Adjamagbo Determinant and Serre Conjecture
for linear groups over Weyl algebras

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Abstract: Thanks to the theory of determinants over an Ore domain, also called Adjamagbo determinant by the Russian school of non commutative algebra, we extend to any Weyl algebra over a field of characteristic zero Suslin theorem solving what Suslin himself called the $K_1$-analogue of the well-known Serre Conjecture and asserting that for any integer $n$ greater than 2, any $n$ by $n$ matrix with coefficients in any algebra of polynomials over a field and with determinant one is the product of elementary matrices with coefficients in this algebra.

Introduction

Let $A$ be a ring, $n$ a positive integer, $M_n(A)$ the ring of $n$ by $n$ matrices with coefficients in $A$, $GL_n(A)$ the group of invertible elements of $M_n(A)$ and $E_n(A)$ its subgroup generated by its elementary matrices.

Let us consider the following natural question: Does there exist a function of the coefficients of the elements of $M_n(A)$ with values somewhere which allows the characterization of elements of $E_n(A)$ between those of $M_n(A)$ and not only between those of $GL_n(A)$ and such that this function is “effectively computable” whenever the ring $A$ is “effective” in the sense of [3]?

Let us first assume $A$ commutative. In this case, we think naturally to the determinant function of matrices in $M_n(A)$. Indeed, for any element $a$ of $M_n(A)$, if $a \in E_n(A)$, then it is obvious that $\det a = 1$. But the converse is false in general, as shown by the famous counter-example of P.M. Cohn [10], prop. 7.3:

$$a = \begin{pmatrix} 1 - XY & -Y^2 \\ XY & 1 + XY \end{pmatrix} \in M_2(\mathbb{Q}[X, Y])$$

Nevertheless, it is well known that this converse is true if $A$ is a field, or an euclidian ring or a semi-local one, or if $A$ is a noetherian ring whose Krull dimension is lower than $n - 1$, according to Bass-Milnor-Serre theorem on products of elementary matrices published in 1967 [9].

On the other hand, Suslin proved ten years later in [18] that this last theorem can be improved by taking $n$ only greater than 2 (instead of 1 plus the Krull dimension of $A$) and by choosing $A$ as any algebra of polynomials over a field. In the introduction of this paper, Suslin himself presented this result as the solution to “the $K_1$-analogue of the well-known serre Problem recently solved completely by the author, and independently by Daniel Quillen”. Indeed, according to the triviality of the special Withehead group $SK_1A$ of such algebra, for a square matrix $a$ with entries in $A$, $\det a = 1$ means that this matrix is “stably” (i.e. after augmentation of the matrix by adding some 1’s on the diagonal and 0 outside) a product of elementary square matrices with entries in $A$, and the problem is to know if it is “actually” a product of elementary square matrices with entries in $A$. This justifies Suslin’s analogy with
Serre problem which ask if any “stably” (i.e. after addition by a finite free $A$-module) free $A$-module of finite type is “actually” free.

Thanks to the theory of determinants over an Ore domain developed in [4], summed up in [7], more briefly in [8], A.III, and already called “Adjagambo determinant” by Russian school of non commutative algebra following A. Mikhalev and A. Guterman (see for instance [14], [15], [11], [12], [13]), we extended in [5], this theorem of Bass-Milnor-Serre to the case where $A$ is a non-commutative “classical filtered ring”, i.e. a ring endowed with an increasing $\mathbb{N}$-filtration $F$ whose associated graded ring is a commutative regular domain flat over its subring $F(0)$ which has a trivial special Whitehead group $SK_1 F(0)$. It is in particular the case of the enveloping algebra of a Lie algebra of finite dimension over a field. It is also the case of a classical or a formal (resp. analytic) Weyl algebra over a field (resp. the field of real or complex numbers), see for instance [5], p. 404.

Exactly as Suslin did for Bass-Milnor-Serre theorem, the aim of this note is to improve this generalization of Bass-Milnor-Serre theorem, by taking $n$ only greater than 2 (instead of 1 plus the Krull dimension of $A$) and by choosing $A$ as any Weyl algebra over a field of characteristic zero, thanks to Stafford Main Theorem on the “module structure of Weyl algebras” [17] and to Varerstein $K_1$-stability theorem [19], th. 3.2.

Finally, we formulate a natural conjecture about filtered rings which would prove that this Suslin Theorem follows from the forthcoming theorem. We end with other open problems related to this theorem.

Recall (on the canonical determinant over filtered Weyl algebras, see for instance [5])

1) $K$ being a commutative field and $m$ a positive integer, let us denote by $(X_i, Y_i)_{1 \leq i \leq m}$ a system of indeterminates over $K$, $K[X_1, \ldots, X_m]$ the $K$-algebra of polynomials in indeterminates $X_1, \ldots, X_m$, $A_m(K)$ the Weyl algebra over $K$ of index $m$, i.e. the $K$-algebra $K[X_1, \ldots, X_m] [\partial/\partial X_1, \ldots, \partial/\partial X_m]$ of $K$-linear differential operators over $K[X_1, \ldots, X_m]$, $F$ the differential filtration on $A_m(K)$, i.e. $F(0) = K[X_1, \ldots, X_m]$ and $F(j + 1) = F(j) + \sum_{1 \leq i \leq m} F(j) \partial/\partial X_i$ for each natural integer $j$, $gr_F A_m K$ the associated graded ring, $gr_F$ the canonical map from $A_m(K)$ to $gr_F A_m K$ and $gr_F(A_m(K))$ the image of $gr_F$, i.e. the set of homogeneous elements of the graded ring $gr_FA_m K$, or in terms of partial differential equations, the set of principal symbols of the differential operators belonging to $A_m(K)$.

2) The ring $gr_F A_m(K)$ being isomorphic to the regular ring of polynomials over $K$ in indeterminates $X_1, \ldots, X_m, Y_1, \ldots, Y_m$ and the group $SK_1 F(0) = SK_1 K[X_1, \ldots, X_m]$, isomorphic to $SK_1 K$ according to Quillen Theorem [16], 6, th. 7, being trivial, $F$ is therefore a “classic regular filtration” on $A_m(K)$ making the later be a “classic regular ring” and in particular an Ore domain.

3) According to the theory of determinants over Ore domains, there exists an unique map from the set of square matrices of elements of $A_m(K)$ and with values in $gr_F(A_m(K))$, denoted by $\det_F$ and called “the canonical determinant over the filtered Weyl algebra over $K$ of index $m$” or “the principal determinant over the
Weyl algebra over $K$ of index $m$, such that, for any square matrices $a, b$ of the same size with coefficients in $A_m(K)$ and any diagonal matrix $\text{diag}(x, 1, \ldots, 1)\) with diagonal coefficients $x, 1, \ldots, 1$ in $A_m(K)$, we have:

$$\det_F(ab) = \det_F(a)\det_F(b) \quad \text{(the homomorphism axiom)}$$

$$\det_F(\text{diag}(x, 1, \ldots, 1)) = \text{gr}_F(x) \quad \text{(the prolongation axiom)}$$

4) The first fundamental property following from this homomorphism axiom is that for any elementary $e$ with coefficients in $A_m(K)$, we have:

$$\det_F(e) = 1 \quad \text{(the elementary property)}$$

5) The second fundamental property following easily from these two axioms and this elementary property is that for any triangular matrix $t$ with coefficient in $A_m(K)$, $\det_F(t)$ is the product of the principal symbols of its diagonal coefficients (the triangular property)

6) The third fundamental property following from these two axioms and this triangular property is that for any $n$ by $n$ matrix $a$ with coefficients in $A_m(K)$, $\det_F(a)$ can be computed in a practical way by “Gauss method”. Indeed, thanks to the left commun multiple property of the Ore domain $A_m(K)$ and to suitable combinations on the lines of square matrices with coefficients in $A_m(K)$, it is possible to find $n$ by $n$ elementary or cancellable diagonal matrices $p_1, \ldots, p_r$ with coefficients in $A_m(K)$ such that $p_1 \ldots p_r a$ is an upper triangular matrix $t$. Then, thanks to the homomorphism axiom and the triangular property, we obtain the following explicit expression of $\det_F(a)$ as quotient of two principal symbols of elements of $A_m(K)$ which can finally be reduced to the principal symbol of a element of $A_m(K)$ thanks to the factoriality of the ring $\text{gr}_F A_m(K)$ and to the “regularity theorem”:

$$\det_F(a) = \prod_{1 \leq i \leq n} \text{gr}_F(t(i, i)) / \prod_{1 \leq k \leq r} \prod_{1 \leq i \leq n} \text{gr}_F(p_k(i, i))$$

7) If the field $K$ is “effective” in the sense of [2], then it follows from this last formula and from the “effectivity” of the Ore property of the “effective ring” $A_m(K)$ that the restriction of $\det_F$ to each “effective ring” of square matrices with coefficients in $A_m(K)$ is an “effectively computable” function, as proved in [1].

8) It also follows from this formula that if $a$ and $b$ are two square matrices with coefficients in $A_m(K)$ and if

$$a \oplus b = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$$

then

$$\det_F(a \oplus b) = \det_F(a)\det_F(b)$$

9) The link between the classical determinant over a commutative ring and $\det_F$ is that if $B$ is any commutative sub-ring of $A_m(K)$ and $\det$ the classical determinant over $B$, then:

$$\det_F|_B = \text{gr}_F \circ \det$$

10) Finally, one of the most remarkable analogies already discovered between the classical determinant over a commutative ring and $\det_F$ is that an element $a$ of a ring of square matrices over $A_m(K)$ is invertible if and only if $\det_F(a)$ is invertible
in $\text{gr}_F A_m(K)$, i.e. a non zero element of $K$.

**Theorem**

For any integer $n$ greater than 2, a $n$ by $n$ matrix $a$ with coefficients in any Weyl algebra over any field of characteristic zero is a product of elementary matrices with coefficient in this algebra if and only its canonical determinant over this algebra is 1.

**Proof**

1) Let $A_m(K)$ be such a Weyl algebra and $F$ its differential filtration.

2) If $a$ is a product of elementary matrices with coefficients in $A_m(K)$, then according to the homomorphism axiom and the elementary property of $\text{det}_F$, it is obvious that $\text{det}_F(a) = 1$.

3) So let us assume conversly that $\text{det}_F(a) = 1$. According the above characterization of elements of $GL_n(A_m(K))$ between those of $M_n(A_m(K))$, $a \in GL_n(A_m(K))$.

4) $F$ being a classical regular filtration as we remarked above, it follows from the cited Quillen theorem that the canonical map from $K_1 F(0)$ in $K_1 A_m(K)$ is an isomorphism. So there exists a positive integer $p$, a $p$ by $p$ matrix $b$ with coefficients in $F(0)$ and integers $r$ and $s$ such that $n + r = p + s$ and :

\[
(*) \quad (a \oplus i_r)^{-1}(b \oplus i_s) \in E_{n+r}(A_m(K))
\]

where $i_j$ denotes the unit of the group $GL_j(A_m(K))$.

5) According to $(*)$ and to the points 3), 4), 8) and 9) of the recall, we have :

\[
\text{det}_F(b \oplus i_s) = \text{det}(b \oplus i_s) = 1
\]

6) Since $F$ is a classical regular filtration, in particular since $SK_1 F(0)$ is trivial, it follows from this last equality that there exists a positif integer $t$ such that :

\[
b \oplus i_{s+t} \in E_{p+s+t}(F(0))
\]

7) So according to $(*)$, we have :

\[
a \oplus i_{r+t} \in E_{n+r+t}(A_m(K))
\]

8) On the other hand, according to Stafford cited theorem, the stable range of $A_m(K)$ is 2. Thanks to Vaserstein cited theorem, it follows from the last relation as desired that :

\[
a \in E_n(A_m(K))
\]

Q.E.D.

**Remark 1**

1) The previous theorem may by interpreted in terms of systems of partial differential equations with solutions in any $K[X_1, \ldots, X_m]$-algebra $B$ which is a $A_m(K)$-left-module, more precisely in terms of such a system which is “elementary resoluble by derivation”, following and refining [1].
2) The statement of the previous theorem is clearly the faithful generalization to Weyl algebras over a field of characteristic zero of Suslin theorem for algebras of polynomials over such a field.

3) Furthermore, it solves the $K_1$-analogue of Serre Conjecture over Weyl algebras, since according to points (2) and (7) of the previous proof, for any square matrix $a$ with entries in a Weyl algebra over any field of characteristic zero $A$, $\det_F(a) = 1$ means that this matrix is “stably” a product of elementary square matrices with entries in $A$.

4) On the other hand, Suslin theorem would follow from the previous theorem, with a non commutative algebraic proof completely different from Suslin original commutative algebraic proof, if the following “natural” conjecture could be confirmed:

**Conjecture**

If $A$ is a ring endowed with an increasing $\mathbb{N}$ - filtration $F$ such that the associated graded ring is a domain, then for any integer $n \geq 2$, the sub-ring $F(0)$ of $A$ verifies:

$$GL_n(F(0)) \cap E_n(A) = E_n(F(0))$$

**Remark 2**

1) The assumption on the associated graded ring means that the associated degree function $deg_F$ is ”additive”, i.e. is an homomorphism from $(A - \{0\}, \times)$ to $(\mathbb{N}, +)$. Moreover, this degree function is such that $F(0) - \{0\} = deg_F^{-1}(0)$.

2) Using this degree function, it seems that the proof of the conjecture could be purely formal, as this statement could be easily checked for $n = 2$ in the case where the considered element of $GL_n(F(0)) \cap E_n(A)$ is a product of at most five elements of $A$.

3) The main interest of this conjecture is that thanks to it, a non trivial property of a commutative ring (Suslin theorem) could be deduced from the similar property of a “simple” non commutative extension of this ring (previous theorem), in an analogous way as some deep properties of the field of real numbers could be deduced from the similar ones of an “algebraically closed” extension of this field like the field of complex numbers.

4) This kind of contribution of non commutative algebra to commutative algebra seems non common in mathematics. A confirmed example of such a contribution is the fact that the famous Jacobian Conjecture, which claim that any endomorphism of an algebra of polynomials over a field of characteristic zero with a non zero jacobian in this field is an automorphism, could be deduced from Dixmier Conjecture, which claims that any endomorphism of a Weyl algebra over a field of characteristic zero is an automorphism (see for instance [7], p. 297).

5) Another interest of the proposed conjecture is that it would prove that Cohn counter-example cited in the introduction is even better than one thinks now, in the sense that it is not in $E_2(A_2(\mathbb{Q}))$, showing in this way that the lower bound 3 for $n$ in the previous theorem is the finest as in Suslin theorem.
6) Conformly to the “effectiveness” problem evocated in the introduction and the recall, a natural question risen from the previous theorem is the following:

For an integer \( n \) greater than 2, how to split “effectively” a \( n \times n \) matrix with coefficients in a Weyl algebra over an “effective” field of characteristic zero and with principal determinant 1 as a product of elementary ones?

7) According to the prominent part that the cited Stafford stable rank theorem plays in the proof of the previous theorem, it is easy to conjecture that this last “effectiveness” problem should need the resolution of the following one:

Given three elements \( a, b, c \) of a Weyl algebra over an “effective” field, how to find “effectively” two elements \( d \) and \( e \) of this algebra such that \( a + dc \) and \( b + ec \) generate that same left ideal of this algebra as \( a, b \) and \( c \)?

8) Let us now consider a last question risen from the previous theorem.

Indeed, for the problems met in the algebraic theory of partial differential equations, the working noetherian domains of differential operator are not Weyl algebras, but what could be called “formal (resp. convergent)” Weyl algebras, deduced from “classical” Weyl algebras by replacing polynomials by formal (resp. convergent) power series (see for instance [5], p. 404). Since it is natural to hope that the previous theorem works also for these “working” Weyl algebras, the proof of this theorem lead us to the following question:

Is the stable rank of a “formal” (resp. “convergent”) Weyl algebra over a field of characteristic zero (resp. a sub-field of the field of complex numbers) also 2?

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