Solving the two-mode squeezed harmonic oscillator and the $k$th-order harmonic generation in Bargmann–Hilbert spaces

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Received 31 May 2013, in final form 16 September 2013
Published 24 October 2013
Online at stacks.iop.org/JPhysA/46/455302

Abstract

We analyze the two-mode squeezed harmonic oscillator and the $k$th-order harmonic generation within the framework of Bargmann–Hilbert spaces of entire functions. For displaced single-mode squeezed and two-mode squeezed harmonic oscillators, we derive exact closed-form expressions for their energies and wave functions. For the $k$th-order harmonic generation with $k \geq 3$, our result indicates that it does not have eigenfunctions and is thus ill-defined in the Bargmann–Hilbert space.

PACS numbers: 03.65.Ge, 02.30.Ik, 42.50.Pq

(Some figures may appear in colour only in the online journal)

1. Introduction

Recently, there has been renewed interest in formulating and solving dynamical systems involving harmonic modes in the framework of Bargmann–Hilbert spaces [1–3]. For example, in [4, 5], we applied the Bargmann–Hilbert space technique to obtain the exact solutions of families of quantum nonlinear optical as well as spin-boson models. In [6–10], the authors applied the technique to the quantum Rabi model, a simple system describing the interaction of a two-level atom with a harmonic mode. A Bargmann–Hilbert space is a Hilbert space of entire functions introduced by Bargmann and Segal. It is a vector space with typical orthonormal basis $z^n \sqrt{n!}$, $n = 0, 1, 2, \ldots$. Elements in the space are entire functions and the space is equipped with a well-defined Hermitian scalar product,

$$ (f, h) = \int f(z) h(z) \, d\mu(z) \quad (1.1) $$

for any two elements $f(z), h(z)$ in the space, where $d\mu(z) = \frac{1}{\pi} e^{-|z|^2} \, dx \, dy$. In a Bargmann–Hilbert space, the harmonic creation and annihilation operators $a^\dagger, a$ can be realized as $a^\dagger \rightarrow z, a \rightarrow \frac{\partial}{\partial z}$. This realization enables one to convert the time-independent Schrödinger...
equation of a dynamical system into a differential equation. Solutions to the differential equation are entire functions.

In this paper, we apply Bargmann–Hilbert spaces to analyze the two-mode squeezed harmonic oscillator and the kth-order harmonic generation. For the cases of displaced single-mode squeezed and two-mode squeezed harmonic oscillators, we derive exact closed-form expressions for their energies and wave functions. For the kth-order harmonic generation with \( k \geq 3 \), our result shows that it does not have eigenfunctions that are entirely in the Bargmann–Hilbert space. The rest of this paper is as follows. In section 2, we exactly solve the two-mode squeezed harmonic oscillator. In section 3, we report our results from our investigation of the solvability of the kth-order harmonic generation. We draw our conclusions in section 4.

2. Two-mode squeezed harmonic oscillator

The Hamiltonian of the two-mode squeezed harmonic oscillator reads

\[
H = \omega (a_1^\dagger a_1 + a_2^\dagger a_2) + g (a_1^\dagger a_2 + a_2^\dagger a_1),
\]

where we assume that the boson modes are degenerate with the same frequency \( \omega \) and \( g \) is a real constant. In terms of the operators \( K_\pm, K_0 \) defined by

\[
K_+ = a_1^\dagger a_2^\dagger, \quad K_- = a_1 a_2, \quad K_0 = \frac{1}{2} (a_1^\dagger a_1 + a_2^\dagger a_2 + 1),
\]

the Hamiltonian (2.1) can be written as

\[
H = 2 \omega (K_0 - \frac{1}{2}) + g (K_+ + K_-).
\]

The operators \( K_\pm, K_0 \) form the su(1, 1) Lie algebra. The quadratic Casimir of the algebra, \( C = K_+ K_- - K_0 (K_0 - 1) \), has eigenvalue \( \kappa (1 - \kappa) \) in the infinite-dimensional unitary irreducible representation of su(1, 1) known as the positive discrete series \( D^+(\kappa) \). The parameter \( \kappa > 0 \) is the so-called Bargmann index. For the two-mode boson realization (2.2), \( \kappa \) can take any positive integers or half integers, i.e. \( \kappa = 1/2, 1, 3/2, \ldots \) Thus the Fock–Hilbert space decomposes into the direct sum of infinite subspaces \( \mathcal{H}_\kappa \) labeled by \( \kappa = 1/2, 1, 3/2, \ldots \).

The basis state in the subspace \( \mathcal{H}_\kappa \), denoted as \( |\kappa, n\rangle \), \( n = 0, 1, 2, \ldots \), has the form

\[
|\kappa, n\rangle = (a_1^\dagger)^{n+2\kappa-1} (a_2^\dagger)^n |0\rangle,
\]

and the action of \( K_\pm, K_0 \) in this representation is given by

\[
K_0 |\kappa, n\rangle = (n + \kappa) |\kappa, n\rangle, \quad K_+ |\kappa, n\rangle = \sqrt{(n + 2\kappa)(n + 1)} |\kappa, n + 1\rangle, \quad K_- |\kappa, n\rangle = \sqrt{(n + 2\kappa - 1)(n + 1)} |\kappa, n - 1\rangle.
\]

Using the Fock–Bargmann correspondence

\[
a^\dagger \rightarrow z, \quad a \rightarrow \frac{d}{dz}, \quad |0\rangle \rightarrow 1,
\]

we can show that the infinite set of monomials

\[
\Psi_{\kappa,n}(z) = \frac{z^n}{\sqrt{(n + 2\kappa - 1)n!}}, \quad n = 0, 1, 2, \ldots
\]

form the basis in the Bargmann–Hilbert subspace associated with the representation (2.5). Thus the operators \( K_\pm, K_0 \) (2.2) have the single-variable differential realization in the subspace labeled by the Bargmann index \( \kappa \),

\[
K_0 = z \frac{d}{dz} + \kappa, \quad K_+ = z, \quad K_- = z^2 \frac{d^2}{dz^2} + 2\kappa z \frac{d}{dz}, \quad \kappa = 1/2, 1, 3/2, \ldots
\]
By means of this differential representation (2.8), we can express the Hamiltonian (2.3) (i.e. (2.1)) as the 2nd-order differential operator in each Bargmann–Hilbert subspace labeled by \( \kappa \),

\[
H = 2\omega \left( \frac{d^2}{dz^2} + \kappa - \frac{1}{2} \right) + g \left( z + \frac{d^2}{dz^2} + 2\kappa \frac{d}{dz} \right). \tag{2.9}
\]

Then the time-independent Schrödinger equation gives the differential equation for wave function \( \psi(z) \),

\[
g z \frac{d^2}{dz^2} \psi(z) + 2(\omega z + g\kappa) \frac{d}{dz} \psi(z) + \left[ g z + 2\omega \left( \kappa - \frac{1}{2} \right) - E \right] \psi(z) = 0. \tag{2.10}
\]

With the substitution

\[
\psi(z) = e^{-\frac{z^2}{2\Lambda}} \phi(z), \quad \Lambda = \sqrt{1 - \frac{g^2}{\omega^2}}, \tag{2.11}
\]

where \( \left| \frac{z}{\omega} \right| < 1 \), it follows,

\[
\mathcal{L} \phi \equiv \left\{ g z \frac{d^2}{dz^2} + 2[\omega \Lambda z + g\kappa] \frac{d}{dz} + 2\kappa \omega \Lambda - \omega - E \right\} \phi = 0. \tag{2.12}
\]

This differential equation is exactly solvable. This is seen as follows. First of all, let us recall the characterization of the exact solvability of a differential operator. A linear differential operator \( \mathcal{L} \) is exactly solvable if it preserves an infinite flag of finite-dimensional functional spaces,

\[ \mathcal{V}_1 \subset \mathcal{V}_2 \subset \cdots \subset \mathcal{V}_M \subset \cdots, \]

whose bases admit explicit analytic forms, that is, there exists a sequence of finite-dimensional invariant subspaces \( \mathcal{V}_M, M = 1, 2, 3, \ldots \),

\[ \mathcal{L} \mathcal{V}_M \subset \mathcal{V}_M, \quad \dim \mathcal{V}_M < \infty, \quad \mathcal{V}_M = \text{span}\{\xi_1, \ldots, \xi_{\dim \mathcal{V}_M}\}. \]

In our case, we have, for any positive integer \( n \),

\[ \mathcal{L}z^n = [(2n + 2\kappa)\omega \Lambda - \omega - E]z^n + n(n + 2\kappa - 1)gz^{n-1}. \tag{2.13} \]

It follows that \( \mathcal{L} \) preserves an infinite flag of finite dimensional spaces \( \mathcal{V}_1 \subset \mathcal{V}_2 \subset \cdots \subset \mathcal{V}_M \subset \cdots \), with explicitly determined subspaces \( \mathcal{V}_M = \{1, z, z^2, \ldots, z^M\} \), and exact solutions are polynomials in \( z \) in the Bargmann–Hilbert space. We thus seek solutions of the form to the differential equation (2.12),

\[
\phi(z) = \prod_{i=1}^{M}(z - z_i), \quad \mathcal{M} = 0, 1, 2, \ldots, \tag{2.14}
\]

where \( \phi(z) \equiv 1 \) for \( \mathcal{M} = 0 \), \( \mathcal{M} \) is the degree of the polynomial and \( z_i \) are the roots of the polynomial to be determined. Substituting into (2.12) and dividing both sides by \( \phi(z) \) gives rise to

\[
E + \omega - 2\kappa \omega \Lambda = gz \sum_{i=1}^{M} \frac{1}{z - z_i} \sum_{j \neq i}^{M} \frac{2}{z_i - z_j} + 2[\omega \Lambda z + g\kappa] \sum_{i=1}^{M} \frac{1}{z - z_i}
\]

\[ = 2n\omega \Lambda + \sum_{i=1}^{M} \text{Res}_{z=z_i}, \tag{2.15} \]

where \( \text{Res}_{z=z_i} \) are the residues of the right-hand side of the first equality at the simple poles \( z = z_i \), i.e.,

\[
\text{Res}_{z=z_i} = gz \sum_{j \neq i}^{M} \frac{2}{z_i - z_j} + 2\omega \Lambda z_i + 2g\kappa. \tag{2.16}
\]
The left-hand side (2.15) is a constant and the right-hand side is a meromorphic function with simple poles at \( z = z_i \). The right-hand side is a constant if and only if the coefficient of all the residues at the simple poles are vanishing. We thus obtain the energies
\[
E = -\omega + \left[ 2M + 2(k - \frac{1}{2}) + 1 \right] \omega \Lambda,
\]  
and the system of algebraic equations satisfied by the roots \( z_i \),
\[
\sum_{j \neq i} \frac{1}{z_i - z_j} + \frac{\omega}{g} \Lambda + \frac{\kappa}{z_i} = 0, \quad i = 1, 2, \ldots, M.
\]  
The corresponding wave functions are given by
\[
\psi(z) = e^{-\frac{z}{\Lambda} \left( 1 - \Lambda z \right)} \prod_{i=1}^{M} (z - z_i).
\]  
As examples, we list the first three eigenstates. For \( M = 0 \), we have \( \psi(z) = e^{-\frac{z}{\Lambda} \left( 1 - \Lambda z \right)} \). For \( M = 1 \), we obtain from (2.18) the root \( z_1 = -\frac{\kappa g}{\omega \Lambda} \) and from (2.19) the corresponding wave function \( \psi(z) = e^{-\frac{z}{\Lambda} \left( 1 - \Lambda z \right)} \left( z + \frac{\kappa g}{\omega \Lambda} \right) \). For \( M = 2 \), the roots \( z_1, z_2 \) satisfy the system of algebraic equations
\[
\frac{1}{z_1 - z_2} + \frac{\omega \Lambda}{g} z_1 + \frac{\kappa}{z_1} = 0, \quad \frac{1}{z_2 - z_1} + \frac{\omega \Lambda}{g} z_2 + \frac{\kappa}{z_2} = 0.
\]  
Solving the two equations simultaneously gives
\[
z_1 = \frac{-(1 + 2\kappa)}{2\omega \Lambda} + \sqrt{1 + 2\kappa} \frac{g}{2}, \quad z_2 = \frac{-(1 + 2\kappa)}{2\omega \Lambda} - \sqrt{1 + 2\kappa} \frac{g}{2}.
\]  
The corresponding wave function is given by
\[
\psi(z) = e^{-\frac{z}{\Lambda} \left( 1 - \Lambda z \right)} \left[ z^2 + \frac{(1 + 2\kappa)g}{\omega \Lambda} z + \frac{\kappa(1 + 2\kappa)g^2}{2\omega^2 \Lambda^2} \right].
\]  
In figure 1, we plot some of these eigenstates over a disc of radius 3 in the complex plane.

### 3. kth-order harmonic generation

The Hamiltonian of the kth-order harmonic generation reads
\[
H = \omega a^\dagger a + g [(a^\dagger)^k + a^k],
\]  
where \( k = 1, 2, \ldots \) is any positive integer and \( g \) is a real constant. The \( k = 1 \) and \( k = 2 \) cases of (3.1) give the Hamiltonians of the displaced and single-mode squeezed harmonic oscillators, respectively. These two oscillator models can be solved by the single-mode Bogoliubov transformation [11]. For \( k \geq 3 \), (3.1) gives models with higher order harmonic generation.

Introduce the operators \( Q_+, Q_0, Q_- \) in terms of the harmonic mode,
\[
Q_+ = \frac{1}{(\sqrt{k})^k} (a^\dagger)^k, \quad Q_- = \frac{1}{(\sqrt{k})^k} a^k, \quad Q_0 = \frac{1}{k} \left( a^\dagger a + \frac{1}{k} \right).
\]  
Then in terms of \( Q_+, Q_0, Q_- \), the Hamiltonian (3.1) can be written as
\[
H = k \omega \left( Q_0 - \frac{1}{k^2} \right) + g \sqrt{k} (Q_+ + Q_-).
\]  
It can be shown [4] that the operators \( Q_+, Q_0, Q_- \) form a polynomial algebra of degree \( k - 1 \), defined by the commutation relations
\[
[Q_0, Q_\pm] = \pm Q_\pm, \quad [Q_+, Q_-] = \phi^{(k)}(Q_0) - \phi^{(k)}(Q_0 - 1),
\]  
\[4\]
Figure 1. Wave functions for $M = 0, 1, 2$ for the two-mode squeezed harmonic oscillator with $\omega / g = 2$. The domain is the disc of radius 3 in the complex plane. The height of the surface is the real part of $\psi(z)$. The color of the surface is the imaginary part of $\psi(z)$. (a) $\kappa = 1/2$. (b) $\kappa = 1$. (d) $\kappa = 1/2$. (e) $\kappa = 1$. 
where
\[ \phi^{(k)}(Q_0) = -\prod_{i=1}^{k} \left( Q_0 + i - \frac{1}{k^2} \right) + \prod_{i=1}^{k} \left( i - \frac{1}{k} - \frac{1}{k^2} \right) \]  
(3.5)

is a \( k \)th-order polynomial in \( Q_0 \). The Casimir operator of the algebra is given by
\[ C = Q_+ Q_+ + \phi^{(k)}(Q_0) = Q_+ Q_- + \phi^{(k)}(Q_0 - 1). \]  
(3.6)

For \( k = 1 \) and \( k = 2 \), (3.4) reduces to the Heisenberg and \( su(1, 1) \) algebras, respectively. Thus, the algebra (3.4) can be viewed as the polynomial deformation of \( su(1, 1) \) and Heisenberg Lie algebras.

The realization (3.2) provides a unitary irreducible representation of the polynomial algebra, which for \( k = 2 \) reduces to the well-known positive discrete series of \( su(1, 1) \). In the realization, the Casimir (3.6) takes the particular value,
\[ C = \prod_{i=1}^{k} \left( i - k - \frac{1}{k^2} \right). \]  
(3.7)

If we use \( q \) to denote the Bargmann index which labels the basis states of this representation in the Fock–Hilbert space \( \mathcal{H}_b \), then it can be shown that \( q \) takes \( k \) values,
\[ q = \frac{1}{k}, \frac{2}{k}, ..., \frac{(k-1)k}{k^2}. \]  
(3.8)

For \( k = 2 \), then \( C = \frac{q}{16} \) and \( q \) equals to \( \frac{1}{2}, \frac{3}{2} \), as expected. Thus the single-mode boson realization (3.2) corresponds to the infinite-dimensional unitary representation with particular \( q \) values (3.8) and the Fock space \( \mathcal{H}_b \) decomposes into the direct sum \( \mathcal{H}_b = \mathcal{H}_b^{\frac{1}{2}} \oplus \cdots \oplus \mathcal{H}_b^{\frac{(k-1)k}{k^2}} \) of \( k \) irreducible components \( \mathcal{H}_b^{\frac{1}{2}}, ..., \mathcal{H}_b^{\frac{(k-1)k}{k^2}} \).

The basis state \( |q, n\rangle, n = 0, 1, \ldots \), in the irreducible representation space \( \mathcal{H}_b^{q} \) is then given by [4]
\[ |q, n\rangle = \frac{1}{\sqrt{(k(n+q-1/21))!}} d^{i(k(n+q-1/21))} |0\rangle, \]  
(3.9)

The action \( Q_0, Q_\pm \) in this representation reads
\[ Q_0 |q, n\rangle = (q + n) |q, n\rangle, \]
\[ Q_+ |q, n\rangle = \prod_{i=1}^{k} \left( n + q + \frac{ik - 1}{k^2} \right) |q, n + 1\rangle, \]
\[ Q_- |q, n\rangle = \prod_{i=1}^{k} \left( n + q + \frac{(i-1)k + 1}{k^2} \right) |q, n - 1\rangle. \]  
(3.10)

Using the Fock–Bargmann correspondence (2.6), we can make the following association
\[ |q, n\rangle \longrightarrow \Psi_{q,n}(z) = \frac{z^n}{\sqrt{(k(n+q-1/21))!}}. \]  
(3.11)

It can then be shown that in the Bargmann–Hilbert subspace with basis vectors \( \Psi_{q,n}(z) \), the operators \( Q_\pm \), (3.2) are realized by single-variable differential operators
\[ Q_0 = z \frac{dz}{dz} + q, \]
\[ Q_+ = \frac{z}{(\sqrt{k})^1} \]
\[ Q_- = z^{-1} (\sqrt{k})^1 \prod_{j=1}^{k} \left( z \frac{dz}{dz} + q + \frac{(j-1)k + 1}{k^2} \right), \]  
(3.12)
Thus the time-independent Schrödinger equation for the model yields

\[ H = k\omega \left( \frac{d}{dz} + q - \frac{1}{k^2} \right) + g \left[ z + k^2 z^{-1} \prod_{j=1}^{k} \left( \frac{d}{dz} + q - \frac{(j-1)k+1}{k^2} \right) \right]. \tag{3.13} \]

Thus the asymptotic structure of solutions to the Puiseux diagram formed with the points \( h_{k-1} \) that converges in the entire complex plane, i.e. solution \( \psi(z) \) which is entire.

Substituting (3.15) into (3.14), we obtain the three-step recurrence relation,

\[
\begin{align*}
K_1(E) + A_0 K_0(E) &= 0, \\
K_{n+1}(E) + A_n K_n(E) + B_n K_{n-1}(E) &= 0, \quad n \geq 1, \tag{3.16}
\end{align*}
\]

where

\[
\begin{align*}
A_n &= \frac{\omega (n + q - \frac{1}{k^2} - \frac{E}{k^2})}{g k^{-1} \prod_{j=1}^{k-1} \left( n + 1 + q - \frac{(j-1)k+1}{k^2} \right)}, \\
B_n &= \frac{1}{k^k \prod_{j=1}^{k} \left( n + 1 + q - \frac{(j-1)k+1}{k^2} \right)}. \tag{3.17}
\end{align*}
\]

The coefficients \( A_n, B_n \) have the behavior when \( n \to \infty \),

\[
\begin{align*}
A_n &\sim an^a, \quad B_n \sim bn^b \tag{3.18}
\end{align*}
\]

with

\[
\begin{align*}
a &= \frac{\omega}{gk^{k-1}}, \quad \alpha = -k + 1, \quad b = \frac{1}{k^2}, \quad \beta = -k. \tag{3.19}
\end{align*}
\]

Thus the asymptotic structure of solutions to the \( n \geq 1 \) part of (3.16) depends on the Newton–Puiseux diagram formed with the points \( P_0(0, 0), P_1(1, -k + 1), P_2(-k) \) \[12\]. Let \( \sigma \) be the slope of \( P_0P_1 \) and \( \tau \) the slope of \( P_0P_2 \) so that \( \sigma = \alpha \) and \( \tau = \beta - \alpha \). Applying the Perron–Kreuser theorem (i.e. theorem 2.3 of \[12\]), we have

The \( k = 1 \) case: the displaced harmonic oscillator: \( \sigma = 0, \tau = -1 \), that is \( \sigma > \tau \). In this case, the truly three-term part (i.e. the \( n \geq 1 \) part) of the recurrence relation (3.16) has two linearly independent solutions \( K_{n,1}, K_{n,2} \) for which, when \( n \to \infty \)

\[
\begin{align*}
\frac{K_{n+1,1}}{K_{n,1}} &\sim -\frac{\omega}{g}, \quad \frac{K_{n+1,2}}{K_{n,2}} \sim -\frac{g}{\omega} n^{-1}. \tag{3.20}
\end{align*}
\]
This case belongs to the one treated in [7]. So \( K_n^{\text{min}} \equiv K_n^{\alpha, 2} \) is a minimal solution of the truly three-term part of (3.16) with \( k = 1 \). The corresponding infinite power series solution is generated by substituting \( K_n^{\text{min}} \) for the \( K_n \)'s in (3.15) and converges in the whole complex plane, i.e. it is entire.

The \( k = 2 \) case: the squeezed harmonic oscillator. \( \sigma = -1, \tau = -1 \), and so \( \sigma = \tau = \alpha \). The characteristic equation of the \( n \geq 1 \) part of (3.16), \( t^2 = 0 \), has two equal solutions \( t_1 = t_2 = 0 \). Then all solutions of (3.16) behave similarly as \( n \to \infty \), viz,

\[
\lim_{n \to \infty} \sup |(K_n | n!)|^{1/2} = 0
\]

for all non-trivial solutions of the second equation of (3.16) with \( k = 2 \). The zero limit means that the \( n \geq 1 \) part (i.e. the truly three-term part) of the recurrence (3.16) possesses a minimal solution \( K_n^{\text{min}} \) and the corresponding infinite power series expansion, obtained by substituting \( K_n^{\text{min}} \) for the \( K_n \)'s in (3.15), converges in the whole complex plane, i.e. it is entire.

The \( k \geq 3 \) case: anharmonic oscillators. \( \sigma = -k + 1, \tau = -1 \), and thus point \( P_1 \) lies below the line segment \( P_0 P_1 \) in the Newton–Puiseux diagram. Then

\[
\lim_{n \to \infty} \sup |(K_n | n!)|^{1/2} = \frac{1}{\sqrt{k}}
\]

for all non-trivial solutions of the second equation of (3.16). This indicates that solutions to the truly three-term part of the recurrence (3.16) with \( k \geq 3 \) are dominant and the corresponding infinite series expansion (3.15) has a finite radius of convergence proportional to \( \sqrt{k} \). It follows an entire solution to (3.14) with \( k \geq 3 \) does not exist. We thus conclude that the \( k \)th-order harmonic generation model with \( k \geq 3 \) does not have eigenfunctions (and is ill-defined) in the Bargmann–Hilbert space. This implies that the Hamiltonian (3.1) cannot be diagonalized for \( k \geq 3 \) using the basis states \( \{q, n\} \) (3.9) in the Hilbert space \( \mathcal{H}_b \) because its eigenstate \( |\psi\rangle \) is not normalizable (due to the fact that the corresponding eigenfunction \( \psi(z) \) has a finite radius of convergence in the Bargmann–Hilbert space).

The above analysis still holds for the \( k \)-photon Rabi model with Hamiltonian

\[
H_{kR} = \Delta \sigma_z + \omega a^\dagger a + g \sigma_x [(a^\dagger)^k + a^k],
\]

where \( \sigma_z, \sigma_x \) are Pauli matrices describing two atomic levels. For degenerate atomic level \( \Delta = 0 \), (3.23) has the form of (3.1):

\[
H_{kR}^{(\Delta=0)} = \omega a^\dagger a \pm g[(a^\dagger)^k + a^k],
\]

where \( \pm \) signs correspond to the two eigenvalues of \( \sigma_z \). Now the first term \( \Delta \sigma_z \) in (3.23) is a bounded spin operator which obviously does not affect the analytic property of eigenfunctions in the Bargmann–Hilbert space. In other words, eigenfunctions of (3.23) and (3.24) share the same analytic properties and have the same radius of convergence. Thus another physical consequence of our result above is that the \( k \)-photon Rabi model is also non-diagonalizable for \( k \geq 3 \). This is in sharp contrast to the \( k \)-photon Jaynes–Cummings model which can be exactly solved for all \( k \) [4, 5].

In what follows, we will focus on the \( k = 1, 2 \) cases. By the Pincherle theorem, i.e. theorem 1.1 of [12], the ratios of successive elements of the minimal solution sequences \( K_n^{\text{min}} \) for the \( k = 1, 2 \) cases are expressible in terms of infinite continued fractions. Proceeding in the direction of increasing \( n \), we find

\[
R_n = \frac{K_n^{\text{min}}}{K_{n+1}^{\text{min}}} = \frac{B_{n+1}}{A_{n+1}} - \frac{B_{n+2}}{A_{n+2}} + \frac{B_{n+3}}{A_{n+3}} - \cdots
\]

\( 8 \)

\[
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which for \( n = 0 \) gives

\[
R_0 = \frac{K_{0}^{\text{min}}}{K_{0}^{\text{min}}} = -\frac{B_1}{A_1} - \frac{B_2}{A_2} - \frac{B_3}{A_3} - \cdots. \tag{3.26}
\]

Note that the ratio \( R_0 = \frac{K_{0}^{\text{min}}}{K_{0}^{\text{min}}} \) involves \( K_{0}^{\text{min}} \), although the above continued fraction expression is obtained from the truly three-term part of (3.16), i.e. the recurrence (3.16) for \( n \geq 1 \). However, for single-ended sequences such as those appearing in the infinite power series expansion (3.15), the ratio \( R_0 = \frac{K_{0}^{\text{min}}}{K_{0}^{\text{min}}} \) of the first two terms of a minimal solution is unambiguously fixed by the \( n = 0 \) part (i.e. the first equation) of the recurrence (3.16), namely,

\[
R_0 = -A_0 = -\frac{\omega \left( q - \frac{1}{z^2} \right) - \frac{E}{\omega z}}{g^{k-1} \prod_{j=1}^{k} \left( 1 + q - \frac{(j-1)(j+1)}{k} \right)}. \tag{3.27}
\]

In general, the \( R_0 \) computed from (3.26) is not the same as that from (3.27) (i.e. (3.26) and (3.27) are not both satisfied) for arbitrary values of recurrence coefficients \( A_n \) and \( B_n \). As a result, general solutions to the recurrence (3.16) are dominant and are usually generated by simple forward recursion from a given value of \( K_0 \). Physical meaningful solutions are those that are entire in the Bargmann–Hilbert spaces. They can be obtained if \( E \) can be adjusted so that equations (3.26) and (3.27) are both satisfied. Then the resulting solution sequence \( K_0(E) \) will be purely minimal and the power series expansion (3.15) will converge in the whole complex plane.

Therefore, if we define the function \( F(E) = R_0 + A_0 \) with \( R_0 \) given by the continued fraction in (3.26), then the zeros of \( F(E) \) correspond to the points in the parameter space where the condition (3.27) is satisfied. In other words, \( F(E) = 0 \) yields the eigenvalue equation, which may be solved for \( E \) by standard nonlinear root-search techniques. Only for the denumerable infinite values of \( E \), which are the roots of \( F(E) = 0 \), do we get entire solutions of the differential equations.

As a matter of fact, the spectra for the \( k = 1, 2 \) cases can be determined explicitly. As will be seen in the next two subsections, the infinite power series in (3.15) actually truncates for these two cases, so that their solutions are given by polynomials in Bargmann–Hilbert spaces.

### 3.1. Displaced harmonic oscillator

The displaced harmonic oscillator is the \( k = 1 \) special case of the \( k \)th-order harmonic generation. By (3.14), the time-independent Schrödinger equation in the Bargmann–Hilbert space of analytic functions reads [1]

\[
(\omega z + g) \frac{d}{dz} \psi + \left( gz - E \right) \psi = 0. \tag{3.28}
\]

With the substitution

\[
\psi(z) = e^{-gz/\omega} \phi(z), \tag{3.29}
\]

the above differential equation reduces to

\[
\left( (\omega z + g) \frac{d}{dz} - \left( E + \frac{g^2}{\omega} \right) \right) \phi(z) = 0. \tag{3.30}
\]

This differential equation is exactly solvable; exact solutions are polynomial of the form

\[
\phi(z) = \prod_{i=1}^{N} (z - z_i), \quad N = 0, 1, 2, \ldots, \tag{3.31}
\]
where $\phi(z) \equiv 1$ for $N = 0$, $N$ is the degree of the polynomial and $z_i$ are the roots of the polynomial to be determined. Following a similar procedure to that in the last section, we obtain the energies of the system,

$$E = \omega \left( N - \frac{g^2}{\omega^2} \right),$$

and the set of algebraic equations determining the roots $z_i$, $\omega z_i + g = 0$, $i = 1, 2, \ldots, N$. It follows that $z_i = -\frac{g}{\omega}$ and the solution \eqref{eq:3.31} has the form

$$\phi(z) = \prod_{i=1}^{N} \left(z + \frac{g}{\omega}\right) = \left(z + \frac{g}{\omega}\right)^N.$$  \hfill (3.33)

Thus the wave function of the model is given by

$$\psi(z) = e^{-\frac{g}{\omega}z} \phi(z).$$  \hfill (3.34)

These expressions for the energies and wave function agree with those in [1] by a different approach.

### 3.2. Single-mode squeezed harmonic oscillator

The Hamiltonian of the single-mode squeezed harmonic oscillator is given by [11]

$$H = \omega a^\dagger a + g (a^\dagger)^2 + a^2,$$

which corresponds to the $k = 2$ case of the $k$th-order harmonic generation Hamiltonian \eqref{eq:3.1}. In the Bargmann–Hilbert space, the time-independent Schrödinger equation reads

$$4g^2 \frac{d^2}{dz^2} \psi(z) + \left(2\omega z + 8gq \right) \frac{d}{dz} \psi(z) + \left[ g z + 2 \omega \left(q - \frac{1}{4}\right) - E \right] \psi(z) = 0,$$

where $q = \frac{1}{2}, \frac{3}{2}$ are the Bargmann index of $su(1, 1)$. This equation is the $k = 2$ special case of the $k$th-order differential equation \eqref{eq:3.14}.

With the substitution

$$\psi(z) = e^{-\frac{g}{\omega}z} \phi(z), \quad \Omega = \sqrt{1 - \frac{4g^2}{\omega^2}},$$

where $\left|\frac{2g}{\omega}\right| < 1$, it follows,

$$4g^2 \frac{d^2}{dz^2} [2\omega\Omega z + 8gq] \frac{d}{dz} + 2qg\omega - \frac{1}{2} \omega - E \right] \phi(z) = 0.$$  \hfill (3.38)

This differential equation is exactly solvable; exact solutions are polynomials in $z$ which are automatically entire functions in the Bargmann–Hilbert space. We thus seek solution of the form

$$\phi(z) = \prod_{i=1}^{M} (z - z_i), \quad M = 0, 1, 2, \ldots,$$

where $\phi(z) \equiv 1$ for $M = 0$, $M$ is the degree of the polynomial solution and $z_i$ are the roots of the polynomial to be determined. Following a similar procedure to that in the last section, we obtain the energy eigenvalues,

$$E = -\frac{1}{2} \omega + \left[ 2M + 2(q - \frac{1}{4}) + \frac{1}{4} \right] \omega \Omega,$$

and the set of algebraic equations which determine the roots $z_i$,

$$\sum_{j \neq i}^{M} \frac{2}{z_i - z_j} + \frac{\omega}{2g} \Omega + \frac{2g}{z_i} = 0, \quad i = 1, 2, \ldots, M.$$  \hfill (3.41)
Figure 2. Wave functions for $\mathcal{M} = 0, 1, 2$ for the squeezed harmonic oscillator with $\omega/2\chi = 2$. The domain is the disc of radius 3 in the complex plane. The height of the surface is the real part of $\psi(z)$. The color of the surface is the imaginary part of $\psi(z)$. (b) $q = 1/4$. (c) $q = 3/4$. (d) $q = 1/4$. (e) $q = 3/4$. 
The corresponding wave functions are

$$\psi(z) = e^{-\frac{q}{2}(1-\Omega)z} \prod_{i=1}^M (z - z_i).$$

(3.42)

Some remarks are in order. The spectrum (3.40) coincides with the corresponding result in [11]. This is seen by noting that (3.40) is the energy in the Bargmann–Hilbert subspaces labeled by \(q = 1/4, 3/4\). When \(q = 1/4\), we have \(2M + 2(q - 1/4) = 2M\), which corresponds to even integer \(n\) in [11]; while when \(q = 3/4\), we have \(2M + 2(q - 1/4) = 2M + 1\), which corresponds to odd \(n\) in that reference.

As examples, let us list the first three eigenstates. For \(M = 0\), we have \(\psi(z) = e^{-\frac{q}{2}(1-\Omega)z}\). For \(M = 1\), we obtain from (3.41) the root \(z_1 = -\frac{4q}{\omega/\Omega_1}\) and from (3.42) the corresponding wave function \(\psi(z) = e^{-\frac{q}{2}(1-\Omega)z} \left(\frac{z + \frac{4q}{\omega/\Omega_1}}{z_1}\right)\). For \(M = 2\), the roots \(z_1, z_2\) satisfy the system of algebraic equations

$$\frac{2}{z_1 - z_2} + \frac{\omega\Omega}{2g} + \frac{2q}{z_1} = 0, \quad \frac{2}{z_2 - z_1} + \frac{\omega\Omega}{2g} + \frac{2q}{z_2} = 0.$$  
(3.43)

Solving the two equations simultaneously gives

$$z_1 = \frac{-(1 + 2q) + \sqrt{1 + 2q} \sqrt{2g}}{\omega\Omega}, \quad z_2 = \frac{-(1 + 2q) - \sqrt{1 + 2q} \sqrt{2g}}{\omega\Omega}.$$  
(3.44)

The corresponding wave function is given by

$$\psi(z) = e^{-\frac{q}{2}(1-\Omega)z} \left[ \frac{2}{\omega\Omega} \right]^2 \left(\frac{z^2 + \frac{4(1 + 2q)g}{\omega\Omega}}{z_1} + \frac{8q(1 + 2q)g^2}{\omega^2\Omega^2}\right).$$  
(3.45)

In figure 2, we plot some of these eigenstates over a disc of radius 3 in the complex plane.

4. Conclusions

We have reported our results on solutions of the two-mode squeezed oscillator and \(k\)th-order harmonic generation models. These have been achieved through the application of algebraizations and Bargmann–Hilbert spaces. We have seen that the algebraizations via either \(\text{su}(1, 1)\) Lie algebra or its polynomial deformations decompose the Fock–Hilbert spaces of states into direct sums of independent subspaces, thus partially diagonalizing the Hamiltonians of the models by bringing them into block-diagonal forms. The block-diagonal sectors of the Hamiltonians can be realized as differential operators in Bargmann–Hilbert spaces. We have investigated the eigenvalues and eigenfunctions of the Hamiltonians in these sectors by applying the theory of Bargmann–Hilbert spaces. For displaced single-mode squeezed and two-mode squeezed harmonic oscillators, we have obtained exact closed-form expressions for their energies and wave functions. For the \(k\)th-order harmonic generation with \(k \geq 3\), we have shown that it does not have entire eigenfunctions and thus is ill-defined in the Bargmann–Hilbert space. We have argued that the same conclusion also holds for the \(k\)-photon Rabi model with \(k \geq 3\). It is not difficult to see that the \(k\)-photon Rabi model Hamiltonian (3.23) possess two (one discrete and one continuous) degrees of freedom and each of its states in the block-diagonal sector can be labeled by two quantum numbers (energy and parity). Despite this fact, the \(k \geq 3\) case cannot be diagonalized due to the lack of normalizable eigenstates. Thus this case seems to provide a counter-example to the criteria of quantum integrability proposed recently by Braak in [6]. A thorough investigation of this point is underway and the results will be reported elsewhere.
Acknowledgments

This work was supported by the Australian Research Council through Discovery Projects grant DP110103434.

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