FIELD THEORY FOR MULTIPLE INTEGRALS
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Abstract
New constructions in the theory of fields for multiple integrals are designed. Generalizations of the Legendre - Weyl - Caratheodory transforms and corresponding invariant integrals are introduced and explored. Connection and curvature of bundles induced by a field of extremals are calculated.

1 Introduction
Let $\mathcal{N}$ be a domain on a smooth $n$-dimensional Riemannian manifold, and let $\rho : \xi \to \mathcal{N}$ be a $\nu$-dimensional vector bundle over the base $\mathcal{N}$. A fiber of the bundle over a point $t \in \mathcal{N}$, i.e. the full inverse image of the point $t$ under the map $\rho$, is an $\nu$-dimensional linear subspace. Local coordinates on $\mathcal{N}$ are denoted by $t = (t^1, ... t^n)$; local coordinates on fibers are denoted by $x = (x^1, ... x^\nu)$. Usual convention on summation over two-fold occurring indices is used. Latin indices correspond to coordinates on the base and vary from 1 to $n$, Greek ones correspond to coordinates on fibers and vary from 1 to $\nu$. Multi-indices will be denoted by capital Latin and Greek indices respectively. Collection of all indices from 1 to $n$ is denoted by $I$. The ordered exterior product of differential entering into the multi-index $K$ is denoted by $dt^K := dt^{i_1} \wedge dt^{i_2} \wedge ... dt^{i_k}$ (the letter $K$ shows that $|K| = k$, where the absolute value of multi-index means the number of its indices). The symbol wedge for exterior product (the symbol $\wedge$) will be omitted for the sake of brevity; it always will be implied while dealing with product of differentials.

Consider the functional whose part related to the chart $V \subset \mathcal{N}$, is

$$\mathcal{F} = \int_V f\left(t, x, \frac{Dx}{Dt}\right) dt^I. \quad (1)$$

Subsequent calculations will be produced in coordinates of the chart $V$. Let us denote by $J_1(\xi)$ the bundle of 1-jets over $\xi$ and let

$$q^\alpha_i := \frac{\partial x^\alpha}{\partial t^i} = g^\alpha_i(t, x) \quad (2)$$

be a section of $J_1(\xi)$. Such section can be considered as a slope field $\mathcal{G}$, i.e. as a distribution of $n$-dimensional planes in the space $\xi$. We shall say that a manifold $M = \{x = \hat{x}(t)\}$ embedded in the slope field $\mathcal{G}$, if $\frac{dx^\alpha}{dt^i} = g^\alpha_i(t, x)$.

We distinguish three standpoints concerning the arguments in our functions:
1. If $t^i, x^\alpha, \frac{\partial x^\alpha}{\partial t^i}$ are taken as independent variables, as for instance in the function $L$, then the derivatives with respect to these variables are marked by attaching the respective variable as an index.
2. By using a given slope field $\mathcal{G}$, the arguments $\frac{dx^\alpha}{dt^i}$ are replaced by functions $g^\alpha_i(t, x)$ that depends on the $t^i$ and $x^\alpha$ only. The partial derivatives with respect to the arguments $t^i$ and $x^\alpha$ are then denoted by $\frac{\partial}{\partial t^i}$ and $\frac{\partial}{\partial x^\alpha}$. 


3. The substitution \( x(t) \) and its derivatives referring to a given surface \( V \) changes functions which appeared in the second (or the first) standpoint, into functions of the \( t \) alone. Their derivation with respect to \( t \) is denoted by \( \frac{d}{dt} \).

The vanishing of the first variation is expressed by Euler’s equations

\[
\frac{d}{dt} \frac{\partial f}{\partial x^\alpha} - f_x^\alpha = 0.
\]

Functions \( x(\cdot) \) which satisfy this equation are called extremals. The Jacobi matrix of variables \( S^K \) relative to the arguments \( t^I \) is denoted by \( \frac{D[S^K]}{D[t^J]} \). The calligraphical \( D \) means the derivative with respect to the explicitly entering argument, while the direct \( D \) (say \( \frac{D[S^K]}{D[t^J]} \)) means the full derivative taking into account the dependance \( x(t) \). The determinant of this matrix is denoted by \( |\frac{D[S^K]}{D[t^J]}| \). The identity matrix is denoted by \( I \) (its dimension is implicitly defined by the formula in question).

Nonnegativity of the second variation is the natural necessary condition of minimum \([8]\). The investigation of conditions of the nonnegativity begins with works of A.Clebsch \([13]\) who explored the Dirichlet functional

\[
\delta^2 F = \int_V \left[ \frac{\partial^2 \hat{f}}{\partial (\frac{\partial x^\alpha}{\partial t^i}) \partial (\frac{\partial x^\beta}{\partial t^j})} \frac{\partial h^\alpha}{\partial t^i} \frac{\partial h^\beta}{\partial t^j} + 2 \frac{\partial^2 \hat{f}}{\partial (\frac{\partial x^\alpha}{\partial t^i}) \partial x_\beta} \frac{\partial h^\alpha}{\partial t^i} h^\beta + \frac{\partial^2 \hat{f}}{\partial x_\alpha \partial x_\beta} h^\alpha h^\beta \right] dt^I.
\]

The idea of Clebsch was to reduce this functional to the integral from its main part, i.e. to the rearrange quadratic terms relative to first derivatives. The reduction was realized by adding under the integral sign a closed differential form having the type of divergency. It seems that Clebsch presumes that for multiple integral a direct analog of Legendre condition is valid: if the second variation is nonnegative then the quadratic terms relative to first derivatives that is defined on the space of \((n \times \nu)\)-matrices

\[
q_i^\alpha = \frac{\partial x^\alpha}{\partial t^i}
\]

must be nonnegative for all values of \( t \).

But J.Hadamard \([20]\) almost through half a year after the work of Clebsch shows that it is not correct. This quadratic form is nonnegative not for all matrices. The correct necessary condition (which is called Hadamard-Legendre condition) is the following.

**Theorem 1** (Hadamard). Let the functional

\[
\delta^2 F = \int_V \left[ a_{ij}^\alpha(t) \frac{\partial x^\alpha}{\partial t^i} \frac{\partial x^\beta}{\partial t^j} + 2b_{i\beta}^\alpha(t) \frac{\partial x^\alpha}{\partial t^i} x^\beta + b_{\alpha\beta}(t) x^\alpha x^\beta \right] dt^I \tag{3}
\]

be nonnegative for functions \( x(\cdot) \) that meet boundary conditions

\[
x|_{\partial V} = 0.
\]

Then for all values of \( t \in V \) the quadratic form \( a_{ij}^\alpha(t)q_i^\alpha q_j^\beta \) takes nonnegative values on \((n \times \nu)\)-matrices having the form \( q_i^\alpha = \xi^\alpha \eta_i \) (that is for matrices of rank 1)
The assertion of the theorem can be reformulated as follows: the biquadratic form
\[ a_{\alpha\beta}(t)\xi^\alpha\xi^\beta\eta_i\eta_j \]
nonnegative for all values of \( t \in V \) and \( \xi \in \mathbb{R}^\nu, \eta \in (\mathbb{R}^n)^* \).

Let us remark that a simple sufficient condition for optimality of small pieces of extremals is the condition of convexity of \( f \) relative to variables \( Dx/Dt \). But this assumption is much more strong than the necessary condition of Hadamard-Legendre.

The work of Hadamard stimulates whole series of works that aim for decreasing the gap between necessary and sufficient conditions for optimality. One of the bright work of this cycle was that of Terpstra [30] where abstract algebraic questions suggested by this themes were considered.

Namely, let
\[ a^{ij}_{\alpha\beta}(t)q_i^\alpha q_j^\beta \] (4)
be a quadratic form on \((n \times \nu)\)-matrices \( \|q^\alpha_i\| \). Consider the cone of rank 1 matrices, that is ones having the form \( q^\alpha_i = \xi^\alpha \otimes \eta_i \). On this cone the form (4) turns into a biquadratic form
\[ a^{ij}_{\alpha\beta}(t)\xi^\alpha\xi^\beta\eta_i\eta_j, \] (5)
which is defined on pairs of vectors \( \xi \in \mathbb{R}^\nu, \eta \in (\mathbb{R}^n)^* \). Assume that the form (4) is nonnegative on matrices of rank 1. Terpstra set up the question: is it possible to turn (4) into a positive form on the space of all matrices by adding terms
\[ r^{ij}_{\alpha\beta}(q^\alpha_i q^\beta_j - q^\beta_i q^\alpha_j), \]
where tensor \( r^{ij}_{\alpha\beta} \) is skew-symmetric relative to both \( i, j \), and \( \alpha, \beta \)? Similar additions to the main terms arise by adding a closed differential form to the integrand. These additional terms give zero on rank 1 matrices, and the biquadratic form (5) do not changes. Terpstra shows that it is possible under the following condition

**Condition 1.** The form (5) admits decomposition into a sum of squares of bilinear forms from which as a minimum \( n\nu \) forms are linearly independent.

The question on a decomposition of even forms into a sum of squares was investigated by Hilbert. The final result is due to Van-der-Waerden who proves that a decomposition is realizable if \( \text{min}(\nu, n) \leq 2 \). If this minimum is more or equal to 3 then there exist indecomposable positively definite forms. The first explicit example of such forms was constructed by Terpstra [30]. More simple example was suggested later by D.Serre [24], [25].

This subject appears closely related with problems of existence of minima. The leading part in the proofs of existence plays the condition of lower semicontinuity of a functional in question.

C.Morrey [23] and later J.Ball [9] show that necessary and sufficient condition for lower semicontinuity of second variation is the Hadamard-Legendre condition. And they explored an interesting generalization of the notion of convexity for multiple integral that is called polyconvexity. To define it, consider an integrand \( f(t, x, \dot{x}) \) as a function of matrix
\[ q = \left\| \frac{\partial x^\alpha}{\partial t^i} \right\| \]
where \( t \) and \( x \) are fixed. One corresponds to each \((n \times \nu)\)-matrix the set of elements of its exterior powers, that is all \((l \times l)\)-minors of the matrix \( 1 \leq l \leq \min(n, \nu) \). One obtains a point of \( r \)-dimensional space denoted by \( \tau(q) \). It is easy to calculate that \( r = \binom{n+\nu}{n} - 1 \). The mapping \( \tau \) transfers \( \mathbb{R}^{n\nu} \) into an algebraic subset \( K \) of the space \( \mathbb{R}^r \) which is defined by Plücker relations on minors of the matrix \( q \). Thus the function \( f \) appears to be defined on the Plücker cone \( K \) of the space \( \mathbb{R}^r \). The mapping \( \tau \) transfers \( \mathbb{R}^{n\nu} \) into an algebraic subset \( K \) of the space \( \mathbb{R}^r \) which is defined by Plücker relations on minors of the matrix \( q \). Thus the function \( f \) appears to be defined on the Plücker cone \( K \) of the space \( \mathbb{R}^r \).

The function \( f \) is called polyconvex if it admits a convex extension on the whole space \( \mathbb{R}^r \). The polyconvexity of \( f \) is a sufficient (but not a necessary) condition of lower semicontinuity of the integral (see [23], [19], [26], [10]). It is interesting that functions constructed by Terpstra and Serre gives examples of lower semicontinuous but not polyconvex functionals. The gap between the necessary and the sufficient conditions for optimality was essentially shorten by Van-Hove [32] who proved that natural strengthening of the Hadamard-Legendre condition

\[
\frac{\partial^2 \hat{f}}{\partial \left( \frac{\partial x^\alpha}{\partial t^i} \right) \partial \left( \frac{\partial x^\nu}{\partial t^j} \right)} \xi^\alpha \xi^\beta \eta^i \eta^j \geq \varepsilon |\xi|^2 |\eta|^2 \tag{6}
\]

gives the locally sufficient condition for \( C^1 \)-minimum. The expression ”locally sufficient” means that the domain of the integration is sufficiently small.

The idea of the Van-Hove’s proof is the following. First, we make coefficients to be frozen, i.e. we fixe arguments \((t = t_0, x = x_0)\) in coefficients of the quadratic form of the integrand [3]. It does not affects on the estimations since it can be taken as a domain of integration an arbitrarily small neighborhood of the point \((t = t_0, x = x_0)\). Second, we apply the Fourier transform to the functional obtained and use Parseval equality.

This construction disclose the internal reason why the Hadamard-Legendre condition includes only rank 1 matrices. Namely, Fourier transform transfers the operation of differentiation into the operation of multiplication by the corresponding independent variable. So, if Fourier image of a function \( x^\alpha(t) \) is \( \xi^\alpha(\eta) \) then the image of its derivative \( \frac{\partial x^\alpha}{\partial t^i} \) will be \( \xi^\alpha \eta_i \). As a result, the biquadratic form which stand at the left side of the formula [6] appears under the integral sign of the Parseval equality, and the inequality [6] guarantee the positive definiteness of the functional in question.

To prove optimality conditions on ”large” parts of the manifold, one needs the theory of index of the functional, and the generalization of the notion of the conjugate point. It was the subject of big series of works (see, for example, Dennemeyer [16], Simons [27], Smale [28], Swanson [29], Ühlenbeck [31]). We do not concern this theme here.

The positivity of the second variation is insufficient to prove the sufficient conditions of the strong minimum. One needs the approach connected with the field theory and invariant Hilbert integral [18], [17]. The first variant of this theory was suggested by C.Carathéodory [12]. Another (relatively more simple) variant with more strong demands on functions was constructed by H.Weyl [33]. We will shortly describe both approaches in a suitable form for subsequent presentation.

In variational calculus for multiple integrals one needs as many principal functions of Hamilton (action-functions) as there are independent variables. Let we have \( n \) action-functions: \( S^i(t, x), \ (i = 1, \ldots n) \). The Weyl construction is based on invariant integral using divergence of a vector \( S^i \):

\[
\Omega = \sum_{i=1}^{n} dt^1 \ldots dt^{i-1} dS^i dt^{i+1} \ldots dt^n = \frac{dS^i}{dt^i} dt^I.
\]
Let us recall that under the expression $\frac{dS^i}{dt^i}$ we mean the full derivative of the function $S^i$ taking into account the dependence $x = x(t)$. For functions of matrices $q^\alpha_i = \partial x^\alpha / \partial t^i$ Weyl suggests to use the direct analog of Legendre transform in which the part of the scalar product plays the trace of the product of matrices.

$$f(t, x, q) \leftrightarrow f^*(t, x, p^*) = -f + \text{tr}(p^* q), \quad (p^*)^\alpha_i = \frac{\partial f}{\partial q^\alpha_i}.$$ 

Here the value $q$ as a function of $p^*$ has to give global maximum to the Weierstrass function $-f + \text{tr}(p^* q)$. So one needs the condition of convexity of the function $f$ on the space of matrices $q$ with fixed $t$ and $x$. The condition for the minimum of the Weierstrass function is an equality

$$\frac{\partial S^i}{\partial x^\alpha} = \frac{\partial f}{\partial (\frac{\partial x^\alpha}{\partial t^i})}.$$ 

The construction of Caratheodory is based on an invariant integral of the determinant from derivatives of functions $S^i$.

$$\Omega = \det \left| \frac{\partial S^i}{\partial t^j} \right| dt^i.$$ 

Caratheodory considered the Legendre transform as an algebraic mapping of the space of quadruples. Let $(f, \phi, q^\alpha_i, p^\alpha_i) 1 \leq i \leq n, 1 \leq \alpha \leq \nu$ be a set, consisting of $2(n\nu + 1)$ elements, where $f$ is the initial function, $\phi$ is the dual function, $q^\alpha_i$ are the initial independent variables that correspond to "velocities" $(\partial x^\alpha / \partial t^i)$, and $p^\alpha_i$ are the dual variables that correspond to moments $(\partial f / \partial (\partial x^\alpha / \partial t^i))$. It is supposed that all these variables are bound together by the relation

$$f + \phi = \text{tr}(pq). \quad (7)$$

We shall speak that a transformation is tangential if it transfers the canonical differential form $df - pdq$ on the bundle of 1-jets into proportional to it canonical form $df^* - q^* dp^*$. New variables of the standard Legendre transform are obtained from the old ones by simple permutation

$$f^* = \phi, \quad \phi^* = f, \quad q^* = p, \quad p^* = q.$$ 

It is evident that this transformation is tangential, birational, involutory, and it preserves the relation $(7)$.

Caratheodory has constructed a new transformation in the space of quadruples for the theory of invariant integrals having the type of determinant. The original text of Caratheodory is difficult, because of lack of motivations. Big series of many cumbersome formulas with numerous indices, which at the end bring to desirable results, give the impression of magic. So, we will give here the proof of the main theorem of Caratheodory in convenient for us matrix-form.

Let us introduce an auxiliary square matrix $A$

$$A = \|f I - pq\|.$$ 

(8)
Introduce the notation $\frac{f^n}{\det A} = \gamma$. The transpose of a matrix $B$ will be denoted by $B^t$.

Define the mapping $Z$ of a quadruple $(f, \phi, p, q)$ by the formulae

$$f^* = \gamma f; \ \phi^* = \gamma \phi; \ (p^*)^t = \gamma qA; \ (q^*)^t = A^{-1}p. \quad (9)$$

It follows

$$p^*q^* = ApqA^{-1}\gamma.$$

Using permutability of matrices $A$ and $pq$, one has

$$\frac{p^*q^*}{f^*} = \frac{pq}{f}.$$

Hence, taking into account the condition $f + \phi = \text{tr}(pq)$, one obtains

$$f^* + \phi^* = \text{tr}(p^*q^*).$$

**Theorem 2** (Caratheodory).

The transformation $Z$ is tangential, birational, involutory, and it preserves the relation (7).

**Proof.**

Rewrite the formula (8) as $pq = f\mathbb{I} - A$ and express $(q^*)^tq$ using (9). One has

$$(q^*)^tq = A^{-1}pq = A^{-1}(f\mathbb{I} - A) = A^{-1}f - \mathbb{I}. \quad \text{Hence,} \quad \mathbb{I} + (q^*)^tq = fA^{-1}.$$ It follows

$$\det(\mathbb{I} + (q^*)^tq) = \frac{f^n}{\det A} = ff^*.$$ \quad (10)

Calculate the differential of (10) using the following formula for differential of determinant

$$d(\det g) = (\det g) \text{tr}(g^{-1}dg).$$

We have

$$fdf^* + f^*df = ff^* \left( \text{tr} \left( \frac{A^t}{f}(dq^*)^t + (q^*)^tdq \right) \right) =$$

$$f^* \left( \text{tr} \left( \frac{A^t}{f}(dq^*)^t \right) + \text{tr} \left( A(q^*)^tdq \right) \right) =$$

$$ff^*(\text{tr} \left( \frac{p^*dq^*}{f} \right) + \text{tr} \left( \frac{pdq}{f} \right)) =$$

$$f \text{tr}(pq^*) + f^* \text{tr}(pdq). \quad (11)$$

By combining the first and the last members in the chain (11), one has

$$f(df^* - \text{tr}(pdq)) + f^*(df - \text{tr}(pq)) = 0. \quad (12)$$

Now let us prove that the transformation $Z$ is birational and involutory. Indeed. Repeat the transformation $Z \circ Z$. Introduce the auxiliary matrix $A^* = f^*\mathbb{I} - p^*q^*$. Using (8) one obtains

$$A^* = \gamma f\mathbb{I} - (\gamma qA)^t(A^{-1}p)^t = \gamma ||f\mathbb{I} - (A^{-1}pqA)^t||.$$
The matrix \( pq \) commute with the matrix \( A \), hence
\[
A^* = \gamma ||f\mathbb{I} - (pq)^t|| = \gamma A^t, \quad \gamma^* = \left(\frac{(f^*)^n-2}{\det A^*}\right) = \gamma^{n-2} f^{n-2} = 1/\gamma. \tag{13}
\]
Using (9), (13) successively obtain
\[
\begin{align*}
\phi^{**} &= \gamma^* \phi^* = \phi;\\
p^{**} &= \gamma^*(q^tA^t)^t = \gamma^* \gamma AA^{-1}p = p;\\
q^{**} &= ((A^*)^{-1}p^*)^t = \gamma (A^tq^t)^t A^{-1} = q. \tag{14}
\end{align*}
\]
□

Later on Th.De Donde [15], J.T.Lepage [21], H.Börner [11] a.o. connect approaches of H.Weyl and C.Caratheodory with the technique of differential forms. They tried to find the most general differential forms giving the field theory.

We think that one has to find only invariant differential forms having direct geometrical meaning (such as forms of Weyl and Caratheodory). Action-functions define simultaneous parametrization of all the manifolds which define "the flow" of solutions generated a field of extremals. This is the reason why differential forms that characterize "the flow" in the bundle \( \xi \) have to be invariant relative to choice of coordinates for a mapping \( \xi \to S' \). Besides, it is not imperative that unknown manifolds admit one-to-one projection onto the space of variables \( (t^1, \ldots, t^n) \). This fact counts in favour of invariance.

The main goal of this work is to find and to explore new constructions of invariant integrals and the corresponding tangential transformations. It will be designed a series of transformations using tangent planes of \( f(Dx/Dt) \) in spaces of different exterior powers of matrices of jets. Note that Caratheodory uses only one (the highest) exterior power — the determinant.

To construct a field theory one has to add to an integrand a closed differential form, that does not changes the value of the integral but turns the integrand into nonnegative function (analog of the Weierstrass function). As in the theory of characteristic classes [6], to design differential forms it will be reasonable to use invariant polynomials on the matrix algebra. Such polynomial were the trace and the determinant, i.e. forms suggested by Weyl and Caratheodory respectively. We consider remaining generators of the ring of invariant symmetrical polynomials from roots of matrices \( ||\frac{\partial^k}{\partial t^k}|| \).

2 \ k-Lagrangian Manifolds

Let \( C \) be a square \((n \times n)\) -matrix.

**Definition 1.**

The minor of a matrix \( C \) is called the principal one if the indices of its rows coincide with indices of its columns.

**Lemma 1.**

The sum of all principal minors having the order \( k \) for a matrix \( C \) is equal to the coefficient of the characteristic polynomial \( P(\lambda) \) standing before the term \( \lambda^{n-k} \).
Proof.
It is sufficient only to see on the matrix

\[ C = \|e_j^i - \lambda \delta_j^i\| \]

and to note that the coefficient before \( \lambda^{n-k} \) is obtained by elimination rows and columns of any \( k \) diagonal elements (that leads to principal minors), and subsequent summation over all such elements.

\( \square \)

Suppose that \( n \) action-functions \( S^i(t,x), (i = 1, \ldots n) \) are given. To construct an analog of a Hilbert integral, consider the following closed differential form

\[ \mathcal{S}_k = \sum_{|K|=k} (-1)^{r(K)} dS^{i_1} \cdots dS^{i_k} dt^{I\setminus K}, \quad (15) \]

where \( K = (i_1, \ldots, i_k) \). By \( r(K) \) we denote the number of permutations needed to put members of the exterior product of differential forms \( dt^{i_1} \cdots dt^{i_k} dt^{I\setminus K} \) in ascending order. It is easy to calculate that

\[ r(K) = \sum_{i_s \in K} i_s - \frac{k(k+1)}{2}. \]

The differential form \( \mathcal{S}_k \) corresponds to the coefficient of \( \lambda^{n-k} \) of the characteristic polynomial \( P_C(\lambda) \) of the matrix

\[ C = \left\| \frac{dS^i}{dt^j} \right\| = \left\| \frac{\partial S^i}{\partial t^j} + \frac{\partial S^i}{\partial x^\alpha} \frac{dx^\alpha}{dt^j} \right\| \]

Indeed, each of the summands of the matrix \( \mathcal{S}_k \) has the following Jacobian that corresponds to the principal minor of the matrix \( C \) as a coefficient

\[ \frac{|D[S^K]|}{|D[t^K]|}. \]

For \( k = 1 \) we have Weyl theory, for \( k = n \) Caratheodory theory.

To represent the differential form \( \mathcal{S}_k \) in terms of the integrand \( f \) we need a small regression.

Consider \((n \times n)\)-matrix \( C = \Phi + \Psi \), where \( \Phi = \text{diag}(\phi^k) \) is a diagonal matrix, \( \Psi = \|a^i b_j\| \) is a rank 1 matrix being the tensor product of a contravariant vector \( a^i \) by a covariant vector \( b_j \). We will consider only the case (needed in what follows) when \( \phi^k, a^i \) are differential forms of the first order, and \( b_j \) are elements of the basic field. Due to the noncommutativity of the exterior product of differential forms, we will conceive determinant consisted in that elements in a special, not usual, sense. While expanding the determinant, we will order factors of each terms in accordance with numbers of its rows. Taking such rule, a determinant with two equal column (but not rows!) will be equal to zero. Hence, it is possible without changing of the determinant to add its columns (but one cannot add rows!).

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Lemma 2. The determinant of the matrix $C$ is equal

$$\det C = \prod_{i=1}^{n} \phi^i + \sum_{j=1}^{n} b_j \phi^1 ... \phi^{j-1} \phi^{j+1} ... \phi^n.$$ 

Proof.

Since among the components of the vector $b_j$ there exists at least one distinct from zero (otherwise $\det C = 0$), we can set without loss of generality that $b_1 \neq 0$. Preserving the first column of the matrix $C$ we subtract from each column with the number $i$ the first one being multiplied by $b_i/b_1$. The expansion of the matrix obtained relative to the first row gives the needed formulae.

□

To find the analog of the Poincare-Cartan form, we rewrite integrand of (1) as follows

$$\frac{1}{(n)_f} \sum_{|K|=k} (-1)^{r(K)} \sum_{i \in K} \det(\text{diag} f dt^i) dt^I \setminus K = \frac{1}{(n)_f} \sum_{|K|=k} \left( (-1)^{r(K)} \prod_{i \in K} f dt_i \right) dt^I \setminus K.$$

Then we add to each factor, standing under the sign of product, the canonical form for the distribution (2): $\omega^\alpha := dx^\alpha - \sum_{j=1}^{n} g_j^\alpha dt^j$ being multiplied by the corresponding momentum $p^\alpha_i := f q^\alpha_i$. The lemma 2 gives the reason to consider as a Poincare-Cartan form the following expression

$$\Delta = \sum_{|K|=k} \Delta_K,$$ (16)

where

$$\Delta_K = \frac{(-1)^{r(K)}}{(n)_f} \prod_{i \in K} \left[ f dt^i + \frac{n}{k} \sum_{\alpha=1}^{\nu} p^\alpha_i (dx^\alpha - g_i^\alpha(t, x) dt^i) \right] dt^I \setminus K.$$ (17)

Here $\prod$ stands for exterior product of differential forms, and into the function $f$ we substitute the distribution (2). In parentheses of the formulae (17) only the summand $g_i^\alpha dt^i$ takes part, because other summands of the type $g_j^\alpha dt^j$ give zero for $j \neq i$, since the term $dt^j$ meet twice in the exterior product.

The choice of differential form (16) can be substantiated as follows. Each summand of the differential form $\sum_K dS^K dt^I \setminus K$ is a simple multivector of an order $k$. We can write it as an exterior product of one-dimensional forms. Using the invariance, we select in each factor, entering in $dS^K$, independent summands and take it as $dt^i$. The rest can be justify as follows. The differential form $\Delta$ is reduced to $f dt$ on the integral manifold of the distribution (2). Besides, the derivatives of integrands of both functionals relative to $q^\alpha_i$ coincide. These facts are provide by the choice of numerical coefficients $(n)_f$ and $n/k$. It is necessary to speak about minimum of the Weierstrass function that will be built using the differential form (16) (see below the section ”Weierstrass function”).

Note 1. The exchange of the determinant by this product is equivalent, in essence, to the restriction of a symmetrical multilinear $Ad$-invariant form, defined on a Lie algebra, to the Cartan subalgebra. This restriction uniquely defines this form (1).
Suppose that the distribution (2) is integrable. Then the manifold $\mathcal{M} \subset J_1(\xi)$, $\dim \mathcal{M} = \nu + n$ defined by equations (2) is fibred by $n$-dimensional fibers. Since $\mathcal{M}$ (due to its definition) has one-to-one projection onto the space $\xi$, the foliation on $\mathcal{M}$ induces a foliation of $\xi$ by $n$-dimensional fibers $\Phi$.

**Definition 2.**

A manifold $\mathcal{M} \subset J_1(\xi)$, $\dim \mathcal{M} = \nu + n$ is called $k$-Lagrangian if the restriction of the differential form $\Delta$ to $\mathcal{M}$ is the closed form.

The differential form $\Delta$ lead to the natural generalization of the Legendre transform. It corresponds to describing of the function $f$ from matrix variable as an envelope of the family of all tangent planes to the surface $f : \mathbb{R}^n \rightarrow \mathbb{R}$ considered as a function of exterior $k$-power of its argument $\frac{\partial f}{\partial x}$.  

**Theorem 3.** Let the distribution (2) be integrable, and the manifold $\mathcal{M}$ be $k$-Lagrangian. Then the fibers $\Phi \subset \xi$ are solutions to the Euler equations

$$
\frac{d}{dt^I}(f^{\alpha}) + f_x^{\alpha} = 0. 
$$

**Proof.**

Calculate the differential of the form

$$
\Delta_K = \frac{1}{k!} f dt + \frac{1}{(k-1)!} \sum_{\alpha=1}^{\nu} \sum_{s=1}^{k} [f^{\alpha}_{ij} dt^i ... dt^{i+s-1}(dx^\alpha - g^{\alpha}_{ij} dt^{j+s})dt^{i+s+1} ... dt^{iK}]dt^{I \setminus K}.
$$

Summands which contain more than one form $\omega^\alpha$ are omitted in this formula because after differentiation and subsequent substitution the distribution (2) they give zero. While differentiating, we consider only summands of the type $dx^\lambda dt^I$. The result will be presented as four groups.

$$
\phi_1 = \frac{1}{(k)!} \frac{\partial f}{\partial x^\lambda} dx^\lambda dt^I, \\
\phi_2 = \frac{1}{(k-1)!} f^{\alpha}_{ij} \frac{\partial \omega^\alpha}{\partial x^\lambda} dx^\lambda dt^I, \\
\phi_3 = \sum_{s=1}^{k} \frac{1}{(k-1)!} f^{\alpha}_{ij} \left( -\frac{\partial \omega^\alpha}{\partial x^\lambda} \right) dx^\lambda dt^I, \\
\phi_4 = \sum_{\nu=1}^{k} \frac{1}{(k-1)!} \frac{dt^{i+1} ... dt^{iK}}{dt^I} f^{\alpha}_{ij} dt^{i+1} dx^\lambda dt^{i+1} ... dt^{iK} dt^{I \setminus K}.
$$

The sums $\phi_1$ and $\phi_2$ are generated from differentiation with respect to $x^\lambda$ of the summand $\frac{1}{(k)!} f dt^I$. The sum $\phi_3$ is generated from differentiation with respect to $x^\lambda$ of the factor $\omega^\alpha$. In the sum $\phi_4$ stand the full derivatives with respect to $t^{i+1}$. From $\omega^\alpha$ is retained only $dx^\lambda$. To order differentials in $\phi_4$, it is necessary to permute $dt^{i+1}$ and $dx^\lambda$, so the sum $\phi_4$ changes its sign.

Further we have to sum over $K$ the expression obtained. The expression $\sum_K \phi_1$ as well as $\sum_K \phi_2$ consist of equal summands, and its number is equal to the number of groups $K$. Hence, the coefficients $\binom{n}{k}$ are cancelled. Each summand in the expression $\sum_I \phi_3$ as well as in $\sum_I \phi_4$ is met as many times as there are groups that it contain. The number of such groups equal $\binom{n-1}{k-1}$. Hence, the coefficients $\frac{1}{(k-1)!}$ are cancelled. After this reduction $\sum_K \phi_2 + \sum_K \phi_3 = 0$. The remaining summands $\sum_K \phi_1 + \sum_K \phi_4$ give
\[-\frac{d}{dt}(f_{q_i}) + f_{x^i}) \, dx^i dt = 0.\]

\[□\]

**Note 2.** For \( n = 1 \) the set of solutions to Euler equation is finite-dimensional. For \( n > 1 \) there are functional freedom to choose solutions. These solutions could be combined in the "Lagrangian manifolds" variously. This is the reason why there are many different kinds of fields of extremals for the case of multiple integral.

### 3 Weierstrass function

Having invariant integral of the Hilbert type corresponding to the form (16) one can build the analog of the Weierstrass function for the distribution of the slope field \( g^α_i(t, x) \). To have nonnegative Weierstrass function for all values of derivatives it is necessary that: first, the value of the integrand corresponding to Hilbert integral on manifolds imbedded into the slope field \( g^α_i(t, x) \) was equal to the value of \( f \), and second, its derivatives with respect to \( Dx/Dt \) were equal to that of the function \( f \). Canonical forms \( ω^α \) vanish on manifolds imbedded into the slope field in question. Hence, calculating differential one can ignore summands containing products of form \( ω^α_j = dx^α dt - g^α_j(t, x) \). Let us denote by \( \hat{f} \) the value of the function \( f \) after substitution of the slope field (2).

\[\mathcal{E}(t, x, \frac{dx}{dt}, g) = f(t, x, \frac{dx}{dt}) - \sum_{|K|=k} \frac{1}{\binom{n}{k} k^{k-1}} \det \| \hat{f} || + \frac{n}{k^m} \hat{p}_m(ω^α_j).\]

It is evident that the coefficient \( \frac{1}{\binom{n}{k} k^{k-1}} \) standing before the sum provides the condition \( \hat{f} = f \). Coefficients standing inside of determinants provide coincidence of derivatives. Indeed, let us expand the determinant in (19). Summands containing \( ω^α_j = \frac{dx^α}{dt} - g^α_j(t, x) \) in the first degree arise if we take only one factor standing on diagonal. The derivative \( \frac{dx^α}{dt} \) stands on the \( m \) place of diagonal, and its coefficient is \( \frac{n^m}{k^m} \). This does not depend on \( K \). This coefficient meets as many times as there are multiindices \( K \) containing index \( m \), that is \( \binom{n-1}{k-1} / \binom{n}{k} = \frac{k}{n} \). This gives the coincidence of derivatives of the function \( f \) and of the subtrahend in the formula (19). Hence, the following relations will be valid

\[\mathcal{E}(t, x, g, g) = 0; \quad \frac{\partial \mathcal{E}}{\partial (\frac{dx^α}{dt})}(t, x, g, g) = 0.\]  

**Definition 3.** A slope field \( g^α_i(t, x) \) is called geodesic for the differential form \( \mathcal{E}_k \) in a domain \( V \subset \xi \) if the minimum of the Weierstrass function (19) is reached at each point \( (t, x) \in V \) for \( \frac{dx^α}{dt} = g \).

**Note 3.** Let us recall that the Legendre transform in case of simple integrals and the Weyl transform in case of multiple integrals describe a function \( f(\cdot, \cdot, Dx/Dt) \) with the help of support planes to the graph \( f : \mathbb{R}^{n\nu} \to \mathbb{R} \). By contrast, the transforms \( Z_k \), defined below...
in section 6, will describe \( f \) with the help of multilinear (relative to variables \( \partial x^\alpha / \partial t^i \)) support manifolds to the graph \( f : \mathbb{R}^{m+n} \to \mathbb{R} \). These support manifolds can be regarded as planes in the space of multivectors of dimension \( k \).

**Theorem 4.**

If a manifold \( \hat{x}(\cdot) \) is imbedded into the geodesic field \( g^\alpha_i(t, x) \) in the domain \( \mathcal{V} \) then the functional in question reaches on this manifold the minimal value relative to any manifold with the same boundary lying in the domain \( \mathcal{V} \).

In addition, the local minimum relative to \( dx/dt \) gives the sufficient condition for the weak local minimum while the global minimum gives the sufficient condition for the strong minimum in \( \mathcal{V} \).

**Proof.**

The proof follows by the standard way from the invariance of the Hilbert integral and from the positivity of the Weierstrass function in the given domain.

□

Note that in this theorem we do not suppose that the slope field \( g^\alpha_i(t, x) \) is integrable. Nevertheless, the imbedding into a geodesic field gives minimum for one individual manifold. If the field is integrable then there exists a manifold with the slope defined by the field which passes through each point of the domain \( \mathcal{V} \). Due to the theorem 2 it gives the minimum, and, a fortiori, it is the extremal. Hence, in the integrable case, the manifold \( \hat{x}(\cdot) \) is imbedded into a field of extremals. Through each point of the domain \( \mathcal{V} \) passes one and only one \( n \)-dimensional extremal giving minimum to the functional. The \( n \)-parametric family of \( \nu \)-dimensional level surfaces of action-functions \( S^i = \text{Const} \) transversally cross these extremals. (Transversality conditions will be found below in section 5.) The German classical literature on variational calculus awards to the described geometrical object the name "Perfect or complete picture" (eine vollstädige Figure).

### 4 Condition of solvability

Let us find conditions for minimum of function \( \mathcal{E} \) at a point \( g \). The first derivative at a point \( g \) must be zero. We differentiate \( \mathcal{E} \) by \( dx^\lambda / dt^m \). This argument enters only into the \( m \)-column of each determinant. The derivative will be equal to the same determinant in which the \( m \)-column is changed by

\[
\frac{n}{k} \tilde{p}_\lambda^i.
\]

By expanding the determinant relative to this column one obtains

\[
\tilde{p}_\lambda^i - \frac{1}{\binom{n}{k} f_{k-1}} \sum_{|K|=k, K \ni l} \sum_{i \in K} \left( \text{adj}_l^i (C(K)) \frac{n}{k} \tilde{p}_\lambda^i \right) = 0. \tag{21}
\]

Denote by \( C(K) \) the matrix corresponding to the principal minor with the index \( K \) in formula [19]. Denote by \( \text{adj}_l^i (C(K)) \) the adjunct (the cofactor) of the element \( (i,l) \) of the matrix \( C(K) \). Denote by \( \text{adj}_{l/m}^i (C(K)) \) the adjunct of the minor standing in the columns \( (l,m) \) and in the rows \( (i,j) \) of the matrix \( C(K) \).
The second derivative of the function $E$ at a point $g$ is obtained by the exchange two columns by coefficients of the corresponding $dx/dt$. The expansion of these determinant relative to pair of columns $(l, m)$ gives the following condition of minimum: The quadratic form with the coefficients

$$\frac{\partial^2 f}{\partial q^l \partial q^m} - \frac{1}{(k-1)} \sum_{|K|=k,K \ni (lm)} \sum_{(i,j) \in K} \frac{n^2}{k^2} (\text{adj}^m_{ij}(C(K))) [\hat{p}^i_j \hat{p}^j_i - \hat{p}^i_j \hat{p}^j_i]$$

(22)
onumber

on the space $(n\nu \times n\nu)$-matrix must be nonnegative. Let us rearrange coefficients of quadratic form (22) which adds to the first summand

$$\frac{\partial^2 f}{\partial q^l \partial q^m}.$$  

(23)

If the minor $\text{adj}^m_{ij} C(K)$ is not the principal one $(l, m) \neq (i, j)$ then, after the substitution $q = g$, it will have zero row or zero column. If $(l, m) = (i, j)$ then it turns into the diagonal matrix with the diagonal elements $\hat{f}$. So, we have $\text{adj}^m_{ij} C(K) = \hat{f}^{k-2}$. Hence, all summands of coefficients in question appear the same. The number of these summands is equal to the number of minors $K$ containing the pair of indices $l, m$. So the total coefficient is equal to $\binom{n-2}{k-2}$. Hence, we have

$$\frac{\partial^2 f}{\partial q^l \partial q^m}.$$

(24)

The expression $(p^l_m p^m_\lambda - p^l_\mu p^m_\mu)$ defines the skew-symmetric form vanishing on matrices of rank 1. Hence, its addition do not violates Hadamard-Legendre condition. The formula (24) gives precisely such skew-symmetrical forms which has to be added to the integrand to obtain the invariant integral of one or other degree $k$.

\section{5 Transversality condition}

Consider moving boundary problems of minimization of the functional (11). For example we can put that the boundary $\partial V$ of unknown solution $\hat{x}(\cdot)$ belongs to a fixed manifold $\mathcal{X}$. It is evident that the solution to this problem meets the necessary conditions for optimality for the problem with the fixed boundary. Find additional necessary conditions caused by the possibility to vary the boundary.

Suppose that each point of the manifold $\hat{x}(\cdot)$ moves in space $(t, x)$ along trajectories of a vector field $T^i(t, x), X^\alpha(t, x)$ that tangent to the manifold $\mathcal{X}$. Denote by $\theta$ the time of translation. The solution of the system

$$\begin{cases}
\dot{i} = T(t, x) \\
\dot{x} = X(t, x)
\end{cases}$$

(25)

with initial conditions $(t_0, x_0) \in \hat{x}(\cdot)$ will be denoted by $T(\theta; t_0, x_0), \hat{x}(\theta; t_0, x_0)$. By $h(t)$ we denote the derivative of $x$ with respect to parameter $\theta$ for given $t$.

$$h(t) = \frac{\partial x}{\partial \theta}(0; t, \hat{x}(t)).$$

13
After substitution $T(\theta), x(\theta)$ we obtain the function $F(\theta)$. Find

$$
\frac{d}{d\theta}F(0) = \int_V \left( \hat{f}_x h^\alpha + \hat{f}_{2x} \frac{\partial h^\alpha}{\partial t^i} \right) dt^i + \int_{\partial V} \hat{T}u dS. \tag{26}
$$

Here $T_u$ is the projection of the vector $T$ on $u$ (on the normal to the boundary $\partial V$), and $dS$ is the element of the volume of the boundary. Integrate by parts the summand including derivatives of $h$. Under the integral on $V$ we get the left-hand side of the Euler equation on the manifold $\hat{x}(\cdot)$ that gives zero. Under the integral on $\partial V$ it is added the summand $\hat{f}u \partial x \partial t^i \partial h^\alpha \partial t^i$. Let us express $h^\alpha$ at points $\partial V$ through components of the vector field $(T,X)$.

Differentiation of the identity $x(\theta) = x(T(\theta), \theta)$ gives

$$
\frac{\partial x^\alpha}{\partial \theta} + \frac{\partial \hat{x}^\alpha}{\partial t^j} \frac{\partial}{\partial \theta} = X^\alpha
$$
or

$$
h^\alpha = X^\alpha - \frac{\partial \hat{x}^\alpha}{\partial t^j} T^j. \tag{27}
$$

Substitute (27) into (26) and take into account that $T_u dS = T^i (-1)^{i-1} dt^{1\ldots i}$. We obtain

$$
\frac{d}{d\theta}F(0) = \int_{\partial V} \left[ \hat{f}_{2x} X^\alpha + \left( \hat{f}_{\delta^i_j} - \hat{f}_{2x} \frac{\partial \hat{x}^\alpha}{\partial t^j} \right) T^j \right] (-1)^{i-1} dt^{1\ldots i} \tag{28}
$$

**Theorem 5.** *The necessary condition for optimality for moving boundary problem is

$$
\hat{f}_{\delta^i_j} X^\alpha + \left( \hat{f}_{\delta^i_j} - \hat{f}_{2x} \frac{\partial \hat{x}^\alpha}{\partial t^j} \right) T^j = 0
$$

for any vector field $T(t,x), X(t,x)$ which tangent to the manifold $\mathcal{X}$.*

**Proof.**

Let the theorem be violated for a vector $T(t_0,x_0), X(t_0,x_0)$ which tangent to $\mathcal{X}$ at a point $(t_0,x_0)$. Extend this vector to a smooth field which tangent to $\mathcal{X}$ and which is nonzero only in the sufficiently small neighborhood of the point $(t_0,x_0)$. For the variation corresponding to the shift along this vector field we get

$$
\frac{d}{d\theta}F(0) < 0,
$$

that contradicts to minimality of $\hat{x}$.

□

The transversality conditions give $n$ equations on the vector $(T,X)$. The number of conditions is equal to the number of action-functions. Clarify the meaning of these conditions in case $k = n$ (the case of Caratheodory). We have

$$
\int_V f dt^i = \int_V \det \left[ \frac{\partial S^i}{\partial t^j} + \frac{\partial S^i}{\partial x^\alpha} \frac{\partial x^\alpha}{\partial t^j} \right] dt^i.
$$

Denote the matrix standing under the sign of determinant by $F$. Denote by $a^r_s$ the matrix appearing in the transversality condition.
\[ a_{s}^{r} = f \delta_{s}^{r} - f \sum_{i=1}^{n} \left( F^{-1}\right)_{i}^{r} \frac{\partial S^{i}}{\partial x^{\alpha}} \frac{\partial x^{\alpha}}{\partial t^{s}} \]

The transversality condition is

\[ a_{s}^{r} T^{s} + p_{\alpha}^{i} X^{\alpha} = 0. \]

In our case

\[
\left[ f \delta_{s}^{r} - f \sum_{i=1}^{n} \left( F^{-1}\right)_{i}^{r} \frac{\partial S^{i}}{\partial x^{\alpha}} \frac{\partial x^{\alpha}}{\partial t^{s}} \right] T^{s} + f \sum_{i=1}^{n} \left( F^{-1}\right)_{i}^{r} \frac{\partial S^{i}}{\partial x^{\alpha}} X^{\alpha} = 0.
\]

Divide by \( f \) and multiply from the left by the matrix \( F_{r}^{m} \):

\[
\left[ F_{s}^{m} - \frac{\partial S^{m}}{\partial x^{\alpha}} \right] T^{s} + \frac{\partial S^{m}}{\partial x^{\alpha}} X^{\alpha} = 0.
\]

Substitute the value of \( F \)

\[
\frac{\partial S^{m}}{\partial t^{s}} T^{s} + \frac{\partial S^{m}}{\partial x^{\alpha}} X^{\alpha} = 0.
\]

Consequently, the vector \((T, X)\) is lying on the intersection of manifolds \( S^{m} = \text{Const.} \)

### 6 Generalization of Legendre-Weyl-Caratheodory transforms

We fix value \((t, x)\) as for the classical Legendre transform. These variables will not appears in subsequent formulas, and by \( f(q) \) we always understand \( f(t, x, q) \). Recall that

\[ q_{i}^{\alpha} = \frac{\partial x^{\alpha}}{\partial t^{i}} \]

are the main arguments of the function \( f \), and

\[ p_{i}^{\alpha} = \frac{\partial f}{\partial q_{i}^{\alpha}} \]

are the corresponding moments. Denote by \( \Lambda^{k} R \) exterior \( k \)-power of the matrix \( R \). It is the matrix consisted from minors of order \( k \) of the matrix \( R \). The function that corresponds to the invariant integral \[ 15 \] will be written in the form

\[
\tilde{\Delta} = \frac{1}{\binom{\nu}{k}} \text{tr} \Lambda^{k} \left\| \hat{f} \delta_{j}^{i} + \frac{n}{k} p_{i}^{\alpha} q_{j}^{\alpha} \right\|.
\]

In view of lemma 1 it is natural to relate the expression \( \Delta \) with the function

\[
\tilde{\Theta} = \text{tr} \Lambda^{k} \left\| \frac{\partial S^{i}}{\partial \theta^{j}} + \sum_{\alpha=1}^{\nu} \frac{\partial S^{i}}{\partial x^{\alpha}} \frac{\partial x^{\alpha}}{\partial \theta^{j}} \right\|. \]
The conjugate variables (with the variable \( q^\alpha_i \)) which will play the part of new independent variable in our analog of Legendre transform must be tied with \( \frac{\partial S^i}{\partial x^\alpha} \). Keeping this in mind we introduce the matrix

\[
(q^*)^\mu_m(t, x, q) = \left( \frac{\partial \tilde{\Delta}}{\partial (q^\mu_m)} \right)^t,
\]

(31)

The differentiation of (29) gives

\[
(q^*)^\mu_m = \frac{1}{\binom{n}{k} \binom{f^{k-1}}{k}} \sum_{K \ni (m, l)} \left( \text{adj}^{\mu}_{t} \left[ \tilde{f} \delta^i_j + \frac{n}{k} p^i_\alpha q^\alpha_j \right] \right) \left[ \frac{n}{k} (p^i_{t})^\mu \right].
\]

(32)

The subscript \( K \) means that entries of the matrix have indices belonging to \( K \). Using the auxiliary matrix

\[
[A^{-1}]^l_m = \frac{1}{\binom{n}{k} \binom{f^{k-1}}{k}} \sum_{K \ni (m, l)} \left( \text{adj}^{\mu}_{t} \left[ f \delta^i_j + \frac{n}{k} p^i_\alpha q^\alpha_j \right] \right) \frac{n}{k}
\]

we write \((q^*)^\mu_m\) in the form

\[
(q^*)^\mu_m = [p^i_{t}]^\mu [A^{-1}]^l_m.
\]

(33)

Denote the canonical function corresponding to \( f \) by

\[
H = f^*(q^*) = \gamma(\tilde{\Delta} - f).
\]

(34)

The definition of this function will be given later in the formula (38).

Let us express the differential form \( \Omega \) related to the function \( \tilde{\Delta} \) in canonical coordinate. We carry out of brackets the function \( f^*(q^*) \). Each simple multi-vector entering in the sum can be represented as a product of \( k \) differential forms of the first order. Select in each form the summand \( dt^i \), \( i \in K \). The number of these simple multivectors is \( \binom{n}{k} \) and we cancel \( \Omega \) by this factor. The coefficient of \( q^\alpha_i \) has to be equal to \( \hat{f}_{q^\alpha_i} \). Consequently, coefficients of \( dx^\alpha \) in factors of each monomial include \( \binom{n-1}{k-1} \) — the number of minors \( K \) containing the index \( i \). In view of our cancellation by \( \binom{n}{k} \) this factor is \( \frac{n}{k} \). As a result, under the integral appears multilinear function (relative to variables \( q^\alpha_i \)). Its derivative with respect to \( q^\mu_m \) gives \((q^*)^\mu_m\). Consequently, this function coincides with the function \( \Delta \). We arrive to the following differential form

\[
\Omega = H \sum_{|K|=k} (-1)^r(K) \left( \prod_{i \in K} (dt^i + \sum_{\alpha=1}^\nu \frac{n[(q^*)^\mu]_{i\alpha}}{kH} dx^\alpha) dt^I \setminus K \right).
\]

(35)

Normalize variables by putting

\[
Q^i_{\alpha} = \frac{n[(q^*)^\mu]_{i\alpha}}{kH}.
\]

(36)

The formula (35) takes the form

\[
\Omega = H \sum_{|K|=k} (-1)^r(K) \left( \prod_{i \in K} (dt^i + \sum_{\alpha=1}^\nu Q^i_{\alpha} dx^\alpha) dt^I \setminus K \right).
\]

(37)
The condition of minimum of the Weierstrass function (21) can be written as the following equation

\[ p_i^\alpha = \frac{\partial \tilde{\Delta}}{\partial q_i^\alpha}, \]

or

\[ q^* = p^\alpha A^{-1}. \]

In view of (24) the equation (31) can be resolved relative to \( q \). It gives the dependence

\[ q^* = \phi(q^*). \]

Define the transform \( Z_k \) following the Carathéodory’s approach. Consider quadruples \( \{f, \phi, q_i^\alpha, p_i^\alpha\} \), which satisfy the condition

\[ f + \phi = \tilde{\Delta}. \]

Introduce the auxiliary matrix \( A \) by using (32). In view of (24) the matrix \( A \) on extremals has to be nonsingular. The formula (33) gives

\[ \frac{\partial \tilde{\Delta}}{\partial q_i^\alpha} = [p^m_i]^n(A^{-1})^i_m. \]

Denote by \( \gamma \) the scalar coefficient

\[ \gamma = \frac{\hat{\gamma}^{k-2}}{\det A}. \]

Variables that are images of quadruples under the transform \( Z_k \) will be marked by superscript *'. Define \( Z_k \) by formulas

\[ f^* = \gamma \phi; \quad \phi^* = \gamma f; \quad (p^*) = \gamma Aq^*; \quad (q^*) = p^\alpha A^{-1}. \]  

The transformed function will be \( f^*(q^*) \).

**Theorem 6.** The transform \( Z_k : f(q) \mapsto f^*(q^*) \) is birational and involutory.

**Proof.**

The fact that \( Z_k \) is birational is evident.

The proof of involutority.

\[ q^*p^* = \gamma(p^\alpha A^{-1})(Aq^*) = \gamma p^\alpha q^*. \]

Since the transposition does not changes determinants we have

\[ \tilde{\Delta}^* = \gamma \tilde{\Delta}. \] 

One obtains by multiplication of the formula \( f + \phi = \tilde{\Delta} \) by \( \gamma \)

\[ f^* + \phi^* = \tilde{\Delta}^*. \]

The auxiliary matrix \( A^{-1} \) was defined through differentiation of the function (29)

Therefore the matrix \( (A^{-1})^* \) will be defined through differentiation of the matrix

\[ \frac{1}{(k^*)^{k-1}} \text{tr} A^k \left\| (f^*)^i_j \delta_j^i + \frac{n}{k} (p^*)^i_\alpha (q^*)^\alpha_j \right\|. \]

In view of the formulas (38), (39), this matrix equals

\[ \frac{\gamma}{(k^*)^{k-1}} \text{tr} A^k \left\| f \delta_j^i + \frac{n}{k} p_i^\alpha q^\alpha_i \right\|. \]
Hence \( A^* = \gamma A^t \), and

\[
\gamma^* = \frac{(f^*)^{k-2}}{\det A^*} = \frac{\gamma^{k-2}(f)^{k-2}}{\gamma^k \det A} = \frac{1}{\gamma}.
\]

The formulas obtained allow us to find the second iteration of the transpose \( Z_k \):

\[
\begin{align*}
    f^{**} &= \gamma^*(\phi^*) = f; \quad \phi^{**} = \gamma^* f^* = \phi; \\
    p^{**} &= \gamma^* A^*(q^*)^t = \gamma^* \gamma A^t (A^{-1})^t p = p; \\
    q^{**} &= (p^*)^t (A^*)^{-1} = \gamma (qA^t)^t (A^{-1})^t = q.
\end{align*}
\]

(40)

The theorem is proved. □

In full agreement with the fact that \( Z_k \) is involutory and tangential transform we have that the form \( \Omega \) is obtained from the differential form corresponding to the function \( f \) by exchange all the variables by its dual (in the sense of the transform \( Z_k \)).

**Note 4.** There is an intimacy between the transforms \( Z_k \) and the condition of polyconvexity of function \( f \). Indeed, if the function \( f \) admits a convex extension to the space \( V^k(Dx/Dt) \) then it can be described as an envelope of support planes in the space of multivectors (see the note 3 in the section 3). Hence, after the addition of corresponding expression of the type \( (24) \), we can apply to \( f \) the transform \( Z_k \).

## 7 Formulas for action-functions

Let us compare two expression of the integrand of invariant integral. The first is in terms of action-function

\[
\mathcal{S} = \sum_{|K|=k} \det \left[ \begin{array}{c} \partial S^i \partial x^\alpha \\ \partial t^j \partial u^\gamma \\
\end{array} \right] + \sum_{\alpha=1}^\nu \partial x^\alpha \partial u^\gamma.
\]

(41)

The second is in terms of initial integrand \( f \) being written as integrand of differential form \( \Omega \).

\[
H \sum_{|K|=k} \left( \prod_{i \in K} (1 + \sum_{\alpha=1}^\nu Q^\alpha_i \frac{\partial x^\alpha}{\partial u^\gamma}) \right).
\]

(42)

We expand each of determinants of the formula \( (41) \) in the sum of its columns and gather terms with the determinants of the same order. To write the corresponding formulas we put that \( J \subset K \) is a multiindex, and \( \Xi \subset \{1,2,\ldots,\nu\} \) is a multiindex of the same order as the order of \( J \).

\[
\mathcal{S} = \sum_{|K|=k} \left( \frac{D[S^K]}{D[t^K]} \right) + \sum_{s=1}^{\min(k,\nu)} \sum_{|J|=|\Xi|=s} \left( \frac{D[S^K\setminus J]}{D[t^K\setminus J]} \left| \frac{D[S^J]}{D[x^\Xi]} \right| \left| \frac{D[x^\Xi]}{D[t^J]} \right| \right).
\]

(43)

Multiply brackets under the sign of the product in the formula \( (36) \), taking into account that the Jacobian \( \frac{D[x^\Xi]}{D[t^J]} \) is equal to the coefficient arising by expressing of \( dx^\Xi \) through \( dt \). Comparing \( (38) \) with \( (43) \) we obtain the series of formulas.
\[
\sum_{|K|=k} \left| \frac{D[S^K]}{D[t^K]} \right| = H.
\]

Finally, to write the general formula we make denotation more concrete. We fix the sets \( J = \{j_1, \ldots, j_s\} \) and \( \Xi = \{\alpha_1, \ldots, \alpha_s\} \). Then

\[
\sum_{K \supset J} \left| \frac{D[S^K]}{D[t^K, x^\Xi]} \right| = H \sum_{K \supset J} \|Q^i_\alpha\| = \left( \frac{n - s}{k - s} \right) H^{1-s} \det \|\{q^i_\alpha\}_\Xi^J\|.
\]

Expressions \( \frac{D[S^K]}{D[t^K, x^\Xi]} \) are the Plücker coordinates of \( k \)-multivectors composed from gradients of the action-functions \( S^i \) (relative to both dependent and independent arguments \( (x^\alpha, t^i) \)). Summarize: The canonical variables \( H \) and \( Q \) are sums of the Plücker coordinates of gradients of the action-functions in the bundle \( \xi \). It is the usual gradient-vector in the Weyl construction, and it is only one multivector of maximum dimension \( n \) — the determinant — in the construction of Caratheodory.

Now we find the corresponding expression using initial function \( f \). Recall that

\[
\Delta = \sum_{|K|=k} \frac{1}{(n)^{k-1}} \det \left\{ f^i_\alpha - \frac{n}{k} \sum_{\alpha=1}^\nu (\delta^i_\alpha (q^j_\alpha - g^j_\alpha)) \right\}. \quad (44)
\]

With the determinant in the formula \(44\) we shall carry out the same operation as with the function \(43\). We expand each determinant into the sum of its columns and gather determinants of the same order together.

We calculate coefficients of the corresponding minors. Coefficient of \( f \) under the sign of the sum over \( K \) is equal \( \frac{n}{k} \), and this sum include the same number of identical summands. So, the total coefficient equals to 1. Coefficient of \( f^0 p^i_\alpha \frac{\partial x^\alpha}{\partial t^j} \) under the sign of sum over \( K \) is equal to \( \frac{n}{k-1} \) and the sum over \( K \) includes the same number of identical summands, since it is just the number of \( k \)-sets including the index \( i \). So, the total coefficient equals to 1 too. We have \( \frac{n-1}{k-2} \) summands including the pair of indices \((i,j)\), and the coefficient of \( f^{-1} \frac{D[x^\alpha, t^j]}{D[p^j_\alpha]} \) is equal to \( \frac{k-1}{n-1} \). The coefficient of the general member \( f^{-s+1} \frac{D[x^\Xi]}{D[p^j_\alpha]} \) is equal to \( \frac{n-s}{k-s} / \frac{n-1}{k-1} \).

The minor composed from \( \partial f / \partial p^j_\alpha \), where \( \alpha \in \Xi, i \in J \) will be denoted by

\[
\frac{D[f]}{D[p^j_\Xi]}.
\]

We have the formula

\[
\Delta = f + p^i_\alpha \frac{\partial x^\alpha}{\partial t^i} + \sum_{s=2}^{\min(k, \nu)} \sum_{|J|=|\Xi|=s} \left( f^{-s+1} \frac{\binom{n-s}{k-s} \binom{n-1}{k-1}}{\binom{n}{k-1}} \frac{D[f]}{D[p^j_\Xi]} \frac{D[x^\Xi]}{D[t^j]} \right). \quad (45)
\]

If we equate coefficients of \( \frac{D[x^\Xi]}{D[t^j]} \) in formulas \(44\) and \(45\) we obtain equations for action-functions in terms of \( f \).
8 The canonical equations

It is natural to write the analog of the Jacobi equation in the canonical coordinates, i.e. in terms of Plücker coordinates of multivectors composed from gradients of the action-functions. To obtain these equations it is sufficient to write the conditions of closeness of the differential form $\Omega$:

$$
\Omega = H \sum_{|K|=k} (-1)^{|K|} \left( \prod_{i \in K} (dt^i + \sum_{\alpha=1}^{\nu} Q^i_\alpha dx^\alpha)dt^i \setminus K \right).
$$

We have

$$
d\Omega = \sum_K \left( \frac{\partial H}{\partial t^m} dt^m + \frac{\partial H}{\partial x^\mu} dx^\mu \right) \left( \sum_K \prod (dt^i + Q^i_\alpha dx^\alpha) \right) + H \sum_K \left( \frac{\partial Q^m_\mu}{\partial x^\lambda} dx^\lambda + \frac{\partial Q^m_\mu}{\partial t^m} \right) \prod (dt^i + Q^i_\alpha dx^\alpha) = 0. \tag{46}
$$

All the summands of this product are ordered relative to the order of $dt^i$ but in factors that do not contain the index $i$, i.e. $i \notin K$, we omit all summands adding to $dt^i$. Besides, in factors $dt^i + Q^i_\alpha dx^\mu$ we omit that functions $Q^i_\mu$ the derivatives of which was already calculated. We equate coefficients of the form $d\Omega$ for independent products of differential. The coefficient of $dx^\mu dt^l$ equals

$$
\sum_{K \ni m} \left( \frac{\partial H}{\partial x^\mu} - Q^m_\mu \frac{\partial H}{\partial t^m} - H \frac{\partial Q^m_\mu}{\partial t^m} \right) = 0. \tag{47}
$$

The sign "minus" for two last members in the formula (47) is stipulated by necessity to exchange the order of differentials $dt^m$ and $dx^\mu$.

All summands standing under the sign of the sum are identical, hence each of it is zero. To write equations in a more compact form, it is convenient to introduce an operator

$$
L_\mu = \frac{\partial}{\partial x^\mu} - Q^m_\mu \frac{\partial}{\partial t^m}.
$$

In terms of the operator $L$ the equation (47) takes the form

$$
H \frac{\partial Q^m_\mu}{\partial t^m} = L_\mu H. \tag{48}
$$

**Note 5.** The abbreviated writing $L_\alpha$ is not an occasional one. With the heuristic point of view, it is convenient to imagine $Q^i_\alpha$ as if it were the differential of $t^i$ with respect to $x^\alpha$ do not calling attention to dimensions. Then the operator

$$
L_\alpha = \frac{\partial}{\partial x^\alpha} - Q^m_\alpha \frac{\partial}{\partial t^m}
$$

would be the full derivative with respect to $x^\alpha$ taking into account the imaginary dependence $t(x)$.

Rewrite the equation in another form substituting in it $Q^m_\mu = \frac{n[(q^*)^\mu]^m}{kH}$. We get

$$
\frac{n}{k} H \frac{\partial [(q^*)^\mu]^m}{\partial t^m} - \frac{n}{k} [(q^*)^\mu]^m \frac{\partial H}{\partial x^\mu} = H \frac{\partial H}{\partial x^\mu} - H Q^m_\mu \frac{\partial H}{\partial t^m}.
$$
The second members in both sides of the last equation mutually annihilate. The equation \((48)\) takes the form

\[
\frac{n\partial [(q^*)^t]_m^\mu}{k\partial t^m} = \frac{\partial H}{\partial x^\mu} \tag{49}
\]

The definition \((34)\) of the function \(H\) and the involutivity of the trasform \(Z_k\) give

\[
-\frac{\partial H}{\partial [(q^*)^t]_m^\mu} = q^\mu_m = \frac{\partial x^\mu}{\partial t^m}. \tag{50}
\]

Combining \((49)\) and \((50)\), we get the system

\[
\begin{cases}
\frac{n\partial [(q^*)^t]_m^\mu}{k\partial t^m} = \frac{\partial H}{\partial x^\mu} \\
\frac{\partial x^\mu}{\partial t^m} = -\frac{\partial H}{\partial [(q^*)^t]_m^\mu}.
\end{cases} \tag{51}
\]

**Note 6.** The Euler equations were obtained by equating to zero the coefficient of \(dx^\lambda dt^l\) of the differential form \(d\Delta\). The formula \((49)\) was also obtained by equating to zero the same coefficient of the same form but being written in canonical variables (in terms of Plücker coordinates of gradients of the action-functions). Hence, it is natural to consider the system of equations \((51)\) as a canonical form of the Euler equation. As in the classical case, the equation \((50)\) follows from very definition of the functions \(H\) and \(q^*\). The shape of the canonical system is the same for any \(k\) but the Hamiltonian as well as the function \(q^*\) depends on the choice of the invariant integral.

9 Necessary and sufficient conditions for closeness of the form \(\Omega\)

Calculate the coefficient of \(dx^\mu dx^\lambda dt^l\) at the left hand side of the formula \((46)\).

The summand \(\frac{\partial H}{\partial t^m} dt^m\) gives two members. For one of these members we choose the differential \(dx^\mu\) standing on the m-spot, and differential \(dx^\lambda\) on the l-spot. To reorder we need \(l\) transpositions. For another one we choose the differential \(dx^\mu\) standing on the l-spot, and differential \(dx^\lambda\) on the m-spot. To reorder we need \((l-1)\) transpositions. These two members take the form

\[
(-1)^l \frac{\partial H}{\partial t^m} (Q^l_\lambda Q^m_\mu - Q^l_\mu Q^m_\lambda).
\]

The summand \(\frac{\partial H}{\partial x^\mu} dx^\mu\) gives two members too. By the same reasoning we verify that its sum is

\[
(-1)^l \left( \frac{\partial H}{\partial x^\lambda} Q^l_\mu - \frac{\partial H}{\partial x^\mu} Q^l_\lambda \right).
\]

The summand \(\frac{\partial Q^l_\mu}{\partial x^\lambda} dx^\lambda\) from the second part of the formula \((46)\) gives the following pair of members
\[
(-1)^l H \left( \frac{\partial Q^l_\mu}{\partial x^\lambda} - \frac{\partial Q^l_\lambda}{\partial x^\mu} \right).
\]

Finally, the summand \(\frac{\partial Q^l_\mu}{\partial t^m} dt^m\) gives the fourth pair of members

\[
(-1)^l H \left( \frac{\partial Q^l_\lambda}{\partial t^m} Q^m_\mu - \frac{\partial Q^l_\mu}{\partial t^m} Q^m_\lambda \right).
\]

For the fixed \(l\) the expression standing under the sign of the sum by multiindex \(K\) does not depend on \(K\). Hence, the sum of all four written out pairs equals to zero. We get

\[
\frac{\partial H}{\partial t^m}(Q^l_\lambda Q^m_\mu - Q^l_\mu Q^m_\lambda) + \left( \frac{\partial H}{\partial x^\mu} Q^l_\mu - \frac{\partial H}{\partial x^\mu} Q^l_\lambda \right)
+ H \left( \frac{\partial Q^l_\lambda}{\partial t^m} Q^m_\mu - \frac{\partial Q^l_\mu}{\partial t^m} Q^m_\lambda \right) = 0. \tag{52}
\]

Using the definition of the operator \(L\) and grouping in the due order the summands we can rewrite the formula (52) in the form

\[
Q^l_\lambda L_\mu(H) + HL_\lambda(Q^l_\mu) - Q^l_\mu L_\lambda(H) - HL_\mu(Q^l_\lambda) = 0.
\]

Since \(L\) is the differential operator of the first order we can use the Leibnitz rule for differentiation of product. Hence, the last formula turns into

\[
L_\lambda(Q^l_\mu H) - L_\mu(Q^l_\lambda H) = 0.
\]

Taking into account the definition \(Q^i_\alpha = \frac{n(q^*)^i_\alpha}{kH}\), we get the final formula

\[
L_\lambda(q^*^i_\mu) = L_\mu(q^*^i_\lambda), \tag{53}
\]

which is the natural generalization of the potential condition of the vector of momentum.

The subsequent calculations follow the same pattern. We introduce the results abbreviating a little details of justifications.

Calculate coefficient of \(dx^\mu dx^\lambda dx^\rho dt^{\lambda(lm)}\) at the left hand side of the formula (46).

The first summand \(\frac{\partial H}{\partial t^m} dt^m\) standing before the first sum (46) gives the six members:

\[
(-1)^m \frac{\partial H}{\partial t^m}(Q^m_\rho Q^l_\mu Q^l_\lambda - Q^m_\rho Q^l_\lambda Q^l_\mu + Q^m_\rho Q^l_\lambda Q^l_\rho - Q^m_\rho Q^l_\mu Q^l_\rho).
\]

The second summand \(\frac{\partial H}{\partial x^\mu} dx^\mu\) standing before the first sum (46) gives also the six members:

\[
(-1)^l \left( \frac{\partial H}{\partial x^\mu}(Q^l_\lambda Q^\rho - Q^l_\rho Q^\lambda) + \frac{\partial H}{\partial x^\mu}(Q^l_\rho Q^\mu - Q^l_\mu Q^\rho) + \frac{\partial H}{\partial x^\mu}(Q^l_\mu Q^\lambda - Q^l_\lambda Q^\mu) \right). \tag{55}
\]

The summand \(H \frac{\partial Q^l_\mu}{\partial x^\lambda} dx^\lambda\) standing under the sign of the second sum (46) gives twelve members:
\[
\begin{align*}
( H \frac{\partial Q_i^\rho}{\partial x^\mu} Q_r^\rho + H \frac{\partial Q_i^\rho}{\partial x^\mu} Q_l^\lambda ) - & ( H \frac{\partial Q_i^\rho}{\partial x^\mu} Q_r^\lambda + H \frac{\partial Q_i^\rho}{\partial x^\mu} Q_l^\mu ) + \\
( H \frac{\partial Q_i^\mu}{\partial x^\nu} Q_r^\mu + H \frac{\partial Q_i^\mu}{\partial x^\nu} Q_l^\rho ) - & ( H \frac{\partial Q_i^\mu}{\partial x^\nu} Q_r^\rho + H \frac{\partial Q_i^\mu}{\partial x^\nu} Q_l^\mu ) + \\
( H \frac{\partial Q_i^\nu}{\partial x^\nu} Q_r^\lambda + H \frac{\partial Q_i^\nu}{\partial x^\nu} Q_l^\mu ) - & ( H \frac{\partial Q_i^\nu}{\partial x^\nu} Q_r^\mu + H \frac{\partial Q_i^\nu}{\partial x^\nu} Q_l^\lambda ) .
\end{align*}
\]

The summand \( H \frac{\partial Q_i^\mu}{\partial x^\mu} dt^m \) standing under the sign of the second sum (56) gives twelve members:

\[
\begin{align*}
HQ^m_\lambda \left( \frac{\partial Q_i^\rho}{\partial x^\mu} Q_r^\rho + \frac{\partial Q_i^\rho}{\partial x^\mu} Q_l^\mu \right) - & HQ^m_\lambda \left( \frac{\partial Q_i^\rho}{\partial x^\mu} Q_r^\mu + \frac{\partial Q_i^\rho}{\partial x^\mu} Q_l^\rho \right) + \\
HQ^m_\mu \left( \frac{\partial Q_i^\mu}{\partial x^\nu} Q_r^\mu + \frac{\partial Q_i^\mu}{\partial x^\nu} Q_l^\rho \right) - & HQ^m_\mu \left( \frac{\partial Q_i^\mu}{\partial x^\nu} Q_r^\rho + \frac{\partial Q_i^\mu}{\partial x^\nu} Q_l^\mu \right) + \\
HQ^m_\rho \left( \frac{\partial Q_i^\nu}{\partial x^\nu} Q_r^\lambda + \frac{\partial Q_i^\nu}{\partial x^\nu} Q_l^\mu \right) - & HQ^m_\rho \left( \frac{\partial Q_i^\nu}{\partial x^\nu} Q_r^\mu + \frac{\partial Q_i^\nu}{\partial x^\nu} Q_l^\lambda \right) .
\end{align*}
\]

Gathering together all these summands (54)-(57), and using the definition of the operator \( L_\alpha \), we get

\[
L_\mu [H(Q_i^\rho Q_r^\rho - Q_i^\lambda Q_l^\lambda)] + L_\lambda [H(Q_i^\rho Q_r^\mu - Q_i^\mu Q_r^\mu)] + L_\rho [H(Q_i^\mu Q_r^\lambda - Q_i^\lambda Q_r^\lambda)] = 0. \tag{58}
\]

Let us mark that the expression (58) does not depends on the multi-index \( K \). That is the reason why the right hand side of the formula (58) is zero.

To obtain the general formula we consider two multiindices \( J = \{i_1, \ldots, i_s\} \subset \{1, 2, \ldots, n\} \) of the order \( s \) and \( \Xi = \{\alpha_1, \ldots, \alpha_{s+1}\} \subset \{1, 2, \ldots, n\} \) of the order \( s + 1 \). Let us calculate the coefficient of the member \( dx^\Xi dt^J \) in the form \( d\Omega \). Let us fix the index \( \mu \in \Xi \).

The summand \( \frac{\partial H}{\partial x^\mu} dx^\mu \) standing before the first sum of the formula (46) gives members consisted of products of \( s \) factors, that was obtained from \( Q_i^\alpha dx^\alpha \) where \( \alpha \in \Xi \setminus \mu \). It is necessary to choose the index \( i \) from the multiindex \( J \), since otherwise would arise the factor \( dt^i \) which does not include into the product of differentials under consideration. Each index \( \alpha \in \Xi \setminus \mu \) and \( i \in J \) must be taken once and only once, and moreover, a permutation of a pair of indices \( \alpha_i \) and \( \alpha_j \) leads to the permutation of the differentials \( dx^\alpha_i \) and \( dx^\alpha_j \), and to the changing of the sign of the coefficient of \( \frac{\partial H}{\partial x^\mu} \). Hence, while ordering the product of the differentials we get the sign corresponded to the parity of the permutation. These demands define exactly the determinant

\[
\det \|Q_{\Xi\setminus\mu}^\alpha\|, \tag{59}
\]
composed of elements \( Q_{\Xi\setminus\mu}^\alpha \), where \( i \in J, \alpha \in \Xi \setminus \mu \). To define the sign of this determinant it is suffice to find the sign of one concrete summands among its expansion.

Coefficients of members obtained from the summand \( \frac{\partial H}{\partial x^\mu} dt^m \) standing before the first sum of the formula (46) give members consisting of products of \( s \) factors that is obtained from the summands \( Q_{\alpha}^\mu dx^\alpha \) with the upper index \( m \) (to avoid the arising of the existing differential \( dt^m \)). By repeating the previous reasoning we verify that the sum of all such differentials has as a coefficient

\[
\sum_{\mu=1}^{N} \frac{\partial H}{\partial x^\mu} Q_{\Xi\setminus\mu}^\alpha \det \|Q_{\Xi\setminus\mu}^\alpha\|, \tag{60}
\]
Joining (59) and (60), gives
\[ \sum_{\nu=1}^{\nu} \left( \frac{\partial H}{\partial x^\mu} + \frac{\partial H}{\partial t^m} Q^\mu_\mu \right) \det \| Q^J_{\Xi \mu} \| dx^\Xi dt^I \cap J = \sum_{\nu=1}^{\nu} L_\mu(H) \det \| Q^J_{\Xi \mu} \| dx^\Xi dt^I \cap J. \] (61)

Let us fix the index \( \mu \) and gather the summands that are obtained from the members \( \frac{\partial Q^\lambda_\lambda}{\partial x^\mu} dx^\mu \) of the second sum in the formula (46). Coefficients of these summands are adjuncts of the elements \( Q^\lambda_\lambda \) in the matrix \( \| Q^J_{\Xi \mu} \| \) with the natural cyclic ordering of rows and columns. After its multiplication by \( \frac{\partial Q^\lambda_\lambda}{\partial x^\mu} \) and after summation we get the derivative of the determinant with respect to \( x^\mu \). One obtains
\[ \sum_{\mu=1}^{\nu} H \frac{\partial}{\partial x^\mu} \left( \det \| Q^J_{\Xi \mu} \| \right) dx^\Xi dt^I \cap J. \] (62)

Summands obtained from \( \frac{\partial Q^\lambda_\lambda}{\partial t^m} dt^m \) standing before the second sum of (46) define the derivative of the same determinant over \( t^m \). This leads to the following expression
\[ \sum_{\mu=1}^{\nu} H Q^m_\mu \frac{\partial}{\partial t^m} \left( \det \| Q^J_{\Xi \mu} \| \right) dx^\Xi dt^I \cap J. \] (63)

The addition of (62) and (63) define the action of the operator \( L_\mu \) on the corresponding determinant. We unify (61) and (62) and get
\[ \sum_{\mu=1}^{\nu} \left( L_\mu(H) \det \| Q^J_{\Xi \mu} \| dx^\Xi dt^I \cap J + H L_\mu \left( \det \| Q^J_{\Xi \mu} \| \right) dx^\Xi dt^I \cap J \right). \]

These expressions do not depend on the multiindex \( \mathcal{K} \), hence, they equal to zero. The following theorem was proved

**Theorem 7.** The necessary and sufficient conditions for the differential form \( \Omega \) (38) to be closed is
\[ \sum_{\mu=1}^{\nu} \left( L_\mu(H) \det \| Q^J_{\Xi \mu} \| dx^\Xi dt^I \cap J + H L_\mu \left( \det \| Q^J_{\Xi \mu} \| \right) dx^\Xi dt^I \cap J \right) = 0 \] (64)
for any choice \( \Xi \) and \( J \).

### 10 Connection and Curvature

To find the connection generated by a field of extremals let us turn to the canonical system (51).

\[
\begin{align*}
\frac{\partial x^\mu}{\partial t^m} &= -\frac{\partial H}{\partial \left( q^{(r)} \right)^m_\mu} \\
\frac{\partial (q^{(r)})^m_\mu}{\partial t^m} &= \frac{\partial H}{\partial x^\mu}.
\end{align*}
\]
Consider the variational equations, i.e. the system for the derivatives of solutions with respect to a parameter.

\[
\frac{n \partial [(q^*)^m]}{k \partial \eta^\sigma} = U^m_{\mu \sigma}, \quad \frac{\partial x^\lambda}{\partial \eta^\sigma} = V^\lambda_{\sigma}.
\] (65)

Here the parameters \( \eta^\sigma \) corresponds to coordinates of the fiber (for instance, we can take the value of the function \( x(t) \) at \( t = t_0 \)). It is natural to consider \( \eta^\sigma \) as coordinates of the standard fiber of the bundle \( \xi \). The variational equations on a solution \( \hat{x}(\cdot), \hat{q}^*(\cdot) \) have the form

\[
\begin{aligned}
\frac{\partial V}{\partial t^m} &= -\hat{H}_{x^\rho} V - \hat{H}_{x^\rho}^* U \\
\frac{\partial U}{\partial t^m} &= \hat{H}_{xx^\rho} V + \hat{H}_{x^\rho}^* U
\end{aligned}
\] (66)

Assumption 1. Let \( U, V \) be a solution of variational equations (66) for the canonical system of the Euler equations, defined on a domain \( \mathcal{N} \) of the space \( t \).

Suppose that the matrix \( V \) is defined and invertible in \( \mathcal{N} \),

Consider the matrices \( W^m_{\rho \sigma} \), \( m = 1...n \), which are defined by the formulas \( W^m_{\mu \rho} = U^m_{\mu \sigma}(V^\rho_{\sigma})^{-1} \) on the domain \( \mathcal{N} \). By differentiating \( W^m \) we get

\[
\frac{\partial}{\partial t^m}(UV^{-1}) = \hat{H}_{xx}VV^{-1} + \hat{H}_{x^\rho}^* UV^{-1} + UV^{-1}(\hat{H}_{x^\rho}^* V + \hat{H}_{x^\rho}^* U)V^{-1}.
\]

By substituting here the definition of \( W \) we get the Riccati equation in partial derivatives (1) for the second variation of the functional (1) that corresponds to the Hamiltonian \( H \):

\[
\frac{\partial W^m}{\partial t^m} = \hat{H}_{xx} + \hat{H}_{x^\rho}^* W + W \hat{H}_{x^\rho} + W \hat{H}_{x^\rho}^* W.
\] (67)

This equation defines the changing of \( k \)-Lagrangian planes with the coordinates \( W^m_{\rho \sigma}(t) \) along the extremal surface \( \hat{x}(t) \) (See. [2], [3]).

Theorem 8. Let \( \mathcal{L} \) be a smooth foliation on a fibre space \( \xi \) generated by an invariant integral \( S \), and its fibers have diffeomorphic projection on a domain \( \mathcal{N} \) of the space of variables \( t \). Let the assumption 1 be fulfilled.

Then the functions

\[
W^m_{\rho \sigma} = \frac{\partial}{\partial x^\rho} \left\{ \sum_{K \in (m,i)} \text{adj}_m^{ij} \left\| \frac{\partial S^i}{\partial \xi^j} + \frac{\partial S^i}{\partial x^\alpha} \frac{\partial x^\alpha}{\partial t^\nu} \right\|_K \right\} \frac{\partial S^i}{\partial x^\mu}
\] (68)

defines the solution to the Riccati equation (67).

Proof.
The functions

\[
U^m_{\rho \beta} = \frac{n \partial (q^*)^m}{k \partial \eta^\beta}
\]
define derivatives from \( q^* \) with respect to coordinates of the standard fiber. The matrix \( V \) gives the operator of differentiation of the mapping from the moving fiber into the
standard one of the space \( \xi \). In view of assumption 1, this mapping is invertible and its inverse is defined by the matrix

\[
(V^{-1})^\beta = \frac{\partial \eta^\beta}{\partial x^\sigma}.
\]

The composition \( UV^{-1} \) gives

\[
\frac{\partial [(q^*)^m\rho]}{\partial \eta^\beta} \frac{\partial \eta^\beta}{\partial x^\sigma} = \frac{\partial [(q^*)^m\rho]}{\partial x^\sigma}.
\]

In view of the formula (31) we get

\[
[(q^*)^i\alpha](t, x, q) = \frac{\partial \tilde{\Delta}}{\partial q^i\alpha}.
\]

Functions \( \tilde{\Delta} \), \( \tilde{S} \) are the different form of the same integrand. Hence

\[
\frac{\partial \tilde{\Delta}}{\partial q^i\alpha} = \frac{\partial \tilde{S}}{\partial q^i\alpha}.
\]

The theorem is proved.

\( \square \)

**Corollary 1.** In the case \((k = 1) \) i.e. for the Weyl transform \((33)\) we obtain from \((68)\) that the Hessian of the action-vector

\[
W = UV^{-1} = \frac{\partial^2 S m}{\partial x^\rho \partial x^\sigma}.
\]

is the solution to the Riccati equation \((67)\).

Let us show that \((68)\) defines the connection on the fibre space \( \xi \). With this in mind, it is convenient to return from the canonical variables to the Lagrangian ones. Since the transform \( Z \) is involutori, we have again to make the transform \( Z \) of the quadratic approximation of the Hamiltonian that generates the variational equation \((66)\)

\[
\frac{1}{2}((H_{q^*q^*}U, U) + 2(H_{q^*x}U, V) + (H_{xx}V, V)).
\]

(69)

As a new variable we take the derivative of \((69)\) with respect to \( V \).

\[
H_{q^*q^*}U + H_{q^*x}V.
\]

Then \( W \) transforms into

\[
2Y^\alpha_{\beta} = (H_{q^*q^*}U + H_{q^*x}V)V^{-1} = H_{q^*q^*}W + H_{q^*x}.
\]

(70)

As a differential form of connection we consider

\[
\zeta^\alpha = dx^\alpha - 2Y^\alpha_{\beta}x^\beta dt^i.
\]

(71)

The operator of covariant differentiation associated with the form \((71)\) is
\[ \nabla_{\omega}x^\alpha = v^i \frac{\partial x^\alpha}{\partial t^i} - v^i \gamma_{i\beta}^\alpha x^\beta. \] (72)

The operator of projection of tangent vectors \((dt, dx)\) at a point \((t, x)\) of the bundle \(\xi\) on the fiber has the form

\[ (dt^i, dx^\alpha) \mapsto (0, dx^\alpha - \gamma_{i\beta}^\alpha x^\beta dt^i). \]

Horizontal vectors of the connection \(\nabla\) are vectors \((dt, dx)\) that belongs to the kernel of that operator: \(dx - \gamma_{i}x dt^i = 0\). Hence, the horizontal component of the vector \((dt, dx)\) is

\[ (dt, \gamma_{i}x dt^i). \] (73)

A commutator of matrices \(A\) and \(B\) we will denote, as is customary, by \([A, B]\).

**Theorem 9.** The tensor of curvature of the connection \(\nabla\) equals

\[ \mathfrak{R} = \frac{\partial \gamma_{i}}{\partial t^{j}} - \frac{\partial \gamma_{j}}{\partial t^{i}} - [\gamma_{i}, \gamma_{j}]. \] (74)

**Proof.**

Let us recall that the exterior covariant derivative of the form of connection \(\zeta\) is called the form of curvature of the given connection [5]. The exterior covariant derivative \(D\zeta\) is the value of the exterior derivative \(d\zeta\) on the horizontal components of vectors \((dt, dx)\). Let us calculate it

\[ d\zeta = -\gamma_{m}^{\alpha\beta} dx^\beta \wedge dt^m + \left( \frac{\partial}{\partial t^m} \gamma_{i}^{\alpha\beta} dt^i \wedge dt^j - \frac{\partial}{\partial t^i} \gamma_{j}^{\alpha\beta} dt^j \wedge dt^i \right). \]

By substitution of the horizontal component of the vector \((dt, dx)\) we obtain (74).

\[ \Box \]

A connection with the zero curvature is called flat.

**Theorem 10.** Let the assumption 1 be fulfilled. Then the connection \(\nabla\) that was generated by the field of extremals is flat.

**Proof.**

Let us rearrange the formula (70).

\[ \gamma = \left( H_{q^{*}q^{*}} U + H_{q^{*}x} V \right) V^{-1} = -\frac{\partial V}{\partial t^m} V^{-1}. \] (75)

Substitute it in (74). We get

\[ -\frac{\partial^2 V}{\partial t^i \partial t^j} V^{-1} + \frac{\partial V}{\partial t^i} V^{-1} \frac{\partial V}{\partial t^j} V^{-1} - \left( -\frac{\partial^2 V}{\partial t^i \partial t^j} V^{-1} + \frac{\partial V}{\partial t^i} V^{-1} \frac{\partial V}{\partial t^j} V^{-1} \right) \]

\[ \frac{\partial V}{\partial t^i} V^{-1} \frac{\partial V}{\partial t^j} V^{-1} + \frac{\partial V}{\partial t^j} V^{-1} \frac{\partial V}{\partial t^i} V^{-1} = 0. \]

\[ \Box \]

Hence, it was shown that fields of extremals define in the given chart the flat curvature. It is because that the existence of a field leads to horizontal integrable distribution of planes. Its integral surfaces give the full system of horizontal sections of the bundle \(\xi\).
11 Examples

As an example let us consider the standard Hopf bundle \( S^3 \to S^2 \). Here \( S^3 \) is realized in the space \( \mathbb{C}^2 \) as a unit sphere: \((z_1, z_2; w_1, w_2), |z|^2 + |w|^2 = 1 \). Fibers of the Hopf bundle are defined as big circles \( \{e^{ib}z, e^{ib}w\} \) passing through each point \((z, w)\). The central projection \( \pi \) from the center of the sphere onto the tangent plane \( P \) at the point \((1, 0; 0, 0)\) (at the north pole of \( S^3 \)) turns \( P \) into a 3-dimensional projective space \( \mathbb{R}P^3 \) with the coordinates that corresponds to the three last coordinates of the plane \( P \). It may be considered as a chart \( \mathcal{A} \) on the north hemisphere of \( S^3 \). The metric on the space \( \mathbb{R}P^3 \) induced by the projection \( \pi \), is

\[
d s^2 = (1 + \eta^2 + \zeta_1^2 + \zeta_2^2)^{-2} \left( (d\eta)^2 + (d\zeta_1)^2 + (d\zeta_2)^2 \right) .
\]

Fibers of the Hopf bundle projects on straight lines (one of two families of rectilinear generators of the set of hyperboloid of one sheet — projections of torus on \( S^3 \)). So, we have on \( P \) the family of geodesics \( \Psi \), and through each point passes one and only one geodesic. Each geodesic gives the absolute minimum of the length among all curves lying in the chart \( \mathcal{A} \). Let us show that there is no action-function that simultaneously synchronizes all the extremals of the set \( \Psi \). Indeed, fibers of the Hopf bundle are obtained by the simultaneous rotation about the same angle \( \varphi \) in planes \( z \) and \( w \). The point \((1, \eta; \zeta_1, \zeta_2)\) passes into the point

\[
A = ((\cos \varphi + \eta \sin \varphi), (\sin \varphi - \eta \cos \varphi), (\zeta_1 \cos \varphi + \zeta_2 \sin \varphi), (\zeta_1 \sin \varphi - \zeta_2 \cos \varphi)).
\]

The image of the \( \pi \) projection of the point \( A \) is obtained by normalization — dividing by the first coordinate. So, we get

\[
\left(\frac{\sin \varphi - \eta \cos \varphi}{\cos \varphi + \eta \sin \varphi}, \frac{\zeta_1 \cos \varphi + \zeta_2 \sin \varphi}{\cos \varphi + \eta \sin \varphi}, \frac{\zeta_1 \sin \varphi - \zeta_2 \cos \varphi}{\cos \varphi + \eta \sin \varphi}\right).
\]

Tangent vectors \( X \) to fibers of the Hopf bundle are obtained by differentiation with respect to \( \varphi \) and putting \( \varphi = 0 \). Hence, \( X = ((1 + \eta^2), (-\eta \zeta_1 + \zeta_2), (\zeta_1 + \eta \zeta_2)) \). The transversality condition for \( n = 1 \) is the orthogonality condition in the metric induced on \( P \) by the projection \( \pi \). Orthogonal planes to vectors \( X \) correspond to the space of zeroes of the differential form \( \xi = (1 + \eta^2)d\eta + (\eta \zeta_1 + \zeta_2)d\xi + (\zeta_1 + \eta \zeta_2)d\zeta_2 \). However, \( \xi \wedge d\xi \neq 0 \) and the form \( \xi \) is not integrable. It is the obstacle to design action-function. Consequently, \( \Psi \) does not generate a field of extremals. The corresponding manifold is not a Lagrangian one.

The similar situation takes place for the quaternary Hopf bundle \( S^7 \to S^4 \). It is realized as a unit sphere \( \{(z, w), \|z\|^2 + \|w\|^2 = 1\} \) in the 2-dimensional quaternary space \( \mathbb{H}^2 \). A fiber passing through a point \((z, w)\) is defined as a set of points \( \{\sigma z, \sigma w\} \) where \( \sigma \) runs over the unit quaternary sphere \( \|\sigma\| = 1 \). It is easy to verify that each fiber is the central section of the sphere \( S^7 \). Through each point of the sphere passes one and only one of these fibers. The projection \( \pi \) from the center of the sphere on the tangent plane at the north pole \((1, 0, 0; 0, 0, 0, 0)\) of the sphere turns \( P \) into 7-dimensional projective space \( \mathbb{R}P^7 \) with coordinates \((\eta_1, \eta_2, \eta_3; \zeta_0, \zeta_1, \zeta_2, \zeta_3)\) that corresponds to the seven last coordinates of the plane \( P \). The metric on the space \( \mathbb{R}P^7 \) induced by projection \( \pi \) is
$$ds^2 = (1 + (\eta)^2 + (\zeta)^2)^{-2}((d\eta)^2 + (d\zeta)^2).$$

The projection $\pi$ transfers fibers into the set $\mathcal{P}$ of 3-dimensional planes. Through each point passes one and only one plane. Each such plane gives the absolute minimum to the functional of 3-dimensional volume in $\mathbb{RP}^7$ in the class of variations lying in the considered chart. The normalization relative to the first coordinate defines the parametric equation of the planes $\mathcal{P}$:

$$\eta_i(\sigma) = (\sigma z)^{-1}_i(\sigma z)_i, \quad (i = 1, 2, 3); \quad \zeta_j(\sigma) = (\sigma w)^{-1}_j(\sigma w)_j, \quad (j = 0, 1, 2, 3).$$

Note that an element of $k$-dimensional volume is determined by the length of $k$-dimensional multivector in metric being the tensor $k$-power of the metric of ambient space [7]. The integrand in the situation in question is

$$f = (1 + \eta^2 + \zeta^2)^{-3} \sum_{|I|+|J|=3} \left| \frac{D(\eta^I \zeta^J)}{D(\sigma)} \right|^2.$$  

The function $f$ is not a convex one and the Weyl construction is inapplicable. However, the function $f$ is convex as a function of Jacobians

$$\left| \frac{D(\eta^I \zeta^J)}{D(\sigma)} \right|.$$ 

Hence, we can apply the developed theory taking $k = 3$. Let us use the transversality condition from the section 5. It can be verified that the distribution of planes that are transverse to fibers of the quaternary Hopf bundle is not an integrable one. The family $\mathcal{P}$ does not give the field of extremals and the corresponding manifold is not a 3-Lagrangian one.

Calculations in this case are much more cumbersome and we do not give it here.

It is possible to calculate the differential form of connection for the distribution of normal planes to fibers of the Hopf bundle and the corresponding form of curvature. It may be presumed that we obtain the Chern class of the Hopf bundle.

It would be interesting to consider exotic 7-dimensional Milnor’s spheres for which fibers are 3-dimensional spheres that are obtained by the action of unit quaternary $\sigma$ on $S^7 \subset \mathbb{H}^2$ using the formula $\{\sigma z, a^h w \sigma^j\}$, where $h + j = 1$ (see. [22]). Here is an additional difficulty connected with the finding of the explicit expression for metric on these spheres.

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