Two-Photon Algebra Eigenstates: A Unified Approach to Squeezing

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Abstract

We use the concept of the algebra eigenstates that provides a unified description of the generalized coherent states (belonging to different sets) and of the intelligent states associated with a dynamical symmetry group. The formalism is applied to the two-photon algebra and the corresponding algebra eigenstates are studied by using the Fock-Bargmann analytic representation. This formalism yields a unified analytic approach to various types of single-mode photon states generated by squeezing and displacing transformations.

I. INTRODUCTION

Coherent states (CS) associated with various dynamical symmetry groups are important in many problems of quantum physics [1–3]. Actually, there are three distinct ways in which CS for a Lie group can be defined [3].

In the general group-theoretic approach developed by Perelomov [4] and Gilmore [5], the CS are generated by the action of group elements on a reference state of a group representation Hilbert space. These states (called the generalized CS) have a number of remarkable properties that make them very useful in description of many quantum phenomena [1–3]. The most important features of the coherent-state systems are their overcompleteness and their invariance under the action of group representation operators. The last property means that the CS transform among themselves during the evolution governed by Hamiltonians for which the corresponding Lie group is the dynamical symmetry group.

The second approach deals with states defined as eigenstates of a lowering group generator. Attention was mainly paid to eigenstates of the lowering generator $K_-$ for different realizations of SU(1,1) [6–11].

The third way in which CS can be defined is associated with the optimization of uncertainty relations for Hermitian generators of a group [12–21]. States that minimize uncertainty relations are called intelligent states (IS) or minimum-uncertainty states. Ordinary IS [13] provide an equality in the Heisenberg uncertainty relation while generalized IS [18,19] do so in the Robertson uncertainty relation [22]. The IS are determined by some type of the

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eigenvalue equation \[23,18,19\], and the lowering-generator eigenstates are in fact a particular case of the IS, corresponding to equal uncertainties of two Hermitian generators.

In the special case of the Heisenberg-Weyl group \(H_3\) \[24\] whose generators are the boson annihilation and creation operators \(a\) and \(a^\dagger\) and the identity operator \(I\), the first and second definitions coincide. The Glauber CS \(|\alpha\rangle\) \[23\] can be defined as eigenstates of the lowering generator, \(a|\alpha\rangle = \alpha|\alpha\rangle\), and also as states generated by the displacement operator \(D(\alpha)\) (representing group elements) acting on the vacuum state |0\rangle,

\[|\alpha\rangle = D(\alpha)|0\rangle = \exp(\alpha a^\dagger - \alpha^* a)|0\rangle. \tag{1.1}\]

At the same time, the Glauber CS \(|\alpha\rangle\) are the IS for the field quadratures \(X_1 = (a^\dagger + a)/2\) and \(X_2 = i(a^\dagger - a)/2\), i.e., they minimize the Heisenberg uncertainty relation \(\Delta X_1 \Delta X_2 \geq 1/4\). The uncertainties are equal, \(\Delta X_1 = \Delta X_2 = 1/2\), when the expectation values are calculated for the \(|\alpha\rangle\) states. In this sense, the CS \(|\alpha\rangle\) are a special case of the canonical squeezed states \[26\]. For the squeezed states, the fluctuations in one quadrature are reduced on account of growing fluctuations in the other (conjugate) quadrature. The canonical squeezed states can be considered as the generalized IS for the Heisenberg-Weyl group \[13\].

For more complicated groups, e.g., for \(SU(1,1)\), the different definitions lead to distinct states. The Perelomov CS for the \(SU(1,1)\) Lie group, obtained by the action of the group elements on the reference state \[3\], and the Barut-Girardello states, defined as the eigenstates of the \(SU(1,1)\) lowering generator \(K_-\) \[6\], are quite different. However, the concept of squeezing can be naturally extended to the \(SU(1,1)\) group, and the squeezing properties of the \(SU(1,1)\) ordinary and generalized IS have been widely discussed \[27–29,8,9,15–21\].

In Perelomov’s definition, different sets of the CS are obtained for different choices of the reference state. The usually used sets of the CS (the standard sets, as we refer to them) correspond to the cases when an extreme state of the representation Hilbert space (e.g., the vacuum state of the quantized field mode) is chosen as the reference state \[3\]. In general, this choice of the reference state leads to the sets consisting of states with properties closest to those of classical states \[3\]. On the other hand, the IS show a variety of nonclassical properties, such as squeezing and sub-Poissonian photon statistics. In the case of the \(SU(1,1)\) Lie group, the standard set of Perelomov’s CS and the set of the ordinary IS have an intersection \[27,20\]. Both these types of states form subsets of the generalized IS \[18\].

In this paper we develop a formalism that provides a unified description of different types of coherent and intelligent states. We introduce the concept of algebra eigenstates (AES) which are defined for an arbitrary Lie group as eigenstates of elements of the corresponding complex Lie algebra. We show that different sets of the generalized CS (both standard and nonstandard) can be equivalently defined as the AES. Moreover, the ordinary and generalized IS for Hermitian generators of a Lie group form a subset of the AES associated with this group. On the basis of the algebra-eigenstate formalism, we use analytic methods that enable us to treat different types of states (including the standard and nonstandard CS and the IS) in a unified way. This unified description is also applicable for investigating more complicated states obtained by the action of unitary group transformations on the IS. Such states can be considered as (nonstandard) generalized CS with the reference state being an intelligent state.
In the present work we apply the general formalism to the two-photon group $H_6$ 
that enables us to obtain the unified description of single-mode photon states generated by 
displacing and squeezing transformations. We use the Fock-Bargmann analytic representa-
tion [30] based on the standard set of the Glauber CS. In this analytic representation the 
eigenvalue equation that determines the two-photon AES becomes a linear homogeneous 
differential equation. Then the powerful theory of analytic functions is applied for studying 
various types of photon states and relations between them.

In Sec. 2 we develop the group-theoretic formalism of the AES for an arbitrary Lie group. 
The Fock-Bargmann representation of the two-photon AES is derived in Sec. 3. By using 
this representation, we find entire analytic functions representing different types of photon 
states. In Sec. 4 we consider displaced and squeezed Fock states. The superpositions of 
the Glauber CS (the Schrödinger-cat states) and their squeezed and displaced versions are 
discussed in Sec. 5. The two-photon IS for the SU(1,1) subgroup of $H_6$ are considered in 
Sec. 6. We introduce the states which are generated by squeezing and displacement of the 
IS. We also touch on the question of the production of various two-photon AES.

## II. THE GENERAL THEORY OF THE ALGEBRA EIGENSTATES

Let $G$ be an arbitrary Lie group and $T$ its unitary irreducible representation acting on 
the Hilbert space $\mathcal{H}$. By choosing a fixed normalized reference state $|\Psi_0\rangle \in \mathcal{H}$, one can 
define the system of states $\{|\Psi_g\rangle\}$,

$$
|\Psi_g\rangle = T(g)|\Psi_0\rangle, \quad g \in G,
$$

(2.1)

which is called the coherent-state system.

The isotropy (or maximum-stability) subgroup $H \subset G$ consists of all the group elements 
h that leave the reference state invariant up to a phase factor,

$$
T(h)|\Psi_0\rangle = e^{i\delta(h)}|\Psi_0\rangle, \quad |e^{i\phi(h)}| = 1, \quad h \in H.
$$

(2.2)

For every element $g \in G$, there is a unique decomposition of $g$ into a product of two group 
elements, one in $H$ and the other in the quotient (or coset) space $G/H$,

$$
g = \Omega h, \quad g \in G, \quad h \in H, \quad \Omega \in G/H.
$$

(2.3)

It is clear that group elements $g$ and $g'$ with different $h$ and $h'$ but with the same $\Omega$ produce 
the coherent states which differ only by a phase factor: $|\Psi_g\rangle = e^{i\delta}|\Psi_{g'}\rangle$, where $\delta = \phi(h) - \phi(h')$. Therefore a coherent state $|\Psi_\Omega\rangle$ is determined by a point $\Omega = \Omega(g)$ in the quotient 
space $G/H$.

One can see from this group-theoretic procedure for the construction of the generalized 
CS that the choice of the reference state $|\Psi_0\rangle$ firmly determines the structure of the coherent- 
state set. An important class of coherent-state sets corresponds to the quotient spaces $G/H$ 
which are homogeneous Kählerian manifolds. Then $G/H$ can be considered as the phase 
space of a classical dynamical system, and the mapping $\Omega \rightarrow |\Psi_\Omega\rangle\langle\Psi_\Omega|$ is the quantization 
for this system [31]. It means that the quantization is performed via the CS [3].
Let us consider the Lie algebra \( \mathfrak{g} \) of the group \( G \) (here and in the what follows we will call algebra the complex extension of the real algebra, i.e., the set of all linear combinations of elements of the real algebra with complex coefficients). The isotropy subalgebra \( \mathfrak{b} \) is defined as the set of elements \( \{ b \} \), \( b \in \mathfrak{g} \), such that

\[
 b | \Psi_0 \rangle = \lambda | \Psi_0 \rangle.
\]  

(2.4)

Here \( \lambda \) is a complex eigenvalue. If the isotropy subgroup \( H \) is nontrivial, then the isotropy subalgebra \( \mathfrak{b} \) will be nontrivial too. By acting with \( T(g) \) on both sides of Eq. (2.4), we obtain

\[
 T(g) b T^{-1}(g) | \Psi_0 \rangle = \lambda T(g) | \Psi_0 \rangle.
\]  

(2.5)

This leads to the eigenvalue equation

\[
 g | \Psi_\lambda \rangle = \lambda | \Psi_\lambda \rangle,
\]  

(2.6)

where \( | \Psi_\lambda \rangle = T(g) | \Psi_0 \rangle \) is a coherent state, and the operator \( g = T(g) b T^{-1}(g) \) is an element of the algebra \( \mathfrak{g} \). We see that the generalized CS are the eigenstates of the elements of the complex algebra.

Now, let us choose a basis \( \{ \mathfrak{K}_1, \mathfrak{K}_2, \ldots, \mathfrak{K}_p \} \) for a \( p \)-dimensional Lie algebra \( \mathfrak{g} \). Then an element of the complex algebra can be written as the Euclidean scalar product in the \( p \)-dimensional vector space,

\[
 g = \beta \cdot \mathfrak{K} = \beta_1 \mathfrak{K}_1 + \beta_2 \mathfrak{K}_2 + \cdots + \beta_p \mathfrak{K}_p,
\]  

(2.7)

where \( \beta_1, \beta_2, \ldots, \beta_p \) are arbitrary complex coefficients. Then the AES are defined by the eigenvalue equation:

\[
 \beta \cdot \mathfrak{K} | \Psi(\lambda, \beta) \rangle = \lambda | \Psi(\lambda, \beta) \rangle.
\]  

(2.8)

The comparison of Eqs. (2.6) and (2.8) shows that the generalized CS can be defined as the AES, and a specific set of the CS is obtained for the appropriate choice of the parameters \( \beta \)'s. More precisely, let a state \( | \Psi(\lambda, \beta) \rangle \) belong to a specific set of the CS corresponding to the reference state \( | \Psi_0 \rangle \) that satisfies Eq. (2.4). Then the parameters \( \beta \)'s must satisfy the condition \( \beta \cdot \mathfrak{K} = T(g) b T^{-1}(g), \ \forall g \in G \). Note that the definition (2.8) of the AES does not depend explicitly on the choice of the reference state \( | \Psi_0 \rangle \). Hence it is possible to treat the CS defined as the AES in a quite general way, regardless of the set to which they belong.

An important property of the generalized CS is the identity resolution:

\[
 \int d\mu(\Omega) | \Psi_\Omega \rangle \langle \Psi_\Omega | = I,
\]  

(2.9)

where \( I \) is the identity operator in the Hilbert space \( \mathcal{H} \), and \( d\mu(\Omega) \) is the invariant measure in the homogeneous quotient space \( G/H \). Then any state \( | \Psi \rangle \in \mathcal{H} \) can be expanded in the coherent-state basis \( | \Psi_\Omega \rangle \),

\[
 | \Psi \rangle = \int d\mu(\Omega) f(\Omega) | \Psi_\Omega \rangle,
\]  

(2.10)
where \( f(\Omega) = \langle \Psi_\Omega | \Psi \rangle \), and

\[
\langle \Psi | \Psi \rangle = \int d\mu(\Omega) |f(\Omega)|^2. \tag{2.11}
\]

If we restrict the consideration to the square-integrable Hilbert space then the integral in (2.11) must be convergent. Since the CS are not orthogonal to each other, the CS themselves can be expanded in their own basis.

Now, let us represent all the AES in the standard coherent-state basis. In what follows we will consider only the simplest cases in which the quotient space \( G/H \) corresponding to the standard set is a homogeneous Kählerian manifold that can be parametrized by a single complex number \( z \), so we write the standard generalized CS \( |\Psi_\Omega\rangle \) in the form \( |z\rangle \). Then Eq. (2.10) reads

\[
|\Psi(\lambda, \beta)\rangle = \int d\mu(z) f(\lambda, \beta; z^* ) |z\rangle. \tag{2.12}
\]

The function \( f(\lambda, \beta; z) = \langle z^* | \Psi(\lambda, \beta) \rangle \) can be decomposed into two factors: \( f(\lambda, \beta; z) = \mathcal{N}(z) \Lambda(\lambda, \beta; z) \). Here \( \mathcal{N}(z) \) is a normalization factor such that \( \Lambda(\lambda, \beta; z) \) is an entire analytic function of \( z \) defined on the whole complex plane or on part of it. Such analytic representations are well studied [30,2] for the standard coherent-state bases of the simplest Lie groups. In these simplest cases the elements of the Lie algebra act in the Hilbert space of entire analytic functions as linear differential operators. Then the eigenvalue equation (2.8) is converted into a linear homogeneous differential equation. By solving this equation, we obtain the entire analytic functions \( \Lambda(\lambda, \beta; z) \) representing the AES \( |\Psi(\lambda, \beta)\rangle \) in the standard coherent-state basis \( |z\rangle \).

The standard set of the CS is a particular case of the wide system of the AES. Other particular cases of the AES are the sets of the ordinary and generalized IS. Any two quantum observables (Hermitian operators in the Hilbert space) \( A \) and \( B \) obey the Robertson uncertainty relation [22]

\[
(\Delta A)^2 (\Delta B)^2 \geq \frac{1}{4} (\langle C \rangle^2 + 4\sigma_{AB}^2), \quad C = -i[A, B], \tag{2.13}
\]

where the variance of \( A \) is \( (\Delta A)^2 = \langle A^2 \rangle - \langle A \rangle^2 \), \( (\Delta B)^2 \) is defined similarly, the covariance of \( A \) and \( B \) is \( \sigma_{AB} = \frac{1}{2}(\langle AB \rangle + \langle BA \rangle) - \langle A \rangle \langle B \rangle \), and the expectation values are taken over an arbitrary state in the Hilbert space. When the covariance of \( A \) and \( B \) vanishes, \( \sigma_{AB} = 0 \), the Robertson uncertainty relation reduces to the Heisenberg uncertainty relation,

\[
(\Delta A)^2 (\Delta B)^2 \geq \frac{1}{4} \langle C \rangle^2. \tag{2.14}
\]

The states which provide an equality in the Heisenberg uncertainty relation (2.14) are called the ordinary IS [13] and the states which minimize the Robertson uncertainty relation (2.13) are called the generalized IS [18]. It is clear that the ordinary IS form a subset of the generalized IS. The generalized IS for operators \( A \) and \( B \) are determined from the eigenvalue equation [18,19]

\[
(\eta A + iB)|\lambda, \eta\rangle = \lambda|\lambda, \eta\rangle, \tag{2.15}
\]
where the parameter \( \eta \) is an arbitrary complex number, and \( \lambda \) is a complex eigenvalue. For the particular case of real \( \eta \), the eigenvalue equation (2.15) determines the ordinary IS for operators \( A \) and \( B \). Then the equation can be written in the form [23]

\[
(A + i\gamma B)\{\lambda, \gamma\} = \lambda\{\lambda, \gamma\},
\]

(2.16)

where \( \gamma \) is a real parameter. By comparing Eqs. (2.15) and (2.16) with Eq. (2.8), we see that the IS for any two Hermitian group generators form a subset of the AES of the group.

The generalized IS for the quadratures \( X_1 \) and \( X_2 \) coincide with the canonical squeezed states [18]. The concept of squeezing is naturally related also to the IS associated with the SU(2) and SU(1,1) Lie groups [27–29,8,9,15–21]. At the last years there is a great interest in the IS. The SU(2) and SU(1,1) IS have been shown recently to be useful for improving the accuracy of interferometric measurements [32]. The investigation of the AES yields the most full information on the IS for generators of the corresponding Lie group. It is also possible to consider the states generated by the action of unitary group transformations on the IS. The most convenient way to examine different subsets of the AES and relations between them is via the analytic representation of the AES in the standard coherent-state basis. In the present work the algebra-eigenstate method is applied to the two-photon group \( H_6 \) whose unitary transformations squeeze and displace single-mode photon states.

### III. THE FOCK-BARGMANN REPRESENTATION OF THE TWO-PHOTON ALGEBRA EIGENSTATES

The theoretical analysis [26,33] and experimental realization [34–36] of squeezed states continue to attract considerable attention [37]. Much of the work so far was concerned with the single-mode case whose group-theoretic basis lies in the two-photon Lie group \( H_6 \) [3]. The corresponding Lie algebra is spanned by the six operators \( \{N, a^2, a^{12}, a, a^\dagger, I\} \),

\[
\begin{align*}
[a^2, a^{12}] &= 4N + 2I, & [a, a^\dagger] &= I; \\
[a^{12}, a] &= -2a^\dagger, & [a^2, a^\dagger] &= 2a; \\
[N, a^{12}] &= 2a^{12}, & [N, a^2] &= -2a^2; \\
[N, a^\dagger] &= a^\dagger, & [N, a] &= -a,
\end{align*}
\]

(3.1)

where \( N = a^\dagger a \) is the number operator. All the other commutation relations are zero. The unified group-theoretic description of various types of states associated with the \( H_6 \) transformations can be obtained by means of the algebra-eigenstate method. This provides the analytic representation of single-mode photon states generated by squeezing and displacement group operators.

The representation Hilbert space of \( H_6 \) is the Fock space of the quantum harmonic oscillator. The orthonormal basis in this space is the Fock basis of the number eigenstates \( |n\rangle \) \( (n = 0, 1, \ldots, \infty) \). For any Fock state \( |n\rangle \), the isotropy subgroup is \( U(1) \otimes U(1) \) with the algebra spanned by \( \{N, I\} \). The isotropy subgroup consists of all group elements \( h \) of the form \( h = \exp(i\delta N + i\varphi I) \). Thus \( h|n\rangle = \exp(i\delta n + i\varphi)|n\rangle \). The oscillator group \( H_4 \) is a subgroup of \( H_6 \). The corresponding solvable Lie algebra is spanned by the four operators \( \{N, a, a^\dagger, I\} \). The quotient space \( H_4/U(1) \otimes U(1) \) can be parametrized by an arbitrary complex number \( \alpha \). Then an element \( \Omega \in H_4/U(1) \otimes U(1) \) can be written as the displacement
operator, $\Omega \equiv D(\alpha) = \exp(\alpha a^\dagger - \alpha^* a)$. Note that the same quotient space and hence the same set of the CS is obtained also for the Heisenberg-Weyl group $H_3$. This is a subgroup of $H_4$ ($H_3 \subset H_4 \subset H_6$), and the nilpotent Lie algebra corresponding to $H_3$ is spanned by the three operators $\{a, a^\dagger, I\}$. The quotient space $H_3/U(1)$ is the same as the space $H_4/U(1) \otimes U(1)$.

The standard Glauber set of the CS is obtained when the vacuum state $|0\rangle$ is chosen as the reference state,

$$|\alpha\rangle = D(\alpha)|0\rangle = e^{-|\alpha|^2/2} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} |n\rangle. \quad (3.2)$$

For any state $|\Psi\rangle = \sum_{n=0}^{\infty} c_n |n\rangle$ in the Hilbert space, one can construct the entire analytic function [30]

$$f(\alpha) = e^{i|\alpha|^2/2} \langle \alpha^* | \Psi \rangle = \sum_{n=0}^{\infty} c_n \frac{\alpha^n}{\sqrt{n!}}. \quad (3.3)$$

Then the identity resolution, $(1/\pi) \int d^2 \alpha |\alpha\rangle\langle \alpha| = I$, can be used to expand the state $|\Psi\rangle$ in the coherent-state basis:

$$|\Psi\rangle = \frac{1}{\pi} \int d^2 \alpha e^{-|\alpha|^2/2} f(\alpha^*) |\alpha\rangle. \quad (3.4)$$

It is customary in quantum mechanics to restrict the Hilbert space to consist of normalizable states that satisfy the condition

$$\langle \Psi | \Psi \rangle = \frac{1}{\pi} \int d^2 \alpha e^{-|\alpha|^2} |f(\alpha^*)|^2 < \infty. \quad (3.5)$$

The analytic representation (3.3) is known as the Fock-Bargmann representation [30]. The Glauber coherent state $|\nu\rangle$ is represented by the function

$$\mathcal{F}(\nu; \alpha) = e^{i|\nu|^2/2} e^{i\nu \alpha}. \quad (3.6)$$

The generators of $H_6$ act in the Hilbert space of entire analytic functions $f(\alpha)$ as linear differential operators:

$$a = \frac{d}{d\alpha}, \quad a^\dagger = \alpha, \quad I = 1, \quad N = \alpha \frac{d}{d\alpha}, \quad a^2 = \frac{d^2}{d\alpha^2}, \quad a^\dagger a^2 = \alpha^2. \quad (3.7)$$

The two-photon AES are determined by the eigenvalue equation

$$(\beta_1 N + \beta_2 a^2 + \beta_3 a^\dagger a + \beta_4 a + \beta_5 a^\dagger)|\lambda, \beta\rangle = \lambda|\lambda, \beta\rangle. \quad (3.8)$$

The AES $|\lambda, \beta\rangle$ are represented by the function
\[ \Lambda(\lambda, \beta; \alpha) = e^{\frac{|\alpha|^2}{2}} \langle \alpha^* | \lambda, \beta \rangle, \quad (3.9) \]

and in the Fock-Bargmann representation the eigenvalue equation \((3.8)\) becomes the second-order linear homogeneous differential equation
\[ \beta_2 \frac{d^2 \Lambda}{d\alpha^2} + (\beta_1 \alpha + \beta_4) \frac{d\Lambda}{d\alpha} + (\beta_5 \alpha^2 + \beta_5 \alpha - \lambda) \Lambda = 0. \quad (3.10) \]

By using the transformation
\[ \Lambda(\lambda, \beta; \alpha) = \exp \left( \frac{\Delta - \beta_1}{4 \beta_2} \alpha^2 \right) T(\lambda, \beta; \alpha), \quad (3.11) \]
we get the equation with coefficients that are linear in \(\alpha\),
\[ \beta_2 \frac{d^2 T}{d\alpha^2} + (\Delta \alpha + \beta_4) \frac{dT}{d\alpha} + [\sigma \alpha + \frac{1}{2}(\Delta - \beta_1) - \lambda] T = 0, \quad (3.12) \]
where
\[ \Delta^2 \equiv \beta_1^2 - 4 \beta_2 \beta_3, \quad (3.13) \]
\[ \sigma \equiv \beta_4 \Delta - \frac{\beta_1}{2 \beta_2} + \beta_5. \quad (3.14) \]

Note the double-valuedness of \(\Delta\). Equation \((3.12)\) can be transformed into the Kummer equation for the confluent hypergeometric function or into the Bessel equation, depending on values of the parameters \([38]\).

In the most general case, \(\beta_2 \neq 0, \Delta \neq 0\), two independent solutions of Eq. \((3.12)\) are given by \([38]\)
\[ T_1(\lambda, \beta; \alpha) = \exp \left( -\frac{\sigma}{\Delta} \alpha \right) _1F_1 \left( d \left| \frac{1}{2} \right| - \frac{\Delta}{2 \beta_2} (\alpha - \mu_\Delta) \right), \quad (3.15a) \]
\[ T_2(\lambda, \beta; \alpha) = \sqrt{-\frac{\Delta}{2 \beta_2}} (\alpha - \mu_\Delta) \exp \left( -\frac{\sigma}{\Delta} \alpha \right) _1F_1 \left( d + \frac{1}{2} \left| \frac{3}{2} \right| - \frac{\Delta}{2 \beta_2} (\alpha - \mu_\Delta) \right), \quad (3.15b) \]
where
\[ \mu_\Delta \equiv (2 \beta_2 \beta_5 - \beta_1 \beta_4)/\Delta^2, \quad (3.16) \]
\[ d \equiv \frac{1}{2 \Delta} \left( \beta_2 \frac{\sigma^2}{\Delta^2} - \beta_4 \frac{\sigma}{\Delta} + \frac{\Delta - \beta_1}{2} - \lambda \right), \quad (3.17) \]
and \(_1F_1(d|c|x)\) is the confluent hypergeometric function (the Kummer function). Note that the function \(_1F_1(d|c|x)\) with \(c = 1/2\) or \(c = 3/2\) can be expressed in terms of the parabolic cylinder functions \(D_\nu(\pm x)\) by using the relation \([39]\)
\[ D_\nu(\pm x) = \sqrt{\pi} 2^{\nu/2} e^{-x^2/4} \left[ \frac{1}{\Gamma \left( \frac{1-\nu}{2} \right)} _1F_1 \left( -\frac{\nu}{2} \left| \frac{x^2}{2} \right| \right) \pm \frac{\sqrt{2} x}{\Gamma \left( -\frac{\nu}{2} \right)} _1F_1 \left( \frac{1-\nu}{2} \left| \frac{3}{2} \right| \frac{x^2}{2} \right) \right]. \quad (3.18) \]
The function \( \Lambda(\lambda, \beta; \alpha) \) is manifestly analytic, and the normalization condition (3.5) requires

\[
|\frac{\Delta \pm \beta_1}{2\beta_2}| < 1.
\] (3.19)

The sign ‘−’ in Eq. (3.19) must be taken when \( d \left( d + \frac{1}{2} \right) \) is a nonpositive integer and the sign ‘+’ otherwise.

The physical meaning of the two solutions can be understood by considering the particular case \( \beta_4 = \beta_5 = 0 \) when one-photon processes are excluded. Then \( \sigma = 0 \), and \( \mu_\Delta = 0 \), so \( T_1(\lambda, \beta; \alpha) \) contains only even powers of \( \alpha \) and \( T_2(\lambda, \beta; \alpha) \) contains only odd powers of \( \alpha \). If we recall that the operators \( \{N, a^2, a^{12}\} \) form a realization of the SU(1,1) Lie algebra, it will be clear that the two solutions represent the AES in the two irreducible sectors of SU(1,1).

One-photon processes represented by \( a \) and \( a^{\dagger} \) mix these irreducible sectors, and then the total solution is given by a superposition of \( T_1 \) and \( T_2 \).

In the degenerate case \( \Delta = 0 \). Provided that \( \beta_2 \neq 0, \sigma \neq 0 \), a solution of Eq. (3.12) is given by

\[
T(\lambda, \beta; \alpha) = \exp \left( -\frac{\beta_4}{2\beta_2} \alpha \right) \sqrt{\alpha - \mu_0} J_{1/3} \left( 2 \frac{\sigma}{\beta_2} (\alpha - \mu_0)^{3/2} \right),
\] (3.20)

where

\[
\mu_0 = \frac{4\beta_2 \lambda + 2\beta_1 \beta_2 + \beta_4^2}{4\beta_2 \sigma},
\] (3.21)

and \( J_\nu(x) \) is the Bessel function of the first kind. Another independent solution includes \( J_{-\nu}(x) \) instead of \( J_\nu(x) \) (for a noninteger \( \nu \)). The solution can be also expressed in terms of the Airy functions \[38\]:

\[
\text{Ai}(-x) = \frac{\sqrt{x}}{3} \left[ J_{1/3} \left( \frac{2}{3} x^{3/2} \right) + J_{-1/3} \left( \frac{2}{3} x^{3/2} \right) \right],
\] (3.22a)

\[
\text{Bi}(-x) = \frac{\sqrt{x}}{3} \left[ J_{-1/3} \left( \frac{2}{3} x^{3/2} \right) - J_{1/3} \left( \frac{2}{3} x^{3/2} \right) \right].
\] (3.22b)

The function \( \Lambda(\lambda, \beta; \alpha) \) is manifestly analytic, and the normalization condition (3.5) requires \(|\beta_1/\beta_2| < 2\).

When, apart from \( \Delta = 0 \), also \( \sigma = 0 \), then Eq. (3.12) becomes an equation with constant coefficients. The solution of this equation can be written in terms of elementary functions:

\[
T(\alpha) = C_+ \exp(\omega_+ \alpha) + C_- \exp(\omega_- \alpha),
\] (3.23)

where \( C_\pm \) are the integration constants, and

\[
\omega_\pm = \frac{1}{2\beta_2} \left( -\beta_4 \pm \sqrt{\beta_4^2 + 4\beta_2 \lambda + 2\beta_2 \beta_2} \right).
\] (3.24)

In the case \( \beta_2 = 0, \beta_1 \neq 0 \), the eigenvalue equation (3.10) is a first-order differential equation whose solution is easily found to be
\[ \Lambda(\lambda, \beta; \alpha) = \Lambda_0 \left( \alpha + \frac{\beta_4}{\beta_1} \right)^p \exp \left( -\frac{\beta_5}{\beta_1} \alpha^2 + \frac{\beta_3 \beta_4 - \beta_1 \beta_5}{\beta_1^2} \alpha \right), \] (3.25)

where

\[ p = \frac{[\beta_1^2 \lambda - \beta_4(\beta_3 \beta_4 - \beta_1 \beta_5)]}{\beta_1^3} \] (3.26)

must be a non-negative integer in order to satisfy the analyticity condition. \( \Lambda_0 \) is a normalization factor, and the normalization condition \( [3.3] \) requires \( |\beta_3/\beta_1| < 1 \).

When \( \beta_2 = 0 \) and \( \beta_3 = 0 \), the resulting AES are associated with the oscillator group \( H_4 \). The corresponding analytic function is

\[ \Lambda(\lambda, \beta; \alpha) = \Lambda_0 \left( \alpha + \frac{\beta_4}{\beta_1} \right)^p \exp \left( -\frac{\beta_5}{\beta_1} \alpha \right), \] (3.27)

where

\[ p = \frac{(\beta_1 \lambda + \beta_4 \beta_5)}{\beta_1^2} \] (3.28)

is once again a non-negative integer. The function \( \Lambda(\lambda, \beta; \alpha) \) of Eq. (3.27) represents displaced Fock states [10]. In order to derive the corresponding eigenvalue equation, we start from the equation \( N|n\rangle = n|n\rangle \). By applying the unitary displacement operator \( D(\nu) = \exp(\nu a^\dagger - \nu^* a) \) to this equation, we obtain

\[ (N - \nu^* a - \nu a^\dagger)|n, \nu\rangle = (n - |\nu|^2)|n, \nu\rangle, \] (3.29)

where \( |n, \nu\rangle = D(\nu)|n\rangle \) is the displaced Fock state that reduces to the standard Glauber state for \( n = 0 \). The corresponding analytic function is given by Eq. (3.27). By substituting

\[ \beta_1 = 1, \quad \beta_5 = \beta_4^* = -\nu, \quad \lambda = n - |\nu|^2, \] (3.30)

we find \( p = n \), and

\[ \Lambda(n, \nu; \alpha) = \Lambda_0 (\alpha - \nu^*)^n e^{\nu \alpha}. \] (3.31)

For \( n = 0 \), this function reduces to the function \( \mathcal{F}(\nu; \alpha) \) representing the Glauber CS \( |\nu\rangle \) [cf. Eq. (3.6)]. The normalization factor in this case is \( \Lambda_0 = \exp(-|\nu|^2/2) \). A consequence of Eq. (3.29) is the following equation satisfied by the Glauber CS

\[ (N - \nu^* a - \nu a^\dagger + |\nu|^2)|\nu\rangle = 0. \] (3.32)

By using the Glauber definition \( a|\nu\rangle = \nu|\nu\rangle \), we see that Eq. (3.32) is an identity.

We also consider another example of displaced states. In the case \( \beta_1 = \beta_2 = \beta_3 = 0, \beta_4 \neq 0 \), the resulting AES are associated with the \( H_3 \) group. Then the solution of the eigenvalue equation \( (3.10) \) is

\[ \Lambda(\lambda, \beta; \alpha) = \Lambda_0 \exp \left( -\frac{\beta_5}{2 \beta_4} \alpha^2 + \frac{\lambda}{\beta_4} \alpha \right). \] (3.33)
We see that the analyticity condition is fulfilled. Besides, the normalization condition \( \beta_\text{5}/\beta_\text{4} < 1 \) requires \( |\beta_\text{5}|/\beta_\text{4} < 1 \). By comparing the function \( \Lambda(\lambda, \beta; \alpha) \) of Eq. (3.33) with the function \( \mathcal{F}(\nu; \alpha) \) of Eq. (3.9), we find that the algebra eigenstate \( |\lambda, \beta\rangle \) coincides with the Glauber coherent state \( |\nu\rangle \) for \( \beta_i = 0, i = 1, 2, 3, 5 \). Then \( \nu = \lambda/\beta_\text{4} \), and Eq. (3.8) reduces to the famous Glauber equation \( a|\nu\rangle = \nu|\nu\rangle \). We see that the eigenvalue equation for a state (e.g., for the standard coherent state) can be written in a number of ways, i.e., there is a number of equivalent definitions of the state. Note also that in the case \( \beta_i = 0, i = 1, 2, 3, 4, \beta_\text{5} \neq 0 \), Eq. (3.8) has not any nontrivial solution. The reason is that the creation operator \( a^\dagger \) has not any eigenstate.

The Gaussian form of the function \( \Lambda(\lambda, \beta; \alpha) \) of Eq. (3.33) means that this function represents displaced (canonical) squeezed states of Stoler and Yuen [26]. These states are generated by the action of the squeezing and displacement operators on the vacuum [26],

\[
|\xi, \nu\rangle = D(\nu)S(\xi)|0\rangle,
\]  

(3.34)

where the squeezing operator is

\[
S(\xi) = \exp \left( \frac{1}{2} \xi a^\dagger a - \frac{1}{2} \xi^* a a^\dagger \right).
\]  

(3.35)

By applying the squeezing operator \( S(\xi = s e^{i\theta}) \) to the equation \( a|0\rangle = 0 \), one derives the equation satisfied by the squeezed vacuum \( |\xi\rangle = S(\xi)|0\rangle \),

\[
[(\cosh s) a - (\sinh s e^{i\theta}) a^\dagger]|\xi\rangle = 0.
\]  

(3.36)

By applying the displacement operator \( D(\nu) \) to this equation, one finds the eigenvalue equation satisfied by the displaced squeezed state \( |\xi, \nu\rangle \),

\[
(a - \zeta a^\dagger)|\xi, \nu\rangle = (\nu - \zeta \nu^*)|\xi, \nu\rangle,
\]  

(3.37)

where

\[
\zeta \equiv \frac{\xi}{|\xi|} \tanh |\xi| = \tanh s e^{i\theta}.
\]  

(3.38)

By substituting

\[
\beta_\text{4} = 1, \quad \beta_\text{5} = -\zeta, \quad \lambda = \nu - \zeta \nu^*
\]  

(3.39)

into Eq. (3.33), one obtains the analytic function representing the displaced squeezed states,

\[
\Lambda(\xi, \nu; \alpha) = \Lambda_0 \exp \left[ \frac{1}{2} \zeta \alpha^2 + (\nu - \zeta \nu^*) \alpha \right].
\]  

(3.40)

The normalization factor in this case is [26]

\[
\Lambda_0 = \frac{\exp(-\frac{1}{2} |\xi|^2 - \zeta^* u^2)}{\sqrt{\cosh s}},
\]  

(3.41)

where
It is interesting to note that the displaced squeezed states $|\xi, \upsilon\rangle$ are the standard CS of the group $H_6$ but simultaneously they are nonstandard CS of its subgroup $H_3$. The reference state of this nonstandard set is the squeezed vacuum $|\xi\rangle$. The displaced squeezed states $|\xi, \upsilon\rangle$ are also the generalized IS for the quadratures $X_1$ and $X_2$ that are the Hermitian generators of $H_3$. By putting $a = X_1 + iX_2$ and $a^\dagger = X_1 - iX_2$ in the eigenvalue equation (3.37), one obtains the equation of type (2.15):

\[
\left[ \frac{1 - \zeta}{1 + \zeta} \right] X_1 + iX_2 |\xi, \upsilon\rangle = \left( \frac{\upsilon - \zeta \upsilon^*}{1 + \zeta} \right) |\xi, \upsilon\rangle.
\] (3.43)

(The $X_1$-$X_2$ generalized IS also are known as “correlated coherent states” [41]). For $\theta = 0$ and $\theta = \pi$, $\zeta$ is real and the $|\xi, \upsilon\rangle$ states are the ordinary IS, i.e., they provide an equality in the uncertainty relation $\Delta X_1 \Delta X_2 \geq 1/4$. The Glauber CS $|\upsilon\rangle$ form the zero-squeezing subset of the $X_1$-$X_2$ IS.

### IV. DISPLACED AND SQUEEZED FOCK STATES

The differential equation (3.10) determines analytic functions representing various photon states that can be produced by squeezing and displacement of an initial state. The first candidate to be the initial state is the vacuum. Recently, have been considerable interest in attempts to produce Fock states (photon number eigenstates) $|n\rangle$ with nonzero occupation number [42–47]. Given that Fock states can be generated, it is natural to consider their displacement (by driving the light field by a classical current) and squeezing (by degenerate parametric amplification). Properties of displaced Fock states [40,48–51], squeezed Fock states [52,48,53], and displaced and squeezed Fock states (DSFS) [54,55] have been widely discussed. In this section we consider the DSFS as a characteristic example of the two-photon AES. The general results of the preceding section are used to obtain the Fock-Bargmann analytic representation of the DSFS.

We start from the equation $N|n\rangle = n|n\rangle$. By acting on both sides of this equation with the squeezing operator $S(\xi = se^{i\theta})$, we derive the eigenvalue equation satisfied by the squeezed Fock states $|n, \xi\rangle = S(\xi)|n\rangle$,

\[
(\beta_1 N + \beta_2 a^2 + \beta_3 a^2)|n, \xi\rangle = (n - \sinh^2 s)|n, \xi\rangle,
\] (4.1)

where

\[
\beta_1 = \cosh 2s, \quad \beta_2 = \beta_3^* = -\frac{1}{2} \sinh 2s e^{-i\theta}.
\] (4.2)

Then we apply the displacement operator $D(\upsilon = re^{i\phi})$. The resulting eigenvalue equation reads

\[
(\beta_1 N + \beta_2 a^2 + \beta_3 a^2 + \beta_4 a + \beta_5 a^\dagger)|n, \xi, \upsilon\rangle = \lambda|n, \xi, \upsilon\rangle,
\] (4.3)

where
\[ |n, \xi, v\rangle = D(v)S(\xi)|n\rangle \] (4.4)

are the DSFS. The parameters \( \beta_1, \beta_2 \) and \( \beta_3 \) remain as given above, and
\[
\begin{align*}
\beta_4 &= \beta_5^* = v^* \left( \sinh 2s e^{-i(\theta - 2\phi)} - \cosh 2s \right), \\
\lambda &= n - \sinh^2 s + v^2 \left[ \sinh 2s \cos(\theta - 2\phi) - \cosh 2s \right].
\end{align*}
\] (4.5)

These results can be easily derived by using the general recipe
\[
D(\upsilon)F(a, a^\dagger)D^{-1}(\upsilon) = F(a - \upsilon, a^\dagger - \upsilon^*),
\] (4.7)

where \( F(a, a^\dagger) \) is a power series.

Equation (4.3) is of the general form (3.8) and the corresponding differential equation is of the form (3.10) with solutions given by Eqs. (3.11) and (3.15). A simple calculation yields
\[
\Delta^2 = \beta_1^2 - 4\beta_2\beta_3 = 1,
\] (4.8)

which is a direct consequence of the unitarity of the squeezing operator \( S(\xi) \). Then \( \Delta = \pm 1 \), and we find, respectively,
\[
\sigma = \pm v \left[ \left( \frac{\sinh s}{\cosh s} \right)^{\pm 1} e^{i(\theta - 2\phi)} - 1 \right], \quad \mu_\Delta = v^*, \quad d = \mp \frac{1}{2} \left( n + \frac{1}{2} \mp \frac{1}{2} \right). \] (4.9)

Let us start from \( \Delta = +1 \). Then \( d = -n/2 \), and the normalization condition (3.19) is satisfied by taking \( T_1(\lambda, \beta; \alpha) \) for even values of \( n \) and \( T_2(\lambda, \beta; \alpha) \) for odd values of \( n \). This result is dictated by the fact that the analytic function representing the squeezed Fock states \( |n, \xi \rangle \) contains only even powers of \( \alpha \) for even \( n \) and only odd powers of \( \alpha \) for odd \( n \). By using the relations between the confluent hypergeometric functions and the Hermite polynomials,
\[
\begin{align*}
_{1}F_1 \left( -m \left| \frac{1}{2} \right| x^2 \right) &= \frac{(-1)^m m! H_{2m}(x)}{(2m)!}, \\
_{1}F_1 \left( -m \left| \frac{1}{2} \right| x^2 \right) &= \frac{(-1)^m m! H_{2m+1}(x)}{2(2m+1)!},
\end{align*}
\] (4.10a, 4.10b)

we find the solution:
\[
\Lambda(n, \xi, v; \alpha) = e^{\frac{\lvert \alpha \rvert^2}{2}} \langle \alpha^* | n, \xi, v \rangle = \Lambda_0(n, \xi, v) \exp \left[ \frac{\zeta}{2} \alpha^2 + (v - \zeta v^*) \alpha \right] H_n \left( \frac{\alpha - v^*}{\sqrt{\sinh 2s e^{-i\theta}}} \right). \] (4.11)

As usual, \( \Lambda_0 \) is a normalization factor, and \( \zeta \) is defined by Eq. (3.38). This result is in accordance with the expression for \( \langle \alpha | S(\xi)D(v)|n\rangle \) derived in a different way by Král [54].

The normalization factor is identified to be
\[
\Lambda_0(n, \xi, v) = \frac{(\zeta^*/2)^{n/2}}{\sqrt{n! \cosh s}} \exp \left( -\frac{1}{2} |u|^2 - \zeta^* u^2 \right), \] (4.12)
where $u$ is defined by Eq. (3.42).

It is well known [38] that the confluent hypergeometric function can be written in two equivalent forms which are related by Kummer’s transformation

$$
\text{iF}_1(d|c|x) = e^x \text{iF}_1(c-d|c|\text{e}^{-x}).
$$

(4.13)

It is not difficult to see that the choice $\Delta = -1$ leads to the solution which is related to the function $\Lambda(n, \xi, \nu; \alpha)$ of Eq. (4.11) by Kummer’s transformation (4.13). Then the solution can be written in the form

$$
\Lambda_1(n, \xi, \nu; \alpha) = \Lambda_0^{(1)} \exp \left[ \frac{\alpha^2}{2\xi^*} + (\nu - \nu^*/\xi^*)\alpha \right] \text{iF}_1 \left( \frac{n + 1}{2} \left[ \frac{1}{2} \left| -\frac{(\alpha - \nu^*)^2}{\sinh 2s \text{e}^{-i\theta}} \right| \right] \right)
$$

(4.14a)

for even values of $n$ and

$$
\Lambda_2(n, \xi, \nu; \alpha) = \Lambda_0^{(2)} \exp \left[ \frac{\alpha^2}{2\xi^*} + (\nu - \nu^*/\xi^*)\alpha \right] (\alpha - \nu^*) \text{iF}_1 \left( \frac{n + 2}{2} \left[ \frac{3}{2} \left| -\frac{(\alpha - \nu^*)^2}{\sinh 2s \text{e}^{-i\theta}} \right| \right] \right)
$$

(4.14b)

for odd values of $n$. As usual, $\Lambda_0$ are appropriate normalization factors, and the normalization condition (3.19) is obviously satisfied.

In the particular case $n = 0$, the function $\Lambda(n, \xi, \nu; \alpha)$ given by Eq. (4.11) reduces to the function (3.40) representing the displaced squeezed states. The analytic function representing the squeezed Fock states $|n, \xi\rangle$ is obtained by putting $\nu = 0$ in Eq. (4.11) or in Eqs. (4.14). The displaced Fock states $|n, \nu\rangle$ were discussed in the preceding section and the corresponding analytic function is given by Eq. (3.31).

We finish this section by a short review of basic methods for producing the DSFS. Displacement can be implemented by linear amplification of the light field. A usual method for doing that is by driving the field by a classical current. The use of a linear directional coupler as a displacing device was also discussed [54]. The most frequently used squeezing device in the single-mode case is a degenerate parametric amplifier. These methods of displacement and squeezing are well developed and the main problem remaining is the production of a stable Fock state that will serve as the input state of displacing and squeezing devices. It was demonstrated that it is possible to generate a Fock state of the single-mode electromagnetic field in a micromaser operated under the appropriate conditions [42,43]. Another interesting method for producing Fock states is based on the process of parametric down-conversion in which one pump photon is destroyed and two correlated photons are simultaneously created, one in each of two distinct modes. The state of one mode is then conditioned on the detection of photons in the other mode [44]. It was also shown that a Fock state can be generated by observation of quantum jumps in an ion trap [45], by coupling a cavity to a single three-level atom in a Raman lambda configuration [46], and by using the single-atom interference [47].

V. SQUEEZING AND DISPLACEMENT OF COHERENT SUPERPOSITION STATES

In this section we will consider squeezed and displaced superpositions of the Glauber CS $|\nu\rangle$ and $|-\nu\rangle$, which provide an interesting example of the two-photon AES. These
states belong to the wide class of macroscopic quantum superpositions which are frequently referred to as the Schrödinger-cat states \cite{57}. Properties of different types of the Schrödinger-cat states have been recently studied in a number of works \cite{58,7,29,58– 66}. The problem of the generation of optical superposition states have drawn recently a lot of attention \cite{58,67–75,43,76–78}. It was shown that the Schrödinger-cat states can be produced in various nonlinear processes \cite{58,67–72}, in field-atom interactions \cite{73–75,43,76,77}, and in quantum nondemolition measurements \cite{78}.

We start from the coherent superposition state of the form

\[
|v, \tau, \varphi\rangle = {\cal N} (|v\rangle + \tau e^{i\varphi}| - v\rangle),
\]

where \(|v\rangle\) and \(|-v\rangle\) are the standard Glauber CS, \(\tau\) and \(\varphi\) are real parameters, and

\[
{\cal N} = \left(1 + \tau^2 + 2\tau e^{-2|v|^2} \cos \varphi\right)^{-1/2}
\]

is the normalization factor. The analytic function

\[
\Lambda(v, \tau, \varphi; \alpha) = e^{\alpha^2/2} \langle \alpha^* | v, \tau, \varphi \rangle
\]

can be straightforwardly calculated:

\[
\Lambda(v, \tau, \varphi; \alpha) = {\cal N} e^{-|v|^2/2} \left(e^{\alpha \varphi} + \tau e^{i\varphi} e^{-\alpha \varphi}\right).
\]

The superposition \(|v, \tau, \varphi\rangle\) is a special kind of the two-photon AES since it is the eigenstate of the operator \(a^2\):

\[
a^2|v, \tau, \varphi\rangle = v^2|v, \tau, \varphi\rangle.
\]

In the case \(\tau = 0\), this state reduces to the Glauber coherent state \(|v\rangle\).

Interesting superpositions are even and odd CS \(|v\rangle_e\) and \(|v\rangle_o\) \cite{7}:

\[
|v\rangle_e = |v, \tau = 1, \varphi = 0\rangle = \frac{|v\rangle + |-v\rangle}{\sqrt{2 (1 + e^{-2|v|^2})}},
\]

\[
|v\rangle_o = |v, \tau = 1, \varphi = \pi\rangle = \frac{|v\rangle - |-v\rangle}{\sqrt{2 (1 - e^{-2|v|^2})}}.
\]

The even and odd CS have a number of interesting nonclassical properties. The even CS are highly squeezed in the field quadrature \(X_2\), while the odd CS have sub-Poissonian photon statistics \cite{59,60}. Multimode versions of the even and odd CS have been recently studied \cite{79}.

In the case \(\tau = 1, \varphi = \pi/2\), one obtains the so-called Yurke-Stoler state

\[
|v\rangle_{YS} = \frac{1}{\sqrt{2}} \left(|v\rangle + i|-v\rangle\right),
\]

that can be generated when the Glauber state \(|v\rangle\) propagates through a nonlinear Kerr medium \cite{58}. The \(|v\rangle_{YS}\) states are squeezed in the \(X_2\) field quadrature \cite{50}.
It follows from the eigenvalue equation (5.5) that the superpositions $|\nu, \tau, \varphi\rangle$ are a special case of the two-photon IS. More precisely, let us consider the two-photon realization of the SU(1,1) Lie algebra:

$$K_+ = \frac{1}{2}a^{12}, \quad K_- = \frac{1}{2}a^2, \quad K_0 = \frac{1}{2}N + \frac{1}{4},$$ \hspace{1cm} (5.8)

$$[K_-, K_+] = 2K_0, \quad [K_0, K_\pm] = \pm K_\pm.$$ \hspace{1cm} (5.9)

It is clear that SU(1,1) $\subset H_6$. One can use the Hermitian combinations

$$K_1 = \frac{1}{2}(K_+ + K_-) = \frac{1}{4}(a^{12} + a^2),$$

$$K_2 = \frac{1}{2i}(K_+ - K_-) = \frac{1}{4i}(a^{12} - a^2),$$ \hspace{1cm} (5.10)

which satisfy the commutation relation $[K_1, K_2] = -iK_0$. According to the general formalism of section II D, the $|\nu, \tau, \varphi\rangle$ states are the $K_1$-$K_2$ IS, i.e., they provide an equality in the uncertainty relation

$$(\Delta K_1)^2(\Delta K_2)^2 \geq \frac{1}{4}(K_0)^2.$$ \hspace{1cm} (5.11)

Indeed, a simple calculation yields

$$(\Delta K_1)^2 = (\Delta K_2)^2 = \frac{1}{2}\langle K_0 \rangle = \frac{|\nu|^2(1 + \tau^2 - 2\tau e^{-2|\nu|^2} \cos \varphi)}{4} + \frac{1}{8},$$ \hspace{1cm} (5.12)

when the expectation values are calculated for the superpositions $|\nu, \tau, \varphi\rangle$.

Now, let us recall that the Barut-Girardello states are defined as the eigenstates of the SU(1,1) lowering generator $K_-$. For each unitary irreducible representation of SU(1,1), there is a set of the Barut-Girardello states. In the case of the two-photon realization (5.8), there are two irreducible representations and the two irreducible sectors are spanned by the Fock states $|n\rangle$ with even and odd values of $n$, respectively. The two sets of the Barut-Girardello states are the even and odd CS $|\nu\rangle_e$ and $|\nu\rangle_o$. Their intelligent properties were first recognized by Hillery [29].

Nonclassical properties of displaced even and odd CS were briefly discussed by Xia and Guo [34]. Squeezed coherent superpositions were considered recently by Hach and Gerry [64] and by Xin et al. [65]. We will use the algebra-eigenstate method developed above in order to obtain the Fock-Bargmann analytic representation of the squeezed and displaced superpositions. By applying the squeezing operator $S(\xi = se^{i\theta})$ to Eq. (5.5), we find that the squeezed superpositions

$$|\nu, \tau, \varphi, \xi\rangle = S(\xi)|\nu, \tau, \varphi\rangle$$ \hspace{1cm} (5.13)

satisfy the following eigenvalue equation

$$a^2_\xi|\nu, \tau, \varphi, \xi\rangle = \nu^2|\nu, \tau, \varphi, \xi\rangle,$$ \hspace{1cm} (5.14)
where

\[ a_\xi = S(\xi)aS^{-1}(\xi) = (\cosh s)a - (\sinh s e^{i\theta})a^\dagger. \]  

Equation (5.14) can be written in the standard form (3.8):

\[ (-2\zeta N + a^2 + \zeta^2 a^{2\dagger})|v, \tau, \varphi, \xi\rangle = [v^2(1 - |\xi|^2) + \zeta]|v, \tau, \varphi, \xi\rangle, \]  

where \( \zeta = \tanh se^{i\theta} \) is defined by Eq. (3.38). Now, we apply the displacement operator \( D(z) \). The resulting eigenvalue equation is

\[ a_{\xi,z}^2|v, \tau, \varphi, \xi, z\rangle = v^2|v, \tau, \varphi, \xi, z\rangle, \]  

where

\[ |v, \tau, \varphi, \xi, z\rangle = D(z)S(\xi)|v, \tau, \varphi\rangle \]  

is the displaced and squeezed superposition, and

\[ a_{\xi,z} = D(z)S(\xi)aS^{-1}(\xi)D^{-1}(z) = (\cosh s)(a - z) - (\sinh s e^{i\theta})(a^\dagger - z^*). \]  

The standard form (3.8) of the eigenvalue equation is

\[ (-2\zeta N + a^2 + \zeta^2 a^{2\dagger} - 2\rho a + 2\xi \rho a^\dagger)|v, \tau, \varphi, \xi, z\rangle = [v^2(1 - |\xi|^2) + \zeta - \rho^2]|v, \tau, \varphi, \xi, z\rangle, \]  

where we have defined

\[ \rho \equiv z - \zeta z^*. \]  

A simple calculation yields

\[ \Delta^2 = \beta_1^2 - 4\beta_2\beta_3 = 0, \quad \sigma = \beta_4 \frac{\Delta - \beta_1}{2\beta_2} + \beta_5 = 0. \]  

The solution in this case is given by Eq. (3.23). The analytic function

\[ \Lambda(v, \tau, \varphi, \xi, z; \alpha) = e^{i|\alpha|^2/2}(\alpha^*|v, \tau, \varphi, \xi, z\rangle \]  

is then given by

\[ \Lambda(v, \tau, \varphi, \xi, z; \alpha) = \exp\left(\frac{\zeta}{2}\alpha^2 + \rho\alpha\right)\left[C_+ \exp\left(\frac{v\alpha}{\cosh s}\right) + C_- \exp\left(-\frac{v\alpha}{\cosh s}\right)\right]. \]  

This function is manifestly analytic and normalizable due to the condition \( |\zeta| < 1 \). By putting \( \rho = 0 \) in Eq. (5.22), we obtain the function that represents squeezed superpositions \( |v, \tau, \varphi, \xi\rangle \). The case of zero squeezing is also included in Eq. (5.24). By putting there \( \zeta = 0 \), we find the function that represents displaced superpositions \( |v, \tau, \varphi, z\rangle \).
By comparing the function $\Lambda(\nu, \tau, \varphi, \xi, z; \alpha)$ of Eq. (5.24) with the function $\Lambda(\nu, \tau, \varphi; \alpha)$ of Eq. (5.4), we deduce that $C^- = \tau e^{i\varphi} C^+$, and $C^+ = N e^{-|\nu|^2/2}$ for $\zeta = z = 0$. By using the generating function for the Hermite polynomials

$$e^{2tx - x^2} = \sum_{n=0}^{\infty} H_n(t) \frac{x^n}{n!}, \quad (5.25)$$

we expand the function $\Lambda(\nu, \tau, \varphi, \xi, z; \alpha)$ of Eq. (5.24) into the power series in $\alpha$ and obtain the Fock-state expansion of the displaced and squeezed superpositions:

$$|\nu, \tau, \varphi, \xi, z\rangle = C^+ \sum_{n=0}^{\infty} \frac{(-\zeta/2)^{n/2}}{\sqrt{n!}} H_n \left( \frac{u + v}{\kappa} \right) + \tau e^{i\varphi} H_n \left( \frac{u - v}{\kappa} \right) |n\rangle, \quad (5.26)$$

where we have defined

$$u \equiv \rho \cosh s = \frac{z - \zeta z^*}{\sqrt{1 - |\zeta|^2}}, \quad (5.27)$$

$$\kappa \equiv i \sqrt{2 \cosh s} = i \sqrt{\sinh 2s e^{i\theta}}. \quad (5.28)$$

By using the summation theorem for Hermite polynomials [38], we readily find the normalization factor:

$$C^{-2}_+ = \exp \left\{ |u|^2 + |v|^2 + \text{Re} \left[ \zeta^*(u^2 + v^2) \right] \right\} \frac{e^{2\text{Re} y + \tau^2 e^{-2\text{Re} y} + 2\tau e^{-2|\nu|^2} \cos (\varphi - 2 \text{Im} y)}}{\sqrt{1 - |\zeta|^2}}, \quad (5.29)$$

where

$$y \equiv u^* v + \zeta^* u v = \frac{v z^*}{\cosh s}. \quad (5.30)$$

All the properties of the displaced and squeezed superpositions $|\nu, \tau, \varphi, \xi, z\rangle$ can be calculated by using the analytic function $\Lambda(\nu, \tau, \varphi, \xi, z; \alpha)$ of Eq. (5.24) or the Fock-state expansion of Eq. (5.26).

**VI. SQUEEZING AND DISPLACEMENT OF THE SU(1,1) INTELLIGENT STATES**

Recently, Nieto and Truax [15] proposed a generalization of squeezed states for an arbitrary dynamical symmetry group. They found that the generalized squeezed states are eigenstates of a linear combination of the lowering and raising generators of a group. Actually, these states are the IS for the group Hermitian generators. Connections between the concepts of squeezing and intelligence were further investigated by Trifonov [18]. It turns out that the IS for two Hermitian generators can provide an arbitrarily strong squeezing in either of these observables [18]. In the simplest case of the Heisenberg-Weyl group $H_3$, the quadrature IS determined by the eigenvalue equation (3.43) are the canonical squeezed
states $|\xi, \nu\rangle$ of Stoler and Yuen \cite{26}. By considering the $K_1K_2$ IS, one can generalize the concept of squeezing to the SU(1,1) group \cite{13,17,18}. On the other hand, the usual squeezed vacuum states $|\xi\rangle$ are the generalized CS of SU(1,1). The algebra-eigenstate method enables to treat both the generalized CS and the generalized squeezed states (i.e., the IS) for an arbitrary Lie group in a unified way. Since the SU(1,1) Lie group in the two-photon realization (5.8) is a subgroup of $H_6$, the SU(1,1) IS are a particular case of the two-photon AES. Furthermore, we can consider the states generated by the squeezing transformations $S(\xi)$ and displacement transformations $D(z)$ of the SU(1,1) IS. Such states form a nonstandard set of the generalized two-photon CS.

According to Eq. (2.15), the SU(1,1) IS are determined by the eigenvalue equation

$$ (\eta K_1 - i K_2)|\lambda, \eta\rangle = \lambda|\lambda, \eta\rangle. \quad (6.1) $$

Here $\lambda$ is a complex eigenvalue and the parameter $\eta$ is complex in the general case of the Robertson intelligence [an equality is achieved in Eq. (2.13)] and real in the particular case of the Heisenberg intelligence [an equality is achieved in Eq. (2.14)]. By evaluating the expectation values over the state $|\lambda, \eta\rangle$, one gets \cite{18} (for $\text{Re} \eta \neq 0$)

$$ (\Delta K_1)^2 = \frac{\langle K_0 \rangle}{2 \text{Re} \eta}, \quad (\Delta K_2)^2 = |\eta|^2 \frac{\langle K_0 \rangle}{2 \text{Re} \eta}, \quad \sigma_{12} = \frac{1}{2} \langle K_1K_2 + K_2K_1 \rangle - \langle K_1 \rangle \langle K_2 \rangle = \frac{\text{Im} \eta}{2 \text{Re} \eta} \langle K_0 \rangle. \quad (6.2) $$

In the two-photon realization, the SU(1,1) Hermitian generators $K_1$ and $K_2$ are given by Eq. (5.10). Then the eigenvalue equation (6.1) can be written in the form

$$ \left( \frac{\eta + 1}{4} a^2 + \frac{\eta - 1}{4} a^2 \right) |\lambda, \eta\rangle = \lambda|\lambda, \eta\rangle. \quad (6.3) $$

In the particular case $\eta = 1$, the states $|\lambda, \eta\rangle$ are the eigenstates of the operator $a^2$, i.e., they reduce to the coherent superpositions $|\nu, \tau, \varphi\rangle$ considered in the preceding section. Then, according to Eq. (6.2), the uncertainties of $K_1$ and $K_2$ are equal [cf. Eq. (5.12)]. In more general case of ordinary intelligent states ($\eta$ is real), the states $|\lambda, \eta\rangle$ are squeezed in $K_1$ for $\eta > 1$ and squeezed in $K_2$ for $\eta < 1$.

As usual, we define the entire analytic function

$$ \Lambda(\lambda, \eta; \alpha) = e^{\text{Re} \alpha^2/2} \langle \alpha^* | \lambda, \eta \rangle \quad (6.4) $$

that describes the IS $|\lambda, \eta\rangle$ in the Fock-Bargmann representation. Then Eq. (6.3) becomes a differential equation of the type (3.10). The function $\Lambda(\lambda, \eta; \alpha)$ in this case is given by Eqs. (3.11) and (3.15) with the parameters

$$ \Delta^2 = \frac{1}{4} (1 - \eta^2), \quad \sigma = 0, \quad \mu_\Delta = 0, \quad d = \frac{1}{4} - \frac{\lambda}{2 \Delta}. \quad (6.5) $$

Therefore we obtain
\[
\Lambda_e(\lambda, \eta; \alpha) = A(\alpha) \, _1F_1 \left( \frac{1}{4} - \frac{\lambda}{2\Delta} \left| \frac{1}{2} \right| - \Omega_\eta \alpha^2 \right),
\]
\[
\Lambda_o(\lambda, \eta; \alpha) = \alpha A(\alpha) \, _1F_1 \left( \frac{3}{4} - \frac{\lambda}{2\Delta} \left| \frac{3}{2} \right| - \Omega_\eta \alpha^2 \right),
\]
(6.6a, 6.6b)

where
\[
A(\alpha) \equiv \exp \left( \frac{1}{2} \Omega_\eta \alpha^2 \right),
\]
(6.7)
\[
\Omega_\eta^2 \equiv \frac{\Delta^2}{4\beta_2^2} = \frac{1 - \eta}{1 + \eta}.
\]
(6.8)

The solutions \( \Lambda_e \) and \( \Lambda_o \) represent the states belonging to the SU(1,1) irreducible sectors spanned by the Fock states \(|n\rangle\) with even and odd values of \(n\), respectively. The total solution is given by a superposition of \( \Lambda_e \) and \( \Lambda_o \). Note that the double-valuedness of \( \Delta \) and \( \Omega_\eta \) reflects the invariance of the solution under Kummer’s transformation (4.13). The normalization condition (3.5) requires \(|\Omega_\eta| < 1\) which is satisfied for \(\text{Re} \, \eta > 0\). This is the only restriction on values of \(\eta\). If we express the Kummer functions \(_1F_1 \left( d \left| \frac{1}{2} \right| x \right)\) and \(_1F_1 \left( d + \frac{1}{2} \left| \frac{3}{2} \right| x \right)\) in terms of the parabolic cylinder functions \(D_{-2d}(\pm x)\) by means of Eq. (3.18), we will recover the results of Prakash and Agarwal [17].

An important feature of the algebra-eigenstate method is the possibility to find relations between various types of states. For the SU(1,1) group, the standard set of the generalized CS have an intersection with the set of the ordinary IS [27,20] and is a subset of the wider set of the generalized IS [18]. Let us demonstrate these relations by using the Fock-Bargmann representation of the two-photon AES. The squeezed vacuum states \(|\xi\rangle\) which are the standard CS of SU(1,1) are represented by the function \(\Lambda(\xi, \upsilon; \alpha)\) of Eq. (3.40) with \(\upsilon = 0\), i.e.,
\[
\Lambda(\xi; \alpha) = (1 - |\xi|^2)^{1/4} \exp \left( \frac{1}{2} \zeta \alpha^2 \right).
\]
(6.9)

On the other hand, when
\[
\frac{1}{4} - \frac{\lambda}{2\Delta} = \frac{1}{2},
\]
(6.10)
the formula \(_1F_1(d|d|x) = e^x\) enables us to write Eq. (6.6a) in the (normalized) form
\[
\Lambda_e(\lambda, \eta; \alpha) = (1 - |\Omega_\eta|^2)^{1/4} \exp \left( -\frac{1}{2} \Omega_\eta \alpha^2 \right).
\]
(6.11)

Therefore, the intelligent state \(|\lambda, \eta\rangle\) is the standard coherent state \(|\xi\rangle\) under the condition
\[
\lambda = -\Delta/2 = \pm \frac{1}{4} \sqrt{1 - \eta^2}.
\]
(6.12)
The corresponding coherent-state amplitude is
\[
\zeta = -\Omega_\eta = \pm \sqrt{\frac{1 - \eta}{1 + \eta}}.
\]
(6.13)
The condition $|\zeta| < 1$ is guaranteed by virtue of the normalization requirement $|\Omega_{\eta}| < 1$ ($\text{Re}\eta > 0$). When $\eta$ is complex (the case of the generalized IS), $\zeta$ can acquire any value in the unit disk. It means that the standard CS form a subset of the generalized IS. However, when $\eta$ is real (the case of the ordinary IS), $\zeta$ is real for $\eta < 1$ and pure imaginary for $\eta > 1$. It means that the standard set of the generalized CS has an intersection with the set of the ordinary IS. The standard CS are the ordinary IS squeezed in $K_2$ for real $\zeta$ and squeezed in $K_1$ for pure imaginary $\zeta$.

Now, let us consider the action of the squeezing operator $S(\xi = se^{i\theta})$. By applying $S(\xi)$ to Eq. (6.3), we obtain the eigenvalue equation

$$\left(\frac{\eta + 1}{4}a_\xi^2 + \frac{\eta - 1}{4}a_\xi^{+2}\right)|\lambda, \eta, \xi\rangle = \lambda|\lambda, \eta, \xi\rangle$$

(6.14)

satisfied by the squeezed IS

$$|\lambda, \eta, \xi\rangle = S(\xi)|\lambda, \eta\rangle.$$  

(6.15)

The operator $a_\xi$ is given by Eq. (5.15) and $a_\xi^{+}$ is its Hermitian conjugate. Equation (6.14) can be written in the standard form:

$$(\beta_1 N + \beta_2 a^2 + \beta_3 a^2)|\lambda, \eta, \xi\rangle = \lambda|\lambda, \eta, \xi\rangle,$$

(6.16)

where

$$\beta_1 = -2\zeta(\eta + 1) - 2\zeta^*(\eta - 1), \quad \beta_2 = (\eta + 1) + \zeta^2(\eta - 1), \quad \beta_3 = \zeta^2(\eta + 1) + (\eta - 1),$$

(6.17)

$$\lambda_\xi = 4(1 - |\zeta|^2)\lambda + \zeta(\eta + 1) + \zeta^*(\eta - 1),$$

(6.18)

and $\zeta = \tanh se^{i\theta}$ is defined by Eq. (3.38). The analytic function $\Lambda(\lambda, \eta, \xi; \alpha)$ representing the squeezed IS is given by Eqs. (3.11) and (3.15) with the parameters

$$\Delta^2 = 4(1 - \eta^2)(1 - |\zeta|^2), \quad \sigma = 0, \quad \mu_\Delta = 0, \quad d = \frac{1}{4} - 2(1 - |\zeta|^2)\lambda/\Delta.$$

(6.19)

The unitary squeezing operator $S(\xi)$ is an element of the SU(1,1) group and, therefore, Eq. (6.14) does not include the first-order operators $a$ and $a^+$ which represent one-photon processes. Then the total solution $\Lambda(\lambda, \eta, \xi; \alpha)$ can be written as a superposition of two solutions $\Lambda_e$ and $\Lambda_o$ which represent the two irreducible sectors of SU(1,1).

At the next step, we apply the displacement operator $D(z)$. The resulting eigenvalue equation is

$$\left(\frac{\eta + 1}{4}a^2_{\xi,z} + \frac{\eta - 1}{4}a^{+2}_{\xi,z}\right)|\lambda, \eta, \xi, z\rangle = \lambda|\lambda, \eta, \xi, z\rangle,$$

(6.20)

where

$$|\lambda, \eta, \xi, z\rangle = D(z)S(\xi)|\lambda, \eta\rangle.$$  

(6.21)
are the displaced and squeezed IS. The operator \(a_{\xi,z}\) is given by Eq. (5.19) and \(a_{\xi,z}^\dagger\) is its Hermitian conjugate. Equation (6.20) can be written in the standard form:

\[
(\beta_1 N + \beta_2 a^2 + \beta_3 a^\dagger a + \beta_4 a + \beta_5 a^\dagger)|\lambda, \eta, \xi, z\rangle = \lambda_{\xi,z}|\lambda, \eta, \xi, z\rangle,
\]

(6.22)

where \(\beta_1, \beta_2\) and \(\beta_3\) remain as given by Eq. (6.17), and

\[
\beta_4 = -2\rho(\eta + 1) + 2\zeta^*\rho^*(\eta - 1), \quad \beta_5 = 2\zeta\rho(\eta + 1) - 2\rho^*(\eta - 1),
\]

(6.23)

\[
\lambda_{\xi,z} = 4(1 - |\zeta|^2)\lambda + (\zeta - \rho^2)(\eta + 1) + (\zeta^* - \rho^*\eta)(\eta - 1).
\]

(6.24)

Here \(\rho = z - \zeta z^*\) is defined by Eq. (5.21). The analytic function \(\Lambda(\lambda, \eta, \xi, z; \alpha)\) representing the displaced and squeezed IS is given by general equations (3.11) and (3.15). The corresponding parameters are

\[
\Delta^2 = 4(1 - \eta^2)(1 - |\zeta|^2),
\]

(6.25)

\[
\sigma = \frac{1}{2}z^*(1 - |\zeta|^2)\Delta^2 + [-\rho(\eta + 1) + \zeta^*\rho^*(\eta - 1)]\Delta
\]

(6.26)

\[
\mu_\Delta = \rho^* + \zeta^*\rho = z^*(1 - |\zeta|^2),
\]

(6.27)

and \(d\) can be found from the general expression (3.17).

The analytic function \(\Lambda(\lambda, \eta, z; \alpha)\) that represents the displaced IS

\[
|\lambda, \eta, z\rangle = D(z)|\lambda, \eta\rangle
\]

(6.28)

can be found by taking \(\zeta = 0\) in the expressions for the displaced and squeezed IS. Then we obtain

\[
\beta_1 = 0, \quad \beta_2 = (\eta + 1), \quad \beta_3 = (\eta - 1),
\]

\[
\beta_4 = -2z(\eta + 1), \quad \beta_5 = -2z^*(\eta - 1),
\]

(6.29)

\[
\lambda_z = 4\lambda - z^2(\eta + 1) - z^*\eta(\eta - 1).
\]

(6.30)

By using these results, we find the usual set of the parameters:

\[
\Delta^2 = 4(1 - \eta^2), \quad \sigma = 2z^*(1 - \eta) - z\Delta, \quad \mu_\Delta = z^*,
\]

\[
d = \frac{1}{4} + [z^2(\eta + 1) - 2\lambda]/\Delta.
\]

(6.31)

Let us finish this discussion by a brief review of possibilities for the generation of the SU(1,1) IS. Gerry and Hach [70] demonstrated a possibility to generate coherent superposition states for the long-time evolution of the competition between two-photon absorption and two-photon parametric processes for a special initial state. This method can be also applied to the production of the SU(1,1) IS more general than the coherent superposition states. Prakash and Agarwal [17] proposed to use the degenerate down-conversion of coherent light in presence of a broadband squeezed field in the cavity (see also Ref. [21] where the same idea was applied to the generation of the two-mode SU(1,1) IS). This method is based on an earlier proposal [80] introduced in the context of the SU(2) group.
VII. CONCLUSIONS

We have shown that almost all photon states known in the context of the two-photon algebra can be considered as the AES. Therefore, the algebra-eigenstate formalism unifies the description of various types of states within a common frame. This helps in understanding of relations between different kinds of states and of the physical basis of their mathematical properties. The theory of the AES is in general applicable to an arbitrary Lie group and will be useful for a unified description of generalized coherence and squeezing in a wide class of quantum systems. In the present work we have concentrated on the basic quantum optical phenomena, such as usual displacement and squeezing of the quantized single-mode light field. The mathematical formulation of these physical processes is based on the two-photon group $H_6$. The corresponding two-photon AES form an extremely wide set that includes as particular cases various types of photon states whose properties have drawn recently considerable attention. The standard CS of Glauber, two-photon CS (canonical squeezed states) of Stoler and Yuen, displaced and squeezed Fock states, displaced and squeezed coherent superpositions, and displaced and squeezed SU(1,1) IS are incorporated into the set of the two-photon AES. The Fock-Bargmann analytic representation of all the particular subsets is obtained from the general differential equation (3.10) that is common for all the kinds of the AES. Then the powerful theory of analytic functions can be used for investigating properties of various types of states and relations between them.

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