Variational inequalities
for the spectral fractional Laplacian

Roberta Musina* and Alexander I. Nazarov†

Abstract

In this paper we study the obstacle problems for the Navier (spectral) fractional
Laplacian \((-\Delta_\Omega)^s\) of order \(s \in (0, 1)\), in a bounded domain \(\Omega \subset \mathbb{R}^n\).

1 Introduction

Let \(\Omega\) be a bounded and Lipschitz domain in \(\mathbb{R}^n, n \geq 1\). Given \(s \in (0, 1)\), a measurable
function \(\psi\) on \(\Omega\) and \(f \in \tilde{H}^s(\Omega)'\), we consider the variational inequality

\[
\langle (-\Delta_\Omega)^s u - f, v - u \rangle \geq 0 \quad \forall v \in K^s_\psi.
\]

Here \((-\Delta_\Omega)^s\) is the spectral (or Navier) fractional Laplacian, that is the \(s\)-th power of
the standard Laplacian in the sense of spectral theory, and

\[
K^s_\psi = \left\{ v \in \tilde{H}^s(\Omega) \mid v \geq \psi \ \text{a.e. on} \ \Omega \right\}.
\]

We will always assume that the closed and convex set \(K^s_\psi\) is not empty, also when not
explicitly stated.

* Dipartimento di Matematica ed Informatica, Università di Udine, via delle Scienze, 206 - 33100
Udine, Italy. Email: roberta.musina@uniud.it. Partially supported by Miur-PRIN 201274FYK7_004.
† St.Petersburg Department of Steklov Institute, Fontanka 27, St.Petersburg, 191023, Russia,
and St.Petersburg State University, Universitetskii pr. 28, St.Petersburg, 198504, Russia. E-mail:
al.il.nazarov@gmail.com. Supported by RFBR grant 14-01-00534.
Problem $\mathcal{P}_\Omega(\psi, f)$ admits a unique solution $u$, that can be characterized as the unique minimizer for
\[
\inf_{v \in K_\psi} \frac{1}{2}\langle (-\Delta_\Omega)^s v, v \rangle - \langle f, v \rangle.
\] (1) 

The variational inequality $\mathcal{P}_\Omega(\psi, f)$ is naturally related to the free boundary problem
\[
\begin{cases}
    u \geq \psi, 
    (-\Delta_\Omega)^s u \geq f & \text{in } \Omega \\
    (-\Delta_\Omega)^s u = f & \text{in } \{u > \psi\} \\
    u = 0 & \text{in } \mathbb{R}^n \setminus \Omega
\end{cases}
\] (2)
as well. In fact, it is easy to show that any solution $u \in \tilde{H}^s(\Omega)$ to (2) satisfies $\mathcal{P}_\Omega(\psi, f)$, see Remark 4.4. The converse needs more care. One of the main motivations of the present paper was indeed to find out mild regularity assumptions on the data, to have that the solution $u$ to $\mathcal{P}_\Omega(\psi, f)$ solves the free boundary problem (2).

Problem (2) has been largely investigated in case $(-\Delta_\Omega)^s$ is replaced by the “Dirichlet” Laplacian $(-\Delta)^s$, that is defined via the Fourier transform by
\[
\mathcal{F}[(-\Delta)^s u](\xi) = |\xi|^{2s} \mathcal{F}[u](\xi) = \frac{|\xi|^{2s}}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{-i\xi \cdot x} u(x) \, dx,
\] (3)
see Section 6 for details. On this subject, we cite the pioneering paper [24] by Louis E. Silvestre, [2, 4, 6, 10, 20, 21, 23] and references there-in, with no attempt to provide a complete list.

As far as we know, the variational inequality $\mathcal{P}_\Omega(\psi, f)$ and the free boundary problem (2) has never been discussed before. Actually, the ”Navier” case is more challenging, because of the dependence of the differential operator $(-\Delta)^s$ on the domain. This extra difficulty led us to investigate in Section 2 the dependence $\Omega \mapsto (-\Delta_\Omega)^s$. The results there, and in particular Lemmata 2.2, 2.3 might have an independent interest and could open new research directions, see Remark 2.4.

In Section 3 we focus our attention on the action of $(-\Delta_\Omega)^s$ on truncations $v \mapsto v^{\pm}$. The results here are essentially used in the remaining part of the paper, as they provide the needed tools to construct test functions for $\mathcal{P}_\Omega(\psi, f)$. Once we have developed the above mentioned tools, we indicate how to modify the arguments in [20] to find out useful equivalent formulations and continuous dependence results for $\mathcal{P}_\Omega(\psi, f)$, see Section 4.

Section 5 is entirely dedicated to regularity results. Most of the proofs here follows the outlines of the proofs in [20]. However here again more attention is needed because
of the dependence $\Omega \mapsto (-\Delta)_{\Omega}^s$; the preliminary results in Section 2 will be crucially used in the proof of our main regularity result, that is stated in Theorem 5.1.

In the last section we take $f = 0$ and compare the solution to $P_{\Omega}(\psi, f)$ with the solution of the corresponding variational inequality with Dirichlet fractional operator $(-\Delta)_{\Omega}^s$. The main result is stated in Theorem 6.3.

**Notation.** For a bounded and Lipschitz domain $\Omega \subset \mathbb{R}^n$ we denote by $-\Delta_{\Omega}$ the conventional Dirichlet Laplacian in $\Omega$, that is the self-adjoint operator in $L^2(\Omega)$ defined by its quadratic form
\[ \langle -\Delta_{\Omega} u, u \rangle = \| \nabla u \|_2^2, \quad u \in H^1_0(\Omega). \]

We denote by $\lambda_j, j \geq 1$, the eigenvalues of $-\Delta_{\Omega}$ arranged in a non-decreasing unbounded sequence, according to their multiplicities. Corresponding eigenfunctions $\varphi_j \in H^1_0(\Omega), -\Delta \varphi_j = \lambda_j \varphi_j, j \geq 1,$ form an orthogonal bases in $L^2(\Omega)$ and in $H^1_0(\Omega)$, and we assume them orthonormal in $L^2(\Omega)$.

For $u \in H^1_0(\Omega)$ we have
\[ u = \sum_{j=1}^{\infty} \left( \int_{\Omega} u \varphi_j \right) \varphi_j, \quad -\Delta_{\Omega} u = \sum_{j=1}^{\infty} \lambda_j \left( \int_{\Omega} u \varphi_j \right) \varphi_j, \]
where the first series converges in $H^1_0(\Omega)$, while the second one has to be intended on the sense of distributions. Thus
\[ \| u \|_2^2 = \sum_{j=1}^{\infty} \left( \int_{\Omega} u \varphi_j \right)^2, \quad \langle -\Delta_{\Omega} u, u \rangle = \sum_{j=1}^{\infty} \lambda_j \left( \int_{\Omega} u \varphi_j \right)^2. \]

Next, take $s \in (0, 1)$. The “Navier” (or spectral) fractional Laplacian of order $s$ on $\Omega$ is defined by the series (in the sense of distributions)
\[ (-\Delta)_{\Omega}^s u = \sum_{j=1}^{\infty} \lambda_j^s \left( \int_{\Omega} u \varphi_j \right) \varphi_j. \]

It is known that the domain of the corresponding quadratic form $\langle (-\Delta)_{\Omega}^s u, u \rangle$ is the space
\[ \tilde{H}^s(\Omega) := \{ u \in H^s(\mathbb{R}^n) \mid u \equiv 0 \text{ on } \mathbb{R}^n \setminus \Omega \}, \]
see for instance [18, Lemma 1]. The standard reference for the Sobolev space $H^s(\mathbb{R}^n)$ is the monograph [26] by Triebel. We endow $\tilde{H}^s(\Omega)$ with the Hilbertian norm
\[ \| u \|_{\tilde{H}^s(\Omega)}^2 = \langle (-\Delta)_{\Omega}^s u, u \rangle = \| (-\Delta_{\Omega}^s) u \|_2^2. \]

Notice that $\varphi_j$ is the eigenfunction of $(-\Delta)_{\Omega}^s$ corresponding to the eigenvalue $\lambda_j^s$. That is, $(-\Delta)_{\Omega}^s$ is the $s$-th power of $-\Delta_{\Omega}$ in the sense of spectral theory.

We recall here some basic facts from [23]. For $u \in \tilde{H}^s(\Omega)$ we put
\[ \mathcal{E}(w) = \int_{\mathbb{R}^n} y^{1-2s} |\nabla w(x, y)|^2 \, dx \, dy, \]
\[ \mathcal{W}_u = \left\{ w(x, y) \mid \mathcal{E}(w) < \infty, \quad w|_{y=0} = u, \quad w(\cdot, y) \equiv 0 \text{ on } \mathbb{R}^n \setminus \Omega \right\}. \]
The minimization problem
\[
\inf_{w \in \mathcal{W}_\Omega u} \mathcal{E}(w) \quad (\mathcal{M}_u) \tag{eq:CSmini}
\]
has a unique solution \( w_\Omega^u : \mathbb{R}^n \times \mathbb{R}_+ \to \mathbb{R} \), that solves the Dirichlet problem
\[
\begin{align*}
-\text{div}(y^{1-2s}\nabla w) &= 0 \quad \text{in } \Omega \times \mathbb{R}_+; \\
w(\cdot, y) &\equiv 0 \quad \text{on } (\mathbb{R}^n \setminus \overline{\Omega}) \times \mathbb{R}_+, \quad w|_{y=0} = u. \\
\end{align*} \tag{eq:ST} \tag{L:ST}
\]
The results in \cite[Theorem 1.1]{25} (see also Section 2 therein), and integration by parts imply that
\[
\langle (-\Delta_\Omega)^s u, u \rangle = c_s \mathcal{E}(w_\Omega^u), \tag{eq:ST_energy} \tag{4}
\]
for an explicitly known constant \( c_s > 0 \). In addition, we have that
\[
(-\Delta_\Omega)^s u(x) = -c_s \lim_{y \to 0^+} y^{1-2s} \partial_y w_\Omega^u(x, y) = -2sc_s \lim_{y \to 0^+} \frac{w_\Omega^u(x, y) - u(x)}{y^{2s}}, \tag{eq:neumann} \tag{5}
\]
where the limits have to be intended in the sense of distributions.

From (\ref{eq:ST}) and (\ref{eq:neumann}) it follows that for any function \( w \) on \( \mathbb{R}^n \times \mathbb{R}_+ \) with finite energy \( \mathcal{E}(w) \), one has
\[
c_s \int_0^\infty \int_{\mathbb{R}^n} y^{1-2s} \nabla w_\Omega^u \cdot \nabla w \, dx \, dy = \langle (-\Delta_\Omega)^s u, u|_{y=0} \rangle. \tag{eq:w_variation} \tag{6}
\]

2 Dependence of the Navier Laplacian on \( \Omega \)

The following statement was in fact proved in \cite{18}, see also \cite{10}. We give it with full proof for reader’s convenience.

**Lemma 2.1** Let \( \Omega, \tilde{\Omega} \) be bounded and Lipschitz domains in \( \mathbb{R}^n \), with \( \overline{\Omega} \subset \tilde{\Omega} \). Let \( u \in \tilde{H}^s(\tilde{\Omega}) \), \( u \not\equiv 0 \). Then
\[
\langle (-\Delta_{\tilde{\Omega}})^s u, u \rangle < \langle (-\Delta_\Omega)^s u, u \rangle. \tag{eq:monotonicity} \tag{7}
\]

Moreover, if \( u \geq 0 \) then \( (-\Delta_{\tilde{\Omega}})^s u < (-\Delta_\Omega)^s u \) in the distributional sense on \( \Omega \).

**Proof.** Let \( w_{\Omega}^u \), \( w_{\tilde{\Omega}}^u \) be the solutions to the minimization problems \((\mathcal{M}_u^\Omega), (\mathcal{M}_u^{\tilde{\Omega}})\), respectively. Since \( \tilde{H}^s(\tilde{\Omega}) \subset \tilde{H}^s(\tilde{\Omega}) \), then \( \mathcal{W}_\Omega^u \subset \mathcal{W}_u^{\tilde{\Omega}} \) and therefore \( \mathcal{E}(w_u^{\tilde{\Omega}}) \leq \mathcal{E}(w_u^\Omega) \).

Now notice that \( w_{\tilde{\Omega}}^u \) is nontrivial and analytic on \( \tilde{\Omega} \times \mathbb{R}_+ \), because it solves \((\mathcal{L}_u^{\tilde{\Omega}})\). Hence it can not vanish in \((\tilde{\Omega} \setminus \Omega) \times \mathbb{R}_+ \), that trivially gives \( w_{\tilde{\Omega}}^u \not\equiv w_{\Omega}^u \) and \( \mathcal{E}(w_{u}^{\tilde{\Omega}}) < \mathcal{E}(w_u^\Omega) \).

Inequality (\ref{eq:monotonicity}) readily follows from (\ref{eq:CSmini}).
Notice that if $u \geq 0$ then $w_\Omega^\Omega$ is positive on $\tilde{\Omega} \times \mathbb{R}_+$ by the maximum principle. The function $W := w_\Omega^\Omega - w_\Omega^\Omega$ solves
\[
\begin{cases}
-\text{div}(y^{1-2s}\nabla W) = 0 & \text{in } \Omega \times \mathbb{R}_+ \\
W(\cdot, y) > 0 & \text{on } (\tilde{\Omega} \setminus \Omega) \times \mathbb{R}_+, \quad W|_{y=0} = 0,
\end{cases}
\]
and the maximum principle gives $W > 0$ on $\Omega \times \mathbb{R}_+$. Applying the Hopf-Oleinik boundary point lemma (see [17], [1]) to the function $W(x, t^2)$, we obtain
\[
0 < 2s \lim_{y \to 0^+} \frac{W(x, y) - W(x, 0)}{y^{2s}} = \lim_{y \to 0^+} y^{1-2s}(\partial_y w_\Omega^\Omega - \partial_y w_\Omega^\Omega), \quad x \in \Omega.
\]
The conclusion readily follows from (5). \qed

Now let $\Omega_h, h \geq 1$, be a sequence of uniformly Lipschitz domains such that
\[
\overline{\Omega} \subset \Omega_h \subset \left\{ x \in \mathbb{R}^n \mid d(x, \Omega) < \frac{1}{h} \right\}.
\] (8) domains

It is convenient to regard at $\tilde{H}^s(\Omega), s \in (0, 1]$, as a subspace of $\tilde{H}^s(B_R)$, where $B_R$ is an open ball containing $\overline{\Omega}_h$, so that we have continuous embeddings $\tilde{H}^s(\Omega) \hookrightarrow \tilde{H}^s(\Omega_h) \hookrightarrow \tilde{H}^s(B_R)$. In particular, $H^1_0(\Omega) \hookrightarrow H^1_0(\Omega_h) \hookrightarrow H^1_0(B_R)$.

It is easy to show that the domains $\Omega_h \gamma$-converge to $\Omega$ as $h \to \infty$. That is, for any $f \in L^2(B_R)$, we have $v_h \to v$ in $H^1_0(B_R)$, where the functions $v_h \in H^1_0(\Omega_h)$ and $v \in H^1_0(\Omega)$ are defined via
\[
-\Delta_{\Omega_h} v_h = f \quad \text{in } \Omega_h, \quad -\Delta_{\Omega} v = f \quad \text{in } \Omega.
\]
Let us recall some facts from [9, Lemma XI.9.5], see also [12, Theorem 2.3.2] and [3, Example 2.1]. The $\gamma$-convergence of the domains implies that the eigenvalues and eigenfunctions of $-\Delta_{\Omega_h}$ converge, respectively, to the eigenvalues and eigenfunctions of $-\Delta_{\Omega}$. More precisely, it turns out that
\[
\lambda^h_j \to \lambda_j, \quad \varphi^h_j \to \varphi_j \quad \text{in } H^1_0(B_R) \quad \text{as } h \to \infty,
\] (9) eq:1 provided that eigenfunctions corresponding to multiple eigenvalues are suitably chosen.

Now we start to study the behavior of the fractional Laplacian $(-\Delta_{\Omega_h})^s$ as $h \to \infty$. The next lemma, of independent interest, will be crucially used in the proof of our regularity results.

Lemma 2.2 Let $u_h \in \tilde{H}^s(\Omega_h)$ be a bounded sequence in $\tilde{H}^s(\Omega_h)$ such that $u_h \to u$ in $L^2(B_R)$. Then $u \in \tilde{H}^s(\Omega)$ and
\[
\langle (-\Delta_{\Omega})^s u, u \rangle \leq \liminf_{h \to \infty} \langle (-\Delta_{\Omega_h})^s u_h, u_h \rangle.
\]
Proof. Clearly, $u$ is the weak limit of the sequence $u_h$ in $\tilde{H}^s(B_R)$ and $u_h \to u$ almost everywhere. Hence $u \in \tilde{H}^s(\Omega)$. Next, for any integer $m \geq 1$ we have that

$$\langle (-\Delta_{\Omega_h})^s u_h, u_h \rangle \geq \sum_{j=1}^{m} \lambda_j^s \left( \int_{\Omega} u_h \varphi_j^h \right)^2 = \sum_{j=1}^{m} \lambda_j^s \left( \int_{\Omega} u \varphi_j \right)^2 + o_h(1)$$

by (9). Thus

$$\liminf_{h \to \infty} \langle (-\Delta_{\Omega_h})^s u_h, u_h \rangle \geq \sum_{j=1}^{\infty} \lambda_j^s \left( \int_{\Omega} u \varphi_j \right)^2 = \langle (-\Delta_{\Omega})^s u, u \rangle,$$

that ends the proof. $\square$

Lemma 2.3 Let $u \in \tilde{H}^s(\Omega)$. Then

\begin{enumerate}[(i)]
  \item $\lim_{h \to \infty} \langle (-\Delta_{\Omega_h})^s u, u \rangle = \langle (-\Delta_{\Omega})^s u, u \rangle$;
  \item $(-\Delta_{\Omega_h})^s u \to (-\Delta_{\Omega})^s u$ weakly in $\tilde{H}^s(\Omega)'$.
\end{enumerate}

Proof. The first claim is an immediate consequence of Lemmata 2.1 and 2.2. Now take any test function $v \in \tilde{H}^s(\Omega)$ and use (i) to get

$$4\langle (-\Delta_{\Omega_h})^s u, v \rangle = \langle (-\Delta_{\Omega_h})^s (u + v), (u + v) \rangle - \langle (-\Delta_{\Omega_h})^s (u - v), (u - v) \rangle$$

$$= \langle (-\Delta_{\Omega})^s (u + v), (u + v) \rangle - \langle (-\Delta_{\Omega})^s (u - v), (u - v) \rangle + o_h(1)$$

$$= 4\langle (-\Delta_{\Omega})^s u, v \rangle + o_h(1).$$

Remark 2.4 Let us introduce the functionals $L^2(B_R) \to \mathbb{R} \cup \{\infty\}$,

$$Q^\Omega_s(u) = \begin{cases} 
  \langle (-\Delta_{\Omega})^s u, u \rangle & \text{if } u \in \tilde{H}^s(\Omega); \\
  \infty & \text{otherwise.}
\end{cases}$$

Lemma 2.2 and (i) in Lemma 2.3 say that $Q^\Omega_s$ is the $\Gamma$-limit of the sequence $Q^\Omega_{s_h}$. One can wonder if this fact holds for any sequence of perturbing domains $\Omega_h$ that $\gamma$-converges to $\Omega$. For $s = 1$, answer is positive, see [8, Theorem 13.12]. Differently from the fractional case $s \in (0,1)$, for $s = 1$ the quadratic forms $Q^\Omega_{s_h}$ and $Q^\Omega_1$ coincide on the intersection of their domains.
3 Truncations

Truncation operators play an important role in studying obstacle problems. For measurable functions $v, w$ we put

$$v \lor w = \max \{v, w\}, \quad v \land w = \min \{v, w\}, \quad v^+ = v \lor 0, \quad v_- = -(v \land 0),$$

so that $v = v^+ - v^-$ and $|v| = v^+ + v^-$. It is well known that $v \lor w \in H^s(\mathbb{R}^n)$ and $v \land w \in H^s(\mathbb{R}^n)$ if $v, w \in H^s(\mathbb{R}^n)$. In addition it holds that

$$(v + m)^-, \quad (v - m)^+, \quad v \land m \in \tilde{H}^s(\Omega)$$

for any $v \in \tilde{H}^s(\Omega)$, $m \geq 0$, see [20] Lemma 2.4.

**Lemma 3.1** Let $v \in \tilde{H}^s(\Omega)$ and $m \geq 0$.

i) If $(v + m)$ changes sign, then

$$\langle (-\Delta_{\Omega})^s v, (v + m)^- \rangle + \| (-\Delta_{\Omega})^{s/2} (v + m)^- \|_2^2 < 0;$$

ii) If $(v - m)$ changes sign, then

$$\langle (-\Delta_{\Omega})^s v, (v - m)^+ \rangle - \| (-\Delta_{\Omega})^{s/2} (v - m)^+ \|_2^2 > 0;$$

iii) If $(v - m)$ changes sign, then

$$\| (-\Delta_{\Omega})^{s/2} (v \land m) \|_2^2 < \| (-\Delta_{\Omega})^{s/2} v \|_2^2 - \| (-\Delta_{\Omega})^{s/2} (v - m)^+ \|_2^2.$$

**Proof.** Let $w_v^\Omega$ be the solution to the minimization problem $(\mathcal{M}_v^\Omega)$, and let $w_{(v+m)^-}^\Omega$ be the solution to $(\mathcal{M}_{(v+m)^-}^\Omega)$. Since $(w_v^\Omega + m)^- \in \mathcal{W}_{(v+m)^-}^\Omega$, we have

$$\mathcal{E}((w_v^\Omega + m)^-) \geq \mathcal{E}(w_{(v+m)^-}^\Omega).$$

(10) \hspace{1cm} \text{eq:CSstrict}

In addition, from (9) we get

$$c_s \int_0^\infty \int_{\mathbb{R}^n} y^{1-2s} \nabla w_v^\Omega \cdot \nabla (w_v^\Omega + m) - dxdy = \langle (-\Delta_{\Omega})^s v, (v + m)^- \rangle. \hspace{1cm} (11) \hspace{1cm} \text{eq:WW}$$

It is well known that $\nabla w_v^\Omega \cdot \nabla (w_v^\Omega + m)^- = -|\nabla (w_v^\Omega + m)^-|^2$ a.e. in $\Omega \times \mathbb{R}_+$. Thus (11), (10) and (9) give

$$\langle (-\Delta_{\Omega})^s v, (v + m)^- \rangle = - c_s \mathcal{E}(w_v^\Omega + m)^- \leq - c_s \mathcal{E}(w_{(v+m)^-}^\Omega) \leq - \| (-\Delta_{\Omega})^{s/2} (v + m)^- \|_2^2, \hspace{1cm} (12)$$
and i) with a large inequality follows.

Now assume that equality holds in i). We have to show that \( v + m \) is nonnegative or nonpositive. Since equality holds everywhere in (12), then \( \mathcal{E}((w^\Omega_v + m)^-) = \mathcal{E}(w^\Omega_{(v+m)^-}) \).
We infer that \( (w^\Omega_v + m)^- = w^\Omega_{(v+m)^-} \), as the minimization problem \( \mathcal{M}^\Omega_{(v+m)^-} \) admits a unique solution. Whence, \( (w^\Omega_v + m)^- \) solves \( \mathcal{L}^\Omega_{(v+m)^-} \) with nonnegative boundary datum \( (v + m)^- \). By the maximum principle either \( (w^\Omega_v + m)^- \equiv 0 \), i.e. \( v + m \geq 0 \); or \( (w^\Omega_v + m)^- > 0 \) in \( \Omega \times \mathbb{R}_+ \), that is, \( w^\Omega_v + m < 0 \) in \( \Omega \times \mathbb{R}_+ \) and \( v + m \leq 0 \) on \( \Omega \). This gives i). To check ii) notice that \( (v - m)^+ = ((-v) + m)^- \) and then use i) with \( -(v) \) instead of \( v \).

Finally, we write \( v \wedge m = v - (v - m)^+ \) and use ii) to get
\[
\|(-\Delta)^s (v \wedge m)^2 \rangle = \|(-\Delta)^s v\|^2_2 - 2(-\Delta)^s v, (v - m)^+ \rangle + \|(-\Delta)^s (v - m)^+\|^2_2
\[
< \|(-\Delta)^s v\|^2_2 - \|(-\Delta)^s (v - m)^+\|^2_2.
\]
Thus iii) holds true, and the lemma is completely proved.

**Remark 3.2** For the Dirichlet fractional Laplacian the inequalities i)–iii) were proved in [20, Lemma 2.4] with large signs. Arguing as above and using the Caffarelli-Silvestre extension [2] instead of [20] we can get complete “Dirichlet” analog of Lemma 4.1.

**Remark 3.3** Taking \( m = 0 \) in Lemma 3.1 we obtain the “Navier” counterpart of [20, Lemma 2.1]. The statement iii) in this case can be rewritten as follows: for \( v \in \tilde{H}^s(\Omega) \) with \( v^+, v^- \neq 0 \)
\[
\langle (-\Delta)^s v^+, v^- \rangle = \langle (-\Delta)^s v^-, v^+ \rangle < 0.
\]
Next, we notice that ii) in Lemma 3.1 with \( m = 0 \) gives the well known weak maximum principle for \( (-\Delta)^s \). A strong maximum principle was proved in [7, Lemma 2.4]. Namely, if \( u \in H^s(\Omega) \setminus \{0\} \) and \( (-\Delta)^s u \geq 0 \) in \( \Omega \) then \( u \) is bounded away from zero on every compact set \( K \subset \Omega \).

**Remark 4.4** Assume \( v_h \to v \) in \( \tilde{H}^s(\Omega) \). Then \( \|(-\Delta)^s (v_h^+ - v)^\|_2 \to 0 \). For the proof, recall that \( \|(-\Delta)^s \|_2 \) equivalent to the norm in \( \tilde{H}^s(\Omega) \) that is induced by \( H^s(\mathbb{R}^n) \), see [18, Corollary 1]; then use the continuity of truncation operators \( v \mapsto v^\pm \) in \( H^s(\mathbb{R}^n) \), see for instance [22, Theorem 5.5.2/3].

## 4 Equivalent formulations

and continuous dependence results

We start by recalling the notion of (distributional) supersolution.
Definition 4.1 A function $U \in \tilde{\mathcal{H}}^s(\Omega)$ is a supersolution for $(-\Delta)^s v = f$ if

$$\langle (-\Delta)^s U - f, \varphi \rangle \geq 0 \quad \text{for any} \ \varphi \in \tilde{\mathcal{H}}^s(\Omega), \ \varphi \geq 0.$$  

Thanks to the results in the previous section, the arguments in [20, Section 3] can be easily adapted to cover the problem $P^\Omega(\psi, f)$. We start by pointing out some equivalent formulations for $P^\Omega(\psi, f)$. For the proof, argue as for [20, Theorem 3.2].

Theorem 4.2 Let $u \in K^s_\psi$. The following sentences are equivalent.

a) $u$ is the solution to the problem $P^\Omega(\psi, f)$;

b) $u$ is the smallest supersolution for $(-\Delta)^s v = f$ in the convex set $K^s_\psi$. That is, $U \geq u$ almost everywhere in $\Omega$, for any supersolution $U \in K^s_\psi$;

c) $u$ is a supersolution for $(-\Delta)^s v = f$ and

$$\langle (-\Delta)^s u - f, (v - u)^- \rangle = 0 \quad \text{for any} \quad v \in K^s_\psi.$$  

d) $\langle (-\Delta)^s v - f, v - u \rangle \geq 0$ for any $v \in K^s_\psi$.

The next corollary is an immediate consequence of a) $\Rightarrow$ b) in Theorem 4.2.

Corollary 4.3 Let $f_1, f_2 \in \tilde{\mathcal{H}}^s(\Omega)'$ and let $u_i$ be the solution to $P^\Omega(\psi, f_i)$, $i = 1, 2$. If $f_1 \geq f_2$ in the sense of distributions, then $u_1 \geq u_2$ a.e. in $\Omega$.

Remark 4.4 Let $u \in K^s_\psi$ be such that $(-\Delta)^s u \geq f$ in $\Omega$. Then $(-\Delta)^s u - f$ can be identified with a nonnegative Radon measure on $\Omega$. Assume that the support of this measure is contained in the coincidence set $\{u = \psi\}$, so that $u$ solves the free boundary problem [2]. Let $v \in K^s_\psi$. Since $(v - u)^-$ vanishes on $\{u = \psi\}$, we have $\langle (-\Delta)^s u - f, (v - u)^- \rangle = 0$. Hence $u$ solves $P^\Omega(\psi, f)$ by Theorem 4.2.

Now we can state our continuous dependence results. The proof of the next theorem is totally similar to the proof of Theorem 4.1 in [20], and we omit it.

Theorem 4.5 Let $\psi_1, \psi_2$ be given obstacles, $f \in \tilde{\mathcal{H}}^s(\Omega)'$ and let $u_i$ be solutions to $P^\Omega(\psi_i, f)$, $i = 1, 2$. If $\psi_1 - \psi_2 \in L^\infty(\Omega)$, then the difference $u_1 - u_2$ is bounded, and

i) $\|u_1 - u_2\|^\infty \leq 1\|\psi_1 - \psi_2\|^\infty,$ ii) $\|u_1 - u_2\|^\infty \leq 1\|\psi_1 - \psi_2\|^\infty.$

In particular, $\|u_1 - u_1\|^\infty \leq 1\|\psi_1 - \psi_2\|^\infty.$
**Corollary 4.6** Let $\psi \in L^\infty(\Omega)$ and $f \in L^p(\Omega)$, with $p \in (1, \infty)$, $p > n/2s$. Let $u \in \tilde{H}^s(\Omega)$ be the solution to $\mathcal{P}_\Omega(\psi, f)$. Then $u \in L^\infty(\Omega)$ and
\[
\psi \vee \omega_f \leq u \leq \|\psi^+\|_\infty + c\|f^+\|_p \quad \text{a.e. in } \Omega,
\] (13) \hspace{1cm} \text{eq:Linf}
where $\omega_f$ solves the problem
\[
(-\Delta_\Omega)^s \omega_f = f \quad \text{in } \Omega, \quad \omega_f \in \tilde{H}^s(\Omega),
\] (14) \hspace{1cm} \text{eq:e}
and $c$ depends only on $n, s, p$ and $\Omega$. In particular, if $f = 0$ then
\[
\psi^+ \leq u \leq \|\psi^+\|_\infty.
\]
**Proof.** Notice that $f \in \tilde{H}^s(\Omega)'$ by Sobolev embedding theorem. Since $u$ is supersolution of (14), the first inequality in (13) follows by the maximum principle in Remark 3.3. To prove the second inequality in (13) we introduce the functions $\omega_{f^+}^N, \omega_{f^+}^D \in \tilde{H}^s(\Omega)$ via
\[
(-\Delta_\Omega)^s \omega_{f^+}^N = (-\Delta)^s \omega_{f^+}^D = f^+ \quad \text{in } \Omega.
\]
It has been proved in [20], proof of Corollary 4.2, that $\omega_{f^+}^D \leq c\|f^+\|_p$, where the constant $c > 0$ does not depend on $f$.

Next, $\omega_{f^+}^D \geq 0$ by the maximum principle. Therefore, $(-\Delta_\Omega)^s \omega_{f^+}^D \geq (-\Delta)^s \omega_{f^+}^D$ in $\Omega$ by [18, Theorem 1]. Thus $(-\Delta_\Omega)^s (\omega_{f^+}^D - \omega_{f^+}^N) \geq 0$ in $\Omega$, that implies $\omega_{f^+}^D \geq \omega_{f^+}^N$ by the maximum principle in Remark 3.3. In particular we have $\omega_{f^+}^N \leq c\|f^+\|_p$ a.e. in $\Omega$.

Now let $u_1$ be the unique solution of $\mathcal{P}_\Omega(\psi, f^+)$. We can consider $\omega_{f^+}^N$ as the solution of the problem $\mathcal{P}_\Omega(\omega_{f^+}, f^+)$, so that Theorem 4.5 gives
\[
u \leq (u_1 - \omega_{f^+}^N)^+ + \omega_{f^+}^N \leq \|\psi^+ - \omega_{f^+}^N\|_\infty + \omega_{f^+}^N,
\]
and the second inequality in (13) readily follows. \hfill \Box

To prove the next continuous dependence results, argue as in [20], proofs of Theorems 4.3 and 4.4, respectively.

**Theorem 4.7** Let $\psi_h \in L^\infty(\Omega)$ be a sequence of obstacles and let $f \in \tilde{H}^s(\Omega)'$ be given. Assume that there exists $v_0 \in \tilde{H}^s(\Omega)$, such that $v_0 \geq \psi_h$ for any $h$.

Denote by $u_h$ the solution to the obstacle problem $\mathcal{P}_\Omega(\psi_h, f)$. If $\psi_h \to \psi$ in $L^\infty(\Omega)$, then $u_h \to u$ in $\tilde{H}^s(\Omega)$, where $u$ is the solution to the limiting problem $\mathcal{P}_\Omega(\psi, f)$.

**Theorem 4.8** Let $\psi_h \in H^s(\mathbb{R}^n)$ be a sequence of obstacles such that $\psi_h^+ \in \tilde{H}^s(\Omega)$, and let $f_h$ be a sequence in $\tilde{H}^s(\Omega)'$. Assume that
\[
\psi_h \to \psi \quad \text{in } H^s(\mathbb{R}^n), \quad f_h \to f \quad \text{in } H^s(\mathbb{R}^n)'.
\]

Denote by $u_h$ the solution to the obstacle problem $\mathcal{P}_\Omega(\psi_h, f_h)$. Then $u_h \to u$ in $\tilde{H}^s(\Omega)$, where $u$ is the solution of the limiting obstacle problem $\mathcal{P}_\Omega(\psi, f)$.\hfill 10
5 Regularity results

Let $u$ be the solution to $P_{\Omega}(\psi, f)$. In this Section we provide estimates of the Radon measure $(-\Delta_{\Omega})^s u - f \geq 0$ in $\Omega$ and a regularity result for $u$. The results in Section 2 will be largely used.

Recall that $\omega_f$ is the solution to the boundary value problem (14).

Theorem 5.1 Assume that $f \in \tilde{H}^s(\Omega)'$ and that $f, \psi$ satisfy the following conditions.

A1) $(\psi - \omega_f)^+ \in \tilde{H}^s(\Omega)$;

A2) $(-\Delta_{\Omega})^s (\psi - \omega_f)^+ - f$ is a locally finite signed measure on $\Omega$.

Let $u \in \tilde{H}^s(\Omega)$ be the solution to $P_{\Omega}(\psi, f)$. Then

$$0 \leq (-\Delta_{\Omega})^s u - f \leq ((-\Delta_{\Omega})^s (\psi - \omega_f)^+ - f)^+ \quad \text{in the distributional sense on } \Omega.$$  

Proof. The first step is quite similar to the Dirichlet case ([20, Theorem 1.1]) and is based on the penalty method by Lewy-Stampacchia [15].

Notice that we can assume $f = 0$, $\psi \in \tilde{H}^s(\Omega)$ and $\psi \geq 0$ in $\Omega$. In fact, since $u - \omega_f \in \tilde{H}^s(\Omega)$ and $(-\Delta_{\Omega})^s (u - \omega_f) \geq 0$, then $u - \omega_f \geq 0$ in $\Omega$ by the maximum principle in Remark 3.3. Thus $u - \omega_f \geq \psi \vee \omega_f - \omega_f = (\psi - \omega_f)^+$ and $u - \omega_f$ solves the obstacle problem $P_{\Omega}((\psi - \omega_f)^+, 0)$.

In conclusion, we only have to show that

$$0 \leq (-\Delta_{\Omega})^s u \leq ((-\Delta_{\Omega})^s \psi)^+ \quad \text{in the distributional sense on } \Omega,$$

where $u$ solves $P_{\Omega}(\psi, 0)$ and $\psi$ is a nonnegative obstacle in $\tilde{H}^s(\Omega)$, such that $(-\Delta_{\Omega})^s \psi$ is a measure on $\Omega$. The first inequality in (15) holds by Theorem 4.2.

Now we prove the following claim:

Assume $(-\Delta_{\Omega})^s \psi \in L^p(\Omega)$ for any $p > 1$. Then (15) holds. (16)

We take $p \geq \frac{2n}{n+2s}$, that is needed only if $n > 2s$. Then $\tilde{H}^s(\Omega) \hookrightarrow L^p(\Omega)$ and $L^p(\Omega) \subset \tilde{H}^s(\Omega)'$ by Sobolev embeddings. In particular $((-\Delta_{\Omega})^s \psi)^+ \in \tilde{H}^s(\Omega)'$. Take a function $\theta_\varepsilon \in C^\infty(\mathbb{R})$ such that $0 \leq \theta_\varepsilon \leq 1$, and $\theta_\varepsilon(t) = 1$ for $t \leq 0$, $\theta_\varepsilon(t) = 0$ for $t \geq \varepsilon$. Let $u_\varepsilon \in \tilde{H}^s(\Omega)$ be the unique solution to

$$(-\Delta_{\Omega})^s u_\varepsilon = \theta_\varepsilon(u_\varepsilon - \psi) ((-\Delta_{\Omega})^s \psi)^+ \quad \text{in } \Omega.$$  


Notice that \((1 - \theta_\varepsilon(u_\varepsilon - \psi))(\psi - u_\varepsilon)^+ = 0\) a.e. in \(\Omega\). Therefore, using \(ii)\) in Lemma 3.1 with \(v = \psi - u_\varepsilon\) and \(m = 0\) one gets \((\psi - u_\varepsilon)^+ \equiv 0\). In particular we infer that \(u_\varepsilon \in K^*_\psi\). On the other hand, \((-\Delta_\Omega)^s u_\varepsilon \geq 0\); thus \(u_\varepsilon \geq u\) by \(b)\) in Theorem 4.2.

Next, notice that \((-\Delta_\Omega)^s (u_\varepsilon - u) \leq (-\Delta_\Omega)^s u_\varepsilon\) and that \(\theta_\varepsilon(u_\varepsilon - \psi)(u_\varepsilon - u - \varepsilon)^+ = 0\) a.e. in \(\Omega\). Then again \(ii)\) in Lemma 3.1 plainly implies \((u_\varepsilon - u - \varepsilon)^+ \equiv 0\). In conclusion, we have \(u \leq u_\varepsilon \leq u + \varepsilon\), hence \(\|u_\varepsilon - u\|_\infty \to 0\) as \(h \to \infty\). Therefore, for any nonnegative test function \(\varphi \in C^\infty_0(\Omega)\) we have that

\[
\langle (-\Delta_\Omega)^s u, \varphi \rangle = \int_\Omega u (-\Delta_\Omega)^s \varphi \, dx = \int_\Omega u_\varepsilon (-\Delta_\Omega)^s \varphi \, dx + o_\varepsilon(1)
\]

Thus, \((-\Delta_\Omega)^s u \leq ((-\Delta_\Omega)^s \psi)^+\) in the distributional sense in \(\Omega\), and (16) is proved.

The second step uses an approximation argument that requires to enlarge the domain \(\Omega\). It needs more care than in the Dirichlet case, because of the dependence of the Navier quadratic form on the domain. Let \(\Omega_h \supset \overline{\Omega}\), \(h \geq 1\), be a sequence of uniformly Lipschitz domains satisfying (8). The convex set

\[
K_h(\psi) = \{ v \in \tilde{H}^s(\Omega_h) \mid v \geq \psi \text{ a.e. on } \mathbb{R}^n \}
\]

contains \(K^*_\psi\), hence it is not empty. Let \(u_h \in \tilde{H}^s(\Omega_h)\) be the solution to

\[
u_h \in K_h(\psi), \quad \langle (-\Delta_\Omega_h)^s u_h, v - u_h \rangle \geq 0 \quad \forall v \in K_h(\psi) . \quad (P_{\Omega_h})
\]

We claim that

\[
(-\Delta_\Omega_h)^s u_h \leq ((-\Delta_\Omega)^s \psi)^+ \text{ in the distributional sense on } \Omega . \quad (18)
\]

Fix \(h\), and approximate \(\psi\) with a sequence of smooth obstacles \(\psi^k = \psi * \rho_k\), where \(\text{supp}(\rho_k) \subset B_{1/2^k}\). For \(k\) large enough we have \(\psi^k \in C^\infty_0(\Omega_h)\). In addition \(\psi^k \to \psi\) in \(\tilde{H}^s(B_R)\) as \(k \to \infty\), where \(B_R\) is any ball containing \(\Omega_h\). Now, let \(u_h^k \in \tilde{H}^s(\Omega_h)\) be the solution to the obstacle problem

\[
u_h^k \in K_h(\psi^k), \quad \langle (-\Delta_\Omega_h)^s u_h^k, v - u_h^k \rangle \geq 0 \quad \forall v \in K_h(\psi^k) . \quad (P^k_{\Omega_h})
\]

Then \(u_h^k \to u_h\) in \(\tilde{H}^s(\Omega_h)\) as \(k \to \infty\) by Theorem 4.8 and (16) gives

\[
(-\Delta_\Omega_h)^s u_h^k \leq ((-\Delta_\Omega_h)^s \psi^k)^+ \text{ in the distributional sense on } \Omega . \quad (19)
\]

Next, \(((\Delta_\Omega_h)^s \psi)^+ * \rho_k\) is a nonnegative smooth function, and

\[
((\Delta_\Omega_h)^s \psi)^+ * \rho_k \geq ((\Delta_\Omega_h)^s \psi) * \rho_k = (-\Delta_\Omega_h)^s \psi^k .
\]
Thus \((-\Delta_{\Omega_h}^s \psi)^* \ast \rho_h \geq ((-\Delta_{\Omega_h}^s \psi^k)^+\), and (19) implies
\((-\Delta_{\Omega_h}^s) u_h^k \leq ((-\Delta_{\Omega_h}^s \psi)^+ \ast \rho_k \text{ in the distributional sense on } \Omega.
\]
Now, as \(k \to \infty\) we have that \(((\Delta_{\Omega_h}) \psi)^+ \ast \rho_k \to ((\Delta_{\Omega_h}) \psi)^+\) in the sense of measures, and \((-\Delta_{\Omega_h}) u_h^k \to (-\Delta_{\Omega_h}) u_h\) in the sense of distributions. Thus
\((-\Delta_{\Omega_h}) u_h \leq ((-\Delta_{\Omega})^s)^+\) in the distributional sense on \(\Omega.
\]
Since \(\psi \in \tilde{H}^s(\Omega)\) is nonnegative, Lemma 2.1 gives \(((\Delta_{\Omega_h}) \psi)^+ \leq ((\Delta_{\Omega})^s)^+\), and (18) follows.

The last step is the passage to the limit in (18) as the domains \(\Omega_h\) shrink to \(\Omega\). It makes the main difference with respect to the Dirichlet case. We notice that \(u \in \tilde{H}^s(\Omega_h)\) and in particular \(u \in K(\psi)\). Therefore, using the variational characterization of \(u_h\) as the solution to \((\mathcal{P}_{\Omega_h})\) and Lemma 2.1 we find

\[\langle ((-\Delta_{\Omega_h})^s u_h, u_h) \rangle \leq \langle ((-\Delta_{\Omega_h})^s u, u) \rangle \leq \langle ((-\Delta_{\Omega})^s u, u) \rangle.\]  (20) \[\text{eq:uff]\]

Lemma 2.1 gives also \(\langle ((-\Delta_{B_R})^s u_h, u_h) \rangle \leq \langle ((-\Delta_{\Omega_h})^s u_h, u_h) \rangle.\) Thus (20) implies that the sequence \(u_h\) is bounded in \(\tilde{H}^s(B_R)\), and therefore we can assume that \(u_h \to \tilde{u}\) weakly in \(\tilde{H}^s(B_R)\). Using Lemma 2.2 and (20) we readily get \(\tilde{u} \in \tilde{H}^s(\Omega)\) and

\[\langle ((-\Delta_{\Omega})^s \tilde{u}, \tilde{u}) \rangle \leq \liminf_{h \to \infty} \langle ((-\Delta_{\Omega_h})^s u_h, u_h) \rangle \leq \limsup_{h \to \infty} \langle ((-\Delta_{\Omega_h})^s u_h, u_h) \rangle \leq \langle ((-\Delta_{\Omega})^s u, u) \rangle.\]

(21)

That is, \(\langle ((-\Delta_{\Omega})^s \tilde{u}, \tilde{u}) \rangle \leq \langle ((-\Delta_{\Omega})^s u, u) \rangle.\) On the other hand, \(u_h \to \tilde{u}\) almost everywhere and \(u_h \geq \psi\) on \(\Omega\). Thus \(\tilde{u} \in K(\psi)\). Using the characterization of \(u\) as the unique solution to the minimization problem \((\Pi)\) (with \(f \equiv 0\)), we first get \(\tilde{u} = u\). Then we use (21) to infer that \(u_h \to u\) in \(\tilde{H}^s(B_R)\).

To conclude take any nonnegative function \(\eta \in C^\infty(\Omega)\). Using \(\Omega \subset \Omega_h \subset \bar{\Omega}_h \subset B_R\) and Lemma 2.1 we see that \((\Delta_{B_R}) \eta \leq ((\Delta_{\Omega_h})^s \eta \leq ((\Delta_{\Omega})^s \eta\) in \(B_R\). Thus \((\Delta_{\Omega_h})^s \eta\) is a bounded sequence in \(L^2(B_R) \hookrightarrow \tilde{H}^s(B_R)^'\). In addition, recall that \(u_h \to u\) in \(\tilde{H}^s(B_R)\) and \((\Delta_{\Omega_h})^s \eta \to ((\Delta_{\Omega})^s \eta\) weakly in \(\tilde{H}^s(\Omega)\), see Lemma 2.3. Thus, we can use (18) to estimate

\[\langle ((-\Delta_{\Omega})^s \psi)^+, \eta \rangle \geq \langle ((-\Delta_{\Omega_h})^s u_h, \eta) \rangle = \langle ((-\Delta_{\Omega_h})^s \eta, u_h) \rangle = \langle ((-\Delta_{\Omega_h})^s \eta, u) + o_h(1) \rangle = \langle ((-\Delta_{\Omega})^s \eta, u) + o_h(1) \rangle = \langle ((-\Delta_{\Omega})^s u, \eta) + o_h(1) \rangle.
\]

The proof is complete. \[\square\]
In the next results we adopt "pointwise" definitions of the contact set and of the non-contact set (compare with Definition 6.1 below for a different notion), that is,

\[ \{ u = \psi \} := \{ x \in \Omega \mid u(x) = \psi(x) \}, \quad \{ u > \psi \} := \{ x \in \Omega \mid u(x) > \psi(x) \}. \quad (22) \]

Clearly, \( \{ u = \psi \} \) and \( \{ u > \psi \} \) are determined up to negligible sets.

**Theorem 5.2** Let \( \psi \) and \( f \) satisfy assumptions of Theorem 5.1 and

A3) \( ((-\Delta_\Omega)^s (\psi - \omega_f)^+ - f)^+ \in L^p_{loc}(\Omega) \) for some \( p \in [1, \infty] \).

Let \( u \in \tilde{H}^s(\Omega) \) be the solution to \( \mathcal{P}_{\Omega}(\psi, f) \). Then the following facts hold.

i) \( (-\Delta_\Omega)^s u - f \in L^p_{loc}(\Omega) \);

ii) \( 0 \leq (-\Delta_\Omega)^s u - f \leq ((-\Delta_\Omega)^s (\psi - \omega_f)^+ - f)^+ \) a.e. on \( \Omega \);

iii) \( (-\Delta_\Omega)^s u = f \) a.e. on \( \{ u > \psi \} \).

In particular, \( u \) solves the free boundary problem (2).

**Proof.** We follow the proof of [20, Theorem 1.1]. Statements i) and ii) hold by Theorem 5.1. Let us prove the last claim. As before, we assume \( f \equiv 0 \). Then \( (-\Delta_\Omega)^s u \geq 0 \). If \( u \equiv 0 \) then iii) is evident. Otherwise the strong maximum principle, see Remark 3.3, gives \( u > 0 \) in \( \Omega \). In particular, \( u \geq \psi^+ \) and \( \{ u > \psi \} = \{ u > \psi^+ \} \).

Use c) in Theorem 4.2 with \( v = \psi^+ \in \tilde{H}^s(\Omega) \), to get \( \langle ((-\Delta_\Omega)^s u, u - \psi^+) \rangle = 0 \). Let \( \varphi \in C_0^\infty(\Omega) \) be any nonnegative cut off function; for \( m \geq 1 \) put \( g_m = (u - \psi^+) \wedge m \). Since \( (-\Delta_\Omega)^s u \geq 0 \) and \( u - \psi^+ \geq \varphi g_m \), we have

\[ 0 = \langle ((-\Delta_\Omega)^s u, u - \psi^+) \rangle \geq \langle ((-\Delta_\Omega)^s u, \varphi g_m) \rangle = \int_\Omega (-\Delta_\Omega)^s u \cdot (\varphi g_m) dx. \]

The last equality holds because \( (-\Delta_\Omega)^s u \in L^1_{loc}(\Omega) \) and \( \varphi g_m \in L^\infty(\Omega) \) has compact support in \( \Omega \). Thanks to the monotone convergence theorem we get

\[ 0 \geq \lim_{m \to \infty} \int_\Omega (-\Delta_\Omega)^s u \cdot \varphi g_m dx = \int_\Omega ((-\Delta_\Omega)^s u \cdot (u - \psi^+)) \varphi dx. \]

Now \( ((-\Delta_\Omega)^s u \cdot (u - \psi^+)) \varphi \geq 0 \) a.e. in \( \Omega \), that gives \( ((-\Delta_\Omega)^s u \cdot (u - \psi^+)) \varphi = 0 \) a.e. in \( \Omega \). Since \( \varphi \) was arbitrarily chosen, we conclude that \( (-\Delta_\Omega)^s u \cdot (u - \psi^+) = 0 \) a.e. in \( \Omega \), and iii) is proved. \( \square \)
Remark 5.3 To obtain better regularity results for $u$, one can apply the regularity theory for $$(-\Delta)^s u = g \in L^p(\Omega), \quad u \in \tilde{H}^s(\Omega).$$ In particular, if $p > \frac{n}{2s}$, then $u$ is Hölder continuous in $\Omega$, see [11, Corollary 3.5].

We conclude this section by giving a sufficient condition for the continuity of $u$.

**Theorem 5.4** Let $\psi \in C^0(\Omega)$ be a given obstacle, such that $K^s_\psi$ is not empty and $\psi \leq 0$ on $\partial \Omega$. Let $f \in L^p(\Omega)$, with $p > n/2s$. Then $u$ is continuous on $\Omega^n$.

**Proof.** The argument is the same as in [20, Theorem 1.2]. Fix a small $\varepsilon > 0$. We can assume that $\psi - \varepsilon \in C^0_0(\mathbb{R}^n)$ and $\psi - \varepsilon \leq 0$ outside $\Omega$. Let $\psi^\varepsilon_h$ be a sequence in $C^\infty_0(\mathbb{R}^n)$ such that $\psi^\varepsilon_h \to \psi - \varepsilon$ uniformly on $\mathbb{R}^n$, as $h \to \infty$.

By Theorem 5.2 the solution $u^\varepsilon_h \in \tilde{H}^s(\Omega)$ to $P^{\text{N}}(\psi^\varepsilon_h, f)$ satisfies $$(-\Delta)^s u^\varepsilon_h \in L^p(\Omega)$$ and therefore $u^\varepsilon_h$ is Hölder continuous, see Remark 5.3. Moreover, the estimates in Theorem 4.5 imply that $u^\varepsilon_h \to u^\varepsilon$ uniformly on $\Omega$, where $u^\varepsilon$ solves $P^{\text{D}}(\psi - \varepsilon, f)$. In particular, $u^\varepsilon \in C^0(\Omega)$. Finally, use again Theorem 4.5 to get that $u^\varepsilon \to u$ uniformly, and conclude the proof. \qed

6 Comparing the Navier and the Dirichlet problems

In this section we compare the unique solutions $u_N, u_D$ to the variational inequalities

$$u_N \in K^s_\psi, \quad \langle (-\Delta)^s u_N, v - u_N \rangle \geq 0 \quad \forall v \in K^s_\psi, \quad (P^{\text{N}}(\psi))$$  \tag{eq:N}  

$$u_D \in K^s_\psi, \quad \langle (-\Delta)^s u_D, v - u_D \rangle \geq 0 \quad \forall v \in K^s_\psi, \quad (P^{\text{D}}(\psi))$$  \tag{eq:D}  

respectively. Here $(-\Delta)^s$ is the "Dirichlet" (or restricted) Laplacian, that has been already introduced in [3].

Problem $P^{\text{D}}(\psi)$ has been investigated in [20]. Recall that

$$(-\Delta)^s u_N \geq 0, \quad (-\Delta)^s u_D \geq 0 \quad \text{on} \quad \Omega \quad (23)$$  \tag{eq:L_positive}  

in the sense of distributions, see Theorem 4.2 and [20 Theorem 3.2], so that $u_N, u_D$ are nonnegative by the maximum principle, and $u_N, u_D \geq \psi^+$. Thus, $u_N, u_D$ solve problems $P^{\text{N}}(\psi^+), P^{\text{D}}(\psi^+)$, respectively. Hence we can assume without loss of generality that $\psi \geq 0$ a.e. in $\Omega$. Since $\psi \equiv 0$ easily implies $u_N \equiv u_D \equiv 0$, we assume also that $\psi \not\equiv 0$. 

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By the strong maximum principle (see, respectively, [7, Lemma 2.4] and [13, Theorem 2.5]) we have
\[ u_N > 0 \text{ and } u_D > 0 \text{ on } \Omega, \]
in the sense that \( u_N, u_D \) are bounded away from 0 on every compact set \( K \subset \Omega \).

In this section we need to refine the notion of contact set introduced in (22). We essentially use an idea due to Lewy and Stampacchia [15, 16], see also [14, Section 6]. We start with some preliminaries.

**Definition 6.1** Let \( v \) be a nonnegative measurable function on the open set \( \Omega \), and let \( x \in \Omega \). We say that \( v(x) > 0 \) if there exist \( \rho, \varepsilon > 0 \) such that \( B_\rho(x) \subset \Omega \) and
\[ v - \varepsilon \geq 0 \text{ almost everywhere in } B_\rho(x). \]

For any measurable function \( v \) on \( \Omega \) we define
\[ P[v] = \{ x \in \Omega \mid v(x) > 0 \} \]
and we put \( I[v] = \Omega \setminus P[v] \). The set \( P[v] \) is clearly open; thus \( I[v] \) is closed in \( \Omega \).

**Lemma 6.2** Let \( v \geq 0 \) be a measurable function on \( \Omega \) and let \( K \subset P[v] \) be a compact set. Then there exists \( \varepsilon_0 > 0 \) such that \( v - \varepsilon_0 \geq 0 \) a.e. on \( K \).

In particular, \( v > 0 \) a.e. in \( P[v] \).

**Proof.** If \( K \) is not empty, for any \( x \in K \) there exists a ball \( B_x \) about \( x \) such that \( B_x \subset \Omega \) and \( v \geq \varepsilon_x > 0 \) on \( B_x \). Since \( K \) is compact, we can find a finite number of points \( x_i \in K \) such that \( K \) is covered by the finite family \( B_{x_i} \). Let \( \varepsilon_0 = \min_i \varepsilon_{x_i} \). Then \( v \geq \varepsilon_0 \) a.e. on \( K \) and the first claim is proved.

Now put \( N = \{ x \in P[v] \mid v(x) = 0 \} \). By the first part of the proof, any compact set \( K \subset N \) must have null measure. Thus \( N \) is a negliglible set. \( \square \)

By Lemma 6.2 we have the inclusions
\[ P[u_N - \psi] \subseteq \{ u_N > \psi \}, \quad P[u_D - \psi] \subseteq \{ u_D > \psi \}. \]

It might happen that \( \{ u_N > \psi \} \) and \( \{ u_D > \psi \} \) have positive measure but \( P[u_N - \psi] \) and \( P[u_D - \psi] \) are empty, see Remark 6.4 below.

**Theorem 6.3** The following facts hold true.
\[ i) \ ( -\Delta_\Omega )^s u_N = 0 \text{ on } P[u_N - \psi] \text{ and } ( -\Delta )^s u_D = 0 \text{ on } P[u_D - \psi]; \]

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ii) \((-\Delta_\Omega)^s u_D > 0\) in the distributional sense on \(\Omega\);

iii) \(u_N \leq u_D\);

iv) \(u_N < u_D\) on \(P[u_N - \psi]\);

v) \(u_D\) is the solution to the obstacle problem \(\mathcal{P}_D(u_N)\);

vi) \(\|(-\Delta)^\frac{s}{2} u_D\|_2 \leq \|(-\Delta)^\frac{s}{2} u_N\|_2 \leq \|(-\Delta_\Omega)^\frac{s}{2} u_D\|_2\), and all signs are strict, unless \(u_N \equiv u_D\).

**Proof.** Fix any nonnegative \(\varphi \in C^\infty_0(P[u_N - \psi])\). By Lemma 6.2 we have that \(u_N \mp t\varphi \in K_\psi^s\) for sufficiently small \(t \in \mathbb{R}\). Thus \( \pm t((-\Delta_\Omega)^s u_N, \varphi) \geq 0\), that proves i) for \(u_N\). The argument for \(u_D\) is the same.

To prove ii) use [18, Theorem 1], that gives \((-\Delta_\Omega)^s u_D > (-\Delta)^s u_D \geq 0\) by (23).

By b) in Theorem 4.2, we know that \(u_N\) is the smallest supersolution to \((-\Delta_\Omega)^s v = 0\) in the set \(K_\psi^s\). Thus \(u_N \leq u_D\) by ii), and iii) is proved.

On the open set \(P[u_N - \psi]\) both \(u_N\) and \(u_D\) are smooth by i). Assume by contradiction that there exists \(x \in P[u_N - \psi]\) such that \((u_D - u_N)(x) = 0\). Then from iii) we see that

\[
(-\Delta)^s(u_D - u_N)(x) = C \cdot V.P. \int_{\mathbb{R}^n} \frac{(u_D - u_N)(x) - (u_D - u_N)(y)}{|x - y|^{n+2s}} dy \\
= C \cdot V.P. \int_{\mathbb{R}^n} \frac{- (u_D - u_N)(y)}{|x - y|^{n+2s}} dy \leq 0.
\]

But this is impossible, as \((-\Delta)^s(u_D - u_N) > (-\Delta)^s u_D - (-\Delta_\Omega)^s u_N = 0\) on \(P[u_N - \psi]\) by [18, Theorem 1] and i). Claim iv) is proved.

To check v) we use [20, Theorem 3.2], that characterizes \(u_D\) as the smallest supersolution to \((-\Delta)^s u = 0\) in \(K_\psi^s\). Since \(K_{u_N}^s \subseteq K_\psi^s\), and \(u_D \in K_{u_N}^s\) by iii), we see that \(u_D\) is the smallest supersolution to \((-\Delta)^s u = 0\) in \(K_{u_N}^s\) and the conclusion follows again by [20, Theorem 3.2].

Finally, Theorem 2 in [18] gives \(\|(-\Delta)^\frac{s}{2} u_N\|_2 < \|(-\Delta_\Omega)^\frac{s}{2} u_N\|_2\). The remaining inequalities in vi) follow from the the fact that \(u_N\) and \(u_D\) are unique solutions to \(\mathcal{P}_N(\psi)\) and \(\mathcal{P}_D(\psi)\) respectively, and from variational formulations of these problems, see [11] and [20, (1.2)] (with \(f = 0\)).
Remark 6.4 Take a smooth function $\eta \in \tilde{H}^s(\Omega)$ satisfying $(-\Delta)^s \eta \geq 0$, $\eta > 0$ in $\Omega$. Let $\kappa \subset \Omega$ be a compact set, having positive measure but empty interior. Consider the obstacle $\psi = \eta \chi_{\Omega \setminus \kappa}$ and the solutions $u_N, u_D$ to $P_N(\psi)$, $P_D(\psi)$ respectively. Clearly $\eta \in K^s_\psi$. Thus $\eta \geq u_D$ because $u_D$ is the smallest supersolution for $(-\Delta)^s v = 0$ in $K^s_\psi$. But then we have $\eta \geq u_D \geq u_N \geq \psi = \eta \chi_{\Omega \setminus \kappa}$. In particular, $u_D = u_N = \psi$ on $\Omega \setminus \kappa$. Actually $\{u_D = \psi\} = \{u_N = \psi\} = \Omega \setminus \kappa$, because $u_N, u_D$ are positive in $\Omega$. In particular the sets $\{u_D > \psi\}, \{u_N > \psi\}$ have positive measure as they coincide with $\kappa$, but $P[u_N - \psi] = P[u_D - \psi] = \emptyset$.

Remark 6.5 Sufficient conditions in order to have that $P[u_N - \psi]$ is not empty can be easily obtained. For instance, if $\psi$ vanishes on an open set $B \subset \Omega$, then $B \subseteq P[u_N - \psi]$, since $u_N$ is positive by the strong maximum principle in $[7]$. If $\psi$ is continuous on $\overline{\Omega}$ then $u_N$ is continuous as well by Theorem 5.4. Thus either $u_N \equiv \psi$, or $P[u_N - \psi] = \{u_N > \psi\}$ is not empty.

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