Existence and Regularity of a Nonhomogeneous Transition Matrix under Measurability Conditions

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Abstract: This paper is about the existence and regularity of the transition probability matrix of a nonhomogeneous continuous-time Markov process with a countable state space. A standard approach to prove the existence of such a transition matrix is to begin with a continuous (in $t$) and conservative matrix $Q(t) = [q_{ij}(t)]$ of nonhomogeneous transition rates $q_{ij}(t)$, and use it to construct the transition probability matrix. Here we obtain the same result except that the $q_{ij}(t)$ are only required to satisfy a mild measurability condition, and $Q(t)$ may not be conservative. Moreover, the resulting transition matrix is shown to be the minimum transition matrix and, in addition, a necessary and sufficient condition for it to be regular is obtained. These results are crucial in some applications of nonhomogeneous continuous-time Markov processes, such as stochastic optimal control problems and stochastic games, which motivated this work in the first place.

Key words: Nonhomogeneous continuous–time Markov chains, nonhomogeneous transition rates, Kolmogorov equations, minimum transition matrix.

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1 INTRODUCTION

Nonhomogeneous continuous-time Markov processes have applications in a wide variety of contexts, including stochastic control problems [9, 10, 11, 13, 14, 16, 20], stochastic games [7, 8, 17, 18], and queueing systems and stochastic networks [3, 17, 19], to name a few. As is well known, such a Markov process is uniquely determined by its transition probability matrix $P(s, t) = [P_{ij}(s, t)]$ (for $i, j \in S$ and $0 \leq s \leq t$), which in turn is usually constructed from a given matrix $Q(t) = [q_{ij}(t)]$ of transition rates $q_{ij}(t)$, for $t \geq 0$. Therefore, a natural question is, under what conditions on $Q(t)$ is the transition matrix $P(s, t)$ uniquely determined?

To answer this question, a standard approach—which can be traced back to Feller’s 1940 paper [5]—is to assume that the transition rates $q_{ij}(t)$ are continuous in $t \geq 0$; see, e.g., [3, 6, 7, 8, 9, 10, 13, 15, 20]. This continuity requirement, however, imposes severe restrictions in some applications, for instance, in stochastic control and game theory, where discontinuous “policies” typically lead to discontinuous transition rates. To illustrate this situation, which was the main motivation for this work, let us consider the following example.

**Example.** Consider a single-server queueing system in which the state variable $i$ denotes the number of jobs in the system at each time $t \geq 0$. Suppose that a controller wants to control the system’s service rate $\mu$ according to the current state $i \in S := \{0, 1, \ldots\}$. When the state is $i$ at some time $t$, the controller takes a service rate $\mu$ from a given finite set $A(i)$ of available “actions” at state $i$; then the system transfers to another state $j$ according to the transition rate induced by the chosen $\mu$, and the process is repeated. The choice of service rates is done according to so-called control policies $\pi = \{\pi_t, t \geq 0\}$. If the controller is using a particular policy $\pi = \{\pi_t, t \geq 0\}$ and the state at time $t$ is $i \in S$, then the controller takes the service rate $\pi_t(i) \in A(i)$. To be more specific, consider the policy $\pi$ given by

$$\pi_t(i) := \sum_{k=0}^{\infty} \mu_k(i) 1_{[k, k+1)}(t),$$

with $\mu_k(i) \in A(i)$ for all $i \in S$ and $k \geq 0$. Hence, when using this policy, if the present state is $i$, then the controller chooses the action $\mu_k(i) \in A(i)$ during the time interval $[k, k+1)$. Obviously, $\pi_t(i)$ is measurable in $t \geq 0$, but not continuous. Similarly, if $q_{ij}^\mu$ denotes the transition rate from $i$ to $j$ under $\mu \in A(i)$, then the matrix $Q^\pi(t) = [q_{ij}^\pi(t)]$ of transition rates when using $\pi$ has elements

$$q_{ij}^\pi(t) := \sum_{k=0}^{\infty} q_{ij}^{\mu_k(i)} 1_{[k, k+1)}(t).$$

Therefore, the transition rates are measurable in $t \geq 0$ but not continuous, and so we cannot use the standard Markov chain theory to show the existence of transition probability functions $P^\pi_{ij}(s, t)$ induced by the discontinuous control policy $\pi$ defined in (1.1). An analogous situation occurs in stochastic game problems [7, 8], where discontinuous “strategies” usually lead to discontinuous transition rates. This is precisely what motivated our paper—to establish the existence and regularity of a nonhomogeneous transition matrix without requiring the transition rates to be continuous.
Summarizing, our main objective is to replace the above-mentioned continuity requirement by a mild measurability condition under which we obtain the existence and uniqueness of a transition matrix \( P(s,t) \), even if the \( Q(t) \) matrix is not conservative. (See Section 2 for definitions.) In fact, the existence and uniqueness of \( P(s,t) \) under a measurability condition have been considered in [13], but the results there are stated without proofs and assuming, in addition, that \( Q(t) \) is conservative. We also obtain some new results (see, for instance, Theorems 2(i) and 2(iii), Theorem 3(ii)).

In this paper, firstly, we introduce a precise definition of a nonhomogeneous transition matrix (Definition 1), which is weaker than previous definitions, e.g., as in [3, 6, 12]. Even in this weaker context, we can obtain some key properties of the transition matrix (Theorem 1). Secondly, given a \( Q(t) \) matrix satisfying our measurability condition, we construct a nonhomogeneous transition matrix (Theorem 2), and, finally, we give a necessary and sufficient condition for this transition matrix to be unique and regular (Theorem 3).

The rest of this paper is organized as follows. In Section 2 we present the definitions and main results concerning nonhomogeneous transition matrices. The proofs of our results are all given in Section 3. In Section 4 we state some conclusions.

2 MAIN RESULTS

2.1 Nonhomogeneous Pretransition Matrices

Throughout the following \( S \) denotes a given countable set.

**Definition 1.** A real-valued matrix \( P(s,t) = (P_{ij}(s,t), \ i, j \in S) \), defined for all \( 0 \leq s \leq t < \infty \), is called a nonhomogeneous pretransition matrix if it satisfies the following for every \( i, j \in S \) and \( 0 \leq s \leq t < \infty \):

(i) \[ P_{ij}(s,t) \geq 0, \quad \text{and} \quad \sum_{j \in S} P_{ij}(s,t) \leq 1; \quad (2.1) \]

(ii) \[ P_{ij}(s,t) = \sum_{k \in S} P_{ik}(s,u)P_{kj}(u,t) \quad \forall s \leq u \leq t; \quad (2.2) \]

(iii) \[ \lim_{h \to 0^+} |P_{ij}(s,s+h) - \delta_{ij}| = 0 \quad \text{uniformly in} \ j \in S, \quad \text{and} \quad P_{ij}(s,s) = \delta_{ij}, \quad (2.3) \]

where \( \delta_{ij} \) stands for the Kronecker symbol (\( \delta_{ij} = 0 \) if \( i \neq j \); \( \delta_{ij} = 1 \) if \( i = j \)).

If in addition \( \sum_{j \in S} P_{ij}(s,t) \equiv 1 \) for every \( i \in S \), then \( P(s,t) \) is said to be a nonhomogeneous transition probability matrix, and its elements \( P_{ij}(s,t) \) are called transition functions.

The equation (2.2) is known as the Chapman-Kolmogorov (C–K) equation. Our definition of a pretransition matrix is weaker than that in [3, 6, 12], but still we can obtain most of the standard results. In particular, the following theorem is essentially the same as Theorem 1.1 and Theorem 1.2 in Chapter 3 of [12].
Theorem 1. For any nonhomogeneous pretransition probability matrix $P(s,t)$ we have:

(i) \[ P_{ii}(s,t) > 0 \quad \forall \ i \in S, \ 0 \leq s \leq t < \infty; \quad (2.4) \]

(ii) \[ |P_{ij}(u,t) - P_{ij}(v,t)| \leq 1 - P_{ii}(u \land v, u \lor v) \quad \forall \ i,j \in S, 0 \leq u, v \leq t, \]
where $u \land v = \min(u,v)$, $u \lor v = \max(u,v)$;

(iii) $P_{ij}(s,t)$ is continuous in $s \in [0,t]$ (right-continuous in 0, left-continuous in $t$), and uniformly in $t \geq 0$ and $j \in S$;

(iv) $P_{ij}(s,t)$ is continuous in $t \in [s, +\infty)$ (right-continuous in $s$), and uniformly in $j \in S$;

(v) The following holds:
\[
\lim_{t \to s+} \frac{P_{ii}(s,t) - 1}{t - s} = \lim_{t \to s-} \frac{P_{ii}(s,t) - 1}{t - s} =: q_{ii}(s) \leq 0, \\
\lim_{t \to s+} \frac{P_{ij}(s,t)}{t - s} = \lim_{t \to s-} \frac{P_{ij}(s,t)}{t - s} =: q_{ij}(s) \geq 0 \text{ for } j \neq i; \\
\]

(vi) For every $i \in S$ and $s \geq 0$, $\sum_{j \neq i} q_{ij}(s) \leq -q_{ii}(s)$. (Hence, $q_{ij}(s) \leq q_i(s)$ for all $i, j \in S$ and $s \geq 0$, where $q_i(s) := -q_{ii}(s) \geq 0$.)

Proof. See subsection 3.1.

\[ \square \]

2.2 Nonhomogeneous $Q(t)$-Transition Matrices

In this subsection we introduce the definition of a nonhomogeneous $Q(t)$-matrix and a nonhomogeneous transition matrix induced by $Q(t)$, which are related by our key hypothesis, Assumption A.

Definition 2. For each $i, j \in S$, let $q_{ij}(t)$ be a real-valued function defined on $[0, +\infty)$. The matrix function $Q(t) = (q_{ij}(t), i, j \in S)$ is said to be a nonhomogeneous $Q(t)$-matrix on $S$ if for every $i, j \in S$ and $t \geq 0$ satisfies that:

(i) \[ 0 \leq q_{ij}(t) < \infty \text{ if } i \neq j, \text{ and } 0 \leq -q_{ii}(t) < \infty; \quad (2.6) \]

(ii) \[ \sum_{j \in S} q_{ij}(t) \leq 0 \quad \forall \ i \in S. \quad (2.7) \]

If in addition $\sum_{j \in S} q_{ij}(t) = 0$ for all $i \in S$ and $t \geq 0$, then we say that $Q(t)$ is conservative.
We now introduce our measurability-integrability assumption with respect to the Lebesgue measure on $[0, +\infty)$.

**Assumption A** Let $Q(t)$ be a given nonhomogeneous $Q(t)$-matrix. For every $b \geq a \geq 0$ and $i, j \in S$, we have: $q_{ij}(t)$ is Borel-measurable in $t \in [a, b]$ if $i \neq j$, and $q_{ii}(t)$ is integrable on $[a, b]$.

The following definition relates a $Q(t)$-matrix and a pretransition matrix. (The abbreviations $a.e.$ (almost everywhere) and $a.a.$ (almost all) refer to the Lebesgue measure.)

**Definition 3.** Let $Q(t) = (q_{ij}(t), i, j \in S)$ ($t \geq 0$) be a nonhomogeneous $Q(t)$-matrix satisfying Assumption A. If a nonhomogeneous pretransition matrix $P(s, t) = (P_{ij}(s, t), i, j \in S)$ ($0 \leq s \leq t < \infty$) satisfies that, for every $i, j \in S$ and a.a. $s \geq 0$, the partial derivatives

$$
\frac{\partial P_{ij}(s, t)}{\partial s} \quad \text{and} \quad \frac{\partial}{\partial t} P_{ij}(s, t)|_{t=s^+} = \lim_{h \to 0^+} \frac{P_{ij}(s, s + h) - \delta_{ij}}{h} = q_{ij}(s)
$$

exist, then $P(s, t)$ is called a nonhomogeneous $Q(t)$-transition matrix. Further, such a $Q(t)$-transition matrix is said to be regular if $P(s, t)$ is a transition probability matrix and it is the unique transition probability matrix that satisfies (2.8). We call $P_{ij}(s, t)$ a nonhomogeneous $Q(t)$-transition function or simply a $Q(t)$-function.

### 2.3 Existence of a Nonhomogeneous $Q(t)$-Transition Matrices

The question now is, given a nonhomogeneous $Q(t)$-matrix, how can we ensure the existence of a nonhomogeneous $Q(t)$-transition matrix? The following Theorem 2 shows that this can be done if the $Q(t)$-matrix satisfies Assumption A. To obtain a regular $Q(t)$-transition matrix, however, we have to go a bit further and construct a minimum $Q(t)$-transition matrix on which we can readily impose conditions for it to be regular. This is done in Theorem 3.

**Theorem 2.** Given a nonhomogeneous $Q(t)$-matrix satisfying Assumption A, let

$$
d_{ij}(u) := \delta_{ij}(-q_{ii}(u)) \quad \forall \ i, j \in S, \ u \geq 0,
$$

and for each $i, j \in S$ and $0 \leq s \leq t < \infty$, define recursively

$$
P_{ij}^{(0)}(s, t) := \delta_{ij} e^{\int_s^t q_{ii}(u)du},
$$

$$
P_{ij}^{(n+1)}(s, t) := \int_s^t \sum_{k \in S} e^{\int_u^t q_{ik}(v)dv} [q_{ik}(u) + d_{ik}(u)] P_{kj}^{(n)}(u, t)du \quad \forall n \geq 0.
$$

Let

$$
P_{ij}(s, t) := \sum_{n=0}^{\infty} P_{ij}^{(n)}(s, t).
$$
Similarly, for each \(i, j \in S\) and \(0 \leq s \leq t < \infty\), define recursively
\[
Q^{(0)}_{ij}(s, t) := \delta_{ij} e^{\int_s^t q(u) du},
\]
\[
Q^{(n+1)}_{ij}(s, t) := \int_s^t \sum_{k \in S} Q^{(n)}_{ik}(s, u) e^{\int_u^t q_j(v) dv}[q_{kj}(u) + d_{kj}(u)] du \quad \forall n \geq 0.
\]
Let
\[
Q_{ij}(s, t) := \sum_{n=0}^\infty Q^{(n)}_{ij}(s, t).
\]

Then, for every \(i, j \in S\) and \(0 \leq s \leq t < \infty\), we have
(i) \(P^{(n)}_{ij}(s, t) = Q^{(n)}_{ij}(s, t)\) for all \(n \geq 0\), and so \(P_{ij}(s, t) = Q_{ij}(s, t)\);
(ii) \(P_{ij}(s, t) = \int_s^t \sum_{k \in S} P_{ik}(s, v)q_{kj}(v)dv + \delta_{ij}\);
(iii) \(P_{ij}(s, t) = \int_s^t \sum_{k \in S} q_{ik}(v)P_{kj}(v, t)dv + \delta_{ij}\);
(iv) \(\mathcal{P}(s, t) = (\mathcal{P}_{ij}(s, t), i, j \in S)\) is a nonhomogeneous \(Q(t)\)-transition matrix.

\textbf{Proof.} See subsection 3.2. \hfill \blacksquare

The following theorem states our main results in this paper. It shows that, whether \(Q(t)\) is conservative or not, the \(Q(t)\)-transition matrix \(\mathcal{P}(s, t)\) constructed in Theorem 2 is minimum; see (2.19). Moreover, it gives a reasonably mild necessary and sufficient condition for \(\mathcal{P}(s, t)\) to be regular; see (2.20).

\textbf{Theorem 3.} Assume that \(Q(t)\) is a nonhomogeneous \(Q(t)\)-matrix satisfying Assumption \(A\), and let \(\mathcal{P}(s, t)\) be the nonhomogeneous \(Q(t)\)-transition matrix in Theorem 2. Then:

(i) \(\mathcal{P}(s, t)\) is the minimum \(Q(t)\)-transition matrix, that is, for any nonhomogeneous \(Q(t)\)-transition matrix \(P(s, t) = (P_{ij}(s, t), i, j \in S)\), we have
\[
P_{ij}(s, t) \geq \mathcal{P}_{ij}(s, t) \quad \forall i, j \in S, \ 0 \leq s \leq t < \infty.
\]
(ii) \(\mathcal{P}(s, t)\) is regular if and only if
\[
\sum_{j \in S} \int_s^t \sum_{k \in S} \mathcal{P}_{ik}(s, v)q_{kj}(v)dv \equiv 0 \quad \forall i \in S, \ 0 \leq s \leq t < \infty.
\]

\textbf{Proof.} See subsection 3.3. \hfill \blacksquare
3 PROOFS

3.1 Proof of Theorem 1

To prove Theorem 1 we will use the following lemma in which, for \( h > 0 \) and an integer \( m \geq 1 \), we denote by \( A P_{i,B}(s, s + mh) \) the probability of transition from the state \( i \) at time \( s \) to the set \( B \) at time \( s + mh \) while avoiding the set \( A \) at times \( s + kh \) for \( k = 1, \ldots, m - 1 \). Observe that

\[
A P_{i,B}(s, s + h) = P_{i,B}(s, s + h),
\]

and, for \( m \geq 2 \),

\[
A P_{i,B}(s, s + mh) = \sum_{r \not\in A} A P_{ir}(s, s + (m - 1)h) P_{r,B}(s + (m - 1)h, s + mh).
\]

Lemma 1. Let \( P(s, t) = (P_{ij}(s, t), i, j \in S) \) be a nonhomogeneous pretransition matrix and, for \( 0 \leq s < t \), let \( 0 < h < t - s \), \( n := [h^{-1}(t - s)] \), and \( 1 \leq m \leq n \). Then:

(i) For every \( i \in S \) and \( B \subset S \),

\[
P_{i,B}(s, t) = \sum_{l=1}^{m} \sum_{k \in A} A P_{ik}(s, s + lh) P_{k,B}(s + lh, t)
+ \sum_{k \not\in A} A P_{ik}(s, s + mh) P_{k,B}(s + mh, t). \tag{3.1}
\]

(ii) For every \( 0 < \varepsilon < 1/3 \), there exists \( 0 < \delta < 1 \) such that when \( t - s < \delta \), we have

\[
P_{ij}(s, t) \geq (1 - 3\varepsilon) \sum_{l=1}^{n} P_{ij}(s + (l - 1)h, s + lh) \quad \forall j \neq i. \tag{3.2}
\]

Proof. (i) We will use induction on \( m \). For \( m = 1 \), (3.1) holds by the definition of \( A P_{i,B}(s, s + mh) \) and the C-K equation (2.2). Now assume that (3.1) holds for \( m - 1 \). We will prove that it holds for \( m \). Indeed, by the definition of \( A P_{i,B}(s, s + mh) \), the C-K equation (2.2) again and the induction hypothesis, we have

\[
P_{i,B}(s, t)
= \sum_{l=1}^{m-1} \sum_{k \in A} A P_{ik}(s, s + lh) P_{k,B}(s + lh, t)
+ \sum_{k \not\in A} A P_{ik}(s, s + (m - 1)h) P_{k,B}(s + (m - 1)h, t)
= \sum_{l=1}^{m} \sum_{k \in A} A P_{ik}(s, s + lh) P_{k,B}(s + lh, t) - \sum_{k \in A} A P_{ik}(s, s + mh) P_{k,B}(s + mh, t)
+ \sum_{k \not\in A} A P_{ik}(s, s + (m - 1)h) P_{k,B}(s + (m - 1)h, t)
\]
\[ \sum_{k \not\in A} A P_k(s, s + mh) P_{k,B}(s + mh, t) - \sum_{k \not\in A} A P_k(s, s + mh) P_{k,B}(s + mh, t) = \sum_{k \not\in A} A P_k(s, s + lh) P_{k,B}(s + lh, t) + \sum_{k \not\in A} A P_k(s, s + mh) P_{k,B}(s + mh, t) \\
+ \sum_{k \not\in A} A P_k(s, s + (m - 1)h) P_{k,B}(s + (m - 1)h, t) \\
- \sum_{k \in S} A P_k(s, s + mh) P_{k,B}(s + mh, t) = \sum_{l=1}^{m} \sum_{k \not\in A} A P_k(s, s + lh) P_{k,B}(s + lh, t) + \sum_{k \not\in A} A P_k(s, s + mh) P_{k,B}(s + mh, t). \]

This completes the induction.

(ii) Taking \( B = A = \{j\} \) and \( m = n \) in (3.1), we obtain

\[ P_{ij}(s, t) = \sum_{l=1}^{n} j P_{ij}(s, s + lh) P_{jj}(s + lh, t) + \sum_{k \not= j} j P_{ik}(s, s + nh) P_{kj}(s + nh, t). \]  

On the other hand, (2.3) implies that for any given \( 0 < \varepsilon < 1/3 \) and \( j \not= i \), there exists \( 0 < \delta < 1 \) such that when \( 0 < t - s < \delta \), we have \( P_{ii}(s, t) > 1 - \varepsilon \), \( P_{jj}(s, t) > 1 - \varepsilon \), and \( P_{ij}(s, t) < \varepsilon \). These facts together with (3.3) imply that for \( h < t - s < \delta \) and \( j \not= i \)

\[ \varepsilon > 1 - P_{ii}(s, t) \geq \sum_{k \not= i} P_{ik}(s, t) \geq \sum_{l=1}^{n} j P_{ij}(s, s + lh) P_{jj}(s + lh, t) \geq (1 - \varepsilon) \sum_{l=1}^{n} j P_{ij}(s, s + lh), \]

so

\[ \sum_{l=1}^{n} j P_{ij}(s, s + lh) \leq \frac{\varepsilon}{1 - \varepsilon}. \]

Note that, for each \( 1 \leq l \leq n \),

\[ P_{ii}(s, s + lh) = j P_{ii}(s, s + lh) + \sum_{m=1}^{l-1} j P_{ij}(s, s + mh) P_{ji}(s + mh, s + lh). \]

Thus

\[ j P_{ii}(s, s + lh) = P_{ii}(s, s + lh) - \sum_{m=1}^{l-1} j P_{ij}(s, s + mh) P_{ji}(s + mh, s + lh) \geq P_{ii}(s, s + lh) - \sum_{m=1}^{l-1} j P_{ij}(s, s + mh) \geq 1 - \varepsilon - \frac{\varepsilon}{1 - \varepsilon}, \]
and so

\[ P_{ij}(s, t) \geq \sum_{l=1}^{n} j P_{ij}(s, s + lh) P_{jj}(s + lh, t) \]

\[ = \sum_{l=1}^{n} \sum_{r \neq j} j P_{ir}(s, s + (l-1)h) P_{rj}(s + (l-1)h, s + lh) P_{jj}(s + lh, t) \]

\[ \geq \sum_{l=1}^{n} j P_{ij}(s, s + (l-1)h) P_{ij}(s + (l-1)h, s + lh) P_{jj}(s + lh, t) \]

\[ \geq (1 - \varepsilon - \frac{\varepsilon}{1 - \varepsilon}) \sum_{l=1}^{n} P_{ij}(s + (l-1)h, s + lh)(1 - \varepsilon) \]

\[ \geq (1 - 3\varepsilon) \sum_{l=1}^{n} P_{ij}(s + (l-1)h, s + lh). \]

This completes the verification of (3.2) and also the proof of the lemma. \( \square \)

With Lemma 1, we can easily prove Theorem 1.

**Proof of Theorem 1**

(i) By the C-K equation (2.2), we obtain

\[ P_{ii}(s, t) \geq \prod_{k=1}^{n} P_{ii} \left( s + \frac{k-1}{n}(t-s), s + \frac{k}{n}(t-s) \right) \forall n \geq 1. \]

Hence, for \( n \) sufficiently large, (2.4) follows from (2.3).

(ii) Let \( 0 \leq u \leq v \leq t < \infty \). By the C-K equation (2.2)

\[ P_{ij}(u, t) - P_{ij}(v, t) = \sum_{k \neq i} P_{ik}(u, v) P_{kj}(v, t) + (P_{ii}(u, v) - 1)P_{ij}(v, t). \]

Applying (2.1), we obtain

\[ P_{ij}(u, t) - P_{ij}(v, t) \geq (P_{ii}(u, v) - 1)P_{ij}(v, t) \geq P_{ii}(u, v) - 1, \]

\[ P_{ij}(u, t) - P_{ij}(v, t) \leq \sum_{k \neq i} P_{ik}(u, v) P_{kj}(v, t) \leq \sum_{k \neq i} P_{ik}(u, v) \leq 1 - P_{ii}(u, v). \]

These inequalities yield (2.5).

(iii) Using (2.5) and (2.3), for each \( h \geq 0 \) we have

\[ |P_{ij}(s + h, t) - P_{ij}(s, t)| \leq 1 - P_{ii}(s, s + h) \to 0 \text{ as } h \to 0^+, \]

\[ |P_{ij}(s, t) - P_{ij}(s - h, t)| \leq 1 - P_{ii}(s - h, s) \to 0 \text{ as } h \to 0^+, \]

which together with (2.3) yield (iii).

(iv) By the C-K equation (2.2), for each \( h \geq 0 \) we have

\[ P_{ij}(s, t + h) - P_{ij}(s, t) = \sum_{k \neq j} P_{ik}(s, t) P_{kj}(t, t + h) - (1 - P_{jj}(t, t + h))P_{ij}(s, t). \]
Therefore

\[ P_{ij}(s, t + h) - P_{ij}(s, t) \geq -\sum_{k \in S} (1 - P_{kk}(t, t + h))P_{ik}(s, t), \]

and

\[ P_{ij}(s, t + h) - P_{ij}(s, t) \leq \sum_{k \neq j} P_{ik}(s, t)P_{kj}(t, t + h) \leq \sum_{k \in S} P_{ik}(s, t)(1 - P_{kk}(t, t + h)). \]

Hence, as \( h \to 0^+ \),

\[ |P_{ij}(s, t + h) - P_{ij}(s, t)| \leq \sum_{k \in S} P_{ik}(s, t)(1 - P_{kk}(t, t + h)) \to 0, \]

by (2.3) and the Dominated Convergence Theorem [2]. It follows that \( P_{ij}(s, t) \) is right-continuous in \( t \in [s, +\infty) \), uniformly in \( j \in S \).

Similar arguments show that \( P_{ij}(s, t) \) is left-continuous in \( t \in [s, +\infty) \), uniformly in \( j \in S \).

(v) To avoid trivial situations we suppose that \( i \in S \) is not an absorbing state, i.e., \( P_{ii}(s, t) \neq 0 \). For \( 0 \leq s \leq t < +\infty \), let \( f(s, t) := -\log P_{ii}(s, t) \), which is a well-defined function, nonnegative and finite. Since \( P_{ii}(s, t) \geq P_{ii}(s, u)P_{ii}(u, t) \), for \( 0 \leq s \leq u \leq t < \infty \), we have \( f(s, t) \leq f(s, u) + f(u, t) \). Now for each \( s \geq 0 \), let \( q_i(s) := \sup_{0 < t < \infty} \frac{f(s, t)}{t - s} \). We will next prove that the limit of \( \frac{f(s, t)}{t - s} \) exists and equals \( q_i(s) \).

Obviously, by definition of \( q_i(s) \),

\[ \lim_{t \to s^+} \sup_{t \to s^+} \frac{f(s, t)}{t - s} \leq q_i(s). \]

Therefore, it is sufficient to argue that \( \liminf_{t \to s^+} \frac{f(s, t)}{t - s} \geq q_i(s) \). Given any \( 0 < h < t - s \), take \( n \) such that \( t - s = nh + \varepsilon \), with \( 0 \leq \varepsilon < h \). Then

\[ \frac{f(s, t)}{t - s} \leq \frac{nh}{t - s} \frac{f(s, s + nh)}{nh} + \frac{f(s + nh, t)}{t - s}. \]

Now take the limit of both sides as \( h \to 0^+ \), and note that \( nh \to t - s \) as \( \varepsilon \to 0^+ \), so that the continuity of \( P_{ij}(s, t) \) implies that \( f(s + nh, t) = f(t - \varepsilon, t) \to 0 \). Hence we have

\[ \frac{f(s, t)}{t - s} \leq \liminf_{h \to 0^+} \frac{f(s, s + nh)}{nh} = \liminf_{t \to s^+} \frac{f(s, t)}{t - s}, \]

and so

\[ \lim_{t \to s^+} \frac{f(s, t)}{t - s} = q_i(s). \]

Finally, recalling the definition of \( f(s, t) \), we have

\[ \lim_{t \to s^+} \frac{1 - P_{ii}(s, t)}{t - s} = \lim_{t \to s^+} \frac{1 - e^{-f(s, t)}}{t - s} = \lim_{t \to s^+} \frac{1 - e^{-f(s, t)}}{f(s, t)} \frac{f(s, t)}{t - s} = q_i(s). \]

This proves the first part of (v). Next we prove the second part.
To this end, first note that (2.3) implies that for any given $0 < \varepsilon < 1/3$ and $j \neq i$, there exists $0 < \delta < 1$ such that when $0 < t - s < \delta$, we have $P_{ii}(s, t) > 1 - \varepsilon$, $P_{jj}(s, t) > 1 - \varepsilon$, and $P_{ij}(s, t) < \varepsilon$. Since (2.3) holds uniformly in $j \in S$, for $0 < h < t - s$, $A \subset S$, and $i \not\in A$, it follows from (3.2) that

$$P_{i,A}(s, t) \geq (1 - 3\varepsilon) \sum_{l=1}^{n} P_{i,A}(s + (l - 1)h, s + lh). \quad (3.4)$$

In particular, taking $A = S - \{i, j\}$ in (3.4), we obtain

$$P_{i,S-(i,j)}(s, t) \geq (1 - 3\varepsilon) \sum_{l=1}^{n} P_{i,S-(i,j)}(s + (l - 1)h, s + lh). \quad (3.5)$$

On the other hand, taking $B = A = S - \{i\}$ and $m = n$ in (3.1), it follows that

$$P_{i,S-(i)}(s, t) = \sum_{l=1}^{n} \sum_{k \in S - \{i\}} A P_{ik}(s, s + lh) P_{k,S-(i)}(s + lh, t)$$

$$+ A P_{ii}(s, s + nh) P_{i,S-(i)}(s + nh, t)$$

$$\leq \sum_{l=1}^{n} \sum_{k \in S - \{i\}} A P_{ik}(s, s + lh) + A P_{ii}(s, s + nh) P_{i,S-(i)}(s + nh, t)$$

$$= \sum_{l=1}^{n} A P_{i,S-(i)}(s, s + lh) + A P_{ii}(s, s + nh) P_{i,S-(i)}(s + nh, t)$$

$$= \sum_{l=1}^{n} A P_{ii}(s, s + (l - 1)h) P_{i,S-(i)}(s + (l - 1)h, s + lh)$$

$$+ A P_{ii}(s, s + nh) P_{i,S-(i)}(s + nh, t) \quad (\text{since } A = S - \{i\});$$

consequently,

$$P_{i,S-(i)}(s, t) \leq \sum_{l=1}^{n} P_{i,S-(i)}(s + (l - 1)h, s + lh)$$

$$+ A P_{ii}(s, s + nh) P_{i,S-(i)}(s + nh, t). \quad (3.6)$$

Subtracting (3.5) from (3.6), and using (3.5) again, we obtain

$$P_{ij}(s, t) \leq \sum_{l=1}^{n} P_{ij}(s + (l - 1)h, s + lh) + 3\varepsilon \sum_{l=1}^{n} P_{i,S-(i,j)}(s + (l - 1)h, s + lh)$$

$$+ A P_{ii}(s, s + nh) P_{i,S-(i)}(s + nh, t)$$

$$\leq \sum_{l=1}^{n} P_{ij}(s + (l - 1)h, s + lh) + \frac{3\varepsilon}{1 - 3\varepsilon} P_{i,S-(i,j)}(s, t)$$

$$+ A P_{ii}(s, s + nh) P_{i,S-(i)}(s + nh, t). \quad (3.7)$$
Recalling that \( \varepsilon \) was arbitrary, taking the limit of both sides of (3.7) as \( \varepsilon \to 0 \), we see that
\[
P_{ij}(s, t) \leq \sum_{l=1}^{n} P_{ij}(s + (l - 1)h, s + lh) + A P_{ii}(s, s + nh) P_{i,S-{i}}(s + nh, t). \tag{3.8}
\]

Summarizing, by (3.2)
\[
(1 - 3\varepsilon) \sum_{l=1}^{n} \frac{P_{ij}(s + (l - 1)h, s + lh)}{nh} \leq \frac{P_{ij}(s, t)}{nh}, \tag{3.9}
\]
whereas by (3.8)
\[
\frac{P_{ij}(s, t)}{nh} \leq \sum_{l=1}^{n} \frac{P_{ij}(s + (l - 1)h, s + lh)}{nh} + A P_{ii}(s, s + nh) P_{i,S-{i}}(s + nh, t). \tag{3.10}
\]

To conclude, note that \( nh \to t - s \) as \( h \to 0^+ \). Hence, using (2.3), (3.9), and (3.10) we obtain
\[
\limsup_{h \to 0^+} \sum_{l=1}^{n} \frac{P_{ij}(s + (l - 1)h, s + lh)}{nh} = \limsup_{t-s \to 0^+} \frac{P_{ij}(s, t)}{t-s}. \tag{3.11}
\]

From this equality and (3.9) it follows that
\[
\limsup_{t-s \to 0^+} \frac{P_{ij}(s, t)}{t-s} \leq \frac{1}{1 - 3\varepsilon} \frac{P_{ij}(s, t)}{t-s}.
\]

Hence, taking the limit infimum of both sides we obtain
\[
\limsup_{t-s \to 0^+} \frac{P_{ij}(s, t)}{t-s} \leq \frac{1}{1 - 3\varepsilon} \liminf_{t-s \to 0^+} \frac{P_{ij}(s, t)}{t-s}.
\]

Finally, letting \( \varepsilon \to 0 \) we conclude the proof of part (v).

(vi) By (2.1), \( P_{ii}(s, t) + \sum_{j \neq i} P_{ij}(s, t) \leq 1 \) for all \( i, j \in S \) and \( t \geq s \geq 0 \), or, equivalently, \( \sum_{j \neq i} P_{ij}(s, t) \leq 1 - P_{ii}(s, t) \). Hence (vi) follows from (v) and Fatous’ Lemma. \( \square \)

### 3.2 Proof of Theorem 2

**Proof.** (i) By Assumption A and the definition of \( q_{ij}(t) \), the functions \( P_{ij}^{(n)}(s, t) \) and \( Q_{ij}^{(n)}(s, t) \) are well defined for every \( n \geq 0 \).

To prove (2.16), we use induction on \( n \geq 0 \). Obviously, \( P_{ij}^{(0)}(s, t) = Q_{ij}^{(0)}(s, t) \) and \( P_{ij}^{(1)}(s, t) = Q_{ij}^{(1)}(s, t) \), by (2.11) and (2.14). Now assume that (2.16) holds for some
\(n \geq 0\); we will show that it holds for \(n + 1\). By the induction hypothesis and (2.11) and (2.14), we have

\[
\mathbb{P}^{(n+1)}_{ij}(s, t) = \int_s^t \sum_{k \in S} e^{\int_s^u q_{ik}(v)dv} [q_{ik}(u) + d_{ik}(u)]\overline{Q}^{(n)}_{kj}(u, t)du
\]

\[
= \int_s^t \sum_{k \in S} e^{\int_s^u q_{ik}(v)dv} [q_{ik}(u) + d_{ik}(u)] \cdot \left[ \int_u^t \sum_{l \in S} \overline{Q}^{(n-1)}_{kl}(u, x) e^{\int_x^t q_{lj}(v)dv} [q_{lj}(x) + d_{lj}(x)]dx \right] du
\]

\[
= \int_s^t \sum_{l \in S} e^{\int_s^t q_{kj}(v)dv} [q_{kj}(x) + d_{kj}(x)] \cdot \left[ \int_s^x \sum_{k \in S} e^{\int_s^u q_{ik}(v)dv} [q_{ik}(u) + d_{ik}(u)]\overline{Q}^{(n-1)}_{kl}(u, x)du \right] dx
\]

\[
= \int_s^t \sum_{l \in S} \overline{Q}^{(n)}_{il}(s, x) e^{\int_s^t q_{lj}(v)dv} [q_{lj}(x) + d_{lj}(x)]dx
\]

\[
= \overline{Q}^{(n+1)}_{ij}(s, t),
\]

Consequently, (2.16) holds for \(n + 1\). This completes the induction and verifies (i).

(ii) To prove (2.17), we first prove, by induction on \(n\), the following statement: for every \(n \geq 0\) and \(t \geq s \geq 0\), we have

\[
\int_s^t \sum_{k \in S} \mathbb{P}^{(n)}_{ik}(s, v) d_{kj}(v) dv = \int_s^t \sum_{k \in S} \mathbb{P}^{(n)}_{ik}(s, v) [q_{kj}(v) + d_{kj}(v)] dv - \mathbb{P}^{(n+1)}_{ij}(s, t). \tag{3.12}
\]

Indeed, by (2.10) and (2.9), for \(n = 0\) a direct calculation gives

\[
\int_s^t \sum_{k \in S} \mathbb{P}^{(0)}_{ik}(s, v) d_{kj}(v) dv = \delta_{ij} - \mathbb{P}^{(0)}_{ij}(s, t),
\]

and so (2.11) together with (2.10) and (2.9) gives

\[
\int_s^t \sum_{k \in S} \mathbb{P}^{(1)}_{ik}(s, v) d_{kj}(v) dv
\]

\[
= \int_s^t \sum_{k \in S} \int_s^v \sum_{l \in S} e^{\int_s^u q_{il}(x)dx} [q_{il}(u) + d_{il}(u)]\delta_{ik} e^{\int_u^v q_{li}(x)dx} \delta_{kj}(-q_{kk}(v))dudv
\]

\[
= \int_s^t \sum_{k \in S} \int_s^v e^{\int_s^u q_{il}(x)dx} [q_{ik}(u) + d_{ik}(u)]\delta_{kj} e^{\int_u^v q_{ik}(x)dx}(-q_{kk}(v))dudv
\]

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Consequently, (3.12) holds for \( n = 0 \).

Now assume that (3.12) holds for some \( n \geq 0 \); we will show that it holds for \( n + 1 \).

By the induction hypothesis and (2.11), we obtain

\[
\int_s^t \sum_{k \in S} e^{f_s^u q_{ik}(x)dx} [q_{ik}(u) + d_{ik}(u)]s, t \sum_{l \in S} e^{f_s^u q_{il}(x)dx} [q_{il}(u) + d_{il}(u)]\delta_{kj} \left[ \int_u^t e^{f_u^v q_{kk}(x)dx} (-q_{kk}(v))dv \right] du
\]

\[
= \int_s^t \sum_{k \in S} e^{f_s^u q_{ik}(x)dx} [q_{ik}(u) + d_{ik}(u)]s, t \int_u^t e^{f_u^v q_{kk}(x)dx} (1 - e^{f_u^v q_{kk}(x)dx})du
\]

\[
= \int_s^t \sum_{k \in S} \delta_{ik} e^{f_s^u q_{ik}(x)dx} [q_{kj}(u) + d_{kj}(u)]du
\]

\[- \int_s^t \sum_{k \in S} e^{f_s^u q_{ik}(x)dx} [q_{ik}(u) + d_{ik}(u)] \bar{P}_{kk}^{(0)}(u, t)du
\]

\[
= \int_s^t \sum_{k \in S} \bar{P}_{ik}^{(0)}(s, u)[q_{kj}(u) + d_{kj}(u)]du - \bar{P}_{ij}^{(1)}(s, t).
\]

Hence, (3.12) holds for \( n = 0 \).

By the induction hypothesis and (2.11), we obtain

\[
\int_s^t \sum_{k \in S} \bar{P}_{ik}^{(n+2)}(s, v)d_{kj}(v)dv
\]

\[
= \int_s^t \sum_{k \in S} \left[ \int_s^u \sum_{l \in S} e^{f_s^u q_{il}(x)dx} [q_{il}(u) + d_{il}(u)] \bar{P}_{lk}^{(n+1)}(u, v) du \right] d_{kj}(v)dv
\]

\[
= \int_s^t \sum_{l \in S} e^{f_s^u q_{il}(x)dx} [q_{il}(u) + d_{il}(u)] \left[ \int_u^t \sum_{k \in S} \bar{P}_{lk}^{(n+1)}(u, v)d_{kj}(v)dv \right] du
\]

\[
= \int_s^t \sum_{l \in S} \left[ \int_u^t \sum_{k \in S} e^{f_u^v q_{il}(x)dx} [q_{il}(u) + d_{il}(u)] \bar{P}_{lk}^{(n)}(u, v)d_{kj}(v) + q_{kj}(v) \right] dvdu
\]

\[- \int_s^t \sum_{l \in S} e^{f_s^u q_{il}(x)dx} [q_{il}(u) + d_{il}(u)] \bar{P}_{lj}^{(n+1)}(u, t)du
\]

\[
= \int_s^t \sum_{k \in S} \left[ \int_s^u \sum_{l \in S} e^{f_s^u q_{il}(x)dx} [q_{il}(u) + d_{il}(u)] \bar{P}_{lk}^{(n)}(u, v)du \right] \cdot
\]

\[
[d_{kj}(v) + q_{kj}(v)]dv - \bar{P}_{ij}^{(n+2)}(s, t)
\]

\[
= \int_s^t \sum_{k \in S} \bar{P}_{ik}^{(n+1)}(s, v)[d_{kj}(v) + q_{kj}(v)]dv - \bar{P}_{ij}^{(n+2)}(s, t).
\]

Consequently, (3.12) holds for \( n + 1 \) and this completes the induction.
Now note that \((3.12)\) gives
\[
\sum_{n=0}^{\infty} P_{ij}^{(n+1)}(s, t) = \int_s^t \sum_{k \in S} \sum_{n=0}^{\infty} P_{ik}^{(n)}(s, v)[d_{kj}(v) + q_{kj}(v)]dv - \int_s^t \sum_{k \in S} \sum_{n=0}^{\infty} P_{ik}^{(n+1)}(s, v)d_{kj}(v)dv.
\]
This equality and \((2.12)\) yield
\[
\sum_{n=0}^{\infty} P_{ij}^{(n)}(s, t) = \delta_{ij} - \int_s^t \sum_{k \in S} P_{ik}(s, v)d_{kj}(v)dv + \sum_{n=0}^{\infty} P_{ij}^{(n+1)}(s, t)
\]
\[
= \delta_{ij} + \int_s^t \sum_{k \in S} P_{ik}(s, v)(d_{kj}(v) + q_{kj}(v))dv - \int_s^t \sum_{k \in S} P_{ik}(s, v)d_{kj}(v)dv
\]
\[
= \int_s^t \sum_{k \in S} P_{ik}(s, v)q_{kj}(v)dv + \delta_{ij},
\]
which proves \((2.17)\).

(iii) The proof of \((2.18)\) is quite similar to that of \((2.17)\). We first prove, by induction on \(n\), the following statement, which is analogous to \((3.12)\): for every \(n \geq 0\) and \(t \geq s \geq 0\),
\[
\int_s^t \sum_{k \in S} d_{ik}(v)\overline{Q}_{kj}^{(n+1)}(v, t)dv = \int_s^t \sum_{k \in S} [g_{ik}(v) + d_{ik}(v)]\overline{Q}_{kj}^{(n)}(v, t)dv - \overline{Q}_{ij}^{(n+1)}(s, t). \tag{3.13}
\]
Indeed, by \((2.13)\), for \(n = 0\) we obtain
\[
\int_s^t \sum_{k \in S} d_{ik}(v)\overline{Q}_{kj}^{(0)}(v, t)dv = \delta_{ij} - \overline{Q}_{ij}^{(0)}(s, t).
\]
This fact and \((2.14)\) yield
\[
\int_s^t \sum_{k \in S} d_{ik}(v)\overline{Q}_{kj}^{(1)}(v, t)dv
\]
\[
= \int_s^t \sum_{k \in S} d_{ik}(v) \left[ \int_v^t \sum_{l \in S} \overline{Q}_{kl}^{(0)}(v, u) e_{l}^{u} q_{lj}(x)dx [q_{lj}(u) + d_{lj}(u)] du \right] dv
\]
\[
= \int_s^t \int_v^t \sum_{l \in S} \delta_{il}(-q_{ii}(v)) e_{l}^{u} q_{ii}(x)dx e_{l}^{u} q_{jj}(x)dx [q_{lj}(u) + d_{lj}(u)] du dv
\]
\[
= \int_s^t \sum_{l \in S} \delta_{il} \left[ \int_s^u (-q_{ii}(v)) e_{l}^{u} q_{ii}(x)dx dv \right] e_{l}^{u} q_{jj}(x)dx [q_{lj}(u) + d_{lj}(u)] du
\]
\[
= \int_s^t \sum_{l \in S} \delta_{il} (1 - e_{l}^{u} q_{ii}(x)dx) e_{l}^{u} q_{jj}(x)dx [q_{lj}(u) + d_{lj}(u)] du
\]
\[
= \int_s^t \sum_{l \in S} [q_{ii}(u) + d_{ii}(u)] \delta_{ij} e_{l}^{u} q_{ii}(x)dx du
\]
- \int_s^t \sum_{l \in S} Q_{il}^{(0)}(s, u) e^{f_u^l q_{ij}(x)} dx [q_{ij}(u) + d_{ij}(u)] \, du \\
= \int_s^t \sum_{l \in S} [q_{il}(u) + d_{il}(u)] Q_{ij}^{(0)}(u, t) \, du - Q_{ij}^{(1)}(s, t).

Hence, (3.13) holds for \( n = 0 \).

Now assume that (3.13) holds for some \( n \geq 0 \); we will show that it holds for \( n + 1 \). By the induction hypothesis and (2.14), we obtain

\[
\int_s^t \sum_{k \in S} d_{ik}(v) Q_{kj}^{(n+2)}(v, t) \, dv
= \int_s^t \sum_{k \in S} d_{ik}(v) \left[ \int_v^u \sum_{l \in S} Q_{kl}^{(n+1)}(v, u) e^{f_u^l q_{ij}(x)} dx [q_{ij}(u) + d_{ij}(u)] \, dv \right] \, du
= \int_s^t \sum_{k \in S} \left[ \int_v^u \sum_{l \in S} d_{ik}(v) Q_{kl}^{(n+1)}(v, u) \, dv \right] e^{f_u^l q_{ij}(x)} dx [q_{ij}(u) + d_{ij}(u)] \, du
\]

\[
= \int_s^t \sum_{l \in S} \left[ \int_v^u \sum_{k \in S} [q_{ik}(v) + d_{ik}(v)] Q_{kl}^{(n)}(v, u) \, dv \right] e^{f_u^l q_{ij}(x)} dx [q_{ij}(u) + d_{ij}(u)] \, du
- \int_s^t \sum_{l \in S} Q_{il}^{(n+1)}(s, u) e^{f_u^l q_{ij}(x)} dx [q_{ij}(u) + d_{ij}(u)] \, du
= \int_s^t \sum_{k \in S} [q_{ik}(v) + d_{ik}(v)] \left[ \int_v^t \sum_{l \in S} Q_{kl}^{(n)}(v, u) e^{f_u^l q_{ij}(x)} dx [q_{ij}(u) + d_{ij}(u)] \, du \right] \, dv
- Q_{ij}^{(n+2)}(s, t)
= \int_s^t \sum_{k \in S} [q_{ik}(v) + d_{ik}(v)] Q_{kj}^{(n+1)}(v, t) \, dv - Q_{ij}^{(n+2)}(s, t).
\]

Consequently, (3.13) holds for \( n + 1 \) and this completes the induction.

Now note that (3.13) gives

\[
\sum_{n=0}^{\infty} Q_{ij}^{(n+1)}(s, t) = \int_s^t \sum_{k \in S} [q_{ik}(v) + d_{ik}(v)] \sum_{n=0}^{\infty} Q_{kj}^{(n)}(v, t) \, dv - \int_s^t \sum_{k \in S} d_{ik}(v) \sum_{n=0}^{\infty} Q_{kj}^{(n+1)}(v, t) \, dv.
\]

Therefore, by (2.15),

\[
\sum_{n=0}^{\infty} Q_{ij}^{(n)}(s, t) = \delta_{ij} - \int_s^t \sum_{k \in S} d_{ik}(v) Q_{ij}^{(0)}(v, t) \, dv + \sum_{n=0}^{\infty} Q_{ij}^{(n+1)}(s, t)
\]
\[
\delta_{ij} + \int_s^t \sum_{k \in S} [q_{ik}(v) + d_{ik}(v)]Q_{kj}(v,t)dv - \int_s^t \sum_{k \in S} d_{ik}(v)Q_{kj}(v,t)dv
\]

\[
= \int_s^t \sum_{k \in S} q_{ik}(v)Q_{kj}(v,t)dv + \delta_{ij}.
\]

This equality and (2.16) give (2.18).

(iv) To prove that \( P(s,t) \) is a nonhomogeneous \( Q(t) \)-transition matrix we need to show that \( P_{ij}(s,t) \) satisfies (2.1)–(2.3), and that the partial derivatives in (i)–(ii) of Definition 3 exist.

To prove (2.1) we already know that \( P^{(0)}_{ij}(s,t) \geq 0 \), by (2.10). Suppose now that \( P^{(n)}_{ij}(s,t) \geq 0 \) for some \( n \). To prove that this holds for \( n + 1 \), we use (2.11) and (2.6) to obtain

\[
P^{(n+1)}_{ij}(s,t) = \int_s^t \sum_{k \in S} e^{\int_s^u q_{ik}(v)dv} [q_{ik}(u) + d_{ik}(u)]P^{(n)}_{kj}(u,t)du
\]

\[
= \int_s^t \sum_{k \neq i \atop k \in S} e^{\int_s^u q_{ik}(v)dv} q_{ik}(u)P^{(n)}_{kj}(u,t)du \geq 0 \quad \text{(by (2.9))}.
\]

Hence \( P_{ij}(s,t) \geq 0 \).

To prove that \( P_{ij}(s,t) \) satisfies the second part of (2.1), it suffices to show that

\[
\sum_{j \in S} P^{(n)}_{ij}(s,t) \leq 1 \quad \forall n \geq 0,
\]

because then in a similar manner we can prove show that

\[
\sum_{j \in S} \sum_{n=0}^N P^{(n)}_{ij}(s,t) \leq 1 \quad \forall N \geq 0.
\]

Therefore, since

\[
P_{ij}(s,t) = \sum_{n=0}^{\infty} P^{(n)}_{ij}(s,t) = \lim_{N \to \infty} \sum_{n=0}^N P^{(n)}_{ij}(s,t),
\]

(2.1) follows. Now, to prove (3.14), we use induction on \( n \). For \( n = 0 \), (3.14) trivially holds, by (2.10).

Suppose now that (3.14) holds for some \( n \), that is,

\[
\sum_{j \in S} P^{(n)}_{ij}(s,t) \leq 1.
\]

To see that this holds for \( n + 1 \), we use (2.11) and monotone convergence to obtain

\[
\sum_{j \in S} P^{(n+1)}_{ij}(s,t) = \int_s^t \sum_{k \in S} e^{\int_s^u q_{ik}(v)dv} [q_{ik}(u) + d_{ik}(u)] \left[ \sum_{j \in S} P^{(n)}_{kj}(u,t) \right] du
\]

\[
\leq \int_s^t e^{\int_s^u q_{ik}(v)dv} \sum_{k \in S} q_{ik}(u)du + \int_s^t \sum_{k \in S} e^{\int_s^u q_{ik}(v)dv} d_{ik}(u)du.
\]
Hence, by (2.7) and (2.9),

\[ \sum_{j \in S} P_{ij}^{(n+1)}(s, t) \leq \int_s^t \sum_{k \in S} e^{-\int_{u}^{t} q_{ki}(v) \, dv} \delta_{ik}(-q_{ii}(u)) \, du = 1 - e^{-\int_{s}^{t} q_{ii}(u) \, du} \leq 1, \]

which yields (3.14).

Now we will verify that \( P_{ij}(s, t) \) satisfies the C-K equation (2.2). Observe that this holds if and only if, for every \( n \geq 0 \) and \( s \leq u \leq t \),

\[
P_{ij}^{(n)}(s, t) = \sum_{m=0}^{n} \sum_{k \in S} P_{ik}^{(m)}(s, u) P_{kj}^{(n-m)}(u, t). \tag{3.16}
\]

We will prove (3.16) by induction. In fact, for \( n = 0 \) (3.16) follows from (2.10). Suppose now that (3.16) holds for some \( n \geq 0 \). To prove (3.16) for \( n + 1 \), we use the induction hypothesis and (2.11), to obtain, for any \( s \leq r \leq t \),

\[
P_{ij}^{(n+1)}(s, t) = \int_s^t \sum_{k \in S} e^{-\int_{s}^{u} q_{ki}(v) \, dv} \left[ q_{ik}(u) + d_{ik}(u) \right] P_{kj}^{(n)}(u, t) \, du
\]

\[= \int_r^t \sum_{k \in S} e^{-\int_{s}^{u} q_{ki}(v) \, dv} \left[ q_{ik}(u) + d_{ik}(u) \right] P_{kj}^{(n)}(u, t) \, du + \int_s^r \sum_{k \in S} e^{-\int_{s}^{u} q_{ki}(v) \, dv} \left[ q_{ik}(u) + d_{ik}(u) \right] P_{kj}^{(n)}(u, t) \, du
\]

\[= \sum_{m=0}^{n} \sum_{l \in S} P_{il}^{(m+1)}(s, r) P_{lj}^{(n-m)}(r, t) + B,
\]

where

\[B = \int_r^t \sum_{k \in S} e^{-\int_{s}^{u} q_{ki}(v) \, dv} \left[ q_{ik}(u) + d_{ik}(u) \right] P_{kj}^{(n)}(u, t) \, du.
\]

Since (3.16) holds for \( n = 0 \), recalling (2.10) we then have

\[B = P_{ii}^{(0)}(s, r) P_{ij}^{(n+1)}(r, t) = \sum_{l \in S} P_{il}^{(0)}(s, r) P_{lj}^{(n+1)}(r, t),
\]

and so

\[P_{ij}^{(n+1)}(s, t) = \sum_{m=0}^{n+1} \sum_{l \in S} P_{il}^{(m)}(s, r) P_{lj}^{(n+1-m)}(r, t).
\]

Hence, (3.16) holds for all \( n \), and, as already noted, (2.2) follows for \( P_{ij}(s, t) \).

To see that \( P_{ij}(s, t) \) satisfies (2.3), using (2.9)- (2.12), (2.6), (2.7) and (3.15), we obtain

\[
| P_{ij}(s, t) - \delta_{ij} | \leq \left( 1 - e^{-\int_{s}^{t} q_{ii}(u) \, du} \right) + \sum_{n=0}^{\infty} P_{ij}^{(n+1)}(s, t)
\]

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\[
(1 - e^{f_s^t q_{ii}(v) dv}) + \int_s^t \sum_{k \in S} e^{f_s^u q_{ii}(v) dv} [q_{ik}(u) + d_{ik}(u)] \sum_{n=0}^{\infty} P_{kj}^{(n)}(u, t) du
\]

\[
\leq (1 - e^{f_s^t q_{ii}(v) dv}) + \int_s^t \sum_{k \in S} e^{f_s^u q_{ii}(v) dv} [q_{ik}(u) + d_{ik}(u)] du
\]

\[
\leq (1 - e^{f_s^t q_{ii}(v) dv}) + \int_s^t e^{f_s^u q_{ii}(v) dv} [-q_{ii}(u)] du
\]

\[
\rightarrow 0 \text{ as } t \rightarrow s^+.
\]

This implies the desired result.

To summarize, we have just shown that \(\mathcal{P}(s, t) = (P_{ij}(s, t), \ i, j \in S)\) is a nonhomogeneous pretransition matrix. Therefore, to complete the proof that it is a nonhomogeneous \(Q(t)\)-transition matrix, it only remains to verify that \(\mathcal{P}(s, t)\) satisfies (i) and (ii) in Definition 3. But in fact from (2.17), (2.18) and the fundamental theorem of calculus for Lebesgue integrals we can obtain a bit more than (i), namely, Kolmogorov’s forward and backward equations

\[
\frac{\partial P_{ij}(s, t)}{\partial t} = \sum_{k \in S} P_{ik}(s, t)q_{kj}(t),
\]

\[
\frac{\partial P_{ij}(s, t)}{\partial s} = -\sum_{k \in S} q_{ik}(s)P_{kj}(s, t)
\]

for a.a. \(t \geq s \geq 0\). Moreover, if we take \(t = s\) in the forward equation, we obtain (2.8). This verifies (iv) and it also completes the proof of Theorem 2. \(\square\)

### 3.3 Proof of Theorem 3

To prove Theorem 3 we need the following facts.

**Lemma 2.** Assume that \(Q(t)\) is a nonhomogeneous \(Q(t)\)-matrix satisfying Assumption A. Then for any nonhomogeneous \(Q(t)\)-transition matrix \(P(s, t) = (P_{ij}(s, t), \ i, j \in S)\) we have, for every \(i, j \in S\) and \(0 \leq s < t < \infty\),

(i) \[
\frac{\partial P_{ij}(s, t)}{\partial s} \leq -\sum_{k \in S} q_{ik}(s)P_{kj}(s, t); \quad (3.17)
\]

(ii) \[
P_{ij}(s, t) \geq \int_s^t \sum_{k \in S} e^{f_s^u q_{ii}(v) dv} [q_{ik}(u) + d_{ik}(u)] P_{kj}(u, t) du + \delta_{ij} e^{f_s^t q_{ii}(v) dv}. \quad (3.18)
\]

**Proof.** (i) To prove (3.17), we use the C-K equation (2.2) to obtain

\[
\frac{1}{h}[P_{ij}(s + h, t) - P_{ij}(s, t)]
\]

\[
= \frac{1}{h}[1 - P_{ii}(s, s + h)]P_{ij}(s + h, t) - \sum_{k \in S} \frac{1}{h} P_{ik}(s, s + h)P_{kj}(s + h, t). \quad (3.19)
\]
Hence, by Fatou’s Lemma, (2.3) and Theorem 1(iii) we have

$$\liminf_{h \to 0^+} \sum_{k \in S} \frac{1}{h} P_{ik}(s, s + h)P_{kj}(s + h, t) \geq \sum_{k \neq i, j \in S} q_{ik}(s)P_{kj}(s, t). \quad (3.20)$$

Then (i) follows from Definition 3, (3.19), and (3.20).

(ii) By (3.17), we have

$$\int_s^t \sum_{k \in S} e^{f_s u} q_{ij}(v)dv [q_{ik}(u) + d_{ik}(u)]P_{kj}(u, t)du$$

$$= \int_s^t \sum_{k \in S} e^{f_s u} q_{ij}(v)dv q_{ik}(u)P_{kj}(u, t)du + \int_s^t \sum_{k \in S} e^{f_s u} q_{ij}(v)dv d_{ik}(u)P_{kj}(u, t)du$$

$$\leq -\int_s^t e^{f_s u} q_{ij}(v)dv \partial P_{ij}(u, t)\partial u + \int_s^t e^{f_s u} q_{ij}(v)dv (-q_{ii}(u))P_{ij}(u, t)du$$

$$= -\delta_{ij} e^{f_s u} q_{ij}(v)dv + P_{ij}(s, t).$$

This yields (3.18), and completes the proof of Lemma 2. \(\square\)

With Lemma 2 we can easily prove Theorem 3.

**Proof of Theorem 3**

(i) Let \(P_{ij}^{(m)}(s, t)\) and \(P_{ij}(s, t)\) be as defined in Theorem 2 and let \(P_{ij}(s, t)\) be as in Lemma 2. We will prove (2.19) by showing that (3.21), below, holds for all \(n \geq 0\).

By Lemma 2 we know that \(P_{ij}(s, t)\) satisfies (3.17), which for \(i = j\) becomes

$$\frac{\partial P_{ii}(u, t)}{\partial u} \leq -\sum_{k \in S} q_{ik}(u)P_{ki}(u, t) \quad \forall i \in S, \quad 0 \leq u \leq t < \infty.$$ 

Since \(q_{ij}(u) \geq 0\) for \(i \neq j\), we have \(\frac{\partial P_{ii}(u, t)}{\partial u} \leq -q_{ii}(u)P_{ii}(u, t)\), and so \(-\frac{\partial P_{ii}(u, t)}{P_{ii}(u, t)} \geq q_{ii}(u)du\). Integrating from \(s\) to \(t\) on both sides and using (2.10), we obtain \(P_{ii}(s, t) \geq \frac{\partial P_{ii}^{(0)}(s, t)}{\partial u}\). Hence

$$P_{ij}(s, t) \geq P_{ij}^{(0)}(s, t) \quad \forall i, j \in S.$$ 

Suppose now that for some \(n \geq 0\),

$$P_{ij}(s, t) \geq \sum_{m=0}^n P_{ij}^{(m)}(s, t). \quad (3.21)$$

To prove that this holds for \(n + 1\), we use (3.18) and the induction hypothesis to obtain

$$P_{ij}(s, t) \geq \delta_{ij} e^{f_s u} q_{ij}(u)du + \sum_{m=0}^n \int_s^t \sum_{k \in S} e^{f_s u} q_{ij}(v)dv [q_{ik}(u) + d_{ik}(u)]P_{kj}(u, t)du$$

$$\geq \delta_{ij} e^{f_s u} q_{ij}(u)du + \sum_{m=0}^n \int_s^t \sum_{k \in S} e^{f_s u} q_{ij}(v)dv [q_{ik}(u) + d_{ik}(u)]P_{ij}^{(m)}(u, t)du$$

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\[ P_{ij}(s, t) = T_{ij}^{(0)}(s, t) + \sum_{m=0}^{n} T_{ij}^{(m+1)}(s, t) \quad \text{(by (2.10), (2.11))} \]
\[ = \sum_{m=0}^{n+1} T_{ij}^{(m)}(s, t). \]

Therefore, (3.21) holds for all \( n \geq 0 \), and letting \( n \to \infty \) in (3.21), we obtain (2.19).

(ii) By Definition 3 and (2.17), we know that \( P(s, t) = (P_{ij}(s, t), \quad i, j \in S) \) is regular if and only if

\[ \sum_{j \in S} \left[ \int_{\mathbb{S}} \sum_{k \in S} T_{ik}(s, v)q_{kj}(v)dv + \delta_{ij} \right] = 1, \]

that is
\[ \sum_{j \in S} \left[ \int_{s}^{t} \sum_{k \in S} T_{ik}(s, v)q_{kj}(v)dv \right] = 0, \]

which is the same as (2.20). This completes the proof of Theorem 3. \( \square \)

4 CONCLUSIONS

In this paper we have presented a fairly detailed, self-contained, exposition of the construction of a \( Q(t) \)-transition matrix starting from a nonhomogeneous \( Q(t) \)-matrix that satisfies a very mild measurability condition. Moreover, such a transition matrix is in fact the minimum \( Q(t) \)-transition matrix and we have presented a necessary and sufficient condition for it to be unique and regular. In short, this paper efficiently generalizes the main results of a nonhomogeneous \( Q(t) \)-transition matrix with continuous and conservative transition rates \( q_{ij}(t) \), to the case in which the \( q_{ij}(t) \) are measurable and may not be conservative.

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