δ-function spin-$\frac{1}{2}$ fermions in a one-dimensional potential well

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The quantum-mechanical problem of $N$ fermions with δ-function interaction in a one-dimensional potential well of finite depth is solved. It is shown that there exists exact wave function of Bethe-ansatz form in the case that a single particle tunnels outside of the well. The Bethe-ansatz like secular equations for the spectrum are obtained.

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I. INTRODUCTION

One-dimensional quantum-mechanical systems of electrons have been a subject of great importance in recent years because they serve as tractable limits of quasi-one-dimensional systems [1] as well as models for the conductive properties of the one-dimensional quantum wire [2]. A model Hamiltonian describing the system is a $N$-body problem with delta-function interaction. For convenience, let $\hbar$ and the mass of the electron be one. The Hamiltonian of the electron gas in one dimension then reads

$$H = -\sum_{i=1}^{N} \frac{\partial^2}{\partial x_i^2} + c \sum_{j>i=1}^{N} \delta(x_i - x_j)$$

where $c$ stands for the interaction strength. The quantum-mechanical problem of the above Hamiltonian were solved in [4-8] with periodic boundary conditions, and in [9] with the boundary condition for a potential well of infinite depth. Preceding a very narrow and very deep attractive potential at one side of the well Ref. [9] considered another kind of boundary condition. In the discussion of [9], a phase shift between the incident and reflected waves, which depends on a phenomenological parameter, was introduced. The problem of many particles with δ-function interaction in a potential well of finite depth is more complicated due to the fact that the wave function outside of the well is nonzero and should be taken into account. In [10] one of the author solved the problem of $N$ bosons with delta-function interaction in a potential well of finite depth. The more important physical problem should be $N$ fermions in a potential well of finite depth. So we will discuss this problem in the present paper. The Hamiltonian of spin-$1/2$ $N$-particle system interacting by a δ-function in the presence of a square well is

$$H = -\sum_{i=1}^{N} \frac{\partial^2}{\partial x_i^2} + \sum_{i=1}^{N} V(x_i) + 2c \sum_{i>j=1}^{N} \delta(x_i - x_j),$$

(1.1)

where

$$V(x_i) = \begin{cases} 0 & |x_i| < \frac{L}{2} \\ V_0^2 & |x_i| > \frac{L}{2} \end{cases}$$

in which $L$ is the width of the well. When $V_0 = 0$, the Hamiltonian is known to be exactly solvable by the Bethe ansatz in the case of periodic boundary condition. When $V_0 \neq 0$, the Hamiltonian is not invariant under translation, and the total momentum of the system is not conserved. However, the system is still invariant under the action of the permutation group $S_N$. We will diagonalize the Hamiltonian (1.1) by a generalized Bethe ansatz which involves

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all non-diffraction scattering waves. When the system is considered as a canonical ensemble in which almost all the particles are confined in the well, except for one particle which is able to tunnel out of the well, we will show that the diagonalization can be exactly realized and the consistent Bethe ansatz equations obtained.

In section II we consider the two-particle system to illustrate the idea of the Bethe ansatz [11]. In section III we analyze the generic system of $N$ fermions and solve the wave function of Bethe ansatz form. In section IV we diagonalize the $S$-matrix and obtain secular equations for the spectrum. In section V we give some discussions and remarks.

II. TWO-PARTICLE STATES

Before exploring the generalized Bethe ansatz for a system of $N$ fermions, we discuss the case of two fermions. The quantum-mechanical problem of one particle in a potential well of finite depth is well known. Because there is no external field, the spin degrees of freedom $a = \pm, -$ are degenerate. The wave functions $\psi_a(x)$ are

$$\psi_a(x) = \begin{cases} A^L_\kappa e^{i\kappa x}, & x < -\frac{L}{2}, \\ A^+_\alpha e^{ikx} + A^-_\alpha e^{-ikx}, & |x| \leq \frac{L}{2}, \\ A^R_\kappa e^{-i\kappa x}, & x > \frac{L}{2}. \end{cases}$$

(2.1)

where $\kappa > 0$ and $k^2 = \kappa^2 + V_0^2 = E$ is the energy of the system. The amplitudes $A$’s are related via the boundary conditions at $x = \pm \frac{L}{2}$, which say that the wave function and its derivative are continuous at the ends of the well.

One can easily find that the spectrum is determined by $\left(\frac{\kappa + ik}{\kappa - ik}\right)^2 = e^{-2ikL}$.

Because of the $\delta$-function interaction between the two particles, we should divide the $(x_1, x_2)$ plane into two regions, namely, the region $C(id) = \{(x_1, x_2) \mid x_1 < x_2\}$ and the region $C(\sigma_1) = \{(x_1, x_2) \mid x_2 < x_1\}$. The definition of $\sigma_1$ and other terminology will be given in the next section. In the square well $x_1, x_2 \in [-\frac{L}{2}, \frac{L}{2}]$, the wave function takes the form of plane waves of all the non-diffraction scatterings. We suppose that in the region $C(id)$, the wave function reads

$$\psi_{a_1a_2}^{id}(x_1, x_2) = \sum_{\alpha, \beta = \pm, -} \left[ e^{i(\alpha k_1 x_1 + \beta k_2 x_2)} A_{a_1a_2}(\alpha, \beta) e^{i(\alpha k_1 x_2 + \beta k_2 x_1)} B_{a_2a_1}(\alpha, \beta) \right].$$

(2.2)

The amplitudes $A(\alpha, \beta), B(\alpha, \beta)$ are so chosen that in the region $C(\sigma_1)$, the wave function reads

$$\psi_{a_1a_2}^{\sigma_1}(x_1, x_2) = \sum_{\alpha, \beta = \pm, -} \left[ e^{i(\alpha k_1 x_1 + \beta k_2 x_2)} B_{a_1a_2}(\alpha, \beta) e^{i(\alpha k_1 x_2 + \beta k_2 x_1)} A_{a_2a_1}(\alpha, \beta) \right].$$

(2.3)

Obviously, the requirement of antisymmetry for the fermion wave function $\psi_{a_1a_2}^{id}(x_1, x_2) = -\psi_{a_2a_1}^{id}(x_2, x_1)$ is satisfied by (2.2) and (2.3).

When the interaction strength is not zero, the wave functions are continuous while its derivative has a finite difference at the $\delta$-function singularity

$$\frac{\partial}{\partial x_1} - \frac{\partial}{\partial x_2} \left( \psi_{a_1a_2}^{id}(x_1, x_2) \right)_{x_1 = x_2} = \psi_{a_1a_2}^{\sigma_1}(x_1, x_2)_{x_1 = x_2} = \psi_{a_1a_2}^{id}(x_1, x_2)_{x_1 = x_2},$$

(2.4)

Substituting (2.2) and (2.3) into (2.4), we obtain a relation between the amplitudes $A$ and $B$

$$B_{a_1a_2}(\alpha, \beta) = S_{a_1a_2}^{\sigma_1}(\alpha, \beta) A_{a_1a_2}^{\sigma_1}(\alpha, \beta)$$

(2.5)

where $S_{a_1a_2}^{\sigma_1}(\alpha, \beta) = \frac{(\alpha k_1 - \beta k_2) \delta_{a_1a_2}^{\sigma_1} + 2i c \delta_{a_1a_2}^{\sigma_1} \delta_{a_1a_2}^{\sigma_1}}{\alpha k_1 - \beta k_2 + 2i c}$ is the two-body scattering matrix.

When one of the fermions moves out of the potential well, there is an additional restriction that the wave function should vanish at infinity. Thus the wave function on the left side of the well takes the form $\psi_{L_{a_1a_2}}^{id}(x_1, x_2) = \sum_{\beta = \pm, -} \left[ e^{i k_1 x_1} e^{i k_2 x_2} A_{L_{a_1a_2}}^{L}(\alpha, \beta) e^{i k_1 x_2} e^{i k_2 x_1} B_{L_{a_2a_1}}^{L}(\alpha, \beta) \right]$ for $x_1 < -\frac{L}{2}$, $x_2 < \frac{L}{2}$, and $\psi_{L_{a_1a_2}}^{id}(x_1, x_2) = \sum_{\beta = \pm, -} \left[ e^{i k_2 x_2} e^{i k_1 x_1} B_{L_{a_1a_2}}^{L}(\alpha, \beta) e^{i k_2 x_1} e^{i k_1 x_2} A_{L_{a_2a_1}}^{L}(\alpha, \beta) \right]$ for $x_2 < -\frac{L}{2}$, $x_1 < \frac{L}{2}$. The former is the antisymmetric counterpart of the later and vice versa. Similarly, the wave function on the right side of the well should
be \( \psi^R_{a_1a_2}(x_1, x_2) = \sum_{\beta=+,-,} \left[ e^{-\kappa_2 x_2 e^{i\beta k_2 x_1} A^R_{a_1a_2}(\beta)} - e^{-\kappa_1 x_1 e^{i\beta k_2 x_1} B^R_{a_2a_1}(\beta)} \right] \) for \(-\frac{L}{2} < x_1 < \frac{L}{2} < x_2\), and
\( \psi^R_{a_1a_2}(x_1, x_2) = \sum_{\beta=+,-,} \left[ e^{-\kappa_1 x_1 e^{i\beta k_2 x_2} B^R_{a_1a_2}(\beta)} - e^{-\kappa_2 x_2 e^{i\beta k_2 x_2} A^R_{a_2a_1}(\beta)} \right] \) for \(-\frac{L}{2} < x_2 < \frac{L}{2} < x_1\).

The amplitudes \( A^{R,L}(\alpha, \beta), B^{R,L}(\alpha, \beta) \) are related to the amplitudes \( A(\alpha, \beta), B(\alpha, \beta) \) via the boundary conditions at \( x = \pm \frac{L}{2} \), namely
\[
\psi^{L}_{a_1a_2}(\frac{-L}{2}, x_2) = \psi^{R}_{a_1a_2}(\frac{-L}{2}, x_2),
\]
\[
\frac{\partial}{\partial x_1} \psi^{L}_{a_1a_2}(x_1, x_2)|_{x_1=\frac{-L}{2}} = \frac{\partial}{\partial x_1} \psi^{R}_{a_1a_2}(x_1, x_2)|_{x_1=\frac{-L}{2}}
\]
and
\[
\psi^{R}_{a_1a_2}(\frac{L}{2}, x_2) = \psi^{L}_{a_1a_2}(\frac{L}{2}, x_2),
\]
\[
\frac{\partial}{\partial x_2} \psi^{R}_{a_1a_2}(x_1, x_2)|_{x_2=\frac{L}{2}} = \frac{\partial}{\partial x_2} \psi^{L}_{a_1a_2}(x_1, x_2)|_{x_2=\frac{L}{2}}.
\]

Thus we get
\[
e^{-ik_1 L} \frac{k_1 - ik_1}{k_1 + ik_1} = \frac{A_{a_1a_2}(\beta)}{A_{a_2a_1}(\beta)}, \tag{2.6}
\]
\[
e^{-ik_2 L} \frac{k_2 - ik_2}{k_2 + ik_2} = \frac{B_{a_2a_1}(\beta)}{B_{a_1a_2}(\beta)}, \tag{2.7}
\]
\[
e^{-ik_1 L} \frac{k_1 - ik_1}{k_1 + ik_1} = \frac{B_{a_2a_1}(\beta)}{B_{a_2a_1}(\beta)}, \tag{2.8}
\]
and
\[
e^{-ik_2 L} \frac{k_2 - ik_2}{k_2 + ik_2} = \frac{A_{a_1a_2}(\beta)}{A_{a_2a_1}(\beta)}, \tag{2.9}
\]
where \( \beta = +, - \). We observe that the dependence of \( A^{R,L}, B^{R,L} \) on \( A, B \) are the same as what occurred in the one particle situation.

The equations (2.6-2.9) define the secular equations for the spectrum of quasi momentum (charge) \( k_1, k_2 \). This is due to the fact that
\[
S^{-1}(k_1, k_2)S(k_1, -k_2) = -S^{-1}(-k_2, k_1)S(k_2, k_1). \tag{2.10}
\]

The secular equation for the spectrum is
\[
S^{-1}(-, \beta)S(+, \beta)A(-, \beta) = f_1^2 A(-, \beta),
\]
\[
S^{-1}(+, \beta)S(-, \beta)A(\beta, +) = f_2^2 A(\beta, +), \beta = +, - \tag{2.11}
\]
where \( f_j = \frac{k_j - ik_j}{k_j + ik_j} \), \( j = 1, 2 \). After diagonalizing the scattering operators \( S \) and \( S^{-1} \), one can obtain the Bethe ansatz equations for \( k_1, k_2 \). This will be discussed in section [V].

### III. THE CASE OF N FERMIONS

We first introduce some terminology which is helpful for avoiding the ambiguities which have appeared in some previous literature. In a Euclidean space \( \mathbb{R}^N \) with Cartesian coordinates \( x = (x_1, x_2, \ldots, x_N) \), the set of hyperplanes \( \{ x|x_i - x_j = 0 \}; \ i, j = 1, 2, \ldots, N \} \) partition \( \mathbb{R}^N \) into finitely many regions. We use the convention that the scalar product of two \( N \)-dimensional vectors is written as \( (x|y) = \sum_{i=1}^N x_i y_i \). The above-mentioned hyperplanes are Weyl reflection hyperplanes \( x|\alpha_{i_j} = 0 \) where \( \alpha_{i_j} = e_i - e_j \) are the roots of the Lie algebra \( A_{N-1} \). We denote a
Weyl reflection hyperplane as $P_\alpha := \{ x \mid (x|\alpha) = 0 \}$, in which $\alpha$ is a root of $A_{N-1}$. The connected components of $R^N \setminus \{ P_\alpha \}$ are called Weyl Chambers $[12]$ of the Lie algebra $A_{N-1}$. We denote the Weyl group of $A_{N-1}$ as $W_A$, the basic elements of which are defined by $\sigma_i : (x_1, x_2, \cdots, x_N) \mapsto (x_1, x_2, \cdots, x_{i+1}, x_{i+1}, x_i, \cdots, x_N)$. The Weyl group of $B_N$, denoted by $W_B$, has one more basic element $\sigma_0$ besides the previously defined $\sigma_i (i = 1, 2, \cdots N - 1)$. The definition of $\sigma_N$ is given by $\sigma_N : (x_1, \cdots, x_N, x_N) \mapsto (x_1, \cdots, x_N, -x_N)$. An element of the Weyl group obviously maps one Weyl chamber onto another and all the chambers can be obtained from any given chamber via the actions of the whole Weyl group. Thus Weyl chambers can be specified by the elements of Weyl group.

Now we consider the case of $N$ fermions. Obviously, the Hamiltonian $[11]$ is invariant under the action of the permutation group $S_N$, but it is not invariant under translation. Thus the total momentum of the system is not conserved. The Schrödinger equation of the Hamiltonian $[11]$ on the domain $R^N \setminus \{ P_\alpha \}$ becomes

$$\sum_{i=1}^{N} \left[ -\frac{\partial^2}{\partial x_i^2} + V(x_i) \right] \psi_a(x) = E \psi_a(x),$$

where $a := (a_1, a_2, \cdots, a_N)$ and $a_i \in \{ +, - \}$. Solutions of the equation are plane waves. Considering that the total momentum is not conserved, we adopt the following Bethe $[11]$ ansatz form

$$\psi_a(x) = \sum_{\sigma \in W_B} A_a(\sigma, \tau) e^{i(\sigma k|x|)}, \quad x \in \mathcal{C}(\tau) \quad (3.1)$$

where $\sigma k$ stands for the image of a given $k := (k_1, k_2, \cdots, k_N)$ by a mapping $\sigma \in W_B$ and the coefficients $A(\sigma, \tau)$ are functionals on $W_B \odot W_A$. We emphasize that the sum runs over the Weyl group of the Lie algebra $B_N$ but the wave function is defined on various Weyl chambers corresponding to the Weyl group of the Lie algebra $A_{N-1}$. This is different from the situation of periodic boundary condition.

For a fermionic system, the wave function is supposed to be anti-symmetric under any permutation of both coordinates and spin states, i.e.

$$(\sigma_i \psi)_a(x) = -\psi_a(x). \quad (3.2)$$

Here $(\sigma_i \psi)_a$ is well defined by $\psi_{\sigma a}(\sigma^{-1} x)$. Therefore both sides of (3.2) can be written out by using (3.1). Furthermore, using the evident identity $(\sigma k|\sigma^{-1} x) = (\sigma k|x)$ and the rearrangement theorem of group theory, we obtain the following consequence from (3.2)

$$A_a(\sigma, \sigma \tau) = -A_a(\sigma, \sigma \tau). \quad (3.3)$$

The $\delta$-function term in the Hamiltonian $[11]$ contributes a boundary condition at hyperplane $P_\alpha$ ($\alpha$ is a root of Lie algebra $A_{N-1}$), namely a discontinuity of the derivative of wave function along the normal of Weyl hyperplane:

$$\lim_{\epsilon \to 0^+} [\alpha \cdot \nabla \psi_a(x(\alpha) + \epsilon \alpha) - \alpha \cdot \nabla \psi_a(x(\alpha) - \epsilon \alpha)] = 2c \psi_a(x(\alpha)), \quad \lim_{\epsilon \to 0^+} [\psi_a(x(\alpha) + \epsilon \alpha) - \psi_a(x(\alpha) - \epsilon \alpha)] = 0, \quad (3.4)$$

where $x(\alpha) \in P_\alpha$ and $\nabla := \sum_{i=1}^{N} e_i (\partial / \partial x_i)$.

Substituting (3.1) into (3.4), we find that

$$i[(\sigma k)_i - (\sigma k)_{i+1}] [A_a(\sigma, \sigma \tau) - A_a(\sigma \sigma, \sigma \tau) - A_a(\sigma \sigma, \sigma \tau) - A_a(\sigma \sigma, \sigma \tau) + A_a(\sigma \sigma, \sigma \tau) + A_a(\sigma \sigma, \sigma \tau)] = 2c[A_a(\sigma, \tau) + A_a(\sigma \sigma, \sigma \tau)]. \quad (3.5)$$

By making use of the relation (3.3), we can obtain the following relations from (3.4)

$$A_a(\sigma \sigma, \sigma \tau) = S^i_{a, a'}(\sigma k) A_{a'}(\sigma, \tau), \quad (3.6)$$

$$S^i_{a, a'}(\sigma k) = \frac{c \delta_{a, a'} - i[(\sigma k)_i - (\sigma k)_{i+1}] P_{a, a'}}{c - i[(\sigma k)_i - (\sigma k)_{i+1}]}. \quad (3.7)$$

where $a' = \sigma \sigma$ and $P_{a, a'}$ stands for the matrix elements of the spinor representation of permutation group. The relation (3.3) provides for the coefficients $A$ a relation between different Weyl chambers. Eq. (3.4) provides a connection between those coefficients which are related via any element of Weyl group $W_A$ in the same Weyl chamber. Although the $2^N N!$ coefficients are determined by (3.6) only up to $2^N$ arbitrary constants, we will see
in next section that the secular equation for the spectrum is determined uniquely. The basic elements of the Weyl group $\mathcal{W}_2$ obey $\sigma^2 = 1$ and $\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}$ as identities. These identities must involve some relations i.e. $A_u(\sigma^2 \sigma, \tau) = A_u(\sigma, \tau)$ and $A_u(\sigma_i \sigma_{i+1} \sigma_i \sigma_{i+1}, \tau) = A_u(\sigma_{i+1} \sigma_i \sigma_{i+1}, \tau)$. Using (3.4) repeatedly, one can obtain the following relations:

$$S'(\sigma, \sigma) S'(\sigma) = I,$$

$$S'(\sigma_i \sigma_{i+1} \sigma_i) S'(\sigma_i) S'(\sigma_{i+1} \sigma_i) S'(\sigma_{i+1}) = S'(\sigma_{i+1} \sigma_{i+1} \sigma_{i+1}) S'(\sigma_{i+1} \sigma_i) S'(\sigma_{i+1}),$$

(3.8)

where we have adopted the conventions $S = \text{matrix}(S_{ab})$, $S' = S \otimes I$, $S'^{+1} = I \otimes S$ (I is a $2 \times 2$ unit matrix). These relations are consistency conditions for the S-matrix. The second relation is called a Yang-Baxter equation. The concrete S-matrix in (3.7) verifies these relations.

**IV. DIAGONALIZATION OF THE S-MATRIX AND THE SECULAR EQUATION FOR THE SPECTRUM**

Because the antisymmetric property provides a relation for wave function between different Weyl chambers, we only need to discuss the problem in one chamber. For simplicity we consider

\begin{align*}
\text{(i)} & \quad \text{In the region } \bar{x} < x_1 < x_2 < \cdots < x_N < \frac{L}{2}, \quad -\frac{L}{2} < x_1 < x_2 < \cdots < x_N < \frac{L}{2} \quad \text{and} \quad -\frac{L}{2} < x_1 < x_2 < \cdots < x_{N-1} < \frac{L}{2} < x_N
\end{align*}

respectively. The cases of more than one particle being outside of the interval $[-\frac{L}{2}, \frac{L}{2}]$ are not important since those regions are not next to the region $-\frac{L}{2} < x_1 < x_2 < \cdots < x_N < \frac{L}{2}$.

We first make some notation conventions: $\mathcal{W}' := \{\sigma_2, \sigma_3, \ldots, \sigma_{N-1}, \sigma_N\}$ and $\mathcal{W}'' := \{\sigma_1, \sigma_2, \ldots, \sigma_{N-2}, \sigma_{N-1} \sigma_N\}$ are two subgroups of the Weyl group of $\mathbb{R}_N$; two particular cycles are $q_j' : (x_1, \ldots, x_j, \ldots, x_N) \mapsto (x_1, x_1, \ldots, x_{j-1}, x_{j+1}, \ldots, x_N)$ and $q_j'' : (x_1, \ldots, x_j, \ldots, x_N) \mapsto (x_1, \ldots, x_{j-1}, x_{j+1}, \ldots, x_N, x_j)$. Then we can write (3.1) into a sum of pairs and get the following expressions (because there will be no confusion here, we adopt the notation $\psi_a$ instead of $\psi_a^{id}$):

(i) In the region $-\frac{L}{2} < x_1 < x_2 < \cdots < x_N < \frac{L}{2}$,

$$\psi_a(x) = \sum_{j=1}^N \sum_{\alpha' \in \mathcal{W}'} A_a(\alpha' q_j', id) e^{ik_j x_1} + A_a(\bar{\alpha} \sigma' q_j', id) e^{-ik_j x_1} \right| e^{i(\sigma' q_j' k | x')}$$

$$= \sum_{j=1}^N \sum_{\alpha' \in \mathcal{W}'} A_a(\alpha'' q_j', id) e^{i(\sigma'' q_j' k | x')}$$

(4.1)

here and the following $x' := (0, x_2, \ldots, x_N)$, $x'' := (x_1, x_2, \ldots, x_{N-1}, 0)$ and $\bar{\alpha} := \sigma_1 \sigma_2 \cdots \sigma_{N-1} \sigma_N \sigma_{N-1} \cdots \sigma_1$.

(ii) In the region $x_1 < -\frac{L}{2} < x_2 < \cdots < x_N < \frac{L}{2}$,

$$\psi_a^L(x) = \sum_{j=1}^N \sum_{\sigma' \in \mathcal{W}'} A_a^L(\sigma' q_j', id) e^{\kappa_j x_1} e^{i(\sigma' q_j' k | x')}$$

(4.2)

here and the following $\kappa_j > 0$, $\kappa_j^2 + k_j^2 = V_0^2$;

(iii) In the region $-\frac{L}{2} < x_1 < x_2 < \cdots < x_{N-1} < \frac{L}{2} < x_N$,

$$\psi_a^R(x) = \sum_{j=1}^N \sum_{\sigma' \in \mathcal{W}''} A_a^R(\sigma'' q_j'', id) e^{-\kappa_j x_N} e^{i(\sigma'' q_j'' k | x''')}$$

(4.3)

Because of the finite depth of potential well, both the wave function and its derivative are continuous at the ends of the well. At the left end, we have

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Thus we get \( A_\sigma'q', id \) = \( A_\sigma(q', id) e^{-ik_j - \kappa_j}L/2 + A_\sigma(\bar{\sigma}_1\sigma'q', id) e^{(ik_j + \kappa_j)L/2} \) and
\[
\frac{\kappa_j - ik_j}{\kappa_j + ik_j} = \frac{A_\sigma(\bar{\sigma}_1\sigma'q', id)}{A_\sigma(q', id)} e^{ik_j L}.
\]

Similarly, the conditions at the right end read
\[
\frac{\kappa_j + ik_j}{\kappa_j - ik_j} = \frac{A_\sigma(\bar{\sigma}_N\sigma''q''', id)}{A_\sigma(q''', id)} e^{-ik_j L}.
\]

By making use of (4.1) and (3.6), we obtain the matrix relations
\[
A(\sigma q', id) = S_1(\sigma \cdots \sigma_{N-1}\sigma q''_k) S_2(\sigma_3 \cdots \sigma_{N-1}\sigma q''_k) \cdots S_{N-2}(\sigma_{N-1}\sigma q''_k) S_{N-1}(\sigma q''_k) \cdot A(\sigma q''_k, id) \tag{4.8}
\]
\[
A(\bar{\sigma}_1q', id) = S_1(\sigma_2 \cdots \sigma_{N-1}\bar{\sigma}_N\sigma q''_k) S_2(\sigma_3 \cdots \sigma_{N-1}\bar{\sigma}_N\sigma q''_k) \cdots S_{N-2}(\sigma_{N-1}\bar{\sigma}_N\sigma q''_k) S_{N-1}(\bar{\sigma}_N\sigma q''_k) \cdot A(\bar{\sigma}_N\sigma q''_k, id). \tag{4.9}
\]

Here and in the following equations the spin labels are omitted, hence they should be understood as matrix equations. From (4.5, 4.7) together with (4.8, 4.9) we obtain an eigenequation in the spin space:
\[
\bar{T}_j^{-1}T_j A(\bar{\sigma}_N\sigma q''_k, id) = f^2_j A(\bar{\sigma}_N\sigma q''_k, id) \tag{4.10}
\]
where
\[
T_j = S^{ij-1} \cdots S^{j1} S^j N \cdots S^{jj+1}, \quad \bar{T}_j = \bar{S}^{ij-1} \cdots \bar{S}^{j1} \bar{S}^j N \cdots \bar{S}^{jj+1} \tag{4.11}
\]
and
\[
S^{ji} \equiv S_i(\sigma_{i+1} \cdots \sigma_{N-1}\sigma q''_k) = -\frac{[(\sigma k)_j - (\sigma k)_i]P - ic}{(\sigma k)_j - (\sigma k)_i + ic} \tag{4.12}
\]
\[
\bar{S}^{ji} \equiv S_i(\sigma_{i+1} \cdots \sigma_{N-1}\bar{\sigma}_N\sigma q''_k) = -\frac{[-(\sigma k)_j - (\sigma k)_i]P - ic}{-(\sigma k)_j - (\sigma k)_i + ic}. \tag{4.13}
\]

In order to solve the eigenequation (4.10), we observe the highest weight state corresponding to \( M \) down-spins and \( N - M \) up-spins,
\[
\Phi^M = \sum_{\{m_\mu\}} \varphi_{\{m_\mu\}} \prod_{\mu=1}^M \hat{s}_{m_\mu}^{++ \cdots +} \tag{4.14}
\]

where \( \varphi_{++ \cdots +} \) stands for the state of \( N \) up-spins, \( \hat{s}_{m_\mu} \) flips a up-spin to a down-spin at the \( m_\mu \)th position. The state (4.14) is checked to be a common eigenstate of both operators \( T_j \) and \( \bar{T}_j \) if \( \varphi \) takes the following form
\[
\varphi_{m_1 m_2 \cdots m_M} = \sum_\{\tilde{s}\} \Gamma(\tilde{s}) \prod_{\mu=1}^M F(\tilde{s}_\lambda, m_\mu). \tag{4.15}
\]
where
\[
\frac{\Gamma(\hat{\delta}, \bar{\delta}\lambda)}{\Gamma(\bar{\delta}\lambda)} = \frac{i(\hat{\delta}\lambda)_{i+1} - (\bar{\delta}\lambda)_i - c}{i(\hat{\delta}\lambda)_{i+1} - (\bar{\delta}\lambda)_i + c}, \quad \frac{\Gamma(\bar{\delta}, \hat{\delta}\lambda)}{\Gamma(\bar{\delta}\lambda)} = -1.
\] (4.16)

and
\[
F(\lambda; m) = \frac{1}{i(k_m - \lambda) + c/2} \sum_{j=1}^{m-1} \frac{i(k_j - \lambda) - c/2}{i(k_j - \lambda) + c/2},
\] (4.17)

where \(\hat{\delta}\) stands for the elements of the Weyl group of the Lie algebra \(B_M\); \(\hat{\delta}_i (i = 1, 2, \cdots, M)\) and \(\bar{\delta}_M\) stands for its basic elements. The parameter \(\lambda_\mu (\mu = 1, 2, \cdots, M)\) for the spin wave function is supposed to satisfy
\[
\prod_{\nu \neq \mu}^{M} \frac{\lambda_\mu - \lambda_\nu + ic}{\lambda_\mu - \lambda_\nu - ic} \frac{\lambda_\mu + \lambda_\nu + ic}{\lambda_\mu + \lambda_\nu - ic} = \prod_{j=1}^{N} \frac{\lambda_\mu - k_j - ic/2}{\lambda_\mu - k_j + ic/2} \frac{\lambda_\mu + k_j - ic/2}{\lambda_\mu + k_j + ic/2}
\] (4.18)

This is a generalization of Bethe-Yang ansatz [5]. The eigenvalues corresponding to the operators \(T_j\) and \(\bar{T}_j^{-1}\) are
\[
z_j = \prod_{\nu = 1}^{M} \frac{\lambda_\nu - k_j - ic/2}{\lambda_\nu - k_j + ic/2}, \quad z_j^{-1} = \prod_{\nu = 1}^{M} \frac{\lambda_\nu + k_j - ic/2}{\lambda_\nu + k_j + ic/2}
\] (4.19)

respectively. Hence we get the coupled Bethe ansatz equations for the variables \(\lambda_\mu, (\mu = 1, 2, \cdots, M)\): and the quasi-momenta \(z_j z_j^{-1} = f_j^2\), explicitly
\[
k_j = \frac{\pi}{L} J_j - \frac{1}{L} \sin^{-1} \left( \frac{k_j}{V_0} \right) - \frac{1}{L} \sum_{\nu = 1}^{M} \left[ \tan^{-1} \left( \frac{k_j - \lambda_\nu}{c/2} \right) + \tan^{-1} \left( \frac{k_j + \lambda_\nu}{c/2} \right) \right]
\] (4.20)

Taking the logarithm of (4.20) and (4.18) we obtain the secular equations
\[
\sum_{j=1}^{N} \left[ \tan^{-1} \left( \frac{\lambda_\mu - k_j}{c/2} \right) + \tan^{-1} \left( \frac{\lambda_\mu + k_j}{c/2} \right) \right] = \pi J_\mu + \sum_{\nu \neq \mu}^{M} \left[ \tan^{-1} \left( \frac{\lambda_\mu - \lambda_\nu}{c} \right) + \tan^{-1} \left( \frac{\lambda_\mu + \lambda_\nu}{c} \right) \right].
\] (4.22)

where both \(J_j\) and \(J_\mu\) are integers, that play the roles of quantum numbers. These equations determine the spectrum \(\{k_j\}\) of the system with total spin \(S_z = \frac{1}{2} N - M\) and the total energy \(E = \sum_{j=1}^{N} k_j^2\).

V. CONCLUSIONS AND REMARKS

In the previous sections, we solved the problem of \(N\) fermions with \(\delta\)-function interaction in a one dimensional potential well of finite depth. In our discussion, we considered the special case in which a single particle tunnels out of the potential well. In this case, we showed that there exists a Bethe ansatz like exact solution for the problem. We also obtained the secular equations which determine the spectrum of \(N\)-body system. Differing from the usual \(\delta\)-function problem, there is a contribution to the secular equations by the reflection of the particles at the ends of the potential well. This contribution affects both the charge and spin excitations. A new feature of present model is the additional term proportional to \(\sin^{-1}(k_j/V_0)\) in (4.22). As a phase shift of one particle at the barrier of potential well of finite depth, this term contributes to charge excitation only. The phase shift vanishes when \(V_0 \to \infty\), which agrees with the result for the potential well of infinite depth. In the thermodynamic limit \((L \to \infty, N \to \infty\) and \(M \to \infty\) but \(N/L - M/L\) keeps a finite value), equations (4.21) and (4.22) will become integral equations. Certainly, one can discuss the ground state properties and its thermodynamics in the spirit of [13].

We known that the \(N\) particles were thought of as situated on a circle in the case of periodic boundary conditions. For the problem of \(N\) electrons moving in a quantum wire of finite length, the periodic boundary condition is no
longer available. Then we may consider the boundary as a potential well. If all the $N$ electrons are confined in an interval, the potential well is supposed to have infinite depth. Strictly speaking, electrons are able to leave the ends of the wire due to the tunneling effect. Then the more appropriate boundary condition is a potential well of finite depth. Different from the case of infinite depth in which one needs only to consider the wave function in the potential well, we should take into account the possibility of a non-vanishing wave function outside of the well in the case of finite depth. The single particle tunneling process is the simplest non-trivial one for the problem of the potential well of finite depth. The result of present paper is the first step toward the exact solution of this problem.

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