KMS STATES ON FINITE-GRAPH C*-ALGEBRAS

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ABSTRACT. We study KMS states on finite-graph C*-algebras with sinks and sources. We compare finite-graph C*-algebras with C*-algebras associated with complex dynamical systems of rational functions. We show that if the inverse temperature \( \beta \) is large, then the set of extreme \( \beta \)-KMS states is parametrized by the set of sinks of the graph. This means that the sinks of a graph correspond to the branched points of a rational function from the point of KMS states. Since we consider graphs with sinks and sources, left actions of the associated bimodules are not injective. Then the associated graph C*-algebras are realized as (relative) Cuntz-Pimsner algebras studied by Katsura. We need to generalize Laca-Neshveyev’s theorem of the construction of KMS states on Cuntz-Pimsner algebras to the case that left actions of bimodules are not injective.

KEYWORDS: KMS states, graph C*-algebras, C*-correspondences
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1. INTRODUCTION

KMS states on C*-algebras are originated from the equilibrium states in statistical physics. Olsen-Pederson [26] studied KMS states on Cuntz-algebra \( \mathcal{O}_n \) ([2]) of \( n \) generators with respect to the gauge action, and they proved that a \( \beta \)-KMS state exists if and only if \( \beta = \log n \) and the \( \beta \)-KMS state is unique. Evans [5] extended the result to certain quasi-free automorphisms. Enomoto-Fujii-Watatani [4] studied KMS states on Cuntz-Krieger algebras \( \mathcal{O}_A \) ([3]) associated with finite graphs with no sinks nor sources. If the 0-1 matrix \( A \) is irreducible and not a permutation, then a \( \beta \)-KMS state exists if and only if \( \beta = \log r(A) \) and the \( \beta \)-KMS state is unique, where \( r(A) \) is the spectral radius of \( A \). Exel-Laca [7] studied KMS states on Exel-Laca algebras and Toeplitz extensions. They introduced KMS states of finite type and infinite type, which are useful in our study. They showed that there occur phase transitions. Exel [6] considered KMS states on his C*-algebras by endomorphism with a transfer operator. Kumjian and Renault [14] studied KMS states on C*-algebras associated with expansive maps.

Pimsner [27] introduced a general construction of C*-algebras through Hilbert C*-bimodules or C*-correspondances. Many C*-algebras are known to be expressed as Cuntz-Pimsner C*-algebras.

The above results on KMS states are extended to KMS states on C*-algebras associated with subshifts in Matsumoto-Watatani-Yoshida [23] and Cuntz-Pimsner algebras associated with bimodules of finite basis in Pinzari-Watatani-Yonetani [28].
Laca-Neshevyev [23] gave a theorem of construction of KMS states for general Cuntz-Pimsner algebras. Using their theorem, we classified KMS states on C*-algebras associated with the complex dynamical systems on the Riemann sphere \( \hat{C} \) given by iteration of rational functions \( R \) and C*-algebras associated with self-similar sets in [10] with Izumi. In particular, we showed that there exists a phase transition at \( \beta = \log \text{deg} R \). If the inverse temperature \( \beta > \log \text{deg} R \), then the set of extreme \( \beta \)-KMS states is parametrized by the set of branched points.

On the other hand, Cuntz-Krieger algebras are generalized as graph C*-algebras associated with general graphs having sinks and sources, which are studied for example in Kumjian-Pask-Raeburn [21], Kumjian-Pask-Raeburn-Renault [22] and Fowler-Laca-Raeburn [8]. They consider relations between graphs and the associated graph C*-algebras. See [29] by I. Raeburn to know a total aspect of graph C*-algebras.

In this paper we study KMS states on finite-graph C*-algebras associated with graphs having sinks and sources. We show that if the inverse temperature \( \beta \) is large, then the set of extreme \( \beta \)-KMS states is parametrized by the set of sinks of the graph. We compare finite-graph C*-algebras with C*-algebras associated with complex dynamical systems of rational functions. Our result suggests that the sinks of a graph correspond to the branched points of a rational function from the point of KMS states.

Relative Cuntz-Pimsner algebras are a generalization of Cuntz-Pimsner algebras and are defined in [25] and studied in [9]. As in Katsura [19], the graph C*-algebras associated with graphs having sources and sinks can be constructed as relative Cuntz-Pimsner algebras of bimodules such that left actions are not injective. Hence we shall generalize Laca-Neshevyev’s theorem of KMS states to that of relative Cuntz-Pimsner algebras associated with general C*-correspondences with a countable basis. Our proof is more constructive than that of Laca-Neshevyev. We need to investigate the structure of cores of relative Cuntz-Pimsner algebras to study it.

The contents of the present paper is as follows. In section 2, we present the fundamental matters of C*-correspondences, relative Cuntz-Pimsner algebras and the structure of cores of relative Cuntz-Pimsner algebras. In section 3, we present properties of countable basis, the degree of C*-correspondences. We prove a theorem of construction of KMS states on relative Cuntz-Pimsner algebras which generalize the theorem of Laca-Neshevyev. In section 4, we present a classification of KMS states on finite-graph C*-algebras, and show that sinks correspond to KMS states if the inverse temperature is sufficiently large.

2. C*-CORRESPONDENCES AND THE STRUCTURE OF CORES

In this section, we present fundamental matters of C*-correspondences, the construction of associated C*-algebras, and investigate the structure of the cores of relative Cuntz-Pimsner algebras using some results of Katsura [19], [20].
Definition 2.1. Let $A$ be a $C^*$-algebra. A linear space $X$ is called a Hilbert $A$-module if the following conditions hold:

1. There exist an $A$-valued hermitian, positive definite inner product $\langle \cdot | \cdot \rangle_A$ and a right action of $A$ which is compatible with the $A$-inner product.
2. $X$ is complete with respect to the norm $\|x\| = \|(x|x)_A\|^{1/2}$.

If the linear span of $A$-inner product is dense in $A$, then $X$ is called full.

Let $A$ be a $C^*$-algebra and $X$ a Hilbert $A$-module. We denote by $\mathcal{L}(X)$ the set of linear operators on $X$ which are adjointable with respect to the $A$-valued inner product. For $x$ and $y \in X$, put $\theta_{x,y}z = x(y|z)_A$ for $z \in X$. We denote by $K(X)$ the norm closure of the linear span of $\{ \theta_{x,y} | x, y \in X \}$ in $\mathcal{L}(X)$. If there exists an $^*$-homomorphism $\phi$ from $A$ to $\mathcal{L}(X)$, then we call the pair $(X, \phi)$ (or simply $X$) a $C^*$-correspondence over $A$. We assume neither that $X$ is full, that $\phi$ is non-degenerate nor that $\phi$ is isometric. Let $J_X = \phi^{-1}(K(X)) \cap (\ker \phi)^\perp$, and $J$ be a closed two sided ideal of $A$ contained in $J_X$.

A representation $\pi$ of a $C^*$-correspondence $(X, \phi)$ on a Hilbert space $\mathcal{H}$ consists of representations $\pi_A$ and $\pi_X$ of $A$ and $X$ i.e. $\pi_A$ is a $^*$-homomorphism from $A$ to $B(\mathcal{H})$ and $\pi_X$ is a linear map from $X$ to $B(\mathcal{H})$ satisfying

$$\pi_X(x)^*\pi_X(y) = \pi_A((x|y)_A), \quad \pi_X(x)\pi_A(a) = \pi_X(xa), \quad \pi_A(a)\pi_X(x) = \pi_X(\phi(a)x),$$

for $x, y \in X$ and $a \in A$. When $\pi_A$ is injective, $\pi_X$ is isometric.

For a representation $\pi = (\pi_A, \pi_X)$ of $(X, \phi)$, there corresponds a representation $\pi_K$ of $K(X)$ satisfying $\pi_K(\theta_{x,y}) = \pi_X(x)\pi_X(y)^*$. The representation $\pi_X^{(n)}$ of $X^{\otimes n}$, $\pi_K^{(n)}$ of $K(X^{\otimes n})$ are also defined naturally. We use the notation $K(X^{\otimes 0}) = A$ and $\pi_K^0 = \pi_A$ for convenience.

Definition 2.2. (Fowler-Muhly-Raeburn [9], Katsura [20]) Let $J$ be a closed two sided ideal of $A$ contained in $J_X$. A representation $\pi = (\pi_A, \pi_X)$ of a $C^*$-correspondence $(X, \phi)$ is said to be $J$-covariant if

$$\pi_A(a) = \pi_K(\phi(a)) \quad \text{for any } a \in J. \quad (1)$$

Let $\pi = (\pi_A, \pi_X)$ be the representation of $(X, \phi)$ which is universal for all $J$-covariant representations. The relative Cuntz-Pimsner algebra $\mathcal{O}_X(J) = C^*(\pi)$ is the $C^*$-algebra generated by $\pi_A(A)$ and $\pi_X(X)$ for the universal representation $\pi$. We note that $\pi_A$ of the universal representation $\pi$ is known to be injective (Katsura [19] Proposition 4.11).

Lemma 2.3. (T.Katsura [19] Proposition 3.3) Let $\pi = (\pi_A, \pi_X)$ be a representation of $(X, \phi)$. Assume that $\pi$ is $J$-covariant and $\pi_A$ is injective. Take $a$ in $A$. If $\pi_A(a)$ is in $\pi_K(K(X))$, then $a$ is in $J_X$ and $\pi_A(a) = \pi_K(\phi(a))$.

Lemma 2.4. Let $\pi = (\pi_A, \pi_X)$ be a representation of $(X, \phi)$. Assume that $\pi$ is $J$-covariant and $\pi_A$ is injective. Then for $a \in A$, $\pi_A(a)$ is in $\pi_K(K(X))$ if and only if $a$ is in $J$. 

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Proof. For $a \in A$, assume that $\pi_A(a) \in \pi_K(K(X))$. By Lemma 2.3, we have $a \in J_X$ and $\pi_A(a) = \pi_K(\phi(a))$.

By Katsura [20] Corollary 11.4, if $(\pi_A, \pi_X)$ is a representation of $(X, \phi)$ satisfying the equation (1), we have

$$\{a \in A \mid \phi(a) \in K(X), \pi_A(a) = \pi_K(\phi(a))\} = J.$$ 

This shows the conclusion. \hfill \qed

We define subalgebras $B_n \ n \geq 1$ and $B_0$ by

$$B_n = \pi_K^{(n)}(K(X^{\otimes n})), \quad B_0 = \pi_A(A).$$

These are C*-subalgebras of $O_X(J)$. We put

$$\mathcal{F}^{(n)} = B_0 + B_1 + \cdots + B_n.$$ 

For integers $n, i$ we introduce the notation $(n, i)$ by

$$(n, i) = \begin{cases} n - i & n \geq 1, \ i \geq 1 \\ n - 1 & n \geq 1, \ i = 0 \\ 0 & n = 0, \ i = 0. \end{cases}$$

Let $k \in K(X^{\otimes i}) \ i \geq 1$. For $\xi_1 \in X^{\otimes i}, \ \xi_2 \in X^{\otimes n-i}$, we define $k \otimes id_{(n, i)}$ by

$$(k \otimes id_{(n, i)})(\xi_1 \otimes \xi_2) = k\xi_1 \otimes \xi_2.$$ 

Then $k \otimes id_{(n, i)}$ is an element of $L(X^{\otimes n})$. The notation $a \otimes id_{(m, 0)}$ means $\phi(a) \otimes id_{n-1}$. When $n = 0$, $a \otimes id_{(0, 0)}$ is a left multiplication representation of $A$ on a Hilbert $A$-module $A$.

**Lemma 2.5.** For each $m$, $B_m$ is an ideal in $\mathcal{F}^{(m)}$, and $\mathcal{F}^{(m)}$ is a C*-subalgebra.

Proof. We assume $1 \leq m \leq n$, $k \in K(X^{\otimes m}), \ k' \in K(X^{\otimes n})$. Since $k \otimes id_{(n, m)} \in L(X^{\otimes n})$, we have $(k \otimes id_{(n, m)})k' \in K(X^{\otimes n})$. By Katsura [19] Lemma 5.4, we have

$$\pi_K^{(m)}(k)\pi_K^{(n)}(k') = \pi_K^{(n)}((k \otimes id_{(n, m)})k').$$

This shows that $B_n$ is an ideal in $\mathcal{F}^{(n)}$. We can check the case $m = 0$ separately. \hfill \qed

We need to investigate $\mathcal{F}^{(n-1)} \cap B_n$ for a proof of the theorem of constructing KMS states.

Note that $(X^{\otimes n}, J)$ is a right $A$-submodule of $X_A$. By considering the embedding of ”rank-one” operators, we may have an inclusion $K(X^{\otimes n}) \subset K(X^{\otimes n})$, and we have $\theta_{\xi,j, \eta, j'} \in K(X^{\otimes n}, J)$ for $j, j' \in J$.

**Lemma 2.6.** An element $k \in K(X^{\otimes n})$ is in $K(X^{\otimes n}, J)$ if and only if

$$(\xi | k\eta)_A \in J \quad \forall \xi, \eta \in X^{\otimes n}.$$
Proof. We refer Fowler-Muhly-Raeburn [9] Lemma 1.6 and Katsura [20] for quotient modules $X_J$. The notation $[a]_J \in A/J$ for an element $a$ of a $C^*$-algebra $A$ means the quotient image of $a \in A$ by $J$.

Since $T \in \mathcal{L}(X)$ leaves $X_J$ invariant, we can consider an operator $[T]_J \in X/XJ = X_J$. The map $k \in K(X^{\otimes n}) \rightarrow [k]_J$ is an onto map from $K(X^{\otimes n})$ to $K(X^{\otimes n}_J)$, and its kernel is $k$'s such that $k \in K(X^{\otimes n})$ (Lemma 1.6). Then $k \in K(X^{\otimes n})$ is contained in $K(X^{\otimes n}_J)$ if and only if $[k]_J = 0$. Moreover we have

\[(\xi|k\eta)_A \in J \quad \text{if and only if} \quad [(\xi|k\eta)_A]_J = 0\]

\[= \lim_{n \to \infty} x_\Lambda \in A/J\]

If it holds for each $\xi$, $\eta$, then we have $[k]_J = 0$, and this means $k \in K(X^{\otimes n}_J)$. □

We put $B'_n = \pi_K^{(n)}((X^{\otimes n}_J))$, $(n \geq 1)$ and $B'_0 = \pi_A(J)$. For the case $J = J_X$, the following Lemma is presented in Katsura [19]. It also holds for the case $J \subset J_X$.

The following lemmas are T.Katsura [19] Lemma 5.10 and T. Katsura [19] Proposition 5.11 for the case $J = J_X$. The proof for general cases is the same as the case $J = J_X$.

Lemma 2.7. Let $k \in K(X^{\otimes n+1})$. Then for an approximate unit $\{u_\Lambda\}_{\Lambda \in \Lambda}$ in $K(X^{\otimes n})$, we have that $k = \lim_{n \to \infty} (u_\Lambda \otimes id_1)k$.

Lemma 2.8. We have that $\mathcal{F}^{(n)} \cap B_{n+1} \subset B_n$.

Proposition 2.9. We have that $B_n \cap B_{n+1} = B'_n$ and $\mathcal{F}^{(n)} \cap B_{n+1} = B'_n$.

Proof. First we show that $B_n \cap B_{n+1} = B'_n$. This is Katsura [19] Proposition 5.9 for $J = J_X$. Let $n = 0$. Then we have

\[\pi_A(A) \cap B_1 = \pi_A(A) \cap \pi_K(K(X))\]

By Lemma 2.4, we have $\pi_A(A) \cap B_1 = \pi_A(J)$. Moreover the proposition holds for $n = 0$ because $B'_0 = \pi_A(J)$. We may assume $n \geq 1$. Let $a, b \in J$, $\xi, \eta \in X^{\otimes n}$. Then we have $\pi_A(a) \in B_1 = \pi_K(K(X))$, and

\[\pi_K^{(n)}(\theta_{\xi a, \eta b}) = \pi_X^{(n)}(\xi a)\pi_X^{(n)}(\eta b)^* = \pi_X^{(n)}(\xi)\pi_A(a)\pi_A(b)^* \pi_X^{(n)}(\eta)^*\]

Since $\pi_A(a)\pi_A(b)^* \in B_1$, the left hand side is contained in $B_{n+1}$. Then we have

\[B'_n \subset B_n \cap B_{n+1}\]

Let $x \in B_n \cap B_{n+1}$. There exists $k \in K(X^{\otimes n})$ such that $\pi_K^{(n)}(k) = x$. For $\xi$, $\eta \in X^{\otimes n}$, we have

\[\pi_A([(\xi|k\eta)_A] = \pi_X^{(n)}(\xi)^* \pi_K^{(n)}(k) \pi_X^{(n)}(\eta) = \pi_X^{(n)}(\xi)^* x \pi_X^{(n)}(\eta)\]
By \( x \in B_{n+1} \), the last expression is contained in \( B_1 \). Since for \( \xi, \eta \in X^\otimes n \), \( \pi_A((\xi|k\eta)_A) \in B_1 \), we have \( (\xi|k\eta)_A \in J \) for \( \xi, \eta \in X^\otimes n \) by Lemma 2.4. Then we have \( k \in K(X^\otimes n, J) \), and we have \( x = \pi^{(n)}_K(k) \in B_n' \).

Lastly, we shall show that \( F(n) \cap B_{n+1} = B_n' \). By Lemma 2.8 \( F(n) \cap B_{n+1} \subset B_n \).

We have
\[
F(n) \cap B_{n+1} = (F(n) \cap B_{n+1}) \cap B_n = (F(n) \cap B_n) \cap B_{n+1}
\]
\[
= B_n \cap B_{n+1} = B_n'.
\]
This completes the proof. \( \square \)

3. KMS states on relative Cuntz-Pimsner algebras

In this section, we generalize a theorem of the construction of KMS states of Cuntz-Pimsner algebras in Laca-Neshevyev [23] to relative Cuntz-Pimsner algebras.

Let \( A \) be a \( \sigma \)-unital \( \text{C}^* \)-algebra and \( X \) be a countably generated Hilbert \( A \)-module.

**Definition 3.1.** A sequence \( \{u_i\}_{i=1}^\infty \) of a Hilbert (right) \( \text{C}^* \)-module \( X \) over \( A \) is called a countable basis (or normalized tight frame) of \( X \) if
\[
x = \sum_{i=1}^\infty u_i(u_i|x)_A
\]
for each \( x \in X \), where the right hand side converges in norm.

As in Remark after [13] Proposition 1.2 ,

For \( a, b \in K(X), x \in X \) with \( 0 \leq a \leq b \leq I \),
\[
\|x - bx\|^2 \leq \|x\| \|x - ax\|. \tag{3}
\]
This inequality implies that the right hand side of (2) converges unconditionally in the following sense: For every \( \varepsilon > 0 \), there exists a finite subset \( F_0 \) of \( \mathbb{N} \) such that for every finite subset \( F \) of \( \mathbb{N} \) with \( F_0 \subset F \) we have
\[
\|x - \sum_{i \in F} u_i(u_i|x)_A\| < \varepsilon.
\]
Since \( \sum_{i \in F} \theta u_i u_i \leq I \) for each finite subset \( F \in \mathbb{N} \), it is sufficient to prove (2) for each \( x \) in some norm dense subset of \( X \). We often write it as \( x = \sum_{i \in \mathbb{N}} u_i(u_i|x)_A \) to express unconditionally convergence. More generally, for any countable set \( \Omega \), the notation
\[
x = \sum_{i \in \Omega} u_i(u_i|x)_A. \tag{4}
\]
makes sense as unconditional convergence.

We can show the following Lemma:

**Lemma 3.2.** Let \( A \) be a \( \text{C}^* \)-algebra, \( Y \) a \( \text{C}^* \)-correspondence over \( A \) and \( X \) a Hilbert \( A \)-module. Let \( \{u_i\}_{i \in \Omega_1} \) be a countable basis of \( X \) and \( \{v_j\}_{j \in \Omega_2} \) a countable basis of \( Y \). Then \( \{u_i \otimes v_j\}_{(i,j) \in \Omega_1 \times \Omega_2} \) is a countable basis of the inner tensor product module \( X \otimes_A Y \) of \( X \) and \( Y \).
Proof. Let \( \varepsilon > 0 \). We fix a nonzero \( x \otimes y \in X \otimes_A Y \). Let \( \delta = \varepsilon^2 / \| x \otimes y \| \). be a positive number. We take a finite subset \( F \) of \( \Omega_1 \) such that
\[
\| \sum_{i \in F} u_i (u_i | x)_A - x \| < \frac{\delta}{2\| y \|}.
\]
Put \( s \) be the cardinality of \( F \). For each \( i \ (i = 1, \ldots, s) \) we take a finite subset \( G_i \subset \Omega_2 \) such that if \( G' \) is a finite subset containing \( G_i \) then it holds that
\[
\| \sum_{j \in G'} v_j (v_j | (u_i | x)_A y) \| - (u_i | x)_A y \| < \frac{\delta}{2s\| u_i \|}.
\]
Let \( G \) be a finite subset containing \( \bigcup_{i=1}^{s} G_i \). Then we have
\[
\| x \otimes y - \sum_{(i,j) \in F \times G} u_i \otimes v_j (u_i \otimes v_j | x \otimes y)_A \|
= \| x \otimes y - \sum_{i \in F} \sum_{j \in G} u_i \otimes v_j (v_j | (u_i | x)_A y) \|
\leq \| x \otimes y - \sum_{i \in F} u_i (u_i | x)_A \otimes y \| + \sum_{i \in F} \| u_i \| \| \sum_{j \in G} v_j (v_j | (u_i | x)_A y) \| - (u_i | x)_A y \|
< \delta.
\]
Using (\[\text{[3]}\]), for each finite subset \( H \) of \( \Omega_1 \times \Omega_2 \) such that \( H \supset F \times G \) we have that
\[
\| x \otimes y - \sum_{(i,j) \in H} u_i \otimes v_j (u_i \otimes v_j | x \otimes y)_A \| < \varepsilon.
\]
Hence \( x \otimes y = \sum_{(i,j) \in \Omega_1 \times \Omega_2} u_i \otimes v_j (u_i \otimes v_j | x \otimes y)_A \). If \( z = \sum_{p \text{ finite}} x_p \otimes y_p \), then
\[
\sum_{(i,j) \in \Omega_1 \times \Omega_2} u_i \otimes v_j (u_j \otimes v_j | z)_A = z.
\]
Since the subset of elements of the form \( \sum_{p \text{ finite}} x_p \otimes y_p \) is dense in \( X \otimes_A Y \), \( \{ u_i \otimes v_j \}_{\Omega_1 \times \Omega_2} \) constitute a basis of \( X \otimes_A Y \). \( \Box \)

We fix a C*-correspondence \( X \) over a C*-algebra \( A \), and a countable basis \( \{ u_i \}_{i=1}^{\infty} \) of \( X \). Let \( J \) be a closed two sided ideal of \( A \) which is contained in \( J_\infty \). \( O_X(J) \) denotes the relative Cuntz-Pimsner algebra constructed from \( X \) and \( J \) in section 2.

Lemma 3.3. Let \( \tau \) be a tracial state on \( A \). Then the possibly infinite positive number
\[
\sup_n \sum_{i=1}^{n} \tau((u_i | u_i)_A) \text{ does not depend on the choice of a countable basis } \{ u_i \}_{i=1}^{\infty}.
\]
Proof. Let \( \{ v_i \}_{j=1}^{\infty} \) be another countable basis of \( X \). Since \( u_i = \lim_{m \to \infty} \sum_{j=1}^{m} v_j (v_j | u_i)_A \), we have

\[
\sup_{n} \sum_{i=1}^{n} \tau((u_i | u_i)_A) = \sup_{n} \lim_{m \to \infty} \sum_{i=1}^{m} \sum_{j=1}^{n} \tau((v_j (v_j | u_i)_A | u_i)_A) \\
= \sup_{n} \sup_{m} \sum_{i=1}^{m} \sum_{j=1}^{n} \tau((v_j (v_j | u_i)_A | u_i)_A) = \sup_{n} \sum_{i=1}^{n} \sum_{j=1}^{m} \tau((v_j (v_j | u_i)_A | u_i)_A) \\
= \sup_{m} \lim_{n \to \infty} \sum_{i=1}^{m} \sum_{j=1}^{n} \tau((v_j (v_j | u_i)_A | u_i)_A) = \sup_{m} \lim_{n \to \infty} \sum_{j=1}^{m} \sum_{i=1}^{n} \tau((v_j (v_j | u_i)_A | u_i)_A) \\
= \sup_{m} \sum_{j=1}^{m} \tau((v_j | v_j)_A).
\]

Therefore we may put \( d_r = \sup_{n} \sum_{i=1}^{n} \tau((u_i | u_i)_A) \leq \infty \).

We denote by \( \mathcal{T}(A) \) the set of tracial states on \( A \).

**Definition 3.4.** The degree \( d(X) \) of a \( C^* \)-correspondence \( X \) is defined by \( d(X) := \sup\{ d_r | \tau \in \mathcal{T}(A) \} \). We say that \( X \) is of finite-degree type if \( d(X) < \infty \).

**Lemma 3.5.** If \( A \) is commutative, then \( d(X) = \sup_{n} \| \sum_{i=1}^{n} (u_i | u_i)_A \| \) for any countable basis \( \{ u_i \}_{i=1}^{\infty} \).

**Proof.** We assume that \( A \) is commutative. By the method similar as in the proof of Lemma \( 3.3 \) we can show that \( \sup_{n} \| \sum_{i=1}^{n} (u_i | u_i)_A \| \) does not depend on the choice of a countable basis \( \{ u_i \}_{i=1}^{\infty} \).

Since

\[
\sum_{i=1}^{n} \tau((u_i | u_i)_A) = \tau(\sum_{i=1}^{n} (u_i | u_i)_A) \leq \| \sum_{i=1}^{n} (u_i | u_i)_A \|,
\]

we have \( d(X) \leq \sup_{n} \| \sum_{i=1}^{n} (u_i | u_i)_A \| \).

We fix a countable basis \( \{ u_i \}_{i=1}^{\infty} \). For every \( \varepsilon > 0 \), there exists an \( n_0 \) such that for each \( n \geq n_0 \),

\[
\| \sum_{i=1}^{n} (u_i | u_i)_A \| > \sup_{n} \| \sum_{i=1}^{n} (u_i | u_i)_A \| - \varepsilon.
\]

There exists a tracial state \( \tau \) such that

\[
\tau(\sum_{i=1}^{n} (u_i | u_i)_A) > \| \sum_{i=1}^{n} (u_i | u_i)_A \| - \varepsilon.
\]

Thus we have \( d(X) \geq \sup_{n} \| \sum_{i=1}^{n} (u_i | u_i)_A \| \). \( \square \)

Let \( R \) be a rational function of degree \( N \) and \( A \) a commutative \( C^* \)-algebra \( C(\hat{\mathbb{C}}) \). Consider a \( C^* \)-correspondence \( X \) over \( A \) associated with the complex dynamical
system given by $R$ on $\hat{\mathcal{C}}$. As described in [11], we can choose a concrete countable basis such that we can compute explicitly as

$$\sum_{i=1}^{n} (u_i|u_i)_A(y) = \# \{ R^{-1}(y) \}. $$

This equation is also shown in [16] for any basis. Thus we have

$$\sup_n \| \sum_{i=1}^{\infty} (u_i|u_i)_A \| = N. $$

Therefore the degree of $X$ coincides with the degree of $R$. Similar formulas hold for the case of self-similar maps.

Let $Y$ be a $C^*$-correspondence over a $C^*$-algebra $B$ of finite-degree type with $d(Y) = N$. Let $\beta > \log N$. For a tracial state $\tau$ on $B$, we can define a bounded linear functional $\hat{\tau}_1$ on $L(Y)$ by

$$\hat{\tau}_1(k) = e^{-\beta} \sum_{i=1}^{\infty} \tau((u_i|T u_i)_A),$$

for $T \in L(Y)$.

We need an elementary fact as follows:

**Lemma 3.6.** $\hat{\tau}_1$ is a trace and does not depend on the choice of a basis $\{u_i\}_{i=1}^{\infty}$.

**Proof.** Let $\{v_j\}_{j=1}^{\infty}$ be another basis of $X$, and $T$ be a positive element in $L(Y)$. As in the proof of Lemma 3.3, we have

$$\sup_n \sum_{i=1}^{n} e^{-\beta} \tau((T^{1/2}u_i|T^{1/2}u_i)_A) = \sup_m \sum_{j=1}^{m} \sum_{i=1}^{n} e^{-\beta} \tau((v_j|T^{1/2}u_i(T^{1/2}u_i|v_j)_A)_A)$$

$$= \sup_m \sum_{j=1}^{m} \sum_{i=1}^{n} e^{-\beta} \tau((v_j|T^{1/2}u_i(T^{1/2}u_i|v_j)_A)_A)$$

$$= \sup_m \sum_{j=1}^{m} e^{-\beta} \tau((T^{1/2}v_j|T^{1/2}v_j)_A).$$

This shows that the definition of $\hat{\tau}_1$ does not depend on the choice of basis. Let $U$ be a unitary in $L(Y)$. Then $\{U u_i\}_{i=1}^{\infty}$ is also a basis of $Y$. For $T \in L(Y)$, we have

$$\sum_{i=1}^{\infty} e^{-\beta} \tau(U u_i|T U u_i)_A = \sum_{i=1}^{\infty} e^{-\beta} \tau(u_i|T u_i)_A.$$ 

Then $\hat{\tau}_1(U^* TU) = \hat{\tau}_1(T)$, and it follows that $\hat{\tau}_1$ is a trace. \qed
Let $I$ be an closed two sided ideal of a C*-algebra $B$, and $\varphi$ be a state on $I$. Consider the GNS representation $(\pi_\varphi, H_\varphi, \xi_\varphi)$. Let $\pi : A \to B(H_\varphi)$ be the extension of $\pi_\varphi$ to $A$. Recall that the canonical extension $\varphi$ of $\varphi$ to $B$ is defined as $\varphi(a) = (\pi(a)\xi_\varphi, \xi_\varphi)$. Then $\varphi(a) = \lim_i \varphi(aa_i)$, for any approximate unit $\{e_i\}_i$ in $I$ as in [1]. Prop. 6.4.16.

Let $\tau_1$ be the restriction of $\hat{\tau}_1$ on $\mathcal{K}(Y)$. We note that the C*-algebra $\mathcal{K}(Y)$ is a closed two sided ideal of $\mathcal{L}(Y)$.

**Lemma 3.7.** The canonical extension $\mathcal{T}_1$ of $\tau_1$ to $\mathcal{L}(Y)$ is given by $\hat{\tau}_1$.

**Proof.** We note that $\hat{\tau}_1(\theta_{x,y}) = e^{-\beta}(y|x)_A$. If $\{u_i\}_{i=1}^\infty$ is a basis of $Y$, then $\{\theta_{u_i,u_i}\}_{i=1}^\infty$ is an approximate unit in $\mathcal{K}(Y)$. Therefore for $T \in \mathcal{L}(Y)$, we have

$$\mathcal{T}_1(T) = \lim_{m\to\infty} \sum_{j=1}^m \tau_1(T\theta_{u_j,u_j}) = \lim_{m\to\infty} \sum_{j=1}^m \tau_1(\theta_{u_j,u_j})$$

$$= \lim_{m\to\infty} \sum_{j=1}^m e^{-\beta}(u_j|Tu_j)_A = \hat{\tau}_1(T)$$

$\square$

Let $A$ be a C*-algebra and $X$ be a C*-correspondence over $A$ with a countable basis $\{u_i\}_{i=1}^\infty$. Since we use tensor products of correspondences and their bases frequently, we use the notations of multi index. Namely, for $p = (i_1, i_2, \ldots, i_n) \in \mathbb{N}^n$, we write $u_p = u_{i_1} \otimes u_{i_2} \otimes \cdots \otimes u_{i_n}$.

We assume that $X$ is of finite-degree type. We can also define a bounded tracial linear functional $\hat{\tau}^{(n)}$ on $\mathcal{L}(X^{\otimes n})$ and its restriction $\tau^{(n)}$ to $\mathcal{K}(X^{\otimes n})$ using the Hilbert $A$-module $X^{\otimes n}$ and its basis $\{u_p\}_{p \in \mathbb{N}^n}$ as

$$\hat{\tau}^{(n)}(T) = e^{-n\beta} \sum_{p \in \mathbb{N}^n} \tau((u_p|Tu_p)_A)$$

for $T \in \mathcal{L}(X^{\otimes n})$.

**Definition 3.8.** Let $J$ be a closed two-sided ideal of $A$ such that $J \subset J_X$, and $\beta$ be a positive real number. A tracial state $\tau$ on $A$ satisfies $\beta$-condition if it satisfies the following two conditions:

(\beta_1) $\sum_{i=1}^\infty \tau((u_i|\phi(a)u_i)_A) = e^{\beta} \tau(a) \quad \forall a \in J$,

(\beta_2) $\sum_{i=1}^\infty \tau((u_i|\phi(a)u_i)_A) \leq e^{\beta} \tau(a) \quad \forall a \in A^+$.

Since $B_n$ is isomorphic to $\mathcal{K}(X^{\otimes n})$ by $\pi_K^{(n)}$ for each $n$, we can define a bounded linear tracial functional $\sigma^{(n)}$ on $B_n$ by

$$\sigma^{(n)} = \tau^{(n)} \circ (\pi_K^{(n)})^{-1}.$$  

For convenience, we put $\tau^{(0)} = \tau$, $\sigma^{(0)} = \tau \circ \pi_A^{-1}$.

**Proposition 3.9.** We assume that a tracial state $\tau$ on $A$ satisfies (\beta_1). Then, for $x \in \mathcal{F}^{(n)} \cap B_{n+1} = B_n \cap B_{n+1}$, we have

$$\sigma^{(n+1)}(x) = \sigma^{(n)}(x).$$

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Proof. We put \( p = (i_1, i_2, \ldots, i_n) \), \( u_p = u_{i_1} \otimes u_{i_2} \otimes \cdots \otimes u_{i_n} \), \( p' = (i_1, i_2, \ldots, i_n, i_{n+1}) \) and \( u_{p'} = u_{i_1} \otimes u_{i_2} \otimes \cdots \otimes u_{i_n} \otimes u_{i_{n+1}} = u_p \otimes u_{i_{n+1}} \).

Due to \( x \in \mathcal{F}^{(n)} \cap B_{n+1} = B_n \cap B_{n+1} = B'_n \), we can write as \( x = \pi_K^{(n)}(k) \), \( k \in K(X^\otimes n) \), which shows \( (u_p|ku_p)_A \in J \) for each \( u_p \).

On the other hand, we can write \( x = \tau_K^{(n+1)}(k') \), \( k' \in K(X^\otimes n+1) \) because \( x \in B_{n+1} \).

Then we have
\[
\pi_A((u_p|k'u_{p'})_A) = \pi_X(u_{i_{n+1}}) \cdots \pi_X(u_{i_1}) x \pi_X(u_{i_1}) \cdots \pi_X(u_{i_{n+1}}) = \pi_X(u_{i_{n+1}}) \cdots \pi_X(u_{i_1}) \pi_X(u_{i_{n+1}}) = \pi_X(u_{i_{n+1}}) \pi_X(\phi((u_p|ku_p)_A)u_{i_{n+1}}) = \pi_A((u_{i_{n+1}}|\phi((u_p|ku_p)_A)u_{i_{n+1}})_A).
\]

Then we have
\[
\tau^{(n+1)}(k') = e^{-(n+1)\beta} \sum_{p \in \mathbb{N}^n} \sum_{i_{n+1}=1}^{\infty} \tau((u_{i_{n+1}}|\phi((u_p|ku_p)_A)u_{i_{n+1}})_A).
\]

Using (\( \beta_1 \))
\[
\tau^{(n+1)}(k') = e^{-n\beta} \sum_{p \in \mathbb{N}^n} \tau((u_p|ku_p)_A) = \tau^n(k).
\]

By this, we have \( \sigma^{(n+1)}(x) = \sigma^n(x) \) for \( x \in \mathcal{F}^{(n)} \cap B_{n+1} \). \( \square \)

We assume that \( \tau \) satisfies (\( \beta_2 \)). \( \tau^{(n+1)} \) is a tracial bounded linear functional on \( K(X^\otimes n+1) \).

We assume \( n \geq 1 \). We denote by \( \mathcal{F}(\Sigma) \) the set of finite subsets of \( \Sigma \). Let \( e_F = \sum_{p \in F} \theta_{u_p} u_p \) for a finite subset \( F \) of \( \mathbb{N}^n \). Then \( \{e_F\}_{F \in \mathcal{F}(\mathbb{N}^n)} \) is an approximate unit of \( K(X^\otimes n) \). The canonical extension \( \overline{\tau^{(n)}} \) of \( \tau^{(n)} \) to \( \mathcal{L}(X^\otimes n) \) satisfies
\[
\overline{\tau^{(n)}}(T) = e^{-n\beta} \lim_{F} \sum_{p \in F} \tau((u_p|Te_F u_p)_A),
\]
where \( T \in \mathcal{L}(X^\otimes n) \), and it is expressed by Lemma 3.7 as
\[
\overline{\tau^{(n)}}(T) = e^{-n\beta} \sum_{q \in \mathbb{N}^n} \tau((u_q|Tu_q)_A)
\]
for \( T \in \mathcal{L}(X^\otimes n) \). Then the following Lemma holds.

**Lemma 3.10.** We assume that \( \tau \) satisfies (\( \beta_2 \)). Let \( n \geq 1 \) and \( 0 \leq i \leq n \). For \( k \in K(X^\otimes i) \) we have
\[
\overline{\tau^{(n)}}(k \otimes id_{(n,i)}) = e^{-n\beta} \sum_{q \in \mathbb{N}^n} \tau((u_q|(k \otimes id_{(n,i)})u_q)_A).
\]
Since $B_{n+1}$ is an ideal of $\mathcal{F}^{(n+1)}$, there exists the canonical extension $\overline{\sigma}^{(n+1)}$ on $\mathcal{F}^{(n+1)}$. For a finite subset $F$ of $\mathbb{N}^{n+1}$, we put $\hat{e}_F = \sum_{q \in F} \pi_X^{(n+1)}(u_q)\pi_X^{(n+1)}(u_q)^* = \sum_{q \in F} \sigma_K^{(n+1)}(\theta_{u_q,u_q})$. Then $\{\hat{e}_F\}_{F \in \mathcal{F}(n^{n+1})}$ is an approximate unit of $B_{n+1}$. Then we have
\[
\sigma^{(n+1)}(x) = \lim_F \sigma^{(n+1)}(x\hat{e}_F),
\]
for $x \in \mathcal{F}^{(n+1)}$.

Let $x \in B_i$ for $0 \leq i \leq n$. We write as $x = \pi_i^{(n)}(k)$ where $k \in \mathcal{K}(X^{\otimes i})$. Then we have
\[
x\hat{e}_F = \sum_{q \in F} \pi_K^{(n+1)}((k \otimes id_{(n+1,i)})\theta_{u_q,u_q}).
\]
Using this,
\[
\overline{\sigma}^{(n+1)}(x) = \lim_F \sigma^{(n+1)}(x\hat{e}_F) = \lim_F \tau^{(n+1)}((k \otimes id_{(n+1,i)})\sum_{q \in F} \theta_{u_q,u_q}) = \tau^{(n+1)}(k \otimes id_{(n,i)}).
\]

**Lemma 3.11.** We assume that $\tau$ satisfies (β2). Let $x \in \mathcal{F}^{(n)}$ with $x = \sum_{i=0}^{n} x_i$, where $x_i \in B_i$. Take $k_i \in \mathcal{K}^{(i)}(X^{\otimes i})$ such that $x_i = \pi_K^{(i)}(k_i)$. Then we have
\[
\overline{\sigma}^{(n+1)}(x) = e^{-(n+1)\beta} \sum_{q \in [n]+1} \tau(\sum_{i=1}^{n} (k_i \otimes id_{(n+1,i)})u_q)_A).
\]

**Proof.** Using Lemma 3.10, we have
\[
\overline{\sigma}^{(n+1)}(x) = \sum_{i=0}^{n} \overline{\sigma}^{(n+1)}(x_i) = \sum_{i=0}^{n} \tau^{(n+1)}(k_i \otimes id_{(n+1,i)})
\]
\[
= e^{-(n+1)\beta} \sum_{q \in [n]+1} \tau(\sum_{i=1}^{n} (k_i \otimes id_{(n+1,i)})u_q)_A).
\]

\[\Box\]

**Proposition 3.12.** We assume that $\tau$ satisfies (β2). For $x \in (\mathcal{F}^{(n)})^+$, we have
\[
\overline{\sigma}^{(n+1)}(x) \leq \overline{\sigma}^{(n)}(x).
\]

**Proof.** We take $x \in (\mathcal{F}^{(n)})^+$. Then we can write as $x = y^*y$ where $y \in \mathcal{F}^{(n)}$. We also write as $y = \sum_{i=0}^{n} y_i$ where $y_i \in B_i$, and write as $y_i = \pi_K^{(i)}(h_i)$, $h_i \in \mathcal{K}(X^{\otimes i})$. 

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Then, by Lemma 5.4 in [19], we have

\[
x = \sum_{i=0}^{n} \sum_{j=0}^{n} y_i^* y_j = \sum_{i=0}^{n} \sum_{j=0}^{n} \pi_K^{(i)}(h_i)^* \pi_K^{(j)}(h_j)
\]

\[
= \sum_{i=0}^{n} \pi_K^{(i)} \left( \sum_{j=0}^{i} (h_j \otimes \text{id}(i,j))^* h_i + h_i^* \sum_{j=0}^{i-1} (h_j \otimes \text{id}(i,j)) \right) = \sum_{i=0}^{n} \pi_K^{(i)}(k_i),
\]

where

\[
k_i = \sum_{j=0}^{i} (h_j \otimes \text{id}(i,j))^* h_i + h_i^* \sum_{j=0}^{i-1} (h_j \otimes \text{id}(i,j)).
\]

We put

\[
k = \sum_{i=0}^{n} k_i \otimes \text{id}_{(n,i)}.
\]

\[
k = \sum_{i=0}^{n} \left( \sum_{j=0}^{i} (h_j \otimes \text{id}(i,j))^* h_i + h_i^* \sum_{j'=0}^{i-1} h_{j'} \otimes \text{id}(i,j') \right) \otimes \text{id}_{(n,i)}
\]

\[
= \sum_{i=0}^{n} \left( \sum_{j=0}^{i} (h_j \otimes \text{id}_{(i,j)})^* (h_i \otimes \text{id}_{(n,i)}) + \sum_{j'=0}^{i-1} (h_i \otimes \text{id}_{(n,i)})^* (h_{j'} \otimes \text{id}_{(n,j')}) \right)
\]

\[
= \left( \sum_{i=0}^{n} (h_i \otimes \text{id}_{(n,i)})^* \right) \left( \sum_{j=0}^{n} (h_j \otimes \text{id}_{(n,j)}) \right) \geq 0.
\]

As in the proof of Proposition 3.9, we put \( p = (i_1, i_2, \ldots, i_n) \), \( u_p = u_{i_1} \otimes u_{i_2} \otimes \cdots \otimes u_{i_n}, \)
\( p' = (i_1, i_2, \ldots, i_n, i_{n+1}) \) and \( u_{p'} = u_{i_1} \otimes u_{i_2} \otimes \cdots \otimes u_{i_n} \otimes u_{i_{n+1}} = u_p \otimes u_{i_{n+1}} \).

We prepare the following: For \( k \in \mathcal{K}(X^{\otimes n}) \) (0 ≤ i ≤ n), we have

\[
e^{-(n+1)\beta} \sum_{p \in \mathbb{N}^{n+1}} \tau((u_p \otimes (k \otimes \text{id}_{(n+1,i)}))u_{p'}, A) = e^{-(n+1)\beta} \sum_{p \in \mathbb{N}^{n}} \tau((u_p \otimes u_{i_{n+1}})(k \otimes \text{id}_{(n+1,i)})(u_p \otimes u_{i_{n+1}}), A) = e^{-(n+1)\beta} \sum_{p \in \mathbb{N}^{n}} \sum_{i_{n+1}=1}^{\infty} \tau((u_{i_{n+1}} \otimes (u_p \otimes (k \otimes \text{id}_{(n,i)}))u_{p'}, A) \leq e^{\beta} \tau((u_p | T u_p) A).
\]

If \( \tau \) satisfies (32) and \( T \in \mathcal{L}(X^{\otimes n}) \) is positive, we have

\[
\sum_{i_{n+1}=1}^{\infty} \tau((u_{i_{n+1}} | (u_p | T u_p) A) u_{i_{n+1}} A) \leq e^{\beta} \tau((u_p | T u_p) A).
\]
Using these, we prove the Proposition. Let $x \in (F^{(n)})^+$. We express $x = \sum_{i=1}^{n} x_i$ where $x_i \in B_i$. For $x_i$ we take $k_i$ such that $x_i = \pi^i_k(k_i)$ and write as $k = \sum_{i=0}^{n}(k_i \otimes id_{(n+1,i)})$. Then by Lemma 3.15 and by the fact that $\tau$ satisfies (β2), we have

$$
\overline{\sigma^{[n+1]}}(x) = e^{-(n+1)\beta} \sum_{p' \in \mathbb{N}^{n+1}} \tau((u_{p'}| \sum_{i=0}^{n}(k_i \otimes id_{(n+1,i)}))u_{p'})_A
$$

$$
= e^{-(n+1)\beta} \sum_{p' \in \mathbb{N}^{n+1}} \tau((u_{p'}(k \otimes id_1))u_{p'})_A
$$

$$
= e^{-(n+1)\beta} \sum_{p' \in \mathbb{N}^{n+1}} \sum_{i_{n+1}=1}^{\infty} \tau((u_{p'} \otimes u_{i_{n+1}})((kuf_p) \otimes u_{i_{n+1}}))_A
$$

$$
= e^{-(n+1)\beta} \sum_{p' \in \mathbb{N}^{n}} \sum_{i_{n+1}=1}^{\infty} \tau((u_{n+1} \otimes \phi((u_{p'}kuf_p)_A)u_{i_{n+1}}))_A
$$

$$
\leq e^{-n\beta} \sum_{p' \in \mathbb{N}^{n}} \tau(u_{n+1} \otimes \phi(u_{p'}kuf_p)_A)
$$

$$
= \sigma^{(n)}(x).
$$

□

Let $A$ be a C*-algebra, $\alpha$ be an automorphic action of one dimensional torus $\mathbb{T}$ on $A$. $A^{\text{anal}}$ denotes the set $a \in A$ such that $t \rightarrow \alpha_t(a)$ has an analytic extension to $\mathbb{C}$.

**Definition 3.13.** Let $\beta > 0$. A state $\varphi$ of $A$ is called a $\beta$-KMS state on $A$ with respect to $\alpha$ if

$$
\varphi(x \alpha_t(y)) = \varphi(yx)
$$

for $x \in A$ and $y \in D$, where $D$ is a dense *-subalgebra contained in $A^{\text{anal}}$.

We denote by $E$ the conditional expectation of $O_X(J)$ onto the fixed point algebra $O_X(J)^\mathbb{T}$ by the gauge action. We denote by $O_X(J)^{(n)}$ the $n$-spectral subspace with respect to the gauge action.

**Lemma 3.14.** ([28] Proposition 1.3 ) Fix $\beta > 0$. If $\varphi$ is a $\beta$-KMS state on $O_X(J)$, then for $x, y \in O_X(J)^{(n)}$,

$$
\varphi(x^*y) = e^{n\beta} \varphi(yx^*).
$$

Conversely, if a tracial state $\varphi$ on $O_X(J)^\mathbb{T}$ satisfies the equation (5) for $x, y \in O_X(J)^{(n)}$, then $\tau \circ E$ is a $\beta$-KMS state on $O_X(J)$. The correspondence is one to one and conserves extreme points.

**Lemma 3.15.** (Exel-Laca [1] Proposition 12.5) Let $B$ be a unital C*-algebras, $A$ a C*-subalgebra of $B$ containing unit, and $I$ a closed two sided ideal of $B$ such that $B = A + I$. Let $\varphi$ be a state on $A$ and $\psi$ a positive linear functional on $I$. We assume that $\varphi(x) = \psi(x)$ for $x \in A \cap I$ and that $\overline{\psi(x)} \leq \varphi(x)$ for $x \in A$. Then there exists a unique state $\Phi$ on $B$ such that $\Phi|_A = \varphi$ and $\Phi|_I = \psi$.
Corollary 3.16. Let $B$ be a unital C*-algebras, $A$ be a C*-subalgebra of $B$ containing unit, and $I$ be a closed two sided ideal of $B$ such that $B = A + I$. Let $\varphi$ be a tracial state on $A$ and $\psi$ a trace on $I$. We assume that $\varphi(x) = \psi(x)$ for $x \in A \cap I$ and that $\overline{\psi(x)} \leq \varphi(x)$ for $x \in A$. Then there exists a unique tracial state $\Phi$ on $B$ such that $\Phi|_A = \varphi$ and $\Phi|_I = \psi$.

Proof. Let $\Phi$ be the state extension on $B$ constructed in Lemma 3.15. All we have to show is that $\Phi$ is tracial. Consider GNS representation $(\pi_\psi, H_{\psi}, \xi_\psi)$ of $I$. Let $\pi : B \to B(H_\psi)$ be the extension of $\pi_\psi$. The canonical extension $\overline{\psi}$ of $\psi$ to $B$ is defined as $\overline{\psi}(b) = (\pi(b) \xi_\psi | \xi_\psi)$ for $b \in B$. Define a state $\psi'$ on the von Neumann algebra $\pi(I)'$ by $\psi'(m) = (m \xi_\psi | \xi_\psi)$, for $m \in \pi(I)'$. Since $\psi$ is tracial, $\psi'$ is also tracial. Since $\overline{\psi}(b) = (\pi(b) \xi_\psi | \xi_\psi)$, the canonical extension $\Phi$ is also tracial. Hence for $a, b \in A$ and $x, y \in I$, we have
\[
\Phi((a + x)(b + y)) = \Phi(ab + xb + ay + xy) = \varphi(ab) + \overline{\psi}(xb + ay + xy) = 
\varphi(ba) + \overline{\psi}(bx + ya + yx) = \Phi((b + y)(a + x)),
\]
because $\varphi$ and $\overline{\psi}$ are tracial. Thus $\Phi$ is also tracial.

Under these preparations, we shall generalize Laca-Neshveyev's theorem of the construction of KMS states on Cuntz-Pimsner algebras as follows:

Theorem 3.17. Let $X$ be a C*-correspondence over $A$ of finite-degree type with degree $N = d(X)$ and $\{u_i\}_{i=1}^\infty$ a countable basis of $X$. Let $J$ be an ideal of $A$ contained in $J_X$. Let $\beta > 0$. Let $\varphi$ be a $\beta$-KMS state on a relative Cuntz-Pimsner algebra $O_X(J)$ with respect to the gauge action $\gamma$. Then the restriction of $\varphi$ to $\pi_A(A)$ is a tracial state on $A$ satisfying $\beta$-condition. Conversely, a tracial state on $A$ satisfying $\beta$-condition extends to a $\beta$-KMS state on $O_X(J)$. The correspondence between the $\beta$-KMS states on $O_X(J)$ and the tracial states on $A$ satisfying $\beta$-condition given by $\varphi \to \varphi|_{\pi_A(A)} \circ \pi_A$ is bijective and affine.

Proof. Let $\varphi$ be a $\beta$-KMS state on $O_X(J)$. The restriction of $\varphi$ to $O_X(J)$ is a tracial state and satisfies the condition (5) in Lemma 3.14. For $a \in J$, we have $\phi(a) \in K(X)$ and $\pi_A(a) = \pi_K(\phi(a))$. Since the equation
\[
\sum_{i=1}^n \pi_X(u_i) \pi_X(u_i)^* \pi_A(a) = \pi_K(\sum_{i=1}^n \theta_{u_i, u_i} \phi(a))
\]
holds and $\{\sum_{i=1}^n \theta_{u_i, u_i}\}_{n=1}^\infty$ is an approximate unit of $K(X)$, we have
\[
\lim_{n \to \infty} \sum_{i=1}^n \pi_X(u_i) \pi_X(u_i)^* \pi_A(a) = \pi_A(a).
\]
Using this, we have
\[ \sum_{i=1}^{\infty} \varphi(\pi_A(\phi(a)u_i)) = \sum_{i=1}^{\infty} \varphi(\pi_X(u_i)^*\pi_A(a)\pi_X(u_i)) \]
\[ = e^{\beta} \sum_{i=1}^{\infty} \varphi(\pi_X(u_i)\pi_X(u_i)^*\pi_A(a)) \]
\[ = e^{\beta} \varphi \left( \lim_{n \to \infty} \left( \left( \sum_{i=1}^{n} \pi_X(u_i)\pi_X(u_i)^* \right) \pi_A(a) \right) \right) \]
\[ = e^{\beta} \varphi(\pi_A(a)). \]
This shows that \( \varphi \circ \pi_A \) satisfies (\( \beta_1 \)). Let \( a \in A^+ \). We have
\[ \sum_{i=1}^{n} (\phi \circ \pi_A)((u_i|\phi(a)u_i)) = \sum_{i=1}^{n} \varphi(\pi_X(u_i)^*\pi_A(a)\pi_X(u_i)) \]
\[ = e^{\beta} \sum_{i=1}^{n} \varphi(\pi_A(a)\pi_X(u_i)^*\pi_X(u_i)) \]
\[ = e^{\beta} \varphi \left( \pi_A(a)^{1/2} \left( \sum_{i=1}^{n} \pi_X(u_i)\pi_X(u_i)^* \right) \pi_A(a)^{1/2} \right) \]
\[ \leq e^{\beta} \varphi(\pi_A(a)). \]
As \( n \to \infty \), we can show that \( \varphi \circ \pi_A \) satisfies (\( \beta_2 \)).

Conversely, we take a tracial state \( \tau \) on \( A \) satisfying (\( \beta_1 \)) and (\( \beta_2 \)). We construct a tracial state \( \omega\) on \( \mathcal{O}_X(J)^T \) satisfying the condition (5) in Lemma 3.14 and \( \omega|_{\pi_A(A)} \circ \pi_A = \tau \) holds.

We construct a tracial state \( \omega^{(n)} \) on \( \mathcal{F}^{(n)} \) for each natural integer \( n \) inductively. We put \( \omega^{(0)} = \tau \circ \pi_A^{-1} \). We assume that there exists a tracial state \( \omega^{(n)} \) on \( \mathcal{F}^{(n)} \) such that
\[ \omega^{(n)}|_{B_n} = \tau^{(n)} \circ (\pi_K^{(n)})^{-1} = \sigma^{(n)} \]
and
\[ \sigma^{(n)} \leq \omega^{(n)} \quad \text{on} \quad \mathcal{F}^{(n-1)}. \]
By Proposition 3.9, for \( x \in \mathcal{F}^{(n)} \cap B_{n+1} = B_n \cap B_{n+1} \) we have
\[ \sigma^{(n+1)}(x) = \sigma^{(n)}(x) = \omega^{(n)}(x). \]
For \( x \in \mathcal{F}^{(n)} \), by Proposition 3.12, we have
\[ \sigma^{(n+1)}(x^*x) \leq \sigma^{(n)}(x^*x). \] (6)
By the assumption of induction, for \( x \in \mathcal{F}^{(n-1)} \),
\[ \sigma^{(n)}(x^*x) \leq \omega^{(n)}(x^*x). \]
Let $x \in \mathcal{F}^{(n)}$ be written as $x = y + z$ where $y \in \mathcal{F}^{(n-1)}$ and $z \in B_n$. Then we have
\[
\sigma^{(n)}(x^*x) = \sigma^{(n)}((y + z)^*(y + z))
\]
\[
= \sigma^{(n)}(y^*y + y^*z + z^*y + z^*z)
\]
\[
= \sigma^{(n)}(y^*y) + \sigma^{(n)}(y^*z + z^*y + z^*z)
\]
\[
= \sigma^{(n)}(y^*y) + \sigma^{(n)}(y^*z + z^*y + z^*z)
\]
\[
\leq \omega^{(n)}(y^*y) + \omega^{(n)}(y^*z + z^*y + z^*z)
\]
\[
= \omega^{(n)}(y^*y + y^*z + z^*y + z^*z)
\]
\[
= \omega^{(n)}(x^*x).
\]
Using (6) and (7), for $x \in \mathcal{F}^{(n)}$ we have
\[
\sigma^{(n+1)}(x^*x) \leq \omega^{(n)}(x^*x).
\]
By Lemma 3.15, there exists a state $\omega^{(n+1)}$ on $\mathcal{F}^{(n+1)} = \mathcal{F}^{(n)} + B_{n+1}$ such that $\omega^{(n+1)}|_{\mathcal{F}^{(n)}} = \omega^{(n)}$ and $\omega^{(n+1)}|_{B_{n+1}} = \sigma^{(n+1)}$. Since $\omega^{(n)}$ and $\sigma^{(n+1)}$ are traces, we have that $\omega^{(n+1)}$ is a trace by Lemma 3.16.

We note that that $\sigma^{(n+1)} \leq \omega^{(n+1)}$ on $\mathcal{F}^{(n)}$ because $\omega^{(n+1)} = \omega^{(n)}$ on $\mathcal{F}^{(n)}$.

By a mathematical induction argument, there exists a desired sequence $\{\omega^{(n)}\}_{n=1,2,...}$ of tracial states on $\mathcal{F}^{(n)}$. We define $\omega$ on $\bigcup_{n=0}^{\infty} \mathcal{F}^{(n)}$ by $\omega|_{\mathcal{F}^{(n)}} = \omega^{(n)}$, and extend it to the closure $\mathcal{O}_X(J)^T$ by continuity. Then $\omega$ is a tracial state on $\mathcal{O}_X(J)^T$.

Since $\omega(\pi_A(a) + \pi_K^{(1)}(k_1) + \cdots + \pi_K^{(n)}(k_n)) = \tau(a) + \tau^{(1)}(k_1) + \cdots + \tau^{(n)}(k_n)$ for $a \in A$ and $k_i \in \mathcal{K}(X^{\otimes i})$, $\omega$ does not depend on the choice of the basis $\{u_k\}_{k=1}^{\infty}$ we have used in the construction.

From $\omega(\theta_{x_1,\ldots,x_n,y_1,\ldots,y_n}) = e^{-n\beta}T((y_1 \otimes \cdots \otimes y_n|x_1 \otimes \cdots \otimes x_n)_A)$, $\omega$ satisfies the condition (6) of Lemma 3.14. Let $E : \mathcal{O}_X(J) \to \mathcal{O}_X(J)^T$ be the canonical conditional expectation. Put $\varphi = \omega \circ E$. Then $\varphi$ is a $\beta$-KMS state of $\mathcal{O}_X(J)$ such that its restriction to $A$ is $\tau$.

4. KMS states on the C*-algebra associated with finite graphs

Cuntz-Krieger algebras are generalized as graph C*-algebras associated with general graphs having sinks and sources, which are studied for example in Kumjian-Pask-Raeburn [21], Kumjian-Pask-Raeburn-Renault [22] and Fowler-Laca-Raeburn [8]. As in Katsura [19] C*-algebras associated with graphs with possibly sources and sinks are expressed as C*-algebras associated with C*-correspondences canonically constructed from graphs. But the left actions are not necessarily injective.

Using the construction and Theorem 3.17 in the preceding section, we can describe KMS states on finite-graph C*-algebras.

Let $E = (E^0, E^1)$ be a finite graph without multiple edges. We denote by $s$ the source map and by $r$ the range map of $E$. A vertex $v \in E^0$ is called a sink if $s^{-1}(v) = \emptyset$ and $v \in E^0$ is called a source if $r^{-1}(v) = \emptyset$.
Definition 4.1. The graph C*-algebra $C^*(E)$ of a finite graph $E$ is the universal C*-algebra generated by mutually orthogonal projections $\{p_e\}_{e \in E^0}$ and partial isometries $\{q_e\}_{e \in E^1}$ with orthogonal ranges, such that $q_e^*q_e = p_{r(e)}$, $q_eq_e^* \leq p_{s(e)}$ for $e \in E^1$ and

$$p_e = \sum_{v \in s^{-1}(v)} q_vq_v^* \quad \text{for} \quad 0 < |s^{-1}(v)|.$$ 

We put $E_r^0 = \{ v \in E^0 \mid |s^{-1}(v)| > 0 \}$, $E_s^0 = \{ v \in E^0 \mid |s^{-1}(v)| = 0 \}$.

We note that $E_s^0$ is the set of sinks of $E$, Let $A = C(E^0)$ and $X = C(E^1)$. For $\xi, \eta \in X$ and $f \in A$, we put

$$(\xi|\eta)_A(v) = \sum_{e \in s^{-1}(v)} \xi(e)\eta(e) \quad \forall v \in E^0$$

$$(\xi f)(e) = \xi(e)f(r(e)) \quad \forall e \in E^1.$$ 

Then $X$ is a Hilbert C*-module over $A$.

We define $\phi(f)$ for $f \in A$ by

$$\phi(f)\xi(e) = f(s(e))\xi(e),$$

where $\xi \in X$. Then $\phi$ is a *-representation of $A$ in $\mathcal{L}(X_A)$ and $(X, \phi)$ is a C*-correspondence over $A$. As in atsura [19], it holds that $\phi^{-1}(K(X)) = C(E^0) = A$, $\ker(\phi) = C(E_0^0)$ and $J_X = C(E_s^0)$. The left action $\phi$ on $X$ is injective if $E$ has no sinks. The C*-correspondence $X$ is full if $E$ has no sources. We denote by $\mathcal{O}_E$ the (relative) Cuntz-Pimsner algebra of the C*-correspondence $X$ with $J = J_X$. Then $\mathcal{O}_E$ is isomorphic to the graph C*-algebra $C^*(E)$ ([18]).

We denote by $\gamma$ the gauge action of $\mathbb{T}$ on $\mathcal{O}_E$. Let $\beta$ be a positive number. We consider $\beta$-KMS states on the C*-algebra $\mathcal{O}_E$ with respect to the gauge action $\gamma$.

We number vertices of $E$ from 1 to $n$. Let $e$ be an edge such that $s(e) = i$ and $r(e) = j$. Since multiples edges are not permitted, we write $e$ as $(i, j)$. We denote by $\chi_{(i,j)}$ the characteristic function of the edge $(i, j)$. Then $\{\chi_{(i,j)} \mid (i, j) \in E^1\}$ constitutes a basis of $X$ on $A$.

By Theorem 3.11, there exists a bijective correspondence between the $\beta$-KMS states on $\mathcal{O}_E$ and the tracial states $\tau$ on the commutative C*-algebra $A$ such that

$$\sum_{(i,j) \in E^1} \tau((\chi_{(i,j)}\phi(f)\chi_{(i,j)})_A) = e^{\beta\tau(f)} \quad f \in J_X = C(E_r^0)$$

$$\sum_{(i,j) \in E^1} \tau((\chi_{(i,j)}\phi(f)\chi_{(i,j)})_A) \leq e^{\beta\tau(f)} \quad f \in A^+ = C(E_s^0)^+.$$ 

The correspondence is affine.

We denote by $D = [a_{i,j}]_{1 \leq i, j \leq n}$ the adjacency matrix of the graph $E$ i.e.

$$a_{ij} = \begin{cases} 1 & \text{there exists } e \in E^1 \text{ such that } s(e) = j, r(e) = i \\ 0 & \text{otherwise,} \end{cases}$$
and we denote by $\chi_j$ the characteristic function of the vertex $j$.

Then we have

$$(\chi_{i,j})\phi(\chi_i)\chi_{i,j})A(k) = \sum_{r(e) = k} \chi_{i,j}(e)\phi(\chi_i)\chi_{i,j}(e)$$

$$= \delta_{i,j}a_{k,l}.$$

We put $f = \sum_{i=1}^{n} f_i\chi_i$. Then we have

$$(\chi_{i,j})\phi(f)\chi_{i,j})A(k) = \sum_{l=1}^{n} f_l\delta_{i,l}\delta_{j,k}a_{k,l}.$$

For $\tau \in A^*$, we write as $\tau = \iota(\tau_1, \ldots, \tau_n)$. We rewrite $(\beta)$ as:

$$\sum_{(i,j) \in E^1} \tau((\chi_{i,j})\phi(f)\chi_{i,j})A) = \sum_{(i,j) \in E^1} \sum_{k=1}^{n} \tau_k \sum_{l=1}^{n} f_l\delta_{i,l}\delta_{j,k}a_{k,l}$$

$$= \sum_{k=1}^{n} \sum_{l=1}^{n} \tau_k f_l a_{k,l} = \sum_{l=1}^{n} \left( \sum_{k=1}^{n} a_{k,l} \tau_k \right) f_l.$$

We put $B = \iota D$. Let $\tau$ be a tracial state on $A$. Then $(\beta)$ and $(\beta_2)$ are written as

$$(B\tau|f) = e^\beta(\tau|f) \quad f \in C(E_1^0)$$

$$(B\tau|f) \leq e^\beta(\tau|f) \quad f \in C(E_0^0)^+.$$  \hspace{1cm} (8) \hspace{1cm} (9)

**Lemma 4.2.** If $E$ is a finite graph, $(\beta_1)$ implies $(\beta_2)$.

**Proof.** If $E$ has no sink, the Lemma is trivial.

Let $i$ be a sink and put $f = \chi_i$. The left hand side of (1) is

$$\sum_{l=1}^{n} a_{k,l}f_l = a_{k,i}.$$

If $i$ is a sink then it becomes 0. \hfill $\square$

If $E$ has no sink the conditions $(\beta_1)$ and $(\beta_2)$ are made into one condition:

$$(\beta) \quad (B\tau|f) = e^\beta(\tau|f) \quad f \in A.$$

Then $\tau$ is a Perron-Frobenius eigenvector and $e^\beta$ is the Perron-Frobenius eigenvalue of $B$.

We assume that $E$ has a sink. We denote by $E_1^0$ the set of vertices such that there exists an infinite path from them, and denote by $E_4^0$ the set of vertices such that there exists no infinite path from them. We note that $E_1^0$ is not empty if and only if the graph $E$ has a loop because $E$ is a finite graph.

We note that $E_0^0 = E_1^0 \cup E_4^0$ and $E_1^0 \cap E_4^0 = \emptyset$. The set $E_4^0$ contains all sinks, and all paths which start from the vertices in $E_4^0$ must end at sinks.
We note that there exists no edge from vertices in \( E_4^0 \) to vertices in \( E_1^0 \). If such an edge exists, it holds that there exists an infinite path which starts from a vertex in \( E_4^0 \).

**Lemma 4.3.** We can number \( E^0 \) as follows:

1. The numbers of vertices in \( E_1^0 \) are larger than that of vertices in \( E_4^0 \).
2. There exists no edge from \( j \) to \( i \) where \( i < j \) and \( i \) and \( j \) are edges in \( E_1^0 \).
3. The numbers of sinks are larger than the numbers of vertices which are not sinks.

*Proof.* We note that \( E_1^0 \) contains sinks. First, we number vertices of \( E \) so that the numbers of sinks are larger than the numbers of vertices which are not sink. We denote by \( E(1) \) the graph obtained by removing sinks and edges whose ranges are sinks. If \( E(1) \) has no sink, the proof is completed. If \( E(1) \) has sinks, we renumber vertices of \( E(1) \) so that the numbers of sinks in \( E(1) \) are larger than the numbers of vertices which are not sinks. We get graphs \( E(0), E(1), E(2), \ldots \), inductively. We has put \( E(0) = E \) for the convenience. Since \( E \) is a finite graph, there exists a non negative integer \( r \) such that \( E(r) \) contains a sink and \( E(r + 1) \) is empty or \( E(r + 1) \) has no sink.

The vertices in \( E_1^0 \) are not removed, because there exists a infinite path from starting from vertices in \( E_1^0 \). On the other hand, vertices in \( E_4^0 \) are removed because a path starting from vertex in \( E_0^1 \) must reaches a sink, and if such a path remains, then a sink is also remained. \( \square \)

We denote by \( F \) the graph obtained by removing vertices in \( E_4^0 \) and edges whose ranges are in \( E_4^0 \). We call \( F \) the core of the graph \( E \).

We put \( E_3^0 = E_s^0 \) and put \( E_2^0 = E_4^0 \setminus E_3^0 \). Then \( E^0 \) is expressed as the disjoint union of \( E_1^0, E_2^0 \) and \( E_3^0 \). Using this dividing of vertices, we write \( f = [f_1, f_2, f_3], f_i \in C(E_i^0) \) \((i = 1, 2, 3)\) for \( f \in A = C(E^0) \), and \( \tau = [\tau_1, \tau_2, \tau_3] \) for a tracial state \( \tau \) of \( A \). We rewrite (3) using the above block expression. We use the notation \((\tau, f)\) of dual paring instead of \( \tau(f) \).

We write the following equation

\[
(Br, f) = e^\beta(\tau, f) \quad f \in C(E_s^0).
\]

using the above block notation as follows:

\[
\begin{pmatrix}
B_{11} & B_{12} & B_{13} \\
O & B_{22} & B_{23} \\
O & O & 0
\end{pmatrix}
\begin{pmatrix}
\tau_1 \\
\tau_2 \\
\tau_3
\end{pmatrix},
\begin{pmatrix}
f_1 \\
\tau_2 \\
\tau_3
\end{pmatrix} =
\begin{pmatrix}
e^\beta \tau_1 \\
e^\beta \tau_2 \\
e^\beta \tau_3
\end{pmatrix},
\begin{pmatrix}
f_1 \\
f_2 \\
0
\end{pmatrix}.
\]

Since \( f_1 \) and \( f_2 \) are arbitrary, we have

\[
B_{11}\tau_1 + B_{12}\tau_2 + B_{13}\tau_3 = e^\beta \tau_1 \tag{10}
\]

\[
B_{22}\tau_2 + B_{23}\tau_3 = e^\beta \tau_2 \tag{11}
\]

From (11), we have

\[
\tau_2 = e^{-\beta}(B_{22}\tau_2 + B_{23}\tau_3).
\]
Since \((i, j)\) element in \(B_{22}\) is 0 for \(i > j\), we can determine all elements of \(\tau_2\) for a given nonnegative \(\tau_3\) for every \(\beta > 0\).

We assume \(E\) has a loop, and \(E_0^1\) is not empty. We note that \(B_{11}\) is the transpose of the adjacency matrix of \(F\). Let \(\lambda_0\) be the Perron-Frobenius eigenvalue of \(B_{11}\).

Using (10), we have

\[
(e^\beta I - B_{11})\tau_1 = B_{12}\tau_2 + B_{13}\tau_3.
\]

If \(e^\beta\) is greater than \(\lambda_0\), for nonnegative \(\tau_2, \tau_3\) we can determine nonnegative \(\tau_1\) by

\[
\tau_1 = (e^\beta I - B_{11})^{-1}(B_{12}\tau_2 + B_{13}\tau_3).
\]

For a sink \(v\), let \(\tau_3\) be the state corresponding to the Dirac measure \(\delta_v\) on \(v\). we can determine \(\tau_2\) and \(\tau_1\), and we get a tracial state \(\tau_v\) by the normalization. The tracial state \(\tau_v\) on \(A\) gives the \(\beta\)-KMS state \(\varphi_v\) on \(O_E\).

We summarize the results as the following theorem.

**Theorem 4.4.** (1) We assume that \(E\) has a loop. We denote by \(\lambda_0\) the Perron-Frobenius eigenvalue of the transposed of the adjacency matrix of the core \(F\). If \(\beta > \log \lambda_0\), the set of the extreme \(\beta\)-KMS states on \(O_E\) with respect to the gauge action correspond to the Dirac measures on the set of sinks in \(E\).

(2) We assume that \(E\) has no loop. Then for every \(\beta > 0\), the set of the extreme \(\beta\)-KMS states on \(O_E\) with respect to the gauge action correspond to the Dirac measures on the set of sinks in \(E\).

**Proposition 4.5.** Each extreme KMS state in Theorem 4.4 generates a type I factor.

**Proof.** We write as \(\tau = [\tau_1 \tau_2 \tau_3]\). Using the equation (10) and (11), we can write as

\[
(I - e^{-\beta} B) \begin{bmatrix} \tau_1 \\ \tau_2 \\ \tau_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \tau_3 \end{bmatrix}.
\]

Then we have

\[
\begin{bmatrix} \tau_1 \\ \tau_2 \\ \tau_3 \end{bmatrix} = (I - e^{-\beta} B)^{-1} \begin{bmatrix} 0 \\ 0 \\ \tau_3 \end{bmatrix}.
\]

If \(\beta > \log \lambda_0\), then \(\sum_{i=0}^{\infty} e^{i\beta} B^i\) is convergent, and we have

\[
\tau = \sum_{i=0}^{\infty} e^{i\beta} B^i \begin{bmatrix} 0 \\ 0 \\ \tau_3 \end{bmatrix}.
\]

Let \(v\) be a sink and \(\tau_3\) be the state corresponding to the Dirac measure \(\delta_v\) on \(v\). We can determine \(\tau_2\) and \(\tau_1\), and we get a tracial state \(\tau_v\) by the normalization of \(\tau\).

The \(\beta\)-KMS state \(\varphi_v\) extending \(\tau_v\) is of finite type in [23], and generates type I factor ([10]).
We assume that $E$ has a loop and $E_1^0$ is not empty. If $\beta = \log \lambda_0$, there exists a $\beta$-KMS state on $\mathcal{O}_E$ which is of infinite type in [23]. These KMS state are essentially the same as that given in [4].

**Proposition 4.6.** We assume $\beta = \log \lambda_0$. Let $\hat{\tau}_1$ be the normalized Perron Frobenius eigenvector of $B_{11}$. Then $(\hat{\tau}_1, 0, 0)$ is a $\beta$-KMS state on $\mathcal{O}_E$. It corresponds to a $\beta$-KMS state of the graph $C^*$-algebra associated graph $F$.

The KMS states in Proposition 4.6 generates type III von Neumann algebra under some condition ([4]).

**Remark 4.1.** KMS states on Exel-Laca algebras are classified in [7] and graph $C^*$-algebras are known to be strongly Morita equivalent to some Exel-Laca algebra by adding tails to sinks. But KMS states of finite graphs with sinks can not be obtained directly from that of Exel-Laca algebras because the relation of KMS states on strongly Morita equivalent $C^*$-algebras are not known.

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