WHEN DEGREE OF ROUGHNESS IS A NEIGHBORHOOD OVER
LOCALLY SOLID RIESZ SPACES

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Abstract. In this paper we introduce the notion of rough weighted $I$-$\tau$-limit points set and weighted $I$-$\tau$-cluster points set in a locally solid Riesz space which are more generalized version of rough weighted $I$-limit points set and weighted $I$-cluster points set in a $\theta$-metric space respectively. Successively to compare with the following important results of Fridy [Proc. Amer. Math. Soc. 118 (4) (1993), 1187-1192] and Das [Topology Appl. 159 (10-11) (2012), 2621-2626], respectively be stated as

(i): Any number sequence $x = \{x_n\}_{n \in \mathbb{N}}$, the statistical cluster points set of $x$ is closed,

(ii): In a topological space the $I$-cluster points set is closed,

we show that in general, the weighted $I$-$\tau$-cluster points set in a locally solid Riesz space may not be closed. The resulting summability method unfollows some previous results in the direction of research works of Aytar [Numer. Funct. Anal. Optim. 29 (3-4) (2008) 291-303], Dündar [Numer. Funct. Anal. Optim. 37 (4) (2016) 480-491], Ghosal [Math. Slovaca 70 (3) (2020) 667-680] and Savaş, Et [Period. Math. Hungar. 71 (2015) 135-145].

1. Introduction

The idea of convergence of a sequence in a norm linear space $(X, ||.||)$ had been extended to rough convergence first by Phu [20] as follows: Let $r$ be a non-negative real number, a sequence $x = \{x_n\}_{n \in \mathbb{N}}$ in $X$ is said to be rough convergent to $x_*$ w.r.t the roughness of degree $r$, denoted by $x_n \xrightarrow{r} x_*$ provided that

$$\forall \varepsilon > 0 \exists n_\varepsilon \in \mathbb{N} : n \geq n_\varepsilon \Rightarrow ||x_n - x_*|| \leq r + \varepsilon.$$ 

The set $LIM^r x = \{x_* \in X : x_n \xrightarrow{r} x_*\}$ is called the $r$-limit set of the sequence $x = \{x_n\}_{n \in \mathbb{N}}$. Phu studied the set $LIM^r x$ of all such points and showed that this set is bounded, closed and convex.

In the year 2013, the idea of rough $I$-convergence was introduced by Pal et al. [19] as a generalization of rough convergence [20, 21], statistical convergence [8, 28], rough statistical convergence [4] and $I$-convergence [14, 16] which is based on the structure of the ideal $I$ of subsets of the set $\mathbb{N}$ as: Let $r$ be a non-negative real number. A sequence $x = \{x_n\}_{n \in \mathbb{N}}$ in $X$ is said to be rough $I$-convergent to $x_*$, denoted by $x_n \xrightarrow{I, r} x_*$, provided for any $\varepsilon > 0$ the set

$$\{n \in \mathbb{N} : ||x_n - x_*|| \geq r + \varepsilon\} \in I.$$ 

The basic properties of this interesting concept were studied by Pal et al. [19] in an arbitrary norm linear space. We could follow references [3, 7, 9, 15, 24, 26] related to

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the concepts of statistical convergence, rough convergence and others.

One of the most impressive generalizations of the notion of rough $\mathcal{I}$-convergence is the concept rough weighted $\mathcal{I}$-convergence. Motivated from the definitions of rough weighted statistical limit points set and weighted statistical cluster points set [9], recently Ghosal et al. [11] introduced the notion of rough weighted $\mathcal{I}$-limit points set and weighted $\mathcal{I}$-cluster points set on $\theta$-metric space $(X, d_\theta)$ by using the weighted sequence of real numbers $\{t_n\}_{n \in \mathbb{N}}$ (i.e., $t_n > \delta$, for all $n \in \mathbb{N}$ for some positive real number $\delta$) as follows:

**Definition 1.1.** [11] Let $r$ be a non-negative real number and $\{t_n\}_{n \in \mathbb{N}}$ be a weighted sequence. A sequence $x = \{x_n\}_{n \in \mathbb{N}}$ in a $\theta$-metric space $X$ is said to be rough weighted $\mathcal{I}$-convergent to $x_\ast \in X$ w.r.t the roughness of degree $r$ if for every $\varepsilon > 0$,

$$\{n \in \mathbb{N} : t_n d_\theta(x_n, x_\ast) \geq r + \varepsilon\} \in \mathcal{I}.$$ 

In this case we write $x_n \xrightarrow{\mathcal{I}}^r x_\ast$. The set $\mathcal{W}_r^\mathcal{I} - \text{LIM}^r x = \{x_\ast \in X : x_n \xrightarrow{\mathcal{I}}^r x_\ast\}$ is called the rough weighted $\mathcal{I}$-limit set of the sequence $x = \{x_n\}_{n \in \mathbb{N}}$ with degree of roughness $r$. The sequence $x = \{x_n\}_{n \in \mathbb{N}}$ is said to be rough weighted $\mathcal{I}$-convergent provided that $\mathcal{W}_r^\mathcal{I} - \text{LIM}^r x \neq \emptyset$. Visit [13] [17] for more references related this topic.

**Definition 1.2.** [11] Let $\{t_n\}_{n \in \mathbb{N}}$ be a weighted sequence and $c^\ast \in X$ is called a weighted $\mathcal{I}$-cluster point of a sequence $x = \{x_n\}_{n \in \mathbb{N}}$ in a $\theta$-metric space $X$ if for every $\varepsilon > 0$,

$$\{n \in \mathbb{N} : t_n d_\theta(x_n, c^\ast) < \varepsilon\} \notin \mathcal{I}.$$ 

We denote the set of all weighted $\mathcal{I}$-cluster points of the sequence $x = \{x_n\}_{n \in \mathbb{N}}$ by $\mathcal{W}_r^\mathcal{I}(x_\ast)$.

The notion of Riesz space was first introduced by Riesz [22] in 1928 and since then it has found several applications in measure theory, operator theory, optimization. It is well known that a topology on a vector space that makes the operations of addition and scalar multiplication continuous is called a linear topology and a vector space endowed with a linear topology is called a topological vector space. A Riesz space is an ordered vector space which is also a lattice, endowed with a linear topology. Further if it has a base consisting of solid sets at zero then it is known as a locally solid Riesz space. We briefly recall some of the basic notions in the theory of Riesz space and we refer readers to [2] [15] [23] [25] for more details.

**Definition 1.3.** [1] Let $L$ be a real vector space and $\leq$ be a partial order on this space. $L$ is said to be an ordered vector space if it satisfies the following properties:

(i) If $x, y \in L$ and $y \leq x$, then $y + z \leq x + z$ for each $z \in L$.

(ii) If $x, y \in L$ and $y \leq x$, then $\lambda y \leq \lambda x$ for each $\lambda \geq 0$.

In addition, if $L$ is a lattice with respect to the partial ordering, $L$ is said to be a Riesz space (or a vector lattice).

For an element $x$ in a Riesz space $L$ the positive and negative parts of $x$ are defined by $x^+ = x \vee \theta$ and $x^- = (-x) \vee \theta$ respectively. The absolute value of $x$ by $|x| = x \vee (-x)$, where $\theta$ is the element zero of $L$.

A subset $S$ of a Riesz space $L$ is said to be solid if $y \in S$ and $|x| \leq |y|$ imply $x \in S$.

A topology $\tau$ on a real vector space $L$ that makes the addition and the scalar multiplication continuous is said to be a linear topology, that is, the topology $\tau$ makes the functions

$$(x, y) \rightarrow x + y \ (\text{from} \ (L \times L, \tau \times \tau) \rightarrow (L, \tau)),$$

$$(\lambda, x) \rightarrow \lambda x \ (\text{from} \ (\mathbb{R} \times L, \sigma \times \tau) \rightarrow (L, \tau))$$
continuous, where \( \sigma \) is the usual topology on \( \mathbb{R} \). In this case, the pair \((L, \tau)\) is called a topological vector space.

Every linear topology \( \tau \) on a vector space \( L \) has a base \( \mathcal{N} \) for the neighborhoods of \( \theta \) (zero) satisfying the following properties:
\[(a) \text{ Each } V \in \mathcal{N} \text{ is a balanced set, that is, } \lambda x \in V \text{ holds for all } x \in V \text{ and every } \lambda \in \mathbb{R} \text{ with } |\lambda| \leq 1. \]
\[(b) \text{ Each } V \in \mathcal{N} \text{ is an absorbing set, that is, for every } x \in L, \text{ there exists a } \lambda > 0 \text{ such that } \lambda x \in V. \]
\[(c) \text{ For each } V \in \mathcal{N} \text{ there exists some } W \in \mathcal{N} \text{ with } W + W \subseteq V. \]

**Definition 1.4.** [1] A linear topology \( \tau \) on a Riesz space \( L \) is said to be locally solid if \( \tau \) has a base at zero consisting of solid sets. A locally solid Riesz space \((L, \tau)\) is a Riesz space \( L \) equipped with a locally solid topology \( \tau \).

The symbol \( \mathcal{N}_{\text{sol}} \) will stand for a base at zero consisting of solid sets and satisfying the properties (a), (b) and (c) in a locally solid topology.

**Definition 1.5.** [12] A subset \( A \) of a Riesz space \((L, \tau)\) is said to be topologically bounded or \( \tau \)-bounded if for every neighborhood \( U \) of zero there exists some \( \lambda > 0 \) such that \( A \subseteq \lambda U \). If \( A \) is not \( \tau \)-bounded then it is called \( \tau \)-unbounded.

**Definition 1.6.** [12] A subset \( B \) of a Riesz space \((L, \tau)\) is said to be order bounded if it is contained in some order interval.

**Definition 1.7.** [5] A sequence \( \{x_n\} \) in a topological space \((X, \tau)\) is said to be \( I \)-convergent to \( x \in X \) if for any \( U \in \tau \) containing \( x \), \( \{n \in \mathbb{N}: x_n \notin U\} \subseteq I \).

Naturally a prominent question may arise that, does there exists any notion of convergence in a topological vector space which could transform the core factor **degree of roughness** ‘\( \tau \)' to a **neighborhood** ‘\( V \)' of a topological vector space \((X, \tau)\). For answering this question, a different aspect of rough weighted \( I \)-convergence been prominently discussed, following the concepts of Definition 1.1 and 1.2, specifically by replacing ‘\( \theta \)-metric space \( X \)', ‘\( \tau \)' and ‘\( \varepsilon \)' by the ‘locally solid Riesz space \( L \)', ‘\( V \)' (where \( V \) is a fixed \( \tau \)-neighborhood of the zero element of \( L \)) and ‘\( U \)' (where \( U \) is any arbitrary \( \tau \)-neighborhood of the zero element of \( L \)) respectively. We introduce the following definitions:

**Definition 1.8.** Let \( I \) be an admissible ideal of \( \mathbb{N} \) and \( t = \{t_n\}_{n \in \mathbb{N}} \) be a weighted sequence of real numbers. A sequence \( x = \{x_n\} \) in a locally solid Riesz space \((L, \tau)\) is said to be rough weighted \( I \)-convergent to \( x_\ast \in L \) w.r.t the roughness of degree \( V \) (where \( V \) is a \( \tau \)-neighborhood of \( \theta \)) if for every \( \tau \)-neighborhood \( U \) of \( \theta \), denoted by \( x_n \xrightarrow{W I \tau \ast} x_\ast \), the following expression holds,
\[\{n \in \mathbb{N} : t_n(x_n - x_\ast) \notin V + U\} \subseteq I.\]

We shall write \( W I \tau - LIM^V x = \{x_\ast \in X : x_n \xrightarrow{W I \tau \ast} x_\ast\} \) to denote the set of all rough weighted \( I \)-limit points of the sequences \( x = \{x_n\} \) with degree of roughness \( V \).

**Definition 1.9.** An element \( c \in L \) is called weighted \( I \)-cluster point of a sequence \( x = \{x_n\} \) in \( L \) for every \( \tau \)-neighborhood \( U \) of \( \theta \), denoted by \( x_n \xrightarrow{W I \Gamma \ast} c \), the set
\[\{n \in \mathbb{N} : t_n(x_n - c) \notin U\} \notin I.\]

The set of all weighted \( I \)-cluster points of the sequence \( x = \{x_n\}_{n \in \mathbb{N}} \) is denoted by \( W I \Gamma \ast_x \).
Our main objective is to interpret the topological structure of the new convergence and characterize the rough weighted $\mathcal{I}_r$-limit set and weighted $\mathcal{I}_r$-cluster points set in a locally solid Riesz space. In addition, we give some results about the relationship between the sets $W\mathcal{I}_r - LIM^V x$ and $W\mathcal{I}_{\tau}^r$.

2. Main Results

Followed by the Definition 1.8 of rough weighted $\mathcal{I}_r$-convergence over locally solid Riesz spaces we give the necessary condition for $W\mathcal{I}_r - LIM^V x$ to be convex.

**Theorem 2.1.** If $V$ is convex then the set $W\mathcal{I}_r - LIM^V x$ is convex.

**Proof.** Let $U$ be an arbitrary $\tau$-neighborhood of zero. Then there exists a $W \in \mathcal{N}_{sol}$ such that $W + W \subseteq U$. Since $W$ is a balanced set, we get $\lambda W \subseteq W$ and $(1 - \lambda)W \subseteq W$ for all $0 < \lambda < 1$. If $x_s, y_s \in W\mathcal{I}_r - LIM^V x$ then we have $A = \{k \in \mathbb{N} : t_k(x_k - x_s) \in V + W\} \in \mathcal{F}(\mathcal{I})$, $B = \{k \in \mathbb{N} : t_k(x_k - y_s) \in V + W\} \in \mathcal{F}(\mathcal{I})$, where $\mathcal{F}(\mathcal{I}) = \{M \subseteq \mathbb{N} : M \in \mathcal{I}\}$ is a filter associated with the ideal $\mathcal{I}$. Consequently $A \cap B \subseteq \mathcal{F}(\mathcal{I})$. Further let $k \in A \cap B$. Thus

$$t_k[x_k - \{\lambda x_s + (1 - \lambda)y_s\}] = \lambda t_k(x_k - x_s) + (1 - \lambda)t_k(x_k - y_s) \in V + W + W \subseteq V + U.$$ 

This implies $A \cap B \subseteq \{k \in \mathbb{N} : t_k[x_k - \{\lambda x_s + (1 - \lambda)y_s\}] \in V + U\}$. So it follows that $W\mathcal{I}_r - LIM^V x$ is convex. □

The converse of Theorem 2.1 is not true in general. To prove this important fact, we consider an example as follows:

**Example 1.** Let $\mathcal{I}$ be the ideal of subsets of $\mathbb{N}$ of natural density zero. Let us consider the locally solid Riesz Space $(\mathbb{R}^2, || \cdot ||)$ with the max norm $|| \cdot ||$ and coordinate-wise ordering. The family $\mathcal{N}_{sol}$ of all $U(\varepsilon)$ defined as $U(\varepsilon) = \{\alpha \in \mathbb{R}^2 : ||\alpha|| < \varepsilon\}$ where $\varepsilon > 0$, constitutes a base at $\theta = (0,0)$. Let us define the sequence $x = \{x_n\}_{n \in \mathbb{N}}$, the weighted sequence $t = \{t_n\}_{n \in \mathbb{N}}$ and $\tau$-neighborhood of $\theta$, say $V$, in the following manner:

$$x_n = \begin{cases} 
(1,0) & \text{if } n \neq m^2 \text{ for all } m \in \mathbb{N}, \\
(n,n) & \text{otherwise},
\end{cases}$$

$$t_n = n \text{ for all } n \in \mathbb{N}, \text{ and } V = \{(\xi,\eta) \in \mathbb{R}^2 : ||(\xi,\eta)|| \leq 1\} \cup \{(\xi,\eta) \in \mathbb{R}^2 : \xi = \eta\}.$$ 

Therefore $W\mathcal{I}_r - LIM^V x = \{(\xi,\eta) \in \mathbb{R}^2 : \xi - \eta = 1\}$ is a convex set but not $V$. □

Eliminating the condition of convexity from the aforementioned theorem, then the result may not be true.

**Example 2.** Consider the locally solid Riesz Space $(\mathbb{R}^2, || \cdot ||)$, where $|||\xi,\eta||| = |\xi| + |\eta|$ for $(\xi,\eta) \in \mathbb{R}^2$ and coordinate-wise ordering. The family $\mathcal{N}_{sol}$ of all $U(\varepsilon)$ defined as $U(\varepsilon) = \{\alpha \in \mathbb{R}^2 : ||\alpha|| < \varepsilon\}$ where $\varepsilon > 0$, constitutes a base at $\theta = (0,0)$. Consider the similar sequence as in Example 1, weighted sequence $t_n = e$ for all $n \in \mathbb{N}$ and $V = \{(\xi,\eta) \in \mathbb{R}^2 : ||(\xi,\eta)|| \leq \pi\} \cup \{(\xi,\eta) \in \mathbb{R}^2 : \eta = \sin(\xi)\}$. Taking the similar ideal as in example 1, we get $W\mathcal{I}_r - LIM^V x = \{(\xi,\eta) \in \mathbb{R}^2 : \xi = e\eta, |\xi| + |\eta| \leq \pi\} \cup \{(\xi,\eta) \in \mathbb{R}^2 : e\eta = \sin(e\xi - e)\}$, which is not a convex set. We conclude that if $V$ is not a convex set then the set $W\mathcal{I}_r - LIM^V x$ may or may not be convex.

Ghosal et al. [11, Theorem 3.1], had shown that if the weighted sequence is not $\mathcal{I}$-bounded the the set $W\mathcal{I} - LIM^r x$ contains at most one element. While reformulating the above theorem based over locally solid Riesz Space, the object of “$W\mathcal{I} - LIM^r x$
contains at most one element” violates. We exemplify this assertion below i.e., if the weighted sequence is not \( I \)-bounded then the set \( W_{\mathcal{I}} - LIM^V x \) may not be singleton in fact it may be infinite and \( \tau \)-unbounded.

**Example 3.** Consider the ideal \( \mathcal{I} = \{ A \subset \mathbb{N} : \sum_{a \in A} a^{-1} < \infty \} \) and the locally solid Riesz Space \((\mathbb{R}^2, || \cdot ||)\), where \( || \cdot || \) is the Euclidean norm. Let us define the sequence \( x = \{ x_n \}_{n \in \mathbb{N}} \) and the weighted sequence \( t = \{ t_n \}_{n \in \mathbb{N}} \) in the following manner:

\[
x_n = \begin{cases} 
((-1)^n, 0) & \text{if } n \neq m^p \text{ for all } m \in \mathbb{N} \text{ and } p \in \mathbb{N} \setminus \{1\}, \\
(n, 0) & \text{otherwise},
\end{cases}
t_n = n \text{ for all } n \in \mathbb{N}.
\]

It is very obvious that the sequence \( x = \{ x_n \}_{n \in \mathbb{N}} \) is not \( \mathcal{I} \)-convergent to any point of \( \mathbb{R}^2 \). Let \( V = \{ (\xi, \eta) \in \mathbb{R}^2 : -1 < \eta < 1 \} \). Then \( W_{\mathcal{I}} - LIM^V x = \{ (\xi, \eta) \in \mathbb{R}^2 : \eta = 0 \} \). This example shows that the sequence \( \{ x_n \}_{n \in \mathbb{N}} \) is not \( \mathcal{I} \)-convergent to any point but the rough weighted \( \mathcal{I} \)-limit set is \( \{ (\xi, \eta) \in \mathbb{R}^2 : \eta = 0 \} \) which is infinite and \( \tau \)-unbounded. \( \square \)

**Theorem 2.2.** If the weighted sequence \( t = \{ t_n \}_{n \in \mathbb{N}} \) is not \( \mathcal{I} \)-bounded and \( V \) be \( \tau \)-bounded then the rough weighted \( \mathcal{I} \)-limit set \( W_{\mathcal{I}} - LIM^V x \) of a sequence \( x = \{ x_n \}_{n \in \mathbb{N}} \) in locally solid Hausdorff Riesz spaces can have at most one element.

**Proof.** If \( W_{\mathcal{I}} - LIM^V x = \emptyset \), the theorem is obvious. So assuming \( W_{\mathcal{I}} - LIM^V x \neq \emptyset \). If possible let \( x_\ast \) and \( y_\ast \) be two distinct elements in \( W_{\mathcal{I}} - LIM^V x \). Let \( \alpha = x_\ast - y_\ast \).

If \( \alpha \neq \theta \) then there exists \( U \in \mathcal{N}_{sol} \) such that \( \theta \in U \) but \( \alpha \notin U \) (since the Riesz space is \( T_2 \)).

Now \( V \) is \( \tau \)-bounded so is \( B = V + V + V \). So there exists a positive real number \( p \) such that \( B \subset pU \). Let \( A_1 = \{ k \in \mathbb{N} : t_k \geq p \} \). Since \( t = \{ t_n \}_{n \in \mathbb{N}} \) is not \( \mathcal{I} \)-bounded so \( A_1 \notin \mathcal{I} \). Since \( V \) is a \( \tau \)-neighborhood of \( \theta \) so there exists \( W \in \mathcal{N}_{sol} \) such that \( W + W + W \subset V \).

Let

\[
A_2 = \{ k \in \mathbb{N} : t_k(x_\ast - x_\ast) \in V + W \}
\]

and

\[
A_3 = \{ k \in \mathbb{N} : t_k(x_\ast - y_\ast) \in V + W \}.
\]

then \( A_2, A_3 \in \mathcal{F}(\mathcal{I}) \). Let \( A = A_1 \cap A_2 \cap A_3 \) then \( A \neq \emptyset \) as well as \( A \) is an infinite subset of \( \mathbb{N} \).

Therefore \( A \subseteq \{ k \in \mathbb{N} : t_k \alpha \in B \} \) and so \( \{ k \in \mathbb{N} : t_k \alpha \in B \} \neq \emptyset \), i.e., \( t_k \alpha \in B \) for all \( k \in A \).

Since \( \alpha \notin U \), we get \( t_k \alpha \notin t_k U \) for all \( k \in A \).

\[
\{ t_k \alpha : k \in A \} \notin t_1 U \Rightarrow B \subseteq t_l U \text{ for no } l \in A,
\]

which is a contradiction. Hence \( \alpha = \theta \) and our result is established. \( \square \)

Our next example proves that if \( V \) be \( \tau \)-bounded then the set \( W_{\mathcal{I}} - LIM^V x \) may not be an order bounded set.

**Example 4.** Let \( \mathcal{I} \) is the ideal of subsets of \( \mathbb{N} \) of natural density zero and \( L \) be the space of all Lebesgue measurable functions on \( I = [0, 1] \) with the usual point-wise ordering, i.e., for \( x, y \in L \), we define \( x \leq y \) if and only if \( x(t) \leq y(t) \) for every \( t \in I \). Consider the map \( || \cdot || : L \to \mathbb{R} \) defined by \( ||x|| = (\int_I x^2(s) ds)^{1/2} \), where \( x \in L \). Then \( (L, \tau) \) forms a locally solid Riesz space. Put \( V = \{ x \in L : ||x|| \leq 1 \} \). Again we consider the weighted sequence \( t = \{ t_n \}_{n \in \mathbb{N}} \) and the sequence \( x = \{ x_n \}_{n \in \mathbb{N}} \) in the following manner:

\[
t_n = \begin{cases} 
2 + \frac{1}{n} & \text{if } n \neq m^2 \text{ for all } m \in \mathbb{N}, \\
3 & \text{otherwise},
\end{cases}
\]

and

\[
x_n(s) = \begin{cases} 
\frac{1}{n} & \text{if } n \neq m^2 \text{ for all } m \in \mathbb{N}, \\
1 & \text{otherwise},
\end{cases}
\]


for all \( s \in I, n \in \mathbb{N} \). Thus it follows that \( W \mathcal{I}_\tau - \text{LIM}^V x = \frac{1}{2}V \) is \( \tau \)-bounded but not order bounded. \( \square \)

**Theorem 2.3.** Let \((L, \tau)\) be a Hausdorff locally solid Riesz space and \( V \in \mathcal{N}_{sol} \) such that \( V + V + U \in \mathcal{N}_{sol} \) for all \( U \in \mathcal{N}_{sol} \). For a sequence \( x = \{x_n\}_{n \in \mathbb{N}} \) in \( L \), we have

\[
x_* - y_* \in \frac{V + V}{\inf t_n} \text{ for all } x_*, y_* \in W \mathcal{I}_\tau - \text{LIM}^V x.
\]

In addition, if \( V \) is \( \tau \)-bounded then \( W \mathcal{I}_\tau - \text{LIM}^V x \) is a \( \tau \)-bounded set.

**Proof.** Let \( U \) be an arbitrary \( \tau \)-neighborhood of zero. Then there exist \( U_0, U_1 \in \mathcal{N}_{sol} \) such that \( U_0 \subseteq U \) and \( U_1 + U_1 \subseteq U_0 \). Thus \( K_1, K_2 \in \mathcal{F}(\mathcal{I}) \), where \( K_1 = \{k \in \mathbb{N} : t_k(x - x_*) \in V + U_1\} \) and \( K_2 = \{k \in \mathbb{N} : t_k(x - y_*) \in V + U_1\} \). Consequently \( K_1 \cap K_2 \in \mathcal{F}(\mathcal{I}) \).

Moreover if \( k \in K_1 \cap K_2 \) then \( t_k(x_* - y_*) = t_k(x_* - x_k) + t_k(x_k - y_*) \in V + U_1 + V + U_1 \subseteq V + V + U_0 \). Thus we get \( t_k(x_* - y_*) \in V + V + U_0 \) for all \( k \in K_1 \cap K_2 \). Also for \( k \in K_1 \cap K_2 \),

\[
\inf t_n |(x_* - y_*)| \leq t_k |(x_* - y_*)| \in V + V + U_0.
\]

This implies \( (x_* - y_*) \in \frac{V + V}{\inf t_n} + \frac{U_0}{\inf t_n} \subseteq \frac{V + V}{\inf t_n} + \frac{U}{\inf t_n} \). Hence \( (x_* - y_*) \in \frac{V + V}{\inf t_n} \) since \((L, \tau)\) is Hausdorff and the intersection of all \( \tau \)-neighborhoods \( U \) of zero is the singleton \( \{0\} \).

Now we proceed to the second part of the theorem. Let \( W \mathcal{I}_\tau - \text{LIM}^V x \neq \emptyset \) and \( x_* \in W \mathcal{I}_\tau - \text{LIM}^V x \). From the above argument we get \( W \mathcal{I}_\tau - \text{LIM}^V x \subseteq \frac{V + V}{\inf t_n} + \{x_*\} \). \( \square \)

**Remark 1.** Naturally a question may arise that, does there exist any neighborhood \( V \in \mathcal{N}_{sol} \) in a Hausdorff locally solid Riesz space such that \( V + V + U \in \mathcal{N}_{sol} \) for all \( U \in \mathcal{N}_{sol} \)?

For answering the above question we consider the Hausdorff locally solid Riesz space same as in Example 1 and define the neighborhood \( V \in \mathcal{N}_{sol} \) in the following manner:

\( V = U(1) \in \mathcal{N}_{sol} \) where \( U(\varepsilon) = \{\alpha \in \mathbb{R}^2 : ||\alpha|| < \varepsilon\} \) for all \( \varepsilon > 0 \) and it is clear that \( V + V + U(\varepsilon) \in \mathcal{N}_{sol} \) for all \( U(\varepsilon) \in \mathcal{N}_{sol} \).

Further we discuss the closeness of rough weighted \( \mathcal{I}_\tau \)-limit set over locally solid Riesz spaces. Our next example assures that rough weighted \( \mathcal{I}_\tau \)-limit set is not closed.

**Example 5.** Consider the locally solid Riesz Space \( \mathbb{R}^2 \) and the ideal \( \mathcal{I} \) as in Example 3. Let us define the sequence \( x = \{x_n\}_{n \in \mathbb{N}} \) and the weighted sequence \( t = \{t_n\}_{n \in \mathbb{N}} \) in the following manner;

\[
x_n = \begin{cases} (1, -\frac{2}{n}) & \text{if } n \neq m^2 \text{ for all } m \in \mathbb{N}, \\ (n, -n) & \text{otherwise} \end{cases}
\]

and \( t_n = n \) for all \( n \in \mathbb{N} \).

Let \( V = \{||\xi, \eta|| \in \mathbb{R}^2 : -1 < \xi < 1, \eta \geq -1\} \). Therefore \( W \mathcal{I}_\tau - \text{LIM}^V x = \{||\xi, \eta|| \in \mathbb{R}^2 : \xi = 1, \eta < 0\} \) is not closed. In fact it is not an open set. \( \square \)

A careful inspection of the previous examples exhibit how the set \( W \mathcal{I}_\tau - \text{LIM}^V x \) could be generalized. We do so in the next theorem.
Theorem 2.4. Let \((L, \tau)\) be a Hausdorff locally solid Riesz space. Hence we assert

\[
W\mathcal{I}_\tau - \text{LIM}^V x \begin{cases} 
\text{is closed; if weighted sequence be } \mathcal{I}\text{-bounded,} \\
\text{is closed; if weighted sequence is not } \mathcal{I}\text{-bounded and } V \text{ be } \tau\text{-bounded,} \\
\text{has no definite conclusion; if weighted sequence is not } \mathcal{I}\text{-bounded and } V \text{ be } \tau\text{-unbounded.} 
\end{cases}
\]

Proof. Case 1: Let the weighted sequence \(t = \{t_n\}_{n \in \mathbb{N}}\) be \(\mathcal{I}\)-bounded. In this case there exists a positive real number \(M\) such that \(K_1 = \{k \in \mathbb{N} : t_k < M\} \subseteq \mathcal{F}(\mathcal{I})\). Assume \(p_* \in W\mathcal{I}_\tau - \text{LIM}^V x\). Then there exists a sequence \(\{p_n\}_{n \in \mathbb{N}}\) in \(W\mathcal{I}_\tau - \text{LIM}^V x\) such that \(p_n \to p_*\) as \(n \to \infty\) (by using 1st countable property of \(L\)). Naturally \(M p_n \to M p_*\) as \(n \to \infty\).

Again we consider \(U\) is any neighborhood of \(\theta\) and corresponding to \(U\) there exists \(U_0 \in \mathcal{N}_{\text{col}}\) such that \(U_0 + U_0 \subset U\). Hence \(M(p_n - p_*) \in U_0\) for all \(n \geq k_0\), where \(k_0\) is a positive integer depends on \(U_0\).

On the other hand if \(k \in K_1\) then \(t_k < M\) implies \(t_k|p_{k_0} - p_*| \leq M|p_{k_0} - p_*|\) and finally \(|t_k(p_{k_0} - p_*)| \leq |M(p_{k_0} - p_*)|\).

Since \(M(p_{k_0} - p_*) \in U_0\) and \(U_0\) is a solid set so \(t_k(p_{k_0} - p_*) \in U_0\) for all \(k \in K_1\).

We define the set \(K_2 = \{k \in \mathbb{N} : t_k(x_k - p_{k_0}) \in V + U_0\}\). Then \(p_{k_0} \in W\mathcal{I}_\tau - \text{LIM}^V x\) follows \(K_2 \subseteq \mathcal{F}(\mathcal{I})\). For \(k \in K_1 \cap K_2\), \(t_k(x_k - p_*) = t_k(x_k - p_{k_0}) + t_k(p_{k_0} - p_*) \in V + U_0 + U_0 \subset V + U\).

This implies \(K_1 \cap K_2 \subseteq \{k \in \mathbb{N} : t_k(x_k - p_*) \in V + U\}\). Clearly \(\{k \in \mathbb{N} : t_k(x_k - p_*) \in V + U\} \subseteq \mathcal{F}(\mathcal{I})\). So we conclude case 1.

Case 2: If the weighted sequence \(t = \{t_n\}_{n \in \mathbb{N}}\) is not \(\mathcal{I}\)-bounded and \(V\) is \(\tau\)-bounded then from Theorem 2.2, the set \(W\mathcal{I}_\tau - \text{LIM}^V x\) becomes either singleton or empty. Hence it is closed.

Case 3: From Examples 1, 2 and 5 it is clear that the the set \(W\mathcal{I}_\tau - \text{LIM}^V x\) neither open nor closed. \(\square\)

We initiate this section with the definitions of weighted \(\tau\)-boundedness and weighted \(\mathcal{I}\tau\)-boundedness over a locally solid Riesz space \(L\).

Definition 2.1. Let \((L, \tau)\) be a locally solid Riesz space and \(t = \{t_n\}_{n \in \mathbb{N}}\) is a weighted sequence. A sequence \(x = \{x_n\}_{n \in \mathbb{N}}\) in \(L\) is said to be weighted \(\tau\)-bounded if for every \(\tau\)-neighborhood \(U\) of \(\theta\) there exists some \(\lambda > 0\) such that \(\lambda t_k x_k \notin U\) at most for finitely many \(k\).

Definition 2.2. Let \((L, \tau)\) be a locally solid Riesz space and \(t = \{t_n\}_{n \in \mathbb{N}}\) is a weighted sequence. A sequence \(x = \{x_n\}_{n \in \mathbb{N}}\) in \(L\) is said to be weighted \(\mathcal{I}\tau\)-bounded if for every \(\tau\)-neighborhood \(U\) of \(\theta\) there exists some \(\lambda > 0\) such that \(\{k \in \mathbb{N} : \lambda t_k x_k \notin U\} \subseteq \mathcal{I}\).

Theorem 2.5. If a sequence \(x = \{x_n\}_{n \in \mathbb{N}}\) be weighted \(\mathcal{I}\tau\)-bounded then for every \(\tau\)-neighborhood \(V\) of \(\theta\) there exists a positive real number \(\mu\) such that \(W\mathcal{I}_\tau - \text{LIM}^\mu V x \neq \emptyset\).

Proof. Since \(x = \{x_n\}_{n \in \mathbb{N}}\) is weighted \(\mathcal{I}\tau\)-bounded then for every \(\tau\)-neighborhood \(V\) of \(\theta\) there exists some \(\lambda > 0\) such that

\[
\{k \in \mathbb{N} : \lambda t_k x_k \in V\} = \{k \in \mathbb{N} : \lambda t_k(x_k - \theta) \in V\} \subseteq \mathcal{F}(\mathcal{I}).
\]

Therefore \(\{k \in \mathbb{N} : t_k(x_k - \theta) \in \frac{1}{\lambda}V + U\} \subseteq \mathcal{F}(\mathcal{I})\), where \(U\) is an arbitrary \(\tau\)-neighborhood of \(\theta\). Setting \(\mu = \frac{1}{\lambda}\). Hence the set \(W\mathcal{I}_\tau - \text{LIM}^\mu V x\) contains the null element of \(L\). \(\square\)
But the converse is not true. We choose an example to emphasis our assertion.

**Example 6.** Consider the ideal $\mathcal{I}$ and the locally solid Riesz Space $\mathbb{R}^2$ as defined in Example 3. Let us define the sequence $x = \{x_n\}_{n \in \mathbb{N}}$ and the weighted sequence $t = \{t_n\}_{n \in \mathbb{N}}$ in the respective order:

$$
x_n = \begin{cases}
(2 + \frac{1}{n}, 1 + \frac{3}{n}) & \text{if } n \neq m^2 \text{ for all } m \in \mathbb{N}, \\
(5, -5) & \text{otherwise}
\end{cases}
$$

and $t_n = \sqrt{n}$ for all $n \in \mathbb{N}$.

If $V$ be any arbitrary $\tau$-neighborhood of $\theta$ and $\mu$ be any positive real number then obviously $(2, 1) \in W_{\mathcal{I}} - \text{LIM}^\mu V x$ and so $W_{\mathcal{I}} - \text{LIM}^\mu V x \neq \varnothing$. For any positive real number $\lambda$ we get

$$
\lambda t_n x_n = \begin{cases}
(2\lambda \sqrt{n} + \frac{1}{n}, \lambda \sqrt{n} + \frac{3}{\sqrt{n}}) & \text{if } n \neq m^2 \text{ for all } m \in \mathbb{N}, \\
(5\lambda \sqrt{n}, -5\lambda \sqrt{n}) & \text{otherwise}.
\end{cases}
$$

In this case $\{k \in \mathbb{N} : \lambda t_k x_k \notin V\} \notin \mathcal{I}$ Therefore the sequence $x = \{x_n\}_{n \in \mathbb{N}}$ is not weighted $\mathcal{I}_\tau$-bounded although $W_{\mathcal{I}} - \text{LIM}^\mu V x \neq \varnothing$. □

If we consider the sequence $x_n = (\frac{1}{n}, 0)$ for all $n \in \mathbb{N}$ in $\mathbb{R}^2$ as taken in Example 3, $t_n = n^2$ for all $n \in \mathbb{N}$ and $\mathcal{I}$ an arbitrary ideal. Hence the sequence $x = \{x_n\}_{n \in \mathbb{N}}$ is $\mathcal{I}_\tau$-bounded as well as $\tau$-bounded, but $W_{\mathcal{I}} \Gamma_x^\tau = \varnothing$.

As an immediate consequence the following example shows that the set $W_{\mathcal{I}} \Gamma_x^\tau$ is empty even if the sequence $x = \{x_n\}_{n \in \mathbb{N}}$ is weighted $\mathcal{I}_\tau$-bounded.

**Example 7.** Consider the infinite dimensional normed space $l^2$ space with the norm

$$
||\alpha|| = \left(\sum_{j=1}^{\infty} |\xi_j|^2\right)^{\frac{1}{2}} \text{ where } \alpha = (\xi_1, \xi_2, \ldots) \in l^2
$$

with coordinatewise ordering. So $l^2$ is a locally solid Riesz Space. The family $\mathcal{N}_{sol}$ of all $U(\varepsilon)$ defined as $U(\varepsilon) = \{\alpha \in l^2 : ||\alpha|| < \varepsilon\}$ where $\varepsilon > 0$, constitutes a base at $\theta = (0, 0, \ldots)$. We consider the ideal $\mathcal{I} = \{K \subset \mathbb{N} : \lim_{n \rightarrow \infty} \frac{|K \cap \{1, 2, 3, \ldots, n\}|}{n} = 0\}$, the sequence $\{x_n\}_{n \in \mathbb{N}}$ in $l^2$ such that $x_n = e_n$, where $e_n$ has $n^{th}$ term 1 and other terms are 0 and $t_n = 2$ for all $n \in \mathbb{N}$. Therefore the sequence $x = \{x_n\}_{n \in \mathbb{N}}$ is weighted $\mathcal{I}_\tau$-bounded as well as $\mathcal{I}_\tau$-bounded but $W_{\mathcal{I}} \Gamma_x^\tau = \varnothing$. □

In the above example, if we reassume $t_n = n$ for all $n \in \mathbb{N}$ keeping the space $l^2$ and the sequence $x = \{x_n\}_{n \in \mathbb{N}}$ unaltered, the sequence $x = \{x_n\}_{n \in \mathbb{N}}$ is $\mathcal{I}_\tau$-bounded as well as $\tau$-bounded but $W_{\mathcal{I}} \Gamma_x^\tau = \varnothing$.

On the other hand, another important question may arise that does the set $W_{\mathcal{I}} \Gamma_x^\tau$ is compact if the space is infinite dimensional? Answer is no. We sketch an important example below to answer this question.

**Example 8.** Let $\mathbb{N} = \bigcup_{j=1}^{\infty} \Delta_j$ where $\Delta_j = \{2^{j-1}(2s - 1) : s \in \mathbb{N}\}$ and $\mathcal{I} = \{A \subset \mathbb{N} : A \cap \Delta_j \neq \varnothing \text{ for finitely many } j\}$. Then $\mathcal{I}$ forms an admissible ideal. Consider another decomposition of $\mathbb{N}$, i.e., $D_r = \{p_r^s : s \in \mathbb{N}\}$, for all $r \in \mathbb{N} \setminus \{1\}, \{p_2 < p_3 < p_4, \ldots\}$ is a sequence of distinct primes and $D_1 = \mathbb{N} \setminus \bigcup_{r=2}^{\infty} D_r$. Setting $t_n = n$, if $n \in \mathbb{N}$ and the sequence $x = \{x_n\}_{n \in \mathbb{N}}$ such that $x_n = e_r$, for all $n \in D_r$ (in the locally solid Riesz
space $l^2$ as above). For each $0 < \varepsilon < 1$ and $r \in \mathbb{N}$, \{k \in \mathbb{N} : t_k(\ell_k - e_r) \in U(\varepsilon)\} \notin I.
This shows that $e_r \in W_2\Gamma_x^r$ for all $r \in \mathbb{N}$. Let $A = \{e_1, e_2, e_3, \ldots\}$. Then, $A(\subset W_2\Gamma_x^r)$ is closed but not compact. So the set $W_2\Gamma_x^r$ is not compact. □

In [10] Fridy shown that for any number sequence $x = \{x_n\}_{n \in \mathbb{N}}$, the statistical cluster points set of $x$ is closed. Also in [5] Das had shown that in a topological space the $I$-cluster points set is closed. But the following example shows that in general, the weighted $I_{\tau}$-cluster points set in a locally solid Riesz space may not be closed.

Example 9. Consider two decompositions of $\mathbb{N}$, as in previous Example 8 and the locally solid Riesz Space $\mathbb{R}^2$ same as Example 3. Let us define the sequence $x = \{x_n\}_{n \in \mathbb{N}}$ and the weighted sequence $t = \{t_n\}_{n \in \mathbb{N}}$ in the following manner, $x_k = \left(\frac{1}{k^2} + \frac{1}{k^2}, 0\right)$ for all $k \in D_j$ (where $j = 1, 2, 3, \ldots$) and $t_k = k$ for all $k \in \mathbb{N}$.

Let $U$ be a $\tau$-neighborhood of $\theta$, so there exists some $U(\varepsilon) \in \mathcal{N}_{\text{sol}}$, $\varepsilon > 0$ such that $U(\varepsilon) \subset U$. Then for each $j \in \mathbb{N}$, we get \{k \in \mathbb{N} : t_k(x_k - (\frac{1}{j}, 0)) \in U(\varepsilon)\} \notin I$. This shows that $(\frac{1}{j}, 0) \in W_2\Gamma_x^r$ for all $j \in \mathbb{N}$. Next we assume $k \in \mathbb{N}$ then there exists an integer $j \in \mathbb{N}$ such that $k \in D_j$ for some $j \in \mathbb{N}$. If $k \in D_j$ for some $j \in \mathbb{N} \setminus \{1\}$, then $k$ is of the form $k = p_j^s$ where $s \in \mathbb{N}$ then

$$t_k(x_k - \theta) = p_j^s \left(\frac{1}{j} + \frac{1}{p_j^s}, 0\right) = \left(\frac{p_j^s}{j} + \frac{1}{p_j^s}, 0\right) \geq (1, 0).$$

Then \{k \in \mathbb{N} : t_k(x_k - \theta) \in U(\alpha)\} \in I$, where $0 < \alpha < 1$. This implies $\theta \notin W_2\Gamma_x^r$. On the other hand if $k \in D_1$, then \{k \in \mathbb{N} : t_k(x_k - \theta) \in U(1)\} \in I$. Therefore the set $W_2\Gamma_x^r$ is not closed.

Theorem 2.6. For a sequence $x = \{x_n\}_{n \in \mathbb{N}}$, the weighted $I_{\tau}$-cluster points set $W_2\Gamma_x^r$ is closed if the weighted sequence $\{t_n\}_{n \in \mathbb{N}}$ is $I$-bounded.

Proof. As the weighted sequence $\{t_n\}_{n \in \mathbb{N}}$ is $I$-bounded, there exists a non negative real number $M$ such that the set $A = \{n \in \mathbb{N} : t_n < M\} \in \mathcal{F}(I)$. Let $U$ is any neighborhood of $\theta$ and so there exists a $U_0 \in \mathcal{N}_{\text{sol}}$ such that $U_0 + U_0 \subset U$. Consider a sequence $\{p_n\}_{n \in \mathbb{N}} \in W_2\Gamma_x^r$ such that $p_n \to p$. Hence $M(p_n - p) \in U_0$ for all $n \geq k_0$, where $k_0$ is a positive integer depends on $U_0$.

Also $p_{k_0} \in W_2\Gamma_x^r$ implies $B = \{n \in \mathbb{N} : t_n(x_n - p_{k_0}) \in U_0\} \notin I$. Therefore $A \cap B \notin I$ otherwise $A^c \cup (A \cap B) \in I$ (since $A \in \mathcal{F}(I)$) we end up with a contradiction that $B \in I$. For $k \in A \cap B$, we have $t_k(x_k - p) = t_k(x_k - p_{k_0}) + t_k(p_{k_0} - p) \in U_0 + U_0 \subset U$.
Thus $A \cap B \subseteq \{n \in \mathbb{N} : t_n(x_n - p) \in U\}$ demonstrates that $p \in W_2\Gamma_x^r$ and the closeness of $W_2\Gamma_x^r$ is established. □

Theorem 2.7. Let $(L, \tau)$ be a locally solid Riesz space, $x_n \xrightarrow{W_2\Gamma_x^r} c$ and $x_n \xrightarrow{W_2\Gamma_x^r} d$.

Then 

(i) $|x_n| \xrightarrow{W_2\Gamma_x^r} |c|$, (ii) $x_n^+ \xrightarrow{W_2\Gamma_x^r} c^+$, (iii) $x_n^- \xrightarrow{W_2\Gamma_x^r} c^-$,

(iv) $x_n^\tau \xrightarrow{W_2\Gamma_x^r} c \cap d$, (v) $x_n \xrightarrow{W_2\Gamma_x^r} c \cap d$.

Proof. Proof of (i), (ii) and (iii) are similar to proof of Theorem 4.2 [1], so omitted.

For the proof of (iv) and (v) we consider the locally solid Riesz Space $\mathbb{R}^2$ as in Example 3. Let us define the sequence $x = \{x_n\}_{n \in \mathbb{N}}$ and the weighted sequence $t = \{t_n\}_{n \in \mathbb{N}}$ in the manner;

$$x_n = \begin{cases} 
(1, -2) & \text{if } n \in \{2^p : p \in \mathbb{N}\}, \\
(-1, 2) & \text{otherwise} 
\end{cases} \quad \text{and } t_n = n \text{ for all } n \in \mathbb{N} \text{ respectively}.$$
Consider the ideal $\mathcal{I}$ as in Example 8. Then the weighted $\mathcal{I}_x$-cluster points set of the sequence $x$, $W_{\mathcal{I}_x} \Gamma_x = \{(1, -2), (-1, 2)\}$. So $x_n \xrightarrow{W_{\mathcal{I}_x}} (1, -2)$ and $x_n \xrightarrow{W_{\mathcal{I}_x}} (-1, 2)$ but $(1, 2), (-1, -2) \notin W_{\mathcal{I}_x} \Gamma_x$. Hence the results.

From the above Theorem 2.6 it is clear that the set $W_{\mathcal{I}_x} \Gamma_x$ may not be a convex set.

We finally draw a significant relationship between the sets $W_{\mathcal{I}_x} - LIM^V_x$ and $W_{\mathcal{I}_x} \Gamma_x$.

**Theorem 2.8.** Let $(L, \tau)$ be a Hausdorff locally solid Riesz space and $V \in \mathcal{N}_{sol}$ such that $V + U \in \mathcal{N}_{sol}$ for all $U \in \mathcal{N}_{sol}$. If $x = \{x_n\}_{n \in \mathbb{N}}$ be a sequence in $L$, then

$$x_* - c \in \frac{V}{\inf_{n \in \mathbb{N}} t_n} \text{ where } x_* \in W_{\mathcal{I}_x} - LIM^V_x \text{ and } c \in W_{\mathcal{I}_x} \Gamma_x.$$  

Moreover if $V$ is $\tau$-bounded, $V + V + U \in \mathcal{N}_{sol}$ for all $U \in \mathcal{N}_{sol}$ and $W_{\mathcal{I}_x} - LIM^V_x$ is non-empty then $W_{\mathcal{I}_x} \Gamma_x$ is $\tau$-bounded.

**Proof.** Let us consider $U$ be an arbitrary $\tau$-neighborhood of zero. Then there exist $U_0, U_1 \in \mathcal{N}_{sol}$ such that $U_0 \subseteq U$ and $U_1 + U_1 \subseteq U_0$. We know that $K_1 \in \mathcal{F}(\mathcal{I})$ and $K_2 \notin \mathcal{I}$, where $K_1 = \{k \in \mathbb{N} : t_k(x_k - x) \in V + U_1\}$ and $K_2 = \{k \in \mathbb{N} : t_k(x_k - c) \in U_1\}$. So it follows that $K_1 \cap K_2$ is nonempty and an infinite set. Further $k \in K_1 \cap K_2$ then $t_k(x_* - c) = t_k(x_* - x_k) + t_k(x_k - c) \in V + U_1 + U_1 \subseteq V + U_0$. Thus we get $t_k(x_* - c) \in V + U_0$ for all $k \in K_1 \cap K_2$. For $k \in K_1 \cap K_2$,

$$\inf_{n \in \mathbb{N}} |(x_* - c)| \leq | \inf_{k \in K_1 \cap K_2} t_k(x_* - c)|. $$

This implies $(x_* - c) \subseteq \frac{V}{\inf_{n \in \mathbb{N}} t_n} + \frac{U_0}{\inf_{n \in \mathbb{N}} t_n} \subseteq \frac{V}{\inf_{n \in \mathbb{N}} t_n} + \frac{U}{\inf_{n \in \mathbb{N}} t_n}$. Since $(L, \tau)$ is Hausdorff and the intersection of all $\tau$-neighborhoods $U$ of zero is the singleton set $\{\theta\}$. This shows that $(x_* - c) \subseteq \frac{V}{\inf_{n \in \mathbb{N}} t_n}$.

To start the second part of the theorem, let $x_* \in W_{\mathcal{I}_x} - LIM^V_x$. Applying the above argument as well as Theorem 2.3, we get $W_{\mathcal{I}_x} \Gamma_x \subseteq \frac{V}{\inf_{n \in \mathbb{N}} t_n} + W_{\mathcal{I}_x} - LIM^V_x$ is $\tau$-bounded. Hence the results conclude.

**Remark 2.** In retrospect to Remark 1, if we consider $V = U(1) \in \mathcal{N}_{sol}$ where $U(\varepsilon) = \{\alpha \in \mathbb{R}^2 : ||\alpha|| < \varepsilon\}$ for all $\varepsilon > 0$ prominently $V + U(\varepsilon), V + V + U(\varepsilon) \in \mathcal{N}_{sol}$ for all $U(\varepsilon) \in \mathcal{N}_{sol}$. So the existence of such neighbourhood $V$ is guaranteed.

**Example 10.** In view of the above Theorem 2.8 one would naturally like to seek an example for which $W_{\mathcal{I}_x} \Gamma_x$ will be $\tau$-unbounded, i.e., in general $W_{\mathcal{I}_x} \Gamma_x$ is not $\tau$-bounded. If we replace the sequence $x = \{x_n\}_{n \in \mathbb{N}}$ by the sequence $z = \{z_n\}_{n \in \mathbb{N}}$ where $z_n(t) = j$ for all $n \in \Delta_j$ (where $j = 1, 2, 3, ...$), for all $t \in I$ and $t_n = n$ for all $n \in \mathbb{N}$ as in Example 4 then we get $\{z_n : n \in \mathbb{N}\} \subset W_{\mathcal{I}_x} \Gamma_x$ where $\mathcal{I}$ is the ideal as in Example 8.

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