Complexifications of infinite-dimensional manifolds and new constructions of infinite-dimensional Lie groups

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Abstract

Let \( M \) be a real analytic manifold modeled on a locally convex space and \( K \subseteq M \) be a non-empty compact set. We show that if an open neighborhood \( \Omega \) of \( K \) in \( M \) admits a complexification \( \Omega^* \) which is a regular topological space, then the germ of \( \Omega^* \) around \( K \) (as a complex manifold) is uniquely determined. If \( M \) is regular and the complexified modeling space of \( M \) is normal, then a regular complexification \( \Omega^* \) exists for some \( \Omega \). For each regular \( K \)-analytic manifold \( M \) modeled on a metrizable locally convex space over \( \mathbb{K} \in \{ \mathbb{R}, \mathbb{C} \} \) and each \( K \)-analytic Banach-Lie group \( H \), this enables the group \( \text{Germ}(K, H) \) of germs of \( \mathbb{K} \)-analytic \( H \)-valued maps around \( K \) in \( M \) to be turned into a \( \mathbb{K} \)-analytic Lie group which is regular in Milnor’s sense (and, actually, \( C^0 \)-regular). Notably, this provides a \( C^0 \)-regular real analytic Lie group structure on the group \( C^\omega(M, H) \) of \( H \)-valued real analytic maps on a compact real analytic manifold \( M \) (which, previously, had only been treated in the convenient setting of analysis). Combining our results concerning Lie groups of germs with an idea by Neeb and Wagemann, it is also possible to obtain a \( C^0 \)-regular Lie group structure on \( C^\omega(\mathbb{R}, H) \).

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Introduction and statement of results

It is well known that every paracompact, finite-dimensional real analytic manifold $M$ admits a complexification, whose germ around $M$ (as a complex manifold) is uniquely determined. Two classical constructions for such complexifications are known. The first one, by Bruhat and Whitney [5], is based on a glueing construction: complex analytic extensions of real analytic charts are pasted together in a suitable way. The second approach was devised by Grauert [16]. Here the manifold is embedded in its cotangent bundle which supports a complex analytic structure (of a Stein manifold) on a neighborhood of the embedding. These neighborhoods are the famous “Grauert tubes,” and one advantage is that these can be chosen contractible to $M$.

In this paper, we investigate complexifications around a non-empty compact subset of an infinite-dimensional real analytic manifold. One cannot expect to adapt Grauert’s construction to infinite-dimensional manifolds, and, in fact, one would not always expect to find (global) complexifications, in the natural sense:

$\textbf{Definition.}$ Let $M$ be a real analytic manifold modeled on a locally convex space $E$. A complex analytic manifold $M^*$ modeled on $E$ is called a complexification of $M$ if $M \subseteq M^*$ and each $x \in M$ is contained in the domain $U^*$ of a chart $\psi: U^* \to U' \subseteq E$ of $M^*$ such that $\psi(M \cap U^*) = E \cap U'$ and $\psi|_{M \cap U^*}: M \cap U^* \to E \cap U'$ is a chart for $M$.

Nonetheless, we shall see that a complexification can be constructed at least for an open neighborhood of a compact set, if the manifold and its model space satisfy certain regularity properties. Our argument adapts the line of thought developed by Bruhat and Whitney in the classical paper [5]. In particular, we obtain the following theorem:

$\textbf{Theorem A}$ Let $M$ be a real analytic manifold modeled on the locally convex space $E$. Assume that as topological spaces, $M$ is regular and $E$ is normal. For each non-empty compact subset $K$ of $M$, there is an open neighborhood $\Omega$ of $K$ in $M$ such that $\Omega$ admits a complexification $\Omega^*$. The complex analytic manifold $\Omega^*$ constructed is a regular topological space.

We also obtain a uniqueness result for complexifications:

$\textbf{Theorem B}$ Let $M$ be a real analytic manifold modeled on a locally convex space.

(a) If $M_j^*$ is a complexification of $M$ for $j \in \{1, 2\}$ such that $M$ is a closed subset of $M_j^*$ and $M_j^*$ is paracompact, then there exist open neighborhoods $U_j$ of $M$ in $M_j^*$ and a complex analytic diffeomorphism $\psi: U_1 \to U_2$ such that $\psi|_{M} = \text{id}_{M}$.

(b) If $K \subseteq M$ is a non-empty compact set and $\Omega_j^*$, for $j \in \{1, 2\}$, is a regular complexification of an open neighborhood $\Omega_j$ of $K$ in $M$ such that $\Omega_j \subseteq \Omega_j^*$, then there exist open neighborhoods $U_j \subseteq \Omega_j^*$ of $K$ and a complex analytic diffeomorphism $\psi: U_1 \to U_2$ such that $\psi|_{U_1 \cap \Omega_1} = \text{id}_{U_1 \cap \Omega_1}$.

Later, we shall find it convenient to allow also complexifications which are only diffeomorphic to those just described, see Definition 1.8.
We mention that closedness of \( M \) in a complexification \( M^* \) is often easily achieved. Indeed, if \( M^* \) is a complexification of \( M \) and each open neighborhood of \( M \) in \( M^* \) contains a paracompact open neighborhood of \( M \), then there exists a paracompact open neighborhood \( U \) of \( M \) in \( M^* \) such that \( M \) is closed in \( U \) (Proposition 2.3(a)). The paracompactness is automatic if \( M^* \) is metrizable (see [11] Theorem 5.1.3). If \( M \) is closed in \( M^* \) and \( M^* \) is paracompact, then there exists an anti-holomorphic involution \( \sigma : U \to U \) on an open neighborhood \( U \) of \( M \) in \( M^* \) such that

\[
M = \{ z \in U : \sigma(z) = z \}.
\]

A similar result is available for complexifications around compact sets (see Proposition 2.3(b) and (c) for details).

Our applications in infinite-dimensional Lie theory concern Lie groups of analytic Lie group-valued mappings (or germs of such), and their regularity properties. A Lie group \( G \) modeled on a locally convex space is called \( C^0\)-regular if each continuous curve \( \gamma : [0, 1] \to L(G) \) in its Lie algebra \( L(G) = T_1(G) \) admits a continuously differentiable left evolution \( \eta = \eta_\gamma : [0, 1] \to G \) determined by

\[
\eta(0) = 1 \quad \text{and} \quad (\forall t \in [0, 1]) \quad \eta'(t) = \eta(t) \cdot \gamma(t),
\]

and the evolution map \( \operatorname{evol}_G : C^0([0, 1], L(G)) \to G, \operatorname{evol}(\gamma) := \eta_\gamma(1) \) is smooth (i.e., \( C^\infty \)). Here, the dot denotes the natural left action \( G \times TG \to TG \) of \( G \) on its tangent bundle (see [12], [14], [21]; cf. [19] and [20] for the related weaker notion of “regularity,” and some of its applications).

Let \( H \) be a Banach-Lie group over \( \mathbb{K} \in \{ \mathbb{R}, \mathbb{C} \} \), \( L(H) \) be its Lie algebra, \( M \) a \( \mathbb{K}\)-analytic manifold modeled on a metrizable locally convex space \( X \), and \( K \subseteq M \) a non-empty, compact subset. We write \( \text{Germ}_\mathbb{K}(K, H) \) for the group of all germs \( [\gamma] \) around \( K \) of \( \mathbb{K}\)-analytic maps \( \gamma : U \to H \), defined on an open neighborhood \( U \subseteq M \) of \( K \).

If \( \mathbb{K} = \mathbb{C} \), then \( \text{Germ}_\mathbb{C}(K, L(H)) \) carries a natural locally convex vector topology (see Proposition 1.2 making it an (LB)-space, namely the locally convex direct limit of the Banach spaces \( \text{BH}^\mathbb{C}(U_\alpha, L(H)) \) of complex analytic, bounded \( L(H) \)-valued functions on a basis \( U_1 \supseteq U_2 \supseteq \cdots \) of neighborhoods of \( K \) in \( M \). We show:

**Theorem C** \( \text{Germ}_\mathbb{C}(K, H) \) can be made a \( C^0\)-regular complex analytic Lie group with Lie algebra \( \text{Germ}_\mathbb{C}(K, L(H)) \).

If \( \mathbb{K} = \mathbb{R} \) and \( M \) is regular as a topological space, then a complexification \( M^* \) of \( M \) can be used to make \( \text{Germ}_\mathbb{R}(K, L(H)_\mathbb{R}) \) a complex locally convex space. We turn \( \text{Germ}_\mathbb{R}(K, L(H)) \) into a real locally convex space such that \( \text{Germ}_\mathbb{R}(K, L(H)_\mathbb{R}) = \text{Germ}_\mathbb{R}(K, L(H))_{\mathbb{R}} \). Then the following results can be obtained (proofs of which will be provided in a later version of this preprint):

**Theorem D** Let \( M \) be a real analytic manifold modeled on a metrizable locally convex space, \( K \subseteq M \) be a non-empty compact set and \( H \) be a Banach-Lie group over \( \mathbb{K} \in \{ \mathbb{R}, \mathbb{C} \} \). If \( M \) is regular as a topological space, then \( G := \text{Germ}_\mathbb{R}(K, H) \) can be made a \( C^0\)-regular \( \mathbb{K}\)-analytic Lie group with Lie algebra

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Theorem E. Let $M$ be a compact real analytic manifold and $H$ be a Banach-Lie group over $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$. Then the group $G := C^0(M, H)$ of all real analytic $H$-valued maps on $M$ can be made a $C^0$-regular $\mathbb{K}$-analytic Lie group with Lie algebra $C^0(M, L(H))$, such that the evolution map $\text{evol}: C^0([0,1], L(G)) \to G$ is $\mathbb{K}$-analytic.

In the special case that $M = X$ is a metrizable locally convex space, the Lie group structure on $\text{Germ}_X(K, H)$ was already provided in [11]. The $C^0$-regularity was established in [7, Theorem 5.1.1], if $M = X$ is a Banach space (using tools from [6]). For $M$ a finite-dimensional $\mathbb{K}$-analytic manifold, the Lie group structure on $\text{Germ}_X(K, H)$ was constructed in [17]. For real analytic compact $M$, a Lie group structure on $C^\infty(M, H)$ was provided in [18], which is real analytic in the (weaker) sense of convenient differential calculus.

Recently, Neeb and Wagemann [22] obtained a regular infinite-dimensional Lie group structure on $C^\infty(\mathbb{R}, H)$ for each regular Lie group $H$, and more generally on $C^\infty(\mathbb{R}^n \times K, H)$ if $n \in \mathbb{N}$ and $K$ is a compact smooth manifold. Since $C^\infty(\mathbb{R}, H) = C^\infty(\mathbb{R}, H)_* \times H$ with $C^\infty(\mathbb{R}, H)_* := \{\gamma \in C^\infty(\mathbb{R}, H) : \gamma(0) = 1\}$, the main point is to make $C^\infty(\mathbb{R}, H)_*$ a Lie group. To achieve this, Neeb and Wagemann noted that the left logarithmic derivative $(\delta^t \gamma)(t) := \gamma(t)^{-1}\gamma'(t)$ provides a bijection

$$\delta^t: C^\infty(\mathbb{R}, H)_* \to C^\infty(\mathbb{R}, L(H)), \quad \gamma \mapsto \delta^t \gamma$$

which can be used as a global chart for a smooth manifold structure on $C^\infty(\mathbb{R}, H)_*$ that turns the group operations into smooth maps. Combining a variant of this idea with our results concerning spaces of germs, one can show:

Theorem F. If $H$ is a real Banach-Lie group, then the group $C^\omega(\mathbb{R}, H)$ of all real analytic maps $\gamma: \mathbb{R} \to H$ is a $C^0$-regular smooth Lie group with Lie algebra $C^\omega(\mathbb{R}, L(H))$. If $\mathbb{K} = \mathbb{C}$ and $H$ is a complex Banach-Lie group or $\mathbb{K} = \mathbb{R}$ and $H$ is a real Banach-Lie group such that $L(H)_C = L(H_\mathbb{C})$ for some complex Banach-Lie group $H_\mathbb{C}$, then $G := C^\omega(\mathbb{R}, H)$ is a $C^0$-regular $\mathbb{K}$-analytic Lie group and the evolution map $\text{evol}: C^0([0,1], L(G)) \to G$ is $\mathbb{K}$-analytic.

The article is structured as follows: After a preliminary Section 1 we discuss existence and uniqueness of complexifications in Sections 2 and 3. Locally convex spaces of complex analytic germs and their regularity (as a locally convex direct limit) are discussed in Section 4 and then used in Section 5 to construct the $C^0$-regular Lie group of complex analytic germs. For later use, also the case of local Lie groups $H$ (see Definition 5.1) needs to be addressed there.

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2Recall from [19] pp. 104–105 that such mappings need not be real analytic, as they need not admit Taylor expansions. The counter-example is defined on an (LB)-space.

3Compare [23] for a detailed and streamlined discussion of the special case $C^\omega(\mathbb{R}, H)$ (as well as weighted analogs), and [11] for a treatment of $C^k(\mathbb{R}, H)$.
1 Preliminaries and notation

In this section, we recall preliminaries and fix some notation, for later use.

By a locally convex space, we mean a locally convex Hausdorff topological vector space. As usual, if $X$ is a topological space and $Y \subseteq X$, then a subset $U \subseteq X$ is called an open neighborhood of $Y$ in $X$ if $U$ is open in $X$ and $Y \subseteq U$.

1.1 Definition Let $E$ be a real locally convex space. Endow the real locally convex space $E_C := E \times E$ with the operation

$$(x + iy)(u, v) := (xu - yv, xv + yu) \quad \text{for } x, y \in \mathbb{R}, u, v \in E.$$ 

The complex locally convex space $E_C$ obtained in this way is called the complexification of $E$. We identify $E$ with the subspace $E \times \{0\} \subseteq E_C$.

1.2 Definition (Complex analytic maps [3]) Let $E$ and $F$ be complex locally convex spaces. A map $f : U \to F$ on an open subset $U \subseteq E$ is complex analytic if it is continuous and admits locally a power series expansion around each $a \in U$, i.e. there exist continuous homogeneous polynomials $p_k : E \to F$ of degree $k$, such that

$$f(x) = \sum_{k=0}^{\infty} p_k(x - a)$$

pointwise for all $x$ in a neighborhood of $a$ in $U$.

See [10] and [15] for the following definition (cf. also [19]).

1.3 Definition (Real analytic maps) Let $E$ and $F$ be locally convex spaces over $\mathbb{R}$. A map $f : U \to F$ on an open subset $U \subseteq E$ is called real analytic or $C^\omega_R$ if it extends to a complex analytic map $g : W \to F_C$ on some open neighborhood $W \subseteq E_C$ of $U$.

Throughout the article, the word “analytic manifold” means an analytic manifold modeled on a locally convex space (as discussed, e.g., in [10] of [15]).

1.4 Definition If $F$ is a complex locally convex space, we write $F_{\text{op}}$ for $F$, endowed with the opposite complex structure (thus multiplication with $i$ on $F_{\text{op}}$ is given by multiplication with $-i$ in $F$). If $M$ is a complex analytic manifold modeled on $F$, with atlas $\mathcal{A}$ of charts $\phi : U_\phi \to V_\phi \subseteq F$, we write $M_{\text{op}}$ for $M$, endowed with the complex analytic manifold structure given by atlas of charts $\phi : U_\phi \to V_\phi \subseteq F_{\text{op}}$ for $\phi \in \mathcal{A}$. A mapping $f : M \to N$ between complex analytic manifolds is called anti-holomorphic if it is complex analytic as a map $M \to N_{\text{op}}$ (or, equivalently, as a map $M_{\text{op}} \to N$).

1.5 Remark Let $M$ be a complex analytic manifold modeled on a complex locally convex space $F$ such that $F = E_C$ for a real locally convex space $E$. Let $\tau : E_C \to E_C$ denote the complex conjugation given by $\tau(x + iy) := x - iy$ for all $x, y \in E$. Then $\tau : (E_C)_{\text{op}} \to E_C$ is an isomorphism of complex locally convex
spaces. The maps $\tau \circ \phi: U_\phi \to \tau(V_\phi) \subseteq F$ form an atlas for $M_{op}$ (modeled on $F$), if we let $\phi: U_\phi \to V_\phi \subseteq F$ range through an atlas for $M$.

This readily entails:

1.6 Remark If $M$ is a real analytic manifold modelled on $E$ and $M^*$ a complexification of $M$, then also $(M^*)_{op}$ is a complexification of $M$. Indeed, if $x \in M$ and $\phi: U^* \to U'$ is a chart of $M^*$ around $x$ with $\phi(M \cap U^*) = E \cap U'$, then $\tau \circ \phi: U^* \to \tau(U')$ is a chart of $(M^*)_{op}$ around $x$ which takes $M \cap U^*$ onto $E \cap \tau(U')$.

We recall a version of the well-known Identity Theorem for analytic functions (cf. [15]).

1.7 Lemma (Identity Theorem) (a) Let $M$ and $N$ be $\mathbb{K}$-analytic manifolds modeled on locally convex spaces (where $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$) and $f_j: M \to N$ be $\mathbb{K}$-analytic maps for $j \in \{1, 2\}$. If $M$ is connected and $f_1|_U = f_2|_U$ for some non-empty open set $U \subseteq M$, then $f_1 = f_2$.

(b) If $E$ and $F$ are real locally convex spaces, $U \subseteq E_\mathbb{C}$ an open connected subset with $U \cap E \neq \emptyset$ and $f_j: U \to F_\mathbb{C}$ complex analytic mappings for $j \in \{1, 2\}$ such that $f_1|_{E \cap U} = f_2|_{E \cap U}$, then $f_1 = f_2$.

A subset $M$ of a topological space $X$ is called locally closed if each $x \in M$ has a neighborhood $U \subseteq X$ such that $U \cap M$ is a relatively closed subset of $U$. For example, every closed subset of $X$ is locally closed. A real analytic manifold $M$ admitting a complexification $M^*$ (with $M \subseteq M^*$) is always locally closed in $M^*$.

In fact, for $x \in M$ and $\psi: U^* \to U'$ as in the above definition, the set $E \cap U'$ is relatively closed in $U'$ (as $E$ is closed in $E_\mathbb{C}$) and thus $M \cap U^* = \psi^{-1}(E \cap U')$ is relatively closed in $U^*$.

Complexifications were already defined in the introduction. In Section 3 we shall find it convenient to loosen the concept of a complexification (by allowing passage to diffeomorphic copies of complexifications in the earlier sense)\(^\text{[6]}\).

1.8 Definition Let $\Omega$ be a real analytic manifold modeled on a real locally convex space $E$. A complexification of $\Omega$ is a pair $(\Omega^*, \varphi)$ such that

- $\Omega^*$ is a complex analytic manifold modeled on $E_\mathbb{C}$,
- $\varphi: \Omega \to \varphi(\Omega) \subseteq \Omega^*$ is a real analytic diffeomorphism onto a real analytic submanifold $\varphi(\Omega)$ of $\Omega^*$,
- for each $x \in \Omega$, there are an open neighborhood $U^*$ of $x$ in $\Omega^*$ and a complex analytic diffeomorphism $U^* \to U'$ onto an open subset $U' \subseteq E_\mathbb{C}$ which takes $\varphi(\Omega) \cap U^*$ onto $E \cap U'$.

\(^6\)Outside Section 3 the earlier definition will be retained to simplify the notation (without loss of mathematical substance).
2 Uniqueness of complexifications

In this section, we discuss the existence of complex analytic extensions of real analytic mappings between real analytic manifolds to open neighborhoods in given complexifications. As a consequence, we shall obtain a proof for Theorem B. We begin with purely topological considerations, which are the foundation for the extension results.

2.1 Lemma Let X be a Hausdorff topological space and \((U_i)_{i \in I}\) be a family of open subsets of X.

(a) If X is regular, M a compact subset of X and \((U_i)_{i \in I}\) a cover of M, then there exists a cover \((W_j)_{j \in J}\) of M by open sets \(W_j \subseteq X\) such that

\[
(\forall j, k \in J) \quad W_j \cap W_k \neq \emptyset \quad \Rightarrow \quad (\exists i \in I) \quad W_j \cup W_k \subseteq U_i.
\]

(b) If X is paracompact and \((U_i)_{i \in I}\) a cover of M, then there exists an open cover \((W_j)_{j \in J}\) of X such that (1) holds.

(c) If X is paracompact, M a closed subset of X and \((U_i)_{i \in I}\) a cover of M, then there exists a cover \((W_j)_{j \in J}\) of M by open subsets \(W_j\) of X such that (1) holds.

(d) If M is a locally closed subset of X, every open neighborhood of M in X contains a paracompact open neighborhood of M in X and \((U_i)_{i \in I}\) is a cover of M, then there exists a cover \((W_j)_{j \in J}\) of M by open subsets \(W_j\) of X such that (1) holds.

Proof. (a) For each \(x \in M\), there is \(i_x \in I\) such that \(x \in U_{i_x}\). There is a closed neighborhood \(A_x\) of \(x\) in X such that \(A_x \subseteq U_{i_x}\), by regularity of X. By compactness of M, we find a finite subset \(Z \subseteq M\) such that \(M \subseteq \bigcup_{z \in Z} A_0^z\), where \(A_0^z\) means the interior of \(A_z\) in X. For \(x \in M\), we define

\[
W_x := \bigcap_{x \in A_0^z} A_0^z \cap \bigcap_{x \in \partial A_z} U_{i_x} \cap \bigcap_{x \notin A_z} (X \setminus A_z),
\]

where the indices range through all \(z \in Z\) with the respective property, and intersections over empty index sets are defined as X. There is \(z(x) \in Z\) such that \(x \in A_0^z(x)\). Then \(W_x \subseteq A_0^{z(x)} \subseteq U_{i_x(z(x))}\). If \(y \in M\) and \(W_x \cap W_y \neq \emptyset\), then \(y \in A_z(x)\). In fact, if this was wrong, then \(y \notin A_z(x)\), entailing that \(W_y \subseteq X \setminus A_z(x)\). Since \(W_x \subseteq A_z(x)\), we deduce that \(W_x \cap W_y = \emptyset\), a contradiction. Now \(y \in A_z(x)\) entails that \(y \in A_0^{z(x)}\) or \(y \in \partial A_z(x)\). In the first case, \(W_y \subseteq A_0^{z(x)} \subseteq U_{i_x(z(x))}\); in the second case, \(W_y \subseteq U_{i_x(z(x))}\). Thus \(W_x \cup W_y \subseteq U_{i_x(z(x))}\). Hence \((W_x)_{x \in M}\) has the desired properties.

(b) For each \(x \in X\), there is \(i_x \in I\) such that \(x \in U_{i_x}\). As every paracompact space is normal and hence regular, there is a closed neighborhood \(B_x\) of \(x\) in X such that \(B_x \subseteq U_{i_x}\). Since X is paracompact, there is a locally finite open cover \((V_z)_{z \in Z}\) of X subordinate to \((B_0^z)_{z \in X}\). Thus, for each \(z \in Z\), there is \(x(z) \in X\) such that \(V_z \subseteq B_0^{x(z)}\). For the closure \(A_z := \overline{V_z}\) in X, we get \(A_z \subseteq B_{x(z)} \subseteq U_{i_x(z)}\). For \(x \in X\), define

\[
W_x := \bigcap_{x \in V_z} V_z \cap \bigcap_{x \in \partial V_z} U_{i_x(z)} \cap \bigcap_{x \notin A_z} (X \setminus A_z).
\]

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Note that, since the family \((V_k)_{k \in K}\) is locally finite, also the family \((A_k)_{k \in K}\) of the closures is locally finite \([9, \text{Theorem 1.1.13}]\). Therefore the first two intersections in (2) are finite intersections. Moreover, \(\bigcup_{x \in A_k} A_k\) is closed as a locally finite union of closed sets (see \([9, \text{Corollary 1.1.12}]\)). Therefore its complement \(\bigcap_{x \in A_k} (X \setminus A_k)\) is open and hence \(W_x\) is an open neighborhood of \(x\). Choose \(z \in Z\) such that \(x \in V_z\). We now see as in the proof of (a) that \(W_z \cap W_y \neq \emptyset\) implies \(W_z \cup W_y \subseteq U_{i(z)}\).

(c) Let \(\mathcal{V}\) be the open cover of \(X\) obtained by joining the sets \((U_i)_{i \in I}\) and the open set \(X \setminus M\). Applying (b) to \(\mathcal{V}\), we find an open cover \((W'_j)_{j \in J'}\) of \(X\) such that \(W'_j \cap W'_k \neq \emptyset\) for \(j, k \in J'\) implies that \(W'_j \cup W'_k \subseteq U\) for some set \(U\) in \(\mathcal{V}\). Note that \(U \neq X \setminus M\) and thus \(U = U_i\) for some \(i \in I\) if we assume that \(W_j\) and \(W_k\) meet \(M\). Thus, if we define

\[
J := \{ j \in J' : W'_j \cap M \neq \emptyset \},
\]

then \((W'_j)_{j \in J}\) has the desired properties.

(d) Since \(M\) is locally closed, each \(x \in M\) has an open neighborhood \(P_x\) in \(X\) such that \(M \cap P_x\) is relatively closed in \(P_x\). Thus \(P_x \setminus M\) is open in \(X\). The set \(\bigcup_{x \in M} P_x\) is an open neighborhood of \(M\) in \(X\). By hypothesis, there exists a paracompact open neighborhood \(P \subseteq \bigcup_{x \in M} P_x\) of \(M\) in \(X\). After replacing \(P_x\) with \(P_x \cap P\), we may assume that \(P = \bigcup_{x \in M} P_x\). Let \(\mathcal{V}\) be the open cover of \(P\) obtained by joining the sets \((P \cap U_i)_{i \in I}\) and the open set \(P \setminus M = \bigcup_{x \in M} (P_x \setminus M)\). Applying (b) to \(P\) and \(\mathcal{V}\) in place of \(X\) and \((U_i)_{i \in I}\), we find an open cover \((W'_j)_{j \in J'}\) of \(P\) such that \(W'_j \cap W'_k \neq \emptyset\) for \(j, k \in J'\) implies that \(W'_j \cup W'_k \subseteq U\) for some set \(U\) in \(\mathcal{V}\). Note that \(U \neq P \setminus M\) and thus \(U = P \cap U_i \subseteq U_i\) for some \(i \in I\) if we assume that \(W_j\) and \(W_k\) meet \(M\).

Thus, if we define \(J := \{ j \in J' : W'_j \cap M \neq \emptyset \}\), then \((W'_j)_{j \in J}\) has the desired properties. ☐

2.2 Lemma Let \(M\) and \(N\) be real analytic manifolds modeled on locally convex spaces and \(f: M \to N\) be a real analytic map. Let \(M^*\) and \(N^*\) be complexifications of \(M\) and \(N\), respectively, such that \(M \subseteq M^*\) and \(N \subseteq N^*\).

(a) If \(K \subseteq M\) is compact and \(M^*\) is a regular topological space, then there exists a complex analytic map \(g: U \to N^*\), defined on an open neighborhood \(U\) of \(K\) in \(M^*\), such that \(g|_{U \cap M} = f|_{U \cap M}\).

(b) If every open neighborhood of \(M\) in \(M^*\) contains a paracompact open neighborhood of \(M\) in \(M^*\), then there exists a complex analytic extension \(g: U \to N^*\) of \(f\), defined on an open neighborhood \(U\) of \(M\) in \(M^*\).

(c) If \(M^*\) is paracompact and \(M\) is closed in \(M^*\), then there exists a complex analytic extension \(g: U \to N^*\) of \(f\), defined on an open neighborhood \(U\) of \(M\) in \(M^*\).

(d) If \(g_j: U_j \to N^*\) are complex analytic extensions of \(f\) for \(j \in \{1, 2\}\), defined on open neighborhoods \(U_j\) of \(M\) in \(M^*\), then the union \(U\) of all connected components \(C\) of \(U_1 \cap U_2\) such that \(C \cap M \neq \emptyset\) is an open neighborhood of \(M\) in \(M^*\), and \(g_1|_U = g_2|_U\).

(e) Let \(f\) be a real analytic diffeomorphism in the sense that \(f^{-1}\) exists and is real analytic. Assume that \(f\) admits a complex analytic extension \(g: U \to N^*\)
N^*$, defined on an open neighborhood $U$ of $M$ in $M^*$, and that $f^{-1}$ admits
a complex analytic extension $h: V \rightarrow M^*$, defined on an open neighborhood
$V$ of $N$ in $N^*$. Then there exist an open neighborhood $W$ of $M$ in $U$
such that $g(W)$ is an open subset of $N^*$ contained in $V$, and $g|_W: W \rightarrow
\overline{g(W)}$ is a complex analytic diffeomorphism with inverse $h|_{\overline{g(W)}}$.

Proof. Let $E$ be the modeling space of $M$ and $F$ be the modeling space of $N$.

(a) For $x \in K$, let $\psi_x: U'_x \rightarrow U'_x$ be a chart of $M^*$ around $x$ and $\theta_x: V'_x \rightarrow V'_x$
be a chart of $N^*$ around $f(x)$ such that $\psi_x(M \cap U'_x) = E \cap U'_x$ and $\theta_x(N \cap V'_x) = F \cap V'_x$. After shrinking $U_x$, we may assume that $f(M \cap U'_x) \subseteq V'_x$. Thus

$$E \cap U'_x \rightarrow F \cap V'_x, \quad y \mapsto \theta_x(f(\psi_x^{-1}(y)))$$

is a real analytic map, which admits a complex analytic extension $h_x: P_x \rightarrow V'_x$
on some open neighborhood $P_x$ of $E \cap U'_x$ in $U'_x$. Then the open sets $U_x :=
\psi_x^{-1}(P_x) \subseteq M^*$, for $x \in K$, form a cover of $K$, and the maps

$$g_x := \theta_x^{-1} \circ h_x \circ \psi_x|_{U_x}: U_x \rightarrow V'_x \subseteq N^*$$

are complex analytic and satisfy $g_x|_{M \cap U_x} = f|_{M \cap U_x}$. Using Lemma 2.1(a), we
find a cover $(W_j)_{j \in J}$ of $K$ by open subsets of $M^*$ such that $W_j \cap W_k \neq \emptyset$ implies
the existence of some $(j, k) \in K$ such that $W_j \cap W_k \subseteq U_{(j,k)}$. After replacing
the sets $W_j$ by smaller neighborhoods of the points in $M \cap W_j$, we may assume
that each $W_j$ is connected and $M \cap W_j \neq \emptyset$. Define

$$g_j := g_{x(j,j)}|_{W_j}: W_j \rightarrow N^*.$$ 

If $W_j \cap W_k \neq \emptyset$, we have

$$g_j = g_{x(j,k)}|_{W_j}$$

by the Identity Theorem (Lemma 1.7), because $W_j$ is connected and both sides of
4 are complex analytic extensions of $f|_{M \cap W_j}$ (where $M \cap W_j \neq \emptyset$). Likewise,
$g_k = g_{x(k,k)}|_{W_k}$, and thus

$$g_j|_{W_j \cap W_k} = g_{x(j,k)}|_{W_j \cap W_k} = g_k|_{W_k \cap W_j}.$$ 

Now $U := \bigcup_{j \in J} W_j$ is an open neighborhood of $K$ in $M^*$. By the preceding,
the map $g: U \rightarrow N^*$ defined via $g(z) := g_j(z)$ if $z \in W_j$ is well defined. By
construction, it is a complex analytic extension of $f|_{M \cap W}$. 

(b) Using Lemma 2.1(d) instead of Lemma 2.1(a), we can repeat the proof of
(a) with $K := M$.

(c) Using Lemma 2.1(c) instead of Lemma 2.1(a), we can repeat the proof of
(a) with $K := M$.

(d) Since each $C$ (as in the statement) is connected and $g_1|_{M \cap C} = g_2|_{M \cap C}$
with $M \cap C \neq \emptyset$, the Identity Theorem shows that $g_1|_C = g_2|_C$. The assertion
follows.

(e) Let $W$ be the union of all connected components $C$ of the open set
$g^{-1}(V)$ such that $M \cap C \neq \emptyset$. Then $h \circ g|_W$ defines a complex analytic map
$W \rightarrow M^*$ such that $h(g(x)) = x$ for all $x \in M \cap W$. Hence $h \circ g|_W = \text{id}_W$,
by the Identity Theorem. In particular, $W \subseteq h(V)$, and $g|_W: W \rightarrow g(W)$ is a bijection with $(g|_W)^{-1} = h|_{g(W)}$. Let $Q$ be the union of all connected components $D$ of $h^{-1}(W)$ such that $N \cap D \neq \emptyset$. Arguing as before, we see that
Because $g(W) \cap h(Q) = \text{id}_Q$. In particular, $Q \subseteq g(W)$. If $C \subseteq W$ is as above, then $g(C)$ is a connected subset of $N$ such that $N \cap g(C) \neq \emptyset$. Since $h(g(C)) = C \subseteq W$, we have $g(C) \subseteq h^{-1}(W)$ and thus $g(C) \subseteq D$ for some $D$ as above. Hence $g(C) \subseteq Q$, entailing that $C = h(g(C)) \subseteq h(Q)$. We deduce that $W = h(Q)$. Thus $h|_Q: Q \to W$ is a bijection with inverse $g|_W$.

Proof of Theorem B. (a) By Lemma 2.2(c), the real analytic map $\text{id}_M: M \to M$ extends to complex analytic mappings $g: U \to M^*_1$ and $h: V \to M^*_1$, defined on open neighborhoods $U \subseteq M^*_1$ and $V \subseteq M^*_2$, respectively, of $M$. By Lemma 2.2(e), after shrinking $U$ and $V$, the map $g$ is a complex analytic diffeomorphism from $U$ onto $V$ (with inverse $h$).

(b) Applying Lemma 2.2(a) to the identity map $\text{id}_M: M \to M$, we find open neighborhoods $U$ and $V$ of $K$ in $M^*_1$ and $M^*_2$, respectively, and complex analytic mappings $g: U \to M^*_2$ and $h: V \to M^*_1$ such that $g|_{M^*_1} = \text{id}_{M^*_1}$ and $h|_{M^*_2} = \text{id}_{M^*_2}$. Then $N := (M \cap U) \cap (M \cap V)$ is an open neighborhood of $K$ in $M$, and both $M^*_1$ and $M^*_2$ are complexifications of $N$. By Lemma 2.2(e), after shrinking $U$ and $V$, the map $g$ is a complex analytic diffeomorphism from $U$ onto $V$ (with inverse $h$).

2.3 Proposition Let $M^*$ be a complexification of a real analytic manifold $M$.

(a) If every open neighborhood of $M$ in $M^*$ contains a paracompact open neighborhood of $M$ in $M^*$, then there exists a paracompact open neighborhood $U$ of $M$ in $M^*$ such that $M$ is closed in $U$.

(b) If $M^*$ is paracompact and $M$ is closed in $M^*$, then there exists an open neighborhood $U$ of $M$ in $M^*$ and an anti-holomorphic map $\sigma: U \to V$ such that $\sigma \circ \sigma = \text{id}_U$ and $M \cap U = \{ z \in U : \sigma(z) = z \}$.

(c) If $K \subseteq M$ is a non-empty compact set and $M^*$ is regular as a topological space, then there exists an open neighborhood $U$ of $K$ in $M^*$ and an anti-holomorphic map $\sigma: U \to V$ such that $\sigma \circ \sigma = \text{id}_U$ and $M \cap U = \{ z \in U : \sigma(z) = z \}$. In particular, $M \cap U$ is closed in $U$.

Proof. (a) Let $E$ be the modeling space of $M$ and $\tau: E_C \to E_C$ be complex conjugation, $\tau(x + iy) := x - iy$ for $x, y \in E$. By Remark 1.6 and Lemma 2.2(b), $\text{id}_M$ admits a complex analytic extension $g: V \to (M^*)_{\text{op}}$ on an open neighborhood $V$ of $M$ in $M^*$. For $x \in M$, let $\psi_x: U^*_x \to U^*_x$ be a chart for $M^*$ such that $\psi_x(M \cap U^*_x) = E \cap U^*_x$ and $U^*_x \subseteq V$. Then $\tau \circ \psi_x$ is a chart for $(M^*)_{\text{op}}$. Let $Q_x \subseteq U^*_x$ be a connected neighborhood of $x$ such that $g(Q_x) \subseteq U^*_x$. Then $P_x := \psi_x(Q_x)$ is a connected open subset of $E_C$ such that $\psi_x(x) \in E \cap P_x$. Because $\tau \circ \psi_x \circ g \circ \psi_x^{-1}|_{P_x}$ and $\text{id}_{P_x}$ are complex analytic maps $P_x \to E_C$ which coincide with $\text{id}_{E \cap P_x}$ on $E \cap P_x$, the Identity Theorem shows that $\tau \circ \psi_x \circ g \circ \psi_x^{-1}|_{P_x} = \text{id}_{P_x}$ and thus $\psi_x \circ g \circ \psi_x^{-1}|_{P_x} = \tau|_{P_x}$. As the latter map moves each vector in $P_x \setminus E$, we deduce that $g(z) \neq z$ for all $z \in Q_x \setminus M$. Hence, after replacing $V$ with its open subset $Q := \bigcup_{x \in M} Q_x$, we may assume that $g(z) \neq z$ for all $z \in V \setminus M$. By Lemma 2.2(e), there is an open neighborhood $W$ of $M$ in $V$ such that $g(W)$ is open and $g: W \to g(W) \subseteq (M^*)_{\text{op}}$ is a complex analytic diffeomorphism, with inverse $g|_{g(W)}$. Henceforth, we may consider $g$ as an anti-holomorphic diffeomorphism $W \to g(W) \subseteq M^*$. Now $P := W \cap g(W)$ is an open
such that $\forall \phi \in \mathcal{M}$, we construct open neighborhoods $V^\phi_\phi$ such that $\mathcal{M} \subseteq \{ z \in P : \sigma(z) = z \}$ and actually $\mathcal{M} = \{ z \in P : \sigma(z) = z \}$ (as we ensured by passing to $Q$ that $\sigma$ moves all points outside $M$). In particular, $M$ is closed in $P$. By hypothesis, there exists a paracompact open neighborhood $U$ of $M$ in $P$. By the preceding, $M$ is closed in $U$.

(b) Using Lemma 2.2(c) instead of Lemma 2.2(b), we can repeat the proof of (a) to find an open neighborhood $V$ of $M$ in $M^*$ and an anti-holomorphic involution $\sigma : P \rightarrow P$ with fixed point set $\mathcal{M}$.

(c) Using Lemma 2.2(a) instead of Lemma 2.2(b), we obtain $g : V \rightarrow (M^*)_{\text{op}}$ on an open neighborhood $V$ of $K$ in $M^*$. We can now continue as in the proof of (a), replacing each occurrence of "$M$" with "$V \cap M$" and taking $U := P$ at the end.

3 Existence of complexifications around a compact set

In this section, we prove Theorem 1.2. The proof splits into a proposition, where the complexification is constructed, and a corollary. The latter establishes regularity of the complexification.

3.1 Proposition Let $M$ be a real analytic manifold modeled on a locally convex space $E$. Assume that the topological space $M$ is regular and $E_{\mathbb{C}}$ is normal. For each compact subset $K$ of $M$, there is an open neighbourhood $\Omega$ of $K$ in $M$ such that the open submanifold $\Omega$ admits a complexification $(\Omega^*, \varphi)$.

Proof. We construct the neighborhood $\Omega$ of $K$ as a union of finitely many chart domains: For $x \in K$, choose a manifold chart $\varphi_x : M \supseteq T_x' \rightarrow T_x$ of $M$ with $x \in T_x'$. Since $E$ is regular as a topological vector space and $M$ is regular by assumption, [[24, Proposition 1.5.5]] allows us to choose an open neighborhood $W_x'$ of $x$ with closure $\overline{W_x'} \subseteq T_x'$ in $M$. Then $\varphi_x(\overline{W_x'})$ is a closed (with respect to the subspace topology) neighborhood of $\varphi_x(x)$ in $T_x$. Moreover, we can choose an open neighborhood $W_x$ of $\varphi_x(x)$ with closure $\overline{W_x} \subseteq T_x$ in $E$. Define the open sets $U_x := \varphi_x(W_x') \cap W_x$ and $U_x^\prime := \varphi_x^{-1}(U_x) = W_x' \cap \varphi_x^{-1}(W_x)$

and observe that the set $\overline{U_x} \subseteq \overline{W_x}$ is closed in $E$ and $\overline{U_x} \subseteq \overline{W_x}$ is closed in $M$. Analogously, we construct open neighborhoods $V_x'$ and $V_x$ with $\varphi_x(V_x') = V_x$, such that $\overline{V_x} \subseteq U_x'$ and $\overline{V_x} \subseteq U_x$. The open sets $(V_x')_{x \in K}$ cover $K$. Since $K$ is compact, we obtain finite families of open subsets $(V_i')_{i \in I}$, $(U_i')_{i \in I}$ and $(T_i')_{i \in I}$ of $M$ with the following properties for each $i \in I$:

There is a map $\varphi_i : T_i' \rightarrow T_i \subseteq E$ such that $(T_i', \varphi_i)$ is a manifold chart for $M$.

The sets are ordered via $\overline{V_i} \subseteq U_i' \subseteq \overline{U_i} \subseteq T_i'$ and the sets $\overline{V_i}, \overline{U_i}$ are closed in $M$. The compact set $K$ is contained in $\Omega := \bigcup_{i \in I} V_i' \subseteq M$. Furthermore, the sets $\varphi_i(V_i')$ and $\varphi_i(U_i')$ are closed in $E$. Notice that $\Omega$ is an open real analytic submanifold of $M$, such that the charts $(V_i', \varphi_i|_{V_i'})_{i \in I}$ form an atlas for $\Omega$.

We now construct a complexification for the open submanifold $\Omega$. To shorten
the notation, we define the following sets for $i, j \in I$:

\[
U_i := \varphi_i(U_i'), \quad V_i := \varphi_i(V_i'), \quad T_i := \varphi_i(T_i') \subseteq E,
\]

\[
U_{i,j} := \varphi_i(U_i' \cap U_j'), \quad V_{i,j} := \varphi_i(V_i' \cap V_j'), \quad T_{i,j} := \varphi_i(T_i' \cap T_j').
\]

The change of charts $\varphi_j \circ \varphi_i^{-1}|_{T_{i,j}} : T_{i,j} \to T_{j,i}$ is a real analytic diffeomorphism. Hence, by Lemma 2.2(e), there are open neighborhoods $T_{i,j}^*$ of $T_{i,j}$ and $T_{j,i}^*$ of $T_{j,i}$ in $E_C$ together with a complex analytic diffeomorphism $\psi_{i,j} : T_{i,j}^* \to T_{j,i}^*$ which extends $\varphi_j \circ \varphi_i^{-1}|_{T_{i,j}}$. Adjusting choices, we can achieve that $T_{i,j}^* \cap E = T_{i,j}$ and $\psi_{i,j} = \id|_{T_{i,j}}$ for all $i, j \in I$. Without loss of generality, $T_{i,j}^*$ is the empty set if $T_{i,j}$ is empty and

\[
\psi_{i,j} = \varphi_{i,j}^{-1} \quad \text{for each pair } (i, j) \in I \times I. \tag{5}
\]

For $(i, j) \in I \times I$, the open set $U_{i,j}$ satisfies $U_{i,j} \subseteq U_i \cap U_j \subseteq T_{i,j}$. Here the closure in $T_{i,j}$ coincides with the closure in $E_C$. Hence $U_{i,j} \times \{0\}$ is a closed subset of $T_{i,j}^*$. By normality of $E_C$, we can choose an open neighborhood $O_{i,j}$ of $U_{i,j} \times \{0\}$ whose closure with respect to $E_C$ satisfies $O_{i,j} \subseteq T_{i,j}^*$. The open subset $U_{i,j}^* := O_{i,j} \cap (U_{i,j} \times E)$ of $T_{i,j}^*$ satisfies

\[
\overline{U}_{i,j} \subseteq T_{i,j}, \quad U_{i,j}^* \cap E = U_{i,j} \quad \text{and} \quad \overline{U}_{i,j} \cap E = U_{i,j}. \tag{6}
\]

Shrinking the sets $U_{i,j}^*$, we can achieve that $\psi_{i,j}(U_{i,j}^*) = U_{i,j}^*$, since by construction we have $\psi_{i,j}(U_{i,j} \times \{0\}) = \varphi_{i,j}^{-1}(U_{i,j}) \times \{0\} = U_{i,j} \times \{0\}$. The set $\overline{V}_{i} \cap \varphi_{i,j}^{-1}((\overline{V}_{j} \cap \overline{U}_{j,i}) \times \{0\})$ is closed in $U_{i,j}$. As above, we can choose an open neighborhood $W_{i,j}^*$ of $(\overline{V}_{i} \cap \varphi_{i,j}^{-1}((\overline{V}_{j} \cap \overline{U}_{j,i}) \times \{0\}))$ whose closure (with respect to $E_C$) is contained in $U_{i,j}^*$. Adjusting choices, we obtain $\psi_{i,j}(W_{i,j}^*) = W_{i,j}^*$ for each pair $(i, j) \in I \times I$.

Observe that $\psi_{i,j}((\overline{V}_{j} \cap \overline{U}_{j,i}) \times \{0\})$ is closed in $E_C$, since it is relatively closed in $T_{i,j}$ and contained in the $E_C$-closed set $U_{i,j} \times \{0\} \subseteq T_{i,j}$. Thus $(\overline{V}_{i} \times \{0\}) \cap W_{i,j}^*$ and $\psi_{i,j}((\overline{V}_{j} \cap \overline{U}_{j,i}) \times \{0\}) \cap W_{i,j}^*$ are disjoint closed subsets of $E_C$. As $E_C$ is a normal space, there are disjoint open subsets $A_{i,j}^*$ and $B_{i,j}^*$ of $E_C$ such that

\[
\overline{V}_{i} \subseteq A_{i,j}^* \cup W_{i,j}^* \quad \text{and} \quad \psi_{i,j}((\overline{V}_{j} \cap \overline{U}_{j,i}) \times \{0\}) \subseteq W_{i,j}^* \cup B_{i,j}^*. \tag{7}
\]

An argument analogous to the one which produced (3) yields an open subset $A_{i,j}^*$ of $E_C$ with the following properties:

\[
A_{i,j}^* \cap E = V_i, \quad \overline{A}_{i,j} \cap E = \overline{V}_i, \quad \text{and} \quad A_{i,j}^* \subseteq \bigcap_{j \in I, T_{i,j} \neq 0} A_{i,j}^* \cup W_{i,j}^*. \tag{8}
\]

By construction, $\overline{A}_{i,j} \cap U_{i,j}$ is a closed subset of $U_{i,j} \subseteq T_{i,j}$, whence the diffeomorphism $\psi_{i,j}$ satisfies $\psi_{i,j}(A_{i,j}^* \cap U_{i,j}^*) = \psi_{i,j}(\overline{A}_{i,j} \cap U_{i,j})$. Combine (5) and (8) to obtain

\[
\psi_{i,j}(A_{i,j}^* \cap U_{i,j}^*) \cap E = \psi_{i,j}(\overline{A}_{i,j} \cap U_{i,j}) \cap E = \psi_{i,j}((\overline{V}_i \cap \overline{U}_{i,j}) \times \{0\}) \subseteq \psi_{i,j}((\overline{V}_i \cap \overline{U}_{i,j}) \cap E). \tag{10}
\]

Since $I$ is a finite set, we may choose for each $i \in I$ and $x \in U_i$ an open neighborhood $U_{i,x} \subseteq E_C$ such that the following properties are satisfied:
(a) For each $j \in I$ with $x \in U_{i,j}$, we have $U_{i,x}^* \subseteq U_{i,j}^*$.

(b) If $x \in \psi_{i,j}((V_j \cap U_{i,j}) \times \emptyset)$, then $U_{i,x}^* \subseteq W_{i,j}^* \cup B_{i,j}^*$ (cf. (7)).

(c) For each $j \in I$ with $\varphi_{i,x}^{-1}(x) \notin V_j$, the intersection $U_{i,x}^* \cap \psi_{i,j}(A_i^* \cap U_{i,j}^*)$ is empty.

(d) For each $(j,k) \in I \times I$ with $x \in U_{i,j} \cap U_{i,k}$, i.e. $\varphi_{i,x}^{-1}(x) \in U_i^* \cap U_j^* \cap U_k^*$, we have

$$U_{i,x}^* \subseteq \psi_{i,j}(U_{i,j}^* \cap U_{i,k}^*) \cap \psi_{k,i}(U_{k,i}^* \cap U_{k,j}^*),$$

such that the cocycle condition $\psi_{i,j}|_{U_{i,x}^*} = \psi_{k,j} \circ \psi_{i,k}|_{U_{i,x}^*}$ holds\(^3\).

It is possible to choose neighborhoods with property (c) because of the following observations: If $U_{i,j}^* = \emptyset$, then the property is trivially satisfied. For $j \in I$ with $U_{i,j}^* \neq \emptyset$, the condition $\varphi_{i,x}^{-1}(x) \notin V_j$ implies $x \notin \psi_{i,j}(A_i^* \cap U_{i,j}^*)$ by (10).

Define the set $U_i^* := \bigcup_{x \in U_i^*} U_{i,x}^*$ and observe that $\bigcup_{i \in I} \{V_i \times \emptyset\}$ is a closed set contained in $U_i^*$. By (8), there is an open neighborhood $V_i^*$ of $V_i$ which is contained in $A_i^* \cap U_i^*$. As $E_C$ is normal, we can choose $V_i^*$ such that its closure with respect to $E_C$ is contained in $U_i^*$. The identities (9) then yield $V_i^* \cap E = V_i$ and $\bigcup_{i \in I} E = V_i$. Define the following sets for $i, j, k \in I$:

$$V_{i,j}^* := V_i^* \cap \psi_{i,j}(V_j^* \cap U_{j,i}^*) \quad \text{and} \quad V_{i,j,k}^* := V_{i,j}^* \cap V_{i,k}^*.$$  

By construction, $V_{i,j}^* \subseteq U_{i,j}^*$ is satisfied, since $\psi_{i,j}$ maps $U_{i,j}^*$ to $U_{i,j}^*$. In particular, by (11), $\psi_{i,j}$ restricts to a map $\psi_{i,j}|_{V_{i,j}^*}$ which is a complex analytic diffeomorphism. Finally, we remark that $V_{i,i}^* = V_i^*$.

Each point $y \in V_{i,j,k}^*$ is contained in an open set $U_{i,x}^*$ for some $x \in U_i$. From (11), we infer $y \in U_{i,x}^* \cap \psi_{i,j}(V_j^* \cap U_{j,i}^*) \cap \psi_{k,i}(V_k^* \cap U_{k,i}^*)$. Thus, by choice of $V_i^*$ and $V_k^*$, the intersection $U_{i,x}^* \cap \psi_{i,j}(V_j^* \cap U_{j,i}^*) \cap \psi_{k,i}(V_k^* \cap U_{k,i}^*)$ is not empty. We conclude from property (c) for $U_{i,x}^*$, the following condition:

$$x \in \varphi_i(V_j \cap V_i^*) \cap \varphi_i(V_k \cap V_i^*) \subseteq \varphi_i(U_i^* \cap U_j^*) \cap \varphi_i(U_i^* \cap U_k^*) = U_{i,j} \cap U_{i,k}.$$

Hence property (d) for $U_{i,x}^*$ yields $\psi_{k,i} \circ \psi_{i,j}(y) = \psi_{i,j}(y)$. The point $z := \psi_{i,j}(y)$ is contained in $V_{i,j}^*$ since $\psi_{i,j}$ maps $V_{i,j}^*$ into this set. In particular, $z \in V_i^*$ holds. Furthermore, by definition of $V_{i,j,k}^*$, we obtain $\psi_{i,j}(y) \in V_k^* \cap U_{k,i}^*$ (cf. (11)). Hence we deduce $z = \psi_{i,j}(y) \in V_{i,j,k}^*$. As $y \in V_{i,j,k}^*$ was arbitrary, $\psi_{i,j}$ maps $V_{i,j,k}^*$ into $V_{i,j,k}^*$. An analogous argument yields $\psi_{i,j}(V_{i,j,k}^*) \subseteq V_{i,j,k}^*$. From (9), we deduce that $\psi_{i,j}|_{V_{i,j,k}^*}$ is an analytic diffeomorphism whose inverse is $\psi_{i,j}|_{V_{i,j,k}^*}^{-1}$; moreover, the identity $\psi_{i,j}|_{V_{i,j,k}^*} = \psi_{i,j} \circ \psi_{i,k}|_{V_{i,j,k}^*}$ holds.

Consider the disjoint union (topological sum) $\tilde{\Omega} := \bigsqcup_{i \in I} V_i^*$ and recall $V_i^* = V_{i,i}^*$. We declare a relation “∽” on $\tilde{\Omega}$. For two points $x \in V_i^*$ and $y \in V_j^*$, define:

$x \sim y \iff x \in V_i^*$, $y \in V_j^*$, and $y = \psi_{i,j}(x)$.

\(^3\)This is possible as the identity already holds on $(U_{i,j} \cap U_{i,k}) \times \emptyset$.}
From the construction above, it is clear that \( \sim \) is an equivalence relation on \( \tilde{\Omega} \).
Define \( \Omega^* \) to be the quotient space \( \tilde{\Omega}/\sim \). Denote the associated quotient map by \( q: \tilde{\Omega} \to \Omega^* \). Then the composition \( \kappa_i := q \circ (V_i^* \to \tilde{\Omega}) \) (with the natural embedding) is injective. The pairs \((V_i^*, \kappa_i)_{i \in I}\) form a family of complex charts for \( \Omega^* \). Indeed this family is an atlas for \( \Omega^* \) whose change of chart maps for \((i,j) \in I \times I \) are the complex analytic maps \( \psi_{i,j} \). Hence \( \Omega^* \) with this atlas becomes a (possibly non-Hausdorff) complex analytic manifold.

**Claim:** \( \Omega^* \) is a Hausdorff topological space.
If the claim is true then \( \Omega^* \) is a Hausdorff complex analytic manifold. Furthermore, the map \( \bigsqcup_{i \in I} \varphi_i|_{V_i^*}: \bigsqcup_{i \in I} V_i^* \to \Omega \) is a real analytic embedding whose image is contained in \( \Omega \cap E \) (identifying \( E \) with the real subspace \( E \times \{0\} \)). The change of chart maps for \( \Omega^* \) are complex analytic extensions \( \psi_{i,j} \) of the changes of charts \( \varphi_j \circ \varphi_i^{-1} \). Hence \( \bigsqcup_{i \in I} \varphi_i|_{V_i^*} \) factors to a real analytic embedding \( \varphi: \Omega \to \Omega^* \) whose image is a real analytic submanifold of \( \Omega^* \). By construction, the pair \((\Omega^*, \varphi)\) is a complexification of \( \Omega \). Hence the proof will be complete, if we can verify that \( \Omega^* \) is a Hausdorff space.

**Proof of the claim:** As a first step, we prove \( \overline{V}_{i,j}^* \subseteq U_{i,j} \) for \((i,j) \in I \times I \).
We already know \( \overline{W}_{i,j} \subseteq U_{i,j}^* \), whence it suffices to prove \( V_{i,j}^* \subseteq W_{i,j} \) for \((i,j) \in I \times I \). Moreover, it suffices to consider the case \( T_{i,j} \neq 0 \), since otherwise \( V_{i,j}^* \) is empty. Let \( y \in V_{i,j}^* \). By construction, \( V_{i,j}^* \subseteq V_{i}^* \subseteq U_{i}^* \) holds and there is some \( x \in U_i \) with \( y \in U_{i,x}^* \). If \( x \) is not contained in \( \psi_{i,j}((V_j \cap U_{j,i}) \times \{0\}) \), we deduce that \( \varphi_i^{-1}(x) \) is not contained in \( \overline{V}_{i,j} \). Hence property (c) for \( U_{i,x}^* \) implies \( y \not\in \psi_{j,i}(A_{j}^* \cap U_{j,i}^* \cup U_{j,x}^*) \). As \( V_{i,j}^* \subseteq A_{j}^* \) holds, we obtain \( y \not\in \psi_{j,i}(V_{j}^* \cap U_{j,i}^*) \). However, this contradicts \( y \in V_{i,j}^* \subseteq \psi_{j,i}(V_{j,i}^*) \) (cf. (11)). Therefore we must have \( x \in \psi_{j,i}((V_j \cap U_{j,i}) \times \{0\}) \).

Then property (b) for \( U_{i,x}^* \) implies \( y \in W_{i,j} \cup B_{i,j}^* \). On the other hand, \( y \in V_{i,j}^* \subseteq A_{j}^* \subseteq A_{j,i}^* \cup W_{i,j}^* \) by (9) and the definition of \( V_{i,j}^* \). Since \( A_{j,i}^* \) and \( B_{i,j}^* \) are disjoint, we obtain \( y \in W_{i,j}^* \). Thus \( V_{i,j}^* \subseteq W_{i,j}^* \) is satisfied.

To prove the Hausdorff property, let \( x', y' \in \Omega^* \) be two distinct points. We choose \( i, j \in I \) and points \( x \in V_{i}^* \) and \( y \in V_{j}^* \) with \( q(x) = x' \) and \( q(y) = y' \).

We have to construct neighborhoods of \( x \) in \( V_{i}^* \) and of \( y \) in \( V_{j}^* \) which contain no equivalent points. To reach a contradiction, suppose to the contrary that this was not possible. We could then find nets \((x_\alpha)_{\alpha \in A} \) and \((y_\alpha)_{\alpha \in A} \) indexed by a set \( A \) with \( x_\alpha \to x \) and \( y_\alpha \to y \) such that, for each \( \alpha \in A \), we have \( x_\alpha \in V_{i,j}^* \) and \( y_\alpha \in V_{j,i}^* \) and \( x_\alpha = \psi_{j,i}(y_\alpha) \). Observe that this implies \( x \in \overline{V}_{i,j} \subseteq U_{i,j}^* \) and \( y \in \overline{V}_{j,i} \subseteq U_{j,i}^* \).

The mapping \( \psi_{j,i}|_{U_{j,i}^*} \) is continuous, whence \( x = \psi_{j,i}(y) \) follows. We deduce that \( y \in V_{j}^* \cap U_{j,i}^* \) and thus \( x \in V_{i,j}^* \cap \psi_{j,i}(V_{j}^* \cap U_{j,i}^*) = V_{i,j}^* \).
Analogously, one derives \( y \in V_{j,i}^* \). Now \( x' = y' \) follows from the definition of \( \sim \). This contradicts our choices of \( x' \) and \( y' \), whence \( \Omega^* \) must be a Hausdorff space.

\[ \square \]

**3.2 Corollary** The manifold \( \Omega^* \) constructed in Proposition 3.1 is a regular topological space.

**Proof.** Let \( x \in \Omega^* \) and consider a closed set \( C \subseteq \Omega^* \) with \( x \not\in C \). We construct
open disjoint neighborhoods of $x$ and $C$, respectively. Let $q: \tilde{\Omega} \to \Omega^*$ be the quotient map and define $C_i := q^{-1}(C) \cap V^*_i$. Then $C_i$ is relatively closed in $V^*_i$. Furthermore, there is a non-empty subset $J \subseteq I$ such that $q^{-1}(x) \cap V^*_i$ is non-empty if and only if $i \in J$. Since $q|_{V^*_i}$ is injective, there is a unique $x_i \in q^{-1}(x)$ for each $i \in J$. Observe that this implies for $i \in J, j \in I$ that $x_i$ is contained in $V^*_{i,j}$ if and only if $j \in J$. The space $E_C$ is normal, hence a regular topological space. The subsets $V^*_i \subseteq E_C$ are thus regular spaces by [9, Theorem 2.1.6]. We construct a family of pairwise disjoint open sets $X_i, Y_i \subseteq V^*_i$ for each $i \in I$ as follows:

**Step 1:** Let $i \in J$. The space $V^*_i$ is regular, whence there are disjoint open subsets $X_i, Y_i \subseteq V^*_i$ such that $x_i \in X_i$ and $C_i \subseteq Y_i$.

**Step 2:** Consider $j \in I$ such that $x$ is not contained in $\overline{q(V^*_j)}$. In other words, $x_i$ is not contained in $\overline{V^*_{i,j}}$ (here closure is taken with respect to $V^*_j$) for each $i \in J$ and in particular $j \in I \setminus J$. For each $j$ with this property, define $X_j := \emptyset$ and $Y_j := V^*_j$. We recall $C_j \subseteq Y_j$. As $I$ is a finite set, we can shrink each $X_i, i \in J$ such that $X_i \cap \overline{V^*_{i,j}} = \emptyset$ holds for each $j \in I$ with $x \notin q(V^*_j)$.

**Step 3:** Consider $j \in I$ with $x \in \partial q(V^*_j) = q(V^*_j) \setminus q(V^*_j)$, i.e. $q^{-1}(x) \cap V^*_{i,j} = \emptyset$ is satisfied. Since $x \in \partial q(V^*_j)$, there is $i \in J$ with $x_i \in \overline{V^*_{i,j}}$ (the sets $V^*_j$ are the domains of the change of chart mappings $\psi_{i,j}|_{V^*_j}$).

Consider the closure $\overline{C_j}$ of $C_j \subseteq V^*_j$ with respect to $E_C$. We prove that $\psi_{i,j}(x_i)$ is not contained in $\overline{C_j}$. To reach a contradiction, suppose this was wrong. We could then find a net $(y_\sigma)_{\sigma \in \Sigma} \subseteq C_j$ with $y_\sigma \to \psi_{i,j}(x_i)$. Recall from the proof of Proposition 3.1 that $\overline{V^*_{i,j}} \subseteq U^*_{i,j}$ is satisfied and the mapping $\psi_{i,j}|_{\overline{V^*_{i,j}}}$ is a real analytic diffeomorphism. Hence $\psi_{i,j}(X_i \cap U^*_{i,j})$ is an open neighborhood of $\psi_{i,j}(x_i)$ in $U^*_{i,j}$, which contains $\psi_{i,j}(x_i)$. Thus the following holds:

$$(y_\sigma)_{\sigma \in \Sigma} \subseteq C_j \cap \psi_{i,j}(X_i \cap U^*_{i,j}) \subseteq V^*_j \cap \psi_{i,j}(X_i \cap U^*_{i,j}) \subseteq V^*_j \cap \psi_{i,j}(V^*_i \cap U^*_{i,j}) = V^*_j.$$ 

Now $q(\psi_{i,j}(y_\sigma)) = q(y_\sigma) \in C$ yields $\psi_{i,j}(y_\sigma) \in C_i$. Furthermore, we observe $\psi_{i,j}(y_\sigma) \subseteq \psi_{i,j}(X_i \cap U^*_{i,j}) \subseteq X_i$. But this contradicts the choice of $X_i$ for $i \in J$, as $C_i \cap X_i = \emptyset$ (cf. Step 1). In conclusion, $\psi_{i,j}(x_i)$ is not contained in $\overline{C_j}$. By regularity of $E_C$, there are open disjoint neighborhoods $X'_j$ of $\psi_{i,j}(x_i)$ and $Y'_j$ of $\overline{C_j}$: Define $X_j := V^*_j \cap X'_j$ and $Y_j := V^*_j \cap Y'_j$.

Shrinking the open sets obtained for $i \in J$, we may achieve the following:

(a) $\psi_{i,j}(X_i) = X_j$ for all $i, j \in J$.

(b) For each $j \in I \setminus J$, i.e. $x_i \notin V^*_j$, the set $X_i \cap U_{i,j}$ satisfies $\psi_{i,j}(X_i \cap U_{i,j}) \subseteq X_j$. Observe that by Step 2 this condition is trivially satisfied if $x \notin q(V^*_j)$.

Define neighborhoods $X := \bigcup_{i \in I} q(X_i)$ and $Y := \bigcup_{i \in I} q(Y_i)$. These sets are open in $\Omega^*$ and $X$ is a neighborhood of $x$. Since $q^{-1}(C) = \bigcup_{i \in I} C_i \subseteq \bigcup_{i \in I} Y_i$, the open set $Y$ contains $C$.

We claim that $X$ and $Y$ are disjoint. If this is true, then $\Omega^*$ is a regular
We conclude that there cannot be such an index $j$ such that $\tilde{z} \in q^{-1}(z) \cap V_i^*$. Since $z \in X \cap Y$, we obtain:

$$\tilde{z} \in q^{-1}(X) \cap q^{-1}(Y) \cap V_i^* \subseteq X_i \cap \bigcup_{i \in I} \psi_{j,i}(Y_j \cap V_{i,j}^*).$$

The inclusion above holds for $i \in I \setminus J$ by property (b) for the $X_i$. By construction, the sets $X_i$ and $Y_i = \psi_{i,j}(Y_j \cap V_{i,j}^*)$ are disjoint, whence there must be $j \neq i$ with $\tilde{z} \in \psi_{j,i}(Y_j \cap V_{i,j}^*)$. For such $i$, we obtain $\psi_{i,j}(\tilde{z}) \in Y_j$. However, $\psi_{i,j}(\tilde{z})$ is the uniquely determined preimage of $z$ with respect to $q|_{V_i^*}$. Hence $\psi_{i,j}(\tilde{z})$ is an element of $X_j$ as $z \in X$ holds. Again, this contradicts $Y_j \cap X_j = \emptyset$. We conclude that there cannot be such an index $j$. Summing up, the set $X \cap Y$ must be empty.

Sticking to the general theme of constructing complexifications for infinite-dimensional manifolds we now turn to complexifications of locally convex vector bundles. For finite-dimensional vector bundles these results seem to be part of the folklore. The notation of vector bundles we use follows [13, Section 3].

3.3 Definition Let $(E, \pi, M)$ be a real analytic locally convex vector bundle. We say that $(E, \pi, M)$ admits a bundle complexification, if there exists a complex analytic bundle $(E^*, p, M^*)$ such that $M^*$ is a complexification of $M$, $E^*$ is a complexification of $E$ and the bundle projection $p$ restricts to the bundle projection $\pi$ on $E \subseteq E^*$.

3.4 Lemma Let $N$ be a real analytic manifold with a complexification $N_C$.

(a) The bundle $(TN, \pi_{TN}, N)$ admits a bundle complexification, it is given by the bundle $(T(N_C), \pi_{T(N_C)}, N_C)$.

(b) If $N_C$ is paracompact and $N$ is modeled on a metrisable space, then $TN \subseteq T(N_C)$ is a paracompact complexification which is unique in the sense of Theorem B (a). Note that in this case $N$ is a paracompact manifold.

Proof. (a) Let $E$ be the model space of $N$ and $\Psi: N_C \supseteq U_{\Psi} \to V_{\Psi} \subseteq E_C$ be an adapted chart of $N_C$, i.e. $\psi := \Psi|_{U_{\Psi} \cap N}$ is a chart for $N$. Then each $T\Psi$ is a bundle trivialisation for the complex analytic bundle which restricts to the bundle trivialisations $T\psi$ of $TN \to N$. Note that the inclusion $\iota E \to E_C$ is real analytic. Hence for each adapted chart $T\Psi$ of $N_C$ the map $I_{\Psi} : \pi_{TN}^{-1}(X_{\Psi}) \to \pi_{T(N_C)}^{-1}(X_{\Psi}), I_{\Psi} := T\Psi^{-1} \circ \iota \circ T\psi$ is real analytic. Fix a family $F$ of adapted charts for $N_C$ whose domain covers $N$. Then the family $(I_{\Psi})_{\Psi \in F}$ glues to a real analytic map $I: TN \to T(N_C)$ which is fibre-wise the inclusion $\iota$. Moreover it is clear from the construction that $I$ takes $TN$ diffeomorphically to a real analytic submanifold of $T(N_C)$. We conclude that (up to identification) $T(N_C)$ is a complexification of $TN$.

(b) Let now $N$ be modeled on a metrisable space and $N_C$ be paracompact. Hence $E_C$ is also metrisable. Now $N_C$ is paracompact and locally metrizable and the fibre of $T(N_C)$ is metrizable, the paracompactness of $T(N_C)$ follows from [13] Proposition 29.7.
To prove the uniqueness of the germ in the sense of Theorem B (a) it suffices to see that the image of $I$ is closed in $T(N_C)$. As $N_C$ is paracompact and modeled on a metrizable space, $N_C$ is metrizable. Hence Proposition 2.3(a) shows that there is a paracompact open neighborhood $\Omega \subseteq N_C$ of $N$ such that $N$ is a closed subset of $\Omega$. Now assume that without loss of generality $X_\Psi \subseteq \Omega$ for all $\Psi \in F$. By paracompactness of $\Omega$, we shrink the family $F$ such that the family of open sets $(X_\Psi)_F$ becomes a locally finite cover of $N$. Following [16] Lemma 5.1.6, we can choose a locally finite closed cover $C$ of $\Omega$ subordinate to the locally finite open $\{X_\Psi\}_{\Psi \in F} \cup \{\Omega \setminus N\}$. Denote for all $\Psi \in F$ by $A_\Psi$ the element in $C$ subordinate to the set $X_\Psi$. We note that $N \subseteq \bigcup_{\Psi \in F} A_\Psi$ holds. Then $\text{im} I = \bigcup_{\Psi}(T \Psi^{-1}(A_\Psi \cap N) \times (E \times \{0\}))$ is closed as a union of a locally finite family of closed sets thus completing the proof.

Let us complete the picture for bundle trivialisations with finite dimensional paracompact base.

3.5 Proposition Let $(E,\pi,M)$ be a real analytic locally convex bundle with finite dimensional base $M$. If $M$ is paracompact, then $(E,\pi,M)$ admits a bundle complexification. Moreover, the bundle complexification is unique.

Proof. Let $F$ be the typical fibre of the bundle $(E,\pi,M)$. We describe the bundle structure of $(E,\pi,M)$ via cocycles. To this end fix an atlas $F$ of real analytic bundle trivialisations for $(E,\pi,M)$ indexed by a set $I$. For each pair $(i,j) \in I^2$ we obtain a cocycle $g_{ij} : X_{ij} := X_{\psi_i} \cap X_{\psi_j} \times F \to F$, i.e. a real analytic mapping which is linear in the second component. We will now extend the cocycles to complex analytic mappings whose domains glue to a complex analytic vector bundle.

As $M$ is a finite dimensional and paracompact real analytic manifold, $M$ admits a complexification $M^*$. Following [16], we can realize $M^*$ as a Grauert tube, i.e. $M^* \subseteq T^*M$ is an open $\mathbb{R}$-balanced neighborhood of the zero-section. Thus there is a canonical scalar multiplication on $M^*$ such that $[-1,1] \cdot M^* \subseteq M^*$ holds.

Each cocycle $g_{ij}, i,j \in I$ extends to a complex analytic map $\tilde{g}_{ij} : \tilde{W} \to F_{\mathbb{C}}$ where $\tilde{W} \subseteq M^* \times F_{\mathbb{C}}$ is an open neighborhood of $U_{ij}$. Shrinking $\tilde{W}$ we can assume that $\tilde{W} = \tilde{U}_{ij} \times \Omega \subseteq M^* \times F_{\mathbb{C}}$, where $\tilde{U}_{ij} \subseteq M^*$ is an $\mathbb{R}$-balanced neighborhood of $U_{ij}$ and $\Omega \subseteq F_{\mathbb{C}}$ is a zero-neighborhood. Now we shrink $\Omega$ such that it becomes a $C^1$-balanced set, i.e. $\overline{B_1^2(0)} \cdot \Omega = \Omega$. We consider the restriction of $\tilde{g}_{ij}$ to $\tilde{U}_{ij} \times \Omega$ which we also denote by $\tilde{g}_{ij}$. Then the complex analytic maps

$$\tilde{U}_{ij} \times \Omega \times \overline{B_1^2(0)} \to F_{\mathbb{C}}, (x,v,z) \mapsto \tilde{g}_{ij}(x,zv) \text{ and}$$

$$\tilde{U}_{ij} \times \Omega \times \overline{B_1^2(0)} \to F_{\mathbb{C}}, (x,v,z) \mapsto z\tilde{g}_{ij}(x,v)$$

coincide on the the total real subset $U_{ij} \times (\Omega \cap F) \times [0,1]$ as $g_{ij}$ is (real)-linear. The identity theorem for real analytic maps thus shows that both maps coincide and we obtain the formula $\tilde{g}_{ij}(x,zv) = z\tilde{g}_{ij}(x,v)$ for all $(x,v,z) \in \tilde{U}_{ij} \times \Omega \times \overline{B_1^2(0)}$. A similar argument shows that for $v,w \in \Omega$ with $v+w \in \Omega$ the

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formula $\tilde{g}_{ij}(x,v + w) = \tilde{g}_{ij}(x,v) + \tilde{g}_{ij}(x,w)$ holds. Thus for each pair $(i,j) \in I^2$ we obtain a well defined complex analytic map

$$\tilde{h}_{ij}: \tilde{U}_{ij} \times F_C \to F_C, \tilde{h}_{ij}(x,v) := \tilde{g}_{ij}(x,zv) \cdot \frac{1}{z} \text{ for } z \in \mathbb{C}^* \text{ with } zv \in \Omega$$

which is complex linear in the second component.

We now replace $M^*$ with the open subset $\bigcup_{i,j \in I} \tilde{U}_{ij}$. Then the $\tilde{h}_{ij}$ are cocycles for a complex analytic bundle $E^* \to M^*$ To see this we have to establish the cocycle condition. Consider $i,j,k \in I$ with $\tilde{U}_{ij} \cap \tilde{U}_{jk} \neq \emptyset$. For each $z \in \tilde{U}_{ij} \cap \tilde{U}_{jk}$ we have $0 \cdot z \in U_{ij} \cap U_{jk} \subseteq \tilde{U}_{ij} \cap \tilde{U}_{jk}$. From the cocycle condition for the maps $g_{ij}$ we derive $\tilde{h}_{ij}(0z, g_{hk}(0z,v)) = \tilde{g}_{ij}(0z, g_{jk}(0z,v)) = \tilde{g}_{ik}(0z, v) = \tilde{h}_{ik}(0z,v)$ for all $v \in \Omega \cap F$. Again the identity theorem for complex analytic maps yields $\tilde{h}_{ij}(z, \tilde{h}_{jk}(z,v)) = \tilde{h}_{ik}(z,v)$ for all $z \in \tilde{U}_{ij} \cap \tilde{U}_{jk} \times \Omega$. We conclude that the cocycles $\tilde{h}_{ij}$, $i,j \in I$ give rise to a complex analytic bundle $\pi^*: E^* \to M^*$ which is a bundle complexification of $(E, \pi, M)$.

To see that this bundle complexification is unique, let $(E^*, p, M^*)$ be another bundle complexification. As the complexification of the finite-dimensional paracompact manifold $M$ is unique in the sense of Theorem B (a), without loss of generality we may assume $M^* = M^*$. Let $f_{\alpha\beta}: X_{\alpha\beta} \times F_C \to F_C$ with $\alpha, \beta \in I$ be a family of cocycles which defines the bundle structure for $(E^*, pM^*)$. Note that $X_{\alpha\beta} \subseteq M^*$ for all $\alpha, \beta \in I$. Going to open subsets of $E^*$ and $M^*$, we may assume that $X_{\alpha\beta}$ is also $\mathbb{R}$-balanced. In particular this implies for $i,j \in I$ and $\alpha, \beta \in I$ that the intersection $\tilde{U}_{ij} \cap X_{\alpha\beta}$ is $\mathbb{R}$-balanced. Hence if $\tilde{U}_{ij} \cap X_{\alpha\beta} \neq \emptyset$ the cocycles $f_{\alpha\beta}$ and $\tilde{h}_{ij}$ restrict on the real subset $X_{\alpha\beta} \cap \tilde{U}_{ij} \times F$ to cocycles of the bundle $E \to M$. Thus the identity theorem for complex analytic maps shows that the composition of $f_{\alpha\beta}$ and $\tilde{h}_{ij}$ is a cocycle for both complex bundles $E^* \to M^*$ and $E^* \to M^*$. We conclude that there is an open neighborhood $W$ of $E$ in $E^*$ together with a complex analytic diffeomorphism $\varphi: W \to \varphi(W) \subseteq E^*$ such that $\varphi(W)$ is an open neighborhood of $E$ in $E^*$ and $\varphi|_E = \text{id}_E$. 

4 The compactly regular locally convex vector space $\text{Germ}_\mathbb{C}(K, Z)$

Let $M$ be a $\mathbb{K}$-analytic manifold modeled on a metrizable locally convex topological $\mathbb{K}$-vector space $X$ and let $K \subseteq M$ be a compact subset. For a $\mathbb{K}$-analytic Banach-Lie group $H$, we consider all $\mathbb{K}$-analytic functions $\gamma: U_\gamma \to H$, defined on an open neighborhood $U_\gamma$ of $K$ in $M$. Identifying two such functions if they agree on a common smaller neighborhood, we obtain the set $\text{Germ}_\mathbb{K}(K, H)$ of equivalence classes $[\gamma]$ (the germs around $K$). With the operation $[\gamma][\eta] := [\gamma \eta]$ inherited by pointwise multiplication of representatives, the set $\text{Germ}_\mathbb{K}(K, H)$ becomes a group. In the special case that $Z$ is a Banach space over $\mathbb{K}$, we define $z[\gamma] := [z\gamma]$ for $z \in \mathbb{K}$ and $[\gamma] \in \text{Germ}_\mathbb{K}(K, Z)$. In this way, $\text{Germ}_\mathbb{K}(K, Z)$ becomes a vector space over $\mathbb{K}$.

In this and the following section, we shall always assume that $\mathbb{K} = \mathbb{C}$; all manifolds and vector spaces will be complex. In Section 5 we show that the group $\text{Germ}_\mathbb{C}(K, H)$ carries a $C^0$-regular Lie group structure. In the present section, we consider the easier case of germs with values in a Banach space instead of a Banach Lie group:
To this end, for the rest of this section, let $Z$ be a complex Banach space. Then, as just explained, the set $\text{Germ}_C(K, Z)$ carries a natural vector space structure. Our first step is to define a suitable topology on it. We start by observing that each function $\gamma : U \to Z$ defined on an open neighborhood of $K$ maps $K$ to a compact, hence bounded subset of the space $Z$. Therefore $\gamma(K)$ is contained in an open ball $B^Z_R(0)$ for an $R > 0$. The preimage $V := \gamma^{-1}(B^Z_R(0))$ is now an open neighborhood of $K$ such that $\gamma|_V$ is bounded. This means that each germ $[\gamma] \in \text{Germ}_C(K, Z)$ can be represented by a \textit{bounded} analytic function on a suitable open neighborhood of $K$.

For each open neighborhood $U \subseteq X$, the space $\text{BHol}(U, Z)$ of bounded $C$-analytic functions from $U$ to $Z$ is a closed vector subspace of the Banach space $(BC(U, Z), \|\cdot\|_\infty)$ of bounded continuous maps endowed with the sup-norm and hence, $\text{BHol}(U, Z)$ is a Banach space as well.

This allows us to endow the space $\text{Germ}_C(K, Z)$ with the locally convex inductive limit topology with respect to all Banach spaces $\text{BHol}(U, Z)$, where $U$ ranges through all open neighborhoods of $K$. Note that at this moment, we do not know if this topology is Hausdorff. Before examining this space in greater detail, we state a topological lemma which will be helpful later on:

\textbf{4.1 Lemma} Let $K$ be a compact subset of a manifold $M$, modeled on a metrizable space $X$.

(a) The set $K$ can be written as a finite union of (not necessarily disjoint) compact sets $K_1, \ldots, K_m$ such that each $K_j$ is contained in some chart.

(b) There exists a sequence $U_1 \supseteq U_2 \supseteq U_3 \supseteq \cdots$ of open subsets, forming a basis of neighborhoods of $K$ in $M$, i.e. each open neighborhood of $K$ contains one of the sets $U_n$ as a subset. This sequence may be chosen in a way such that each connected component of each $U_n$ has a nonempty intersection with $K$.

\textbf{Proof.} (a): The domains of charts for the manifold $M$ form an open cover of the compact set $K$. Therefore, $K$ is contained in a finite union of open chart domains $U_1, \ldots, U_m$. We will only show the statement for the case $m = 2$, the general case follows by an easy induction argument. To this end, assume that $K \subseteq U_1 \cup U_2$. The compact sets $A_1 := K \setminus U_2$ and $A_2 := K \setminus U_1$ are disjoint in the Hausdorff space $M$ and thus have disjoint open neighborhoods $W_1$ and $W_2$, respectively. Now, we have the decomposition $K = K_1 \cup K_2$ with $K_1 := K \setminus W_2$ and $K_2 := K \setminus W_1$.

(b): By part (a), we can write the compact set $K$ as a finite union $K = \bigcup_{j=1}^m K_j$ of compact sets each of which is contained in a chart domain $U_j$. For each $K_j$, which is a compact subset of a metrizable open set $U_j$, we can find a sequence \( \left( U_n^{(j)} \right)_{n \in \mathbb{N}} \), which is a basis of open neighborhoods of $K_j$ using a metric on $U_j$.

By taking the union $\widehat{U}_n := \bigcup_{j=1}^m U_n^{(j)}$, we obtain a sequence of open neighborhoods of $K$. With this construction, it is not automatic that each connected component of each $\widehat{U}_n$ has a nonempty intersection with $K$. However, by setting $U_n$ as the union of all connected components of $\widehat{U}_n$ intersecting $K$ nontrivially, we obtain a sequence $\left( U_n \right)_{n \in \mathbb{N}}$ having the desired properties.
From now on, we fix a basis of open neighborhoods \((U_n)_{n \in \mathbb{N}}\) of \(K\) as in part(b) of the preceding lemma and consider the Banach spaces \(BH\ell_0(U_n, Z)\) with the corresponding bonding maps

\[ \iota_{m,n} : BH\ell_0(U_m, Z) \to BH\ell_0(U_n, Z) : \gamma \mapsto \gamma|_{U_m} \text{ for } m \leq n. \]

These mappings are obviously continuous (with operator norm at most one) and since each \(U_n\) meets the compact set \(K\), we obtain the injectivity of the bonding maps by the Identity Theorem (Lemma 1.7).

Since the sequence \((U_n)_{n \in \mathbb{N}}\) is cofinal in the directed set of all open neighborhoods, we obtain the following

4.2 Proposition The locally convex vector topology on the space \(Germ_{\mathcal{C}}(K, Z)\) defined above makes it the locally convex direct limit of the sequence

\[ BH\ell_0(U_1, Z) \xrightarrow{\iota_{1,2}} BH\ell_0(U_2, Z) \xrightarrow{\iota_{2,3}} BH\ell_0(U_3, Z) \xrightarrow{\iota_{3,4}} \cdots \]

In particular, \(Germ_{\mathcal{C}}(K, Z)\) is an \((LB)\)-space, i.e. a direct limit of an ascending sequence of Banach spaces with injective continuous bonding maps. The topology on \(Germ_{\mathcal{C}}(K, Z)\) does not depend on the choice of the sequence \((U_n)_{n \in \mathbb{N}}\).

In general, \((LB)\)-spaces need not be well-behaved, for example an \((LB)\)-space need not be Hausdorff and even if it is Hausdorff, it need not be complete. However, for the class of compact regular \((LB)\)-spaces, the situation is nicer:

4.3 Lemma (Criterion for compact regularity) Let \(E := \bigcup_{n=1}^{\infty} E_n\) be a locally convex direct limit of Banach spaces. Consider the following statements:

(i) For every \(n \in \mathbb{N}\), there is an \(m \geq n\) such that for each \(\ell \geq m\), there is an open absolutely convex 0-neighborhood \(\Omega\) of \(E_n\) such that \(E_\ell\) and \(E_m\) induce the same topology on the set \(\Omega\).

(ii) \(\forall n \in \mathbb{N} \exists m \geq n \forall \varepsilon > 0, \ell \geq n \exists \delta > 0 : B_{E_m}^m(0) \cap B_{E_\ell}^\ell(0) \subseteq B_{E_n}^n(0)\).

(iii) The sequence \((E_n)_{n \in \mathbb{N}}\) is compactly regular, i.e. for every compact subset \(C\) of \(E\) there is an index \(n \in \mathbb{N}\) such that \(C\) is compact in \(E_n\).

(iv) The locally convex vector space \(E\) is Hausdorff and complete.

Then (i), (ii) and (iii) are equivalent and imply (iv).

Proof. The equivalence of (i) and (ii) is clear, since (ii) is just a restatement of (i). The rest of this proposition follows from statements in [24]: To be more concrete: [24] Theorem 6.4] says that (i) is equivalent to (iii) and that both are equivalent to a property of \((E_n)_{n \in \mathbb{N}}\) called acyclicity. [24] Proposition 6.3] shows that the limit topology of an acyclic sequence is Hausdorff and [24] Corollary 6.5] tells us that the limit of an acyclic sequence is complete. All of this is true in the more general setting of (LF)-spaces but we shall not use this here.

In the rest of this section, we show that \(Germ_{\mathcal{C}}(K, Z)\) is compactly regular. This will not only guarantee that the space is a complete Hausdorff topological
vector space but also that the (local and global) Lie groups we will construct later in Theorem 5.3 and Theorem 5.5 are $C^\alpha$-regular.

As a first step, we consider the case where there is no manifold $M$ involved, i.e. $K$ is a compact subset of the space $X$. The following fact will be useful:

4.4 Lemma (Factorization of bounded holomorphic functions) Let $X$ be a complex locally convex space and let $p : X \to [0, +\infty]$ be a continuous seminorm with open unit ball $B^p \subseteq X$. Let $X_p$ be the Banach space obtained as the completion of the normed space $(X/p^{-1}(\{0\}), p)$ and let $\pi_p : X \to X_p$ denote the canonical continuous linear map. For a subset $K \subseteq X$, we consider the open set $U := K + B^p$. Then the following map is an isometric isomorphism of Banach spaces:

$$\text{BHol}(W, Z) \to \text{BHol}(U, Z) : \gamma \mapsto \gamma \circ \pi_p,$$

where $W := \pi_p(K) + B^{X^p}(0)$ is the corresponding open subset in the Banach space $X_p$.

In particular, every bounded complex analytic map defined on $U$ factors through the Banach space $X_p$.

Proof. It is clear that $\pi_p(U)$ is dense in $W$. This implies that the map

$$\text{BHol}(W, Z) \to \text{BHol}(U, Z) : f \mapsto f \circ \pi_p,$$

is an isometric embedding. It remains to show the surjectivity. To this end, let $\eta \in \text{BHol}(U, Z)$ be given. We will show that there is a $\gamma \in \text{BHol}(W, Z)$ such that $\eta = \gamma \circ \pi_p$.

Let $a \in K$ be a point. Then $\eta$ admits a power series expansion around $a$, i.e. for $x \in a + B^p$, we have:

$$\eta(x) = \sum_{k=0}^{\infty} \beta_k(x - a, \ldots, x - a)$$

with continuous symmetric $k$-linear maps $\beta_k : X^k \to Z$.

Let $v \in X$ be a fixed vector and set $R := \frac{1}{p(v)} \in [0, +\infty]$. Then we define the following function of one complex scalar variable:

$$h : B^C_R(0) \to Z : z \mapsto f(a + zv) = \sum_{k=0}^{\infty} \beta_k(v, \ldots, v) \cdot z^k.$$

The coefficients of this series can be computed via Cauchy’s integral formula:

$$\beta_k(v, \ldots, v) = \frac{1}{2\pi i} \int_{|z|=r} \frac{h(z)}{z^{k+1}} \, dz = \frac{1}{2\pi i} \int_{|z|=r} \frac{f(a + zv)}{z^{k+1}} \, dz.$$

Applying the norm on both sides, we get the estimate:

$$\left\| \beta_k(v, \ldots, v) \right\|_Z = \left\| \frac{1}{2\pi i} \int_{|z|=r} \frac{f(a + zv)}{z^{k+1}} \, dz \right\|_Z \leq \frac{1}{2\pi} \cdot \frac{2\pi r \|f\|_{\infty}}{r^{k+1}}$$

$$= \|f\|_{\infty} \cdot \frac{1}{r^k} \leq \|f\|_{\infty}(p(v))^k.$$
With use of the polarization formula (e.g. [2, Theorem A]), we obtain that each $k$-linear map $\beta_k : X^k \to Z$ is continuous with respect to the seminorm $p$. This implies that it factors through the normed space $X/p_1(0)$ to a continuous $k$-linear map and using the completeness of the range space $Z$, we obtain a continuous extension $\tilde{\beta}_k$ to the Banach space $X_p$.

The power series $\sum_{k=0}^{\infty} \tilde{\beta}_k$ so obtained converges on the open ball $B_1^{X_p}(\pi_p(a))$ to a complex analytic $Z$-valued function.

Now, we let the point $a \in K$ vary and obtain a complex analytic map on each ball $B_1^{X_p}(\pi_p(a))$. By construction, it is clear that the functions so obtained agree on intersecting balls. Glueing together these functions, we get the function $\gamma : W \to Z$ with the desired properties. 

The preceding lemma enables us to restrict our attention to the case where the domain is a Banach space. This is useful because for functions defined on Banach spaces, we have the following tool which can be found in [6, Lemma 1.5]:

4.5 Lemma (Absolute convergence of families of bounded power series) Let $K \subseteq X$ be a nonempty subset of a complex normed vector space $X$. Let $W := K + B_R(0) = \bigcup_{a \in K} B_R^X(a)$ be a union of open balls with fixed radius $R > 0$. Now, consider a set $M$ of bounded complex analytic mappings from $W$ to a normed space $Z$ such that $\sup_{\gamma \in M} \|\gamma\|_{\infty} < \infty$. Then we have for all $r < \frac{R}{2e}$ the following estimate:

$$\sum_{k=0}^{\infty} \sup_{\gamma \in M} \sup_{a \in K} \|\gamma^{(k)}(a)\|_{op} \leq \frac{R}{R - 2er} \cdot \sup_{\gamma \in M} \|\gamma\|_{\infty}.$$  

4.6 Lemma (Compact Regularity in the case that $M = X$) Let $X$ be a metrizable complex locally convex vector space and let $K \subseteq X$ be a nonempty compact subset. Let $Z$ be a complex Banach space. Then the locally convex direct limit

$$G_{\text{C}}(K,Z) = \bigcup_{n \in \mathbb{N}} B_{\text{Hol}}(U_n, Z)$$

is Hausdorff and compactly regular. Here $(U_n)_{n \in \mathbb{N}}$ is as in Lemma 4.1(b).

Proof. For the whole proof, we fix a real number $r > 0$ which is strictly less than $\frac{1}{2e}$, where $e$ denotes Euler’s number.

Since the space $X$ is metrizable and locally convex, its topology is generated by a sequence of seminorms $(p_n)_{n \in \mathbb{N}}$. Replacing each $p_n$ by the seminorm

$$\tilde{p}_n := \frac{1}{p_n^r} \sum_{k \leq n} p_k,$$

we obtain a new sequence of seminorms, still generating the same topology, but with the additional property that $\tilde{p}_n \leq r \cdot \tilde{p}_{n+1}$. To simplify notation, we call this new sequence of seminorms again $(p_n)_{n \in \mathbb{N}}$. Let $B_1^{X_p}(0)$ be an open ball around zero with respect to the seminorm $p_k$. Since $r < 1$, there is a number $n \geq k$ such that $r^{n-k} < \varepsilon$ and therefore, the given ball $B_1^{X_p}(0)$ contains the unit
ball $B^{p_n} = B_{1}^{p_n}(0)$. This shows that the sequence of unit balls $(B^{p_n})_{n \in \mathbb{N}}$ is a basis of 0-neighborhoods in $X$.

From now on, we will use this basis of 0-neighborhoods to construct a basis of open neighborhoods of the compact set $K$ by setting

$$U_n := K + B^{p_n}.$$ 

By construction, each component of $U_n$ intersects $K$ non-trivially, so $(U_n)_{n \in \mathbb{N}}$ is a sequence of the form in Lemma 4.1(b).

In analogy to the notation in Lemma 4.3 we let $X_n$ denote the Banach space obtained by completing the normed space $(X/p^{-1}_n(0), p_n)$. The unit ball in $X_n$ is denoted by $B_n$, the canonical map by $\pi_n : X \rightarrow X_n$ and the bonding maps by $\eta_{k_1,k_2} : X_{k_2} \rightarrow X_{k_1}$:

$$\begin{align*}
\pi_1 & \quad \pi_2 & \quad \pi_3 & \quad \pi_4 & \quad \cdots \\
X_1 & \quad X_2 & \quad X_3 & \quad X_4 & \quad \cdots \\
\eta_{1,2} & \quad \eta_{2,3} & \quad \eta_{3,4} & \quad \cdots
\end{align*}$$

By construction of the spaces $X_n$, it is clear that the maps $\eta_{n,n+1} : X_{n+1} \rightarrow X_n$ are continuous linear with operator norm $\|\eta_{n,n+1}\|_{\text{op}} \leq r$ and with dense image.

Let $K_n := \pi_n(K)$ be the compact subset of $X_n$, and $W_n := K_n + B^n$ be the corresponding open neighborhood of $K_n$ in the Banach space $X_n$.

We want to show that the direct limit

$$\text{BHol}(U_1, Z) \rightarrow \text{BHol}(U_2, Z) \rightarrow \cdots$$

is compactly regular. By Lemma 4.3 each $\gamma \in \text{BHol}(U_n, Z)$ factors through $\pi_n$ and we may identify the Banach spaces $\text{BHol}(U_n, Z) \cong \text{BHol}(W_n, Z)$. It remains to show that the direct limit

$$E_1 := \text{BHol}(W_1, Z) \rightarrow E_2 := \text{BHol}(W_2, Z) \rightarrow \cdots$$

is compactly regular. We denote the bonding maps by

$$\iota_{k_1,k_2} : E_{k_1} \rightarrow E_{k_2} : \gamma \mapsto \gamma \circ \eta_{k_1,k_2}.$$

We will use (ii) in the characterisation in Lemma 4.3, i.e. let $n \in \mathbb{N}$ be given and set $m := n + 1$. Let $\ell \geq n + 1$ and $\varepsilon > 0$ be given. It remains to show that there is a number $\delta > 0$ such that

$$(B^{E_m}_{1}(0) \cap B^{E_n}_{1}(0)) \subseteq B^{E_m}_{\delta}(0).$$

We may apply Lemma 4.5 to the subset $K_n$ of the Banach space $X_n$, the set $M := \overline{B^{E_n}_{1}(0)}$, the number $R := 1$ and $r > 0$ as already defined at the beginning of this proof. Then we may conclude the convergence of the series

$$\sum_{k=0}^{\infty} s_k r^k \quad \text{with} \quad s_k := \sup_{\gamma \in \overline{B^{E_n}_{1}(0)}} \frac{\|\gamma(a)\|_{\text{op}}}{k!},$$

Since this sum is convergent, there is an index $k_0 \in \mathbb{N}$ such that $\sum_{k > k_0} s_k r^k \leq \frac{\varepsilon}{2}$. Now, we are able to define the desired number $\delta$ as:

$$\delta := (1 - 2\varepsilon r)^{-k_0} \cdot \frac{\varepsilon}{2}.$$
It remains to show that \( \mathcal{B}^{E_n}_1(0) \cap \mathcal{B}^{E_n}_\delta(0) \subseteq \mathcal{B}^{E_n}_\epsilon(0) \). To this end, let \( [\gamma] \in \mathcal{B}^{E_n}_1(0) \cap \mathcal{B}^{E_n}_\delta(0) \) be given. Since \( [\gamma] \in \mathcal{B}^{E_n}_1(0) \), we can view \( [\gamma] \) as an element \( \gamma_n \in E_n = \text{BHol}(W_n, Z) \) with \( \|\gamma_n\|_{E_n} \leq 1 \). By \( [\gamma] \in \mathcal{B}^{E_n}_\delta(0) \), we know that \( \gamma_\ell := t_n, x(\gamma_n) \) is an analytic function on \( W_\ell \) and \( \|\gamma_\ell\|_{E_\ell} < \delta \). We want to show that \( \|\gamma_{n+1}\|_{E_{n+1}} = \sup_{x \in W_{n+1}} \|\gamma_{n+1}(x)\| \leq \epsilon \). To this end, let \( x_{n+1} \in W_{n+1} \) be given. It remains to show that \( \|\gamma_{n+1}(x_{n+1})\| \leq \epsilon \).

Since \( \eta_{n+1, \ell}(W_\ell) \) is dense in \( W_{n+1} \), we may assume that \( x_{n+1} = \eta_{n+1, \ell}(x_\ell) \) for an \( x_\ell \in W_\ell = K_\ell + B_\ell^\ell \), i.e. the element \( x_\ell \) can be written as

\[ x_\ell = a_\ell + v_\ell \text{ with } a_\ell \in K_\ell \text{ and } \|v_\ell\|_{X_\ell} < 1. \]

Now, we can estimate the value of \( \gamma_{n+1}(x_{n+1}) \):

\[
\|\gamma_{n+1}(x_{n+1})\|_Z = \|\gamma_n(x_n)\|_Z \leq \sum_{k \leq k_0} \|\gamma^{(k)}_n(a_n)\|_Z (v_n, \ldots, v_n) \|
+ \sum_{k > k_0} \|\gamma^{(k)}_n(a_n)\|_Z (v_n, \ldots, v_n) \|
\leq \epsilon
\]

\((*)\)

The first sum of (4) can be estimated by:

\[
\sum_{k \leq k_0} \|\gamma^{(k)}_n(a_n)\|_Z (v_n, \ldots, v_n) \|
\leq \sum_{k \leq k_0} \frac{\|\gamma^{(k)}_\ell(a_\ell)\|_{\text{op}}}{k!} \|v_\ell\|_{E_\ell} \cdot r^{k} \cdot r^{-k_0} \leq 1
\]

\[
\leq \left( \sum_{k=0}^\infty \frac{\|\gamma^{(k)}_\ell(a_\ell)\|_{\text{op}}}{k!} r^{k} \right) \cdot r^{-k_0} \]

If we apply Lemma 4.3 to the Banach space \( X_\ell \), the one element family \( M = \{\gamma_\ell\} \) and parameters \( R = 1 \) and \( r > 0 \) as above, we obtain that this sum can be bounded above by

\[
\frac{1}{1 - 2er} \leq \delta \cdot r^{-k_0} = \frac{1}{1 - 2er} \cdot \delta \cdot r^{-k_0} = \frac{\epsilon}{2}
\]

by the choice of \( \delta \).

The second sum of (4) is equal to:

\[
\sum_{k > k_0} \|\gamma^{(k)}_n(a_n)\|_Z (v_n, \ldots, v_n) \|
\leq \sum_{k > k_0} \frac{\|\gamma^{(k)}_n(a_n)\|_{\text{op}}}{k!} \|\eta_{n,n+1}(v_{n+1})\|_{X_n}^k
\leq \sum_{k > k_0} s_k \left( \frac{\|\eta_{n,n+1}\|_{\text{op}}}{\epsilon} \frac{\|v_{n+1}\|_{X_{n+1}}}{r} \right)^k \leq \frac{\epsilon}{2}
\]

This finishes the proof. ∎
Let us now return to the general case of \( K \) being a compact subset of a manifold \( M \) which is modeled on the metrizable locally convex space \( X \). We will use the following easy tools from the theory of locally convex direct limits:

**4.7 Lemma** (Products and subspaces of compact regular direct limits)

(a) Let \( E = \bigcup_n E_n \) and \( F = \bigcup_n F_n \) be compactly regular direct limits of Banach spaces \((E_n)_{n \in \mathbb{N}}\) and \((F_n)_{n \in \mathbb{N}}\) respectively. Then the product \( E \times F \) can be written as the compactly regular direct limit

\[
E \times F = \bigcup_n (E_n \times F_n).
\]

(b) Consider two sequences of locally convex spaces \((E_n)_{n \in \mathbb{N}}\) and \((F_n)_{n \in \mathbb{N}}\) respectively, together with topological embeddings \( \tau_n : E_n \to F_n \) such that the following diagram commutes:

\[
\begin{array}{cccc}
E_1 & \to & E_2 & \to & \cdots \\
\downarrow \tau_1 & & \downarrow \tau_2 & & \\
F_1 & \to & F_2 & \to & \cdots
\end{array}
\]

Assume that \( E_m \cap \tau_n^{-1}(F_n) \subseteq E_n \) for all \( m, n \in \mathbb{N} \) and that the sequence \((F_n)_{n \in \mathbb{N}}\) is compactly regular. Then \((E_n)_{n \in \mathbb{N}}\) is compactly regular as well. (Note that we do not state that \( \bigcup_n E_n \) is a topological subspace of \( \bigcup_n F_n \) as this will not be true in general.)

**Proof.** (a):
Since the product of two locally convex vector spaces agrees with the direct sum whose topology is a final locally convex topology (as are locally convex direct limit topologies), we may use the well-known transitivity of final locally convex topologies and obtain that the product topology on \( E \times F \) agrees with the locally convex direct limit topology:

\[
E \times F = \bigcup_n (E_n \times F_n).
\]

Let \( C \subseteq E \times F \) be a compact set. Then the projections on the two factors \( E \) and \( F \) yield compact subsets \( C_E \) and \( C_F \) respectively. By compact regularity of \( E \) and \( F \), we find a number \( n \in \mathbb{N} \) such that \( C_E \) is a compact subset of \( E_n \) and \( C_F \) is a compact subset of \( F_n \). This implies that the product \( C_E \times C_F \) is a compact subset of \( E_n \times F_n \), and since \( C \subseteq C_E \times C_F \), the claim follows.

(b):
Since every \( \tau_n : E_n \to F_n \) is continuous linear, we obtain a continuous linear map \( \tau : E \to F \) by the universal property of the direct limit. The compact set \( C \subseteq E \) is then mapped onto the compact set \( \tau(C) \subseteq F \) which is a compact subset of \( F_n \) for a number \( n \in \mathbb{N} \) by the compact regularity of \((F_n)_{n \in \mathbb{N}}\). Let \( x \in C \subseteq E \) be given. Then there exists a number \( m \in \mathbb{N} \) such that \( x \in E_m \). By the hypothesis, this implies that \( x \in E_n \). Since \( x \in C \) was arbitrary, this implies that \( C \) is a subset of \( E_n \). Using that \( \tau_n : E_n \to F_n \) is a topological embedding, \( C \) is compact in \( E_n \) since \( \tau_n(C) \) is compact in \( F_n \). \( \square \)
Now, we are able to show the main result of this section:

4.8 Proposition (Compact Regularity of \( \text{Germ}_\mathbb{C}(K, Z) \)) Let \( M \) be a manifold modeled on the metrizable locally convex space \( X \). Let \( Z \) be a complex Banach space. Then for each compact subset \( K \subseteq M \) the space \( \text{Germ}_\mathbb{C}(K, Z) \) is the compact regular direct limit of the spaces \( \text{BHol}(U_n, Z) \) for \( n \in \mathbb{N} \), where \( (U_n)_{n \in \mathbb{N}} \) is any sequence of open neighborhoods as in part (b) of Lemma 4.1. In particular, the locally convex space \( \text{Germ}_\mathbb{C}(K, Z) \) is Hausdorff and complete.

Proof. In the case that \( K \) is so small that it is contained in one chart, we may assume that \( K \subseteq X \) and hence \( \text{Germ}_\mathbb{C}(K, Z) \) is compactly regular by Proposition 4.6. Consider the following

Claim: Let \( K', K'' \subseteq M \) be compact subsets such that \( \text{Germ}_\mathbb{C}(K', Z) \) and \( \text{Germ}_\mathbb{C}(K'', Z) \) are compactly regular. Then \( \text{Germ}_\mathbb{C}(K' \cup K'', Z) \) is compactly regular as well.

If this claim is true, then the proposition follows, as each compact set \( K \) can be written as a finite union of compact sets each of which is contained in one chart by part (a) of Lemma 4.1.

It remains to show the claim: To this end, let \( K' \) and \( K'' \) be two compact subsets with compactly regular direct limits \( \text{Germ}_\mathbb{C}(K', Z) \) and \( \text{Germ}_\mathbb{C}(K'', Z) \), respectively. By part (b) of Lemma 4.1, we obtain open neighborhoods \( (U_n)_{n \in \mathbb{N}} \) of \( K_1 \) and \( (U'_n)_{n \in \mathbb{N}} \) of \( K_2 \) respectively. Let us denote \( F_n' := \text{BHol}(U_n, Z) \) and \( F'_n := \text{BHol}(U'_n, Z) \) and \( E_n := \text{BHol}(U_n \cup U'_n, Z) \). The space \( E_n \) can be embedded in the product \( F_n' \times F'_n \) via

\[
\tau_n : E_n \rightarrow F_n' \times F'_n : \gamma \mapsto (\gamma|_{U_n'}, \gamma|_{U'_n}).
\]

The direct limit \( \bigcup_n (F_n' \times F'_n) \) is compactly regular by 4.7(a). There are two cases to consider: If \( K' \cap K'' = \emptyset \), then we may assume that for each \( n \in \mathbb{N} \), the two open sets, \( U_n' \) and \( U'_n \), are disjoint. Therefore the mappings \( \tau_n \) defined above are isomorphisms. This shows that \( \bigcup_n E_n \cong \bigcup_n (F_n' \times F'_n) \) is compactly regular.

Let us assume now that \( K' \cap K'' \neq \emptyset \). Let \( m, n \in \mathbb{N} \) be given. If we are able to show that \( E_m \cap \tau^{-1}(F_n) \subseteq E_n \), the compact regularity of \( \bigcup_n E_n \) follows by part (b) of Lemma 4.1. To this end, let \( \gamma \in E_n \), i.e. a function \( \gamma : U_n' \cup U_n'' \rightarrow Z \) be given and assume that \( \sigma(\gamma) = (\gamma|_{U'} , \gamma|_{U''}) \in (F_n' \times F'_n) \). This means that the function \( \gamma \) can be extended holomorphically to a function \( \gamma' : U_n' \rightarrow Z \) and to a function \( \gamma'' : U_n'' \rightarrow Z \). The domains \( U_n' \) and \( U_n'' \) are not disjoint since they both contain the nonempty set \( K' \cap K'' \). Hence, by the Identity Theorem (Lemma 4.7), the two extensions \( \gamma' : U_n' \rightarrow Z \) and \( \gamma'' : U_n'' \rightarrow Z \) can be combined to obtain an extension on the union \( U_n' \cup U_n'' \). This shows that \( \gamma \in E_n \) and finishes the proof.

5 Construction of a regular Lie group structure on \( \text{Germ}_\mathbb{C}(K, H) \)

In this section, we show Theorem C stated in the introduction, namely that there exists a \( C^\omega \)-regular Lie group structure on the space of Lie group valued
germs. Before we consider global Lie groups, we first recall the notion of a local Lie group:

5.1 Definition (Local Lie group) (a) Let $G$ be a smooth manifold, $D \subseteq G \times G$ an open subset, $1 \in G$, and let $m_G : D \rightarrow G : (x,y) \mapsto x \ast y, \eta_G : G \rightarrow G : x \mapsto x^{-1}$ be smooth maps. We call $(G, D, m_G, 1_G, \eta_G)$ a (smooth) local Lie group if

(Loc1) Assume that $(x, y), (y, z) \in D$. If $(x \ast y, z)$ or $(x, y \ast z) \in D$, then both are contained in $D$ and $(x \ast y) \ast z = x \ast (y \ast z)$.

(Loc2) For each $x \in G$ we have $(x, 1_G), (1_G, x) \in D$ and $x \ast 1_G = 1_G \ast x = x$.

(Loc3) For each $x \in G$ we have $(x, x^{-1}), (x^{-1}, x) \in D$ and $x \ast x^{-1} = x^{-1} \ast x = 1_G$.

(Loc4) If $(x, y) \in D$, then $(y^{-1}, x^{-1}) \in D$.

(b) The Lie algebra $L(G) := T_{1_G}G$ of a local Lie group $G = (G, D, m_G, 1_G, \eta_G)$ is the tangent space of the manifold $G$ at point $1_G$. As in the case of (global) Lie groups, there is a natural structure of a locally convex Lie algebra on $L(G)$.

(c) A local Lie group $G$ is called $C^0$-regular if there is an open $0$-neighborhood $\Omega \subseteq C([0, 1], L(G))$ such that each continuous curve $\gamma \in \Omega$ admits a continuously differentiable left evolution $\eta = \eta_\gamma : [0, 1] \rightarrow G$ determined by

$$\eta(0) = 1 \quad \text{and} \quad (\forall t \in [0, 1]) \quad \eta'(t) = \eta(t) \cdot \gamma(t),$$

and the evolution map $\text{evol}_G : \Omega \rightarrow G, \text{evol}(\gamma) := \eta_\gamma(1)$ is smooth (i.e., $C^\infty$).

Real analytic and complex local Lie groups are defined analogously. In these cases, smoothness is replaced with real and complex analyticity, respectively.

See [20, Definition II.1.10] for more details on local Lie groups. The following fact (see [4]) gives a description of local Lie groups modeled on Banach spaces:

5.2 (Local Banach Lie groups) Let $\mathfrak{g}$ be a complex Banach Lie algebra. Then there exits an absolutely convex open $0$-neighborhood $W$ such that the Baker-Campbell-Hausdorff-series (BCH-series for short) converges to a complex analytic map $*: W \times W \rightarrow \mathfrak{g}$. By setting $D := \{(x, y) \in W \times W | x \ast y \in W\}$, we obtain a local Banach Lie group $(W, D, *, 0, -\text{id}_W)$. Furthermore, every local Banach Lie group is locally isomorphic to one which is obtained in this fashion.

Now, we return to the setting of Section 4. Let $M$ be a complex analytic manifold modeled on a complex metrizable locally convex space $X$ and let $K$ be a compact non-empty subset. We fix a basis of open neighborhoods $(U_n)_{n \in \mathbb{N}}$ as in Lemma 4.1(b). As in the last section, all vector spaces will be complex.

5.3 Theorem (Regularity of the local group of germs) Let $\mathfrak{g}$ be a complex Banach-Lie algebra. The pointwise Lie bracket of $\mathfrak{g}$-valued germs in $\text{Germ}_C(K, \mathfrak{g})$
is continuous and turns $\text{Germ}_C(K, \mathfrak{h})$ into a complete locally convex topological Lie algebra.

Furthermore, there is an open 0-neighborhood $\Omega \subseteq \text{Germ}_C(K, \mathfrak{h})$ such that the BCH-series converges to a complex analytic map $*: \Omega \times \Omega \rightarrow \text{Germ}_C(K, \mathfrak{h})$. The open set $\Omega$ together with this multiplication becomes a complex local Lie group which is $C^0$-regular.

Proof of Theorem 5.3. For each $n \in \mathbb{N}$ the Banach space $\mathfrak{g}_n := \text{BHol}(U_n, \mathfrak{h})$ admits a pointwise Lie bracket and becomes a Banach-Lie algebra in its own right with a corresponding BCH-multiplication which corresponds to the pointwise BCH-multiplication of analytic maps.

By [8, Theorem 4.5 (a)], we conclude that the locally convex direct limit $\text{Germ}_C(K, \mathfrak{h})$ becomes a locally convex topological Lie algebra (the completeness of the space of germs was already shown in Proposition 4.8) admitting an open zero-neighborhood such that the BCH-multiplication defines a local Lie group structure. Furthermore, since the sequence $(\mathfrak{g}_n)_{n \in \mathbb{N}}$ is compactly regular by Proposition 4.8, we may apply part (c) of [8, Theorem 4.5] to conclude that the local Lie group so obtained is $C^0$-regular (which is called strongly $C^0$-regular there).

Finally, we will now consider germs of mappings with values in a (global) Banach Lie group. Therefore, let $H$ be a complex Banach Lie group and let $\mathfrak{h} := L(H)$ be its Lie algebra, endowed with a norm compatible with the Lie bracket. We denote the adjoint representation of $H$ on $\mathfrak{h}$ by $\text{Ad}_H: H \rightarrow \text{Aut}(\mathfrak{h}) \leq \text{GL}(\mathfrak{h})$. Before we endow the group $\text{Germ}_C(K, H)$ with a manifold structure, we will study the natural action of it on the locally convex Lie algebra we just defined:

5.4 Proposition The pointwise adjoint action

$$\text{AD}: \text{Germ}_C(K, H) \times \text{Germ}_C(K, \mathfrak{h}) \rightarrow \text{Germ}_C(K, \mathfrak{h}),$$

$$[\gamma], [\eta] := [x \mapsto \text{Ad}_H(\gamma(x)).\eta(x)]$$

of the abstract group $\text{Germ}_C(K, H)$ on the locally convex Lie algebra $\text{Germ}_C(K, \mathfrak{h})$ is well-defined and for each fixed $[\gamma] \in \text{Germ}_C(K, H)$, the linear map

$$\text{AD}(\gamma): \text{Germ}_C(K, \mathfrak{h}) \rightarrow \text{Germ}_C(K, \mathfrak{h}): [\eta] \mapsto [\gamma] \cdot [\eta]$$

is continuous.

Proof. Let a group element $[\gamma] \in \text{Germ}_C(K, H)$ and a Lie algebra element $[\eta] \in \text{Germ}_C(K, \mathfrak{h})$ be given. Then these germs can be represented by complex analytic functions: $\gamma: U_m \rightarrow H$ and $\eta: U_n \rightarrow \mathfrak{h}$, with $m, n \in \mathbb{N}$. We define the action of the group element $[\gamma]$ on the Lie algebra element $[\eta]$ as the germ $[\gamma].[\eta]$ of the following function:

$$\gamma.\eta: U_n \cap U_m \rightarrow \mathfrak{h}: x \mapsto \text{Ad}_H(\gamma(x)).\eta(x).$$

This function is analytic as a composition of analytic maps and it is clear that the germ $[\gamma.\eta]$ does not depend on the chosen functions $\gamma$ and $\eta$ but only on the corresponding germ.
It remains to show continuity. Let \( [\gamma] \in \text{Germ}_C(K, H) \) be an element in the abstract group. It can be represented by an analytic function

\[
\gamma : U \rightarrow H
\]

for \( U \) an open neighborhood of \( K \). Now, consider the composition with the adjoint representation of the Banach Lie group \( H \) which is an analytic map

\[
\text{Ad}_H : H \rightarrow \text{Aut}(h) \subseteq (\mathcal{L}(h), \|\cdot\|_{\text{op}}).
\]

Thus, the composition \( \text{Ad}_H \circ \gamma : U \rightarrow \mathcal{L}(h) \) is an analytic function on \( U_m \) with values in the Banach space \( \mathcal{L}(h) \) of bounded operators on \( h \). Hence, by the same arguments as in the last section, we may shrink the domain \( U \) to a smaller neighborhood \( V \) and assume that \( \text{Ad}_H \circ \gamma|_V \) is bounded by an \( R > 0 \) on the smaller set. We may assume that \( V = U_m \) for a number \( m \in \mathbb{N} \).

Now, for each \( n \geq m \), we consider the linear map

\[
A_n : \text{BH} \mathcal{O}(U_n, h) \rightarrow \text{BH} \mathcal{O}(U_n, h) : \eta \mapsto \gamma. \eta
\]

with \( (\gamma. \eta)(x) := \text{Ad}_H(\gamma(x)). \eta(x) \). This linear map is continuous, since for each \( \eta \in \text{BH} \mathcal{O}(U_n, h) \), we have

\[
\|A_n(\eta)\|_{\infty} = \sup_{x \in U_n} \|\text{Ad}_H(\gamma(x)). \eta(x)\|_h \leq \sup_{x \in U_n} \|\text{Ad}_H(\gamma(x))\|_{\text{op}} \|\eta(x)\|_h \leq R \|\eta\|_{\infty}.
\]

This shows that each \( A_n \) is continuous linear (with \( \|A_n\|_{\text{op}} \leq R \)) and by the universal property of the locally convex direct limit, the direct limit map

\[
\text{AD}([\gamma]) : \text{Germ}_C(K, h) \rightarrow \text{Germ}_C(K, h) : [\eta] \mapsto [\gamma] \cdot [\eta]
\]

is continuous as well.

Now, we are able to show the main result about the group \( \text{Germ}_C(K, H) \), in particular that it carries a natural Lie group structure modelled on the \((\text{LB})-\)space \( \text{Germ}_C(K, h) \):

**5.5 Theorem** On the group \( \text{Germ}_C(K, H) \), there exists a unique locally convex complex Lie group structure such that the map

\[
\text{EXP} : \text{Germ}_C(K, h) \rightarrow \text{Germ}_C(K, H) : [\eta] \mapsto [\exp_H \circ \eta]
\]

becomes a complex analytic local diffeomorphism. The Lie algebra of this Lie group is the complete \((\text{LB})\)-space \( \text{Germ}_C(K, h) \) and its exponential map is the map \( \text{EXP} : \text{Germ}_C(K, h) \rightarrow \text{Germ}_C(K, H) \) just defined.

Furthermore, the Lie group \( \text{Germ}_C(K, H) \) is \( C^0 \)-regular. In particular, it is regular in Milnor’s sense.

To show Theorem 5.5, we shall use the following fact, which can be found in [7, Corollary 1.3.16] and is based on a general construction principle (see e.g. [3, Chapter III,§1.9, Prop. 18]).

**5.6 (Construction of a Lie group with a given exponential function)***
(a) Let $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$ and let $\mathfrak{g}$ be a Hausdorff locally convex $\mathbb{K}$-Lie algebra and let $\Omega_1$ and $\Omega_2$ be open symmetric $0$-neighborhoods in $\mathfrak{g}$ such that the BCH-series converges on $U \times U$ and defines a $C^\omega_\mathbb{K}$-map $*: \Omega_1 \times \Omega_1 \longrightarrow \Omega_2$. Let $\Phi: \mathfrak{g} \longrightarrow \mathcal{G}$ be an map into an abstract group $\mathcal{G}$ satisfying

- $\Phi(\Omega_2)$ is injective.
- $\Phi(\Omega_1) = (\Phi(x))^n$ for $n \in \mathbb{N}, x \in \mathfrak{g}$.
- $\Phi(x \cdot y) = \Phi(x) \cdot \Phi(y)$ for $x, y \in \Omega_1$.

Then there exists a unique $C^\omega_\mathbb{K}$-Lie group structure on $\mathcal{G}_0 := \langle \Phi(\mathfrak{g}) \rangle = \langle \Phi(\Omega_1) \rangle = \langle \Phi(\Omega_2) \rangle$ such that $\Phi|_{\Omega_1}: \Omega_1 \longrightarrow \Phi(\Omega_1)$ becomes a diffeomorphism onto an open subset.

Furthermore, the linear map $T_0 \Phi: \mathfrak{g} \longrightarrow L(\mathcal{G}_0)$ is an isomorphism of locally convex Lie algebras and after identifying $\mathfrak{g}$ with $L(\mathcal{G}_0)$, we obtain that $\mathcal{G}_0$ admits a $C^\omega$ exponential function and we have $\exp_\mathcal{G} = \Phi$.

(b) Let $\mathcal{G}$ be an abstract group and let $\mathcal{N} \subseteq \mathcal{G}$ be a normal subgroup, carrying a $C^\omega_\mathbb{K}$-Lie group structure. We assume that for each $a \in \mathcal{G}$ the conjugation map

$$\mathcal{N} \longrightarrow \mathcal{N}: g \mapsto a \cdot g \cdot a^{-1}$$

is $C^\omega_\mathbb{K}$. Then $\mathcal{G}$ carries a unique $C^\omega_\mathbb{K}$-structure such that $\mathcal{N}$ is open in $\mathcal{G}$.

Now, we can prove Theorem 5.5.

**Proof of Theorem 5.5** We set $\mathfrak{g} := \text{Germ}_C(K, \mathfrak{h})$ and let $\mathcal{G} := \text{Germ}_C(K, H)$ and let

$$\Phi := \text{EXP}: \text{Germ}_C(K, \mathfrak{h}) \longrightarrow \text{Germ}_C(K, H): [\eta] \mapsto [\exp_H \circ [\eta]].$$  

Since $H$ is a Banach Lie group, there is an open ball $B^K_\mathfrak{h}(0)$ such that $\exp_H|_{B^K_\mathfrak{h}(0)}$ is injective. This implies that the map $\Phi$ is injective on the set

$$\Omega_2 := \{ [\gamma] \in \text{Germ}_C(K, \mathfrak{h}) | \gamma(K) \subseteq B^K_\mathfrak{h}(0) \}$$

which is clearly an open neighborhood in $\mathfrak{g} = \text{Germ}_C(K, \mathfrak{h})$. Furthermore, we know by Theorem 5.3 that there is a symmetric open $0$-neighborhood in $\Omega_1 \subseteq \mathfrak{g}$ such that the BCH-multiplication on $\mathfrak{g}$ converges on $\Omega_1 \times \Omega_1$ to a $C^\omega_\mathbb{K}$-map $*: \Omega_1 \times \Omega_1 \longrightarrow \mathfrak{g}$. By shrinking $\Omega_1$ if necessary, we may also assume that $\Omega_1 \cdot \Omega_1 \subseteq \Omega_2$. These neighborhoods $\Omega_1$ and $\Omega_2$ now satisfy the hypotheses of part (a) of Proposition 5.6. Therefore, we obtain an analytic Lie group structure on the group $\mathcal{G}_0$ generated by the image of $\Phi$ such that $\Phi$ maps the $0$-neighborhood $\Omega_1 \subseteq \mathfrak{g}$ diffeomorphically to the open identity neighborhood $\Phi(\Omega_1) \subseteq \mathcal{G}_0$.

Next, we want to use part (b) of Proposition 5.6 to extend the Lie group structure on $\langle \Phi(\mathfrak{g}) \rangle$ to the whole group $\mathcal{G} = \text{Germ}_C(K, H)$. To this end, let $[\gamma] \in \mathcal{G}$ and let $[\eta] \in \mathfrak{g}$ be given. Then we have the formula

$$[\gamma] \cdot \Phi([\eta]) \cdot ([\gamma])^{-1} = \Phi(\text{AD}([\gamma]). [\eta])$$

which is easily checked. This shows that $\Phi(\mathfrak{g})$ is invariant under the conjugation with elements of the group $\mathcal{G}$, hence $\langle \Phi(\mathfrak{g}) \rangle$ is a normal subgroup of $\mathcal{G}$. It remains to show that the map

$$\langle \Phi(\mathfrak{g}) \rangle \longrightarrow \langle \Phi(\mathfrak{g}) \rangle: [\xi] \mapsto [\gamma] \cdot [\xi] \cdot ([\gamma])^{-1}$$

(12)
is analytic for each fixed group element $[\gamma] \in G$. Since $\Phi$ is an analytic local diffeomorphism at 0 (taking 0 to the identity element) and $\text{AD}([\gamma])$ is continuous linear (and hence analytic) by Proposition 5.4 we deduce from (3) that the conjugation map $\Phi$ is analytic on an identity neighborhood and hence analytic (being a group homomorphism). Therefore, by part (b) of Proposition 5.6 there is a unique Lie group structure on the group $\text{Germ}_{C}(K,H)$ such that $\langle \Phi(\text{Germ}_{C}(K,H)) \rangle$ is an open subgroup.

It remains to show that this Lie group is $C^0$-regular. By Lemma 9.5 in [21], we know that a Lie group is $C^0$-regular if and only if it has an open identity neighborhood which is a $C^0$-regular local Lie group. Since $\Phi(\Omega_1)$ is an open identity neighborhood of $G_0 = \langle \Phi(\mathfrak{g}) \rangle$ which itself is open in $\text{Germ}_{C}(K,H)$, the regularity of $\text{Germ}_{C}(K,H)$ follows from the regularity of the local group $(\Omega_1, \ast)$ which was shown in Theorem 5.3.

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