On extensions of $J$-skew-symmetric and $J$-isometric operators.

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1 Introduction.

Last years an increasing number of papers was devoted to the investigations of operators related to a conjugation in a Hilbert space, see, e.g. [1], [2], [3], [4] and references therein. A conjugation $J$ in a Hilbert space $H$ is an antilinear operator on $H$ such that $J^2x = x$, $x \in H$, and $(Jx, Jy)_H = (y, x)_H$, $x, y \in H$. The conjugation $J$ generates the following bilinear form:

$$[x, y]_J := (x, Jy)_H, \quad x, y \in H.$$  

A linear operator $A$ in $H$ is said to be $J$-symmetric ($J$-skew-symmetric) if

$$[Ax, y]_J = [x, Ay]_J, \quad x, y \in D(A),$$  \hspace{1cm} (1)

or, respectively,

$$[Ax, y]_J = -[x, Ay]_J, \quad x, y \in D(A).$$  \hspace{1cm} (2)

A linear operator $A$ in $H$ is said to be $J$-isometric if

$$[Ax, Ay]_J = [x, y]_J, \quad x, y \in D(A).$$  \hspace{1cm} (3)

If $D(A) = H$, then conditions (1), (2) and (3) are equivalent to the following conditions:

$$JAJ \subseteq A^*, \quad \text{(4)}$$

$$JAJ \subseteq -A^*, \quad \text{(5)}$$

and

$$JA^{-1}J \subseteq A^*,$$  \hspace{1cm} (6)

respectively. A linear operator $A$ in $H$ is called $J$-self-adjoint ($J$-skew-self-adjoint, or $J$-unitary) if

$$JAJ = A^*,$$  \hspace{1cm} (7)

$$JAJ = -A^*,$$  \hspace{1cm} (8)

or

$$JA^{-1}J = A^*,$$  \hspace{1cm} (9)

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respectively.

We shall prove that each densely defined $J$-skew-symmetric operator (each $J$-isometric operator with $D(A) = R(A) = H$) in a Hilbert space $H$ has a $J$-skew-self-adjoint (respectively $J$-unitary) extension in a Hilbert space $\tilde{H} \supseteq H$. We shall follow the ideas of Galindo in [5] with necessary modifications. In particular,Lemma in [5] can not be applied in our case, since its assumptions can never be satisfied with $T$: $T^2 = I$. In fact, in this case $T$ would be a conjugation in $H$. Choosing an element $f \in H$ of an orthonormal basis in $H$ which corresponds to $T$, we would get $(f, Tf) = (f, f) = 1 \neq 0$. Moreover, an exit out of the original space can appear in our case.

We notice that under stronger assumptions on a $J$-skew-symmetric operator the existence of a $J$-skew-self-adjoint extension was proved by Kalinina in [6].

**Notations.** As usual, we denote by $\mathbb{R}$, $\mathbb{C}$, $\mathbb{N}$, $\mathbb{Z}$, the sets of real numbers, complex numbers, positive integers, integers and non-negative integers, respectively. Set $\overline{0,d} = \{0,1,...,d\}$, if $d \in \mathbb{N}$; $\overline{0,\infty} = \mathbb{Z}_+$. If $H$ is a Hilbert space then $(\cdot, \cdot)_H$ and $\| \cdot \|_H$ mean the scalar product and the norm in $H$, respectively. Indices may be omitted in obvious cases. For a linear operator $A$ in $H$, we denote by $D(A)$ its domain, by $R(A)$ its range, and $A^*$ means the adjoint operator if it exists. If $A$ is invertible then $A^{-1}$ means its inverse. For a set $M \subseteq H$ we denote by $\overline{M}$ the closure of $M$ in the norm of $H$. By Lin $M$ we denote the set of all linear combinations of elements of $M$, and $\text{span} M := \text{Lin} M$. By $E_H$ we denote the identity operator in $H$, i.e. $E_H x = x$, $x \in H$. In obvious cases we may omit the index $H$. All appearing Hilbert spaces are assumed to be separable.

## 2 Extensions of $J$-skew-symmetric and $J$-isometric operators.

We shall make use of the following lemma.

**Lemma 1** Let $H$ be a Hilbert space with a positive even or infinite dimension, and $J$ be a conjugation on $H$. Then there exists a subspace $M$ in $H$ such that

$$M \oplus JM = H.$$ 

**Proof.** Let $\{f_n\}_{n=0}^{2d+1}$ be an orthonormal basis in $H$ corresponding to $J$, i.e. such that $Jf_n = f_n$, $0 \leq n \leq d$; $d \in \mathbb{Z}_+ \cup \{+\infty\}$ ($2d + 2 = \dim H$). Set $f^+_{2k,2k+1} = \frac{1}{\sqrt{2}} (f_{2k} + if_{2k+1})$, $f^-_{2k,2k+1} = \frac{1}{\sqrt{2}} (f_{2k} - if_{2k+1})$, $k \in \overline{0,d}$. 

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It is easy to see that \( \{ f_{2k,2k+1}^+, f_{2k,2k+1}^- \}_{k=0}^d \) is an orthonormal basis in \( H \). Set \( M := \text{span}\{ f_{2k,2k+1}^+ \}_{k=0}^d \). It remains to notice that \( JM = \text{span}\{ f_{2k,2k+1}^- \}_{k=0}^d \).

\( \blacksquare \)

**Theorem 1.** Let \( H \) be a Hilbert space and \( J \) be a conjugation on \( H \). Let \( A \) be a \( J \)-skew-symmetric (\( J \)-isometric) operator in \( H \). Suppose that \( D(A) = H \) (respectively \( D(A) = R(A) = H \)). Then there exists a \( J \)-skew-self-adjoint (respectively \( J \)-unitary) extension of \( A \) in a Hilbert space \( \tilde{H} \supseteq H \) (with an extension of \( J \) to a conjugation on \( \tilde{H} \)).

**Proof.** Let \( A \) be such an operator as that in the statement of the theorem. The operator \( A \) admits the closure which is \( J \)-skew-symmetric (respectively \( J \)-isometric) (see, e.g. \cite{[3], p. 18}). Thus, without loss of generality we shall assume that \( A \) is closed. In what follows, in the case of a \( J \)-skew-symmetric \((J\)-isometric\) \( A \), we shall say about case (a) (respectively case (b)). Set \( H_2 = H \oplus H \), and consider the following transformations on \( H_2 \):

\[ J_2 \{ x, y \} = \{ Jx, Jy \}, \quad V \{ x, y \} = \{ y, -x \}, \quad U \{ x, y \} = \{ y, x \}, \quad \forall \{ x, y \} \in H_2, \]

and \( R := UJ_2 = J_2U, \quad K := VR \). Observe that \( R \) and \( K \) are conjugations on \( H_2 \). The graph of an arbitrary linear operator \( C \) in the Hilbert space \( H \) will be denoted by \( G_C \subseteq H_2 \). Observe that

\[ J_2G_C = G_{JCJ}, \quad RG_C = UG_{JCJ}. \tag{10} \]

If \( D(C) = H \), then

\[ G_C^* = H_2 \oplus VG_C. \tag{11} \]

In the case (a) we may write:

\[ (\{ x, Ax \}, \{ JAy, y \}) = (x, JAJy) + (Ax, y) = 0, \quad \forall x \in D(A), y \in D(JAJ). \]

Then

\[ G_A \perp RG_A. \tag{12} \]

In the case (b), we have

\[ (\{ x, Ax \}, \{ JA^{-1}y, -y \}) = 0, \quad \forall x \in D(A), y \in D(JA^{-1}J), \]

and therefore

\[ G_A \perp KG_A. \tag{13} \]

Set \( D = \begin{cases} H_2 \oplus [G_A \oplus RG_A] & \text{in the case (a)} \\ H_2 \oplus [G_A \oplus KG_A] & \text{in the case (b)} \end{cases} \). If \( D = \{0\} \) then it means that \( A \) is \( J \)-skew-self-adjoint (respectively \( J \)-unitary), see considerations for
the operator $B$ below. In the opposite case, we have $RD = D$ (respectively $KD = D$).

At first, suppose that $D$ has a positive even or infinite dimension. By Lemma \[1\] we obtain that there exists a subspace $X \subseteq D$ such that $X \oplus RX = D$ (respectively $X \oplus KX = D$). Since each element of $X$ is orthogonal to $RG_A = VG_{-JAJ}$ ($KG_A = VJG_{JA^{-1}J}$), by \[11\] it follows that

$$X \subseteq G_{-JAJ} \quad \text{(respectively } X \subseteq G_{JA^{-1}J}). \quad (14)$$

Set $G' = G_A \oplus X$. Suppose that $\{0, y\} \in G'$. Then there exist $\{x, Ax\} \in G_A$ such that $\{0, y\} - \{x, Ax\} = \{-x, y - Ax\} \in X$. By \[11\] we get $y - Ax = JA^*Jx$ (respectively $y - Ax = -J(A^{-1})^*Jx$), and therefore $y = 0$. Thus, $G'$ is a graph $G_B$ of a densely defined linear operator $B$. Moreover, we have

$$G_B \oplus RG_B = H_2 \quad \text{(respectively } G_B \oplus KG_B = H_2).$$

In the case (a) we get

$$UG_B \oplus URG_B = H_2;$$

$$G_{(-B)^*} = H_2 \oplus VG_{-B} = H_2 \oplus UG_B = URG_B = J_2G_B = G_{JB}. \quad \text{In the case (b) we get}$$

$$VG_B \oplus VKG_B = H_2;$$

$$G_{B^*} = H_2 \oplus VG_B = VKG_B = -RG_B = G_{JB^{-1}}.$$

Suppose now that $D$ has a positive odd dimension. In this case we consider a linear operator $A = A \oplus A$, with $D(A) = D(A) \oplus D(A)$, in a Hilbert space $\mathcal{H} = H \oplus H$ with a conjugation $\mathcal{J} = J \oplus J$. Observe that $A$ is a closed $\mathcal{J}$-skew-symmetric ($\mathcal{J}$-isometric) operator with $\overline{D(A)} = \mathcal{H}$ (respectively $\overline{D(A)} = \overline{R(A)} = \mathcal{H}$). Its graph $G_A$ in a Hilbert space $\mathcal{H}_2 = \mathcal{H} \oplus \mathcal{H}$ may be identified with $G_A \oplus G_A$ in $H_2 \oplus H_2$:

$$G_A = \{(f, Af), (g, Ag)\}, \ f, g \in D(A).$$

Let $R$, $K$ be constructed for $A$ as $R$ and $K$ for $A$. In the case (a) we see that

$$\mathcal{H}_2 \oplus [G_A \oplus RG_A] = (H_2 \oplus [G_A \oplus RG_A]) \oplus (H_2 \oplus [G_A \oplus RG_A]),$$

has a positive even dimension. In the case (b), $\mathcal{H}_2 \oplus [G_A \oplus KG_A]$ has a positive even dimension. Thus, we may apply the above construction with $A$ instead of $A$.

\[\square\]
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In this paper it is proved that each densely defined J-skew-symmetric operator (or each J-isometric operator with $D(A) = R(A) = \mathcal{H}$) in a Hilbert space $\mathcal{H}$ has a J-skew-self-adjoint (respectively J-unitary) extension in a Hilbert space $\tilde{\mathcal{H}} \supseteq \mathcal{H}$. We follow the ideas of Galindo in [A. Galindo, On the existence of J-self-adjoint extensions of J-symmetric operators with adjoint, Communications on pure and applied mathematics, Vol. XV, 423-425 (1962)] with necessary modifications.