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Dynamic Spatial Autoregressive Models with Autoregressive and Heteroskedastic Disturbances

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Abstract

We propose a new class of models specifically tailored for spatio–temporal data analysis. To this end, we generalize the spatial autoregressive model with autoregressive and heteroskedastic disturbances, i.e. SARAR(1,1), by exploiting the recent advancements in Score Driven (SD) models typically used in time series econometrics. In particular, we allow for time–varying spatial autoregressive coefficients as well as time–varying regressor coefficients and cross–sectional standard deviations. We report an extensive Monte Carlo simulation study in order to investigate the finite sample properties of the Maximum Likelihood estimator for the new class of models as well as its flexibility in explaining several dynamic spatial dependence processes. The new proposed class of models are found to be economically preferred by rational investors through an application in portfolio optimization.

Keywords: SARAR, time varying parameters, spatio–temporal data, score driven models.

1. Introduction

Modeling spatio–temporal data have recently received an increasing amount of attention, with applications that span from time geography to spatial panel data econometrics (see An et al. (2015)). Specifically to the econometric field, researchers were focused on how to manage the raising availability of panel data by proposing a new class of dynamic spatial autoregressive models able to deal with (i) serial dependence between the observations on each spatial unit over time, (ii) spatial dependence between the observations at each point in time, (iii) unobservable spatial and/or time-period-specific effects, (iv) endogeneity of one or more of the regressors other than dependent variables lagged in space and/or time (see Elhorst (2012)). According to the type of restriction that we impose, one may obtain several dynamic spatial sub-models. For instance, a time–space dynamic model can be obtained if we impose restriction on the spatio–temporal evolution of the

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regressors, or a time-space recursive model if we ignore spatial autocorrelation but we account for time/space–
lagged dependent variable and eventually for spatially–lagged regressors (see Elhorst (2010), LeSage and Pace
(2009)). As Anselin et al. (2008) stressed, however, the sub–general time–space dynamic model still may
suffer from identification problems, which led to the suggestion of setting the autoregressive coefficient of the
time/space-lagged dependent variable equal to zero, and then forced researchers to choose between a time-space
simultaneous or a time-space recursive specification. Moreover, most of the contributions rely on cases in which
the cross–sectional/spatial dimension $N$ vastly exceeds the time dimension $T$, i.e. $N >> T$, since allowing for
large $T$ might cause the incidental parameter problem (see Lee and Yu (2010)).

In this paper we propose a dynamic spatial (first-order) autoregressive model with (first-order)
autoregressive and heteroskedastic disturbances – Heteroskedastic DySARAR(1,1) – in order to introduce
a new generalized class of spatio–temporal models in the spatial econometrics literature. We generally consider
the opposite situation in which $T >> N$, with the possibility of increase the spatial dimension by imposing some
constrains on our dynamic general spatial model. This new class of dynamic spatial models are based on the
Score Driven (SD) framework recently introduced by Harvey (2013) and Creal et al. (2013). The SD framework
of Harvey (2013) and Creal et al. (2013) allows us to update a set of time–varying unobserved parameters using
the information contained in the scaled score of the conditional distribution of the observables. Score driven
models can be seen as filters for unobserved component models of Harvey (1989). Furthermore, the use of the
score to track the conditional distribution of a random variable over time has been proved to be optimal in a
realised Kullback–Leibler sense, see e.g. Blasques et al. (2015). Generally speaking, SD models belong to the
class of observation-driven models in which parameters are perfectly predictable given the past information.
Given the high flexibility in selecting several appropriate functions of the past data with also the advantage
of defining the entire density for the updating process instead of simply considering the first– or second–order
moments, SD models are becoming rapidly popular in many applied research fields.

Blasques et al. (2014b) have recently developed a dynamic extension of the spatial autoregressive models,
i.e. SAR(1), also relying on the SD framework. In their paper, they also provide proofs of the consistency and
asymptotic normality of the MLE, with an application to credit default swaps (CDS) in EU over the period
2009–2014. Our model specification is a generalization of the Blasques et al.’s model, by considering global\footnote{In spatial econometrics, the term “global” refers to those autoregressive effects which lead to a new steady-state equilibrium. We will use this term several times throughout the paper.} observed and unobserved spatial spillover effects, with an empirical application in portfolio optimization. In
recent years, first efforts of introducing spatial econometric techniques into financial systems have been made.
Spatial spillover effects in empirical finance can take the meaning of credit risk propagation (Keiler and Eder
(2013)), return co-movements over time (Asgharian et al. (2013)), or risk premium propagation among firms
(Fernandez (2011)). However, most of these emerging analyses are typically based on panel data with no
time–varying spatial spillover effects as in Blasques et al. (2014b). In line with Blasques et al. (2014b) we move for a dynamic structure of general spatial models with time–varying autoregressive coefficients.

The choice of a general spatial model is quite complicated in spatial econometrics. When we only consider cross–sectional dependencies the most general specification that we can currently consider is the so–called Manski model. Due to its identification problems (Anselin et al. (2008), Manski (1993)), the model choice is restricted to a spatial autoregressive model with autoregressive disturbances (SARAR), a spatial autoregressive model with spatially-lagged regressors (spatial Durbin model – SDM), and a spatial autoregressive error model with spatially-lagged regressors (spatial Durbin error model – SDEM), which are unfortunately not nested. Among the Durbins’ supporters, the justification is typically that the cost of ignore/omit relevant (spatially-lagged) explanatory variables produces biased and inconsistent estimates due to omitted variable problems. In general, the right decision can be driven by (i) an economic theory justification or (ii) a model selection with information criteria. In this paper we use a combination of both (i) and (ii). We focus our attention on the SARAR model because we assume there are no local spatial spillovers effects (i.e. spatially-lagged regressors) into our financial empirical application, leading to an exclusion of the Durbin models, and we then choose the best model specification with Akaike and Bayesian information criteria. The reason why we prefer a SARAR model is also sustained by suspected unobserved shocks, e.g. consumers’ perceptions, that can have indirect effects on the entire financial system.

The remainder of the paper is organized as follows. Section 2 introduces our general heteroskedastic dynamic spatial model and its Maximum Likelihood (ML) estimation procedure. A short subsection 2.1 on dynamic/static spatial-nested models is also included. Section 3 reports two different Monte Carlo experiment to assess the statistical properties of our model: approximation of stochastic nonlinear dynamics and finite sample properties of the ML estimator. In Section 4 we illustrate the empirical application in portfolio optimization. Finally, Section 5 concludes.

2. Dynamic General Spatial Models

In this section we extend a (first-order) spatial autoregressive model with (first-order) autoregressive and heteroskedastic disturbances, SARAR(1,1), by allowing for dynamic spatial effects as well as dynamic cross–sectional variances and regressor coefficients. It proves helpful to first introduce the following notation. Let $y_t = (y_{it}; i = 1, \ldots, N)'$ be a $N$–dimensional stochastic vector of spatial variables at time $t$, and $X_t$ an exogenous matrix at time $t$ with $j$–th column $x_{j,t}$. Then, a Heteroskedastic DySARAR(1,1) model can be written as

$$y_t = \rho_t W_1 y_t + X_t \beta_t + u_t, \quad u_t = \lambda_t W_2 u_t + \epsilon_t, \quad \epsilon_t \sim N_N(0, \Sigma_t), \quad t = 1, \ldots, T \quad (1)$$

where $\rho_t$ and $\lambda_t$ are a time–varying autocorrelation parameters, $\Sigma_t$ is a diagonal matrix whose elements are the time–conditional heteroskedastic variances of the spatial independent innovations at time $t$ ($\epsilon_t$), i.e.
$\Sigma_t = \text{diag} (\sigma^2_t; i = 1, \ldots, N)$, $X_t = (x_{j,t}; j = 1, \ldots, K)$ is a $N \times K$ matrix of exogenous covariates with associated time–varying vector of coefficients $\beta_t = (\beta_{j,t}; j = 1, \ldots, K)'$, $W_1$ and $W_2$ are $N \times N$ spatial weighting matrices, and $u_t = (u_{i,t}; i = 1, \ldots, N)$ is a $N$–dimensional vector of (first-order) autoregressive error terms.

In order to ensure stable spatial processes we have to introduce the following assumptions in line with Kelejian and Prucha (2010). Let first introduce the following Lemma 2.1.

**Lemma 2.1.** Let $\tau$ denote the spectral radius of the square $N$–dimensional $W_1$ ($W_2$) matrix, i.e.:

$$\tau = \max \{ |\omega_1|, \ldots, |\omega_N| \}$$

where $\omega_1, \ldots, \omega_N$ are the eigenvalues of $W_1$ ($W_2$). Then, $(I_N - \rho_t W_1)^{-1} \left( (I_N - \lambda t W_2)^{-1} \right)$ is non singular for all values of $\rho_t$ ($\lambda_t$) in the interval $(-1/\tau, 1/\tau)$.

**Assumption 1.** (a) All diagonal elements of $W_1$ and $W_2$ are zero. (b) $\rho_t \in (-1/\tau, 1/\tau)$ and $\lambda_t \in (-1/\tau, 1/\tau)$.

Assumption 1(a) means that each spatial unit is not viewed as its own neighbor, whereas Assumption 1(b) ensures that the model in (1) can be uniquely defined by Lemma 2.1. Note that if all eigenvalues of $W_1$ ($W_2$) are real and $(\omega < 0, \overline{\omega} > 0)$, where $\omega = \min\{\omega_1, \ldots, \omega_N\}$ and $\overline{\omega} = \max\{\omega_1, \ldots, \omega_N\}$, we are in the particular case in which $\rho_t$ ($\lambda_t$) lies in the interval $(1/\omega, 1/\overline{\omega})$ (see Kelejian and Prucha (2010), note 6).

**Assumption 2.** The rows and the columns of both $W_1$ and $W_2$ before row-normalization should be uniformly bounded in absolute value as $N$ goes to infinity, ensuring that the correlation between two spatial units should converge to zero as the distance separating them increases to infinity.

The concept behind Assumption 2 will be got back in the interpretation of the infinite series expansions in equation (5). In this paper we specify row-standardized exogenous $W_1$ and $W_2$ weighting matrices in (1) to ensure the above stationarity conditions, with a general definition of the space metric among all the possible pairs of spatial units. The typical row-normalization of $W$ matrices ensures that $\overline{\omega} = +1$ for each of them and that the model can be written in reduced form as in equation (2), with appropriate inverse matrices which are nonsingular for all values of $\lambda_t$ and $\rho_t$ that lie in the interval $(-1, +1)^2$. Moreover, $X_t$ may contains past values of $y_t$, i.e. $x_{j,t} = y_{t-h_j}$ for some $h_j > 0$ and $j = 1, \ldots, p \leq K$, implying that (1) behaves as a usual autoregressive model in time. In this case, additional restrictions need to be imposed on the parameters of the model to ensure covariance stationarity of the spatio–temporal process (see e.g. Hays et al. (2010) and Elhorst (2012)).

The inclusion of spatially-lagged dependent variables $W_1 y_t$ typically causes an endogeneity problem, which

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2 According to the type of spatial statistical units we can specify several types of weighting matrices, i.e. contiguity matrices, geographical distance matrices, see e.g. Getis and Aldstadt (2010). For some particular spatial structures with complex eigenvalues, e.g. asymmetric $W$ matrices before row-normalization, we may find that $\lambda_t, \rho_t < -1$ leading to an explosive spatial process. In this paper we do not consider such cases, and the readers are referred e.g. to LeSage and Pace (2009) for details on particular $W$ structures.
in turn produce inconsistency of ordinary least squares estimators. This problem is referred to the bi-
directionality nature of spatial dependence in which each site, say \( i \), is a second-order neighbor of itself,
implying that spatial spillover effects have the important meaning of feedback/indirect effects also on the site
where the shock may have had origin. Due to the simultaneous nature of spatial autoregressive processes,
spatial models are typically specified in reduced forms. In order to see, let us first define
\[ A_t = (I_N - \rho_t W_1) \]
and \[ B_t = (I_N - \lambda_t W_2) \]. Then, model (1) can be specified in a reduced form as
\[ y_t = A_t^{-1} X_t \beta_t + A_t^{-1} u_t, \quad u_t = B_t^{-1} \epsilon_t, \quad \epsilon_t \sim N(0, \Sigma_t), \quad t = 1, \ldots, T \]  
(2)
By substituting \( u_t \), we obtain
\[ y_t = A_t^{-1} X_t \beta_t + A_t^{-1} B_t^{-1} \epsilon_t, \quad \epsilon_t \sim N(0, \Sigma_t), \quad t = 1, \ldots, T \]  
(3)
implying that the conditional density of \( y_t \) is equal to
\[ y_t|F_{t-1} \sim N_N \left( y_t; A_t^{-1} X_t \beta_t, A_t^{-1} B_t^{-1} \Sigma_t A_t^{-1} B_t^{-1} \right) \]  
(4)
where \( F_{t-1} \) represents the past history of the process \( \{ y_s, s > 0 \} \) up to time \( t - 1 \) and the exogenous covariates
up to time \( t \), i.e. \( X_t \in F_{t-1} \). It is notable that, when both matrices \( A_t \) and \( B_t \) in equation (2) are functions of
the same matrix \( W^3 \), i.e. \( W_1 = W_2 = W \), then distinguishing among the two spatial effects may be difficult, with
possible identification problems of the autoregressive parameters. Sufficient conditions to ensure identifiability
of the model is that \( X_t \) makes a material contribution towards explaining variation in the dependent variable
(see Kelejian and Prucha (2007)). The above inverse matrices \( (A_t^{-1}, B_t^{-1}) \) can be written by using the infinite
series expansion as
\[ A_t^{-1} = (I_N - \rho_t W_1)^{-1} = I_N + \rho_t W_1 + \rho_t^2 W_1^2 + \rho_t^3 W_1^3 + \ldots \]
\[ B_t^{-1} = (I_N - \lambda_t W_2)^{-1} = I_N + \lambda_t W_2 + \lambda_t^2 W_2^2 + \lambda_t^3 W_2^3 + \ldots \]  
(5)
which leads up to a useful interpretation of the spatial indirect effects: every location\(^3\), say \( i \), is correlated
with every other location in the system but closer locations more so (see Anselin (2003)). Differently from
the so-called global spillover effects \( \rho_t W_1 y_t \), in our paper we also consider the global diffusion of shocks to the
disturbances, i.e. \( \lambda_t W_2 \epsilon_t \), which means that a change in the disturbance of a single location \( i \) can produce
impacts on disturbances of the neighborhood. Since the powers of both \( W_1 \) and \( W_2 \) corresponds to observations
themselves (zero–order), immediate (first–order) neighbors, second-order neighbors etc., then the impacts can
be observed for each order of “proximity” . If both the conditions \( |\rho_t| < 1 \) and \( |\lambda_t| < 1 \) are satisfied, then the

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\(^3\)This is a frequently equivalence in the spatial econometrics literature, especially if geographic distance criteria are considered.

\(^4\)Here for “location” we intend a general spatial unit or a statistical unit that can be interconnected with the others through
the Cliff-Ord-type models (see e.g. Ord (1975)).
impacts also decay with the order of neighbors. However, stronger spatial dependence reflected in larger values of \( \rho_t \) and \( \lambda_t \) leads to a larger role for the higher order neighbors (LeSage and Pace (2009)). This concept will be got back in our empirical application in subsection 4.3.

Finally, following Anselin (1988), the contribution of \( \nu_t \) to the log likelihood of the model is proportional to

\[
\ell_t (y_t; \cdot) \propto - (1/2) \ln |\Sigma_t| + \ln |B_t| + \ln |A_t| - (1/2) \nu_t^\prime \nu_t, \tag{6}
\]

where

\[
\nu_t = (A_t y_t - X_t \beta_t)^\prime B_t^\prime \Sigma_t^{-1} B_t (A_t y_t - X_t \beta_t). \tag{7}
\]

In this paper we propose a general dynamic spatial model in order to temporally update the set of parameters \( \beta_t, \rho_t, \lambda_t \) and \( \sigma_{jt} \) for \( j = 1, \ldots, N \) by using the score of the conditional distribution of \( y_t \) in (4), exploiting the recent advantages for score driven models of Creal et al. (2013) and Harvey (2013). To this end, we define \( \theta_t = (\rho_t, \lambda_t, \beta_t, \sigma_{jt}; j = 1, \ldots, N) \) to be a vector containing all the time–varying parameters, such that \( \theta_t \in \Omega \subseteq \mathbb{R}^{N+K+2} \). Furthermore, we define \( h : \mathbb{R}^{K+N+2} \rightarrow \Omega \) to be a \( F_{t-1} \) measurable vector valued mapping function such that \( h \in C^2 \) and \( h (\tilde{\theta}_t) = \theta_t \), where \( \tilde{\theta}_t = (\tilde{\rho}_t, \tilde{\lambda}_t, \tilde{\beta}_t, \tilde{\sigma}_{jt}; j = 1, \ldots, N \) is a time–varying vector of unrestricted parameters defined in \( \mathbb{R}^{N+K+2} \). In our context, a convenient choice for the mapping function \( h (\cdot) \) is

\[
h (\tilde{\theta}_t) : \begin{cases} 
\rho_t = \omega_\rho + \frac{\sigma_\rho - \omega_\rho}{1 - \exp(\rho_t)}, \\
\lambda_t = \omega_\lambda + \frac{\sigma_\lambda - \omega_\lambda}{1 - \exp(\lambda_t)}, \\
\beta_t = h_\beta (\tilde{\beta}_t), \\
\sigma_{jt} = \exp (\tilde{\sigma}_{jt}), \quad \text{for} \quad j = 1, \ldots, N,
\end{cases} \tag{8}
\]

where \( h_\beta (\cdot) \) holds the same properties of \( h (\cdot) \), and maps \( \tilde{\beta}_t \) in \( \beta_t \). The updating equation for the vector of reparametrised parameters \( \tilde{\theta}_t \) is given by

\[
\tilde{\theta}_{t+1} = (I_{N+K+2} - R) \kappa + F \tilde{s}_t + R \tilde{\theta}_t, \tag{9}
\]

where \( \kappa = (\kappa_\rho, \kappa_\lambda, \kappa_{\beta_j}, \kappa_{\sigma_j}; j, \ldots, N) \in \mathbb{R}^{N+K+2} \) is a vector representing the unconditional mean of the process and \( F \) and \( R \) are \((N+K+2) \times (N+K+2)\) matrices of coefficients to be estimated. To avoid problems of parameters proliferation, for the rest of the paper we define a diagonal structure for \( F \) and \( R \), i.e. we impose \( F = \text{diag} (f_\rho, f_\lambda, f_{\beta_j}, f_{\sigma_j}; j, \ldots, N) \) and \( R = \text{diag} (r_\rho, r_\lambda, r_{\beta_j}, r_{\sigma_j}; j, \ldots, N) \), respectively. The quantity \( \tilde{s}_t \) is the scaled score with respect to \( \tilde{\theta}_t \) of the reparametrized conditional distribution of \( y_t \), i.e.

\[
\tilde{s}_t = \tilde{f} (\tilde{\theta}_t)^\prime \nabla (\tilde{\theta}_t), \tag{10}
\]
where \( \gamma \) usually takes value in \{0, -1/2, -1\} and,

\[
\tilde{\nabla} (y_t, \hat{\theta}_t) = \left. \frac{\partial f (y_t; \hat{\theta})}{\partial \theta} \right|_{\hat{\theta} = \hat{\theta}_t} \tag{11}
\]

\[
\tilde{I} (\hat{\theta}_t) = E_{t-1} \left[ \tilde{\nabla} (y_t, \hat{\theta}) \times \tilde{\nabla} (y_t, \hat{\theta})' \right]_{\hat{\theta} = \hat{\theta}_t},
\tag{12}
\]

are the score and the Fisher information matrix of (4) with respect to \( \hat{\theta}_t \), respectively. It is worth noting that, simply exploiting the chain rule, it is possible to define \( \tilde{\nabla} (y_t, \hat{\theta}_t) \) and \( \tilde{I} (\hat{\theta}_t) \) as

\[
\tilde{\nabla} (y_t, \hat{\theta}_t) = J (\hat{\theta}_t)' \nabla (y_t, \theta_t) \tag{13}
\]

\[
\tilde{I} (\hat{\theta}_t) = J (\hat{\theta}_t)' I (\theta_t) J (\hat{\theta}_t), \tag{14}
\]

where again, \( \nabla (y_t, \theta_t) \) and \( I (\theta_t) \) are the score and the information matrix of (4) with respect to the original set of parameters \( \theta_t \), respectively. In equation (13), the \((N + K + 2) \times (N + K + 2)\) matrix \( J (\hat{\theta}_t) \) represents the Jacobian of the mapping function \( h (\cdot) \). According to our specification of \( h (\cdot) \) reported in equation (8), the \((h, l)\)-th element of the Jacobian matrix \( J (\hat{\theta}_t) \) is given by

\[
J (\hat{\theta}_t)_{(h, l)} = \begin{cases} 
(\frac{\pi_i - \omega_i}{1 - \exp (\lambda_i)})^2, & \text{if } h = l = 1 \\
(\frac{\pi_i - \omega_i}{1 - \exp (\lambda_i)})^2, & \text{if } h = l = 2 \\
\frac{\partial h_{ij}}{\partial \beta_j} (\hat{\theta}), & \text{if } 2 < h = l \leq K + 2 \\
\frac{\partial h_{ij}}{\partial \beta_j} (\hat{\theta}), & \text{if } h \neq l \land 2 < h, l \leq K + 2 \\
\exp (\tilde{\sigma}^2_{h, t}), & \text{if } h = l > K + 2 \\
0, & \text{otherwise},
\end{cases}
\tag{15}
\]

where \( h_{\beta_j} (\cdot) \) is the \( j \)-th element of \( h_{\beta} \). Finally, the score \( \nabla (y_t, \theta_t) \) can be partitioned as \( \nabla (y_t, \theta_t) = \left( \nabla^\phi (y_t, \theta_t), \nabla^\lambda (y_t, \theta_t), \nabla^\beta (y_t, \theta_t)' \right)' \), \( \nabla^\sigma_j (y_t, \theta_t); j = 1, \ldots, N \)'', where

\[
\nabla^\phi (y_t, \theta_t) = \nu \Sigma^{-1/2} B_t W_1 y_t - \text{tr} [A_t^{-1} W_1] \tag{16}
\]

\[
\nabla^\lambda (y_t, \theta_t) = \nu \Sigma^{-1/2} W_1 (A_t y_t - X_t \beta_t) - \text{tr} [B_t^{-1} W_2] \tag{17}
\]

\[
\nabla^\beta (y_t, \theta_t) = \nu \Sigma^{-1/2} B_t X_t \tag{18}
\]

\[
\nabla^\sigma_j (y_t, \theta_t) = -\frac{1}{2 \sigma_t^2} + \frac{1}{2} (A_t y_t - X_t \beta_t)' B_t^j \Sigma^{-1} \nu_j \Sigma^{-1} B_t (A_t y_t - X_t \beta_t), \tag{19}
\]

where \( \nu_j \) is a vector of length \( N \) of zeros except for its \( j \)-th element which is equal to 1.

The estimation of the Heteroskedastic DySARAR(1,1) in (1) can be easily performed via ML. Given a series of spatio–temporal endogenous and exogenous variables \( \{y_t, X_t; t = 1, \ldots, T\} \), we can define the partitioned vector as \( \Theta = (\nu', \text{diag} (F)'', \text{diag} (R)'')' \) which contains the \( 3(N + K + 2) \) coefficients of the model, where \( N \)
is the spatial cross-sectional sample, $K$ is equal to the number of exogenous variables, 2 corresponds to the pairs of autoregressive parameters $(\rho_t, \lambda_t)$. Then, the ML estimate of $\Theta$ is given by

$$
\hat{\Theta} = \arg \max_{\Theta} \sum_{t=1}^{T} \ell_t(\theta_t; y_t, X_t),
$$

where $\ell_t(\theta_t; y_t, X_t)$ is the likelihood contribution of $y_t$ at time $t$ conditional on $F_{t-1}$, given the filtered values for the parameter $\theta_t$. Standard errors can be easily computed by inverting the Hessian matrix of the likelihood at its optimum value.

The properties of the ML estimator for SD models is an ongoing topic of research. Several results in a general setting are given by Blasques et al. (2014a), while Blasques et al. (2014b) shows the specific case for their Time-Varying SAR model. In the next subsection we will show several dynamic/static spatial models that nests our Heteroskedastic DySARAR(1,1) model.

2.1. Heteroskedastic DySARAR(1,1)–nested specifications

In this section we show all the possible nested models that can be used after setting a series of constrains on both autoregressive coefficients and heteroskedastic disturbances. Let us first consider our Heteroskedastic DySARAR(1,1) in (1). Then, we can obtain a class of dynamic spatial–nested models according to the type of constrains that we set

1. DySAR(1) model: if $\lambda_t = 0$ for all $t = 1, \ldots, T$
   
   (a) StSAR(1) model: if also $\rho_t = \rho$ and $\Sigma_t = \Sigma$ and $\beta_t = \beta$ for all $t = 1, \ldots, T$

2. DySAE(1) model: if $\rho_t = 0$ for all $t = 1, \ldots, T$
   
   (a) StSAE(1) model: if also $\lambda_t = \lambda$ and $\Sigma_t = \Sigma$ and $\beta_t = \beta$ for all $t = 1, \ldots, T$

3. DyOLS model: if $\rho_t = \lambda_t = 0$ for all $t = 1, \ldots, T$
   
   (a) StOLS model: if also $\Sigma_t = \Sigma$ and $\beta_t = \beta$ for all $t = 1, \ldots, T$

where DySAR(1) model stands for Dynamic Spatial (first-order) Autoregressive model (as in Blasques et al. (2014b)), DySAE(1) for Dynamic Spatial (first-order) Autoregressive Error model and DyOLS for a simple dynamic linear model without spatial effects, whereas the “St” is the acronym for their Static counterparts.

It is worth noting that all the static specifications necessarily imply time and dynamic homoscedasticity. To this purpose, we also define three different types of heteroskedasticity that can be captured by the proposed Heteroskedastic DySARAR(1,1) model.

- Time (Homo)Heteroskedasticity (THo)THe: $y_t$ displays (THo)THe if the matrix $\Sigma_t$ is (constant) time-varying. Sufficient constraints for THo are $f_{\sigma_j} = r_{\sigma_j} = 0$, for all $j = 1, \ldots, N$.

- Cross (Homo)Heteroskedasticity (CHO)CHe: $y_t$ displays (CHO)CHe if $(E[\sigma_j^2] = E[\sigma_i^2]) E[\sigma_j^2] \neq E[\sigma_i^2]$ for all $j \neq i$ in $\{1, \ldots, N\}$ and for all $t = 1, \ldots, T$. Sufficient constraints for CHO are $\kappa_{\sigma_j} = \kappa_{\sigma_i}$, for all $i \neq j$ in $\{1, \ldots, N\}$. 


Dynamic Heteroskedasticity (DHo)DHe displays

(DHo)DHe if \((\frac{\partial \sigma_{i,t}}{\partial x_{j,t}} = \frac{\partial \sigma_{i,t}}{\partial x_{j,t-1}} = \frac{\partial \sigma_{i,t}}{\partial x_{j,t-1}}) \neq \frac{\partial \sigma_{j,t}}{\partial x_{j,t}} \neq \frac{\partial \sigma_{j,t}}{\partial x_{j,t-1}} \neq \frac{\partial \sigma_{j,t}}{\partial x_{j,t-1}}, \) for all \(j \neq i\) in \(\{1, \ldots, N\}\) and for all \(t = 1, \ldots, T\). Here \(\sigma_{i,t}\) represents the element of \(\sigma_t\) associated to \(\sigma_{j,t}^r\). Sufficient constraints for DHo are \(f_{\sigma_j} = f_{\sigma_i} \wedge r_{\sigma_j} = r_{\sigma_i}\), for all \(i \neq j\) in \(\{1, \ldots, N\}\).

Finally, it is notable that setting \(\lambda_t = 0, \rho_t = \rho\) for all \(t = 1, \ldots, T\), a generalization of the Time–Space Simultaneous model in Anselin et al. (2008) can be specified. In the same way, if we impose \(\lambda_t = \mu_t = 0\) for all \(t = 1, \ldots, T\) and \((y_{t-1}, W y_{t-1}) \in X_t\), we obtain a generalization of the Time–Space Recursive model in Anselin et al. (2008)\(^5\). On the contrary, we do not consider model specifications that directly contain spatially lagged \(X\) (see Elhorst (2012), LeSage and Pace (2009)).

3. Simulation Studies

In this section we report an extensive simulation study to investigate the Heteroskedastic DySARAR(1,1) model properties. To this purpose we perform two simulation studies. The former aims to demonstrate the flexibility of the proposed Dynamic Spatial SARAR (S–SARAR) specification. With S–SARAR specifications we intend those dynamic SARAR models for which a nonlinear dynamic stochastic evolution is assumed for the parameters of the model. These kind of specifications, within the spatial statistics literature, have been employed for example by Hsu et al. (2012) and have the drawback of being usually estimated relaying on computer intensive simulation procedures.

Specifically, we assume that the vector of spatial units at time \(t\), \(y_t\), is generated according to the following S–SARAR specification

\[
y_t = \rho_t W_t y_t + X_t \beta_t + \epsilon_t, \quad \epsilon_t \sim N(0, \Sigma_{\epsilon,t}),
\]

with \(\Sigma_{\epsilon,t} = \text{diag}(\sigma_{j,t}^2; j = 1, \ldots, N)\) and \(\theta_t = (\rho_t, \lambda_t, \beta_t, \sigma_{j,t}^2; j = 1, \ldots, N)'\) which is implicitly defined by \(\theta_t = h(\tilde{\theta}_t)\), where \(\tilde{\theta}_t = (\tilde{\rho}_t, \tilde{\lambda}_t, \tilde{\beta}_t, \tilde{\sigma}_{j,t}^2; j = 1, \ldots, N)'\) evolves according to

\[
\tilde{\theta}_t = (I_{N+K+2} - \Phi) \mu + \Phi \tilde{\theta}_{t-1} + \zeta_t, \quad \zeta_t \sim N_{N+K+2}(0, \Sigma),
\]

\(^5\)Following Anselin et al.,’s notation, the generalizations refer to the coefficient \(\rho_t, \phi_t\) (of \(y_{t-1}\)) and \(\eta_t\) (of \(W y_{t-1}\)) which would be time-varying in our case.
with \( \mathbf{\mu} = (\mu_\rho, \mu_\lambda, \mu_{\beta_1}, \mu_{\sigma_1}, \ldots, \mu_{\sigma_J}; j = 1, \ldots, N)' \), \( \Phi = \text{diag}(\phi_\rho, \phi_\lambda, \phi_{\beta_1}, \phi_{\sigma_1}, \ldots, \phi_{\sigma_J}; j = 1, \ldots, N) \) and \( \mathbf{U} = \text{diag}(u_\rho, u_\lambda, u_{\beta_1}, u_{\sigma_1}, \ldots, u_{\sigma_J}; j = 1, \ldots, N) \) are diagonal matrices containing the autoregressive coefficients and variances, respectively. The first column of the \( N \times K \) matrix \( \mathbf{X}_t \), is a vector of ones allowing for a common temporal trend captured by the first element of the vector \( \beta_t = (\beta_{1,t}; i = 1, \ldots, K)' \), i.e. \( \beta_{1,t} \). The mapping function \( h(\cdot) \) is the same reported in equation (8), with \( h_\beta(\cdot) \) equals to the identity map such that \( h_\beta(\hat{\beta}_t) = \hat{\beta}_t = \beta_t \). For our simulation study we set, \( N = 4 \), \( K = 2 \), and

\[
\begin{pmatrix}
\mu_\rho \\
\mu_\lambda \\
\mu_{\beta_1} \\
\mu_{\beta_2} \\
\mu_{\sigma_1} \\
\mu_{\sigma_2} \\
\mu_{\sigma_3} \\
\mu_{\sigma_4}
\end{pmatrix} =
\begin{pmatrix}
0.010 \\
-0.004 \\
1.000 \\
2.000 \\
0.986 \\
0.944 \\
0.289 \\
-0.421
\end{pmatrix},
\begin{pmatrix}
\phi_\rho \\
\phi_\lambda \\
\phi_{\beta_1} \\
\phi_{\beta_2} \\
\phi_{\sigma_1} \\
\phi_{\sigma_2} \\
\phi_{\sigma_3} \\
\phi_{\sigma_4}
\end{pmatrix} =
\begin{pmatrix}
0.997 \\
0.997 \\
0.997 \\
0.997 \\
0.997 \\
0.997 \\
0.997 \\
0.997
\end{pmatrix},
\begin{pmatrix}
u_\rho \\
u_\lambda \\
u_{\beta_1} \\
u_{\beta_2} \\
u_{\sigma_1} \\
u_{\sigma_2} \\
u_{\sigma_3} \\
u_{\sigma_4}
\end{pmatrix} =
\begin{pmatrix}
0.010 \\
0.010 \\
0.010 \\
0.010 \\
0.010 \\
0.010 \\
0.010 \\
0.010
\end{pmatrix},
\]

(23)

with

\[
\mathbf{X}_t = \begin{bmatrix} \mathbf{1} & \mathbf{\eta}_{t-1} \end{bmatrix}, \quad \mathbf{\eta}_{t-1} \overset{iid}{\sim} \mathcal{N}_4(\mathbf{0}, \mathbf{I}_4),
\]

(24)

where \( \mathbf{1} = (1, 1, 1, 1)' \), such that \( \mathbf{X}_t \) is assumed to be available and observable at time \( t = 1 \). The above values ensure a smooth time evolution of the reparametrised vector of parameters \( \hat{\theta}_t \). Without loss of generality, in this experiment the matrices \( \mathbf{W}_1 \) and \( \mathbf{W}_2 \) are assumed to be equal, i.e. \( \mathbf{W}_1 = \mathbf{W}_2 = \mathbf{W} \). We simulate a unique symmetric \( \mathbf{W} \) in such a way that it has all real eigenvalues, with generic distance measure inside. Finally we row–normalize.

To perform our simulation study we generate from (22) \( T = 10000 \) values for \( \theta_t \), then, for each \( t \), we simulate \( B = 1000 \) values according to (21), collecting each resulting spatio–temporal series \( \mathbf{y}^{(b)} = (\mathbf{y}_t^{(b)}; t = 1, \ldots, T)' \), into \( B \) vectors of proper dimension. As previously detailed, the exogenous regressors \( \mathbf{X}_t, \quad t = 1, \ldots, T \) are assumed to be known at time \( t = 1 \), and are the same across the \( B \) generated samples. We estimate on each generated series \( \mathbf{y}^{(b)}, \quad b = 1, \ldots, B \), the Heteroskedastic DySARAR(1,1) model detailed in Section 2. Then, we compare the filtered values for \( \hat{\theta}_t \) with those previously simulated from the nonlinear autoregressive system provided in equation (22).

Figure A.1 shows the results that we obtain in form of fan charts around the true value for \( \hat{\theta}_t, \quad t = 1, \ldots, T \). As we can see, the proposed Heteroskedastic DySARAR(1,1) model have very high filtering ability when a S–SARAR model is assumed for the evolution of the spatial units \( y_t \). More precisely, the accuracy of the Heteroskedastic DySARAR(1,1) model can be better understood by looking at the confidence bars, which
suggest very low dispersion across the true values $\theta_t, \quad t = 1, \ldots, T$. The medians across the $B$ samples at each point in time $t$ (in purple), are very close to the true value, for all the simulated dynamics.

### 3.2. Finite sample properties of the ML estimator

The ML estimator (MLE) has been proved by Blasques et al. (2014b) to be consistent and asymptotically normal for the special case of DySAR(1) (with only time–varying spatial effects), whereas, in the spatial literature, Bao and Ullah (2007) have derived the finite sample properties of the MLE for a StSAR(1) model and Lee (2004) the asymptotic distributions of Quasi-MLEs for the same model specification. The investigation of the asymptotic theory for the general Heteroskedastic DySARAR(1,1) model is beyond the scope of this paper. However, in this subsection we provide an extensive Monte Carlo experiment in order to investigate the finite sample properties of the ML estimator.

The experiment consists of generating $M$ series of length $T$ according to the Heteroskedastic DySARAR(1,1) model presented in Section 2. The Heteroskedastic DySARAR(1,1) model is estimated on each $m, \ m = 1, \ldots, M$ series by ML, and the estimated coefficient are stored. To investigate the different properties of the ML estimator depending on the sample size of the available time series, we choose $T$ equal to 1000, 5000, 10000 and $M = 1000$. We set $N = 6$ and we impose empirical relevant values for the model coefficients, such as persistent dynamics for the conditional volatility processes as well as for the spatial autoregressive parameters. The values are listed in the first row of Table B.1. Differently from the previous experiment, in this case we explore the ML finite-sample properties by assuming there are no significant effects carried out by $X_t$, so we simply set $X_t = 0$. Therefore, for model identification issues, we must impose the spatial weighting matrices $(W_1, W_2)$ sufficiently different in order to ensure that the infinite series of spatial effects in (5) are functions of different pre-specified, i.e. exogenous, spatial structures. As in the previous experiment, we simulate symmetric matrices to guarantee the presence of all real eigenvalues, and finally we row–normalize.

Figure A.2 shows the empirical density associated to each parameter. The empirical densities are evaluated using a Gaussian kernel on the $R$ coefficients estimates for all the considered sample sizes $T = \{1000, 5000, 10000\}$. We note that, for all the coefficients, the ML estimator provides unbiased estimates. Furthermore, also the variance of the estimated coefficients decreases when the sample size increases, suggesting that the ML estimator for Heteroskedastic DySARAR(1,1) models is asymptotically consistent. Table B.1 shows summary statistics for the ML estimator for all the considered sample sizes. As suggested from the graphical investigation, the ML estimator seems to be unbiased in finite samples and displays decreasing variance as long as the sample size increases.

### 4. Empirical Application

Every financial application designed into a fully multivariate environment heavily depends on the dependence structure characterising assets’ returns. Indeed, the evolution of the dependence structure over
time is one of the most relevant stylised fact affecting multivariate financial returns (see e.g. McNeil et al. (2015)). Unfortunately, although there is a large agreement about the role that the dependence structure has in finance, spatial econometric models, which generally deal with such dependences, are rarely used to solve financial problems. Notable exceptions are given by Fernandez (2011) who proposes the Spatial Capital Asset Pricing Model (S–CAPM), Arnold et al. (2013) who investigate on global and local dependencies as well as dependence effects inside industrial branches of financial returns, and finally Blasques et al. (2014b) and Keiler and Eder (2013) who focus on spillover effects across financial markets into CDS modeling. Differently from the previous studies, our empirical investigation is based on portfolio optimisation theory by exploiting the classical Markowitz (1952)’s Mean–Variance (MV) framework. In his seminal paper, Markowitz demonstrated that if the investors’ utility function is quadratic or asset returns are normally distributed, the optimal allocation of wealth only depends on the mean and the covariance matrix of future assets returns. Following this general theory, we employ our DySARAR model to predict the first two centered moments of future assets returns.

The empirical investigation is composed by two parts. The first part aims at investigating which spatial econometric model, between those nested in our general DySARAR specification reported in Section 2.1, is the most adequate to model financial returns. To this purpose, we perform model choice by using both AIC and BIC. The second part concerns the portfolio optimisation study and compares our DySARAR model with several alternatives usually employed in finance for assets allocation problems.

4.1. Data

Our data set consists of log returns for 18 US economic sectorial indexes recorded from 2nd January, 2002 to 5th January, 2016 for a total of 3'513 observations per index. Specifically, we use the “super sector” classified indexes constructed by the Dow Jones and available from Datastream, the sectors are: Oil & Gas (EN), Chemicals (CH), Basic Resources (BS), Construction & Materials (CN), Industrial Goods & Services (IG), Automobiles & Parts (AP), Food & Beverage (FB), Personal & Household Goods (NG), Health Care (HC), Media (ME), Travel & Leisure (CG), Telecommunications (TL), Utilities (UT), Banks (BK), Insurance (IR), Real Estate (RE), Financial Services (FI) and Technology (TC). The whole sample is divided into two sub–samples which represent an in sample period from 2nd January, 2002 till 20th January 2010 with 2,013 observations and an out of sample period from 21th January 2010 to the end of the time series with 1,500 observations. Table B.2 reports summary statistics for the in sample as well as for the out of sample periods. As expected, according to the Jarque–Bera statistic, we found empirical evidence of the departure from the normal distribution for each of the indexes. Furthermore, each series displays negative skewness and excess of kurtosis for both the considered periods. We also found empirical evidence of little negative serial autocorrelation for some indexes, suggesting that a very low portion of future returns may be predicted using an autoregressive model. Table B.3 shows the empirical correlation matrix for the two periods. We note that, the correlations between the US sectors indexes range from 0.5 to 0.9 and increase over the sample period. To
further investigate the time–variation of the correlation structure, we use the test of Tse (2000) which provides a value of about 3,513, which strongly goes against the null of constant correlation. In order to capture the common sector reaction to past US equity market information, we employ the S&P500 logarithmic differences as an exogenous regressor, \( x_t \). The exogenous covariate is lagged by one period, such that at time \( t \), \( x_{t+1} \) is known.

4.2. Distance in finance

Although the notion of distance in space is already more general than the pure geographical distance, even in the spatial econometric literature there is a huge discussion on the appropriate definition of the weighting matrix to avoid possible consequences on estimation and inference (see e.g. LeSage and Pace (2014)). Robustness checks and carefully structured arguments coming from theory should be the ordinary case (Arbia and Fingleton (2008)), or otherwise one may consider endogenous \( W \) matrices (Kelejian and Piras (2014), Qu and Lee (2015)). Moreover, complications on the definition of \( W \) may arise even more if we consider dynamic spatial panel data models (Baltagi et al. (2014)).

In finance the choice of the weighting matrix is not easy at all, mainly due to the immateriality of the notion of distance. The ideal situation would be that of defining an economic measure of distance. For example, Blasques et al. (2014b) use a weighting matrix by exploiting countries cross–border debt data for their application in CDS. In studies on sector index returns, however, the issue of finding appropriate economic information is more complicated. Moreover, the use of economic distances should be carefully supervised since “basing \( W \) on economic variables may lead to some forms of interaction between \( W \) and \( X_t \) that are difficult to detect...”, with complications in the interpretation of the weighting matrix if its elements change with \( X_t \) (LeSage and Pace (2014), page 247).

In our paper, we then follow Fernandez (2011) and build our weighting matrix by using a measure of concordance among financial returns. Since our empirical results are based on a substantive \( X_t \) effect, we can set \( W_1 = W_2 = W \), without any identification problem. In particular, we construct the matrix \( W \) by using the empirical spearman correlation matrix estimated on the data. Formally, the \((i, j)\)–th element of \( W \) is given by

\[
w_{(i,j)} = \begin{cases} 
\exp(-d_{i,j}) / \sum_{k=1}^{N} \exp(-d_{i,k}), & \text{if } i \neq j \\
0, & \text{otherwise,}
\end{cases}
\]

(25)

where \( \rho_{i,j} \) is the empirical spearman correlation coefficient between returns \( i \) and \( j \) and \( d_{i,j} = \sqrt{2(1 - \rho_{i,j}^2)} \) is the defined metric among pairs of spatial units. Note that the above definition of weights already includes the row-standardization rule, such that \( \sum_j w_{i,j} = 1 \).

4.3. In sample analysis

The first two conditional moments of multivariate financial returns displays well known stylised facts. For example, the first conditional moment of assets returns generally displays absence or very little serial correlation,
indeed, returns are usually assumed to behave as a martingale difference sequence. On the contrary, the second conditional moment displays very high persistence over time. Furthermore, periods of high volatility are followed by periods of high volatility and vice versa. This is usually referred to as the so-called volatility clustering phenomenon, see, for example, McNeil et al. (2015). Consequently, the spatial specification used to model financial returns needs to account for these empirical evidences.

The issue of choosing between several alternative dynamic spatial panel data models has been analysed for example by Anselin et al. (2008), Elhorst (2010) and Elhorst (2012). As detailed in subsection 2.1, our general DySARAR specification nests a large number of spatial models already available in the literature. Moreover, as previously detailed, we can also discriminate between different types of cross and time heteroskedasticity assumed for the assets return. In order to assess which is the most adequate model specification for our panel of financial returns, we estimate both the static (St) and dynamic (Dy) versions of the SARAR, SAR, SAE and OLS models. Furthermore, we also specify different assumptions for the evolution of the second conditional moments of our series. Specifically, for the static models, we discriminate between Cross–Heteroskedastic (CHe) and Cross–Homoscedastic (CHo) models. Concerning the dynamic specifications, we also discriminate between Dynamic–Heteroskedastic (DHe) and Dynamic–Homoscedastic (DHo) models. In conclusion, we consider 8 different static specifications, namely StOLS–CHo, StOLS–CHe, StSAR–CHo, StSAR–CHe, StSARAR–CHo, StSARAR–CHe, StSAE–CHo, StSAE–CHe and 12 different dynamic specifications, namely DyOLS–DHo.CHo, DyOLS–DHe.CHe, DyOLS–DHo.CHe, DySAR–DHo.CHo, DySAR–DHe.CHe, DySAR–DHo.CHe, DySARAR–DHo.CHo, DySARAR–DHe.CHe, DySARAR–DHo.CHe, DySAE–DHo.CHo, DySAE–DHe.CHe, DySAE–DHo.CHe, for a total of 20 different nested specifications.

Table B.4 shows the values of AIC and BIC as well as the number of estimated coefficients and the log-likelihood evaluated at its optimum for all the 20 different model specifications. The first important result to note is that, as widely expected, a dynamic specification for the conditional distribution of assets returns is strongly required by the data. Indeed, static models are clearly suboptimal compared with dynamic counterparts in terms of goodness of fit. SAE and SAR seem to perform in a similar way, especially if we consider the dynamic cases. This is a typical problem in the spatial econometrics literature, which is based on the need of choosing the best specification between these two or other types of non-nested models (see e.g. Kelejian (2008)). The DySARAR specification outperforms both of them, independently form the presence of Dynamic–Heteroskedasticity (DHe). Therefore, a SARAR specification should be used to model financial returns according to both AIC and BIC rankings, rather than using OLS, SAE and SAR specifications that have been used so far. For the rest of the empirical application we will then employ the DySARAR-DHo.CHe parametrisation of the general DySARAR model presented in Section 2.

Before moving to the out of sample investigation, we test if there is empirical evidence of time–variation of the spatial coefficients $\rho_t$ and $\lambda_t$. To this end, we estimate three constraint versions of the DySARAR-DHo.CHe specification ($M_{c1}, M_{c2}, M_{c3}$), assuming static spatial autoregressive parameters. Specifically, we define the
following restricted models as a combination of constraints described in subsection 2.1:

1. $M_{c1}$ for which $\rho_t = \rho$ for all $t = 1, \ldots, T$, imposing $f_\rho = r_\rho = 0$.
2. $M_{c2}$ for which $\lambda_t = \lambda$ for all $t = 1, \ldots, T$, imposing $f_\lambda = r_\lambda = 0$.
3. $M_{c3}$ for which $\rho_t = \rho \land \lambda_t = \lambda$ for all $t = 1, \ldots, T$, imposing $f_\rho = r_\rho = f_\lambda = r_\lambda = 0$.

We compare each of the above restricted models with the unrestricted DySARAR-DHo.CHe specification in terms of the Likelihood Ratio (LR) statistic. The values of LRs are 19.47, 20.32, and 238.65 for $M_{c1}$ vs. DySARAR-DHo.CHe, $M_{c2}$ vs. DySARAR-DHo.CHe and $M_{c3}$ vs. DySARAR-DHo.CHe, respectively, which strongly adverses the null of a restricted specification. This provides a statistical evidence in favour of the unrestricted DySARAR-DHo.CHe model in all three cases.

Figures A.3 and A.4 show the dynamics of the filtered parameters by using the DySARAR-DHo.CHe specification. In particular, Figure A.3 shows the evolution of the regressor coefficient, $\beta_t$, and the spatial dependence parameters, $\rho_t$ and $\lambda_t$. First of all we can observe that both the regressor coefficient, $\beta_t$, and the spatial autoregressive parameter, $\rho_t$, fluctuate around their means, whereas $\lambda_t$ reveals approximately a linear upward trend.

Looking at the second and third panel of Figure A.3, we note that the unconditional mean of $\rho_t$ is about 0.56, revealing a medium/high spatial indirect effect on the entire financial system over the whole period, while $\lambda_t$ increases over time in a value range approximately equals to (0.46, 0.62). A more interesting result is that both the spatial autoregressive coefficients are always greater than zero, suggesting that the SARAR process is not inhibitory, so that financial returns of one sector positively and directly affects the probability of higher returns in the other sectors of the entire system. While this is true for $\rho_t$, in the case of $\lambda_t$ a different interpretation can be made. Unobserved factors, i.e. systemic events that can propagate through indirect channels like financial institutions balance sheet and the credit market, may produce indirect global effects on the entire financial system through the spatial disturbances. Consequently, a shock in one sector will indirectly affect also the disturbances associated to the other sectors, with a higher effect over time. In other words, $\rho_t$ and $\lambda_t$ display a different time–varying behaviour, especially in terms of the reported persistence. Indeed, $\lambda_t$ evolves much more persistently than $\rho_t$, suggesting that past information affects the spatial dependence of the model residuals more heavily than that of the dependent variables. As suggested in Section 2, higher absolute values of $\rho_t$ and $\lambda_t$ are revealed in a larger role of higher order neighbors in the financial system. It is interesting to note that we obtain “larger-radius” effects in correspondence to the recent financial crisis, with simultaneous picks showed by both the spatial autoregressive parameters.

The first panel of Figure A.3 is referred to the $\beta_t$ coefficient, which linearly affects the conditional mean of the returns distribution and so it measures the contribution that the exogenous regressor has in predicting future returns. Similarly to Timmermann (2008), we found that this contribution changes over time, i.e. there are periods when financial returns are easier to predict and periods when this task becomes incredibly difficult.
According to our estimates, and similarly to Welch and Goyal (2008), we found that the \( \beta_t \) coefficient displays higher deviations from its unconditional level during periods of financial turmoil such as the dot–com bubble of early 2000 and the Global Financial Crisis of 2007-2008 that highly affected the US economy.

4.3.1. The effect of spatial dependence on individual volatilities

Figure A.4 reports the cross-sectional conditional standard deviations \( \sigma_{i,t}^\epsilon = \text{diag} \left( \Sigma_t^{1/2} \right) \), the spatial conditional standard deviations \( \sigma_{i,t}^u = \text{diag} \left( \left(B_t^{-1} \Sigma_t B_t^{-1'}\right)^{1/2} \right) \) implied by unobserved correlated shocks (i.e. through \( \lambda_t \)), and the total spatial conditional standard deviations \( \sigma_{i,t}^y = \text{diag} \left( \left(A_t^{-1} B_t^{-1} \Sigma_t B_t^{-1'} A_t^{-1'}\right)^{1/2} \right) \) (i.e. due to both spillover effects, \( \rho_t \), and unobserved correlated shocks, \( \lambda_t \)), delivered by the DySARAR-DHo.CHe model. A useful interpretation of sectoral risk can be made by decomposing the total volatility displayed by each sector as the sum of two components. The former represents the systematic part of the total risk, i.e. \( \sigma_{i,t}^\epsilon \) for the \( i \)-th sector, whereas the latter represents the part of risk implied by the overall spatial dependence through \( \rho_t \) and \( \lambda_t \), i.e. the systemic part of the risk given by \( \sigma_{i,t}^{sys} = \sigma_{i,t}^y - \sigma_{i,t}^\epsilon \) for \( i = 1, \ldots, N \) and \( t = 1, \ldots, T \).

We can also define the normalized quantity as \( \%\sigma_{i,t}^{sys} = \sigma_{i,t}^{sys} / \sigma_{i,t}^y \), which represents the portion of total risk of each sector implied by the spatial dependence. This quantity is the cost (in terms of risk) that each sector pays due to its interdependence with other sectors. It is worth noting that, the quantities \( \sigma_{i,t}^{sys} \) and \( \%\sigma_{i,t}^{sys} \) do not represent the systemic importance of sector \( i \), but instead are informative about the way in which spatial dependence affects the total riskiness of sector \( i \).

Figure A.5 depicts the series \( \%\sigma_{i,t}^{sys} \) for each \( i = 1, \ldots, N \) and \( t = 1, \ldots, T \). Interestingly, we note that the influence of spatial dependence in terms of risk is quite heterogeneous across the considered sectors and also varies over time. For instance, 51% of the total risk of IG is due to its interdependence with other sectors, while only 22% in the case of BS. We found that the contribution of spatial dependence in terms of risk increased over time, especially after the turbulent period of 2008–2009.

4.4. Out of sample analysis

After having assessed the in sample properties of the proposed DySARAR specification we move to our out of sample analysis. We consider a portfolio optimisation problem where a rational investor recursively takes an investment decision at each point in time using past information. The investment decision is taken under the classical Markowitz’s Mean–Variance framework, selecting the tangency portfolio between the Capital Market Line (CML) and the efficient frontier, see e.g. Elton et al. (2009) for a textbook treatment of this topic. We allow for short sales and we set the risk–free rate equal to 0. We estimate the DySARAR-DHo.CHe model using the data of the in sample period, then we perform a rolling one step ahead forecast for the whole out of sample period of length 1500. Model parameters are updated each 100 observations using a fixed moving window.

Formally, let \( \hat{\Omega}_{t+1} \) and \( \hat{\mu}_{t+1} \) be the one step ahead conditional covariance matrix and means vector.
prediction of the assets returns at time $t$. According to our DySARAR model, these quantities are given by

$$
\hat{\Omega}_{t+1} = \hat{A}_{t+1}^{-1} \hat{B}_{t+1}^{-1} \hat{\Sigma}_{t+1} \hat{A}^{-1}_{t+1} \hat{B}^{-1}_{t+1},
$$

$$
\hat{\mu}_{t+1} = \hat{A}^{-1}_{t+1} X_{t+1} \hat{\beta}_{t+1},
$$

where we recall that $X_{t+1}$ belongs to the information set at time $t$ since we use past market returns. Under this setting, the optimal portfolio weights for the investment period $(t, t + 1]$ are available in closed form as

$$
\hat{w}_{t+1} = \frac{\hat{\Omega}^{-1}_{t+1} \hat{\mu}_{t+1}}{1' \hat{\Omega}^{-1}_{t+1} \hat{\mu}_{t+1}},
$$

where $1$ is a $N$–valued vector of ones and $\hat{w}_{t+1} = (\hat{w}_{j,t+1}; j = 1, \ldots, N)'$ is the vector containing the optimal portfolio weights.

In order to assess the performance of the resulting portfolio investment strategy, we also perform a comparative study. Specifically, we repeat the same investment strategy, but using the conditional vector of means and covariance matrices predicted by the Dynamic Conditional Correlation (DCC) model of Engle (2002) and Tse and Tsui (2002). The DCC model is the natural extension of GARCH (Engle, 1982; Bollerslev, 1986) models to the multivariate case and represents a benchmark for multivariate volatility modeling. To keep the strategy resulting from the DCC model comparable in terms of the available information set, we include the same exogenous regressor in the conditional mean specification of each marginal distribution. The DCC model is estimated using the two step QML estimation procedure detailed in Engle (2002). DCC parameters are updated each 100 observations using a rolling window as for the DySARAR specification. Similarly to De Lira Salvatierra and Patton (2015) and Jondeau and Rockinger (2012), portfolios comparison is reported in terms of management fee, which is the quantity that a rational investor is willing to pay to switch from a portfolio that she is currently holding to an alternative. In formula, assuming a power utility function $U(x) = (1 - \nu)^{-1} x^{1-\nu}$, where $\nu > 1$ is the relative risk aversion coefficient, the management fee coincides with the solution of the following equality

$$
S^{-1} \sum_{t=F+1}^{F+S} U\left(1 + \lambda^{x}_{t+1} r_{t+1}\right) = S^{-1} \sum_{t=F+1}^{F+S} U\left(1 + \lambda^{B}_{t+1} r_{t+1} - \vartheta\right),
$$

where $F = 2013$ and $S = 1500$ are the length of the in sample and out of sample periods, respectively. From equation (28), it is easy to see that if $\vartheta > 0$, the investor is willing to pay in order to switch from portfolio $A$ to portfolio $B$. On the contrary, if $\vartheta < 0$, the investor is going to ask a higher return from portfolio $B$ in order to compensate the loss in utility for switching from $A$ to $B$. Finally, if $\vartheta = 0$ the two portfolios give the same utility to the investor, leaving the investor indifferent between the two options.

Table B.5 reports the management fees for switching between the DCC and the DySARAR model, under different values of relative risk aversion coefficient $\nu$. We note that all the fees are positive and statistically
different from zero, indicating that the DySARAR model is to be preferred against the DCC model for rational investors. Table B.6 shows, instead, several portfolio backtest measures. We note that the strategy resulting from the DySARAR specification stochastically dominates the one resulting from the DCC one since it reports higher annualised return and lower annualised standard deviation. Furthermore, we also note that the DySARAR model should be preferred even from a risk management viewpoint since it results in more conservative Value–at–Risk and Expected Shortfall statistics than the DCC. Finally, the DySARAR models deliver portfolio weights with lower turnover than the DCC one, implying less transaction cost.

5. Conclusions

In this paper we present a new flexible spatio–temporal dynamic model named DySARAR. We allow for time–varying spatial dependence as well as for time–varying and cross–sectional heteroskedasticity. We let the time–varying model parameters to be updated using the scaled score of the spatial conditional distribution relying on the recently proposed score driven updating mechanism (see e.g. Creal et al. (2013), Harvey (2013)). Our model generalizes the dynamic SAR model recently proposed by Blasques et al. (2014b), by allowing for time–varying spatial dependence in the residuals as well as for a time–varying coefficient of the regressors. The model is enough flexible to nest several previously proposed static and dynamic spatial models. We detail the model characteristics and we asses the finite sample properties of the Maximum Likelihood Estimator for the DySARAR model. The flexibility of the proposed model is also investigated in a simulation study. Specifically, we found that the DySARAR model is able to adequately approximate the time–varying SARAR models with stochastic nonlinear autoregressive evolving parameters.

The paper also contributes under an empirical prospective reporting an application in portfolio optimization using financial time series. In this respect, the paper illustrates the usefulness of SARAR models in finance suggesting to employ these kind of specifications instead of SAR and SAE models as researches have done so far, see e.g. Fernandez (2011). The superior ability of the DySARAR specification is illustrated in an extensive in sample study. We found that accounting for time–varying spatial dependence for returns and residuals as well as for time heteroskedasticity is of primary importance for financial time series. The out of sample analysis illustrates the usefulness of the DySARAR model for asset allocations purposes. Indeed, we report an application in portfolio optimisation theory under the classical Markowitz’s Mean–Variance framework. Specifically, we consider a rational investor who performs a dynamic portfolio optimisation strategy by allocating her wealth over 18 US sectorial indexes. The resulting strategy is then compared with that implemented according to the conditional mean and variance predictions delivered by the DCC model of Engle (2002), which represents the industry standard for multivariate volatility modeling. Our results suggest that the DySARAR model should be chosen against the DCC model under both a mean variance criterion and a risk management prospective.
Future studies should aim to implement Score Driven (SD) models for other types of general spatial models, which has been briefly mentioned in our introduction. For instance, we intend to detect the usefulness of the SD framework into spatial Durbin model specifications in order to assess time-varying local spatial spillover effects, i.e. $WX_t\beta_{2,t}$, under some model restrictions (e.g. time-homoscedasticity or cross-homoscedasticity). The recent literature on the use of appropriate spatial weighting matrices also suggest a research work on time-varying spatial “connections”, by defining $W_t$. Finally, since spatial discrete choice or nonlinear models have received an increasing attention in the last few years, we will also propose a modification of the above procedure to directly deal with categorical data analysis.

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Appendix A. Figures
Figure A.1: S–SARAR approximation using the Heteroskedastic DySARAR(1,1) model. Black dotted lines represent the paths for the conditional parameters simulated from the Data Generating Process defined in equation (22). Purple lines are the medians across the 1000 estimates delivered by the Heteroskedastic DySARAR(1,1) model using data simulated from (21) accordingly to the previously simulated paths. Red bands are 10%-90% quantiles evaluated at each point in time $t$ using the 1000 estimates.
Figure A.2: Gaussian Kernel density for the Maximum Likelihood estimated coefficients for the DySARAR(1,1) model. Vertical red dashed lines represent the true parameters values.
Figure A.3: Filtered $\beta_t$, $\rho_t$ and $\lambda_t$ delivered by the DySARAR-DHo.CHe model. Blue vertical bands indicate periods of European recession according to the OECD Recession Indicators. Red vertical bands represent periods of US recession according to the Recession Indicators Series available from the Federal Reserve Bank of St. Louis. Red dashed vertical lines represent relevant market episodes of the recent GFC like: Freddie Mac announces that it will no longer buy the most risky subprime mortgages and mortgage-related securities (February 27, 2007), S&P announces it may cut ratings on $12$bn of subprime debt (July 10, 2007), the collapse of the 2 Bear Sterns hedge funds (August 5th, 2007), the global stock markets suffer their largest fall since September 2001 (January 21, 2008), the Bear Stearns acquisition by JP Morgan Chase (March 16, 2008), Fannie Mae and Freddie Mac are nationalized (September 7, 2008), the Lehman’s failure (September 15, 2008), the peak of the onset of the recent GFC (March 9, 2009), the S&P downgrading of US sovereign debt (August 05, 2011).
Figure A.4: Cross-sectional conditional standard deviations $\text{diag} \left( \Sigma_{1/2} \right)$ (blue), spatial conditional standard deviations $\text{diag} \left( \left( B^{-1}_{t} \Sigma_{t} B^{-1}_{t} \right)^{1/2} \right)$ (red), and total spatial conditional standard deviation $\left( A^{-1}_{t} B^{-1}_{t} \Sigma_{t} B^{-1}_{t} A^{-1}_{t} \right)^{1/2}$ (black) delivered by the DySARAR-DHo.CHe model. Blue vertical bands indicate periods of European recession according to the OECD Recession Indicators. Red vertical bands represent periods of US recession according to the Recession Indicators Series available from the Federal Reserve Bank of St. Louis. Red dashed vertical lines represent relevant market episodes of the recent GFC like: Freddie Mac announces that it will no longer buy the most risky subprime mortgages and mortgage-related securities (February 27, 2007), S&P announces it may cut ratings on $12bn of subprime debt (July 10, 2007), the collapse of the 2 Bear Sterns hedge funds (August 5th, 2007), the global stock markets suffer their largest fall since September 2001 (January 21, 2008), the Bear Stearns acquisition by JP Morgan Chase (March 16, 2008), Fannie Mae and Freddie Mac are nationalized (September 7, 2008), the Lehman’s failure (September 15, 2008), the peak of the onset of the recent GFC (March 9, 2009), the S&P downgrading of US sovereign debt (August 05, 2011).
Figure A.5: Portion of total risk implied by the spatial dependence of each sector ($\%\sigma_{i,t}^{\text{sys}} = \frac{\sigma_{i,t}^{\text{sys}}}{\sigma_{i,t}^{\text{y}}}$, for $i = 1, \ldots, N$). Blue vertical bands indicate periods of European recession according to the OECD Recession Indicators. Red vertical bands represent periods of US recession according to the Recession Indicators Series available from the Federal Reserve Bank of St. Louis. Blue dashed horizontal lines represent the sample mean. Red dashed vertical lines represent relevant market episodes of the recent GFC like: Freddie Mac announces that it will no longer buy the most risky subprime mortgages and mortgage-related securities (February 27, 2007), S&P announces it may cut ratings on $12bn of subprime debt (July 10, 2007), the collapse of the 2 Bear Sterns hedge funds (August 5th, 2007), the global stock markets suffer their largest fall since September 2001 (January 21, 2008), the Bear Stearns acquisition by JP Morgan Chase (March 16, 2008), Fannie Mae and Freddie Mac are nationalized (September 7, 2008), the Lehman’s failure (September 15, 2008), the peak of the onset of the recent GFC (March 9, 2009), the S&P downgrading of US sovereign debt (August 05, 2011).
Appendix B. Tables
Table B.1: Summary statistics for the ML estimates of the DySAR ARAR coefficients considering different sample sizes for the simulated time series of observations. The rows MSE and MAD report the empirical mean square error and the mean absolute deviation of the estimated coefficients from the true values. The row SD reports the standard deviation between the estimates.

| K | True Value | \( \rho_0 \) | \( \rho_1 \) | \( \rho_2 \) | \( \rho_3 \) | \( \rho_4 \) | \( \rho_5 \) | \( \rho_6 \) | \( \lambda_0 \) | \( \lambda_1 \) | \( \lambda_2 \) | \( \lambda_3 \) | \( \lambda_4 \) | \( \lambda_5 \) | \( \lambda_6 \) | \( \sigma_1 \) | \( \sigma_2 \) | \( \sigma_3 \) | \( \sigma_4 \) | \( \sigma_5 \) | \( \sigma_6 \) |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| K = 1000 | | | | | | | | | | | | | | | | | | | | | |
| Mean | 0.9000 | 0.2000 | -0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 |
| Median | 0.9000 | 0.2000 | -0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 |
| SD | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 |
| MSE | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 |
| MAD | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 |

| K = 3000 | | | | | | | | | | | | | | | | | | | | | |
| Mean | 0.8978 | 0.1986 | -0.0850 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 |
| Median | 0.8978 | 0.1986 | -0.0850 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 |
| SD | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 |
| MSE | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 |
| MAD | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 |

| K = 10000 | | | | | | | | | | | | | | | | | | | | | |
| Mean | 0.8993 | 0.2007 | -0.0784 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 |
| Median | 0.8993 | 0.2007 | -0.0784 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 |
| SD | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 |
| MSE | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 |
| MAD | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 |
Table B.2: Summary statistics the US sectorial indexes log returns in percentage points, for the period form 2nd January, 2002 till 5th January, 2016. The seventh column, denoted by “1% Str. Lev.” is the 1% empirical quantile of the returns distribution, while the eight column, denoted by “JB” is the value of the Jarque-Berá test-statistics. The last column denoted by $\rho (1)$ is the first empirical autocorrelation coefficient, apexes $a, b, c$ denote statistical difference from zero at confidence levels of 1%, 5%, 10%, respectively.
### In-sample, from 02/01/2002 to 20/01/2010

|    | EN  | CHe | BS  | CN  | IG  | AP  | FB  | NG  | HC  | ME  | CG  | TL  | UT  | BK  | IR  | RE  | FI  | TC  |
|----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|
| EN | 0.71| 0.79| 0.72| 0.68| 0.52| 0.54| 0.59| 0.56| 0.60| 0.52| 0.66| 0.47| 0.56| 0.48| 0.58| 0.55|
| CHe|     | 0.82| 0.84| 0.85| 0.71| 0.65| 0.72| 0.66| 0.73| 0.72| 0.64| 0.64| 0.69| 0.60| 0.72| 0.7 |
| BS |     |     | 0.81| 0.76| 0.64| 0.54| 0.59| 0.53| 0.64| 0.60| 0.51| 0.58| 0.50| 0.59| 0.63| 0.61|
| CN |     |     |     | 0.85| 0.74| 0.63| 0.75| 0.63| 0.74| 0.76| 0.58| 0.62| 0.66| 0.72| 0.68| 0.77| 0.69|
| IG |     |     |     |     | 0.8 | 0.69| 0.81| 0.75| 0.84| 0.83| 0.7 | 0.67| 0.71| 0.79| 0.68| 0.83| 0.82|
| AP |     |     |     |     | 0.54| 0.67| 0.57| 0.72| 0.73| 0.57| 0.54| 0.63| 0.68| 0.72| 0.62| 0.72| 0.66|
| FB |     |     |     |     |     | 0.77| 0.7 | 0.63| 0.63| 0.56| 0.6 | 0.52| 0.62| 0.51| 0.61| 0.54|
| NG |     |     |     |     |     |     | 0.74| 0.71| 0.77| 0.62| 0.65| 0.64| 0.72| 0.62| 0.73| 0.65|
| HC |     |     |     |     |     |     |     | 0.71| 0.67| 0.62| 0.63| 0.55| 0.68| 0.51| 0.67| 0.66|
| ME |     |     |     |     |     |     |     |     | 0.75| 0.7 | 0.63| 0.64| 0.74| 0.6 | 0.76| 0.78|
| CG |     |     |     |     |     |     |     |     |     | 0.59| 0.56| 0.68| 0.72| 0.68| 0.78| 0.7 |
| TL |     |     |     |     |     |     |     |     |     |     | 0.57| 0.55| 0.61| 0.48| 0.65| 0.68|
| UT |     |     |     |     |     |     |     |     |     |     |     | 0.49| 0.6 | 0.51| 0.6 | 0.56|
| BK |     |     |     |     |     |     |     |     |     |     |     |     | 0.82| 0.74| 0.88| 0.56|
| IR |     |     |     |     |     |     |     |     |     |     |     |     |     | 0.72| 0.87| 0.64|
| RE |     |     |     |     |     |     |     |     |     |     |     |     |     |     | 0.79| 0.52|
| FI |     |     |     |     |     |     |     |     |     |     |     |     |     |     |     | 0.7 |
| TC |     |     |     |     |     |     |     |     |     |     |     |     |     |     |     |     | 0.0 |

### Out-of-sample, from 21/01/2010 to 05/01/2016

|    | EN  | CHe | BS  | CN  | IG  | AP  | FB  | NG  | HC  | ME  | CG  | TL  | UT  | BK  | IR  | RE  | FI  | TC  |
|----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|
| EN | 0.84| 0.82| 0.82| 0.7 | 0.61| 0.77| 0.74| 0.78| 0.62| 0.63| 0.65| 0.87| 0.81| 0.7 | 0.75| 0.77| 0.69|
| CHe|     | 0.8 | 0.73| 0.69| 0.6 | 0.73| 0.77| 0.84| 0.64| 0.73| 0.61| 0.86| 0.82| 0.7 | 0.8 | 0.84| 0.65|
| BS |     |     | 0.68| 0.74| 0.71| 0.84| 0.75| 0.74| 0.62| 0.71| 0.78| 0.71| 0.62| 0.67| 0.78| 0.74| 0.56|
| CN |     |     |     | 0.67| 0.76| 0.88| 0.81| 0.85| 0.68| 0.92| 0.84| 0.66| 0.56| 0.78| 0.79| 0.84| 0.65|
| IG |     |     |     |     | 0.66| 0.91| 0.8 | 0.81| 0.68| 0.84| 0.76| 0.83| 0.77| 0.69| 0.78| 0.8 | 0.72|
| AP |     |     |     |     |     | 0.83| 0.66| 0.79| 0.57| 0.73| 0.85| 0.9 | 0.82| 0.75| 0.66| 0.64| 0.61|
| FB |     |     |     |     |     |     | 0.6 | 0.79| 0.52| 0.8 | 0.81| 0.79| 0.72| 0.72| 0.63| 0.69| 0.57|
| NG |     |     |     |     |     |     |     | 0.73| 0.68| 0.76| 0.88| 0.9 | 0.83| 0.18| 0.71| 0.74| 0.9 |
| HC |     |     |     |     |     |     |     |     | 0.72| 0.83| 0.79| 0.87| 0.81| 0.8 | 0.79| 0.8 | 0.72|
| ME |     |     |     |     |     |     |     |     |     | 0.83| 0.85| 0.69| 0.86| 0.81| 0.69| 0.74| 0.91|
| CG |     |     |     |     |     |     |     |     |     |     | 0.82| 0.76| 0.77| 0.72| 0.79| 0.82| 0.74|
| TL |     |     |     |     |     |     |     |     |     |     |     | 0.72| 0.75| 0.7 | 0.77| 0.83| 0.79|
| UT |     |     |     |     |     |     |     |     |     |     |     |     | 0.73| 0.71| 0.83| 0.62| 0.92|
| BK |     |     |     |     |     |     |     |     |     |     |     |     |     | 0.81| 0.69| 0.64| 0.8 |
| IR |     |     |     |     |     |     |     |     |     |     |     |     |     | 0.62| 0.71| 0.77| 0.7 |
| RE |     |     |     |     |     |     |     |     |     |     |     |     |     |     | 0.65| 0.69| 0.83|
| FI |     |     |     |     |     |     |     |     |     |     |     |     |     |     |     | 0.83| 0.0 |

**Table B.3:** Empirical linear correlation matrix for the in sample and out of sample periods
Table B.4: AIC, BIC, number of estimated parameters (Np) and log-likelihood (LLK) of different spatial specifications for asset returns.

| Model       | Method | AIC      | BIC      | Np | LLK      |
|-------------|--------|----------|----------|----|----------|
| Dynamic     | CHo OLS | 156618.44 | 156655.42 | 6  | -78303.22 |
|             | SAR    | 100444.67  | 100500.15  | 9  | -50213.34 |
|             | SAE    | 100446.97  | 100502.45  | 9  | -50214.49 |
|             | SARAR  | 100046.16  | 100120.13  | 12 | -50011.08 |
| DHe.CHe     | OLS    | 156481.06  | 156832.42  | 57 | -78183.53 |
|             | SAR    | 100152.12  | 100521.97  | 60 | -50016.06 |
|             | SAE    | 100154.58  | 100524.43  | 60 | -50017.29 |
|             | SARAR  | 99751.45   | 100019.8   | 63 | -49812.73 |
| DHo.CHe     | OLS    | 156521.1   | 156662.88  | 23 | -78237.55 |
|             | SAR    | 100325.53  | 100485.79  | 26 | -50136.76 |
|             | SAE    | 100328.45  | 100488.72  | 26 | -50138.22 |
|             | SARAR  | 99926.45   | 100103.4   | 29 | -49933.32 |
| Static      | CHo OLS | 189344.51  | 189356.83  | 2  | -94670.25 |
|             | SAR    | 137530.78  | 137549.27  | 3  | -68762.39 |
|             | SAE    | 134217.54  | 134236.03  | 3  | -67105.77 |
|             | SARAR  | 133700.89  | 133725.54  | 4  | -66846.44 |
| CHe         | OLS    | 156517.17  | 156646.62  | 21 | -78237.59 |
|             | SAR    | 101803.77  | 101939.38  | 22 | -50879.88 |
|             | SAE    | 100549.2   | 100684.81  | 22 | -50252.6  |
|             | SARAR  | 100151.33  | 100293.11  | 23 | -50052.67 |

Table B.5: Management fee that a rational investor is willing to pay for switching between the DySARAR model and the DCC model. The apexes “a”, “b” and “c”, denote the rejection of the null hypothesis of not significance of the corresponding parameter, at different confidence levels 1%, 5% and 10%. P-values are obtained using a block bootstrap procedure as in De Lira Salvaterra and Patton (2015).
### Table B.6:

Annualised means (Mean) and standard deviations (St.Dev) of the empirical portfolio return distributions for the optimal strategies implied by the DySARAR and DCC specifications. The third and the fourth columns report the maximum loss and the maximum gain in percentage returns faced by the investor during the investment strategy. The annualised Sharpe Ratio (SR) is calculated assuming a zero risk free rate. The seventh and the eighth columns report the empirical Value–at–Risk (VaR) and Expected Shortfall (ES) evaluated at the 5% confidence level. The VaR\(5\%\) and ES\(5\%\) quantities are evaluated as the solution of \(P(r_p < \text{VaR}_{5\%}) = 0.05\) and \(\text{ES}_{5\%} = E(r_p | r_p < \text{VaR}_{5\%})\), respectively, where \(r_p\) is the portfolio return. The last column report the Turnover of the portfolio evaluated as \(\frac{100}{T} \sum_{t=2}^{T} \sum_{i=1}^{N} |\omega_{i,t} - \omega_{i,t-1}|\).

| model  | Mean | St.Dev. | MaxLoss | MaxGain | SR  | VaR\(5\%) | ES\(5\%) | Turnover |
|--------|------|---------|---------|---------|-----|-----------|----------|----------|
| DySARAR | 17.37 | 13.14   | -3.86   | 3.7     | 1.32| -1.3      | -1.83    | 21.02    |
| DCC   | 9.83  | 16.03   | -4.45   | 4.41    | 0.59| -1.64     | -2.23    | 72.76    |
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