MUTATIONS AND SHORT GEODESICS IN HYPERBOLIC 3-MANIFOLDS

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ABSTRACT. Ruberman has shown that mutations of hyperelliptic surfaces inside hyperbolic 3-manifolds preserve volume. Here, we provide geometric and topological conditions under which such mutations also preserve the initial length spectrum. This work requires us to analyze when least area surfaces could intersect short geodesics in a hyperbolic 3-manifold. As a corollary of this result, we show that the number of hyperbolic knot complements with the same volume and the same initial length spectrum grows at least factorially fast with the volume and the number of twist regions. Furthermore, we show that the knots used for this construction are pairwise incommensurable by analyzing their cusp shapes.

1. INTRODUCTION

The work of Mostow and Prasad implies that every finite volume hyperbolic 3-manifold admits a unique hyperbolic structure, up to isometry [29], [26]. Thus, geometric invariants of a hyperbolic manifold, such as volume and geodesic lengths, are also topological invariants. It is natural to ask: how effective can such invariants be at distinguishing hyperbolic 3-manifolds? Furthermore, how do these invariants interact with one another?

In this paper, we will study how mutations along hyperelliptic surfaces inside of a hyperbolic 3-manifold affect such invariants. A hyperelliptic surface \( F \) is a surface admitting a hyperelliptic involution: an order two automorphism of \( F \) which fixes every isotopy class of curves in \( F \). A mutation of a hyperelliptic surface \( F \) in a hyperbolic 3-manifold \( M \) is an operation where we cut \( M \) along \( F \), and then reglue by a hyperelliptic involution \( \mu \) of \( F \), often producing a new 3-manifold, \( M^\mu \). While a mutation can often change the global topology of a manifold, the action is subtle enough that many geometric, quantum, and classical invariants are preserved under mutation; see [7] for details. In particular, Ruberman showed that mutating hyperbolic 3-manifolds along incompressible, \( \partial \)-incompressible surfaces preserves hyperbolicity and volume in [35].

Here, we investigate under which conditions such mutations preserve the systole length, and even the smallest \( n \) values of the length spectrum, the \textit{initial length spectrum}. The \textit{systole} of a manifold is the shortest geodesic in that manifold. The \textit{length spectrum} of a manifold, \( M \), is the set of all lengths of closed geodesics in \( M \) counted with multiplicites. We say that two manifolds are \textit{iso-length spectral} if they have the same length spectrum. We will also consider the \textit{complex length spectrum} of \( M \): the set of all complex lengths of closed geodesics in \( M \) counted with multiplicities. Given a closed geodesic \( \gamma \subset M \), the \textit{complex length} of \( \gamma \) is the complex number \( \ell_C(\gamma) = \ell(\gamma) + i\theta \) where \( \ell(\gamma) \) denotes the length of \( \gamma \) and \( \theta \) is the angle of rotation incurred by travelling once around \( \gamma \).
Throughout this paper, any surface will be connected, orientable, and of finite complexity, unless stated otherwise. Any hyperbolic 3-manifold $M$ will have finite volume and be connected, complete, and orientable. Our investigation requires a surface that we mutate along to be a least area surface in $M$, or a close variant, to be defined later.

**Definition 1.1** (Least Area Surface in $M$). Let $F \subset M$ be a proper, embedded surface in a 3-manifold $M$. Then $F$ is called a least area surface if $F$ minimizes area in its homotopy class.

Least area surfaces inside closed 3-manifolds were first analyzed by Freedman–Hass–Scott in [9] and Hass and Scott in [13]. Their work showed that incompressible surfaces can always be realized as smoothly immersed least area surfaces. Ruberman expanded this analysis to noncompact surfaces in noncompact hyperbolic 3-manifolds in [35], where he provided conditions for the existence, uniqueness, and embeddedness of least area surfaces in a hyperbolic 3-manifold.

Our main result gives three possible properties of a hyperbolic 3-manifold that can help determine whether or not a closed geodesic $\gamma$ intersects a least area surface. The geometric properties we consider are the maximal embedded tube radius $r$ of a neighborhood of $\gamma$, denoted $T_r(\gamma)$, and the length of $\gamma$, denoted $\ell(\gamma)$. The topological property to be analyzed is the normalized length of a Dehn filling, which we will now describe.

**Definition 1.2** (Dehn Surgery). Let $M$ be a 3-manifold with torus boundary $\partial M$ and $s$ a slope on $\partial M$, that is, an isotopy class of simple closed curves on $\partial M$. The manifold obtained by gluing a solid torus $S^1 \times D^2$ to $\partial M$ in such a way that the slope $s$ bounds a disc in the resulting manifold is called a Dehn surgery along $s$ or a Dehn filling along $s$.

**Definition 1.3** (Normalized Length). Given a Euclidean torus $T$, the normalized length of a slope $s$ is defined to be:

$$\hat{L}(s) = \frac{\text{Length}(s)}{\sqrt{\text{Area}(T)}},$$

where $\text{Length}(s)$ is defined to be the length of a geodesic representative of $s$ on $T$.

Note that, normalized length is scale invariant and well-defined for cusps of $M$.

By a closed curve $n \cdot \gamma$, we mean a simple closed curve that is in the homotopy class of $[n \cdot \gamma] \in \pi_1(\partial T_r(\gamma))$. We can now state one of our main results.

**Theorem 1.4.** Let $M$ be a hyperbolic manifold with $F \subset M$ an embedded least area surface that is incompressible and $\partial$-incompressible with $|\chi(F)| \leq 2$. Let $\gamma \subset M$ be a closed geodesic with embedded tubular radius $r$. Assume

1. $r > 2 \ln(1 + \sqrt{2})$, or
2. $\ell(\gamma) < 0.015$, or
3. $\gamma$ is the core of a solid torus added by Dehn filling $N \cong M \setminus \gamma$ along a slope of normalized length $\hat{L} \geq 22$.

Then $\gamma$ can be isotoped disjoint from $F$. Furthermore, either $\gamma \cap F = \emptyset$ without any isotopy or $n \cdot \gamma$ is isotopic into $F$ for some $n \in \mathbb{N}$.

A few remarks about this theorem:
This theorem is stated in full generality in Theorem 3.11 where no constraints are made on the Euler characteristic. We mainly care about $|\chi(F)| \leq 2$ because every hyperelliptic surface has Euler characteristic $-1$ or $-2$. Also, Theorem 3.11 is stated in terms of almost least area surfaces, which generalize least area surfaces; see Definition 3.1.

(2) implies (1) by the work of Meyerhoff stated in Theorem 3.6 and (3) implies (1) by the work of Hodgson and Kerckhoff [14] and Purcell [31] on cone deformations.

(3) can be stated in terms of Dehn filling multiple curves each of which satisfy $\hat{L} \geq 22$; see Proposition 3.9.

The proof of Theorem 1.4 relies on both the topology and geometry of $F \cap T_r(\gamma)$, where $T_r(\gamma)$ is the embedded tubular neighborhood of radius $r$ around $\gamma$. Since $F$ is incompressible, components of $F \cap T_r(\gamma)$ must be disks or annuli. If a component of $F \cap T_r(\gamma)$ that intersects $\gamma$ is a disk, $D_r$, then we work to get an area contradiction. Specifically, if $r$ is sufficiently large, then the area of $D_r$ inside of this neighborhood will be too big, and so, $\gamma$ must be disjoint from $F$ in this case. As mentioned in the remarks, conditions (2) and (3) each imply (1), so all of our cases rely on a sufficiently large tube radius in the end. If a component of $F \cap T_r(\gamma)$ that intersects $\gamma$ is an annulus, then this annulus must be parallel to the boundary torus $\partial T_r(\gamma)$. Here, $\gamma$ can be isotoped disjoint from $A_r$, and more generally, isotoped disjoint from $F$.

The following corollary to this theorem tells us when the initial length spectrum is preserved under mutation.

**Corollary 1.5.** Let $F \subset M$ be a properly embedded surface that is incompressible, $\partial$-incompressible, and admits a hyperelliptic involution $\mu$. Suppose that $M$ has exactly $n$ geodesics shorter than some constant $L < 0.015$. Then $M$ and $M^\mu$ have the same $n$ initial values of their respective (complex) length spectra.

Under these hypotheses, any sufficiently short geodesic $\gamma$ in $M$ can be isotoped disjoint from $F$. After this isotopy, if we mutate $M$ along $(F, \mu)$ to obtain $M^\mu$, then there will also be a closed curve in $M^\mu$ corresponding with $\gamma$. We just need to analyze the representations $\rho: \pi_1(M) \to \text{PSL}(2, \mathbb{C})$ and $\rho^\mu: \pi_1(M^\mu) \to \text{PSL}(2, \mathbb{C})$ to see that $[\gamma]$, as an element of either $\pi_1(M)$ or $\pi_1(M^\mu)$, has the same representation in $\text{PSL}(2, \mathbb{C})$, and so, the same (complex) length associated to it in either case.

This corollary gives us a tool to produce non-isometric hyperbolic 3-manifolds that have at least the same initial length spectrum. Over the past 35 years, there have been a number of constructions for producing iso-length spectral, non-isometric hyperbolic 3-manifolds. Vignéras in [39] used arithmetic techniques to produce the first known constructions of such manifolds. Sunada developed a general method for constructing iso-length spectral manifolds [37], which helped him produce many iso-length spectral, non-isometric Riemann surfaces. This technique produces covers of a manifold $M$ that are iso-length spectral by finding certain group theoretic conditions on subgroups of $\pi_1(M)$. We will refer to any such group theoretic construction for producing covers that have either the same length spectrum or some variation of this as a Sunada-type construction. Sunada type constructions always produce commensurable manifolds, that is, manifolds that share a common finite-sheeted cover. In fact, by the work of Reid [33], iso-length spectral, non-isometric
arithmetic hyperbolic 3-manifolds are also *always* commensurable. To date, there are no known examples of iso-length spectral, non-isometric hyperbolic 3-manifolds that are not commensurable.

Since Sunada’s original work, many Sunada-type constructions have been developed. These constructions often have very interesting relations to volume. McReynolds uses a Sunada-type construction in [22] to build arbitrarily large sets of closed, iso-length spectral, non-isometric hyperbolic manifolds. Furthermore, the growth of size of these sets of manifolds as a function of volume is super-polynomial. In contrast, Leininger–McReynolds–Neumann–Reid in [19] also use a Sunada-type construction to show that for any closed hyperbolic 3-manifold $M$, there exists infinitely many covers $\{M_j, N_j\}$ of $M$, such that the length sets of these pairs are equal but $\frac{\text{vol}(M_j)}{\text{vol}(N_j)} \to \infty$. Here, the length set of a manifold is the set of all lengths of closed geodesics counted without multiplicities. Thus, volume can behave drastically differently for hyperbolic 3-manifolds that are iso-length spectral as compared with hyperbolic 3-manifolds with the same length set.

The work on producing iso-length spectral, non-isometric hyperbolic 3-manifolds over the past three decades raises a number of questions.

**Question 1.6.** *Are there any constructions for producing iso-length spectral hyperbolic 3-manifolds that do not use Sunada-type constructions or arithmetic methods? Must these constructions always be commensurable?*

Here, we construct large families of mutant pretzel knot complements which have the same initial length spectrum, the same volume, and are pairwise incommensurable. Our construction does not use arithmetic methods or a Sunada-type construction, but rather, the simple cut and paste operation of mutating along Conway spheres. This work is highlighted in the following theorem.

**Theorem 1.7.** *For each $n \in \mathbb{N}$, $n > 2$, there exist $\left(\frac{2n}{2}\right)!$ non-isometric hyperbolic pretzel knot complements that differ by mutation, $\{M_{2n+1}^\sigma\}$, such that these manifolds:*

- *have the same $2n + 1$ shortest geodesic (complex) lengths,*
- *are pairwise incommensurable,*
- *have the same volume,* and
- *$\left(\frac{2n-1}{2}\right) \nu_{\text{oct}} \leq \text{vol}(M_{2n+1}^\sigma) \leq (4n + 2) \nu_{\text{oct}}$, where $\nu_{\text{oct}} (\approx 3.6638)$ is the volume of a regular ideal octahedron.*

In fact, we are able to show that a large class of hyperbolic pretzel knot complements are pairwise incommensurable; see Section 6 for full details. See Section 4 for the definition of a pretzel knot.

**Theorem 1.8.** *Let $n \geq 2$ and let $q_1, \ldots, q_{2n+1}$ be integers such that only $q_1$ is even, $q_i \neq q_j$ for $i \neq j$, and all $q_i$ are sufficiently large. Then the complement of the hyperbolic pretzel knot $K\left(\frac{1}{q_1}, \frac{1}{q_2}, \ldots, \frac{1}{q_{2n+1}}\right)$ is the only knot complement in its commensurability class. In particular, any two of these hyperbolic pretzel knot complements are incommensurable.*

Proving that a particular knot complement is the only knot complement in its commensurability class is generally not an easy task. Only two large classes of knot complements are
known to have this property. Reid and Walsh in [34] have shown that hyperbolic 2-bridge knot complements are the only knot complements in their respective commensurability classes, and similarly, Macasieb and Mattman in [21] have shown this for the complements of hyperbolic pretzel knots of the form $K\left(\frac{1}{2}, \frac{1}{3}, \frac{1}{n}\right)$, $n \in \mathbb{Z} \setminus \{7\}$. Usually the hardest part of this work is showing that these knot complements have no hidden symmetries, that is, these knot complements are not irregular covers of orbifolds. We are able to rule out hidden symmetries by analyzing the cusp shapes of certain untwisted augmented links (see Section 5) that we Dehn fill along to obtain our pretzel knot complements.

Now, let us outline the rest of this paper. In Section 2 we prove the monotonicity of the mass ratio for least area disks in $\mathbb{H}^3$. This result helps give a lower bound on the area of a least area disk inside a ball in $\mathbb{H}^3$. Section 3 gives the proof of Theorem 1.4 and Corollary 1.5 and states these result in their full generality. This section is broken down into subsections, each dealing with one of the conditions to be satisfied for Theorem 1.4. In Section 4 we construct and describe our class of hyperbolic pretzel knots which are mutants of one another. We also highlight a theorem from our past work [24] that describes how many of these mutant pretzel knot complements are non-isometric and have the same volume. In Section 5 we analyze the geometry of our pretzel knots by realizing them as Dehn fillings of untwisted augmented links, whose complements have a very simple polyhedral decomposition. In particular, this analysis allows us to put a lower bound on the normalized lengths of the Dehn fillings performed to obtain our pretzel knot complements, and also, helps determine the cusp shapes of the pretzel knots themselves. In Section 6 we prove that these knots are pairwise incommensurable. In Section 7 we apply Corollary 1.5 to show that our class of pretzel knot complements have the same initial length spectrum. We also give an application to closed hyperbolic 3-manifolds with the same initial length spectrum. Putting all these results together gives Theorem 1.7 in Section 7.

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2. MONOTONICITY OF THE MASS RATIO FOR LEAST AREA DISKS IN $\mathbb{H}^3$

Throughout this section, $\ell(-)$ will denote hyperbolic length, $A(-)$ will denote hyperbolic area, and $B(a,r) \subset \mathbb{H}^3$ will denote a ball of radius $r$ centered at $a$. Here, we establish a useful result for least area disks in $\mathbb{H}^3$.

**Definition 2.1 (Least Area Disk in $\mathbb{H}^3$).** Let $D \subset \mathbb{H}^3$ be a smoothly immersed disk and let $c$ be a simple closed curve in $\mathbb{H}^3$ such that $\partial D = c$. Then $D$ is called a least area disk in $\mathbb{H}^3$, if $D$ minimizes area amongst all smoothly immersed disks with boundary $c$.

The compactness theorem in [25] Theorem 5.5] guarantees that this infimum is always realized for disks in $\mathbb{R}^n$, and similarly, it can be shown for disks in $\mathbb{H}^n$. The following definition will be useful for analyzing least area disks in $\mathbb{H}^3$.

**Definition 2.2 (Mass Ratio and Density).** Let $a \in \mathbb{H}^3$ and consider $A(D \cap B(a,r))$, the area of a disk inside a ball. Define the mass ratio to be
\[ \Theta(D, a, r) = \frac{A(D \cap B(a, r))}{4\pi \sinh^2(\frac{r}{2})}. \]

Define the density of \( D \) at \( a \) to be
\[ \Theta(D, a) = \lim_{r \to 0} \Theta(D, a, r). \]

A few comments about the above definition. First, \( 4\pi \sinh^2(\frac{r}{2}) \) is the area of a disk of radius \( r \) in \( \mathbb{H}^n \). Also, for smoothly immersed surfaces, \( \Theta(D, a) \geq 1 \) at any point \( a \in D \). For an embedded surface we actually have \( \Theta(D, a) = 1 \). If \( D \) is not embedded at a point \( a \in D \), then restricting to a subset of \( D' \) of \( D \) so that \( D' \cap B(a, r) \) is an embedding only decreases the numerator of the mass ratio.

The monotonicity of the mass ratio was proved in the case for Euclidean geometry by Federer [8] and a proof can also be found in Morgan [25, Theorem 9.3]. Here, we obtain a similar result in \( \mathbb{H}^3 \) by using the same techniques as the proof given in Morgan.

**Theorem 2.3.** Let \( D \) be a least area disk in \( \mathbb{H}^3 \). Let \( a \in \mathcal{D} \subseteq \mathbb{H}^3 \). Then for \( 0 < r < d(a, \partial D) \), the mass ratio \( \Theta(D, a, r) \) is a monotonically increasing function of \( r \).

To prove this theorem, we need the following basic fact in hyperbolic trigonometry:

**Lemma 2.4.** \[ \frac{\sinh(\frac{r}{2})}{\cosh(\frac{r}{2})} = \frac{\cosh(r) - 1}{\sinh(r)}, \text{ for } r > 0. \]

**Proof.** \[ \frac{\sinh(\frac{r}{2})}{\cosh(\frac{r}{2})} \leftrightarrow \sinh(\frac{r}{2}) \sinh(r) = (\cosh(r) - 1) \cosh(\frac{r}{2}) \leftrightarrow \sinh(\frac{r}{2})(2 \sinh(\frac{r}{2}) \cosh(\frac{r}{2})) = (\cosh(r) - 1) \cosh(\frac{r}{2}) \leftrightarrow 2 \sinh^2(\frac{r}{2}) = \cosh(r) - 1 \leftrightarrow 2 \sinh^2(\frac{r}{2}) + 1 = \cosh(r), \text{ which holds.} \]

**Proof of Theorem 2.3.** For \( 0 < r < d(a, \partial D) \), let \( f(r) \) denote \( A(D \cap B(a, r)) \). Also, set \( \gamma_r = \partial(D \cap B(a, r)) \) and \( \gamma = \partial(D \cap B(a, 1)) \). Obviously, \( f \) is monotonically increasing, which implies that \( f'(r) \) exists almost everywhere. Now,
\[ f(r) = A(D \cap B(a, r)) = \int_0^r \ell(\gamma_t) ds \geq \int_0^r \ell(\gamma_t) dt, \]
where \( ds \) denotes arc length. Thus,
\[ (1) \quad \ell(\gamma_t) \leq f'(r), \]
by the Fundamental Theorem of Calculus. Since \( D \) is area-minimizing, \( A(D \cap B(a, r)) \leq A(C) \), where \( C \) is the cone over \( \gamma_r \) to \( a \).

**Claim:** \( A(C) = \ell(\gamma_r) \frac{\cosh(r) - 1}{\sinh(r)}. \)

Our area form is \( dA = \sinh(R) dsdR \), where \( ds \) is the change in radius of a hyperbolic sphere and \( dR \) is arc length. We have that
\[ A(C) = \int_0^r \int_0^l(\gamma_t) \sinh(R) dsdR = \ell(\gamma) \int_0^r \sinh(R) dR = \ell(\gamma)(\cosh(r) - 1). \]
In order to rescale to make \( A(C) \) a function of \( \ell(\gamma_r) \), we use the fact that \( \ell(\gamma_r) = \ell(\gamma) \sinh(r) \) to get that \( A(C) = \ell(\gamma_r) \frac{\cosh(r) - 1}{\sinh(r)}. \)
Putting (1) together with the previous claim and Lemma 2.4 gives:
The hyperbolic cone $C$ over $\gamma_r$ to $a$

\[ f(r) \leq A(C) = \ell(\gamma_r) \frac{\cosh(r) - 1}{\sinh(r)} \leq f'(r) \frac{\cosh(r) - 1}{\sinh(r)} = f'(r) \frac{\sinh(\frac{\gamma}{2})}{\cosh(\frac{\gamma}{2})}. \]

Consequently,

\[ \frac{d}{dr} \left[ 4\pi \Theta(D, a, r) \right] = \frac{d}{dr} \left[ f(r) \sinh^{-2}(\frac{\gamma}{2}) \right] = \frac{f'(r)}{\sinh^2(\frac{\gamma}{2})} - \frac{f(r) \cosh(\frac{\gamma}{2})}{\sinh^3(\frac{\gamma}{2})} = \frac{\cosh(\frac{\gamma}{2})}{\sinh(\frac{\gamma}{2})} \left[ f'(r) \frac{\sinh(\frac{\gamma}{2})}{\cosh(\frac{\gamma}{2})} - f(r) \right] \geq 0 \]

since $\frac{\cosh(\frac{\gamma}{2})}{\sinh(\frac{\gamma}{2})} \geq 0$ for any $r > 0$. \qed

The following corollary will play a pivotal role in Section 3.

**Corollary 2.5.** Suppose $D \subset \mathbb{H}^3$ is a least area disk and $a \in \hat{D}$. Then $A(D \cap B(a, r)) \geq 4\pi \sinh^2(\frac{\gamma}{2})$, for any $r$, $0 < r \leq d(a, \partial D)$.

**Proof.** Let $D \subset \mathbb{H}^3$ be a least area surface and $a \in \hat{D} \subset \mathbb{H}^3$. Since $\Theta(D, a, r)$ is increasing with $r$, we have that:

\[ \Theta(D, a) = \lim_{t \to 0} \Theta(D, a, t) \leq \Theta(D, a, r) = \frac{A(D \cap B(a, r))}{4\pi \sinh^2(\frac{\gamma}{2})}, \]

for any $0 < r < d(a, \partial D)$. By continuity of the area function, we can extend this up to $r = d(a, \partial D)$.

Now, being smoothly immersed implies that $\Theta(D, a) \geq 1$ for all $a \in \hat{S}$. By the above, we have that $A(D \cap B(a, r)) \geq 4\pi \sinh^2(\frac{\gamma}{2})$, for any $0 < r \leq d(a, \partial D)$, as desired. \qed

**3. Least area surfaces and short geodesics in hyperbolic 3-manifolds**

First, let us set some notation. Let $M$ be a hyperbolic 3-manifold. The universal cover of $M$ is $\mathbb{H}^3 = \{(x, y, z) \mid z > 0\}$, and there exists a covering map $p : \mathbb{H}^3 \to M$. Let $T_r(\gamma)$ denote an embedded tubular neighborhood of radius $r$ about a closed geodesic $\gamma \subset M$. $\gamma$ lifts to a
geodesic, \( \gamma \), in \( \mathbb{H}^3 \), and we will assume that the endpoints of \( \gamma \) are 0 and \( \infty \). Let \( T_r(\gamma) \) be the corresponding lift of \( T_r(\gamma) \).

Let \( F \) be a surface in \( M \) realized by the map \( \varphi : S \rightarrow F \). Suppose \( \gamma \cap F \neq \emptyset \), and say \( p_0 = \varphi(s_0) \in \gamma \cap F \subset M \). Let \( S \) be the universal cover of \( S \), and denote by \( p_1 \) the covering map \( p_1 : S \rightarrow S \). Let \( s_0 \in S \) be a point with \( p_1(s_0) = s_0 \) and let \( \tilde{\varphi} : S \rightarrow \mathbb{H}^3 \) be a lift of \( \varphi \) such that \( \tilde{p}_0 = \tilde{\varphi}(s_0) \) is a point in \( \tilde{\gamma} \). We have the following commutative diagram.

\[
\begin{array}{ccc}
(\tilde{S}, \tilde{s}_0) & \xrightarrow{\varphi} & (\mathbb{H}^3, \tilde{p}_0) \\
\downarrow p_1 & & \downarrow p \\
(S, s_0) & \xrightarrow{\varphi} & (M, p_0)
\end{array}
\]

The focus of the following subsections is to prove a number of propositions that can tell us when \( \gamma \) can be isotoped disjoint from \( F \) based on a variety of geometric and topological properties. Specifically, we will be interested in the tube radius of \( \gamma \), the length of \( \gamma \), and particular Dehn filling slopes. We will then use these conditions to show when the initial length spectrum can be preserved under mutation. We will always be working with an almost least area surface \( F \) that is incompressible and \( \partial \)-incompressible in a hyperbolic 3-manifold \( M \). The existence and embeddedness of such surfaces is provided by the following result of Ruberman. First, we define an almost least area surface.

**Definition 3.1 (Almost Least Area Surface in \( M \)).** Let \( F \subset M \) be a proper, embedded surface in a 3-manifold \( M \). A surface \( F \subset M \) is called almost least area if \( F \) is either a least area surface, or is the boundary of an \( \varepsilon \)-neighborhood of a one-sided embedded least area surface \( F' \).

**Remark:** Theorems about almost least area surfaces hold for all \( \varepsilon \) sufficiently small.

**Theorem 3.2.** [35, Theorem 1.6] Let \( F \subset M \) be a properly embedded surface that is incompressible and \( \partial \)-incompressible. Then \( F \) can be properly isotoped to an almost least area surface.

In the next three subsections, we shall be using the following functions to describe when \( \gamma \cap F = \emptyset \):

**Definition 3.3 (Important Functions).**

- \( f(x) = 2 \sinh^{-1}\left(\sqrt{\frac{x}{2}}\right) \),
- \( k(\ell(\gamma)) = \cosh\left(\sqrt{\frac{\pi \ell(\gamma)}{\sqrt{3}}}\right) - 1, \)
- \( g(x) = 2 \sinh^2(2 \sinh^{-1}\left(\sqrt{\frac{x}{2}}\right)) + 1, \)
- \( F(w) = \frac{-(1+4w+6w^2+w^4)}{(w+1)(1+w^2)^2}, \)
- \( I(r) = \frac{2(2\pi)^2}{5.3957(1-r)} \exp\left(\int_1^r F(w) dw\right), \)
- \( h(x) = I(f(x)). \)

### 3.1. Least area surfaces and the tube radius of \( \gamma \)

The following proposition tells us that a closed geodesic \( \gamma \) can be isotoped disjoint from a least area surface, if \( \gamma \) has a sufficiently
large embedded tubular radius. This fact can also be shown using [11, Lemma 4.3]. However, here we provide additional topological and geometric information about $\gamma \cap F$. Recall that by a closed curve $n \cdot \gamma$, we mean a simple closed curve that is in the homotopy class of $[n \cdot \gamma] \in \pi_1(\partial T_r(\gamma))$.

**Proposition 3.4.** Let $\gamma \subset M$ be a closed geodesic with embedded tubular radius $r$, and let $F$ be an almost least area surface in $M$ that is incompressible and $\partial$-incompressible. Set $f(x) = 2 \ln(\sqrt{x^2} + \sqrt{\frac{x^2}{2} + 1}) = 2 \sinh^{-1}(\sqrt{x^2})$. Assume $r > f(|\chi(F)|)$. Then $\gamma$ can be isotoped disjoint from $F$. Furthermore, either $\gamma \cap F = \emptyset$ without any isotopy or $n \cdot \gamma$ is isotopic into $F$ for some $n \in \mathbb{N}$. In particular, if $|\chi(F)| \leq 2$, then our result holds whenever $r > 2 \ln(1 + \sqrt{2})$.

In order to prove this proposition, we will need the following lemma, which gives a lower bound on the area of a least area disk inside a tubular neighborhood of a geodesic.

**Lemma 3.5.** Let $\gamma \subset M$ be a closed geodesic with embedded tubular neighborhood $T_r(\gamma)$. Suppose $D_r$ is a least area disk in $M$ such that $\gamma \cap D_r \neq \emptyset$ and $\partial D_r \subset \partial T_r(\gamma)$. Then $A(D_r \cap T_r(\gamma)) \geq 4 \pi \sinh^2(\frac{r}{2})$.

**Proof.** Since $\pi_1(D_r)$ is trivial, $D_r$ lifts isometrically to a disk $\tilde{D}_r \subset T_r(\gamma) \subset \mathbb{H}^3$, with $\partial \tilde{D}_r \subset \partial T_r(\gamma)$ and $\tilde{p}_0 \in \tilde{D}_r \cap \tilde{\gamma}$. Since $D_r$ is least area and $D_r$ lifts isometrically to $\tilde{D}_r$, $\tilde{D}_r$ is a least area disk in $\mathbb{H}^3$ for the boundary curve $c = \partial \tilde{D}_r$. See figure 2. By Corollary 2.5, $A(\tilde{D}_r \cap B(\tilde{p}_0, r)) \geq 4 \pi \sinh^2(\frac{r}{2})$. Therefore,

$$A(D_r \cap T_r(\gamma)) = A(\tilde{D}_r) \geq 4 \pi \sinh^2(\frac{r}{2}),$$

as desired. \hfill \square

![Figure 2. The lift of a disk $D_r$ to $\mathbb{H}^3$](image)

**Proof of Proposition 3.4.** Set $F_r = F \cap T_r(\gamma)$. Assume that $\gamma \cap F \neq \emptyset$.

**Claim:** Each component of $F_r$ is a disk or an annulus.
Recall that $F$ is an orientable surface in an orientable 3-manifold $M$. Theorem 3.2 tells us that $F$ is embedded as an almost least area surface. Since $F$ is a two-sided surface that is incompressible in $M$, the loop theorem tells us that $\pi_1(F_r) \leq \pi_1(T_r(\gamma)) \cong \mathbb{Z}$. Each component of $F_r$ is an orientable surface with boundary on $\partial T_r(\gamma)$. The only such surfaces that meet our fundamental group condition are disks and annuli.

Now, we will prove the two possibilities for $\gamma \cap F$. Then we will show that in either case, $\gamma$ can be isotoped disjoint from $F$.

**Case 1:** A component of $F_r$ is a disk that intersects $\gamma$. Say $D_r$ is a disk component of $F_r$ that intersects $\gamma$. We have the following area inequality:

$$2\pi |\chi(F)| \geq A(F) > A(F_r) \geq A(D_r \cap T_r(\gamma)) \geq 4\pi \sinh^2\left(\frac{r}{2}\right).$$

The first inequality comes from the Gauss-Bonnet Theorem, combined with properties of minimal surfaces (see Futer-Purcell [11, Lemma 3.7]). If $F$ has been homotoped from a least area surface to an $\varepsilon$-neighborhood of a one-sided embedded least area surface $F'$, then we choose $\varepsilon$ sufficiently small so that $A(F)$ changes by an arbitrarily small amount, and so, our area inequality will still hold. The last inequality comes from Lemma 3.5. This gives us that $\sqrt{\frac{|\chi(F)|}{2}} \geq \sinh\left(\frac{r}{2}\right)$. Recall that $\sinh^{-1}(y) = \ln(y + \sqrt{y^2 + 1})$ and that $\sinh(x)$ is an increasing function. Thus,

$$f(|\chi(F)|) = 2\sinh^{-1}\left(\sqrt{\frac{|\chi(F)|}{2}}\right) = 2\ln\left(\sqrt{\frac{|\chi(F)|}{2}} + \sqrt{\frac{|\chi(F)|}{2} + 1}\right) \geq r.$$

So, if $\gamma$ has a large enough embedded tubular radius, we will have a contradiction, specifically, if $r > f(|\chi(F)|)$. In particular, if $|\chi(F)| \leq 2$, then $r > 2\ln(1 + \sqrt{2}) = f(|\chi(F)|)$ will provide the necessary area contradiction, and so, $\gamma \cap F = \emptyset$.

**Case 2:** Every component of $F_r$ that intersects $\gamma$ is an annulus.

Suppose $A_r$ is an annulus component of $F_r$ that intersects $\gamma$. In this case, the inclusion map $i : A_r \to T_r(\gamma)$, induces an injective homomorphism $i_* : \pi_1(A_r) \hookrightarrow \pi_1(T_r(\gamma))$ with $[\alpha] \mapsto [n \cdot \gamma]$ for some $n \in \mathbb{N}$, where $[\alpha]$ is the homotopy class of the core of the annulus $A_r$. Now, $[\alpha]$ can be represented by a curve $\alpha$ on a component of $\partial A_r$, with $A_r$ providing the isotopy between the core and the boundary component. Since $\partial A_r \subset \partial T_r(\gamma)$, $\alpha$ is isotopic into the boundary torus $\partial T_r(\gamma)$, providing a satellite knot of the form $n \cdot \gamma$ on $\partial T_r(\gamma)$.

The only thing left to be shown is that $\gamma$ can be isotoped disjoint from $F$ in both cases. Obviously, if $\gamma \cap F = \emptyset$, then no isotopy needs to even take place. So, suppose $n \cdot \gamma$ is isotopic into $F$. The proof of case 2 explains the topology of such a situation. Specifically, the annulus $A_r$ is boundary parallel to $\partial T_r(\gamma)$, and so, could be isotoped disjoint from $\gamma$. Equivalently, we could keep $A_r$ fixed (since it is part of our least area surface $F$) and isotope $\gamma$ so that this closed curve is disjoint from $A_r$, and more generally, disjoint from $F$. □

It is important to note that case 2 of Proposition 3.4 is certainly a possibility and can be an obstruction to a useful lower bound estimate on $A(F)$. Techniques similar to the proof of Theorem 2.3 can be used to find a lower bound for $A(F \cap T_r(\gamma))$ when every component is an annulus, but the lower bound is of the form $C_0 \cdot \ell(\gamma) \cdot \sinh(r)$, where $C_0 > 0$ is a constant. It is possible to put a hyperbolic metric on a surface $F$ so that a specific geodesic is arbitrarily short and contains an embedded collar of area $2\ell(\gamma) \cdot \sinh(r)$. So, if $\ell(\gamma)$ is
sufficiently short and $\gamma$ actually lies on $F$, then the quantity $C_0 \cdot \ell(\gamma) \cdot \sinh(r)$ could be too small to be useful for our purposes.

3.2. Least area surfaces and the length of $\gamma$. Next, we will examine when $\gamma$ can be isotoped disjoint from $F$ based on the length of $\gamma$. To do this, we will need to use the Collar Lemma, which essentially says that the shorter the length of a closed geodesic in a hyperbolic 3-manifold, the larger the embedded tubular neighborhood of that geodesic. This result comes from Meyerhoff [23]:

**Theorem 3.6 (Collar Lemma).** Let $\gamma \subset M$ be a closed geodesic in a hyperbolic 3-manifold with (real) length $\ell(\gamma)$. Suppose $\ell(\gamma) < \frac{\sqrt{2}}{4\pi} \left[ \ln(\sqrt{2} + 1) \right]^2 \approx 0.107$. Then there exists an embedded tubular neighborhood around $\gamma$ whose radius $r$ satisfies

$$\sinh^2(r) = \frac{1}{2} \left( \frac{\sqrt{1 - 2k}}{k} - 1 \right)$$

where $k(\ell(\gamma)) = \cosh\left( \sqrt{\frac{4\pi\ell(\gamma)}{\sqrt{3}}} \right) - 1$.

**Proposition 3.7.** Let $\gamma \subset M$ be a closed geodesic, and let $F$ be an almost least area surface in $M$ that is incompressible and $\partial$-incompressible. Set $g(x) = 2 \sinh^2(2 \min^{-1}(\sqrt{x})) + 1$. Assume $\frac{\sqrt{1 - 2k(\ell(\gamma))}}{k(\ell(\gamma))} > g(|\chi(F)|)$. Then $\gamma$ can be isotoped disjoint from $F$. Furthermore, either $\gamma \cap F = \emptyset$ without any isotopy or $n \cdot \gamma$ is isotopic into $F$ for some $n \in \mathbb{N}$. In particular, if $|\chi(F)| \leq 2$ our result holds whenever $\ell(\gamma) < 0.015$.

**Proof.** We will use the Collar Lemma to show that if $\ell(\gamma)$ is sufficiently small, then the tube radius $r$ is sufficiently large. Then Proposition 3.4 will give us the desired result. So, we need to see when $r > \max(|\chi(F)|) = 2 \sinh^{-1}(\sqrt{\frac{|\chi(F)|}{2}})$. Assume that $\ell(\gamma) < 0.107$, so the Collar Lemma applies. Then we have $\sinh^2(r) = \frac{1}{2} \left( \frac{\sqrt{1 - 2k}}{k} - 1 \right)$ where $k(\ell(\gamma)) = \cosh\left( \sqrt{\frac{4\pi\ell(\gamma)}{\sqrt{3}}} \right)$.

Now, $k(\ell(\gamma))$ is an increasing function on $0 < \ell(\gamma) < \infty$ with $k(\ell(\gamma)) \to 0$ as $\ell(\gamma) \to 0$, while $\frac{1}{2} \left( \frac{\sqrt{1 - 2k}}{k} - 1 \right)$ is a decreasing function $(0 < k \leq \frac{1}{2})$, which heads to $\infty$ as $k \to 0$. So, as $\ell(\gamma) \to 0$, $\sinh^2(r) = \frac{1}{2} \left( \frac{\sqrt{1 - 2k}}{k} - 1 \right) \to \infty$.

Specifically, we need the following inequality to hold:

$$r = \sinh^{-1}\left( \frac{1}{2} \left( \frac{\sqrt{1 - 2k}}{k} - 1 \right) \right) > 2 \sinh^{-1}\left( \sqrt{\frac{|\chi(F)|}{2}} \right),$$

$$\frac{1}{2} \left( \frac{\sqrt{1 - 2k}}{k} - 1 \right) > \sinh^2\left( 2 \sinh^{-1}\left( \sqrt{\frac{|\chi(F)|}{2}} \right) \right),$$

$$\frac{\sqrt{1 - 2k}}{k} > 2 \sinh^2\left( 2 \sinh^{-1}\left( \sqrt{\frac{|\chi(F)|}{2}} \right) \right) + 1 = g(|\chi(F)|).$$

For the case when $|\chi(F)| \leq 2$, we need $r > 2 \ln(1 + \sqrt{2})$ from Proposition 3.4 and so, we check when the inequality

$$\left( \frac{\sqrt{1 - 2k}}{k} \right) \geq 2 \sinh^2\left( 2 \ln(1 + \sqrt{2}) \right) + 1 = 17$$
is satisfied. This occurs when $\ell(\gamma) < 0.015$, giving the desired result. \qed

3.3. Least area surfaces and Dehn filling slopes. Now, we would like to examine when $\gamma \cap F = \emptyset$ based on certain Dehn filling slopes. In order to do this, we need to go over some background on Dehn fillings and cone deformations.

Given a hyperbolic 3-manifold $M$ with a cusp corresponding to a torus boundary on $\partial M$, we choose a basis $\langle m, l \rangle$ for the fundamental group of the torus. After this choice of basis, we can form the manifold $M(p, q)$ obtained by doing a $(p, q)$-Dehn surgery on the cusp, where $(p, q)$ is a coprime pair of integers. A $(p, q)$-Dehn surgery maps the boundary of the meridian disk to $s = pm + ql$. Similarly, we can form the manifold $M((p_1, q_1), \ldots, (p_k, q_k))$ by performing a $(p_i, q_i)$-Dehn surgery on the $i^{th}$ cusp of $M$, for each $i$, $1 \leq i \leq k$.

Thurston has shown that $M((p_1, q_1), \ldots, (p_k, q_k))$ is in fact a hyperbolic 3-manifold for all $((p_1, q_1) \ldots (p_k, q_k))$ near $(\infty, \ldots, \infty)$; see [38]. Following Thurston’s work, many people have developed techniques to more explicitly understand the change in geometry under Dehn surgery. Cone deformations are one such tool, which we will use here. This machinery has been developed by Hodgson and Kerckhoff in [14] and expanded upon in Purcell [31]. We are interested in the change in geometry under Dehn surgery when we fill certain cusps of a hyperbolic 3-manifold. If this Dehn surgery can be realized via a cone deformation, then many interesting geometric bounds can be obtained. Specifically, if the normalized lengths of the slopes on which Dehn fillings were performed were sufficiently large, then Hodgson and Kerckhoff showed that these Dehn fillings can be realized by a hyperbolic cone deformation [14].

The following lemma is essentially Lemma 6.6 from [31].

**Lemma 3.8.** Let $M$ and $N$ be hyperbolic 3-manifolds such that $M = N((p_1, q_1), \ldots, (p_k, q_k))$ and this Dehn filling is realized by a cone deformation. Fix $r \geq 0.531$. Then the tube radius is greater than or equal to $r$ throughout the deformation provided that all Dehn filling slopes satisfy

$$\hat{L}^2 \geq \frac{2(2\pi)^2}{3.3957(1 - \tanh r)} \exp(\int_1^{\tanh r} F(w)dw) = I(r),$$

where $F(w) = \frac{-(1 + 4w + 6w^2 + w^4)}{w(w+1)(1+w^2)^2}$.

For our purposes, we want to make sure the resulting tube radii are sufficiently large in order to apply Proposition 3.4 and obtain the following result. In what follows, $I(x)$ is the function from Lemma 3.8 and $f(x)$ is the function from Proposition 3.4. The definitions of these functions can be found in Definition 3.3.

**Proposition 3.9.** Let $\{\gamma_1, \ldots, \gamma_k\} \subset M$ be a set of closed geodesics which came from Dehn filling cusps of a hyperbolic 3-manifold $N$. Let $F$ be an almost least area surface in $M$ that is incompressible and $\partial$-incompressible. Set $h(x) = I(f(x))$. Assume that $L(\gamma_i) \geq h(|\chi(F)|)$ for each $i$, $1 \leq i \leq k$. Then each $\gamma_i$ can be isotoped disjoint from $F$. Furthermore, either $\gamma_i \cap F = \emptyset$ without any isotopy or $n \cdot \gamma_i$ is isotopic into $F$ for some $n \in \mathbb{N}$. In particular, if $|\chi(F)| \leq 2$, then our result holds whenever each $L(\gamma_i) \geq 22$. 

Proof. Here, we just need to make sure that the tube radius, \( r \), satisfies \( r > f(\chi(\mathcal{F})) \) to apply Proposition 3.4. Lemma 3.8 provides a formula to find a lower bound on tube radius in terms of normalized length. When \( |\chi(\mathcal{F})| \leq 2 \), we consider \( r > 2\ln(1 + \sqrt{2}) \) and use Lemma 3.8. Doing the explicit calculation gives the desired result.

Remark: Proposition 3.9 gives us conditions for \( \gamma \cap \mathcal{F} = \emptyset \) based on Dehn filling slopes. Rather than applying Lemma 3.8 and then using Proposition 3.4, we could have obtained a similar result via different work. Specifically, we could apply [15, Corollary 5.13] to bound the length of the core geodesics that come from Dehn filling in terms of normalized length. We could then use Proposition 3.7 to obtain the desired result. However, the inequality reached using this method would not be as sharp as the one stated in Proposition 3.9.

3.4. Hyperelliptic surfaces and mutations that preserve short geodesics. We will prove in this section that mutating along hyperelliptic surfaces inside hyperbolic 3-manifolds will provide a method for preserving the initial length spectrum. In what follows, let \( S_{g,n} \) denote a surface of genus \( g \) and \( n \) boundary components.

Recall that a hyperelliptic surface \( S \) is a surface that admits at least one non-trivial involution \( \mu \) of \( S \) so that \( \mu \) fixes every isotopy class of curves in \( S \). The hyperelliptic surfaces are \( S_{2,0}, S_{1,2}, S_{1,1}, S_{0,3}, \) and \( S_{0,4} \). These are all surfaces with Euler characteristic \(-1\) or \(-2\). For our constructions in Section 4, we will examine hyperelliptic surfaces that arise in hyperbolic knot complements. An \( S_{0,4} \) in a knot complement is called a Conway sphere.

For a Conway sphere there are three hyperelliptic (orientation preserving) involutions, given by 180° rotations about the x-axis, y-axis, and z-axis, respectively, as shown in figure 3.

**Figure 3.** A standard Conway sphere.

**Definition 3.10 (Mutation).** A mutation of a hyperelliptic surface \( S \) in a 3-manifold \( M \) is the process of cutting \( M \) along \( S \) and then regluing by one of the nontrivial involutions of \( S \) to obtain the 3-manifold \( M^\mu \). If \( K \) is a knot in \( S^3 \) with a Conway sphere \( S \), then cutting \((S^3,K)\) along \((S,S \cap K)\) and regluing by a mutation, \( \mu \), yields a knot \( K^\mu \subset S^3 \).

In order to determine when geodesic lengths are preserved under mutation, we first summarize the conditions under which \( \gamma \) can be isotoped disjoint from \( F \). Recall that the definitions of all of the following functions can be found in Definition 3.3.
Theorem 3.11. Let $M$ be a hyperbolic manifold with $F \subset M$ an almost least area surface that is incompressible and $\partial$-incompressible. Let $\gamma \subset M$ be a closed geodesic with embedded tubular radius $r$. Assume

(1) $r > f(|\chi(F)|)$, or 
(2) $\sqrt{\frac{2L-k(L)}{k(L)}} > g(|\chi(F)|)$, or 
(3) $\gamma$ is the core of a solid torus added by Dehn filling $N \cong M \setminus \gamma$ along a slope of normalized length $\beta \geq h(|\chi(F)|)$.

Then $\gamma$ can be isotoped disjoint from $F$. Furthermore, either $\gamma \cap F = \emptyset$ without any isotopy or $n \cdot \gamma$ is isotopic into $F$ for some $n \in \mathbb{N}$.

In particular, if $|\chi(F)| \leq 2$, then our result holds whenever

(1) $r > 2 \ln(1 + \sqrt{2})$, or 
(2) $\ell(\gamma) < 0.015$, or 
(3) $\beta \geq 22$.

Remark: Condition (3) can be stated in terms of Dehn filling multiple cusps. See Proposition 3.9.

Theorem 3.11 will help us analyze which mutant hyperbolic 3-manifolds have the same initial length spectra. To do this, we first need to see how $\pi_1(M)$ and $\pi_1(M^\mu)$ are related as amalgamated products and HNN-extensions along $\pi_1(F)$. What follows, is a standard application of van Kampen’s Theorem. In fact, Kuessner in [17] gives a different proof of Ruberman’s result about mutations and volume that uses this decomposition of $\pi_1(M)$ and $\pi_1(M^\mu)$ along with the Maskit combination theorem and homological arguments.

Lemma 3.12. Let $F \subset M$ be a properly embedded surface that is incompressible, $\partial$-incompressible, and admits a hyperelliptic involution $\mu$. If $F$ separates $M$ into two submanifolds, $M_a$ and $M_b$, then $\pi_1(M) = \pi_1(M_a) \ast \pi_1(F) \pi_1(M_b)$ and $\pi_1(M^\mu) = \pi_1(M_a) \ast \pi_1(F) \beta \pi_1(M_b) \beta^{-1}$, for some $\beta \in \text{PSL}(2, \mathbb{C})$. If $F$ is non-separating, then $\pi_1(M) = \pi_1(N) \ast \psi_c$ and $\pi_1(M^\mu) = \pi_1(N) \ast \psi_{c, \mu}$.

Proof: First, suppose that $F$ is separating in $M$. Then $\pi_1(M) = \pi_1(M_a) \ast \pi_1(F) \pi_1(M_b)$. $M^\mu$ is also constructed by cutting $M$ along $F$, and then gluing the pieces $M_a$ and $M_b$ back together along $F$. However, we now rotate one of these pieces, say $M_b$, before gluing it back to $M_a$ along $F$. Thus, there exists $\beta \in \text{PSL}(2, \mathbb{C})$ such that $\pi_1(M^\mu) = \pi_1(M_a) \ast \pi_1(F) \beta \pi_1(M_b) \beta^{-1}$.

Here, $\beta$ is an order two element of $\text{PSL}(2, \mathbb{C})$ that corresponds with $\mu$.

Suppose that $F$ is non-separating in $M$. Then there is a manifold $N$ with $\partial N = F_1 \cup F_2$ such that $M$ is the quotient of $N$ under some homeomorphism $\psi : F_1 \to F_2$. Choose a basepoint $x_0$ in $F_1$, and let $u$ be a path in $N$ from $x_0$ to $\psi(x_0)$. Let $H \subset \pi_1(N)$ be the subgroup consisting of all homotopy classes of loops of the form $u \cdot \alpha \cdot u^{-1}$, where $\alpha$ is a loop in $F_2$ based at $\psi(x_0)$. Our attaching map $\psi$ induces an isomorphism $\psi_* : \pi_1(F_1) \to H$ defined by $\psi_*([\sigma]) = [u \cdot \psi(\sigma) \cdot u^{-1}]$. Then $\pi_1(M) = \pi_1(N) \ast \psi_c$. Now, $\mu$ induces an isomorphism $\mu_* : \pi_1(F) \to \pi_1(F)$. When we attach $N$ along its boundary to obtain $M^\mu$, we don’t attach by just $\psi*$, but rather, $\psi \circ \mu$. This gives $\pi_1(M^\mu) = \pi_1(N) \ast \psi_{* \mu}$. \qed

Set $G_L(M) = \{ \gamma \subset M : \gamma$ is a closed geodesic and $\ell(\gamma) < L \}$. 
Corollary 3.13. Let $F \subset M$ be a properly embedded surface that is incompressible, $\partial$-incompressible, and admits a hyperelliptic involution $\mu$. Suppose that $M$ has exactly $n$ geodesics shorter than some constant $L < 0.015$. Then $M$ and $M^\mu$ have the same $n$ initial values of their respective (complex) length spectra. In other words, there exists a bijective correspondence between $G_L(M)$ and $G_L(M^\mu)$.

Proof. Let $\{\gamma_i\}_{i=1}^n = G_L(M)$. By Theorem 3.11, we can isotope any such $\gamma_i$ disjoint from $F$, and assume we have performed these isotopies. We claim that for each $\gamma_i$, mutation along $F$ will produce a closed geodesic $\gamma_i'$ in $M^\mu$, such that $\ell_C(\gamma_i) = \ell_C(\gamma_i')$.

Proof of claim: First, suppose that $F$ separates $M$. By Lemma 3.12, we have that $\pi_1(M) = \pi_1(M) *_{\pi_1(F)} \pi_1(M)$ and $\pi_1(M^\mu) = \pi_1(M) *_{\pi_1(F)} \beta \pi_1(M)$ $\beta^{-1}$. Since any $\gamma_i \subset M$ has been isotope disjoint from $F$, $\gamma_i \in M_a$, or $\gamma_i \in M_b$. Without loss of generality, assume $[\gamma_i] \in \pi_1(M)$, i.e., $\gamma_i$ now lies in $M_a$. $[\gamma_i] \in \pi_1(M)$ has a unique (complex) length associated to it, $\ell_C(\gamma_i)$, coming from the representation $\rho : \pi_1(M) \to \text{PSL}(2, \mathbb{C})$. This (complex) length is determined by the trace of its representation. Specifically, $\cosh(\frac{\ell_C(\gamma_i)}{2}) = \pm tr(\gamma)$, where $tr(\gamma)$ denotes the trace of the representation of $\gamma$. Since we have isotope $\gamma_i$ disjoint from $F$, mutating along $F$ to obtain $M^\mu$ will produce a corresponding homotopy class $[\gamma_i'] \in \pi_1(M^\mu)$. Similarly, $[\gamma_i'] \in \pi_1(M^\mu)$ also has a unique (complex) length associated to it, coming from $\rho_\mu : \pi_1(M^\mu) \to \text{PSL}(2, \mathbb{C})$. For both $\pi_1(M)$ and $\pi_1(M^\mu)$, $[\gamma_i]$ and $[\gamma_i']$ have the same representation in $\text{PSL}(2, \mathbb{C})$. This is true because in both cases these representations come from the restriction to $\pi_1(M_a)$, and these representations of $\pi_1(M_a)$ into $\text{PSL}(2, \mathbb{C})$ are the same since $\mu_*$ acts by conjugation in $\text{PSL}(2, \mathbb{C})$ [35, Theorem 2.2]. So, the same complex length is associated to $\gamma_i$ and $\gamma_i'$, as desired.

If $F$ is non-separating in $M$, then Lemma 3.12 gives us that $\pi_1(M) = \pi_1(N) *_{\psi'} \pi_1(N)$ and $\pi_1(M^\mu) = \pi_1(N) *_{\psi_\mu} \pi_1(N)$. Since $\gamma_i$ has been isotope disjoint from $F$, we once again have that $[\gamma_i]$ and $[\gamma_i']$, as elements of $\pi_1(M)$ and $\pi_1(M^\mu)$ respectively, have the same representation in $\text{PSL}(2, \mathbb{C})$, and so, the same complex length associated to them.

It remains to show that there exists a bijective correspondence between $G_L(M)$ and $G_L(M^\mu)$, since this will imply that the $n$ shortest geodesics in $M^\mu$ are exactly the set $\{\gamma_i'\}_{i=1}^n$ coming from mutating the set $\{\gamma_i\}_{i=1}^n = G_L(M)$. Let $f : G_L(M) \to G_L(M^\mu)$ be the function defined by $f(\gamma_i) = \gamma_i'$, for each $\gamma_i \in G_L(M)$. This map is obviously one-to-one: if $\gamma_i' = f(\gamma_i) = f(\gamma_j) = \gamma_j'$, then mutating $M^\mu$ along $(F, \mu)$ to obtain $M$ implies $\gamma_i = \gamma_j$. Now, suppose $f$ is not onto, and so, there exists some $\gamma' \in G_L(M^\mu)$ such that $\gamma' \notin \{\gamma_i\}_{i=1}^n$. Mutate $M^\mu$ by $(F, \mu)$ to obtain $M$. Since $\gamma' \in G_L(M^\mu)$, $\ell(\gamma') < L < 0.015$, which implies that $\gamma'$ can be isotope disjoint from $F$. As the proof of our claim shows, this implies that there is a corresponding $\gamma \in M$ with the same complex length as $\gamma'$. However, then $\gamma < L$, i.e. $\gamma \in G_L(M)$, which is a contradiction. Thus, $f$ gives a bijective correspondence between $G_L(M)$ and $G_L(M^\mu)$, as desired. 

Remark: This corollary uses the length condition from Theorem 3.11 to determine when $M$ and its mutant $M^\mu$ have the same initial length spectra. We also get similar corollaries (highlighted below) using this proof, based upon the tube radius condition and the normalized length condition. However, with the tube radius condition, we cannot guarantee that these common geodesic lengths are the shortest ones in the length spectra of $M$ and $M^\mu$. 

\[\square\]
since there can exist geodesics with a very large embedded tube radius that are not very short. Thus, we can only say that a portion of these length spectra are the same, not necessarily the initial length spectra. Fortunately, for the normalized length condition, we can still get a corollary that determines when $M$ and $M^\mu$ have the same initial length spectra.

**Corollary 3.14.** Let $F \subset M$ be a properly embedded surface that is incompressible, $\partial$-incompressible, and admits a hyperelliptic involution $\mu$. Suppose that $M$ has exactly $n$ geodesics with embedded tubular radius larger than some constant $R > 2 \ln(1 + \sqrt{2})$. Then $M$ and $M^\mu$ have at least $n$ common values in their respective (complex) length spectra.

**Corollary 3.15.** Let $F \subset M$ be a properly embedded surface that is incompressible, $\partial$-incompressible, and admits a hyperelliptic involution $\mu$. Suppose that $M$ has exactly $n$ geodesics that are the core geodesics coming from Dehn filling a hyperbolic 3-manifold $N$. Let $\hat{L}(\gamma_i)$ denote the normalized slope of the $i^{th}$ Dehn filling.

- If $\hat{L}(\gamma_i) \geq 22$ for each $i$, $1 \leq i \leq n$, then $M$ and $M^\mu$ have at least $n$ common values in their respective (complex) length spectra.
- There exists a constant $D > 0$ such that if $\hat{L}(\gamma_i) \geq D$ for each $i$, $1 \leq i \leq n$, then $M$ and $M^\mu$ have the same $n$ initial values of their respective (complex) length spectra.

**Proof.** The remark following Proposition 3.9 tells us that if the normalized slopes of a Dehn filling are sufficiently long, then the lengths of the resulting core geodesics can be made arbitrarily short. An explicit constant $D$ could be determined so that the lengths of the resulting core geodesics are all smaller than some constant $L$, as in Corollary 3.13. Finally, Thurston’s Dehn Surgery Theorem [2] implies that as the slopes of these Dehn fillings head to $\infty$, the other geodesic lengths stabilize. So, if our normalized slopes are sufficiently long, we can guarantee our core geodesics are the only geodesics in $M$ shorter than the cut off length $L$. Corollary 3.13 now gives the desired result. \[\square\]

4. Hyperbolic Pretzel Knots: $\{K_{2n+1}\}_{n=2}^\infty$

Here, we construct a specific class of pretzel knots, $\{K_{2n+1}\}_{n=2}^\infty$. We will be able to show that for each $n \geq 2$, $K_{2n+1}$ generates a large number of mutant pretzel knots whose complements all have the same volume and initial length spectrum. This section describes pretzel links, their classification, and the basic properties of $\{K_{2n+1}\}_{n=2}^\infty$.

4.1. Pretzel Links. We shall describe vertical tangles and see how they can be used to construct pretzel links. Afterwards, we will give a simple classification of pretzel links.

**Definition 4.1** (Pretzel link). The vertical tangles, denoted by $\frac{1}{n}$, are made of $n$ vertical half-twists, $n \in \mathbb{Z}$, as depicted in figure 4. A pretzel link, denoted $K\left(\frac{1}{q_1}, \frac{1}{q_2}, \ldots, \frac{1}{q_n}\right)$, is defined to be the link constructed by connecting $n$ vertical tangles in a cyclic fashion, reading clockwise, with the $i^{th}$-tangle associated with the fraction $\frac{1}{q_i}$.

$K$ in figure 7 is the pretzel link $K = K(\frac{1}{4}, \frac{1}{7}, \frac{1}{9})$. Note that, each vertical tangle corresponds with a twist region for a knot diagram of a pretzel link. Twist regions are defined at the beginning of Section 5.1.
Now, we state the classification of pretzel links, which is a special case of the classification of Montesinos links. The classification of Montesinos links was originally proved by Bonahon in 1979 \cite{5}, and another proof was given by Boileau and Siebenmann in 1980 \cite{3}. A proof similar to the one done by Boileau and Siebenmann can be found in \cite{6, Theorem 12.29}. Here, we state the theorem solely in terms of pretzel links.

**Theorem 4.2.** \cite{5} The pretzel links $K\left(\frac{1}{q_1}, \frac{1}{q_2}, \ldots, \frac{1}{q_n}\right)$ and $K'\left(\frac{1}{q'_1}, \frac{1}{q'_2}, \ldots, \frac{1}{q'_n}\right)$ with $n \geq 3$ and $\sum_{j=1}^{n} \frac{1}{q_j} \leq n - 2$, are classified by the ordered set of fractions $\left(\frac{1}{q_1} \mod 1, \ldots, \frac{1}{q_n} \mod 1\right)$ up to the action of the dihedral group generated by cyclic permutations and reversal of order, together with the rational number $\sum_{j=1}^{n} \frac{1}{q_j}$.

### 4.2. Our Construction

Consider the pretzel link $K_{2n+1} = K\left(\frac{1}{q_1}, \frac{1}{q_2}, \ldots, \frac{1}{q_{2n+1}}\right)$, where each $q_i > 6$, $q_1$ is even, each $q_i$ is odd for $i > 1$, and $q_i \neq q_j$ for $i \neq j$. We will always work with the diagram of $K_{2n+1}$ that is depicted below in figure 5. Each $R_i$ in this diagram of $K_{2n+1}$ represents a twist region in which the vertical tangle $\frac{1}{q_i}$ takes place. For $n \geq 2$, $K_{2n+1}$ has the properties listed below; details can be found in \cite{24}. Though our construction here is slightly different, it still retains all the same key properties listed below.

1. Each $K_{2n+1}$ is a knot (link with a single component).
2. Each $K_{2n+1}$ is a hyperbolic knot.
3. This diagram of $K_{2n+1}$ is alternating.
4. This diagram of $K_{2n+1}$ is prime and twist-reduced (definitions can be found in \cite{10}).

We are focused on keeping our constructions knots because by the Gordon–Luecke Theorem \cite{12}, we know that knots are determined by their complements.
4.3. Mutations of $K_{2n+1}$ that preserve volume. In this subsection, we will see how mutations can be useful for preserving the volume of a large class of hyperbolic 3-manifolds $\{M^\sigma_{2n+1}\}$, with $M^\sigma_{2n+1} = S^3 \setminus K^\sigma_{2n+1}$. $K^\sigma_{2n+1}$ is one of our hyperbolic pretzel knots constructed in Section 4.2 and the upper index $\sigma$ signifies a combination of mutations along Conway spheres, which we will now describe.

Given a $K_{2n+1}$, consider the set $\{(S_a, \sigma_a)\}_{a=1}^{2n}$ where $S_a$ is a Conway sphere that encloses only $R_a$ and $R_{a+1}$ on one side, and $\sigma_a$ is the mutation along $S_a$ which rotates about the $y$-axis. On one of our pretzel knots, such a mutation $\sigma_a$ interchanges the vertical tangles $R_a$ and $R_{a+1}$, as depicted in figure 6. In terms of our pretzel knot vector, such a mutation just switches $\frac{1}{q_a}$ and $\frac{1}{q_{a+1}}$.

In [24], we used the following theorem proved by Ruberman to construct many hyperbolic knot complements with the same volume.

**Theorem 4.3.** [35] Let $\mu$ be any mutation of an incompressible and $\partial$-incompressible hyperelliptic surface in a hyperbolic 3-manifold $M$. Then $M^\mu$ is also hyperbolic, and $\text{vol}(M^\mu) = \text{vol}(M)$.

Ruberman’s proof of this theorem requires the hyperelliptic surface $S$ to be isotoped into least area form in order to perform a volume-preserving mutation of a hyperbolic 3-manifold $M$ along $S$. This fact will be crucial, considering the conditions for Theorem 3.11.

By the proof of [24 Theorem 2], for a given $M_{2n+1} = S^3 \setminus K_{2n+1}$, performing combinations of mutations along the collection $\{(S_a, \sigma_a)\}_{a=1}^{2n}$ produces a large number of non-isometric hyperbolic knot complements with the same volume, and this number grows as $n$ increases. Specifically, we have:

**Theorem 4.4.** [24] For each $n \in \mathbb{N}$, $n > 2$, there exist $\frac{(2n)!}{2}$ distinct hyperbolic pretzel knots, $\{K^\sigma_{2n+1}\}$, obtained from each other via mutations along the Conway spheres $\{(S_a, \sigma_a)\}$. Furthermore, for each such $n$,

- their knot complements have the same volumes, and
- $\left(\frac{2n-1}{2}\right) v_{\text{oct}} \leq \text{vol}(M^\sigma_{2n+1}) \leq (4n+2) v_{\text{oct}}$, where $v_{\text{oct}} (\approx 3.6638)$ is the volume of a regular ideal octahedron.

5. The Geometry of Untwisted Augmented Links

The goal of this section is to better understand the geometry and topology of our pretzel knots by realizing them as Dehn fillings of untwisted augmented links. Recall that $K_{2n+1} =$
with $q_1$ even, while the rest are odd and distinct. We can realize each $K_{2n+1}$ as a Dehn surgery along specific components of a hyperbolic link $L_{2n+1}$. We want to find a lower bound on the normalized length of the Dehn filling slopes along each of these components in order to apply Corollary 3.15. We also can understand the cusp shape of $K_{2n+1}$ by studying the cusp shape of this knot in $L_{2n+1}$. This will be helpful for determining that these knots are pairwise incommensurable in Section 6. The following analysis will help us determine the properties we are interested in.

5.1. Augmented Links. First, we will go over some basic properties of knots. We usually visualize a knot by its diagram. A diagram of a knot can be viewed as a 4-valent planar graph $G$, with over-under crossing information at each vertex. Here, we will need to consider the number of twist regions in a given diagram. A twist region of a knot diagram is a maximal string of bigons arranged from end to end. A single crossing adjacent to no bigons is also a twist region. We also care about the amount of twisting done in each twist region. We describe the amount of twisting in terms of half twists and full twists. A half twist of a twist region of a diagram consists of a single crossing of two strands. A full twist consists of two half twists. Now, we can define augmented links, which were introduced by Adams [1] and have been studied extensively by Futer and Purcell in [10] and Purcell in [30], [31]. For an introduction to augmented links, we suggest first reading [32].

**Definition 5.1 (Augmented Links).** We augment a diagram of a knot or link $K$ by inserting a simple closed curve encircling each twist region. These simple closed curves are called crossing circles. An augmented link is a link which is formed from a knot or link diagram by augmenting each twist region, giving a new diagram $L'$. The top two diagrams in figure 7 show a link $K$ with three twist regions and then the corresponding augmented link $L'$. Suppose in the diagram of $L'$, the $i^{th}$ twist region consists of $t_i$ full twists, plus possibly a half twist. Obtain a new link $L$ by removing all $t_i$ full twists from the diagram of $L'$, for each $i$. Then $L$ has a diagram consisting of crossing circle components bounding components from the link. Near each crossing circle, the link component is embedded in the projection plane if the corresponding twist region contained only full twists. Otherwise, there is a single half twist. We shall refer to the link $L$ as the untwisted augmented link. The untwisted augmented link $L$ is made up of two types of components: the crossing circles and the other components coming from the original link $K$. We shall refer to these other components as the knot components of $L$. When $K$ is a knot, there is a single knot component in $L$, which will be the case for our work. If we remove all of the remaining single crossings from the twist regions, then we form the flat augmented link, $J$. The bottom two diagrams of figure 7 show the corresponding untwisted augmented link and flat augmented link.

The 3-manifolds $S^3 \setminus L$ and $S^3 \setminus L'$ actually are homeomorphic. Performing $t_i$ full twists along the punctured disk bound by a crossing circle and then regluing this disk gives a homeomorphism between link exteriors. Thus, if either $S^3 \setminus L$ or $S^3 \setminus L'$ is hyperbolic, then Mostow-Prasad rigidity implies that the two manifolds are isometric.

Next, we shall examine the polyhedral decompositions of certain untwisted augmented links. We will do this by first examining such structures on the corresponding flat augmented links, which are almost the same, but easier to initially analyze. We will also
analyze the geometry of a knot $K$ by realizing $K$ as a Dehn filling of its corresponding untwisted augmented link $L$.

5.2. **Ideal Polyhedral Decompositions of Untwisted Augmented Links.** The polyhedral decompositions of untwisted augmented link complements have been thoroughly described in [10]. This polyhedral decomposition was first described by Agol and Thurston in the appendix of [18], and many of its essential properties are highlighted in the following theorem.

**Theorem 5.2.** Let $L$ be the untwisted augmented link corresponding to a link $K$. Assume the given diagram of $K$ is prime, twist-reduced, and $K$ has at least two twist regions. Then $S^3 \setminus L$ has the following properties:

1. $S^3 \setminus L$ has a complete hyperbolic structure.
2. This hyperbolic 3-manifold decomposes into two identical ideal, totally geodesic polyhedra, $I$ and $I'$, all of whose dihedral angles are $\frac{\pi}{2}$.
3. The faces of $I$ and $I'$ can be checkerboard colored, shaded and white.
(4) *Shaded faces come in pairs on each polyhedron, and they are constructed by peeling apart a single 2-punctured disc bounded by a crossing circle.*

(5) *White faces come from portions of the projection plane bounded by knot strands.*

Here, we will briefly describe this decomposition and the resulting circle packings, with emphasis on our untwisted augmented link complements, \( N_{2n+1} = S^3 \setminus L_{2n+1} \). We direct the reader to [31, Sections 6 and 7] for more details on cusp shape analysis of untwisted augmented link complements.

**Figure 8.** The untwisted augmented link \( L_{2n+1} \) and the flat augmented link \( J_{2n+1} \).

First, consider \( S^3 \setminus J_{2n+1} \), where \( J_{2n+1} \) is the flat augmented link, whose diagram is shown in figure 8. In the diagram of \( J_{2n+1} \), the knot strands all lie on the projection plane. To subdivide \( S^3 \setminus J_{2n+1} \) into polyhedra, first slice it along the projection plane, cutting \( S^3 \) into two identical 3-balls. These identical polyhedra are transformed into ideal polyhedra by collapsing strands of \( J_{2n+1} \) to ideal vertices. These ideal polyhedra have two types of faces: shaded faces and white faces, described in the above theorem.

To go from an ideal polyhedral decomposition of \( S^3 \setminus J_{2n+1} \) to one for \( S^3 \setminus L_{2n+1} \), we just have to introduce a half-twist into our gluing at each shaded face where a crossing circle bounds a single twist. Depicted in figure 9 below is an ideal polyhedral decomposition of the flat augmented pretzel knot, \( J_{2n+1} \).

These polyhedra actually have totally geodesic faces in \( \mathbb{H}^3 \), meeting in dihedral angles of \( \frac{\pi}{2} \). Such totally geodesic polyhedra are constructed by finding an appropriate circle packing via Andreev’s theorem, and cutting away half spaces bounded by hemispheres in \( \mathbb{H}^3 \). The following theorem is a corollary of Andreev’s theorem noted by Thurston in [38].
Theorem 5.3 (Andreev). Let \( \gamma \) be a triangulation of \( S^2 \) such that each edge has distinct ends and no two vertices are joined by more than one edge. Then there is a circle packing of \( S^2 \) whose nerve is isotopic to \( \gamma \). This circle packing is unique up to Möbius transformation.

The nerve of a circle packing is the graph obtained by associating a vertex to each circle, and an edge connecting two vertices if and only if the corresponding circles are tangent. Our polyhedral decomposition gives a triangulation of \( S^2 \). Place a vertex on each white face, and connect two vertices by an edge if and only if the two corresponding white faces meet at a vertex of the polyhedron. This gives a triangulation of \( S^2 \). In [31, Section 6], it is shown that this triangulation satisfies the conditions of Andreev’s theorem. Figure 10 shows the corresponding nerve for \( J_{2n+1} \) and figure 11 shows the resulting circle packing.

The decomposition of \( S^3 \setminus L_{2n+1} \) is determined by this circle packing. First, slice off half-spaces bounded by geodesic hemispheres in \( \mathbb{H}^3 \) corresponding to each circle in the circle packing. These give the geodesic white faces of the polyhedron. The shaded faces are obtained by slicing off hemispheres in \( \mathbb{H}^3 \) corresponding to each circle of the dual circle.
packing. Finally, we just need to make sure we glue up most of the shaded faces with a half-twist. Only the two shaded faces corresponding to the first twist region are glued up without a half-twist.

5.3. Normalized Lengths on Cusps. For this section, we will specialize our analysis to just our pretzel knot complements \( M_{2n+1} = \mathbb{S}^3 \setminus K_{2n+1} \) which result from Dehn filling the \( 2n + 1 \) crossing circles, \( \{ C_i \}_{i=1}^{2n+1} \), of \( N_{2n+1} = \mathbb{S}^3 \setminus L_{2n+1} \). Recall that \( K_{2n+1} \) has \( 2n + 1 \) twist regions with \( q_i \) crossings in the \( i \)th twist region, and in \( L_{2n+1} \), exactly \( 2n \) of these crossing circle encloses a single crossing since \( 2n \) of our \( q_i \) are odd. To apply Corollary 3.15 we will need the following information on normalized lengths of particular slopes on cusps in \( N_{2n+1} \).

**Proposition 5.4.** On the cusps of \( N_{2n+1} \) corresponding to crossing circles, we have the following normalized lengths: Let \( s_i \) be the slope such that Dehn filling \( N_{2n+1} \) along \( s_i \) re-inserts the \( q_i - 1 \) or \( q_i \) crossings at that twist region. Then \( \hat{L}(s_i) \geq \sqrt{\frac{(2n-1)(1+q_i^2)}{4n}}. \) In particular, if \( n \geq 2 \), we have that \( \hat{L}(s_i) \geq \sqrt{\frac{3(1+q_i^2)}{8}}. \)

**Proof.** Pictured in figure 11 is a circle packing for \( J_{2n+1} \) coming from the white faces. There also exists a circle packing for the shaded faces, which is dual to the circle packing coming from the white faces. These two circle packings also determines the same circle packings for \( L_{2n+1} \) since the only difference between \( L_{2n+1} \) and \( J_{2n+1} \) is how the two ideal polyhedra are glued together. Much of what follows in the next two paragraphs is done in [10, Sections 2 and 3]. In their work, the cusp shapes are analyzed with respect to any augmented link, while we will specialize to our \( L_{2n+1} \).

First, let us recall our polyhedra obtained in Section 5.2. Each cusp will be tiled by rectangles given by the intersection of the cusp with the totally geodesic white and shaded...
faces of the polyhedra. Two opposite sides of each of these rectangles come from the intersection of the cusp with shaded faces of the polyhedra (corresponding with the 2-punctured disc in the diagram of $L_{2n+1}$), and the other two sides from white faces. Call these sides shaded sides and white sides, respectively. We can make an appropriate choice of cusp neighborhoods as in [10, Section 3]. This allows us to consider the geometry of our rectangles tiling a cusp.

Our crossing circle cusp is tiled by two rectangles, each rectangle corresponding with a vertex in one of the polyhedra. In terms of our circle packing of $S^2$, this vertex corresponds with a point of tangency of two circles. Consider the point of tangency given by $P_i \cap P_{i+1}$, which corresponds to one of the two identical rectangles making up the crossing circle cusp $C_{i+1}$. By the rotational symmetry of the circle packing in figure [11] all of these rectangles (along with their circle packings) are in fact isometric. Thus, taking a step along a shaded side will be the same for any such rectangle, and similarly for stepping along a white side. Let $s$ represent taking one step along a shaded face and $w$ represent taking one step along a white face. Each torus cusp, $T$, has universal cover $\tilde{T} = \mathbb{R}^2$. $\tilde{T}$ contains a rectangular lattice coming from the white and shaded faces of our polyhedron. We let $(s, w)$ be our choice of basis for this $\mathbb{Z}^2$ lattice.

Now, we shall examine the normalized length in terms of our longitudes and meridians of the cusps corresponding to crossing circles. Lemma 2.6 from [10] tells us that the meridian is given by $w \pm s$ when there is a half-twist, and the meridian is $w$ without the half-twist. In either case, the longitude is given by $2s$. When $q_i$ is odd, $\frac{q_i - 1}{2}$ full twists were removed in constructing $L_{2n+1}$, so the surgery slope for the $i^{th}$ crossing circle will be $(1, \frac{q_i - 1}{2})$. Thus, the slope $s_i$ is given by $(w \pm s) \pm \frac{q_i - 1}{2}(2s) = w \pm q_is$, when $q_i$ is odd. For the single even $q_i$, the surgery slope is $(1, \frac{q}{2})$ and the slope is given by $w \pm \frac{q}{2}(2s) = w \pm q_is$; see [10, Theorem 2.7]. In either case, the normalized length of $s_i$ is:

$$\hat{L}(s_i) = \sqrt{\ell(w)^2 + q_i^2\ell(s)^2} \over 2\ell(w)\ell(s).$$

Here, $\ell(w)$ and $\ell(s)$ denote the lengths of $w$ and $s$ respectively, on our choice of cusp neighborhoods. To bound the normalized length, we need to bound $\ell(w)$ and $\ell(s)$. We shall use our circle packing to obtain such bounds. Consider the tangency given by $P_i \cap P_{i+1}$, which corresponds to one of the two rectangles making up our cusp. Note that, $P_i$ is also tangent to circles $P_{i-1}, A$, and $B$, while $P_{i+1}$ is also tangent to $P_{i+2}, A$, and $B$ (values taken mod $2n+1$). Apply a Möbius transformation taking $P_i \cap P_{i+1}$ to infinity. This takes the two tangent circles $P_i$ and $P_{i+1}$ to parallel lines, as in figure [12]. This also gives the similarity structure of the rectangle under consideration. Our choice of cusp neighborhoods results in $\ell(s) = 1$. This makes the circles $A$ and $B$ lying under the dashed lines in figure [12] have diameter 1. Since circles in our circle packing can not overlap, this forces $\ell(w) \geq 1$. Note that, the dashed lines come from our dual circle packing corresponding to shaded faces.

Now, we just need to find an upper bound for $\ell(w)$. Again, consider figure [12]. Since $P_j$ is tangent to $A, B, P_{j-1}$, and $P_{j+1}$ for $1 \leq j \leq 2n+1$, all the circles $P_1, \ldots, P_{i-1}, P_{i+1}, \ldots, P_{2n+1}$ lie in between our parallel lines and in between $A$ and $B$, stacked together as depicted in figure [12] to meet our tangency conditions. Notice, that this circle packing of one of
these rectangles has two lines of symmetry: the line $l_w$ going through the two $w$ sides in their respective midpoints, and the line $l_s$ going through the two $s$ sides in their respective midpoints. $l_w$ is a translate of $s$ and $l_s$ is a translate of $w$. Reflecting across either of these lines preserves our circle packing. In particular, if $P_{i} \cap P_{i+1}$ to $\infty$.

Next, the fact that we have symmetries about both $l_s$ and $l_w$ and an odd number of $P_j$ packed in between $A$ and $B$ implies that one of our $P_j$'s is centered at $l_s \cap l_w$. Call this circle $P_j^*$. Note that, $l_s$ intersects $A$, $B$, and $P_j^*$ in their respective centers. Thus, $\ell(w) = \ell(l_s) = \frac{D(A)}{2} + \frac{D(B)}{2} + D(P_j^*) = 1 + D(P_j^*)$.

Now, we claim that $P_j^*$ has the minimal diameter amongst $P_j$, $j \neq i, i+1$. This follows from our tangency conditions: each such $P_j$ must be tangent to both $A$ and $B$. The diameter of $P_j^*$ obviously minimizes the distance between $A$ and $B$. For any other $P_j$, consider the line $l_j$ in $P_j$ from $P_j \cap A$ to $P_j \cap B$. Then we have that $D(P_j^*) \leq \ell(l_j) \leq D(P_j)$. The first inequality holds because $D(P_j^*)$ minimizes distance from $A$ to $B$, while the second inequality is obviously true for any circle. So, $D(P_j^*)$ must be the smallest such diameter.

Finally, we have $1 = \ell(s) = \sum_{j \neq i,i+1} D(P_j) \geq \sum_{j \neq i,i+1} D(P_j^*) = (2n-1)D(P_j^*)$, which implies that $D(P_j^*) < \frac{1}{2n-1}$. This helps give us the desired upper bound on $\ell(w)$:

$$\ell(w) = \ell(l_s) = 1 + D(P_j^*) \leq 1 + \frac{1}{2n-1} = \frac{2n}{2n-1}.$$  

With these bounds, we have that

$$\tilde{L}(s_i) = \frac{\sqrt{\ell(w)^2 + q_i^2} \ell(s_i)}{2\ell(w)\ell(s)} = \sqrt{\frac{\ell(w)^2 + q_i^2}{\ell(w)}} \geq \frac{\sqrt{1+q_i^2}}{\sqrt{\frac{2n}{2n-1}}} = \sqrt{\frac{2n-1}{2n}}.$$  

In particular, if $n \geq 2$, we have that $\tilde{L}(s_i) \geq \sqrt{\frac{3(1+q_i^2)}{8}}$. □
We will also need to analyze the cusp shape of the one cusp $C$ corresponding to the knot component of $L_{2n+1}$. Such an analysis will play an important role in determining that our knot complements are not commensurable with one another. We will see that the tiling of the cusp $C$ by rectangles which come from truncating certain vertices of our ideal polyhedral decomposition has a number of nice properties, highlighted in the following proposition.

**Proposition 5.5.** Let $C$ be the cusp corresponding to the knot component of $L_{2n+1}$. This cusp has the following properties:

1. There are $4(2n + 1)$ rectangles tiling this cusp, half of which come from each ideal polyhedron.
2. This cusp shape is rectangular.
3. All of these rectangles, along with their circle packings, are isometric to one another.

**Proof.** As in the previous proposition, we can make an appropriate choice of cusp neighborhoods as in [10, Section 3], which allows us to fix the geometry of our cusp $C$.

1. Consider the ideal polyhedral decomposition in figure 9 for $J_{2n+1}$. There are $2n + 1$ disks corresponding to crossing circles, and we peel each of these disks apart to obtain $2(2n + 1)$ shaded faces on each polyhedron. For each shaded face, there are two vertices corresponding to rectangles that tile the knot component cusp $C$; specifically, the two vertices meeting $A$ or the two vertices meeting $B$, depending on the face. Since each of these vertices is shared by exactly two shaded faces, we obtain $2(2n + 1)$ rectangles from each polyhedron, or $4(2n + 1)$ total such rectangles.

2. This holds if there are no half-twists under any of the crossing circles, as in $J_{2n+1}$; see [10, Section 2]. However, $L_{2n+1}$ has $2n$ half-twists in its diagram. A half-twist shifts the gluing of the rectangles making up the cusp. Since $K$ is a knot, it must go through each crossing circle twice, and so, it will pass through an even number of half-twists. Thus, from Lemma 2.6 in [10], the fundamental domain for this torus is given by the meridian $2s$ and the longitude $2(2n + 1)w + 2ks$, for some integer $k$. By a change of basis, we can see that this cusp shape is once again rectangular.

3. Consider the circle packing depicted in figure 11. The rectangles tiling our cusp $C$ come from mapping $P_i \cap A$ to $\infty$ or mapping $P_i \cap B$ to $\infty$ for $i = 1, \ldots, 2n + 1$. By the rotational symmetry of this circle packing, any $P_i \cap A$ and $P_j \cap A$ will determine isometric rectangles, and similarly for $P_i \cap B$ and $P_j \cap B$. In fact, $P_i \cap A$ and $P_j \cap B$ will also determine isometric rectangles. The circle packings of these rectangles are exactly the same except the roles of $A$ and $B$ have been switched; see figure 13. □

Without loss of generality, we will assume any such rectangle coming from the tiling of our knot cusp looks like the one depicted in figure 13, i.e., we assume $P_i \cap A$ is mapped to $\infty$. 
Figure 13. The cusp shape of any one of the rectangles tiling our knot cusp C.

Lemma 5.6. Let R be any rectangle from the tiling of C. Let $P_j^*$ be the smallest such $P_j$ in the circle packing of this rectangle. Then for all $n \geq 2$, the circle packing of R has the following size bounds:

1. $\ell(s) = 1$ and $1 < \ell(w) < 2$,
2. $\frac{n-2}{n-1} < D(B) < 1$,
3. $D(B) > \frac{1}{2}$,
4. $D(P_j^*) < \frac{1}{n-1}$.

Proof. As before, our choice of cusp neighborhood results in $\ell(s) = 1$. Then $D(P_1) = D(P_3) = 1$. We will assume our rectangle is the one depicted in figure 13. By part 3 of Proposition 5.5, all such rectangles tiling our cusp, along with their circle packings, are isometric to this one, up to relabelling.

First, we claim that for any $L_{2n+1}$, $1 < \ell(w) < 2$. The lower bound follows from the fact that $D(P_1) = D(P_3) = 1$, and $P_1$ and $P_3$ can not be tangent to one another. If $\ell(w) > 2$, then $D(B) > 1$ in order to be tangent to both $P_1$ and $P_3$. However, since $\ell(s) = 1$, $B$ would not be tangent to $A$ and $P_2$. If $\ell(w) = 2$, then $D(B) = 1$ in order to meet its tangency conditions. Since $\ell(s) = 1$, $B$ must separate our rectangle into two parts, one to the right of $B$ and one to the left of $B$. This violates the tangency conditions of the $P_j$, for $j = 4, \ldots, 2n+1$. So, $1 < \ell(w) < 2$ and $D(B) < 1$.

Take the vector $w$ and translate it vertically so it intersects $P_j^*$ in its center. This line will intersect all the $P_j$ in some segment $l(P_j)$, which must be at least as large as $D(P_j^*)$. This can easily be seen by translating $P_j^*$ horizontally along this line so that its point of tangency with $A$ is $P_j \cap A$. Note that, there are exactly $2n-2$ circles $\{P_j\}_{j=4}^{2n+1}$ packed under $B$. This gives the following inequality:

$$2 > \ell(w) > \sum_{j=4}^{2n+1} l(P_j) \geq \sum_{j=4}^{2n+1} D(P_j^*) = (2n-2)D(P_j^*).$$

This gives us that $D(P_j^*) < \frac{2}{2n-2} = \frac{1}{n-1}$.
Now, for any such \( j \), \( D(B) + D(P_j) \geq 1 \). Combining with the previous result, we have that
\[
D(B) \geq 1 - D(P_j)^n > 1 - \frac{1}{n-1} = \frac{n-2}{n-1},
\]
as desired.

Finally, we need to show that \( D(B) > \frac{1}{2} \). This is already true if \( n > 2 \) since \( \frac{n-2}{n-1} < D(B) \).

So, assume \( n = 2 \), which means there are exactly two circles, \( P_4 \) and \( P_5 \), packed under \( B \).

Suppose \( D(B) \leq \frac{1}{2} \). Then \( D(P_4) > \frac{1}{2} \) since \( D(B) + D(P_4) > 1 \). Also, \( \ell(w) \leq D(B) + \frac{D(P_4)}{2} + \frac{D(P_3)}{2} = \frac{3}{2} \). Take the vector \( w \) and translate it vertically so that it intersects \( P_4 \) in its center, and take the vector \( s \) and translate it horizontally so that it intersects \( P_3 \) in its center. We shall still refer to the translates of these vectors as \( w \) and \( s \), respectively. Now consider the right triangle with vertices at the center of \( P_3 \), \( w \cap s \), and the left end point of \( P_4 \cap w \).

The hypotenuse, \( c \), of this triangle has length at least \( \frac{1}{2} \) since \( \frac{D(P_4)}{2} = \frac{1}{2} \). The height, \( a \), has length less than \( \frac{1}{2} \) since \( \frac{D(P_3)}{2} = \frac{1}{2} \) and \( \frac{D(P_3)}{2} \leq \frac{1}{2} \). The base, \( b \), has length less than \( \frac{1}{2} \) since \( \ell(w) \leq \frac{3}{4} \) and \( D(P_4) > \frac{1}{2} \). This gives us that \( \frac{1}{4} \leq \ell(c)^2 = \ell(a)^2 + \ell(b)^2 \leq \frac{1}{16} + \frac{1}{16} = \frac{1}{8} \), which is a contradiction. Thus, \( D(B) > \frac{1}{2} \).

\[ \square \]

6. Commensurability Classes of Hyperbolic Pretzel Knot Complements

Recall that two hyperbolic 3-manifolds \( M_1 = \mathbb{H}^3 / \Gamma_1 \) and \( M_2 = \mathbb{H}^3 / \Gamma_2 \) are called commensurable if they share a common finite-sheeted cover. In terms of fundamental groups, this definition is equivalent to \( \Gamma_1 \) and a conjugate of \( \Gamma_2 \) in \( \text{PSL}(2, \mathbb{C}) \) sharing some finite index subgroup. The commensurability class of a hyperbolic 3-manifold \( M \) is the set of all 3-manifolds commensurable with \( M \).

We are interested in the case when \( M = \mathbb{S}^3 \setminus K \), where \( K \) is a hyperbolic knot. It is conjectured in [34] that there are at most three knot complements in the commensurability class of a hyperbolic knot complement. In particular, Reid and Walsh show that when \( K \) is a hyperbolic 2-bridge knot, then \( M \) is the only knot complement in its commensurability class. Their work provides a criterion for checking whether or not a hyperbolic knot complement is alone in its commensurability class. Specifically, we have the following theorem coming from Reid and Walsh’s work in [34] Section 5]; this version of the theorem can be found at the beginning of [21].

\[ \text{Theorem 6.1. Let } K \text{ be a hyperbolic knot in } \mathbb{S}^3. \text{ If } K \text{ admits no hidden symmetries, has no lens space surgery, and admits either no symmetries or else only a strong inversion and no other symmetries, then } \mathbb{S}^3 \setminus K \text{ is the only knot complement in its commensurability class.} \]

Macasieb and Mattman use this criterion in [21] to show that for any hyperbolic pretzel knot of the form \( K \left( \frac{1}{2}, \frac{1}{2}, \frac{1}{n} \right), n \in \mathbb{Z} \setminus \{7\} \), its knot complement \( \mathbb{S}^3 \setminus K \left( \frac{1}{2}, \frac{1}{2}, \frac{1}{n} \right) \) is the only knot complement in its commensurability class. The main challenge in their work was showing that these knots admit no hidden symmetries.

\[ \text{Definition 6.2. Let } \Gamma \text{ be a finite co-volume Kleinian group. The normalizer of } \Gamma \text{ is } \]
\[ N(\Gamma) = \{ g \in \text{PSL}(2, \mathbb{C}) : g\Gamma g^{-1} = \Gamma \}. \]
The commensurator of $\Gamma$ is

$$C(\Gamma) = \{ g \in \text{PSL}(2, \mathbb{C}) : |\Gamma \cap g\Gamma^{-1}| < \infty \text{ and } |g\Gamma^{-1} : \Gamma \cap g^{-1}\Gamma g| < \infty \}.$$  

If $N(\Gamma)$ is strictly smaller than $C(\Gamma)$, then $\Gamma$ and $\mathbb{H}^3/\Gamma$ are said to have hidden symmetries. If $\mathbb{H}^3/\Gamma \cong S^3 \setminus K$, then we also say that $K$ admits hidden symmetries.

Here, we would also like to apply Reid and Walsh’s criterion to show that our hyperbolic pretzel knot complements are the only knot complements in their respective commensurability classes. The following proposition immediately takes care of symmetries and lens space surgeries. Given a knot $K \subset S^3$, $K$ admits a strong inversion if there exists an involution $t$ of $(S^3, K)$ such that the fixed point set of $t$ intersects the knot in exactly two points.

**Proposition 6.3.** Let $M = S^3 \setminus K$, where $K = K \left( \frac{1}{q_1}, \frac{1}{q_2}, \ldots, \frac{1}{q_n} \right)$ is a hyperbolic pretzel knot with all $q_i$ distinct, exactly one $q_i$ even, and $K \neq K \left( \frac{1}{n}, \frac{1}{3}, \frac{1}{7} \right)$. Then $M$ admits no lens space surgeries, and a strong inversion is its only symmetry. In particular, any $M^g_{2n+1}$ admits no lens space surgeries, and a strong inversion is its only symmetry.

**Proof.** All pretzel knots admitting lens space surgeries have been classified by Ichihara and Jong in [16], and this classification is also implied by the work of Lidman and Moore in [20]. Both works show that the only hyperbolic pretzel knot that admits any lens space surgeries is $K \left( \frac{1}{2}, \frac{1}{3}, \frac{1}{7} \right)$.

To deal with symmetries, we first note that the work of Boileau and Zimmermann [4] implies that $\text{Sym}(S^3, K) = \mathbb{Z}_2$. It is easy to see that the one non-trivial symmetry of any $K$ is a strong inversion. Consider the knot diagram of $K^g_{2n+1}$ as shown in figure [5]. Recall that exactly one twist region $R_i$ has an even number of crossings. Consider the involution of our knot in $S^3$ whose axis cuts directly through the middle of all of our twist regions. This involution will intersect $K^g_{2n+1}$ in exactly two points, always inside the one twist region with an even number of crossings. In the other twist regions, this axis will miss the knot, passing in between two strands at a crossing. This process for finding the strong involution generalizes to any pretzel knot $K$ with exactly one $q_i$ even. \hfill $\square$

It remains to rule out hidden symmetries. In [21], Macasieb and Mattman do this by arguing that the invariant trace field of any $K \left( \frac{1}{n}, \frac{1}{3}, \frac{1}{7} \right)$ has neither $\mathbb{Q}(i)$ nor $\mathbb{Q}(\sqrt{-3})$ as a subfield. This criterion for the existence of hidden symmetries is supplied by Neumann and Reid [27]. Here, we use a geometric approach to show that our knots do not admit hidden symmetries. We will also use a criterion for the existence of hidden symmetries provided by Neumann and Reid in [27], stated below.

**Lemma 6.4.** [27, Proposition 9.1] Let $\mathbb{H}^3/\Gamma$ be a hyperbolic knot complement which is not the figure-8 knot complement. Then $\mathbb{H}^3/\Gamma$ admits hidden symmetries if and only if $\mathbb{H}^3/C(\Gamma)$ has a rigid Euclidean cusp cross-section.

The rigid Euclidean orbifolds are $S^2(2,4,4)$, $S^2(3,3,3)$, and $S^2(2,3,6)$, and are named so because their moduli spaces are trivial. The following proposition will imply that our hyperbolic pretzel knot complements do not admit hidden symmetries, and so, they are
the only knot complements in their respective commensurability classes. In what follows, \( \mathbb{H}^3 = \{(x, y, z) | z > 0\} \).

**Proposition 6.5.** For all \( n > 2 \) and \( q_i \) sufficiently large, the hyperbolic knot complement \( M = \mathbb{S}^3 \setminus K = N_{2n+1}((1, q_1), \ldots, (1, q_{2n+1})) \) admits no hidden symmetries.

**Proof.** We will show that any such hyperbolic knot complement does not cover a 3-orbifold that admits a rigid cusp 2-orbifold, and so, by Lemma 6.4 these knot complements admit no hidden symmetries. First, we shall analyze the cusp of \( N_{2n+1} \) corresponding to the knot component of \( L_{2n+1} \), and then expand this analysis to the cusp shape of any such \( M \). In particular, we will show that this cusp of \( N_{2n+1} \) does not cover any rigid 2-orbifold. Then, by taking sufficiently long Dehn surgeries along all of the crossing circles of \( L_{2n+1} \), we can make sure that the cusp of \( M \) also does not cover any rigid 2-orbifold.

Throughout this proof, let \( C \) denote the cusp of \( N_{2n+1} \) that corresponds to the knot component of \( L_{2n+1} \). Lift to \( \mathbb{H}^3 \) so that one of the lifts of the cusp \( C \) is a horoball centered at \( \infty \), denoted \( H_\infty \). There will be a collection of disjoint horoballs in \( \mathbb{H}^3 \) associated with each cusp in \( N_{2n+1} \). We expand our horoballs according to the procedure given in [10, Section 3.2]. Specifically, we pick an order for our cusps, and expand the horoball neighborhood of a cusp until it either meets another horoball or meets the midpoint of some edge of one of the polyhedra; see [10, Definition 3.6]. This procedure allows us to expand \( H_\infty \) to height \( z = 1 \), since any other horoballs will have diameter at most 1 under these expansion instructions; see [10, Theorem 3.8]. We shall refer to a horoball of diameter 1 as a **maximal horoball**. This procedure from [10, Theorem 3.8] results in maximal horoballs sitting at each vertex of a rectangle tiling our cusp cross-section \( C \).

![Figure 14](image-url)

**Figure 14.** The cusp tiling of a cross-section of \( C \). The red circles denote the shadows of maximal horoballs from \( C \), and the green circles denote the shadows of maximal horoballs coming from crossing circles.

By Proposition 5.5 and Lemma 5.6 the cusp cross-section of \( C \) is tiled by a collection of rectangles in a very particular fashion. All of these rectangles have the same dimensions: \( \ell(s) \) by \( \ell(w) \), with \( \ell(s) = 1 \) and \( 1 < \ell(w) < 2 \). Furthermore, the circle packing for each of...
these rectangles is exactly the same. These $4(2n+1)$ rectangles are glued together to form a $2 \times 2(2n+1)$ block of rectangles. Expand this tiling of the cusp cross-section to cover the entire plane. From our view at $\infty$, we will see the shadow of a maximal horoball centered at each vertex. Specifically, each of the $2n+1$ crossing disks gives three vertices, two of which correspond to horoballs coming from our cusp $C$. In terms of our $2 \times 2(2n+1)$ block of rectangles, the vertices along the middle row correspond with maximal horoballs of our crossing circles. Vertices along the top and bottom rows of the block correspond with maximal horoballs from $C$. We claim that they are in fact the only maximal horoballs of $C$. See figure 14 for a diagram showing the maximal horoballs of $C$.

![Figure 15. The local picture of our cusp tiling of a cross section of C. The red circles denote the shadows of maximal horoballs from C, and the green circles denote the shadows of maximal horoballs coming from crossing circles.](image)

Our circle packing analysis of the rectangles tiling $C$ from Lemma 5.6 will help us prove this claim. Figure 15 shows two adjacent rectangles coming from the tiling of $C$, along with their circle packings. This figure also includes the shadows of the maximal horoballs located at vertices. See figure 13 for a picture of one of these rectangles without the horoball shadows. Suppose there exists another maximal horoball of $C$, call it $H$. We know $H$ can not intersect the other maximal horoballs, except possibly in points of tangencies. Also, $H$ must be centered at a point either outside of the circles or on the boundary of one of the circles from our circle packing since in constructing our link complement, we cut away hemispheres bound by these circles. On our cusp cross-section of $C$, there are two lines of symmetry that will be useful here: the line $A$ and the line $l_w$, which cuts through the vector...
w in its midpoints. Our horoball packing admits reflective symmetries about both of these lines. We shall now consider two cases.

**Case 1:** $H$ is centered along $l_w$. Since the center of $H$ can not be contained in $B$, $H$ is either centered at $x_0 = P_2 \cap B$ or some $y$ that lies below $B$ and above $A$ on $l_w$. First, suppose $H$ is centered at $x_0$. Since $\ell(w) < 2$ and there are maximal horoballs at the corners of any such rectangle, $H$ can not be maximal. Now, suppose $H$ is centered at some $y$ as described above. By applying the reflection along $A$, $H$ will get mapped to another maximal horoball. For $n \geq 2$, we know that $D(B) > \frac{1}{2}$ by Lemma 5.6. Thus, for $n \geq 2$, the distance from the center of $H$ to $l_w \cap A$ is less than $\frac{1}{2}$. In this case, $H$ will overlap with its image. In order to meet our tangency conditions, $H$ must map to itself. This implies that $H$ is centered at $y_0 = l_w \cap A$. Once again, since $\ell(w) < 2$ and there are maximal horoballs at the corners of any such rectangle, $H$ can not be maximal.

**Case 2:** Assume $H$ is not centered along $l_w$. Then the reflection along $l_w$ maps $H$ to some other maximal horoball, $H'$. Now, if $H'$ and $H$ intersect, it must be at a point of tangency. So, both $H$ and $H'$ each must be centered a distance at least $\frac{1}{2}$ from $l_w$. This implies that the center of $H$ is at most a distance $\frac{1}{2}$ from the $s$ side of the rectangle closest to $H$. Also, since $\ell(s) = 1$, the center of $H$ will be at most a distance $\frac{1}{2}$ from a $w$ side of a rectangle. Therefore, the center of $H$ will be at most a distance of $\sqrt{(\frac{1}{2})^2 + (\frac{1}{2})^2} = \frac{\sqrt{2}}{2} < 1$ away from a corner of a rectangle, which is also a center of a maximal horoball. This implies that $H$ will overlap with a maximal horoball at one of the corners, which can not happen. Thus, the only maximal horoballs of $C$ occur at the corners of our rectangles as specified above.

We now claim that the horoball packing corresponding to the cusp $C$ of $N_{2n+1}$ does not admit an order three or order four rotational symmetry. We fail to have such symmetries because of the shape of our rectangles. Pick any maximal horoball $H$ of $C$ such that $H \neq H_\infty$. The distance from the center of $H$ to the center of any other maximal horoball of $C$ in the $s$ direction is an integer multiple of $2 \ell(s) = 2$, and the distance from the center of $H$ to the center of any other maximal horoball of $C$ in the $w$ direction is an integer multiple of $\ell(w)$, where $\ell(w) < 2$. Next, note that the distance across the diagonal of the $2s \times 2w$ rectangle from the center of $H$ to the center of another maximal horoball of $C$ is $\sqrt{(2\ell(s))^2 + (2\ell(w))^2} = \sqrt{4 + 4\ell(w)^2} > \sqrt{5} > 2$ since $\ell(w) > 1$. This implies that the two closest maximal horoballs of $C$ are a distance $\ell(w)$ in the $w$ direction (one to the left of $H$ and one to the right of $H$). Any rotational symmetry would have to map these horoballs either to themselves or to one another. Thus, the only possible rotational symmetry would be order two. So, the horoball packing of $C$ does not admit an order three or order four rotational symmetry. Thus, this cusp does not cover a 2-orbifold that has an order three or order four cone point. But any rigid cusp 2-orbifold has an order three or order four cone point. Therefore, $C$ does not cover any rigid cusp 2-orbifold.

Since the cusp cross-section of $N_{2n+1}$ corresponding to the knot component of $L_{2n+1}$ does not admit order three or order four rotational symmetries, we can now show that the cusp cross-section of $M$ also doesn’t have these symmetries. This is made possible by taking sufficiently long Dehn fillings along the crossing circles. As $q_i \to \infty$, any such $M$ converges to $N_{2n+1}$ in the geometric topology. So, choose all $q_i$ sufficiently large so that the geometry of $C'$, the cusp of $M$ corresponding to the knot $K$, is sufficiently close to the
geometry of $C$. For sufficiently small $\delta$, we choose $q_i$ large enough so that any horoball of $C'$ of diameter at least $1 - \delta$ is centered at a vertex; we call these horoballs *almost maximal horoballs*. Let $H'$ be an almost maximal horoball of $C'$. A symmetry of $C'$ would have to map a closest almost maximal horoball of $C'$ to another closest almost maximal horoball of $C'$ (with respect to $H'$). We can ensure this is impossible by minimizing the change in the lengths and widths of the boundary rectangles under Dehn filling, i.e., for some $\varepsilon > 0$ that is sufficiently small, we have $|\ell(w) - \ell(w')| < \varepsilon$ and $|\ell(l) - \ell(l')| < \varepsilon$, where $\ell(w')$ and $\ell(l')$ are the widths and lengths of any one the parallelograms tiling $C'$. Note that, while the rectangles of $C$ all have the same shape, each of these parallelograms of $C'$ could have different widths and lengths. If anything, this further breaks any possible symmetry in our cusp tiling. At the least, we can ensure that these size changes still results in $2\ell(s') > \ell(w')$ and that the distance across the diagonal of a $2s' \times w'$ block still is sufficiently large (greater than $2\ell(s')$). Under these conditions, the same justification can be given to show that the horoball packing of the cusp $C'$ of $M$ also does not admit order three or order four rotational symmetries. Thus, the one cusp of $M$ cannot cover a rigid 2-orbifold, and so, $M$ does not admit hidden symmetries. 

Combining Proposition 6.5 with Proposition 6.3 shows that we have covered the three criterion in Reid and Walsh’s theorem. This gives the following theorem, which applies to our pretzel knots $K_{2n+1} = K\left(\frac{1}{q_1}, \frac{1}{q_2}, \ldots, \frac{1}{q_{2n+1}}\right)$, if we assume that all $q_i$ are sufficiently large.

**Theorem 6.6.** Let $n \geq 2$ and let $q_1, \ldots, q_{2n+1}$ be integers such that only $q_1$ is even, $q_i \neq q_j$ for $i \neq j$, and all $q_i$ are sufficiently large. Then the complement of the hyperbolic pretzel knot $K\left(\frac{1}{q_1}, \frac{1}{q_2}, \ldots, \frac{1}{q_{2n+1}}\right)$ is the only knot complement in its commensurability class. In particular, any two of these hyperbolic pretzel knot complements are incommensurable.

The work of Schwartz [36, Theorem 1.1] tells us that two cusped hyperbolic 3-manifolds are commensurable if and only if their fundamental groups are quasi-isometric. This immediately gives the following corollary.

**Corollary 6.7.** If two pretzel knot complements as described in Theorem 6.6 are non-isometric, then they do not have quasi-isometric fundamental groups.

7. MUTATIONS AND SHORT GEODESICS COMING FROM DEHN FILLINGS

In this section, we shall analyze the behavior of short geodesics in the set of knot complements $\{M_{2n+1}^\sigma\}$. If there is enough vertical twisting in each twist region, i.e., if each $q_i$ is sufficiently large, then we can easily figure out which geodesic are the shortest. This analysis is possible by realizing our pretzel knot complements as Dehn surgeries along untwisted augmented link complements. We shall also see that if each $q_i$ is sufficiently large, then the initial length spectrum is actually preserved under mutation, and so, we will be able to generate a large class of hyperbolic knot complements with both the same volume and the same initial length spectrum. Here, we also give an application to closed hyperbolic 3-manifolds that come from Dehn filling $M_{2n+1}^\sigma$ along $K_{2n+1}^\sigma$. For each $n \in \mathbb{N}$, $n > 2$
these sets of closed manifolds will have the same volume and the same initial length spectrum. We end this section by raising some questions about the effectiveness of geometric invariants of hyperbolic 3-manifolds.

7.1. Mutations of $K_{2n+1}$ with the same initial length spectrum. The following proposition can help determine the shortest geodesics in a hyperbolic 3-manifold.

Proposition 7.1. Let $M$ be a hyperbolic 3-manifold such that $M = N((p_1, q_1), \ldots, (p_n, q_n))$. Then there exists a constant $Q > 0$ dependent only on $N$, such that if for each $i$, $1 \leq i \leq n$, we have $p_i^2 + q_i^2 > Q$, then the $n$ shortest geodesics of $M$ are the core geodesics resulting from the Dehn surgeries along the cusps of $N$.

Proof. For each $i$, $1 \leq i \leq n$, let $\gamma_i$ be the core geodesic in $M$ coming from Dehn filling the $i^{th}$ cusp of $N$. By the work of Neumann and Zagier [28], as $p_i^2 + q_i^2 \to \infty$, $\ell(\gamma_i) \to 0$, while the other geodesic lengths stabilize. Thus, if all of our $p_i^2 + q_i^2$ are sufficiently large, then the geodesic cores of these solid tori filled in from Dehn surgery will be our shortest geodesics.

Remark: How large $p_i^2 + q_i^2$ needs to be depends on the systole length of $N$, since we will essentially need our $\ell(\gamma_i)$ to be smaller than that value. Once we have determined the systole length of $N$, this constant $Q$ can be made explicit by using the work of Futer-Purcell-Schleimer (in preparation).

Given the untwisted augmented link complement $N_{2n+1} = S^3 \setminus L_{2n+1}$, we form $M_{2n+1} = S^3 \setminus K_{2n+1}$ by performing Dehn surgeries $(1, \frac{2k-1}{2})$ along $2n$ of the crossing circle cusps, and one Dehn surgery $(1, \frac{k}{2})$ along the crossing circle cusp not enclosing a half-twist, i.e.,

$$M_{2n+1} = N_{2n+1}(1, \frac{2k}{2}, 1, \frac{2k-1}{2}, \ldots, 1, \frac{2k-n-1}{2}).$$

Similarly, any mutation $M_{2n+1}^\sigma$ is obtained by performing the same Dehn surgeries on $N_{2n+1}$, just with some of the surgery coefficients permuted. We now show that if each $q_i$ is sufficiently large, then short geodesics of $M_{2n+1}$ are exactly these core geodesics, and they are preserved under mutation.

Theorem 7.2. Let $\gamma_i^{2n+1}$ be the $2n+1$ geodesics in $M_{2n+1}^\sigma$ that came from Dehn filling the crossing circles of $N_{2n+1}$. There exists a constant $Q' > 0$, such that if each $q_i \geq Q'$, then $\gamma_i^{2n+1}$ are the shortest geodesics in their respective hyperbolic 3-manifold and every $M_{2n+1}^\sigma$ has the same shortest $2n+1$ (complex) geodesic lengths.

Proof. Given $M_{2n+1}$, we must show that the result holds for a mutation $\sigma_a$ along $S_a$, and the general result will follow by induction. Set $Q' = \max\left\{\sqrt{Q}, \sqrt{\frac{8}{3}D^2 - 1}\right\}$, where $D$ is the constant from Corollary [3.15]. Since $q_i \geq \sqrt{Q}$, $p_i^2 + q_i^2 = 1 + q_i^2 \geq 1 + Q > Q$. Then Proposition 7.1 implies that the $2n + 1$ shortest geodesics in $M_{2n+1}$ are given by the set $\gamma_i^{2n+1}$, and similarly for $\gamma_i^{2n+1} \sigma_a$ in $M_{2n+1}^{\sigma_a}$. By Proposition 5.1, we know that the normalized length of the $i^{th}$ filling slope satisfies $\hat{L}(s_i) \geq \sqrt{\frac{3(1+q_i^2)}{8}}$. Since $q_i \geq \sqrt{\frac{8}{3}D^2 - 1}$,
each \( \hat{L}(s_i) \geq D \). Then Corollary 3.15 tells us that \( M \) and \( M_{\sigma} \) have the same \( 2n + 1 \) shortest (complex) geodesic lengths, which are given by \( \{ \ell_C(y_i) \}_{i=1}^{2n+1} = \{ \ell_C(y_i^{\sigma}) \}_{i=1}^{2n+1} \). \( \square \)

**Remark:** The constant \( Q' \) can obviously be made explicit by determining \( Q \) and \( D \), as discussed in previous remarks.

The following theorem comes from combining Theorem 7.2, Theorem 6.6, and Theorem 4.4 and requires all \( q_i \) to be chosen sufficiently large. This theorem shows that there are large classes of geometrically similar pretzel knots – they have non-isometric knot complements, but a large number of their geometric invariants are the same.

**Theorem 7.3.** For each \( n \in \mathbb{N}, n > 2 \), there exist \( \frac{(2n)!}{2} \) non-isometric hyperbolic pretzel knot complements, \( \{ M_{\sigma}^{2n+1} \} \), such that these manifolds:
- have the same \( 2n + 1 \) shortest geodesic (complex) lengths,
- are pairwise incommensurable,
- have the same volume, and
- \( \left( \frac{2n-1}{2} \right) v_{\text{oct}} \leq \text{vol}(M_{\sigma}^{2n+1}) \leq (4n + 2) v_{\text{oct}} \), where \( v_{\text{oct}} \approx 3.6638 \) is the volume of a regular ideal octahedron.

### 7.2. Closed hyperbolic 3-manifolds with the same volume and initial length spectrum.

Let \( M = S^3 \setminus K \) and let \( M(p, q) \) denote the closed manifold obtained by performing a \((p, q)\)-Dehn surgery along the knot \( K \). In [24, Theorem 3], we show that for each \( n \in \mathbb{N}, n > 2 \), and for \((p, q)\) sufficiently large, \( M_{\sigma}^{2n+1}(p, q) \) and \( M_{\sigma'}^{2n+1}(p, q) \) have the same volume and are non-isometric closed hyperbolic 3-manifolds, whenever \( M_{\sigma}^{2n+1} \) and \( M_{\sigma'}^{2n+1} \) are non-isometric. This proof relies on another result of Ruberman’s [35, Theorem 5.5] which shows that corresponding Dehn surgeries on a hyperbolic knot \( K \) and its fellow mutant \( K^{\mu} \) will often result in manifolds with the same volume. Specifically, this happens when a Conway sphere and its mutation are unlinked.

**Definition 7.4 (Unlinked).** Let \( K \) be a knot in \( S^3 \) admitting a Conway sphere \( S \). Observe that a specific choice of a mutation \( \mu \) gives a pair of \( S^0 \)'s on the knot such that each \( S^0 \) is preserved by \( \mu \). We say that \( \mu \) and \( S \) are unlinked if these \( S^0 \)'s are unlinked on \( K \).

Being unlinked allows one to tube together the boundary components of a Conway sphere that are interchanged by \( \mu \) to create a closed surface of genus two, which we call \( S' \). \( S' \) is also a hyperelliptic surface, and its involution is the same as the involution \( \mu \) of our Conway sphere. Dehn surgeries on \( S^3 \setminus K \) and its mutant \( S^3 \setminus K^{\mu} \) differ by mutating along this closed surface. Thus, Ruberman’s result for preserving volume will apply to these closed manifolds.

Combining our work in [24] with Corollary 3.13 gives the following.

**Theorem 7.5.** For each \( n \in \mathbb{N}, n > 2 \), and any \((p, q)\) sufficiently large, there exist \( \frac{(2n-1)!}{2} \) non-isometric closed hyperbolic 3-manifolds \( \{ M_{\sigma}^{2n+1}(p, q) \} \) such that these manifolds:
- have the same \( 2n + 2 \) shortest (complex) geodesic lengths,
- have the same volumes, and
- \( \text{vol}(M_{\sigma}^{2n+1}(p, q)) < (4n + 2) v_{\text{oct}} \).
Proof. In [24], we constructed our $K_{2n+1}$ so that all Conway spheres in $\{(S_α, σ_α)\}_{α=1}^{2n}$ are unlinked. However, here we have slightly modified this construction of each $K_{2n+1}$. Specifically, we now have one twist region with an even number of twists in $K_{2n+1}$. As a result, $(S_1, σ_1)$ is not unlinked. Thus, we will only mutate along the other Conway spheres: $\{(S_α, σ_α)\}_{α=2}^{2n}$. These combinations of mutations create $\frac{(2n)!}{2(2n)}$ non-isometric, hyperbolic pretzel knots; see [24, Theorem 2] for more details.

Let $σ$ and $σ'$ be any combination of mutations along our unlinked Conway spheres resulting in non-isometric knot complements. Now, $M_{2n+1}^σ(p, q)$ and $M_{2n+1}^q(p, q)$ have the same volume by Ruberman’s work. In [24, Theorem 3], we show that $M_{2n+1}^σ(p, q)$ and $M_{2n+1}^q(p, q)$ are non-isometric by choosing $(p, q)$ sufficiently large so that the core geodesics resulting from this Dehn filling are the systoles of their respective manifolds. This comes from the work of Neumann-Zagier [28]. Furthermore, the other closed geodesic lengths stabilize by Thurston’s Dehn Surgery Theorem. So, for $(p, q)$ sufficiently large, any $M_{2n+1}^σ(p, q)$ will have exactly $2n+2$ closed geodesics shorter than a constant $L < 0.015$. $2n+1$ of these geodesics come from Dehn filling our crossing circles of $L_{2n+1}$, and the systole comes from then Dehn filling the knot component. We can apply Corollary 3.15 to these closed manifolds to show that they have the same $2n+2$ shortest geodesic lengths. The upper bound on volume follows from the proof of [24, Theorem 3]. □

7.3. Closing Remarks. The fact that the manifolds $\{M_{2n+1}^σ\}$ are constructed by mutating knots that are pairwise incommensurable sharply contrasts any of the known constructions for building large classes of hyperbolic 3-manifolds that are iso-length spectral. Also, there is a general recipe for this type of construction and we did not necessarily need to use pretzel knots. In order to construct a large number of non-isometric hyperbolic manifolds with the same volume and the same initial length spectrum, you need the following key ingredients.

- An initial hyperbolic 3-manifold $M$ with:
  - a large number of hyperelliptic surfaces in $M$ to mutate along to create the set of manifolds $\{M^σ\}$, and
  - a way to determine your shortest geodesics in $M$ and make sure they are sufficiently short, i.e., realize them as the cores of sufficiently long Dehn fillings.
- A simple method to determine how much double counting you are doing, i.e., a method to determine if any $M^σ$ and $M^q$ are isometric or not.

Given this recipe, you want to maximize the number of hyperelliptic surfaces in $M$ to mutate along and maximize the number of sufficiently short geodesics, while minimizing the double counting. It would be interesting to examine how well we did with maximizing and minimizing these parameters. Such an examination leads us to consider the function $N(v, s)$, which counts the number of hyperbolic 3-manifolds with same volume $v$ and the same $s$ shortest geodesic lengths. We can also consider the restriction of this counting function to specific classes of hyperbolic 3-manifolds. Let $N_K(v, s)$ denote the restriction of $N(v, s)$ to hyperbolic knot complements and $N_{Cl}(v, s)$ denote the restriction of $N(v, s)$ to closed hyperbolic 3-manifolds. An immediate corollary of Theorem 7.3 and Theorem 7.5 gives the following lower bounds on the growth rates of $N_K(v, s)$ and $N_{Cl}(v, s)$ as functions...
of $v$. The proof of this corollary is the same as the proof of [24, Theorem 1], except we can now take the short geodesic lengths into account.

**Corollary 7.6.** There are sequences $\{(v_n, s_n)\}$ and $\{(x_n, t_n)\}$ with $(v_n, s_n), (x_n, t_n) \to (\infty, \infty)$ such that

$$N_K((v_n, s_n)) \geq (v_n)^{\frac{5}{8}}$$

and

$$N_{Cl}((x_n, t_n)) \geq (x_n)^{\frac{5}{8}}$$

for all $n \gg 0$.

This corollary tells us that the counting function $N((v, s))$ grows at least factorially fast with $v$, and immediately raises some questions.

**Question 7.7.** Can a Sunada-type construction or an arithmetic method be applied to also show $N((v, s))$ grows at least factorially fast with $v$? Also, are there sequences $\{(v_n, s_n)\}$ with $v_n \to \infty$ such that $N((v_n, s_n))$ grows faster than factorially with $v_n$?

It would be interesting to find a construction realizing a growth rate faster than the one given in Corollary 7.6 or show that a factorial growth rate is actually the best we can do.

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