Non-Hermitian Many-Body Localization of Coupled Hatano-Nelson Chains

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The explorations of non-Hermiticity have been devoted to study disorder-induced localization, but little is known about the sensitivity of non-Hermitian many-body phenomena to the spatial boundary conditions. In this work, we investigate the eigenspectrum and localization properties of the interacting coupled Hatano-Nelson open chains in the presence of random disorder potential. In the delocalized regime, a complex-real spectral transition occurs, accompanied with the nearest-level-spacing distribution of eigenergies transforming from the Ginibre random matrix ensemble to the Gaussian orthogonal ensemble. In contrast, a disorder-induced complex-real transition occurs with Ginibre to real Poisson level-spacing distribution transition. Moreover, we characterize wavefunctions through the (biorthogonal) inverse participation ratio and fractal dimension. Unlike the delocalized regime, we reveal that for strong disorder, the non-Hermitian skin effect is suppressed and the system is insensitive to the boundary conditions. Finally, the many-body localization is corroborated by the dynamics of entanglement entropy and spin imbalance.

Introduction— When traditional quantum mechanics postulates Hermiticity, numerous developments have been made for exploring non-Hermitian quantum physics in various fields of physics like condensed-matter, cold atoms, open quantum systems, as well as classical systems which can emulate non-Hermitian wave physics [1–6]. The recent experimental advances provide access to engineering the non-Hermitian Hamiltonians with the dissipation and non-reciprocal tunneling [7–10]. These developments allow us to explore fundamental physics of localization. The interplay between disorder and non-Hermiticity due to asymmetric hopping was first investigated by the pioneer works of Hatano and Nelson, which reveal a real-complex transition of single-particle spectrum [11–13]. Moreover, it has been shown that the random potential can suppress the complex eigenenergies of an interacting single Hatano-Nelson chain with periodic boundary conditions (PBCs) and the spectrum transition is accompanied with non-Hermitian many-body localization (MBL) transition [14]. The similar phenomena have also been found for quasiperiodic potential [15]. Since the disordered single-chain model respecting time-reversal symmetry belongs to the symmetry class AI, its localized phase follows the real Poisson ensemble while the delocalized phase follows the Ginibre ensemble [16–18].

The non-Hermitian systems could depend crucially on the imposed boundary conditions. One outstanding phenomenon that has no Hermitian counterparts is the non-Hermitian skin effect. It describes the anomalous localization for an extensive number of bulk modes which can occur at the boundaries of non-Hermitian open lattices [19–25]. The skin effect has been realized in various experimental setups of photonics [8, 26], electrical circuits [9, 10, 27], metamaterials [28–30], and ultracold atoms [31, 32]. The breakdown of the conventional bulk-boundary correspondence for non-Hermitian Hamiltonians, fundamentally challenging our knowledge of the band theory, has been successful to explain the concept of non-Bloch band theory. Furthermore, most recently the fate of skin modes in fermionic and bosonic systems with many-body atomic correlations has been investigated [33–41]. However, little is known about the role of skin effect in interacting systems with disorder.

Last few years witnessed enormous interest to understand the coupled chains with non-reciprocal hoppings in detail. This is because several non-trivial behaviors can arise from the two-chain model [42–44]. For example, a hybridized non-interacting Hatano-Nelson chain system exhibits a pseudo mobility edge separating extended states to localized states (induced by skin effect) [45]. Along this line, whether richer and more interesting localization physics will emerge in the coupled chain system when interactions come into play?

Aiming to answer above questions, in this Letter we study the many-body coupled Hatano-Nelson chains in the presence of random disorder. In particular, we investigate the role of the imposed boundary conditions in non-Hermitian interacting system. For certain parameter space of inter-chain coupling, a complex to real spectral transition is observed for finite open chains, while the spectrum for PBC maintains complex. The spectral transition is indicated by the transition in the nearest-level spacing distribution of eigenergies and in the average complex spacing ratio between nearest- and next-to-nearest-neighbour spacings. As disorder strength increases, the system enters the many-body localized phase which possesses a real eigenspectrum. We characterize the MBL phase by inverse participation ratio (IPR) and fractal dimension (FD) and find this phase, unlike the delocalized one insensitive to the boundary conditions. Finally, the time evolution of entanglement entropy and imbalance are also investigated.

Model— We consider the two coupled Hatano-Nelson chains in the presence of disorder potential and interaction [See Fig. 1(a)]. In the two-chain geometry, two chains
can be interpreted as two spin states of spin-1/2 fermions. The interacting model reads as
\[
\hat{H} = -\sum_{j,\sigma} J \left( e^{-g_{\sigma}} \hat{c}_{j,\sigma}^\dagger \hat{c}_{j+1,\sigma} + e^{g_{\sigma}} \hat{c}_{j+1,\sigma}^\dagger \hat{c}_{j,\sigma} \right) - \sum_{j} \left( K \hat{c}_{j,\sigma}^\dagger \hat{c}_{j,\sigma} + \text{H.c.} \right) + U \sum_{j} \hat{n}_{j,\uparrow} \hat{n}_{j,\downarrow} + \sum_{j,\sigma} \epsilon_{j\sigma} \hat{n}_{j,\sigma},
\]

where \( j \) represents the spatial indices and \( \sigma = \uparrow, \downarrow \) spin (chain) indices. Here \( \hat{c}_{j,\sigma}^\dagger (\hat{c}_{j,\sigma}) \) creates (annihilates) fermions with spin \( \sigma \) at \( j \)th site, and the occupation number operator \( \hat{n}_{j,\sigma} = \hat{c}_{j,\sigma}^\dagger \hat{c}_{j,\sigma} \). The parameter \( J \) is the hopping strength between neighbouring lattice sites on the same chain and \( g_{\sigma} \) is the non-Hermiticity of \( \sigma \)-spin. Here, \( K \) is the spin-flip (inter-chain) coupling strength, \( U \) is the Hubbard interaction strength and \( \epsilon_{j\sigma} \) is the random disorder potential at \( j \)th site for \( \sigma \)-spin chosen between \([-W, W]\) with \( W \) being the disorder strength. In the present work, we primarily consider the uncorrelated (spin-dependent) disorder, which breaks SU(2) spin-symmetry of the model and is known to induce full localization in the Hermitian system. However, we also discuss the contrast behaviour due to spin-independent disorder potential (\( \epsilon_{\uparrow\downarrow} = \epsilon_{\downarrow\uparrow} \)) for some specific cases. It is also worth noting, according to the symmetry class of a non-Hermitian system [46–48], the model considered here belongs to class AI. This class preserves the time reversal symmetry (\( H = H^\dagger \)) and breaks the transposition symmetry (\( H \neq H^T \)).

The Non-Hermitian system with non-reciprocal hoppings is extremely sensitive to the imposed boundary conditions. In the two-chain model with dissimilar non-Hermiticity parameter \( g_{\sigma} \), the inter-chain coupling \( K \) can further yield non-trivial interference between chains. When \( K = 0 \), the open boundary condition (OBC) allows us to remove \( g_{\sigma} \) by an imaginary gauge transformation (IGT) [49], \( c_{j,\sigma} \rightarrow e^{i\gamma_{j}} c_{j,\sigma} \) and \( \hat{c}_{j,\sigma}^\dagger \rightarrow e^{-i\gamma_{j}} \hat{c}_{j,\sigma}^\dagger \). As a result, Eq. (1) becomes a Hermitian disordered Hubbard model and all the many-body eigenenergies are real under the OBC. For any \( K \neq 0 \), if \( g_{\uparrow} \neq g_{\downarrow}, g_{\sigma} \) cannot be removed by such a transformation, and imaginary parts could appear in the spectrum. While the complex (real) spectrum of the single Hatano-Nelson chain is unique to the PBC (OBC), much richer eigenspectrum and many-body dynamics arise from the two-chain model under OBC.

Real-complex transition of eigenspectrum—We first show the characteristic properties of the change in the eigenspectrum as a function of the coupling strength \( K \) and non-Hermiticity parameters \( g_{\sigma} \). To manifest the complex spectrum, we define a fraction \( f_{im} = \frac{D_{im}}{D} \), where \( D_{im} \) is the number of complex eigenenergies and \( D \) is the total number of eigenenergies. The eigenenergies are defined as complex if \( |\text{Im}\{E\}| \geq C \) with a cut-off \( C = 10^{-13} \), which is identified based on the machine error. The underline denotes the disorder average. We will focus on the eigenspectrum with OBC in the main text and the role of the boundary conditions are discussed in the Supplementary Material [50].

We first discuss the eigenenergies for lower disorder strengths. Fig. 1(b) represents \( f_{im} \) as a function of \( K \) and \( g_{\uparrow}/g_{\downarrow} \) for \( W = 2 \). The value of \( f_{im} \) is obtained after disorder average of 200 samples. We fix \( U = 1 \) and set \( J = 1 \) as the unit of the energy. The phase diagram can be divided in various regimes (i) \( K = 0 \rightarrow K \neq 0 \), (ii) \( g_{\uparrow} \neq g_{\downarrow} \rightarrow g_{\uparrow} = g_{\downarrow} \), (iii) \( K \gg 1 \). (i) When tuning \( K = 0 \) to \( K \neq 0 \), we numerically verify the phase diagram, except for \( g_{\uparrow}/g_{\downarrow} = 1 \), exhibits a real-complex phase transition of many-body eigenenergies. (ii) We further discuss the effects of \( g_{\uparrow}/g_{\downarrow} \) on the eigenspectrum. A smooth complex-real energy transition is observed as the ratio \( g_{\uparrow}/g_{\downarrow} \) approaches \( g_{\uparrow}/g_{\downarrow} = 1 \). We note the OBC eigenspectrum is real at \( g_{\uparrow}/g_{\downarrow} = 1 \) because IGT is valid at this special point. As the strength of \( K \) increases, the critical ratio \( (g_{\uparrow}/g_{\downarrow})_c \) of the complex-real transition decreases, as evident in Fig. 1(b). (iii) At sufficiently higher \( K \), the effect of the non-Hermiticity is suppressed and the system is driven by the inter-chain coupling. Hence, in the limit \( K \gg 1 \), the system possess real spectra irrespective of any \( g_{\uparrow}/g_{\downarrow} \).

We have provided the quantitative variation of \( f_{im} \) for different system sizes in the Supplementary Material, which confirms the robustness of the phase diagram with \( L \) [51]. At strong disorder strengths, the system is expected to be in the many-body localized phase. The eigenspectrum for \( W = 30 \) as a function of \( K \) and \( g_{\uparrow}/g_{\downarrow} \) is shown in Fig. 1(c). The eigenenergies remain real in entire \( K-g_{\uparrow}/g_{\downarrow} \) plane since the strong disorder destroys the effects of the non-Hermiticity [52]. The disorder-induced complex-real transi-
tions share the similarity to that of the periodic single Hatano-Nelson chain [14]. With periodic boundary condition, the eigenspectra of coupled Hatano-Nelson chains remain complex for all \( K \) and \( g_t/g_L \). However, the disorder-driven complex-real transition is still present [53].

**Spectral statistics**—The characteristic transition in eigenenergies and the disorder-induced localization can be further investigated by the energy level statistics. In this Letter, we employ the nearest-level spacing distribution of (unfolded) eigenenergies and the complex spacing ratio between nearest- and next-to-nearest-neighbor spacings to characterize the spectral properties. The energy level spacing is defined as \( s_i \equiv |E_i - E_{i+1}^{NN}| \), where \( E_{i}^{NN} \) is a complex-valued eigenvalue nearest to \( E_i \) in the complex energy plane. The complex spacing ratio is a dimensionless variable [54], \( z_i \equiv (|E_i - E_{i+1}^{NN}|/|E_i - E_{i-1}^{NN}|) \) with the amplitude \( |z_i| \equiv r_i \). Here, \( E_{i}^{NN} \) is a complex-valued eigenvalue that is the next-nearest-neighbour to \( E_i \) in the complex plane. The mean level spacing ratio \( \langle r \rangle \) is obtained by the average of \( r_i \) over the energy window and number of disorder realizations. Here, we consider the energy window to be 10% eigenvalues around the center of the eigenspectrum in the complex energy plane. This allows us to obtain a large number of eigenvalues for the level statistics and ascertain that their eigenstates share similar localization properties. The number of disorder realizations is chosen such that the total number of eigenvalues is \( \sim 10^6 \).

We first investigate the \( \langle r \rangle \) for weak disorder \( (W = 2) \) with \( K \) for several \( g_t/g_L \) ratios [Fig. 2(a)]. The \( \langle r \rangle \) for small coupling \( K \) and \( g_t \neq g_L \) attains a constant value \( \approx 0.74 \), which is characterized by the universal value corresponding to the Ginibre ensemble. As a representative case, the probability density of unfolded normalized level spacing [46] for \( K = g_t/g_L = 0.5 \) is shown in Fig. 2(c). With the increase of \( K \), the level distribution transforms from Ginibre level statistics to Gaussian orthogonal ensemble (GOE) statistics, as shown in Fig. 2(d,e). Fig. 2(a) shows a smooth transition to \( \langle r \rangle \approx 0.56 \) for the GOE distribution when \( K \) increases to the strong coupling \( (K \gg 1) \). This transition is consistent with the corresponding complex-real transition of the eigenspectrum [Fig. 1(a)]. In Fig. 2(a), it is evident that the critical \( K \) demarcating Ginibre and GOE statistics lowers as the \( g_t/g_L \) ratio approaches unity. For \( g_t/g_L = 1 \), the \( \langle r \rangle \) remains nearly GOE, because the non-Hermiticity can be removed by the imaginary gauge transformation under OBC. It is interesting to note for \( g_t/g_L = 1 \), the correlated (spin-independent) disorder leads to a Poisson level distribution due to inherent SU(2) spin symmetry. Therefore, to have GOE statistics, we need the uncorrelated disorder potential which breaks this symmetry [55].

In Fig. 2(b) we further present \( \langle r \rangle \) as a function of the disorder strength \( W \) with various system sizes. Here, we fix \( K = 1 \) and \( g_t/g_L = 0.5 \). The disorder-induced complex-real transition of system is accompanied by the corresponding change in \( \langle r \rangle \) from Ginibre to real Poisson statistics. It is important to stress that in the present non-Hermitian model, we define \( \langle r \rangle \approx 0.5 \) of the real Poisson statistics, which is different from the conventional \( \langle r \rangle \approx 0.38 \) of the Hermitian many-body systems [56], even both being characterized for real spectra. Our result coincides with the recent studies of the spectral analysis of the directed random networks [57]. In addition, the evolution of the level distributions from Ginibre to Poisson with disorder is illustrated in Fig. 2(f,g,h). The critical disorder strength of the localization transition \( W_c \approx 9 \) is obtained by inspecting the crossing of \( \langle r \rangle \)-curves of two largest systems. In contrast to the above behaviour, for strong coupling \( K \) or \( g_t/g_L = 1 \), the disorder leads to a transition from a GOE to the Poisson distribution. At strong \( W \), the effects of the disorder potential prevails and \( \langle r \rangle \)-value reaches a stationary value of real Poisson statistics.

**Localization properties**—To further understand the interplay between the localization and the skin effect in real space, the quantities we have chosen to characterize wavefunctions are inverse participation ratio and fractal dimension. For non-
Hermitian systems, the IPR can be defined in two ways: one using the \( n \)th left or right eigenstates and other is defined under biorthogonal basis from both left and right eigenstates. These are defined as

\[
I_{n,s} = \sum_{i,\sigma} |\psi_{i,\sigma}^n|^2 \quad \text{and} \quad I_{n,B,s} = \sum_{i,\sigma} \left| \tilde{\psi}_{i,\sigma}^n \right|^2,
\]

where \( s \) is the disorder realization, the subscript \( nB \) represents the biorthogonal IPR, \( \psi_{i,\sigma}^n \equiv (\langle n | b_i, \sigma \rangle) (n | b_i, \sigma \rangle \) and \( \tilde{\psi}_{i,\sigma}^n \equiv (\langle \tilde{n} | b_i, \sigma \rangle) (\tilde{n} | b_i, \sigma \rangle \) with \( n \) and \( \tilde{n} \) are the right and corresponding left eigenstates, and \( | b_i, \sigma \rangle \) are Fock space chosen as computational basis. The IPRs are further averaged over numerous disorder realizations to get the mean IPRs, which are defined as

\[
I_{\text{avg}} = \frac{1}{n} \sum_{nB} I_{n,B,s},
\]

for delocalized states, the IPR approaches zero in the thermodynamic limit while saturates to a finite value \((\approx 1)\) for localized states. The

that the value of \( I_{\text{avg}} \) is nonzero even in the delocalized limit, whereas \( I_{\text{B,avg}} \) is zero [60]. This suggests the biorthogonal density distributions do not suffer the skin effects. This fact can be understood from the similarity transformation \( | n' \rangle = S^{-1} | n \rangle \) and \( \langle n' | = \langle \tilde{n} | S \), with \( | n' \rangle \) being the eigenstate of the corresponding Hermitian system. This transformation leads to the biorthogonal density distribution devoid of the non-Hermitian skin effect. In addition, the choice of eigenstates for IPR and FD is only important for open chains. Under PBC the non-Hermitian skin effect is absent. Furthermore, the corresponding contrast behaviour in \( \eta \) and \( \eta_B \) is also noted in Fig. 3(a).

The evolution of \( I_{\text{B,avg}} \) and \( \eta_B \) with \( K \) confirms the delocalization of system and is consistent to the \( \langle r \rangle \) as shown in Fig. 2(a). While the system remains delocalized at \( W = 2 \), it possesses the complex-real transition as \( K \) increases. Hence, contrary to the single-chain case, such eigenspectrum transition in a coupled-chain model does not coincide to the localization transition. We further present the evolution of mean IPR and FD as a function of \( W \) in Fig. 3(b). At lower \( W \), the \( I_{\text{B,avg}} \) is small but \( I_{\text{avg}} \) acquires a finite value due to the interference of the non-Hermitian skin effects. The biorthogonal FD \( \eta_B \) approaches unity at lower \( W \) representing the delocalization. The value of IPR (FD) increases (decreases) with \( W \) confirms the disorder-driven localization of two-chain model.

The observed behaviours of IPR and FD show \( W_c \approx 9 \), which is consistent with the prediction from spectral statistics [Fig. 2(b)]. In addition, this localization transition with \( W \) overlaps to the complex-real eigenspectrum transition.

Finally, we investigate the time evolution of the spin imbalance \( I_s(t) \) and half-chain entanglement entropy \( S_E(t) \), which are shown in Figs. 3(c) and 3(d). Here we choose the Néel ordered state \( | \uparrow \downarrow \uparrow \cdots \rangle \) at initial time \( t = 0 \). This state has a unity spin imbalance that reads

\[
I_s = \frac{1}{L} \sum_{j=1}^{L} (-1)^{j-1} (n_j \uparrow - n_j \downarrow),
\]

whose long-time stationary value effectively serves as an order parameter of the MBL phase. In Fig. 3(c), \( I_s(t) \) relaxes to zero as time evolves with weak disorder \( (W = 2) \), losing memory of initial ordering, suggests delocalization of the system. However, at strong disorder \( I_s(t) \) shows a non-vanishing stationary value at long times. This initial state memory retention indicates the disorder-driven MBL. In addition, we find that large \( K \) suppresses the stationary value of \( I_s(t) \). This is because the \( K \) term couples different spin sectors and scrambles the initial spin ordering. We further calculate the entanglement entropy \( S_E(t) \) between the left and right halves of the system, which is shown in Fig. 3(d). For strong disorder, the long-time evolution of \( S_E(t) \) exhibits a logarithmic growth, which is reminiscent of the non-ergodicity in the Hermitian many-body localized systems [61, 62]. For weak disorder, however, \( S_E(t) \) can decrease after \( t \approx 5 \) in the complex eigenenergy phase \((K = 0.5)\), but it does not in the real eigenenergy one \((K = 6)\) [63]. This long-time behavior of \( S_E(t) \) signifies the real-complex transition which is unique to

![Fig. 3.](image-url)
Conclusions—We have investigated the eigenspectrum, level statistics, and localization properties of non-Hermitian Hatano-Nelson chains. We chart out several regimes of the interplay of non-reciprocal hopping coefficients and the inter-chain coupling. At weak disorder, while the coupled Hatano-Nelson open-chains remain delocalized, they possess a complex-real phase transition, which is absent when periodic chains are considered. Furthermore, we corroborated the inter-chain coupling and disorder-induced eigenspectrum transition using level statistics, IPR, and FD. In particular, the localized phase is not sensitive to the boundary conditions. The many-body localization of the non-Hermitian coupled chains is confirmed by the dynamical evolution of spin imbalance and entanglement entropy. Our findings pave the way to the understanding of the critical phenomena due to the non-Hermitian skin effects of many-body systems and disorder-driven localization. It would further be interesting to investigate the interplay of the boundary conditions and the many-body dynamics in different universality classes.

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1 C. M. Bender and S. Boettcher, Phys. Rev. Lett. 80, 5243 (1998).
2 S. Longhi, Phys. Rev. Lett. 103, 123601 (2009).
3 Y. Ashida, S. Furukawa, and M. Ueda, Nat. Commun. 8, 15791 (2017).
4 Z. Gong, Y. Ashida, K. Kawabata, K. Takasan, S. Higashikawa, and M. Ueda, Phys. Rev. X 8, 031079 (2018).
5 Y. Ashida, Z. Gong, and M. Ueda, Adv. Phys. 69, 249 (2020).
6 E. J. Bergholtz, J. C. Budich, and F. K. Kunst, Rev. Mod. Phys. 93, 015005 (2021).
7 J. M. Zeuner, M. C. Rechtsman, Y. Plotnik, Y. Lumer, S. Nolte, M. S. Rudner, M. Segev, and A. Szameit, Phys. Rev. Lett. 115, 040402 (2015).
8 L. Xiao, T. Deng, K. Wang, G. Zhu, Z. Wang, W. Yi, and P. Xue, Nat. Phys. 16, 761 (2020).
9 T. Helbig, T. Hofmann, S. Imhof, M. Abdelghany, T. Kiessling, L. W. Molenkamp, C. H. Lee, A. Szameit, M. Greiter, and R. Thomale, Nat. Phys. 16, 747 (2020).
10 T. Hofmann, T. Helbig, F. Schindler, N. Salgo, M. Brzezińska, M. Greiter, T. Kiessling, D. Wolf, A. Vollhardt, A. Kabaši, C. H. Lee, A. Bilušić, R. Thomale, and T. Neupert, Phys. Rev. Res. 2, 023265 (2020).
11 N. Hatano and D. R. Nelson, Phys. Rev. Lett. 77, 570 (1996).
12 N. Hatano and D. R. Nelson, Phys. Rev. B 56, 8651 (1997).
13 N. Hatano and D. R. Nelson, Phys. Rev. B 58, 8384 (1998).
14 R. Hamazaki, K. Kawabata, and M. Ueda, Phys. Rev. Lett. 123, 090603 (2019).
15 L.-J. Zhai, S. Yin, and G.-Y. Huang, Phys. Rev. B 102, 064206 (2020).
16 J. Ginibre, J. Math. Phys. 6, 440 (1965).
17 R. groove, F. Haake, and H.-J. Sommers, Phys. Rev. Lett. 61, 1899 (1988).
18 F. Haake, Quantum Signatures of Chaos (Springer, 2013).
19 S. Yao and Z. Wang, Phys. Rev. Lett. 121, 086803 (2018).
20 V. M. Martinez Alvarez, J. E. Barrios Vargas, and L. E. F. Foa Torres, Phys. Rev. B 97, 121401 (2018).
21 N. Okuma, K. Kawabata, K. Shiozaki, and M. Sato, Phys. Rev. Lett. 124, 086801 (2020).
22 K. Zhang, Z. Yang, and C. Fang, Phys. Rev. Lett. 125, 126402 (2020).
23 K. Kawabata, M. Sato, and K. Shiozaki, Phys. Rev. B 102, 205118 (2020).
24 R. Okugawa, R. Takahashi, and K. Yokomizo, Phys. Rev. B 102, 241402 (2020).
25 Y. Fu, J. Hu, and S. Wan, Phys. Rev. B 103, 045420 (2021).
26 S. Weidemann, M. Kremer, T. Helbig, T. Hofmann, A. Stegmaier, M. Greiter, R. Thomale, and A. Szameit, Science 368, 311 (2020).
27 S. Liu, R. Shao, S. Ma, L. Zhang, O. You, H. Wu, Y. J. Xiang, T. J. Cui, and S. Zhang. Research 2021, 5608038 (2021).
28 C. Coulais, D. Sounas, and A. Ali, Nature 542, 461 (2017).
29 M. Brandenbourger, X. Locsin, E. Lerner, and C. Coulais, Nat. Commun. 10, 4608 (2019).
30 A. Ghatak, M. Brandenbourger, J. van Wezel, and C. Coulais, Proc. Natl. Acad. Sci. 117, 29561 (2020).
31 W. Gou, T. Chen, D. Xie, T. Xiao, T.-S. Deng, B. Gadway, W. Yi, and B. Yan, “Observation of non-hermitian skin effect and topology in ultracold atoms,” (2022), arXiv:2201.09478 [cond-mat.quant-gas].
32 E. Lee, H. Lee, and B.-J. Yang, Phys. Rev. B 101, 121109 (2020).
33 S. Mu, C. H. Lee, L. Li, and J. Gong, Phys. Rev. B 102, 081115 (2020).
34 T. Yoshida, Phys. Rev. B 103, 125145 (2021).
35 W. Zhang, F. Di, H. Yuan, H. Wang, X. Zheng, L. He1, H. Sun, and X. Zhang, “Observation of non-hermitian many-body skin effects in hilbert space,” (2021), arXiv:2109.08334 [cond-mat.mes-hall].
36 T. Liu, J. J. He, T. Yoshida, Z.-L. Xiang, and F. Nori, Phys. Rev. B 102, 235151 (2020).
37 K. Cao, Q. Du, X.-R. Wang, and S.-P. Kou, “Physics of many-body nonreciprocal model: Quantum system with maxwell’s pressure demon,” (2021), arXiv:2109.03690 [cond-mat.other].
38 D.-W. Zhang, Y.-L. Chen, G.-Q. Zhang, L.-J. Lang, Z. Li, and S.-L. Zhu, Phys. Rev. B 101, 235130 (2020).
39 Z. Xu and S. Chen, Phys. Rev. B 102, 035153 (2020).
40 F. Alsallom, L. Herviou, O. V. Yazyev, and M. Brzezińska, “Fate of the non-hermitian skin effect in many-body fermionic systems,” (2021), arXiv:2110.13164 [cond-mat.mes-hall].
41 L. Li, C. H. Lee, and J. Gong, Nat. Commun. 11, 5491 (2020).
42 L. Li, C. H. Lee, and J. Gong, Commun. Phys. 4, 42 (2021).
43 K. Yokomizo and S. Murakami, Phys. Rev. B 104, 165117 (2021).
S. Mu, L. Zhou, L. Li, and J. Gong, “Non-hermitian pseudo mobility edge in a coupled chain system,” (2021), arXiv:2111.11914 [cond-mat.mes-hall].

R. Hamazaki, K. Kawabata, N. Kura, and M. Ueda, Phys. Rev. Res. 2, 023286 (2020).

K. Kawabata, K. Shiozaki, M. Ueda, and M. Sato, Phys. Rev. X 9, 041015 (2019).

H. Zhou and J. Y. Lee, Phys. Rev. B 99, 235112 (2019).

N. Hatano and H. Obuse, Ann. Phys. , 168615 (2021).

See Supplementary Material for the discussion on the role of boundary conditions on the eigenspectrum.

The finite-size effect on the role of $K$ for lower disorder strength $W = 2$ is discussed in the Supplementary Material. The complex-real eigenspectrum transition pushed to larger $K$ as $L$ increases, although the difference between $K_c$ for $L = 7$ and $L = 8$ is marginal.

See Supplementary Material for the discussion on the variation of $f_{\text{im}}$ as a function of $W$ for different $g_\uparrow/g_\downarrow$ ratios.

See Supplementary Material for the details of the eigenspectrum with PBC.

L. Sá, P. Ribeiro, and T. Prosen, Phys. Rev. X 10, 021019 (2020).

See Supplementary Material for the nearest-neighbour level distributions with correlated and uncorrelated disorder.

V. Oganesyan and D. A. Huse, Phys. Rev. B 75, 155111 (2007).

T. Peron, B. M. F. de Resende, F. A. Rodrigues, L. d. F. Costa, and J. A. Méndez-Bermúdez, Phys. Rev. E 102, 062305 (2020).

N. Macé, F. Alet, and N. Laflorencie, Phys. Rev. Lett. 123, 180601 (2019).

A. Bäcker, M. Haque, and I. M. Khaymovich, Phys. Rev. E 100, 032117 (2019).

We refer reader to the Supplementary Material for the discussion on the Disorder-averaged IPR as a function of eigenstate index.

M. Žnidarič, T. Prosen, and P. Prelovšek, Phys. Rev. B 77, 064426 (2008).

J. H. Bardarson, F. Pollmann, and J. E. Moore, Phys. Rev. Lett. 109, 017202 (2012).

See Supplementary Material for the time dynamics of the local particle density at each site for weak and strong inter-chain coupling regime.

H. Markum, R. Pullirsch, and T. Wettig, Phys. Rev. Lett. 83, 484 (1999).

K. Mochizuki, N. Hatano, J. Feinberg, and H. Obuse, Phys. Rev. E 102, 012101 (2020).

A. J. Daley, Adv. Phys. 63, 77 (2014).
SUPPLEMENTAL MATERIAL for “Non-Hermitian Many-Body Localization of Coupled Hatano-Nelson Chains”

We here discuss the spectral phase transition of the eigenspectrum with disorder, the role of boundary conditions, and the corresponding level statistics. Finally, we discuss the quench dynamics of the half-chain entanglement entropy and spin imbalance.

COMPLEX-REAL TRANSITION OF EIGENSPECTRUM AND BOUNDARY CONDITIONS

![Complex-Real Transition of Eigenspectrum and Boundary Conditions](image)

FIG. S1. Eigenenergies of the non-Hermitian coupled Hatano-Nelson chain model for $L = 8$ as a function of (a,b,c) the inter-chain coupling $K$ and (d,e,f) the disorder strength $W$. These are shown corresponding to Fig. 2(c-h) of the main text. The decrease in complex energies and complex-real transition with increase in $K$ at $W = 2$ ($W$ at $K = 1$) are evident. (g,h) The disorder-averaged complex energy fraction $f_{im}$ as a function of $K$ and $W$ for different system size and $g_{\uparrow}/g_{\downarrow} = 0.2$. The observed behaviour of the eigenspectrum represent the robustness of complex-real transition with system size. The evolution of $f_{im}$ reveal the critical coupling strength $K_c$ of complex-real phase transition does not change with system size.

The eigenenergies of the model Hamiltonian [Eq.(1) in main-text] with $L = 8$ and for different $K$ and $W$ are shown in Fig. S1(a-f). The complex eigenenergies remain symmetric (with respect to the real axis) owing to the time-reversal symmetry and AI class of the system. With the increase in $K$ [Fig. S1(a,b,c)] and $W$ [Fig. S1(d,e,f)], the imaginary parts of the energies are suppressed. The complex-real transition is shown for $g_{\uparrow} \neq g_{\downarrow}$ while in the case of $g_{\uparrow} = g_{\downarrow}$, the spectrum remains real with $K$ and $W$. This is due to the applicability of the imaginary gauge transformation (IGT). It is important to note that the aforementioned transition is unique to open chains, while closed or periodic Hatano-Nelson chains possess complex eigenspectrum.

We further examine the finite-size effects on $f_{im}$ of the eigenspectrum. The evolution of $f_{im}$ with $K$ is shown in Fig. S1(g). The eigenenergies of the single Hatano-Nelson open chain remains real, however the interplay of $g_{\sigma}$’s and $K$ in two-chain system can result into complex spectrum. An infinitesimal coupling between the chains and $g_{\uparrow} \neq g_{\downarrow}$ lead to complex energies. The corresponding fraction for smaller $K$’s at $g_{\uparrow}/g_{\downarrow} = 0.2$ is evident from Fig. S1(g). The $f_{im}$ shifts to larger value with $L$, however the qualitative feature of complex-real transition at larger $K$ is justified. Moreover, the $K_c$ of the transition does not change significantly with $L$. The variation of $f_{im}$ as a function of $W$ is shown in Fig. S1(h). At weak disorder strengths, the system is delocalized possessing complex energies. The strong disorder potential affects the dynamical stability of the system and lead to complex-real phase transition. This feature is also represented by a single Hatano-Nelson chain [14]. In Fig. S1(h), the $f_{im}$ shown corresponds to the evolution of average level spacing shown in Fig. 2(b). We find the critical disorder strength of the localization transition $W_c \approx 9$ which is obtained by inspecting the crossing of curves for two largest $L$ in Fig. S1(h). This value is consistent to $W_c$ obtained using spectral statistics and the eigenstate properties (in the main text).
We now discuss the role of boundary conditions in terms of the model parameters $K$ and $W$. The disorder-averaged $f_{im}$ for $L = 6$ are shown in Fig. S2. For larger system sizes, the qualitative behaviour of the energies does not change, however the critical value of the transitions may vary. With OBC, as discussed previously (and in the main text), the inter-chain coupling and disorder induced complex-real phase transition is evident in Fig. S2(a,b,c). For open coupled-chains with unequal non-Hermiticity or hybridized chains ($g_τ = 0, g_↓ ≠ 0$ and vice-versa) the system possess complex energies. As the $g_τ/g_↓$ approaches unity, the number of complex energies decreases and at equal $g_σ$'s the system is devoid of dynamical instability, exhibiting real spectrum. For periodic chains, the eigenspectrum remains complex as a function of $K$ for lower disorder strengths, as illustrated in Fig. S2(d,e,f). It is worth noting that the sensitivity of the boundary conditions on the non-Hermitian system has already been recognized in the original works of Hatano-Nelson [11, 12]. The complex nature of the energies with PBC is related to the plane-wave character of the eigen wavefunction, while under OBC this is no longer true due to the emergence of skin effect. In the presence of weak disorder ($W < W_c$), the wavefunction are extended, and hence the spectrum is complex (with PBC) due to the prevailing role of non-reciprocal tunnelings. At strong disorder ($W > W_c$), the wavefunction is localized (ascertain by eigenstate properties in the main text), and delocalization-localization transition coincides to the complex-real transition. In short, the $K$-driven transition is absent in PBC case while the $W$-driven is present for both boundary cases.

**COMPLEX LEVEL SPACING RATIO AND DISTRIBUTIONS**

We study the level-spacing statistics to unveil the universal features of the eigenspectrum of coupled Hatano-Nelson model. We first perform an unfolding of the spectrum to obtain the histogram of the Euclidean distance between nearest-neighbour eigenvalues and normalized level spacing. We use the procedure put forward in Ref. [18, 46]. It is important to note that the unfolding procedure is used for both complex and real spectrum of the model. To get the unfolding spectrum, we first compute the nearest-neighbour distance of the eigenvalues

$$d_{1,i} \equiv \min_j |E_i - E_j|.$$  \hspace{1cm} (S1)

Next, the local mean density of the eigenvalues is computed as

$$\bar{\rho} = \frac{n}{\pi d_{n,i}^2},$$  \hspace{1cm} (S2)

where $n$ is sufficiently larger than unity ($\approx 30$), which is very small compared to the Hamiltonian matrix size. Here, $d_{n,i}$ is the $n$th nearest-neighbour distance from $E_i$. The rescaled nearest-neighbour distance $s_i$ is obtained as

$$s_i = d_{1,i} \sqrt{\bar{\rho}},$$  \hspace{1cm} (S3)

**FIG. S2.** The disorder-averaged $f_{im}$ as a function of $K$ and $W$ for different $g_τ/g_↓$ ratios. The eigenspectrum distributions of open coupled chains are shown for (a) $g_τ/g_↓ = 0.2$, (b) $g_τ/g_↓ = 0.6$, (c) $g_τ/g_↓ = 1.0$. In the lower panel (d,e,f), the eigenspectrum distributions are shown with periodic boundary condition. With OBC, the $g_τ/g_↓ = 1.0$ case (c) can be gauged out and hence the spectrum remains real in entire $K$-$W$ plane. For open non-Hermitian coupled-chain system, the coupling also drives the system into real eigenspectrum. Here, the eigenspectrum is averaged over 500 disorder realizations.
which removes the dependence of the local density of eigenvalues on the level spacing. Finally, the statistics of nearest-neighbour spacings are computed from \( s_n \). We have provided the nearest-neighbour level distribution as a function of \( K \) and \( W \). For weak disorder, the complex energy spectrum with \( g_\uparrow \neq g_\downarrow \) obeys the Ginibre distribution \( P_{\text{Gin}}(s) = cp(s) \) which describes the ensemble of non-Hermitian Gaussian random matrices \([18, 46, 54]\). Here,

\[
p(s) = \lim\limits_{N \to \infty} \left[ \prod_{n=1}^{N-1} e_n(s^2) e^{-s^2} \right] \sum_{n=1}^{N-1} \frac{2s^{2n+1}}{n! e_n(s^2)}
\]

with \( e_n(x) = \sum_{m=0}^{n} \frac{x^m}{m!} \) and \( c = \int_0^\infty ds \ p(s) = 1.1429 \ [18, 64]\). At larger \( K \) and for equal \( g_\uparrow \)'s of two chain, the real eigenspectrum of weak disorder case follows the level statistics of Gaussian orthogonal ensemble (GOE). The level-spacing distribution of GOE is

\[
P_{\text{GOE}}(s) = \frac{\pi}{2} s \exp(-\pi s^2/4),
\]

as shown in Fig. 2(e). Hence, the general features of the non-Hermitian Hamiltonian with purely real eigenvalues can be mapped to a Hermitian Hamiltonian. Recently, a similar mapping is also shown to be present in non-Hermitian Su-Schrieffer-Heeger model \([65]\). On the other hand, at strong disorder case, the localized phase with real spectrum characterized by the Poisson level distribution \( P_{\text{PO}}(s) = \exp(-s) \). In the main text, we have discussed the spectral statistics using the Euclidean distance and generalized level spacing of the spectrum.

![FIG. S3. The average value of \( \cos(\theta) \) as a function of the (a) inter-chain coupling strength \( K \) at \( W = 2 \) and (b) disorder strengths for \( K = 1 \) and \( g_\uparrow/g_\downarrow = 0.2 \). Here, the evolution with \( K \) is shown for various non-Hermiticity coefficient ratios and (b) is presented for three system sizes. The corresponding evolution of the average level spacing \( \langle r \rangle \) is shown in Fig. 2(a,b). Here, the system size \( L = 8 \) and disorder average is performed for 100 realizations.](image)

![FIG. S4. The nearest neighbour level spacing distributions as a function of the inter-chain coupling strength \( K \) for \( g_\uparrow = g_\downarrow \). The upper (lower) panel represents the distributions with correlated (uncorrelated) disorder for \( L = 8 \). Here, the disorder strength \( W = 2 \).](image)

We further extract the angular distribution and examine its evolution with the model parameters. Likewise, the average level spacing, the single-number signature of \( \langle \cos(\theta) \rangle \) also distinguish the phases. The evolution of the disorder-averaged mean \( \cos(\theta) \) with \( K \) and \( W \) are shown in Fig. S3. We first examine the evolution of \( \langle \cos(\theta) \rangle \) with \( K \) for weak disorder. At lower \( K \), the \( -\langle \cos(\theta) \rangle \) takes positive values for Ginibre level distribution, the constant value \( -\langle \cos(\theta) \rangle \approx 0.2 \) is consistent to the similar
analysis for single Hatano-Nelson chain [54]. As $K$ increases, the average value decreases and becomes negative, which further increases with $K$. The dip in $-\langle \cos(\theta) \rangle$ demarcates two different phases, moreover the minima for larger $g_\uparrow/g_\downarrow$ (for $g_\uparrow \neq g_\downarrow$) occurs at smaller $K$, which is in consonance to eigenspectrum transition and evolution of $\langle r \rangle$. On the other hand, for $g_\uparrow = g_\downarrow$ case the distribution of $-\langle \cos(\theta) \rangle$ does not change sign. We further show the variation of $\langle \cos(\theta) \rangle$ as a function of disorder strength at $K = 1$ and $g_\uparrow/g_\downarrow = 0.2$ in Fig. S3(b). The $-\langle \cos(\theta) \rangle \approx 0.2$ corresponding to the Ginibre level distribution for two largest system sizes. With $W$, the average value lowers, becomes negative and at strong disorder $\langle \cos(\theta) \rangle = 0$ for the Poisson statistics. Hence, the average angular distribution also confirms the localization of interacting coupled chains.

We further show the role of disorder potential on the nearest-neighbour level distributions. For $g_\uparrow = g_\downarrow$ case, the IGT maps the system to the Hermitian case. In the presence of correlated (spin-independent) disorder potential, the delocalized regime shows Poisson behaviour due to SU(2) spin symmetry. This is evident from the level distributions shown for $W = 2$ in the upper panel of Fig. S4. This shows strong dependence of the realized symmetries on the spectral properties of the system. The presence of uncorrelated disorder breaks the SU(2) spin symmetry of the model for $g_\uparrow = g_\downarrow$. This leads to GOE level distribution for delocalized regime ($W = 2$) as a function of $K$, as illustrated in Fig. S4 (lower panel).

![Inverse Participation Ratio](image)

**FIG. S5.** The disorder-averaged inverse participation ratio obtained using $n$th eigenstates ($\bar{I}_n$) and biorthogonal IPR ($\bar{I}_{nB}$) for the system size $L = 8$. Here, the averaged-IPR is shown for two disorder strengths corresponding to delocalized ($W = 2$) and localized ($W = 30$) regime. Here, the OBC is considered. The other parameters of the model are $K = 1$ and $g_\uparrow/g_\downarrow = 0.5$.

**INVERSE PARTICIPATION RATIO**

To gain further insight into the localization behaviour, we show the disorder averaged inverse participation ratio. As discussed in the main text, for the non-Hermitian systems, this observable can be defined in two ways: one is using $n$th left or right eigenstate $\bar{I}_n$ while the other choice is the biorthogonal IPR ($\bar{I}_{nB}$), where the overline denotes the disorder average. The behaviour of $\bar{I}_{nB}$ and $\bar{I}_n$ in whole energy spectrum under OBC are plotted for $K = 1$, $g_\uparrow/g_\downarrow = 0.5$ and $L = 8$. At weak disorder ($W = 2$), the eigenstates show small values of IPR, suggest delocalization. However, the standard IPR $\bar{I}_n$ show a finite value which is attributed to the role of skin effects at low disorder. Since, the biorthogonal density distributions do not suffer from the skin effects, hence $\bar{I}_{nB} \approx 0$ for $W = 2$. At strong $W$, the larger values of IPR (using both definitions) for eigenstates suggest many-body localization.

**DYNAMICAL PROPERTIES**

We finally study the non-equilibrium time dynamics of the non-Hermitian system from the perspective of quantum trajectories with no-jump condition [66], which corresponds to the continuously measured system. By choosing an arbitrary initial state $|\psi_0\rangle$ at $t = 0$, the time dynamics is encoded in the wavefunction

$$|\psi_t\rangle = \frac{e^{-i\mathcal{H}t/\hbar} |\psi_0\rangle}{\sqrt{\langle \psi_0 | e^{i\mathcal{H}t/\hbar} e^{-i\mathcal{H}t/\hbar} | \psi_0 \rangle}}, \quad (S6)$$

which is governed solely by the non-Hermitian effective Hamiltonian $\mathcal{H}$. With this time-dependent wavefunction, all the dynamical properties can be explored. For instance, the half-chain entanglement entropy $S_E(t)$ discussed in the main text is given
FIG. S6. (a) Schematic of the initial state $\hat{c}_1 \dagger \hat{c}_2 \dagger \hat{c}_3 \dagger \hat{c}_4 \dagger \cdots |\text{vac}\rangle$. The red (blue) dots are initially occupied (unoccupied) sites, and the dark to light gradients denote the spatial position from left to right. (b,c) Calculated local particle density at each site for weak ($K = 0.5$, upper panel) and strong ($K = 6$, lower panel) inter-chain couplings and for weak [$W = 2$, (b)] and strong [$W = 20$, (c)] disorder strengths. We find that the saturation value of local particle density in (c) coincides with the corresponding configuration of the considered initial state. In addition, in both the delocalized and localized phases, the non-reciprocal coupling gives rise to the larger local particle density at the site closer to the rightmost site. The plots are obtained by averaging over 500 disorder realizations, and the other parameters are $J = 1$, $g_\uparrow/g_\downarrow = 0.5$, and $U = 1$.

FIG. S7. Asymptotic spin imbalance $I_s$ as a function of the inter-chain coupling $K$ and the disorder strength $W$. For a fixed non-Hermiticity parameter of the second chain $g_\downarrow = 1$, the simulation results at $g_\uparrow = 0.5$. The asymptotic value is obtained by averaging over 50 disorder realizations at $t = 30$, and the other parameters are $J = U = 1$.

by the von Neumann entropy

$$S(t) = -\text{tr}[\rho_A \ln \rho_A],$$  \hspace{1cm} (S7)

where the reduced density matrix is defined as

$$\rho_A = \text{tr}_B[|\psi_t\rangle \langle \psi_t|].$$  \hspace{1cm} (S8)

Apart from the spin imbalance and entanglement entropy in the main text, here we show the dynamics of local particle density for different inter-chain couplings $K$ and disorder strengths $W$ in Fig. S6. At weak disorder regime, the saturation values of local
particle densities with the same spatial indices coincide in Fig. S6(b), which implies the zero spin imbalance. On the other hand, the coincidence between the local particle density at long times in Fig. S6(c) and the corresponding configuration of the given initial state reflects the initial-state memory retention and signifies the MBL. In addition, although the initial state is delocalized in spatial dimension, we observe that in all cases the local particle density shows the spatial localization that stems from the non-reciprocal coupling.

To further investigate the role of this inter-chain coupling $K$, we calculate the asymptotic (at time $t = 30$) spin imbalance $I_s$ at several inter-chain coupling $K$ and disorder strength $W$ in Fig. S7, where the disorder-induced localization transition begins to occur at higher disorder strength as the inter-chain coupling $K$ increases. This result reflects that $K$ couples spins at the same site and destroys the initial spin ordering.