POLYNOMIAL LOG-VOLUME GROWTH AND THE GK-DIMENSIONS OF TWISTED HOMOGENEOUS COORDINATE RINGS

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Abstract. Let $f$ be a zero entropy automorphism of a compact Kähler manifold $X$. We study the polynomial log-volume growth $\text{Plov}(f)$ of $f$ in light of the dynamical filtrations introduced in our previous work with T.-C. Dinh. We obtain new upper bounds and lower bounds of $\text{Plov}(f)$. As a corollary, we completely determine $\text{Plov}(f)$ when $\dim X = 3$, extending a result of Artin–Van den Bergh for surfaces. When $X$ is projective, $\text{Plov}(f) + 1$ coincides with the Gelfand–Kirillov dimensions $\text{GKdim}(X, f)$ of the twisted homogeneous coordinate rings associated to $(X, f)$. Reformulating these results for $\text{GKdim}(X, f)$, we improve Keeler’s bounds of $\text{GKdim}(X, f)$ and provide effective upper bounds of $\text{GKdim}(X, f)$ which only depend on $\dim X$.

Contents

1. Introduction 1
2. Polynomial log-volume growth 5
3. Quasi-nef sequences and dynamical filtrations 16
4. Upper bounds of $\text{Plov}(f)$: beginning of the proof of Theorem 1.2 22
5. A refined lower bound: end of the proof of Theorem 1.2 and Corollary 1.6 27
6. Complex tori: Proof of Theorem 1.4 and a few remarks 28
7. Three dimensional case and explicit examples 32
8. Twisted homogeneous coordinate rings and GK-dimensions: Proofs of Theorem 1.8 and Corollary 8.6 34
References 39

1. INTRODUCTION

1.1. Zero entropy automorphisms.

Let $X$ be a compact Kähler manifold and let $f : X \circlearrowleft$ be an automorphism (i.e. biholomorphic self-map) of $X$. The topological entropy $h_{\text{top}}(f)$ is an invariant measuring the complexity of the dynamical system $f : X \circlearrowleft$. Thanks to Gromov–Yomdin’s theorem [12, 26], we have

$$h_{\text{top}}(f) = \log r(f) \geq 0,$$

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where $r(f)$ is the spectral radius of $f^*: H^*(X, \mathbb{C}) \circlearrowright$.

This paper focuses on automorphisms $f$ with zero entropy $h_{\text{top}}(f) = 0$ (cf. Lemma 2.3). In the context of complex dynamics of compact Kähler manifolds, they have recently been investigated in various works (see e.g. [4, 21, 31, 11, 8]). In these works, more refined invariants of them are studied, such as the polynomial entropy, the polynomial log-volume growth $\text{Plov}(f)$ [5], and the polynomial growth $k(f)$ of the pullbacks $f^*: H^1(X, \mathbb{C}) \circlearrowright$:

\[ \| (f^m)^*: H^1(X, \mathbb{C}) \circlearrowright \| \asymp m \rightarrow \infty m^{k(f)}. \]

New structures of $f^*: H^*(X, \mathbb{C}) \circlearrowright$ have also been discovered such as the dynamical filtrations [8, §3]. Below is one such consequence, which is also relevant to the present work.

**Theorem 1.1** ([8, Theorem 1.1, Remark 3.9.(1)] and Corollary 3.7). Let $f \in \text{Aut}(X)$ be an automorphism of zero entropy. Assume that $d := \text{dim} X \geq 1$. Then $k(f)$ is an even integer which satisfies

\[ (1.2) \quad k(f) \leq 2(d - 1) \]

and

\[ (1.3) \quad k(f) \leq 2(d - \kappa(X)), \]

where $\kappa(X)$ is the Kodaira dimension of $X$. Moreover, these estimates are optimal.

The upper bound (1.2) is the most essential part and was proven in [8, Theorem 1.1]. We will prove the refinement (1.3) in Corollary 3.7 based on the approach developed in [8].

### 1.2. Polynomial log-volume growths.

The main goal of this paper is to study the polynomial log-volume growth $\text{Plov}(f)$ of an automorphism $f: X \circlearrowright$. We first recall its definition. For every $n \geq 1$, let $\Gamma_f(n) \subset X^{n+1}$ be the graph of

\[ f \times f^2 \times \cdots \times f^n : X \to X^n \]

and let $\text{Vol}_\omega(\Gamma_f(n))$ be the volume of $\Gamma_f(n)$ defined with respect to a Kähler form $\omega$ on $X$. We then define

\[ \text{Plov}(f) := \text{Plov}(X, f) := \limsup_{n \to \infty} \frac{\log \text{Vol}_\omega(\Gamma_f(n))}{\log n} \in [0, \infty]. \]

This invariant of $f$ is independent of the choice of $\omega$ (Lemma 2.1).

We will study upper bounds and lower bounds of $\text{Plov}(f)$ in terms of $d = \text{dim} X$ and $k(f)$ introduced in §1.1. Using dynamical filtrations, we obtain the following estimates.

**Theorem 1.2.** Let $X$ be a compact Kähler manifold of dimension $d$ and let $f \in \text{Aut}(X)$ be a zero entropy automorphism.

- (0) (Corollary 2.17) $\text{Plov}(f) = d$ if and only if $k(f) = 0$.
- (1) (Proposition 4.1 and Theorem 5.1) Suppose that $k(f) > 0$. Then we have
  \[ d + 2k(f) - 2 \leq \text{Plov}(f) \leq k(f)(d - 1) + d. \]
- (2) (Theorem 4.2) Suppose that $k(f) > 0$ and $d \geq 3$. Then we have
  \[ \text{Plov}(f) \leq k(f)(d - 1) + d - 2. \]
By Theorem 1.1, we have

\[ k(f) \in \{ 0, 2, \ldots, 2d-2 \}. \]

Also \( \text{Plov}(f) \) has the same parity as \( \dim X \); see Corollary 2.18. These together with Theorem 1.2 immediately determine \( \text{Plov}(f) \) when \( d = 2, 3 \).

**Corollary 1.3.**

1. If \( d = 2 \), then

\[
\text{Plov}(f) = \begin{cases} 
2 & \text{if } k(f) = 0 \\
4 & \text{if } k(f) = 2.
\end{cases}
\]

2. If \( d = 3 \), then

\[
\text{Plov}(f) = \begin{cases} 
3 & \text{if } k(f) = 0 \\
5 & \text{if } k(f) = 2 \\
9 & \text{if } k(f) = 4.
\end{cases}
\]

Together with Theorem 1.1, Theorem 1.2 implies that \( \text{Plov}(f) \leq 2d^2 - 3d \) whenever \( d \geq 3 \). When \( d \geq 4 \), we will further improve this upper bound to

\[
(1.4) \quad \text{Plov}(f) \leq 2d^2 - 3d - 2;
\]

see Proposition 4.4.

**1.3. A conjectural upper bound.**

When \( X \) is a complex torus, we determine \( \text{Plov}(f) \) in terms of the pullback \( f^* : H^{1,0}(X) \otimes \).

**Theorem 1.4.** Let \( X \) be a complex torus of dimension \( d \) and \( f \in \text{Aut}(X) \) an automorphism of zero entropy. Assume that the Jordan canonical form of \( f^* : H^{1,0}(X) \otimes \) consists of Jordan blocks of sizes \( k_1, \ldots, k_p \), counted with multiplicities. Then

\[
\text{Plov}(f) = \sum_{i=1}^{p} k_i^2.
\]

In particular, we have \( \text{Plov}(f) \leq d^2 \), and this upper bound is optimal among complex tori.

Theorem 1.4 also shows that the quadratic order of the upper bounds with respect to \( d \) in (1.4) is optimal. We will also compute \( \text{Plov}(X, f) \) for other examples including threefolds; see §7. As we fail to construct examples of \( f : X \otimes \) such that \( d^2 < \text{Plov}(f) < \infty \) where \( d = \dim X \), presumably the upper bound in (1.4) when \( d \geq 4 \) is still not optimal. Taking Corollary 1.3 and Theorem 1.4 into account, it seems reasonable to ask the following questions.

**Question 1.5.** Let \( X \) be a compact Kähler manifold of dimension \( d \geq 1 \). Let \( f \in \text{Aut}(X) \) be a zero entropy automorphism.

1. Is \( \text{Plov}(f) \leq d^2 \)?
2. More precisely, are possible values of \( \text{Plov}(X, f) \) always realizable by \( d \)-dimensional tori?
Question 1.5 (1) is the analogous question of [3, Question 4.1] asked for polynomial entropy by Cantat–Paris-Romaskevich. By [3, (2.7)], a positive answer to Question 1.5 (1) also answers [3, Question 4.1] in the affirmative.

The following partial answer to Question 1.5 is a direct consequence of the above theorems.

**Corollary 1.6.** (see §5) Let $X$ be a compact Kähler manifold of dimension $d \geq 3$ and let $f \in \text{Aut}(X)$ be a zero entropy automorphism.

1. If $k(f) \leq d$, then $\text{Plov}(f) \leq d^2 - 2$. In particular, $\text{Plov}(f) \leq d^2 - 2$, whenever $\kappa(X) \geq d/2$.

2. Question 1.5 has positive answers when $\dim X \leq 3$.

We also give a positive answer to Question 1.5 (1) in dimension 4.

**Proposition 1.7.** (see Proposition 4.4) Let $X$ be a compact Kähler manifold of dimension 4 and let $f \in \text{Aut}(X)$ be a zero entropy automorphism. Then we have

$$\text{Plov}(f) \leq 16.$$ 

Question 1.5 is open for $d \geq 5$.

1.4. Gelfand–Kirillov dimension.

Finally, we notice that when $X$ is projective, the polynomial log-volume growth $\text{Plov}(f)$ actually coincides with some known invariant of $f$ studied in noncommutative algebra. The following identification is implicit in the seminal work of Keeler [17].

**Theorem 1.8.** Let $X$ be a complex smooth projective variety and let $f \in \text{Aut}(X)$ be a zero entropy automorphism. Then

$$\text{GKdim } B(X, f, L) - 1 = \text{Plov}(X, f).$$

Here, GKdim $B(X, f, L)$ is the Gelfand–Kirillov dimension (or GK-dimension for short) of the twisted homogeneous coordinate ring $B(X, f, L)$ associated to $f : X \looparrowright$ and any ample line bundle $L$.

We refer to Section 8 for the definition and basic properties of GKdim $B(X, f, L)$, as well as the proof of Theorem 1.8.

In this regard, two of our results are not new for projective varieties. The first one is the upper bound in Theorem 1.2 (1), as the estimate

$$\text{GKdim } B(X, f, L) - 1 \leq k(f)(d - 1) + d$$

has already been proven in [17, Lemma 6.13]. The second one is Corollary 1.3 (1), due to Artin–Van den Bergh [1, Theorem 1.7]. Our approach based on dynamical filtrations is however completely different, and extends both results in a non-trivial way.

Thanks to Theorem 1.8, the main results we prove for Plov($f$) also translate to new results about the GK-dimension of $B(X, f, L)$; see Corollary 8.6 for some instances. So far, the GK-dimension has been studied mostly by specialists in noncommutative algebras. We hope that the dynamical properties of $(X, f)$ might provide a better understanding of the algebraic structure of $B(X, f, L)$, and vice versa.
1.5. Organization of the paper and a few remarks to the readers.

We start with §2, proving basic properties of polynomial log-volume growth (see e.g. Proposition 2.5). In §3 we recall the construction of quasi-nef sequences and dynamical filtrations, together with their fundamental properties. We also prove several auxiliary results related to them, which will be useful in the study of upper and lower bounds of Plov(f). §3 also contains a proof of the statement in Theorem 1.1 involving the Kodaira dimension. §4 and §5 are devoted to upper and lower bounds of Plov(f) respectively, all together implying Theorem 1.2 and Corollary 1.6. §6 and §7, we study explicit examples, which contain complete descriptions of Plov(f) for tori (Theorem 1.4) and threefolds (Proposition 7.2) respectively. §8 starts with a brief review of twisted homogeneous coordinate rings and their GK-dimensions. We recall some fundamental results proven in [17, 1] (Theorem 8.2) and derive Theorem 1.8 as a direct consequence. We finish §8 by Corollary 8.6, translating results from Plov(f) to GK-dimensions.

Notations and conventions.

All manifolds are assumed to be connected. Let $X$ be a compact Kähler manifold. Write $H^{i,i}(X,\mathbb{R}) := H^{i,i}(X) \cap H^{2i}(X,\mathbb{R})$.

For every $\alpha \in H^{i,i}(X,\mathbb{R})$, if $\alpha \cdot H_{1,1}(X,\mathbb{R})^{d-i} = 0$ (where $d = \dim X$), we write $\alpha \equiv 0$.

We follow [6] for the basic terminology, like positive classes and cones.

2. POLYNOMIAL LOG-VOLUME GROWTH

2.1. Definition and basic properties.

Let $X$ be a compact Kähler manifold of dimension $d \geq 1$ and let $f \in \text{Aut}(X)$. Let $\omega \in H^{1,1}(X,\mathbb{R})$ be a Kähler class. For every integer $n \geq 1$, the volume of the graph $\Gamma_f(n) \subset X^{n+1}$ of $f \times f^2 \times \cdots \times f^n : X \to X^n$

with respect to any Kähler metric in the class $\omega$ is equal to

$$\text{Vol}_\omega(\Gamma_f(n)) = \int_{\Gamma_f(n)} \frac{1}{d!} \left( \sum_{i=1}^{n+1} \text{pr}_i^*\omega \right)^d = \frac{1}{d!} \Delta_n(f,\omega)^d,$$

where $\text{pr}_i : X^{n+1} \to X$ is the projection to the $i$th factor and

$$\Delta_n(f,\omega) := \sum_{i=0}^{n} (f^i)^*\omega \in H^{1,1}(X,\mathbb{R}).$$

Note that the class $\Delta_n(f,\omega)$ and the invariant $\text{Vol}_\omega(\Gamma_f(n))$ are defined more generally for any $\omega \in H^{1,1}(X,\mathbb{R})$.

Lemma 2.1. For every nef $\omega \in \text{Nef}(X) \subset H^{1,1}(X,\mathbb{R})$, define

$$\text{Plov}(f,\omega) := \limsup_{n \to \infty} \frac{\log \text{Vol}_\omega(\Gamma_f(n))}{\log n} = \limsup_{n \to \infty} \frac{\log \Delta_n(f,\omega)^d}{\log n} \in [0, \infty].$$

Then Plov(f,\omega) is a constant in $\omega$ whenever $\omega$ is nef and big.
Lemma 2.1 justifies the well-definedness of the polynomial logarithmic volume growth of $f$ in the introduction, which is defined to be

$$Plov(f) := Plov(X, f) := Plov(f, \omega),$$

where $\omega$ is any nef and big class.

To prove Lemma 2.1, we need the following easy but useful result.

**Lemma 2.2.** Let $X$ be a compact Kähler manifold of dimension $d$, and let

$L_1, \ldots, L_d, M_1, \ldots, M_d$

be nef classes in $H^{1,1}(X, \mathbb{R})$ such that $M_i \geq L_i$, i.e., $M_i - L_i$ is pseudo-effective. Then

$$(M_1 \cdots M_d) \geq (L_1 \cdots L_d).$$

In particular, $(M_i^d) \geq (L_i^d)$.

**Proof.** Inductively, we have

$$(M_1 \cdots M_d) \geq (L_1 \cdot M_2 \cdots M_d) \geq \cdots \geq (L_1 \cdots L_j \cdot M_{j+1} \cdots M_d) \geq \cdots \geq (L_1 \cdots L_d),$$

which proves Lemma 2.2. $\square$

**Proof of Lemma 2.1.** Let $\omega$ and $\omega'$ be two nef and big classes. Then there exists some $\varepsilon > 0$ such that $\omega - \varepsilon \omega'$ is pseudo-effective. Accordingly, $\Delta_n(f, \omega) - \varepsilon \Delta_n(f, \omega')$ is pseudo-effective, so $\Delta_n(f, \omega)^d \geq \varepsilon^d \Delta_n(f, \omega')^d$ by Lemma 2.2 and therefore $Plov(f, \omega) \geq Plov(f, \omega')$. By symmetry, we have $Plov(f, \omega) = Plov(f, \omega')$. $\square$

The following is an immediate consequence of Lemma 2.1.

**Corollary 2.3.** Let $X$ and $Y$ be compact Kähler manifolds with automorphisms $f \in \text{Aut}(X)$ and $g \in \text{Aut}(Y)$. Suppose that there exists a $\mathbb{C}$-linear isomorphism

$$\phi : H^\bullet(X, \mathbb{C}) \sim \rightarrow H^\bullet(Y, \mathbb{C})$$

of the cohomology rings such that the following conditions are satisfied:

(i) $\phi \circ f^* = g^* \circ \phi$;

(ii) There exists a Kähler class $\omega \in H^{1,1}(X)$ on $X$ such that $\phi(\omega)$ is Kähler on $Y$.

Then

$$Plov(f) = Plov(g).$$

The similar statement holds if $\phi$ is replaced by a $\mathbb{C}$-linear isomorphism of the subalgebras

$$\psi : \bigoplus_i H^{i,i}(X) \sim \rightarrow \bigoplus_i H^{i,i}(X').$$

The same argument as in the proof of Lemma 2.2 proves the following.

**Lemma 2.4.** For every nef class $\alpha \in H^{1,1}(X, \mathbb{R})$, we have

$$Plov(f, \alpha) \leq Plov(f).$$

**Proof.** Take a Kähler class $\omega$ such that $\omega \geq \alpha$. By Lemma 2.2, we have $\Delta_n(f, \omega)^d \geq \Delta_n(f, \alpha)^d$ for every integer $n \geq 0$. Hence $Plov(f, \alpha) \leq Plov(f)$. $\square$

Now we prove some basic dynamical properties of $Plov(X, f)$ summarised in the following, which will be frequently used in this paper.
Proposition 2.5. Let \( f : X \to \) be an automorphism of a compact Kähler manifold.

1. (Independence of the metric; Lemma 2.3) The invariant \( \text{Plov}(f) \) is independent of \( \omega \in H^{1,1}(X, \mathbb{R}) \) whenever \( \omega \) is nef and big.

2. (Finiteness and integrality; Lemmas 2.8 and 2.16) We have \( \text{Plov}(f) < \infty \) if and only if \( f^* : H^{1,1}(X) \to \) is quasi-unipotent. In this case, \( \text{Plov}(f) \) is an integer.

3. (Finite index; Lemma 2.6) We have \( \text{Plov}(f) = \text{Plov}(f^N) \) for any integer \( N \neq 0 \).

4. (Product; Lemma 2.7) Let \( X_i \) \((i = 1, 2)\) be compact Kähler manifolds and let \( f_i \in \text{Aut}(X_i) \). Then
   \[
   \text{Plov}(f_1 \times f_2) = \text{Plov}(f_1) + \text{Plov}(f_2)
   \]
   for the product automorphism \( f_1 \times f_2 \in \text{Aut}(X_1 \times X_2) \).

5. (Invariance under generically finite maps; Lemma 2.10) Let \( X \) and \( Y \) be compact Kähler manifolds and let \( f_X \in \text{Aut}(X) \) and \( f_Y \in \text{Aut}(Y) \). Let \( \phi : X \to Y \) be a generically finite dominant meromorphic map such that \( f_Y \circ \phi = \phi \circ f_X \). Then
   \[
   \text{Plov}(f_X) = \text{Plov}(f_Y).
   \]

6. (Restriction; see Lemma 2.11 also for the precise definition of \( \text{Plov}(f|_W) \) when \( W \) is not smooth) Let \( W \subset X \) be a closed subvariety such that \( f(W) = W \). Then \( \text{Plov}(f|_W) \leq \text{Plov}(f) \) for the automorphism \( f|_W \in \text{Aut}(W) \) induced from \( f \) by restriction.

First, we prove that \( \text{Plov}(X, f) \) is invariant under taking finite iterations.

Lemma 2.6. For every integer \( N \neq 0 \), we have \( \text{Plov}(f^N) = \text{Plov}(f) \).

Proof. Since
   \[
   \left( \sum_{i=0}^{N} (f^{-i})^* \omega \right)^d = \left( (f^{-n})^* \sum_{i=0}^{n} (f^i)^* \omega \right)^d = \left( \sum_{i=0}^{n} (f^i)^* \omega \right)^d,
   \]
we have \( \text{Plov}(f^{-1}) = \text{Plov}(f) \). So it suffices to prove Lemma 2.6 for \( N > 0 \).

For every integers \( r \) and \( j > 0 \) such that \( 0 \leq r < N \), consider the Kähler form \( \omega_{r,j} := \sum_{i=r}^{r+j-1} (f^i)^* \omega \). Then
   \[
   \omega_{r,mN} \leq \omega_{0,r} + \omega_{r,mN} = \omega_{0,r+mN} \leq \omega_{r-N(m+1)N}.
   \]
So
   \[
   \text{Vol}_{\omega_{r,mN}}(X) \leq \text{Vol}_{\omega_{0,r+mN}}(X) \leq \text{Vol}_{\omega_{r-N(N+1)N}}(X)
   \]
by Lemma 2.2, and thus
   \[
   \text{Vol}_{\sum_{j=0}^{N-1} (f^{r+j})^* \omega}(\Gamma f^N(m)) \leq \text{Vol}_{\omega}(\Gamma f(r+mN)) \leq \text{Vol}_{\sum_{j=1}^{N} (f^{r-j})^* \omega}(\Gamma f^N(m + 1)).
   \]
By Lemma 2.1, we have
   \[
   \lim_{m \to \infty} \frac{\log \text{Vol}_{\sum_{j=0}^{N-1} (f^{r+j})^* \omega}(\Gamma f^N(m))}{\log m} = \text{Plov}(f^N) = \lim_{m \to \infty} \frac{\log \text{Vol}_{\sum_{j=1}^{N} (f^{r-j})^* \omega}(\Gamma f^N(m + 1))}{\log(m + 1)},
   \]
so for every integer \( r \) such that \( 0 \leq r < N \), we have
   \[
   \lim_{m \to \infty} \frac{\log \text{Vol}_{\omega}(\Gamma f(r+mN))}{\log(r+mN)} = \lim_{m \to \infty} \frac{\log \text{Vol}_{\omega}(\Gamma f(r+mN))}{\log(m)} = \text{Plov}(f^N).
   \]
Hence \( \text{Plov}(f) = \text{Plov}(f^N) \). \( \square \)
The polynomial logarithmic volume growth is also compatible with product.

**Lemma 2.7.** Let $X_i (i = 1, 2)$ be compact Kähler manifolds and let $f_i \in \text{Aut}(X_i)$. Then

$$\text{Plov}(f_1 \times f_2) = \text{Plov}(f_1) + \text{Plov}(f_2).$$

**Proof.** Let $\omega_i$ be a Kähler metric on $X_i$ and let $pr_i : X_1 \times X_2 \to X_i$ be the projection to the $i$-th factor. Then

$$\text{Vol}_{pr_1^*\omega_1 + pr_2^*\omega_2}(\Gamma_{f_1 \times f_2}(n)) = \text{Vol}_{\omega_1}(\Gamma_{f_1}(n))\text{Vol}_{\omega_2}(\Gamma_{f_2}(n)),$$

which proves Lemma 2.7. \qed

We can characterize whether a holomorphic automorphism $f \in \text{Aut}(X)$ has zero entropy based on the finiteness of $\text{Plov}(f)$.

**Lemma 2.8.** Let $X$ be a compact Kähler manifold of dimension $d \geq 1$ and let $f \in \text{Aut}(X)$. Then the following conditions are equivalent.

1. $f^* : H^1(X, \mathbb{C}) \circlearrowleft$ is quasi-unipotent, i.e., a positive power of it is unipotent.
2. $f^* : H^{1,1}(X) \circlearrowleft$ is quasi-unipotent.
3. the first dynamical degree $d_1(f) = 1$.
4. the topological entropy $h_{\text{top}}(f) = 0$.
5. $\text{Plov}(f) < \infty$.

Here we recall that for $1 \leq i \leq d$, the $i$-th dynamical degree of $f$ is defined as

$$d_i(f) := \lim_{n \to \infty} \frac{(\omega^{d-i} \cdot (f^n)^*\omega^i)^{1/n}},$$

where $\omega \in H^{1,1}(X)$ is a Kähler class \[10\]; these $d_i(f)$ are independent of $\omega$.

**Proof.** The equivalence of the first four conditions is well-known and is obtained as follows. By Gromov–Yomdin’s theorem (cf. \[12\], \[26\]; see also \[23\] Theorem 3.6), we have

$$h_{\text{top}}(f) = \text{lov}(f) = \log r(f^*) = \log(\max_{1 \leq i \leq d} \{d_i(f)\}),$$

where $r(f^*)$ is the spectral radius of $f^* : H^*(X, \mathbb{C}) \circlearrowleft$, and

$$\text{lov}(f) := \limsup_{n \to \infty} \frac{\log \text{Vol}_\omega(\Gamma_{f^n}(n))}{n}.$$ Together with the log concavity of dynamical degrees $d_i(f)$ (which follows from Khovanskii–Teissier’s inequality), this implies that $h_{\text{top}}(f) > 0$ if and only if $d_i(f) > 1$ for some (and hence all) $i \in \{1, \ldots, d - 1\}$. Thus the equivalence of the first four assertions follows from Kronecker’s theorem. Also, since $\text{Plov}(f) \geq \text{lov}(f) \geq \log(d_1(f))$, (5) implies these assertions.

To see that (2) implies (5), recall that in order to compute $\text{Plov}(f)$, by Lemma 2.6 we can replace $f$ by some iteration of it, so that $f^* : H^{1,1}(X) \circlearrowleft$ is unipotent. Hence $\text{Plov}(f) < \infty$ is a consequence of Lemma 2.16 below. \qed

Next, we prove the invariance of $\text{Plov}(f)$ under generically finite meromorphic maps.
Lemma 2.9. Let $X$ and $Y$ be compact Kähler manifolds of dimension $d \geq 1$ and $f_X \in \text{Aut}(X)$ and $f_Y \in \text{Aut}(Y)$. Let $\phi : X \rightarrow Y$ be a generically finite dominant meromorphic map such that $f_Y \circ \phi = \phi \circ f_X$. Then

$$\text{Plov}(f_X) = \text{Plov}(f_Y).$$

Proof. First we reduce to the case where $\phi$ is holomorphic. Let $\Gamma$ be the Zariski closure of the graph of $\phi$ in $X \times Y$. Let $p_X : \Gamma \rightarrow X$ and $p_Y : \Gamma \rightarrow Y$ be the projection. Since $f_X \in \text{Aut}(X)$ and $f_Y \in \text{Aut}(Y)$, it follows that

$$f_\Gamma := (f_X \times f_Y)|_\Gamma \in \text{Aut}(\Gamma)$$

and $f_\Gamma$ and $f_X$ (resp. $f_\Gamma$ and $f_Y$) are equivariant with respect to a generically finite surjective morphism $p_X$ (resp. $p_Y$). By the existence of functorial resolution of singularities [2] (see also [19, Theorem 3.45]), there exists a Kähler desingularization $\nu : \widetilde{\Gamma} \rightarrow \Gamma$ such that $f_\Gamma \circ \nu = \nu \circ f_{\widetilde{\Gamma}}$ for some $f_{\widetilde{\Gamma}} \in \text{Aut}(\widetilde{\Gamma})$. If Lemma 2.9 holds whenever $\phi$ is holomorphic, then $\text{Plov}(f_X) = \text{Plov}(f_{\widetilde{\Gamma}}) = \text{Plov}(f_Y)$.

For every Kähler class $\omega$ on $Y$, since

$$\left(\sum_{i=0}^n (f_X^i)^* (\phi^* \omega)\right)^d = \phi^* \left(\sum_{i=0}^n (f_Y^i)^* \omega\right)^d = \deg(\phi) \cdot \left(\sum_{i=0}^n (f_Y^i)^* \omega\right)^d,$$

we have $\text{Plov}(f_X, \phi^* \omega) = \text{Plov}(f_Y, \omega) = \text{Plov}(f_Y)$. As $\phi^* \omega$ is nef and big, it follows from Lemma 2.1 that $\text{Plov}(f_X) = \text{Plov}(f_Y)$. \qed

Lemma 2.10. Let $W \subset X$ be a closed subvariety such that $f(W) = W$. Then $\text{Plov}(f|_W) \leq \text{Plov}(f)$ for the automorphism $f|_W \in \text{Aut}(W)$ induced from $f$ by restriction. Here we define

$$\text{Plov}(f|_W) := \text{Plov}(\overline{W}, \tilde{f}),$$

where $\tau : \overline{W} \rightarrow W$ is any Kähler desingularization of $W$ such that $f|_W \circ \tau = \tau \circ \tilde{f}$ for some $\tilde{f} \in \text{Aut}(\overline{W})$, which does not depend on the choice of $\overline{W}$ by Lemma 2.9.

Proof. Let $\nu : \overline{W} \rightarrow X$ be the composition of $\tau$ with the inclusion $W \hookrightarrow X$. Let $d := \dim X$ and $e := \dim W$. Let $\omega$ be a Kähler class of $X$. Up to replacing $\omega$ by some positive multiple of it, we can assume that

$$(\omega^{d-e} - [W]) \cdot \beta \geq 0$$

for every $\beta$ in the closed convex cone generated by products of $e$ Kähler classes.

Since $\sum_{i=0}^n (f^i)^* (\omega^e) - \omega$ is Kähler when $n \geq 1$, by Lemma 2.2 we have,

$$\left(\sum_{i=0}^n (\tilde{f}^i)^* (\nu^* \omega)\right)^e = \left(\left(\sum_{i=0}^n (f^i)^* (\omega^e)\right)^e \cdot [W]\right)^e$$

$$\leq \left(\left(\sum_{i=0}^n (f^i)^* (\omega^e)\right)^e \cdot \omega^{d-e}\right) \leq \left(\sum_{i=0}^n (f^i)^* (\omega^e)\right)^d.$$

So $\text{Plov}(\tilde{f}, \nu^* \omega) \leq \text{Plov}(f, \omega) = \text{Plov}(f)$. As $\nu^* \omega$ is nef and big, we have $\text{Plov}(\tilde{f}) = \text{Plov}(\tilde{f}, \nu^* \omega)$ by Lemma 2.1. Hence $\text{Plov}(f|_W) \leq \text{Plov}(f)$. \qed
2.2. Cohomological polynomial growth $k(f)$.

From now on until the end of §2 we assume that

$$f^* : H^{1,1}(X) \circlearrowleft$$

is unipotent, unless otherwise specified. The operator

$$N := f^* - \text{Id} : H^{1,1}(X) \to H^{1,1}(X)$$

is thus nilpotent; let $k(f)$ be the largest integer such that $(f^* - \text{Id})^k \neq 0$. Equivalently, $k(f) + 1$ is the maximal size of the Jordan blocks of the Jordan canonical form of $f^* : H^{1,1}(X) \circlearrowleft$.

The following result implies in particular that $k(f)$ is invariant under bimeromorphic modifications.

**Proposition 2.11.** Let $\pi : X \dasharrow Y$ be a dominant, generically finite meromorphic map between compact Kähler manifolds. Let $f_X \in \text{Aut}(X)$ and $f_Y \in \text{Aut}(Y)$ be automorphisms such that $\pi \circ f_X = f_Y \circ \pi$. Assume that $f_X^* : H^{1,1}(X) \circlearrowleft$, or equivalently $f_Y^* : H^{1,1}(Y) \circlearrowleft$ is quasi-unipotent (cf. [9, Theorem 1.1] and Lemma 2.8). Then

$$k(f_X) = k(f_Y).$$

We shall also prove the following.

**Lemma 2.12.** Let $X$ and $Y$ be compact Kähler manifolds. Let $f_X \in \text{Aut}(X)$ and $f_Y \in \text{Aut}(Y)$. Then we have

$$k(f_X \times f_Y) = \max \{ k(f_X), k(f_Y) \}.$$

To prove both Proposition 2.11 and Lemma 2.12 we need the following result in linear algebra.

**Lemma 2.13.** Let $V$ be a finite dimensional vector space over $\mathbb{R}$ and let $\phi \in \text{GL}(V)$ be a unipotent operator. Let $N := \phi - \text{Id}_V$ and let $k$ denote the largest integer such that $N^k \neq 0$. Assume that $\phi$ preserves a closed salient convex cone $C \subset V$ with nonempty interior. Then for every $v \in \text{Int}(C)$, the following assertions hold.

1. We have $N^k(v) \in C \setminus \{0\}$ and

$$\phi^n(v) \sim_{n \to \infty} C_v n^k \cdot N^k(v).$$

for some $C_v > 0$.

2. For every linear form $\chi : V \to \mathbb{R}$ such that $\chi(C \setminus \{0\}) > 0$, we have

$$\chi(\phi^n(v)) \sim_{n \to \infty} C'_v n^k$$

for some $C'_v > 0$.

**Proof.** We can assume $\phi \neq \text{Id}$. Then $\ker N^k \neq V$, and for every $w \in V \setminus \ker N^k$, developing $\phi^n(w) = (\text{Id}_V + N)^n(w)$ shows that

$$\phi^n(w)/n^k \sim_{n \to \infty} C_w N^k(w).$$
for some \( C_w > 0 \). If moreover \( w \in \mathcal{C} \), then \( \phi(\mathcal{C}) \subset \mathcal{C} \) and \( \mathcal{C} \) being closed, imply \( N^k(w) \in \mathcal{C} \). Assume the contrary that there exists some \( x \in \text{Int}(\mathcal{C}) \) such that \( N^k(x) = 0 \). Then there exists some \( \varepsilon \in \mathcal{V} \) such that

\[
x \pm \varepsilon \in \mathcal{C} \quad \text{and} \quad N^k(x \pm \varepsilon) \neq 0.
\]

As \( x \pm \varepsilon \in \mathcal{C} \) and \( \phi(\mathcal{C}) \subset \mathcal{C} \), both \( N^k(x \pm \varepsilon) = \pm N^k(\varepsilon) \) are limits of elements in \( \mathcal{C} \), which contradicts the assumptions that \( \mathcal{C} \) is closed and salient. This proves (1).

Since \( N^k(v) \in \mathcal{C} \setminus \{0\} \), we have \( \chi(N^k(v)) > 0 \). Thus (2) follows from (1).

\[\tag{1}\]

Proof of Lemma 2.12. Assume that \( k(f_X) = \infty \) or \( k(f_Y) = \infty \). Then Lemma 2.12 follows from the product formula of the first dynamical degree ([9, Theorem 1.1]) together with Lemma 2.8.

Assume that both \( k(f_X) \) and \( k(f_Y) \) are finite. By Lemma 2.8 up to replacing \( f_X \) and \( f_Y \) by a common positive power, we can assume

\[
f_X : H^{1,1}(X, \mathbb{R}) \circlearrowleft, \quad f_Y : H^{1,1}(Y, \mathbb{R}) \circlearrowleft, \quad \text{and} \quad (f_X \times f_Y) : H^{1,1}(X \times Y, \mathbb{R}) \circlearrowleft
\]

are unipotent. Fix Kähler classes \( \omega_X \in H^{1,1}(X, \mathbb{R}) \) and \( \omega_Y \in H^{1,1}(Y, \mathbb{R}) \). Let \( p_X : X \times Y \to X \) and \( p_Y : X \times Y \to Y \) be the projections. Applying Lemma 2.13 to \( H^{1,1}(\bullet, \mathbb{R}) \) and the nef cone therein shows that \( k(f_X \times f_Y) \) (resp. \( k(f_X) \) and \( k(f_Y) \)) is the polynomial growth rate of

\[
((f_X \times f_Y)^*)(p_X^*\omega_X + p_Y^*\omega_Y) = p_X^*(f_X^*)^n(\omega_X) + p_Y^*(f_Y^*)^n(\omega_Y)
\]

(resp. \( (f_X)^n(\omega_X) \) and \( (f_Y)^n(\omega_Y) \)).

Hence \( k(f_X \times f_Y) = \max \{ k(f_X), k(f_Y) \} \).

\[\tag{2.4}\]

Proof of Proposition 2.11. As in Lemma 2.9 up to replacing \( X \) by an equivariant desingularization of the graph of \( \pi \), we can assume that \( \pi \) is holomorphic. By assumption together with Lemmas 2.3 and 2.8 \( f_X^* : H^{1,1}(X) \circlearrowleft \) and \( f_Y^* : H^{1,1}(Y) \circlearrowleft \) are both quasi-unipotent, so up to replacing \( f_X \) and \( f_Y \) by some positive iterations, we can assume that they are both unipotent.

Applying Lemma 2.13 to the nef cone in \( H^{1,1}(X, \mathbb{R}) \), we see that for every pair of Kähler classes \( \omega, \eta \) on \( X \), we have

\[
(f_X^*)^n(\omega) \cdot \eta^{\dim X - 1} \sim_{n \to \infty} Cn^{k(f_X)}\tag{2.3}
\]

for some \( C > 0 \). Similarly, for every pair of Kähler classes \( \omega', \eta' \) on \( Y \), we have

\[
(f_Y^*)^n(\pi^*\omega') \cdot \pi'\eta'^{\dim Y - 1} = \deg(\pi) \cdot (f_Y^*)^n(\omega') \cdot \eta'^{\dim Y - 1} \sim_{n \to \infty} C'n^{k(f_Y)}\tag{2.4}
\]

for some \( C' > 0 \). Since the classes \( \omega, \eta, \pi^*\omega', \pi^*\eta' \) are all nef and big, with the notation of Lemma 2.2 we have

\[
c_1\pi^*\omega' \leq \omega \leq c_2\pi^*\omega' \quad \text{and} \quad c_3\pi^*\eta' \leq \eta \leq c_4\pi^*\eta'
\]

for some positive real numbers \( c_i \). It follows from Lemma 2.2 that the growth rates of (2.3) and (2.4) have the same polynomial order. Hence \( k(f_X) = k(f_Y) \). \[\square\]
2.3. Bounding the polynomial log-volume growth.

We continue to assume that $f^* : H^{1,1}(X) \supset \rightarrow g_1$ is unipotent, unless otherwise specified. For every $\alpha \in H^{1,1}(X, \mathbb{R})$, recall that

$$\Delta_n(f, \alpha) := \sum_{i=0}^{n} (f^i)^* \omega \in H^{1,1}(X, \mathbb{R}).$$

The following lemma shows that $\Delta_n(f, \alpha)$ has polynomial expressions in $n$ for both ranges $n \in \mathbb{Z}_{\geq 0}$ and $n \in \mathbb{Z}_{\leq 0}$ (but these two polynomials are usually different).

Lemma 2.14. We have

$$\Delta_n(f, \alpha) = \begin{cases} \Delta_n^+(f, \alpha) & \text{if } n \geq 0 \\ \Delta_{n-1}^+(f, \alpha) & \text{if } n \leq 0, \end{cases}$$

where

$$\Delta_n^+(f, \alpha) := \sum_{j=0}^{k(f)} \binom{n+1}{j+1} N^j \omega.$$ 

Proof. By definition of $k := k(f)$, for every $\alpha \in H^{1,1}(X, \mathbb{R})$ and every $n \in \mathbb{Z}$:

$$\Delta_n(f, \alpha) = \sum_{i=0}^{n} (f^i)^* (\alpha) = \sum_{i=0}^{n} \sum_{j=0}^{k} \binom{i}{j} N^j \omega = \sum_{j=0}^{k} \sum_{i=0}^{n} \binom{i}{j} N^j \omega.$$ 

If $n \geq 0$, then

$$\sum_{i=0}^{n} \binom{i}{j} = \binom{n+1}{j+1}$$

by the hockey-stick identity. Similarly, if $n \leq 0$, then

$$\sum_{i=0}^{n} \binom{i}{j} = (-1)^j \sum_{i=0}^{n} \binom{j-i-1}{j} = (-1)^j \binom{j-n}{j+1} = -\binom{n}{j+1}.$$ 

Hence Lemma 2.14 follows. \qed

The following lemma will be useful to prove results on lower bounds of Plov$(f)$ in this paper. In the projective setting, this lemma was due to Keeler [17, Lemma 6.5(4)] and was applied in his work to prove his lower bound.

Lemma 2.15. For every integer $i \in [1, d]$, let

$$P_{f, \omega, i}(n) := \Delta_n^+(f, \omega)^i \omega^{d-i} = \left( \sum_{j=0}^{k} \binom{n+1}{j+1} N^j \omega \right)^i \omega^{d-i},$$ 

which is a polynomial in $n$ of degree $\deg_n P_{f, \omega, i}(n)$. Then we have

$$\deg_n P_{f, \omega, i}(n) > \deg_n P_{f, \omega, i-1}(n).$$

Proof. For every non-negative integer $m$, define

$$P_{f, \omega, i, m}(n) := \Delta_n^+(f, \omega)^{i-1} \cdot (f^m)^* \omega \cdot \omega^{d-i} = \left( \sum_{j=0}^{k} \binom{n+1}{j+1} N^j \omega \right)^{i-1} \cdot (\text{Id} + N)^m(\omega) \cdot \omega^{d-i},$$

which is a polynomial in $n$ of degree $\deg_n P_{f, \omega, i, m}(n)$. Note that since both $\omega$ and $(f^m)^* \omega$ are Kähler, we have

$$C_1 \omega \leq (f^m)^* \omega \leq C_2 \omega.$$
for some \(C_1, C_2 > 0\), so
\[
C_1 P_{f, \omega, i, 0}(n) \leq P_{f, \omega, i, m}(n) \leq C_2 P_{f, \omega, i, 0}(n)
\]
by Lemma 2.2 and therefore
\[
\deg_n P_{f, \omega, i, m}(n) = \deg_n P_{f, \omega, i, 0}(n) = \deg_n P_{f, \omega, i-1}(n).
\]
In particular, \(\deg_n P_{f, \omega, i, m}(n)\) is independent of \(m\).

For every \(m\), since \(P_{f, \omega, i, m}(n) > 0\), the leading coefficient \(C_{f, \omega, i}(m)\) of the polynomial \(P_{f, \omega, i, m}\) satisfies \(C_{f, \omega, i}(m) > 0\). As \(C_{f, \omega, i}(m)\) is a polynomial in \(m\) (because \(N\) is nilpotent), the minimum of
\[
\{C_{f, \omega, i}(m) \mid m \in \mathbb{Z}_{\geq 0}\}
\]
exists; let \(\ell \in \mathbb{Z}_{\geq 0}\) such that \(C_{f, \omega, i}(\ell)\) is the minimum.

By construction, we have
\[
\lim_{n \to \infty} \frac{P_{f, \omega, i, m}(n)}{P_{f, \omega, i, \ell}(n)} = \frac{C_{f, \omega, i}(m)}{C_{f, \omega, i}(\ell)} \geq 1
\]
for every \(m \in \mathbb{Z}\). So
\[
\frac{P_{f, \omega, i}(n)}{P_{f, \omega, i, \ell}(n)} = \sum_{m=0}^{n} \frac{P_{f, \omega, i, m}(n)}{P_{f, \omega, i, \ell}(n)} \geq n \to \infty \gamma,
\]
for any \(\gamma > 0\), which shows that \(\deg_n P_{f, \omega, i, \ell}(n) < \deg_n P_{f, \omega, i}(n)\). Hence
\[
\deg_n P_{f, \omega, i-1}(n) = \deg_n P_{f, \omega, i, \ell}(n) < \deg_n P_{f, \omega, i}(n).
\]

\[\square\]

Lemma 2.16. Plov\((f)\) is equal to the degree of the polynomial
\[
n \mapsto P_{f, \omega}(n) := P_{f, \omega, d}(n) = \Delta^+_n(f, \omega)^d = \left(\sum_{j=0}^{k(f)} \binom{n+1}{j} N^j \omega \right)^d
\]
for any Kähler class \(\omega\) on \(X\). As a consequence, Plov\((f)\) is a positive integer satisfying
\[
k(f) + d \leq \text{Plov}(f) \leq d + \max \left\{\sum_{j=1}^{d} i_j \mid i_j \in \mathbb{Z}_{\geq 0}, (N^i_1 \omega) \cdots (N^i_d \omega) \neq 0\right\},
\]
where \(d = \dim X\).

Proof. The first claim is clear by the definition of \(P_{f, \omega}(n)\). Then, the upper bound of Plov\((f)\) is clear by the equation (2.6).

For the lower bound, by Lemma 2.15 with the notations therein, we have
\[
\text{Plov}(f) = \deg_n P_{f, \omega, d}(n) > \deg_n P_{f, \omega, d-1}(n) > \cdots > \deg_n P_{f, \omega, 1}(n).
\]
As
\[
N^{k(f)} \omega = k(f)! \lim_{m \to \infty} \frac{(f^*)^m(\omega)}{m^{k(f)}}
\]
is nef and nonzero by definition of \(k(f)\), we have \((N^{k(f)} \omega) \cdot \omega^{d-1} \neq 0\). So \(\deg_n P_{f, \omega, 1}(n) \geq k(f) + 1\), which shows that Plov\((f)\) \(\geq k(f) + d\).

\[\square\]

Corollary 2.17. Let \(X\) be a compact Kähler manifold of dimension \(d\) and let \(f \in \text{Aut}(X)\) be a zero entropy automorphism. Then Plov\((f) = d\) if and only if \(k(f) = 0\).
We have an analogous statement of Lemma 2.16 with \( N \) we have Since \( k(f) = 0 \) implies \( N = 0 \), and \( \text{Plov}(f) = d \) by Lemma 2.16. Again by Lemma 2.16 \( \text{Plov}(f) = d \) implies \( k(f) = 0 \).

Another consequence of Lemma 2.16 is the following.

**Corollary 2.18.** \( \text{Plov}(f) \) has the same parity as \( d = \dim X \).

**Proof.** Since \( \Delta_n(f, \omega) \) is Kähler, we have \( \Delta_n(f, \omega)^d > 0 \) for all \( n \in \mathbb{Z} \). So by Lemma 2.14 we have

\[
\begin{cases}
\Delta_n^+(f, \omega)^d > 0 & \text{for } n \gg 0 \\
(-1)^d\Delta_{n-1}^+(f, \omega)^d > 0 & \text{for } n \ll 0.
\end{cases}
\]

It follows that the degree of the polynomial \( n \mapsto \Delta_n^+(f, \omega)^d \), which is also \( \text{Plov}(f) \) by Lemma 2.16 has the same parity as \( d \).

The following lemma provides another way to compute \( \text{Plov}(f) \), and turns out to be useful. Define

\[
\Delta_n'(f, \omega) := \sum_{i=0}^{n} ((f^i)^* + (f^{-i})^*) \omega.
\]

**Lemma 2.19.** \( \text{Plov}(f) \) is also the degree of the polynomial

\[
n \mapsto \Delta_n'(f, \omega)^d = \left( \omega + \sum_{i=-n}^{n} (f^i)^* \omega \right)^d.
\]

**Proof.** Recall that \( \text{Plov}(f) \) is defined as the polynomial degree of \( n \mapsto (\sum_{i=0}^{n} (f^i)^* \omega)^d \), which is also the polynomial degree of \( n \mapsto \left( \sum_{i=0}^{2n} (f^i)^* \omega \right)^d \) as well as \( n \mapsto \left( \sum_{i=0}^{2n} 2(f^i)^* \omega \right)^d \). Hence Lemma 2.19 follows from

\[
\left( \omega + \sum_{i=-n}^{n} (f^i)^* \omega \right)^d = \left( (f^{-n})^* \left( (f^n)^* \omega + \sum_{i=0}^{2n} (f^i)^* \omega \right) \right)^d = \left( (f^n)^* \omega + \sum_{i=0}^{2n} (f^i)^* \omega \right)^d
\]

and

\[
\left( \sum_{i=0}^{2n} (f^i)^* \omega \right)^d \leq \left( (f^n)^* \omega + \sum_{i=0}^{2n} (f^i)^* \omega \right)^d \leq \left( \sum_{i=0}^{2n} 2(f^i)^* \omega \right)^d.
\]

\( \square \)

As \( f^* : H^{1,1}(X) \otimes \) is unipotent, \( (f^{-1})^* : H^{1,1}(X) \otimes \) is also unipotent. Set

\[
N' := (f^{-1})^* - \text{Id} : H^{1,1}(X) \to H^{1,1}(X)
\]

and let

\[
N_m := N' + N^m.
\]

We have an analogous statement of Lemma 2.16 with \( N^m \) replaced by \( N_m \).

**Lemma 2.20.** We have

\[
\text{Plov}(f) \leq d + \max \left\{ \sum_{j=1}^{d} i_j \mid i_j \in \mathbb{Z}_{\geq 0}, (N_{i_1} \omega) \cdots (N_{i_d} \omega) \neq 0 \right\}.
\]
Proof. Lemma 2.20 follows from Lemma 2.19 together with
\[ \Delta'_n(f, \omega) = \sum_{i=0}^{n} (f^i)^* \omega + \sum_{i=-n}^{0} (f^i)^* \omega = \sum_{i=0}^{k(f)} (n+1) \left( N^i + N^m \right)(\omega) = \sum_{i=0}^{k(f)} (n+1) N_i(\omega). \]

\[ \square \]

Lemma 2.21. For every integer \( i \in [0, d] \), let
\[ P'_{f,\omega,i}(n) := \Delta'_n(f, \omega)_{i}^\omega \omega^{d-i} = \left( \sum_{j=0}^{k} \left( \frac{n+1}{j+1} N_j(\omega) \right)^i \right) \omega^{d-i}, \]
which is a polynomial in \( n \). Then we have
\[ \deg P'_{f,\omega,i} > \deg P'_{f,\omega,i-1}. \]

Proof. As in the proof of Lemma 2.19, for every non-negative integer \( m \), we define
\[ P'_{f,\omega,i,m}(n) := \Delta'_n(f, \omega)^{i-1} \cdot ((f^m)^* \omega + (f^{-m})^* \omega) \cdot \omega^{d-i}. \]
The same argument in Lemma 2.19 shows that
\[ \deg_n P'_{f,\omega,i,m}(n) = \deg_n P'_{f,\omega,i,0}(n) = \deg_n P'_{f,\omega,i-1}(n) \]
for every \( m \), and there exists \( \ell \in \mathbb{Z}_{>0} \) such that the leading coefficient \( C_{f,\omega,i}(\ell) > 0 \) of \( P'_{f,\omega,i,\ell} \) is minimum among all \( \ell \in \mathbb{Z}_{>0} \). Since \( \Delta'_n(f, \omega)^{i-1} \cdot \omega \cdot \omega^{d-i} > 0 \) (because \( \omega \) is Kähler), it follows that
\[ \frac{P'_{f,\omega,i}(n)}{P'_{f,\omega,i,\ell}(n)} = \sum_{m=0}^{n} \frac{P'_{f,\omega,i,m}(n)}{P'_{f,\omega,i,\ell}(n)} \leq_{n \to \infty} \gamma, \]
for any \( \gamma > 0 \), and we conclude the proof as in Lemma 2.19. \( \square \)

Let \( \omega \in H^{1,1}(X) \). For all integer \( 0 \leq p \leq d \), consider the following polynomial in \( n \) with coefficients in \( H^{p,p}(X) \):
\[ Q_{f,\omega,p} : n \mapsto ((f^n)^* \omega + (f^{-n})^* \omega)^p = \left( \sum_{i=0}^{k} \left( \frac{n}{i} \right) N_i(\omega) \right)^p. \]

Let \( \lambda_p(f, \omega) \) denote the polynomial degree of \( Q_{f,\omega,p}(n) \).

Remark 2.22. Note that for any product \( \Omega \in H^{d-p,d-p}(X) \) of \( d-p \) Kähler classes, \( \lambda_p(f, \omega) \) is also the polynomial degree of
\[ n \mapsto \Omega \cdot ((f^n)^* \omega + (f^{-n})^* \omega)^p. \]
The same argument proving Lemma 2.1 shows that the polynomial degree \( \lambda_p(f, \omega) \) is independent of the choice of \( \omega \) whenever \( \omega \) is nef and big.

We will use the next lemma in the proof of Lemma 4.5.

Lemma 2.23. For every integer \( p \), we have
\[ \lambda_p(f, \omega) \leq \max \left\{ r_i \in \mathbb{Z} \mid \| (f^n)^* \otimes H^{r_i}(X) \| \sim_{n \to \infty} C_i n^{r_i} \text{ for some } C_i > 0, 0 \leq i \leq p \right\}. \]
In particular,
\[ \lambda_p(f, \omega) \leq k(f) \left| \frac{d}{2} \right| \leq \frac{k(f)d}{2} \leq d(d-1). \]
Proof. The first statement follows from
\[
\left( (f^n)^* \omega + (f^{-n})^* \omega \right)^p = \sum_{j=0}^p \binom{p}{j} (f^n)^* \omega^j \cdot (f^{-n})^* \omega^{p-j} = \sum_{j=0}^p \binom{p}{j} ((f^{2n})^* \omega^j) \cdot \omega^{p-j}.
\]
For the last statement, the first inequality follows from [7, Proposition 5.8] and the last inequality from Theorem 1.1 \(\square\)

3. Quasi-nef sequences and dynamical filtrations

3.1. Dynamical filtrations and proof of the upper bound \(1.3\) in Theorem 1.1

First we recall the definitions and basic properties of quasi-nef sequences and dynamical filtrations. We then prove some useful lemmas, and finally the optimal upper bound \(1.3\) in Theorem 1.1 (see Corollary 3.7).

Let \(X\) be a compact Kähler manifold of dimension \(d \geq 1\). For every \(\alpha \in H^{i,i}(X, \mathbb{R})\), if \(\alpha \cdot H^{1,1}(X, \mathbb{R})^{d-i} = 0\), we write\(\alpha \equiv 0\) as in Notations. Let \(\mathcal{K}(X) \subset H^{i,i}(X, \mathbb{R})\) be the closed convex cone generated by classes of smooth positive \((i,i)\)-forms. We have \(\mathcal{K}(X) = \text{Nef}(X)\), which is the nef cone of \(X\). For every \(\alpha \in \mathcal{K}(X)\), define
\[
\text{Nef}(\alpha) := \alpha \cdot \text{Nef}(X) \subset H^{i,i+1}(X, \mathbb{R}).
\]
As \(\text{Nef}(X)\) is a convex cone, so is \(\text{Nef}(\alpha)\). Since \(\text{Nef}(\alpha) \subset \mathcal{K}(X)\) and \(\mathcal{K}(X)\) is salient, \(\text{Nef}(\alpha)\) is a closed salient cone.

Construction 3.1 (Quasi-nef sequence [27]). Let \(f \in \text{Aut}(X)\) be an automorphism of \(X\) such that \(f^*: H^{i,1}(X) \supset 0\) is unipotent. A quasi-nef sequence (with respect to \(f\)) is a sequence
\[
M_1, \ldots, M_d \in H^{1,1}(X, \mathbb{R})
\]
constructed recursively as follows. Suppose that \(M_1, \ldots, M_i \in H^{1,1}(X, \mathbb{R})\) are constructed, then \(M_{i+1} \in H^{1,1}(X, \mathbb{R})\) is an element such that
\begin{itemize}
  \item \(f^*(M_1 \cdots M_{i+1}) = M_1 \cdots M_{i+1} \neq 0\),
  \item \(M_1 \cdots M_i M_{i+1} \in \text{Nef}(M_1 \cdots M_i)\).
\end{itemize}
Since \(f^*: H^{1,1}(X) \supset 0\) is unipotent, the existence of \(M_{i+1}\) follows from Birkhoff’s Perron–Frobenius theorem [3] applied to \(\text{Nef}(M_1 \cdots M_i)\). See also [18, Theorem 1.1] for a generalization. We set \(L_0 := 1 \in H^0(X, \mathbb{R})\) and define \(L_i := M_1 \cdots M_i \in H^{i,i}(X, \mathbb{R})\).

Note that \(M_1, \ldots, M_d \in H^{1,1}(X, \mathbb{R})\) is also a quasi-nef sequence with respect to \(f^{-1}\) \(\square\)

Given a quasi-nef sequence \(M_1, \ldots, M_d \in H^{1,1}(X, \mathbb{R})\) with respect to an automorphism \(f \in \text{Aut}(X)\) such that \(f^*: H^{1,1}(X) \supset 0\) is unipotent, define
\[
F_i := \{ \alpha \in H^{1,1}(X, \mathbb{R}) \mid L_i \alpha \equiv 0 \}
\]
and let \(F'_i\) be the subspace of \(F_i\) spanned by
\[
\{ \alpha \in F_i \mid L_{i-1} \alpha \equiv \beta \text{ for some } \beta \in \text{Nef}(L_{i-1}) \}.
\]
Recall from [3] that these subspaces form an \(f^*\)-stable filtration
\[
0 = F_0 \subset F'_1 \subset F_1 \subset \cdots \subset F'_{d-1} \subset F_{d-1} \subset F_d = H^{1,1}(X, \mathbb{R}).
\]
We note that the filtration (3.1) depends on the choice of a quasi-nef sequence $M_1, \ldots, M_d \in H^{1,1}(X, \mathbb{R})$. These are examples of dynamical filtrations studied in [8].

Let us recall some fundamental properties of these filtrations proven in [8].

**Proposition 3.2** ([8, Theorem 1.3]).

1. We have $\dim(F'_i/F_{i-1}) \leq 1$ and 
   \[ F_i' = \{ \gamma \in F_i \mid L_{i-1}\gamma^2 = 0 \}. \]
   Moreover the following conditions are equivalent:
   (i) $F_{i-1} \neq F'_i$;
   (ii) $F'_i = F_{i-1} \oplus (\mathbb{R} \cdot M_i)$;
   (iii) $L_{i-1}M_i^2 = 0$.

2. There exist an integer $r \in [1, d - 1]$ and a strictly decreasing sequence of integers
   
   \[ d - 1 \geq s_1 > \cdots > s_r \geq 1 \]
   such that for every Kähler class $\omega \in H^{1,1}(X, \mathbb{R})$ and every integer $j \in [1, r]$, 
   
   \[ (f^* - \text{Id})^{2j-1}\omega \in F_{s_j}\setminus F'_{s_j} \quad \text{and} \quad (f^* - \text{Id})^{2j}\omega \in F'_{s_j}\setminus F_{s_j-1}, \]
   
   and $(f^* - \text{Id})^{2r+1}\omega = 0$. In particular, $(f^* - \text{Id})^{2r+1} = 0 \in \text{End}(H^{1,1}(X, \mathbb{R}))$.

The sequence $s_1 > \cdots > s_r$ in Proposition 3.2 (2) depends on $f$ and is unique for a given quasi-nef sequence. The inverse $f^{-1}$ defines the same sequence with respect to the same quasi-nef sequence by the next lemma.

**Lemma 3.3.** Let $s_1 > \cdots > s_r$ be the sequence in Proposition 3.2 (2) associated to $f$. Then for every Kähler class $\omega \in H^{1,1}(X, \mathbb{R})$ and every integer $j \in [1, r]$, we have 

\[ ((f^{-1})^* - \text{Id})^{2j-1}\omega \in F'_{s_j}\setminus F_{s_j} \quad \text{and} \quad ((f^{-1})^* - \text{Id})^{2j}\omega \in F_{s_j}\setminus F'_{s_j-1}, \]
and $(f^{-1})^* - \text{Id})^{2r+1}\omega = 0$.

**Proof.** Since both $F_{s_j}$ and $F'_{s_j}$ are $f^*$-invariant, we have 

\[ ((f^{-1})^* - \text{Id})^{2j-1}\omega = (-1)^{2j-1}(f^{1-2j})^*(f^* - \text{Id})^{2j-1}\omega \in F_{s_j}\setminus F'_{s_j}. \]

The same argument shows that $(f^{-1})^* - \text{Id})^{2j}\omega \in F'_{s_j}\setminus F_{s_j-1}$ and $(f^{-1})^* - \text{Id})^{2r+1}\omega = 0$. \qed

The following two lemmas are both consequences of Proposition 3.2 (1).

**Lemma 3.4.** For $i \in [1, d] \cap \mathbb{Z}$, take $\eta_i \in F'_i$. Let $p \in [1, d] \cap \mathbb{Z}$ and $j \in [0, p] \cap \mathbb{Z}$. Then:

1. There exists some $C \in \mathbb{R}$ such that 
   \[ L_j\eta_{j+1} \cdots \eta_p \equiv CL_p. \]

2. For any $\eta \in F_p$, we have 
   \[ L_j\eta_{j+1} \cdots \eta_p \eta \equiv 0. \]

**Proof.** Since either $F'_p = F_{p-1}$ or $F'_p/F_{p-1}$ is a line spanned by $M_p + F_{p-1}$ by Proposition 3.2 (1), there exists some $C_p \in \mathbb{R}$ such that $\eta_p - C_pM_p \in F_{p-1}$. As $L_{p-1}F_{p-1} = 0$, we have 

\[ L_{p-1}\eta_p \equiv C_pL_{p-1}M_p = C_pL_p. \]

Induction proves that $L_j\eta_{j+1} \cdots \eta_p \equiv CL_p$ for some $C \in \mathbb{R}$.

Since $L_pF_p \equiv 0$, (2) follows from (1) and the definition of $F_p$. \qed
Lemma 3.5. Assume that $M_1 = \cdots = M_i \in H^{1,1}(X, \mathbb{R})$. Then
\[ F_j' = F_{j-1} \]
for every $j \leq i - 1$.

Proof. Assume to the contrary that $F_j' \neq F_{j-1}$ for some $j \leq i - 1$. By Proposition 3.2 (1), we would have
\[ L_{j+1} = L_{j-1}M_jM_{j+1} = L_{j-1}M_j^2 = 0, \]
which is impossible. Hence $F_j' = F_{j-1}$ for every $j \leq i - 1$. \qed

As a consequence of these results, we obtain the following refinements of Theorem 1.1.

Corollary 3.6. Let $\phi : X \to B$ be a surjective morphism with connected fibers between compact Kähler manifolds. Let $f ∈ \text{Aut}(X)$ such that $f^* : H^{1,1}(X, \mathbb{R}) ⊆$ is unipotent and $\phi^*ω_B$ is $f^*$-invariant for some Kähler class $ω_B ∈ H^{1,1}(B, \mathbb{R})$. Then
\[ k(f) ≤ 2(\dim X - \dim B). \]

Here, we recall that $k(f)+1$ is the maximal size of the Jordan blocks of the Jordan canonical form of the unipotent $f^*|_{H^{1,1}(X,\mathbb{R})}$.

Proof. Let $m := \dim B$. As $\phi^*ω_B$ is an $f^*$-invariant nef class and $\phi^*ω_B^m ≠ 0$, we can complete
\[ M_1 = \cdots = M_m = \phi^*ω_B \]
to a quasi-nef sequence $M_1, \ldots, M_d$. By Lemma 3.5, we have $F_j' = F_{j-1}$ for every $j \leq m - 1$. So according to Proposition 3.2 (2) and the notation therein, necessarily $s_r ≥ m$, so $r ≤ \dim X - \dim B$. Hence $(f^* - \text{Id})^{2(\dim X - \dim B)+1}(ω) = 0$ for every $ω ∈ H^{1,1}(X, \mathbb{R})$. \qed

Corollary 3.7. Let $X$ be a compact Kähler manifold of dimension $d ≥ 1$ and of Kodaira dimension $κ(X)$. Let $f ∈ \text{Aut}(X)$ be an automorphism of zero entropy.

(1) We have
\[ k(f) ≤ 2(\dim X - κ(X)). \]

In other words,
\[ \|(f^m)^* : H^{1,1}(X) ⊆ \| = O\left(m^{2(d-κ(X))}\right) \]
as $m → ∞$ for any norm of $\text{End}_C(H^{1,1}(X))$.

(2) The estimate in (1) is optimal, in the sense that for every $d ≥ 1$ and $κ ≥ 1$, there exist some $X$ and $f ∈ \text{Aut}(X)$ such that $\dim(X) = d$, $κ(X) = κ$, and
\[ \|(f^m)^* : H^{1,1}(X) ⊆ \| \sim_{m→∞} Cm^{2(d-κ(X))} \]
for some $C > 0$.

We prove first Corollary 3.7 (1). We will prove Corollary 3.7 (2) in §7 by constructing explicit examples.

Proof of Corollary 3.7 (1). By an equivariant Kähler desingularization, there exists a bimeromorphic morphism $ν : \bar{X} → X$ form a compact Kähler manifold $\bar{X}$ such that $f$ lifts to an automorphism $\tilde{f} ∈ \text{Aut}(\bar{X})$ and that $\bar{X}$ admits a surjective morphism $\phi : \bar{X} → B$ to a projective manifold as a model of its Iitaka fibration. As $\phi$ is an Iitaka fibration of $\bar{X}$, $\tilde{f}$ descends to a bimeromorphic self-map of $B$ of finite order by [22, Theorem A]. Up to
replacing $f$ by a finite iteration of it, we can assume that $\phi$ is $\tilde{f}$-invariant. In particular, $\phi^*\omega_B$ is $\tilde{f}^*$-invariant for every Kähler class $\omega_B \in H^{1,1}(X, \mathbb{R})$.

Since $f$ has zero entropy, we have $d_1(\tilde{f}) = 1$ by Lemma 2.8 and [9, Theorem 1.1] for the invariance under a generically finite map. Replacing $f$ by its finite iteration, we can assume that $\tilde{f}^* : H^{1,1}(\tilde{X}, \mathbb{C}) \otimes \sim$ is unipotent by Lemma 2.8. Thus by Corollary 3.6, we have

$$\|(f^m)^* : H^{1,1}(\tilde{X}, \mathbb{C}) \otimes \|_{m \to \infty} O\left(m^{2(d-\kappa(X))}\right).$$

As $H^{1,1}(X, \mathbb{C}) \hookrightarrow H^{1,1}(\tilde{X}, \mathbb{C})$ is $\tilde{f}^*$-stable and the restriction of $\tilde{f}^* : H^{1,1}(\tilde{X}, \mathbb{C}) \otimes$ to $H^{1,1}(X, \mathbb{C})$ is $f^* : H^{1,1}(X, \mathbb{C}) \otimes$, we have

$$\|(f^m)^* : H^{1,1}(X, \mathbb{C}) \otimes \|_{m \to \infty} O\left(m^{2(d-\kappa(X))}\right).$$

$\square$

3.2. Some vanishing lemmas.

From now on till the end of §3, $f \in \text{Aut}(X)$ is an automorphism such that $f^* : H^{1,1}(X) \otimes$ is unipotent. Under this assumption, $(f^{-1})^* : H^{1,1}(X) \otimes$ is also unipotent. Recall that in §2 we have defined

$$N := f^* - \text{Id} \in \text{End}(H^{1,1}(X, \mathbb{R})) \quad \text{and} \quad N' := (f^{-1})^* - \text{Id} \in \text{End}(H^{1,1}(X, \mathbb{R})),
$$

and also $N_m := N^m + N'^m$.

In this subsection, we will prove some vanishing results of intersections of $(1,1)$-classes which are images of $N_m$ or $N'^m$. Let us start with the following lemma.

Lemma 3.8. Let $\alpha \in H^{1,1}(X, \mathbb{R})$.

(1) Let $d - 1 \geq s_1 > \cdots > s_r \geq 1$ be the sequence associated to $f$ as in Proposition 3.2.

(2) Then we have

$$N_{2i-1}(\omega), N_{2i}(\omega) \in F'_{s_i} \setminus F_{s_i-1}$$

for any Kähler class $\omega$. In particular,

$$N_{2i-1}(\alpha), N_{2i}(\alpha) \in F'_{d-i_i}.$$

(2) Both $N_{k(f)-1}(\alpha)$ and $N_{k(f)}(\alpha)$ are $f^*$-invariant.

(3) If $\omega$ is nef, then both $N_{k(f)}(\omega)$ and $N_{k(f)-1}(\omega)$ are nef.

Proof. First we prove (1). Note that $N + N' = -NN'$, so

$$N_{2i-1} = N^{2i-1} + N'^{2i-1} = (N + N')^{2i-2} \sum_{j=0}^{2i-2} (-1)^j N^j N'^{2i-2-j}$$

$$= \sum_{j=0}^{2i-2} (-1)^{j+1} N^{j+1} N'^{2i-1-j}$$

$$= \sum_{j=0}^{2i-2} (-1)^{j+1} (f^* - \text{Id})^{j+1} \left((f^{-1})^* - \text{Id}\right)^{2i-j-1}$$

$$= \sum_{j=0}^{2i-2} (f^* - \text{Id})^{2i} (f^{-2i+j+1})^* = \sum_{j=0}^{2i-2} N^{2i} \circ (f^{-2i+j+1})^*$$
Since $\sum_{j=0}^{2i-2}(f^{-2i+j+1})^*\omega$ is Kähler, Proposition 3.2 (2) implies

$$N_{2i-1}(\omega) = N^{2i} \left( \sum_{j=0}^{2i-2} (f^{-2i+j+1})^*\omega \right) \in F'_{s_i} \setminus F_{s_i-1}.$$  

By Proposition 3.2 (2) and Lemma 3.3, we have $N^{2i}(\omega), N^{2i}(\omega) \in F'_{s_i},$ so $N_{2i}(\omega) \in F'_{s_i}.$ Since $N^{2i}\omega \in F'_{s_i},$ we have

$$L_{s_i-1}N^{2i}\omega \equiv CL_{s_i}$$

for some $C \in \mathbb{R}$ by Lemma 3.4. Since $N^{2i}\omega \notin F_{s_i-1},$ we have $C \neq 0.$ Moreover, as $L_{s_i-1}N^{2i}\omega \equiv 0$ for every $p > 2i$ by Proposition 3.2 (2), we have

$$L_{s_i-1}N^{2i}\omega = (2i)! \cdot \lim_{m \to \infty} L_{s_i-1} \left( \frac{(f^*)^m(\omega)}{m^{2i}} \right) \in \mathcal{K}^s(X)/\equiv$$

where $\mathcal{K}^s(X)/\equiv$ denotes the image of $\mathcal{K}^s(X)$ in $H^{s,s}(X, \mathbb{R})/\equiv.$ Since $L_{s_i} \in \mathcal{K}^s(X),$ necessarily $C > 0.$ Since $s_1 > \cdots > s_r$ is also the sequence associated to $f^{-1}$ by Lemma 3.3, the same argument shows that there exists $C' > 0$ such that

$$L_{s_i-1}N^{2i}\omega \equiv C'L_{s_i}.$$  

Hence

$$L_{s_i-1}N_{2i}(\omega) \equiv (C + C')L_{s_i} \neq 0,$$

namely $N_{2i}(\omega) \notin F_{s_i-1}.$ The last part follows from $F'_{s_i} \subset F_{d-i},$ noting that $s_i \leq d - i.$

For (2), recall that $k(f)$ is an even number (Theorem 1.1) so we can write $k(f) = 2i.$ Since $N^{2i+1} = 0$ and $f^* = \text{Id} + N,$ we have

$$N^{2i}/(2i)! = \lim_{m \to \infty} (f^*)^m/m^{2i}$$

whose image is hence $f^*$-invariant. Since $f^*$ commutes with $N,$ and $N' = -N(f^{-1})^*,$ we have $N^{2i} = N^{2i}(f^{-2i})^*$ whose image is hence $f^*$-invariant too. Thus the images of $N_{2i} = N^{2i} + N^{2i}$ and $N_{2i-1} = \sum_{j=0}^{2i-2} N^{2i} \circ (f^{-2i+j+1})^*$ are also $f^*$-invariant.

For (3),

$$N^{2i}(\omega) = (2i)! \cdot \lim_{m \to \infty} (f^*)^m(\omega)/m^{2i}, \quad N_{2i-1}(\omega) = \sum_{j=0}^{2i-2} N^{2i}((f^{-2i+j+1})^*(\omega))$$

are clearly all nef. \hfill \Box

**Corollary 3.9.** Let $\omega$ be a Kähler class. Assume that $k(f) = 2d - 2.$ Then for every integer $\ell \geq 2,$ we have

$$N_{i_1}(\omega) \cdots N_{i_\ell}(\omega) \equiv 0$$

whenever

$$i_j \geq 2(d - j) - 1 \text{ for all } j \leq \ell - 2, \text{ and } i_{\ell-1}, i_\ell \geq 2(d - \ell + 1) - 1.$$  

Moreover, whenever

$$i_j \in \{2(d - j), 2(d - j) - 1\} \text{ for all } j,$$

there exists some $C \in \mathbb{R}$ such that

$$N_{i_1}(\omega) \cdots N_{i_j}(\omega) \equiv CN_{2d-2}(\omega) \cdots N_{2}(d-j)(\omega).$$
Proof. Corollary 3.9 follows directly from Lemma 3.4 and Lemma 3.8. Indeed, by the assumption and Lemma 3.8, we have

\[ N_1(\omega) \in F', \quad N_2(\omega) \in F', \quad \ldots, \quad N_{i-2}(\omega) \in F'_{i-2}, \quad N_{i-1}(\omega), N_i(\omega) \in F'_{i-1}. \]

Thus the first assertion follows from Lemma 3.4 (2).

Similarly, by the assumption and Lemma 3.8, we have

\[ N_1(\omega), N_{2d-2}(\omega) \in F', \quad N_2(\omega), N_{2d-4}(\omega) \in F', \quad \ldots, \quad N_j(\omega), N_{(d-j)}(\omega) \in F'_{j}. \]

Thus the second assertion follows from Lemma 3.4 (1).

\[
\text{Lemma 3.10. Let } m \text{ be a positive integer and let } \\
\Sigma := \left\{ N^{k(f)}(\omega), N_{k(f)-1}(\omega) \mid \omega \in H^{1,1}(X, \mathbb{R}) \text{ Kähler} \right\}. \\
\text{Then the following conditions are equivalent:} \\
\begin{enumerate}
\item \( M_1 \cdot \cdots \cdot M_m \neq 0 \) for some \( M_1, \ldots, M_m \in \Sigma \).
\item \( M_1 \cdot \cdots \cdot M_m \neq 0 \) for every \( M_1, \ldots, M_m \in \Sigma \).
\end{enumerate}
\]

Proof. Fix a positive integer \( m \). It suffices to prove that (1) implies (2). To this end, it suffices to prove that given \( M_1, \ldots, M_m, M'_m \in \Sigma \),

\[ M_1 \cdot \cdots \cdot M_{m-1}M_m \neq 0 \implies M_1 \cdot \cdots \cdot M_{m-1}M'_m \neq 0. \]

Then we can replace each factor of \( M_1 \cdot \cdots \cdot M_m \) by any choice of \( m \) elements \( M'_1, \ldots, M'_m \in \Sigma \) one by one and obtain \( M'_1 \cdot \cdots \cdot M'_m \neq 0 \).

Since every element of \( \Sigma \) is nef and \( f^* \)-invariant by Lemma 3.8, the sequence \( M_1, \ldots, M_{m-1} \) can be completed to a quasi-nc sequence. Since \( L_{m-1}M_m = M_1 \cdot \cdots \cdot M_{m-1}M_m \neq 0 \) and \( M'_m \in \Sigma \) by assumption, Proposition 3.2 (2) and Lemmas 3.3 and 3.8 (1) then imply that

\[ M_1 \cdot \cdots \cdot M_{m-1}M'_m = L_{m-1}M'_m \neq 0. \]

As for when we have \( (N^{k(f)}\omega)^i = 0 \) whenever \( 2i > d \).

\[
\text{Lemma 3.11. } (N^{k(f)}\omega)^i = 0 \text{ whenever } 2i > d.
\]

Proof. Let \( j \in \mathbb{Z}_{\geq 0} \). Since \( \|(f^*n)^j \| : H^{1,1}(X) \odot \| = O(n^{k(f)}) \), we have

\[ \|(f^*n)^j : H^{1,1}(X) \odot \| = O(n^{j(k(f)}) \]

by [4, Proposition 5.8]. Suppose that \( (N^{k(f)}\omega)^j \neq 0 \), then

\[ \|(f^*n)^j : H^{1,1}(X) \odot \| \sim Cn^{j(k(f)}. \]

As

\[ \|(f^*n)^j : H^{1,1}(X) \odot \| \sim \|(f^*n)^j : H^{d-j,d-j}(X) \odot \|, \]

necessarily \( (N^{k(f)}\omega)^j = 0 \) whenever \( 2i > d \).

\[
\text{Corollary 3.12. Let } m \text{ be a non-negative integer such that } N^{k(f)}(\omega_0)^m \equiv 0 \text{ (or equivalently } N_{k(f)}(\omega_0)^m \equiv 0 \text{ by Lemma 2.14) for some Kähler class } \omega_0. \quad \text{Then for every } \omega \in H^{1,1}(X, \mathbb{R}), \quad \text{we have} \\
N_{k(f)}(\omega)^i N_{k(f)-1}(\omega)^j \equiv 0 \\
\text{whenever } i + j \geq m.
\]
As a consequence, for every \( \omega \in H^{1,1}(X, \mathbb{R}) \) and every pair of non-negative integers \( i \) and \( j \) such that \( 2i + 2j > \min(d, 2d - k(f)) \), we have

\[
N_{k(f)}(\omega)^i N_{k(f)}^{-1}(\omega)^j \equiv 0.
\]

**Proof.** Since the vanishing \( N_{k(f)}(\omega)^i \) is a Zariski closed condition for \( \omega \in H^{1,1}(X, \mathbb{R}) \) and since the Kähler cone is Zariski dense in \( H^{1,1}(X, \mathbb{R}) \), we can assume that \( \omega \in H^{1,1}(X, \mathbb{R}) \) is Kähler. Then the first statement follows from Lemma 3.10.

Now we prove the second statement. Once again, we can assume that \( \omega \) is Kähler. Recall that \( k(f) = 2\ell \) is an even number (see e.g. Theorem 1.1). By the first statement, it suffices to show that

\[
N_{k(f)}(\omega)^d - \ell + 1 \equiv 0,
\]

as we already know that \( N_{k(f)}(\omega)^i \equiv 0 \) if \( 2i > d \) by Lemma 3.11. To this end, we can assume that \( N_{k(f)}(\omega)^d \not\equiv 0 \) and complete

\[
M_1 = \cdots = M_{d-\ell} := N_{k(f)}(\omega)
\]

to a quasi-nef sequence. Then Lemma 3.8 implies that

\[
N_{k(f)}(\omega) = N_{2\ell}(\omega) \in F'_{d-\ell} \subset F_{d-\ell}.
\]

Hence

\[
N_{k(f)}(\omega)^d - \ell + 1 = L_{d-\ell} N_{k(f)}(\omega) \equiv 0.
\]

\qed

4. Upper bounds of \( Plov(f) \): Beginning of the proof of Theorem 1.2

Let us first prove Keeler’s upper bound in Theorem 1.2 (1).

**Proposition 4.1.** Let \( X \) be a compact Kähler manifold of dimension \( d \) and let \( f \in \text{Aut}(X) \) be a zero entropy automorphism. Suppose that \( k(f) > 0 \). Then we have

\[
Plov(f) \leq k(f)(d - 1) + d.
\]

We will first provide a sketch of Keeler’s original proof, then an alternative proof using Corollary 3.12.

**Keeler’s proof of Proposition 4.1.** Recall that \( Plov(f) \) is the degree of the polynomial \( P_{f, \omega}(n) \) which is the same as the polynomial \( (\Delta_n(f, L)^d) \) in Theorem 3.2 (6) if we replace the ample class \( L \) by the Kähler class \( \omega \). Therefore, by setting \( D = \omega \) and \( P = f^* \) in the proof of [17 Lemma 6.13], the purely cohomological proof of [17 Lemma 6.13] works without any further change, which proves the result.

**Second proof of Proposition 4.1.** By Lemmas 2.6 and 2.8 we can assume that \( f^* : H^{1,1}(X) \odot \) is unipotent. Let \( \alpha \in H^{1,1}(X, \mathbb{R}) \) and let

\[
k(f) \geq i_1 \geq \cdots \geq i_d \geq 0
\]

be \( d \) integers such that

\[
\sum_{j=1}^d i_j > k(f)(d - 1).
\]
Write the product $N_{i_1}(\alpha) \cdots N_{i_d}(\alpha)$ in the form
\[ \Pi := N_{k(f)}(\alpha)^a N_{k(f)-1}(\alpha)^b N_{i_{a+b+1}}(\alpha) \cdots N_{i_d}(\alpha) \]
with $i_{a+b+1} \leq k(f) - 2$. Then $2a + 2b > 2d - k(f)$ by the assumption. It follows from Corollary 3.12 that $\Pi = 0$. Thus $\Plov(f) \leq k(f)(d - 1) + d$ by Lemma 2.20.

The main result of this section is the following sharpened upper bound of $\Plov(f)$.

**Theorem 4.2.** Let $X$ be a compact Kähler manifold of dimension $d$ and let $f \in \Aut(X)$ be a zero entropy automorphism. Assume that $d \geq 3$ and $k(f) > 0$. Then
\[ \Plov(f) \leq k(f)(d - 1) + d - 2. \]

When $k(f) = 2$, we have the optimal upper bound:
\[ \Plov(f) \leq \begin{cases} 2d & \text{if } d \text{ is even}; \\ 2d - 1 & \text{if } d \text{ is odd}. \end{cases} \]

The above inequality for $k(f) = 2$ was originally due to F. Hu with a different proof [13]. We will prove Theorem 4.2 based on results in §3 about the dynamical filtrations. Let us first prove Theorem 4.2 when $k(f) = 2$.

**Proof of Theorem 4.2 when $k(f) = 2$**. By Lemmas 2.6 and 2.8, we can assume that $f^* : H^{1,1}(X) \cap 0$ is unipotent. Let $\omega$ be a Kähler class and let $i$ be the largest integer such that $(N^2\omega)^i \neq 0$. By Lemma 3.11 we have $i \leq \lfloor d/2 \rfloor$. Since $(N^2\omega)^{i+1} = 0$, it follows from Corollary 3.12 that
\[ (N^2\omega)^a(N^1\omega)^b \equiv 0 \]
whenever $a + b > i$. Hence by Lemma 2.20
\[ \Plov(f) \leq d + 2i \leq d + 2\lfloor d/2 \rfloor. \]

For optimal examples, let $S$ be any compact Kähler surface and $f \in \Aut(S)$ any automorphism with $k(f) = 2$ (see e.g. [8, §4.1] for an example where $S$ is a torus). Then $\Plov(f) = 4$ by Corollary 4.3. If $d = 2m$, then $k(f^{\times m}) = 2$ for $f^{\times m} \in \Aut(S^m)$ by Lemma 2.12 and $\Plov(f^{\times m}) = 4m = 2d$ by Lemma 2.7. If $d = 2m + 1$, then we consider $f^{\times m} \times \Id_C \in \Aut(S^m \times C)$ where $C$ is any smooth projective curve.

The proof of Theorem 4.2 when $k(f) > 2$ follows from a different argument. In Lemmas 4.3 and 4.5 below, let $X$ be a compact Kähler manifold of dimension $d \geq 1$ and $f \in \Aut(X)$ an automorphism such that $f^* : H^{1,1}(X) \cap 0$ is unipotent.

**Lemma 4.3.** Assume $k(f) > 0$. Let $(a, b)$ be a pair of non-negative integers such that $2a + 2b \geq 2d - k(f)$. Let $i_1 \geq \cdots \geq i_{d'} \geq 0$ be $d'$ integers. When $2a + 2b = 2d - k(f)$ we assume
\[ \sum_{j=1}^{d'} i_j > (k(f) - 4)d' + 2. \]

Then
\[ N_{k(f)}(\alpha)^a N_{k(f)-1}(\alpha)^b N_{i_1}(\alpha) \cdots N_{i_{d'}}(\alpha) \equiv 0 \]
for every $\alpha \in H^{1,1}(X, \mathbb{R})$. 

\[ \text{POLYNOMIAL LOG-VOLUME GROWTH AND GK-DIMENSION 23} \]
Proof. If $2a + 2b > 2d - k(f)$, then we already have

$$N_{k(f)}(\alpha)^a N_{k(f) - 1}(\alpha)^b \equiv 0$$

by Corollary 3.12. So we can assume that $2a + 2b = 2d - k(f)$, and that $i_1 \leq k(f) - 2$. We can also assume that $N_{k(f)}(\alpha)^a N_{k(f) - 1}(\alpha)^b \not\equiv 0$ and that $\alpha \in H^{1,1}(X, \mathbb{R})$ is Kähler. By Lemma 3.8 both $N_{k(f)}(\alpha)$ and $N_{k(f) - 1}(\alpha)$ are $f^*$-invariant nef, so we can complete

$$M_1 = \cdots = M_a := N_{k(f)}(\alpha), \ M_{a+1} = \cdots = M_{a+b} := N_{k(f) - 1}(\alpha)$$
to a quasi-nef sequence.

Since $\sum_{j=1}^{d'} i_j > (k(f) - 4)d' + 2$, we have

$$i_1, i_2 \in [k(f) - 3, k(f) - 2].$$

Indeed, otherwise we would have $i_2 \leq k(f) - 4$ and

$$\sum_{j=1}^{d'} i_j \leq (k(f) - 2) + (d' - 1)(k(f) - 4) = (k(f) - 4)d' + 2.$$ 

It follows from Lemma 3.8 that

$$N_{i_1}(\alpha), N_{i_2}(\alpha) \in F'_{d-k(f)+1} = F'_{a+b+1}.$$ 

So

$$N_{k(f)}(\alpha)^a N_{k(f) - 1}(\alpha)^b N_{i_1}(\alpha) N_{i_2}(\alpha) = L_{a+b} N_{i_1}(\alpha) N_{i_2}(\alpha) \equiv 0$$

by Lemma 3.4 (2), which proves Lemma 4.3. □

End of proof of Theorem 4.2. Recall that $k(f)$ is an even number (Theorem 1.1), and we already proved the statement for $k(f) = 2$. It remains to prove Theorem 4.2 for $k(f) \geq 4$.

By Lemmas 2.6 and 2.8 we can assume that $f^* : H^{1,1}(X) \otimes$ is unipotent. Let $\alpha \in H^{1,1}(X, \mathbb{R})$ and let

$$k(f) \geq i_1 \geq \cdots \geq i_d \geq 0$$

be $d$ integers such that

$$\sum_{j=1}^{d} i_j > k(f)(d - 1) - 2.$$

Then the product $N_{i_1}(\alpha) \cdots N_{i_d}(\alpha)$ is of the form

$$\Pi := N_{k(f)}(\alpha)^a N_{k(f) - 1}(\alpha)^b N_{a+b+1}(\alpha) \cdots N_{i_d}(\alpha)$$

with $i_{a+b+1} \leq k(f) - 2$ We now show that $\Pi = 0$. We have

$$(a + b)k(f) + \sum_{j=a+b+1}^{d} i_j \geq \sum_{j=1}^{a+b} i_j + \sum_{j=a+b+1}^{d} i_j = \sum_{j=1}^{d} i_j > k(f)(d - 1) - 2.$$ 

So if $d' := d - a - b$, then

$$d'(k(f) - 2) \geq \sum_{j=a+b+1}^{d} i_j > k(f)(d - a - b - 1) - 2 = k(f)d' - k(f) - 2,$$
which implies \( k(f) + 2 > 2d' \). As \( k(f) \) is even, we have \( k(f) \geq 2d' \), namely \( 2a + 2b \geq 2d - k(f) \). Assume that \( 2a + 2b = 2d - k(f) \), namely \( 2d' = k(f) \), then since \( 2d' = k(f) \geq 4 \) by assumption, we have
\[
\sum_{j=a+b+1}^{d} i_j > k(f)d' - k(f) - 2 \geq (k(f) - 4)d' + 2.
\]
It follows from Lemma 4.3 that \( \Pi = 0 \), and then from Lemma 2.20 that
\[
Plov(f) \leq d + k(f)(d - 1) - 2.
\]

\[\square\]

We finish this section by the following upper bound of \( Plov(f) \) when \( d \geq 4 \), which improves the upper bound \( Plov(f) \leq 2d^2 - 3d \) obtained by combining Theorem 4.2 and \( k(f) \leq 2d - 2 \) in Theorem 1.1.

**Proposition 4.4.** Let \( X \) be a compact Kähler manifold of dimension \( d \geq 4 \) and let \( f \in Aut(X) \) such that \( d_1(f) = 1 \). Then
\[
Plov(f) \leq 2d^2 - 3d - 2.
\]
Moreover if \( d = 4 \), we can further improve the upper bound to
\[
Plov(f) \leq 16,
\]
which is optimal.

We first prove the following.

**Lemma 4.5.** Assume that \( k := k(f) = 2d - 2 \) and \( d \geq 4 \). Take \( d \) integers
\[
k \geq i_1 \geq \cdots \geq i_d \geq 0
\]
such that
\[
\sum_{j=1}^{d} i_j \geq (k - 2)d - 1.
\]
Then for every \( \alpha \in H^{1,1}(X) \), we have
\[
(4.1) \quad N_{i_1}(\alpha) \cdots N_{i_d}(\alpha) = 0.
\]
Moreover if \( d = 4 \), we have (4.1) as long as
\[
\sum_{j=1}^{4} i_j \geq 12.
\]

**Proof.** First we assume that \( i_1 \geq k - 1 \). We have
\[
(4.2) \quad \sum_{j=2}^{d} i_j = \left( \sum_{j=1}^{d} i_j \right) - i_1 \geq (k - 2)d - 1 - k = 2d^2 - 6d + 1
\[
\geq 2d^2 - 8d + 8 = (k - 4)(d - 1) + 2,
\]
where the second inequality follows from \( d \geq 4 \). So \( N_{i_1}(\alpha) \cdots N_{i_d}(\alpha) = 0 \) by Lemma 4.3.

Assume that \( i_1 \leq k - 2 \). Since \( \sum_{j=1}^{d} i_j \geq (k - 2)d - 1 \) and the sequence \( i_j \) is decreasing, necessarily
\[
i_1 = \cdots = i_{d-1} = k - 2 \quad \text{and} \quad i_d = k - 2 \text{ or } k - 3.
\]
Since we have already proven that \( N_{j_1}(\alpha) \cdots N_{j_d}(\alpha) = 0 \) whenever \( j_1 \geq k - 1 \), in particular whenever

\[
\sum_{l=1}^{d} j_l > (k - 2)d,
\]

we have

\[
Q_{f,\alpha,d}(n) := \left( \sum_{i=0}^{k-2} \binom{n}{i} N_i(\alpha) \right)^d \tag{4.3}
\]

\[
= n \to \infty \left( \frac{n}{k-2} \right)^d N_{k-2}(\alpha)^d + d \left( \frac{n}{k-2} \right)^{d-1} \left( \frac{n}{k-3} \right)^{N_{k-2}(\alpha)^d-1} N_{k-3}(\alpha) + O(n^{(k-2)d-2}).
\]

Recall that \( \deg(Q_{f,\alpha,d}) \leq (d-1) \) by Lemma \([\ref{2.23}]\). Since \( d(d-1) \leq (k-2)d - 2 \) (because \( d \geq 4 \)), it follows from (4.3) that

\[
N_{k-2}(\alpha)^d = 0
\]

and then

\[
N_{k-2}(\alpha)^{d-1} N_{k-3}(\alpha) = 0,
\]

which proves the first statement of Lemma \([\ref{4.5}]\).

Finally, assume that \( d = \dim X = 4 \) (so \( k = 6 \)). By the first statement of Lemma \([\ref{4.5}]\), it suffices to show that

\[
N_{i_1}(\omega) N_{i_2}(\omega) N_{i_3}(\omega) N_{i_4}(\omega) = 0
\]

as long as \( i_1 + i_2 + i_3 + i_4 = 13 \) or 14. Note then that \( i_1 \in \{4, 5, 6\} \) and \( i_2 \geq 3 \). By Corollary \([\ref{3.9}]\) we have

\[
N_{i_1}(\omega) N_{i_2}(\omega) \equiv 0 \text{ for } i_1, i_2 \geq 5
\]

\[
N_{i_1}(\omega) N_{i_2}(\omega) N_{i_3}(\omega) \equiv 0 \text{ for } i_1 \geq 5 \text{ and } i_2, i_3 \geq 3.
\]

\[
N_{i_1}(\omega) N_{i_2}(\omega) N_{i_3}(\omega) N_{i_4}(\omega) = 0 \text{ for } i_1 \geq 5, \text{ and } i_2 \geq 3, \text{ and } i_3, i_4 \geq 1.
\]

So it remains to show the vanishing (4.4) for the following five terms:

\[
N_4(\omega)^2 N_3(\omega)^2
\]

\[
N_4(\omega)^3 N_2(\omega)
\]

\[
N_4(\omega)^3 N_1(\omega)
\]

\[
N_4(\omega)^2 N_3(\omega) N_2(\omega)
\]

\[
N_4(\omega) N_3(\omega)^3.
\]

By Corollary \([\ref{3.9}]\) whenever we have

\[
i_1 \in \{5, 6\}, i_2 \in \{3, 4\}, i_3 \in \{1, 2\},
\]

there exists some \( C \in \mathbb{R} \) such that

\[
N_{i_1}(\omega) N_{i_2}(\omega) N_{i_3}(\omega) \equiv C N_6(\omega) N_4(\omega) N_2(\omega).
\]
Together with the above vanishings, we obtain
\[
Q_{f,\omega,3}(n) := \left( \sum_{i=0}^{k=6} \binom{n}{i} N_i(\omega) \right)^3
\]
\[
\equiv_{n \to \infty} \left( \frac{n}{4} \right)^3 N_4(\omega)^3 + 6 \left( \frac{n}{4} \right) \left( \frac{n}{2} \right) N_6(\omega) N_4(\omega) N_2(\omega)
\]
\[
+ 3 \left( \frac{n}{4} \right) \left( \frac{n}{3} \right) N_4(\omega)^2 N_3(\omega) + C' n^{11} \cdot N_6(\omega) N_4(\omega) N_2(\omega) + O(n^{10}).
\]
for some $C' \in \mathbb{R}$ as a polynomial of $n$ with coefficients in $H^{3,3}(X, \mathbb{R})$. By Lemma 2.23 and [3, Remark 3.9.(2)], we have $\deg \, \omega \omega = 6$ by Lemma 2.23, we finally have
\[
\deg \, Q_{f,\omega,4}(n) = \deg \, N_4(\omega) N_3(\omega) = 0.
\]
As $\deg \, Q_{f,\omega,4} \leq 12$ by Lemma 2.23, we finally have
\[
N_4(\omega) N_3(\omega)^3 = 0.
\]
\[\square\]

**Proof of Proposition 4.4.** By Theorem 1.1, we have $k(f) \leq 2d - 2$ and $k(f)$ is even. Since $d \geq 4$, Proposition 4.4 in the case $k(f) < 2d - 2$ (resp. $k(f) = 2d - 2$) follows from Theorem 1.2 (resp. Lemmas 2.20 and 4.3).

\[\square\]

5. A refined lower bound: end of the proof of Theorem 1.2 and Corollary 1.6

In this section we prove the following lower bound of $Plov(f)$. At the end we will conclude the proof of Theorem 1.2 together with Corollary 1.6.

**Theorem 5.1.** Let $X$ be a compact Kähler manifold of dimension $d > 0$ and let $f \in \text{Aut}(X)$ be a zero entropy automorphism. Then we have
\[
Plov(f) \geq d + 2k(f) - 2.
\]

**Proof.** We can assume that $\dim X \geq 2$, otherwise $k(f) = 0$, and Theorem 5.1 holds trivially.

By Lemmas 2.6 and 2.8 we can assume that $f^* : H^{1,1}(X) \cap \omega$ is unipotent. Let $\omega$ be a Kähler class. Recall that we have \[
\Delta'_n(f, \omega) := \sum_{i=0}^{k(f)} \left( (f^i)^* \omega + (f^{-i})^* \omega \right) = \sum_{i=0}^{k(f)} \binom{n}{i+1} N_i(\omega)
\]
from the computation in the proof of Lemma 2.20. By Lemma 2.21 and using the notations therein, we have
\[
Plov(f) = \deg_n P'_{f,\omega,d}(n) > \deg_n P'_{f,\omega,d-1}(n) > \cdots > \deg_n P'_{f,\omega,2}(n).
\]
Therefore it suffices to show that
\[ \deg_n P'_{f,\omega,2}(n) \geq 2k(f). \]

Recall that
\[ P'_{f,\omega,2}(n) = \Delta'_n(f, \omega)^2\omega^{d-2} = \left( \sum_{j=0}^{\deg_n} \binom{n+1}{j+1} N_j \omega \right)^2 \omega^{d-2}. \]

Assume that \((N_{k(f)}\omega)^2 \neq 0\). Since \(N_{k(f)}\omega\) is nef by Lemma 3.8 (3), we have \((N_{k(f)}\omega)^2 \cdot \omega^{d-2} \neq 0\). Hence \(\deg_n P'_{f,\omega,2}(n) \geq 2k(f) + 2\) by (5.1).

Now assume that \((N_{k(f)}\omega)^2 = 0\). Then
\[ (N_{k(f)}\omega)^2 \equiv 0, \quad (N_{k(f)}\omega)(N_{k(f)}\omega) \equiv 0 \]
by Lemma 3.10. Since \(N_{k(f)}\omega\) is nef and \(f^*\)-invariant by Lemma 3.8, we can construct a quasi-nef sequence \(M_1, \ldots, M_d\) with \(M_1 = N_{k(f)}\omega\). Suppose that \((N_{k(f)}\omega)(N_{k(f)}\omega) \equiv 0\). Then \((N_{k(f)}\omega) \equiv 1\), and we would have \((N_{k(f)}\omega) = 0\) by Lemma 3.8 (1), which contradicts the assumption that \(N_{k(f)}\omega \neq 0\). Hence \((N_{k(f)}\omega)(N_{k(f)}\omega) \neq 0\). Together with the vanishing (5.2) and (5.1), we obtain \(\deg_n P'_{f,\omega,2}(n) = 2k(f)\).

**Proof of Theorem 1.2.** The upper bound and lower bound of Plov\((f)\) in Theorem 1.2 follows from Theorems 4.2 and 5.1 respectively.

**Proof of Corollary 1.6.** The main statement of Corollary 1.6 follows from Theorem 1.2 (resp. Theorem 1.2) when \(k(f) > 0\) (resp. \(k(f) = 0\)). Together with Theorem 1.1, it follows that \(\kappa(X) \geq d/2\) implies Plov\((f) \leq d^2 - 2\).

6. Complex tori: Proof of Theorem 1.4 and a few remarks

In this section, we prove Theorem 1.4. See Remark 6.5 for further discussion.

**Proof of Theorem 1.4.** First we perform some reduction. By Lemmas 2.6 and 2.8, up to replacing \(f\) by some finite iteration of it, we can assume that \(f^* : H^{1,0}(X) \circlearrowleft\) is unipotent. Fix a basis
\[ dz_{1,1}, dz_{1,2}, \ldots, dz_{1,k_1}, \ldots, dz_{p,1}, \ldots, dz_{p,k_p} \]
of \(H^{1,0}(X)\) such that for every \(i = 1, \ldots, p\),
\[ f^*dz_{i,j} = \begin{cases} dz_{i,1} & \text{if } j = 1 \\ dz_{i,j} + dz_{i,j-1} & \text{if } 2 \leq j \leq k_i. \end{cases} \]

As the \(f^*\)-action on \(H^{1,0}(X)\) determines the \(f^*\)-action on \(H^*(X, \mathbb{C})\) when \(X\) is a torus, by Corollary 2.3 we can assume that \(X = E^d\) with \(E\) being an elliptic curve, or even \(E = \mathbb{C}/\mathbb{Z}[\sqrt{-1}]\), and \((z_{i,j})_{1 \leq i \leq p, 2 \leq j \leq k_i}\) are the global coordinates of \(E^d\), so that
\[ f : E^d = \prod_{i=1}^{p} E^{k_i} \to \prod_{i=1}^{p} E^{k_i} = E^d \]
is the product of \(E^{k_i} \to E^{k_i}\) defined by the unipotent Jordan matrix of size \(k_i\). By the product formula (Proposition 2.5 (4)), it suffices to prove Theorem 1.4 for the case \(p = 1\). So, from now on until the end of proof, we assume that \(p = 1\). Namely \(f^* : H^{1,0}(X) \circlearrowleft\) has only one Jordan block.
Set \( e_i = dz_i \) and \( \bar{e}_i = d\bar{z}_i \). For every \( \sigma = \sum_{i,j} a_{ij} e_i \wedge \bar{e}_j \in H^{1,1}(X, \mathbb{C}) \setminus \{0\} \), define
\[
\omega(\sigma) := \max \{i + j \mid a_{ij} \neq 0\},
\]
and for every \( p = 2, \ldots, 2d \), define
\[
\sigma(p) := \sum_{i+j=p} a_{ij} e_i \wedge \bar{e}_j.
\]

Note that \( \sum_{i=1}^d w_i(\sigma_i) \leq d(d + 1) \) by definition. We need the following.

**Lemma 6.1.** Let \( \sigma_1, \ldots, \sigma_d \in H^{1,1}(X, \mathbb{C}) \setminus \{0\} \) and let \( w_i := \omega(\sigma_i) \).

1. If \( \sum_{i=1}^d w_i < d(d + 1) \), then \( \sigma_1 \wedge \cdots \wedge \sigma_d = 0 \).
2. If \( \sum_{i=1}^d w_i = d(d + 1) \), then \( \sigma_1 \wedge \cdots \wedge \sigma_d = \sigma_1(w_1) \wedge \cdots \wedge \sigma_d(w_d) \).

**Proof.** By multi-linearity of \( \sigma_1 \wedge \cdots \wedge \sigma_d \), it is clear that (1) implies (2), and that it suffices to prove (1) for \( \sigma_1, \ldots, \sigma_d \) of the form \( \sigma_i = e_{i_1} \wedge \bar{e}_{j_1}, \ldots, \sigma_d = e_{i_d} \wedge \bar{e}_{j_d} \). If \( \sigma_1 \wedge \cdots \wedge \sigma_d \neq 0 \), then necessarily
\[
\{i_1, \ldots, i_d\} = \{1, \ldots, d\} = \{j_1, \ldots, j_d\},
\]
so \( \sum_{i=1}^d w_i = d(d + 1) \). \( \square \)

We return to the proof of Theorem 1.3. Let \( N := f^* - \text{Id} \) and let
\[
\omega := \sqrt{-1} \sum_{i=1}^d e_i \wedge \bar{e}_i,
\]
which is a Kähler class on \( X \). For every \( q = 0, \ldots, 2d - 2 \), by induction on \( q \) we have
\[
(N^q \omega)(p) = 0
\]
for every \( p > 2d - q \) and
\[
(N^q \omega)(2d - q) = \sqrt{-1} (N^q(e_d \wedge \bar{e}_d))(2d - q) = \sqrt{-1} \sum_{i+j=q} \binom{q}{i} e_{d-i} \wedge \bar{e}_{d-j} \neq 0.
\]
Therefore,
\[
w(N^q \omega) = 2d - q.
\]
Let \( q_1, \ldots, q_d \geq 0 \) be non-negative integers. If \( \sum_{i=1}^d q_i > d^2 - d \), then by Lemma 6.1
\[
(N^{q_1} \omega) \cdots (N^{q_d} \omega) = 0,
\]
so \( \text{Plov}(f) \leq d^2 \) by (2.7).

It remains to prove that \( \text{Plov}(f) \geq d^2 \). Note that since \( \omega(2d) = \sqrt{-1} e_d \wedge \bar{e}_d \) is nef, by Lemma 2.4 we have \( \text{Plov}(f) \geq \text{Plov}(f, \omega(2d)) \). Until the end of the proof, we formally define \( e_i \wedge \bar{e}_j = 0 \) whenever \( i \) and \( j \) are integers such that \( i \notin [1, d] \) or \( j \notin [1, d] \).

**Claim 6.2.** We have
\[
N^q(e_d \wedge \bar{e}_d) = \sum_{i+j \leq q} \binom{q}{i,j,q-i-j} e_{d-i} \wedge \bar{e}_{d-j}.
\]
Proof. Let $V := \mathbb{C}[X,Y]/(X^d,Y^d)$. We have an isomorphism of $\mathbb{C}$-vector spaces $V \cong H^{1,1}(X)$ sending each $X^iY^j$ to $e_{d-i} \wedge e_{d-j}$. Under this isomorphism, $N : H^{1,1}(X) \to H^{1,1}(X)$ becomes

$$N : V \to V \quad \quad \quad P \mapsto (XY + X + Y)P \mod (X^d,Y^d),$$

so

$$(6.1) \quad N^q(1) = (XY + X + Y)^q = \sum_{i+j \leq q} \binom{n}{i,j} X^{q-i}Y^{q-j} \mod (X^d,Y^d).$$

Translating (6.1) back to $N : H^{1,1}(X) \to H^{1,1}(X)$ proves the claim. \hfill \Box

For every integer $n > 0$, by Claim 6.3 we have

$$\Omega := \sum_{q=0}^{2d-2} \binom{n}{q+1} N^q(\omega(2d)) = \sqrt{-1} \sum_{q=0}^{2d-2} \sum_{i+j \leq q} \binom{n}{q+1} \binom{q}{i,j} \cdot e_{d-q+i} \wedge e_{d-q+j}.$$

For each pair of integers $1 \leq i,j \leq d$, define the polynomial $P_{i,j}(n)$ in $n$ by

$$(6.2) \quad \Omega = \sqrt{-1} \sum_{1 \leq i,j \leq d} P_{i,j}(n) e_i \wedge e_j.$$

Claim 6.3. The polynomial $P_{d-i,d-j}(n)$ in $n$ has degree $i+j+1$ and leading coefficient

$$\frac{1}{(i+j+1)!} \binom{i+j}{i}.$$

Proof. As $e_{d-i} \wedge e_{d-j} = e_{d-q+i} \wedge e_{d-q+j}$, by construction we have (with $q$ varying in the sum)

$$P_{d-i,d-j}(n) = \sum_{(q-i)+(q-j) \leq n} \binom{n}{q+1} \binom{q}{i,j} \cdot e_{d-q+i} \wedge e_{d-q+j}.$$

So the degree and the leading coefficient of $P_{d-i,d-j}$ are equal to those of the polynomial $\binom{n}{q+1} \binom{q}{i,j} \cdot e_{d-q+i} \wedge e_{d-q+j}$ in $n$ when $q$ is maximal and satisfying $(q-i) + (q-j) \leq q$ (that is, when $q = i + j$). This proves the claim. \hfill \Box

By (6.2), we have

$$\Omega^d = \left( \sqrt{-1} \right)^d d! P(n) (e_1 \wedge e_1) \wedge \cdots \wedge (e_d \wedge e_d)$$

where $P(n)$ is the determinant of the matrix $(P_{i,j}(n))_{1 \leq i,j \leq d}$. By Claim 6.3, we have $\deg_n(P(n)) \leq d^2$ and the coefficient in front of $n^{d^2}$ is $\det M$, where $M = (M_{i+1,j+1})_{0 \leq i,j \leq d-1}$ is the $(d \times d)$ matrix defined by

$$M_{i+1,j+1} = \frac{1}{(i+j+1)!} \binom{i+j}{i} = \frac{1}{i!} \frac{1}{(i+j+1)!}.$$

We have

$$\det M = \frac{1}{\left( \prod_{p=0}^{d-1} p! \right)^2} \det \left( \frac{1}{(i+j+1)!} \right)_{0 \leq i,j \leq d-1} = \frac{\prod_{p=0}^{d-1} p!}{\prod_{p=0}^{2d-1} p!} \neq 0,$$

where the second equality follows from the determinant of the Cauchy matrix (see e.g. [16 (1.1)]). Since Plov$(f,\omega(2d)) = \deg_n(P(n))$ by (2.6) and the definition of $\Omega$, it thus follows that

$$\text{Plov}(f) \geq \text{Plov}(f,\omega(2d)) = \deg_n(P(n)) = d^2.$$
This completes the proof of the main statement of Theorem 1.4. The optimality of the upper bound is provided by Example 6.4 below.

Example 6.4. Let $E$ be a complex elliptic curve and let $X = E^d$. Let

$$f : X \to X$$

$$\quad (x_1, \ldots, x_d) \mapsto (x_1, x_2 + x_1, \ldots, x_d + x_{d-1})$$

Then $f^* : H^{1,0}(X) \cap \mathcal{O}$ is represented by the $(d \times d)$-Jordan matrix, and we have

$$\text{Plov}(X, f) = d^2$$

as a consequence of the main statement of Theorem 1.4.

Remark 6.5.

(1) Consider $f \in \text{Aut}(E^3)$ with $f(x_1, x_2, x_3) = (x_1, x_1 + x_2, x_3)$. Then $f^* : H^{1,0}(E^3) \cap \mathcal{O}$ has two Jordan blocks, of sizes 2 and 1 respectively. By Theorem 1.4, we have

$$\text{Plov}(f) = 2^2 + 1^2 = 5,$$

which is also consistent with [17, Example 6.14].

(2) The upper bound in Theorem 1.4 is also asserted in the proof of [5, Proposition 4.3] (without optimality). However, the estimates (4.6)-(4.7) using $\ell_1$ in their proof have to be suitably modified, otherwise as we can see that if $f$ is the identity, the estimate $\text{Vol}(\Gamma(n)) \leq Cn^2$ in [5, (4.6)-(4.7)] would imply that $\text{Vol}(\Gamma(n))$ grows at most linearly in $n$, which contradicts the equality $\text{Plov}(X) = d$.

Finally, note that in this paper, whenever we prove that $\text{Plov}(f)$ is bounded from above by some constant $C$ for an automorphism $f \in \text{Aut}(X)$ such that $f^* : H^{1,1}(X) \cap \mathcal{O}$ is unipotent, we actually prove that the right hand side of the inequality in Lemma 2.16 or 2.20 is bounded by $C$. In view of Question 1.5, we ask the following.

Question 6.6. Let $X$ be a compact Kähler manifold of dimension $d \geq 1$ and let $f \in \text{Aut}(X)$ such that $f^* : H^{1,1}(X) \cap \mathcal{O}$ is unipotent. For every $\omega \in H^{1,1}(X)$, do we have

$$(N_{i_1} \omega) \cdots (N_{i_d} \omega) = 0$$

and

$$(N_i \omega) \cdots (N_i \omega) = 0$$

whenever the $i_j$ are non-negative integers satisfying $\sum_{j=1}^d i_j > d(d-1)$?

The following more refined question is asked by Fei Hu.

Question 6.7 (Hu). Do we have the vanishings in Question 6.6 as long as

$$\sum_{j=1}^d i_j > \frac{dk(f)}{2}.$$ 

So long one of the vanishings in Question 6.6 holds for every such $d$-tuple of integers $i_1, \ldots, i_d$, Question 1.5 can be answered readily in the affirmative by Lemma 2.16 or 2.20.
7. THREE DIMENSIONAL CASE AND EXPLICIT EXAMPLES

We know that $\text{Plov}(f) = 1$ for a compact Riemann surface, $\text{Plov}(X, f) = 2$ or $4$ for a compact Kähler surface by Corollary 1.3 and $\text{Plov}(X, f) = d$ for a projective variety $X$ of dimension $d$ whose desingularization $\tilde{X}$ is of general type (as $|\text{Bir}(\tilde{X})| < \infty$ by [25 Corollary 14.3]). Besides complex tori and these three cases, we can also determine $\text{Plov}(f)$ for some other classes of compact Kähler manifolds $X$ (Proposition 7.1), and especially threefolds (Proposition 7.2). We also prove Corollary 3.7 (2) in this section.

**Proposition 7.1.**

1. Let $X$ be a compact hyper-Kähler manifold of dimension $2d$ and let $f \in \text{Aut}(X)$ such that $d_1(f) = 1$. Then $\text{Plov}(f) = 2d$ if $f$ is of finite order, and $\text{Plov}(f) = 4d$ if $f$ is of infinite order; both cases are realizable, with $X$ projective.

2. Let $X$ be a smooth projective variety whose nef cone is a finite rational polyhedral cone. Let $\dim X = d$ and $f \in \text{Aut}(X)$. Then $f$ is quasi-unipotent and $\text{Plov}(f) = d$. In particular, this is the case when $X$ is a Mori dream space, especially when $X$ is a toric variety or a Fano manifold.

**Proof.** (1) The reader is referred to [14] for basics about compact hyper-Kähler manifolds. Note that a compact hyper-Kähler manifold has no global vector field other than $0$. Hence $\text{Aut}(X)$ is discrete. Thus $f$ is of finite order if and only if $f^* : H^{1,1}(X, \mathbb{R}) \odot$ is of finite order.

So, replacing $f$ by its power and using Proposition 2.5 (2), we can assume that $f = \text{id}_X$ or $f^* : H^{1,1}(X, \mathbb{R}) \odot$ is unipotent of infinite order. The result is clear when $f = \text{id}_X$. In the rest, we will assume that $f^* : H^{1,1}(X, \mathbb{R}) \odot$ is unipotent of infinite order.

Let $q_X(x)$ be Beauville-Bogomolov-Fujiki’s quadratic form on $H^{1,1}(X, \mathbb{R})$. The signature of $q_X(x)$ is $\langle 1, h^{1,1}(X) - 1 \rangle$.

Let $\omega$ be a Kähler class on $X$. Then the degree of the polynomial $q_X(P_f,\omega(n))$ in Lemma 2.16 with respect to $n$ is $2^d = 4$ by [1 Lemma 5.4]. The first part of (1) then follows from Fujiki’s relation below (with positive constant $c_X > 0$):

$$ (x^{2d})_X = c_X(q_X(x))^d. $$

For the realization part of (1), let $\varphi : S \rightarrow \mathbb{P}^1$ be a projective elliptic K3 surface whose Mordell-Weil group $\text{MW}(\varphi)$ has an element of infinite order, say $f$. There are plenty of such K3 surfaces. Then $f \in \text{Aut}(S)$ and it induces an automorphism $f^{[d]} \in \text{Aut}(\text{Hilb}^d(S)/\mathbb{P}^d)$ of infinite order. Here the Hilbert scheme $X := \text{Hilb}^d(S)$ is a projective hyper-Kähler manifold of dimension $2d$ with the Lagrangian fibration $\text{Hilb}^d(S) \rightarrow \mathbb{P}^d$ induced by $\varphi$. Hence $d_1(f^{[d]}) = 1$ as it preserves the pullback $h$ of the hyperplane class of $\mathbb{P}^d$, which is non-zero nef class on $X$ such that $q_X(h) = 0$. Thus $(X, f^{[d]})$ provides an example such that $\text{Plov}(f^{[d]}) = 2 \dim X = 4d$. This completes the proof of (1).

(2) By the assumption, $f^* : N^1(X) \odot$ is always of finite order (even though the order of $f$ itself can be often infinite). Thus we have $\text{Plov}(f) = d$ by Theorem 1.2.

We finish this section with proofs of Corollary 3.7 (2) and Proposition 7.2 by constructing explicit examples. The examples that we will construct also appear in other complex dynamical contexts [24, 25].
Proof of Corollary 3.7 (2). Let \( X_d = E^d \) \((d \geq 2)\) be the \( d \)-fold self-product of an elliptic curve \( E \) and \( f_d \) the automorphism of \( X_d \) defined by
\[
f_d(x_1, x_2, \ldots, x_d) = (x_1, x_2 + x_1, \ldots, x_d + x_{d-1}),
\]

as in Example 6.4. We have \( k(f_d) = 2d - 2 \) \[8\] \( §4.1\].

Let \( C \) be a smooth projective curve of genus \( g(C) \geq 2 \) with a surjective morphism
\[
\pi : C \to E.
\]

For instance, the double cover of \( E \) branched at two points is such an example. Let
\[
Y_d := C \times E^{d-1}.
\]

Then \( Y_d \) is a smooth projective variety with \( \dim Y_d = d \) and Kodaira dimension \( \kappa(Y_d) = 1 \). We define \( g_d \in \text{Aut}(Y_d) \) by
\[
g_d(P, x_2, x_3, \ldots, x_d) = (P, x_2 + \pi(P), x_3 + x_2 \ldots, x_d + x_{d-1})
\]

We also define
\[
p : Y_d \to X_d ; \ (P, x_2, x_3, \ldots, x_d) \mapsto (\pi(P), x_2, x_3, \ldots, x_d).
\]

Then \( p \) is a finite surjective morphism such that
\[
f_d \circ p = p \circ g_d,
\]

so \( k(g_d) = k(f_d) = 2d - 2 \) by Proposition 2.11. Finally, for every smooth projective variety \( V \) with \( \kappa(V) = \dim V \), let \( V_d := Y_d \times V \) and consider \( \phi_d := g_d \times \text{Id}_V \in \text{Aut}(V_d) \). We have
\[
2(d - 1) = k(g_d) \leq k(g_d \times \text{Id}_V) \leq 2(\dim V_d - \kappa(V_d)) = 2(d - 1),
\]

where the second inequality follows from the first statement of Corollary 3.7. So
\[
k(\phi_d) = 2(\dim V_d - \kappa(V_d)).
\]

When \( d \) and \( V \) vary, any pair of positive integers \( \dim V_d \geq 1 \) and \( \kappa(V_d) \geq 1 \) is realizable, which finishes the proof.

\[\Box\]

**Proposition 7.2.** Let \( X \) be a compact Kähler manifold of dimension 3 and let \( f \in \text{Aut}(X) \).

1. If \( \kappa(X) = 3 \), then \( \text{Plov}(f) = 3 \).
2. If \( \kappa(X) = 2 \), then \( \text{Plov}(f) \in \{3, 5\} \). Each number is realizable by some automorphism \( f \) of some smooth projective threefold \( X \).
3. We have \( \text{Plov}(f) \in \{3, 5, 9\} \), all realizable by pairs \((X, f)\), where \( X \) can be taken from any of the following classes of smooth projective threefolds.
   - \( X \) is rational;
   - \( X \) is an abelian threefold;
   - \( X \) is a simply connected Calabi-Yau manifold;
   - \( X \) has Kodaira dimension 1.
4. Without any assumption on \( \dim X \), for any integer \( d \geq 1 \), we can construct \((X, f)\) such that \( \dim X = d \), \( \text{Plov}(f) = d^2 \), \( \kappa(X) = 1 \), and \( X \) is projective.

**Proof.** Recall that by Corollary 1.3, we have \( \text{Plov}(f) \in \{3, 5, 9\} \) when \( \dim X = 3 \). If \( \kappa(X) = 2 \) (resp. \( \kappa(X) = 3 \)), then \( k(f) \in \{0, 2\} \) (resp. \( k(f) = 0 \)) by Theorem 1.1 so \( \text{Plov}(f) \in \{5, 9\} \) (resp. \( \text{Plov}(f) = 9 \)) by Theorems 1.2 and 5.1.
Now we construct explicit examples for Proposition 7.2, starting with Theorem 7.2 (4). We continue to use the automorphisms constructed in the previous proof of Corollary 3.7, together with the same notations. Since 

\[ f_d \circ p = p \circ g_d, \]

we have

\[ \text{Plov}(Y_d, g_d) = \text{Plov}(X_d, f_d) = d^2, \]

by Proposition 2.5 (5) and Example 6.4. Hence \((Y_d, g_d)\) provides an example satisfying description (4) in Proposition 7.2.

Note that since \(\text{Plov}(\text{Id}_{X^d}) = 3\) for every threefold \(X\), all we need to construct for Theorem 7.2 (1), (2), (3) are examples with \(\text{Plov}(f_d) = 5\) or \(9\). The automorphism \((\text{Id}_{C^d} \times g_2) : C^d \rightarrow C^d\) satisfies \(\kappa(C^d \times Y_2) = 2\) and \(\text{Plov}(\text{Id}_{C^d} \times g_2) = 5\) by Proposition 2.5 (4), which provides an example for Theorem 7.2 (2). We also have \(\text{Plov}(f_3) = 9\) and \(\text{Plov}(f'_3) = 5\); here \(f'_d := f_{d-1} \times \text{Id}_E \in \text{Aut}(E^d)\). These automorphisms provide examples when \(X\) is an abelian threefold for Theorem 7.2 (3).

Next, let \(E\) be an elliptic curve admitting an automorphism \(\alpha : E \rightarrow E\) which lifts to the multiplication \(C \rightarrow C\) by a primitive \(m\)-th root of unity (so \(m = 2, 3, 4, 6\)). Then the diagonal automorphism \(\tau := \alpha \times \cdots \times \alpha \in \text{Aut}(E^d) = \text{Aut}(X^d)\) has finite order. Let 

\[ \nu : Z_{d,\alpha} \rightarrow X_{d,\alpha} := X_d / \langle \tau \rangle \]

be the blowup at the maximal ideals of the image of the fixed points of \(\langle \tau \rangle\), which consists of finitely many isolated fixed points. Then \(Z_{d,\alpha}\) is a smooth projective variety of dimension \(d\) and Kodaira dimension \(\kappa(Z_{d,\alpha}) \leq 0\).

Let 

\[ \varphi : X_d \rightarrow Z_{d,\alpha} \]

denote the dominant rational map induced by the quotient morphism and \(\nu\). Clearly, \(\varphi\) is generically finite. As \(f_d \circ \tau = \tau \circ f_d\), by the universal property of blowup, \(f_d\) induces \(h_d \in \text{Aut}(Z_{d,\alpha})\) such that \(\varphi \circ f_d = h_d \circ \varphi\). Once again, we have 

\[ \text{Plov}(Z_{d,\alpha}, h_d) = \text{Plov}(X_d, f_d) = d^2 \]

by Proposition 2.5 (5) and Example 6.4.

As is well known (see e.g. [25, Example 16.12]), \(Z_{d,\alpha}\) is a (simply connected) Calabi-Yau threefold when \(d = 3\), \(m = 3\), and \(E\) is an elliptic curve of period \(\zeta_3\). By [24], \(Z_{d,\alpha}\) is a rational threefold when \(d = 3\), \(m = 6\), and \(E\) is an elliptic curve of period \(\zeta_3\). These two cases provide examples of \(\text{Plov} = 9\) satisfying description (3) in Proposition 7.2.

Following the same argument, replacing \(f_d\) with \(f'_d = f_{d-1} \times \text{Id}_E \in \text{Aut}(E^d)\) with \(d = 3\) in the above constructions gives examples of \(\text{Plov} = 5\) satisfying Proposition 7.2 (3). □

8. Twisted homogeneous coordinate rings and GK-dimensions: Proofs of Theorem 1.8 and Corollary 8.6

In this section, we first relate the polynomial log-volume growths \(\text{Plov}(f)\) to the GK-dimensions \(\text{GKdim}(X, f)\) of twisted homogeneous coordinate rings through Keeler’s work [17]. Then we prove Theorem 1.8 and Corollary 8.6 explaining how the results of \(\text{Plov}(f)\) imply the analogous statements for \(\text{GKdim}(X, f)\).
8.1. Recollection of Keeler’s work [17].

Following [17], we recall the definition of twisted homogeneous coordinate rings and related notions, together with the fundamental properties proven in [1] and [17].

Let \( X \) be an irreducible projective variety defined over an algebraically closed field \( k \) of characteristic 0. Let \( f \in \text{Aut}(X) \) be an automorphism. We say that a line bundle \( L \) on \( X \) is \( f \)-ample if for any coherent sheaf \( F \) on \( X \), there is a positive integer \( m_F \) such that

\[
H^q(X, F \otimes L \otimes f^* L \otimes \ldots \otimes (f^m)^* L) = 0
\]

for any integer \( q > 0 \) and for any integer \( m > m_F \). A Cartier divisor \( D \) is called \( f \)-ample if \( \mathcal{O}(D) \) is \( f \)-ample.

Let \( f \in \text{Aut}(X) \). For any integer \( n \in \mathbb{Z}_{\geq 0} \) and any line bundle \( L \) on \( X \), define

\[
\Delta_n(f, L) := L \otimes f^* L \otimes \ldots \otimes (f^n)^* L,
\]

and

\[
B_{n+1}(X, f, L) := H^0(X, \Delta_n(f, L)), \quad B_0 = H^0(X, \mathcal{O}_X) = k.
\]

Recall that the twisted homogeneous coordinate ring [1] of \( X \) associated to \( (f, L) \) is the (noncommutative) associative graded \( k \)-algebra

\[
B(X, f, L) := \bigoplus_{n \in \mathbb{Z}_{\geq 0}} B_n(X, f, L),
\]

with product defined by

\[
ab := a \otimes (f^n)^* b
\]

for \( a \in B(X, f, L)_m \) and \( b \in B(X, f, L)_n \).

**Remark 8.1.** By the definition, \( L \) is \( \text{Id}_X \)-ample precisely when \( L \) is ample in the usual sense and \( B(X, \text{Id}_X, L) \) is nothing but the projective coordinate ring of \( X \) associated to the ample line bundle \( L \).

For an associative \( k \)-algebra \( A \), its Gelfand–Kirillov dimension (or GK-dimension for short) is defined as

\[
\text{GKdim } A := \sup \limsup_{N \to \infty} \frac{\log \dim_k V^{\leq N}}{\log N} \in [0, \infty].
\]

Here \( V \) runs through all finite dimensional vector subspaces of the \( k \)-vector space \( A \), and \( V^{\leq N} \) is the vector subspace of \( A \), generated by the monomials

\[
x_1 x_2 \cdots x_\nu \ (0 \leq \nu \leq N, x_1, x_2, \ldots, x_\nu \in V).
\]

The supremum is reached by any finite dimensional subspace \( V \) generating \( A \) [20], p.14, Remark].

The Gelfand–Kirillov dimension is a natural noncommutative generalization of Krull dimension in commutative algebra, in the sense that when \( A \) is a finitely generated commutative \( k \)-algebra, we have \( \text{GKdim } A = \dim \text{Spec } A \) [20] Theorem 4.5]. In particular,

\[
\text{GKdim } B(X, \text{Id}_X, L) = \dim X + 1
\]

for any ample line bundle \( L \) on a projective scheme \( X \). From the perspective of projective geometry, it is thus natural to introduce the reduced Gelfand–Kirillov dimension defined as

\[
\text{rGKdim } B(X, f, L) := \text{GKdim } B(X, f, L) - 1.
\]
for twisted homogeneous coordinate rings. While the Krull dimension is an integer (possibly $+\infty$), any real number $r \geq 2$ can be realized as the the GK-dimension of a $k$-algebra [15]. The reader is referred to [20] for further properties about GK-dimension.

The study of twisted homogeneous coordinate rings was initiated by Artin and Van den Bergh [1]. Together with the seminal work of Keeler [17], here is a list of fundamental properties they proved about twisted homogeneous coordinate rings. In the statements, $N^1(X) := \text{NS}(X)/(\text{torsion})$.

**Theorem 8.2** (Keeler, Artin–Van den Bergh). Let $X$ be a projective variety of dimension $d > 0$ and let $f \in \text{Aut}(X)$. Then we have the assertions (1) – (4) below.

1. $f$-ample line bundles exist if and only if $f^*: N^1(X) \otimes$ is quasi-unipotent.

In the following, let $L$ be a line bundle and let $B := B(X, f, L)$ be the twisted homogeneous coordinate ring associated to $(f, L)$. Assume that $f^*: N^1(X) \otimes$ is quasi-unipotent. Then:

2. $L$ is $f$-ample if and only if there exists an integer $n > 0$ such that $\Delta_n(f, L)$ is ample. In particular, any ample line bundle is $f$-ample.

3. The twisted homogeneous coordinate ring $B$ is a finitely generated $k$-algebra, which is both right and left noetherian.

4. The GK-dimension $\text{GKdim } B$ is independent of the choice of an $f$-ample line bundle $L$. We therefore define the GK-dimension as

$$\text{GKdim } (X, f) := \text{GKdim } B(X, f, L)$$

and the reduced GK-dimension as

$$r\text{GKdim } (X, f) := r\text{GKdim } B(X, f, L) := \text{GKdim } B(X, f, L) - 1$$

for any choice of $f$-ample line bundle $L$.

5. For any positive integer $N$, we have

$$\text{GKdim } (X, f^N) = \text{GKdim } (X, f).$$

6. The GK-dimension is a positive integer. More precisely, after replacing $f$ by its suitable positive power so that $f^*: N^1(X) \otimes$ is unipotent, the self intersection number $(\Delta_n(f, L)^d)$ is a polynomial in $n$ and its degree satisfies:

$$r\text{GKdim } B = \text{deg}_n(\Delta_n(f, L)^d).$$

**Proof.** Let us indicate the references where these statements are proven. Statements (1) and (3) are contained in [17, Theorem 1.2]. Statements (2), (4), and (6) follow from [11, Lemma 4.1], and [17, Proposition 6.11, Theorem 6.1.(1)], respectively. Statement (5) is also contained in [11, 17]. Indeed, since $\Delta_N(f, L)$ is $f$-ample, we have

$$\text{GKdim } (X, f^N) = \text{GKdim } B(X, f^N, \Delta_{N-1}(f, L)) = \text{GKdim } B(X, f, L) = \text{GKdim } (X, f)$$

where the equality in the middle follows from [17, (6.3)].

When $X$ is a complex projective manifold, Keeler’s work implies Theorem 1.8 as an immediate corollary that the reduced GK-dimension of $(X, f)$ coincides with the polynomial log-volume growth of $f$. Together with Theorem 8.2, this suggests unexpected relations between noncommutative algebra and complex dynamics of automorphisms of zero entropy.
Proof of Theorem 1.8. Since $f^* : N^1(X) \otimes$ is quasi-unipotent, by Theorem 8.2 (6) and the definition of $\text{Plov}(f)$, given any ample line bundle $L$ on $X$, we have

$$r\text{GKdim} B = \deg_n(\Delta_n(f, L)^d) = \limsup_{n \to \infty} \frac{\log \Delta_n(f, c_1(L))^d}{\log n} = \text{Plov}(f).$$

\[\Box\]

8.2. From Kähler to projective.

Let $X$ be a projective variety over $\mathbb{C}$ and let $f \in \text{Aut}(X)$. Let $\nu : \tilde{X} \to X$ be a projective desingularization of $X$ such that

$$f \circ \nu = \nu \circ \tilde{f}$$

for some $\tilde{f} \in \text{Aut}(\tilde{X})$ (see e.g. [19, Theorem 3.45] for the existence). Before we prove Corollary 8.6, first we identify some dynamical properties and invariants of $(X, f)$ as a projective variety with those of $(\tilde{X}, \tilde{f})$ as a compact Kähler manifold.

Lemma 8.3. The following conditions are equivalent.

(1) $f^* : N^1(X) \otimes$ is quasi-unipotent.
(2) $\tilde{f}^* : N^1(\tilde{X}) \otimes$ is quasi-unipotent.
(3) $\tilde{f}$ has zero entropy.

Proof. We define $d_1(f)$ as in (2.1) but replacing $\omega$ by an ample divisor. Then the same proof of [22, Proposition A.2 and Lemma A.7] says that in the definition $d_1(f)$, we can assume that $\omega$ is a nef and big divisor instead, and $d_1(f)$ is the spectral radius of $f^* : N^1(X) \otimes$. Note that $d_1(f) = 1$ if and only if $f^* : N^1(X) \otimes$ is quasi-unipotent by Kronecker’s theorem. Then projection formula shows $d_1(f) = d_1(\tilde{f})$, hence (1) and (2) are equivalent. The equivalence between (2) and (3) follows from Lemma 2.8, as we can compute $d_1(\tilde{f})$ using ample classes (which also lie in $N^1(\tilde{X})$). \[\Box\]

Assume that $f^* : N^1(X) \otimes$ is quasi-unipotent. Then

$$\|(f^m)^* : N^1(X) \otimes \mathbb{C} \otimes \sim_{m \to \infty} Cm^{k_{NS}(f)}$$

for some $k_{NS}(f) \in \mathbb{Z}_{\geq 0}$ and $C > 0$.

Lemma 8.4. We have

$$k_{NS}(f) = k(\tilde{f}).$$

Proof. First of all, the same argument proving Proposition 2.11 shows that

$$k_{NS}(f) = k_{NS}(\tilde{f}).$$

It suffices to show that $k_{NS}(\tilde{f}) = k(\tilde{f})$.

Since $k(\tilde{f})$ is invariant under finite iterations, by Lemma 8.3 and Lemma 2.8 we can assume that $\tilde{f}^* : H^{1,1}(\tilde{X}) \otimes$ is unipotent. As the ample cone of $\tilde{X}$ spans $\text{NS}(\tilde{X})_\mathbb{R}$ we can thus find an ample class $\omega$ of $\tilde{X}$ such that

$$(\tilde{f}^* - \text{Id})^{k_{NS}(\tilde{f})}(\omega) \neq 0 \quad \text{and} \quad (\tilde{f}^* - \text{Id})^{k_{NS}(\tilde{f})+1}(\omega) = 0.$$
Lemma 8.5. We have

$$\text{GKdim} (X, f) = \text{GKdim} B(X, f, L)$$

for any big and nef line bundle $L$. As a consequence,

$$\text{rGKdim} (X, f) = \text{rGKdim} (\tilde{X}, \tilde{f}) = \text{Plov} (\tilde{f})$$

and $\text{GKdim} (X, f)$ is a birational invariant.

Proof. Since $\text{GKdim} (X, f)$ is the polynomial degree of $n \mapsto \deg_n (\Delta_n (f, L)^d)$ by Theorem 8.2 (6), the same argument of Lemma 2.1 proves the first assertion of Lemma 8.5.

Let $L$ be an ample line bundle on $X$, since

$$\left( \sum_{i=0}^{n} (\tilde{f}^i)^* (\nu^* c_1 (L)) \right)^d = \nu^* \left( \sum_{i=0}^{n} (f^i)^* c_1 (L) \right)^d = \left( \sum_{i=0}^{n} (f^i)^* c_1 (L) \right)^d,$$

by Theorem 8.2 (6) we have

$$\text{GKdim} B(\tilde{X}, \tilde{f}, \nu^* L) = \text{GKdim} B(X, f, L) = \text{GKdim} (X, f)$$

As $\nu^* L$ is nef and big, it follows from the first statement that $\text{GKdim} B(\tilde{X}, \tilde{f}, \nu^* L) = \text{GKdim} (\tilde{X}, \tilde{f})$, which finishes the proof of Lemma 8.5. \hfill \square

In the corollary below, $k(f)$ is defined with $f^*: H^{1,1} (X) \otimes$ replaced by $f^*: N^1 (X) \otimes$ (denoted as $k_{NS} (f)$ in §8.2).

Corollary 8.6. Let $X$ be a projective variety of dimension $d > 0$ over $k$, not necessarily smooth, and let $f \in \text{Aut}(X)$ be an automorphism such that $f^*: N^1 (X) \otimes$ is quasi-unipotent. Then the analogous statements of Theorem 1.2, Corollary 1.3, inequality (1.4) (Proposition 4.4), etc., hold with $\text{Plov}(f)$ replaced by $\text{rGKdim}(X, f)$ under the same assumptions on $d$ and $k(f)$. In particular, we have

$$\text{rGKdim} (X, f) \in \{3, 5, 9\}$$

if $d = 3$ (by Corollary 1.3), and

$$\text{rGKdim} (X, f) \leq 2d^2 - 3d - 2$$

whenever $d \geq 4$ (by Proposition 4.4).

Proof. By the Lefschetz principle, we can assume that the pair $(X, f)$ is defined over $k = \mathbb{C}$. Corollary 8.6 then follows the existence of equivariant projective desingularization (see e.g. [19, Theorem 3.45]), together with the comparison results Lemma 8.5 and Lemma 8.4. \hfill \square

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