On the spatially homogeneous Boltzmann equation for Bose-Einstein particles with balanced potentials

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Abstract

The paper is concerned with the spatially homogeneous isotropic Boltzmann equation for Bose-Einstein particles with quantum collision kernel where the interaction potential $\phi(x)$ can be approximately written as the delta function plus a certain attractive potential such that the Fourier transform $\hat{\phi}$ of $\phi$ behaves like $0 \leq \hat{\phi}(\xi) \leq \text{const.}|\xi|^n$ for $|\xi| << 1$ for some constant $n \geq 1$. We prove that in this case, there is no condensation in finite time for all temperatures and all solutions, and thus it is completely different from the case $\hat{\phi}(\xi) \geq \text{const.}|\xi|^n$ for $|\xi| << 1$ with $0 \leq n < 1/4$ as considered in [6]. For a class of initial data that have some nice integrability near the origin, we also get some regularity, stability and $L^\infty$ estimate.

Key words: Bose-Einstein particles, balanced potentials, non-condensation in finite time, negative order of moment, regularity and stability.

1 Introduction

We study the spatially homogeneous Boltzmann equation for Bose-Einstein particles:

$$\frac{\partial}{\partial t} f(v, t) = \int_{\mathbb{R}^3 \times S^2} B(v - v_s, \omega) \left( f' f'_s (1 + f) (1 + f_s) - f f_s (1 + f') (1 + f'_s) \right) d\omega dv_s$$

with $(v, t) \in \mathbb{R}^3 \times (0, \infty)$. This equation (which is now well-known) describes time-evolution of a dilute and space homogeneous gas of bosons. Derivations of this equation can be found for instance in [25], [31], [4], [7], [8], [23].

In Equation (1.1), $f = f(v, t) \geq 0$ is the number density of particles at time $t$ with the velocity $v$, and $f_s = f(v_s, t), f' = f(v', t), f'_s = f(v'_s, t)$ where $v, v_s$ and $v', v'_s$ are velocities of two particles before and after their collision and the particle collision is assumed to be elastic:

$$v' + v'_s = v + v_s, \quad |v'|^2 + |v'_s|^2 = |v|^2 + |v_s|^2,$$

which can be written as an explicit form:

$$v' = v - ((v - v_s) \cdot \omega) \omega, \quad v'_s = v_s + ((v - v_s) \cdot \omega) \omega, \quad \omega \in S^2$$

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As before we assume that the interaction potential $\phi(\cdot)$ of particles is real and is of the central form, i.e. $\phi(x) = \phi(|x|)$. According to [4] and [8] in the weak-coupling regime, $B(v - v_s, \omega)$ and $\phi$ has the following relation (after normalizing physical parameters)

$$B(v - v_s, \omega) = \frac{1}{(4\pi)^2} |(v - v_s) \cdot \omega| \Phi(|v - v'|, |v - v'_s|)$$

where

$$\Phi(r, \rho) = (\hat{\phi}(r) + \hat{\phi}(\rho))^2, \quad r, \rho \geq 0$$

$\hat{\phi}$ is the Fourier transform of $\phi$:

$$\hat{\phi}(r) := \hat{\phi}(|\xi| = r) = \int_{\mathbb{R}^3} \phi(|x|) e^{-i\xi \cdot x} d|\xi| = r.$$

In this paper, the function $r \mapsto \hat{\phi}(r)$ is often assumed to be continuous and bounded on $\mathbb{R}_{\geq 0}$:

$$\hat{\phi} \in C_b(\mathbb{R}_{\geq 0}).$$

A special case is that $\phi(|x|) = \frac{1}{2} \delta(x)$ i.e. $\hat{\phi}(r) \equiv \frac{1}{2}$ (hence $\Phi(r) \equiv 1$), where $\delta(x)$ is the three dimensional Dirac delta function concentrating at $x = 0$. In this case, (1.4) becomes the hard sphere model:

$$B(v - v_s, \omega) = \frac{1}{(4\pi)^2} |(v - v_s) \cdot \omega|$$

which has been concerned in many papers about Eq. (1.1). Since in general $B(v - v_s, \omega)$ is a nonnegative Borel function of $|v - v_s|$ and $|(v - v_s) \cdot \omega|$ only, we sometimes also use the notation:

$$B(v - v_s, \omega) \equiv B(|v - v_s|, \cos \theta), \quad \theta = \arccos(|(v - v_s) \cdot \omega|/|v - v_s|).$$

Note that by canceling the common terms $f' f_s' f_s$, Eq. (1.1) becomes

$$\frac{\partial}{\partial t} f(v, t) = \int_{\mathbb{R}^3 \times S^2} B(v - v_s, \omega) (f' f'_s (1 + f + f_s) - f f_s (1 + f' + f'_s)) d\omega dv_s$$

which still has the cubic nonlinear terms. Due to the strong nonlinear structure and the effect of condensation, there have been no results on global in time existence of solutions of Eq. (1.1) for the general anisotropic initial data without additional assumptions. See [5] for local in times existence without smallness assumption on the initial data and [13] for global in times existence with a relative smallness assumption on the initial data. For global in time solutions with general initial data, in particular for the case of low temperature, so far one has to consider weak solutions $f$ which are solutions of the following equation

$$\frac{d}{dt} \int_{\mathbb{R}^3} \psi(v) f(v, t) dv = \frac{1}{2} \int_{\mathbb{R}^3 \times \mathbb{R}^3 \times S^2} (\psi + \psi_s - \psi' - \psi'_s) B(v - v_s, \omega) f' f'_s dv dv_s d\omega$$

$$+ \int_{\mathbb{R}^3 \times \mathbb{R}^3 \times S^2} (\psi + \psi_s - \psi' - \psi'_s) B(v - v_s, \omega) f f'_s dv dv_s d\omega$$

(1.9)
for all test functions $\psi$ and all $t \in [0, \infty)$. In general however the cubic integral of $B(\mathbf{v} - \mathbf{v}_s, \omega)f f'_s d\mathbf{v}_s d\omega$, etc. are divergent (see e.g. [16]). A subclass of $f$ that has no such divergence is the isotropic (i.e. radially symmetric) functions: $f(\mathbf{v}) = f(|\mathbf{v}|^2/2)$. By changing variables $x = |\mathbf{v}|^2/2, y = |\mathbf{v}'|^2/2, z = |\mathbf{v}'|^2/2$, one has

$$B(\mathbf{v} - \mathbf{v}_s, \omega)f(|\mathbf{v}|^2/2)f(|\mathbf{v}'|^2/2)d\mathbf{v}_s d\omega = 4\pi \sqrt{2} W(x, y, z) dF(x) dF(y) dF(z)$$

where $dF(x) = f(x)\sqrt{x} dx$, etc., and $x, y, z \in \mathbb{R}_{\geq 0}$ in the right side are independent variables. This is the main reason that almost all results obtained so far are concerned with isotropic initial data hence isotropic solutions, see e.g. [14], [16], [17] for the global existence of isotropic solution, moment production and long time weak convergence to equilibrium; [21], [22], [6] for long time strong convergence to the equilibrium; [12], [24], [28], [29], [30] for self-similar structure and deterministic numerical methods; [31], [3], [9], [10], [11], [19] for singular solutions and the formation of blow-up and condensation in finite time; and [1], [2], [26] for general discussions and basic results for similar models on low temperature evolution of condensation.

Having done researches on the case of hard-sphere like models, the case of other interaction models is naturally concerned. Recently we found that if the interaction potential $\phi$ is balanced i.e. $\tilde{\phi}(r) = O(r^{\eta})$ for small $r > 0$ with $\eta \geq 1$ (see below for details), then there will be no spontaneous condensation in finite time for all temperatures and all solutions, see Theorem 1.7. This is completely different from those of the hard sphere interaction model.

Before stating the main result of the paper we introduce some notations and definitions. Let $L^1_s(\mathbb{R}^3)$ with $s \geq 0$ be the linear space of the weighted Lebesgue integrable functions defined by $L^0_0(\mathbb{R}^3) = L^1(\mathbb{R}^3)$ and

$$L^1_s(\mathbb{R}^3) = \{ f \in L^1(\mathbb{R}^3) \mid \| f \|_{L^1_s} := \int_{\mathbb{R}^3} \langle v \rangle^s |f(v)| dv < \infty \}, \quad \langle v \rangle := (1 + |v|^2)^{1/2}.$$ 

Let $B_k(X)$ ($k \geq 0$) be the linear space of signed real Borel measures $F$ on a Borel set $X \subset \mathbb{R}^d$ satisfying \( \int_X (1 + |x|)^k d|F|(x) < \infty \), where $|F|$ is the total variation of $F$. Let

$$B^+_k(X) = \{ F \in B_k(X) \mid F \geq 0 \}. $$

For the case $k = 0$ we also denote $B(X) = B_0(X), B^+(X) = B^+_0(X)$. In this paper we only consider two cases $X = \mathbb{R}^3$ and $X = \mathbb{R}_{\geq 0}$, and in many cases we consider isotropic measures $\tilde{F} \in B_{2k}(\mathbb{R}^3)$, which define and can be defined by measures $F \in B_k(\mathbb{R}_{\geq 0})$ in terms of the following relations:

$$F(A) = \frac{1}{4\pi \sqrt{2}} \int_{\mathbb{R}^3} 1_A(|v|^2/2) d\tilde{F}(v), \quad A \subset \mathbb{R}_{\geq 0} \tag{1.10}$$

$$\tilde{F}(B) = 4\pi \sqrt{2} \int_{\mathbb{R}_{\geq 0}} \left( \frac{1}{4\pi} \int_{S^2} 1_B(\sqrt{2x} \omega) d\omega \right) dF(x), \quad B \subset \mathbb{R}^3 \tag{1.11}$$
for all Borel measurable sets $A,B$. For any $k \geq 0$ let
\[\|F\|_k = \int_{\mathbb{R}_{\geq 0}} (1 + x)^k |F|(x), \quad F \in \mathcal{B}_k(\mathbb{R}_{\geq 0}).\]

We will also use a semi-norm:
\[\|F\|_1^0 = \int_{\mathbb{R}_{\geq 0}} x |F|(x).\]

Including negative orders, moments for a positive Borel measure $F$ on $\mathbb{R}_{\geq 0}$ are defined by
\[M_p(F) = \int_{\mathbb{R}_{\geq 0}} x^p dF(x), \quad p \in (-\infty, \infty).\]  \hspace{1cm} (1.12)

Here for the case $p < 0$ we adopt the convention $0^p = (0^+)^p = \infty$, and we recall that $\infty \cdot 0 = 0$. Then it should be noted that
\[M_p(F) < \infty \quad \text{and} \quad p < 0 \quad \implies \quad F(\{0\}) = 0.\]  \hspace{1cm} (1.13)

Moments of orders 0,1 correspond to the mass and energy and are particularly denoted as
\[N(F) = M_0(F), \quad E(F) = M_1(F).\]  \hspace{1cm} (1.14)

In order to be able to study long time behavior of solutions of Eq.(1.1) for low temperature, we first consider weak solutions of the Eq.(1.1) and in fact we could so far only define weak solution for isotropic initial data. A test function space for defining weak solution is chosen
\[C^{1,1}_b(\mathbb{R}_{\geq 0}) = \left\{ \varphi \in C^1_b(\mathbb{R}_{\geq 0}) \left| \frac{d}{dx} \varphi \in \text{Lip}(\mathbb{R}_{\geq 0}) \right. \right\} \]

For isotropic functions $f = f(|v|^2/2) \geq 0$, $\varphi = \varphi(|v|^2/2)$ with $f(| \cdot |^2/2) \in L^1_2(\mathbb{R}^3)$, $\varphi \in C^{1,1}_b(\mathbb{R}_{\geq 0})$, and for the measure $F$ defined by $dF(x) = f(x) \sqrt{\pi} dx$, the collision integrals in (1.9) can be rewritten

\[\frac{1}{2} \int_{\mathbb{R}^3 \times \mathbb{R}^3 \times S^2} (\varphi + \varphi_* - \varphi' - \varphi'_*) Bf f'_s d\nu d\nu_s d\omega = 4\pi \sqrt{2} \int_{\mathbb{R}^3_{\geq 0}} J[\varphi] d^2 F;\]
\[\int_{\mathbb{R}^3 \times \mathbb{R}^3 \times S^2} (\varphi + \varphi_* - \varphi' - \varphi'_*) B f f'_s d\nu d\nu_s d\omega = 4\pi \sqrt{2} \int_{\mathbb{R}^3_{\geq 0}} K[\varphi] d^3 F\]

where $B = B(v - v_s, \omega)$ is given by (1.4) with (1.6), $d^2 F = dF(y) dF(z)$, $d^3 F = dF(x) dF(y) dF(z)$, and $J[\varphi], K[\varphi]$ are linear operators defined as follows:

\[J[\varphi](y,z) = \frac{1}{2} \int_0^{y+z} K[\varphi](x,y,z) \sqrt{x} dx, \quad K[\varphi](x,y,z) = W(x,y,z) \Delta \varphi(x,y,z),\]  \hspace{1cm} (1.15)
\[\Delta \varphi(x,y,z) = \varphi(x) + \varphi(x_s) - \varphi(y) - \varphi(z) = (x-y)(x-z) \int_0^1 \varphi''(\xi) d\xi dt\]  \hspace{1cm} (1.16)
\[ \xi = y + z - x + t(x - y) + s(x - z), \quad x, y, z \geq 0, \quad x_* = (y + z - x)_+, \]

\[ W(x, y, z) = \frac{1}{4\pi \sqrt{xyz}} \int_{|x| = \sqrt{y} \vee \sqrt{z}}^{(\sqrt{x} + \sqrt{y}) \vee (\sqrt{x} + \sqrt{z})} ds \int_0^{2\pi} \Phi(\sqrt{2s}, \sqrt{2Y_*}) d\theta \quad \text{if} \quad x_* y z > 0, \quad (1.17) \]

\[
W(x, y) = \begin{cases} 
\frac{1}{\sqrt{yz}} \Phi(\sqrt{2y}, \sqrt{2z}) & \text{if } x = 0, \ y > 0, \ z > 0 \\
\frac{1}{\sqrt{zx}} \Phi(\sqrt{2x}, \sqrt{2(z - x)}) & \text{if } y = 0, \ z > x > 0 \\
\frac{1}{\sqrt{xy}} \Phi(\sqrt{2(y - x)}, \sqrt{2x}) & \text{if } z = 0, \ y > x > 0 \\
0 & \text{others} 
\end{cases} 
\quad (1.18)
\]

\[ Y_* = Y_*(x, y, z, s, \theta) = \begin{cases} 
\sqrt{\left( z - \frac{(x - y + s^2)^2}{4s^2}\right)_+} + e^{i\theta} \sqrt{\left( x - \frac{(x - y + s^2)^2}{4s^2}\right)_+} & \text{if } s > 0 \\
0 & \text{if } s = 0 
\end{cases} \quad (1.19) \]

where \( \Phi(r, \rho) \) is given in (1.6), \( (u)_+ = \max\{u, 0\} \), \( a \vee b = \max\{a, b\} \), \( a \wedge b = \min\{a, b\} \), \( i = \sqrt{-1} \).

**Remark 1.1.** It is easily seen that if \( s > 0 \) and \(|\sqrt{x} - \sqrt{y}| \vee |\sqrt{x_*} - \sqrt{z}| \leq s \leq (\sqrt{x} + \sqrt{y}) \wedge (\sqrt{x_*} + \sqrt{z})\), then

\[ x - \frac{(x - y + s^2)^2}{4s^2} \geq 0, \quad z - \frac{(x - y + s^2)^2}{4s^2} \geq 0. \quad (1.20) \]

Based on the existence results (see [16]), we introduce directly the concept of measure-valued isotropic solutions of Eq. (1.1) in the weak form:

**Definition 1.2.** Let \( B(v - v_*, \omega) \) be given by (1.4), (1.5), (1.6) and let \( F_0 \in B^+_1(\mathbb{R}_0) \). We say that a family \( \{F_t\}_{t \geq 0} \subset B^+_1(\mathbb{R}_0) \), or simply \( F_t \), is a conservative measure-valued isotropic solution of Eq. (1.1) on the time-interval \([0, \infty)\) with the initial datum \( F_t|_{t=0} = F_0 \) if

(i) \( N(F_t) = N(F_0) \), \( E(F_t) = E(F_0) \) for all \( t \in [0, \infty) \),

(ii) for every \( \varphi \in C^{1,1}_b(\mathbb{R}_0) \), \( t \mapsto \int_{\mathbb{R}_0} \varphi(x) dF_t(x) \) belongs to \( C^1([0, \infty)) \),

(iii) for every \( \varphi \in C^{1,1}_b(\mathbb{R}_0) \)

\[ \frac{d}{dt} \int_{\mathbb{R}_0} \varphi dF_t = \int_{\mathbb{R}_0} J[\varphi] d^2F_t + \int_{\mathbb{R}_0} K[\varphi] d^3F_t \quad \forall t \in [0, \infty). \quad (1.21) \]

**Remark 1.3.** (1) The transition from (1.17) to (1.18) in defining \( W \) is due to the identity

\[ (\sqrt{x} + \sqrt{y}) \wedge (\sqrt{x_*} + \sqrt{z}) - |\sqrt{x} - \sqrt{y}| \vee |\sqrt{x_*} - \sqrt{z}| = 2 \min\{\sqrt{x}, \sqrt{x_*}, \sqrt{y}, \sqrt{z}\} \quad (1.22) \]
from which one sees also that if $\Phi(r, \rho) \equiv 1$, then $W(x, y, z)$ becomes the function corresponding to the hard sphere model. In the case for hard sphere model, We use notation $W_H$ to replace $W$ and:

$$W_H(x, y, z) = \frac{1}{\sqrt{xyz}} \min \{\sqrt{x}, \sqrt{y}, \sqrt{z}, \sqrt{x^*}\}$$

(2) By [16] and Appendix of [6], we conclude from Theorem 1 (Weak Stability), Theorem 2 (Existence) and Theorem 3 in [16] that for any $F_0 \in B_1^+ (\mathbb{R}_{\geq 0})$, the Eq. (1.1) has always a conservative measure-valued isotropic solution $F_t$ on the time-interval $[0, \infty)$ with the initial datum $F_t|_{t=0} = F_0$.

In order to get regularity results and $L^\infty$ estimates, we also need the definition of mild solutions as follows.

**Definition 1.4.** Let $B(v - v, \omega)$ be given by (1.4), (1.5), (1.6). Let $f(x, t)$ be a nonnegative measurable function on $\mathbb{R}_{\geq 0} \times [0, T_\infty)$ ($0 < T_\infty \leq \infty$). We say that $f(\cdot, t)$ is a mild solution of Eq. (1.1) on the time-interval $[0, T_\infty)$ if $f$ satisfies

(i) $\sup_{t \in [0, T]} \int_{\mathbb{R}^3} (1 + x)f(x, t)\sqrt{xdx} < \infty \quad \forall 0 < T < T_\infty$,

(ii) there is a null set $Z \subset \mathbb{R}_{\geq 0}$ which is independent of $t$ such that for all $x \in \mathbb{R}_{\geq 0} \setminus Z$ and all $t \in [0, T_\infty)$, $\int_0^t d\tau \int_{\mathbb{R}_{\geq 0}} W(x, y, z)[f'f^*_s(1 + f + f^*_s) + f f^*_s(1 + f' + f^*_s)]\sqrt{y}\sqrt{z}dydz < \infty$ and

$$f(x, t) = f_0(x) + \int_0^t Q(f)(x, \tau)d\tau.$$

Here $f_0 = f(\cdot, 0)$ denotes the initial datum of $f(\cdot, t)$ and

$$Q(f)(x) = \int_{\mathbb{R}_{\geq 0}^2} W(x, y, z)[f'f^*_s(1 + f + f^*_s) - f f^*_s(1 + f' + f^*_s)]\sqrt{y}\sqrt{z}dydz.$$

It is obvious that if $f(\cdot, t)$ is a mild solution of Eq. (1.1), then the measure $F_t$, defined by $dF_t(x) = f(x, t)\sqrt{xdx}$, is a distributional solution of Eq. (1.1).

**Kinetic Temperature.** Let $F \in B_1^+ (\mathbb{R}_{\geq 0})$, $N = N(F)$, $E = E(F)$ and suppose $N > 0$. If $m$ is the mass of one particle, then $m4\pi\sqrt{2}N$, $m4\pi\sqrt{2}E$ are total mass and kinetic energy of the particle system per unite space volume. The kinetic temperature $T$ and the kinetic critical temperature $T_c$ are defined by (see e.g. [16] and references therein) $T = \frac{2mE}{3k_B N}$, $T_c = \zeta(5/2)(2\pi)^{5/2}[\zeta(3/2)]^{5/3} \frac{2m}{k_B} N^{2/3}$ where $k_B$ is the Boltzmann constant, $\zeta(s) = \sum_{n=1}^\infty n^{-s}$, $s > 1$. Keeping in mind the constant $m4\pi\sqrt{2}$, there will be no confusion if we also call $N$ and $E$ the mass and energy of a particle system.

**Regular-Singular Decomposition.** According to measure theory (see e.g. [27]), every finite positive Borel measure can be uniquely decomposed into regular part and singular part.
with respect to the Lebesgue measure. For instance if $F \in B^+_1(\mathbb{R}_{\geq 0})$, then there exists unique
$0 \leq f \in L^1(\mathbb{R}_{\geq 0}, (1 + x)\sqrt{x}dx), \nu \in B^+_1(\mathbb{R}_{\geq 0})$ and a Borel set $Z \subset \mathbb{R}_{\geq 0}$ such that
\[ dF(x) = f(x)\sqrt{x}dx + d\nu(x), \quad mes(Z) = 0, \quad \nu(\mathbb{R}_{\geq 0} \setminus Z) = 0. \]
We call $f$ and $\nu$ the regular part and the singular part of $F$ respectively.\footnote{Strictly speaking the product $f(x)\sqrt{x}$ is the regular part of $F$. The reason that we only mention $f$ is because $f(x)\sqrt{x}$ comes from the 3D-isotropic function $f = f(|x|^2/2)$.}

**Bose-Einstein Distribution.** According to Theorem 5 of \cite{16} and its equivalent version proved in the Appendix of \cite{19} we know that for any $N > 0$, $E > 0$ the Bose-Einstein distribution $F_{be} \in B^+_1(\mathbb{R}_{\geq 0})$, which is the unique equilibrium solution of Eq.~(1.21) satisfying $N(F_{be}) = N, E(F_{be}) = E$, is given by
\[ dF_{be}(x) = \begin{cases} 
\frac{1}{Ae^{x/\kappa} - 1}\sqrt{x}dx, & A > 1, \quad \text{if } \frac{T}{T_c} > 1, \\
\frac{1}{e^{x/\kappa} - 1}\sqrt{x}dx + (1 - (\frac{T}{T_c})^{3/5})N\delta(x)dx, & \text{if } \frac{T}{T_c} \leq 1 \end{cases} \]
where $\delta(x)$ is the Dirac delta function concentrated at $x = 0$, and functional relations of the coefficients $A = A(N, E), \kappa = \kappa(N, E)$ can be found in for instance Proposition 1 in \cite{17}. The positive number $(1 - (\frac{T}{T_c})^{3/5})N$ is called the Bose-Einstein condensation (BEC) of the equilibrium state of Bose-Einstein particles at low temperature $T < T_c$.

**Entropy.** The entropy functional for Eq.~(1.1) is
\[ S(f) = \int_{\mathbb{R}^3} \left( (1 + f(v)) \log(1 + f(v)) - f(v) \log f(v) \right)dv, \quad 0 \leq f \in L^1_2(\mathbb{R}^3). \]
As in \cite{6}, we define the entropy $S(F)$ of a measure $F \in B^+_1(\mathbb{R}^3)$ by
\[ S(F) := \sup_{\{f_n\}_{n=1}^\infty} \limsup_{n \to \infty} S(f_n) \]
where $\{f_n\}_{n=1}^\infty$ under the sup is taken all sequences in $L^1_2(\mathbb{R}^3)$ satisfying
\[ f_n \geq 0, \quad \sup_{n \geq 1} \|f_n\|_{L^1_2} < \infty; \]
\[ \lim_{n \to \infty} \int_{\mathbb{R}^3} \psi(v)f_n(v)dv = \int_{\mathbb{R}^3} \psi(v)dF(v) \quad \forall \psi \in C_0(\mathbb{R}^3). \]
Let $0 \leq f \in L^1_2(\mathbb{R}^3)$ be the regular part of $F$, i.e. $dF(v) = f(v)dv + d\nu(v)$ with $\nu \geq 0$ the singular part of $F$. By Lemma 3.2 of \cite{6} we have
\[ S(F) = S(f) \]
which shows that the singular part of $F$ has no contribution to the entropy $S(F)$ and that $F$ is non-singular if and only if $S(F) > 0$. For any $0 \leq f \in L^1(\mathbb{R}_{\geq 0}, (1 + x)\sqrt{x}\,dx)$, the entropy $S(f)$ is defined by $S(f) = S(\tilde{f})$ with $\tilde{f}(v) := f(|v|^2/2)$, so that (using (1.24) and change of variable)

$$S(f) = S(\tilde{f}) = 4\pi\sqrt{2} \int_{\mathbb{R}_{\geq 0}} \left( (1 + f(x)) \log (1 + f(x)) - f(x) \log f(x) \right) \sqrt{x}\,dx. \quad (1.29)$$

In general, the entropy $S(F)$ for a measure $F \in \mathcal{B}_1^+(\mathbb{R}_{\geq 0})$ is defined by $S(F) = S(\tilde{F})$ where $\tilde{F} \in \mathcal{B}_1^+(\mathbb{R}^3)$ is defined by $F$ through (1.11) and $S(\tilde{F})$ is defined by (1.20) or (1.28).

In a recent work [6], condensation in finite time and strong convergence to equilibrium of $F_t$ have been proven for the collision kernel $B$ that is similar to hard sphere model. We summarize them as the following theorem.

**Theorem 1.5 ([6]).** Suppose $B(v - v_\ast, \omega)$ is given by (1.4), (1.5) where the Fourier transform $r \mapsto \hat{\phi}(r)$ (of a radially symmetric interaction potential $x \mapsto \phi(|x|)$) is continuous and non-decreasing on $\mathbb{R}_{\geq 0}$, and there are constants $0 < b_0 \leq 1/2, 0 \leq \eta < \frac{1}{4}$ such that

$$b_0 \frac{r^\eta}{1 + r^\eta} \leq \hat{\phi}(r) \leq \frac{1}{2} \quad \forall r \geq 0. \quad (1.30)$$

Let $F_0 \in \mathcal{B}_1^+(\mathbb{R}_{\geq 0})$ satisfy $N(F_0) > 0, E(F_0) > 0$, let $F_{be}$ be the unique Bose-Einstein distribution with the same mass $N = N(F_0)$ and energy $E = E(F_0)$, and let $1/20 < \lambda < \frac{1}{19}$. Then there exists a conservative measure-valued isotropic solution $F_t$ of Eq. (1.1) on $[0, \infty)$ with the initial datum $F_0$ such that $S(F_{be}) \geq S(F_t) \geq S(F_0), S(F_t) > 0$ for all $t > 0$ and

$$S(F_{be}) - S(F_t) \leq C(1 + t)^{-\lambda}, \quad \|F_t - F_{be}\|_1 \leq C(1 + t)^{-\frac{(1-\eta)\lambda}{2(4-\eta)}} \quad \forall t \geq 0. \quad (1.31)$$

In particular if $T/T_c < 1$ then

$$|F_t(\{0\}) - (1 - (T/T_c)^{3/5})N| \leq C(1 + t)^{-\frac{(1-\eta)\lambda}{2(4-\eta)}} \quad \forall t \geq 0. \quad (1.31)$$

Here the constant $C > 0$ depends only on $N, E, b_0, \eta$ and $\lambda$.

Theorem 1.5 tells us that if the Fourier transform of the interaction potential $\hat{\phi}(r)$ satisfies (1.30) (which may be viewed as a small perturbation of the hard sphere model $\hat{\phi}(r) \equiv 1/2$), finite time condensation and strong convergence to equilibrium hold true. It is natural to ask whether or not they are still true for an opposite case $0 \leq \hat{\phi}(r) \leq b_0 \frac{r^\eta}{1 + r^\eta}$? In this paper we show that this is false if $\eta \geq 1$. More precisely, we introduce the following Assumption.

**Assumption 1.6.** The collision kernel $B(v - v_\ast, \omega)$ is given by (1.4), (1.5), where the Fourier transform $r \mapsto \hat{\phi}(r)$ (of a radially symmetric interaction potential) is continuous on $[0, \infty)$, and there are constants $b_0 > 0, \eta \geq 1$ such that

$$0 \leq \hat{\phi}(r) \leq b_0 \frac{r^\eta}{1 + r^\eta} \quad \forall r \geq 0 \quad (1.32)$$
and there is a function $k \in C^1([1, \sqrt{2}])$ with $k(1) = 1$, such that

$$\hat{\phi}(ar) \leq k(a)\hat{\phi}(r) \quad \forall r > 0, \forall 1 < a \leq \sqrt{2}.$$ 

In this paper we always denote $q_1 := \max_{x \in [1, \sqrt{2}]} \max\{2k(x)k'(x), 0\}$.

If $\hat{\phi}$ satisfies Assumption 1.6 then we say that $\phi$ is a balanced potential since Assumption 1.6 implies that

$$\int_{\mathbb{R}^3} \phi(|x|)dx = \hat{\phi}(0) = 0.$$ 

Generally if $\eta > n \in \mathbb{N}$ and $(1 + |x|^n)\phi \in L^1(\mathbb{R}^3)$, then

$$\int_{\mathbb{R}^3} x_1^{\alpha_1}x_2^{\alpha_2}x_3^{\alpha_3}\phi(|x|)dx = i^{3\alpha}\hat{\phi}(0) = 0$$ 

for all indices $\alpha$ with $|\alpha| = \alpha_1 + \alpha_2 + \alpha_3 \leq n$. Roughly speaking, the higher $\eta$ is, the more balanced $\phi$ becomes.

There are many examples of balanced potentials. For instance $\phi(|x|) = \frac{1}{2}(\delta(x) - U(|x|))$ where $U(|x|) \geq 0$ is 3D Yukawa potential $U(|x|) = \frac{1}{4\pi|x|}e^{-|x|}$, $x \in \mathbb{R}^3$, then $\hat{\phi}(r) = \frac{r^2}{1 + r^2}$ satisfies Assumption 1.6. More generally, given any $\eta > \frac{3}{2}$, $g(r) = \frac{1}{1 + r^\eta}$. $\eta > \frac{3}{2}$ implies $g \in L^2(\mathbb{R}^3)$, then one can use basic knowledge of Fourier transform to get a function $U_\eta \in L^2(\mathbb{R}^3)$ such that $\hat{U}_\eta(r) = \frac{1}{1 + r^\eta}$, so that $\hat{\phi}(r) = \frac{r^2}{1 + r^\eta}$ satisfies Assumption 1.6.

Main Results. The main results of the paper are as follows:

**Theorem 1.7.** Suppose $B(v - v_\star, \omega)$ satisfy Assumption 1.6. Let $F_0 \in \mathcal{B}^+_1(\mathbb{R}_{\geq 0})$ with mass $N = N(F_0) > 0$ and energy $E = E(F_0) > 0$ and let $F_t$ be a conservative measure-valued isotropic solution $F_t$ of Eq. (1.1) on $[0, \infty)$ with the initial datum $F_0$ (the existence of $F_t$ has been insured by Remark 1.3). Then we have

$$F_t(\{0\}) \leq e^{ct}F_0(\{0\}) \quad \forall t \geq 0$$

where $c = 8^{2+\eta}c^2_0(1 + q_1)N^2$. In particular if $F_0(\{0\}) = 0$, then $F_t(\{0\}) = 0$ for all $t \geq 0$.

**Theorem 1.8.** Suppose $B(v - v_\star, \omega)$ satisfy Assumption 1.6 with $\eta \geq \frac{3}{2}$. Let $F_t$ be a conservative measure-valued isotropic solution of Eq. (1.1) on $[0, \infty)$ whose initial datum $F_0$ is regular and satisfies $M_{-1/2}(F_0) < \infty$, Then $F_t$ is regular for all $t \in [0, \infty)$ and its density $f(\cdot, t)$ is a mild solution of Eq. (1.1) on $[0, \infty)$ satisfying $f \in C([0, \infty); L^1(\mathbb{R}_+))$ and $f(\cdot, 0) = f_0$, where $f_0$ is the density of $F_0$. In particular if $F_t$ is conservative, so is $f(\cdot, t)$ on $[0, \infty)$.

For any given $F_0 \in \mathcal{B}^+_1(\mathbb{R}_{\geq 0})$ we define a function $\Psi_{F_0}(\varepsilon)$ on $\varepsilon \in [0, \infty)$ by

$$\Psi_{F_0}(\varepsilon) = \varepsilon + \sqrt{\varepsilon} + \int_{\frac{\varepsilon}{\sqrt{\varepsilon}}}^{\infty} xdF_0(x), \quad \varepsilon > 0; \quad \Psi_{F_0}(0) = 0. \quad (1.33)$$

Here $\int_{\frac{\varepsilon}{\sqrt{\varepsilon}}}^{\infty}$ can be understood as either $\int_{(\frac{\varepsilon}{\sqrt{\varepsilon}}, \infty)}$ or $\int_{(\frac{1}{\sqrt{\varepsilon}}, \infty)}$. Now we can introduce stability theorem.
Theorem 1.9. Suppose $B(v - v_*, \omega)$ satisfy Assumption (1.6) with $\eta \geq \frac{3}{2}$, moreover assume $\phi$ satisfy $\phi(r) = a_0 r^{-\beta} > 0$ for all $r \geq R$ with $R \geq 0$ and $0 \leq \beta < \frac{1}{2}$. Let $F_0, G_0 \in R^+_0(\mathbb{R}_{\geq 0})$ satisfy $M_{-1/2}(F_0) < \infty, M_{-1/2}(G_0) < \infty$, and $F_t, G_t$ be conservative measure-valued isotropic solution to Eq. (1.1) on $[0, \infty)$ with their initial data $F_0, G_0$ respectively. Then
\[ \|F_t - G_t\|_1 \leq C\Psi_{F_0}(\|F_0 - G_0\|_1)e^{ct} \quad \forall t \in [0, \infty) \] (1.34)
where $\Psi_{F_0}(\cdot)$ is defined in (1.33) and $C, c$ are finite positive constants depending only on $N(F_0), E(F_0), N(G_0), E(G_0), a_0, b_0, \beta, \eta, q_1, R, M_{-1/2}(F_0), M_{-1/2}(G_0)$.

In particular if $F_0 = G_0$, then $F_t = G_t$ for all $t \in [0, \infty)$.

Theorem 1.10. Let the collision kernel $B(v - v_*, \omega)$ be given by (1.4), (1.5), (1.6) and suppose $\phi(r)$ satisfy
\[ a(r) \leq \phi(r) \quad \forall r \geq 0. \] (1.35)
where $a(\cdot)$ is a non-decreasing continuous function on $[0, \infty)$ satisfying $a(r) > 0$ for $r > 0$ and $b_1$ is a constant. Given any $F_0 \in R^+_0(\mathbb{R}_{\geq 0})$ with $N = N(F_0) > 0$ and $E = E(F_0) > 0$ and let $F_t$ be a conservative measure-valued isotropic solution $F_t$ of Eq. (1.1) on $[0, \infty)$ with the initial datum $F_0$ (the existence of $F_t$ has been insured by Remark 1.3). Let $F_{be}$ be the unique Bose-Einstein distribution with the same mass and energy as $F_0$. Then
\[ \lim_{t \to \infty} S(F_t) = S(F_{be}), \quad \lim_{t \to \infty} \|F_t - F_{be}\|_1 = 0. \]
Consequently it holds the weak convergence:
\[ \lim_{t \to \infty} \int_{\mathbb{R}_{\geq 0}} \varphi(x)dF_t(x) = \int_{\mathbb{R}_{\geq 0}} \varphi(x)dF_{be}(x) \quad \forall \varphi \in C_0(\mathbb{R}_{\geq 0}). \]

The rest of the paper is organized as follows. In Section 2 we prove Theorem 1.7 and Theorem 2.5: non-condensation in finite time and propagation of $M_{-p}(F_t)$ for $0 < p \leq \frac{1}{2}$. In Section 3, we prove moment production, positive lower bound of entropy and weak convergence. In Section 4, we use propagation of $M_{-1/2}(F_t)$ to get regularity, stability (uniqueness) of $F_t$ if $M_{-1/2}(F_0) < \infty$. We also prove the global existence of mild solution and strong solution of Eq. (1.1) if $M_{-1/2}(F_0) < \infty$ and get $L^\infty$ estimate about the mild solutions.

2 Non-condensation in finite time and propagation of negative order of moment

In this section we prove non-condensation in finite time and propagation of $M_{-p}(F_t) < \infty$ for $0 < p \leq \frac{1}{2}$. To prove them, we need the following lemma about $W(x, y, z)$.
Lemma 2.1. Suppose $B(\mathbf{v} - \mathbf{v}_*, \omega)$ satisfy Assumption 1.6, then the following estimates about $W(x, y, z)$ hold

$$W(x, y, z) \leq 4b_0^2 \min\{1, \max\{8x, 8y, 8z\}\} \min\{\sqrt{x}, \sqrt{y}, \sqrt{z}, \sqrt{x_0}\}, \quad \forall x, y, z > 0$$ (2.1)

$$W(0, y, z) \leq 4b_0^2 \min\{1, \max\{8y, 8z\}\} \frac{\sqrt{y}}{\sqrt{z}}, \quad y, z > 0$$ (2.2)

$$W(x, 0, z) \leq 4b_0^2 \min\{1, (8z)^{\eta}\}, \quad z > x > 0$$ (2.3)

$$W(x, y, 0) \leq 4b_0^2 \min\{1, (8y)^{\eta}\}, \quad y > x > 0$$ (2.4)

$$W(x, y, z) \leq (1 + q_1 \frac{y}{z})W(y, x, z), \quad \forall 0 \leq x \leq y \leq \frac{z}{2}$$ (2.5)

where $b_0, \eta$ and $q_1$ are defined in Assumption 1.6.

Proof. First we need to estimate $\Phi(\sqrt{s}, \sqrt{y}, \sqrt{z})$. By (1.19) and (2.22), for the case of $|\sqrt{x} - \sqrt{y}| \vee |\sqrt{x_0} - \sqrt{z}| \leq s \leq (\sqrt{x} + \sqrt{y}) \wedge (\sqrt{x_0} + \sqrt{z}), s > 0$, we have

$$s \leq 2 \max\{\sqrt{x}, \sqrt{y}, \sqrt{z}\},$$

$$Y_* \leq \sqrt{\frac{z - (x - y + s^2)^2}{4s^2}} + \sqrt{\frac{x - (x - y + s^2)^2}{4s^2}} \leq 2 \max\{\sqrt{x}, \sqrt{y}, \sqrt{z}\}.$$

So we obtain

$$\Phi(\sqrt{2s}, \sqrt{2Y_*}) = \left(\frac{1}{\sqrt{2}}(\sqrt{2s} + \sqrt{2Y_*})\right)^2 \leq b_0^2 \left(\frac{(\sqrt{2s})^\eta}{1 + (\sqrt{2s})^\eta} + \frac{(\sqrt{2Y_*})^\eta}{1 + (\sqrt{2Y_*})^\eta}\right)^2$$

$$\leq b_0^2 \left(\frac{2\sqrt{2} \max\{\sqrt{x}, \sqrt{y}, \sqrt{z}\}}{1 + (2\sqrt{2} \max\{\sqrt{x}, \sqrt{y}, \sqrt{z}\})^\eta} + \frac{2\sqrt{2} \max\{\sqrt{x}, \sqrt{y}, \sqrt{z}\}}{1 + (2\sqrt{2} \max\{\sqrt{x}, \sqrt{y}, \sqrt{z}\})^\eta}\right)^2$$

$$\leq 4b_0^2 \min\{1, \max\{8x, 8y, 8z\}\}^\eta.$$

Together with (1.18), (1.22), (1.32), this yields

$$W(x, y, z) = \frac{1}{4\pi \sqrt{xyz}} \int_{|\sqrt{x} - \sqrt{y}| \wedge (\sqrt{x} + \sqrt{z})}^{(\sqrt{x} + \sqrt{y}) \wedge (\sqrt{x} + \sqrt{z})} \int_0^{2\pi} \Phi(\sqrt{2s}, \sqrt{2Y_*})d\theta$$

$$\leq \frac{b_0^2}{\pi \sqrt{xyz}} \int_{|\sqrt{x} - \sqrt{y}| \wedge (\sqrt{x} + \sqrt{z})}^{(\sqrt{x} + \sqrt{y}) \wedge (\sqrt{x} + \sqrt{z})} \int_0^{2\pi} \min\{1, \max\{8x, 8y, 8z\}\}^\eta d\theta$$

$$= 4b_0^2 \min\{1, \max\{8x, 8y, 8z\}\} \min\{\sqrt{x}, \sqrt{y}, \sqrt{z}, \sqrt{x_0}\}. \quad \forall x, y, z > 0$$

Thus we complete the proof of (2.1). The proofs of (2.2), (2.3), (2.4) are analogous.

In order to prove (2.5), first we prove an useful inequality:

$$1 \leq \frac{Y_*}{Y_0^{\frac{\eta}{2}}} \leq \frac{\sqrt{x}}{\sqrt{y} - y} \quad \forall 0 \leq x \leq y < z$$ (2.6)
where $Y^2_z = Y_s(y, x, z, s, \theta)$. To prove this inequality, recalling that

$$Y_s = Y_s(x, y, z, s, \theta) = \left\{ \begin{array}{ll} \left| \sqrt{z - \frac{(x - y + s^2)^2}{4s^2}} + e^{i\theta} \sqrt{\left( x - \frac{(x - y + s^2)^2}{4s^2} \right)} \right| & \text{if } s > 0 \\ 0 & \text{if } s = 0, \end{array} \right.$$ 

we have $\frac{Y_s}{Y^2_z} = \frac{\sqrt{z-y}}{\sqrt{z-y}}$ for $0 = x \leq y < z$. For the case $x \neq 0$, denote $U = z - \frac{(x - y + s^2)^2}{4s^2}, V = z - \frac{(x - y + s^2)^2}{4s^2}$, $O = x - \frac{(x - y + s^2)^2}{4s^2} = y - \frac{(y-x+s^2)^2}{4s^2}$. By (2.20) we know $u \leq x$ if $s \in [\sqrt{y-y}, \sqrt{x+y}], 0 < x \leq y < z$, thus we obtain

$$\frac{Y_s}{Y^2_z} = \left| \frac{\sqrt{U} + e^{i\theta} \sqrt{O}}{\sqrt{V} + e^{i\theta} \sqrt{O}} \right| = \frac{\sqrt{U + O + \sqrt{UO}(e^{i\theta} + e^{-i\theta})}}{\sqrt{V + O + \sqrt{VO}(e^{i\theta} + e^{-i\theta})}} \leq \frac{\sqrt{U + O + UO(e^{i\theta} + e^{-i\theta})}}{\sqrt{V + O + VO(e^{i\theta} + e^{-i\theta})}} \geq 1$$

and

$$\frac{Y_s}{Y^2_z} = \left| \frac{\sqrt{U} + e^{i\theta} \sqrt{O}}{\sqrt{V} + e^{i\theta} \sqrt{O}} \right| = \frac{\sqrt{U + O + \sqrt{UO}(e^{i\theta} + e^{-i\theta})}}{\sqrt{V + O + \sqrt{VO}(e^{i\theta} + e^{-i\theta})}} \geq 1$$

Now we are ready to prove inequality (2.25); using (2.6), (1.5), (1.17), (1.18), it suffices to prove

$$\hat{\phi}(\sqrt{2}Y_s) \leq \sqrt{1 + q_1 \frac{y}{z} \hat{\phi}(\sqrt{2}Y^2_z)} \quad \forall 0 \leq x \leq y \leq \frac{z}{2}$$

If $\hat{\phi}(\sqrt{2}Y^2_z) = 0$, by Assumption 1.6, this inequality is obvious. If $\hat{\phi}(\sqrt{2}Y^2_z) \neq 0$

$$\left( \frac{\hat{\phi}(\sqrt{2}Y_s)}{\hat{\phi}(\sqrt{2}Y^2_z)} \right)^2 \leq (k \frac{Y_s}{Y^2_z} - 1) \max_{x \in [1, \frac{z}{2}]} 2k(x)k'(x) = 1 + \left( \frac{Y_s}{Y^2_z} - 1 \right) \max_{x \in [1, \frac{z}{2}]} 2k(x)k'(x) \leq 1 + \left( \frac{z}{z-y} - 1 \right) \max_{x \in [1, \frac{z}{2}]} 2k(x)k'(x) \leq 1 + (\frac{z}{z-y} - 1)q_1 \leq 1 + q_1 \frac{y}{2(z-y)} \leq 1 + q_1 \frac{y}{z}$$

for all $0 \leq x \leq y < \frac{z}{2}$, so we complete the proof. □

Remark 2.2. Combining with [6], we know for $\hat{\phi}(r) = b_0 \frac{e^r}{1+r^p}$ the following estimates hold

$$\frac{b_0^2 z^n}{8 \sqrt{y} \sqrt{z}} \leq W(x, y, z) \leq 4b_0^2 \frac{(z^n)^n}{\sqrt{y} \sqrt{z}} \quad \forall 0 < x \leq y \leq z \leq 1.$$

In this case, $W$ is still unbounded.

Proof of Theorem 1.7. Denote $\varphi_\epsilon(x) = (1 - \frac{x}{\epsilon})^2$. By the definition of weak solution,

$$\int_{R^2_0} \varphi_\epsilon(x) dF(x) = \int_{R^2_0} \varphi_\epsilon(x) dF_0(x) + \int_0^t d\tau \int_{R^2_0} \mathcal{J}[\varphi_\epsilon](y, z) dF_\tau(y) dF_\tau(z) + \int_0^t d\tau \int_{R^2_0} \mathcal{K}[\varphi_\epsilon] d^3F_\tau. \quad (2.8)$$
By the fact that \( W(x, y, z) \leq 4b_0^2W_H(x, y, z) \), we have
\[
\mathcal{J}[\varphi_\varepsilon](y, z) \leq \frac{1}{2} \int_0^{y+z} W(x, y, z)(\varphi_\varepsilon(x) + \varphi_\varepsilon(y + z - x))\sqrt{x}dx
\]
\[
\leq 2b_0^2 \int_0^{y+z} W_H(x, y, z)(\varphi_\varepsilon(x) + \varphi_\varepsilon(y + z - x))\sqrt{x}dx = 4b_0^2 \int_0^{y+z} W_H(x, y, z)\varphi_\varepsilon(x)\sqrt{x}dx.
\]
Combining with the fact that \( W_H(x, y, z)\sqrt{x} \leq \sqrt{\frac{2}{y+z}} \) for all \( 0 < x < y+z \) and \( \sup_{r>0} \sqrt{\frac{1}{r}} \int_0^r \varphi_1(x)dx \leq 1 \), this leads to
\[
\int_{R^3 \geq 0} \mathcal{J}[\varphi_\varepsilon](y, z)d^2F_t \leq 4\sqrt{2}b_0^2\varepsilon \int_{y, z \geq 0, y+z > 0} \sqrt{\frac{\varepsilon}{y+z}}dF_t(y)dF_t(z) \int_0^{y+z} \varphi_1(x)dx
\]
\[
\leq 4\sqrt{2}b_0^2N^2\varepsilon. \tag{2.9}
\]
The term \( \int_{R^3 \geq 0} \mathcal{K}[\varphi_\varepsilon]d^3F_t \) can be decomposed into the following parts (see [6]):
\[
\int_{R^3 \geq 0} \mathcal{K}[\varphi_\varepsilon]d^3F_t = \left( 2 \int_{0 \leq x < y < z} + 2 \int_{0 \leq y < x < z} + \int_{0 \leq x < y < z} \right) W(x, y, z)\Delta\varphi_\varepsilon(x, y, z)d^3F_t
\]
\[
= \int_{0 \leq x < y \leq z} \chi_{y,z}W(x, y, z)\Delta\text{sym}\varphi_\varepsilon(x, y, z)d^3F_t
\]
\[
+ 2 \int_{0 \leq x < y < z} (W(y, x, z) - W(x, y, z))\Delta\varphi_\varepsilon(y, x, z)d^3F_t
\]
\[
+ \int_{0 < y, z < x} W(x, y, z)\Delta\varphi_\varepsilon(x, y, z)d^3F_t
\]
\[
= \int_{0 < x < y \leq z} \chi_{y,z}W(x, y, z)\Delta\text{sym}\varphi_\varepsilon(x, y, z)d^3F_t
\]
\[
+ 2 \int_{0 < x < y < z} (W(y, x, z) - W(x, y, z))\Delta\varphi_\varepsilon(y, x, z)d^3F_t
\]
\[
+ \int_{0 < y, z < x} W(x, y, z)\Delta\varphi_\varepsilon(x, y, z)d^3F_t
\]
\[
+ F_\tau(\{0\}) \int_{0 < y \leq z} \chi_{y,z}W(x, y, z)\Delta\text{sym}\varphi_\varepsilon(0, y, z)d^2F_t
\]
\[
+ 2F_\tau(\{0\}) \int_{0 < y < z} (W(y, 0, z) - W(0, y, z))\Delta\varphi_\varepsilon(y, 0, z)d^2F_t
\]
\[:= I_1(\tau) + I_2(\tau) + I_3(\tau) + I_4(\tau) + I_5(\tau), \tag{2.10}\]
where
\[
\Delta\text{sym}\varphi(x, y, z) = \varphi(z + y - x) + \varphi(z + x - y) - 2\varphi(z)
\]
\[
= (y - x)^2 \int_0^1 \int_0^1 \varphi''(z + (s-t)(y-x))dudsdt, \quad 0 \leq x, y, z \leq z. \tag{2.11}
\]
\[
\chi_{y,z} = \begin{cases} 2 & \text{if } y < z, \\ 1 & \text{if } y = z. \end{cases}
\]

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Now we are going to prove that

\[
\limsup_{\varepsilon \to 0^+} \int_0^t \left( I_1(\tau) + I_2(\tau) + I_3(\tau) \right) d\tau = 0 \tag{2.13}
\]

\[
\limsup_{\varepsilon \to 0^+} \int_0^t \left( I_4(\tau) + I_5(\tau) \right) d\tau \leq \int_0^t F_{\tau}(\{0\}) 8^{2+\eta} b_0^2 (1 + q_1) N^2 d\tau \tag{2.14}
\]

It is easy to deduce that

\[
\lim_{\varepsilon \to 0^+} W(x, y, z) \Delta_{\text{sym}} \varphi_\varepsilon(x, y, z) = 0 \quad \text{for all} \quad 0 < x < y \leq z,
\]

\[
\lim_{\varepsilon \to 0^+} (W(y, x, z) - W(x, y, z)) \Delta \varphi_\varepsilon(y, x, z) = 0 \quad \text{for all} \quad 0 < x < y \leq z,
\]

\[
\lim_{\varepsilon \to 0^+} W(x, y, z) \Delta \varphi_\varepsilon(x, y, z) = 0 \quad \text{for all} \quad 0 < y, z < x < y + z,
\]

This triggers us to use dominated convergence theorem to prove (2.13). If we can prove

\[
W(x, y, z) \Delta_{\text{sym}} \varphi_\varepsilon(x, y, z) \leq 8^{1+\eta} b_0^2 \quad \text{for all} \quad 0 < x < y \leq z, \tag{2.15}
\]

\[
(W(y, x, z) - W(x, y, z)) \Delta \varphi_\varepsilon(y, x, z) \leq 8^{1+\eta} q_1 b_0^2 \quad \text{for all} \quad 0 < x < y \leq z, \tag{2.16}
\]

\[
W(x, y, z) \Delta \varphi_\varepsilon(x, y, z) = 0 \quad \text{for all} \quad 0 < y, z < x < y + z, \tag{2.17}
\]

then we can use dominated convergence theorem to prove (2.13). To prove (2.15), by the convexity of \( \varphi_1 \) we have \( 0 \leq -\varphi_1'(x) \leq \frac{\varphi_1(0) - \varphi_1(x)}{x} \leq \frac{1}{x} \), thus

\[
\Delta_{\text{sym}} \varphi_\varepsilon(x, y, z) \leq \varphi_1(\frac{z + x - y}{\varepsilon}) - \varphi_1(\frac{z}{\varepsilon}) \leq -\varphi_1'(\frac{z + x - y}{\varepsilon}) \frac{y - x}{\varepsilon} \leq \frac{y - x}{z + x - y} \quad \forall 0 \leq x < y \leq z
\]

So we have for \( 0 \leq x < y \leq \frac{z}{2} \),

\[
W(x, y, z) \Delta_{\text{sym}} \varphi_\varepsilon(x, y, z) \leq 4b_0 \min\{1, \{8z\}^\eta\} \frac{y - x}{\sqrt{y} \sqrt{z} (z - y + x)} \leq 8^{1+\eta} b_0^2,
\]

and for \( 0 \leq x < y \leq z, y > \frac{z}{2} \),

\[
W(x, y, z) \Delta_{\text{sym}} \varphi_\varepsilon(x, y, z) \leq 4b_0 \min\{1, \{8z\}^\eta\} \frac{y - x}{\sqrt{y} \sqrt{z}} \leq 8^{1+\eta} b_0^2.
\]

For the term \( (W(y, x, z) - W(x, y, z)) \Delta \varphi_\varepsilon(y, x, z) \), if \( W(y, x, z) \geq W(x, y, z) \), then \( (W(y, x, z) - W(x, y, z)) \Delta \varphi_\varepsilon(y, x, z) \leq 0 \) for \( 0 \leq x < y < z \). If \( W(y, x, z) \leq W(x, y, z) \), and \( 0 \leq x < y < \frac{z}{2} \), then

\[
(W(y, x, z) - W(x, y, z)) \Delta \varphi_\varepsilon(y, x, z) \leq q_1 \frac{y}{z} W(y, x, z) \varphi_\varepsilon(x) \leq q_1 \frac{y}{z} 4b_0 ^2 \min\{1, \{8z\}^\eta\} \sqrt{y} \sqrt{z} \leq 8^{1+\eta} q_1 b_0^2.
\]

For \( W(y, x, z) \leq W(x, y, z) \) and \( 0 \leq x < y < z, y \geq \frac{z}{2} \), we have

\[
(W(y, x, z) - W(x, y, z)) \Delta \varphi_\varepsilon(y, x, z) \leq 4b_0 ^2 \min\{1, \{8z\}^\eta\} \varphi_\varepsilon(x) \leq 8^{1+\eta} b_0^2.
\]
So we have proved (2.15), (2.16), (2.17) thus (2.13) holds. The proof of (2.14) is analogous, in fact we can use the same method to prove that

\[ \lim_{\epsilon \to 0^+} W(0, y, z) \Delta_{\text{sym}} \varphi_{\epsilon}(0, y, z) = 0 \quad \text{for all} \quad 0 < y < z, \]
\[ \lim_{\epsilon \to 0^+} (W(y, 0, z) - W(0, y, z)) \Delta \varphi_{\epsilon}(y, x, z) = 0 \quad \text{for all} \quad 0 < y < z, \]
\[ W(0, y, z) \Delta_{\text{sym}} \varphi_{\epsilon}(y, 0, z) \leq 8^{1+\eta} b_0^2 \quad \text{for all} \quad 0 < y \leq z, \]
\[ ((W(y, 0, z) - W(0, y, z)) \Delta \varphi_{\epsilon}(y, 0, z) \leq 8^{1+\eta} q_1 b_0^2 \quad \text{for all} \quad 0 < y \leq z. \]

The only difference is that we can not prove

\[ \lim_{\epsilon \to 0^+} W(0, y, z) \Delta_{\text{sym}} \varphi_{\epsilon}(0, y, z) = 0 \quad \text{for all} \quad 0 < y = z, \]
\[ \lim_{\epsilon \to 0^+} (W(y, 0, z) - W(0, y, z)) \Delta \varphi_{\epsilon}(y, x, z) = 0 \quad \text{for all} \quad 0 < y = z, \]

Combining (2.9), (2.13), (2.14), and taking sup limits in (2.8) as \( \epsilon \to 0^+ \) we have

\[ F_t(\{0\}) \leq F_0(\{0\}) + \int_0^t F_\tau(\{0\}) 8^{2+\eta} b_0^2 (1 + q_1) N^2 d\tau. \]

So by Gronwall inequality we conclude

\[ F_t(\{0\}) \leq e^{8^{2+\eta} b_0^2 (1 + q_1) N^2 t} F_0(\{0\}), \quad t \geq 0. \]

\( \square \)

**Remark 2.3.** The above inequality, i.e. \( F_t(\{0\}) \leq e^{Ct} F_0(\{0\}) \), is very special and has an obvious physics meaning: under the assumption about balanced potential, if there is no seed of condensation at the origin, then there is always no condensation at the origin. However this property does not hold for a set away from the origin, i.e. the inequality like \( F_t(\{x\}) \leq e^{Ct} F_0(\{x\}) \) may not hold for \( x > 0 \). In the following we only show this phenomenon for \( x \) belonging to a set of positive integers. The proof for other \( x \in (0, \infty) \) is essentially the same.

**Example (propagation of singularity away from the origin).**

Let \( F_0 \in B_0^+(\mathbb{R} \geq 0) \) satisfy \( F_0(\{1\}) > 0, F_0(\{2\}) > 0 \). Let \( F_t \in B_0^+(\mathbb{R} \geq 0) \) with the initial datum \( F_0 \) be a conservative measure-valued solution of Eq. (1.1) where the collision kernel \( B \) together with \( F_0 \) satisfies one of the following two conditions:

(a) \( B(\mathbf{v} - \mathbf{v}_s, \omega) \) satisfies Assumption 1.6 with \( \inf_{R \geq R} \hat{\varphi}(r) > 0 \) for all \( R > 0 \), and \( M_{-1/2}(F_0) < \infty \);

(b) \( B(\mathbf{v} - \mathbf{v}_s, \omega) = \frac{1}{(4\pi)^2} |(\mathbf{v} - \mathbf{v}_s) \cdot \omega| \) (the hard sphere model) and \( M_{-1/2}(F_0) \leq \frac{1}{(4\pi)^2} [\varphi(F_0) E(F_0)]^{1/4} \).

Then \( F_t(\{n\}) > 0 \) for all \( n \in \mathbb{N} \) and all \( t > 0 \).

**Proof.** In the proof we will use some notations and results in Section 3 and Section 4. First of all we note that each of the conditions (a), (b) implies that the solution \( F_t \) is unique ( see Theorem 1.9 and Theorem 3.2 of [20]), and this allows us to use approximate solutions.
Part (a): Denote \( a(r) = \inf_{l \geq r} \hat{\delta}(l) \), \( u = F_0(\{1\}), v = F_0(\{2\}), H_0 = F_0 - u\delta(-1) - v\delta(-2) \), where \( \delta(\cdot) \) is the Dirac measure concentrated at \( x = 0 \). For any \( 2 \leq k \in \mathbb{N} \), let

\[
f_{0,k}(x) = \frac{k^2}{2} 1_{[-\frac{1}{k}, \frac{1}{k}]} + \frac{k^2}{2\sqrt{2}} 1_{[\frac{1}{k}, \frac{1}{k}]}(x) + \tilde{f}_{0,k}(x), \quad x \in (0, \infty)
\]

where \( \tilde{f}_{0,k}(\cdot) \) converges to \( H_0 \) weakly, i.e.

\[
\lim_{k \to \infty} \int_{\mathbb{R}^2} \varphi \tilde{f}_{0,k}(x) \sqrt{d}x = \int_{\mathbb{R}^2} \varphi H_0(x) \quad \forall \varphi \in C_0(\mathbb{R}^2).
\]

We can choose \( \tilde{f}_{0,k} \) appropriately such that

\[
\frac{1}{2} N(F_0) \leq N(f_{0,k}) \leq 2 N(F_0), \quad \frac{1}{2} E(F_0) \leq E(f_{0,k}) \leq 2 E(F_0),
\]

\[
\frac{1}{2} M_{-1/2}(F_0) \leq M_{-1/2}(f_{0,k}) \leq 2 M_{-1/2}(F_0) \quad \forall k \geq 2.
\]

(2.18)

It is obvious that \( f_{0,k}(\cdot) \) converges weakly to \( F_0 \). Using Lemma 2.2 and Lemma 2.3 in [2], \( W(x, y, z) \leq 4b_0^2 W_H(x, y, z) \), Theorem 2.5 and Theorem 1.8, we know there exist unique conservative mild solutions \( f_k \) on \( \mathbb{R}^2 \times [0, \infty) \) with initial data \( f_{0,k} \) and satisfies for any \( T \in [0, \infty) \),

\[
\sup_{\tau \in [0, T], x \leq 5, k \geq 2} L(f_k)(x, \tau)
\]

\[
\leq \sup_{\tau \in [0, T], x \leq 5, k \geq 2} 4b_0^2 (\sqrt{x} N(f_k) + M_{1/2}(f_k)(\tau) + 2[M_{-1/2}(f_k)]^2(\tau)) := C_T < \infty.
\]

So by Proposition 4.3 we get

\[
f_k(x, t) = f_{0,k}(x)e^{-\int_0^t L(f_k)(x, \tau)d\tau} + \int_0^t Q^+(f_k)(x, \tau)e^{-\int_0^\tau L(f_k)(x, s)ds}d\tau
\]

\[
\geq f_{0,k}(x)e^{-\int_0^t L(f_k)(x, \tau)d\tau} \geq f_{0,k}(x)e^{-\int_0^t C_Td\tau} = f_{0,k}(x)e^{-C_Tt}
\]

for all \( x \in [0, 4], k \geq 2 \) and \( t \in [0, T] \). By (1.18), we calculate

\[
W(x, y, z) \sqrt{y} \sqrt{z} = \frac{1}{4\pi \sqrt{x}} \int_{[\sqrt{x}+\sqrt{y}] \backslash [\sqrt{x}-\sqrt{y}, \sqrt{x}-\sqrt{z}]} ds \int_0^{2\pi} \Phi(\sqrt{2}s, \sqrt{2}Y_\tau)d\theta
\]

\[
\geq \frac{1}{4\pi \sqrt{x}} \int_{[\sqrt{x}-\sqrt{y}] \backslash [\sqrt{x}-\sqrt{z}, \sqrt{x}-\sqrt{z}]} ds \int_0^{2\pi} a^2 \frac{1}{8}d\theta = \frac{\sqrt{x}}{2\sqrt{x}} a^2 \frac{1}{8} \geq \frac{1}{2\sqrt{13}} a^2 \frac{1}{8} := c_1
\]

for all \( x \in [\frac{11}{4}, \frac{13}{4}], y \in [\frac{7}{4}, \frac{9}{4}], z \in [\frac{7}{4}, \frac{9}{4}] \). So for all \( x \in [3 - \frac{1}{2k}, 3 + \frac{1}{2k}] \)

\[
Q^+(f_k)(x, \tau) \geq \int_{\mathbb{R}^2} W(x, y, z) f_k(y, \tau) f_k(z, \tau) f_k(x, \tau) \sqrt{y} \sqrt{z} dydz
\]

\[
\geq \int_{y,z \in [2 - 1/4, 2 + 1/4]} c_1 f_k(y, \tau) f_k(z, \tau) f_k(x, \tau) dydz
\]

\[
\geq \int_{y,z \in [2 - 1/4, 2 + 1/4]} e^{-3C_T \tau} c_1 k^3 u v^2 16 dydz = c_1 e^{-3C_T \tau} c_1 k u v^2 64 \quad \tau \in [0, T].
\]
Thus for all \( x \in [3 - \frac{1}{2k}, 3 + \frac{1}{2k}] \), \( t \in [0, T] \), we have
\[
f_k(x, t) = f_{0, k}(x)e^{-\int_0^t L(f_k(x, \tau))d\tau} + \int_0^t Q^+(f_k)(x, \tau)e^{-\int_\tau^t L(f_k(x, s))ds} \geq \int_0^t e^{-3Ct}c_1kuv^2 \frac{64}{e^{Ct}} - e^{-3Ct}c_1kuv^2 \frac{128}{64\sqrt{2}} ds = \frac{e^{-Ct} - e^{-3Ct}c_1kuv^2}{C_T}.
\]
This leads to
\[
F_{k, t}([3 - \frac{1}{2k}, 3 + \frac{1}{2k}]) = \int_{3 - \frac{1}{2k}}^{3 + \frac{1}{2k}} f_k(x, t) \sqrt{3} dx \geq \int_{3 - \frac{1}{2k}}^{3 + \frac{1}{2k}} e^{-Ct} - e^{-3Ct}c_1kuv^2 \frac{64\sqrt{2}}{C_T} \sqrt{3} dx.
\]
\[
= e^{-Ct} - e^{-3Ct}c_1kuv^2 \frac{64\sqrt{2}}{C_T}, \quad \tau \in [0, T].
\]
From Theorem 3.2 and Theorem 1.9 (uniqueness), we conclude that the unique conservative measure-valued solution \( F_t \) satisfies
\[
F_t([3 - \frac{1}{2k}, 3 + \frac{1}{2k}]) \geq \frac{e^{-Ct} - e^{-3Ct}c_1kuv^2}{64\sqrt{2}} t \in [0, T].
\]
Since \( k \) can be arbitrarily large, it follows that
\[
F_t([3]) \geq \frac{e^{-Ct} - e^{-3Ct}c_1kuv^2}{64\sqrt{2}} > 0 \quad t \in [0, T].
\]
So far we know \( F_t([1]) > 0, F_t([2]) > 0, F_t([3]) > 0 \) for all \( t > 0 \). In particular for any \( 0 < T < \infty \), we know \( F_T([3]) > 0, F_T([2]) > 0, M_{-1/2}(F_T) < \infty \). So we can use \( F_T \) as initial datum and use the same method to get \( F_t([4]) > 0 \) for all \( \frac{T}{2} < t \leq T \). And we can use \( F_T \) as initial datum to get \( F_t([5]) > 0 \) for all \( \frac{3T}{4} < t \leq T \). By induction we can get \( F_t([n]) > 0 \) for all \( T - \frac{T}{2n} < t \leq T \). In particular we can choose \( t = T \), then \( F_T([n]) > 0 \). Since \( T > 0 \) is arbitrary, we get the conclusion.

Part(b): We use notations and choose \( f_{0, k} \) just the same as in (I). By (2.18), we have \( \|f_{0, k}\|_{L^1} \leq \frac{1}{80}[N(f_{0, k})E(f_{0, k})]^{\frac{1}{2}} \). Using Lemma 2.2, Lemma 2.3 and Theorem 4.1 in [20], we know there exist unique conservative mild solutions \( f_k \) on \( \mathbb{R}_{\geq 0} \times [0, \infty) \) with initial data \( f_{0, k} \) and satisfies
\[
\sup_{\tau \in [0, \infty], x \leq 5, k \geq 2} \mathcal{L}(f_k)(x, \tau) \\
\leq \sup_{\tau \in [0, \infty], x \leq 5, k \geq 2} \sqrt{3}N(f_k) + M_{1/2}(f_k)(\tau) + 2[M_{-1/2}(f_k)]^2(\tau) := C < \infty.
\]
In a way similar to the proof of (I) and using \( W_H(x, y, z)\sqrt{y^2} = \frac{\min(\sqrt{x}, \sqrt{x}, \sqrt{y}, \sqrt{z})}{\sqrt{2}} \), we get
\[
F_{k, t}([3 - \frac{1}{2k}, 3 + \frac{1}{2k}]) \geq \frac{e^{-Ct} - e^{-3Ct}c_1kuv^2}{C_T} \frac{192\sqrt{2}}{192\sqrt{2}}.
\]
By Theorem 3.2 and Theorem 3.2 in [20] (uniqueness), we conclude that the conservative measure-valued solution $F_t$ satisfies

$$F_t([3 - \frac{1}{2k}, 3 + \frac{1}{2k}]) \geq \frac{e^{-Ct} - e^{-3Ct}}{C} \frac{uv^2}{192\sqrt{2}}.$$ 

Since $k$ can be arbitrarily large, we obtain

$$F_t(\{3\}) \geq \frac{e^{-Ct} - e^{-3Ct}}{C} \frac{uv^2}{192\sqrt{2}} > 0.$$ 

Now we have proved $F_t(\{1\}) > 0, F_t(\{2\}) > 0, F_t(\{3\}) > 0$ for all $t > 0$. The rest of the parts of prove is just the same as in Part (a).

**Remark 2.4.** This Example also tells us that for many initial data $F_0$, there is no hope for $F_t$ (with $t > 0$) to be decomposed as $dF_t(x) = f(x, t)\sqrt{x}dx + n(t)\delta(x)dx$ where $0 \leq f(\cdot, t) \in L^1(\mathbb{R}_0, \sqrt{x}dx)$, $n(t) \geq 0$ and $\delta(\cdot)$ is the Dirac delta function concentrated at $x = 0$.

**Theorem 2.5.** Let $F_0 \in B_+^+(\mathbb{R}_0)$ with mass $N = N(F_0)$ and energy $E = E(F_0)$. Given any $0 < p \leq \frac{1}{2}$, suppose $B(v - v_*, w)$ satisfy the Assumption 1.0 with $\eta \geq 1 + p$ and the initial $F_0$ satisfy $M_{-p}(F_0) < \infty$. Let $F_t$ be a conservative measure-valued isotropic solution $F_t$ of Eq.(1.1) on $[0, \infty)$ with the initial datum $F_0$. Then $M_{-p}(F_t) < \infty$ for all $t > 0$. More precisely we have

$$M_{-p}(F_t) \leq (at + M_{-p}(F_0))e^{bt} \quad \forall t \geq 0 \quad (2.19)$$

where $a = 8^2b_0^2N^{\frac{4}{2} + p}E\frac{1}{2 - p} + 8^{2 + \eta}b_0^3N^3$, $b = 8^{3 + \eta}b_0^3N^2(1 + q_1)$.

**Proof.** Denote $\varphi_{\varepsilon, p}(x) = \frac{1}{(x + \varepsilon)^p}$ and $M_{\varepsilon, p}(F_t) = \int_{\mathbb{R}_0} \varphi_{\varepsilon, p}(x)dF_t(x)$. To prove (2.19), first we prove the following differential inequality of $M_{\varepsilon, p}(F_t)$:

$$\frac{d}{dt}M_{\varepsilon, p}(F_t) \leq a + bM_{\varepsilon, p}(F_t).$$

Recalling that $W_H(x, y, z) = \frac{1}{\sqrt{xyz}} \min\{\sqrt{x}, \sqrt{y}, \sqrt{z}, \sqrt{xy}\}$ and $W(x, y, z) \leq 4b_0^2W_H(x, y, z)$, for $0 < y \leq z$ we have

$$\mathcal{J}[\varphi_{\varepsilon, p}](y, z) \leq \frac{1}{2} \int_0^{y+z} W(x, y, z)(\varphi_{\varepsilon, p}(x) + \varphi_{\varepsilon, p}(y + z - x))\sqrt{x}dx$$

$$\leq 2b_0^2 \int_0^{y+z} W_H(x, y, z)(\varphi_{\varepsilon, p}(x) + \varphi_{\varepsilon, p}(y + z - x))\sqrt{x}dx = 4b_0^2 \int_0^{y+z} W_H(x, y, z)\varphi_{\varepsilon, p}(x)\sqrt{x}dx$$

and

$$\int_0^{y+z} W_H(x, y, z)\varphi_{\varepsilon, p}(x)\sqrt{x}dx \leq \int_0^{y} \frac{1}{\sqrt{y}z} dx + \int_0^{z} \frac{1}{\sqrt{2}x^p} dx + \int_z^{y+z} \frac{1}{\sqrt{y}z} \frac{1}{x^p} dx \leq 2\frac{(y + z)^{1-p} - z^{1-p}}{\sqrt{z}} \leq 4z^{\frac{1}{2} - p}.$$
By symmetry of $y, z$ we obtain
$$
\int_0^{y+z} W_H(x, y, z) \varphi_{\varepsilon, p}(x) \sqrt{x} dx \leq 4(y^{\frac{1}{2}-p} + z^{\frac{1}{2}-p}) \quad \text{for all } 0 \leq y, z.
$$
Thus we can use Hölder inequality to get
$$
\int_{\mathbb{R}^2 \geq 0} J[\varphi_{\varepsilon, p}](y, z) d^2 F_t \leq \int_{\mathbb{R}^2 \geq 0} 16(y^{\frac{1}{2}-p} + z^{\frac{1}{2}-p}) b_0^2 d^2 F_t \leq 32 b_0^2 N^{\frac{1}{2}+p} E^{\frac{1}{2}-p}. \quad (2.20)
$$
For the cubic term, we use the following decomposition again
$$
\int_{\mathbb{R}^3 \geq 0} K[\varphi_{\varepsilon, p}] d^3 F_t = \left(2 \int_{0 \leq x < y < z} + 2 \int_{0 \leq y < x < z} + \int_{0 \leq x < y < z} \right) W(x, y, z) \Delta \varphi_{\varepsilon, p}(x, y, z) d^3 F_t
$$
$$
= \int_{0 \leq x < y \leq z} \chi_{y,z} W(x, y, z) \Delta_{\text{sym}} \varphi_{\varepsilon, p}(x, y, z) d^3 F_t
$$
$$
+ 2 \int_{0 \leq x < y < z} (W(y, x, z) - W(x, y, z)) \Delta \varphi_{\varepsilon, p}(y, x, z) d^3 F_t.
$$
$$
+ \int_{0 < y, z < x} W(x, y, z) \Delta \varphi_{\varepsilon, p}(x, y, z) d^3 F_t. \quad (2.21)
$$
Using Lemma 2.1 (2.11) and the fact that $\eta \geq \frac{3}{2}$, $0 < p \leq \frac{1}{2}$, $\varphi_{\varepsilon, p}$ is convex and decreasing we can get the following estimates:
$$
\int_{0 \leq x < y \leq z} \chi_{y,z} W(x, y, z) \Delta_{\text{sym}} \varphi_{\varepsilon, p}(x, y, z) d^3 F_t \leq 2 \int_{0 \leq x < y \leq z} W(x, y, z) \Delta_{\text{sym}} \varphi_{\varepsilon, p}(x, y, z) d^3 F_t
$$
$$
= 2 \int_{0 \leq x < y < z} W(x, y, z) \Delta_{\text{sym}} \varphi_{\varepsilon, p}(x, y, z) d^3 F_t + 2 \int_{0 \leq x < y < z} W(x, y, z) \Delta_{\text{sym}} \varphi_{\varepsilon, p}(x, y, z) d^3 F_t
$$
$$
\leq \int_{0 \leq x < y < \frac{z}{2}} 8 b_0^2 \min \{1, (8z)^\eta\} \frac{\varphi_{\varepsilon, p}(z + y - x)}{\sqrt{y z}} (y - x)^2 \frac{1}{p+1} + \frac{(y - x)^2}{(\varepsilon + z + x - y)^{p+2}} d^3 F_t
$$
$$
+ \int_{0 \leq x < y < \frac{z}{2}} 8 b_0^2 \min \{1, (8z)^\eta\} \frac{\varphi_{\varepsilon, p}(z + y - x)}{\sqrt{y z}} (y - x)^2 \frac{1}{p+1} + \frac{(y - x)^2}{(\varepsilon + z + x - y)^{p+2}} d^3 F_t
$$
$$
\leq \int_{0 \leq x < y < \frac{z}{2}} 8 b_0^2 \min \{1, (8z)^\eta\} \frac{\varphi_{\varepsilon, p}(z + y - x)}{\sqrt{y z}} (y - x)^2 \frac{1}{p+1} + \frac{(y - x)^2}{(\varepsilon + z + x - y)^{p+2}} d^3 F_t
$$
$$
\leq 8^2 + \eta b_0^2 N^3 + 8^2 + \eta b_0^2 N^2 M_{-p}(F_t). \quad (2.22)
$$
Combining (2.20), (2.21), (2.22), (2.23), (2.24), we prove the following inequality:

\[ \epsilon M \leq \min\{1, \{8x\}^q\} \sqrt{x} \Delta \varphi\varepsilon, p(x, y, z) d^3 F \leq \frac{8b_0^2}{\sqrt{x} \sqrt{y} \sqrt{z}} \varphi\varepsilon, p(x, y, z) d^3 F \leq 8b_0^2 \frac{\min\{1, \{8x\}^q\}}{\sqrt{x} \sqrt{y} \sqrt{z}} \varphi\varepsilon, p(x, y, z) d^3 F \]

Combining (2.20), (2.21), (2.22), (2.23), (2.24), we prove the following inequality:

\[\frac{d}{dt} M_{\varepsilon, p}(F_t) \leq 8b_0^2N^3 + 8^{3+q_1}N^2(1+q_1)M_{\varepsilon, p}(F_t) = a + b M_{\varepsilon, p}(F_t).\]

Solving this differential inequality we obtain

\[ M_{\varepsilon, p}(F_t) \leq (8b_0^2N^3 + 8^{3+q_1}N^2(1+q_1)t)e^{8^{3+q_1}N^2(1+q_1)t} M_{\varepsilon, p}(F_0). \]

Let \( \epsilon \to 0^+ \) and using the monotone convergence theorem, the above inequality yields

\[ M_{\varepsilon, p}(F_t) \leq (8b_0^2N^3 + 8^{3+q_1}N^2(1+q_1)t)e^{8^{3+q_1}N^2(1+q_1)t} M_{\varepsilon, p}(F_0) < \infty \quad \forall t \geq 0, \]

which is the desired result. \( \Box \)

### 3 Moment Production and Weak Convergence

To prove moment production and positive lower bound of entropy, as the same in [6], we introduce the following definition of a class of approximate solutions:

**Definition 3.1.** Let \( B(v - v_s, \omega) \) be given by (1.4), (1.3). We say that \( \{B_K(v - v_s, \omega)\}_{K \in \mathbb{N}} \) is a sequence of approximation of \( B(v - v_s, \omega) \) if \( B_K(v - v_s, \omega) \) are such Borel measurable functions on \( \mathbb{R}^3 \times S^2 \) that they are functions of \( |v - v'|, |v - v'_s| \) only and satisfy

\[ B_K(v - v_s, \omega) \geq 0, \quad \lim_{K \to \infty} B_K(v - v_s, \omega) = B(v - v_s, \omega) \]

for a.e \( (v - v_s, \omega) \in \mathbb{R}^3 \times S^2 \). Let \( Q_K(f) \) be the collision integral operators corresponding to the approximate kernels \( B_K \), i.e.

\[ Q_K(f)(v) = \int_{\mathbb{R}^3 \times S^2} B_K(v - v_s, \omega)(f'(1 + f + f_s) - ff_s(1 + f' + f'_s)) d\omega dv_s. \quad (3.1) \]
Given any \( K \in \mathbb{N} \) and \( 0 \leq f^K_0 \in L^1_2(\mathbb{R}^3) \). We say that \( f^K = f^K(\mathbf{v}, t) \) is a conservative approximate solution of Eq. (1.1) on \( \mathbb{R}^3 \times [0, \infty) \) corresponding to the approximate kernel \( B_K \) with the initial datum \( f^K_0 \) if \( (\mathbf{v}, t) \mapsto f^K(\mathbf{v}, t) \) is a nonnegative Lebesgue measurable function on \( \mathbb{R}^3 \times [0, \infty) \) satisfying

(i) \( \sup_{t \geq 0} \| f^K(t) \|_{L^1_2} < \infty \) (here and below \( f^K(t) := f^K(\cdot, t) \)) and

\[
\int_0^T \int_{\mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{S}^2} B_K(\mathbf{v} - \mathbf{v}_s, \omega)(f^K)'(f^K)'(1 + f^K + f^K_*) \sqrt{1 + |\mathbf{v}|^2 + |\mathbf{v}_s|^2} \, d\omega d\mathbf{v}_s < \infty \quad (3.2)
\]

for all \( 0 < T < \infty \).

(ii) There is a null set \( Z \subset \mathbb{R}^3 \) which is independent of \( t \) such that

\[
f^K(\mathbf{v}, t) = f^K_0(\mathbf{v}) + \int_0^t Q_K(f^K)(\mathbf{v}, \tau) \, d\tau \quad \forall \, t \in [0, \infty), \, \forall \, \mathbf{v} \in \mathbb{R}^3 \setminus Z. \quad (3.3)
\]

(iii) \( f^K \) conserves the mass, momentum, and energy, and satisfies the entropy equality, i.e.

\[
\int_{\mathbb{R}^3} (1, \mathbf{v}, |\mathbf{v}|^2/2) f^K(\mathbf{v}, t) \, d\mathbf{v} = \int_{\mathbb{R}^3} (1, \mathbf{v}, |\mathbf{v}|^2/2) f^K_0(\mathbf{v}) \, d\mathbf{v} \quad \forall \, t \geq 0 \quad (3.4)
\]

\[
S(f^K(t)) = S(f^K_0) + \int_0^t D_K(f^K(\tau)) \, d\tau \quad \forall \, t \geq 0. \quad (3.5)
\]

Here \( Q_K(f^K)(\mathbf{v}, t) = Q_K(f^K(\cdot, t))(\mathbf{v}) \), \( D_K(f) \) is the entropy dissipation corresponding to the approximate kernel \( B_K(\mathbf{v} - \mathbf{v}_s, \omega) \), i.e.

\[
D_K(f) = \frac{1}{4} \int_{\mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{S}^2} B_K(\mathbf{v} - \mathbf{v}_s, \omega) \Pi(f)(g'g'_s, gg_s) \, d\omega d\mathbf{v}_s d\mathbf{v} \quad (3.6)
\]

Where

\[
\Gamma(a, b) = \begin{cases} 
(a - b) \log \left( \frac{a}{b} \right) & \text{if } a > 0, b > 0 \\
\infty & \text{if } a > 0 = b \text{ or } a = 0 < b \\
0 & \text{if } a = b = 0
\end{cases} \quad (3.7)
\]

\[
\Pi(f) = (1 + f)(1 + f'_s)(1 + f)'(1 + f'_s), \quad g = \frac{f}{1 + f}. \quad (3.8)
\]

If a conservative approximate solution \( f^K \) is isotropic, i.e. if \( f^K(\mathbf{v}, t) \equiv f^K(|\mathbf{v}|^2/2, t) \), then \( f^K \) is called a conservative isotropic approximate solution of Eq. (1.1). In this case, if we define \( h^K(x) = f^K(|\mathbf{v}|^2/2, t) \) for \( x = |\mathbf{v}|^2/2 \), then \( h^K \) is a mild solution in the sense of Definition (1.4).

A suitbale class of \( B_K \) that was often be used is

\[
B_K(\mathbf{v} - \mathbf{v}_s, \omega) = \min \{ B(\mathbf{v} - \mathbf{v}_s, \omega), K|\mathbf{v} - \mathbf{v}'|^2|\mathbf{v} - \mathbf{v}_s|^2 \}, \quad K \geq 1. \quad (3.9)
\]

An important Theorem will often be used below is Theorem 1 in [16] (weak stability). Notice that the condition \( \int_0^1 B(V, \tau) \, d\tau > 0 \) for all \( V > 0 \) was not used in the proof of Theorem 1 in [16] (weak stability), so using Appendix of [8] (Equivalence of Solutions) we would like to rephrase that Theorem into the following form.
Theorem 3.2. \cite{16}(Weak Stability). Let $B, B_K$ be collision kernels satisfying the conditions

(i) $B(\cdot, \cdot) \in C(\mathbb{R}_0 \times [0, 1])$,

(ii) $\sup_{v \geq 0, \tau \in [0, 1]} \frac{B(v, \tau)}{1 + v} < \infty$, $\sup_{v \geq 0} \frac{B(V, \tau)}{1 + V} \to 0$ as $\tau \to 0^+$.

and either $B_K \equiv B$ ($\forall K \geq 1$) or $B_K$ be the cutoff of $B$ given by $\eqref{3.9}$. Let $F_0, F^K_0 \in B^+_1(\mathbb{R}_0)$ satisfying

$$\sup_{K \geq 1} \int_{\mathbb{R}_0} (1 + x) dF^K_0(x) < \infty$$

and

$$\lim_{n \to \infty} \int_{\mathbb{R}_0} \phi(x) dF^K_0(x) = \int_{\mathbb{R}_0} \phi(x) dF_0(x) \quad \forall \phi \in C_0(\mathbb{R}_0)$$

Let $F^n_t$ be conservative distributional solutions of Eq.$\eqref{1.1}$ with kernel $B_n$ and initial datum $F^n_0$. Then there exist a subsequence $\{F^K_j\}_{j=1}^\infty$ and a conservative distributional $F_t$ of Eq.$\eqref{1.1}$ with the kernel $B$ and initial datum $F_0$ such that

$$\lim_{n \to \infty} \int_{\mathbb{R}_0} \phi(x) dF^K_j(x) = \int_{\mathbb{R}_0} \phi(x) dF_t(x) \quad \forall t \geq 0, \quad \phi \in C_0(\mathbb{R}_0)$$

(therefore $F_t$ conserves the mass) and

$$\int_{\mathbb{R}_0} xdF_t(x) = \liminf_{j \to \infty} \int_{\mathbb{R}_0} xdF^K_j(x).$$

Furthermore if

$$\lim_{K \to \infty} \int_{\mathbb{R}_0} xdF^K_0(x) = \int_{\mathbb{R}_0} xdF_0(x),$$

then the solution $F_t$ also conserves the energy:

$$\int_{\mathbb{R}_0} xdF_t(x) = \int_{\mathbb{R}_0} xdF_0(x).$$

Proposition 3.3. Let the collision kernel $B(v - v_*, \omega)$ be given by $\eqref{1.4}, \eqref{1.5}, \eqref{1.6}$ where the Fourier transform $r \mapsto \hat{\phi}(r)$ is non-negative on $[0, \infty)$ and satisfies

$$a_0 r^{-\beta} 1_{[R, \infty)}(r) \leq \hat{\phi}(r) \leq b_1 \quad \forall r > 0 \quad \eqref{3.10}$$

for some constants $a_0 > 0, 0 \leq \beta < 1/2, 0 < R < \infty$. Given any $N > 0, E > 0$.

(1) Let $B_K(v - v_*, \omega)$ be given by $\eqref{3.9}$, let $\{f^K_0 = f^K_0(|v|^2/2)\}_{K \in \mathbb{N}}$ be any sequence of nonnegative isotropic functions in $L^1_2(\mathbb{R}^3)$ satisfying

$$\int_{\mathbb{R}^3} (1, |v|^2/2) f^K_0(|v|^2/2) dv = 4\pi \sqrt{2}(N, E) \quad \forall K \in \mathbb{N}. \quad \eqref{3.11}$$

Then for every $K \in \mathbb{N}$, there exist a unique conservative isotropic approximate solution $f^K = f^K(|v|^2/2, t)$ of Eq.$\eqref{1.1}$ on $\mathbb{R}^3 \times [0, \infty)$ corresponding to the approximate kernel $B_K$ such that $f^K|_{t=0} = f^K_0$, and it holds the moment production:

$$\sup_{K \in \mathbb{N}} \|f^K(t)\|_{L^s_1} \leq C_s(1 + 1/t)^{s-2} \quad \forall t > 0, \forall s > 2 \quad \eqref{3.12}$$
where the constant $0 < C_s < \infty$ depends only on $N, E, s, a_0, b_1, R$ and $\beta$.

(II) Let $F_0 \in B^1_1(\mathbb{R}_{>0})$ satisfy $N(F_0) = N, E(F_0) = E$. Then there exists a conservative measure-valued isotropic solution $F_t$ of Eq. (1.1) on $[0, \infty)$ with the initial datum $F_0$, such that
\[
M_p(F_t) \leq C_p(1 + 1/t)^{2(p-1)} \quad \forall t > 0, \forall p > 1
\]  
where the constant $0 < C_p < \infty$ depends only on $N, E, p, a_0, b_1, R$ and $\beta$.

**Proof.** We only need to prove part (I), part (II) is only an application of part (I) and . The existence of the conservative isotropic approximate solutions $f^K$ has been proven in Theorem 3 of [14]. So we only need to prove moment production and uniqueness. Without loss of generality, we can assume $R \geq 1$.

For notation convenience we denote (with $K$ fixed)
\[
f^K(|v|^2/2, t) := f^K(|v|^2/2, t).
\]

to prove (3.12), we prove it holds for the case that $\|f_0\|_{L^1_2} < \infty$ for all $s > 2$, then we can get the general case by Theorem 3.2. By Theorem 3 in [14] and further cut-off $B_{K,n}(v, -v_s, \omega) = B_K(v, -v_s, \omega) \wedge n$, there exists a conservative solution $f$ such that $\sup_{t \in [0, \infty]} \|f(\cdot, t)\|_{L^1_2} < \infty$ for all $t_1 > 0$ and $s > 2$. By the same reason in the proof of Theorem 4 in [16], we only need to prove the case for $s \geq 4$. We also need the following version of Povzne-Elmroth inequality (see e.g. [15] and Appendix of [6] (Equivalence of Solutions), we obtain
\[
\langle v' \rangle^s + \langle v_s' \rangle^s - \langle v \rangle^s - \langle v_s \rangle^s \leq 2^{s+1}(\langle v \rangle^{s-1} + \langle v_s \rangle^{s-1}) - 2 \cos^2 \theta \sin^2 \theta \langle v \rangle^s.
\]

Now we can compute
\[
\frac{d\|f(\cdot, t)\|_{L^1_2}}{dt} = \frac{1}{2} \int_{\mathbb{R}^3 \times \mathbb{R}^3} B_K f f_s(\langle v' \rangle^s + \langle v_s' \rangle^s - \langle v \rangle^s - \langle v_s \rangle^s) d\omega dv dv_s
\]
\[
+ 4\pi \sqrt{2} \int_{\mathbb{R}^3} K_{B_K}[\varphi](x, y, z) dF_1(x) dF_1(y) dF_1(z)
\]
\[
\leq 2^s \int_{\mathbb{R}^3 \times \mathbb{R}^3} B_K f f_s(\langle v \rangle^{s-1} + \langle v_s \rangle^{s-1}) d\omega dv dv_s
\]
\[
- \int_{\mathbb{R}^3 \times \mathbb{R}^3} B_K \cos^2 \theta \sin^2 \theta f f_s(\langle v \rangle^s) d\omega dv dv_s
\]
\[
+ \int_{\mathbb{R}^3} K_{B_K}[\varphi](x, y, z) dF_1(x) dF_1(y) dF_1(z)
\]
\[
:= 2^s I_1^{(n)} - I_2^{(n)} + 4\pi \sqrt{2} I_3^{(n)}
\]
where $\varphi(r) = (1+2r)^{\frac{s}{2}}$. Let $A = 4\pi \int_0^{\infty} \cos^2 \theta \sin^3 \theta \min\{\cos^2 \theta \sin \theta, \frac{\theta^2}{(4\pi)^2} \cos \theta\} d\theta$ and $C_s^{(1)}, C_s^{(2)}, ...$ denote finite and strictly positive constants that depend only on $a_0, b_1, s, R$ and $\beta$. Using the same method [16] and Appendix of [6] (Equivalence of Solutions), we obtain
\[
I_1^{(n)} \leq C_s^{(1)} \|f(\cdot, t)\|_{L^1_2} \|f(\cdot, t)\|_{L^1_2}.
\]

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For $|v - v_*| \geq 2R$, using the condition $\tilde{\phi}(r) \geq a_0 r^{-\beta} 1_{[R,\infty)}(r)$, we calculate

\[
\int_{S^2} B_K(v - v_*, \omega) \cos^2 \theta \sin^2 \theta d\omega \geq |v - v_*| \int_{S^2} \cos^2 \theta \sin^2 \theta \min \{\cos^2 \theta \sin \theta, \frac{1}{(4\pi)^2} \cos \theta (\tilde{\phi}(|v - v'|) + \tilde{\phi}(|v - v_*|))^2 \} d\omega \geq |v - v_*|^{1-2\beta} \int_{S^2} \cos^2 \theta \sin^2 \theta \min \{\cos^2 \theta \sin \theta, \frac{a_0^2}{(4\pi)^2} \cos \theta \} d\omega = |v - v_*|^{1-2\beta} A.
\]

Using lemma 10 of [14], we have

\[
I_2^{(n)} \geq A \int_{|v - v_*| \geq 2R} f(v, t) \langle v \rangle^s f(v, t)|v - v_*| A_* dvdv_* \geq A \int_{|v| \geq 2\sqrt{\frac{E}{N}} + 2R} f(v, t) \langle v \rangle^s dv \int_{|v_*| \leq 2\sqrt{\frac{E}{N}}} f(v, t)|v - v_*|^{1-2\beta} dv_* \geq \frac{1}{2} A \int_{|v| \geq 2\sqrt{\frac{E}{N}} + 2R} f(v, t) \langle v \rangle^s dv \int_{|v_*| \leq 2\sqrt{\frac{E}{N}}} f(v, t)(|v|^2 + |v_*|^2)^{\frac{1}{2} - \beta} dv_* \geq \frac{3\pi \sqrt{2} NA_*}{2} \int_{|v| \geq 2\sqrt{\frac{E}{N}} + 2R} f(\langle v \rangle^{s+1-2\beta} - \langle v \rangle^s) dv \geq C_s^{(2)} \||f(\cdot, t)||_{L^1} (\||f(\cdot, t)||_{L^1_{2+1-2\beta}} - (2\sqrt{\frac{E}{N}} + 2R + 2) \||f(\cdot, t)||_{L^1})\).
\]

Using Hölder inequality we obtain

\[
\||f(\cdot, t)||_{L^1_{2+1-2\beta}} \geq \left(\||f(\cdot, t)||_{L^1_{2}}\right)^{-\frac{1-2\beta}{1-\frac{1}{2}}} \left(\||f(\cdot, t)||_{L^1}\right)^{1-\frac{1}{2} + \frac{1}{2}}.
\]

This gives

\[
2^s I_1^{(n)} - I_2^{(n)} \leq C_s^{(3)} \||f(\cdot, t)||_{L^1_{2}} \||f(\cdot, t)||_{L^1_{2}} \||f(\cdot, t)||_{L^1_{2}} \left(\||f(\cdot, t)||_{L^1}\right)^{-\frac{1-\beta}{1-\frac{1}{2}}} \left(\||f(\cdot, t)||_{L^1}\right)^{1-\frac{1}{2} + \frac{1}{2}}.
\]

For $I_3$, using [14], [15], [16], [17], [18] and the condition $|\tilde{\phi}(r)| \leq b_1$, we deduce that

\[
|\mathcal{K}_{B_\infty}[\phi](x, y, z)| \leq 4b_1^2 s^2 (1 + y + z)^{\frac{1}{2} - 1}.
\]

So we obtain

\[
4\pi \sqrt{2} I_3^{(n)} \leq C_s^{(5)} \left(\||f(\cdot, t)||_{L^1}\right)^2 \||f(\cdot, t)||_{L^1_{2}}.
\]

Now we can see $\||f(\cdot, t)||_{L^1_{2}}$ satisfies the following differential inequality

\[
\frac{d}{dt} \||f(\cdot, t)||_{L^1_{2}} \leq C_s^{(6)} (1 + \||f(\cdot, t)||_{L^1_{2}}) \||f(\cdot, t)||_{L^1_{2}} \||f(\cdot, t)||_{L^1_{2}} \leq C_s^{(4)} \left(\||f(\cdot, t)||_{L^1}\right)^{\frac{2\beta-1}{\beta-2}} \left(\||f(\cdot, t)||_{L^1}\right)^{1+\frac{1-2\beta}{\beta-2}}.
\]

which implies that (see [32])

\[
M_s(f(\cdot, t)) < \||f(\cdot, t)||_{L^1_{2}} \leq C_s^{\frac{4-}\beta} (1 + \frac{1}{a})^{\frac{3-2\beta}{2}} (1 + \frac{1}{t})^{\frac{1}{\beta-2}}.
\]

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Then for every constant in (3.12) for we only need to prove the positive lower bound of entropy. As the same in Proposition (3.3), a solution of Eq.(1.1) on a and t for all 

Proof. Part (I): By Proposition 3.3, we have the positive lower bound of entropy as follows:

\[ S(f^K(t)) \geq S(f^K(t_0)) = S_*(t_0) \quad \forall t \geq t_0, \forall K \in \mathbb{N}. \]  

(3.15)

Where

\[ S_*(t_0) = \min \left\{ \frac{7\pi a^3}{24}, \frac{4\pi^2 E^2}{5C(1 + 2/t_0)^2}, \min \left\{ 4m^2, \left( \frac{4\pi}{a} \right)^2 a^2 \right\}, \frac{7\pi^4 \sqrt{2} a^3 E^5 t_0}{96C^3(1 + 2/t_0)^6} \right\} \]  

(3.16)

and

\[ a = \frac{1}{2} \sqrt{E/N}, b = \left( \frac{C}{2\pi \sqrt{2E}} \right) \left( 1 + 2/t_0 \right)^{1/2}, \quad m = \inf_{a \leq 2a + b} \hat{\phi}(r) > 0, \quad 0 < C = C_4 < \infty \text{ is the constant in (3.13) for } s = 4 \text{ so that } C \text{ depends only on } N, E, a_0, b_1, R \text{ and } \beta. \]

(II) Let \( F_0 \in B^+_1(\mathbb{R}_+^0) \) satisfy \( N(F_0) = N, E(F_0) = E \). Then there exists a conservative measure-valued isotropic solution \( F_t \) of Eq.(1.1) on \( [0, \infty) \) with the initial datum \( F_0 \), such that \( F_t \) satisfies moment production (3.13) and

\[ S(F_t) \geq S_*(t_0) \quad \forall t \geq t_0 \]  

(3.17)

for all \( t_0 > 0 \).

Proof. Part (I): By Proposition 3.3, \( f^K \) is unique and satisfies moment production (3.12). So we only need to prove the positive lower bound of entropy. As the same in Proposition (3.3),
we omit the superscript $K$ in $f^K$. Since $t \mapsto S(f(t))$ is non-decreasing, it is sufficient to prove $S(f(t_0)) \geq S_*(t_0)$. To do this we may assume that

$$S(f(t_0)) \leq \min \left\{ \frac{7\pi a^3}{24}, \frac{4\pi^2 E^2}{5C(1 + 2/t_0)^2} \right\}.$$  

Let

$$\mathcal{V}_t = \left\{ (v, v_*, \omega) \in \mathbb{R}^3 \times \mathbb{R}^3 \times S^2 \mid a/2 \leq |v| \leq a, 2a \leq |v'| \leq b, 2a \leq |v_*| \leq b, \right. \quad f(|v|^2/2, t) \leq 1/3, f(|v'|^2/2, t) \geq 9, f(|v_*|^2/2, t) \geq 9, \quad t \geq t_0/2.$$  

Then for all $(v, v_*, \omega) \in \mathcal{V}_t$ we have $a \leq |v - v'| \leq 2a + b, a \leq |v - v_*| \leq 2a + b$ and so

$$B_K(v - v_*, \omega) \geq \frac{|(v - v_*) \cdot \omega|}{(4\pi)^2} \min \left\{ 4m^2, (4\pi)^2 a^2 \right\}.$$  

Using the same method in Proposition 3.4 of [6], we obtain for $t \geq t_0/2$

$$D_K(f(t)) \geq \frac{1}{4} \int_{\mathcal{V}_t} B_K(v - v_*, \omega) \Pi(f) \Gamma(g' g_*, g g_*) d\omega dv_* dv \geq \frac{\min \left\{ 4m^2, (4\pi)^2 a^2 \right\}}{8(4\pi)^2 b^2} \int_{\mathcal{V}_t} |v - v_*| \cdot \omega |v'| f(|v'|^2/2, t) |v_*| f(|v_*|^2/2, t) d\omega dv_* dv \geq \frac{\min \left\{ 4m^2, (4\pi)^2 a^2 \right\}}{8(4\pi)^2 b^2} \times \frac{7\pi a^3}{12} \left( \frac{4\pi^2 E^2}{C(1 + 2/t_0)^2} \right)^2.$$  

Thus we have

$$D_K(f(t)) \geq \frac{\min \left\{ 4m^2, (4\pi)^2 a^2 \right\}}{8(4\pi)^2 b^2} \frac{7\pi a^3}{12} \left( \frac{4\pi^2 E^2}{C(1 + 2/t_0)^2} \right)^2 \geq \min \left\{ 4m^2, (4\pi)^2 a^2 \right\} \frac{7\pi^4 \sqrt{2a^3 E^5}}{48C^3(1 + 2/t_0)^6},$$

$$S(f(t_0)) = S(f(t_0/2)) + \int_{t_0/2}^{t_0} D_K(f(t)) dt \geq \int_{t_0/2}^{t_0} D_K(f(t)) dt \geq \min \left\{ 4m^2, (4\pi)^2 a^2 \right\} \frac{7\pi^4 \sqrt{2a^3 E^5}}{48C^3(1 + 2/t_0)^6} t_0/2 \geq S_*(t_0).$$

This proves (3.15).

Part (II): Since by part (I) we know $f^K$ satisfies the moment production (3.12) and positive lower bound of entropy (3.15), we can use Lemma 3.2 of [6] and Theorem 3.2 to prove the result. □

In order to prove the weak or semi-strong convergence to equilibrium, we need to assume that the function $\hat{\phi}(r)$ has a lower bound function $a(r)$ which is positive, bounded and non-decreasing in $(0, \infty)$. For instance one may take $a(r) = a_0 r^{\eta} \mathbb{1}_{1+r^\eta}$ for some constants $a_0 > 0, \eta \geq 1$ so that it includes many cases of balanced potentials, e.g. the case where $\hat{\phi}(r)$ satisfies $\hat{\phi}(r) = b_0 r^{\eta} \mathbb{1}_{1+r^\eta}$. In
In this section, we use Theorem 2.5 (the most important case is $p$ of Theorem 1 of [17]. Using Theorem 3.2 and Lemma 2.1 of [6] we get the result.

Thus we can choose the same approximation solution $f^K$ with $K \geq b_1^2$ in Theorem 1 of [14] (with $b_1^2 \equiv b_1^2$, $\Psi(r) = a^2(\frac{1}{\sqrt{2}} r)/b_1^2$). Since we have proved moment production and positive lower bound of entropy, we can show $f^K$ ($K \geq b_1^2$) satisfies the same results in Theorem 1 of [17]. Using Theorem 3.2 and Lemma 2.1 of [6] we get the result.

4 Regularity and Stability

In this section, we use Theorem 2.5 (the most important case is $p = \frac{1}{2}$.) and some results of [20] to get regularity and stability.

First we define the working space $B_{p,1}(\mathbb{R}_{\geq 0})$ as

$$B_{p,1}(\mathbb{R}_{\geq 0}) = \{ F \in B_1(\mathbb{R}_{\geq 0}) \mid M_p(|F|) < \infty \}, \quad B_{p,1}^+(\mathbb{R}_{\geq 0}) = B_{p,1}(\mathbb{R}_{\geq 0}) \cap B^+(\mathbb{R}_{\geq 0}).$$

It is easily seen that

$$p < q < 0 \implies B_{p,1}(\mathbb{R}_{\geq 0}) \subset B_{q,1}(\mathbb{R}_{\geq 0}), \quad B_{p,1}^+(\mathbb{R}_{\geq 0}) \subset B_{q,1}^+(\mathbb{R}_{\geq 0}).$$

Let us define

$$M_{p,q}(|F|) = M_p(|F|) + M_q(|F|), \quad -\infty < p, q < \infty.$$ And as usual the notations $F \otimes G, F \otimes G \otimes H$ stand for the product measures of $F, G, H$. As the same in [20], we introduce the following lemma.

**Lemma 4.1.** Let the collision kernel $B(v - v_s, \omega)$ be given by (1.4), (1.5), (1.6). Assume $|\hat{\phi}(r)| \leq b_0$ on $[0, \infty)$, we have

(a) Let $F, G, H \in B_{-1/3,1}(\mathbb{R}_{\geq 0}), k \in [0, 1]$. Then

$$\int_{\mathbb{R}^3_{\geq 0}} Wd(|F| \otimes |G| \otimes |H|) \leq 4b_0^2 M_{-1/3}(|F|) M_{-1/3}(|G|) M_{-1/3}(|H|),$$

$$\int_{\mathbb{R}^3_{\geq 0}} (1 + y^k + z^k) Wd(|F| \otimes |G| \otimes |H|) \leq 4b_0^2 M_{-1/3,k-1/3}(|F|) M_{-1/3,k-1/3}(|G|) M_{-1/3,k-1/3}(|H|).$$
Furthermore, if $F, G, H \in \mathcal{B}_{-1/2,1}(\mathbb{R}_{\geq 0})$, then
\[
\int_{\mathbb{R}^3_{\geq 0}} (1 + y^k + z^k) W(|F| \otimes |G| \otimes |H|) \leq a(F, G, H) \min\{\|F\|, \|G\|, \|H\|\}
\]
(4.1)
where
\[
a(F, G, H) := 4b_0^2[M_{-1/2,1/2}(|F|) + M_{-1/2,1/2}(|G|) + M_{-1/2,1/2}(|H|)^2].
\]
(4.2)

(b) Let $\varphi$ be any Borel function on $\mathbb{R}_{\geq 0}$ satisfying $\sup_{x \geq 0} |\varphi(x)|(1 + x)^{-k} \leq 1$ with $k \in [0, 1]$. Then for all $F, G \in \mathcal{B}_{k+1/2}(\mathbb{R}_{\geq 0})$,
\[
\int_{\mathbb{R}^3_{\geq 0}} |\mathcal{J}^{\pm}[\varphi]| d(|F| \otimes |G|) \leq 4b_0^2 \|F\|_{k+1/2} \|G\|_{k+1/2},
\]
(4.3)
\[
\int_{\mathbb{R}^3_{\geq 0}} |\mathcal{K}^{\pm}[\varphi]| d(|F| \otimes |G| \otimes |H|) \leq 8b_0^2 M_{-1/3,0}(|F|) M_{-1/3,1}(|G|) M_{-1/3,0}(|H|).
\]
(4.4)

Furthermore, if $F, G, H \in \mathcal{B}_{-1/2,1}(\mathbb{R}_{\geq 0})$, then
\[
\int_{\mathbb{R}^3_{\geq 0}} |\mathcal{K}^{\pm}[\varphi]| d(|F| \otimes |G| \otimes |H|) \leq 2a(F, G, H) \min\{\|F\|, \|G\|, \|H\|\}.
\]
(4.5)

Proof. This is an immediate consequence of Lemma 2.1 and the fact that $W(x, y, z) \leq 4b_0^2 W(x, y, z)$. □

By Lemma 4.1 as the same in [20], we can define Borel measures $\mathcal{Q}^{\pm}_{2}(F, G) \in \mathcal{B}_k(\mathbb{R}_{\geq 0})$ for $F, G \in \mathcal{B}_{k+1/2}(\mathbb{R}_{\geq 0})$ ($k \in [0, 1]$) and $\mathcal{Q}^{\pm}_{3}(F, G, H) \in \mathcal{B}_1(\mathbb{R}_{\geq 0})$ for $F, G, H \in \mathcal{B}_{-1/3,1}(\mathbb{R}_{\geq 0})$ through Riesz representation theorem by
\[
\int_{\mathbb{R}^3_{\geq 0}} \varphi(x) d\mathcal{Q}^{\pm}_{2}(F, G)(x) = \int_{\mathbb{R}^3_{\geq 0}} \mathcal{J}^{\pm}[\varphi] d(F \otimes G),
\]
(4.6)
\[
\int_{\mathbb{R}^3_{\geq 0}} \varphi(x) d\mathcal{Q}^{\pm}_{3}(F, G, H)(x) = \int_{\mathbb{R}^3_{\geq 0}} \mathcal{K}^{\pm}[\varphi] d(F \otimes G \otimes H)
\]
(4.7)
for all $\varphi \in C_b(\mathbb{R}_{\geq 0})$. It is obvious that $(F, G) \mapsto \mathcal{Q}^{\pm}_{2}(F, G)$ and $(F, G, H) \mapsto \mathcal{Q}^{\pm}_{3}(F, G, H)$ are bounded bilinear and trilinear operators from $[\mathcal{B}_{k+1/2}(\mathbb{R}_{\geq 0})]^2$ to $\mathcal{B}_k(\mathbb{R}_{\geq 0})$ and from $[\mathcal{B}_{-1/3,1}(\mathbb{R}_{\geq 0})]^3$ to $\mathcal{B}_1(\mathbb{R}_{\geq 0})$ respectively ($k \in [0, 1]$) and
\[
\|\mathcal{Q}^{\pm}_{2}(F, G)\|_k \leq 4b_0^2 \|F\|_{k+1/2} \|G\|_{k+1/2},
\]
(4.8)
\[
\|\mathcal{Q}^{\pm}_{3}(F, G, H)\|_0 \leq 8b_0^2 M_{-1/3,0}(|F|) M_{-1/3,1}(|G|) M_{-1/3,0}(|H|),
\]
(4.9)
\[
\|\mathcal{Q}^{\pm}_{3}(F, G, H)\|_k \leq 8b_0^2 M_{-1/3,0}(|F|) M_{-1/3,1}(|G|) M_{-1/3,0}(|H|),
\]
(4.10)
\[
\|\mathcal{Q}^{\pm}_{3}(F, G, H)\|_k \leq 2a(F, G, H) \min\{\|F\|, \|G\|, \|H\|\}.
\]
(4.11)
Here in the third inequality we assume further that $F, G, H \in \mathcal{B}_{-1/2,1}(\mathbb{R}_{\geq 0})$ so that $a(F, G, H) < \infty$. 28
In connecting with the equation Eq. (1.9) we define
\[
Q_2^+(F) = Q_2^+(F, F), \quad Q_2(F) = Q_2^+(F) - Q_2^-(F), \\
Q_3^+(F) = Q_3^+(F, F, F), \quad Q_3(F) = Q_3^+(F) - Q_3^-(F), \\
Q(F) = Q_2(F) + Q_3(F).
\]

We then deduce from
\[
F \otimes F - G \otimes G = \frac{1}{2}(F - G) \otimes (F + G) + \frac{1}{2}(F + G) \otimes (F - G),
\]
\[
Q_2^+(F) - Q_2^+(G) = \frac{1}{2}Q_2^+(F - G, F + G) + \frac{1}{2}Q_2^+(F + G, F - G),
\]
and (4.8) that for all \(F, G \in B_{k+1/2}(\mathbb{R}_{\geq 0})\) (with \(k \in [0, 1]\))
\[
\|Q_2^+(F) - Q_2^+(G)\|_k \leq 4t_0^2\|F + G\|_{k+1/2}\|F - G\|_{k+1/2}. \tag{4.12}
\]

Similarly we deduce from
\[
F \otimes F \otimes F - G \otimes G \otimes G = (F - G) \otimes F \otimes F + G \otimes (F - G) \otimes F + G \otimes G \otimes (F - G),
\]
\[
\|Q_2^+(F) - Q_3^+(G)\|_k \leq \|Q_3^+(F - G, F, F)\|_k + \|Q_3^+(F, F - G, F)\|_k + \|Q_3^+(G, F - G, F)\|_k
\]
and (4.10), (4.11) that
\[
\|Q_3^+(F) - Q_3^+(G)\|_0 \leq 8b_0^2[M_{-1/3}(|F|) + M_{-1/3}(|G|)]^2 M_{-1/3}(|F - G|), \tag{4.13}
\]
\[
\|Q_3^+(F) - Q_3^+(G)\|_k \leq b(F, G)\|F - G\|_k, \quad k \in [0, 1] \tag{4.14}
\]
where for the inequality (4.14) we assume that \(F, G \in B_{-1/2,1}(\mathbb{R}_{\geq 0})\) so that
\[
b(F, G) := 144t_0^2[M_{-1/2,1/2}(|F|) + M_{-1/2,1/2}(|G|)]^2 < \infty. \tag{4.15}
\]

In order to prove Theorem 1.8 and Theorem 1.9 we shall introduce the concept of strong solutions.

**Definition 4.2.** Let \(F_t\) be a distributional solution of Eq. (1.7) on \([0, \infty)\). Let \(0 < T_\infty \leq \infty\).

We say that \(F_t\) is a strong solution of Eq. (1.7) on \([0, T_\infty)\) if it satisfies the following (i)-(iii):

(i) \(t \mapsto F_t\) belongs to \(C^1([0, T_\infty); B_0(\mathbb{R}_{\geq 0}))\),

(ii) \(t \mapsto Q_2^+(F_t), t \mapsto Q_3^+(F_t)\) belong to \(C([0, T_\infty); B_0(\mathbb{R}_{\geq 0}))\), and

(iii)
\[
\frac{d}{dt} F_t = Q(F_t) \quad \text{in} \quad (B_0(\mathbb{R}_{\geq 0}), \| \cdot \|_0) \quad \forall t \in [0, T_\infty). \tag{4.16}
\]

Besides, if \(F_t\) also conserves the energy on \([0, T_\infty)\), then \(F_t\) is also called a conservative strong solution of Eq. (1.7) on \([0, T_\infty)\).

Strong solutions can be also defined on a finite closed time-interval by replacing \([0, T_\infty)\) with \([0, T]\) for \(0 < T < \infty\).
Remark 4.3. Under the condition (ii), the conditions (i),(iii) are equivalent to the integral equation:

\[ F_t = F_0 + \int_0^t Q(F_\tau) d\tau \quad \forall t \in [0, T_\infty) \quad (4.17) \]

where the integral is taken as the Riemann integral defined with the norm \( \| \cdot \|_0 \). This then implies that, under the condition (ii), the integral equation (4.16) is equivalent to its dual form:

\[ \int_{\mathbb{R}^d} \psi dF_t = \int_{\mathbb{R}^d} \psi dF_0 + \int_0^t d\tau \int_{\mathbb{R}^d} \psi dQ(F_\tau) \quad \forall \psi \in L^\infty(\mathbb{R}_\geq 0) \quad (4.18) \]

for all \( t \in [0, T_\infty) \).

Proposition 4.4. Suppose \( B(v - v_*, \omega) \) satisfy Assumption 1.7 with \( \eta \geq \frac{3}{2} \). Let \( F_t \) be a distributional solution of Eq. (1.1) on \([0, \infty)\) with the initial datum \( F_0 \) satisfying \( M_{-1/2}(F_0) < \infty \). Then \( F_t \) is a strong solution of Eq. (1.1) on \([0, \infty)\).

Proof. Take any \( T \in (0, \infty) \). Using Theorem 2.5 for \( p = \frac{1}{2} \) we know \( \sup_{t \in [0, T]} M_{-1/2}(F_t) < \infty \). Combining with the estimates (4.8), (4.11) for \( k = 1/2 \) we have \( \| Q_2^+ (F_t) \|_{1/2}, \| Q_3^+ (F_t) \|_{1/2} \leq C_T \) for all \( t \in [0, T] \), where \( C_T \) \( \infty \) depends only on \( \sup_{t \in [0, T]} M_{-1/2}(F_t) \) \( \infty \) and \( \sup_{t \in [0, T]} \| F_t \|_1 \). From this and the integral equation (1.21) which also reads

\[ \int_{\mathbb{R}^d} \varphi d(F_t - F_s) = \int_s^t d\tau \int_{\mathbb{R}^d} \varphi dQ(F_\tau) \quad \forall \varphi \in C_b^{1,1}(\mathbb{R}_\geq 0) \quad (4.19) \]

we obtain \( \| F_t - F_s \|_0 \leq C_T |t - s| \) for all \( t, s \in [0, T] \). Since \( \| F_t - F_s \|_{1/2} \leq \| F_t - F_s \|_{1/2}^1 \| F_t - F_s \|_0^{1/2} \) by Cauchy-Schwarz inequality, it follows that \( t \mapsto F_t \) also belongs to \( C([0, \infty); B_{1/2}(\mathbb{R}_\geq 0)) \) and thus we conclude from (4.12)-(4.14) with \( k = 0 \) that \( t \mapsto Q_2^+ (F_t), t \mapsto Q_3^+ (F_t) \) hence \( t \mapsto Q(F_t) \) all belong to \( C([0, \infty); B_0(\mathbb{R}_\geq 0)) \). Next for any \( T \in (0, \infty) \), using \( \sup_{t \in [0, T]} M_{-1/2}(F_t) \) \( \infty \) and smooth approximation it is easily deduced that (4.19) with \( s = 0 \) and \( t \in [0, T] \) holds for all bounded Borel functions \( \varphi \) on \( \mathbb{R}_\geq 0 \), in particular it holds for all characteristic functions \( \varphi(x) = 1_E(x) \) of Borel sets \( E \subset \mathbb{R}_\geq 0 \). Therefore \( F_t \) satisfies the integral equation (4.16) and so, according to the equivalent definition of strong solutions discussed in Remark 4.3, \( F_t \) is a strong solutions of Eq. (1.1) on \([0, \infty)\).

The proofs of Theorem 1.8 and Theorem 1.9 is essentially the same as those of Proposition 4.1 and Theorem 3.1 in [20]. The only difference is that we can use Theorem 2.5 and Proposition 4.4 to ensure the propagation of \( M_{-1/2}(F_t) \) so as to obtain a global in time strong solution. For the sake of completeness, we provide complete proofs below.

Proof of Theorem 1.8. Using Theorem 2.5 we know \( \sup_{t \in [0, T]} M_{-1/2}(F_t) \) \( \infty \) for all \( T \in [0, \infty) \). Recalling Proposition 4.4 that \( F_t \) is a strong distributional solution on \([0, \infty)\) and relation
we have $F_t(\{0\}) = 0$ for all $t \in [0, \infty)$, which means that the origin $x = 0$ has no contribution with respect to the measure $F_t$ and thus the integration domain $\mathbb{R}_+ = \mathbb{R}_{>0}$. Let
\[ V_t(\delta) = \sup_{\text{mes}(U) < \delta} F_t(U), \quad t \in [0, \infty) \]
where $E \subset \mathbb{R}_{\geq 0}$ is any Borel set, $U$ is chosen from all open sets in $\mathbb{R}_+$, and $\text{mes}(\cdot)$ denotes the Lebesgue measure on $t \in [0, \infty)$. Take any open set $U \subset \mathbb{R}_+$ satisfying $\text{mes}(U) < \delta$. Applying the integral equation (1.13) to a monotone sequence $0 \leq \varphi_n \in C_b(\mathbb{R}_{\geq 0})$ satisfying
\[ \varphi_n(x) \nearrow \psi_U(x) := 1_U(x) \quad (n \to \infty) \quad \forall x \in \mathbb{R}_+ \]
for instance $\varphi_n(x) = (1 - \exp(-n\text{dist}(x, U^c)))$, and then omitting negative parts we deduce from monotone convergence that
\[ F_t(U) \leq F_0(U) + \int_0^t \int_{\mathbb{R}_+^2} J^+[\psi_U]d^2F_t + \int_0^t \int_{\mathbb{R}_+^3} K^+[\psi_U]d^3F_t, \quad t \in [0, \infty) \]
where
\[ J^+[\varphi](y, z) = \frac{1}{2} \int_0^{y+z} K^+[\varphi](x, y, z) \sqrt{x}dx, \]
\[ K^+[\varphi](x, y, z) = W(x, y, z)[\varphi(x) + \varphi(x^*)]. \]

Next we compute for all $x, y, z > 0$
\[ J^+[\psi_U](y, z) \leq \frac{1}{2} \int_0^{y+z} W(x, y, z)(1_U(x) + 1_U(y + z - x)) \sqrt{x}dx \]
\[ \leq \int_0^{y+z} 2b_0^2 \min\{1, \max\{8x, 8y, 8z\}\} \min\{\sqrt{x}, \sqrt{y}, \sqrt{z}, \sqrt{x^*}\}(1_U(x) + 1_U(y + z - x))dx, \]
\[ \leq \int_0^{y+z} 2b_0^2 8^n(y + z)^\frac{1}{2} \min\{\sqrt{x}, \sqrt{y}, \sqrt{z}, \sqrt{x^*}\}(1_U(x) + 1_U(y + z - x))dx \leq 8^{1+n}b_0^2 \delta, \]

\[ \int_{0 \leq x, y, z} W(x, y, z)1_U(x)dx^3 \leq 2 \left( \int_{0 \leq x \leq y \leq z} W(x, y, z)1_U(x)dx^3 + \int_{0 \leq y \leq x \leq z} W(x, y, z)1_U(x)dx^3 \right) \]
\[ \leq 8b_0^2 \left( \int_{0 \leq x \leq y \leq z} \frac{\min\{1, (8z)^n\}}{\sqrt{y} \sqrt{z}} 1_U(x)dx^3 + \int_{0 \leq y \leq z \leq x} \frac{\min\{1, (8x)^n\}}{\sqrt{x} \sqrt{z}} 1_U(x)dx^3 \right) \]
\[ \leq 8^{1+n}b_0^2 \left( \int_{0 \leq x \leq y \leq z} \frac{1}{\sqrt{y}} 1_U(x)dx^3 + \int_{0 \leq y \leq x \leq z} \frac{1}{\sqrt{y}} 1_U(x)dx^3 + \int_{0 \leq y \leq z \leq x} \frac{1}{\sqrt{z}} 1_U(x)dx^3 \right) \]
\[ \leq 3 \cdot 8^{1+n}b_0^2 NM_{-1/2}(F_t)V_\tau(\delta), \]
and
\[ \int_{0 \leq x, y, z} W(x, y, z)1_U(y + z - x)d^3F = 2(\int_{0 \leq x, y \leq z} W(x, y, z)1_U(y + z - x)d^3F + \int_{0 \leq y \leq z} W(x, y, z)1_U(y + z - x)d^3F) \]
\[ \leq 8b_0^2(\int_{0 \leq x \leq y \leq z} \min\{1, (8z)^{\eta}\} \frac{1}{\sqrt{y}} \sqrt{z} d^3F + \int_{0 \leq y \leq z} \min\{1, (8z)^{\eta}\} \frac{1}{\sqrt{y}} \sqrt{z} d^3F) \]
\[ \leq 8^{1 + \eta}b_0^2(\int_{0 \leq x \leq y \leq z} \frac{1}{\sqrt{y}} d^3F + \int_{0 \leq y \leq z} \frac{1}{\sqrt{z}} d^3F) \]
\[ \leq 3 \cdot 8^{1 + \eta}b_0^2 N M_{-1/2}(F_t)V_r(\delta). \]

It follows that
\[ F_t(U) \leq V_0(\delta) + 8^{1 + \eta}b_0^2 \delta N^2 t + 8^{2 + \eta}b_0^2 N \int_0^t M_{-1/2}(F_r)V_r(\delta)d\tau. \]

Taking \( \sup_{\text{mes}(U) < \delta} \) leads to
\[ V_t(\delta) \leq V_0(\delta) + 8^{1 + \eta}b_0^2 \delta N^2 t + 8^{2 + \eta}b_0^2 N \int_0^t M_{-1/2}(F_r)V_r(\delta)d\tau, \quad t \in [0, \infty) \]
and so, by Gronwall inequality,
\[ V_t(\delta) \leq \left( V_0(\delta) + 8^{1 + \eta}b_0^2 \delta N^2 t \right) \exp \left( 8^{2 + \eta}b_0^2 N \int_0^t M_{-1/2}(F_r)d\tau \right), \quad t \in [0, \infty). \]

Since \( F_0 \) is regular implies \( \lim_{\delta \to 0^+} V_0(\delta) = 0 \) and since \( t \mapsto M_{-1/2}(F_t) \) is locally bounded on \([0, \infty)\) (see Theorem 2.5), it follows that \( \lim_{\delta \to 0^+} V_t(\delta) = 0 \) for all \( t \in [0, \infty) \). This proves that \( V_t \) is absolutely continuous with respect to the Lebesgue measure for every \( t \in [0, \infty) \), and thus there is a unique \( 0 \leq f(\cdot, t) \in L^1(\mathbb{R} \geq 0) \) such that \( dF_t(x) = f(x, t)\sqrt{x}dx \). That is, we have proved that \( F_t \) is regular for all \( t \in [0, \infty) \) and its density \( f(\cdot, t) \) belongs to \( L^1(\mathbb{R} \geq 0) \) for all \( t \in [0, \infty) \).

Since \( \|f(t)\|_{L^1} = M_{-1/2}(F_t) \), it follows from \( \sup_{t \in [0, T]} M_{-1/2}(F_t) < \infty \) and \( W(x, y, z)\sqrt{y}\sqrt{z} \leq 4b_0^2 W_H(x, y, z)\sqrt{y}\sqrt{z} = 4b_0^2 \min_{t \in [0, T]} \sqrt{x}\sqrt{y}\sqrt{z} \) that for all \( 0 < T < \infty \)
\[ \sup_{0 \leq t \leq T} \int_{\mathbb{R}^3_+} W(x, y, z)[f'f_*(1 + f + f_*)] + f'f_*(1 + f' + f_*)]\sqrt{y}\sqrt{z}dx dy dz \]
\[ \leq 16b_0^2 \sup_{0 \leq t \leq T} \left( M_{1/2}(f(t))\|f(t)\|_{L^1} + \|f(t)\|_{L^1}^3 \right) < \infty \]  \hspace{1cm} (4.20)
where \( M_{1/2}(f(t)) = \int_0^\infty xf(x,t)dx \). From this and that \( F_t \) is a strong solution of Eq. (1.1) on \([0, \infty)\) we conclude that the equation

\[
\begin{align*}
\int_{\mathbb{R}_{\geq 0}} \psi(x)\left(f(x,t) - f_0(x) - \int_0^t Q(f(x,\tau))d\tau\right)\sqrt{x} \, dx \\
= \int_{\mathbb{R}_{\geq 0}} \psi(x)d\left(F_t - F_0 - \int_0^t Q(F_\tau)d\tau\right)(x) = 0
\end{align*}
\]

holds for all \( t \in [0, \infty) \) and all bounded Borel functions \( \psi \) on \( \mathbb{R}_{\geq 0} \). Thus for any \( t \in [0, \infty) \), there is a null set \( Z_t \subset \mathbb{R}_{\geq 0} \) such that

\[
f(x,t) = f_0(x) + \int_0^t Q(f(x,\tau))d\tau \quad \forall x \in \mathbb{R}_{\geq 0} \setminus Z_t.
\]

In order to get a common null set \( Z \) independent of \( t \), we consider \( \tilde{f}(\cdot,t) := |f_0 + \int_0^t Q(f(\cdot,\tau))d\tau| \). The advantage of \( \tilde{f}(\cdot,t) \) is that there is a null set \( Z \) which is independent of \( t \) such that \( t \mapsto \tilde{f}(x,t) \) is continuous in \( t \in [0, \infty) \) for all \( x \in \mathbb{R}_+ \setminus Z \). Also, since \( \tilde{f}(x,t) = f(x,t) \) for all \( t \in [0, \infty) \) and all \( x \in \mathbb{R}_+ \setminus Z_t \), it follows from Fubini theorem that \( \tilde{f}(\cdot,t) \) is a mild solution to Eq. (1.1) on \([0, \infty)\). Again since \( \tilde{f}(\cdot,0) = f_0 \) and \( \tilde{f}(x,t) = f(x,t) \) for all \( t \in [0, \infty) \) and all \( x \in \mathbb{R}_+ \setminus Z_t \), it follows that \( \tilde{f}(\cdot,t) \) is also the same density of \( F_t \) for \( t \in [0, \infty) \). Thus by rewriting \( \tilde{f}(\cdot,t) \) as \( f(\cdot,t) \) we conclude that the density \( f(\cdot,t) \) of \( F_t \) is a mild solution of Eq. (1.1) on \([0, \infty)\).

Finally for any \( T \in (0, \infty) \), let \( C_T \) be the left hand side of (1.20). Then we deduce from (1.20) and the definition of mild solutions that \( \|\tilde{f}(\tau) - f(t)\|_{L^1} \leq 2C_T|s - t| \) for all \( s, t \in [0, T] \). Therefore \( f \in C([0, \infty); L^1(\mathbb{R}_{\geq 0})) \).

**Proof of Theorem 1.9.** First according to Proposition 4.4, \( F_t, G_t \) are strong solutions on \([0, \infty)\). The proof is divided into three steps. First we assume that \( F_t \) has the moment production (3.13) for all \( t \in (0, \infty) \). The existence of such \( F_t \) is assured by Proposition 3.4. Let us denote

\[
H_t = F_t - G_t.
\]

By conservation of mass we have \( \|F_t \pm G_t\|_1 \leq \|F_0\|_1 + \|G_0\|_1 \) for all \( t \geq 0 \). So if \( \|H_0\|_1 \geq 1 \), then \( \|H_t\|_1 \leq 2\|F_0\|_1 + \|H_0\|_1 \leq (2\|F_0\|_1 + 1)\|H_0\|_1 \) for all \( t \geq 0 \). Therefore to prove (1.34) we can assume \( \|H_0\|_1 < 1 \).

**Step 1.** Given any \( s \in (0, t) \), we prove that

\[
\begin{align}
\|H_t\|_0 &\leq \|H_0\|_0 + C_1(t) \int_0^t \|H_\tau\|_1 d\tau, \quad (4.21) \\
\|H_t\|_1 &\leq \|H_s\|_1 + C_0 \int_s^t (1 + 1/\tau)\|H_\tau\|_0 d\tau + C_1(t) \int_s^t \|H_\tau\|_1 d\tau. \quad (4.22)
\end{align}
\]
Here and below the constant $0 < C_0 < \infty$ depends only on $N(F_0), E(F_0), \beta, a_0, b_0$ and $R$, and

$$C_1(t) = 288b_0^2(2at + M_{-1/2}(F_0) + M_{-1/2}(G_0) + \|F_0\|_1 + \|G_0\|_1 + 1)^2 e^{2bt},$$

$$a = 8^2b_0^2 \max\{N(F_0), N(G_0)\}^2 + 8^{2+\eta}b_0^2 \max\{N(F_0), N(G_0)\}^3,$$

$$b = 8^{3+\eta}b_0^2 \max\{N(F_0), N(G_0)\}^2(1 + q_1).$$

The inequality\(^{(4.21)}\) follows from $H_t = H_0 + \int_0^t[Q(F_\tau) - Q(G_\tau)]d\tau$, Theorem\(^{2.5}\) and the estimates\(^{(4.12)}, (4.14)}\) for $k = 0$. To prove\(^{(4.22)}\) we first use the identity $|H_t| = -H_t + 2(H_t)^+$ (recall that $H_t = F_t - G_t$) and the conservation of mass and energy to write

$$\|H_t\|_1 = \|G_s\|_1 - \|F_s\|_1 + 2\|(H_t)^+\|_1, \quad t \geq s.$$  \(^{(4.23)}\)

Let $x \mapsto \kappa_t(x) \in \{0, 1\}$ be the Borel function on $\mathbb{R}_{\geq 0}$ such that $\kappa_t(x)dx = d(H_t)^+(x)$. Since $t \mapsto Q(F_t) - Q(G_t)$ belongs to $C([0, \infty); B_0(\mathbb{R}_{\geq 0}))$, applying Lemma 5.1 of\(^{18}\) to the measure equation $H_t = H_s + \int_s^t[Q(F_\tau) - Q(G_\tau)]d\tau, \ t \in [s, \infty)$, we have

$$\int_{\mathbb{R}_{\geq 0}} \psi(x)dx\,(H_t)^+(x) = \int_{\mathbb{R}_{\geq 0}} \psi(x)dH_s(x) + \int_s^t \int_{\mathbb{R}_{\geq 0}} \psi(x)\kappa_\tau(x)d(Q(F_\tau) - Q(G_\tau))(x)$$

for all $t \in [s, \infty)$ and all bounded Borel functions $\psi$ on $\mathbb{R}_{\geq 0}$. In particular we have

$$\int_{\mathbb{R}_{\geq 0}} \psi(x)d(H_t)^+(x) \leq \|(H_s)^+\|_1 + \int_s^t \int_{\mathbb{R}_{\geq 0}} \psi(x)\kappa_\tau(x)d(Q(F_\tau) - Q(G_\tau))(x).$$

Next applying\(^{(3.13)}\) with $p = 3/2$ we see that the function $t \mapsto \|F_t\|_{3/2} < C_0(1 + 1/t)$ is integrable on $[s, T)$. Using the estimate that is analogous to Lemma 3.5 of\(^{20}\) and the reverse Fatou’s Lemma we deduce

$$\limsup_{n \to \infty} \int_s^t \int_{\mathbb{R}_{\geq 0}} (1 + x \wedge n)\kappa_\tau(x)d(Q(F_\tau) - Q(G_\tau))(x)$$

$$\leq C_0 \int_s^t (1 + 1/\tau)\|H_\tau\|_0d\tau + C_1(t) \int_s^t \|H_\tau\|_1d\tau.$$

Letting $n \to \infty$ we conclude

$$\|(H_t)^+\|_1 \leq \|(H_s)^+\|_1 + C_0 \int_s^t (1 + 1/\tau)\|H_\tau\|_0d\tau + C_1(t) \int_s^t \|H_\tau\|_1d\tau.$$  \(^{(4.24)}\)

This together with\(^{(4.23)}\) and $\|G_s\|_1 - \|F_s\|_1 + 2\|(H_s)^+\|_1 = \|H_s\|_1$ gives\(^{(4.22)}\).

**Step 2.** We prove that for any $R_1 \geq 1$

$$\|H_t\|_1 \leq 5R_1\|H_0\|_1 + C_1(t)Rt + 2 \int_{x > R_1} xdF_0(x) \quad (4.24)$$

In fact using $|H_t| = G_t - F_t + 2(H_t)^+$ and conservation of mass and energy we have

$$\|H_t\|_1 \leq \|H_0\|_1 + 4R_1\|H_t\|_0 + 2 \int_{x > R_1} xdF_t(x) \quad (4.25)$$
and applying (4.18) to the bounded function $\psi(x) = 1_{\{x \leq R_1\}}$ we deduce

$$\int_{x > R_1} x dF_t(x) = E(F_0) - \int_{R_2 > 0} 1_{\{x \leq R_1\}} x dF_t(x)$$

$$= \int_{x > R} x dF_0(x) - \int_0^t d\tau \int_{R_2 > 0} 1_{\{x \leq R_1\}} x dQ(F_\tau)(x)$$

$$\leq \int_{x > R_1} x dF_0(x) + R_1 \int_0^t \|Q(F_\tau)\|_0 d\tau \leq \int_{x > R_1} x dF_0(x) + C_1(t) R_1 t.$$ 

This together with (4.25) and (4.26) gives

$$R_B y$$

By Gronwall inequality we then obtain

$$C_2$$

where $C_2(t) = C_1(t) + 5$. Suppose now $\|H_0\|_1 < t$ and let $\varepsilon > 0$ satisfy $\|H_0\|_1 \leq \varepsilon < 1$. Taking $R_1 = \frac{1}{\sqrt{\varepsilon}}$ and using (4.24) we have

$$\|H_t\|_1 \leq C_2(t) \sqrt{\varepsilon} + 2 \int_{x > \frac{1}{\sqrt{\varepsilon}}} x dF_0(x) \leq C_2(\varepsilon) \Psi_{F_0}(\varepsilon), \quad \forall \tau \in [0, \varepsilon]$$

(4.26)

In particular this inequality holds for $\tau = \varepsilon$. Thus using (4.22) for $s = \varepsilon$ gives

$$\|H_t\|_1 \leq C_2(\varepsilon) \Psi_{F_0}(\varepsilon) + C_0 \int_{\varepsilon}^{t} (1 + 1/\tau) \|H_t\|_0 d\tau + C_2(t) \int_{\varepsilon}^{t} \|H_t\|_1 d\tau, \quad t \in [\varepsilon, 1]$$

(4.27)

Next, using (4.21) we know for $\|H_0\|_0 \leq \varepsilon \leq t \leq 1$,

$$\int_{\varepsilon}^{t} (1 + 1/\tau) \|H_\tau\|_0 d\tau \leq 2 \varepsilon \log(1/\varepsilon) + 2 C_1(1) \int_{\varepsilon}^{1} \frac{1}{\tau} \int_{0}^{\tau} \|H_u\|_1 du d\tau$$

$$\leq 2 \sqrt{\varepsilon} + 2 C_1(1) \int_{0}^{t} \|H_u\|_1 \log u|du, \quad t \in [\varepsilon, 1].$$

This together with (4.27) and (4.26) gives

$$\|H_t\|_1 \leq 2 C_2(1) \Psi_{F_0}(\varepsilon) + 2 (C_2(1))^2 \int_{0}^{\varepsilon} (1 + \log |\tau|) \|H_\tau\|_1 d\tau, \quad t \in [0, 1].$$

(4.29)

By Gronwall inequality we then obtain

$$\|H_t\|_1 \leq 2 C_2(1) \Psi_{F_0}(\varepsilon) \exp \left(2 (C_2(1))^2 \int_{0}^{\varepsilon} (1 + \log |\tau|) d\tau\right) = C_3 \Psi_{F_0}(\varepsilon), \quad t \in [0, 1]$$

(4.30)

where $C_3 = 2 C_2(1) \exp \left(2 (C_2(1))^2 \int_{0}^{\varepsilon} (1 + \log |\tau|) d\tau\right)$. Now if $t \leq 1$, then (4.34) follows from (4.30). Suppose $t > 1$, using (4.30) we know $\|H_1\|_1 \leq C_3 \Psi_{F_0}(\varepsilon)$. On the other hand from (4.22) with $s = 1$ we have $\|H_t\|_1 \leq \|H_1\|_1 + (C_1(t) + 2 C_0) \int_{0}^{t} \|H_\tau\|_1 d\tau$ for all $t \in [1, \infty)$ and
so \( \|H_t\|_1 \leq \|H_1\|_1 e^{c(t-1)} \leq C_3 \Psi F_0(\varepsilon) e^{(C_1(t)+2C_0)t} \) for all \( t \in [1, \infty) \) by Gronwall Lemma. This together with the estimate for \( t \in [0,1] \) leads to

\[
\|H_t\|_1 \leq C_3 \Psi F_0(\varepsilon) e^{(C_1(t)+2C_0)t}, \quad t \in [0, \infty).
\]

Using the representation of \( C_1(t) \), we can choose some constant \( C, c \) appropriately such that

\[
\|H_t\|_1 \leq C \Psi F_0(\varepsilon) e^{ct}, \quad t \in [0, \infty).
\] (4.31)

Where \( C, c \) depend only on \( N(F_0), E(F_0), N(G_0), E(G_0), a_0, b_0, \eta, \eta_1, R, M_{1/2}(F_0), M_{1/2}(G_0) \).

Finally if \( \|H_0\|_1 > 0 \) then taking \( \varepsilon = \|H_0\|_1 \) in (4.31) gives (1.34). If \( \|H_0\|_1 = 0 \), then in letting \( \varepsilon \to 0^+ \) we conclude \( \|H_t\|_1 = 0 \) for all \( t \in [0, T] \) and thus (1.34) still holds true. This proves (1.34) for the case where \( F_t \) has the moment production \( (3.13) \). The general case is still true since if \( F_0 = G_0, F_t \) has the moment production, then (1.34) tells us \( F_t = G_t \) for all \( t \in [0, \infty) \).

As did for the classical Boltzmann equation, the collision integral \( Q(f) \) can be decomposed as positive and negative parts:

\[
Q(f)(x) = Q^+(f)(x) - Q^-(f)(x),
\] (4.32)

\[
Q^+(f)(x) = \int_{\mathbb{R}_{>0}^2} W(x, y, z) f(y) f(z)(1 + f(x_1 + f(x)) \sqrt{y} \sqrt{z}) dy dz,
\] (4.33)

\[
Q^-(f)(x) = f(x) L(f)(x),
\] (4.34)

\[
L(f)(x) = \int_{\mathbb{R}_{>0}^2} W(x, y, z) [f(x_1)(1 + f(y) + f(z))] \sqrt{y} \sqrt{z} dy dz.
\] (4.35)

By the fact that \( W(x, y, z) \leq 4b_0^2 W_H(x, y, z) \), it is easy to deduce that for any \( 0 \leq f \in L^1(\mathbb{R}_+, \sqrt{x} dx) \), the function \( x \mapsto L(f)(x) \) is well-defined and

\[
0 \leq L(f)(x) \leq 4b_0^2 (\sqrt{x} N(f) + M_{1/2}(f) + 2[M_{-1/2}(f)]^2).
\] (4.36)

Where the moments for a nonnegative measurable function \( f \) on \( \mathbb{R}_{\geq 0} \) are defined in consistent with the case of measures: \( M_p(f) = M_p(F) \) with \( dF(x) = f(x) \sqrt{x} dx \), i.e.

\[
M_p(f) = \int_{\mathbb{R}_+} x^p f(x) \sqrt{x} dx, \quad p \in (-\infty, \infty).
\] (4.37)

We also denote \( N(f) = M_0(f), E(f) = M_1(f) \). And notice that \( M_{-1/2}(f) = \int_{\mathbb{R}_+} f(x) dx \). The following proposition gives an exponential-positive representation (i.e. Duhamel’s formula) for a class of mild solutions. It has been used in the above Example.

**Proposition 4.5.** Suppose \( B(v - v_*, \omega) \) satisfy Assumption \( \ref{assumption} \) with \( \eta \geq \frac{3}{2} \). Let \( 0 \leq f_0 \in L^1(\mathbb{R}_{\geq 0}) \) have finite mass and energy. There exists a unique conservative mild solution \( f \in \)
Proof. Since \( f_0 \in L^1(\mathbb{R}_{\geq 0}) \) means \( M_{-1/2}(f_0) < \infty \). So using Theorem 1.8, Theorem 1.9 we know there there exists a unique conservative mild solution \( f \in C([0, \infty); L^1(\mathbb{R}_{\geq 0})) \) of Eq. 1.1 on \([0, \infty)\) satisfying \( f(\cdot, 0) = f_0 \). By definition of mild solutions and \( Q(f) = Q^+(f) - fL(f) \) there is a null set \( Z \subset \mathbb{R}_{\geq 0} \) which is independent of \( t \) such that for every \( x \in \mathbb{R}_{\geq 0} \) \( Z \)

\[
\frac{\partial}{\partial t} f(x, t) = Q^+(f)(x, t) - f(x, t)L(f)(x, t) \quad (4.39)
\]

for almost every \( t \in [0, \infty) \). Applying (4.36) and \( f \in C([0, \infty); L^1(\mathbb{R}_{\geq 0})) \) we have

\[
\sup_{t \in [0, T]} L(f)(x, t) \leq \sup_{t \in [0, T]} (\sqrt{2}N(f(t)) + M_{1/2}(f(t)) + 2\|f(t)\|_{L^1}^2) < \infty \quad (4.40)
\]

for all \( T \in (0, \infty) \) and all \( x > 0 \). Therefore, for every \( x \in \mathbb{R}_{\geq 0} \setminus Z \), the function \( t \mapsto f(x, t)e^{-\int_0^t L(f)(x, \tau)d\tau} \) is also absolutely continuous on \([0, T]\) for all \( T \in (0, \infty) \) and thus the Duhamel’s formula (4.38) follows from the differential equation (4.39).

We give a \( L^\infty \) estimates for bounded mild solutions to end this section.

**Proposition 4.6.** Suppose \( B(v - v_*, \omega) \) satisfy Assumption 1.6 with \( \eta \geq \frac{3}{2} \). Let \( 0 \leq f_0 \in L^1(\mathbb{R}_{\geq 0}) \) have finite mass and energy and let \( f \in C([0, \infty); L^1(\mathbb{R}_{\geq 0})) \) be the unique conservative mild solution of Eq. 1.1 on \([0, \infty)\) satisfying \( f(\cdot, 0) = f_0 \). Suppose in addition \( f_0 \in L^\infty(\mathbb{R}_{\geq 0}) \). Then \( f(\cdot, t) \in L^\infty(\mathbb{R}_{\geq 0}) \) for all \( t \in [0, \infty) \) and there holds the following estimate: for all \( t \in [0, \infty) \),

\[
\|f(t)\|_{L^\infty} \leq (1 + \|f_0\|_{L^\infty}) \exp \left(8b_0^2 \int_0^t \|f(\tau)\|_{L^1}^2 d\tau \right), \quad (4.41)
\]

**Proof.** Let \( K(t) \) be the right hand side of (4.41), i.e.

\[
K(t) := (1 + \|f_0\|_{L^\infty})e^{2 \int_0^t a(\tau)d\tau}, \quad a(t) := 4b_0^2 \|f(t)\|_{L^1}^2, \quad t \in [0, \infty).
\]

By definition of mild solutions and \( f(x, 0) = f_0(x) \leq K(0) \) for all \( x \in \mathbb{R}_{\geq 0} \setminus Z \) (here and below \( Z \subset \mathbb{R}_{\geq 0} \) denotes any null set which is independent of time variable) we have for all \( t > 0 \)

\[
(f(x, t) - K(t))_+ = \int_0^t (Q(f)(x, \tau) - 2K(\tau)a(\tau))1_{\{f(x, \tau) > K(\tau)\}} d\tau.
\]
Taking integration with respect to $x \in \mathbb{R}_+$ and omitting the negative part $Q^{-}(f) \geq 0$ gives
\[
\int_{\mathbb{R}_{\geq 0}} (f(x,t) - K(t))_+ \, dx \leq \int_{0}^{t} \, d\tau \int_{\mathbb{R}_{\geq 0}} Q^{+}(f)(x,\tau) \mathbf{1}_{\{f(x,\tau) > K(\tau)\}} \, dx
\]
\[
- \int_{0}^{t} \, d\tau \int_{\mathbb{R}_{\geq 0}} 2K(\tau)a(\tau) \mathbf{1}_{\{f(x,\tau) > K(\tau)\}} \, dx.
\]
For the integrand $Q^{+}(f)(x,\tau)$, we have
\[
f(y,\tau)f(z,\tau)(1 + f(x_{*},\tau) + f(x,\tau)) \leq f(y,\tau)f(z,\tau)(f(x_{*},\tau) - K(\tau))_+
\]
\[
+ f(y,\tau)f(z,\tau)(f(x,\tau) - K(\tau))_+ + 2K(\tau)f(y,\tau)f(z,\tau).
\]
Using the fact that $W(x,y,z)\sqrt{y}\sqrt{z} \leq 4b_{0}^{2}$, we can obtain that
\[
\int_{0}^{t} \, d\tau \int_{\mathbb{R}_{\geq 0}} Q^{+}(f)(x,\tau) \mathbf{1}_{\{f(x,\tau) > K(\tau)\}} \, dx \leq \int_{0}^{t} \, 2a(\tau) \, d\tau \int_{\mathbb{R}_{\geq 0}} (f(x,\tau) - K(\tau))_+ \, dx
\]
\[
+ \int_{0}^{t} \, d\tau \int_{\mathbb{R}_{\geq 0}} 2K(\tau)a(\tau) \mathbf{1}_{\{f(x,\tau) > K(\tau)\}} \, dx.
\]
It follows that for all $t \in [0,\infty)$
\[
\int_{\mathbb{R}_{\geq 0}} (f(x,t) - K(t))_+ \, dx \leq \int_{0}^{t} \, 2a(\tau) \, d\tau \int_{\mathbb{R}_{\geq 0}} (f(x,\tau) - K(\tau))_+ \, dx.
\]
By Gronwall inequality we conclude $\int_{\mathbb{R}_{\geq 0}} (f(x,t) - K(t))_+ \, dx = 0$ for all $t \in [0,\infty)$. This implies $f(\cdot, t) \in L^{\infty}(\mathbb{R}_{\geq 0})$ and $\|f(t)\|_{L^{\infty}} \leq K(t)$ for all $t \in [0,\infty)$, i.e. (4.41) holds true. \hfill \square

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