The Game of Cops and Robbers on Directed Graphs with Forbidden Subgraphs

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Abstract The traditional game of cops and robbers is played on undirected graph. Recently, the same game played on directed graph is getting attention by more and more people. We knew that if we forbid some subgraph we can bound the cop number of the corresponding class of graphs. In this paper, we analyze the game of cops and robbers on $\vec{H}$-free digraphs. However, it is not the same as the case of undirected graph. So we give a new concept ($\vec{H}$-free digraph) to get a similar conclusion about the case of undirected graph.

Keywords cops and robbers; directed graph; induced subgraphs
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1 Introduction

The game of Cops and robbers first introduced by Nowakowski and Winkler[10] and independently by Quilliot[11] is a two player game played on a graph $G = (V, E)$. The first player, called the cop player has $k$ ($k \geq 1$) pieces and the second player, the robber, has only one piece. In the beginning, the cop player places his $k$ pieces on some vertices (not necessarily distinct) of the graph and the robber places his piece on one vertex of the same graph. In the first round, the cop player can move all his pieces before the robber. Each piece of the cop can be moved to an adjacent vertex or stay idle. In the robber’s move, he can also move his piece to an adjacent vertex or place it still. After the cop player and the robber finish their moves, one round is finished. It’s the time for the next round, the two players do the same thing like the previous round alternately. The game ends when some piece of the cop and the robber are on the same vertex (that is, the cops catch the robber). In this case, we say the cop player wins or cops win. The robber wins if he can never be caught by the cops. Both players have complete information, which means they know the graph and the positions of all the pieces.

Nowakowski and Winkler[10] and Quilliot[11] considered the case for $k = 1$. That means one cop catches one robber. Later, Aigner and Fromme[1] generalized the game to several cops. The key problem for this game is to know how many cops are needed to catch the robber. We call the minimum integer of cops required to capture the robber the cop number and denoted by $c(G)$. Many mathematicians have already done some research for the cop number. Frankl[3] conjectured that for any connected $n$-vertex graph $G$ it holds that $c(G) = O(\sqrt{n})$ in 1987. This conjecture, known as Meyniel’s conjecture still remains open. The best known upper bound, provided independently in [4, 9, 14], said that the cop number of any graph on $n$ vertices is upper bounded by $n2^{-1+o(1)}\sqrt{n}$. That means we even cannot give a loose upper bound $O(n^{1-o(1)})$ so far. Aigner and Fromme[1] proved that $c(G) \leq 3$ for any planar graph $G$ in 1984. Quilliot[12] gave an upper bound $2g + 3$ about the cop number for any graph $G$ of genus $g$ in

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1985. Schroeder\cite{13} improved this bound to $\left\lceil \frac{2g}{3} \right\rceil + 3$ and conjectured that $c(G) \leq g + 3$ for any graph of genus $g$ (minimum number of handles that must be added to the plane to embed the graph without any crossings) in 2001.

In [1], a simple forbidden subgraph condition was given to ensure that $c(G) \geq \delta(G)$. Joret, Kamiński and Theis\cite{6} gave an important theorem about the relationship between the class of $\bar{H}$-free graphs and bounded cop number. More recently, the game of cops and robber has been considered on directed graphs, or digraphs, for short. In [5], upper bounds were provided for directed abelian Cayley graphs. In [4], Frieze, Krivelevich and Loh proved that $c(\bar{D}) = O(n\frac{(\log \log n)^2}{\log n})$ for any strongly connected digraph $\bar{D}$. For more research about the game of cops and robbers on directed graph, please refer to [7].

In this paper, we discuss the relationship between the class of $\bar{H}$-free digraphs and the bounded cop number. We find that we cannot find a subgraph $\bar{H}$ such that the class of $\bar{H}$-free digraphs has bounded cop number, even if the directed graphs are strongly connected.

**Theorem 1.1.** For any directed graph $\bar{H}$, the class of $\bar{H}$-free digraphs has unbounded cop number.

**Theorem 1.2.** For any directed graph $\bar{H}$, the class of $\bar{H}$-free strongly connected digraphs has unbounded cop number.

That means we cannot find any given structures to guarantee that $\bar{H}$-free digraphs have bounded cop number. Maybe the condition of $\bar{H}$-free is too strict and we find another way to forbid the subgraphs and we call it $\bar{H}^*$-free. We get a conclusion about the $\bar{H}^*$-free and the bounded cop number.

**Theorem 1.3.** Let $\bar{D}$ be a $\bar{P}_k^*$-free strongly connected digraph for $k \geq 3$, then $c(\bar{D}) \leq k - 2$.

## 2 Preliminaries

When we mention a graph $G$, we mean that $G$ is simple, finite, connected and undirected. When we mention a digraph $\bar{D}$, we mean that $\bar{D}$ has no loops or parallel arcs (arcs with the same head and the same tail), and is weakly connected (its underlying undirected graph is connected). The oriented graph is a directed graph without 2-cycles, which means its underlying undirected graph has no multiple edges.

Let $\bar{H}$ be a digraph. A digraph is called $\bar{H}$-free if it does not contain $\bar{H}$ as an induced subgraph. The adjacency matrix of $\bar{H}$ of order $n$ is the $n \times n$ matrix $(a_{uv})$, where $a_{uv} = 1$ or 0 according to whether there exists an arc in $\bar{H}$ with tail $u$ and head $v$. Let $\bar{P}_k$ be an oriented path of order $k$. A digraph $\bar{D}$ is called $\bar{P}_k^*$-free if $\bar{D}$ does not contain any induced subgraph whose upper triangular part of its adjacency matrix is the same as $\bar{P}_k$.

If a digraph $\bar{D}$ does not contain $\bar{P}_k$ as a subgraph (not necessary an induced subgraph), then $\bar{D}$ is $\bar{P}_k^*$-free. If $\bar{D}$ is $\bar{P}_k^*$-free, then $\bar{D}$ must be $\bar{P}_k$-free.

We say that the cop number is bounded for a class of digraphs, if there exits a constant $C$ such that the cop number of every digraph in this class is at most $C$; otherwise the cop number is unbounded for this class.

Let $\bar{D}$ be a digraph. The neighbourhood of a vertex $v$ in $\bar{D}$, denoted by $N(v)$, is the set of vertices adjacent to $v$ in its underlying undirected graph. Let $N^-(v)$ (resp. $N^+(v)$) be the set of vertices who only has arc with head $v$ (resp. with tail $v$) and $N^\pm(v)$ be the set of vertices who has both arcs with head $v$ and with tail $v$ in $\bar{D}$. Then $N^-(v), N^+(v), N^\pm(v)$ are pairwise disjoint and $N(v) = N^-(v) \cup N^+(v) \cup N^\pm(v)$. Let $N^-(v) = \{x_1^-, \ldots, x_{i_1}^-\}$, $N^+(v) = \{x_1^+, \ldots, x_{i_2}^+\}$ and $N^\pm(v) = \{x_1^\pm, \ldots, x_{i_3}^\pm\}$. A clique substitution at a vertex $v$ consists in replacing $v$ with new vertices $\{y_1^-, \ldots, y_{i_1}^-\} \cup \{y_1^+, \ldots, y_{i_2}^+\} \cup \{y_1^\pm, \ldots, y_{i_3}^\pm\}$ satisfies (i) there is one arc with tail
$x_i^-$ and head $y_i^+$, 1 ≤ $i$ ≤ $i_1$, there is one arc with tail $y_i^+$ and head $x_i^+$, 1 ≤ $i$ ≤ $i_2$ and there are two opposite arcs between $x_i^+$ and $y_i^+$, 1 ≤ $i$ ≤ $i_3$; (ii) {y$_1^i$, · · · , y$_i^i_1$}, {y$_i^i_1$, · · · , y$_i^i_2$} and {y$_i^i_1$, · · · , y$_i^i_3$} are three cliques with two opposite arcs, respectively; (iii) there are two opposite arcs between $x_i^\pm$ and $y_i^\pm$, 1 ≤ $i$ ≤ $i_3$; (ii) {y$_1^i$, · · · , y$_i^i_1$}, {y$_i^i_1$, · · · , y$_i^i_2$} and {y$_i^i_1$, · · · , y$_i^i_3$} are three cliques with two opposite arcs, respectively; (iii) there are two opposite arcs between $y_i^\pm$ for 1 ≤ $i$ ≤ $i_3$ and all $y_i^\pm$ ∈ {y$_1^i$, · · · , y$_i^i_1$} ∪ {y$_i^i_1$, · · · , y$_i^i_2$} and there is one arc with tail $y_i^\prime$ and head $y_i^\prime\prime$ for all $y_i^\prime$ ∈ {y$_1^i$, · · · , y$_i^i_1$} and $y_i^\prime\prime$ ∈ {y$_i^i_1$, · · · , y$_i^i_2$}. The digraph obtained from a digraph $\vec{D}$ by substituting a clique at each vertex of $\vec{D}$ will be denoted by $\vec{D}^+$. An arc substitution at an arc $a$ is to replace $a$ by a same directed path of any length (see Figure 2.2 for example). We can find that a strongly connected directed graph is still strongly connected after the operation of clique substitution or arc substitution.

Figure 2.1. Example of clique substitution at vertex $v$

Figure 2.2. Example of arc substitution at arc $a$

3 Forbidden Induced Subgraph

**Theorem 3.1.** For any directed graph $\vec{H}$, the class of $\vec{H}$-free digraphs has unbounded cop number.

*Proof.* Suppose not, let $\vec{H}$ be a directed graph such that the class of $\vec{H}$-free graphs has bounded cop number. First, suppose the underlying graph of $\vec{H}$ contains a cycle. We know that the class of directed graphs whose underlying graphs are trees is contained in the class of $\vec{H}$-free graphs. However, the class of directed graphs whose underlying graphs are trees is cop-unbounded since the cop number for any digraph $\vec{D}$ is at least as large as the number of its sources (vertices of indegree zero), a contradiction. Now we consider the case that the underlying graph of $\vec{H}$ does not contain a cycle. Then the underlying graph of $\vec{H}$ must be a forest. Since $\vec{H}$ is a forest, $\vec{H}$ must contain an induced arc $\vec{P}_2$. However, we can find a class of $\vec{P}_2$-free graphs which is cop-unbounded. In fact we just replace each edge in the incidence graphs of finite projective plane by two oppositely oriented arcs with the same ends. This graph is $\vec{P}_2$-free since every arc has the oppositely oriented arc with the same ends. The cop number of this graph is the same with the corresponding undirected graph which is $\sqrt{n}$ [2]. The number $n$ can be arbitrarily large based on the vertex number, so we can have a class of $\vec{P}_2$-free graphs which is cop-unbounded, that is a contradiction. Finally the class of $\vec{H}$-free digraphs has unbounded cop number.

We know that the cop number of any directed graph can reduce to the cop number of its strongly connected components. So we restrict the directed graph to the strongly connected directed graph. However, we still cannot get the desired conclusion. First, we need four lemmas.
Lemma 3.2. For any directed graph, clique substitution does not decrease the cop number.

Proof. Let $\bar{D}$ be a directed graph. Denote the graph by clique substituting at every vertex in $\bar{D}$ by $\bar{D}^+$. To each vertex $v \in V(\bar{D})$ there corresponds a clique in $\bar{D}^+$, which we denote by $\phi(v)$. We simultaneously play two games: one on $\bar{D}$ and another on $\bar{D}^+$. We assume that we have a winning strategy for the cop player on $\bar{D}^+$ and we simulate her moves on $\bar{D}$. On the other hand, the robber is playing on $\bar{D}$ and we simulate his moves on $\bar{D}^+$.

At the beginning, according to the strategy that the cops are placed on $\bar{D}^+$, the corresponding cops in $\bar{D}$ are placed in the obvious way: if a cop in $\bar{D}^+$ is on a vertex of $\phi(v)$ for $v \in V(\bar{D})$, then the corresponding cop in $\bar{D}$ is put on $v$. Then, we put the robber in $\bar{D}^+$ on an arbitrary vertex of the clique $\phi(u)$ if the robber is in $u$ in $\bar{D}$.

In the first round, we let all the cops in $\bar{D}$ stay idle and the corresponding cops in $\bar{D}^+$ also stay idle. Then it’s the robber’s turn to move in $\bar{D}$. The robber has two choices: stay idle or move. If the robber stays idle, then we do not move the robber in $\bar{D}^+$. Assume the robber moves in $\bar{D}$, for example, from $u$ to $v$. Let $u'v'$ be the (unique) arc in $\bar{D}^+$, where $u' \in \phi(u)$ and $v' \in \phi(v)$. We will consider two cases.

Case 1. In $\bar{D}^+$, the robber is on $u'$.

In this case, we move the robber to $v'$ (the arc works by the definition of clique substitution) and the cops in $\bar{D}^+$ have one round to move by winning strategy. Assume one of cops, say $c_1$, moves from $x'$ to $y'$, where $x' \in \phi(x)$ and $y' \in \phi(y)$. If $x = y$, then $c_1$ does nothing in $\bar{D}$; otherwise $c_1$ moves from $x$ to $y$.

Case 2. In $\bar{D}^+$, the robber is on another vertex of $\phi(u)$, say $u''$.

In this case, we move the robber first from $u''$ to $u'$ by one step (it works by the definition of clique substitution). Then the cops in $\bar{D}^+$ have one round to move by winning strategy. Assume one of cops, say $c_1$, moves from $x'$ to $y'$, where $x' \in \phi(x)$ and $y' \in \phi(y)$. In the next round in $\bar{D}^+$, we can move the robber from $u'$ to $v'$ and the cops can also move for another round by winning strategy. In this round, we assume $c_1$ moves from $y'$ to $z'$, where $z' \in \phi(z)$. By the definition of clique substitution, we have $x = y = z$ or $x = y \neq z$ or $x \neq y = z$ in $\bar{D}$. If $x = y = z$, then $c_1$ does nothing in $\bar{D}$; otherwise $c_1$ moves from $x$ to $z$.

Now we get a strategy for the cops in $\bar{D}$. By our assumption, the robber will be caught in $\bar{D}^+$. That means at least one cop and the robber are on the clique $\phi(v)$ for some vertex $v \in V(\bar{D})$ and the corresponding cop and the robber in $\bar{D}$ will be on the same vertex by our translation.

Lemma 3.3. For any directed graph, arc substitution does not decrease the cop number.

Proof. Let $\bar{D}^{++}$ be a digraph obtained from $\bar{D}$ by subdividing all arcs of $\bar{D}$ at the same time. We can construct two maps $\iota: V(\bar{D}) \to V(\bar{D}^{++})$ and $\psi: V(\bar{D}^{++}) \to V(\bar{D})$ such that $\iota: V(\bar{D}) \to V(\bar{D}^{++})$ is the natural map so that the vertices in $V(\bar{D}^{++}) \setminus \iota(V(\bar{D}))$ are these new vertices added to $\bar{D}$ to obtain $\bar{D}^{++}$. Suppose that $uv$ is an arc in $\bar{D}$, then there is a directed path $P_{u,v}$ in $\bar{D}^{++}$ joining $\iota(u)$ with $\iota(v)$. The map $\psi$ sends the vertices of $P_{u,v}$ except $\iota(u)$ to the vertex $v \in V(\bar{D})$, and vertex $\iota(u)$ to the vertex $u \in V(\bar{D})$.

Now we know that the corresponding relationship between $V(\bar{D})$ and $V(\bar{D}^{++})$. We use the “copy” strategy to copy the winning strategy in $\bar{D}^{++}$ and will catch the robber in digraph $\bar{D}$ just like Lemma 3.1.

Now we know some powerful lemmas and we can explore the relationship between the class of $H$-free strongly connected directed graphs and the bounded cop number.
Lemma 3.4. The class of $\vec{H}$-free strongly connected directed graphs has unbounded cop number, where $\vec{H}$ is one of the four structures in Figure 3.1.

**Proof.** Let $\mathcal{D}$ be any class of strongly connected digraphs with unbounded cop number and $\mathcal{D}^+ := \{\vec{D}^+ | \vec{D} \in \mathcal{D}\}$. Notice that all digraphs in $\mathcal{D}^+$ are also strongly connected and $\vec{H}$-free. Applying Lemma 1, we draw a conclusion that the cop number of digraphs in $\mathcal{D}^+$ is unbounded.

Lemma 3.5. The class of strongly connected digraphs with undirected girth at least $l$ ($l \geq 2$) has unbounded cop number.

**Proof.** Let $\mathcal{D}$ be any class of strongly connected digraphs with unbounded cop number and $\mathcal{D}^{++} := \{\vec{D}^{++} | \vec{D} \in \mathcal{D}\}$, $\vec{D}^{++}$ is obtained by subdividing arcs of $\vec{D}$ sufficient times to make the undirected girth of $\vec{D}^{++}$ at least $l$. The digraphs in $\mathcal{D}^{++}$ are also strongly connected. Applying Lemma 2, we find that the cop number of digraphs in $\mathcal{D}^{++}$ is unbounded.

Now we are ready to complete the proof of Theorem 1.2.

**Theorem 3.6.** For any directed graph $\vec{H}$, the class of $\vec{H}$-free strongly connected digraphs has unbounded cop number.

**Proof.** Let $\vec{H}$ be a directed graph such that the class of $\vec{H}$-free strongly connected digraphs has bounded cop number. By Lemma 4, we have that the underlying graph of $\vec{H}$ must be a forest.

Now suppose that the underlying graph of $\vec{H}$ is not a path, which means that $\vec{H}$ must contains at least one of four structures mentioned in Lemma 3 as an induced subgraph. However, we know that is also impossible by Lemma 3. So the underlying graph of $\vec{H}$ must be a path, which must contain $\vec{P}_2$ and makes a contradiction like the proof of Theorem 1.

**4 $\vec{P}_k^*$-free**

We now turn our attention to the class of $\vec{P}_k^*$-free graphs. We can get the following theorem.

**Theorem 4.1.** Let $\vec{D}$ be a $\vec{P}_k^*$-free strongly connected digraph for $k \geq 3$, then $c(\vec{D}) \leq k - 2$.

**Proof.** $\vec{P}_k^*$-free means that $\vec{D}$ does not contain $\vec{P}_k$ as an induced subgraph. Moreover, $\vec{D}$ also does not contain any induced subgraphs which has the same upper triangular part of adjacency matrix as $\vec{P}_k$.

We define the distance from the cops to the robber as the minimum distance from a cop to the robber and we measure the distance after the robber’s and before the cops’ move. We will give a winning strategy for $k - 2$ cops. Initially all $k - 2$ cops are on the same arbitrary vertex denoted by $u$ and the robber is on vertex $v$. The distance from $u$ to $v$ is $d$ and $2 \leq d \leq k - 2$, otherwise the cops will catch the robber in the next round or $\vec{D}$ is not $\vec{P}_k^*$-free.

We order one cop to travel along the shortest directed path from $u$ to $v$ and then follow the exact route the robber took from vertex $v$. We instruct the other cops to follow the first cop
in single file; the cops should form a directed path of length \( k - 2 \). Since \( \vec{D} \) is \( P_5^* \)-free, the distance from the cops to the robber will decrease after at most \( k - d - 1 \) moves. Since \( \vec{D} \) is finite, we can repeat this process until the distance drop to 1.

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\begin{array}{c}
\includegraphics{figure}\n\end{array}
\]

**Figure 4.1.** Some examples who have the same upper triangular part of adjacency matrix as \( P_k^* \)

We knew that we have the similar conclusion (\( k - 2 \) cops can catch the robber for any \( P_k \)-free undirected graph) in [6]. So we provide a new perspective about how to forbid the subgraphs and get a similar conclusion in directed graphs.

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