On the Laxton Group

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Abstract. In 1969, Laxton defined a multiplicative group structure on the set of rational sequences satisfying a fixed linear recurrence of degree two. He also defined some natural subgroups of the group, and determined the structures of their quotient groups. Nothing has been known about the structure of Laxton’s whole group and its interpretation. In this paper, we redefine his group in a natural way and determine the structure of the whole group, which clarifies Laxton’s results on the quotient groups. This definition makes us possible to use the group to show various properties of such sequences.

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1 The Laxton group

Let $P$ and $Q$ are integers with $Q \neq 0$.

We consider linear recurrence sequences associated to the characteristic polynomial $f(t) := t^2 - Pt + Q$. Namely, they are determined by $w_{n+2} - Pw_{n+1} + Qw_n = 0$ with the first rational numbers $w_0$ and $w_1$. Let $\theta_1$ and $\theta_2$ be the roots of $f(t)$. We assume

\[ D := \text{disc}(f) = P^2 - 4Q = (\theta_1 - \theta_2)^2 \neq 0. \]

We define an equivalence relation $\sim^*$ on the set of the linear recurrence sequences. For $w = (w_n)$, $v = (v_n)$, we write $w \sim^* v$ if there exists $\lambda \in \mathbb{Q}^\times$ and $\nu \in \mathbb{Z}$ such that $w_n = \lambda v_{n+\nu}$ for any $n \in \mathbb{Z}$. Laxton [3] considered the following quotient set using this relation:

\[ G^*(f) := \{(w_n)_{n \in \mathbb{Z}} \mid w_0, w_1 \in \mathbb{Q} \text{ with } A(w_1, w_0) \neq 0, \ w_{n+2} - Pw_{n+1} + Qw_n = 0 \]

for any $n \in \mathbb{Z}$, and there exists $\nu \in \mathbb{Z}$ such that $w_k \in \mathbb{Z}$ for any $k \geq \nu$ } $/ \sim^*$,

where $A(w_1, w_0) := w_1^2 - Pw_0w_1 + Qw_0^2$. Laxton introduced a product on $G^*(f)$ as follows. For classes $W \in G^*(f)$, let $(w_n)$ and $(v_n)$ are representatives of the classes $W$ and $V$, respectively. The product $W \times V$ of $W$ and $V$ is, then, defined by the class of $(u_n)$ with $u_n = (A\theta_1^n - BD\theta_2^n)/((\theta_1 - \theta_2))^n$, where $A = w_1 - w_0\theta_2$, $B = w_1 - w_0\theta_1$, $C = \nu_1 - \nu_0\theta_2$ and $D = \nu_1 - \nu_0\theta_1$. We call $G^*(f)$ the Laxton group. Let $p$ be a prime number with $p \nmid Q$. For $w = (w_n)$, if $w_n \in p\mathbb{Z}_{(p)}$
for some \( n \), we say \( p \) is a divisor of \( w \) and write \( p \mid w \), where \( \mathbb{Z}_{(p)} \) is the localization of the integers at \( p \). Laxton defined the following subgroups of \( G^*(f) \):

\[
G^*(f, p)_L := \{ w \in G^*(f) \mid p \nmid w \text{ for all } w \in W \},
\]

\[
K^*(f, p)_L := \{ w \in G^*(f) \mid \text{there exists } w \in W \text{ for which } w_0, w_1 \in \mathbb{Z} \text{ and } p \nmid A(w_1, w_0) \},
\]

\[
H^*(f, p)_L := \{ w \in G^*(f) \mid W^n \in G^*(f, p)_L \text{ for some } n \in \mathbb{Z} \}.
\]

Laxton proved the following theorem on the quotient groups.

**Theorem 1 (Laxton [53, Theorem 3.7])** Let \( p \) be a prime number with \( p \nmid Q \), and \( r(p) \) be the rank of the Lucas sequence \( F \) (see Definitions 2 and 4).

1. Assume \( p \nmid D \). If \( \mathbb{Q} (\theta_1) \neq \mathbb{Q} \) and \( p \) is inert in \( \mathbb{Q} (\theta_1) \), then \( G^*(f) = H^*(f, p)_L = K^*(f, p)_L \) and \( G^*(f)/G^*(f, p)_L \) is a cyclic group of order \((p + 1)/r(p)\).

2. Assume \( p \nmid D \). If \( \mathbb{Q}(\theta_1) = \mathbb{Q} \) or \( \mathbb{Q}(\theta_1) \neq \mathbb{Q} \) and \( p \) splits in \( \mathbb{Q}(\theta_1) \), then \( G^*(f)/H^*(f, p)_L \) is an infinite cyclic group, and \( H^*(f, p)_L = K^*(f, p)_L \) and \( H^*(f, p)_L \) is a cyclic group of order \((p - 1)/r(p)\).

3. If \( p \mid D \) and \( p^2 \nmid D \), then \( G^*(f) = H^*(f, p)_L \) and \( K^*(f, p)_L = G^*(f, p)_L \). Furthermore, if \( p \neq 2 \), then \( G^*(f)/G^*(f, p)_L \) is a cyclic group of order two.

Laxton made no assumption in \( p^2 \nmid D \) of Theorem 1 [3]. Suwa [3] recently pointed out that this assumption is necessary and gave counterexamples that did not hold in the case \( p^2 \nmid D \) of Theorem 1 [3], and gave the correct formulation above and the proof for the case (Corollary 2 [4]). He also gives in his paper an interpretation of linear recurrence sequences of degree two from a viewpoint of the theory of group schemes.

Although Laxton studied structures of the quotient groups of \( G^*(f) \), he did not study the group \( G^*(f) \) itself. The aims of this paper are to redefine Laxton’s group in a natural way (Theorems 3 and 4) and to give structure theorems of the group itself and the subgroups. One of our main results is the following theorem.

**Theorem 2 (Theorems 6, 7 and 8)** Notations being as above. Put \( D = p^s D_0 \) with \( s \geq 0 \), \( p \nmid D_0 \).

1. If \( f(t) \) is irreducible over \( \mathbb{Q} \), then we have

\[
G^*(f) \twoheadrightarrow \mathbb{Q}(\theta_1)^\times / \mathbb{Q}(\theta_1)^\times.
\]

If \( f(t) \) is reducible over \( \mathbb{Q} \), then we have

\[
G^*(f) \twoheadrightarrow \mathbb{Q}^\times / (\theta_1 \theta_2^{-1}).
\]

2. There exists the following exact sequence

\[
1 \rightarrow G^*(f, p)_L \rightarrow K^*(f, p)_L \xrightarrow{\text{red}_p} G^*_{\mathbb{F}_p}(f) \rightarrow 1
\]

where \( G^*_{\mathbb{F}_p}(f) \) is the group of equivalence classes of linear recurrence sequences over the finite field \( \mathbb{F}_p \).

3. If \( f(t) \) is irreducible over \( \mathbb{Q} \), then we have

\[
K^*(f, p)_L \twoheadrightarrow \mathbb{Z}(\theta_1)^\times / \mathbb{Z}(\theta_1)^\times.
\]

If \( f(t) \) is reducible over \( \mathbb{Q} \), then we have

\[
K^*(f, p)_L \twoheadrightarrow \begin{cases} 
\mathbb{Z}(\theta_1)^\times / (\theta_1 \theta_2^{-1}) & \text{if } p \nmid D, \\
(1 + p^2 \mathbb{Z}(\theta_1))/(\theta_1 \theta_2^{-1}) & \text{if } p | D.
\end{cases}
\]
On the Laxton Group

The content of this paper is as follows. In §2 we begin with redefining the group law on the set of linear recurrence sequences, which gives a natural interpretation of Laxton’s product. In §3 we determine the group structures of the group of the set of linear recurrence sequences according to the irreducibility of the characteristic polynomial \( f(t) \). In §4 we introduce two relations on the group of linear recurrence sequences, and determine the group structures of the quotient groups by these relations. In particular, we can determine that of the Laxton group \( G^*(f) \). In §5 we redefine the subgroups \( G^*(f,p)_L \), \( K^*(f,p)_L \) and \( H^*(f,p)_L \) for a fixed prime number \( p \). From our new definitions, we see that \( G^*(f,p)_L \) is the kernel of the reduction map from \( K^*(f,p)_L \) to the group \( G^*_p(f) \) of equivalence classes of linear recurrence sequences of the finite field \( \mathbb{F}_p \). Furthermore, we study the relations between these subgroups and the rank \( r(p) \) of the Lucas sequence, which is a classical invariant concerning Artin’s conjecture on primitive roots. In §6 and 7 we determine the group structures of \( K^*(f,p)_L \), \( G^*(f)/K^*(f,p)_L \) and \( K^*(f,p)/G^*(f,p) \), respectively by using \( p \)-adic logarithm functions. The results in these sections yield Laxton’s (Theorems 1) and Suwa’s theorems. Suwa first gave a proof in the case \( p^2|D \) from a viewpoint of the theory of group schemes.

2 Group laws of Linear recurrence sequences

Let \( R \) be an integral domain. In order to discuss in the general situation, we consider the sequences \( (w_n) \) of \( R \) determined by

\[
w_{n+2} - Pw_{n+1} + Qw_n = 0,
\]

for fixed elements \( P, Q \in R \) with \( Q \in R^\times \). Let \( \theta_1 \) and \( \theta_2 \) be the roots of the characteristic polynomial

\[
f(t) := t^2 - Pt + Q \in \mathbb{R}[t]
\]

in an algebraic closure \( \overline{K}_R \) of the quotient field \( k_R \) of \( R \). Put

\[
d := \text{disc}(f) = P^2 - 4Q \in R.
\]

Define

\[
\mathcal{J}(f, R) := \{(w_n)_{n \in \mathbb{Z}} \mid w_0, w_1 \in R, \ w_{n+2} - Pw_{n+1} + Qw_n = 0 \text{ for any } n \in \mathbb{Z}\}.
\]

By the assumption \( Q \in R^\times \), any sequence \( (w_n) \in \mathcal{J}(f, R) \) satisfies \( w_n \in R \) for any \( n \in \mathbb{Z} \). All sequences in \( \mathcal{J}(f, R) \) have the characteristic polynomial \( f(t) \), and the set \( \mathcal{J}(f, R) \) has a natural \( R \)-module structure and is isomorphic to a free \( R \)-module of rank 2:

\[
V_f(R) := \left\{ \begin{pmatrix} a_1 \\ a_0 \end{pmatrix} \mid a_1, a_0 \in R \right\}
\]

with an isomorphism given by

\[
\phi_R : \mathcal{J}(f, R) \xrightarrow{\sim} V_f(R), \quad (w_n) \mapsto \begin{pmatrix} w_1 \\ w_0 \end{pmatrix}.
\]

Furthermore, we define an isomorphism of \( R \)-modules \( \varphi_R \) by

\[
\varphi_R : V_f(R) \xrightarrow{\sim} \mathcal{O}_R, \quad \begin{pmatrix} a_1 \\ a_0 \end{pmatrix} \mapsto a_1 - a_0 t \mod f(t),
\]

where \( \mathcal{O}_R := R[t]/(f(t)) \) is the quotient ring of \( R[t] \). We define ring structures on \( V_f(R) \) and \( \mathcal{J}(f, R) \) via the maps \( \phi_R \) and \( \varphi_R \) induced from that of \( \mathcal{O}_R \). Put

\[
\mathcal{B} := \begin{pmatrix} P - Q \\ P \\ 0 \end{pmatrix}.
\]

Since \( \det(\mathcal{B}) = Q \in R^\times \), we have \( \mathcal{B} \in \text{GL}_2(R) \). There is a natural left action of the group \( \langle \mathcal{B} \rangle \) on \( V_f(R) \) because the matrix \( \mathcal{B} \) satisfies

\[
\begin{pmatrix} w_{n+1} \\ w_n \end{pmatrix} = \mathcal{B}^n \begin{pmatrix} w_1 \\ w_0 \end{pmatrix},
\]

(2.3)
for any $n \in \mathbb{Z}$. We define left actions of $\mathcal{B}$ on $\mathcal{S}(f, R)$ and $\mathcal{O}_R$ via the maps $\varphi_R$ and $\varphi_R$, respectively. From

$$\varphi_R \left( \mathcal{B} \begin{pmatrix} a_1 \\ a_0 \end{pmatrix} \right) = (Pa_1 - Qa_0) - a_1t = (P - t)(a_1 - a_0t), \quad (2.4)$$

for any $\begin{pmatrix} a_1 \\ a_0 \end{pmatrix} \in V_f(R)$, the action of $\mathcal{B}$ on $\mathcal{O}_R$ is given by the multiplication by $P - t$. In particular, we get

$$\mathcal{B} \begin{pmatrix} a_1 \\ a_0 \end{pmatrix} \ast \begin{pmatrix} b_1 \\ b_0 \end{pmatrix} = \begin{pmatrix} a_1 \\ a_0 \end{pmatrix} \ast \mathcal{B} \begin{pmatrix} b_1 \\ b_0 \end{pmatrix} = \mathcal{B} \left\{ \begin{pmatrix} a_1 \\ a_0 \end{pmatrix} \ast \begin{pmatrix} b_1 \\ b_0 \end{pmatrix} \right\}, \quad (2.5)$$

for any $\begin{pmatrix} a_1 \\ a_0 \end{pmatrix}, \begin{pmatrix} b_1 \\ b_0 \end{pmatrix} \in V_f(R)$. Furthermore, we get

$$\mathcal{B}^n \begin{pmatrix} a_1 \\ a_0 \end{pmatrix} = \begin{pmatrix} P \\ 1 \end{pmatrix} \ast \begin{pmatrix} a_1 \\ a_0 \end{pmatrix}, \quad (2.6)$$

for any $n \in \mathbb{Z}$. For a class of $a_1 - a_0t$ in $\mathcal{O}_R$, we have

$$(a_1 - a_0t)(1 - t) = (1 - t) \begin{pmatrix} a_1 \\ a_0 \end{pmatrix} = a_1 - a_0Q \in (\mathcal{O}_R) \quad (2.7)$$

Define the norm of a class of $a_1 - a_0t \in \mathcal{O}_R$ by

$$N(a_1 - a_0t) := \det \begin{pmatrix} a_1 \\ a_0 \end{pmatrix} = \begin{pmatrix} a_1 - a_0Q \\ a_0 \end{pmatrix} \in \mathcal{O}_R \quad (2.8)$$

Remark 1 If $f(t)$ is irreducible over the quotient field $k_R$ of $R$, then $N(a_1 - a_0t) \neq 0$ if and only if $a_0 \neq 0$ or $a_1 \neq 0$.

By the definition of the norm and $(2.7), the norm is multiplicative, namely we have

$$N((a_1 - a_0t)(b_1 - b_0t)) = N(a_1 - a_0t)N(b_1 - b_0t),$$

for any classes of $a_1 - a_0t, b_1 - b_0t$ of $\mathcal{O}_R$. From the fact, we can see that a class of $a_1 - a_0t$ is invertible in $\mathcal{O}_R$ if and only if $N(a_1 - a_0t) \in \mathcal{O}_R \times$. In particular, since

$$(P - t)(1 - t) = (1 - t) \mathcal{B},$$

we have $N(P - t) = \det(\mathcal{B}) = Q \in \mathcal{O}_R \times$, and hence $P - t$ is invertible in $\mathcal{O}_R$. Since the norm is multiplicative, $(2.8)$ and $(2.9)$ yield

$$N(w_{n+1} - w_nt) = N(P - t)^nN(w_{t} - w_0t) = Q^nN(w_{t} - w_0t), \quad (2.9)$$

for any integer $n$ and any sequence $(w_n) \in \mathcal{S}(f, R)$. We can endow the inverse image of $\mathcal{O}_R \times$ by $\varphi_R:

$$V_f(R)^\times := \varphi_R^{-1}(\mathcal{O}_R \times) = \left\{ \begin{pmatrix} a_1 \\ a_0 \end{pmatrix} \in V_f(R) \bigg| A(a_1, a_0) \in \mathcal{O}_R \times \right\}$$

where $A(a_1, a_0) := N(a_1 - a_0t)$, and its inverse image by $\varphi_R:

$$\mathcal{S}(f, R)^\times := \varphi_R^{-1}(V_f(R)^\times) = \left\{ (w_n) \in \mathcal{S}(f, R) \big| A(w_1, w_0) \in \mathcal{O}_R \times \right\}$$

with the structure of a multiplicative group induced from $\mathcal{O}_R$. If $\begin{pmatrix} a_1 \\ a_0 \end{pmatrix}$ and $\begin{pmatrix} b_1 \\ b_0 \end{pmatrix}$ are elements of $V_f(R)^\times$, then the corresponding product in $\mathcal{O}_R$ is

$$(a_1 - a_0t)(b_1 - b_0t) \equiv (a_1b_1 - a_0b_0Q) - (a_0b_1 + a_1b_0 - Pa_0b_0)t \pmod{f(t)}.$$
Thus the multiplication in $V_f(R)^\times$ is given by
\[
\begin{pmatrix} a_1 \\ a_0 \end{pmatrix} \ast \begin{pmatrix} b_1 \\ b_0 \end{pmatrix} = \begin{pmatrix} a_1b_1 - Qa_0b_0 \\ a_0b_1 + a_1b_0 - P_a0b_0 \end{pmatrix}.
\]

(2.10)

The identity element is \( \begin{pmatrix} 1 \\ 0 \end{pmatrix} \) and \( \begin{pmatrix} a_1 \\ a_0 \end{pmatrix}^{-1} = A(a_1, a_0)^{-1} \begin{pmatrix} a_1 - P_a0 \\ -a_0 \end{pmatrix} \). If \( \begin{pmatrix} a_1 \\ a_0 \end{pmatrix} \in V_f(R)^\times \), then we have \( \varphi_R \left( B \begin{pmatrix} a_1 \\ a_0 \end{pmatrix} \right) \in O_R \), from (2.4) and \( P - t \in O_R^\times \), and hence \( B \begin{pmatrix} a_1 \\ a_0 \end{pmatrix} \in V_f(R)^\times \). Therefore, the multiplicative groups \( V_f(R)^\times, O_R^\times \) and \( \mathcal{S}(f, R)^\times \) have the left actions of \( \langle B \rangle \).

The roots \( \theta_1 \) and \( \theta_2 \) of \( f(t) \) are the eigenvalues of \( B \) and the corresponding eigenspaces are \( \langle \theta_1 \rangle \) and \( \langle \theta_2 \rangle \) respectively. Hence, if \( d \neq 0 \), then \( B \) is diagonalized by \( \mathcal{P} := \begin{pmatrix} \theta_1 & 0 \\ 0 & \theta_2 \end{pmatrix} \in \text{GL}_2(\mathbb{K}_R) \) as
\[
\mathcal{P}^{-1}B\mathcal{P} = \begin{pmatrix} \theta_1 & 0 \\ 0 & \theta_2 \end{pmatrix}.
\]

Let \( w = (w_n)_{n \in \mathbb{Z}} \in \mathcal{S}(f, R) \). The general term is given by
\[
w_n = \frac{A\theta_1^n - B\theta_2^n}{\theta_1 - \theta_2},
\]
where \( A = w_1 - w_0\theta_2 \) and \( B = w_1 - w_0\theta_1 \). We have
\[
AB = A(w_1, w_0) \in R.
\]

**Definition 1** We call the sequence \( \mathcal{F} = (F_n) \in \mathcal{S}(f, R) \) with \( F_0 = 0 \) and \( F_1 = 1 \) the Lucas sequence, and the sequence \( \mathcal{L} = (L_n) \in \mathcal{S}(f, R) \) with \( L_0 = 2 \) and \( L_1 = P \) the companion Lucas sequence after Lucas, who first introduced them in [5]. Their general terms are \( F_n = \frac{\theta_1^n - \theta_2^n}{\theta_1 - \theta_2} \) and \( L_n = \theta_1^n + \theta_2^n \).

Next, we recall a product on the sets \( \mathcal{S}(f, R)^\times \) introduced by Laxton [3] (see also [1]).

**Definition 2** Let \( w = (w_n), v = (v_n) \in \mathcal{S}(f, R)^\times \). Write
\[
w_n = \frac{A\theta_1^n - B\theta_2^n}{\theta_1 - \theta_2}, \quad v_n = \frac{C\theta_1^n - D\theta_2^n}{\theta_1 - \theta_2},
\]
where \( A = w_1 - w_0\theta_2, B = w_1 - w_0\theta_1, C = v_1 - v_0\theta_2 \) and \( D = v_1 - v_0\theta_1 \). Laxton defined the product \( w \times v = u = (u_n) \) by
\[
u_n = \frac{AC\theta_1^n - BD\theta_2^n}{\theta_1 - \theta_2},
\]
for any \( n \in \mathbb{Z} \). In particular, \( u_0 \) and \( u_1 \) are given by \( u_0 = w_0v_1 + w_1v_0 - P_vw_0 \), \( u_1 = w_1v_1 - Q_vw_0 \). We get \( u \in \mathcal{S}(f, R)^\times \) since \( A(u_1, u_0) = ABCD = A(w_1, w_0)A(v_1, v_0) \in R^\times \). The associativity is trivial, and the identity is the Lucas sequence \( \mathcal{F} = (F_n) \). The inverse element of
\[
w = (w_n), \quad w_n = \frac{A\theta_1^n - B\theta_2^n}{\theta_1 - \theta_2}
\]
is given by
\[
w^{-1} = (u_n), \quad u_n = A(w_1, w_0)^{-1} \frac{B\theta_1^n - A\theta_2^n}{\theta_1 - \theta_2}.
\]

The multiplicative group structure on \( \mathcal{S}(f, R)^\times \) defined by Laxton coincides with one induced from \( O_R^\times = (R[t]/(f(t)))^\times \) via maps \( \varphi_R \) and \( \varphi_R \) in [21] and [22], respectively. We get the following theorem.
**Theorem 3** Let $R$ be an integral domain. The group structure of $\mathcal{J}(f, R)^\times$ defined by Laxton coincides with one induced from $\mathcal{O}_R^\times = (R[t]/(f(t)))^\times$ via the maps $\phi_R$ and $\varphi_R$:

$$
\mathcal{J}(f, R)^\times \xrightarrow{\phi_R} V_f(R)^\times \xrightarrow{\varphi_R} \mathcal{O}_R^\times = (R[t]/(f(t)))^\times,
$$

where $w = (w_n) \mapsto \left(\begin{smallmatrix} w_1 \\ w_0 \end{smallmatrix}\right) \mapsto w_1 - w_0t$.

Furthermore, these group isomorphisms are compatible with the following action of $(B)$:

(i) For $w = (w_n) \in \mathcal{J}(f, R)^\times$ and $v \in \mathbb{Z}$,

$$
B^v \cdot w = (v_n),
$$

where $v_n = w_{n+v}$ for any $n \in \mathbb{Z}$.

(ii) For $(w_1, w_0) \in V_f(R)^\times$ and $v \in \mathbb{Z}$,

$$
B^v \cdot \left(\begin{smallmatrix} w_1 \\ w_0 \end{smallmatrix}\right) = \left(\begin{smallmatrix} w_1 \\ w_0 \end{smallmatrix}\right) + \left(\begin{smallmatrix} v_1 \\ 0 \end{smallmatrix}\right) = \left(\begin{smallmatrix} w_1 + v_1 \\ w_0 + v_0 \end{smallmatrix}\right)
$$

(the right-hand side is the ordinary matrix product).

(iii) For $w_1 - w_0t \in \mathcal{O}_R^\times$ and $v \in \mathbb{Z}$,

$$
B^v (w_1 - w_0t) = (P - t)^v (w_1 - w_0t).
$$

3 Group structures of $\mathcal{J}(f, R)^\times$

In this section, we assume that $R$ is a unique factorization domain, and study the group structures of

$$
\mathcal{J}(f, R)^\times \xrightarrow{\phi_R} V_f(R)^\times \xrightarrow{\varphi_R} \mathcal{O}_R^\times = (R[t]/(f(t)))^\times,
$$

according to the irreducibility of the polynomial $f(t)$.

**Theorem 4** (1) If $f(t)$ is irreducible over $R$, then we have an isomorphism of $R$-algebras

$$
\psi_R : V_f(R) \xrightarrow{\sim} R[\theta_1], \quad \left(\begin{smallmatrix} a_1 \\ a_0 \end{smallmatrix}\right) \mapsto a_1 - a_0\theta_1.
$$

This yields a group isomorphism

$$
\mathcal{J}(f, R)^\times \simeq V_f(R)^\times \simeq R[\theta_1]^\times.
$$

(2) Assume that $f(t)$ is reducible over $R$, and hence $d = \text{disc}(f) \in \mathbb{R}^2$.

(i) The case where $f(t)$ has no double root in $R$: Let $H_R$ be an $R$-subalgebra of $R \times R$ defined by

$$
H_R := \{(x, y) \in R \times R \mid x \equiv y \pmod{\sqrt{d}}\},
$$

(if $d \in \mathbb{R}^\times$, then $H_R = R \times R$). We have an isomorphism of $R$-algebras

$$
\psi_R : V_f(R) \xrightarrow{\sim} H_R, \quad \left(\begin{smallmatrix} a_1 \\ a_0 \end{smallmatrix}\right) \mapsto (a_1 - a_0\theta_1, a_1 - a_0\theta_2).
$$

This yields a group isomorphism

$$
\mathcal{J}(f, R)^\times \simeq V_f(R)^\times \simeq H_R^\times = \{(x, y) \in \mathbb{R}^\times \times \mathbb{R}^\times \mid x \equiv y \pmod{\sqrt{d}}\}.
$$
(ii) The case where $f(t)$ has a double root $\theta$ in $R$: We have an isomorphism of $R$-algebras
\[
\psi_R : V_f(R) \simto R[\varepsilon]/(\varepsilon^2), \quad \left(\frac{a_1}{a_0}\right) \mapsto a_1 - a_0(\varepsilon + \theta) \mod \varepsilon^2.
\]

This yields a group isomorphism
\[
\mathcal{J}(f, R) \cong V_f(R)[\varepsilon]/(\varepsilon^2) \cong \{x + y\varepsilon \mod \varepsilon^2 \mid x \in R^\times, y \in R\}.
\]

Proof The assertion (1) follows from (2.2) and the isomorphism
\[
\mathcal{O}_R = R[t]/(f(t)) \simto R[\theta_1], \quad g(t) \mod f(t) \mapsto g(\theta_1).
\]
Next, we show the assertion (2).

In the case (1), we have $\theta_1 \neq \theta_2$ in $R$. Consider the following homomorphism of $R$-algebras
\[
\mathcal{O}_R = R[t]/(f(t)) \longrightarrow R[t]/(t - \theta_1) \times R[t]/(t - \theta_2) \simto R \times R,
\]

(3.2)
\[
g(t) \mod f(t) \mapsto (g(t) \mod t - \theta_1, g(t) \mod t - \theta_2) \mapsto (g(\theta_1), g(\theta_2)).
\]

The map is injective, and the image is in the set $H_R$ since $\theta_1 \equiv \theta_2 \mod \sqrt{\theta}$. Conversely, we have
\[
g(t) = \frac{1}{\theta_2 - \theta_1}(g(t - \theta_1) - x(t - \theta_2)) \in R[t]
\]
for any $(x, y) \in H_R$, and $g(t) \mod f(t)$ maps to $(x, y)$ by the map (3.2), we get the isomorphism
\[
R[t]/(f(t)) \cong H_R,
\]

and the assertion follows from the isomorphism and (2.2).

In the case (2), we have $\theta_1 = \theta_2$ in $R$. The assertion follow from (2.2) and the following isomorphism of $R$-algebras
\[
\mathcal{O}_R = R[t]/(f(t)) \simto R + R\varepsilon, \quad a_1 - a_0t \mod f(t) \mapsto a_1 - a_0(\varepsilon + \theta).
\]

\[\Box\]

4 Equivalence classes

Let $R$ be a unique factorization domain, and assume $Q \in R^\times$. In this section, we introduce two relations $\sim$ and $\sim^*$ on the set $\mathcal{J}(f, R)$, and consider the quotient sets of $\mathcal{J}(f, R)^\times$ by these relations. The group structure of $\mathcal{J}(f, R)^\times$ defined in 2 naturally induce the group structures on the quotient sets. Note that we have $w_n \in R$ for any $n \in \mathbb{Z}$ by the assumption $Q \in R^\times$.

Definition 3 Let $w = (w_n)$, $v = (v_n) \in \mathcal{J}(f, R)$.

1. We define $w \sim v$ if there exists $\lambda \in R^\times$ such that $w_n = \lambda v_n$ for any $n \in \mathbb{Z}$.
2. We define $w \sim^* v$ if there exists $\lambda \in R^\times$ and $\nu \in \mathbb{Z}$ such that $w_n = \lambda v_{n+\nu}$ for any $n \in \mathbb{Z}$.

These relations are equivalence relations on the set $\mathcal{J}(f, R)$. By the isomorphism (2.1), we can introduce the corresponding relations $\sim$ and $\sim^*$ on the set $V_f(R)$. Let $\left(\frac{a_1}{a_0}\right), \left(\frac{b_1}{b_0}\right) \in V_f(R)$.

1. We have $\left(\frac{a_1}{a_0}\right) \sim_{R} \left(\frac{b_1}{b_0}\right)$ if there exists $\lambda \in R^\times$ such that $\left(\frac{a_1}{a_0}\right) = \lambda \left(\frac{b_1}{b_0}\right)$.
2. We have $\left(\frac{a_1}{a_0}\right) \sim^*_{R} \left(\frac{b_1}{b_0}\right)$ if there exist $\lambda \in R^\times$ and $\nu \in \mathbb{Z}$ such that $\left(\frac{a_1}{a_0}\right) = \lambda \beta^\nu \left(\frac{b_1}{b_0}\right)$. 
If \((a_1/a_0) \sim^* (b_1/b_0)\), then there exist \(\lambda \in R^\times\) and \(\nu \in \mathbb{Z}\) such that \((a_1/a_0) = \lambda B^\nu (b_1/b_0)\), hence we have from (2.4)

\[
A(a_1, a_0) = N(a_1 - a_0 t) = \lambda^2 N(P - t)\nu N(b_1 - b_0 t) = \lambda^2 Q^\nu A(b_1, b_0).
\]

We conclude that \((a_1/a_0) \in V_f(R)^\times\) if and only if \((b_1/b_0) \in V_f(R)^\times\).

**Definition 4** Define two quotient sets of \(\mathcal{S}(f, R)^\times\) using the relations above:

\[G_R(f) := \mathcal{S}(f, R)^\times / \sim,\quad \text{and} \quad G^*_R(f) := \mathcal{S}(f, R)^\times / \sim^*.\]

We define the products on the sets \(G_R(f)\) and \(G^*_R(f)\) induced by the abelian group \(\mathcal{S}(f, R)^\times\). We can see that the products are well-defined as follows. First, recall that the product in \(\mathcal{S}(f, R)^\times\) is induced by that of \(\mathcal{O}_R^\times = (R[t]/(f(t)))^\times\) via the isomorphisms (3.1). The products on the sets \(G_R(f)\) and \(G^*_R(f)\) are well-defined because the multiplication by \(\lambda \in R^\times\) on \(V_f(R)^\times\) is equivalent to that on \(\mathcal{O}_R^\times\), and the action of \(B^\nu\) \((\nu \in \mathbb{Z})\) on \(V_f(R)^\times\) is interpreted as the multiplication of \((P - t)\nu\) on \(\mathcal{O}_R^\times\) from (2.4). The identity elements of \(G_R(f)\) and \(G^*_R(f)\) are the class of the Lucas sequence \(f\), the inverse element of the class \([w]\), \(u = (w_n)\), \(w_n = \frac{A^\theta_1^n - A^\theta_2^n}{\theta_1 - \theta_2}\)

**Remark 2** There exists a natural bijection between \(G^*_Q(f)\) and the Laxton group \(G^*(f)\) in (7)

We denote by \([a_1/a_0]\) the class of \(V_f(R)^\times / \sim\) containing \((a_1/a_0)\). For \(0/1 \in V_f(R)^\times / \sim\), we have

\[
\begin{bmatrix} 0 \\ 1 \end{bmatrix}^{-1} = \begin{bmatrix} P \\ 1 \end{bmatrix}
\]

and hence (2.10) yields the action of \(B\) on \(V_f(R)^\times / \sim\):

\[
B^n \begin{bmatrix} a_1 \\ a_0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}^{-n} \begin{bmatrix} a_1 \\ a_0 \end{bmatrix}, \quad (4.1)
\]

for any \(n \in \mathbb{Z}\) and \([a_1/a_0] \in V_f(R)^\times / \sim\).

We identify the group \(G_R(f)\) with \(V_f(R)^\times / \sim\) and the group \(G^*_R(f)\) with \(V_f(R)^\times / \sim^*\) via the group isomorphisms induced by (2.4):

\[
G_R(f) \sim \rightarrow V_f(R)^\times / \sim, \quad [(w_n)] \mapsto \begin{bmatrix} w_1 \\ w_0 \end{bmatrix},
\]

\[
G^*_R(f) \sim \rightarrow V_f(R)^\times / \sim^*, \quad [(w_n)] \mapsto \begin{bmatrix} w_1 \\ w_0 \end{bmatrix}.
\]

Therefore, we denote a class of \(G_R(f)\) or \(G^*_R(f)\) by \([w_1/w_0]\) instead of \([(w_n)]\). Consider a natural surjection

\[
\pi : G_R(f) \longrightarrow G^*_R(f), \quad \begin{bmatrix} w_1 \\ w_0 \end{bmatrix} \mapsto \begin{bmatrix} w_1 \\ w_0 \end{bmatrix}, \quad (4.3)
\]

and the image of a subgroup \(N\) of \(G_R(f)\).

**Lemma 1** Let \(N\) be a subgroup of \(G_R(f)\) and \(\pi : G_R(f) \longrightarrow G^*_R(f)\) be the natural surjection. Then we have a group isomorphism:

\[
N/N \cap \langle [a] \rangle \sim \rightarrow \pi(N), \quad \begin{bmatrix} w_1 \\ w_0 \end{bmatrix} \mod N \cap \langle [a] \rangle \mapsto \begin{bmatrix} w_1 \\ w_0 \end{bmatrix}.
\]
Proof The kernel of the restriction map \( \pi |_N \) : \( N \to \pi(N) \) is the subgroup \( N \cap \{ B^\nu \begin{bmatrix} 1 \\ 0 \end{bmatrix} \mid \nu \in \mathbb{Z} \} \).

Since \( B^\nu \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}^{−\nu} \) from (4.1), we get the group isomorphism:

\[
N/N \cap \langle \begin{bmatrix} 1 \\ 1 \end{bmatrix} \rangle \simeq \pi(N).
\]

\( \square \)

Note that \( \pi(N) \) is the quotient of \( N \) by the action of \( \langle B \rangle \). In particular, we get

\[
G_R(f)/\langle \begin{bmatrix} 1 \\ 0 \end{bmatrix} \rangle \overset{\sim}{\longrightarrow} G_R^\nu(f), \quad \begin{bmatrix} w_1 \\ w_0 \end{bmatrix} \mod \langle \begin{bmatrix} 1 \\ 0 \end{bmatrix} \rangle \mapsto \begin{bmatrix} w_1 \\ w_0 \end{bmatrix}.
\] (4.4)

Assume that \( f \) is irreducible over \( R \). We get the following group isomorphism from Theorem 4:

\[
\Psi_R : G_R(f) \overset{\sim}{\longrightarrow} R[\theta_1]^\times / R^\times, \quad \begin{bmatrix} w_1 \\ w_0 \end{bmatrix} \mapsto w_1 - w_0\theta_1 \mod R^\times.
\] (4.5)

Since \( \Psi_R \left( \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right) = \theta_1 \mod R^\times \), the isomorphism induces the following isomorphism

\[
\Psi_R^\ast : G_R^\ast(f) \overset{\sim}{\longrightarrow} R[\theta_1]^\times / R^\times \langle \theta_1 \rangle, \quad \begin{bmatrix} w_1 \\ w_0 \end{bmatrix} \mapsto w_1 - w_0\theta_1 \mod R^\times \langle \theta_1 \rangle.
\] (4.6)

Assume that \( f \) is reducible over \( R \) and \( f \) has no double root in \( R \). Furthermore, we assume that \( R \) is a local ring. Let \( H_R^\times = \{(x, y) \in R^\times \times R^\times \mid x \equiv y \pmod{\sqrt{d}} \} \) be the group in Theorem 4. (if \( d \in R^\times \), then \( H_R^\times = R^\times \times R^\times \)). We get a group isomorphism

\[
H_R^\times / I_R \overset{\sim}{\longrightarrow} \begin{cases} R^\times & (\text{if } d \in R^\times), \\ 1 + \sqrt{d} R & (\text{if } d \not\in R^\times), \end{cases} (x, y) \pmod{I_R} \mapsto xy^{-1}
\] (4.7)

where \( I_R := \{(x, x) \mid x \in R^\times \} \) and \( 1 + \sqrt{d} R := \{1 + \sqrt{d} z \mid z \in R \} \). Then we get the following group isomorphism from Theorem 4. (4.2), (4.3) and (4.7).

\[
\Psi_R : G_R(f) \overset{\sim}{\longrightarrow} \begin{cases} R^\times & (\text{if } d \in R^\times), \\ 1 + \sqrt{d} R & (\text{if } d \not\in R^\times), \end{cases} \begin{bmatrix} w_1 \\ w_0 \end{bmatrix} \mapsto (w_1 - w_0\theta_1)(w_1 - w_0\theta_2)^{-1}.
\] (4.8)

Since \( \Psi_R \left( \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right) = \theta_1\theta_2^{-1} \), the isomorphism induces the following isomorphism

\[
\Psi_R^\ast : G_R^\ast(f) \overset{\sim}{\longrightarrow} \begin{cases} R^\times / \langle \theta_1\theta_2^{-1} \rangle & (\text{if } d \in R^\times), \\ (1 + \sqrt{d} R)/\langle \theta_1\theta_2^{-1} \rangle & (\text{if } d \not\in R^\times), \end{cases} \begin{bmatrix} w_1 \\ w_0 \end{bmatrix} \mapsto (w_1 - w_0\theta_1)(w_1 - w_0\theta_2)^{-1} \pmod{\langle \theta_1\theta_2^{-1} \rangle}.
\] (4.9)

Assume that \( f \) has a double root \( \theta \) in \( R \). From Theorem 4. (4.2), (4.3) and the following surjective group homomorphism

\[
(R[\theta]/(\theta^2))^\times \longrightarrow R, \quad x + y\theta \pmod{\theta^2} \mapsto x^{-1}y,
\]

we get the group isomorphism

\[
\Psi_R : G_R(f) \overset{\sim}{\longrightarrow} R, \quad \begin{bmatrix} w_1 \\ w_0 \end{bmatrix} \mapsto -w_0(w_1 - w_0\theta)^{-1}.
\] (4.10)
The surjectivity follows from the fact that \( 1 + y\theta^\ast \in G_R(f) \) maps to \( y \in R \). Since \( \Psi_R \left( \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right) = \theta^{-1} \), the isomorphism induces the following isomorphism

\[
\Psi_R^\ast : G_R^\ast(f) \sim \rightarrow R/(\theta^{-1}), \quad \left[ \begin{array}{c} w_1 \\ w_0 \end{array} \right] \mapsto -w_0(w_1 - w_0\theta)^{-1} \mod (\theta^{-1}).
\] (4.11)

In the case \( R = \mathbb{Q} \), we denote the discriminant \( d \) of \( f \) by \( D \):

\[ D := p^2 - 4Q \in \mathbb{Q}^\times. \]

By summarizing the above discussion, we obtain the following theorem.

**Theorem 5** Let \( p \) be a prime number with \( p \nmid Q \).

1. If \( f(t) \) is irreducible over \( \mathbb{Q} \), then we have

\[
G_Q(f) \sim \rightarrow \mathbb{Q}\{\theta_1\}^\times / \mathbb{Q}^\times, \quad \left[ \begin{array}{c} w_1 \\ w_0 \end{array} \right] \mapsto w_1 - w_0\theta_1 \mod \mathbb{Q}^\times.
\]

If \( f(t) \) is reducible over \( \mathbb{Q} \), then we have

\[
G_Q(f) \sim \rightarrow \mathbb{Q}^\times, \quad \left[ \begin{array}{c} w_1 \\ w_0 \end{array} \right] \mapsto (w_1 - w_0\theta_1)(w_1 - w_0\theta_2)^{-1}.
\]

2. If \( f(t) \) is irreducible over \( \mathbb{Q} \), then we have

\[
G_{\mathbb{Z}(p)}(f) \sim \rightarrow \mathbb{Z}(p)\{\theta_1\}^\times / \mathbb{Z}^\times(p), \quad \left[ \begin{array}{c} w_1 \\ w_0 \end{array} \right] \mapsto w_1 - w_0\theta_1 \mod \mathbb{Z}^\times(p).
\]

If \( f(t) \) is reducible over \( \mathbb{Q} \), then we have

\[
G_{\mathbb{Z}(p)}(f) \sim \rightarrow \begin{cases} \mathbb{Z}^\times(p) \\ 1 + p\mathbb{Z}^\times(p) \end{cases} \text{ if } p \nmid D, \quad \left[ \begin{array}{c} w_1 \\ w_0 \end{array} \right] \mapsto (w_1 - w_0\theta_1)(w_1 - w_0\theta_2)^{-1}.
\]

3. (i) If \( f(t) \) mod \( p \) is irreducible over \( \mathbb{F}_p \), then we have

\[
G_{\mathbb{F}_p}(f) \sim \rightarrow \mathbb{F}_p\{\theta_1\}^\times / \mathbb{F}^\times_p, \quad \left[ \begin{array}{c} w_1 \\ w_0 \end{array} \right] \mapsto w_1 - w_0\theta_1 \mod \mathbb{F}^\times_p.
\]

(ii) Assume that \( f(t) \) is reducible over \( \mathbb{F}_p \).

If \( p \nmid D \), then we have

\[
G_{\mathbb{F}_p}(f) \sim \rightarrow \mathbb{F}^\times_p, \quad \left[ \begin{array}{c} w_1 \\ w_0 \end{array} \right] \mapsto (w_1 - w_0\theta_1)(w_1 - w_0\theta_2)^{-1}.
\]

If \( p|D \), then we have

\[
G_{\mathbb{F}_p}(f) \sim \rightarrow \mathbb{F}_p, \quad \left[ \begin{array}{c} w_1 \\ w_0 \end{array} \right] \mapsto -w_0(w_1 - w_0\theta)^{-1},
\]

where \( \theta \) is the double root of \( f(t) \) mod \( p \).

We also have the following theorem for the group \( G_R^\ast(f) \) as well.

**Theorem 6** Let \( p \) be a prime number with \( p \nmid Q \).
In this section, we define natural subgroups $G_5$ of $(3)$ is well-defined. Therefore, the reduction map $f$ of $(2)$ is reducible over $Q$, then we have

$$G_5^\ast (f) \cong \mathbb{Q}(\theta_1)^\times / \mathbb{Q}_2^\times \langle \theta_1 \rangle,$$

where $w_0 \mapsto (w_1 - w_0 \theta_1)(w_1 - w_0 \theta_2)^{-1} \bmod (\theta_1 \theta_2^{-1})$.

(2) If $f(t)$ is irreducible over $Q$, then we have

$$G_5^\ast (f) \cong \mathbb{Z}_{(p)}[\theta_1]^\times / \mathbb{Z}_{(p)}^\times \langle \theta_1 \rangle,$$

where $w_0 \mapsto (w_1 - w_0 \theta_1)(w_1 - w_0 \theta_2)^{-1} \bmod (\theta_1 \theta_2^{-1})$.

(3) (i) If $f(t) \bmod p$ is irreducible over $\mathbb{F}_p$, then we have

$$G_5^\ast (f) \cong \mathbb{F}_p(\theta_1)^\times / \mathbb{F}_p^\times \langle \theta_1 \rangle,$$

where $w_0 \mapsto (w_1 - w_0 \theta_1)(w_1 - w_0 \theta_2)^{-1} \bmod (\theta_1 \theta_2^{-1})$.

(ii) Assume that $f(t)$ is reducible over $\mathbb{F}_p$.

If $p \nmid D$, then we have

$$G_5^\ast (f) \cong \mathbb{F}_p^\times / \theta_1^\times \langle \theta_1 \rangle,$$

where $w_0 \mapsto (w_1 - w_0 \theta_1)(w_1 - w_0 \theta_2)^{-1} \bmod (\theta_1 \theta_2^{-1})$.

If $p|D$, then we have

$$G_5^\ast (f) \cong 0.

5 Subgroups of $G_5(f)$ and $G_5^\ast (f)$

In this section, we define natural subgroups $G(f, p)$, $K(f, p)$, $H(f, p)$ of $G_5(f)$, and $G^\ast (f, p)$, $K^\ast (f, p)$, $H^\ast (f, p)$ of $G^\ast (f)$, the definitions of these groups give natural interpretation of Laxton’s subgroups $G^\ast (f, p)_L$, $K^\ast (f, p)_L$ and $H^\ast (f, p)_L$ in §1.

Let $p$ be a prime number with $p \nmid Q$. Consider the projective line over $\mathbb{F}_p$:

$$\mathbb{P}^1(\mathbb{F}_p) := \mathbb{F}_p^2 / \sim = \left\{ \begin{bmatrix} a_1 \\ a_0 \end{bmatrix} \in \mathbb{F}_p^2 \mid a_0 \in \mathbb{F}_p^\times \text{ or } a_1 \in \mathbb{F}_p^\times \right\},$$

where $\begin{bmatrix} a_1 \\ a_0 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_0 \end{bmatrix}$ if and only if there exists $c \in \mathbb{F}_p^\times$ such that $a_0 = cb_0$, $a_1 = cb_1$. For any class $\begin{bmatrix} w_1 \\ w_0 \end{bmatrix} \in G_5(f)$, we can choose the representative $\begin{bmatrix} w_1 \\ w_0 \end{bmatrix}$ so that $w_0, w_1 \in \mathbb{Z}$ and $(w_0, w_1) = 1$. Therefore, the reduction map

$$\text{red}_p : G_5(f) \longrightarrow \mathbb{P}^1(\mathbb{F}_p), \quad \begin{bmatrix} w_1 \\ w_0 \end{bmatrix} \mapsto \begin{bmatrix} w_1 \\ w_0 \end{bmatrix}$$

(5.1)

is well-defined. The group $G_5(f) \left( \cong V_f(\mathbb{F}_p)^\times / \sim \right)$ is a subset of $\mathbb{P}^1(\mathbb{F}_p)$. 

On the Laxton Group 11
Definition 5 Define subsets $G(f, p)$, $K(f, p)$ and $H(f, p)$ of $G_\mathbb{Q}(f)$ by

$$G(f, p) := \left\{ \begin{bmatrix} w_1 \\ w_0 \end{bmatrix} \in G_\mathbb{Q}(f) \mid \text{red}_p \begin{bmatrix} w_1 \\ w_0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\},$$

$$K(f, p) := \left\{ \begin{bmatrix} w_1 \\ w_0 \end{bmatrix} \in G_\mathbb{Q}(f) \mid \text{red}_p \begin{bmatrix} w_1 \\ w_0 \end{bmatrix} \in G_\mathbb{Z}_p(f) \right\},$$

$$H(f, p) := \left\{ \begin{bmatrix} w_1 \\ w_0 \end{bmatrix} \in G_\mathbb{Q}(f) \mid \text{red}_p \begin{bmatrix} w_1 \\ w_0 \end{bmatrix}^n = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \text{ for some } n \in \mathbb{Z} \right\}.$$

From the definition of $K(f, p)$, we can see that $K(f, p)$ is a subgroup of $G_\mathbb{Q}(f)$. The reduction map \([5.1]\) induces the group homomorphism

$$\text{red}_p : K(f, p) \longrightarrow G_\mathbb{Z}_p(f), \quad \begin{bmatrix} w_1 \\ w_0 \end{bmatrix} \mapsto \begin{bmatrix} w_1 \\ w_0 \end{bmatrix}.$$  \hspace{1cm} (5.2)

The set $G(f, p)$ is a subgroup of $K(f, p)$ because $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ is the identity element of $G_\mathbb{Z}_p(f)$. The set $H(f, p)$ is a subgroup of $G_\mathbb{Q}(f)$ since $G(f, p)$ is a group and $\begin{bmatrix} w_1 \\ w_0 \end{bmatrix} \in H(f, p)$ if and only if $\begin{bmatrix} w_1 \\ w_0 \end{bmatrix}^n \in G(f, p)$ for some $n \in \mathbb{Z}$. Furthermore, we have $K(f, p) \subset H(f, p)$ since the image of $K(f, p)$ by the reduction map is in the finite group $G_\mathbb{Z}_p(f)$. We get the following sequence of subgroups:

$$G(f, p) \leq K(f, p) \leq H(f, p) \leq G_\mathbb{Q}(f)$$  \hspace{1cm} (5.3)

By the definition of the subgroups, we have an exact sequence of groups:

$$1 \longrightarrow G(f, p) \longrightarrow K(f, p) \longrightarrow G_\mathbb{Z}_p(f) \longrightarrow 1.$$  \hspace{1cm} (5.4)

This exact sequence is an analogue of the elliptic curve exact sequence (\([8\text{ VII, Proposition 2.1}]\)).

Next, we define the corresponding subgroups of $G_\mathbb{Q}^*(f)$ by the natural map $\pi : G_\mathbb{Q}(f) \rightarrow G_\mathbb{Q}^*(f)$.

Definition 6 Define subsets $G^*(f, p)$, $K^*(f, p)$ and $H^*(f, p)$ of $G_\mathbb{Q}^*(f)$ by

$$G^*(f, p) := \pi(G(f, p)),$$

$$K^*(f, p) := \pi(K(f, p)),$$

$$H^*(f, p) := \pi(H(f, p)).$$

These subsets are subgroups of $G_\mathbb{Q}^*(f)$ since the map $\pi$ is a group homomorphism and we get the following sequence of subgroups:

$$G^*(f, p) \leq K^*(f, p) \leq H^*(f, p) \leq G_\mathbb{Q}^*(f).$$  \hspace{1cm} (5.5)

Let $p$ be a prime number, and assume $p \nmid Q$. For the Lucas sequence $\mathcal{F} = (\mathcal{F}_n) \in \mathcal{F}(f, Q)$, we have $\mathcal{F}_n \in \mathbb{Z}^*_p$ for any $n \in \mathbb{Z}$. Lucas showed that there exists a positive integer $n$ satisfying $p\mid \mathcal{F}_n$ in this case (\([8\text{ IV.18, IV.19 and p67}]\)).

Definition 7 Assume $p \nmid Q$. We denote the rank of the Lucas sequence $\mathcal{F} = (\mathcal{F}_n) \in \mathcal{F}(f, Q)$ by $r(p)$. Namely, it is the smallest positive integer $n$ satisfying $p\mid \mathcal{F}_n$.

We can easily check $r(2) = 2$ if $P$ is even, and $r(2) = 3$ if $P$ is odd. If $p \neq 2$, then we know that $r(p)$ divides $p - \left( \frac{D}{p} \right)$ from the results of Lucas, where $\left( \frac{\cdot}{\cdot} \right)$ is the Legendre symbol.

Lemma 2 Assume $p \nmid Q$. Let $r(p)$ be the rank of the Lucas sequence $\mathcal{F} = (\mathcal{F}_n)$. Then $r(p)$ is equal to the order of $\begin{bmatrix} 0 \\ 1 \end{bmatrix} \in G_\mathbb{Z}_p(f)$. 

Lemma 3 Let \( r(p) \) be the rank of the Lucas sequence \( F = (F_n) \). Then we have

\[
G(f, p) \cap \langle \left[ \begin{array}{c} 0 \\ 1 \end{array} \right] \rangle = \langle \left[ \begin{array}{c} 0 \\ 1 \end{array} \right] \rangle^{r(p)}.
\]

Proof The assertion follows from Lemma 2. □

Theorem 7 Let \( G^*(f) \) be the Laxton group and \( G^*(f, p) \), \( K^*(f, p) \), \( H^*(f, p) \) be the subgroups defined in §1. For the natural isomorphism of groups \( \iota : G^*(f) \rightarrow G^*_G(f) \), we have \( \iota(G^*(f, p)) = G^*(f, p) \), \( \iota(K^*(f, p)) = K^*(f, p) \), and \( \iota(H^*(f, p)) = H^*(f, p) \). Furthermore, we have the following exact sequence of groups:

\[
1 \rightarrow G^*(f, p) \rightarrow K^*(f, p) \rightarrow G^*_G(f) \rightarrow 1.
\]
6 Group structures of $K(f, p)$ and $K^*(f, p)$

In this section, we determine the group structures of $K(f, p)$ and $K^*(f, p)$ defined in Definitions and Let $p$ be a prime number with $p \nmid Q$.

**Lemma 4** We have the following group isomorphisms.

(1) \[ \rho : G_{Z(p)}(f) \xrightarrow{\sim} K(f, p), \quad \begin{bmatrix} w_1 \\ w_0 \end{bmatrix} \mapsto \begin{bmatrix} w_1 \\ w_0 \end{bmatrix} \]

(2) \[ \rho^* : G_{Z(p)}^*(f) \xrightarrow{\sim} K^*(f, p), \quad \begin{bmatrix} w_1 \\ w_0 \end{bmatrix} \mod (\begin{bmatrix} p \\ 1 \end{bmatrix}) \mapsto \begin{bmatrix} w_1 \\ w_0 \end{bmatrix} \mod (\begin{bmatrix} p \\ 1 \end{bmatrix}) \]

**Proof** [1] We only show that $\rho$ is injective because the other part is trivial. If $\rho \left( \begin{bmatrix} w_1 \\ w_0 \end{bmatrix} \right) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ for $\begin{bmatrix} w_1 \\ w_0 \end{bmatrix} \in G_{Z(p)}^*(f)$, then we get $w_0 = 0, w_1 \in \mathbb{Q}^\times$. Furthermore, we have $w_1 \in Z_{Z(p)}^\times$ since $A(w_1, w_0) = w_1^2 - p w_1 + Q w_0^2 \in Z_{Z(p)}^\times$. We get $\begin{bmatrix} w_1 \\ w_0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ in $G_{Z(p)}^*(f)$, and hence $\rho$ is injective.

[2] We get the assertion from [1] since the kernels of the natural surjection $K(f, p) \rightarrow K^*(f, p)$ and $G_{Z(p)}(f) \rightarrow G_{Z(p)}^*(f)$ are the subgroups generated by $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

By Lemma [4], Theorems [5](2) and [6](2), we get the following theorem.

**Theorem 8** Put $D = p^s D_0$ with $s \geq 0$, $p \nmid D_0$.

(1) If $f(t)$ is irreducible over $\mathbb{Q}$, then we have

\[ K(f, p) \xrightarrow{\sim} Z_{Z(p)}[\theta_1]^\times / Z_{Z(p)}^\times, \quad \begin{bmatrix} w_1 \\ w_0 \end{bmatrix} \mapsto w_1 - w_0 \theta_1 \mod Z_{Z(p)}^\times, \]

and

\[ K^*(f, p) \xrightarrow{\sim} Z_{Z(p)}[\theta_1]^\times / Z_{Z(p)}^\times(\theta_1), \quad \begin{bmatrix} w_1 \\ w_0 \end{bmatrix} \mapsto w_1 - w_0 \theta_1 \mod Z_{Z(p)}^\times(\theta_1). \]

(2) If $f(t)$ is reducible over $\mathbb{Q}$, then we have

\[ K(f, p) \xrightarrow{\sim} \begin{cases} Z_{Z(p)}^\times \\ 1 + p^s Z_{Z(p)} \end{cases} \text{ if } p \nmid D, \quad \begin{bmatrix} w_1 \\ w_0 \end{bmatrix} \mapsto (w_1 - w_0 \theta_1)(w_1 - w_0 \theta_2)^{-1}, \]

and

\[ K^*(f, p) \xrightarrow{\sim} \begin{cases} Z_{Z(p)}^\times / (\theta_1 \theta_2^{-1}) \\ (1 + p^s Z_{Z(p)}) / (\theta_1 \theta_2^{-1}) \end{cases} \text{ if } p \mid D, \quad \begin{bmatrix} w_1 \\ w_0 \end{bmatrix} \mapsto (w_1 - w_0 \theta_1)(w_1 - w_0 \theta_2)^{-1} \mod (\theta_1 \theta_2^{-1}). \]
7 Group structures of $G_\mathbb{Q}(f)/K(f, p)$ and $G_\mathbb{Q}^*(f)/K^*(f, p)$

In this section, we determine the structures of $G_\mathbb{Q}(f)/K(f, p)$ and $G_\mathbb{Q}^*(f)/K^*(f, p)$. Put $F := \mathbb{Q}(\theta_1)$ and denote the ring of integers of $F$ by $\mathcal{O}_F$. Let $p$ be a prime number and $p$ be a prime ideal of $F$ above $p$. Put $\mathcal{O}_p := S_p^{-1}\mathcal{O}_F$ and $\mathcal{O}_{(p)} := S_p^{-1}\mathcal{O}_F = \bigcap_{\mathfrak{p}|p}\mathcal{O}_{F, \mathfrak{p}}$, where $S_p := \mathcal{O}_F \setminus \mathfrak{p}$ and $S_p := \mathbb{Z} \setminus p\mathbb{Z}$.

We have $\mathcal{O}_{(p)}^\times = \{ \alpha \in F^\times \mid v_p(\alpha) = 0 \text{ for any } \mathfrak{p} \text{ above } p \}$. Put $D = ma^2$ with $a \in \mathbb{N}$ and a squarefree integer $m$. We have $F = \mathbb{Q}(\sqrt{m})$, and if $p \neq 2$, then $\mathcal{O}_{(p)} = \mathbb{Z}_{(p)}[\sqrt{m}]$, $\mathbb{Z}_{(p)}[\theta_1] = \mathbb{Z}_{(p)}[\sqrt{D}]$.

**Lemma 5** If $p^2 \mid D$, then we have $\mathcal{O}_{(p)} = \mathbb{Z}_{(p)}[\theta_1]$.

**Proof** If $F = \mathbb{Q}$, then the assertion is trivial. Assume $F \neq \mathbb{Q}$. In the case $p \neq 2$, the assertion follows from

$\mathcal{O}_{(p)} = \mathbb{Z}_{(p)}[\sqrt{m}] = \mathbb{Z}_{(p)}[\sqrt{D}] = \mathbb{Z}_{(p)}[\theta_1]$.

Assume $p = 2$. Since $2 \mid D = P^2 - 4Q$, we have $2 \mid P$. Furthermore, since $\theta_1 = (P \pm \sqrt{D})/2 = (P + a\sqrt{m})/2 \in \mathcal{O}_F$, we have $m \equiv 1 \pmod{4}$ and $2 \mid a$, and hence $\mathcal{O}_F = \mathbb{Z}[(1 + \sqrt{m})/2]$. We get

$$\frac{1 + \sqrt{m}}{2} = \theta_1 = \frac{(P - 1) \pm (a - 1)\sqrt{m}}{2}.$$ 

Therefore, we have $\mathcal{O}_{(p)} = \mathbb{Z}_{(p)}[\theta_1]$. \hfill \Box

**Lemma 6** If $f(t)$ is irreducible over $\mathbb{Q}$, then we have

$$F^\times / \mathbb{Z}_{(p)}[\theta_1]^\times \mathbb{Q}^\times \simeq \begin{cases} \mathcal{O}_{(p)}^\times / \mathbb{Z}_{(p)}[\theta_1]^\times & \text{if } p \text{ is inert in } F, \\ \mathbb{Z} \times (\mathcal{O}_{(p)}^\times / \mathbb{Z}_{(p)}[\theta_1]^\times) & \text{if } p \text{ splits in } F, \\ \mathbb{Z}/2\mathbb{Z} \times (\mathcal{O}_{(p)}^\times / \mathbb{Z}_{(p)}[\theta_1]^\times) & \text{if } p \text{ is ramified in } F. \end{cases}$$

**Proof** If $p$ is inert in $F$, then the assertion follows from

$$F^\times / \mathbb{Z}_{(p)}[\theta_1]^\times \mathbb{Q}^\times = \mathcal{O}_{(p)}^\times / \mathbb{Z}_{(p)}[\theta_1]^\times \langle p \rangle \simeq \mathcal{O}_{(p)}^\times / \mathbb{Z}_{(p)}[\theta_1]^\times.$$ 

Next, we consider the case $p$ splits in $F$. Put $\text{Gal}(F/\mathbb{Q}) = \langle \sigma \rangle$ and $(p) = \mathfrak{p}_p^\sigma$. Consider a split surjection

$$\mathcal{V} : F^\times / \mathbb{Z}_{(p)}[\theta_1]^\times \mathbb{Q}^\times \to \mathbb{Z}, \quad \mathcal{V}(\alpha) = v_p(\alpha^{1-\sigma}) = v_{\mathfrak{p}_p^\sigma}(\alpha) - v_{\mathfrak{p}_p^\sigma}(\alpha).$$

We have $\alpha \in \ker \mathcal{V}$ if and only if $\alpha = p^n \beta$ for some $n \in \mathbb{Z}$ and $\beta \in \mathcal{O}_{(p)}^\times$. Therefore, we get

$$\ker \mathcal{V} = \mathcal{O}_{(p)}^\times / \mathbb{Z}_{(p)}[\theta_1]^\times \mathbb{Q}^\times = \mathcal{O}_{(p)}^\times (p) / \mathbb{Z}_{(p)}[\theta_1]^\times (p) \simeq \mathcal{O}_{(p)}^\times / \mathbb{Z}_{(p)}[\theta_1]^\times,$$

and hence

$$F^\times / \mathbb{Z}_{(p)}[\theta_1]^\times \mathbb{Q}^\times \simeq \mathbb{Z} \times (\mathcal{O}_{(p)}^\times / \mathbb{Z}_{(p)}[\theta_1]^\times).$$

Finally, we consider the case $p$ is ramified in $F$. Put $(p) = \mathfrak{p}^2$, and choose $\pi \in \mathfrak{p} \setminus \mathfrak{p}^2$. The assertion follows from the isomorphism

$$\mathcal{W} : F^\times \xrightarrow{\sim} \mathcal{O}_{(p)}^\times \times \mathbb{Z}, \quad \mathcal{W}(\alpha) = (\pi^{-v_{\mathfrak{p}^2}(\alpha)} \mathfrak{p}(\alpha), \mathfrak{p}(\alpha)),$$

and

$$\mathcal{W}(\mathbb{Z}_{(p)}[\theta_1]^\times \mathbb{Q}^\times) = \mathcal{W}(\mathbb{Z}_{(p)}[\theta_1]^\times (p)) = \mathbb{Z}_{(p)}[\theta_1]^\times \times 2\mathbb{Z}.$$ 

\hfill \Box

**Lemma 7** Assume that $f(t)$ is irreducible over $\mathbb{Q}$. Put $D = p^sD_0$ with $s \geq 0$, $p \nmid D_0$.

(1) If $p$ is inert in $F$, then we have

$$\mathcal{O}_{(p)}^\times / \mathbb{Z}_{(p)}[\theta_1]^\times \simeq \begin{cases} 0 & \text{if } s = 0, \\ \mathbb{Z}/p^s/2-1(p + 1)\mathbb{Z} & \text{if } s \neq 0, s \equiv 0 \pmod{2} \text{ and } p \neq 2. \end{cases}$$
(2) If \( p \) splits in \( F \), then we have
\[
\mathcal{O}_F \times / \mathbb{Z}(p)[\theta_1]^\times \simeq \begin{cases} 
0 & \text{if } s = 0, \\
\mathbb{Z}/p^{s/2-1}(p-1)\mathbb{Z} & \text{if } s \neq 0, \ s \equiv 0 \pmod{2} \text{ and } p \neq 2.
\end{cases}
\]

(3) If \( p \) is ramified in \( F \), then we have
\[
\mathcal{O}_F \times / \mathbb{Z}(p)[\theta_1]^\times \\
\simeq \begin{cases} 
0 & \text{if } s = 1, \\
\mathbb{Z}/p^{s/2}\mathbb{Z} & \text{if } s \neq 1, \ s \equiv 1 \pmod{2} \text{ and } p \neq 2, 3, \\
\mathbb{Z}/p^{s/2}\mathbb{Z} \times \mathbb{Z}/p^{s/2-1}\mathbb{Z} & \text{if } s \neq 1, \ s \equiv 1 \pmod{2} \text{ and } p = 3.
\end{cases}
\]

Proof The assertions in the cases \( s = 0, 1 \) follow from Lemma \( \text{[p]} \). Consider the cases \( s \neq 0, 1 \) and \( p \neq 2 \). We have \( \mathcal{O}_F = \mathbb{Z}(p)[\sqrt{m}] \), \( \mathbb{Z}(p)[\theta_1] = \mathbb{Z}(p)[\sqrt{|D|}] \) since \( p \neq 2 \). Put \( k = [s/2](\geq 1) \). From the following commutative diagram:
\[
\begin{array}{cccccc}
1 & \rightarrow & 1 + p^k \mathcal{O}_F & \rightarrow & \mathcal{O}_F^\times & \rightarrow & (\mathcal{O}_F / (p^k))\times & \rightarrow & 1 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
1 & \rightarrow & 1 + p^k \mathbb{Z}(p)[\sqrt{m}] & \rightarrow & \mathbb{Z}(p)[\theta_1]^\times & \rightarrow & (\mathbb{Z}/p^k\mathbb{Z})^\times & \rightarrow & 1,
\end{array}
\]
where the middle and right vertical maps are injective, we get
\[
\mathcal{O}_F^\times / \mathbb{Z}(p)[\theta_1]^\times \simeq (\mathcal{O}_F / (p^k))\times / (\mathbb{Z}/p^k\mathbb{Z})^\times.
\]

(1) The assertion in the case where \( p \) is inert in \( F \) follows from the isomorphisms:
\[
(\mathcal{O}_F / (p^k))\times \simeq (\mathcal{O}_F / (p))\times \times (1 + (p)) / (1 + (p^k)) \rightarrow (\mathcal{O}_F / (p))\times \times (\mathcal{O}_F / (p^{k-1}))
\]
and \( (\mathcal{O}_F / (p^k))\times \simeq (\mathbb{Z}/p^k\mathbb{Z})^\times \times (1 + p\mathbb{Z}) / (1 + p^k\mathbb{Z}) \rightarrow (\mathbb{Z}/p\mathbb{Z})^\times \times \mathbb{Z}/p^{k-1}\mathbb{Z}, \)
and \( (\mathcal{O}_F / (p^k))\times \simeq (\mathcal{O}_F / (p))\times \times (1 + p^k\mathbb{Z}) / (1 + (p^\sigma)^k) \).

(2) Consider the case where \( p \) splits in \( F \). Put \( \text{Gal}(F / \mathbb{Q}) = \langle \sigma \rangle \) and \( (p) = pp^\sigma \). Then we have
\[
(\mathcal{O}_F / (p^k))\times \simeq (\mathcal{O}_F / (p))\times \times (1 + p^k) / (1 + p^k) \times (1 + p^\sigma) / (1 + (p^\sigma)^k)
\]
and hence
\[
(\mathcal{O}_F / (p^k))\times \simeq (\mathcal{O}_F / (p))\times \times (\mathcal{O}_F / (p^{k-1})),
\]
and
\[
(\mathbb{Z}/p^k\mathbb{Z})^\times \simeq (\mathbb{Z}/p\mathbb{Z})^\times \times \mathbb{Z}/p^{k-1}\mathbb{Z}.
\]

The assertion follows from these isomorphisms and \( (\mathcal{O}_F / (p^k))\times \simeq (\mathcal{O}_F / (p))\times \times (\mathcal{O}_F / (p^{k-1})) \).
(3) Consider the case where $p$ is ramified. Put $(p) = p^2$,

$$\ell := \begin{cases} 1 & \text{if } p \neq 2, 3, \\ 2 & \text{if } p = 3, \end{cases}$$

and choose $\pi \in p^\ell \setminus p^{\ell+1}$. We have a commutative diagram:

$$1 \longrightarrow (1 + p^\ell)/(1 + p^{2k}) \longrightarrow (O_F/p^{2k})^\times \longrightarrow (O_F/p^\ell)^\times \longrightarrow 1$$

$$1 \longrightarrow (1 + p\mathbb{Z})/(1 + p^k\mathbb{Z}) \longrightarrow (\mathbb{Z}/p^k\mathbb{Z})^\times \longrightarrow (\mathbb{Z}/p\mathbb{Z})^\times \longrightarrow 1,$$

where all the vertical maps are injective, and

$$(O_F/p^\ell)^\times/(\mathbb{Z}/p\mathbb{Z})^\times \simeq \begin{cases} 0 & \text{if } p \neq 2, 3, \\ \mathbb{Z}/p\mathbb{Z} & \text{if } p = 3. \end{cases}$$

Define the injection $\iota : \mathbb{Z}/p^{k-1}\mathbb{Z} \to O_F/p^{2k-\ell}$ by

$$\iota(\alpha) = \begin{cases} \pi\alpha & \text{if } p \neq 2, 3, \\ \alpha & \text{if } p = 3. \end{cases}$$

We have

$$\text{Coker } \iota \simeq \begin{cases} \mathbb{Z}/p\mathbb{Z} & \text{if } p \neq 2, 3, \\ \mathbb{Z}/p^{k-1}\mathbb{Z} & \text{if } p = 3. \end{cases}$$

The assertion follows from (7.1), (7.2) and the following commutative diagram:

$$\log_p (1 + p^\ell)/(1 + p^{2k}) \simeq \log_p p^\ell/p^{2k} \overset{\pi}{\rightarrow} O_F/p^{2k-\ell}$$

$$\uparrow \quad \uparrow \quad \iota \uparrow$$

$$(1 + p\mathbb{Z})/(1 + p^k\mathbb{Z}) \longrightarrow p\mathbb{Z}/p^k\mathbb{Z} \leftarrow \mathbb{Z}/p^{k-1}\mathbb{Z},$$

where all the vertical maps are injective. □

**Corollary 1** Put $D = p^sD_0$ with $s \geq 0$, $p \nmid D_0$.

(1) Assume that $f(t)$ is irreducible over $\mathbb{Q}$. We have

$$G_\mathbb{Q}(f)/K(f,p) \simeq G_\mathbb{Q}^+(f)/K^+(f,p) \simeq F^\times/\mathbb{Z}[\theta_1]^\times \mathbb{Q}^\times.$$

(i) If $p$ is inert in $F$, then

$$G_\mathbb{Q}(f)/K(f,p) \simeq \begin{cases} 0 & \text{if } s = 0, \\ \mathbb{Z}/p^{s/2-1}(p + 1)\mathbb{Z} & \text{if } s \neq 0, s \equiv 0 \pmod{2} \text{ and } p \neq 2. \end{cases}$$

(ii) If $p$ splits in $F$, then

$$G_\mathbb{Q}(f)/K(f,p) \simeq \begin{cases} \mathbb{Z} & \text{if } s = 0, \\ \mathbb{Z} \times \mathbb{Z}/p^{s/2-1}(p - 1)\mathbb{Z} & \text{if } s \neq 0, s \equiv 0 \pmod{2} \text{ and } p \neq 2. \end{cases}$$

(iii) If $p$ is ramified in $F$, then

$$G_\mathbb{Q}(f)/K(f,p) \simeq \begin{cases} \mathbb{Z}/2\mathbb{Z} & \text{if } s = 1, \\ \mathbb{Z}/2p^{s/2}\mathbb{Z} & \text{if } s \neq 1, s \equiv 1 \pmod{2} \text{ and } p \neq 2, 3, \text{ or } \mathbb{Z}/2p\mathbb{Z} \times \mathbb{Z}/p^{s/2-1}\mathbb{Z} & \text{if } s \neq 1, s \equiv 1 \pmod{2} \text{ and } p \neq 3. \end{cases}$$
We have 

\[
G_{\mathbb{Q}}(f)/K(f, p) \cong G^*_\mathbb{Q}(f)/K^*(f, p)
\]

\[
\cong \begin{cases} 
\mathbb{Q}^\times/\mathbb{Z}^\times_{(p)} \cong \mathbb{Z} & \text{if } s = 0, \\
\mathbb{Q}^\times/(1 + p^{s/2}\mathbb{Z})_{(p)} \cong \mathbb{Z} \times \mathbb{Z}/(p - 1)p^{s/2-1}\mathbb{Z} & \text{if } s \neq 0 \text{ and } p \neq 2.
\end{cases}
\]

Proof: (1) The first assertion follows from Theorems 5 and 8 and the others follow from Lemmas 3 and 4.

(2) We get from Theorems 5, 6 and 8, and the others follow from Lemmas 3 and 4.

In this section, we determine the structure of the quotient groups \(K(f, p)/G(f, p)\) and \(K^*(f, p)/G^*(f, p)\), where \(K(f, p)\) and \(G(f, p)\) are defined in Definition 3 and \(K^*(f, p)\) and \(G^*(f, p)\) are defined in Definition 4. By the results of this section and §7, we get Laxton’s theorem (Theorem 4 in §4), and a result proved by Suwa in the case \(p||D\). Let \(p\) be a prime number with \(p \not| Q\). From exact sequences (8.1) and (8.2), we get group isomorphisms

\[
K(f, p)/G(f, p) \cong G^*_p(f), \quad K^*(f, p)/G^*(f, p) \cong G^*_p(f).
\]

By Lemma 1, we have

\[
G^*_p(f) \cong G^*_p(f)/[\theta_1^n].
\]

From (8.1), (8.2), Theorems 3 and 6 and Lemma 2, we get the following theorem.

Theorem 9 (1) Assume that \(f(t) \mod p\) is irreducible over \(\mathbb{F}_p\). We have

\[
K(f, p)/G(f, p) \cong \mathbb{F}_p(\theta_1)^\times/\mathbb{F}_p^\times \cong \mathbb{Z}/(p + 1)\mathbb{Z},
\]

\[
\begin{bmatrix} w_1 \\ w_0 \end{bmatrix} \mod G(f, p) \mapsto w_1 - w_0\theta_1 \mod \mathbb{F}_p^\times,
\]

and

\[
K^*(f, p)/G^*(f, p) \cong \mathbb{F}_p(\theta_1)^\times/\mathbb{F}_p^*(\theta_1)^\times \cong \mathbb{Z}/(p + 1)/r(p)\mathbb{Z},
\]

\[
\begin{bmatrix} w_1 \\ w_0 \end{bmatrix} \mod G^*(f, p) \mapsto w_1 - w_0\theta_1 \mod \mathbb{F}_p^*(\theta_1),
\]
(2) Assume that \( f(t) \mod p \) is reducible over \( \mathbb{F}_p \).
If \( p \nmid D \), then we have
\[
K(f, p)/G(f, p) \simeq \mathbb{F}_p^\times \simeq \mathbb{Z}/(p - 1)\mathbb{Z},
\]
and
\[
K^*(f, p)/G^*(f, p) \simeq \mathbb{F}_p^\times /\langle \theta_1 \theta_2^{-1} \rangle \simeq \mathbb{Z}/((p - 1)/r(p))\mathbb{Z},
\]
\[
\begin{bmatrix} w_1 \\ w_0 \end{bmatrix} \mod G(f, p) \mapsto (w_1 - w_0 \theta_1)(w_1 - w_0 \theta_2)^{-1},
\]
and
\[
K(f, p)/G(f, p) \simeq \mathbb{F}_p, \quad \begin{bmatrix} w_1 \\ w_0 \end{bmatrix} \mod G(f, p) \mapsto -w_0(w_1 - w_0 \theta)^{-1},
\]
where \( \theta \) is the double root of \( f(t) \mod p \), and
\[
K^*(f, p)/G^*(f, p) \simeq 0.
\]

Since the groups in \( \mathbb{F}_p \) are finite groups, we can see
\[
\text{Tor}_2(G_Q(f)/K(f, p)) = H(f, p)/K(f, p), \quad (8.3)
\]
\[
\text{Tor}_2(G_Q^*(f)/K^*(f, p)) = H^*(f, p)/K^*(f, p).
\]

By \( \mathbb{F}_p \), Corollary \( \text{H} \) and Theorem \( \text{E} \) we see that our results lead Laxton’s (Theorem \( \text{I} \) and Suwa’s theorems.

Corollary 2 Let \( p \) be a prime number with \( p \nmid Q \), and \( r(p) \) be the rank of the Lucas sequence \( \mathcal{F} \).

(1) Assume \( p \nmid D \). If \( Q(\theta_1) \neq \mathbb{Q} \) and \( p \) is inert in \( Q(\theta_1) \), then \( G_Q^*(f) = H^*(f, p) = K^*(f, p) \) and \( G^*(f)/G^*(f, p) \) is a cyclic group of order \( (p + 1)/r(p) \).

(2) Assume \( p \nmid D \). If \( Q(\theta_1) = \mathbb{Q} \), or \( Q(\theta_1) \neq \mathbb{Q} \) and \( p \) splits in \( Q(\theta_1) \), then \( G_Q^*(f)/H^*(f, p) \) is an infinite cyclic group, and \( H^*(f, p) \) is not a cyclic group, and \( K^*(f, p) \) and \( H^*(f, p)/G^*(f, p) \) is a cyclic group of order \( (p + 1)/r(p) \).

(3) If \( p|D \) and \( p^2 \nmid D \), then \( G_Q^*(f) = H^*(f, p) \) and \( K^*(f, p) = G^*(f, p) \). Furthermore, if \( p \neq 2 \), then \( G_Q^*(f)/G^*(f, p) \) is a cyclic group of order two.

(4) Assume \( D = p^sD_0 \) with \( s \geq 2 \) and \( p \nmid D_0 \).

(i) Assume \( s \equiv 1 \mod 2 \). We have \( G_Q^*(f) = H^*(f, p) \) and \( K^*(f, p) = G^*(f, p) \). Furthermore, if \( p \neq 2,3 \), then \( G_Q^*(f)/G^*(f, p) \) is a cyclic group of order \( 2p^{(s/2)} \), and if \( p = 3 \), then \( G_Q^*(f)/G^*(f, p) \) is a cyclic group of order \( 2p^{(s/2)} \) or a direct product of two cyclic groups of order \( 2p \) and \( p^{(s/2) - 1} \).

(ii) Assume \( s \equiv 0 \mod 2 \). If \( Q(\theta_1) \neq \mathbb{Q} \) and \( p \) is inert in \( Q(\theta_1) \), then \( G_Q^*(f) = H^*(f, p) \) and \( K^*(f, p) = G^*(f, p) \). Furthermore, if \( p \neq 2 \), then \( G_Q^*(f)/G^*(f, p) \) is a cyclic group of order \( (p + 1)p^{(s/2) - 1} \).

(iii) Assume \( s \equiv 0 \mod 2 \). If \( Q(\theta_1) = \mathbb{Q} \) or \( Q(\theta_1) \neq \mathbb{Q} \), \( p \) splits in \( Q(\theta_1) \), then \( K^*(f, p) = G^*(f, p) \). Furthermore, if \( p \neq 2 \), then \( G_Q^*(f)/H^*(f, p) \) is a cyclic group of order \( (p - 1)p^{(s/2) - 1} \).
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