On The Communication Complexity of Finding an (Approximate) Stable Marriage

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Abstract

In this paper, we consider the communication complexity of protocols that compute stable matchings. We work within the context of Gale and Shapley’s original stable marriage problem\(^3\): \(n\) men and \(n\) women each privately hold a total and strict ordering on all of the members of the opposite gender. They wish to collaborate in order to find a stable matching—a pairing of the men and women such that no unmatched pair mutually prefer each other to their assigned partners in the matching. We show that any communication protocol (deterministic, nondeterministic, or randomized) that correctly outputs a stable matching requires \(\Omega(n^2)\) bits of communication. Thus, the original algorithm of Gale and Shapley is communication-optimal up to a logarithmic factor. We then introduce a “divorce metric” on the set of all matchings, which allows us to consider approximately stable matchings. We describe an efficient algorithm to compute the “distance to stability” of a given matching. We then show that even under the relaxed requirement that a protocol only yield an approximate stable matching, the \(\Omega(n^2)\) communication lower bound still holds.

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1 Introduction

Our story is the confluence of two classical narratives in theoretical computer science. First is the stable marriage problem, which was initially formalized and studied by Gale and Shapley in 1962, see [3]. Second is the theory of communication complexity introduced by Yao in 1979, see [11]. Each of these celebrated works has had tremendous impact on theoretical computer science. In this essay, we use tools from communication complexity in order to study the stable marriage problem. Our work shows that communication is a major impediment to efficiently solving an instance of the stable marriage problem. We begin with a brief overview of the stable matching problem and some results we will require from communication complexity.

The study of the communication complexity of the stable marriage problem was initiated by [10]. Their work showed that for exact computation of a stable matching, \( \Omega(n^2) \) communication is required for deterministic and nondeterministic protocols. Our generalization of this result to randomized protocols is of interest because randomized protocols can be exponentially more efficient than their deterministic or nondeterministic counterparts.

In [2], the authors consider the communication complexity of approximate stable matchings. They use an “unstable partners” notion of approximate stability. They prove a lower bound of \( \Omega(n^2 \log n) \) communication for a deterministic sketching model of communication. In their communication model, a central processor queries the players about their preferences where all computation and messages are deterministic.

In this paper we generalize the \( \Omega(n^2) \) communication lower bound of [10] to account for randomized communication protocols—see theorem 2. Next, we consider a relaxation of the stable marriage problem to protocols which compute approximate stable matchings. We define a notion of approximate stable matching (divorce distance to stability) and show that even for this relaxation an \( \Omega(n^2) \) lower bound still holds for randomized algorithms. Our communication model is more general than that considered in [2], but our notion of approximate stability is more restrictive. Nonetheless, we believe our result (theorem 5) is the first such lower bound for interactive communication with any notion of approximate stability.

1.1 Stable Marriage Problem

Let \( X \) and \( Y \) be sets of \( n \) women and \( n \) men, respectively, who wish to form couples. We denote the men \( Y = \{m_1, m_2, \ldots, m_n\} \) and the women \( X = \{w_1, w_2, \ldots, w_n\} \). A matching \( M \) is a one-to-one correspondence between men and women:

\[
M = \{(m_1, w_{i_1}), (m_2, w_{i_2}), \ldots, (m_n, w_{i_n})\}
\]

where each woman \( w_i \) appears exactly once in \( M \). We call a pair \((m, w) \in M\) a couple in \( M \). Notice that we can describe a matching \( M \) by a permutation \( \sigma : [n] \rightarrow [n] \) where \( M = \{(m_i, w_{\sigma(i)}) \mid 1 \leq i \leq n\} \).
In our model, each man and woman holds a ranking of all the players of the opposite gender. The ranking is a complete list where all preferences are strict—i.e., a linear order. We say man \( m \) prefers \( w \) to \( w' \) and write
\[
w <_m w'
\]
if \( w \) appears before \( w' \) in \( m \)'s ranking, and symmetrically for the women. Given a matching \( M \), we call a pair \((m, w)\) blocking if \((m, w) \notin M\), but \( m \) and \( w \) mutually prefer each other to their partners in \( M \). Specifically, \((m, w) \notin M\) is blocking if \((m, w'), (m', w) \in M\) with
\[
w <_m w'
\text{ and } m <_w m'.
\]
A matching \( M \) is stable if it contains no blocking pairs. The stable marriage problem is to find a stable matching given the preferences of the men and women.

Gale and Shapley showed that given any instance of the stable marriage problem (i.e., any list of preferences for the men and women) there exists a stable matching. They accomplished this by describing the following “deferred acceptance” algorithm which computes a stable matching: in the first round, the men all propose to their most preferred woman. Each woman receiving proposals rejects all but her most preferred among the men who proposed to her, but defers accepting any proposals until the end of the algorithm. In the second round, the rejected men propose to their second-most preferred women, and again the women reject all but their most favored proposal so far. The algorithm terminates when all of the women have been proposed to. At this time, all of the women accept their deferred proposals. It is straightforward to show (see [3]) that this algorithm yields a stable matching and that the process terminates in \( O(n^2) \) steps.

Notice that in an execution of the Gale-Shapley (GS) algorithm, the preferences of each man are revealed by the sequence of their proposals, while the women’s preferences are revealed by their rejections. In particular, it may be the case that a constant fraction of the men must reveal a constant fraction of their preferences. We will use communication complexity to show that any correct protocol for the stable marriage problem requires this much (\( \Omega(n^2) \)) communication.

1.2 Communication Complexity

In its simplest incarnation, communication complexity is concerned with the following problem: suppose two players, Alice and Bob wish to collaborate to compute a boolean function \( f(x, y) \), where
\[
f : X \times Y \to \{0, 1\}.
\]
Alice and Bob know \( x \) and \( y \), respectively, but each has no information about the other’s input. How many bits must Alice and Bob exchange in order to
determine $f(x, y)$? Yao first formalized the notion of a communication protocol and gave methods for proving lower bounds on communication complexity in [11]. We quantify communication complexity of $f$ by the number of bits Alice and Bob must exchange (in the worst case) to compute $f(x, y)$. For our purposes, communication protocols may be randomized. It is interesting to note that randomized protocols for a function $f$ may require exponentially less communication than their deterministic or nondeterministic counterparts. See [7] for a delightful introduction to communication complexity.

Of particular interest to us is the disjointness function: DISJ($x, y$). Alice and Bob hold subsets $A, B \subset [n]$, respectively. The value of the disjointness function is 1 if $A \cap B = \emptyset$, and 0 otherwise. We can encode DISJ as a boolean function by associating to $A$ and $B$ their characteristic vectors:

$$ x_i = 1 \iff i \in A \quad \text{and} \quad y_j = 1 \iff j \in B. $$

Thus, we can express DISJ in terms of the boolean formula

$$ \text{DISJ}(x, y) = \neg \bigvee_{i=1}^{n} (x_i \land y_i). \quad (1) $$

We will rely heavily on the following result:

**Theorem 1** ([6], [9]). Any (randomized) protocol which correctly computes DISJ($x, y$) with probability at least $2/3$ requires $\Omega(n^2)$ communication. In fact, this lower bound holds even if we require that inputs $x$ and $y$ are either disjoint or uniquely intersecting: $|x \cap y| \in \{0, 1\}$.

Our results on communication lower bounds will follow from defining suitable embeddings of DISJ into instances of the stable marriage problem such that finding a stable matching reveals the value of DISJ. We will assume that the input to DISJ satisfies the promise that $|x \cap y| \in 0, 1$.

## 2 Lower Bound for Exactly Stable Matches

The goal of this section is to prove the following theorem.

**Theorem 2.** Any protocol (deterministic, nondeterministic, or randomized) that for any set of preferences finds a stable matching requires $\Omega(n^2)$ bits of communication between the men and women.

Notice that up to a logarithmic factor, this lower bound is tight. Indeed, the players of each gender hold a total of $n$ sets of preferences, where each preference list is a permutation of $n$. Thus there are $(n!)^{2n}$ distinct stable marriage instances of size $n$. Therefore, specifying a single instance requires

$$ \log((n!)^{2n}) = \Theta(n^2 \log n) $$

bits in any reasonable encoding. The trivial communication protocol of, say, the men sending all of their preferences then uses $O(n^2 \log n)$, which up the the factor of $\log n$ matches our lower bound.
2.1 The Special Case \( n = 2 \)

As mentioned before, we will prove theorem 2 by embedding any instance of DISJ into a stable marriage problem such that the structure of a stable matching reveals the value of DISJ. Before describing the embedding in full generality, we look at the special case of DISJ where Alice and Bob each hold a single bit. Thus DISJ\((x, y)\) = 1 if and only if at least one of the bits is 0. In this case, we will have \( n = 2 \) men and women. We use Alice’s bit to determine the women’s preferences, and Bob’s bit to determine the men’s preferences. Here is our embedding for the women’s preferences:

| player | preferences if \( x = 1 \) | preferences if \( x = 0 \) |
|--------|---------------------------|---------------------------|
| \( w_1 \) | 1, 2                      | 2, 1                      |
| \( w_2 \) | 1, 2                      | 1, 2                      |

The men’s preferences are identical replacing \( w \)s with \( m \)s and \( x \) with \( y \). It is easy to show (by brute force) that for these preferences the unique stable matching is \( M = \{(1, 1), (2, 2)\} \) if \( x = y = 1 \) and \( M = \{(1, 2), (2, 1)\} \) otherwise. In particular, the (unique) stable matching contains the couple \((1, 1)\) if and only if DISJ\((x, y)\) = 0.

2.2 The General Case

We are now ready to describe how to construct the women’s preferences from a vector \( x \) whose entries are indexed by the \((i, j)\) with \( 1 \leq i, j \leq n \). The preference structure for \( w \) depends on her parity.

**even women** (\( w = w_{2j} \)): In this case \( w \)’s preferences are independent of \( x \).

Her preferences are

\[
m_1 <_w m_3 <_w \cdots <_w m_{2i-1} <_w \cdots <_w m_{n-1} <_w m_n <_w m_1 <_w m_3 <_w \cdots <_w m_n.
\]

In particular, \( w \) prefers all odd men to even men.

**odd women** (\( w = w_{2j-1} \)): In this case \( w \)’s preferences are assigned according to the following groups:

1. Odd men \( m_{2i-1} \) such that \( x_{(i,j)} = 1 \)
2. Even men \( m_{2i} \)
3. Odd men \( m_{2i-1} \) such that \( x_{(i,j)} = 0 \)

Then \( w \) prefers all men in group 1 to group 2, and all men in group 2 to group 3. Within the groups, \( w \)’s preferences are increasing in \( i \). That is \( m_i <_w m_{i'} \) if and only if \( i < i' \).

The preferences for the men are determined analogously to the women’s preferences: just replace \( w \)s with \( m \)s and \( x \) with \( y \).
2.3 Proof of Theorem 2

Theorem 2 will follow immediately from the following claim. We say that a couple \((m, w) \in M\) is **odd** if \(m = m_{2i+1}\) and \(w = w_{2j+1}\) for some \(i, j \in \mathbb{N}\).

**Claim 3.** Assume that \(x, y \in \{0, 1\}^{n^2/4}\) satisfy \(|x \cap y| \in \{0, 1\}\). Given the preferences described above, a stable matching \(M\) contains an odd couple if and only if \(\text{DISJ}(x, y) = 0\).

**Proof.** We first consider the case where \(\text{DISJ}(x, y) = 0\). That is, there is a unique index \((i, j)\) such that \(x_{(i, j)} = y_{(i, j)} = 1\). We will show that \((m_{2i-1}, w_{2j-1}) \in M\) for any stable matching. To this end, suppose to contrary that \((m_{2i-1}, w_{2j-1}) \notin M\) for some stable matching \(M\). Since \(M\) is stable, \((m_{2i-1}, w_{2j-1})\) is not a blocking pair. Without loss of generality assume that \(w_{2j-1}\) is matched with someone she prefers to \(m_{2i-1}\). Since \(x_{(i, j)} = 1\), the only men she could prefer to \(m_{2i-1}\) are \(m_{2i'}-1\) with \(x_{(i', j)} = 1\) and \(i' < i\). Since \((i, j)\) is the unique element in the intersection of \(x\) and \(y\), we must have \(y_{(i', j)} = 0\), hence \(m_{2i'-1}\) prefers all even women to \(w_{(2j-1)}\). Since there are \(n/2\) even women, \(n/2\) odd men and \(m_{2i'-1}\) is matched with an odd woman, some even woman \(w_{2j'}\) is matched with an even man. But even women prefer all odd men to even men. Therefore \((m_{2j'-1}, w_{2j'})\) forms a blocking pair, contradicting the assumption that \(M\) was stable. Therefore, \((m_{2i-1}, w_{2j-1})\) is contained in every stable matching. In particular, every stable matching contains an odd couple.

We now consider the case where \(\text{DISJ}(x, y) = 1\). We will show that no stable matching \(M\) contains an odd couple. Suppose to the contrary that \((m_{2i-1}, w_{2j-1}) \in M\) for some \((i, j)\). Since \(x\) and \(y\) are disjoint, \(x_{(i, j)} = 0\) or \(y_{(i, j)} = 0\). Without loss of generality, assume that \(x_{(i, j)} = 0\), so that \(w_{2j-1}\) prefers all even men to \(m_{2i-1}\). As before, some even man \(m_{2i'}\) is paired with an even woman. But then \(w_{2j-1}\) and \(m_{2i'}\) mutually prefer each other to their partners in \(M\), hence \(M\) is not stable. Therefore, no stable marriage contains an odd couple.

**Proof of Theorem 2** Suppose that a protocol \(\Pi\) which given an instance of the stable marriage problem finds a stable matching using \(B\) bits of communication. Then given an instance of \(\text{DISJ}\) of size \(n/2\), using the embedding described above, we can compute \(\text{DISJ}(x, y)\) using \(B\) bits of communication: when \(\Pi\) terminates, the men and women can both detect whether or not there is an odd couple. By \(B = \Omega(n^2)\), as desired.

**Remark.** In the above proof, we only account for communication between the men and women, thus allowing, for example, the men to communicate among each other free of charge. In the distributed setting, the preferences of each player would be known only to that player. Yet our result shows that allowing collaboration among the men and women still yields the near-optimal \(\Omega(n^2)\) lower bound.
3 Approximately Stable Matchings

Since we have proven that finding a stable matching essentially requires exchanging the entire input, we ask the following natural question: are there more efficient protocols for the stable marriage problem if we are only required to produce an “approximately stable” matching? Before addressing this question, we must define what it means for a matching to be “approximately stable.” There is no consensus in the literature on how to measure the instability of a matching. We use a measure which we refer to as divorce distance.

**Definition 4.** For any pair of matchings $M, M'$ of size $n$, we define the **divorce distance** between $M$ and $M'$ to be

$$d(M, M') = n - |M \cap M'|.$$ 

Note that $d$ measures the minimum number of divorces required to convert $M$ to $M'$ (or vice versa). By abuse of notation, we denote the **divorce distance to stability** of a matching $M$ to be

$$d(M) = \min_{M' \in \mathcal{M}} d(M, M')$$

where $\mathcal{M}$ is the set of all stable matchings for a given instance of the stable marriage problem. Thus $d(M)$ is the minimum number of divorces required to convert $M$ into a stable matching.

We find the divorce distance appealing because it is derived from a metric on the set of all matchings between men and women. For the lower bound arguments that follow, we will exclusively use the divorce distance.

### 3.1 Computing $d(M)$

Computing $d(M)$ for an arbitrary instance of the stable matching problem and matching $M$ is highly nontrivial. A brute force approach would be to find all stable matchings $M' \in \mathcal{M}$ and compute $d(M) = \min \{d(M, M') | M' \in \mathcal{M} \}$. However, $|\mathcal{M}|$ can be exponentially large in $n$, so this process is very inefficient.

It turns out that it is possible to compute $d(M)$ in polynomial time. The idea is to exploit the structure of the set $\mathcal{M}$. Following [4] we find that $\mathcal{M}$ is a distributive lattice under a partial order where $M \leq M'$ if every man (weakly) prefers his partner in $M$ to his partner in $M'$. Using this structure, we can find a directed acyclic graph $G(\mathcal{M})$ whose closed subsets are in one-to-one correspondence with the elements of $\mathcal{M}$. Further, this bijection respects $\leq$ where the order relation on the closed subsets of $G(\mathcal{M})$ is set containment. The graph $G(\mathcal{M})$ is polynomial in size and be constructed from the players’ preferences in polynomial time (see [4].

Each vertex in $G(\mathcal{M})$ is labelled by a rotation $\rho$. A **rotation** is an ordered tuple of partners

$$(m_{i_1}, w_{j_1}), (m_{i_2}, w_{j_2}), \ldots, (m_{i_k}, w_{j_k}) \in M \in \mathcal{M}$$
such that the matching $M'$ by pairing $m_{i_\alpha}$ with $w_{j_{\alpha+1}}$ for $\alpha = 1, 2, \ldots, k$ and maintaining the other pairs form $M$ is stable. Using these rotations, we can interpret the bijection of the previous paragraph in the following way. The matching $M$ associated to the closed subset $F \subset G(M)$ is obtained by applying the rotations $\rho$ in $F$ to the male-optimal stable matching $M_0$ obtained from the Gale-Shapley algorithm.

To compute $d(M)$ from $G(M)$, we assign a weight to each rotation $\rho$ given by

$$f_M(\rho) = |M \cap \{(m_{i_\alpha}, w_{j_{\alpha+1}})\}| - |M \cap \rho|$$

Thus $f_M(\rho)$ measures how much the distance $d(M, M')$ changes when we apply the rotation $\rho$ to $M' \in \mathcal{M}$. The distance $d(M)$ can then be found by computing the maximum weight of a closed subset of $G(M)$. Specifically

$$d(M) = d(M, M_0) + \max_{S \subseteq G(M)} \left\{ \sum_{\rho \in S} f_M(\rho) \mid S \text{ is closed} \right\}.$$

Once $M_0$ and $G(M)$ are known, $d(M)$ can be computed in polynomial time by linear programming (see [4]) or reduction to a max-flow/min-cut problem (see [5]).

4 Lower Bound for Approximately Stable Matchings

In this section, we will prove that any protocol which output an approximately stable matching requires $\Omega(n^2)$ communication. We say that a matching $M$ is $(1 - \epsilon)$-stable or $\epsilon$-unstable if $d(M) \leq \epsilon n$.

**Theorem 5.** Suppose a protocol $\Pi$ produces a $(1 - \epsilon)$ stable matching for any instance of the stable matching problem with probability at least $2/3$ and $0 < \epsilon < \frac{1}{2}$. Then $\Pi$ requires $\Omega(n^2)$ communication between the men and women.

The idea of our proof is the same as the proof for exactly stable matchings: embed a suitably large instance of DISJ into the stable matching problem such that the players can infer the value of DISJ from the matching produced by $\Pi$. Essentially, we show that small changes in the preferences of the players can yield very large changes in the global structure of the stable matchings for those preferences.

4.1 Description of the preferences

We divide the players into three groups: high, mid and low which we denote $Y_h, Y_m$ and $Y_l$ respectively for the men and $X_h, X_m, X_l$ for the women. These groups have size

$$|X_h| = |Y_h| = \frac{1}{2} \delta n, \quad |X_m| = |Y_m| = \frac{1}{2} (1 - \delta)n, \quad |X_l| = |Y_l| = \frac{1}{2} n$$
where $\delta$ is a parameter with $0 < \delta < 1$. The low and mid players preferences will be fixed, while we will use the preferences of the high players to embed an instance of disjointness of size $(\delta n)^2$. We assume that the players are

$$X = \{w_1, w_2, \ldots, w_n\}, \quad Y = \{m_1, m_2, \ldots, m_n\}$$

where in both cases the first $\delta n/2$ players are high, the next $(1 - \delta)n/2$ are mid and the remaining players are low. Since the low and mid players’ preferences are the same for all instances, we describe those first. As before, the players’ preferences are symmetric in the sense that the men and women’s preferences are constructed analogously.

**low players** The low women prefer all of the men in order: $m_1 > m_2 > \cdots > m_n$ (and symmetrically for low men). In particular, each low player prefers all high players to mid players to low players.

**mid players** The mid players prefer low players to high players to mid players. Within each group, the preferences are “in order.” Specifically, the mid women have preferences

$$m_{n/2+1} > m_{n/2+2} > \cdots > m_n > m_1 > m_2 > \cdots > m_{n/2}$$

and symmetrically for the men.

**high players** We assume that each of the high players holds a bit vector of length $\delta n$. Together, the men and women’s preferences thus encode an instance of DISJ of size $(\delta n)^2$. For each $w_j \in X_h$, we denote her bit vector $x_j$. She then constructs her preferences as follows. She prefers in order

1. men $m_i \in Y_h$ such that $x_{(j,i)}(i) = 1$;
2. men $m \in Y_l$;
3. men $m \in Y_m$;
4. men $m_i \in Y_h$ such that $x_{(j,i)}(i) = 0$.

Within each group, the women prefer men in numerical order. The men’s preferences are constructed analogously.

### 4.2 Proof of the lower bound

Now that we have described the players’ preferences, we can prove theorem 5. We break the proof up into several lemmas.

**Lemma 6.** Any instance of the stable marriage problem with preferences described above corresponding to $\text{DISJ}(x, y) = 1$ has a unique stable matching $M_0$ given by

$$M_0 = \{ (m_i, w_{i+n/2}) \mid i = 1, 2, \ldots, n/2 \} \cup \{ (m_{i+n/2}, w_i) \mid i = 1, 2, \ldots, n/2 \}.$$

See figure [1]
Figure 1: The (unique) stable matching for disjoint instances.

Proof. We first argue that every high and mid player is matched with a low player. Suppose to the contrary that some $m = m_i$ with $i \leq n/2$ is matched with some $w = w_j$ for $j \leq n/2$ in a matching $M_0$. By the definition of the preferences and the assumption that $\text{DISJ}(x, y) = 0$, at least one of $m$ and $w$ prefers every low player to their partner. Assume without loss of generality that $m$ prefers all $w' = w_{j'}$ with $j' > n/2$ to $w$. That is, $m$ prefers all low women to his partner $w$. Since $m$ is paired with a medium or high woman, there must be some low $w'$ that is paired with a low man $m'$. But $w'$ prefers all high and medium men to $m'$. In particular she prefers $m$ to $m'$. Therefore, $(m, w')$ is a blocking pair, so $M_0$ is not stable. Thus any stable matching must match low players with mid or high players and vice versa.

Now we argue that if $(m_i, w_{j+n/2}) \in M_0$, then we must have $i = j$. The argument for pairs $(m_{i+n/2}, w_j)$ is identical. Suppose that $(m_i, w_{j+n/2}) \in M_0$ with $i < j$. Then there is some $j' < j$ such that $w' = w_{j'+n/2}$ is matched with $m' = m_{i'}$ with $i' > i$. But then $(m, w')$ mutually prefer each other, contradicting the stability of $M_0$. We arrive at a similar contradiction of $i > j$, hence we must have $i = j$, as desired. 

Lemma 7. Suppose we have stable matching instance with preferences described above corresponding to $\text{DISJ}(x, y) = 0$ with $x$ and $y$ uniquely intersecting. Let $x_{\alpha, \beta} = y_{\alpha, \beta} = 1$ be the uniquely intersecting entry in the vectors.
Then the unique stable matching $M_1$ is given by

$$M_1 = \{(m_\alpha, w_\beta)\} \cup \{(m_i, w_{i+n/2}) \mid i < \alpha\} \cup \{(m_{i+n/2}, w_i) \mid i < \beta\}
\cup \{(m_i, w_{i+n/2-1}) \mid i < \alpha \leq n/2\} \cup \{(m_{i+n/2-1}, w_i) \mid i < \beta \leq n/2\}
\cup \{(m_n, w_n)\}$$

See figure 2.

Proof. We will first argue that $(m_\alpha, w_\beta) \in M_1$ for any stable matching $M_1$ for the preferences described above. Since $M$ is stable, if $(m_\alpha, w_\beta) \notin M$, at least one of $m_\alpha$ and $w_\beta$, say $m_\alpha$, must be matched with someone he prefers to $w_\beta$. From $m_\alpha$’s preferences, this implies that $(m_\alpha, w) \in M_1$ for some $w = w_j$ with $j < \beta$ for which $y_{\alpha,j} = 1$. Since the instance of DISJ is uniquely intersecting, we must have $x_{\alpha,j} = 0$. Thus $w$ prefers all low men to $m_\alpha$. Since at most $n/2 - 1$ medium and high women are paired with low men (indeed $w$ is a high woman paired with a high man) and there are $n/2$ low men, some low man $m$ is paired with a low woman. But then $m$ and $w$ mutually prefer each other, hence form a blocking pair. Thus, we must have $(m_\alpha, w_\beta) \in M$.

The remainder of the proof of the lemma is identical to the proof of lemma if we remove $m_\alpha$ and $w_\beta$ from all the players’ preferences.
Lemma 8. The matchings $M_0$ and $M_1$ from the previous two lemmas satisfy
\[ d(M_0, M_1) \geq (1 - \delta)n. \]

Proof. This follows from the following two observations:
1. All mid men and women $Y_m \cup X_m$ have different partners in $M_0$ and $M_1$.
2. No mid men are matched with mid women in either $M_0$ or $M_1$.

From these facts, we can conclude that
\[ d(M_0, M_1) = n - |M_0, M_1| \geq |Y_m| + |X_m| = (1 - \delta)n. \]

Proof of theorem 5. Suppose a protocol $\Pi$ correctly outputs a matching $M$ which is a $(1 - \varepsilon)$-stable matching using $B$ bits of communication. Notice that for the preference structure described above and $\varepsilon < (1 - \delta)/2$, we then have $d(M, M_0) \leq \varepsilon n < (1 - \delta)n/2$ if and only if $\text{DISJ}(x, y) = 0$. Since the players can compute $d(M, M_0)$ without communication, they can determine the value of DISJ. Thus, $B = \Omega(n^2)$, as desired.

Corollary 9. Given a matching $M$ and preferences for men and women, it requires $\Omega(n^2)$ communication to decide if $M$ is $(1 - \varepsilon)$-stable for any $\varepsilon > 0$.

Proof. Take $M = M_0$ and choose preferences that embed DISJ as above.

5 Commentary

5.1 Time-space tradeoffs

Communication lower bounds naturally give time-space tradeoffs for any distributed model of computation. For the stable marriage problem, it is natural to consider a distributed setting where each player is represented by a separate processor that knows only its own preferences. The different processors then need to communicate in order to find an (approximate) stable matching. There are many different models of distributed computation [8], but most models have two complementary notions of complexity:

1. Capacity—which we denote $C$—which is a measure of the length of individual messages sent between the various processors;
2. Time—denoted $T$—which measures the number of communication rounds required to carry out a computation.

The communication complexity of a distributed computational problem gives a lower bound on the product $CT$, showing that both capacity and time cannot be simultaneously optimized.
We derived our communication lower bounds for the setting where a single processor knows all the men’s preferences while another knows all of the women’s preferences. However, the bounds also apply to bounding the total communication in the fully distributed case where there are $2n$ processors each holding a single preference list. In fact, the two player scenario is equivalent to the $2n$ scenario where all players of the same gender are allowed to communicate free of charge.

In [1] the authors consider a restricted version of the stable matching problem where all preferences are governed by weights associated to each edge. They consider a “billboard” model of communication where in each round, every player is allowed to broadcast a single message of length $O(\log n)$ to a billboard to which all players have access. The authors show that for a restricted model of deterministic computation, computing a stable matching in this way requires $\Omega(n)$ rounds of posting. Our communication lower bound naturally generalizes this result to randomized and nondeterministic protocols and approximate stable matchings. Indeed, in each round, the $2n$ players can post at most $O(2n \log n)$ bits to the billboard. Hence $\Omega(n/\log n)$ rounds are required to find an approximate stable matching. This is clearly optimal up to logarithmic factor, as the players could simply post their entire input in $n$ rounds.

5.2 Work to be done

Up to logarithmic factors, theorems 2 and 5 are asymptotically tight. A natural question is whether the logarithmic gap can be closed. In a different vein, the lower bound in theorem 5 could be improved by considering a coarser notion of approximate stability.

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