Eikonal contributions to ultra high energy neutrino-nucleon cross sections in low scale gravity models

E. M. Sessolo and D. W. McKay

Department of Physics and Astronomy, University of Kansas, Lawrence, KS 66045

Abstract

We calculate low scale gravity effects on the cross section for neutrino-nucleon scattering at center of mass energies up to the Greisen-Zatsepin-Kuzmin (GZK) scale, in the eikonal approximation. We compare the cases of an infinitely thin brane embedded in 5 compactified extra-dimensions, and of a brane with a physical tension $M_S = 1 \text{ TeV}$ and $M_S = 10 \text{ TeV}$. The extra dimensional Planck scale is set at $10^3 \text{ GeV}$ and $2 \times 10^3 \text{ GeV}$. We also compare our calculations with pre-existing neutral current standard model calculations in the same energy range. New physics effects enhance the cross section by orders of magnitude on average, though they are quite sensitive to parameter choices. Moreover, in the thin brane limit, the full eikonal approximation results in a cross section noticeably larger than in the corresponding saddle point approximation.

1 Introduction

Since neutrinos interact only weakly with matter, neutrino observatories provide a powerful tool for exploring the deepest reaches of stars and galaxies. In recent years there has been an upswell in the number of experiments, either running or about to run, that are aimed at detecting ultra-high energy (UHE) neutrinos, neutrinos with energies from the multi-TeV range to beyond the EeV range. Their existence is predicted by a number of theoretical models. Although the production mechanism can be different - either as a byproduct of creation and decay of pions in the interaction between high energy primary cosmic rays and the cosmic microwave background, as predicted by GZK \[1\], or as products of the same astrophysical sources that generate the cosmic rays observed in the highest energy air showers \[2\] -there is wide expectation in the community that neutrinos of galactic or cosmic origin will be detected in the UHE energy range in the near future. Many experimental limits on astrophysical neutrino fluxes have already been established. Frejus, Baikal, AMANDA and MACRO \[3\] have reported limits on neutrinos from astrophysical sources in the $10^3 - 10^6 \text{ GeV}$ range, while Fly’s Eye, AGASA, AMANDA and RICE \[4\] have given limits in the
range $10^6 - 10^9$ GeV, and AGASA, Fly’s Eye, RICE, Forte, ANITA and GLUE [5] have reported limits in the range above $10^9$ GeV. In the near future, IceCube [6], ANITA and AUGER [7] can be expected to release stronger results based on more recent data. In the longer term, proposed expansions of IceCube, such as AURA [8], or a salt-based radio telescope like SALSA [9] could afford substantially enhanced sensitivity. Further in the future, an orbiting telescope like the proposed EUSO project [10] opens the possibility to achieve a huge effective volume.

The design of neutrino telescopes depends critically on the estimates of neutrino cross sections in the UHE regime, most of which is far beyond currently available data. Moreover, the systematic study of the scattering of such high energy neutrinos with baryonic matter might prove to be an essential instrument to test new physics effects, such as those given by models of low scale gravity (LSG). Models of extra dimensions [11, 12, 13] have lowered the characteristic quantum-gravity scale to energies comparable with the electro-weak scale, enhancing the expected neutrino cross sections at GZK-energies and above, even for conservative parameter choices.

Efforts at realistic calculations of the UHE neutrino-nucleon cross section have been performed in the framework of QCD and the standard model [14, 15], and many speculative calculations based on new physics effects have been presented over the past decade or more. The only detailed calculations of neutrino event rates within the framework of the LSG models that include the neutrino-proton eikonal cross section with LSG graviton exchange, rely upon the saddle point approximation [16]. In this article we present in detail the calculation for the full eikonal approximation of the LSG neutrino-proton cross section, including a study of the effects of a finite tension on the brane [17, 18].

In Sec. II we summarize the main procedure of the eikonal approximation (EA) in quantum field theory and apply it to the calculation of the desired differential cross sections at the parton level in the Arkani-Hamed, Dimopoulos and Dvali (ADD) model of extra dimensions [11]. We consider the cases of an infinitely thin standard model brane and the one where it has finite thickness. In Sec. III we present the full EA calculation and compare the results with those in the saddle point approximation, and with the standard model cross section. Sec. IV is dedicated to the summary of results, comments on the sensitivity of results to choice of gravity scale and brane tension and on the origin of the discrepancy between the saddle point and full EA calculation. We also comment on the impact of our cross section results on neutrino telescope event rate expectations. The Appendix presents details of the representation of the eikonal amplitude in terms of Meijer G-functions.

2 Theory

The eikonal approximation is a technique widely used in quantum mechanics and wave physics to derive an expression for elastic scattering in the limit of large center of mass (CM) energies (see for
example Reference [19]). In quantum field theory the amplitude can be obtained by summation of all the ladder and cross-ladder diagrams for boson exchange at all order [20][21], as exemplified in Fig. 1.

![Figure 1: Sum of the ladder and cross-ladder diagrams in the eikonal approximation](image)

Since we are dealing with center of momentum energies that are high with respect to $Q^2$, the absolute magnitude of the momentum transfer $q^2$, the high energy scattering angle is small and one can neglect all but the leading order dependence in the propagator $1/q^2$. In the vertex and external legs calculation one can neglect the mass of the incoming particles with respect to their four-momentum and higher order terms in the transferred momentum $q$. The result is thus independent from the spin of the incoming particle, and reduces to simply $2p^\mu$.

The tree-level amplitude (Fig. 2a), or Born term for the exchange of one intermediate vector or tensor boson is given by:

$$A_{\text{Born}}(t) = g^2 s^r \frac{1}{t - m^2}, \quad (1)$$

where $g$ is the dimension-dependent interaction coupling constant, $s$ and $t$ are the usual Mandelstam variables for the 2→2 scattering amplitude, $m$ is the mass of the exchanged particle and $r = 1, 2$ is its spin.

![Figure 2: 2a: Born term diagram. 2b: One loop diagram with matter propagators put on shell by the Cutkowsky rule](image)

At one-loop level (Fig. 2b), the EA simplifies the matter propagators:

$$\frac{i}{(p \pm q')^2 - M^2 + i\epsilon} \rightarrow \frac{i}{\pm 2p \cdot q' + i\epsilon}, \quad (2)$$
where $M$ is the mass of the incoming particles. The one-loop eikonal amplitude, obtained when the above propagators are put on shell by the Cutkowsky rule, is imaginary. The integral over the transferred four-momentum reduces to an integral over its components in the plane perpendicular to the momenta of the incoming particles, $q_{\perp}$. One gets for the eikonal at one loop:

$$A_{\text{loop}} = -\frac{i}{2} \frac{(g^2 s)^2}{4} \int \frac{d^4 q'}{(2\pi)^4} \frac{1}{q'^2 - m^2} \frac{1}{(q - q')^2 - m^2} (2p_1 \cdot q') \delta(2p_2 \cdot q')$$

$$= \frac{i}{4s} \int \frac{d^2 q_{\perp}}{(2\pi)^2} A_{\text{Born}}(q_{\perp}^2) A_{\text{Born}}[(q_{\perp} - q_{\perp}')^2]$$

$$= -2si \int d^2 b_{\perp} e^{iq_{\perp} \cdot b_{\perp}} \left(-\frac{1}{2} \chi^2\right)$$

where the eikonal phase

$$\chi(b_{\perp}) = \frac{1}{2s} \int \frac{d^2 q_{\perp}}{(2\pi)^2} e^{-iq_{\perp} \cdot b_{\perp}} A_{\text{Born}}(q_{\perp}^2)$$

is the Fourier transform of the Born term.

The procedure can be repeated with higher order diagrams [20], such that one can sum all the terms in $\chi$ to obtain:

$$A_{\text{eik}} = A_{\text{Born}} + A_{\text{loop}} + \ldots + A_{n-\text{loops}} + \ldots = -2si \int d^2 b_{\perp} e^{iq_{\perp} \cdot b_{\perp}} \left(\sum_{n=1}^{\infty} (i\chi)^n / n!\right)$$

$$= -2si \int d^2 b_{\perp} e^{iq_{\perp} \cdot b_{\perp}} (e^{i\chi} - 1).$$

It is worth noting at this point of our discussion that the validity of the EA in describing the asymptotic behavior of elastic scattering has been long debated. References [20] and [22] argue that the eikonal is not a correct description of the high energy limit in the cases of scalar particle exchange, abelian vector exchange and non-abelian vector exchange. In the cases of vector particle exchange, when we put $r = 1$ in Eq. (1), the EA describes accurately the asymptotic behavior of the sum of exchange diagrams, but fails to consider the inelastic diagrams related by unitarity to particle production, that have been shown by [20] to give a dominant contribution to the elastic scattering. On the other hand, the $s^2$ dependence of the graviton-exchange Born term in the quantum gravity EA renders the sum of exchange diagrams dominant with respect to the inelastic unitarity diagrams ([22] and references therein). Consistent with the state of present knowledge, we will consider the eikonal a good approximation of the asymptotic behavior.

In models of LSG with large extra dimensions [11, 13], gravitons exchange ($r = 2$) is ultraviolet divergent already at tree-level [23, 24]. This is due to the presence of an infinite tower of massive Kaluza-Klein modes over which the amplitude has to be summed. For a number $n$ of extra-dimensions we have, approximating the sum by an integral,
\[ A_{\text{Born}} = \frac{s^2}{M_D^{n+2}} \int \frac{d^n m}{t - m^2} \equiv \frac{s^2}{M_D^{n+2}} S_n \int_0^\infty \frac{m^{n-1} dm}{t - m^2}, \]  

where \( M_D \) is the \( D \)-dimensional Plank mass (where \( D \equiv 4 + n \)) and \( S_n \) is the surface of a \( n \)-dimensional unit sphere. As the notation indicates, this sum is adopted as the "Born input" to the eikonal calculation \[25\].

In the literature, there are two common solutions to the problem of tree-level divergencies. One can introduce a cut-off of the order of the low gravity scale, as this is the limit below which the effective theory is deemed to be valid \[23, 24, 26\]. Or, one can consider the physically reasonable case where the brane has a finite tension \( M_S \), corresponding to an extension \( 1/M_S \) of the standard model fields along the extra dimensions \[17, 18\]. In this case, if the extra-dimensional part \( \vec{y} \) of the standard model wave function has a gaussian cut-off of the kind

\[
\psi(\vec{y}) = \left( \frac{M_S}{\sqrt{2\pi}} \right)^{\frac{n}{2}} e^{-\frac{|\vec{y}|^2 M_S^2}{4}},
\]

Eq. (6) is modified to

\[
A_{\text{Born}} = \frac{s^2}{M_D^{n+2}} S_n \int_0^\infty e^{-\frac{m^2}{M_S^2}} m^{n-1} \frac{dm}{t - m^2}.
\]

Looking back to Eq. (6), it can be seen that when the brane is infinitely thin the regularization procedure is not important, as it gets absorbed in the eikonal phase calculation. The Born term can be calculated by dimensional regularization \[26\]:

\[
A_{\text{Born}}(q_\perp^2) = \frac{\pi^{\frac{n}{2}} \Gamma \left( 1 - \frac{n}{2} \right)}{M_D^{n+2}} (q_\perp^2)^{\frac{n}{2} - 1} s^2.
\]

Substitution of the latter into Eq. (4) yields

\[
\chi = \frac{\pi^{n-1} \Gamma(1-n/2)s}{4M_D^{n+2}} \int_0^\infty dq q^{n-1} J_0(qb) = \left[ \frac{(4\pi)^{\frac{n}{2} - 1}s\Gamma(\frac{n}{2})}{2M_D^{n+2}} \right] \frac{1}{b^n} \equiv \left( \frac{b_c}{b} \right)^n
\]

where we’ve followed the commonly simplified notation \( q = q_\perp, b = b_\perp \). The critical impact parameter \( b_c \) separates the region of space where the Born term is dominant \( (b \gg b_c) \) from the one where higher order terms in \( \chi \) become relevant. Expression (10) is clearly finite for all \( b > 0 \).

The eikonal amplitude is then given by

\[
A_{\text{eik}} = 4\pi s b_c^2 F_n(b_c q),
\]

with

\[
F_n(\eta) = -i \int_0^\infty d\xi \xi J_0(\xi \eta)(e^{i\xi^n} - 1),
\]
and $J_0$ the Bessel function of order zero.

As mentioned in passing in [26, 27], but not pursued, the $F_n$ can be expressed in terms of Meijer G-functions. We implement our calculation explicitly in these terms, providing details in the Appendix. The result is

$$F_n(\eta) = 2^{-\frac{2}{n}-1} \pi \frac{1}{2} n^{-1} (R_n(\eta) + i I_n(\eta)),$$

(13)

where the real and imaginary part are given by

$$I_n(\eta) = G^{n+1,0}_{0,2(n+1)} \left( \frac{\eta^{2n}}{2^{2n+2} n^{2n}} \right) \left[ \begin{array}{c} 0, \frac{1}{n}, \ldots, \frac{n-1}{n}, \frac{n-1}{n}, -\frac{1}{n}, 0, \frac{n-2}{2n}, \frac{n-2}{2n}, \ldots, 1 \end{array} \right]$$

(14)

and

$$R_n(\eta) = -G^{n+1,0}_{0,2(n+1)} \left( \frac{\eta^{2n}}{2^{2n+2} n^{2n}} \right) \left[ \begin{array}{c} 0, \frac{n-2}{2n}, \frac{1}{n}, \ldots, \frac{n-1}{n}, -\frac{1}{n}, 0, \frac{n-1}{n}, \frac{n-2}{2n}, \ldots, 1 \end{array} \right].$$

(15)

where $G_{ij}^{kl}$ are the Meijer G-functions [28].

In the limit of large momentum transfer with respect to the inverse impact parameter ($q \gg 1/b_c$), the amplitude of Eq. (11) can be approximated analytically by the steepest descent method ([16, 25, 26, 27]) and it reads

$$A_{eik}(q) = A_n e^{i \phi_n} \left[ \frac{s}{q M_D} \right]^{\frac{n+2}{n+1}},$$

(16)

where

$$A_n = \frac{(4\pi)^{\frac{3n}{2}+2}}{\sqrt{n+1}} \left[ \Gamma \left( \frac{n}{2} + 1 \right) \right]^{\frac{1}{n+1}},$$

(17)

$$\phi_n = \frac{\pi}{2} + (n + 1) \left[ \frac{q b_c}{n} \right]^{\frac{1}{n+1}},$$

(18)

and $b_c$ is given by Eq. (10). The location of the saddle point, $b_s = b_c \left( \frac{n}{q b_c} \right)^{\frac{1}{n+1}}$, should satisfy $b_s \ll b_c$ in order for the saddle point to give the dominant contribution to the amplitude.

On the other hand, in the case of a finite tension $M_S$ in the brane, substitution of Eq. (8) into Eq. (4) yields [18]:

$$\chi(b) = s M_S^2 M_D^n \Gamma \left( \frac{n}{2} \right) \frac{8}{n+2} \left[ 4 U \left( \frac{n}{2}, 1; \frac{M_S^2 b^2}{4} \right) - \left( \frac{b_c M_S}{2} \right)^n \right] U \left( \frac{n}{2}, 1; \frac{M_S^2 b_c^2}{4} \right) ,$$

(19)

where $U(a, c; x)$ is a confluent hypergeometric function of the second kind [28].

By substitution of Eq. (19) into Eq. (5) the eikonal amplitude can be calculated easily:

$$A_{eik} = -4\pi i s \int_0^\infty db b J_0(q b) \left[ e^{i \left( \frac{b_c M_S}{2} \right)^n \left[ U \left( \frac{n}{2}, 1; \frac{M_S^2 b_c^2}{4} \right) - 1 \right]} \right],$$

(20)
where in the last expression we have used the functional identity:

\[
\int d^2\eta e^{i\vec{\xi} \cdot \vec{\eta}} f(\eta) = 2\pi \int_0^\infty d\eta J_0(\xi\eta) f(\eta),
\]  

(21)

with \( \xi = |\vec{\xi}|, \eta = |\vec{\eta}| \).

The differential cross section at the parton level can now be calculated with the usual substitution \(-t \equiv -q^2 \equiv Q^2 = xys\), where \( x \) is the fraction of momentum of the proton carried by the parton and \( y \) is the inelasticity. We get for the infinitely thin brane:

\[
\frac{d^2\sigma}{dx dy} = \sum_i x f_i(x, xys) s\pi b_c^4(\hat{s}) |F_n(b_c(\sqrt{xys}))|^2,
\]  

(22)

with \( F_n(\eta) \) given by Eq. (12).

For the brane \( 1/M_S \)-thick:

\[
\frac{d^2\sigma}{dx dy} = \sum_i x f_i(x, xys) s\pi |G_n(\sqrt{xys})|^2,
\]  

(23)

with

\[
G_n(\zeta) = -i \int_0^\infty db b J_0(b\zeta) \left[ e^{\left(\frac{b^2(\hat{s}M_S)}{2}\right)} U\left(\frac{1}{2},\frac{M_W^2}{\hat{s}}\right) - 1 \right].
\]  

(24)

The \( f_i(x, Q^2) \) are the usual parton distribution functions (PDF) summed over all the quark and antiquark flavors and the gluons. Note here that the parameter \( b_c(s, n) \) of Eq. (10) has to be calculated with respect to the center of mass energy squared of the neutrino-parton system \( \hat{s} = xs \).

3 Results and Discussion

The total cross section is obtained by integration of Eqs. (22) and (23). The \( y \)-integral has been performed with Mathematica using the CTEQ5 PDFs [29], in the interval \([Q_0^2/(xs), 1/(x s R_S^2)]\), where \( Q_0^2 = 0.01M_W^2 \), \( M_W \) is the mass of the \( W \)-boson and \( R_S^2 \propto s^{1/(n+1)} / M_D^{2(n+2)/(n+1)} \) is the Schwarzschild radius for black-hole production in the \( n \)-dimensional theory [26]. We adopt \( \sqrt{Q^2} \) as the scale parameter in the PDF’s. The upper limit of integration is introduced since the eikonal cross section is not applicable for \( Q^2 \gtrsim 1/R_S^2 \). The lower limit, as in standard model calculations, allows us to be safely inside the perturbative QDC regime. The \( x \)-integral has been performed by dividing the interval \([Q_0^2/s, 1]\) into 38 logarithmic subintervals and performing Simpson’s rule. This procedure gives an accurate evaluation of the cross section, which is reasonably smooth in \( x \).

We consider the case of \( n = 5 \) extra-dimensions, which is sufficient to illustrate all our points. In Fig. 3 we show the comparison between our calculation of the proton-neutrino cross section in the full EA with the saddle point approximation of the same cross-section given by Eqs. (16), (17) and
The brane is here assumed to be infinitely thin, with the (4+5)-dimensional Planck scale set at \( M_D = 1 \text{ TeV} \). In the range \( E_\nu > 10^7 \text{ GeV} \), where the EA is expected to be valid, the full eikonal cross section is significantly larger than its evaluation at the saddle point. The difference ranges from a factor 2.6 at the low end to about 1.5 at the high end. As discussed in the next section, the conditions for the dominance of the saddle point evaluation are not satisfied until extremely high energies, explaining the slow convergence of the saddle point value to the full eikonal result.

![Figure 3: Solid light line: Full EA corrected by the Born term, \( M_D = 10^3 \text{ GeV}, M_S = \infty \), \( n = 5 \). Dashed light line: Saddle point approximation corrected by the Born term. Solid dark line: Standard model neutral current](image)

As also shown in Fig. 3, even with the saddle point approximation to the LSG eikonal, the neutral current cross section in this model exceeds the standard model values \cite{15} at CM energies above the gravity scale, corresponding to roughly \( 10^6 \text{ GeV} \) for \( M_D = 1 \text{ TeV} \) \cite{16}. The dominance is even more striking when the full eikonal is used: at neutrino energies greater than \( 10^{10} \text{ GeV} \) new physics effects raise the cross section by almost three orders of magnitude.

The presence of a thick brane with a tension \( M_S = 1 \text{ TeV} \) sharply reduces the cross section, due to the effective gaussian cut-off in the Born amplitude of Eq. \cite{8}. The lower tension suppresses the pure eikonal cross section by an order of magnitude at the low energy end and a factor of two at the high end, as shown in Fig. 4. As the tension rises, and the brane gets thinner, the cross section should approach the infinitely thin limit. As we can see in Fig. 5 we already obtain an almost perfect overlapping when the tension is \( M_S = 10 \text{ TeV} \).

Fig. 6 and Fig. 7 show the reduction to the cross sections due to a higher Planck scale \( M_D = 2 \text{ TeV} \). We here display the cases of the infinitely thin brane and of the \( M_S = 1 \text{ TeV} \)-tension brane. We see that the increase in scale to \( 2 \text{ TeV} \) suppresses the cross section an order of magnitude over

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\[1\] For the treatment of the Born term, see the first paper of Ref. \cite{16}.
Figure 4: Pure eikonal comparison at high energies for branes with different tension. Solid light: \( M_D = 10^3 \) GeV, \( M_S = \infty \), \( n = 5 \). Dashed light: \( M_D = 10^3 \) GeV, \( M_S = 10^3 \) GeV, \( n = 5 \). Solid dark: Standard model NC

Figure 5: Pure eikonal comparison at LSG \( M_D = 10^3 \) GeV for branes with different tension: \( M_S = \infty \) and \( M_S = 10^4 \) GeV, \( n = 5 \).

the entire energy range for the thin brane case (Fig. 6). The relative suppression is even bigger in the case of a thick brane (Fig. 7), where the cross section is lowered to values beneath the standard model calculation at energies less than \( 10^9 \) GeV.

4 Summary, Further Discussion and Conclusions

We have numerically calculated the parton level cross-sections for UHE neutrino-nucleon gravitational scattering in the eikonal approximation, under the assumption that the standard model
particles are confined in a 4-dimensional brane embedded in a higher number of compactified extra-dimensions (ADD Model). The analytical calculations are presented in the most general case of $n$ extra-dimensions, but the numerical calculations were performed for $n = 5$ to illustrate the magnitude of the effects for value of the extra dimensions that is only weakly constrained by current laboratory and astrophysical data. The cross sections are only mildly dependent on $n$, at any rate. Previous applications of the ADD-type models to the UHE neutrino cross section and expectations for event rates relied on the saddle point estimation of the eikonal amplitude, and only in the thin
brane limit. Our application completes the picture of the impact of low scale gravity on the predictions of high energy cross sections. We reach the same qualitative conclusions as previous studies: the effects are sensitive to the scale, or radius, of the extra dimensions but much less sensitive to the actual number of dimensions. As in other applications, the brane tension, or thickness, greatly affects the size of LSG influence on UHE neutrino scattering, weakening sharply as brane tension decreases, or equivalently as brane thickness increases.

Quantitatively speaking, we have found that the presence of new physics at LSG $M_D = 10^3$ GeV enhances the "neutral current" cross section compared to the standard model cross section by three order of magnitude at CM energies of $10^{10}$ GeV and above, and that the full eikonal calculation of the cross section gives values consistently higher than the corresponding saddle point approximation, when the brane is considered to be infinitely thin. The enhancement above the standard model expectation is not nearly as marked in the case of a brane physically extended $10^{-3}$ GeV$^{-1}$ along the extra-dimensions, or in the case where the LSG Planck Mass is increased by a factor of two. However, we found that calculations of the cross sections performed with brane tensions of $10$ TeV and above are indistinguishable, indicating that a ratio of tension to gravity scale of $10:1$ is effectively "infinite" for these purposes.

To understand the difference between the UHE cross section evaluated in the saddle point approximation versus the full eikonal calculation, we recall that the saddle point dominance of the integral in question requires $q b_c \gg 1$. In our eikonal evaluation, as in those in Ref.[16], we required $q \leq 1/R_S$. Let us check the consistency of these two requirements. Defining $q_{max} = 1/R_S$, we find that $0.77 \leq q_{max} \times b_c \leq 5.1$ as $E_\nu$ runs from $10^5$ GeV to $10^{12}$ GeV for $M_D = 10^3$ GeV and $n = 5$. Moreover, the saddle point value of the impact parameter, $b_s = b_c(qb_c/n)^{-1/(n+1)}$ is only slightly less than $b_c$ for these parameter choices, failing to satisfy the condition $b_s \ll b_c$ needed for saddle point dominance of the phase integral, Eq. (12). These considerations tell us that, even with the maximal $q$ consistent with use of the eikonal, the strong coupling, saddle point conditions are, at best, only marginally satisfied even at the highest, super GZK energies we consider.

The sensitivity of the calculation to $M_S$ and $M_D$ values is much more important to the estimate of event rates than the sensitivity to the use of the saddle point approximation, rather than the full EA. Since collisions that satisfy the EA conditions tend to be highly elastic, the consequences for event rate estimates based on the model assumptions are not severe, since the fraction of deposited energy is small, and the corresponding events are mixed with highly inelastic events at neutrino energies an order of magnitude or more lower, where the flux is presumably much larger. Thus the factor of ten difference in cross section between the $M_D = 1$ TeV and 2 TeV, or between the infinite and finite tension at fixed LSG is, in a sense, part of uncertainty in a "background" correction to inelastic processes, such as black hole formation, where the neutrino energy is nearly all converted
to visible energy. Only if black hole formation is severely suppressed, does the enhanced signal from elastic processes become a sensitive tool to detect LSG.

In summary, our conclusions are that, in agreement with other studies in collider settings, the LSG effects are sensitively dependent on the choice of scale parameters - the gravity scale and the brane tension. The eikonal neutrino-nucleon cross section always rises sharply above the standard model versions, but the energy at which the onset occurs and the degree to which the new physics dominates, is strictly dictated by the input scales. In the regime we consider, $10^6 - 10^{12}$ GeV, the saddle point dominance conditions are not well satisfied, and the approximation, though lending itself to simple expressions and intuitive pictures, noticeably underestimates the cross section. Having said that, the uncertainty in $M_D$ and $M_S$ values is nevertheless the dominant effect.

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Appendix

The integral of Eq. (12) can be expressed as a Mellin convolution:

$$ F_n(y) = i \int_0^\infty \frac{dx}{x} f_1(x) f_2 \left( \frac{y}{x} \right), \quad (25) $$

where

$$ f_1(x) = \frac{1}{x^2} \left( e^{ix^n} - 1 \right) \quad (26) $$

$$ f_2(x) = J_0(x). \quad (27) $$

We can calculate the Mellin transforms of Eqs. (26), (27):

$$ f_1^*(s) = \int_0^\infty f_1(x)x^{s-1}dx = -\frac{1}{n} \Gamma \left( \frac{s-2}{n} \right) e^{i\pi \left(n-1+\frac{s}{2}\right)} \quad (28) $$

$$ f_2^*(s) = \int_0^\infty f_2(x)x^{s-1}dx = \frac{2^{s-1} \Gamma \left( \frac{s}{2} \right)}{\Gamma \left( 1 - \frac{s}{2} \right)}. \quad (29) $$

Upon defining $f^*(s) = f_1^*(s)f_2^*(s)$, the integral of Eq. (25) turns out to be a Mellin-Barnes integral:

$$ -iF_n(y) = \frac{1}{2\pi i} \oint f^*(s)y^{-s}ds \quad (30) $$

that can be expressed explicitly in terms of Meijer G-functions.
The real part of Eq. (30) is

$$\Re F_n(y) = \frac{1}{2\pi i} \oint \Re f^*(s)y^{-s}ds$$

$$= \frac{1}{2\pi i} \int -\frac{2^{s-1}\Gamma\left(\frac{3}{2}\right)\Gamma\left(\frac{s-2}{n}\right)\sin\left(\frac{\pi}{2n}(n-1+\frac{s}{2})\right)}{\Gamma\left(1-\frac{s}{2}\right)}y^{-s}ds$$

$$= \frac{1}{2\pi i} \int n\Gamma\left(1-\frac{s}{2}\right)\Gamma\left(\frac{n-1}{n}+\frac{s}{2n}\right)\Gamma\left(1-\frac{n-1}{n}-\frac{s}{2n}\right)y^{-s}ds,$$  \hspace{1cm} (31)

where in the last step we have used Euler’s duplication formula ($\sin(\pi x) = \pi/\Gamma(x)\Gamma(1-x)$).

In the standard definition of Meijer G-functions \[28\] the coefficient multiplying $s$ inside the Gamma functions has to be $\pm 1$. We therefore perform the substitution $s \to 2ns$ to get

$$\Re F_n(y) = \frac{1}{2\pi i} \int \frac{\pi\Gamma(ns)\Gamma\left(2\left(s-\frac{1}{n}\right)\right)}{\Gamma(1-ns)\Gamma\left(\frac{n-1}{n}+s\right)\Gamma\left(1-\frac{n-1}{n}-s\right)}\left[\left(\frac{y}{2}\right)^{2n}\right]^{-s}ds,$$  \hspace{1cm} (32)

and use some known identities of the Gamma function to recast Eq. (32) into

$$\Re F_n(y) = -2^{-\frac{3}{n}-1}\pi^\frac{1}{2}n^{-1} \times$$

$$\frac{1}{2\pi i} \int \frac{\Gamma\left(s-\frac{1}{n}+\frac{2}{n}\right)\Gamma\left(s+\frac{1}{n}\right)\Gamma\left(s+\frac{2}{n}\right)\Gamma\left(s+n-\frac{1}{n}\right)}{\Gamma(1-s)\Gamma\left(\frac{n}{n}+s\right)\Gamma\left(\frac{n}{n}-s\right)\Gamma\left(1+n-s\right)}\left(\frac{y^{2n}}{2^{2n+2n^{2n}}}\right)^{-s}ds.$$ \hspace{1cm} (33)

The integral is now in the form of a Meijer G-function:

$$\Re F_n(y) = -2^{-\frac{3}{n}-1}\pi^\frac{1}{2}n^{-1} \times$$

$$\text{MeijerG}\left[\{\},\{\},\left\{\left\{0,\frac{n-2}{2n},\frac{1}{n},...,\frac{n-1}{n}\right\}\right\},\left\{\left\{-\frac{1}{n},0,\frac{n-1}{n},\frac{n-2}{n},...,\frac{1}{n}\right\}\right\},\frac{y^{2n}}{2^{2n+2n^{2n}}}\right]\equiv$$

$$-2^{-\frac{3}{n}-1}\pi^\frac{1}{2}n^{-1} \times G_{0,2(1+n)}^{n+1,0}\left(\frac{y^{2n}}{2^{2n+2n^{2n}}}\right)_{0,n-2,\frac{1}{n},...,\frac{n-1}{n},-\frac{1}{n},0,n-2,\frac{1}{n},...,\frac{1}{n}}.$$ \hspace{1cm} (34)

A similar calculation for the imaginary part yields

$$\Im F_n(y) = 2^{-\frac{3}{n}-1}\pi^\frac{1}{2}n^{-1} \times$$

$$\text{MeijerG}\left[\{\},\{\},\left\{\left\{0,\frac{n}{n},...,\frac{n-1}{n},\frac{n-1}{n}\right\}\right\},\left\{\left\{-\frac{1}{n},0,\frac{n-2}{2n},\frac{n-2}{n},...,\frac{1}{n}\right\}\right\},\frac{y^{2n}}{2^{2n+2n^{2n}}}\right]\equiv$$

$$2^{-\frac{3}{n}-1}\pi^\frac{1}{2}n^{-1} \times G_{0,2(1+n)}^{n+1,0}\left(\frac{y^{2n}}{2^{2n+2n^{2n}}}\right)_{0,\frac{1}{n},...,\frac{n-1}{n},-\frac{1}{n},0,\frac{n-2}{2n},\frac{n-2}{n},...,\frac{1}{n}}.$$ \hspace{1cm} (35)

In Eqs. (34) and (35), the first line refers to the Mathematica input necessary to perform the calculation, while the second line refers to the standard mathematical notation of Eqs. (14) and (15) \[28\].
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