Exactly solvable piecewise analytic double well potential \( V_D(x) = \min[(x + d)^2, (x - d)^2] \) and its dual single well potential \( V_S(x) = \max[(x + d)^2, (x - d)^2] \)

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Abstract

By putting two harmonic oscillator potential \( x^2 \) side by side with a separation \( 2d \), two exactly solvable piecewise analytic quantum systems with a free parameter \( d > 0 \) are obtained. Due to the mirror symmetry, their eigenvalues \( \{E\} \) for the even and odd parity sectors are determined exactly as the zeros of certain combinations of the confluent hypergeometric function \( \text{$_1F_1$} \) of \( d \) and \( E \), which are common to \( V_D \) and \( V_S \) but in two different branches. The eigenfunctions are the piecewise square integrable combinations of \( \text{$_1F_1$} \), the so called \( U \) functions. By comparing the eigenvalues and eigenfunctions for various values of the separation \( d \), vivid pictures unfold showing the tunneling effects between the two wells.

Published

Journal of Mathematical Physics 64 (2023) 022102

1 Introduction

Double well potentials in quantum mechanics are discussed in various contexts, \( \text{e.g.} \) tunneling or spontaneous symmetry breaking, etc. Most commonly studied are the quartic potentials \( V_Q(x) = x^4 - ax^2 + bx, (a > 0) \), whose eigenfunctions are quite complicated and the system is far from exactly solvable. Some solvable examples are the square double well and the double Dirac delta potential. Recently Miloslav Znojil introduced an interesting double and single well potentials \( (2.1), (2.2) \) \[1\]. They are just mirror symmetrically separated harmonic oscillator potentials, which are not analytic at the origin but exactly solvable due to the harmonic oscillator nature. One motivation of this paper is to supplement the pioneering work of Znojil.
Another profound motivation is to disseminate the possibility to enlarge the list of exactly solvable quantum mechanical systems by combining the technique of constructing mirror symmetric non-analytic solvable potentials [1, 2, 3, 4] with other known methods of solvability such as the factorisation [5, 6], shape-invariance [7], the exceptional and multi-indexed polynomials [8, 9, 10, 11], non-polynomial extensions [12], etc, in particular, the Krein-Adler deformations [13, 14, 15]. By incorporating the present solvable model construction method to the simplest Krein-Adler deformation of the harmonic oscillator potential [16], a new double well potential is proposed in §5 (5.3). It is expected to be the breakthrough point for constructing a multitude of similarly exactly solvable potentials.

The present paper is prepared in a plain style so that non-experts can easily understand. This paper is organised as follows. In section two, after a brief introduction of the double and single well potentials, the simplification of the connection conditions in mirror symmetric potentials due to the separation into the even and odd sectors is recapitulated. The symmetric relationships between the connection conditions of the double and single well potentials are stated as Theorem 2.1. A simple lower bound of the eigenvalues of the single well potential is mentioned as Remark 2.2. In section three, elementary polynomial type solutions are briefly surveyed. The connection conditions determining the eigenvalues of the even and odd sectors are expressed as the zeros of the Hermite polynomials and the derivatives in Theorem 3.1. The polynomial type eigenfunctions are displayed in Theorem 3.2 together with the explicit expressions of the connection conditions in Tables 1 and 2. The relationship with the results in [11] is remarked. The upper bound of the greatest zeros of the Hermite polynomials is mentioned in connection with Remark 2.2. In §4, starting with the Kummer differential equation, the piecewise square integrable combination of the confluent hypergeometric functions are introduced. Theorem 4.1 states that the eigenvalues are obtained as the zeros of the connection conditions for the even and odd sectors. The ‘duality’ of the connection conditions for the double and single well potentials is alluded in Remark 4.2. Seven lowest eigenvalues for the double and single well potentials for a few small values of d’s are shown in Tables 3 and 4. Several graphs of some lower eigenfunctions are shown in Figures 1–4. A simple interpretation of the tunneling effects on the even and odd sector eigenvalues are presented. In §4.2 it is shown that various quantities and expressions in §4.1 are simplified for odd integer eigenvalues. In section five, after a very brief summary, two new piecewise analytic double and single well potentials are proposed (5.1), (5.4) together
with the graphs of the potentials in Figure 5 and 6.

2 Mirror symmetric and piecewise analytic potential

Here we explore a new exactly solvable double well potential $V_D(x)$ and its dual single well potential $V_S(x)$ with $d > 0$,

$$V_D(x) = \min[(x+d)^2, (x-d)^2] = \begin{cases} (x-d)^2 & x \geq 0 \\ (x+d)^2 & x \leq 0 \end{cases}, \quad V_D(x) = V_D(-x), \quad (2.1)$$

$$V_S(x) = \max[(x+d)^2, (x-d)^2] = \begin{cases} (x+d)^2 & x \geq 0 \\ (x-d)^2 & x \leq 0 \end{cases}, \quad V_S(x) = V_S(-x), \quad (2.2)$$

which were recently introduced by Miloslav Znojil in a pioneering work [1]. This paper will be cited as I hereafter. For $d = 0$, $V_D$ and $V_S$ reduce to the well known harmonic oscillator $V_D(x) = V_S(x) = x^2$. Since the min, max definitions of $V_D(x)$ and $V_S(x)$ are obviously symmetric with $d \leftrightarrow -d$, we have restricted to $d > 0$ and the $d$-dependence of the potentials, the wavefunctions, eigenvalues etc. is usually suppressed for the simplicity of presentation.

These potentials are obviously mirror symmetric $V(x) = V(-x)$ and analytic on either half line $x > 0$ and $x < 0$ and the non-analyticity occurs only at the origin $x = 0$. That is, the wavefunctions on either half line are analytic functions. That is, their wavefunctions $\psi_D(x)$ and $\psi_S(x)$ of the Schrödinger equations

$$-\frac{d^2\psi_D(x, E)}{dx^2} + V_D(x)\psi_D(x, E) = E\psi_D(x, E), \quad -\frac{d^2\psi_S(x, E)}{dx^2} + V_S(x)\psi_S(x, E) = E\psi_S(x, E), \quad (2.3)$$

are piecewise analytic

$$\psi_D(x, E) = \begin{cases} \psi_D^+(x, E) & x > 0 \\ \psi_D^-(x, E) & x < 0 \end{cases}, \quad \psi_S(x, E) = \begin{cases} \psi_S^+(x, E) & x > 0 \\ \psi_S^-(x, E) & x < 0 \end{cases}.$$  

Let us assume that the above wavefunctions are piecewise square integrable for both D and S,

$$\int_0^\infty (\psi_D^+(x, E))^2 \, dx < \infty, \quad \int_{-\infty}^0 (\psi_D^-(x, E))^2 \, dx < \infty.$$

This selects one solution in the two-dimensional solution space of the above Schrödinger equations (2.3) for generic $E$. Like other one-dimensional quantum mechanical systems with piecewise analytic potentials, we require the continuity of the wavefunctions and their first derivatives

$$\psi_D^+(0, E) = \psi_D^-(0, E), \quad \frac{d\psi_D^+(0, E)}{dx} = \frac{d\psi_D^-(0, E)}{dx}. \quad (2.4)$$
These select the eigenvalues \{E_n\}, \(n = 0, 1, \ldots\), since the continuous wavefunctions are square integrable eigenfunctions \{\psi(x, E_n)\},

\[
\int_{-\infty}^{\infty} \psi(x, E_n)^2 dx < \infty, \quad n = 0, 1, \ldots
\]

Thanks to the mirror symmetry, this solution process is simplified extensively as demonstrated in other similar examples, \(V(x) = -g^2\exp(-|x|)\) \[2\], \(V(x) = g^2\exp(2|x|)\) \[3\] and symmetric Morse potential \[4\], etc. Due to the mirror symmetry of the potentials the wavefunctions are split into the even and odd parity sectors

\[
even : \psi(+) (x, E) = \psi(-) (-x, E), \quad \text{odd : } \psi(+) (x, E) = -\psi(-) (-x, E).
\] (2.5)

The continuity of the wavefunctions and their first derivatives provides the equations determining the eigenvalue \(E\), which have the same forms for D and S,

\[
even : \quad \psi(+) (0, E) = \psi(-) (0, E), \quad \frac{d\psi(+) (0, E)}{dx} = 0, \quad \frac{d\psi(-) (0, E)}{dx} = 0, \quad \psi(+) (0, E) = 0.
\] (2.6)

\[
\text{odd : } \quad \frac{d\psi(+) (0, E)}{dx} = \frac{d\psi(-) (0, E)}{dx}, \quad \psi(+) (0, E) = 0.
\] (2.7)

Thanks to the mirror symmetry, the first condition is trivially satisfied for both sectors by fixing the relative scales of the \(\psi(+) (x)\) and \(\psi(-) (x)\), The second condition determines the eigenvalues \{\(E_n\)\} \(n = 0, 1, \ldots\) as functions of the system parameters. In the present case they are \(d\). The second conditions can be replaced by the equivalent one \(\frac{d\psi(-) (0, E)}{dx} = 0\) for the even sector and \(\psi(-) (0, E) = 0\) for the odd sector. Obviously the even parity condition (2.6) is the Neumann boundary condition and the odd parity one (2.7) is the Dirichlet boundary condition. This is the rare occasion that the Neumann b.c. appears in quantum mechanics. The Dirichlet b.c. appears wherever an impenetrable barrier stands.

Another simplification is built in due to the forms of the double and single well potentials \(V_D(x)\) (2.1) and \(V_S(x)\) (2.2). On the positive half line \(x > 0\), \(V_D(x) = (x-d)^2\), \(V_S(x) = (x+d)^2\) and they interchange by \(d \leftrightarrow -d\). The same situation happens on the negative half line, \(x < 0\), too. Therefore, when the Neumann b.c. equation (2.6) is written down for the \(V_D(x)\) wavefunctions, the equation for the \(V_S(x)\) wavefunctions is simply obtained by changing \(d\) into \(-d\), and vice versa. The situation is the same for the Dirichlet b.c. equation (2.7). Let us write down the equations determining the eigenvalues \{\(E\)\} due to the Neumann b.c. (2.6) (for the even sector) and due to the Dirichlet b.c. (2.7) (for the odd sector) of the \(V_D\) and
The following theorem states their close relationship.

**Theorem 2.1** For the even and odd sectors, the functions for the double well and single well are simply related by

\[
even: C_D^{(e)}(x, E) = C_S^{(e)}(-x, E), \quad \text{odd: } C_D^{(o)}(x, E) = C_S^{(o)}(-x, E), \quad x \in \mathbb{R}, \tag{2.10}
\]

up to some irrelevant constant factors.

This is why we call \(V_S(x)\) is the dual potential of \(V_D(x)\), and vice versa. Without determining these functions, we can safely make the following statement concerning the lower bounds of the eigenvalues of the \(V_S\) system.

**Remark 2.2** For both even and odd sectors, the eigenvalues of the single well system are greater than \(d^2\),

\[
V_S(x) \geq d^2 \implies E > d^2. \tag{2.11}
\]

### 3 Polynomial type solutions

The most basic result of one-dimensional quantum mechanics is that the Hermite polynomials \(\{H_n(x)\}\) provide the complete set of eigenfunctions of the quadratic potential \(x^2\). This means,

\[
-\frac{d^2\Psi_p(x, 2n+1)}{dx^2} + (x+d)^2\Psi_p(x, 2n+1) = (2n+1)\Psi_p(x, 2n+1), \quad n \in \mathbb{Z}_{\geq 0}, \tag{3.1}
\]

\[
-\frac{d^2\Psi_m(x, 2n+1)}{dx^2} + (x-d)^2\Psi_m(x, 2n+1) = (2n+1)\Psi_m(x, 2n+1), \quad n \in \mathbb{Z}_{\geq 0}, \tag{3.2}
\]

in which \(\alpha_n\) is a constant. Here the subscript \(p\) means ‘plus’ \(d\), *i.e.* \((x+d)^2\) potential and \(m\) means ‘minus’ \(d\), \((x-d)^2\) potential. The degree \(n\) Hermite polynomial \(H_n(x)\) has the parity \(H_n(-x) = (-1)^nH_n(x)\). This means that for \(E = 2n+1\), for example, \(\psi_{S}^{(+)}(x, 2n+1) = e^{-(x+d)^2/2}H_n(x+d)\) is a piecewise square integrable wavefunction of the single well system on the right half line \(x > 0\). Thus we arrive at a theorem.
Theorem 3.1  The Neumann \((2.6)\) and Dirichlet \((2.7)\) b.c. provide the equations

\[
\text{even : } H'_n(d) - dH_n(d) = 0, \quad \text{odd : } H_n(d) = 0, \quad (H'_n(d) = 2nH_{n-1}(d)), \quad (3.3)
\]
determining a finite number of \(\{d\}\)’s with which the continuous connection with the left half line wavefunction \(\psi_S^{(-)}(x,2n+1)\) is realised.

Due to the parity of the Hermite polynomial, the contents of these equations are the same when \(d\) is changed to \(-d\), meaning that the above equations apply to the double well system, too. According to Theorem \([2.1]\) we arrive at the following theorem.

Theorem 3.2  To each positive odd integer \(2n+1\) \((n \in \mathbb{N})\) correspond two sets of distinct positive parameters \(\{d^e_j\}, j = 1, \ldots, [\langle n \rangle/2]\), and \(\{d^o_j\}, j = 1, \ldots, [(n)/2]\), satisfying \(H'_n(d^e_j) - d^e_jH_n(d^e_j) = 0\) and \(H_n(d^o_j) = 0\) \((3.3)\), respectively. For the even type \(d^e_j\), the Schrödinger equations with \(V_D(x)\) and \(V_S(x)\) potential have an even parity eigenstate with the eigenvalue \(2n+1\),

\[
\psi^{(e)}_{D,j}(x,2n+1) = \begin{cases} 
  e^{-(x+d^e_j)^2/2}H_n(x+d^e_j) & -\infty < x \leq 0 \\
  (-1)^n e^{-(x-d^e_j)^2/2}H_n(x-d^e_j) & 0 \leq x < \infty
\end{cases}, \quad (3.4)
\]

\[
\psi^{(e)}_{S,j}(x,2n+1) = \begin{cases} 
  (-1)^n e^{-(x-d^e_j)^2/2}H_n(x-d^e_j) & -\infty < x \leq 0 \\
  e^{-(x+d^e_j)^2/2}H_n(x+d^e_j) & 0 \leq x < \infty
\end{cases}. \quad (3.5)
\]

For the odd type \(d^o_j\), the Schrödinger equations with \(V_D(x)\) and \(V_S(x)\) potential have an odd parity eigenstate with the eigenvalue \(2n+1\),

\[
\psi^{(o)}_{D,j}(x,2n+1) = \begin{cases} 
  -e^{-(x+d^o_j)^2/2}H_n(x+d^o_j) & -\infty < x \leq 0 \\
  (-1)^n e^{-(x-d^o_j)^2/2}H_n(x-d^o_j) & 0 \leq x < \infty
\end{cases}, \quad (3.6)
\]

\[
\psi^{(o)}_{S,j}(x,2n+1) = \begin{cases} 
  (-1)^n e^{-(x-d^o_j)^2/2}H_n(x-d^o_j) & -\infty < x \leq 0 \\
  -e^{-(x+d^o_j)^2/2}H_n(x+d^o_j) & 0 \leq x < \infty
\end{cases}. \quad (3.7)
\]

Here \([a]\) denotes the greatest integer not exceeding \(a\) and \(\langle n \rangle = n+1\) for odd \(n\) and \(\langle n \rangle = n\) for even \(n\). Likewise \(\langle n \rangle = n+1\) for even \(n\) and \(\langle n \rangle = n\) for odd \(n\). Obviously these numbers are distinct \(\{d^e_j\} \cap \{d^o_j\} = \emptyset\). Here we list the explicit expressions of the connection conditions \((3.3)\) and the corresponding values of \(\{d^e_j\}\) and \(\{d^o_j\}\) for \(n\) up to 6. It is straightforward to verify that the even connection condition \((3.3)\) is the same as \((1.21)\) of Znojil’s paper [1] and the odd condition \((3.3)\) agrees with \((1.24)\).
Table 1: Even parameters

| n  | \(-(H_n'(d) - dH_n(d))\) | \(d_j^e\): six digits |
|----|--------------------------|-----------------------|
| 1  | 2(-1 + d^2)              | 1                     |
| 2  | 2d(-5 + 2d^2)            | 1.58114               |
| 3  | 4(3 - 9d^2 + 2d^4)       | 0.602114, 2.03407     |
| 4  | 4d(27 - 28d^2 + 4d^4)    | 1.07461, 2.41769      |
| 5  | 8(-15 + 75d^2 - 40d^4 + 4d^6) | 0.476251, 1.47524, 2.75624 |
| 6  | 8d(-195 + 330d^2 - 108d^4 + 8d^6) | 0.881604, 1.82861, 3.06251 |

Table 2: Odd parameters

| n  | \(H_n(d)\) | \(d_j^o\): six digits |
|----|-------------|-----------------------|
| 2  | 4(-1 + 2d^2) | 0.707107 |
| 3  | 24d(-3 + 2d^2) | 1.22474 |
| 4  | 96(3 - 12d^2 + 4d^4) | 0.524648, 1.65068 |
| 5  | 960d(15 - 20d^2 + 4d^4) | 0.958572, 2.02018 |
| 6  | 5760(-15 + 90d^2 - 60d^4 + 8d^6) | 0.436077, 1.33585, 2.3506 |

The upper bound of the zeros of the Hermite polynomial \(H_n(x)\) is known [17] (6.32.6),

\[ H_n(x_j^{(n)}) = 0, \quad x_j^{(n)} < \sqrt{2n + 1} - \frac{c}{(2n + 1)^{1/6}}, \quad c = 1.85575 \ldots \] (3.8)

This means that the eigenvalues of these explicitly known odd states are greater than the corresponding \((d_j^o)^2\),

\[ (d_j^o)^2 < 2n + 1, \] (3.9)

which is consistent with Remark [2.2]. We do not know a corresponding bound for the even sector, that is the zeros of \(H_n'(d) - dH_n(d) = 0\).

We would not call these exactly solvable states QES (quasi-exactly solvable states) [18]. A quantum mechanical system with a quasi-exactly solvable potential has a finitely many exactly solvable states. In most cases these states are related by \(sl(2, R)\) algebra [19]. In the present case, a double (2.1) or single well (2.2) potential with the parameter \(d\) being the zeros of (3.3) has only one exactly solvable state. This is a totally different situation from QES.

We will come back to the topic of the integer eigenvalues in the second half of the subsequent section. Before closing this section, let us emphasise the fact that the above
connection conditions (3.3), the Neumann and Dirichlet b.c. including the fact that they are identical for the double and single well systems, are intuitively quite easy to understand.

4 Non-polynomial exact eigenfunctions

4.1 Confluent hypergeometric functions

It is well known that the one dimensional Schrödinger equation with the quadratic potential \( x^2 \) can be rewritten as an equation of the confluent hypergeometric function \( \varphi(z, E) \),

\[
-\frac{d^2 \psi(x, E)}{dx^2} + x^2 \psi(x, E) = E \psi(x, E), \quad \psi(x, E) = e^{-x^2/2} \varphi(z, E), \quad z \overset{\text{def}}{=} x^2,
\]

\[
\Rightarrow z \frac{d^2 \varphi(z, E)}{dz^2} + (b - z) \frac{d \varphi(z, E)}{dz} - a \varphi(z, E) = 0, \quad (4.1)
\]

\[
a = \frac{(1 - E)}{4}, \quad b = \frac{1}{2}. \quad (4.2)
\]

The two fundamental solutions of the above Kummer’s differential equation (4.1) are

\[
\varphi_1(z, E) = _1F_1(a, b; z) = \sum_{k=0}^{\infty} \frac{(a)_k z^k}{(b)_k k!}, \quad (4.3)
\]

\[
\varphi_2(z, E) = z^{1-b} _1F_1(a + 1 - b, 2 - b; z) = z^{1-b} \sum_{k=0}^{\infty} \frac{(a + 1 - b)_k (2 - b)_k}{(b)_k k!} z^k, \quad (4.4)
\]

in which \((a)_n\) is the shifted factorial,

\[
(a)_n \overset{\text{def}}{=} \frac{\Gamma(a + n)}{\Gamma(a)} = \prod_{k=0}^{n-1} (a + k) = a(a + 1) \cdots (a + n - 1). \quad (4.5)
\]

The well-known piecewise square integrable combination of the fundamental solutions is

\[
U(a, b; z) \overset{\text{def}}{=} \frac{\Gamma(1 - b)}{\Gamma(a - b + 1)} _1F_1(a, b; z) + \frac{\Gamma(1 - b)}{\Gamma(a)} z^{1-b} _1F_1(a - b + 1, 2 - b; z). \quad (4.6)
\]

Since \(b = \frac{1}{2}, \Gamma(-\frac{1}{2}) = -2\Gamma(\frac{1}{2}) = -2\sqrt{\pi},\) we introduce

\[
\bar{U}(a, \frac{1}{2}; z) \overset{\text{def}}{=} \frac{1}{\Gamma(a + \frac{1}{2})} _1F_1(a, \frac{1}{2}; z) - \frac{2}{\Gamma(a)} z^{1/2} _1F_1(a + \frac{1}{2}, \frac{3}{2}; z), \quad (4.7)
\]

for simplicity of presentation. For the present case, corresponding to the two types of \(z\) as a function of \(x\), \(\sqrt{z_p} = \pm(x + d), \sqrt{z_m} = \pm(x - d)\), we choose the branch of \(\Psi_p(x, E)\) and \(\Psi_m(x, E)\) in such a way \(\sqrt{z_p} > 0\) and \(\sqrt{z_m} > 0\) at infinity, so that the wavefunctions are damped at plus and minus infinity,

\[
-\frac{d^2 \psi_p(x, E)}{dx^2} + (x + d)^2 \psi_p(x, E) = E \psi_p(x, E), \quad z_p \overset{\text{def}}{=} (x + d)^2,
\]

\[
\psi_m(x, E) = e^{-x^2/2} \varphi(z, E), \quad z \overset{\text{def}}{=} x^2,
\]

\[
\Rightarrow z \frac{d^2 \varphi(z, E)}{dz^2} + (b - z) \frac{d \varphi(z, E)}{dz} - a \varphi(z, E) = 0, \quad (4.1)
\]

\[
a = \frac{(1 - E)}{4}, \quad b = \frac{1}{2}. \quad (4.2)
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The two fundamental solutions of the above Kummer’s differential equation (4.1) are

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\varphi_1(z, E) = _1F_1(a, b; z) = \sum_{k=0}^{\infty} \frac{(a)_k z^k}{(b)_k k!}, \quad (4.3)
\]

\[
\varphi_2(z, E) = z^{1-b} _1F_1(a + 1 - b, 2 - b; z) = z^{1-b} \sum_{k=0}^{\infty} \frac{(a + 1 - b)_k (2 - b)_k}{(b)_k k!} z^k, \quad (4.4)
\]

in which \((a)_n\) is the shifted factorial,

\[
(a)_n \overset{\text{def}}{=} \frac{\Gamma(a + n)}{\Gamma(a)} = \prod_{k=0}^{n-1} (a + k) = a(a + 1) \cdots (a + n - 1). \quad (4.5)
\]

The well-known piecewise square integrable combination of the fundamental solutions is

\[
U(a, b; z) \overset{\text{def}}{=} \frac{\Gamma(1 - b)}{\Gamma(a - b + 1)} _1F_1(a, b; z) + \frac{\Gamma(1 - b)}{\Gamma(a)} z^{1-b} _1F_1(a - b + 1, 2 - b; z). \quad (4.6)
\]

Since \(b = \frac{1}{2}, \Gamma(-\frac{1}{2}) = -2\Gamma(\frac{1}{2}) = -2\sqrt{\pi},\) we introduce

\[
\bar{U}(a, \frac{1}{2}; z) \overset{\text{def}}{=} \frac{1}{\Gamma(a + \frac{1}{2})} _1F_1(a, \frac{1}{2}; z) - \frac{2}{\Gamma(a)} z^{1/2} _1F_1(a + \frac{1}{2}, \frac{3}{2}; z), \quad (4.7)
\]

for simplicity of presentation. For the present case, corresponding to the two types of \(z\) as a function of \(x\), \(\sqrt{z_p} = \pm(x + d), \sqrt{z_m} = \pm(x - d)\), we choose the branch of \(\Psi_p(x, E)\) and \(\Psi_m(x, E)\) in such a way \(\sqrt{z_p} > 0\) and \(\sqrt{z_m} > 0\) at infinity, so that the wavefunctions are damped at plus and minus infinity,
The boundary values of the double well wavefunctions are
\[ \Psi_p(x, E) = \begin{cases} \Psi_p^{(+)}(x, E) = e^{-z_p/2}U_p^{(+)}(a, \frac{1}{2}; z_p) & x \geq 0, \ \sqrt{z_p} = x + d, \\ \Psi_p^{(-)}(x, E) = \alpha e^{-z_p/2}U_p^{(-)}(a, \frac{1}{2}; z_p) & x \leq 0, \ \sqrt{z_p} = -(x + d), \end{cases} \tag{4.8} \]
\[ -d^2\Psi_m(x, E) + (x - d)^2\Psi_m(x, E) = E\Psi_m(x, E), \quad z_m \overset{\text{def}}{=} (x - d)^2, \]
\[ \Psi_m(x, E) = \begin{cases} \Psi_m^{(+)}(x, E) = e^{-z_m/2}U_m^{(+)}(a, \frac{1}{2}; z_m) & x \geq 0, \ \sqrt{z_m} = x - d, \\ \Psi_m^{(-)}(x, E) = \beta e^{-z_m/2}U_m^{(-)}(a, \frac{1}{2}; z_m) & x \leq 0, \ \sqrt{z_m} = d - x, \end{cases} \tag{4.9} \]
in which \( \alpha \) and \( \beta \) are constants to be determined later. Within the interval \(-d < x < d\), which include the connection point \( x = 0 \), the wavefunctions \( \psi_D^{(\pm)}(x, E) \) and \( \psi_S^{(\pm)}(x, E) \) are expressed by
\[ \begin{align*} 
\psi_D^{(+)}(x, E) &= \Psi_m^{(+)}(x, E), & \psi_D^{(-)}(x, E) &= \Psi_p^{(-)}(x, E), \\
\psi_S^{(+)}(x, E) &= \Psi_p^{(+)}(x, E), & \psi_S^{(-)}(x, E) &= \Psi_m^{(-)}(x, E). 
\end{align*} \tag{4.10, 4.11} \]

The boundary values of the double well wavefunctions are
\[ e^{d^2/2}\Psi_D^{(+)}(0, E) = \frac{1}{\Gamma(a + \frac{1}{2})} F_1(a, \frac{1}{2}; d^2) + \frac{2d}{\Gamma(a)} F_1(a + \frac{1}{2}, \frac{3}{2}; d^2), \]
\[ e^{d^2/2}\Psi_D^{(+)}(0, E) = \frac{d}{\Gamma(a + \frac{1}{2})} \left\{ F_1(a, \frac{1}{2}; d^2) - 2d_1F_1(a, \frac{1}{2}; d^2) \right\} \]
\[ - \frac{2}{\Gamma(a)} \left\{ (1 - d^2) \cdot F_1(a + \frac{1}{2}, \frac{3}{2}; d^2) + 2d_1F_1(a + \frac{1}{2}, \frac{3}{2}; d^2) \right\}, \]
in which
\[ _1F_1(a, b; z) \overset{\text{def}}{=} \frac{d_1F_1(a, b; z)}{dz} = \frac{a}{b} \cdot F_1(a + 1, b + 1; z). \tag{4.12} \]

According to Theorem 2.1 the corresponding quantities for the single well wavefunctions are obtained by changing \( d \) to \(-d\), These lead to the following theorem.

**Theorem 4.1** The connection conditions for the double and single well wavefunctions are

\[ \begin{align*} 
\text{D, even :} & \quad \alpha = 1, \\
& \quad \frac{d}{\Gamma(a + \frac{1}{2})} \left\{ F_1(a, \frac{1}{2}; d^2) - 2d_1F_1(a, \frac{1}{2}; d^2) \right\} \\
& \quad - \frac{2}{\Gamma(a)} \left\{ (1 - d^2) \cdot F_1(a + \frac{1}{2}, \frac{3}{2}; d^2) + 2d_1F_1(a + \frac{1}{2}, \frac{3}{2}; d^2) \right\} = 0, \tag{4.13} \\
\text{D, odd :} & \quad \alpha = -1, \\
& \quad \frac{1}{\Gamma(a + \frac{1}{2})} F_1(a, \frac{1}{2}; d^2) + \frac{2d}{\Gamma(a)} F_1(a + \frac{1}{2}, \frac{3}{2}; d^2) = 0. \tag{4.14} \\
\text{S, even :} & \quad \beta = 1, \\
& \quad \frac{d}{\Gamma(a + \frac{1}{2})} \left\{ F_1(a, \frac{1}{2}; d^2) - 2d_1F_1(a, \frac{1}{2}; d^2) \right\} \\
& \quad + \frac{2}{\Gamma(a)} \left\{ (1 - d^2) \cdot F_1(a + \frac{1}{2}, \frac{3}{2}; d^2) + 2d_1F_1(a + \frac{1}{2}, \frac{3}{2}; d^2) \right\} = 0. \tag{4.15} 
\end{align*} \]
The zeros of these equations provide the eigenvalues \( \{E_n\} \) of the \( V_D \) and \( V_S \) systems in the even and odd sectors. The corresponding eigenfunctions are those listed in (4.10) and (4.11).

**Remark 4.2** Roughly speaking, the functions in the \( V_S \) system (4.15), (4.16) are the other branches of the corresponding \( \dot{U} \) and \( U \) functions in the \( V_D \) system (4.13), (4.14) and vice versa.

We show the seven lowest eigenvalues, 4 from the even sector and 3 from the odd sector, of the double (Table 3) and single (Table 4) well potentials for a selected small values of \( d \).

For each value of \( d \), the eigenvalues of the odd sectors are greater than the corresponding ones in the even sectors, as dictated by the oscillation theorem. Each specific eigenvalue of the \( V_S \) system increases monotonically with the parameter \( d \). The \( d \)-dependence of the states of the \( V_D \) system is quite interesting. As \( d \) increases above 3, the split between the even and odd sectors diminishes appreciably. One could say for almost safely that when \( d \) increases the tunneling effects of the lowest \( n \) \( (n < d^2/2) \) states in each sector disappear. This would mean that for large \( d \), the eigenvalues of \( V_D(x) \) approach to \( E_n^e = 1 + 2n - \epsilon_n \), \( E_n^o = 1 + 2n + \epsilon'_n \) with very small \( \epsilon_n, \epsilon'_n > 0 \). It is a good challenge to find out the asymptotic behaviours of \( E_n^e(d) \) and \( E_n^o(d) \) of the \( V_S(x) \) system.

| \( d \) | \( E_0^e \) | \( E_0^o \) | \( E_1^e \) | \( E_1^o \) | \( E_2^e \) | \( E_2^o \) | \( E_3^e \) |
|---|---|---|---|---|---|---|---|
| 0 | 1 | 1 | 3 | 3 | 5 | 5 | 7 |
| 1/10 | 0.895426 | 2.78209 | 4.72612 | 6.6950 | 8.62731 | 10.5849 | 12.5497 |
| 1/4 | 0.768973 | 2.48392 | 4.34603 | 6.20358 | 8.09868 | 9.99237 | 11.9046 |
| 1/2 | 0.635529 | 2.06077 | 3.79417 | 5.50548 | 7.29817 | 9.08421 | 10.9098 |
| 3/4 | 0.590301 | 1.72471 | 3.34471 | 4.90343 | 6.59770 | 8.27404 | 10.0146 |
| 1 | 0.618919 | 1.46847 | 3 | 4.39493 | 5.99720 | 7.56038 | 9.21846 |
| 3/2 | 0.801494 | 1.15748 | 2.64868 | 3.64627 | 5.10400 | 6.41679 | 7.92382 |
| 2 | 0.951419 | 1.03576 | 2.73504 | 3.22301 | 4.67082 | 5.64089 | 7.04349 |
| 3 | 0.999551 | 1.00039 | 2.99252 | 3.00604 | 4.94552 | 5.03982 | 6.79866 |
| 4 | 0.999999 | 1.000000 | 2.99998 | 3.00001 | 4.99977 | 5.00020 | 6.99802 |

A few remarks on the numerical calculations of the eigenvalues. For \( z \to +\infty \), \( U(a,b;z) \) (4.6) behaves asymptotically \( \sim z^{-a} \). The expressions in Theorem 4.1 (4.13)–(4.16) increase drastically \( \sim d^{E/2} \) as \( d \) and \( E \) increase. With certain reduction factors the zeros (the
Table 4: 7 lowest eigenvalues of $V_S$

| $d$  | $E_0^e$ | $E_0^o$ | $E_1^e$ | $E_1^o$ | $E_2^e$ | $E_2^o$ | $E_3^e$ |
|------|---------|---------|---------|---------|---------|---------|---------|
| 1/10 | 1.12121 | 3.23353 | 5.29034 | 7.34657 | 9.38899 | 11.4312 | 13.4665 |
| 1/4  | 1.33487 | 3.61368 | 5.75688 | 7.89681 | 10.0032 | 12.1086 | 14.1970 |
| 1/2  | 1.77790 | 4.32871 | 6.61797 | 8.89589 | 11.1096 | 13.3194 | 15.4967 |
| 3/4  | 2.33218 | 5.14812 | 7.58472 | 9.99898 | 12.3203 | 14.6339 | 16.9002 |
| 1    | 3       | 6.07439 | 8.65856 | 11.2076 | 13.6366 | 16.0533 | 18.4086 |
| 3/2  | 4.68276 | 8.25537 | 11.1329 | 13.9472 | 16.5907 | 19.2113 | 21.7441 |
| 2    | 6.83597 | 10.8843 | 14.0506 | 17.1244 | 19.9803 | 22.8017 | 25.5108 |
| 5/2  | 9.46595 | 13.9704 | 17.4196 | 20.7471 | 23.8127 | 26.8318 | 29.7154 |

Eigenvectors (of these expressions) of these expressions can be determined as precisely as wanted for a specified parameter $d$. This preciseness propagates to the preciseness of the eigenfunctions. This is why the systems with $V_D(x)$ and $V_S(x)$ belong to the category of potentials of non-polynomial exact solvability [1].

In order to share the vivid images of polynomial and non-polynomial type eigenfunctions of the low lying eigenstates we present four figures. The two lowest eigenfunctions of the $V_S(x)$ potential with $d = 1$ are shown in Fig.1, the ground state $E_0^e = 3$ and the first excited state $E_0^o = 6.07439$. For comparison, we show the three lowest energy states of the $V_D$ system with $d = 1$, $E_0^e = 0.618919$, $E_0^o = 1.46846$ and $E_1^e = 3$ in Fig.2. The two lowest ones are not of the polynomial type. Two $E = 5$ eigenstates in the odd sector with $d = 1/\sqrt{2}$ are shown in Fig.3 and those in the even sector with $d = \sqrt{5/2}$ are displayed in Fig.4. Those of the $V_S$ system have red lines and those in $V_D$ blue.

![Figure 1](image1.png)  
Figure 1: $d = 1$, $E = 3$, 6.07, S

![Figure 2](image2.png)  
Figure 2: $d = 1$, $E = 0.619, 1.468$, 3, D

Fig.2 shows the effect of splitting due to tunneling. Let us introduce a pair of potentials...
They have the same set of eigenvalues. The eigenfunctions are restricted to the right and left line satisfying the Dirichlet b.c. at the origin. The function vanishing on the right half line and take the odd line of Fig.2 at $x > 0$ is the ground state eigenfunction of $V_{DR}(x)$ with $E = 1.46846 > 1$, $d = 1$, since $V_{DR}(x)|_{d=1} > (x - 1)^2$ on the left half line. Likewise the vanishing on the right half line and take the odd line of Fig.2 at $x < 0$ is the ground state eigenfunction of $V_{DL}(x)$ with $E = 1.46846$, $d = 1$. When the infinite barrier is removed and $V_{DR}(x)$ and $V_{DL}(x)$ merge to become $V_D(x)$, the odd combination of these states becomes the first excited states with the same eigenvalue $E = 1.46846$. The removal of the infinite barrier at the origin could be rephrased as the addition of an infinitely deep and narrow well at the origin. This has no effect on the odd combination of the original ground state eigenfunctions as they vanish at the origin. However, this has the effect of increasing the even combination of the wavefunctions at the origin to the point of satisfying the Neumann b.c. and thus decreasing the eigenvalue. This mechanism applies to all the eigenfunctions of the $V_{DR}(x)$ and $V_{DL}(x)$ systems. Therefore, the splitting of the eigenlevels in the $V_D(x)$ potential means pushing down the even states whereas the odd states stay at the original eigenvalues of $V_{DL}$ and $V_{DR}$. The situation is essentially the same for the $V_S(x)$ potential. Instead of (4.17)

$$V_{SR}(x) = \begin{cases} (x + d)^2 & x > 0 \\ +\infty & x = 0 \end{cases}, \quad V_{SL}(x) = \begin{cases} +\infty & x = 0 \\ (x - d)^2 & x < 0 \end{cases}, \quad d > 0. \quad (4.18)$$

are the potentials to be considered.
4.2 Revisiting polynomial type solutions

Since the parameter $a$ in the Kummer’s differential equation (4.1) is $a = (1 - E)/4$ (4.2), various quantities and expressions in the previous subsection simplify a lot for the odd integer eigenvalues $E$,

$$E = 4n + 1, 4n + 3, \iff a = -n, a + \frac{1}{2} = -n, \quad n \in \mathbb{Z}_{\geq 0}.$$ (4.19)

For these values the confluent hypergeometric functions $\text{$_1F_1$(}a, b; z\text{)}$ and $\text{$_1F_1$(}a + \frac{1}{2}, b; z\text{)}$ terminate and become polynomials in $z$. Many expressions in §4.1 reduce to those in §3. Here we list them for comparison.

(i) $E = 4n + 1$ This means

$$a = -n \Rightarrow \frac{1}{\Gamma(a)} = \frac{1}{\Gamma(-n)} = 0, \quad U(-n, \frac{1}{2}; z) = \frac{\Gamma(\frac{1}{2})}{\Gamma(-n + \frac{1}{2})} \text{$_1F_1$(} -n, \frac{1}{2}; z\text{)}.$$ here \text{$_1F_1$(} -n, \frac{1}{2}; x^{2}\text{)} is a degree $n$ polynomial in $x^{2}$,

$$\text{$_1F_1$(} -n, \frac{1}{2}; x^{2}\text{)} = \frac{n!}{(\frac{1}{2})_{n}} L_{n}^{(-\frac{1}{2})}(x^{2}), \quad H_{2n}(x) = (-1)^{n} n! 2^{2n} L_{n}^{(-\frac{1}{2})}(x^{2}),$$ (4.20)

in which $L_{n}^{(-\frac{1}{2})}(x)$ is the Laguerre polynomial.

(ii) $E = 4n + 3$ This means

$$a + \frac{1}{2} = -n \Rightarrow \frac{1}{\Gamma(a + \frac{1}{2})} = \frac{1}{\Gamma(-n)} = 0, \quad U(-n - \frac{1}{2}, \frac{1}{2}; z) = \frac{\Gamma(-\frac{1}{2})}{\Gamma(-n - \frac{1}{2})} \sqrt{z} \text{$_1F_1$(} -n, \frac{3}{2}; z\text{)}.$$ Here \text{$_1F_1$(} -n, \frac{3}{2}; x^{2}\text{)} is a degree $n$ polynomial in $x^{2}$,

$$\text{$_1F_1$(} -n, \frac{3}{2}; x^{2}\text{)} = \frac{n!}{(\frac{3}{2})_{n}} L_{n}^{(-\frac{1}{2})}(x^{2}), \quad H_{2n+1}(x) = (-1)^{n} n! 2^{2n+1} x L_{n}^{(-\frac{1}{2})}(x^{2}),$$ (4.21)

By using these relations, one can easily verify that (4.13)–(4.16) reduce to (3.3).

5 Summary and Comments

Some basic facts, expressions and numbers related with the eigenvalues and eigenfunctions of the piecewise analytic and exactly solvable potentials $V_{D}(x)$ and $V_{S}(x)$ are explored. For
the applications, the norms of some of the lower lying eigenstates would be needed. This would require a substantial work.

It is well known that a piecewise linear potential \( V_L(x) = g^3|x| \) is exactly solvable. It is expected that the potentials

\[
V_{LD}(x) = \min[V_L(x + d), V_L(x - d)], \quad V_{LS}(x) = \max[V_L(x + d), V_L(x - d)],
\]

would be exactly solvable by the same procedures as those used in this paper. More interesting would be the double and single well versions of a Krein-Adler deformation [13, 14, 15] of the harmonic oscillator potential,

\[
V_{KA}(x) = x^2 + 3 + \frac{32x^2}{(2x^2 + 1)^2} - \frac{8}{2x^2 + 1},
\]

\[
V_{KAD}(x) = \min[V_{KA}(x + d), V_{KA}(x - d)], \quad V_{KAS}(x) = \max[V_{KA}(x + d), V_{KA}(x - d)], \quad d > 0.
\]

Dubov et al. [16] introduced the exactly solvable \( V_{KA}(x) \), which has the regular singular points at \( x = \pm \frac{i}{\sqrt{2}} \) with the characteristic exponents \((-1, 2)\). The complete set of the eigenvalues and eigenfunctions are

\[
\psi_{KA,n}(x) = \frac{e^{-x^2/2}W[H_1, H_2, H_n](x)}{4(2x^2 + 1)}, \quad \mathcal{E}(n) = 2n, \quad n \in \mathbb{Z}_{\geq 0}\setminus\{1, 2\},
\]

in which

\[
W[f_1, \ldots, f_m](x) \overset{\text{def}}{=} \det \left( \frac{d^{j-1}f_k(x)}{dx^{j-1}} \right)_{1 \leq j, k \leq m},
\]

is the Wronskian of functions \( \{f_1, \ldots, f_m\} \). At least we can find the polynomial type eigenfunctions of \( V_{KAD}(x) \) and \( V_{KAS}(x) \) quite easily. If these \( V_{KAD}(x) \) and \( V_{KAS}(x) \) turn out to be exactly solvable, we would have an infinitely many similar potentials by the Krein-Adler prescriptions [15].
Acknowledgements

R.S. thanks Milosh Znojil for sending [1] just after publication.

Informed Consent Statement: Not applicable.

Data Availability Statement: No new data created

Conflicts of Interest: There is no conflict of interests.

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