Financial accumulation implies ever-increasing wealth inequality

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Abstract
Wealth inequality is an important matter for economic theory and policy. The recent rise in wealth inequality has been discussed in connection with the recent development of active global financial markets. The existing literature on wealth distribution links wealth inequality to a variety of drivers. Our approach develops a minimalist modelling strategy that combines three featuring mechanisms: active financial markets, individual wealth accumulation and compound interest structure. We provide mathematical proof that accumulated financial investment returns involve ever-increasing wealth concentration and inequality across individual investors most of the time. This cumulative effect over space and time depends on financial accumulation processes, including under efficient financial markets, which generate a fair investment game that individual investors repeatedly play through time.

Keywords Inequality · Economic process · Compound return · Simple return · Minimal institution

JEL Classification C46 · D31 · D63 · E02 · E21

1 Introduction
Wealth inequality is an important matter for economic theory and policy. The recent rise in wealth inequality has been discussed in connection with the recent development of active global financial markets (Beck et al. 2007; Piketty 2013; Krugman 2013; Stiglitz 2012; Solow 2014). While some have argued for the role of financial
development in reducing income inequality and benefiting the poorer, others criticise the increasing concentration of financial capital and related income as the main driver of rising inequality. Moreover, issues of wealth distribution were raised by the 99% movement in the USA following the Global Financial Crisis of 2007–2008. They have claimed that increased financialisation of the economy and society in recent decades has involved growing appropriation of wealth by the richest 1% of the population to the detriment of the remaining 99%, implying a more unequal and unfair wealth distribution.

The existing literature on wealth distribution links wealth inequality to a variety of drivers (Bertola et al. 2006; Snowdon and Vane 2005). Already at the beginning of the twentieth century, pointing to wealth concentration in the economy and society, economist and sociologist V. Pareto suggested the so-called Pareto wealth distribution as an empirical regularity (Pareto 1897), while economic statistician C. Gini developed ingenious statistical techniques to represent wealth inequality through the so-called Gini Index (Gini 1912). In this context, a stream of the relevant literature draws upon Champernowne (1953) and Rutherford (1955) to develop an elegant formal modelling strategy that explains wealth concentration and the Pareto wealth distribution under conditions of financial market efficiency, involving stochastic distribution of financial returns across individuals investing in that market (Levy 2005; Levy and Levy 2003). This modelling strategy considers financial investment as a multiplicative process closely related to a Kesten process (Kesten 1973; Redner 1990).

The existing literature and current debate on wealth inequality suggest some connection between wealth concentration and the financial investment process through active financial markets. Our approach involves disentangling the minimal components of this process in order to formalise its working and infer general results as to its impact on the economy and society. In particular, we identify two minimal institutions that surround individual investment strategies: compound return structure and active financial markets.

Compound return structure is central to this financial investment process. It implies that individual investors keep reinvesting financial proceeds together with previous capital stock through time and circumstances. Meanwhile, active financial markets ensure that this financial investment process is played as a fair game, involving some market order over return-seeking individual strategies. In particular, under efficient financial markets, individual investors extract their realised returns from the same distribution, while individual results remain independent through space and time, preventing arbitrage opportunities (see Samuelson 1965, 1973; Fama 1995; Biondi and Righi 2017 for a further literature review).

Our mathematical modelling strategy formalises the financial investment process in line with both minimal institutions, in order to study its relationship to relative wealth concentration across individuals and through time in the economy and society. In particular, we prove that, in the absence of countering forces, financial accumulation implies ever-increasing relative wealth inequality most of the time. We define financial accumulation as the peculiar financial investment process in which financial proceeds are systematically reinvested together with previous capital stock through time and circumstances.
The rest of the article is organised as follows. The next section summarises our modelling strategy. Sections 3 and 4 provide mathematical proof of the relationship between wealth concentration and the financial investment process under compound and simple return structures, respectively. Section 5 develops some implications of and perspectives on our results. Section 6 concludes.

2 Modelling strategy

The existing models of economic process generally combine a multiplicative factor called capital with an additive factor called labour, in order to explain economic growth and study aggregate wealth distribution (Nirei and Souma (2007) providing further references). Our modelling strategy focuses only on the capital factor, so as to disentangle featuring financial investment mechanisms and their impact on the economy and society.

Generally speaking, financial investment process under efficient capital markets implies that each and every individual investment strategy generates independent and independently distributed returns extracted from the same distribution for all investors and at every point of time, although real financial markets factually show some correlation through time and cross-sectionally. Our modelling strategy relaxes this strong set of assumptions while featuring cumulative effects through space and time. Our results do not require financial returns to be independent and independently distributed over time and across agents. Only some decorrelation through time is required. Although some previous studies have noticed the non-stationarity of wealth distribution driven by financial investment (Kalecki 1945; Biondi and Righi 2019), its cumulative effects and implications have generally been neglected.

Cumulative effects relate to the financial accumulation process embedded in compound return structure. Throughout this process, every investor starts investing an initial capital value through time, realising one given return at each point of time. Individual wealth investment is then repeatedly submitted to a series of realised returns that singles out each and every individual investment portfolio trajectory. Compound return structure further ensures that current-period financial investment proceeds are added to the past-period capital stock to be reinvested in the next period, generating individual financial accumulation through time.

Our modelling strategy highlights this cumulative effect by featuring its impact both on individual financial accumulation through time and on relative wealth concentration across individuals at a given point of time. Our minimalist modelling strategy helps disentangle key features of the financial accumulation process that characterises a large set of real-world mechanisms such as saving deposits and mutual investment funds. In this context, it must be stressed that compound return is generally taken as the benchmark for financial investment performance.
3 Model and proof under compound return structure

Consider a population of \( n \) individual investors \( i = 1, \ldots, n \), holding financial wealth available for investment \( W_i(t) \) at time \( t \). Here, we consider discrete time dynamics, so \( t \) is integer-valued.

We formalise the financial investment process according to the familiar structure of compound return where the wealth \( W_i(t+1) \) of investor \( i \) at time \( t+1 \) depends on the past wealth stock \( W_i(t) \) invested at the return \( r_{i,t+1} \) as follows:

\[
W_i(t+1) = W_i(t)(1 + r_{i,t+1}) \tag{1}
\]

where \( \{r_t = (r_{1,t}, \ldots, r_{n,t})\}_{t \in \mathbb{N}} \) is an infinite sequence of identically distributed random vectors in \( \mathbb{R}^n \). Components of these random vectors are not required to be independent. Without loss of generality, we may impose \( r_{i,t} \geq -1 \). In fact, we only impose conditions required for mathematical proof. For the sake of clarity, if \( r_{i,t} \geq 0 \), investor \( i \) maintains or increases its past wealth stock invested at time \( t - 1 \), while that investor loses it at least partly if \( r_{i,t} < 0 \).

This setting generates the following cumulative process through time from initial time \( t = 1 \) to time \( t = T \):

\[
W_i(T) = W_i(0) \prod_{t=1}^{T} (1 + r_{i,t}) = W_i(0) \exp \left\{ \sum_{t=1}^{T} Z_{i,t} \right\}, \tag{2}
\]

where \( Z_{i,t} = \log[1 + r_{i,t}] \). This logarithmic transformation defines, for any \( i = 1, \ldots, n \), the individual series of realised financial returns \( \{Z_{i,t}; t = 1, \ldots, T\} \) as a sequence of identically distributed random variables; we require the following conditions over these series:

(i) strictly positive finite second moments \( 0 < \mathbb{E} Z_{i,t}^2 < +\infty \),
(ii) \( |\mathbb{E}(Z_{i,t}Z_{j,t})|^2 < \mathbb{E} Z_{i,t}^2 \mathbb{E} Z_{j,t}^2 \) for \( i \neq j \) and any \( t \geq 1 \).

Defining \( \{Y_{i,j}(t) = Z_{i,t} - Z_{j,t}, t \in \mathbb{N}\} \), for \( i \neq j \), by (ii) we have that \( \mathbb{E}(Y_{i,j}(t)^2) > 0 \). Furthermore, we assume the following condition on the temporal dependence of \( Y_{i,j}(t) \):

\[
\inf_{i \neq j} \left| \sum_{t=1}^{T} Y_{i,j}(t) \right| \xrightarrow{T \to \infty} +\infty \quad \text{in probability}, \tag{3}
\]

which means that for any \( 0 < M < +\infty \):

\[
\lim_{T \to \infty} \mathbb{P} \left( \inf_{i \neq j} \left| \sum_{t=1}^{T} Y_{i,j}(t) \right| \geq M \right) = 1.
\]

There are many explicit conditions that imply (3) for different reasons:

- If \( \mathbb{E}(Z_{i,t}) \neq \mathbb{E}(Z_{j,t}) \) for all \( i \neq j \).
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- If \{Y_{i,j}(t)\}_t are martingale increments, i.e. \(\mathbb{E}(Y_{i,j}(t)|Y_{i,j}(s), s < t) = 0\).
- If \{Y_{i,j}(t)\}_t is strongly mixing, satisfying a condition for a limit theorem (cf. Ibragimov 1971).

Let define \(X_i(T) = \sum_{t=1}^{T} Z_{i,t}, W_i(T) = W_i(0) e^{X_i(T)}\) and 
\[
\theta_i(T) = \frac{W_i(T)}{\sum_j W_j(T)}, \quad i = 1, \ldots, n,
\]
while \(\tilde{\theta}_i^T(T)\) is the corresponding ranked sequence at time \(T\): \(\tilde{\theta}_i(T) \leq \tilde{\theta}_{i+1}(T)\). This indicator ranks all relative individual wealth \(W_i(T)\). In particular, \(\tilde{W}_i^T(T)\) is the initial position for the ranked sequence at time \(T\).

**Theorem 3.1** Under the above conditions, the following limit holds in probability:
\[
\lim_{T \to \infty} \tilde{\theta}_n(T) = 1, \quad \lim_{T \to \infty} \tilde{\theta}_i(T) = 0 \quad \forall i \neq n
\]  

**Proof** Let \(\tilde{X}_i(T), \tilde{W}_i(T)\) be the corresponding ranked sequences at time \(T\),
\[
\tilde{\theta}_n(T) = \left(1 + \sum_{j \neq n} \tilde{W}_j(T) \tilde{W}_n(T)^{-1}\right)^{-1} = \left(1 + \sum_{j \neq n} \frac{\tilde{W}_j(T)}{\tilde{W}_n(T)} e^{\tilde{X}_j(T) - \tilde{X}_n(T)}\right)^{-1}
\] 
\[
\geq \left(1 + n \left(\sup_{j \neq n} \frac{W_j(0)}{W_n(0)}\right) e^{-\inf_{j \neq n} (\tilde{X}_n(T) - \tilde{X}_j(T))}\right)^{-1}.
\]

Since \(W_j(0) > 0\) for all \(j\), we have that \(\sup_{j \neq n} \frac{W_j(0)}{W_n(0)} < \infty\). Furthermore,
\[
\inf_{j \neq n} (\tilde{X}_n(T) - \tilde{X}_j(T)) \geq \inf_{i \neq j} |X_i(T) - X_j(T)| = \inf_{i \neq j} \left| \sum_{t=1}^{T} Y_{i,j}(t) \right|
\]
which implies \(\mathbb{P}\left(\tilde{\theta}_n(T) \geq 1 - \epsilon\right) \to_T \infty 1\) for any \(\epsilon > 0\). \(\square\)

**Remark 3.2** Notice that the convergence is only in probability, and not almost sure. This means that a single realisation of a trajectory of \((\tilde{\theta}_n(t), t \geq 0)\) oscillates almost surely, but the probability of finding \(\tilde{\theta}_n(t)\) strictly smaller than one converges to 0 as \(t \to \infty\). A more accurate analysis shows that
\[
\limsup_{T \to \infty} \tilde{\theta}_n(T) = 1, \quad \liminf_{T \to \infty} \tilde{\theta}_n(T) = 0, \quad \text{almost surely.}
\]  

This result implies that financial accumulation through time by individual investors leads to ever-increasing wealth concentration and inequality across them most of the time. This result holds for any initial wealth distribution and any mean return for financial investments over time.

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The result does not imply that one single individual will own all the wealth indefinitely; over a large time-scale, wealth may move from one investor to another. The result says that, most the time, wealth remains concentrated in the hands of one individual.

### 4 Model and proof under simple return structure

The previous section proved the link between financial accumulation through time and ever-increasing wealth concentration across individual investors most of the time. This latter structure section provides a complementary proof by studying simple return structure. This excludes financial accumulation from the financial investment process. Financial proceeds are then consumed or held in precaution (if positive), and not reinvested together with the previous capital stock of wealth. Simple return structure corresponds to the historical cost accounting rule of financial performance, which assumes an additive process through time between invested capital stock and generated income flow. The capital stock is thus reproduced rather than accumulated. Financial capital is remunerated as a productive factor but not financially accumulated over time. We formalise this additive process as follows:

\[
W_i(T) = W_i(0) \left(1 + \sum_{t=1}^{T} r_{i,t}\right) \tag{6}
\]

assuming that \( r_{i,t} \geq -1 \). For the sake of simplicity, we do not impose further restrictions which would make this investment process more realistic. Note that individual wealth may become negative at some point of time. Again, we include only conditions that are required for mathematical proof. This formalisation defines \( \{r_t = (r_{1,t}, \ldots, r_{n,t}), t = 1, \ldots, T\} \) as an ergodic sequence of positive random vectors with finite expectation: \( E r_{i,t} = \mu_i < +\infty \).

**Theorem 4.1** With probability one, we have

\[
\lim_{T \to \infty} \frac{W_i(T)}{\sum_{j=1}^{n} W_j(T)} = \frac{W_i(0)\mu_i}{\sum_{j=1}^{n} W_j(0)\mu_j}. \tag{7}
\]

**Proof** This is a direct consequence of the ergodic theorem. In fact with probability one

\[
\theta_i(T) = \frac{W_i(T)/T}{\sum_j W_j(T)/T} \xrightarrow{T \to \infty} \frac{W_i(0)\mu_i}{\sum_j W_j(0)\mu_j}.
\]

Corollary 4.2 If \( \mu_j = \mu \) for any \( j \), then \( \theta_i(T) \xrightarrow{T \to \infty} \theta_i(0) \).

**Remark 4.3** No condition is required on correlations between components of the random vector \( r_t = (r_{1,t}, \ldots, r_{n,t}) \).
Remark 4.4 Note that the convergence with probability one in (7) is stronger than the convergence in probability in (4).

5 Discussion and implications

Our result concerning wealth concentration is twofold. In the simple interest return scenario (Sect. 4), wealth concentration converges—with probability one—to certain values determined by the initial conditions. In contrast, in the compound interest return scenario (Sect. 3), the probability that maximum wealth concentration does not occur tends towards zero. More precisely, most of the time, there is one individual who owns all the wealth, though not always the same individual through time: over a very large time-scale, wealth may move from one lucky individual to another, passing through a fast redistribution phase.

According to our results, financial accumulation through time implies ever-increasing wealth concentration and inequality across individuals most of the time. Wealth distribution tends to degenerate so that, at a certain point of time, a single individual eventually owns virtually all wealth, relatively speaking. This result holds for whatever initial wealth distribution, and it critically depends on some variance in financial returns and some return decorrelation through time. Moreover, this result does not depend on any specific distribution of returns as long as the distribution involves some finite positive variance. Therefore, even efficient financial markets—implying a fair investment game that individual investors repeatedly play through time—involves ever-increasing wealth inequality and maximum wealth concentration most of the time. Since this result depends only on some temporal decorrelation through time, it may be extended to include mutual investment opportunities that deliver a joint return to subsets of individual investors at one point in time. Furthermore, our analysis could be extended by studying the likely transition time patterns that generate wealth concentration and inequality over time.

Our analysis identifies the financial accumulation process as an important driver of increasing wealth inequality in the economy and society. This effect does not depend on the condition that capital investment is remunerated as a productive factor, but on the peculiar opportunity that financial proceeds from that capital investment are systematically reinvested, compounding financial returns. Ever-increasing inequality depends on some temporal decorrelation between individual return trajectories.

Further mechanisms may be introduced which counter this wealth inequality effect. Countering forces may include aggregate mechanisms that redistribute wealth $W_{i,t}$ across individuals at some point of time, compensating for increasing wealth concentration and inequality over time. An obvious candidate mechanism is taxation. Another countering force may emerge in the form of a limit to the economic process. Although this is generally assumed by economic models, it may be unrealistic to assume compound returns that last indefinitely over time and circumstances (Voinov and Farley 2007; Biondi 2011). This would implicitly maintain constant returns at scale for whatever involved size, while some decreasing returns may occur after some threshold due to natural and artificial limits to growth. Furthermore, recent studies suggest that higher inequality may reduce growth, limiting financial investment opportunities (Berg and
Ostry 2013; Ostry et al. 2014). However, according to our analysis, as long as compound returns apply, financial accumulation at those returns involves ever-increasing wealth inequality and maximum wealth concentration most of the time.

6 Concluding remarks

Our approach focuses only on one dimension of the economic process, namely its capital factor and related financial investment through time and circumstances. Through formal modelling, we disentangle its featuring mechanisms and study the financial investment process, respectively, under compound and simple return structures. By construction, our modelling strategy neglects interaction with other dimensions that may affect its impact on inequality. Mathematical proofs identify the set of conditions under which financial accumulation generates ever-increasing wealth concentration and inequality. Left alone without countering forces, compound return structure then constitutes an important driver of inequality in the economy and society.

Recent evolutions in the economy and society have given rise to increasing financial investment opportunities through active capital markets around the world. Our result corroborates the claim that this phenomenon may have contributed to increased wealth concentration and inequality across individuals in recent decades.

From an epistemological viewpoint, our approach points to interaction between microscopic and macroscopic dimensions of the economic system under investigation. One cumulative effect concerns compound financial returns through time at the level of every individual investment portfolio. Another cumulative effect concerns wealth concentration across populations of individual investors at every point of time.

Our approach therefore contributes to including space heterogeneity and time evolution in economic modelling. Concerning time, financial accumulation has been somewhat neglected because modelling strategies considered one-period or two-period horizons, while the compound interest structure plays its featuring role when at least three successive periods are considered. Concerning space, our approach introduces a population of heterogeneous individuals. Individual financial investment trajectories constitute a heterogeneous set of financial accumulation processes performed under active financial markets through time and circumstances.

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