Cost Analysis of $M/M/1/N$ queue with working breakdowns and a two-phase services

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Abstract. In this article, we analyze the finite capacity Markovian queueing model with working breakdowns and a two-phase service. The server provides the primary service as well as the secondary elective service. The cost optimization problem is to find out the minimum cost per unit time by a direct search method to optimize service rates.

1. Introduction

There are some situations in real life where it may not be necessary for arriving customers to receive only one service and may want to choose some optional service from the services available in the system. The motivation for this sort of model comes from some computer and communication networks where messages are processed in two phases by a single server. This led to the two-phase queueing system and is characterized by the feature that all arrivals demand the first, “essential service” whereas only some of them decide to demand the second optional phase service. These kinds of queueing systems were first considered by Madan [15]. Medhi [17] has discussed a more general case of Madan [16], where the service times, of the essential and optional channels, are independent, having general distributions. He also discussed the Takacs integral equation for the LST of the normal period. Choudhury [1] generalized the results of Madan [15] by deriving the steady-state queue size distribution the stationary points of time for general second optional service times. Wang [23] studied the steady analysis of an unreliable server with Poisson arrivals where the service times follow general service time distribution. Yang et al. [25] studied the optimization of a randomized control problem of a single server queue in which the unreliable server may provide two phases of heterogeneous services to all arriving customers. Wang and Xu [24] discussed the well-posedness of the steady-state solution of $M/G/1$ queueing models with the second optional service and server breakdowns. Choudhury and Tadj [2] discussed the steady-state behavior of the $M/G/1$ queueing system with an additional second phase of optional service subject and delayed repair. Yang et al. [27] discussed an $M/G/1$ queueing models with variety of the features such the second optional service, $< N,p >$ policy, server breakdowns and general startup times. Dequan Yue et al. [4] studied a two-phase queueing system with impatient customers where the enjoy multiple vacations. Sharma [20] provided an overview of queues where the server provides service in phase-type and references therefore in.

In the theory of queueing systems, many authors have discussed the queueing models with...
the server breakdowns. There is a sizeable literature on this topic and we guide the intended readers to study the queueing systems with failures to Krishnamoorthy et al. [11]. It is more often considered that the server stops the service entirely during the breakdown period. In many of the real-life queueing systems, there is some situation where the server provides service at a slower rate rather than entirely stopping the service during the breakdown period. These kinds of server breakdowns are called as working breakdowns. Kalidass and Kasturi[9] first introduced the concept and studied the steady-state analysis of the \(M/M/1\) queue with working breakdowns. Kim and Lee [10] examined a single server disaster queueing model with a working breakdown where the service times follow a general distribution. Liou[13] applied the matrix-geometric method to examine a finite capacity Markovian queue with an unreliable server subject to working breakdowns and impatient customers. Kalidass and Pavithra [12] analyzed the steady-state analysis of \(M/M/1/N\) queue with working breakdowns and Bernoulli feedbacks. Vijayalakshmi and Kalidass[22] established the single server vacation queue with geometric abandonments and Bernoulli’s feedbacks. Yang and Chen [26] optimized the \(M/M/1\) queue with the second optional service and working breakdowns. Deepa and Kalidass [3] discussed the steady-state behavior of Markovian queue with finite capacity, working breakdowns, and server vacations. Dong-Yuh et al. [5] considered an \(M/M/1\) queueing models with second optional service, in which the server is subject to working breakdowns. Recently, Vijayalakshmi et al. [21] therefore considered an \(M/M/1/N\) queueing system with a working breakdown and two-phase services.

The article contains as follows. In section 2, we describe the model. In section 3, the steady-state distribution is obtained by the matrix analytic method. We derive a minimum expected cost per unit time by direct search method for the different service rates, in section 3. The next section concluded.

2. The model

We consider an \(M/M/1/N\) with second optional and slow services. The basic presumptions of our queueing model are considered as follows. Customers join the system with a Poisson process. The arrival rate of customers into the system is \(\lambda\). Customers setups a single queue in the system and the service provider provides service for a customer who is in the head of the queue. Primary service times and the second optional service times follow an exponential distribution with rate \(\mu > 0\) and \(\mu_1 > 0\) respectively. While providing service (either principal or secondary optional) to the customers the server may get a partial failure. The failure times of the server are assumed to be exponentially distributed with a parameter \(\gamma\) and \(\alpha\) whenever the server providing the first phase and second phase respectively. Due to these partial failures, the server reduces the service rate and providing the principal service with the rate \(\mu_b < \mu\) and the second optional service with the rate \(\mu_{b1} < \mu_1\). The repair times of the server when the server is in primary and secondary partial breakdown follows an exponential distribution with parameter \(\beta\) and \(\xi\) respectively. Once the partial crashed server is fixed, the server immediately backs to serve customers with a regular service rate. Let \(C(t)\) denoted by server state at time \(t\). Then

\[
C(t) = \begin{cases} 
1, & \text{if server is idle (or) a regular primary service takes place at time } t, \\
2, & \text{if a regular secondary service takes place at time } t \\
3, & \text{if server is idle (or) a working breakdown primary service takes place at time } t, \\
4, & \text{if a working breakdown secondary service takes place at time } t.
\end{cases}
\]
Let \( N(t) \) represents by the number of customers in the system at time \( t \). Then the bivariate process \( \{(C(t), N(t)), t \geq 0\} \) is a continuous time Markov chain. Let

\[
P_{i,n}(t) = \text{Prob}\{C(t) = i, N(t) = n\}, \quad i = 1, 2, 3, 4 \text{ and } n \geq 0.
\]

3. Steady state analysis

In this model studied by Vijayalakshmi et al. [21]. The steady-state probability equations are as follows:

\[
\begin{align*}
\mu p P_{1,1} + \mu_1 P_{2,1} &= \lambda P_{1,0} \quad (1) \\
\lambda P_{1,n-1} + \gamma P_{3,n} + \mu p P_{1,n+1} + \mu_1 P_{2,n+1} &= (\lambda + \beta + \mu) P_{1,n}, \; 1 \leq n \leq N - 1 \quad (2) \\
\lambda P_{1,n-1} + \gamma P_{3,n} &= (\beta + \mu) P_{1,n}, \; n = N \quad (3) \\
\mu q P_{1,1} + \xi P_{1,1} &= (\mu_1 + \alpha + \lambda) P_{2,1} \quad (4) \\
\lambda P_{2,n-1} + \mu q P_{1,n} + \xi P_{4,n} &= (\mu_1 + \alpha + \lambda) P_{2,n}, \; 2 \leq n \leq N \quad (5) \\
\lambda P_{2,n-1} + \mu q P_{1,n} + \xi P_{4,n} &= (\mu_1 + \alpha) P_{2,n}, \; n = N \quad (6) \\
\mu q P_{3,1} + \mu_2 P_{4,1} &= \lambda P_{3,0} \quad (7) \\
\mu_1 q P_{3,1} + \lambda P_{3,n-1} + \beta P_{1,n} + \mu_3 P_{4,n+1} &= (\gamma + \lambda + \mu_1) P_{3,n}, \; 1 \leq n \leq N - 1 \quad (8) \\
\lambda P_{3,n-1} + \beta P_{1,n} &= (\gamma + \mu_1) P_{3,n}, \; n = N \quad (9) \\
\alpha P_{2,1} + \mu_1 q P_{3,1} &= (\xi + \lambda + \mu_3) P_{4,1} \quad (10) \\
\alpha P_{2,n} + \mu_1 q P_{3,n} + \lambda P_{4,n-1} &= (\xi + \mu_3) P_{4,n}, \; 2 \leq n \leq N \quad (11) \\
\alpha P_{2,n} + \mu_1 q P_{3,n} + \lambda P_{4,n-1} &= (\xi + \mu_3) P_{4,n}, \; n = N \quad (12)
\end{align*}
\]

From the normalizing condition

\[
P_{1,0} + P_{1} e + P_{2,0} + P_{2} e + P_{3,0} + P_{3} e + P_{4,0} + P_{4} e = 1
\]

where \( e \) indicate the unit column vector of dimension \( N \), we have

\[
P_{1,0} = \frac{1}{\Psi}
\]

where \( \Psi = 1 - A_{12} A_{22}^{-1} e + \Theta_7 A_{62} A_{22}^{-1} e + \Theta_8 e + \Theta_7 e + \Theta_5 \Theta_6^{-1} e \) After \( P_{1,0} \) is resolved, the remaining steady-state probabilities \( P_{i} \) \((i = 1, 2, 3, 4)\) can be obtained from equations.

4. Cost optimization analysis

In this section, we study the total expected cost function with decision variables \((\mu, \mu_1, \mu_2, \mu_3)\) for the above discussed model. Our main aim is obtain the optimal values for the continuous variables say \((\mu^*, \mu^*_1, \mu^*_2, \mu^*_3)\) in order to minimize the cost. Let us denote the following cost elements.

\( C_1 \) : holding rate per unit time for every customer present in the system.
\( C_2 \) : cost per unit period for the FES during normal service time.
\( C_3 \) : cost per unit period for the SOS during normal service time.
\( C_4 \) : cost per unit period for the FES during slow service time.
\( C_5 \) : cost per unit period for the SOS during slow service time.
\( C_6 \) : waiting rate per unit time when one customer waits for normal or slow service time.
\( C_7 \) : stable cost for fast service rate and
\( C_8 \) : stable cost for slow service rate.
By definition of these cost elements itemized above, the expected cost function per unit time is known by
\[ F(\mu, \mu_b, \mu_{b1}) = C_1E(N) + C_2P_1 + C_3P_2 + C_4P_3 + C_5P_4 + C_6W(1) + C_7\mu + C_8\mu_b \]
The cost minimization problem can be formulated as
\[ F(\mu^*, \mu_1^*, \mu_b^*, \mu_{b1}^*) = \text{Minimize} \ F(\mu, \mu_1, \mu_b, \mu_{b1}) \]
Subject to : \( \mu > \mu_b \) and \( \rho < 1 \)
The complexity of terms \( P_1, P_2, P_3 \) and \( P_4 \) complicate the cost function. Unfortunately, it is not possible to derive the analytic solutions for the optimal service rates at the minimum expected cost. Thus, we progress the approximations to achieve the optimal service rates \( \mu, \mu_1, \mu_b, \mu_{b1} \) by direct search method.

4.1. Direct search method
We assume the following cost parameters as \( C_1 = 25, C_2 = 85, C_3 = 80, C_4 = 350, C_5 = 40, C_6 = 12, C_7 = 2, C_8 = 40. \)

Numerical examples are presented to determine the optimal value \( \mu \) by means of the direct search method. Also, we fix \( \alpha = 0.002, \gamma = 0.004, \beta = 0.002, \xi = 0.003, p = 0.8, q = 0.2, \mu_b = 0.25 \) and \( \mu_{b1} = 0.10 \). Vary \( \mu \) from 0.5 to 10, and choose different values of \( \lambda \). The minimum expected cost \( F(\mu, \mu_1) \) are shown in table 1. and For \( \lambda = 0.05, 0.06, 0.07 \) the minimum expected cost Rs 235 is achieved at \( \mu = 2.5 \), for \( \lambda = 0.05 \), Rs 291 is achieved at \( \mu = 2.5 \) for \( \lambda = 0.06 \) and Rs 348 is achieved at \( \mu = 2.5 \) for \( \lambda = 0.07 \). From figure 2, it is seen that initially the total cost decreases and starts increasing with the growth of \( \mu \) for fixed values of \( \lambda \). The convex nature of the cost function with respect to \( \mu \) shows the trend for the optimum cost by increasing the normal service domain of the customers.

From table 2, we concluded that the minimum expected cost Rs 235 is attained at \( \mu = 2.5 \), for \( \alpha = 0.002 \), Rs 240 is attained at \( \mu = 2.5 \), for \( \alpha = 0.003 \) and Rs 163 is attained at \( \mu = 2.5 \), for \( \alpha = 0.004 \) for fixed \( \lambda = 0.05, \gamma = 0.004, \beta = 0.002, \xi = 0.003, p = 0.8, q = 0.2, \mu_b = 0.25 \) and \( \mu_{b1} = 0.10 \). From figure 3, it is evident that the total cost function decreases first and then increases. Consequently, the convex nature arises in the total cost function. This confirms the possibility of obtaining the optimum service rates.

We concluded in the table 3, the minimum expected rate Rs 244 is attained at \( \mu = 2.5 \), for \( \gamma = 0.002 \), Rs 240 is attained at \( \mu = 2.5 \), for \( \gamma = 0.003 \) and Rs 235 is attained at \( \mu = 2.5 \), for \( \gamma = 0.004 \) for fixed \( \lambda = 0.05, \alpha = 0.002, \beta = 0.002, \xi = 0.003, p = 0.8, q = 0.2, \mu_b = 0.25 \) and \( \mu_{b1} = 0.10 \). As we hoped, the figure 4 shows the convexity in the total cost function.

In table 4, we obtained that the minimum expected rate Rs 235 is attained at \( \mu = 2.5 \), for \( \beta = 0.002 \), Rs 217 is attained at \( \mu = 1.5 \), for \( \beta = 0.003 \), and Rs 198 is attained at \( \mu = 1.5 \) for \( \beta = 0.004 \) for fixed \( \lambda = 0.05, \alpha = 0.002, \gamma = 0.004, \xi = 0.003, p = 0.8, q = 0.2, \mu_b = 0.25 \) and \( \mu_{b1} = 0.10 \). From figure 5 we seen that the total cost function behaves like convex function.

In table 5, we found that the minimum expected rate Rs 237 is attained at \( \mu = 2.5 \), for \( \xi = 0.001 \), Rs 233 is attained at \( \mu = 2.5 \) for \( \xi = 0.005 \), and Rs 230 is attained at \( \mu = 2.5 \), for \( \xi = 0.010 \) for fixed \( \lambda = 0.05, \alpha = 0.002, \gamma = 0.004, \beta = 0.002, p = 0.8, q = 0.2, \mu_b = 0.25 \) and \( \mu_{b1} = 0.10 \). From figure 6, it is seen that the total cost function is a convex function.

5. Conclusion
In this article, we described an \( M/M/1/N \) queue with working breakdowns and two-phase services. The model can be noticed as an extension of the model [21]. The cost model was derived, and a direct search method was used to obtain the best values for service rates to minimize the average cost per unit period. In the future extend the model to consider general service times.
Figure 1. Transition rate diagram

Figure 2. Cost function versus $\mu$
**Figure 3.** cost function versus $\mu$

**Figure 4.** cost function versus $\mu$
Figure 5. cost function versus $\mu$

Figure 6. cost function versus $\mu$

Table 1. cost function versus $\mu$

| $\lambda$ | 0.05 | 0.06 | 0.07 |
|-----------|------|------|------|
| $\mu = 0.5$ | 309 | 391 | 474 |
| $\mu = 1.5$ | 236 | 295 | 354 |
| $\mu = 2.5$ | 242 | 297 | 353 |
| $\mu = 3.5$ | 262 | 306 | 361 |
| $\mu = 4.5$ | 273 | 316 | 371 |
| $\mu = 5.5$ | 281 | 326 | 381 |
| $\mu = 6.5$ | 295 | 337 | 392 |
| $\mu = 7.5$ | 307 | 349 | 404 |
| $\mu = 8.5$ | 307 | 349 | 404 |
| $\mu = 9.5$ | 307 | 349 | 404 |
### Table 2. cost function versus $\mu$

| $\mu$ | 0.5 | 1.5 | 2.5 | 3.5 | 4.5 | 5.5 | 6.5 | 7.5 | 8.5 | 9.5 |
|-------|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|
| $\alpha = 0.002$ | 309 | 236 | 235 | 242 | 252 | 262 | 273 | 281 | 295 | 307 |
| $\alpha = 0.003$ | 287 | 202 | 200 | 206 | 214 | 224 | 236 | 247 | 258 | 269 |
| $\alpha = 0.004$ | 264 | 168 | 163 | 169 | 177 | 187 | 197 | 208 | 220 | 231 |

### Table 3. cost function versus $\mu$

| $\mu$ | 0.5 | 1.5 | 2.5 | 3.5 | 4.5 | 5.5 | 6.5 | 7.5 | 8.5 | 9.5 |
|-------|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|
| $\gamma = 0.002$ | 324 | 246 | 244 | 251 | 260 | 270 | 281 | 292 | 303 | 315 |
| $\gamma = 0.003$ | 317 | 241 | 240 | 247 | 256 | 266 | 277 | 288 | 299 | 311 |
| $\gamma = 0.004$ | 309 | 236 | 235 | 242 | 252 | 262 | 273 | 281 | 295 | 307 |

### Table 4. cost function versus $\mu$

| $\mu$ | 0.5 | 1.5 | 2.5 | 3.5 | 4.5 | 5.5 | 6.5 | 7.5 | 8.5 | 9.5 |
|-------|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|
| $\beta = 0.002$ | 309 | 236 | 235 | 242 | 252 | 262 | 273 | 281 | 295 | 307 |
| $\beta = 0.003$ | 265 | 217 | 222 | 232 | 243 | 254 | 266 | 278 | 287 | 301 |
| $\beta = 0.004$ | 219 | 198 | 210 | 222 | 235 | 247 | 259 | 271 | 283 | 296 |

### Table 5. cost function versus $\mu$

| $\mu$ | 0.5 | 1.5 | 2.5 | 3.5 | 4.5 | 5.5 | 6.5 | 7.5 | 8.5 | 9.5 |
|-------|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|
| $\xi = 0.001$ | 313 | 238 | 237 | 244 | 253 | 264 | 275 | 286 | 297 | 308 |
| $\xi = 0.005$ | 306 | 234 | 233 | 241 | 250 | 260 | 271 | 283 | 294 | 305 |
| $\xi = 0.010$ | 300 | 231 | 230 | 237 | 247 | 258 | 269 | 280 | 291 | 303 |
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