Unified framework for generalized quantum statistics: canonical partition function, maximum occupation number, and permutation phase of wave function

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ABSTRACT: Beyond Bose and Fermi statistics, there still exist various kinds of generalized quantum statistics. Two ways to approach generalized quantum statistics: (1) in quantum mechanics, generalize the permutation symmetry of the wave function and (2) in statistical mechanics, generalize the maximum occupation number of quantum statistics. The connection between these two approaches, however, is obscure. In this paper, we suggest a unified framework to describe various kinds of generalized quantum statistics. We first provide a general formula of canonical partition functions of ideal $N$-particle gases obeying various kinds of generalized quantum statistics. Then we reveal the connection between the permutation phase of the wave function and the maximum occupation number, through constructing a method to obtain the permutation phase and the maximum occupation number from the canonical partition function. We show that the permutation phase of the wave function is closely related to the higher dimensional representation of the permutation group. In our scheme, for generalized quantum statistics, the permutation phase of wave functions is generalized to a matrix phase, rather than a number. The permutation phase of Bose or Fermi wave function, $e^{\phi_0} = 1$ or $e^{i\pi} = -1$, is regarded as $1 \times 1$ matrices, as special cases of generalized statistics. It is commonly accepted that different kinds of statistics are distinguished by the maximum number. We show that the maximum occupation number is not sufficient to distinguish different kinds of generalized quantum statistics. As examples, we discuss a series of generalized quantum statistics in the unified framework, giving the corresponding canonical partition functions, maximum occupation numbers, and the permutation phase of wave functions. Especially, we propose three new kinds of generalized quantum statistics which seem to be the missing pieces in the puzzle. The mathematical basis of the scheme are the mathematical theory of the invariant matrix, the Schur-Weyl duality, the symmetric function, and the representation theory of the permutation group and the unitary group. The result in this paper builds a bridge between the statistical mechanics and such mathematical theories.

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Contents

1 Introduction

2 The integer partition and the symmetric function: a brief review
   2.1 The integer partition
   2.2 The symmetric function

3 The canonical partition function for various kinds of statistics: a general result

4 The maximum occupation number for various kinds of statistics
   4.1 Obtaining the maximum occupation number from the canonical partition function
   4.2 The failure of distinguishing different statistics by the maximum occupation number

5 The permutation phase of the wave function for various kinds of statistics
   5.1 The statistics with the Hamiltonian invariant under permutations: the permutation phase
   5.2 The statistics with the Hamiltonian noninvariant under permutations

6 A unified framework
   6.1 The N-distinguishable-particle gas system: Boltzmann statistics
      6.1.1 The Hilbert subspace
      6.1.2 The maximum occupation number and the distinguishability of the particle.
   6.2 Gentile statistics
      6.2.1 Discussions on the permutation phase of the wave function
   6.3 Parastatistics
      6.3.1 The maximum occupation number
      6.3.2 The permutation phase of the wave function
   6.4 The immannons and Gentileonic statistics
      6.4.1 The canonical partition function
      6.4.2 Discussions on the permutation phase of the wave function
      6.4.3 The maximum occupation number
   6.5 generalized quantum statistics dual to Gentile statistics: GD-ons
      6.5.1 The canonical partition function
      6.5.2 Discussions on the permutation phase of the wave function
      6.5.3 The maximum occupation number
   6.6 generalized quantum statistics corresponding to the monomial symmetric function: M-ons
      6.6.1 The canonical partition function
1 Introduction

The principle of indistinguishability requires that exchanging two identical particles does not lead to any observable effect [1–3]. In quantum mechanics, a physical system is described by a complex wave function, but the observable is a real number. The principle of indistinguishability allows a change on wave functions after exchanging two identical particles so long as the observable does not change. Consequently, the wave function may change a phase factor after exchanging identical particles. It comes naturally Bose-Einstein statistics and Fermi-Dirac statistics whose phase factors change $e^{i0} = 1$ and $e^{i\pi} = -1$, respectively. Beside Bose-Einstein statistics and Fermi-Dirac statistics, however, one can still consider other kinds of quantum statistics so long as it does not violate the principle of indistinguishability, i.e., there are no changes on observables after exchanging identical particles. Generalizing quantum statistics along this line is to consider phase factors between 0 and $\pi$, e.g., anyons are successful in explaining the fractional quantum Hall effect [4, 5].

In statistical mechanics, macroscopic systems are treated averagely. The number of microstates is the key in the calculation of the average value. Particles are indistinguishable, so exchanging particles occupying different states does not lead to new microstates. In quantum statistics allowed by the principle of indistinguishability, there is no new microstate after exchanging identical particles. From the view of statistical mechanics, the difference between various quantum statistics is reflected in the maximum occupation numbers, e.g., for Fermi-Dirac statistics the maximum occupation number is 1 and for Bose-Einstein statistics there is no limitation on the maximum occupation number. Generalizing statistics
along this line is to generalize the maximum occupation number to arbitrary number, e.g., the spin wave satisfies Gentile statistics [6].

The above analyses shows two approaches of constructing generalized statistics: (1) in quantum mechanics, generalize the permutation symmetry of the wave function and (2) in statistical mechanics, generalize the maximum occupation number. The connection between those two approaches, however, is obscure.

In statistical mechanics, a system with the fixed particle number should be considered in the canonical ensemble and the canonical partition function is the key. It is because all the thermodynamic information is embedded in the canonical partition function. For example, the eigenvalue spectrum can be obtained from the canonical partition function [7]. However, to calculate the canonical partition function is difficult. It is because one has to deal with the inter-particle interactions and at the same time takes the constraint of fixed particle number into consideration. For example, the previous work [8] gives the canonical partition function for ideal Bose, Fermi, and Gentile statistics.

In this paper, based on the mathematical theory of the invariant matrix [9], the Schur-Weyl duality [10], and the symmetric function [9, 11], we suggest a unified framework to describe various kinds of generalized quantum statistics. We first provide a general formula of canonical partition functions of ideal $N$-particle gases who obeying various kinds of generalized quantum statistics. Then we reveal the connection between the permutation phase of the wave function and the maximum occupation number, through constructing a method to obtain the permutation phase and the maximum occupation number from the canonical partition function.

We show that the permutation phase of the wave function is closely related to the higher dimensional representation of the permutation group. In our scheme, for generalized quantum statistics, the permutation phase of wave functions is generalized to a matrix, rather than a number. The permutation phase of Bose or Fermi wave function, 1 or $-1$, is regarded as $1 \times 1$ matrices, as special cases of generalized quantum statistics.

It is commonly accepted that different kinds of statistics are distinguished by the maximum number. We show that the maximum occupation number is not sufficient to distinguish different kinds of generalized quantum statistics.

As examples, we describe various kinds of statistics in a unified framework, including parastatistics proposed in 1952 by H. Green [12, 13], the intermediate statistics or Gentile statistics proposed in 1940 by G. Gentile Jr [14, 15], Gentileonic statistics proposed by Cattani and Fernandes in 1984 [16], and the immannons proposed by Tichy in 2017 [17], etc, where the canonical partition function, the maximum occupation number, and the permutation phase of the wave function are given. Especially, we propose three new kinds of generalized statistics which seem to be the missing pieces in the puzzle.

The mathematical basis of the scheme are the mathematical theory of the invariant matrix [9], the Schur-Weyl duality [10], the symmetric function [9, 11], and the representation theory of the permutation group and the unitary group [18, 19]. The result in this paper builds a bridge between the quantum statistical mechanics and such mathematical theories and enables one to use the fruitful result in the theory of the symmetric function to solve the problem in quantum statistical mechanics.
There are discussions on various kinds of generalized quantum statistics. For example, the generalized quantum statistics such as parastatistics and Gentile statistics are discussed in Ref. [20]. The distinctions between the intermediate statistics, parastatistics, and Okayama statistics are discussed in Ref. [21]. The connection between the irreducible representation of $S_N$ and the parastatistics is given by Okayama [22]. The operator realization of Gentile statistics is given in Ref. [23] and the statistical distribution of various intermediate statistics is calculated from operator relations in Ref. [24]. The relation between properties of Gentile statistics and the fractional statistics of anyons is discussed in Ref. [25]. The operator realization of intermediate statistics is discussed in Ref. [26]. Statistical distributions for generalized ideal gas of fractional-statistics gases are given in Ref. [27].

This paper is organized as follows. In Sec. 2, we briefly review the mathematical theory of integer partitions and symmetric functions. In Sec. 3, we give a general formula of the canonical partition function of an ideal $N$-particle gases under various kinds of statistics. In Sec. 4, we give a method to obtain the maximum occupation number and show that the maximum occupation number is not sufficient to distinguish different statistics. In Sec. 5, we give a method to obtain the permutation phase from the canonical partition function and discuss the permutation phase of the wave function for different statistics. In Sec. 6, we give a unified framework to describe a series of generalized statistics. The canonical partition function, the maximum occupation number, and the permutation phase of the wave function are given. The intermediate statistics such as Gentile statistics, parastatistics, the immanonns, Gentileonic statistics, and the new proposed generalized quantum statistics is discussed as examples. In Sec. 7, the conclusion is given. Some details of the calculation are given in appendixes.

2 The integer partition and the symmetric function: a brief review

The main result of the present paper involves some basic knowledges of the mathematical theory of the integer partition and the symmetric function. In this section, we give a brief review on the theory. For more details, one can refer to Refs. [9, 11, 28].

2.1 The integer partition

The integer partition and the length of an integer partition. An integer $N$ can be represented as a sum of other integers:

$$N = \lambda_1 + \lambda_2 + \ldots + \lambda_l,$$

where $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_l > 0$. The integer partition of $N$ is denoted by the notation $(\lambda) = (\lambda_1, \lambda_2, \ldots, \lambda_l)$. The number of the integer in $(\lambda)$ is the length of $(\lambda)$, denoted by $l(\lambda)$. $N$ is the size of $(\lambda)$. For example, for an integer partition $(\lambda) = (3, 2, 1)$, the length is $l(\lambda) = 3$ and the size is 6.

The unrestricted partition function $P(N)$ and arranging integer partitions in a prescribed order. An integer $N$ has many integer partitions and the unrestricted partition function $P(N)$ counts the number of integer partitions [3]. For a given $N$, one arranges the
integer partition in the following order: $(\lambda), (\lambda)',$ when $\lambda_1 > \lambda'_1$; $(\lambda), (\lambda)'$, when $\lambda_1 = \lambda'_1$ but $\lambda_2 > \lambda'_2$; and so on. One keeps comparing $\lambda_i$ and $\lambda'_i$ until all the integer partitions of $N$ are arranged in a prescribed order. $(\lambda)_I$ is the $I$th integer partition function. For example, the integer partitions of $4$ are $(\lambda)_1 = (4), (\lambda)_2 = (3,1), (\lambda)_3 = (2^2), (\lambda)_4 = (2,1^2),$ and $(\lambda)_5 = (1^4)$, where, e.g., the superscript in $1^4$ means $1$ appearing twice, the superscript in $2^2$ means $2$ appearing twice, and so on.

The conjugate integer partition. For an integer partition $(\lambda) = (\lambda_1, \lambda_2, \ldots, \lambda_l)$, there is one and only one integer partition $(\lambda)^*$ that is conjugate to $(\lambda)$. To get the conjugate integer partition $(\lambda)^*$ from $(\lambda)$, an efficient method is to use the Young diagram. For example, the conjugate integer partition of $(\lambda) = (3,1)$, as shown in Fig. (1), is $(\lambda)^* = (2,1^2)$.

$$\lambda = (3,1) \quad \lambda^* = (2,1^2)$$

Figure 1. An example of the method to obtain the conjugate integer partition by Young diagram.

### 2.2 The symmetric function

The symmetric function, first studied by Hall in 1950s, is an important issue in algebraic combinatorics. It is closely related to the integer partition in number theory and plays an important role in the theory of group representations.

In this section, we give a brief review on several important types of symmetric functions, such as the S-function $s_{(\lambda)}(x_1, x_2, x_3, \ldots)$, the monomial symmetric function $m_{(\lambda)}(x_1, x_2, x_3, \ldots)$, and the power sum symmetric function $p_{(\lambda)}(x_1, x_2, x_3, \ldots)$.

The S-function $s_{(\lambda)}(x_1, x_2, x_3, \ldots)$. The S-function, also called the Schur function, is an important type of the symmetric function. For an integer partition $(\lambda)$ of the integer $N$, the S-function is defined by

\[
s_{(\lambda)}(x_1, x_2, \ldots, x_n) = \frac{\det(x^j+n-j)}{\det(x^{n-j})},
\]

where $\det(A)$ represents the determinate of matrix $A$. There is also another definition of $s_{(\lambda)}(x_1, x_2, \ldots)$ without limitation on the number of variables:

\[
s_{(\lambda)}_I(x_1, x_2, \ldots) = \sum_{J=1}^{P(N)} \frac{g_{IJ}}{N!} \chi^I J \sum_{m=1}^{k} x^m_i \sum_{j} a_{j,m},
\]

where $g_{IJ}$ is a generating function.
where \((\lambda)_I\) is the \(I\)th integer partition of the integer \(N\) and \(a_{J,m}\) represents the times \(m\) occurring in \((\lambda)_J\). \(\chi^I_J\) is the simple characteristic of the permutation group of order \(N\). \(g_I\) satisfies
\[
g_I = N! \left( \prod_{j=1}^{N} j^{a_{I,j}} \right)^{-1}.
\] (2.4)

The monomial symmetric function \(m(\lambda) (x_1, x_2, \ldots)\). For an integer partition \((\lambda)\), there is a corresponding monomial symmetric polynomial, defined by [9, 11]
\[
m(\lambda) (x_1, x_2, \ldots, x_n) = \sum_{\text{perm}} x_1^{\lambda_1} x_2^{\lambda_2} \ldots.,
\] (2.5)
where \(\sum_{\text{perm}}\) indicates that the summation runs over all possible monotonically increasing permutations of \(x_i\).

The power sum symmetric function \(p(\lambda) (x_1, x_2, \ldots)\). For an integer partition \((\lambda)\), there is a symmetric polynomial, defined by [9, 11]:
\[
p(\lambda) (x_1, x_2, \ldots, x_n) = \prod_{i=1}^{l(\lambda)} m(\lambda_i).
\] (2.6)

The relation among \(s(\lambda) (x_1, x_2, \ldots, x_n)\), \(m(\lambda) (x_1, x_2, \ldots, x_n)\), and \(p(\lambda) (x_1, x_2, \ldots, x_n)\). There is a relation between \(m(\lambda) (x_1, x_2, \ldots, x_n)\) and \(s(\lambda) (x_1, x_2, \ldots)\): the S-function can be represented as a linear combination of the monomial symmetric polynomial [9, 11], i.e.,
\[
s(\lambda)_K (x_1, x_2, \ldots) = \sum_{I=1}^{P(N)} k^I_K m(\lambda)_I (x_1, x_2, \ldots),
\] (2.7)
where \(k^I_K\) is the Kostka number [9, 11]. There is a relation between \(p(\lambda) (x_1, x_2, \ldots, x_n)\) and \(s(\lambda) (x_1, x_2, \ldots)\): the S-function \(s(\lambda) (x_1, x_2, \ldots)\) can be represented as a linear combination of the power sum symmetric function \(p(\lambda) (x_1, x_2, \ldots)\) [30]:
\[
p(\lambda)_K (x_1, x_2, \ldots) = \sum_{I=1}^{P(N)} \chi^I_K s(\lambda)_I (x_1, x_2, \ldots),
\] (2.8)
where \(\chi^I_K\) is the simple characteristics of \(S_N\) [28, 30].

3 The canonical partition function for various kinds of statistics: a general result

In statistical mechanics, the canonical partition function carries all the thermodynamic information and plays a central role. In this section, we provide a general formula of the canonical partition function of various kinds of generalized quantum statistics. The canonical partition function of Bose, Fermi, and Gentile statistics considered in the previous work [8] is the special case of the formula.

The result in this section is a bridge between the maximum occupation number and the permutation phase of the wave function.
Theorem 1  (1) The canonical partition function of an ideal gas consisting of \( N \)-identical particles is

\[
Z(\beta,N) = \sum_{I} C_{I}s(\lambda_{I}) \left( e^{-\beta \varepsilon_{1}}, e^{-\beta \varepsilon_{2}}, ... \right),
\]

where \( C_{I} \) is an nonnegative integer and \( s(\lambda_{I}) \left( e^{-\beta \varepsilon_{1}}, e^{-\beta \varepsilon_{2}}, ... \right) \) is the S-function with \( \varepsilon_{i} \) the eigenvalue of single particles.

(2) The Hilbert subspace describing the system is

\[
D = \bigoplus_{I=1}^{P(N)} (V^{I})^{\oplus C_{I}},
\]

where \( V^{I} \) carries the inequivalent and irreducible representation of \( S_{N} \) corresponding to the integer partition \( (\lambda)_{I} \).

Proof. The direct sum decomposition of the \( N \)-particle Hilbert space \( V^{\otimes N} \) and the trace of the operator \( e^{-\beta H} \) in the subspace of \( V^{\otimes N} \) are the basises of the proof. Before proving Eqs. (3.1) and (5.1), we prove that the \( N \)-particle Hilbert space \( V^{\otimes N} \) can be decomposed into subspaces labeled by the integer partition \( (\lambda) \) of \( N \) and the trace of the operator \( e^{-\beta H} \) in the subspace is the S-function.

A brief review on the mathematical theory of the Schur-Weyl duality. Let \( V \) be a linear space of the dimension \( m \). Let \( g \in GL(V) \) be a linear operator on \( V \). The action of \( g \) on a vector \( e_{i_{1}} \otimes e_{i_{2}} \otimes ... \otimes e_{i_{N}} \) in \( V^{\otimes N} \) is

\[
g(e_{i_{1}} \otimes e_{i_{2}} \otimes ... \otimes e_{i_{N}}) \equiv ge_{i_{1}} \otimes ge_{i_{2}} \otimes ... \otimes ge_{i_{N}},
\]

where, \( \{e_{1}, e_{2}, ...\} \) is a basis in \( V \). For \( \sigma \in S_{N} \), the action of \( \sigma \) on the vector \( e_{i_{1}} \otimes e_{i_{2}} \otimes ... \otimes e_{i_{N}} \) is

\[
\sigma(e_{i_{1}} \otimes e_{i_{2}} \otimes ... \otimes e_{i_{N}}) \equiv e_{\sigma_{i_{1}}} \otimes e_{\sigma_{i_{2}}} \otimes ... \otimes e_{\sigma_{i_{N}}}.\]

Eqs. (3.3) and (3.4) imply that the operator \( g \) commute with \( \sigma \) on \( V^{\otimes N} \). For \( m \geq N \), the space \( V^{\otimes N} \) can be decomposed into a direct sum of the subspace \( V^{I} \) [10, 31]. The number of \( V^{I} \) is \( P(N) \) and each integer partition \( (\lambda)_{I} \) of \( N \) corresponds to a subspace \( V^{I} \) [10, 31], i.e.,

\[
V^{\otimes N} = \bigoplus_{I=1}^{P(N)} V^{I}.\]

The subspace \( V^{I} \) carries irreducible representations for \( S_{N} \) and \( GL(V) \) with the dimension \( f_{I} \) and \( R_{I} \) respectively, where

\[
f_{I} = N! \prod_{i,j} (\lambda_{I,i} - \lambda_{I,j} - i + j) \prod_{i} [(l(\lambda) + \lambda_{I,i} - i)!]^{-1},
\]

and

\[
R_{I} = \prod_{i,j} (a_{I,i} - a_{I,j} + j - i) (j - i)^{-1}
\]
with \( a_{I,1} = \lambda_{I,1}, a_{I,2} = \lambda_{I,2}, \ldots, a_{I,d(\lambda)} = \lambda_{I,d(\lambda)}, \) \( a_{I,d(\lambda)+1} = 0, \ldots, \) and \( a_{I,m} = 0 \) \([10, 31]\).

The dimension of \( V^I \) is \([10, 31]\)

\[
\dim (V^I) = f_I R_I.
\]

(3.8)

For \( S_N \), the inequivalent and irreducible representation with the dimension \( f_I \) occurs \( R_I \) times in \( V^I \) \([10, 31]\). For \( GL(V) \), the inequivalent and irreducible representation of the dimension \( R_I \) occurs \( f_I \) times in \( V^I \) \([10, 31]\). It can be verified that

\[
\dim (V^{\otimes N}) = m^N = \sum_{I=1}^{P(N)} \dim (V^I) = \sum_{I=1}^{P(N)} R_I f_I.
\]

(3.9)

A brief review on the mathematical theory of the invariant matrix. For an \( m \)-dimensional matrix group \( G \). Let \( A \) be an \( m \)-dimensional matrix in \( G \). Let \( T(A) \) be a function of \( A \). \( T(A) \) is an invariant matrix \([9]\) if

\[
T(A) T(B) = T(AB),
\]

(3.10)

where \( B \) is also an \( m \)-dimensional matrix in \( G \). The invariant matrix gives a representation of the group \( G \). If \( T \) is reducible, then for any \( A \) in \( G \), \( T(A) \) can be diagonalized in the same way and the matrix in diagonal is a new invariant matrix of \( G \) \([9]\). The \( N \) times direct product of \( G \), \( G^{\otimes N} \), is an invariant matrix \([9]\). The \( G^{\otimes N} \) can be decomposed into \( P(N) \) irreducible invariant matrices. An integer partition \( (\lambda) \) corresponds to an irreducible invariant matrix, denoted by \( T^I(G) \). For \( A \) in \( G \), the trace of \( T^I(A) \) is \([9]\)

\[
tr [T^I(A)] = s_{(\lambda)I}(a_1, a_2, \ldots),
\]

(3.11)

where \( a_i \) is the eigenvalue of \( A \).

The direct sum decomposition of the \( N \)-particle Hilbert space \( V^{\otimes N} \). By using the Schur-Weyl duality, the Hilbert space of a \( N \)-particle system \( V^{\otimes N} \) can be decomposed into a direct sum of subspaces \( V^I \). The number of the subspace is \( P(N) \) and each integer partition \( (\lambda) \) of \( N \) corresponds to a subspace \( V^I \). The dimension of the subspace \( V^I \) is \( R_I f_I \). The space \( V^I \) gives \( f_I \) equivalent and irreducible representations with the dimension \( R_I \) for the Hamiltonian and \( R_I \) equivalent and irreducible representations with the dimension \( f_I \) for \( S_N \).

The trace of the operator \( e^{-\beta H} \) in the subspace. Let \( H \) be the Hamiltonian of a single particle and \( V \) be the Hilbert space of a single particle. One can give the matrix expression of the operator \( e^{-\beta H} \) on \( V \)

\[
e^{-\beta H} = \sum_{i=1}^{N} e^{-\beta \epsilon_i} |\phi_i\rangle\langle\phi_i|,
\]

(3.12)

where \( |\phi_i\rangle \) is the eigenfunction of the Hamiltonian \( H \) and \( \epsilon_i \) is the corresponding eigenvalue. \( e^{-\beta H_N} = (e^{-\beta H})^{\otimes N} \) is an operator on \( V^{\otimes N} \), where \( H_N = \bigoplus_{i=1}^{N} H_i \) is the Hamiltonian of an \( N \)-identical-particle gas system. Since \( e^{-\beta H_N} \) is an invariant matrix of \( e^{-\beta H} \), by using the mathematical theory of the invariant matrix, \( e^{-\beta H_N} \) can be decomposed into \( P(N) \)
irreducible invariant matrices. Each irreducible invariant matrix, denoted by $D_l \left( e^{-\beta H} \right)$, corresponds to an integer partition $(\lambda)_l$ of $N$. The trace of $D_l \left( e^{-\beta H} \right)$ is

$$\text{tr} \left[ D_l \left( e^{-\beta H} \right) \right] = s_{(\lambda)_l} \left( e^{-\beta \varepsilon_1}, e^{-\beta \varepsilon_2}, \ldots \right).$$

(3.13)

By the mathematical theory of the Schur-Weyl duality, we recognize that Eq. (3.13) is the trace of $e^{-\beta H_N}$ in the subspace $V''_I$ with $V''_I$ the inequivalent and irreducible representation corresponding to the integer partition $(\lambda)_l$. That is, for a complete basis $|\Phi\rangle$ in $V''_I$, one has

$$\sum \langle \Phi | e^{-\beta H_N} | \Phi \rangle = s_{(\lambda)_l} \left( e^{-\beta \varepsilon_1}, e^{-\beta \varepsilon_2}, \ldots \right).$$

(3.14)

According to the Schur-Weyl duality, the inequivalent and irreducible representations $V''_I$ occurs $f_I$ times in $V_I$, that is, for a complete basis $|\Psi\rangle$ in $V_I$, the equation

$$\sum \langle \Psi | e^{-\beta H_N} | \Psi \rangle = f_I s_{(\lambda)_l} \left( e^{-\beta \varepsilon_1}, e^{-\beta \varepsilon_2}, \ldots \right)$$

(3.15)

holds. In Eq. (3.15), the coefficient $f_I$ can be canceled by setting $|\Psi'\rangle = \frac{1}{\sqrt{f_I}} |\Psi\rangle$.

(3.16)

Thus, we make no distinguish between the subspace $V''_I$ and $V_I$ in the rest discussion of the present paper.

The proof of Eq. (3.1). An identical-particle system is described in a Hilbert subspace $D$. The space $D$ can be decomposed into subspace $V_I$ that carries the equivalent and irreducible representation of $S_N$ [19], i.e.,

$$D = \bigoplus_{I=1}^{P(N)} (V_I)^{\oplus C_I},$$

(3.17)

where $C_I$ are nonnegative integers representing the times of $V_I$ occuring in $D$. By the definition of the canonical partition function, $Z(\beta, N) = \text{tr} \left[ D \left( e^{-\beta H} \right) \right]$, and Eq. (3.13), we give the canonical partition function of an $N$-identical-particle system:

$$Z(\beta, N) = \text{tr} \left[ D \left( e^{-\beta H} \right) \right]$$

$$= \sum_{I=1}^{P(N)} C_I \text{tr} \left[ D_l \left( e^{-\beta H} \right) \right]$$

$$= \sum_{I=1}^{P(N)} C_I s_{(\lambda)_l} \left( e^{-\beta \varepsilon_1}, e^{-\beta \varepsilon_2}, \ldots \right),$$

(3.18)

where $D \left( e^{-\beta H} \right)$ is the representation of $e^{-\beta H}$ on $D$. Therefore, Eq. (3.18) proves Eq. (3.1). For a basis $|\Phi\rangle$ in $V_I$, it gives the equivalent and irreducible representation of $S_N$. Therefore, Eq. (3.13) requires that Eqs. (5.1) and (5.2) hold.

In the proof of Theorem (1), we show that the S-function $s_{(\lambda)}(x_1, x_2, \ldots)$ in mathematics is closely related to the Hilbert subspace: if the canonical partition function is written
as a linear combination of the S-function $s_{(\lambda)}(x_1, x_2, \ldots)$, the coefficient gives the Hilbert subspace.

An example of decomposing the Hilbert space $V^N$. For the sake of clarity, we give an example of the decomposition of the $V^\otimes N$. Let the dimension of the Hilbert space of a single particle $V$ be 6. For a system consists of 5 particles, the Hilbert space $V^\otimes 5$ is a 7776-dimensional space. It can be decomposed into 7 subspaces, as shown in Table. (1) : the subspace $V^{(5)}$ corresponds to the integer partition (5) and the dimension of $V^{(5)}$ is 252. It gives a one-dimensional representation of $S_N$, and a 252-dimensional representation of the Hamiltonian, and so on.

| Table 1. An example of decomposing the Hilbert space |
|-----------------------------------------------|
| subspace $V^{(\lambda)}$ | dim $(V^{(\lambda)})$ | $f_{(\lambda)}$ | $R_{(\lambda)}$ |
|-------------------------|----------------------|----------------|---------------|
| $V^{(5)}$                | 252                  | 1              | 252           |
| $V^{(4,1)}$              | 2016                 | 4              | 504           |
| $V^{(3,2)}$              | 2100                 | 5              | 420           |
| $V^{(3,1^2)}$            | 2016                 | 6              | 336           |
| $V^{(2^2,1)}$            | 1050                 | 5              | 210           |
| $V^{(2,1^3)}$            | 336                  | 4              | 84            |
| $V^{(1^5)}$              | 6                    | 1              | 6             |

4 The maximum occupation number for various kinds of statistics

In statistical mechanics, it is commonly believed that quantum statistics is determined by the maximum occupation number. Different maximum occupation numbers lead to different kinds of quantum statistics, e.g., setting the maximum occupation number to 1 leads to Fermi statistics. Therefore, the maximum occupation number is an important issue. For example, Ref. [24] shows that a number of quantization schemes in quantum field theory corresponds to different maximum occupation numbers in quantum statistical mechanics. Ref. [15] considers quantum statistics where the maximum occupation number for different states is different.

In this section, (1) we provide a method to obtain the maximum occupation number from the canonical partition function, and (2) we point out that the maximum occupation number is not sufficient to distinguish different kinds of statistics.

4.1 Obtaining the maximum occupation number from the canonical partition function

In this section, we show that as long as the canonical partition function is expressed in terms of $m_{(\lambda)}(x_1, x_2, \ldots)$, the maximum occupation number can be obtained directly.

**Theorem 2** For an ideal gas consisting of $N$-identical particles, if the canonical partition function $Z(\beta, N)$ can be written in terms of the monomial symmetric polynomial
$m_{(\lambda)}(x_1, x_2, \ldots)$ with nonnegative-integers coefficient, i.e.,

$$Z(\beta, N) = m_{(\lambda)^q}(e^{-\beta_1}, e^{-\beta_2}, \ldots) + \sum_{\lambda_1 \leq q}^{P(N)} M^I m_{(\lambda)_I}(e^{-\beta_1}, e^{-\beta_2}, \ldots) \quad (4.1)$$

with $(\lambda)^q$ denoting the integer partition with $\lambda_1 = q$ and $M^I$ a nonnegative integer, then the maximum occupation number is $q$.

Proof. The canonical partition function is

$$Z(\beta, N) = \sum_E \omega(E, N) e^{-\beta E}, \quad (4.2)$$

where $\omega(E, N)$ is the number of the microstate in the macrostate $(N, E)$ [1, 2]. By letting $x_i = e^{-\beta_1}$ with $\varepsilon_i$ the energy of quantum state in Eq. (2.5), the monomial symmetric polynomial becomes

$$m_{(\lambda)}(e^{-\beta_1}, e^{-\beta_2}, \ldots) = \sum_{\text{perm}} e^{-\beta \varepsilon_1 \lambda_1} e^{-\beta \varepsilon_2 \lambda_2} \ldots \quad (4.3)$$

If the canonical partition function of a quantum system $Z(\beta, N)$ is the monomial symmetric polynomial $m_{(\lambda)}(e^{-\beta_1}, e^{-\beta_2}, \ldots)$, i.e.,

$$Z(\beta, N) = m_{(\lambda)}(e^{-\beta_1}, e^{-\beta_2}, \ldots), \quad (4.4)$$

Then, substituting Eqs. (4.2) and (4.3) into Eq. (4.4) gives

$$\sum_E \omega(E, N) e^{-\beta E} = \sum_{\text{perm}} e^{-\beta \varepsilon_1 \lambda_1} e^{-\beta \varepsilon_2 \lambda_2} \ldots, \quad (4.5)$$

where $\omega(E, N)$ in Eq. (4.5) counts the number of microstate where there are $\lambda_1$ particles occupying a quantum state, $\lambda_2$ particles occupying another quantum states, and so on [8, 32]. In this case, the maximum occupation number is $\lambda_1$.

If the canonical partition function is a linear combination of $m_{(\lambda)}(e^{-\beta_1}, e^{-\beta_2}, \ldots)$, say, Eq. (4.1), then the corresponding $\omega(E, N)$ satisfies

$$\sum_E \omega(E, N) e^{-\beta E} = m_{(\lambda)^q}(e^{-\beta_1}, e^{-\beta_2}, \ldots) + \sum_{\lambda_1 \leq q}^{P(N)} M^I m_{(\lambda)_I}(e^{-\beta_1}, e^{-\beta_2}, \ldots). \quad (4.6)$$

Introducing $\omega_I(E, N)$ that satisfies

$$\omega(E, N) = \sum_{I=1}^{P(N)} \omega_I(E, N) \quad (4.7)$$

and substituting Eq. (4.7) into Eq. (4.6) give

$$\sum_E \omega_{(\lambda)^q}(E, N) e^{-\beta E} = m_{(\lambda)^q}(e^{-\beta_1}, e^{-\beta_2}, \ldots), \quad (4.8)$$

$$\sum_{\lambda_1 \leq q}^{P(N)} \sum_E \omega_I(E, N) e^{-\beta E} = \sum_{\lambda_1 \leq q}^{P(N)} M^I m_{(\lambda)_I}(e^{-\beta_1}, e^{-\beta_2}, \ldots). \quad (4.9)$$


Since $\omega(I(E,N))$ counts the number of microstate where there are $\lambda_{I,1}$ particles occupying a quantum state, $\lambda_{I,2}$ particles occupying another quantum state, and so on. Thus, the maximum occupation number of the system is the largest $\lambda_{J,1}$, for $M^J \neq 0$. In Eq. (4.6), the largest $\lambda_{J,1}$ is $q$. Therefore the maximum occupation number of the system is $q$.

In the proof of Theorem (2), we show that the monomial symmetric polynomials $m_{(\lambda)}(x_1, x_2, \ldots)$ in mathematics is closely related to the maximum occupation number in physics: if the canonical partition function is written as a linear combination of the monomial symmetric polynomials $m_{(\lambda)}(x_1, x_2, \ldots)$, the maximum element in the integer partition with non-zero coefficient gives the maximum occupation number.

Examples: Bose-Einstein, Fermi-Dirac, and Gentile statistics. As examples, we give a brief discussion on Bose-Einstein, Fermi-Dirac, and Gentile statistics as examples. The canonical partition functions for ideal Bose, Fermi, and Gentile gases are given in the previous work [8]:

$$Z_B(N, \beta) = m_{(N)} \left( e^{-\beta \epsilon_1}, e^{-\beta \epsilon_2}, \ldots \right) + \sum_{\lambda_{I,1} < N} m_{(\lambda)} \left( e^{-\beta \epsilon_1}, e^{-\beta \epsilon_2}, \ldots \right), \tag{4.10}$$

$$Z_F(N, \beta) = m_{(1N)} \left( e^{-\beta \epsilon_1}, e^{-\beta \epsilon_2}, \ldots \right), \tag{4.11}$$

$$Z_q(N, \beta) = m_{(\lambda)q} \left( e^{-\beta \epsilon_1}, e^{-\beta \epsilon_2}, \ldots \right) + \sum_{\lambda_{I,1} \leq q} m_{(\lambda)} \left( e^{-\beta \epsilon_1}, e^{-\beta \epsilon_2}, \ldots \right). \tag{4.12}$$

From Eqs. (4.10), (4.11), and (4.12), we can directly obtain the maximum occupation number: for Fermi cases, the maximum occupation number is 1; for the Gentile cases, the maximum occupation number is $q$; for Bose cases, there is no limitation on the maximum occupation number.

4.2 The failure of distinguishing different statistics by the maximum occupation number

It is commonly accepted that various statistics are distinguished by the maximum occupation number. In this section, we show that the maximum occupation number is not sufficient to distinguish different kinds of statistics. It is the maximum occupation number together with the coefficient of the monomial symmetric polynomial $m_{(\lambda)}(e^{-\beta \epsilon_1}, e^{-\beta \epsilon_2}, \ldots)$ in Eqs. (4.1) determines the kind of statistics. A discussion of para-Fermi statistics and Gentile statistics is given as examples.

Corollary 3 (1) The maximum occupation number is not sufficient to distinguish different statistics. The nonnegative-integers coefficient $M^I$ in Eq. (4.1) distinguishes statistics with the same maximum occupation number $q$.

(2) The microstate where there are $\lambda_{I,1}$ particles occupying a quantum state, $\lambda_{I,2}$ particles occupying another quantum state, and so on, will be counted $M^I$ times in the number of the microstate $\omega(E,N)$.

The proof of Corollary. (3) is embedded in the proof of Theorem (2). We give examples to illustrate Corollary. (3).
Example: Para-Fermi and Gentile statistics. For the sake of clarity, we give explicit expressions of the canonical partition function of para-Fermi statistics and Gentile statistics as examples. The detail of the calculation is given in the following section. For example, the canonical partition function for para-Fermi statistics with parameter $q = 2$ is

$$Z_{PF}^2(\beta, 5) = m_{(2, 1)} + 3m_{(2, 1^3)} + 10m_{(1^5)}$$

and the canonical partition function for Gentile statistics of maximum occupation number $q = 2$ is

$$Z_G^2(\beta, 5) = m_{(2, 1)} + m_{(2, 1^3)} + m_{(1^5)},$$

where we denote $m_{(\lambda)} = m_{(\lambda)}(e^{-\beta \varepsilon_1}, e^{-\beta \varepsilon_2}, ...)$ for convenience. Eqs. (4.13) and (4.14) show the difference between the para-Fermi statistics and Gentile statistics: although the maximum occupation numbers both are 2, the weight for microstate corresponding to the integer partition $(2, 1^3)$ and $(1^5)$ are different. The microstate with less number of particles occupying the same states has larger weight in para-Fermi statistics. This is because exchanging two different-state particles that obey para-Fermi statistics will lead to a new microstate.

5 The permutation phase of the wave function for various kinds of statistics

Unlike that in Bose and Fermi statistics, the permutation phase of the wave function of some kinds of generalized quantum statistics with a given maximum occupation number is unknown. For example, the permutation phase of the wave function corresponding to Gentile statistics which is determined by a maximum occupation number is obscure.

In this section, we show that for generalized quantum statistics with the Hamiltonian invariant under permutations, the permutation phase is generalized to a matrix, rather than a number. We provide a method to obtain the permutation phase of the wave function. The permutation phase of Bose or Fermi wave function, 1 or $-1$, is regarded as $1 \times 1$ matrices, as special cases in the scheme.

We also point out that there is generalized quantum statistics with the Hamiltonian noninvariant under permutations. For such statistics, the permutation phase of the wave function can not be constructed. We provide a method to distinguish them.

5.1 The statistics with the Hamiltonian invariant under permutations: the permutation phase

In this section, for various kinds of statistics with the Hamiltonian invariant under permutations, we provide a method to obtain the permutation phase from the canonical partition function.

**Theorem 4** If the Hamiltonian is invariant under permutations, i.e., $[H, S_N] = 0$, then the wave function after exchanging two particles $\sigma_{ij}|\Phi\rangle$ satisfies

$$\sigma_{ij}|\Phi\rangle = D(\sigma_{ij})|\Phi\rangle,$$

(5.1)
where $\sigma_{ij}|\Phi\rangle$ represents exchanging the $i$th particle and the $j$th particle in the wave function $|\Phi\rangle$. $D(\sigma_{ij})$ here is the permutation phase,

$$D(\sigma_{ij}) = \bigoplus_{I=1}^{P(N)} [D^I(\sigma_{ij})] \oplus C_I, \quad (5.2)$$

where $D^I$ is the inequivalent and irreducible representation of $S_N$ corresponding to the integer partition $(\lambda)_I$, and $D^I$ occurs $C_I$ times with $C_I$ given in Eq. (3.1).

Proof. Since the Hamiltonian of the system is invariant under permutations. The wave function $|\Phi\rangle$ forms a base of the space $D$ that carries the representation of $S_N$. By using Theorem (1), one can find that if the canonical partition function is written in the form Eq. (3.1), then the space $D$ can be decomposed into subspace $V^I$ that carries the equivalent and irreducible representation of $S_N$, and $V^I$ occurs $C_I$ times with $C_I$ given in Eq. (3.1). Thus, Eq. (5.2) holds. ■

It is obvious that the permutation phase given in Eq. (5.2) is a matrix. For Bose and Fermi cases, the permutation phase, $D(\sigma_{ij})$, recovers $\pm 1$.

5.2 The statistics with the Hamiltonian noninvariant under permutations

A physics system is described by a Hamiltonian. The canonical partition function which is obtained by taking statistical averages, however, does not contain all the information of the physics system. Consequently, the invariance of the Hamiltonian under permutations ensures the invariance of the canonical partition function under permutations, but not vice versa.

In this section, we provide a method to distinguish statistics with the Hamiltonian noninvariant under permutations. For such statistics, the permutation phase can not be constructed in the scheme.

Theorem 5 If the canonical partition function of an ideal gas consisting of $N$-identical particles, $Z(\beta,N)$, can not be written in the form Eq. (3.1) with non-negative-integer coefficient, then the Hamiltonian is noninvariant under permutations, i.e., $[H,S_N] \neq 0$.

Proof. Theorem (5) is the reverse proposition of Theorem (1). Thus Theorem (5) holds. ■

Examples such as Gentile statistics will be given in the following section.

6 A unified framework

For Bose statistics, there is no limitation on the maximum occupation and the wave function is symmetric with the permutation phase $e^{i0} = 1$. For Fermi statistics, the maximum occupation number is 1 and the wave function is anti-symmetric with the permutation phase $e^{i\pi} = -1$. The Hilbert subspace describing Bose statistics is the symmetric subspace, as shown in Fig. (3). The Hilbert subspace describing Fermi statistics is the anti-symmetric subspace, as shown in Fig. (4).
In this section, as examples, we describe a series of generalized quantum statistics, such as parastatistics [12, 13], the intermediate statistics or Gentile statistics [14, 15], Gentileonic statistics [16], and the immannons [17], in a unified framework. For these kinds of statistics, the canonical partition function, the maximum number, and the permutation phase of the wave function are given. Especially, three new generalized quantum statistics, which seem to be the missing pieces in the puzzle are proposed.

The Hilbert subspace helps to illustrate the difference between generalized quantum statistics intuitively. Therefore, we also give the Hilbert subspace for these kinds of statistics.

6.1 The $N$-distinguishable-particle gas system: Boltzmann statistics

The system consisting of distinguishable particles obeys Boltzmann statistics. The canonical partition function for an ideal $N$-distinguishable-particle gas is [1, 2]

$$Z_{cl}(\beta, N) = \left( \sum_i e^{-\beta \varepsilon_i} \right)^N. \quad (6.1)$$

In this section, by discussing the $N$-distinguishable-particle gas in the scheme, we suggest an unconventional perspective to study the Hilbert subspace and the maximum occupation number of the system. The distinguishability of the particle of Boltzmann statistics appears automatically.

6.1.1 The Hilbert subspace

One can verify that Eq. (6.1) can be expressed as a linear combination of the S-function:

$$Z_{cl}(\beta, N) = \left( \sum_i e^{-\beta \varepsilon_i} \right)^N = \sum_{I=1}^{P(N)} f_I s(\lambda)_I \left( e^{-\beta \varepsilon_1}, e^{-\beta \varepsilon_2}, \ldots \right), \quad (6.2)$$

where the coefficient $f_I$ is defined in Eq. (3.6). By using Theorem (1), Eq. (6.2) implies that the Hilbert space describing the $N$-distinguishable-particle system is

$$D = \bigoplus_{I=1}^{P(N)} (V_I')^\oplus f_I = \bigoplus_{I=1}^{P(N)} V_I' \otimes N = V^\otimes N, \quad (6.3)$$

where we use the fact that $V_I'$ occurs $f_I$ times in $V_I$, i.e., $V_I' = (V_I')^\oplus f_I$. Therefore, as shown in Fig. (2), the Hilbert space $V^\otimes N$ describes an $N$-distinguishable-particle gas system.
Figure 2. The space $V^\otimes N$ describing the $N$-distinguishable-particle gas system. Each cube represents a subspace corresponding to an integer partition.

Figure 3. The Hilbert subspace describing Bose-Einstein statistics.
6.1.2 The maximum occupation number and the distinguishability of the particle.

One can verify that Eq. (6.1) can be expressed as a linear combination of the monomial symmetric polynomial $m_{\lambda}(e^{-\beta \varepsilon_1}, e^{-\beta \varepsilon_2}, \ldots)$:

$$Z_{cl}(\beta, N) = \left( \sum_i e^{-\beta \varepsilon_i} \right)^N = \sum_{I=1}^{P(N)} z_I m_{\lambda(I)}(e^{-\beta \varepsilon_1}, e^{-\beta \varepsilon_2}, \ldots), \quad (6.4)$$

where $z_I$ satisfies

$$z_I = N! \left( \prod_{j=1}^{N} \lambda_{I,j}! \right)^{-1}, \quad (6.5)$$

By using Theorem (2), one can find that there is no limitation on the maximum occupation number for an $N$-distinguishable-particle gas system, since the first term in Eqs. (6.4) is $m_{\lambda(N)}(e^{-\beta \varepsilon_1}, e^{-\beta \varepsilon_2}, \ldots)$.

A direct manifestation of the distinguishability of the particle is given: the coefficient $z_I$ appears because the number of microstates with $\lambda_{I,1}$ distinguishable particles occupying a quantum state, $\lambda_{I,2}$ distinguishable particles occupying another quantum state, and so on, is exactly $z_I$.

6.2 Gentile statistics

Gentile statistics is a generalization of Bose-Einstein and Fermi-Dirac statistics. The maximum occupation number of Gentile statistics is an integer $q [14, 15, 33]$. 
The canonical partition function of an ideal Gentile gas is [8]

\[ Z^G_q (\beta, N) = \sum_{I=1}^{P(N)} Q^I (q) s(\lambda), \quad \left( e^{-\beta \varepsilon_1}, e^{-\beta \varepsilon_2}, \ldots \right), \quad (6.6) \]

where the coefficient \( Q^I (q) \) is given as

\[ \sum_{J=1}^{P(N)} \gamma^J (q) \left( k^J_I \right)^{-1} = Q^I (q). \quad (6.7) \]

\( \left( k^J_I \right)^{-1} \) satisfies \( \left( k^J_I \right)^{-1} k^J_L = \delta^J_L \) and \( \gamma^J (q) \) satisfies

\[ \gamma^J (q) = \begin{cases} 1, & \text{if } \lambda_I, 1 \leq q, \\ 0, & \text{otherwise}. \end{cases} \quad (6.8) \]

In this section, we give a discussion on Gentile statistics in the scheme. Especially, we show that the Hamiltonian of a Gentile-statistics system is noninvariant under permutations.

6.2.1 Discussions on the permutation phase of the wave function

By using Theorem (5), one can find that, for Gentile statistics, the Hamiltonian is noninvariant under permutations. The permutation phase can not be constructed in the scheme. This can be verified, from Eq. (6.6), that the coefficient, \( Q^I (q) \), is not nonnegative. As examples, we list the explicit expression of the canonical partition function of \( N = 3, 4, 5, \) and 6 with various maximum occupation numbers \( q \).

**N = 3,**

\[ Z^G_2 (\beta, 3) = s(2, 1) - s(1^3), \quad (6.9) \]

where we denote \( s(\lambda) = s(\lambda) \left( e^{-\beta \varepsilon_1}, e^{-\beta \varepsilon_2}, \ldots \right) \) for convenience.

**N = 4,**

\[ Z^G_2 (\beta, 4) = s(2^2) - s(1^4), \quad (6.10) \]

\[ Z^G_3 (\beta, 4) = s(3, 1) - s(2, 1^2) + s(1^4). \quad (6.11) \]

**N = 5,**

\[ Z^G_2 (\beta, 5) = s(2^2, 1) - s(2, 1^3), \quad (6.12) \]

\[ Z^G_3 (\beta, 5) = s(3, 2) - s(2^2, 1) + s(1^5), \quad (6.13) \]

\[ Z^G_4 (\beta, 5) = s(4, 1) - s(3, 1^2) + s(2, 1^3) - s(1^5). \quad (6.14) \]

**N = 6,**

\[ Z^G_2 (\beta, 6) = s(2^3) - s(2, 1^4) + s(1^6), \quad (6.15) \]

\[ Z^G_3 (\beta, 6) = s(3, 3) - s(2^2, 1^2) + s(1^4), \quad (6.16) \]

\[ Z^G_4 (\beta, 6) = s(4, 2) - s(3, 2, 1) + s(2^2, 1^2) - s(1^6), \quad (6.17) \]

\[ Z^G_5 (\beta, 6) = s(5, 1) - s(4, 1^2) + s(3, 1^3) - s(2, 1^4) + s(1^6). \quad (6.18) \]
The Hilbert subspace describing Gentile statistics is the subspaces with nonzero coefficient $Q^I(q)$, as shown in Figs. (5)-(7). One can find that the subspace becomes complicated as $N$ increases.

**Figure 5.** The Hilbert subspace describing Gentile statistics with maximum occupation number $q=2$. The cube with horizontal lines means that the corresponding coefficient is negative.

**Figure 6.** The Hilbert subspace describing Gentile statistics with maximum occupation number $q=3$. 
6.3 Parastatistics

Parastatistics is proposed by H. Green as a generalization of Bose and Fermi statistics in 1953 [12, 20]. In Green’s generalization, a trilinear relation of the algebra of creation and annihilation operators is proposed [34, 35]. Moreover, one already knows that parastatistics corresponding to the higher dimensional representation of the permutation group [4]. For instance, Okayama (1952) [22] suggests that all irreducible representations associated with Young diagrams of at most $p$ columns yield parastatistics [36].

The canonical partition function for para-Bose and para-Fermi statistics with $q$ the parameter is [34]

$$Z_{q}^{PB}(\beta, N) = \sum_{I=1, I(\lambda) \leq q}^{P(N)} s(\lambda)_I \left( e^{-\beta \varepsilon_1}, e^{-\beta \varepsilon_2}, \ldots \right),$$  \hspace{1cm} (6.19)

$$Z_{q}^{PF}(\beta, N) = \sum_{I=1, \lambda_I \leq q}^{P(N)} s(\lambda)_I \left( e^{-\beta \varepsilon_1}, e^{-\beta \varepsilon_2}, \ldots \right).$$  \hspace{1cm} (6.20)

In this section, we discuss parastatistics in the scheme. The permutation phase for parastatistics is discussed in Refs. [4, 22, 36]. Our method is new and gives results such as the dimension of the permutation phase of the wave function corresponding to parastatistics.
6.3.1 The maximum occupation number

The canonical partition function of parastatistics, Eqs. (6.19) and (6.20), can be rewritten in terms of \( m_{(\lambda)} \) \( (e^{-\beta \epsilon_1}, e^{-\beta \epsilon_2}, ...) \):

\[
Z_{q}^{PB} (\beta, N) = \sum_{I,J=1}^{P(N)} P_{b}^{I} (q) k_{I}^{J} m_{(\lambda)} \left( e^{-\beta \epsilon_1}, e^{-\beta \epsilon_2}, ... \right), \tag{6.21}
\]

\[
Z_{q}^{PF} (\beta, N) = \sum_{I,J=1}^{P(N)} P_{f}^{I} (q) k_{I}^{J} m_{(\lambda)} \left( e^{-\beta \epsilon_1}, e^{-\beta \epsilon_2}, ... \right), \tag{6.22}
\]

where

\[
P_{b}^{I} (q) = \begin{cases} 
1, & \text{for } l_{(\lambda)} \leq q, \\
0, & \text{otherwise}
\end{cases}, \tag{6.23}
\]

\[
P_{f}^{I} (q) = \begin{cases} 
1, & \text{for } \lambda_{I,1} \leq q, \\
0, & \text{otherwise}
\end{cases}. \tag{6.24}
\]

Since the Kostka number is a lower triangular matrix \([9, 11]\), for para-Fermi statistics, the first term in the canonical partition function is \( m_{(\lambda)} \) \( (e^{-\beta \epsilon_1}, e^{-\beta \epsilon_2}, ...) \) with \( \lambda_{I,1} = q \). For the para-Bose statistics, the first term in the canonical partition function is \( m_{(N)} \) \( (e^{-\beta \epsilon_1}, e^{-\beta \epsilon_2}, ...) \) with \( \lambda_{I,1} = N \). Eqs. (6.21) and (6.22) can be written in the form

\[
Z_{q}^{PB} (\beta, N) = m_{(N)} \left( e^{-\beta \epsilon_1}, e^{-\beta \epsilon_2}, ... \right) + \sum_{\lambda_{I,1} < N}^{P(N)} M^{I} m_{(\lambda)} \left( e^{-\beta \epsilon_1}, e^{-\beta \epsilon_2}, ... \right), \tag{6.25}
\]

\[
Z_{q}^{PF} (\beta, N) = m_{(\lambda)} \left( e^{-\beta \epsilon_1}, e^{-\beta \epsilon_2}, ... \right) + \sum_{\lambda_{I,1} < q}^{P(N)} M^{I} m_{(\lambda)} \left( e^{-\beta \epsilon_1}, e^{-\beta \epsilon_2}, ... \right). \tag{6.26}
\]

By using Theorem (2), one can find that for para-Fermi statistics with parameter \( q \), the maximum occupation number is \( q \). For para-Bose statistics, there is no limitation on the maximum occupation number.

For the sake of clarity, we list some of the explicit expression of the canonical partition function for parastatistics of \( N = 3, 4, 5 \) with different \( q \) (for \( q = 1 \) parastatistics recovers Bose and Fermi statistics). The detail of the calculation can be found in appendixes.

\( N = 3 \), for para-Bose cases,

\[
Z_{2}^{PB} (\beta, 3) = m_{(3)} + 2m_{(2,1)} + 3m_{(1^3)}, \tag{6.27}
\]

\[
Z_{3}^{PB} (\beta, 3) = m_{(3)} + 2m_{(2,1)} + 3m_{(1^3)}. \tag{6.28}
\]

For para-Fermi cases,

\[
Z_{2}^{PF} (\beta, 3) = m_{(2,1)} + 3m_{(1^3)}, \tag{6.29}
\]

\[
Z_{3}^{PF} (\beta, 3) = m_{(3)} + 2m_{(2,1)} + 4m_{(1^3)}. \tag{6.30}
\]
For para-Fermi cases,

\[ Z_2^{PF}(\beta, 4) = m_{(4)} + 2m_{(3,1)} + 3m_{(2^2)} + 4m_{(2,1^2)} + 6m_{(1^4)}, \]  
\[ Z_3^{PF}(\beta, 4) = m_{(4)} + 2m_{(3,1)} + 3m_{(2^2)} + 5m_{(2,1^2)} + 9m_{(1^4)}, \]  
\[ Z_4^{PF}(\beta, 4) = m_{(4)} + 2m_{(3,1)} + 3m_{(2^2)} + 5m_{(2,1^2)} + 10m_{(1^4)}. \]  

For para-Bose cases,

\[ Z_3^B(\beta, 4) = \left( \begin{array}{c} \frac{1}{2} \end{array} \right) (3^2) \]  
\[ Z_4^B(\beta, 4) = \left( \begin{array}{c} \frac{1}{2} \end{array} \right) (3^2) \]  
\[ Z_5^B(\beta, 4) = \left( \begin{array}{c} \frac{1}{2} \end{array} \right) (3^2) \]
From Eqs. (6.27) - (6.44), one can see that the coefficient distinguishes parastatistics from Gentile statistics. For example, for para-Fermi statistics with parameter $q$, although the maximum occupation number is $q$, however, it does not yield Gentile statistics, because the weight, represented by the coefficient, distinguishes those two kinds of statistics, e.g., $Z_{2}^{PF}(\beta, 5) = m(2,1) + 3m(2,1^3) + 10m(1^5)$ and $Z_{2}^{G}(\beta, 5) = m(2,1) + m(2,1^3) + m(1^5)$. In Gentile statistics, the microstate with the same occupation number only be counted once.

6.3.2 The permutation phase of the wave function

The canonical partition function of parastatistics, Eqs. (6.19) and (6.20), shows, using Theorem (1), that the Hilbert subspace describing para-Bose statistics is a direct sum of those spaces corresponding to $(\lambda)$ with length smaller than $q$, as shown in Figs. (10) and (11). The Hilbert subspace describing para-Fermi statistics is a direct sum of those spaces corresponding to $(\lambda)$ with $\lambda_1$ smaller than $q$, as shown in Figs. (8) and (9). That is,

\[ \bigoplus_{l(\lambda) \leq q} V^{(\lambda)} \text{ for para-Bose statistics with parameter } q, \]
\[ \bigoplus_{\lambda_{r,1} \leq q} V^{(\lambda)} \text{ for para-Fermi statistics with parameter } q. \]

![Figure 8. The Hilbert subspace describing para-Fermi statistics with $q=2$.](image-url)
Figure 9. The Hilbert subspace describing para-Fermi statistics with q=3.

Figure 10. The Hilbert subspace describing para-Bose statistics with q=2.
For parastatistics, using Theorem (4), the permutation phase of the wave function can be given as
\[ \sigma_{ij} |\Phi\rangle = D(\sigma_{ij}) |\Phi\rangle, \tag{6.45} \]
where
\[ D = \begin{cases} \bigoplus_{l(\lambda) \leq q} D^l & \text{for para-Bose statistics with parameter } q, \\ \bigoplus_{\lambda_1,1 \leq q} D^l & \text{for para-Fermi statistics with parameter } q. \end{cases} \tag{6.46} \]
This is a multi-dimensional representation. By using the dimension of the representation of the permutation group \([18]\), we give the dimension of the permutation phase. For para-Bose statistics with parameter \(q\),
\[ \dim(D) = \sum_{l(\lambda) \leq q} N! \prod_{i=1,i<j} (\lambda_{i,j} - \lambda_{i,j} - i + j) \prod_{i=1} \left( (l(\lambda) + \lambda_{i,j}) - i \right)!^{-1}. \tag{6.47} \]
For para-Fermi statistics with parameter \(q\),
\[ \dim(D) = \sum_{\lambda_1,1 \leq q} N! \prod_{i=1,i<j} (\lambda_{i,j} - \lambda_{i,j} - i + j) \prod_{i=1} \left( (l(\lambda) + \lambda_{i,j} - i) \right)!^{-1}. \tag{6.48} \]

6.4 The immannons and Gentileonic statistics

Gentileonic statistics is proposed by Cattani and Fernandes in 1984 \([16]\). The immannons is proposed by Tichy in 2017 \([17]\). They both are generalized quantum statistics corresponding to the higher dimensional representation of permutation groups \([16, 17]\).

In this section, we discuss the immannons and Gentileonic statistics in the scheme. The result shows that in statistical mechanics, the immannons and Gentileonic statistics are essentially equivalent.
6.4.1 The canonical partition function

**Theorem 6** The canonical partition function of the immanonns and Gentileonic statistics is

\[ Z^{IG}_{(\lambda)}(\beta, N) = s(\lambda) \left( e^{-\beta \varepsilon_1}, e^{-\beta \varepsilon_2}, \ldots \right). \]  

(6.49)

**Proof.** Gentileonic statistics is related to the higher dimensional representation of \( S_N \) and the wave function is given as [16, 20]

\[ |\Psi_{(\lambda)}\rangle = \frac{1}{\sqrt{f_I}} \begin{pmatrix} \Phi_1^{(\lambda)} \\ \Phi_2^{(\lambda)} \\ \vdots \\ \Phi_f^{(\lambda)} \end{pmatrix}, \]  

(6.50)

where \( \Phi_i^{(\lambda)} \) is the operator associated with the Young shapes [19] corresponding to the integer partition \( (\lambda) \). Since the operator associate with the Young shape is one of the constructions for the basis of the subspace that carries the irreducible representation for \( S_N \) [19], that is, the subspace spanned by the wave function \( |\Psi_{(\lambda)}\rangle \) is \( V^{(\lambda)} \). The canonical partition function is

\[ Z^{IG}_{(\lambda)}(\beta, N) = \sum \langle \Psi_{(\lambda)} | e^{-\beta H_N} | \Psi_{(\lambda)} \rangle \]  

(6.51)

\[ = \frac{1}{f_I} \sum_{i=1}^{f_I} \langle \Phi_i^{(\lambda)} | e^{-\beta H_N} | \Phi_i^{(\lambda)} \rangle \]  

(6.52)

where \( \langle \Phi_i^{(\lambda)} | e^{-\beta H_N} | \Phi_i^{(\lambda)} \rangle = \langle \Phi_j^{(\lambda)} | e^{-\beta H_N} | \Phi_j^{(\lambda)} \rangle = s(\lambda) \left( e^{-\beta \varepsilon_1}, e^{-\beta \varepsilon_2}, \ldots \right) \) is used. \( \Phi_i^{(\lambda)} \) and \( \Phi_j^{(\lambda)} \) give the equivalent and irreducible representation for \( H_N \).

The immanonns [17] is a kind of generalized quantum statistics, of which, the inner product of the wave function gives the immanant, labeled by an integer partition \( (\lambda) \) [17]. It recovers Bose-Einstein statistics for \( (\lambda) = (N) \), and Fermi-Dirac statistics for \( (\lambda) = (1^N) \). The wave function of the immanonns is [17]

\[ |\Phi_{(\lambda)}\rangle \propto \hat{\mathcal{P}}_{\lambda} |\psi_1, \psi_2, \ldots, \psi_N\rangle, \]  

(6.53)

where

\[ \hat{\mathcal{P}}_{\lambda} = \frac{\chi(\lambda)(e)}{N!} \sum_{\sigma \in S_N} \chi(\lambda)(\sigma) \hat{Q}(\sigma) \]  

(6.54)

with \( \chi(\lambda)(\sigma) \) the simple characteristic of \( \sigma \) and \( \hat{Q}(\sigma) \) an operator satisfying [17]

\[ \hat{Q}(\sigma) |\psi_1, \psi_2, \ldots, \psi_N\rangle = |\psi_{\sigma_1}, \psi_{\sigma_2}, \ldots, \psi_{\sigma_N}\rangle. \]  

(6.55)

The wave function, Eq. (6.53), is a construction of the basis for the subspace \( V^{(\lambda)} \). Thus the wave function, Eq. (6.53), directly yields

\[ Z^{IG}_{(\lambda)}(\beta, N) = \langle \Phi_{(\lambda)} | e^{-\beta H_N} | \Phi_{(\lambda)} \rangle = s(\lambda) \left( e^{-\beta \varepsilon_1}, e^{-\beta \varepsilon_2}, \ldots \right). \]  

(6.56)
Eq. (6.49) shows that in statistical mechanics, the immanonns and Gentileonic statistics share the same canonical partition function and thus are essentially the same statistics.

6.4.2 Discussions on the permutation phase of the wave function

The Hilbert subspace describing Gentileonic statistics and the immanonns are slightly difference: the subspace describing the Gentileonic statistics is $V^{(\lambda)}$ and the subspace describing the immanonns is $V'^{(\lambda)}$, i.e.,

$$D = \begin{cases} V^{(\lambda)}, & \text{for Gentileonic statistics,} \\ V'^{(\lambda)}, & \text{for the immanonns.} \end{cases}$$  \hspace{1cm} (6.57)$$

We have shown in the proof of Theorem (1) that the subspace $V'^{(\lambda)}$ occurs $f^{(\lambda)}$ times in $V^{(\lambda)}$, thus these two subspaces are essentially the same.

The immanonns does not possess a permutation symmetry, because $V'^{(\lambda)}$, spanned by wave function Eq. (6.53), does not carry a representation for $S_N$. However, for Gentileonic statistics the permutation phase of the wave function is

$$\sigma_{ij} |\Phi\rangle = D^I (\sigma_{ij}) |\Phi\rangle.$$  \hspace{1cm} (6.58)$$

![Figure 12](image)

**Figure 12.** The Hilbert subspace describing the immanonns and Gentileonic statistics labeled by the integer partition $(4,1^2)$. 
6.4.3 The maximun occupation number

We express the canonical partition function of the immanons and the Gentileonic statistics, Eq. (6.49), in terms of $m_{(\lambda)}$, $(x_1, x_2, ...)$:

$$Z_{I}^{\delta \gamma}(\gamma, \lambda) = \frac{P(\gamma)P(\lambda)}{P(\gamma)} \sum_{I = 1}^{\Delta_{(\lambda)}} \sum_{I = 0}^{\Delta_{(\lambda)}} \delta_{I, K}k_{I}^{I}m_{(\lambda)}I\left( e^{-\beta \varepsilon_{1}}, e^{-\beta \varepsilon_{2}}, \ldots \right) \equiv \sum_{I = 1}^{\Delta_{(\lambda)}} k_{I}^{I}m_{(\lambda)}I\left( e^{-\beta \varepsilon_{1}}, e^{-\beta \varepsilon_{2}}, \ldots \right).$$  (6.59)

Since the Kostka number is a lower triangular matrix, i.e., $k_{J}^{I} = 0$ if $I < J$ [9, 11], the first term of in Eq. (6.59) is always

$$Z_{I}^{\delta \gamma}(\gamma, \lambda) = m_{(\lambda)}I\left( e^{-\beta \varepsilon_{1}}, e^{-\beta \varepsilon_{2}}, \ldots \right) + \ldots,$$  (6.60)

which, using Theorem (2), implies that the maximum occupation number for the immanons and Gentileonic statistics is $\lambda_{J,1}$.

For example, the explicit expression of the canonical partition function for the immanons and Gentileonic statistics labeled by the integer partition $(3, 1^{2})$ is

$$Z_{I}^{\delta \gamma}(3, 1^{2}) = m_{(3, 1^{2})} + m_{(2^{2}, 1)} + 3m_{(2, 1^{3})} + 6m_{(1^{5})}.$$  (6.61)

The coefficient in Eq. (6.59) distinguishes the immanons and Gentileonic statistics from Gentile statistics.

6.5 generalized quantum statistics dual to Gentile statistics: GD-ons

In this section, we propose a new kind of generalized quantum statistics dual to Gentile statistics. For convenience, we denote the particle GD-ons. By dual we mean that the relation between GD-ons and Gentile statistics is an analog to the relation between para-Bose and para-Fermi statistics.

6.5.1 The canonical partition function

The canonical partition function for GD-ons labeled by the integer partition $(\lambda)$ is

$$Z_{q}^{*}(N, \beta) \equiv \sum_{I = 1}^{\Delta_{(\lambda)}} \sum_{I = 1}^{\Delta_{(\lambda)}} m_{(\lambda)}I\left( e^{-\beta \varepsilon_{1}}, e^{-\beta \varepsilon_{2}}, \ldots \right).$$  (6.62)

GD-ons recovers bosons when $q = N$.

For Gentile statistics, the canonical partition function is given in Eq. (4.12). From Eqs. (6.62) and (4.12), one can see that the relation between GD-ons and Gentile statistics is an analog to the relation between para-Bose and para-Fermi statistics of which the canonical partition function is given in Eqs. (6.19) and (6.20).
6.5.2 Discussions on the permutation phase of the wave function

Expressing the canonical partition function, Eq. (6.62), in terms of the S-function gives

\[ Z_q^* (N, \beta) = \sum_{J=1}^{P(N)} G^J(q) s_{(\lambda)_I} \left( e^{-\beta \varepsilon_1}, e^{-\beta \varepsilon_2}, \ldots \right), \quad (6.63) \]

where the coefficient \( G^J(q) \) is given as

\[ \sum_{J=1}^{P(N)} \Gamma^J(q) (k^I_J)^{-1} = G^I(q) \quad (6.64) \]

with

\[ \Gamma^J(q) = \begin{cases} 1, \text{if } l_{(\lambda)_I} \leq q \\ 0, \text{otherwise} \end{cases} \quad (6.65) \]

By calculating the coefficient \( G^I(q) \), we find that \( G^I(q) \) is not nonnegative. For example, the explicit expressions for the canonical partition function of GD-ons with \( N = 5 \) are

\[ Z_1^* (5, \beta) = s(5) - s(4,1) + s(3,1^2) - s(2,1^3) + s(1^5), \quad (6.66) \]
\[ Z_2^* (5, \beta) = s(5) - s(4,1) + 2s(2,1^3) - 3s(1^5), \quad (6.67) \]
\[ Z_3^* (5, \beta) = s(5) - s(2,1^3) + 3s(1^5), \quad (6.68) \]
\[ Z_4^* (5, \beta) = s(5) - s(1^5), \quad (6.69) \]
\[ Z_5^* (5, \beta) = s(5) + s(4,1) + s(3,2) + s(3,1^2) + s(2,1^2) + s(2,1^3) + s(1^5). \quad (6.70) \]

Thus, for GD-ons, using Theorem (5), the Hamiltonian is noninvariant under permutations. The permutation phase of GD-ons can not be constructed in the scheme.

The Hilbert subspace describing GD-ons is the space with nonzero coefficients.

6.5.3 The maximum occupation number

Since the first term in the canonical partition function of GD-ons, Eqs. (6.62), is \( m_{(N)} \left( e^{-\beta \varepsilon_1}, e^{-\beta \varepsilon_2}, \ldots \right) \), i.e.,

\[ Z_q^* (N, \beta) = m_{(N)} \left( e^{-\beta \varepsilon_1}, e^{-\beta \varepsilon_2}, \ldots \right) + \sum_{I=2, l_{(\lambda)_I} \leq q}^{P(N)} m_{(\lambda)_I} \left( e^{-\beta \varepsilon_1}, e^{-\beta \varepsilon_2}, \ldots \right), \quad (6.71) \]

the maximum occupation number, using Theorem (2), is \( N \), that is, there is no limitation on the maximum occupation number for GD-ons.
6.6 generalized quantum statistics corresponding to the monomial symmetric function: M-ons

We have shown that the S-function is closely related to the permutation phase and the monomial symmetric function is closely related to the maximum occupation number. The generalized quantum statistics corresponding to the S-function is given as the immannons and Gentileonic statistics. In this section, we propose a kind of generalized quantum statistics corresponding to the monomial symmetric function \( m(\lambda) \). For the sake of convenience, we denote the particle M-ons.

6.6.1 The canonical partition function

The canonical partition function of M-ons labeled by the integer partition \( (\lambda) \) is

\[
Z^M_{(\lambda)} (\beta, N) = m(\lambda) \left( e^{-\beta \varepsilon_1}, e^{-\beta \varepsilon_2}, ... \right).
\] (6.72)

The M-ons recover fermions when \( (\lambda) = (1^N) \).

6.6.2 Discussions on the permutation phase of the wave function

Expressing the canonical partition function, Eq. (6.72), in terms of the S-function gives

\[
Z^M_{(\lambda)} (\beta, N) = \sum_{K=1}^{P(N)} \sum_{I=1}^{P(N)} \delta_{J,K} \left( k^I_J \right)^{-1} s(\lambda)_I \left( e^{-\beta \varepsilon_1}, e^{-\beta \varepsilon_2}, ... \right). \] (6.73)

The coefficient in Eq. (6.73) is not nonnegative. For example, the explicit expressions of the canonical partition function for M-ons labeled by \( (\lambda) = (5), (4,1), \ldots \), are

\[
Z^M_{(5)} (\beta, 5) = s(5) - s(4,1) + s(3,1^2) - s(2,1^3) + s(1^5), \] (6.74)

\[
Z^M_{(4,1)} (\beta, 5) = s(4,1) - s(3,2) - s(3,1^2) + s(2^2,1) + s(2,1^3) - 2s(1^5), \] (6.75)

\[
Z^M_{(3,2)} (\beta, 5) = s(3,2) - s(3,1^2) - s(2^2,1) + 2s(2,1^3) - 2s(1^5), \] (6.76)

\[
Z^M_{(3,1^2)} (\beta, 5) = s(3,1^2) - s(2^2,1) - s(2,1^3) + 3s(1^5), \] (6.77)

\[
Z^M_{(2^2,1)} (\beta, 5) = s(2^2,1) - 2s(2,1^3) + 3s(1^5), \] (6.78)

\[
Z^M_{(2,1^3)} (\beta, 5) = s(2,1^3) - 4s(1^5). \] (6.79)

For M-ons, using Theorem (5), the Hamiltonian is noninvariant under permutations. The permutation phase for M-ons can not be constructed in the scheme.

The Hilbert subspace describing the M-ons is the space with nonzero coefficients.

6.6.3 The maximum occupation number

One can find, by using Theorem (2), that the maximum occupation number is \( q = \lambda_1 \). In the case of M-ons, the only counted microstate is that there are \( \lambda_1 \) particles occupying a quantum state, \( \lambda_2 \) particles occupying another quantum state, and so on.
6.7 generalized quantum statistics corresponding to the power sum symmetric function: P-ons

In this section, we propose a kind of generalized quantum statistics corresponding to the power sum symmetric function \( p_{(\lambda)} \). For the sake of convenience, we denote the particle P-ons.

6.7.1 The canonical partition function

The canonical partition function of P-ons labeled by the integer partition \((\lambda)\) is

\[
Z_{(\lambda)}^P (\beta, N) = p_{(\lambda)} \left( e^{-\beta \epsilon_1}, e^{-\beta \epsilon_2}, ... \right). \tag{6.80}
\]

Interestingly, one can verify that the P-ons recovers the distinguishable particle when \((\lambda) = (1^N)\).

6.7.2 Discussions on the permutation phase of the wave function

Expressing the canonical partition function, Eq. (6.80), in terms of the S-function gives

\[
Z_{(\lambda)K}^P (\beta, N) = p_{(\lambda)K} \left( e^{-\beta \epsilon_1}, e^{-\beta \epsilon_2}, ... \right) = \sum_{I=1}^{P(N)} \chi_{K, I} s_{(\lambda)I} (x_1, x_2, ...). \tag{6.81}
\]

The coefficient in Eq. (6.81) is not nonnegative. For example, the explicit expressions of the canonical partition function for P-ons labeled by \((\lambda) = (5), (4, 1), \ldots\), are

\[
\begin{align*}
Z_{(5)}^P (5, \beta) &= s_{(5)} - s_{(4, 1)} + s_{(3, 1^2)} - s_{(2, 1^3)} + s_{(1^5)}, \tag{6.82} \\
Z_{(4, 1)}^P (5, \beta) &= s_{(5)} - s_{(3, 2)} + s_{(2^2, 1)} - s_{(1^5)}, \tag{6.83} \\
Z_{(3, 2)}^P (5, \beta) &= s_{(5)} - s_{(4, 1)} + s_{(3, 2)} - s_{(2^2, 1)} + s_{(2, 1^3)} - s_{(1^5)}, \tag{6.84} \\
Z_{(3, 1^2)}^P (5, \beta) &= s_{(5)} + s_{(4, 1)} - s_{(3, 2)} - s_{(2^2, 1)} + s_{(2, 1^3)} + s_{(1^5)}, \tag{6.85} \\
Z_{(2^2, 1)}^P (5, \beta) &= s_{(5)} + s_{(3, 2)} - 2s_{(3, 1^2)} + s_{(2^2, 1)} + s_{(1^5)}, \tag{6.86} \\
Z_{(2, 1^3)}^P (5, \beta) &= s_{(5)} + s_{(4, 1)} - s_{(3, 2)} - s_{(2^2, 1)} + s_{(2, 1^3)} + s_{(1^5)}, \tag{6.87} \\
Z_{(1^5)}^P (5, \beta) &= s_{(5)} + 4s_{(4, 1)} + 5s_{(3, 2)} + 6s_{(3, 1^2)} + 5s_{(2^2, 1)} + 4s_{(2, 1^3)} + s_{(1^5)}. \tag{6.88}
\end{align*}
\]

Thus, the Hamiltonian, using Theorem (5), is noninvariant under permutations. The permutation phase of P-ons can not be constructed in the scheme.

The Hilbert subspace describing the P-ons is the space with nonzero coefficients.
6.7.3 The maximum occupation number

Expressing the canonical partition function, Eq. (6.80), in terms of the monomical symmetric function gives

\[ Z^K_{\lambda \beta}(\beta, N) = \sum_{j=1}^{P(N)} \left( \sum_{I=1}^{P(N)} \chi^K_I k^I_J \right) m(\lambda)_j \left( e^{-\beta \epsilon_1}, e^{-\beta \epsilon_2}, \ldots \right). \] (6.89)

The maximum occupation number can be obtained from the non-zero coefficient in Eq. (6.89). Based on the property of the simple characteristics of the \( S_N \) and the Kostka number \( k^I_J \), Eq. (6.89) can be written as

\[ Z^K_{\lambda \beta}(\beta, N) = m(N) \left( e^{-\beta \epsilon_1}, e^{-\beta \epsilon_2}, \ldots \right) + \ldots. \] (6.90)

For example, the explicit expression of the canonical partition function for P-ons labeled by \( (\lambda) = (5), (4,1), \ldots \), are

\[ Z^K_{(5)}(5, \beta) = m(5), \] (6.91)

\[ Z^K_{(4,1)}(5, \beta) = m(5) + m(4,1), \] (6.92)

\[ Z^K_{(3,2)}(5, \beta) = m(5) + m(3,2), \] (6.93)

\[ Z^K_{(3,1^2)}(5, \beta) = m(5) + 2m(4,1) + m(3,2) + 2m(2^2,1), \] (6.94)

\[ Z^K_{(2^2,1)}(5, \beta) = m(5) + m(4,1) + 2m(3,2) + 2s(2^2,1), \] (6.95)

\[ Z^K_{(2,1^3)}(5, \beta) = m(5) + 2m(4,1) + m(3,2) + 2m(3,1^2), \] (6.96)

\[ Z^K_{(1^5)}(5, \beta) = m(5) + 5m(4,1) + 10m(3,2) + 20m(3,1^2) + 30m(2^2,1) + 60m(2,1^3) + 120m(1^5). \] (6.97)

One can find, by using Theorem (2), that for P-ons, there is no limitation on the maximum occupation number.

7 Conclusions

In this paper, we give the unified framework to describe various kinds of generalized quantum statistics. We provide a general formula of canonical partition functions of ideal \( N \)-particle gases obeying various kinds of generalized quantum statistics. We reveal the connection between the permutation phase of the wave function and the maximum occupation number, through constructing a method of obtaining the permutation phase and the maximum occupation number from the canonical partition function.

We show that for generalized quantum statistics, the permutation phase of wave functions should be generalized to a matrix, rather than a number. The permutation phase of Bose or Fermi wave function, 1 or -1, is regarded as \( 1 \times 1 \) matrices, as special cases of
generalized quantum statistics. We suggest a method to distinguish generalized quantum statistics with the Hamiltonian noninvariant under permutations, in which the permutation phase can not be constructed in the scheme.

It is commonly accepted that different kinds of statistics are distinguished by the maximum number. We show that the maximum occupation number is not sufficient to distinguish different kinds of generalized quantum statistics.

As examples, we describe various kinds of statistics in a unified framework, including parastatistics, Gentile statistics, Gentileonic statistics, and the immannons. The canonical partition function, the maximum occupation number, the permutation phase of the wave function, and the Hilbert subspace are given. Especially, we propose three new kinds of generalized statistics which seem to be the missing pieces in the puzzle.

The mathematical basis of the scheme are the mathematical theory of the invariant matrix, the Schur-Weyl duality, the symmetric function, and the representation theory of the permutation group and the unitary group. The result in this paper, together with our previous works [8, 32], builds a bridge between the quantum statistical mechanics and such mathematical theories. This enables one to use the fruitful result in such theories to solve the problem in quantum statistical mechanics.

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9 Appendix

9.1 The Kostka number

$N = 3$. The Kostka number $k^L_J$ is $k^{(3)}_1 = k^1_1 = 1$, $k^{(3)}_2 = k^1_2 = 0$, $k^{(3)}_3 = k^1_3 = 0$, $k^{(2,1)}_1 = k^1_1 = 1$, $k^{(2,1)}_2 = k^2_1 = 1$, $k^{(1)}_3 = k^3_1 = 1$, and $k^{(1)}_3 = k^3_2 = 2$, and $k^{(1)}_3 = k^3_3 = 1$ [9, 11]. For clarity, we rewrite the Kostka number in a matrix form:

$$k = \begin{pmatrix}
1 & 0 & 0 \\
1 & 1 & 0 \\
1 & 2 & 1
\end{pmatrix},$$

(9.1)

where we take the upper index of $k^L_J$ as a row index and the lower index as a column index.

$N = 4$. The Kostka number is [9, 11]

$$k = \begin{pmatrix}
1 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 \\
1 & 1 & 1 & 0 \\
1 & 2 & 1 & 1 \\
1 & 3 & 2 & 3
\end{pmatrix}.$$
$N = 5$. The Kostka number is \[9, 11\]

$$k = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 & 0 & 0 \\
1 & 1 & 1 & 0 & 0 & 0 \\
1 & 2 & 1 & 1 & 0 & 0 \\
1 & 2 & 2 & 1 & 1 & 0 \\
1 & 3 & 3 & 2 & 1 & 0 \\
1 & 4 & 5 & 6 & 5 & 4 \\
1 & 4 & 5 & 6 & 5 & 4 \\
\end{pmatrix}. \quad (9.3)$$

### 9.2 The simple characteristic of $S_N$

$N = 3$. The simple characteristic $\chi^J$ is $\chi^{(3)} = \chi_1^1 = 1$, $\chi^{(3)}_{(2,1)} = \chi_2^1 = 1$, $\chi^{(3)}_{(1^3)} = \chi_3^1 = 1$, $\chi^{(2,1)} = \chi_1^2 = 2$, $\chi^{(2,1)}_{(2,1)} = \chi_2^2 = 0$, $\chi^{(2,1)}_{(1^3)} = \chi_3^2 = -1$, $\chi^{(1^3)}_{(3)} = \chi_1^3 = 1$, $\chi^{(1^3)}_{(2,1)} = \chi_2^3 = -1$, and $\chi^{(1^3)}_{(1^3)} = \chi_3^3 = 1$ [19]. For clarity, we can rewrite the simple characteristic in a matrix form:

$$\chi = \begin{pmatrix}
1 & 1 & 1 \\
2 & 0 & -1 \\
1 & -1 & 1 \\
\end{pmatrix}. \quad (9.4)$$

$N = 4$. The $\chi$ is [19]

$$\chi = \begin{pmatrix}
1 & 1 & 1 & 1 \\
3 & 0 & -1 & 1 & -1 \\
2 & 1 & 2 & 0 & 0 \\
3 & 0 & -1 & -1 & 1 \\
1 & 1 & 1 & -1 & -1 \\
\end{pmatrix}. \quad (9.5)$$

$N = 5$. The $\chi$ is [19]

$$\chi = \begin{pmatrix}
1 & 1 & 1 & 1 & 1 & 1 \\
-1 & 0 & -1 & 1 & 0 & 1 & 4 \\
0 & -1 & 1 & -1 & 1 & -1 & 5 \\
1 & 0 & 0 & 0 & -2 & 0 & 6 \\
0 & 1 & -1 & -1 & 1 & -1 & 5 \\
-1 & 0 & 1 & 1 & 0 & 1 & 4 \\
1 & -1 & -1 & -1 & 1 & 1 & 1 \\
\end{pmatrix}. \quad (9.6)$$

### 9.3 The calculation of the maximum occupation number of the parastatistics: an example of calculation the coefficient in the canonical partition function

The canonical partition function of an $N$-particle gas under various kinds of generalized quantum statistics can be expressed in terms of symmetric functions such as the S-function, the monomial symmetric function, and so on. The procedure of the calculation of the coefficient in the canonical partition function involves the transformation of the coefficient by the matrix of Kostka number or the simple characteristic. In this appendix, we give details of the calculation of the coefficient for parastatistics as an example.
\( N = 3, q = 1 \), for para-Bose statistics, the coefficient in Eq. (6.21) is

\[
\begin{pmatrix}
1 & 0 & 0 \\
1 & 1 & 0 \\
1 & 2 & 1
\end{pmatrix}
\begin{pmatrix}
0 \\
0 \\
1
\end{pmatrix}
= 
\begin{pmatrix}
1 \\
1 \\
1
\end{pmatrix},
\]

(9.7)

that is,

\[
Z_{1}^{PB}(\beta, 3) = m_{(3)} + m_{(2, 1)} + m_{(1^3)};
\]

for para-Fermi statistics, the coefficient in Eq. (6.22) is

\[
\begin{pmatrix}
1 & 0 & 0 \\
1 & 1 & 0 \\
1 & 2 & 1
\end{pmatrix}
\begin{pmatrix}
0 \\
0 \\
1
\end{pmatrix}
= 
\begin{pmatrix}
0 \\
0 \\
1
\end{pmatrix},
\]

(9.8)

that is,

\[
Z_{1}^{PB}(\beta, 3) = m_{(1^3)}.
\]

In the following, the corresponding expression of the canonical partition function is already given in Eqs. (6.27)-(6.44), thus, we only give the coefficient. \( q = 2 \), for para-Bose statistics, the coefficient in Eq. (6.21) is

\[
\begin{pmatrix}
1 & 0 & 0 \\
1 & 1 & 0 \\
1 & 2 & 1
\end{pmatrix}
\begin{pmatrix}
1 \\
1 \\
0
\end{pmatrix}
= 
\begin{pmatrix}
1 \\
2 \\
3
\end{pmatrix};
\]

(9.9)

for para-Fermi statistics, the coefficient in Eq. (6.22) is

\[
\begin{pmatrix}
1 & 0 & 0 \\
1 & 1 & 0 \\
1 & 2 & 1
\end{pmatrix}
\begin{pmatrix}
0 \\
1 \\
1
\end{pmatrix}
= 
\begin{pmatrix}
0 \\
1 \\
3
\end{pmatrix}.
\]

(9.10)

\( q = 3 \), for para-Bose statistics, the coefficient in Eq. (6.21) is

\[
\begin{pmatrix}
1 & 0 & 0 \\
1 & 1 & 0 \\
1 & 2 & 1
\end{pmatrix}
\begin{pmatrix}
1 \\
1 \\
1
\end{pmatrix}
= 
\begin{pmatrix}
1 \\
2 \\
3
\end{pmatrix};
\]

(9.11)

for para-Fermi statistics, the coefficient in Eq. (6.22) is

\[
\begin{pmatrix}
1 & 0 & 0 \\
1 & 1 & 0 \\
1 & 2 & 1
\end{pmatrix}
\begin{pmatrix}
1 \\
1 \\
1
\end{pmatrix}
= 
\begin{pmatrix}
1 \\
2 \\
3
\end{pmatrix}.
\]

(9.12)

\( N = 4, q = 1 \), for para-Bose statistics, the coefficient in Eq. (6.21) is
for para-Fermi statistics, the coefficient in Eq. (6.22) is

\[
\begin{pmatrix}
1 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 \\
1 & 1 & 1 & 0 \\
1 & 2 & 1 & 1 \\
1 & 3 & 2 & 3
\end{pmatrix}
\begin{pmatrix}
1 \\
0 \\
0 \\
0 \\
1
\end{pmatrix} = \begin{pmatrix}
1 \\
1 \\
1 \\
2 \\
3
\end{pmatrix} ; \quad (9.13)
\]

\(q = 2\), for para-Bose statistics, the coefficient in Eq. (6.21) is

\[
\begin{pmatrix}
1 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 \\
1 & 1 & 1 & 0 \\
1 & 2 & 1 & 1 \\
1 & 3 & 2 & 3
\end{pmatrix}
\begin{pmatrix}
1 \\
1 \\
1 \\
0 \\
0
\end{pmatrix} = \begin{pmatrix}
1 \\
2 \\
3 \\
4 \\
6
\end{pmatrix} . \quad (9.14)
\]

for para-Fermi statistics, the coefficient in Eq. (6.22) is

\[
\begin{pmatrix}
1 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 \\
1 & 1 & 1 & 0 \\
1 & 2 & 1 & 1 \\
1 & 3 & 2 & 3
\end{pmatrix}
\begin{pmatrix}
0 \\
0 \\
1 \\
0 \\
1
\end{pmatrix} = \begin{pmatrix}
0 \\
0 \\
1 \\
2 \\
9
\end{pmatrix} ; \quad (9.16)
\]

\(q = 3\), for para-Bose statistics, the coefficient in Eq. (6.21) is

\[
\begin{pmatrix}
1 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 \\
1 & 1 & 1 & 0 \\
1 & 2 & 1 & 1 \\
1 & 3 & 2 & 3
\end{pmatrix}
\begin{pmatrix}
1 \\
1 \\
1 \\
1 \\
0
\end{pmatrix} = \begin{pmatrix}
1 \\
2 \\
3 \\
4 \\
9
\end{pmatrix} ; \quad (9.17)
\]

for para-Fermi statistics, the coefficient in Eq. (6.22) is

\[
\begin{pmatrix}
1 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 \\
1 & 1 & 1 & 0 \\
1 & 2 & 1 & 1 \\
1 & 3 & 2 & 3
\end{pmatrix}
\begin{pmatrix}
0 \\
1 \\
1 \\
0 \\
1
\end{pmatrix} = \begin{pmatrix}
0 \\
1 \\
2 \\
4 \\
9
\end{pmatrix} . \quad (9.18)
\]
$q = 4$, for para-Bose statistics, the coefficient in Eq. (6.21) is
\[
\begin{pmatrix}
1 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 \\
1 & 1 & 1 & 0 \\
1 & 2 & 1 & 1 \\
1 & 3 & 2 & 3
\end{pmatrix}
\begin{pmatrix}
1 \\
1 \\
1 \\
1 \\
1
\end{pmatrix}
= \begin{pmatrix}
1 \\
2 \\
3 \\
4 \\
5
\end{pmatrix};
\] (9.19)

for para-Fermi statistics, the coefficient in Eq. (6.22) is
\[
\begin{pmatrix}
1 & 0 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 & 0 \\
1 & 1 & 1 & 0 & 0 \\
1 & 2 & 1 & 1 & 0 \\
1 & 3 & 2 & 3 & 1
\end{pmatrix}
\begin{pmatrix}
1 \\
1 \\
1 \\
1 \\
1
\end{pmatrix}
= \begin{pmatrix}
1 \\
2 \\
3 \\
4 \\
5
\end{pmatrix}.
\] (9.20)

$N = 5$, $q = 1$, for para-Bose statistics, the coefficient in Eq. (6.21) is
\[
\begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 & 0 & 0 \\
1 & 1 & 1 & 0 & 0 & 0 \\
1 & 2 & 1 & 1 & 0 & 0 \\
1 & 2 & 2 & 1 & 1 & 0 \\
1 & 3 & 3 & 3 & 2 & 1 \\
1 & 4 & 5 & 6 & 5 & 4 & 1
\end{pmatrix}
\begin{pmatrix}
1 \\
0 \\
0 \\
0 \\
1 \\
0 \\
0
\end{pmatrix}
= \begin{pmatrix}
1 \\
0 \\
0 \\
0 \\
0
\end{pmatrix};
\]

for para-Fermi statistics, the coefficient in Eq. (6.22) is
\[
\begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 1 & 0 & 0 & 0 & 0 \\
1 & 2 & 1 & 1 & 0 & 0 & 0 \\
1 & 2 & 2 & 1 & 1 & 0 & 0 \\
1 & 3 & 3 & 3 & 2 & 1 & 0 \\
1 & 4 & 5 & 6 & 5 & 4 & 1
\end{pmatrix}
\begin{pmatrix}
0 \\
0 \\
0 \\
0 \\
1 \\
0 \\
1
\end{pmatrix}
= \begin{pmatrix}
0 \\
0 \\
0 \\
0 \\
1 \\
0
\end{pmatrix}.
\]

$q = 2$, for para-Bose statistics, the coefficient in Eq. (6.21) is
\[
\begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 1 & 0 & 0 & 0 & 0 \\
1 & 2 & 1 & 1 & 0 & 0 & 0 \\
1 & 2 & 2 & 1 & 1 & 0 & 0 \\
1 & 3 & 3 & 3 & 2 & 1 & 0 \\
1 & 4 & 5 & 6 & 5 & 4 & 1
\end{pmatrix}
\begin{pmatrix}
1 \\
1 \\
1 \\
1 \\
1 \\
1 \\
1
\end{pmatrix}
= \begin{pmatrix}
1 \\
2 \\
3 \\
4 \\
5 \\
7 \\
10
\end{pmatrix};
\]
for para-Fermi statistics, the coefficient in Eq. (6.22) is

\[
\begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 & 0 & 0 \\
1 & 1 & 1 & 0 & 0 & 0 \\
1 & 2 & 1 & 1 & 0 & 0 \\
1 & 2 & 2 & 1 & 1 & 0 \\
1 & 3 & 3 & 3 & 2 & 1 \\
1 & 4 & 5 & 6 & 5 & 4 \\
1 & 0 & 0 & 0 & 0 & 0 \\
\end{pmatrix}
\begin{pmatrix}
0 \\
0 \\
0 \\
0 \\
1 \\
1 \\
1 \\
1 \\
\end{pmatrix}
= 
\begin{pmatrix}
0 \\
0 \\
0 \\
0 \\
1 \\
1 \\
3 \\
10 \\
\end{pmatrix}
\].

\( q = 3 \), for para-Bose statistics, the coefficient in Eq. (6.21) is

\[
\begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 & 0 & 0 \\
1 & 1 & 1 & 0 & 0 & 0 \\
1 & 2 & 1 & 1 & 0 & 0 \\
1 & 2 & 2 & 1 & 1 & 0 \\
1 & 3 & 3 & 3 & 2 & 1 \\
1 & 4 & 5 & 6 & 5 & 4 \\
1 & 0 & 0 & 0 & 0 & 0 \\
\end{pmatrix}
\begin{pmatrix}
1 \\
1 \\
1 \\
1 \\
1 \\
0 \\
0 \\
0 \\
\end{pmatrix}
= 
\begin{pmatrix}
1 \\
2 \\
3 \\
5 \\
7 \\
12 \\
21 \\
\end{pmatrix}
; \]

for para-Fermi statistics, the coefficient in Eq. (6.22) is

\[
\begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 & 0 & 0 \\
1 & 1 & 1 & 0 & 0 & 0 \\
1 & 2 & 1 & 1 & 0 & 0 \\
1 & 2 & 2 & 1 & 1 & 0 \\
1 & 3 & 3 & 3 & 2 & 1 \\
1 & 4 & 5 & 6 & 5 & 4 \\
1 & 0 & 0 & 0 & 0 & 0 \\
\end{pmatrix}
\begin{pmatrix}
0 \\
0 \\
1 \\
0 \\
1 \\
1 \\
1 \\
0 \\
\end{pmatrix}
= 
\begin{pmatrix}
0 \\
0 \\
1 \\
2 \\
3 \\
4 \\
9 \\
21 \\
\end{pmatrix}
. \]

\( q = 4 \), for para-Bose statistics, the coefficient in Eq. (6.21) is

\[
\begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 & 0 & 0 \\
1 & 1 & 1 & 0 & 0 & 0 \\
1 & 2 & 1 & 1 & 0 & 0 \\
1 & 2 & 2 & 1 & 1 & 0 \\
1 & 3 & 3 & 3 & 2 & 1 \\
1 & 4 & 5 & 6 & 5 & 4 \\
1 & 0 & 0 & 0 & 0 & 0 \\
\end{pmatrix}
\begin{pmatrix}
1 \\
1 \\
1 \\
1 \\
1 \\
0 \\
0 \\
0 \\
\end{pmatrix}
= 
\begin{pmatrix}
1 \\
2 \\
3 \\
5 \\
7 \\
13 \\
25 \\
\end{pmatrix}
; \]
for para-Fermi statistics, the coefficient in Eq. (6.22) is
\[
\begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 & 0 & 0 \\
1 & 1 & 1 & 0 & 0 & 0 \\
1 & 2 & 1 & 1 & 0 & 0 \\
1 & 2 & 2 & 1 & 1 & 0 \\
1 & 3 & 3 & 2 & 1 & 0 \\
1 & 4 & 5 & 6 & 5 & 4 \\
\end{pmatrix}
\begin{pmatrix}
0 \\
1 \\
1 \\
1 \\
1 \\
1 \\
1 \\
\end{pmatrix}
= \begin{pmatrix}
0 \\
1 \\
2 \\
4 \\
6 \\
12 \\
25 \\
\end{pmatrix}.
\]

$q = 5$, for para-Bose statistics, the coefficient in Eq. (6.21) is
\[
\begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 & 0 & 0 \\
1 & 1 & 1 & 0 & 0 & 0 \\
1 & 2 & 1 & 1 & 0 & 0 \\
1 & 2 & 2 & 1 & 1 & 0 \\
1 & 3 & 3 & 2 & 1 & 0 \\
1 & 4 & 5 & 6 & 5 & 4 \\
\end{pmatrix}
\begin{pmatrix}
1 \\
1 \\
1 \\
1 \\
1 \\
1 \\
1 \\
\end{pmatrix}
= \begin{pmatrix}
1 \\
2 \\
3 \\
5 \\
7 \\
13 \\
26 \\
\end{pmatrix}.
\]

for para-Fermi statistics, the coefficient in Eq. (6.22) is
\[
\begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 & 0 & 0 \\
1 & 1 & 1 & 0 & 0 & 0 \\
1 & 2 & 1 & 1 & 0 & 0 \\
1 & 2 & 2 & 1 & 1 & 0 \\
1 & 3 & 3 & 2 & 1 & 0 \\
1 & 4 & 5 & 6 & 5 & 4 \\
\end{pmatrix}
\begin{pmatrix}
1 \\
1 \\
1 \\
1 \\
1 \\
1 \\
1 \\
\end{pmatrix}
= \begin{pmatrix}
1 \\
2 \\
3 \\
5 \\
7 \\
13 \\
26 \\
\end{pmatrix}.
\]

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