A Log-Dagum Weibull Distribution: Properties and Characterization
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Abstract
Developments of new probability models for data analysis are keen interest of importance for all fields. The log-Dagum distribution has a prominent role in the theory and practice of statistics. In this article, a new family of continuous distributions generated from a log Dagum random variable called the log-Dagum Weibull distribution is proposed. The key properties of the proposed distribution are derived. Its density function can be symmetrical, left-skewed, right-skewed and reversed-J shaped and can have increasing, decreasing, bathtub hazard rates shaped. The model parameters are estimated by the method of maximum likelihood and illustrate its importance by means of applications to real data sets.

Keywords: probability distributions; log-dagum distribution; parameter estimation; weibull distribution

1. Introduction

Statistical distributions are extensively used in literature for modelling and forecasting real life phenomena. The recent literature has suggested several ways of extending well-known distributions. There has been an increased interest in defining new classes of univariate continuous distributions by introducing one or more additional shape parameter(s) to the baseline distribution. This induction of parameter(s) has been proved useful in exploring tail properties and also for improving the goodness-of-fit of the generator family. The well-known families are: the beta-G [9], Kumaraswamy-G [6], McDonald-G [3], Gamma-X [2], Gamma-G (type 1) [18], Gamma-G (type 2) [15], Gamma-G (type 3) [17], Log-Gamma-G [4], Logistic-G [16], Exponentiated Generalized-G [7], Transformed-transformer [2], Exponentiated T-X [2], Weibull-G [5], etc.
The proposed new distribution generalizes the log-dagum Weibull distributions. Some structural properties of this distribution are obtained and estimation the parameters via the Method of maximum likelihood presented.
This paper is organized as follows. In section 2; we present the generalized distribution including the corresponding probability density functions (pdf), survival function hazard
functions shape of hazard function and concavity. In section 3; Rth moments, L- moments quantile function and order statistics are presented. Section 4 contains the Shannon entropy and Renyi entropy. Section 5 Bonferroni and Lorenz curves. Section 6 is concerned with characterization via Hazard function, reverse hazard function and truncated moments of distribution. Estimation of model parameters is presented in section 7. Evaluation Measures and Practical Data Examples of the proposed model to real data are given in section 8, followed by concluding remarks.

Let \( r(t) \) be the probability density function (pdf) of a random variable \( T \in [a; b] \) for \(-\infty \leq a < b < \infty\) and let \( W[G(x)] \) be a function of the cumulative distribution function (cdf) of a random variable \( X \) such that \( W[G(x)] \) satisfies the following conditions:

1. \( W[G(x)] \in [a; b] \);
2. \( W[G(x)] \) is differentiable and monotonically non-decreasing, and
3. \( W[G(x)] \to a \) as \( x \to -\infty \) and \( W[G(x)] \in b \) as \( x \to \infty \)

Recently, Alzaatreh et al. (2013) defined the \( T-X \) family of distributions by

\[
F(x) = \int_0^{W[G(x)]} r(t) \, dt
\]

Where \( W[G(x)] \) satisfies the condition (1). The pdf corresponding to (2) is given by

\[
f(x) = \left[ \frac{d}{dx} W[G(x)] \right] r(W[G(x)])
\]

In Table 1, we provide the \( W[G(x)] \) functions for some members of the \( T-X \) family of distributions.

**Table 1: Different \( W[G(x)] \) functions for special models of the \( T-X \) family**

| S.No. | \( W[G(x)] \) | Range of \( T \) | Members of \( T-X \) family |
|-------|-----------------|-----------------|-----------------------------|
| 1     | \( G(x) \)      | \([0,1]\)        | Beta-G (Eugene et al., 2002) |
|       |                 |                 | Kw-G type 1 (Cordeiro and de Castero, 2011) |
|       |                 |                 | Mc-G (Alexander et al., 2012) |
|       |                 |                 | Exp-G (Kw-G type 2) (Cordeiro et al., 2013) |
| 2     | \(- \log [G(x)]\) | \([0,\infty]\) | Gamma-G Type-2 (Risti´c and Balakrishnan, 2012) |
|       |                 |                 | Log-Gamma-G Type-2 (Amini et al., 2012) |
| 3     | \(- \log [1 - G(x)]\) | \([0,\infty]\) | Gamma-G Type-1 (Zografos and Balakrishnan, 2009) |
|       |                 |                 | Log-Gamma-G Type-1 (Amini et al., 2012) |
|       |                 |                 | Weibull-X (Alzaatreh et al., 2013) |
|       |                 |                 | Gamma-X (Alzaatreh et al., 2014) |
| 4     | \(- \log [1 - G^\alpha(x)]\) | \([0,\infty]\) | Exponentiated T-X (Alzaghal et al., 2013) |
| 5     | \( \log [- \log [1 - G(x)]] \) | \([-\infty, \infty]\) | Logistic-X |
The aim of this paper is to propose a new family of continuous distributions, called the log Dagum Weibull distribution, and to study some of its mathematical properties.

### 2. The Log Dagum Weibull Distribution

A random variable $T$ has the log Dagum distribution with shape parameter $\beta > 0$ and $\lambda > 0$ if its cumulative distribution function (cdf) is given by

$$\pi(t) = (1 + e^{-\lambda t})^{-\beta} \quad t \in R, \quad \beta > 0, \quad \lambda > 0$$

and its corresponding probability density function (pdf) can be expressed as

$$r(t) = \beta \lambda e^{-\lambda t} (1 + e^{-\lambda t})^{-\beta-1} \quad t \in R, \quad \beta > 0, \quad \lambda > 0$$

Let $G(x)$ and $\hat{G}(x) = 1 - G(x)$ be the baseline cdf and survival function (sf) by replacing $W[G(x)]$ by $\log\left(\frac{G(x)}{1-G(x)}\right)$ and $r(t)$ with (5) in equation (2), we define the cdf of the Log Dagum-X family by

$$F(x) = \left[1 + \left(\frac{G(x)}{1-G(x)}\right)^{-\lambda}\right]^{-\beta}$$

The Log Dagum family pdf is expressed as

$$f(x) = \left[1 + \left(\frac{G(x)}{1-G(x)}\right)^{-\lambda}\right]^{-\beta-1} \left(\frac{G(x)}{1-G(x)}\right)^{-\lambda-1} \frac{\lambda \beta g(x)}{(1-G(x))^2}$$

Henceforth, we denote by $X$ a random variable having density function (7). The basic motivations for using the Log Dagum-x family in practice are to construct heavy tailed distributions that are not longer-tailed for modelling real data, to generate distributions with symmetric, left-skewed, right-skewed and reversed-J shaped, to define special models with all types of the hazard rate function (hrf), to provide consistently better fits than other generated models under the same baseline distribution. The fact is well-demonstrated by fitting the log Degum Weibull distribution to two real data sets. However, we expect that there are other contexts in which the LX special models can produce worse fits than other generated distributions. Clearly, the results indicate that the new family is a very competitive class to other widely known generators with one extra shape parameter.
The corresponding cumulative density function (cdf) probability density function (pdf), hazard function (hrf) and survival function are given as

\[ F_{LDW}(x, \lambda, \theta, \beta) = (1 + (e^{\theta x^\beta} - 1)^{-\lambda})^{-\beta} \]  

(Figure 1 about here.)

Figure 1 gives the plots of the cumulative distribution function of the LDW distribution. The plots of this figure shows that for fixed \( \lambda \) and \( \beta \) and changing \( \theta \) the curve stretch out insignificantly towards right as \( \theta \) increases. However, for fixed \( \beta \) and \( \theta \) and changing \( \lambda \) the curve stretch out towards right significantly as \( \lambda \) increases.

And

\[ f_{LDW}(x, \lambda, \theta, \beta) = (1 + (e^{\theta x^\beta} - 1)^{-\lambda})^{-\beta-1}(-1 + e^{\theta x^\beta})^{-\lambda-1} e^{\theta x^\beta} \theta \lambda \beta^2 x^\beta-1 \]  

(Figure 2 about here.)

Plots of Figure 2 display the density functions of the LDW distribution. Figure 2 portrays that changing \( \lambda \) against the fixed \( \beta \) and \( \theta \) the density function decreases. but changing \( \beta \) against the fixed \( \lambda \) the nature of the curve towards right as \( \theta \) increases, however in case of changing \( \theta \) with fixed \( \beta \) and \( \lambda \) shift the curve towards left.

\[ S_{LDW}(x, \lambda, \theta, \beta) = 1 - (1 + (e^{\theta x^\beta} - 1)^{-\lambda})^{-\beta} \]  

(Figure 4 about here.)

The graph of survival function increases for different values of parameters then suddenly starts gradually decreases and converges to zero.

\[ h_{LDW}(x, \lambda, \theta, \beta) = \frac{(1+(e^{\theta x^\beta} - 1)^{-\lambda})^{-\beta-1}(-1 + e^{\theta x^\beta})^{-\lambda-1} e^{\theta x^\beta} \theta \lambda \beta^2 x^\beta-1}{1-(1 + (e^{\theta x^\beta} - 1)^{-\lambda})^{-\beta}} \]  

(Figure 3 about here.)

Hazard function is a significant indicator for observing the declining circumstance of a product which ranges from increasing, decreasing, bathtub (BT) shapes. So in this regard Figure 3 speaks out itself and justifies the potential of the model. Moreover, the hazard function plots in Figure 3 also portray the declining circumstance of the product as time increases in terms of impulsive spikes at the end of either increasing or decreasing hazard rate. This implies that the hazard function is sensitive against different combinations of the parameters as time changes,
which seems to be a refine image of non stationary process and hence the hazard curve does not remain stable as times passes. Moreover, Figure 3 displays increasing, decreasing bathtub hazard shapes.

**Shape of hazard function:**

Shape of the density function can be described analytically. The critical point of the LDW density are the root of the equation

\[
\frac{1+e^{\theta x^\beta-1}}{1-(e^{\theta x^\beta-1})^{-\beta}} - \frac{1+e^{\theta x^\beta}}{1-\lambda (e^{\theta x^\beta-1})^{-\beta}} e^{\theta x^\beta} x^2 \beta^2 \theta^2 \beta^3 (-1-\lambda) \lambda
\]

There may be more than one root.

**Concavity:**

The concavity of hazard rate function \(h''(x) = 0\)

\[
\frac{e^{\theta x^\beta}(1+e^{\theta x^\beta})^{-\lambda}}{1-(e^{\theta x^\beta})^{-\lambda}} - \frac{e^{\theta x^\beta}}{1-(e^{\theta x^\beta})^{-\lambda}} e^{\theta x^\beta} x^2 \beta^2 (-1-\lambda) \theta^2 \beta^2 \lambda^2 + \frac{e^{\theta x^\beta}(1+e^{\theta x^\beta})^{-\lambda}}{1-(e^{\theta x^\beta})^{-\lambda}} e^{\theta x^\beta} x^2 \beta^2 (1+e^{\theta x^\beta})^{-\lambda} \lambda
\]

\[
\frac{e^{\theta x^\beta}(1+e^{\theta x^\beta})^{-\lambda}}{1-(e^{\theta x^\beta})^{-\lambda}} e^{\theta x^\beta} x^2 \beta^2 (-1-\lambda) \theta^2 \beta^2 \lambda^3 + \frac{e^{\theta x^\beta}(1+e^{\theta x^\beta})^{-\lambda}}{1-(e^{\theta x^\beta})^{-\lambda}} e^{\theta x^\beta} x^2 \beta^2 (1+e^{\theta x^\beta})^{-\lambda} \lambda
\]

\[
\frac{e^{\theta x^\beta}(1+e^{\theta x^\beta})^{-\lambda}}{1-(e^{\theta x^\beta})^{-\lambda}} e^{\theta x^\beta} x^2 \beta^2 (-1-\lambda) \theta^2 \beta^3 \lambda + \frac{e^{\theta x^\beta}(1+e^{\theta x^\beta})^{-\lambda}}{1-(e^{\theta x^\beta})^{-\lambda}} e^{\theta x^\beta} x^2 \beta^2 (1+e^{\theta x^\beta})^{-\lambda} \lambda
\]
\[3e^{2\theta x} \left(-1 + e^{\theta x}\right)^{2\lambda - 2} \left( 1 + e^{\theta x} \right)^{-1 - \lambda} \frac{\beta - 1}{1 - (1 + (e^{\theta x} - 1)^{-\lambda})^{-\beta}} \times 3^\beta - 3 \beta^4 \theta^3 \beta^3 (1 - \lambda) \lambda \]

\[e^{3\theta x} \left(-1 + e^{\theta x}\right)^{2\lambda - 2} \left( 1 + e^{\theta x} \right)^{-1 - \lambda} \frac{\beta - 1}{1 - (1 + (e^{\theta x} - 1)^{-\lambda})^{-\beta}} \times 3^\beta - 3 \beta^4 \theta^3 (1 - \lambda) \lambda \]

\[e^{2\theta x} \left(-1 + e^{\theta x}\right)^{2\lambda - 2} \left( 1 + e^{\theta x} \right)^{-1 - \lambda} \frac{\beta - 1}{1 - (1 + (e^{\theta x} - 1)^{-\lambda})^{-\beta}} \times 3^\beta - 3 \beta^4 \theta^3 (1 - \lambda) \lambda \]

\[e^{2\theta x} \left(-1 + e^{\theta x}\right)^{2\lambda - 2} \left( 1 + e^{\theta x} \right)^{-1 - \lambda} \frac{\beta - 1}{1 - (1 + (e^{\theta x} - 1)^{-\lambda})^{-\beta}} \times 3^\beta - 3 \beta^4 \theta^3 (1 - \lambda) \lambda \]

\[3e^{2\theta x} \left(-1 + e^{\theta x}\right)^{2\lambda - 2} \left( 1 + e^{\theta x} \right)^{-1 - \lambda} \frac{\beta - 1}{1 - (1 + (e^{\theta x} - 1)^{-\lambda})^{-\beta}} \times 3^\beta - 3 \beta^5 \theta^3 \lambda^2 \]

\[e^{3\theta x} \left(-1 + e^{\theta x}\right)^{2\lambda - 2} \left( 1 + e^{\theta x} \right)^{-1 - \lambda} \frac{\beta - 1}{1 - (1 + (e^{\theta x} - 1)^{-\lambda})^{-\beta}} \times 3^\beta - 3 \beta^5 \theta^3 (2 - 2 \lambda) \lambda^2 \]

\[e^{3\theta x} \left(-1 + e^{\theta x}\right)^{2\lambda - 2} \left( 1 + e^{\theta x} \right)^{-1 - \lambda} \frac{\beta - 1}{1 - (1 + (e^{\theta x} - 1)^{-\lambda})^{-\beta}} \times 3^\beta - 3 \beta^5 \theta^3 (2 - 2 \lambda) \lambda^2 \]

\[e^{3\theta x} \left(-1 + e^{\theta x}\right)^{2\lambda - 2} \left( 1 + e^{\theta x} \right)^{-1 - \lambda} \frac{\beta - 1}{1 - (1 + (e^{\theta x} - 1)^{-\lambda})^{-\beta}} \times 3^\beta - 3 \beta^5 \theta^3 (2 - 2 \lambda) \lambda^2 \]

\[e^{3\theta x} \left(-1 + e^{\theta x}\right)^{2\lambda - 2} \left( 1 + e^{\theta x} \right)^{-1 - \lambda} \frac{\beta - 1}{1 - (1 + (e^{\theta x} - 1)^{-\lambda})^{-\beta}} \times 3^\beta - 3 \beta^5 \theta^3 (2 - 2 \lambda) \lambda^2 \]

\[e^{3\theta x} \left(-1 + e^{\theta x}\right)^{2\lambda - 2} \left( 1 + e^{\theta x} \right)^{-1 - \lambda} \frac{\beta - 1}{1 - (1 + (e^{\theta x} - 1)^{-\lambda})^{-\beta}} \times 3^\beta - 3 \beta^5 \theta^3 (2 - 2 \lambda) \lambda^2 \]
for different value of parameters the hazard function is concave up and concave down where the point concavity change is called point of inflection.

3. Some Statistical Properties

In this section, we study some statistical properties of the LDW distribution, including Rth moments, L- moments quantile function and order statistics.

3.1 Moments of LDWD.

Let X is a particularly continuous non-negative random variable with PDF \( f(X) \), and then the Rth ordinary moment of the (LDW) distribution is given by:

\[
E(X^r) = \int_0^\infty X^r f(X)dX
\]

\[
E(X^r) = \int_0^\infty X^r (1 - (1 - e^{\theta x})^{-\lambda})^{-\beta-1}(1 - e^{\theta x})^{-\lambda-1}e^{\theta x} \theta \lambda \beta^2 x^{\beta-1} dx
\]

\[
E(X^r) = \sum_{h=0}^\infty \frac{(-1)^h}{h!} \int_0^\infty X^r ((1 - e^{\theta x})^{-\lambda})^{-\beta-1}(1 - e^{\theta x})^{-\lambda-1}e^{\theta x} \theta \lambda \beta^2 x^{\beta-1} dx
\]

\[
E(X^r) = \int_0^\infty X^r (1 - e^{\theta x})^{-\lambda} \theta \lambda \beta^2 x^{\beta-1} dx
\]

Let \( y = e^{\theta x} \)

\[
dy = e^{\theta x} \theta \beta x^{\beta-1} dx \quad x = \frac{1}{\theta} \ln(y)^{\frac{1}{\beta}}
\]

\[
E(X^r) = \lambda \beta \sum_{h=0}^\infty \frac{(-k)_h (-1)^h}{h!} \int_0^\infty \left(\frac{1}{\theta} \ln(y)^{\frac{1}{\beta}}\right)^r (1 - y)^{\lambda \beta - 1} dy
\]

\[
E(X^r) = \sum_{h=0}^\infty \frac{(-k)_h (-1)^h}{h!} \sum_{m=0}^\infty \frac{(-1)^m}{m!} \int_0^\infty (\ln y)^{\frac{r}{\beta}} y^m dy
\]

Let \( \ln y = -z \quad y = e^{-z} \quad dy = -e^{-z} dz \)
\[ E(x^r) = \sum_{h=0}^{\infty} \frac{(-k)_h(-1)^h}{h!} \sum_{m=0}^{\infty} \frac{(-1)^m}{m!} \int_0^\infty (-z)^r \beta e^{-zm}(-e^{-z})dz \]

\[ E(x^r) = \lambda \beta \frac{1}{\theta^r} \sum_{h=0}^{\infty} \frac{(-k)_h(-1)^h}{h!} \sum_{m=0}^{\infty} \frac{(-1)^m}{m!} \int_0^\infty (z)^r \beta e^{-x(m+1)}dz \]

Let \[ z(m+1) = g \quad z = \frac{g}{(m+1)} \quad dy = \frac{dg}{(m+1)} \]

\[ E(x^r) = \lambda \beta \frac{1}{\theta^r} \sum_{h=0}^{\infty} \frac{(-k)_h(-1)^h}{h!} \sum_{m=0}^{\infty} \frac{(-1)^m}{m!} \frac{1}{(m+1)^{r+1}} \Gamma \left( \frac{r}{\beta} + 1 \right) \]

### 3.2 L-Moments:

\[ B_k = E(x(F(x))^k) \]

Where \[ F(x) = (1 + (e^{\theta x}_\beta - 1)^-\lambda)^{-k\beta} \]

\[ B_k = E(x(1 + (e^{\theta x}_\beta - 1)^-\lambda)^{-k\beta}) \]

\[ B_k = \int_0^\infty x \left( 1 - (1 - e^{\theta x}_\beta)^{-\lambda} \right)^{-k\beta} f(x)dx \]

\[ B_k = \int_0^\infty x \left( 1 - (1 - e^{\theta x}_\beta)^{-\lambda} \right)^{-k\beta} \left( (1 - (1 - e^{\theta x}_\beta)^{-\lambda})^{-1} (1 - e^{\theta x}_\beta)^{-\lambda-1} e^{\theta x}_\beta \theta \lambda \beta^2 x^{-1} \right) dx \]

\[ = \int_0^\infty x \left( 1 - (1 - e^{\theta x}_\beta)^{-\lambda} \right)^{-\beta(k+1)-1} (1 - e^{\theta x}_\beta)^{-\lambda-1} e^{\theta x}_\beta \theta \lambda \beta^2 x^{-1} dx \]

Let \[ e^{\theta x}_\beta = y, \quad e^{\theta x}_\beta \theta x^{-1} dx = dy, \quad x = \frac{1}{\theta^r} (lny)_1^\infty \]

When \[ x \to 0, \quad y \to 1 \quad \text{and} \quad x \to \infty, \quad y \to \infty \]

\[ B_k = \frac{\lambda \beta}{\theta^r} \sum_{h=0}^{\infty} \frac{(-k)_h(-1)^h}{h!} \int_0^\infty (lny)_1^\infty (1 - y)^{\beta(k+1)-1} dy \]

Expanding \( (1 - y)^{\beta(k+1)-1} \) using binomial expansion

\[ B_k = \frac{\lambda \beta}{\theta^r} \sum_{h=0}^{\infty} \frac{(-k)_h(-1)^h}{h!} e^{m+1} \sum_{k=0}^{\infty} \sum_{m=0}^{\infty} \left( \lambda \beta(k+1)-1 \right)^{-2m} \int_1^\infty (lny)_1^\infty (y)^m dy \]

Let \[ lny = -z \quad y = e^{-z} \quad dy = -e^{-z} dz \]

When \[ y \to 1, \quad z \to 0 \quad \text{when} \quad y \to \infty, \quad z \to \infty \]

\[ = \frac{\lambda \beta}{\theta^r} \sum_{h=0}^{\infty} \frac{(-k)_h(-1)^h}{h!} e^{m+1} \sum_{k=0}^{\infty} \sum_{m=0}^{\infty} \left( \lambda \beta(k+1)-1 \right)^{-2m} \int_1^\infty (lny)_1^\infty (y)^m dy \]
\[
\frac{\lambda \beta}{\theta^\beta} \sum_{h=0}^{\infty} \frac{(-k)^h (-1)^h}{h!} e^{m+1} \sum_{k=0}^{\infty} \sum_{m=0}^{(\lambda \beta (k+1)-1)} \left( \lambda \beta (k+1) - 1 \right) \left( \frac{1}{m} \right) \left( -1 \right)^{2m} \int_0^{\infty} e^{-z m} \frac{1}{z} e^{-\frac{1}{z}} dz
\]

\[
\frac{\lambda \beta}{\theta^\beta} \sum_{h=0}^{\infty} \frac{(-k)^h (-1)^h}{h!} e^{m+1} \sum_{k=0}^{\infty} \sum_{m=0}^{(\lambda \beta (k+1)-1)} \left( \lambda \beta (k+1) - 1 \right) \left( \frac{1}{m} \right) \left( -1 \right)^{2m} \int_0^{\infty} e^{-z (m+1)} \frac{1}{z} e^{-\frac{1}{z}} dz
\]

Let \( z(m+1) = g \)

\[
\frac{g}{z(m+1)} = z
\]

\[
\frac{dg}{z(m+1)} = dz
\]

When \( z \to 0 \), \( g \to 0 \) \quad \text{When} \quad z \to \infty \), \( g \to \infty \)

\[
\frac{\lambda \beta}{\theta^\beta} \sum_{h=0}^{\infty} \frac{(-k)^h (-1)^h}{h!} e^{m+1} \sum_{k=0}^{\infty} \sum_{m=0}^{(\lambda \beta (k+1)-1)} \left( \lambda \beta (k+1) - 1 \right) \left( \frac{1}{m} \right) \left( -1 \right)^{2m} \left( \frac{1}{m+1+1} \right) \int_0^{\infty} e^{-z} \frac{1}{g} e^{-\frac{1}{g}} dg
\]

\[
B_k = \frac{\lambda \beta}{\theta^\beta} \sum_{h=0}^{\infty} \frac{(-k)^h (-1)^h}{h!} e^{m+1} \sum_{k=0}^{\infty} \sum_{m=0}^{(\lambda \beta (k+1)-1)} \left( \lambda \beta (k+1) - 1 \right) \left( \frac{1}{m} \right) \left( -1 \right)^{2m} \left( \frac{1}{m+1+1} \right)
\]

### 3.3 Quantile function

The quantile function is another way of describing a probability distribution. It can also be called the inverse cdf. It can be used to generate random samples for probability distributions and thereby can serve as an alternative to the pdf. In general, it is given as:

\[
G(x) = u
\]

\[
(1 + (e^{\theta x}) - 1)^{-\beta} = u
\]

\[
1 + (e^{\theta x}) - 1^{-\lambda} = u^{-1/\lambda}
\]

\[
(e^{\theta x}) - 1^{-\lambda} = u^{-1/\lambda} - 1
\]

\[
e^{\theta x} - 1 = \left( u^{-\frac{1}{\theta}} - 1 \right)^{-1/\lambda}
\]

\[
e^{\theta x} = \left( u^{-\frac{1}{\theta}} - 1 \right)^{-1/\lambda} + 1
\]

\[
x = \left[ \frac{1}{\theta} \log \left( u^{-\frac{1}{\theta}} - 1 \right) \right]^{-1/\beta}
\]

### 3.4 Ordered Statistic

The pdf of the jth order statistic for a random sample of size n from a distribution function \( F(x) \) and an associated pdf \( f(x) \) is given by:
\[ f_{j,n}(x) = \frac{n!}{(j-1)(n-j)!} \left[ F(x) \right]^{j-1} \left[ 1 - F(x) \right]^{n-j} f(x) \]

where \( f(x) \) and \( F(x) \) are the pdf and cdf of the LDWD, respectively. The pdf of the jth order statistics for a random sample of size \( n \) from the LDW distribution is, however, given as follows

\[ f_{j,n}(x) = \frac{n!}{(j-1)(n-j)!} \left[ \left( 1 + \left( e^{\theta x^\beta} - 1 \right)^{-\lambda} \right)^{-\beta} \right]^{j-1} \left[ 1 - \left( 1 + \left( e^{\theta x^\beta} - 1 \right)^{-\lambda} \right)^{-\beta} \right]^{n-j-1} \]

\[ (1 + \left( e^{\theta x^\beta} - 1 \right)^{-\lambda} \)^{-\beta-1} (1 - 1 + e^{\theta x^\beta})^{-\lambda-1} e^{\theta x^\beta} \theta \lambda^2 x^\beta^{-1} \]

So, the pdf of minimum order statistics is obtained by substituting \( j = 1 \) we have:

\[ f_{j,n}(x) = \frac{n!}{(j-1)(n-j)!} \left[ 1 - \left( 1 + \left( e^{\theta x^\beta} - 1 \right)^{-\lambda} \right)^{-\beta} \right]^{n-1} \left( 1 + \left( e^{\theta x^\beta} - 1 \right)^{-\lambda} \right)^{-\beta-1} \]

\[ (1 + e^{\theta x^\beta})^{-\lambda-1} e^{\theta x^\beta} \theta \lambda^2 x^\beta^{-1} \]

While the corresponding pdf of maximum order statistics is obtained by making the substitution of \( j = n \) in above equation

\[ f_{j,n}(x) = \frac{n!}{(j-1)(n-j)!} \left[ \left( 1 + e^{\theta x^\beta} - 1 \right)^{-\lambda} \right]^n \left( 1 + e^{\theta x^\beta} - 1 \right)^{-\beta-1} \]

\[ (1 + e^{\theta x^\beta})^{-\lambda-1} e^{\theta x^\beta} \theta \lambda^2 x^\beta^{-1} \]

4. Entropies

Entropy is the measure of uncertainty. It is actually a concept of physics.

4.1 Shannon Entropy:

\[ S = -\int_0^\infty f(x) \ln f(x) \, dx \]

\[ f(x) = (1 + (e^{\theta x^\beta} - 1)^{-\lambda})^{-\beta-1} (1 - 1 + e^{\theta x^\beta})^{-\lambda-1} e^{\theta x^\beta} \theta \lambda^2 x^\beta^{-1} \]

\[ \ln f(x) = -(\beta + 1) \ln (1 + (e^{\theta x^\beta} - 1)^{-\lambda}) - (\lambda + 1) \ln (1 - 1 + e^{\theta x^\beta}) + (\lambda + 1) \ln (\theta \lambda \beta^2) + (\beta - 1) \ln x \]

\[ \ln f(x) = \lambda (\beta + 1) \ln (e^{\theta x^\beta} - 1) - (\lambda + 1) \ln (1 + e^{\theta x^\beta}) + \theta x^\beta + \ln (\theta \lambda \beta^2) + (\beta - 1) \ln x \]

\[ S = (1 - \lambda \beta) \int_0^\infty f(x) \ln (e^{\theta x^\beta} - 1) \, dx - \theta \int_0^\infty f(x) x^\beta \, dx - \lambda \beta \int_0^\infty f(x) \ln (\theta \lambda \beta^2) \, dx - (\beta - 1) \int_0^\infty f(x) \ln x \, dx \]

Where \( \int_0^\infty f(x) \, dx = 1 \), \( \int_0^\infty f(x) \ln x \, dx = E(\ln x) \), \( \int_0^\infty f(x) x^\beta \, dx = E(x^\beta) \)

\[ \ln (e^{\theta x^\beta} - 1) = \sum_{k=0}^{\infty} \frac{(-\theta)^k x^\beta}{k!} \]

\[ S = (1 - \lambda \beta) \sum_{k=0}^{\infty} \frac{(-\theta)^k x^\beta}{k!} - \theta E(x^\beta) - \ln (\theta \lambda \beta^2) - (\beta - 1) \ln E(\ln x) \]
4.2 Renyi Entropy:

\[ R = \frac{1}{1-k} \int_0^\infty (f(x))^k \, dx \]

\[ R = \frac{1}{1-k} \int_0^\infty \left( 1 + (e^{\theta x^\beta} - 1)^{-\lambda} - 1 + e^{\theta x^\beta} - \lambda e^{\theta x^\beta} \theta \right)^k \, dx \]

\[ R = \frac{1}{1-k} \int_0^\infty \left( 1 + (1 - e^{\theta x^\beta})^{-\lambda} - k(\lambda + 1) (1 - e^{\theta x^\beta})^{-k(\lambda + 1)} e^{k \theta x^\beta} x^k \right) \, dx \]

Let \( e^{\theta x^\beta} = y \), \( e^{\theta x^\beta} \beta \theta x^{\beta - 1} \, dx = dy, \quad x = \frac{1}{\theta^\beta} \left( \ln y \right) \, dy \)

When \( x \to 0, \quad y \to 1, \) and \( x \to \infty, \quad y \to \infty \)

\[ R = \frac{1}{1-k} \int_0^\infty \left( 1 + (1 - y)^{-\lambda} y^{k-1} \right) \frac{1}{\theta^{(k-1)}(1-\beta)} \left( \ln y \right)^{(1-1/\beta)}(k-1) dy \]

\[ R = \frac{1}{1-k} \int_0^\infty \sum_{h=0}^\infty \frac{(-k)_h(-1)_k}{h!} \sum_{k=0}^\infty \frac{k(\lambda - 1)_m}{m!} \sum_{m=0}^\infty (-1)^m \left( \int_1^\infty \left( \sum_{m+k-1} y^{m+k-1} \right) \frac{1}{\theta^{(k-1)}(1-\beta)} \right) \, dy \]

Let \( \ln y = -z \)

\[ y = e^{-z}, \quad dy = -e^{-z} \, dz \]

When \( y \to 1, \quad z \to 0, \) when \( y \to \infty, \quad z \to \infty \)

\[ R = \frac{1}{1-k} \int_0^\infty \sum_{h=0}^\infty \frac{(-k)_h(-1)_k}{h!} \sum_{k=0}^\infty \frac{k(\lambda - 1)_m}{m!} \sum_{m=0}^\infty (-1)^m \left( \int_0^\infty -z^{(m+k-1)} \right) \frac{1}{\theta^{(k-1)}(1-\beta)} \, dx \]

Let \( z((m+k-2)) = g \)

\[ g \cdot \frac{dz}{z(m+k-2)} = dz \]

When \( z \to 0, \quad g \to 0, \) when \( z \to \infty, \quad g \to \infty \)

\[ R = \frac{1}{1-k} \int_0^\infty \sum_{h=0}^\infty \frac{(-k)_h(-1)_k}{h!} \sum_{k=0}^\infty \frac{k(\lambda - 1)_m}{m!} \sum_{m=0}^\infty (-1)^m \frac{1}{(m+k-2) \left( \frac{1}{\theta^{(k-1)}(1-\beta)} \right)} \, dg \]

\[ R = \frac{1}{1-k} \int_0^\infty \sum_{h=0}^\infty \frac{(-k)_h(-1)_k}{h!} \sum_{k=0}^\infty \frac{k(\lambda - 1)_m}{m!} \sum_{m=0}^\infty (-1)^m \frac{1}{(m+k-2) \left( \frac{1}{\theta^{(k-1)}(1-\beta)} \right)} \, dg \]

5. Bonferroni and Lorenz Curves

In 1905, Max O. Lorenz represented a model for inequality of wealth distribution and C.E. Bonferroni in 1930 proposed a measure of income inequality. Both are used in financial mathematics to check equal distribution of wealth.
Bonferroni and Lorenz curves are defined as follows:

\[ E(p) = \frac{1}{p\mu} \int_0^q x \ f(x) \, dx \]

and

\[ L(p) = \frac{1}{\mu} \int_0^q x \ f(x) \, dx \]

respectively, where \( p, q \in (0,1) \).

\[ E(p) = \frac{1}{p\mu} \int_0^q x \ \left(1 - \left(1 - e^{\theta x^\beta} \right) - \lambda - \beta - 1\right) \left(-1 + e^{\theta x^\beta}\right)^{-\lambda-1} e^{\theta x^\beta} \, dx \]

Let \( (1 - e^{\theta x^\beta})^{-\lambda} = y \), \( (1 - e^{\theta x^\beta})^{-\lambda-1} e^{\theta x^\beta} \lambda \beta 0x^\beta - 1 \, dx = dy \),

\[ x = \frac{1}{\theta^\beta} \left(1 - y^{-\frac{1}{\lambda}}\right)^\frac{1}{\beta} \]

When \( x \to 0 \), \( y \to 0 \) and \( x \to q \), \( y \to (1 - e^{\theta qx^\beta})^{-\lambda} \)

\[ E(p) = \frac{1}{p\mu} \int_0^q \ln \left(1 - y^{-\frac{1}{\lambda}}\right) \, dy \]

\[ E(p) = \frac{1}{p\mu} \frac{1}{\beta^\beta} \sum_{n=0}^\infty \frac{(a)_n}{n!} \int_0^q \ln \left(1 - y^{-\frac{1}{\lambda}}\right) \, dy \]

Let \( \ln(1 - y^{-\frac{1}{\lambda}}) = z \), \( 1 - y^{-\frac{1}{\lambda}} = e^z \)

\[ y = (1 - e^z)^{-\lambda} \quad dy = -\lambda(1 - e^z)^{-\lambda-1} \, dz \]

When \( y \to 0 \), \( z \to 0 \), \( y \to (1 - e^{\theta q x^\beta})^{-\lambda} \), \( z \to \theta q^\beta \)

\[ E(p) = \frac{1}{p\mu} \frac{1}{\beta^\beta} \sum_{n=0}^\infty \frac{(a)_n}{n!} \int_0^q \ln(1 - e^z)^{-\lambda n} \, dz \]

Expanding \( (1 - e^z)^{-\lambda(n+1)-1} \) by using binomial expansion

\[ E(p) = \frac{1}{p\mu} \frac{1}{\beta^\beta} \sum_{n=0}^\infty \frac{(a)_n}{n!} \sum_{m=0}^\infty \frac{(-1)^m}{m!} \frac{\lambda(n+1)-1}{n} \int_0^q \frac{z^\beta}{n} e^{-nz} \, dz \]

\[ E(p) = \frac{1}{p\mu} \frac{1}{\beta^\beta} \sum_{n=0}^\infty \frac{(a)_n}{n!} \sum_{m=0}^\infty \frac{(-1)^m}{m!} \frac{\lambda(n+1)-1}{n} \int_0^q z^\beta e^{-z(n-1)} \, dz \]
\[
E(p) = \frac{1}{p^\mu} \sum_{n=0}^{\infty} \frac{(\alpha)_n}{n!} \sum_{m=0}^{\infty} \frac{(-1)^m}{m!} \left(\frac{\lambda(n+1)-1}{n}\right) \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} (n-1)^k \int_0^{\theta \mu} \frac{1}{z^{\beta+k}} \, dz
\]

\[
E(p) = \frac{1}{p^\mu} \sum_{n=0}^{\infty} \frac{(\alpha)_n}{n!} \sum_{m=0}^{\infty} \frac{(-1)^m}{m!} \left(\frac{\lambda(n+1)-1}{n}\right) \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} (n-1)^k \left(\frac{\theta q^\beta}{\beta+k+1}\right)^{\frac{1}{\beta+k+1}}
\]

\[
E(p) = \frac{1}{p^\mu} \left[ \lambda \theta \mu^{k+1} \sum_{n=0}^{\infty} \frac{(\alpha)_n}{n!} \sum_{m=0}^{\infty} \frac{(-1)^m}{m!} \left(\frac{\lambda(n+1)-1}{n}\right) \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} (n-1)^k \left(\frac{\theta q^\beta}{\beta+k+1}\right)^{\frac{1}{\beta+k+1}} \right]
\]

and

\[
L(p) = \frac{1}{\mu} \left[ \lambda \theta \mu^{k+1} \sum_{n=0}^{\infty} \frac{(\alpha)_n}{n!} \sum_{m=0}^{\infty} \frac{(-1)^m}{m!} \left(\frac{\lambda(n+1)-1}{n}\right) \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} (n-1)^k \left(\frac{\theta q^\beta}{\beta+k+1}\right)^{\frac{1}{\beta+k+1}} \right]
\]

6. Characterization

In order to develop a stochastic function for a certain problem, it is necessary to know whether function fulfills the theory of specific underlying probability distribution, it is required to study characterizations of specific probability distribution. Different characterization techniques have developed. Glanzel (1987, 1988 and 1990), Hamedani . (1993, 2002, 2011 and 2015), Ahsanullah and Hamedani (2007, 2012), Ahsanullah et al. (2013), Shakil et al. (2014), and Merovci et al. (2016) have worked on characterization.

6.1 Characterization based on Hazard function

Definition: Let \( x \) be a continuous random variable with pdf \( f(x) \) if and only if the hazard function \( h(x) \) of a twice differentiable function \( f \), satisfies equation

\[
\frac{d}{dx} \ln f(x) = \frac{h(x)}{f(x)} - h(x)
\]

For random variable \( X \) having LDW distribution with hazard rate function we obtain the following equation

\[
h(x)(1 - (1 + e^{\theta x^\beta} - 1)^{-\lambda})^{-\beta-1} - h(x)(1 + (-1 + e^{\theta x^\beta})^{-\lambda})^{-\beta-1}(-1 + e^{\theta x^\beta})^{-\lambda-1} =
\]

\[
\frac{d}{dx} \left[ (1 + e^{\theta x^\beta} - 1)^{-\lambda} \right]^{-\beta-1} (-1 + e^{\theta x^\beta})^{-\lambda-1} e^{\theta x^\beta} \theta \lambda^2 x^\beta - 1
\]

\[
\frac{d}{dx} \left[ h(x)(1 + e^{\theta x^\beta})^{-\lambda} \right]^{-\beta-1} = \frac{d}{dx} \left[ (1 + e^{\theta x^\beta} - 1)^{-\lambda} \right]^{-\beta-1} (-1 + e^{\theta x^\beta} - 1) e^{\theta x^\beta} \theta \lambda^2 x^\beta - 1
\]

\[
h(x) = \frac{(1 + e^{\theta x^\beta} - 1)^{\lambda}}{(1 + e^{\theta x^\beta})^{-\lambda} \theta \lambda^2 x^\beta - 1}
\]
After manipulation, integrating and simplifying, we obtain as

\[ F(x) = \left( 1 + (e^{\theta x^\beta} - 1)^{-\lambda} \right)^{-\beta} \]

This is the cdf of LDW distribution.

6.2 Characterization based on reversed hazard Function:

\[ \frac{d}{dx} \{\ln f(x)\} = \frac{\varphi'(x)}{\varphi(x)} - \varphi(x) \]

For random variable \( X \) having LDW distribution with hazard rate function we obtain the following equation

\[ \varphi'(x)(1 + (e^{\theta x^\beta} - 1)^{-\lambda})^{-\beta} + \varphi(x) \left(1 + (e^{\theta x^\beta} - 1)^{-\lambda}\right)^{-\beta-1} (-1 + e^{\theta x^\beta}) -\lambda^{-1}e^{\theta x^\beta} \theta \lambda^2 x^\beta^{-1} = \frac{d}{dx}\left[ \left( 1 + (e^{\theta x^\beta} - 1)^{-\lambda} \right)^{-\beta} \right] \]

After manipulation, integrating and simplifying, we obtain as

\[ F(x) = \left( 1 + (e^{\theta x^\beta} - 1)^{-\lambda} \right)^{-\beta-1} \]

6.3 Characterization through Ratio of Truncated Moments

In this section, we characterize WD distribution using Theorem 1 (Glanzel; 1987) on the basis of simple relationship between two functions of \( X \). Theorem 1 is given in appendix A.

Preposition

Suppose that random variable \( X: \Omega \rightarrow (0, \infty) \) is continuous. Let

\[ h_1(x) = \frac{1}{\lambda \beta} \left( 1 + (e^{\theta x^\beta} - 1)^{-\lambda} \right)^{\beta+1} (e^{\theta x^\beta} - 1)^{\lambda+1} \quad \text{and} \quad h_2(x) = \frac{2e^{\theta x^\beta}}{\left(1 + (e^{\theta x^\beta} - 1)^{-\lambda}\right)^{-\beta-1} \left( e^{\theta x^\beta} - 1 \right)^{-\lambda-1}} \]

\( x > 0 \)

The pdf of \( X \) is (9) if and only if \( p(x) \) has the form \( p(x) = e^{-\theta x^\beta}, \quad x > 0 \)

Proof:

For random variable \( X \) having LDW distribution with pdf (9) and cdf (8), we proceed as

\[ (1 - f(X))E(h_1(x)/X \geq x) = e^{-\theta x^\beta} \]

\[ (1 - f(X))E(h_2(x)/X \geq x) = e^{-2\theta x^\beta} \]
\[ p(x) = e^{-\theta x^\beta} \quad \text{and} \quad p'(x) = e^{-\theta x^\beta} \theta x^{\beta - 1} \]

\[ s'(t) = \frac{p'(t)h_2(t)}{p(t)h_2(t) - h_1(t)} = -2\theta \beta x^{\beta - 1} \quad \text{and} \quad s(t) = -2\theta x^\beta \]

Therefore in the light of Theorem 1, X has pdf (9)

### 7. Maximum Likelihood Estimation

Several approaches for parameter estimation have been proposed in the literature but the maximum likelihood method is the most commonly employed.

Let \( X_1, X_2, \ldots, X_m \) be a random sample of size \( n \) of the LDW distribution then the total log-likelihood (LL) function is given

\[
L(\lambda, \theta, \beta) = \beta^{2n} \prod_{i=1}^{n} \left( 1 + \left( e^{\theta x_i^\beta} - 1 \right) \right)^{-\beta - 1} \lambda^n
\]

\[
\prod_{i=1}^{n} (-1 + e^{\theta x_i^\beta})^{-\lambda - 1} \theta^n \prod_{i=1}^{n} e^{\theta x_i^\beta} \prod_{i=1}^{n} x_i^{\beta - 1},
\]

\[
L(\lambda, \theta, \beta) = 2n \log[\beta] + n \log[\theta] + n \log[\lambda] + \theta \sum_{i=1}^{n} x_i^\beta + (\beta - 1) \sum_{i=1}^{n} \log[x_i]
\]

\[
-(\lambda + 1) \sum_{i=1}^{n} \log[e^{x_i^\beta \theta} - 1] + \lambda(\beta + 1) \sum_{i=1}^{n} \log(1 + (e^{x_i^\beta \theta} - 1)),
\]

The First derivatives of the log-likelihood function are given as follow

\[
\frac{\delta L(\lambda, \theta, \beta)}{\delta \beta} = \frac{2n}{\beta} + \theta \beta \sum_{i=1}^{n} x_i^{\beta - 1} + \sum_{i=1}^{n} \log[x_i] - (\lambda + 1) \sum_{i=1}^{n} \theta x_i^{\beta - 1} + 2\lambda \sum_{i=1}^{n} \beta x_i^{\beta - 1} \theta,
\]

\[
\frac{\delta L(\lambda, \theta, \beta)}{\delta \lambda} = \frac{n}{\lambda} - \sum_{i=1}^{n} \log[e^{x_i^\beta \theta} - 1] + \lambda(\beta + 1) \sum_{i=1}^{n} \log(1 + (e^{x_i^\beta \theta} - 1)),
\]

\[
\frac{\delta L(\lambda, \theta, \beta)}{\delta \theta} = \frac{n}{\theta} + \sum_{i=1}^{n} x_i^\beta - (\lambda + 1) \sum_{i=1}^{n} x_i^\beta + \lambda(\beta + 1) \sum_{i=1}^{n} x_i^\beta,
\]

Equating equations to zero and solving them numerically, one can obtain the estimates of the unknown parameters.
8 Simulation Study

This section deals the simulation study. In proposed model we generated random variables by using CDF of LDWD with four different value of parameters for $n=25,50,100,200$. Parameters are estimated with method of MLE by using each generated random variable. In statistical study, bias states to the tendency of a measurement process to over or under estimate the value of population parameters. Bias of MLEs can be fundamental. Squared error is a function which obtained from square values of bias. MSE is always constructive. Bias shows the contrasts between estimated values of parameter variation from true value of parameter. By using the estimated parameters, we calculated Bias and MSE of LDWD and also calculate their average mean which are shown in different graphs. The resulting Behaviour of these Bias and MSE of these estimated parameters are also shown below with the help of graphs. All simulations were done on computational software ‘Mathematica 8.0’. The analysis computes the coming values:

- Average bias of the simulated estimates:
  \[ \frac{1}{1000} \sum_{i=1}^{1000} (\theta^* - \theta) \]

- Average mean square error (MSE) of the simulated estimates:
  \[ \frac{1}{1000} \sum_{i=1}^{1000} (\theta^* - \theta)^2 \]

The results are reported in Tables 1 and 2.

[Table 2 about here.]
[Table 3 about here.]

9. Evaluation Measures and Practical Data Examples

We illustrate the usefulness of the Log Dagum Weibull distribution and compare the results with the WD GD LD EED and NEED distributions by means of four real data sets. One of which is data of leukaemia-free survival times of 50 patients with Autologous transplant obtained from [11] and the second data set contains Lifetime of 50 devices [13], Third data set
consists of 100 uncensored data on breaking stress of carbon fibres (in Gba) [32], fourth data consist times to failure of eighteen electronic devices [33].

In this section we illustrate the usefulness of the Log Dagum Weibull distribution. We estimate the unknown parameters of LDWD using MLE method and compare the log likelihood with some other distributions including EED, WD, GD, NEED and LD. We will check goodness of fit of our model with some test statistics like AD test, CVM test, K-S test and p-value. All calculations are executed on computational software MATHEMATICA 11.0.

**Numerical measures**

In order to demonstrate the proposed methodology, we consider four different practical data sets described below with their analysis. They represent different level of skewness ranging from negative skewness to positive skewness along with various demonstration of failure rate pattern, like increasing, decreasing, and bathtub shape. Moreover, perfection of competing models is also tested via the Kolmogrov-Smirnovov (K S), the Anderson Darling (A∗) and the Cramer-von Misses (W∗) statistics. The mathematical expressions for the statistics are given by

\[ KS = \max \left\{ \frac{i}{m} - z_i, z_i - \frac{i - 1}{m} \right\} \]

\[ A^* = \left(\frac{2.25}{m^2} - \frac{0.75}{m} + 1\right) \left\{-1 - \frac{1}{m} \sum_{i=1}^{m} (2i - 1) \ln(z_i(z_{m-i+1}))\right\} \]

and

\[ w^* = \sum_{i=1}^{m} \left( z_i - \frac{2i - 1}{2m} \right)^2 + \frac{1}{12m} \]

where \( m \) denotes the number of classes, \( z_i = FX(x_i) \), the \( x_i \)'s being the ordered observations respectively.

**Data set 1.** The first data set obtained from [12] of leukemia-free survival times of 50 patients with Autologous transplant. Data sets are presented in the following tables:

- 0.030, 0.493, 0.855, 1.184, 1.283, 1.480, 1.776, 2.138, 2.500, 2.763, 2.993, 3.224, 3.421, 4.178, 4.441, 5.691, 5.855, 6.941, 6.941, 7.993, 8.882, 8.882, 9.145, 11.480, 11.513, 12.105, 12.796, 12.993, 13.849, 16.612, 17.138, 20.066, 20.329, 22.368, 26.776, 28.717, 28.717, 32.928, 33.783, 34.211, 34.770, 39.539, 41.118, 45.033, 46.053, 46.941, 48.289, 57.401, 58.322, 60.625.
The measures of goodness of fit including the Akaike information criterion (AIC), consistent Akaike information criterion (CAIC), Bayesian information criterion (BIC), Hannan-Quinn information criterion (HQIC), Anderson-Darling ($A^*$), Cramér–von Mises ($W^*$) and Kolmogrov-Smirnov (K-S) statistics are computed to compare the fitted models. The statistics $A^*$ and $W^*$ are described in details in [8]. In general, the smaller values of these statistics, better fit to the data. The required computations are carried out in the Mathematica 11.0.

[Table 4 about here]
[Table 5 about here]

Analysis:
Table 4 and 5 represents the results of test statistics and information criterion respectively for data set 1. Minimum test statistics and information criterion shows the goodness of fit of our designed model.

Data set 2: Second data set Lifetime of 50 devices are obtained from [14].

0.1, 0.2, 1, 1, 1, 1, 1, 2, 3, 6, 7, 11, 12, 18, 18, 18, 18, 21, 32, 36, 40, 45, 46, 47, 50, 55, 60, 63, 63, 67, 67, 67, 67, 72, 75, 79, 82, 82, 83, 84, 84, 84, 85, 85, 85, 85, 85, 85, 86, 86.

We fit the LDW model and other competitive models such as the Exponentiated exponential distribution (EED) [10], Weibull distribution (WD) [11], Gamma distribution (GD) [10], NEED Nadarajah Exponentiated exponential distribution and Lomax distribution (LD) to data sets.

[Table 6 about here.]
[Table 7 about here]

Analysis:
Tables 6 represents the results of test statistics for data set 2. Minimum test statistics shows the goodness of fit of our designed model. It can be easily seen that our model has better fitness than other five models.

Data set 3: This data set consists of 100 uncensored data on breaking stress of carbon fibres (in Gba), [32].

0.39, 0.81, 0.85, 0.98, 1.08, 1.12, 1.17, 1.18, 1.22, 1.25, 1.36, 1.41, 1.47, 1.57, 1.57, 1.59, 1.59, 1.61, 1.61, 1.69, 1.69, 1.71, 1.73, 1.8, 1.84, 1.84, 1.87, 1.89, 1.92, 2.2, 2.03, 2.03, 2.05, 2.12, 2.17, 2.17, 2.17, 2.35, 2.38, 2.41, 2.43, 2.48, 2.48, 2.5, 2.53, 2.55, 2.55, 2.56, 2.59, 2.67, 2.73, 2.74, 2.76, 2.77, 2.79, 2.81, 2.81, 2.82, 2.83, 2.85, 2.87, 2.88, 2.93, 2.95, 2.96, 2.97, 2.97, 3.09, 3.11, 3.11, 3.15, 3.15, 3.19, 3.19, 3.22, 3.22,
3.27, 3.28, 3.31, 3.31, 3.33, 3.39, 3.39, 3.51, 3.56, 3.6, 3.65, 3.68, 3.68, 3.68, 3.7, 3.75, 4.2, 4.38, 4.42, 4.7, 4.9, 4.91, 5.08, 5.56.

[Table 8 about here.]
[Table 9 about here.]

**Analysis:**

Table 8 and 9 represents the results of test statistics and information criterion respectively for data set 3. Minimum test statistics and information criterion shows the goodness of fit of our designed model

**Data set3:** This data consist times to failure of eighteen electronic devices [33] used to show how the proposed distribution can be applied in practice.
5, 11, 21, 31, 46, 75, 98, 122, 145, 165, 196, 224, 245, 293, 321, 330, 350, 420

[Table 10 about here.]
[Table 11 about here.]

**Concluding Remarks:**

There has been an increased interest in defining new generated classes of univariate continuous distributions. The extended distributions have attracted several statisticians to develop new models. In this paper we propose the new log Dagum-$X$ family of distributions. We study some of its mathematical properties. The maximum likelihood method is employed for estimating the model parameters. One special model, the distribution Weibull is considered and its properties are studied. It is fitted to two real data sets. The proposed special model consistently better fit than other competing models. We hope that the new family and its generated models will attract wider application in several areas such as engineering, survival and lifetime data, hydrology, economy.
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**Appendix:**

**Theorem 1:**

Suppose that probability space \((\mathcal{Q}, F, P)\) and interval \([g_1, g_2]\) with \(g_1 < g_2\), \((g_1, = -\infty, g_2, = \infty)\) are given. Let continuous random variable \(X: \mathcal{Q} \rightarrow [g_1, g_2]\) has distribution function \(F\). Let real functions \(g_1\) and \(g_2\) be continuous on \([g_1, g_2]\) such that \(\frac{e(g_1(x)/X=x)}{e(g_2(x))/X=x}\) is real function in simple form. Assume that \(g_1, g_2 \in c([g_1, g_2]), p(x) \in c^2([g_1, g_2])\) and \(F\) is two times continuously differentiable and strictly monotone function on \([g_1, g_2]\): As a final point, assume that the equation \(g_2p(x) = g_1\) has no real solution in \([g_1, g_2]\].

then \(F(x)=\int_{\text{ln}k}^{x} \frac{p'(t)}{\frac{p(t)g_2(t)}{g_1(t)}-g_1(t)} \exp(-st) dt\) is obtained from the function \(g_1, g_2, p(t)\) and \(s(t)\).where \(S(t)\) is obtained from equation \(s'(t) = \frac{p'(t)g_2(t)}{p(t)g_2(t)-g_1(t)}\) and \(k\) is constant, picked to make \(\int_{g_1}^{g_2} df = 1\).
Figure 1: Cumulative Distribution plot of LDWD

Figure 2: Density plots of LDWD
Figure 3: Survival plot of LDWD
Figure 4: Hazard plot of LDWD

Figure 5: Shape of Hazard function

Table 2: Average mean of Bias and MSE values for estimators $\hat{\theta}$, $\hat{\beta}$, and $\hat{\lambda}$ of data 1
Table 3: Average mean of Bias and MSE values for estimators $\hat{\theta}$, $\hat{\beta}$ and $\hat{\lambda}$ of data 2

| $n$ | Bias($\hat{\theta}$) | Bias($\hat{\beta}$) | Bias($\hat{\lambda}$) | MSE($\hat{\theta}$) | MSE($\hat{\beta}$) | MSE($\hat{\lambda}$) |
|-----|----------------------|----------------------|-----------------------|---------------------|-------------------|---------------------|
| 25  | 0.65234              | -0.08837             | 0.56572               | 0.56572             | 0.40559           | 0.86286             |
| 50  | 0.35310              | 0.23613              | 0.17950               | 0.29754             | 0.52918           | 0.15562             |
| 100 | 0.27740              | 0.46278              | -0.01169              | 0.11882             | 0.72056           | 0.04713             |
| 250 | 0.44889              | -0.03051             | 0.07439               | 0.24297             | 0.20479           | 0.03035             |

Table 4: AD, CVM, The K-S statistics and p-values for the data set 1

| Distributions | $A^*$  | $W^*$  | K-S     | p-value  |
|---------------|--------|--------|---------|----------|
| LDWD          | 0.403996 | 0.0651719 | 0.076948 | 0.943568 |
| EED           | 0.362828 | 0.0483839 | 0.084435 | 0.868171 |
| WD            | 0.411538 | 0.0562415 | 0.0868536 | 0.845013 |
| GD            | 0.369975 | 0.0496265 | 0.0847622 | 0.86513  |
| LD            | 2.504843 | 0.3799524 | 0.19666206 | 0.04182  |
| NEED          | 0.666096 | 0.0962511 | 0.0906376 | 0.805953 |

Table 5: Information Criteria of Different Distributions for Data 1
### Table 6: The K-S statistics and p-values for the data sets 2

| Distributions | A*       | W*       | K-S   | p-value |
|---------------|----------|----------|-------|---------|
| LDWD          | 0.41395  | 0.06328  | 0.07134 | 0.9135  |
| EED           | 0.36282  | 0.04838  | 0.08444 | 0.8682  |
| WD            | 0.41153  | 0.05624  | 0.08685 | 0.8450  |
| GD            | 0.36997  | 0.04962  | 0.08476 | 0.8651  |
| LD            | 8.09533  | 1.66869  | 0.3377 | 0.00002 |
| NEED          | 8.11488  | 1.67229  | 0.32272 | 0.00006 |

### Table 7: Information Criteria of Different Distributions for Data 2

| Model | AIC    | AICC   | BIC    | CAIC   |
|-------|--------|--------|--------|--------|
| LDWD  | 455.064| 455.586| 460.800| 455.586|
| EED   | 483.99 | 484.246| 487.814| 484.246|
| WD    | 486.004| 486.259| 489.828| 486.259|
| GD    | 484.38 | 484.636| 488.204| 484.636|
| NEED  | 516.033| 519.857| 516.289| 516.857|
| LD    | 474.0873| 474.3427| 477.9114| 474.3427|

### Table 8: Information Criteria of Different Distributions for Data 3
| Model | AIC   | BIC   | AICC  | HQIC  | CAIC  |
|-------|-------|-------|-------|-------|-------|
| LDWD  | 288.62| 296.43| 288.8685| 296.4883| 296.4883|
| GD    | 290.4673| 295.6775| 290.5909| 292.576| 290.5909|
| WD    | 289.06| 296.87| 289.3086| 292.2217| 289.3086|
| EED   | 296.3646| 301.574| 296.4883| 298.4733| 296.4883|
| NEED  | 393.8472| 399.0575| 393.9709| 395.9559| 393.9709|
| LD    | 474.0873| 477.9114| 474.3427| 475.54356| 474.3427|

Table 9: The K-S statistics and p-values for the data sets 3

| Distributions | $A^*$  | $W^*$  | K-S   | p-value         |
|---------------|--------|--------|-------|-----------------|
| LDWD          | 0.39666| 0.06508| 0.0618| 0.8395          |
| EED           | 1.2341 | 0.2303 | 0.1077| 0.19618         |
| WD            | 18.9521| 3.7772 | 0.3341| $4.02837 \times 10^{-10}$ |
| GD            | 200.5016| 32.9885| 0.9996| $2.22044 \times 10^{-16}$ |
| LD            | 79.3018| 17.3623| 0.8210| $-2.22044 \times 10^{-16}$ |
| NEED          | 16.9307| 3.35163| 0.3170| $3.73137 \times 10^{-9}$ |

Table 10: The K-S statistics and p-values for the data sets 4
| Distributions | $A^*$  | $W^*$  | K-S  | p-value |
|---------------|--------|--------|------|---------|
| LDWD          | 0.1725 | 0.02361| 0.0840| 0.9996  |
| EED           | 0.4456 | 0.07077| 0.12138| 0.9535  |
| WD            | 0.4609 | 0.0644 | 0.1132| 0.9752  |
| GD            | 0.4487 | 0.06986| 0.1206| 0.956104|
| LD            | 28.2328| 5.0981 | 0.9157| $1.5487 \times 10^{-13}$ |
| NEED          | 2.46950| 0.4826 | 0.28141| 0.115548|

Table 11: Information Criteria of Different Distributions for Data 4

| Model | AIC   | BIC   | HQIC  | AICC  | CAIC  |
|-------|-------|-------|-------|-------|-------|
| LDWD  | 208.2915 | 210.9626 | 208.6598 | 210.0058 | 210.0057 |
| GD    | 226.1 | 227.9 | 226.9 | 229.9 | 229.9 |
| WD    | 395.433 | 397.214 | 395.6789 | 396.233 | 396.233 |
| EED   | 225.2528 | 227.0335 | 225.4983 | 226.05277 | 226.0527 |
| NEED  | 237.8595 | 239.6403 | 238.10512 | 238.6596 | 238.65956 |
| LD    | 341.4154 | 343.1962 | 341.6609 | 342.2154 | 342.2154 |