DISTINGUISHED ORBITS AND THE L-S CATEGORY OF SIMPLY CONNECTED COMPACT LIE GROUPS

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ABSTRACT. We show that the Lusternik-Schnirelmann category of a simple, simply connected, compact Lie group $G$ is bounded above by the sum of the relative categories of certain distinguished conjugacy classes in $G$ corresponding to the vertices of the fundamental alcove for the action of the affine Weyl group on the Lie algebra of a maximal torus of $G$.

1. Introduction

1.1. The (normalized) Lusternik-Schnirelmann category of a topological space $X$, denoted $\text{cat}(X)$, is the least integer $m$ such that $X$ can be covered by $m + 1$ open sets that are contractible in $X$. One of the problems on Ganea’s list ([3]) from 1971 asks to find the L-S category of (compact) Lie groups. In 1975, Singhof ([9]) proved that $\text{cat}(\text{SU}(n + 1)) = n$. For the other families of simply connected compact Lie groups, the answer is only known when the rank is small (cf. [7] for a nice summary of what is known for simply connected and non-simply connected compact Lie groups of small rank.)

1.2. The purpose of this short note is to show that the L-S category of a simple, simply connected, compact Lie group $G$ is bounded above by the sum of the relative categories of certain distinguished conjugacy classes in $G$. More precisely, suppose $\{v_0, \ldots, v_n\}$ are the vertices of the fundamental alcove for the action of the affine Weyl group on the Lie algebra of a maximal torus of $G$. For $0 \leq k \leq n$, let $O_k$ be the conjugacy class of $\exp v_k$ in $G$. Then we will show in Section 4 that

$$\text{cat}(G) + 1 \leq \sum_{k=0}^{n} \left( \text{cat}_{G}(O_k) + 1 \right),$$

where $\text{cat}_{G}(O_k)$ is the relative L-S category of $O_k$ in $G$. (If $Y \subseteq X$ is a topological subspace, $\text{cat}_{X}(Y)$ is the least integer $m$ such that there there is a covering of $Y$ by $m + 1$ open subsets of $X$, each contractible in $X$.)

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1.3. For $G = \text{SU}(n + 1)$, the conjugacy classes $O_k$ turn out to be the points of the center of $G$ and we recover Singhof’s result that $\text{cat}(\text{SU}(n + 1)) \leq n$. For $G = \text{Sp}(n)$, we conjecture that $\text{cat}_G(O_k) \leq \min\{k, n - k\}$ (with respect to an appropriate numbering) which would imply that

$$\text{cat}(\text{Sp}(n)) \leq \left\lfloor \frac{(n + 2)^2}{4} \right\rfloor - 1.$$ 

Thus for $n = 1, 2, 3, 4, 5, 6, \text{etc.}$ our conjectured upper bound is $1, 3, 5, 8, 11, 15, \text{etc.}$ For $n = 1, 2, 3, 4$ it is known ([2]) that $\text{cat}(\text{Sp}(n)) = 1, 3, 5$. Also, for $n = 1, 2, 3, 4, 5, 6$ it is known ([5]) that $\text{cat}(\text{Spin}(2n + 1)) = 1, 3, 5, 8$. Based on this small set of data, we conjecture that $\text{cat}(\text{Sp}(n)) = \text{cat}(\text{Spin}(2n + 1))$ and that the inequality above is in fact an equality. We remark that the best known lower bound is $\text{cat}(\text{Sp}(n)) \geq n + 2$ for $n \geq 3$ ([2], [6]).

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2. Notation

2.1. Let $G$ be a simple, simply connected, compact Lie group with Lie algebra $\mathfrak{g}$. Let $T$ be a maximal torus of $G$ with Lie algebra $\mathfrak{t}$. Then $\mathfrak{h} = \mathfrak{t}_C$ is a Cartan subalgebra of $\mathfrak{g}_C$ with $\mathfrak{h}_\mathbb{R} = \mathfrak{t}$. Write $\Delta = \Delta(\mathfrak{g}_C, \mathfrak{h})$ for the set of roots and choose a positive system $\Delta^+$ with corresponding set of simple roots $\Pi = \{\alpha_1, \ldots, \alpha_n\}$. With respect to this system, write $\alpha_0$ for the highest root. For the classical Lie groups and with respect to standard notation, $\Pi$ and $\alpha_0$ can be taken as in the following table:

| $G$         | $\Pi$                          | $\alpha_0$      |
|------------|--------------------------------|-----------------|
| $\text{SU}(n + 1)$ | $\{\alpha_i = \varepsilon_i - \varepsilon_{i+1} \mid 1 \leq i \leq n\}$ | $\varepsilon_1 - \varepsilon_{n+1}$ |
| $\text{Sp}(n)$ | $\{\alpha_i = \varepsilon_i - \varepsilon_{i+1} \mid 1 \leq i \leq n - 1\} \cup \{\alpha_n = 2\varepsilon_n\}$ | $2\varepsilon_1$ |
| $\text{Spin}(2n + 1)$ | $\{\alpha_i = \varepsilon_i - \varepsilon_{i+1} \mid 1 \leq i \leq n - 1\} \cup \{\alpha_n = \varepsilon_n\}$ | $\varepsilon_1 + \varepsilon_2$ |
| $\text{Spin}(2n)$ | $\{\alpha_i = \varepsilon_i - \varepsilon_{i+1} \mid 1 \leq i \leq n - 1\} \cup \{\alpha_n = \varepsilon_{n-1} + \varepsilon_n\}$ | $\varepsilon_1 + \varepsilon_2$ |

2.2. Write $R^\vee$ for the coroot lattice in $\mathfrak{h}$ (which is the same as the dual to the weight lattice in $\mathfrak{h}^*$) so that

$$R^\vee = \text{span}_\mathbb{Z}\{h_\alpha \mid \alpha \in \Delta\}.$$ 

Here $h_\alpha = 2u_\alpha / B(u_\alpha, u_\alpha) \in \mathfrak{h}_\mathbb{R}$ where $B(\cdot, \cdot)$ is the Killing form and $u_\alpha \in \mathfrak{h}_\mathbb{R}$ is uniquely determined by the equation $\alpha(H) = B(H, u_\alpha)$ for all $H \in \mathfrak{h}_\mathbb{R}$. Since $G$ is simply connected,
it follows that
\[ \ker(\exp|_t) = 2\pi i R^\vee. \]

2.3. The connected components of
\[ \{ t \in t \mid \alpha(t) \notin 2\pi i \mathbb{Z} \text{ for } \alpha \in \Delta \} \]
are called *alcoves*. Write \( W = W(G,t) \) for the Weyl group of \( G \) with respect to \( t \) viewed as acting on \( t \) (and extended to \( \mathfrak{h} \) as needed). The *affine Weyl group*, \( \hat{W} \), is the group generated by the transformations of \( t \) of the form \( t \mapsto \omega t + z \) for \( \omega \in W \) and \( z \in \ker(\exp|_t) \). It acts simply transitively on the set of alcoves. The *fundamental alcove*, \( A_0 \), is the alcove given by
\[ A_0 = \{ t = i H \in t \mid 0 < \alpha(H) < 2\pi \text{ for } \alpha \in \Delta^+ \} = \{ t = i H \in t \mid \alpha_0(H) < 2\pi \text{ and } 0 \leq \alpha_j(H) \text{ for } 1 \leq j \leq n \}. \]
The closure of the fundamental alcove, \( \overline{A}_0 \), is a fundamental domain for the \( \hat{W} \)-action (cf. [4, Thm. 4.8]). For \( G = \text{Sp}(2) \), the roots and the fundamental alcove are shown in Fig. [1].

3. Cells

3.1. Define \( v_0 = 0 \in t \) and for \( 1 \leq k \leq n \), define \( v_k \in t \) by the equations
\[ \alpha_j(v_k) = \begin{cases} 2\pi i & \text{if } j = 0 \\ 0 & \text{if } 1 \leq j \leq n \text{ and } j \neq k. \end{cases} \]
Then \( \{ v_0, \ldots, v_n \} \) is the set of vertices of the \( n \)-simplex \( \overline{A}_0 \). Notice that if we write \( \alpha_0 = \sum_{j=1}^{n} m_j \alpha_j \) with \( m_j \in \mathbb{N} \), we get \( 2\pi i = \alpha_0(v_k) = \sum_{j=1}^{n} m_j \alpha_j(v_k) = m_k \alpha_k(v_k) \). Therefore,
\[ \alpha_k(v_k) = \frac{2\pi i}{m_k} \text{ for } 1 \leq k \leq n. \]
(For classical \( G \), the \( m_k \in \{1,2\} \); however, for exceptional \( G \), the \( m_k \) can be as large as 6.)

3.2. Define
\[ F_0 = \{ t = i H \in t \mid \alpha_0(t) = 2\pi i \text{ and } 0 \leq \alpha_j(H) \text{ for } 1 \leq j \leq n \} \]
and for \( 1 \leq k \leq n \),
\[ F_k = \{ t = i H \in t \mid \alpha_0(H) \leq 2\pi, 0 \leq \alpha_j(H) \text{ for } 1 \leq j \leq n \text{ with } j \neq k, \text{ and } 0 = \alpha_k(t) \} \]
Then \( \{F_0, \ldots, F_n\} \) is the set of faces of \( \overline{A}_0 \). For \( 0 \leq k \leq n \), we will call \( F_k \) the \textit{face opposite to} \( v_k \). In the following, we will write \( r_k \in \hat{W} \) for the reflection across \( F_k \). Explicitly, \( r_0(t) = t - (\alpha_0(t) - 2\pi i)h_{\alpha_0} \) and \( r_k(t) = t - \alpha_k(t)h_{\alpha_k} \) for \( 1 \leq k \leq n \).

### 3.3

For \( 0 \leq k \leq n \), let \( \hat{W}_k \) be the stabilizer of \( v_k \),

\[
\hat{W}_k = \{ w \in \hat{W} \mid w(v_k) = v_k \}. 
\]

**Lemma 1.** For \( 0 \leq k \leq n \), the group \( \hat{W}_k \) is generated by \( \{r_j \mid 0 \leq j \leq n \text{ and } j \neq k\} \) and \( \{ \text{alcoves } A \text{ such that } v_k \in \overline{A}\} = \{w(A_0) \mid w \in \hat{W}_k\} \).

**Proof.** For the first statement, recall that it is well known (cf. [2, Ch. 4]) that the stabilizer of any point in \( \overline{A}_0 \) is generated by the set of reflections across the alcove faces that contain the point. In particular, \( v_k \) lies on every face except \( F_k \) and the result follows. For the second statement, observe that any alcove \( A \) can be uniquely written as \( A = w(A_0) \) for some \( w \in \hat{W} \). Since the vertices of \( w(A_0) \) are \( \{w(v_j) \mid 0 \leq j \leq n\} \), it follows that \( v_k \in \overline{A} \) if and only if \( v_k = w(v_j) \) for some \( j, 0 \leq j \leq n \). Since \( \overline{A}_0 \) is a fundamental domain for the action of \( \hat{W} \), \( v_k = w(v_j) \) if and only if \( k = j \) if and only if \( w \in \hat{W}_k \) as desired. \( \square \)

### 3.4

For \( 0 \leq k \leq n \), define

\[
C_k = \bigcup_{w \in \hat{W}_k} w \left( \overline{A}_0 \setminus F_k \right). 
\]

For \( G = \text{Sp}(2) \), the cells are shown in Fig 2.

By Lemma 1 and construction, the following result is immediate.

**Proposition 2.** (a) \( C_k \) is an open neighborhood of \( v_k \) that is contractible to \( v_k \) via a straight line contraction.
(b) Each alcove wall having nonempty intersection with $C_k$ contains $v_k$.
(c) Suppose $u_1, u_2 \in C_k$ satisfy $u_2 = w(u_1)$ for some $w \in \hat{W}$. Then $v_k = w(v_k)$.
(d) $\mathfrak{A}_0 \subseteq \bigcup_{k=0}^{n} C_k$.

\[\Box\]

Figure 2. The cells $C_0$, $C_1$, and $C_2$ for $\text{Sp}(2)$

4. A Cover of $G$

4.1. For $0 \leq k \leq n$, define

\[U_k = \{ c_g(\exp t) \mid g \in G, \ t \in C_k \} \quad \text{and} \quad \mathcal{O}_k = \{ c_g(\exp v_k) \mid g \in G \},\]

where $c_g(x) = gxg^{-1}$ for $g, x \in G$.

**Theorem 3.**

(a) $\{U_k \mid 0 \leq k \leq n\}$ is an open cover of $G$.

(b) $\mathcal{O}_k$ is a deformation retract of $U_k$.

**Proof.** Since $\exp(C_k)$ is open in $T$ and since conjugation takes the exponential of the closure of an alcove onto $G$, part (a) is automatic. For part (b), we claim the deformation retract is given by $R_k : U_k \times I \rightarrow U_k$ where $I = [0, 1]$ and

\[R_k(c_g(\exp t), s) = c_g(\exp ((1-s)t + sv_k)).\]

It remains to see that $R_k$ is actually well defined.

Suppose $c_{g_1}(\exp t_1) = c_{g_2}(\exp t_2)$ for $g_j \in G$ and $t_j \in C_k$. Writing $c_{g_2^{-1}g_1}(\exp t_1) = \exp t_2$, there exists $h \in Z_G(\exp t_2)^0$ so that $\bar{w} = h g_2^{-1} g_1 \in N_G(T)$ (cf. [S, Section 6.4].) Let

\[\Sigma_{t_2} = \{ \alpha \in \Delta \mid \alpha(t_2) \in 2\pi i \mathbb{Z} \},\]

i.e., the set of $\alpha$ for which $t_2$ lies on an $\alpha$-alcove wall.
Then \(Z_G(\exp t_2)^0\) is the exponential of the direct sum of \(t_2\) and all \(su(2)\)-triples corresponding to roots in \(\Sigma_{t_2}\). Since \(v_k\) also lies on all such \(\alpha\)-alcove walls, it follows that \(h \in Z_G(\exp ((1 - s) t + s v_k))^0\).

Setting \(w = \text{Ad}_{\tilde{w}} \in W\), we have \(c_{\tilde{w}}(\exp t_1) = \exp t_2\). Thus \(\exp(w t_1) = \exp(t_2)\) so that \(t_2 = w t_1 + z\) for some \(z \in \ker (\exp |_A)\). By Proposition 2, it follows that \(v_k = w v_k + z\). Then

\[
c_{g_1} (\exp ((1 - s) t_1 + s v_k)) = c_{g_2 h^{-1} \tilde{w}} (\exp ((1 - s) t_1 + s v_k)) = c_{g_2 h^{-1}} (\exp ((1 - s) w t_1 + s w v_k)) = c_{g_2 h^{-1}} (\exp ((1 - s) (t_2 - z) + s (v_k - z))) = c_{g_2 h^{-1}} (\exp ((1 - s) t_2 + s v_k - z)) = c_{g_2} (\exp ((1 - s) t_2 + s v_k))
\]

and we are finished. \(\square\)

4.2. The results of the previous subsection give immediately the following main result.

**Theorem 4.**

\[
\text{cat}(G) + 1 \leq \sum_{k=0}^{n} (\text{cat}_G (O_k) + 1).
\]

\(\square\)

5. The Orbits \(O_k\)

We present some remarks and explicit realizations for the \(O_k\) in the classical cases.

5.1. **\(G = SU(n + 1)\).** Trivial calculations show that

\[
v_k = \frac{2\pi i}{n + 1} (n + 1 - k, \ldots, n + 1 - k, -k, \ldots, -k)
\]

for \(0 \leq k \leq n\). Therefore \(\exp v_k = e^{-\frac{2\pi i k}{n+1}} \text{Id}\). In particular, \(O_k = \{e^{-\frac{2\pi i k}{n+1}} \text{Id}\}\) and so \(\text{cat}(O_k) = 0\) for all \(0 \leq k \leq n\). Thus, Theorem 4 implies \(\text{cat}(SU(n + 1)) \leq n\), i.e., we recover Singhof’s result [9].

5.2. **\(G = Sp(n)\).** Let \(\mathbb{H}\) denote the division algebra of quaternions \(q = a + bi + cj + dk, a, b, c, d \in \mathbb{R}\). View \(\mathbb{H}^n\) as a right vector space and identify the set of quaternionic matrices, \(M_n(\mathbb{H})\), with the set of \(\mathbb{H}\)-linear endomorphisms of \(\mathbb{H}^n\) via standard matrix multiplication on
the left. Write \( \nu : M_n(\mathbb{H}) \to \mathbb{R} \) for the reduced norm. In particular, if \( \varphi : M_n(\mathbb{H}) \to M_{2n}(\mathbb{C}) \) is the \( \mathbb{C} \)-linear injective homomorphism given by

\[
\varphi(A + jB) = \begin{pmatrix} A & B \\ -B & A \end{pmatrix}
\]

for \( A, B \in M_n(\mathbb{C}) \), then \( \nu = \det \circ \varphi \). We then realize \( \text{GL}(n, \mathbb{H}) = \{ g \in M_n(\mathbb{H}) \mid \nu(g) \neq 0 \} \), \( \text{SL}(n, \mathbb{H}) = \{ g \in M_n(\mathbb{H}) \mid \nu(g) = 1 \} \), and

\[
G = \text{Sp}(n) = \{ g \in \text{SL}(n, \mathbb{H}) \mid gg^* = I_n \},
\]

where \( g^* \) denotes the quaternionic conjugate transpose of \( g \). We also fix the maximal torus

\[
T = \{ \text{diag}(e^{i\theta_1}, \ldots, e^{i\theta_n}) \mid \theta_j \in \mathbb{R} \}.
\]

With this set-up, it is straightforward to check that

\[
v_k = i\pi \text{diag}(1, \ldots, 1, 0, \ldots, 0)
\]

for \( 0 \leq k \leq n \). Therefore

\[
\exp v_k = \begin{pmatrix} -I_k & 0 \\ 0 & I_{n-k} \end{pmatrix}.
\]

In particular, \( O_0 = \{ \text{Id} \} \) and \( O_n = \{ -\text{Id} \} \) so that \( \text{cat}(O_0) = \text{cat}(O_n) = 0 \).

The other \( O_k \) require more work, though they are easy to identify. For this we realize the quaternionic Grassmannian of \( k \)-planes in \( \mathbb{H}^n \), \( \text{Gr}_k(\mathbb{H}^n) \), by \( \{ x \in M_{n \times k}(\mathbb{H}) \mid \text{rk}(x) = k \} \) equipped with the equivalence relation \( x \sim xh \) where \( x \in M_{n \times k}(\mathbb{H}^n) \) and \( h \in \text{GL}(k, \mathbb{H}) \).

The following result is immediate.

**Lemma 5.** Let \( 1 \leq k \leq n - 1 \) and set \( d_k = \min\{k, n-k\} \). Then there is a diffeomorphism \( \tau_k : O_k \to \text{Gr}_{d_k}(\mathbb{H}^n) \),

\[
O_k \cong \text{Sp}(n) / (\text{Sp}(k) \times \text{Sp}(n-k)) \cong \text{Gr}_{d_k}(\mathbb{H}^n),
\]

given by

\[
\tau_k (c_g(\exp v_k)) = g \begin{pmatrix} I_k & 0 \\ 0 & 0_{(n-k) \times (n-k)} \end{pmatrix}
\]

when \( d_k = k \) and by

\[
\tau_k (c_g(\exp v_k)) = g \begin{pmatrix} 0_{k \times (n-k)} & 0 \\ 0_{(n-k) \times k} & I_{n-k} \end{pmatrix}
\]

when \( d_k = n-k \).

\( \square \)

**Conjecture 1.** \( \text{cat}_{\text{Sp}(n)}(O_k) = d_k \).
As we observed already in the introduction, if the conjecture is true, then Theorem 4 quickly shows that
\[
\text{cat } (\text{Sp}(n)) \leq \left\lfloor \frac{(n+2)^2}{4} \right\rfloor - 1.
\]

In terms of trying to show that \(\text{cat}_{\text{Sp}(n)}(\mathcal{O})_k \leq d_k\), there is an obvious choice of a cover of \(\mathcal{O}_k\). For this, we introduce the following notation. For the sake of clarity, we assume we are in the case of \(d_k = k\), i.e., \(1 \leq k \leq n/2\).

For \(1 \leq j \leq k + 1\), write \(x \in \text{Gr}_{k-1}(\mathbb{H}^{n-1})\) as
\[
x = \begin{pmatrix}
x_{j,1} \\
x_{j,2}
\end{pmatrix}
\]
with \(x_{j,1} \in M_{(j-1) \times (k-1)}(\mathbb{H})\) and \(x_{j,2} \in M_{(n-j) \times (k-1)}(\mathbb{H})\). Let \(X_{j,k} \cong \text{Gr}_{k-1}(\mathbb{H}^{n-1}) \subseteq \text{Gr}_k(\mathbb{H}^n)\) be given by
\[
\{ \begin{pmatrix}
0_{(j-1) \times 1} & x_{j,1} \\
1 & 0_{1 \times (k-1)}
\end{pmatrix} \mid x \in \text{Gr}_{k-1}(\mathbb{H}^{n-1}) \}.
\]

Write \(y \in \text{Gr}_k(\mathbb{H}^{n-1})\) as
\[
y = \begin{pmatrix}
y_{j,1} \\
y_{j,2}
\end{pmatrix}
\]
with \(y_{j,1} \in M_{(j-1) \times k}(\mathbb{H})\) and \(y_{j,2} \in M_{(n-j) \times k}(\mathbb{H})\). Let \(Y_{j,k} \cong \text{Gr}_k(\mathbb{H}^{n-1}) \subseteq \text{Gr}_k(\mathbb{H}^n)\) be given by
\[
\{ \begin{pmatrix}
y_{j,1} \\
0_{1 \times k}
\end{pmatrix} \mid y \in \text{Gr}_k(\mathbb{H}^{n-1}) \}.
\]

**Proposition 6.**

(a) \(\{\text{Gr}_k(\mathbb{H}^n) \setminus X_{j,k} \mid 1 \leq j \leq k + 1\}\) is an open cover of \(\text{Gr}_k(\mathbb{H}^n)\).

(b) \(Y_{j,k}\) is a deformation retract of \(\text{Gr}_k(\mathbb{H}^n) \setminus X_{j,k}\).

(c) Written in \((j-1) \times 1 \times (n-j)\) block form, \(\tau_k^{-1}(Y_{j,k})\) is
\[
\{ \begin{pmatrix}
A & B \\
1 & C
\end{pmatrix} \begin{pmatrix}
A & B \\
C & D
\end{pmatrix} \in \text{Sp}(n-1) \text{ and conjugate to } \exp v_{k-1,n-1} \}
\]

where \(v_{k,n} = i \text{diag}(\pi, \ldots, \pi, 0, \ldots, 0)\).
Proof. For part (a), simply observe that a $k$-plane in $X_{1,k} \cap \cdots \cap X_{k+1,k}$ would have to contain $k + 1$ independent vectors which is impossible. For part (b), observe that $Gr_k(\mathbb{H}^n) \setminus X_{j,k}$ is the of the set of

$$
\begin{pmatrix}
x(j-1)x_k \\
y_1x_k \\
z(n-j)x_k
\end{pmatrix} \in Gr_k(\mathbb{H}^n) \text{ so that } 
\begin{pmatrix}
x(j-1)x_k \\
y_1x_k \\
z(n-j)x_k
\end{pmatrix} \in Gr_k(\mathbb{H}^{n-1}).
$$

Therefore, the retraction $R : Gr_k(\mathbb{H}^n) \setminus X_{j,k} \times I \to X_{j,k}$ given by

$$
R \left( \begin{pmatrix} x(j-1)x_k \\ y_1x_k \\ z(n-j)x_k \end{pmatrix}, s \right) = \begin{pmatrix} x(j-1)x_k \\ (1-s)y_1x_k \\ z(n-j)x_k \end{pmatrix}
$$

does the trick. For part (c), observe that $\tau_{k}^{-1}(Y_{j,k})$ can be written in $(j - 1) \times 1 \times (n - k)$ block form as

$$
\{ g = \begin{pmatrix} \alpha & \beta & \gamma \\ 0 & \delta & \zeta \\ \eta & \iota & \kappa \end{pmatrix} \in G \}.
$$

Making note that $gg^* = I$, part (c) follows immediately by explicit matrix multiplication using $(j - 1) \times 1 \times (k - j) \times (n - j)$ block form when $j \leq k$ and by using $k \times 1 \times (n - k - 1)$ block form when $j = k + 1$.

\[ \square \]

**Proposition 7.** If the sets $\tau_{k}^{-1}(Y_{j,k})$ are contractible in $\text{SL}(n, \mathbb{H})$, then $\text{cat}_{\text{Sp}(n)}(O_k) \leq k$.

Proof. Let $F_1 : \tau_{k}^{-1}(Y_{j,k}) \times I \to \text{SL}(n, \mathbb{H})$ be a contraction that takes $\tau_{k}^{-1}(Y_{j,k})$ to a point. Using the Cartan decomposition, there is a diffeomorphism $\text{SL}(n, \mathbb{H}) \cong G \times \mathfrak{p}$ where $\mathfrak{p}$ is the the $-1$ eigenspace of the Cartan involution corresponding to $\mathfrak{sp}(n)$, i.e., the involution given by $\theta(x) = -x^\ast$. For $g \in \text{SL}(n, \mathbb{H})$, uniquely write $g = \kappa(g) \exp(\rho(g))$ with $\kappa(g) \in G$ and $\rho(g) \in \mathfrak{p}$. Finally, define $F_2 : \tau_{k}^{-1}(Y_{j,k}) \times I \to G$ by $F_2(g, s) = \kappa(F_1(g, s))$. By construction, $F_2$ contracts $\tau_{k}^{-1}(Y_{j,k})$ to a point. Thus, if the sets $\tau_{k}^{-1}(Y_{j,k})$ are contractible in $\text{SL}(n, \mathbb{H})$ then they are also contractible in $G = \text{Sp}(n)$. The proposition then follows from Proposition 6. \[ \square \]

At the present time, we do not know whether $\tau_{k}^{-1}(Y_{j,k})$ is contractible in $\text{SL}(n, \mathbb{H})$. It is worth noting that a similar result can be obtained by showing that $\tau_{k}^{-1}(Y_{j,k})$ is contractible in $\text{Sp}(2n, \mathbb{C})$. This too is unknown.
5.3. \( G = \text{Spin}(2n + 1) \). Write the tensor algebra over \( \mathbb{R}^m \) as \( T_m(\mathbb{R}) \). Then the Clifford algebra is \( C_m(\mathbb{R}) = T_m(\mathbb{R})/I \) where \( I \) is the ideal of \( T_m(\mathbb{R}) \) generated by \( \{(x \otimes x + \|x\|^2) \mid x \in \mathbb{R}^m\} \). By way of notation for Clifford multiplication, write \( x_1x_2 \cdots x_k \) for the element \( x_1 \otimes x_2 \otimes \cdots \otimes x_k + I \in C_m(\mathbb{R}) \) where \( x_1, x_2, \ldots, x_m \in \mathbb{R}^m \). Write \( C_m^+(\mathbb{R}) \) for the subalgebra of \( C_m(\mathbb{R}) \) spanned by all products of an even number of elements of \( \mathbb{R}^m \). Conjugation, an anti-involution on \( C_m(\mathbb{R}) \), is defined by \( (x_1x_2 \cdots x_k)^* = (-1)^k x_k \cdots x_2x_1 \) for \( x_i \in \mathbb{R}^m \).

Then

\[ \text{Spin}(m) = \{ g \in C_m^+(\mathbb{R}) \mid gg^* = 1 \text{ and } gxg^* \in \mathbb{R}^m \text{ for all } x \in \mathbb{R}^m \}. \]

In fact, it is the case that \( \text{Spin}(m) = \{ x_1x_2 \cdots x_{2k} \mid x_i \in S^{m-1} \text{ for } 2 \leq 2k \leq 2m \} \). If we write \( (Ag)x = gxx^* \) when \( g \in \text{Spin}(m) \) and \( x \in \mathbb{R}^m \), then \( A \) gives the double cover of \( \text{SO}(m) \):

\[
\{1\} \to \{\pm 1\} \to \text{Spin}(m) \xrightarrow{A} \text{SO}(m) \to \{I_m\}.
\]

A maximal torus \( T_0 \) for \( \text{SO}(2n + 1) \) is given by

\[
T_0 = \left\{ \begin{pmatrix}
\cos \theta_1 & \sin \theta_1 \\
-\sin \theta_1 & \cos \theta_1 \\
\vdots & \\
\cos \theta_n & \sin \theta_n \\
-\sin \theta_n & \cos \theta_n \\
1
\end{pmatrix} \mid \theta_i \in \mathbb{R} \right\}
\]

with Lie algebra

\[
t_0 = \left\{ \begin{pmatrix}
0 & \theta_1 \\
-\theta_1 & 0 \\
\vdots & \\
0 & \theta_n \\
-\theta_n & 0 \\
0
\end{pmatrix} \mid \theta_i \in \mathbb{R} \right\}.
\]

We write \( \exp_{\text{SO}(2n+1)} \) for the exponential map from \( t_0 \) onto \( T_0 \) and condense notation by writing \( E_k \) for the element of \( t \) given by

\[
E_k = \text{blockdiag} \left( \begin{array}{cccc}
0 & 0 & \cdots & 0 \\
0 & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & 0
\end{array} \right), \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \ldots, \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, 0).
\]
Writing $e_k$ for the $k^{th}$ standard basis vector in $\mathbb{R}^n$, observe that $A(\cos \theta - \sin \theta e_{2k-1}e_k)$ acts by the rotation \[
\begin{pmatrix}
\cos 2\theta & \sin 2\theta \\
-\sin 2\theta & \cos 2\theta
\end{pmatrix}
\]
in the $e_{2k-1}e_k$ plane. It follows that
\[
T = \{(\cos \theta_k - \sin \theta_k e_1 e_2) \cdots (\cos \theta_n - \sin \theta_n e_{2n-1}e_{2n}) \mid \theta_k \in \mathbb{R}\}
\]
is a maximal torus of $\text{Spin}(2n+1)$. If we identify $t$ with the Lie algebra of $T$ and write $\exp$ for the exponential map of $\text{Spin}(2n+1)$ taking $t$ onto $T$, then $\exp_{\text{SO}(n)} = A \circ \exp$. It follows that
\[
\exp(\theta E_k) = (\cos(\theta/2) - \sin(\theta/2) e_{2k-1}e_{2k}).
\]

Using the definitions, it is straightforward to check that
\[
\begin{align*}
v_0 &= 0 \\
v_1 &= 2\pi E_1 \\
v_k &= n \sum_{j=1}^{k} E_j
\end{align*}
\]
for $2 \leq k \leq n$. Therefore $\exp v_0 = 1$, $\exp v_1 = -1$, and $\exp v_k = (-1)^k \prod_{j=1}^{k} e_{2j-1}e_j$. Of course, $O_0 = \{1\}$ and $O_1 = \{-1\}$ so $\text{cat}(O_0) = \text{cat}(O_1) = 0$.

The other orbits are easy to describe, though calculating $\text{cat}_G(O_k)$ is not easy.

**Proposition 8.** For $2 \leq k \leq n$,
\[
O_k \cong \text{Spin}(2k)/\text{Spin}(2k)\text{Spin}(2n+1-2k) \cong \text{SO}(2n+1)/\left(\text{SO}(2k) \times \text{SO}(2n+1-2k)\right) \cong \text{Gr}_{2k}(\mathbb{R}^{2n+1}),
\]
the Grassmannian of oriented $2k$-planes in $\mathbb{R}^{2n+1}$.

**Proof.** Since $A(\exp v_k) = \begin{pmatrix} -I_{2k} \\ I_{n-2k} \end{pmatrix}$,
\[
A(O_k) \cong \text{SO}(2n+1)/S(O(2k) \times O(2n+1-2k)) \cong \text{Gr}_{2k}(\mathbb{R}^{2n+1}),
\]
the Grassmannian of $2k$-planes in $\mathbb{R}^{2n+1}$. Moreover, $A : O_k \to A(O_k)$ is a double cover. To see this, observe that there is a Weyl group (isomorphic to $S_n \ltimes \mathbb{Z}_2^n$) element taking $v_k$ to $-\pi E_1 + \pi \sum_{j=2}^k E_j$ which exponentiates to $-\exp v_k$.

To prove the proposition, first observe that the stabilizer of $\exp v_k$ under conjugation must be contained in $S = A^{-1}(S(O(2k) \times O(2n+1-2k))) = S(\text{Pin}(2k)\text{Pin}(2n+1-2k))$. Since $\text{Pin}(2k) \cap \text{Pin}(2n+1-2k) \subseteq \mathbb{R}$, it follows that the connected component of the identity of $S$ is $S_0 = \text{Spin}(2k)\text{Spin}(n-2k) \cong \text{Spin}(2k) \times \text{Spin}(n-2k)/\{\pm(1,1)\}$ and the
other component is diffeomorphic to \( \text{Pin}(2k)_1 \times \text{Pin}(2n + 1 - 2k)_1 \) where \( \text{Pin}(j)_1 \) is the non-identity component of \( \text{Pin}(j) \). Recalling that the center of \( \text{Spin}(2k) \) is \( \{ \pm 1, \pm \exp v_k \} \), it follows that \( S_0 \) is contained in the stabilizer of \( \exp v_k \). However, \( \text{Pin}(2k)_1 \) anticommutes with \( \exp v_k \) while \( \text{Pin}(2n + 1 - 2k)_1 \) commutes. Therefore, the stabilizer of \( \exp v_k \) is \( S_0 \). Finally, since \( S_0 = A^{-1}(\text{SO}(2k) \times \text{SO}(n - 2k)) \), the proof is complete. \( \square \)

The relative cat calculation of \( O_k \) in \( \text{Spin}(2n + 1) \) is not known.

5.4. \( G = \text{Spin}(2n) \). A maximal torus \( T_0 \) for \( \text{SO}(2n) \) is given by

\[
T_0 = \left\{ \begin{pmatrix} \cos \theta_1 & \sin \theta_1 \\ -\sin \theta_1 & \cos \theta_1 \\ \vdots \\ \cos \theta_n & \sin \theta_n \\ -\sin \theta_n & \cos \theta_n \end{pmatrix} \mid \theta_i \in \mathbb{R} \right\}
\]

with Lie algebra

\[
t = \left\{ \begin{pmatrix} 0 & \theta_1 \\ -\theta_1 & 0 \\ \vdots \\ 0 & \theta_n \\ -\theta_n & 0 \end{pmatrix} \mid \theta_i \in \mathbb{R} \right\}.
\]

As before, write

\[
E_k = \text{blockdiag} \left( \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \ldots, \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \ldots, \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \right).
\]

From the definitions, it is straightforward to check that

\[
v_0 = 0
\]
\[
v_1 = 2\pi E_1
\]
\[
v_k = \pi \sum_{j=1}^{k} E_j
\]
\[
v_{n-1} = \pi \sum_{j=1}^{n-1} E_j - \pi E_n
\]
for $2 \leq k \leq n$, $k \neq n - 1$. Therefore $\exp v_0 = 1$, $\exp v_1 = -1$, $\exp v_k = (-1)^k \prod_{j=1}^{k} e_{2j-1} e_j$, and $\exp v_{n-1} = (-1)^{n-1} \prod_{j=1}^{n} e_{2j-1} e_j$. Of course, $O_0 = \{1\}$ and $O_1 = \{-1\}$ so $\text{cat}(O_0) = \text{cat}(O_1) = 0$. As in Proposition 8, the remaining conjugacy classes are
\[
O_k \cong \text{Spin}(2k) / \text{Spin}_{2k}(\mathbb{R}) \text{Spin}(2n - 2k)
\]
\[
\cong \text{SO}(2n) / \text{SO}(2k) \times \text{SO}(2n - 2k) \cong \tilde{G}r_{2k}(\mathbb{R}^n),
\]
the Grassmannian of oriented $2k$-planes in $\mathbb{R}^{2n}$. Again, the relative category in $\text{Spin}(2n)$ is not known.

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