Generalized Toda Theories from WZNW Reduction
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Abstract

We reconsider the, by Brink and Vasiliev, recently proposed generalized Toda field theories using the framework of WZNW → Toda reduction. The reduced theory has a gauge symmetry which can be fixed in various ways. We discuss some different gauge choices. In particular we study the \( W \) algebra associated with the generalized model in some different realizations, corresponding to different gauge choices. We also investigate the mapping between the Toda field and a free field and show the relation between the \( W \) algebra generators expressed in terms of the two different fields. All results apply also to the case of ordinary Toda theories.

1 Introduction

Recently Brink and Vasiliev \[1\] proposed a model generalizing the Toda (field) theories based on the simple Lie algebras \( A_N \) (the cases of \( B_N \) and \( C_N \) were also treated). The model involves a continuous parameter, such that when this parameter takes certain discrete values the model reduces to the ordinary \( A_N \) Toda theories. In this sense the model is a universal theory for the \( A_N, B_N, \) and \( C_N \) (Toda) theories. The approach used by Brink and Vasiliev is based on a field \( \phi \) which is taken to depend not only on the two dimensional space-time coordinates, but also on an \( sl_2 \) algebra generator, \( T^0 \). The field can be expanded as \( \phi(T^0, z) = \sum n \varphi_n(z) h_n(T^0) \), where \( h_n \) is a certain \( n \)th order polynomial in \( T^0 \). It is possible to define a trace operation \[2\] \[3\] for the \( h_n \)'s such that \( \text{tr}(h_n h_m) \propto \delta_{nm} \). Using this trace operation, an action for the model can be constructed. In this paper we will reconsider this model from a different perspective. We will confine ourselves to the study of the classical model.

One of the interesting features of the Toda theories, as was first shown by Bilal and Gervais \[4\] \[5\] \[6\], is the fact that the Toda theories possess a \( W \) symmetry. The generalized infinite-dimensional model carries a \( W \) symmetry algebra generalizing the \( W_N \) algebra, in the sense that the \( W \) algebra reduces to the \( W_N \) algebra, for a particular choice of the continuous parameter in the model. The \( W \) algebra is a universal algebra for the \( W_N \) algebras. In \[1\] two different realizations of the general \( W \) algebra were given. In the first realization the \( W \) currents were expressed in terms of the field \( \phi \). The \( i \)th generator, \( W_i \), was shown to be the trace of an \( i \)th order polynomial in \( \phi \). The \( W \) algebra is the Poisson bracket algebra of the \( W_i \)'s using the canonical commutation relations of the \( \phi \)-field.
In the other realization of the $W$ algebra a field $\mu$, depending on (besides the space-time coordinates) the $sl_2$ generator $T^-$, was introduced. It was argued that the generators of the $W$ algebra were equal to the components of the field $\mu$. The $\mu$-field bracket used for calculating commutators of the $W$ algebra, however, is not of the standard Darboux form; rather, as we will see in section 5, it is a Dirac bracket.

It has been shown [7] [8] [9] that the Toda theories can be obtained from a WZNW model by a (hamiltonian) reduction. The constraints imposed on the WZNW model are first class which means that we have gauge invariance in the theory. In this paper we will apply the methods given in [9] to the generalization proposed by Brink and Vasiliev.

In sections 2 and 3 we will review the basic facts about WZNW → Toda reduction, following [1]. In section 4 we will describe the work of Brink and Vasiliev, with emphasis on the infinite-dimensional algebra underlying their work. The $W$ algebra appears naturally in the WZNW reduction approach as a Dirac bracket algebra. Different gauge choices lead to different (isomorphic) realizations of the $W$ algebra. In section 5 we will discuss two particular gauge choices, these will be shown to lead to the two realizations of the $W$ algebra used by Brink and Vasiliev. We will obtain a natural explanation of the bracket for the $\mu$-field as the Dirac bracket in one of the two gauges. Finally, in section 6 we will examine the connection between the two gauge choices mentioned above; this will lead us to the governing equation of ref. [1]. This equation connects the $W$ algebra generators in the two realizations (gauge choices). We will also touch upon the subject of Bäcklund transformations. We end with a short discussion.

2 Review of WZNW → Toda Reduction

Some years ago it was realized that the Toda theories could be obtained from a WZNW model by a (hamiltonian) reduction [7] [8] [9]. We will here briefly review some aspects of this development in a way which can easily be extended to the infinite-dimensional generalization which we will discuss in later sections. The WZNW action is [10] [11] [12]

$$S_W(g) = \frac{\kappa}{2} \int \eta^{\mu\nu} \text{tr} (g^{-1} \partial_\mu g)(g^{-1} \partial_\nu g) d^2z - \frac{\kappa}{3} \int_{B_3} \text{tr} (g^{-1} dg)^3,$$

(2.1)

where as usual $B_3$ is a three-dimensional manifold whose boundary is space-time. The field $g(z)$ is taken to be valued in a connected Lie group, whose associated Lie algebra, $\mathcal{G}$, is assumed to be simple and maximally non-compact. The maximally non-compact real form of a complex simple Lie algebra is the real algebra obtained by choosing the Cartan-Weyl basis in which all the structure constants are real numbers; the generators in this basis are then taken to span a real algebra, which is the maximally non-compact one. As an example, the maximally non-compact form of $A_N$ is $sl(N+1, \mathbb{R})$. Among the various $sl_2$ subalgebras of $\mathcal{G}$ we will in particular be interested in the so called principal embedding\(^3\). This special embedding will be described in more detail later. The details will be important in the derivation of the Toda theory. We denote the generators of this

\(^2\)Signature of metric, $\eta_{\mu\nu} = \text{diag}(1, -1)$; level $k = -4\pi \kappa$.

\(^3\)Also known as the maximal embedding.
particular embedded $sl_2$ by $T^0, T^\pm$; they satisfy the commutation relations

$$[T^0, T^\pm] = \pm T^\pm, \quad [T^-, T^+] = 2T^0. \quad (2.2)$$

$T^0$ defines an integral gradation of the algebra by the eigenvalues of $ad_{T^0} = [T^0, \cdot]$. We may split the algebra into eigenspaces of this operator. Let $\mathcal{G}_m$ be the eigenspace corresponding to the eigenvalue $m$; for future reference we introduce the notation

$$\mathcal{G}_- = \bigoplus_{m<0} \mathcal{G}_m, \quad \mathcal{G}_+ = \bigoplus_{m>0} \mathcal{G}_m, \quad (2.3)$$

where the direct sum is to be considered as a sum of vector spaces. We denote the basis elements of $\mathcal{G}$ by $E^s_n$, where $s$ is the eigenvalue of $ad_{T^0}$, and $n$ is an additional label needed to completely specify the element. The subset consisting of basis elements $E^s_n$ with $s > 0$ ($s < 0$) form a basis of $\mathcal{G}_+$ ($\mathcal{G}_-$). The set of elements $h_n = E^0_n$ form a basis of $\mathcal{G}_0$. Elements which lie in $\ker ad_{T^+}$ ($\ker ad_{T^-}$)\footnote{The kernel of an operator $A$, $\ker A$, is defined as the set of all $x$ such that $Ax = 0$.} will be called highest (lowest) weight.

With the exception of (the maximally non-compact form of) $D_{2n}$, it can be shown that there is no more than one highest (lowest) weight elements for each fixed $T^0$ grade. The basis elements can in all cases be chosen to satisfy

$$\text{tr}(E^s_n E^r_m) = \delta_{s+r} \delta_{nm}, \quad (2.4)$$

which means that only elements which have zero grade with respect to $ad_{T^0}$ can have a non-zero trace. The WZNW currents are $J_+ = \kappa \partial_+ gg^{-1}$ and $J_- = -\kappa g^{-1} \partial_- g$; the equation of motion is, $\partial_+ J_- = 0$, or equivalently $\partial_- J_+ = 0$.\footnote{Conventions: $z^\pm = \frac{1}{\sqrt{2}}(t \pm x)$, $\partial^\pm = \frac{1}{\sqrt{2}}(\partial_t \pm \partial_x)$.} The two equations are not independent since $\partial_- J_+ = -g \partial_+ J_- g^{-1}$. To derive the Toda theory, we impose the constraints\footnote{We use a different convention compared to ref. \textsuperscript{[9]}: $T^+ = M_-, T^- = -M_+$, and $T^0 = -M_0$. This convention is chosen in order to facilitate comparison with ref. \textsuperscript{[9]} later on.}

$$\gamma_{\alpha\pm} = \langle E_{\alpha\pm}, J_{\pm} - \kappa T^\pm \rangle \approx 0, \quad (2.5)$$

where $E_{\alpha\pm}$ form an orthogonal set of basis elements of $\mathcal{G}_\pm$, and $\langle A, B \rangle = \text{tr}(AB)$. The currents are thus constrained to have the form

$$J_\pm(z) = \kappa T^\pm + j_\pm(z), \quad j_\pm(z) \in (\mathcal{G}_0 + \mathcal{G}_\pm). \quad (2.6)$$

It can be proven that the constraints (2.5) are consistent with the dynamics of the WZNW model. The Poisson bracket between the components of the currents is given as

$$\{\langle \alpha, J_+(z) \rangle, \langle \beta, J_+(w) \rangle \} = \langle [\alpha, \beta], J_+(z) \rangle \delta(z^+ - w^+) + \kappa \langle \alpha, \beta \rangle \partial_+ \delta(z^+ - w^+), \quad (2.7)$$

where $\alpha$ and $\beta$ are arbitrary elements of $\mathcal{G}$. We will in most of the rest of the text set $\kappa = 1$, and occasionally denote $J_+$ by $J$. By computing the Poisson brackets between the constraints we see that the constraints are first class, e.g.

$$\{\gamma_{\alpha+}, \gamma_{\beta+} \} = \langle [E_{-\alpha}, E_{-\beta}], J \rangle + \langle E_{-\alpha}, E_{-\beta} \rangle \approx 0, \quad (2.8)$$
where, in the first step, we have used (2.5) and (2.7); we have also suppressed the delta functions. In the second step we have used \( \langle E_{-\alpha}, E_{-\beta} \rangle = 0 \), and the fact that \([E_{-\alpha}, E_{-\beta}] \in \mathcal{G}_{m<2}\); then since \( J \approx T_+ + j \), where \( j \in \mathcal{G}_0 \oplus \mathcal{G}_- \), it follows that \( \langle [E_{-\alpha}, E_{-\beta}], J \rangle \approx 0 \), by using the properties of the trace (2.4).

We now assume that we can make a generalized Gauss decomposition of the form \( g = g_0 g_+ \), where \( g_0 = e^{-\phi}, \phi = \sum \varphi_n h_n \), and \( g_\pm \) are the parts of the group which correspond to the parts \( \mathcal{G}_\pm \) of the algebra. More precisely, we assume that this can be done locally (close to the identity); this means that we neglect global effects of the group. Because of these global effects the constrained WZNW theory is larger than the associated Toda theory. For a discussion about the global effects, see [13]. Using the Gauss decomposition we see that, after a similarity transformation with \( g_+ \), the equation of motion \( \partial \cdot J_+ = 0 \), can be written as a zero commutation relation of the form

\[
[\partial_+ - A_+, \partial_- - A_-] = 0. \tag{2.9}
\]

Here \( A_- = g^{-1} \partial_- g_-, \) and \( A_+ = \partial_+ g_0 g_+^{-1} + g_0 \partial_+ g_+ g_+^{-1} g_0^{-1} \). We can impose the constraints directly into the equation of motion. As before, this procedure is consistent since the WZNW dynamics do not affect the constraint surface. We obtain \( A_+ = -\partial_+ \phi + T^+, \) and \( A_- = e^{-\phi} T^- e^\phi \). The equation of motion then becomes

\[
\partial_- \partial_+ \phi = [T^+, e^{-\phi} T^- e^\phi], \tag{2.10}
\]

which can be shown to be equivalent to the usual Toda equations of motion. We now proceed to show this equivalence. We will make use of a few relations and definitions from the theory of Lie algebras; we will be brief. For more details, see e.g. [14]. Instead of the basis used earlier it will now be convenient to use a Chevalley basis with the following normalizations

\[
\begin{align*}
[E_{\alpha}, E_{-\beta}] &= \delta_{\alpha\beta} H_{\alpha} \quad \alpha \in \Delta_+ \\
[H_{\alpha}, E_{\beta}] &= 2 \delta_{(\alpha,\beta)} E_{\beta} \quad \alpha, \beta \text{ roots} \\
[H_{\alpha}, E_{\pm \beta}] &= \pm K_{\alpha \beta} E_{\pm \beta} \quad \alpha, \beta \in \Delta_+.
\end{align*}
\tag{2.11}
\]

\( \Delta_+ \) denotes the set of (positive) simple roots, and \( K_{\alpha \beta} \) is the Cartan matrix. The generators of the principal \( sl_2 \) embedding can, in the basis (2.11), be written

\[
T^0 = \frac{1}{2} \sum_{\alpha \in \Phi_+} H_\alpha, \quad T^+ = \sum_{\alpha \in \Delta_+} c_\alpha E_\alpha, \quad T^- = - \sum_{\alpha \in \Delta_+} E_{-\alpha}, \tag{2.12}
\]

where \( c_\alpha = 2 \sum_{\beta \in \Delta_+} K_{\alpha \beta}^{-1} \), and \( \Phi_+ \) is the set of positive roots. The Weyl vector, \( \rho \), is defined as half the sum of the positive roots, i.e.

\[
\rho = \frac{1}{2} \sum_{\alpha \in \Phi_+} \alpha. \tag{2.13}
\]

\( T^0 \) is the Cartan subalgebra element corresponding to \( \rho \) under the isomorphism between the Cartan subalgebra and the root space. The Weyl vector have the following important

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7The same notation ‘\( E_\alpha \)’ is used for other basis elements in later sections, hopefully no confusion will occur.
property \((\rho, \alpha^\vee) = 1\), where \(\alpha^\vee = \frac{\alpha}{(\alpha, \alpha)}\), and \(\alpha \in \Delta_+\). The dual Weyl vector is defined through

\[
\rho^\vee = \frac{1}{2} \sum_{\alpha \in \Phi_+} \alpha^\vee = \frac{1}{2} \sum_{\alpha \in \Phi_+} \frac{2\alpha}{(\alpha, \alpha)}.
\] (2.14)

In general, \(\rho \neq \rho^\vee\). The dual Weyl vector satisfies \((\rho^\vee, \alpha) = 1\). Using the above definitions and properties it can be checked that \(T_0\) and \(T_\pm\) satisfy the commutation relations (2.2).

Expanding \(\phi\) in the following manner:

\[
\phi = \sum_{\alpha \in \Delta_+} \phi_\alpha H_\alpha,
\]

and using the Baker-Hausdorff formula, the equation of motion (2.10) can be seen to reduce to

\[
\partial_+ \partial_- \varphi_\alpha = e^{-\sum_\beta K_{\alpha\beta} \varphi_\beta},
\] (2.15)

which are the equations of motion of the usual finite dimensional Toda theory \cite{12}, in a suitable normalization. To obtain the form (2.15) we also had to make a redefinition of the fields, viz. \(\varphi_\alpha \rightarrow -\varphi_\alpha - \sum_\beta K_{\alpha\beta}^{-1}(\ln(\sum_\gamma 2K_{\beta\gamma}))\).

We now return to our main discussion. The equation of motion (2.10) can be obtained from an effective action

\[
S_{\text{eff}} = S_W(g_0) + 2 \int \text{tr}(T^+ e^{-\phi} T^- e^\phi) d^2 z = \int \text{tr}(\partial_+ \phi \partial_- \phi) d^2 z + 2 \int \text{tr}(T^+ e^{-\phi} T^- e^\phi) d^2 z.
\] (2.16)

The first class constraints generate infinitesimal gauge transformations

\[
\delta_\alpha A = \{ \int \gamma_\alpha(z) dz, A \},
\] (2.17)

where \(A\) is arbitrary. In particular, the currents have the (infinitesimal) gauge transformations on the space of classical solutions

\[
\delta J = [\alpha(z^+), J] + \partial_+ \alpha(z^+),
\] (2.18)

where \(\alpha \in G_-\). Notice that the transformations (2.18) do not affect the Toda field \(\phi\), which thus is gauge invariant. The energy momentum tensor is given by the standard Sugawara expression

\[
T_{++}(z) = \frac{\langle J(z) J(z) \rangle}{2}.
\] (2.19)

The Sugawara energy momentum tensor is not gauge invariant, this can be rectified by adding an improvement term

\[
\Theta_{++}(z) = T_{++}(z) + \langle T^0 \partial_+ J(z) \rangle.
\] (2.20)

3 Lagrangean Realization of the Reduction

There is also an equivalent lagrangean realization of the WZNW \(\rightarrow\) Toda reduction, as a gauged WZNW model. We now proceed to describe this approach. We use the left-right asymmetric gauge transformations,

\[
g \rightarrow h g \bar{h}^{-1}, \quad h = e^{\alpha(z^+, z^-)}, \quad \bar{h}^{-1} = e^{-\beta(z^+, z^-)},
\] (3.1)
where $\alpha \in G_-$, $\beta \in G_+$. The canonical choice for a gauge invariant action is

$$S(A_-, A_+, g) = S_W(g) + 2 \int \text{tr}[A_- (\partial_+ gg^{-1}) + A_+ (g^{-1} \partial_-. g) + A_- gA_+ g^{-1}] d^2 z. \quad (3.2)$$

This action is invariant under the gauge transformations \((3.1)\), together with

$$A_- \to e^{\alpha} A_- e^{-\alpha} + e^\alpha (\partial_- e^{-\alpha}) , \quad A_+ \to e^{\beta} A_+ e^{-\beta} + (\partial_+ e^{\beta}) e^{-\beta}. \quad (3.3)$$

The action \((3.2)\) is “derived” in the usual way; the natural starting point is $S_W(hg\bar{h}^{-1})$, which may be rewritten using the Polyakov-Wiegmann identity \([16], [17]\)

$$S_W(abc) = S_W(a) + S_W(b) + S_W(c) + 2 \int \text{tr}(a^{-1} \partial_- a)(\partial_+ bb^{-1}) d^2 z +$$

$$+ 2 \int \text{tr}((b^{-1} \partial_- b)(\partial_+ cc^{-1}) + (a^{-1} \partial_- a)b(\partial_+ c)c^{-1}b^{-1}) d^2 z. \quad (3.4)$$

(Notice that $S(h) = 0$, and $S(\bar{h}^{-1}) = 0$, because of the properties of tr). By making the identification of the gauge fields according to, $A_+ = \partial_+ \bar{h}^{-1} \bar{h}$, $A_- = h^{-1} \partial_- h$, the gauge invariant action \((3.2)\) follows. We have potential problems with anomalies; an infinitesimal gauge variation of $S$ gives

$$\delta S = 2\text{tr}(\beta \partial_+ A_+ - \alpha \partial_+ A_-). \quad (3.6)$$

The variation vanishes, however, as a consequence of the properties of the trace. The above action is not quite what we want. We want to constrain some of the components of the WZNW currents to be constant (not zero). To achieve this feature we add a gauge invariant term to $S$, viz.

$$- 2 \int \text{tr}(A_- T_+ - A_+ T_-) d^2 z. \quad (3.7)$$

It can be checked that this transforms as a total derivative under an infinitesimal gauge transformation. Summarizing, we get

$$S = S_W(g) + 2 \int \text{tr}[A_- (\partial_+ gg^{-1} - T_+) + A_+ (g^{-1} \partial_- g + T_-) + A_- gA_+ g^{-1}] d^2 z, \quad (3.8)$$

which is invariant under the gauge transformations \((3.3)\). We can use the gauge freedom to set $A_\pm = 0$. If we eliminate the $A$‘s using their Euler-Lagrange equations and remember the gauge choice, we recover the constraints \((2.5)\), and the equations of motion of the WZNW model. This does not, however, completely fix the gauge, we still have the freedom to make gauge transformations of the form $g \to e^{\alpha(z^+)} ge^{-\beta(z^-)}$. This will give us back the gauge transformations \((2.18)\).

4 The General Algebra

In section \((2)\) we based our discussion on a simple maximally non-compact Lie algebra. In this section we will describe a generalization of the simple Lie algebras $sl_N$ and the
associated generalized Toda theories. The algebra we have in mind originated in higher spin theories \[2\] \[3\], and is defined as the Lie algebra obtained from the (associative) algebra of all monomials of two operators, \(a^\pm\), satisfying the commutation relations

\[
[a^-, a^+] = 1 + 2\nu K,
\{K, a^\pm\} = 0, \quad K^2 = 1.
\] (4.1)

Here \(\nu\) is a free parameter. A general element of the algebra can be written as

\[
B = \sum_{A=0}^{1} \sum_{n=0}^{\infty} \frac{1}{n!} b_{a_1...a_n}^A K^A a^{\alpha_1}...a^{\alpha_n}.
\] (4.2)

We can choose \(b_{a_1...a_n}^A\) to be totally symmetric in the lower indices, so the basis elements can be chosen to be the Weyl ordered products

\[
E_{nm} = \left((a^+)^n(a^-)^m\right)_{\text{Weyl}} = \frac{1}{(n+m)!}((a^+)^n(a^-)^m + ((n+m)!) - 1)\text{ permutations},
\] (4.3)

together with the elements \(KE_{nm}\). We would like to point out that in general there does not exist a generalization of the Chevalley basis described in section 2 to the infinite-dimensional algebra considered in this section. The quadratic combinations of \(a^\pm\)

\[
T^\pm = \frac{1}{2}(a^\pm)^2, \quad T^0 = \frac{1}{4}\{a^+, a^-\},
\] (4.4)

satisfy the commutation relations (2.2), and thus form an \(sl_2\) subalgebra. This realization of \(sl_2\) dates back to Wigner \[18\], see also \[19\] \[20\]. The algebra under discussion can be shown to be isomorphic to the universal enveloping algebra of \(osp(1, 2), U(osp(1, 2))\), divided by a certain ideal \[21\]. When \(d = N\), where \(d = \frac{2\nu + 1}{2}\) and \(N\) is a positive integer, the algebra becomes, after dividing out an ideal, isomorphic to \(gl(N, N - 1)\), when considered as a Lie algebra. The algebra has a unique trace operation \[2\] \[3\], defined as

\[
\text{str}(B) = b^0 - 2\nu b^1
\] (4.5)

with the properties,

\[
\text{str}(AB) = (-1)^{\pi(A)} \text{str}(BA) = (-1)^{\pi(B)} \text{str}(BA),
\] (4.6)

where \(\pi(A)\) is defined to be equal to 0 or 1 for monomials with even or odd powers \(n\) in (4.2), respectively. In the bosonic subalgebra (spanned by all even monomials, i.e. all \(E_{nm}\)’s, with \(n + m = \text{even}\)) we can use the projection operators \(P_{\pm} = \frac{1\pm K}{2}\) to split the algebra into two parts. The two sectors carry the same amount of information so without loss of generality we will in the sequel only work in the \(P_-\) sector of the bosonic subalgebra. The element \(T^0\) defines an integral gradation of the bosonic algebra, hence the algebra can be split as (\(\mathcal{G}\) denotes the bosonic algebra)

\[
\mathcal{G} = \mathcal{G}_- \oplus \mathcal{G}_0 \oplus \mathcal{G}_+,
\] (4.7)
where the $G_+ \ (G_-)$ are the parts of the algebra with positive (negative) grade with respect to $T^0$, and $G_0$ is the zero grade part. From now on we write tr instead of $str$. It can be proven \[3\] that

$$\text{tr}(E_{nm}E_{rs}) = \delta_{ms}\delta_{ns}f_{nm}(\nu), \quad (4.8)$$

where $f_{nm}(\nu)$ is a certain function of $\nu$. The bilinear form $\langle A, B \rangle = \text{tr}(AB)$ is not positive definite in general. When $d = N$, where $N$ is a positive integer, the bilinear form $\langle \cdot, \cdot \rangle$ becomes degenerate; all $E_{nm}$'s with $n + m \geq 2(N + 1)$ have $f_{nm} = 0$, and hence decouple under the trace; they form the ideal mentioned earlier. The ideal can be divided out, leaving us with a finite set of basis elements for which the bilinear form is non-degenerate. In fact, the residual set of elements form the algebra $\text{gl}_N$. In order to make the correspondence with the formulas is section 2 as close as possible we could use the following notation for the basis elements: $E^s_n = E_{kl}$, with $n = \frac{k+l-|k-l|}{2}$, and $s = \frac{k-l}{2}$.

In the following sections we will make use of the operator $t$, defined through

$$[T^+, t(x)] = x - \Pi_-(x), \quad t([T^+, x]) = x - \Pi_+(x). \quad (4.9)$$

Here $\Pi_-$ ($\Pi_+$) is the projector onto the subset consisting of lowest (highest) weight elements, and $x$ is an arbitrary element in the algebra. $t$ lowers the grade one step, and when $t$ hits a lowest weight element it gives zero. We will also use the following relation

$$\langle At(B) \rangle = -\langle t(A)B \rangle. \quad (4.10)$$

The action of $t$ on the basis elements is

$$t(E_{n,m}) = -\frac{1}{m+1}E_{n-1,m+1}, \quad t(E_{0,m}) = 0. \quad (4.11)$$

We will in what follows assume that the ideal $P_-E^0_0$ has been factored out, which implies that we get $\text{sl}_N$ when $d = N$. We will now make contact with section 2. Let us first note that in section 4 (most of) the results did not depend on the specific form of the Lie algebra used. It only depended on the fact that the Lie algebra had an $\text{sl}_2$ subalgebra, which induced an integral gradation of the algebra. We can thus use the $P_-$ part of the bosonic sector of the algebra described above as a basis for constructing a generalized Toda theory. We have one problem though: As always, it is problematic to integrate an infinite-dimensional algebra (representation) into a group. In the infinite-dimensional case we do not in general have the simple relation (exponentiation) between the algebra and a group, so whether we have an ‘WZNW’ action which gives the equations of motion in the general case is an interesting problem. We will not address this problem here, we will simply base our discussion on the equation of motion, together with the (first class) constraints (2.5) which utilises only the algebra. Using this approach we obtain the model of ref. 1 by noting that the (effective) action (2.16) and the associated equation of motion (2.10) coincide with the ones used in 1, if the underlying algebra is taken to be the (bosonic) algebra described in this section.
5 Different Gauges in the Reduced WZNW model

We now return to the discussion of the WZNW → Toda reduction, with the understanding that the underlying Lie algebra is the $P_{-}$ part of the bosonic sector of the algebra described in the previous section (with the ideal $P_{0}$ factored out), which encompasses e.g. $sl_N$ as a special case. As discussed in sections 2 and 3 we have a gauge invariance in our system (cf. (2.18)). The generators of the $W$ symmetry are gauge invariant polynomials in the components of the WZNW current $J$; they form a closed algebra. There are several possible ways of fixing the gauge. A useful gauge choice is the lowest weight gauge. The lowest weight gauge is defined by gauge fixing $j = J - T^+$ to lie in ker $ad_{T^-}$. This completely fixes the gauge. The lowest weight gauge has a remarkable feature [9] which makes the $W$ algebra particularly transparent, viz. if we write

$$J = T^+ + \sum_i j_i \frac{(T^-)^i}{\langle(T^+)^i(T^-)^i\rangle}$$

(5.1)

then the $j_i$’s form a basis of primary fields of the $W$ algebra (with the exception of $j_1$, which is the energy momentum tensor; here we have the usual central charge). That is, on the constraint surface, we have $W_i = j_i$, where $W_i$ is the $i$th generator of the $W$ symmetry.

Another gauge choice of interest is the diagonal gauge. This gauge is specified by demanding that $j$ should lie in ker $ad_{T^0}$. Thus, in this gauge we can write $J(z) = T^+ - \partial \theta$, where $\theta \in G_0$. Notice that $\theta$ here is not the Toda field, but a free field. In the diagonal gauge the Dirac bracket is equal to the original Poisson bracket, as was shown in ref. [9], using methods similar to those we use in section 6. Summarizing, the field $\theta$ satisfies canonical commutation relations, as well as $\partial_- \partial_+ \theta = 0$. The $W$ generators are traces of polynomials in the components of the current $J$, i.e. polynomials in $\vartheta_n$, if we use the expansion $\theta = \sum_n \vartheta_n h_n$.

An important issue is whether the gauges are accessible. The lowest weight gauge is accessible, as was shown in ref. [9]; the diagonal gauge on the other hand is only locally accessible, this however is sufficient for our purposes since we will only be concerned with calculating Dirac brackets which are local objects.

We will now derive an (explicit) expression for the Dirac bracket in the lowest weight gauge. For convenience we will, in this and the following section, use $E_{\alpha m} / \sqrt{f_m(n)}$ as our basis elements, with the understanding that when $d = N$ only the finite number of elements with $f_{nm} \neq 0$ should be included. The rest automatically decouples under the trace operation. The fact that the rescaling makes some of the basis elements imaginary is not a problem considering the way that they enter into the definition of the constraints. The end result will be insensitive to the rescaling of the basis elements. We will in what follows denote the rescaled basis elements by $E_{\alpha}$. They satisfy

$$\text{tr}(E_{\alpha} E_{-\beta}) = \delta_{\alpha \beta}.$$  

(5.2)

Here $E_{-\beta}$ is the dual of $E_{\beta}$, which means that if $E_{\beta}$ equals $E_{\beta} / \sqrt{f_{\beta}(n)}$, then $E_{-\beta}$ equals

---

This is the highest weight gauge of [9], in the conventions we use.
\[ E_{\alpha} \approx \sqrt{\gamma(t, n)}, \]

Recall that the first class constraints were given by

\[ \gamma_\alpha = \langle E_\alpha, J - T^+ \rangle \approx 0, \tag{5.3} \]

for all \( E_\alpha \in \mathcal{G}_-. \) Together with the gauge fixing conditions \( c_\beta = \langle E_\beta, J - T^+ \rangle \), where \( E_\beta \in \mathcal{G}_0 \oplus \mathcal{G}_+ \) but is not highest weight, the constraints become second class. These constraints can be written

\[ \chi_\alpha = \langle tE_\alpha, J - T^+ \rangle \approx 0, \tag{5.4} \]

for all (except lowest weight) basis elements \( E_\alpha \) of the algebra. The operator \( t \) has been defined in section \( \ref{sec:1} \). On the constraint surface \( \chi_\alpha = 0 \), \( J \) will thus be of the form \( J = T^+ - \mu \), where \( \mu \in \ker ad_{T^+} \). We now recall that the Dirac bracket is given by

\[ \{\cdot, \cdot\}^* = \{\cdot, \cdot\} - \sum_{\alpha\beta} \int\int \{\cdot, \chi_\alpha(x)\} C_{\alpha\beta}^{-1}(x, y)\{\chi_\beta(y), \cdot\} dxdy, \tag{5.5} \]

where \( C_{\alpha\beta}(x, y) = \{\chi_\alpha(x), \chi_\beta(y)\} \). Using the explicit expression for the constraints \( \{\cdot, \cdot\}^* \) and the commutation relations \( \{\cdot, \cdot\} \), together with \( (4.10) \) and \( (4.9) \) we get, on the constraint surface \( \chi_\alpha = 0 \)

\[ C_{\alpha\beta}(x, y) = \langle E_\alpha tU_0^{-1}E_\beta\rangle \delta(x - y). \tag{5.6} \]

Here \( E_\alpha \) and \( E_\beta \) are basis elements of the algebra which are not lowest weight, and \( U_0^{-1} = 1 - D_0t \), where \( D_0 = \partial_+ + [\mu, \cdot] \). We need to calculate the inverse of the matrix, \( \langle E_\alpha tU_0^{-1}E_\beta\rangle \). We notice that \( t \) does not have an inverse, the closest we can get is \( (4.9) \). However, when sandwiched between basis elements as below, we can effectively use \( t^{-1} = [T^+, \cdot] \). We will only use \( t^{-1} \) as a short-hand notation for \( [T^+, \cdot] \). We now show that the inverse of \( \langle E_\alpha tU_0^{-1}E_\beta\rangle \) is given by \( \langle E_{-\gamma} U_0 t^{-1} E_{-\gamma} \rangle \), where \( E_{-\alpha} \) and \( E_{-\beta} \) are not highest weight elements, explicitly

\[ \sum_{\beta} \langle E_\alpha tU_0^{-1}E_\beta\rangle \langle E_{-\beta} U_0 t^{-1} E_{-\gamma} \rangle = \delta_{\alpha\gamma}, \tag{5.7} \]

where \( E_\alpha \) is not lowest weight and \( E_{-\gamma} \) is not highest weight. The sum in \( (5.7) \) runs over values such that the basis elements \( E_\beta \) are not lowest weight, but the sum can be extended to the entire algebra, as follows from the fact that the terms where \( E_\beta \) is lowest weight automatically are zero, because of the \( t \) in \( tU_0^{-1} \). We can then use the closure relation (the sum runs over all \( \beta \))

\[ \sum_{\beta} E_{\beta} \langle E_{-\beta} \rangle = 1. \tag{5.8} \]

Equation \( (5.7) \) then follows, since \( (4.9) \) implies that \( \langle E_\alpha t^{-1} E_{-\gamma} \rangle = \delta_{\alpha\gamma} \), when \( E_\alpha \) is not lowest weight and \( E_{-\gamma} \) is not highest weight. \( C_{\alpha\beta}^{-1}(x, y) \) is defined through

\[ \sum_{\gamma} \int C_{\alpha\gamma}(x, z)C_{\gamma\beta}^{-1}(z, y)dz = \delta_{\alpha\beta}\delta(x - y). \tag{5.9} \]

Hence, using the above results we get

\[ C_{\alpha\beta}^{-1}(x, y) = \langle E_{-\alpha} U_0 t^{-1} E_{-\beta} \rangle \delta(x - y). \tag{5.10} \]
We are now in a position to calculate the Dirac bracket between the components of the \( \mu \)-field. Since the components of the \( \mu \)-field are the generators of the \( \mathcal{W} \) algebra we will thus obtain an expression for the commutators of the \( \mathcal{W} \) algebra. We would like to stress that although we derive all formulas in the general case they apply also to the finite-dimensional cases (ordinary \( A_N \) Toda theories and \( \mathcal{W}_N \) algebras); we only have to choose the parameter \( \nu \) appropriately. The basis elements can in these cases be realized as finite dimensional matrices. To calculate the Dirac brackets we start by recalling that 
\[
\mu = -J + T^+,
\]
which gives
\[
\begin{align*}
\{ \int \langle u, \mu \rangle dz, \chi_\alpha \} &= -\langle tD_0u, E_\alpha \rangle, \\
\{ \chi_\beta, \int \langle v, \mu \rangle dw \} &= \langle E_\beta, tD_0v \rangle,
\end{align*}
\]
(5.11)
where we have used (5.4), (4.10) and (4.9) together with the ubiquitous (2.7). In (5.11), \( u \) and \( v \) are arbitrary highest weight elements, and \( E_\alpha, E_\beta \) are not lowest weight elements.

The final piece of information we need is, \( \{ \langle u, \mu \rangle, \langle v, \mu \rangle \} = 0 \). Collecting the above results together, we get the following expression for the Dirac brackets between the components of the \( \mu \)-field
\[
\begin{align*}
\{ \int \langle u, \mu \rangle dz, \int \langle v, \mu \rangle dw \}^* &= \int \sum \langle tD_0u, E_\alpha \rangle \langle E_{-\alpha}U_0t^{-1}E_{-\beta} \rangle \langle E_\beta tD_0v \rangle = \\
&= \int \langle uD_0U_0D_0v \rangle = \int \langle uU_0D_0v \rangle.
\end{align*}
\]
(5.12)
We now outline the method used to prove the steps in (5.12). The sums run over all \( \alpha \) and \( \beta \) such that \( E_\alpha, E_\beta \) never are lowest weight elements. To show the second equality we insert \( U_0^{-1} \sum \gamma E_\gamma \langle E_{-\gamma}U_0 = 1 \) between \( t \) and \( D_0 \) in the third bracket, and use
\[
\sum \langle E_{-\alpha}U_0t^{-1}E_{-\beta} \rangle \langle E_\beta tU_0^{-1}E_\gamma \rangle = \delta_{\alpha\gamma},
\]
(5.13)
which is true if \( E_\gamma \) is not lowest weight, otherwise the result is zero. The sum in (5.13) runs over all \( \beta \) such that \( E_\beta \) is not a lowest weight element, but it is easy to see that it can be extended to the entire algebra. The sum which remains after (5.13) has been used can similarly also be extended to the whole algebra. The final equality in (5.12) follows from \( \langle u, D_0v \rangle = 0 \), together with the definition \( U_0 = (1 - D_0t)^{-1} \), which implies that \( D_0U_0 = U_0 - 1 \). The result we have obtained is in agreement with the expression for the \( \mu \)-bracket derived in [1]. Another way to obtain this bracket is through use of the governing equation of the next section; this path was the one followed in ref. [1].

The explicit expression for the Dirac bracket furnishes us with an algorithmic method for computing the \( \mathcal{W} \) algebra. The \( \mathcal{W} \) algebra under discussion was denoted \( \mathcal{W}_G \) in [9]. In our case \( G \) is the algebra of section [4].

6 The Governing Equation

In this section we will be concerned with the connection between the diagonal gauge, and the lowest weight gauge. In the diagonal gauge the Dirac bracket is simple (equal
to the original Poisson bracket), but the $\mathcal{W}$ generators are complicated (polynomials in the components of the current). In the lowest weight gauge, on the other hand, we have

the opposite situation, here the $\mathcal{W}$ generators are simple (equal to the components the current, see the previous section), but the Dirac bracket is nonlinear.

In order to establish a connection between the two gauges we will exploit the fact that the $\mathcal{W}$ currents are gauge invariant. The first class constraints are as before given by

$$\gamma_\alpha = \langle E_\alpha, J - T^+ \rangle \approx 0,$$

for all $E_\alpha \in \mathcal{G}_-$. We make a partial gauge fixing of the first class constraints. We gauge-fix all constraints $\gamma_\alpha$, which are not of the form $\langle th, J - T^+ \rangle$, where $h$ is an arbitrary zero grade element. The gauge fixing conditions are chosen to be, $c_\alpha = \langle tE_\alpha, J - T^+ \rangle$ where $tE_\alpha \notin \mathcal{G}_0$, and $E_\alpha$ is not a lowest weight basis element. The second class constraints are thus

$$\chi_\alpha = \langle tE_\alpha, J - T^+ \rangle \approx 0,$$

where $tE_\alpha$ is not of the form $h$ or $th$ (where $h$ is an arbitrary grade zero element) and $E_\alpha$ is not a lowest weight basis element. We notice that we can reach both the diagonal and the lowest weight gauge by further gauge fixing the remaining first class constraints

$$\varrho_\alpha = \langle th_\alpha, J - T^+ \rangle \approx 0,$$

where $h_\alpha$ form a set of orthonormal basis elements of $\mathcal{G}_0$. On the constraint surface we can write $J = T^+ - \partial_+ \theta - \mu$. To be precise this is a slight abuse of notation. What we have called $\mu$ here really only corresponds to $\mu$ in the previous section if we set $\theta = 0$ i.e. if we go to the lowest weight gauge. A similar statement holds for $\theta$. In analogy with the calculation in the previous section we can derive the following expression, on the constraint surface, for the constraint matrix (suppressing the delta function)

$$C_{\alpha\beta} = \{\chi_\alpha, \chi_\beta\} = \langle E_\alpha tU^{-1} E_\beta \rangle.$$

The range of indices in $C_{\alpha\beta}$ are such that $E_\alpha$, $E_\beta$ do not have grade 0 or 1, and are not lowest weight elements. We have introduced the notation $U^{-1} = 1 - Dt$, where $D = \partial_+ + [\partial_+ \theta, \cdot] + [\mu, \cdot]$. The constraint matrix has the generic block form,

$$C_{\alpha\beta} = \begin{pmatrix} 0 & A \\ -A^T & B \end{pmatrix}.$$

The matrix elements are labeled by $\alpha$ and $\beta$ in $C_{\alpha\beta} = \langle E_\alpha tU^{-1} E_\beta \rangle$. When we say e.g. that $\alpha$ has zero grade we really mean that $E_\alpha$ has zero grade. This rule of using $\alpha$ and $E_\alpha$ interchangeably will occasionally be used in the remaining part of this section. The range of the indices in $A_{\alpha\beta} = \langle E_\alpha tU^{-1} E_\beta \rangle$, are such that $\alpha$ has grade $\leq -1$ (but is not lowest weight), and $\beta$ has grade $\geq 2$. The inverse of the constraint matrix is

$$C_{\alpha\beta}^{-1} = \begin{pmatrix} (A^T)^{-1} BA^{-1} & -(A^T)^{-1} \\ A^{-1} & 0 \end{pmatrix}.$$

\footnote{The fact that these really are first class constraints can be shown by computing their Dirac Brackets, $\{\varrho_\alpha, \varrho_\beta\}^* \approx 0$.}
Thus, in order to be able to calculate the inverse we need \( A^{-1} \). The inverse of \( A_{\alpha\beta} \) is given by, \( A_{\alpha\beta}^{-1} = \langle E_{-\beta} U t^{-1} \rangle \), where \( E_{-\alpha} \) has grade \( \leq -2 \), and \( E_{-\beta} \) has grade \( \geq 1 \) but is not highest weight. To show that

\[
\sum_{\beta} A_{\alpha\beta} A_{\beta\gamma}^{-1} = \sum_{\beta} \langle E_{\alpha} U t^{-1} E_{\beta} \rangle \langle E_{-\beta} U t^{-1} E_{-\gamma} \rangle = \delta_{\alpha\gamma},
\]

where the sum runs over all \( \beta \) with grade \( \geq 2 \), we use the fact that \( t U^{-1} \) decreases the grade when acting on elements which are not lowest weight. Because of this property \( A_{\alpha\beta} \) is zero unless the grade of \( \beta \) is \( \geq 2 \) (since \( \alpha \) has grade \( \leq -1 \)), using the properties of the trace. The sum over \( \beta \) in (6.7) can thus be extended to the entire algebra. Finally, the closure relation can be invoked, and equation (6.7) follows.

The remaining first class constraints generate gauge transformations. Since the \( \mathcal{W} \) currents are gauge invariant, they satisfy \( \delta \mu = \{ \mu, W \} / \approx 0 \), for all first class constraints \( \mu \) of the form (5.3). Here \( W \) is an arbitrary \( \mathcal{W} \) current. \( W \) is given as a trace of a polynomial in \( J \), which implies (weakly) that it is a function of \( \partial_+ \theta \) and \( \mu \). This gives us (summation over \( i \) is understood)

\[
\delta W = \left[ \int \{ \mu, \mu_i \}^{*} \frac{\delta}{\delta \mu_i} + \int \{ \mu, \partial_+ \vartheta_i \}^{*} \frac{\delta}{\delta \partial_+ \vartheta_i} \right] W \approx 0.
\]

Here \( \mu_i = \langle E_i, \mu \rangle \) and \( \partial_+ \vartheta_i = \langle h_i, \partial_+ \vartheta \rangle \), where the set of orthonormal basis elements \( E_i \) span the subspace of highest weight elements, and the orthonormal set \( h_i \) span \( G_0 \). We have also used the property, \( \frac{\delta \mu_i(x)}{\delta \mu_i(y)} = \delta_{ij} \delta(x - y) \), and analogously for \( \partial_+ \vartheta_i \). Remembering that \( J \approx T^+ - \partial_+ \theta - \mu \), we see that in order to discover the implications of the gauge invariance of \( W \), we need to calculate \( \{ \mu, \langle h, J \rangle \}^{*} \) and \( \{ \mu, \langle u, J \rangle \}^{*} \), where \( h \) has zero grade and \( u \) is a highest weight element. Here we choose \( \mu = \langle J, t \xi(z)dz, J - T^+ \rangle \) where \( \xi \) is an arbitrary zero grade element. We start with \( \{ \mu, \langle h, J \rangle \}^{*} \). The second term in the Dirac bracket is schematically

\[
\sum_{\alpha\beta} \{ \mu, \chi_\alpha \} C_{\alpha\beta}^{-1} \{ \chi_\beta, \langle h, J \rangle \}.
\]

We now show that (5.5) is zero. First recall that the second class constraints were given by (5.4). The first bracket in (5.9) can be calculated to give the result \( -\langle \xi t De_\alpha \rangle \). Using the fact that \( t \) lowers the grade, we see that \( \alpha \) is constrained to have grade \( \geq 2 \) (remember that \( \xi \) is a zero grade element). Similarly, the other bracket can be calculated to give \( -\langle E_{\beta} t Dh \rangle \). Its implications are to constrain \( \beta \) to have grade \( \geq 2 \) (remember that there are no constraints \( \chi_\beta \), with \( E_\beta \) a grade 1 element). Using the block form of \( C_{\alpha\beta}^{-1} \) (5.4), the result follows (\( C_{\alpha\beta}^{-1} \) is zero in the appropriate sector). The first term in the expression for the Dirac bracket can be calculated to give \( \{ \mu, \langle h, J \rangle \} = \langle \xi, h \rangle \); hence

\[
\{ \mu, \langle h, J \rangle \}^{*} = \langle \xi, h \rangle.
\]

We now turn to \( \{ \mu, \langle u, J \rangle \}^{*} \). The second term in the Dirac bracket is

\[
\iint dxdy \sum_{\alpha\beta} \{ \mu, \chi_\alpha(x) \} C_{\alpha\beta}^{-1}(x, y) \{ \chi_\beta(y), \langle u, J \rangle \}.
\]
Here, as above, the first bracket is non-zero only if \( \alpha \) has grade \( \geq 2 \), so (6.11) becomes using (6.6), and reinstating the delta function

\[
\sum_{\alpha\beta} \langle \xi t D t E_\alpha \rangle \langle E_{-\alpha} U t^{-1} E_{-\beta} \rangle \langle E_\beta t D u \rangle.
\] (6.12)

Here \( E_{-\beta} \) has grade \( \geq 1 \) (but is not highest weight), and \( E_{-\alpha} \) has grade \( \leq -2 \). We can use the same trick as in the previous section, and insert \( U^{-1} \sum_\gamma E_\gamma \langle E_{-\gamma} U \rangle = 1 \), between \( t \) and \( D \). The fact that \( E_{-\alpha} \) has grade \( \leq -2 \), means that we can restrict the sum over \( \gamma \) to run over values such that \( E_\gamma \) has grade \( \geq 2 \) (as follows from the fact that \( t U^{-1} \) decreases the grade). We get

\[
\sum_{\alpha\beta\gamma} \langle \xi t D t E_\alpha \rangle A_{\alpha\beta}^{-1} A_{\beta\gamma} \langle E_{-\gamma} U D u \rangle = \sum_{\alpha} \langle \xi t D t E_\alpha \rangle \langle E_{-\alpha} U D u \rangle.
\] (6.13)

Since the first bracket is zero for elements \( E_\alpha \), which do not have grade \( \geq 2 \), we see that the sum over \( \alpha \) can be extended to the entire algebra. Then, using the closure relation and the definition of \( U \), we get

\[
\sum_{\alpha} \langle \xi t D t E_\alpha \rangle \langle E_{-\alpha} U D u \rangle = \langle \xi, \sum_{n=2}^{\infty} (t D)^n u \rangle.
\] (6.14)

The first term in the expression for the Dirac bracket is \( \{ \varrho, \langle u, J \rangle \} \) = \( -\langle \xi, t D u \rangle \), including this term we get

\[
\{ \varrho, \langle u, J \rangle \}^* = -\langle \xi, t D u \rangle - \langle \xi, \sum_{n=2}^{\infty} (t D)^n u \rangle = -\langle \sum_{n=1}^{\infty} (D t)^n \xi, u \rangle = -\langle U \xi, u \rangle.
\] (6.15)

In the last step we have used \( \langle \xi, u \rangle = 0 \). We have also done an integration by parts, which is permissible since our expression is to be integrated over, cf. (6.8). Collecting, the gauge invariance property of the \( W \) currents (6.8) can be seen to reduce to the condition

\[
\left[ \int \langle \xi, \frac{\delta}{\delta \partial_+ \theta} \rangle - \int \langle U \xi, \frac{\delta}{\delta \mu} \rangle \right] W = 0.
\] (6.17)

Here we have used the definition \( \frac{\delta}{\delta \mu} = \sum_i E_i \frac{\delta}{\delta \mu_i} \), where the orthonormal set of \( E_i \)'s span the highest weight elements, which implies

\[
\int \langle \eta, \frac{\delta}{\delta \mu} \rangle \mu = \Pi_{-}(\eta),
\] (6.18)

and similarly for \( \frac{\delta}{\delta \partial_+ \theta} \). We see that (6.17) is the governing equation of [1] with one important difference. In (6.17) the \( \theta \)-field is not the Toda field but a free field. The governing equation gives the connection between the \( W \) generators in the two gauges. Equation (6.17) was solved in [1] with the result (P denotes path ordering)

\[
W(\partial_+ \theta, \mu) = P \exp \left[ \int_0^1 \left[ \int \langle U(s \partial_+ \theta) \partial_+ \theta, \frac{\delta}{\delta \mu} \rangle ds \right] \right] W(\mu).
\] (6.19)

\[\text{In [1] the notation } J^s \text{ was used on the left hand side instead of } W, \text{ and } j \text{ was used on the right hand side.}\]
Hence, \( W(\partial_+ \theta) = W(\partial_+ \theta, \mu)|_{\mu=0} \). In our case this is a (Miura) transformation, which expresses the \( W \) generators in terms of free fields satisfying canonical commutation relations. We once again would like to point out that (by choosing the parameter \( \nu \) appropriately) all results apply also to the finite-dimensional cases. How is \( W(\partial_+ \theta) \) related to the \( W \) currents expressed in terms of the Toda field \( \phi \), \( W(\partial_+ \phi) \)? From the gauge invariance of \( W \), it follows that \( W(\phi) \) and \( W(\theta) \) are equal. Notice that \( \phi \) is gauge invariant so \( W(\phi) \) is the same in all gauges. To further investigate the relation between the two realizations of \( W \), we first note that 

\[
J = \partial_+ \eta T^- + e^{\eta T^-} (T^+ - \partial_+ \phi) e^{-\eta T^-} - \partial_- \xi T^+ + e^{-\xi T^+} (T^+ + \partial_- \phi) e^{\xi T^+},
\]

(6.22)

where \( \phi = \varphi T^0 \). In the diagonal gauge we have

\[
J_+ = T^+ - \partial_+ \theta,
J_- = T^- - \partial_- \theta,
\]

(6.23)

where \( \theta = \vartheta T^0 \). It follows from consistency arguments that the gauge choice for \( J_- \) is the only possible choice, if we demand that \( J_- - T^- \in G_0 \). If we do not choose \( J_- = T^- - \partial_- \theta \) \((6.25)\) will lead to an inconsistency. In order for \((6.22)\) to be compatible with \((6.23)\) we obtain the equations

\[
\partial_+ \varphi = \frac{1}{\eta} \partial_+ \eta + \eta, \\
\partial_+ \vartheta = \partial_+ \varphi - 2\eta.
\]

(6.24)
The equations involving $\xi$ are similar. The equations can be solved to give (up to an irrelevant integration constant which can be set equal to 1), $\eta = e^{\frac{\varphi + \vartheta}{2}}$ and $\xi = e^{\frac{\varphi - \vartheta}{2}}$. As a further requirement we get the following two equations connecting the two fields

$$\partial_+ \varphi = \partial_+ \vartheta + 2e^{\frac{\varphi + \vartheta}{2}},$$
$$\partial_- \varphi = -\partial_- \vartheta - 2e^{\frac{\varphi - \vartheta}{2}}. \quad (6.25)$$

We recognize (6.25) as the Bäcklund transformation [22] [23] for the Liouville model, which maps the Liouville field $\varphi$ onto the free field $\vartheta$. In order for the two equations in (6.25) to be consistent we obtain

$$\partial_+ \partial_- \varphi + 2e^{\varphi} = 0,$$
$$\partial_+ \partial_- \vartheta = 0, \quad (6.26)$$

which are the equations of motion for the Liouville field $\varphi$ and the free field $\vartheta$, respectively.

The canonical transformation between the two fields implicit in (6.25) is not one-to-one. For a discussion of this fact in the “particle limit”, see ref. [24]. The fact that the mapping is not one-to-one is important and means that, although there exists a canonical transformation relating the Liouville system to a free system, the Liouville system is not trivial. The procedure outlined above can in principle also be used to derive relations similar to (6.25) (Bäcklund transformations) for the Toda theories, although the calculations quickly get complex.

### 7 Discussion

We would like to mention that although we have only considered the generalized $A_N$ Toda theories the cases $B_N$ and $C_N$ could also be treated within the framework presented in this paper. In this paper we have exclusively dealt with the classical model. It should be possible to carry out quantization along the usual lines. In particular it should be possible to calculate the central charge [23]. However, we get seemingly infinite sums, which need to be properly defined. Ultimately we will face the same problems as in the Liouville theory. For a lucid discussion of the problems encountered in the quantization see [26]. It should also be possible to study the quantum version of the $W$ algebra along the lines of the finite dimensional cases.

Although we have derived the formulas in sections 5 and 6 in the general case they apply also to the finite dimensional cases (ordinary Toda theories and $W_N$ algebras); we only have to choose the parameter $\nu$ appropriately.

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