Instantons and Donaldson–Thomas Invariants

Michele Cirafici\(^{(a)}\), Annamaria Sinkovics\(^{(b)}\) and Richard J. Szabo\(^{(c)}\)

\(^{(a)}\) Institute for Theoretical Physics and Spinoza Institute
Utrecht University, 3508 TD Utrecht, The Netherlands
Email: M.Cirafici@uu.nl

\(^{(b)}\) Department of Applied Mathematics and Theoretical Physics
Centre for Mathematical Sciences, University of Cambridge
Wilberforce Road, Cambridge CB3 0WA, UK
Email: A.Sinkovics@damtp.cam.ac.uk

\(^{(c)}\) Department of Mathematics, Heriot–Watt University and
Maxwell Institute for Mathematical Sciences
Colin Maclaurin Building, Riccarton, Edinburgh EH14 4AS, UK
Email: R.J.Szabo@ma.hw.ac.uk

Abstract

We review some recent progress in understanding the relation between a six dimensional topological Yang–Mills theory and the enumerative geometry of Calabi–Yau threefolds. The gauge theory localizes on generalized instanton solutions and is conjecturally equivalent to Donaldson–Thomas theory. We evaluate the partition function of the \(U(N)\) theory in its Coulomb branch on flat space by employing equivariant localization techniques on its noncommutative deformation. Geometrically this corresponds to a higher dimensional generalization of the ADHM formalism. This formalism can be extended to a generic toric Calabi–Yau.

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1 Introduction

Topological field and string theories have been the focus of extensive investigation in the last two decades. These models are more tractable than their physical counterparts but still capture some interesting physical quantities, in particular those related to the vacuum structure of the full quantum theory. Due to the topological nature of the model these quantities can be often computed exactly. The underlying reason is their deep relation to geometric and topological invariants of the physical space where the original model is defined.

In this note we will focus on the Donaldson–Thomas (DT) invariants \[1\]. From the viewpoint of the topological string they can be defined as follows. One starts with a Calabi–Yau manifold \(X\) on which the topological A–model is defined. Then the DT invariant corresponds to the number of bound states formed by a single D6 brane wrapping the full Calabi–Yau manifold with a D2 brane wrapping a 2–cycle \(C \subset X\) in homology class \(\beta\) and \(m\) D0 branes. This configuration is encoded in a mathematical object called an ideal sheaf and the set of all possible configurations is described by the moduli space of ideal sheaves \(I_m(X, \beta)\). This space is also known as the Hilbert scheme of points and curves of the threefold \(\text{Hilb}^m(X, \beta)\). Then the DT invariant \(D^m_\beta(X)\) is defined as the “volume” of this moduli space.

If the Calabi–Yau is toric all the geometric information can be essentially encoded in a combinatorial problem and the topological string has a reformulation in terms of the classical statistical mechanics of a melting crystal \[2\]. In this more physical setting the DT invariants parametrize the atomic configurations of the melting crystal. This leads to a very non–trivial conjecture that the geometrical information captured by the DT invariants is equivalent to Gromov–Witten theory, since they are two different expansions of the same topological string amplitude. So far this conjecture has been proven in a number of cases \[3\].

A detailed understanding of DT theory on Calabi–Yaus could sharpen our knowledge about the geometrical meaning of the topological string and thus about the vacuum structure of the full string theory. In this note we will report about some progress towards this ambitious goal \[4\]. Namely we will only consider local toric threefolds on which the DT problem can be conjecturally rephrased as a topological gauge theory \[5\]. We will put this conjecture on firmer grounds and by employing the techniques of equivariant localization show how to set the ground for explicit computations. We will apply our formalism to higher rank DT invariants on the Coulomb branch of the gauge theory.

2 The Topological Gauge Theory and Equivariant Localization

Let us consider a local toric threefold \(X\). In this case DT theory can be (conjecturally) described by a six dimensional abelian topological gauge theory living on the worldvolume of the D6 brane wrapping \(X\) \[5\]. This gauge theory is the topologically twisted version of maximally supersymmetric Yang–Mills in six dimensions \[6\]–\[9\]. Its bosonic matter content consists of a gauge field \(A_\mu\), a complex Higgs field \(\Phi\) and a \((3, 0)\) form \(\rho^{3,0}\) along with their complex conjugates. Essentially its action has the form of the topological density

\[
\frac{1}{2} \int_X \text{Tr} \left( F_A \wedge F_A \wedge k_0 + \frac{2}{3} F_A \wedge F_A \wedge F_A \right),
\]

supplemented by a gauge fixing term. Here \(k_0\) is the background Kähler two-form of \(X\) and \(\vartheta\) is the six-dimensional theta-angle which will be identified with the topological string coupling \(g_s\). This gauge theory has a BRST symmetry and hence localizes onto the moduli space of solutions of the fixed point equations

\[
F_A^{2,0} = \overline{J}_A^I \rho ,
\]
\( F_A^{1,1} \wedge k_0 \wedge k_0 + [\rho, \overline{\rho}] = \lambda k_0 \wedge k_0 \wedge k_0 \),
\( d_A \Phi = 0 \).

The solutions of these equations minimize the gauge theory action and we will therefore call them generalized instantons or just instantons. On a Calabi–Yau manifold we can set the field \( \rho \) to zero. Then the first two equations reduce to the Donaldson–Uhlenbeck–Yau (DUY) equations which are conditions of stability for holomorphic bundles over \( X \) with finite characteristic classes.

The introduction of this auxiliary gauge theory essentially reformulates DT theory as a (generalized) instanton counting problem. The gauge theory localizes onto the moduli space \( M(X) \) of holomorphic bundles (or coherent sheaves) on \( X \) and the instanton multiplicities in the instanton expansion of the path integral represent the DT invariants. Note that in the gauge theory language it is immediate to generalize DT theory to a non–abelian \( U(N) \) setting, with an arbitrary number of D6 branes (corresponding to generic rank \( N \) bundles).

So far we have reduced the difficult algebro–geometrical problem of counting sheaves to a more tractable path integral. Unfortunately the theory as it stands is not very manageable since moduli spaces of instantons suffer from non-compactness problems arising both from singularities where instantons shrink to zero size as well from the non-compactness of the ambient space \( X \) on which the gauge theory is defined. The way out comes from an analogous issue in instanton counting in four dimensional twisted \( N = 2 \) theory. In \[10\] Nekrasov proposed that equivariant localization techniques could be used in combination with a noncommutative deformation of the theory to evaluate directly the instanton factors. This idea has turned out to be very powerful allowing for explicit computations in the four dimensional setting and can be applied to our six dimensional case \[5\]. Here we will only consider the case of flat space \( X = \mathbb{C}^3 \) unless explicitly mentioned. The noncommutative deformation resolves the small instanton singularities of the moduli space \( M(X) \) and provides a natural compactification of \( M(X) \). Also working equivariantly can be easily implemented: on \( \mathbb{C}^3 \) there is naturally the action of the torus \( T^3 \) coming from the maximal torus of the \( U(3) \) group generating rotational isometries that preserve the Kähler form of \( \mathbb{C}^3 \). On the coordinates of \( \mathbb{C}^3 \) this torus acts as \( z_i \to z_i e^{t_i}, i = 1, 2, 3 \).

The equivariant model can be obtained by modifying the BRST operator so that it becomes an equivariant differential with respect to this toric action. In other words we restrict attention to field configurations that are annihilated by the old BRST operator only up to a toric action.

After these modifications the gauge theory localizes onto the fixed points of the equivariant BRST operator. One can show that these fixed points are isolated and their contribution to the path integral can be computed by direct equivariant integration, by using the Duistermaat–Heckman formula or its generalizations. The problem of computing the path integral is now reduced to two simpler ones, namely the classification of the critical points of the equivariant BRST differential and the actual evaluation of the instanton factor.

These goals can be accomplished in two distinct but ultimately equivalent ways as we are about to see.

3 The Noncommutative Theory

The path integral of the noncommutative field theory can be evaluated directly by using equivariant localization. After the noncommutative deformation we can think of the theory as an infinite–dimensional matrix model where the fields are replaced by operators acting on a separable Hilbert space. This approach has the advantage that some explicit instanton solutions can be constructed and it provides a natural compactification of the instanton moduli space. In terms of the noncommutative fields the instanton equations become

\[ [Z^i, Z^j] + \epsilon^{ijk} [Z^i, \rho] = 0, \]
\[ [Z^i, Z^j] + [\rho, \rho^j] = 3 \delta_{ij}, \]
\[ [Z_i, \Phi] = \epsilon_i Z_i, \tag{3.1} \]

where in the last equation there is no sum over the index \( i \) and the right–hand side reflects explicitly the equivariant deformation.

These sets of equations can be solved by three-dimensional harmonic oscillator algebra. The unique irreducible representation of this algebra is provided by the Fock module
\[
\mathcal{H} = C[\alpha_1^+, \alpha_2^+, \alpha_3^+]|0, 0, 0\rangle = \bigoplus_{n_1, n_2, n_3 \in \mathbb{N}_0} C|n_1, n_2, n_3\rangle, \tag{3.2} \]
where \(|0, 0, 0\rangle\) is the Fock vacuum with \(\alpha_i|0, 0, 0\rangle = 0\) for \(i = 1, 2, 3\), and the orthonormal basis states \(|n_1, n_2, n_3\rangle\) are connected by the usual action of the creation and annihilation operators \(\alpha_i^+\) and \(\alpha_i\).

The operators \(Z^i\) may then be taken to act on the Hilbert space \(\mathcal{H}_W = W \otimes \mathcal{H}\) where \(W \cong C^N\) is a Chan-Paton multiplicity space of dimension \(N\), the number of D6-branes (and the rank of the gauge theory). The space \(W\) carries the nonabelian degrees of freedom and we understand \(Z^i\) and \(\Phi\) as \(N \times N\) matrices of operators acting on \(\mathcal{H}\).

We can diagonalize the field \(\Phi\) using the \(U(N)\) gauge symmetry. One can now classify the fixed points of the nonabelian gauge theory by generalizing the arguments of [5] [11]. We are prescribed to compute the path integral over configurations of the Higgs field whose asymptotic limit is \(a = \text{diag}(a_1, \ldots, a_N) \in u(1)^N\). With this choice of boundary condition the noncommutative field \(\Phi\) has the form \(\Phi = a \otimes 1_{\mathcal{C}} + 1_{N, N} \otimes \Phi_{\mathcal{C}}\). The degeneracies of the asymptotic Higgs vevs breaks the gauge group \(U(N) \to \prod_i U(k_i)\) with \(\sum_k k_i = N\). Correspondingly, the Chan-Paton multiplicity space \(W\) decomposes into irreducible representations \(W = \bigoplus_i W_i\) with \(\dim W_i = k_i\).

Due to the equivariant deformation the theory now localizes on \(U(1)^N\) noncommutative instantons. These correspond to ideals \(J\) of codimension \(k\) in \(C[z_1, z_2, z_3]\) that are associated, via partial isometries, to subspaces of the full Hilbert space of the form \(\bigoplus_{j \in J} f(\alpha_1^+, \alpha_2^+, \alpha_3^+)|0, 0, 0\rangle\).

These ideals are generated by monomials \(z^i z^j z^k\) and are in one-to-one correspondence with three-dimensional partitions, with the triplet \((i, j, k)\) corresponding to boxes of the partition. More precisely, the set of solutions can be completely classified in terms of coloured partitions \(\vec{\pi} = (\pi_1, \ldots, \pi_N)\), which are rows of \(N\) ordinary three-dimensional partitions \(\pi_l\) labelled by \(a_l\).

We can now write the full path integral as a sum over critical points and compute the fluctuation factor around each critical point. This factor assumes the form of a ratio of functional determinants
\[
\det(\text{ad} \Phi) \det(\text{ad} \Phi + \epsilon_1 + \epsilon_2) \det(\text{ad} \Phi + \epsilon_1 + \epsilon_3) \det(\text{ad} \Phi + \epsilon_2 + \epsilon_3) \det(\text{ad} \Phi + \epsilon_1 + \epsilon_2 + \epsilon_3)\tag{3.3} \]

where the \(\epsilon_i\) parametrize the toric action. This ratio can be computed explicitly to give a factor of \((-1)^{|\#|}\).

\[ Z^{U(1)^N}_{\text{DT}}(\mathbb{C}^3) = \sum_{\vec{\pi}} (-1)^{(N+1)|\#|} q^{\#|\#|}, \tag{3.4} \]

where \(q = -e^{i \theta} = e^{-g_s}\).

4 Matrix Quantum Mechanics and Coherent Sheaves

The second approach consists in the introduction of an appropriate topological matrix quantum mechanics [11]. From the string theoretical point of view this corresponds to the effective action on a gas of D0 branes that bound the original D6 on \(\mathbb{C}^3\). From the perspective of the gauge theory it
arises as quantization of the collective coordinates around each instanton solution and provides a higher dimensional generalization of the ADHM construction of instantons. Indeed one can see this explicitly by parametrizing each holomorphic bundle (or coherent sheaf) on the projective space $\mathbb{P}^3$ that corresponds to a compactification of the physical space $\mathbb{C}^3$ in term of a set of algebraic matrix equations that we’ll call generalized ADHM equations. This can be done by using Beilinson’s theorem which states that for any coherent sheaf $E$ on $\mathbb{P}^3$ there is a spectral sequence $E_r^{p,q}$ with $E_1$-term $E_1^{p,q} = H^q(\mathbb{P}^3, \mathcal{E}(\mathbb{P}^3) \otimes \mathcal{O}_{\mathbb{P}^3}(p))$ for $p \leq 0$ that converges to the original sheaf (here $\mathcal{O}_{\mathbb{P}^3}$ and $\mathcal{O}_{\mathbb{P}^3}$ are respectively the sheaf of differential forms and the structure sheaf). By the appropriate set of boundary conditions this spectral sequence degenerates at the $E_2$ term.

The outcome of this procedure is that the original sheaf can be described as the only non-vanishing cohomology of a four term complex. The associated conditions yield a particular set of matrix equations plus stability conditions. One can show that this system boils down to the following set of generalized ADHM equations

$$[B_1, B_2] + IJ = 0 \ , \quad [B_1, B_3] + IK = 0 \ , \quad [B_2, B_3] = 0 \ , \quad (4.1)$$

where $B_i \in \text{End}(V)$, $i = 1, 2, 3$, $I \in \text{Hom}(W, V)$ and $J, K \in \text{Hom}(V, W)$ and a suitable stability condition has to be imposed. The vector spaces $V$ and $W$ arise in the geometrical construction outlined above as particular cohomology groups of the sheaf $\mathcal{E}$. These equations are naturally in correspondence with the noncommutative instantons described in the previous section. In particular in the abelian case $N = 1$ the stability conditions allow us to set $J = K = 0$ and the cohomology sheaf $\mathcal{E}$ is isomorphic to the ideal 3 that enters in the description of the Hilbert scheme in terms of noncommutative instantons. As we localize the theory onto its $U(1)^N$ phase this is the relevant case. One can easily construct a cohomological matrix model starting from these equations. In this framework $V$ with $\dim V = k$ represent the gas of $k$ D0 branes (or the charge $k$ topological sector in the gauge theory) while $W$ stands for the D6 branes and its dimension is the rank of the gauge theory $N$.

The matrices $B_i$ arise from 0–0 strings and represent the position of the coincident D0-branes inside the D6-branes. On the other hand, the field $I$ describes open strings stretching from the D6-branes to the D0-branes. It characterizes the size and orientation of the D0-branes inside the D6-branes. Other fields are necessary to close the equivariant BRST algebra and localize the theory on the generalized ADHM equations but we refer the reader to [4] for a complete treatment.

In the abelian case the generalized ADHM equations ensure that the critical points can be expressed by a certain sequence of maps between the spaces $V$ and $W$. This configuration can be explicitly mapped into a three dimensional partition thus recovering the classification of the fixed points that we found in the noncommutative setting. The generalization to the $U(1)^N$ theory is simple and corresponds to $N$–tuples of three dimensional partitions. We will denote a generic fixed point $f$ as $\vec{\pi} = (\pi_1, \ldots, \pi_N)$. Accordingly at the fixed points the vector spaces $V$ and $W$ have the following weight decompositions

$$V_f = \sum_{l=1}^{N} e^{i\alpha_l} \sum_{(i,j,k) \in \pi_l} \rho_i e^{-i(j-1)k-1}, \quad W_f = \sum_{l=1}^{N} e^{i\alpha_l}, \quad (4.2)$$

where $t_i = e^{-i\alpha_i}$ generate the $\mathbb{T}^3$ action.

The computation of the instanton factors proceeds as in the four dimensional case [10, 12]. For every fixed point we can describe the local structure of the moduli space via an equivariant complex that encodes the linearization of the generalized ADHM equations up to linearized (complexified) gauge invariance. The character of this complex

$$\chi_f(\mathbb{C}^3)^k = W_f^* \otimes V_f - \frac{V_f^* \otimes W_f}{t_1 t_2 t_3} + V_f^* \otimes V_f \frac{(1-t_1)(1-t_2)(1-t_3)}{t_1 t_2 t_3} \ , \quad (4.3)$$
contains all the local information needed in the localization formula and can be used to compute explicitly the instanton factors following [10, 12]. In (4.3) the subscript \( f \) is stressing that the computation only holds at a particular fixed point and the conjugation acts on the elements of the weight decomposition as \( t_i^f = t_i^{-1} \). A straightforward computation gives the fluctuation factor \( (-1)^N |\vec{\pi}| \). To get the partition function the last missing ingredient is the instanton action (2.1). This can be obtained by writing the universal sheaf \( \mathcal{E} \) on the moduli space as \( \mathcal{E} = W \oplus V \otimes (S^- \otimes S^+) \) where \( S^\pm \) are the positive/negative chirality spinor bundles over \( \mathbb{P}^3 \). By using the correspondence between spinors and differential forms we can decompose its Chern character at a given fixed point as

\[
\text{ch}(\mathcal{E}_\vec{\pi}) = W_{\vec{\pi}} - (1 - t_1)(1 - t_2)(1 - t_3)V_{\vec{\pi}}. \tag{4.4}
\]

Collecting all pieces of information one can write down the full partition function

\[
Z_{\text{UT}}^{U(1)^N}(X) = \sum_{\vec{\pi}} \left( -1 \right)^N \prod_{i,j=1}^{N} (1 - t_i - t_j) e^{i \vartheta} |\pi_i|^{1} |\pi_j|^{1} \left( m_{1,i} + m_{2,j} \right), \tag{5.1}
\]

that agrees precisely with (3.4).

5 The Coulomb Branch on a Toric Manifold

The construction just outlined carries on to the case of a general toric manifold \( X \). The geometric information of a toric manifold is encoded in a trivalent graph \( \Delta \). The vertices \( f \) of \( \Delta \) are locally isomorphic to \( \mathbb{C}^3 \) while the edges \( e \) correspond to rational curves of Kähler area \( t_e \). By using the localization procedure on the physical space the gauge theory localizes onto a sum of contributions associated with each vertex corresponding to three dimensional partitions and a set of propagators associated with the edges. Each propagator depends on the area \( t_e \) of the rational curve and on a two dimensional partition that arises when gluing together two different three dimensional partitions as a section of the common leg. In the rank 1 case one recovers the Calabi–Yau crystal picture directly from the gauge theory. In the more general rank \( N \) setting on the Coulomb branch one finds

\[
Z_{\text{UT}}^{U(1)^N}(X) = \sum_{\vec{\pi}} q^{\vec{I}} \prod_{e \in \text{edge}} \left( -1 \right)^{\sum_{l=1}^{N} |\lambda_{l,e}|} e^{i \vartheta} \left( m_{1,e} + m_{2,e} \right), \tag{5.2}
\]

where \( q = -e^{i \vartheta} \) and

\[
I = \sum_{f} \sum_{l=1}^{N} |\pi_{l,f}| \sum_{e \in \text{edge}} \sum_{l=1}^{N} \sum_{i,j=1}^{\lambda_{l,e}} \left( m_{1,e}(i - 1) + m_{2,e}(j - 1) + 1 \right). \tag{5.2}
\]

The integers \( m_{1,e} \) and \( m_{2,e} \) determine the normal bundle to the rational curve corresponding to the edges.

6 Conclusions

In [4] we have studied the relationship with a six dimensional topological Yang–Mills theory and Donaldson–Thomas invariants. As a first step one can use equivariant localization to write the partition function of the noncommutative deformation of the theory as a sum over point–like instantons. These noncommutative instantons can be interpreted in purely geometrical terms as certain coherent sheaves on \( \mathbb{P}^3 \) through a higher dimensional generalization of the ADHM formalism. In turn this can be used to construct a topological matrix quantum mechanics that dynamically describes the stable coherent sheaves. This formalism can be used, for example, to compute the rank \( N \) partition function on a toric manifold; the result is the \( N \)-th power of the abelian result with an
dependent sign shift. This shift can be absorbed in a redefinition of the string coupling constant \( g_s \rightarrow g_s - N\pi \). This modification is natural from the point of view of the OSV conjecture \cite{13} that relates the entropy of a BPS black hole with the topological string amplitude. The parameters that enter in the topological string amplitude are functions of the D–brane charges at the attractor point of the BPS moduli space. In the presence of D6 branes this relation is consistent with the above shift.

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