LOCAL GROMOV-WITTEN INVARIANTS OF BLOWUPS OF FANO SURFACES

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Abstract. In this paper, using the degeneration formula we obtain a blowup formulae of local Gromov-Witten invariants of Fano surfaces. This formula makes it possible to compute the local Gromov-Witten invariants of non-toric Fano surfaces from toric Fano surface, such as del Pezzo surfaces. This formula also verified an expectation of Chiang-Klemm-Yau-Zaslow in the section 8.3 of [CKYZ].

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1. INTRODUCTION

Local del Pezzo surface used to play an important role in physics. Local del Pezzo surfaces are usually associated to phase transitions in the Kähler moduli space of various string, M-theory, and F-theory compactifications. More precisely, del Pezzo contractions in Calabi-Yau threefolds are related to quantum field theories in four and five dimensions [DKV] [KKV] [MS] via geometric engineering. Non-toric del Pezzo surfaces seem to be related to exotic physics in four, five and six dimensions such as nontrivial fixed points of the renormalization group [GMS] without lagrangian description and strongly interacting noncritical strings. There is also a relation between non-toric del Pezzo surfaces and string junctions in F-theory [KMV] [LMW]. Certain problems of physical interest such as counting of BPS states reduce to questions related to topological strings on local del Pezzo surfaces.

“Local mirror symmetry” mathematically refers to a specialization of mirror symmetry techniques to address the geometry of Fano surfaces within Calabi-Yau manifolds. The procedure produces certain “invariants” associated to surfaces.

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Let $S$ be a Fano surface and $K_S$ its canonical bundle. For $\beta \in H_2(S, \mathbb{Z})$, denote by $\overline{M}_{g,k}(S, \beta)$ the moduli space of $k$-pointed stable maps of degree $\beta$ to $S$. Then the following diagram
\[
\overline{M}_{g,1}(S, \beta) \xrightarrow{ev} S \\
\rho \downarrow \\
\overline{M}_{g,0}(S, \beta)
\]
defines the obstruction bundle $R^1\rho_*ev^*K_S$ whose fiber over a stable map $f : C \to S$ is given by $H^1(C, f^*K_S)$.

One can define the local Gromov-Witten invariants [CKYZ] of $K_S$ by
\[K^S_{g,\beta} = \int_{[\overline{M}_{g,0}(S, \beta)]_{vir}} e(R^1\rho_*ev^*K_S).\]

Yang-Zhou [YZ] generalized this definition of local Gromov-Witten invariants to the canonical line bundles of toric surfaces, not necessarily Fano. When $S$ is toric, one can use the localization technique to compute the local Gromov-Witten invariants, see [CKYZ, KZ, YZ]. When $S$ is non-toric Fano, the localization technique is no longer valid. Therefore few results on local Gromov-Witten invariants of non-toric Fano surface are known. In this paper, we will study how to compute the local Gromov-Witten invariants of some non-toric surfaces from that of some toric surfaces.

Denote by $Y_S = \mathbb{P}(K_S \oplus \mathcal{O})$ the projective bundle completion of the total space of the canonical bundle $K_S$. Then $Y_S$ has two canonical sections $S^+, S^-$ with normal bundle $N_{S^+|Y_S} \cong K_S$ and $N_{S^-|Y_S} \cong -K_S$ respectively. Then the normal bundle of a surface $S^+$ inside $Y_S$ is negative. If the image of a stable map lies in $S^+$, it is not able to deform it outside of $S^+$. This means if we denote also by $\beta$ the image of a class $\beta \in H_2(S, \mathbb{Z})$ under the inclusion map $S \hookrightarrow Y_S$ via the section $S^+$, then one has $\overline{M}_{g,0}(Y_S, \beta) = \overline{M}_{g,0}(S, \beta)$. By the constructions of the virtual fundamental cycles, we have
\[\overline{M}_{g,0}(Y_S, \beta)_{vir} = [\overline{M}_{g,0}(S : \beta)]_{vir} \cap e(R^1\rho_*ev^*K_S).\]

Denote the Gromov-Witten invariant of $Y_S$ of degree $\beta$ by
\[n^Y_{g,\beta} = \int_{[\overline{M}_{g,0}(Y_S, \beta)]_{vir}} 1.\]

Therefore, from (1) and (2), we have
\[K^S_{g,\beta} = n^Y_{g,\beta}.\]

Denote by $p : \tilde{S} \to S$ the natural projection of the blow-up of $S$ at a smooth point $p_0 \in S$. Let $\beta \in H_2(S, \mathbb{Z})$ and $p!(\beta) = PDp^*PD(\beta) \in H_2(\tilde{S}, \mathbb{Z})$. In [CKYZ], the authors computed the genus zero local Gromov-Witten invariants of $\mathbb{P}^2$ and the Hirzebruch surface $\mathbb{F}_1$ via the localization technique in the case of lower degrees. They observed that the genus zero
local Gromov-Witten invariants of $K_{P^2}$ of degree $\beta$ are equal to the genus zero local Gromov-Witten invariants of $K_{F_1}$ of degree $p^!(\beta)$. In this paper, we use the degeneration formula to study the change of local Gromov-Witten invariants under the blowup of the Fano surfaces and verify their observation and generalize it to any genus case. Our main theorem is

**Theorem 1.1.** Suppose that $S$ is a Fano surface and its blowup, $\tilde{S}$, of $S$ at a smooth point $p$ is also Fano. Let $\beta \in H_2(S, \mathbb{Z})$. Then for any genus $g$, we have

$$(5) \quad K_{g,\beta}^S = K_{g,p^!(\beta)}^{\tilde{S}},$$

where $p : \tilde{S} \to S$ is the natural projection of the blowup.

**Remark 1.2.** Theorem 1.1 confirmed the Chiang-Klemm-Yau-Zaslow’s expectation about the genus zero local Gromov-Witten invariants of $K_{P^2}$ and $K_{F_1}$ and generalized their expectation to any genus. In particular, our theorem also make it possible to compute the local Gromov-Witten invariants of nontoric del Pezzo surfaces $\tilde{P}^2_r$, $4 \leq r \leq 8$ from the local Gromov-Witten invariants of toric del Pezzo surfaces $\mathbb{P}^2_r$, $1 \leq r \leq 3$.

**Remark 1.3.** In [LLW], the authors consider the genus zero open Gromov-Witten invariants.

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## 2. Gromov-Witten invariants

We use [CK], [LR] as our general reference on moduli spaces of stable maps, Absolute/relative Gromov-Witten invariants and its degeneration formula. Let $X$ be a smooth complex projective manifold and $\beta \in H_2(X, \mathbb{Z})$. Let $\mathcal{M}_{g,n}(X, \beta)$ be the moduli space of $n$-pointed stable maps $f : (\Sigma; x_1, \cdots, x_n) \to X$ from a nodal curve $\Sigma$ with arithmetic genus $g(\Sigma) = g$ and degree $[f(\Sigma)] = \beta$. Let $e_i : \mathcal{M}_{g,n}(X, \beta) \to X$ be the evaluation maps $f \mapsto f(x_i)$. The Gromov-Witten invariant for classes $\alpha_i \in H^*(X)$, $1 \leq i \leq n$, is given by

$$\langle \alpha_1, \cdots, \alpha_n \rangle^X_{g,n,\beta} := \int_{[\mathcal{M}_{g,n}(X, \beta)]^{\text{vir}}} e_1^* \alpha_1 \cdots e_n^* \alpha_n.$$

The degeneration formula [LR], [IP], [Li] provides a rigorous formulation about the change of Gromov-Witten invariants under the semi-stable degeneration, or symplectic cutting. The formula related the absolute Gromov-Witten invariant of $X$ to the relative Gromov-Witten invariants of two smooth pairs.

Now we recall the relative invariants of a smooth pair $(X, Z)$ with $Z \to X$ a smooth divisor. Let $\beta \in H_2(X, \mathbb{Z})$ and $\mu = \{\mu_1, \cdots, \mu_{\ell(\mu)}\} \in \mathbb{N}^{\ell(\mu)}$ be a
partition of $|\mu| := \sum_{i=1}^{\ell(\mu)} \mu_i = \beta \cdot Z$. Let $\Gamma = (g, n, \beta, \mu)$ be a relative graph. For $A \in H^*(X)^{\otimes n}$ and $\delta_\mu \in H^*(Z)^{\otimes \ell(\mu)}$, the relative invariant of stable maps with topological type $\Gamma$ (i.e. with contact order $\mu_i$ in $Z$ at the $i$-th relative point) is
\[
\langle A | \delta_\mu \rangle_{\Gamma(X,Z)} := \int_{[\mathcal{M}_\Gamma(X,Z)]^{vir}} e^*_X A \cup e^*_Z \delta_\mu
\]
where $e_X : \mathcal{M}_\Gamma(X,Z) \rightarrow X^n$, $e_Z : \mathcal{M}_\Gamma(X,Z) \rightarrow Z^{\ell(\mu)}$ are evaluation maps on absolute marked points and relative marked points respectively.

If $\Gamma = \prod_\pi \Gamma^\pi$, the relative invariants (with disconnected domain curves)
\[
\langle A | \delta_\mu \rangle_{\Gamma} := \prod_\pi \langle A | \delta_\mu \rangle_{\Gamma^\pi}
\]
is defined to be the product of each connected component.

In the following, we shall discuss the degeneration formula which is the main tool employed in this paper.

Let $\pi : \chi \rightarrow D$ be a smooth 4-fold over a disk $D$ such that $\chi_t = \pi^{-1}(t) \cong X$ for $t \neq 0$ and $\chi_0$ is a union of two smooth 3-folds $X_1$ and $X_2$ intersecting transversely along a smooth surface $Z$. We write $\chi_0 = X_1 \cup_Z X_2$. Assume that $Z$ is simply connected.

Consider the natural maps
\[
i_t : X = \chi_t \rightarrow \chi, \quad i_0 : \chi_0 \rightarrow \chi,
\]
and the gluing map
\[
g = (j_1, j_2) : X_1 \coprod X_2 \rightarrow \chi_0.
\]
We have
\[
H_2(X) \xrightarrow{i_{t*}} H_2(\chi) \xleftarrow{i_{0*}} H_2(\chi_0) \xleftarrow{g_*} H_2(X_1) \oplus H_2(X_2),
\]
where $i_{0*}$ is an isomorphism since there exists a deformation retract from $\chi$ to $\chi_0$ (see [C]) and $g_*$ is surjective from Mayer-Vietoris sequence. For $\beta \in H_2(X)$, there exist $\beta_1 \in H_1(X_1)$ and $\beta_2 \in H_2(X_2)$ such that
\[
i_{t*}(\beta) = i_{0*}(j_{1*}(\beta_1) + j_{2*}(\beta_2)).
\]
For simplicity, we write $\beta = \beta_1 + \beta_2$ instead.

Since the family $\chi \rightarrow D$ comes from a trivial family, all cohomology classes $\alpha \in H^*(X)^{\otimes n}$ have global liftings and the restriction $\alpha(t)$ on $\chi_t$ is defined for all $t$.

For $\{\delta_1\}$ a basis of $H^*(Z)$ with $\{\delta^i\}$ its dual basis and a partition $\mu$, denote $\delta_\mu = \delta_1 \otimes \cdots \otimes \delta_{\ell(\mu)}$ and its dual $\delta^\mu = \delta^1 \otimes \cdots \otimes \delta^{\ell(\mu)}$. The degeneration formula expresses the absolute invariants of $X$ in terms of the relative invariants of the two smooth pairs $(X_1, Z)$ and $(X_2, Z)$:
\[
\langle \alpha \rangle_{g,n,\beta}^X = \sum_{\mu} \sum_{\eta \in \Omega_\beta} C_{\eta} j_1^*(\alpha(0)) | \delta_\mu \rangle_{\Gamma_1} \langle j_2^* \alpha(0) | \delta_\mu \rangle_{\Gamma_2} X_{g,n,\beta}.
\]
Here η = (Γ₁, Γ₂, I_ℓ(µ)) is an admissible triple which consists of (possibly disconnected) topological types

\[ Γ_i = \prod_{\pi=1}^{\lvert Γ_i \rvert} \Gamma_i^\pi \]

with the same contact order partition µ under the identification I_µ of relative marked points. The gluing Γ₁ + I_ℓ(µ) has type (g, n, β) and is connected. In particular, ℓ(µ) = 0 if and only if that one of the Γ_i is empty. The total genus g, total number of absolute marked points n, and the total degree β imply the splitting relations

\[ g = g_1 + g_2 + ℓ(µ) + 1 - \lvert Γ_1 \rvert - \lvert Γ_2 \rvert, \]

\[ n_1 + n_2 = n \quad \text{and} \quad β_1 + β_2 = β. \]

The constants C_η = m(µ)/|Aut η|, where m(µ) = \( \prod \mu_i \) and Aut η = {σ ∈ \( S_{\ell}(µ) \mid \eta^\sigma = η \}). We denote by Ω the equivalence class of all admissible triples, also by Ω_β and Ω_µ the subset with fixed degree β and fixed contact order µ respectively.

For the dimensions of the related moduli spaces in the degeneration formula, we have

**Lemma 2.1.** With the assumption as above,

\[ \dim_c \overline{M}_{Γ_1} + \dim_c \overline{M}_{Γ_2} = \dim_c \overline{M}_Γ + 2ℓ(µ). \]

### 3. Projective completion

In this section, we describe how to obtain \( Y_\hat{S} \) from \( Y_S \) by the degenerations. This makes it possible to find some relations between the local Gromov-Witten invariants of \( \hat{S} \) and \( S \).

Let \( S \) be a smooth surface and \( Y_\hat{S} = \mathbb{P}(K_S \oplus \mathcal{O}) \) the projective completion of its canonical bundle \( K_S \). Pick a smooth point \( p_0 \in S \) and blow it up, then we obtain the blowup \( \hat{S} \) of \( S \) at the point \( p_0 \) with the natural projection \( p : \hat{S} \to S \) and denote by \( E \) the exceptional divisor in \( \hat{S} \). Since \( Y_S \) is the bundle \( \mathbb{P}(K_S \oplus \mathcal{O}) \) over \( S \), one can pull this bundle back to \( \hat{S} \) using the projection \( p \). It is easy to see that the pullback bundle is the same thing as blowing up the fiber over \( p_0 \). Denote by \( \hat{Y}_S \) the blowup of \( Y_S \) along the fiber \( F_{p_0} \cong \mathbb{P}^1 \) over \( p_0 \), and the exceptional divisor in \( \hat{Y}_S \) is denoted by \( D_1 := E \times \mathbb{P}^1 = \mathbb{P}_{\mathbb{P}^1}(\mathcal{O} \oplus \mathcal{O}) \). In \( \hat{Y}_S \), take a section, \( σ \), corresponding to \( \mathcal{O} \to \mathcal{O} \oplus K_S \), of the exceptional divisor \( D_1 \) over \( E \) and blow it up. Denote by \( Z \) the blown-up manifold, then \( Z \) has a natural projection \( π \) to \( \hat{S} \) given by the composition of the blowup projection \( Z \to \hat{Y}_S \) and the bundle projection \( \hat{Y}_S \to \hat{S} \). It is easy to see that the fiber \( π^{-1}(E) \) has two normal crossing components: \( D_1 \cong \mathbb{P}_0 \) and \( D_2 \cong \mathbb{P}_1 \) intersecting along a section \( σ \) with the normal bundle \( N_{σ|F_0} \cong \mathcal{O} \) and \( N_{σ|F_1} \cong \mathcal{O}(-1) \) respectively.

Next, we consider the projective completion \( Y_\hat{S} \). Since the restriction \( K_\hat{S} \mid_E \) of the canonical bundle \( K_S \) to the exceptional divisor \( E \) in \( \hat{S} \) is isomorphic to \( \mathcal{O}(-1) \), so we can pick up a section, \( σ_1 \), of the restriction of
to know that the blown-up manifold is \( \tilde{Z} \). Then we blow up this section \( \sigma_1 \) up, and it is easy to know that the blown-up manifold is \( Z \).

Let \( \tilde{\mathbb{P}}_r^2 \) be the blowup of \( \mathbb{P}^2 \) at \( r \) points. Pick one more point \( p \) and blow it up, then we obtain \( \tilde{\mathbb{P}}_r^2 + 1 \) with the map \( p : \tilde{\mathbb{P}}_r^2 + 1 \to \tilde{\mathbb{P}}_r^2 \) and denote by \( E \) the exceptional divisor in \( \tilde{\mathbb{P}}_r^2 + 1 \). It is well-known that for \( 0 \leq r \leq 3 \), \( \tilde{\mathbb{P}}_r^2 \) is toric, but for \( 4 \leq r \leq 8 \), \( \tilde{\mathbb{P}}_r^2 \) is non-toric. In [CKYZ], the authors computed the genus zero local Gromov-Witten invariants of \( \tilde{\mathbb{P}}_r^2 \) with \( 0 \leq r \leq 3 \) of lower degree. As opposed to toric del Pezzo surfaces, one can not directly use localization with respect to a torus action because there is no torus action on a generic del Pezzo surface. Localization with respect to a torus action because there is no torus action on a generic del Pezzo surface \( \tilde{\mathbb{P}}_r^2, 4 \leq r \leq 8 \). Our Theorem \ref{thm:main} implies that for some degrees, we could compute any genus local Gromov-Witten invariants of non-toric surfaces \( \tilde{\mathbb{P}}_r^2 \) with \( 4 \leq r \leq 8 \) from the local Gromov-Witten invariants of \( \tilde{\mathbb{P}}_r^2 \) with \( 0 \leq r \leq 3 \).

4. Main theorem

In this section, we will study the change of local Gromov-Witten invariants under the blowup of surface \( S \). Throughout this section, we assume that \( S \) and its blowup surface all are Fano surfaces.

If we blow up \( Y_S \) along the fiber over the point \( p_0 \in S \), by the blowup formula of Gromov-Witten invariant, see Theorem 1.5 of [H1], for the genus zero invariants we have \( n_{Y_S} = n_{0,p_0(\beta)} \). In this section, we first want to generalize this result to the case of any genus.

**Lemma 4.1.** Suppose that \( S \) and its blowup \( \tilde{S} \) are Fano surfaces. Let \( \tilde{Y}_S \) be the blowup of \( Y_S \) along the fiber over \( p_0 \in S \). Then for any \( \beta \in H_2(S, \mathbb{Z}) \), we have

\[
n_{Y_S} = \langle |\emptyset| \rangle \tilde{Y}_S, D_1 \rangle_{g,p_0(\beta)}
\]

where \( D_1 = P_{\mathbb{P}^1}(O \oplus O) \cong \mathbb{P}^1 \times \mathbb{P}^1 \) is the exceptional divisor in \( \tilde{Y}_S \), \( p_0(\beta) = PDp^*PD(\beta) \) and \( p : \tilde{S} \to S \) is the natural projection of the blowup.

**Proof.** We degenerate \( Y_S \) along the fiber \( F_{p_0} \cong \mathbb{P}^1 \) over the blown-up point \( p_0 \). We obtain two smooth 3-folds

\[
X_1 = \tilde{Y}_S, \quad X_2 = P_{\mathbb{P}^1}(O \oplus O \oplus O) \cong \mathbb{P}^2 \times \mathbb{P}^1,
\]

with the common divisor \( D_1 \cong \mathbb{P}^1 \times \mathbb{P}^1 \).

Now we apply the degeneration formula to \( n_{Y_S} \), then we have

\[
n_{Y_S} = \sum \eta C_\mu(\delta_\mu)_{g_1, \beta_1}(\delta_\mu)_{g_2, \beta_2} \chi_{X_2, D_1}, \tag{8}
\]

where the summation runs over all admissible configurations \( \eta = (\Gamma_1, \Gamma_2, I_{\mu}) \) and \( C_\mu = m(\mu)|\text{Aut}(\mu)| \).

Next we consider the contribution to the Gromov-Witten invariants of each gluing component \( \eta = (\Gamma_1, \Gamma_2, I_{\mu}) \) where \( \Gamma_1 = (g_1, \beta_1, \mu) \) and \( \Gamma_2 = (g_2, \beta_2, \mu) \). According to our convention, the \( X_2 \)-component \( u^+ : C^+ \to X_2 \)
may have many connected components \( u_i^+ : C_i^+ \rightarrow X_2 \), \( i = 1, \ldots, l^+ \). Denote by \([u_i^+]\) the homology class in \( X_2 \) represented by \( u_i^+(C_i^+) \). Then we have

\[
\dim \mathcal{M}_{\Gamma_2} = \sum_{i=1}^{l^+} C_1[u_i^+] + \ell(\mu) - \sum \mu_i,
\]

where \( C_1 \) is the first Chern class of \( X_2 \).

Let \( V \) be a complex rank \( r \) vector bundle over a complex manifold \( M \), and \( \pi : \mathbb{P}(V) \rightarrow M \) be the corresponding projective bundle. Let \( \xi_V \) be the first Chern class of the tautological bundle in \( \mathbb{P}(V) \). A simple calculation shows

\[
C_1(\mathbb{P}(V)) = \pi^*C_1(M) + \pi^*C_1(V) - r\xi_V.
\]

Applying (9) to \( X_2 \), we obtain

\[
C_1(X_2) = \pi^*\mathcal{O}(2) - 3\xi.
\]

Therefore, we have

\[
\dim \mathcal{M}_{\Gamma_2} \geq \ell(\mu) + 2\sum \mu_i.
\]

From Lemma 2.1, we have

\[
\dim \mathcal{M}_{\Gamma_1} \leq \ell(\mu) - 2\sum \mu_i \leq -\sum \mu_i.
\]

This implies that for any nontrivial partition \( \mu \), we have

\[
\langle |\delta_\mu\rangle_{Y_S,D_1} \rangle_{\tilde{g},\beta} = 0.
\]

This means that the only summand with trivial partition \( \mu = \emptyset \) has the nonzero contribution to the right hand side of (8). Therefore we have

\[
n_{g,\beta}^{Y_S} = \langle |\emptyset\rangle_{\tilde{g},\beta} \rangle_{\tilde{g},p(\beta)}.
\]

This completes the proof of the lemma. \( \square \)

**Lemma 4.2.** For any \( \beta \in H_2(S,\mathbb{Z}) \), we have

\[
n_{g,p(\beta)}^{\tilde{Y}_S,D_1} = \langle |\emptyset\rangle_{\tilde{g},p(\beta)} \rangle_{\tilde{g},p(\beta)}.
\]

**Proof.** We degenerate \( \tilde{Y}_S \) along the exceptional divisor \( D_1 \cong \mathbb{P}^1 \times \mathbb{P}^1 \). Then we obtain two smooth 3-folds

\[
X_1 = \tilde{Y}_S, \quad X_2 = \mathbb{P}(N_{D_1} \oplus \mathcal{O}),
\]

where the normal bundle of the divisor \( D_1 \) in \( \tilde{Y}_S \) is \( N_{D_1} = \mathcal{O}(-1,-1) \).
Therefore, we have

\[ n_{g,p,l}(\beta) = \sum_{\eta} C_{\mu}(\eta) \tilde{\gamma}, \]

where the summation runs over all admissible configurations \( \eta = (\Gamma_1, \Gamma_2, I_\ell(\mu)) \) and \( C_\mu = m(\mu)|\text{Aut}(\mu)| \).

Now we consider the contribution to \( n_{g,p,l}(\beta) \) of each gluing component \( \eta = (\Gamma_1, \Gamma_2, I_\ell(\mu)) \) with \( \Gamma_i = (g_i, \beta_i, \mu), \ i = 1, 2 \). Assume that the \( X_2 \)-component \( u^+_1 : C^+ \rightarrow X_2 \) has many connected components \( u^+_i : C^+_i \rightarrow X_2, \ i = 1, \cdots, l^+ \). Denote by \([u^+_i]\) the homology class represented by \( u^+_i(C^+_i) \). Therefore we have

\[ \dim_{\mathbb{C}} \overline{\mathcal{M}}_{l^+} = \sum_{i=1}^{l^+} C_1[u^+_i] + \ell(\mu) - \sum_{i} \mu_i, \]

where \( C_1 \) denotes the first Chern class of \( X_2 \).

Note that \( X_2 = \mathbb{P}_{D_1}(N_{D_1} \oplus O) \) and \( D_1 = \mathbb{P}_2^1(O \oplus O) \). Denote by \( F_p \cong \mathbb{P}^1 \) the fiber of \( Y_2 \) at the point \( p_0 \). Applying (8) to \( X_2 \) and \( D_1 \), we obtain

\[ C_1(X_2) = \pi^*C_1(D_1) + \pi^*C_1(N_{D_1}) - 2\xi \]
\[ = \pi^*C_1(F_{p_0}) + \pi^*C_1(N_{F_{p_0}Y_2}) - 2\xi_1 + \pi^*C_1(N_{D_1}) - 2\xi, \]

where \( \xi_1 \) and \( \xi \) are the first Chern classes of the tautological bundles in \( \mathbb{P}(N_{F_{p_0}Y_2}) \) and \( \mathbb{P}(N_{D_1} \oplus O) \) respectively. Here we denote the Chern class and its pullback by the same symbol. It is well-known that the normal bundle to \( D_1 \) in \( Y_2 \) is just the tautological line bundle on \( D_1 \cong \mathbb{P}(N_{F_{p_0}Y_2}) \). Therefore \( C_1(N_{D_1}) = \xi_1 \). So we have

\[ C_1(X_2) = \pi^*C_1(F_{p_0}) - \xi_1 - 2\xi. \]

Note that \( X_2 \) is a projective bundle over \( D_1 \) with fiber \( \mathbb{P}^1 \). Let \( L \) be the class of a line in the fiber \( \mathbb{P}^1 \) and \( e \) be the class of a line in the fiber \( \mathbb{P}^1 \) in \( D_1 = \mathbb{P}(N_{F_{p_0}Y_2}) \). Denote by \([u^+_i]\) the homology class of the projection in \( F_{p_0} \) of the curve \( u^+_i \). Denote by \([u^+_i]^f \) the difference of \([u^+_i] \) and \([u^+_i]^f \), i. e., \([u^+_i]^f = [u^+_i] - [u^+_i]^f \). Then it is easy to know \([u^+_i]^f = aL + be \). Since \( \xi \cdot [u^+_i] = \sum \mu_j \), where the summation runs over ends of \( u^+_i \), and \( D_1 \cdot [u^+_i] = 0 \), so we have \( \xi \cdot [u^+_i]^f = a = \sum \mu_j \) and \( D_1 \cdot [u^+_i]^f = a - b = 0 \). Therefore, we have \( a = b = \sum \mu_j \). So we have \([u^+_i]^f = \sum \mu_j(L + e) \). Since \( C_1(F_{p_0}) + C_1(N_{F_{p_0}Y_2}) = C_1(F_{p_0}) \geq 0 \), we have

\[ \sum_{i=1}^{l^+} C_1[u^+_i] \geq 4 \sum \mu_j. \]

Therefore we have

\[ \dim_{\mathbb{C}} \overline{\mathcal{M}}_{l^+} \geq 3 \sum \mu_j + \ell(\mu). \]
From Lemma 2.1, we have
\[ \dim C_{\mathcal{M}_{\Gamma_1}} \leq \ell(\mu) - 3 \sum \mu_j. \]

Therefore, for any nontrivial partition \( \mu \), we have \( \dim C_{\mathcal{M}_{\Gamma_1}} < 0 \). This implies that the only nonzero summand in the right hand side of (10) must have the trivial partition \( \mu = \emptyset \). Therefore, we have
\[ n_{g,p}(\beta) = \langle | \emptyset \rangle Y_{g,p}(\beta) \].

This proves the lemma.

Summarizing Lemma 4.1 and Lemma 4.2, we have

**Theorem 4.3.**
\[ n_{g,\beta} = n_{g,p}(\beta). \]

Next, we want to compare the Gromov-Witten invariants \( n_{g,p}(\beta) \) of \( \tilde{Y}_S \) to the Gromov-Witten invariants of \( Z \). In fact, we have

**Theorem 4.4.**
\[ n_{g,p}(\beta) = n_{g,p}(\beta). \]

**Proof.** In \( \tilde{Y}_S \), take a section of the exceptional divisor \( D_1 \) over the old exceptional divisor \( E \), then \( \sigma \cong \mathbb{P}^1 \) and the normal bundle to \( \sigma \) in \( \tilde{Y}_S \) is \( N_{\sigma|\tilde{Y}_S} = \mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O} \). We degenerate \( \tilde{Y}_S \) along this section \( \sigma \), then we obtain two smooth 3-folds
\[ X_1 = Z, \quad X_2 = \mathbb{P}_\sigma(\mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O} \oplus \mathcal{O}), \]

with the Hirzebruch surface \( \mathbb{F}_1 = \mathbb{P}_\sigma(\mathcal{O}(-1) \oplus \mathcal{O}) \) as the common divisor.

Applying the degeneration formula to \( n_{g,p}(\beta) \), then we have
\[ \sum_{\eta} C_{\mathcal{M}_{\Gamma_1}} = \sum_{\eta} C_{\mathcal{M}_{\Gamma_2}} + \sum_{\mu} C_{\mathcal{M}_{\Gamma_2}} \]

where the summation runs over all admissible configurations \( \eta = (\Gamma_1, \Gamma_2, I_{\ell(\mu)}) \) and \( C_{\mathcal{M}_{\Gamma_1}} = m(\mu)|\text{Aut}(\mu)| \).

Similar to the proof of Lemma 4.1, we consider the contribution to \( n_{g,p}(\beta) \) of each gluing component \( \eta = (\Gamma_1, \Gamma_2, I_{\ell(\mu)}) \). Assume that the \( X_2 \)-component \( u^+ : C^+ \rightarrow X_2 \) has \( l^+ \) components \( u^+_i : C_{i}^+ \rightarrow X_2, \quad i = 1, \ldots , l^+ \). Denote by \( [u^+_i] \) the homology class in \( X_2 \) represented by \( u^+_i(C_{i}^+) \). Then we have
\[ \dim C_{\mathcal{M}_{\Gamma_2}} = \sum_{i=1}^{l^+} C_1[u^+_i] + \ell(\mu) - \sum \mu_i, \]

where \( C_1 \) is the first Chern class of \( X_2 \). From (9), it is easy to know
\[ C_1(X_2) = \pi^*C_1(\mathcal{O}_\sigma(1)) - 3\xi, \]
where $\xi$ is the first Chern class of the tautological line bundle over $X_2$. Therefore we have
\[
\sum_{i=1}^{\ell^+} C_1[u^+_i] \geq 3 \sum \mu_i.
\]
Therefore, we have
\[
\dim_\mathbb{C} \overline{\mathcal{M}}_{\Gamma_2} \geq \ell(\mu) + 2 \sum \mu_i.
\]
From Lemma 2.1 we have
\[
\dim_\mathbb{C} \overline{\mathcal{M}}_{\Gamma_1} \leq \ell(\mu) - 2 \sum \mu_i \leq - \sum \mu_i.
\]
This means that for any nontrivial partition $\mu$, $\dim_\mathbb{C} \overline{\mathcal{M}}_{\Gamma_1} < 0$. This implies that the only nonzero summand in the right hand side of (11) must be the trivial partition $\mu = \emptyset$. Therefore, we have
\[
n_{g,p}(\beta) = \langle |\emptyset| \rangle_{\mathcal{M}_{\Gamma_1}^1}.
\]
Now it remains to prove
\[
n_{Z,g,p}(\beta) = \langle |\emptyset| \rangle_{\mathcal{M}_{\Gamma_1}^1}.
\]
To prove this, we degenerate $Z$ along the exceptional divisor $F_1$. Then we obtain two smooth 3-folds
\[
X_1 = Z, \quad X_2 = \mathbb{P}_{F_1}(N_{F_1} \oplus O),
\]
intersecting along the exceptional divisor $F_1$ in $Z$ and the infinite section of the $\mathbb{P}^1$-bundle $X_2$.

Applying the degenerate formula to $n_{g,p}(\beta)$, we have
\[
n_{Z,g,p}(\beta) = \sum_{\eta} C_\mu(\delta_\mu)_{g_1,\beta_1} \langle |\emptyset\rangle_{g_2,\beta_2},
\]
where the summation runs over all admissible configurations $\eta = (\Gamma_1, \Gamma_2, T_{(\ell(\mu))})$ and $C_\mu = m(\mu)|\text{Aut}(\mu)|$.

Similar to the proof of Lemma 4.2, we consider the contribution to $n_{g,p}(\beta)$ of each gluing component $\eta = (\Gamma_1, \Gamma_2, T_{(\ell(\mu))})$ with $\Gamma_i = (g_i, \beta_i, \mu)$, $i = 1, 2$. Assume that the $X_2$-component $u^+_i : C^+ \to X_2$ has many connected components $u^+_i : C^+_i \to X_2$, $i = 1, \ldots, \ell^+$. Denote by $[u^+_i]$ the homology class represented by $u^+_i(C^+_i)$. Therefore we have
\[
\dim_\mathbb{C} \overline{\mathcal{M}}_{\Gamma_2} = \sum_{i=1}^{\ell^+} C_1[u^+_i] + \ell(\mu) - \sum \mu_i,
\]
where $C_1$ denotes the first Chern class of $X_2$.

Note that $X_2 = \mathbb{P}_{F_1}(N_{F_1} \oplus O)$ and $F_1 = \mathbb{P}_\sigma(O(-1) \oplus O)$. Applying (9) to $X_2$ and $F_1$, we obtain
\[
C_1(X_2) = \pi^*C_1(F_1) + \pi^*C_1(N_{F_1}) - 2\xi = \pi^*C_1(O_\sigma(1) - \xi_1 - 2\xi,
\]
where $\xi_1$ and $\xi$ are the first Chern classes of the tautological bundles in $\mathbb{P}_s(\mathcal{O}(-1) \oplus \mathcal{O})$ and $\mathbb{P}(N_{\mathcal{F}_1} \oplus \mathcal{O})$ respectively. Here we denote the Chern class and its pullback by the same symbol. The same calculation as in the proof of Lemma 4.2 shows that

$$\sum_{i=1}^{l^+} C_1[u_i^+] \geq 4 \sum \mu_i.$$ 

Therefore we have

$$\dim \overline{\mathcal{M}}_{\Gamma_2} \geq 3 \sum \mu_i + \ell(\mu).$$

From Lemma 2.1 we have

$$\dim \overline{\mathcal{M}}_{\Gamma_1} \leq \ell(\mu) - 3 \sum \mu_j.$$ 

Therefore, for any nontrivial partition $\mu$, we have $\dim \overline{\mathcal{M}}_{\Gamma_1} < 0$. This implies that the only nonzero summand in the right hand side of (12) must have the trivial partition $\mu = \emptyset$. Therefore we have

$$n_{g,p(\beta)}^Z = \langle |\emptyset\rangle \mathcal{F}_1 \mathcal{Z}, \mathcal{F}_0 \rangle_{g,p(\beta)}.$$

This proves the theorem.

Finally, we want to prove that $n_{g,p(\beta)}^Y = n_{g,p(\beta)}^Z$.

**Theorem 4.5.**

$$n_{g,p(\beta)}^Y = n_{g,p(\beta)}^Z$$

**Proof.** Take a section $\sigma_1 \cong \mathbb{P}^1$ of $Y^S_\mathcal{F} |_{\mathcal{E}} \cong \mathcal{F}_1$ such that $\sigma_1^2 = -1$. Then we degenerate $Y^S_\mathcal{F}$ along the section $\sigma_1$. Then we obtain two 3-folds, see Section 3

$$X_1 = Z, \quad X_2 = \mathbb{P}_{\sigma_1}(\mathcal{O}(-1) \oplus \mathcal{O}(-1) \oplus \mathcal{O}),$$

with the common divisor $F_0 = \mathbb{P}_{\sigma_1}(\mathcal{O}(-1) \oplus \mathcal{O}(-1))$.

Applying the degenerate formula to $n_{g,p(\beta)}^Y$, we have

$$n_{g,p(\beta)}^Y = \sum_\eta C_\mu \langle |\delta_{\mu}^1 \mathcal{F}_0 \rangle_{g_1,\beta_1}, |\delta_{\mu}^2 \mathcal{F}_0 \rangle_{g^2,\beta_2} \mathcal{X}_2 \mathcal{F}_0,$$

where the summation runs over all admissible configurations $\eta = (\Gamma_1, \Gamma_2, T_l(\mu))$ and $C_\mu = m(\mu)|Aut(\mu)|$.

Similar to the proof of Lemma 4.1, we need to prove that the summand with nonzero contribution in the right hand side of (13) must have trivial partition $\mu = \emptyset$. Using the same notation as before, we have

$$\dim \overline{\mathcal{M}}_{\Gamma_2} = \sum_{i=1}^{l^+} C_1[u_i^+] + \ell(\mu) - \sum \mu_i,$$

where $C_1$ denotes the first Chern class of $X_2$.

Note that $X_2 = \mathbb{P}_{\sigma_1}(\mathcal{O}(-1) \oplus \mathcal{O}(-1) \oplus \mathcal{O})$. From (9), it is easy to know

$$C_1(X_2) = \pi^* C_1(\mathcal{F}_1) + \pi^* C_1(\mathcal{O}(-1) \oplus \mathcal{O}(-1)) - 3\xi = -3\xi,$$
where \( \xi \) is the first Chern class of the tautological line bundle over \( X_2 \).

Therefore we have
\[
\dim \mathcal{M}_{\Gamma_2} = \ell(\mu) + 2 \sum \mu_j.
\]

From Lemma 2.1 we have
\[
\dim \mathcal{M}_{\Gamma_1} = \ell(\mu) - 2 \sum \mu_j \leq - \sum \mu_j.
\]

This means that for any nontrivial partition \( \mu \), \( \dim \mathcal{M}_{\Gamma_1} < 0 \). This implies that the only nonzero summand in the right hand of (13) must have the trivial partition \( \mu = \emptyset \). Therefore we have
\[
n_{g,p}(Y_\emptyset) = (|\emptyset|)_{\mathbb{P}_{\mathbb{P}_{\emptyset}}(\mathcal{Z})}.
\]

Now it remains to prove
(14)
\[
n_{g,p}(\beta) = (|\emptyset|)_{\mathbb{P}_{\mathbb{P}_{\emptyset}}(\mathcal{Z})}.
\]

To prove this, we degenerate \( Z \) along the exceptional divisor \( \mathbb{F}_0 \). Then we obtain two 3-folds
\[
X_1 = Z, \quad X_2 = \mathbb{P}_{\mathbb{F}_0}((N_{\mathbb{F}_0} \oplus \mathcal{O})).
\]

Similar to the proof of Theorem 4.4 we consider the contribution of each gluing component \( \eta = (\Gamma_1, \Gamma_2, I_{\ell(\mu)}) \). Using the same notation as before, we have
\[
\dim \mathcal{M}_{\Gamma_2} = \sum_{i=1}^{l^+} C_1[u_i^+] + \ell(\mu) - \sum \mu_j,
\]
where \( C_1 \) denotes the first Chern class of \( X_2 \).

Note that \( X_2 = \mathbb{P}_{\mathbb{F}_0}((N_{\mathbb{F}_0} \oplus \mathcal{O}) \oplus \mathcal{O}) \) and \( \mathbb{F}_0 = \mathbb{P}_{\sigma_1}(\mathcal{O}(-1) \oplus \mathcal{O}(-1)) \). Applying (10) to \( X_2 \) and \( \mathbb{F}_0 \), we have
\[
c_1(X_2) = \pi^* c_1(\mathbb{F}_0) + \pi^* c_1(N_{\mathbb{F}_0}) - 2 \xi
= \pi^* c_1(\mathcal{O}(-1) \oplus \mathcal{O}(-1)) - 2 \xi_1 + \pi^* c_1(N_{\mathbb{F}_0}) - 3 \xi
= - \xi_1 - 2 \xi,
\]
where \( \xi_1 \) and \( \xi \) are the first Chern classes of the tautological bundles in \( \mathbb{P}_{\sigma_1}(\mathcal{O}(-1) \oplus \mathcal{O}(-1)) \) and \( \mathbb{P}(N_{\mathbb{F}_0} \oplus \mathcal{O}) \) respectively. The same calculation as in the proof of Lemma 4.2 shows that
\[
\dim \mathcal{M}_{\Gamma_2} = \ell(\mu) + 3 \sum \mu_j.
\]

From Lemma 2.1 we have
\[
\dim \mathcal{M}_{\Gamma_1} = \ell(\mu) - 3 \sum \mu_j.
\]
As before, this implies (14). This completes the proof of the theorem.

Remark 4.6. From (4), Theorem 4.4 and Theorem 4.5, it is easy to know that Theorem 1.1 holds.
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