EXPONENTIAL STABILITY OF SOLUTIONS FOR RETARDED STOCHASTIC DIFFERENTIAL EQUATIONS WITHOUT DISSIPATIVITY

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Abstract. This work focuses on a class of retarded stochastic differential equations that need not satisfy dissipative conditions. The principle technique of our investigation is to use variation-of-constants formula to overcome the difficulties due to the lack of the information at the current time. By using variation-of-constants formula and estimating the diffusion coefficients we give sufficient conditions for $p$-th moment exponential stability, almost sure exponential stability and convergence of solutions from different initial value. Finally, we provide two examples to illustrate the effectiveness of the theoretical results.

1. Introduction. Stochastic dynamical systems have been used to represent the real world behavior and they can reveal the uncertainty of the environment in which the model is operating. Because of their wide application in various sciences such as physics, mechanical engineering, control theory and economics, the theory of stochastic dynamical systems has attracted extensive attention. Moreover, it is because stochastic dynamical systems often run for an extended time that the study of stability properties is considerably important and has been one of the most active areas in stochastic analysis. Especially, there has been much interesting in studying stochastic dynamical systems whose evolution in time is governed by random forces.
as well as intrinsic dependence on the state over a finite interval of its history. Such systems can be called as retarded stochastic differential equations (SDEs); for example, see the monograph [19] for more details. Because of many systems involving retarded arguments, a large number of interesting results on the existence, uniqueness, stability, other quantitative and qualitative properties of solutions have been reported (see, e.g., [4, 9, 13, 21, 27]).

In response to the great needs, there is an extensive literature on stability for retarded SDEs. So far there are numerous approaches to investigate various stability (e.g., moment stability, sample path stability, stability in distribution, stability in probability, etc) for retarded SDEs; see, for instance, [10, 12, 13, 15] by exploiting Razumikhin-type theorems, [5, 6, 22, 24] by making use of the weak convergence method, [16] by employing the semimartingale convergence theorem, [17] by applying the LaSalle-type theorems, and [18] by taking advantage of Borel-Cantelli lemma, to name a few. Nevertheless, most of the existing literature focuses on stability for retarded SDEs under certain dissipativity, which is normally assured by imposing information of the current time with certain decay conditions. In contrast to the rapid progress in stability for SDEs with dissipativity, the study for retarded SDEs without dissipativity is still scarce. Compared with retarded SDEs without dissipativity, as far as SDEs without dissipativity, one of the outstanding issues is the lack of the information at the current time, which makes the goal of investigating stability a very difficult task. Whereas, this work aims to take the challenges as well as intrinsic dependence on the state over a finite interval of its history. Such systems can be called as retarded stochastic differential equations (SDEs); for example, see the monograph [19] for more details. Because of many systems involving retarded arguments, a large number of interesting results on the existence, uniqueness, stability, other quantitative and qualitative properties of solutions have been reported (see, e.g., [4, 9, 13, 21, 27]).

The rest of this paper is structured as follows. In section 2 we provide some preliminary results and recall definitions of p-th moment exponential stability and almost sure exponential stability, which lay a good foundation for stability analysis; Section 3 focuses on stability for retarded SDEs driven by Brownian motions; Section 4 is devoted to the stability for retarded SDEs of neutral type. In the last section, to show the effectiveness of our theory, two illustrative examples are provided.

2. Preliminary. For any integer $n > 0$, let $\mathbb{R}^n$ be an $n$-dimensional Euclidean space endowed with the inner product $\langle u, v \rangle := \sum_{i=1}^{n} u_i v_i$ and the Euclidean norm $|u| := \langle u, u \rangle^{\frac{1}{2}}$ for $u, v \in \mathbb{R}^n$. Denote $\mathbb{R}^n \otimes \mathbb{R}^m$ by the set of all $n \times m$ matrices $A$ endowed with Hilbert-Schmidt norm $\|A\| := \sqrt{\text{trace}(A^T A)}$, in which $A^T$ is the transpose of $A$. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space with a filtration $\{\mathcal{F}_t\}_{t \geq 0}$ satisfying the usual conditions (i.e., $\mathcal{F}_t = \mathcal{F}_{t+} := \bigcap_{s \geq t} \mathcal{F}_s$ and $\mathcal{F}_0$ contains all $\mathbb{P}$-null sets). Let $\{W(t)\}_{t \geq 0}$ be an $m$-dimensional Brownian motion defined on $(\Omega, \mathcal{F}, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0})$. Fix $\tau \in (0, \infty)$, which will be referred to as the delay or time lag. Let $\mathcal{C} = C([-\tau, 0]; \mathbb{R}^n)$ be the family of all continuous functions $\xi: [-\tau, 0] \to \mathbb{R}^n$ equipped with the uniform norm $\|\xi\|_{\infty} := \sup_{-\tau \leq \theta < 0} |\xi(\theta)|$ for $\xi \in \mathcal{C}$. For $X(\cdot) \in C([-\tau, \infty]; \mathbb{R}^n)$, define the segment process $X_t \in \mathcal{C}$ by $X_t(\theta) := X(t + \theta)$, $\theta \in [-\tau, 0]$, $t \geq 0$. Let $\mu(\cdot)$ and $\rho(\cdot)$ be $\mathbb{R}^n \otimes \mathbb{R}^n$-valued finite signed measure on $[-\tau, 0]$, $\nu(\cdot)$ a measure on $[-\tau, 0]$, $\mathcal{C}$ the set of all complex numbers and $\Re(z)$ the real part of $z \in \mathbb{C}$. Let $\|\rho\| = \sup_{1 \leq k \leq n} \sqrt{\sum_{1 \leq j \leq n} \|\rho_{kj}\|^2_{\text{var}}}$, where $\|\rho_{kj}\|_{\text{var}}$ is the total variation of $\rho_{kj}$.

To begin with, consider the following deterministic linear retarded equation:

$$dY(t) = \left( \int_{[-\tau, 0]} \mu(d\theta) Y(t + \theta) \right) dt, \quad t > 0, \quad Y_0 = \xi \in \mathcal{C}. \quad (2.1)$$
By the variation-of-constants formula (see, e.g., [8, Theorem 1.2, p.170]), Eq. (2.1) has a unique explicit solution

\[ Y(t; \xi) = r(t)\xi(0) + \int_{[-\tau, 0]} \mu(d\theta) \int_{\theta}^{0} r(t + \theta - s)\xi(s)ds, \]

where \( r(t) \) is the fundamental solution of (2.1) with the initial value \( r(0) = I_{n \times n} \) and \( r(\theta) = 0_{n \times n} \) for \( \theta \in [-\tau, 0) \). Set

\[ v_1 := \sup \left\{ \text{Re}(\lambda) : \lambda \in \mathbb{C}, \det \left( \lambda I_{n \times n} - \int_{[-\tau, 0]} e^{\lambda \theta} \mu(d\theta) \right) = 0 \right\}, \]

where \( \det(A) \) denotes the determinant of an \( A \in \mathbb{R}^{n} \otimes \mathbb{R}^{n} \). Then according to [8, Theorem 3.2, p.271], for any \( \alpha_1 > v_1 \), there exists a constant \( c_{\alpha_1} > 0 \) such that

\[ \|r(t)\| \leq c_{\alpha_1} e^{\alpha_1 t}, \quad t \geq -\tau, \]

where \( \| \cdot \| \) denotes the operator norm of the matrix. If \( v_1 < 0 \), by [11, Lemma 1], for any \( \alpha \in (0, -v_1) \), there exists a constant \( C_{\alpha} > 0 \) such that

\[ \|r(t)\| \leq C_{\alpha} e^{-\alpha t}, \quad t \geq -\tau. \] (2.2)

Next, the deterministic linear retarded equation of neutral type

\[ d \left( Y(t) - \int_{[-\tau, 0]} \rho(d\theta) Y(t + \theta) \right) = \left( \int_{[-\tau, 0]} \mu(d\theta) Y(t + \theta) \right) dt, \] (2.3)

with the initial value \( Y_0 = \xi \in \mathcal{C} \), by [8, Theorem 1.1, p.256], has a unique solution \( \{ Y(t; \xi) \}_{t \geq -\tau} \). Then by virtue of [11, Theorem 2.2], for any \( \xi \in W^{1,2}([-\tau, 0]; \mathbb{R}^{n}) \) (the Sobolev space consisting of functions \( f : [-\tau, 0] \rightarrow \mathbb{R}^{n} \) such that \( f(\cdot) \) and its distributional derivative \( f'(\cdot) \) belong to \( L^{2}([-\tau, 0]; \mathbb{R}^{n}) \)), \( Y(t; \xi) \) can be expressed explicitly by

\[ Y(t; \xi) = G(t)\xi(0) - \int_{[-\tau, 0]} \rho(d\theta) G(t + \theta)\xi(0) + \int_{[-\tau, 0]} \mu(d\theta) \int_{\theta}^{0} G(t + \theta - s)\xi(s)ds \]
\[ + \int_{[-\tau, 0]} \rho(d\theta) \int_{\theta}^{0} G(t + \theta - s)\xi'(s)ds, \] (2.4)

where \( G(t) \) is the fundamental solution of (2.3) with the initial value \( G(0) = I_{n \times n} \) and \( G(\theta) = 0_{n \times n} \) for \( \theta \in [-\tau, 0) \). Set

\[ v_2 := \sup \left\{ \text{Re}(\lambda) : \lambda \in \mathbb{C}, \det \left( \lambda I_{n \times n} - \int_{[-\tau, 0]} e^{\lambda \theta} \rho(d\theta) - \int_{[-\tau, 0]} e^{\lambda \theta} \mu(d\theta) \right) = 0 \right\}. \]

According to [8, Theorem 3.2, p.271], for any \( \alpha_2 > v_2 \) there exists a constant \( c_{\alpha_2} > 0 \) such that

\[ \|G(t)\| \leq c_{\alpha_2} e^{\alpha_2 t}, \quad t \geq -\tau. \]

If \( v_2 < 0 \), according to [2, Lemma 1], for any \( \beta \in (0, -v_2) \), there exists a constant \( C_{\beta} > 0 \) such that

\[ \|G(t)\| \leq C_{\beta} e^{-\beta t}, \quad t \geq -\tau. \] (2.5)

Before we end this section, we recall some notions on stability and lemmas for later purpose.
Definition 2.1. The solution process \( \{X(t)\} \) is said to be \( p \)-th moment exponentially stable if there is a pair of positive constants \( \kappa \) and \( K \), for any \( p > 0 \) and \( \xi \in \mathcal{C} \) such that
\[
\mathbb{E}|X(t; \xi)|^p \leq K\|\xi\|_\infty^p e^{-\kappa pt}.
\]
For \( p = 2 \), it is named as exponential stability in mean square.

Definition 2.2. The solution process \( \{X(t)\} \) is said to be almost surely exponentially stable if for any \( \xi \in \mathcal{C} \) such that
\[
\limsup_{t \to \infty} \frac{1}{t} \log |X(t; \xi)| < 0 \quad \text{a.s.}
\]

Lemma 2.3. (\cite{19} Theorem 2.3) Let \( \{f(t)\}_{t \geq 0} \) be an \( \mathbb{R}^n \otimes \mathbb{R}^m \)-valued predictable process. Then, for all \( p \geq 1 \) and \( t > 0 \)
\[
\left( \mathbb{E} \left[ \int_0^t |r(t-s) f(s) dW(s)|^{2p} \right] \right)^{\frac{1}{2p}} \leq p(2p - 1) \int_0^t \mathbb{E} |r(t-s) f(s)|^{2p} ds,
\]
provided that the integral on the right hand side is finite for each \( t > 0 \).

3. Exponential stability for retarded SDEs. In this section, we first consider a semi-linear retarded SDE of the form
\[
dX(t) = \left( \int_{[-\tau,0]} \mu(d\theta) X(t + \theta) \right) dt + \sigma(X_t) dW(t), \quad t > 0, \quad X_0 = \xi \in \mathcal{C}, \quad (3.1)
\]
where \( \sigma: \mathcal{C} \mapsto \mathbb{R}^n \otimes \mathbb{R}^m \) such that \( \sigma(0) = 0_{n \times m} \) is Borel measurable. Throughout this section, we assume that, for any \( \phi, \varphi \in \mathcal{C} \), there exists an \( L > 0 \) such that
\[
\|\sigma(\phi) - \sigma(\varphi)\|^2 \leq L \left( |\phi(0) - \varphi(0)|^2 + \int_{[-\tau,0]} |\phi(\theta) - \varphi(\theta)|^2 \nu(d\theta) \right), \quad (3.2)
\]
Under (3.2), by \cite{19} Theorem 2.1, p.36, Eq. (3.1) admits a unique strong solution \( \{X(t; \xi)\}_{t \geq -\tau} \) with the initial value \( X_0 = \xi \in \mathcal{C} \). It can be represented explicitly by
\[
X(t; \xi) = r(t)\xi(0) + \int_{[-\tau,0]} \mu(d\theta) \int_0^t r(t + \theta - s)\xi(s) ds + \int_0^t r(t - s)\sigma(X_s(\xi)) dW(s). \quad (3.3)
\]
Indeed, for fixed \( t \geq 0 \), by means of the chain rule, we have
\[
d(r(t-s)X(s)) = d(r(t-s))X(s) + r(t-s)dX(s), \quad s \in [0,t].
\]
Integrating from 0 to \( t \), together with (2.1) and (3.1), leads to
\[
X(t) = r(t)\xi(0)
= -\int_0^t \left( \int_{[-\tau,0]} \mu(d\theta) r(t-s+\theta) \right) X(s) ds
+ \int_0^t r(t-s) \left( \int_{[-\tau,0]} \mu(d\theta) X(s+\theta) \right) ds + \int_0^t r(t-s)\sigma(X_s)dW(s)
= -\int_{[-\tau,0]} \mu(d\theta) \int_0^t r(t-s+\theta)X(s) ds + \int_{[-\tau,0]} \mu(d\theta) \int_0^{t+\theta} r(t-s+\theta)X(s) ds
+ \int_0^t r(t-s)\sigma(X_s)dW(s)
\]
In the sequel, we fix $\gamma$.

It follows from (3.3) and (3.6) that

$$\int_{[-\tau,0]} r(t-s+\theta)X(s)ds - \int_{[-\tau,0]} r(t-s+\theta)X(s)ds + \int_0^t r(t-s)\sigma(X_s)dW(s)$$

Next, we shall investigate the $p$-th moment exponential stability and almost sure exponential stability for the solution $X(t)$ to Eq. (3.1).

**Theorem 3.1.** Let $p \geq 2$ and $v_1 < 0$. Assume further that (3.2) holds with $L > 0$ such that $p(p-1)LC_2^2(1+\|\nu\|_{\text{var}})e^{2\alpha\tau} < 4\alpha$, for $\alpha \in (0,-v_1)$, where $C_\alpha > 0$ is introduced in (2.2). Then, for any initial value $\xi \in \mathscr{C}$, the solution of (3.1) is $p$-th moment exponentially stable, i.e. there exist constants $K > 0$ and $\tilde{\alpha} > 0$ such that

$$E|X(t;\xi)|^p \leq K\|\xi\|\infty e^{-\tilde{\alpha}t}, \; t \geq 0. \quad (3.4)$$

**Proof.** In what follows, for notation simplicity, we write $X(t)$ in lieu of $X(t;\xi)$. For any $p \geq 2$, due to $p(p-1)LC_2^2(1+\|\nu\|_{\text{var}})e^{2\alpha\tau} < 4\alpha$, we choose $\gamma_2 > 1$ such that

$$p(p-1)\gamma_2^2 LC_2^2(1+\|\nu\|_{\text{var}})e^{2\alpha\tau} < 4\alpha. \quad (3.5)$$

In the sequel, we fix $\gamma_2 > 1$ such that (3.5). Then there exists a $\gamma_1 > 1$ such that

$$(a+b+c)^2 \leq \gamma_1(a^q + b^q) + \gamma_2 c^q, \; q > 1, \; a, b, c \in \mathbb{R}^+. \quad (3.6)$$

By the elementary inequality:

$$(a+b)^\theta \leq a^\theta + b^\theta, \; 0 < \theta < 1, \; a, b \in \mathbb{R}^+, \quad (3.7)$$

it follows from (3.3) and (3.6) that

$$\left( E|X(t)|^p \right)^{\frac{2}{p}} \leq \gamma_1^2 \left( E|\mu(t)\xi(0)|^2 + E \int_{[-\tau,0]} \mu(d\theta) \int_0^t r(t+\theta-s)\xi(s)ds \right)^{\frac{2}{p}}$$

$$+ \gamma_2^2 \left( E \int_0^t r(t-s)\sigma(X_s)dW(s) \right)^{\frac{2}{p}} \quad (3.8)$$

$$= \sum_{i=1}^3 J_i.$$

In what follows, we estimate the terms $J_1$, $J_2$ and $J_3$, one-by-one. For the first term, by virtue of (2.2), one has

$$J_1 \leq \gamma_1^2 C_2^2 e^{-2\alpha t}\|\xi\|^2_{\infty}.$$ 

For the term $J_2$, by the Hölder inequality and (2.2), it is readily seen that

$$J_2 \leq \gamma_1^2 \|\mu\|_{\tau} \int_{[-\tau,0]} \mu(d\theta) \left( \int_0^t r(t+\theta-s)\xi(s)ds \right)^{\frac{2}{p}}$$

$$\leq \gamma_1^2 \|\mu\|_{\tau} \int_{[-\tau,0]} \mu(d\theta) \left( \int_0^t r(t+\theta-s)\|\xi(s)\|^2ds \right)^{\frac{2}{p}}$$

$$\leq \gamma_1^2 \|\mu\|^2 C_2^2 \tau \int_0^\tau e^{2\alpha s}ds \|\xi\|^2_{\infty} e^{-2\alpha t}.$$
Now we estimate the last term of (3.8). By applying Lemma 2.3 and the Hölder inequality, it follows from (2.2) and (3.2) that

\[
\left( E \left| \int_0^t r(t-s)\sigma(X_s) dW(s) \right|^p \right)^{\frac{2}{p}} \leq \frac{P}{2}(p-1) \int_0^t \left( E\|r(t-s)\sigma(X_s)\|^p \right)^{\frac{2}{p}} ds
\]

\[
\leq \frac{P}{2}(p-1)C_2^2 \int_0^t e^{-2\alpha(t-s)} \left( E\|\sigma(X_s)\|^p \right)^{\frac{2}{p}} ds
\]

\[
\leq \frac{L}{2} \frac{P}{2}(p-1)C_2^2 \int_0^t e^{-2\alpha(t-s)} \left[ E \left( |X(s)|^2 + \int_{-\tau,0} |X(s+\theta)|^2 \nu(d\theta) \right) \right]^{\frac{2}{p}} ds
\]

\[
\tau,
\]

\[
\leq \frac{L}{2} \frac{P}{2}(p-1)C_2^2 \int_0^t e^{-2\alpha(t-s)} \left( \left( E|X(s)|^p \right)^{\frac{2}{p}} + \int_{-\tau,0} E|X(s+\theta)|^p \nu(d\theta) \right) ds
\]

\[
(3.9)
\]

where in the fourth step we have used Minkovskii’s inequality.

Substituting the previous estimates into (3.8), it gives that

\[
\left( E|X(t)|^p \right)^{\frac{2}{p}} \leq \gamma_1 \int_0^t \left( C_2^2 e^{-2\alpha t} \|\xi\|_{\infty}^2 + \|\|\| \right) ds\int_0^t e^{2\alpha s} ds\|\|_{\infty}^2 e^{-2\alpha t}
\]

\[
+ \gamma_2 \frac{L}{2} \frac{P}{2}(p-1)C_2^2 e^{-2\alpha t} \nu\|\var \|^{2}(2\alpha)^{-1} \|\|_{\infty}^2
\]

\[
+ \gamma_2 \frac{L}{2} \frac{P}{2}(p-1)C_2^2 (1 + e^{2\alpha t} \nu\|\var \|) e^{-2\alpha t}
\]

\[
\times \int_0^t e^{2\alpha s} \left( E|X(s)|^p \right)^{\frac{2}{p}} ds.
\]

\[
(3.10)
\]

Multiplying by \( e^{2\alpha t} \) on both side of (3.10) gives that

\[
e^{2\alpha t} \left( E|X(t)|^p \right)^{\frac{2}{p}} \leq C_1 \|\xi\|_{\infty}^2 + \gamma_2 \frac{L}{2} \frac{P}{2}(p-1)C_2^2 (1 + e^{2\alpha t} \nu\|\var \|) \int_0^t e^{2\alpha s} \left( E|X(s)|^p \right)^{\frac{2}{p}} ds,
\]
where
\[ C_1 := \gamma_1^2 C^2 + \gamma_1^2 \|\mu\|^2 C^2 \int_0^\tau e^{2s} ds + \gamma_2^2 L_{\bar{\nu}} \frac{p}{2} (p-1) C^2 \|\nu\|_{\text{var}} e^{2\alpha \tau} (2\alpha)^{-1} > 0. \]

So, the Gronwall inequality (see, e.g., [18, Theorem 8.1, p.45]) leads to
\[ (E|X(t)|^\bar{p})^{\frac{1}{\bar{p}}} \leq C_1|\xi|^2 e^{-2\bar{\alpha} t}, \quad t \geq 0, \]
where \(2\bar{\alpha} := 2\alpha - \gamma_2^2 L_{\bar{\nu}} (p-1) C^2 (1 + e^{2\alpha \tau} \|\nu\|_{\text{var}}).\) This further gives
\[ E|X(t)|^p \leq K|\xi|^2 e^{-p\bar{\alpha} t}, \quad t \geq 0. \]

Note that (3.5) implies \(\bar{\alpha} > 0,\) therefore the result (3.4) is established in the case of \(p \geq 2.\)

Noting
\[ (E|X(t)|^p)^{\frac{1}{p}} \leq (E|X(t)|^\bar{p})^{\frac{1}{\bar{p}}}, \text{ for } 0 < \bar{p} < 2, \quad p \geq 2, \]
together with (3.4), we see that the \(p\)-th moment exponential stability implies the \(\bar{p}\)-th moment exponential stability. Taking \(p = 2\) yields to the estimate for \(E|X(t)|^\bar{p}(0 < \bar{p} < 2).\) Therefore, we have the following corollary.

**Corollary 3.2.** Let \(0 < p < 2\) and \(v_1 < 0.\) Assume further that (3.2) holds with \(L > 0\) such that \(LC_\alpha^2 (1 + \|\nu\|_{\text{var}} e^{2\alpha \tau}) < 2\alpha,\) for \(\alpha \in (0, -v_1),\) where \(C_\alpha > 0\) is introduced in (2.2). Then, for any initial value \(\xi \in \mathcal{C},\) the solution of (3.1) is \(p\)-th moment exponentially stable, i.e. there exist constants \(K > 0\) and \(\bar{\alpha} > 0\) such that
\[ E|X(t; \xi)|^p \leq K|\xi|^2 e^{-p\bar{\alpha} t}, \quad t \geq 0. \]

Carrying out similar arguments as Theorem 3.1 and Corollary 3.2 respectively, we can show that the solution \(X(t)\) of Eq. (3.1) has the properties as follows:

**Theorem 3.3.** Let the conditions of theorem 3.1 hold. Then for any different initial values \(\xi, \eta \in \mathcal{C},\) there exists a pair of positive constants \(K\) and \(\bar{\alpha}\) such that
\[ E|X(t; \xi) - X(t; \eta)|^p \leq K|\xi - \eta|^2 e^{-p\bar{\alpha} t}, \quad t \geq 0, \quad p \geq 2, \]
where \(\bar{\alpha}\) is given in Theorem 3.1.

**Corollary 3.4.** Let the conditions corollary 3.2 hold. Then for any different initial values \(\xi, \eta \in \mathcal{C},\) there exists a pair of positive constants \(K\) and \(\bar{\alpha}\) such that
\[ E|X(t; \xi) - X(t; \eta)|^p \leq K|\xi - \eta|^2 e^{-p\bar{\alpha} t}, \quad t \geq 0, \quad 0 < p < 2, \]
where \(\bar{\alpha}\) is given in Theorem 3.1.

In a stable system, by virtue of the results of Theorem 3.3 and Corollary 3.4, trajectories of solutions corresponding to different initial values become closer in the \(p\)-th moment after a long time.

In general, moment exponential stability and almost sure exponential stability do not imply each other. However, if some conditions are required, moment exponential stability implies almost sure exponential stability. The following result demonstrates this point.

**Theorem 3.5.** Let the conditions of theorem 3.1 hold. Then for any initial values \(\xi \in \mathcal{C},\) there exists a constant \(\gamma > 0\) such that
\[ \limsup_{t \to \infty} \frac{1}{t} \ln |X(t)| \leq -\gamma. \]
Proof. To show this assertion, it is sufficient to show that there exists a constant $\delta > 0$ such that
\[
\mathbb{E}(\sup_{n-1 \leq t \leq n} |X(t)|^2) \leq Ke^{-2\delta n}, \quad \forall \ n \geq 1. \tag{3.11}
\]
Indeed, if (3.11) is true, using the Chebyshev inequality, we have for any $\gamma < \delta$
\[
P\left(\sup_{n-1 \leq t \leq n} |X(t)|^2 > e^{-2\gamma n}\right) \leq e^{2\gamma n}\mathbb{E}(\sup_{n-1 \leq t \leq n} |X(t)|^2) \leq Ke^{-2(\delta - \gamma)n}.
\]
Since $\sum_{n=1}^{\infty} e^{-2(\delta - \gamma)n} < \infty$, in view of Borel-Cantelli lemma, there exists an $\Omega_0 \in \Omega$ with $P(\Omega_0) = 1$ such that for any $\omega \in \Omega_0$ there exists an integer $n_0(\omega)$, for $n \geq n_0(\omega)$ and $n-1 \leq t \leq n$,
\[
|X(t)|^2 \leq e^{-2\gamma n} \leq e^{-2\gamma t},
\]
which implies the desired conclusion. The remainder of the proof is to check (3.11).

For $n-1 \leq t \leq n$, $n \geq 1 + \tau$, $X(t)$ can be represented as
\[
X(t) = X(n-1) + \int_{n-1}^{t} \left( \int_{[-\tau,0]} \mu(d\theta)X(s + \theta) \right) ds + \int_{n-1}^{t} \sigma(X_s)dW(s).
\]
By the elementary inequality: $(a + b + c)^2 \leq 3a^2 + 3b^2 + 3c^2$, it follows from (3.4) that
\[
\mathbb{E}(\sup_{n-1 \leq t \leq n} |X(t)|^2) \leq 3\mathbb{E}|X(n-1)|^2 
+ 3\mathbb{E}\left(\int_{n-1}^{t} \left( \int_{[-\tau,0]} \mu(d\theta)X(s + \theta) \right) ds \right)^2
+ 3\mathbb{E}\left(\sup_{n-1 \leq t \leq n} \left| \int_{n-1}^{t} \sigma(X_s)dW(s) \right|^2 \right)
\leq 3K\|\xi\|_\infty^2 e^{-2\delta(n-1)} + 3N_2 + 3N_3. \tag{3.12}
\]
Observe from Hölder’s inequality and (3.4) that
\[
N_2 \leq \mathbb{E} \int_{n-1}^{t} \left( \int_{[-\tau,0]} \mu(d\theta)X(s + \theta) \right)^2 ds
\leq ||\mu||\mathbb{E} \int_{n-1}^{t} \int_{[-\tau,0]} |X(s + \theta)|^2 \mu(d\theta) ds
\leq ||\mu||^2 \int_{n-1-\tau}^{n} \mathbb{E}|X(s)|^2 ds
\leq ||\mu||^2 K\|\xi\|_\infty^2 (2\delta)^{-1} e^{2\delta \tau} e^{-2\delta(n-1)}. \tag{3.13}
\]
In terms of the Burkholder-Davis-Gundy inequality and (3.2), one obtains that
\[
N_3 \leq 4\mathbb{E} \int_{n-1}^{t} \|\sigma(X_s)\|^2 ds
\leq 4\mathbb{E} \int_{n-1}^{t} \left( |X(s)|^2 + \int_{[-\tau,0]} |X(s + \theta)|^2 \nu(d\theta) \right) ds
\leq 4\int_{n-1}^{t} \mathbb{E}|X(s)|^2 ds + 4\int_{n-1-\tau}^{n} \mathbb{E}|X(s)|^2 ds
\leq 4\int_{n-1-\tau}^{n} \mathbb{E}|X(s)|^2 ds
\leq 4(1 + \|\nu\|_\text{var}) \int_{n-1-\tau}^{n} \mathbb{E}|X(s)|^2 ds
\leq 4(1 + \|\nu\|_\text{var}) K\|\xi\|_\infty^2 (2\delta)^{-1} e^{2\delta \tau} e^{-2\delta(n-1)}. \tag{3.14}
\]
Thus, combining (3.12) with (3.13) and (3.14) leads to

\[
E(\sup_{n-1 \leq t \leq n} |X(t)|^2) \leq 3K\|\xi\|_\infty^2 e^{-2\tilde{\alpha}(n-1)} + 3\|\mu\|_\infty^2 K\|\xi\|_\infty^2 (2\tilde{\alpha})^{-1} e^{2\tilde{\alpha}r} e^{-2\tilde{\alpha}(n-1)}
\]

\[
+ 12L(1 + \|\nu\| \text{var}) K\|\xi\|_\infty^2 (2\tilde{\alpha})^{-1} e^{2\tilde{\alpha}r} e^{-2\tilde{\alpha}(n-1)}
\]

\[
\leq Ke^{-2\tilde{\alpha}n}.
\]

Let \( \delta = \tilde{\alpha} \), then the required assertion follows.

\[ \square \]

**Remark 1.** It is worth pointing out that the right-hand sides of (3.1) do not involve information on current time. The techniques used in [23] and [5] do not work. To overcome this difficulty, by use of the variation-of-constants formula we can deduce both \( p \)-th moment and almost sure exponential stability of solution. From Theorem 3.5, under some conditions we obtain that \( p \)-th moment exponential stability implies almost sure exponential stability.

**Remark 2.** Due to the fact that

\[
\int_0^t r(t-s)\sigma(X_s)dW(s)
\]

is not a martingale, when \( p > 2 \) the estimate of this term makes the analysis more difficult. It cannot be obtained directly from [7] Lemma 7.7, p.194. And some ideas of the aforementioned reference cannot be used. Lemma 2.3 overcome this difficulty. The established method in the estimate of (3.8) can be extended to study the \( p \)-th moment exponential stability for a wide range of SDEs.

**Remark 3.** D. Nguyen (2014) in [20] gave a method for the estimate for diffusion process. They used the following derivation: Let \( Y \) denote a random variable following an \( N(0,a^2) \). Then for any \( p > 0 \)

\[
E|Y|^p = \frac{2^p \Gamma(p+1)}{\Gamma(\frac{p}{2})} a^p.
\]

However, there is a minor problem in the proof. Let \( Y \sim N(0,a^2) \). Then, by means of the characteristic function, it follows that

\[
E|Y|^p \leq \begin{cases} a^{2k}(2k-1)!! & p = 2k, \ k = 1,2,... \\ 0 & p = 2k - 1, \ k = 1,2,.... \end{cases}
\]

And by a close scrutiny of the argument, this derivation in [20] should be revised to

\[
E|Y|^p \leq \frac{2^p \Gamma(\frac{p+1}{2})}{\Gamma(\frac{1}{2})} a^p.
\]

If the diffusion coefficient \( \sigma(\cdot) \) is a deterministic function of time \( t \), the approach of D. Nguyen [20] can be successfully used in investigating the \( p \)-th moment exponential stability for the solution. However, this approach seems hard to work for the diffusion coefficient \( \sigma(\cdot) \) involving the retarded elements. This difficulty from the diffusion coefficient can be resolved by using the above method applied in Theorem 3.1. Therefore, the established method in proof of Theorem 3.1 and Corollary 3.2 can be extend to study the stability of a class of SDEs with the diffusion coefficient involving the retarded element. So it can be used to improve those results given in [20].
4. Exponential stability for retarded SDEs of neutral type. Next we proceed to extend Eq. (3.1) to retarded SDE of neutral type in the form

\[ d\left( X(t) - \int_{[-\tau,0]} \rho(\theta) X(t + \theta) \right) = \left( \int_{[-\tau,0]} \mu(\theta) X(t + \theta) \right) dt + \sigma(X_t) dW(t). \]  

(4.1)

**Theorem 4.1.** Let \( p \geq 2 \) and \( v_2 < 0 \). Assume further that \([3.2]\) holds with \( L > 0 \) such that \( p/(p-1)LC_{\beta}^2(1 + e^{2\beta t\|\nu\|_{\text{var}}}) < 4\beta \), for \( \beta \in (0, v_2) \), where \( C_{\beta} > 0 \) is introduced in \((2.5)\). Then, for any initial value \( \xi \in W^{1,2}([-\tau,0];\mathbb{R}^n) \), the solution of \((4.1)\) is \( p \)-th moment exponentially stable, i.e. there exist constants \( K > 0 \) and \( \beta > 0 \) such that

\[ E|X(t;\xi)|^p \leq K\|\xi\|_\infty^p e^{-p\beta t}, \quad t \geq 0. \]  

(4.2)

**Proof.** By \((2.4)\) and \([2, \text{Theorem 1}]\), for any \( \beta > 0 \), we have

\[ E|X(t;\xi)|^p \leq \beta \left( E\|\tilde{X}_t\|_{\text{var}}^2 \right)^{\frac{p}{2}} \]  

(4.3)

By carrying out a similar argument of \((3.9)\), one finds that

\[ E\left( E\|\tilde{X}_t\|_{\text{var}}^2 \right)^{\frac{p}{2}} \leq L_p^2(p-1)C_{\beta}^2(1 + e^{2\beta t\|\nu\|_{\text{var}}}) e^{-2\beta t} \int_0^t e^{2\beta t}(E|X(s)|^p)^{\frac{p}{2}} ds \]  

(4.4)

And following a similar argument for estimate \( J_2 \) in the proof of Theorem 3.1, we obtain that

\[ \beta \left( E\|\tilde{X}_t\|_{\text{var}}^2 \right)^{\frac{p}{2}} \leq L_p^2(p-1)C_{\beta}^2 e^{-2\beta t\|\nu\|_{\text{var}} e^{2\beta t(2\beta)^{-1}\|\xi\|_\infty^2}}. \]  

(4.5)
Combining (4.4) and (4.5) with (4.6) and taking (2.5) and (3.7) into consideration, the following inequality is obtained:

\[
(\mathbb{E}|X(t)|^p)^{\frac{1}{p}} 
\leq \gamma_1 \|C_\sigma^2 e^{-2\beta t}\|\|\xi\|_\infty^2 + \gamma_1 \|\rho\|_{\infty}^2 e^{2\beta r} C_\sigma^2 e^{-2\beta t}\|\xi\|_\infty^2
\]

\[
+ \gamma_1 \|\rho\|_{\infty}^2 e^{2\beta r} C_\sigma^2 e^{-2\beta t}\|\xi\|_\infty^2 + \tau C_\sigma^2 e^{-2\beta t}\|\xi\|_\infty^2
\]

\[
+ \frac{\gamma_2}{2} \left\{ \int_0^t e^{2\beta s}(\mathbb{E}|X(s)|^p)^{\frac{1}{p}} ds \right\}^{\frac{1}{p}} \quad \text{and} \quad \frac{\gamma_2}{2} \left\{ \int_0^t e^{2\beta s}(\mathbb{E}|X(s)|^p)^{\frac{1}{p}} ds \right\}^{\frac{1}{p}}
\]

which further implies that

\[ e^{2\beta t}(\mathbb{E}|X(t)|^p)^{\frac{1}{p}} \leq K_1 + \frac{\gamma_2}{2} \left\{ \int_0^t e^{2\beta s}(\mathbb{E}|X(s)|^p)^{\frac{1}{p}} ds \right\}^{\frac{1}{p}} \quad \text{where}
\]

\[ K_1 = \gamma_1 \|\rho\|_{\infty}^2 e^{2\beta r} C_\sigma^2 e^{-2\beta t}\|\xi\|_\infty^2 + \frac{\gamma_2}{2} \left\{ \int_0^t e^{2\beta s}(\mathbb{E}|X(s)|^p)^{\frac{1}{p}} ds \right\}^{\frac{1}{p}}
\]

As \( K_1 \) is a positive constant, by the Gronwall inequality, we have

\[ (\mathbb{E}|X(t)|^p)^{\frac{1}{p}} \leq K_1 e^{-2\beta t}, \quad t \geq 0,
\]

where \( 2\bar{\beta} = 2\beta - \frac{\gamma_2}{2} L_2 (p - 1) C_\sigma^2 (1 + e^{2\beta r} \|\xi\|_{\infty}) \). This further gives that for \( p \geq 2 \)

\[ \mathbb{E}|X(t)|^p \leq K e^{-p\bar{\beta}t}, \quad t \geq 0.
\]

For any \( p \geq 2 \), due to \( p(p - 1) LC_\sigma^2 (1 + \|\xi\|_{\infty} e^{2\beta r}) < 4\beta \), we choose \( L > 0 \) sufficiently small and \( \gamma_2 > 1 \) such that

\[ p(p - 1) \gamma_2 \frac{2}{p} LC_\sigma^2 (1 + \|\xi\|_{\infty} e^{2\beta r}) < 4\beta.
\]

Therefore the result (4.2) is established in the case of \( p \geq 2 \). \( \square \)

Moreover, the higher moment can estimate the lower moment, so we can obtain the following results.

**Theorem 4.4.** Let \( 0 < p < 2 \) and \( \nu_2 < 0 \). Assume further that (4.2) holds with \( L > 0 \) such that \( LC_\sigma^2 (1 + e^{2\beta r} \|\xi\|_{\infty}) < 2\beta \), for \( \beta \in (0, -\nu_2) \), where \( C_\beta > 0 \) is introduced in (2.5). Then, for any initial value \( \xi \in W^{1,2}([-\tau, 0]; \mathbb{R}^n) \), the solution of (4.1) is \( p \)-th moment exponentially stable, i.e., there exist constants \( K > 0 \) and \( \bar{\beta} > 0 \) such that

\[ \mathbb{E}|X(t; \xi)|^p \leq K \|\xi\|_{\infty}^p e^{-p\bar{\beta}t}, \quad t \geq 0.
\]

**Theorem 4.3.** Let the conditions of Theorem 4.4 (resp. Theorem 4.2) hold. Then for any different initial values \( \xi, \eta \in W^{1,2}([-\tau, 0]; \mathbb{R}^n) \), there exists a constant \( K \) such that

\[ \mathbb{E}|X(t; \xi) - X(t; \eta)|^p \leq K \|\xi - \eta\|_{\infty}^p e^{-p\bar{\beta}t}, \quad t \geq 0, \quad p \geq 2 \quad (\text{resp.} \quad 0 < p < 2),
\]

where \( \bar{\beta} \) is given in Theorem 4.4.
Theorem 4.4. Let the conditions of theorem 4.3 hold. Then for any initial values \( \xi \in W^{1,2}([-\tau, 0]; \mathbb{R}^n) \), there exists a constant \( \bar{\gamma} > 0 \) such that

\[
\limsup_{t \to \infty} \frac{1}{t} \ln |X(t)| \leq -\bar{\gamma},
\]

where \( \bar{\gamma} = \min \{ 2\bar{\beta}, \tau^{-1} \ln \frac{1}{\| \rho \|} \} \), \( \bar{\beta} \) is given in Theorem 4.1.

Proof. For any integer \( n \geq 1 \), using the Doob martingale inequality and Hölder’s inequality, together with (3.2), (4.1) and (4.2), we have

\[
\mathbb{E} \left( \sup_{0 \leq \theta \leq \tau} |X(n\tau + \theta) - \int_{[-\tau, 0]} \rho(d\theta) X(n\tau + \theta + \theta)\right)^2 \leq 3E|X(n\tau) - \int_{[-\tau, 0]} \rho(d\theta) X(n\tau + \theta)|^2 + 3E \int_{n\tau}^{(n+1)\tau} \left( \int_{[-\tau, 0]} \mu(d\theta) X(s + \theta) \right) ds^2 + 3E \left( \sup_{0 \leq \theta \leq \tau} \left| \int_{n\tau}^{n\tau + \theta} \sigma(X_s) dW(s) \right|^2 \right)
\]

\[
\leq 6E|X(n\tau)|^2 + 6E \int_{[-\tau, 0]} \rho(d\theta) X(n\tau + \theta)^2 \right)^2 + 3E \mathbb{E} \left( \int_{n\tau}^{(n+1)\tau} \mu(d\theta) X(s + \theta) \right) ds^2 + 12 \int_{n\tau}^{(n+1)\tau} \mathbb{E} \| \sigma(X(s + \theta)) \|^2 ds
\]

\[
\leq 6E|X(n\tau)|^2 + 6\| \rho \| \int_{[-\tau, 0]} \mathbb{E} |X(n\tau + \theta)|^2 \rho(d\theta) + 3\tau \| \mu \|
\]

\[
x \mathbb{E} \int_{n\tau}^{(n+1)\tau} |X(s + \theta)|^2 \mu(d\theta) ds + 12L \mathbb{E} \int_{n\tau}^{(n+1)\tau} \left( |X(s)|^2 + \int_{[-\tau, 0]} |X(s + \theta)|^2 \nu(d\theta) \right) ds
\]

\[
\leq 6E|X(n\tau)|^2 + 6\| \rho \| \sup_{-\tau \leq s \leq 0} \mathbb{E} |X(n\tau + \theta)|^2 + 3\tau \| \mu \|
\]

\[
\times \int_{n\tau}^{(n+1)\tau} \mu(d\theta) \int_{n\tau}^{(n+1)\tau} \mathbb{E} |X(s + \theta)|^2 ds + 12L \int_{n\tau}^{(n+1)\tau} \left( \int_{[-\tau, 0]} \mathbb{E} |X(s + \theta)|^2 \nu(d\theta) \right) ds
\]

\[
\leq 6E|X(n\tau)|^2 + 6\| \rho \| \sup_{-\tau \leq s \leq 0} \mathbb{E} |X(n\tau + \theta)|^2 + 3\tau \| \mu \| \int_{n\tau}^{(n+1)\tau} \mathbb{E} |X(s)|^2 ds + 12L \| \nu \| \text{var} \int_{n\tau}^{(n+1)\tau} \mathbb{E} |X(s)|^2 ds
\]

\[
\leq 6K \| \xi \|^2 e^{-\gamma n\tau} + 6\| \rho \|^2 K \| \xi \|^2 e^{-\gamma (n\tau - \tau)} + 3\tau \| \mu \|^2 K \| \xi \|^2 \int_{n\tau}^{(n+1)\tau} e^{-\gamma s} ds + 12LK \| \xi \|^2 \int_{n\tau}^{(n+1)\tau} e^{-\gamma s} ds + 12L \| \nu \| \text{var} K \| \xi \|^2 \int_{n\tau}^{(n+1)\tau} e^{-\gamma s} ds
\]
Through a straightforward mathematical computation, we get
\[ \leq 6K\|\xi\|^2_{\infty}e^{\gamma nT} + 6\|\rho\|^2_{\infty}K\|\xi\|^2_{\infty}e^{\gamma T} + 3\|\mu\|^2_{\infty}K\|\xi\|^2_{\infty}e^{\gamma nT} + 12LK\|\xi\|^2_{\infty}e^{\gamma T} + 12L\|\nu\|\var K\|\xi\|^2_{\infty}e^{\gamma T} \]
\[ = Ce^{-\gamma nT}, \]
where \( C = K\|\xi\|^2_{\infty}[6 + 6\|\rho\|^2_{\infty}e^{\gamma T} + 3\|\mu\|^2_{\infty}e^{\gamma T}] + 12L\|\nu\|\var K\|\xi\|^2_{\infty}e^{\gamma T}. \)
For any \( \varepsilon \in (0, \gamma) \), using the Chebyshev inequality, we have
\[ P\left( \omega: \sup_{0 \leq \theta \leq T} \left| X(nT + \theta) - \int_{[-\tau,0]} \rho(d\theta)X(nT + \theta + \theta) \right|^2 > e^{-(\gamma - \varepsilon)nT} \right) \leq Ce^{-\varepsilon nT}. \]
The Borel-Cantelli lemma yields that for almost all \( \omega \in \Omega \), there exists an integer \( n_0(\omega) \) such that
\[ \sup_{0 \leq \theta \leq T} \left| X(nT + \theta) - \int_{[-\tau,0]} \rho(d\theta)X(nT + \theta + \theta) \right|^2 \leq e^{-(\gamma - \varepsilon)nT}, \quad n \geq n_0. \]
Consequently, for almost all \( \omega \in \Omega \), if \( t \geq n_0T \),
\[ \left| X(t) - \int_{[-\tau,0]} \rho(d\theta)X(t + \theta) \right|^2 \leq e^{-(\gamma - \varepsilon)(t - \tau)}. \]
Moreover, for \( t \in [0, n_0T] \), \( X(t) - \int_{[-\tau,0]} \rho(d\theta)X(t + \theta) \) is finite. So, for almost all \( \omega \in \Omega \), there exists a finite constant \( H = H(\omega) \), if \( t \geq 0 \),
\[ \left| X(t) - \int_{[-\tau,0]} \rho(d\theta)X(t + \theta) \right|^2 \leq He^{-(\gamma - \varepsilon)t}. \]
On the other hand, set \( e^{\gamma T}\|\rho\|^2_{\infty} < \varepsilon < 1 \). For \( t \geq 0 \), note that
\[ \left| X(t) - \int_{[-\tau,0]} \rho(d\theta)X(t + \theta) \right|^2 \]
\[ \geq |X(t)|^2 - 2|X(t)||\int_{[-\tau,0]} \rho(d\theta)X(t + \theta)| + \left| \int_{[-\tau,0]} \rho(d\theta)X(t + \theta) \right|^2 \]
\[ \geq (1 - \varepsilon)|X(t)|^2 + (\varepsilon^{-1} - 1)\left| \int_{[-\tau,0]} \rho(d\theta)X(t + \theta) \right|^2. \]
Hence, we see that
\[ |X(t)|^2 \leq \frac{1}{1 - \varepsilon} |X(t) - \int_{[-\tau,0]} \rho(d\theta)X(t + \theta)|^2 + \frac{1}{\varepsilon} \left| \int_{[-\tau,0]} \rho(d\theta)X(t + \theta) \right|^2. \]
Also, for each \( T > 0 \),
\[ \sup_{0 \leq t \leq T} |e^{(\gamma - \varepsilon)t}|X(t)|^2| \leq \frac{H}{1 - \varepsilon} + \frac{1}{\varepsilon} \sup_{0 \leq t \leq T} \left| e^{(\gamma - \varepsilon)t}\int_{[-\tau,0]} \rho(d\theta)X(t + \theta) \right|^2 \]
\[ \leq \frac{H}{1 - \varepsilon} + \frac{e^{(\gamma - \varepsilon)T}\|\rho\|^2_{\infty}}{\varepsilon} \sup_{-\tau \leq t \leq T} |e^{(\gamma - \varepsilon)t}|X(t)|^2|. \]
Through a straightforward mathematical computation, we get
\[ \left( 1 - \frac{e^{(\gamma - \varepsilon)T}\|\rho\|^2_{\infty}}{\varepsilon} \right) \sup_{0 \leq t \leq T} |e^{(\gamma - \varepsilon)t}|X(t)|^2| \leq \frac{H}{1 - \varepsilon} + \frac{e^{(\gamma - \varepsilon)T}\|\rho\|^2_{\infty}}{\varepsilon} \|\xi\|^2_{\infty}. \]
which implies that
\[ \limsup_{t \to \infty} \frac{1}{t} \ln |X(t)| \leq -\frac{\gamma - \varepsilon}{2} \text{ a.s.} \]
Let \( \varepsilon \to 0 \). The required result is obtained. \( \square \)
Remark 4. In the beginning of this paper, we suppose that \(\sigma(0) = 0_{n \times m}\). This assumption plays a key role in our stability analysis as above. From the assumption (3.2) on the diffusion coefficient we deduce that for any \(\phi \in \mathcal{C}\) there exists an \(L > 0\) such that
\[
\|\sigma(\phi)\|^2 \leq L \left( |\phi(0)|^2 + \int_{[-\tau,0]} |\phi(\theta)|^2 \nu(d\theta) \right),
\]
which has been used in the proof of stability theory. It guarantees that \(X(t)\) admits the property as the form
\[
E|X(t)|^p \leq a e^{-bt}.
\]
Otherwise, we arrive at
\[
E|X(t)|^p \leq c + a e^{-bt} (a, b, c \in \mathbb{R}),
\]
and the constants \(c \neq 0\) cannot be dumped. Moreover, this method is widely applied in the stability analysis. For example, Zhu [26] studied asymptotic stability in the \(p\)th moment for SDEs with Lévy noise; Zhou and Yang [25] gave the criterion of mean square exponential stability for delayed neural networks with Lévy noise. However, if we only seek convergence of solutions from different initial value, this assumption can be removed.

5. Examples. In this section, we consider a couple of examples to verify the theories established in the previous section.

Example 1. Consider a semi-linear retarded SDE
\[
dX(t) = -X(t-1)dt + \sigma(X(t-1))dW(t), X_0 = \xi \in \mathcal{C}.
\]
(5.1)
It is impossible to choose constants \(b_1 > b_2 > 0\) such that
\[
-2xy + \sigma^2(y) \leq c - b_1|x|^2 + b_2|y|^2, \quad x, y \in \mathbb{R}.
\]
So, (5.1) does not satisfy a dissipative condition. In view of the corresponding characteristic equation \(\lambda + e^{-\lambda} = 0\), we deduce that the unique root is \(\lambda = -0.3181 + 1.3372i\). So, we have \(\nu = -0.3181\). Taking \(p = 2\) and by Theorem 3.1 when the Lipschitz constant \(L\) of \(\sigma\) such that \(LC_0^2(1 + e^{2\alpha\tau}) < 2\alpha\) for \(\alpha \in (0, 0.3181)\), the solution \(X(t)\) of (5.1) is almost surely exponentially stable and exponentially stable in mean square.

Example 2. Consider a semi-linear retarded SDE
\[
d\left(X(t) + \frac{1}{3}X(t-1)\right) = -X(t-1)dt + a \int_{-1}^{0} X(t+\theta)dW(t), X_0 = \xi \in \mathcal{C},
\]
(5.2)
where \(a \in \mathbb{R}\) and \(W(t)\) is a real-valued Brownian motion. It is easy to see that the corresponding characteristic equation is
\[
\lambda + (1 + \frac{1}{3}\lambda)e^{-\lambda} = 0, \lambda \in \mathbb{C}.
\]
(5.3)
A simple calculation by Matlab yields that the unique root of (5.3) is \(\lambda = -2.313474269\). Then, by the results in section 4 we deduce that the solution \(X(t)\) of (5.2) is almost surely exponentially stable and moment exponentially stable if \(a \in \mathbb{R}\) is sufficiently small.

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