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Dirac-Kähler equation in curved space-time,
relation between spinor and tensor formulations†

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A common view is that generalization of a wave equation on Riemannian space-time is substantially determined by what a particle is – boson or fermion. As a rule, they say that tensor equations for bosons are extended in a simpler way then spinor equations for fermions. In that context, a very interesting problem is of extension a wave equation for Dirac–Kähler field (Ivanenko–Landau field was historically first term, also the term a vector field of general type was used).

The article relates a generally covariant tensor formalism to a spinor one when these both are applied to description of the Dirac-Kähler field in a Riemannian space-time. Both methods are taken to be equivalent and the tensor equations are derived from spinor ones. It is shown that, for characterization of Dirac-Kähler’s tensor components, two alternative approaches are suitable: these are whether a tetrad-based pseudo tensor classification or a generally coordinate pseudo tensor one. By imposing definite restrictions on the the Dirac-Kähler function, we have produced the general covariant form of wave equations for scalar, pseudoscalar, vector, and pseudovector particles.

1 Introduction

Mathematical description of the concept of elementary particles as certain relativistically invariant objects was found in the frames of 4-dimensional Minkowski space-time. It is assumed that for any particle there are given definite transformation properties of a corresponding field and a wave equation to which that field obeys; wave equation must be Lorentz (or Poincaré) invariant: Wigner [1], Pauli [2], Bhabha [3], Harish-Chandra [4], Gel’fand – Yaglom [5], Corson [6], Umezawa [7], Shirokov [8], Bogush – Moroz [9], Fedorov [10]).

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A common view is that generalization of a wave equation on Riemannian space-time is substantially determined by what a particle is – boson or fermion. As a rule, they say that tensor equations for bosons are extended in a simpler way than spinor equations for fermions. This believing evidently correlates with the fact: concepts of both flat and curved space model are based on the notion of a vector.

In that context, a very interesting problem is of extension a wave equation for Dirac–Kähler field (there are used other terms as well: Ivanenko—Landau field, or a vector field of general type).

Scientific literature consecrated with this field is enormous, it started early in the development of quantum mechanical wave equations theory, just after the concept of a particle with spin 1/2 arises. In particular, news objects themselves, spinors, seemed mysterious and obscure in comparison with familiarized tensors.

The main feature of the Ivanenko—Landau field [11] was that it seemingly gave possibility to perform smoothly transition from tensors to spinors, in a sense it was an attempt to eliminate spinors at all. Different aspect of that relation were investigated by many authors: , [11], Lanczos [12, 13], Juvet [14, 15], Einstein – Mayer [16, 17, 18, 19], Frenkel [20], Whittaker [21], Proca [22], Ruse [23], Taub [24, 25], Belinfante [26, 27], Ivanenko – Sokolov [28], Feshbach – Nikols [29], Kähler [30, 31], Leutwyler [32], Klauder [33], Penney [34], Cereignani [35], Streater – Wilde [36], Pestov [37, 38, 39], Osterwalder [40], Crumeyrolle [41], Durand [42], Strazhev et al [43, 44, 45, 46, 78, 48, 49, 50, 51, 52, 53, 54], Graf [55], Benn – Tucker [56, 57, 58, 59, 60, 61], Banks et al [62], Garbaczewski [63], letjuxov – Strazhev [64, 65, 66, 67], Holland [68], Ivanenko et al [69, 70], Bullinaria [71], Blau [72], Jourjine [73], Krolkowski [74, 75], Howe [76], Nikitin et al [77, 78, 79, 80, 81], Marchuk [82, 83, 84, 85, 86, 87, 88], Krivskij et al [89].

Three most interesting points in connection of general covariant extension of the wave equation for this field are: in flat Minkowski space there exist tensor and spinor formulations of the theory; in the initial tensor form there are presented tensors with different intrinsic parities; there exist different views about physical interpretation of the object: whether it is a composite boson or a set of four fermions. These three point will be of primary importance in the treatment below.

2 Spinor and tensor forms of the wave equation

In Minkowski space-time, the Dirac–Kähler field is described by 16-component wave function with transformation properties of 2-rank 4-bispinor $U(x)$ or by equivalent set of elementary tensor constituents

$$U(x) \quad \text{or} \quad \{ \Psi(x), \Psi_i(x), \tilde{\Psi}(x), \tilde{\Psi}_i(x), \Psi_{mn}(x) \} ,$$

where $\Psi(x)$ is a scalar; $\Psi_i(x)$ is a vector; $\tilde{\Psi}(x)$ is a pseudoscalar; $\tilde{\Psi}_i(x)$ represents a pseudovector; $\Psi_{mn}(x)$ is an anti-symmetric tensor. Correspondingly, we have two representations for the wave equation

$$[i \gamma^a \partial_a - m] U(x) = 0 ,$$

and

$$\partial_l \Psi + m \Psi_l = 0 , \quad \partial_l \tilde{\Psi} + m \tilde{\Psi}_l = 0 , \quad \partial_l \Psi + \partial_a \Psi_{la} - m \Psi_l = 0 ,$$

$$\partial_l \tilde{\Psi} - \frac{1}{2} \epsilon_l^{amn} \partial_a \Psi_{mn} - m \tilde{\Psi}_l = 0 ,$$
\[ \partial_m \psi_n - \partial_n \psi_m + \epsilon_{mn}^{ab} \partial_a \bar{\psi}_b - m \psi_{mn} = 0. \]  
(2.2)

Let us detail relation between 2-rank bispinor \( U(x) \) and corresponding tensors. It is well known that any \((4 \times 4)\)-matrix can be expanded on 16 Dirac matrices; and for that expanding it does not matter whether the matrix \( U \) is a 2-rank bispinor or not. However, if it is so, coefficients arise arising \{\psi, \psi_t, \bar{\psi}, \bar{\psi}_t, \psi_{mn}\} will posses quite definite tensorial properties with respect to the Lorentz group. Let such a 2-rank bispinor \( U(x) \) is parameterized according to

\[ U(x) = \left[ -i \psi + \gamma^l \psi_t + i \sigma^{mn} \psi_{mn} + \gamma^5 \bar{\psi} + i \gamma^5 \bar{\psi}_t \right] E^{-1}; \]  
(2.3a)

here \( E \) stands for a metrical bispinor matrix with simple properties

\[ E = \begin{vmatrix} \epsilon & 0 \\ 0 & \epsilon^{-1} \end{vmatrix}, \quad E^2 = -I, \quad \bar{E} = -E, \quad \text{Sp} E = 0, \quad \bar{\sigma}^{ab} E = -E \sigma^{ab}. \]  
(2.3b)

Inverse to (2.3a) relations are

\[ \psi(x) = -\frac{1}{4i} \text{Sp} \left[ E U(x) \right], \quad \bar{\psi}(x) = \frac{1}{4} \text{Sp} \left[ E\gamma^5 U(x) \right], \]
\[ \psi_t(x) = \frac{1}{4i} \text{Sp} \left[ E\gamma_t U(x) \right], \quad \bar{\psi}_t(x) = \frac{1}{4i} \text{Sp} \left[ E\gamma^5\gamma_t U(x) \right], \]
\[ \psi_{mn}(x) = -\frac{1}{2i} \text{Sp} \left[ E\sigma_{mn} U(x) \right]. \]  
(2.3c)

Below we will use also a 2-component spinor formalism; to this end, it sufﬁces to choose Dirac matrices in spinor Weyl basis and specify additionally notation for constituents of \( U(x) \):

\[ U(x) = \begin{vmatrix} \xi_{\alpha\beta}(x) & \Delta_{\alpha\beta}(x) \\ H_{\alpha\beta}(x) & \eta_{\alpha\beta}(x) \end{vmatrix}. \]  
(2.4a)

Thus, instead of (2.3a) we obtain

\[ \Delta(x) = [ \psi_t(x) + i \bar{\psi}_t(x) ] \sigma^{-1}, \quad H(x) = [ \psi_t(x) - i \bar{\psi}_t(x) ] \bar{\sigma}^{l} \sigma^{-1}, \]  
(2.4b)

\[ \xi(x) = [ -i \psi(x) - \bar{\psi}(x) + i \Sigma^{mn} \psi_{mn}(x) ] \sigma^{-1}, \]
\[ \eta(x) = [ -i \psi(x) + \bar{\psi}(x) + i \Sigma^{mn} \psi_{mn}(x) ] \sigma^{-1}, \]

and inverse relations

\[ \psi_t(x) + i \bar{\psi}_t(x) = \frac{1}{2} \text{Sp} \left[ \sigma^{-1} \sigma_t \Delta(x) \right], \quad \psi_t(x) - i \bar{\psi}_t(x) = \frac{1}{2} \text{Sp} \left[ \sigma \bar{\sigma}_t H(x) \right], \]
\[ -i \psi(x) - \bar{\psi}(x) = \frac{1}{2} \text{Sp} \left[ \sigma \xi(x) \right], \quad -i \psi(x) + \bar{\psi}(x) = \frac{1}{2} \text{Sp} \left[ \sigma^{-1} \xi(x) \right], \]
\[ -i \psi^{kl}(x) + \frac{1}{2} \epsilon^{klmn} \psi_{mn}(x) = \text{Sp} \left[ \sigma \Sigma^{kl} \xi(x) \right], \]
\[ -i \psi^{kl}(x) - \frac{1}{2} \epsilon^{klmn} \psi_{mn}(x) = \text{Sp} \left[ \sigma^{-1} \Sigma^{kl} \xi(x) \right]. \]  
(2.4c)
Dirac–Kähler equation in 2-spinor form looks as follows
\[ i\sigma^a \partial_a \xi(x) = m H(x) , \quad i\sigma^a \partial_a H(x) = m \xi(x) , \]
\[ i\tilde{\sigma}^a \partial_a \eta(x) = m \Delta(x) , \quad i\tilde{\sigma}^a \partial_a \Delta(x) = m \eta(x) . \] (2.5)

Now let us consider a general covariant form. First, we turn to the 4-spinor approach – according to the well known recipe by Tetrode–Weyl–Fock–Ivanenko eq. (2.1) should be changed into
\[ [ i\gamma^\alpha(x) (\partial_\alpha + B_\alpha(x)) - m ] U(x) = 0 ; \] (2.6)
connection \( B_\alpha(x) \) is defined by
\[ B_\alpha(x) = \frac{1}{2} J^{a\beta} e_{(a)}^\beta(x) \nabla_\alpha e_{(b)\beta}(x) = \Gamma_\alpha(x) \otimes I + I \otimes \Gamma_\alpha(x) , \]
where \( J^{a\beta} = [\sigma^{ab} \otimes I + I \otimes \sigma^{ab}] \) stand for generators for bispinor representation of the Lorentz group. From (2.6) it follow 2-spinor form of equations for Dirac–Kähler field
\[ i\sigma^a (\partial_a + \Sigma_\alpha(x) \otimes I + I \otimes \Sigma_\alpha(x)) \zeta(x) = m H(x) , \]
\[ i\tilde{\sigma}^a (\partial_a + \Sigma_\alpha(x) \otimes I + I \otimes \Sigma_\alpha(x)) H(x) = m \xi(x) , \]
\[ i\tilde{\sigma}^a (\partial_a + \Sigma_\alpha(x) \otimes I + I \otimes \Sigma_\alpha(x)) \eta(x) = m \Delta(x) , \]
\[ i\tilde{\sigma}^a (\partial_a + \Sigma_\alpha(x) \otimes I + I \otimes \Sigma_\alpha(x)) \Delta(x) = m \eta(x) . \] (2.7)

Eqs. (2.6) and (2.7) posse symmetry with respect to local Lorentz group: if \( U(x) \) is subject to local Lorentz transformation
\[ U'(x) = [ S(k(x), k^*(x)) \otimes S(k(x), k^*(x)) ] U(x) , \] (2.8a)
then the new field function \( U'(x) \), or set of new 2-spinors \( [ \xi'(x), \eta'(x), \Delta'(x), H'(x) ] \), will obey a wave equation of the same type as before
\[ [ i\gamma'^\alpha(x) (\partial_\alpha + B'_\alpha(x)) - m ] U'(x) = 0 , \] (2.8b)
where primed \( \gamma'^\alpha(x) \) and \( B'_\alpha(x) \) are constructed with the help of primed tetrad \( e'^\alpha_{(b)}(x) \), related to the initial one by Local Lorentz transformation
\[ e'^\alpha_{(b)}(x) = L_b^a(k(x), k^*(x)) e^\alpha_{(a)}(x) . \]
This symmetry prove correctness of the equation under consideration: the symmetry describes a gauge freedom in choosing an explicit form of the tetrad.

In addition, there exists discrete symmetry. Indeed, if \( U(x) \) is subject to the following discrete operation
\[ U'(x) = [ i\gamma^0 \otimes i\gamma^0 ] U(x) \quad \text{or} \quad \begin{bmatrix} \xi'(x) & \Delta'(x) \\ H'(x) & \eta'(x) \end{bmatrix} = \begin{bmatrix} -\eta(x) & -H(x) \\ -\Delta(x) & -\xi(x) \end{bmatrix} , \] (2.9a)
then the new wave function \( U'(x) \) (new set of 2-spinors) will obey an equation of the same form (2.6) or (2.7)), but now constructed on the base of a tetrad \( e'^\alpha_{(b)}(x) \), -reflected to the initial
\[ e'^\alpha_{(b)}(x) = L_b^{(p)a} e^\alpha_{(a)}(x) , \quad L_b^{(p)a} = \text{diag} (+1, -1, -1, -1) . \] (2.9b)
With respect to general coordinate transformations, the wave function $U(x)$ behaves as a scalar (similarly as a wave function $\Psi(x)$ in the Dirac equation does) Correspondingly, the term $\partial_a U(x)$ represents a general covariant vector and eq. (2.6) is correct in the sense of general covariance.

Now we turn to extending the tensor form of equations (2.2). We face here a rather specific problem. Indeed, a formal change

$$\partial_t \implies \nabla_\alpha, \quad \xi_i(x) \implies \Psi_\alpha(x),$$

leads to appearing some vagueness: it is not clear how we should distinguish between two functions $\Psi_\alpha(x)$ and $\tilde{\Psi}_\alpha(x)$ – because they have one the same index $\alpha$, the sign of covariant vector. Nevertheless, making such a formal generalization we get the system

$$\nabla^\alpha \Psi_\alpha(x) + m \Psi_\alpha(x) = 0, \quad \nabla^\alpha \tilde{\Psi}_\alpha(x) + m \tilde{\Psi}_\alpha(x) = 0,$$

$$\nabla_\alpha \Psi_\alpha(x) + \nabla^\beta \Psi_\alpha\beta(x) - m \Psi_\alpha(x) = 0,$$

$$\nabla_\alpha \tilde{\Psi}_\alpha(x) - \nabla^\beta \Psi_\alpha\beta(x) + \epsilon_\alpha^\beta\rho\sigma(x) \nabla^\rho \tilde{\Psi}_\sigma(x) - m \tilde{\Psi}_\alpha(x) = 0.$$  

(2.10)

Resolving the problem of distinguishing between $\Psi_\alpha(x)$ and $\tilde{\Psi}_\alpha(x)$, also $\Psi(x)$ and $\tilde{\Psi}(x)$, also determining a covariant Levi-Civita object, can be found for comparing eq. (2.10) with eq. (2.6).

We will demonstrate that from eq. (2.6) it follows eqs. (2.10), if instead of $U(x)$ in (2.6) we substitute expansion of the matrix $U(x)$ in terms of tetrad tensor constituents and then translate equations to covariant tensors according to

$$\Psi_\alpha(x) = e^{(i)}_\alpha(x) \xi_i(x), \quad \tilde{\Psi}_\alpha(x) = e^{(i)}_\alpha(x) \tilde{\xi}_i(x),$$

$$\Psi_\alpha\beta(x) = e^{(m)}_\alpha(x) e^{(n)}_\beta(x) \Psi_{mn}(x),$$  

(2.11a)

and a covariant Levi–Civita object is defined as follows

$$\epsilon^{\alpha\beta\rho\sigma}(x) = \epsilon^{abcd} e^{(a)}_\alpha(x) e^{(b)}_\beta(x) e^{(c)}_\rho(x) e^{(d)}_\sigma(x).$$  

(2.11b)

At this we note that the relevant similar functions entering eqs. (2.10) differ in their transformation properties with respect to tetrad $P$-reflection: $\Psi(x)$, $\Psi_\alpha(x)$, $\Psi_\alpha\beta(x)$ are tetrad scalars; $\tilde{\Psi}(x)$, $\tilde{\Psi}_\alpha(x)$ are tetrad pseudoscalars.

Let us explain calculations proving this. First, eq. (2.6) is written in the form (the symbol $\sim$ designates matrix transposition)

$$[i \gamma^\alpha \partial_\alpha U + i \gamma^\alpha \Gamma_\alpha(x) U + i \gamma^\alpha \tilde{\Gamma}_\alpha U \tilde{\Gamma}_\alpha - m U] = 0,$$  

(2.12a)

then eq. (2.12a) is translated to

$$[i \gamma^c e^{(c)}_\alpha \partial_\alpha U + i \gamma_{abc} \gamma^c \sigma^{ab} U + i \gamma_{abc} \gamma^c U \tilde{\sigma}^{ab} - m U] = 0.$$  

(2.12b)
Further, into (2.12b) we substitute expansion for $U$ in term of local tetrad tensors

$$
\begin{align}
&\{i\gamma^\alpha e_\alpha^\beta \partial_\beta [ -i\Psi + \gamma^I \Psi_I + i\sigma^{mn} \Psi_{mn} + \gamma^5 \tilde{\Psi} + i\gamma^5 \tilde{\Psi}_I ] E^{-1} \\
+ \frac{i}{2} \gamma_{abc} \gamma^c \sigma^{ab} [ -i\Psi + \gamma^I \Psi_I + i\sigma^{mn} \Psi_{mn} + \gamma^5 \tilde{\Psi} + i\gamma^5 \tilde{\Psi}_I ] E^{-1} \\
+ \frac{i}{2} \gamma_{abc} \gamma^c [ -i\Psi + \gamma^I \Psi_I + i\sigma^{mn} \Psi_{mn} + \gamma^5 \tilde{\Psi} + i\gamma^5 \tilde{\Psi}_I ] E^{-1} \\
- m [ -i\Psi + \gamma^I \Psi_I + i\sigma^{mn} \Psi_{mn} + \gamma^5 \tilde{\Psi} + i\gamma^5 \tilde{\Psi}_I ] E^{-1} \} = 0 .
\end{align}
$$

(2.12c)

Now, acting subsequently from the left by operators

$$\text{Sp} (E \times , \text{Sp} (E \gamma^5 \times , \text{Sp} (E \gamma^k \times , \text{Sp} (E \gamma^5 \gamma^k \times , \text{Sp} (E \sigma^{kd} \times \text{Sp} (E \gamma^5 \times ,
$$

and using known formulas for traces of relevant combinations of Dirac matrices we arrive at

$$
e^{(l)\alpha} \partial_\alpha \psi_I + \gamma^c_{\alpha c} \Psi^I + m \Psi = 0 ,
$$

$$
e^{(l)\alpha} \partial_\alpha \tilde{\psi}_I + \gamma^c_{\alpha c} \tilde{\Psi}^I + m \tilde{\Psi} = 0 ,
$$

$$
e^{(k)\alpha} \partial_\alpha \Psi + e^{(c)}_\alpha \partial_\alpha \Psi_{kc} + \gamma^k_{mn} \Psi_{mn} + \gamma^c_{\alpha c} \Psi_{kl} - m \Psi^I = 0 ,
$$

$$
e^{(k)\alpha} \partial_\alpha \Psi - \frac{1}{2} e^{kmn} e^{(c)}_\alpha \partial_\alpha \Psi_{mn} + e^{kmn} \gamma^c_{bc} \Psi_{mn} - m \Psi_{kl} = 0 ,
$$

$$
e^{(d)\alpha} \partial_\alpha \Psi^k - e^{(k)\alpha} \partial_\alpha \Psi^d + (\gamma^l_{dk} - \gamma^l_{kd}) \Psi^I +
$$

$$+ e^{dkl} e^{(c)}_\alpha \partial_\alpha \Psi_I + e^{abcd} \gamma_{b c e \Psi^b} - m \Psi^I = 0 .
$$

(2.12d)

they represent written in tetrad components (2.11a, b) eqs. (2.10).

One important point should be specially emphasized: during calculation, a Levi–Civita object $e^{abcd}$ arose in (2.12d) as a direct result of the use of a trace formula for product of three Dirac matrices, so this quantity $e^{abcd}$ is not a tensor with respect to the lorentz group, it is rather just a fixed 4-index object.

It is readily to demonstrate that the combination

$$
\epsilon^{\alpha \beta \rho \sigma} (x) = e^{abcd} e^{\alpha}_{(a)} (x) e^{\beta}_{(b)} (x) e^{\rho}_{(c)} (x) e^{\sigma}_{(d)} (x)
$$

(2.13a)

represents a tetrad pseudoscalar. Indeed, let us compare $\epsilon^{\alpha \beta \rho \sigma} (x)$ and $\epsilon^{\alpha \beta \rho \sigma} (x)$, constructed on the base of tetrads $e^{\alpha}_{(a)} (x)$ and $e^{\alpha}_{(a)} (x)$ respectively. We have

$$
\epsilon^{\alpha \beta \rho \sigma} (x) = e^{abcd} e^{\alpha}_{(a)} (x) e^{\beta}_{(b)} (x) e^{\rho}_{(c)} (x) e^{\sigma}_{(d)} (x) ,
$$

or

$$
\epsilon^{\alpha \beta \rho \sigma} (x) = e^{abcd} L_a^i (x) L_b^j (x) L_c^m (x) L_d^n (x) e^{\alpha}_{(i)} (x) e^{\beta}_{(j)} (x) e^{\rho}_{(m)} (x) e^{\sigma}_{(n)} (x) =
$$

$$
\left[ e^{abcd} L_a^{(x)} (x) L_b^{(x)} (x) L_c^{(x)} (x) L_d^{(x)} (x) \right] e^{(i)\alpha} (x) e^{(j)\beta} (x) e^{(m)\rho} (x) e^{(n)\sigma} (x) .
$$
With the use of the known identity \( \epsilon^{abcd} A_{ai} A_{bj} A_{cm} A_{dn} = - \det[A_{ab}] \times \epsilon_{ijmn}, \)

we get a transformation law for \( \epsilon^{\alpha\beta\rho\sigma}(x) \) with respect to local tetrad transformations:

\[
e^{\alpha\beta\rho\sigma}(x) = - \det[L_{ai}(x)] \epsilon^{\alpha\beta\rho\sigma}(x).
\]

(2.13b)

From (2.13b) it follows that under tetrad \( P \)-reflection covariant Levi-Civita object (2.13a) behaves as a tetrad pseudoscalar

\[
e^{(p)\alpha\beta\rho\sigma}(x) = (-1) \epsilon^{\alpha\beta\rho\sigma}(x).
\]

(2.13c)

One can notice that in each equation in (2.10), there are combined terms with equal transformation properties with respect to the tetrad \( P \)-reflection.

The system (2.10) can be translated to the form in which all the component of the wave function are tetrad scalars:

\[
\Phi(x) = \{ \Psi(x), \Psi_{\alpha}(x), \Psi_{\alpha\beta}(x), \Psi_{\alpha\beta\rho}(x) = \epsilon_{\alpha\beta\rho\sigma}(x) \tilde{\Psi}^\sigma(x), \Psi_{\alpha\beta\rho\sigma}(x) = \epsilon_{\alpha\beta\rho\sigma}(x) \tilde{\Psi}(x) \}.
\]

(2.14a)

then the Dirac–Kähler equation reads

\[
\nabla^\rho \Psi_{\rho} - m \Psi = 0, \\
\nabla^\rho \Psi_{\rho\alpha} + \nabla_{\alpha} \Psi + m \Psi_{\alpha} = 0, \\
\nabla^\rho \Psi_{\rho\alpha\beta} + \nabla_{\alpha} \Psi_{\beta} - \nabla_{\beta} \Psi_{\alpha} - m \Psi_{\alpha\beta} = 0, \\
\nabla^\rho \Psi_{\rho\alpha\beta\sigma} + \nabla_{\alpha} \Psi_{\beta\sigma} - \nabla_{\beta} \Psi_{\alpha\sigma} - \nabla_{\sigma} \Psi_{\beta\alpha} + m \Psi_{\alpha\beta\sigma}(x) = 0, \\
\nabla^\rho \Psi_{\rho\alpha\beta\sigma} - \nabla_{\alpha} \Psi_{\rho\beta\sigma} - \nabla_{\beta} \Psi_{\rho\alpha\sigma} - \nabla_{\sigma} \Psi_{\alpha\beta\rho} - m \Psi_{\rho\alpha\beta\sigma} = 0.
\]

(2.14b)

Deriving (2.14b) from (2.10), one should take into account that covariant derivative of the covariant Levi–Civita tensor vanishes identically

\[
\nabla_{\mu} \epsilon^{\alpha\beta\rho\sigma}(x) = 0.
\]

(2.15)

Let us prove it. By symmetry reason, it suffices to prove only one relation \( \nabla_{\mu} \epsilon_{0123}(x) = 0 \). In accordance with definition we have

\[
\nabla_{\mu} \epsilon_{0123}(x) = \left[ \partial_{\mu} \epsilon_{0123}(x) - \left( \Gamma_{\mu0}^{\nu} \epsilon_{\nu123}(x) + \Gamma_{\mu1}^{\nu} \epsilon_{0\nu23}(x) \right) \right. \\
\left. + \Gamma_{\mu2}^{\nu} \epsilon_{01\nu3}(x) + \Gamma_{\mu3}^{\nu} \epsilon_{012\nu}(x) \right]\right] = \partial_{\mu} \epsilon_{0123}(x) - \Gamma_{\mu\alpha}^{\nu} \epsilon_{0123}(x).
\]

Let us specify the first term \( \partial_{\mu} \epsilon_{0123}(x) \), where

\[
\epsilon_{0123}(x) = - \epsilon_{0123} \det[e_{(a)\alpha}(x)];
\]

with the use of the known identity \( \partial_{\mu} A = A (A_{ij}^{-1} \partial_{\mu} A_{ij}), \quad A = \det[A_{ij}] \)
and allowing for that the inverse to \( e(a)\alpha \) is a matrix \( e^{\beta(b)} \), we get

\[
e(x) = \det[e(a)\alpha(x)], \quad \partial_\mu e(x) = e(x) e^{\alpha(a)}(x) \partial_\mu e(a)\alpha(x).
\]

Therefore,

\[
\partial_\mu \epsilon_{0123}(x) = \epsilon_{0123}(x) \left[ e^{\alpha(a)}(x) \partial_\mu e(a)\alpha(x) \right].
\]

In turn, for \( \Gamma^\alpha_{\mu\alpha}(x) \) we have

\[
\Gamma^\alpha_{\mu\alpha}(x) = \frac{1}{2} g^{\rho\sigma}(x) \Gamma_{\rho,\mu\alpha}(x) = \frac{1}{2} g^{\rho\sigma}(x) \left[ \partial_\mu g_{\rho\alpha}(x) + \partial_\alpha g_{\rho\mu}(x) + \partial_\rho g_{\mu\alpha}(x) \right]
\]

\[
= \frac{1}{2} g^{\alpha\beta} \left[ \partial_\mu \left( e(i)\rho(x) e^{(i)}(x) \right) + \partial_\alpha \left( e(i)\rho(x) e^{(i)}(x) \right) + \partial_\rho \left( e(i)\mu(x) e^{(i)}(x) \right) \right],
\]

from whence after simple calculation we derive

\[
\Gamma^\alpha_{\mu\alpha}(x) = e^{\alpha(i)}(x) \partial_\mu e^{(i)}(x).
\]

Thus, we prove the needed identity

\[
\nabla_\mu e^{\alpha\beta\rho\sigma}(x) = 0.
\]

3 On two different covariant Levi-Civita objects

Let us recall a standard view on covariant Levi-Civita object – it is defined \([90]\) as follows

\[
E_{\alpha\beta\rho\sigma}(x) \equiv +\sqrt{-g(x)} \epsilon_{\alpha\beta\rho\sigma}, \quad E^{\alpha\beta\rho\sigma}(x) \equiv \frac{1}{+\sqrt{-g(x)}} \epsilon^{\alpha\beta\rho\sigma},
\]

\[
E^{\alpha\beta\rho\sigma}(x) = g^{\alpha\mu}(x) g^{\beta\nu}(x) g^{\rho\gamma}(x) g^{\sigma\delta}(x) E_{\mu\nu\gamma\delta}(x), \quad (3.1a)
\]

where \( g(x) \) is a determinant of a metric tensor \( g_{\alpha\beta}(x) \); and \( E_{0123}(x) = +\sqrt{-g(x)} \). This definition does not depend on tetrads at all, which means that \( E_{\alpha\beta\rho\sigma}(x) \) is a tetrad scalar. To have the covariant Levi-Civita object invariant with respect to arbitrary coordinate changes we must assume that the object \( E_{\alpha\beta\rho\sigma}(x) \) transform as a pseudotensor, that is we add in relevant transformation law an additional a special factor sgn \( \Delta(x) \)

\[
\text{sgn } \Delta(x) = \frac{\Delta(x)}{\left| \Delta(x) \right|}, \quad \Delta(x) \equiv \det \left[ \frac{\partial x^\alpha}{\partial x'^\alpha} \right],
\]

\[
E_{\alpha'\beta'\rho'\sigma'} = \frac{\Delta(x)}{\left| \Delta(x) \right|} \frac{\partial x'^\alpha}{\partial x^\alpha} \frac{\partial x'^\beta}{\partial x^\beta} \frac{\partial x'^\rho}{\partial x^\rho} \frac{\partial x'^\sigma}{\partial x^\sigma} E_{\alpha\beta\rho\sigma}. \quad (3.1b)
\]

Above, in the frames of the tetrad formalism, the quantity \( \epsilon_{\alpha\beta\rho\sigma} \) was introduced by (2.11b); so it is an ordinary covariant tensor with 4 indices and in the same time it is a tetrad pseudoscalar. Two objects, \( \epsilon_{\alpha\beta\rho\sigma}(x) \) and \( E_{\alpha\beta\rho\sigma} \) were defined independently from each other, therefore they may not coincide. However, quite definite relation between them exists, let us detail this point.

First of all, let us transform the tetrad based Levi-Civita tensor \( \epsilon_{\alpha\beta\rho\sigma}(x) \) to a different form similar to (3.1a):

\[
\epsilon_{\alpha\beta\rho\sigma}(x) = e^{abcd} e(a)\alpha(x) e(b)\beta(x) e(c)\rho(x) e(d)\sigma(x) = - e(x) \epsilon_{\alpha\beta\rho\sigma},
\]

8
\( e(x) \equiv \det \left[ e_{(a)\alpha}(x) \right] . \)  

(3.2)

For instance, in the case of flat Minkowski space, using a diagonal tetrad \( e_{(a)\alpha}^\alpha(x) = \delta_a^\alpha \), we get \( e(x) = -1 \), and further derive \( \epsilon_{\alpha\beta\rho\sigma}(x) = +\epsilon_{\alpha\beta\rho\sigma} \).

It is easy to obtain relation relating determinants of the tetrad and metric tensor

\[
e_{(a)\alpha}(x) \ e_{(b)\beta}(x) \ g^{ab} = g_{\alpha\beta}(x) \implies -e^2(x) = g(x),
\]

from whence it follows

\[
e(x) = +\sqrt{-g(x)} \quad \text{and} \quad e(x) = -\sqrt{-g(x)}.
\]

Taking solution as \( e(x) = -\sqrt{-g(x)} \), we arrive at the tetrad based definition for Levi-Civita tensor (2.3); so it is equivalent to definition according to (3.1a). However, a tetrad determinant can be positive as well, in this case two definition are not equivalent – they differ in sign.

Note a useful formula

\[
\epsilon_{\mu\nu\rho\sigma}(x) = -e(x) \ \det \left[ g_{\alpha\beta}(x) \right] \epsilon_{\alpha\beta\rho\sigma} = +\epsilon(x) \ \frac{1}{g(x)} \ e^{\mu\nu\rho\sigma} = -\frac{1}{e(x)} \ e^{\mu\nu\rho\sigma}.
\]  

(3.3)

Let us specify transformation properties for \( e(x) = \det \left[ e_{(a)\alpha}(x) \right] \). Under general coordinate transformations it behaves

\[
x^{\alpha} \implies x'^{\alpha} , \ e'(x') = \det \left[ \frac{\partial x'^{\alpha}}{\partial x^{\alpha}} \right] e_{(a)\alpha}(x) = \frac{1}{\Delta(x)} e(x) ;
\]  

(3.4a)

with respect to tetrad changes it is a pseudoscalar

\[
e_{(a)\alpha}(x) \implies e'_{(a)\alpha}(x) , \ e'(x) = \det \left[ L_a^b e_{(b)\alpha}(x) \right] = \det \left[ L_a^b \right] e(x) .
\]  

(3.4b)

Let us introduce special quantity

\[
J(e) = -\frac{\det \left[ e_{(a)\alpha}(x) \right]}{\det \left[ e_{(a)\alpha}(x) \right]} = -\frac{e(x)}{e(x)}.
\]  

(3.5a)

which transforms as follows

\[
x^{\alpha} \implies x'^{\alpha} , \ J[e'(x')] = \frac{\Delta(x)}{\Delta(x)} J[e(x)] ,
\]  

(3.5b)

\[
e_{(a)\alpha}(x) \implies e'_{(a)\alpha}(x) , \ J[e'(x')] = \det \left[ L_a^b \right] J[e(x)] ;
\]  

(3.5c)

\( J[e(x)] \) is a tetrad pseudoscalar, and a coordinate pseudoscalar

Collecting results together

\[
\epsilon_{\alpha\beta\rho\sigma}(x) = -e \ \epsilon_{\alpha\beta\rho\sigma} ,
\]

\[
E_{\alpha\beta\rho\sigma}(x) = \sqrt{-g} \ \epsilon_{\alpha\beta\rho\sigma} ,
\]

\[
e^2 = -g , \quad +\sqrt{-g} = \frac{e}{|e|} e ,
\]
we readily find relation between two Levi-Civita tensors
\[ E_{\alpha\beta\rho\sigma}(x) = \frac{e}{|e|} \epsilon_{\alpha\beta\rho\sigma}(x) = \frac{e}{|e|} \epsilon_{\alpha\beta\rho\sigma}(x) = J[e(x)] \epsilon_{\alpha\beta\rho\sigma}(x) . \]

Let us turn back to (2.11a). Instead of \( \tilde{\Psi}(x), \tilde{\Psi}_\alpha(x) \) one can introduce new variables
\[ \bar{\Psi}(x) = J(e) \tilde{\Psi}(x), \quad \bar{\Psi}_\alpha(x) = J(e) \epsilon^{(a)}_\alpha(x) \tilde{\Psi}_a(x) , \] (3.6a)
and instead of \( \epsilon_{\alpha\beta\rho\sigma}(x) \) (2.11b) one may determine another quantity
\[ E_{\alpha\beta\rho\sigma}(x) = J[e(x)] \epsilon_{\alpha\beta\rho\sigma}(x) . \] (3.6b)

Correspondingly, the main system of tensor equations can be presented as follows (compare with (2.10))

\[ \nabla^\alpha \Psi_\alpha + m \Psi = 0 , \]
\[ \nabla^\alpha \bar{\Psi}_\alpha + m \bar{\Psi} = 0 , \]
\[ \nabla_\alpha \Psi + \nabla^\beta \Psi_{\alpha\beta} - m \Psi_\alpha = 0 , \]
\[ \nabla_\alpha \bar{\Psi} - \frac{1}{2} E_{\alpha\beta\rho\sigma}(x) \nabla^\beta \Psi_{\rho\sigma} - m \bar{\Psi}_\alpha = 0 , \]
\[ \nabla_\alpha \bar{\Psi}_\beta - \nabla_\beta \bar{\Psi}_\alpha + E_{\alpha\beta\rho\sigma}(x) \nabla^\rho \bar{\Psi}_\sigma - m \bar{\Psi}_{\alpha\beta} = 0 . \] (3.7)

Here \( \Psi(x), \Psi_\alpha(x), \Psi_{\alpha\beta}(x) \) are general covariant tensors, whereas \( \bar{\Psi}(x), \bar{\Psi}_\alpha(x), E^{\rho\sigma\alpha\beta}(x) \) are general covariant pseudotensors; all six objects are tetrad scalars.

Thus, when describing tensor components for Dirac–Kähler field one can use alternatively both methods. Evidently, classification of the components through their tetrad properties is more preferable because it has clear Lorentzian status (as spin and mass).

It should be noted additionally that classification for tensor quantities in the frames of the full Lorentz group within Minkowski space-time assumes four different possibilities distinguished by adding special factors in transformation low
\[ 1 , \quad \det(L^a_b) , \quad \text{sgn}(L^0_0) , \quad \text{sgn}(L^0_0) \det(L^a_b) . \]

It is not clear how that Lorentz group based classification can be described in terms of a pure general covariant theory without tetrad formalism.

### 4 On fermion interpretation for Dirac–Kähler field

The Dirac–Kähler equation in arbitrary curved space-time
\[ [ i \gamma^\alpha(x) \left( \partial_\alpha + B_\alpha(x) \right) - m ] U(x) = 0 . \] (4.1a)
does not split up into four independent equations for particles with spin 1/2:
\[ [ i \gamma^\beta(x) \left( \partial_\beta + \Gamma_\beta(x) \right) - m ] \Psi^{(i)}(x) = 0 , \quad (i = 1, 2, 3, 4) ; \] (4.1b)
In other words, these two models are completely different in any curved space-time model.

Let us consider eqs. (4.1b) in more detail. Relevant four local bispinor fields can be developed into (4 × 4)-matrix \( V(x) \) according to \( V(x) = (\Psi^{(1)}, \Psi^{(2)}, \Psi^{(3)}, \Psi^{(4)}) \); then eqs. (4.1b) read

\[
[i\gamma^\alpha(x)(\partial_\alpha + \Gamma_\alpha(x)) - m]V(x) = 0 .
\]  

(4.2a)

Matrix \( V(x) \) can be decomposed as

\[
V(x) = [-i\Phi(x) + \gamma^l\tilde{\Phi}_l(x) + i\sigma^{mn}\Phi_{mn}(x) + \gamma^5\tilde{\Phi}(x) + i\gamma^l\gamma^5\tilde{\Phi}_l(x)]E^{-1} ;
\]  

(4.2b)

however involved quantities \( \Phi(x) \), \( \tilde{\Phi}(x) \), \( \Phi_l(x) \), \( \Phi_{mn}(x) \) do not posses transformation properties of tensor nature with respect to the local Lorentz group. In the same time, some quasi-tensor equations can be derived from (4.2a). To this end, one should act in the same manner as above. For instance, turning to (2.12c)

\[
\{i\gamma^c\epsilon^\beta\partial_\beta [ -i\Psi + \gamma^l\Psi_l + i\sigma^{mn}\Psi_{mn} + \gamma^5\tilde{\Psi} + i\gamma^l\gamma^5\tilde{\Psi}_l ] E^{-1} +
\]

\[
i\frac{1}{2}\gamma_{abc}\gamma^{ab} [ -i\Psi + \gamma^l\Psi_l + i\sigma^{mn}\Psi_{mn} + \gamma^5\tilde{\Psi} + i\gamma^l\gamma^5\tilde{\Psi}_l ] E^{-1} +
\]

\[
i\frac{1}{2}\gamma_{abc}\gamma^c [ -i\Psi + \gamma^l\Psi_l + i\sigma^{mn}\Psi_{mn} + \gamma^5\tilde{\Psi} + i\gamma^l\gamma^5\tilde{\Psi}_l ] E^{-1}\sigma^{ab} -
\]

\[
m [ -i\Psi + \gamma^l\Psi_l + i\sigma^{mn}\Psi_{mn} + \gamma^5\tilde{\Psi} + i\gamma^l\gamma^5\tilde{\Psi}_l ] E^{-1} \} = 0
\]  

(4.3)

Multiplying eq. (4.3) from the left by \( E \) and taking the trace of the result (with the use of the rule \( E\sigma^{ab} = -\sigma^{ab}E \)), which results in

\[
i\epsilon^\beta\epsilon_{(c}\text{Sp} (\gamma^c\gamma^l)\partial_\beta\Psi_l + i\gamma_{abc}\text{Sp} (\gamma^c\sigma^{ab}\gamma^l)\Psi_l + i\text{Sp} (\gamma^c\sigma^{ab}\gamma^l\gamma^5)\tilde{\Psi}_l) +
\]

\[
\frac{1}{2}\gamma^{ab}[-\text{Sp} (\gamma^c\gamma^l\sigma^{ab})\Psi_l - i\text{Sp} (\gamma^c\gamma^l\gamma^5\sigma^{ab})\tilde{\Psi}_l] + 4i\Psi = 0 .
\]

Further, allowing for the known formulas

\[
\frac{1}{2}\gamma_{abc}\text{Sp} (\gamma^c\sigma^{ab}\gamma^l)\Psi_l = +\frac{1}{2}\gamma^c_l,
\]

\[
\frac{1}{2}\gamma_{abc}\text{Sp} (\gamma^c\gamma^l\sigma^{ab})\Psi_l = +\frac{1}{2}\gamma^c_l,
\]

\[
\frac{1}{2}\gamma_{abc}\text{Sp} (\gamma^c\sigma^{ab}\gamma^l\gamma^5)\tilde{\Psi}_l = +\frac{i}{4}\gamma_{abc}\epsilon^{abcd}\tilde{\Psi}_l,
\]

\[
\frac{1}{2}\gamma_{abc}\text{Sp} (\gamma^c\gamma^l\gamma^5\sigma^{ab})\tilde{\Psi}_l = -\frac{i}{4}\gamma_{abc}\epsilon^{abcd}\tilde{\Psi}_l .
\]

for \( U \)- and \( V \)-fields respectively we find

for \( U \)-field

\[
e^{(l)\alpha}_\beta\partial_\alpha\Psi_l + \gamma^c_l\tilde{\Psi}_l + m\Psi = 0 ,
\]  

(4.4a)

for \( V \)-field

\[
e^{(l)\alpha}_\beta\partial_\alpha\Phi_l + \frac{1}{2}\gamma^c_l(x)\Phi_l - \frac{1}{4}\gamma_{abc}\epsilon^{abcd}\tilde{\Phi}_l + m\Phi = 0 .
\]  

(4.4b)
It should be emphasized that because $\Phi$-fields are not of tensor nature under the local Lorentz group eq. (4.4b), cannot be presented in pure covariant tensor form whereas eq. (3.4a) does.

In similar manner, acting on eq. (4.3) by operator $\text{Sp} (E\gamma^5 \times)$ we get

for $U$-field

$$ e^{(l)\alpha} \partial_\alpha \tilde{\Psi} + \gamma^c \left[ \tilde{\Psi} + m \tilde{\Psi} \right] = 0 ; \quad (4.5a) $$

for $V$-field

$$ e^{(l)\alpha} \partial_\alpha \tilde{\Phi} + \frac{1}{2} \gamma^c \left[ \tilde{\Phi} + \frac{1}{4} \gamma_{abc} \epsilon^{abcd} \Phi \right] + m \tilde{\Phi} = 0 . \quad (4.5b) $$

One more case is when multiplying (4.3) by $\text{Sp} (E\gamma^k)$:

$$ i\epsilon^{\beta} \partial_\beta \left[ -i \text{Sp} (\gamma^k \gamma^c) \Psi + i \text{Sp} (\gamma^k \gamma^c \sigma^{mn}) \Psi_{mn} \right] + \frac{i}{2} \gamma_{abc} \left[ -i \text{Sp} (\gamma^k \gamma^c \sigma^{ab}) \Psi + \right. $$

$$ i \text{Sp} (\gamma^k \gamma^c \sigma^{ab} \sigma^{mn}) \Psi_{mn} + \text{Sp} (\gamma^k \gamma^c \sigma^{ab} \sigma^5) \tilde{\Psi} \right] + \frac{i}{2} \gamma_{abc} \left[ +i \text{Sp} (\gamma^k \gamma^c \sigma^{ab}) \Psi - \right. $$

$$ \text{Sp} (\gamma^k \gamma^c \sigma^{mn} \sigma^{ab}) \Psi_{mn} - \text{Sp} (\gamma^k \gamma^c \sigma^{5} \sigma^{ab}) \tilde{\Psi} \right] - m \text{Sp} (\gamma^k \gamma^c) \Psi_{kl} = 0 , $$

which results in

for $U$-field

$$ e^{(k)\alpha} \partial_\alpha \Psi + e^{(c)\alpha} \partial_\alpha \Psi^{kc} + \gamma^k \Psi_{mn} + \gamma^c \Psi^{kl} - m \tilde{\Psi} = 0 ; \quad (4.6a) $$

for $V$-field

$$ e^{(k)\alpha} \partial_\alpha \Phi + e^{(c)\alpha} \partial_\alpha \Phi^{kc} + \frac{1}{2} \gamma^k \Phi_{mn} + \frac{1}{2} \gamma^c \Phi^{kl} + $$

$$ \frac{1}{2} \gamma^c \epsilon(x) \Phi + \frac{1}{4} \gamma_{abc} \epsilon^{abc} \Phi + \frac{1}{4} \gamma_{mn} \Phi_{mn} + m \Phi = 0 . \quad (4.6b) $$

Eqs. (4.6a) and (4.6b) substantially differ from each other, only the first is reduced to covariant tensor form. Remaining equations can be treated similarly, the main results are the same: only for $U$-field there arise covariant tensor equations.

Else one remark about interpretation of the Dirac–Kähler field in flat Minkowski space as a set of four Dirac particles should be given. The matter is that any particle as a relativistic object is determined not only by explicitly given wave equation but also determined by a relevant operation of charge conjugation. The latter, in turn, is fixed by transformation properties of the wave function under the Lorentz group. Evidently, the Dirac–Kähler object and the system four Dirac fields assume their own and different charge conjugations. In particular, having introduced a definition for a particle and antiparticle in accordance with four fermions interpretation, one immediately see that such a particle-antiparticle separating turns to be non-invariant with respect to tensor transformation rules of the Dirac-Kähler constituents. Thus, even in the flat Minkowski space, the four fermion interpretation for this field cannot be evolved with success.
5 Bosons with different intrinsic parities in curved space-time

From the Dirac–Kähler theory, by imposing special linear restrictions, one can derive more simple equations for particle with single value of spin: ordinary bosons of spin 0 or 1 with different intrinsic parity.

First, let us consider tensor equations in flat Minkowski space with four different additional constraints:

\[
\tilde{S} = 0 \quad \tilde{\Phi} = 0, \quad \tilde{\Phi}_\alpha = 0, \quad \Phi_{\alpha\beta} = 0,
\]
\[
\partial^l\tilde{\Phi}_l + m\tilde{\Phi} = 0, \quad \partial^l\Phi_l - m\Phi_l = 0, \quad \partial^k\Phi^k - \partial^k\Phi^d = 0; \quad (5.1a)
\]
\[
\tilde{S} = \tilde{0} \quad \Phi = 0, \quad \Phi_\alpha = 0, \quad \Phi_{\alpha\beta} = 0,
\]
\[
\partial^l\tilde{\Phi}_l + m\tilde{\Phi} = 0, \quad \partial^k\Phi_l - m\Phi_l = 0; \quad (5.1b)
\]
\[
\tilde{S} = 1 \quad \Phi = 0, \quad \tilde{\Phi} = 0, \quad \tilde{\Phi}_l = 0,
\]
\[
\partial^l\tilde{\Phi}_l + m\tilde{\Phi} = 0, \quad \partial^k\Phi_l - m\Phi_l = 0; \quad (5.2a)
\]
\[
\tilde{S} = \tilde{1} \quad \Phi = 0, \quad \tilde{\Phi} = 0, \quad \Phi_l = 0,
\]
\[
\partial^l\tilde{\Phi}_l = 0, \quad \partial^l\Phi_{kl} = 0, \quad \frac{1}{2}\epsilon^{kcmn}\partial_c\Psi_{mn} + m\Phi_l = 0, \quad \epsilon^{dkcl}\partial_c\tilde{\Phi}_l - m\Phi^d = 0. \quad (5.2b)
\]

Let us describe additional constraints in spinor form. For a scalar particle we get

\[
\tilde{S} = 0 \quad \begin{vmatrix} \xi & \Delta \\ H & \eta \end{vmatrix} = \begin{vmatrix} -\Phi\sigma^2 & +i\Phi_l\sigma^2 \\ -i\Phi_l\sigma^2 & +\Phi\sigma^2 \end{vmatrix},
\]
\[
\Delta^{tr} = +H, \quad \xi = -\eta, \quad \xi^{tr} = -\xi, \quad \eta^{tr} = -\eta, \quad (5.3)
\]
the symbol of \(tr\) stands for a matrix transposition. For a pseudoscalar particle we get

\[
\tilde{S} = \tilde{0} \quad \begin{vmatrix} \xi & \Delta \\ H & \eta \end{vmatrix} = \begin{vmatrix} +i\tilde{\Phi}\sigma^2 & -\tilde{\Phi}_l\sigma^2 \\ -\tilde{\Phi}_l\sigma^2 & +i\tilde{\Phi}\sigma^2 \end{vmatrix},
\]
\[
\tilde{\Delta} = -H, \quad \xi = +\eta, \quad \tilde{\xi} = -\xi, \quad \tilde{\eta} = -\eta. \quad (5.4)
\]

For a vector particle, we will have

\[
\tilde{S} = 1 \quad \begin{vmatrix} \xi & \Delta \\ H & \eta \end{vmatrix} = \begin{vmatrix} \sum_{mn}\sigma^2\Phi_{mn} & +i\sigma^2\Phi_l \\ -i\sigma^2\Phi_l & -\sum_{mn}\sigma^2\Phi_{mn} \end{vmatrix},
\]
\[
\tilde{\Delta} = +H, \quad \tilde{\xi} = +\xi, \quad \tilde{\eta} = +\eta. \quad (5.5a)
\]

Here each of symmetrical spinors \(\xi\) and \(\eta\) depends on three independent variables:

\[
\begin{align*}
\xi + \eta &= -2i \left( \sigma^1\Phi_{23} + \sigma^2\Phi_{31} + \sigma^3\Phi_{12} \right) \sigma^2, \\
\xi - \eta &= 2 \left( \sigma^1\Phi_{01} + \sigma^2\Phi_{02} + \sigma^3\Phi_{03} \right) \sigma^2.
\end{align*}
\quad (5.5b)
Finally, a pseudovector case is given by

\[
S = \begin{vmatrix}
\xi & \Delta \\
H & \eta
\end{vmatrix} = \begin{vmatrix}
\Sigma_{mn} \sigma^2 \Phi_{mn} & -\sigma' \sigma^2 \tilde{\Phi}_l \\
-\sigma' \sigma^2 \tilde{\Phi}_l & -\Sigma_{mn} \sigma^2 \Phi_{mn}
\end{vmatrix},
\]

\[\tilde{\Delta} = -H, \ \tilde{\xi} = +\xi, \ \tilde{\eta} = +\eta. \quad (5.6)\]

We are to extend this approach to general covariant case

\[
\nabla^\alpha \Psi_\alpha + m \Psi = 0, \quad \tilde{\nabla}^\alpha \tilde{\Psi}_l + m \tilde{\Psi} = 0,
\]

\[
\nabla_\alpha \Psi + \nabla^\beta \Psi_{\alpha\beta}(x) - m \Psi_\alpha = 0,
\]

\[
\tilde{\nabla}_\alpha \Psi - \frac{1}{2} \epsilon^\beta_{\rho\sigma}(x) \nabla_\beta \Psi_{\rho\sigma} - m \tilde{\Psi}_\alpha = 0,
\]

\[
\nabla_\alpha \Psi_\beta - \nabla_\beta \Psi_\alpha + \epsilon_{\alpha\beta\rho}(x) \nabla_\rho \Psi_\sigma - m \Psi_{\alpha\beta} = 0. \quad (5.7)
\]

First let it be

\[
S = 0, \quad \nabla^\alpha \tilde{\Psi}_\alpha + m \tilde{\Psi} = 0,
\]

\[
\nabla_\alpha \Psi - m \Psi_\alpha = 0, \quad \nabla_\alpha \Psi_\beta - \nabla_\beta \Psi_\alpha = 0, \quad (5.8)
\]

two first are the Proca equations for scalar particle, the last equation holds identically

\[
\partial_\alpha \partial_\beta \Psi - \Gamma^\mu_{\alpha\beta} \partial_\mu \Psi - \partial_\beta \partial_\alpha \Psi + \Gamma^\mu_{\beta\alpha} \partial_\mu \Psi = 0.
\]

For a pseudoscalar field we have

\[
S = \tilde{0}, \quad \nabla^\alpha \tilde{\Psi}_\alpha + m \tilde{\Psi} = 0,
\]

\[
\nabla_\alpha \tilde{\Psi} - m \tilde{\Psi}_\alpha = 0, \quad \epsilon_{\alpha\beta} \nabla_\rho \Psi_\sigma = 0; \quad (5.9)
\]

here the last equation holds identically. Now, let only \(\Psi_\alpha \neq 0, \ \Psi_{\alpha\beta}(x) \neq 0\), then

\[
S = 1, \quad \nabla^\alpha \Psi_\alpha = 0, \quad \nabla^\beta \Psi_{\alpha\beta} - m \Psi_\alpha = 0,
\]

\[
-\frac{1}{2} \epsilon^\beta_{\rho\sigma}(x) \nabla_\beta \Psi_{\rho\sigma} = 0, \quad \nabla_\alpha \Psi_\beta - \nabla_\beta \Psi_\alpha = m \Psi_{\alpha\beta}. \quad (5.10)
\]

Here the first and third equation hold identically:

\[
\nabla^\alpha \Psi_\alpha = \frac{1}{m} \nabla^\alpha \nabla^\beta \Psi_{\alpha\beta} = \frac{1}{2m} \left[ \Psi_{\alpha\nu} R^{\nu}_{\beta\alpha} + \Psi_{\beta\nu} R^{\nu}_{\alpha\beta} \right],
\]

\[
-\Psi_{\beta\nu} R^{\nu}_{\alpha\beta} = \frac{1}{2m} \left[ -\Psi_{\alpha\nu} R^{\nu}_{\alpha\beta} - \Psi_{\beta\nu} R^{\nu}_{\alpha\beta} \right] = 0,
\]

\[
-\frac{1}{2m} \epsilon^\beta_{\rho\sigma}(x) \nabla_\beta \left[ \nabla_\rho \Psi_\sigma - \nabla_\sigma \Psi_\rho \right] = -\frac{1}{4m} \epsilon^\beta_{\rho\sigma}(x) \left[ \left( \nabla_\beta \nabla_\rho - \nabla_\rho \nabla_\beta \right) \Psi_\sigma - \left( \nabla_\beta \nabla_\sigma - \nabla_\sigma \nabla_\beta \right) \Psi_\rho \right] = -\frac{1}{4m} \epsilon^\beta_{\rho\sigma}(x) \left( \Psi^\nu R_{\nu\rho\beta} - \Psi^\nu R_{\nu\rho\beta} \right) = 0.
\]
Now, let $\Psi(x) = \tilde{\Psi} = \Psi_\alpha = 0$, then

$$S = \tilde{1}, \quad \nabla^\alpha \tilde{\Psi}_\alpha = 0, \quad \nabla^\beta \tilde{\Psi}_{\alpha\beta} = 0,$$

$$\frac{1}{2} \epsilon^\beta_\alpha \rho_\sigma (x) \nabla_\beta \Psi_\rho_\sigma + m \tilde{\Psi} = 0,$$

$$\epsilon^\beta_\alpha \rho_\sigma (x) \nabla_\rho \tilde{\Psi}_\sigma - m \Psi_\alpha_\beta = 0. \quad (5.11)$$

The first and the second equations hold identically:

$$\nabla^\alpha \tilde{\Psi}_\alpha = -\frac{1}{2m} \nabla^\alpha \epsilon^\beta_\alpha \rho_\sigma (x) \nabla_\beta \Psi_\rho_\sigma = \frac{1}{2m} \epsilon^\beta_\alpha \rho_\sigma (x) \nabla^\alpha \nabla_\beta \Psi_\rho_\sigma$$

$$= -\frac{1}{2m} \epsilon^\beta_\alpha \rho_\sigma (x) \left[ \Psi_\nu_\sigma \, R^\nu_\beta_\rho_\alpha (x) + \Psi_\rho_\nu \, R^\nu_\sigma_\beta_\alpha \right],$$

$$\nabla^\beta \Psi_\alpha_\beta (x) = \frac{1}{m} \nabla^\beta \epsilon^\rho_\alpha \rho_\sigma (x) \nabla_\rho \Psi_\sigma = \frac{1}{2m} \epsilon^\rho_\alpha \rho_\sigma (x) \Psi^\sigma \, R_{\nu_\sigma_\rho_\beta}.$$

Constraints separating four boson fields are the same as in the case of Minkowski space:

$$S = 0, \quad \tilde{\Delta} = +H, \quad \xi = -\eta, \quad \tilde{\xi} = -\xi, \quad \tilde{\eta} = -\eta.$$

$$S = \tilde{0}, \quad \tilde{\Delta} = -H, \quad \xi = +\eta, \quad \tilde{\xi} = -\xi, \quad \tilde{\eta} = -\eta.$$

$$S = 1, \quad \tilde{\Delta} = +H, \quad \tilde{\xi} = +\xi, \quad \tilde{\eta} = +\eta.$$

$$S = \tilde{1}, \quad \tilde{\Delta} = -H, \quad \tilde{\xi} = +\xi, \quad \tilde{\eta} = +\eta. \quad (5.12)$$

### 6 Discussion

We may conclude that the use of tetrad formalism permit us to apply results on classification of the particles with respect to discrete Lorentzian transformations (including intrinsic parity of bosons) when treating relevant particle fields on the background of arbitrary curved space-time model.

### References

[1] Wigner E.P. On unitary representations of the inhomogeneous Lorentz group. Ann. of Math. 1939. Vol. 40. P. 149 – 204.

[2] Pauli W. Relativistic field theories of elementary particles. Rev. Mod. Phys. 1941. Vol. 13. P. 203 – 232.

[3] Bhabha H.J. On the postulational basis of the theory of elementary particles. Rev. Mod. Phys. 1949. Vol. 21, no 3. P. 451 – 462.

[4] Harish-Chandra. On relativistic wave equation. Phys. Rev. 1947. Vol. 71, no 11. P. 793 – 805.
[5] Gel’fand I.M., Yaglom A.M. General relativistically invariant equation and infinite infinite-dimensional, representations of the Lorent group JETP. 1948. Vol. 18, no 8. P. 703 – 733.

[6] Corson E.M. Introduction to tensors, spinors and relativistic wave equations. London: Blackie and Son, 1953.

[7] Umezawa H. Quantum Field Theory. Amsterdam (North-Holland), 1956.

[8] Yu.M. Shirokov. Group theory consideration of the bases of relativistic quantum mechanics I. JETP. 1957. Vol. 33. No 4. P. 861-872; II. JETP. 1957. Vol. 33. No 5. P. 1196-1207; III. JETF. 1957. Vol. 33. No 5. P. 1208-1214; IV. JETP. 1958. Vol. 34. No 3. P.

[9] Bogush A.A., Moroz L.G. Introduction to klassical field theory. Minsk, Nauka i tekhnika, 1968.

[10] Fedorov F.I, The Lorentz group. Moskow, 1979.

[11] Ivanenko D., Landau L. Zur theorie des magnetischen electrons. Zeit. Phys. 1928. Bd. 48, N 8. S. 340 – 348.

[12] Lanczos C. The tensor analytical relationships of Dirac’s equation. Zeit. Phys. 1929. Bd. 57. S. 447 – 473.

[13] Lanczos C. The covariant formulation of Dirac’s equation. Zeit. Phys. 1929. Bd. 57. S. 474 – 483; The conservation laws in the field theoretical representation of Dirac’s theory. Zeit. Phys. 1929. Bd. 57. S. 484 – 493.

[14] Juvet G. Opérateurs de Dirac et équations de Maxwell. Comm. Math. Helv. 1930. Vol. 2. P. 225 – 235.

[15] Juvet G., Schidlof A. Sur les nombres hypercomplexes de Clifford et leurs applications à l’analyse vectorielle ordinaire, à l’électromagnétisme de Minkowski et à la théorie de Dirac. Bull. Soc. Sci. Nat. Neuchâtel. 1932. Vol. 57. P. 127 – 141.

[16] Einstein A., Mayer V. Semivektoren und Spinoren. Sitz. Ber. Preuss. Akad. Wiss. Berlin. Phys.-Math. Kl. 1932. S. 522 – 550.

[17] Einstein A., Mayer W. Die Diracgleichungen für Semivektoren. Proc. Akad. Wet. Amsterdam. 1933. Bd. 36. S. 497 – 516.

[18] Einstein A., Mayer W. Spaltung der Natürlichsten Feldgleichungen für Semi-Vektoren in Spinor-Gleichungen von Diracschen Tipus. Proc. Akad. Wet. Amsterdam. 1933. Bd. 36. S. 615 – 619.

[19] Einstein A., Mayer W. Darstellung der Semi-Vektoren als gewöhnliche Vektoren von Besonderem Differentiations Charakter. Ann. of Math. 1934. Vol. 35, N 1. P. 104 – 110.

[20] Frenkel Ya.I. Electrodynamics. Vol. I, 1934; Vol. II, 1935.

[21] Whittaker E.T. On the relations of the tensor-calculus to the spinor-calculus. Proc. Roy. Soc. London. A. 1937. Vol. 158. P. 38 – 46.
[22] Proca A. Sur un article de M.E. Whittaker, intitulé "Les relations entre le calcul tensoriel et le calcul des spineurs". J. Phys. et Radium. 1937. Vol. 8. P. 363 – 365.

[23] Ruse H.S. On the geometry of Dirac’s equations and their expression in tensor form. Proc. Roy. Soc. Edin. 1936. Vol. 57. P. 97 – 127.

[24] Taub A.H. Tensors equations equivalent to the Dirac equations. Ann. Math. 1939. Vol. 40. P. 937.

[25] Taub A.H. Spinor equations for the meson and their solution when no field is present. Phys. Rev. 1939. Vol. 56, N 8. P. 799 – 810.

[26] Belinfante F.J. The undor equation of the meson field. Physica. 1939. Vol. 6. P. 870.

[27] Belinfante F.J. Spin of Mesons. Physica. 1939. Vol. 6. P. 887 – 898.

[28] Ivanenko D., Sokolov A. Quantum field theory. Moscov, 1951.

[29] Feshbach H., Nickols W. A wave equation for a particle of maximum spin one. Ann. Phys. N.Y. 1958. Vol. 4, N 4. P. 448 – 458.

[30] Kähler E. Innerer and äusserer Differentialkalkül. Abh. Dt. Akad. Wiss. Berlin. Kl. Math.-Phys. u. Techn. 1960. N 4.

[31] Kähler E. Die Dirac-Gleichung. Abh. Dt. Akad. Wiss. Berlin, Kl. Math.-Phys. u. Techn. 1961, N 1.

[32] Leutwyler H. Generally covariant Dirac equation and associated boson fields. // Nuovo Cimento. 1962. Vol. 26, N 5. P. 1066.

[33] Klauder J.R. Linear representation of spinor fields by antysymmetric tensors. J. Math. Phys. 1964. Vol. 5, N 9. P. 1204 – 1214.

[34] Penney R. Tensorial description of neutrinos. J. Math. Phys. 1965. Vol. 6, N 7. P. 1026 – 1028.

[35] Cereignani C. Linear representations of spinors by tensors. J. Math. Phys. 1967. Vol. 8, N 3. P. 417 – 422.

[36] Streater R.F., Wilde I.F. Fermion states of a boson field. Nucl. Phys. B. 1970. Vol. 24. P. 561.

[37] Pestov A.B. Connection between Dirac and Maxwell equations. Dubna, 1971. 18 pages (Preprint P2-5798).

[38] Pestov A.B. Relativistic equations defined by exterior derivative operators and extended divergence. TMP. 1978. Vol. 34, no 1. P. 48 – 57.

[39] Pestov A.B. On the group of internal symmetry of the wave equation defined by exterior derivative operators. Dubna, 1983. (Preprint P2-83-506).

[40] Osterwalder K. Duality for free Bose fields. Commun. Math. Phys. 1973. Vol. 29, N 1. P. 1 – 14.
[41] Crumeyrolle A. Une théorie de Einstein – Dirac en spin maximum 1. Ann. Inst. H. Poincaré. A. 1975. Vol. 22. P. 43.

[42] Durand E. 16-component theory of the spin-1 particle and its generalization to arbitrary spin. Phys. Rev. D. 1975. Vol. 11, N 12. P. 3405 – 3416.

[43] Strazhev V.I. On the symmetry group of extended equations for a vector field. Izvestiya Vuzov. Fizika. 1977, no 8. P. 45 – 48.

[44] Kruglov S.I., Strazhev V.I. Internal symmetries and conservation lows in classical theory of a vector field of general type. Izvestiya Vuzov. Fizika. 1978, no 4. P. 77 – 81.

[45] Strazhev V.I. On dyad symmetry of a vector fielded of general type. Acta Phys. Pol. B. 1978. Vol. 9 P. 449 – 458.

[46] Bogush A.A., Kruglov S.I., Strazhev V.I. On the group of internal symmetry of 16-component theory of a vector particles. Doklady AN BSSR. 1978. Vol. 22, no 10. P. 893 – 895.

[47] Bogush A.A., Kruglov S.I. On equation of vector field of general type. Proceeding of Academy of Sciences of BSSR. ser. phys.-mat. 1978, no 4. P. 58 – 65.

[48] Satikov I.A., Strazhev V.I. On quantum description of the Dirac–Kähler field. TMP. 1987. Vol. 73, no 1. P. 16 – 25.

[49] Strazhev V.I., Pletjuxov, Fedorov F.I. On connection of spin and statistics in the theory of relativistic wave equations with intrinsic degrees of freedom. Minsk, 1988. 36 pages. (Preprint no 517 / IP AN BSSR).

[50] Strazhev V.I., Berezin A.V., Satikov I.A. Dirac–Kähler equations and quantum theory of the Dirac field with internal symmetry group SU(2, 2). Minsk, 1988. 20 pages. (Preprint no 522 / IP AN BSSR).

[51] Strazhev V.I., Tsionenko D.A. On Dirac–Kähler gauge field theory in a curved space-time. Vestnik BGU. ser. I, fiz.-mat.-inform. . 2002, no 2. P. 15 – 21.

[52] Tsionenko D.A. - // i . i . 1 . . 2002, no 4 . 75 – 83.

[53] Tsionenko D.A. Dirac–Kähler equation in non-Euclidean space-time. Proceeding of National Academy of Sciences of Belarus. phys.-mat. 2003, no 1. P. 81 – 85.

[54] Strazhev V.I., Satikov I.A., Tsionenko D.A. Dirac–Kähler equation, classical theory. Minsk, BGU, 2007.

[55] Graf W. Differential forms as spinors. Ann. Inst. H. Poincaré. A. 1978. Vol. 29, N 1. P. 85 – 109.

[56] Benn I.M., Tucker R.W. A generation model based on Kähler fermions. Phys. Lett. B. 1982. Vol. 119, N 4-6. P. 348 – 350.

[57] Benn I.M., Tucker R.W. Fermions without spinors. Commun. Math. Phys. 1983. Vol. 89, N 3. P. 341 – 362.
[58] Benn I.M., Tucker R.W. Kähler fields and five-dimensional Kaluza – Klein theory. J. Phys. A. 1983. Vol. 16, N 4. P. 123 – 125.

[59] Benn I.M., Tucker R.W. Clifford analysis of exterior forms and Fermi-Bose symmetry. J. Phys. A. 1983. Vol. 16, N 17. P. 4147 – 4153.

[60] Benn I.M., Tucker R.W. A local right-spin covariant Kähler equation. Phys. Lett. B. 1983. Vol. 130, N 3-4. P. 177 – 178.

[61] Tucker R.W., Benn I.M. The differential approach to spinors and their symmetries. Nuovo Cim. A. 1985. Vol. 88. Ser. 2, N 3. P. 273 – 285.

[62] Banks T., Dothan Y., Horn D. Geometric fermions. Phys. Lett. B. 1982. Vol. 117, N 6. P. 413 – 417.

[63] Garbaczewski P. Quantization of spinor fields. Meaning of ”bosonization” in (1+1) and (1+3) dimensions. J. Math. Phys. 1982. Vol. 23, N 3. P. 442 – 450.

[64] Pletjuxov V.A., Strazhev V.I. On Dirac like wave equation for particles with maximal spin 1 Doklady AN BSSR. 1982. Vol. 26, no 8. P. 691 – 693.

[65] Pletjuxov V.A., Satikov I.A., Strazhev V.I. Relativistic wave equations and massless Dirac–Kähler field. Covariant methods in theoretical physics. Elementary particle physics and relativity theory. Minsk, Institute of Physics, 1986. P. 31 – 35.

[66] Pletjuxov V.A., Strazhev V.I. On possible extensions of the Dirac–Kähler field Vesti AN BSSR. ser. fiz.-mat. 1987, no 5. P. 87 – 92.

[67] Pletjuxov V.A., Strazhev V.I. Tensorial equations and Dirac particles with internal degrees of freedom. Yadernaya Fizika. 1989. Vol. 49. P. 1505 – 1514.

[68] Holland P.R. Tensor conditions for algebraic spinors. J. Phys. A. 1983. Vol. 16, N 11. P. 2363 – 2374.

[69] Ivanenko D.D., Obukhov Yu.N., Solodukhin S.N. On antisymmetric tensor representation of the Dirac equation. Trieste, 1985 (Preprint IC/85/2. ICTP).

[70] Obukhov Yu.N., Solodukhin S.N. Reduction of the Dirac equation and its connection to Ivanenko–Landau–Kähler equation. TMP. 1993. Vol. Vol. 94 . P. 276 – 295.

[71] Bullinaria J.A. Kähler fermions in arbitrary space-times, their dimensional reduction and relation to spinorial fermions. Ann. Phys. (N.Y.). 1986. Vol. 168, N 2. P. 301 – 343.

[72] Blau M. Clifford algebras and Kähler – Dirac spinors. Ph.D. dissertation, Report UWTH Ph 198616. Universitat Wien, 1986. 200 p.

[73] Jourjine A.N. Space-time Dirac – Kähler spinors. Phys. Rev. D. 1987. Vol. 35, N 2. P. 757 – 758.

[74] Krolikowski W. Dirac equation with hidden extra spin: a generalization of kähler equation. I. Acta Phys. Polon. B. 1989. Vol. 20, no 10. P. 849 – 858.
[75] Krolikowski W. Dirac equation with hidden extra spin: a generalization of Kähler equation. II. Acta Phys. Polon. B. 1990. Vol. 21, no 3. P. 201 – 207.

[76] Howe P. A particle mechanics description of antisymmetric tensor fields. Class. Quant. Grav. 1989. Vol. 6. P. 1125.

[77] Beckers J., Debergh N., Nikitin A.G. On parasupersymmetries and relativistic descriptions for spin one particles. I. The free context. Fortschr. Phys. 1995. Vol. 43, N 1. P. 67 – 80; II. The interacting context (with electromagnetic fields) Fortschr. Phys. 1995. Vol. 43, N 1. P. 81 – 96.

[78] Bogush A.A., Kruglov S.I. On equations for a vector field of general type. Vesti AN BSSR. ser. fiz.-mat. 1978, no 4. P. 58 – 65.

[79] Kruglov S.I. Symmetry and electromagnetic interactions of fields with multispin. N.Y.: Nova Science Pub. Inc., Hauppauge, 2000.

[80] Kruglov S.I. Dirac – Kähler equations. Intern. J. Theor. Phys. 2002. Vol. 41. P. 653 – 687.

[81] Kruglov S.I. On the generalized Dirac equation for fermions with two mass states. Ann. Fond. L. de Broglie. 2004. Vol. 29, Hors série 2. P. 1005 – 1016.

[82] Marchuk N.G. Dirac gamma-equation, classical gauge fields and Clifford algebra. Adv. Appl.Clifford Alg. 1998. Vol. 8. P. 181 – 2242.

[83] Marchuk N.G. Gauge fields of the matrix Dirac equation. Nuovo Cim. B. 1998. Vol. 113. P. 1287 – 1295.

[84] Marchuk N.G. A gauge model with spinor group for a description of local interaction of a fermion with electromagnetic and gravitational fields. Nuovo Cim. B. 2000. Vol. 115. P. 11 – 25.

[85] Marchuk N.G. A tensor form of the Dirac equation. Nuovo Cim. B. 2001. Vol. 116, N 10. P. 1225 – 1248.

[86] Marchuk N.G. Dirac-type tensor equations with non-Abelian gauge symmetries on pseudo-Riemannian space. Nuovo Cim. B. 2002. Vol.117. P. 95 – 120.

[87] Marchuk N.G. The Dirac equation vs. the Dirac type tensor equation. Nuovo Cim. B. 2002. Vol. 117. P. 511 – 520.

[88] Marchuk N. A concept of Dirac-type tensor equations. arXiv:math-ph/0212006

[89] Krivskij I.Yu., Lompej R.R. Simulik B.M. On symmetries of complex Dirac – Kähler equation. TMP. 2005. Vol. 143. P. 64 – 82.

[90] Landau L.D., Lifshitz E.M. Theoretical physics, II. Field theory. Moscow, 1973.