POSITIVITY OF CERTAIN CLASS OF FUNCTIONS RELATED TO THE FOX H-FUNCTIONS AND APPLICATIONS

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ABSTRACT. In this present investigation, we found a set of sufficient conditions to be imposed on the parameters of the Fox H-functions which allow us to conclude that it is non-negative. As applications, various new facts regarding the Fox-Wright functions, including complete monotonicity, logarithmic completely monotonicity and monotonicity of ratios are considered.

1. INTRODUCTION

The Fox H-function plays an important role in various branches of applied mathematics, many areas of theoretical physics, statistical distribution theory, and engineering sciences. More recently, the Fox H-function are special functions of fractional calculus, as well as in their applications, including non-Gaussian stochastic processes and phenomena of nonstandard (i.e. anomalous) relaxation and diffusion see e.g. [2, 9, 10, 11, 12]. For the reader’s convenience, let us start with the definition. To simplify the notation we introduce

\[(a_p, A_p) = ((a_1, A_1), (a_2, A_2), \ldots, (a_p, A_p))\]

or the set of parameters appearing in the definition of Fox H-functions. The Fox H-function is defined via a Mellin-Barnes type integral as

\[H_{q,p}^{n,m}[z] := H_{q,p}^{n,m}[z] = H_{q,p}^{n,m}[z] (b_q, B_q), (a_p, A_p) = \frac{1}{2\pi i} \int_{\mathcal{L}} \mathcal{H}_{q,p}^{n,m}(s) z^{-s} ds,\]

where

\[\mathcal{H}_{q,p}^{n,m}(s) = \prod_{j=1}^{n} \frac{\Gamma(A_j s + a_j) \prod_{j=1}^{m} \Gamma(1 - b_j - B_j s)}{\prod_{j=m+1}^{p} \Gamma(B_j s + b_k) \prod_{j=m+1}^{1} \Gamma(1 - a_j - A_j s)}\]

is the Mellin transform of the Fox H-function \(H_{q,p}^{n,m}[z]\) and \(\mathcal{L}\) is the infinite contour in the complex plane which separates the poles

\[a_{il} = -\frac{a_i + l}{A_i}, \quad (i = 1, \ldots, n; l = 0, 1, 2, \ldots)\]

of the Gamma function \(\Gamma(a_i + A_i s)\) to the left of \(\mathcal{L}\) and the poles

\[b_{jk} = \frac{1 - b_j + k}{B_j}, \quad (j = 1, \ldots, m; k = 0, 1, 2, \ldots)\]

to the right of \(\mathcal{L}\). Here, and in what follows, we use the Fox-Wright (generalized hypergeometric ) function \(p\Psi_q[],\) with \(p\) numerator parameters \(a_1, \ldots, a_p\) and \(q\) denominator parameters \(b_1, \ldots, b_q\), which are defined by [10] p. 4, Eq. (2.4)

\[(p\Psi_q[;], z) = \sum_{k=0}^{\infty} \prod_{j=0}^{p} \frac{\Gamma(a_j + k A_i j)}{k!} z^k,\]

\[(a_i, b_j \in \mathbb{C}, \text{ and } A_i, B_j \in \mathbb{R}^+ (i = 1, \ldots, p, j = 1, \ldots, q)),\]

where, as usual,

\[\mathbb{N} = \{1, 2, 3, \ldots\}, \quad \mathbb{N}_0 = \mathbb{N} \cup \{0\},\]

\(\mathbb{R}, \mathbb{R}^+\) and \(\mathbb{C}\) stand for the sets of real, positive real and complex numbers, respectively. The convergence conditions and convergence radius of the series at the right-hand side of (1.2) immediately follow from

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the known asymptotic of the Euler Gamma–function. The defining series in (1.2) converges in the whole complex \( z \)-plane when

\[
\Delta = \sum_{j=1}^{q} B_j - \sum_{i=1}^{p} A_i > -1.
\]

If \( \Delta = -1 \), then the series in (1.2) converges for \(|z| < \rho\), and \(|z| = \rho\) under the condition \( \Re(\mu) > \frac{1}{2} \), (see [13] for details), where

\[
\rho = \left( \prod_{i=1}^{p} A_i^{-A_i} \right) \left( \prod_{j=1}^{q} B_j^B_j \right), \quad \mu = \sum_{j=1}^{q} b_j - \sum_{k=1}^{p} a_k + \frac{p - q}{2}
\]

If, in the definition (1.2), we set

\[A_1 = \ldots = A_p = 1 \quad \text{and} \quad B_1 = \ldots = B_q = 1,\]

we get the relatively more familiar generalized hypergeometric function \( _pF_q(\cdot) \) given by

\[
_pF_q\left[a_1, \ldots, a_p \mid b_1, \ldots, b_q \right] z := \frac{\prod_{i=1}^{p} \Gamma(b_i)}{\prod_{i=1}^{p} \Gamma(a_i)} _p\Psi_q \left(a_1, \ldots, a_p \mid b_1, \ldots, b_q \right) z^{b_1 - a_1 - 1}.
\]

The three-parameter Mittag-Leffler type function \( E_{\alpha,\beta}(z) \) is defined by (see [15])

\[
E_{\alpha,\beta}(z) := \sum_{k=0}^{\infty} \frac{\Gamma(\gamma + k)}{\Gamma(\beta + k\alpha)} \frac{z^k}{k!}, \quad \alpha > 0, \beta, \gamma \in \mathbb{C}.
\]

It is easily seen from the definition (1.6) that

\[
E_{\alpha,\beta}(z) = \frac{1}{\Gamma(\gamma)} \Psi_{1}^{\gamma,1}_{1,\alpha} \left[ z^{\beta} \right].
\]

The definition of the H-function is still valid when the \( A_i \)'s and \( B_j \)'s are positive rational numbers. Therefore, the H-function contains, as special cases, all of the functions which are expressible in terms of the G–function. More importantly, it contains the Fox-Wright generalized hypergeometric function defined in (1.2), the generalized Mittag-Leffler functions, etc. For example, the function \( _p\Psi_q(\cdot) \) is one of these special case of H-function. By the definition (1.2) it is easily extended to the complex plane as follows [1, Eq. 1.31],

\[
_p\Psi_q\left[\left(a_p, A_p \right) \mid \left(b_q, B_q \right) \right] z = H_{p,q+1}^{1,q+1} \left( -z^{|(A_p,1-a_p)\mid_{(1,1),(1-\beta,\alpha)}} \right).
\]

The representation (1.8) holds true only for positive values of the parameters \( A_i \) and \( B_j \).

It is straightforward to verify that

\[
E_{\alpha,\beta}(z) = \frac{1}{\Gamma(\gamma)} H_{1,2}^{1,1} \left[ z^{\left(1-\gamma,1\right)_{(0,1),(1-\beta,\alpha)}} \right].
\]

The special case for which the H-function reduces to the Meijer G-function is when \( A_1 = \ldots = A_p = B_1 = \ldots = B_q = 1, \quad A > 0 \). In this case,

\[
H_{q,p}^{m,n} \left( z^{(B_q, B_q) \mid (A_p, a_p)} \right) = \frac{1}{A} G_{q,p}^{m,n} \left( z^{1/A \mid A} \right).
\]

Additionally, when setting \( A_i = B_j = 1 \) in (1.8) (or \( A = 1 \) in (1.10)), the H- and Fox-Wright functions turn readily into the Meijer G-function.

For the reader’s convenience, we first recall the definition of the completely monotone functions: A real valued function \( f \), defined on an interval \( I \), is called completely monotonic on \( I \), if \( f \) has derivatives of all orders and satisfies

\[
(-1)^n f^{(n)}(x) \geq 0, \quad n \in \mathbb{N}_0, \quad x \in I.
\]

The basic property of the completely monotone functions is given by the Bernstein theorem (see, e.g., [8]) that says that a function \( f : (0, \infty) \to \mathbb{R} \) is completely monotone if and only if it can be represented as the Laplace transform of a non-negative measure (non-negative function or generalized function).

An infinitely differentiable function \( f : I \to [0, \infty) \) is called a Bernstein function on an interval \( I \), if \( f' \) is completely monotonic on \( I \).
A positive function $f$ is said to be logarithmically completely monotonic on an interval $I$ if its logarithm $\log f$ satisfies
\begin{equation}
(-1)^n(\log f)^{(n)}(x) \geq 0, \quad n \in \mathbb{N}, \quad \text{and} \quad x \in I.
\end{equation}

In [2, Theorem 1.1] and [13, Theorem 4], it was found and verified once again that a logarithmically completely monotonic function must be completely monotonic, but not conversely.

Motivated by the papers [3, 4], our aim in this investigation is to give sufficient conditions so that some class of functions related to the Fox H-functions are non-negative functions. Applying this results, various new facts regarding the Fox-Wright functions, including complete monotonicity, logarithmic completely monotonicity and monotonicity of ratios are established.

2. SOME LEMMAS

In the proof of the main results we will need the following lemmas, which we collect in this section. The first Lemma is about some properties for the Fox H-functions, see [5] for more details.

**Lemma 1. Properties of Fox H-functions:**

**Property 1.**

\[ H_{q,p}^{n,m} \left[ \begin{array}{c} (b_q,B_q) \\ \vdots \\ (a_p,A_p) \end{array} \right] = H_{p,q}^{m,n} \left[ \begin{array}{c} (1-a_p,A_p) \\ \vdots \end{array} \right]. \]

**Property 2.**

\[ H_{q,p}^{n,m} \left[ \begin{array}{c} (b_q,B_q) \\ \vdots \\ (a_p,A_p) \end{array} \right] = k H_{q,p}^{n,m} \left[ \begin{array}{c} (b_q,B_q) \\ \vdots \end{array} \right], \quad k > 0. \]

**Property 3.**

\[ H_{q,p}^{n,m} \left[ \begin{array}{c} (b_q,B_q) \\ \vdots \\ (a_p,A_p) \end{array} \right] = H_{q,p}^{n,m} \left[ \begin{array}{c} (b_q,B_q) \\ \vdots \end{array} \right]. \]

**Property 4.** For $x, b > 0$, we have
\begin{align*}
\int_0^\infty & r^{\rho-1} J_\rho(xr) H_{q,p}^{n,m} \left[ \begin{array}{c} (b_q,B_q) \\ \vdots \end{array} \right] dr \\
& = \frac{2^{\rho-1}}{x^\rho} H_{q+2,p}^{n,m} \left[ \begin{array}{c} (b_q,B_q) \\ \vdots \end{array} \right] dx.
\end{align*}

**Property 5.** For $\sigma \in \mathbb{C}$, the following relation holds
\[ z^\sigma H_{q,p}^{n,m} \left[ \begin{array}{c} (b_q,B_q) \\ \vdots \end{array} \right] = H_{q,p}^{n,m} \left[ \begin{array}{c} (b_q+B_\sigma B_q,B_q) \\ \vdots \end{array} \right]. \]

In the following three lemmas we present some integral representations for the Fox-Wright functions, which plays a crucial role in the proof of some Theorems given in Section 3, see [3, 4] for a proofs.

**Lemma 2.** Suppose that $\mu > 0$, $\gamma = \min_{1 \leq j \leq p} (a_j/A_j) \geq 1$, and $\sum_{j=1}^p A_j = \sum_{k=1}^q B_k$. Then, the following integral representation
\begin{equation}
\rho + 1 \Psi \left[ \frac{(a_p,A_p)}{(b_q,B_q)} \right] = \int_0^\rho \frac{e^{zt} H_{q,p}^{0,0} \left[ \frac{(B_q,\beta_q)}{(A_p,\alpha_p)} \right]}{t} \, dt, \quad (z \in \mathbb{R}),
\end{equation}

hold true. Moreover, the function
\[ z \mapsto \rho + 1 \Psi \left[ \frac{(a_p,A_p)}{(b_q,B_q)} \right] \]
is completely monotonic on $(0, \infty)$, if and only if, the function $H_{q,p}^{0,0}(z)$ is non-negative on $(0, \rho)$.

**Lemma 3.** Assume that the assumption of Lemma 2 and the function $H_{q,p}^{0,0}(z)$ is non-negative. Then, the following Stieltjes transform holds true:
\begin{equation}
\rho + 1 \Psi \left[ \frac{(\sigma,1)}{(\beta_q,B_q)} \right] = \int_0^\rho \frac{d\mu(t)(1+zt)^\gamma}{(1+zt)^\gamma},
\end{equation}

where
\begin{equation}
d\mu(t) = H_{q,p}^{0,0} \left[ \frac{(B_q,\beta_q)}{(A_p,\alpha_p)} \right] \, dt.
\end{equation}
Moreover, the function
\[ z \mapsto p_+ \Psi_q \left[ (\alpha, \beta; A_p, B_p) \right] - z \]
is logarithmically completely monotonic on \((0, 1)\).

**Lemma 4.** Suppose that \( \mu = 0, \gamma \geq 1 \) and \( \sum_{i=1}^{p} A_i = \sum_{j=1}^{q} B_j \). Then, the Fox–Wright function \( p \Psi_q \) possesses the following integral representation
\[ (2.16) \quad p \Psi_q \left[ (\alpha, \beta; A_p, B_p) \right] - z = \int_0^\omega e^{-zt} d\omega(t), \quad z \in \mathbb{R} \]
where
\[ (2.17) \quad d\omega(t) = H^{p,0}_{q,p} \left( \left( \frac{(B_j + \beta_j)}{(A_j + \alpha_j)} \right) dt \right) + \eta \delta_p(t), \quad \eta = (2\pi)^{i} \prod_{i=1}^{p} A_i^{\alpha_i - \frac{1}{2}} \prod_{j=1}^{q} B_j^{\beta_j - \frac{1}{2}}. \]

Moreover, if the function \( H^{p,0}_{q,p} \) is non-negative, then the functions
\[ z \mapsto p \Psi_q \left[ (\alpha, \beta; A_p, B_p) \right] - z, \quad \text{and} \quad z \mapsto p \Psi_q \left[ (\alpha, \beta; A_p, B_p) \right] - z - \eta e^{-\rho^2}, \]
are completely monotonic on \((0, \infty)\).

The following Lemma is the so-called the Chebyshev integral inequality [5, p. 40]

**Lemma 5.** If \( f, g : [a, b] \rightarrow \mathbb{R} \) are synchronous (both increasing or decreasing) integrable functions, and \( p : [a, b] \rightarrow \mathbb{R} \) is a positive integrable function, then
\[ (2.18) \quad \int_a^b p(t)f(t)dt \int_a^b p(t)g(t)dt \leq \int_a^b p(t)dt \int_a^b p(t)f(t)g(t)dt. \]

Note that if \( f \) and \( g \) are asynchronous (one is decreasing and the other is increasing), then (2.18) is reversed.

3. **Positivity of some class of functions related to the Fox H-functions**

Our first main result is asserted in the following Theorem.

**Theorem 1.** Under the conditions
\[ (H_1): 0 < \tau \leq 2, \alpha, \beta \in (0, 1), \gamma > 0, \beta \geq \alpha \gamma, \]
the function
\[ (3.19) \quad H^{1,2}_{1,1} \left[ \xi^{-\tau} \left( (1 - \frac{\mu}{2} + \frac{\tau}{2}), (1 - \gamma), \frac{(a, \beta)}{(1, \alpha - \beta + 1, \alpha)} \right) \right], \quad \xi \in \mathbb{R}^d, \]
is non-negative.

**Proof.** Putting
\[ (3.20) \quad F^{\gamma, \tau}_{\alpha, \beta}(x) = \frac{t^{\beta - 1}}{\Gamma(\gamma)} H^{1,2}_{1,1} \left[ t^{\alpha} |x|^\tau \left( (1 - \gamma, 1), (0, 1), (1, \alpha - \beta + 1, \alpha) \right) \right], \quad t > 0, \quad x \in \mathbb{R}^d. \]
By using the fact that \( F^{\gamma, \tau}_{\alpha, \beta} \) is radial function in \( x \), and since the radial Fourier transform in \( d \) dimensions is given in terms of the Hankel transform, that is
\[ \mathcal{F}(f)(|\xi|) = |\xi|^{\frac{d-1}{2}} \int_0^\infty r^{\frac{d-1}{2}} J_{\frac{d-1}{2}} (r|\xi|) f(r)dr, \]
where \( J_{\frac{d-1}{2}}(\cdot) \) is the Bessel function. In view of the above formula and Lemma 11 we get
\[ \mathcal{F}(F^{\gamma, \tau}_{\alpha, \beta}(t, .))(|\xi|) = \frac{t^{\beta - 1}}{\Gamma(\gamma)} \int_0^\infty r^{\frac{d-1}{2}} J_{\frac{d-1}{2}} (r|\xi|) H^{1,2}_{1,1} \left[ t^{\alpha} |r|^\tau \left( (1 - \gamma, 1), (0, 1), (1, \alpha - \beta + 1, \alpha) \right) \right] dr \]
\[ = \frac{\tau^{\gamma - 1} |\xi|^{\tau - 1}}{\Gamma(\gamma)} H^{1,2}_{1,1} \left[ t^{\alpha} |r|^\tau \left( (1 - \gamma, 1), (0, 1), (1, 3, \alpha - \beta + 1, \alpha) \right) \right] \]
\[ = \frac{\tau^{\gamma - 1} |\xi|^{\tau - 1}}{\Gamma(\gamma)} H^{1,2}_{1,1} \left[ t^{\alpha} |r|^\tau \left( (1 - \gamma, 1), (0, 1), (1, \alpha - \beta + 1, \alpha) \right) \right]. \]
Corollary 1. The following functions:

\[ H_{1,2}^{2,0} \left[ r \right. \left. \frac{(\beta - \alpha)}{(\xi - \frac{1}{2})}, \gamma, r > 0, \beta \geq \alpha \gamma \right], \]

\[ H_{2,2}^{2,0} \left[ r \right. \left. \frac{(\beta - \alpha)}{(\xi - \frac{1}{2})}, (1, 1), \beta \geq \alpha, 0 < \tau \leq 2, d \geq 1, r > 0 \right], \]

are non-negatives.

Proof. Letting \( \tau = 2 \) in (3.19). Applications of Property 3, Property 1 and Property 2 of Lemma 1 yields that

\[ H_{1,2}^{2,0} \left[ \xi \right. \left. \frac{(1 - \frac{1}{2})}{(\xi - \frac{1}{2})}, \gamma, r > 0, \beta \geq \alpha \gamma \right] = H_{1,2}^{2,0} \left[ \xi \right. \left. \frac{(1 - \frac{1}{2})}{(\xi - \frac{1}{2})}, (1, 1), \beta \geq \alpha + \alpha \gamma \right] = H_{1,2}^{2,0} \left[ \xi \right. \left. \frac{(1 - \frac{1}{2})}{(\xi - \frac{1}{2})}, (1, 1), \beta \geq \alpha + \alpha \gamma \right] = 2^{-1} H_{1,2}^{2,0} \left[ \xi \right. \left. \frac{(1 - \frac{1}{2})}{(\xi - \frac{1}{2})}, (1, 1), \beta \geq \alpha + \alpha \gamma \right]. \]

Now, setting \( \gamma = 1 \) in (3.19). By repeating the above procedure, we get that the function

\[ H_{1,2}^{2,0} \left[ r \right. \left. \frac{(\beta - \alpha)}{(\xi - \frac{1}{2})}, (1, 1), \beta \geq \alpha, 0 < \tau \leq 2, d \geq 1, r > 0 \right], \]

is non-negative. This proves the second statement. \[ \square \]

Now, putting \( \gamma = 3/2 \) and \( \tau = 1 \) respectively, in the first and second assertion of Corollary 1 and taking the relation

\[ \Gamma(1 + s) = \pi^{-1/2} 2^s \Gamma((1 + s)/2) \Gamma((2 + s)/2) \]

into account, we are led to the following

Corollary 2. The following functions

\[ H_{2,2}^{2,0} \left[ r \right. \left. \frac{(\beta - \alpha)}{(\xi - \frac{1}{2})}, (1, 1), \beta \geq \alpha, 0 < \tau \leq 2, d \geq 1, r > 0 \right], \]

are non-negatives.

Theorem 2. Under the conditions

\[ (H_2) : \tau \in (0, 2], \alpha \in (0, 1], \frac{1}{\alpha} - 1 < \beta, \gamma \in \mathbb{R}, \]

the function

\[ H_{1,2}^{2,0} \left[ \xi \right. \left. \frac{(1 - \frac{1}{2})}{(\xi - \frac{1}{2})}, (1, 1), \beta \geq \alpha, \gamma \in \mathbb{R} \right] \]

is non-negative.

Under the hypotheses \((H_1)\), Tomovski et al. [18], proved that the function \( e^{\gamma}_{\alpha,\beta}(t, \lambda) \) defined by

\[ e^{\gamma}_{\alpha,\beta}(t, \lambda) = t^\beta 1 E^{\gamma}_{\alpha,\beta}(-\lambda t^\alpha), \]

is completely monotonic on \((0, \infty)\). Let \( 0 < \tau < 2 \). Then the function \( g(\lambda) = \lambda^{\tau} \) is a Bernstein function. By using the fact that the composition of a completely monotone function and a Bernstein function is completely monotone, we deduce that the function \( e^{\gamma}_{\alpha,\beta}(t, g(\lambda)) = t^\beta 1 E^{\gamma}_{\alpha,\beta}(-\lambda t^\alpha) \) is completely monotone for \( 0 < \tau < 2, \alpha, \beta \in (0, 1), \gamma > 0, \beta \geq \alpha \gamma \) and consequently the function \( e^{\gamma}_{\alpha,\beta}(t, \lambda^{\tau}) \) is completely monotone under the hypotheses \((H_1)\). Therefore, by using the Schoenberg Theorem (see [17], Theorem 7.14), we gave that the function

\[ \mathcal{F}(\tau, x) \]

is positive definite on \( \mathbb{R}^d \). But the Fourier transform of positive definite function is non-negative. This implies that \( \mathcal{F}(\tau, x) \) is non-negative. So, the proof of Theorem 1 is complete. \( \square \)
Proof. In [19], Luchko and Kiryakova proved that the function
\[
1 \Psi_1 \left( \begin{bmatrix} 1 + \frac{\alpha}{2} + \frac{1}{2} \cdot \frac{\beta}{\gamma} \\ 1 + \frac{\alpha}{2} + \frac{1}{2} \cdot \frac{\beta}{\gamma} \end{bmatrix} - z \right) = H_{1,2}^{1,1} \left( z \right| \begin{bmatrix} 1 - \frac{1 + \alpha + \beta}{2} \\ (0,1), (1 - \frac{1 + \alpha + \beta}{2}) \end{bmatrix} \right)
\]
is completely monotonic on \((0, \infty)\) under the hypotheses \((H_2)\). Therefore, the function
\[
H_{1,2}^{1,1} \left( z \right| \begin{bmatrix} 1 - \frac{1 + \alpha + \beta}{2} \\ (0,1), (1 - \frac{1 + \alpha + \beta}{2}) \end{bmatrix} \right),
\]
is completely monotonic on \((0, \infty)\), and using the fact that the Fourier transform for a function positive definite function is non-negative, we obtain
\[
\text{Proof.}
\]
\[
(3.24)
\]
is non-negative. Finally, taking into account Property 2 of Lemma 1, we get
\[
\text{Finally, using the fact that the Fourier transform for a function positive definite function is non-negative, we deduce that the function } \mathcal{F}(G_{\alpha, \beta}^{\gamma, \tau})(\xi) \text{ is non-negative and this completes the proof of Theorem 2.}
\]

**Corollary 3.** The following Fox H-functions
\[
H_{1,2}^{1,1} \left( \tau \right| \begin{bmatrix} 1 + \frac{\alpha}{2} + \frac{1}{2} \cdot \frac{\beta}{\gamma} \\ (0,1), (1 - \frac{1 + \alpha + \beta}{2}) \end{bmatrix} \right), \quad \left( \alpha \in (0, 1], \frac{1}{\alpha} - 1 < \beta, \gamma \in \mathbb{R}, r > 0 \right),
\]
\[
H_{1,2}^{2,0} \left( \tau \right| \begin{bmatrix} 0, \frac{\alpha}{2} \\ (\frac{\alpha}{2} - \frac{\beta}{\gamma}, 1 - \frac{\beta}{\gamma}) \end{bmatrix} \right), \quad \left( \tau \in (0, 2], \alpha \in (0, 1], r > 0 \right),
\]
\[
H_{1,2}^{2,0} \left( \tau \right| \begin{bmatrix} 0, \frac{\alpha}{2} \\ (\frac{\alpha}{2} - \frac{\beta}{\gamma}, 1 - \frac{\beta}{\gamma}) \end{bmatrix} \right), \quad \left( 0 < \tau \leq 2, \frac{\tau}{2} - 1 < \beta, r > 0 \right),
\]
are non-negative.

**Proof.** Upon setting \(\tau = 2\), \((\alpha = \frac{1}{3}, \gamma = -\beta)\) and \(\alpha = -\frac{\alpha}{2} = \frac{2}{3}\), respectively in Theorem 2. Hence in view of Property 3 and Property 1 of Lemma 1 we obtain
\[
H_{1,2}^{2,0} \left( \right| \begin{bmatrix} 1 + \frac{\alpha}{2} + \frac{1}{2} \cdot \frac{\beta}{\gamma} \\ (0,1), (1 - \frac{1 + \alpha + \beta}{2}) \end{bmatrix} \right),
\]
\[
H_{1,2}^{2,0} \left( \right| \begin{bmatrix} 0, \frac{\alpha}{2} \\ (\frac{\alpha}{2} - \frac{\beta}{\gamma}, 1 - \frac{\beta}{\gamma}) \end{bmatrix} \right),
\]
\[
H_{1,2}^{2,0} \left( \right| \begin{bmatrix} 0, \frac{\alpha}{2} \\ (\frac{\alpha}{2} - \frac{\beta}{\gamma}, 1 - \frac{\beta}{\gamma}) \end{bmatrix} \right)
\]
are non-negative. Finally, taking in account the Property 2 of Lemma 1 in the above functions we get the desired results.

**Corollary 4.** The following functions
\[
H_{2,2}^{2,0} \left( \right| \begin{bmatrix} 1 + \frac{\alpha}{2} + \frac{1}{2} \cdot \frac{\beta}{\gamma} \\ (0,1), (1 - \frac{1 + \alpha + \beta}{2}) \end{bmatrix} \right), \quad \left( \alpha \in (0, 1], \frac{1}{\alpha} - 1 < \beta, \gamma \in \mathbb{R}, r > 0 \right),
\]
\[
H_{2,2}^{2,0} \left( \right| \begin{bmatrix} 0, \frac{\alpha}{2} \\ (\frac{\alpha}{2} - \frac{\beta}{\gamma}, 1 - \frac{\beta}{\gamma}) \end{bmatrix} \right), \quad \left( \alpha \in (0, 1], r > 0 \right),
\]
\[
H_{2,2}^{2,0} \left( \right| \begin{bmatrix} 0, \frac{\alpha}{2} \\ (\frac{\alpha}{2} - \frac{\beta}{\gamma}, 1 - \frac{\beta}{\gamma}) \end{bmatrix} \right), \quad \left( d \in \{2, 3\}, \frac{d}{2} - 3/2 < \beta, r > 0 \right),
\]
are non-negatives.

**Proof.** Taking \(d = 3\), \(\tau = 1\) and \(\tau = d - 1\) in the first, second and third functions defined in Corollary 3 and applications of identity (3.22) we get the desired results.
4. Applications: Monotonicity properties for some class of functions related to the Fox–Wright functions

The purpose of this section is twofold. First we derive the monotonicity of ratios for some class of functions related to the Fox–Wright functions. Second, we give sufficient conditions for some functions involving the Fox–Wright functions to be completely monotonic. The following Theorem are powerful tools to treat the monotonicity of ratios between two Fox–Wright functions.

Theorem 3. Let \( \psi : [a, b] \rightarrow (0, \infty) \) be a twice differentiable mapping on \([a, b], 0 \leq a < b\), such that the function \( t \mapsto t^{\psi(t)/t} \) is increasing (decreasing) on \([a, b]\). We define the function \( K_{n,m}^{q,p}[\cdot] \) by

\[
K_{q,p}^{n,m} (\alpha_p, \beta_p) \sigma, \delta; z = \frac{\int_a^b t^{-1} H_{n,m}^{q,p} (\frac{\alpha_p + \delta A_p}{\beta_p}, \beta_p) t^{(\psi(t))^{\sigma}} dt}{\int_a^b t^{-1} H_{q,p}^{n,m} (\alpha_p, \beta_p) \sigma, \delta; z}, \quad z, \delta > 0, \sigma \in \mathbb{R} - \{0\}.
\]

Assume that the function \( H_{n,m}^{q,p}[\cdot] \) is non-negative. Then the function \( K_{n,m}^{q,p}[z] \) is increasing (decreasing) if \( \sigma > 0 \) and decreasing (increasing) if \( \sigma < 0 \).

Proof. By means of Property 5 of Lemma \[1\] we can write the function \( K_{n,m}^{q,p}[z] \) in the following form:

\[
(4.26) \quad K_{n,m}^{q,p} (\alpha_p, \beta_p) \sigma, \delta; z = \frac{\int_a^b t^{-1} H_{n,m}^{q,p} (\frac{\alpha_p + \delta A_p}{\beta_p}, \beta_p) t^{(\psi(t))^{\sigma}} dt}{\int_a^b t^{-1} H_{q,p}^{n,m} (\alpha_p, \beta_p) \sigma, \delta; z}.
\]

Therefore,

\[
\left[ \int_a^b t^{-1} H_{q,p}^{n,m} (\frac{\alpha_p}{\beta_p}) t^{(\psi(t))^{\sigma}} dt \right]^2 \frac{\partial}{\partial z} K_{n,m}^{q,p} (\alpha_p, \beta_p) \sigma, \delta; z
\]

\[
= \sigma \left( \int_a^b t^{\delta-1} H_{n,m}^{q,p} (\frac{\alpha_p}{\beta_p}) \psi'(t) [\psi(t)]^{\sigma-1} dt \right) \left( \int_a^b t^{-1} H_{q,p}^{n,m} (\frac{\alpha_p}{\beta_p}) \psi'[t] [\psi(t)]^{\sigma} dt \right)
\]

\[
- \sigma \left( \int_a^b t^{\delta-1} H_{n,m}^{q,p} (\frac{\alpha_p}{\beta_p}) \psi'(t) [\psi(t)]^{\sigma} dt \right) \left( \int_a^b t^{-1} H_{q,p}^{n,m} (\frac{\alpha_p}{\beta_p}) \psi'[t] [\psi(t)]^{\sigma-1} dt \right).
\]

(4.27)

Putting

\[ p(t) = t^{-1} [\psi(t)]^{\sigma} H_{q,p}^{n,m} (\alpha_p, \beta_p) t, \quad f(t) = t^\delta, \quad g(t) = t^{\psi(t)/t}/\psi(t), \]

In the case when the function \( g \) is increasing, the function \( f \) and \( g \) are synchronous. Thus, by Lemma \[3\] we obtain

\[
\left( \int_a^b t^{-1} H_{q,p}^{n,m} (\frac{\alpha_p}{\beta_p}) \psi'(t) [\psi(t)]^{\sigma} dt \right) \left( \int_a^b t^{\delta-1} H_{q,p}^{n,m} (\frac{\alpha_p}{\beta_p}) \psi'[t] [\psi(t)]^{\sigma-1} dt \right)
\]

\[
(4.28) \quad \leq \left( \int_a^b t^{\delta-1} H_{q,p}^{n,m} (\frac{\alpha_p}{\beta_p}) [\psi(t)]^{\sigma} dt \right) \left( \int_a^b t^{-1} H_{q,p}^{n,m} (\frac{\alpha_p}{\beta_p}) \psi'[t] [\psi(t)]^{\sigma-1} dt \right).
\]

Then, keeping \( \[4.27\] \) and \( \[4.28\] \) in mind, we deduce that the function \( K_{n,m}^{q,p}[z] \) is increasing if \( \sigma > 0 \) and decreasing if \( \sigma < 0 \). Moreover, if the function \( g \) is decreasing then the inequality \( \[4.28\] \) is reversed and consequently the function \( K_{n,m}^{q,p}[z] \) is decreasing if \( \sigma > 0 \) and increasing if \( \sigma < 0 \). This completes the proof of Theorem \[3\].

Remark 1. We note that in the case when \( \delta < 0 \) and the function \( t \mapsto t^{\psi(t)/t}/\psi(t) \) is decreasing, we obtain the same monotonicity property of \( K_{n,m}^{q,p}[z] \) as in Theorem \[3\].

Theorem 4. Let \( \delta > 0 \). Under the conditions

\[ (H_3) : 0 < a_1 \leq \ldots \leq a_p, 0 < b_1 \leq \ldots \leq b_p, \sum_{j=1}^{k} b_j - \sum_{j=1}^{k} a_j \geq 0, k = 1, \ldots, p. \]

Then the ratio:

\[ z \mapsto \int_{p+1} \psi_p \left( \sigma, (a_p + \delta A_p) | \int_{p+1} \psi_p (\sigma, a_p) - z \right), \sigma > 0 \]
is decreasing on $(0, 1)$.

**Proof.** In [3, Theorem 4], the author proved that the Fox-Wright function $p+1 \Psi_p[z]$ possesses the following integral representation

\[
(4.29) \quad p+1 \Psi_p \left[ \frac{(\sigma, 1), (a_p, A_p)}{(b_p, A_p)} \right] - z = \int_0^1 H_{p, 0}^{p, 0} \left[ \left( \frac{(b_p, A_p)}{(a_p, A_p)} \right) \right] \frac{dt}{t(1 + z t)^{\sigma}}, \quad 0 < z < 1,
\]

under the conditions $(H_3)$, such that the Fox $H$-function $H_{p, 0}^{p, 0}(b_p, A_p)[t]$ is non-negative (see [3, Remark 2]). In our case, we let $\psi(t) = (1 + t)^{-1}$, it is clear that the function $t \mapsto t \psi'(t)/\psi(t)$ is decreasing on $(-1, \infty)$. So, applying Theorem [3] we deduce that the assertions asserted by Theorem [4] holds true. □

**Example 1.** The following function

\[
z \mapsto \varphi^\tau(b + \delta \tau, c + \delta \tau, -z), \quad (c > b > 0, \tau, \delta > 0, |z| < 1)
\]

is decreasing on $(0, 1)$ where $\varphi^\tau$ is the $\tau$-Kummer hypergeometric, defined by [6]

\[
\varphi^\tau(b, c, z) = \sum_{k=0}^{\infty} \frac{\Gamma(b + k \tau)}{\Gamma(c + k \tau)} \frac{z^k}{k!}, \quad (c > b > 0, \tau > 0, |z| < 1).
\]

**Theorem 5.** Assume that $\mu, \delta > 0$, $\sum_{i=1}^{p} A_i = \sum_{j=1}^{q} B_j$ and $\gamma \geq 1$. If the Fox $H$-function $H_{p, 0}^{p, 0}[]$ is non-negative, then the function

\[
z \mapsto p+1 \Psi_q \left[ \frac{(\sigma, \delta, 1), (a_p, \delta A_p, A_p)}{(b_p, \delta A_p, A_p)} \right] - z \right] / p+1 \Psi_q \left[ \frac{(a_p, A_p)}{(b_p, A_p)} \right] - z
\]

is decreasing on $(0, 1)$.

**Proof.** By virtue of Theorem [3] and Lemma [5], we can complete the proof of the above-asserted result immediately. □

**Corollary 5.** Let $\delta, \sigma > 0$. Assume that

\[
(H_4) : \tau \in (0, 1), d - \tau \geq 1, \beta > \frac{d}{2} + \frac{1}{2}.
\]

The function

\[
z \mapsto 3 \Psi_1 \left[ \frac{(\sigma, \delta, 1), (\delta - \delta \tau, \delta, \beta, \delta)}{(\beta - \tau, 1)} \right] - z \right] / 3 \Psi_1 \left[ \frac{(\sigma, 1), (\delta - \delta \tau, \beta, \delta)}{(\beta - \tau, 1)} \right] - z
\]

is decreasing on $(0, 1)$. Moreover, the function

\[
z \mapsto 3 \Psi_1 \left[ \frac{(\sigma, 1), (\delta - \delta \tau, \beta, \delta)}{(\beta - \tau, 1)} \right] - z
\]

is logarithmically completely monotonic on $(0, 1)$.

**Proof.** We consider the second function defined in Corollary [1]. In particular, the function

\[
H_{1, 2}^{2, 0} \left[ \frac{(\delta - \delta \tau, \beta, \delta)}{(\beta - \tau, 1)} \right],
\]

is non-negative. So, in this case the hypotheses of Theorem [5] is equivalent to the hypotheses $(H_4)$. Now, applying Theorem [3] and Lemma [5] respectively, we deduce that the above assertions holds. This ends the proof. □

**Corollary 6.** Let $\sigma, \delta > 0$. In addition assume that the hypotheses

\[
(H_5) : \begin{cases} \gamma \in \mathbb{R}, \alpha \in \left( \frac{1 + \gamma}{2}, 1 \right), \\
\min \left( d - 2, \frac{1 + \gamma}{\alpha} \right) \geq 1, \\
\frac{5}{2} > \frac{1 + \gamma}{\alpha} + \frac{d}{2}
\end{cases}
\]

holds true. Then the function

\[
z \mapsto 3 \Psi_1 \left[ \frac{(\sigma, \delta, 1), (\delta - \delta \tau, \beta, \delta)}{(1 + \gamma(1 - \alpha)) \frac{1 + \gamma}{\alpha} \delta} \right] - z \right] / 3 \Psi_1 \left[ \frac{(\sigma, 1), (\delta - \delta \tau, \beta, \delta)}{(1 + \gamma(1 - \alpha)) \frac{1 + \gamma}{\alpha} \delta} \right] - z
\]

is non-negative.
is decreasing on $(0,1)$. Moreover, the function
\[
\begin{aligned}
z \mapsto 3\Psi_1 \left[ \left( \sigma, 1 \right), \left( \frac{\gamma - 1}{\alpha}, \frac{\beta}{\alpha} \right), \left( 1 + \frac{\beta(1 - \alpha)}{\alpha}, \frac{\beta}{\alpha} \right) \right] - z 
\end{aligned}
\]
is logarithmically completely monotonic on $(0,1)$.

Proof. Again, by using Corollary 8, we have that the function
\[
H_{\alpha}^2 \left[ 
\begin{array}{c}
\left( \frac{\gamma - 1}{\alpha}, \frac{\beta}{\alpha} \right), \\
\left( 1 + \frac{\beta(1 - \alpha)}{\alpha}, \frac{\beta}{\alpha} \right)
\end{array}
\right] r > 0,
\]
is non-negative. In this case, the hypotheses of Theorem 5 is equivalent to the hypotheses of $(H_5)$. Applying Theorem 8 and Lemma 10 leads to the desired results.

Obviously, by repeating the same calculations as above with Theorem 5, Corollary 3 (third function) and Lemma 8 we can deduce the following result:

**Corollary 7.** Let $\sigma, \delta > 0$. Suppose also that
\[
(H_6): \begin{cases}
\tau \in (0,2], \frac{r}{2} - 1 < \beta, \frac{1}{\alpha} = \frac{1}{2} + \frac{1}{\sigma d}, \\
\tau > \frac{d}{2} + \frac{1}{2},
\end{cases}
\]
Then, the function
\[
z \mapsto 3\Psi_1 \left[ \left( \sigma, \delta \right), \left( \frac{\gamma - 1}{\alpha}, \frac{\beta}{\alpha} \right), \left( 1 + \frac{\beta(1 - \alpha)}{\alpha}, \frac{\beta}{\alpha} \right) \right] - z
\]
is decreasing on $(0,1)$. In addition, the function
\[
z \mapsto 3\Psi_1 \left[ \left( \sigma, 1 \right), \left( \frac{\gamma - 1}{\alpha}, \frac{\beta}{\alpha} \right), \left( 1 + \frac{\beta(1 - \alpha)}{\alpha}, \frac{\beta}{\alpha} \right) \right] - z
\]
is logarithmically completely monotonic on $(0,1)$.

By repeating the procedure of the proofs of the above Corollary, and make use Theorem 5 Corollary 8 (first function) and Lemma 8 leads us to the asserted results in Corollary 8.

**Corollary 8.** Let $\delta, \sigma > 0$. Under the conditions
\[
(H_7): \begin{cases}
\alpha \in (0,1], \gamma \in \mathbb{R}, 2(\gamma + \beta) \geq 1 \\
\frac{1}{\alpha} = 1 + \frac{1}{\alpha \alpha}, \frac{1}{\alpha} - 1 < \beta, \\
1 + \frac{\beta(1 - \alpha)}{\alpha} = 1 + \frac{1}{\alpha \alpha} > \frac{1}{\alpha},
\end{cases}
\]
The function
\[
z \mapsto 3\Psi_2 \left[ \left( \sigma, \delta \right), \left( 1, 1 \right), \left( \frac{\gamma - 1}{\alpha}, \frac{\beta}{\alpha} \right), \left( 1 + \frac{\beta(1 - \alpha)}{\alpha}, \frac{\beta}{\alpha} \right) \right] - z
\]
is decreasing on $(0,1)$. Furthermore, the function
\[
z \mapsto 3\Psi_2 \left[ \left( \sigma, 1 \right), \left( 1, 1 \right), \left( \frac{\gamma - 1}{\alpha}, \frac{\beta}{\alpha} \right), \left( 1 + \frac{\beta(1 - \alpha)}{\alpha}, \frac{\beta}{\alpha} \right) \right] - z
\]
is logarithmically completely monotonic on $(0,1)$.

**Theorem 6.** The following assertions are true:
1. The function
\[
z \mapsto 2\Psi_1 \left[ \left( \frac{\gamma - 1}{\alpha}, \frac{\beta}{\alpha} \right), \left( 1 + \frac{\beta(1 - \alpha)}{\alpha}, \frac{\beta}{\alpha} \right) \right] - z
\]
is completely monotonic on $(0, \infty)$ under the hypotheses of corollary 8.
2. The function
\[
z \mapsto 2\Psi_1 \left[ \left( \frac{\gamma - 1}{\alpha}, \frac{\beta}{\alpha} \right), \left( 1 + \frac{\beta(1 - \alpha)}{\alpha}, \frac{\beta}{\alpha} \right) \right] - z
\]
is completely monotonic on $(0, \infty)$ under the hypotheses of corollary 8.
3. The function
\[
z \mapsto 2\Psi_1 \left[ \left( \frac{\gamma - 1}{\alpha}, \frac{\beta}{\alpha} \right), \left( 1 + \frac{\beta(1 - \alpha)}{\alpha}, \frac{\beta}{\alpha} \right) \right] - z
\]
is completely monotonic on $(0, \infty)$, under the hypotheses of corollary 4.

4. The function

\[ z \mapsto 2\Psi_{2} \left[ (1, 1), \left( \frac{\gamma + \beta}{2\alpha \beta}, \frac{\beta}{2\beta} \right), (1, \frac{1}{2}) \right] - z, \]

is completely monotonic on $(0, \infty)$, under the hypotheses of corollary 8.

**Proof.** The above assertions follows immediately by combining Lemma 2 with Corollary 1 Corollary 3 respectively, under some restrictions on the parameters of the Fox H-functions which allow us to conclude that it is non-negative. \qed

**Theorem 7.** Letting $\eta_{1} = \sqrt{\pi 2^{2 - \frac{d}{2}} \frac{\gamma + \beta}{2\alpha \beta}}$, Assume that

\[ (H_{5}) : \tau \in (0, 1), d - \tau \geq 1. \]

Then, the functions

\[ g_{1} := z \mapsto 2\Psi_{1} \left[ \left( \frac{\gamma + \beta}{2\alpha \beta}, \frac{\beta}{2\beta}, \frac{1}{2}, (1 - \tau, 1) \right) \right] - z, \quad \text{and} \quad g_{2} := z \mapsto 2\Psi_{1} \left[ \left( \frac{\gamma + \beta}{2\alpha \beta}, \frac{\beta}{2\beta}, \frac{1}{2}, (1 - \tau, 1) \right) \right] - z = \eta_{1} e^{-\eta_{1}z}, \]

are completely monotonic on $(0, \infty)$.

**Proof.** We consider the second function defined in Corollary 1. In particular, the function

\[ H_{2,0}^{2,0} \left[ r \left( \frac{\gamma + \beta}{2\alpha \beta}, \frac{\beta}{2\beta}, \frac{1}{2}, (1 - \tau, 1) \right) \right] \]

is non-negative. Here, the parameters of the above function and the hypotheses $(H_{5})$ satisfies the hypotheses of Lemma 3. Now, applying Lemma 3 we obtain that the function $g_{1}$ and $g_{2}$ are completely monotonic on $(0, \infty)$. \qed

**Theorem 8.** Let

\[ \eta_{2} = \sqrt{2\pi} \left( \frac{1}{2} \right)^{\frac{d}{2}} \frac{\gamma + \beta}{2\alpha \beta} \left( \frac{1 - \alpha}{2\alpha} \right)^{\frac{2 + 2\gamma + (1 - \alpha - \frac{1}{2})}{2\gamma + \beta}}, \quad \rho_{2} = \sqrt{2} \left( \frac{1}{2\alpha} \right)^{\frac{\gamma - \frac{1}{2}}{2\gamma + \beta}}. \]

Assume that the hypotheses

\[ (H_{2}) : \begin{cases}
\alpha \in \left[ \frac{3 - \sqrt{5}}{2}, 1 \right], \gamma \in \mathbb{R}, \\
\min \left( d - 2, 2 \left( \gamma + \frac{1}{2\alpha} \right) \right) \geq 1, \\
\frac{1 + \gamma (1 - \alpha)^{2}}{\alpha} + \frac{d}{2} \leq \frac{5}{2},
\end{cases} \]

holds true. Then, the functions

\[ g_{3} := z \mapsto 2\Psi_{1} \left[ \left( \frac{\gamma + \beta}{2\alpha \beta}, \frac{\beta}{2\beta}, \frac{1}{2}, (1 - \tau, 1) \right) \right] - z, \quad \text{and} \quad g_{4} := z \mapsto 2\Psi_{1} \left[ \left( \frac{\gamma + \beta}{2\alpha \beta}, \frac{\beta}{2\beta}, \frac{1}{2}, (1 - \tau, 1) \right) \right] - z = \eta_{2} e^{-\rho_{2}z}, \]

are completely monotonic on $(0, \infty)$.

**Proof.** By using the Corollary 3 the function

\[ H_{2,0}^{2,0} \left[ r \left( \frac{1 + \gamma (1 - \alpha)^{2}}{\alpha}, \frac{1 - \alpha}{2\alpha}, (1 - \frac{1}{2}) \right) \right] r > 0, \]

is non-negative and your parameters with hypotheses $(H_{2})$ satisfies the statements of Lemma 4 and consequently the function $g_{3}$ and $g_{4}$ are completely monotonic on $(0, \infty)$.

**Theorem 9.** Let

\[ \eta_{3} = \sqrt{2} \left( \frac{1}{2\alpha} \right)^{\frac{\gamma + \beta}{2\alpha \beta}} \left( \frac{1 - \alpha}{2\alpha \beta} \right)^{\frac{\gamma + \beta}{2\alpha \beta}}, \quad \rho_{3} = \left( \frac{1}{2} \right)^{\frac{\gamma + \beta}{2\alpha \beta}} \left( \frac{1}{2\alpha \beta} \right)^{\frac{\gamma + \beta}{2\alpha \beta}}, \]

such that the following hypotheses

\[ (H_{10}) : \begin{cases}
\alpha \in (0, 1], \frac{1}{\alpha} - 1 < \beta, \gamma \in \mathbb{R}, \\
2(\gamma + \beta) \geq 1, \\
1 + \frac{\gamma}{\beta} \left( 1 - \frac{1}{\alpha} \right) = 1 + \frac{1}{\alpha \beta} = \frac{1}{\alpha},
\end{cases} \]

holds true. Then the functions

\[ g_{5} := z \mapsto 2\Psi_{2} \left[ (1, 1), \left( \frac{\gamma + \beta}{2\alpha \beta}, \frac{\beta}{2\beta}, \frac{1}{2}, (1 - \frac{1}{\alpha}, \frac{1}{2}) \right) \right] - z, \quad \text{and} \quad g_{6} := z \mapsto 2\Psi_{2} \left[ (1, 1), \left( \frac{\gamma + \beta}{2\alpha \beta}, \frac{\beta}{2\beta}, \frac{1}{2}, (1 - \frac{1}{\alpha}, \frac{1}{2}) \right) \right] - z = \eta_{3} e^{-\rho_{3}z}, \]
are completely monotonic on \((0, \infty)\).

**Proof.** Is an applications of Lemma \ref{lem:1}, just we observe that the parameter of the first function of Corollary \ref{cor:1} and the hypotheses \((H_{10})\) satisfies the hypotheses of Lemma \ref{lem:1}. \(\square\)

**References**

[1] A.M. Mathai, R. K. Saxena H. J. Haubold The \(H\)-functions: Theory and applications, Springer (2010).

[2] C. Berg, Integral representation of some functions related to the gamma function, Mediterr. J. Math. 1 (2004) 433–439.

[3] K. Mehrez, New Integral representations for the Fox-Wright functions and its applications, J. Math. Anal. Appl. 468 (2018), 650–673.

[4] K. Mehrez, New Integral representations for the Fox-Wright functions and its applications II, arXiv:2439586.

[5] D. S. Mitrinović, J. E. Pečarić, A. M. Fink, Classical and New Inequalities in Analysis, Kluwer Academic Publishers (1993), D. S. Mitrinović, Analytic Inequalities, Springer-Verlag, Berlin, 1970.

[6] N. Virchenko, On some generalizations of the functions of hypergeometric type, Fract. Calc. Appl. Anal., 2 (3) (1999), 233–244.

[7] V.V. Anh, N.N. Leonenko, Spectral analysis of fractional kinetic equations with random data, J. Statist. Phys. 104 (2001), 1349–1387.

[8] R.L. Schilling, R. Song, Z. Vondracek, Bernstein Functions. Theory and Applications, De Gruyter, Berlin, 2010.

[9] R. Hilfer, Fractional time evolution, in: R. Hilfer (Ed.), Applications of Fractional Calculus in Physics, World Scientific, Singapore, 2000, pp. 87-130.

[10] A.A. Kilbas, M. Saigo, On the \(H\) functions, J. Appl. Math. Stochastic Anal. 12 (1999) 191204.

[11] V. Kiryakova, Generalized Fractional Calculus and Applications, Pitman Research Notes in Mathematics, vol. 301, Longman, Harlow, 1994.

[12] A.M. Mathai, R.K. Saxena, The \(H\)-function with Applications in Statistics and Other Disciplines, Wiley Eastern Ltd., New Delhi, 1978.

[13] A. A. Kilbas, H. M. Srivastava, J. J. Trujillo, Theory and Applications of Fractional Differential Equations, Elsevier, Amsterdam, 2006.

[14] B.-N. Guo, F. Qi, A property of logarithmically absolutely monotonic functions and the logarithmically complete monotonicity of a power-exponential function, Politehn. Univ. Bucharest Sci. Bull. Ser. A Appl. Math. Phys. 72 (2) (2010) 21–30.

[15] T. R. Prabhakar, A singular integral equation with a generalized Mittag-Leffler function in the kernel, Yokohama Math. J. 19 (1971), 7–15.

[16] E. M. Wright, The asymptotic expansion of the generalized hypergeometric function, Journal London Math. Soc. 10 (1935), 287–293.

[17] H. Wendland, Scattered Data Approximation, Cambridge University Press, 2005.

[18] Zivorad Tomovski, Tibor K. Pogány, H.M. Srivastava, Laplace type integral expressions for a certain three-parameter family of generalized mittag-leffler functions with applications involving complete monotonicity, J. Franklin Institute, 351 (12) (2014), 5437–5454.

[19] Yu. Luchko, V. Kiryakova, The Mellin integral transform in fractional calculus, Fract. Calc. Appl. Anal. 16 (2013), 405–430.

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