ON BOUNDED ELEMENTARY GENERATION FOR $SL_n$ OVER POLYNOMIAL RINGS

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Abstract. Let $\mathbb{F}[X]$ be the polynomial ring over a finite field $\mathbb{F}$. It is shown that, for $n \geq 3$, the special linear group $SL_n(\mathbb{F}[X])$ is boundedly generated by the elementary matrices.

1. Introduction

The special linear group $SL_n(\mathbb{Z})$ is generated by the elementary matrices, that is, matrices which differ from the identity by at most one non-zero off-diagonal entry. Far more remarkable is the following fact:

Theorem 1 (Carter - Keller [6]). Let $n \geq 3$. Then $SL_n(\mathbb{Z})$ is boundedly generated by the elementary matrices.

This means that, for some positive integer $\nu_n$, every matrix in $SL_n(\mathbb{Z})$ is a product of at most $\nu_n$ elementary matrices. The Carter - Keller theorem provides, in fact, the explicit bound $\nu_n = \frac{1}{2}(3n^2 - n) + 36$. See [2] for a variation on the Carter - Keller argument, with a slightly worse bound. In [5], Carter and Keller extend their argument to rings of integers in algebraic number fields.

A different approach to bounded elementary generation for $SL_n$ over rings of integers, based on unpublished work of Carter, Keller, and Paige, can be found in [12]. The novelty is the use of model-theoretic ideas. Unlike the original Carter - Keller approach, this is a non-explicit argument. One proves the existence of a bound on the number of elementary matrices needed to express a matrix in $SL_n$.

Elementary generation of $SL_n$ also holds for polynomial rings over fields. However, bounded elementary generation may fail. In [8] van der Kallen shows, by means of algebraic K-theory, that $SL_n(\mathbb{C}[X])$, $n \geq 2$, is not boundedly generated by the elementary matrices. From an arithmetical viewpoint, however, the closest relative of $\mathbb{Z}$ is a polynomial ring over a finite field. The purpose of this note is to show the following analogue of Theorem 1 which appears to be new (cf., e.g., [11, p.523]).

Theorem 2. Let $\mathbb{F}$ be a finite field, and let $n \geq 3$. Then $SL_n(\mathbb{F}[X])$ is boundedly generated by the elementary matrices.

The proof is an adaptation of the Carter - Keller argument. Here is one technical difference. A crucial role in [6], and also in [2], is played by a ‘power lemma’ [6 Lemma 1] whose origins lie in properties of the so-called Mennicke symbols. We use instead a...
simple ‘swindle lemma’, Lemma 4 herein. A version of this swindle was used in [9, §2.3].

The proof of Theorem 2 yields the explicit bound \( \nu_n = \frac{1}{2}(3n^2 - n) + 29 \). As this bound does not depend on the size of the finite field \( \mathbb{F} \), it follows that Theorem 2 holds, more generally, whenever \( \mathbb{F} \) is an algebraic extension of a finite field.

One cannot take \( n = 2 \) in Theorem 2: \( \text{SL}_2(\mathbb{F}[X]) \) is not boundedly generated by the elementary matrices. This fact, and the reason behind it, are analogous to what happens for \( \text{SL}_2(\mathbb{Z}) \). The principal congruence subgroup of \( \text{SL}_2(\mathbb{F}[X]) \) corresponding to the ideal \((X)\), in other words the kernel of the surjective homomorphism \( \text{SL}_2(\mathbb{F}[X]) \to \text{SL}_2(\mathbb{F}) \) given by the evaluation \( X = 0 \), has a free product structure.

2. Proof of Theorem 2

Throughout, an elementary operation will be called, simply, a move. We allow the degenerate move of multiplying by the identity matrix. We write \( \sim \) for the equivalence relation of being connected by a finite number of moves.

2.1. Reduction to a framed \( \text{SL}_2 \) matrix. The first step is to reduce a matrix in \( \text{SL}_n \), \( n \geq 3 \), to a matrix of the following form:

\[
\begin{pmatrix}
a & b \\
c & d \\
I_{n-2}
\end{pmatrix}
\]

This is a standard reduction which works over any principal domain \( A \). The general concept underpinning this procedure is Bass’s stable range [3]. For the sake of completeness, let us sketch the argument for \( n = 3 \). Let \((u, v, w)\) be the last row of a matrix in \( \text{SL}_3(A) \). Thus, \( u, v, \) and \( w \) are relatively prime, and we may assume that either \( u \) or \( v \) is non-zero. A suitable move takes us to a matrix whose last row is \((u', v', w)\), and such that \( u' \) and \( v' \) are relatively prime. The key fact here is that, if \( \gcd(u, v, w) = 1 \) and \( u \) is non-zero, then \( \gcd(u, v + tw) = 1 \) for some \( t \in A \). (An explicit choice for \( t \) is the product of all primes dividing \( u \) but not \( v \). More precisely, we take one prime per associate class. We set \( t = 1 \) if there are no such primes.) Now \( w - 1 \) is a combination of \( u' \) and \( v' \), so two moves turn \( w \) into 1. Four additional moves clear the last row and the last column. In summary, we have reached a framed \( \text{SL}_2 \) matrix, as desired, in 7 moves. More generally, this argument reduces an \( \text{SL}_n \) matrix to a framed \( \text{SL}_2 \) matrix in \( \frac{1}{2}(3n^2 - n) - 5 \) moves.

The remainder of the argument is devoted to showing that 34 moves are sufficient in order to reduce, in \( \text{SL}_3 \), a framed \( \text{SL}_2 \) matrix to the identity. For the purposes of the next step, we assume that \( a \neq 0 \); otherwise, the reduction is trivial and quick, in only 3 moves.

2.2. A convenient anti-diagonal. The second step will use the following analogue of Dirichlet’s theorem on primes in arithmetic progressions.

**Theorem 3** (Kornblum - Artin). If \( a, b \in \mathbb{F}[X] \) are relatively prime and \( a \neq 0 \), then there are infinitely many primes congruent to \( b \) mod \( a \). Furthermore, such a prime can have arbitrary degree, provided the degree is sufficiently large.

The first part is due to Kornblum (1919). The second part is a sharpening due to Artin (1921). See [10, Chapter 4] for a modern treatment.
Consider a matrix
\[
\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}_2(\mathbb{F}[X]).
\]

As \(a\) and \(b\) are relatively prime, the first part of Theorem 3 ensures that there is a prime \(b' \in \mathbb{F}[X]\) satisfying \(b' \equiv b \mod a\). Similarly, there is a prime \(c' \in \mathbb{F}[X]\) satisfying \(c' \equiv c \mod a\). Thus
\[
\begin{pmatrix} a & b \\ c & d \end{pmatrix} \sim \begin{pmatrix} a & b' \\ c' & d' \end{pmatrix}
\]
in 2 moves. Furthermore, we can assume that \(b'\) and \(c'\) have relatively prime degrees: once \(b'\) has been chosen, we use the second part of Theorem 3 to pick \(c'\) of suitable degree.

2.3. The main step. Let
\[
\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}_2(\mathbb{F}[X])
\]
be a matrix enjoying the property granted by the previous step: the anti-diagonal entries \(b\) and \(c\) are prime, with relatively prime degrees.

Let \(q\) denote the number of elements in \(\mathbb{F}\). Then the integers
\[
\delta(b) := \frac{q^{\deg(b)} - 1}{q - 1}, \quad \delta(c) := \frac{q^{\deg(c)} - 1}{q - 1}
\]
are relatively prime, as well. Let \(x\) and \(y\) be positive integers satisfying \(x\delta(b) - y\delta(c) = 1\).

We write
\[
\begin{pmatrix} a & b \\ c & d \end{pmatrix} = XY^{-1},
\]
where
\[
X = \begin{pmatrix} a & b^{x\delta(b)} \\ c & d \end{pmatrix}^{x\delta(b)}, \quad Y = \begin{pmatrix} a & b^{y\delta(c)} \\ c & d \end{pmatrix}^{y\delta(c)}.
\]

We aim to reduce \(X\) and \(Y\) independently in \(\text{SL}_3\). More precisely, we will show that
\[
\left( \begin{array}{c} Y \\ 1 \end{array} \right) \sim \left( \begin{array}{c} D(u) \\ -1 \end{array} \right), \quad D(u) := \left( \begin{array}{cc} -u & 0 \\ 0 & u^{-1} \end{array} \right)
\]
in 14 moves, for some \(u \in \mathbb{F}^*\). The same will hold for \(Y^{-1}\) in place of \(Y\), and \(u^{-1}\) in place of \(u\), by inverting. It also holds for \(X\) in place of \(Y\), by interchanging \(b\) and \(c\), and then transposing, with respect to some other unit \(v \in \mathbb{F}^*\). However, \(D(u) \sim D(u)\) in \(\text{SL}_2\), in 4 additional moves. We can then deduce that
\[
\left( X Y^{-1} \right) = \left( X \begin{array}{c} Y^{-1} \\ 1 \end{array} \right) \sim \left( D(u) \begin{array}{c} -1 \\ -1 \end{array} \right) = I_3
\]
in \(14 + 4 + 14 = 32\) moves. Along the way, we are using the fact that diagonal matrices normalize the elementary matrices. Taking into account the second step, we conclude that 34 moves are sufficient in order to reduce a framed \(\text{SL}_2\) matrix to the identity.

Let us turn to proving (†). By the Cayley - Hamilton theorem, there are \(e, f \in \mathbb{F}[X]\) such that:
\[
\begin{pmatrix} a & b^{y\delta(c)} \\ c & d \end{pmatrix} = eI_2 + f \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} e + fa & fb \\ fc & e + fd \end{pmatrix}
\]
Modulo $c$, the above matrices become upper triangular. So $e + fa \equiv a y^\delta(c) \mod c$. On the other hand, $a y^\delta(c) \mod c$ is in $\mathbb{F}^*$. This follows by viewing the finite field $\mathbb{F}[X]/(c)$ as an extension of $\mathbb{F}$ of degree $\deg(c)$. Thus $e + fa \equiv u \in \mathbb{F}^* \mod c$. A similar argument applies to the lower diagonal entry. Keeping in mind that the determinant is 1, we find that $e + f a \equiv u \in \mathbb{F}^* \mod c$. At this point, we would like to replace the lower entry, $f c$, by $c$ so as to be able to perform reductions.

These considerations motivate the following lemma. Roughly speaking, it provides a way of swindling factors across the diagonal.

**Lemma 4.** Let $A$ be a principal domain, and let

$$\begin{pmatrix} a & b \\ sc & d \end{pmatrix} \in \text{SL}_2(A)$$

where $a \equiv d \mod s$. Then

$$\begin{pmatrix} a & b \\ sc & d \end{pmatrix} \sim \begin{pmatrix} \pm a & -sb \\ c & \mp d \end{pmatrix}$$

in 11 moves.

**Proof.** The degenerate case $s = 0$ is easily seen to hold, so let us assume that $s \neq 0$. The hypotheses imply that $a^2 \equiv ad \equiv 1 \mod s$. So there are $s_1, s_2$ and $k_1, k_2$ in $A$ such that

$$s = s_1 s_2, \quad a = k_1 s_1 + 1 = k_2 s_2 - 1.$$  

Now

$$\begin{pmatrix} a & b & 0 \\ sc & d & 0 \\ 0 & 0 & 1 \end{pmatrix} \sim \begin{pmatrix} a & b & 0 \\ sc & d & 0 \\ s_1 & 0 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & b & -k_1 \\ 0 & d & -s_2 c \\ 0 & s_1 b & a \end{pmatrix} \sim \begin{pmatrix} 1 & b & -k_1 \\ 0 & d & -s_2 c \\ 0 & -s_1 b & a \end{pmatrix}$$

by a column move, two row moves, and one more row move. We have basically swindled $s_1$ across the diagonal, and we now go for $s_2$. Firstly,

$$\begin{pmatrix} 1 & b & -k_1 \\ 0 & d & -s_2 c \\ 0 & -s_1 b & a \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & s_2 \\ 0 & d & -s_2 c \\ 0 & s_1 b & a \end{pmatrix}$$

by two column moves. Next,

$$\begin{pmatrix} 1 & 0 & s_2 \\ 0 & d & -s_2 c \\ 0 & s_1 b & a \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & s_2 \\ c & d & 0 \\ -k_2 & -s_1 b & -1 \end{pmatrix} \sim \begin{pmatrix} -a & -sb & 0 \\ c & d & 0 \\ -k_2 & -s_1 b & -1 \end{pmatrix}$$

by two row moves, another row move, and two column moves. Overall, we have performed 11 moves, as claimed.

For the other choice of signs on the diagonal, one could ‘pivot’ around $d$ instead of $a$. Alternatively, start from the above choice of signs, invert both matrices, interchange $a$ and $d$, and switch the signs of $b$ and $c$.  

$\square$
Applying the above lemma, we obtain
\[
\begin{pmatrix}
  e + fa & fb \\
  fc & e + fd
\end{pmatrix}
\begin{pmatrix}
  1 \\
  1
\end{pmatrix}
\sim
\begin{pmatrix}
  -(e + fa) & -f^2b \\
  c & e + fd
\end{pmatrix}
\sim
\begin{pmatrix}
  -u & \ldots \\
  c & u^{-1}
\end{pmatrix}
\]
in $11 + 2 = 13$ moves. Taking the determinant reveals that the missing entry of the last matrix is 0. One additional move, bringing the total to 14, clears out the entry $c$. This completes the argument for (†), and so for Theorem 2 as well.

3. Further remarks

3.1. The notion of bounded generation is commonly used for the property that a group is a product of finitely many cyclic subgroups. For $\text{SL}_n(\mathbb{Z})$, $n \geq 3$, bounded cyclic generation follows from bounded elementary generation. This is no longer the case over $\mathbb{F}[X]$. In fact, bounded cyclic generation fails for $\text{SL}_n(\mathbb{F}[X])$, $n \geq 3$. The idea that bounded cyclic generation is essentially a characteristic 0 phenomenon, is crystallized by the following result from [1]: if a linear group in positive characteristic enjoys bounded cyclic generation, then the group is virtually abelian.

3.2. Lemma 4 can also be used over $A = \mathbb{Z}$. In this case, it leads to a simplification of the original Carter - Keller argument for $\text{SL}_n(\mathbb{Z})$, and to the better bound $\nu_n = \frac{1}{2}(3n^2 - n) + 25$. The improved bound is irrelevant from an asymptotic perspective, but it becomes interesting in the case $n = 3$. The question, which seems to us quite appealing, is how many elementary operations are needed to reduce a matrix in $\text{SL}_3(\mathbb{Z})$ to the identity? Carter and Keller have shown that 48 operations are sufficient. We have reduced this number to 37. We challenge the reader to reduce this bound even further.

3.3. There is an interesting issue of effectiveness in using the Kornblum - Artin theorem. The usual Dirichlet theorem, used in [6], is made effective by a result of Linnik and its modern improvements (see, for instance, [7]). Theorem 3 is also effective, since the Riemann hypothesis in the function field context is already known.

See [4] for further instances of Dirichlet-type theorems over polynomial rings.

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