Wave Solutions of Evolution Equations and Hamiltonian Flows on Nonlinear Subvarieties of Generalized Jacobians

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Abstract

The algebraic-geometric approach is extended to study solutions of $N$-component systems associated with the energy dependent Schrödinger operators having potentials with poles in the spectral parameter, in connection with Hamiltonian flows on nonlinear subvarieties of Jacobi varieties. The systems under study include the shallow water equation and Dym type equation. The classes of solutions are described in terms of theta-functions and their singular limits by using new parameterizations. A qualitative description of real valued solutions is provided.

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1 Introduction.

The quasi-periodic solutions of most classical integrable PDEs can be obtained using the inverse scattering transform (IST) (see, for example, Ablowitz and Segur [1981], Newell [1985] and Ablowitz and Clarkson [1991]). This is done by establishing a connection with an isospectral eigenvalue problem for an associated Schrödinger operator.

The solution of nonlinear evolution equations using algebraic-geometric techniques was initially developed to handle $N$-phase wave trains. This approach can be summarized as follows. By using the trace formula, families of quasi-periodic and soliton solutions are associated with Hamiltonian flows on finite dimensional phase spaces. These flows are described by using so called $\mu$-variable representations leading to an Abel–Jacobi mapping which include holomorphic and, in some cases, meromorphic differentials (see amongst others, Ablowitz and Ma [1981], Dubrovin [1981], Ercolani [1989] and Alber and Alber [1985]). Then the mapping is inverted in terms of Riemann theta-functions and their singular limits. Many well-known nonlinear equations such as KdV, sine-Gordon, focusing and defocusing nonlinear Schrödinger equations, which describe a wide variety of important phenomena in physics, optics, biology and engineering, were studied by using this approach.

Recently special attention was given to the shallow water (SW) equation derived in Cammassa and Holm [1993] in the context of the Hamiltonian structure,

$$U_t + 3UU_x = U_{xxt} + 2U_xU_{xx} + UU_{xxx} - 2\kappa U_x,$$

and the Dym type equation (see Cewen [1990], Hunter and Zheng [1994] and Alber et al. [1994, 1995])

$$U_{xxt} + 2U_xU_{xx} + UU_{xxx} - 2\kappa U_x = 0, \quad \kappa = \text{const.}$$

(1.2)
Camassa and Holm [1993] described classes of $n$-peakon soliton-type solutions for an integrable (SW) equation (1.1). In particular, they obtained a system of completely integrable Hamiltonian equations for the locations of the “peaks” of the solution, the points at which its spatial derivative changes sign. In other words, each peakon solution can be associated with a mechanical system of moving particles. Calogero [1995] and Calogero and Francoise [1996] further extended the class of mechanical systems of this type.

The problem of describing complex traveling wave and quasi-periodic solutions of the equations (1.1) and (1.2) can be reduced to solving finite-dimensional Hamiltonian systems on symmetric products of hyperelliptic curves. Namely, according to Alber et al [1994,1995,1999], such solutions can be represented in the following form

$$U(x,t) = \mu_1 + \cdots + \mu_g - M,$$

where $g$ is a positive integer, $M$ is a constant and the evolution of the variables $\mu$ is given by the equations

$$\sum_{i=1}^{g} \frac{\mu_i^k d\mu_i}{2\sqrt{R(\mu_i)}} = \begin{cases} 0 & k = 1, \ldots, g - 2, \\ \frac{dt}{dt} & k = g - 1, \\ \frac{dx}{dx} & k = g. \end{cases}$$

(1.3)

Here $R(\mu)$ is a polynomial of degree $2g + 2$ (for shallow water equation (1.1)) or $2g + 1$ (for the Dym type equation (1.2)). Also $M = 0$ for the Dym type equation.

In contrast to the finite-dimensional reductions of such equations as KdV and sine-Gordon equations, system (1.3) contains a meromorphic differential, also the number of holomorphic differentials is less than the genus $g$ of the corresponding hyperelliptic curve: $W^2 = R(\mu)$. This implies that the problem of inversion (1.3) can not be solved in terms of meromorphic functions of $x$ and $t$. Examples of such equations arise in several problems of mechanics. These were considered in Vanhaecke [1995] and Abenda and Fedorov [1999], where a connection was established with the flows on nonlinear subvarieties of hyperelliptic Jacobian varieties, so-called strata. In Alber et al. [1997] a whole class of $N$-component systems with poles was shown to be integrable by reducing them to similar nonstandard inversion problems which contained meromorphic differentials. Therefore $N$-component systems can be overdetermined, implying that the genus of the spectral curve can be higher then the number of $\mu$-variables.

$N$-component systems can be briefly described as follows. For the KdV equation, the spectral parameter appears linearly in the potential of the corresponding Schrödinger equation: $V = u - \lambda$, in the context of the IST method. In contrast, Antonowicz and Fordy [1987a,b, 1988, 1989] and Antonowicz, Fordy and Liu [1991] investigated potentials with poles in the spectral parameter for what they refer to as energy dependent Schrödinger operators connected to certain systems of evolution equations. Specifically, they obtained multi-Hamiltonian structures for $N$-component integrable systems of equations related to the following isospectral eigenvalue problem:

$$L\psi = \left( \frac{\partial^2}{\partial x^2} + \frac{V}{K} \right) \psi = 0 ,$$

(1.4)

$$K = \sum_{j=0}^{M} k_j \lambda^j; \quad V = \sum_{j=0}^{N} v_j(x,t) \lambda^j ,$$

(1.5)

where the $k_j$ are constants and the $v_j(x,t)$ are functions of the variable $x$, the parameter $t$ and the spectral parameter $\lambda$ is complex. This includes the coupled KdV and Dym systems.
In Alber et al. [1994, 1995, 2000a], the presence of a pole in the potential was shown to be necessary for the existence of weak billiard solutions of nonlinear equations. Billiard solutions of nonlinear PDE’s have been related to finite-dimensional integrable dynamical systems with reflections including ellipsoidal Birkhoff billiards. It turned out that the existence of billiard solutions and the presence of monodromy effects is a specific feature of the whole class of \( N \)-component systems with poles (see Alber et al. [1997]).

Quasi-periodic solutions of the Dym equation were studied in Dmitrieva [1993a] and Novikov [1999] by using a connection with KdV equation and introducing additional phase functions. Soliton solutions of the Dym type equation were studied in Dmitrieva [1993b]. Periodic solutions of the shallow water equation were discussed in McKean and Constantin [1999]. Beals et al. [1999, 2000] used Stieltjes theorem on continued fractions and the classical moment problem for studying multi-peakon solutions of the (SW) equation.

The main goal of this paper is to describe explicit formulae in terms of theta-functions and their singular limits for the solutions to the shallow water equation (1.1). We also explain the role of the mysterious phase functions used by Dmitrieva [1993a] when studying SW equation and equation of the Dym type. This phase is present for the whole class of WKI hierarchy. (see Wadati et al. [1979]).

In the present paper the traveling wave, soliton, peakon, cuspon and quasi-periodic solutions are considered. Usually in the case of integrable evolution equations quasi-periodic flows are liberalized on the Jacobi varieties. In this paper we show that in the case of N-component systems with poles the \( x \)- and \( t \)-flows take place on nonlinear subvarieties (strata) of generalized (noncompact) Jacobians. For this reason, the term “liberalization” is no longer applicable here. This makes the above nonlinear equations quite different from such well known equations as KdV, sine-Gordon and nonlinear Schrödinger equations. For the sake of clarity, in this paper we start with solutions related to (hyper)elliptic curves of at most genus 2. The case of arbitrary genus is only notationally more complicated and we provide complete formulae. In addition, we give a complete classification of real bounded solutions \( U(x, t) \) in the above cases and provide corresponding plots.

Notice that the complex geometry of the traveling wave solutions, cusp and peakon solutions was previously studied in Alber et al [1994, 1995, 2000] in connection with geodesic flows with reflections on Riemannian manifolds and in Li and Olver [1998] from the point of view of singularity analysis.

The Contents of the Paper. In Section 2 we demonstrate the main difference between the nonlinear SW equation and the KdV equation from the point of view of the algebraic-geometric approach by obtaining traveling wave solutions as a result of inverting elliptic integrals of the second and third kind. Here we express the amplitude \( U \) and the phase \( X \) as meromorphic functions of an auxiliary variable parameterizing the elliptic curve.

In Section 3 we apply different singular limits to the problem of inversion resulting in formulae for different periodic and solitary solutions. In particular, peakon solutions are obtained as limits of the traveling wave solutions and are related to various singularizations of the elliptic curves.

Section 4 provides explicit expressions in terms of theta-functions for the stationary quasi-periodic solutions. This is done by using new complex parametrizations on the related associated hyperelliptic curve of genus 2.

In Section 5 we find time-dependent solutions by integrating and inverting equations (1.3) in the genus 2 case. We show that these equations can be extended to a standard Abel–Jacobi mapping of a symmetric product of the hyperelliptic curve to its generalized Jacobian. The original system (1.3) then defines a mapping onto a 2-dimensional nonlinear stratum of the Jacobian, a generalized theta-divisor, where the dynamics actually takes place. By fixing \( t \) in the expression
for the solution in terms of theta-functions, we then recover the stationary quasi-periodic solutions obtained in Section 4.

Section 6 contains qualitative analysis of real bounded solutions for the case when the Weierstrass points of the spectral curve are real.

In a forthcoming paper we will consider different singular limits of the quasi-periodic solutions when the spectral curve becomes singular and its arithmetic genus drops to zero. The solutions are then expressed in terms of purely exponential tau-functions and, in the real bounded case, they describe, in particular, a quasi-periodic train of peakons tending to a periodic one at infinity.

2 Traveling Wave Solutions.

The main difference between the nonlinear SW or Dym equations and the KdV equation from the point of view of the algebraic-geometric approach can be demonstrated already on the level of traveling wave solutions.

The traveling wave solutions of the KdV equation are obtained by inversion of an elliptic integral of the first kind, i.e., the integral of a holomorphic differential, which results in a meromorphic function.

In contrast to this, after substituting $U(x,t) = \lambda(x - ct)$ into the Dym type equation (1.2) and using a simple transformation (see Alber et al [1995]), we arrive at the problem of inversion of the following Abelian integral of the second kind

$$
\int_{U_0}^U \sqrt{\lambda - c} d\lambda = \int_{U_0}^U \frac{\lambda - c}{\sqrt{\kappa \sqrt{R_3(\lambda)}}} d\lambda = x - ct = X,
$$

(2.1)

defined on the elliptic curve $E = \{W^2 = R_3(\lambda)\}$, where

$$
R_3(\lambda) = (\lambda - a_1)(\lambda - a_2)(\lambda - c)
$$

and $U_0$ being a constant. Here we suppose that the roots of $R_3(\lambda)$ are distinct. The differential in (2.1) has a double pole at infinity $\lambda = \infty$ and a double zero at the Weierstrass point $\lambda = c$ on $E$. It follows that the complex inverse function $U(X)$ must have two independent periods on $\mathbb{C}$, the periods of the differential along two nontrivial homology cycles on $E$. On the other hand, $U(X)$ blows up only at $X = \infty$. There are no meromorphic functions with such properties (see e.g. Markushevich [1977]). Moreover, because of the double zero of the differential, the solution $U(X)$ has moving branch points of the form

$$
U - c = O((X - X_0)^{2/3}).
$$

(2.2)

One can show that $U(X)$ has infinitely many such branch points.

To deal with this we describe the complex function $U(X)$ in a new parametric form by introducing a new complex variable $u$ which gives a parameterization of the curve $E$:

$$
\int_{\lambda}^{\infty} \frac{d\lambda}{2\sqrt{R_3(\lambda)}} = u.
$$

(2.3)

Then, according to the theory of elliptic functions,

$$
\lambda(u) = \varphi(u) + \Delta = -\frac{d^2}{du^2} \log \theta_{11}(z) + \text{const}, \quad z = 2\pi i u/\omega_3
$$

(2.4)
where \( \varphi(u) \) is the elliptic Weierstrass function with periods \( 2\omega_1 \) and \( 2\omega_3 \) depending on the coefficients of \( R_3(\lambda) \) and \( \Delta = (a_1 + a_2 + c)/3 \). The \( \theta_{11}(z) \) is the quasi-periodic Riemann theta-function which vanishes at the points of the period lattice 

\[ \Lambda = \{ z = 2\pi i Z + B Z \}, \quad B = \pi i \omega_1 / \omega_3 \text{ (see, e.g. Dubrovin [1981])}: \]

\[ \theta_{11}(z) = \sum_{M \in \mathbb{Z}} \exp \left( \frac{1}{2} B (M + 1/2)^2 + (M + 1/2)(z + \pi i) \right). \]

Here we use two variables \( u, z \) because of different normalizations of (quasi)-periods of \( \varphi \) and \( \theta \). The curve \( \mathcal{E} \) can be identified with the factor \( \mathbb{C} / \Lambda \). Now the integral (2.1) can be transformed as follows

\[ \int_{u_0}^{u} (\varphi(u) + \Delta - c) \, du = \sqrt{\kappa} X + \text{const}, \quad u_0 = \text{const}, \quad (2.5) \]

which yields

\[ \sqrt{\kappa} X + \text{const} = \zeta(u) + (\Delta - c) u = \frac{d}{dz} \log \theta_{11}(z) + \Delta' z, \quad \Delta' = \text{const}, \quad (2.6) \]

where \( \zeta(u) \) is the Riemann zeta-function with parameters \( \eta_1 \) and \( \eta_2 \) such that for any \( u \in \mathbb{C} \)

\[ \zeta(u) = \int_{u}^{\infty} \varphi(u) \, du, \quad \zeta(u + 2\omega_1) = \zeta(u) + 2\eta_1, \quad \zeta(u + 2\omega_3) = \zeta(u) + 2\eta_3. \]

The constants \( 2\eta_1, 2\eta_3 \) are interpreted as the periods of the differential

\[ \lambda \, d\lambda / 2\sqrt{R_3(\lambda)} = \lambda(u) \, du. \]

Then, in view of (2.6),

\[ \sqrt{\kappa} X(u + 2\omega_1) = \sqrt{\kappa} X(u) + 2\eta_1 + 2(\Delta - c)\omega_1, \]
\[ \sqrt{\kappa} X(u + 2\omega_3) = \sqrt{\kappa} X(u) + 2\eta_3 + 2(\Delta - c)\omega_3. \]

Thus we have expressed the amplitude \( U = \lambda \) and the phase \( X \) in terms of the auxiliary complex variable \( u \).

In what follows we study real solutions \( U(X) = \lambda(u(X)) \) which correspond to the case of all roots of the polynomial \( R_3(\lambda) \) being real. We choose the half-period \( \omega_1 \) to be real and \( \omega_3 \) to be purely imaginary. The existence of branch points of \( U(X) \) (see (2.2)) implies that the real solutions may have cusps. To demonstrate this we consider two different cases.

**Case 1:** \( \kappa > 0, \quad a_1 < a_2 < c < \infty \). According to the theory of elliptic functions, parameterization (2.3) yields that

\[ \lambda(\omega_1) = c, \quad \lambda(\omega_3) = a_1, \quad \lambda(\omega_1 + \omega_3) = a_2, \quad \lambda(0) = \infty. \quad (2.7) \]

Along the real axis \( \Re u \) and the line \( \{ u = \omega_3 + u' \} \) \( u' \in \mathbb{R} \), \( \lambda(\omega_3 + u') \) is real valued. Since we are interested in nonsingular traveling wave solutions and \( \lambda(u) \) has a pole at the origin, let us consider \( \lambda(\omega_3 + u') \) as a function of \( u' \). It is periodic with period \( 2\omega_3 \) and takes values in the interval \( [a_1, a_2] \). The differentials in (2.1) and (2.3) do not have poles or zero’s on the line \( \{ u = \omega_3 + u' \} \). Hence, \( X(\omega_3 + u') \) and the inverse \( u'(X) \) are monotonic functions. As a result, due to (2.7), the composition function \( U(X) = \lambda(\omega_3 + u'(X)) \) is a regular periodic function with period \( (2\eta_3 + 2(\Delta - c)\omega_3)/\sqrt{\kappa} \), and its graph is merely a distortion of that of \( \lambda(\omega_3 + u') \).
Case 2: \( \kappa > 0, a_1 < c < a_2 < \infty \). Here we have

\[
\lambda(\omega_1) = a_2, \quad \lambda(\omega_3) = a_1, \quad \lambda(\omega_1 + \omega_3) = c, \quad \lambda(0) = \infty.
\] (2.8)

Again, \( \lambda(u) \) is real valued along the real axis \( \Re u \) and the line \( \{ u = \omega_3 + u'|u' \in \mathbb{R} \} \). Since \( \lambda(u) \) has a pole along the real axis, we again consider only the real function \( \lambda(\omega_1 + u'), u' \in \mathbb{R} \) which is \( 2\omega_3 \)-periodic and now takes values in the interval \([c, a_2]\). In this case, the differential (2.3) has a double zero at \( u = \omega_1 + \omega_3 \) \((u' = \omega_1)\), where the derivative \( d\lambda/du \) has a simple zero. As a result, the derivative \( dU/dX = d\lambda/du \cdot du/dX \) blows up for the corresponding value of \( X \), which implies that the graph of the function \( U(X) \) has cusps with periodicity \((2\eta_3 + 2(\Delta - c)\omega_3)/\sqrt{\kappa}\). This phenomenon was first detected in Alber et al [1994,1995].

**SW Equation.** Traveling wave solutions for the shallow water equation have a similar description. Substituting \( U(x, t) = \mu(x - ct) \) into (1.4), we obtain an Abelian integral of the third kind

\[
\int_{U_0}^U \sqrt{\kappa(\mu-a_1)(\mu-a_2)(\mu-a_3)} \, d\mu = \int_{U_0}^U \frac{\mu-c}{\sqrt{\kappa R_4(\mu)}} \, d\mu = x - ct = X,
\] (2.9)
defined on the even order elliptic curve

\[
\tilde{E} = \{ W^2 = R_4(\mu) \}, \quad R_4(\mu) = (\mu-a_1)(\mu-a_2)(\mu-a_3)(\mu-c),
\]
where the roots of \( R_4 \) are supposed to be distinct. The differential in (2.9) has a pair of simple poles at the infinite points \( \infty_- \), \( \infty_+ \) on \( \tilde{E} \). Then the complex inverse function \( U(X) \) must have three independent periods on \( \mathbb{C} \): the periods of the differential along two nontrivial homology cycles on \( \tilde{E} \) and along a homology zero cycle around one of the infinite points. A meromorphic function with such a property does not exist.

Let \( A, B \) be canonically conjugated cycles on \( \tilde{E} \) and \( \bar{\omega} \) be the normalized holomorphic differential

\[
\bar{\omega} = \frac{d\mu}{\Pi \sqrt{R_4(\mu)}},
\]
where the multiplier \( \Pi \) is chosen from the condition \( \oint_A \bar{\omega} = 2\pi i \). Introduce a new variable \( z \) parameterizing \( \tilde{E} \) as follows

\[
\int_{\mu}^{\tilde{c}} \frac{d\mu}{\Pi \sqrt{R_4(\mu)}} = z,
\] (2.10)
where \( a_1, a_2 \) and \( c \) denote the corresponding Weierstrass (branch) points on \( \tilde{E} \). Then we get the expression

\[
\mu(z) = \rho - d \frac{d}{dz} \log \frac{\theta[\delta](z - q/2)}{\theta[\delta](z + q/2)},
\] (2.11)
\[
\rho = \oint_A \mu \bar{\omega}, \quad q = 2 \int_{c}^{\infty} \frac{d\mu}{\Pi \sqrt{R_4(\mu)}},
\]
Thus \( \mu(z) \) is an elliptic function with periods \( 2\pi i \), \( B = \oint_B \bar{\omega} \). The integral (2.9) takes the form

\[
\sqrt{\kappa}X = \int_{z_0}^{z} (\mu(z) - c) \, dz = (\rho - c)z - \log \frac{\theta_{11}(z - q/2)}{\theta_{11}(z + q/2)} + \text{const}.
\] (2.12)


As a result, we have expressed the function $U = \mu$ and its argument $X$ in terms of $z$.

Since the differential in (2.9) has a double zero at the Weierstrass point $\mu = c$, similar to the case of the Dym equation, the inversion of (2.9) yields $U(X)$ with infinitely many branch points of the form (2.2). This results in real solutions having cusps.

Now we describe 2 different types of real traveling wave solutions of (1.1) assuming that $\kappa > 0$ and all roots of $R_4(\mu)$ are real. In this case the period $B$ is real as well.

**Case 1.** $a_1 < a_2 < a_3 < c$. According to the parameterization (2.10), we have

$$q \in \mathbb{R}, \quad \Pi = \int_{a_1}^{a_2} \frac{d\mu}{R_4(\mu)}, \quad B = 2 \int_c^a \frac{d\mu}{\Pi \sqrt{R_4(\mu)}}, \quad \mu(q/2) = U(-q/2) = \infty,$$

$$\mu(0) = c, \quad \mu(B/2) = a_1, \quad \mu(\pi i + B/2) = a_2, \quad \mu(\pi i) = a_3.$$

The function $\mu(z)$ is real along the real axis $\Re u$ and the line $\ell = \{z = \pi i + z'| z' \in \mathbb{R}\}$, but it is finite only in the second case. The function $\mu(\pi i + z')$ changes periodically between $a_2$ and $a_3$ with period $B$. Along the line $\ell$ the differential in (2.3) has no zeros, therefore $X(\pi i + z')$ and the inverse real function $z'(X)$ are strictly monotonic functions. According to (2.12), the composition function $U(X) = \mu(\pi i + z'(X))$ is a regular periodic function with real period

$$\frac{1}{\sqrt{\kappa}} \left[ (\rho - c)B - \log \frac{\theta_{11}(B - q/2)}{\theta_{11}(B + q/2)} \right].$$

Such a function is shown in figure 1a.

**Case 2.** $a_1 < a_2 < c < a_3$. Now $q, \Pi, B$ have the same expressions as above and

$$\mu(0) = c, \quad \mu(B/2) = a_2, \quad \mu(\pi i + B/2) = a_1, \quad \mu(\pi i) = a_3.$$

The function $\mu(z)$ is real and finite only along the real axis, on which the differential has a double zero. As a result, the composition $U(X) = \mu(z(X))$ varies between $a_2$ and $c$, with the same period as above and has cusps for $U = c$. See figure 1b.

The cases of other positions of $c$ among the real roots $(a_1, a_2, a_3)$ lead to either one of the above two types of solutions or to unbounded solutions.

### 3 Periodic and solitary peakon solutions.

In this section we describe different periodic and solitary solutions to the Dym type and shallow water SW equations obtained as limits of the traveling wave solutions and associated with various singularizations of the elliptic curves described in the previous section. In particular, we encounter periodic and solitary peakon solutions, i.e. solutions with discontinuous derivatives.

In contrast to traveling wave solutions given above globally in a parametric form, peakon solutions can be given explicitly only in certain intervals.

**Periodic peakon solution for the Dym type equation.** Consider again the integral (2.1). Assuming $\kappa = -1$, in the limit $a_2 \to c$ the latter can be written in the equivalent differential form

$$\frac{d\lambda}{\sqrt{a_1 - \lambda}} = dX, \quad \text{or} \quad \frac{d\lambda}{(\lambda - c)\sqrt{a_1 - \lambda}} = dX', \quad dX = (\lambda - c) dX', \quad (3.1)$$
$X'$ being a new variable. Now putting $\lambda = a_1 - \nu^2$, $a_1 - c = \alpha^2$ and integrating the second differential in (3.1), we find that

$$\frac{1}{\alpha} \log \left( \frac{\nu - \alpha}{\nu + \alpha} \right) = X' + C', \quad C' = \text{const}, \quad \text{i.e.}, \quad \nu = \alpha \frac{1 + e^{\alpha X'}}{1 - e^{\alpha X'}}.$$

Let us choose here $C' = \pi i / \alpha$. Then we have

$$\lambda(X') - c = \alpha^2 - \alpha^2 \left( \frac{e^{-\alpha X'/2} - e^{\alpha X'/2}}{e^{-\alpha X'/2} + e^{\alpha X'/2}} \right)^2, \quad \text{or, equivalently,}
$$

$$\lambda(X') - c = 4\alpha^2, \quad \log \tau(X'), \quad \tau(X') = e^{-\alpha X'/2} + e^{\alpha X'/2}. \quad (3.2)$$

This function is $2\pi i$ periodic in $\alpha X$ and takes real finite values along the real axis.

Next, using the expression (3.2), we integrate the relation $dX = (\lambda(X') - c) dX'$ and get the connection between $X$ and $X'$:

$$X = 4\alpha \log \tau(X') + X_0 = 2\alpha \frac{e^{-\alpha X'/2} - e^{\alpha X'/2}}{e^{-\alpha X'/2} + e^{\alpha X'/2}} + X_0, \quad X_0 = \text{const}. \quad (3.3)$$

Notice that as $X' \to -\infty$ or $+\infty$, $X$ has finite limits which differ by $4\alpha$. Thus, expressions (3.2) and (3.3) give the solution $U(X) = \lambda(X'(X))$ in a parametric form only in the interval $[X_0 - 2\alpha, X_0 + 2\alpha]$. As follows from (3.2) and (3.1), at the endpoints of the interval, $U = c$ and $dU/dX = \pm \sqrt{a_1 - c}$.

Now we define the global solution $U(X)$ for all values of $X$ by using periodic extension, i.e., by gluing an infinite number of pieces corresponding to $X_0 = 4\alpha N$, $N \in \mathbb{Z}$ in (3.3). At the endpoint of each interval the derivative of the solution changes sign resulting in a peak, and we obtain a periodic peakon solution.

On the other hand, a direct integration of the first differential in (3.1) yields the following solution which holds between subsequent peaks

$$\lambda(X) = \alpha_1 - \frac{1}{4}(X - X_0)^2, \quad X \in [X_0 - 2\alpha, X_0 + 2\alpha].$$

Thus, in contrast to $\lambda(X')$ and $X(X')$, the profile of $U(X)$ between peaks is not exponential, but a quadratic one.

**Periodic peakon solution for the SW equation.** In a similar way, consider a limit of the periodic solution of the SW equation by putting $a_3 = \alpha$ in (2.9). Then the curve $\tilde{E}'$ becomes singular having a double point at $\mu = c$. Let $\tilde{E}'$ be the corresponding regularized curve. The integral (2.9) gives rise to the following differentials

$$\frac{d\mu}{\sqrt{(\mu - a_1)(\mu - a_2)}} = dX, \quad \frac{d\mu}{(\mu - c)\sqrt{(\mu - a_1)(\mu - a_2)}} = dX', \quad (3.4)$$

where, as before, $X = x - ct, dX = (\mu - c) dX'$. Integrating and inverting (3.4) we obtain respectively

$$\mu(X) = \frac{a_1 - a_2}{4}(e^X + e^{-X}) + \frac{a_1 + a_2}{2}, \quad (3.5)$$
\[ \mu(X') - c = \frac{4}{(e^{Z/2} + e^{-Z/2})^2/(a_1 - c) + (e^{Z/2} - e^{-Z/2})^2/(a_2 - c)}, \] (3.6)

The last expression is \(2\pi i\)-periodic in \(Z\) and provides a parameterization of the regularized curve \(\tilde{\mathcal{E}}\). Integrating (3.6) with respect to \(X'\), we find that

\[ X = \log \frac{(a_1 - a_2)e^{-Z} - (a_1 + a_2) + 2c - 2\sqrt{(c - a_1)(c - a_2)}}{(a_1 - a_2)e^{-Z} - (a_1 + a_2) + 2c + 2\sqrt{(c - a_1)(c - a_2)}} + \text{const}. \] (3.7)

Now suppose that \(a_1, a_2, c\) are real and \(c < a_1 < a_2\) or \(a_1 < a_2 < c\). As follows from (3.6), (3.7), for real parameter \(X'\), the variables \(\mu\) and \(X\) are real as well. As \(X' \to \pm\infty\), the function \(X(X')\) varies between different finite limits which we denote by \(X_1\) and \(X_2\), whereas, by (3.6), \(\mu(X') - c\) tends to zero. As a result, the composition function \(U(X) = \mu(X'(X))\) is defined only in the interval \([X_1, X_2]\), where it takes values in \([c, a_1]\) or \([a_2, c]\). According to (3.4), at the endpoints of the interval, \(U = c\), \(dU/dX = \pm \sqrt{(c - a_1)(c - a_2)}\). Like for the Dym equation, we define the global solution \(U(X)\) by periodic extension of \(\mu(X'(X))\). This implies that \(U(X)\) has periodic peaks on the infinite interval \((-\infty, \infty)\).

The expression in (3.3) provides a piece of the solution between two subsequent peaks. Solving the equation \(\mu(X) = c\) or using (3.7) directly, we find the period of the peakon solution to be

\[ X_2 - X_1 = \log \frac{2c - (a_1 + a_2) + 2\sqrt{(c - a_1)(c - a_2)}}{2 - (a_1 + a_2) - 2\sqrt{(c - a_1)(c - a_2)}}. \] (3.8)

See figure 1c. This completes the description of the periodic peakon solution.

One can show that the other possible case: \(a_1 < c < a_2\) does not provide a real bounded solution. For details about algebraic geometric approach to describing n-peakon solutions see Alber and Miller [2000] and Alber et al. [2000a].

**Solitons for the SW equation.** Consider another possible degeneration of the integral (2.3), assuming that \(a_2 = a_3 = a\) and \(a, a_1, c\) are distinct. Then we have

\[ \frac{(\mu - c) d\mu}{(\mu - a)\sqrt{(\mu - a_1)(\mu - c)}} = dX, \quad \frac{d\mu}{(\mu - a)\sqrt{(\mu - a_1)(\mu - c)}} = dX', \] (3.9)

\[ X = x - ct + \text{const}, \quad dX = (\mu - c)dX'. \]

The first differential has 2 pairs of simple poles on the rational curve \(\{\nu^2 = (\mu - a_1)(\mu - c)\}\) corresponding to \(\mu = a\) and \(\mu = \infty\) and a double zero for \(\mu = c\). Therefore the inverse function \(U(X)\) again has branching of the form (2.2).

Integrating the second differential in (3.9) and inverting, we obtain

\[ \mu(X') - a = \frac{4}{(e^{Z/2} + e^{-Z/2})^2/(c - a) + (e^{Z/2} - e^{-Z/2})^2/(a_1 - a)}, \] (3.10)

\[ \tilde{Z} = \sqrt{(a - a_1)(a - c)}X'. \]

Then, integrating the relation between \(X\) and \(X'\) yields

\[ X = (a - c)X' + \log \frac{(a_1 - c)e^{-Z} - (a_1 + c) + 2a - 2\sqrt{(a - a_1)(a - c)}}{(a_1 - c)e^{-Z} - (a_1 + c) + 2a + 2\sqrt{(a - a_1)(a - c)}} + \text{const}. \] (3.11)

As before, let us consider possible real solutions assuming that \(a, a_1, c\) are real. According to (3.10), \(\mu(\tilde{Z})\) is \(2\pi i\)-periodic and it takes real values along the lines \(\text{Im}\tilde{Z} = N\pi i, N \in \mathbb{Z}\).
Case 1. $a < a_1 < c$. The function $\mu(\tilde{Z})$ is real and finite only along the lines $\text{Im} \tilde{Z} = \pi i + 2N\pi i$, where it describes a smooth solitary wave tending to $a$ as $RX$ or $R\tilde{Z}$ tends to $\pm \infty$ and having the maximum $a_1$. Along these lines the derivative $dX/dX' = \mu - c$ is always real, negative and separated from zero. Therefore, the composition function $U(X) = \mu(X'(X))$ gives a smooth solitary wave as well.

Case 2. $a < c < a_1$. Now $\mu(\tilde{Z})$ is real and finite only along the lines $\text{Im} \tilde{Z} = 2N\pi i$, where it again describes a smooth solitary wave tending to $a$ as $RX \to \pm \infty$ and having the maximum $c$. On the other hand, along these lines the derivative $dX/dX' = \mu - c$ is always real and negative except when $\mu = c$. At this point $dX/dX'$ has a double zero. As a result, for $\mu = c$ the derivative $dU/dX$ blows up and the composition function $U(X) = \mu(X'(X))$ describes a solitary cusp (cuspon).

Solitary peakon for the SW equation. As seen from (3.8), in the limit $a_1 = a_2 = a$ the period of the above peakon solution tends to infinity. This gives us a solitary peakon solution, which can also be regarded as the separatrix between the smooth soliton and cuspon solutions. Indeed, in this case the algebraic curve $\mathcal{E}$ has 2 double points with $\mu = a$ and $\mu = c$. Its reauthorization $\mathcal{E}'$ consists of two disjoint copies of $\mathbb{P} = \{\mu\} \cup \infty$. On them the differentials (3.4) take the following simple form

$$
\frac{d\mu}{\mu - a} = l \, dX, \quad \frac{d\mu}{(\mu - c)(\mu - a)} = l \, dX',
$$

(3.12)

where, as above, $dX = (\mu - c)dX'$ and $l = \pm 1$, regarding to the copy of $\mathbb{P}$. First, we suppose $l = 1$. After integration in $\mu$ and inversion this yields

$$
\mu(X) = a + e^{X + C_2}, \quad C_2 = \text{const}, \quad \mu(X') = \frac{a - ce^{(a-c)X'}}{1 - e^{(a-c)X'}}.
$$

(3.13)

The function $\mu(X')$ has period $2\pi i/(a - c)$ and its fundamental domain gives a parameterization of the cylinder $\mathbb{P} \setminus \{\mu = a\} \setminus \{\mu = c\}$.

Integrating the differential relation between $X$ and $X'$, we obtain

$$
X(X') = - \int_{X_0'}^{X'} \frac{(a-c)e^{-(a-c)X'}}{1 - e^{-(a-c)X'}} \, dX' = - \log(e^{(c-a)X'} - 1) + C_1, \quad X', C_1 = \text{const}.
$$

Suppose that $a, c$ are real and, for definiteness, put $a < c$. The function $\mu(X')$ is $2\pi i/(a - c)$ periodic, it has a pole along the real axis, whereas along the line $\text{Im} X' = \pi i/(c - a)$ it varies in the interval $(a, c)$. Let us put $C_1 = \pi i$. Then $X(X')$ is real along this line. We notice that as $RX' \to \infty$, $X$ tends to $-\infty$ and $\mu$ tends to $a$, whereas for $RX' \to -\infty$, we have $X \to 0$ and $\mu \to c$. Thus the composition function $\mu(X'(X))$ is defined on the interval $[-\infty, 0]$ only, where it is also given by the first expression in (3.13) for $C_2 = \log(c - a)$.

Now assuming $l = -1$, we obtain the same expressions with $X$ replaced by $-X$. Let us choose the same values of the integration constants $C_1, C_2$. Then the composition function $\mu(X'(X))$ is defined on the interval $[0, \infty]$, where it has the form $\mu(X) = a + (c - a)e^{-X}$. As a result, the two copies of $\mathbb{P}$ give rise to two branches of the real continuous solution

$$
U(X) = a + (c - a)e^{-|X|}, \quad X \in \mathbb{R},
$$

which has a peak at the origin.
Remark. As seen from the above peakon solutions, the amplitude of peaks coincides with the velocity \( c \), which is a specific property of soliton propagation.

4 Stationary quasi-periodic solutions.

Stationary solutions provide profiles of the quasi-periodic wave solutions. For the sake of clarity, in this paper we restrict ourselves to the simplest nontrivial case \( g = 2 \). All the formulae and solutions below (except structure of real solutions) can be easily extended to the arbitrary \( g \)-dimensional case which is only notationally more complicated.

Stationary quasi-periodic solutions for the SW equation. According to the trace formula, in the genus 2 case we have

\[
U(x, t) = \mu_1 + \mu_2 - \sum_{j=1}^{5} a_j, \tag{4.1}
\]

and equations (1.3) take the form

\[
\begin{align*}
\frac{\mu_1 d\mu_1}{2\sqrt{R_6(\mu_1)}} + \frac{\mu_2 d\mu_2}{2\sqrt{R_6(\mu_2)}} &= dt, \\
\frac{\mu_1^2 d\mu_1}{2\sqrt{R_6(\mu_1)}} + \frac{\mu_2^2 d\mu_2}{2\sqrt{R_6(\mu_2)}} &= dx,
\end{align*}
\tag{4.2}
\]

where

\[ R_6(\mu) = -\kappa \mu (\mu - a_1) \cdots (\mu - a_5), \quad a_1, \ldots, a_5 = \text{const.} \]

Here we suppose that all the roots of \( R_6(\mu) \) are distinct. The variables \( \mu_1, \mu_2 \) must be regarded as coordinates of points \( P_1 = (\mu_1, w_1), P_2 = (\mu_2, w_2) \) on the genus 2 hyperelliptic curve \( \Gamma = \{ w^2 = R_6(\mu) \} \). Equations (4.2) involve one holomorphic differential and one meromorphic differential of the third kind having a pair of simple poles at the infinite points \( \infty_-, \infty_+ \) on \( \Gamma \). Integrating (4.2), we obtain the mapping of the symmetric product \( \Gamma^{(2)} \) to \( \mathbb{C}^2 = (t, x) \)

\[
\begin{align*}
\int_{P_0}^{P_1} &\frac{\mu d\mu}{2\sqrt{R_6(\mu)}} + \int_{P_0}^{P_2} \frac{\mu d\mu}{2\sqrt{R_6(\mu)}} = t, \\
\int_{P_0}^{P_1} &\frac{\mu^2 d\mu}{2\sqrt{R_6(\mu)}} + \int_{P_0}^{P_2} \frac{\mu^2 d\mu}{2\sqrt{R_6(\mu)}} = x.
\end{align*}
\tag{4.3}
\]

where \( P_0 \) is a fixed basepoint of the mapping. Notice that for \( P_1 \) or \( P_2 = \infty_-, \infty_+ \), we have \( x = \infty \).

Let us fix a canonical basis of cycles \( A_1, A_2, B_1, B_2 \) on \( \Gamma \) in a standard way (see, for example, Mumford [1978]). The mapping has four independent (over the reals) 2-dimensional vectors of periods of the above differentials along the cycles. In addition, it has one extra period vector corresponding to a homology zero cycle around \( \infty_- \) or \( \infty_+ \). As a result, the mapping (4.3) has 5 period vectors in \( \mathbb{C}^2 \) hence its inversion is not well defined: there do not exist meromorphic functions on \( \mathbb{C}^2 \) with five periods. In particular, \( U(x, t) \) is not a meromorphic or single valued complex function of \( t, x \).
In order to describe properties of $U(x, t)$, we, first, fix time by putting $t = t_0 \ (dt = 0)$ and consider stationary solutions $U(x, t_0)$. Introduce a new coordinate $x'$ such that
\[ dx = \mu_1 \mu_2 \, dx'. \quad (4.4) \]
Then equations (4.2) lead to the Abel–Jacobi mapping of $\Gamma^{(2)}$ to the Jacobian variety $\text{Jac}(\Gamma)$ of $\Gamma$, which includes holomorphic differentials only:
\[
\int_{P_0}^{P_1} \frac{d\mu}{2\sqrt{R_6(\mu)}} + \int_{P_0}^{P_2} \frac{d\mu}{2\sqrt{R_6(\mu)}} = u_1, \\
\int_{P_0}^{P_1} \frac{\mu d\mu}{2\sqrt{R_6(\mu)}} + \int_{P_0}^{P_2} \frac{\mu d\mu}{2\sqrt{R_6(\mu)}} = u_2, \quad (4.5)
\]
where $u_1, u_2$ are coordinates on the universal covering $C^2$ of $\text{Jac}(\Gamma)$.

Let $\tilde{\omega}_1, \tilde{\omega}_2$ be the dual basis of normalized holomorphic differentials on $\Gamma$ with respect to the above choice of cycles and $z_1, z_2$ be the corresponding coordinates on the universal covering of $\text{Jac}(\Gamma)$:
\[
\tilde{\omega}_1 = \frac{d_{11} + d_{12} \mu}{2\sqrt{R_6(\mu)}} \, d\mu, \quad \tilde{\omega}_2 = \frac{d_{21} + d_{22} \mu}{2\sqrt{R_6(\mu)}} \, d\mu, \\
z_1 = d_{11} u_1 + d_{12} u_2, \quad z_2 = d_{21} u_1 + d_{22} u_2. \quad (4.6)
\]
Here the normalizing constants $d$ are uniquely determined by the conditions $\oint_{A_i} \tilde{\omega}_j = \delta_{ij}$.

Recall that the standard theta-function related to a Riemann surface of genus $g$ and theta-functions with characteristics $\alpha = (\alpha_1, \ldots, \alpha_g)$, $\beta = (\beta_1, \ldots, \beta_g) \in \mathbb{R}^g$ have the form
\[
\theta(z|B) = \sum_{M \in \mathbb{Z}^g} \exp \left( \frac{1}{2}(BM, M) + (M, z) \right), \\
(M, z) = \sum_{i=1}^{g} M_i z_i, \quad (BM, M) = \sum_{i,j=1}^{g} B_{ij} M_i M_j, \\
\theta \left[ \frac{\alpha}{\beta} \right] (z|B) = \exp \{ (B\alpha, \alpha)/2 + (z + 2\pi i \beta, \alpha) \} \theta(z + 2\pi i \beta + B\alpha|B), \quad (4.7)
\]
$B$ being the $g \times g$ period matrix of $\Gamma$. In the sequel we shall omit it in the notation.

Now we choose the basepoint $P_0$ of the mapping (4.5) to be the last Weierstrass point $(a_5, 0)$ on $\Gamma$. Then, according to the trace formula for even order hyperelliptic curves (see e.g., Clebsch and Gordan [1866], Dubrovin [1981])
\[
U = \mu_1 + \mu_2 - \sum_{j=1}^{5} a_j = \text{const} - \partial_W \log \frac{\theta[\delta](z - q/2)}{\theta[\delta](z + q/2)}, \quad (4.8)
\]
where $z = (z_1, z_2)$, $q = (q_1, q_2)^T$, $q_i = \int_{\infty}^{-\infty} \tilde{\omega}_i$. 

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where, in view of the normalizing change \( \bar{\zeta} \), \( z_1 = d_{11} x' + \text{const} \), \( z_2 = d_{21} x' + \text{const} \), \( \partial_W \) is the derivative along a tangent vector \( W \) of \( \Gamma \subset \text{Jac}(\Gamma) \) at \( \infty_+ \), namely, in the coordinates \((u_1, u_2)\), \( W = (0, 1)^T \), and in the coordinates \((z_1, z_2)\), \( W = (d_{12}, d_{22})^T \). Finally, \( \delta = (\delta'', \delta')^T \), \( \delta'', \delta' \in \frac{1}{2} \mathbb{Z}^g / \mathbb{Z}^g \) is the half-integer theta-characteristic corresponding to the vector of Riemann constants (see Mumford [1983]). For the chosen standard canonical basis of cycles and the basepoint \( P_0 = (a_5, 0) \)

\[
\delta' = (1/2, \ldots, 1/2)^T, \quad \delta'' = (g/2, (g - 1)/2, \ldots, 1, 1/2)^T \pmod{1}. \tag{4.9}
\]

Thus, in our case

\[
\delta = \begin{pmatrix}
0 & 1/2 \\
1/2 & 1/2
\end{pmatrix}.
\]

The function \( U(z_1, z_2) \) is meromorphic on \( \text{Jac}(\Gamma) \) and it has simple poles along 2 translates of the theta-divisor \( \Theta = \{ \theta(z) = 0 \} \subset \text{Jac}(\Gamma) \):

\[
\Theta_- = \{ \theta[\delta](z - q/2) = 0 \}, \quad \Theta_+ = \{ \theta[\delta](z + q/2) = 0 \},
\]

which are tangent to each other at the origin \( \{ z = 0 \} \). Thus, \( U(z_1(x'), z_2(x')) \) is a quasi-periodic function of the complex variable \( x' \). Notice that a quasi-periodic genus 2 solution of the nonlinear mKdV equation has the same form.

We also notice that the point \( E_0 = (\mu = 0, w = 0) \) is a Weierstrass (branch) point on \( \Gamma \). Then, following Clebsch and Gordan [1866], we have the following expression for the symmetric polynomial

\[
\mu_1 \mu_2 = \theta^2 \frac{\theta^2[\delta + \eta_0](z)}{\theta[\delta](z - q/2) \theta[\delta](z + q/2)}, \quad \theta = \text{const}, \tag{4.10}
\]

where \( \eta_0 \) is the half-integer theta-characteristic corresponding to the branch point \( E_0 \):

\[
\eta_0 = (\eta''_1, \eta'_1)^T, \quad \int_{P_0}^{E_0} (\bar{\omega}_1, \bar{\omega}_2)^T = 2\pi i \bar{\eta}''_1 + B\eta'_0 \in \mathbb{C}^2. \tag{4.11}
\]

Thus, the product \( \mu_1 \mu_2 \) is a meromorphic function on \( \text{Jac}(\Gamma) \) having simple poles along \( \Theta_- \), \( \Theta_+ \) and a double zero along another translate of the theta-divisor \( \Theta, \Theta_0 = \{ \theta[\delta + \eta_0](z) = 0 \} \), passing through the origin and intersecting each of the translates \( \Theta_- \), \( \Theta_+ \) at two points. The translate \( \Theta_0 \) can also be interpreted as the image of the curve \( \Gamma \) itself under the Abel–Jacobi mapping \( \Gamma \):

\[
\Theta_0 = \left\{ \int_{P_0}^{P} (\bar{\omega}_1, \bar{\omega}_2)^T + \int_{P_0}^{E_0} (\bar{\omega}_1, \bar{\omega}_2)^T \left| P \in \Gamma \right. \right\}.
\]

It follows from \([1, 3]\), \(\eta_1, \eta_2\) that generically the derivative of the function \( x(x') \) is equal to \( \mu_1 \mu_2 \) and that it has a double zero each time when the complex \( x' \)-flow intersects \( \Theta_0 \), i.e., when \( \theta[\delta + \eta_0](z) \) vanishes, except the points where the flow is tangent to \( \Theta_0 \), i.e., when \( \theta[\delta + \eta_0](z) \) has a higher vanishing order in \( x' \) and \( \mu_1 \mu_2 \) too. This takes place only at the origin of \( \text{Jac}(\Gamma) \). Since at the origin the solution \([1, 3]\) blows up, we conclude that for bounded solutions the function \( \mu_1 \mu_2 \) may have only a double zero and \( x(x') \) a simple zero in \( x' \).

On the other hand, in view of the second equation of \([1, 3]\), the original variable \( x \) is a sum of Abelian integrals of third kind. Introduce the normalized differentials of third kind \( \Omega_{\infty_-, \infty_+} \) on \( \Gamma \) having poles at \( \infty_-, \infty_+ \) with residues \( \pm 1 \):

\[
\Omega_{\infty_-, \infty_+} = \frac{\mu^2 \, d\mu}{\sqrt{R_0(\mu)}} + h_1 \bar{\omega}_1 + h_2 \bar{\omega}_2, \tag{4.12}
\]
where \( h_1, h_2 \) are normalizing constants specified by the conditions for \( \Omega_{\infty -\infty} \) to have zero \( A \)-periods on \( \Gamma \).

According to Clebsch and Gordan [1866],

\[
\int_{P_0}^{P_1} \Omega_{\infty -\infty} + \int_{P_0}^{P_2} \Omega_{\infty -\infty} = \log \frac{\theta[\delta](z + q/2)}{\theta[\delta](z - q/2)} + \text{const.} \tag{4.13}
\]

Then, in view of the second equation in (4.12), we get

\[
x(x') = \log \frac{\theta[\delta](z + q/2)}{\theta[\delta](z - q/2)} - h_1z_1 - h_2z_2 + \text{const}, \tag{4.14}
\]

As a result, we expressed the stationary quasi-periodic solution \( U \) and the argument \( x \) in terms of the auxiliary complex variable \( x' \). The algebraic geometrical structure of the general solution \( U(x,t) \) and the behaviour of real solutions will be considered in the next sections.

**Stationary quasi-periodic solutions for the Dym equation.** Now we pass to the Dym equation (1.2) and seek its solutions again in the form (4.1). In this case the variables \( \mu_1, \mu_2 \) again change according to equations of the form (4.2) with the only difference being that the order of the polynomial defining the corresponding hyperelliptic curve is odd:

\[
\frac{\mu_1 d\mu_1}{2\sqrt{R_5(\mu_1)}} + \frac{\mu_2 d\mu_2}{2\sqrt{R_5(\mu_2)}} = dt,
\]

\[
\frac{\mu_1^2 d\mu_1}{2\sqrt{R_5(\mu_1)}} + \frac{\mu_2^2 d\mu_2}{2\sqrt{R_5(\mu_2)}} = dx, \tag{4.15}
\]

\[
R_5(\mu) = -\kappa \mu(\mu - a_1) \cdots (\mu - a_4),
\]

hence the corresponding hyperelliptic curve \( \Gamma = \{ w^2 = R_5(\mu) \} \) has just one infinite point \( \infty \). As a consequence, the equations (4.15) contain one holomorphic differential and one differential of the second kind.

As before, we first consider stationary solutions by putting \( t = t_0 \ (dt = 0) \) and assuming \( \kappa = 1 \). Notice that under these conditions, (4.15) has the same structure as quadratures for the Jacobi problem on geodesics on a triaxial ellipsoid \( Q \), where \( \mu_1, \mu_2 \) play the role of ellipsoidal coordinates on \( Q \), parameters \( a_1, a_2, a_3 \) the squares of the semi-axes of \( Q \), \( a_4 \) the constant of motion, and \( x \) the length of a geodesic.

Under the change of parameter (4.4), we arrive at the Abel–Jacobi mapping

\[
\int_{P_0}^{P_1} \frac{d\mu}{2\sqrt{R_5(\mu)}} + \int_{P_0}^{P_2} \frac{d\mu}{2\sqrt{R_5(\mu)}} = u_1, \tag{4.16}
\]

\[
\int_{P_0}^{P_1} \frac{\mu d\mu}{2\sqrt{R_5(\mu)}} + \int_{P_0}^{P_2} \frac{\mu d\mu}{2\sqrt{R_5(\mu)}} = u_2,
\]

\[
u_1 = x' + \text{const}, \quad u_2 = \text{const},
\]

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This change was first made by Weierstrass [1878] in order to find the theta-functional solution for the geodesic problem (see also Cewen [1990]). Next we introduce normalized holomorphic differentials \( \omega_1, \omega_2 \) on \( \Gamma \) and coordinates \( z_1, z_2 \) on the universal covering of \( \text{Jac}(\Gamma) \) according to (4.6) and, in addition, the normalized differential of the second kind having a double pole at \( \infty \)

\[
\Omega^{(1)}_{\infty} = \frac{\mu_i^2 d\mu_i}{2 \sqrt{R_5(\mu_i)}} + h_1^{'} \omega_1 + h_2^{'} \omega_2.
\]  

(4.17)

Similarly to (4.12), the constants \( h_1^{'}, h_2^{'} \) are uniquely defined by the condition that \( \Omega^{(1)}_{\infty} \) have zero \( A \)-periods on \( \Gamma \).

Then, instead of the expressions (4.8), and (4.10), we have (see, e.g., Dubrovin [1981], Dubrovin et al. [1985])

\[
U(x') = \mu_1 + \mu_2 = \text{const} - \partial_V \theta[\delta](z),
\]

(4.18)

where \( \partial_V \) is the derivative along the tangent vector \( V \in \text{Jac}(\Gamma) \) at \( \infty \): \( V = (d_{12}, d_{22})^T \), and, respectively,

\[
\mu_1 \mu_2 = \kappa \frac{\theta^2[\delta + \eta_0](z)}{\theta^2[\delta](z)}, \quad \kappa = \text{const},
\]

(4.19)

where the characteristic \( \eta_0 \) is defined in (4.11). In addition, in contrast to (4.13), the sum of Abelian integrals of second kind has the form

\[
\int_{P_1}^{P_2} \Omega^{(1)}_{\infty} = \int_{P_0}^{P_2} \Omega^{(1)}_{\infty} = \text{const} - \partial_V \log \theta[\delta](z).
\]

(4.20)

Comparing this with (4.17), we find that the analog of the relation (4.14) between the parameters \( x \) and \( x' \) has the form

\[
x(x') = -\partial_V \log \theta[\delta](z) - h_1 z_1 - h_2 z_2 + \text{const},
\]

(4.21)

\[
z_1 = d_{11} x' + \text{const}, \quad z_2 = d_{21} x' + \text{const}.
\]

This expression can be regarded as a 2-dimensional generalization of the Weierstrass zeta-function in (2.3).

Thus, we have expressed the stationary solution \( U \) and the argument \( x \) in terms of the auxiliary complex variable \( x' \). Various types of real solutions defined by the above expressions will be considered in Section 6.

5 Time-dependent quasi-periodic solutions.

The solutions for the SW equation. In order to obtain general time-dependent solutions \( U(x, t) \) of the SW equation given by (4.1), one has to invert the mapping (4.3). However, as already mentioned, the problem of inversion is unsolvable in terms of meromorphic functions.

To describe the structure of general solutions, let us first consider a divisor of 3 points \( P_i = (\mu_i, w_i), i = 1, 2, 3 \) on \( \Gamma \setminus \{ \infty_-, \infty_+ \} \) and the following extended equations

\[
\sum_{i=1}^{3} \frac{d\mu_i}{2 \sqrt{R_6(\mu_i)}} = dy, \quad \sum_{i=1}^{3} \frac{\mu_i d\mu_i}{2 \sqrt{R_6(\mu_i)}} = dt, \quad \sum_{i=1}^{3} \frac{\mu_i^2 d\mu_i}{2 \sqrt{R_6(\mu_i)}} = dx,
\]

(5.1)
including the extra variable \( y \), two holomorphic differentials and one differential of the third kind on \( \Gamma \). The latter are linear combinations of the normalized differentials \( \tilde{\omega}_1, \tilde{\omega}_2, \Omega_{\pm \infty} \) defined in \((1.6)\) and \((4.12)\). According to Clebsch and Gordan [1866], Fedorov [1999], equations \((5.1)\) describe a differential of a well defined mapping of the symmetric product \((\Gamma \setminus \{\infty_-, \infty_+\})^3\) to \textit{generalized Jacobian variety} \( \text{Jac}(\Gamma, \infty_{\pm}) \), a \textit{noncompact} algebraic group represented as the quotient of \( \mathbb{C}^3 \) by a lattice \( \Lambda \) generated by five vectors of periods of the differentials \( \tilde{\omega}_1, \tilde{\omega}_2, \Omega_{\pm \infty} \) on \( \Gamma \). Topologically, \( \text{Jac}(\Gamma, \infty_{\pm}) \) is the product of the 2 dimensional variety \( \text{Jac}(\Gamma) \) and the cylinder \( \mathbb{C}^* = \mathbb{C} \setminus \{0\} \). An analytical and algebraic-geometrical description of generalized Jacobians can be found in Clebsch and Gordan [1866], Belokolos et al [1994], Fedorov [1999], Gavrilov [1999].

Let \( z_1, z_2, Z \) be coordinates on the universal covering of \( \text{Jac}(\Gamma, \infty_{\pm}) \) such that

\[
\begin{align*}
\sum_{i=1}^{3} \int_{P_0}^{P_i} \tilde{\omega}_1 &= z_1, \\
\sum_{i=1}^{3} \int_{P_0}^{P_i} \tilde{\omega}_2 &= z_2, \\
\sum_{i=1}^{3} \int_{P_0}^{P_i} \Omega_{\pm \infty} &= Z,
\end{align*}
\]

where, as above, \( P_0 = (a_5, 0) \). Then, according to \((1.6)\), \((1.12)\),

\[
\begin{align*}
z_1 &= d_{11}y + d_{12}t + \text{const}, \\
z_2 &= d_{21}y + d_{22}t + \text{const}, \\
Z &= x + h_1(d_{11}y + d_{12}t) + h_2(d_{21}y + d_{22}t) + \text{const}.
\end{align*}
\]

The problem of inversion of Abel–Jacobi mappings including differentials of the third and second kind is solved in terms of \textit{generalized theta-functions} which are finite sums of products of customary theta-functions, rational functions, and exponentials (see Ercolani [1989], Fedorov [1999], Gagnon et al. [1987]). To invert the mapping \((5.2)\), we shall make use of the following theta-functions

\[
\begin{align*}
\tilde{\theta}(z, Z) &= e^{Z/2} \theta(z + q/2) + e^{-Z/2} \theta(z - q/2), \\
\tilde{\theta}[\eta](z, Z) &= e^{Z/2} \theta[\eta](z + q/2) + e^{-Z/2} \theta[\eta](z - q/2),
\end{align*}
\]

\[
\begin{align*}
z &= (z_1, z_2), \\
q &= (q_1, q_2)^T, \\
q_1 &= \int_{\infty_-}^{\infty_+} \tilde{\omega}_1, \\
q_2 &= \int_{\infty_-}^{\infty_+} \tilde{\omega}_2,
\end{align*}
\]

where \( \theta(z), \theta[\eta](z) \) are customary theta-functions associated with the curve \( \Gamma \) with half-integer theta-characteristics \( \eta \). Like \( \theta[\eta](z) \), generalized theta-functions have a quasi-periodic property: a shift of the argument \( (z, Z) \) by any period vector of the generalized Jacobian results in multiplication of \( \tilde{\theta}[\eta](z, Z) \) by a constant factor.

Now consider the dissection \( \tilde{\Gamma} \) of \( \Gamma \) along the canonical cycles \( A_1, A_2, B_1, B_2 \), which is a one-connected domain having the form of an octagon. In addition, we cut \( \tilde{\Gamma} \) along the paths joining a point \( O \) on the boundary \( \partial \tilde{\Gamma} \) of \( \tilde{\Gamma} \) to the points \( \infty_-, \infty_+ \). On the obtained domain \( \tilde{\Gamma}' \) we introduce the single-valued function \( \tilde{F}(P) = \tilde{\theta}[\delta](\tilde{A}(P) - (z, Z)^T) \), where

\[
\tilde{A}(P) = \left( \int_{P_0}^{P} \tilde{\omega}_1, \int_{P_0}^{P} \tilde{\omega}_2, \int_{P_0}^{P} \Omega_{\pm \infty} \right)^T,
\]

and the characteristic \( \delta \) is defined in \((1.9)\). Then the following analog of the Riemann theorem holds (see e.g., Fedorov [1999], Gagnon et al. [1985]).

**Theorem 5.1**: Let the coordinates \( z, Z \) be such that the function \( \tilde{F}(P) \) does not vanish identically on \( \tilde{\Gamma}' \). Then it has precisely 3 zeros \( P_1, P_2, P_3 \) giving a unique solution to the inversion of the generalized mapping \((5.2)\).
Now let us consider the logarithmic differential $\mu(P)d\log \tilde{F}(P)$. By Theorem 5.1, the sum of the residues of its poles in the domain $\Gamma'$ equals $\mu(P_1) + \mu(P_2) + \mu(P_3)$. Applying the residue theorem, after calculations, we get the following compact “trace formula”

$$\mu_1 + \mu_2 + \mu_3 = \text{const} - \frac{e^Z\theta[\delta](z + q) + e^{-Z}\theta[\delta](z - q)}{\theta[\delta](z)}$$ (5.5)

with the characteristic $\delta$ specified in (4.4).

The principal difference between the extended mappings (5.1) or (5.2) and the system (1.3) is that the latter contains only 2 points on $\Gamma \setminus \{\infty_-, \infty_+\}$. On the other hand, (5.1) reduces to (1.2) by fixing $P_3 \equiv P_0$ ($\mu_3 \equiv a_5$, $d\mu_3 \equiv 0$). Under this condition, (5.2) describes the embedding of the symmetric product ($\Gamma \setminus \{\infty_-, \infty_+\})^2$ into $\text{Jac}(\Gamma, \infty_\pm)$. Its image is a 2-dimensional nonlinear analytic subvariety (stratum) $W_2$. Like the generalized Jacobian itself, it is a noncompact variety.

**Remark 5.1.** In case of customary Jacobian varieties, the corresponding nonlinear subvarieties and their stratification have been studied in Gunning [1972] and Vanhaecke [1995]. Such varieties or their open subsets often appear as (coverings of) complex invariant manifolds of finite-dimensional integrable systems (see Vanhaecke [1995], Abenda and Fedorov [1999]).

It follows from the above that on the stratum $W_2$ the variables $z_1, z_2, Z$ play the role of excessive (abundant) coordinates, hence they cannot be independent there. The analytic structure of $W_2$ is explicitly described by the following theorem (see e.g., Fedorov [1999], Gagnon et al. [1987]).

**Theorem 5.2** The subvariety $W_2 \subset \text{Jac}(\Gamma, \infty_\pm)$ coincides with the zero locus of the generalized theta-function:

$$W_2 = \{e^{Z/2}\theta[\delta](z + q/2) - e^{-Z/2}\theta[\delta](z - q/2) = 0\}. \quad (5.6)$$

On the other hand, in view of relations (5.3), the coordinates $z, Z$ are linear functions of the variables $x, t, y$. Thus, equation (5.4) can also be regarded as a constraint on them. It follows that on fixing $P_3 = P_0$, $y$ becomes a transcendent function of $x, t$.

Now we notice that the sum $\mu_1 + \mu_2 + a_5 = \mu(P_1) + \mu(P_2) + \mu(P_0)$ coincides with the restriction of the total sum $\mu(P_1) + \mu(P_2) + \mu(P_3)$, as a function on $\text{Jac}(\Gamma, \infty_\pm)$, onto $W_2$. Then, using expression (5.5), we conclude that the 2-phase solution of the SW equation has the form

$$U(x, t) = \text{const} - \frac{e^Z\theta[\delta](z + q) + e^{-Z}\theta[\delta](z - q)}{\theta[\delta](z)}$$ (5.7)

$$z_1 = d_{11}y + d_{12}t, \quad z_2 = d_{21}y + d_{22}t, \quad Z = x + h_1(d_{11}y + d_{12}t) + h_2(d_{21}y + d_{22}t),$$

where the extra variable $y$ depends on $x, t$ according to (5.6). As a result, we arrive at the following algebro-geometric description of motion:

The $x$-flow (t-flow) defined by equations (4.3) evolves on the nonlinear variety $W_2 \subset \text{Jac}(\Gamma, \infty_\pm)$ in such a way that $y$ is a nonlinear transcendent function of $x$ (respectively, of $t$). In this sense the flow is nonlinear.

We emphasize that the solution $U(x, t)$ is neither meromorphic in $x$, nor in $t$. 
Remark 5.2. Let us consider the $x$-flow by putting $t = \text{const}$. It turns out that, up to an additive constant, the extra variable $y$ can now be identified with the auxiliary variable $x'$ introduced in (4.4) when we considered stationary solutions. Indeed, in view of (5.3), in this case the condition in (5.6) becomes

$$Z = x + h_1 z_1 + h_2 z_2 + \text{const} = \log \frac{\theta[\delta](z - q/2)}{\theta[\delta](z + q/2)} + \text{const},$$

which is equivalent to the relation (4.14) between $x$ and $x'$. In view of (5.8) and the addition theorem for theta-functions, the solution (5.7) reduces to the stationary solution (4.8).

In contrast to $x$, the parameter $t$ enters both expressions for $z$ and $Z$ in (5.7). Therefore, in the case of the $t$-flow, $t$ cannot be explicitly expressed in terms of $y$ as in the case of the $x$-flow. This implies that solutions $U(x_0, t)$, $x_0 = \text{const}$ must have different properties in comparison with (4.8).

Remark 5.3. We notice that the subvariety $W_2$ of the generalized Jacobian becomes linear in rare cases when the curve $\Gamma$ enjoys some nontrivial involutions, i.e., when it can be regarded as a covering of an elliptic curve. (Various examples of the involutions can be found in Belokolos et al. [1994].) In such cases $U(x, t)$ becomes a meromorphic function of its arguments.

The solutions for the Dym equation. Now we proceed to the problem of inversion of the reduction (4.13) of the Dym equation which is related to the odd order hyperelliptic curve $\Gamma = \{ w^2 = R_5(\mu) \}$. As in the case of the reduction of the SW equation, in order to describe the function $U(x, t) = \mu_1 + \mu_2$, we first consider an “excessive” divisor of 3 points $P_i = (\mu_i, w_i)$, $i = 1, 2, 3$ on $\Gamma \setminus \{ \infty \}$ and the extended equations

$$(5.9) \quad \sum_{i=1}^{3} \frac{d\mu_i}{2\sqrt{R_5(\mu_i)}} = dy, \quad \sum_{i=1}^{3} \frac{\mu_i d\mu_i}{2\sqrt{R_5(\mu_i)}} = dt, \quad \sum_{i=1}^{3} \frac{\mu_i^2 d\mu_i}{2\sqrt{R_5(\mu_i)}} = dx,$$

including 2 holomorphic differentials and one differential of the second kind having a double pole at $\infty \in \Gamma$. They are linear combinations of the normalized differentials $\omega_1, \omega_2, \Omega^{(1)}_\infty$ defined in (4.6) and (4.17).

In contrast to (5.4), equations (5.9) describe a differential of a well defined mapping of the symmetric product $(\Gamma \setminus \{ \infty \})^{(3)}$ to the generalized Jacobian variety $\text{Jac}(\Gamma, \infty)$, the quotient of $\mathbb{C}^3$ by the lattice generated by four period vectors of the differentials $\omega_1, \omega_2, \Omega^{(1)}_\infty$ on $\Gamma$. Topologically, this variety is a product of the 2 dimensional variety $\text{Jac}(\Gamma)$ and the complex plane $\mathbb{C}$ (see Clebsch [1866], Gavrilov [1999]).

Let us introduce coordinates $z_1, z_2, Z$ by the mapping

$$\sum_{i=1}^{3} \int_{E_0}^{P_i} \omega_1 = z_1, \quad \sum_{i=1}^{3} \int_{E_0}^{P_i} \omega_2 = z_2, \quad \sum_{i=1}^{3} \int_{E_0}^{P_i} \Omega^{(1)}_\infty = Z$$

(5.10)

with the basepoint $E_0 = (0, 0)$ (we cannot choose the basepoint to be $\infty$ as in the previous section, since it is the pole of $\Omega^{(1)}_\infty$). Next, comparing (4.6), (4.17) with (5.9), we find the following relations

$$z_1 = d_{11} y + d_{12} t + \text{const}, \quad z_2 = d_{21} y + d_{22} t + \text{const}$$

$$Z = x + h'_1(d_{11} y + d_{12} t) + h'_2(d_{21} y + d_{22} t) + \text{const}.$$  

(5.11)
Like (5.2), the mapping (5.10) is invertible in terms of meromorphic functions. The inversion problem is solved by means of the following rational degeneration of the customary theta-function

\[ \hat{\theta}(z, \tilde{\omega}) = Z \theta[\delta](z) + \partial \nu \theta[\delta](z), \]

(5.12)

where \( \partial \nu \) is defined in (4.18) (compare with the generalized theta-functions (5.4)). Like (5.4), the function (5.12) enjoys the quasi-periodic property.

Consider again the dissection \( \tilde{\Gamma} \) of \( \Gamma \) and cut it along a path joining a point \( O \) on the boundary \( \partial \Gamma \) to \( \infty \). In the obtained domain we introduce the single-valued function

\[ \hat{F}(P) = \left( z - \int_{E_0}^{P} \frac{\theta[\delta]}{\tilde{\omega}} \left( z - \int_{E_0}^{P} \hat{\theta}[\delta] \left( z - \int_{E_0}^{P} \hat{\theta}[\delta] \right) \right) + \partial \nu \theta[\delta] \left( z - \int_{E_0}^{P} \hat{\theta}[\delta] \right) \right). \]

Using a modification of Theorem 5.1 and calculating the logarithmic differential \( \mu(P) \, d \log \hat{F}(P) \), we obtain

\[ \mu_1 + \mu_2 + \mu_3 = \text{const} - (Z + \partial \nu \theta[\delta + \eta_0](z))^2 - \partial \nu \log \theta[\delta + \eta_0](z) \]

\[ = \text{const} - 2Z^2 - \frac{2Z \partial \nu \theta[\delta + \eta_0](z) - \partial \nu \theta[\delta + \eta_0](z)}{\theta[\delta + \eta_0](z)}, \]

(5.13)

where \( \eta_0 = (\eta_0, \eta_0)^T \in \frac{1}{2} Z^2 / Z^2 \), such that \( 2\pi i \eta_0 + \nu P = \int_{E_0}^{\infty}(\hat{\omega}_1, \hat{\omega}_2)^T \). Now, similarly to the case of the SW equation, we fix \( P_3 \equiv E_0 (\mu_3 = 0, d\mu_3 = 0) \) in the mapping (5.10). In this case its image becomes a 2-dimensional nonlinear noncompact analytic subvariety \( \tilde{\hat{W}}_2 \subset \text{Jac}(\Gamma, \infty) \). Comparing the third sum in (5.10) and expression (4.20), we find

\[ \tilde{\hat{W}}_2 = \{ Z + \text{const} + \partial \nu \log \theta[\delta + \eta_0](z) = 0 \}. \]

(5.14)

Finally, taking into account the trace formula (5.13), we conclude that the solution of the Dym equation has the form

\[ U(x, t) = \mu_1 + \mu_2 = \text{const} - \partial \nu \log \theta[\delta + \eta_0](z), \]

(5.15)

\[ z_1 = d_{11}y + d_{12}t + \text{const}, \quad z_2 = d_{21}y + d_{22}t + \text{const}, \]

where the extra variable \( y \) depends on \( x, t \) in a transcendental way according to the constraint (5.14) and the expression for \( Z \) in (5.14). The solution \( U(x, t) \) is not meromorphic with respect to its arguments.

**Remark 5.4.** As in the case of SW equation, the stationary solutions for the Dym equation given in the previous section can be obtained from (5.15) by putting \( t = \text{const} \). Then \( y \) can be identified with the auxiliary variable \( x' \) defined in (4.14) and the condition in (5.14) becomes equivalent to the relation (4.14) between \( x \) and \( x' \). As a result, (5.15) gives precisely the stationary solution (4.18).

### 6 Real bounded stationary 2-phase solutions.

In this section we impose reality conditions on the stationary complex solutions obtained in Section 4.

Let \( \sigma \) be the antiholomorphic involution on a hyperelliptic curve \( \Gamma = \{w^2 = P(\mu)\} \) of genus \( g \). The part of \( \Gamma \) which is invariant with respect to \( \sigma \) is called the real part \( \Gamma(\mathbb{R}) \). On the plane
\( \mathbb{R}^2 = (\mathbb{R} \mu, \mathbb{R} \nu) \) it is either the empty set or a union of ovals. By the Abel–Jacobi mapping, the involution \( \sigma \) lifts to \( \text{Jac}(\Gamma) \). By \( \text{Jac}_\mathbb{R}(\Gamma) \) we denote the real part of \( \text{Jac}(\Gamma) \) that is invariant under \( \sigma \). One can show that the elementary symmetric functions of the variables \( \mu_1, \ldots, \mu_g \) take real values on \( \text{Jac}_\mathbb{R}(\Gamma) \) and only there.

**Theorem 6.1** (Comessatti [1924]). Let \( s \) be the number of connected components of \( \Gamma(\mathbb{R}) \) and \( L \) be the number of connected components of \( \text{Jac}_\mathbb{R}(\Gamma) \). If \( s \neq 0 \), then \( L = 2^{s-1} \). If \( s = 0 \), then \( L = 1 \) provided the degree of \( R(\mu) \) is even and \( L = 2 \) in case the degree is odd.

**Shallow water equation.** Suppose all the roots of the polynomial \( R_6(\mu) \) in \( \mathbb{R}[\mu] \) arising in the reduction of the SW equation are real, i.e., \( \Gamma(\mathbb{R}) \) consists of 3 ovals about the segments \([0,e_1]\), \([e_2,e_3]\), and \([e_4,e_5]\). By Theorem 5.1, \( \text{Jac}_\mathbb{R}(\Gamma) \) has 4 connected components. They are characterized by the following behavior of \( \mu \)-variables, which reflects in different properties of real stationary solutions \( U(x,t_0) \):

**Case 1.** The variables \( \mu_1, \mu_2 \) are real and \( \mu_1 \in [a_2,a_3] \), \( \mu_2 \in [a_4,a_5] \). The sum \( U = \mu_1 + \mu_2 \) is thus a real quasi-periodic function of \( x' \) having no poles and zeros. The product \( \mu_1 \mu_2 \) has the same properties. Geometrically this means that the corresponding component of \( \text{Jac}_\mathbb{R}(\Gamma) \) does not intersect the translates \( \Theta_-, \Theta_+, \) and \( \Theta_0 \). In view of (4.2), \( x(x') \) and \( x'(x) \) are strictly monotonic real functions. Therefore the composition \( U(x,t_0) = U(x'(x)) \) is a quasi-periodic regular function.

**Cases 2,3.** \( \mu_1 \in [0,a_1] \), whereas \( \mu_2 \in [a_2,a_3] \) or \([a_4,a_5]\) \( \). The function \( U(x') \) has the same properties as above, whereas \( \mu_1 \mu_2 \) does not blow up, but has zeros. As found in Section 4, the derivative \( dx/dx' \) vanishes with a second order with respect to \( x' \) as one of the \( \mu \)-variables vanishes. This happens when the real \( x' \)-flow on the considered components of \( \text{Jac}_\mathbb{R}(\Gamma) \) intersects \( \Theta_0(\mathbb{R}) = \Theta_2 \cap \text{Jac}_\mathbb{R}(\Gamma) \). If follows that at this moment the derivative \( dU/dx = dU/dx' \cdot dx'/dx \) blows up and the graph of the function \( U(x,t_0) \) has an inflection point with vertical tendency line.

In addition, when \( (\mu_1,\mu_2) = (0,a_2) \) or \( (0,a_4) \), i.e., when the real \( x' \)-flow passes a half-period on \( \text{Jac}_\mathbb{R}(\Gamma) \), the function \( U(x') \) has an extremum, which implies that the graph of \( U(x,t_0) \) has a cusp. Due to quasi-periodicity of the flow, \( U(x,t_0) \) has an infinite quasi-periodic sequence of cusps.

**Case 4.** Now the variables \( \mu_1, \mu_2 \) are complex conjugated. Using equations (4.2), we show that they cannot reach real axis. It follows that the product \( \mu_1 \mu_2 \) is always nonzero and \( U(x,t_0) \) is again a quasi-periodic regular function.

In a similar way one can show that when some of the roots of \( R_6(\mu) \) are complex conjugate, the qualitative behavior of the real solution \( U(x,t_0) \) coincides with one of the above four cases.

In a forthcoming paper we will consider different singular limits of the quasi-periodic solutions when the spectral curve becomes singular and its arithmetic genus drops to zero. The solutions are then expressed in terms of purely exponential tau-functions and, in the real bounded case, they describe an interaction of the two smooth solitons or cuspons, or a quasi-periodic train of peakons tending to a periodic one at infinity.
7 Peakon-soliton solutions and elliptic billiards.

In this section we continue studying degenerate solutions of the Dym equation. Consider another possible confluence of roots of the polynomial \( R_5(\mu) \) in (4.13) by putting
\[
a_1 = 0, \quad a_2 = a_3 = b, \quad a_4 = a, \quad \kappa = 1. \tag{7.1}
\]
As before, we first analyze stationary solutions (x-flow) by setting \( dt = 0 \). Passing to the new variable \( x' \) according to the change
\[
dx = \mu_1\mu_2 \, dx', \tag{7.2}
\]
from (4.15) we get
\[
\frac{d\mu_1}{\mu_1\sqrt{a - \mu_1}} + \frac{d\mu_2}{\mu_2\sqrt{a - \mu_2}} = -b \, dx', \tag{7.3}
\]
(7.3) gives rise to the generalized Abel–Jacobi equations
\[
\int_{P_0}^{P_1} \Omega_1 + \int_{P_0}^{P_2} \Omega_1 = z_1, \quad \int_{P_0}^{P_1} \Omega_2 + \int_{P_0}^{P_2} \Omega_2 = z_2, \quad P_1 = (\mu_i, \xi_i), \tag{7.4}
\]
where we put \( P_0 = (a,0) \). These describe a well defined mapping of the symmetric product \( (\mathbb{P} \setminus \{Q_1, Q_1^+, Q_2, Q_2^+\})^2 \) to the generalized Jacobian \( \text{Jac}(\mathbb{P}, Q_1^\pm, Q_2^\pm) \).

As a result of inversion of (7.4), one finds the following expressions for symmetric polynomials of \( \mu_1 \) and \( \mu_2 \)
\[
\mu_1 + \mu_2 = U(z_1, z_2) = \partial_W \log \tau(z_1, z_2) + \beta_1^2 - \beta_2^2 = 4(\beta_1^2 - \beta_2^2) \frac{\beta_1^2(e^{z_1} + e^{-z_1}) + \beta_2^2(e^{z_1} + e^{-z_1}) + 2(\beta_1^2 - \beta_2^2)}{\tau^2(z_1, z_2)} + \beta_1^2 - \beta_2^2, \tag{7.6}
\]
\[
\mu_1\mu_2 = -\frac{1}{b_1} \partial_{z_1} \partial_W \log \tau(z_1, z_2) \quad \text{or} \quad \mu_1\mu_2 = 4ab \frac{e^{z_1/2} + e^{-z_1/2}}{\tau^2(z_1, z_2)},
\]
where
\[
\partial_W = 2\beta_1 \frac{\partial}{\partial z_1} + 2\beta_2 \frac{\partial}{\partial z_2}, \quad z_1 = -b_1 \, x' + z_{10}, \quad z_{10}, z_2 = \text{const},
\]
and \( \tau(z_1, z_2) \) is the 2-dimensional tau-function with \( \alpha_1, \alpha_2 \) replaced by \( \beta_1, \beta_2 \). The latter admits decomposition in the following sum of one-dimensional tau-functions
\[
\tau(z_1, z_2) = e^{z_1/2} \tau(z_2 + q/2) - e^{-z_1/2} \tau(z_2 - q/2),
\]
\[
\tau(z_2) = e^{z_2/2} + e^{-z_2/2}, \quad q = \log \left( \frac{\beta_1 - \beta_2}{\beta_1 + \beta_2} \right)^2. \tag{7.7}
\]
Lastly, by using the second expression in (7.6) and the relation between \( z_1 \) and \( x' \) in (7.3), we find

\[
x = \int \mu_1 \mu_2 \, dx' = \partial_W \log (z_1, z_2) + \text{const}
\]

or, in view of decomposition (7.7),

\[
x(x') = \beta_1 e^{z_1/2} \tau (z_2 + q/2) + e^{-z_1/2} \tau (z_2 - q/2) + \text{const},
\]

\[
+ \beta_2 e^{z_1/2} \partial_{z_2} \tau (z_2 + q/2) - e^{-z_1/2} \partial_{z_2} \tau (z_2 - q/2) + \text{const},
\]

\[
z_1 = -b \beta_1 x' + z_{10}, \quad z_2 = \text{const}.
\]  

(7.8)

Remark 8.1. As mentioned in Remark 4.1, equations (4.13) with \( dt = 0 \) describing the quasiperiodic stationary solutions have the same structure as quadratures for the geodesic motion on an triaxial ellipsoid \( \mathcal{E} \) (more generally, a quadric) in \( \mathbb{R}^3 \). Parameter \( x \) plays the role of length of a geodesic. Under the limit (7.1) one of the semiaxes of \( \mathcal{E} \) tends to zero whereas the geodesic motion passes to the asymptotic billiard motion inside the ellipse

\[
\mathcal{E} = \{ X_2^2/a + X_2^2/b = 1 \} \subset \mathbb{R}^2 = (X_1, X_2).
\]

Geodesics on \( \mathcal{E} \) transform to straight line segments passing through a focus of the ellipse between each subsequent elastic reflections (impacts) along \( \mathcal{E} \). As \( x \to \pm \infty \), the billiard motion tends to oscillations along the bigger axis of the ellipse. Now the variables \( \mu_1, \mu_2 \) play the role of elliptic coordinates in \( \mathbb{R}^2 \) such that

\[
X_1^2 = \frac{(a - \mu_1)(a - \mu_2)}{a - b}, \quad X_2^2 = \frac{(b - \mu_1)(b - \mu_2)}{b - a}.
\]

Along \( \mathcal{E} \) one of the variables equals zero.

It follows that equations (7.3) can be regarded as describing the straight line motion of a point mass inside \( \mathcal{E} \). When the point meets the ellipse, one of the \( \mu \)-variables, say \( \mu_1 \), vanishes, and the corresponding point \( P_1 = (\mu_1, \sqrt{R(\mu_1)}) \) on the Riemann surface \( \mathbb{P} \) coincides with one of the poles \( Q^1, Q^2 \) of the differential \( \Omega \). Then, as follows from the mapping (7.4), \( z_1 \) and \( x' \) become infinite. On the other hand, as \( x' \), \( z_1 \to \pm \infty \), the second expression in (7.6) vanishes, whereas the first one has finite limits giving the values of \( \mu_2 \) at the subsequent impact points. The variable \( z_2 \) plays the role of a constant phase defining position of the segment between the points.

According to (7.8), (7.4)

\[
x(-\infty, z_2) = -\beta_1 + \beta_2 \frac{\partial_{z_2} \tau (z_2 + q/2)}{\tau (z_2 + q/2)} + c_x,
\]

\[
x(\infty, z_2) = \beta_1 + \beta_2 \frac{\partial_{z_2} \tau (z_2 + q/2)}{\tau (z_2 + q/2)} + c_x, \quad c_x = \text{const}.
\]

(7.9)

and

\[
U(-\infty, z_2) = \frac{4 \beta_2^2 (\beta_1^2 - \beta_2^2)}{(\beta_1 - \beta_2)^2 e^{-z_2} + (\beta_1 + \beta_2)^2 e^{z_2} + 2 (\beta_1^2 - \beta_2^2)} + \beta_1^2 - \beta_2^2,
\]

\[
U(\infty, z_2) = \frac{4 \beta_2^2 (\beta_1^2 - \beta_2^2)}{(\beta_1 - \beta_2)^2 e^{-z_2} + (\beta_1 + \beta_2)^2 e^{z_2} + 2 (\beta_1^2 - \beta_2^2)} + \beta_1^2 - \beta_2^2.
\]

(7.10)
At the moment of impact, billiard trajectory.

inside $\bar{\gamma}$ respectively. All this results in the following algebro-geometric description: As the point mass inside $\mathcal{E}$ moves from one impact to the next one, the point $P_1$ on $\mathcal{P}$ moves from the pole $Q_1^+$ to $Q_1^-$. At the moment of impact, $P_1$ jumps from $Q_1^+$ back to $Q_1^-$, whereas the phase $z_2$ in (7.6) increases by $q$. Then the story repeats.

Using this property, by induction, from (7.10) the elliptic coordinates of the whole sequence of impact points are found in form

$$
\mu_1 = 0, \quad \mu_2 = \frac{4\beta_2(\beta_1^2 - \beta_2^2)}{(\beta_1 - \beta_2)^2 e^{2z_2N} + (\beta_1 + \beta_2)^2 e^{-2z_2N} + 2(\beta_1^2 - \beta_2^2)} + \beta_1^2 - \beta_2^2, \quad (7.11)
$$

$N \in \mathbb{N}$ being the number of impact and the constant $z_{20}$ is the same for all the segments of the billiard trajectory.

In addition, from (7.3) we find the length of the $N$-th segment of the billiard trajectory in form

$$
x(\infty, z_{2N}) - x(-\infty, z_{2N}) = 2\beta_1 + 2\beta_2 \frac{e^{q/2} - e^{-q/2}}{\exp(z_{2N}) - \exp(-z_{2N}) + e^{q/2} - e^{-q/2}}, \quad (7.12)
$$

$z_{20}$ being the same as in (7.11).

According to the trace formula $U(x, t_0) = \mu_1 + \mu_2$, expressions (7.6), (7.8) provide us stationary peakon solutions of Dym equation in a parametric form. Here the phase $z_2$ must be regarded as a certain function of $t_0$. Namely, the expressions describe one piece of the profile $U(x, t_0)$ corresponding to trajectory of the point mass between subsequent impacts. The other pieces are obtained by changing the phase $z_2$ in (7.6), (7.8) by $q$ and adding $2\beta_1$ to $x$. The pieces are glued at peak points, where the spatial derivative of $U$ changes sign and which correspond to impacts in the billiard problem. The profile $U(x, t_0)$ thus consists of an infinite sequence of peaks and knots between them. For this reason we call this solution the soliton-peakon solution. Notice that, in contrast to exponentials profiles of the functions $U(x')$, $x(x')$, any piece of $U(x, t_0)$ has quadratic profile, as will be explained below.

The heights $U_N$ of $N$-th peak is given by (7.11). The distance between subsequent peaks along x-axis is a quasiperiodic function of $N$ given by (7.12).

We emphasize that, in contrast to the peakon traveling wave solution considered in Section 3, now the x-distance between subsequent peaks is not constant. However, as seen from (7.12), for $N \to \pm \infty$ it tends to $2\beta_1$, the doubled bigger semi-axis of the ellipse, whereas the pieces tend to identical ones corresponding to periodic billiard motion along $X_1$-axis.

Remark 8.2. Expressions (7.6), (7.8) describe an asymptotic motion of an elliptic as well as a hyperbolic billiard. In the first case the initial phase $z_{10}$ is pure imaginary whereas in the second case it is real. According to the trace formula, the hyperbolic billiard corresponds to unbounded stationary solutions of HD equation, which is out of interest of this paper.

Remark 8.3. The above considerations can be extended to multi-dimensional case. Namely, following similar approach one can consider genus $g$ solution of HD equation described by equations (1.3), then its asymptotic stationary limit which is related to a generalized Abel–Jacobi mapping including $g$ meromorphic differentials of 3rd kind. Consider a billiard inside a $g$-dimensional ellipsoid. Then such a limit solution corresponds to asymptotic billiard trajectories that intersect $g - 1$ focal quadrics of the ellipsoid between any subsequent impacts. The resulting stationary solution
$U(x,t_0)$ consists of an infinite series of peaks and between each subsequent peaks there are $g - 1$ knots.

In order to study time-dependent soliton-peakon solutions, we consider the system (4.15) under the limits (7.1) without changing the variable $x$. As a result, we arrive at

$$
\frac{d\mu_1}{2(\mu_1 - b)\sqrt{a - \mu_1}} + \frac{d\mu_2}{2(\mu_2 - b)\sqrt{a - \mu_2}} = dt,
$$

$$
\frac{d\mu_1}{2\sqrt{a - \mu_1}} + \frac{d\mu_2}{2\sqrt{a - \mu_2}} = dx - b dt.
$$

(7.13)

The latter equations include one differential of third kind $\Omega_2 = \frac{\beta_2 dz}{(\mu - b)^2}$ having simple poles $Q^+_2$, $Q^-_2$ and one differential of second kind $\Omega^{(1)}_\infty$ having a double pole at infinity on the Riemann surface $\mathbb{P} = \{\xi^2 = a - \mu\}$. Consider the mapping

$$
\int_{P_0}^{P_1} \Omega^{(1)}_\infty + \int_{P_0}^{P_2} \Omega^{(1)}_\infty = z_1, \quad \int_{P_0}^{P_1} \Omega_2 + \int_{P_0}^{P_2} \Omega_2 = z_2, \quad P_i = (\mu_i, \xi_i),
$$

(7.14)

where, $z_1 = \beta_2 t + Z_{z_1}, \ z_2 = x + bt + z_{20}, \ z_{10}, z_{20} = \text{const}$ and, as above, $P_0 = (a, 0)$. Integrating it explicitly, we obtain

$$
\frac{(\xi_1 - \beta_2)(\xi_2 - \beta_2)}{(\xi_1 + \beta_2)(\xi_2 + \beta_2)} = e^{z_2}, \quad \xi_1 + \xi_2 = z_1.
$$

(7.15)

Inverting these relations yields the following formal solution for the HD equation

$$
U(x,t) = \mu_1 + \mu_2 = (\xi_1 + \xi_2)^2 - 2\xi_1\xi_2 = z_1^2 - 2\beta_2(z_2 - z_1)\frac{e^{z_2/2} + e^{-z_2/2}}{e^{z_2/2} - e^{-z_2/2}},
$$

$$
\beta_2 + \xi_2 = z_1.
$$

(7.16)

It is seen that $U$ depends on $x$ rationally (quadratically, as already mentioned above) and $U$ is unlimited as $x \to \pm \infty$.

However, this solution does not take into account the reflection phenomenon described above: when the variable $\mu_1$ vanishes, the corresponding point $P_1 \in \mathbb{P}$ jumps from the pole $Q^+_2$ of the differential $\Omega_2$ to $Q^-_2$. According to mapping (7.14), this results in changing the phases $z_1, z_2$ in (7.10) by the constants

$$
\int_{Q^-_2}^{Q^+_2} \Omega^{(1)}_\infty = 2\beta_1, \quad \text{respectively} \quad q = \int_{Q^-_2}^{Q^+_2} \Omega_2 = 2 \log \frac{\beta_1 - \beta_2}{\beta_1 + \beta_2},
$$

(7.17)

the latter being already defined in (7.4). It follows that the actual solution $U(x,t)$ to HD equation consists of an infinite number of pieces described by (7.16) with

$$
\int_{Q^-_2}^{Q^+_2} \Omega^{(1)}_\infty = 2\beta_1, \quad \text{respectively} \quad q = \int_{Q^-_2}^{Q^+_2} \Omega_2 = 2 \log \frac{\beta_1 - \beta_2}{\beta_1 + \beta_2},
$$

and glued along peak lines $\{x = q_N(t)\}$ in the plane $(x,t)$, where for a fixed time $t$, the function $q_N(t)$ gives $x$-coordinate of $N$-th peakon. Now if we assume $z_{20}$ to be imaginary and $z_{10}$ real, the series of pieces will provide a real bounded peakon solution.

The functions $q_N(t)$ can be found as follows. Along the peak lines one of the variables $\mu$, say $\mu_1$, vanishes implying $d\mu_1 = 0, \ \xi_1 \equiv \beta_2$. Putting this into (7.13), we find

$$
\frac{\xi_2 - \beta_2}{\xi_2 + \beta_2} = e^{z_2 - q/2}, \quad \beta_2 + \xi_2 = z_1.
$$
Substituting here (7.17), putting \( x = q_N(t) \), and eliminating \( \xi_2 \), we obtain the sought expression

\[
q_N(t) = -bt - z_{10} - 2\beta_1 N + \beta_1 + \beta_2 \frac{1 + e^{z_2 - q/2}}{1 - e^{z_2 - q/2}}, \quad z_2 = \beta_2 t + Nq + z_{20}.
\]

It follows that for \( |t| \gg 1 \) any peak moves with constant velocity \(-b\), and as \( t \) evolves from \(-\infty\) to \( \infty \) the peaks undergo the phase shift \( x \to x - 2\beta_2 \).
Bibliography.

Abenda, S. and Fedorov, Yu [1999], On the weak Kowalewski–Painlevé property for hyperelliptically separable systems, *Acta Appl. Math.* (to appear).

Ablowitz, M.J. and Segur, H [1981], Solitons and the Inverse Scattering Transform, SIAM, Philadelphia.

Alber, M.S. and Alber, S.J. [1985], Hamiltonian formalism for finite-zone solutions of integrable equations, *C. R. Acad. Sc. Paris* 301, 777-781.

Alber, M.S., R. Camassa, D.D. Holm and J.E. Marsden [1994], The geometry of peaked solitons and billiard solutions of a class of integrable pde’s, *Lett. Math. Phys.* 32 137-151.

Alber, M.S., Camassa, R., Holm, D.D., and Marsden, J.E. [1995], On the link between umbilic geodesics and soliton solutions of nonlinear PDE’s, *Proc. Roy. Soc* 450 677-692.

Alber, M.S., R. Camassa, Y. Fedorov, D.D. Holm and J.E. Marsden [1999], On Billiard Solutions of Nonlinear PDE’s, *Phys. Lett. A* 264 171–178.

Alber, M.S., and C. Miller [1999], On Peakon Solutions of the Shallow Water Equation, *Appl.Math. Lett.* (to appear).

Alber, M.S., Camassa, R., Fedorov, Yu., Holm, D.D., and Marsden J.E. [1999], The geometry of new classes of weak billiard solutions of nonlinear PDE’s. (subm.)

Belokolos, E.D., A.I. Bobenko, V.Z. Enol’sii, A.R. Its, and V.B. Matveev [1994] *Algebro-Geometric Approach to Nonlinear Integrable Equations*. Springer-Verlag series in Nonlinear Dynamics.

Beals, R., D.H. Sattinger, J. Szmigielski [1999], Multi-peakons and a theorem of Stietjes, Inverse Problems 15 L1–L4.

Beals, R., D.H. Sattinger, J. Szmigielski [2000], Multipeakons and the Classical Moment, Advances in Mathematics (to appear)

Camassa, R. and Holm, D.D. [1993], An integrable shallow water equation with peaked solitons, *Phys. Rev. Lett.*, 71, 1661-1664.

Camassa, R., Holm, D.D., and Hyman, J.M. [1994], A new integrable shallow water equation. *Adv. Appl. Mech.*, 31, 1–33.

Cewen, C. [1990], Stationary Harry-Dym’s equation and its relation with geodesics on ellipsoid, *Acta Math. Sinica* 6, 35–41.

Clebsch, A., Gordan, P. [1866], Theorie der abelschen Funktionen. Teubner, Leipzig

Comessatti, A. and Sulle varietá abeliane reale I. *Ann. Math. pure Appl.*, 2, (1924) 67–106.

Dmitrieva, L.A. [1993a] Finite-gap solutions of the Harry Dym equation, *Phys.Lett.A* 182 (2) 65–70.

Dmitrieva, L.A. [1993b], The higher -times approach to multisoliton solutions of the Harry Dym equation, *J.Phys.A* 26 6005–6020.
Dubrovin, B. [1981], Theta functions and nonlinear equations. *Russ. Math. Surveys*, **36**, 11–92.

Dubrovin, B.A., Novikov S.P., Krichiver, I.M. [1985] *Integrable Systems. I. Itogi Nauki i Tekhniki. Sovr.Probl.Math. Fund.Naprav. 4*, VINITI, Moscow. English transl.: Encyclopaedia of Math.Sciences, Vol. 4, Springer-Verlag, Berlin 1989.

Ercolani, N. [1989], Generalized theta functions and homoclinic varieties, *Proc. Symp. Pure Appl. Math.*, **49**, 87–100.

Fedorov, Yu. [1999], Classical integrable systems and billiards related to generalized Jacobians, *Acta Appl. Math.*, **55**, 3, 151–201

Gagnon, L., Harnad, J., Hurtubise, J. and Winternitz, P. [1985], Abelian integrals and the reduction method for an integrable Hamiltonian system, *J.Math.Phys.* **26**, 1605–1612.

Gavrilov, L. [1999], Generalized Jacobians of spectral curves and completely integrable systems, *Math.Z.* (to appear).

Gunning, R. [1972], Lectures on Riemann Surfaces. Jacobi varieties. Princeton University Press.

Hunter, J.K., and Zheng, Y.X. [1994], On a completely integrable nonlinear hyperbolic variational equation, *Physica D* **79**, 361–386.

Li, Y.A. and Olver P.J. [1998], Convergence of solitary-wave solutions in a perturbed bi-Hamiltonian dynamical system. *Discrete and continuous dynamical systems*, **4**, 159–191.

Markushevich, A. I. [1977], Theory of Functions of a Complex Variable, Chelsea Publishing Company: New York.

McKean, H.P. and A. Constantin [1999], A Shallow Water Equation on the Circle, *Comm.Pure Appl.Math. Vol LII* 949–982.

Mumford, D. [1983], Tata Lectures on Theta II, Birkhauser-Verlag.

Novikov, D.P. [1999], Algebraic geometric solutions of the Harry Dym Equations, *Siberian Math. J.* **40** 136–140.

Previato E. [1985], Hyperelliptic quasi-periodic and soliton solutions of the nonlinear Schrödinger equation *Duke Math. J.* **52**, I, 329–377.

Vanhaecke, P. [1995], Integrable systems and symmetric products of algebraic curves, *Math. Z.* **40** 143–172.

Weierstrass, K. [1878], Über die geodätischen Linien auf dem dreiachsigen Ellipsoid, Mathematische Werke I, 257–266.

Wadati M., Konno, H., Ichikawa, Y.H. [1979], *J. Phys. Soc. Japan* **47**, 1698.