Estimates for solutions of Burgers type equations and some applications

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Abstract. We obtain precise large time asymptotics for the Cauchy problem for Burgers type equations satisfying shock profile condition. The proofs are based on the exact a priori estimates for (local) solutions of these equations and the result of [7].

Résumé. Nous trouvons asymptotiques précises en temps grand des solutions du problème de Cauchy pour d'équations de type de Burgers admettant des profils de chocs. Les preuves sont basées sur les résultats de [7] et sur l'estimations a priori précises des solutions de ces équations.

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Mot-clés: Burgers type équations, asymptotiques intermédiaires, ondes progressives.

1. Introduction. The Burgers type equations have been introduced for studying different models of fluids ([1],[3],[4],[10]). The difference-differential analogues of these equations have been proposed in some models of economic development ([5],[6]).

One of the most useful versions of the Burgers type equations is the following ([4],[11],[13])

\[
\frac{\partial f}{\partial t} + \varphi(f)\frac{\partial f}{\partial x} = \varepsilon \frac{\partial^2 f}{\partial x^2},
\]

(1.1)

where \( \varepsilon > 0, (x, t) \in \Omega \subset \mathbb{R}^2 \).

One of the most interesting difference-differential analogues of equation (1.1) is the following ([5],[6])

\[
\frac{\partial F}{\partial t} + \varphi(F)\frac{F(x, t) - F(x - \varepsilon, t)}{\varepsilon} = 0,
\]

(1.2)

where \( \varepsilon > 0, (x, t) \in \Omega \subset \mathbb{R}^2 \).

The interesting and difficult problems, related with equations (1.1), (1.2), are the following.

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Problem I ([4],[11]). Find asymptotic \((t \to \infty)\) of the solution \(f(x,t), x \in \mathbb{R}, t \geq t_0\), of the equation (1.1) with initial condition:

\[
\alpha \leq f(x,t_0) \leq \beta, \quad \int_{-\infty}^{0} (f(x,t_0) - \alpha) dx + \int_{0}^{\infty} (\beta - f(x,t_0)) dx < \infty. \tag{1.3}
\]

Problem II ([6]). Find asymptotic \((t \to \infty)\) of the solution \(F(n,t), n \in \mathbb{Z}, t \geq t_0\), of the equation (1.2) with \(\varepsilon = 1\) and initial condition

\[
\alpha \leq F(n,t_0) \leq \beta, \quad \sum_{-\infty}^{0} (F(n,t_0) - \alpha) + \sum_{0}^{\infty} (\beta - F(n,t_0)) < \infty. \tag{1.4}
\]

See [7], [13] for a review of several recent results on these problems.

In this paper we present a complete solution of these problems for the special case of equations, satisfying the shock profile condition. The detailed study of this special case is highly important for solving these problems (see [6], [7]).

**Definition.** The equation (1.1) (correspondingly (1.2)) satisfies \((\alpha,\beta)\)-shock profile condition, if there exist wave-train solutions of this equation of the form \(f = \tilde{f}(x-Ct)\) (corr. \(F = \tilde{F}(x-Ct)\)) such that \(\tilde{f}(x) \to \beta, x \to +\infty, \tilde{f}(x) \to \alpha, x \to -\infty\) (correspondingly \(\tilde{F}(x) \to \beta, x \to +\infty, \tilde{F}(x) \to \alpha, x \to -\infty\)).

From the results of [4],[12] it follows that equation (1.1) with positive \(\varphi\) satisfies (0,1)-shock profile condition iff

\[
\frac{1}{u} \int_{0}^{u} \varphi(y) dy > C = \int_{0}^{1} \varphi(y) dy, \quad \forall \ u \in (0,1). \tag{1.5}
\]

From the results of [5],[2] it follows that equation (1.2) with positive \(\varphi\) satisfies (0,1)-shock profile condition iff

\[
\frac{1}{u} \int_{0}^{u} \frac{dy}{\varphi(y)} < \frac{1}{C} = \int_{0}^{1} \frac{dy}{\varphi(y)}, \quad \forall \ u \in (0,1). \tag{1.6}
\]

Let further \(\varphi\) be a positive piecewise twice continuously differential function on the interval \([0,1]\).

**Theorem 1.**

i) Let equation (1.1) satisfy (0,1)-shock profile condition (1.5); \(\varphi'(0) \neq 0\) if \(\varphi(0) = C\); \(\varphi'(1) \neq 0\) if \(\varphi(1) = C\). Let \(f(x,t)\) be a solution of (1.1) with initial condition (1.3), where \(\alpha = 0, \beta = 1\). Then there exists constants \(\gamma_0\) and \(d_0\) such that

\[
\sup_{x \in \mathbb{R}} |f(x,t) - \tilde{f}(x-Ct + \varepsilon \gamma_0 \ln t + d_0)| \to 0, \quad t \to \infty, \tag{1.7}
\]

where \(\tilde{f}(x-Ct)\) is a wave-train solution of (1.1),

\[
\gamma_0 = \begin{cases} 
0, & \text{if } \varphi(0) > C > \varphi(1), \\
\frac{1}{\varphi'(0)}, & \text{if } \varphi(0) > C = \varphi(1), \\
-\frac{1}{\varphi'(1)}, & \text{if } \varphi(0) = C > \varphi(1), \\
\frac{1}{\varphi'(1)} - \frac{1}{\varphi'(0)}, & \text{if } \varphi(0) = C = \varphi(1). 
\end{cases}
\]
such initial conditions that $f, F$ where the following. Stability results presented in the paper of D. Serre [13]. Results of [13] give in particular where $\tilde{\alpha} \neq 0$ if $\varphi(1) = C$. Let $F(n, t)$ be a solution of (1.2) with initial condition (1.3), where $\alpha = 0$, $\beta = 1$ and $\Delta F(n, t_0) \geq 0$ Then there exist constants $\Gamma_0$ and $D_0$ such that

$$
\sup_{n \in \mathbb{Z}} |F(n, t) - \tilde{F}(n - Ct + \Gamma_0 \ln t + D_0)| \to 0, \ t \to \infty,
$$

(1.8)

where $\tilde{F}(x - Ct)$ is a wave-train solution of (1.2), $\Delta F(n, t) \overset{\text{def}}{=} F(n, t) - F(n - 1, t)$

$$
\Gamma_0 = \begin{cases} 
0, & \text{if } \varphi(0) > C > \varphi(1), \\
\frac{C}{2\varphi'(1)}, & \text{if } \varphi(0) > C = \varphi(1), \\
-\frac{C}{2\varphi(0)}, & \text{if } \varphi(0) = C > \varphi(1), \\
\frac{C}{2}(\varphi'(1) - \varphi'(0)), & \text{if } \varphi(0) = C = \varphi(1).
\end{cases}
$$

Remarks.

1. In the case $\varphi(0) > C > \varphi(1)$ the statement i) of Theorem 1 is the main result of [9] and the statement ii) of Theorem 1 is the main result of [5].

2. For the other cases when $\varphi(0) = C$ or $\varphi(1) = C$ or $\varphi(0) = \varphi(1) = C$ in the previous work [7] it was already obtained the existence of the shift-functions $\gamma(t) = O(\ln t)$ and $\Gamma(t, \{x\}) = O(\ln t)$ with the properties

$$
\sup_x |f(x, t) - \tilde{f}(x - Ct + \varepsilon \gamma(t))| \to 0 \quad \text{and} \quad \sup_x |F(x, t) - \tilde{F}(x - Ct + \varepsilon \Gamma(t, \{x\}))| \to 0, \ t \to \infty,
$$

where $f, F$ - solutions of (1.1), (1.2) under conditions (1.3), (1.4), $\{x\}$ is the fractional part of $x \in \mathbb{R}$.

3. It is interesting to compare the statements i), ii) of Theorem 1 with the $L^1$-stability results presented in the paper of D. Serre [13]. Results of [13] give in particular the following.

Let $f(x, t)$ and $F(n, t)$ be solutions of equations (1.1) and (1.2) correspondingly with such initial conditions that

$$
\int_{-\infty}^{\infty} |f(x, 0) - \tilde{f}(x)|dx < \infty, \quad \sum_{-\infty}^{\infty} |F(n, 0) - \tilde{F}(n)| < \infty,
$$

where $\tilde{f}(x - Ct)$ and $\tilde{F}(n - Ct)$ are wave-trains solutions of (1.1), (1.2). Then

$$
\int_{-\infty}^{\infty} |f(x, t) - \tilde{f}(x - Ct + d_0)|dx \to 0, \quad \sum_{-\infty}^{\infty} |F(n, t) - \tilde{F}(n - Ct + D_0)| \to 0, \ t \to \infty,
$$

where constants $d_0$ and $D_0$ are being calculated from equations

$$
\int_{-\infty}^{\infty} (f(x, 0) - \tilde{f}(x + d_0))dx = 0 \quad \text{and} \quad \sum_{-\infty}^{\infty} \int_{\tilde{F}(n + D_0)}^{F(n, 0)} \frac{dy}{\varphi(y)} = 0.
$$
The proof of Theorem 1 is based on the results of [7] and the following crucial a priori estimates of (local) solutions for (1.1) and (1.2).

Without loss of generality we will put further parameter \( \varepsilon = 1 \). Otherwise, we make substitution: \( t \to \frac{t}{\varepsilon}, \ x \to \frac{x}{\varepsilon} \).

**Theorem 2.**

Let in (1.1), (1.2) parameter \( \varepsilon = 1 \). Let \( C = \varphi(0) > 0, \gamma_0 > |\varphi'(0)|, \bar{x} = \frac{x-Ct}{\sqrt{Ct}}, \)

\[ \Omega_\sigma = \{(x,t) : a_1 < \bar{x} < a_2 + \sigma\sqrt{Ct}, \ 0 < a_1 < a_2 < \infty, \ \sigma \geq 0. \]  

i) If function \( f(x,t) \) defined in the domain \( \Omega_0 \) satisfies equation (1.1) and

\[ |f(x,t)| \leq \frac{\gamma}{\sqrt{Ct}}, \ (x,t) \in \Omega_0, \ t \geq t_0, \]  

then the following estimate holds

\[ \left| \frac{\partial f}{\partial x}(x,t) \right| \leq \frac{b\gamma}{Ct}, \ (x,t) \in \Omega_0, \ t \geq t_0, \]  

where

\[ b = \frac{b_0}{C} \left( \gamma \gamma_0 + \frac{1}{\delta} \right) \left( 1 + \ln \frac{\gamma \gamma_0 + 1/\delta}{\sqrt{C}} \right), \]

\( d = \min (\bar{x} - a_1, a_2 - \bar{x}, a_2/2), \ \delta = \min (1,d), \ b_0 \) is absolute constant.

ii) If function \( F(x,t) \) defined in the domain \( \Omega_\sigma, \sigma > 0, \) satisfies equation (1.2), \( \Delta F(x,t) \) defined as

\[ \Delta F(x,t) = F(x,t) - F(x-1,t) \geq 0, \ t \geq t_0 \]

then the following estimate holds

\[ |F(x,t)| \leq \frac{\Gamma \cdot \bar{x}}{\sqrt{Ct}}, \ where \ \bar{x} \in (a_1,a_2 + \sigma\sqrt{Ct}), \ t \geq t_0, \]

\[ 0 \leq \Delta F(x,t) \leq \frac{B\Gamma \cdot \bar{x}}{Ct}, \ where \ \bar{x} \in (a_1,a_2 + \sigma_0\sqrt{Ct}), \sigma > \sigma_0, \ t \geq t_0 \geq a_1^2, \]

\[ B = B_0 \left[ \frac{\Gamma + \sigma}{\sqrt{\sigma - \sigma_0}} + \frac{\gamma_0 \Gamma}{\sqrt{C}} + \frac{1}{d} + \frac{\gamma_0 \Gamma \cdot a_1}{C} \right], \ d = \bar{x} - a_1, \]

\( B_0 \) is absolute constant.

ii') If function \( F(x,t) \) defined in the domain \( \Omega_0, \) satisfies equation (1.2), \( \varphi'(0) \geq 0, \)

\( \Delta F(x,t) \geq 0, \ t \geq t_0 \) and

\[ 0 \leq |F(x,t)| \leq \frac{\Gamma}{\sqrt{Ct}}, \ (x,t) \in \Omega_0, \ t \geq t_0, \]

\[ \Delta F(x,t_0) \equiv F(x,t_0) - F(x-1,t_0) \geq 0, \]

\[ \Delta F(x,t_0) \]
then the following estimate holds

\[ 0 \leq \Delta F(x,t) \leq \frac{B \Gamma}{Ct}, \quad (x,t) \in \Omega_0, \quad t \geq t_0, \tag{1.12}' \]

where

\[ B = B_0 \left[ a_2 + \left( \frac{1}{d} + \frac{\gamma_0 \Gamma}{C} \right) (1 + \ln (1 + a_2)) \right], \quad d = \min (\bar{x} - a_1, a_2 - \bar{x}), \]

\( B_0 \) is absolute constant.

Remarks.

1. Theorem 2ii)' in the weak form (condition \( 0 \leq F(x,t) \leq O(1/\sqrt{t}) \) for \( \bar{x} \in [a_1, a_2] \) implies the estimate \( 0 \leq \Delta F(x,t) \leq O(1/t) \) for \( \bar{x} \in [\bar{a}_1, \bar{a}_2] \subset (a_1, a_2) \) was formulated in [7] (with the reference to the present paper) and was essentially used in [7].

2. Theorem 2ii) is used for the proof of Theorem 1ii) of this paper. Theorem 2i) is needed for the proof of Theorem 1i).

3. Theorem 2 can be applied to the problems I,II, because the necessary conditions (1.9), (1.11) are always satisfied due to [14], [6].

4. Theorem 2 can be applied also to the study of Problems I,II in other cases. For example, in the important case \( \alpha = \beta = 0 \) the necessary conditions (1.9), (1.11) are valid globally: \( |f(x,t)| = O(1/\sqrt{t}) \), \( |F(x,t)| = O(1/\sqrt{t}) \), \( x \in \mathbb{R}, t > 0 \) (see [8],[14],[6]).

Theorem 1ii) is proved in Section 2. The proofs of Theorem 2ii) and sketch of the proof of Theorem 2ii)' are given in Section 3. Theorem 1i) and Theorem 2i) will be proved in the another paper.

2. Asymptotics for solutions of Burgers type equations with shock profile conditions.

The detailed proof of Theorem 1ii) will be given below only in the principal case: \( \alpha = 0, \beta = 1, \varepsilon = 1, \varphi(0) > C = \varphi(1), x = n \in \mathbb{Z} \). Other cases can be proved by very similar arguments.

Let \( F(n,t), n \in \mathbb{Z}, t \in \mathbb{R}_+ \), be a solution of the equation

\[ \frac{dF(n,t)}{dt} = \varphi(F(n,t))(F(n-1,t) - F(n,t)), \tag{2.1} \]

under initial conditions: \( F(n-1,t_0) \leq F(n,t_0), n \in \mathbb{Z}, \)

\[ \sum_{-\infty}^{0} F(n,t_0) + \sum_{0}^{\infty} (1 - F(n,t_0)) < \infty. \tag{2.2} \]

By the shock profile condition there exists a wave-train solution \( \tilde{F}(n - Ct) \) for (2.1) with overfall \((0,1)\).

Let \( \Phi(F) = \int_{F}^{1} dy/\varphi(y) \). Let \( d_A(t) \), \( A > 0 \), be such function that
\[
\sum_{k=-\infty}^{[Ct + A\sqrt{t}]} \left(\Phi(F(k, t) - \Phi(\tilde{F}(k - Ct + d_A(t)))) + (Ct + A\sqrt{t} - [Ct + A\sqrt{t}])\times \right)
\]
\[
(\Phi(F([Ct + A\sqrt{t}] + 1, t)) - \Phi(\tilde{F}([Ct + A\sqrt{t}] + 1 - Ct + d_A(t)))) = 0.
\]

By Theorem 1 from [7] for any \(A > 2\sqrt{C}\) we have
\[
\frac{\Gamma}{t} = d'_{A}(t) \overset{\text{def}}{=} \frac{d}{dt}d_A(t) \leq \frac{\Gamma_+}{t},
\]
where \(0 < \Gamma_- \leq \Gamma_+ < \infty, t > t_0 > 0\) and
\[
\sup_n |F(n, t) - \tilde{F}(n - Ct + d_A(t))| \to 0, \quad t \to \infty.
\]

To prove Theorem 1ii) we use statement (2.5) and the following crucial improvement of the statement (2.4).

**Proposition 1.** Let \(A > 2\sqrt{C}\). Then the shift-function, defined by (2.3), has the following asymptotic behavior
\[
d_A(t) = \frac{1}{2} \frac{C}{\varphi'(1)} \ln t + \text{const} + o(1), \quad t \to \infty.
\]

The proof of Proposition 1 is based on the appropriate comparison of statements for Burgers type equations and on Theorem 2ii) proved in Section 3. Besides known comparison results [5],[7] we need also the following new one.

**Lemma 1.** Let
\[
\psi(z) = \frac{C}{\varphi'(1)} \exp \left(-\frac{z^2}{2}\right) \left(\int_{-\infty}^{z/2} \exp(-2y^2)dy\right)^{-1}.
\]

For any solution \(F(n, t)\) of the Cauchy problem (2.1),(2.2) and for any \(0 < \delta_0 < \delta < 1\) and \(A > 2\sqrt{C}\) there exist \(t_0 > 0, T > 0\), such that
\[
F(n, t - T) > 1 - \frac{1}{\sqrt{t}}\psi\left(\frac{n - Ct - 2\sqrt{Ct} - \delta\sqrt{Ct}}{\sqrt{Ct}}\right),
\]
if \(Ct + 2\sqrt{Ct} + (\delta - \delta_0)\sqrt{Ct} < n < Ct + A\sqrt{Ct}, t > t_0\).

**Remark.** The function \(u(\xi, t) = 1 - \frac{1}{\sqrt{t}}\psi\left(\frac{\xi}{\sqrt{t}}\right)\) is one of the most important (in fluid mechanics) solutions of the classical Burgers equation: \(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial \xi} = \frac{1}{2} \frac{\partial^2 u}{\partial \xi^2}\) (see [10]).

For the proving Lemma 1 we need additional lemmas about subsolutions for the equation (2.1) and about patching of these subsolutions.
The next lemma shows that the function $1 - \frac{1}{\sqrt{t}} \psi\left(\frac{x-Ct}{\sqrt{Ct}}\right)$, being a solution of classical Burgers equation, is also the subsolution for the equation (2.1) in the domains

$$\{(x, t) : B < \frac{x - Ct}{\sqrt{Ct}} < A, \ t > t_0\}, \ t_0 = t_0(A, B).$$

This subsolution will be called asymptotic subsolution.

**Lemma 2.** For any $B < A$ and increasing function $D(t) = O(\sqrt{t})$ there exists $t_0 > 0$ such that for $t \geq t_0$ and $x \in (Ct + B\sqrt{Ct}, Ct + A\sqrt{Ct})$ the function $\hat{F}(x, t) = 1 - \frac{1}{\sqrt{t}} \psi\left(\frac{x-Ct-D(t)}{\sqrt{Ct}}\right)$ satisfies inequality

$$\frac{\partial \hat{F}}{\partial t}(x, t) \leq \varphi(\hat{F}(x, t))(\hat{F}(x-1, t) - \hat{F}(x, t)). \quad (2.8)$$

**Remark.**

For the proof of Lemma 1 we will use Lemma 2 in the domain

$$\{(x, t) : 2 - \delta < \frac{x - Ct}{\sqrt{Ct}} < A\}$$

for $D(t) = (2 + \delta_0)\sqrt{Ct}$, $\delta_0 < \delta < 1$.

In other domains $\{(x, t) : 1 < \frac{x - Ct}{\sqrt{Ct}} \leq 2 - \delta\}$ and $\{(x, t) : \frac{x - Ct}{\sqrt{Ct}} \leq 1\}$ we will need other subsolutions for (2.1): so called diffusion subsolution $\hat{F}(x, t) = \varphi((-1)\left(\frac{x-2\sqrt{Ct}}{t}\right)$ and wave-train subsolution $\tilde{F}_\sigma(x-C\sigma t)$ with overfall $[-\sigma, 1]$, $\sigma > 0$ (see the properties of these subsolutions in [5],[6]).

**Proof of Lemma 2.** We will use the equality

$$\frac{\partial \hat{F}}{\partial t}(x, t) = \frac{1}{2t^{3/2}} \hat{\psi}(\frac{x-Ct}{2\sqrt{t}}) - \frac{1}{\sqrt{t}} \frac{d\hat{\psi}(\frac{x-Ct}{2\sqrt{t}})}{dx} \cdot (-\frac{x}{4t^{3/2}} - \frac{C}{4\sqrt{t}}),$$

where

$$\hat{\psi}(\bar{x}) = \frac{C}{\varphi'(1)} \exp\left(-\frac{2}{C} \bar{x}^2\right) \left(\int_{-\infty}^{\bar{x}} \exp\left(-\frac{2}{C} y^2\right) dy\right)^{-1}, \ \bar{x} = \frac{x-Ct}{2\sqrt{t}}.$$

Let us fixe $\beta > 0$. Then for $\bar{x} = \frac{x-Ct}{2\sqrt{t}} \geq -\beta$ and $t \rightarrow +\infty$ we have

$$\varphi(\hat{F}(x, t)) = C - \frac{\varphi'(1)}{\sqrt{t}} \hat{\psi}\left(\frac{x-Ct}{2\sqrt{t}}\right) + O\left(\frac{\psi^2(-\beta)}{t}\right),$$

$$\hat{F}(x-1, t) - \hat{F}(x, t) = -\frac{\partial \hat{F}(x, t)}{\partial x} + \frac{1}{2} \frac{\partial^2 \hat{F}(x, t)}{\partial x^2} + \ldots =$$

$$2\left(\frac{1}{2\sqrt{t}}\right)^2 \frac{d\hat{\psi}(\frac{x-Ct}{2\sqrt{t}})}{dx} - \left(\frac{1}{2\sqrt{t}}\right)^3 \frac{d^2 \hat{\psi}(\frac{x-Ct}{2\sqrt{t}})}{dx^2} + O(1/t^2).$$

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Hence, for \( \bar{x} \geq -\beta \) we obtain

\[
\frac{\partial \hat{F}(x,t)}{\partial t} - \varphi(\hat{F}(x,t))(\hat{F}(x-1,t) - \hat{F}(x,t)) = \\
\frac{1}{2t^{3/2}} \hat{\psi}(\bar{x}) + \frac{1}{t^2} \frac{d\hat{\psi}(\bar{x})}{d\bar{x}} \left( \frac{2Ct + 2\bar{x}\sqrt{t}}{4} \right) - \\
(C - \frac{\varphi'(1)}{\sqrt{t}} \hat{\psi}(\bar{x})) \left( \frac{1}{2t} \frac{d\hat{\psi}(\bar{x})}{d\bar{x}} - \frac{1}{8t^{3/2}} \frac{d^2\hat{\psi}(\bar{x})}{d\bar{x}^2} \right) + O(1/t^2) = \\
\frac{1}{2t^{3/2}} \left( \hat{\psi}(\bar{x}) + x \frac{d\hat{\psi}(\bar{x})}{d\bar{x}} + \varphi'(1) \hat{\psi}(\bar{x}) \frac{d\hat{\psi}(\bar{x})}{d\bar{x}} + \frac{C}{4} \frac{d^2\hat{\psi}(\bar{x})}{d\bar{x}^2} \right) + O(1/t^2). 
\]

(2.9)

By direct differentiation with respect to \( \bar{x} \) we obtain

\[
\frac{d^2\hat{\psi}}{d\bar{x}^2} + \left( \frac{4}{C} \bar{x} + \frac{2\varphi'(1)}{C} \hat{\psi}(\bar{x}) \right) \frac{d\hat{\psi}(\bar{x})}{d\bar{x}} + \frac{4}{C} \hat{\psi}(\bar{x}) = 0.
\]

Hence,

\[
\hat{\psi}(\bar{x}) + \bar{x} \frac{d\hat{\psi}(\bar{x})}{d\bar{x}} + \varphi'(1) \hat{\psi}(\bar{x}) \frac{d\hat{\psi}(\bar{x})}{d\bar{x}} + \frac{C}{4} \frac{d^2\hat{\psi}(\bar{x})}{d\bar{x}^2} = \\
\hat{\psi}(\bar{x}) + \bar{x} \frac{d\hat{\psi}(\bar{x})}{d\bar{x}} + \varphi'(1) \hat{\psi}(\bar{x}) \frac{d\hat{\psi}(\bar{x})}{d\bar{x}} - \bar{x} \frac{d\hat{\psi}(\bar{x})}{d\bar{x}} - \frac{\varphi'(1)}{2} \hat{\psi}(\bar{x}) \frac{d\hat{\psi}(\bar{x})}{d\bar{x}} - \hat{\psi}(\bar{x}) = \\
\frac{\varphi'(1)}{2} \hat{\psi}(\bar{x}) \frac{d\hat{\psi}(\bar{x})}{d\bar{x}}.
\]

Let us check the inequality

\[
\frac{d\hat{\psi}(\bar{x})}{d\bar{x}} < 0, \quad \forall \bar{x} \in \mathbb{R}.
\]

(2.10)

By direct differentiation we have

\[
\frac{d\hat{\psi}(\bar{x})}{d\bar{x}} = -\frac{4}{C} \bar{x} \hat{\psi}(\bar{x}) - \frac{\varphi'(1)}{C} \hat{\psi}^2(\bar{x}).
\]

This implies the equality

\[
\frac{d\hat{\psi}(\bar{x})}{d\bar{x}} = \\
- \hat{\psi}(\bar{x}) \left( \int_{-\infty}^{\bar{x}} \exp \left( -\frac{2}{C} y^2 \right) dy \right)^{-1} \left( \int_{-\infty}^{\bar{x}} \exp \left( -\frac{2}{C} y^2 \right) dy + \frac{C \exp \left( -2\bar{x}^2 / C \right)}{4\bar{x}} \right) \frac{4\bar{x}}{C}.
\]

Hence, (2.8) is equivalent to the inequality

\[
\frac{4}{C} \bar{x} \int_{-\infty}^{\bar{x}} \exp \left( -\frac{2}{C} y^2 \right) dy + \exp \left( -2\bar{x}^2 / C \right) > 0.
\]
For $\bar{x} \geq 0$ this inequality is obvious. For $\bar{x} < 0$ this inequality follows from the relations
\[
\lim_{\bar{x} \to -\infty} \left( \int_{-\infty}^{\bar{x}} \exp \left( -\frac{2}{C} y^2 \right) dy + \frac{C \exp \left( -2\bar{x}^2/C \right)}{4\bar{x}} \right) = 0 \quad \text{and}
\]
\[
\frac{d}{d\bar{x}} \left( \int_{-\infty}^{\bar{x}} \exp \left( -\frac{2}{C} y^2 \right) dy + \frac{C \exp \left( -2\bar{x}^2/C \right)}{4\bar{x}} \right) = -\frac{C \exp \left( -2\bar{x}^2/C \right)}{4\bar{x}^2} < 0.
\]

From (2.9), (2.10) it follows that there exists $\sigma > 0$ such that
\[
\sup \{ \hat{\psi}(\bar{x}) \frac{d\hat{\psi}(\bar{x})}{d\bar{x}} \mid -\beta \leq \bar{x} \leq \alpha \} < -\sigma.
\]
Hence, for $x \in [Ct - \beta\sqrt{t}, Ct + \alpha\sqrt{t}]$ we obtain the estimate:
\[
\frac{\partial \hat{F}(x, t)}{\partial t} - \varphi(\hat{\psi}(x, t) (\hat{F}(x-1,t) - \hat{F}(x,t)) \leq -\frac{\varphi'(1)}{4t^{3/2}} \sigma + O(1/t^2).
\]

It means that there exists $t_0 > 0$ such that for $t \geq t_0$ and $x \in [Ct - \beta\sqrt{t}, Ct + \alpha\sqrt{t}]$ the inequality (2.8) is valid if $\hat{F}(x, t) = 1 - \exp \left( \frac{1}{\sqrt{t}} \psi \left( \frac{x-Ct}{\sqrt{Ct}} \right) \right)$. This inequality is also valid if $\hat{F}(x, t) = 1 - \frac{1}{\sqrt{t}} \psi \left( \frac{x-Ct-D(t)}{\sqrt{Ct}} \right)$ for $x \in [Ct + D(t) - \beta\sqrt{t}, Ct + D(t) + \alpha\sqrt{t}]$ because $t \mapsto D(t)$ is increasing function. The next lemma gives conditions for patching diffusion subsolutions and asymptotic subsolutions.

**Lemma 3.** For any $\delta \in (0, 1)$ and constant $\Gamma > 0$ there exists $t_0 > 0$ such that for $t \geq t_0$ and $n \in [Ct + (2 - \delta)\sqrt{Ct} - \Gamma, Ct + (2 - \delta)\sqrt{Ct} + \Gamma]$ the following inequality is valid
\[
\varphi^{-1} \left( \frac{n - 2\sqrt{Ct}}{t} \right) > 1 - \frac{1}{\sqrt{t}} \psi \left( \frac{n - Ct - 2\sqrt{Ct}}{\sqrt{Ct}} \right).
\]  

**Proof of Lemma 3.** We have equalities
\[
\lim_{\bar{x} \to -\infty} \frac{1}{\bar{x}} \exp \left( -2\bar{x}^2 \right) \left( \int_{-\infty}^{\bar{x}} \exp \left( -2y^2 \right) dy \right)^{-1} = -4 \quad \text{and}
\]
\[
\lim_{\bar{x} \to 0} \frac{1}{\bar{x}} \exp \left( -2\bar{x}^2 \right) \left( \int_{-\infty}^{\bar{x}} \exp \left( -2y^2 \right) dy \right)^{-1} = -\infty.
\]
Hence, for any $\varepsilon \in (0, 1)$ there exists $\bar{x}^*(\varepsilon) < 0$ such that
\[
\exp \left( -2(\bar{x}^*)^2 \right) \left( \int_{-\infty}^{\bar{x}^*} \exp \left( -2y^2 \right) dy \right)^{-1} = -\frac{4\bar{x}^*(\varepsilon)}{1 - \varepsilon}.
\]  

Besides, $\bar{x}^*(\varepsilon) \to 0$ when $\varepsilon \to 1$. Let us take $n \in [Ct + (2 + 2\bar{x}^*)\sqrt{Ct} - \Gamma, Ct + (2 + 2\bar{x}^*)\sqrt{Ct} + \Gamma]$. Then $\frac{n - Ct - 2\sqrt{Ct}}{\sqrt{Ct}} = 2\bar{x}^* + O(1/\sqrt{t})$. 

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We have now from one side
\[ 1 - \varphi^{-1}\left(\frac{n - 2\sqrt{Ct}}{t}\right) = \frac{C}{\varphi'(1)} \frac{(-2\bar{x}^*)}{\sqrt{Ct}} + O(1/t). \]

From the other side we obtain using (2.12)
\[ \frac{1}{\sqrt{t}} \psi\left(\frac{n - Ct - 2\sqrt{Ct}}{\sqrt{Ct}}\right) = \frac{1}{\sqrt{t}} \psi(2\bar{x}^* + O(1/\sqrt{t})) = \]
\[ \frac{C}{\varphi'(1)\sqrt{t}} \exp\left(-2(\bar{x}^* + O(1/\sqrt{t}))^2\right) \left( \int_{-\infty}^{\infty} \exp(-2y^2)dy \right)^{-1} = \]
\[ \frac{C}{\varphi'(1)\sqrt{t}} \frac{(-4\bar{x}^*)}{1 - \varepsilon} + O(1/t). \]

If \(-\frac{2}{\sqrt{C}} < \frac{4}{1 - \varepsilon}\) then there exists \(t_0 > 0\) such that
\[ 1 - \varphi^{-1}\left(\frac{n - 2\sqrt{Ct}}{t}\right) < \frac{1}{\sqrt{t}} \psi\left(\frac{n - Ct - 2\sqrt{Ct}}{\sqrt{Ct}}\right). \]

Besides, if \((1 - \varepsilon)\) small enough we have \(-2\bar{x}^* \in [0, 1]\). So, we can finish the proof by putting \(\delta = -2\bar{x}^*\).

**Proof of Lemma 1.** Let the function \(\varphi(F)\) be extended for negative values of \(F\) as a smooth strictly decreasing function. Then there exists a wave-train solution \(\tilde{F}_\sigma(n - C\sigma t)\) for (2.1) with overfall \((-\sigma, 1), \sigma > 0\). Put \(\sigma = \sigma(t) = \exp(-t^{1/3})\). Proposition 1, Lemma 5, Lemma 6 from [7] together with Lemma 2, Lemma 3 above imply the following statement.

For any \(\delta \in (0, 1), l > 1, A > 2\sqrt{C}\) there exist \(t_0 > 0\) and increasing functions \(\gamma_1(t) = O(t^{1/3}), \gamma_2(t) = 2\sqrt{Clt + a(l)}\) such that

\[
F^-(n, t) = \begin{cases}
\tilde{F}_{\sigma(t)}(n - Ct - \gamma_1), & n \leq Ct + \sqrt{Clt + a(l)}, \\
\varphi^{-1}\left(\frac{n - Ct - \gamma_2}{\sqrt{Ct}}\right), & Ct + \sqrt{Clt + a(l)} < n < Ct + \gamma_1 + \gamma_2 - \delta\sqrt{Ct}, \\
1 - \frac{1}{\sqrt{t}} \psi\left(\frac{n - Ct - \gamma_2}{\sqrt{Ct}}\right), & Ct + \gamma_1 + \gamma_2 - \delta\sqrt{Ct} \leq n < Ct + A\sqrt{Ct}, \\
1 - \delta, & n \geq Ct + A\sqrt{t}
\end{cases}
\]

(2.13)
is a subsolution for (2.1), if \(t \geq t_0\).

This statement and comparison principle from [6] imply that for any solution \(F(n, t)\) of the Cauchy problem (2.1), (2.2) there exists \(T > 0\) such that
\[ F(n, t) > F^-(n, t + T), \]

(2.14)

if \(n \in \mathbb{Z}, t \geq -T + t_0\).

Lemma 1 follows from (2.13) and (2.14).
Proof of Proposition 1. Put \( \kappa(t) = \{Ct + A\sqrt{t}\}, 0 \leq \kappa(t) \leq 1, N(t) = [Ct + A\sqrt{t}], F = F(N(t), t), F_1 = F(N(t) + 1, t), \tilde{F} = \tilde{F}(N(t) - Ct + d_A(t)), \tilde{F}_1 = \tilde{F}(N(t) + 1 - Ct + d_A(t)) \). Proposition 3 from [7] implies the following asymptotic formula for

\[
d'_A(t) \overset{\text{def}}{=} \frac{d}{dt}(d_A(t)) = (1 + O(1/\sqrt{t})) = C(1 - \kappa)(F_1 - F) - C(1 - \kappa)(\tilde{F}_1 - \tilde{F}) + \frac{A(1 - \tilde{F}_1)}{2\sqrt{t}} + \frac{1}{2}\varphi'(1)(1 - \tilde{F}_1)^2 - (\frac{A(1 - F_1)}{2\sqrt{t}} + \frac{1}{2}\varphi'(1)(1 - F_1)^2). \tag{2.15}
\]

Let us estimate now all terms of (2.15). The assumption \( \Delta F(n, t_0) \geq 0 \) implies (by Theorem 1 in [5]) that \( \Delta F(n, t_0) \geq 0 \ \forall \ t \geq t_0 \). From this and from inequality (2.7) it follows for \( n \geq N(t) + 1 \) and \( t \geq t_0 \):

\[
0 \leq 1 - F(n, t) \leq 1 - F_1 \leq 1 - F(N(t) + 1, t + T) \leq \frac{1}{\sqrt{t}}\psi(A - \delta_0 - 2) \leq \frac{C}{\varphi'(1)\sqrt{t}}\exp(-(A - \delta_0 - 2)^2/2)\left(\int_0^\infty \exp(-2y^2)dy\right)^{-1} \leq O\left(\frac{1}{\sqrt{t}}\exp(-(A - \delta_0 - 2)^2/2)\right). \tag{2.16}
\]

From (2.16) and from inequality (1.12) of Theorem 2ii) for \( t \geq t_0 \geq A^2, n \geq N(t) \) we obtain the crucial inequality

\[
F_1 - F = O\left(\frac{A}{t}\exp(-(A - \delta_0 - 2)^2/2)\right). \tag{2.17}
\]

From [5] (Theorems 2, 2') and [6] (Theorems 6.1, 6.2) it follows asymptotic formula

\[
\tilde{F}_1 = 1 - \frac{C}{\varphi'(1)(A\sqrt{t} + d_A(t))} + O\left(\frac{1}{(A\sqrt{t} + d_A(t))^2}\right).
\]

This formula and estimate \( d_A(t) \geq 0 \) (see (2.4)) gives inequalities

\[
0 < 1 - \tilde{F}_1 \leq O\left(\frac{1}{A\sqrt{t}}\right), \tag{2.18}
\]

\[
\tilde{F}_1 - \tilde{F} = O\left(\frac{1}{A^2t}\right).
\]

Let us put estimates (2.16)-(2.18) into formula (2.15). We obtain

\[
d'_A(t)(1 + O(1/\sqrt{t})) = C(1 - \kappa)(F_1 - F) - C(1 - \kappa)(\tilde{F}_1 - \tilde{F}) + \frac{C}{2\varphi'(1)t} + \frac{C^2}{2\varphi'(1)A^2t} - \frac{(1 - F_1)}{2}\left(\frac{A}{\sqrt{t}} + \varphi'(1)(1 - F_1)\right) = (1 - \kappa)O\left(\frac{A}{t}\exp(-(A - \delta_0 - 2)^2/2)\right) + (1 - \kappa)O\left(\frac{1}{A^2t}\right) + \frac{C}{2\varphi'(1)t} + \frac{C^2}{2\varphi'(1)A^2t} + O\left(\frac{A}{t}\exp(-(A - \delta_0 - 2)^2/2)\right) = \frac{C}{2\varphi'(1)t} + O\left(\frac{1}{A^2t}\right) + O\left(\frac{A}{t}\exp(-(A - \delta_0 - 2)^2/2)\right). \tag{2.19}
\]
Estimate (2.9) implies asymptotic formula

\[ d_A(t) = \frac{C}{2\varphi'(1)} \ln t + O(1/A^2) \ln t + \text{const}. \]

From result (2.5) it follows that for any \( A_1 > 2\sqrt{C} \) and \( A_2 > 2\sqrt{C} \) we have \( d_{A_1}(t) - d_{A_2}(t) \to 0, \ t \to \infty. \)

Hence,

\[ d_A(t) = \frac{C}{2\varphi'(1)} \ln t + \text{const} + o(1). \]

3. A priori estimates for local solutions of Burgers type equations.

Without loss of generality we will put further \( C = 1 \) and \( \varepsilon = 1 \). Otherwise we make substitutions: \( t \to Ct/\varepsilon, \ x \to x/\varepsilon \) for the equation (1.2) and \( t \to C^2t/\varepsilon, \ x \to Cx/\varepsilon \) for the equation (1.1). We will give here a complete proof of Theorem 2ii) which is sufficient for all current applications and a sketch of the proof of Theorem 2ii)’. Theorem 2i) will be proved in a separate paper.

The first step in the proof of Theorem 2ii) is the Green-Poisson type representation formula (for function \( u \) in \( \Omega_\sigma \)) associated with operator \( u \mapsto u_t + \Delta u \), where \( \Delta u \overset{\text{def}}{=} u(x,t) - u(x-1,t), \ u_t = \frac{\partial u(x,t)}{\partial t}. \)

Let \( \chi_0 : \mathbb{R} \to \mathbb{R} \) be a smooth cut-off function such that

\[ 0 \leq \chi_0 \leq 1, \ \chi_0 \bigg|_{(-\infty,a_1]} \equiv 0, \ \chi_0 \bigg|_{[\tilde{a}_1, +\infty]} \equiv 1, \ 0 < a_1 < \tilde{a}_1 < \infty, \]

\[ |\chi_0'| \leq \frac{A_0}{\delta} \quad \text{and} \quad |\chi_0''| \leq \frac{A_0}{\delta^2}, \]

where \( \delta = \tilde{a}_1 - a_1 \). Put \( \chi(x,t) = \chi_0 \left( \frac{x-t}{\sqrt{t}} \right). \)

**Proposition 2.** Let function \( u(x,t) \) be defined in the domain \( \Omega_\sigma = \{(x,t) : a_1 < x < a_2 + \sigma \sqrt{t}, \sigma > 0 \} \) and \( \tilde{u}(x,t) = u(x,t) \cdot \chi(x,t) \). Let \( 0 < \sigma_0 < \sigma \) and \( \alpha \in (\frac{1}{1+\sigma_0},1) \). Then function \( \tilde{u} \) can be represented in \( \Omega_{\sigma_0} \) by the following formula of the Green-Poisson type

\[ \tilde{u}(x,t) = \int_{-\infty}^{\infty} G(x-\xi, t-\alpha t) \tilde{u}(\xi, \alpha t) d\xi + \int_{\alpha t}^{t} d\tau \int_{-\infty}^{\infty} G(x-\xi, t-\tau) (\tilde{u}'_\tau + \Delta \tilde{u})(\xi, \tau) d\xi, \]

where

\[ G(x,t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \exp(-i\xi x) \exp([e^{i\xi} - 1]t) \, d\xi. \]
Besides,

\[ G(x, t) = \sum_{n=-\infty}^{\infty} G_n(t) \delta(n - x), \quad (3.3) \]

where

\[
\begin{cases}
G_n(t) = 0, & \text{if } n < 0, \\
G_n(t) = \frac{t^n}{n!} e^{-t}, & \text{if } n \geq 0
\end{cases}
\]
is Poisson-distribution.

This statement is certainly classical but we did not find the precise reference. So, we will indicate the abridge proof.

The operator \( \frac{\partial}{\partial t} + \Delta \) can be considered as a parabolic operator of infinite order in \( x \) and it can be represented by the following formula

\[
\frac{\partial}{\partial t} + \Delta = \frac{\partial}{\partial t} + (1 - \exp(-\frac{\partial}{\partial x})).
\]

We will apply further to the Cauchy problem for this operator the same Fourier method as for parabolic operator of finite order and we will obtain (3.2). The formula (3.3) is the Fourier inversion formula for the classical Poisson distribution through its characteristic function.

It is important to remark that the function \( \tilde{\mu}(\xi, \tau) \) is well defined for \((\xi, \tau) : \xi < \tau + a_2 \sqrt{\tau + \sigma \tau}, \tilde{u}(\xi, \tau) \equiv 0 \) for \( \xi \leq \tau + a_1 \sqrt{\tau} \) and function \( \xi \mapsto G(x - \xi, t - \tau) \) is equal to zero for \( \xi > x = t + \tilde{x} \sqrt{t} \). So, the function \( \xi \mapsto \tilde{u}(\xi, \tau) \cdot G(x - \xi, t - \tau) \) can be naturally interpreted in the formula (3.2) as a function with compact support in \( \mathbb{R} \) if the following inequality is satisfied

\[
\tau + a_2 \sqrt{\tau + \sigma \tau} \geq x = t + \tilde{x} \sqrt{t} \quad \text{for} \quad \tilde{x} \in (a_1, a_2 + \sigma_0 \sqrt{t}), \quad \sigma_0 < \sigma \quad \text{and} \quad \tau \geq \alpha t \geq t_0(\sigma, \sigma_0).
\]

In order to satisfy these inequalities we choose \( \alpha \in (0, 1) \) such that for \( t > t_0(\sigma, \sigma_0) \) the following inequality is valid

\[
\alpha t + a_2 \sqrt{\alpha t + \sigma \alpha t} > t + a_2 \sqrt{t + \sigma_0 t},
\]
i.e. we must take \( \alpha > \frac{1 + \sigma_0}{1 + \sigma} \).

**Corollary** (Integral representation for \( \Delta u(x, t) \)). Let function \( u(x, t) \) satisfy (1.2) in \( \Omega_\sigma \) with \( \varphi(0) = C = 1 \) and \( \varepsilon = 1 \). Put \( \varphi_0 = \varphi - C \). Then in assumption of Proposition 2 for

\[
(x, t) \in \tilde{\Omega}_{\sigma_0} = \{(x, t) \in \Omega_{\sigma_0} : x \geq t + \tilde{a}_1 \sqrt{t} \}, \quad \sigma_0 < \sigma, \quad t > t^* = \alpha t \geq t_0, \quad \alpha \in (\frac{1 + \sigma_0}{1 + \sigma}, 1)
\]

we have the equality

\[
\Delta u(x, t) = I_0 u + I_1 u + I_2 u + I_3 u + I_4 u, \quad (3.4)
\]
where

\[
I_0 u(x, t) = -\int_{t^*}^t d\tau \int_{\xi > \tilde{a}_1} \Delta_x G(x - \xi, t - \tau) \varphi_0(u) \Delta u(\xi, \tau) d\xi,
\]

\[
I_1 u(x, t) = -\int_{t^*}^t d\tau \int_{[a_1, \tilde{a}_1]} \Delta_x G(x - \xi, t - \tau) \varphi_0(u) \Delta u(\xi, \tau) \chi(\xi, \tau) d\xi,
\]

\[
I_2 u(x, t) = \int_{\xi \geq a_1} \Delta_x G(x - \xi, t - t^*) u(\xi, t^*) \chi(\xi, t^*) d\xi,
\]

\[
I_3 u(x, t) = \int_{t^*}^t d\tau \int_{[a_1, \tilde{a}_1]} \Delta_x G(x - \xi, t - \tau) (u\chi_\tau' + u\Delta \chi)(\xi, \tau) d\xi,
\]

\[
I_4 u(x, t) = -\int_{t^*}^t d\tau \int_{[a_1, \tilde{a}_1]} \Delta_x G(x - \xi, t - \tau) \Delta u(\xi, \tau) \Delta \chi(\xi, \tau) d\xi.
\]

**Remark.** We will use below several times the following simple relation: let \( u = u(x), \ v = v(x), \) then \( \Delta(u \cdot v) = u \cdot \Delta v + v(x - 1)\Delta u, \) where \( \Delta u \overset{\text{def}}{=} u(x) - u(x - 1). \)

**Proof of Corollary.** We have relations

\[
\tilde{u}(\xi, \tau) = u(\xi, \tau) \cdot \chi(\xi, \tau),
\]

\[
\tilde{u}_\tau' = (u \cdot \chi)' = u_\tau' \cdot \chi + u \chi_\tau',
\]

\[
\Delta \tilde{u} = \Delta(u \cdot \chi) = \Delta u \cdot \chi(\xi - 1, t) + u \cdot \Delta \chi = \Delta u \cdot \chi + u(\xi - 1, t) \Delta \chi.
\]

Using (1.2) we obtain

\[
(u_\tau' + \Delta u) \cdot \chi = -\varphi_0(u) \Delta u \chi = -\varphi_0(u)(\Delta \tilde{u} - u(\xi - 1, t) \cdot \Delta \chi),
\]

\[
(\tilde{u}_\tau' + \Delta \tilde{u}) = -\varphi_0(u) \Delta \tilde{u} + \varphi_0(u) \cdot u(\xi - 1, \tau) \Delta \chi + u(\chi_\tau' + \Delta \chi) - \Delta u \cdot \Delta \chi =
\]

\[
- \varphi_0(u) \Delta u \cdot \chi + u \cdot (\chi_\tau' + \Delta \chi) - \Delta u \cdot \Delta \chi.
\]

Plugging these relations into (3.2) and using the equality \( \tilde{u}(\xi, \tau) = u(\xi, \tau) \) for \( \tilde{\xi} > \tilde{a}_1 \) we obtain (3.4).

For the estimates of terms \( I_1 u, I_2 u, I_3 u, I_4 u \) in formula (3.4) we will use elementary estimates for cut-off function \( \chi(x, t) \) and rather precise estimates for Green-Poisson function \( G(x, t) \).

**Lemma 4.** Let \( \chi(x, t) \) be cut-off function defined by (3.1). Then the following estimates for derivatives of \( \chi \) are valid

\[
|\Delta \chi(x, t)| \leq \frac{A_0}{\delta \sqrt{t}}, \quad |\Delta^2 \chi(x, t)| \leq \frac{A_0}{\delta^2 t} \quad \text{and}
\]

\[
|(\chi_\tau' + \Delta \chi)(x, t)| \leq \frac{A_0}{t} \left( \frac{1}{\delta^2} + \frac{\tilde{a}_1}{2\delta} \right),
\]

where \( (x, t) \in \Omega_\sigma, \ \delta = \tilde{a}_1 - a_1. \)
Proof. We have

\[ \chi'(x,t) = -\left( \frac{1}{\sqrt{t}} + \frac{x-t}{2t^{3/2}} \right) \chi'_0\left( \frac{x-t}{\sqrt{t}} \right); \]

\[ \Delta \chi(x,t) = \frac{1}{\sqrt{t}} \int_{x-1}^{x} \chi'_0\left( \frac{y-t}{\sqrt{t}} \right) dy; \]

\[ (\chi' + \Delta \chi)(x,t) = -\frac{1}{\sqrt{t}} \int_{x-1}^{x} \left( \chi'_0\left( \frac{x-t}{\sqrt{t}} \right) - \chi'_0\left( \frac{y-t}{\sqrt{t}} \right) \right) dy - \chi'_0\left( \frac{x-t}{\sqrt{t}} \right) \frac{x-t}{2t^{3/2}} = \]

\[ -\frac{1}{t} \int_{x-1}^{x} \int_{y}^{x} \chi''_0\left( \frac{z-t}{\sqrt{t}} \right) dz dy - \chi'_0\left( \frac{x-t}{\sqrt{t}} \right) \frac{x-t}{2t^{3/2}}. \]

From these relations and from estimates (3.1) for \( \chi_0 \) we obtain necessary estimates for \( \chi(x,t) \).

**Lemma 5.** (Estimates for Green-Poisson distribution \( G(x,t) \)). Let \( G(x,t) = \sum_{n=0}^{\infty} G_n(t) \delta(n-x) \) be the Poisson distribution (3.3). The following estimates for \( \{G_n(t)\} \) are valid

i) if \( p = n - t \geq 0 \) then

\[ G_n(t) \leq \frac{1}{\sqrt{2\pi n}} e^{-p^2/(2n)}. \]

ii) if \( q = t - n > 0, q \leq t \) then

\[ G_n(t) \leq \frac{1}{\sqrt{2\pi n}} e^{-q^2/(2t)}. \]

iii) if \( n = t + a\sqrt{t} \) then

\[ G_n(t) = \frac{1}{\sqrt{2\pi n}} \exp\left( -\frac{(n-t)^2}{2t} \right) \left( 1 + O\left( \frac{(n-t)^3}{t^2} \right) \right) = \]

\[ \frac{1}{\sqrt{2\pi t}} \exp\left( -\frac{a^2}{2} \right) \left( 1 + O\left( \frac{a^3}{\sqrt{t}} \right) \right). \]

Proof. By Stirling’s formula we have

\[ n! = \sqrt{2\pi n} \left( \frac{n}{e} \right)^n \left( 1 + O\left( \frac{1}{n} \right) \right). \]

Then

\[ G_n(t) = \frac{1}{\sqrt{2\pi n}} e^{\ln t - n \ln n + n - t} \left( 1 - O\left( \frac{1}{n} \right) \right). \]

If \( p = n - t > 0 \) then

\[ \ln \frac{t}{n} = \ln \left( 1 - \frac{p}{n} \right) = -\frac{p}{n} - \frac{p^2}{2n^2} - \ldots. \]
If \( q = t - n > 0 \) then
\[
\ln \frac{n}{t} = \ln \left(1 - \frac{q}{t}\right) = -\frac{q}{t} - \frac{q^2}{2t^2} - \ldots
\]
Hence,
\[
G_n(t) = \frac{1}{\sqrt{2\pi n}} e^{-p^2/(2n)} \left(1 - O\left(\frac{1}{n}\right)\right), \quad \text{if } p = n - t > 0
\]
and
\[
G_n(t) = \frac{1}{\sqrt{2\pi n}} e^{-q^2/(2t) - (1/2 - 1/3)(q^3/t^2) - \ldots} \left(1 - O\left(\frac{1}{n}\right)\right),
\]
if \( q = t - n > 0, \ q < t. \)

These relations give i), ii) and iii).

**Lemma 6.** (Estimates for \( \Delta G(x, t) \)). Let \( G(x, t) = \sum_{n=0}^{\infty} G_n(t) \delta(n - x) \) be the Poisson distribution. We put
\[
\Delta G_n(t) = G_n(t) - G_{n-1}(t),
\]
\[
\Delta_x G(x - \xi, t - \tau) = G(x - \xi, t - \tau) - G(x - 1 - \xi, t - \tau),
\]
\[
\bar{\xi} = \frac{\xi - \tau}{\sqrt{\tau}}, \quad \bar{x} = \frac{x - t}{\sqrt{t}}.
\]

Then the following estimates are valid

i) \[
\Delta G_n(t) = G_n(t)\left(\frac{t - n}{t}\right), \quad \text{and as consequence}
\]
\[
\Delta G_n(t) > 0, \quad \text{if } n < t, \quad \Delta G_n(t) < 0, \quad \text{if } n > t;
\]
\[
\Delta^2 G_n(t) = G_n(t)\left(1 - \frac{2n}{t} + \frac{n(n - 1)}{t^2}\right), \quad \text{and as consequence}
\]
\[
\Delta^2 G_n(t) < 0, \quad \text{if } n - t - \frac{1}{2} \in (-\sqrt{t + 1/4}, +\sqrt{t + 1/4}),
\]
\[
\Delta^2 G_n(t) \geq 0, \quad \text{if } n - t - \frac{1}{2} \notin (-\sqrt{t + 1/4}, +\sqrt{t + 1/4});
\]

ii) \( \forall \ s \geq 0 \) and \( p \geq 0 \) we have inequalities
\[
- \Delta G_{p+s}(s) \leq A_1 s^{-3/2} p \exp\left(-\frac{p^2}{4s}\right), \quad \text{if } p < s,
\]
\[
- \Delta G_{p+s}(s) \leq A_1 p^{-1/2} e^{-p/4}, \quad \text{if } p > s;
\]

iii) \( \forall \ s \geq 0 \) and \( q \in (0, s) \) we have inequalities
\[
\Delta G_{s-q}(s) \leq A_1 \frac{q}{s \sqrt{s - q}} \exp\left(-\frac{q^2}{2s}\right);
\]
iv) 
\[ \sum_{n=-\infty}^{\infty} |\Delta G_n(t)| = \min \{2, \frac{2}{\sqrt{2\pi t}}(1 + O\left(\frac{1}{\sqrt{t}}\right))\}; \]
\[ \sum_{n=-\infty}^{\infty} |\Delta^2 G_n(t)| = \min \{4, \frac{4}{\sqrt{2\pi \epsilon t}}(1 + O\left(\frac{1}{\sqrt{t}}\right))\}; \]

v) \( \forall \bar{x} > \bar{a} \) and \( t > \tau > \alpha t \) we have inequality
\[ I = \int_{\xi \geq \bar{a}} |\Delta_x G(x - \xi, t - \tau)| \left(1 + \ln_+ \frac{1}{\xi - \bar{a}}\right)(1 + \bar{\xi})d\xi \leq \frac{A_1}{\sqrt{t - \tau}} \left(1 + \sqrt{(1 - \alpha)/\alpha}\right) \left(1 + \ln_+ \frac{1}{\bar{x} - \bar{a}}\right)(1 + \bar{x}/\sqrt{\alpha}), \]
where \( A_1 \) is absolute constant.

**Remark.** We will use further several times the differential and integral relations:

- \( \Delta_\xi (G(x - \xi - 1, t - \tau)u(\xi)) = G(x - \xi, t - \tau)\Delta u(\xi) + \Delta_\xi G(x - \xi - 1, t - \tau) \cdot u(\xi); \)
- \( \Delta_\xi G(x - \xi - 1, t - \tau) = \Delta_x G(x - \xi, t - \tau); \)

if \( G(x - \xi - 1, t - \tau) \cdot u(\xi) \) has compact support with respect to \( \xi \) then
\[ -\int_{\xi \in \mathbb{R}} \Delta_\xi (G(x - \xi - 1, t - \tau) \cdot u(\xi)) d\xi = 0 \quad \text{and hence} \]
\[ -\int_{\xi \in \mathbb{R}} \Delta_x G(x - \xi, t - \tau) \cdot u(\xi) d\xi = \int_{\xi \in \mathbb{R}} G(x - \xi, t - \tau) \Delta u(\xi) d\xi. \]

**Proof of Lemma 6.**

i) We have from (3.3)
\[ \Delta G_n(t) = \left(\frac{t^n}{n!} - \frac{t^{n-1}}{(n-1)!}\right)e^{-t} = G_n(t)\left(\frac{t - n}{t}\right), \]
\[ \Delta^2 G_n(t) = \left(\frac{t^n}{n!} - 2\frac{t^{n-1}}{(n-1)!} + \frac{t^{n-2}}{(n-2)!}\right)e^{-t} = G_n(t) \left(1 - \frac{2n}{t} + \frac{n(n-1)}{t^2}\right); \]

ii) follows from i) and Lemma 5i).
iii) follows from i) and Lemma 5ii).
iv) Putting in i) \( p = n - \tau = a\sqrt{t} \) and using Lemma 5iii) we obtain
\[ \Delta G_n(t) = \frac{1}{\sqrt{2\pi t}} e^{-a^2/2} \left(-\frac{a}{\sqrt{t}}\right) \left(1 - O\left(\frac{a^3}{\sqrt{t}}\right)\right) \]
and as consequence \( \Delta G_n(t) = 0 \) if \( a = 0. \)
So,
\[
\sum_{n=-1}^{\infty} |\Delta G_n(t)| = \left( \sum_{n \geq t} \Delta G_n(t) - \sum_{n \leq t} \Delta G_n(t) \right).
\]

Then
\[
\sum_{n=-1}^{\infty} |\Delta G_n(t)| = [G_{[t]}(t) - G_{-\infty}(t)] - [G_{+\infty}(t) - G_{[t]}(t)] = 2G_{[t]}(t) = \frac{2}{\sqrt{2\pi t}} \left( 1 + O\left( \frac{1}{\sqrt{t}} \right) \right), \quad t \geq t_0.
\]

For all \( t > t_0 \) we have
\[
\sum_{n=-1}^{\infty} |\Delta G_n(t)| = \min \{ 2, \frac{2}{\sqrt{2\pi t}} \left( 1 + O\left( \frac{1}{\sqrt{t}} \right) \right) \}.
\]

By similar arguments we have
\[
\sum_{n=-2}^{\infty} |\Delta^2 G_n(t)| = \sum_{-\infty}^{[t-\sqrt{t}]} \Delta^2 G_n(t) - \sum_{[t-\sqrt{t}]+1}^{[t+\sqrt{t}]} \Delta^2 G_n(t) + \sum_{[t+\sqrt{t}]+1}^{\infty} \Delta^2 G_n(t) =
\]
\[
2\Delta G_{[t-\sqrt{t}]}(t) + 2|\Delta G_{[t+\sqrt{t}]}(t)| = \frac{4}{t\sqrt{2\pi e}} \left( 1 + O\left( \frac{1}{\sqrt{t}} \right) \right), \quad t \geq t_0.
\]

For all \( t > 0 \) we have
\[
\sum_{n=-2}^{\infty} |\Delta^2 G_n(t)| = \min \{ 4, \frac{4}{t\sqrt{2\pi e}} \left( 1 + O\left( \frac{1}{\sqrt{t}} \right) \right) \}.
\]

v) Put \( t - \tau = s, \ x - \xi = y \). We have \( p = y - s = \bar{x}\sqrt{t} - \bar{\xi}\sqrt{\tau} \). Put \( I = I_+ + I_- \), where
\[
I_\pm = \int_{\pm\Delta_x G < 0} |\Delta_x G(x - \xi, t - \tau)| \left( 1 + \ln_+ \frac{1}{(\xi - \bar{a}_1)} \right)(1 + \tilde{\xi})d\xi.
\]

By part i) \( \Delta_x G(x - \xi, t - \tau) < 0 \) iff \( p = (x - \xi) - (t - \tau) > 0 \).

Hence, \( I_+ = I'_+ + I''_+ \), where
\[
I'_+ = -\int_{\xi > \bar{a}_1 : 0 < p < s} \Delta_x G(x - \xi, t - \tau) \left( 1 + \ln_+ \frac{1}{(\xi - \bar{a}_1)} \right)(1 + \tilde{\xi})d\xi,
\]
\[
I''_+ = -\int_{\xi > \bar{a}_1 : p > s} \Delta_x G(x - \xi, t - \tau) \left( 1 + \ln_+ \frac{1}{(\xi - \bar{a}_1)} \right)(1 + \tilde{\xi})d\xi.
\]

Put \( p_1 = \bar{x}\sqrt{t} - \bar{a}_1\sqrt{\tau} \). We have \( \bar{a}_1 - \tilde{\xi} = \frac{p - p_1}{\sqrt{\tau}} < 0 \) and \( p = p_1 \) iff \( \tilde{\xi} = \bar{a}_1 \).
For $I'_+$ when $p \in (0, s)$ we use ii) and obtain

$$I'_+ \leq A s^{-3/2} \int_0^{p_1} e^{-p^2/(4s)} p \left(1 + \ln+ \frac{\sqrt{\tau}}{p_1 - p}\right) (1 + \tilde{a}_1 + p_1 - p)dp \leq$$

(putting $p = \rho \sqrt{s}$)

$$As^{-1/2} \int_0^{p_1/\sqrt{s}} e^{-p^2/4} \rho \left(1 + \ln+ \frac{\sqrt{s}}{(p_1 - \rho)}\right) (1 + \tilde{a}_1 + p_1 - \rho)dp \leq$$

(by Lemma $A_1$ of Appendix)

$$As^{-1/2} (1 + \ln+ \frac{\sqrt{\tau}}{p_1}) (1 + \tilde{a}_1 + p_1) \leq As^{-1/2} (1 + \ln+ \frac{1}{x/\sqrt{t/\tau} - \tilde{a}_1}) (1 + \bar{x}/\sqrt{\alpha}).$$

For $I''_+$ when $p > s$ we use ii) and obtain

$$I''_+ \leq As^{-1/2} \int_0^{p_1} e^{-p^2/4} (1 + \ln+ \frac{\sqrt{\tau}}{p_1 - p}) (1 + \tilde{a}_1 + p_1 - p)dp \leq$$

$$As^{-1/2} (1 + \ln+ \frac{\sqrt{\tau}}{p_1}) (1 + \tilde{a}_1 + p_1) \leq As^{-1/2} (1 + \ln+ \frac{1}{x/\sqrt{t/\tau} - \tilde{a}_1}) (1 + \bar{x}/\sqrt{\alpha}).$$

Let us estimate now integral $I_-$. Put $q = (t - \tau) - (x - \xi)$. By part i) $\Delta_x G(x - \xi, t - \tau) > 0$ iff $q \in (0, s)$. We use now part iii) and obtain

$$I_- \leq \frac{A}{s} \int_0^{s} e^{-q^2/(2s)} \frac{q}{\sqrt{s - q}} (1 + \ln+ \frac{\sqrt{\tau}}{p_1 + q}) (1 + \tilde{a}_1 + p_1 + q) \frac{1}{\sqrt{\alpha}} \int_0^{s} e^{-q^2/(2s)q^2} dq \leq$$

$$\frac{A}{s} \int_0^{s} \left(1 + \frac{1}{x/\bar{a}_1}\right) \left(1 + \bar{x}/\sqrt{\alpha}\right) \int_0^{s} e^{-q^2/(2s)} \frac{q}{\sqrt{s - q}} dq + \frac{1}{\sqrt{s}} \int_0^{s} e^{-q^2/(2s)q^2} dq, \quad \text{where}$$

$$\int_0^{s} \frac{e^{-q^2/(2s)q^2}}{\sqrt{s - q}} dq \leq \int_0^{s/2} \frac{e^{-q^2/(2s)q^2}}{\sqrt{s/2}} dq + \int_{s/2}^{s} \frac{e^{-s/8s}}{\sqrt{s - q}} dq \leq$$

$$\sqrt{2s} \int_0^{s/8} e^{-q} dq + 2se^{-s/8} \sqrt{s/2} = \sqrt{2s} (1 - e^{-s/2} + se^{-s/8}) = O(\sqrt{s}),$$

$$\int_0^{s} \frac{e^{-q^2/(2s)q^2}}{\sqrt{s - q}} dq \leq \int_0^{s/2} \frac{e^{-q^2/(2s)q^2}}{\sqrt{s/2}} dq + \int_{s/2}^{s} \frac{e^{-s/8s^2}}{\sqrt{s - q}} dq \leq$$

$$2s \cdot \int_0^{s/8} \sqrt{ye^{-q}} dq + 2s^2 e^{-s/8} \sqrt{s/2} = O(s).$$

Hence,

$$I_- \leq \frac{A_2}{\sqrt{s}} \left(1 + \ln+ \frac{1}{x/\bar{a}_1}\right) \left(1 + \sqrt{(1 - \alpha)/\alpha}\right) \left(1 + \bar{x}/\sqrt{\alpha}\right).$$
Lemma 6 is proved.

Now we are ready to estimate terms \( I_{2u} \) and \( I_{3u} \) of formula (3.4).

\textbf{Lemma 7.} Let function \( F = u \) satisfy the conditions of Theorem 2ii) and \( \Delta u \) is represented in \( \Omega_\sigma \) by formula (3.4), \( \alpha \geq \sup\{1/2, \frac{1+\sigma_0}{1+\sigma}\} \), \( \sigma_0 < \sigma \). Then terms \( I_{2u} \) and \( I_{3u} \) of formula (3.4) admit the following estimates

\[
|I_{2u}(x,t)| \leq A_2 \frac{\Gamma \cdot \bar{x}}{\sqrt{(1-\alpha)t}}, \quad (3.5)
\]

\[
|I_{3u}(x,t)| \leq A_0 \frac{\Gamma \cdot \bar{a}_1}{t^{3/2}} \left( \frac{1}{\delta^2} + \frac{\bar{a}_1}{2\delta} \right) K^+, \quad (3.6)
\]

where \( K^+ = \int_t^\infty d\tau \int_{\xi \in [a_1, \bar{a}_1]} |\Delta_x G(x - \xi, t - \tau)| d\xi, \)

\( A_2 \) is absolute constant, \( \bar{x} \in (\bar{a}_1, a_2 + \sigma_0 \sqrt{t}), t > t_0(\sigma_0, \sigma) \).

\textbf{Remark.} \( I_{2u} \) is the only term in representation (3.4), where \( (1-\alpha) \) is in the denominator.

\textbf{Proof.} The definitions of \( I_{2u} \) and \( I_{3u} \), condition (1.11) and Lemma 4 imply estimates:

\[
|I_{2u}(x,t)| \leq \Gamma \int_{\xi > a_1} |\Delta_x G(x - \xi, t - \alpha t)| \frac{\bar{\xi} d\xi}{\sqrt{\alpha t}}, \quad (3.7)
\]

where \( \bar{\xi} = \frac{\xi - \alpha t}{\sqrt{\alpha t}} \),

\[
|I_{3u}(x,t)| \leq \Gamma A_0 \left( \frac{1}{\delta^2} + \frac{\bar{a}_1}{2\delta} \right) \int_t^\infty d\tau \int_{\xi \in (a_1, \bar{a}_1)} |\Delta_x G(x - \xi, t - \tau)| \frac{\bar{\xi} d\xi}{\sqrt{\tau}}, \quad (3.8)
\]

where \( \bar{\xi} = \frac{\xi - \tau}{\sqrt{\tau}} \).

Using Lemmas 6i), 5i), 6iv) (see also (3.14)) we obtain further from (3.7)

\[
|I_{2u}(x,t)| \leq \frac{\Gamma}{\sqrt{\alpha t}} \left[ \int_{\xi < \bar{x} \sqrt{t/(\alpha t)}} \Delta_x G(x - \xi, t - \alpha t) \bar{\xi} d\xi - \int_{\xi > \bar{x} \sqrt{t/(\alpha t)}} \Delta_x G(x - \xi, t - \alpha t) \bar{\xi} d\xi \right] \leq
\]

\[
\frac{\Gamma}{\sqrt{\alpha t}} \left[ - \int_{\xi < \bar{x} \sqrt{\alpha}} G \cdot \Delta_x \bar{\xi} d\xi + \int_{\xi > \bar{x} \sqrt{\alpha}} G \cdot \Delta_x \bar{\xi} d\xi + G\xi_{a_1}^{x / \sqrt{\alpha}} - G\xi_{\bar{x} / \sqrt{\alpha}}^{x / \sqrt{\alpha} + (1-\alpha) \sqrt{t} / \sqrt{\alpha}} \right] \leq
\]

\[
\frac{\Gamma}{\sqrt{\alpha t}} \left[ \int_{\xi > \bar{x} \sqrt{\alpha}} \frac{1}{\sqrt{\alpha t}} G(x - \xi, t - \alpha t) d\xi + 2G(t - \alpha t, t - \alpha t) \frac{\bar{x}}{\sqrt{\alpha}} \right] \leq
\]

\[
\frac{\Gamma}{\sqrt{\alpha t}} \left( \frac{1}{\sqrt{\alpha t}} + \frac{2}{\sqrt{2\pi (t - \alpha t)}} \left( \frac{\bar{x}}{\sqrt{\alpha}} \right) \right) \leq
\]

\[
\frac{A_2 \Gamma}{\sqrt{(1-\alpha) \alpha}} \frac{1}{t} \left( \frac{\bar{x}}{\sqrt{\alpha}} \right), \quad \text{if} \quad t \geq t_0.
\]
From (3.8) we deduce

$$|I_3 u(x, t)| \leq \frac{\Gamma A_0}{t^{3/2}} \left( \frac{1}{\delta^2} + \tilde{a}_1 \right) \tilde{a}_1 \int_0^t d\tau \int_{\xi \in (\bar{a}_1, \tilde{a}_1)} |\Delta_x G(x - \xi, t - \tau)| d\xi.$$ 

We have proved (3.5), (3.6).

We will estimate now the terms $I_1 u$ and $I_4 u$ of (3.4).

**Lemma 8.** Let function $u$ satisfy conditions of Theorem 2ii) and $\Delta u$ be represented in $\Omega_\sigma$ by formula (3.4). Then terms $I_1 u$ and $I_4 u$ of formula (3.4) admit the following (preliminary) estimates for $\bar{x} \geq \tilde{a}_1$ and $t \geq t_0$:

$$|I_1 u| \leq \frac{4 \gamma_0 \Gamma^2 \cdot \tilde{a}_1^2}{a_t} (K^- + K_1), \quad (3.9)$$

$$|I_4 u| \leq \frac{2A_0 \Gamma \cdot \tilde{a}_1}{\delta a_t} (K^- + K_1), \quad (3.10)$$

where

$$K^- = \int_{\xi = a_-}^{\xi = \tilde{a}_1} |\Delta_x G(x - \xi, t - \tau)| d\xi,$$

$$K_1 = \int_{\xi = a_1}^{\xi = \bar{a}_1} |\Delta_x G(x - \xi, t - \tau)| d\xi,$$

$$a_- = \bar{x} \sqrt{t/\tau} - \frac{1}{2 \sqrt{\tau}} - \sqrt{(t - \tau)/\tau + 1/(4\tau)}.$$

**Proof.** If $t_0$ is large enough and $\tau \geq t_0$ we have using (1.11) inequalities

$$|u(\xi, \tau)| \leq \frac{\Gamma \cdot \xi}{\sqrt{\tau}}, \quad |\varphi_0(u)| \leq 2\gamma_0 |u|, \quad |\Delta_x \chi(\xi, \tau)| \leq \frac{A_0}{\delta \sqrt{\tau}}.$$

From these relations and from definitions of $I_1 u$, $I_4 u$ it follows (using also that $\bar{\xi} \leq \tilde{a}_1 \leq \bar{x}$):

$$|I_1 u| \leq \frac{2\gamma_0 \Gamma \cdot \tilde{a}_1}{\sqrt{a_t}} I_5 u \quad \text{and} \quad |I_4 u| \leq \frac{A_0}{\delta \sqrt{a_t}} I_5 u, \quad (3.11)$$

where

$$I_5 u = \int_{\xi = a_1}^{\xi = \tilde{a}_1} d\xi \int_{\xi \in [a_1, \tilde{a}_1]} |\Delta_x G(x - \xi, t - \tau)| \cdot |\Delta_x u(\xi, \tau)| d\xi. \quad (3.12)$$

The assumption of Theorem 2ii) implies that

$$\Delta_x u(\xi, \tau) \geq 0 \quad \forall \tau \geq \tau_0. \quad (3.13)$$

By Lemma 6 we have also inequalities

$$\Delta_x G(x - \xi, t - \tau) < 0 \quad \text{iff} \quad \bar{\xi} < \bar{x} \sqrt{t/\tau},$$

$$\Delta_x G(x - \xi, t - \tau) > 0 \quad \text{iff} \quad \bar{\xi} > \bar{x} \sqrt{t/\tau}. \quad (3.14)$$
From (3.12)-(3.14) we deduce
\[ I_5 u = - \int_{\alpha t}^{t} d\tau \int_{\xi \in [a_1, \tilde{a}_1]} \Delta_x G \Delta_\xi u d\xi = \]
\[ - \int_{\alpha t}^{t} d\tau \left( \int_{\xi \in [a_1, \tilde{a}_1]} \Delta_x^2 G \cdot u d\xi + \Delta_x G \cdot u \bigg|_{\xi = \tilde{a}_1} - \Delta_x G \cdot u \bigg|_{\xi = a_1} \right). \]

Using inequality \(|u(\xi, \tau)| \leq \frac{\Gamma \xi}{\sqrt{\tau}}\) we obtain
\[ |I_5 u| \leq \frac{\Gamma \cdot \tilde{a}_1}{\sqrt{\alpha t}} \int_{\alpha t}^{t} d\tau \left( \int_{\xi \in [a_1, \tilde{a}_1]} |\Delta_x^2 G| d\xi + |\Delta_x G|_{\xi = a_1} + |\Delta_x G|_{\xi = \tilde{a}_1} \right). \] (3.15)

From Lemma 6 we have
\[ \Delta_x^2 G(x - \xi, t - \tau) < 0, \quad \text{iff} \quad \tilde{\xi} \in (a_-, a_+), \]
where \( a_\pm = \bar{x} \sqrt{t/\tau} - \frac{1}{2\sqrt{\tau}} \pm \sqrt{(t - \tau)/\tau + 1/(4\tau)}. \) (3.16)

If \( t_0 \) is large enough and \( \tilde{a}_1 > a_1 \sqrt{\alpha} + \sqrt{1 - \alpha} \) we have inequality: \( a_- > a_1. \)

Put \( \xi_- = \inf \{a_1, a_\pm\}. \)

From (3.14), (3.15), (3.16) we deduce
\[ |I_5 u| \leq \frac{\Gamma \cdot \tilde{a}_1}{\sqrt{\alpha t}} \int_{\alpha t}^{t} d\tau \left( \int_{\xi \in [a_1, \tilde{a}_1]} \Delta_x^2 G d\xi - \int_{\xi \in [\xi_-, \tilde{a}_1]} \Delta_x^2 G d\xi - \Delta_x G \bigg|_{\xi = a_1} - \Delta_x G \bigg|_{\xi = \tilde{a}_1} \right) \leq \]
\[ \frac{\Gamma \cdot \tilde{a}_1}{\sqrt{\alpha t}} \int_{\alpha t}^{t} d\tau \left[ \Delta_x G \bigg|_{\xi = a_1} - \Delta_x G \bigg|_{\xi = \xi_-} + \Delta_x G \bigg|_{\xi = \tilde{a}_1} - \Delta_x G \bigg|_{\xi = \xi_-} - \Delta_x G \bigg|_{\xi = \xi_-} - \Delta_x G \bigg|_{\xi = \tilde{a}_1} \right] \]
\[ \leq \frac{2 \Gamma \cdot \tilde{a}_1}{\sqrt{\alpha t}} (K^- + K_1). \]

The last estimate together with estimates (3.12) imply (3.9), (3.10).

The following lemma gives more precise estimates for terms \( I_1 u, I_3 u, I_4 u. \)

**Lemma 9.** In conditions and notations of Lemmas 7,8 we have estimates:

\[ |I_3 u| \leq A_2 A_0 \Gamma \cdot \tilde{a}_1 \left( \frac{\sqrt{1 - \alpha}}{\delta^2} + \frac{1}{\delta} + \frac{\tilde{a}_1}{\delta \sqrt{\xi}} \right), \] (3.17)

\[ |I_1 u| \leq A_2 \gamma_0 \Gamma \cdot \tilde{a}_1^3 \left( 1 + \ln \frac{\sqrt{1 - \alpha}}{\bar{x} - a_1} \right), \] (3.18)

\[ |I_4 u| \leq A_2 \frac{A_0 \Gamma \cdot \tilde{a}_1}{\delta t} \left( 1 + \ln \frac{\sqrt{1 - \alpha}}{\bar{x} - a_1} \right), \] (3.19)

where \( A_2 \) is absolute constant, \( \alpha \) is sufficiently close to 1.
Proof. In order to prove (3.17), (3.18), (3.19) it is sufficient to prove estimates:

\[ K^- \leq A(1 + \ln_+ \frac{\sqrt{1 - \alpha}}{\bar{x} - \bar{a}_1}), \quad (3.20) \]
\[ K_1 \leq A, \quad (3.21) \]
\[ K^+ \leq A\sqrt{t} \inf \{\sqrt{1 - \alpha}, \frac{1}{\bar{a}_1} + \frac{1}{\sqrt{t}}\}, \quad (3.22) \]

where \( K^+, K^-, K_1 \) are integrals from (3.6), (3.9), (3.10).

Let us prove firstly (3.21). Put \( \varepsilon = \bar{x} - \bar{a}_1 \), indicating that it can be arbitrary small, \( y = x - \xi, s = t - \tau \). We have

\[ p = y - s = \bar{x}\sqrt{t} - \bar{a}_1\sqrt{t} = \varepsilon\sqrt{t} + \bar{a}_1 s \]
\[ \frac{\varepsilon\sqrt{t} + \bar{a}_1(\sqrt{t} - \sqrt{t} - s)}{2\theta\sqrt{t}} > 0, \]

where \( \theta(s) = \sqrt{t + \sqrt{t} - s} - \frac{1 + \sqrt{\alpha}}{2} \leq \theta < 1. \)

Since \( 0 \leq s \leq (1 - \alpha)t \) we have

\[ K_1 = \int_{\alpha t}^{t} |\Delta G|_{\xi = \bar{a}_1} d\tau = -\int_{0}^{(1-\alpha)t} \Delta G(p + s, s)ds = K_{10} + K_{11}, \]

where \( K_{10} = -\int_{s < p} \Delta G(p + s, s)ds, \quad K_{11} = -\int_{s > p} \Delta G(p + s, s)ds. \)

Note that \( s < p \) iff \( s < \varepsilon\sqrt{t}(1 - \frac{\bar{a}_1}{2\theta\sqrt{t}})^{-1} \) and \( \frac{\bar{a}_1}{2\theta\sqrt{t}} < 1. \) Hence, inequality \( s < p \) implies \( s < 2\varepsilon\sqrt{t}, \) if \( t > t_0 \) and inequality \( s > p \) implies \( s > \varepsilon\sqrt{t}, \) if \( t > t_0. \)

Using Lemma 6ii) we obtain

\[ K_{10} \leq \int_{0}^{2\varepsilon\sqrt{t}} \frac{1}{\sqrt{s}} e^{-s/4} ds \leq \int_{0}^{\infty} \frac{1}{\sqrt{s}} e^{-s/4} ds \leq A_2 \quad \text{and} \]
\[ K_{11} \leq A \int_{\varepsilon\sqrt{t}}^{(1-\alpha)t} s^{-3/2} p e^{-p^2/(4s)} ds \leq \]

\[ \leq A \int_{\varepsilon\sqrt{t}}^{1-\alpha} \eta^{-3/2}(\varepsilon + \frac{\bar{a}_1 \eta}{2\theta}) \exp\left(-\left(\varepsilon + \frac{\bar{a}_1 \eta}{2\theta}\right)^2/(4\eta)\right) d\eta \leq \]
\[ A\left(\int_{0}^{1-\alpha} \frac{\bar{a}_1}{2\theta \sqrt{\eta}} \exp\left(-\frac{\bar{a}_1^2 \eta}{16}\right) d\eta + \int_{0}^{1} \varepsilon \eta^{-3/2} e^{-\varepsilon^2/(4\eta)} d\eta\right) \leq \]

\[ \leq A \left(\frac{1}{2\theta} \int_{0}^{\infty} r^{-1/2} e^{-r/16} dr + \int_{0}^{\infty} \rho^{-3/2} e^{-1/(4\rho)} d\rho\right) \leq A_2. \]
Inequality (3.21) is proved.
Let us prove now (3.20). Let us find interval of variable \( s \) in which \( \bar{\xi} - \alpha < \bar{a} \), i.e.
\[
a_\alpha = \bar{x} \sqrt{t/\tau} - \frac{1}{2 \sqrt{\tau}} - \sqrt{(t-\tau)/\tau} + 1/(4\tau) < \bar{a},
\]
i.e. \( \bar{x} \sqrt{t/\tau} - \sqrt{(t-\tau)/\tau} < \bar{a}, \) where \( \bar{a} = \bar{a}_1 (1 + O(1/\sqrt{\tau})) \).

Put \( \eta = \frac{t-\tau}{t} = \frac{\xi}{t} \). We obtain
\[
\bar{x} - \sqrt{\eta} < \bar{a}_1 \sqrt{1 - \eta}, \text{ i.e.} \n\bar{x}^2 - 2\bar{x} \sqrt{\eta} + \eta < \bar{a}_1^2 (1 - \eta), \text{ i.e.} \n(\sqrt{\eta} - \frac{\bar{x}}{1 + \bar{a}_1})^2 < \bar{a}_1^2 (1 + \bar{a}_1^2 - \bar{x}^2), \text{ i.e.} \n\sqrt{\eta} - \frac{\bar{x}}{1 + \bar{a}_1} < \frac{\bar{a}_1 \sqrt{1 + \bar{a}_1^2 - \bar{x}^2}}{1 + \bar{a}_1^2}, \text{ where} \n\sqrt{\eta} = \frac{\bar{x}}{1 + \bar{a}_1} - \frac{\bar{a}_1 \sqrt{1 + \bar{a}_1^2 - \bar{x}^2}}{1 + \bar{a}_1^2} \text{ ; } \sqrt{\eta} = \frac{\bar{x}}{1 + \bar{a}_1} + \frac{\bar{a}_1 \sqrt{1 + \bar{a}_1^2 - \bar{x}^2}}{1 + \bar{a}_1^2}.
\]
The interval is not empty if \( \bar{x} \leq \sqrt{1 + \bar{a}_1^2} \). In addition we have
\[
\bar{x} - \bar{a}_1 \sqrt{1 + \bar{a}_1^2 - \bar{x}^2} \geq \bar{x} - \bar{a}_1 (1 + \frac{\bar{a}_1^2 - \bar{x}^2}{2}) = \n(\bar{x} - \bar{a}_1) (1 + \frac{\bar{a}_1 (\bar{x} + \bar{a}_1)}{2}) \geq (\bar{x} - \bar{a}_1)(1 + \bar{a}_1^2).
\]
Hence \( \sqrt{\eta} > \bar{x} - \bar{a}_1 \). The condition \( \bar{\xi} = \alpha \) implies that
\[
y = (x - \xi) = (t - \tau) + \sqrt{t - \tau} + O(1) = s + \sqrt{s} + O(1).
\]
From Lemmas 5,6 we deduce
\[
-\Delta G|_{\bar{\xi} = \alpha} \leq \frac{\sqrt{e}}{s \sqrt{2\pi}} (1 + O(1/\sqrt{s})) \leq \frac{A}{s}.
\]
Hence,
\[
K^- \leq \int_{at}^{t} |\Delta_x G|_{\bar{\xi} = \alpha} \, d\tau \leq \int_{\eta_1 t}^{(1-\alpha) t} \frac{A}{s} \, ds \leq A_2 \ln \frac{\sqrt{1 - \alpha}}{s \sqrt{2\pi} (x - \bar{a}_1)}, \text{ } t \geq t_0.
\]
Let us prove (3.22). Using definition of \( K^+ \) and (3.14) we obtain
\[
K^+ = -\int_{at}^{t} d\tau \int_{\bar{\xi} = \alpha} |\Delta_x G(x - \xi, t - \tau) d\xi \leq \int_{at}^{t} G(x - \xi, t - \tau) d\xi \rightarrow \bar{a}_1 \, d\tau.
\]
Put (as in the proof of (3.21)) \( \varepsilon = \bar{x} - \tilde{a}_1, y = x - \xi, s = t - \tau, p = y - s. \)

We have

\[
\int_{\alpha t}^{t} G(\bar{\xi} = \bar{a}_1) d\tau = \int_{s < p} G(p + s, s) ds + \int_{s > p} G(p + s, s) ds.
\]

Because \( s < p \) implies \( s < 2\varepsilon \sqrt{t}, t \geq t_0 \) and using Lemma 5i) we obtain

\[
\int_{s < p} G(p + s, s) ds \leq \int_{0}^{2\varepsilon \sqrt{t}} \frac{1}{\sqrt{2\pi(p + s)}} \exp\left(-\frac{p^2}{2(p + s)}\right) ds \leq \int_{0}^{2\varepsilon \sqrt{t}} \frac{1}{\sqrt{2\pi p}} e^{-p/4} ds \leq \int_{0}^{2\varepsilon \sqrt{t}} \frac{1}{\sqrt{2\pi s}} e^{-s/4} ds \leq A_2.
\]

Because \( s > p \) implies \( s \in (\varepsilon \sqrt{t}, (1 - \alpha)t) \) and using Lemma 5i) we obtain

\[
\int_{s > p} G(p + s, s) ds \leq \int_{\varepsilon \sqrt{t}}^{(1-\alpha)t} \frac{1}{\sqrt{2\pi s}} \exp\left(-\frac{p^2}{4s}\right) ds \leq \frac{1}{\sqrt{2\pi}} \int_{0}^{(1-\alpha)t} s^{-1/2} \exp\left(-\frac{p^2}{4s}\right) ds.
\]

Using \( p = \sqrt{t}(\varepsilon + \frac{\tilde{a}_1 s}{2\sqrt{t}}) \) and putting \( \rho = \frac{\tilde{a}_1^2 s}{2\sqrt{t}} \), we obtain further

\[
\int_{s > p} G(p + s, s) ds \leq \frac{\sqrt{t}}{\sqrt{2\pi \tilde{a}_1}} \int_{0}^{\tilde{a}_1^2(1-\alpha)} \frac{1}{\sqrt{\rho}} e^{-\rho/16} d\rho \leq \sqrt{\frac{t}{2\pi}} \inf \left\{ \frac{2\sqrt{1 - \alpha}}{\alpha \tilde{a}_1}, \frac{1}{\tilde{a}_1} \int_{0}^{\infty} \frac{1}{\sqrt{\rho}} e^{-\rho/16} d\rho \right\}.
\]

Hence, \( K^+ \leq A\sqrt{t} \inf \{ \sqrt{1 - \alpha}, \frac{1}{\sqrt{t}} + \frac{1}{\tilde{a}_1} \} \).

Lemma 9 is proved.

**Proof of Theorem 2ii).** From formula (3.4) and estimates (3.5),(3.17),(3.18), (3.19) we deduce the following inequality under condition that \( x \in (\tilde{a}_1, a_2 + \sigma_0 \sqrt{t}) \), \( \sigma_0 < \sigma, t \geq \tilde{a}_1^2 \) and \( \alpha > \frac{1 + \sigma_0}{1 + \sigma} \):

\[
\Delta u(x, t) \leq \frac{A_3 \Gamma \cdot \bar{x}}{t} \left[ \frac{1}{\sqrt{1 - \alpha}} + \frac{\sqrt{1 - \alpha}}{\delta^2} + \frac{1}{\delta} + \frac{\gamma_0 \Gamma \cdot \bar{a}_1 + \frac{1}{\delta}}{(1 + \ln_{\bar{a}_1} + \frac{\sqrt{1 - \alpha}}{\bar{x} - \tilde{a}_1})} \right] + \gamma_0 \Gamma \cdot \bar{x} \int_{\alpha t}^{t} d\tau \int_{\xi \geq \bar{a}_1} |\Delta_x G(x - \xi, t - \tau)| \frac{|\Delta u(\xi, \tau)|}{\sqrt{\tau}} d\xi.
\]

Put

\[
v(t) = t \cdot \max_{\bar{x} \in (\tilde{a}_1, a_2 + \sigma_0 \sqrt{t})} \frac{\Delta u(x, t)}{g(\bar{x})}.
\]
where
\[ g(\bar{x}) = B_1 + B_2(1 + \ln_+ \frac{\sqrt{1-\alpha}}{\bar{x} - \bar{a}_1}), \]
\[ B_1 = \bar{x}\left(\frac{1}{\sqrt{1-\alpha}} + \frac{\sqrt{1-\alpha}}{\delta^2} + \frac{1}{\delta}\right); \]
\[ B_2 = \bar{x}(\gamma_0 \Gamma \cdot \bar{a}_1 + 1). \]

Then we have \( \Delta u(x, t) \leq \frac{v(t) \cdot g(\bar{x})}{t} \). From this relation and from (3.23) we obtain
\[ v(t) \leq A_3 \Gamma + A_4 \gamma_0 \Gamma \int_{\alpha t}^{t} \int_{\xi > \bar{a}_1} |\Delta x G| \cdot g(\bar{x}) d\xi. \]

By Lemma 6v) we have
\[ \int_{\xi > \bar{a}_1} |\Delta x G| \cdot g(\bar{x}) d\xi \leq A_4 g(\bar{x}) (1 + \sqrt{(1-\alpha)/\alpha}) (1/\sqrt{\alpha}). \]

From the last two inequalities, putting \( \tau = \rho t \), we get
\[ v(t) \leq A_3 \Gamma + A_4 \gamma_0 \Gamma \int_{\alpha t}^{t} \int_{\xi > \bar{a}_1} |\Delta x G| \cdot g(\bar{x}) d\xi. \]

Choose \( \alpha_1 \) so close to 1 that \( \alpha_1 > \frac{1+\sigma_0}{1+\sigma} \) and
\[ (1 + \sqrt{(1-\alpha_1)/\alpha_1})(1/\sqrt{\alpha_1}) A_4 \gamma_0 \Gamma \int_{\alpha_1}^{1} \frac{d\rho}{\rho^{3/2} \sqrt{1-\rho}} < 1. \]

It means that \( \frac{1}{\sqrt{1-\alpha_1}} \) must be of order \( O\left(\frac{\sqrt{1+\sigma}}{\sqrt{\sigma-\sigma_0}} + \gamma_0 \Gamma\right) \). Using Lemma A_2 of Appendix we obtain
\[ \Delta u \leq \frac{v(t) \cdot g(\bar{x})}{t} \leq A_5 \Gamma \left( B_1 + B_2(1 + \ln_+ \frac{\sqrt{1-\alpha}}{\bar{x} - \bar{a}_1}) \right), \]

where \( \bar{x} \in (\bar{a}_1, a_2 + \sigma_0 \sqrt{t}) \), \( t \geq t_0 \geq \bar{a}_1^2 \). Put now \( \sqrt{1-\alpha} = \min\{\delta, \sqrt{1-\alpha_1}\} \). Then we obtain
\[ \Delta u \leq \frac{A_5 \Gamma \cdot \bar{x}}{t} \left[ \frac{1}{\sqrt{1-\alpha}} + (\gamma_0 \Gamma \cdot \bar{a}_1 + 1) \right] (1 + \ln_+ \frac{\sqrt{1-\alpha}}{\bar{x} - \bar{a}_1}). \]

Now let \( \bar{x} > a_1 \) be fixed and take \( \bar{a}_1 = \frac{a_1 + \bar{x}}{2}, \delta = \frac{\delta}{2} \). We obtain
\[ \Delta u \leq \frac{A_6 \Gamma \cdot \bar{x}}{t} \left[ \frac{1}{\sqrt{1-\alpha}} + \gamma_0 \Gamma + (\gamma_0 \Gamma \cdot \bar{a}_1 + 1) \right]. \]

Theorem 2ii) is proved.

**Sketch of the proof of Theorem 2ii)'**

Step 1. Let function \( u \) satisfy equation (1.2) in \( \Omega_0 \) with \( \varphi(0) = C = 1 \) and \( \varepsilon = 1 \). Put \( \varphi_0 = \varphi - C \). We use again the Green-Poisson type representation formulas for \( u \) of
such that $0 \leq \chi_0 \leq 1$, $\chi_0|_{[\tilde{a}_1, \tilde{a}_2]} \equiv 1$, $\chi|_{(-\infty, a_1)} \equiv 0$, $\chi|_{(a_2, \infty)} \equiv 0$, $0 < a_1 < \tilde{a}_1 < \tilde{a}_2 < a_2$, inequalities (3.1) are valid with $\delta = \min\{\tilde{a}_1 - a_1, a_2 - \tilde{a}_2\}$. We obtain representation ($\bar{x} \in [\tilde{a}_1, \tilde{a}_2], t > \alpha t$)

$$\Delta u = I_0 u + I_1 u + I_2 u + I_3 u + I_4 u,$$  

(3.4)

where

$$I_0 u = -\int_\alpha^t d\tau \int_{\xi \in (\tilde{a}_1, \tilde{a}_2)} \Delta G \cdot \varphi_0(u) \cdot \Delta u d\xi,$$

$$I_1 u = -\int_\alpha^t d\tau \int_{\xi \in [a_1, a_2]\setminus[\tilde{a}_1, \tilde{a}_2]} \Delta G \cdot \varphi_0(u) \cdot \Delta u \chi d\xi,$$

$$I_2 u = \int_{\xi \in [a_1, a_2]} \Delta G (x - \xi, t - \alpha t) u(\xi, \alpha t) \chi(\xi, \alpha t) d\xi,$$

$$I_3 u = \int_\alpha^t d\tau \int_{\xi \in [a_1, a_2]\setminus[\tilde{a}_1, \tilde{a}_2]} \Delta G (u\chi' + u\Delta \chi) d\xi,$$

$$I_4 u = -\int_\alpha^t d\tau \int_{\xi \in [a_1, a_2]\setminus[\tilde{a}_1, \tilde{a}_2]} \Delta G \cdot \Delta u \cdot \Delta \chi d\xi.$$

Step 2. Let $u$ satisfy conditions of Theorem 2)ii)' and $\Delta u$ be represented in $\Omega_0$ by formula (3.4)', $\alpha > 1/2$. Using Lemmas 4,5,6 we obtain Lemma 7' and 9':

**Lemma 7'.** For $\bar{x} \in [a_1, a_2]$ and $t \geq t_0$ the following estimates are valid

$$|I_2 u(x, t)| \leq \frac{A_2 \Gamma}{\sqrt{(1 - \alpha)t}},$$  

(3.5)'

$$|I_3 u(x, t)| \leq \frac{A_2 \Gamma}{t} \left(\frac{1}{\delta^2} + \frac{\tilde{a}_2}{2\delta}\right) \sqrt{1 - \alpha}.$$  

(3.6)'

**Lemma 9'.** For $\bar{x} \in [\tilde{a}_1, \tilde{a}_2]$ and $t \geq t_0$ the following estimates are valid

$$|I_1 u(x, t)| \leq A_2 \frac{\gamma_0 \Gamma^2}{t} \left(1 + \ln_+ \frac{\sqrt{1 - \alpha}}{\bar{x} - \tilde{a}_1} + \ln_+ \frac{\sqrt{1 - \alpha}}{\tilde{a}_2 - \bar{x}}\right),$$  

(3.18)'

$$|I_4 u(x, t)| \leq A_2 \frac{A_0 \Gamma}{\delta t} \left(1 + \ln_+ \frac{\sqrt{1 - \alpha}}{\bar{x} - \tilde{a}_1} + \ln_+ \frac{\sqrt{1 - \alpha}}{\tilde{a}_2 - \bar{x}}\right).$$  

(3.19)'

Step 3. From formula (3.4)' and estimates (3.5)', (3.6)', (3.18)', (3.19)' we deduce the following inequality ($\bar{x} \in [\tilde{a}_1, \tilde{a}_2]$)

$$\Delta u \leq \frac{A_3 \Gamma}{t} \left[\frac{1}{\sqrt{1 - \alpha}} + \frac{\sqrt{1 - \alpha}}{2\delta^2} + \frac{\tilde{a}_2 \sqrt{1 - \alpha}}{2\delta}\right] +$$

$$(\gamma_0 \Gamma + \frac{1}{\delta}) \left(1 + \ln_+ \frac{\sqrt{1 - \alpha}}{\bar{x} - \tilde{a}_1} + \ln_+ \frac{\sqrt{1 - \alpha}}{\tilde{a}_2 - \bar{x}}\right) -$$

$$\int_\alpha^t d\tau \int_{\xi \in (\tilde{a}_1, \tilde{a}_2)} \Delta G \varphi_0(u) \Delta u d\xi.$$
By assumption of Theorem 2ii)’ we have \( \Delta \xi(\xi, \tau) \geq 0 \). If in assumptions of Theorem 2ii)’ we have additional positivity conditions \( \varphi'(0) \geq 0 \) and \( u \geq 0 \) then we can replace the integral term in (3.23)’ by the following bigger one

\[
-\gamma_0 \Gamma \int_{at}^t \frac{d\tau}{\sqrt{\tau}} \int_{\xi : \Delta G < 0} \Delta_x G \cdot \Delta u \, d\xi.
\]

Following further the proof of Theorem 2ii) and applying again Lemma 6v) we obtain the statement of Theorem 2ii)’ with constant \( B = B_0(a_2 + \frac{1}{d} + \gamma_0 \Gamma^2) \).

Without additional positivity conditions the statement of Theorem 2ii)’ is also valid but for the proof of it more hard version of Lemma 6v) is needed where the weight \( (1 + \ln \frac{b}{a_2 - \xi}) \) is replaced by \( (1 + \ln \frac{b}{a_2 - \tilde{x}}) \).

**Lemma 6v)’.** Let \( 0 < \bar{x} < \tilde{a}_2 \). Then

\[
\int_{\xi < \tilde{a}_2} |\Delta G(x - \xi, t - \tau)| \left( 1 + \ln \frac{1}{a_2 - \xi} \right) d\xi \leq \frac{A_1}{\sqrt{t - \tau}} \left( 1 + \ln \tilde{a}_2 + \ln \frac{1}{\tilde{a}_2 - \bar{x}} \right).
\]

**Appendix.** Integral inequalities.

**Lemma A1.** Let \( 0 \leq \psi(x) = O\left( \frac{1}{x} \right), x \geq 0, \text{ and } \int_0^\infty \psi(x) dx < \infty \). Then

\[
\int_0^a \psi(x) \ln_+ \frac{b}{a - x} dx \leq A_\psi \left( 1 + \ln \frac{b}{a} \right).
\]

**Proof.** Let \( a < b \). Then

\[
\int_0^a \psi(x) \ln_+ \frac{b}{a - x} dx = \int_0^{a/2} \psi(x) \ln_+ \frac{b}{a - x} dx + \int_{a/2}^a \psi(x) \ln_+ \frac{b}{a - x} dx \leq \ln_+ \frac{2b}{a} \int_0^\infty \psi(x) dx + \max \psi(x) \int_0^a \ln_+ \frac{b}{a - x} dx = A_\psi \left( \frac{1}{2} \ln_+ \frac{2b}{a} + \ln_+ \frac{b}{a} + 1 \right) \leq A_\psi \left( \ln_+ \frac{b}{a} + 1 \right).
\]

Let \( a > b \). Then

\[
\int_0^a \psi(x) \ln_+ \frac{b}{a - x} dx = \int_{a-b}^a \psi(x) \ln_+ \frac{b}{a - x} dx = \int_{a-b}^{a-b/2} \psi(x) \ln_+ \frac{b}{a - x} dx + \int_{a-b/2}^a \psi(x) \ln_+ \frac{b}{a - x} dx \leq \ln_+ \frac{b}{x} \int_0^\infty \psi(x) dx + \max \psi(x) \int_0^{b/2} \ln_+ \frac{b}{x} dx \leq A_\psi.
\]
Lemma A2. Let $v(t)$ be a continuous function satisfying the inequality

$$v(t) \leq A + \int_{\alpha}^{1} h(\rho)v(\rho t)d\rho, \ t \geq t_0,$$

where

$$0 < \int_{\alpha}^{1} h(\rho)d\rho < 1, \ h \geq 0, \ \alpha \in (0, 1).$$

Then $\exists \ m > 0, \ M > 0$ such that $v(t) \leq A_1 + Mt^{-m}$, $t \geq t_0$, where

$$A_1 = A(1 - \int_{\alpha}^{1} h(\rho)d\rho)^{-1}.$$

Proof. Find $A_1 \in \mathbb{R}$ such that $v_1(t) = A_1$ satisfies the equation

$$v_1(t) = A + \int_{\alpha}^{1} h(\rho)v_1(\rho t)d\rho.$$ 

We get

$$A_1 = A(1 - \int_{\alpha}^{1} h(\rho)d\rho)^{-1}.$$ 

Let us find $m > 0$ such that $v_0(t) = 1/t^m$ satisfies the equation

$$v_0(t) = \int_{\alpha}^{1} h(\rho)v_0(\rho t)d\rho.$$ 

This holds iff $\int_{\alpha}^{1} \frac{h(\rho)}{\rho^m} d\rho = 1$. Since $I(m) = \int_{\alpha}^{1} \frac{h(\rho)}{\rho^m} d\rho$ is a continuous function of $m$, $I(m) \to +\infty$ as $m \to +\infty$, $I(0) < 1$, then there exists $m$ such that $I(m) = 1$.

Choose $M$ large enough such that

$$V(t) = v(t) - v_1(t) - Mv_0(t) < 0$$

for $t_0 < t \leq t_0/\alpha = t_1$. We claim that $V(t) < 0 \ \forall \ t \geq t_0$.

Indeed, let $t^* = \sup \{t \geq t_0 : V(t) < 0\}$. By the choice of $M$ and continuity of $V$ we have $t^* > t_1$.

If $t^*$ is finite then

$$V(t^*) \leq \int_{\alpha}^{1} h(\rho)V(\rho t^*)d\rho < 0.$$ 

Since $V$ is continuous, $V < 0$ holds in a neighborhood of $t^*$, but this contradicts the definition of $t^*$. 

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