Counting Nambu-Goldstone modes of higher-form global symmetries

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We discuss the counting of Nambu-Goldstone (NG) modes associated with the spontaneous breaking of higher-form global symmetries. Effective field theories of NG modes are developed based on symmetry breaking patterns, using a generalized coset construction for higher-form symmetries. We derive a formula of the number of gapless NG modes, which involves expectation values of the commutators of conserved charges, possibly of different degrees.

Introduction.—Spontaneous symmetry breaking (SSB) is a common thread of modern physics running through various fields from condensed-matter to high-energy physics. When continuous symmetries are spontaneously broken, gapless excitations appear and they are called the Nambu-Goldstone (NG) modes [1–3] (see Ref. 4 for a recent review). Due to their gapless nature, they dominate the low-energy physics. In relativistic systems, there is a one-to-one correspondence between a broken generator and a gapless mode. However, the Lorentz symmetry is not present in many physically-interesting situations, especially in condensed-matter systems. In the absence of Lorentz invariance, the one-to-one correspondence no longer holds [5–8] and the number of NG modes \( N_{\text{NG}} \) can be smaller than the number of broken symmetry generators \( N_{\text{BS}} \) [9–14],

\[
N_{\text{NG}} = N_{\text{BS}} - \frac{1}{2} \text{rank} \rho_{ab},
\]

where \( \rho_{ab} \) is the matrix of the expectation value of charge commutators, \( \rho_{ab} \propto [\{Q_a, Q_b\}] \).

Recently, those symmetries are understood as a part of a wider class of symmetries, called higher-form symmetries [15] (see also [16–27]). The defining feature of higher-form symmetries is that charged objects under those symmetries are extended: The charged objects for a \( p \)-form symmetry are \( p \)-dimensional. From this point of view, an ordinary symmetry corresponds to a 0-form symmetry, whose charged objects are point-like. This concept provides us with a unifying perspective, and has been used in describing topological orders [17, 28, 31], formulating the magnetohydrodynamics [32, 33], and realizing a new class of Symmetry Protected Topological phases [35–37]. Similarly to the ordinary symmetries, when a continuous higher-form symmetry is spontaneously broken, gapless excitations appear [15, 38]. In fact, photons are understood as the NG bosons associated with the spontaneous breaking of a \( U(1) \) 1-form symmetry [15, 10].

The question we would like to address in this Letter is: How many gapless NG modes appear when the higher-form symmetries are spontaneously broken? To answer this, we develop effective field theories by extending the coset construction [41, 42], which is widely used for ordinary symmetries. In the case of higher-form symmetries, the counting of the number of physical NG modes becomes involved because of the gauge degrees of freedom. We derive a generalized counting formula of gapless NG modes for systems that are not necessarily Lorentz-invariant, which reduces to Eq. (4) when all the symmetries are 0-form symmetries.

SSB and coset construction.—We consider the spontaneous breaking of continuous internal symmetries that can include higher-form symmetries in \( D \)-dimensional Minkowski spacetime, \( \mathbb{R}^{1,D-1} \). We assume that the translational symmetry is unbroken. We label each of the broken generators by \( A \), that generates a \( p_A \)-form symmetry. For \( p_A \geq 1 \), the symmetry is always Abelian. Let us denote the the charged object for the \( p_A \)-form symmetry by \( W_A(C_{p_A}) \), which is supported on a closed \( p_A \)-dimensional submanifold \( C_{p_A} \). In the case of a \( U(1) \) symmetry, the action of the symmetry is

\[
W_A(C_{p_A}) \mapsto e^{i \phi} W_A(C_{p_A}).
\]

The spontaneous breaking of the symmetry \( A \) is diagnosed by Coulomb or perimeter behavior the vacuum expectation value,

\[
\langle W_A(C_{p_A}) \rangle \sim 1 \quad \text{or} \quad \langle W_A(C_{p_A}) \rangle \sim e^{-T \text{perimeter}[C_{p_A}]},
\]

where perimeter \([C_{p_A}]\) is the \( p_A \)-dimensional volume of \( C_{p_A} \), and \( T \) is the tension.

We would like to construct the low-energy effective field theory based on the symmetry breaking patterns of higher-form symmetries. The coset construction is a powerful technique to write down the action of the effective theory systematically for ordinary symmetries [41–44] and also for spacetime symmetries [45, 46]. The basic building block for this method is the Maurer-Cartan
1-forms. To apply the coset construction to higher-form symmetries, we need the corresponding object. For a $p_A$-form symmetry breaking, we define a generalized Maurer-Cartan form $f_A$, which is a $(p_A + 1)$-form, by

$$e^{i f_A} f_A := W_A^i (C_{p_A}) W_A(C'_{p_A}),$$

where $W_A (C_{p_A}) = \exp(i f_{C_{p_A}} a_A)$ is a generalization of coset variables, and $X$ is a $(p_A + 1)$-dimensional subspace such that $\partial X = C'_{p_A} \cup (-C'_{p_A})$. For a 0-form symmetry, the left-hand side of Eq. (4) should be understood as a path-ordered product. The generalized Maurer-Cartan form $f_A$ should satisfy the flatness condition $df_A = 0$, since the right-hand side of Eq. (4) is independent of the choice of the interpolating manifold $X$. The corresponding equation for a 0-form symmetry is the Maurer-Cartan equation. Since $f_A$ is a closed $(p_A + 1)$-form, it can be regarded as the conserved charge associated with the symmetry, which is emergent in the symmetry-broken phase. The coset variable has a gauge redundancy under $a_A \mapsto a_A + \theta A$, where $\theta A$ is a $\mathbb{Z}$-valued gauge parameter, because $A$ is integrated over a closed subspace $C_{p_A}$. A symmetry transformation shifts the field $a_A$ by a flat connection, $a_A \mapsto a_A + \lambda$. Using the generalized Maurer-Cartan form as a building block, we can obtain effective actions by writing down the terms consistent with the spacetime symmetry of the systems of interest. More details on this construction will be given elsewhere [54].

**Generalized counting rule.**—We here present the generalized counting rule of gapless NG modes associated with spontaneously broken higher-form symmetries. In Lorentz-invariant systems, the number of gapless modes for a broken generator $A$ of a $p_A$-form symmetry is given by

$$N_{D,A} = D - 2C_{p_A},$$

in $D$-dimensional spacetime. In the absence of Lorentz invariance, the number can be reduced. In this Letter, we show that the total number of gapless NG modes is given by

$$N_{NG} = \sum_A N_{D,A} - \frac{1}{2} \text{rank } M_{\alpha \beta},$$

where the summation is over all the broken generators. Here, $M_{\alpha \beta}$ is an anti-symmetric matrix, whose definition will be given below. Let us first define a quantity proportional to the expectation values of charge commutators,

$$M_{AB}(V, V_B) := \frac{\langle [\mathcal{Q}_{AB}^{[p_A]}(V), \mathcal{Q}_{[p_B]}^B(V_B)] \rangle}{\text{vol} \left[ V_A \cap V_B \right]}.$$  

(7)

Here, we denote the conserved charge associated with the generator $A$ as $\mathcal{Q}_{[p_A]}^{[p_A]}(V_A)$, which is supported on a $(D - p_A - 1)$-dimensional subspace $V_A$ located in the spatial slice $\Sigma$, and $\text{vol} \left[ V \right]$ indicates the volume of a subspace $V$. To account for the independent choices of subspaces $V_A, V_B$, we shall consider the following matrix,

$$M_{AB}^{i_1 \ldots i_{p_A}, j_1 \ldots j_{p_B}} := \lim_{V_{i_1 \ldots i_{p_A}} \rightarrow \infty} M_{AB}(V_{i_1 \ldots i_{p_A}}, V_{j_1 \ldots j_{p_B}}),$$  

(8)

where indices of $V_{i_1 \ldots i_{p_A}}$ specify a $(D - p_A - 1)$-dimensional plane placed inside the $(D - 1)$-dimensional spatial manifold $\Sigma$. The indices $i_1, \ldots$ only take spatial ones and is ordered, $i_1 < i_2 < \cdots < i_{p_A}$. For example, in the case of a 1-form symmetry in 4-spacetime dimensions, i.e., $D = 4$ and $p_A = 1$, the symmetry generator is 2-dimensional plane and $V$ represents the plane perpendicular to the $i$-axis for $i = x, y, z$. In taking the limit, we first take $V_{i_1 \ldots i_{p_A}} \rightarrow \infty$ and then take $V_{j_1 \ldots j_{p_B}} \rightarrow \infty$. We collectively denote the indices using the Greek letters as $\alpha := (A, (i_1, \cdots, i_{p_A}))$. With this notation, the matrix $M_{\alpha \beta}$ is denoted as $M_{\alpha \beta}$, that appears in the formula (8). $M_{\alpha \beta}$ is a real anti-symmetric square matrix of dimension $\sum_A D - 1C_{p_A}$. Its rank is always even, and $(\text{rank } M_{\alpha \beta})/2$ is an integer. For a higher-form symmetry, a generator is supported on a closed subspace of dimension less than $D - 1$. When we place them in the spatial slice $\Sigma$, we have to specify how to place it, which is the role of the indices $(i_1, \cdots, i_{p_A})$. Thus, the matrix $M_{\alpha \beta}$ contains the information of the expectation values of charge commutators, including how to place them, and that determines the number of gapless modes. The formula (6) is a natural generalization of the one for 0-form symmetries [11]. When all the symmetries are 0-form symmetries, $p_A = 0$ for all $A$, then $\sum_A N_{D,A} = \sum_A 1 = N_{BS}$, and all the symmetry generators are supported on a $(D - 1)$-dimensional subspace $Q_{[A]}^{[0]}(V_{D - 1})$. Then, the formula (6) reduces to the counting rule (1) for 0-form symmetries.

Let us make a comment on the dispersion relations. For 0-form symmetries, the NG modes are classified into type A and B, and they typically have linear and quadratic dispersion relations, respectively. When higher-form symmetries are involved, it is still possible to classify the modes into type A and B, but it is not

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*1 For a 0-form symmetry, $a_A$ is a 0-form and the symmetry transformation is a constant shift, to the leading order in the number of fields.

*2 In the summation, one does not need to include the dual symmetry, since it does not exist in the UV and has rather emerged as a consequence of SSB. For example, in the Maxwell theory, the $U(1)_{m}^{[1]}$ magnetic 1-form symmetry is broken as well as $U(1)_{m}^{[3]}$ symmetry, and there is a mixed ’t Hooft anomaly between them. The breaking pattern and the anomaly can be reproduced by introducing an NG mode of either $U(1)_{m}^{[1]}$ or $U(1)_{m}^{[3]}$ symmetries.

*3 When the space-time symmetry is involved, there are examples where the charge commutator is not anti-symmetric [53].
clear if we can associate this to the behavior of dispersion relations in general. For example, whether the dispersion is linear or quadratic may depend on the direction of the propagation, as can be seen in an example discussed later [see Eq. (14)].

Examples.—To illustrate the coset construction and the counting rule, let us confirm that the photons are NG modes associated with a $U(1)$ 1-form symmetry, which we denote by $U(1)_{\epsilon}$. When $U(1)_{\epsilon}$ is spontaneously broken, the coset variable is defined on a loop $C$ and can parametrized as $W(C) = e^{i\epsilon_x \alpha}$, which is nothing but the Wilson loop. Since it is defined on a loop, the field $a$ has a gauge redundancy, $a \mapsto a + d\theta_0$, with a $U(1)$ 0-form parameter $e^{i\theta_0}$. Therefore, we can identity $a$ as a $U(1)$ 1-form gauge field. The $U(1)_{\epsilon}$ transformation of $a$ is locally given by $a \mapsto a + \lambda_1$ with a flat 1-form $\lambda_1$. The generalized Maurer-Cartan form $f$ is defined through $e^{i f} := W^{-1}(C) W(C')$, where $C$ and $C'$ are loops, and $S$ is a 2-dimensional surface with $\partial S = C' \cup (-C)$. We can identify $f$ as $f = da$, and the flatness condition $df = 0$ is nothing but the Bianchi identity. In the Lorentz-invariant case, the gauge-invariant term with the lowest number of derivatives is given by

$$- \frac{1}{2e^2} f \wedge *f,$$

where $e$ is a coupling constant, and $*$ is the Hodge star operator. This is nothing but the Maxwell theory. There are $d-2C_1$ gapless excitations in $D$ spacetime dimensions (two photons for $D = 4$), which is consistent with the formula (9).

As a second example, let us consider photons in the presence of the gradient of the $\theta$ angle in $(3 + 1)$ dimensions [54], whose Lagrangian is given by

$$- \frac{1}{2e^2} f \wedge *f + \frac{c}{2} \epsilon f \wedge f,$$

where $c$ is a constant. The theory has $U(1)_{\epsilon}^{[1]}$ symmetry as in the case of the Maxwell theory, and the corresponding conserved charge is identified as $Q_\epsilon^{[1]}(S) = \frac{1}{2} \int_S *f - c \int_S \theta f$. This symmetry is spontaneously broken, and $Q_\epsilon^{[1]}(S)$ is a broken generator. Due to the presence of the nonvanishing $d\theta$, the charge $Q_\epsilon^{[1]}(S)$ becomes non-commutative,

$$\{[Q_\epsilon^{[1]}(S_1), Q_\epsilon^{[1]}(S_2)]\} \propto \int_{S_1 \cap S_2} d\theta \neq 0,$$

where $S_1, S_2 \subset \Sigma$ are 2-dimensional surfaces located inside the spatial manifold $\Sigma$. Let us organize the charges as a vector, $Q_i = \left(Q_\epsilon^{[1]}(S_x), Q_\epsilon^{[1]}(S_y), Q_\epsilon^{[1]}(S_z)\right)^T$, where $S_i$ is the surface perpendicular to $i$-axis. The charge commutator matrix is given by

$$M_{ij} = \{[Q_i, Q_j]\} \propto \epsilon_{ijk} \partial_k \theta,$$

where the gradient $\partial_k \theta$ is constant. The rank of this matrix is 2, and the number of gapless NG modes via the formula (9) is

$$N_{NG} = 2 - \frac{1}{2} \text{rank } M_{ij} = 1.$$

This coincides with the result obtained by explicitly solving the equations of motion (EOM). There are one gapless and one gapped modes [54], whose gap squared is given by $\omega^2(k = 0) = C^2$. Here, $C := |C|$ is the length of the vector $C := -c\epsilon^2 \nabla \theta$. The behavior of the dispersion relation depends on the direction of the momentum $k$. When the angle of $k$ and $C$ is $\ldots$, the dispersion relation at small $k = |k|$ is (see the Supplementary Material)

$$\omega^2(k) = \begin{cases} (\sin^2 \phi) k^2 + \frac{\cos^4 \phi}{c^2} k^4 + O(k^6), & C^2 + (2 - \sin^2 \phi) k^2 + O(k^4). \end{cases}$$

The dispersion relation of the gapless mode is quadratic, $\omega \sim k^2$, only when $\sin \phi = 0$ ($k$ is parallel to $C$), and is linear, $\omega \sim k$, for other angles.

A similar interpretation is possible for the system of neutral pions in the presence of background magnetic field [53] [56], where the commutators of 0-form and 1-form symmetry generators acquire expectation values.

Derivation.—Let us now give the derivation of the counting formula (9). For $p_A \geq 1$, the higher-form symmetry is Abelian, so $f_A$ can be written as $da_A$. For a 0-form non-Abelian symmetry, $f_A$ includes non-linear terms, that represent the interaction between NG modes. For the counting of the modes, it suffices to consider the linear order. Thus, we can also express the Maurer-Cartan form for 0-form symmetry as $f_A = da_A$, and we include indices of the Lie-algebra in $A$.

For Lorentz invariant systems, the kinetic terms are written, to the lowest order in derivatives and fields, as

$$\mathcal{L}_0(da_A) = -\frac{1}{2} F_{AB}^2 f_A \wedge *f_B = -\frac{1}{2} F_{AB}^2 da_A \wedge *da_B,$$

which is invariant under $a_A \mapsto a_A + \lambda A$. Here, $F_{AB}^2 = F_{CA} F_{CB}$ is a positive-definite symmetric matrix, and $F_{AB}$ corresponds to the decay-constant matrix. The Lagrangian can also have topological terms of the form $G_{AB} f_A \wedge f_B$, where $G_{AB}$ are flat coefficients. Those terms do not contribute to the EOM, so they can be dropped. Since we can diagonalize $F_{AB}^2$, it suffices to count the modes when one $p_A$-form symmetry is spontaneously broken. The EOM and the flatness condition read

$$d^1 f_A = 0, \quad df_A = 0,$$

$^4$ The form $f_A$ is closed also for a non-Abelian 0-form symmetry since we focus on the free part for the counting.
where $d^\dagger$ is the codifferential. One can immediately see that the field strength satisfies $\Delta f_A = 0$, where $\Delta := dd^\dagger + d'd$ is the Hodge Laplacian. In the momentum space, the Hodge Laplacian is given by $-\omega^2 + k^2$. Each component of the field strength satisfies the wave equation, but not all of them are independent. The Bianchi identity and EOM have $d-2C_{pA+1}$ and $d-2C_{pA-1}$ relations without time derivatives, and these are the constraints. The number of physical modes is given by the number of components of the field strength, $dC_{pA+1}$, minus the number of constraints. Since the physical modes consist of pairs of these degrees of freedom, we find

$$N_{D,pA} = \frac{1}{2}(dC_{pA+1} - d-2C_{pA+1} - d-2C_{pA-1}) = d-2C_{pA}. \tag{17}$$

Thus, the number of gapless modes in the Lorentz invariant system is given by the sum of this number over each generator,

$$N_{\text{gapless}} = \sum_A N_{D,pA}. \tag{18}$$

Now let us discuss the case without Lorentz invariance. The breaking of the Lorentz invariance means that there are special directions in spacetime. We still assume that the transversal symmetry is not broken. A key observation here is that the absence of the Lorentz symmetry can be represented as the presence of external objects placed in the spacetime. This allows us to write additional terms with one derivative,

$$\mathcal{L}_1(a_A, f_A) := \frac{1}{2} f_A \wedge a_B \wedge \Omega_{AB}, \quad \tag{19}$$

where $\Omega_{AB}$ is a flat $d$-form with $d := D - p_A - p_B - 1$. The components of $\Omega_{AB}$ satisfy $\Omega_{AB} = -(-)^{p_A p_B} \Omega_{BA}$ so that they are not a total derivative. Those terms are invariant under symmetry transformations only up to a total derivative, and are reminiscent of the Wess-Zumino term. We assume here that the external defect is located spatially, by which we mean that $\Omega_{AB}$ does not involve $dx^0$. Unless both $A$ and $B$ are both 0-form symmetries, the external defect $\Omega_{AB}$ explicitly breaks the spatial rotation symmetry, which can lead to direction-dependent dispersion relations, as is seen in Eq. (11). The terms with two derivatives can also be more generic compared to the relativistic case. Although such a deformation leads to more generic dispersion relations, it does not change the number of constraints, so the number of physical gapless modes is unaffected. Thus, for the purpose of the counting of the modes, it suffices to use the following Lagrangian,

$$\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_1, \tag{20}$$

where $\mathcal{L}_0$ is given by Eq. (15).

We can identify the existence of the terms (19) with the non-vanishing expectation values of charge commutators.

The Noether current associated with the generator $A$ is given by

$$\star J_A = -F_{AB}^2 \star f_B + a_B \wedge \Omega_{AB}. \tag{21}$$

The symmetry generator is obtained by integrating the current over a submanifold $V_{D-pA-1} \subset \Sigma$,

$$Q_A^{[pA]}(V_{D-pA-1}) = \int_{V_{D-pA-1}} \star J_A. \tag{22}$$

Since the action is quadratic, the current commutator can be readily computed using the canonical commutation relations or the Ward-Takahashi identity as

$$\langle [iQ_A^{[pA]}(V_{D-pA-1}), J_B^{[pB]}(V_{D-pB-1})] \rangle \propto \int_{V_{D-pA-1} \cap V_{D-pB-1}} \Omega_{AB}. \tag{23}$$

Now we are ready to discuss the number of physical gapless modes. The EOM of the Lagrangian (20) are written in the form of current conservation, $d \star J_A = 0$. Let us write this with gauge-invariant variables $f_A = f_A F_B$ as

$$d \star f_A = \tilde{f}_B = \tilde{f}_B \wedge \tilde{\Omega}_{AB}, \tag{24}$$

where $\tilde{\Omega}_{AB} := F_A C_D F_B D$. If we write out Eq. (21) explicitly with indices,

$$-\partial^\alpha f_{\mu_1 \cdots \mu_pA}^{\nu_1 \cdots \nu_pB} = \frac{\epsilon_{\mu_1 \cdots \mu_pA}^{\nu_1 \cdots \nu_pB} + \epsilon_{\mu_1 \cdots \mu_pA}^{\nu_1 \cdots \nu_pB+1}}{(p_B + 1)!} \langle \tilde{\Omega}_{AB} \rangle^{\nu_1 \cdots \nu_pB}, \tag{25}$$

In the relativistic case, we have $\sum_A d-2C_{pA}$ gapless modes, but the presence of $\Omega_{AB}$ makes a part of them gapped. We shall here count the number of gapped modes. For this purpose, we write Eq. (26) in the energy/momentum space and take $k = 0$ and $\omega \neq 0$,

$$-\omega (\tilde{f}_A)_{i_1 \cdots i_pA}^{0} j_1 \cdots j_{pB} 0k_1 \cdots k_d (\tilde{\Omega}_{AB})_{k_1 \cdots k_d} (\tilde{f}_B)_{j_1 \cdots j_{pB} 0} \tag{26}$$

where the Roman indices $i_1, \cdots, j_1, \cdots, k_1, \cdots$ are spatial ones. The sum of the indices is implicitly taken such that they satisfy $k_1 < k_2 < \cdots < k_d$, and $j_1 < j_2 < \cdots < j_{pB}$, which removes the redundancy in the sum. Let us introduce a matrix notation

$$\tilde{M}_{i_1 \cdots i_pA}^{j_1 \cdots j_{pB}} = \epsilon_{i_1 \cdots i_pA}^{j_1 \cdots j_{pB} 0} 0k_1 \cdots k_d (\tilde{\Omega}_{AB})_{k_1 \cdots k_d}, \tag{27}$$

Then we have

$$-i\omega (\tilde{f}_A)_{i_1 \cdots i_pA}^{0} 0 \tilde{M}_{i_1 \cdots i_pA}^{j_1 \cdots j_{pB}} (\tilde{f}_B)_{j_1 \cdots j_{pB} 0} \tag{28}$$

By using the short-hand notation, $\alpha := (A, (i_1, \cdots, i_pA))$, it can be further written as $-i\omega \tilde{f}_\alpha = \tilde{M}_{\alpha \beta} \tilde{f}_\beta$. The matrix
\[ \tilde{M}_{\alpha \beta} \text{ is antisymmetric, and it can be always transformed to the following form by an orthogonal matrix } O, \]
\[ O \tilde{M} O^T = \begin{pmatrix} i \sigma_2 \lambda_1 & \cdots & i \sigma_2 \lambda_m \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \end{pmatrix}, \]
(29)
where \( \sigma_2 \) is the second Pauli matrix, \( m := (\text{rank } \tilde{M}_{\alpha \beta})/2 \), and \( \lambda_i \neq 0 \) for \( i = 1, \ldots, m \). Each 2 \times 2 sector in the upper-left part gives a gapped mode. For example,
\[ -i \omega \left( \begin{array}{c} f_{\alpha=1} \\ f_{\alpha=2} \end{array} \right) = \left( \begin{array}{cc} 0 & \lambda_1 \\ -\lambda_1 & 0 \end{array} \right) \left( \begin{array}{c} f_{\alpha=1} \\ f_{\alpha=2} \end{array} \right), \]
(30)
represents a gapped excitation whose gap is given by \( \omega^2 = \lambda_1^2 \). Thus, the number of gapped modes is given by
\[ N_{\text{gapped}} = \frac{1}{2} \text{rank } \tilde{M}_{\alpha \beta} = \frac{1}{2} \text{rank } M_{\alpha \beta}, \]
(31)
where \( M_{\alpha \beta} \) is defined similarly to \( \tilde{M}_{\alpha \beta} \) by replacing \( \tilde{\Omega}_{AB} \) with \( \Omega_{AB} \) in Eq. (27), and we used the fact that \( F_{AB} \) is invertible. The matrix \( M_{\alpha \beta} \) is nothing but the matrix \( \mathfrak{g} \).

Since the presence of \( L_1 \) does not change the number of physical degrees of freedom, the number of gapless modes in the absence of Lorentz invariance is given by Eq. (18) minus Eq. (31), which is the formula \( \mathfrak{d} \).

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**SUPPLEMENTARY MATERIAL**

**Photons under a constant \( \theta \) gradient**

Here, we detail on the dispersion relation of a model of photons under a space-dependent \( \theta \) angle. The Lagrangian of the system we consider is
\[ \mathcal{L} = -\frac{1}{2} e^2 f \wedge \star f + \frac{c}{2} \theta f \wedge f \]
\[ = -\frac{1}{4 e^2} f_{\mu \nu} f^{\mu \rho} f^{\nu \beta} \rho \beta d^2 x - \frac{c}{8} \varepsilon^{\mu \nu \alpha \beta} f_{\mu \nu} f_{\alpha \beta} d^2 x, \]
where \( e \) is the coupling constant, \( f_{\mu \nu} = \partial_{\mu} a_{\nu} - \partial_{\nu} a_{\mu} \) is the field strength with the photon field \( a_{\mu} \), \( c \) is a constant, and \( \varepsilon^{\mu \nu \rho \sigma} \) is the totally antisymmetric tensor with \( \varepsilon^{0123} = -1 \). The corresponding equations of motion are given, in the non-relativistic notation, by
\[ \nabla \cdot e = -C \cdot b, \]
(33)
\[ -\dot{e} + \nabla \times b = C \times e, \]
(34)
\[ \nabla \cdot b = 0, \]
(35)
\[ \dot{b} + \nabla \times e = 0. \]
(36)
where \( e = (f^{01}, f^{02}, f^{03})/e \) and \( b = (f_{23}, f_{31}, f_{12})/e \) are electric and magnetic fields, \( C = -c e^2 \nabla \theta \) is taken to be a constant vector, and we used \( \partial_\theta \theta = 0 \). From those equations, we can derive the following equations with second-order derivatives,
\[ -\ddot{e} + \nabla^2 e - \nabla(\nabla \cdot e) - C \times \dot{e} = 0, \]
(37)
\[ -\ddot{b} + \nabla^2 b + C(\nabla \cdot e) - (C \cdot \nabla)e = 0. \]
(38)
The corresponding equations in energy/momentum space are
\[ (\omega^2 - k^2)e + k(k \cdot e) + i \omega C + e = 0, \]
(39)
\[ (\omega^2 - k^2)b + iC(k \cdot e) - i(C \cdot k)e = 0, \]
(40)
where \( k := |k| \). Let us here use the basis \( \{ \hat{k}, \epsilon_{(1)}, \epsilon_{(2)} \} \), where \( \hat{k} \) is the unit vector in the direction of \( k \), and two transverse unit vectors are defined by
\[ \epsilon_{(1)} := \frac{1}{\sin \phi} \hat{k} \times \hat{C}, \]
(41)
\[ \epsilon_{(2)} := \frac{1}{\sin \phi} \left( \cos \phi \hat{k} - \hat{C} \right) \propto \hat{k} \times (\hat{k} \times \hat{C}), \]
(42)
where \( \phi \) is the angle between \( C \) and \( k \), and \( \hat{C} := C/C \) with \( C := |C| \). The set \( \{ \hat{k}, \epsilon_{(1)}, \epsilon_{(2)} \} \) forms a right-handed triple. We later use relations,
\[ C \cdot \hat{k} = C \cos \phi, \]
\[ C \cdot \epsilon_{(1)} = 0, \]
\[ C \cdot \epsilon_{(2)} = -C \sin \phi, \]
\[ C \times \hat{k} = -C \sin \phi \epsilon_{(1)}, \]
\[ C \times \epsilon_{(1)} = C \left( \sin \phi \hat{k} + \cos \phi \epsilon_{(2)} \right), \]
\[ C \times \epsilon_{(2)} = -C \cos \phi \epsilon_{(1)}. \]
(43)

Since the equations of motion for the electric fields are closed, let us use them to identify the dispersion relations. We decompose \( e \) with the basis \( \{ \hat{k}, \epsilon_{(1)}, \epsilon_{(2)} \} \) as
\[ e = \chi_0 \hat{k} + \chi_1 \epsilon_{(1)} + \chi_2 \epsilon_{(2)}. \]
(44)
Noting that
\[ C \times e = C \times \left( \chi_0 \hat{k} + \chi_1 \epsilon_{(1)} + \chi_2 \epsilon_{(2)} \right) \]
\[ = -C \sin \phi \chi_0 \epsilon_{(1)} + C \left( \sin \phi \hat{k} + \cos \phi \epsilon_{(2)} \right) \chi_1 \]
\[ - C \cos \phi \epsilon_{(1)} \chi_2 \]
\[ = C \sin \phi \chi_0 \hat{k} - C \sin \phi \chi_0 + \cos \phi \chi_2 \epsilon_{(1)} + C \cos \phi \chi_1 \epsilon_{(2)}, \]
(45)
we can write the equations of motion of the electric field as
\[ (\omega^2 - k^2)e + k^2 \chi \hat{k} \]
\[ + i \omega C \left( \sin \phi \chi \hat{k} - (\sin \phi \chi_0 + \cos \phi \chi_2) \epsilon^{(1)} + \cos \phi \chi_1 \epsilon^{(2)} \right) \]
\[ = (\omega^2 \chi_0 + i \omega C \sin \phi \chi) \hat{k} \]
\[ + \left[ (\omega^2 - k^2) \chi_1 - i \omega C (\sin \phi \chi_0 + \cos \phi \chi_2) \right] \epsilon^{(1)} \]
\[ + \left[ (\omega^2 - k^2) \chi_2 + i \omega C \cos \phi \chi_1 \right] \epsilon^{(2)} \]
\[ = 0. \quad (46) \]

Since the basis vectors are independent, we have
\[
\begin{pmatrix}
\omega^2 & i \omega C \sin \phi & 0 \\
-i \omega C \sin \phi & \omega^2 - k^2 & -i \omega C \cos \phi \\
0 & i \omega C \cos \phi & \omega^2 - k^2
\end{pmatrix}
\begin{pmatrix}
\chi_0 \\
\chi_1 \\
\chi_2
\end{pmatrix}
= 0. \quad (47)
\]

We can identify the dispersion relations of two physical modes as
\[ \omega^2(k) = \frac{1}{2} \left( C^2 + 2k^2 \pm \sqrt{C^4 + 2C^2k^2(1 + \cos 2\phi)} \right). \quad (48) \]

At small \( k \), they behave as
\[ \omega^2(k) = \begin{cases} C^2 + (2 - \sin^2 \phi)k^2 + O(k^4) \\ \sin^2 \phi k^2 + \frac{\cos^2 \phi}{C^2} k^4 + O(k^6) \end{cases}. \quad (49) \]

One mode is gapped and the other one is gapless. The dispersion relation of the gapless mode is quadratic, \( \omega \sim k^2 \), only when \( \sin \phi = 0 \), and is linear, \( \omega \sim k \), for other angles.

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