Asymptotic AdS String Solutions for Null Polygonal Wilson Loops in $R^{1,2}$

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Abstract

For the asymptotic string solution in $AdS_3$ which is represented by the $AdS_3$ Poincare coordinates and yields the planar multi-gluon scattering amplitude at strong coupling in arXiv:0904.0663, we express it by the $AdS_4$ Poincare coordinates and demonstrate that the hexagonal and octagonal Wilson loops surrounding the string surfaces take closed contours consisting of null vectors in $R^{1,2}$ owing to the relations of Stokes matrices. For the tetragonal Wilson loop we construct a string solution characterized by two parameters by solving the auxiliary linear problems and demanding a reality condition, and analyze the asymptotic behavior of the solution in $R^{1,2}$. The freedoms of two parameters are related with some conformal SO(2,4) transformations.
1 Introduction

The AdS/CFT correspondence has more and more revealed the deep relations between the $\mathcal{N} = 4$ super Yang-Mills (SYM) theory and the string theory in $AdS_5 \times S^5$, where classical string solutions play an important role \cite{1,2,3}. Alday and Maldacena \cite{4} have evaluated the planar four-gluon scattering amplitude at strong coupling in the $\mathcal{N} = 4$ SYM theory by computing the four-cusp Wilson loop composed of four lightlike gluon momenta from a certain open string solution in $AdS$ space whose worldsheet surface is related by a conformal SO(2,4) transformation to the one-cusp Wilson loop surface found in \cite{5}. The dimensionally regularized four-gluon amplitude at strong coupling agrees with the BDS conjectured form regarding the all-loop iterative structure and the IR divergence of the perturbative gluon amplitude \cite{6}.

Inspired by this investigation there have been a lot of works about the string theory computations of the gluon amplitudes and the constructions of open string solutions in $AdS$ with null boundaries \cite{7,8,9,10,11,12,13}.

On the other hand based on the perturbative approach in the SYM theory various studies have been made about duality between the planar gluon amplitudes \cite{14} and the null Wilson loops where both have the same (dual) conformal symmetry, and the anomalous conformal Ward identity which constrains the Wilson loops \cite{15}.

Using the Pohlmeyer reduction \cite{16,17} several string solutions in $AdS$ have been constructed from the soliton solutions in the generalized Sinh-Gordon model \cite{10,18}. Within the Pohlmeyer reduction the string configurations with the timelike and spacelike minimal surfaces in $AdS$ \cite{19} and the closed strings in SL(2,R) \cite{20} have been studied.

Recently the multi-gluon scattering amplitudes at strong coupling have been investigated \cite{21}, where the asymptotic form of string configuration in $AdS_3$ is constructed in the complex worldsheet $z$ plane by using the Pohlmeyer reduction and solving approximately the auxiliary left and right linear problems in large $z$ involving a single field $\alpha$ which obeys a generalized Sinh-Gordon equation. Through the Stokes phenomenon as $z \to \infty$ which is expressed by the Stokes matrices, the various cusps for the Wilson loop appear in the various angular sectors of the $z$ plane. The problem to compute the minimal area for the octagonal Wilson loop is reduced to the study of SU(2) Hitchin equations \cite{22}. The octagonal gluon amplitude has been computed by using the asymptotic form of the solution in large $z$ and the remainder function has been constructed to be expressed in terms of the spacetime cross ratios.

Starting with the genus one finite-gap form for the string solution in $AdS$, a classification of the allowed solutions has been performed by solving the reality and Virasoro conditions \cite{23}, where there is a construction of a class of solutions with six null boundaries, among which two pairs are collinear.

The asymptotic string solution for the multi-gluon amplitude in ref. \cite{21} is represented by the $AdS_3$ Poincare coordinates $(r, x^+, x^-)$, where a polygonal Wilson loop is going around the cylinder that is identified with the two dimensional space $R^{1,1}$, namely, the $(x^+, x^-)$ space. We will regard this string configuration as living in $AdS_3$ subspace of $AdS_4$ and express it in terms of the $AdS_4$ Poincare coordinates $(\tilde{r}, \tilde{x}_0, \tilde{x}_1, \tilde{x}_2)$ and see how the hexagonal and octagonal Wilson loops at the $AdS_4$ boundary are constructed by connecting a sequence of null vectors in a closed form in the three dimensional space $R^{1,2}$, the $(\tilde{x}_0, \tilde{x}_1, \tilde{x}_2)$ space.
We will demonstrate that in this loop formation the parameters of Stokes matrices play an important role.

For the four-gluon amplitude case specified by $\alpha = 0$ we will construct a general string solution by combining two linearly independent solutions with arbitrary two coefficients in the left and right linear problems respectively. The reality and normalization conditions for the general solutions constrain the left and right coefficients to be characterized by two parameters. We will examine how the freedoms expressed by two parameters are related with some conformal SO(2,4) transformations. We will demonstrate that the string configuration determined from the solutions of the linear problems indeed satisfies the string equations of motion and the Virasoro constraints in the embedding coordinates $(Y_{-1}, Y_0, Y_1, Y_2)$ as well as the $AdS_3$ global coordinates $(t, \rho, \phi)$. Using only the asymptotic form of the exact general string solution we will extract the figure of a generic tetragonal Wilson loop composed of four null vectors in $R^{1,2}$.

2 Asymptotic string solutions for the hexagonal and octagonal Wilson loops

We consider the approximate string solution with Euclidean worldsheet in $AdS_3$ which gives the multi-gluon amplitude in planar $N = 4$ SYM theory at strong coupling [21]. Through a Pohlmeyer type reduction, the problem of string moving $AdS_3$ whose worldsheet is paramerized by complex coordinates $z, \bar{z}$ is transformed to the auxiliary linear problem involving a single field $\alpha(z, \bar{z})$ and a holomorphic polynomial $p(z)$ whose degree $n - 2$ determines the number of the cusps of Wilson loop to be $2n$. The field $\alpha(z, \bar{z})$ satisfies a generalized Sinh-Gordon equation

$$\partial \bar{\partial} \alpha(z, \bar{z}) - e^{2\alpha(z, \bar{z})} + |p(z)|^2 e^{-2\alpha(z, \bar{z})} = 0$$

so that the following SL(2) connections $B^L, B^R$ are flat

$$B^L_z = \begin{pmatrix} \frac{1}{2} \partial_\alpha & -e^\alpha \\ -e^{-\alpha} p(z) & -\frac{1}{2} \partial_\alpha \end{pmatrix}, \quad B^R_z = \begin{pmatrix} -\frac{1}{2} \bar{\partial}_\alpha & -e^{-\alpha} \bar{p}(\bar{z}) \\ -e^\alpha & \frac{1}{2} \bar{\partial}_\alpha \end{pmatrix},$$

$$B^L_{\bar{z}} = \begin{pmatrix} -\frac{1}{2} \partial_\alpha & e^{-\alpha} p(z) \\ -e^\alpha & \frac{1}{2} \partial_\alpha \end{pmatrix}, \quad B^R_{\bar{z}} = \begin{pmatrix} -\frac{1}{2} \bar{\partial}_\alpha & -e^\alpha \\ e^{-\alpha} \bar{p}(\bar{z}) & -\frac{1}{2} \bar{\partial}_\alpha \end{pmatrix}.$$  

(2)

For the connections $B^L,R$ the auxiliary left and right linear problems are given by

$$\partial \psi^L_\alpha + (B^L_z)_\alpha \psi^L_\beta = 0, \quad \bar{\partial} \psi^L_\alpha + (B^L_{\bar{z}})_\alpha \psi^L_\beta = 0,$$

$$\partial \psi^R_\alpha + (B^R_z)_\alpha \psi^R_\beta = 0, \quad \bar{\partial} \psi^R_\alpha + (B^R_{\bar{z}})_\alpha \psi^R_\beta = 0,$$

(3)

whose two linearly independent solutions $\psi^L_{\alpha,a}, a = 1, 2$ and $\psi^R_{\alpha,\tilde{a}}, \tilde{a} = 1, 2$ are normalized as

$$e^{\alpha_\alpha} \psi^L_{\alpha,a} \psi^L_{\beta,b} = \epsilon_{ab}, \quad e^{\tilde{\alpha}_{\tilde{a}}} \psi^R_{\tilde{a},\tilde{a}} \psi^R_{\beta,b} = \epsilon_{\tilde{a}\tilde{b}}.$$  

(4)

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These left and right solutions lead to the spacetime configuration of the string surface which is expressed as
\[ Y_{a \dot{a}} = \left( \begin{array}{c} Y_{-1} + Y_2 \\ Y_1 + Y_0 \\ Y_{-1} - Y_2 \end{array} \right)_{a, \dot{a}} = \psi^L_{a, \dot{a}} M^{\alpha \dot{\beta}}_{1} \psi^R_{\beta, \dot{\alpha}}, \quad M^{\alpha \dot{\beta}}_1 = \left( \begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array} \right), \tag{5} \]
where the embedding coordinates \( Y_{\mu} \) describe the AdS3 space by \(-Y_{-1}^2 - Y_0^2 + Y_1^2 + Y_2^2 = 1\). Introducing the complex coordinates \( w \) and \( \tilde{w} \) by \( dw = \sqrt{p(z)}dz, d\tilde{w} = \sqrt{\tilde{p}(\tilde{z})d\tilde{z}} \) the generalized Sinh-Gordon equation (11) is simplified to be
\[ \partial_w \partial_{\tilde{w}} \alpha e^{2\tilde{\alpha}} + e^{-2\tilde{\alpha}} = 0, \quad \tilde{\alpha} \equiv \alpha - \frac{1}{4} \log p \tilde{p}. \tag{6} \]
The left and right linear problems (3) are also expressed by complex variables \( w, \tilde{w} \) and the approximate left and right solutions for large \( w \) yield the following string solution
\[ Y_{a \dot{a}} = \frac{1}{\sqrt{2}} (c^{L+}_{a} + c^{R-}_{a} e^{u} + c^{L-}_{a} - c^{R+}_{a} e^{-u} - c^{L-}_{a} - c^{R-}_{a} e^{-v} + c^{L+}_{a} c^{R+}_{a} e^{v} + c^{L-}_{a} c^{R-}_{a} e^{-v}), \tag{7} \]
where
\[ u = w + \tilde{w} + \frac{w - \tilde{w}}{i}, \quad v = -(w + \tilde{w}) + \frac{w - \tilde{w}}{i}. \tag{8} \]
The expression (7) shows the asymptotic solution such that in each quadrant of the \( w \) plane only one of these terms dominates and characterizes the spacetime string configuration near each cusp. Owing to the Stokes phenomenon the coefficients in (7) in the five Stokes sectors for the left problem are specified as [21]

\[ \begin{align*}
[12] : & \quad c^{L+}_{[12],a} = b^{+}_{a}, \quad c^{L-}_{[12],a} = b^{-}_{a}, \\
[23] : & \quad c^{L+}_{[23],a} = b^{+}_{a} + \gamma^{L}_{2} b^{-}_{a}, \quad c^{L-}_{[23],a} = b^{-}_{a}, \\
[34] : & \quad c^{L+}_{[34],a} = b^{+}_{a} + \gamma^{L}_{2} b^{-}_{a}, \quad c^{L-}_{[34],a} = b^{-}_{a} + \gamma^{L}_{3} (b^{+}_{a} + \gamma^{L}_{2} b^{-}_{a}), \\
[45] : & \quad c^{L+}_{[45],a} = b^{+}_{a} + \gamma^{L}_{2} b^{-}_{a} + \gamma^{L}_{4} (b^{+}_{a} + \gamma^{L}_{3} (b^{+}_{a} + \gamma^{L}_{2} b^{-}_{a})), \\
& \quad c^{L-}_{[45],a} = b^{-}_{a} + \gamma^{L}_{3} (b^{+}_{a} + \gamma^{L}_{2} b^{-}_{a}), \\
[56] : & \quad c^{L+}_{[56],a} = b^{+}_{a} + \gamma^{L}_{2} b^{-}_{a} + \gamma^{L}_{4} (b^{+}_{a} + \gamma^{L}_{3} (b^{+}_{a} + \gamma^{L}_{2} b^{-}_{a})), \\
& \quad c^{L-}_{[56],a} = b^{-}_{a} + \gamma^{L}_{3} (b^{+}_{a} + \gamma^{L}_{2} b^{-}_{a}) + \gamma^{L}_{5} (b^{+}_{a} + \gamma^{L}_{2} b^{-}_{a} + \gamma^{L}_{4} (b^{+}_{a} + \gamma^{L}_{3} (b^{+}_{a} + \gamma^{L}_{2} b^{-}_{a}))), \\
\end{align*} \tag{9} \]
where \( \gamma^{L}_{i} \) is the Stokes parameter for the left problem and the parameters \( b^{r}_{a} (r = +, -) \) satisfy
\[ b^{+}_{a} b^{-}_{a} - b^{-}_{a} b^{+}_{a} = \epsilon^{ab}. \tag{10} \]
The coefficients of the asymptotic solution for the right problem are also expressed by the right Stokes parameter \( \gamma^{R}_{i} \) and the correspending quantities with tildes \( \tilde{b}^{r}_{a} \) which also satisfy
\[ \tilde{b}^{+}_{a} \tilde{b}^{-}_{a} - \tilde{b}^{-}_{a} \tilde{b}^{+}_{a} = \epsilon^{\dot{a} \dot{b}}. \tag{11} \]
Using the minimal surface string solution in $AdS_3$ \( \text{(7)} \) together with \( \text{(9)} \) we describe the $AdS_3$ subspace of $AdS_4$ in terms of the $AdS_4$ embedding coordinates with $Y_3 = 0$ and express the $AdS_4$ embedding of the surface as

\[
\frac{1}{r} = Y_1 Y_2, \quad \tilde{x}_0 = \frac{Y_0}{Y_1 Y_2}, \quad \tilde{x}_{1,2} = \frac{Y_{1,2}}{Y_1 Y_2}.
\]  

We consider the spacetime feature of the string configuration in the $AdS_4$ Poincare coordinates \((\tilde{r}, \tilde{x}_0, \tilde{x}_1, \tilde{x}_2)\). In the first quadrant at $\Re w > 0, \Im w > 0$ which belongs to [12] Stokes sector for the left problem and [01] Stokes sector for the right problem where $c_{[01], \tilde{a}} = \tilde{b}_a^+$, the first term in \( \text{(7)} \) dominates so that we combine \( \text{(5)} \) with \( \text{(7)} \) and \( \text{(9)} \) to obtain

\[
\left(\begin{array}{cc}
Y_{-1} + Y_2 & Y_1 - Y_0 \\
Y_1 + Y_0 & Y_{-1} - Y_2
\end{array}\right) = \frac{e^u}{\sqrt{2}} \left(\begin{array}{cc}
b_1^+ b_1^+ & b_1^+ b_2^+
\end{array}\right).
\]  

The asymptotic solution in the first cusp labelled by \((1,1)\) is expressed as

\[
\frac{1}{r} = \frac{1}{2\sqrt{2}} X_1^1 e^u, \quad \tilde{x}_0 = \frac{Y_1}{X_1^1}, \quad \tilde{x}_1 = \frac{Y_1}{X_1^1}, \quad \tilde{x}_2 = \frac{X_1^1}{X_1^1}
\]  

with $X_1^1 = b_1^+ b_1^+ \pm b_2^+ b_2^+$, $Y_1^1 = b_2^+ b_1^+ \pm b_1^+ b_2^+$. For large $u$ the string approaches the boundary of $AdS_4$ specified by $\tilde{r} = 0$. In the second quadrant at $\Re w < 0, \Im w > 0$ which belongs to \([12]\) Stokes sectors for the left and right problems the third term in \( \text{(7)} \) becomes a big term so that the asymptotic solution in the second cusp labelled by \((2,1)\) is given by

\[
\frac{1}{r} = \frac{1}{2\sqrt{2}} X_2^2 e^v, \quad \tilde{x}_0 = \frac{Y_2}{X_2^2}, \quad \tilde{x}_1 = \frac{Y_2}{X_2^2}, \quad \tilde{x}_2 = \frac{X_2^2}{X_2^2}
\]  

with $X_2^2 = b_1^- b_1^+ \pm b_2^- b_2^+$, $Y_2^2 = b_2^- b_1^+ \pm b_1^- b_2^+$. In the third quadrant at $\Re w < 0, \Im w < 0$ which belongs to \([23]\) Stokes sector for the left problem and \([12]\) Stokes sector for the right problem the second term in \( \text{(7)} \) dominates and the string near the third \((2,2)\) cusp is specified by

\[
\frac{1}{r} = \frac{1}{2\sqrt{2}} X_3^3 e^{-u}, \quad \tilde{x}_0 = \frac{Y_3}{X_3^3}, \quad \tilde{x}_1 = \frac{Y_3}{X_3^3}, \quad \tilde{x}_2 = \frac{X_3^3}{X_3^3}
\]  

with $X_3^3 = b_1^- b_1^- \pm b_2^- b_2^-$, $Y_3^3 = b_2^- b_1^- \pm b_1^- b_2^-$. In the fourth quadrant at $\Re w > 0, \Im w < 0$ which belongs to both \([23]\) Stokes sectors for the left and right problems the fourth term in \( \text{(7)} \) dominates and the string near the fourth \((3,2)\) cusp is specified by

\[
\frac{1}{r} = \frac{1}{2\sqrt{2}} X_4^4 e^{-v}, \quad \tilde{x}_0 = \frac{Y_4}{X_4^4}, \quad \tilde{x}_1 = \frac{Y_4}{X_4^4}, \quad \tilde{x}_2 = \frac{X_4^4}{X_4^4}
\]  

with

\[
X_4^4 = (b_1^+ + \gamma_2 b_1^-)b_1^- \pm (b_2^+ + \gamma_2 b_2^-)b_2^- \quad \text{and} \quad Y_4^4 = (b_2^+ + \gamma_2 b_2^-)b_1^- \pm (b_1^+ + \gamma_2 b_1^-)b_2^-.
\]  

Succeeding the solutions in the fifth \((3,3)\) cusp, the sixth \((4,3)\) cusp, the seventh \((4,4)\) cusp and the eighth \((5,4)\) cusp are respectively described in order as

\[
\frac{1}{r} = \frac{1}{2\sqrt{2}} X_k^k f_k(u, v), \quad \tilde{x}_0 = \frac{Y_k}{X_k^k}, \quad \tilde{x}_1 = \frac{Y_k}{X_k^k}, \quad \tilde{x}_2 = \frac{X_k^k}{X_k^k}, \quad k = 5, 6, 7, 8
\]  

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where
\[
\begin{align*}
  f_k(u, v) &= (e^u, -e^v, e^{-u}, e^{-v}), \\
  X_\pm^k &= B_1^k \tilde{B}_1^k \pm B_2^k \tilde{B}_2^k, \quad Y_\pm^k = B_1^k \tilde{B}_1^k \pm B_2^k \tilde{B}_2^k, \\
  B_a^5 &= b_a^+ + \gamma_2^L b_a^-, \\
  B_a^6 &= b_a^- + \gamma_2^R (b_a^+ + \gamma_2^L b_a^-), \\
  B_a^7 &= b_a^- + \gamma_3^L (b_a^+ + \gamma_2^L b_a^-), \\
  B_a^8 &= b_a^+ + \gamma_2^R b_a^- + \gamma_4^L (b_a^- + \gamma_3^L (b_a^+ + \gamma_2^L b_a^-)), \\
  B_a^9 &= \tilde{b}_a^+ + \gamma_2^R \tilde{b}_a^- + \gamma_4^L (\tilde{b}_a^- + \gamma_3^L (\tilde{b}_a^+ + \gamma_2^L \tilde{b}_a^-)). 
\end{align*}
\]

From these expressions we can confirm that the vectors \(\tilde{x}_\mu^k - \tilde{x}_\mu^{k+1}\) are the null vectors
\[
(\tilde{x}_\mu^k - \tilde{x}_\mu^{k+1})^2 = 0, \quad k = 1, 2, \ldots, 6 
\]
where the normalization conditions of the parameters \(b_a^r\) and \(\tilde{b}_a^r\) in (10) and (11) are used.

Here for the hexagonal Wilson loop with six cusps to be constructed in a closed form at the AdS_4 boundary we require \(\tilde{x}_\mu^7 = \tilde{x}_\mu^1\) which gives the following relations for the Stokes parameters of the left and right problems
\[
1 + \gamma_2^L \gamma_3^L = 0, \quad 1 + \gamma_2^R \gamma_3^R = 0. 
\]

In ref. [21] by analyzing the behavior of the approximate left and right solutions when we go around once in the \(z\) plane ( or \(n/2\) times in the \(w\) plane ), the following relations for the Stokes matrices for both left and right problems are presented
\[
\begin{align*}
  S_p(\gamma_1)S_n(\gamma_2)S_p(\gamma_3) \cdots S_p(\gamma_n) &= i(-1)^{\frac{n-1}{2}} \sigma^2 \quad \text{n odd}, \\
  S_p(\gamma_1)S_n(\gamma_2) \cdots S_n(\gamma_n)e^{\sigma^3 (w_\alpha + \bar{w}_\alpha)} &= -(-1)^{n/2} \quad \text{n even}
\end{align*}
\]
with the Pauli matrices \(\sigma^i\), where
\[
S_p = \begin{pmatrix} 1 & \gamma \\ 0 & 1 \end{pmatrix}, \quad S_n = \begin{pmatrix} 1 & 0 \\ \gamma & 1 \end{pmatrix}
\]
and the shift parameters \(w_\alpha, \bar{w}_\alpha\) are related with the spacetime cross ratios. For the hexagonal Wilson loop case we express the first relation in (23) with \(n = 3\) as
\[
\begin{pmatrix}
  1 + \gamma_1 \gamma_2 & \gamma_1 + \gamma_3 + \gamma_1 \gamma_2 \gamma_3 \\
  \gamma_2 & 1 + \gamma_2 \gamma_3 
\end{pmatrix} = \begin{pmatrix}
  0 & 1 \\
  -1 & 0 
\end{pmatrix},
\]
whose diagonal components directly yield (22). Alternatively this observation implies that the segment between the first cusp and the sixth cusp is described by a lightlike vector.

In order to consider the \(n = 4\) case, that is, the octagonal Wilson loop case we write down the solution in the ninth (5,5) cusp as the forms of \([19]\) and \([20]\) for \(k = 9\) with \(f_9 = e^u\) and
\[
\begin{align*}
  B_a^9 &= b_a^+ + \gamma_2^R b_a^- + \gamma_4^L (b_a^- + \gamma_3^L (b_a^+ + \gamma_2^L b_a^-)), \\
  \tilde{B}_a^9 &= \tilde{b}_a^+ + \gamma_2^R \tilde{b}_a^- + \gamma_4^L (\tilde{b}_a^- + \gamma_3^L (\tilde{b}_a^+ + \gamma_2^L \tilde{b}_a^-)).
\end{align*}
\]
For the eight-cusp Wilson loop to be of a closed form we require \( \mathbf{\tilde{x}}^9 = \mathbf{x}^1 \) which leads to the constraints for the Stokes parameters

\[
\gamma_2^L + \gamma_4^L + \gamma_2^L \gamma_3^L \gamma_4^L = 0, \quad \gamma_2^R + \gamma_4^R + \gamma_2^R \gamma_3^R \gamma_4^R = 0.
\]  
(27)

The second relation in (23) with \( n = 4 \) is given by

\[
\begin{pmatrix}
(1 + \gamma_2 \gamma_3)(1 + \gamma_3 \gamma_4) + \gamma_1 \gamma_4 e^{w_s + \bar{w}_s} \\
(\gamma_2 + \gamma_4 + \gamma_2 \gamma_3 \gamma_4)e^{w_s + \bar{w}_s}
\end{pmatrix}
= - \begin{pmatrix}
1 & 0 \\
0 & 1
\end{pmatrix},
\]
(28)

whose off-diagonal components directly yield (27).

For the hexagonal Wilson loop case in the projected \((\mathbf{x}_1, \mathbf{x}_2)\) plane the square of the position vector \( \mathbf{\tilde{x}}^1 = (\mathbf{x}_1^1, \mathbf{x}_2^1) \) for the first cusp is larger than unity

\[
(\mathbf{\tilde{x}}^1)^2 = \frac{\mathbf{\tilde{b}}_L^2 \mathbf{\tilde{b}}_R^2}{(\mathbf{b}_L \cdot \mathbf{b}_R)^2} \geq 1,
\]
(29)

where two vectors \( \mathbf{\tilde{b}}_L \) and \( \mathbf{\tilde{b}}_R \) are defined by \( \mathbf{\tilde{b}}_L = (b_1^+, b_2^+) \), \( \mathbf{\tilde{b}}_R = (\tilde{b}_1^+, \tilde{b}_2^+) \). In similar expressions including an inner product of two vectors we see \( (\mathbf{\tilde{x}}^k)^2 \geq 1, \ k = 1, 2, \cdots, 6 \). For example the sixth cusp \((k = 6)\) is characterized by

\[
\mathbf{\tilde{b}}_L = (B_1^6, B_2^6), \quad \mathbf{\tilde{b}}_R = (\tilde{B}_1^6, \tilde{B}_2^6).
\]
(30)

Thus we observe that in the \((\mathbf{x}_1, \mathbf{x}_2)\) plane each cusp is located outside the unit circle.

An arbitrary point \( P_k \) \((k = 1, 2, \cdots, 6)\) between \( \mathbf{x}^k \) and \( \mathbf{x}^{k+1} \) defines a vector

\[
\mathbf{P}_k = \mathbf{\tilde{x}}^k + (\mathbf{\tilde{x}}^{k+1} - \mathbf{\tilde{x}}^k)t_k
\]
(31)

with \( \mathbf{\tilde{x}}^7 = \mathbf{\tilde{x}}^1 \), whose parameter \( t_k \) is fixed as

\[
t_k = \frac{(\mathbf{\tilde{x}}^k)^2 - \mathbf{\tilde{x}}^k \cdot \mathbf{\tilde{x}}^{k+1}}{(\mathbf{\tilde{x}}^k - \mathbf{\tilde{x}}^{k+1})^2}
\]
(32)

by demanding the orthogonal condition \( \mathbf{P}_k \cdot (\mathbf{\tilde{x}}^k - \mathbf{\tilde{x}}^{k+1}) = 0 \). Therefore eliminating \( t_k \) we have

\[
\mathbf{P}_k^2 = \frac{(\mathbf{\tilde{x}}^k)^2(\mathbf{\tilde{x}}^{k+1})^2 - (\mathbf{\tilde{x}}^k \cdot \mathbf{\tilde{x}}^{k+1})^2}{(\mathbf{\tilde{x}}^k - \mathbf{\tilde{x}}^{k+1})^2},
\]
(33)

which can be shown to become unity through (10) and (11), so that each point \( P_k \) is located on the unit circle. Thus we demonstrate that for \( n = 3 \) the sides of hexagonal Wilson loop are tangent to the unit circle in the projected \((\mathbf{x}_1, \mathbf{x}_2)\) plane. In the \((\mathbf{x}_0, \mathbf{x}_1, \mathbf{x}_2)\) space the time component of the \( k \)-th point \( P_k \) is defined by \( \mathbf{x}^0_k + (\mathbf{x}^{k+1}_0 - \mathbf{x}^k_0)t_k \) which turns out to be zero through (32). Thus in the \((\mathbf{x}_0, \mathbf{x}_1, \mathbf{x}_2)\) space the cusps of hexagonal Wilson loop are going alternating up and down outside the unit circle. It implies that the even sides of the Wilson loop allow the successive lightlike vectors to form a closed contour.

Let us examine how the asymptotic solution (7) constructed from the auxiliary linear problems in the \( w \) plane satisfies the string equations of motion and the Virasoro constraints
in the $z$ plane for the octagonal Wilson loop case ($n = 4$). For large $z$ or large $w$, the polynomials $p(z)$, $\bar{p}(\bar{z})$ are approximately given by $p(z) \approx z^2$, $\bar{p}(\bar{z}) \approx \bar{z}^2$ so that $w \approx z^2/2$, $\bar{w} \approx \bar{z}^2/2$. We use a notation $z = x + iy$, $\bar{z} = x - iy$ to express $u, v$ as $u \approx x^2 - y^2 + 2xy$, $v \approx -x^2 + y^2 + 2xy$. The conformal gauge equations of motion and the Virasoro constraints in the $z$ plane are expressed in terms of the embedding coordinates $Y_\mu$ as

$$\partial \bar{\partial} Y_\mu - (\partial Y_\nu \bar{\partial} Y^\nu) Y_\mu = 0, \quad \partial Y_\mu \partial Y^\nu = \bar{\partial} Y_\mu \bar{\partial} Y^\nu = 0. \quad (34)$$

Here we begin with the asymptotic solutions in the first, second, third and fourth quadrants

$$(Y^{-1}_k, Y^0_k, Y^1_k, Y^2_k) = \frac{1}{2\sqrt{2}}(X^k_+, Y^k_-, Y^k_+, X^k_-) f_k(x, y), \quad k = 1, 2, 3, 4 \quad (35)$$

where

$$f_1 = e^{x^2-y^2+2xy}, \quad f_2 = -e^{-x^2+y^2+2xy}, \quad f_3 = e^{-x^2+y^2-2xy}, \quad f_4 = e^{x^2-y^2-2xy}. \quad (36)$$

For large $w$ the parameter $\hat{\alpha}$ becomes zero so that $\partial Y_\nu \bar{\partial} Y^\nu = 2e^{2\alpha} \approx 2\sqrt{p\bar{p}} \approx 2(x^2 + y^2)$. This is compared with the four-side Wilson loop case ($n = 2$) where $\hat{\alpha} = \alpha = 0$ and the $w$ plane is identical to the $z$ plane. The equations of motion for large $z$

$$\frac{1}{4} (\partial_x^2 + \partial_y^2) Y^k - 2(x^2 + y^2) Y^k_\mu = 0 \quad (37)$$

are satisfied by (35) with the four kinds of functions $f_k(x, y)$ in (36). The Virasoro constraints also hold owing to the relations $(X^k_+)^2 - (X^k_-)^2 = (Y^k_+)^2 - (Y^k_-)^2$, $k = 1, 2, 3, 4$ which follow from (14), (15), (16) and (17). For the other quadrants each dominant solution in (7) for large $z$ satisfies the equations of motion and the Virasoro constraints through the same relations $(X^k_+)^2 - (X^k_-)^2 = (Y^k_+)^2 - (Y^k_-)^2$, $k = 5, 6, 7, 8$ which are obeyed by the expressions in (20). These relations are attributed to the expression that the coefficient of the dominant term for each quadrant in (7) is given in a product form as $c_a^{L, \pm} c_a^{R, \pm}$.

## 3 Asymptotic behaviors of the general string solutions for the tetragonal Wilson loop

We consider the four-side Wilson loop in the $z$ plane which is specified by $\hat{\alpha} = \alpha = 0, p = 1$. The left linear problem in (3) becomes

$$\partial_x \psi^L_1 = \psi^L_2, \quad \partial_y \psi^L_1 = \psi^L_2, \quad \partial_x \psi^L_2 = \psi^L_1, \quad \partial_y \psi^L_2 = \psi^L_1. \quad (38)$$

The two equations in (38) combine to be $\partial^2_x \psi^L_1 = \psi^L_1$ which gives two linearly independent solutions $\psi^L_{1a}$, $a = 1, 2$ as expressed by $\psi^L_{11} = c_1(z)e^z$, $\psi^L_{12} = c_2(z)e^{-z}$. The two solutions should satisfy $\partial^2_x \psi^L_1 = \psi^L_1$ which follows from (38) so that $c_1(z), c_2(z)$ are expanded by $c_1(z) = c_{11}e^z + c_{12}e^{-z}$, $c_2(z) = c_{21}e^z + c_{22}e^{-z}$. The substitution of $\psi^L_{1a}$ into (38) gives $\psi^L_{21} = c_1(z)e^z$, $\psi^L_{22} = -c_2(z)e^{-z}$. The expressions $\psi^L_{11}$ and $\psi^L_{21}$ are substituted into (39) and
we have $c_{12} = 0$, while the insertions of $\psi_{12}^L$ and $\psi_{22}^L$ into (39) lead to $c_{21} = 0$. Thus we obtain two linearly independent solutions for the left problem

$$\psi_{\alpha 1}^L = c_{11} \left( \frac{e^{z+\bar{z}}}{e^{z+\bar{z}}} \right), \quad \psi_{\alpha 2}^L = c_{22} \left( \frac{e^{-z-\bar{z}}}{-e^{-z-\bar{z}}} \right),$$

(40)

whose coefficients are fixed as $c_{11} = c_{22} = 1/\sqrt{2}$ through the normalization condition (3).

The right linear problem in (3) is also expressed as

$$\partial_z \psi_1^R = -\psi_2^R, \quad \partial_z \psi_2^R = \psi_1^R, \quad \bar{\partial}_z \psi_1^R = \psi_2^R, \quad \bar{\partial}_z \psi_2^R = -\psi_1^R,$$

(41)

which give two linearly independent solutions $\psi_{11}^R = \tilde{c}_1(\bar{z})e^{-iz}$, $\psi_{12}^R = \tilde{c}_2(\bar{z})e^{iz}$ and $\psi_{21}^R = \tilde{i}\tilde{c}_1(\bar{z})e^{-iz}$, $\psi_{22}^R = -\tilde{i}\tilde{c}_2(\bar{z})e^{iz}$ where the coefficients are expanded as $\tilde{c}_1(\bar{z}) = \tilde{c}_{11}e^{iz} + \tilde{c}_{12}e^{-iz}$ and $\tilde{c}_2(\bar{z}) = \tilde{c}_{21}e^{iz} + \tilde{c}_{22}e^{-iz}$ but with $\tilde{c}_{12} = \tilde{c}_{21} = 0$. The two linearly independent solutions for the right problem are expressed as

$$\psi_{\alpha 1}^R = \tilde{c}_{11} \left( \frac{e^{-i\bar{z}}}{i e^{i\bar{z}}} \right), \quad \psi_{\alpha 2}^R = \tilde{c}_{22} \left( \frac{e^{-i\bar{z}}}{-i e^{i\bar{z}}} \right),$$

(42)

whose coefficients are fixed as $\tilde{c}_{11} = 1/\sqrt{2}$, $\tilde{c}_{22} = -i/\sqrt{2}$ through the normalization condition (4).

For the left and right problems the general solutions are given by the linear combination of two independent solutions

$$\psi_{\alpha a}^L = d_{ab}\psi_{ab}^L, \quad \psi_{\alpha a}^R = \bar{d}_{ab}\psi_{ab}^R.$$  

(43)

If we substitute these expressions into the normalization condition (4) for $\psi'$ we have

$$d_{11}d_{22} - d_{12}d_{21} = 1, \quad \bar{d}_{11}\bar{d}_{22} - \bar{d}_{12}\bar{d}_{21} = 1.$$  

(44)

Substitutions of the general solutions (43) into (5) yield

$$Y_{-1} + Y_2 = \frac{1}{2} [d_{11}(1+i)(\bar{d}_{11}e^u - \bar{d}_{12}e^{-u}) + d_{12}(1-i)(\bar{d}_{11}e^u + \bar{d}_{12}e^{-u})],$$

$$Y_{-1} - Y_2 = \frac{1}{2} [d_{21}(1+i)(\bar{d}_{21}e^u - \bar{d}_{22}e^{-u}) + d_{22}(1-i)(\bar{d}_{21}e^u + \bar{d}_{22}e^{-u})],$$

$$Y_1 + Y_0 = \frac{1}{2} [d_{21}(1+i)(\bar{d}_{21}e^u - \bar{d}_{22}e^{-u}) + d_{12}(1-i)(\bar{d}_{21}e^u + \bar{d}_{22}e^{-u})],$$

$$Y_1 - Y_0 = \frac{1}{2} [d_{11}(1+i)(\bar{d}_{11}e^u - \bar{d}_{12}e^{-u}) + d_{12}(1-i)(\bar{d}_{11}e^u + \bar{d}_{12}e^{-u})],$$

(45)

where $u = z + \bar{z} + (z - \bar{z})/i$, $v = -(z + \bar{z}) + (z - \bar{z})/i$. It is noted that we should choose the parameters $d_{ab}$ such that $d_{11}, d_{21}$ include a factor $(1 - i)$ and $d_{22}, d_{12}$ have $(1 + i)$ because $Y_{\mu}$ is real. Therefore we use the conditions in (44) to parametrize $d_{ab}, \bar{d}_{ab}$ as

$$d_{ab} = \frac{1}{\sqrt{2}} \left( \begin{array}{cc} (1 - i) \cos \beta & (1 + i) \sin \beta \\ -(1 - i) \sin \beta & (1 + i) \cos \beta \end{array} \right), \quad \bar{d}_{ab} = \left( \begin{array}{cc} \cosh \gamma & \sinh \gamma \\ \sinh \gamma & \cosh \gamma \end{array} \right).$$

(46)
Here if we express the complex variables $z, \bar{z}$ as $z = (\sigma + i\tau)/2$, $\bar{z} = (\sigma - i\tau)/2$ the string configuration is obtained by

\[
\begin{pmatrix}
Y_{-1} \\
Y_1
\end{pmatrix} = \frac{1}{\sqrt{2}}
\begin{pmatrix}
\cosh\gamma & \sinh\gamma \\
\sinh\gamma & \cosh\gamma
\end{pmatrix}
\begin{pmatrix}
\cos\beta \cosh(\sigma + \tau) + \sin\beta \cosh(\sigma - \tau) \\
-\sin\beta \sinh(\sigma + \tau) - \cos\beta \sinh(\sigma - \tau)
\end{pmatrix},
\]

(47)

\[
\begin{pmatrix}
Y_0 \\
Y_2
\end{pmatrix} = \frac{1}{\sqrt{2}}
\begin{pmatrix}
\cosh\gamma & -\sinh\gamma \\
-\sinh\gamma & \cosh\gamma
\end{pmatrix}
\begin{pmatrix}
-\sin\beta \cosh(\sigma + \tau) + \cos\beta \cosh(\sigma - \tau) \\
\cos\beta \sinh(\sigma + \tau) - \sin\beta \sinh(\sigma - \tau)
\end{pmatrix}.
\]

For the $\gamma = 0, \beta \neq 0$ case the string solution is expressed as

\[
\begin{pmatrix}
Y_{-1} \\
Y_0
\end{pmatrix} = \begin{pmatrix}
\cos\beta & \sin\beta \\
-\sin\beta & \cos\beta
\end{pmatrix}
\begin{pmatrix}
\frac{1}{\sqrt{2}} \cosh(\sigma + \tau) \\
\frac{1}{\sqrt{2}} \cosh(\sigma - \tau)
\end{pmatrix},
\]

(48)

\[
\begin{pmatrix}
Y_1 \\
Y_2
\end{pmatrix} = \begin{pmatrix}
\cos\beta & -\sin\beta \\
\sin\beta & \cos\beta
\end{pmatrix}
\begin{pmatrix}
-\frac{1}{\sqrt{2}} \sinh(\sigma - \tau) \\
\frac{1}{\sqrt{2}} \sinh(\sigma + \tau)
\end{pmatrix}.
\]

The basic solution specified by $\gamma = 0, \beta = \pi/4$ is given by

\[
Y_{-1} = \cosh\tau \cosh\sigma, \quad Y_0 = -\sinh\tau \sinh\sigma, \quad Y_1 = -\cosh\tau \sinh\sigma, \quad Y_2 = \sinh\tau \cosh\sigma.
\]

(49)

The sign change as $\sigma \rightarrow -\sigma$ for (49) leads to the expression in ref. [4] associated with the square Wilson loop with four cusps which is obtained by making a scaling limit of the spinning folded string with Minkowski worldsheet in $AdS_3$ and changing to Euclidean worldsheet $[11]$. On the other hand the $\gamma = \beta = 0$ solution is written by

\[
Y_{-1} = \frac{1}{\sqrt{2}} \cosh(\sigma + \tau), \quad Y_0 = \frac{1}{\sqrt{2}} \cosh(\sigma - \tau),
\]

\[
Y_1 = -\frac{1}{\sqrt{2}} \sinh(\sigma - \tau), \quad Y_2 = \frac{1}{\sqrt{2}} \sinh(\sigma + \tau),
\]

(50)

which is transformed through $\sigma \rightarrow -\sigma$ and a discrete SO(2,4) interchange $Y_{-1} \leftrightarrow Y_0$ to the solution $[11]$ which is obtained by making an SO(2) rotation in the $(\tau, \sigma)$ plane for the basic one-cusp solution $[5]$. Thus we have observed that the linear coefficients $d_{ab}$ for the left problem are parametrized to be associated with the simultaneous SO(2) rotations with opposite angles in the $(Y_{-1}, Y_0)$ and $(Y_1, Y_2)$ planes, while those $\tilde{d}_{ab}$ for the right problem with the simultaneous SO(2,4) boosts with opposite boost parameters in the $(Y_{-1}, Y_1)$ and $(Y_0, Y_2)$ planes. As the other parametrizations we take $\beta = \pi/4$ with

\[
\tilde{d}_{ab} = \begin{pmatrix}
e^{\tau_1} & 0 \\
0 & e^{-\tau_1}
\end{pmatrix}, \quad \tilde{d}_{\bar{a}b} = \begin{pmatrix}
0 & e^{\tau_2} \\
-e^{-\tau_2} & 0
\end{pmatrix}
\]

(51)

to derive the string solutions respectively as

\[
Y_{-1} = \cosh(\tau + \tau_1) \cosh\sigma, \quad Y_0 = -\sinh(\tau + \tau_1) \sinh\sigma, \quad Y_1 = -\cosh(\tau + \tau_1) \sinh\sigma, \quad Y_2 = \sinh(\tau + \tau_1) \cosh\sigma
\]

(52)

and

\[
Y_{-1} = \sinh(\tau - \tau_2) \sinh\sigma, \quad Y_0 = \cosh(\tau - \tau_2) \cosh\sigma, \quad Y_1 = -\sinh(\tau - \tau_2) \cosh\sigma, \quad Y_2 = -\cosh(\tau - \tau_2) \sinh\sigma.
\]

(53)
The configuration (52) shows only a shift of $\tau$ by $\tau_1$ for the basic solution (49).

The Virasoro constraints in (34) are given by

$$\left(\partial_{Y_1} \partial Y_1\right) \left(\begin{array}{c} -1 & 0 \\ 0 & 1 \end{array}\right) \left(\partial_{Y_1} \partial Y_1\right) + \left(\partial_{Y_0} \partial Y_2\right) \left(\begin{array}{c} -1 & 0 \\ 0 & 1 \end{array}\right) \left(\partial_{Y_0} \partial Y_2\right) = 0$$

(54)

and the equation which we get by the replacement of $\partial \rightarrow \bar{\partial}$, which are confirmed to be satisfied by the matrix representation of the general solution (47). Similarly using the matrix representation (47) we obtain $\partial Y_\nu \bar{\partial} Y^\nu = 2$, that is, $\alpha = 0$ consistently so that the equations of motion in (34) become

$$(\partial_\sigma^2 + \partial_\tau^2) Y_\mu - 2Y_\mu = 0 \quad \text{or} \quad \frac{1}{4}(\partial_\sigma^2 + \partial_\tau^2) Y_\mu - 2Y_\mu = 0,$$

(55)

which are simply satisfied by the string configuration (47) and compared with the asymptotic equations of motion (37) for large $z$ associated with the string configuration for the hexagonal Wilson loop case.

The embedding coordinates $Y_\mu$ are related to the standard global coordinates $(t, \rho, \phi)$ on $AdS_3$ by

$$Y_{-1} + iy_0 = \cosh \rho e^{it}, \quad Y_1 + iy_2 = \sinh \rho e^{i\phi}.$$  

(56)

We analyse the general solution (47) in the global coordinates, which is expressed as

$$\cosh \rho = \frac{1}{\sqrt{2}}[(\cosh \gamma \cosh \sigma_+ - \sinh \gamma \sinh \sigma_-)^2 + (\sinh \gamma \sinh \sigma_+ - \cosh \gamma \cosh \sigma_-)^2]^{1/2}$$

$$= [\cosh^2 \tau \cosh^2(\sigma - \gamma) + \sinh^2 \tau \sinh^2(\sigma + \gamma)]^{1/2},$$

$$\sinh \rho = \frac{1}{\sqrt{2}}[(\sinh \gamma \cosh \sigma_+ - \cosh \gamma \sinh \sigma_-)^2 + (\cosh \gamma \sinh \sigma_+ - \sinh \gamma \cosh \sigma_-)^2]^{1/2}$$

$$= [\cosh^2 \tau \sinh^2(\sigma - \gamma) + \sinh^2 \tau \cosh^2(\sigma + \gamma)]^{1/2},$$

$$\tan(t + \beta) = -\frac{\sinh \gamma \sinh \sigma_+ + \cosh \gamma \cosh \sigma_-}{\cosh \gamma \cosh \sigma_+ - \sinh \gamma \sinh \sigma_-},$$

$$\tan(\phi - \beta) = -\frac{\cosh \gamma \sinh \sigma_+ - \sinh \gamma \cosh \sigma_-}{\sinh \gamma \cosh \sigma_+ - \cosh \gamma \sinh \sigma_-}$$

(57)

with $\sigma_\pm = \sigma \pm \tau$. Since there are compact differential expressions derived from (57)

$$\partial_\tau t = -\frac{\sinh(\sigma + \gamma) \cosh(\sigma - \gamma)}{\cosh^2 \rho}, \quad \partial_\sigma t = -\frac{\cosh 2\gamma \sinh 2\tau}{2 \cosh^2 \rho},$$

$$\partial_\tau \phi = -\frac{\sinh(\sigma - \gamma) \cosh(\sigma + \gamma)}{\sinh^2 \rho}, \quad \partial_\sigma \phi = \frac{\cosh 2\gamma \sinh 2\tau}{2 \sinh^2 \rho},$$

(58)

the equations of motion for $t$ and $\phi$ from the conformal gauge string Lagrangian given by

$$\partial_\tau (\cosh^2 \rho \partial_\tau t) + \partial_\sigma (\cosh^2 \rho \partial_\sigma t) = 0,$$

$$\partial_\tau (\sinh^2 \rho \partial_\tau \phi) + \partial_\sigma (\sinh^2 \rho \partial_\sigma \phi) = 0$$

(59)

are simply satisfied. The other symmetric differential expressions

$$\partial_\tau \rho = \frac{\cosh 2\gamma \sinh 2\tau \cosh 2\sigma}{\sinh 2\rho}, \quad \partial_\sigma \rho = \frac{\cosh^2 \tau \sinh 2(\sigma - \gamma) + \sinh^2 \tau \sinh 2(\sigma + \gamma)}{\sinh 2\rho}$$

(60)
with (58) make the Virasoro constraint \( T_{\tau \sigma} = 0 \),
\[
\partial_\tau \rho \partial_\sigma \rho = \cosh^2 \rho \partial_\tau t \partial_\sigma t - \sinh^2 \rho \partial_\tau \phi \partial_\sigma \phi
\]  
(61)
hold through the suitable variables \( \sigma \pm \gamma \). The other Virasoro constraint \( T_{\tau \tau} - T_{\sigma \sigma} = 0 \) for the Euclidean worldsheet described by
\[
(\partial_\tau \rho)^2 - (\partial_\sigma \rho)^2 = \cosh^2 \rho ((\partial_\tau t)^2 - (\partial_\sigma t)^2) - \sinh^2 \rho ((\partial_\tau \phi)^2 - (\partial_\sigma \phi)^2)
\]  
(62)
is also satisfied. The equation of motion for \( \rho \)
\[
\partial^2_\tau \rho + \partial^2_\sigma \rho = \frac{1}{2} \sinh 2\rho ((\partial_\tau \phi)^2 + (\partial_\sigma \phi)^2 - (\partial_\tau t)^2 - (\partial_\sigma t)^2)
\]  
(63)
can be confirmed to be satisfied by using \( \cosh^2 (A \pm B) + \sinh^2 (A \mp B) = \cosh 2A \cosh 2B \) and the \( \cosh 2\tau \) expression of \( \partial_\rho \rho \) in (60).

The general string solution (17) is expressed in terms of the \( AdS_4 \) Poincare coordinates (12) as
\[
\frac{1}{\tilde{r}} = \frac{1}{\sqrt{2}} \cosh \gamma (\cos \beta \cosh \sigma_+ + \sin \beta \cosh \sigma_-) - \sinh \gamma (\sin \beta \sinh \sigma_+ + \cos \beta \sinh \sigma_-),
\]
\[
\tilde{x}_0 = \frac{\tilde{r}}{\sqrt{2}} \cosh \gamma (-\sin \beta \cosh \sigma_+ + \cos \beta \cosh \sigma_-) - \sinh \gamma (\cos \beta \sinh \sigma_+ - \sin \beta \sinh \sigma_-),
\]
\[
\tilde{x}_1 = \frac{\tilde{r}}{\sqrt{2}} \sinh \gamma (\cos \beta \cosh \sigma_+ + \sin \beta \cosh \sigma_-) - \cosh \gamma (\sin \beta \sinh \sigma_+ + \cos \beta \sinh \sigma_-),
\]
\[
\tilde{x}_2 = \frac{\tilde{r}}{\sqrt{2}} \sinh \gamma (\sin \beta \cosh \sigma_+ - \cos \beta \cosh \sigma_-) + \cosh \gamma (\cos \beta \sinh \sigma_+ - \sin \beta \sinh \sigma_-).
\]  
(64)
For the basic \( \gamma = 0, \beta = \pi/4 \) solution we have
\[
\tilde{r} = \frac{1}{\cosh \tau \cosh \sigma}, \quad \tilde{x}_0 = - \tanh \tau \tanh \sigma, \quad \tilde{x}_1 = - \tanh \sigma, \quad \tilde{x}_2 = \tanh \tau,
\]  
(65)
where the eliminations of \( \sigma, \tau \) lead to \( \tilde{r} = \sqrt{(1 - \tilde{x}_1^2)(1 - \tilde{x}_2^2)} \), \( \tilde{x}_0 = \tilde{x}_1 \tilde{x}_2 \) whose expressions directly give the positions of the four cusps for the square Wilson loop. For the \( \gamma = \beta = 0 \) solution we have
\[
\tilde{r} = \frac{\sqrt{2}}{\cosh \sigma_+}, \quad \tilde{x}_0 = \frac{\cosh \sigma_-}{\cosh \sigma_+}, \quad \tilde{x}_1 = - \frac{\sinh \sigma_-}{\cosh \sigma_+}, \quad \tilde{x}_2 = \tanh \sigma_+.
\]  
(66)
The eliminations of variables \( \sigma_+ \) and \( \sigma_- \) yield
\[
\tilde{r} = \sqrt{2(1 - \tilde{x}_2^2)}, \quad \tilde{x}_0 = \sqrt{1 + \tilde{x}_1^2 - \tilde{x}_2^2},
\]  
(67)
which provide the locations of two cusps as \( (\tilde{x}_0, \tilde{x}_1, \tilde{x}_2) = (0, 0, 1), \ (0, 0, -1) \) at which two semi infinite lightlike lines intersect transversely (see the first ref. in [12]).

On the other hand we cannot eliminate the \( \sigma_+, \sigma_- \) variables for the general solution (64) but it is possible to extract the cusp positions in the \( (\tilde{x}_0, \tilde{x}_1, \tilde{x}_2) \) space by analyzing
the asymptotic behavior of (64). The $AdS_4$ boundary is characterized by $\bar{r} = 0$, which is achieved by taking the four limits as $\sigma_+ \to \pm \infty$ and $\sigma_- \to \pm \infty$. Defining the variables $y_i, i = 1, 2, \cdots, 6$ as

$$y_1 = \frac{\tan \beta - \tanh \gamma}{1 - \tan \beta \tanh \gamma}, \quad y_2 = \frac{\tan \beta + \tanh \gamma}{1 + \tan \beta \tanh \gamma}, \quad y_3 = \frac{1 + \tan \beta \tanh \gamma}{1 - \tan \beta \tanh \gamma},$$

$$y_4 = \frac{\tan \beta - \tanh \gamma}{\tan \beta + \tanh \gamma}, \quad y_5 = \frac{1 - \tan \beta \tanh \gamma}{\tan \beta + \tanh \gamma}, \quad y_6 = \frac{1 + \tan \beta \tanh \gamma}{\tan \beta - \tanh \gamma},$$

we express the first cusp obtained by taking $\sigma_+ \to \infty$ limit as $\vec{x}_\mu^1 = (-1/y_5, -y_1, y_3)$, the second one in the $\sigma_- \to -\infty$ limit as $\vec{x}_\mu^2 = (y_5, 1/y_2, y_4)$, the third one in the $\sigma_+ \to -\infty$ limit as $\vec{x}_\mu^3 = (-1/y_6, y_2, -1/y_3)$ and the fourth one in the $\sigma_- \to \infty$ limit as $\vec{x}_\mu^4 = (y_6, -1/y_1, -1/y_4)$. For the $\gamma = 0, \beta = \pi/4$ case the four cusp positions reduce to $\vec{x}_\mu^1 = (-1, -1, 1), \vec{x}_\mu^2 = (1, 1, 1), \vec{x}_\mu^3 = (-1, 1, -1), \vec{x}_\mu^4 = (1, -1, -1)$ which are cusp positions of square Wilson loop (65). For the $\gamma = \beta = 0$ case the four cusps degenerate to be two cusps specified by $\vec{x}_\mu^1 = (0, 0, 1), \vec{x}_\mu^3 = (0, 0, -1)$ which are just two cusp positions of (67), where the remaining two cusps $\vec{x}_\mu^2$ and $\vec{x}_\mu^4$ are sented away to infinity.

Using the expressions in (68) we can show

$$\vec{x}_\mu^k - \vec{x}_{\mu+1}^k = 0, \quad k = 1, 2, 3, 4$$

with $\vec{x}_\mu^5 = \vec{x}_\mu^1$, so that the four-cusp Wilson loop indeed consists of four lightlike vectors. In the projected $(\vec{x}_1, \vec{x}_2)$ plane the square of the position vector $\vec{x}_k$ for the $k$-th cusp is larger than unity

$$(\vec{x}_k)^2 = \frac{\vec{b}_k \cdot \vec{b}'_k}{(\vec{b}_k \cdot \vec{b}'_k)^2} \geq 1,$$

where

$$\vec{b}_1 = (1, \tanh \gamma), \quad \vec{b}_1 = (1, -\tan \beta), \quad \vec{b}_2 = (1, \tan \gamma), \quad \vec{b}_2 = (1, \frac{1}{\tan \beta}),$$

$$\vec{b}_3 = (1, \tan \gamma), \quad \vec{b}_3 = (1, \tan \beta), \quad \vec{b}_4 = (1, \tan \gamma), \quad \vec{b}_4 = (1, -\frac{1}{\tan \beta}).$$

The expression (70) shows the same factorized parametrization as (29) for the hexagonal Wilson loop case. Each cusp is located outside the unit circle and the figure of the tetragonal Wilson loop in the projected $(\vec{x}_1, \vec{x}_2)$ plane is characterized by different values of $|\vec{x}_k|, k = 1, 2, 3, 4$ and $\vec{x}_1 \cdot \vec{x}_2 = \vec{x}_3 \cdot \vec{x}_4 = 0$. The point $P_k$ between $\vec{x}_k$ and $\vec{x}_{k+1}$ defined by $P_k \cdot (\vec{x}_k - \vec{x}_{k+1}) = 0$ is specified by (31) with the parameter $t_k$ of (32). Here we express $t_k$ as

$$t_1 = \frac{(\tan \beta + \tanh \gamma)^2}{(1 + \tanh^2 \gamma) \sec^2 \beta}, \quad t_2 = 1 - \frac{\tan^2 \beta \tanh^2 \gamma}{(1 - \tanh^2 \gamma) \sec^2 \beta}, \quad t_3 = \frac{(\tan \beta - \tanh \gamma)^2}{(1 + \tanh^2 \gamma) \sec^2 \beta}.$$
4 Conclusion

Based on the asymptotic solution of the string with Euclidean worldsheet in $AdS_3$ which was constructed by solving the auxiliary linear problems approximately associated with the generalized Sinh-Gordon model for the field $\alpha$ in the complex $w$ plane, we have expressed the approximate forms of the string solution near the cusps in the $AdS_3$ Poincare coordinates to extract the cusp positions in $R^{1,2}$ for the hexagonal and octagonal Wilson loops. A sequence of segments for these Wilson loops living in $R^{1,2}$ have been confirmed to be described by the null vectors. From these expressions of cusp positions we have observed that the necessary conditions for the hexagonal and octagonal Wilson loops to take closed contours are satisfied by using the relations for the Stokes matrices.

We have demonstrated that the hexagonal Wilson loop is tangent to the unit circle in the projected $(\tilde{x}_1, \tilde{x}_2)$ space and going alternating up and down in $R^{1,2}$, where successive sign changings of cusp’s $\tilde{x}_0$ are allowed only for the Wilson loop with an even number of cusps. The asymptotic string configuration derived from the large $w$ asymptotic solutions for the auxiliary linear left and right problems in the $w$ plane has been confirmed to satisfy the string equations of motion and the Virasoro constraints in the $z$ plane.

For the $\alpha = 0$ case we have solved the auxiliary left and right linear problems in the $z$ plane to construct the general solutions by making linear combinations of two independent solutions with arbitrary coefficients. Owing to the reality and normalization conditions for the general solutions, the coefficients for the left problem are characterized by appropriate complex values multiplying the trigonometrical functions of variable $\beta$, while those for the right problem are expressed by the hyperbolic functions of variable $\gamma$. In the embedding coordinates $Y_\mu$, the former characterization turns out to be two SO(2) rotations with angles $\beta$ and $-\beta$ in the $(Y_{-1}, Y_0)$ plane and the $(Y_1, Y_2)$ plane, while the latter one becomes two SO(2,4) boosts with boost parameters $\gamma$ and $-\gamma$ in the $(Y_{-1}, Y_1)$ plane and the $(Y_0, Y_2)$ plane.

We have demonstrated that the string configuration with parameters $\beta$ and $\gamma$ derived from the exact solutions of the linear problems satisfies the equations of motion and the Virasoro constraints for the string with Euclidean worldsheet labelled by $(z, \bar{z})$ in the embedding coordinates $Y_\mu$. Alternatively using the $AdS_3$ global coordinates $(t, \rho, \phi)$ we have confirmed that the $\beta, \gamma$ string configuration satisfies the equations of motion and the Virasoro constraints for the string with Euclidean worldsheet labelled by $(\tau, \sigma)$. The string solution with $\beta = \pi/4, \gamma = 0$ is related to the square Wilson loop with four cusps and the $\beta = \gamma = 0$ string solution is associated with the four semi infinite Wilson lines with two cusps, where in this demonstration we note that two cusps are located at infinity. In order to capture the figure of the Wilson loop surrounding the $\beta, \gamma$ string surface we extract the asymptotic solution near each cusp from the exact string solution. The four cusp locations in $R^{1,2}$ for the tetragonal Wilson loop are determined as functions of $\beta, \gamma$ in the same way as the hexagonal and octagonal Wilson loop cases by using only the asymptotic solution. For the $\beta = \pi/4, \gamma = 0$ string solution and the $\beta = \gamma = 0$ one we have observed that the cusp positions determined from the asymptotic solution indeed agree with those fixed by the exact solution.
References

[1] S.S. Gubser, I.R. Klebanov and A.M. Polyakov, “A semi-classical limit of the gauge/string correspondence,” Nucl. Phys. B636 (2002) 99 [arXiv:hep-th/0204051].

[2] A.A. Tseytlin, “Spinning strings and AdS/CFT duality,” arXiv:hep-th/0311139 “Semiclassical strings and AdS/CFT,” arXiv:hep-th/0409296

[3] J. Plefka, “Spinning strings and integrable spin chains in AdS/CFT correspondence,” arXiv:hep-th/0507136

[4] L.F. Alday and J. Maldacena, “Gluon scattering amplitudes at strong coupling,” JHEP 0706 (2007) 064 [arXiv:0705.0303[hep-th]].

[5] M. Kruczenski, “A note on twist two operators in $\mathcal{N} = 4$ SYM and Wilson loops in Minkowski signature,” JHEP 0212 (2002) 024 [arXiv:hep-th/0210115].

[6] Z. Bern, L.J. Dixon and V.A. Smirnov, “Iteration of planar amplitudes in maximally supersymmetric Yang-Mills theory at three loops and beyond,” Phys. Rev. D72 (2005) 085001 [arXiv:hep-th/0505205]; C. Anastasiou, Z. Bern, L.J. Dixon and D.A. Kosower, “Planar amplitudes in maximally supersymmetric Yang-Mills theory,” Phys. Rev. Lett. 91 (2003) 251602 [arXiv:hep-th/0309040].

[7] S. Abel, S. Forste and V.V. Khoze, “Scattering amplitudes in strongly coupled $\mathcal{N} = 4$ SYM from semiclassical strings in AdS,” JHEP 0802 (2008) 042 [arXiv:0705.2113[hep-th]]; E.I. Buchbinder, “Infrared limit of gluon amplitudes at strong coupling,” Phys. Lett. B654 (2007) 46 [arXiv:0706.2015[hep-th]]; Z. Komargodski and S.S. Razamat, “Planar quark scattering at strong coupling and universality,” JHEP 0801 (2008) 044 [arXiv:0707.4367[hep-th]]; J. McGreevy and A. Sever, “Quark scattering amplitudes at strong coupling,” JHEP 0802 (2008) 015 [arXiv:0710.0393[hep-th]]; A. Popolitov, “On coincidence of Alday-Maldacena-regularized sigma-model and Nambu-Goto areas of minimal surfaces,” arXiv:0710.2073[hep-th]; G. Yang, “Comment on the Alday-Maldacena solution in calculating scattering amplitude via AdS/CFT,” JHEP 0803 (2008) 010 [arXiv:0711.2828[hep-th]]; Y. Oz, S. Theisen and S. Yankielowicz, “Gluon scattering in deformed $\mathcal{N} = 4$ SYM,” Phys. Lett. B662 (2008) 297 [arXiv:0712.3491[hep-th]]; Z. Komargodski, “On collinear factorization of Wilson loops and MHV amplitudes in $\mathcal{N} = 4$ SYM,” JHEP 0805 (2008) 019 [arXiv:0801.3274[hep-th]]; M. Kruczenski and A.A. Tseytlin, “Spiky strings, light-like Wilson loops and pp-wave anomaly,” Phys. Rev. D77 (2008) 126005 [arXiv:0802.2039[hep-th]]; R. Ishizuki, M. Kruczenski and A.A. Tseytlin, “New open string solutions in AdS$_5$,” Phys. Rev. D77 (2008) 126018 [arXiv:0804.3438[hep-th]]; C.M. Sommerfield and C.B. Thorn, “Classical worldsheets for string scattering on flat and AdS spacetime,” Phys. Rev. D78 (2008) 046005 [arXiv:0805.0388[hep-th]].

[8] D. Astefanesei, S. Dobashi, K. Ito and H. Nastase, “Comments on gluon 6-point scattering amplitudes in $\mathcal{N} = 4$ SYM at strong coupling,” JHEP 0712 (2007) 077 [arXiv:0710.1684[hep-th]]; K. Ito, H. Nastase and K. Iwasaki, “Gluon scattering in
\( N = 4 \) super Yang-Mills at finite temperature,” Prog. Theor. Phys. 120 (2008) 99 [arXiv:0711.3532[hep-th]]; S. Dobashi, K. Ito and K. Iwasaki, “A numerical study of gluon scattering amplitudes in \( N = 4 \) super Yang-Mills theory at strong coupling,” JHEP 0807 (2008) 088 [arXiv:0805.3594[hep-th]]; S. Dobashi and K. Ito, “Discretized minimal surface and the BDS conjecture in \( N = 4 \) super Yang-Mills theory at strong coupling,” Nucl. Phys. B819 (2009) 18 [arXiv:0901.3046[hep-th]].

[9] L.F. Alday and J. Maldacena, “Comments on gluon scattering amplitudes via AdS/CFT,” JHEP 0711 (2007) 068 [arXiv:0710.1060[hep-th]]; A. Mironov, A. Morozov and T.N. Tomaras, “On \( n \)-point amplitudes in \( N = 4 \) SYM,” JHEP 0711 (2007) 021 [arXiv:0708.1625[hep-th]]; “Some properties of the Alday-Maldacena minimum,” Phys. Lett. B659 (2008) 723 [arXiv:0711.0192[hep-th]]; H. Itoyama, A. Mironov and A. Morozov, “Boundary ring: a way to construct approximate NG solutions with polygon boundary conditions: I. \( Z_n \)-symmetric configurations,” Nucl. Phys. B808 (2009) 365 [arXiv:0712.0159[hep-th]]; ‘Anomaly’ in \( n = \infty \) Alday-Maldacena duality for wavy circle,” JHEP 0807 (2008) 024 [arXiv:0803.1547[hep-th]]; H. Itoyama and A. Morozov, “Boundary ring or a way to construct approximate NG solutions with polygon boundary conditions. II. Polygons which admit an inscribed circle,” Prog. Theor. Phys. 120 (2008) 231 [arXiv:0712.2316[hep-th]]; A. Morozov, “Alday-Maldacena duality and AdS Plateau problem,” Int. J. Mod. Phys. A23 (2008) 2118 [arXiv:0803.2431[hep-th]]; D. Galakhov, H. Itoyama, A. Mironov and A. Morozov, “Deviation from Alday-Maldacena duality for wavy circle,” Nucl. Phys. B823 (2009) 289 [arXiv:0812.4702[hep-th]].

[10] A. Jevicki, C. Kalousios, M. Spradlin and A. Volovich, “Dressing the giant gluon,” JHEP 0712 (2007) 047 [arXiv:0708.0818[hep-th]]; A. Jevicki, K. Jin, C. Kalousios and A. Volovich, “Generating AdS string solutions,” JHEP 0803 (2008) 032 [arXiv:0712.1193[hep-th]].

[11] M. Kruczenski, R. Roiban, A. Tirziu and A.A. Tseytlin, “Strong-coupling expansion of cusp anomaly and gluon amplitudes from quantum open strings in \( AdS_5 \times S^5 \),” Nucl. Phys. B791 (2008) 93 [arXiv:0707.4254[hep-th]]; R. Roiban and A.A. Tseytlin, “Strong-coupling expansion of cusp anomaly from quantum superstring,” JHEP 0711 (2007) 016 [arXiv:0709.0681[hep-th]].

[12] S. Ryang, “Conformal SO(2,4) transformations of the one-cusp Wilson loop surface,” Phys. Lett. B659 (2008) 894 [arXiv:0710.1673[hep-th]]; “Conformal SO(2,4) transformations for the helical AdS string solution,” JHEP 0805 (2008) 021 [arXiv:0803.3853].

[13] L.F. Alday and R. Roiban, “Scattering amplitudes, Wilson loops and the string/gauge theory correspondence,” Phys. Rept. 468 (2008) 153 [arXiv:0807.1889[hep-th]].

[14] F. Cachazo, M. Spradlin and V. Volovich, “Iterative structure within the five-particle two-loop amplitude,” Phys. Rev. D74 (2006) 045020 [arXiv:hep-th/0602228]; Z. Bern, M. Czakon, D.A. Kosower, R. Roiban and V.A. Smirnov, “Two-loop iteration of five-point \( N = 4 \) super Yang-Mills amplitudes,” Phys. Rev. Lett. 97 (2006) 181601 [arXiv:hep-th/0604074]; Z. Bern, L.J. Dixon, D.A. Kosower R. Roiban, M. Spradlin, C.
Vergu and V. Volovich, “The two-loop six-gluon MHV amplitude in maximally super-symmetric Yang-Mills theory,” Phys. Rev. D78 (2008) 045007 [arXiv:0803.1465[hep-th]].

[15] J.M. Drummond, J. Henn, G.P. Korchemsky and E. Sokatchev, “On planar gluon amplitudes/Wilson loops duality,” Nucl. Phys. B795 (2008) 52 [arXiv:0709.2368[hep-th]]; “Conformal Ward identities for Wilson loops and a test of the duality with gluon amplitudes,” [arXiv:0712.1223[hep-th]]; “The hexagon Wilson loop and the BDS ansatz for the six-gluon amplitudes,” Phys. Lett. B662 (2008) 456 [arXiv:0712.4138[hep-th]]; “Hexagon Wilson loop = six-gluon MHV amplitude,” Nucl. Phys. B815 (2009) 142 [arXiv:0803.1469[hep-th]]; C. Anastasiou, A. Brandhuber, P. Heslop, V.V. Khoze, B. spence and G. Travaglini, “Two-loop polygon Wilson loops in $\mathcal{N} = 4$ SYM,” [arXiv:0902.2245[hep-th]].

[16] K. Pohlmeyer, “Integral Hamiltonian systems and interactions through quadratic constraints,” Commun. Math. Phys. 46 (1976) 207.

[17] H.J. de Vega and N. Sanchez, “Exact integrability of strings in D-dimensional de Sitter spacetime,” Phys. Rev. D47 (1993) 3394; M. Grigoriev and A.A. Tseytlin, “Pohlmeyer reduction of $AdS_5 \times S^5$ superstring sigma model,” Nucl. Phys. B800 (2008) 450 [arXiv:0711.0155[hep-th]]; “On reduced models for superstrings on $AdS_n \times S^n$,” Int. J. Mod. Phys. A23 (2008) 2107 [arXiv:0806.2623[hep-th]]; J.L. Miramontes, “Pohlmeyer reduction revisited,” JHEP 0810 (2008) 087 [arXiv:0808.3365[hep-th]].

[18] A. Jevicki and K. Jin, “Solitons and AdS string solutions,” Int. J. Mod. Phys. A23 (2008) 2289 [arXiv:0804.0412[hep-th]]; “Moduli dynamics of $AdS_3$ strings,” JHEP 0906 (2009) 064 [arXiv:0903.3389[hep-th]].

[19] H. Dorn, G. Jorjadze and S. Wuttke, “On spacelike and timelike minimal surfaces in $AdS_n$,” [arXiv:0903.0977[hep-th]]; H. Dorn, “Some comments on spacelike surfaces with null polygonal boundaries in $AdS_m$,” [arXiv:0910.0934[hep-th]].

[20] G. Jorjadze, “Singular Liouville fields and spiky strings in $R^{1,2}$ and SL(2,R),” [arXiv:0909.0350[hep-th]].

[21] J.F. Alday and J. Maldacena, “Null polygonal Wilson loops and minimal surfaces in Anti-de-Sitter space,” [arXiv:0904.0663[hep-th]].

[22] D. Gaiotto, G.W. Moore and A. Neitzke, “Four-dimensional wall-crossing via three-dimensional field theory,” [arXiv:0807.4723[hep-th]]; “Wall-crossing, Hitchin systems, and the WKB approximation,” [arXiv:0907.3987[hep-th]].

[23] K. Sakai and Y. Satoh, “A note on string solutions in $AdS_3$,” HEP 0910 (2009) 001 [arXiv:0907.5259[hep-th]].