Representation of Small Conformal Algebra in $\kappa$-basis

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Abstract

In [hep-th/0202087] it was argued that the operator $L_0$ is bad defined in $\kappa$-basis as a kernel operator. Indeed, we show that $L_0$ is a difference operator. We also find a representation of $L_1$ and $L_{-1}$ in a class of difference operators.

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1 Introduction

The basic ingredients in the construction of the covariant string field theory are Witten’s star product [1] and BRST operator. Recent progress in the diagonalization [2] of the Neumann matrices $M^{rs}$ defining star product allows one to identify this product with the continuous Moyal product [3, 4]. We have

$$\sum_{n=1}^{\infty} M^{rs}_{mn} v_n^{(\kappa)} = \mu^{rs}(\kappa) v_m^{(\kappa)}$$

where $-\infty < \kappa < \infty$, the eigenvalues $\mu^{rs}(\kappa)$ are

$$\mu^{rs}(\kappa) = \frac{1}{1 + 2 \cosh \frac{\pi \kappa}{2}} \left[ 1 - 2 \delta_{r,s} + e^{\frac{\pi \kappa}{2}} \delta_{r+1,s} + e^{-\frac{\pi \kappa}{2}} \delta_{r,s+1} \right].$$

The eigenvectors $v_n^{(\kappa)}$ are given by their generating function

$$f^{(\kappa)}(z) = \sum_{n=1}^{\infty} v_n^{(\kappa)} z^n = \frac{1}{\kappa \sqrt{N(\kappa)}} (1 - e^{-\kappa \tan^{-1} z})$$

where

$$N(\kappa) = \frac{2}{\kappa} \sinh \left( \frac{\pi \kappa}{2} \right).$$

In spite of the fact that we understand the nature of the star product, it is still unclear how BRST operator acts in the basis in which star product is simple. Moreover we do not know a representation of Virasoro generators in this basis. The authors of [3] tried to construct generator $L_0$ as a kernel operator in the $\kappa$-basis. They show that this kernel is not defined as a distribution. In this paper I show that small conformal algebra $\{L_{-1}, L_0, L_1\}$ is represented by a certain difference operators in the $\kappa$-basis.

Let me briefly remind a construction of the Fock space at hand [5]. In the discrete basis we have creation and annihilation operators $a_n^\dagger$ and $a_n$, $n = 1, 2, \ldots$. Introduce the corresponding operators in the continuous basis via

$$a_\kappa^\dagger = \sum_{n=1}^{\infty} v_n^{(\kappa)} a_n^\dagger \quad \text{and} \quad a_\kappa = \sum_{n=1}^{\infty} v_n^{(\kappa)} a_n.$$  

(1.2)

Let $f_1(\kappa), \ldots, f_m(\kappa)$ be functions from Schwartz space, then the states

$$a_0^\dagger(f_1) \ldots a_0^\dagger(f_m)|0\rangle, \quad \text{where} \quad a_0^\dagger(f) = \int_{-\infty}^{\infty} d\kappa f(\kappa) a_\kappa^\dagger$$

form a dense subset in this Fock space. Since the operator

$$L_0 = \sum_{n=1}^{\infty} n a_n^\dagger a_n$$

(1.4)


contains only one creation and one annihilation operator it is completely determined by specifying its action on one particle states. Therefore we can translate the action of operator $L_0$ on states to its action on functions from the Schwartz space via

$$[L_0, a^\dagger(f)] = a^\dagger(L_0[f]).$$ (1.5)

In the next section I show that

$$L_0[f](\kappa) = \frac{1}{4} \left[ \sqrt{\kappa(\kappa+i2^-)} f(\kappa+i2^-) + \sqrt{\kappa(\kappa-i\frac{1}{2})} f(\kappa-i2^-) \right],$$ (1.6)

where $2^- = 2 - 0$. The operator $L_0$ is defined on a class of holomorphic functions in the strip $-2 < \Im \kappa < 2$. These functions may have poles on the boundary, and the term $i2^-$ shows how one approaches these poles. I.e. one should approach the pole on the line $\kappa + 2i$ from the bottom and the pole on the line $\kappa - 2i$ from the top. If it happens that the function $f$ is holomorphic in strip $-4 < \Im \kappa < 4$ one can apply operator $L_0$ once again, etc.

Notice that the operators $L_1$ and $L_{-1}$ also contain only one creation and one annihilation operator, and therefore they are completely determined by specifying their action on one particle subspace. We can write

$$[L_{\pm 1}, a^\dagger(f)] = a^\dagger(L_{\pm 1}[f]),$$ (1.7)

where

$$L_1[f] = -\frac{\kappa}{2} f(\kappa) + \frac{i}{4} \left[ \sqrt{\kappa(\kappa+i2^-)} f(\kappa+i2^-) - \sqrt{\kappa(\kappa-i2^-)} f(\kappa-i2^-) \right]$$ (1.8a)

$$L_{-1}[g] = -\frac{\kappa}{2} f(\kappa) - \frac{i}{4} \left[ \sqrt{\kappa(\kappa+i2^-)} f(\kappa+i2^-) - \sqrt{\kappa(\kappa-i2^-)} f(\kappa-i2^-) \right]$$ (1.8b)

The paper is organized as follows. In Sections 2 and 3 we give a derivation of the equations (1.6) and (1.8) correspondingly. In Section 4 we check that the generators (1.6) and (1.8) defining in a class of difference operators have indeed correct commutation relations. The Appendix contains the technical information.

## 2 Operator $L_0$

Using the fact that

$$[L_0, a_n^\dagger] = n a_n^\dagger,$$ (2.1)

one can easily obtain that $L_0$ acts on the generating function $[\hat{f}(z)]$ (it should be considered as function of $\kappa$; $z$ is an external parameter) in the following way

$$L_0[\hat{f}(\kappa)] = z \frac{d}{dz} \hat{f}(\kappa)(z).$$ (2.2)
From this one can easily obtain a formal expression for the kernel of the operator $L_0$

$$L_0(\kappa, \kappa') \sim \sum_{n=1}^{\infty} n_v(\kappa) v_n(\kappa'). \quad (2.3)$$

It was argued in [3] that this kernel is not defined as a distribution. This means that the operator $L_0$ is not a kernel operator in the $\kappa$-basis.

It was explained in the Introduction that operator $L_0$ is completely determined by its restriction onto the one-particle Fock space. To actually find it we will use the following strategy. First, find the inverse of the operator $L_0$ on one particle Fock subspace. In other words we are looking for an operator $G_0$ such that

$$G_0 L_0 |\psi\rangle = L_0 G_0 |\psi\rangle = |\psi\rangle$$

for any state $|\psi\rangle = a^\dagger(f)|0\rangle$ specifying by function $f$ from Schwartz space. Using relation (1.3) one can formulate this statement in the Schwartz space as

$$G_0[L_0[f]](\kappa) = L_0[G_0[f]](\kappa) = f(\kappa) \quad (2.4)$$

for any Schwartz function $f(\kappa)$. From (2.3) one can easily obtain a formal expression for the kernel of operator $G_0$

$$G_0(\kappa, \kappa') = \sum_{n=1}^{\infty} \frac{1}{n} n_v(\kappa) v_n(\kappa'). \quad (2.5)$$

Straight forward calculations of this kernel, which I present in Appendix [A], show that operator $G_0$ is indeed a kernel operator and

$$G_0(\kappa, \kappa') = \left[ \frac{\theta(\kappa)}{\kappa} \right]^{1/2} \left[ \frac{\theta(\kappa')}{\kappa'} \right]^{1/2} \frac{1}{4 \cosh \left[ \frac{\pi}{4} (\kappa - \kappa') \right]}.$$

Here $\theta(\kappa) = 2 \tanh \frac{\pi \kappa}{4}$ is a non-commutativity parameter specifying the continuous Moyal algebra [4].

Second, we find the inverse of operator $G_0$ on the one-particle Fock space. In other words we are going to solve equation (2.4). The resulting operator will coincide with the operator $L_0$ on the one-particle Fock space. But $L_0$ is completely determined by its action on this subspace and therefore we will actually find operator $L_0$ on the whole Fock space.

So we need to solve the equation $G_0[L_0[f]] = f$:

$$\int_{-\infty}^{\infty} d\kappa' \left[ \frac{\theta(\kappa)}{\kappa} \right]^{1/2} \frac{1}{4 \cosh \left[ \frac{\pi}{4} (\kappa - \kappa') \right]} \left[ \frac{\theta(\kappa')}{\kappa'} \right]^{1/2} g(\kappa') = f(\kappa), \quad (2.7)$$

where $g = L_0[f]$. It is convenient to introduce new functions $G$ and $F$ via

$$G(\kappa') = \left[ \frac{\theta(\kappa')}{\kappa'} \right]^{1/2} g(\kappa') \quad \text{and} \quad F(\kappa) = \left[ \frac{\theta(\kappa)}{\kappa} \right]^{-1/2} f(\kappa). \quad (2.8)$$
Then equation (2.7) takes a from
\[
\int_{-\infty}^{\infty} d\kappa' \frac{1}{4 \cosh \left[ \frac{\pi}{4} (\kappa - \kappa') \right]} G(\kappa') = F(\kappa). \tag{2.9}
\]
This equation can be easily solved by using Fourier transformation method:
\[
F(\kappa') = \frac{1}{2\pi} \int_{-\infty}^{\infty} d\omega e^{i\omega \kappa'} \tilde{F}(\omega)
\]
Using that
\[
\int_{-\infty}^{\infty} d\kappa' \frac{e^{i\omega \kappa'}}{4 \cosh \left[ \frac{\pi}{4} (\kappa - \kappa') \right]} = \frac{e^{i\omega \kappa}}{\cosh 2\omega}
\]
one obtains
\[
\tilde{G}(\omega) = \cosh 2\omega \tilde{F}(\omega).
\]
Finally the solution to (2.9) is of the form
\[
G(\kappa) = \frac{1}{2\pi} \int_{-\infty}^{\infty} d\omega \int_{-\infty}^{\infty} d\kappa' e^{i\omega (\kappa - \kappa')} \cosh 2\omega F(\kappa'). \tag{2.10}
\]
The integral over \(\omega\) does not exist (this once again confirms that \(L_0\) is not a kernel operator) and therefore we can not simply switch integrals over \(\omega\) and over \(\kappa'\). One can easily see that the integral operator in the right hand side is well defined at least on functions on \(\kappa'\) with Gaussian fall off at infinity. One can give a precise mathematical meaning to this operator if instead of Fourier transform one will use a Laplace transform and define \(0 < \kappa < \infty\). In this case we can rigorously prove that\footnote{Of course, expression (2.11) can be obtained by a formal integration of (2.10).}
\[
G(\kappa) = \frac{1}{2} \left[ F(\kappa + 2i - i0) + F(\kappa - 2i + i0) \right]. \tag{2.11}
\]
The function \(F\) here is supposed to be holomorphic function in the strip \(-2 < \Im \kappa < 2\). It is allowed to have poles on the boundary of the strip. The \(\pm i0\) in the formula above tells us how one should approach this poles. Namely one has to approach the pole on the line \(\kappa + 2i\) from the bottom and the one on the line \(\kappa - 2i\) from the top. Substitution of (2.8) into (2.11) yields expression (1.6) for the operator \(L_0\).

Let us now check that (2.11) is indeed a solution to equation (2.9):
\[
\int_{-\infty}^{\infty} d\kappa' \frac{1}{4 \cosh \left[ \frac{\pi}{4} (\kappa - \kappa') \right]} \frac{1}{2} \left[ F(\kappa + 2i - i0) + F(\kappa - 2i + i0) \right]
\]
\[
= \frac{1}{8} \int_{-\infty}^{\infty} d\kappa' F(\kappa') \left[ \frac{i}{\sinh \frac{\pi}{4} (\kappa - \kappa') + i\varepsilon \cosh \frac{\pi}{4} (\kappa - \kappa')} + \frac{-i}{\sinh \frac{\pi}{4} (\kappa - \kappa') - i\varepsilon \cosh \frac{\pi}{4} (\kappa - \kappa')} \right]
\]
\[
= F(\kappa).
\]
Here in the middle line we have made a shift of integration variable, which is allowed due to the fact that $F(\kappa)$ is holomorphic function on the strip $-2 < \Im \kappa < 2$. This completes the derivation of formula (1.6) defining the operator $L_0$ in the class of difference operators.

Let me show now that the operator $L_0$ defining through (1.6) acts onto generating function $f(\kappa)$ in the way it requires by (2.2). Indeed,

$$L_0[f(\kappa)(z)] = \frac{1}{4} \left[ \frac{\sqrt{\kappa(\kappa + 2i)}}{(\kappa + 2i)\sqrt{2\sinh(\pi\kappa / 2)e^{i\pi}}(1 - e^{-(\kappa+2i)\arctan z})} + \frac{\sqrt{\kappa(\kappa - 2i)}}{(\kappa - 2i)\sqrt{2\sinh(\pi\kappa / 2)e^{-i\pi}}(1 - e^{-(\kappa-2i)\arctan z})} \right]$$

$$= \frac{i}{4} e^{-\kappa \arctan z} \left( e^{-2i \arctan z} - e^{2i \arctan z} \right) = \frac{z}{1 + z^2} e^{-\kappa \arctan z} = z\frac{d}{dz} f(\kappa)(z). \quad (2.12)$$

3 Operators $L_1$ and $L_{-1}$

The operators $L_1$ and $L_{-1}$ are related by hermitian conjugation. Therefore it is enough to find the operator $L_{-1}$. As it is done in the previous section we restrict our considerations to the zero momentum sector. The action of $L_{-1}$ on the generating function $f(\kappa)(z)$ is given by the formula

$$L_{-1}[f(\kappa)(z)] = -\frac{1}{N(\kappa)} + \frac{d}{dz} f(\kappa)(z). \quad (3.1)$$

Therefore formally its kernel can be written as

$$L_{-1}(\kappa, \kappa') \sim \sum_{m=1}^{\infty} v_{m+1}^{(\kappa)} \sqrt{(m+1)m} v_m^{(\kappa')} \quad (3.2)$$

As it is explained in the Introduction the operator $L_{-1}$ is completely determined by its action onto the one-particle Fock space. Therefore in order to find it we will use the same method as the one used in the previous section. There will be only one modification of the method related to the fact that the operator $L_1$ (conjugated to the operator $L_{-1}$) has a one-dimensional kernel. First, we find an operator $G_{-1}$ such that it “inverts” $L_{-1}$ on the one-particle Fock space. More precisely $G_{-1}$ satisfies the equation

$$G_{-1}[L_{-1}[f]](\kappa) = f(\kappa) \quad \text{and} \quad L_{-1}[G_{-1}[f]](\kappa) = P_1[f](\kappa), \quad (3.3a)$$

where $P_1$ is a projector on the subspace on which $L_1$ is a non-degenerate operator:

$$P_1[f](\kappa) = f(\kappa) - v_1^{(\kappa)} \int_{-\infty}^{\infty} dk' v_1^{(k')} f(k'). \quad (3.3b)$$
From equation (3.2) one can easily obtain the formal expression for the kernel of operator \( G_{-1} \):

\[
G_{-1}(\kappa, \kappa') = \sum_{m=1}^{\infty} \frac{v_m^{(\kappa)} v_{m+1}^{(\kappa')}}{\sqrt{m(m+1)}}.
\]  

(3.4)

Straightforward calculations (very similar to the ones presented in Appendix A) show that this expression does indeed define a distribution:

\[
G_{-1}(\kappa, \kappa') = \frac{1}{\kappa \sqrt{\mathcal{N}(\kappa)\mathcal{N}(\kappa')}} - \frac{1}{2\kappa} \sqrt{\frac{\mathcal{N}(\kappa)}{\mathcal{N}(\kappa')}} \frac{\kappa - \kappa'}{\sinh \frac{\pi}{2}(\kappa - \kappa')}
\]

(3.5)

and therefore \( G_{-1} \) is a kernel operator.

Second, we find the “inverse” of operator \( G_{-1} \) on the one particle Fock space. In other words we are going to solve the first equation in (3.3a). The resulting operator will coincide with the operator \( L_{-1} \) on the one-particle Fock space. Since \( L_{-1} \) is completely determined by its action on this subspace we will actually find operator \( L_{-1} \) on the whole Fock space.

So we need to solve the equation

\[
\int_{-\infty}^{\infty} dk' \frac{1}{\kappa \sqrt{\mathcal{N}(\kappa)\mathcal{N}(\kappa')}} g(k') - \frac{1}{2\kappa} \int_{-\infty}^{\infty} dk' \sqrt{\frac{\mathcal{N}(\kappa)}{\mathcal{N}(\kappa')}} \frac{\kappa - \kappa'}{\sinh \frac{\pi}{2}(\kappa - \kappa')} g(k') = f(k),
\]  

(3.6)

where \( g(k') = L_{-1}[f](k') \). Notice now that by the construction \( g \) is a function that belongs to the image of \( L_{-1} \), therefore it satisfies the equation \( P_1[g] = g \). This equation just reflects the fact that there is no state \( a_1^{\dagger}|0\rangle \) in the image of operator \( L_{-1} \) restricted to the one-particle subspace. Eventually the first term in the equation is identically zero.

It is useful to introduce new functions \( F \) and \( G \)

\[
G(k') = \sqrt{\mathcal{N}(k')} g(k') \quad \text{and} \quad F(k) = \kappa \sqrt{\mathcal{N}(k)} f(k)
\]  

(3.7)

for which the equation takes extremely simple form

\[
-\frac{1}{2} \int_{-\infty}^{\infty} dk' \frac{k' - k}{\sinh \frac{\pi}{2}(k' - k)} G(k') = F(k).
\]

This equation can be solved using method of Fourier transform. Using that

\[
\int_{-\infty}^{\infty} dk' e^{i\omega k'} \frac{k'}{\sinh \frac{\pi}{2} k'} = \frac{2}{\cosh^2 \omega}
\]

one can easily obtain the solution

\[
G(k) = -\frac{1}{2} F(k) - \frac{1}{4\pi} \int_{-\infty}^{\infty} d\omega \int_{-\infty}^{\infty} dk' e^{i\omega(k-k')} \cosh 2\omega F(k').
\]

\[2\] We use the fact that \( v_1^{(\kappa')} = \frac{1}{\sqrt{\mathcal{N}(\kappa')}} \)
Using the arguments we presented equation (2.10) one obtains

$$G(\kappa) = -\frac{1}{2} F(\kappa) - \frac{1}{4} \left[ F(\kappa + i2^{-}) + F(\kappa - i2^{-}) \right].$$

Here the function $F(\kappa)$ is supposed to be holomorphic on the strip $-2 < \Im \kappa < 2$. Now using the relations (3.7) one obtains expression (1.8) defining the operator $L_{-1}$ in a class of difference operators. An expression for the operator $L_{1}$ is easily obtained by hermitian conjugation.

Using the same technique as we used in deriving (2.12) we can show that operators $L_{1}$ and $L_{-1}$ defined via (1.8) act on the generating function (1.1) as follows

$$L_{1}[f^{(\kappa)}(z)] = z^2 \frac{d}{dz} f^{(\kappa)}(z) \quad (3.8a)$$

$$L_{-1}[f^{(\kappa)}(z)] = -\frac{1}{N(\kappa)} + \frac{d}{dz} f^{(\kappa)}(z) \quad (3.8b)$$

### 4 Commutation relations

In this section we are going to show that operators $L_{0}$, $L_{\pm 1}$ represented by the difference operators (1.6) and (1.8) satisfy the standard commutation relations.

First, let me demonstrate how to calculate the commutator of $L_{1}$ and $L_{-1}$. From (1.8) we obtain the identities

$$L_{1}[L_{-1}[g]](\kappa) = -\frac{\kappa}{2} L_{-1}[g] + \frac{i}{4} \left[ \sqrt{\kappa(\kappa + 2i)} L_{-1}[g](\kappa + 2i) - \sqrt{\kappa(\kappa - 2i)} L_{-1}[g](\kappa - 2i) \right];$$

$$L_{-1}[L_{1}[g]](\kappa) = -\frac{\kappa}{2} L_{1}[g] - \frac{i}{4} \left[ \sqrt{\kappa(\kappa + 2i)} L_{1}[g](\kappa + 2i) - \sqrt{\kappa(\kappa - 2i)} L_{1}[g](\kappa - 2i) \right].$$

By subtracting the last equation from the first one one obtains

$$[L_{1}, L_{-1}][g](\kappa) = \frac{\kappa}{2} (L_{1}[g] - L_{-1}[g])(\kappa)$$

$$+ \frac{i}{4} \left[ \sqrt{\kappa(\kappa + 2i)} (L_{1}[g] + L_{-1}[g])(\kappa + 2i) - \sqrt{\kappa(\kappa - 2i)} (L_{1}[g] + L_{-1}[g])(\kappa - 2i) \right].$$

Substitution of (1.8) and simplification yield

$$[L_{1}, L_{-1}][g] = 2L_{0}[g]. \quad (4.2)$$

Second, we check commutation relations between $L_{0}$ and $L_{1}$.

$$[L_{0}, L_{1}][g](\kappa) = \frac{\kappa}{2} L_{0}[g](\kappa)$$

$$+ \frac{i}{4} \left[ \sqrt{\kappa(\kappa + 2i)} (L_{1}[g] - iL_{0}[g])(\kappa + 2i) + \sqrt{\kappa(\kappa - 2i)} (L_{1}[g] + iL_{0}[g])(\kappa - 2i) \right]. \quad (4.3)$$
Substitution of (1.6) and (1.8) and simplification yield

\[ [L_0, L_1][g] = -L_1[g]. \]  

(4.4)

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Note added: While this paper was nearing completion, the paper [10] appeared, which contains in Section 5 similar expressions for \( L_0 \) and \( L_{\pm 1} \). The reason for publishing this paper is mainly to give another point of view on the Virasoro operators in the \( \kappa \)-basis.
Appendix

A Calculations of $G_0$

The kernel for operator $G_0$ can be written as the following contour integral

$$G_0(\kappa, \kappa') = \frac{1}{\kappa \kappa'} \sqrt{N(\kappa)N'(\kappa')} \oint_C \frac{dz}{z} (1 - e^{-\kappa \arctan \frac{1}{z}})(1 - e^{-\kappa' \arctan z}).$$  \hspace{1cm} (A.1)

Notice that this integral has two logarithmic singularities at the points $\pm i$. Therefore the complex plane has two cuts starting at those points. We choose them to lie on the imaginary axis (for more details on the contour see Appendix in [8]). To calculate the integral we deform the contour $C$ in such way that it will go along the cuts. We introduce $z = ix - \epsilon, z = -ix + \epsilon$ and $z = -ix - \epsilon$ on contours $C_+, C_-, C'_+$ and $C'_-$ respectively

$$\kappa \kappa' \sqrt{N(\kappa)N'(\kappa')} G_0(\kappa, \kappa') = \frac{1}{2\pi i} \int_1^\infty \frac{idx}{ix - \epsilon} \left[ (1 - e^{-\kappa \arctan \frac{1}{ix-\epsilon}})(1 - e^{-\kappa' \arctan(ix-\epsilon)}) + \frac{-idx}{ix + \epsilon} \left(1 - e^{-\kappa \arctan \frac{1}{ix+\epsilon}}\right)(1 - e^{-\kappa' \arctan(ix+\epsilon)}) + \frac{-idx}{-ix + \epsilon} \left(1 - e^{-\kappa \arctan \frac{1}{-ix+\epsilon}}\right)(1 - e^{-\kappa' \arctan(-ix+\epsilon)}) + \frac{idx}{-ix - \epsilon} \left(1 - e^{-\kappa \arctan \frac{1}{-ix-\epsilon}}\right)(1 - e^{-\kappa' \arctan(-ix-\epsilon)}) \right]$$

Using that $\text{arctanh}(ix \pm \epsilon) = \pm \frac{\pi}{2} + i \coth^{-1} x$ and $\text{arctanh}(ix \pm \epsilon)^{-1} = -i \coth^{-1} x$ one obtains

$$= \frac{1}{2\pi i} \int_1^\infty \frac{dx}{x} \left[ (1 - e^{-\kappa(-i \coth^{-1} x)})(1 - e^{-\kappa'(-\frac{\pi}{2} + i \coth^{-1} x)}) - (1 - e^{-\kappa(-i \coth^{-1} x)})(1 - e^{-\kappa'(-\frac{\pi}{2} + i \coth^{-1} x)}) + (1 - e^{-\kappa i \coth^{-1} x})(1 - e^{-\kappa'(-\frac{\pi}{2} - i \coth^{-1} x)}) - (1 - e^{-\kappa i \coth^{-1} x})(1 - e^{-\kappa'(-\frac{\pi}{2} - i \coth^{-1} x)}) \right]$$

Now the change of variable

$$x = \coth u \quad \text{and} \quad dx = -\frac{du}{\sinh^2 u}$$

yields

$$= \frac{1}{\pi i} \int_0^\infty \frac{du}{\sinh 2u} \left[ (1 - e^{iu})\left(1 - e^{-\frac{\pi}{2} - iu}\right) - (1 - e^{iu})\left(1 - e^{-\frac{\pi}{2} + iu}\right) \right]$$
Simplification yields

\[
\frac{2}{\pi i} \sinh \frac{\pi \kappa'}{2} \int_0^{\infty} \frac{du}{\sinh 2u} \left[ e^{iu} - e^{i(\kappa' - \kappa)u} - e^{-iu} + e^{-i(\kappa' - \kappa)u} \right]
\]

\[
= \frac{2}{\pi i} \sinh \frac{\pi \kappa'}{2} \int_{-\infty}^{\infty} du \mathcal{P} \frac{1}{\sinh 2u} \left[ e^{iu} + e^{-i(\kappa' - \kappa)u} \right]
\]  

(A.2)

Using the fact that

\[
\frac{1}{\pi i} \int_{-\infty}^{\infty} du \mathcal{P} \frac{1}{\sinh 2u} e^{i\beta u} = \frac{1}{2} \tanh \frac{\pi \beta}{4}
\]  

(A.3)

one obtains

\[
= \sinh \frac{\pi \kappa'}{2} \left[ \tanh \frac{\pi \kappa'}{4} + \tanh \frac{\pi (\kappa - \kappa')}{4} \right] = 2 \sinh \frac{\pi \kappa}{4} \sinh \frac{\pi \kappa'}{4} \cosh \frac{\pi (\kappa - \kappa')}{4}.
\]  

(A.4)

Finally one gets

\[
G_0(\kappa, \kappa') = \left[ \frac{\theta(\kappa)}{\kappa} \right]^{1/2} \left[ \frac{\theta(\kappa')}{\kappa'} \right]^{1/2} \frac{1}{4 \cosh \left[ \frac{\pi}{4} (\kappa - \kappa') \right]}.
\]  

(A.5)
References

[1] E. Witten, “Noncommutative Geometry And String Field Theory,” Nucl. Phys. B 268, 253 (1986).

[2] L. Rastelli, A. Sen, B. Zwiebach, *Star Algebra Spectroscopy*, hep-th/0111281

[3] M. Douglas, H. Liu, G. Moore and B. Zwiebach, *Open String Star as a Continuous Moyal Product*, hep-th/0202087

[4] D. M. Belov and A. Konechny, *On continuous Moyal product structure in string field theory*, hep-th/0207174

[5] F.A. Berezin, *The method of second quantization*, New York, Academic Press, 1966

[6] B. Feng, Y.-H. He, N. Moeller, *The Spectrum of the Neumann Matrix with Zero Modes*, hep-th/0202176

[7] D. Belov, *Diagonal representation of open string star and Moyal product*, hep-th/0204164

[8] E. Fuchs, M. Kroyter and A. Marcus, *Squeezed State Projectors in String Field Theory*, hep-th/0207001

[9] K. Okuyama, *Ghost Kinetic Operator of Vacuum String Field Theory*, hep-th/0201015

[10] E. Fuchs, M. Kroyter and A. Marcus, *Virasoro operators in the continuous basis of string field theory*, hep-th/0210155