Emergent AdS$_3$ in the Zero Entropy Extremal Black Holes

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Abstract

We investigate the zero entropy limit of the near horizon geometries of $D = 4$ and $D = 5$ general extremal black holes with $\text{SL}(2,\mathbb{R}) \times \text{U}(1)^{D-3}$ symmetry. We derive some conditions on the geometries from expectation of regularity. We then show that an AdS$_3$ structure emerges in a certain scaling limit, though the periodicity shrinks to zero. We present some examples to see the above concretely. We also comment on some implications to the Kerr/CFT correspondence.

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1 Introduction

The (generalized) Kerr/CFT correspondence is one of the promising candidates for the framework to understand quantum aspects of black holes in a broader class, including
realistic ones. It has already provided a fairly general framework for finding the dual theories (which are 2D chiral CFT’s) to extremal black holes, not limited to the ones embedded in string or supersymmetric theories. Deeper understanding and wider generalizations of the Kerr/CFT would reveal some general aspects and practical applications of gauge/gravity correspondence and quantum gravity.

However, the understanding of the Kerr/CFT is still rather poor and there are many seemingly strange assumptions in it. One of the most important mysteries of the Kerr/CFT is why it works well itself. Unlike the AdS/CFT’s based on brane setup in string/M theories [14], the interpretation of the dual chiral CFT is, generically, quite obscure. Furthermore, the boundary condition imposed on the geometry is rather violent, which implies that the dual CFT does not correspond to one fixed macroscopic background, but a series of many backgrounds with macroscopically different charges [15]. It in turn suggests that there is no dynamics in a fixed extremal background, as is pointed out in both holographic [15, 16] and gravitational [17, 18] points of view. This fact also makes difficult a clear understanding of the Kerr/CFT.

Against the problem above, a hopeful conjecture proposed up to now is that the Kerr/CFT comes from AdS$^3$/CFT$^2$ correspondence [15, 16]. It has been argued that some AdS$^3$ structures would be hidden behind the near horizon geometries of extremal black holes, and that the chiral CFT$_2$ would appear as some limit of the non-chiral CFT$_2$ dual to the AdS$^3$ [19], though no explicit realizations were proposed.

Very recently, a related and very interesting result was reported [20]. They investigated the maximal charge limit of 5D extremal Kerr-Newman (or BMPV [21–23]) black hole [24–27], where the entropy goes to zero, and showed that the total space including graviphoton fiber in the near horizon limit is locally the same as the one for the BPS black string at zero left and right temperatures, AdS$^3 \times S^3$. They then argued that the Kerr/CFT (with central charge $c = 6J_L$) for this system is embedded in string theory which provides the microscopic realization [28, 29] under the maximal charge limit, and also discussed some deformation from there.

From this, on one hand, one may hope that there is an AdS$^3$ structure in some points in the parameter space of general extremal black holes with rotational symmetries, although their simple realizations in string theory are difficult to imagine generally. On the other hand, the 5D black holes considered in [20] can be uplifted to the 6D black string

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1 Generalizations of the Kerr/CFT to non-extremal black holes are also proposed and the existence of dual non-chiral CFT’s is suggested [7–9]. However, appropriate boundary conditions have not been found for it yet, although there are some attempts in this direction [10–13].

2 For a recent related attempt, see [30].
solution with an AdS$_3$ structure in the near horizon geometry, which is not expected for general extremal black holes. Thus, the emergence of AdS$_3$ is seemingly due to the special properties of the solutions.

In this paper, however, we find that the AdS$_3$ structure indeed emerges just by taking the zero entropy limit, $S_{BH} = 0$, of the near horizon geometries of almost general extremal black holes with axial $U(1)^{D-3}$ in $D = 4$ and $D = 5$. Here, the zero entropy limit, we mean, does not include the small black holes and massless limits — that is, the near horizon AdS$_2$ structure must not collapse.

From a holographic point of view, an extremal black hole should correspond to the ground states of a superselection sector with fixed charges in the boundary theory. Therefore, some decoupled infrared theory is expected to live there, even when the degeneracy of the ground states vanishes. It suggests the existence of some scaling limit where the near horizon geometry remains regular$^3$ while the entropy goes to zero. This expectation requires some additional conditions on the general form of the near horizon geometry$^3$ in this zero entropy limit. In fact, these conditions are satisfied in the concrete examples which we will investigate later.

Those conditions, in turn, prove to lead to the emergence of an AdS$_3$ structure$^4$. We will see it in the concrete examples, too, and finally discuss some implications of our theorem to the Kerr/CFT.

Organization of this paper is as follows. In §2 and §3, we start with general form of near horizon geometry of extremal black holes with $\text{SL}(2, \mathbb{R}) \times U(1)^{D-3}$ isometry for $D = 4$ and $D = 5$ cases respectively. We then derive some conditions on the geometry in the zero entropy limit from the expectation of regularity, and show that they lead to the emergence of an AdS$_3$ structure. In §4, we consider the zero entropy limit for some concrete examples, including 5D Myers-Perry black hole$^3$, 5D Kaluza-Klein black hole$^3$ and extremal black holes in 5D supergravity. In §5, following the argument of [20], we explain some relation between chiral CFT$_2$ appearing in the Kerr/CFT and non-chiral CFT$_2$ expected to be dual to AdS$_3$ emerging in the zero entropy limit. In §6, we end up with the conclusions and discussions.

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$^3$ Strictly speaking, this “regular” means “regular almost everywhere”. It will always be the case henceforth in this paper.

$^4$ This emergence of AdS$_3$ is the same one as has already been observed in some special systems, in [20],[24],[27] above and also in [34]. In each of them, the near horizon limit is taken first, and after that the zero entropy limit is taken. On the other hand, a similar but inverse-ordered limit is investigated for some black holes in [35] and [36],[37], for example. In those cases, an AdS$_3$ structure does also emerge, but in a slightly different manner.
2 Four Dimensions

First we investigate the 4D extremal black holes. In §2.2 we will derive some conditions on the behavior of the near horizon geometry in the zero entropy limit, based on the expectation that the geometry should remain regular. In §2.3 we will show that those conditions guarantee that an AdS$_3$ structure always emerges as a covering space in the limit.

Note again that, we do not consider the cases of massless or small black holes, where the near horizon AdS$_2$ structure collapses. For example, Kerr and Kerr-Newman black holes are excluded, since they become inevitably massless when the entropy goes to zero. We will give one of the simplest examples in 4D at the beginning of §4.

2.1 Near horizon geometry

We consider the near horizon geometry of the extremal and axial symmetric black holes. We assume the near horizon metric has the form as follows:

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu = A(\theta)^2 \left[ -r^2 dt^2 + \frac{dr^2}{r^2} + B(\theta)^2 (d\phi - krdt)^2 \right] + F(\theta)^2 d\theta^2,$$  

(2.1)

where $A(\theta), B(\theta), F(\theta) > 0$, $0 \leq \theta \leq \pi$ and $\phi \sim \phi + 2\pi$. This geometry has an AdS$_2$ structure with an S$^1$ fiber on it, and in turn it fibrates on an interval parameterized by $\theta$. This “standard form” of the near horizon metric is proven to be the general one for arbitrary theories with Abelian gauge fields and uncharged scalars, including higher-derivative interactions in general [31], so it is quite a general one. We will focus on the case without higher-derivative interactions for simplicity. The Bekenstein-Hawking entropy $S_{BH}$ for (2.1) is given as

$$S_{BH} = \frac{\text{Area(horizon)}}{4G_4} = \frac{\pi}{2G_4} \int d\theta A(\theta)B(\theta)F(\theta),$$  

(2.2)

where $G_4$ is the 4D Newton constant. On the other hand, the volume element $d^4V$ is

$$d^4V = \sqrt{-g} d^4x = A(\theta)^3 B(\theta) F(\theta) dt dr d\theta d\phi.$$  

(2.3)

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5 Kerr black hole can, though, be embedded in 5D as “Kerr black string”, that is, Kerr $\times$ S$^1$. It can be regarded as a fast rotating Kaluza-Klein black hole with the Kaluza-Klein electric and magnetic charges $Q = P = 0$. By increasing $Q$ and $P$ from there, we can obtain a zero entropy black hole.

6 We turned the sign of $k$ from [31][42][43], so that it agrees with that of the rotation in terms of $\phi$.

7 The functions $A(\theta), B(\theta), F(\theta)$ and the constant $k$ are not arbitrary, because we assume that the geometry satisfies the equation of motion with appropriate matters.
2.2 Regularity conditions in zero entropy limit

Now we consider the zero entropy limit, $S_{BH} \to 0$, for the geometry (2.1). We are interested in black holes, and so the geometry should remain both regular and nontrivial in this limit. For this purpose, rescalings of the coordinates are allowed in general. But along the angular directions, we look at the whole region of the geometry since we regard it as a black hole. Therefore we focus on the scale where, generically, $d\theta \sim 1, d\phi \sim 1$.

Furthermore, since (2.1) has a scaling symmetry (as a subgroup of $SL(2,\mathbb{R})$)

$$t \to t, \quad r \to \lambda r, \quad (2.4)$$

for an arbitrary constant $\lambda$, we can fix the scale of the coordinate $t$ by this transformation. So we always take $dt \sim 1$. Under these conditions above for scales, $A(\theta) \sim 1, F(\theta) \sim 1$ is obviously required to prevent the geometry from collapse. Therefore we have to take $B(\theta) \to 0$ for $S_{BH} = 0$. But at the same time, the volume element (2.3) has to remain nonzero and finite. The only way under the current conditions is to take

$$B(\theta) = \epsilon B'(\theta), \quad r = \frac{r'}{\epsilon}, \quad B'(\theta) \sim 1, \quad r' \sim 1, \quad \epsilon \to 0. \quad (2.5)$$

By using these new variables, (2.1) becomes

$$ds^2 = A(\theta)^2 \left[ \frac{dr'^2}{r'^2} + \left( B'(\theta)^2 k^2 - \frac{1}{\epsilon^2} \right) r'^2 dt'^2 - 2 B'(\theta)^2 \epsilon k r' dt d\phi + B'(\theta)^2 \epsilon^2 d\phi^2 \right] + F(\theta)^2 d\theta^2$$

$$= A(\theta)^2 \left[ \frac{dr'^2}{r'^2} + \left( B'(\theta)^2 k^2 - \frac{1}{\epsilon^2} \right) r'^2 dt'^2 - 2 B'(\theta)^2 \epsilon k r' dt d\phi \right] + F(\theta)^2 d\theta^2 + O(\epsilon^2). \quad (2.6)$$

In order that the geometry does not collapse, the $dt d\phi$ term has to remain nonzero and it implies

$$k = \frac{k'}{\epsilon}, \quad k' \sim 1. \quad (2.7)$$

However, in that case the $dt^2$ term becomes

$$\left( B'(\theta)^2 k^2 - \frac{1}{\epsilon^2} \right) r'^2 dt'^2 = \epsilon^{-2} (B'(\theta)^2 k^2 - 1) r'^2 dt'^2, \quad (2.8)$$

which generically diverges. We can escape from this divergence only when

$$B'(\theta) k' = 1 + \epsilon^2 b(\theta), \quad b(\theta) = O(1), \quad (2.9)$$

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8 In this paper, $X \sim Y$ means that $\lim X/Y$ is a nonzero finite value, while $\lim X/Y$ may be 0 for $X = O(Y)$. Although we also use $\sim$ to describe periodic identifications, they can usually be distinguished clearly from the context.
that is, \( B'(\theta) \) should go to a \( \theta \)-independent constant \( B' \equiv 1/k' \) in the \( \epsilon \to 0 \) limit. In terms of the original parameters, the zero entropy limit consistent with regularity implies

\[
B(\theta) \to 0, \quad k \to \infty, \quad \text{while} \quad B(\theta)k \to 1.
\]

(2.10)

Using (2.7) and (2.9), the metric (2.6) is finally written in the \( \epsilon \to 0 \) limit as

\[
ds^2 = A(\theta)^2 \left( \frac{dr'^2}{r^2} + 2b(\theta)r'^2dt^2 - \frac{2}{k'} r'dt'd\phi \right) + F(\theta)^2d\theta^2,
\]

(2.11)

showing the regularity manifestly. (Especially, if \( b(\theta) \to 0 \), this metric has a local AdS structure in the null selfdual orbifold form [44].) Then we can conclude that (2.10) is the general condition for the near horizon geometry (2.1) to have vanishing entropy while keeping itself regular.

2.3 AdS\(_3\) emergence

Let us return to the original metric (2.1), and define a new coordinate \( \phi' \) as

\[
\phi' = \frac{\phi}{k},
\]

(2.12)

whose periodicity is given by

\[
\phi' \sim \phi' + \frac{2\pi}{k}.
\]

(2.13)

Using this coordinate, the metric (2.1) is written as

\[
ds^2 = A(\theta)^2 \left[ -r'^2dt'^2 + \frac{dr'^2}{r'^2} + B(\theta)k^2(\phi' - rdt)^2 \right] + F(\theta)^2d\theta^2.
\]

(2.14)

Under the limit (2.10), while formally regarding \( dr \sim 1 \) and \( d\phi' \sim 1 \), the metric becomes

\[
ds^2 = A(\theta)^2 \left[ -r'^2dt'^2 + \frac{dr'^2}{r'^2} + (\phi' - rdt)^2 \right] + F(\theta)^2d\theta^2,
\]

(2.15)

where we find an (not warped or squashed) AdS\(_3\) structure, fibrated on the \( \theta \) direction.

Note that, however, to be really an AdS\(_3\), the coordinate \( \phi' \) has to run from \(-\infty\) to \(\infty\), while the period of \( \phi' \) is \( 2\pi/k \to 0 \) here. This makes (2.15) singular, and so the precise meaning of the form (2.15) is quite subtle and we leave it for future investigation. In the present stage, we interpret our “AdS\(_3\)” above to emerge as a covering space of the original geometry. In other words, the AdS\(_3\) is orbifolded by an infinitesimally narrow period.

\footnote{This \( \phi' \) is essentially similar one to the \( y \) in [20]. Our \( k \) corresponds to \( \frac{1}{2\pi T_Q} \) there.}
This orbifolding may be regarded as a zero temperature limit of BTZ black hole, as was adopted in [20].

The AdS$_3$ structure can also be obtained directly from the zero entropy regular geometry (2.11). When we take infinitesimal $r'$, the $dt^2$ term turns to be subleading in $r'$ expansion, by considering the diagonalization or eigenequation for the metric. Therefore the metric becomes

$$ ds^2 \approx A(\theta)^2 \left( \frac{dr'^2}{r'^2} - \frac{2}{k'} r' dt d\phi \right) + F(\theta)^2 d\theta^2, $$

(2.16)
in the first order of $r'$. On the other hand, if we use $\phi$ again and regard $d\phi \sim 1$ in (2.15), the metric is rewritten as

$$ ds^2 = A(\theta)^2 \left( \frac{dr'^2}{r^2} - \frac{2\epsilon}{k^'} r' dt d\phi + \frac{\epsilon^2}{k^2} d\phi^2 \right) + F(\theta)^2 d\theta^2. $$

(2.17)

Similarly to the discussion above, the $d\phi^2$ term proves to be subleading, and so this metric also becomes (2.16) in the first order of $\epsilon$, by using $r' = \epsilon r$. It means that the zero entropy regular geometry (2.11) itself has the infinitesimally orbifolded AdS$_3$ structure in the infinitesimal $r'$ region.

3 Five Dimensions

After the 4D case in §2 in this section we will examine the 5D case. Although it is more complicated than 4D case because of the existence of two rotational directions, we successfully show a similar theorem to 4D case, warranting the emergence of an AdS$_3$ structure.

3.1 Near horizon geometry

Let us consider 5D extremal black holes with two axial symmetries. We assume the metric has the form of [31]

$$ ds^2 = g_{\mu\nu} dx^\mu dx^\nu = A(\theta)^2 \left[ -r^2 dt^2 + \frac{dr^2}{r^2} + B(\theta)^2 (d\phi_1 - k_1 r dt)^2 
+ C(\theta)^2 (d\phi_2 - k_2 r dt + D(\theta)(d\phi_1 - k_1 r dt))^2 \right] + F(\theta)^2 d\theta^2, $$

(3.18)

where $A(\theta), B(\theta), C(\theta), F(\theta) > 0$, $k_1, k_2 \geq 0$, $0 \leq \theta \leq \pi$ and

$$ \phi_1 \sim \phi_1 + 2\pi, \quad \phi_2 \sim \phi_2 + 2\pi. $$

(3.19)

$^{10}$ We can also obtain the same form as (2.16), by defining $t' \equiv \epsilon t$ for (2.17).
This is proven to be the general form under the same condition as the 4D case. The Bekenstein-Hawking entropy $S_{BH}$ for (3.18) is given as

$$S_{BH} = \frac{\text{Area(horizon)}}{4G_5} = \frac{\pi^2}{G_5} \int d\theta A(\theta)^2 B(\theta) C(\theta) F(\theta),$$

(3.20)

where $G_5$ is the 5D Newton constant. The volume element $d^5V$ is

$$d^5V = \sqrt{-g} \, d^5x = A(\theta)^4 B(\theta) C(\theta) F(\theta) \, dt \, d\theta \, d\phi_1 \, d\phi_2.$$

(3.21)

**Modular transformations** Since $\phi_1$ and $\phi_2$ forms a torus $T^2$ in the coordinate space, we can act the modular transformation group $\text{SL}(2, \mathbb{Z})$ while keeping the metric to be the form of (3.18) and the periodicities (3.19). If necessary for keeping $k_1, k_2 > 0$, we redefine the signs of $\phi_1$ and $\phi_2$ at the same time. The generators $S$ and $T$ of this $\text{SL}(2, \mathbb{Z})$ are defined by

$$S : (\tilde{\phi}_1, \tilde{\phi}_2) = (-\phi_2, \phi_1),$$

(3.22)

$$T : (\tilde{\phi}_1, \tilde{\phi}_2) = (\phi_1, \phi_1 + \phi_2),$$

(3.23)

respectively, although $S$ always acts together with the redefinition of the sign of $\tilde{\phi}_1$, behaving as a mere swapping

$$S' : (\tilde{\phi}_1, \tilde{\phi}_2) = (\phi_2, \phi_1).$$

(3.24)

Under $S'$ and $T$, the functions and parameters in (3.18) are transformed as,

$$S' : \quad \tilde{B}(\theta)^2 = \frac{B(\theta)^2 C(\theta)^2}{B(\theta)^2 + C(\theta)^2 D(\theta)^2}, \quad \tilde{C}(\theta)^2 = B(\theta)^2 + C(\theta)^2 D(\theta)^2,$$

$$\tilde{D}(\theta) = \frac{C(\theta)^2 D(\theta)}{B(\theta)^2 + C(\theta)^2 D(\theta)^2}, \quad \tilde{k}_1 = k_2, \quad \tilde{k}_2 = k_1,$$

(3.25)

$$T : \quad \tilde{B}(\theta)^2 = B(\theta)^2, \quad \tilde{C}(\theta)^2 = C(\theta)^2,$$

$$\tilde{D}(\theta) = D(\theta) - 1, \quad \tilde{k}_1 = k_1, \quad \tilde{k}_2 = k_2 + k_1.$$  

(3.26)

### 3.2 Regularity conditions in zero entropy limit

Now we consider the regularity conditions in $S_{BH} \to 0$ limit for (3.20). Here we explain the outline and the results. The details are given in Appendix A.

From exactly a similar discussion to that in §2.2 we take

$$A(\theta) \sim 1, \quad F(\theta) \sim 1, \quad d\theta \sim 1, \quad d\phi_1 \sim 1, \quad d\phi_2 \sim 1, \quad dt \sim 1,$$

(3.27)
and then it is required that
\[ B(\theta)C(\theta) \sim \epsilon, \quad r = \frac{r'}{\epsilon}, \quad r' \sim 1, \quad \epsilon \to 0. \] (3.28)

Obviously, divergence of \( B(\theta) \) or \( C(\theta) \) causes the \( d\phi_2^2 \) or \( d\phi_2^2 \) term of the metric (3.18) to diverge, and so it is not allowed. Therefore, to realize (3.28), there are three possibilities for the behaviors of \( B(\theta) \) and \( C(\theta) \), depending on ether (or both) of them goes to 0 in the limit. However, by using the swapping transformation \( S' \) given in (3.24)(3.25), we can show that all the cases are resulted in the case of
\[ B(\theta) \sim \epsilon, \quad C(\theta) \sim 1. \] (3.29)

In this case, similarly to the 4D result (2.10),
\[ B(\theta)k_1 = 1 + \mathcal{O}(\epsilon^2) \] (3.30)
has to be satisfied. In other words,
\[ B(\theta) = \epsilon B'(\theta), \quad k_1 = \frac{k'_1}{\epsilon}, \quad k'_1B'(\theta) = 1 + \epsilon^2 b(\theta), \quad B'(\theta) \sim 1, \quad k'_1 \sim 1, \quad b(\theta) = \mathcal{O}(1). \] (3.31)

Finally, for remaining parameters \( k_2 \) and \( D(\theta) \), the condition proves to be \( D(\theta) = \mathcal{O}(1) \) and \( k_2 + D(\theta)k_1 = \mathcal{O}(\epsilon) \). This is satisfied if and only if
\[ k_2 = \frac{k'_2}{\epsilon}, \quad D(\theta) = -\frac{k'_2}{k'_1} + \epsilon^2 d(\theta), \quad k'_2 = \mathcal{O}(1), \quad d(\theta) = \mathcal{O}(1). \] (3.32)

Then \( D(\theta) \) goes to a constant \( D \equiv -k'_2/k'_1 = -k_2/k_1 \), which may or may not be 0.

Under (3.28) (3.29) (3.31) (3.32), the metric (3.18) becomes
\[ ds^2 = A(\theta)^2 \left[ \left( 2b(\theta) + k'_1 C(\theta)^2 d(\theta)^2 \right) r'^2 dt^2 + \frac{dr'^2}{r'^2} 
- 2 \left( \frac{1}{k'_1} + C(\theta)^2 d(\theta)D \right) r' dt d\phi_1 - 2C(\theta)^2 d(\theta)r' dt d\phi_2 
+ C(\theta)^2 D^2 d\phi_1^2 + 2C(\theta)^2 Dd\phi_1 d\phi_2 + C(\theta)^2 d\phi_2^2 \right] + F(\theta)^2 d\theta^2, \] (3.33)

which is indeed regular. It is simply rewritten by defining
\[ \phi'_2 = \phi_2 + D\phi_1, \] (3.34)
as
\[ ds^2 = A(\theta)^2 \left[ \left( 2b(\theta) + k'_1 C(\theta)^2 d(\theta)^2 \right) r'^2 dt^2 + \frac{dr'^2}{r'^2} 
- 2 \frac{k'_1}{k'_1} r' dt d\phi_1 - 2C(\theta)^2 d(\theta)r' dt d\phi'_2 + C(\theta)^2 d\phi'_2^2 \right] + F(\theta)^2 d\theta^2. \] (3.35)
### 3.3 AdS$_3$ emergence

Now let us look at the metric (3.18), under the same limit (3.29) (3.31) (3.32), for the parameters as (3.35), but different scalings for the coordinates. Using the original coordinates, the metric is written as

$$ds^2 = A(\theta)^2 \left[ -r^2 dt^2 + \frac{dr^2}{r^2} + \left( \frac{d\phi_1}{k_1} - r dt + \frac{e^3 b(\theta)}{k_1} d\phi_1 - e^2 b(\theta) r dt \right)^2 \right. + C(\theta)^2 \left( d\phi_2 + D d\phi_1 + \epsilon^2 d(\theta) d\phi_1 - e k_1' d(\theta) r dt \right)^2 + F(\theta)^2 d\theta^2. \quad (3.36)$$

Now we switch the coordinates from $(\phi_1, \phi_2)$ to $(\phi'_1, \phi'_2)$, where $\phi'_2$ is (3.34) and $\phi'_1$ is defined as

$$\phi'_1 = \frac{\phi_1}{k_1}. \quad (3.37)$$

We regard $d\phi'_1 \sim 1$, $d\phi'_2 \sim 1$ and $dr \sim 1$, together with $dt \sim 1$, $d\theta \sim 1$. Then in the $\epsilon \to 0$ limit, (3.36) goes to

$$ds^2 = A(\theta)^2 \left[ -r^2 dt^2 + \frac{dr^2}{r^2} + \left( d\phi'_1 - r dt \right)^2 + C(\theta)^2 d\phi'^2_2 \right] + F(\theta)^2 d\theta^2. \quad (3.38)$$

Manifestly, (3.38) has locally a form of a product of AdS$_3$ and S$^1$, fibered on the $\theta$-interval $11$

Of course, the periodicities of $\phi'_1$ and $\phi'_2$ are problematic. As for $\phi'_1$ in the AdS$_3$, the situation is exactly the similar to the 4D case explained in §2.3. What about the S$^1$ coordinate $\phi'_2$? When $D$ is an integer, there is no problem. In fact, in this case we could use the modular transformation $T$ (3.23) (3.26) (or $T^{-1}$) repeatedly, to make $D = 0$ in advance. On the other hand, in case $D$ is not an integer, it may cause some new problem, perhaps depending on whether $D \in \mathbb{Q}$ or not. For, when we take a covering space over $\phi_1$ and have a full AdS$_3$, the $D \phi_1$ term will do nothing on the S$^1_{\phi'_2}$, regardless of the value of $D$.

In a similar way to the 4D case, the AdS$_3$ structure can be obtained from (3.35). When we take infinitesimal $r'$, the metric becomes

$$ds^2 = A(\theta)^2 \left( \frac{dr'^2}{r'^2} - \frac{2}{k_1} r' dt d\phi_1 + C(\theta)^2 d\phi'^2_2 \right) + F(\theta)^2 d\theta^2, \quad (3.39)$$

in the leading order of $r'$, and this coincides with that of (3.38).

$^{11}$ At the same time, we notice that the rotation along the $\phi'_2$ direction vanishes here and then the geometry is static. Therefore, we can say that the near horizon geometry in the zero entropy limit always results in the static AdS$_3$ case classified by §31, though the periodicity goes to zero here.
4 Examples

In the previous sections, we systematically argued the form of the near horizon geometry and the emergence of the AdS$_3$ structure in the zero entropy limit. In this section, we consider some interesting examples of extremal black holes to demonstrate our discussions above. We see that the regularity conditions we derived are indeed realized in each case, and an AdS$_3$ emerges as a result of it.

We deal with a class of 5D vacuum extremal black holes and the ones in the 5D supergravity. The former includes the extremal Myers-Perry black hole and the extremal slow rotating Kaluza-Klein black hole as we will explain, while the latter includes the setup discussed in [20].

Because the examples below are complicated, we give one of the simplest examples here. Let us consider the 4D extremal slow rotating dyonic black hole in Einstein-Maxwell-dilaton theory [46]. In the near horizon limit, the geometry is written as

$$ds^2 = \frac{2G_4J(u^2-1)\sin^2\theta}{\sqrt{u^2 - \cos^2\theta}} \left( d\phi - \frac{rdt}{\sqrt{u^2 - 1}} \right)^2 + 2G_4J\sqrt{u^2 - \cos^2\theta} \left( -r^2 dt^2 + \frac{dr^2}{r^2} + d\theta^2 \right) + 2G_4J \sin \theta d\theta^2,$$

(4.40)

and the entropy is expressed as $S_{BH} = 2\pi J\sqrt{u^2 - 1}$. Here $\tilde{Q}$, $\tilde{P}$, $J$ are the electric charge, the magnetic charge and the angular momentum respectively, and $u = \tilde{P}\tilde{Q}/G_4J$. The dilaton and the gauge field are regular. For the concrete expression of them, see [46]. Let us define $\phi' = \sqrt{u^2 - 1}\phi$. Then, by taking a zero entropy limit $u \to 1$ with $J$ and $\phi'$ fixed to order one, the geometry turns out to be

$$ds^2 = 2G_4J \sin \theta \left( -r^2 dt^2 + \frac{dr^2}{r^2} + (d\phi' - rdt)^2 \right) + 2G_4J \sin \theta d\theta^2.$$

(4.41)

This form is exactly the one we found in the previous section and we can see the AdS$_3$ structure in the first term. Notice that, by U-duality, this black hole is related to a broad class of extremal black holes appearing in string theory. Therefore, we can say AdS$_3$ structure emerges in the zero entropy limit of them, too.

Extremally rotating NS5-brane is another example and we can also see the emergence of AdS$_3$ structure in the zero entropy limit [34].

4.1 Vacuum 5D black holes

For the purpose above, we consider 5D pure Einstein gravity with zero cosmological constant and then analyze vacuum 5D extremal black holes with two U(1) symmetries.
After Kaluza-Klein reduction along a $S^1$ fiber and switching to the Einstein frame, these black holes reduce to the extremal black holes in 4D Einstein-Maxwell-dilaton gravity discussed above. Here we concentrate on the cases with zero cosmological constant for simplicity, but generalization to the case with cosmological constant is straightforward.

When $SL(2, \mathbb{R}) \times U(1)^2$ symmetry is assumed, the explicit form of the near horizon geometry of these black holes is classified by [42]. As for slow rotating black holes whose angular momenta are bounded from above and the topology of horizon is $S^3$, it is written as

$$ds^2 = \frac{\Gamma(\sigma)}{c_0^2} \left(-r^2 dt^2 + \frac{dr^2}{r^2}\right) + \frac{\Gamma(\sigma)}{Q(\sigma)} d\sigma^2 + \gamma_{ij} \left(dx^i - \frac{\bar{k}_x^i}{c_0^2} r dt\right) \left(dx^j - \frac{\bar{k}_x^j}{c_0^2} r dt\right), \quad (4.42)$$

where

$$Q(\sigma) = -c_0^2 \sigma^2 + c_1 \sigma + c_2, \quad (4.43)$$

$$\gamma_{ij} dx^i dx^j = \frac{P(\sigma)}{\Gamma(\sigma)} \left(dx^1 + \frac{\sqrt{-c_1 c_2}}{c_0 P(\sigma)} dx^2\right)^2 + \frac{Q(\sigma)}{P(\sigma)} (dx^2)^2, \quad (4.44)$$

$$P(\sigma) = c_0^2 \sigma^2 - c_2, \quad (4.45)$$

$$\Gamma = \sigma, \quad (4.46)$$

and $\bar{k}_x^1 = 1$, $\bar{k}_x^2 = 0$. Parameters satisfy $c_1 > 0$, $c_2 < 0$ and $c_0 > 0$ and $\sigma$ takes $\sigma_1 < \sigma < \sigma_2$, where $\sigma_1$ and $\sigma_2$ are roots of $Q(\sigma) = 0$. Explicit forms of these quantities are written as

$$\sigma_1 = \frac{1}{2 c_0^2} \left(1 - \sqrt{1 + \frac{4 c_0 c_1^2}{c_1^2}}\right), \quad \sigma_2 = \frac{1}{2 c_0^2} \left(1 + \sqrt{1 + \frac{4 c_0 c_1^2}{c_1^2}}\right). \quad (4.47)$$

For example, Myers-Perry black hole and the slow rotating Kaluza-Klein black hole have this near horizon geometry in the extreme as we will explain. As for the detailed relation between parameters, see Appendix [B].

In order to make the regularity of the geometry manifest, we apply coordinate transformation so that new coordinates $\phi_1, \phi_2$ have periodicity $2\pi$:

$$x^1 = \frac{2 \sqrt{-c_2}}{c_0^2 (\sigma_2 - \sigma_1)} (\phi_1 - \phi_2), \quad x^2 = -\frac{2 \sqrt{c_1}}{c_0^2 (\sigma_2 - \sigma_1)} (\sigma_1 \phi_1 - \sigma_2 \phi_2). \quad (4.48)$$

Then the corresponding parameter $\bar{k}_{\phi_1}, \bar{k}_{\phi_2}$ are determined by a relation $\bar{k} = \bar{k}_x^1 \partial_{x^1} + \bar{k}_x^2 \partial_{x^2} = \bar{k}_{\phi_1} \partial_{\phi_1} + \bar{k}_{\phi_2} \partial_{\phi_2}$, and the explicit forms are

$$\bar{k}_{\phi_1} = \frac{c_0^3}{2 \sqrt{-c_2}} \sigma_2, \quad \bar{k}_{\phi_2} = \frac{c_0^3}{2 \sqrt{-c_2}} \sigma_1. \quad (4.49)$$
By using the new coordinate, $\gamma_{ij}$ is written as

$$\gamma_{ij} dx^i dx^j = f(\sigma)(d\phi_1)^2 + 2g(\sigma) d\phi_1 d\phi_2 + h(\sigma)(d\phi_2)^2, \quad (4.50)$$

where

$$f(\sigma) = \frac{-4c_2}{c_0^2(\sigma_2 - \sigma_1)^2} P \left(1 - \frac{c_1\sigma_1}{P}\right)^2 + \frac{Q}{P} \frac{4c_1}{c_0^2(\sigma_2 - \sigma_1)^2}\sigma_1^2, \quad (4.51)$$

$$g(\sigma) = \frac{4c_2}{c_0^2(\sigma_2 - \sigma_1)^2} P \left(1 - \frac{c_1\sigma_1}{P}\right) \left(1 - \frac{c_1\sigma_2}{P}\right) - \frac{Q}{P} \frac{4c_1}{c_0^2(\sigma_2 - \sigma_1)^2}\sigma_1\sigma_2, \quad (4.52)$$

$$h(\sigma) = \frac{-4c_2}{c_0^2(\sigma_2 - \sigma_1)^2} P \left(1 - \frac{c_1\sigma_2}{P}\right)^2 + \frac{Q}{P} \frac{4c_1}{c_0^2(\sigma_2 - \sigma_1)^2}\sigma_2^2. \quad (4.53)$$

Then the total metric is

$$ds^2 = \frac{\sigma}{c_0^2} \left[-r^2 dt^2 + \frac{dr^2}{r^2} + \frac{c_0^2}{Q} f h - g^2}{h} (d\phi_1 - k_{\phi_1} r dt)^2$$

$$+ \frac{c_0^2 h}{\sigma} \left(d\phi_2 + \frac{g}{h} d\phi_1 - \left(k_{\phi_2} + \frac{g}{h} k_{\phi_1}\right) r dt\right)^2 \right], \quad (4.54)$$

where

$$k_{\phi_1} = \frac{k_{\phi_1}}{c_0^2} = \frac{c_0}{2\sqrt{-c_2}} \sigma_2, \quad k_{\phi_2} = \frac{k_{\phi_2}}{c_0^2} = \frac{c_0}{2\sqrt{-c_2}} \sigma_1. \quad (4.55)$$

This is the “standard form” introduced in the previous section under the identification

$$A(\theta)^2 = \frac{\sigma}{c_0^2}, \quad B(\theta)^2 = \frac{c_0^2 f h - g^2}{h}, \quad (4.56)$$

$$C(\theta)^2 = \frac{c_0^2 h}{\sigma}, \quad D(\theta)^2 = \frac{g}{h}, \quad F(\theta)^2 d\theta^2 = \frac{d\sigma^2}{Q(\sigma)}. \quad (4.57)$$

As we explained the near horizon geometry, we consider the zero entropy limit of it. For this purpose, let us write down the expression of the entropy and Frolov-Thorne temperatures corresponding to $\phi_1$-cycle and $\phi_2$-cycle:

$$S_{BH} = \frac{4\pi^2}{4G_5} \int_{\sigma_1}^{\sigma_2} d\sigma \sqrt{\frac{\sigma(h f - g^2)}{Q}}, \quad (4.58)$$

$$T_{\phi_1} = \frac{1}{2\pi k_{\phi_1}}, \quad T_{\phi_2} = \frac{1}{2\pi k_{\phi_2}}. \quad (4.59)$$

According to the discussion of previous section, zero entropy limit corresponds to $hf - g^2 = 0$. As a nontrivial example, we consider $c_2 \to 0$. This corresponds to the zero
entropy limit of the extremal Myers-Perry black hole as can be seen by using the list of identification in Appendix B. By expanding with respect to $c^2$, we have

$$\sigma_1 = -\frac{c_2}{c_1} + \mathcal{O}(c_2^2), \quad \sigma_2 = \frac{c_1}{c_0^2} + \frac{c_2}{c_1} + \mathcal{O}(c_2^2) ,$$

(4.60)

and then

$$k_{\phi_1} = \frac{c_1}{2c_0\sqrt{-c_2}} - \frac{c\sqrt{-c_2}}{2c_1} + \mathcal{O}\left((-c_2)^{3/2}\right), \quad k_{\phi_2} = \frac{c_0\sqrt{-c_2}}{2c_1} + \mathcal{O}\left((-c_2)^{3/2}\right).$$

(4.61)

Moreover, since

$$f(\sigma) = -\frac{4\sigma}{c_1^2}c_2 + \frac{4(c^2\sigma - c_1)^2}{c_1^2} + \mathcal{O}(c_2^3) ,$$

(4.62)

$$g(\sigma) = \frac{4(c_0^2\sigma - c_1)}{c_0^2c_1^2}c_2 + \mathcal{O}(c_2^2) ,$$

(4.63)

$$h(\sigma) = -\frac{4c_1(c_0^2\sigma - c_1)}{c_0^3}\sigma + \mathcal{O}(c_2) ,$$

(4.64)

we have

$$\frac{fh - g^2}{h} = -\frac{4\sigma}{c_1^2}c_2 + \frac{4(c_0^4\sigma^2 + 3c_0^2c_1\sigma - c_1^2)c_2^2}{c_1^2} + \mathcal{O}(c_2^3) ,$$

(4.65)

$$\frac{g}{h} = -\frac{c_0^4\sigma}{c_1^2}c_2 + \mathcal{O}(c_2^2) .$$

(4.66)

Now from (4.56) (4.57) (4.60) (4.61) (4.64) (4.65) (4.66), it is easily shown that all of the regularity conditions (3.29) (3.31) (3.32) are indeed satisfied in the current limit, with $\epsilon \sim \sqrt{-c_2}$. In order to see the regularity manifestly, we rescale the radial coordinate $r = k_{\phi_1}r'$, rewrite the metric by using $r'$ and then take $c_2 \to 0$ limit. Explicitly, the metric turns out to be a regular form

$$ds^2 = \frac{c_0^4\sigma^2 + c_1c_0^2\sigma - c_1^2}{4c_1^2c_0}dr^2 + \frac{\sigma}{c_0^2} \left(-2r'dtd\phi_1 + \frac{dr'^2}{r'^2}\right)$$

$$\quad + \frac{1}{(c_1 - c_0^2\sigma)}d\sigma^2 + \frac{4c_1(c_1 - c_0^2\sigma)}{c_0^3\sigma} \left(d\phi_2 - \frac{c_1^2}{4c_1}\sigma r'dt\right)^2 .$$

(4.67)

Here, due to some annoying terms, AdS$_3$ factor does not appear but the regularity is manifest. The similar situation occurs in the setup of [20] when the radial coordinate is rescaled as above.

Let us next return to the coordinate (4.54) and introduce $\psi = \phi_1, \phi = \phi_2$ corresponding to the angular variables of Myers-Perry black hole (see Appendix B). By regarding $d\psi' =
$d\psi/k_{\phi_1}$ and $d\phi$ as order one quantities and taking $c_2 \to 0$, we obtain the metric in the zero entropy limit as

$$ds^2 = \frac{\sigma}{c_0^2} \left( -r^2 dt^2 + \frac{dr^2}{r^2} + (d\psi' - r dt)^2 \right) + \frac{1}{(c_1 - c_0^2 \sigma)} d\sigma^2 + \frac{4c_1 (c_1 - c_0^2 \sigma)}{c_0^2 \sigma} d\phi^2. \quad (4.68)$$

Here $0 \leq \sigma \leq c_1/c_0^2$, $\phi \sim \phi + 2\pi$ and $\psi'$-cycle shrink to zero, as explained in the previous section. Therefore, up to the $\sigma$-dependent overall factor, the first term is AdS$_3$ with vanishing periodicity in $\psi'$ direction.

As for the zero entropy limit, there is another possibility corresponding to the extremal slow rotating Kaluza-Klein black hole. We first introduce $L$ as a sufficiently large quantity and consider a limit

$$c_0^2 = c_0^2 L, \quad c_1 = c_1' L^2, \quad c_2 = c_2' L \quad (L \to \infty) \quad (4.69)$$

where quantities with prime are order one. In this case, up to leading order in $1/L$ expansion,

$$\sigma_1 = -\frac{c_2'}{c_1} L^{-1}, \quad \sigma_2 = \frac{c_1'}{2c_0^2 L}. \quad (4.70)$$

and

$$k_{\phi_1} = \frac{c_1'}{4c_0' \sqrt{-c_2'}} L, \quad k_{\phi_2} = \frac{\sqrt{-c_2' c_0'}}{2c_1'} L^{-1}. \quad (4.71)$$

By introducing $\sigma' = \sigma/L$, we also have

$$f = -\frac{16c_0'}{c_1'^2} \sigma L^{-2}, \quad g = \frac{8c_1' (c_0'^2 \sigma' - c_1')}{c_1'^2 c_0^2} L^{-2}, \quad h = -\frac{4c_1' (c_0'^2 \sigma' - c_1')}{c_0'^6 \sigma'}. \quad (4.72)$$

in the leading order. Therefore, also in this limit, we can make sure that $(4.56)$ $(4.57)$ $(4.71)$ $(4.72)$ satisfy $(3.29)$ $(3.31)$ $(3.32)$, with $\epsilon \sim L^{-1}$.

Let us next introduce a new coordinate

$$\phi = \phi_1 - \phi_2, \quad y = 2\tilde{P}(\phi_2 + \phi_1), \quad (4.73)$$

corresponding to the angular variables of the extremal slow rotating Kaluza-Klein black hole, and set $\phi \sim \phi + 2\pi$ and $y \sim y + 8\pi \tilde{P}$, before taking the zero entropy limit. Here $\tilde{P}$ is the magnetic charge of the Kaluza-Klein black hole. Detailed relations to the extremal slow rotating Kaluza-Klein black hole are summarized in Appendix B.
Then by regarding \( d\phi' = d\phi/k_\phi = d\phi/(k_{\phi_1} - k_{\phi_2}) \) and \( dy - 2\hat{P}d\phi' = 4\hat{P}d\phi' \) as order one quantities and taking the zero entropy limit as above, we have

\[
ds^2 = \frac{\sigma'}{c_0'^2} \left( -r^2 dt^2 + \frac{dr^2}{r^2} + (d\phi' - rdt)^2 \right) + \frac{1}{c_1' - c_0'^2} d\sigma'^2 + \frac{1}{16\hat{P}^2} \frac{4c_1'(c_1' - c_0'^2)}{c_0'^4} \left( dy - 2\hat{P}d\phi' \right)^2.
\]

(4.74)

Again, the period of \( \phi' \) shrinks to zero in this limit.\(^\text{12}\)

### 4.2 Black holes in 5D supergravity

Next we consider with the black holes in 5D supergravity, obtained from dimensional reduction of the rotating D1-D5-P black holes in the compactified IIB supergravity. They include the ones dealt in \[20\], where they took the zero entropy limit from the fast rotating range. Unlike \[20\], we do not require \( J_R = 0 \) and work on the purely 5D reduced theory to clarify the emergence of the AdS\(_3\), because this system always has an AdS\(_3\) structure along the uplifted Kaluza-Klein direction.

We will take the zero entropy limit from the slow rotating range. The 6D form of the near horizon metric of this system was given in \[15\] as

\[
ds_{(6)}^2 = \frac{\lambda^2}{4} \left[ -r^2 dt^2 + \frac{dr^2}{r^2} + 4r_+^2 \left( dy - \frac{r}{2r_+} dt \right)^2 + 4 \left( d\theta^2 + \sin^2 \theta \left( d\phi - \frac{2G_6}{\pi^2 \lambda^4} J_\psi dy \right)^2 + \cos^2 \theta \left( d\psi - \frac{2G_6}{\pi^2 \lambda^4} J_\phi dy \right)^2 \right) \right],
\]

(4.75)

where

\[
0 \leq \theta \leq \frac{\pi}{2}, \quad \phi \sim \phi + 2\pi, \quad \psi \sim \psi + 2\pi,
\]

(4.76)

\[
\lambda^4 = Q_1Q_5,
\]

(4.77)

\[
r_+ = \frac{G_6}{\pi^2 \lambda^4} S_{BH},
\]

(4.78)

\[
S_{BH} = 2\pi \sqrt{\left( \frac{\pi^2 R}{2G_6} \right)^2 Q_1Q_5Q_p - J_\phi J_\psi},
\]

(4.79)

\[
Q_1, \ Q_5, \ Q_p > 0, \quad J_\phi, \ J_\psi > 0.
\]

(4.80)

\(^\text{12}\) In this case in the zero entropy limit, actually we encounter another singularity than the one due to the shrink of S\(^1\)-cycle in the AdS\(_3\) factor. It appears at \( \sigma = 0 \) for (4.68) and at \( \sigma' = 0 \) for (4.74).

\(^\text{13}\) The current authors called it “very near horizon” geometry there, to distinguish it from the AdS\(_3\) decoupling limit. We redefined some signs from there to agree with our convention here.
Here $G_6$ is 6D Newton constant, $R$ is the Kaluza-Klein radius, $J_\phi$ and $J_\psi$ are angular momenta, $Q_1$, $Q_5$, $Q_p$ are D1, D5 and Kaluza-Klein momentum charges respectively, and $S_{BH}$ is the Bekenstein-Hawking entropy. The zero entropy limit we want to take here is characterized by cancellation of the two terms in the root in (4.79), while keeping all the charges and angular momenta to be nonzero finite together with $G_6$ and $R$. From (4.78), it immediately means the limit of

$$r_+ \to 0.$$  

(4.81)

After some short algebra, the metric (4.75) can be rewritten as

$$ds^2_{(6)} = \frac{\lambda^2}{4} \left[ -r^2 dt^2 + \frac{dr^2}{r^2} + B(\theta)^2 e_\phi^2 + \Phi(\theta)^2 \left( dy - \mathcal{A} \right)^2 \right] + \lambda^2 d\theta^2 + \Phi(\theta)^2 \left( dy - \mathcal{A} \right)^2,$$  

(4.82)

where

$$\Phi(\theta)^2 = \frac{4G_6^2(J_\phi^2 \cos^2 \theta + J_\psi^2 \sin^2 \theta)}{\pi^4 \lambda^6} + \lambda^2 r_+^2,$$  

(4.83)

$$A = k_\phi r dt + \frac{2\pi^2 G_6 \lambda^4}{4G_6^2(J_\phi^2 \cos^2 \theta + J_\psi^2 \sin^2 \theta) + \pi^4 \lambda^8 r_+^2} \left( J_\psi \sin^2 \theta e_\phi + J_\phi \cos^2 \theta e_\psi \right),$$  

(4.84)

$$B(\theta)^2 = \frac{4\pi^4 \lambda^8 r_+^2 \sin^2 \theta}{4G_6^2 J_\phi^2 \sin^2 \theta + \pi^4 \lambda^8 r_+^2},$$  

(4.85)

$$C(\theta)^2 = \frac{4(4G_6^2 J_\psi^2 \sin^2 \theta + \pi^4 \lambda^8 r_+^2) \cos^2 \theta}{4G_6^2(J_\phi^2 \cos^2 \theta + J_\psi^2 \sin^2 \theta) + \pi^4 \lambda^8 r_+^2},$$  

(4.86)

$$D(\theta) = \frac{4G_6^2 J_\phi J_\psi \sin^2 \theta}{4G_6^2 J_\psi^2 \sin^2 \theta + \pi^4 \lambda^8 r_+^2},$$  

(4.87)

$$e_\phi = d\phi - k_\phi r dt, \quad e_\psi = d\psi - k_\psi r dt,$$  

(4.88)

$$k_\phi = \frac{G_6 J_\psi}{\pi^2 \lambda^4 r_+}, \quad k_\psi = \frac{G_6 J_\phi}{\pi^2 \lambda^4 r_+}, \quad k_y = \frac{1}{2r_+}.$$  

(4.89)

Because the $y$-cycle never shrinks or blows up from (4.83), we can safely reduce the system into 5D theory, with the dilatonic field $\Phi(\theta)$, Kaluza-Klein gauge field $A$, 5D metric

$$ds^2_{(5)} = \frac{\lambda^2}{4} \left[ -r^2 dt^2 + \frac{dr^2}{r^2} + B(\theta)^2 e_\phi^2 + \Phi(\theta)^2 \left( dy - \mathcal{A} \right)^2 \right] + \lambda^2 d\theta^2,$$  

(4.90)

and the Newton constant being $G_5 = G_6/2\pi R$. Note that the gauge field $\mathcal{A}$ (4.84) is finite in this limit, although it is obvious from the finiteness of (4.75). Manifestly (4.90) has the very form of (3.18), and in the limit (4.81), the behaviors of $B(\theta), C(\theta), D(\theta)$
and \( k_1, k_2 \) in (4.85) (4.86) (4.87) (4.89) do satisfy the conditions (3.29) (3.31) (3.32), under an identification \( \epsilon = r_+ \). In fact, in the limit the 5D metric goes to just the form of (3.38), with

\[
A(\theta) = \frac{\lambda}{2}, \quad F(\theta) = \lambda,
\]

\[
C(\theta)^2 = \frac{4J_\psi^2 \sin^2 \theta \cos^2 \theta}{J_\phi^2 \cos^2 \theta + J_\psi^2 \sin^2 \theta},
\]

\[
\phi'_1 = \frac{\phi}{k_\phi}, \quad \phi'_2 = \psi + D\phi, \quad k_\phi = \frac{-2RG_5 J_\psi}{\pi \lambda^4 \epsilon}, \quad D = -\frac{J_\phi}{J_\psi}.
\]

This is a very special case of emergent AdS\(_3\), in that \( A(\theta) \) is constant and so the AdS\(_3\) is not fibered\(^{14}\). Furthermore, we see from (4.93) that, if \( J_\phi \) is \( J_\psi \) times an integer, \( \phi'_2 \) has a proper periodicity as we discussed in §3.3. In that case the spacetime is a direct product of an (infinitesimally orbifolded) AdS\(_3\) and a squashed S\(_2\). Finally, the most special case is \( J_\phi = J_\psi \), that is, zero-entropy BMPV. In this case \( C(\theta) = \sin 2\theta \) and so we obtain a non-squashed AdS\(_3\) × S\(_2\), as was seen in [20,24].

5 Implications to the Kerr/CFT

In this section we shortly consider the correspondence between the central charges of the Kerr/CFT and AdS\(_3\)/CFT\(_2\) in the zero entropy limit. For simplicity we work on the 4D case here, but 5D case is almost the same.

5.1 Kerr/CFT in the zero entropy limit

Let us consider again the 4D metric (2.1),

\[
ds^2 = A(\theta)^2 \left[ -r^2 dt^2 + \frac{dr^2}{r^2} + B(\theta)^2 (d\phi - k\sqrt{r^2} dt)^2 \right] + F(\theta)^2 d\theta^2.
\]

First of all, for clarification of the discussions below, we transform this metric from Poincaré form to the global form,

\[
ds^2 = A(\theta)^2 \left[ -(1 + \tilde{r}^2) d\tilde{t}^2 + \frac{d\tilde{r}^2}{1 + \tilde{r}^2} + B(\theta)^2 (d\tilde{\phi} - k\tilde{r} d\tilde{t})^2 \right] + F(\theta)^2 d\theta^2.
\]

\(^{14}\) The metric is written in the string frame here. In the Einstein frame, \( A(\theta) \) has a \( \theta \)-dependence unless \( J_\phi = J_\psi \).
This coordinates transformation \((t, r, \phi) \rightarrow (\tilde{t}, \tilde{r}, \tilde{\phi})\) can be carried out without any change on the forms and values of \(A(\theta), B(\theta), F(\theta)\) and \(k\). For this geometry, the usual Kerr/CFT procedure with the chiral Virasoro generators

\[
\xi_n = -inre^{-in\tilde{\phi}}\partial_{\tilde{r}} - e^{-in\tilde{\phi}}\partial_{\tilde{\phi}},
\]

(5.96) gives the central charge

\[
c_{\text{Kerr/CFT}} = \frac{3k}{G_4} \int d\theta A(\theta)B(\theta)F(\theta).
\]

(5.97)

The corresponding Frolov-Thorne temperature is

\[
T_{FT} = \frac{1}{2\pi k},
\]

(5.98) as usual. Therefore the entropy is

\[
S_{\text{Kerr/CFT}} = \frac{\pi^2}{3} c_{\text{Kerr/CFT}} T_{FT} = \frac{\pi}{2G_4} \int d\theta A(\theta)B(\theta)F(\theta),
\]

(5.99) which of course agrees with \(S_{BH} (2.2)\). In particular, near the zero entropy limit (2.10), the central charge (5.97) is expanded as

\[
c_{\text{Kerr/CFT}} = \frac{3}{G_4} \int d\theta A(\theta)F(\theta) + \mathcal{O}\left(\frac{1}{k^2}\right),
\]

(5.100) and the entropy (5.99) becomes

\[
S_{\text{Kerr/CFT}} = \frac{\pi}{2kG_4} \int d\theta A(\theta)F(\theta) + \mathcal{O}\left(\frac{1}{k^3}\right).
\]

(5.101)

### 5.2 AdS\(_3\)/CFT\(_2\) in the emergent AdS\(_3\)

In the last subsection, we examined the Kerr/CFT near the zero entropy limit. In this limit, the metric (5.95) itself becomes

\[
ds^2 = A(\theta)^2 \left[ -(1 + \tilde{r}^2) d\tilde{t}^2 + \frac{d\tilde{r}^2}{1 + \tilde{r}^2} + (d\tilde{\phi}' - \tilde{r} d\tilde{t})^2 \right] + F(\theta)^2 d\theta^2,
\]

(5.102)

\[
\tilde{\phi}' = \frac{\tilde{\phi}}{k}, \quad \tilde{\phi}' \sim \tilde{\phi}' + \frac{2\pi}{k}, \quad (k \rightarrow \infty)
\]

(5.103) in exactly the same way as (2.15). Since an AdS\(_3\) structure is included, it is expected that this geometry has a non-chiral dual theory and the Kerr/CFT above is a chiral part of it.\(^{15}\) We will partly demonstrate it below.

\(^{15}\) Chiral CFT\(_2\) and an AdS\(_3\) structure were discussed in \cite{16}.
First we stress again that the AdS$_3$ is orbifolded by an infinitesimally narrow period (5.103). In this form of the metric, if we change the coordinates range as
\[ \tilde{t} \sim \tilde{t} + 4\pi, \quad -\infty < \tilde{r} < \infty, \quad -\infty < \tilde{\phi} < \infty, \] (5.104)
they cover the whole AdS$_3$ as a hyperbolic hypersurface in $\mathbb{R}^{2,2}$\[16\]. Even under the replaced periodicity of $\tilde{\phi}$ in (5.104), the results for the central charge of the Kerr/CFT (5.97) (5.100) remain true. Generally speaking, multiplying the period of the $S^1$ coordinate by $n$ alters the corresponding Frolov-Thorne temperature $T_{FT} = 1/2\pi k$ to $T'_{FT} = T_{FT}/n$, but leaves the central charge unchanged. It is consistent with holography, because the central charges are local quantities in the dual CFT and so should not depend on the periodicity.

From this observation, we carry out a coordinates transformation and map the metric (5.102) from the form of an $S^1$ fibered AdS$_2$ to a more conventional AdS$_3$ form,
\[ ds^2 = 4A(\theta)^2 \left[ -(1 + \rho^2)d\tau^2 + \frac{d\rho^2}{1 + \rho^2} + \rho^2 d\psi^2 \right] + F(\theta)^2 d\theta^2. \] (5.105)
Corresponding to the coordinates range (5.104) for (5.102), this coordinate system again covers the whole AdS$_3$ as a hyperbolic hypersurface, when we take
\[ \tau \sim \tau + 2\pi, \quad \rho \geq 0, \quad \psi \sim \psi + 2\pi. \] (5.106)
Now in the coordinate system $(\tau, \rho, \psi, \theta)$, we can adopt, as the asymptotic symmetry generators for (5.105), the Virasoro generators obtained in [47], or those recently proposed in [48],
\[ \xi_n^{(R)} = \frac{1}{2} \left( e^{in(\tau+\psi)}\partial_\tau - ine^{in(\tau+\psi)}\partial_\rho + e^{in(\tau+\psi)}\partial_\psi \right), \] (5.107)
\[ \xi_n^{(L)} = \frac{1}{2} \left( e^{in(\tau-\psi)}\partial_\tau - ine^{in(\tau-\psi)}\partial_\rho - e^{in(\tau-\psi)}\partial_\psi \right). \] (5.108)
By explicit calculation, both choices lead to the same result
\[ c^{(R)} = c^{(L)} = \frac{3}{G_A} \int d\theta A(\theta) F(\theta). \] (5.109)
This value exactly agrees with that of (5.100). It suggests that there are indeed some relations between the Kerr/CFT and the AdS$_3$/CFT$_2$ coming from our emergent AdS$_3$. Although they have been calculated under the periodicity (5.106), they are expected to be

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\[16\] The boundary of AdS$_3$ corresponds to $\tilde{r} = \pm \infty$ or $\tilde{\phi} = \pm \infty$. It is a special property of the unorbifolded case. When $\tilde{\phi}$ has a period, the $\tilde{r} = -\infty$ region is identified with the $\tilde{r} = \infty$ region, and so the boundary is described simply by $\tilde{r} = \infty$. 

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independent of it for the same reason as above — the central charges are local quantities in the dual CFT.

From the periodicity (5.103), the Frolov-Thorne temperatures of the system could be identified as $T_L = 1/2\pi k$ and $T_R = 0$, as in [20]. Then the entropy computed from Cardy formula is

$$S_{\text{AdS}_3} = \frac{\pi^2}{3} c^{(L)} T_L + \frac{\pi^2}{3} c^{(R)} T_R = \frac{\pi}{2k G_4} \int d\theta A(\theta) F(\theta).$$

(5.110)

It agrees with $S_{\text{Kerr/CFT}}$ (5.101), up to $O(1/k^3)$ correction terms. This result is reasonable, or better than expected. For, from the first, (5.110) is reliable only for leading order of $1/k$ expansion, because the metric will be changed in higher order in $1/k$.

6 Conclusions and Discussions

In this paper, we studied the zero entropy limit for near horizon geometries of $D = 4$ and $D = 5$ general extremal black holes with $SL(2,\mathbb{R}) \times U(1)^{D-3}$ symmetry. We derived the conditions on the near horizon geometries of the black holes in the zero entropy limit, based on the expectation that they should remain regular. Then we found that they have AdS$_3$ structure in general, although the periodicity shrinks to zero. We presented some concrete examples, including extremal 5D Myers-Perry black hole, 5D Kaluza-Klein black hole and black holes in 5D supergravity to see the emergence. We also discussed some relation between the chiral CFT$_2$ appearing in the Kerr/CFT and the non-chiral CFT$_2$ expected to be dual to AdS$_3$ emerging in the zero entropy limit.

There are possible generalizations of our consideration to other setups. For example, generalization to higher dimensional cases would be valuable, and finding more concrete examples would be also interesting.

Of course, there are many important points which should be addressed in order to understand the Kerr/CFT correspondence from the AdS$_3$ structures generally investigated in this paper. One of those which was not studied in this paper is what is an appropriate boundary condition. In particular, it is totally unclear how such boundary condition will be changed under the deformation to non-zero entropy black holes. We hope to return to these problems in near future.

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A Derivation of the Regularity Conditions in 5D

In this appendix, we give the detail of the derivation of the regularity conditions explained in §3.2.

Using \( r' \) in (3.28), the metric (3.18) is written as

\[ ds^2 = A(\theta)^2 \left[ \frac{B(\theta)^2 k_1^2 + C(\theta)^2 (k_2 + D(\theta)k_1)^2 - 1}{\epsilon^2} r'^2 dt^2 + \frac{dr'^2}{r'^2} \right. \]

\[ \left. - \frac{B(\theta)^2 k_1 + C(\theta)^2 D(\theta)(k_2 + D(\theta)k_1)}{\epsilon} 2r' dtd\phi_1 - \frac{C(\theta)^2 (k_2 + D(\theta)k_1)}{\epsilon} 2r' dtd\phi_2 \right. \]

\[ \left. + (B(\theta)^2 + C(\theta)^2 D(\theta)^2) d\phi_1^2 + 2C(\theta)^2 D(\theta) d\phi_1 d\phi_2 + C(\theta)^2 d\phi_2^2 \right] + F(\theta)^2 d\theta^2 \]

(A.111)

Each term here must not diverge, therefore, from the finiteness of the \( dt^2, dtd\phi_1, dtd\phi_2, d\phi_1^2, d\phi_2^2 \) terms respectively,

\[ B(\theta)^2 k_1^2 + C(\theta)^2 (k_2 + D(\theta)k_1)^2 - 1 = \mathcal{O}(\epsilon^2), \]  
\[ B(\theta)^2 k_1 + C(\theta)^2 D(\theta)(k_2 + D(\theta)k_1) = \mathcal{O}(\epsilon), \]  
\[ C(\theta)^2 (k_2 + D(\theta)k_1) = \mathcal{O}(\epsilon), \]  
\[ B(\theta)^2 + C(\theta)^2 D(\theta)^2 = \mathcal{O}(1), \]  
\[ C(\theta)^2 D(\theta) = \mathcal{O}(1), \]  
\[ C(\theta)^2 = \mathcal{O}(1). \]

(A.112a) - (A.112f)

In particular, (A.112d) means

\[ B(\theta) = \mathcal{O}(1), \quad C(\theta) D(\theta) = \mathcal{O}(1), \]

(A.112g)

then (A.112c) is always satisfied under (A.112f) and (A.112g).
A.1 Generality of $B(\theta) \to 0$, $C(\theta) \sim 1$

To achieve $B(\theta)C(\theta) \to 0$ while satisfying the above conditions (A.112f) (A.112d), there are three cases to be considered:

- $B(\theta) \to 0$, $C(\theta) \sim 1$. (A.113a)
- $B(\theta) \sim 1$, $C(\theta) \to 0$. (A.113b)
- $B(\theta) \to 0$, $C(\theta) \to 0$. (A.113c)

Here we will examine the cases of (A.113b) and (A.113c), and show that they can be resulted in the case of (A.113a).

A.1.1 $B(\theta) \sim 1$, $C(\theta) \to 0$

This case is expressed with $\epsilon$ in (3.28) as

$$B(\theta) \sim 1, \quad C(\theta) \sim \epsilon. \quad (A.114)$$

Now (A.112g) leads to

$$B(\theta)^2 + C(\theta)^2D(\theta)^2 \sim 1, \quad C(\theta)^2D(\theta) = \mathcal{O}(\epsilon). \quad (A.115)$$

so after acting the swapping transformation $S'$ (3.25), we obtain

$$\tilde{B}(\theta) \sim \epsilon, \quad \tilde{C}(\theta) \sim 1, \quad \tilde{D}(\theta) = \mathcal{O}(\epsilon). \quad (A.116)$$

Therefore this is clearly the case of (A.113a), especially with $D = 0$.

A.1.2 $B(\theta) \to 0$, $C(\theta) \to 0$

In this case, we introduce new small parameters $\epsilon_1$, $\epsilon_2$ and rewrite the condition as

$$B(\theta) \sim \epsilon_1, \quad C(\theta) \sim \epsilon_2, \quad \epsilon = \epsilon_1\epsilon_2, \quad \epsilon_1 \to 0, \quad \epsilon_2 \to 0. \quad (A.117)$$

With $\epsilon_1$ and $\epsilon_2$, (A.112b), (A.112c), (A.112g) lead to

$$B(\theta)^2k_1 + C(\theta)^2D(\theta)(k_2 + D(\theta)k_1) = \mathcal{O}(\epsilon_1\epsilon_2), \quad (A.118)$$

$$k_2 + D(\theta)k_1 \sim \epsilon_1/\epsilon_2, \quad (A.119)$$

$$D(\theta) = \mathcal{O}(1/\epsilon_2). \quad (A.120)$$
Notice that, since $d\phi_1 d\phi_2$ and $d\phi_2^2$ terms go to zero in this case, $dt d\phi_2$ term must not vanish for regularity and so (A.119) fixes the scaling order exactly, rather than $O(\epsilon_1/\epsilon_2)$. We see that, from (A.117) and (A.119),

$$C(\theta)^2(k_2 + D(\theta)k_1)^2 \sim \epsilon_1^2,$$

(A.121)

therefore by (A.112a),

$$B(\theta)^2 k_2^2 = 1 + O(\epsilon_1^2),$$

(A.122)

which is equivalent to

$$B(\theta) = \frac{1}{k_1} + O(\epsilon_1).$$

(A.123)

Thus $B(\theta)^2 k_1 \sim \epsilon_1 \gg \epsilon_1 \epsilon_2$, so (A.118) means

$$C(\theta)^2 D(\theta)(k_2 + D(\theta)k_1) = -B(\theta)^2 k_1 + O(\epsilon_1 \epsilon_2) \sim \epsilon_1,$$

(A.124)

which leads to, by using (A.119),

$$D(\theta) \sim 1/\epsilon_2.$$  

(A.125)

Now it is easy to see, from (A.117) and (A.125), that $S'$ transforms $B(\theta)$, $C(\theta)$ and $D(\theta)$ into

$$\tilde{B}(\theta) \sim \epsilon_1 \epsilon_2 = \epsilon, \quad \tilde{C}(\theta) \sim 1, \quad \tilde{D}(\theta) \sim \epsilon_2.$$  

(A.126)

Therefore the current case (A.113c) is also transformed to the case of (A.113a), again with $D = 0$.

**A.2 Conditions under $B(\theta) \rightarrow 0$, $C(\theta) \sim 1$**

Under (3.29),

$$B(\theta) \sim \epsilon, \quad C(\theta) \sim 1,$$

(A.127)

the finiteness conditions (A.112a),(A.112b),(A.112c) and (A.112g) become, respectively,

$$B(\theta)^2 k_1^2 + C(\theta)^2(k_2 + D(\theta)k_1)^2 - 1 = O(\epsilon^2),$$

(A.128)

$$B(\theta)^2 k_1 + C(\theta)^2 D(\theta)(k_2 + D(\theta)k_1) = O(\epsilon),$$

(A.129)

$$k_2 + D(\theta)k_1 = O(\epsilon),$$

(A.130)

$$D(\theta) = O(1).$$

(A.131)
Then $C(\theta)^2(k_2 + D(\theta)k_1)^2 = \mathcal{O}(\epsilon^2)$ from (A.130), and so substituting it into (A.128) yields
\[ B(\theta)^2k_1^2 = 1 + \mathcal{O}(\epsilon^2), \] (A.132)
or equivalently,
\[ B(\theta)k_1 = 1 + \mathcal{O}(\epsilon^2), \] (A.133)
where we can immediately see that
\[ k_1 \sim \frac{1}{\epsilon}. \] (A.134)
Thus from (A.130) and (A.134),
\[ D(\theta) = -\frac{k_2}{k_1} + \mathcal{O}(\epsilon^2), \] (A.135)
which leads to, using (A.131) and (A.134),
\[ k_2 = \mathcal{O}(\epsilon^{-1}). \] (A.136)
Therefore we have successfully shown (3.31) and (3.32) to be the regularity condition for the geometry in the zero entropy limit.

B Some Relations

In this appendix we summarize the relations between the parameters $c_1$, $c_2$, $c_0^2$ appearing in §4.1 and physical quantities in the extremal Myers-Perry black hole and the extremal slow rotating Kaluza-Klein black hole [42].

B.1 Myers-Perry black hole

The relation to Myers-Perry black hole is as follows. Let us first use a scaling symmetry
\[ c_0^2 \rightarrow \lambda c_0^2, \quad c_1 \rightarrow \lambda^2 c_1, \quad c_2 \rightarrow \lambda^3 c_2, \quad x_1 \rightarrow \lambda^{-1} x_1, \] (B.137)
to set $c_0^2 = c_1$ and introduce
\[ a = \frac{1}{\sqrt{c_1}} + \frac{\sqrt{c_1 + 4c_2}}{c_1}, \quad b = \frac{1}{\sqrt{c_1}} - \frac{\sqrt{c_1 + 4c_2}}{c_1}. \] (B.138)
By changing coordinates

$$\cos^2 \theta = \frac{\sigma - \sigma_1}{\sigma_2 - \sigma_1}, \quad x^1 = -\frac{\sqrt{ab}(a + b)^2}{2(a - b)} (-\psi + \phi), \quad x^2 = \frac{a + b}{a - b} (-b\psi + a\phi), \quad (B.139)$$

so that $0 \leq \theta \leq \pi/2$, $\phi \sim \phi + 2\pi$ and $\psi \sim \psi + 2\pi$. The horizon data are summarized as follows:

$$\gamma_{ij}dx^idx^j = \frac{1}{\rho_+^2}\left[(r_+^2 + a^2)^2 \sin^2 \theta d\phi^2 + (r_+^2 + b^2)^2 \cos^2 \theta d\psi^2\right]$$

$$+ \frac{1}{r_+^2 \rho_+^2} \left[b(r_+^2 + a^2) \sin^2 \theta d\phi + a(r_+^2 + b^2) \cos^2 \theta d\psi \right]^2, \quad (B.140)$$

$$\frac{\sigma}{Q(\sigma)}d\sigma^2 = \rho_+^2 d\theta^2, \quad \Gamma = \frac{\rho_+^2 r_+^2}{(r_+^2 + a^2)(r_+^2 + b^2)}, \quad (B.141)$$

$$\tilde{k}_\phi = \frac{2ar_+}{(r_+^2 + a^2)^2}, \quad \tilde{k}_\psi = \frac{2br_+}{(r_+^2 + b^2)^2}. \quad (B.142)$$

Here $r_+^2 = ab$ and $\rho_+^2 = r_+^2 + a^2 \cos^2 \theta + b^2 \sin^2 \theta$. This corresponds to the near horizon geometry of the extremal Myers-Perry black hole. The two angular momenta $J_\phi$, $J_\psi$ corresponding to $\phi$, $\psi$ are

$$J_\phi = \frac{\pi}{4} a(a + b)^2, \quad J_\psi = \frac{\pi}{4} b(a + b)^2, \quad (B.143)$$

respectively.

**B.2 Slow rotating Kaluza-Klein black hole**

On the other hand the relation to the extremal slow rotating Kaluza-Klein black hole is as follows. Let us first define $p$, $q$ and $j$ such that

$$p = \frac{1}{c_0^2} \sqrt{c_1 \left(1 - \frac{c_2}{c_0^2}\right)}, \quad q^2 = \frac{c_1}{c_2^2} \left(1 - \frac{c_2}{c_0^2}\right), \quad j^2 = 1 + \frac{4c_0^2 c_2^2}{c_1^2}. \quad (B.144)$$

By changing coordinates as

$$\cos \theta = \frac{2\sigma - \sigma_1 - \sigma_2}{\sigma_2 - \sigma_1}, \quad (B.145)$$

$$x^1 = \sqrt{\frac{1 - j^2}{c_0^2j}} \phi, \quad x^2 = -\frac{2}{c_0^2q} \sqrt{\frac{(p + q)}{p(1 - j^2)}} \left(\frac{\phi}{\eta} - \sqrt{\frac{p + q}{p^2}} y\right), \quad (B.146)$$
the horizon data is written as

\[ \gamma_{ij}dx^i dx^j = \frac{H_q}{H_p} (dy - A_\phi d\phi)^2 + \frac{(pq)^3(1 - j^2) \sin^2 d\phi^2}{4(p + q)^2 H_q}, \quad (B.147) \]

\[ \frac{\sigma}{Q(\sigma)} d\sigma^2 = H_p d\theta^2, \quad \Gamma = \frac{2(p + q)}{(pq)^{3/2}(1 - j^2)^{1/2}} H_p, \quad (B.148) \]

\[ \bar{k}_\phi = \frac{2(p + q)}{(pq)^{3/2}(1 - j^2)}, \quad \bar{k}_y = \frac{2}{1 - j^2} \sqrt{\frac{p + q}{q^3}}, \quad (B.149) \]

with

\[ H_p = \frac{p^2 q}{2(p + q)} (1 + j \cos \theta), \quad H_q = \frac{pq^2}{2(p + q)} (1 - j \cos \theta), \quad (B.150) \]

\[ A_\phi = \frac{q^2 p^{5/2}}{2(p + q)^{3/2} H_q} (j - \cos \theta). \quad (B.151) \]

When periodicities \( \phi \sim \phi + 2\pi, y \sim y + 8\pi \tilde{P} \) are imposed, this corresponds to the near horizon geometry of the extremal slow rotating Kaluza-Klein black hole. Then magnetic charge \( \tilde{P} \), electric charge \( \tilde{Q} \) and angular momentum \( J \) are written as

\[ \tilde{P}^2 = \frac{p^3}{4(p + q)}, \quad \tilde{Q}^2 = \frac{q^3}{4(p + q)}, \quad G_4 J = \frac{(pq)^{3/2}}{4(p + q)} j. \quad (B.152) \]

References

[1] M. Guica, T. Hartman, W. Song, and A. Strominger, *The Kerr/CFT Correspondence*, Phys. Rev. **D80** (2009) 124008, [arXiv:0809.4266](http://arxiv.org/abs/0809.4266).

[2] H. Lu, J. Mei, and C. N. Pope, *Kerr-AdS/CFT Correspondence in Diverse Dimensions*, JHEP **04** (2009) 054, [arXiv:0811.2225](http://arxiv.org/abs/0811.2225).

[3] T. Azeyanagi, N. Ogawa, and S. Terashima, *Holographic Duals of Kaluza-Klein Black Holes*, JHEP **04** (2009) 061, [arXiv:0811.4177](http://arxiv.org/abs/0811.4177).

[4] T. Hartman, K. Murata, T. Nishioka, and A. Strominger, *CFT Duals for Extreme Black Holes*, JHEP **04** (2009) 019, [arXiv:0811.4393](http://arxiv.org/abs/0811.4393).

[5] G. Compere, K. Murata, and T. Nishioka, *Central Charges in Extreme Black Hole/CFT Correspondence*, JHEP **05** (2009) 077, [arXiv:0902.1001](http://arxiv.org/abs/0902.1001).

[6] T. Azeyanagi, G. Compere, N. Ogawa, Y. Tachikawa, and S. Terashima, *Higher-Derivative Corrections to the Asymptotic Virasoro Symmetry of 4d Extremal Black Holes*, Prog. Theor. Phys. **122** (2009) 355–384, [arXiv:0903.4176](http://arxiv.org/abs/0903.4176).
[7] I. Bredberg, T. Hartman, W. Song, and A. Strominger, *Black Hole Superradiance From Kerr/CFT*, JHEP 04 (2010) 019, [arXiv:0907.3477](http://arxiv.org/abs/0907.3477).

[8] M. Cvetic and F. Larsen, *Greybody Factors and Charges in Kerr/CFT*, JHEP 09 (2009) 088, [arXiv:0908.1136](http://arxiv.org/abs/0908.1136).

[9] A. Castro, A. Maloney, and A. Strominger, *Hidden Conformal Symmetry of the Kerr Black Hole*, Phys. Rev. D82 (2010) 024008, [arXiv:1004.0996](http://arxiv.org/abs/1004.0996).

[10] Y. Matsuo, T. Tsukioka, and C.-M. Yoo, *Another Realization of Kerr/CFT Correspondence*, Nucl. Phys. B825 (2010) 231–241, [arXiv:0907.0303](http://arxiv.org/abs/0907.0303).

[11] Y. Matsuo, T. Tsukioka, and C.-M. Yoo, *Yet Another Realization of Kerr/CFT Correspondence*, Europhys. Lett. 89 (2010) 60001, [arXiv:0907.4272](http://arxiv.org/abs/0907.4272).

[12] J. Rasmussen, *Isometry-preserving boundary conditions in the Kerr/CFT correspondence*, Int. J. Mod. Phys. A25 (2010) 1597–1613, [arXiv:0908.0184](http://arxiv.org/abs/0908.0184).

[13] J. Rasmussen, *A near-NHEK/CFT correspondence*, [arXiv:1004.4773](http://arxiv.org/abs/1004.4773).

[14] J. M. Maldacena, *The large N limit of superconformal field theories and supergravity*, Adv. Theor. Math. Phys. 2 (1998) 231–252, [hep-th/9711200](http://arxiv.org/abs/hep-th/9711200).

[15] T. Azeyanagi, N. Ogawa, and S. Terashima, *The Kerr/CFT Correspondence and String Theory*, Phys. Rev. D79 (2009) 106009, [arXiv:0812.4883](http://arxiv.org/abs/0812.4883).

[16] V. Balasubramanian, J. de Boer, M. M. Sheikh-Jabbari, and J. Simon, *What is a chiral 2d CFT? And what does it have to do with extremal black holes?*, JHEP 02 (2010) 017, [arXiv:0906.3272](http://arxiv.org/abs/0906.3272).

[17] A. J. Amsel, G. T. Horowitz, D. Marolf, and M. M. Roberts, *No Dynamics in the Extremal Kerr Throat*, JHEP 09 (2009) 044, [arXiv:0906.2376](http://arxiv.org/abs/0906.2376).

[18] O. J. C. Dias, H. S. Reall, and J. E. Santos, *Kerr-CFT and gravitational perturbations*, JHEP 08 (2009) 101, [arXiv:0906.2380](http://arxiv.org/abs/0906.2380).

[19] A. Strominger, *Black hole entropy from near-horizon microstates*, JHEP 02 (1998) 009, [hep-th/9712251](http://arxiv.org/abs/hep-th/9712251).

[20] M. Guica and A. Strominger, *Microscopic Realization of the Kerr/CFT Correspondence*, [arXiv:1009.5039](http://arxiv.org/abs/1009.5039).
[21] J. C. Breckenridge, R. C. Myers, A. W. Peet, and C. Vafa, D-branes and spinning black holes, Phys. Lett. B391 (1997) 93–98, hep-th/9602065.

[22] J. C. Breckenridge, D. A. Lowe, R. C. Myers, A. W. Peet, A. Strominger, and C. Vafa, Macroscopic and Microscopic Entropy of Near-Extremal Spinning Black Holes, Phys. Lett. B381 (1996) 423–426, hep-th/9603078.

[23] M. Cvetic and D. Youm, General Rotating Five Dimensional Black Holes of Toroidally Compactified Heterotic String, Nucl. Phys. B476 (1996) 118–132, hep-th/9603100.

[24] M. Guica and A. Strominger, Wrapped M2/M5 duality, JHEP 10 (2009) 036, hep-th/0701011.

[25] G. W. Gibbons and C. A. R. Herdeiro, Supersymmetric rotating black holes and causality violation, Class. Quant. Grav. 16 (1999) 3619–3652, hep-th/9906098.

[26] N. Alonso-Alberca, E. Lozano-Tellechea, and T. Ortin, The near-horizon limit of the extreme rotating d = 5 black hole as a homogeneous spacetime, Class. Quant. Grav. 20 (2003) 423–430, hep-th/0209069.

[27] L. Dyson, Studies of the over-rotating BMPV solution, JHEP 01 (2007) 008, hep-th/0608137.

[28] A. Strominger and C. Vafa, Microscopic Origin of the Bekenstein-Hawking Entropy, Phys. Lett. B379 (1996) 99–104, hep-th/9601029.

[29] J. M. Maldacena, A. Strominger, and E. Witten, Black hole entropy in M-theory, JHEP 12 (1997) 002, hep-th/9711053.

[30] G. Compere, W. Song, and A. Virmani, Microscopics of Extremal Kerr from Spinning M5 Branes, arXiv:1010.0685.

[31] H. K. Kunduri, J. Lucietti, and H. S. Reall, Near-horizon symmetries of extremal black holes, Class. Quant. Grav. 24 (2007) 4169–4190, arXiv:0705.4214.

[32] S. Hollands and A. Ishibashi, All Vacuum Near-Horizon Geometries in D-dimensions with (D-3) Commuting Rotational Symmetries, Annales Henri Poincare 10 (2010) 1537–1557, arXiv:0909.3462.

[33] H. K. Kunduri and J. Lucietti, An infinite class of extremal horizons in higher dimensions, arXiv:1002.4656.
[34] Y. Nakayama, Emerging AdS from Extremally Rotating NS5-branes, Phys. Lett. B673 (2009) 272–278, [arXiv:0812.2234].

[35] J. M. Bardeen and G. T. Horowitz, The extreme Kerr throat geometry: A vacuum analog of $AdS_2 \times S^2$, Phys. Rev. D60 (1999) 104030, [hep-th/9905099].

[36] V. Balasubramanian, J. de Boer, V. Jejjala, and J. Simon, Entropy of near-extremal black holes in $AdS_5$, JHEP 05 (2008) 067, [arXiv:0707.3601].

[37] R. Fareghbal, C. N. Gowdigere, A. E. Mosaffa, and M. M. Sheikh-Jabbari, Nearing Extremal Intersecting Giants and New Decoupled Sectors in $N = 4$ SYM, JHEP 08 (2008) 070, [arXiv:0801.4457].

[38] R. C. Myers and M. J. Perry, Black Holes in Higher Dimensional Space-Times, Ann. Phys. 172 (1986) 304.

[39] G. W. Gibbons and D. L. Wiltshire, Black Holes in Kaluza-Klein Theory, Ann. Phys. 167 (1986) 201.

[40] D. Rasheed, The Rotating dyonic black holes of Kaluza-Klein theory, Nucl. Phys. B454 (1995) 379–401, [hep-th/9505038].

[41] F. Larsen, Rotating Kaluza-Klein black holes, Nucl. Phys. B575 (2000) 211–230, [hep-th/9909102].

[42] H. K. Kunduri and J. Lucietti, A classification of near-horizon geometries of extremal vacuum black holes, J. Math. Phys. 50 (2009) 082502, [arXiv:0806.2051].

[43] H. K. Kunduri and J. Lucietti, Uniqueness of near-horizon geometries of rotating extremal $AdS_4$ black holes, Class. Quant. Grav. 26 (2009) 055019, [arXiv:0812.1576].

[44] O. Coussaert and M. Henneaux, Selfdual solutions of (2+1) Einstein gravity with a negative cosmological constant, [hep-th/9407181].

[45] F. Loran and H. Soltanpanahi, 5D Extremal Rotating Black Holes and CFT duals, Class. Quant. Grav. 26 (2009) 155019, [arXiv:0901.1595].

[46] D. Astefanesei, K. Goldstein, R. P. Jena, A. Sen, and S. P. Trivedi, Rotating attractors, JHEP 10 (2006) 058, [hep-th/0606244].
[47] J. D. Brown and M. Henneaux, *Central Charges in the Canonical Realization of Asymptotic Symmetries: An Example from Three-Dimensional Gravity*, Commun. Math. Phys. 104 (1986) 207–226.

[48] A. P. Porfyriadis and F. Wilczek, *Effective Action, Boundary Conditions, and Virasoro Algebra for AdS$_3$*, arXiv:1007.1031