In this study, a theory analogous to both the theories of polynomial-like mappings and Smale's real horseshoes is developed for the study of the dynamics of mappings of two complex variables.

In partial analogy with polynomials in a single variable there are the Hénon mappings in two variables as well as higher dimensional analogues. From polynomial-like mappings, Hénon-like and quasi-Hénon-like mappings are defined following this analogy. A special form of the latter is the complex horseshoe.

The major results about the real horseshoes of Smale remain true in the complex setting. In particular:

1. Trapping fields of cones (which are sectors in the real case) in the tangent spaces can be defined and used to find horseshoes.
2. The dynamics of a horseshoe is that of the two-sided shift on the symbol space on some number of symbols which depends on the type of the horseshoe.
3. Transverse intersections of the stable and unstable manifolds of a hyperbolic periodic point guarantee the existence of horseshoes.

Biographical Sketch

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To my parents

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Acknowledgments

I wish to thank Professor John Hamal Hubbard for his guidance, inspiration, confidence, and friendship. At a time when I thought mathematics had died, his work made it come back to life for me. His energy and accessibility, day and night, have been very much appreciated.

I thank Professors Adrien Douady, John Guckenheimer, and John Smillie for their interest and comments. I thank Professor Clifford Earle for serving on my Special Committee for the entire duration of my stay at Cornell.

I thank my parents, to whom this is dedicated, for their patience and encouragement despite their doubts.

I thank Homer Smith for his expert computer and photographic assistance and for generously loaning me some of his equipment.

Last, but by no means least, I thank my friends for providing the many diversions, including bridge, squash, and tennis, as well as the comic relief needed to make life bearable.

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1.1 Periodic cycles of Hénon-like mappings of degree 2 .... Page 17
The study of the subject of the dynamics of complex analytic functions of one variable goes back to the early 1900’s with the publication of several papers by Pierre Fatou and Gaston Julia, including the long memoirs [F1] and [J], published around 1920. These papers proved to be the definitive works in the subject for a long time to come. For the most part, new interesting results did not appear again until 1982. (For a survey of the subject, see [Bl].)

Somewhat surprisingly, this recent resurgence of the field has been spurred not so much by mathematical developments but by advances in computer graphics. However the original impetus at the beginning of the century was strictly mathematical. Besides the obvious connection between differential equations and dynamical systems, the theory of complex analytic dynamics was very much based upon the developments in complex analysis. In particular, the theory of normal families and the work of P. Montel in that regard had given rise to much of the work of Fatou and Julia.

Compared with the theory of complex analytic functions of a single variable, the theory of complex analytic mappings of several, that is, two or more, variables is quite different. In particular, the theory of omitted values, including normal families, is not paralleled in the several complex variables case. This difference was really exposed by Fatou and L. Bieberbach in the 1920’s in [F2] and [Bi].

They showed the existence of what we refer to as Fatou-Bieberbach domains: open subsets of \( \mathbb{C}^n \) whose complements have nonempty interior and yet are the images of \( \mathbb{C}^n \) under an injective analytic mapping. This is contrary to the one variable case where the image of every non-constant analytic function on \( \mathbb{C} \) omits at most a single point.

The present work started with an attempt to understand such Fatou-Bieberbach domains. These arise naturally as the basins of attractive fixed points of analytic automorphisms of \( \mathbb{C}^n \). The basins are then the image of the mapping conjugating the given automorphism to its linear part at the given fixed point.

This remains true even when the Jordan canonical form of the linearization is not diagonal and when there are resonances in general. This latter result is apparently new. (See [HO].) For example, consider

\[
F : \begin{bmatrix} x \\ y \end{bmatrix} \mapsto \begin{bmatrix} x^2 + 9/32 - y/8 \\ x \end{bmatrix}.
\]
This has two fixed points, of which $(3/8, 3/8)$ is attractive with its linear part having resonant eigenvalues $1/4$ and $1/2$ (that is, $1/4 = (1/2)^2$). Moreover, none of the points in the region

$$\{ (x, y) \mid |y| < 4|x|^2/3, |x| > 4 \}$$

remain bounded under iteration of $F$. So the basin of $(3/8, 3/8)$ is not all of $\mathbb{C}^2$.

As was the case with one dimensional complex analytic dynamic, two dimensional complex dynamics had been a rather dormant field until recently. In analogy, the present work started with a computer investigation of a specific Fatou-Bieberbach domain.

In the time between Fatou and the present, most of the attention of those studying dynamical systems has been limited to mappings in the real. This is somewhat surprising for two reasons. First, small perturbations of the coefficients of polynomial terms of

$$F_a, c : \mathbb{R}^2 \to \mathbb{R}^2,$$

with $a \neq 0$, has received much attention. Hénon, in [H1] and [H2], first studied these numerically and they have become known as the Hénon mappings. This family contains, up to conjugation, most of the most interesting of the simplest nonlinear polynomial mappings of two variables. However, Hénon mappings are still rather poorly understood and indeed the original question concerning the existence of a strange attractor for any values of the parameters is still unresolved today.

Despite the differences between the real and complex theories and the one variable and several variable theories, much of the development of the subject of complex analytic dynamics in several variables has been conceived through analogy.

Recently, Hénon mappings have started to be examined in the complex, that is, with both the variables and the parameters being complex ([HO], [FM]). Besides this exposition, a joint paper with John Hubbard [HO] detailing the part of this work which concerns Hénon mappings is in preparation. It has also been noted that there exist analogous mappings of higher degree of the form

$$\begin{bmatrix} x \\ y \end{bmatrix} \mapsto \begin{bmatrix} p(x) - ay \\ x \end{bmatrix},$$

where $p$ is a polynomial of degree at least two and $a \neq 0$ [FM]. Note that these are always invertible with inverses given by

$$\begin{bmatrix} x \\ y \end{bmatrix} \mapsto \begin{bmatrix} y \\ (p(y) - x)/a \end{bmatrix}.$$
Since Hénon mappings are of this form, these are called the generalized Hénon mappings. For polynomials $p$ of degree $d$, they are called Hénon mappings of degree $d$.

The major development in the theory of real two dimensional dynamical systems has been the horseshoe mapping of Smale $[S]$. The main contribution of this thesis is the study of the generalization of horseshoe mappings to the complex setting.

Recently, A. Douady and J. Hubbard $[DH]$ developed the theory of polynomial-like mappings. Polynomial-like mappings are designed to capture, in their topology, the essence of polynomial mappings. These have helped to inspire an analogous class of mappings of several variables.

The following is a summary of the contents of this thesis:

Chapter 1 gives the definition of Hénon-like mappings of degree $d$, which are intended to capture the dynamical essence of Hénon mappings, in analogy with polynomial-like mappings. The relationship between Hénon, Hénon-like, and polynomial-like mappings is explored.

Chapter 2 gives the definition of complex horseshoes, the complex analog of Smale’s real horseshoes, as a special class of quasi-Hénon-like mappings, which themselves generalize Hénon-like mappings. A criterion called a trapping field of cones is given which guarantees horseshoes. This is used to show that a large number of Hénon mappings are complex horseshoes.

Chapter 3 shows that, in analogy with the real case, complex horseshoes of degree $d$ are conjugate to the two-sided shift on symbol space on $d$ symbols.

Chapter 4 shows that complex horseshoes are ubiquitous in the subject of two dimensional complex analytic dynamics by showing that they appear almost every time a mapping has a hyperbolic periodic point.

1. Hénon-Like Mappings

In this chapter, a class of mappings is defined which attempt to capture in their topology the dynamics of Hénon mappings of all degrees. These mappings will be called Hénon-like. The definition of Hénon-like mappings is inspired by the definition of polynomial-like mappings (see $[DH]$), which was designed to capture the topological essence of polynomials on some disc or, more generally, on some open subset of $\mathbb{C}$ isomorphic to a disc.

A polynomial-like mapping of degree $d$ is a triple $(U, U', f)$, where $U$ and $U'$ are open subsets of $\mathbb{C}$ isomorphic to discs, with $U'$ relatively compact in $U$, and $f : U' \to U$ analytic and proper of degree $d$. Note that it is convenient to think of polynomial-like of degree $d$ as meaning an analytic mapping $f : U \to \mathbb{C}$ such that $f(\partial U) \subset \mathbb{C} \setminus U$ and $f|_{\partial U}$ of degree $d$ (cf. $[DH]$).

Figure 1.1 gives examples of the behavior, pictured in $\mathbb{R}^2$, which should be captured by the definition of Hénon-like mappings of degree 2. (In each case, the crescent-shaped region is the image of the square with $A'$ the image of $A$, etc.)
It seems clear that the behaviors described by (a) and (b) versus (c) and (d) in Figure 1.1 must be described differently, albeit analogously, trading “horizontal” for “vertical.”

We set some notation: \( d \) will always be an arbitrary fixed integer greater than one. Let \( \pi_1, \pi_2 : \mathbb{C}^2 \to \mathbb{C} \) be the projections onto the first and second coordinates, respectively. We will consider a bidisc \( B = D_1 \times D_2 \subset \mathbb{C}^2 \), where \( D_1, D_2 \subset \mathbb{C} \) are discs. Vertical and horizontal “slices” of \( B \) are denoted by

\[
V_x = \{x\} \times D_2 \quad \text{and} \quad H_y = D_1 \times \{y\},
\]

for all \( x \in D_1 \) and for all \( y \in D_2 \), respectively. We will be considering mappings of the bidisc, \( F : \overline{B} \to \mathbb{C}^2 \) together with a mapping denoted by \( F^{-1} : \overline{B} \to \mathbb{C}^2 \) which is the inverse of \( F \) where that makes sense. Now define, for each \((x, y) \in B\),

\[
F_{1,y} = \pi_1 \circ F \circ (\text{Id} \times y) : D_1 \to \mathbb{C},
\]

\[
F_{2,x} = \pi_2 \circ F^{-1} \circ (x \times \text{Id}) : D_2 \to \mathbb{C},
\]

\[
F_{2,x}^{-1} = \pi_2 \circ F \circ (x \times \text{Id}) : D_2 \to \mathbb{C},
\]

\[
F_{1,y}^{-1} = \pi_1 \circ F^{-1} \circ (\text{Id} \times y) : D_1 \to \mathbb{C}.
\]
Definition 1.1. \( F : \overline{B} \to \mathbb{C}^2 \) is a Hénon-like mapping of degree \( d \) if there exists a mapping \( G : \overline{B} \to \mathbb{C}^2 \) such that

1. Both \( F \) and \( G \) are injective and continuous on \( \overline{B} \) and analytic on \( B \).
2. \( F \circ G = 1d \) and \( G \circ F = 1d \) where each makes sense.

Hence, rename \( G \) as \( F^{-1} \).

3. For all \( x \in D_1 \) and \( y \in D_2 \), either
   
   (a) \( F_{1,y} \) and \( F_{2,x}^{-1} \) are polynomial-like of degree \( d \), or
   
   (b) \( F_{2,x} \) and \( F_{1,y}^{-1} \) are polynomial-like of degree \( d \).

Depending on whether \( F \) satisfies condition (a) or (b), call it horizontal or vertical.

Remarks. (1) It would be correct and, perhaps, cleaner to define a Hénon-like pair as a triple \((F, G, B)\) and then call \( F \) a Hénon-like mapping. However, this would put the emphasis in the wrong place.

(2) When the degree of a Hénon-like mapping is either clear from context, or is not of primary interest, or is two, we will sometimes fail to mention the degree.

(3) Note that conditions (a) and (b) of (3) of Definition 1.1 are dual in the sense that if a Hénon-like mapping \( F \) satisfies condition (a), then \( F^{-1} \) satisfies condition (b) and vice versa. In Figure 1.1, (a) and (b) correspond with horizontal Hénon-like mappings while (c) and (d) correspond with vertical Hénon-like mappings. In general, unless otherwise specified, it will be assumed that Hénon-like mappings are horizontal, i.e., satisfying condition (a).

(4) Just as for polynomial-like mappings, it does not make sense, in general, to say that a mapping \( F : \mathbb{C}^2 \to \mathbb{C}^2 \) is Hénon-like as this may be ambiguous. \( F \) may exhibit different Hénon-like behavior in different regions. The domain \( \overline{B} \) is part of the definition of Hénon-like mappings. Of course, the context may be used to resolve such ambiguities.

(5) Let
\[
\partial B_V = \partial \overline{D}_1 \times \overline{D}_2 \quad \text{and} \quad \partial B_H = \overline{D}_1 \times \partial \overline{D}_2.
\]
the “vertical and horizontal boundaries.”

Proposition 1.2. If \( F : \overline{B} \to \mathbb{C}^2 \) is a Hénon-like mapping, then either

\[
F(\partial B_V) \subset \mathbb{C}^2 \setminus \overline{B} \quad \text{and} \quad F^{-1}(\partial B_H) \subset \mathbb{C}^2 \setminus \overline{B}
\]

or

\[
F^{-1}(\partial B_V) \subset \mathbb{C}^2 \setminus \overline{B} \quad \text{and} \quad F(\partial B_H) \subset \mathbb{C}^2 \setminus \overline{B}.
\]

Proof. This follows from the fact that the boundary of a polynomial-like mapping is mapped outside of the closure of its domain.

QED
Note that these are equivalent to

\[
F(\partial B_V) \cap \overline{B} = \emptyset \quad \text{and} \quad F^{-1}(\partial B_H) \cap \overline{B} = \emptyset
\]
or

\[
F^{-1}(\partial B_V) \cap \overline{B} = \emptyset \quad \text{and} \quad F(\partial B_H) \cap \overline{B} = \emptyset.
\]

In the next chapter we will consider an alternative to Hénon-like mappings where conditions (a) and (b) of Definition 1.1(3) are replaced by such conditions on the vertical and horizontal boundaries.

(6) The class of polynomial-like mappings is stable under small perturbations [DH] and the same is true for Hénon-like mappings:

**Proposition 1.3.** Suppose \( F : \overline{B} \to \mathbb{C}^2 \) is Hénon-like of degree \( d \). Let \( H : \overline{B} \to \mathbb{C}^2 \) be injective, continuous on \( B \), and analytic on \( B \). If \( \|H\| \) is sufficiently small, then \( F + H \) is also Hénon-like of degree \( d \).

**Proof.** Recall that \( \partial B_V = \partial \overline{D}_1 \times \overline{D}_2 \). By Proposition 1.2, we can choose an \( \varepsilon > 0 \) so that \( \varepsilon < d(\overline{B}, F(\partial B_V)) \). If \( \|H(x,y)\| < \varepsilon \) for all \( (x,y) \in \partial B_V \), then \( F + H \) is Hénon-like of degree \( d \).

QED

Of course, the Hénon mappings themselves provide the obvious examples of Hénon-like mappings:

**Proposition 1.4.** For every Hénon mapping, \( F \), of every degree, there exists an \( R \), such that \( F : \overline{D}_R^2 \to \mathbb{C}^2 \) is a Hénon-like mapping of the same degree.

**Proof.** This follows immediately from the fact that all polynomials are polynomial-like of the same degree on sufficiently large discs.

QED

**Examples.** (1) A simple computation shows that if \( F \) is the Hénon mapping (of degree 2) with parameters \( a \) and \( c \), and \( D_R \) is the disc of radius \( R \), with

\[
R > (1/2)(1 + |a| + \sqrt{(1 + |a|)^2 + 4|c|}),
\]

then \( F : \overline{D}_R^2 \to \mathbb{C}^2 \) is a horizontal Hénon-like mapping of degree 2. This \( R \) is exactly what is required so that \( F(\partial \overline{D}_R \times \overline{D}_R) \cap \overline{D}_R^2 = \emptyset \) and \( F^{-1}(\overline{D}_R \times \partial \overline{D}_R) \cap \overline{D}_R^2 = \emptyset \). Of course, their inverses are vertical Hénon-like mappings.
(2) More generally, consider the mappings \( G : \mathbb{C}^2 \to \mathbb{C}^2 \) of the form
\[
G : \begin{bmatrix} x \\ y \end{bmatrix} \mapsto \begin{bmatrix} x^d + c - ay \\ x \end{bmatrix},
\]
with \( a \neq 0 \) and \( d \geq 2 \). When \( d = 2 \), we are back to the previous example. The lower bound on \( R \) came from solving the inequality
\[
R^d - (1 + |a|)R - 4|c| > 0
\]
for \( R \). Note that when \( d = 2 \) we already had \( R > 1 \). Therefore, the same lower bound will work here as well. Of course, better lower bounds can be found.

Analogous to the invariant sets defined for Hénon mappings [HO], we can define the following sets for Hénon-like mappings:
\[
K_+ = \left\{ z \in B \mid F_T^n(z) \in B \quad \text{for all} \quad n > 0 \right\},
\]
\[
K_- = \left\{ z \in B \mid F_T^{-n}(z) \in B \quad \text{for all} \quad n > 0 \right\},
\]
\[
J_\pm = \partial K_\pm, \quad K = K_+ \cap K_-, \quad J = J_+ \cap J_-.
\]

At least some of the dynamics of Hénon mappings is captured by the notion of a Hénon-like mapping. When the Hénon-like mapping

\[
\text{is actually a Hénon mapping, as in Proposition 1.4, a simple computation shows that the set } K \text{ for the Hénon mapping actually is contained in } \overline{D_{R^2}} \text{ and thus equals the corresponding } K \text{ defined above. Moreover:}
\]

**Proposition 1.5.** For every \( d \), all Hénon-like mappings of degree \( d \) have the same number of periodic cycles, counted with multiplicity, as a polynomial of degree \( d \).

In the following, in order to simplify the language, the phrase “counted with multiplicity” is omitted but implied and the word “period” means the lowest period.

**Proof.** Let \( F : \overline{B} \to \mathbb{C}^2 \) be an arbitrary Hénon-like mapping of degree \( d \), with coordinate functions \( f_1 = \pi_1 \circ F \) and \( f_2 = \pi_2 \circ F \). We may suppose that \( F \) is horizontal. Consider the family of mappings, \( F_\varepsilon : \overline{B} \to \mathbb{C}^2 \), for \( 0 < \varepsilon < 1 \), defined by
\[
F_\varepsilon(x, y) = (f_1(x, y), \varepsilon f_2(x, y)).
\]
Each of the mappings \( F_\varepsilon \) has the same number of periodic cycles. As \( \varepsilon \to 0 \), the periodic points converge to the plane \( \mathbb{C} \times \{0\} \).

Hence we can consider the map on this plane induced by the limit mapping, \( x \mapsto f_1(x, 0) \). By definition, this mapping is polynomial-like of degree \( d \) on \( H_0 = \{ x \mid (x, 0) \in B \} \). Such a mapping
is quasi-conformally equivalent to a polynomial of degree $d$ and all polynomials of degree $d$ have the same number of periodic cycles.

QED

**Remarks.** Moreover, all Hénon mappings of degree $d$ have the same number of periodic cycles. In particular, they can be counted explicitly.

There are $d$ fixed points. The number of periodic cycles of prime period $p$ is $d(d^{p-1} - 1)/p$. For period $n \geq 4$, with $n$ not prime, there is a recursive algorithm for determining the number of periodic points: the sum, taken over all $m$ which divide $n$, of the number of points of period $m$ equals $d^n$. Table 1.1 shows the number of periodic cycles of Hénon-like mappings of degree 2 for periods 1 through 12.

Of course, there are other examples of Hénon-like mappings besides actual Hénon mappings and their perturbations. This is much the same as for polynomial-like mappings.

It is tempting to conjecture that every Hénon-like mapping of degree 2, or, in fact, any degree, is conjugate to a Hénon mapping, as in the case of polynomial-like mappings. Unfortunately, this is false.

**Example.** Find an actual Hénon mapping $F$ with $a = 1$ and two periodic points, say of period $k$, at each of which the linearization of $F^\circ k$ has complex conjugate eigenvalues, of absolute value one. Then choose $R > 1 + \sqrt{1 + |c|}$ and a mapping

$$H : \overline{D}_R^2 \rightarrow \mathbb{C}^2$$

which is small on $\overline{D}_R^2$, vanishes at the two periodic points, and so that

$$F + H : \overline{D}_R^2 \rightarrow \mathbb{C}^2$$

is a Hénon-like mapping with these points still periodic, but one being attractive and the other being repulsive. This cannot be conjugate to a Hénon mapping.

However, we cannot find such a Hénon mapping $F$ where the periodic points are fixed points are fixed points even if $a = 1$ is replaced with an arbitrary complex parameter $a$ with $|a| = 1$. 

\begin{table}[h]
\begin{tabular}{|c|c|c|c|c|c|}
\hline
Period & Cycles & Period & Cycles & Period & Cycles \\
\hline
1 & 2 & 5 & 6 & 9 & 56 \\
2 & 1 & 6 & 9 & 10 & 99 \\
3 & 2 & 7 & 18 & 11 & 186 \\
4 & 3 & 8 & 30 & 12 & 335 \\
\hline
\end{tabular}
\caption{Periodic cycles of Hénon-like mappings of degree two}
\end{table}
The problem is that when a Hénon mapping has distinct fixed points, then at least one of them is hyperbolic.

Perhaps with the added requirement that $F$ contracts volumes, it is reasonable to hope that Hénon-like mappings are conjugate to Hénon mappings.

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2. Complex Horseshoes

In this chapter and the next, complex analogs of Smale horseshoes (cf. [S], [Mo]) are defined and analyzed. Using a criterion analogous to the one given by Moser [Mo] in the real case, we will show that many Hénon mappings are complex horseshoes. In particular, actual Hénon mappings (of degree 2) are complex horseshoes when $|c|$ is sufficiently large.

In Figure 1.1, only (a) and (c) appear to be horseshoes. Basically, we would like to say that a horizontal Hénon-like mapping, $F$, of degree $d$ is a complex horseshoe of degree $d$ if the projections

$$
\pi_1 : \bigcap_{0 \leq m \leq n} F^m(B) \to \mathbb{C} \quad \text{and} \quad \pi_2 : \bigcap_{0 \leq m \leq n} F^{n-m}(B) \to \mathbb{C}
$$

are trivial fibrations with fibers disjoint unions of $d^n$ discs.

However, this is not general enough for our purposes. When we investigate the existence of horseshoes in the setting of homoclinic points in chapter 4, the natural domains for these mappings become quite “wiggly” and the meaning of $\pi_1$ and $\pi_2$ as horizontal and vertical projections outside of the domain of definition, $B$, becomes unclear and the notion of any such projections becomes unusable. Figure 4.1 in Chapter 4 should convince the reader of this.

We now give a definition with weaker conditions, inspired by Proposition 1.2, which encompasses the Hénon-like mappings of the previous chapter. Now, instead of requiring $B$ to be an actual bidisc as in Chapter 1, $B$ may be an embedded bidisc. More precisely, letting $D \subset \mathbb{C}$ be the open unit disc, assume that there is an embedding, $\varphi : D^2 \to \mathbb{C}^2$, which is analytic on $D^2$ and such that $B = \varphi(D^2)$ and, naturally, $\overline{B} = \varphi(D^2)$. Set the notation

$$
\partial B_H = \varphi(D \times \partial D) \quad \text{and} \quad \partial B_V = \varphi(\partial D \times D)
$$

for the horizontal and vertical boundaries of $B$. Also, define horizontal and vertical slices: $H_y = \varphi(D \times \{y\})$ and $V_x = \varphi(\{x\} \times D)$ for all $x, y \in D$.

Consider mappings $F : \overline{B} \to \mathbb{C}^2$ which are injective and continuous on $\overline{B}$ and analytic on $B$, and such that either

$$
(2.1) \quad F(\overline{B}) \cap \partial B_H = \emptyset \quad \text{and} \quad \overline{B} \cap F(\partial B_V) = \emptyset
$$

or

$$
(2.2) \quad \overline{B} \cap F(\partial B_H) = \emptyset \quad \text{and} \quad F(\overline{B}) \cap \partial B_V = \emptyset.
$$

Under these conditions, we have the following:
Lemma 2.1. For all $y \in D$, $\pi_1 \circ \varphi^{-1} : F(H_y) \cap B \to D$ is proper.

Proof. Consider the preimage, $S$, of a compact subset, $C$, of $D$. Suppose a sequence in $S$ converges to a point $x$ which is not in $S$. The second part of either condition (2.1) or (2.2) above implies that $x$ is not in $B$ while the first part implies that $x$ is not in $\partial B_H$. Compactness of $C$ implies $x$ is not in $\partial B_V$.

QED

Such a proper mapping has a degree and since the degree is integer-valued and continuous in $y$, this defines a constant, the degree of such a mapping $F$. Now a class of mappings generalizing the Hénon-like mappings of Chapter 1 can be defined.

Definition 2.2. $F : \overline{B} \to \mathbb{C}^2$ is a quasi-Hénon-like mapping of degree $d$ if there exists a mapping $G : \overline{B} \to \mathbb{C}^2$ such that

1. Both $F$ and $G$ are injective and continuous on $\overline{B}$ and analytic on $B$.
2. $F \circ G = 1d$ and $G \circ F = 1d$ where each makes sense.
   Hence, rename $G$ as $F^{-1}$.
3. Either
   (a) $F(\overline{B}) \cap \partial B_H = \emptyset$ and $\overline{B} \cap F(\partial B_V) = \emptyset$ or
   (b) $\overline{B} \cap F(\partial B_H) = \emptyset$ and $F(\overline{B}) \cap \partial B_V = \emptyset$.

The degree from Lemma 2.1 is $d \geq 2$. Moreover, call $F$ either horizontal or vertical according to whether it satisfies (a) or (b), respectively.

Remark. Note that, as in Definition 1.1 for Hénon-like mappings, the conditions (a) and (b) of Definition 2.2 are dual in the sense that (a) and (b) are equivalent to

1. $\overline{B} \cap F^{-1}(\partial B_H) = \emptyset$ and $F^{-1}(\overline{B}) \cap \partial B_V = \emptyset$ and
2. $F^{-1}(\overline{B}) \cap \partial B_H = \emptyset$ and $\overline{B} \cap F^{-1}(\partial B_V) = \emptyset$, respectively.

Of course, Proposition 1.2 gives the following:

Proposition 2.3. Hénon-like mappings of degree $d$ are quasi-Hénon-like of degree $d$.

Now, using the notion of quasi-Hénon-like mappings, complex horseshoes may be defined as suggested above.

Definition 2.4. A complex horseshoe of degree $d$ is a quasi-Hénon-like mapping of degree $d$, $F : \overline{B} \to \mathbb{C}^2$, such that, for all integers $n > 0$, depending on if $F$ is horizontal or vertical, then either the projections

$$\pi_1 \circ \varphi^{-1} : \bigcap_{0 \leq m \leq n} F^m(B) \to \mathbb{C} \quad \text{and} \quad \pi_2 \circ \varphi^{-1} : \bigcap_{0 \leq m \leq n} F^{n-m}(B) \to \mathbb{C}$$

or

$$\pi_2 \circ \varphi^{-1} : \bigcap_{0 \leq m \leq n} F^m(B) \to \mathbb{C} \quad \text{and} \quad \pi_1 \circ \varphi^{-1} : \bigcap_{0 \leq m \leq n} F^{n-m}(B) \to \mathbb{C},$$
respectively, are trivial fibrations with fibers disjoint unions of $d^n$ discs.

Remarks. (1) This definition should be compared to real Smale horseshoes, where the corresponding projections are bundles of $d^n$ intervals. In the real definition, hyperbolicity conditions must be imposed. Being in the complex domain, we get them for free.

(2) As for Hénon-like mappings, we can define sets $K$, $J$, etc. The set $B$ would ideally be a small neighborhood of $K$.

In the context of Hénon-like mappings, the following results will show the close relation between complex horseshoe mappings of degree $d$ and polynomial-like mappings of degree $d$ whose critical points escape immediately.

The following definition, borrowed from Moser [Mo], is the key tool in this study of complex horseshoes. Apparently, this concept is due to V. Alekseev [A].

Let $M$ be a differentiable manifold, $U \subset M$ an open subset, and $f : U \to M$ a differentiable mapping.

**Definition 2.5.** A field of cones $C = (C_x \subset T_x M)_{x \in U}$ on $U$ is an $f$-trapping field if

1. $C_x$ depends continuously on $x$ and
2. whenever $x \in U$ and $f(x) \in U$, then $d_x f(C_x) \subset C_{f(x)}$.

Now consider the connection between trapping fields of cones and complex horseshoes.

**Theorem 2.6.** Let $F : \overline{B} \to \mathbb{C}^2$ be a quasi-Hénon-like mapping of degree $d$. The following are equivalent:

1. $F : \overline{B} \to \mathbb{C}^2$ is a complex horseshoe of degree $d$

2. There exist continuous, positive functions $\alpha(z)$ and $\beta(z)$ on $B$ such that the field of cones

   $$C_z = \{ (\xi_1, \xi_2) \mid |\xi_2| < \alpha(z)|\xi_1| \}$$

   is $F$-trapping and the field of cones

   $$C'_z = \{ (\xi_1, \xi_2) \mid |\xi_1| < \beta(z)|\xi_2| \}$$

   is $F^{-1}$-trapping.

3. $F(\overline{B}) \cap \overline{B}$ and $F^{-1}(\overline{B}) \cap \overline{B}$ both have $d$ connected components.

Remarks. Note that (2) $\Rightarrow$ (1) is borrowed from Moser and that the implication (1) $\Rightarrow$ (2) is what the contractive nature of complex analytic mappings gives us for free. (3) arises naturally in the proof of (2) $\Rightarrow$ (1). When considering a mapping which is actually Hénon-like, consideration of the critical points of $F_{1,y}$ and $F_{2,x}^{-1}$ (or $F_{2,x}$ and $F_{1,y}^{-1}$) in light of the equivalences above yields the following.
Corollary 2.7. Let $F : \overline{D}_1 \times \overline{D}_2 \to \mathbb{C}^2$ be a Hénon-like mapping of degree $d$. The following are equivalent:

1. $F : \overline{D}_1 \times \overline{D}_2 \to \mathbb{C}^2$ is a complex horseshoe of degree $d$.
2. For all $(x, y) \in \overline{D}_1 \times \overline{D}_2$, the critical values of the polynomial-like mappings $F_{1,y}$ and $F^{-1}_{2,x}$ (or $F_{2,x}$ and $F^{-1}_{1,y}$) lie outside of $\overline{D}_1$ and $\overline{D}_2$, respectively.

Proof of Theorem 2.6. Without loss of generality we can assume that $B = D^2$ and that $F$ satisfies condition (a) of Definition 2.2 [or Definition 1.1 in case it is actually Hénon-like].

First we shall prove that (1) $\Rightarrow$ (2). Let $F : \overline{D}^2 \to \mathbb{C}^2$ be a complex horseshoe of degree $d$, let $z = (x, y) \in D^2$ and set $\mathcal{C}_z = \{ (\xi_1, \xi_2) \mid |\xi_2|_y < |\xi_1|_x \}$, where $|\xi|_x$ means the Poincaré length at $x$, i.e.,

$$|\xi|_x = |\xi|/ (1 - |x|^2),$$

so that this field of cones is defined by

$$\alpha(z) = \frac{1 - |y|^2}{1 - |x|^2}.$$

We will show that this field is $F$-trapping. First note that (1) $\Rightarrow$ (3). $F^{-1}(D^2) \cap D^2$ has $d$ connected components—call them $U_1, U_2, \ldots, U_d$—and $F(D^2) \cap D^2$ also has $d$ connected

components, $F(U_1), F(U_2), \ldots, F(U_d)$. Moreover, $F : U_i \to F(U_i)$ is an analytic isomorphism for each $i = 1, \ldots, d$.

Now suppose that $x \in D^2, F(x) \in D^2$, and $\xi \in \mathcal{C}_x \subset T_x D^2$. Take an analytic $\alpha : D \to D$ with $\xi$ tangent to the graph of $\alpha$, which we write as $\text{gr}(\alpha)$, at $x$. Suppose that $x \in \text{gr}(\alpha) \cap U_i$ and consider $F(\text{gr}(\alpha) \cap U_i)$. Define $\beta : D \to D$ such that $\text{gr}(\beta) = F(\text{gr}(\alpha) \cap U_i)$. So $d_x F(\xi)$ is tangent to the graph of $\beta$. Since $F(\text{gr}(\alpha))$ has $d$ components, $\beta$ is not surjective. By Schwarz’ Lemma, $\beta$ is contracting in the Poincaré metric and $d_x F(\xi) \in \mathcal{C}_F(x)$. Thus, (1) $\Rightarrow$ (2).

We next show that (2) $\Rightarrow$ (3). Suppose that $\mathcal{C}_x$ and $\mathcal{C}'_x$ are $F$-trapping and $F^{-1}$-trapping fields of cones, respectively. Consider $\pi_1 \circ F : H_y \to \mathbb{C}$ for an arbitrary $y \in D$. The critical values of this mapping are outside $D$, since otherwise the critical points $z_i \in H_y$ and their images would lie in $D^2$, and the image of horizontal vectors at $z_i$ would be vertical vectors at $F(z_i)$, and this is forbidden by the existence of the trapping field. By Lemma 2.2 and the definition of quasi-Hénon-like of degree $d$, $F(\overline{B}) \cap \overline{B}$ has $d$ components. This proves that (2) $\Rightarrow$ (3).
It remains to show that \((3) \Rightarrow (1)\). So, assume that \(F(B) \cap \overline{B}\) has \(d\) components. Suppose by induction that, for all \(n = 1, 2, \ldots, N - 1\),

\[(1) \quad \pi_2 : \bigcap_{0 \leq m \leq n} F^{o-m}(D^2) \to D\]

is a trivial fibration with fiber \(d^n\) discs and

(2) for all \(y \in D\) and each component \(U\) of

\[H_y \cap \bigcap_{0 \leq m \leq n} F^{o-m}(D^2)\]

the map \(\pi_1 \circ F^o : U \to D\) is an analytic isomorphism.

Let us prove it for \(n = N\). Choose such a component \(U\) for \(n = N - 1\), and consider \(F \circ U\).

Since \(\pi_1 \circ F^{oN-1}\) is an isomorphism from \(U\) to \(D\) and \(F(D^2) \cap D^2\) has \(d\) components, \(F^{oN}(U) \cap D^2\) has \(d\) components also, \(U_1, U_2, \ldots, U_d\), each homeomorphic to a disc, and for each the mapping \(\pi_1 : U_i \to D\) is an analytic isomorphism. So, \((3) \Rightarrow (1)\).

QED

Proof of Corollary 2.7. Since polynomial-like of degree \(d\) with critical points escaping implies an image with \(d\) components, (2) implies that \(F(D^2) \cap D^2\) has \(d\) components. So, \((2) \Rightarrow (1)\) follows from \((3) \Rightarrow (1)\) of Theorem 2.6.

By \((1) \Rightarrow (2)\) of Theorem 2.6, there exist \(F\)-trapping and \(F^{-1}\)-trapping fields of cones. Consider \(F_{1,y} = \pi_1 \circ F : H_y \to \mathbb{C}\) for an arbitrary \(y \in D\). This is polynomial-like of degree \(d\). The critical values of this mapping are outside \(D\), since otherwise the critical points \(z_i \in H_y\) and their images would lie in \(D^2\), and the image of horizontal vectors at \(z_i\) would be vertical vectors at \(F(z_i)\), and this is forbidden by the existence of the trapping field. Therefore, \((1) \Rightarrow (2)\).

QED

Remarks. The definition of a trapping field of cones made above is designed to be simple and easily verifiable, but is not the strongest definition which will still give the result above. Such a definition would require the field of cones to be defined only on the set \(\cap_{n \geq 0} f^{o_n}(U)\) and could be used to obtain sharper results (cf.Proposition 2.10 and the remarks following it).

Proposition 2.8. The diameters of the discs in the fibers

\(30\)

above tend to 0 with \(n\).

The essential part of the proof of this proposition is the following lemma on polynomial-like mappings. Let \(D\) be an open disc and let \(\alpha : D \to D\) be an analytic injection. The mapping \(f = f_\alpha : D \to \mathbb{C}\) defined by \(f(x) = f_\alpha(x) = \pi_1(F(x, \alpha_j(x)))\) is polynomial-like of degree \(d\) with the critical points escaping immediately, i.e., with the critical values in \(\mathbb{C} \setminus \overline{D}\). So, there exist \(d\) components, \(U_1, U_2, \ldots, U_d\), of \(f^{-1}(D)\) and \(d\) mappings, \(g_i = g_{\alpha,i} : D \to U_i\), which are inverses of \(f|_{U_i}\), for \(i = 1, 2, \ldots, d\).
Lemma 2.9. Independent of $\alpha$ and $i$, the diameter of the image of a connected subset $U$ of $D$ under $g_i$ as compared with the diameter of $U$ shrinks by a constant factor less than one.

Proof. By the definition of polynomial-like mappings it follows that the distance between the boundary of $D$ and the boundary of $g_i(D)$ is bounded away from 0. Now, the distance between two points, $x$ and $y$, of $g_i(D)$ equals the distance between the points $f(x)$ and $f(y)$ when measured in the Poincaré metrics on $g_i(D)$ and $D$, respectively. Since $g_i(D) \subset D$, when measured in $D$, the distance between $x$ and $y$ is always less than the distance between $f(x)$ and $f(y)$.

There exists a constant $K < 1$ such that $d_D(x, y) \leq K d_D(f(x), f(y))$. So, $K$ is the shrinking factor.

QED

Proof of Proposition 2.8. Consider a component, $V$, of a fiber over $y$ of $\pi_2$:

$$V = V_n \cap V_{n-1} \cap V_{n-2} \cap \cdots$$

where each $V_j$ is the component of $H_y \cap F^{0-j}(D^2)$ which contains $V$. Each $V_j$ has the property that $F^j(V_j)$ is the graph of an analytic injection $\alpha_j : D \to D$. Now there exist such mappings $\alpha_1$, $\alpha_2$, $\ldots$, $\alpha_n : D \to D$ and integers $i_1$, $i_2$, $\ldots$, $i_n$, which are between 1 and $d$, inclusive, such that

$$\pi_2(V) = g_{\alpha_n,i_n} \circ \cdots \circ g_{\alpha_2,i_2} \circ g_{\alpha_1,i_1}(D).$$

By Lemma 2.9, each $g_{\alpha_j,i_j}$ contracts the diameter of $D$ by a constant factor $K < 1$ independent of $\alpha_j$ and $i_j$.

QED

This criterion can be used to show that for each $a$ there exists $r(a)$ such that if $|c| > r(a)$, then the Hénon mapping $F_{a,c}$ is a complex horseshoe. Of course, in the real locus this was known (cf. [DN], [N]), except that then $c$ has to be taken very negative. When $c$ is large and positive, all the “horseshoe behavior” is complex.

Proposition 2.10. For each $a \neq 0$ and each $c$ such that $|c| > (5/4 + \sqrt{5}/2)(1 + |a|)^2$, there exists an $R$ such that $F_{a,c} : \overline{D}_R^2 \to \mathbb{C}^2$ is a complex horseshoe.

Proof. Set $F = F_{a,c}$ and set

$$R = \alpha \left( 1 + |a| + \sqrt{(1 + |a|)^2 + 4|c|} \right)$$

with $\alpha > 1/2$. By Proposition 1.4 and example (1) which follows it, $F : \overline{D}_R^2 \to \mathbb{C}^2$ is quasi-Hénon-like. By Theorem 2.6, this proposition can be proved by showing that, for appropriate values of $a$, $c$, and $\alpha$, the constant field of cones defined simply by

$$\{ (\xi_1, \xi_2) \mid |\xi_2| < |\xi_1| \}$$
is $F$-trapping, and, similarly, that the field of cones

$$\{ (\xi_1, \xi_2) \mid |\xi_1| < |\xi_2| \}$$

is $F^{-1}$-trapping.

The key to showing that the former field of cones is $F$-trapping is the observation that $F(x, y) \in \overline{D_R^2}$ implies that $|x^2 + c - ay| \leq R$ which implies that

$$|x|^2 \geq |c - R(1 + |a|)|.$$

(2.4)

Since $d_{(x,y)}F(\xi_1, \xi_2) = (2x\xi_1 - a\xi_2, \xi_1)$, we want to show that

$$|\xi_1| > |\xi_2| \quad \text{implies} \quad |2x\xi_1 - a\xi_2| > |\xi_1|.$$

Using (2.4) to substitute for $x$, it suffices to show that

$$2\sqrt{|c| - R(1 + |a|)} - |a| > 1.$$

(2.5)

A similar analysis for $F^{-1}$ and the latter field of cones yields the same inequality (2.5).

After using (2.3) to substitute into (2.5) for $R$ and setting $|c| > \beta(1 + |a|)^2$, a computation gives

$$|c| > (\alpha + \frac{1}{\beta} + \alpha\sqrt{1 + 4\beta})(1 + |a|)^2,$$

which is satisfied when

$$\beta \geq 2\alpha^2 + \alpha + \frac{1}{\alpha} + \alpha\sqrt{4\alpha^2 + 4\alpha + 2}.$$

As $\alpha$ goes to $\frac{1}{2}$, the lower bound for $\beta$ goes to $5/4 + \sqrt{5}/2$.

QED

Remarks. (1) The above result says essentially everything about Hénon mappings in the parameter range to which it applies.

(2) In the next section, it is shown that a complex horseshoe of degree $d$, restricted to the set $K$, is conjugate to the full shift on $d$ symbols. Milnor [M] has used a completely different method to show the existence of an embedding of the shift for $\beta = 2$. The degenerate case, $a = 0$, shows that this is the strongest result given by the condition $|c| > \beta(1 + |a|)^2$ which we could hope for.

(3) As was indicated in the remarks after the proof of Theorem 2.6, if we strengthen the definition of trapping fields, then we can improve Proposition 2.10 so that we have complex horseshoes whenever $|c| > \beta(1 + |a|)^2$ for some $\beta$ between 2 and $5/4 + \sqrt{5}/2 \approx 2.368$. By requiring that the trapping field be defined only on the set of points which remain in the bidisc for
additional iterations, we could get a sequence of improvements for the lower bound of $\beta$. However, it was intended that the existence of trapping fields be easily verified and hence they were defined on the bidisc rather than on $K_+$ or some compromise between these two. Nevertheless, with much inequality manipulating sharper results can be achieved.

(4) Allowing fields of larger cones such as

$$\{ (\xi_1, \xi_2) \mid \gamma |\xi_2| < |\xi_1| \},$$

with $0 < \gamma < 1$, it is possible to get better estimates for lower bounds depending on $|a|$ for $|c|$ so that $F$ is a complex horseshoe. For large $|a|$ it is not difficult to get sharper estimates than those given above. For example, for $|a| = 1$, it is possible to use very large cones to see that horseshoes exist for $|c| > 7.2$, as compared with $|c| > 9.48$ using the result above (or $|c| > 8$ using Milnor’s result).

(5) Using Example (2) following Proposition 1.4 and the same procedure as in the proof of Proposition 2.10 above, we can get examples of complex horseshoes of higher degrees. Of course, sharper results are possible here also.

3. Complex Horseshoes as Shift Dynamical Systems

Suppose $d$ is an integer greater than or equal to two. Let $S_d = \{0, 1, \ldots, d\}^\mathbb{Z}$ and $\tau_d : S_d \to S_d$ be the shift on $d$ symbols. Let $\mathbb{Z}_+$ be the positive integers and $\mathbb{Z}_- = \mathbb{Z} \setminus \mathbb{Z}_+$. Let

$$S_d^+ = \{0, 1, \ldots, d\}^{\mathbb{Z}_+} \quad \text{and} \quad S_d^- = \{0, 1, \ldots, d\}^{\mathbb{Z}_-}.$$ 

Let $\tau_d^+ : S_d^+ \to S_d^+$ and $\tau_d^- : S_d^- \to S_d^-$ be the corresponding one-sided shifts.

Let $F : \overline{D}^2 \to \mathbb{C}^2$ be a complex horseshoe of degree $d$. The following results allow a complete understanding of the invariant subsets for a complex horseshoe, essentially identical with what you get for real horseshoes.

**Theorem 3.1.** There exists a homeomorphism $\Phi : K \to S_d$ which conjugates $F$ to the full shift $\tau_d$.

**Remark.** In the case where $d = 2$, it will be seen that $\Phi$ is unique up to the automorphism of $S_2$ exchanging 0 and 1.

In the following we will fix $d$ and set $\mathcal{S} = S_d$, $\mathcal{S}_+ = S_d^+$, $\mathcal{S}_- = S_d^-$, $\tau = \tau_d$, $\tau_+ = \tau_d^+$, and $\tau_- = \tau_d^-$. In fact, assume that $d = 2$ without loss of generality.

**Theorem 3.2.** There exist homeomorphisms

$$\Gamma_+ : K_+ \to \mathcal{S}_+ \times D \quad \text{and} \quad \Gamma_- : K_- \to \mathcal{S}_- \times D.$$
which can be written as
\[ \Gamma_+(x,y) = (\gamma_+(x,y), y) \quad \text{and} \quad \Gamma_-(x,y) = (x, \gamma_-(x,y)), \]
such that the following diagrams commute:
\[
\begin{array}{ccc}
K_+ \cap F^{-1}(D^2) & \xrightarrow{\gamma_+} & S_+ \\
\downarrow F & & \downarrow \tau_+ \\
K_+ & \xrightarrow{\gamma_+} & S_+ \\
\end{array}
\quad \text{and} \quad
\begin{array}{ccc}
K_- \cap F(D^2) & \xrightarrow{\gamma_-} & S_- \\
\downarrow F^{-1} & & \downarrow \tau_- \\
K_- & \xrightarrow{\gamma_-} & S_- \\
\end{array}
\]

The inclusions \( K \subset K_+ \) and \( K \subset K_- \) induce the canonical projections \( S \to S_+ \) and \( S \to S_- \).

We will define in Lemma 3.3 models for \( S_+ \) and \( S_- \) which are defined which are more closely related to the dynamics of the horseshoe and then

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restate the result as Theorem 3.4 in a form which uses this model and such that the new statement is strictly stronger than Theorem 3.2.

Choose a polynomial-like mapping \( f : U' \to U \) of degree 2 such that the critical value lies in \( U \setminus U' \) (for instance, \( U' = D_3(0) \), \( U = D_9(4) \), and \( f(z) = z^2 + 4 \)). Since the critical value is in \( U \setminus U' \), the set \( f^{-1}(U') \) consists of two open sets, \( U_0 \) and \( U_1 \), homeomorphic to discs, and each \( f|_{U_i} : U_i \to U' \) is an isomorphism. Let \( g_i : U' \to U_i \) be the corresponding inverses.

**Lemma 3.3.** (a) For each sequence \( \varepsilon = (\varepsilon_1, \varepsilon_2, \ldots) \) of 0’s and 1’s, the nested intersection
\[
p_+(\varepsilon) = \bigcap_{n \geq 0} g_{\varepsilon_n} \cdots g_{\varepsilon_1}(U')
\]
reduces to a single point.

(b) The mapping \( p_+ : S_+ \to U \) is a homeomorphism onto \( J_f \), the Julia set of \( f \), which conjugates \( f \) to the one-sided shift \( t_+ : S_+ \to S_+ \).

(c) There is a similar homeomorphism \( p_- : S_- \to J_f \)

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which conjugates \( f \) to the one-sided shift \( \tau_- : S_- \to S_- \).

**Proof.** Part (a) is proved by contraction as in Proposition 2.8 and is simpler. Actually, the proof of Proposition 2.8 was adapted from this lemma. Part (b) follows from the fact that \( J_f = K_f \) when the critical value is not in \( U' \).

QED
**Theorem 3.4.** There exist homeomorphisms

\[ \Psi_+ : D^2 \to U \times D \quad \text{and} \quad \Psi_- : D^2 \to U \times D \]

which can be written as

\[ \Psi_+(x,y) = (\psi_+(x,y), y) \quad \text{and} \quad \Psi_-(x,y) = (x, \psi_-(x,y)) \]

such that the following diagrams commute:

\[
\begin{array}{ccc}
D^2 \cap F^{-1}(D^2) & \overset{\psi_+}{\longrightarrow} & U' \\
\downarrow F & & \downarrow f \\
D^2 & \overset{\psi_+}{\longrightarrow} & U
\end{array}
\quad \quad
\begin{array}{ccc}
D^2 \cap F(D^2) & \overset{\psi_-}{\longrightarrow} & U' \\
\downarrow F^{-1} & & \downarrow f \\
D^2 & \overset{\psi_-}{\longrightarrow} & U
\end{array}
\]

**Proof.** First choose the restriction

\[ \psi_+ : \partial D \times D \to \partial U \]

This is induced from a homeomorphism \( \partial D \to \partial U \). Next label the two components \( V_0 \) and \( V_1 \) of \( D^2 \cap F^{-1}(D^2) \), and define \( \psi_+ \) on \( \partial V_i \) to be \( g_i \circ \psi_+ \circ F \).

**Lemma 3.5.** The restricted mapping

\[ \Psi_+ : \partial V_i \to \partial U_i \times D \quad \text{defined by} \quad \Psi_+(x,y) = (\psi_+(x,y), y) \]

is a homeomorphism.

**Proof.** It is enough to prove that for each \( y \in D \), the restriction

\[ \psi_+ : \partial V_i \cap D_y \to \partial U_i \]

is a homeomorphism. It is of degree 1, and locally injective because \( F(\partial V_i \cap D_y) \) is a simple closed curve with tangent vectors in the trapping field, hence transversal to the (vertical) fibers of \( \psi_+ \) on \( \partial D \times D \). But an immersion of degree 1 from a simple closed curve to a simple closed curve is a homeomorphism.

**QED for Lemma 3.5**

The following is a simple result on surface topology.
Lemma 3.6. If $X_1$ and $X_2$ are spheres with three holes and $h : \partial X_1 \to \partial X_2$ is an orientation-preserving homeomorphism, then there exists a homeomorphism $\hat{h} : X_1 \to X_2$ extending $h$.

Completion of Proof of Theorem 3.4. We can extend $\psi_+$ to give a homeomorphism

$$\Psi_+ : D^2 \setminus (V_0 \cup V_1) \to (U \setminus (U_0 \cup U_1)) \times D$$

by the Cerf fibration theorem.

Let $W = D^2 \setminus (V_0 \cup V_1)$. Now $W \cap F^{-1}(W)$ has two components, $W_0 \subset V_0$ and $W_1 \subset V_1$. Extend $\psi_+$ to $W_i$ by $\psi_+(x,y) = g_i(\psi_+(F(x,y)))$. Again we must prove that $(x,y) \mapsto (\psi_+(x,y),y)$ is a homeomorphism $W_i \to U \setminus (U_0 \cup U_1) \times D$; the proof is just as before.

Continue to extend in exactly the same way to $W \cap F^{o-2}(W)$, $W \cap F^{o-3}(W)$, . . . , verifying at each stage that the extension is a homeomorphism.

When we are done, we will have a mapping

$$\psi_+ : D^2 \setminus K_+ \to U \setminus J_f$$

such that the mapping

$$\Psi_+ : D^2 \setminus K_+ \to (U \setminus J_f) \times D \quad \text{defined by} \quad \Psi_+(x,y) = (\psi_+(x,y),y)$$

is a homeomorphism and

$$\psi_+ \circ F(x,y) = f \circ \psi_+(x,y) \quad \text{on} \quad (D^2 \cap F^{-1}(D^2)) \setminus K_+.$$

In each slice $D_y$, the set $D_y \cap K_+$ is a Cantor set, so that $\overline{D_y} \cap K_+$ is the set of ends of $\overline{D_y} \setminus K_+$. In the same way, $J_f$ is the set of ends of $\overline{U} \setminus J_f$. Since the end-point compactification is functorial, the map $\psi_+$ extendsto$D^2$, and since the extension is a homeomorphism $D_y \to U$ for each $y$, the map $\Psi_+$ is a homeomorphism.

QED for Theorem 3.4

Remark. It looks as if there were two independent choices of labeling $V_0$ and $V_1$, the two components of $D^2 \cap F^{-1}(D^2)$, and $V_0'$ and $V_1'$, the two components of $D^2 \cap F(D^2)$. However,

$$F : D^2 \cap F^{-1}(D^2) \to D^2 \cap F(D^2)$$

is a homeomorphism. So choose the labelings so that $F(V_0) = V_0'$.

Now Theorem 3.2 follows immediately from the strictly stronger Theorem 3.4. We return to the theorem on conjugating $F$ on $K$ to the full shift on 2 symbols and use Theorem 3.2.

Proof of Theorem 3.1. Now the mapping given by $(x,y) \mapsto (\gamma-(x,y),\gamma+(x,y))$ clearly has all the properties required.

QED for Theorem 3.1

We will now show that the dynamics of the projective limit under $f$ underlies the dynamics of the horseshoe mapping. Define $U'_- = \lim(U'_+, f)$ and let $\pi : U_- \to U$ be the projection defined by $(\ldots, z_2, z_1, z_0) \mapsto z_0$. Also, let $U'_- = \pi^{-1}(U') \subset U_-$. 
Theorem 3.7. There exist homeomorphisms $\eta_{\pm} : K_{\pm} \to U_{-}$ such that the following diagrams commute:

$K_+ \cap F^{-1}(D^2) \xrightarrow{\eta_+} U'_- \quad K_- \cap F(D^2) \xrightarrow{\eta_-} U'_-$

and

$K_+ \xrightarrow{\eta_+} U_- \quad K_- \xrightarrow{\eta_-} U_-.$

Proof. Define $\eta_+$ by

$$\eta_+(x, y) = (\ldots, \psi_-(F^{\circ 2}(x, y)), \psi_-(F(x, y)), \psi_-(x, y)).$$

QED

4. The Ubiquity of Complex Horseshoes

Suppose that $F : \mathbb{C}^2 \to \mathbb{C}^2$ is an analytic mapping. A point $q$ is a homoclinic point of $F$ if there exists a positive integer $k$ such that

$$(*) \quad \lim_{n \to \infty} F^{\circ kn}(q) = \lim_{n \to \infty} F^{\circ -kn}(q) = p$$

where, in particular, the limits exist.

Note that the limit point, $p$, in $(*)$ is a hyperbolic periodic point of $F$ of period least $k$ satisfying $(*)$. To rephrase this definition, $q$ is in both the stable and unstable manifolds of $F$ at $p$, which will be denoted by $W^s$ and $W^u$, respectively. We will call a homoclinic point transversal if these invariant manifolds intersect transversally.

Note that the invariant manifolds $W^s$ and $W^u$ tend to intersect in lots of points in $\mathbb{C}^2$ unless $F$ is linear or affine. This is quite different from the case of these invariant manifolds in $\mathbb{R}^2$. These intersections are almost always transversal.

In the real, Smale [S] (cf. [Mo]) showed that in a neighborhood of a transversal homoclinic point there exist horseshoes. In this chapter we give an analogous result in the case of complex horseshoes.

Throughout this chapter we will assume that $F$ is analytic, $p$ is a hyperbolic periodic point of period $k$, and $q$ is a transversal homoclinic point as described above.
**Theorem 4.1.** For every positive integer $d \geq 2$, there exists an embedded bidisc, $B_d$, centered at $p$ and a positive integer $N = N(d)$ such that $F^{\circ kN} : B_d \to \mathbb{C}^2$ is a complex horseshoe of degree $d$.

**Remark.** By Theorem 3.1, there exist homeomorphisms $\Phi_d : B_d \to S_d$ which conjugate $F^{\circ kN(d)}$ to the full shifts $\tau_d$ on $d$ symbols.

Without loss of generality and in order to simplify the notation, assume that $k = 1$, i.e., that $p$ is a fixed point. Also, fix a positive integer $d \geq 2$.

Essentially, the proof of Theorem 4.1 involves two steps.

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First, it must be shown that there are quasi-Hénon-like mappings floating around and, second, that these yield complex horseshoes. We proceed by doing the essential part of the second step first, that is, showing the existence of trapping fields of cones.

**Lemma 4.2.** There exists a neighborhood $U$ of $p$ such that the field of cones

$$C_z = \{ (\xi_1, \xi_2) \mid |\xi_2| < |\xi_1| \}$$

is $F$-trapping on $U$ and the field of cones

$$C'_z = \{ (\xi_1, \xi_2) \mid |\xi_1| < |\xi_2| \}$$

is $F^{-1}$-trapping on $U$.

**Proof.** Suppose that the eigenvalues of $dpF$ are $\mu$ and $\lambda$ with $0 < |\mu| < 1 < |\lambda|$. In a small neighborhood, $U$, of $p$, we can use a set of local coordinates $(u, v)$ such that

$$F(u, v) = (f(u, v), g(u, v))$$

with the properties that

$$f(0, v) = g(u, 0) = 0$$

and the partial derivatives

$$f_u(0, 0) = \lambda \quad \text{and} \quad f_v(0, 0) = \mu.$$
and

\[ |\eta_1| = |g_u(\xi_0) + g_v(\eta_0)| \]
\[ \leq |g_u(\xi_0)| + |g_v(\eta_0)| \]
\[ \leq O(r)|\xi_0| + |\mu||\eta_0| \]
\[ \leq (|\mu| + O(r))|\xi_0|. \]

So, for \( r \) small enough, \( |\xi_1| > |\eta_1| \) and \( d_{(u_0,v_0)}F(\xi_0,\eta_0) \in \mathcal{C}_{F_{(u_0,v_0)}} \).

QED

Now, we show that some iterate of \( F \) is quasi-Hénon-like of degree \( d \) on a bidisc contained in \( U \).

**Figure 4.1**: Horseshoes from transverse homoclinic points

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**Lemma 4.3.** There exist \( D_s \subset W^s \) and \( D_u \subset W^u \) isomorphic to discs with \( D_s, D_u \subset U \) and a positive integer \( n \) such that \( F^n(D_u) \) intersects \( D_s \) in exactly \( d \) points and the following conditions hold:

\[ F^n(D_u) \cap \partial D_s = \emptyset \quad \text{and} \quad F^n(\partial D_u) \cap D_s = \emptyset \]
Proof. Choose subsets \( D_s \subset W^s \cap U \) and \( D_u \subset W^u \cap U \) such that \( D_s \cap D_u = \{ p \} \) and both \( D_s \) and \( D_u \) are isomorphic to discs. Since \( W^s = \bigcup_{j>0} F^j(D_u) \) and \( W^s \) and \( D_u \) intersect in homoclinic points near \( p \), there exists a smallest positive integer \( n \) such that \( F^n(D_u) \cap D_s \) contains at least \( d \) points, including \( p \).

Note that the set \( F^n(D_u) \cap D_s \) must consist of finitely many points for all positive integer \( n \).

If \( F^n(D_u) \cap \partial D_s \neq \emptyset \), then take \( D_s \) to be slightly smaller. If \( F^n(\partial D_u) \cap D_s \neq \emptyset \), then take \( D_u \) to be slightly smaller. In either case, make the adjustments to keep the same number of points in \( F^n(D_u) \cap D_s \).

If there are more than \( d \) points in \( F^n(D_u) \cap D_s \), then deform \( D_u \) slightly, making it smaller, to exclude some points from \( F^n(D_u) \cap D_s \). This can be done in an orderly way. In particular, there is a natural metric on \( D_s \)—the Poincaré metric of a disc and consider the point \( x \) of \( F^n(D_u) \cap D_s \) which is farthest from \( p \) in this metric (or one such point if more than one has this property). Now deform \( D_u \) by taking out \( F^{-n}(x) \) and staying clear of the other preimages of points in \( F^n(D_u) \cap D_s \).

QED

\[ \text{Lemma 4.4. There exists a nonnegative integer } m \text{ such that, if } D_{u,m} = F^{-m}(D_u) \text{ and } U_m = D_{u,m} \times D_s, \text{ then } F^{n+m}|_{U_m} \text{ is a Hénon-like mapping of degree } d. \]

Proof. For nonnegative integers \( j \), set \( D_{u,j} = F^{-j}(D_u) \subset D_u \) and \( U_m = D_{u,m} \times D_s \). Let \( m \) be the smallest nonnegative integer such that \( U_m \cap F^{n+m}(\partial D_{u,m} \times D_s) = \emptyset \). Taking \( m \) larger, if necessary, Lemma 4.3 and transversality guarantee that \( F^{n+m}(U_m) \cap (D_{u,m} \times \partial D_s) = \emptyset \).

QED

Set \( N = N(d) = m + n \).

Proof of Theorem 4.1. Now by using the method of the proof of Lemma 4.2 inductively, the same fields of cones

\[ C_x = \{ (\xi_1, \xi_2) \mid |\xi_2| < |\xi_1| \} \]

and

\[ C'_x = \{ (\xi_1, \xi_2) \mid |\xi_1| < |\xi_2| \} \]

are \( F^o N \)-trapping and \( F^o - N \)-trapping on \( U_m \), respectively.

QED

**List Of References**

[A] Alekseev, V., *Quasirandom dynamical systems I, II, III*, Math. USSR Sbornik, 5 (1968), pp. 73–128; 6 (1968), pp. 505–60; 7 (1969), pp. 1–43.
Bieberbach, L., Beispiel zweier ganzer Funktionen zweier komplexer Variablen, welche eine schlicht volumetreue Abbildung des $\mathbb{R}^4$ auf einen Teil seiner selbst vermitteln, Sitzungsber. Preuss. Akad. Wiss. Berlin, Phys.-math. Kl. (1933), pp. 476–79.

Blanchard, P., Complex analytic dynamics on the Riemann sphere, Bull. (New Ser.) AMS 11 (1984), pp. 85–141.

Devaney, R., and Nitecki, Z., Shift automorphisms in the Hénon mapping, Comm. math. Phys. 67 (1979), pp. 137–48.

Douady, A., and Hubbard, J., On the dynamics of polynomial-like mappings, Ann. scient. Éc. Norm. Sup., 4e ser. 18 (1985), pp. 287–343.

Fatou, P., Sur les équations fonctionnelles, Bull. Soc. math. France 47 (1919), pp. 161–271; 48 (1920), pp. 33–94; 48 (1920), pp. 208–314.

Douady, A., and Hubbard, J.,, Sur les fonctions méromorphes de deux variables and Sur certaines fonctions uniformes de deux variables, C. R. Acad. Sc. Paris 175 (1922), pp. 862-65, 1030–33.

Friedland, S., and Milnor, J., Dynamical properties of plane polynomial automorphisms (in preparation).

Hénon, M., Numerical study of quadratic area preserving mappings, Q. Appl. Math. 27 (1969), pp. 291–312.

Hénon, M., A two-dimensional mapping with a strange attractor, Commun. math. Phys. 50 (1976), pp. 69–77.

Hubbard, J., and Oberste-Vorth, R., Hénon mappings in the complex domain (in preparation).

Julia, G., Mémoires sur l’itération des fonctions rationelles, J. Math. 8 (1918), pp. 47–245 (See also Œuvres de Gaston Julia, Gauthier-Villars, Vol. I, Paris, 1968, pp. 121–319.)

Milnor, J., private communication.

Moser, J., Stable and Random Motions in Dynamical Systems, Princeton Univ. Press, Princeton, N.J., 1973.

Newhouse, S., Lectures on dynamical systems in Dynamical Systems, C.I.M.E. Lectures, Bressanone, Italy, June 1978, Birkhauser-Boston, 1980, pp. 1–114.

Smale, S., Diffeomorphisms with many periodic points (in Differential and Combinatorial Topology, S. Cairns, ed.), Princeton Univ. Press, Princeton, N.J., 1965, pp. 63–80.
ADDENDUM

I would like to make several remarks:

(1) The content of this document is intended to be an unedited copy of the original thesis. A few corrections of minor spelling and other typographical errors have been made. However, this document is a reconstruction in TeX of the word-processed original. This has been paginated to reduce the number of pages. The original thesis had 65 double spaced pages including the title page; horizontal lines indicate the original pagination and numbering.

(2) References to [HO] in the thesis refer to a preprint. In actual publication, the reference on page 2 is to [5], the references on page 5 are generally to [1] and [2], and the reference on page 15 is to [1].

(3) Crossed mappings—a typical Hénon-like mapping—are studied also in [2].

(4) The content of this thesis has been published in [3], [4], and [6]:
   - [3] includes the content of chapter 1 and much of chapter 2 using crossed mappings;
     (Note that [3] contains a typographical error: the constant $\frac{5}{4} + \sqrt{\frac{5}{2}}$ of Proposition 2.10 of the thesis is incorrectly given as $\frac{5}{4} + \sqrt{\frac{5}{2}}$ in Proposition 2 of [3].)
   - [4] includes some of the content of chapter 3; and
   - [6] includes the content of chapters 3 and 4.

References

[1] J. Hubbard and R. Oberste-Vorth, Hénon mappings in the complex domain I: the global topology of dynamical space, Publ. Math. IHES 79 (1994), pp. 5-46.
[2] ———–, Hénon mappings in the complex domain II: projective and inductive limits of polynomials, in Real & Complex Dynamical Systems, B. Branner and P. Hjorth, eds., Kluwer, 1995, pp. 89-132.
[3] R. Oberste-Vorth, Horseshoes among Hénon mappings, in Recent Advances in Applied and Theoretical Mathematics, N. Mastorakis, ed., WSES Press, 2000, pp. 116–121.
[4] ———–, Horseshoes as projective limits, in Conference Proceedings: 2002 WSEAS MMACTEE, WAMUS, NOLASC, Vouliagmeni, Athens, Greece, Dec. 29-31, 2002 (CD-ROM), N. Mastorakis, M. Er, C. D’Attelis, eds., WSEAS Press, 2003.
[5] ———–, Normal forms and Fatou-Bieberbach domains, WSEAS Trans. on Math. 3 (2004), pp. 253-258.
[6] ———–, Complex horseshoes, in Proceedings of the Sixth WSEAS International Conference on Applied Mathematics (CD-ROM), N. Mastorakis, ed., WSEAS Press, 2004.

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July 1, 2005