Solution of motion parameters of natural fragments based on monte carlo subdivision projection simulation

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Abstract. In order to get the average windward area of natural fragments of non-metallic materials more accurately and efficiently, and to further simulate the motion parameters, a simulation calculation model of the average windward area of natural fragments is established based on the Monte Carlo subdivision projection method. By the finite element modeling, coordinate translation, random rotation, plane projection and triangulation, the average windward area of natural fragments with arbitrary irregular shape is simulated. Then, based on aerodynamics, the dispersion model of natural fragments is established. The simulation results of the two-dimensional trajectory and the farthest distance of natural fragments are obtained by Runge-Kutta method and cubic spline interpolation, which are compared with the empirical formula of the average windward area of regular fragments. The results show that the value by the Monte Carlo split projection simulation are more consistent with the experimental value, compared with the empirical formula of the windward area of spherical fragment, cube fragment, short cylinder fragment, cylinder fragment, rhombus fragment and cuboid fragment. The error of the farthest distance is between 13% and 25%. It is proved that the result of the average windward area based on Monte Carlo Subdivision Projection Simulation is more suitable for solving the dispersion model of natural fragments of low-speed and non-metallic materials.

1. Introduction
The shell of traditional warhead is mainly made of metal materials, which will produce a large number of metal fragments and form a large killing area after explosion. In the field of long-distance flight, velocity attenuation and average windward area of warhead fragments, relevant scholars have started theoretical and experimental research for a long time. Based on the experimental data, empirical formulas of windward area of high-speed and regular fragments have been computed, such as cylinder, sphere, cuboid and rhombus [1,2]. However, the above research mainly focuses on regular fragments with high initial velocity, while the research on natural fragments with low velocity and non-metal is relatively less, such as the shell of stun grenade used for anti-terrorism and stability maintenance. Currently, the average windward area of natural fragments is mainly solved by the test method of uniform orientation theory [3-9], and the simulation research on the average windward area of natural fragments has not been reported. The establishment and improvement of simulation calculation model of average windward area and the dispersion model of non-metallic natural fragments is not only an
important way to realize the high efficiency and automatic evaluation of non-lethal characteristics of fragments, but also an objective requirement for the optimal design and safe application of ammunition.

In this paper, based on Monte Carlo subdivision projection simulation (MC-SPS for short), a simulation model of the average windward area of natural fragments is established. Through the finite element modeling, coordinate translation, random rotation, plane projection and triangulation, the average windward area of natural fragments with arbitrary irregular shape is obtained. Based on aerodynamics, a natural fragment dispersion model is established, and the average windward area obtained from MC-SPS is applied to the solution of the dispersion model. By using the fourth-order and fifth-order Runge-Kutta (RK4 for short) method and the cubic spline interpolation algorithm, the simulation results of the two-dimensional dispersion trajectory and the farthest dispersion distance of fragments can be obtained.

2. Modeling

2.1. Calculation model of average windward area based on MC-SPS

2.1.1. Model analysis and establishment. Due to the irregular shape of natural fragments, the flight attitude will rotate continuously in space, and the windward area at any point of the flight trajectory is a random quantity. Therefore, the average windward area is often used to characterize the windward area of the whole fragment trajectory. For regular fragments such as sphere, cube, cylinder, rhombus, rectangle, etc., the empirical formula of windward area can be derived from the test [2], that is

\[ S = K \phi m_f^{2/3} \]  

(1)

Where, \( \phi \) is the fragment shape coefficient; \( m_f \) is the fragment mass; \( S \) is the average windward area of regular fragment; \( K \) is the correction coefficient, which is usually 1.08 to 1.12 for regular fragments. The shape factor \( \phi \) value of steel fragment is shown in table 1. And the average windward area of irregular fragment is usually calculated by the measurement of optical projection system and the theory of icosahedron uniform orientation.

| Fragment shape | \( \phi / m_f^{2/3} \cdot kg^{-2/3} \) |
|---------------|---------------------------------|
| Spherical     | \( 3.07 \times 10^{-3} \)         |
| Cubic         | \( 3.09 \times 10^{-3} \)         |
| Cylindrical   | \( 3.35 \times 10^{-3} \)         |
| Rhombic       | \( 3.2 \sim 3.6 \times 10^{-3} \) |
| Rectangular   | \( 3.3 \sim 3.8 \times 10^{-3} \) |

Table 1. Shape factor \( \phi \) value of steel fragment.

In order to further obtain the average windward area of arbitrary irregular fragments, the simulation method of the MC-SPS is proposed.

Firstly, the finite element model of the natural fragments is established to obtain the node coordinates of the segmentation elements of the fragments. The test fragments 1 to 5 and their finite element models is shown in figure 1. The node number of test fragments 1 to 5 is 13081, 8977, 7721, 9028 and 2495 respectively, and the average element size is 1mm to 2mm.
Then, the mean of projection area $A_{ji}$ can be obtained by translation, random rotation, projection and triangulation of the node coordinates of N times. When the N tends to infinity, the mean value can be approximated as the average windward area of the fragment, which can be defined as follows:

$$ A_j = \lim_{N \to \infty} \frac{1}{N} \sum_{i=1}^{N} A_{ji} $$

Where, $A_j$ is the average windward area of the jth natural fragment; $A_{ji}$ is the windward area of the jth fragment after the ith MC-SPS solution; N is the number of MC-SPS.

The schematic diagram of translation, random rotation and plane projection of fragment coordinates is shown in figure 2. The area of the plane projection nodes needs to be solved by triangulation algorithm.

2.1.2. Translation, random rotation and projection of fragment node coordinates. Coordinate translation: As shown in figure 2(b), the computation load of coordinate matrix in random rotations can be greatly reduced by establishing local coordinate system, which translating the origin of the global coordinate system to the centroid coordinate of the fragment.

Random rotation: As shown in figure 2(c), the random angular variables $(\alpha_{ji}, \beta_{ji}, \gamma_{ji})$ rotating around three coordinate axes are obtained by Monte Carlo method, which obey the uniform distribution of $[0, 2\pi]$. And according to the linear transformation of matrix rotation, coordinates of any node of fragment rotating around three local axes in turn can be deduced as follows:

$$
\begin{bmatrix}
    x'_{\text{ref}} \\
    y'_{\text{ref}} \\
    z'_{\text{ref}}
\end{bmatrix} =
\begin{bmatrix}
    \cos \gamma_{ji} & -\sin \gamma_{ji} & 0 \\
    \sin \gamma_{ji} & \cos \gamma_{ji} & 0 \\
    0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
    \cos \beta_{ji} & 0 & \sin \beta_{ji} \\
    0 & 1 & 0 \\
    -\sin \beta_{ji} & 0 & \cos \beta_{ji}
\end{bmatrix}
\begin{bmatrix}
    \cos \alpha_{ji} & 0 & -\sin \alpha_{ji} \\
    0 & 1 & 0 \\
    \sin \alpha_{ji} & 0 & \cos \alpha_{ji}
\end{bmatrix}
\begin{bmatrix}
    x' \\
    y' \\
    z'
\end{bmatrix}
$$

Figure 1. Test fragments 1 to 5 and their finite element models.

Figure 2. Schematic diagram of translation, random rotation and plane projection of fragment coordinates.
In the formula, \((x', y', z')\) is the nodes coordinates after fragment translation; \((\alpha_{ji}, \beta_{ji}, \gamma_{ji})\) is the random angular variables of the jth fragment rotating around the three local axes at the ith time; 
\((x'_{rot}, y'_{rot}, z'_{rot})\) is the nodes coordinates after random rotation of fragment.

**Plane projection:** as shown in figure 2(d), the rotated nodes coordinates are projected to a plane. Set the two-dimensional coordinates points on the projection plane as \((x_{proj}, y_{proj}, z_{proj})\). According to the point-normal equation, the parallel principle of normal vector and projection vector, the following expressions can be derived:

\[
\begin{align*}
x_{s} \left( x_{proj} - x_{0} \right) + y_{s} \left( y_{proj} - y_{0} \right) + z_{s} \left( z_{proj} - z_{0} \right) &= 0 \\
y_{proj} &= \frac{x_{s}}{x_{s}} \left( x_{proj} - x'_{rot} \right) + y'_{rot} \\
z_{proj} &= \frac{z_{s}}{x_{s}} \left( x_{proj} - x'_{rot} \right) + z'_{rot}
\end{align*}
\]

(4)

Where, \((x_{proj}, y_{proj}, z_{proj})\) is the node coordinate after projection; \((x_{0}, y_{0}, z_{0})\) is the coordinate of a point on the projection plane; \((x_{s}, y_{s}, z_{s})\) is the normal vector of the projection plane.

2.1.3. Delaunay triangulation for projection area. Because the polygon figure enclosed by the fragment projection points set is unpredictable in advance, in order to obtain the area of the plane discrete point set, the Delaunay triangulation method is proposed for solving the problem. For the area of convex hull, Delaunay triangulation can be directly used for calculation. For the area of concave polygon, invalid triangular edges in the convex hull which exceed the constraint value \(R\) need to be subtracted, and the algorithm steps are as follows:

**Step 1:** Triangulate discrete points set of concave polygon with Delaunay triangulation;

**Step 2:** Initialize the triangle mesh edge of points set, and calculate the length of each edge and the number of adjacent triangles. Among them, the edge of two adjacent triangles is the internal edge, the edge of one triangle is the boundary edge, and the edge without triangle is the degenerate edge in the calculation process;

**Step 3:** Add all boundary edges whose length is greater than the constraint value \(R\) to the queue, and loop the following process when the queue is not empty: take one edge \(E\) from the queue, and get the unique adjacency triangle \(T\) of \(E\); find the other two sides \(E_{1}, E_{2}\) in \(T\), delete their adjacency triangle set \(T\); add the newly formed boundary edges whose length is greater than \(R\) in \(E_{1}\) and \(E_{2}\) to the queue; set \(E\) as the invalid mark, and if \(E_{1}\) and \(E_{2}\) exist the degenerate edges, set as \(E_{1}\) and \(E_{2}\) invalid mark. Collect all the effective boundary edges to form the edge list of concave graph points set, and further obtain the area of the enclosed graph.

The results of \(N\) times projection area of the jth fragment can be obtained by cycling, and the average windward area of the fragment can be obtained by substituting the results into formula (2). With the increase of the total number \(N\), the average windward area will gradually converge to a certain value.

2.2. Dispersion model

2.2.1. Model analysis and establishment. When the shell is broken, with the continuous attenuation of the shock wave, the thrust on the fragments will gradually weaken until it no longer plays a significant role. At this time, the natural fragments fly forward freely along their initial azimuth and initial angle of fire, which are affected by the air resistance and their own gravity. During this period, due to the
moment of rotation formed by the center of pressure and the center of mass, the flying attitude continuously rotates in space until the fragment contacts the target or lands.

According to the above model, each fragment can be regarded as the motion of the center of mass under the action of air resistance, self gravity and rotation moment. It is assumed that the flight path of the center of mass is always in the same plane, that is, the azimuth angle is constant, and the air resistance direction is always in the opposite direction of the velocity vector of the fragment mass center. The force analysis of any point on fragment 2D trajectory is shown in figure 3.

![Figure 3. Schematic diagram of force decomposition along tangential and normal directions.](image)

According to Newton's second law, the transformation relationship of velocity, acceleration and included angle, the ordinary differential equations of the velocity, horizontal angle, horizontal displacement and vertical displacement of the jth fragment in its two-dimensional plane can be derived by force decomposition along the tangent and normal directions of the point [10]:

\[
\begin{align*}
\frac{dv_j}{dt} &= -\frac{1}{2}C_d\rho S_j v_j^2 - g \sin \theta_j \\
\frac{d\theta_j}{dt} &= -\frac{\cos \theta_j}{v_j} \\
\frac{dx_j}{dt} &= v_j \cos \theta_j \\
\frac{dy_j}{dt} &= v_j \sin \theta_j
\end{align*}
\]

(5)

Where, \( C_d \) is the air resistance coefficient, which can be solved according to the logistic fitting curve and resistance law [11]; \( \rho \) is the density of fragment material, which is 1050kg/m3; \( S \) is the average windward area of irregular fragment, which can be solved according to MC-SPS; \( g \) is the local gravity acceleration.

The initial conditions of each fragment can be obtained according to the high-speed photography system [12,13], which are shown in table 2. In addition, the constraint condition needs to be set to \( y_{ja} \geq 0 \).

| Fragment number | \( v_0/\text{m} \cdot \text{s}^{-1} \) | \( \theta_0/^{\circ} \) | \( x_0/\text{m} \) | \( y_0/\text{m} \) | \( m/\text{kg} \) |
|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|
| #1              | 212             | 23              | 1               | 2.46            | 0.0037          |
2.2.2. Model solution. For the solution of equation (5), the RK4 algorithm can be used preferentially because the equation (5) is a nonlinear Nonstiff ordinary differential equation. RK4 method uses difference instead of integral. On the premise of knowing the derivative and initial value of the equation, the discrete numerical solutions of each variable of ordinary differential equations with high accuracy can be obtained by programming iteration. The specific expression is as follows:

\[ y_{n+1} = y_n + \frac{h}{6}(k_1 + 2k_2 + 2k_3 + k_4) \]  

(6)

\[ k_1 = f(t_n, y_n) \]  

(7)

\[ k_2 = f\left(t_n + \frac{h}{2}, y_n + \frac{h}{2}k_1\right) \]  

(8)

\[ k_3 = f\left(t_n + \frac{h}{2}, y_n + \frac{h}{2}k_2\right) \]  

(9)

\[ k_4 = f\left(t_n + h, y_n + hk_3\right) \]  

(10)

Where, \( y_{n+1} \) is the current iteration results, which is determined by the previous iteration \( y_n \), step size \( h \) and estimated slope. In the estimated slope, \( k_1 \) is the slope at the beginning of the time period; \( k_2 \) is the slope at the midpoint of the time period, which is determined by the value of \( h \) at the point; \( k_3 \) is also the slope at the midpoint, which is determined by the value of \( h \) at the point; \( k_4 \) is the slope at the end of the time period, which is determined by \( k_1 \) and \( h \).

The equations (6) to (10) are substituted into the equations (5), and the motion parameter matrix of each fragment in the two-dimensional plane can be obtained finally according to the initial conditions and constraint condition. And then, the complete two-dimensional flight path and v-x curve can be obtained by spline interpolation.

3. Result analysis and discussion

3.1. Average windward area

In order to verify the accuracy of each fragment points set in the process of translation, rotation and projection, an MC-SPS process of fragment 1 was analyzed, as shown in the figure 4. It can be seen from the figure that the polygon area enclosed by convex hull or concave domain points set can be more accurately obtained.
In order to analyze the distribution of MC-SPS results, 20000 MC-SPS results of each fragment were analyzed, as shown in figure 5. It can be seen from the figure that the distribution of random projected area obtained by MC-SPS method is normal, which is consistent with the distribution law under the law of large numbers.
Figure 5. Frequency histogram and probability density distribution function of fragments 1 to 5 after 20,000 times MC-SPS method.

In order to further verify the influence of projection times on convergence of average area, the results of MC-SPS are recorded at different times, as shown in figure 6. It can be seen from the figure that the average windward area gradually converges to a certain value as the number of times increases. When the number of times is more than 5000, the random number no longer has a significant impact on the results.

In this paper, 20000 times MC-SPS results are selected as the average windward area of each fragment, which are shown in table 3.

Table 3. The average windward area of fragments 1 to 5 by 20000 times MC-SPS.

| Fragment shape | $\bar{S} / m^2$   |
|----------------|-------------------|
| 1#             | $1.473 \times 10^{-3}$ |
| 2#             | $1.090 \times 10^{-3}$ |
| 3#             | $8.598 \times 10^{-4}$ |
| 4#             | $6.957 \times 10^{-4}$ |
| 5#             | $3.384 \times 10^{-4}$ |
3.2. Dispersion trajectory and farthest distance

Compare the two-dimensional trajectory obtained by MC-SPS method with the two-dimensional trajectory obtained by empirical formula of windward area of spherical fragment, cube fragment, short cylinder fragment, cylinder fragment, diamond fragment and cuboid fragment, as shown in Figure 7. It can be seen from the figure that the farthest distance of two-dimensional trajectory obtained by the empirical formula of regular fragments is quite different from that obtained by MC-SPS under the same initial conditions; the farthest distance of MC-SPS is about 20m, diamond and rectangular fragments are more than 70m, cylindrical and square fragments are more than 80m, and spherical fragments are more than 130m; the farthest distance of diamond fragments and rectangular fragments are relatively similar, and the results of cube, short cylindrical and cylindrical fragments are similar.

In order to verify the correctness of MC-SPS, the farthest flying distances of MC-SPS and empirical formula are recorded and compared with the test results, as shown in Table 4. It can be seen from the table that the result of MC-SPS is closer to the test value of PVC shell fragment, while the results of empirical formula of regular fragment are quite different from the test results. The errors between the results of MC-SPS and the test results are 13.3%, 17.8%, 18.7%, 21.3%, 24.7% respectively. There are two main reasons for the error. The one is that the average windward area calculation model with MC-SPS is established by the limit theory when the projection times tend to infinity. In the real case, the rolling times of fragment are relatively less than MC-SPS method, thus the average projected area is affected by randomness, which causes error. The other one is that it is an approximate method to use the projected area instead of the windward area. The real windward area of fragment is more complex.
Figure 7. Comparison of two-dimensional trajectory obtained by MC-SPS method and empirical formula of regular fragments.

Table 4. Initial conditions of fragment 1 to 5.

| Fragment number | Spherical fragment | Cube fragment | Short cylinder fragment | Cylinder fragment | Rhombus fragment | Rectangular fragment | Natural fragment | Experiment fragment |
|-----------------|-------------------|---------------|-------------------------|------------------|------------------|---------------------|------------------|----------------------|
| #1              | 132.40            | 91.69         | 93.23                   | 89.03            | 77.09            | 73.28               | 7.77             | 6.86                 |
| #2              | 151.65            | 107.58        | 108.66                  | 109.54           | 90.03            | 87.82               | 9.71             | 8.24                 |
| #3              | 148.28            | 107.01        | 107.90                  | 109.28           | 91.08            | 87.30               | 11.65            | 9.81                 |
| #4              | 156.26            | 116.31        | 119.30                  | 120.65           | 101.19           | 97.33               | 13.28            | 10.95                |
| #5              | 149.40            | 122.38        | 127.70                  | 128.95           | 105.67           | 102.28              | 21.44            | 17.20                |

4. Conclusion

Based on the MC-SPS method, the motion parameters of the natural fragments are in good agreement with the test results, and the error of the farthest dispersion distance is between 13% and 25%. The results of the empirical formula of regular fragments are quite different from those of the experiment, which is not suitable for solving the motion parameters of natural fragments. The process of solving the motion parameters is based on the combination of finite element method and programming method, and the average windward area of natural fragments of non-metallic materials can be obtained more efficiently.

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