PARTIAL SUMS OF NORMALIZED WRIGHT FUNCTIONS

MUHEY U DIN\(^1\), MOHSAN RAZA\(^2\), NIHAT YAĞMUR\(^3\)∗

Abstract. In this paper we find the partial sums of two kinds normalized Wright functions and the partial sums of Alexander transform of these normalized Wright functions.

1. Introduction and preliminaries

Let \(A\) be the class of functions \(f\) of the form

\[ f(z) = z + \sum_{m=2}^{\infty} a_m z^m \]

analytic in the open unit disc \(U = \{z : |z| < 1\}\). Consider the Alexander transform given as:

\[ \mathcal{A}[f](z) = \int_0^z \frac{f(t)}{t} dt = z + \sum_{m=2}^{\infty} \frac{a_m}{m} z^m. \]

The surprising use of Hypergeometric function in the solution of the Bieberbach conjecture has attracted many researchers to study the special functions. Many authors who study on geometric functions theory are interested in some geometric properties such as univalency, starlikeness, convexity and close-to-convexity of special functions. Recently, several researchers have studied the geometric properties of hypergeometric functions [12, 28], Bessel functions [1, 2, 3, 4, 5, 6, 22, 23, 24], Struve functions [14, 30], Lommel functions [8]. Motivated by the above works Prajpat [19] studied some geometric properties of Wright function

\[ W_{\lambda,\mu}(z) = \sum_{m=0}^{\infty} \frac{z^m}{m! \Gamma(\lambda m + \mu)}, \quad \lambda > -1, \quad \mu \in \mathbb{C}. \]

This series is absolutely convergent in \(\mathbb{C}\), when \(\lambda > -1\) and absolutely convergent in open unit disc \(U\) for \(\lambda = -1\). Furthermore this function is entire. The Wright functions were introduced by Wright [29] and have been used in the asymptotic theory of partitions, in the theory of integral transforms of the Hankel type and in Mikusinski operational calculus. Recently, Wright functions have been found in the solution of partial differential equations of fractional order. It was found that the corresponding Green functions can be represented in terms of the Wright functions [18, 21]. For positive rational number \(\lambda\), the Wright functions can be

\(\text{Date:}\)

\(\ast\) Corresponding author

2010 Mathematics Subject Classification. 30C45, 30C50, 33C10.

Key words and phrases. Partial sums, Analytic functions, Normalized Wright functions.

1
represented in terms of generalized hypergeometric functions. For some details see [10, section 2.1]. In particular, the functions \( W_{1,v+1}(-z^2/4) \) can be expressed in terms of the Bessel functions \( J_v \), given as:

\[
J_v(z) = \left( \frac{z}{2} \right)^2 W_{1,v+1}(-z^2/4) = \sum_{m=0}^{\infty} \frac{(-1)^m (z/2)^{2m+v}}{m! \Gamma(m + v + 1)}.
\]

The Wright function generalizes various functions like Array functions, Whittaker functions, entire auxiliary functions, etc. For the details, we refer to [10]. Prajapat discussed some geometric properties of the following normalizations of Wright functions in [19]

\[
W_{\lambda,\mu}(z) = \Gamma(\mu) z W_{\lambda,\mu}(z) = z + \sum_{m=1}^{\infty} \frac{\Gamma(\mu)}{m! \Gamma(\lambda m + \mu)} z^{m+1}, \lambda > -1, \mu > 0, \ z \in \mathcal{U}, \quad (1.1)
\]

\[
W_{\lambda,\mu}(z) = \Gamma(\lambda + \mu) \left[ W_{\lambda,\mu}(z) - \frac{1}{\Gamma(\mu)} \right] = z + \sum_{m=1}^{\infty} \frac{\Gamma(\lambda + \mu)}{(m + 1)! \Gamma(\lambda m + \lambda + \mu)} z^{m+1}, \quad (1.2)
\]

where \( \lambda > -1, \lambda + \mu > 0 \). The Pochhammer (or Appell) symbol, defined in terms of Euler’s gamma functions is given as \((x)_n = \Gamma(x+n)/\Gamma(x) = x(x+1)...(x+n-1)\).

In this note, we study the ratio of a function of the forms (1.1) and (1.2) to its sequence of partial sums \((W_{\lambda,\mu})_n(z) = z + \sum_{m=1}^{n} \frac{\Gamma(\mu)}{m! \Gamma(\lambda m + \mu)} z^{m+1}\) when the coefficients of \( W_{\lambda,\mu} \) satisfy certain conditions. We determine the lower bounds of

\[
\text{Re} \left\{ \frac{W_{\lambda,\mu}(z)}{(W_{\lambda,\mu})_n(z)} \right\}, \quad \text{Re} \left\{ \frac{(W_{\lambda,\mu})_n(z)}{W_{\lambda,\mu}(z)} \right\}, \quad \text{Re} \left\{ \frac{W_{\lambda,\mu}^\prime(z)}{(W_{\lambda,\mu})_n(z)} \right\}, \quad \text{Re} \left\{ \frac{(W_{\lambda,\mu})_n(z)}{W_{\lambda,\mu}^\prime(z)} \right\},
\]

\[
\text{Re} \left\{ \frac{\mathcal{A}[W_{\lambda,\mu}](z)}{(\mathcal{A}[W_{\lambda,\mu}])_n(z)} \right\}, \quad \text{Re} \left\{ \frac{(\mathcal{A}[W_{\lambda,\mu}])_n(z)}{\mathcal{A}[W_{\lambda,\mu}](z)} \right\},
\]

where \( \mathcal{A}[W_{\lambda,\mu}] \) is the Alexander transform of \( W_{\lambda,\mu} \). Some similar results are obtained for the function \( \mathbb{W}_{\lambda,\mu}(z) \). For some works on partial sums, we refer [7, 11, 13, 15, 17, 25, 26, 27].

**Lemma 1.1.** Let \( \lambda, \mu \in \mathbb{R} \) and \( \lambda > -1, \mu > 0 \). Then the function \( W_{\lambda,\mu} : \mathcal{U} \to \mathbb{C} \) defined by (1.1) satisfies the following inequalities:

(i) If \( \mu > \frac{1}{2} \), then

\[
|W_{\lambda,\mu}(z)| \leq \frac{2\mu + 1}{2\mu - 1}, \quad z \in \mathcal{U}.
\]

(ii) If \( \mu > 1 \), then

\[
|W_{\lambda,\mu}^\prime(z)| \leq \frac{\mu + 1}{\mu - 1}, \quad z \in \mathcal{U}.
\]

(iii) If \( \mu > \frac{1}{2} \), then

\[
|\mathcal{A}[W_{\lambda,\mu}](z)| \leq \frac{2\mu}{2\mu - 1}, \quad z \in \mathcal{U}.
\]
\textbf{Proof.} (i) By using the well-known triangle inequality
\[ |z_1 + z_2| \leq |z_1| + |z_2| \]
with the inequality \( \frac{\Gamma(\mu + m)}{\Gamma(\lambda m + \mu)} \leq \frac{1}{\mu (\mu + 1) \ldots (\mu + m - 1)} = \frac{1}{(\mu)_m}, m \in \mathbb{N} \) and the inequalities
\[ (\mu)_m \geq \mu^m, \ m! \geq 2^{m-1}, m \in \mathbb{N}, \]
we obtain
\[
|W_{\lambda, \mu}(z)| = \left| z + \sum_{m=1}^{\infty} \frac{\Gamma(\mu)}{m! \Gamma(\lambda m + \mu)} z^{m+1} \right| \leq 1 + \sum_{m=1}^{\infty} \frac{\Gamma(\mu)}{m! \Gamma(\lambda m + \mu)}
\]
\[
\leq 1 + \sum_{m=1}^{\infty} \frac{1}{m! (\mu)_m}
\]
\[
\leq 1 + \frac{1}{\mu} \sum_{m=1}^{\infty} \left( \frac{1}{2\mu} \right)^{m-1}
\]
\[
= \frac{2\mu + 1}{2\mu - 1}, \ \mu > 1/2, \ z \in U.
\]

(ii) To prove (ii), we use the well-known triangle inequality with the inequality
\[ \frac{\Gamma(\mu)}{\Gamma(\lambda m + \mu)} \leq \frac{1}{\mu (\mu + 1) \ldots (\mu + m - 1)} = \frac{1}{(\mu)_m}, m \in \mathbb{N} \] and the inequalities
\[ (\mu)_m \geq \mu^m, \ m! \geq \frac{m+1}{2}, m \in \mathbb{N}, \]
we have
\[
|W'_{\lambda, \mu}(z)| = \left| 1 + \sum_{m=1}^{\infty} \frac{\Gamma(\mu) (m+1)}{m! \Gamma(\lambda m + \mu)} z^m \right| \leq 1 + \sum_{m=1}^{\infty} \frac{\Gamma(\mu) (m+1)}{m! \Gamma(\lambda m + \mu)}
\]
\[
\leq 1 + \sum_{m=1}^{\infty} \frac{m+1}{m! (\mu)_m}
\]
\[
\leq 1 + \frac{2}{\mu} \sum_{m=1}^{\infty} \left( \frac{1}{\mu} \right)^{m-1}
\]
\[
= \frac{\mu + 1}{\mu - 1}, \ \mu > 1, \ z \in U.
\]

(iii) Making the use of triangle inequality with \( \frac{\Gamma(\mu)}{\Gamma(\lambda m + \mu)} \leq \frac{1}{(\mu)_m} \) and the inequalities
\[ (\mu)_m \geq \mu^m, (m+1)! \geq 2^m, m \in \mathbb{N}, \]
we have

\[ |\mathbb{A}[W_{\lambda,\mu}](z)| = \left| z + \sum_{m=1}^{\infty} \frac{\Gamma(\mu)}{(m+1)!\Gamma(\lambda m + \mu)} z^{m+1} \right| \]

\[ \leq 1 + \sum_{m=1}^{\infty} \frac{\Gamma(\mu)}{(m+1)!\Gamma(\lambda m + \mu)} \]

\[ \leq 1 + \sum_{m=1}^{\infty} \frac{1}{(m+1)! (\mu)_m} \]

\[ \leq 1 + \frac{1}{2\mu} \sum_{m=1}^{\infty} \left( \frac{1}{2\mu} \right)^{m-1} \]

\[ = \frac{2\mu}{2\mu-1}, \quad \mu > 1/2, \quad z \in \mathbb{U}. \]

\[ \square \]

**Lemma 1.2.** Let \( \lambda, \mu \in \mathbb{R} \) and \( \lambda > -1, \lambda + \mu > 0 \). Then the function \( W_{\lambda,\mu} : \mathbb{U} \to \mathbb{C} \) defined by (1.2) satisfies the following inequalities:

(i) If \( \lambda + \mu > \frac{1}{2} \), then

\[ |W_{\lambda,\mu}(z)| \leq \frac{2(\lambda + \mu)}{2(\lambda + \mu) - 1}, \quad z \in \mathbb{U}. \]

(ii) If \( \lambda + \mu > \frac{1}{2} \), then

\[ |W'_{\lambda,\mu}(z)| \leq \frac{2(\lambda + \mu) + 1}{2(\lambda + \mu) - 1}, \quad z \in \mathbb{U}. \]

**Proof.** (i) By using the well-known triangle inequality

\[ |z_1 + z_2| \leq |z_1| + |z_2| \]

with the inequality \( \Gamma(\lambda + \mu + m) \leq \Gamma(m\lambda + \lambda + \mu), \ m \in \mathbb{N}, \) which is equivalent to

\[ \frac{\Gamma(\lambda+\mu)}{\Gamma(m\lambda+\lambda+\mu)} \leq \frac{1}{(\lambda+\mu)(\lambda+\mu+1)\cdots(\lambda+\mu+m-1)} = \frac{1}{(\lambda+\mu)_m}, \ m \in \mathbb{N} \]

and the inequalities

\[ (\lambda + \mu)_m \geq (\lambda + \mu)^m, \ m! \geq 2^{m-1}, \ m \in \mathbb{N}, \]

we obtain

\[ |W_{\lambda,\mu}(z)| = \left| z + \sum_{m=1}^{\infty} \frac{\Gamma(\lambda + \mu)}{m!\Gamma(\lambda m + \lambda + \mu)} z^{m+1} \right| \leq 1 + \sum_{m=1}^{\infty} \frac{\Gamma(\lambda + \mu)}{m!\Gamma(\lambda m + \lambda + \mu)} \]

\[ \leq 1 + \sum_{m=1}^{\infty} \frac{1}{m! (\lambda + \mu)_m} \]

\[ \leq 1 + \frac{1}{\lambda + \mu} \sum_{m=1}^{\infty} \left( \frac{1}{2(\lambda + \mu)} \right)^{m-1} \]

\[ = \frac{2(\lambda + \mu) + 1}{2(\lambda + \mu) - 1}, \quad \lambda + \mu > 1/2, \quad z \in \mathbb{U}. \]
(ii) By using the well-known triangle inequality with the inequality \( \frac{\Gamma(\lambda+\mu)}{\Gamma(m\lambda+\lambda+\mu)} \leq \frac{1}{(\lambda+\mu)^m}, m \in \mathbb{N} \) and the inequalities
\[
(\lambda + \mu)_m \geq (\lambda + \mu)^m, \ m! \geq \frac{m+1}{2}, \ m \in \mathbb{N},
\]
we have
\[
|W_{\lambda,\mu}(z)| = \left|1 + \sum_{m=1}^{\infty} \frac{\Gamma(\lambda+\mu)(m+1)}{m!\Gamma(\lambda m + \lambda + \mu)} z^m\right| \leq 1 + \sum_{m=1}^{\infty} \frac{\Gamma(\lambda+\mu)(m+1)}{m!\Gamma(\lambda m + \lambda + \mu)}
\]
\[
\leq 1 + \sum_{m=1}^{\infty} \frac{m+1}{m! (\lambda+\mu)_m}
\]
\[
\leq 1 + \frac{2}{(\lambda+\mu)} \sum_{m=1}^{\infty} \left(\frac{1}{\lambda+\mu}\right)^{m-1}
\]
\[
= \frac{(\lambda+\mu) + 1}{(\lambda+\mu) - 1}, \ (\lambda + \mu) > 1, \ z \in \mathcal{U}.
\]

2. Partial Sums of \( W_{\lambda,\mu}(z) \)

**Theorem 2.1.** Let \( \lambda, \mu \in \mathbb{R} \) such that \( \lambda > -1, \mu > \frac{3}{2} \). Then
\[
\text{Re} \left\{ \frac{W_{\lambda,\mu}(z)}{(W_{\lambda,\mu})_n(z)} \right\} \geq \frac{2\mu - 3}{2\mu - 1}, \ z \in \mathcal{U}. \tag{2.1}
\]
and
\[
\text{Re} \left\{ \frac{(W_{\lambda,\mu})_n(z)}{W_{\lambda,\mu}(z)} \right\} \geq \frac{2\mu - 1}{2\mu + 1}, \ z \in \mathcal{U}. \tag{2.2}
\]

**Proof.** By using (i) of Lemma 1.1, it is clear that
\[
1 + \sum_{m=1}^{\infty} |a_m| \leq \frac{2\mu + 1}{2\mu - 1},
\]
which is equivalent to
\[
\frac{2\mu - 1}{2} \sum_{m=1}^{\infty} |a_m| \leq 1.
\]
where $a_m = \frac{\Gamma(\mu)}{m!\Gamma(\lambda m+\mu)}$. Now, we may write

$$
\frac{2\mu - 1}{2} \left\{ \frac{W_{\lambda,\mu}(z)}{(W_{\lambda,\mu})_n(z)} - \frac{2\mu - 3}{2\mu - 1} \right\} \\
1 + \sum_{m=1}^{n} a_m z^m + \left( \frac{2\mu - 1}{2} \right) \sum_{m=n+1}^{\infty} a_m z^m \\
= \frac{1 + \sum_{m=1}^{n} a_m z^m}{1 + \sum_{m=1}^{n} a_m z^m}
$$

Then it is clear that

$$
w(z) = \frac{\left( \frac{2\mu - 1}{2} \right) \sum_{m=n+1}^{\infty} |a_m| z^m}{2 + 2 \sum_{m=1}^{n} a_m z^m + \left( \frac{2\mu - 1}{2} \right) \sum_{m=n+1}^{\infty} a_m z^m}
$$

and

$$|w(z)| \leq \frac{\left( \frac{2\mu - 1}{2} \right) \sum_{m=n+1}^{\infty} |a_m|}{2 - 2 \sum_{m=1}^{n} |a_m| - \left( \frac{2\mu - 1}{2} \right) \sum_{m=n+1}^{\infty} |a_m|}.
$$

This implies that $|w(z)| \leq 1$ if and only if

$$2 \left( \frac{2\mu - 1}{2} \right) \sum_{m=n+1}^{\infty} |a_m| \leq 2 - 2 \sum_{m=1}^{n} |a_m|.
$$

Which further implies that

$$\sum_{m=1}^{n} |a_m| + \left( \frac{2\mu - 1}{2} \right) \sum_{m=n+1}^{\infty} |a_m| \leq 1. \quad (2.3)
$$

It suffices to show that the left hand side of (2.3) is bounded above by $\left( \frac{2\mu - 1}{2} \right) \sum_{m=1}^{\infty} |a_m|$, which is equivalent to

$$\frac{2\mu - 3}{2} \sum_{m=1}^{n} |a_m| \geq 0.$$
To prove (2.2), we write
\[
\frac{2\mu + 1}{2} \left\{ \left( \frac{\mathcal{W}_{\lambda,\mu}(z)}{\mathcal{W}_{\lambda,\mu}(n)} \right) \frac{2\mu - 1}{2\mu + 1} \right\}
\]
\[
= 1 + \sum_{m=1}^{n} a_m z^m - \left( \frac{2\mu - 1}{2} \right) \sum_{m=n+1}^{\infty} a_m z^m
\]
\[
= 1 + \frac{w(z)}{1 - w(z)}.
\]
Therefore
\[
|w(z)| \leq \frac{\left( \frac{2\mu + 1}{2} \right) \sum_{m=n+1}^{\infty} |a_m|}{2 - 2 \sum_{m=1}^{n} |a_m| - \left( \frac{2\mu - 1}{2} \right) \sum_{m=n+1}^{\infty} |a_m|} \leq 1.
\]
The last inequality is equivalent to
\[
\sum_{m=1}^{n} |a_m| + \left( \frac{2\mu - 1}{2} \right) \sum_{m=n+1}^{\infty} |a_m| \leq 1.
\]
(2.4)
Since the left hand side of (2.4) is bounded above by \( \left( \frac{2\mu - 1}{2} \right) \sum_{m=1}^{\infty} |a_m| \), this completes the proof.

\[\square\]

**Theorem 2.2.** Let \( \lambda, \mu \in \mathbb{R} \), with \( \lambda > -1 \) and \( \mu > 3 \). Then
\[
\text{Re} \left\{ \frac{\mathcal{W}_{\lambda,\mu}'(z)}{\mathcal{W}_{\lambda,\mu}(n)} \right\} \geq \frac{\mu - 3}{\mu - 1}, \quad z \in \mathcal{U}.
\]
(2.5)
\[
\text{Re} \left\{ \frac{\mathcal{W}_{\lambda,\mu}'(z)}{\mathcal{W}_{\lambda,\mu}(z)} \right\} \geq \frac{\mu - 1}{\mu + 1}, \quad z \in \mathcal{U}.
\]
(2.6)
**Proof.** From part (ii) of Lemma 1.1, we observe that
\[
1 + \sum_{m=1}^{\infty} (m+1) |a_m| \leq \frac{\mu + 1}{\mu - 1},
\]
where \( a_m = \frac{\Gamma(\mu)}{m! \Gamma(\lambda m + \mu)} \). This implies that
\[
\left( \frac{\mu - 1}{2} \right) \sum_{m=1}^{\infty} (m+1) |a_m| \leq 1.
\]
Consider
\[
\left( \frac{\mu - 1}{2} \right) \left\{ \frac{\mathcal{W}_{\lambda,\mu}'(z)}{\mathcal{W}_{\lambda,\mu}'(z)} - \frac{\mu - 3}{\mu - 1} \right\}
\]
\[
= 1 + \sum_{m=1}^{n} (m + 1)a_m z^m + \left( \frac{\mu - 3}{2} \right) \sum_{m=n+1}^{\infty} (m + 1)a_m z^m
\]
\[
= 1 + \sum_{m=1}^{n} (m + 1)a_m z^m
\]
\[
= 1 + w(z) \quad \frac{1 - w(z)}{1 - w(z)}.
\]
Therefore
\[
|w(z)| \leq \frac{(\frac{\mu - 1}{2}) \sum_{m=n+1}^{\infty} (m + 1) |a_m|}{2 - 2 \sum_{m=1}^{n} (m + 1) |a_m| - \frac{\mu - 3}{2} \sum_{m=n+1}^{\infty} (m + 1) |a_m|} \leq 1.
\]
The last inequality is equivalent to
\[
\sum_{m=1}^{n} (m + 1) |a_m| + \left( \frac{\mu - 1}{2} \right) \sum_{m=n+1}^{\infty} (m + 1) |a_m| \leq 1.
\]
(2.7)
It suffices to show that the left hand side of (2.7) is bounded above by
\[
(\frac{\mu - 1}{2}) \sum_{m=1}^{\infty} |a_m| (m + 1). \quad \text{Which is equivalent to } \frac{\mu - 3}{2} \sum_{m=1}^{n} (m + 1) |a_m| \geq 0.
\]
To prove the result (2.6), we write
\[
\left( \frac{\mu + 1}{2} \right) \left\{ \frac{\mathcal{W}_{\lambda,\mu}'(z)}{\mathcal{W}_{\lambda,\mu}'(z)} - \frac{\mu - 1}{\mu + 1} \right\}
\]
\[
= \frac{1 + w(z)}{1 - w(z)}.
\]
Therefore
\[
|w(z)| \leq \frac{(\frac{\mu + 1}{2}) \sum_{m=n+1}^{\infty} (m + 1) |a_m|}{2 - 2 \sum_{m=1}^{n} (m + 1) |a_m| - \frac{\mu - 3}{2} \sum_{m=n+1}^{\infty} (m + 1) |a_m|} \leq 1.
\]
The last inequality is equivalent to
\[
\sum_{m=1}^{n} |a_m| (m + 1) + \frac{\mu - 1}{2} \sum_{m=n+1}^{\infty} (m + 1) |a_m| \leq 1.
\]
(2.8)
It suffices to show that the left hand side of (2.8) is bounded above by
\[ \frac{\mu - 1}{2} \sum_{m=1}^{\infty} (m + 1) |a_m|, \text{ the proof is complete.} \]

**Theorem 2.3.** Let \( \lambda, \mu \in \mathbb{R} \), with \( \lambda > -1 \) and \( \mu > 1 \). Then

\[
\text{Re}\left\{ \frac{\mathbb{A} [\mathcal{W}_{\lambda,\mu}](z)}{(\mathbb{A} [\mathcal{W}_{\lambda,\mu}])_n(z)} \right\} \geq \frac{2\mu - 2}{2\mu - 1}, \quad z \in \mathcal{U}, \tag{2.9}
\]

and

\[
\text{Re}\left\{ \frac{(\mathbb{A} [\mathcal{W}_{\lambda,\mu}])_n(z)}{(\mathbb{A} [\mathcal{W}_{\lambda,\mu}])_n(z)} \right\} \geq \frac{2\mu - 1}{2\mu}, \quad z \in \mathcal{U}, \tag{2.10}
\]

where \( \mathbb{A} [\mathcal{W}_{\lambda,\mu}] \) is the Alexander transform of \( \mathcal{W}_{\lambda,\mu} \).

**Proof.** To prove (2.9), we consider from part (iii) of Lemma 1.1 so that

\[
1 + \sum_{m=1}^{\infty} \frac{|a_m|}{(m + 1)} \leq \frac{2\mu}{2\mu - 1},
\]

which is equivalent to

\[
(2\mu - 1) \sum_{m=1}^{\infty} \frac{|a_m|}{(m + 1)} \leq 1,
\]

where \( a_m = \frac{\Gamma(\mu) m!}{m! \Gamma(\lambda m + \mu)} \). Now, we write

\[
(2\mu - 1) \left\{ \frac{\mathbb{A} [\mathcal{W}_{\lambda,\mu}](z)}{(\mathbb{A} [\mathcal{W}_{\lambda,\mu}])_n(z)} - \frac{2\mu - 2}{2\mu - 1} \right\}
\]

\[= 1 + \sum_{m=1}^{n} \frac{a_m}{(m + 1)} z^m + (2\mu - 1) \sum_{m=n+1}^{\infty} \frac{a_m}{(m + 1)} z^m\]

\[= \frac{1 + w(z)}{1 - w(z)},\]

where

\[|w(z)| \leq \frac{(2\mu - 1) \sum_{m=n+1}^{\infty} \frac{|a_m|}{(m + 1)}}{2 - 2 \sum_{m=1}^{n} \frac{|a_m|}{(m + 1)} - (2\mu - 1) \sum_{m=n+1}^{\infty} \frac{|a_m|}{(m + 1)}} \leq 1.\]

The last inequality is equivalent to

\[
\sum_{m=1}^{n} \frac{|a_m|}{(m + 1)} + (2\mu - 1) \sum_{m=n+1}^{\infty} \frac{|a_m|}{(m + 1)} \leq 1. \tag{2.11}
\]

It suffices to show that the left hand side of (2.11) is bounded above by

\[
(2\mu - 1) \sum_{m=1}^{\infty} \frac{|a_m|}{(m + 1)} \geq 0. \]

This completes the proof.
The proof of (2.10) is similar to the proof of Theorem 2.1.

\[ \square \]

**Remark 2.4.** For \( \lambda = 1, \mu = 5/2 \) we get \( W_{1,5/2}(-z) = \frac{3}{4} \left( \frac{\sin(2\sqrt{z})}{2\sqrt{z}} - \cos(2\sqrt{z}) \right) \), and for \( n = 0 \), we have \( (W_{1,5/2})_0(z) = z \), so,

\[
\text{Re} \left( \frac{\sin(2\sqrt{z}) - 2\sqrt{z}\cos(2\sqrt{z})}{2z\sqrt{z}} \right) \geq \frac{2}{3} \quad (z \in U), \tag{2.12}
\]

and

\[
\text{Re} \left( \frac{2z\sqrt{z}}{\sin(2\sqrt{z}) - 2\sqrt{z}\cos(2\sqrt{z})} \right) \geq \frac{1}{2} \quad (z \in U). \tag{2.13}
\]

The image domains of \( f(z) = \frac{\sin(2\sqrt{z}) - 2\sqrt{z}\cos(2\sqrt{z})}{2z\sqrt{z}} \) and \( g(z) = \frac{2z\sqrt{z}}{\sin(2\sqrt{z}) - 2\sqrt{z}\cos(2\sqrt{z})} \) are shown in Figure 1.

3. **Partial Sums of** \( W_{\lambda,\mu}(z) \)

**Theorem 3.1.** Let \( \lambda, \mu \in \mathbb{R} \), with \( \lambda > -1 \) and \( \mu + \lambda > 1 \). Then

\[
\text{Re} \left\{ \frac{W_{\lambda,\mu}(z)}{(W_{\lambda,\mu})_n(z)} \right\} \geq \frac{2(\lambda + \mu) - 2}{2(\lambda + \mu) - 1}, \quad z \in U, \tag{3.1}
\]

and

\[
\text{Re} \left\{ \frac{(W_{\lambda,\mu})_n(z)}{W_{\lambda,\mu}(z)} \right\} \geq \frac{2(\lambda + \mu) - 1}{2(\lambda + \mu)}, \quad z \in U, \tag{3.2}
\]

where \( W_{\lambda,\mu}(z) \) is the normalized Wright function.

**Proof.** By using Lemma 1.2 (i), It is clear that

\[
1 + \sum_{m=1}^{\infty} |a_m| \leq \frac{2(\lambda + \mu)}{2(\lambda + \mu) - 1},
\]

where \( a_m = \frac{\Gamma(\lambda+\mu)}{(m+1)!\Gamma(\lambda m + \lambda + \mu)} \). This implies that

\[
\{2(\lambda + \mu) - 1\} \sum_{m=1}^{\infty} |a_m| \leq 1.
\]
Now we may write

\[
\left\{ 2 (\lambda + \mu) - 1 \right\} \left\{ \frac{\mathbb{W}_{\lambda,\mu}(z)}{(\mathbb{W}_{\lambda,\mu})_n(z)} - \frac{2 (\lambda + \mu) - 2}{2 (\lambda + \mu) - 1} \right\}
\]

\[
1 + \sum_{m=1}^{n} a_m z^m + \left\{ 2 (\lambda + \mu) - 1 \right\} \sum_{m=n+1}^{\infty} a_m z^m
\]

\[
= \frac{1 + \sum_{m=1}^{n} a_m z^m}{1 + \sum_{m=1}^{n} a_m z^m}
\]

\[
= \frac{1 + w(z)}{1 - w(z)}.
\]

It is clear that

\[
w(z) = \frac{\left\{ 2 (\lambda + \mu) - 1 \right\} \sum_{m=n+1}^{\infty} a_m z^m}{2 + 2 \sum_{m=1}^{n} a_m z^m + \left\{ 2 (\lambda + \mu) - 1 \right\} \sum_{m=n+1}^{\infty} a_m z^m},
\]

and

\[
|w(z)| \leq \frac{\left\{ 2 (\lambda + \mu) - 1 \right\} \sum_{m=n+1}^{\infty} |a_m|}{2 - 2 \sum_{m=1}^{n} |a_m| - \left\{ 2 (\lambda + \mu) - 1 \right\} \sum_{m=n+1}^{\infty} |a_m|}.
\]

This implies that \(|w(z)| \leq 1\) if and only if

\[
\sum_{m=1}^{n} |a_m| + \left\{ 2 (\lambda + \mu) - 1 \right\} \sum_{m=n+1}^{\infty} |a_m| \leq 1. \tag{3.3}
\]

It suffices to show that the left hand side of (3.3) is bounded above by

\[
\left\{ 2 (\lambda + \mu) - 1 \right\} \sum_{m=1}^{\infty} |a_m|, \text{ which is equivalent to } \left\{ 2 (\lambda + \mu) - 2 \right\} \sum_{m=1}^{\infty} |a_m| \geq 0.
\]

To prove (3.2), we consider that

\[
2 (\lambda + \mu) \left\{ \frac{(\mathbb{W}_{\lambda,\mu})_n(z)}{\mathbb{W}_{\lambda,\mu}(z)} - \frac{2 (\lambda + \mu) - 2}{2 (\lambda + \mu)} \right\}.
\]

\[
= \frac{1 + \sum_{m=1}^{n} a_m z^m + \left\{ 2 (\lambda + \mu) - 1 \right\} \sum_{m=n+1}^{\infty} a_m z^m}{1 + \sum_{m=1}^{\infty} a_m z^m}
\]

\[
= \frac{1 + w(z)}{1 - w(z)}.
\]
Therefore
\[
|w(z)| \leq \frac{\{2(\lambda + \mu)\} \sum_{m=n+1}^{\infty} |a_m|}{2 - 2 \sum_{m=1}^{n} |a_m| - \{2(\lambda + \mu) - 2\} \sum_{m=n+1}^{\infty} |a_m|}.
\]

The last inequality is equivalent to
\[
\sum_{m=1}^{n} |a_m| + \{2(\lambda + \mu) - 1\} \sum_{m=n+1}^{\infty} |a_m| \leq 1. \tag{3.4}
\]

Since the left hand side of (3.4) is bounded above by \(\{2(\lambda + \mu) - 1\} \sum_{m=1}^{\infty} |a_m|\), the proof is complete. \(\square\)

Similarly, we have the following result.

**Theorem 3.2.** Let \(\lambda, \mu \in \mathbb{R}\), with \(\lambda > -1\) and \(\mu + \lambda > \frac{3}{2}\). Then
\[
\text{Re} \left\{ \frac{\mathbb{W}_{\lambda,\mu}'(z)}{(\mathbb{W}_{\lambda,\mu})'(z)} \right\} \geq \frac{2(\lambda + \mu) - 3}{2(\lambda + \mu) - 1}, \quad z \in \mathcal{U}, \tag{3.5}
\]
and
\[
\text{Re} \left\{ \frac{(\mathbb{W}_{\lambda,\mu})'_n(z)}{(\mathbb{W}_{\lambda,\mu})'_n(z)} \right\} \geq \frac{2(\lambda + \mu) - 1}{2(\lambda + \mu) + 1}, \quad z \in \mathcal{U}, \tag{3.6}
\]
where \(\mathbb{W}_{\lambda,\mu}(z)\) is the normalized Wright function.

**Proof.** Proof is similar to the Theorem 2.2. \(\square\)

Recently Ravichandran [20] presented a survey article on geometric properties of partial sums of univalent functions. Using Noshiro-Warschawski Theorem [9] for \(n = 0\) in the inequalities (2.5) of Theorem 2.2 and (3.5) of Theorem 3.2, the functions \(\mathbb{W}_{\lambda,\mu}(z)\) and \(\mathbb{W}_{\lambda,\mu}(z)\) are univalent and also close to convex. Noshiro [16] showed that the radius of starlikeness of \(f_n\) (the partial sums of the function \(f \in \mathcal{A}\)) is \(1/M\) if \(f\) satisfies the inequality \(|f'(z)| \leq M\). This implies that by using the parts (ii) of Lemma 1.1 and Lemma 2.1, the radii of starlikeness of the functions \((\mathbb{W}_{\lambda,\mu})_n(z)\) and \((\mathbb{W}_{\lambda,\mu})_n(z)\) are \(\frac{\mu-1}{\mu+1}\) and \(\frac{2(\lambda+\mu)-1}{2(\lambda+\mu)+1}\) respectively.

**Acknowledgement:** The research of N. Yağmur is supported by Erzincan University Rectorship under "The Scientific and Research Project of Erzincan University", Project No: FEN-A-240215-0126.

**References**

[1] Á. Baricz, Functional inequalities involving special functions, J. Math. Anal. Appl., 319 (2006), 450-459.
[2] Á. Baricz, Functional inequalities involving special functions. II, J. Math. Anal. Appl., 327 (2007), 1202-1213.
[3] Á. Baricz, Generalized Bessel Functions of the First Kind, Lecture Notes in MathematicsVol. 1994, Springer-Verlag, Berlin, 2010.
[4] Á. Baricz, Geometric properties of generalized Bessel functions, Publ. Math. Debrecen, 73 (2008), 155-178.
[5] Á. Baricz, Some inequalities involving generalized Bessel functions, Math. Inequal. Appl., 10 (2007), 827-842.
[6] Á. Baricz and S. Ponnusamy, Starlikeness and convexity of generalized Bessel functions, Integral Transforms Spec. Funct., 21,9 (2010), 641-653.
[7] L. Brickman, D. J. Hallenbeck, T. H. Macgregor and D. Wilken. Convex hulls and extreme-points of families of starlike and convex mappings. Trans. Amer. Math. Soc. 185 (1973), 413–428.
[8] M. Çağlar and E. Deniz, Partial sums of the normalized Lommel functions, Math. Inequal. Appl. 18, 3 (2015), 1189-1199.
[9] A. W. Goodman, Univalent functions, Vol 1. Marinar Publi. Comp., Tempa Florida, 1984.
[10] R. Gorenflo, Y. Luchko and F. Maimardi. Analytic properties and applications of Wright functions. Frac. Cal. App. Anal. 2, 4 (1999), 383-414.
[11] L.J. Lin and S. Owa. On partial sums of the Libera integral operator. J. Math. Anal. Appl. 213, 2 (1997), 444–454.
[12] S. S. Miller, P. T Mocanu, Univalence of Gaussian and confluent hypergeometric functions, Proc Amer Math Soc., 110, 2 (1990), 333–342.
[13] H. Orhan and E. Gunes. Neighborhoods and partial sums of analytic functions based on Gaussian hypergeometric functions. Indian J. Math. 51, 3 (2009), 489–510.
[14] H. Orhan and N. Yağmur, Geometric properties of generalized Struve functions, An. Stiint. Univ. Al. I. Cuza Iasi. Mat. (N.S.) (doi: 10.2478/aicu-2014-0007).
[15] H. Orhan and N. Yağmur. Partial sums of generalized Bessel functions. J. Math.Ineq. 8, 4 (2014), 863-877.
[16] K. Noshiro, On the starshaped mapping by analytic function. Proc. Imp. Acad. 8, 7 (1932), 275-277.
[17] S. Owa, H. M. Srivastava and N. Saito. Partial sums of certain classes of analytic functions. Int. J. Comput. Math. 81, 10 (2004), 1239–1256.
[18] I. Podlubny, Fractional differential equations. San Diego: Academic press, 1999.
[19] J. K. Prajapat. Certain geometric properties of the Wright function. Integ. Trans. Spec Func. 26, 3 (2015), 203-212.
[20] V. Ravichandran, Geometric properties of partial sums of univalent functions. arXiv: 1207.4302v1.
[21] S. G. Samko, A. A. Kilbas and O. I. Marichev. Fractional integrals and derivatives: theory and applications. Gordan and Breach: New York, 1993.
[22] V. Selinger, Geometric properties of normalized Bessel functions, Pure Math. Appl. 6 (1995), 273–277.
[23] R. Szász, About the starlikeness of Bessel functions, Integral Transforms Spec. Funct. 25, 9 (2014), 750-755.
[24] R. Szász and P. Kupán, About the univalence of the Bessel functions, Stud. Univ. Babeş-Bolyai Math. 54(1) (2009), 127–132.
[25] T. Sheil-Small. A note on partial sums of convschlicht functions. Bull. London Math. Soc. 2 (1970), 165–168.
[26] H. Silverman. Partial sums of starlike and convex functions. J. Math. Anal. Appl. 209 (1997), 221–227.
[27] E. M. Silvia. On partial sums of convex functions of order α.Houston J. Math. 11 (1985), 397–404.
[28] V. Singh St, Ruscheweyh, On the order of starlikeness of hypergeometric functions, J Math Anal Appl., 113(1986), 1–11.
[29] E. M. Wright. On the coefficients of power series having exponential singularities. J. London Math. Soc. 8 (1933),71-79.
[30] N. Yağmur and H. Orhan. Partial sums of generalized Struve functions. Miskolc Math. Notes (accepted).
1 Department of Mathematics, Government College University Faisalabad, Pakistan
   E-mail address: muheyudin@gmail.com.

2 Department of Mathematics, Government College University Faisalabad, Pakistan.
   E-mail address: mohsan976@yahoo.com

3 Department of Mathematics, Erzincan University, Erzincan 24000, Turkey
   E-mail address: nhtyagmur@gmail.com