Roughness as Classicality Indicator of a Quantum State

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We define a new quantifier of classicality for a quantum state, the Roughness, which is given by the $L^2(\mathbb{R}^2)$ distance between Wigner and Husimi functions. We show that the Roughness is bounded and therefore it is a useful tool for comparison between different quantum states for single bosonic systems. The state classification via the Roughness is not binary, but rather it is continuous in the interval $[0,1]$, being the state more classic as the Roughness approaches to zero, and more quantum when it is closer to the unity. The Roughness is maximum for Fock states when its number of photons is arbitrarily large, and also for squeezed states at the maximum compression limit. On the other hand, the Roughness reaches its minimum value for thermal states at infinite temperature and, more generally, for infinite entropy states. The Roughness of a coherent state is slightly below one half, so we may say that it is more a classical state than a quantum one. Another important result is that the Roughness performs well for discriminating both pure and mixed states. Since the Roughness measures the inherent quantumness of a state, we propose another function, the Dynamic Distance Measure (DDM), which is suitable for measure how much quantum is a dynamics. Using DDM, we studied the quartic oscillator, and we observed that there is a certain complementarity between dynamics and state, i.e. when dynamics becomes more quantum, the Roughness of the state decreases, while the Roughness grows as the dynamics becomes less quantum.

Keywords: Classical Limit, Wigner Function, classicality indicator, Negativity, Entropy

I. INTRODUCTION

To determine if the system is classical or quantum is one of the most intriguing physics questions of the last decades. The first challenging question was to measure the quantum state. Much effort in this direction was made by several researchers, with many advances, both theoretical and experimental. The first approach to this problem was based on Ehrenfest theorem which states that, under certain conditions, the centroid of a wave-packet state will follow a classical trajectory. Zurek and Paz argue that the quantum system is never isolated, and thus the dynamics of a macroscopic object is modified by the surrounding objects that interact with it. This is the Decoherence Approach to Classical Limit of Quantum Mechanics. Up to our knowledge, Ballentine and collaborators where the first to address the question of which classical dynamics would be reproduced by Quantum Mechanics, a trajectory or an ensemble of them. Their response to this question was that in a coarse grain approach, the quantum state may behave classically if we consider an ensemble of trajectories. Those results were later confirmed by others. Ballentine and collaborators also argue that the decoherence is not necessary if we take into account the experimental limitations. This is the Coarse Grained Approach to Classical Limit of Quantum Mechanics. In fact, both approaches are necessary, since there is a combination of factors that must be considered in order to reproduce the classical regime.

large actions, the interaction with the environment and experimental observation limitations. In fact, if Quantum Mechanics domain includes Classical Mechanics domain, then Quantum Mechanics must reproduce all classical experiments and observations, including individual systems like a planet or a star. The action of the measurement apparatus on the system is closely related to the decoherence program, but there is a subtle difference: if we consider a situation where the action of the environment is negligible, the system is almost isolated, and if we perform continuous simultaneous measurements of position and momentum, then the information about the quantum nature of the particle will be lost and the Newtonian regime is achieved. Those results can be summarized in a simple way: decoherence and experimental limitations are responsible for achieving the Liouville classical regime, while the continuous monitoring of the system leads to Newtonian regime.

Despite the great advances on the Classical Limit problem, quantifying the degree of classicality of a quantum state is still an open question. In the core of the Decoherence program is the assumption that the environment is usually composed of a large number of particles, thus, due to the thermodynamic limit, the environment (thermal bath) is essentially classical. Paradoxically, it has been shown that an interaction with one degree of freedom system can lead the system to behave as it was classical, an example of a small quantum system whose classical counterpart is chaotic and able to produces decoherence-like behavior; similar results can be found in references. Oliveira and Magalhães have shown that a single degree of freedom system is, in the context of decoherence, equivalent to a $n$-degree of freedom system. This equivalence is quanti-
fied by the effective Hilbert space size, which is "as the Hilbert-space size of the phase state that generates purity loss equivalently as the other particular environmental states". Therefore, the effective Hilbert space size is a quantifier of the effectiveness of a system as an environment, i.e. the effectiveness of a specific model mimicking a bath is closely related to the classicality of such state.

Given the richness of possible physical systems and the complicated structure of the quantum state space, it is no surprise that various notions of classicality have been defined. It seems impossible to grasp the variety of quantum states with a unique parameter, especially in infinite-dimensional Hilbert spaces and, therefore, different classicality quantifiers should be considered as complementary rather than competitive.

In the context of harmonic oscillator potential, many classicality quantifiers were defined in terms of how a given state differs from a coherent one. These approaches follow from the postulate that coherent states are the only pure classical states in this situation. Some examples are Mandel Q-parameter and its various generalizations. Another approach is to use the distance of the state to the closest classical state defined in Ref. Also, used in Refs. These approaches to quantify nonclassicality strongly depend on the chosen set of states used as reference classical states and the norms or metrics used to define the distances. Another quantifier of nonclassicality is based on the convolution of the P-function with the amount of thermal noise needed to get a non-negative phase-space function. Other measures are based on the entanglement potential of non-classical states. In Refs., the amount of nonclassicality is quantified in terms of the minimal number of coherent states that are needed to be superposed in order to represent the state under study.

It is a member of a general class of algebraic measures, applying to different notions of nonclassicality. A moment-based approach was introduced to formulate measurable witnesses for the degree of nonclassicality. Another approach is to determine the degree of nonclassicality based on the Fourier transform of the Glauber-Sudarshan P-function, the characteristic function. In reference, the authors quantify the classicality of mixed states from the perspective of representation theory of semi-simple Lie groups and give a group theoretic characterization of cases when it is possible to give an explicit, closed form criterion for a mixed state to be classical. Again, the definition of classicality is heavily dependent on the criteria that coherent states are the most classical states.

This approach can not be easily generalized to other potentials, since the coherent states of the harmonic oscillator are not attainable for them and, therefore, cannot serve as the reference set of classical states. The standard coherent states can be generalized for arbitrary potentials in different nonequivalent ways and it is not clear which class of states should be considered classical, and hence it is not clear what is the best set of reference states for the determination of the nonclassicality of states in other potentials.

The nonclassicality of quantum states phase space is also connected with measures based on information theory. In reference, Ferraro et al. show that there are distinct notions of classicality, and, under their considerations, that there exist quantum correlations that are not accessible by information-theoretic arguments. Shahandeh et al. show that the only known classicality criterion violated by a non-local boson sampling protocol is the phase-space nonclassicality. Baumgratz et al. investigated the quantities of resource theory of quantum coherence. In reference, the authors investigated non-classical light, and they show that quantum resource is the same of Glauber light; the non-classical light can be interpreted as a form of coherence, their procedure is based on the negativity of P-distribution.

The rest of paper is organized as follows: in section we define a new measure, the Roughness, and prove that it is bounded between [0, 1]. Given two states, we say that the one with a larger value of is more non-classical than the other. In section we address some important quantum states, and evaluate the Roughness for each one of them. We stress that we could find, both for lower and upper bounds, examples of states that, in limit case, achieve those values. In section we compare the Roughness with another classicality measure, the Negativity. First, is not a bounded function, so it can be more difficult to compare any two given states. Also, we show that there are some states with (said to be totally classical), but with , i.e. the Roughness can find some quantumness in such cases. Particularly, we study a convex mixing between a thermal and a Fock state, and supported by entropy, we show that the Roughness is more reliable, especially for small temperatures. At last, in section we define another classicality measure, the Dynamic Distance Measure. While evaluates the inherent quantumness of a state, DDM quantifies how much a quantum dynamics is far from a classical one. We numerically evaluate both and for the quartic model, and we find a complementary behavior between them for such model.

II. ROUGHNESS: DEFINITION AND BOUNDS

The Wigner quasiprobability distribution, better known as Wigner function, was introduced in 1932 by Eugene Wigner. It is a real-valued function for any arbitrary quantum state , and it is given by

\[ W_\Psi(q, p) = \frac{1}{2\pi} \int \sqrt{\psi^* \psi} \left( q - \frac{p^*}{2} \right) \left( \psi + \frac{p}{2} \right) dx. \]  

As a distribution, it is normalized, i.e.

\[ \int dq \, dp \, W_\Psi(q, p) = 1, \]
it is also common to say that it has unitary volume. The Wigner function is a real bounded function, with $|W_\Psi(q,p)| \leq \pi^{-1}$ for any $(q,p) \in \mathbb{R}^2$. Moreover, it is square integrable
\[
||W_\Psi||^2 = \langle W_\Psi, W_\Psi \rangle = \int_{\mathbb{R}^2} dq dp |W_\Psi(q,p)|^2 \leq \frac{1}{2\pi},
\]
and the equality above holds when $\Psi$ is a pure state \cite{77}. The inner product above is the canonical one in the $L^2(\mathbb{R}^2)$ space. Among its properties, we emphasize the fact that $W_\Psi(q,p)$ can assume negative values, so it cannot be a regular probability distribution, and for this reason it is seen as a signature of quantumness of the state. Actually, there is a measure for non-classicality based on this property, namely the Negativity \cite{78}.

\[
N(\Psi) = \int_{\mathbb{R}^2} [W_\Psi(q,p) - W_\Psi(q,p')] dq dp = \int_{\mathbb{R}^2} |W_\Psi(q,p)| dq dp - 1,
\]
which evaluates the volume of the negative part for Wigner function. The classicality quantifier $N$ above should not be confused with the negativity measure for entanglement \cite{74}.

Inspired by this same property for Wigner function, we propose here a new measure of how quantum is a state. First we have to refer to another distribution, the Husimi function or $Q$-distribution \cite{77}, which can be evaluated from Wigner function as
\[
Q_\Psi(q,p) = \frac{1}{\pi} \int_{\mathbb{R}^2} dq' dp' W_\Psi(q',p') e^{-[\frac{(q-q')^2}{\pi} + (p-p')^2]}.
\]

In other words, we convolute the W function for any given state with the Gaussian distribution for a vacuum state. It smooths the oscillations of the Wigner function around a point $(q,p)$ in the phase space, as we average $W(q',p')$ values in a circle around this point. Consequently, one can show that $Q_\Psi(q,p)$ is always non-negative, for any $\Psi$, so it is always acceptable as a classical distribution – other quasi-distributions, like the $P$-function \cite{77}, do not have such property. For these reasons, we have chosen Wigner and Husimi functions to propose a new way of measuring non-classicality. The $R$ measure, namely the Roughness, was inspired by standard measures of roughness \cite{81} and is defined as proportional to the $L^2(\mathbb{R}^2)$ distance between both functions. Our idea is that Wigner function, as discussed before, carries some very important information of the quantumness for a given state and since $Q$ smooths the oscillations of $W$, and also it can be seen as a regular probability distribution, we can quantify how much non-classical a state is by evaluating how far those functions are one from another. The Roughness is given by
\[
R(\Psi) = \sqrt{2\pi ||W_\Psi - Q_\Psi||} = \sqrt{\int_{\mathbb{R}^2} dq dp |W_\Psi(q,p) - Q_\Psi(q,p)|^2}.
\]

We classify a state as more non-classical when its Wigner function is more distant form its Husimi function. In other words, given two any states, the one with larger $R$ will be more quantum. From now on, for the sake of simplicity, we drop the $\Psi$ index on $W$ and $Q$ functions notation, unless it is necessary to make it clear.

As a first property, we show that $R$ is bounded
\[
0 \leq R \leq 1,
\]
for any state. The lower bound is obvious from definition \cite{81}. We now prove the upper bound. First we define the symmetric Fourier transform for Wigner function
\[
\hat{W}(u,v) = \frac{1}{2\pi} \int_{\mathbb{R}^2} dq dp e^{-i(uq+vp)}W(q,p),
\]
and in an analogous way, the Fourier transform $\hat{Q}(u,v)$. If we name the Gaussian function $g(q,p) = e^{-(q^2 + p^2)}$, we can see from \cite{74} that the Husimi function is merely the convolution
\[
Q(q,p) = 2(W * g)(q,p).
\]
Thus we have
\[
\hat{Q}(u,v) = \exp\left(-\frac{u^2 + v^2}{4}\right) \hat{W}(u,v).
\]
Using Plancherel theorem \cite{81}, we obtain
\[
R^2 = 2\pi ||W - Q||^2 = 2\pi ||\hat{W} - \hat{Q}||^2
\]
\[
= 2\pi \int_{\mathbb{R}^2} du dv \left(1 - e^{-\frac{u^2 + v^2}{2}}\right)^2 \left|\hat{W}(u,v)\right|^2
\]
\[
\leq 2\pi \int_{\mathbb{R}^2} du dv \left|\hat{W}(u,v)\right|^2 = 2\pi ||\hat{W}||^2 = 2\pi ||W||^2 \leq 1,
\]
where we have used \cite{2} in the last step. As a consequence, given any state $\Psi$, its Roughness $R(\Psi)$ will be always bounded. The Roughness will be closer to one as more non-classical a state is; on the other hand, if a state is more classical, its Roughness will be closer to zero. We will show on section III that there are, for both bounds, states which can be arbitrarily close to these values.

### III. EXAMPLES

We now evaluate the Roughness of some common and important quantum states \cite{77} that will give us some insights about $R$. Also, these states appear in many applications in Quantum Optics \cite{82} and other areas.

#### A. Coherent state

A coherent state is a specific quantum state of the quantum harmonic oscillator, often described as a state whose dynamics most closely resembles the oscillatory
behavior of a classical harmonic oscillator. Its Wigner and Husimi functions respectively are

\[ W_0(q,p) = \frac{1}{\pi} e^{-(q^2+p^2)}, \]

\[ Q_0(q,p) = \frac{1}{2\pi} e^{-\frac{q^2+p^2}{2}}. \]

It is straightforward to evaluate its roughness as

\[ R_0 = \frac{1}{\sqrt{6}} \approx 0.408. \] (10)

Based on our numerical investigations, we conjecture that this is the smallest value for the Roughness of a pure state, and we use this value as a reference to compare to other states. Also, we can see that the coherent state is roughly in the middle of the Roughness scale. Since its Roughness is just below one half, we can say that the coherent state is closer to classical than to quantum, but as it is a pure state, its entropy is zero and therefore its Roughness is greater than that of other non-pure states.

We must emphasize that (9) is the Wigner function for a Fock state, and it is a pure state \( |0\rangle \). Moreover, they are all in a finite open interval \( (0,\nu) \), where this upper bound \( \nu \) is well known \([86]\). It means that \( L_n(x) \) has an oscillatory part on this interval, so it changes its signal \( n \) times, which suggests that, as larger as \( n \) gets, the negative part of \( W_n \) becomes more significant, so its Roughness increases: our results confirm such insight. The Husimi function for a Fock state is \([77]\)

\[ Q_n(q,p) = \frac{1}{2\pi n!} \left( \frac{q^2+p^2}{2} \right)^n e^{-\frac{q^2+p^2}{2}}. \] (12)

Taking \( n = 0 \) in both (11) and (12), we obtain the coherent state (9).

Our calculations for the Roughness are tedious, but straightforward. We give more details on \([8]\). By definition, the Roughness is

\[ R_n^2 = 2\pi \int_{\mathbb{R}^2} dq dp \left[ W_n(q,p) - Q_n(q,p) \right]^2 = R_{W_n}^2 + R_{Q_n}^2 - R_{W_n Q_n}^2, \] (13)

where we have defined

\[ R_{W_n}^2 := 2\pi \int_{\mathbb{R}^2} dq dp \left[ W_n(q,p) \right]^2, \]

\[ R_{Q_n}^2 := 2\pi \int_{\mathbb{R}^2} dq dp \left[ Q_n(q,p) \right]^2, \]

\[ R_{W_n Q_n}^2 := 4\pi \int_{\mathbb{R}^2} dq dp \left[ W_n(q,p) - Q_n(q,p) \right]. \] (14)

We find – details on \([8]\) – that

\[ R_{W_n}^2 = 1, \quad \forall n = 0, 1, 2, \ldots, \] (15a)

\[ R_{Q_n}^2 = \frac{1}{2^{2n+1}} \left( \frac{2n}{n} \right)! = \frac{1}{2} \frac{(2n)!}{2^{2n}(n!)^2} > 0, \] (15b)

\[ R_{W_n Q_n}^2 = \frac{4}{3} \left( -\frac{1}{3} \right)^n \left( -n, n+1; 1; \frac{4}{3} \right) = \frac{4}{3} \left( -\frac{1}{3} \right)^n \sum_{j=0}^{n} \frac{(n+j)!}{j!} \frac{4}{3}^j. \] (15c)

where \( F \) is the hypergeometric function, which becomes a finite sum if either its first or second argument is a negative integer, as it happens on (15c). The results \([10, 15a - 15c]\) were obtained using some known integrals for Laguerre functions \([88]\). It is not straightforward to see on the equations above, but one can check that for \( n = 0 \) we recover \( R_{Q_0}^2 = 1/6 \), as expected. Also, we emphasize that \( R_{W_n}^2 = 1 \) for any \( n \)-Fock state, and it is a typical characteristic for pure states, as we have already said just after Eq. (9).

We also prove in \([8]\) that

\[ 0 < R_{Q_n}^2 < R_{W_n Q_n}^2, \quad \forall n. \] (16)

The inequality above is important: replacing it in \([8,13]\), we can check that our upper bound \([8]\) is respected, as it should be. It is hard to see property \([10]\) from Eq. (15c), but on \([8]\) we rewrite this term as

\[ R_{W_n Q_n}^2 = \frac{4}{3} \left( \frac{1}{9} \right)^n \sum_{j=0}^{n} \binom{n}{j} 2^j > 0. \] (17)

Moreover and more important, we also prove that

\[ \lim_{n \to \infty} R_{Q_n}^2 = 0 = \lim_{n \to \infty} R_{W_n Q_n}^2, \] (18)

and so we have for the Roughness for the Fock state that

\[ \lim_{n \to \infty} R_n = 1. \] (19)
This result is quite remarkable: the Roughness for the Fock state increases as \( n \) becomes larger, and it reaches the upper bound on the limit \( n \to \infty \). In other words, as \( n \) increases, the Fock state \( \rho_n \) becomes more non-classical, and it gets arbitrarily closer to maximum value for the Roughness for a sufficiently large \( n \). This result contradicts those who argue that the Fock state becomes more classical as \( n \) increases – see, for example, [89] and references therein. In figure 1 we show the Roughness of a Fock state dependence on \( n \). Although the Fock state approaches the Roughness upper bound, the convergence to unity is very slow.

As a last comment for this subsection, Fock states can be used as basis for more general states

\[
\rho = \sum_{n,m=0}^{\infty} A_{n,m} |n\rangle \langle m| ,
\]

where, of course, \( A_{n,m} \) are complex constants such that \( \text{Tr} \rho = 1 \), and \( \rho^\dagger = \rho \). In [C] we give detailed analytical results that are useful for evaluating the Roughness for these states [20].

### C. Squeezed states

The squeezed states were presented in 1927, by Kennard [90], as the first example of non-classical states. The Wigner function for a squeezed state is

\[
W_\zeta(q, p) = \frac{1}{\pi} \exp \left[ -\left( e^{2\zeta q^2} + e^{-2\zeta p^2} \right) \right] ,
\]

where \( \zeta \in \mathbb{R} \). Actually, we are only considering states which are squeezed along the principal axes, a more general Gaussian state would take rotations and translations into account. Once again, if \( \zeta = 0 \), we have the coherent state [9]. If \( \zeta > 0 \), we have a narrower Gaussian in \( q \) and a wider one on \( p \), the opposite happens for \( \zeta < 0 \). It is a consequence of Heisenberg’s uncertainty principle: a narrower Gaussian in \( q \) means that we have a larger probability that our state is localized on a small neighborhood of \( q = 0 \), but as a consequence, the wider Gaussian in \( p \) tells us that we have a significant probability to have any velocity. Intuitively, we think that as large as \( |\zeta| \) gets, the squeezed state becomes more non-classical, and our results corroborate this statement.

The Fourier transform for its Wigner function is

\[
\hat{W}_\zeta(u, v) = \frac{1}{2\pi} \exp \left[ -\frac{1}{4} \left( e^{-2\zeta u^2} + e^{2\zeta v^2} \right) \right] ,
\]

and so, from definition, we can straightforward obtain

\[
R(\zeta) = \left[ 1 + \frac{e^{\zeta}}{e^{2\zeta} + 1} - \frac{4e^{\zeta}}{\sqrt{e^{2\zeta} + 2}(2e^{2\zeta} + 1)} \right]^{\frac{1}{2}} .
\]

We can easily check that \( R(0) = 1/\sqrt{6} \). Also, we can prove that

\[
\lim_{\zeta \to \pm \infty} R(\zeta) = 1 ,
\]

so it is another example of a state that reaches the maximal quantumness in our Roughness measure.

In figure 2 we show how the Roughness for squeezed states depends on \( \zeta \). Minimum Roughness occurs at \( \zeta = 0 \), and in this case, the squeezed state is just a coherent one.

### D. Field cat states

Field cat states are usually known as the most common example of a non-classical state [82]. They are given by the superposition of two coherent states, \( |a e^{i\phi}\rangle \),

\[
|C\rangle (\phi, \alpha) = N \left( |\alpha e^{i\phi}\rangle + (-1)^n |\alpha e^{-i\phi}\rangle \right) ,
\]
where $N$ is the normalization constant, $\phi$ is the phase, and $\alpha e^{\pm i\phi}$ determines the center of the coherent state. For $\alpha = -iq_0$ and $\phi = \pi/2$, we have

$$|C\rangle(\phi, \alpha) = |q_0\rangle \pm |-q_0\rangle \sqrt{2},$$

(24)

where the plus (minus) sign refer to even (odd) cat state. We highlight that odd cat state is not defined for $q_0 = 0$; on the other hand, if we take $q_0 = 0$ for even cat state, we recover the coherent state related to Wigner function (9). Indeed, the Wigner functions for (24) are

$$W_{\pm}(q, p) = e^{-\frac{1}{2}(q - q_0)^2 + p^2} + e^{-\frac{1}{2}(q + q_0)^2 + p^2} \pm 2e^{-(q^2 + p^2)} \cos(2q_0 p).$$

(25)

First, we can easily see that $R_+ (0) = \sqrt{1/6}$, as expected. Moreover, we get

$$\lim_{q_0 \to \pm \infty} R_+(q_0) = \lim_{q_0 \to \pm \infty} R_-(q_0) = \sqrt{\frac{7}{12}} \approx 0.764. \quad (27)$$

The result above shows us, alongside with figure 3, that the Roughness for odd cat state is always larger than the even one, and both become more non-classical as $q_0$ increases, but not even on limit they reach the maximum value for the Roughness. As we increase $q_0$, from $q_0 \approx 2$ (for the even cat) or $q_0 > 0$ (for the odd cat), the cat state Roughness is significantly greater than the coherent state, thus corroborating the statement that cat state is non-classical. However, we emphasize that no matters how large we take $q_0$ for a cat state, we always get a more quantum Fock state (for large $n$) or squeezed state (for large $|\zeta|$). In other words, our results show the cat states as non-classical ones, but there are states which are “more quantum”.

### E. Thermal state

A thermal state for the Harmonic Oscillator with frequency $\omega$ is a mixed state given by

$$\rho_T = \left(1 - e^{-\frac{\hbar \omega}{k_B T}}\right) \sum_{n=0}^{\infty} e^{-\frac{\hbar \omega n}{k_B T}} |n\rangle \langle n|$$

(28)

where $\hbar$ and $k_B T$ have their usual meaning, and $|n\rangle$ is the harmonic oscillator eigenstate (the Fock state). Its
Wigner and Husimi functions are
\[ W_n(g, p) = \frac{1}{\pi(2\bar{n} + 1)} \exp \left( -\frac{g^2 + p^2}{2\bar{n} + 1} \right), \]
\[ Q_n(g, p) = \frac{1}{2\pi(\bar{n} + 1)} \exp \left( -\frac{g^2 + p^2}{2(\bar{n} + 1)} \right), \]
where \( \bar{n} = \left( e^{\frac{\bar{\nu}}{kT}} - 1 \right)^{-1} \) is the thermal average number of photons in a mode. For \( \bar{n} = 0 \) we have the coherent state \([9]\) again. Also one can see that, for \( \bar{n} \to \infty \), \( W_n \) and \( Q_n \) have the same limit, so we can expect that the Roughness goes to zero as \( \bar{n} \) increases. It is straightforward to evaluate the Roughness for the thermal state as
\[ R_T(\bar{n}) = \left[ \frac{1}{2 (\bar{n} + 1)(2\bar{n} + 1)(4\bar{n} + 3)} \right]^{\frac{1}{2}}. \] (30)

As expected, if \( \bar{n} = 0 \) we recover \( R_0 = 1 \). Also, we can easily see that \( R_T(\bar{n}) \to 0 \), as \( \bar{n} \to \infty \), and it is a consequence of the fact that the quantum partition function becomes closer to the classical one as \( \bar{n} \to \infty \). Moreover, we can see that it goes to zero as \( \bar{n}^{-3} \).

1. Diagonal State

Another non pure state is the Diagonal state of order \((m + 1)\), which is defined as
\[ \rho_D(m) = \frac{1}{m + 1} \sum_{n=0}^{m} |n\rangle \langle n|. \] (31)
The diagonal state represents a mixed state with uniform distribution. It is easy to see that in the limit case \( m \to \infty \), we will have \( R \to 0 \), since \( R \) is bounded.

The mean photon number for the Diagonal State is \( \bar{n} = \text{Tr}[\hat{N}\rho_D(m)] = m/2 \), where \( \hat{N} \) is the number operator. Again, as in the thermal state case, the Roughness goes to zero when the mean photon number goes to infinity, but in this case, the convergence is slower. The Roughness of the Diagonal state was determined numerically using the results of section [13] and [6].

Now we compare some features for thermal and Diagonal states. In figure [5] we show the Roughness for both states as function of \( \bar{n} \). We can see that even for small values of \( \bar{n} \), \( R \) is already close to zero, which means that it would be very difficult to observe quantum features in these states.

The difference between thermal and Diagonal states is one of those situations where the observable choice determines the system classicality measure. Since the states are not pure, the entropy can be used to quantify their purity. If we consider the same \( \bar{n} \) for both states, we have different values for the Roughness, but the linear entropy (defined as \( \delta = 1 - \text{Tr} \rho^2 \)) is identical for both states, and it is given by
\[ \delta = 1 - \text{Tr} \rho_D^2 = 1 - \text{Tr} \rho_T^2 = \frac{2\bar{n}}{2\bar{n} + 1}. \]

Although they have the same linear entropy, they do not have the same entropy. The entropy for thermal state is
\[ S_T = k_B(1 + \bar{n}) \ln [1 + \bar{n}] - \bar{n}k_B \ln (\bar{n}), \]
while the entropy for Diagonal state is
\[ S_D = k_B \ln (2\bar{n} + 1). \]

As we can see in figure [5], for the same mean photon number \( \bar{n} \), thermal state has a bigger entropy, which explains why its Roughness is smaller. The Roughness is more sensitive to the difference between thermal and diagonal states than linear entropy. This result is an example that the classicality of a system is sensible to which criteria is used to quantify it, and not only on the observable choice [12], since both entropy and linear entropy can be estimated by the same set of measurements.

IV. ROUGHNESS × NEGATIVITY: A COMPARATIVE STUDY

Negativity is widely used as a measure of classicality for a quantum state [78], and it is defined by Eq. [6]. In this section, we compare both quantifiers. First, we remark directly from its definition that Negativity is zero for any state whose Wigner function is positive, i.e. it does not distinguish a thermal state from a coherent state and/or squeezed state, while Roughness can do the trick, as we have seen on section [13]. Another question is about bounds for \( N \), since up to our knowledge it is not a bounded function – actually, it is shown in [78] that the Negativity grows proportionally to \( n^{1/2} \) for Fock states, at least for \( n \) up to 250 –, and there are some results on the integral of the Wigner function over a sub-region of the phase space of a one degree of freedom quantum system which can be less than zero or greater than one on.
directly from the definition (5), is nonlinear on respective Wigner and Husimi functions, its Roughness, although the mixed state (32) is a linear combination for the Diagonal state (red dashed line) as function of mean photon number $\bar{n}$.

FIG. 5. Entropy for a Thermal state (black line) and Entropy for the Diagonal state (red dashed line) as function of mean photon $\bar{n}$.

this sub-region [91]. The lack of known bounds for Negativity can be a problem if one needs to compare different states.

Now, in order to clarify other advantages of Roughness over Negativity, we study a mixture state $\rho_z$ given by the convex combination

$$\rho_z(\beta, M) = (1 - z)\rho_\beta + z |M\rangle \langle M|,$$  \hspace{1cm} (32)

where $\rho_\beta$ is the thermal state at temperature $T = 1/\beta k_B$, and $|M\rangle$ is a $M$ Fock state. We recall here that, although the mixed state (32) is a linear combination between a thermal and a Fock state, and so are their respective Wigner and Husimi functions, its Roughness, directly from the definition (19), is nonlinear on $z$. Moreover, thermal states have positive Wigner functions (29), so $N$ is always zero for them, while $R$ is not, as we can see in Eq. (30). The idea of studying such mixture state is that a Fock state $|M\rangle$ is always a pure state, but it can be as quantum as we want, as we have shown in eq. (19). Alternatively, $\rho_\beta$ is a pure state only in the limit $T \rightarrow 0^+$, namely the coherent state. As the temperature increases, $\rho_\beta$ becomes more non-pure, while its Roughness goes to zero [30]. So, for sufficiently large values of $\beta$, we are practically mixing two pure states, but Roughness for $\rho_\beta$ is given by $R_0$ (10), while we can take a large $M$ to get a Fock state whose Roughness is as close to the unity as we want to. On the other hand, for small values of $\beta$, we have the same Fock state $|M\rangle$, but $\rho_\beta$ is more non-pure and more classical as $T$ gets larger.

In such spirit, we plot in figure 6 Roughness and Negativity for $M = 10$ (taken as large $M$), both for small $\beta = 0.4$ (dotted line) and large $\beta = 10$ (full line). In any case, we expect that $R(z = 1) > R(z = 0)$, since $R_0$ is at the same time a lower bound for Roughness for a Fock state and an upper bound for a thermal state. Nonetheless, for large $\beta$ we have an almost pure state when $z = 0$ and a genuine pure state when $z = 1$, and pure states are typically quantum. So, for small $0 < z < 1$, we might expect that this mixture becomes less quantum, and we can clearly see it in figure 6 as $R(\beta = 10)$ in a non-monotonic function of $z$. For small $\beta$ the initial $z = 0$ thermal state is already non-pure, and for this reason $R$ is monotonic in $z$ in this case. Facing this behavior, the Negativity is always monotonic in $z$, as $N = 0$ for thermal states and $N > 0$ for any Fock state such that $M \geq 1$. Our results

for Roughness for the mixed state (32) are, once again, supported by entropy. Indeed, on figure 7 we plot the entropy $S$ as function on $z$ and $\beta$. We can see that, for small fixed temperatures (large $\beta$), that the entropy is close to zero – as we said, for $z = 0$ the thermal state is almost pure in such case. So $S$ clearly increases as we take small values for $z > 0$, and after $S$ reaches its maximum, it decreases to $S = 0$ at $z = 1$, since now we have a pure Fock state. However, for large temperatures, we already start from a very entropic state, so the entropy is large for $z = 0$. In the inset of figure 7 we show $z_{\text{max}}$, the value of $z$ where entropy $S$ is maximum, as function of $\beta$.

Since the Negativity is not bounded to unity, then in order to compare the curves, we show, in figure 8 the relative Roughness and relative Negativity, respectively defined as \%$R = R(z) / R(1)$ and \%$N = N(z) / N(1)$. Both measures must find their largest values on $z = 1$, since a Fock state for $M = 10$ is more quantum than a thermal state, but $\%R$ is not monotonic on $z$, specially for large values of $\beta$, as we have already discussed. We plot these quantities for a large ($\beta = 0.4$) and a small temperature ($\beta = 10$). We also studied the differences $\Delta R = R(\beta = 0.4) - R(\beta = 10)$ and $\Delta N = N(\beta = 0.4) - N(\beta = 10)$ in figure 9 which are, respectively, the differences between $R$ and $N$ at large
FIG. 7. Entropy as function of temperature \( \beta \) and convex combination parameter \( z \). On the inset we plot \( z_{\max} \) – the value of \( z \) where \( S \) is maximum – dependence on \( \beta \).

FIG. 8. Relative Roughness (\%R) (blue) and Relative Negativity (\%N) (red) as function of \( z \) for the state \( \rho_z \) for \( M = 10 \), \( \beta = 0.4 \) (dashed lines) and \( \beta = 10 \) (full lines).

FIG. 9. Differences \( R(\beta = 0.4) - R(\beta = 10) \) (blue line) and \( N(\beta = 0.4) - N(\beta = 10) \) (red dashed line) as function of \( z \) for the state \( \rho_z \) for \( M = 10 \).

and small temperatures. For \( z = 0 \) we have a thermal state, so they must be quite distinct at different temperatures. On the other hand, when \( z = 1 \), \( \rho_z \) goes to the \( M \) Fock state, no matter if the temperature is small or large, so both \( \Delta R \) and \( \Delta N \) must be zero. As we plot both quantities, we see that \( \Delta R \) is a monotonic function of \( z \), while \( \Delta N \) is not. This means that Negativity has failed to discriminate this mixed quantum state. Again, we appeal to a known quantity, the fidelity \( F \) between two quantum states [92], to support our claim. The fidelity between two mixed states \( \rho_1 \) and \( \rho_2 \) is defined as \( F = Tr(\rho_1 \rho_2)/Tr(\rho_1^2) \). The fidelity between the states \( \rho_z(\beta = 0.4) \) and \( \rho_z(\beta = 10) \) is monotonic on \( z \), so it is the distance \( 1 - F \) between them, as we can see on the inset of figure 4. Its behavior is similar to the one that what we observe on \( R \). The Negativity, however, is not monotonic on \( z \).

In Figures 10, 11 and 12, we plot graphics for both \( R \) and \( N \) as functions of \( M \) and \( z \). Since that Negativity for thermal states is zero for any temperature, so \( N \) is zero for a large set of states, which is shown in the dark blue part of figures. The Roughness, on the other hand, is zero only for large temperatures (small \( \beta \)) and for \( z \approx 0 \), this is evident on figure 11. Comparing figures 10 and 11 we can see that Negativity does not discriminate mixed states with different temperatures, while the Roughness is sensitive to it. This fact is most evident in figure 12.

V. DYNAMIC DISTANCE MEASURE: THE QUARTIC MODEL

As we have shown, the Roughness is a good measure of how quantum is a state, but it does not tell us anything about the dynamics, then we also defined the Dynamic Distance Measure \( (D) \), which is given by

\[
D(\Psi(t)) = \left[ \pi \int \int_B |f(x, p, t) - Q(\Psi(t))|^2 \, dx \, dp \right]^{1/2}
\]

\[
f(x, p, 0) = Q(x, p, 0),
\]

where \( f \) is the classical Liouville evolution for the corresponding classical Hamiltonian. The function \( D \) was constructed to measure quantum aspects of dynamics, and then, identical initial states for the classical and quantum systems must be considered. We observe that \( D \) is not limited in general, but if we exclude states that do not respect Heisenberg’s uncertain principle, then \( D \in [0, 1] \). With classical dissipation, \( f \) can become delta function and in this case \( D \to \infty \). Otherwise, \( D = 1 \) only if \( \int_B f(x, p, t)Q(\Psi(t)) \, dx \, dp \to 0 \), this means that \( f \) and \( Q \) are localized in different regions on phase space. A
A similar classicality measure was used by Toscano and collaborators [93]. The main difference is that they used the Wigner function instead of Husimi function on (33).

In order to investigate the dynamical aspects, we use the quartic oscillator model (Kerr oscillator), which was the object of many investigations [12, 19, 21, 22, 25, 33, 94–100] with expressive experimental results [101]. The Hamiltonian is given by

$$\hat{H}_0 = \omega \hat{a}^\dagger \hat{a} + \lambda \hbar^2 (\hat{a}^\dagger)^2 \hat{a}^2,$$

(34)

where $\hat{a}$ and $\hat{a}^\dagger$ are creation and annihilation operators,

$\omega$ and $\lambda$ are system parameters. Given a general initial state $\rho(0) = \sum_{n,m=0}^{\infty} A_n A_m^* |n\rangle \langle m|$, its time evolution is

$$\rho(t) = \sum_{n,m=0}^{\infty} e^{it(m-n)(\omega + \lambda \hbar|n+m|)} A_n A_m^* |n\rangle \langle m|$$

so the Husimi function is

$$Q(\beta) = e^{-|\beta|^2} \sum_{n,m=0}^{\infty} e^{it(m-n)(\omega + \lambda \hbar|n+m|)} A_n A_m^* \frac{\hbar^{2n}}{2\pi^2} ,$$

(35)

where $\beta = (x + ip)/\sqrt{2}$. If the initial state is a coherent state $|\alpha\rangle$, then its Husimi function is

$$Q(\beta) = g(\alpha, \beta) \sum_{n=0}^{\infty} \frac{\beta^* \alpha e^{-i\omega n} n!}{\sqrt{2\pi}} e^{-it(\omega n + \lambda \hbar n^2)}$$

$$g(\alpha, \beta) = \frac{e^{-|\beta|^2} - |\alpha|^2}{2\pi}.$$
FIG. 12. Roughness (a) and Negativity (b) as function of $z$ and $\beta$ for the state $\rho_z$ for $M = 8$.

The Wigner function is given by

$$W = W(\rho) = \sum_{n,m=0}^{\infty} e^{it(m-n)(\omega + \lambda h(n+m))} A_n A_m^* W(|n\rangle \langle m|)$$  \hspace{1cm} (36)

where $W(|n\rangle \langle m|) = \Pi_{m,n}$ and they are defined in [C2].

A. Classical Liouville evolution

The classical equivalent Hamiltonian is [12, 33]

$$H_{cl} = \omega h |\alpha|^2 + \lambda h^2 |\alpha|^4.$$  \hspace{1cm} (37)

At the initial time we have

$$f_t(\alpha, \beta(x, y), 0) = \frac{1}{2\pi} \exp \left[-|\beta - \alpha|^2\right].$$  \hspace{1cm} (38)

then we have [22]

$$f_t(x, p, t) = \frac{1}{2\pi} \exp \left[-|\alpha - \frac{x + ip}{\sqrt{2}}\exp [it (\omega + \lambda (x^2 + p^2))]|^2\right].$$

In figures 13 and 14 we show the Roughness and the Dynamic Distance Measure (DDM) as function of time, respectively for $\alpha = 2$ and $\alpha = 0.3$. As we can see, the quantum aspects of the dynamics are amplified as we increase the classical action ($S$), since $S \propto |\alpha|^2$, as was previously observed [12]. This feature is attenuated as an environment is included [22] and also when the system is monitored [33].

FIG. 13. Dynamic Distance Measure ($D$, red line) and Roughness ($R$, blue line) as function of time for the Quartic Oscillator with a coherent initial state $\alpha = 2$ and $\omega = 0$.

FIG. 14. Dynamic Distance Measure ($D$, red line) and Roughness ($R$, blue line) as function of time for the Quartic Oscillator with a coherent initial state $\alpha = 0.3$ and $\omega = 0$. 
From our numerical simulations, we conjecture, for the Kerr oscillator, that Roughness and DDM have a complementary aspect in the permanent regime, as Roughness increases DDM decreases and vice-versa. On the other hand, we could not find a simple mathematical relationship between them that would hold for any model.

As a final remark, we emphasize that a non-zero Roughness does not mean that the state is quantum, but it has some quantum characteristic, which is more detectable as Roughness gets closer to its upper bound. For example, we can have a quantum system in a thermal state with high temperature, expected to behave as classical, but with discrete spectrum. On the other hand, a maximum Roughness does not mean that all observable will necessarily have experimental results that diverge from its classic counterpart, but that such probability of detection is maximum. Indeed, that is the case of the Kerr oscillator: for the position expectation value it behaves almost classically, until the revival time \( t_r \approx \pi/(2\lambda) \) \[22\], but as we can see in figure 13, the Roughness has already reached its maximum for times of the order of \( \pi/(4\lambda) \). Moreover, the fact that a system is classical or not depends on the choice of the observable \[12\] and how the measurement is performed \[23, 28, 31, 33, 102, 106\].

**VI. CONCLUSION**

Inspired by the usual definition of roughness, we define the Roughness measure of a state as a distance measure between its Wigner and Husimi functions. As a general result, we emphasize that the Roughness has proved to be an effective measure for characterization of states, being able to discriminate pure states and mixed states. Because the Roughness is bounded, it is possible to compare distinct states by quantifying the degree of their classicality. The Roughness of a state lies in the interval [0,1], so we can say that the state is more classical as its roughness is closer to zero, while it is more quantum as it approaches the unity. The degree of classicality of a state is, in this way, a fuzzy-like measure, and only limit states can be said fully classic or quantum, the others have a degree of classically that varies continuosly between the extremes. Among the states approaching the upper bound, we show that the pure Fock state \(|n\rangle\) at the limit \( n \to \infty \) has maximum Roughness, as well as the squeezed state at the limits of maximum compression. The minimum Roughness value is reached for mixed states at the infinite entropy boundary. Comparing Roughness with Negativity, we believe that Roughness does a better job distinguishing between two given states, since Negativity is zero for any positive state, while Roughness is non-zero for most of them. With the results on \[13\] we can analytically evaluate the Roughness for any state that can be represented using the Fock states as a basis. We also investigated the dynamics of the Roughness for the quartic oscillator model, and we observed that, for the quartic oscillator, there is a certain complementarity relationship between Roughness and the Distance between quantum and classical Liouvillian dynamics.

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**Appendix A: An auxiliary result**

While we were proving the upper bound \( \mathcal{Q} \) for the Roughness, we found an auxiliary result. For any given quantum state, we rewrite Roughness as

\[
R^2 = R^2_{\mathcal{W}} + R^2_{\mathcal{Q}} - R^2_{\mathcal{W} \mathcal{Q}}, \tag{A1}
\]

in the same sense as we did on equations \[14a, 14b\] and \[14c\] for a Fock state. First and second terms on the RHS of the equality are clear positive, but we were not sure about the last one, since the Wigner function is not necessarily positive. Using once again the Fourier transform, as we have used to prove the upper bound, we get

\[
R^2_{\mathcal{W} \mathcal{Q}} := \int_{\mathbb{R}^2} dq dp \; W(q,p)Q(q,p) = (W, Q) = (\hat{W}, \hat{Q}) = \int_{\mathbb{R}^2} du dv \; e^{-\frac{1}{2}(u^2 + v^2)} \left| \hat{W}(u, v) \right|^2 \geq 0, \tag{A2}
\]

which proves that every single term on \[A1\] is positive. As consequence, we have an upper bound for Roughness

\[
R^2 \leq 2\pi \int_{\mathbb{R}^2} dq dp \; (W(q,p)^2 + Q(q,p)^2),
\]

which may be useful sometimes.

**Appendix B: Roughness for a Fock state, analytical results**

We give in this appendix some details for the results obtained on subsection \[12\]. We start proving the first limit on \[15\]. For this, we use Stirling’s approximation \[107\]

\[
\sqrt{2\pi n} \left( \frac{n}{e} \right)^n \leq n! \leq e^{\frac{1}{2}} \sqrt{2\pi n} \left( \frac{n}{e} \right)^n. \tag{B1}
\]

Using it properly on \[15\], we get

\[
\frac{1}{\sqrt{2\pi n}} \leq \frac{1}{2^n} \left( \frac{2n}{n} \right)^n \leq \frac{e^{\frac{1}{2}}}{\sqrt{\pi n}}. \tag{B2}
\]
and since left and right sides of the inequality above goes to 0 as $n \to \infty$, then we finish demonstration by the squeeze theorem.

Now we look to (15c) and rewrite it as

$$R_{W_n Q_n}^2 = \frac{4}{3} \left( \frac{1}{3} \right)^n (-1)^n \sum_{j=0}^{n} \frac{(n+j)!}{j!^2(n-j)!} \left( -\frac{4}{3} \right)^j,$$

$$= : C_n$$  \hspace{1cm} \text{(B3)}

where we just have defined $C_n$. It is not clear above that $C_n > 0$, as it must be\[108\] from (A2).

We rewrite $C_n$ as

$$C_n = (-1)^n \frac{n!}{n!} \sum_{j=0}^{n} (j+n) \ldots (j+1) \frac{j! n!}{(j!)^2 (n-j)!} \left( -\frac{4}{3} \right)^j =$$

$$= \frac{(-1)^n}{n!} \sum_{k=0}^{n} \left( \sum_{k=0}^{n} \frac{n+1}{k+1} \frac{j}{n} \right) \left( \frac{1}{n} \right) \left( -\frac{4}{3} \right)^j ,$$

where $\left[ \begin{array}{l} n \\ k \end{array} \right]$ are the unsigned Stirling numbers of the first kind \[109\], and they appear as coefficients for the polynomials on $j$ from the product above. Indeed, Stirling numbers of the first kind appear on rising factorials definition

$$x^{(n)} := x(x+1) \ldots (x+n-1) \Rightarrow x^{(n)} = \sum_{k=0}^{n} \left[ \begin{array}{l} n \\ k \end{array} \right] x^k ,$$

and many other applications. We reverse the order of summation to get

$$C_n = (-1)^n \frac{n!}{k+1} \sum_{k=0}^{n} \left( \sum_{j=0}^{n} j^k \left( \frac{n}{j} \right) \left( -\frac{4}{3} \right)^j \right).$$

The summation on index $j$ above may be written as derivatives of binomial as

$$C_n = (-1)^n \frac{n!}{k+1} \sum_{k=0}^{n} \left( \frac{d}{dx} \right)^k (1 + x)^n \bigg|_{x=-\frac{1}{4}} ,$$

and we may show that

$$\left( \frac{x}{d x} \right)^k (1 + x)^n = \sum_{j=0}^{k} \left\{ \begin{array}{l} k \\ j \end{array} \right\} \frac{n!}{(n-j)!} x^j (1 + x)^{n-j} ,$$

where $\left\{ \begin{array}{l} n \\ j \end{array} \right\}$ are the Stirling numbers of second kind \[109\]. We replace expression above on (B4), and reversing summation we get

$$C_n = (-1)^n \frac{n!}{k+1} \sum_{k=0}^{n} \left\{ \begin{array}{l} n+1 \\ k+1 \end{array} \right\} \frac{k}{n} \frac{j}{j} \left( -\frac{4}{3} \right)^{n-j} =$$

$$= \left( \frac{1}{3} \right)^n \sum_{j=0}^{n} \left( \sum_{k=0}^{n} \left\{ \begin{array}{l} n+1 \\ k+1 \end{array} \right\} \frac{k}{n} \frac{j}{j} \right) \frac{1}{(n-j)!} \left( -\frac{4}{3} \right)^j \left( -\frac{1}{3} \right)^{n-j} >$$

$$= \left( \frac{1}{3} \right)^n \sum_{j=0}^{n} \left( \sum_{k=0}^{n} \left\{ \begin{array}{l} n+1 \\ k+1 \end{array} \right\} \frac{k}{n} \frac{j}{j} \right) \frac{1}{(n-j)!} 4^j > 0,$$  \hspace{1cm} \text{(B5)}

and so we proved that $C_n > 0$. But, even better, we were able to prove that

$$\sum_{k=j}^{n} \left[ \begin{array}{l} n+1 \\ k+1 \end{array} \right] \left\{ \begin{array}{l} k \\ j \end{array} \right\} = (n-j)! \left( \frac{n}{j} \right)^2 ,$$  \hspace{1cm} \text{(B6)}

and so we can find \[17\].

To continue studying $R_{W_n Q_n}^2$ properties, we match eq. \[17\] to a polynomial $P_n(t)$ whose coefficients are the square of binomial coefficients \[110\], namely

$$P_n(t) := \frac{1}{2\pi} \int_0^{2\pi} d\theta \left( 1 + t^2 - 2t \cos \theta \right)^n =$$

$$= \sum_{j=0}^{n} \left( \frac{n}{j} \right)^2 t^{2j} ,$$  \hspace{1cm} \text{(B7)}

so we get

$$R_{W_n Q_n}^2 = \frac{4}{3} \left( \frac{1}{3} \right)^n \frac{P_n(2)}{2} =$$

$$= \frac{4}{3} \left( \frac{1}{3} \right)^n \frac{1}{2\pi} \int_0^{2\pi} d\theta \left( 5 - 4 \cos \theta \right)^n =$$

$$= \frac{4}{3} \left( \frac{1}{3} \right)^n \frac{1}{2\pi} \int_0^{2\pi} d\theta \left( 9 \sin^2 \frac{\theta}{2} + \cos^2 \frac{\theta}{2} \right)^n >$$

$$> \frac{4}{3} \left( \frac{1}{3} \right)^n \frac{1}{2\pi} \int_0^{2\pi} d\theta \sin^{2n} \theta ,$$

Using the known fact that

$$\frac{1}{\pi} \int_0^{\pi} d\theta \sin^{2n} \theta = \frac{1}{2^{2n}} \left( \frac{2n}{n} \right) ,$$

we have

$$R_{W_n Q_n}^2 > \frac{4}{3} \frac{1}{2^{2n}} \left( \frac{2n}{n} \right) > \frac{1}{2} \frac{1}{2^{2n}} \left( \frac{2n}{n} \right) = R_{Q_n}^2 ,$$  \hspace{1cm} \text{(B8)}
which proves \((16)\). Moreover, we have a lower bound for \(R_{W_n Q_n}^2\). It is quite more technical, but we can find a similar upper bound for \(R_{W_n Q_n}^2\). Indeed, we find a constant \(B > 1\), such that

\[
R_{W_n Q_n}^2 < B \frac{4}{3} \frac{1}{2^{2n}} \left( \frac{2n}{n} \right)^n.
\]

and so, again by using squeeze theorem, we prove that \(R_{W_n Q_n}^2 \to 0\). To do so, we define for each \(n\) the constant \(B_n\) as

\[
B_n := \left[ \frac{1}{2^{2n}} \left( \frac{2n}{n} \right) \right]^{-1} \frac{9^{-n}}{n^2} \mathcal{P}_n(2) = 2^{2n} \left( \frac{2n}{n} \right)^{-1} \frac{1}{n} \int_0^\pi d\theta \left( \sin^2 \theta + \frac{1}{9} \cos^2 \theta \right)^n.
\]

Explicit evaluation shows that \(B_0 = 1\) and \(B_1 = 10/9\). After some tedious calculations, we can show that, for any \(n \geq 1\), we have \(1 \leq B_{n+1} < B_n\), and so, we can conclude that \(B_n \leq 10/9\), for any \(n\), which ends our proof.

**Appendix C: Integrals of \(\Pi_{n,m}\) and \(\Psi_{n,m}\)**

We evaluate here useful quantities to find the Roughness for general states like that on Eq. \((20)\). Since the Wigner transform is linear \([76]\), the Wigner function for this state is

\[
W(q,p) = \sum_{m,n=0}^\infty A_{n,m} \Pi_{m,n}(\alpha),
\]

where \(\alpha = (q + ip)/\sqrt{2}\), and \(\Pi_{m,n}(\alpha)\) is given by

\[
\Pi_{m,n}(\alpha) = \left\{ \begin{array}{ll}
\frac{(-1)^m}{\pi} \sqrt{\frac{m!}{n!}} e^{-2|\alpha|^2} (\alpha)^{n-m} L_m^{n-m}(4|\alpha|^2), & \text{if } n \geq m, \\
\frac{(-1)^n}{\pi} \sqrt{\frac{n!}{m!}} e^{-2|\alpha|^2} (\alpha)^{m-n} L_m^{m-n}(4|\alpha|^2), & \text{if } n < m.
\end{array} \right.
\]

The \(L_m^{n-m}\) are the associated Laguerre functions \([83]\). Analogously, Husimi function for \((20)\) is

\[
Q(q,p) = \sum_{m,n=0}^\infty A_{n,m} \Psi_{m,n}(\alpha),
\]

where

\[
\Psi_{n,m}(\alpha) = \frac{\alpha^n (\alpha^*)^m}{2\pi n! m!} e^{-|\alpha|^2},
\]

where \(\alpha^*\) denotes the complex conjugate. It is important to stress that, for \(n = m\), equations \((C2)\) and \((C4)\), respectively give us functions \((11)\) and \((12)\) for pure states.

From definitions \((C1)-(C2)\), the Roughness for general state \((20)\) is

\[
R^2 = 2\pi \int_{\mathbb{R}^2} dq dp |W(q,p) - Q(q,p)|^2 = \sum_{n,m,n',m'} A_{n,m}^* A_{n',m'} \left[ R_{\Pi_{n,m} \Pi_{m',n'}}^2 + R_{\Psi_{n,m} \Psi_{m',n'}}^2 - \left( R_{\Pi_{n,m} \Psi_{m',n'}}^2 + R_{\Psi_{n,m} \Pi_{m',n'}}^2 \right) \right].
\]

We now present results for the integrals \((C6a)-(C6d)\) above, which are necessary to compute Roughness for general states \((20)\). All these computations were performed analytically, using some well known properties of associated Laguerre functions \([88]\).
\[
R^{2}_{\Pi_{m,n},\Pi_{m',n'}} = \delta_{n,n'}\delta_{m,m'}.
\] (C7)

\[
R^{2}_{\Psi_{m,n},\Psi_{m',n'}} = \frac{\delta_{n-m,n'-m'}}{\sqrt{n!m!n'!m'!}} \left( \frac{1}{2} \right)^{\frac{n+m+n'+m'}{2} + 1} \left( \frac{n + m + n' + m'}{2} \right)!. \] (C8)

Albeit the other two integrals are obtained in the same way, their expressions are a little bit more complicated.

First we must define \( X := \max(n, m) \), \( Y := \min(n, m) \), and similar quantities for prime indices. We get

\[
R^{2}_{\Pi_{m,n},\Pi_{m',n'}} = \frac{2}{3} \delta_{n-m,n'-m'}(-1)^Y \sqrt{Y! \over X!X'n'!} 2^{X-Y} \left( \frac{1}{3} \right)^{X-Y+X'+Y'} \sum_{j=0}^{Y} \left( \frac{X - Y + X' + Y'}{2} + j \right)! \left( -\frac{4}{3} \right)^j \] (C9)

\[
R^{2}_{\Psi_{m,n},\Psi_{m',n'}} = \frac{2}{3} \delta_{n-m,n'-m'}(-1)^{Y'} \sqrt{Y'! \over X'!X'n!} 2^{X'-Y'} \left( \frac{1}{3} \right)^{X-Y+X'+Y'} \sum_{j=0}^{Y'} \left( \frac{X' - Y' + X + Y}{2} + j \right)! \left( -\frac{4}{3} \right)^j \] (C10)

One must see that (C9) and (C10) are the same expression with non-prime and prime indices exchanged.

It may be not easy to evaluate the Roughness by hand for a general state \( (20) \) using equations (C7)-(C10), but we emphasize that we have analytically calculated integrals on Eq. (C5), and it will certainly save lots of computational resources on this task. It is easier for a computer to numerically evaluate sums like those presents on (C7)-(C10) than to compute integrals like (C5).
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[108] Actually, since (A2) was obtained for a general Wigner function, it guarantees that $C_n > 0$. Anyway it is worthwhile to explicitly show this in order to prove [106], since it will be useful to evaluate the second limit on [18].

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