D(x) \end{pmatrix}_a$$

where the matrix entries are operators that act in $\mathcal{H} = h_1 \otimes \cdots \otimes h_L$. To simplify the notation, we have omitted the dependence of the elements of the $M$-matrix on the quantum rapidities $\{z\}$. This dependence is implied from now on. For the purposes of this note, the main aspect of the elements of the $M$-matrix that we need to know is that they can represented in six-vertex model terms as the horizontal lines in Figure 3. The $A$, $B$, $C$ and $D$-lines are the six-vertex model representation of the corresponding elements of the $M$-matrix. This representation is very useful and that is why we introduced it in Section 1.

2.8. Initial and final generic Bethe states. An initial (final) generic Bethe state is represented in six-vertex model terms as a $B$-configuration ($C$-configuration), as defined in Section 1 and illustrated on left (right) hand side of Figure 4. Note that the outcome of the action of the $N B$-lines ($C$-lines) on the initial (final) length-$L$ reference state produces a final (initial) spin system that can assume all possible spin states of net spin $(L - N)$. Each of these definite spin states is weighted by the weight of the corresponding lattice configuration (where when sums over all spins on the bulk segments).

2.9. Bethe eigenstates and Bethe equations. The initial and final reference states $|L\rangle$ and $\langle L|$ are eigenstates of the diagonal elements of the monodromy matrix. The eigenvalues are easy to compute in terms of the vertex weights and will not be listed here as we will not need them. We refer the reader to [2] [3] for these details. This makes these states eigenstates of the transfer matrix $T(x)$, which by definition is the trace of the monodromy $M$-matrix, that is $T(x) = \text{Tr} \left( M(x) \right)$. The rest of the eigenstates $\{\mathcal{O}\}$ of $T(x)$, that is

$$T(x)|\mathcal{O}\rangle_\beta = \left( A(x) + D(x) \right) |\mathcal{O}\rangle_\beta = E_{\mathcal{O}}(x)|\mathcal{O}\rangle_\beta$$

where $E_{\mathcal{O}}(x)$ is the corresponding eigenvalue, are generated using the BA, which is the statement that all eigenstates of $T(x)$ are created in two steps. 1. One acts on the initial reference state with the $B$-element of the monodromy matrix

$$|\mathcal{O}\rangle_\beta = B(u_{\beta N}) \cdots B(u_{\beta 1}) |L\rangle$$
where \( N \leq L \), since acting on \( |L^\lambda\rangle \) with more \( B \)-operators than the number of sites in the spin chain annihilates it.\(^2\) We require that the auxiliary space rapidity variables \( \{ u_{\beta 1}, \ldots , u_{\beta N} \} \) satisfy Bethe equations, hence the use of the subscript \( \beta \). That is, \( |O\rangle \) as well as \( \langle O| \) are eigenstates of \( T(x) \) if and only if

\[
(13) \quad \prod_{j=1}^{L} \frac{a[u_i, z_j]}{b[u_i, z_j]} = \prod_{j \neq i}^{N} \frac{b[u_j, u_i]}{b[u_i, u_j]}
\]

for all \( 1 \leq i \leq N \). Eigenstates of the transfer matrix \( T(x) \) are also eigenstates of the spin-chain Hamiltonian \([29]\). The latter is the spin-chain version of the 1-loop dilatation operator in SYM\(_4\). We construct eigenstates of \( T(x) \) in \( \mathcal{H}^* \) using the \( C \)-element of the \( M \)-matrix

\[
(14) \quad \beta \langle O\rangle = \langle [L^\lambda]\rangle C(u_{\beta 1}) \ldots C(u_{\beta N})
\]

where \( N \leq L \) to obtain a non vanishing result, and requiring that the auxiliary space rapidity variables satisfy the Bethe equations.

2.10. A sequence of scalar products that can be evaluated as determinants. Following [2] [3], we define the scalar product \( S[L, N_1, N_2] \), \( 0 \leq N_2 \leq N_1 \), that involves \( (N_1 + N_2) \) operators, \( N_1 B \)-operators with auxiliary rapidities that satisfy Bethe equations, and \( N_2 C \)-operators with auxiliary rapidities that are free.\(^1\) For \( N_2 = 0 \), we obtain, up to a non-dynamical factor, the domain wall partition function. For \( N_2 = N_1 \), we obtain Slavnov’s scalar product. These scalar products \( S[L, N_1, N_2] \) can be found in [2] [3] The purpose of the exercise is to show that \( S[L, N_1, N_2] \) is the partition function (weighted sum over all internal configurations) of the \( [L, N_1, N_2] \)-configurations introduced in Section 1.\(^4\)

Let \( \{ u \}_{\beta N_1} = \{ u_{\beta 1}, \ldots , u_{\beta N_1} \} \), \( \{ v \}_{N_2} = \{ v_1, \ldots , v_{N_2} \} \), \( \{ z \}_L = \{ z_1, \ldots , z_L \} \) be three sets of variables the first of which satisfies Bethe equations, \( 0 \leq N_2 \leq N_1 \) and \( 1 \leq N_1 \leq L \). We wish to define the scalar products

\[
(15) \quad S[L, N_1, N_2] \left( \{ u \}_{\beta N_1}, \{ v \}_{N_2}, \{ z \}_L \right) = \langle [N_3^X \wedge (L-N_3)^\lambda] \rangle \prod_{j=1}^{N_3} C(v_j) \prod_{j=1}^{N_1} B(u_{\beta j}) \langle [L^\lambda] \rangle \)
\]

where \( 0 \leq N_2 \leq N_1 \), \( N_3 = N_1 - N_2 \).\(^5\) It is clear that for \( N_2 = 0 \), we obtain a domain wall partition function, while for \( N_2 = N_1 \), we obtain Slavnov’s scalar product. In all cases, we assume that the auxiliary rapidities \( \{ u \}_{\beta N_1} \) obey the Bethe equations \([13]\), and use the subscript \( \beta \) to emphasize that, while the auxiliary rapidities \( \{ v \}_{N_2} \) are either free or also satisfy their own set of Bethe equations. When the latter is the case, this fact is not used. The quantum rapidities \( \{ z \}_L \) do not satisfy Bethe equations, and are taken to be equal to the same constant value in the homogeneous limit.

\(^1\)To avoid a proliferation of notation, we use \( N_1, N_2 \) and \( N_3 = N_1 - N_2 \), instead of the corresponding notation used in [2] [3]. The reason is that these variables will match the corresponding ones in Section 3.

\(^2\)Our choice of vertex weights in Equation (3), is such that our \( B \) and \( C \) operators as in Equation (16) are the same as the normalized \( B \) and \( C \) operators of [2]. Our expression for the restricted Slavnov product in Equation (15) agrees with that in [2].
2.11. **The scalar products** $S[L, N_1, N_2]$ **are** $[L, N_1, N_2]$-**configurations.** From the definition of $S[L, N_1, N_2]$, one can easily identify them as the BA versions of the six-vertex $[L, N_1, N_2]$-configurations. We will use this fact from now on.

2.12. **A determinant expression for the** $[L, N_1, N_2]$-**restricted Slavnov scalar product.** Following [2, 3], we consider the $(N_1 \times N_1)$ matrix

\[
\det \begin{pmatrix}
    f_1(z_1) & \ldots & f_1(z_{N_3}) & g_1(v_1) & \ldots & g_1(v_{N_2}) \\
    \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
    f_{N_1}(z_1) & \ldots & f_{N_1}(z_{N_3}) & g_{N_1}(v_1) & \ldots & g_{N_1}(v_{N_2})
\end{pmatrix}
\]

where $N_3 = N_1 - N_2$. Since the auxiliary rapidities $\{u\}_{\beta N_1}$ satisfy Bethe equations [13], following [2, 3]

\[
S[L, N_1, N_2] = \frac{N_S}{D_S} \det S \left( \{u\}_{\beta N_1}, \{v\}_{N_2}, \{z\}_L \right)
\]

\[
N_S = \prod_{i=1}^{N_1} \prod_{j=1}^{N_3} (u_i - z_j + \eta),
\quad
D_S = \prod_{1 \leq i < j \leq N_1} (u_j - u_i) \prod_{1 \leq i < j \leq N_2} (v_i - v_j) \prod_{1 \leq i < j \leq N_3} (z_i - z_j)
\]

To conclude, we have a determinant expression for the $[L, N_1, N_2]$-configurations introduced in Section 1. For the result in this note, we need the homogeneous limit of $S^{\text{hom}}[L, N_1, N_2]$. Taking the limit $z_i \to z$, $i \in \{1, \ldots, L\}$, the result is

\[
S^{\text{hom}}[L, N_1, N_2] = \frac{1}{\prod_{1 \leq i < j \leq N_1} (u_j - u_i) \prod_{1 \leq i < j \leq N_2} (v_i - v_j)} \det S^{\text{hom}} \left( \{u\}_{\beta N_1}, \{v\}_{N_2}, z \right)
\]

\[
S^{\text{hom}} \left( \{u\}_{\beta N_1}, \{v\}_{N_2}, z \right) =
\begin{pmatrix}
    \Phi_1^{(0)}(z) & \ldots & \Phi_1^{(N_3 - 1)}(z) & \Phi_1^{(N_3 - 1)}(v_{N_2}) & \ldots & \Phi_1^{(N_3 - 1)}(v_{N_1}) \\
    \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
    \Phi_{N_1}^{(0)}(z) & \ldots & \Phi_{N_1}^{(N_3 - 1)}(z) & \Phi_{N_1}^{(N_3 - 1)}(v_{N_2}) & \ldots & \Phi_{N_1}^{(N_3 - 1)}(v_{N_1})
\end{pmatrix}
\]
\[ \Phi^{(j)}_i = \frac{1}{j!} \partial_z^{(j)} f_i(z), \]
\[ g_{i}^{\text{hom}}(v_j) = \frac{\eta}{(u_i - v_j)} \left( \frac{v_j - z + \eta}{v_j - z} \right)^L \frac{N_1}{\prod_{k \neq i}(u_k - v_j + \eta)} \frac{N_1}{\prod_{k \neq i}(u_k - v_j - \eta)} \]

2.13. **The Gaudin norm.** Let us consider the original, unrestricted Slavnov scalar product, \( S[L, N_1, N_2 = N_1, N_3 = 0] \left( \{u\}_{\beta N_1}, \{v\}_{N_1}, \{z\}_L \right) \), and set \( \{v\}_{N_1} = \{u\}_{\beta N_1} \) to obtain the Gaudin norm \( N(\{u\}_{\beta N_1}) \) which is the square of the norm of the Bethe eigenstate with auxiliary rapidities \( \{u\}_{\beta N_1} \). It inherits a determinant expression that can be computed starting from that of the Slavnov scalar product that we begin with and taking the limit \( \{v\}_{N_1} \to \{u\}_{\beta N_1} \). Using \( N_i \) for \( N_1 \), and following \[2\], one obtains

\[ (20) \quad \mathcal{N}[L, N_i] \left( \{u\}_{\beta N_i}, \{z\}_L \right) = \eta^{N_i} \prod_{u \neq \beta} \frac{u_i - u_j + \eta}{u_i - u_j} \det \Phi' \left( \{u\}_{\beta N} \right) \]
\[ \Phi'_{ij} \left( \{u\}_{\beta N} \right) = -\partial_{u_j} \ln \left( \frac{u_i + z}{u_i - z} \right)^L \frac{1}{\prod_{k = 1}^{N} \frac{u_k - u_i + \eta}{u_k - u_i - \eta}} \]

We need the Gaudin norm to normalize the Bethe eigenstates that form the 3-point functions whose structure constants we are interested in.

In Section 1 we learned how to construct six-vertex model configurations, using horizontal lines that effectively act on vertical line segments with spin assignments, and defined the \([L, N_1, N_2]\)-configurations. In this section, we saw that all objects introduced in Section 1 have spin-chain analogues, and that the scalar products \( S[L, N_1, N_2] \) are partition functions of the \([L, N_1, N_2]\)-configurations, and that they can be evaluated in determinant form. In the following section, we will see that the structure constants \( c_{ij}^{(0)} \) are nothing but \( S[L, N_1, N_2] \) scalar products, up to simple factors.

3. **The structure constants of SYM**

In this section, we discuss the EGSV expression for the structure constants in view of what learned in Sections 1 and 2.

3.1. **Single-trace operators, normalization factors and pants diagrams.** Following 1, we consider gauge-invariant local single-trace operators \( \{O\} \), with 1-loop conformal dimensions \( \{\Delta_O\} \), that consist of two charged scalar fields that are not conjugates, and thereby map to Bethe eigenstates of an XXX spin-\( \frac{1}{2} \) chain. For example, a single-trace operators in the \( SU(2) \) sector spanned by the charged scalars \( \{Z, X\} \), is in the form \( \text{Tr}(ZZZZZZZZZZZ) \).

Any 2-point function of two operators in \( \{O\} \) is in the form in Equation (1). Any 3-point function of three operators in \( \{O\} \) is in the form in Equation (2). We choose the normalization factor \( N_i \) to be the Gaudin norm of the corresponding Bethe eigenstate.
Figure 10: A schematic representation of a 3-point function. State $O_1$ is at the top. State $O_2$ is at the bottom to the right. State $O_3$ is at the bottom to the left. For further details, please see the text.

\( N_i = \mathcal{N}[N_i, L_i] \left\{ \{ u \}_{\beta N_i}, \{ z \}_{L_i} \right\} \)

3.2. Perturbative expansion of structure constants. The structure constants of these operators have a perturbative expansion in the 't Hooft coupling constant $\lambda$,

\[ C_{ijk} = c_{ijk}^{(0)} + \lambda c_{ijk}^{(1)} + \ldots \]

We restrict the discussion to the leading coefficient $c_{ijk}^{(0)}$. In the limit $\lambda \to 0$, many single-trace operators have the same conformal dimension. This degeneracy is lifted at 1-loop level and certain linear combinations of single-trace operators have definite 1-loop anomalous conformal dimension. Remarkably, these linear combinations correspond to eigenstates of a closed XXX spin-$1/2$ chain. Their anomalous conformal dimensions are the corresponding Bethe eigenvalues. These closed spin chain states correspond to the circles at the boundaries of the pants diagram that can be constructed from Figure 10 as discussed above.

To construct three-point functions at the SYM\(_4\) operator level, the fundamental scalar fields in the operators $O_i$, $i = \{1, 2, 3\}$ are contracted by free propagators. Each propagator connects two fields, hence $L_1 + L_2 + L_3$ is an even number. The number of propagators between $O_i$ and $O_j$ is

\[ l_{ij} = \frac{1}{2}(L_i + L_j - L_k) \]

where $(i, j, k)$ take distinct values in $(1, 2, 3)$. We restrict our attention to the non-extremal case, that is, all $l_{ij}$’s are strictly positive. Following [1], the free propagators reproduce the factor $1/|x_i - x_j|^{{\Delta_i} + {\Delta_j} - {\Delta_k}}$ in Equation (2), where $\Delta_i = \Delta^{(0)}_i$, the tree-level conformal dimension. See Figure 10 for a schematic representation of a three point function of the type discussed in this note. The horizontal line segment between $l_i$ and $r_i$ represents the operator $O_i$. The lines
that start at $O_1$ and end at either $O_2$ or $O_3$ represent one type of propagators. More details are given below.

### 3.3. From single-trace operators to spin-chain states

One represents the single-trace operator $O_i$ of well-defined 1-loop anomalous conformal dimension $\Delta_i$ by a closed spin-chain Bethe eigenstate $|O_i\rangle_\beta$. Its eigenvalue $E_i$ is equal to $\Delta_i$. The number of fundamental fields $L_i$ in the trace is the length of the spin chain.

The single-trace operator $O_i$ is a composite operator built from weighted sums over traces of products of two complex scalar fundamental fields $\{X, Z\}$ and their conjugates. These fundamental fields are mapped to definite (up and down) spin states. A crucial step in [1] is the identification of the operator content of $O_i$, $i \in \{1, 2, 3\}$ with spin-chain spin states as follows.

| Operator | $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ | $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ | $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ |
|----------|--------------------------------|----------------|----------------------------------|
| $O_1$    | $Z$                           | $X$                        | $\bar{Z}$                        | $\bar{X}$                        |
| $O_2$    | $\bar{Z}$                     | $\bar{X}$                   | $Z$                             | $X$                             |
| $O_3$    | $Z$                           | $\bar{X}$                   | $\bar{Z}$                        | $X$                             |

Table 1. Identification of operator content of $O_i$, $i \in \{1, 2, 3\}$ with spin states in initial and final spin chain states

From Table 1, one can read the fundamental-scalar operator content of each single-trace operator $O_i$, $i \in \{1, 2, 3\}$, when it is an initial state and when it is a final state. For example, the fundamental scalar operator content of the initial state $|O_1\rangle$ is $\{Z, X\}$, and that of the corresponding final state $\langle O_1|$ is $\{\bar{Z}, \bar{X}\}$. The content of an initial state and the corresponding final state are related by the ‘flipping’ operation described below.

### 3.4. Remarks

1. Following [1], since we can Wick contract a scalar $f$ only with its conjugate $\bar{f}$, the above is the only choice that is fully contained in the $SU(2)$ sector of the theory and involves non-extremal correlators at the same time.

2. In computing structure constants, we identify the fundamental scalar fields with definite spin states only after we write the structure constants in terms of three scalar products and ignore one of them as trivial. It is only then that the identification becomes unique and simple.

### 3.5. Structure constants in terms of spin chain

Having mapped the single-trace operators $O_i$, $i \in \{1, 2, 3\}$ to spin-chain eigenstates, EGSV construct the structure constants in three steps.

**Step 1.** Split the lattice configurations that correspond to closed spin chain eigenstates into two parts. Consider the open 1-dimensional lattice configuration that corresponds to the $i$-th closed spin chain eigenstate, $i \in \{1, 2, 3\}$. This is schematically represented by a line in Figure 2 that starts at $l_i$ and ends at $r_i$. Split that, at point $c_i$ into left and right sub-lattice configurations of lengths $L_{i,L} = \frac{1}{2}(L_i + L_j - L_k)$ and $L_{i,R} = \frac{1}{2}(L_i + L_k - L_j)$ respectively.
Note that the lengths of the sub-lattices is fully determined by \( L_1, L_2 \) and \( L_3 \) which are fixed.

Following \cite{EGSV}, we express the single lattice configuration of the original closed spin chain state as a weighted sum of tensor products of states that live in two smaller Hilbert spaces. The latter correspond to closed spin chains of lengths \( L_{i,L} \) and \( L_{i,R} \) respectively. That is, \(|\mathcal{O}_i\rangle = \sum H_{L,R} |\mathcal{O}_i\rangle_l \otimes |\mathcal{O}_i\rangle_r\). The factors \( H_{L,R} \) were computed in \cite{EGSV} and were needed in \cite{1}, where one of the scalar products is generic and had to be expressed as an explicit sum. They will not be needed in this work as we use Bethe equations to evaluate this very sum as a determinant.

**Step 2. Map initial states to corresponding final states.** In \cite{1}, EGSV perform the mapping \(|\mathcal{O}_i\rangle_l \otimes |\mathcal{O}_i\rangle_r \rightarrow |\mathcal{O}_i\rangle_l \otimes_r \langle \mathcal{O}_i|\rangle\), using the operator \( \mathcal{F}\). that acts as follows.

\[
\mathcal{F} \left( |f_1 f_2 \cdots f_{L-1} f_L\rangle \right) = \langle f_L f_{L-1} \cdots f_2 f_1 | \langle
\]

In particular,

\[
\langle ZZ \cdots Z | ZZ \cdots Z \rangle = \langle ZZ \cdots Z | ZZ \cdots Z \rangle = 1, \quad \langle ZZ \cdots Z | ZZ \cdots Z \rangle = 0 \tag{24}
\]

More generally

\[
\langle f_1 f_2 \cdots f_n | f_1 f_2 \cdots f_n \rangle \sim \delta i_1 j_1 \delta i_2 j_2 \cdots \delta i_L j_L \tag{26}
\]

The ‘flipping’ operation in Equation \(24\) is the origin of the differences in assignments of fundamental scalar fields to initial and final operator states in Table \(1\). For example, \(|\mathcal{O}_1\rangle\) has field content \(\{Z, X\}\), but \(\langle \mathcal{O}_1|\rangle\) has field content \(\{\bar{Z}, \bar{X}\}\). This agrees with the fact that in computing \(\langle \mathcal{O}_i|\mathcal{O}_i\rangle\), free propagators can only connect charge conjugate scalar fields.

**Step 3. Compute scalar products.** The final step is to Wick contract pairs of initial states \(|\mathcal{O}_i\rangle_r\) and final states \(|\mathcal{O}_{i+1}\rangle_l\), where \(i \in \{1, 2, 3\}\) and \(i + 3 \equiv i\). The spin-chain equivalent of that is to compute the scalar products \(\langle \mathcal{O}_i|\mathcal{O}_{i+1}\rangle_l\), which in six-vertex model terms are \(BC\)-configurations. The most general scalar product that we can consider is the generic scalar product between two generic Bethe states

\[
S_{\text{generic}} \left( \{u\}, \{v\} \right) = \langle 0 | \prod_{j=1}^N \mathcal{C}(v_j) \prod_{j=1}^N \mathcal{B}(u_j) | 0 \rangle \tag{27}
\]

A computationally tractable evaluation of \(S_{\text{generic}}(\{u\}, \{v\})\) using the commutation relations of \(BA\) operators is known \cite{32}. Simpler expressions are obtained when the auxiliary rapidities of one (or both) states satisfies Bethe equations. The

\cite{EGSV} interpret the result of this operation as two open spin chains. In this note, we prefer to interpret it as two open lattice configuration that represents the closed spin-chain eigenstates, and stay clear of open spin chains. This is because the \(BA\) operators used throughout are those that act on lattice configurations that represent closed spin chain states. This is a matter of interpretation, and the final technical result remains the same.

\cite{EGSV} take pains to explain how the flipping operation is not the same as conjugation operation familiar from Quantum Mechanics textbooks. We refer the reader to \cite{1} for details. Further, we will not follow the notation of \cite{1} and add an upper arrow to distinguish a flipped state from a conjugated one as we will not consider any examples of the latter.
result in this case is a determinant. When only one set satisfies Bethe equations, one obtains a Slavnov scalar product. This was discussed in Section 2.

3.6. A preliminary, unevaluated expression. The above three steps lead to the following preliminary, unevaluated expression

\[ c_{123}^{(0)} = \mathcal{N}_{123} \sum_{a,b,c} \langle O_{3a} | O_{1a} \rangle_l \langle O_{1a} | O_{2b} \rangle_l \langle O_{2b} | O_{3c} \rangle_l \]

where the normalization factor, that will turn out to be a non-trivial object that depends on the norms of the Bethe eigenstates, is

\[ \mathcal{N}_{123} = \sqrt{\frac{L_1 L_2 L_3}{N_1 N_2 N_3}} \]

The sum in Equation 28 is to be understood as follows. 1. It is a sum over all possible ways to split the sites of each closed spin chain (represented as a segment in a 1-dimensional lattice) into a left part and a right part. We will see shortly that only one term in this sum survives. 2. It is a sum over all possible ways of partitioning the \( X \) or \( \bar{X} \) content of a spin chain state between the two parts that that spin chain was split into. We will see shortly that only one sum will survive.

3.7. A constraint that leads to simplifications. Wick contracting single-trace operators, we can only contract a fundamental scalar with its conjugate. Given the assignments in Table 1, one can see that 1. All \( Z \) fields in \( | O_3 \rangle \) contract with \( \bar{Z} \) fields in \( O_2 \). The reason is that there are \( \bar{Z} \) fields in \( O_2 \), and none in \( O_1 \). 2. All \( \bar{X} \) fields in \( O_3 \) contract with \( X \) fields in \( O_1 \). The reason is that there \( X \) fields only in \( O_1 \), and none in \( O_2 \). If the total number of scalar fields in \( O_i \) is \( L_i \), and the number of \( \{ X, \bar{X} \} \)-type scalar fields is \( N_i \), then

\[ l_{13} = N_3, \quad l_{23} = L_3 - N_3, \quad l_{12} = L_1 - N_3 \]

and, we have the constraint

\[ N_1 = N_2 + N_3 \]

From Equation 30 and Equation 31, we have the following simplifications. 1. There is only one way to split each lattice configuration that represents a spin chain into a left part and a right part. 2. The scalar product \( \langle O_{2b} | O_{3c} \rangle_l \) involves the fundamental scalar field \( Z \) (and only \( Z \)) in the initial state \( | O_{3c} \rangle_l \) as well as in the final state \( \langle O_{2b} | \). Using Table 1, we find that these states translate to an initial and a final reference state, respectively. This is represented in Figure 10 by the fact that no connecting lines (that stand for propagators of \( \{ X, \bar{X} \} \) states) connect \( O_2 \) and \( O_3 \). The scalar product of two reference states is \( \langle O_{2b} | O_{3c} \rangle_l = 1 \). 3. The scalar product \( \langle O_{1} | O_{3b} \rangle_l \) involves the fundamental scalar fields \( \bar{X} \) (and only \( \bar{X} \)) in the initial state \( | O_{3b} \rangle_l \) as well as in the final state \( \langle O_{1} | \). Using Table 1, we find that these states translate to an initial and a final dual reference state respectively. This is represented in Figure 10 by the high density of connecting lines (that stand for propagators of \( \{ X, \bar{X} \} \) states) between \( O_1 \) and \( O_3 \). The scalar product of two dual reference states is straightforward to evaluate in terms of domain wall partition functions. In the remaining scalar product \( \langle O_{1} | O_{2} \rangle_l \), both the initial state \( | O_{2} \rangle_l \) and the final state \( \langle O_{1} | \) involve \( \{ X, \bar{Z} \} \). These states
translate to up and down spin and the scalar product is generic. Using the BA commutation relations, it can be evaluated as a weighted sum \[29\].

3.8. The EGSV expression. In [1], EGSV put the above facts together and obtain an expression for \(c_{ijk}^{(0)}\) in Equation 28 in the form

\[(32)\]

\[c_{123}^{(0)} = N_{123} \mathcal{F}_1 \sum_{\alpha \cup \bar{\alpha} = \{u\}_{\beta N_1}} \mathcal{F}_2 \ r\langle [N_3^\vee] | \mathcal{O}_1 \rangle_l \ r\langle \mathcal{O}_1 | \mathcal{O}_2 \rangle_l\]

where the normalization factor \(N_{123}\) is defined in Equation \[29\], \(r\langle [N_3^\vee] \rangle\) is a dual reference state of length \(N_3\), and \(\mathcal{F}_1\) and \(\mathcal{F}_2\) are factors the precise form of which need not concern us here\[15\]. The sum in Equation 32 is over all possible ways to partition the rapidities \(\{u\}_{\beta N_1}\) into two sets \(\alpha\) and \(\bar{\alpha}\), with cardinality \(N_2\) and \(N_3\), respectively. In the next section, we organize the computation of \(c_{ijk}^{(0)}\) differently, and obtain a result that evaluates the sum in Equation 32 as a determinant.

4. A DETERMINANT EXPRESSION FOR THE STRUCTURE CONSTANTS

The idea of this note is to identify the expression in Equation 28 up simple factors, with the restricted scalar product \(S[L, N_1, N_2]\), which is the partition function of an \([L, N_1, N_2]\)-configuration, and that can be evaluated as a determinant. This requires two simple steps.

4.1. Step 1. Re-writing one of the scalar products. We use the facts that 1. \(r\langle \mathcal{O}_2 | \mathcal{O}_3 \rangle_l = 1\), and 2. \(r\langle \mathcal{O}_2 | \mathcal{O}_1 \rangle_l = i \langle \mathcal{O}_1 | \mathcal{O}_2 \rangle_r\), which is true for all scalar products, to re-write Equation 28 in the form

\[(33)\]

\[c_{123}^{(0)} = N_{123} \sum_{\alpha \cup \bar{\alpha} = \{u\}_{\beta N_1}} r\langle \mathcal{O}_3 | \mathcal{O}_1 \rangle_l \ i\langle \mathcal{O}_2 | \mathcal{O}_1 \rangle_r = N_{123} \left( r\langle \mathcal{O}_3 | \bigotimes \langle \mathcal{O}_2 \rangle | \mathcal{O}_1 \rangle \right)\]

where the right hand side of Equation 33 is a scalar product of the full initial state \(|\mathcal{O}_1\rangle\) (so we no longer have a sum over partitions of the rapidities \(\{u\}_{\beta N_1}\) since we no longer split the state \(\mathcal{O}_1\)) and two states that are pieces of original states that were split. This right hand side is identical to an \([L, N_1, N_2]\)-configuration, apart from the fact that it includes an \((N_3 \times N_3)\)-domain wall configuration, that corresponds to the dual reference state contribution of \(r\langle [N_3^\vee] \rangle\), that is not included in an \([L, N_1, N_2]\)-configuration.

4.2. Step 2. Accounting for the domain wall partition functions. Accounting for the domain wall partition function, and working in the homogeneous limit where all quantum rapidities are set to \(z = \frac{i}{2} \sqrt{-1}\), we obtain our result for the structure constants, which up to a factor, is in determinant form.

\[(34)\]

\[c_{123}^{(0)} = N_{123} \ Z_{N}^\text{hom} \ \left\{ w \right\}_{N_3}, \ \frac{1}{2} \sqrt{-1} \ \langle S^\text{hom} \ Z_{[L, N_1, N_2]} \ \left\{ u \right\}_{\beta N_1}, \ \left\{ v \right\}_{N_2}, \ \frac{1}{2} \sqrt{-1} \ \rangle\]

\[15\]EGSV obtain their expression in a coordinate Bethe Ansatz basis. This leads to factors relative to the algebraic Bethe Ansatz basis that we use in this note. We collect these factors in \(\mathcal{F}_1\) and \(\mathcal{F}_2\).
Figure 11: The six-vertex lattice configuration that corresponds, up to a normalization factor $N_{123}$, to the structure constant $c_{123}^{(0)}$.

where the normalization $N_{123}$ is defined in Equation [29] the $(N_3 \times N_3)$ domain wall partition function $Z_{\text{dom}}^{N_3} \left\{ \{w\}_{N_3}, \frac{1}{2} \sqrt{-1} \right\}$ is given in Equation [5] $S^{\text{hom}}[L, N_1, N_2] \left\{ \{u\}_{\beta N_1}, \{v\}_{N_2}, \frac{1}{2} \sqrt{-1} \right\}$, is an $(N_1 \times N_1)$ determinant expression of the partition function of an $[L, N_1, N_2]$-configuration, given in Equation [19] Notice that $\{v\}$ and $\{w\}$ are actually $\{v\}_\beta$ and $\{w\}_\beta$, that is, they satisfy Bethe equations, but this fact is not used.

The auxiliary rapidities $\{u\}$, $\{v\}$, and $\{w\}$, are those of the eigenstates $O_1$, $O_2$, and $O_3$, in [1], respectively.

5. Comments

Let us consider Figure [11] which shows the six-vertex representation of $c_{123}^{(0)}$, after a trivial scalar product between two reference states (one came from part of state $O_2$ and the other from part of state $O_3$) is ignored. In [1], EGSV split all three states, so they split state $O_1$ as well. This splitting is represented by the vertical dashed line in Figure [11]. Next they proceed to evaluate the two scalar products (the third is trivial). The $C$ operators in the two final (partial) states are well segregated. But the $N_B$ operators of the initial state $O_1$ must be partitioned into two sets. One of cardinality $(N_3 = N_1 - N_2)$ to match the $C$ operators from the remainder of $O_2$, and one of cardinality $N_2$ to match the $C$ operators from the remainder of $O_3$. There is no unique way to do this, and one can show explicitly that one has to sum over all partitions of the auxiliary rapidities $\{u\}$ of $O_1$. This is the origin of the sum in EGSV expression.

In this note, we do not split $O_1$, but we identify the configuration in Figure [11] as (up to minor modifications) an object that has a known partition function that can be expressed as a determinant. Another way to say it is that by not splitting $O_1$, it remained a Bethe eigenstate and we have effectively used the Bethe equations to put the partition function in determinant form. The Bethe
equations play a crucial role in the proof of the determinant form of this partition function [2, 3].

In [27], the limit where one of the operators is much smaller than the other two was considered. A precise match between weak and strong coupling in the Frolov-Tseytlin classical limit for a general class of classical solutions was obtained. In [28], 3-point functions between one large classical operator and two large BPS operators were computed at weak coupling. In [33], a multiple integral expression for the generic scalar product, and from that a multiple integral version of the EGSV expression was obtained. In [34], a systematic perturbative study of 3-point functions at 1-loop level, involving single-trace operators up to length five, was performed. In [35], a non-trivial numerical check showed that the result in this note agrees with the EGSV expression in [1].

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