Klein-Gordon equation in $q$-deformed Euclidean space

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Abstract
We introduce $q$-versions of the Klein-Gordon equation in the three-dimensional $q$-deformed Euclidean space. We determine plane wave solutions to our $q$-deformed Klein-Gordon equations. We show that these plane wave solutions form a complete orthogonal system. We discuss the propagators of our $q$-deformed Klein-Gordon equations. We derive continuity equations for the charge density, the energy density, and the momentum density of a $q$-deformed spin-zero particle.

1 Introduction
It could be that space-time shows a discrete structure at small distances [1,2], much like a solid is composed of atoms at the microscopic level. A discrete space-time structure could mathematically be described by a noncommutative coordinate algebra such as the $q$-deformed Euclidean space [3–5]. We aim at discussing quantum mechanical wave equations in $q$-deformed Euclidean space. This discussion could provide clues as to whether space-time is discrete at small distances, indeed.

In our former work, we have already considered $q$-versions of the Schrödinger equation for a nonrelativistic particle [6,7]. However, space-time should reveal its discrete structure only at high particle energies. For this reason, we should deal with $q$-versions of relativistic wave equations [8–12]. It is possible to write down a Klein-Gordon equation or a Dirac equation in the so-called $q$-deformed Minkowski space [13]. The $q$-deformed Minkowski space, however, is such that calculations are challenging. So, I am going to handle the problem differently: we introduce and discuss Klein-Gordon equations in $q$-deformed Euclidean space.

As shown in Ref. [14], we can extend the algebra of $q$-deformed Euclidean space by a time element. This time element is a commutative parameter. Thus,

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we obtain Klein-Gordon equations in $q$-deformed Euclidean space from the well-known Klein-Gordon equation if we only replace the usual Laplace operator with its $q$-analog in $q$-deformed Euclidean space. These $q$-versions of the Klein-Gordon equation do not fit with the $q$-deformed Poincaré symmetry [15]. Nevertheless, their discussion can give us an idea of what we might get if we describe space-time by a $q$-deformed quantum space.

We know from Ref. [16] that $q$-deformed partial derivatives can act on wave functions in different ways. Thus, there is more than one $q$-deformed Klein-Gordon equation in the $q$-deformed Euclidean space, as shown in Chap. 2. In Chap. 3 we derive plane wave solutions for our $q$-deformed Klein-Gordon equations. We also show that these solutions form a complete system of orthogonal functions. This fact enables us to write down $q$-versions of the propagator for a spin-zero particle in Chap. 4. In the subsequent chapters, we derive $q$-deformed continuity equations for the probability density, the energy density, and the momentum density of a spin-zero particle. Our reasonings also include a $q$-deformed spin-zero particle interacting with an electromagnetic field. The appendix contains some information about mathematical tools of analysis on $q$-deformed Euclidean space. It will help the reader who is not familiar with our formalism.

2 $q$-Deformed Klein-Gordon equations

A $q$-analog of the Klein-Gordon equation should be invariant under actions of the Hopf algebra $U_q(sl_2)$ (cf. App. D). We assume that for $q$-deformed particles with zero spin and rest mass $m$, the following $q$-version of the Klein-Gordon equation holds:

$$c^{-2}\partial^2_t \varphi_R - \nabla^2_q \varphi_R + (mc)^2 \varphi_R = 0.$$  \hspace{1cm} (1)

The $q$-deformed Laplace operator $\nabla^2_q$ depends on the metric of the three-dimensional $q$-deformed Euclidean space [also see Eq. (169) of App. A]:

$$\nabla^2_q = \partial^A \partial_A = g^{AB} \partial_B \partial_A.$$  \hspace{1cm} (2)

By conjugating Eq. (1), we obtain another $q$-version of the Klein-Gordon equation [see Eq. (201) in App. B]:

$$\varphi_L \bar{\partial}^2 c^{-2} \varphi_L - \varphi_L \bar{\nabla}^2_q + \varphi_L (mc)^2 = 0.$$  \hspace{1cm} (3)

Accordingly, the wave function $\varphi_R$ transforms into $\varphi_L$ by conjugation [see Eq. (187) in App. A]:

$$\varphi_R \leftrightarrow \varphi_L.$$  \hspace{1cm} (4)

There are two types of left-actions and two types of right-actions for $q$-deformed partial derivatives [see Eq. (195) and Eq. (198) in App. B]. Thus, we get further $q$-versions of the Klein-Gordon equation by applying the following substitutions in Eq. (1) or Eq. (3):

$$\triangleright \leftrightarrow \bar{\triangleright}, \quad \bar{\triangleleft} \leftrightarrow \triangleleft, \quad \varphi_R \leftrightarrow \varphi^*_R, \quad \varphi_L \leftrightarrow \varphi^*_L.$$  \hspace{1cm} (5)
This way, we have

\[ c^{-2} \partial_t^2 \bar{\varphi}_R - \nabla_q^2 \bar{\varphi}_R + (mc)^2 \bar{\varphi}_R = 0, \]
\[ \varphi_L^* \triangleleft \partial_t^2 c^{-2} - \varphi_L^* \triangleleft \nabla_q^2 + \varphi_L^* (mc)^2 = 0, \]  

(6)

where \( \varphi_R^* \) transforms into \( \varphi_L^* \) by conjugation:

\[ \bar{\varphi}_R = \varphi_L^*. \]  

(7)

If we want to deal with a charged particle moving in the presence of an electromagnetic field, we have to apply the following substitutions to the \( q \)-deformed Klein-Gordon equations:

\[ \partial_t \rightarrow D^0 = \partial_t + ieA^0, \quad \partial_C \rightarrow D^C = \partial_C - iec^{-1}A^C. \]  

(8)

Thus, the \( q \)-deformed Klein-Gordon equations for a charged particle read

\[ c^{-2} D^0 D^0 \triangleright \varphi_R = D^C D_C \triangleright \varphi_R - (mc)^2 \varphi_R, \]
\[ \varphi_L \triangleleft D^0 D^0 c^{-2} = \varphi_L \triangleleft D^C D_C - \varphi_L (mc)^2, \]  

(9)

and

\[ c^{-2} D^0 D^0 \triangleright \varphi_R^* = D^C D_C \triangleright \varphi_R^* - (mc)^2 \varphi_R^*, \]
\[ \varphi_L^* \triangleleft D^0 D^0 c^{-2} = \varphi_L^* \triangleleft D^C D_C - \varphi_L^* (mc)^2, \]  

(10)

where the ‘action’ of the potentials on the wave functions is defined by the star-product [see App. A]:

\[ A^0 \triangleright \varphi_R = A^0 \otimes \varphi_R, \quad A^C \triangleright \varphi_R = A^C \otimes \varphi. \]
\[ \varphi_L^* \triangleleft A^0 = \varphi_L^* \otimes A^0, \quad \varphi_L^* \triangleleft A^C = \varphi_L^* \otimes A^C. \]  

(11)

The \( q \)-deformed Klein-Gordon equation for \( \varphi_R \) again transforms into the \( q \)-deformed Klein-Gordon equation for \( \varphi_L \) if the potentials \( A^0 \) and \( A^C \) behave as follows under conjugation:

\[ \bar{A}^0 = A_0, \quad \bar{A}^C = A_C. \]  

(12)

The same holds for the \( q \)-deformed Klein-Gordon equations in Eq. (10).

In the following, we show that the \( q \)-deformed Klein-Gordon equations in Eqs. (9) and (10) are invariant under the gauge transformations

\[ e \tilde{A}^C = e A^C + \partial_C \triangleright e \chi = e A^C - e \chi \triangleleft \partial_C, \]
\[ e \tilde{A}^0 = e A^0 - \partial_t \triangleright e \chi = e A^0 + e \chi \triangleleft \partial_t, \]  

(13)

and

\[ \tilde{\varphi}_R(x,t) = \exp(iec^{-1}\chi) \otimes \varphi_R(x,t), \]
\[ \tilde{\varphi}_L^*(x,t) = \varphi_L^*(x,t) \otimes \exp(-iec^{-1}\chi). \]  

(14)
Note that \( \chi \) has to be a central element of the algebra of position space. Additionally, \( \chi \) shows trivial braiding [cf. Eq. (245) in App. D].

That the \( q \)-deformed Klein-Gordon equation for \( \varphi_R \) [cf. Eq. (9)] is invariant under the above gauge transformations is a direct consequence of the following identities:

\[
\tilde{D}^C \triangleright \varphi_R = \exp(ie^{c-1} \chi) \odot D^C \triangleright \varphi_R, \\
\tilde{D}^0 \triangleright \varphi_R = \exp(ie^{c-1} \chi) \odot D^0 \triangleright \varphi_R.
\] (15)

To prove the first identity, we do the following calculation [7]:

\[
\tilde{D}^C \triangleright \varphi_R = \partial^C \triangleright (\exp(ie^{c-1} \chi) \odot \varphi_R) \\
- (ie^{c-1} A^C + i \partial^C \triangleright ec^{c-1} \chi) \odot \exp(ie^{c-1} \chi) \odot \varphi_R \\
= \partial^C \triangleright \exp(ie^{c-1} \chi) \odot \varphi_R + \exp(ie^{c-1} \chi) \odot \partial^C \triangleright \varphi_R \\
- (ieA^C + i \partial^C \triangleright ec^{c-1} \chi) \odot \exp(ie^{c-1} \chi) \odot \varphi_R.
\] (16)

The last step follows from the Leibniz rules of the \( q \)-deformed partial derivatives and the trivial braiding of \( \chi \). For the same reasons, we have

\[
\partial^C \triangleright (ec^{c-1} \chi)^n = \sum_{j=0}^{n-1} (ec^{c-1} \chi)^j \odot (\partial^C \triangleright ec^{c-1} \chi) \odot (ec^{c-1} \chi)^{n-j} \\
= n (\partial^C \triangleright ec^{c-1} \chi) \odot (ec^{c-1} \chi)^{n-1}
\] (17)

with

\[
(ec^{c-1} \chi)^n = ec^{c-1} \chi \odot \ldots \odot ec^{c-1} \chi \text{ n-times}.
\] (18)

From Eq. (17) follows:

\[
\partial^C \triangleright \exp(ie^{c-1} \chi) = \sum_{n=0}^{\infty} \frac{i^n}{n!} \partial^C \triangleright (ec^{c-1} \chi)^n \\
= \sum_{n=1}^{\infty} \frac{i^n}{(n-1)!} (\partial^C \triangleright ec^{c-1} \chi) \odot (ec^{c-1} \chi)^{n-1} \\
= i \partial^C \triangleright ec^{c-1} \chi \odot \sum_{n=0}^{\infty} \frac{i^n}{n!} (ec^{c-1} \chi)^n \\
= i \partial^C \triangleright ec^{c-1} \chi \odot \exp(i ec^{c-1} \chi).
\] (19)

If we insert this result into Eq. (16), we finally get:

\[
\tilde{D}^C \triangleright \varphi_R = \partial^C \triangleright \exp(ie^{c-1} \chi) \odot \varphi_R + \exp(ie^{c-1} \chi) \odot \partial^C \triangleright \varphi_R \\
- (ie^{c-1} A^C + i \partial^C \triangleright ec^{c-1} \chi) \odot \exp(ie^{c-1} \chi) \odot \varphi_R \\
= \exp(ie^{c-1} \chi) \odot \partial^C \triangleright \varphi_R - \exp(ie^{c-1} \chi) \odot i e^{c-1} A^C \odot \varphi_R \\
= \exp(ie^{c-1} \chi) \odot D^C \triangleright \varphi_R.
\] (20)
We can prove the second identity in Eq. (15) with similar reasonings:

\[
\hat{D}^0 \triangleright \varphi_R = \partial_t \triangleright (\exp(iec^{-1} \gamma) \otimes \varphi_R) + ieA^0 \otimes \exp(iec^{-1} \gamma) \otimes \varphi_R
\]

\[
- i\partial_t \triangleright ec^{-1} \gamma \otimes \exp(iec^{-1} \gamma) \otimes \varphi_R
\]

\[
= \partial_t \triangleright \exp(iec^{-1} \gamma) \otimes \varphi_R + \exp(iec^{-1} \gamma) \otimes \partial_t \triangleright \varphi_R
\]

\[
+ ieA^0 \otimes \exp(iec^{-1} \gamma) \otimes \varphi_R - i\partial_t \triangleright ec^{-1} \gamma \otimes \exp(iec^{-1} \gamma) \otimes \varphi_R
\]

\[
= \exp(iec^{-1} \gamma) \otimes D^0 \triangleright \varphi_R.
\]

In the last step of the calculation above, we have taken the definition of \(D^0\) and the following identity into account:

\[
\partial_t \triangleright \exp(iec^{-1} \gamma) = i\partial_t \triangleright ec^{-1} \gamma \otimes \exp(iec^{-1} \gamma).
\]

In the same manner, we can prove the following identities:

\[
\varphi_R^* \bowtie D^0 = \varphi_L^* \bowtie D^0 \otimes \exp(-iec^{-1} \gamma),
\]

\[
\varphi_R^* = \varphi_L^* \bowtie D^C = \varphi_L^* \bowtie D^C \otimes \exp(-iec^{-1} \gamma).
\]

These identities imply that the \(q\)-deformed Klein-Gordon equation for \(\varphi_L^*\) [cf. Eq. (10)] is also invariant under gauge transformations.

Finally, we will show that our \(q\)-deformed Klein-Gordon equations become \(q\)-deformed Schrödinger equations if the kinetic energy is small compared to the rest energy. Our reasonings are in complete analogy to the undeformed case. Accordingly, we separate a phase factor depending on the rest energy \(mc^2\):

\[
\varphi_R = \psi_R \exp(-imc^2 t), \quad \varphi_L = \psi_L \exp(imc^2 t),
\]

\[
\varphi_R^* = \psi_R^* \exp(-imc^2 t), \quad \varphi_L^* = \psi_L^* \exp(imc^2 t).
\]

In the nonrelativistic limit, the total energy \(E\) of a particle is slightly different from its rest energy. Thus, we have

\[
|i\partial_t \triangleright \psi_R| \approx |\psi_R(E - mc^2)| \ll |\psi_R mc^2|,
\]

\[
|\psi_L^* \bowtie \partial_t \psi_L| \approx |(E - mc^2) \psi_L^*| \ll |mc^2 \psi_L^*|.
\]

and

\[
|eA^0 \otimes \psi_R| \ll |mc^2 \psi_R|,
\]

\[
|\psi_L^* \bowtie eA^0| \ll |\psi_L^* mc^2|.
\]

Similar approximations hold for \(\psi_R^*\) and \(\psi_L\), so we can restrict ourselves to the wave functions \(\psi_R\) and \(\psi_L^*\). Due to the conditions in Eqs. (25) and (26), we can make the following approximation:

\[
(\partial_t + ieA^0)(\partial_t + ieA^0) \triangleright \varphi_R =
\]

\[
= \left[ \partial_t \partial_t \triangleright \psi_R - 2i\partial_t \triangleright \psi_R mc^2 - \psi_R mc^2 \right] + i\partial_t \triangleright eA^0 \otimes \psi_R
\]

\[
+ 2ieA^0 \otimes \partial_t \triangleright \psi_R + 2A^0 \otimes \psi_R mc^2 - eA^0 \otimes eA^0 \otimes \psi_R \right] \exp(-imc^2 t)
\]

\[
\approx \left[ - 2i\partial_t \triangleright \psi_R mc^2 - \psi_R mc^2 \right] + i\partial_t \triangleright eA^0 \otimes \psi_R
\]

\[
+ 2eA^0 \otimes \psi_R mc^2 \right] \exp(-imc^2 t).
\]
We can also introduce ‘dual’ momentum eigenfunctions [also see Eq. (235) of App. C] of the q-deformed Klein-Gordon equation for \( \varphi_R \) [cf. Eq. (1)]:
\[
 i \partial_t \psi_R = - (2m)^{-1} (\partial^C - i e c^{-1} A^C)(\partial_C - i e c^{-1} A_C) \psi_R + e A^0 \circ \psi_R + (2mc^2)^{-1} (i \partial_t \circ e A^0) \bowtie \psi_R.
\]
(28)

Similar reasonings lead us to a q-deformed Schrödinger equation for the wave function \( \psi_L^* \):
\[
 \psi_L^* \circ \partial_i = -(2m)^{-1} \psi_L^* \circ (\partial^C - i e c^{-1} A^C)(\partial_C - i e c^{-1} A_C) + \psi_L^* \bowtie e A^0 + (2mc^2)^{-1} \psi_L^* \bowtie (e A^0 \circ \partial_i).
\]
(29)

These q-deformed Schrödinger equations are of the same form as those given in Ref. [6].

3 Plane wave solutions

In Ref. [17], we have introduced q-deformed momentum eigenfunctions (also see App. C):
\[
 u_p(x) = \text{vol}^{-1/2} \exp_q(x \mid ip), \quad u^p(x) = \text{vol}^{-1/2} \exp_q(i^{-1} p \mid x),
\]
\[
 \tilde{u}_p(x) = \text{vol}^{-1/2} \overline{\exp}_q(x \mid ip), \quad \tilde{u}^p(x) = \text{vol}^{-1/2} \overline{\exp}_q(i^{-1} p \mid x).
\]
(30)

The volume element is defined by [also see Eq. (208) in App. B]
\[
 \text{vol} = \int d_q^3 p \int d_q^3 x \exp_q(i^{-1} p \mid x).
\]
(31)

The q-deformed momentum eigenfunctions are subject to the following eigenvalue equations [cf. Eq. (213) of App. C]:
\[
 i^{-1} \partial^A \bowtie u_p(x) = u_p(x) \bowtie p^A, \quad u^p(x) \bowtie \partial^A i^{-1} = p^A \bowtie u^p(x),
\]
\[
 i^{-1} \partial^A \bowtie \tilde{u}_p(x) = \tilde{u}_p(x) \bowtie p^A, \quad \tilde{u}^p(x) \bowtie \partial^A i^{-1} = p^A \bowtie \tilde{u}^p(x).
\]
(32)

We can also introduce ‘dual’ momentum eigenfunctions [also see Eq. (235) of App. C]:
\[
 (u^*)_p(x) = \text{vol}^{-1/2} \overline{\exp}_q^*(ip \mid x), \quad (u^*)^p(x) = \text{vol}^{-1/2} \overline{\exp}_q^*(i^{-1} p \mid x),
\]
\[
 (\tilde{u}^*)_p(x) = \text{vol}^{-1/2} \overline{\exp}_q^*(ip \mid x), \quad (\tilde{u}^*)^p(x) = \text{vol}^{-1/2} \overline{\exp}_q^*(i^{-1} p \mid x).
\]
(33)

The corresponding eigenvalue equations are given by [cf. Eq. (236) of App. C]
\[
 i^{-1} \partial^A \bowtie (u^*)_p(x) = p^A \bowtie (u^*)_p(x),
\]
\[
 i^{-1} \partial^A \bowtie (u^*)^p(x) = (u^*)^p(x) \bowtie p^A,
\]
(34)
(\overline{u}^\ast)_p(x) \circ \hat{\partial}^A i^{-1} = p^A \oplus (\overline{u}^\ast)_p(x),

i^{-1} \hat{\partial}^A \triangleright (\overline{u}^\ast) P(x) = (\overline{u}^\ast)_P(x) \otimes p^A. \quad (35)

To obtain the various expressions and identities for the momentum eigenfunction with a bar, we apply the following substitutions:

\triangleright \leftrightarrow \underline{\circ}, \quad \triangleleft \leftrightarrow \underline{\triangleright}, \quad \partial^A \leftrightarrow \hat{\partial}^A, \quad u \leftrightarrow \overline{u},

\ast \leftrightarrow -, \quad q \leftrightarrow q^{-1}. \quad (36)

For this reason, we need not consider the momentum eigenfunctions \(\overline{u}_p\) and \((\overline{u}^\ast)_p\) or \(u_p\) and \((\overline{u}^\ast) P\) in the following.

We first examine whether the \(q\)-deformed Klein-Gordon equations given in Eqs. (1) and (3) of the previous chapter have plane wave solutions. To this end, we consider the functions

\[ \varphi_p(x, t) = \frac{c}{\sqrt{2}} u_p(x) \circ \exp(-itE_p) \otimes E_p^{-1/2}, \]

\[ \varphi_P(x, t) = \frac{c}{\sqrt{2}} E_p^{-1/2} \otimes \exp(itE_p) \otimes u_P(x), \quad (37) \]

with the time-dependent phase factors

\[ \exp(\pm itE_p) = \sum_{n=0}^{\infty} \frac{(\pm itE_p)^n}{n!}. \quad (38) \]

Note that \(\varphi_p\) and \(\varphi_P\) are subject to the following identities:

\[ i^{-1} \hat{\partial}^A \triangleright \varphi_p(x, t) = \varphi_p(x, t) \circ p^A, \quad \varphi_P(x, t) \circ \hat{\partial}^A i^{-1} = p^A \circ \varphi_P(x, t), \]

\[ i \partial_t \triangleright \varphi_p(x, t) = \varphi_p(x, t) \circ E_p, \quad \varphi_P(x, t) \circ \partial_t i = E_p \circ \varphi_P(x, t). \quad (39) \]

Inserting the functions \(\varphi_p\) or \(\varphi_P\) into our \(q\)-deformed Klein-Gordon equations, we obtain

\[ 0 = c^{-2} \partial_t^2 \triangleright \varphi_p - \nabla_q^2 \triangleright \varphi_p + (mc)^2 \varphi_p \]

\[ = \varphi_p \circ (p^B \circ p_B - c^{-2} E_p \circ E_p + (mc)^2) \quad (40) \]

or

\[ 0 = \varphi_P \circ \partial_t^2 c^{-2} - \varphi_P \circ \nabla_q^2 + (mc)^2 \]

\[ = (p^B \circ p_B - c^{-2} E_p \circ E_p + (mc)^2) \circ \varphi_P. \quad (41) \]

The plane waves in Eq. (37) are solutions to the \(q\)-deformed Klein-Gordon equations if the following energy-momentum relation holds:

\[ c^{-2} E_p \circ E_p = p^B \circ p_B + (mc)^2. \quad (42) \]
Since square mass $m^2$ commutes with square momentum $p^2 (= p^B \otimes p_B)$, we can formally solve the energy-momentum relation for $E_p$. This way, we get an infinite series:

$$E_p = c(p^2 + (mc)^2)^{1/2} = c \sum_{k=0}^{\infty} \left( \frac{1/2}{k} \right) p^{2k} (mc)^{1-2k}.$$  \hspace{1cm} (43)

This result can be generalized as follows ($\alpha \in \mathbb{Q}$):

$$E_p^{2\alpha} = c^{2\alpha} \sum_{k=0}^{\infty} \left( \frac{\alpha}{k} \right) p^{2k} (mc)^{2(\alpha-k)}.$$  \hspace{1cm} (44)

In the formula above, we have to substitute powers of $p^2$ by their normal-ordered expressions [6]:

$$p^{2k} = p^2 \otimes \cdots \otimes p^2 = \sum_{l=0}^{k} q^{2l} (-q - q^{-1})^{k-l} \left( \begin{array}{c} k \\ l \end{array} \right)_q (p_-)^{k-l} (p_+)^{2l} (p_+)^{k-l}. \hspace{1cm} (45)$$

Note that the $q$-deformed binomial coefficients are defined in complete analogy to the undeformed case [also see Eq. (185) in App. A]:

$$\left( \begin{array}{c} n \\ k \end{array} \right)_q = \frac{[n]_q! [k]_q!}{[n-k]_q!}.$$  \hspace{1cm} (46)

The $q$-deformed momentum eigenfunctions form a complete orthogonal system of functions [17, 18]. In the following, we will show that the same applies to the plane wave solutions of our $q$-deformed Klein-Gordon equations.

We recall that the $q$-deformed momentum eigenfunctions in Eqs. (30) and (33) fulfill the orthogonality relation [17]

$$\int d^3_q x (u^*)_p (x) \otimes u_{p'} (x) = \text{vol}^{-1} \int d^3_x \exp_q^*(i p | x) \otimes \exp_q(x | p')$$

$$= \text{vol}^{-1} \delta^3_q ((\otimes \kappa^{-1} p) \oplus p'). \hspace{1cm} (47)$$

or

$$\int d^3_q x u^p (x) \otimes (u^*)_{p'} (x) = \text{vol}^{-1} \int d^3_x \exp_q(i^{-1} p | x) \otimes \exp_q^*(x | i^{-1} p')$$

$$= \text{vol}^{-1} \delta^3_q (p \oplus (\otimes \kappa^{-1} p')). \hspace{1cm} (48)$$

Here $\delta^3_q (p)$ denotes a $q$-deformed version of the three-dimensional delta function:

$$\delta^3_q (p) = \int d^3_q x \exp_q(i^{-1} p | x) = \int d^3_q x \exp_q(x | i^{-1} p). \hspace{1cm} (49)$$

\footnote{We do not discuss conditions for the convergence of this series.}
In analogy to their undeformed counterpart, the $q$-deformed delta function is subject to the following identities:

\[
\int_{-\infty}^{+\infty} d^3q \delta^3_q(y \oplus (\ominus \kappa^{-1} x)) \odot f(x) = 0.
\]

We can write down orthogonality relations for the plane wave solutions in Eq. (37) if we introduce the following functions:

\[
(\phi^*)_p(x, t) = \frac{c}{\sqrt{2}} E_p^{-1/2} \odot \exp(itE_p) \odot (u^*)_p(x),
\]

\[
(\phi^*)_p(x, t) = \frac{c}{\sqrt{2}} (u^*)_p(x) \odot \exp(-itE_p) \odot E_p^{-1/2}.
\]

The functions $(\phi^*)_p$ or $(\phi^*)_p$ are subject to the identities

\[
(\phi^*)_p(x, t) \triangleq \partial^A \ominus 1 = p^A \odot (\phi^*)_p(x, t),
\]

\[
(\phi^*)_p(x, t) \triangleq \partial t \ominus 1 = E_p \odot (\phi^*)_p(x, t),
\]

or

\[
i^{-1} \partial^A \triangleleft (\phi^*)_p(x, t) = (\phi^*)_p(x, t) \odot p^A,
\]

\[
i \partial t \triangleleft (\phi^*)_p(x, t) = (\phi^*)_p(x, t) \odot E_p.
\]

Moreover, $(\phi^*)_p$ or $(\phi^*)_p$ are plane wave solutions to the following $q$-versions of the Klein-Gordon equations:

\[
(\phi^*)_p \triangleq \partial^2 - (\phi^*)_p \triangleq \nabla^2_q + (\phi^*)_p (mc)^2 = 0,
\]

\[
c^{-2} \partial^2 \triangleleft (\phi^*)_p - \nabla^2_q \triangleleft (\phi^*)_p + (mc)^2 (\phi^*)_p = 0.
\]

You can see this by inserting the expressions for $(\phi^*)_p$ or $(\phi^*)_p$ into the above $q$-deformed Klein-Gordon equations. Doing so, we regain the energy-momentum relation in Eq. (42).

Taking the conjugation properties of $q$-deformed exponentials into account [cf. Eq. (240) in App. C], it follows from Eqs. (37) and (51) that our plane wave solutions behave under conjugation as follows:

\[
\overline{\phi^*_p} = \phi^*_p, \quad (\phi^*)_p = (\phi^*)_p.
\]
The results so far enable us to write down orthogonality relations for the plane wave solutions of our $q$-deformed Klein-Gordon equations, i.e.

\[
\begin{align*}
& \text{i}c^{-2} \int d^3q (\varphi^*)_p(x, \pm t) \triangleleft \partial \triangleright \varphi_{p'}(x, \pm t) \\
& + \text{i}c^{-2} \int d^3q (\varphi^*)_p(x, \pm t) \triangleright \partial \triangleright \varphi_{p'}(x, \pm t) \\
& = \pm \text{vol}^{-1} \delta^3((\oplus \kappa^{-1} p) \oplus p') \quad (56)
\end{align*}
\]

and

\[
\begin{align*}
& \text{i}c^{-2} \int d^3q \varphi_p(x, \pm t) \triangleleft \partial \triangleright (\varphi^*)_p'(x, \pm t) \\
& + \text{i}c^{-2} \int d^3q \varphi_p(x, \pm t) \triangleright \partial \triangleright (\varphi^*)_p'(x, \pm t) \\
& = \pm \text{vol}^{-1} \delta^3(p \oplus (\oplus \kappa^{-1} p')).
\end{align*}
\]

If the signs of the time coordinates in the above expressions are different, the integrals will vanish, i.e.

\[
\begin{align*}
& \text{i}c^{-2} \int d^3q (\varphi^*)_p(x, \pm t) \triangleleft \partial \triangleright \varphi_{p'}(x, \mp t) \\
& + \text{i}c^{-2} \int d^3q (\varphi^*)_p(x, \pm t) \triangleright \partial \triangleright \varphi_{p'}(x, \mp t) = 0 \quad (58)
\end{align*}
\]

and

\[
\begin{align*}
& \text{i}c^{-2} \int d^3q \varphi_p(x, \pm t) \triangleleft \partial \triangleright (\varphi^*)_p'(x, \mp t) \\
& + \text{i}c^{-2} \int d^3q \varphi_p(x, \pm t) \triangleright \partial \triangleright (\varphi^*)_p'(x, \mp t) = 0. \quad (59)
\end{align*}
\]

To prove the above orthogonality relation, you only need to evaluate the corresponding integrals. We show this by the following calculation:

\[
\begin{align*}
& \text{i}c^{-2} \int d^3q (\varphi^*)_p(x, \pm t) \triangleleft \partial \triangleright \varphi_{p'}(x, \pm t) = \\
& = \text{i}2^{-1} E^{-1/2}_p \circ \exp(\pm \text{i} E_p) \circ \partial \triangleright \\
& \quad \circ \int d^3q (u^*)_p(x) \circ u_{p'}(x) \circ \exp(\mp \text{i} E_{p'}) \circ E^{-1/2}_{p'} \\
& = \pm 2^{-1} E^{-1/2}_p \circ E_{p'} \circ \exp(\pm \text{i} E_p) \\
& \quad \circ \text{vol}^{-1} \delta^3((\oplus \kappa^{-1} p) \oplus p') \circ \exp(\mp \text{i} E_{p'}) \circ E^{-1/2}_{p'} \\
& = \pm 2^{-1} E^{1/2}_p \circ \exp(\pm \text{i} E_p) \circ \exp(\mp \text{i} E_{p'}) \circ E^{-1/2}_{p'} \\
& \quad \circ \text{vol}^{-1} \delta^3((\oplus \kappa^{-1} p) \oplus p') \\
& = \pm 2^{-1} \text{vol}^{-1} \delta_3^3((\oplus \kappa^{-1} p) \oplus p'). \quad (60)
\end{align*}
\]
First, we have inserted the expressions for \((\varphi^* )_{p}\) and \(\varphi_{p'}\). In the second step, we have calculated the time derivative and applied the orthogonality relation given in Eq. (17). The second last step is a consequence of the identities in Eq. (50).

As was shown in Ref. [17], the \(q\)-deformed momentum eigenfunctions are also subject to the following completeness relations:

\[
\begin{align*}
\int d_q^3 p \, u_p(\mathbf{x}, t) \otimes (u^*)_p(\mathbf{y}, t) &= \text{vol}^{-1} \delta_q^{3}(\mathbf{x} \oplus (\ominus \kappa^{-1} \mathbf{y})), \\
\int d_q^3 p \, (u^*)_p(\mathbf{y}, t) \otimes u_p(\mathbf{x}, t) &= \text{vol}^{-1} \delta_q^{3}((\ominus \kappa^{-1} \mathbf{y}) \oplus \mathbf{x}). \quad (61)
\end{align*}
\]

We can calculate completeness relations for the plane wave solutions of our \(q\)-deformed Klein-Gordon equations as well:

\[
\begin{align*}
\int d_q^3 p \, \varphi_p(\mathbf{x}, t) \otimes (\varphi^*)_p(\mathbf{x}', t) &= \frac{c^2}{2} \int d_q^3 p \, u_p(\mathbf{x}) \otimes E_{p}^{-1} \otimes (u^*)_p(\mathbf{x}') \\
&= \frac{c}{2} \int d_q^3 p \, u_p(\mathbf{x}) \otimes (p^2 + (mc)^2)^{-1/2} \otimes (u^*)_p(\mathbf{x}') \\
&= \frac{c}{2} (-\nabla_q^2 + (mc)^2)^{-1/2} \triangleright \int d_q^3 p \, u_p(\mathbf{x}) \otimes (u^*)_p(\mathbf{x}') \\
&= \frac{c}{2 \text{vol}} (-\nabla_q^2 + (mc)^2)^{-1/2} \triangleright \delta_q^{3}(\mathbf{x} \oplus (\ominus \kappa^{-1} \mathbf{x}')) \\
&= \frac{c}{2 \text{vol}} \mathcal{C}_{-1/2}(\mathbf{x} \oplus (\ominus \kappa^{-1} \mathbf{x}')). \quad (62)
\end{align*}
\]

The second step in the above calculation is due to the energy-momentum relation in Eq. (12), and the third step is a consequence of the eigenvalue equations in Eq. (42). In the second last step, we applied the completeness relations for our \(q\)-deformed momentum eigenfunctions [cf. Eq. (61)]. Finally, we introduced the following distribution:

\[
\mathcal{C}_{-1/2}(\mathbf{x} \oplus (\ominus \mathbf{x}')) = (-\nabla_q^2 + (mc)^2)^{-1/2} \triangleright \delta_q^{3}(\mathbf{x} \oplus (\ominus \mathbf{x}')) \\
= \sum_{k=0}^{\infty} \left( \frac{-1/2}{k} \right) (mc)^{-1/2} \triangleright (-\nabla_q^2)^k \triangleright \delta_q^{3}(\mathbf{x} \oplus (\ominus \mathbf{x}')). \quad (63)
\]

By similar reasoning, we can show the identity

\[
\int d_q^3 p \, (\varphi^*)_p(\mathbf{x}', t) \otimes \varphi_p(\mathbf{x}, t) = \frac{c}{2 \text{vol}} \mathcal{C}_{-1/2}^* ((\ominus \kappa^{-1} \mathbf{x}') \oplus \mathbf{x}), \quad (64)
\]

with

\[
\mathcal{C}_{-1/2}^* ((\ominus \mathbf{x}') \oplus \mathbf{x}) = \delta_q^{3}((\ominus \mathbf{x}') \oplus \mathbf{x}) \triangleright (-\nabla_q^2 + (mc)^2)^{-1/2} \\
= \sum_{k=0}^{\infty} \left( \frac{-1/2}{k} \right) \delta_q^{3}((\ominus \mathbf{x}') \oplus \mathbf{x}) \triangleright (-\nabla_q^2)^k (mc)^{-1/2}. \quad (65)
\]
Next, we consider the general solutions to the $q$-deformed Klein-Gordon equations in Eqs. (1), (3), and (6) of the previous chapter. We can write these general solutions as expansions in terms of plane wave solutions:

$$
\varphi_R(x, t) = \int d^3q \; p \left( \varphi_p(x, t) \oplus f_P^+ \right) + \varphi_p(-x, -t) \ominus f_P^-,
$$

$$
\varphi_L(x, t) = \int d^3q \; p \left( f_P^+ \odot \varphi^P(x, t) + f_P^P \odot \varphi^P(-x, -t) \right).
$$

Likewise, we have

$$
\varphi^*_L(x, t) = \sum_{\varepsilon = \pm} \int d^3q \; p \left[ \varepsilon h_p^{[\varepsilon]} \odot (\varphi^*)_p(e x, e t) \right],
$$

$$
\varphi^*_R(x, t) = \sum_{\varepsilon = \pm} \int d^3q \; p \left[ \varepsilon f_p^{[\varepsilon]} \odot (\varphi^*)_p(e x, e t) \right],
$$

where

$$
\varphi^*_L \triangledown^2 - \varphi^*_L \triangledown^2_q + (mc)^2 = 0,
$$

$$
c^{-2} \triangledown^2_q \circ \varphi^*_R - \triangledown^2_q \circ \varphi^*_R + (mc)^2 \varphi^*_R = 0.
$$

We can calculate the coefficients in the above series expansions by the formulas

$$
\varepsilon h_p^{[\varepsilon]} = ic^{-2} \int d^3q \; x \varphi_L^{[\varepsilon]}(x, t) \odot \partial_t \odot \varphi_p(e x, e t),
$$

$$
+ ic^{-2} \int d^3q \; x \varphi^*_L(x, t) \odot \partial_t \odot \varphi_p(e x, e t),
$$

$$
\varepsilon h_p^{[\varepsilon]} = ic^{-2} \int d^3q \; x \varphi^P(e x, e t) \odot \partial_t \odot \varphi^*_R(x, t),
$$

$$
+ ic^{-2} \int d^3q \; x \varphi^P(e x, e t) \odot \partial_t \odot \varphi^*_R(x, t),
$$

and

$$
\varepsilon f_p^{[\varepsilon]} = ic^{-2} \int d^3q \; x (\varphi^*)_p(e x, e t) \odot \partial_t \odot \varphi_R(x, t),
$$

$$
+ ic^{-2} \int d^3q \; x (\varphi^*)_p(e x, e t) \odot \partial_t \odot \varphi_R(x, t),
$$

$$
\varepsilon f_p^{[\varepsilon]} = ic^{-2} \int d^3q \; x \varphi^*_L(x, t) \odot \partial_t \odot (\varphi^*_P)(e x, e t),
$$

$$
+ ic^{-2} \int d^3q \; x \varphi^*_L(x, t) \odot \partial_t \odot (\varphi^*_P)(e x, e t).
$$

We briefly describe how to check the above formulas for the coefficients. First, you insert the expansions in terms of plane wave solutions [cf. Eq. (66) and
This way, you can use the orthogonality relations given in Eqs. (56)-(59). Taking into account the identities in Eq. (50) will finish the proof.

There is a bilinear form for solutions to our $q$-deformed Klein-Gordon equations. Concretely, we have

$$i c^{-2} \int d^3x \varphi_L^*(x, t) \triangleleft \partial_t \varphi_R(x, t)$$
$$+ i c^{-2} \int d^3x \varphi_L^*(x, t) \triangleright \partial_t \varphi_R(x, t) =$$
$$= \int d^3p \left( h_p^{[+] \odot f_p^{[+]}} - h_p^{[-] \odot f_p^{[-]}} \right)$$

(73)

and

$$i c^{-2} \int d^3x \varphi_R(x, t) \bigtriangleup \partial_t \varphi_R^*(x, t)$$
$$+ i c^{-2} \int d^3x \varphi_R(x, t) \bigtriangledown \partial_t \varphi_R^*(x, t) =$$
$$= \int d^3p \left( f_p^{[+] \odot h_p^{[+]}} - f_p^{[-] \odot h_p^{[-]}} \right).$$

(74)

We briefly explain how we get the expressions in momentum space from those in position space. First, we write each wave function in position space as expansion in terms of plane wave solutions. This way, we can apply the orthogonality relations given in Eqs. (56)-(59). By using the identities in Eq. (50), we finally get the expression in momentum space. Note that the bilinear form in Eq. (73) or Eq. (74) vanishes if we require

$$h_p^{[+]} = h_p^{[-]}, \quad f_p^{[+]} = f_p^{[-]} \quad \text{or} \quad h_p^{[+] \odot h_p^{[-]}} = f_p^{[+] \odot f_p^{[-]}}.$$

(75)

4 Free propagators

The series expansions in Eqs. (66) and (67) of the previous chapter show us that the solutions to our $q$-deformed Klein-Gordon equations consist of two parts with different signs of energy, i.e.

$$\varphi_R = (\varphi_R)^{[+] + (\varphi_R)^{[-]},}$$
$$\varphi_L = (\varphi_L)^{[+] + (\varphi_L)^{[-]},}$$

(76)

and

$$\varphi_R^* = (\varphi_R^*)^{[+] + (\varphi_R^*)^{[-]},}$$
$$\varphi_L^* = (\varphi_L^*)^{[+] + (\varphi_L^*)^{[-]},}$$

(77)

The propagators for the Klein-Gordon equations should take a form so that the positive energy solution runs forward in time while the negative energy solution
runs backward in time. Hence, we seek propagators $\Delta_R$ and $\Delta_L$ that satisfy the identities:

$$\begin{align*}
\text{i}c^{-2} & \int d^3q \, \Delta_R(x', t'; x, t) \triangleq \partial_t \circ \varphi_R(x, t) \\
+ \text{i}c^{-2} & \int d^3q \, \Delta_R(x', t'; x, t) \circ \partial_t \triangleright \varphi_R(x, t) = \\
& = \theta(t' - t) (\varphi_R)^{[+]}(x', t') - \theta(t - t') (\varphi_R)^{[-]}(x', t') \quad (78)
\end{align*}$$

or

$$\begin{align*}
\text{i}c^{-2} & \int d^3q \, \varphi_L(x, t) \triangleq \partial_t \circ \Delta_L(x, t; x', t') \\
+ \text{i}c^{-2} & \int d^3q \, \varphi_L(x, t) \circ \partial_t \triangleright \Delta_L(x, t; x', t') = \\
& = \theta(t' - t) (\varphi_L)^{[+]}(x', t') - \theta(t - t') (\varphi_L)^{[-]}(x', t'). \quad (79)
\end{align*}$$

Propagators with these properties are given by

$$\begin{align*}
\Delta_R(x', t'; x, t) &= \theta(t' - t) \int d^3p \, \varphi_p(x', t') \circ (\varphi^*_p)_{p}(x, t) \\
& \quad + \theta(t - t') \int d^3p \, \varphi_p(x', t') \circ (\varphi^*_p)_{p}(x, -t) \quad (80)
\end{align*}$$

and

$$\begin{align*}
\Delta_L(x, t; x', t') &= \theta(t' - t) \int d^3p \, (\varphi^*_p)_{p}(x, t) \circ \varphi_p(x', t') \\
& \quad + \theta(t - t') \int d^3p \, (\varphi^*_p)_{p}(x, -t) \circ \varphi_p(x', -t'). \quad (81)
\end{align*}$$

To show that the expression in Eq. (80) or Eq. (81) fulfills the identity in Eq. (78) or Eq. (79), we proceed as follows. First, we replace the propagator in Eq. (78) or Eq. (79) by its expression in Eq. (80) or Eq. (81). In the same way, we write the wave functions $\varphi_R$ and $\varphi_L$ as expansions in terms of plane wave solutions [cf. Eq. (66) of the previous chapter]. Then, we can apply the orthogonality relations given in Eqs. (50)-(59) of Chap. 3. Using the identities in Eq. (50) of Chap. 3, we regain the plane wave expansions. We have the additional factor $\theta(t' - t)$ for the plane waves with positive energy. For the plane waves with negative energy, we have the factor $-\theta(t - t')$.

We can also introduce so-called ‘dual’ propagators

$$\begin{align*}
\Delta^*_R(x', t'; x, t) &= \theta(t' - t) \int d^3p \, (\varphi^*_p)_{p}(x', t') \circ \varphi_p(x, t) \\
& \quad + \theta(t - t') \int d^3p \, (\varphi^*_p)_{p}(x', t') \circ \varphi_p(x, -t) \quad (82)
\end{align*}$$

$^3\theta(t)$ stands for the Heaviside step function.
These new propagators are subject to

\[ \Delta^*_L(x, t; x', t') = \theta(t' - t) \int d^3p \varphi_p(x, t) \otimes (\varphi^*_p)(x', t') \]
\[ + \theta(t - t') \int d^3p \varphi_p(x, -t) \otimes (\varphi^*_p)(x', -t'). \]  

These identities follow from Eqs. (80)-(83) if we consider the conjugation properties of \( q^* \)-deformed propagators and \( q^* \)-deformed volume integrals and see Eq. (188) in App. A, Eq. (212) in App. B, and Eq. (55) in Chap. 3.

\( q^* \)-deformed plane wave solutions [also see Eq. (188) in App. A, Eq. (212) in App. B, and Eq. (55) in Chap. 3]

Our \( q^* \)-deformed propagators transform into each other by conjugation:

\[ \overline{\Delta_R(x', t'; x, t)} = \Delta_L(x, t; x', t'), \]
\[ \Delta^*_L(x, t; x', t') = \Delta^*_R(x, t; x', t'). \]  

These identities follow from Eqs. (80)-(83) if we consider the conjugation properties of \( q^* \)-deformed volume integrals and \( q^* \)-deformed plane wave solutions [also see Eq. (188) in App. A, Eq. (212) in App. B, and Eq. (55) in Chap. 3].

Our \( q^* \)-deformed propagators are also subject to the equations

\[ ((mc)^2 + c^{-2} \partial^2_t - \nabla^2_q) \cdot \Delta_R(x, t; x', t') = \]
\[ = -i \text{vol}^{-1} \delta(t - t') \delta^3_q(x \oplus (\ominus \kappa^{-1}x')), \]  

\[ \Delta_L(x', t'; x, t) \ominus (\partial^2_t e^{-2} - \nabla^2_q + (mc)^2) = \]
\[ = i \text{vol}^{-1} \delta^3_q((\ominus \kappa^{-1}x') \oplus x) \delta(t' - t), \]  

or

\[ \Delta^*_L(x', t'; x, t) \ominus (\partial^2_t e^{-2} - \nabla^2_q + (mc)^2) = \]
\[ = i \text{vol}^{-1} \delta^3_q((\ominus \kappa^{-1}x') \oplus x) \delta(t' - t), \]  

\[ ((mc)^2 + c^{-2} \partial^2_t - \nabla^2_q) \cdot \Delta^*_R(x, t; x', t') = \]
\[ = -i \text{vol}^{-1} \delta(t - t') \delta^3_q(x \oplus (\ominus \kappa^{-1}x')). \]  

(83)
We show how the above identities are derived using Eq. (87) as an example. To this end, we write the propagator $\Delta_R$ as

$$\Delta_R(x, t; x', t') = \theta(t - t') D_R(x, t; x', t') + \theta(t' - t) D_R(x, -t; x', -t')$$  \hfill (91)$$

with

$$D_R(x, t; x', t') = \int d^3 p \varphi_p(x, t) \otimes (\varphi^*_p(x', t'). \hfill (92)$$

Using the identity

$$\partial_t \triangleright \theta(t - t') = \delta(t - t') = \delta(t' - t),$$  \hfill (93)$$

we can do the following calculation:

$$\partial_t^2 \triangleright \Delta_R(x, t; x', t') =$$
$$= [\partial_t \triangleright \delta(t - t')] [D_R(x, t; x', t') - D_R(x, -t; x', -t')]
+ 2 \delta(t - t') \partial_t \triangleright [D_R(x, t; x', t') - D_R(x, -t; x', -t')]
+ \theta(t - t') \partial_t^2 \triangleright D_R(x, t; x', t')
+ \theta(t' - t) \partial_t^2 \triangleright D_R(x, -t; x', -t').$$  \hfill (94)$$

With the identities

$$[\partial_t \triangleright \delta(t)] f(t) = f(t) [-\partial_t \triangleright f(t)]$$  \hfill (95)$$

and [cf. Eq. (40) of the previous chapter and Eq. (92)]

$$e^{-2 \partial_t^2} \triangleright D_R(x, t; x', t') = (\Delta_q - (mc)^2) \triangleright D_R(x, t; x', t'),$$  \hfill (96)$$

we can write the result of Eq. (94) as follows:

$$e^{-2 \partial_t^2} \triangleright \Delta_R(x, t; x', t') =$$
$$= e^{-2 \delta(t - t')} \partial_t \triangleright [D_R(x, t; x', t') - D_R(x, -t; x', -t')]
+ \theta(t - t') (\Delta_q - (mc)^2) \triangleright D_R(x, t; x', t')
+ \theta(t' - t) (\Delta_q - (mc)^2) \triangleright D_R(x, -t; x', -t').$$  \hfill (97)$$

If we consider Eq. (92), we can also rewrite the expressions depending on the time derivative:

$$e^{-2 \delta(t - t')} \partial_t \triangleright D_R(x, \pm t; x', \pm t') =$$
$$= e^{-2 \delta(t - t')} \int d^3 p \varphi_p(x, \pm t) \otimes (\mp i E_p) \otimes (\varphi^*_p(x', \pm t')
$$
$$= \mp \frac{i}{2} \delta(t - t') \int d^3 p u_p(x) \otimes (u^*_p(x'))
$$
$$= \mp \frac{i}{2 \text{vol}} \delta(t - t') \delta^3(x \otimes (\mp \kappa^{-1} x')).$$  \hfill (98)$$
We briefly describe the above calculation. The action of the time derivative on the wave function \( \varphi_p \) gives the factor \( \pm i E_p \). Moreover, it holds \( t = t' \) due to the delta function \( \delta(t - t') \). For this reason, the time-dependent phase factors in the wave functions \( \varphi_p \) and \( (\varphi^*)_p \) cancel each other out. In the last step, we applied the completeness relations for our \( q \)-deformed momentum eigenfunctions [cf. Eq. (61) of the previous chapter]. Plugging the result of Eq. (98) into Eq. (97), we finally get:

\[
\begin{align*}
    c^{-2} \partial_t^2 \triangleright \Delta_R(x; t; x', t') &= \\
    &= -i \text{vol}^{-1} \delta(t - t') \delta_q^3(x \oplus (\ominus \kappa^{-1} x')) \\
    &\quad + \Delta_q \triangleright \Delta_R(x; t; x', t') - (mc)^2 \Delta_R(x; t; x', t'). \\
\end{align*}
\]  

(99)

The identities in Eqs. (87) and (88) show that the functions

\[
\begin{align*}
    \phi_R(x, t) &= \varphi_R(x, t) + i \int d^3x' dt' \Delta_R(x, t; x', t') \otimes \varrho(x', t'), \\
    \phi_L(x, t) &= \varphi_L(x, t) - i \int d^3x' dt' \varrho(x', t') \otimes \Delta_L(x', t'; x, t)
\end{align*}
\]  

(100)

are solutions to the following \( q \)-deformed inhomogenous Klein-Gordon equations:

\[
\begin{align*}
    c^{-2} \partial_t^2 \triangleright \phi_R - \nabla_q^2 \triangleright \phi_R + (mc)^2 \phi_R &= \varrho, \\
    c^{-2} \partial_t^2 \varrho \otimes \phi_L - \nabla_q^2 \otimes \phi_L + \phi_L (mc)^2 &= \varrho.
\end{align*}
\]  

(101)

Note that the functions \( \varphi_R \) and \( \varphi_L \) in Eq. (100) are solutions to free \( q \)-deformed Klein-Gordon equations [cf. Eqs. (11) and (3) in Chap. 2]. Similarly, the functions

\[
\begin{align*}
    \phi_R^*(x, t) &= \varphi_R^*(x, t) + i \int d^3x' dt' \Delta_R^*(x, t; x', t') \otimes \varrho(x', t'), \\
    \phi_L^*(x, t) &= \varphi_L^*(x, t) - i \int d^3x' dt' \varrho(x', t') \otimes \Delta_L^*(x', t'; x, t)
\end{align*}
\]  

(102)

satisfy the following \( q \)-deformed versions of the inhomogenous Klein-Gordon equation:

\[
\begin{align*}
    \phi^*_L \otimes \partial_t^2 c^{-2} \triangleright \phi^*_R - \phi^*_L \otimes \Delta_q + \phi^*_L (mc)^2 &= \varrho, \\
    c^{-2} \partial_t^2 \triangleright \phi^*_R - \Delta_q \triangleright \phi^*_R + (mc)^2 \phi^*_R &= \varrho.
\end{align*}
\]  

(103)

Once again, \( \varphi^*_R \) and \( \varphi^*_L \) are solutions to free \( q \)-deformed Klein-Gordon equations [cf. Eq. (11) in Chap. 2].
We can write down a momentum space form of each $q$-deformed Klein-Gordon propagator. Concretely, we have

\[
\Delta_R(x', t'; x, t) = \int dE e^{-i(t' - t)E} \int d^3p \ u_p(x') \odot \Delta_R(E, p) \odot (u^*)_p(x),
\]

\[
\Delta_L(x, t; x', t') = \int dE e^{-i(t' - t)E} \int d^3p \ (u^*)_p(x) \odot \Delta_L(E, p) \odot u^p(x') \quad (104)
\]

with

\[
\Delta_R(E, p) = -\Delta_L(E, p) = i \frac{c^2}{2\pi} \left( E^2 - E_p^2 + i\varepsilon \right)^{-1}. \quad (105)
\]

Note that we can write the expression in Eq. (105) as a series in normal-ordered monomials of momentum coordinates. To get this expansion, we first write the fraction as a power series of $p^2$ and then use the formula in Eq. (45) of the previous chapter.

In the following, we show how to derive the momentum space form of the $q$-deformed Klein-Gordon propagators. We demonstrate our considerations using the example of the propagator $\Delta_R(x', t'; x, t)$. First, we need the following identity for the Heaviside function:

\[
\theta(t) = \frac{i}{2\pi} \lim_{\varepsilon \to 0^+} \int dE \frac{e^{-itE}}{E + i\varepsilon}. \quad (106)
\]

With this identity, we can carry out the following calculation:

\[
\theta(t') \ E_p^{-1} \odot \exp(-i(t' - t)E_p) = \frac{i}{2\pi} \lim_{\varepsilon \to 0^+} \int dE \frac{1}{E + i\varepsilon} \ E_p^{-1} \odot \exp(-i(t' - t)(E + E_p))
\]

\[
= \frac{i}{2\pi} \lim_{\varepsilon \to 0^+} \int dE \ e^{-i(t' - t)E} \left( E - E_p + i\varepsilon \right)^{-1} \odot E_p^{-1}. \quad (107)
\]

As a final step of the above calculation, we have substituted $E$ as the variable of integration by $E + E_p$. In the same way, we get:

\[
\theta(t - t') \ E_p^{-1} \odot \exp(i(t' - t)E_p) = \frac{-i}{2\pi} \lim_{\varepsilon \to 0^+} \int dE \ e^{-i(t' - t)E} \left( E + E_p - i\varepsilon \right)^{-1} \odot E_p^{-1}. \quad (108)
\]

Due to the results in Eqs. (107) and (108), we can rewrite the expression for the propagator $\Delta_R$ in Eq. (101) as follows:

\[
\Delta_R(x', t'; x, t) = \theta(t' - t) \ D_R(x', t'; x, t) + \theta(t - t') \ D_R(x', -t'; x, -t)
\]

\[
= \int d^3p \ u_p(x') \odot \Delta_R(E, p; t', t) \odot (u^*)_p(x) \quad (109)
\]
with
\[ c^{-2} \Delta_R(E, p; t', t) = \]
\[ = \frac{1}{2} \theta(t' - t) E_p^{-1} \exp(-i(t' - t)E_p) \]
\[ + \frac{1}{2} \theta(t - t') E_p^{-1} \exp(i(t' - t)E_p) \]
\[ = \frac{i}{4\pi} \lim_{\epsilon \to 0^+} \int dE \ e^{-i(t' - t)E} (E - E_p + i\epsilon)^{-1} \]
\[ - (E + E_p - i\epsilon)^{-1} \otimes E_p^{-1}. \]
\( (110) \)

We can combine the two fractions in square brackets:
\[ \frac{1}{E - E_p + i\epsilon} - \frac{1}{E + E_p - i\epsilon} = \frac{E + E_p - i\epsilon - E + E_p - i\epsilon}{E^2 - (E_p - i\epsilon)^2} = \frac{2E_p - 2i\epsilon}{E^2 - E_p^2 + 2i\epsilon E_p - \epsilon^2}. \]
\( (111) \)

This way, we end up with the following expression:
\[ \Delta_R(E, p; t', t) = \frac{i^2}{2\pi} \lim_{\epsilon \to 0^+} \int dE (E^2 - E_p^2 + i\epsilon)^{-1} e^{-i(t' - t)E}. \]
\( (112) \)

If we substitute the expression above into Eq. \( 109 \), we can finally read off the momentum space form of the \( q \)-deformed Klein-Gordon propagator \( \Delta_R \).

5 Charge conservation

In this chapter, we derive continuity equations for our \( q \)-deformed Klein-Gordon equations. To this end, we first multiply the Klein-Gordon equation for \( \varphi_R \) [cf. Eq. \( 1 \) in Chap. 2] on the left by \( \varphi^*_L \) and the Klein-Gordon equation for \( \varphi^*_L \) [cf. Eq. \( 6 \) in Chap. 2] on the right by \( \varphi_R \). Subtracting the two equations multiplied by \( \varphi^*_L \) or \( \varphi_R \), we get the following identity:
\[ \varphi^*_L \otimes e^{-2} \partial_t^2 \triangleright \varphi_R - \varphi^*_L \otimes \partial_t^2 e^{-2} \otimes \varphi_R - \varphi^*_L \otimes \nabla^2_q \triangleright \varphi_R + \varphi^*_L \otimes \nabla^2_q \otimes \varphi_R = 0. \]
\( (113) \)

We rewrite the expressions depending on time derivatives as follows:
\[ \varphi^*_L \otimes e^{-2} \partial_t^2 \triangleright \varphi_R - \varphi^*_L \otimes \partial_t^2 e^{-2} \otimes \varphi_R = \]
\[ = \partial_t \triangleright [\varphi^*_L \otimes e^{-2} \partial_t \triangleright \varphi_R + \varphi^*_L \otimes \partial_t e^{-2} \otimes \varphi_R]. \]
\( (114) \)

To rewrite the expressions in Eq. \( 114 \) depending on \( \nabla^2_q \), we need \( q \)-versions of Green’s theorem [7], i.e.
\[ \psi \otimes \nabla^2_q \triangleright \phi = -\partial^C \triangleright [\psi \otimes (\mathcal{L}_q)^A_C \partial^B \otimes \phi + \psi \otimes (\mathcal{L}_q)^A_C \partial^B \triangleright \phi] g_{AB} \]
\( (115) \)
and
\[
\psi \triangleq \nabla_q^2 \phi - \psi \circ \nabla_q^2 \phi = g_{AB} \left[ \psi \circ \partial^A (L_\partial) B^C \ast \phi + q^2 \psi \circ \partial^A (L_\partial) B^C \ast \phi \right] \ast \partial^C. \tag{116}
\]
Substituting \( \psi \) and \( \phi \) by \( \phi^* \) or \( \phi_R \), we get:
\[
\phi^* \circ \nabla_q^2 \ast \phi_R - \phi^* \circ \nabla_q^2 \ast \phi_R = \partial^C \ast \left[ q^{-2} \phi^*_L \circ (L_\partial) C_\partial \circ \phi_R + \phi^*_L \circ (L_\partial) C_\partial \circ \partial_C \ast \phi_R \right]. \tag{117}
\]
By combining this result with that of Eq. (114), we finally get the continuity equation
\[
\partial_t \triangleright \rho(x,t) + \partial^A \triangleright j_A(x,t) = 0 \tag{118}
\]
with
\[
j_A(x,t) = -\frac{ie}{2mc} \left[ q^{-2} \phi^*_L \circ (L_\partial) C_\partial \circ \phi_R + \phi^*_L \circ (L_\partial) C_\partial \circ \partial_C \ast \phi_R \right],
\]
\[
\rho(x,t) = \frac{ie}{2mc} \left[ \phi^*_L \circ \partial_t \triangleright \phi_R + \phi^*_L \circ \partial_t \ast \phi_R \right]. \tag{119}
\]
By conjugating Eq. (118) [cf. Eq. (188) in App. A, Eq. (201) in App. B, Eq. (246) in App. D as well as Eqs. (4) and (7) in Chap. 2], we obtain the continuity equation for the other \( q \)-version of the Klein-Gordon equation, i.e.
\[
\rho^*(x,t) \equiv \rho(x,t) = \frac{ie}{2mc} \left[ \phi^*_L \circ \partial_t \triangleright \phi_R + \phi^*_L \circ \partial_t \ast \phi_R \right], \tag{120}
\]
\[
j_A(x,t) = (j^*)^A(x,t) = -\frac{ie}{2mc} \left[ q^{-2} g_{CD} \phi^*_L \circ \partial^D (\tilde{L}_\partial) D \circ \phi_R \ast g^{EA} \right] = \frac{ie}{2mc} g_{CD} \phi^*_L \circ \partial^D (\tilde{L}_\partial) D \circ \phi_R \ast g^{EA}. \tag{122}
\]
We get further continuity equations for our \( q \)-deformed Klein-Gordon equations by applying the following substitutions to the expressions in Eqs. (118)-(121):
\[
\triangleright \leftrightarrow \triangleright \tilde{\phi}, \quad \circ \leftrightarrow \circ \tilde{\phi}, \quad q \leftrightarrow q^{-1}, \quad \tilde{L}_\partial \leftrightarrow L_\partial, \quad \phi_R \leftrightarrow \phi^*_R, \quad \phi_L \leftrightarrow \phi^*_L. \tag{123}
\]
In analogy to the undeformed case, \( \rho(x,t) \) or \( \rho^*(x,t) \) is the charge density of a \( q \)-deformed spin-zero particle, and \( j_A(x,t) \) or \( j^*_A(x,t) \) is the corresponding current density. By integrating \( \rho(x,t) \) or \( \rho^*(x,t) \) over all space, we regain the bilinear forms in Eq. (73) or Eq. (74) of Chap. 3. In this respect, the condition in Eq. (75) of Chap. 3 describes a \( q \)-deformed spin-zero particle without any
electric charge. Using the continuity equation in Eq. 118 or Eq. 120, we can show that the overall charge of a \( q \)-deformed spin-zero particle is constant over time [cf. Eq. (210) in App. B]:

\[
\frac{i e}{2 mc^2} \partial_t \partial_t \Phi^*_L \circ D^0 \circ \Phi_R - \Phi^*_L \circ D^0 \circ e^{-2} \circ \Phi_R = 0.
\] (124)

We can modify the above reasonings such that they apply to a charged spin-zero particle moving in the presence of an electromagnetic field. To this end, we substitute the time derivative \( \partial_t \) and each partial derivative \( \partial_C \) in Eq. (113) by the corresponding covariant derivative \( D^0 \) or \( D^C \) [cf. Eq. (8) in Chap. 2]:

\[
\Phi^*_L \circ e^{-2} D^0 D^0 \circ \Phi_R - \Phi^*_L \circ D^0 D^0 \circ e^{-2} \circ \Phi_R
\]

\[
- \Phi^*_L \circ D^C D^C \circ \Phi_R + \Phi^*_L \circ D^C D^C \circ \Phi_R = 0.
\] (125)

We have shown in Ref. [7] that the covariant derivatives \( D^C \) are subject to the following identities:

\[
\psi \circ D^C D^C \circ \phi - \psi \circ D^C D^C \circ \phi =
\]

\[
= - \partial F \circ \left[ q^{-2} \psi \circ (L_0)^B(\Phi^*_L \circ D^C \circ \phi + \psi \circ (L_0)^B(\Phi^*_L \circ D^C \circ \phi) \right) \circ g_{BC}
\]

\[
= g_{BC} \left[ \psi \circ (L_0)^F( \Phi^*_L \circ D^C \circ \phi + q^2 \psi \circ (L_0)^F( \Phi^*_L \circ D^C \circ \phi) \right) \circ \partial F.
\] (126)

A direct calculation shows that a similar identity holds for the covariant time derivative \( D^0 \):

\[
\Phi^*_L \circ e^{-2} D^0 D^0 \circ \Phi_R - \Phi^*_L \circ D^0 D^0 \circ e^{-2} \circ \Phi_R =
\]

\[
= \partial_t \partial_t \left[ \Phi^*_L \circ e^{-2} D^0 D^0 \circ \Phi_R + \Phi^*_L \circ D^0 D^0 \circ e^{-2} \circ \Phi_R \right].
\] (127)

The above identities enable us to rewrite Eq. (125) as a continuity equation. We obtain the new charge density and the new current density from Eq. (119) or Eq. (121) by substituting the covariant derivatives \( D^0 \) and \( D^C \) for the time derivative \( \partial_t \) or the \( q \)-deformed partial derivatives \( \partial_C \). We get

\[
\partial_t \circ \rho(x, t) + \partial A \circ j_A(x, t) = 0
\] (128)

with

\[
\rho(x, t) = \frac{ie}{2mc^2} \left[ \Phi^*_L \circ D^0 \circ \Phi_R + \Phi^*_L \circ D^0 \circ \Phi_R \right]
\]

\[
= \frac{ie}{2mc^2} \left[ \Phi^*_L \circ \partial_t \circ \Phi_R + \Phi^*_L \circ \partial_t \circ \Phi_R \right]
\]

\[
- \frac{e^2}{mc^2} \Phi^*_L \circ A^0 \circ \Phi_R
\] (129)
and

\[ j_B(x, t) = -\frac{ie}{2m} \left[ q^{-2} \varphi^*_L \triangleleft (\mathcal{L}_0)^C_B D_C \otimes \varphi_R + \varphi^*_L \triangleleft (\mathcal{L}_0)^C_B \otimes D_C \triangleright \varphi_R \right] \]

Conjugating the above identities and expressions gives us

\[ \rho^*(x, t) \bar{\triangleright} \partial_t + (j^*)^A(x, t) \bar{\triangleright} \partial_A = 0 \] (131)

with

\[ \rho(x, t) = \rho^*(x, t) = \frac{ie}{2mc^2} \left[ \varphi_L \otimes D_0 \triangleright \varphi^*_R + \varphi_L \bar{\triangleleft} D_0 \otimes \varphi^*_R \right], \] (132)

\[ j_A(x, t) = (j^*)^A(x, t) = -\frac{ie}{2m} q^{-2} g_{CD} \varphi_L \otimes D_C (\bar{L}_0) \bar{\triangleright} \varphi^*_R g^{EA} 
- \frac{ie}{2m} g_{CD} \varphi_L \bar{\triangleright} D_C \otimes (\bar{L}_0) \bar{\triangleright} \varphi^*_R g^{EA}. \] (133)

6 Conservation of energy

We interpret the expression [cf. Eqs. (66) and (67) in Chap. 3]

\[ \langle E \rangle \varphi = \frac{1}{2} \sum_{\varepsilon = \pm} \int d^3q \, \hbar^{[\varepsilon]} \otimes E_p \otimes f_p^{[\varepsilon]} \]

\[ = -\frac{1}{2c^2} \int d^3q x \varphi^*_L(x, t) \triangleleft \partial_t \otimes D_0 \triangleright \varphi_R(x, t) 
- \frac{1}{2} \int d^3q x \varphi^*_L(x, t) \triangleleft \partial_A \otimes D_0 \triangleright \varphi_R(x, t) 
+ \frac{1}{2} \int d^3q x \varphi^*_L(x, t) \otimes (mc)^2 \varphi_R(x, t) \] (134)

as the expectation value for the energy of a q-deformed spin-zero particle if the following identity holds (\( \varepsilon = \pm \)):

\[ \hbar^{[\varepsilon]} = f_p^{[\varepsilon]} \] (135)

In addition to this, we require:

\[ f_p^{[\varepsilon]} = f_p^{[\varepsilon]} \quad \text{and} \quad \hbar^{[\varepsilon]} = \hbar^{[\varepsilon]} \] (136)

Due to the above condition, conjugating Eq. (134) gives another expression for the energy of a q-deformed spin-zero particle [cf. also Eq. (188) in App. A]
Eqs. (201) and (212) in App. B as well as Eqs. (4) and (7) in Chap. 2:

\[
\langle E^* \rangle_{\phi} = \frac{1}{2} \sum_{\varepsilon = \pm} \int d^3 p f^P_{[\varepsilon]} \otimes E^P \otimes h^P_{[\varepsilon]}
\]

\[
= - \frac{1}{2c^2} \int d^3 x \varphi_L(x, t) \otimes \partial_t \otimes \partial_t \varphi^*_R(x, t)
\]

\[
- \frac{1}{2} \int d^3 x \varphi_L(x, t) \otimes \partial^A \otimes \partial_A \varphi^*_R(x, t)
\]

\[
+ \frac{1}{2} \int d^3 x \varphi_L(x, t) \otimes (mc)^2 \varphi^*_R(x, t).
\]

(137)

In analogy to Eq. (135), it holds:

\[
h^P_{[\varepsilon]} = f^P_{[\varepsilon]}.
\]

(138)

In Eqs. (134) and (137), we have written down the expectation value for the energy, both in momentum space and in position space. We briefly explain how to get the expression in momentum space from that in position space. As a first step, we have to substitute the wave functions in position space by their series expansions [cf. Eqs. (66) and (67) of Chap. 3]. This way, we can apply the time derivatives and \(q\)-deformed partial derivatives to our \(q\)-deformed plane wave solutions [cf Eqs. (39) and (52) in Chap. 3]. With the help of the energy-momentum relation [cf. Eq. (42) in Chap. 3], the orthogonality relations for \(q\)-deformed plane waves [cf. Eqs. (47) and (48) in Chap. 3], and the identities for \(q\)-deformed delta functions [cf. Eq. (50) of Chap. 3], we finally end up with the expression in momentum space.

Next, we derive continuity equations for the energy density of a free \(q\)-deformed spin-zero particle. A look at Eq. (134) shows that the energy density takes on the following form:

\[
\mathcal{H} = - \frac{1}{2c^2} \varphi^*_L(x, t) \otimes \partial_t \otimes \partial_t \varphi^*_R(x, t)
\]

\[
- \frac{1}{2} \varphi^*_L(x, t) \otimes \partial^C \otimes \partial_C \varphi^*_R(x, t)
\]

\[
+ \frac{1}{2} \varphi^*_L(x, t) \otimes (mc)^2 \varphi^*_R(x, t).
\]

(139)

We can calculate the time derivative of the energy density by applying the usual Leibniz rule:

\[
\partial_t \mathcal{H} = \frac{1}{2c^2} \varphi^*_L \otimes \partial_t \varphi^*_R - \frac{1}{2c^2} \varphi^*_L \otimes \partial_t \varphi^*_R
\]

\[
+ \frac{1}{2} \varphi^*_L \otimes \partial^C \varphi^*_R - \frac{1}{2} \varphi^*_L \otimes \partial_C \varphi^*_R
\]

\[
- \frac{1}{2} \varphi^*_L \otimes (mc)^2 \varphi^*_R + \frac{1}{2} \varphi^*_L \otimes (mc)^2 \varphi^*_R.
\]

(140)
Using the \( q \)-deformed Klein-Gordon equations for \( \varphi_R \) and \( \varphi_L^* \) [cf. Eqs. (1) and (6) in Chap. 2], we can rewrite the above equation as follows:

\[
\partial_t \triangleright H = \frac{1}{2} \varphi_L^* \triangleright \partial^C \triangleright \partial_t \triangleright \varphi_R - \frac{1}{2} \varphi_L^* \triangleright \partial_t \triangleright \partial^C \triangleright \varphi_R \\
+ \frac{1}{2} \varphi_L^* \triangleright \partial_t \triangleright \partial^C \triangleright \partial_t \triangleright \varphi_R - \frac{1}{2} \varphi_L^* \triangleright \partial^C \triangleright \partial_t \triangleright \varphi_R. \tag{141}
\]

The Leibniz rules in Eq. (243) of App. D imply the following identities for actions of \( q \)-deformed partial derivatives (see Ref. [7]):

\[
\psi \triangleright \partial^C \triangleright \phi = \partial^C \triangleright [\psi \triangleright \partial^C \triangleright (1) \triangleright \phi] \\
= \partial^B \triangleright [\psi \triangleright (L_\partial)^{CB} \partial^B \triangleright \phi] + \psi \triangleright \partial^C \triangleright \phi \tag{142}
\]

and

\[
\psi \triangleright \partial^C \triangleright \phi = [\psi \triangleright \partial^C \triangleright (1) \triangleright \phi] \triangleright \partial^C \\
= [\psi \triangleright (L_\partial)^{CB} \triangleright \phi] \triangleright \partial^B + \psi \triangleright \partial^C \triangleright \phi. \tag{143}
\]

With the help of these identities, we can write the right-hand side of Eq. (141) as divergence:

\[
\partial_t \triangleright H = \frac{1}{2} [\varphi_L^* \triangleright \partial_t \triangleright (L_\partial)^{BC} \partial_t \triangleright \varphi_R] \triangleright \partial^C \\
- \frac{1}{2} \partial^C \triangleright [\varphi_L^* \triangleright \partial^B (L_\partial)^{BD} g_{BD} \partial_t \triangleright \varphi_R]. \tag{144}
\]

Eq. (142) together with Eq. (143) implies the following identity:

\[
[\psi \triangleright (L_\partial)^{CB} \triangleright \phi] \triangleright \partial^B = -\partial^B \triangleright [\psi \triangleright (L_\partial)^{CB} \triangleright \phi]. \tag{145}
\]

Moreover, we have shown in Ref. [7] that the L-matrices are subject to

\[
g_{BD} \partial^B (L_\partial)^{CD} = q^{-2} (L_\partial)^{BC} \partial^D g_{BD}. \tag{146}
\]

With the help of Eq. (145) and Eq. (146), we finally get the continuity equation

\[
\partial_t \triangleright H + \partial^C \triangleright S_C = 0 \tag{147}
\]

with the following \textit{energy flux density}:

\[
S_C = \frac{1}{2} \varphi_L^* \triangleright \partial_t (L_\partial)^{BC} \triangleright \partial_t \triangleright \varphi_R \\
+ \frac{1}{2} q^{-2} \varphi_L^* \triangleright (L_\partial)^{BC} \partial_t \triangleright \partial_t \triangleright \varphi_R. \tag{148}
\]

Integrating Eq. (147) over all space leads to a surface term at spatial infinity. However, this surface term will be zero since the wave functions vanish at spatial
infinity. Accordingly, the total energy of a $q$-deformed spin-zero particle is constant over time:

$$\partial_t \int d^3_q x \mathcal{H}(x,t) = - \int d^3_x \partial^C \triangleright S_C = 0.$$  \hfill (149) 

By conjugating Eqs. (139), (147), and (148), we get another expression for the energy density and a corresponding continuity equation. Concretely, we have

$$\mathcal{H}^* \triangleright \partial_t + (S^*)^C \triangleright \partial_C = 0$$ \hfill (150) 

with

$$\mathcal{H}^* = - \frac{1}{2c^2} \varphi_L \triangleright \partial_t \otimes \partial_t \triangleright \varphi_R^* - \frac{1}{2} \varphi_L \triangleright \partial^C \otimes \partial_C \triangleright \varphi_R^*$$

$$+ \frac{1}{2} \varphi_L \otimes (mc)^2 \varphi_R^*$$ \hfill (151) 

and

$$(S^*)^C = \frac{1}{2} g_{BD} \varphi_L \triangleright \partial^B \otimes (\bar{L}_\partial)^D_E \partial_t \triangleright \varphi_R^* g^{EC}$$

$$+ \frac{1}{2} \bar{q}^{-2} g_{BD} \varphi_L \triangleright \partial_t \otimes \partial^{B} (\bar{L}_\partial)^D_E \bar{\varphi}_R^* g^{EC}. $$ \hfill (152)

Once again, we can apply the rules in Eq. (123) of Chap. 5 to the results of this chapter. This way, we get further $q$-versions of the continuity equation for the energy density of a spin-zero particle.

### 7 Conservation of momentum

The expectation value for the momentum of a $q$-deformed spin-zero particle is given by the following expression:

$$\langle p^A \rangle = \frac{1}{2} \sum_{\varepsilon = \pm} \int d^3_q p \ h[p] \otimes p^A \otimes f[p]$$

$$= \frac{1}{2c^2} \int d^3_q x \varphi_L^* (x,t) \triangleright \partial_t \otimes \partial^A \triangleright \varphi_R(x,t)$$

$$+ \frac{1}{2c^2} \int d^3_q x \varphi_L^* (x,t) \triangleright \partial_t \otimes \partial_t \triangleright \varphi_R^* (x,t). $$ \hfill (153) 

Conjugating Eq. (153) leads to another expression for the expectation value of a $q$-deformed spin-zero particle [cf. Eq. (138) in App. A, Eqs. (201) and (212) in App. B, Eqs. (4) and (7) in Chap. 2, and Eq. (136) in Chap. 6]:

$$\langle p^*_A \rangle = \langle p^A \rangle = \frac{1}{2} \sum_{\varepsilon = \pm} \int d^3_q p \ f[p] \otimes p^A \otimes h[p]$$

$$= \frac{1}{2c^2} \int d^3_q x \varphi_L (x,t) \triangleright \partial_t \otimes \partial_A \triangleright \varphi_R^* (x,t)$$

$$+ \frac{1}{2c^2} \int d^3_q x \varphi_L (x,t) \triangleright \partial_A \otimes \partial_t \triangleright \varphi_R^* (x,t).$$ \hfill (154)
In Eqs. (153) and (154), we have written down the expectation value for the momentum of a $q$-deformed spin-zero particle, both in momentum space and in position space. We can derive the expression in momentum space from that in position space by the same reasonings we have already applied to the expectation value for the energy of a $q$-deformed spin-zero particle [cf. Eqs. (134) and (137) of the previous chapter].

We introduce the following expressions for the momentum density of a $q$-deformed spin-zero particle:

\[ i^A = \frac{1}{2c^2} \varphi^*_L \langle \partial_t \odot \partial^A \triangleright \varphi_R + \frac{1}{2c^2} \varphi^*_L \langle \partial^A \odot \partial_t \triangleright \varphi_R, \]

\[ i^*_A = \frac{1}{2c^2} \varphi_L \odot \partial_t \odot \partial_A \triangleright \varphi_R + \frac{1}{2c^2} \varphi_L \odot \partial_A \odot \partial_t \triangleright \varphi_R. \] (155)

Using Eqs. (4) and (7) in Chap. 2 together with Eq. (201) in Chap. B, we can show that the two expressions for the momentum density transform into each other by conjugation:

\[ \overrightarrow{i^A} = i^*_A. \] (156)

Next, we calculate the time derivative of the momentum density $i^A$:

\[ \partial_t \triangleright i^A = - \frac{1}{2c^2} \varphi^*_L \langle \partial_t \odot \partial_t \odot \partial^A \triangleright \varphi_R + \frac{1}{2c^2} \varphi^*_L \langle \partial_t \odot \partial^A \odot \partial_t \triangleright \varphi_R - \frac{1}{2c^2} \varphi^*_L \odot \partial_A \odot \partial_t \triangleright \varphi_R. \] (157)

We use the $q$-deformed Klein-Gordon equations for $\varphi_L$ and $\varphi^*_L$ [cf. Eqs. (1) and (6) in Chap. 2] to rewrite the first and last term on the right-hand side of the above equation:

\[ \partial_t \triangleright i^A = - \frac{1}{2} (\varphi^*_L \langle \partial^B \partial_B - \varphi^*_L (mc)^2 \rangle \odot \partial^A \triangleright \varphi_R + \frac{1}{2} \varphi^*_L \langle \partial^A \odot (\partial^B \partial_B \triangleright \varphi_R - (mc)^2 \varphi_R \rangle + \frac{1}{2c^2} \varphi^*_L \odot \partial^A \odot \partial_t \triangleright \varphi_R. \] (158)

By applying the identities in Eqs. (142) and (143) of the previous chapter, the first expression on the right-hand side takes on the following form:

\[ (\varphi^*_L \langle \partial^B \partial_B - \varphi^*_L (mc)^2 \rangle \odot \partial^A \triangleright \varphi_R = \]

\[ = (\varphi^*_L \langle \partial^A \partial^B \partial_B - \varphi^*_L (mc)^2 \rangle \odot \varphi_R + \partial^D \triangleright [(\varphi^*_L \langle \partial^B \partial_B - \varphi^*_L (mc)^2 \rangle (\mathcal{L}_0)^A_D ) \odot \varphi_R. \] (159)

Similar reasoning leads to

\[ \varphi^*_L \odot \partial_t \odot \partial^A \partial_t \triangleright \varphi_R = \varphi^*_L \odot \partial^A \partial_t \odot \partial_t \triangleright \varphi_R + \partial^D \triangleright [\varphi^*_L \odot \partial_t (\mathcal{L}_0)^A_D \odot \partial_t \triangleright \varphi_R] \] (160)
and
\[
\varphi_L^* \triangleleft \partial_A \otimes \partial_B \triangleright \varphi_R = \varphi_L^* \triangleleft \partial_A \partial_B \triangleright \varphi_R + \partial^D \triangleright [\varphi_L^* \triangleleft \partial_A (\mathcal{L}_D) \otimes \partial_B \triangleright \varphi_R] \\
= \varphi_L^* \triangleleft \partial_A \partial_B \triangleright \varphi_R + \partial^D \triangleright [\varphi_L^* \triangleleft \partial_A \partial_B \triangleright \varphi_R] \\
+ \partial^D \triangleright [\varphi_L^* \triangleleft \partial_A (\mathcal{L}_D) \otimes \partial_B \triangleright \varphi_R].
\] (161)

Putting these results together and considering the identity
\[
(\partial_B \partial_B - (mc)^2)(\mathcal{L}_D)^A_D = q^{-4}(\mathcal{L}_D)^A_D(\partial_B \partial_B - (mc)^2),
\] (162)
we can write the right-hand side of Eq. (158) as divergency. This way, we end up with the continuity equation
\[
\partial_t \triangleright i^A + \partial^D \triangleright T^A_D = 0,
\] (163)
where the stress-tensor takes on the following form:
\[
T^A_D = -\frac{1}{2} \varphi_L^* \triangleleft \partial_A \partial_B (\mathcal{L}_D)^C_D g_{BC} \otimes \varphi_R \\
+ \frac{1}{2q^4} \varphi_L^* \triangleleft (\mathcal{L}_D)^A_D \partial_B \partial_B \triangleright \varphi_R \\
- \frac{1}{2q^4} \varphi_L^* \triangleleft (\mathcal{L}_D)^A_D (mc)^2 \otimes \varphi_R \\
- \frac{1}{2q^2} \varphi_L^* \triangleleft \partial_A (\mathcal{L}_D)^B_D \otimes \partial_B \triangleright \varphi_R \\
- \frac{1}{2c^2} \varphi_L^* \triangleleft \partial_t (\mathcal{L}_D)^A_D \otimes \partial_t \triangleright \varphi_R.
\] (164)
Conjugating Eqs. (163) and (164) gives us another continuity equation, i.e.
\[
(i^*)_A \triangleright \partial_t + (T^*)^D_A \triangleright \partial_D = 0
\] (165)
and
\[
(T^*)^D_A = \frac{1}{2} \varphi_L \otimes (\mathcal{L}_D)^C_B \partial_B \partial_A \triangleright \varphi_R^* g^{CD} \\
+ \frac{1}{2q^4} g_{AE} \varphi_L \otimes \partial^B \partial_B (\mathcal{L}_D)^F_C \otimes \varphi_R^* g^{CD} \\
- \frac{1}{2q^4} g_{AE} \varphi_L \otimes (mc)^2 (\mathcal{L}_D)^F_C \otimes \varphi_R^* g^{CD} \\
- \frac{1}{2q^2} g_{BE} \varphi_L \triangleright \partial^B (\mathcal{L}_D)^F_C \otimes \partial_A \triangleright \varphi_R^* g^{CD} \\
- \frac{1}{2c^2} g_{AE} \varphi_L \triangleright \partial_t (\mathcal{L}_D)^F_C \otimes \partial_t \triangleright \varphi_R^* g^{CD}.
\] (166)
We can get further \(q\)-deformed continuity equations from the above results using the rules in Eq. (123) of Chap. 5.
A Star-products

The three-dimensional $q$-deformed Euclidean space $\mathbb{R}^3_q$ has the generators $X^+$, $X^3$, and $X^-$, subject to the following commutation relations [5]:

\[
\begin{align*}
X^3X^+ &= q^2X^+X^3, \\
X^3X^- &= q^{-2}X^-X^3, \\
X^-X^+ &= X^+X^- + (q - q^{-1})X^3X^3.
\end{align*}
\tag{167}
\]

We can extend the algebra of $\mathbb{R}^3_q$ by a time element $X^0$, which commutes with the generators $X^+$, $X^3$, and $X^-$ [14]:

\[
X^0X^A = X^AX^0, \quad A \in \{+, 3, -\}. \tag{168}
\]

In the following, $\mathbb{R}^{3,t}_q$ denotes the algebra generated by $X^i$ with $i \in \{0, +, 3, -\}$.

There is a $q$-version of the three-dimensional Euclidean metric $g_{AB}$ with its inverse $g^{AB}$ [5] (rows and columns are arranged in the order $+, 3, -$):

\[
g_{AB} = g^{AB} = \begin{pmatrix}
0 & 0 & -q \\
0 & 1 & 0 \\
-q^{-1} & 0 & 0
\end{pmatrix}.
\tag{169}
\]

We can use the $q$-deformed metric to raise and lower indices:

\[
X_A = g_{AB}X^B, \quad X^A = g^{AB}X_B. \tag{170}
\]

The algebra $\mathbb{R}^{3,t}_q$ has a semilinear, involutive, and anti-multiplicative mapping, which we call quantum space conjugation. If we indicate conjugate elements of a quantum space by a bar, we can write the properties of quantum space conjugation as follows ($\alpha, \beta \in \mathbb{C}$ and $u, v \in \mathbb{R}^{3,t}_q$):

\[
\overline{\alpha u + \beta v} = \overline{\alpha}u + \overline{\beta}v, \quad \overline{u} = u, \quad \overline{uv} = \overline{v}\overline{u}.
\tag{171}
\]

The commutation relations in Eq. (167) and Eq. (168) are invariant under conjugation if the following applies [14]:

\[
\overline{X^A} = X_A = g_{AB}X^B, \quad \overline{X^0} = X_0. \tag{172}
\]

We can write each element $F \in \mathbb{R}^{3,t}_q$ as an expansion in terms of normal-ordered monomials (Poincaré-Birkhoff-Witt property):

\[
F = \sum_{n_+, \ldots, n_0} a_{n_+, \ldots, n_0} (X^+)^{n_+}(X^3)^{n_3}(X^-)^{n_-}(X^0)^{n_0}, \quad a_{n_+, \ldots, n_0} \in \mathbb{C}. \tag{173}
\]

There is a vector space isomorphism

\[
\mathcal{W} : \mathbb{C}[x^+, x^3, x^-, t] \to \mathbb{R}^{3,t}_q
\tag{174}
\]

\[\text{A bar over a complex number indicates complex conjugation.}\]
with
\[ W((x^+)^{n_+}(x^3)^{n_3}(x^-)^{n_-}-t^{n_0}) = (X^+)^{n_+}(X^3)^{n_3}(X^-)^{n_-}(X^0)^{n_0}. \] (175)

In general, we have
\[ \mathbb{C}[x^+, x^3, x^-, t] \ni f \mapsto F \in \mathbb{R}^{3,t}, \] (176)
where
\[ f = \sum_{n_+, \ldots, n_0} a_{n_+ \ldots n_0} (x^+)^{n_+}(x^3)^{n_3}(x^-)^{n_-}-t^{n_0}, \]
\[ F = \sum_{n_+, \ldots, n_0} a_{n_+ \ldots n_0} (X^+)^{n_+}(X^3)^{n_3}(X^-)^{n_-}(X^0)^{n_0}. \] (177)

The vector space isomorphism \( W \) is nothing else than the Moyal-Weyl mapping giving an operator \( F \) to a complex-valued function \( f \) \cite{19} \cite{22}.

We can extend this vector space isomorphism to an algebra isomorphism. To this end, we introduce the so-called star-product, subject to the following homomorphism condition:
\[ W(f \odot g) = W(f) \cdot W(g). \] (178)

Since the Moyal-Weyl mapping is invertible, we can write the star-product as follows:
\[ f \odot g = W^{-1}(W(f) \cdot W(g)). \] (179)

The product of two normal-ordered monomials can again be written as an expansion in terms of normal-ordered monomials (see Ref. 23 for details):
\[ (X^+)^{n_+} \ldots (X^0)^{n_0} \cdot (X^+)^{m_+} \ldots (X^0)^{m_0} = \sum_{k=0}^{\infty} B^{n_+ \ldots n_0 m_+ \ldots m_0}_k (X^+)^k \ldots (X^0)^k. \] (180)

This expansion leads to a general formula for the star-product of two power series in commutative space-time coordinates (\( \lambda = q - q^{-1} \)):
\[ f(x, t) \odot g(x, t) = \sum_{k=0}^{\infty} \lambda^k \frac{(x^3)^{2k}}{[k]!} q^{2(n_3 \hat{n}_3 + \hat{n}_- \hat{n}_+)} D_{q^k, x} f(x, t) D_{q^k, x} g(x', t)|_{x' \rightarrow x}. \] (181)

The expression above depends on the operators
\[ \hat{n}_A = x^A \frac{\partial}{\partial x^A} \] (182)
and the so-called Jackson derivatives \cite{24}:
\[ D_{q^k, x} f = \frac{f(q^k x) - f(x)}{q^k x - x}. \] (183)
Moreover, the $q$-numbers are given by

$$[[a]]_q = \frac{1 - q^a}{1 - q},$$

and the $q$-factorials are defined in complete analogy to the undeformed case:

$$[[n]]_q! = [[1]]_q[[2]]_q \cdots [[n - 1]]_q[[n]]_q, \quad [[0]]_q! = 1.$$  \hspace{1cm} (184)

We require that $\mathcal{W}^{-1}$ is a $*$-algebra homomorphism. This way, we can define a new conjugation on the commutative space-time algebra $\mathbb{C}[x^+, x^3, x^-, t]$:

$$\mathcal{W}(\overline{f}) = \overline{\mathcal{W}(f)} \iff \overline{f} = \mathcal{W}^{-1}(\mathcal{W}(f)).$$

(186)

It follows from Eq. (172) and Eq. (186) that $\overline{f}$ takes the following form $[14, 25]$:

$$f(x, t) = \sum_n \overline{a}_{n_+, n_3, n_-} (-qx^-)^{n_+} (x^3)^{n_3} (-q^{-1}x^+)^{n_-} t^{n_0}$$

$$= \sum_n (-q)^{n_- n_+} \overline{a}_{n_-, n_1, n_3, n_0} (x^+)^{n_+} (x^3)^{n_3} (x^-)^{n_-} t^{n_0}$$

$$= \overline{f}(x, t).$$

(187)

The coefficient $\overline{a}_{n_+, n_3, n_-} n_0$ is the complex conjugate of $a_{n_+, n_3, n_-} n_0$. The star-product behaves as follows under quantum space conjugation:

$$\overline{f} \odot g = \overline{g} \odot \overline{f}.$$ \hspace{1cm} (188)

### B Partial derivatives and integrals

There are partial derivatives for $q$-deformed space-time coordinates $[20, 27]$. These $q$-deformed partial derivatives satisfy the same commutation relations as the covariant coordinate generators $X_i$:

$$\partial_0 \partial_+ = \partial_+ \partial_0, \quad \partial_0 \partial_- = \partial_- \partial_0, \quad \partial_0 \partial_3 = \partial_3 \partial_0,$$

$$\partial_+ \partial_3 = q^2 \partial_3 \partial_+, \quad \partial_3 \partial_- = q^2 \partial_- \partial_3,$$

$$\partial_+ \partial_- - \partial_- \partial_+ = \lambda \partial_3 \partial_3.$$ \hspace{1cm} (189)

The commutation relations above are invariant under conjugation if the following conjugation properties hold:

$$\overline{\partial_A} = -\partial^A = -g^{AB} \partial_B, \quad \overline{\partial_0} = -\partial^0 = -\partial_0.$$ \hspace{1cm} (190)

There are two ways of commuting $q$-deformed partial derivatives with $q$-deformed space-time coordinates. Concretely, we have the following $q$-deformed

\footnote{The indices of partial derivatives are raised and lowered in the same way as those of coordinates $[\text{see Eq. (170) in Chap. A}].$}
Leibniz rules\textsuperscript{[14, 20, 27]}:

\[
\begin{align*}
\partial_B X^A &= \delta_B^A + q^4 \hat{R}^{AC BD} X^D \partial_C, \\
\partial_A X^0 &= X^0 \partial_A, \\
\partial_0 X^A &= X^A \partial_0, \\
\partial_0 X^0 &= 1 + X^0 \partial_0.
\end{align*}
\] (191)

\(\hat{R}^{AC BD}\) denotes the vector representation of the R-matrix for the three-dimensional \(q\)-deformed Euclidean space. Introducing \(\hat{\partial}_A = q^\delta \partial_A\) and \(\hat{\partial}_0 = \partial_0\), we can write the Leibniz rules of the second differential calculus in the following form:

\[
\begin{align*}
\hat{\partial}_B X^A &= \delta_B^A + q^{-4} (\hat{R}^{-1})^{AC BD} X^D \hat{\partial}_C, \\
\hat{\partial}_A X^0 &= X^0 \hat{\partial}_A, \\
\hat{\partial}_0 X^A &= X^A \hat{\partial}_0, \\
\hat{\partial}_0 X^0 &= 1 + X^0 \hat{\partial}_0.
\end{align*}
\] (192)

Using the Leibniz rules in Eq. (191) or Eq. (192), we can calculate how partial derivatives act on normal-ordered monomials of noncommutative coordinates. We can carry over these actions to commutative coordinate monomials with the help of the Moyal-Weyl mapping:

\[
\partial^i \triangleright (x^+)^n (x^-)^{n_0} = W^{-1} (\partial^i \triangleright (x^+)^n (x^-)^{n_0}).
\] (193)

Since the Moyal-Weyl mapping is linear, we can apply the action above to a power series in commutative space-time coordinates:

\[
\partial^i \triangleright f(x, t) = W^{-1} (\partial^i \triangleright W(f(x, t))).
\] (194)

If we use the normal-ordered monomials in Eq. (175) of the previous chapter, the Leibniz rules in Eq. (191) lead to the following operator representations\textsuperscript{[10]}:

\[
\begin{align*}
\partial_+ \triangleright f(x, t) &= D_{q^4, x^+} f(x, t), \\
\partial_3 \triangleright f(x, t) &= D_{q^2, x^3} f(q^2 x^3, x^-, t), \\
\partial_- \triangleright f(x, t) &= D_{q^4, x^-} f(x^+, q^2 x^3, x^-, t) + \lambda x^+ D_{q^2, x^3} f(x, t).
\end{align*}
\] (195)

The derivative \(\partial_0\), however, is represented on the commutative space-time algebra by the standard time derivative:

\[
\partial_0 \triangleright f(x, t) = \frac{\partial f(x, t)}{\partial t}.
\] (196)

Using the Leibniz rules in Eq. (192), we get operator representations for the partial derivatives \(\hat{\partial}_i\). The Leibniz rules in Eq. (191) and Eq. (192) are transformed into each other by the following substitutions:

\[
q \rightarrow q^{-1}, \quad X^- \rightarrow X^+, \quad X^+ \rightarrow X^-, \\
\partial^+ \rightarrow \hat{\partial}^-, \quad \partial^- \rightarrow \hat{\partial}^+, \quad \partial^3 \rightarrow \hat{\partial}^3, \quad \partial^0 \rightarrow \hat{\partial}^0.
\] (197)
Thus, we obtain the operator representations of the partial derivatives \( \hat{\partial}_A \) from those of the partial derivatives \( \partial_A \) [cf. Eq. (195)] if we replace \( q \) by \( q^{-1} \) and exchange the indices \( + \) and \( - \):

\[
\begin{align*}
\hat{\partial}_- \triangleright f(x,t) &= D_{q^{-4},x^{-}} f(x,t), \\
\hat{\partial}_3 \triangleright f(x,t) &= D_{q^{-2},x^3} f(q^{-2}x^-, x^3, x^+, t), \\
\hat{\partial}_+ \triangleright f(x,t) &= D_{q^{-4},x^+} f(x^-, q^{-2}x^3, x^+, t) - \lambda x^- D^2_{q^{-2},x^3} f(x,t). 
\end{align*}
\tag{198}
\]

Once again, \( \hat{\partial}_0 \) is represented on the commutative space-time algebra by the standard time derivative:

\[
\hat{\partial}_0 \triangleright f(x,t) = \frac{\partial f(x,t)}{\partial t}.
\tag{199}
\]

Due to the substitutions given in Eq. (197), the actions in Eqs. (198) and (199) refer to normal-ordered monomials different from those in Eq. (175) of the previous chapter:

\[
\tilde{\mathcal{W}} \left( t^{n_0}(x^+)^{n_+}(x^3)^{n_3}(x^-)^{n_-} \right) = (X^0)^{n_0}(X^-)^{n_-}(X^3)^{n_3}(X^+)^{n_+}.
\tag{200}
\]

We can also commute \( q \)-deformed partial derivatives from the right side of a normal-ordered monomial to the left side using the Leibniz rules. This way, we get the so-called right-representations of partial derivatives, for which we write \( f \blacktriangleright \partial_i \) or \( f \blacktriangleleft \hat{\partial}_i \). Conjugation transforms left actions of partial derivatives into right actions and vice versa [16]:

\[
\begin{align*}
\overline{\partial^i} &\triangleright f = -\bar{f} \triangleleft \partial_i, \\
\overline{\partial^i} &\blacktriangleright f = -\bar{f} \triangleright \hat{\partial}_i, \\
\overline{f} &\blacktriangleright \partial_i = -\partial_i \blacktriangleright \bar{f}, \\
\overline{f} &\blacktriangleright \hat{\partial}_i = -\hat{\partial}_i \blacktriangleright \overline{f}.
\end{align*}
\tag{201}
\]

The operator representations in Eqs. (195) and (198) consist of two terms which we call \( \partial_A^{\text{cla}} \) and \( \partial_A^{\text{cor}} \):

\[
\partial^A \triangleright F = (\partial_A^{\text{cla}} + \partial_A^{\text{cor}}) \triangleright F.
\tag{202}
\]

In the undeformed limit \( q \to 1 \), \( \partial_A^{\text{cla}} \) becomes a standard partial derivative, and \( \partial_A^{\text{cor}} \) disappears. We get a solution to the difference equation \( \partial^A \triangleright F = f \) by using the following formula [28]:

\[
F = \left( \partial_A^{\text{cla}} + \partial_A^{\text{cor}} \right)^{-1} \triangleright f = \sum_{k=0}^{\infty} \left[ -\left( \partial_A^{\text{cla}} + \partial_A^{\text{cor}} \right) \right]^k \left( \partial_A^{\text{cla}} \right)^{-1} \triangleright f.
\tag{203}
\]

Applying the above formula to the operator representations in Eq. (195), we get

\[
\begin{align*}
(\partial_+)^{-1} \triangleright f(x,t) &= D_{q^{-1},x^+} f(x,t), \\
(\partial_3)^{-1} \triangleright f(x,t) &= D_{q^{-1},x^3} f(q^{-2}x^+, x^3, x^-, t), \\
(\partial_-)^{-1} \triangleright f(x,t) &= D_{q^{-1},x^-} f(q^{-2}x^-, x^3, x^+, t).
\end{align*}
\tag{204}
\]
and

\[ (\partial_-)^{-1} \triangleright f(x, t) = \sum_{k=0}^{\infty} q^{2k+1} \left( -\lambda x^+ D_{q^2 x^+}^{-1} D_{q^2 x^3}^2 \right)^k D_{q^3 x^-}^{-1} f(x^+, q^{-2(k+1)x^3}, x^-, t). \] (205)

Note that \( D_{q^3 x^-}^{-1} \) stands for a Jackson integral, with \( x \) being the variable of integration \[29\]. The explicit form of this Jackson integral depends on its limits of integration and the value for the deformation parameter \( q \). If \( x > 0 \) and \( q > 1 \), for example, the following applies:

\[ \int_0^x dz f(z) = (q-1)x \sum_{j=1}^{\infty} q^{-j} f(q^{-j}x). \] (206)

The integral for the time coordinate has the same form as in the undeformed case [cf. Eq. (196)]:

\[ (\partial_0)^{-1} \triangleright f(x, t) = \int dt f(x, t). \] (207)

The considerations above also apply to the partial derivatives with a hat. However, we can obtain the representations of \( \hat{\partial}_i \) from those of \( \partial_i \) if we replace \( q \) with \( q^{-1} \) and exchange the indices + and −. Applying these substitutions to the expressions in Eqs. (204) and (205), we immediately get the corresponding results for the partial derivatives \( \hat{\partial}_i \).

By successively applying the integral operators given in Eqs. (204) and (205), we can define an integration over all space \[25,28\]:

\[ \int_{-\infty}^{+\infty} d^3 q x f(x^+, x^3, x^-) = (\partial_-)^{-1}\int_{-\infty}^{+\infty} d^3 q x^+ \int_{-\infty}^{+\infty} d^3 q x^- \int_{-\infty}^{+\infty} d^3 q x^3 \triangleright f. \] (208)

On the right-hand side of the above relation, the different integral operators can be simplified to Jackson integrals \[28,30\]:

\[ \int d^3 q x f = \int_{-\infty}^{+\infty} d^3 q x f(x) = D_{q^3 x^-}^{-1} D_{q^2 x^3}^{-1} D_{q^3 x^-}^{-1} f(x). \] (209)

In the above formula, the Jackson integrals refer to a smaller \( q \)-lattice. Using such a smaller \( q \)-lattice ensures that the integral over all space is a scalar with trivial braiding properties \[18\].

There are \( q \)-deformed versions of Stokes’ theorem for the \( q \)-integral over all space \[25,30\]:

\[ \int_{-\infty}^{+\infty} d^3 q x \partial^A \triangleright f = \int_{-\infty}^{+\infty} d^3 q x f \triangleright \partial^A = 0, \]

\[ \int_{-\infty}^{+\infty} d^3 q x \hat{\partial}^A \triangleright f = \int_{-\infty}^{+\infty} d^3 q x f \triangleright \hat{\partial}^A = 0. \] (210)
Figure 1: Eigenvalue equations of $q$-exponentials.

The $q$-deformed Stokes’ theorem implies rules for integration by parts:

$$
\int_{-\infty}^{+\infty} d^3 q \ x f \odot (\partial^A \triangleright g) = \int_{-\infty}^{+\infty} d^3 q \ x (f \triangleleft \partial^A) \odot g,
$$

$$
\int_{-\infty}^{+\infty} d^3 q \ x f \odot (\hat{\partial}^A \triangleright g) = \int_{-\infty}^{+\infty} d^3 q \ x (f \triangleleft \hat{\partial}^A) \odot g.
$$

Finally, the $q$-integral over all space behaves as follows under quantum space conjugation:

$$
\int_{-\infty}^{+\infty} d^3 q \ x f = \int_{-\infty}^{+\infty} d^3 q \ x \bar{f}.
$$

C Exponentials and Translations

An exponential of a $q$-deformed quantum space is an eigenfunction of each $q$-deformed partial derivative \[31-33\]. In the following, we consider $q$-deformed exponentials that are eigenfunctions for left actions or right actions of partial derivatives:

$$
i^{-1} \partial^A \triangleright \exp_q(x|ip) = \exp_q(x|ip) \odot p^A,
$$

$$
\exp_q(i^{-1}p|x) \triangleleft \partial^{A^{-1}} = p^A \odot \exp_q(i^{-1}p|x).
$$

The above eigenvalue equations are shown graphically in Fig. 1. The $q$-exponentials are defined by their eigenvalue equations and the following normalization conditions:

$$
\exp_q(x|ip)|_{x=0} = \exp_q(x|ip)|_{p=0} = 1,
$$

$$
\exp_q(i^{-1}p|x)|_{x=0} = \exp_q(i^{-1}p|x)|_{p=0} = 1.
$$

Using the operator representation in Eq. (195) of the last chapter, we have found the following expressions for the $q$-exponentials of three-dimensional Eu-
clidean quantum space \([33]\):

\[
\exp_q(i^{-1}p|x|) = \sum_{n=0}^{\infty} \frac{(i^{-1}p^+)^n(i^{-1}p^3)^n(i^{-1}p^-)^n(x_-)^n-(x_3)^n(x_+)^n}{[[n_+]]_{q^n}[[n_3]]_{q^n}[[n_-]]_{q^n}},
\]

\[
\exp_q(x|ip) = \sum_{n=0}^{\infty} \frac{(x^+)^n(x^3)^n(x^-)^n-(ip^-)^n-(ip_3)^n(ip_+)^n}{[[n_+]]_{q^n}[[n_3]]_{q^n}[[n_-]]_{q^n}}.
\]

(215)

If we substitute \( q \) with \( q^{-1} \) in both expressions of Eq. (215), we get two more \( q \)-exponentials which we designate \( \exp_{q^{-1}}(x|ip) \) and \( \exp_{q^{-1}}(i^{-1}p|x) \). We obtain the eigenvalue equations and normalization conditions of these two \( q \)-exponentials by applying the following substitutions to Eqs. (213) and (214):

\[
\exp_q \rightarrow \exp_{q^{-1}}, \quad \triangleright \rightarrow \check{\triangleright}, \quad \triangleright \rightarrow \check{\triangleright}, \quad \partial A \rightarrow \check{\partial} A.
\]

(216)

We can use \( q \)-exponentials to calculate \( q \)-translations \([34]\). If we replace the momentum coordinates in the expressions for \( q \)-exponentials with derivatives, it applies \([25, 31, 35]\):

\[
\exp_q(x|\partial y) \triangleright g(y) = g(x \triangleright y),
\]

\[
\exp_{q^{-1}}(x|\check{\partial} y) \check{\triangleright} g(y) = g(x \check{\triangleright} y),
\]

(217)

and

\[
g(y) \check{\triangleright} \exp_q(-\partial y|x) = g(y \triangleright x),
\]

\[
g(y) \check{\triangleright} \exp_{q^{-1}}(-\check{\partial} y|x) = g(y \check{\triangleright} x).
\]

(218)

In the case of the three-dimensional \( q \)-deformed Euclidean space, for example, we can get the following formula for calculating \( q \)-translations \([36]\):

\[
f(x \triangleright y) = \sum_{i_+ = 0}^{\infty} \sum_{i_3 = 0}^{\infty} \sum_{i_- = 0}^{\infty} \sum_{k = 0}^{\infty} \frac{(-q^{-1}\lambda_{i_+}^k)(x^-)^i-(x^3)^i-k(x^+)^i+k(y^-)^i}{[[2k]]_{q^{-2k}}[[i_+ - k]]_{q^{-k}}[[i_-]]_{q^{-i_-}}} \times (D_{i_+, i_3, k}^{i_-, i_- ; k} \check{\partial}^{i_+} y^k f)(q^{2k-i_3} y^- , q^{-2i_+} f). \]

(219)

In analogy to the undeformed case, \( q \)-exponentials satisfy the addition theorems \([25, 31, 32]\):

\[
\exp_q(x \oplus y|ip) = \exp_q(x|\exp_q(y|ip) \oplus ip),
\]

\[
\exp_q(ix|p \oplus p') = \exp_q(x \oplus \exp_q(x|ip)|ip'),
\]

(220)

and

\[
\exp_{q^{-1}}(x \oplus y|ip) = \exp_{q^{-1}}(x|\exp_{q^{-1}}(y|ip) \oplus ip),
\]

\[
\exp_{q^{-1}}(ix|p \oplus p') = \exp_{q^{-1}}(x \oplus \exp_{q^{-1}}(x|ip)|ip').
\]

(221)
We can obtain further addition theorems from the above identities by substituting position coordinates with momentum coordinates and vice versa. In Fig. 2, we have given graphic representations of the two addition theorems in Eq. (220).

The $q$-deformed quantum spaces considered so far are so-called braided Hopf algebras [37]. From this point of view, the two versions of $q$-translations are nothing else than the representations of two braided co-products $\Delta$ and $\bar{\Delta}$ on the corresponding commutative coordinate algebras [25]:

$$f(x \oplus y) = ((W^{-1} \otimes W^{-1}) \circ \Delta)(W(f)),$$

$$f(x \bar{\oplus} y) = ((W^{-1} \otimes W^{-1}) \circ \bar{\Delta})(W(f)). \tag{222}$$

The braided Hopf algebras have braided antipodes $S$ and $\bar{S}$ as well. We can represent these antipodes on the corresponding commutative algebras, too:

$$f(\ominus x) = (W^{-1} \circ S)(W(f)),$$

$$f(\bar{\ominus} x) = (W^{-1} \circ \bar{S})(W(f)). \tag{223}$$

In the following, we refer to the operations in Eq. (223) as $q$-inversions. In the case of the $q$-deformed Euclidean space, for example, we have found the following operator representation for $q$-inversions [36]:

$$\hat{U}^{-1}f(\ominus x) =$$

$$= \sum_{i=0}^{\infty} (-q^{\lambda\lambda_+})^i \frac{(x^{+}x^{-})^i}{[2i]_{q^{-2}}!!} q^{\frac{2n_+(\bar{n}_+ + \bar{n}_3) - 2n_-(\bar{n}_- + \bar{n}_3) - \bar{n}_3n_3}{2}}$$

$$\times D_{q^{-2}, \bar{x}^3}^{2i} f(-q^{2-4i}x^{-} - q^{1-2i}x^{3} - q^{2-4i}x^{+}). \tag{224}$$

The operators $\hat{U}$ and $\hat{U}^{-1}$ act on a commutative function $f(x^+, x^3, x^-)$ as follows.
[cf. Eq. (182) in App. A]:
\[
\hat{U} f = \sum_{k=0}^{\infty} (-\lambda)^k \left( x^3 \right)^{2k} \left[ \frac{k}{[k]} \right] q^{-2n_3(n_+ + \bar{n}_-) + k} q^{-4n_3(n_+ + \bar{n}_-) + k} \frac{D}{Dx^+} f,
\]
\[
\hat{U}^{-1} f = \sum_{k=0}^{\infty} \lambda^k \left( x^3 \right)^{2k} \left[ \frac{k}{[k]} \right] q^{-2n_3(n_+ + \bar{n}_-) + k} q^{-4n_3(n_+ + \bar{n}_-) + k} \frac{D}{Dx^-} f.
\] (225)

The braided co-products and braided antipodes satisfy the axioms (also see Ref. [37])
\[
m \circ (S \otimes \text{id}) \circ \Delta = m \circ (\text{id} \otimes S) \circ \Delta = \varepsilon,
\]
\[
m \circ (\bar{S} \otimes \text{id}) \circ \bar{\Delta} = m \circ (\text{id} \otimes \bar{S}) \circ \bar{\Delta} = \bar{\varepsilon}.
\] (226)

and
\[
(\text{id} \otimes \bar{\varepsilon}) \circ \bar{\Delta} = \text{id} = (\bar{\varepsilon} \otimes \text{id}) \circ \bar{\Delta},
\]
\[
(\text{id} \otimes \varepsilon) \circ \Delta = \text{id} = (\varepsilon \otimes \text{id}) \circ \Delta.
\] (227)

In the identities above, we denote multiplication on the braided Hopf algebra by $m$. Both co-units $\varepsilon, \bar{\varepsilon}$ of the two braided Hopf structures are linear mappings vanishing on the coordinate generators:
\[
\varepsilon(X^i) = \bar{\varepsilon}(X^i) = 0.
\] (228)

For this reason, we can represent the co-units $\varepsilon, \bar{\varepsilon}$ on a commutative coordinate algebra as follows:
\[
\varepsilon(W(f)) = \bar{\varepsilon}(W(f)) = f(x)|_{x=0} = f(0).
\] (229)

Next, we translate the Hopf algebra axioms in Eqs. (226) and (227) into corresponding rules for $q$-translations and $q$-inversions [25], i.e.
\[
f((\oplus x) \oplus x) = f(x \oplus (\oplus x)) = f(0),
\]
\[
f((\oplus x) \oplus x) = f(x \oplus (\oplus x)) = f(0),
\] (230)

and
\[
f(x \oplus y)|_{y=0} = f(x) = f(y \oplus x)|_{y=0},
\]
\[
f(x \oplus y)|_{y=0} = f(x) = f(y \oplus x)|_{y=0}.
\] (231)

Using $q$-inversions, we are also able to introduce inverse $q$-exponentials:
\[
\exp_q(\bar{\ominus} x|ip) = \exp_q(ix|\bar{\ominus} p).
\] (232)

Due to the addition theorems and the normalization conditions of our $q$-exponentials, the following holds:
\[
\exp_q(ix \oplus \exp_q(\bar{\ominus} x|ip) \oplus p) = \exp_q(x \ominus (\bar{\ominus} x)|ip) = \exp_q(x|ip)|_{x=0} = 1.
\] (233)
In Fig. 3 we have given graphic representations of these identities. The conjugate \( q \)-exponentials are subject to similar rules obtained from the above identities by using the following substitutions:

\[
\exp_q \rightarrow \overline{\exp}_q, \quad \overline{\oplus} \rightarrow \oplus, \quad \overline{\ominus} \rightarrow \ominus.
\]

Next, we describe another way of obtaining \( q \)-exponentials. We exchange the two tensor factors of a \( q \)-exponential using the inverse of the so-called universal R-matrix (also see the graphic representation in Fig. 4):

\[
\exp_q^* (i p | x) = \tau \circ [(\mathcal{R}_{[2]}^{-1} \otimes \mathcal{R}_{[1]}^{-1}) \triangleright \exp_q (i x | \ominus p)],
\]
\[
\exp_q^* (x | i p) = \tau \circ [(\mathcal{R}_{[2]}^{-1} \otimes \mathcal{R}_{[1]}^{-1}) \triangleright \exp_q (\ominus p | i x)].
\]

In the expressions above, \( \tau \) denotes the ordinary twist operator. One can show that the new \( q \)-exponentials satisfy the following eigenvalue equations (see Fig. 4):

\[
\exp_q^* (i p | x) \triangleleft \hat{\partial}^A = ip^A \triangleright \exp_q^* (i p | x),
\]
\[
\hat{\partial}^A \triangleright \hat{\exp}_q^* (x | i^{-1} p) = \exp_q^* (x | i^{-1} p) \triangleright ip^A.
\]

Similar considerations apply to the conjugate \( q \)-exponentials. We only need to modify Eqs. (235) and (236) by performing the following substitutions:

\[
\exp_q^* \rightarrow \overline{\exp}_q^*, \quad \mathcal{R}_{[2]}^{-1} \otimes \mathcal{R}_{[1]}^{-1} \rightarrow \mathcal{R}_{[1]} \otimes \mathcal{R}_{[2]}, \quad \ominus \rightarrow \overline{\ominus},
\]
\[
\overline{\ominus} \rightarrow \ominus, \quad \triangleleft \rightarrow \triangleright, \quad \hat{\partial}^A \rightarrow \hat{\partial}^A.
\]

The \( q \)-exponentials in Eq. (235) are related to the conjugate \( q \)-exponentials. To show this, we rewrite the eigenvalue equations in using the identity \( \hat{\partial}^A = q^6 \partial^A \):

\[
\exp_q^* (i p | x) \triangleleft \hat{\partial}^A = iq^6 p^A \triangleright \exp_q^* (i p | x),
\]
\[
\hat{\partial}^A \triangleright \hat{\exp}_q^* (x | i^{-1} p) = \exp_q^* (x | i^{-1} p) \triangleright iq^6 p^A.
\]

---

\( ^6 \)You find some explanations of this sort of graphical calculations in Ref. [38].
These are the eigenvalue equations for \( \exp_q(i^{-1}q^6p|x) \) and \( \exp_q(x|i^{-1}q^6p) \), so the following identifications are valid:

\[
\begin{align*}
\exp_q^*(ip|x) &= \exp_q(i^{-1}q^6p|x), \\
\exp_q^*(x|i^{-1}p) &= \exp_q(x|i^{-1}q^6p).
\end{align*}
\]

For the sake of completeness, we write down how the \( q \)-exponentials of the \( q \)-deformed Euclidean space behave under quantum space conjugation:

\[
\begin{align*}
\exp_q^*(x|ip) &= \exp_q(i^{-1}p|x), \\
\exp_q^*(ip|x) &= \exp_q(x|i^{-1}p).
\end{align*}
\]

**D Hopf structures and L-matrices**

The three-dimensional \( q \)-deformed Euclidean space \( \mathbb{R}_q^3 \) is a three-dimensional representation of the Drinfeld-Jimbo algebra \( U_q(su_2) \). The latter is a deformation of the universal enveloping algebra of the Lie algebra \( su_2 \) [39]. Accordingly, the algebra \( U_q(su_2) \) has the three generators \( T^+ \), \( T^- \), and \( T^3 \), subject to the following relations [40]:

\[
\begin{align*}
q^{-1}T^+T^- - qT^-T^+ &= T^3, \\
q^2T^3T^+ - q^{-2}T^+T^3 &= (q + q^{-1})T^+, \\
q^2T^-T^3 - q^{-2}T^3T^- &= (q + q^{-1})T^-.
\end{align*}
\]

These relations are compatible with the following conjugation assignment:

\[
\begin{align*}
T^+ &= q^{-2}T^-, \\
T^- &= q^2T^+, \\
T^3 &= T^3.
\end{align*}
\]

The algebra of the \( q \)-deformed partial derivatives \( \partial^A \), \( A \in \{+, 3, -\} \), together with \( U_q(su_2) \) form the cross-product algebra \( \mathbb{R}_q^3 \times U_q(su_2) \) [11, 12]. We know that the algebra \( \mathbb{R}_q^3 \times U_q(su_2) \) is a Hopf algebra [13]. Accordingly, the \( q \)-deformed partial derivatives as elements of \( \mathbb{R}_q^3 \times U_q(su_2) \) have a co-product, an antipode, and a co-unit.

There are two ways of choosing the Hopf structure of the \( q \)-deformed partial derivatives. The two different co-products of the \( q \)-deformed partial derivatives are related to the two versions of Leibniz rules given in Eq. [191] or Eq. [192].
We can generalize these Leibniz rules by introducing the L-matrices $L_\partial$ and $\tilde{L}_\partial$ ($u \in \mathbb{R}^3_q$):

$$\partial^A u = (\partial^A (1) \triangleright u) \partial^A (2) = \partial^A \triangleright u + ((L_\partial)^A_B \triangleright u) \partial^B,$$

$$\hat{\partial}^A u = (\hat{\partial}^A (1) \triangleright u) \hat{\partial}^A (2) = \hat{\partial}^A \triangleright u + ((\tilde{L}_\partial)^A_B \triangleright u) \hat{\partial}^B.$$  \hfill (243)

From the above identities, you can see that the two L-matrices determine the two co-products\textsuperscript{7} of the q-deformed partial derivatives [15]:

$$\partial^A (1) \triangleright \partial^A (2) = \partial^A \otimes 1 + (L_\partial)^A_B \otimes \partial^B,$$

$$\hat{\partial}^A (1) \triangleright \hat{\partial}^A (2) = \hat{\partial}^A \otimes 1 + (\tilde{L}_\partial)^A_B \otimes \hat{\partial}^B.$$  \hfill (244)

The entries of the two L-matrices consist of generators of the Hopf algebra $U_q(su_2)$ and powers of a unitary scaling operator $\Lambda$ [also see Eq. (248)]. For this reason, the L-matrices can act on any element of $\mathbb{R}^3_q$. In this respect, an element of $\mathbb{R}^3_q$ has trivial braiding if the L-matrices act on it as follows:

$$(L_\partial)^A_B \triangleright u = \delta^A_B u,$$

$$(\tilde{L}_\partial)^A_B \triangleright u = \delta^A_B u.$$  \hfill (245)

Moreover, the two L-matrices transform into each other by conjugation:

$$(L_\partial)^A_B = g_{AC} (L_\partial)^C_D g^{DB}, \quad (\tilde{L}_\partial)^A_B = g_{AC} (\tilde{L}_\partial)^C_D g^{DB}.$$  \hfill (246)

In Ref. [16] and Ref. [44], we have written down the co-products of the partial derivatives $\partial^A$ or $\hat{\partial}^A$, $A \in \{+, 3, -\}$. By taking into account Eq. (244), you can read off the entries of the L-matrices $L_\partial$ and $\tilde{L}_\partial$ from these co-products. You find, for example:\textsuperscript{8}

$$(L_\partial)^- = \Lambda^{1/2} \tau^{-1/2} \text{ and } (\tilde{L}_\partial)^+ = \Lambda^{-1/2} \tau^{-1/2}.$$  \hfill (247)

The scaling operator $\Lambda$ acts on the spatial coordinates or the corresponding partial derivatives as follows:

$$\Lambda \triangleright X^A = q^4 X^A, \quad \Lambda \triangleright \partial^A = q^{-4} \partial^A.$$  \hfill (248)

These actions imply the commutation relations

$$\Lambda X^A = q^4 X^A \Lambda, \quad \Lambda \partial^A = q^{-4} \partial^A \Lambda$$  \hfill (249)

if we take into account the Hopf structure of $\Lambda$ [15]:

$$\Delta(\Lambda) = \Lambda \otimes \Lambda, \quad S(\Lambda) = \Lambda^{-1}, \quad \varepsilon(\Lambda) = 1.$$  \hfill (250)

\textsuperscript{7} We write the co-product in the so-called Sweedler notation, i.e. $\Delta(a) = a(1) \otimes a(2)$.

\textsuperscript{8} Instead of $T^3$, one often uses $\tau = 1 - (q - q^{-1}) T^3$. 

40
The Hopf structure of the partial derivatives also includes an antipode and a co-unit. Regarding the co-unit of the partial derivatives, the following holds [14]:

\[ \varepsilon(\partial^A) = 0. \]

We can obtain the antipodes of the partial derivatives from their co-products using the following Hopf algebra axioms:

\[ a_{(1)} \cdot S(a_{(2)}) = \varepsilon(a) = S(a_{(1)}) \cdot a_{(2)}. \] (251)

Due to this axiom, we have:

\[ S(\partial^A) = -S(L_{\partial})^A_B \partial^B, \quad S(\hat{\partial}^A) = -S^{-1}(L_{\partial})^A_B \hat{\partial}^B. \] (252)

For example, we get the following expressions for the antipodes of the partial derivatives \( \partial^- \) and \( \hat{\partial}^+ \) (also see Ref. [16]):

\[ S(\partial^-) = -\Lambda^{-1/2} \mu^{1/2} \partial^-, \quad S(\hat{\partial}^+) = -\Lambda^{1/2} \mu^{-1/2} \hat{\partial}^+. \] (253)

With the help of the antipodes of partial derivatives, we can write the left actions of partial derivatives as right actions and vice versa. Concretely, we have

\[ \partial^A \triangleright f = f \triangleleft S(\partial^A) = -(f \triangleleft S(L_{\partial})^A_B) \triangleleft \partial^B \]
\[ = -(L_{\partial})^A_B \partial^B, \]
\[ \hat{\partial}^A \triangleright f = f \triangleleft S(\hat{\partial}^A) = -(f \triangleleft S(L_{\partial})^A_B) \triangleleft \hat{\partial}^B \]
\[ = -(L_{\partial})^A_B \hat{\partial}^B, \] (254)

and

\[ f \triangleleft \hat{\partial}^A = S^{-1}(\hat{\partial}^A) \triangleright f = -\hat{\partial}^B \triangleright (S^{-1}(L_{\partial})^A_B \triangleright f) \]
\[ = -\hat{\partial}^B \triangleright (f \triangleleft (L_{\partial})^A_B), \]
\[ f \triangleright \partial^A = S^{-1}(\partial^A) \triangleright f = -\partial^B \triangleright (S^{-1}(L_{\partial})^A_B \triangleright f) \]
\[ = -\partial^B \triangleright (f \triangleleft (L_{\partial})^A_B). \] (255)

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