ON GENERALIZED OSCILLATOR ALGEBRAS AND THEIR ASSOCIATED COHERENT STATES

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A unified method of calculating structure functions from commutation relations of deformed single-mode oscillator algebras is presented. A natural approach to building coherent states associated to deformed algebras is then deduced.

Keywords: coherent states; structure function; deformed algebra; Heisenberg algebra.

1. Introduction

Generalization of Heisenberg-Weyl algebra was suggested long before the discovery of quantum groups by Heisenberg to achieve regularization for (nonrenormalizable) nonlinear spinor field theory. The issue was considered as small additions to the canonical commutation relations. Snyder, investigating the infrared catastrophe of soft photons in the Compton scattering, raised this issue and built a noncommutative Lorentz invariant space-time where the noncommutativity of space operators is proportional to nonlinear combinations of phase space operators.

Further modifications of the oscillator algebra and their possible physical interpretations as spectrum generating algebras of non standard statistics were made since the earlier work of Snyder. As matter of citation, let us mention the $q$-oscillator algebras by Coon and coworkers, Kuryshkin, Jannussis and collaborators.

With the development of quantum groups, new aspects of $q$-oscillators were identified as a tool for providing a boson realization of quantum algebra $su_q(2)$ using a $q$-analogue of the harmonic oscillator and the Jordan-Schwinger mapping, and then generalizations in view of unifying or extend-
ing different existing $q$-deformed algebras were elaborated\textsuperscript{11,12}.

Despite all useful properties and applications motivated by various one-parameter deformed algebras, the multi-parameter deformations aroused much interest because of their flexibility when dealing with concrete physical models. See for example\textsuperscript{13,14} and references therein.

Coherent states (CS) have practically followed the same trend as the quantum algebras. They were invented by Schrödinger in 1926 in the context of the quantum harmonic oscillator, and defined as minimum-uncertainty states that exhibit a classical behavior.\textsuperscript{15}

In 1963, coherent states were simultaneously rediscovered by Glauber,\textsuperscript{16,17} Klauder\textsuperscript{18,19} and Sudarshan\textsuperscript{20} in quantum optics of coherent light beams emitted by lasers.

The vast field covered by coherent states motivated their generalizations to other families of states deducible from noncanonical operators and satisfying not necessarily all properties of CS.\textsuperscript{21,22}

The more general class is essentially based on the overcompleteness property of coherent states. This property was the raison d’être of the mathematically oriented construction of generalized coherent states by Ali et al\textsuperscript{21} or of the ones with physical orientations.\textsuperscript{23,24} Numerous publications continue to appear using this property. See for example\textsuperscript{25,26} and references therein.

In this paper we give a method of computing the so-called structure function which is the basis of coherent state construction from a given algebra.

2. Unified deformed single-mode oscillator algebras and associated CS

The Heisenberg-Weyl algebra is a Lie algebra generated by the position $Q$, the momentum $P$ and the unity $\mathbf{1}$ operators satisfying:

$$[Q, P] = \imath \hbar \mathbf{1}, \quad [Q, \mathbf{1}] = 0, \quad [P, \mathbf{1}] = 0. \quad (1)$$

Defining the annihilation and creation operators by $b := \frac{Q + \imath P}{\sqrt{2\hbar}}$ and $b^\dagger := \frac{Q - \imath P}{\sqrt{2\hbar}}$, the Weyl-Heisenberg algebra is generated by $\{b, b^\dagger, \mathbf{1}\}$ satisfying:

$$[b, b^\dagger] = \mathbf{1}, \quad [b, \mathbf{1}] = 0, \quad [b^\dagger, \mathbf{1}] = 0. \quad (2)$$

From (2), one defines the operator $N := b^\dagger b$, also called number operator, with the properties:

$$[N, b] = -b, \quad [N, b^\dagger] = b^\dagger, \quad [N, \mathbf{1}] = 0. \quad (3)$$
The canonical coherent states (CS) are normalized states $|z\rangle$ satisfying one of the following three equivalent conditions:15,16,18,20

\begin{enumerate}[(i)]
  \item $(\Delta Q)(\Delta P) = \frac{\hbar}{2}$,
  \item $b|z\rangle = z|z\rangle$;
  \item $|z\rangle = e^{z\hat{b}^\dagger - \bar{z}\hat{b}}|0\rangle = e^{-\frac{|z|^2}{2}} \sum_{n=0}^{\infty} \frac{z^n}{\sqrt{n!}} |n\rangle$,
\end{enumerate}

where $(\Delta A)^2 := \langle z|A^2 - \langle A\rangle^2|z\rangle$, $\langle A\rangle := \langle z|A|z\rangle$, $b|0\rangle = 0$ and $|0\rangle = 1$.

They belong to the Fock space $\mathcal{F} = \text{span}\{ |n\rangle \mid n \in \mathbb{N} \cup \{0\} \}$, where the states

\begin{equation}
|n\rangle = \frac{1}{\sqrt{n!}} (\hat{b}^\dagger)^n |0\rangle, \quad n = 1, 2, \cdots
\end{equation}

satisfy the orthogonality and completeness conditions:

\begin{equation}
\langle m|n\rangle = \delta_{m,n}, \quad \sum_{n=0}^{\infty} |n\rangle\langle n| = 1.
\end{equation}

The important feature of these coherent states resides in the partition (resolution) of identity:

\begin{equation}
\int_{\mathbb{C}} \frac{|d^2z|}{\pi} |z\rangle\langle z| = 1,
\end{equation}

where we have put $|d^2z| = d(\text{Re}z)d(\text{Im}z)$ for simplicity.

**Definition 2.1.** We call deformed Heisenberg algebra, an associative algebra generated by the set of operators $\{1, a, a^\dagger, N\}$ satisfying the relations

\begin{equation}
[N, a^\dagger] = a^\dagger, \quad [N, a] = -a,
\end{equation}

such that there exists a non-negative analytic function $f$, called the structure function, defining the operator products $a^\dagger a$ and $aa^\dagger$ in the following way:

\begin{equation}
a^\dagger a := f(N), \quad aa^\dagger := f(N + 1),
\end{equation}

where $N$ is a self-adjoint operator, $a$ and its Hermitian conjugate $a^\dagger$ denote the deformed annihilation and creation operators, respectively.

The deformed Heisenberg algebras have a common property characterized by the existence of a self-adjoint number operator $N$, a lowering operator $a$ and its formal adjoint, called raising operator, $a^\dagger$, and differ by the expression of the structure function $f$. 

The associated Fock space $\mathcal{F}$ is now spanned by the orthonormalized eigenstates of the number operator $N$ given by:

$$|n\rangle = \frac{1}{\sqrt{f(n)!}} (a^\dagger)^n |0\rangle, \quad n \in \mathbb{N} \cup \{0\},$$

where $f(n)! = f(n)f(n-1)...f(1)$ with $f(0) = 0$. Moreover,

$$a|n\rangle = \sqrt{f(n)!} |n-1\rangle, \quad a^\dagger |n\rangle = \sqrt{f(n+1)!} |n+1\rangle.$$  \tag{9}

In this context, the structure function $f$ appears as a key unifying methods of coherent state construction corresponding to deformed algebras.

Denote $D_f := \{ z \in \mathbb{C} : |z|^2 < R_f \}$, where $R_f$ is the radius of convergence of the series (the deformed exponential function):

$$N_f(x) := \sum_{n=0}^{\infty} \frac{x^n}{[f(n)]!}. \tag{11}$$

Then, the states

$$|z, f\rangle := (N_f(|z|^2))^{-1/2} \sum_{n=0}^{\infty} \frac{z^n}{\sqrt{[f(n)]!}} |n\rangle, \quad z \in D_f,$$  \tag{12}

are normalized eigenstates of the raising operator $a$ with the eigenvalues $z$. They are not orthogonal to each other. Moreover, the map $z \mapsto |z, f\rangle$ from $D_f \subset \mathbb{C}$ to the Fock space $\mathcal{F}$ is continuous.

The set $\{ |z, f\rangle : z \in D_f \}$ will be called family of coherent states whether there exists a positive measure $\mu_f$ such that\footnote{In the literature, this condition is often referred to as the existence of a positive measure.}

$$\int_{D_f} d\mu_f(\bar{z}, z) |z, f\rangle \langle z, f| = \sum_{n=0}^{\infty} |n\rangle \langle n| = 1. \tag{13}$$

Passing to polar coordinates, $z = \sqrt{x} e^{i\theta}$, where $0 \leq \theta \leq 2\pi$, $0 < x < R_f$, and $d\mu(\bar{z}, z) = d\omega_f(x) d\theta$, this corresponds to the following classical power-moment problem:\footnote{This is a classical power-moment problem that arises in the study of coherent states.}

$$\int_0^{R_f} x^n \frac{2\pi}{N_f(x)} d\omega_f(x) = [f(n)]!, \quad n = 0, 1, 2, \ldots \tag{14}$$

Note immediately that not all deformed algebras lead to coherent states in this construction formalism because the moment problem (14) does not always have a solution.\footnote{This highlights a limitation of the construction formalism.}

Many techniques have been proposed in the literature to determine the structure function $f$ corresponding to a given algebra.\footnote{These include methods such as the use of generating functions and the analysis of the convergence properties of the series.}

Meljanac et al defined the generalized $q$-deformed single-mode oscillator as an algebra generated by the identity operator $1$, a self-adjoint number
operator $N$, a lowering operator $a$ and an operator $\bar{a}$ which is not necessarily conjugate to $a$ satisfying

$$[N, a] = -a, \quad [N, \bar{a}] = \bar{a},$$

(15)

$$a\bar{a} - F(N)\bar{a}a = G(N)$$

(16)

where $F$ and $G$ are arbitrary complex analytic functions.

Such an algebra furnishes an appropriate approach for the unification of classes of deformed algebras known in the literature.

For the purpose, let us start from the relations (15) to get $[N, a\bar{a}] = 0 = [N, \bar{a}a]$ implying the existence of a complex analytic function $\varphi$ such that

$$\bar{a}a = \varphi(N) \quad \text{and} \quad a\bar{a} = \varphi(N + 1).$$

(17)

Eqs (16) and (17) give

$$\varphi(N + 1) - F(N)\varphi(N) = G(N).$$

(18)

Denote now $a^\dagger$ the Hermitian conjugate of the operator $a$. Then,

$$[N, a^\dagger] = a^\dagger, \quad \text{and} \quad \bar{a} = c(N)a^\dagger,$$

(19)

where $c(N)$ is a complex function. For convenience take $c(N) = e^{i\arg \varphi(N)}$. Therefore, from (17) and the fact that $a^\dagger a$ and $aa^\dagger$ are Hermitian operators we necessarily have

$$a^\dagger a = |\varphi(N)| \quad \text{and} \quad aa^\dagger = |\varphi(N + 1)|.$$  

(20)

We now assume the existence of a "vacuum state" $|0\rangle$ such that

$$N|0\rangle = 0, \quad a|0\rangle = 0 \quad \text{and} \quad \langle 0|0 \rangle = 1,$$

(21)

and construct the non normalized eigenvectors, $(a^\dagger)^n|0\rangle$ of the operator $N$. Then, applying (18) to these vectors we obtain

$$\varphi(n) = |F(n - 1)|! \sum_{k=0}^{n-1} \frac{G(k)}{|F(k)|!}, \quad n \geq 1,$$

(22)

unless the initial condition $\varphi(0) = 0$ is satisfied, and where

$$|F(k)|! = \begin{cases} 
  F(k)F(k-1)\cdots F(1) & \text{if } k \geq 1 \\
  1 & \text{if } k = 0
\end{cases}.$$

(23)

Hence, the structure function characterizing a given deformation is defined as follows: $f(n) := \varphi(n)$ if $\varphi(n) \geq 0$, and $f(n) := |\varphi(n)|$, in general. Provided the function $f$, the above formalism can be displayed to construct generalized CS.
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