COMBINED DYNAMIC GRÜSS INEQUALITIES ON TIME SCALES

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Abstract. We prove a more general version of the Grüss inequality by using the recent theory of combined dynamic derivatives on time scales and the more general notions of diamond-$\alpha$ derivative and integral. For the particular case when $\alpha = 1$, one gets a delta-integral Grüss inequality on time scales; for $\alpha = 0$ a nabla-integral Grüss inequality. If we further restrict ourselves by fixing the time scale to the real (or integer) numbers, then the standard continuous (discrete) inequalities are obtained.

1. Introduction

The Grüss inequality is of great interest in differential and difference equations as well as many other areas of mathematics [8, 9, 16, 18, 19]. The classical inequality was proved by G. Grüss in 1935 [13]: if $f$ and $g$ are two continuous functions on $[a, b]$ satisfying $\varphi \leq f(t) \leq \Phi$ and $\gamma \leq g(t) \leq \Gamma$ for all $t \in [a, b]$, then

$$
\left| \frac{1}{b-a} \int_a^b f(t)g(t)dt - \frac{1}{(b-a)^2} \int_a^b f(t)dt \int_a^b g(t)dt \right| \leq \frac{1}{4} (\Phi - \varphi)(\Gamma - \gamma).
$$

The literature on Grüss type inequalities is now vast, and many extensions of the classical inequality (1) have been intensively studied by many authors in the XXI century [3, 5, 10, 17, 22, 23, 24, 25]. Here we are interested in Grüss type inequalities on time scales.

The analysis on time scales is a relatively new area of mathematics that unify and generalize discrete and continuous theories. Moreover, it is a crucial tool in many computational and numerical applications. The subject was initiated by Stefan Hilger [14, 15] and is being applied to many different fields in which dynamic processes can be described with discrete or continuous models: see [1, 4, 6, 7, 12] and references therein. One of the important subjects being developed within the theory of time scales is the study of inequalities [2, 11, 21, 29].

Recently, Bohner and Matthews [5] proved a time scales version of the Grüss inequality (1) by using the delta integral. Roughly speaking, the main result in [5] asserts that (1) continues to be valid on a general time scale by substituting the Riemann integral by the $\Delta$-integral. The main objective of this paper is to use the more general diamond–$\alpha$ dynamic integral.

In 2006, a combined dynamic derivative $\diamondsuit$ was introduced as a linear combination of $\Delta$ and $\nabla$ dynamic derivatives on time scales [28]. The diamond–$\alpha$ derivative reduces to the standard $\Delta$ derivative for $\alpha = 1$ and to the standard $\nabla$ derivative for $\alpha = 0$. On the other hand, it represents a weighted dynamic derivative formula on

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any uniformly discrete time scale when \( \alpha = \frac{1}{m} \). We refer the reader to [20–28] for an account of the calculus with diamond-scales.

In Section 2, we briefly review the necessary definitions and calculus on time scales. Our results are given in Section 3.

2. Preliminaries

A time scale \( \mathbb{T} \) is an arbitrary nonempty closed subset of the real numbers. The time scales calculus was initiated by S. Hilger in his PhD thesis in order to unify discrete and continuous analysis [14, 15]. Let \( \mathbb{T} \) be a time scale with the topology that it inherits from the real numbers. For \( t \in \mathbb{T} \), we define the forward jump operator \( \sigma : \mathbb{T} \to \mathbb{T} \) by

\[
\sigma(t) = \inf \{ s \in \mathbb{T} : s > t \},
\]

and the backward jump operator \( \rho : \mathbb{T} \to \mathbb{T} \) by

\[
\rho(t) = \sup \{ s \in \mathbb{T} : s < t \}.
\]

If \( \sigma(t) > t \) we say that \( t \) is right-scattered, while if \( \rho(t) < t \) we say that \( t \) is left-scattered. Points that are simultaneously right-scattered and left-scattered are said to be isolated. If \( \sigma(t) = t \), then \( t \) is called right-dense; if \( \rho(t) = t \), then \( t \) is called left-dense. Points that are right-dense and left-dense at the same time are called dense. The mappings \( \sigma, \rho : \mathbb{T} \to [0, +\infty) \) defined by \( \mu(t) := \sigma(t) - t \) and \( \nu(t) := t - \rho(t) \) are called, respectively, the forward and backward graininess function.

Given a time scale \( \mathbb{T} \), we introduce the sets \( \mathbb{T}^k \), \( \mathbb{T}_k \), and \( \mathbb{T}_k^k \) as follows. If \( \mathbb{T} \) has a left-scattered maximum \( t_1 \), then \( \mathbb{T}_k = \mathbb{T} - \{ t_1 \} \), otherwise \( \mathbb{T}_k^k = \mathbb{T} \). If \( \mathbb{T} \) has a right-scattered minimum \( t_2 \), then \( \mathbb{T}_k = \mathbb{T} - \{ t_2 \} \), otherwise \( \mathbb{T}_k = \mathbb{T} \). Finally, \( \mathbb{T}_k^k = \mathbb{T}_k \cap \mathbb{T}_k^k \).

Let \( f : \mathbb{T} \to \mathbb{R} \) be a real valued function on a time scale \( \mathbb{T} \). Then, for \( t \in \mathbb{T}_k \), we define \( f^\Delta(t) \) to be the number, if one exists, such that for all \( \epsilon > 0 \), there is a neighborhood \( U \) of \( t \) such that for all \( s \in U \),

\[
|f(\sigma(t)) - f(s) - f^\Delta(t)(\sigma(t) - s)| \leq \epsilon|\sigma(t) - s|.
\]

We say that \( f \) is delta differentiable on \( \mathbb{T}_k \) provided \( f^\Delta(t) \) exists for all \( t \in \mathbb{T}_k \). Similarly, for \( t \in \mathbb{T}_k^k \) we define \( f^\nabla(t) \) to be the number, if one exists, such that for all \( \epsilon > 0 \), there is a neighborhood \( V \) of \( t \) such that for all \( s \in V \),

\[
|f(\rho(t)) - f(s) - f^\nabla(t)(\rho(t) - s)| \leq \epsilon|\rho(t) - s|.
\]

We say that \( f \) is nabla differentiable on \( \mathbb{T}_k \), provided that \( f^\nabla(t) \) exists for all \( t \in \mathbb{T}_k \).

For \( f : \mathbb{T} \to \mathbb{R} \) we define the function \( f^\sigma : \mathbb{T} \to \mathbb{R} \) by \( f^\sigma(t) = f(\sigma(t)) \) for all \( t \in \mathbb{T} \), that is, \( f^\sigma = f \circ \sigma \). Similarly, we define the function \( f^\rho : \mathbb{T} \to \mathbb{R} \) by \( f^\rho(t) = f(\rho(t)) \) for all \( t \in \mathbb{T} \), that is, \( f^\rho = f \circ \rho \).

A function \( f : \mathbb{T} \to \mathbb{R} \) is called rd-continuous, provided it is continuous at all right-dense points in \( \mathbb{T} \) and its left-sided limits finite at all left-dense points in \( \mathbb{T} \). A function \( f : \mathbb{T} \to \mathbb{R} \) is called ld-continuous, provided it is continuous at all left-dense points in \( \mathbb{T} \) and its right-sided limits finite at all right-dense points in \( \mathbb{T} \).

A function \( F : \mathbb{T} \to \mathbb{R} \) is called a delta antiderivative of \( f : \mathbb{T} \to \mathbb{R} \), provided \( F^\Delta(t) = f(t) \) holds for all \( t \in \mathbb{T}_k^k \). Then the delta integral of \( f \) is defined by

\[
\int_a^b f(t) \Delta t = F(b) - F(a).
\]

A function \( G : \mathbb{T} \to \mathbb{R} \) is called a nabla antiderivative of \( g : \mathbb{T} \to \mathbb{R} \), provided \( G^\nabla(t) = g(t) \) holds for all \( t \in \mathbb{T}_k \). Then the nabla integral of \( g \) is defined by

\[
\int_a^b g(t) \nabla t = G(b) - G(a).
\]

For more details on time scales one can see [11, 6, 7].
Thus, we define the diamond-$\alpha$ derivative for more on the associated calculus.

Let $T$ be a time scale and $f$ differentiable on $T$ in the $\Delta$ and $\nabla$ senses. For $t \in T$, we define the diamond-$\alpha$ dynamic derivative $f^{\diamond\alpha}(t)$ by

$$f^{\diamond\alpha}(t) = \alpha f^\Delta(t) + (1 - \alpha)f^\nabla(t), \quad 0 \leq \alpha \leq 1.$$ 

Thus, $f$ is diamond-$\alpha$ differentiable if and only if $f$ is $\Delta$ and $\nabla$ differentiable. The diamond-$\alpha$ derivative reduces to the standard $\Delta$ derivative for $\alpha = 1$, or the standard $\nabla$ derivative for $\alpha = 0$. On the other hand, it represents a “weighted derivative” for $\alpha \in (0, 1)$. Diamond-$\alpha$ derivatives have shown in computational experiments to provide efficient and balanced approximation formulas, leading to the design of more reliable numerical methods.

Let $f, g : T \to \mathbb{R}$ be diamond-$\alpha$ differentiable at $t \in T$. Then,

(i) $f + g : T \to \mathbb{R}$ is diamond-$\alpha$ differentiable at $t \in T$ with

$$(f + g)^{\diamond\alpha}(t) = (f)^{\diamond\alpha}(t) + (g)^{\diamond\alpha}(t).$$

(ii) For any constant $c$, $cf : T \to \mathbb{R}$ is diamond-$\alpha$ differentiable at $t \in T$ with

$$(cf)^{\diamond\alpha}(t) = c(f)^{\diamond\alpha}(t).$$

(iii) $fg : T \to \mathbb{R}$ is diamond-$\alpha$ differentiable at $t \in T$ with

$$(fg)^{\diamond\alpha}(t) = (f)^{\diamond\alpha}(t)g(t) + \alpha f^\alpha(t)g(t)\Delta(t) + (1 - \alpha)f^\alpha(t)(g)\nabla(t).$$

Let $a, t \in T$, and $h : T \to \mathbb{R}$. Then, the diamond-$\alpha$ integral from $a$ to $t$ of $h$ is defined by

$$\int_a^t h(\tau)\diamond_a \tau = \alpha \int_a^t h(\tau)\Delta \tau + (1 - \alpha)\int_a^t h(\tau)\nabla \tau, \quad 0 \leq \alpha \leq 1.$$ 

We may notice the absence of an anti-derivative for the $\diamond_\alpha$ combined derivative. For $t \in T$, in general

$$\left(\int_a^t h(\tau)\diamond_a \tau\right)^{\diamond_\alpha} \neq h(t).$$

Although the fundamental theorem of calculus does not hold for the $\diamond_\alpha$-integral, other properties do. Let $a, b, t \in T$, $c \in \mathbb{R}$. Then,

(i) $\int_a^b \{f(\tau) + g(\tau)\} \diamond_\alpha \tau = \int_a^b f(\tau) \diamond_\alpha \tau + \int_a^b g(\tau) \diamond_\alpha \tau$;
(ii) $\int_a^b cf(\tau) \diamond_\alpha \tau = c \int_a^b f(\tau) \diamond_\alpha \tau$;
(iii) $\int_a^b f(\tau) \diamond_\alpha \tau = \int_a^b f(\tau) \diamond_\alpha \tau + \int_a^b f(\tau) \diamond_\alpha \tau$;
(iv) If $f(t) \geq 0$ for all $t$, then $\int_a^b f(t) \diamond_\alpha \tau \geq 0$;
(v) If $f(t) \leq g(t)$ for all $t$, then $\int_a^b f(t) \diamond_\alpha \tau \leq \int_a^b g(t) \diamond_\alpha \tau$;
(vi) If $f(t) \equiv 0$ if and only if $\int_a^b f(t) \diamond_\alpha \tau = 0$.

In [11], a diamond-$\alpha$ Jensen’s inequality on time scales is proved: let $c$ and $d$ be real numbers, $u : [a, b] \to (c, d)$ be continuous and $h : (c, d) \to \mathbb{R}$ be a convex function. Then,

$$h \left( \frac{\int_a^b u(t) \diamond_\alpha \tau}{b - a} \right) \leq \frac{1}{b - a} \int_a^b h(u(t)) \diamond_\alpha \tau.$$ 

We will use the diamond-$\alpha$ Jensen’s inequality (2) in the proof of our Theorem.
3. Main Results

In the sequel we assume that $\mathbb{T}$ is a time scale, $a, b \in \mathbb{T}$ with $a < b$, and $[a, b]$ denote $[a, b] \cap \mathbb{T}$. We begin by proving some auxiliary lemmas.

Lemma 3.1. If $f \in C([a, b], \mathbb{R})$ satisfy $m \leq f(t) \leq M$ for all $t \in [a, b]$ and $\int_a^b f(t) \ominus t = 0$, then

$$\frac{1}{b-a} \int_a^b f^2(t) \ominus t \leq \frac{1}{4} (M - m)^2.$$  

Proof. Define $\varphi(t) = \frac{f(t) - m}{M - m}$. It follows that $f(t) = m + (M - m)\varphi(t)$. One can easily check that $0 \leq \varphi(t) \leq 1$. Then,

$$\frac{1}{b-a} \int_a^b f^2(t) \ominus t = \frac{1}{b-a} \int_a^b (m + (M - m)\varphi(t))^2 \ominus t$$

$$= \frac{1}{b-a} \int_a^b \left( m^2 + 2m(M - m)\varphi(t) + (M - m)^2\varphi^2(t) \right) \ominus t$$

$$= m^2 + \frac{2m(M - m)}{b-a} \int_a^b \varphi(t) \ominus t + \frac{(M - m)^2}{b-a} \int_a^b \varphi^2(t) \ominus t$$

$$\leq m^2 + \frac{2m(M - m)}{b-a} \int_a^b \varphi(t) \ominus t + \frac{(M - m)^2}{b-a} \int_a^b \varphi(t) \ominus t$$

$$= m^2 + \frac{2m(M - m)}{b-a} \int_a^b f(t) - m \ominus t + \frac{(M - m)^2}{b-a} \int_a^b f(t) - m \ominus t$$

$$= m^2 + 2m(M - m) \left( \frac{-m}{M - m} \right) + (M - m)^2 \left( \frac{-m}{M - m} \right)$$

$$= -mM = \frac{1}{4} ((M - m)^2 - (M + m)^2)$$

$$\leq \frac{1}{4} (M - m)^2.$$  

\[\blacksquare\]

Lemma 3.2. If $f \in C([a, b], \mathbb{R})$ satisfy $m \leq f(t) \leq M$ for all $t \in [a, b]$ and $\int_a^b f(t) \ominus t \neq 0$, then

$$\frac{1}{b-a} \int_a^b f^2(t) \ominus t - \left( \frac{1}{b-a} \int_a^b f(t) \ominus t \right)^2 \leq \frac{1}{4} (M - m)^2.$$  

Proof. Setting

$$\frac{1}{b-a} \int_a^b f(t) \ominus t = I(b-a),$$

$I \in \mathbb{R}$, and $F(t) = f(t) - I(b-a)$, we then have

$$m - I(b-a) \leq F(t) \leq M - I(b-a).$$

Therefore,

$$\frac{1}{b-a} \int_a^b F(t) \ominus t = \frac{1}{b-a} \int_a^b (f(t) - I(b-a)) \ominus t$$

$$= \frac{1}{b-a} \int_a^b f(t) \ominus t - I(b-a)$$

$$= 0.$$

(3)
It follows from Lemma 3.1 that

\[
\frac{1}{b-a} \int_a^b F^2(t) \diamond_\alpha t \leq \frac{1}{4} ((M - I(b - a)) - (m - I(b - a)))^2 \\
= \frac{1}{4}(M - m)^2.
\] (4)

On the other hand, using (3) we have

\[
\frac{1}{b-a} \int_a^b F^2(t) \diamond_\alpha t - \left( \frac{1}{b-a} \int_a^b F(t) \diamond_\alpha t \right)^2 \\
= \frac{1}{b-a} \int_a^b F^2(t) \diamond_\alpha t \\
= \frac{1}{b-a} \int_a^b (f^2(t) - 2I(b-a)f(t) + I^2(b-a)) \diamond_\alpha t \\
= \frac{1}{b-a} \int_a^b f^2(t) \diamond_\alpha t - 2I^2(b-a) + I^2(b-a) \\
= \frac{1}{b-a} \int_a^b f^2(t) \diamond_\alpha t - I^2(b-a) \\
= \frac{1}{b-a} \int_a^b f^2(t) \diamond_\alpha t - \left( \frac{1}{b-a} \int_a^b f(t) \diamond_\alpha t \right)^2.
\] (5)

Then, using (4) and (5), we conclude with

\[
\frac{1}{b-a} \int_a^b f^2(t) \diamond_\alpha t - \left( \frac{1}{b-a} \int_a^b f(t) \diamond_\alpha t \right)^2 \leq \frac{1}{4}(M - m)^2.
\]

\[\square\]

From Lemma 3.1 and Lemma 3.2 we have the following corollary.

**Corollary 3.3.** If \( f \in C([a, b], \mathbb{R}) \) satisfy \( m \leq f(t) \leq M \) for all \( t \in [a, b] \), then

\[
\frac{1}{b-a} \int_a^b f^2(t) \diamond_\alpha t - \left( \frac{1}{b-a} \int_a^b f(t) \diamond_\alpha t \right)^2 \leq \frac{1}{4}(M - m)^2.
\]

We are now in conditions to prove the intended Grüss inequality.

**Theorem 3.4.** (The diamond-\( \alpha \) Grüss inequality on time scales, \( \alpha \in [0,1] \)). Let \( T \) be a time scale, \( a, b \in T \) with \( a < b \). If \( f, g \in C([a, b], \mathbb{R}) \) satisfy \( \varphi \leq f(t) \leq \Phi \) and \( \gamma \leq g(t) \leq \Gamma \) for all \( t \in [a, b] \cap T \), then

\[
\frac{1}{b-a} \int_a^b f(t)g(t) \diamond_\alpha t - \frac{1}{(b-a)^2} \int_a^b f(t) \diamond_\alpha t \int_a^b g(t) \diamond_\alpha t \leq \frac{1}{4} (\Phi - \varphi)(\Gamma - \gamma).
\] (6)

**Proof.** A straightforward computation leads to

\[
\frac{1}{b-a} \int_a^b f(t)g(t) \diamond_\alpha t - \frac{1}{(b-a)^2} \int_a^b f(t) \diamond_\alpha t \int_a^b g(t) \diamond_\alpha t \\
= \frac{1}{4} \left( \frac{1}{b-a} \int_a^b ((f(t) + g(t))^2 - (f(t) - g(t))^2) \diamond_\alpha t \\
- \frac{4}{(b-a)^2} \int_a^b f(t) \diamond_\alpha t \int_a^b g(t) \diamond_\alpha t \right).
\] (7)
If we consider the function \( h(x) = x^2 \), which is obviously convex, then using the diamond-\(\alpha\) Jensen’s inequality on time scales (12), we obtain

\[
\left( \frac{\int_a^b (f(t) - g(t))\diamond_{\alpha} t}{b-a} \right)^2 \leq \frac{1}{b-a} \int_a^b (f(t) - g(t))^2\diamond_{\alpha} t. \tag{8}
\]

Then we have by (7) and (8) that

Gathering (8), (9) and (10), we obtain:

\[
\left( \frac{\int_a^b (f(t) - g(t))\diamond_{\alpha} t}{b-a} \right)^2 \leq \frac{1}{b-a} \int_a^b (f(t) - g(t))^2\diamond_{\alpha} t.
\]

On the other hand, we have \( \phi + \Phi \leq (f + g)(t) \leq \gamma + \Gamma \). Applying Corollary 3.3 to the function \( f + g \), we get

\[
\left( \frac{\int_a^b (f(t) + g(t))\diamond_{\alpha} t}{b-a} \right)^2 - \frac{1}{(b-a)^2} \left( \int_a^b (f(t) + g(t))\diamond_{\alpha} t \right)^2 - \frac{4}{(b-a)^2} \int_a^b f(t)\diamond_{\alpha} t \int_a^b g(t)\diamond_{\alpha} t = 0.
\]

Since

\[
\frac{1}{(b-a)^2} \left( \int_a^b (f(t) + g(t))\diamond_{\alpha} t \right)^2 - \frac{1}{(b-a)^2} \left( \int_a^b (f(t) - g(t))\diamond_{\alpha} t \right)^2 - \frac{4}{(b-a)^2} \int_a^b f(t)\diamond_{\alpha} t \int_a^b g(t)\diamond_{\alpha} t = 0,
\]

we have \( \phi + \Phi \leq (f + g)(t) \leq \gamma + \Gamma \).
We consider now two cases: (i) if $\Phi - \varphi = \Gamma - \gamma$, then

\[
(11) \quad \frac{1}{b-a} \int_{a}^{b} f(t)g(t)\triangle t - \frac{1}{(b-a)^2} \int_{a}^{b} f(t)\triangle t \int_{a}^{b} g(t)\triangle t \leq \frac{1}{4}(\Phi - \varphi)(\Gamma - \gamma);
\]

(ii) if $\Phi - \varphi \neq \Gamma - \gamma$, let us define

\[
\beta = \sqrt{\frac{\Gamma - \gamma}{\Phi - \varphi}}, \quad \mu = \sqrt{\frac{\Phi - \varphi}{\Gamma - \gamma}};
\]

and

\[
f_i(t) = \beta f_i, \quad g_i(t) = \mu g_i.
\]

Note that $\beta \mu = 1$. Then, we have

\[
\overline{m}_1 = \beta \varphi \leq f_1(t) \leq \beta \Phi = \overline{M}_1, \quad \overline{m}_2 = \mu \gamma \leq g_1(t) \leq \mu \Gamma = \overline{M}_2,
\]

and it follows that

\[
(12) \quad \overline{M}_1 - \overline{m}_1 = \beta \Phi - \beta \varphi = \beta(\Phi - \varphi) = \sqrt{(\Phi - \varphi)(\Gamma - \gamma)} = \mu(\Gamma - \gamma) = \overline{M}_2 - \overline{m}_2.
\]

Using the fact that $\beta \mu = 1$, (11) and (12) (with $f_1, g_1$), we get

\[
(13) \quad \frac{1}{b-a} \int_{a}^{b} f(t)g(t)\triangle t - \frac{1}{(b-a)^2} \int_{a}^{b} f(t)\triangle t \int_{a}^{b} g(t)\triangle t
\]

\[
= \frac{1}{b-a} \int_{a}^{b} \beta \mu f(t)g(t)\triangle t - \frac{1}{(b-a)^2} \int_{a}^{b} \beta f(t)\triangle t \int_{a}^{b} \mu g(t)\triangle t
\]

\[
= \frac{1}{b-a} \int_{a}^{b} f_1(t)g_1(t)\triangle t - \frac{1}{(b-a)^2} \int_{a}^{b} f_1(t)\triangle t \int_{a}^{b} g_1(t)\triangle t
\]

\[
\leq \frac{1}{4}(\overline{M}_1 - \overline{m}_1)(\overline{M}_2 - \overline{m}_2) = \frac{1}{4} \beta \mu(\Phi - \varphi)(\Gamma - \gamma) = \frac{1}{4}(\Phi - \varphi)(\Gamma - \gamma).
\]

If we now consider the case of $-f$ we have $-\Phi \leq -f(t) \leq -\varphi$. Using (13),

\[
(14) \quad \frac{1}{b-a} \int_{a}^{b} (-f(t))g(t)\triangle t - \frac{1}{(b-a)^2} \int_{a}^{b} (-f(t))\triangle t \int_{a}^{b} g(t)\triangle t
\]

\[
= - \left\{ \frac{1}{b-a} \int_{a}^{b} f(t)g(t)\triangle t - \frac{1}{(b-a)^2} \int_{a}^{b} f(t)\triangle t \int_{a}^{b} g(t)\triangle t \right\}
\]

\[
\leq \frac{1}{4}(-\varphi(-\Phi))(\Gamma - \gamma) = \frac{1}{4}(\Phi - \varphi)(\Gamma - \gamma).
\]

Then, using (13) and (14), we arrive to

\[
\left| \frac{1}{b-a} \int_{a}^{b} f(t)g(t)\triangle t - \frac{1}{(b-a)^2} \int_{a}^{b} f(t)\triangle t \int_{a}^{b} g(t)\triangle t \right| \leq \frac{1}{4}(\Phi - \varphi)(\Gamma - \gamma).
\]

This completes the proof of our Theorem 3.4. \qed

**Remark 3.1.** When $\alpha = 1$, we have the following Grüss inequality on time scales:

\[
\left| \frac{1}{b-a} \int_{a}^{b} f(t)g(t)\Delta t - \frac{1}{(b-a)^2} \int_{a}^{b} f(t)\Delta t \int_{a}^{b} g(t)\Delta t \right| \leq \frac{1}{4}(\Phi - \varphi)(\Gamma - \gamma).
\]

**Remark 3.2.** For $\mathbb{T} = \mathbb{R}$ Theorem 3.4 gives the classical inequality (1). Applying the Grüss inequality in Theorem 3.4 to the time scale $\mathbb{T} = \mathbb{Z}$ with $a = 0, b = n$, and $f(t) = x_i, i = 1, \ldots, n$, we arrive to the following corollary which improves [10].
Corollary 3.5. If $\varphi \leq x_i \leq \Phi$ and $\gamma \leq y_i \leq \Gamma$ for all $1 \leq i \leq n$, then the following discrete Grüss inequality holds:

$$\frac{1}{n} \sum_{i=1}^{n} x_i y_i - \frac{1}{n^2} \sum_{i=1}^{n} x_i \sum_{i=1}^{n} y_i \leq \frac{1}{4}(\Phi - \varphi)(\Gamma - \gamma).$$

Theorem [5.4] is valid for an arbitrary time scale. For example, let $q > 1$ and $T = \{q^N\}$.

Corollary 3.6. (Quantum Grüss inequality) If $f$ and $g$ satisfy $\varphi \leq f(q^i) \leq \Phi$ and $\gamma \leq g(q^i) \leq \Gamma$ for all $q^i$, $i = m, \ldots, n$, then the following inequality holds:

$$\left| \frac{\sum_{i=m}^{n-1} q^i f(q^{i+1}) g(q^{i+1})}{\sum_{i=m}^{n-1} q^i} - \frac{1}{\left( \sum_{i=m}^{n-1} q^i \right)^2} \left( \sum_{i=m}^{n-1} q^i f(q^{i+1}) \right) \left( \sum_{i=m}^{n-1} q^i g(q^{i+1}) \right) \right| \leq \frac{1}{4}(\Phi - \varphi)(\Gamma - \gamma).$$

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