A POINCARÉ-BIRKHOFF-WITT THEOREM FOR THE UNIVERSAL ENVELOPING ALGEBRA OF A ROTA-BAXTER LIE ALGEBRA

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ABSTRACT. Rota-Baxter associative algebras and Rota-Baxter Lie algebras are both important in mathematics and mathematical physics, with the former a basic structure in quantum field renormalization and the latter a operator form of the classical Yang-Baxter equation. An outstanding problem posed by Gubarev is to determine whether there is a Poincaré-Birkhoff-Witt theorem for the universal enveloping Rota-Baxter associative algebra of a Rota-Baxter Lie algebra. This paper resolves this problem positively, working with operated algebras and applying the method of Gröbner-Shirshov bases.

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1. INTRODUCTION

In this paper, the classical Poincaré-Birkhoff-Witt Theorem for the universal enveloping algebra of a Lie algebra is generalized to Rota-Baxter Lie algebras, by applying the method of Gröbner-Shirshov bases.

1.1. Rota-Baxter associative algebras and Rota-Baxter Lie algebras. The Rota-Baxter operator is a family of linear operators defined on various algebraic structures. Fix a constant \( \lambda \) in the base commutative ring \( \mathbf{k} \) and let \( (A, \ast) \) be a \( \mathbf{k} \)-module with a binary operation \( \ast \). A Rota-Baxter operator of weight \( \lambda \) on \( (A, \ast) \) is a linear map \( P : A \rightarrow A \) satisfying the **Rota-Baxter equation**

\[
P(u) \ast P(v) = P(u \ast P(v)) + P(P(u) \ast v) + \lambda P(u \ast v), \quad \forall u, v \in A.
\]

Then \( (A, P) = (A, \ast, P) \) is called a Rota-Baxter algebra of weight \( \lambda \). The most notable instances of Rota-Baxter algebras are the associative algebra and the Lie algebra.

In the case of associative algebras, the operator was introduced by Glenn Baxter in his probability study [2]. As noted there, the Rota-Baxter equation of weight zero is an abstraction of...
the integration-by-part formula for integrals. Rota-Baxter (associative) algebras were studied in the following years from the perspective of functional analysis and combinatorics. Rota placed this structure at the center of several problems he proposed in combinatorics [27]. At the turn of this century, the Rota-Baxter algebra further showed its significance as one of the main algebraic structures in the Connes-Kreimer approach to the renormalization of quantum field theory [10, 11]. In fact, their fundamental Algebraic Birkhoff Factorization can be reformulated in terms of the Spitzer’s identity for Rota-Baxter algebras [14, 15]. Rota-Baxter algebras also encode the shuffle and stuffle products of multiple zeta values, and gives splitting of the associative multiplication. Since then, Rota-Baxter algebras have experienced rapid developments with broad connections [19].

Rota-Baxter operators on Lie algebras appeared independently in the study of the classical Yang-Baxter equation (CYBE), named after the physicists C. N. Yang and R. J. Baxter [3, 33]. The CYBE is the semi-classical limit of the quantum Yang-Baxter equation and has played a major role in integrable systems and quantum groups [3, 8].

In order to understand the solutions (called r-matrices) of the CYBE in a Lie algebra, their original tensor form was transformed to an operator form which is none other than the Rota-Baxter operators (of weight zero) on the Lie algebra [28]. This operator approach to the CYBE was generalized by Kupershmidt [23] using the notion of $\mathcal{O}$-operators, later also called relative Rota-Baxter operators or generalized Rota-Baxter operators [25, 32].

Rota-Baxter Lie algebras also give the important notions of pre-Lie algebras and post-Lie algebras, and have been studied recently from the perspectives of bialgebras, deformation, cohomology and homotopy [23, 31, 33]. Their integration leads to the notion of Rota-Baxter (Lie) groups with applications to the Yang-Baxter equation [20].

### 1.2. Universal enveloping algebras and PBW type theorems.

Given the vital roles played by Rota-Baxter associative algebras and Rota-Baxter Lie algebras, it is important to study their relationship, expanding the classical relationship between associative algebras and Lie algebras.

Since the classical Poincaré-Birkhoff-Witt (PBW) theorem giving an explicit description of the universal enveloping algebra of a Lie algebra, PBW type theorems have been established in many cases [12]. Obtaining such theorems for Rota-Baxter related structures have been challenging. In the case of the universal enveloping Rota-Baxter algebra of a (tri)dendriform algebra, the notion was introduced in [13]. After some preliminary attempts [6], Chen and Mo [9] proved that dendriform algebra can be embedded into its universal enveloping Rota-Baxter algebra. Eventually, the PBW theorem for the universal enveloping Rota-Baxter algebra of a dendriform algebra was obtained by Gubarev [17], who also established a PBW theorem for pre-Lie and Post-Lie algebra was established in [17].

Back to the case of Rota-Baxter Lie algebras, the Rota-Baxter operator for the associative multiplication becomes the Rota-Baxter operator for the commutator Lie bracket. Thus there is a natural functor from the category of Rota-Baxter associative algebras to the category of Rota-Baxter Lie algebras. For the adjoint functor going in the opposite direction, for a given Rota-Baxter Lie algebra, a universal algebra construction defines its universal enveloping Rota-Baxter algebra. However, an explicit description of the universal enveloping Rota-Baxter algebra, in the sense of a Poincaré-Birkhoff-Witt type theorem, proved to be difficult to establish. Indeed, this problem was recently posed as an open problem by Gubarev [17].

### 1.3. Our approach and an outline of the paper.

The purpose of this paper is to establish the PBW theorem for Rota-Baxter Lie algebras and thus affirmatively resolve the problem posed by
Gubarev. To give the universal enveloping Rota-Baxter algebra of a Rota-Baxter Lie algebra \( g \), the natural choice is to express it as the quotient of the free Rota-Baxter associative algebra generated by the space underlying \( g \), modulo the ideal \( I \) spanned the compatibility of the Lie bracket and the commutator relation. A PBW theorem for Rota-Baxter Lie algebra is to provide a canonical basis for this quotient. A power method for this type of problems is finding a Gröbner-Shirshov basis for the ideal \( I \), extending the work of Gröbner for commutative algebras and Shirshov for Lie algebras. However, previous efforts for Rota-Baxter Lie algebras failed since it is difficult to identify a Gröbner-Shirshov basis of the ideal \( I \). Our strategy is to express the universal enveloping Rota-Baxter algebra as a quotient of the free operated algebra modulo a suitable ideal. Thus both the numerator and the denominator of the quotient are enlarged. Having a larger ideal gives more flexibility for a desired choice of the Gröbner-Shirshov basis for the ideal.

The organization of this paper is as follows. In Section 2, we first recall the needed notions and results on operated algebras and Gröbner-Shirshov bases for free operated algebras. We then state the main theorem giving a Gröbner-Shirshov basis for the ideal defining the universal enveloping Rota-Baxter algebra of a Rota-Baxter Lie algebra (Theorem 2.15). As a consequence, we obtain a PBW theorem for Rota-Baxter Lie algebras (Corollary 2.17). The proof of the main theorem is given in Section 3. In order to show that the given set is a Gröbner-Shirshov basis, we need to verify that all the intersection compositions and including compositions from the leading monomials of the given set are trivial. So we start with a classification of the compositions. Then the verifications of the trivialities of the compositions are carried out in the next two subsections, one for the intersection compositions and one for the including compositions.

**Notation.** Throughout this paper, we will work over a field \( k \) of characteristic zero. It is the base field for all vector spaces, algebras, as well as linear maps. By an algebra we mean a unitary associative algebra, unless otherwise specified.

2. Poincaré-Birkhoff-Witt theorem for Rota-Baxter Lie algebras

After recalling background notions and facts, we state the Poincaré-Birkhoff-Witt theorem for Rota-Baxter Lie algebras.

2.1. Composition-Diamond lemma for free operated algebras. In this section, we review the construction of free operated algebras and Composition-Diamond lemma for free operated algebras.

2.1.1. Free operated algebras. We begin with the concept of operated algebras [18, 19].

**Definition 2.1.** (a) An operated monoid (resp. operated algebra) \( U \) together with a map (resp. linear map) \( P_U : U \to U \).

(b) A homomorphism from an operated monoid (resp. operated algebra) \( (U, P_U) \) to an operated monoid (resp. operated algebra) \( (V, P_V) \) is a monoid (resp. an algebra) homomorphism \( f : U \to V \) such that \( fP_U = P_V f \).

Now we construct free operated algebras. For any set \( Y \), let \( M(Y) \) be the free monoid on \( Y \) with the identity 1 and \([Y] := \{[y] \mid y \in Y \}\) a disjoint copy of \( Y \). Let \( \mathcal{M}_0 = M(X) \) and

\[
\mathcal{M}_1 := M(X \sqcup [\mathcal{M}_0]).
\]

Denote by \( \iota_0 : \mathcal{M}_0 \to \mathcal{M}_1 \) the natural embedding. Assume by induction that for \( n \geq 0 \), we have defined the free monoids \( \mathcal{M}_i, 0 \leq i \leq n + 1 \), with the properties that, for \( 0 \leq i \leq n \), we have
\[ M_{i+1} = M(X \sqcup [M_i]) \] and natural embeddings \( \iota_i : M_i \to M_{i+1} \). Then denote

\[ M_{n+2} := M(X \sqcup [M_{n+1}]). \]

The identity map on \( X \) and the embedding \( \iota_n \) together induce an injection \( \iota_{n+1} : X \sqcup [M_n] \hookrightarrow X \sqcup [M_{n+1}] \), which extends to an embedding (still denoted by \( \iota_{n+1} \)) of free monoids

\[ \iota_{n+1} : M_{n+1} = M(X \sqcup [M_n]) \hookrightarrow M(X \sqcup [M_{n+1}]) = M_{n+2}. \]

This completes our inductive definition of the directed system. Let

\[ M(X) := \lim_{\to n} M_n = \bigcup_{n \geq 0} M_n \]

be the direct limit of the system. Elements in \( M(X) \) are called **bracketed words**. Elements of \( M_n \backslash M_{n-1} \) are said to have **depth** \( n \). Each element \( w \) in \( M(X) \setminus \{1\} \) has a unique factorization

\[ w = w_1 \cdots w_b, \]

for some \( b \) and some \( w_i \in X \sqcup [M(X)] \), for \( 1 \leq i \leq b \). We call \( b \) the **breadth** of \( w \), denoted by \( \text{bre}(w) \). For \( w = 1 \), we define \( \text{bre}(w) = 0 \).

Let \( kM(X) \) be the free \( k \)-module with basis \( M(X) \). Since the basis is a monoid, the multiplication on \( M(X) \) can be extended via linearity to turn the \( k \)-module \( kM(X) \) into an algebra. Similarly, we can extend the operator

\[ \lfloor \rfloor : M(X) \to M(X), \quad w \mapsto \lfloor w \rfloor \]

to a linear operator \( \lfloor \rfloor : kM(X) \to kM(X) \) by linearity and turn the algebra \( kM(X) \) into an operated algebra. Elements in \( kM(X) \) are called **operated polynomials** (or **bracketed polynomials**).

**Lemma 2.2.** ([18]) Let \( X \) be a set. Then, with the structures as above,

(a) the pair \( (M(X), \lfloor \rfloor) \) with the natural embedding \( \iota : X \to M(X) \) is the free operated monoid on \( X \); and

(b) the pair \( (kM(X), \lfloor \rfloor) \) with the natural embedding \( j : X \to kM(X) \) is the free operated algebra on \( X \).

**2.1.2. Composition-Diamond lemma for free operated algebras.** We are going to give a linear basis for the universal enveloping Rota-Baxter algebra of a Rota-Baxter Lie algebra, by the method of Gröbner-Shirshov bases for the free operated algebra \( kM(X) \). For this purpose, we briefly recall the necessary concepts and results [5, 7, 21].

**Definition 2.3.** Let \( X \) be a set, \( \star \) a symbol not in \( X \) and \( X^* = X \sqcup \{\star\} \).

(a) By a **\( \star \)-bracketed word** on \( X \), we mean any bracketed word in \( M(X^*) \) with exactly one occurrence of \( \star \), counting multiplicities. The set of all \( \star \)-bracketed words on \( X \) is denoted by \( M^*(X) \).

(b) For \( q \in M^*(X) \) and \( u \in M(X) \), we define \( q|_u := q|_{\star \to u} \) to be the bracketed word on \( X \) obtained by replacing the unique symbol \( \star \) in \( q \) by \( u \).

(c) For \( q \in M^*(X) \) and \( s = \sum_i c_i u_i \in kM(X) \), where \( c_i \in k \) and \( u_i \in M(X) \), we define \( q|_s := \sum_i c_i q|_{u_i} \).

A monomial order is required in the theory of Gröbner-Shirshov bases.
Definition 2.4. Let $X$ be a set. A monomial order on $\mathcal{M}(X)$ is a well-order $\leq$ on $\mathcal{M}(X)$ such that $u < v \implies q|_u < q|_v$ for all $u, v \in \mathcal{M}(X)$ and $q \in \mathcal{M}^*(X)$.

Here we denote $u < v$ if $u \leq v$ but $u \neq v$.

A linear combination in $k\mathcal{M}(X)$ is called monic if the coefficient of the leading monomial is 1.

Definition 2.5. Let $X$ be a set and $\leq$ a monomial order on $\mathcal{M}(X)$. Let $f, g \in k\mathcal{M}(X)$ be two distinct operated polynomials, each monic with respect to $\leq$.

(a) If there exist $u, v, w \in \mathcal{M}(X)$ such that $w = fu = vg$ with $\max\{\text{bre}(f), \text{bre}(g)\} < \text{bre}(w) < \text{bre}(f) + \text{bre}(g)$, we call the operated polynomial $(f, g)_w := (f, g)_w^u := fu - vg$

the intersection composition of $f$ and $g$ with respect to $(u, v)$.

(b) If there exist $q \in \mathcal{M}^*(X)$ and $w \in \mathcal{M}(X)$ such that $w = f = qg$, we call the operated polynomial $(f, g)_w := (f, g)_w^q := f - qg$

the including composition of $f$ and $g$ with respect to $q$.

In Items $[a]$ and $[b]$ of Definition 2.5, $w$ is called the ambiguity with respect to $f$ and $g$. Now we are ready for the concept of Gröbner-Shirshov bases.

Definition 2.6. Let $X$ be a set and $\leq$ a monomial order on $\mathcal{M}(X)$. Let $S \subseteq k\mathcal{M}(X)$ be monic.

(a) Let $w \in \mathcal{M}(X)$. An element $f \in k\mathcal{M}(X)$ is called trivial modulo $(S, w)$ if $f = \sum c_i q|_{s_i}$, where $c_i \in k$, $q_i \in \mathcal{M}^*(X)$, $s_i \in S$ and $q|_{s_i} < w$.

(b) The set $S$ is called a Gröbner-Shirshov basis in $k\mathcal{M}(X)$ with respect to $\leq$ if every intersection composition $(f, g)_w$ is trivial modulo $(S, w)$, and every including composition $(f, g)_w$ is trivial modulo $(S, w)$.

The Composition-Diamond lemma is the corner stone of the theory of Gröbner-Shirshov bases. For $S \subseteq k\mathcal{M}(X)$, denote by $\text{Id}(S)$ the operated ideal generated by $S$.

Lemma 2.7. (Composition-Diamond lemma [27]) Let $X$ be a set, $\leq$ a monomial order on $\mathcal{M}(X)$ and $S \subseteq k\mathcal{M}(X)$ monic. Then the following conditions are equivalent.

(a) $S$ is a Gröbner-Shirshov basis in $k\mathcal{M}(X)$.

(b) $k\mathcal{M}(X) = k\text{Irr}(S) \oplus \text{Id}(S)$ where $\text{Irr}(S) := \mathcal{M}(X) \setminus \{ q|_{s} \mid q \in \mathcal{M}^*(X), s \in S \}$,

and $\text{Irr}(S)$ is a $k$-basis of the quotient operated algebra $k\mathcal{M}(X)/\text{Id}(S)$.

2.2. Poincaré-Birkhoff-Witt theorem for Rota-Baxter Lie algebras. We first construct the universal enveloping Rota-Baxter algebra of a Rota-Baxter Lie algebra and then provide a PBW-type basis, via the method of Gröbner-Shirshov bases. As a consequence, we prove that a Rota-Baxter Lie algebra can be embedded into its universal enveloping Rota-Baxter algebra.

We begin with the concepts of Rota-Baxter associative algebras and Rota-Baxter Lie algebras.

Definition 2.8. Let $\lambda$ be a fixed element in $k$. 
(a) A Rota-Baxter (associative) algebra of weight $\lambda$ is an (associative) algebra $A$ together with a linear operator $P : A \to A$ such that

$$P(u)P(v) = P(uP(v)) + P(P(u)v) + \lambda P(uv) \text{ for all } u, v \in A.$$  

(b) A homomorphism from a Rota-Baxter algebra $(A, P)$ of weight $\lambda$ to a Rota-Baxter algebra $(B, Q)$ of weight $\lambda$ is an algebra homomorphism $f : A \to B$ such that $fP = Qf$.

Rota-Baxter algebras together with their homomorphisms form a category. Replacing the associative multiplication by the Lie bracket $[\cdot, \cdot]$ in the above definition, we obtain the notions of Rota-Baxter Lie algebras and their homomorphisms, again form a category.

As in the classical case, a Rota-Baxter algebra of weight $\lambda$ is also a Rota-Baxter Lie algebra of weight $\lambda$ under the commutator bracket and the same linear operator, giving rise to a functor from the category of Rota-Baxter algebras to the category of Rota-Baxter Lie algebras. We give a notion for the left adjoint functor.

**Definition 2.9.** Let $(g, P)$ be a Rota-Baxter Lie algebra of weight $\lambda$. The universal enveloping Rota-Baxter algebra of $(g, P)$ is a Rota-Baxter algebra $(U(g), U(P))$ of weight $\lambda$ with a Rota-Baxter Lie algebra homomorphism $i : (g, P) \to (U(g), U(P))$ such that for any Rota-Baxter algebra $(A, Q)$ of weight $\lambda$ and any Rota-Baxter Lie algebra homomorphism $f : (g, P) \to (A, Q)$, there is a unique Rota-Baxter algebra homomorphism $\bar{f} : (U(g), U(P)) \to (A, Q)$ such that $f = \bar{f}i$.

The following is a construction of the universal enveloping Rota-Baxter algebra of a Rota-Baxter Lie algebra $(g, P)$.

Fix a linear basis $X$ of the Lie algebra $g$. Denote the relation from the Rota-Baxter operator by

$$S_{RB} := \left\{ [u][v] - [u][v] - [u][v] - \lambda [uv] \big| u, v \in \mathfrak{g}(X) \right\}.$$

Then $k\mathfrak{g}(X)/\text{Id}(S_{RB})$ is the free Rota-Baxter algebra on $X$\cite{[7], [8]}.  

**Proposition 2.10.** Let $(g, [\cdot, \cdot], P)$ be a Rota-Baxter Lie algebra of weight $\lambda$ with a linear basis $X$. Denote $$S := \left\{ [x] - P(x), \ xy - yx - [x, y], [u][v] - [u][v] - [u][v] - \lambda [uv] \big| x, y \in X, u, v \in \mathfrak{g}(X) \right\}.$$ Then $k\mathfrak{g}(X)/\text{Id}(S)$ is a universal enveloping Rota-Baxter algebra of the Rota-Baxter Lie algebra $(g, [\cdot, \cdot], P)$.

**Proof.** Notice that $$k\mathfrak{g}(X)/\text{Id}(S) \cong (k\mathfrak{g}(X)/\text{Id}(S_{RB}))/\text{Id}(S)/\text{Id}(S_{RB}).$$ Denote by $$j : g \to k\mathfrak{g}(X)/\text{Id}(S_{RB}) \text{ and } \pi : k\mathfrak{g}(X)/\text{Id}(S_{RB}) \to k\mathfrak{g}(X)/\text{Id}(S)$$ the natural embedding as modules and the canonical epimorphism of Rota-Baxter algebras, respectively. Denote $i = \pi j : g \to k\mathfrak{g}(X)/\text{Id}(S)$.

Let $(A, Q)$ be a Rota-Baxter algebra, which as noted above can be viewed as a Rota-Baxter Lie algebra under the commutator bracket with the same Rota-Baxter operator $Q$. Let $f : g \to A$ be a Rota-Baxter Lie algebra homomorphism. Then $f$ is uniquely determined by its restriction $f : X \to A$ to the fixed basis $X$ of $g$. By the universal property of the free Rota-Baxter
algebra \( k\mathfrak{m}(X)/\text{Id}(S_{RB}) \) on \( X \), there exists a unique Rota-Baxter algebra homomorphism \( \tilde{f} : k\mathfrak{m}(X)/\text{Id}(S_{RB}) \to A \) such that \( f = \tilde{f} j \).

Let \( \text{Proposition 2.11.} \)

For \( x, y \in X \) and \( u, v \in \mathfrak{m}(X) \), we have

\[
\tilde{f}(\{x\} - P(x) + \text{Id}(S_{RB})) = \tilde{f}(\{x\} + \text{Id}(S_{RB})) - \tilde{f}(P(x) + \text{Id}(S_{RB})) = Q(\tilde{f}(x + \text{Id}(S_{RB}))) - f(P(x))
\]

(\( \tilde{f} \) a Rota-Baxter algebra homomorphism, \( P(x) \in \mathfrak{g} \) and \( f = \tilde{f} j \))

\[
= Q(f(x)) - f(P(x)) \quad (x \in \mathfrak{g} \text{ and } f = \tilde{f} j)
\]

\[
= Q(f(x)) - Q(f(x)) \quad (f \text{ a Rota-Baxter Lie algebra homomorphism})
\]

\[
= 0,
\]

\[
\tilde{f}(xy - yx - [x, y]_3 + \text{Id}(S_{RB})) = \tilde{f}(xy + \text{Id}(S_{RB})) - \tilde{f}(yx + \text{Id}(S_{RB})) = \tilde{f}([x, y]_3 + \text{Id}(S_{RB}))
\]

(\( \tilde{f} \) a Rota-Baxter algebra homomorphism, \( x, y, [x, y] \in \mathfrak{g} \) and \( f = \tilde{f} j \))

\[
= f(x)f(y) - f(y)f(x) - f([x, y])
\]

(\( f \) a Rota-Baxter Lie algebra homomorphism)

\[
= 0
\]

and

\[
\tilde{f}([u]v - [u]v - [u]v - \lambda uv] + \text{Id}(S_{RB})) = \tilde{f}(0 + \text{Id}(S_{RB})) = 0.
\]

So \( \tilde{f} \) induces a unique Rota-Baxter algebra homomorphism \( \overline{f} : k\mathfrak{m}(X)/\text{Id}(S) \to A \) such that \( \overline{f} \pi = \tilde{f} \). Hence \( \overline{f} \pi j = \tilde{f} j = f \), that is, \( \overline{f} i = f \) for \( i = \pi j \). This verifies the desired universal property. \( \square \)

As we will see in Remark 2.14, the generating set \( S \) is not enough for a Gröbner-Shirshov basis for the ideal \( \text{Id}(S) \). So we need to enlarge \( S \).

**Proposition 2.11.** Let \( (\mathfrak{g}, [\ , \,], P) \) be a Rota-Baxter Lie algebra of weight \( \lambda \) with a linear basis \( X \). Let

\[
T := \{ \{x\} - P(x), xy - yx - [x, y], [u]v - [u]v - [u]v - \lambda uv, P(x)[u] - x[u] - [P(x)]u - \lambda [xu], [u]P(x) - uP(x) - [u]x - \lambda ux, P(x)P(y) - xP(y) - [P(x)]y - \lambda [xy] \mid x, y \in X, u, v \in \mathfrak{m}(X) \}.
\]

Then \( \text{Id}(T) = \text{Id}(S) \) and hence \( k\mathfrak{m}(X)/\text{Id}(T) \) is a universal enveloping Rota-Baxter algebra of \( (\mathfrak{g}, [\ , \,], P) \).
Proof. Obviously $\text{Id}(S) \subseteq \text{Id}(T)$ since $S \subseteq T$. On the other hand, for $x, y \in X$ and $u, v \in \mathcal{M}(X)$, the elements

$$P(x)[v] - [x][v] = [P(x)v] - \lambda[xv],$$
$$[u]P(x) - [uP(x)] - [[u][x] - \lambda[u]x],$$
$$P(x)y - [xP(y)] - [P(x)y] - \lambda[xy]$$

in $T \setminus S$ can be obtained from $[u][v] - [u][v] - [[u][v] - \lambda[u]v]$ by taking either $[u] = [x]$ or $[v] = [x]$ to be $P(x)$, or by taking both $[u] = [x]$ and $[v] = [y]$ to be $P(x)$ and $P(y)$. Thus these elements are in the operated ideal $\text{Id}(S)$. So $\text{Id}(S) = \text{Id}(T)$ and the result follows from Proposition 2.10. \hfill $\square$

Now let $(X, \leq)$ be a well-ordered set. We are going to extend the well-order $\leq$ on $X$ to a monomial order $\leq_{\text{bd}}$ on $\mathcal{M}(X)$. For $u, v \in \mathcal{M}(X)$, we define $u \leq_{\text{bd}} v$ by induction on $\text{deg}(u) + \text{deg}(v) \geq 0$. For the initial step of $\text{deg}(u) + \text{deg}(v) = 0$, we have $u, v \in M(X)$ and write $u = u_1 \cdots u_m, v = v_1 \cdots v_n \in M(X)$, where $u_1, \cdots, u_m, v_1, \cdots, v_n \in X$. Then define

$$u \leq_{\text{bd}} v \iff \begin{cases} \text{bre}(u) < \text{bre}(v), \\ \text{or bre}(u) = \text{bre}(v)(= m) \text{ and } (u_1, \cdots, u_m) \leq (v_1, \cdots, v_m) \text{ lexicographically.} \end{cases}$$

Here we use the convention that the empty word 1 $\leq_{\text{bd}} u$ for all $u \in M(X)$. Then $\leq_{\text{bd}}$ is a well-order on $M(X)$ [1].

For the induction step, we consider several cases. Let $\text{deg}(u)$ denote the number of $x \in X$ and $[\_]$ appearing in $u$, counting multiplicity. If $u \in X$ and $v = [v']$ for some $v' \in \mathcal{M}(X)$, then define $u \leq_{\text{bd}} v$. If $u = [u']$ and $v = [v']$, then define inductively on depth

$$u \leq_{\text{bd}} v \text{ if } u' \leq_{\text{bd}} v'. \quad (2)$$

If $u = u_1 \cdots u_m$ and $v = v_1 \cdots v_n$ with $u_1, \cdots, u_m, v_1, \cdots, v_n \in X \sqcup \mathcal{M}(X)$, then define

$$u \leq_{\text{bd}} v \iff \begin{cases} \text{bre}(u) < \text{bre}(v), \\ \text{or bre}(u) = \text{bre}(v)(= m) \text{ and } \text{deg}(u) < \text{deg}(v), \\ \text{or bre}(u) = \text{bre}(v)(= m), \text{deg}(u) = \text{deg}(v) \text{ and } (u_1, \cdots, u_m) \leq (v_1, \cdots, v_m) \text{ lexicographically.} \end{cases}$$

That is,

$$u \leq_{\text{bd}} v \iff (\text{bre}(u), \text{deg}(u), u_1, \cdots, u_m) \leq (\text{bre}(v), \text{deg}(v), v_1, \cdots, v_n) \text{ lexicographically.} \quad (3)$$

To prove that the order $\leq_{\text{bd}}$ is a monomial order on $\mathcal{M}(X)$, we will make use of the following facts.

**Lemma 2.12.** (a) Let $A$ and $B$ be two sets with well-orders. Then we obtain an extended well-order on the disjoint union $A \sqcup B$ by defining $a < b$ for all $a \in A$ and $b \in B$.

(b) Let $\leq_i$ be a well-order on $Y_i$, with $1 \leq i \leq k$ and $k \geq 1$. Then the lexicographical product order $\leq_{\text{lex}}$ is a well-order on the Cartesian product $Y_1 \times \cdots \times Y_k$.

For $u = u_1 \cdots u_m \in \mathcal{M}(X)$ with $u_i \in X \sqcup \mathcal{M}(X)$, denote $\text{st}(u) := (u_1, \cdots, u_m)$.

**Lemma 2.13.** Let $(X, \leq)$ be a well-ordered set. Then $\leq_{\text{bd}}$ is a monomial order on $\mathcal{M}(X)$.

**Proof.** We first show that the $\leq_{\text{bd}}$ is a well-order on $\mathcal{M}(X)$. The restriction of $\leq_{\text{bd}}$ on $X$ is just $\leq$, which is a well-order. By Eq. (2) and the induction on depth, the order $\leq_{\text{bd}}$ is a well-order on $\mathcal{M}(X)$. So $\leq_{\text{bd}}$ is a well-order on $X \sqcup [\mathcal{M}(X)]$ by Lemma 2.12(a). Notice that bre($u$) and deg($u$) with $u \in \mathcal{M}(X)$ are non-negative integers. Hence by Lemma 2.12(b) and Eq. (3), the order $\leq_{\text{bd}}$ is a well-order on $\mathcal{M}(X)$.\hfill $\square$
We are left to verify that \( \leq_{bd} \) is compatible with the bracket \([\ ]\) and the concatenation product. The compatibility of order \( \leq_{bd} \) and \([\ ]\) is from Eq. (2). So we only need to show
\[
u \leq_{bd} \nu \Rightarrow wu \leq_{bd} wv \quad \text{and} \quad u \leq_{bd} \nu \Rightarrow uw \leq_{bd} vw \quad \text{for each} \quad w \in M(X).
\]
We only prove the first one, as the second one is similar. Suppose \( u \leq_{bd} \nu \). By Eq. (3), we have

**Case 1.** If bre\((u) < \text{bre}(v)\), then
\[
\text{bre}(wu) = \text{bre}(w) + \text{bre}(u) < \text{bre}(w) + \text{bre}(v) = \text{bre}(wv),
\]
so \( wu <_{bd} wv \) by Eq. (3).

**Case 2.** If bre\((u) = \text{bre}(v)\) and deg\((u) < \text{deg}(v)\), then bre\((wu) = \text{bre}(wv)\) and
\[
deg(wu) = deg(w) + deg(u) < deg(w) + deg(v) = deg(wv),
\]
whence \( wu <_{bd} wv \) by Eq. (3).

**Case 3.** If bre\((u) = \text{bre}(v)\), deg\((u) = \text{deg}(v)\) and st\((u) \leq \text{st}(v)\) lexicographically, then bre\((wu) = \text{bre}(wv)\), deg\((wu) = \text{deg}(wv)\) and
\[
\text{st}(wu) = (\text{st}(w), \text{st}(u)) \leq (\text{st}(w), \text{st}(v)) = \text{st}(wv) \quad \text{lexicographically}.
\]
It follows from Eq. (3) that \( wu \leq_{bd} wv \).

** Remark 2.14.** With \([x], xy\) and \([u][v]\) as the leading monomials under \( \leq_{bd} \), the set \( S \) in Proposition [2,11] is not a Gröbner-Shirshov basis in \( kM(X) \). For example, the elements \([x][x^2] - [x][x^2] - \lambda x^3] \text{ and } \lambda x^3] \text{ and } \lambda x^3\) and \([x] - P(x)\) in \( S \) with \( x \in X \) give the including ambiguity \([x][x^2]\). Its corresponding including composition is
\[
- [x][x^2] - [[x][x^2] - \lambda x^3] + P(x)[x^2],
\]
which is not trivial:
\[
- [x][x^2] - [[x][x^2] - \lambda x^3] + P(x)[x^2]
\equiv - [x][x^2] - [P(x)x^2] - \lambda x^3] + P(x)[x^2]
\neq 0 \quad \text{mod} (S, [x][x^2]).
\]

Here the last step employs the facts that
\[
- [x][x^2] \in k(M_2(X) \setminus M_1(X)) \quad \text{and} \quad - [P(x)x^2] - \lambda x^3] + P(x)[x^2] \in kM_1(X).
\]

Now we are left to verify our main results.

**Theorem 2.15.** Let \((\mathfrak{g}, [\ ], \lambda, P)\) be a Rota-Baxter Lie algebra of weight \( \lambda \) with a well-ordered linear basis \( X \). With respect to the monomial order \( \leq_{bd} \), the set \( T \) in Eq. (1) is a Gröbner-Shirshov basis in \( kM(X) \).

The proof the theorem is quite technical and will be given in the next section. For now we apply the theorem to obtain the Poincaré-Birkhoff-Witt theorem for Rota-Baxter Lie algebras, giving an affirmative answer to Gubarev’s question [17].

**Theorem 2.16.** Let \((\mathfrak{g}, [\ ], \lambda, P)\) be a Rota-Baxter Lie algebra of weight \( \lambda \) with a linear basis \( X \). Equip \( X \) with a well-order and extend it to the monomial order \( \leq_{bd} \) on \( M(X) \).

(a) The universal enveloping Rota-Baxter algebra \( kM(X)/\text{Id}(T) \) of \( \mathfrak{g} \) has a linear basis
\[
\text{Irr}(T) := M(X) \setminus \{ qT \mid q \in M^*(X), t \in T \}.
\]
(b) The Rota-Baxter Lie algebra \((\mathfrak{g}, [ , ]_\mathfrak{g}, P)\) can be embedded into its universal enveloping Rota-Baxter algebra \(k\mathfrak{M}(X)/\text{Id}(T)\).

Proof. By Lemma 2.7 and Theorem 2.15, the set

\[ \text{Irr}(T) = \mathfrak{M}(X) \setminus \{ qT \mid q \in \mathfrak{M}^*(X), t \in T \} \]

is a linear basis of \(k\mathfrak{M}(X)/\text{Id}(T)\). Since \(X \subseteq \text{Irr}(T)\), the second result follows. \(\square\)

To explicitly describe the linear basis \(\text{Irr}(T)\), we give some more notations. For \(f \in k\mathfrak{M}(X)\), let \(\bar{f}\) be the leading monomial of \(f\) with respect to the monomial order \(\leq_{\text{bd}}\) and denote

\[ \bar{f} := f - \bar{f} \]

Then for \(x, y \in X\), since \(k\) is a field, we may write

\[
\begin{align*}
P(x) &= \bar{P(x)} + P(x), \\
P(x)P(y) &= \bar{P(x)P(y)} + P(x)\bar{P(y)} + P(x)P(y) = \bar{P(x)P(y)} + P(x)\bar{P(y)},
\end{align*}
\]

and thus \(P(x)P(y) = \bar{P(x)P(y)} + \bar{P(x)P(y)} + P(x)P(y)\). Then elements in \(T\) can be rewritten as

\[
\begin{align*}
(4) & \quad [x] - P(x), \\
(5) & \quad xy - yx - [x, y]_3, \\
(6) & \quad [u][v] - [u]v - [u]v - \lambda[uv], \\
(7) & \quad \bar{P(x)[u]} + \bar{P(x)[u]} - [x][u] - [P(x)u] - \lambda[xu], \\
(8) & \quad [u]\bar{P(x)} + [u]\bar{P(x)} - [u]P(x) - [u]x - \lambda[ux], \\
(9) & \quad \bar{P(x)P(y)} + \bar{P(x)P(y)} - [xP(y)] - [P(x)y] - \lambda[xy], \quad x, y \in X, u, v \in \mathfrak{M}(X).
\end{align*}
\]

Correspondingly, the leading monomials are of the following six forms:

\[
\begin{align*}
(10) & \quad (i) \ [x], \quad (ii) \ xy, \quad (iii) \ [u][v], \quad (iv) \ \bar{P(x)[u]}, \quad (v) \ [u]\bar{P(y)}, \quad (vi) \ \bar{P(x)P(y)},
\end{align*}
\]

where \(x, y \in X, u, v \in \mathfrak{M}(X)\) with \(x >_{\text{bd}} y\) in (ii). Thus by Theorem 2.15, we have the following description of \(\text{Irr}(T)\).

Corollary 2.17. The linear basis \(\text{Irr}(T)\) of the universal enveloping Rota-Baxter algebra of a Rota-Baxter Lie algebra \((\mathfrak{g}, [ , ]_\mathfrak{g}, P)\) consists of bracketed words in \(\mathfrak{M}(X)\) that do not contain any subwords of the forms in Eq. (10).

3. The proof of Theorem 2.15

This section is devoted to the proof of Theorem 2.15, that \(T\) is a Gröbner-Shirshov basis in \(k\mathfrak{M}(X)\) with respect to the monomial order \(\leq_{\text{bd}}\). We first give in Section 3.1 a classification of the possible intersections and including compositions to be considered. We then check that all the intersection compositions and including compositions are trivial, in Sections 3.2 and 3.3 respectively.
3.1. Classification of the compositions. By Definition 2.6, the subset $T$ of $k\mathfrak{W}(X)$ is a Gröbner-Shirshov basis means that all intersection compositions and including compositions from $T$ are trivial. With respect to the monomial order $\leq_{bd}$, the leading monomials of the Gröbner-Shirshov basis $T$ are of the six types in Eq. (10).

The ambiguities from all the possible intersection compositions among these six types of leading monomials are summarized in Table 1.

| Ambiguities w | (i) | (ii) | (iii) | (iv) | (v) | (vi) |
|---------------|-----|------|-------|------|-----|------|
| (i)           | -   | -    | -     | -    | -   | -    |
| (ii)          | -   | xyz  | -     | -    | $xP(y)[u]$ | -    | $xP(y)\overline{P(z)}$ |
| (iii)         | -   | -    | $[u][v][r]$ | -     | $[u][v]\overline{P(x)}$ | -    | -    |
| (iv)          | -   | -    | $\overline{P(x)}[u][v]$ | -    | $\overline{P(x)}[u]\overline{P(y)}$ | -    | -    |
| (v)           | $[u]\overline{P(x)y}$ | -    | $[u]\overline{P(x)}[v]$ | -    | $[u]\overline{P(x)}P(y)$ | -    | -    |
| (vi)          | $\overline{P(x)P(y)}z$ | -    | $\overline{P(x)P(y)}[u]$ | -    | $\overline{P(x)P(y)}P(z)$ | -    | -    |

Table 1. Intersection compositions

The ambiguities from all the possible including compositions among these six leading monomials are summarized in Table 2.

| Ambiguities w | (i) | (ii) | (iii) | (iv) | (v) | (vi) |
|---------------|-----|------|-------|------|-----|------|
| (i)           | -   | -    | -     | $\overline{P(x)}[p_{[x]}[y]]$, | $\overline{P(x)}[p_{[x]}[y]]$, | $\overline{P(x)}[p_{[x]}[y]]$, |
|               | -   | -    | -     | or $[u][p_{P(x)}[v]]$, | or $[u][p_{P(x)}[v]]$, | or $[u][p_{P(x)}[v]]$, |
|               | -   | -    | -     | $[x][v]$ | or $[x][v]$ | or $[x][v]$ |
| (ii)          | -   | -    | -     | $\overline{P(x)}[p_{[x]}[v]]$, | $\overline{P(x)}[p_{[x]}[v]]$, | $\overline{P(x)}[p_{[x]}[v]]$, |
|               | -   | -    | -     | or $[u][p_{P(x)}[v]]$, | or $[u][p_{P(x)}[v]]$, | or $[u][p_{P(x)}[v]]$, |
| (iii)         | -   | -    | -     | $\overline{P(x)}[p_{[x]}[v]]$, | $\overline{P(x)}[p_{[x]}[v]]$, | $\overline{P(x)}[p_{[x]}[v]]$, |
|               | -   | -    | -     | or $[u][p_{P(x)}[v]]$, | or $[u][p_{P(x)}[v]]$, | or $[u][p_{P(x)}[v]]$, |
| (iv)          | -   | -    | -     | $\overline{P(x)}[p_{[x]}[v]]$, | $\overline{P(x)}[p_{[x]}[v]]$, | $\overline{P(x)}[p_{[x]}[v]]$, |
|               | -   | -    | -     | or $[u][p_{P(x)}[v]]$, | or $[u][p_{P(x)}[v]]$, | or $[u][p_{P(x)}[v]]$, |
| (v)           | -   | -    | -     | $\overline{P(x)}[p_{[x]}[v]]$, | $\overline{P(x)}[p_{[x]}[v]]$, | $\overline{P(x)}[p_{[x]}[v]]$, |
|               | -   | -    | -     | or $[u][p_{P(x)}[v]]$, | or $[u][p_{P(x)}[v]]$, | or $[u][p_{P(x)}[v]]$, |
| (vi)          | -   | -    | -     | $\overline{P(x)}[p_{[x]}[v]]$, | $\overline{P(x)}[p_{[x]}[v]]$, | $\overline{P(x)}[p_{[x]}[v]]$, |
|               | -   | -    | -     | or $[u][p_{P(x)}[v]]$, | or $[u][p_{P(x)}[v]]$, | or $[u][p_{P(x)}[v]]$, |

Table 2. Including compositions

For $m, n \in \{i, ii, iii, iv, v, vi\}$, let $(m) \wedge (n)$ denote the intersection (resp. including) composition of the type $(m)$ leading monomial with the type $(n)$ leading monomial. From the above two tables, the 11 pairs of compositions in Table 3 are symmetric in the sense that the ambiguity of one composition in a pair gives the ambiguity of the other composition by taking the opposite multiplication. Thus for each pair, we only need to check the triviality of the first composition.
3.2. Triviality of the intersection compositions. Of the 13 intersection compositions, the compositions (ii)∧(ii) and (iii)∧(iii), namely the compositions from the two ambiguities

\[ \text{xxyz and } [u,v][r, s], \quad x, y, z \in X, u, v, r \in \mathfrak{m}(X) \text{ with } x >_{bd} y >_{bd} z, \]

are trivial by the classical PBW theorem for Lie algebras \([20]\) and by \([7]\), respectively.

We now verify that the remaining 11 intersection compositions in Table 3 are trivial. Thanks to the symmetry noted in Table 3, we only need to verify 7 of them as follows.

**Case 1: (ii)∧(iv).** Here the possible ambiguity is \( w = \overline{xP(y)}[u] = \overline{f_1}[u] = xg \), where

\[ f = xP(y) - P(y)x - [x, P(y)]_g, \]

\[ g = \overline{P(y)[u]} + P(\overline{y}) - [y][u] - [P(y)u] - \lambda[yu] \text{ for some } x, y, u \in \mathfrak{m}(X) \text{ with } x >_{bd} \overline{P(y)}. \]

Then

\[ (f, g)_w = (xP(y) - P(y)x - [x, P(y)]_g)[u] - x(P(y)[u] + \overline{P(y)}[u] - [y][u] - [P(y)u] - \lambda[yu]) \]

\[ = - \overline{P(y)}x[u] - [x, \overline{P(y)}]_g[u] - xP(y)[u] + x[y][u] + xP(y)[u] + \lambda x[yu] \]

\[ = - xP(y)[u] + [P(y), x]_g[u] - [x, P(y)]_g[u] - xP(y)[u] \]

\[ + x[y][u] + xP(y)[u] + \lambda x[yu] \quad \text{(by Eq. (5))} \]

\[ \equiv 0 \quad \text{mod } (T, w). \]

**Case 2: (ii)∧(vi).** Here the possible ambiguity is \( w = \overline{xP(y)} P(z) = \overline{f P(z)} = x\overline{g} \), where

\[ f = xP(y) - P(y)x - [x, P(y)]_g, \]

\[ g = \overline{P(y)[u]} P(z) + P(\overline{y})P(z) - [yP(z)] - [P(y)z] - \lambda[yz] \text{ for some } x, y, z \in X \text{ with } x >_{bd} \overline{P(y)}. \]

The intersection composition is trivial by

\[ (f, g)_w = (xP(y) - P(y)x - [x, P(y)]_g)P(z) - x(P(y) P(z) + P(y)P(z)) \]

\[ \quad \equiv 0 \quad \text{mod } (T, w). \]
Case 3: (iii) ∨ (v). Here the possible ambiguity is $w = [u][v] P(x) = \overline{f P(x)} = [u] g$, where

$$f = [u] [v] - [u] [v] - [u] v - \lambda [uv],$$

$$g = [v] \overline{P(x)} + [v] \overline{P(x)} - [v] P(x) - [v] x - \lambda [vx]$$

for some $x \in X, u, v \in \mathcal{W}(X)$. Then we have

\[
\begin{align*}
(f, g)_w &= ([u][v] - [u][v] - [u] v - \lambda [uv])P(x) \\
&= -[u]([v] P(x) + [v] \overline{P(x)} - vP(x)] - [v] x - \lambda [vx]) \\
&= -[u][v] P(x) - [u][v] \overline{P(x)} - \lambda [uv] P(x) - [u][v] \overline{P(x)} \\
&= -[u][v] P(x) - [u] [v] x - \lambda [uv] x - \lambda^2 [uvx] \\
&= [u][v] P(x) - [u][v] P(x) - [u][v] x + \lambda [u][v] x + [u][v] \overline{P(x)} - [u][v] \overline{P(x)} \\
&= -[u][v][x] - \lambda [u][v] x + \lambda [uv] \overline{P(x)} - \lambda [uv] P(x) - \lambda [uv] x - \lambda^2 [uvx] \\
&= -[u][v][x] - \lambda [u][v] x + \lambda [uv] \overline{P(x)} - \lambda [uv] P(x) + [u][v][x] + [u][v][x] \\
&= \lambda [uv] P(x) + [u][v][x] + [u][v][x] + \lambda [uv] x + \lambda [uv] x + \lambda^2 [uvx] \\
&= \lambda [uv] x + \lambda^2 [uvx] \\
&= [u][v][x] - [u][v][x] - [u][v][x] - [u][v][x] + \lambda [uv] x + \lambda [uv] x \\
&= -[u][v][x] - \lambda [u][v] x + \lambda [uv] \overline{P(x)} - \lambda [uv] P(x) - \lambda [uv] x - \lambda^2 [uvx] \\
&= -[u][v][x] - \lambda [u][v] x + \lambda [uv] \overline{P(x)} - \lambda [uv] P(x) + [u][v][x] + [u][v][x] \\
&= \lambda [uv] P(x) + [u][v][x] + [u][v][x] + \lambda [uv] x + \lambda [uv] x + \lambda^2 [uvx] \\
&= \lambda [uv] x + \lambda^2 [uvx] \\
&= [u][v][x] - [u][v][x] - [u][v][x] - [u][v][x] + \lambda [uv] x + \lambda [uv] x \\
&= \lambda [uv] x + \lambda^2 [uvx] \\
&= [u][v][x] - [u][v][x] - [u][v][x] - [u][v][x] + \lambda [uv] x + \lambda [uv] x \\
&= \lambda [uv] x + \lambda^2 [uvx]
\end{align*}
\]
\[+ \lambda [uv] \overline{P(x)} - \lambda [uvP(x)] - \lambda [uvx] - \lambda [u] \overline{P(x)} - [u] \overline{P(x)}
\]
\[- \lambda [uv] \overline{P(x)} + [u] \overline{P(x)} + [u] \overline{P(x)} + \lambda [uvP(x)] + [u] \overline{P(x)} + [u] \overline{P(x)} + [u] \overline{P(x)} + \lambda [uvx] + \lambda [u] \overline{P(x)} + \lambda [uvx] + \lambda [u] \overline{P(x)} + \lambda [uvx]
\]
(by Eqs. (9) and (10))

\[\equiv 0 \mod (T, w).
\]

**Case 4:** (iv) \& (v). Here the possible ambiguity is \(w = \overline{P(x)}[u] \overline{P(y)} = \overline{f} \overline{P(y)} = \overline{P(x)} \overline{y},\) where
\[f = \overline{P(x)}[u] + \overline{P(x)}[u] - [x][u] - [P(x)u] - \lambda [xu],\]
\[g = [u] \overline{P(y)} + [u] \overline{P(y)} - [u] \overline{P(y)} - [u] \overline{y} - \lambda [uy] \text{ for some } x, y \in X, u \in \mathfrak{M}(X).
\]

Then
\[(f, g) \equiv \left( \overline{P(x)}[u] + \overline{P(x)}[u] - [x][u] - [P(x)u] - \lambda [xu] \overline{P(y)} - [u] \overline{P(y)} + [u] \overline{P(y)} - [u] \overline{P(y)} + [u] \overline{P(y)} + \lambda [xu] \overline{P(y)} + \lambda [xu] \overline{P(y)} \right)
\]
\[\equiv - \overline{P(x)}[u] \overline{P(y)} + \overline{P(x)}[u] \overline{P(y)} + \overline{P(x)}[u] \overline{y} + \lambda \overline{P(x)}[uy] + [x][u] \overline{P(y)} - [x][u] \overline{P(y)}
\]
\[- [x][u][y] - 2 \lambda [xu][y] + [P(x)u] \overline{P(y)} - [P(x)u] \overline{P(y)} - [P(x)u] \overline{P(y)} - [P(x)u] \overline{P(y)} + \lambda [xu] \overline{P(y)}
\]
\[- [P(x)u] \overline{y} - 2 \lambda [xu][y] + [P(x)u] \overline{P(y)} + [x][u] \overline{P(y)} + [P(x)u] \overline{P(y)} + \lambda [xu] \overline{P(y)}
\]
\[- [P(x)u] \overline{y} + [x][u] \overline{P(y)} - [P(x)u] \overline{y} + [x][u] \overline{P(y)} + [P(x)u] \overline{P(y)} + \lambda [xu] \overline{P(y)}
\]
\[- [P(x)u] \overline{y} + [x][u] \overline{P(y)} + [P(x)u] \overline{y} + [x][u] \overline{P(y)} + [P(x)u] \overline{P(y)} + \lambda [xu] \overline{P(y)}
\]
\[\equiv - \overline{P(x)}[u] \overline{P(y)} + \overline{P(x)}[u] \overline{P(y)} + \overline{P(x)}[u] \overline{y} + \lambda \overline{P(x)}[uy] + [x][u] \overline{P(y)} - [x][u] \overline{P(y)}
\]
\[- [x][u][y] - 2 \lambda [xu][y] + [P(x)u] \overline{P(y)} - [P(x)u] \overline{P(y)} - [P(x)u] \overline{P(y)} + \lambda [xu] \overline{P(y)}
\]
\[- [P(x)u] \overline{y} - 2 \lambda [xu][y] + [P(x)u] \overline{P(y)} + [x][u] \overline{P(y)} + [P(x)u] \overline{P(y)} + \lambda [xu] \overline{P(y)}
\]
\[- [P(x)u] \overline{y} + [x][u] \overline{P(y)} - [P(x)u] \overline{y} + [x][u] \overline{P(y)} + [P(x)u] \overline{P(y)} + \lambda [xu] \overline{P(y)}
\]
\[- [P(x)u] \overline{y} + [x][u] \overline{P(y)} + [P(x)u] \overline{y} + [x][u] \overline{P(y)} + [P(x)u] \overline{P(y)} + \lambda [xu] \overline{P(y)}
\]
\[\equiv - \overline{P(x)}[u] \overline{P(y)} + \overline{P(x)}[u] \overline{P(y)} + \overline{P(x)}[u] \overline{y} + \lambda \overline{P(x)}[uy] + [x][u] \overline{P(y)} - [x][u] \overline{P(y)}
\]
\[- [x][u][y] - 2 \lambda [xu][y] + [P(x)u] \overline{P(y)} - [P(x)u] \overline{P(y)} + \lambda [xu] \overline{P(y)}
\]
\[- [P(x)u] \overline{y} - 2 \lambda [xu][y] + [P(x)u] \overline{P(y)} + [x][u] \overline{P(y)} + \lambda [xu] \overline{P(y)}
\]
\[- [P(x)u] \overline{y} + [x][u] \overline{P(y)} - [P(x)u] \overline{y} + [x][u] \overline{P(y)} + \lambda [xu] \overline{P(y)}
\]
\[- [P(x)u] \overline{y} + [x][u] \overline{P(y)} + [P(x)u] \overline{y} + [x][u] \overline{P(y)} + \lambda [xu] \overline{P(y)}
\]
\[\equiv - \overline{P(x)}[u] \overline{P(y)} + \overline{P(x)}[u] \overline{P(y)} + \overline{P(x)}[u] \overline{y} + \lambda \overline{P(x)}[uy] + [x][u] \overline{P(y)} + [x][u] \overline{P(y)}
\]
\[- [x][u][y] - 2 \lambda [xu][y] + [P(x)u] \overline{P(y)} - [P(x)u] \overline{P(y)} + \lambda [xu] \overline{P(y)}
\]
\[- [P(x)u] \overline{y} - 2 \lambda [xu][y] + [P(x)u] \overline{P(y)} + [x][u] \overline{P(y)} + \lambda [xu] \overline{P(y)}
\]
\[- [P(x)u] \overline{y} + [x][u] \overline{P(y)} - [P(x)u] \overline{y} + [x][u] \overline{P(y)} + \lambda [xu] \overline{P(y)}
\]
\[- [P(x)u] \overline{y} + [x][u] \overline{P(y)} + [P(x)u] \overline{y} + [x][u] \overline{P(y)} + \lambda [xu] \overline{P(y)}
\]
\[ P(x)uP(y) + [xuP(y)] + [P(x)uP(y)] + \lambda [xuP(y)] - \lambda [xuP(y)] + [x[u]y] + [x[u]y] \]
\[ P(x)[u]y + [x[u]y] + [P(x)uP(y)] + \lambda [xuP(y)] + [P(x)[u]y] + \lambda [x[u]y] \]
\[ \lambda P(x)[u]y + \lambda [x[u]y] + \lambda [P(x)uP(y)] + \lambda^2 [xuP(y)] \quad \text{(by Eq. (7) and (9))} \]
\[ \equiv 0 \quad \text{mod} \ (T, w). \]

**Case 5:** (v) ∧ (iv). Here the possible ambiguity is \( w = [u]P(x)[v] = \overline{f}[v] = [u]\overline{g}, \) where
\[ f = [u]\overline{P(x)} + [u]\overline{P(x)} - [uP(x)] - [u][x] - \lambda [ux], \]
\[ g = \overline{P(x)}[v] + \overline{P(x)}[v] - [x][v] - [P(x)v] - \lambda [xv] \quad \text{for some} \ x, X, v \in \mathfrak{g}(X). \]
The intersection composition is trivial by
\[ (f, g)_w = \Bigl( [u]\overline{P(x)} + [u]\overline{P(x)} - [uP(x)] - [u][x] - \lambda [ux] \Bigr)[v] - [u][\overline{P(x)}][v] + \overline{P(x)}[v] - [x][v] \]
\[ - \Bigl[ P(x)[v] - \lambda [ux][v] \Bigr] \]
\[ \equiv - [uP(x)][v] - [u][x][v] - \lambda [ux][v] + [u][x][v] + [u][P(x)v] + \lambda [ux][v] \]
\[ \equiv - [uP(x)][v] + [u][P(x)v] - \lambda [uP(x)v] - [u][x][v] - [u][x][v] - \lambda [ux][v] \]
\[ - \lambda [ux][v] - \lambda [ux][v] - \lambda^2 [uxv] + [u][x][v] + [u][x][v] + \lambda [ux][v] \]
\[ + [u][P(x)v] + [u][P(x)v] + \lambda [uP(x)v] + \lambda [ux][v] - \lambda [ux][v] + \lambda^2 [uxv] \]
\[ \quad \text{(by Eq. (3))} \]
\[ \equiv - [u\overline{P(x)}][v] - [u\overline{P(x)}][v] + [u][\overline{P(x)}][v] - [u][x][v] - [u][x][v] - [u][x][v] \]
\[ - \lambda [ux][v] - \lambda [ux][v] - \lambda^2 [uxv] + [u][x][v] + [u][x][v] + \lambda [ux][v] \]
\[ + [u][\overline{P(x)}][v] + [u][\overline{P(x)}][v] + \lambda [uP(x)v] + \lambda [ux][v] + \lambda [ux][v] + \lambda^2 [uxv] \]
\[ \quad \text{(by Eqs. (7) and (9))} \]
\[ \equiv 0 \quad \text{mod} \ (T, w). \]

**Case 6:** (v) ∧ (vi). Here the possible ambiguity is \( w = [u]P(x)\overline{P(y)} = \overline{f}\overline{P(y)} = [u]\overline{g}, \) where
\[ f = [u]\overline{P(x)} + [u]P(x) - [uP(x)] - [u][x] - \lambda [ux], \]
\[ g = \overline{P(x)}P(y) + P(x)\overline{P(y)} - [xP(y)] - [P(x)y] - \lambda [xy] \quad \text{for some} \ x, y, u \in \mathfrak{g}(X). \]

Then
\[ (f, g)_w = \Bigl( [u]\overline{P(x)} + [u]\overline{P(x)} - [uP(x)] - [u][x] - \lambda [ux] \Bigr)\overline{P(y)} - [u]\overline{P(x)}\overline{P(y)} \]
\[
\begin{align*}
\text{Case 7: (vi)} & \land (vi). \text{ Here the possible ambiguity is } w = \overline{P(x)P(y)P(z)} = \overline{fP(z)} = \overline{P(x)g}, \text{ where } \\
f &= \overline{P(x)P(y)} + P(x)\overline{P(y)} - [xP(y)] - [P(x)y] - \lambda [xy], \\
g &= \overline{P(y)P(z)} + \overline{P(y)P(z)} - [yP(z)] - [P(y)z] - \lambda [yz] \text{ for some } x, y, z \in X.
\end{align*}
\]

The composition is trivial by
\[
(f, g)_w = \overline{P(x)P(y)P(z)} - [xP(y)] - [P(x)y] - \lambda [xy] - \overline{P(z)} - P(x)\overline{P(y)P(z)} + P(y)\overline{P(z)} - [yP(z)] - [P(y)z] - \lambda [yz]
\]
\[
= P(x)\overline{P(y)P(z)} - [xP(y)] - P(x)\overline{P(z)} - \lambda [xy] - \overline{P(z)} - \overline{P(x)P(y)P(z)} + \overline{P(x)yP(z)}
\]
\[
\begin{align*}
&+ P(x)[P(y)z] + \lambda P(x)[yz] \\
&= \tilde{P}(x)P(y)P(z) + \tilde{P}(x)P(y)\tilde{P}(z) - [xP(y)]P(z) - [P(x)y]P(z) - \lambda [xy]P(z) \\
&- \tilde{P}(x)P(y)\tilde{P}(z) - \tilde{P}(x)\tilde{P}(y)P(z) + \tilde{P}(x)yP(z) + P(x)[P(y)z] + \lambda \tilde{P}(x)[yz] \\
&\equiv - \tilde{P}(x)\tilde{P}(y)P(z) - \tilde{P}(x)\tilde{P}(y)\tilde{P}(z) - \tilde{P}(x)P(y)\tilde{P}(z) + \tilde{P}(x)\tilde{P}(y)P(z) + P(x)[P(y)z] + \tilde{P}(x)[P(y)z] \\
&+ \lambda \tilde{P}(x)[yz] + P(x)\tilde{P}(y)\tilde{P}(z) + P(x)\tilde{P}(y)P(z) + P(x)P(y)\tilde{P}(z) - [xP(y)]P(z) \\
&- [xP(y)P(z)] - [xP(y)]z - \lambda [xyP(z)] + [P(x)y]P(z) - [P(x)yP(z)] \\
&- [P(z)y]z - [P(x)y]z + \lambda [xyP(z)] - \lambda [xyP(z)] - \lambda^2 [xyz] \\
&- \tilde{P}(x)[yP(z)] + [x[yP(z)]] + [P(x)yP(z)] + \lambda [xyP(z)] - \tilde{P}(x)[P(y)z] \\
&+ [xP(y)z] + [P(x)\tilde{P}(y)z] + \lambda [xyP(z)] - \lambda \tilde{P}(x)[yz] + \lambda [xP(y)] \\
&+ \lambda [P(x)yz] + \lambda^2 [xyz] \quad \text{(by Eqs. (8) and (9))}
\end{align*}
\]

\[
\begin{align*}
&\equiv - \tilde{P}(x)\tilde{P}(y)\tilde{P}(z) - \tilde{P}(x)\tilde{P}(y)\tilde{P}(z) - \tilde{P}(x)P(y)\tilde{P}(z) + \tilde{P}(x)\tilde{P}(y)P(z) + \tilde{P}(x)[P(y)z] + \tilde{P}(x)[P(y)z] \\
&+ \lambda \tilde{P}(x)[yz] + P(x)\tilde{P}(y)\tilde{P}(z) + P(x)\tilde{P}(y)P(z) + P(x)P(y)\tilde{P}(z) - [xP(y)]P(z) \\
&- [xP(y)P(z)] - [xP(y)]z - \lambda [xyP(z)] + [P(x)y]P(z) - [P(x)yP(z)] \\
&- [P(z)y]z - [P(x)y]z + \lambda [xyP(z)] - \lambda [xyP(z)] - \lambda^2 [xyz] \\
&- \lambda [P(x)yz] + \lambda [xyP(z)] - \lambda [xyP(z)] - \lambda [xyP(z)] - \lambda^2 [xyz] - \tilde{P}(x)[yP(z)] \\
&+ [xP(y)z] + [P(x)\tilde{P}(y)z] + \lambda [xyP(z)] - \lambda \tilde{P}(x)[yz] + \lambda [xP(y)] \\
&+ [xP(y)z] + [P(x)\tilde{P}(y)z] + \lambda [xyP(z)] - \lambda \tilde{P}(x)[yz] + \lambda [xP(y)] \\
&- \lambda \tilde{P}(x)[yz] + \lambda [xP(y)] + \lambda [P(x)yz] + \lambda^2 [xyz] \quad \text{(by Eq. (10))}
\end{align*}
\]

\[\equiv 0 \quad \text{mod } (T, w).\]
3.3. Triviality of the including compositions. We finally verify that the 17 including compositions in Table 3 are trivial. Due to the symmetry summarized in Table 3, we only need to verify 10 of them as follows.

Case 1: (i)∧(iii). Here the possible ambiguities are \(w = [p|_{(x)}]|_{(v)}\), \(w = [u]|_{p|_{(x)}}\), \(w = |x|v\) or \(w = |u|x\). By the symmetry given by the opposite multiplication, we only need to consider the cases when \(w = [p|_{(x)}]|_{(v)}\) and \(w = |x|v\).

**Case 1.1.** \(w = [p|_{(x)}]|_{(v)} = \overline{f} = q\pi\), where
\[
f = [p|_{(x)}]|_{(v)} - [p|_{(x)}]|_{(v)} - [p|_{(x)}]|_{(v)} - \lambda[p|_{(x)}]|_{(v)},
\]
\[
g = |x| - P(x) \text{ for some } x \in X, v \in \mathfrak{M}(X), p \in \mathfrak{M}^*(X), q = [p]|_{(v)}.
\]
Then we have
\[
(f, g)_w = [p|_{(x)}]|_{(v)} - [p|_{(x)}]|_{(v)} - [p|_{(x)}]|_{(v)} - \lambda[p|_{(x)}]|_{(v)} - [p|_{(x)}-P(x)]|_{(v)}
\]
\[
= - [p|_{(x)}]|_{(v)} - [p|_{(x)}]|_{(v)} - \lambda[p|_{(x)}]|_{(v)} + [p|_{(x)}]|_{(v)}
\]
\[
\equiv - [p|_{(x)}]|_{(v)} - [p|_{(x)}]|_{(v)} - \lambda[p|_{(x)}]|_{(v)} + [p|_{(x)}]|_{(v)} \quad \text{(by Eq. (3))}
\]
\[
\equiv - [p|_{(x)}]|_{(v)} - [p|_{(x)}]|_{(v)} - \lambda[p|_{(x)}]|_{(v)} + [p|_{(x)}]|_{(v)} + [p|_{(x)}]|_{(v)}
\]
\[
+ \lambda[p|_{(x)}]|_{(v)} \quad \text{(by Eq. (7))}
\]
\[
\equiv 0 \mod (T, w).
\]

**Case 1.2.** \(w = |x|v = \overline{f} = q\pi\), where
\[
f = |x|v - |x|v - |x|v - \lambda|xv|
\]
\[
g = |x| - P(x), \text{ for some } x \in X, v \in \mathfrak{M}(X), q = \pi[v].
\]
Then
\[
(f, g)_w = [x]|v - [x]|v - [x]|v - [x]|v - \lambda[xv] - ([x] - P(x))|v
\]
\[
= - [x]|v - [x]|v - \lambda[xv] + P(x)|v
\]
\[
\equiv - [x]|v - [P(x)|v - \lambda[xv] + P(x)|v] \quad \text{(by Eq. (3))}
\]
\[
\equiv - [x]|v - [P(x)|v - \lambda[xv] + \overline{P(x)}|v + \overline{P(x)}|v
\]
\[
\equiv - [x]|v - [P(x)|v - \lambda[xv] - \overline{P(x)}|v + [x]|v] + [P(x)|v + \lambda|xv]
\]
\[
+ \overline{P(x)}|v \quad \text{(by Eq. (7))}
\]
\[
\equiv 0 \mod (T, w).
\]

Case 2: (i)∧(iv). \(w = \overline{P(x)}|p|_{(y)}\) or \(w = \overline{P(x)}|p|_{(y)}\). Then we have the following two cases.

**Case 2.1.** \(w = \overline{P(x)}|p|_{(y)} = \overline{f} = q\pi\), where
\[
f = \overline{P(x)}|p|_{(y)} + \overline{P(x)}|p|_{(y)} - |x|p|_{(y)} - |P(x)|p|_{(y)} - \lambda|xp|_{(y)},
\]
\[
g = |y| - P(y) \text{ for some } x, y \in X, p \in \mathfrak{M}^*(X), q = \overline{P(x)}|p|.
\]
Then
\[
(f, g)_w = \overline{P(x)}|p|_{(y)} + \overline{P(x)}|p|_{(y)} - |x|p|_{(y)} - |P(x)|p|_{(y)} - \lambda|xp|_{(y)} - \overline{P(x)}|p|_{(y)}-P(x)|p|_{(y)}
\]
Case 2.2. \( w = \overline{P(x)}[y] = \overline{f} = q_{g}, \) where
\[
f = \overline{P(x)}[y] + \overline{P(x)}[y] - [x[y]] - [P(x)y] - \lambda[xy],
g = [y] - P(y) \text{ for some } x, y \in X, q = \overline{P(x)} \star .
\]

Then
\[
(f, g)_{w} = \overline{P(x)}[y] + \overline{P(x)}[y] - [x[y]] - [P(x)y] - \lambda[xy] - \overline{P(x)}(y - P(y))
\]
\[
= \overline{P(x)}[y] - [x[y]] - [P(x)y] - \lambda[xy] + \overline{P(x)}P(y)
\]
\[
= \overline{P(x)}P(y) - [xP(y)] - [P(x)y] - \lambda[xy] + \overline{P(x)}P(y) \text{ (by Eq. (3))}
\]
\[
= \overline{P(x)}\overline{P(x)}P(y) - [xP(y)] - [P(x)y] - \lambda[xy] - \overline{P(x)}\overline{P(x)}P(y) - \overline{P(x)}\overline{P(x)}P(y) - \overline{P(x)}\overline{P(x)}P(y)
\]
\[
+ [xP(y)] + [P(x)y] + \lambda[xy] + \overline{P(x)}\overline{P(x)}P(y) \text{ (by Eq. (3))}
\]
\[
\equiv 0 \mod (T, w).
\]

Case 3: (ii)\(^{\wedge}\) (iii). Here the possible ambiguity is \( w = [p]_{xy}\llbracket v \rrbracket = \overline{f} = q_{g}, \) where
\[
f = [p]_{xy}\llbracket v \rrbracket - [p]_{xy}\llbracket v \rrbracket - [p]_{xy}\llbracket v \rrbracket - \lambda[p]_{xy}v,
g = xy - yx - [x, y]_{g} \text{ for some } x, y \in X, v \in \mathcal{M}(X), p \in \mathcal{M}^{*}(X), q = [p]_{v} \text{ with } x >_{bd} y.
\]

We get
\[
(f, g)_{w} = [p]_{xy}\llbracket v \rrbracket - [p]_{xy}\llbracket v \rrbracket - [p]_{xy}\llbracket v \rrbracket - \lambda[p]_{xy}v - [p]_{xy - yx - [x, y]_{g}}[v]
\]
\[
= - [p]_{xy}\llbracket v \rrbracket - [p]_{xy}\llbracket v \rrbracket - \lambda[p]_{xy}v + \lambda[p]_{xy}[v] + [p]_{xy\llbracket v \rrbracket}
\]
\[
= - [p]_{xy}\llbracket v \rrbracket - [p]_{xy\llbracket v \rrbracket} - \lambda[p]_{xy\llbracket v \rrbracket} + \lambda[p]_{xy\llbracket v \rrbracket} + \lambda[p]_{xy\llbracket v \rrbracket} + \lambda[p]_{xy\llbracket v \rrbracket}
\]
\[
+ [p]_{xy\llbracket v \rrbracket} + \lambda[p]_{xy\llbracket v \rrbracket} + [p]_{xy\llbracket v \rrbracket} + [p]_{xy\llbracket v \rrbracket} + \lambda[p]_{xy\llbracket v \rrbracket} \text{ (by Eq. (3))}
\]
\[
\equiv 0 \mod (T, w).
\]

Case 4: (ii)\(^{\wedge}\) (iv). Here the possible ambiguity is \( w = \overline{P(x)}[p]_{yz} = \overline{f} = q_{g}, \) where
\[
f = \overline{P(x)}[p]_{yz} + \overline{P(x)}[p]_{yz} - [x[p]_{yz}] - [P(x)p]_{yz} - \lambda[xp]_{yz},
g = yz - yz - [y, z]_{g} \text{ for some } x, y, z \in X, p \in \mathcal{M}^{*}(X), q = \overline{P(x)}[p] \text{ with } y >_{bd} z.
\]

Then
\[
(f, g)_{w} = \overline{P(x)}[p]_{yz} + \overline{P(x)}[p]_{yz} - [x[p]_{yz}] - [P(x)p]_{yz} - \lambda[xp]_{yz} - \overline{P(x)}[p]_{yz - yz - [y, z]_{g}}
\]
\[
\begin{align*}
\frac{y}{x} &= P(x)p_{x} - [x[p_{x}]] - [P(x)p_{x}] - \lambda[xp_{x}] + \overline{P(x)p_{y}} + \overline{P(x)p_{y\lambda}} \\
\approx& P(x)p_{x} + P(x)[p_{y\lambda}] - [x[p_{y\lambda}]] - [P(x)p_{y\lambda}] - [P(x)p_{y\lambda}] - [P(x)p_{y\lambda}] \\
&- \lambda[xp_{y\lambda}] - \lambda[xp_{y\lambda}] + \overline{P(x)p_{y\lambda}} + \overline{P(x)p_{y\lambda}} \quad \text{(by Eq. (3))} \\
\approx& P(x)p_{x} + P(x)[p_{y\lambda}] - [x[p_{y\lambda}]] - [P(x)p_{y\lambda}] - [P(x)p_{y\lambda}] \\
&- \lambda[xp_{y\lambda}] - \lambda[xp_{y\lambda}] + \overline{P(x)p_{y\lambda}} + \overline{P(x)p_{y\lambda}} + [P(x)p_{y\lambda}] + \lambda[xp_{y\lambda}] \\
&- \overline{P(x)p_{y\lambda}} + [P(x)p_{y\lambda}] + [P(x)p_{y\lambda}] + \lambda[xp_{y\lambda}] \quad \text{(by Eq. (7))} \\
\equiv& 0 \quad \text{mod } (T, w).
\end{align*}
\]

**Case 5:** \((\text{iii}) \land (\text{iii})\). Here the possible ambiguity is \(w = \overline{P(x)p_{x}} = f = q\), where

\[
f = P(x)p_{x} + P(x)[p_{y\lambda}] - [x[p_{y\lambda}]] - [P(x)p_{y\lambda}] - [P(x)p_{y\lambda}] - [P(x)p_{y\lambda}] - [P(x)p_{y\lambda}] - \lambda[xp_{y\lambda}] - \lambda[xp_{y\lambda}],
\]

\[
g = [u]\lambda - [u]\lambda - [u]\lambda - [u]\lambda \text{ for some } x \in X, u, v \in \mathcal{M}(X), p \in \mathcal{M}(X), q = \overline{P(x)p_{x}}.
\]

We have

\[
(f, g_w) = P(x)p_{x} + P(x)[p_{y\lambda}] - [x[p_{y\lambda}]] - [P(x)p_{y\lambda}] - [P(x)p_{y\lambda}] - [P(x)p_{y\lambda}] - [P(x)p_{y\lambda}] - \lambda[xp_{y\lambda}] - \lambda[xp_{y\lambda}]
\]

\[
+ P(x)p_{x} + P(x)[p_{y\lambda}] + P(x)[p_{y\lambda}] + P(x)[p_{y\lambda}] \quad \text{(by Eq. (7))}
\]

\[
= P(x)p_{x} + P(x)[p_{y\lambda}] + P(x)[p_{y\lambda}] + P(x)[p_{y\lambda}] - [x[p_{y\lambda}]] - [P(x)p_{y\lambda}] - [P(x)p_{y\lambda}] - [P(x)p_{y\lambda}]
\]

\[
- [P(x)p_{y\lambda}] - [P(x)p_{y\lambda}] - [P(x)p_{y\lambda}] - [P(x)p_{y\lambda}] - \lambda[xp_{y\lambda}] - \lambda[xp_{y\lambda}] - \lambda[xp_{y\lambda}] - \lambda[xp_{y\lambda}]
\]

\[
+ P(x)p_{x} + P(x)[p_{y\lambda}] + P(x)[p_{y\lambda}] + P(x)[p_{y\lambda}] \quad \text{(by Eq. (7))}
\]

\[
= P(x)p_{x} + P(x)[p_{y\lambda}] + P(x)[p_{y\lambda}] + P(x)[p_{y\lambda}] - [x[p_{y\lambda}]] - [P(x)p_{y\lambda}] - [P(x)p_{y\lambda}]
\]

\[
- [P(x)p_{y\lambda}] - [P(x)p_{y\lambda}] - [P(x)p_{y\lambda}] - [P(x)p_{y\lambda}] - \lambda[xp_{y\lambda}] - \lambda[xp_{y\lambda}] - \lambda[xp_{y\lambda}] - \lambda[xp_{y\lambda}]
\]

\[
+ P(x)p_{x} + P(x)[p_{y\lambda}] + P(x)[p_{y\lambda}] + P(x)[p_{y\lambda}] \quad \text{(by Eq. (7))}
\]

\[
\equiv 0 \quad \text{mod } (T, w).
\]

**Case 6:** \((\text{iv}) \land (\text{iii})\). Here the possible ambiguity is \(w = [p]\overline{P(x)p_{x}}[v]\) or \(w = [u][p]\overline{P(x)p_{x}}[v]\). By symmetry, we may only consider the former \(w = [p]\overline{P(x)p_{x}}[v]\). By

\[
f = [p]\overline{P(x)p_{x}}[v] - [p]\overline{P(x)p_{x}}[v] - [p]\overline{P(x)p_{x}}[v] - \lambda[p]\overline{P(x)p_{x}}[v],
\]

\[
g = P(x)[u] + P(x)[u] - [x[u]] - [P(x)u] - \lambda[xu] \text{ for some } x \in X, u, v \in \mathcal{M}(X), p \in \mathcal{M}(X), q = [p][v].
\]
Then

\[(f, g)_w = \left[p \big|_{P(y)[u]} \right][v] - \left[p \big|_{P(y)[u]} \right][v] - \left[p \big|_{P(y)[u]} \right][v] - \lambda \left[p \big|_{P(y)[u]} \right][v] \]

\[- \left[p \big|_{P(y)[u]} \right][v] \]

\[= - \left[p \big|_{P(y)[u]} \right][v] + \left[p \big|_{P(y)[u]} \right][v] + \left[p \big|_{P(y)[u]} \right][v] + \left[p \big|_{P(y)[u]} \right][v] \]

\[= \lambda \left[p \big|_{P(y)[u]} \right][v] - \lambda \left[p \big|_{P(y)[u]} \right][v] + \lambda \left[p \big|_{P(y)[u]} \right][v] + \lambda \left[p \big|_{P(y)[u]} \right][v] \]

\[\equiv 0 \mod (T, w).\]

**Case 7: (iv)∧(iv).** Here the possible ambiguity is \(w = \overline{P(x)} \big|_{P(y)[u]} = \overline{x} = q \big|_{P(y)[u]},\) where

\[f = \overline{P(x)} \big|_{P(y)[u]} + \overline{P(x)} \big|_{P(y)[u]} - \left[x \big|_{P(y)[u]} \right] - \left[P(x) \big|_{P(y)[u]} \right] - \lambda \left[x \big|_{P(y)[u]} \right],\]

\[g = \overline{P(y)}[u] + \overline{P(y)}[u] - \left[y \big|_{u} \right] - \left[P(y) \big|_{u} \right] - \lambda \left[y \big|_{u} \right] \text{ for some } x, y \in X, u \in \mathcal{M}(X), p \in \mathcal{M}(X) \text{ and } q = \overline{P(x)}[p].\]

The triviality of the composition follows from

\[(f, g)_w = \overline{P(x)} \big|_{P(y)[u]} + \overline{P(x)} \big|_{P(y)[u]} - \left[x \big|_{P(y)[u]} \right] - \left[P(x) \big|_{P(y)[u]} \right] - \lambda \left[x \big|_{P(y)[u]} \right] \]

\[- \overline{P(x)} \big|_{P(y)[u]} - \overline{P(x)} \big|_{P(y)[u]} + \overline{P(x)} \big|_{P(y)[u]} + \overline{P(x)} \big|_{P(y)[u]} + \overline{P(x)} \big|_{P(y)[u]} \]

\[\equiv - \overline{P(x)} \big|_{P(y)[u]} + \overline{P(x)} \big|_{P(y)[u]} + \overline{P(x)} \big|_{P(y)[u]} + \overline{P(x)} \big|_{P(y)[u]} + \left[x \big|_{P(y)[u]} \right] \]

\[- \left[x \big|_{P(y)[u]} \right] - \left[x \big|_{P(y)[u]} \right] - \left[x \big|_{P(y)[u]} \right] - \left[P(x) \big|_{P(y)[u]} \right] - \left[P(x) \big|_{P(y)[u]} \right] - \lambda \left[x \big|_{P(y)[u]} \right] \]

\[- \lambda \left[x \big|_{P(y)[u]} \right] + \overline{P(x)} \big|_{P(y)[u]} + \overline{P(x)} \big|_{P(y)[u]} + \overline{P(x)} \big|_{P(y)[u]} \]
Case 8: (iv)∧(v). Here the possible ambiguity is $w = [p]_{P(x)\upharpoonright u}P(y) = \overline{f} = q_{\bar{u}}$, where
\[
\overline{f} = [p]_{P(x)\upharpoonright u}P(y) + [p]_{P(x)\upharpoonright u}P(y) - [p]_{P(x)\upharpoonright u}P(y) - \lambda [p]_{P(x)\upharpoonright u}y - \lambda [p]_{P(x)\upharpoonright u}y
\]
\[
g = \overline{P(x)\upharpoonright u} + \overline{P(x)\upharpoonright u} - [x\upharpoonright u] - [P(x)\upharpoonright u] - \lambda [x\upharpoonright u] \text{ for some } x, y \in X, u \in \mathfrak{R}(X), p \in \mathfrak{R}^*(X)
\]
and $q = [p]_{\overline{P}(y)}$.

Then
\[
(f, g)_w = [p]_{\overline{P}(x)\upharpoonright u}P(y) + [p]_{\overline{P}(x)\upharpoonright u}P(y) - [p]_{\overline{P}(x)\upharpoonright u}P(y) - \lambda [p]_{\overline{P}(x)\upharpoonright u}y
\]
\[
- [p]_{\overline{P}(x)\upharpoonright u}P(y) - [p]_{\overline{P}(x)\upharpoonright u}P(y) + [p]_{\overline{P}(x)\upharpoonright u}P(y) + [p]_{\overline{P}(x)\upharpoonright u}P(y) + [p]_{\overline{P}(x)\upharpoonright u}P(y)
\]
\[
- \lambda [p]_{\overline{P}(x)\upharpoonright u}y - \lambda [p]_{\overline{P}(x)\upharpoonright u}y - \lambda [p]_{\overline{P}(x)\upharpoonright u}y - \lambda [p]_{\overline{P}(x)\upharpoonright u}y
\]
\[
= [p]_{\overline{P}(x)\upharpoonright u}P(y) - [p]_{\overline{P}(x)\upharpoonright u}P(y) - [p]_{\overline{P}(x)\upharpoonright u}P(y) - \lambda [p]_{\overline{P}(x)\upharpoonright u}y - \lambda [p]_{\overline{P}(x)\upharpoonright u}y
\]
\[
\equiv - [p]_{\overline{P}(x)\upharpoonright u}P(y) + [p]_{\overline{P}(x)\upharpoonright u}P(y) + [p]_{\overline{P}(x)\upharpoonright u}P(y) + [p]_{\overline{P}(x)\upharpoonright u}P(y) (\text{by Eq. (3)})
\]
\[
\equiv 0 \mod (T, w).
\]

Case 9: (vi)∧(iii). Here the possible ambiguity is $w = [p]_{P(x)\upharpoonright v}P(y)$ or $w = [u]\overline{[p]_{P(x)\upharpoonright v}P(y]}$. By the symmetry from the opposite multiplication, we only need to consider the case of $w = [p]_{P(x)\upharpoonright v}P(y)$ = \overline{f} = q_{\bar{u}}$, where
\[
f = [p]_{P(x)\upharpoonright v}P(y) - [p]_{P(x)\upharpoonright v}P(y) - [p]_{P(x)\upharpoonright v}P(y) - \lambda [p]_{P(x)\upharpoonright v}y,
\]
\[
g = \overline{P(x)\upharpoonright v}P(y) + P(x)\overline{P(x)\upharpoonright v} - [xP(y)] - [P(x)\upharpoonright v] - \lambda [xy] \text{ for some } x, y \in X, v \in \mathfrak{R}(X),
\]
\[
p \in \mathfrak{R}^*(X) \text{ and } q = [p]_{\overline{v}}.
We get

\[(f, g)_w = [p|_{\overline{P}(x)p_{\overline{P}(y)}}]v - [p|_{\overline{P}(x)p_{\overline{P}(y)}}]v - \lambda[p|_{\overline{P}(x)p_{\overline{P}(y)}}]v - [p|_{\overline{P}(x)p_{\overline{P}(y)}}]v - [p|_{\overline{P}(x)p_{\overline{P}(y)}}]v
- [p|_{\overline{P}(x)p_{\overline{P}(y)}}]v + [p|_{\overline{P}(x)p_{\overline{P}(y)}}]v + [p|_{\overline{P}(x)p_{\overline{P}(y)}}]v + [p|_{\overline{P}(x)p_{\overline{P}(y)}}]v
- \lambda[p|_{\overline{P}(x)p_{\overline{P}(y)}}]v - [p|_{\overline{P}(x)p_{\overline{P}(y)}}]v + [p|_{\overline{P}(x)p_{\overline{P}(y)}}]v + [p|_{\overline{P}(x)p_{\overline{P}(y)}}]v
\]

\[(f, g)_w = [p|_{\overline{P}(x)p_{\overline{P}(y)}}]v - [p|_{\overline{P}(x)p_{\overline{P}(y)}}]v - [p|_{\overline{P}(x)p_{\overline{P}(y)}}]v + [p|_{\overline{P}(x)p_{\overline{P}(y)}}]v
- [p|_{\overline{P}(x)p_{\overline{P}(y)}}]v - [p|_{\overline{P}(x)p_{\overline{P}(y)}}]v - [p|_{\overline{P}(x)p_{\overline{P}(y)}}]v - [p|_{\overline{P}(x)p_{\overline{P}(y)}}]v
+ [p|_{\overline{P}(x)p_{\overline{P}(y)}}]v + [p|_{\overline{P}(x)p_{\overline{P}(y)}}]v + [p|_{\overline{P}(x)p_{\overline{P}(y)}}]v + [p|_{\overline{P}(x)p_{\overline{P}(y)}}]v
\]

\[\equiv [p|_{\overline{P}(x)p_{\overline{P}(y)}}]v - [p|_{\overline{P}(x)p_{\overline{P}(y)}}]v - [p|_{\overline{P}(x)p_{\overline{P}(y)}}]v - [p|_{\overline{P}(x)p_{\overline{P}(y)}}]v - [p|_{\overline{P}(x)p_{\overline{P}(y)}}]v + [p|_{\overline{P}(x)p_{\overline{P}(y)}}]v + [p|_{\overline{P}(x)p_{\overline{P}(y)}}]v + [p|_{\overline{P}(x)p_{\overline{P}(y)}}]v
\]

\[\equiv 0 \mod (T, w).
\]

**Case 10: (vi) ∧ (iv).** Here the possible ambiguity is w = \(\overline{P}(x)p_{\overline{P}(y)}\) where

\[f = \overline{P}(x)p_{\overline{P}(y)}p_{\overline{P}(z)} + \overline{P}(x)p_{\overline{P}(y)}p_{\overline{P}(z)} - [x[p_{\overline{P}(y)}p_{\overline{P}(z)}]] - [P(x)p_{\overline{P}(y)}p_{\overline{P}(z)}] - \lambda[xp_{\overline{P}(y)}p_{\overline{P}(z)}],
\]

\[g = \overline{P}(x)p_{\overline{P}(y)}p_{\overline{P}(z)} - [yP(z)] - [P(y)z] - \lambda[yz] \text{ for some } x, y, z \in X, p \in \mathfrak{M}(X)
\]

and \(q = \overline{P}(x)p_{\overline{P}(y)}p_{\overline{P}(z)}\).

The corresponding including composition is trivial by

\[(f, g)_w = \overline{P}(x)p_{\overline{P}(y)}p_{\overline{P}(z)} + \overline{P}(x)p_{\overline{P}(y)}p_{\overline{P}(z)} - [x[p_{\overline{P}(y)}p_{\overline{P}(z)}]] - [P(x)p_{\overline{P}(y)}p_{\overline{P}(z)}] - \lambda[xp_{\overline{P}(y)}p_{\overline{P}(z)}]
- \overline{P}(x)p_{\overline{P}(y)}p_{\overline{P}(z)} - \overline{P}(x)p_{\overline{P}(y)}p_{\overline{P}(z)} + \overline{P}(x)p_{\overline{P}(y)}p_{\overline{P}(z)} + \overline{P}(x)p_{\overline{P}(y)}p_{\overline{P}(z)} + \overline{P}(x)p_{\overline{P}(y)}p_{\overline{P}(z)}
\]

\[\equiv \overline{P}(x)p_{\overline{P}(y)}p_{\overline{P}(z)} - [x[p_{\overline{P}(y)}p_{\overline{P}(z)}]] - [P(x)p_{\overline{P}(y)}p_{\overline{P}(z)}] - \lambda[xp_{\overline{P}(y)}p_{\overline{P}(z)}]
- \overline{P}(x)p_{\overline{P}(y)}p_{\overline{P}(z)} + \overline{P}(x)p_{\overline{P}(y)}p_{\overline{P}(z)} + \overline{P}(x)p_{\overline{P}(y)}p_{\overline{P}(z)} + \overline{P}(x)p_{\overline{P}(y)}p_{\overline{P}(z)}
\]

\[\equiv 0 \mod (T, w).
\]

Now the proof of Theorem 2.15 is completed.
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