EXTENDABLE COHOMOLOGIES FOR COMPLEX ANALYTIC VARIETIES

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Abstract. We introduce a cohomology, called extendable cohomology, for abstract complex singular varieties based on suitable differential forms. Beside a study of the general properties of such a cohomology, we show that, given a complex vector bundle, one can compute its topological Chern classes using the extendable Chern classes, defined via a Chern-Weil type theory. We also prove that the localizations of the extendable Chern classes represent the localizations of the respective topological Chern classes, thus obtaining an abstract residue theorem for compact singular complex analytic varieties. As an application of our theory, we prove a Camacho-Sad type index theorem for holomorphic foliations of singular complex varieties.

Introduction. One of the more important contributions to the study of complex vector bundles over differentiable manifolds has been given by the Chern-Weil theory. Thanks to such a theory it is possible to describe the topological Chern classes of a complex vector bundle on a manifold (which lie in the topological cohomology groups of the manifold) by means of the differentiable Chern classes of the bundle (which belong to the de Rham cohomology groups of the manifold).

By their very definition, the differentiable Chern classes of a complex vector bundle are built starting from suitable differentiable differential forms on the manifold. This is the reason why, until now, it was impossible to achieve a generalization of the Chern-Weil theory allowing to study complex vector bundles over singular varieties. In fact, the hurdles for having such a theory are tied to the difficulties of giving an appropriate definition of differential forms on singular spaces.

In this paper we solve the problem of extending the Chern-Weil theory to the case of abstract complex analytic varieties. Namely, we introduce a suitable notion of differential forms, the extendable differentiable differential forms, we develop a cohomology theory based on such forms, we define the extendable Chern classes for differentiable complex vector bundles over complex analytic varieties and we prove that these classes represent the topological Chern classes of the bundle.

The starting point is the following. In the case of complex analytic varieties, it can be given several natural definitions of holomorphic differential forms. Nevertheless, although remarkable results have been obtained, the development of the theories based on such holomorphic forms did not carry on, because of the failure of the Poincaré lemma. Namely, the cohomologies associated with these holomorphic forms are not, in general, locally trivial (cp. [Fe 1], [Fe 2], [He 1], [He 2], [Bl-He]). Anyway, all these definitions of holomorphic differential form are such that the
sheaves of germs of such forms share the following property: it is a sheaf of modules (over the ring of holomorphic functions on the variety) that, even if it is not necessarily locally free, it is always coherent (cp. [Fe 2]).

On the other hand, we need differential forms that are differentiable but not necessarily holomorphic. So, we define the sheaf of extendable differentiable differential forms we are interested in by tensorizing one of the sheaves of holomorphic differential forms with the sheaf of rings of differentiable functions on the variety. In view of our aim of extending the Chern-Weil theory, the choice of the sheaf of holomorphic differential forms is not important, even if, of course, different choices generally lead to different results. In fact, we only need the coherence of the sheaf of extendable differentiable differential forms.

Extendable differentiable differential forms enjoy many properties of differential forms on smooth manifolds (for example, they always have bounded coefficients), even if the proof of some of these properties is not trivial, because of the presence of singularities (see, for example, Lemma 2, Proposition 5, Theorem 3). On the other hand, the cohomology groups associated with extendable forms (the extendable cohomology groups) are not locally trivial (cp. Example 3).

Let $E \to X$ be a differentiable (holomorphic) $\mathbb{C}$-vector bundle over an abstract finite dimensional complex analytic variety $X$. By using the theory of extendable forms, we introduce the notions of extendable linear connections and extendable curvatures for $E$. Let $\nabla$ be an extendable linear connection for $E$. We define the extendable Chern forms $\left[ c_{\text{ext}}^p(\nabla) \right]$ associated with $\nabla$ and, arguing as in the smooth case, we show that $\left[ c_{\text{ext}}^p(\nabla) \right]$ are closed and only depend on $E$. Then, we define extendable Chern classes of $E$ as the cohomology classes $\left[ c_{\text{ext}}^p(E) \right] = \left[ c_{\text{ext}}^p(\nabla) \right]$.

Next, we may define an operator of integration of extendable forms on simplices (recall that any complex analytic variety is triangulable). Let $H_{\text{ext}}^\bullet(X)$ denote the extendable cohomology groups of $X$. Then the operator of integration induces a homomorphism $H^\bullet : H_{\text{ext}}^\bullet(X) \to H^\bullet(X)$ between extendable and topological cohomology groups (cp. Section 4). We prove the following theorem (cp. Theorem 6).

**Theorem** Let $X$ be an abstract complex analytic variety of complex dimension $n$ and $E \to X$ a differentiable (holomorphic) complex vector bundle of rank $e$. Take $q \in \{1, \ldots, n\}$ with $q \leq e$. Then $c_{\text{top}}^q(E) = H^q \left( c_{\text{ext}}^q(E) \right)$, where $c_{\text{top}}^p(E)$ denote the topological Chern classes of $E$.

One of the topics we deal with in this paper is to prove residue theorems for holomorphic complex vector bundles over compact irreducible abstract complex analytic varieties. Let $X$ be a complex analytic variety of dimension $n$ and $\chi \in H^\bullet(X)$ an element in the topological cohomology groups of $X$. It can happen that the class $\chi$ could represent the first order obstruction to the existence of a certain global object $\sigma$ on $X$ (for example, topological Chern classes represent the first order obstruction to the existence of global frames for complex vector bundles). Generally and very roughly speaking, one asks where on the variety $X$ the existence of $\sigma$ is obstructed. Namely, one asks where on $X$ the class $\chi$ vanishes. Let $S$ denote the loci where $\sigma$ exists ($\chi$ vanishes). It could be possible to make a clever choice of $S$, even if such loci, in general, are not unique. Then, on $X \setminus S$, that is outside $S$, the object $\sigma$ exists and the class $\chi$ vanishes.
Now, assume that $X$ is compact and let $P^*_\bullet : H^\bullet(X) \to H_{2n-\bullet}(X)$ be the Poincaré homomorphism. Suppose that $S$ is an analytic subvariety of $X$ and consider the exact sequence

$$
\cdots \to H^\bullet(X, X \setminus S) \to H^\bullet(X) \to H^\bullet(X \setminus S) \to \cdots
$$

If the image of $\chi \in H^\bullet(X)$ in $H^\bullet(X \setminus S)$ is 0, then there exist $\kappa \in H^\bullet(X, X \setminus S)$ whose image in $H^\bullet(X)$ is $\chi$. Such a $\kappa$ is the localization of $\chi$ at $S$ and, in general, it is not unique. Nevertheless, if $S$ is compact, by taking into account the Alexander-Lefschetz homomorphism $A^*_S, \bullet : H^\bullet(X, X \setminus S) \to H_{2n-\bullet}(S)$ and the commutative diagram

$$
\begin{array}{ccc}
H^\bullet(X, X \setminus S) & \to & H^\bullet(X) \\
\downarrow A^*_{S, \bullet} & & \downarrow \nu^*_\bullet \\
H_{2n-\bullet}(S) & \xrightarrow{i_*} & H_{2n-\bullet}(X),
\end{array}
$$

we get the formula $P^*_2(\chi) = (i_* \circ A^*_{S, \bullet})(\kappa)$. This is an "index theorem" (see [Su 1]).

If $\bullet = 2n$ and $S$ is a finite set of points $\{p_\nu\}$, then $H_0(S) = \oplus p_\nu H_0(p_\nu)$ and $A^*_S(\kappa) = \sum_\nu Res(\kappa, p_\nu)$, where $Res(\kappa, p_\nu) \in H_0(p_\nu)$ is "the residue of $\kappa$ at $p_\nu"$. So, the index theorem can be written as

$$
P^*_2(\chi) = \sum_\nu i_*(Res(\kappa, p_\nu)).
$$

Next, taking into account the homomorphism $H^{2n}_{\text{top}} : H^{2n}_{\text{ext}}(X) \to H^{2n}(X)$ induced by integration on simplices, we have $P^*_2 \circ H^{2n}_{\text{top}} = \int_{[X]}$, where $[X]$ is the fundamental class of $X$. So, if $\chi_{\text{ext}} \in H^{2n}_{\text{ext}}(X)$ is such that $\chi = H^{2n}_{\text{top}}(\chi_{\text{ext}})$, then we get

$$
\int_{[X]} \chi_{\text{ext}} = \sum_\nu i_*(Res(\kappa, p_\nu))
$$

Namely, a "residue theorem". We prove the following theorem (cp. Theorem 7).

**Theorem** Let $X$ be a compact and irreducible complex analytic variety of complex dimension $n$ and $E \to X$ a holomorphic complex vector bundle of rank $e$. Take $q \in \{0, \ldots, n\}$ with $q \leq e$ and set $r = e - q + 1$. Let $s^{(r)}$ be a holomorphic $r$-section of $E$ and $S$ the singular locus of $s^{(r)}$ and $c^{(r)}_{\text{top}}(E, s^{(r)})$ the localization at $S$ of $c^{(r)}_{\text{top}}(E)$ determined by $s^{(r)}$. Set $TopRes_{c^{(r)}_{\text{top}}} (E, s^{(r)}, S) = A^*_S, 2q(c^{(r)}_{\text{top}}(E, s^{(r)}))$. Then

$$
P^*_q \circ H^{2q}_{\text{top}}(c^{(r)}_{\text{ext}}(E)) = i_*(TopRes_{c^{(r)}_{\text{top}}} (E, s^{(r)}, S)).
$$

If $q = n$, then

$$
\int_{[X]} c^{(r)}_{\text{ext}}(E) = i_*(TopRes_{c^{(r)}_{\text{top}}} (E, s^{(r)}, S)).
$$

Actually, a residue theorem becomes really useful only if the residue can be explicitly and easily computed. In order to solve this problem, several people widely and successfully used the theory of Čech-de Rham. Among other authors, we mention J. P. Brasselet, D. Lehmann and T. Suwa, who, furthermore, greatly developed such a theory (see [Le-Su], [Br-Su], [Su 1] and references therein). Indeed, Čech-de Rham theory provides with very handleable tools to explicitly compute the residue, at least in the case of isolated singularities. We should note that, until now, only the cases of manifolds, submanifolds embedded in a manifold and, at most, subvarieties embedded in a manifold were studied (see [Le-Su], [Su 1], [Su 3]). In fact, in these cases, in order to have a Čech-de Rham type theory, the differentiable differential
forms of the ambient are used. On the other hand, our theory allows to take into account the case of abstract complex analytic varieties.

In this paper we also develop a Čech-de Rham type theory for extendable forms. By means of such a theory, we can compute the residue of the localizations of several characteristic classes, at least if the singularities are isolated. For instance, we obtain a generalization of Camacho-Sad index theorem (see Theorem 5 for a more general statement).

**Theorem** Let $X$ be an abstract complex analytic variety of complex dimension 2, $F$ a holomorphic foliation of $X$ and $Y$ an $F$-invariant globally irreducible Cartier divisor of $X$ such that $Y \not\subseteq \text{Sing}(X)$. Set $S = (\text{Sing}(F) \cap Y) \cup \text{Sing}(Y)$ and let $N_Y \to Y$ be the line bundle $\mathcal{O}(Y)$. Then

$$\int_{[Y]} c^{1}_{\text{ext}}(N_Y) = i_*(\text{Res}_{\text{ext}}(N_Y,F,S)),$$

where $c^{1}_{\text{ext}}(N_Y,F,S)$ is the localization of $c^{1}_{\text{ext}}(N_Y)$ at $S$ determined by $F$ and $\text{Res}_{\text{ext}}(N_Y,F,S)$ are, in fact, complex numbers which only depend on the behaviour $F$ of around $S$.

Suppose that $S$ only contains an isolated singular point $p \in \text{Sing}(Y) \cap \text{Sing}(F) \cap \text{Sing}(X)$ and that the stalk $\mathcal{F}_p$ is generated on $\mathcal{O}_{X,p}$ by a single element of $T_X$. Let $(h,y)$ be local coordinates on $X^{\text{Reg}}$ near $p$ such that $y$ is a local coordinate on $Y' = Y \setminus ((\text{Sing}(X) \cap Y) \cup \text{Sing}(Y))$ near each point of $Y' \setminus \{p\}$. If the holomorphic vector field $f \in T_X$ generating $F$ is locally given by $F = a(h,y)h \frac{\partial}{\partial h} + b(h,y) \frac{\partial}{\partial y}$, with $a$ and $b$ holomorphic functions such that $b(0,y)$ is not identically equal to zero, then

$$i_*(\text{Res}_{\text{ext}}(N_Y,F,p)) = \frac{1}{2\pi i} \int_{Lk(p)} \frac{a(0,y)}{b(0,y)} dy,$$

where $Lk(p)$ is the link of the singularity.

The work is organized as follows. In Section 1 we fix some important notations. In Section 2 we define extendable vector bundles on complex analytic varieties and extendable sections of extendable bundles. Then, we study in a deeper way the important case of extendable differentiable differential forms. In Section 3 we define the extendable cohomology groups and we prove several important results concerning these groups. In Section 4 we define a homomorphism between extendable and topological cohomology groups of complex analytic varieties. In Section 5 lie the main results of our work. We use the notion of extendable sections to introduce extendable connections and extendable Chern classes for complex vector bundles over complex analytic varieties. Then, we show that these classes represent the topological Chern classes (defined by means of obstruction theory) via the homomorphism of integration described in Section 4 (cp. Theorem 6). More precisely, we represent the localizations of the topological Chern classes by means of the respective localizations of the extendable Chern classes (cp. Theorem 7). Furthermore, in the compact case, we prove an abstract residue theorem (cp. Theorem 8). Finally, we prove a Camacho-Sad type index theorem for holomorphic foliations of singular complex varieties and, under suitable hypotheses, we explicitly compute the residue at isolated singularities (cp. Theorem 9).

It is in our opinion that the extendable objects we introduced can be successfully used in order to solve problems of continuous and discrete holomorphic dynamics in the setting of singular varieties, avoiding the desingularization processes. Indeed, we think that the theory of extendable differentiable forms which we have developed
lends itself to many uses and applications. In fact, generalizations similar to the
ones of residue theorem (cp. Theorem 8) can be achieved in several contexts.
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1. Main notations

Let $M$ be a complex differentiable manifold. The sheaves of germs of differen-
tiable and holomorphic functions on $M$ are denoted by $\mathcal{C}_M^\infty$ and, re-
spectively, by $\mathcal{O}_M$. The real (holomorphic, antiholomorphic, complexified re-
tal) cotangent and tangent bundles of $M$ are denoted by $T^*M$, $\mathbb{T}^*M$, $T^{C\ast}M$ and,
respectively, by $TM$, $\mathbb{T}M$, $T^{C}M$. For each $p \in \mathbb{N}$ we denote by $\mathcal{E}_M^p$ and
by $\Omega^p_M$ the sheaves of germs of differentiable and, respectively, holomorphic
differential $p$-forms on $M$.

Recall that an abstract complex analytic variety $X$ of complex dimension $n$ is a
second countable, Hausdorff topological space for which there exist an open covering
$\mathcal{C} = \{A_l\}_{l \in L}$ and homeomorphisms
\begin{equation}
F_l : A_l \rightarrow W_l
\end{equation}
between the subsets $A_l \subseteq X$ and holomorphic subvarieties $W_l \subseteq U_l$ of open sets
$U_l \subseteq \mathbb{C}^n$ such that for each nonempty intersection $A_{l_1, l_2} = A_{l_1} \cap A_{l_2}$ the map
\begin{equation}
F_{l_1, l_2} : F_{l_2}(A_{l_1, l_2}) \rightarrow F_{l_1}(A_{l_1, l_2})
\end{equation}
defined by $F_{l_1, l_2} = F_{l_1} \circ F_{l_2}^{-1} | F_{l_2}(A_{l_1, l_2})$ is a biholomorphism such that the regular
part of $X$ is endowed with a structure of a complex manifold of complex dimension
$n$. A covering as $\mathcal{C}$ is a coordinate open covering of $X$ or an atlas of $X$. Sometimes,
to make explicit all the data carried by an atlas $\mathcal{C}$, we write
\begin{equation}
\mathcal{C} = \{(A_l, n_l, U_l, W_l, F_l)\}_{l \in L}.
\end{equation}

Let $X$ be an abstract finite dimensional complex analytic variety. The singular
locus and the regular part of $X$ will be denoted by $Sing(X)$ and, respectively, by
either $X^{Reg}$ or $X'$. Recall that $Sing(X)$ is a complex analytic subvariety of $X$ and
that $X^{Reg}$ is an open and dense subset of $X$. The maximal atlas of $X$ will be
denoted by $\mathcal{A} = \{A_l\}_{l \in L}$ and, given any $x \in X$, the set $\{i \in I : A_i \ni x\} \subseteq I$ will be
denoted by $I(x)$. Finally, the sheaves of germs of differentiable and holomorphic
functions on $X$ will be denoted by $\mathcal{C}_X^\infty$ and, respectively, by $\mathcal{O}_X$.

A finite dimensional complex analytic variety $X$ is a locally compact and para-
compact topological space.

**Lemma 1.** Let $X$ be a finite dimensional complex analytic variety and $\mathcal{V} = \{V_j\}_{j \in J}$
an open covering of $X$. Then
\begin{enumerate}
\item There exists an open covering $\mathcal{V}^* = \{V_j^*\}_{j \in J}$ of $X$ whose set of indices is
still $J$ and such that for any $j \in J$ it holds $\overline{V_j^*} \subseteq V_j$.
\item There exists an open covering $\mathcal{V}^\star = \{V_{\lambda}\}_{\lambda \in \Lambda}$ of $X$ refining $\mathcal{V}$ and such
that for each $\hat{\lambda} \in \Lambda$ there is a finite subset of indices $\Lambda(\hat{\lambda}) \subseteq \Lambda$ such that
$\overline{V_\Lambda} \cap V_{\hat{\lambda}}^\star \neq \emptyset$ if and only if $\lambda \in \Lambda(\hat{\lambda})$.
\end{enumerate}
Proof. \(X\) is a locally compact and paracompact topological space. So, the results follow from General Topology (cp. [Ch-To-Ve]). \(\square\)

Finally, for a general reference on complex analytic varieties, see [Gu], Vol. II.

2. Extendable bundles

2.1. Extendable vector bundles. We begin with the following definition.

**Definition 1.** Let \(X\) be an abstract finite dimensional complex analytic variety and \(E' \to X'\) a differentiable real (complex) vector bundle over \(X'\). We say that \(E'\) is \(S_{E'}\)-extendable if there exists a coherent sheaf \(S_{E'}\) of \(C_X\)-modules over \(X\) such that \(S_{E'}|_{X'} = C_X(E')\). A sheaf as \(S_{E'}\) is associated with \(E'\).

An other definition will be also necessary.

**Definition 2.** Let \(X\) be an abstract finite dimensional complex analytic variety and \(E' \to X'\) an \(S_{E'}\)-extendable differentiable real (complex) vector bundle. A section \(s' \in S_{E'}|_{X'}(X')\) of \(E'\) is \(S_{E'}\)-extendable if there exists a section \(s \in S_{E'}(X)\) such that \(s' = s|_{X'}\). A section as \(s\) is an \(S_{E'}\)-extension of \(s'\).

As a matter of notations and terminologies, the set
\[
\Gamma_{ext}(X', E') = \{s' \in \Gamma(X', E') : s' \text{ is } S_{E'}\text{-extendable}\},
\]
also denoted by \(\Gamma_{ext}(E') = \Gamma_{ext}(X', E')\), is called space of \(S_{E'}\)-extendable differentiable sections of \(E'\). The sheaf of germs of \(S_{E'}\)-extendable differentiable sections of \(E'\) is denoted by \(\Gamma_{ext}E'\).

We present a simple but fundamental example.

**Example 1.** Let \(X\) be an abstract finite dimensional complex analytic variety, \(E \to X\) a differentiable real (complex) vector bundle over the whole of \(X\) and \(E\) the sheaf of germs of differentiable sections of \(E\). Then the bundle \(E' = E|_{X'} \to X'\) is \(E\)-extendable.

Let \(s : X \to E\) be differentiable global section of \(E\). Then \(s' = s|_{X'}\) is an \(E\)-extendable section of \(E'\).

Less trivial examples of extendable vector bundles will be discussed in Subsection 2.2.

**Notation 1.** Let \(X\) be an abstract finite dimensional complex analytic variety and \(S\) a sheaf over \(X\). Given an atlas \(\{(A_k, n_k, U_k, W_k, F_k)\}_{k \in K}\) of \(X\) (cp. Section 1) and, in particular, \(\{(W, k)\}\), we simply denote by \(S_k\) the sheaf \((F_k)_*(S|_{A_k})\) over \(F_k(A_k)\).

Let \(X\) be an abstract finite dimensional complex analytic variety and \(\mathcal{A} = \{A_i\}_{i \in I}\) the maximal atlas of \(X\) (cp. Section 1). For any \(i \in I\) write \(A_i' = A_i \setminus \text{Sing}(X)\). Let \(E' \to X'\) be an \(S_{E'}\)-extendable differentiable real (complex) vector bundle over \(X'\). By the very definition of extendable bundle (cp. Definition 1), the sheaf \(S_{E'}\) is coherent. So, for any \(x \in X\) there exists an index \(l \in I(x)\) such that the restriction \(S_{E'}|_{A_l}\) of \(S_{E'}\) at \(A_l\) is generated by a finite number of sections and such that the sequence
\[
(C_X^\infty)|_{A_l} \to S_{E'}|_{A_l} \to 0,
\]
with \( \nu_t \in \mathbb{N} \), is exact. As a note, the number \( \nu_t \in \mathbb{N} \) is, in general, bigger than the Zariski dimension of the germ \( X_x \) of \( X \) at \( x \). Then the sequence

\[
((C^\infty_X)^{\nu_t}|_{A_t})_t \to (\mathcal{S}_{E'}|_{A_t})_t \to 0
\]

is also exact. Now, taking into account the following exact sequence of sheaves \((C^\infty_{\mathbb{C}})^{\nu_t}|_{U_t} \to ((C^\infty_X)^{\nu_t}|_{A_t})_t \to 0\), we get a diagram of surjective maps

\[
\begin{align*}
(C^\infty_{\mathbb{C}})^{\nu_t}|_{U_t} & \to (C^\infty_X)^{\nu_t}|_{A_t} \to (\mathcal{S}_{E'}|_{A_t})_t \to 0 \\
\downarrow & \downarrow & \downarrow \\
0 & & 0
\end{align*}
\]

(2.2)

In particular, the map \( \zeta_t : (C^\infty_{\mathbb{C}})^{\nu_t}|_{U_t} \to (\mathcal{S}_{E'}|_{A_t})_t \) is surjective.

Next, consider the restriction map \( \epsilon_t : \mathcal{S}_{E'}|_{A_t} \to \mathcal{S}_{E'}|_{A'_t} \) and the map

\[
\epsilon_t : (\mathcal{S}_{E'}|_{A_t})_t \to (\mathcal{S}_{E'}|_{A'_t})_t
\]

(2.3)

induced by \( \epsilon_t \). These maps are not surjective in general. However, if \( a' \in \text{Im}(\epsilon_t) \), then there exists \( \tilde{a} \in (C^\infty_{\mathbb{C}})^{\nu_t}|_{U_t} \) such that \( a' = \epsilon_t(\tilde{a}) \), with \( a' = [F_t]^*(a') \).

This happens, for example, in the case of extendable sections. Indeed, if \( s' \in \mathcal{S}_{E'}|_{X'}(X') \) is an \( \mathcal{S}_{E'} \)-extendable section of \( E' \) and if \( s \in \mathcal{S}_{E'}(X) \) is an \( \mathcal{S}_{E'} \)-extension of \( s' \), then \( s' \) gives rise to an element \( s'_t \) of \( (\mathcal{S}_{E'}|_{A'_t})_t \) which lies in the image of \( \epsilon_t : (\mathcal{S}_{E'}|_{A_t})_t \to (\mathcal{S}_{E'}|_{A'_t})_t \). Denoting by \( s_t \in (\mathcal{S}_{E'}|_{A_t})_t \) the element determined by \( s \), we have \( s'_t = \epsilon_t(s_t) \). So, there exists \( \tilde{s}_t \in (C^\infty_{\mathbb{C}})^{\nu_t}|_{U_t} \) such that

\[
s'_t = \epsilon_t(\tilde{s}_t),
\]

(2.4)

because of the surjectivity of \( \zeta_t : (C^\infty_{\mathbb{C}})^{\nu_t}|_{U_t} \to (\mathcal{S}_{E'}|_{A_t})_t \).

**Remark 1.** Let \( X, A, E' \to X' \) and \( \mathcal{S}_{E'} \) be as in the above discussion. We wish to stress the fact that the sheaf \( \mathcal{S}_{E'} \) determines an atlas \( \mathcal{C}_{E'} \) of \( X \). Namely, the atlas that, using the above notations (cp. [13]), is given by \( \mathcal{C}_{E'} = \{(A_t, u_t, U_t, W_t, F_t)\} \).

An atlas as \( \mathcal{C}_{E'} \) is an atlas associated with \( E' \) or an atlas of trivializing extensions for \( E' \).

We need to introduce some terminology.

**Terminology 1.** Let \( X, E' \to X' \), \( \mathcal{S}_{E'} \) be as above and \( s' \in \Gamma_{ext}(E') \) an \( \mathcal{S}_{E'} \)-extendable section of \( E' \). Let \( \mathcal{C}_{E'} = \{A_t\} \) be an atlas associated with \( E' \).

Let \( Y \) be a subset of \( X \) and \( x \in Y \). We say that \( s' \) is extended by \( \tilde{s}_t \) on \( Y \) around \( x \in Y \) if there exist \( A_t \in \mathcal{C}_{E'} \) such that \( A_t \ni x \) and \( \tilde{s}_t \in (C^\infty_{\mathbb{C}})^{\nu_t}|_{U_t} \) such that \((F_t|_{A_t})^*(s'|_{A'_t}) = \epsilon_t(\tilde{s}_t) \). Let \( Y = A \) be an open subset of \( X \). We say that \( s' \) is completely extendable on \( A \) if for every \( x \in A \) the open set \( A_t \) contains \( A \).

We have the following proposition.

**Proposition 1.** Let \( X_1 \) and \( X_2 \) be finite dimensional complex analytic varieties and \( h : X_1 \to X_2 \) an analytic map. Let \( E \to X_2 \) be a differentiable complex (real) vector bundle defined over the whole of \( X_2 \), \( s' : X'_2 \to E|_{X'_2} \) an extendable section of \( E|_{X'_2} \to X'_2 \) and \( s \in E(X_2) \) an extension of \( s' \). Then \((h|_{X'_1})^*(s) : X_1 \to h^*(E)|_{X'_1} \) is an extendable differentiable section of \( h^*(E)|_{X'_1} \to X'_1 \).
Proof. $h^*(E)$ is a locally free sheaf of $\mathcal{C}^\infty_{X_2}$-modules over $X_1$, because $E$, the sheaf over $X_2$ of germs of differentiable sections of $E \to X_2$, is a locally free sheaf of $\mathcal{C}^\infty_{X_2}$-modules. Moreover, since the map $h$ induces a morphism $h^* : \mathcal{C}^\infty_{X_2} \to \mathcal{C}^\infty_{X_1}$, the sheaf $h^*(E) \otimes_{\mathcal{C}^\infty_{X_2}} \mathcal{C}^\infty_{X_1}$ is a well defined locally free sheaf of $\mathcal{C}^\infty_{X_1}$-modules over $X_1$. Furthermore, $h^*(E) \otimes_{\mathcal{C}^\infty_{X_2}} \mathcal{C}^\infty_{X_1}$ is the sheaf of germs of differentiable sections of the bundle $h^*(E) \to X_1$. So, the section $(h|_{X_1})^*(s) : X_1 \to h^*(E)|_{X_1}$ is extendable, because the section $t \in (h^*(E) \otimes_{\mathcal{C}^\infty_{X_2}} \mathcal{C}^\infty_{X_1})(X_1)$ defined by $t = h^*(s) \otimes 1$ is an extension of it. \hfill \square

The hypotheses of Proposition 1 can be weakened. Let $E' \to X'_2$ be an extendable bundle over $X'_2$ and consider its pull back $(h|_{h^{-1}(X'_2)})^*(E') \to h^{-1}(X'_2)$ via $h|_{h^{-1}(X'_2)}$. A priori, such a bundle is not extendable, because, by the very definition of extendable bundle, a necessary condition for $(h|_{h^{-1}(X'_2)})^*(E')$ to be extendable is to be defined at least on the whole of $X'_1$, the regular part of $X_1$ (cp. Definition 1). So, in order to generalize Proposition 1 we have to assume that $X'_1 \subseteq h^{-1}(X'_2)$, that is

\begin{equation}
(2.5) \quad h^{-1}(Sing(X'_2)) \subseteq Sing(X_1).
\end{equation}

Proposition 2. Let $X_1$ and $X_2$ be finite dimensional complex analytic varieties and $h : X_1 \to X_2$ an analytic map such that $h^{-1}(Sing(X_2)) \subseteq Sing(X_1)$. Let $E' \to X'_2$ be an $\mathcal{S}_{E'}$-extendable differentiable real (complex) vector bundle and $s' : X'_2 \to E'$ be an $\mathcal{S}_{E'}$-extendable section of $E'$. Then the restriction $(h|_{h^{-1}(X'_2)})^*(E')|_{X'_1} \to h^{-1}(X'_2)$ at $X'_1$ is a $(h^*(\mathcal{S}_{E'})) \otimes_{\mathcal{C}^\infty_{X_2}} \mathcal{C}^\infty_{X_1}$-extendable differentiable vector bundle and $(h|_{X'_1})^*(s') : X'_1 \to h^*(E')|_{X'_1}$ is an $(h^*(\mathcal{S}_{E'})) \otimes_{\mathcal{C}^\infty_{X_2}} \mathcal{C}^\infty_{X_1}$-extendable differentiable section of $h^*(E')|_{X'_1} \to X'_1$.

Proof. Since $\mathcal{S}_{E'}$ is a coherent sheaf of $\mathcal{C}^\infty_{X_2}$-modules over $X_2$, its pull back $h^*(\mathcal{S}_{E'})$ via $h$ is a coherent sheaf of $\mathcal{C}^\infty_{X_2}$-modules. Moreover, the map $h$ induces a morphism $h^* : \mathcal{C}^\infty_{X_2} \to \mathcal{C}^\infty_{X_1}$. So, $h^*(\mathcal{S}_{E'}) \otimes_{\mathcal{C}^\infty_{X_2}} \mathcal{C}^\infty_{X_1}$ is a well defined coherent sheaf of $\mathcal{C}^\infty_{X_1}$-modules over $X_1$ whose restrictions at $h^{-1}(X_2)$ and at $X'_1$ are a locally free sheaves of $\mathcal{C}^\infty_{X_1}$-modules. Then the vector bundle $(h|_{h^{-1}(X'_2)})^*(E')|_{X'_1} \to X'_1$ is extendable is $(h^*(\mathcal{S}_{E'})) \otimes_{\mathcal{C}^\infty_{X_2}} \mathcal{C}^\infty_{X_1}$-extendable, because the restriction of $(h^*(\mathcal{S}_{E'})) \otimes_{\mathcal{C}^\infty_{X_2}} \mathcal{C}^\infty_{X_1}$ at $X'_1$ coincides with the locally free sheaf of $\mathcal{C}^\infty_{X_2}$-modules of germs of differentiable sections of $(h|_{h^{-1}(X'_2)})^*(E')|_{X'_1}$. Finally, the section $(h|_{X'_1})^*(s) : X'_1 \to h^*(E')|_{X'_1}$ is extendable, because it admits the extension $h^*(s) \otimes 1 \in (h^*(\mathcal{S}_{E'})) \otimes_{\mathcal{C}^\infty_{X_2}} \mathcal{C}^\infty_{X_1}(X_1)$, with $s \in \mathcal{S}_{E'}(X_2)$ any extension of $s' : X'_2 \to E'$.

\hfill \square

2.2. Extendable differential forms. In this subsection we study the extendable vector bundles we are mainly interested in. Actually, the definition of extendable vector bundles given in Subsection 2.1 (cp. Definition 1) has been based on such examples.

Let $X$ be an abstract finite dimensional complex analytic variety. We need the following observations.

1. Denote by $TX'$ and $T^*X'$ the holomorphic tangent and, respectively, cotangent bundles of the manifold $X'$. Then, in case $Sing(X) \neq \emptyset$, the bundles $TX'$ and $T^*X'$ are not the restriction at $X'$ of any vector bundle defined
on the whole of $X$. The same holds for every their tensor power, for every their non trivial algebraic quotient and for every their vector subbundle. In particular, for any $N$, $N^*$, $p \in \mathbb{N}$ the bundles $TX^{\otimes N} \otimes T^*X^{\otimes N^*}$ and $\Lambda^pT^*X'$ are not the restriction at $X'$ of any bundle defined on $X$.

(2) Let $TX$ and $T^*X$ be the holomorphic tangent and, respectively, holomorphic cotangent varieties of $X$. The following diffeomorphisms (denoted by $\approx$) of real vector bundles hold: $TX' \approx TX|_{X'}$ and $T^*X' \approx T^*X|_{X'}$ (cp. Proposition 6.2 of Chapter I of [Su 1]).

(3) Let $\mathcal{O}_X(TX)$ be the sheaf of germs of holomorphic vector fields on $X$. Then $\mathcal{O}_X(TX)$ is a sheaf of $\mathcal{O}_X$-modules over $X$ that, even if it is not necessarily locally free, is always coherent.

(4) Denote by $\Omega_X$ the sheaf of germs of holomorphic differentials on $X$. Then $\Omega_X$ is a coherent sheaf of $\mathcal{O}_X$-modules over $X$. Furthermore, in case $X$ is reduced and irreducible, $\Omega_X$ is locally free if and only if $X$ is regular (see Theorem 8.15 of [Hs]).

(5) Fix $p \in \mathbb{N}$. We wish to consider the sheaf over $X$ of germs of holomorphic $p$-form on $X$. Actually, several different definitions of such a sheaf can be given. So, for the moment, just choose one of them, call it the sheaf over $X$ of germs of holomorphic $p$-form on $X$ and denote it by $\Omega^p_X$. Anyway, as it will be clear from the following discussion, to our purposes, it is not important which of the several candidates has been chosen. Indeed, what it is really important is the following property common to any sheaf candidate to be the sheaf of germs of holomorphic $p$-form on $X$: it is a sheaf of $\mathcal{O}_X$-modules that, even if it is not necessarily locally free, it is always coherent (cp. [Fe 2]).

Let us recall that $\mathcal{C}^\infty_X$ is the sheaf of germs of differentiable functions on $X$ (see Section [I]).

**Remark 2.** Let $X$ be an abstract finite dimensional complex analytic variety. Then $TX' \rightarrow X'$ is $(\mathcal{O}_X(TX) \otimes_{\mathcal{O}_X} \mathcal{C}^\infty_X)$-extendable and $T^*X' \rightarrow X'$ is $(\Omega_X \otimes_{\mathcal{O}_X} \mathcal{C}^\infty_X)$-extendable. Analogously, every tensor power, every non trivial algebraic quotient and every vector subbundle of $TX'$ and $T^*X'$ is an extendable vector bundle. In particular, for any $N$, $N^*$, $p \in \mathbb{N}$ the bundle $TX^{\otimes N} \otimes T^*X^{\otimes N^*}$ is $((\Omega^\otimes_X \otimes_{\mathcal{O}_X} \mathcal{O}_X(TX) \otimes_{\mathcal{O}_X} \mathcal{C}^\infty_X))$-extendable and the bundle $\Lambda^pT^*X'$ is $(\Omega^p_X \otimes_{\mathcal{O}_X} \mathcal{C}^\infty_X)$-extendable.

We introduce more manageable notations.

**Notation 2.** Let $X$ be an abstract finite dimensional complex analytic variety. For each $N$, $N^*$, $p \in \mathbb{N}$ the sheaf $(\Omega^\otimes_X \otimes_{\mathcal{O}_X} \mathcal{O}_X(TX) \otimes_{\mathcal{O}_X} \mathcal{C}^\infty_X)$ will be denoted by $\mathcal{S}_{N,N^*}$ and the sheaf $\Omega^p_X \otimes_{\mathcal{O}_X} \mathcal{C}^\infty_X$ will be denoted by $\mathcal{S}_p$.

Thus the bundles $TX^{\otimes N} \otimes T^*X^{\otimes N^*}$ and $\Lambda^pT^*X'$ are $\mathcal{S}_{N,N^*}$-extendable and, respectively, $\mathcal{S}_p$-extendable. As a note, both $\mathcal{S}_{0,0}|_{X'}$ and $\mathcal{S}_0|_{X'}$ coincide with $\mathcal{C}^\infty_{X'} = \mathcal{C}^\infty_X|_{X'}$.

**Example 2.** Let $X$ be an abstract finite dimensional complex analytic variety and $E \rightarrow X$ a differentiable real (complex) vector bundle. Then for each $N$, $N^*$, $p \in \mathbb{N}$ the bundle $TX^{\otimes N} \otimes T^*X^{\otimes N^*} \otimes \Lambda^pT^*X' \otimes E|_{X'}$ is $(\mathcal{S}_{N,N^*} \otimes \mathcal{S}_p \otimes E)$-extendable.

The following definition is justified by the very important role played by the $\mathcal{S}_p$-extendable sections of $\Lambda^pT^*X'$.
Definition 3. Let $X$ be an abstract finite dimensional complex analytic variety and $p \in \mathbb{N}$. The space
\begin{equation}
\Gamma_{ext}(\Lambda^p T^* X') = \{ \omega \in \Gamma(X', \Lambda^p T^* X') : \omega \text{ is } S_p\text{-extendable} \}
\end{equation}
of $S_p$-extendable differentiable sections of $\Lambda^p T^* X'$ is called space of extendable differentiable $p$-forms on $X$.

In what follows, we heavily use Notations of Section I (see (1.3)) and Subsection 2.1.

Let $X$ be an abstract finite dimensional complex analytic variety, $p \in \mathbb{N}$ and $C_p = \{(A_l, n_l, U_l, W_l, F_l)\}_{l \in L}$ an atlas of trivializing extension for the $S_p$-extendable bundle $\Lambda^p T^* X'$ (cp. Remark 1). Up to shrink the open sets of $C_p$, if necessary, we can improve (2.2) and get the following commutative diagram of surjective maps (for the notations see Section I).
\begin{equation}
(C^\infty_{n_l}|_{U_l}) \to \mathcal{E}^p_{U_l} \quad \downarrow \quad \downarrow
((C^\infty)^{n_l}|_{A_l}) \to (S_p|_{A_l})
\end{equation}
Indeed, it suffices to observe that for any $x \in X$ the index $l \in I(x)$ can be chosen in such a way that $n_l$ verifies the following inequality $\binom{n_l}{p} \leq n_l$.

Thus, thanks to (2.7), (2.4) can be also improved. Namely, $\omega \in \Gamma(\Lambda^p T^* X')$ is extendable if and only if for each $x \in X$ there are $A_l \in C_p$ and
\begin{equation}
\tilde{\omega}_l \in \Gamma(\Lambda^p T^* U_l)
\end{equation}
such that
\begin{equation}
\omega|_{A_l} = [F_l|_{A_l}]^*(\tilde{\omega}_l).
\end{equation}

We have the following remark (cp. [Pe]).

Remark 3. Let $X$ be a finite dimensional complex analytic variety and $C = \{A_l\}_{l \in L}$ an atlas of $X$. Then $\omega \in \Gamma(\Lambda^p T^* X')$ is extendable on $X$ if and only if it is extendable on $A_l \in C$ for any $l \in L$.

We explicitly note some facts concerning extendable forms.

Remark 4. Let $X$ be a finite dimensional complex analytic variety and $\omega \in \Gamma_{ext}(\oplus_{p \in \mathbb{N}} \Lambda^p T^* X')$ an extendable form on $X$. Then for any point $x \in X$, even singular, there exists a neighborhood $A_x$ of $x$ such that $\omega|_{A_x \cap X_{\text{reg}}}$ is bounded. This follows from the very definition of extendable sections (cp. Definition 3) and from (2.7), (2.8), (2.9).

Remark 5. Let $X_1$, $X_2$ be finite dimensional complex analytic varieties and $h : X_1 \to X_2$ an analytic map. If $\omega \in \Gamma_{ext}(\oplus_{p \in \mathbb{N}} \Lambda^p T^* X'_2)$ is an extendable form on $X_2$, then its pull back $(h|_{h^{-1}(X_2)^{\text{reg}} \cap X_1})^*(\omega)$ is not, in general, an extendable form on $X_1$. Indeed, in order to have the extensibility of $(h|_{h^{-1}(X_2)^{\text{reg}} \cap X_1})^*(\omega)$, Condition (2.9) must be verified (cp. Proposition 2).

Let $X$ be an abstract finite dimensional complex analytic variety and fix $p \in \mathbb{N}$.

Let $Z$ be the closure of a non empty open set of $X$ which is also a polyhedron of $X$ and denote by $i : Z \hookrightarrow X$ the inclusion. As a matter of notations, set $\text{Sing}(Z) = Z \cap \text{Sing}(X)$ and $Z' = Z \cap X'$. Then
\begin{equation}
i^{-1}(\text{Sing}(X)) \subseteq \text{Sing}(Z).
\end{equation}
The set
\[ (2.11) \Gamma_{ext}(\Lambda^p T^* X')_Z = \{ \omega \in \Gamma_{ext}(\Lambda^p T^* X') : (i|Z')^*(\omega) = 0 \} \]
is called space of extendable (differentiable) differential p-forms (on X) vanishing on Z. It is easy to prove that \( \Gamma_{ext}(\Lambda^p T^* X')_Z \) is a subspace of \( \Gamma_{ext}(\Lambda^p T^* X'). \)

Next, consider the vector bundle \( (i|Z')^*(\Lambda^p T^* X') \to Z' \) and look at \((2.10)\). Then, arguing as in the proof of Proposition\(2\) it can be proved that such a bundle is \((i^*(S_{p-1}) \otimes \mathcal{C}^\infty_Z)-extendable. The set
\[ (2.12) \Gamma_{ext}(\Lambda^p i) = \Gamma_{ext}(\Lambda^p T^* X') \oplus \Gamma_{ext}((i|Z')^*(\Lambda^p T^* X')) \]
is called space of extendable (differentiable) differential p-forms relative to the pair \((X, Z)\).

**Remark 6.** Let \( X, Z \) and \( i \) be as above. It is easy to check that \( \Gamma_{ext}(\bigoplus_{p \in \mathbb{N}} \Lambda^p T^* X') \), \( \Gamma_{ext}(\bigoplus_{p \in \mathbb{N}} \Lambda^p T^* X')_Z \) and \( \Gamma_{ext}(\bigoplus_{p \in \mathbb{N}} \Lambda^p i) \) are complex vector spaces endowed with a structure of a graded algebra.

Let \( X \) be a finite dimensional complex analytic variety. The two following short observations are in order.

1. \( \mathcal{C}^\infty X \otimes X \) is an extendable vector bundle. Indeed, \( \mathcal{C}^\infty X \) splits as \( \mathcal{C}^\infty X = TX' \oplus \hat{TX}' \) and both \( TX' \) and \( \hat{TX}' \) are extendable bundles.

2. Up to write holomorphic and \( \mathcal{O}_X \) instead of differentiable and, respectively, \( \mathcal{C}^\infty X \), the above discussion can be repeated word by word in the holomorphic category. The space and the sheaf of germs of \( \mathcal{S}_{E'} \)-extendable holomorphic sections of an \( \mathcal{S}_{E'} \)-extendable holomorphic complex vector bundle \( E' \to X' \) will be denoted by \( \Gamma_{ext}(X', E') \) and, respectively, by \( _{ext} \mathcal{O}_X (E') \).

3. **Extendable cohomologies**

3.1. **Extendable cohomology groups.** In this subsection we introduce the extendable cohomology groups of a complex analytic variety.

1. Let \( X \) be an abstract finite dimensional complex analytic variety. Then
\[ d^p : \Gamma_{ext}(\Lambda^p T^* X') \to \Gamma_{ext}(\Lambda^{p+1} T^* X'), \]
the restriction at \( \Gamma_{ext}(\Lambda^p T^* X') \) of the \( p \)th exterior differential \( d^p : \Gamma(\Lambda^p T^* X') \to \Gamma(\Lambda^{p+1} T^* X') \), is well defined for any \( p \in \mathbb{N} \). So, \( \Gamma_{ext}(\bigoplus_{p \in \mathbb{N}} \Lambda^p T^* X') \) endowed with \( d = \bigoplus_{p \in \mathbb{N}} d^p \) is a cochains complex, because \( d \circ d = 0 \).

**Proof.** Let \( C = \{ A_l \}_{l \in L} \) be an atlas of trivializing extension for \( \bigoplus_{p \in \mathbb{N}} \Lambda^p T^* X' \). Given \( \omega \in \Gamma_{ext}(\Lambda^p T^* X') \), let \( \tilde{\omega} \) be an extension of \( \omega \) on \( A_l \). Then \( d^p(\tilde{\omega}) \) is an extension of \( d^p(\omega) \) on \( A_l \), because \( d^p \) commutes with the pull back operators (see (2.7), (2.8), (2.9)). \( \square \)

2. Let \( X \) be an abstract finite dimensional complex analytic variety, \( Z \) the closure of a nonempty open set of \( X \) which is also a polyhedron of \( X \) and \( i : Z \hookrightarrow X \) the inclusion. Set \( Sing(Z) = Z \cap Sing(X) \) and \( Z' = Z \cap X' \) and let \( \omega|_Z = (i|Z)^*(\omega) \) denote the pull back of \( \omega \in \Gamma(\bigoplus_{p \in \mathbb{N}} \Lambda^p T^* X') \) to \( Z' \). Then

\( a \) The restriction of \( d^p \) at \( \Gamma_{ext}(\Lambda^p T^* X')_Z \) is well defined, because \( d \) is a local operator.
(b) For each $p \in \mathbb{N}$ let $d^p_i : \Gamma_{ext}(\Lambda^p i) \to \Gamma_{ext}(\Lambda^{p+1} i)$ be the operator given by

\begin{equation}
(3.1) \quad d^p_i(\varphi, \psi) = (d^p \varphi, \varphi|_{\mathcal{Z}} - d^{p-1} \psi).
\end{equation}

Set $d_i = \oplus_{p \in \mathbb{N}} d^p_i$. A straightforward computation shows that $d_i \circ d_i = 0$. So, $\Gamma_{ext}(\oplus_{p \in \mathbb{N}} \Lambda^p i)$ endowed with $d_i$ is a cochains complex. As a note, a $d_i$-closed element of $\Gamma_{ext}(\oplus_{p \in \mathbb{N}} \Lambda^p i)$ corresponds to a $d$-closed element of $\Gamma_{ext}(\oplus_{p \in \mathbb{N}} \Lambda^p T^* X)$ whose restriction at $\mathcal{Z}$ is exact.

**Definition 4.** Let $X$ be an abstract finite dimensional complex analytic variety, $\mathcal{Z}$ the closure of a non empty open set which is also a polyhedron of $X$ and $i : \mathcal{Z} \hookrightarrow X$ the inclusion. Fix $p \in \mathbb{N}$.

1. Set $\mathcal{H}^p_{ext}(X) = \ker(d^p|_{\Gamma_{ext}(\Lambda^p T^* X')})$ and $B^p_{ext}(X) = \text{Im}(d^{p-1}|_{\Gamma_{ext}(\Lambda^{p-1} T^* X')})$.
   The group $H^p_{ext}(X) = \frac{\mathcal{H}^p_{ext}(X)}{B^p_{ext}(X)}$

   is the $p$th extendable cohomology group of $X$.

2. Set $\mathcal{H}^p_{ext}(X, \mathcal{Z}) = \ker(d^p|_{\Gamma_{ext}(\Lambda^p T^* X')_\mathcal{Z}})$ and $B^p_{ext}(X, \mathcal{Z}) = \text{Im}(d^{p-1}|_{\Gamma_{ext}(\Lambda^{p-1} T^* X')_\mathcal{Z}})$.
   The group $H^p_{ext}(X, \mathcal{Z}) = \frac{\mathcal{H}^p_{ext}(X, \mathcal{Z})}{B^p_{ext}(X, \mathcal{Z})}$

   is the $p$th extendable cohomology group of $X$ vanishing on $\mathcal{Z}$.

3. Set $\mathcal{H}^p_{ext}(X, \mathcal{Z}) = \ker(d^p|_{\Gamma_{ext}(\Lambda^p T^* X')_\mathcal{Z}})$ and $B^p_{ext}(X, \mathcal{Z}) = \text{Im}(d^{p-1})$.
   The group $H^p_{ext}(X, \mathcal{Z}) = \frac{\mathcal{H}^p_{ext}(X, \mathcal{Z})}{B^p_{ext}(X, \mathcal{Z})}$

   is the $p$th extendable cohomology group relative to the pair $(X, \mathcal{Z})$.

The following example is an adjustment of a real analytic example given by Bloom and Herrera (see pages 287-288 of [Bl-He]).

**Example 3.** Let $z$ denote the coordinate on $\mathbb{C}$ and $(z_1, z_2)$ the coordinates on $\mathbb{C}^2$. Consider the map

\[ f : \mathbb{C} \to \mathbb{C}^2 \]

\[ z \mapsto (z^5, z^6 + z^7) \]

and let $B$ be a neighborhood of $0$ in $\mathbb{C}$ such that $X = f(B)$ is an irreducible complex analytic variety. Recall that, since the complex dimension of $X$ is 1, any holomorphic differential 2-form of type $(2, 0)$ defined on $X_{\text{Reg}}$ is identically zero. Let $\omega$ be a holomorphic differential 1-form of type $(1, 0)$ defined on a neighborhood of $0$ in $\mathbb{C}^2$ and not identically zero on $X_{\text{Reg}}$. Since $d(\omega) = \partial(\omega) + \bar{\partial}(\omega)$, we have $d(\omega) = 0$ on $X_{\text{Reg}}$. Indeed, on one hand, $\bar{\partial}(\omega) = 0$, because $\omega$ is holomorphic, and, on the other hand, $\partial(\omega) = 0$, because $\partial(\omega)$ is a holomorphic differential 2-form of type $(2, 0)$ on $X_{\text{Reg}}$. Let $\omega \in \mathcal{E}^1_{\mathbb{C}^2, 0}$ be the germ at $0$ of $\omega$. Taken $f^*(\omega) \in \mathcal{E}^1_{\mathbb{C}^2, 0}$, it results $df(\omega) = 0$. Indeed, $f^*(\omega)$ is closed, because it is a form of type $(1, 0)$ on $\mathbb{C}$. Then, by Poincaré lemma in $\mathbb{C}$, there exists $h \in \mathcal{E}^0_{\mathbb{C}, 0}$ such that $f^*(\omega) = dh$. In fact, since $\omega$ is holomorphic of type $(1, 0)$, such a germ $h$ is in $\mathcal{O}_{\mathbb{C}, 0}$. If there existed an element $g \in \mathcal{E}^0_{\mathbb{C}^2, 0}$ not identically vanishing on $X_{\text{Reg}}$ and such that $\omega = d(g)$, then we would have $d(f^*(g)) = f^*(d(g)) = f^*(\omega)$ and so $h = f^*(g) + \text{Const.}$.. A necessary condition for $h$ to be of such a form is that the formal power series of $h$
at 0 can be expressed as a power series in $z^5$ and $(z^6 + z^7)$. Now, consider the holomorphic differential 1-form $\omega = z_1 dz_2$ of type $(1,0)$ defined on a neighborhood of 0 in $\mathbb{C}^2$ and not identically zero on $X^{\operatorname{reg}}$. Then, since the resulting $h$ does not have that form, the Poincaré lemma does not hold. So, $[\omega]_{\operatorname{d}^1}(X) \neq 0$ and $H^1_{\operatorname{d}^1}(X) \neq 0$.

The following important remark is in order.

**Remark 7.** Example 3 shows that

1. There exist complex analytic varieties whose extendable cohomology groups are not trivial
2. A Poincaré Lemma for extendable differential forms does not hold.

3.2. A technical lemma. The following technical lemma shows the existence of extendable partition of unity. Notations of Section 1 will be heavily used.

**Lemma 2.** Let $X$ be an abstract finite dimensional complex analytic variety and $\mathcal{B} = \{B_\beta\}_{\beta \in \mathfrak{B}}$ an atlas of $X$ enjoying (2) of Lemma 1. Then there exists a partition of unity $\{\rho_\beta : X \to \mathbb{R}\}_{\beta \in \mathfrak{B}}$ subordinated to $\mathcal{B}$ such that for any $\beta \in \mathfrak{B}$ the function $r_\beta : W_\beta \to \mathbb{R}$ given by

$$\tag{3.2} r_\beta = \rho_\beta |_{A_\beta} \circ F_\beta^{-1}$$

is extendable to a differentiable function $R_\beta : U_\beta \to \mathbb{R}$. $\{\rho_\beta : X \to \mathbb{R}\}_{\beta \in \mathfrak{B}}$ is an extendable partition of unity subordinated to $\mathcal{B}$. As a note, $\rho_\beta |_{X'} : X' \to \mathbb{R}$ is an extendable 0-form for any $\beta \in \mathfrak{B}$.

**Proof.** Let $\mathcal{C} = \{C_\beta\}_{\beta \in \mathfrak{B}}$, $\mathcal{D} = \{D_\beta\}_{\beta \in \mathfrak{B}}$ and $\mathcal{E} = \{E_\beta\}_{\beta \in \mathfrak{B}}$ be open coverings of $X$ all refining $\mathcal{B} = \{B_\beta\}_{\beta \in \mathfrak{B}}$. We can assume without loss of generality that the sets of indices of $\mathcal{C}$, $\mathcal{D}$ and $\mathcal{E}$ coincide with $\mathfrak{B}$, the set of indices of $\mathcal{B}$. Moreover, we can also assume that for each $\beta \in \mathfrak{B}$ it holds

$$E_\beta \subset E_{\beta} \subset D_\beta \subset D_{\beta} \subset C_\beta \subset C_{\beta} \subset B_\beta.$$ 

These facts follow from Lemma 1 (1). Then, $\mathcal{C}$, $\mathcal{D}$ and $\mathcal{E}$ are coordinate open coverings of $X$ enjoying (2) of Lemma 1 as well as $\mathcal{B}$.

For each $\beta \in \mathfrak{B}$ consider the homeomorphism $F_\beta : B_\beta \to W_\beta \subset U_\beta$ and the images in $U_\beta$ of $E_\beta$, $E_{\beta}$, $D_\beta$, $D_{\beta}$, $C_\beta$, $C_{\beta}$, $B_\beta$ via $F_\beta$ (cp. (1.1)). Then, there exist open sets $C_\beta$, $D_\beta$, $E_\beta$ of $U_\beta$ such that $F_\beta(C_\beta) = C_\beta \cap W_\beta$, $F_\beta(D_\beta) = D_\beta \cap W_\beta$ and $F_\beta(E_\beta) = E_\beta \cap W_\beta$, because $F_\beta(C_\beta)$, $F_\beta(D_\beta)$, $F_\beta(E_\beta)$ are open sets of $W_\beta$ and the topology of $W_\beta = F_\beta(B_\beta)$ is induced by Euclidean topology of $U_\beta \subset \mathbb{C}^n$.

For any $\beta \in \mathfrak{B}$ let $G_\beta : U_{\beta} \to \mathbb{R}$ be a positive, bounded, differentiable real valued function such that $E_\beta \subset \text{supp}(G_\beta) \subset D_\beta$, denote by $g_\beta = G_\beta |_{W_\beta} : W_\beta \to \mathbb{R}$ the restriction of $G_\beta$ at $W_\beta$ and consider the function $F_\beta^*(g_\beta) : B_\beta \to \mathbb{R}$ given by $F_\beta^*(g_\beta)(x) = g_\beta \circ F_\beta(x)$. Since $F_\beta^*(g_\beta)$ is identically zero away from $D_\beta$, let us agree that on $X \setminus B_\beta$ the symbol $F_\beta^*(g_\beta)$ denotes the function identically equal to zero.

So, $F_\beta^*(g_\beta) : X \to \mathbb{R}$ is the function defined on the whole of $X$ given by

$$\tag{3.3} F_\beta^*(g_\beta)(x) = \begin{cases} g_\beta \circ F_\beta(x) & \text{for } x \in B_\beta \\ 0 & \text{for } x \in X \setminus B_\beta \end{cases}$$

Moreover, let us agree that for any $(\gamma, \beta) \in \mathfrak{B} \times \mathfrak{B}$ such that $B_\gamma \cap B_\beta = \emptyset$ the symbol $F_{(\gamma, \beta)}^*(g_\gamma)$ denotes the function identically equal to zero (cp. (1.2)).
For each $\beta \in \mathfrak{B}$ let $\rho_{\beta}$ be the function defined by
\[
\rho_{\beta} : X \to \mathbb{R} \\
x \mapsto \rho_{\beta}(x) = \frac{F_\gamma^*(g_\beta)(x)}{\sum_{\gamma \in \mathfrak{B}} F_\beta^* f_{\gamma,\beta}(y_\gamma)(x)}
\]
(3.4)

Again we have $\rho_{\beta} = 0$ away from $\overline{D_\beta}$. So,
\[
\rho_{\beta}(x) = \begin{cases} 
0 & \text{for } x \in X \setminus B_\beta \\
\frac{\sum_{\gamma \in \mathfrak{B}} F_\beta^* f_{\gamma,\beta}(y_\gamma)(x)}{\sum_{\gamma \in \mathfrak{B}} F_\beta^* f_{\gamma,\beta}(y_\gamma)(x)} & \text{for } x \in B_\beta 
\end{cases}
\]
(3.5)

It is easy to check that $\{\rho_{\beta} : X \to \mathbb{R}\}_{\beta \in \mathfrak{B}}$ is a partition of unity subordinate to $\mathfrak{B}$. Thus, it only remains to prove that for each $\beta \in \mathfrak{B}$ the function $r_{\beta} : W_\beta \to \mathbb{R}$ admits a differentiable extension $R_{\beta}$ defined on the whole of $U_{\beta}$.

For this, fix $\beta \in \mathfrak{B}$. It follows from Lemma 1.3 and Lemma 1.4 that there exists a finite subset of indices $\mathfrak{B}(\beta) \subseteq \mathfrak{B}$ such that $B_{\gamma} \cap B_\beta \not= \emptyset$ if and only if $\gamma \in \mathfrak{B}(\beta)$. Note that, by Lemma 1.2 and Lemma 1.4, in order to have a differentiable extension $R_{\beta}$ of $r_{\beta}$, it suffices to find a differentiable extension $L_{(\gamma,\beta)}(G_\gamma) : U_{\beta} \to \mathbb{R}$ of $f_{\gamma,\beta}^*(g_\gamma)$ for any $\gamma \in \mathfrak{B}(\beta)$.

Now, for each $\gamma \in \mathfrak{B}(\beta)$ consider the biholomorphism $F_{(\gamma,\beta)} : F_{\beta}(B_{\beta} \cap C_{\gamma}) \to F_{\gamma}(B_{\gamma} \cap C_{\gamma})$ (see (1.2)) and note that only two cases are possible. Namely, $n_{\beta} \leq n_{\gamma}$ or $n_{\beta} \geq n_{\gamma}$, where $n_{\alpha} = \dim C_{\alpha}$ for any $\alpha \in \mathfrak{B}$ (cp. Section 1).

Suppose $n_{\beta} \leq n_{\gamma}$. Then there exist an open subset $Q_{\beta,\gamma}$ of $U_{\gamma}$ such that $Q_{\beta,\gamma} \cap W_{\beta} = F_{\beta}(B_{\beta} \cap B_{\gamma})$ and a holomorphic injective map $F_{(\gamma,\beta)} : Q_{\beta,\gamma} \to F_{(\gamma,\beta)}(Q_{\beta,\gamma})$ such that $F_{(\gamma,\beta)}$ is the restriction of $F_{(\gamma,\beta)}$ at $F_{\beta}(B_{\beta} \cap B_{\gamma})$. Let $O_{\beta,\gamma}$ be an open subset of $U_{\beta}$ such that $O_{\beta,\gamma} \subseteq Q_{\beta,\gamma}$ and $O_{\beta,\gamma} \cap W_{\beta} = F_{\beta}(B_{\beta} \cap C_{\gamma})$ and consider the restriction $F_{(\gamma,\beta)}|O_{\beta,\gamma} : O_{\beta,\gamma} \to F_{(\gamma,\beta)}(O_{\beta,\gamma})$ of $F_{(\gamma,\beta)}$ at $O_{\beta,\gamma}$. Let $L_{(\gamma,\beta)}(G_{\gamma}) : U_{\beta} \to \mathbb{R}$ be the map defined by
\[
L_{(\gamma,\beta)}(G_{\gamma})(u) = \begin{cases} 
(F_{(\gamma,\beta)}|O_{\beta,\gamma})^*(G_{\gamma}|F_{(\gamma,\beta)}(O_{\beta,\gamma}))(u) & \text{for } u \in O_{\beta,\gamma} \\
0 & \text{for } u \in U_{\beta} \setminus O_{\beta,\gamma}
\end{cases}
\]
(3.6)

It follows from $F_{\gamma} \subseteq \text{supp}(G_{\gamma}) \subseteq D_{\gamma}$ that $L_{(\gamma,\beta)}(G_{\gamma})$ is a well defined, continuous map that is also differentiable, because of the holomorphy of $F_{(\gamma,\beta)}|O_{\beta,\gamma}$ and the differentiability of $G_{\gamma}$. Furthermore, by its very definition, $L_{(\gamma,\beta)}(G_{\gamma})$ is such that
\[
L_{(\gamma,\beta)}(G_{\gamma})|F_{\beta}(B_{\beta} \cap C_{\gamma}) = F_{\beta}^*(g_{\gamma})|F_{\beta}(B_{\beta} \cap C_{\gamma})
\]
(3.7)

Then, since $\rho_{\beta} = 0$ away from $\overline{D_\beta}$, $L_{(\gamma,\beta)}(G_{\gamma})$ is the wanted extension of $F_{(\gamma,\beta)}^*(g_{\gamma})$ and we are done in the case $n_{\beta} \leq n_{\gamma}$.

Suppose $n_{\beta} \geq n_{\gamma}$. Then there exist an open subset $Q_{\beta,\gamma}$ of $U_{\beta}$ such that $Q_{\beta,\gamma} \cap W_{\beta} = F_{\beta}(B_{\beta} \cap B_{\gamma})$ and a holomorphic submersion $F_{(\gamma,\beta)} : Q_{\beta,\gamma} \to F_{(\gamma,\beta)}(Q_{\beta,\gamma})$ such that $F_{(\gamma,\beta)}$ is the restriction of $F_{(\gamma,\beta)}$ at $F_{\beta}(B_{\beta} \cap B_{\gamma})$. Let $O_{\beta,\gamma}$ be an open subset of $U_{\beta}$ such that $O_{\beta,\gamma} \subseteq Q_{\beta,\gamma}$ and $O_{\beta,\gamma} \cap W_{\beta} = F_{\beta}(B_{\beta} \cap C_{\gamma})$ and consider the restriction $F_{(\gamma,\beta)}|O_{\beta,\gamma} : O_{\beta,\gamma} \to F_{(\gamma,\beta)}(O_{\beta,\gamma})$ of $F_{(\gamma,\beta)}$ at $O_{\beta,\gamma}$. By using the local form of submersion, the function $(F_{(\gamma,\beta)}|O_{\beta,\gamma})^*(G_{\gamma}|F_{(\gamma,\beta)}(O_{\beta,\gamma}))$ is an extension of $F_{(\gamma,\beta)}^*(g_{\gamma})|F_{\beta}(B_{\beta} \cap C_{\gamma})$ to the whole of $O_{\beta,\gamma}$.

Now, let $N_{\beta,\gamma}$ be an open subset of $U_{\beta}$ such that $N_{\beta,\gamma} \subseteq \overline{N_{\beta,\gamma}} \subseteq O_{\beta,\gamma}$ and $N_{\beta,\gamma} \cap W_{\beta} \supseteq F_{\beta}(B_{\beta} \cap D_{\gamma})$. Then the restriction $(F_{(\gamma,\beta)}|_{N_{\beta,\gamma}})^*(G_{\gamma}|F_{(\gamma,\beta)}(N_{\beta,\gamma}))$ of $G_{\gamma}|F_{(\gamma,\beta)}(O_{\beta,\gamma})$ at $\overline{N_{\beta,\gamma}}$ is also a differentiable extension of $F_{(\gamma,\beta)}^*(g_{\gamma})|\overline{N_{\beta,\gamma}} \cap W_{\beta}$. Let us agree that the function $(F_{(\gamma,\beta)}|_{N_{\beta,\gamma}})^*(G_{\gamma}|F_{(\gamma,\beta)}(N_{\beta,\gamma}))$ is defined also on $W_{\beta} \setminus N_{\beta,\gamma}$ as the identically vanishing function.
Let $V_{\beta\gamma}$ be an open subset of $U_\beta$ that is a neighborhood of $W_\beta \setminus N_{\beta\gamma}$ such that $V_{\beta\gamma} \subset \overline{V_{\beta\gamma}} \subset U_\beta$, $V_{\beta\gamma} \cap N_{\beta\gamma} \neq \emptyset$ and $\overline{V_{\beta\gamma}} \cap D_\beta = \emptyset$. Then, by Tietze's extension theorem, the function

$$\tag{3.8} \left( F_{(\gamma,\beta)}|_{\overline{V_{\beta\gamma}} \cap N_{\beta\gamma}} \right) : (\overline{V_{\beta\gamma}} \cap N_{\beta\gamma}) \cup (W_\beta \setminus N_{\beta\gamma}) \to \mathbb{R}$$

can be extended to a continuous, non identically vanishing function defined on $(\overline{V_{\beta\gamma}} \cap N_{\beta\gamma}) \cup V_{\beta\gamma}$. Moreover, by the approximation theorem (see [St], 6.7), we can assume without loss of generality that such a function is also differentiable. Thus, we get a differentiable function defined on the closed subset $N_{\beta\gamma} \cup V_{\beta\gamma}$ of $U_\beta$.

By using the same technique (extension and approximation theorems), we get a differentiable function $L_{(\gamma,\beta)}(G_\gamma) : U_\beta \to \mathbb{R}$ such that

$$\tag{3.9} L_{(\gamma,\beta)}(G_\gamma)|_{\overline{D_\beta}} = F_{(\gamma,\beta)}^*(g_\gamma)|_{\overline{D_\beta}}.$$

Then $L_{(\gamma,\beta)}(G_\gamma)$ is the wanted extension of $F_{(\gamma,\beta)}^*(g_\gamma)$, because $\rho_\beta = 0$ away from $\overline{D_\beta}$. We are done also in the case $n_\beta \geq n_\gamma$.

The proof is thereby concluded. Indeed, the function $R_{\beta} : U_\beta \to \mathbb{R}$ defined by

$$\tag{3.10} R_{\beta}(u) = \begin{cases} \sum_{\gamma \in \mathbb{N}} G_{\beta}(u) & \text{for } u \in \bigcup_{\gamma \in \mathbb{N}} O_{\beta\gamma} \\ 0 & \text{for } u \in U_\beta \setminus \bigcup_{\gamma \in \mathbb{N}} O_{\beta\gamma} \end{cases}$$

is a differentiable extension of the function $r_\beta : W_\beta \to \mathbb{R}$ (see [3.2]). □

Lemma [2] has several important consequences. We begin with the following corollary.

**Corollary 1.** Let $X$ be a finite dimensional complex analytic variety. Then the sheaves $S_{N,N^*}$ and $S_p$ are fine and soft for any $N, N^*, p \in \mathbb{N}$.

As a further consequence of Lemma [2] we can prove that the spaces of extendable sections of extendable bundles of Example [2] are not trivial.

**Remark 8.** Let $E \to X$ be a differentiable real (complex) vector bundle defined over an abstract finite dimensional complex analytic variety. Then for each $N, N^*$, $p \in \mathbb{N}$ the bundle $\Gamma_{ext}(TX^{p\otimes N} \otimes T^*X^{p\otimes N*} \otimes N^* T^*X \otimes E|_{X'}) \neq \{0\}$. For this, consider an extendable partition of unity subordinate to a suitable open covering of $X$ and argue locally.

Next, let $X$ be an abstract finite dimensional complex analytic variety, $Z$ the closure of a non empty open set that is also a polyhedron of $X$ and $i : Z \hookrightarrow X$ the inclusion. Let $\alpha^p : \Gamma_{ext}(pT^*X')|_Z \to \Gamma_{ext}(p)\iota$ be the map that to any $\omega \in \Gamma_{ext}(pT^*X')|_Z$ associates $\alpha^p(\omega) = (\omega, 0)$. A straightforward computation shows that $\alpha^p$ induces a homomorphism $\overline{\alpha^p} : H^p_{ext}(X)|_Z \to H^p_{ext}(X,Z)$ in cohomology. Furthermore, the following proposition, whose proof is a direct consequence of Lemma [2] holds.

**Proposition 3.** Let $X$ be a finite dimensional complex analytic variety, $Z$ the closure of a non empty open set which is also a polyhedron of $X$ and $i : Z \hookrightarrow X$ the inclusion. Then $\alpha^p : H^p_{ext}(X)|_Z \to H^p_{ext}(X,Z)$ is an isomorphism for any $p \in \mathbb{N}$.

**Proof (Sketch).** Injectivity of $\alpha^p$.

Let $\omega \in \Gamma_{ext}(pT^*X')|_Z$ be such that $\delta^p(\omega) = 0$ and $\overline{\alpha^p([\omega]|_{\overline{H^p_{ext}(X)|_Z}}) = 0_{H^p_{ext}(X,Z)}$. This means that $\alpha^p(\omega) = (\omega, 0)$ is $d_0$-exact. Then there exists $(\vartheta, \varsigma) \in \Gamma_{ext}(\alpha^{-1}i)$ such that $(\omega, 0) = d_{0}^{-1}(\vartheta, \varsigma) = (d_{p-1}\vartheta, \vartheta|_{Z} - d_{p-2}\varsigma)$. Thus, $\omega$ is an exact global
form and $\partial|_Z - d^{p-2}\zeta = 0$. Actually, this is not enough to conclude, because we want a primitive of $\omega$ vanishing on $Z$. Let $\zeta$ be an extension of $\zeta$ to the whole of $X$. Note that such an extension there exists, because the sheaf $\mathcal{S}_{p-2}$ is soft (cp. Corollary 1 of Lemma 2). Now, set $(\vartheta', \zeta') = (\vartheta - d^{p-2}\zeta, 0) \in \Gamma_{ext}(\Lambda^{p-1})$. Then $d^{p-1}_{\vartheta'}(\vartheta', \zeta') = (\omega, 0)$ and $\partial'|_Z = \partial|_Z - d^{p-2}\zeta = 0$. We are done, because $\partial'$ is a primitive of $\omega$ vanishing on $Z$, because.

Surjectivity of $A^p$.

Let $(\varpi, \upsilon) \in \Gamma_{ext}(\Lambda^p)_Z$ be $d^p$-closed. We want to find a $d^p$-closed $\omega \in \Gamma_{ext}(\Lambda^p T^* X')_Z$ such that $[(\varpi, \upsilon)]_{\mathcal{H}^p_{ext}(X,Z)} = [(\omega, 0)]_{\mathcal{H}^p_{ext}(X,Z)}$. That is, we want to find a $d^p$-closed $\omega \in \Gamma_{ext}(\Lambda^p T^* X')_Z$ and $(\vartheta, \zeta) \in \Gamma_{ext}((\Lambda^p)^{-1})$ such that $(\varpi, \upsilon) = (\omega, 0) + d^{p-1}_{\vartheta} (\vartheta, \zeta)$. Since the sheaf $\mathcal{S}_{p-1}$ is soft, there exists a global extension $\tilde{\upsilon}$ of $\upsilon$. Set $(\vartheta, \zeta) = (\tilde{\upsilon}, \upsilon)$ and define $\omega = \varpi - d^{p-1} \tilde{\upsilon}$. Then $(\varpi, \upsilon) = (\omega, 0) + d^{p-1}_{\vartheta} (\vartheta, \zeta)$. Moreover, $d^p \omega = 0$ and $\varpi|_Z - d^{p-1} \upsilon = 0$, because $(\varpi, \upsilon) \in \Gamma_{ext}(\Lambda^p)$ is $d^p$-closed. Then, by the very definition of $\omega$, it results $\omega|_Z = 0$.

We are done, because $\omega$ is an extendable differentiable form vanishing on $Z$ such that $[(\varpi, \upsilon)]_{\mathcal{H}^p_{ext}(X,Z)} = A^p([\omega])_{\mathcal{H}^p_{ext}(X,Z)}$. □

3.3. Extendable Čech cohomology groups. Let $X$ be a finite dimensional complex analytic variety and $Z$ the closure of a non empty open set that is also a polyhedron of $X$. An open covering $\mathcal{V} = \{V_j\}_{j \in J}$ of $X$ is adapted to $Z$ if there exists a unique $j(Z) \in J$ such that $Z \subset V_{j(Z)}$ and if for any $j \in J \setminus \{j(Z)\}$ it holds $Z \cap V_j = \emptyset$. Sometimes the set $V_{j(Z)}$ is simply denoted by $V_Z$. Let us agree that every open covering of $X$ is adapted to the empty set.

Lemma 3. Let $X$ and $Z$ be as above. If $\mathcal{V}$ is an open covering of $X$ adapted to $Z$, then there exists an open covering $\mathcal{B}$ of $X$ adapted to $Z$ which is a locally finite refinement of $\mathcal{V}$.

Proof. Let $\mathcal{B}^* = \{B^*_\beta\}_{\beta^* \in \mathcal{B}^*}$ be a locally finite refinement of $\mathcal{V}$ and $\lambda : \mathcal{B}^* \to J$ a refinement map associated with $\mathcal{B}^*$ and $\mathcal{V}$. Write $\mathcal{B}^*$, the set of indices $\mathcal{B}^*$, as the disjoint union of $\{\beta^* \in \mathcal{B}^* : Z \cap B^*_\beta \neq \emptyset\}$ and $\{\beta^* \in \mathcal{B}^* : Z \cap B^*_\beta = \emptyset\}$ and note that, if $B^*_\beta \in \mathcal{B}^*$ is such that $Z \cap B^*_\beta \neq \emptyset$, then $\lambda(\beta^*) = j(Z)$, because $\mathcal{V}$ is adapted to $Z$. Now, set $B_Z = \bigcup_{\beta^* \in \mathcal{B}^* : Z \cap B^*_\beta \neq \emptyset} B^*_\beta$, and for any $\beta^* \in \mathcal{B}^*$ such that $Z \cap B^*_\beta = \emptyset$, set $\beta = \beta^*$ and $B_\beta = B^*_\beta$. Write $\mathcal{B} = \{B_\beta \} \cup (\mathcal{B}^* \setminus \{\beta^* \in \mathcal{B}^* : Z \cap B^*_\beta \neq \emptyset\})$. Then $\mathcal{B} = \{B_\beta\}_{\beta \in \mathcal{B}}$ is the wanted covering. □

Let $X$ be a finite dimensional complex analytic variety, $Z$ either the empty set or the closure of a non empty open set that is a polyhedron of $X$ and $\mathcal{V} = \{V_j\}_{j \in J}$ an open covering of $X$ adapted to $Z$.

As a matter of notations, for any $q \in \mathbb{N}$ and $j = (j_0, \ldots, j_q) \in J^{q+1}$ set $V_j = V_{j_0} \cap \ldots \cap V_{j_q}$ and $V'_j = V_j \setminus \text{Sing}(X)$. Moreover, for any $q \in \mathbb{N}$, $m \in \{0, \ldots, q\}$ and $\tilde{j} = (j_0, \ldots, j_q) \in J^{q+1}$ set $\tilde{j}_m = (j_0, \ldots, \widehat{j_m}, \ldots, j_q) \in J^q$ and denote by $\partial^q_{\tilde{j}-1, m} : V_{\tilde{j}} \hookrightarrow V_{\tilde{j}_m}$ the inclusion. Note that, if $q \in \mathbb{N} \setminus \{0\}$ and $j \in J^{q+1}$, then $V_j \cap Z = \emptyset$.

Let $p \in \mathbb{N}$. The set

$$C^p_q(X, \mathcal{V}, V_Z) = \prod_{j \in J^{q+1}, \Gamma_{ext}(\Lambda^p T^* V'_j)_Z}$$

is the space of extendable $p$-forms associated with $\mathcal{V}$ and vanishing on $Z$. Note that, since $\mathcal{V}$ is adapted to $Z$, the unique space really containing forms vanishing on $Z$ is $\Gamma_{ext}(\Lambda^p T^* V'_j(Z))_Z \subset C^p_0(X, \mathcal{V}, V_Z)$. 


The elements of \( \{ \partial_{q-1,m} \}_{q \in \mathbb{N}, m \in \{0, \ldots, q \}} \) give rise to a sequence of families of maps

\[
\left\{ \delta_{q-1,m}^p : \prod_{k \in J_q} \Gamma_{ext}(\Lambda^p T^*V^*_k) \to \prod_{j \in J_{q+1}} \Gamma_{ext}(\Lambda^p T^*V'_j) \right\}_{q \in \mathbb{N}, m \in \{0, \ldots, q \}, p \in \mathbb{N}}
\]

depending on \( p \in \mathbb{N} \). For each \( p \in \mathbb{N} \) the map \( \delta_{q-1,m}^p \) is described as follows. The image \( \left( \delta_{q-1,m}^p(\omega_k)_{k \in J_q} \right)_{j \in J_{q+1}} \) of \( (\omega_k)_{k \in J_q} \in \prod_{k \in J_q} \Gamma_{ext}(\Lambda^p T^*V^*_k) \) is the element of \( \prod_{j \in J_{q+1}} \Gamma_{ext}(\Lambda^p T^*V'_j) \) whose \( j \)-th term \( (\delta_{q-1,m}^p(\omega_k)_{k \in J_q})_j \) is given by \( (\partial_{q-1,m}|_{V'_j}^* \omega_k)_{k \in J_q} \). Sometimes we write \( \omega_{j,m} \) instead of \( (\partial_{q-1,m}|_{V'_j}^* \omega_k)_{k \in J_q} \).

So, \( (\delta_{q-1,m}^p(\omega_k)_{k \in J_q})_j = \omega_{j,m} \).

For any \( j_0 \in J \) denote by \( \partial^{j_0} : V_{j_0} \to X \) the inclusion and define \( \varrho : \prod_{j_0 \in J} V_{j_0} \to X \) by setting \( \varrho|_{V_{j_0}} = \partial^{j_0} \). Then \( \varrho \) gives rise to a sequence of maps \( \{ P^p : \Gamma_{ext}(\Lambda^p T^*X') \to \prod_{j_0 \in J} \Gamma_{ext}(\Lambda^p T^*V'_{j_0}) \}_{p \in \mathbb{N}} \) depending on \( p \in \mathbb{N} \). For each \( p \in \mathbb{N} \) the map \( P^p \) is described as follows. The image of \( \omega \in \Gamma_{ext}(\Lambda^p T^*X') \) is the element \( (P^p(\omega)_{j_0})_{j_0 \in J} \) of \( \prod_{j_0 \in J} \Gamma_{ext}(\Lambda^p T^*V'_{j_0}) \) whose \( j_0 \)-th term is \( (\varrho|_{V'_{j_0}})^* \omega \).

For each \( p \in \mathbb{N} \) and \( q \in \mathbb{N} \setminus \{0\} \) the map

\[
(3.12) \quad \delta_{q-1}^p : \begin{cases} C^p_q(X, \mathcal{V}, V_Z) & \to C^{p+1}_q(X, \mathcal{V}, V_Z) \\ (\omega_k)_{k \in J_q} & \mapsto (\delta_{q-1}^p(\omega_k)_{k \in J_q})_{j \in J_{q+1}} \end{cases}
\]

is called \( p \)-difference operator associated with \( \mathcal{V} \). A straightforward computation shows that for any \( p \in \mathbb{N} \) and \( q \in \mathbb{N} \setminus \{0\} \) it results \( \delta_q^p \circ \delta_{q-1}^p = 0 \) and \( \delta_0^p \circ P^p = 0 \).

**Proposition 4.** Let \( X \) be a finite dimensional complex analytic variety, \( Z \) either the empty set or the closure of a non empty open set that is a polyhedron of \( X \) and \( \mathcal{V} = \{ V_j \}_{j \in J} \) an open covering of \( X \) adapted to \( Z \). Fix \( p \in \mathbb{N} \). Then the \( \delta^p \)-cohomology of \( 0 \to \Gamma_{ext}(\Lambda^p T^*X') \to C^q_0(X, \mathcal{V}, V_Z) \to \ldots \) is identically zero.

**Proof.** By Lemma 3 there exists an open covering \( \mathcal{B} \) of \( X \) adapted to \( Z \) which is a locally finite refinement of \( \mathcal{V} \). Moreover, by Lemma 1 we can assumewithout loss of generality that \( \mathcal{B} \) enjoys (2) of Lemma 2. It follows from Lemma 2 that there exists an extendable partition of unity subordinated to \( \mathcal{B} \). Finally, proceed as in [Gu], Vol. III., Ch. E. \( \square \)

Let \( X \) be a finite dimensional complex analytic variety, \( Z \) either the empty set or the closure of a non empty open set that is a polyhedron of \( X \) and \( \mathcal{V} = \{ V_j \}_{j \in J} \) an open covering of \( X \) adapted to \( Z \). For each \( r \in \mathbb{N} \) set \( K^r(X, \mathcal{V}, V_Z) = \oplus_{p-q=r} C^p_q(X, \mathcal{V}, V_Z) \). The space \( K(X, \mathcal{V}, V_Z) = \oplus_{r \in \mathbb{N}} K^r(X, \mathcal{V}, V_Z) \) is the space of extendable form associated with \( X, Z \) and \( \mathcal{V} \).

For any \( r \in \mathbb{N} \) let \( D^r : K^r(X, \mathcal{V}, V_Z) \to K^{r+1}(X, \mathcal{V}, V_Z) \) be the operator that on each \( C^p_q(X, \mathcal{V}, V_Z) \) with \( p+q = r \) is defined by

\[
(3.14) \quad D^r|_{C^p_q(X, \mathcal{V}, V_Z)} = \delta^p_q + (-1)^q d^p
\]

and that is identically zero otherwise. It easy to prove that \( D^{r+1} \circ D^r = 0 \) for any \( r \in \mathbb{N} \). Denote by \( D : K(X, \mathcal{V}, V_Z) \to K(X, \mathcal{V}, V_Z) \) the operator that on each \( K^r(X, \mathcal{V}, V_Z) \) is given by \( D|_{K^r(X, \mathcal{V}, V_Z)} = D^r \).
4.1. Integration of extendable forms.

Theorem 2. see [Hd] and [Lo]. Then \( X \) is a triangulable topological space.

Remark 9. Let \( X \) be a complex analytic variety of finite complex dimension, \( Z \) either the empty set or the closure of a non empty open set which is a polyhedron of \( X \) and \( V \) an open covering of \( X \) adapted to \( Z \). Fix \( r \in \mathbb{N} \). Then \( P^r : \Gamma_{ext}(\Lambda^r T^* X')_Z \to K^r(X, V, V_Z) \) induces an isomorphism \( P^r_* : H^r_{ext}(X, Z) \to \tilde{H}^r_{ext}(X, V, V_Z) \).

Proof. Proceed as for the proof of Proposition II.8.8 of [Bo-Tu].

We have the following theorem.

Theorem 1. Let \( X \) be a complex analytic variety of finite complex dimension, \( Z \) either the empty set or the closure of a non empty open set which is a polyhedron of \( X \) and \( V \) an open covering of \( X \) adapted to \( Z \). Fix \( r \in \mathbb{N} \). Then \( P^r : \Gamma_{ext}(\Lambda^r T^* X')_Z \to K^r(X, V, V_Z) \) induces an isomorphism \( P^r_* : H^r_{ext}(X, Z) \to \tilde{H}^r_{ext}(X, V, V_Z) \).

Remark 9. Let \( X \) be a complex analytic variety of finite complex dimension, \( Z \) either the empty set or the closure of a non empty open set which is a polyhedron of \( X \) and \( V \) an open covering of \( X \) adapted to \( Z \). Then

1. The possibility of adapting the proofs of Proposition 4 and Theorem 7 to the case of extendable forms is a direct consequence of Lemma 3.
2. The complexes of cochains \( \Gamma_{ext}(\oplus_{\tau \in \mathbb{N}} \Lambda^r T^* X')_Z \) and \( K(X, V, V_Z) \) are chain homotopic. More precisely, let \( B = \{B_0\}_0 \in \mathcal{B} \) be an open covering of \( X \) refining \( V \) and enjoying (2) of Lemma 7. Let \( \{\rho_0 : X \to \mathbb{R}\}_0 \in \mathcal{B} \) be an extendable partition of unity subordinated to \( B \). Then there exist a homotopy operator \( L : K(X, V, V_Z) \to K(X, V, V_Z) \) and a chain map \( \phi = \phi_{X, V, V_Z}(\rho_0) : K(X, V, V_Z) \to \Gamma_{ext}(\oplus_{\tau \in \mathbb{N}} \Lambda^r T^* X')_Z \) such that \( \phi \circ P = \text{id}_{\Gamma_{ext}(\oplus_{\tau \in \mathbb{N}} \Lambda^r T^* X'_Z)} \) and \( P \circ \phi = D \circ L - L \circ D + \text{id}_{K(X, V, V_Z)} \). For a proof, see [Bo-Tu], Ch. II, Sec. 9.

Note that, if \( V = \{V_0, V_1\} \) only contains two open sets, then for each \( r \in \mathbb{N} \) it results

\[
K^r_{ext}(X, V) = \Gamma_{ext}(\Lambda^r T^* V'_0) \oplus \Gamma_{ext}(\Lambda^r T^* V'_1) \oplus \Gamma_{ext}(\Lambda^{r-1} T^* V'_{0,1}).
\]

Moreover, if \( \{\rho_0, \rho_1 : X \to \mathbb{R}\} \) is an extendable partition of unity subordinated to \( B = V \), then \( \phi \) is the map that to \( \hat{\omega} = (\hat{\omega}_0, \hat{\omega}_1, \hat{\omega}_{01}) \in K^r_{ext}(X, V) \) associates the r-form

\[
\omega = \rho_0 \hat{\omega}_0 + \rho_1 \hat{\omega}_1 - d\rho_0 \wedge \hat{\omega}_{01}
\]

As a note, the theory that in this section has been developed for the real bundle \( TX' \) holds also for the vector bundles \( TX', \bar{T}X' \) and \( T^C X' \).

4. Integration

4.1. Integration of extendable forms. For the proof of the following theorem, see [Hr] and [Lo].

Theorem 2. Let \( X \) be an abstract finite dimensional complex analytic variety. Then \( X \) is a triangulable topological space.
As a matter of notations, let $X$ be an abstract finite dimensional complex analytic variety. For each triangulation $\mathcal{T}$ of $X$ the simplicial complex associated with $X$ and $\mathcal{T}$ is denoted by $(X, \mathcal{T})$. Let $h \in \{0, \ldots, \dim_{\mathbb{R}}(X)\}$. The set of the $h$-simplices of $\mathcal{T}$ and the $h$-skeleton of $(X, \mathcal{T})$ are denoted by $\mathcal{T}_h$ and, respectively, $\text{Skel}^h(X, \mathcal{T})$. The groups of $h$-chains and $h$-cochains with coefficients in $\mathbb{C}$ associated with $\mathcal{T}$ are denoted by $C_k^h(X)$ and, respectively, $C^*_k(X)$. The groups $C^h_k(X)$ and $C^*_h(X)$ are endowed with a structure of complex vector spaces. Moreover, $C^h_0(X)$ is the dual space of $C^*_h(X)$. The $h^{th}$ simplicial homology and cohomology spaces of $X$ with coefficients in $\mathbb{C}$ are denoted by $H_h(X)$ and, respectively, by $H^h(X)$. Recall that $H^h(X)$ is the dual space of $H_h(X)$.

Let $\mathcal{C} = \{C_l\}_{l \in L}$ be an open covering of $X$. A triangulation $\mathcal{T}$ of $X$ is $C$-small if for any simplex $\Delta \in \mathcal{T}$ there is $C_l(\Delta) \in \mathcal{C}$ such that $\Delta \subseteq C_l(\Delta)$.

**Lemma 4.** Let $X$ be a triangulable topological space, $\mathcal{C}$ an open covering of $X$ and $\mathcal{T}$ a triangulation of $X$. Then there exists a natural number $b(\mathcal{C}) \in \mathbb{N}$ such that $\mathcal{T}^{b(\mathcal{C})}$, the b($\mathcal{C}$)$^{th}$ barycentric subdivision of $\mathcal{T}$, is $C$-small.

**Proof.** See [Mu], Ch. 4, Sec. 31.\hfill $\square$

Recall that, if $\Delta^h$ denote the standard simplex of real dimension $h$, then

$$H_k(\Delta^h) = H_k(\Delta^h) = \begin{cases} \mathbb{C} & \text{for } k \in \mathbb{N} : k = 0 \\ 0 & \text{for } k \in \mathbb{N} : k > 0 \end{cases}$$

Let $X$ be a finite dimensional complex analytic variety. For any $\omega$ in $\Gamma_{ext}(\Lambda^pT^*X')$ there is an atlas $\mathcal{A}(\omega) = \{(A_l, n_l, U_l, W_l, F_l)\}_{l \in L}$ of trivializing extension for $\Lambda^pT^*X'$ such that $\omega$ is completely extendable on $A_l$ for any $l \in L$ (cp. [1.1], [1.3], Remark 1. Terminology [4]). An atlas as $\mathcal{A}(\omega)$ is an atlas of extensibility of $\omega$. Suppose that an atlas of extensibility $\mathcal{A}(\omega)$ is given for any $\omega \in \Gamma_{ext}(\Lambda^pT^*X')$.

As a matter of terminology, let $X^\diamond \subseteq X$ be a complex analytic subvariety of $X$. A triangulation $\mathcal{T}$ of $X$ is compatible with $X^\diamond$ if $X^\diamond$ is homeomorphic to a simplicial subcomplex of $(X, \mathcal{T})$ via the homeomorphism between $X$ and $(X, \mathcal{T})$.

**Proposition 5.** Let $X$ be an abstract complex analytic variety of complex dimension $n$. Let $\omega \in \Gamma_{ext}(\Lambda^pT^*X')$ be an extendable $p$-form and $\mathcal{A}(\omega)$ an atlas of extensibility of $\omega$. Let $\mathcal{T}$ be an $\mathcal{A}(\omega)$-small triangulation of $X$ compatible with the analytic subvariety $\text{Sing}(X)$ of $X$. Then the map

$$\int \omega : \Delta \mapsto \mathbb{C}$$

$$\int_{\Delta} \omega = \int_{F_{A}(\Delta)} \tilde{\omega},$$

where $A \in \mathcal{A}(\omega)$ is such that $A \supseteq \Delta$ and $\tilde{\omega} \in \Gamma(\Lambda^pT^*U_{A})$ is any extension of $\omega$, is well defined.

**Proof.** We have to show that $\int \omega$ is well defined.

Let $\Delta \in \mathcal{T}$ be such that $\Delta \cap \text{Sing}(X) = \emptyset$. Then the proof goes as in the non singular case. Indeed, on one hand the topological boundary of $\partial \Delta$ is Lebesgue trascurable. On the other hand, $\omega$ is bounded, because it is extendable (see the proof of (1) at the beginning of Subsection 3.1). Then it is possible to define $\int_{\Delta} \omega = \int_{\Delta} \omega$ as in the non singular case. As a note, if $p = 2n$, then $\Delta \cap \text{Sing}(X) = \emptyset$, because $\dim_{\mathbb{R}}(\text{Sing}(X)) \leq 2(n - 1)$.
Next, suppose that $\Delta \in T_p$ is such that $\hat{\Delta} \cap \text{Sing}(X) \neq \emptyset$. We have to prove that the definition of $\int_\Delta \omega$ does not depend neither on $A \in A(\omega)$ such that $A \supseteq \Delta$ nor on the extension $\tilde{\omega} \in \Gamma(\Lambda^p T^* U_A)$ of $\omega$ (cp. (4.2)).

(1) Let $(A, n_A, U_A, W_A, F_A) \in A(\omega)$ be such that $A \supseteq \Delta$ and write $A' = A \setminus \text{Sing}(X)$. If $\tilde{\omega}_1, \tilde{\omega}_2 \in \Gamma(\Lambda^p T^* U_A)$ are of two extensions $\omega$, then $\tilde{\omega}_1|_{F_A(A')} = \tilde{\omega}_2|_{F_A(A')}$, because $(F_A|_{A'})^*(\tilde{\omega}_1) = \omega|_{A'} = (F_A|_{A'})^*(\tilde{\omega}_2)$. Furthermore, it results $\tilde{\omega}_1|_{F_A(A)} = \tilde{\omega}_2|_{F_A(A)}$, because $\tilde{\omega}_1$ and $\tilde{\omega}_2$ are continuous and coincide on the dense subset $F_A(A')$ of $F_A(A)$. Then $\int_{F_A(\Delta)} \tilde{\omega}_1 = \int_{F_A(\Delta)} \tilde{\omega}_2$ and, in this case, the definition of $\int_\Delta \omega$ is independent of the extension of $\omega$.

(2) Let $(A_1, n_1, U_1, W_1, F_1), (A_2, n_2, U_2, W_2, F_2) \in A(\omega)$ be such that both $A_1$ and $A_2$ contain $\Delta$. Set $A_{(1,2)} = A_1 \cap A_2$ and write $A'_1 = A_1 \setminus \text{Sing}(X)$, $A'_2 = A_2 \setminus \text{Sing}(X)$, $A_{(1,2)} = A_{(1,2)} \setminus \text{Sing}(X)$ (see Section 1). If $\tilde{\omega}_1 \in \Gamma(\Lambda^p T^* U_1)$ and $\tilde{\omega}_2 \in \Gamma(\Lambda^p T^* U_2)$ are of two extensions of $\omega$, then $\omega|_{A'_1} = (F_1|_{A'_1})^*(\tilde{\omega}_1)$ and $\omega|_{A'_2} = (F_2|_{A'_2})^*(\tilde{\omega}_2)$. So,

\begin{align}
\tilde{\omega}_1|_{F_1(A_{(1,2)})} &= (F_2(1)|_{F_2(A_{(1,2)})})^*(\tilde{\omega}_2) \\
\tilde{\omega}_2|_{F_2(A_{(1,2)})} &= (F_1(1)|_{F_1(A_{(1,2)})})^*(\tilde{\omega}_1)
\end{align}

and

\begin{align}
d\tilde{\omega}_1|_{F_1(A_{(1,2)})} &= (F_2(1)|_{F_2(A_{(1,2)})})^*(d\tilde{\omega}_2) \\
d\tilde{\omega}_2|_{F_2(A_{(1,2)})} &= (F_1(1)|_{F_1(A_{(1,2)})})^*(d\tilde{\omega}_1)
\end{align}

We claim that, actually, it holds

\begin{align}
\tilde{\omega}_1|_{F_1(A_{(1,2)})} &= \tilde{\omega}_2 \circ F_1(2,1) \\
\tilde{\omega}_2|_{F_2(A_{(1,2)})} &= \tilde{\omega}_1 \circ F_2(1,2)
\end{align}

In order to prove this, let $O_1$ be an open subset of $U_1$ such that $O_1 \cap F_1(A_1) = F_1(A_{(1,2)})$ and $F_2(1,1) : O_1 \rightarrow F_2(1,1)(O_1)$ a holomorphic extension of the biholomorphism $F_1(A_{(1,2)}) \rightarrow F_2(1,1)(O_1)$ (cp. (1.2)). Let $O_2$ be an open subset of $U_2$ such that $O_2 \cap F_2(A_2) = F_2(A_{(1,2)})$. Shrinking $O_1$, if necessary, we can assume without loss of generality that $F_2(1,1)(O_1) \subseteq O_2$. Let $\iota : F_2(1,1)(O_1) \hookrightarrow O_2$ denote the inclusion. Then

\begin{align}
\tilde{\omega}_1|_{F_1(A_{(1,2)})} &= (F_2(1)|_{F_1(A_{(1,2)})})^* \circ \iota^*(\tilde{\omega}_2|_{O_2}) \\
&= \tilde{\omega}_2|_{O_2} \circ \iota \circ F_2(1,1)|_{F_1(A_{(1,2)})}
\end{align}

Indeed, on one hand

$F_2(1,2) = F_2 \circ F_1^{-1} \circ F_1(A_{(1,2)}) \subseteq F_2(1,1)(O_1) \subseteq O_2$

On the other hand, $\tilde{\omega}_1|_{F_1(A_{(1,2)})}$ and $\tilde{\omega}_2|_{O_2} \circ \iota \circ F_2(1,1)|_{F_1(A_{(1,2)})}$ coincide on the dense subset $F_1(A'_{(1,2)})$ of $F_1(A_{(1,2)})$, because of (4.3). Then, it results $\tilde{\omega}_1|_{F_1(A_{(1,2)})} = \tilde{\omega}_2|_{O_2} \circ \iota \circ F_2(1,1)|_{F_1(A_{(1,2)})}$. Then we get $\tilde{\omega}_1|_{F_1(A_{(1,2)})} = \tilde{\omega}_2 \circ F_2 \circ F_1^{-1}|_{F_1(A_{(1,2)})}$. Analogously, $\tilde{\omega}_2|_{F_2(A_{(1,2)})} = \tilde{\omega}_1 \circ F_1 \circ F_2^{-1}|_{F_2(A_{(1,2)})}$. So, (1.4) are proved.

Furthermore, a similar argument shows that

\begin{align}
d(\tilde{\omega}_1|_{F_1(A_{(1,2)})}) &= d(\tilde{\omega}_2|_{F_2(A_{(1,2)})}) \\
d(\tilde{\omega}_2|_{F_2(A_{(1,2)})}) &= d(\tilde{\omega}_1|_{F_1(A_{(1,2)})})
\end{align}

Now, since $\Delta \in T_p$ is such that $\hat{\Delta} \cap \text{Sing}(X) \neq \emptyset$, it results $p = \dim\text{r}(\Delta) \leq 2n$, because the triangulation $T$ is compatible with $\text{Sing}(X)$. Then there exists a
simplex $E \in \mathbb{T}$ such that $\dim_\mathbb{R}(E) \geq \dim_\mathbb{R}(\Delta)$, $\partial(E) \supseteq \Delta$ and $\hat{E} \cap \text{Sing}(X) = \emptyset$. Thus, by the geometry of simplices and (4.1), there are a $p$-chain $\Delta' \subset \hat{E} \subset E$ and a $(p+1)$-chain $\Gamma \subset \hat{E} \subset E$ such that $\Delta', \Gamma \subset A_{(1,2)}$ and $\Delta + \Delta' = \partial(\Gamma)$. Note that $\Delta' \cap \text{Sing}(X) = \emptyset$ and $\Gamma \cap \text{Sing}(X) = \emptyset$, because $\Delta', \Gamma \subset \hat{E}$.

Next, since $F_{(2,1)}|_{F_1(A_{1,2}')} : F_1(A_{1,2}') \rightarrow F_2(A_{1,2}')$ and $F_{(1,2)}|_{F_2(A_{1,2}')} : F_2(A_{1,2}') \rightarrow F_1(A_{1,2}')$ are biholomorphisms between complex manifolds, they preserve orientations. Then

\begin{equation}
\int_{F_1(\Delta')} \hat{\omega}_1 = \int_{F_2(\Delta')} (F_{(1,2)}|_{F_1(A_{1,2}')} \circ F_{(2,1)})^*(\hat{\omega}_1) = \int_{F_2(\Delta')} \hat{\omega}_2
\end{equation}

and

\begin{equation}
\int_{F_1(\Gamma)} d(\hat{\omega}_1) = \int_{F_2(\Gamma)} (F_{(1,2)}|_{F_1(A_{1,2}')} \circ F_{(2,1)})^*(d(\hat{\omega}_1)) = \int_{F_2(\Gamma)} d(\hat{\omega}_2)
\end{equation}

so, by (4.7) and (4.9), it results

\begin{equation}
\int_{F_1(\Delta)+F_1(\Delta')} \hat{\omega}_1 = \int_{F_1(\Gamma)} d(\hat{\omega}_1) = \int_{F_1(\Gamma)} d(\hat{\omega}_2) = \int_{F_2(\Delta)+F_2(\Delta')} \hat{\omega}_2,
\end{equation}

because $F_{(2,1)} \circ F_1(\Gamma) = F_2(\Gamma)$. Then, by (4.5), it results $\int_{F_1(\Delta)} \hat{\omega}_1 = \int_{F_2(\Delta)} \hat{\omega}_2$ and we are done. \qed

Let $X$ an abstract finite dimensional complex analytic variety and take $\omega \in \Gamma_{\text{ext}}(\Lambda^pT^*X')$. By Lemma 3, the operator $\int \omega : \mathbb{T}_p \rightarrow \mathbb{C}$ is well defined for any triangulation $\mathbb{T}$ of $X$ compatible with $\text{Sing}(X)$.

**Theorem 3. (Stokes)** Let $X$ be an abstract finite dimensional complex analytic variety and $\mathbb{T}$ a triangulation of $X$ compatible with $\text{Sing}(X)$. Let $\omega \in \Gamma_{\text{ext}}(\Lambda^pT^*X')$, $C \in \mathcal{C}_{p+1}^{\mathbb{T}}(X)$ and $\iota : \partial C \rightarrow C$ be the inclusion. Then

$$\int_C d\omega = \int_{\partial C} \iota^*(\omega).$$

**Proof (Sketch).** Let $\mathcal{A}(\omega)$ be an atlas of extensibility of $\omega$ and $\mathbb{T}_p$ an $\mathcal{A}(\omega)$-small barycentric subdivision of $\mathbb{T}$ (cp. Lemma 3). Then $C$ belongs to $\mathcal{C}_{p+1}^{\mathbb{T}_p}(X)$. Namely, $C = \sum_{\lambda \in \Lambda} \Delta_\lambda$, with $\Delta_\lambda \in \mathbb{T}_{p+1}$ for any $\lambda \in \Lambda$. Thus, it suffices to show that the thesis holds when $C$ is a simplex $\Delta \in \mathbb{T}_{p+1}$.

If $\Delta \cap \text{Sing}(X) = \emptyset$, then there is nothing to prove, being the classical result. If $\Delta \cap \text{Sing}(X) \neq \emptyset$, then there exist $A \in \mathcal{A}(\omega)$ and $\tilde{\omega} \in \Gamma_{\text{ext}}(\Lambda^pT^*U_A)$ such that $A \supseteq \Delta$ and $\omega|_A = (F_A|_{\text{Sing}(X)})^*(\tilde{\omega})$ (cp. Proposition 3). Set $\Delta_A = F_A(\Delta)$. Then $\Delta_A \subseteq U_A$. Moreover, $\partial \Delta_A = F_A(\partial \Delta) = F_A(\partial \Delta)$, because $F_A$ is a homeomorphism. So, letting $\iota_A : \partial \Delta_A \rightarrow \Delta_A$ denote the inclusion, the wanted result follows from the classical Stokes’ theorem

\begin{equation}
\int_{\Delta_A} d\omega = \int_{\Delta_A} \tilde{\omega} = \int_{\partial \Delta_A} (\iota_A)^*(\tilde{\omega}) = \int_{\partial \Delta} \iota^*(\omega).
\end{equation}

We have the following remark.

**Remark 10.** Let $X$ be a compact irreducible complex analytic variety of complex dimension $n$ and $\mathbb{T}$ a triangulation of $X$ compatible with $\text{Sing}(X)$. Take $\omega \in \Gamma_{\text{ext}}(\Lambda^{2n}T^*X)$ and let $\mathcal{A}(\omega)$ be an atlas of extensibility of $\omega$. Let $b \in \mathbb{N}$ be such that $\mathbb{T}_b$ is $\mathcal{A}(\omega)$-small. Then the number

\begin{equation}
\int_X \omega = \sum_{\Delta \in \mathbb{T}_b} \int_{\Delta} \omega
\end{equation}
is well defined. Indeed, even if in general two triangulations \( T \) of \( X \) do not have a common refinement, actually, \( (4.12) \) does not depend on the chosen triangulation \( T \) of \( X \) compatible with \( \text{Sing}(X) \). In order to prove this, recall that, if \( T_1 \) and \( T_2 \) are triangulations of \( X \) compatible with \( \text{Sing}(X) \), then there exist triangulations \( T_3, T_1, T_{II} \) of \( X \) compatible with \( \text{Sing}(X) \) such that \( T_1 \) is a refinement of both \( T_2 \) and \( T_3 \) and \( T_{II} \) is a refinement of both \( T_2 \) and \( T_3 \). The claimed independence follows from
\[
\sum_{\Delta_1 \in T_1} \Delta_1 = \sum_{\Delta_1 \in T_1} \Delta_1 = \sum_{\Delta_3 \in T_3} \Delta_3 = \sum_{\Delta_{II} \in T_{II}} \Delta_{II} = \sum_{\Delta_2 \in T_2} \Delta_2.
\]
As a matter of terminology, \( \int_X \omega \) is the integral of \( \omega \) on \( X \).

4.2. Integration of extendable cohomology classes. Let \( X \) be a finite dimensional complex analytic variety and \( Z \) either the empty set or the closure of an open set which is a polyhedron of \( X \). For any \( p \in \mathbb{N} \) the \( p \)-th simplicial cohomology group relative to the pair \( (X,Z) \) is denoted by \( H^p(X,Z) \) (cp. Theorem 2). Let \( \omega \in \Gamma_{ext}(\Lambda^p T^*X')_Z \) be an extendable \( p \)-form and \( \Omega \) a triangulation of \( X \) compatible with \( \text{Sing}(X) \) and \( Z \). Extending by linearity \( \int \omega : T_p \rightarrow \mathbb{C} \) to the whole of \( C^p_T(X) \), we get a well defined map \( \int \omega : C^p_T(X) \rightarrow \mathbb{C} \) that, in fact, lies in \( C^p_T(X) \).

So, we get a homomorphism between complex vector spaces
\[
(4.13) \quad \eta^p_Z : \Gamma_{ext}(\Lambda^p T^*X')_Z \rightarrow C^p_T(X) \quad \omega \mapsto \eta^p_Z(\omega) = \int \omega
\]
\( \eta^p_Z \) is called operator of integration of degree \( p \).

Furthermore, for any \( \omega \in \Gamma_{ext}(\Lambda^p T^*X')_Z \) the cochain \( \eta^p_Z(\omega) \) lies in \( C^p_T(X,Z) \), because \( \omega|_Z = 0 \). So,
\[
\eta^p_Z : \Gamma_{ext}(\Lambda^p T^*X')_Z \rightarrow C^p_T(X,Z).
\]
With slight abuses of notations, given any \( \Delta \in C^p_T(X,Z) \), the cochain \( \eta^p_Z(\omega) \) acts as follows \( \Delta + C^p_T(Z) \ni \Delta \mapsto \eta^p_Z(\omega)(\Delta) = \int_\Delta \omega \in \mathbb{C} \).

Let
\[
H^p_Z : H^p_{ext}(X,Z) \rightarrow H^p(X,Z) \quad [\omega] \mapsto [H^p_Z(\omega)] = \int [\omega]
\]
be the operator induced by \( \eta^p_Z \). Then \( H^p_Z([\omega]) \in H^p(X,Z) \) is the map that associates to each \( [C] \in H_p(X,Z) \), \( C \in C + C^p_T(Z) \) the number
\[
(4.15) \quad \int [C][\omega] = \int_C \omega.
\]
In order to prove that \( H^p_Z \) is well defined, we use Stokes’ theorem. Let \( C \in C + C^p_T(Z) \) be a \( p \)-cycle of \( (X,T) \) relative to \( (X,Z) \) and \( \omega \in \Gamma_{ext}(\Lambda^p T^*X')_Z \) a closed extendable \( p \)-form vanishing on \( Z \). Since \( \partial^p C \subseteq Z \) and \( \partial^p \omega = 0 \), \( \omega|_Z = 0 \), for each \( E \in C^p_{p+1}(X,Z) \) and for each \( \sigma \in \Gamma_{ext}(\Lambda^{p-1} T^*X)_Z \) we have
\[
(4.16) \quad \int_{C + \partial^p E + C^p_T(Z)}(\omega + d^{p-1} \sigma) = \int_C \omega + \int_{\partial^p C} \sigma + \int_{C^p_T(Z)}(\omega + d^{p-1} \sigma)|_Z = \int_C \omega + \int_{\partial^p C} \sigma = \int_C \omega,
\]
because \( (\omega + d^{p-1} \sigma)|_Z = 0 \), \( \partial^p C \subseteq Z \) and \( \sigma|_Z = 0 \). So, \( H^p_Z : H^p_{ext}(X,Z) \rightarrow H^p(X,Z) \) is well defined, because \( \int_C[\omega] \) does not depend on the representatives chosen in \( [C]_{H_p(X,Z)} \) and \( [\omega]_{H^p_{ext}(X,Z)} \). Moreover, \( H^p_Z \) is a homomorphism of vector spaces, because such is \( \eta^p_Z \). \( H^p_Z \) is called homomorphism of integration of degree \( p \) relative to \( (X,Z) \).
Lemma 5. Let $X$ be a finite dimensional complex analytic variety and $S$ a closed compact complex analytic subvariety of $X$. Such an $S$ is a polyhedron of $X$. Then there are arbitrarily small open neighborhoods $U_S$ of $S$ in $X$ such that

1. $\overline{U_S}$ and $X \setminus U_S$ are polyhedra in $X$.
2. The inclusions $S \to U_S \to \overline{U_S}$ and $(X \setminus \overline{U_S}) \to (X \setminus U_S) \to (X \setminus S)$ are homotopy equivalences.

Proof. If $X$ is compact, then proceed as in [Mu], Ch. 8, Sec. 72. If $X$ is not compact, we can agree as follows. Let $X^*$ be an open neighborhood of $S$ in $X$ such that $\overline{X^*}$ is a compact polyhedron in $X$ with respect to the same triangulation for which $S$ is a polyhedron. Then the statement holds for $S$ considered as a polyhedron in $\overline{X^*}$. So, the following

$$
\begin{align*}
S & \to U_S \to \overline{U_S} \\
\overline{X^* \setminus U_S} & \to \overline{X^* \setminus U_S} \to \overline{X^* \setminus S}
\end{align*}
$$

are homotopy equivalences, with $U_S$ an open neighborhoods of $S$ in $\overline{X^*}$ and then in $X$. Furthermore, by their very constructions (cp. [Mu], page 429), the homotopy equivalences $\overline{X^* \setminus U_S} \to \overline{X^* \setminus U_S} \to \overline{X^* \setminus S}$ do not involve the boundary of $\overline{X^*}$ in $X$. Then, in order to define the wanted homotopy equivalences

$$
X \setminus \overline{U_S} \to X \setminus U_S \to X \setminus S,
$$

just glue the maps $\overline{X^* \setminus U_S} \to \overline{X^* \setminus U_S} \to \overline{X^* \setminus S}$ and $X \setminus \overline{X^*} \to X \setminus \overline{X^*} \to X \setminus \overline{X^*}$. \qed

Remark 11. Let $X$ be a finite dimensional complex analytic variety, $S$ a closed compact complex analytic subvariety of $X$ and $U_S$ an open neighborhood of $S$ enjoying (1) and (2) of Lemma 4. Set $Z = X \setminus U_S$ and for any $p \in \mathbb{N}$ consider the homomorphism $H^*_p (X, X \setminus U_S) \to H^p (X, X \setminus U_S)$. Then, since $X \setminus S$ and $X \setminus U_S$ are homotopically equivalent, we get a homomorphism $H^*_p (X, X \setminus U_S) \to H^p (X, X \setminus S)$, because the groups $H^p (X, X \setminus S)$ and $H^p (X, X \setminus U_S)$ are isomorphic.

An easy verification shows that the following properties of complex analytic varieties hold. For the necessary background, refer to [Mu].

Remark 12. Let $X$ be a complex analytic variety of complex dimension $n$ and $T$ a triangulation of $X$ compatible with $\text{Sing}(X)$.

1. The pair $(X, \text{Sing}(X))$ is a relative homology $2n$-manifold (cp. [Mu]).
2. If $X$ is also irreducible, then $(X, \text{Sing}(X))$ is a relative pseudo $2n$-manifold, $X$ is the closure of the union of the $2n$-simplices of $T$, any $(2n-1)$ simplex is a face of exactly two $2n$-simplices of $T$ and the $2n$-simplices of $T$ can be coherently oriented (cp. [Mu]).
3. If $X$ is a compact and irreducible, then $(X, \text{Sing}(X))$ is an orientable relative pseudo $2n$-manifold, the groups $H_{2n} (X)$ and $H_{2n} (X, \text{Sing}(X))$ are isomorphic and the $2n$-dimensional simplices of $(X, T)$ can be oriented in such a way that their sum is a non vanishing cycle. Moreover, such a sum is independent of the chosen triangulation (cp. Remark 10). The class $[X]$ represented by such a cycle is the fundamental class of $X$ (cp. [Mu]).

We have the following theorem.
**Theorem 4.** Let $X$ be a compact irreducible complex analytic variety of complex dimension $n$. If $\omega \in \Gamma_{\text{ext}}(\Lambda^pT^*X)$ be such that $d\omega = 0$, then the number
\begin{equation}
\int_{[X]}[\omega] = \int_X \omega
\end{equation}
is well defined. $\int_{[X]}[\omega]$ is the integral of $[\omega]$ on $[X]$.

**Proof (Sketch).** The wanted result follows from (4.15). Indeed, $(X, \text{Sing}(X))$ is an orientable relative pseudomanifold of real dimension $2n$ (cp. Remark [12]). □

As a matter of notations, if $Z = \emptyset$, we omit to write the subscript $Z$ everywhere it should appear. The next remark concerns the injectivity of homomorphisms of integration.

**Remark 13.** Let $X$ be an abstract finite dimensional complex analytic variety and $p \in \mathbb{N}$. Then $H^p : H^p_{\text{ext}}(X) \to H^p(X)$ is generally not injective.

Indeed, let $X$ be as in Example 3. Shrinking the neighborhood $B$ of $0$ in $\mathbb{C}$, if necessary, we can assume that $X$ is topologically contractible (cp. [Go-Ma]). So, $H^1(X) = 0$. Then $H^1 : H^1_{\text{ext}}(X) \to H^1(X)$ is not injective, because $H^1_{\text{ext}}(X) \neq 0$ (cp. Example 3).

The following proposition concerns the surjectivity of homomorphisms of integration. For the necessary background in algebraic topology, see [Mu].

**Proposition 6.** Let $X$ be an abstract finite dimensional complex analytic variety. Fix $p \in \mathbb{N}$. Then $H^p : H^p_{\text{ext}}(X) \to H^p(X)$ is surjective.

**Proof.** Let $T$ be a triangulation of $X$ and $c \in C^p_T(X)$ be a $p$-cocycle. We look for a closed extendable $p$-form $\omega_c \in \Gamma_{\text{ext}}(\Lambda^pT^*X')$ such that $[c]_{H^p(X)} = H^p([\omega_c]|_{H^p_{\text{ext}}(X)})$. Let $\mathcal{V}(T)$ be the open covering of $X$ by the open stars of vertices of $T$. The nerve $N(\mathcal{V}(T))$ of such a covering is an abstract simplicial complex. Moreover, the vertex correspondence $C^0_T(X) \ni \Delta^0 \mapsto St(\Delta^0) \in N(\mathcal{V}(T))$ is an isomorphism between $(X, T)$ and $N(\mathcal{V}(T))$ (cp. [Mu], Theorem 73.2). Then to any $p$-cocycle $c \in C^p_T(X)$ corresponds a unique cocycle $c_c \in C^p(N(\mathcal{V}(T)))$. Furthermore, by the very definition of $C^p(N(\mathcal{V}(T)))$, we have $C^p(N(\mathcal{V}(T))) = C^p_0(X, \mathcal{V})$ (see (3.11)).

Let $\hat{H}^p(X)$ be the $p^{th}$ Čech cohomology group of $X$ (see [Mu], Ch. 8, Sec. 73). Let $T$ be so fine that $\hat{H}^p(X, N(\mathcal{V}(T)))$ is isomorphic to $\hat{H}^p(X)$ and let $B$ be an open covering of $X$ enjoying (2) of Lemma 1 and refining $\mathcal{V}(T)$. Let $\{\rho_\beta\}_{\beta \in \mathbb{B}}$ be an extendable partition of unity subordinated to $B$ and $\phi : K(X, \mathcal{V}(T)) \to \Gamma_{\text{ext}}(\otimes_{\beta \in \mathbb{B}} \mathcal{V}(\Lambda^pT^*X'))$ the chain map associated with $\mathcal{V}(T)$, $B$ and $\{\rho\}$ (see Remark (2) of 9). Then the closed extendable $p$-form $\omega_c \in \Gamma_{\text{ext}}(\Lambda^pT^*X')$ defined via the collating formula $\omega_c = \phi(c_c)$ of Bott and Tu does the desired job (see [Bo-Tu], Ch. II, Sec. 9, Proposition 9.8). □

### 4.3. Integration of extendable Čech cohomology classes.

Let $X$ be an abstract finite dimensional complex analytic variety, $Z$ either the empty set or the closure of an open set that is also a polyhedron of $X$ and $\mathcal{V}$ an open covering of $X$ adapted to $Z$. Let $r \in \mathbb{N}$ and consider the group homomorphism $H^r_Z = H^r_Z \circ (P^*)^{-1} : H^r_{\text{ext}}(X, \mathcal{V}, V_Z) \to H^r(X, Z)$ (cp. Theorem 1).

Let $T$ be a triangulation of $X$ compatible with $\text{Sing}(X)$ and $Z$ and consider the operator $\eta^r_Z : \Gamma_{\text{ext}}(\Lambda^rT^*X)_Z \to C^r_T(X, Z)$. Let $\mathcal{B} = \{B_\beta\}_{\beta \in \mathbb{B}}$ be a refinement of $\mathcal{V}$ enjoying (2) of Lemma 1 $\{\rho\} = \{\rho_\beta : X \to \mathbb{R}\}_{\beta \in \mathbb{B}}$ an extendable partition
of unity subordinated to $B$ and $\phi: K(X, V, V_Z) \rightarrow \bigoplus_{r \in \mathbb{N}} \Gamma^r_{ext}(A^r T^* X, V)$ the chain map associated with $\mathcal{V}$, $\mathcal{B}$ and $\{\rho\}$ (cp. Remark 9 (2)).

Let
\[(4.18) \hat{\eta}_{Z}^r: \tilde{Z}_{ext}^r(X, \mathcal{V}, V_0) \rightarrow Z^r(X, Z)\]
denote the homomorphism defined by
\[(4.19) \quad \hat{\eta}_{Z}^r = \eta_{Z}^r \circ \phi\]
Then $\hat{H}_{Z}^r$ is induced by $\hat{\eta}_{Z}^r$. Indeed, for any $\omega \in \tilde{Z}_{ext}^r(X, \mathcal{V}, V_Z)$ it results
\[(4.20) [\hat{\eta}_{Z}^r(\omega)]^r = [\eta_{Z}^r \circ \phi(\omega)]^r = H_{Z}^r[\phi(\omega)]^r = \hat{H}_{Z}^r[\omega]^r = \hat{H}_{Z}^r[\omega]^r\]

**Definition 6.** Let $X$ be a complex analytic variety of complex dimension $n$ and $\mathcal{V} = \{V_0, V_1\}$ an open covering of $X$. A honeycomb cell system associated with $\mathcal{V}$ is a family $\mathcal{R} = \{R_0, R_1\}$ of subsets of $X$ such that
\begin{enumerate}
  \item For $j \in \{0, 1\}$, $R_j$ is the closure of an $n$-dimensional complex analytic subvariety of $X$ with piecewise differentiable boundary.
  \item It holds $R_0 \subseteq V_0$ and $R_1 \subseteq V_1$.
  \item It holds $R_0 \cup R_1 = X$ and $R_0 \cap R_1 = \emptyset$.
  \item Both $\partial R_0$ and $\partial R_1$ are a complete intersection and $R_0 \cap R_1$ is a variety of real dimension $2n - 1$ with piecewise differentiable boundary.
  \item $R_{(0,1)}$ denotes the hypersurface $R_0 \cap R_1$ oriented in the following way: the orientation on $R_{(0,1)}$ is that one defined by the interior normal of $R_0$ and by the exterior normal of $R_1$.
  \item $R_{(1,0)}$ denoted the hypersurface $R_0 \cap R_1$ oriented in the following way: the orientation on $R_{(1,0)}$ is that one defined by the interior normal of $R_1$ and by the exterior normal of $R_0$.
\end{enumerate}

We have the following remark.

**Remark 14.** Let $X$ be a complex analytic variety of finite dimension and $\mathcal{V} = \{V_0, V_1\}$ an open covering of $X$. Then there exists a honeycomb cell system $\mathcal{R} = \{R_0, R_1\}$ associated with $\mathcal{V}$. Indeed, every analytic variety $X$ admits a triangulation such that each simplex of it corresponds to an analytic subvariety of $X$ (see [La] and [Li]). So, the existence of a honeycomb cell system $\mathcal{R}$ associated to $\mathcal{V}$ can be deduced from Lemma 4.

Next, let $X$ be an abstract finite dimensional complex analytic variety and $Z$ either the empty set or the closure of an open set that is also a polyhedron of $X$. Let $T$ be a triangulation of $X$ compatible with $Sing(X)$ and $Z$, take $r \in \mathbb{N}$ and consider the $(r - 1)$-skeleton $Skel^{r-1}(X, T)$ of $(X, T)$. In what follows we denote by $\sim$ the homotopic equivalence. Let $Z_{r-1}$ be the closure of an open set of $X$ that is a polyhedron of $X$ such that $Z_{r-1} \supseteq Skel^{r-1}(X, T)$ and $Z_{r-1} \sim Skel^{r-1}(X, T)$. Set $Z_* = Z \cup Z_{r-1}$.

Let $V_0$ be an open neighborhood of $Z_*$ such that $V_0 \sim Z_*$ and $V_1$ an open set of $X$ such that $V_1 \cap Z_* = \emptyset$ and $V_0 \cup V_1 = X$. Set $\mathcal{V} = \{V_0, V_1\}$ and note that $\mathcal{V}$ is adapted to $Z_*$. By Proposition 2 there exists an extendable partition of unity $\{\rho\} = \{\rho_0: X \rightarrow \mathbb{R}, \rho_1: X \rightarrow \mathbb{R}\}$ subordinated to $\mathcal{B} = \mathcal{V}$. Moreover, by construction, for any $\Delta \in T$, it results $\rho_1|_{\partial \Delta} \equiv 0$, because $\partial \Delta \subseteq Skel^{r-1}(X, T) \subseteq Z_* \subseteq V_0$. 

EXTENDABLE COHOMOLOGIES 25
Let $\mathcal{R} = \{R_0, R_1\}$ be a honeycomb cell system associated with $\mathcal{V}$. Building $R_0$ by means of very fine triangulations of $X$, we can assume without loss of generality that $R_0 \supseteq Z_\star$, that $R_0 \sim Z_\star$ and that for any $\Delta \in \mathbb{T}_r$ it holds

\[(4.21) \quad \Delta \cap R_{(1,0)} \sim \partial \Delta.\]

So, the inclusions $Z_\star \subseteq R_0 \subseteq V_0$ are homotopic equivalences and $R_1 \cap Z_\star = \emptyset$, because $R_1 \subseteq V_1$.

Let $\phi$ be the chain map associated with $\mathcal{V}$, $B = \mathcal{V}$ and $\{ \rho \}$ (cp. (2) of Remark 9). Take $\omega \in Z_{ext}^r (X, Z)$ and let $\tilde{\omega} = (\tilde{\omega}_0, \tilde{\omega}_1, \tilde{\omega}_{01}) \in \tilde{Z}_{ext}^r (X, \mathcal{V}, V_0)$ be such that $\omega = \phi (\tilde{\omega})$. Namely, by (3.15),

\[(4.22) \quad \omega = \rho_0 \tilde{\omega}_0 + \rho_1 \tilde{\omega}_1 - d\rho_0 \wedge \tilde{\omega}_{01}.\]

Since $\omega \in Z_{ext}^r (X, Z)$ and $\tilde{\omega} \in \tilde{Z}_{ext}^r (X, \mathcal{V}, V_0)$, we have $\omega |_{Z} = 0$ and $\tilde{\omega}_1 |_{Z} = 0$. Moreover, $d\omega = 0$ and $D\tilde{\omega} = 0$. The latter is

\[(4.23) \quad (d\tilde{\omega}_0, d\tilde{\omega}_1, d\tilde{\omega}_{01} - \tilde{\omega}_0 |_{V (0,1)} + \tilde{\omega}_1 |_{V (0,1)}) = (0, 0, 0).\]

In what follows, with slight abuses of notations, for any $\Delta \in C_p^r (X, Z)$ we write $\Delta \in \Delta + C_p^r (Z)$. In the above situation, the integral of $\omega \in Z_{ext}^r (X, Z)$ over an element $\Delta \in C_p^r (X, Z)$ can be written in terms of the integrals of the components of $\tilde{\omega} \in \tilde{Z}_{ext}^r (X, \mathcal{V}, V_0)$ such that $\omega = \phi (\tilde{\omega})$. Indeed, the image of $\tilde{\omega}$ via $\tilde{\eta}_{Z}^r$ (cp. (4.18) and (4.15)) is the map $\tilde{\eta}_{Z}^r (\tilde{\omega}) \in C_p^r (X, Z)$ that to any $\Delta \in \Delta + C_p^r (Z)$ associates the number

\[(4.24) \quad \tilde{\eta}_{Z}^r (\tilde{\omega}_0, \tilde{\omega}_1, \tilde{\omega}_{01}) (\Delta) = \int_{\Delta \cap R_0} \tilde{\omega}_0 + \int_{\Delta \cap R_1} \tilde{\omega}_1 - \int_{\Delta \cap R_{(1,0)}} \tilde{\omega}_{01}\]

In order to prove (4.18), recall (4.24). Then

\[(4.25) \quad \tilde{\eta}_{Z}^r (\tilde{\omega}) (\Delta) = \eta_{Z}^r \circ \phi (\tilde{\omega}) (\Delta) = \int_{\Delta + C_p^r (Z)} \phi (\tilde{\omega})\]

So, by (4.22),

\[
\begin{align*}
\tilde{\eta}_{Z}^r (\tilde{\omega}) (\Delta) &= \int_{(\Delta + C_p^r (Z)) \cap R_0} (\rho_0 \tilde{\omega}_0 + \rho_1 \tilde{\omega}_1) + \int_{(\Delta + C_p^r (Z)) \cap R_1} (\rho_0 \tilde{\omega}_0 + \rho_1 \tilde{\omega}_1) + \int_{(\Delta + C_p^r (Z)) \cap R_0} d\rho_1 \wedge \tilde{\omega}_0 + \int_{(\Delta + C_p^r (Z)) \cap R_1} d\rho_0 \wedge \tilde{\omega}_0 + \int_{(\Delta + C_p^r (Z)) \cap R_0} d(\rho_0 \tilde{\omega}_0) - \rho_1 d\tilde{\omega}_0 + \int_{(\Delta + C_p^r (Z)) \cap R_1} d(\rho_0 \tilde{\omega}_0) - \rho_0 d\tilde{\omega}_0
\end{align*}
\]

Thm. (4.29)
Then, by (4.26),
\[
\tilde{\eta}_Z(\tilde{\omega})(\Delta) = \int_{(\Delta + C_\tau^r(Z)) \cap R_0} \tilde{\omega}_0 + \int_{(\Delta + C_\tau^r(Z)) \cap R_1} \tilde{\omega}_1 + \\
+ \int_{\delta((\Delta + C_\tau^r(Z)) \cap R_0)} \rho_1 \tilde{\omega}_0 - \int_{\delta((\Delta + C_\tau^r(Z)) \cap R_1)} \rho_0 \tilde{\omega}_0 \\
(1) = \int_{(\Delta + C_\tau^r(Z)) \cap R_0} \tilde{\omega}_0 + \int_{(\Delta + C_\tau^r(Z)) \cap R_1} \tilde{\omega}_1 + \\
+ \int_{\Delta \cap R_0} \rho_1 \tilde{\omega}_0 - \int_{\Delta \cap R_1} \rho_0 \tilde{\omega}_0 \\
(2) = \int_{(\Delta + C_\tau^r(Z)) \cap R_0} \tilde{\omega}_0 + \int_{(\Delta + C_\tau^r(Z)) \cap R_1} \tilde{\omega}_1 + \\
+ \int_{\Delta \cap R_0} \rho_1 \tilde{\omega}_0 - \int_{\Delta \cap R_1} \rho_0 \tilde{\omega}_0 \\
(3) = \int_{(\Delta + C_\tau^r(Z)) \cap R_0} \tilde{\omega}_0 + \int_{(\Delta + C_\tau^r(Z)) \cap R_1} \tilde{\omega}_1 \\
- \int_{\Delta \cap R_1} \tilde{\omega}_0,
\]
with $B^r_{\tau-1}(Z) = \partial C^r_\tau(Z)$. Indeed, (1) follows from $\partial(\Delta \cap R_1) = \partial \Delta + \Delta \cap R_1$, and (2) follows from $\rho_1|_{\text{Skel}^{r-1}(X,T)} = 0$ and $B^r_{\tau-1}(Z) \subset \text{Skel}^{r-1}(X,T)$. (3) follows from $R_0(1,0) = -R_1(1,0)$. \rho_0 + \rho_1 \equiv 1.

Actually, the last term of (4.27) is independent of the choice of the representative of $\Delta$ in $\Delta + C^r_\tau(Z)$. Indeed, if $Y \in C^r_\tau(Z)$, then $Y \cap R_0 \subseteq Z$ and $Y \cap R_1 = \emptyset$. So, $\int_{T \cap R_0} \tilde{\omega}_0 + \int_{T \cap R_1} \tilde{\omega}_1$, because $\tilde{\omega}_0|Z = 0$. Then (4.18) follows from (4.23), (4.27) and we are done.

Next, $\tilde{\eta}_Z$ is independent of the chosen honeycomb cell system $\mathcal{R}$ associated with $V$. In fact, by (4.19), $\tilde{\eta}_Z = \eta_Z \circ \phi$ and the right hand side does not depend on the choice of $\mathcal{R}$. Furthermore, $\tilde{\eta}_Z$ is independent of the partition of unity $\{\rho_0, \rho_1\}$ subordinated to $V$. In fact, take into account (4.18) and note that its right hand side does not depend on $\{\rho_0, \rho_1\}$.

As a note, for any $\omega \in Z^r_{\text{ext}}(X,Z)$ and $\Delta \in \Delta + C^r_\tau(Z)$ it results
\[
\int_{\Delta} \omega = \int_{\Delta \cap R_0} \tilde{\omega}_0 + \int_{\Delta \cap R_1} \tilde{\omega}_1 - \int_{\Delta \cap R_1(1,0)} \tilde{\omega}_0,
\]
because of $\eta_Z^{\ast}(\omega)(\Delta) = \eta_Z^{\ast}(\phi(\tilde{\omega})(\Delta) = \tilde{\eta}_Z(\omega)(\Delta)$ (see (4.19)).

**Remark 15.** The explicit expression (4.24) of $\tilde{\eta}_Z$ is strongly related to properties of $R$. Indeed, the hypotheses $R_0 \supseteq Z$, and (4.21) have been given in order to achieve Formula (4.24). If $\mathcal{R}$ is a honeycomb cell system associated with $V$ that does not enjoy any particular property, then the explicit formula for $\tilde{\eta}_Z$ associated with $\mathcal{R}$ is, in general, more complicated than Formula (4.24). For example, it can happen that the explicit expression of $\tilde{\eta}_Z$ depends on the chosen partition of unity subordinated to $V$, even if $\tilde{\eta}_Z$ is independent of it (cp. [Sh 3]).

5. Vector bundles

5.1. Extensible Chern classes. In the following, let $K$ denote either $C$ or $\mathbb{R}$. Let $X$ be an abstract finite dimensional complex analytic variety. Let $E \to X$ be a differentiable (holomorphic) $K$-vector bundle of rank $e$ over $X$ and $C = \{(A_l, \nu_l, U_l, W_l, F_l)\}_{l \in L}$ an atlas of $X$ trivializing $E$. Then for each $l \in L$ there is a differentiable (holomorphic) real (complex) vector bundle $E_l \to U_l$ of rank $e$ such that $E|_{A_l} = (F_l)^{\ast}(E_l|_{W_l})$. For this, just to take the trivial bundle $E_l = U_l \times \mathbb{K}^e \to U_l$. 
In what follows, we will consider $E|_{X'} \to X'$ as an $E$-extendable bundle, with $E$ the sheaf of germs of differentiable sections of $E$ (see Example 1). If this is the case, then $\mathcal{C} = \{A_i\}_{i \in L}$, the atlas trivializing $E$, is associated with $E|_{X'}$ (cp. Remark 1). Shrinking the open sets of $\mathcal{C}$, if necessary, we get the following commutative diagram

$$
\begin{array}{ccc}
(C^\infty_{\mathbb{R}})^n |_{U_i} & \to & C^\infty_{\mathbb{R}}(C^n, \mathbb{R})|_{U_i} \\
\downarrow & & \downarrow \\
((C^\infty_{\mathbb{R}})^n|_{A_i})_t & \to & (E|_{A_i})_t \\
\downarrow & & \downarrow \\
0 & & 0
\end{array}
$$

(5.1)

that improves (2.2) (cp. Notation 1).

Recall that for any $N, N^*, p \in \mathbb{N}$ the bundle $TX^\otimes N \otimes T^* X^\otimes N^* \otimes \Lambda^p T^* X' \otimes E|_{X'}$ is $(\mathcal{S}_N, N^* \otimes \mathcal{S}_p \otimes \mathcal{E})$-extendable (cp. Example 2).

**Definition 7.** Let $X$ be an abstract finite dimensional complex analytic variety and $E \to X$ a differentiable (holomorphic) $\mathbb{K}$-vector bundle on $X$ of rank $e$. An extendable linear connection for $E$ is a $\mathbb{K}$-linear map $\nabla : \Gamma(E|_{X'}) \to \Gamma(\delta^* T^* X' \otimes E|_{X'})$ such that $\forall f \in \Gamma(\delta(T^* X'|_{X'}))$ and $\forall s \in \Gamma(\delta(E|_{X'}))$ it holds $\nabla(fs) = df \otimes s + f\nabla s$.

Let $X$ be an abstract complex analytic variety of complex dimension $n$, $E \to X$ a differentiable (holomorphic) real (complex) vector bundle of rank $e$ over $X$ and $\nabla$ an extendable connection for $E$. If $\mathcal{C} = \{A_i\}_{i \in L}$ is an atlas of $X$ trivializing $E$ that is also associated with $T^* X'$, then for any $i \in L$ there exists a linear connection $^i\nabla : \Gamma(E_i) \to \Gamma(T^* U_i \otimes E_i)$ for $E_i$ such that $^i\nabla|_{A_i} = (F_i|_{A_i})^*(^i\nabla)$. Note that the connection forms of $\nabla$ with respect to any given extendable frames of $E|_{A_i}$ and $T^* X'|_{A_i}$ are extendable differential forms.

For this, let $(\partial) = (\partial_1, ..., \partial_n)$ and $(e) = (e_1, ..., e_e)$ be extendable frames of $T^* X'|_{A_i}$ and, respectively, $E|_{A_i}$. The connection forms $\theta_{\alpha \beta}$ of $\nabla|_{A_i}$ with respect to $(\partial)$ and $(e)$ are defined by $\nabla e_\alpha = \sum_\beta \theta_{\alpha \beta} \otimes e_\beta$. Now, let $(\tilde{\partial}) = (\tilde{\partial}_1, ..., \tilde{\partial}_n)$ and $(\tilde{e}) = (\tilde{e}_1, ..., \tilde{e}_e)$ be frames of $T^* U_i$ and, respectively, $E_i$ extending $(\partial)$ and, respectively, $(e)$ (cp. (2.7) and (1.1)). Then for any $i \in \{1, ..., n\}, v \in \{1, ..., e\}$ we have $\tilde{\partial}_v = (F_i|_{A_i})^*(\partial_v)$ and $\tilde{e}_v = (F_i|_{A_i})^*(e_v)$. Denote by $(\tilde{\theta}_{\alpha \beta})$ the connection forms of $\tilde{\nabla}$ with respect to $(\tilde{\partial})$ and $(\tilde{e})$ and recall that these forms are defined by $^i\tilde{\nabla} \tilde{e}_\alpha = \sum_\beta \tilde{\theta}_{\alpha \beta} \otimes \tilde{e}_\beta$. Then $\theta_{\alpha \beta} = (F_i|_{A_i})^*(\tilde{\theta}_{\alpha \beta})$ for any $\alpha, \beta \in \{1, ..., n\}$.

**Proposition 7.** Let $X$ be an abstract finite dimensional complex analytic variety and $E \to X$ a differentiable (holomorphic) real (complex) vector bundle. Then there exists an extendable linear connection $\nabla$ for $E$.

**Proof.** Choose an extendable partition of unity subordinated to a suitable open covering of $X$ and proceed as for the smooth case. \(\square\)

Next, let $X$, $E$ and $\nabla$ be as above. $\nabla$ induces the $\mathbb{K}$-linear map $\nabla : \Gamma_{\text{ext}}(T^* X' \otimes E|_{X'}) \to \Gamma_{\text{ext}}(\Lambda^2 T^* X' \otimes E|_{X'})$ which to any $\omega \otimes s \in \Gamma_{\text{ext}}(T^* X' \otimes E|_{X'})$ associates $\nabla(\omega \otimes s) = d\omega \otimes s - \omega \wedge \nabla s$ and then it is linearly extended. The map $K^\nabla : \Gamma_{\text{ext}}(E|_{X'}) \to \Gamma_{\text{ext}}(\Lambda^2 T^* X' \otimes E|_{X'})$ defined by $K^\nabla = \nabla \circ \nabla$ is the curvature of $\nabla$. As a note, for any $f \in \Gamma_{\text{ext}}((X \times \mathbb{K})|_{X'})$ and for any $s \in \Gamma_{\text{ext}}(E|_{X'})$ it results $K^\nabla(f s) = f K^\nabla(s)$. So $K^\nabla \in \Gamma_{\text{ext}}(\Lambda^2 T^* X' \otimes E^*|_{X'} \otimes E|_{X'})$. In order to prove
that \( \nabla \) and \( K^\nabla \) are well defined, use the classical, explicit, local expressions of \( \nabla \) and \( K^\nabla \), noting that the differential forms involved in such expressions with respect to any local extendable frame of \( E|_X \) and \( T^\nabla X' \) are extendable (cp. [Su II], Ch. II, Sec. 7).

Let \( n \in \mathbb{N} \). For each \( q \in \{1, \ldots, n\} \) denote by \( \Sigma_q \in \mathbb{C}[t_1, \ldots, t_n] \) the \( q^{th} \) elementary symmetric function in the \( n \) variables \( t_1, \ldots, t_n \) and recall that \( \Sigma_q \) is a polynomial of degree \( q \).

Let \( X \) be an abstract complex analytic variety of complex dimension \( n \), \( E \to X \) a differentiable (holomorphic) complex vector bundle of rank \( e \) over \( X \) and \( \nabla \) an extendable connection for \( E \). Proceeding as for the smooth case, by using the curvature forms of \( \nabla \), that is the extendable forms locally representing \( K^\nabla \), we can associate a global closed extendable form with any elementary symmetric function \( \Sigma_q \in \mathbb{C}[t_1, \ldots, t_n] \) as in the smooth case, such a class is independent of the choice of the connection \( \nabla \).

As in the smooth case, such a class is independent of the choice of the connection \( \nabla \) of \( E \). Indeed, if \( \nabla_I, \nabla_{II} \) are extendable linear connections for \( E \), then there exists \( \nabla_I|_{\nabla_I}, \nabla_{II} \in \Gamma_{ext}(\Lambda^{2q-1}T^\nabla X) \) such that \( \nabla_I|_{\nabla_I}, \nabla_{II} \) and \( d\nabla_I|_{\nabla_I}, \nabla_{II} = \nabla_I|_{\nabla_I}, \nabla_{II} \) and \( d\nabla_I|_{\nabla_I}, \nabla_{II} = \nabla_I|_{\nabla_I}, \nabla_{II} \). In fact, for the existence of \( \nabla_I|_{\nabla_I}, \nabla_{II} \), proceed as for the classical case. The extensibility of \( \nabla_I|_{\nabla_I}, \nabla_{II} \) follows from its explicit, local expression where only extendable differentiable forms are involved (cp. [Ko], Ch. II, Sec. 2, p. 38). \( \nabla_I|_{\nabla_I}, \nabla_{II} \) is the extendable Bott difference form with respect to \( \nabla_I|_{\nabla_I}, \nabla_{II} \).

The extendable cohomology class \( c^q_{ext}(E) \in H^2q_{ext}(X) \) defined by

\[
(5.2) \quad c^q_{ext}(E) = [c^q_{ext}(\nabla)]
\]

is the \( q^{th} \) extendable Chern class of \( E \). Set \( c^0_{ext}(E) = 1 \). The class \( c_{ext}(E) \in \bigoplus_{q=1}^n H^2q_{ext}(X) \) defined by \( c_{ext}(E) = \sum_{q=0}^n c^q_{ext}(E) \) is the total extendable Chern class of \( E \).

**Remark 16.** Let \( X \) be a complex analytic variety of complex dimension \( n \), \( E \to X \) a differentiable (holomorphic) complex vector bundle of rank \( e \) and \( C = \{A_l\}_{l \in L} \) an atlas of \( X \) associated with \( \Lambda^p T^\nabla X' \otimes E|_X \) for any \( p \in \mathbb{N} \). Fix \( l \in L \) and \( q \in \{0, \ldots, n\} \).

1. Let \( \nabla \) be an extendable linear connection for \( E \) and \( ^l\nabla \) a linear connection for \( E_l \) such that \( \nabla|_{A_l} = (F_l|_{A_l})^*(^l\nabla) \). Then \( c^q_{ext}(\nabla) \in \Gamma_{ext}(\Lambda^{2q}T^\nabla X') \) is completely extended on \( A_l \) by the \( q^{th} \) differentiable Chern form \( c^q_{dif}(^l\nabla) \in \Gamma(\Lambda^{2q}T^\nabla U_l) \).

2. Let \( \nabla_I, \nabla_{II} \) be extendable linear connections for \( E \) and \( ^l\nabla_I, ^l\nabla_{II} \) linear connections for \( E_l \) such that \( \nabla_I|_{A_l} = (F_l|_{A_l})^*(^l\nabla_I) \) and \( \nabla_{II}|_{A_l} = (F_l|_{A_l})^*(^l\nabla_{II}) \). Then \( c^q_{ext}(\nabla_I, \nabla_{II}) \in \Gamma_{ext}(\Lambda^{2q-1}T^\nabla X') \) is completely extended on \( A_l \) by the Bott difference form \( c^q_{dif}(^l\nabla_I, ^l\nabla_{II}) \in \Gamma(\Lambda^{2q-1}T^\nabla U_l) \).

The following observation is a consequence of Theorem [I] and Remark [10].
Remark 17. Let \( X \) be an abstract \( n \)-dimensional complex analytic variety, \( E \to X \) a differentiable (holomorphic) complex vector bundle of rank \( e \) and \( V = \{ V_0, V_1 \} \) an open covering of \( X \). Let \( \nabla_0 \) and \( \nabla_1 \) be extendable linear connections for \( E|_{V_0} \to V_0 \) and, respectively, \( E|_{V_1} \to V_1 \). Fix \( q \in \{ 0, \ldots, n \} \) with \( q \leq e \) and consider the class

\[
\hat{c}^q_\text{ext}(E) = P^{2q}_\text{ext}(c^q_\text{ext}(E)) \in \tilde{H}^{2q}_{\text{ext}}(X, \mathcal{V}).
\]

(see Section 3.3). The definition of \( P^{2q}_\text{ext} : \tilde{H}^{2q}_s(X) \to \tilde{H}^{2q}_{\text{ext}}(X, \mathcal{V}) \) implies that \( \hat{c}^q_\text{ext}(E) \in \tilde{H}^{2q}_{\text{ext}}(X, \mathcal{V}) \) is represented by the cocycle

\[
\hat{c}^q_\text{ext}(\nabla_*) = (c^q_\text{ext}(\nabla_0), c^q_\text{ext}(\nabla_1), c^q_\text{ext}(\nabla_0, \nabla_1)).
\]

The following remark is in order.

Remark 18. The construction of extendable Chern classes can be extended to more general cases we do not deal with in this paper. As an example, let \( X \) be an abstract finite dimensional complex analytic variety and \( E' \to X' \) an \( \mathcal{S}_{E'} \)-extendable differentiable (holomorphic) complex vector bundle over the regular part of \( X \). If the coherent sheaf \( \mathcal{S}_{E'} \) associated with \( E' \) admits a finite resolution

\[
0 \to \mathcal{E}_m \to \ldots \to \mathcal{E}_0 \to \mathcal{S}_{E'} \to 0
\]

by locally free sheaves \( \mathcal{E}_0, \ldots, \mathcal{E}_m \) of \( \mathcal{C}^{\infty} \)-modules over \( X \), then we can define the total extendable Chern class \( c_{\text{ext}}(E') \) of \( E' \) by setting \( c_{\text{ext}}(E') = \prod_{i \in \{0, \ldots, m\}} c_{\text{ext}}(E_i)^{(i-1)^r} \), with \( E_i \) the vector bundle associated with \( \mathcal{E}_i \).

Now, we consider the relative case.

Let \( X \) be a complex analytic variety of complex dimension \( n \), \( Z \) the closure of a non empty open set that is also a polyhedron of \( X \) and \( E \to X \) be a differentiable (holomorphic) complex vector bundle of rank \( e \). Let \( T \) be a triangulation of \( X \) compatible with \( \text{Sing}(X) \) and \( Z \) and \( q \in \{ 0, \ldots, n \} \). Set \( r = e - q + 1 \) and consider the \((2q-1)\)-skeleton \( \text{Skel}^{2q-1}(X, T) \) of \((X, T)\). Let \( Z_{2q-1} \) be a polyhedron of \( X \) that is the closure of an open set and that is such that \( Z_{2q-1} \supset \text{Skel}^{2q-1}(X, T) \) and \( Z_{2q-1} \sim \text{Skel}^{2q-1}(X, T) \). Set \( Z\ast = Z_{2q-1} \cup Z \) and let \( V_Z \) be an open neighborhood of \( Z\ast \) such that \( V_Z \sim Z\ast \).

Let \( s^{(r)} \) be a differentiable \( r \)-section of \( E \) whose restriction at \( V_Z \) is an \( r \)-frame. If \( \nabla \) is an extendable linear connection for \( E \) that is \( s^{(r)} \)-trivial on \( V_Z \), then the \( q \)-th extendable Chern form \( c^q_\text{ext}(\nabla) \) vanishes on \( V_Z \). Namely, \( c^q_\text{ext}(\nabla) \in \Gamma_{\text{ext}}((\Lambda^2 q T^\ast C^\ast X')Z) \). To indicate this fact, we write \( c^q_\text{ext}(\nabla, s^{(r)}) \) instead of \( c^q_\text{ext}(\nabla) \). So, \( [c^q_\text{ext}(\nabla, s^{(r)})] \in H^{2q}_{\text{ext}}(X)Z \). Moreover, if \( \nabla' \) is another extendable linear connection for \( E \) that is \( s^{(r)} \)-trivial on \( V_Z \), then \( [c^q_\text{ext}(\nabla', s^{(r)})] = [c^q_\text{ext}(\nabla, s^{(r)})] \) as classes in \( H^{2q}_{\text{ext}}(X)Z \). So, \( [c^q_\text{ext}(\nabla, s^{(r)})] \in H^{2q}_{\text{ext}}(X)Z \) does not depend on the choice of the extendable linear connection \( \nabla \) that is \( s^{(r)} \)-trivial on \( V_Z \). To prove this, proceed as in [Su 1], Ch. III, Sec. 3. The class \( c^q_{\text{ext}}(E, s^{(r)}) = [c^q_\text{ext}(\nabla, s^{(r)})] \) is the localization outside \( Z \) with respect to \( s^{(r)} \) of \( c^q_\text{ext}(E) \). Note that, as a relative class, \( [c^q_\text{ext}(\nabla, s^{(r)})] \) depends on the \( r \)-frame \( s^{(r)} \) (cp. [Su 1], Ch. III, Sec. 3).

Let \( X, Z, E \to X, q, r, T, Z_{2q-1}, Z\ast, V_Z \) and \( s^{(r)} \) be as above. Set \( V_0 = V_Z \) and let \( V_1 \) be an open set in \( X \) such that \( V_1 \cap Z\ast = \emptyset \) and \( V_0 \cup V_1 = X \). Then the open covering \( V = \{ V_0, V_1 \} \) of \( X \) is adapted to \( Z\ast \).

Let \( \nabla_0 \) be an \( s^{(r)} \)-trivial extendable linear connection for \( E|_{V_0} \to V_0 \) and \( \nabla_1 \) any extendable linear connection for \( E|_{V_1} \to V_1 \). By the \( s^{(r)} \)-triviality of \( \nabla_0 \), we get \( c^q_\text{ext}(\nabla_0) = 0 \). So, \( c^q_\text{ext}(\nabla_*) = (c^q_\text{ext}(\nabla_0), c^q_\text{ext}(\nabla_1), c^q_\text{ext}(\nabla_0, \nabla_1)) \) lies in
$K^q(X, \mathcal{V}, V_0)$ (cp. Subsection [3.3]). In order to indicate this, write $c^q_{\text{ext}}(\nabla^e, s^{(r)})$ instead of $c^q_{\text{ext}}(\nabla^e)$. Then $[c^q_{\text{ext}}(\nabla^e, s^{(r)})] \in \check{H}^{2q}_{\text{ext}}(X, \mathcal{V}, V_0)$. Moreover, it can be proved that $[c^q_{\text{ext}}(\nabla^e, s^{(r)})]$ does not depend on both the choices of the $s^{(r)}$-trivial connection $\nabla$ and of the connection $\nabla_1$ (cp. [Mu], Ch. III, Sec. 3).

The class $c^q_{\text{ext}}(E, s^{(r)}) = [c^q_{\text{ext}}(\nabla^e, s^{(r)})] \in \check{H}^{2q}_{\text{ext}}(X, \mathcal{V}, V_0)$ is called the localization outside $Z$ with respect to $s^{(r)}$ of $c^q_{\text{ext}}(E)$. As a reason for this name, note that $c^q_{\text{ext}}(E, s^{(r)})$ is the image of $c^q_{\text{ext}}(E, s^{(r)})$ via $P_{2q}^*: H^{2q}_{\text{ext}}(X, Z) \to \check{H}^{2q}_{\text{ext}}(X, \mathcal{V}, V_0)$. Note that, as a relative class, $[c^q_{\text{ext}}(\nabla^e, s^{(r)})]$ depends on the frame $s^{(r)}$.

Next, we wish to study the following case.

Let $X$ be a complex analytic variety of complex dimension $n$ and $E \to X$ be a holomorphic complex vector bundle of rank $e$. Take $q \in \{0, ..., n\}$ and set $r = e - q + 1$. Let be $s^{(r)}$ a holomorphic $r$-section of $E$ and denote by $S$ its singular locus. Since $s^{(r)}$ is holomorphic, $S$ is a closed complex analytic subvariety of $X$. Furthermore, $S$ is a polyhedron of $X$.

Now, suppose that $S$ is compact and let $U_S$ be an open neighborhood of $S$ in $X$ enjoying (1) and (2) of Lemma 5. We can assume without loss of generality that the closure $\overline{U}_S$ of $U_S$ is also compact. Set $Z = X \setminus U_S$ and note that the restriction $s^{(r)}|_{X \setminus U_S}$ of $s^{(r)}$ to $X \setminus U_S$ is an $r$-frame, because $X \setminus U_S \subseteq X \setminus S$. In the above situation, both $c^q_{\text{ext}}(E, s^{(r)}) \in H^{2q}_{\text{ext}}(X)_{Z}$ and $c^q_{\text{ext}}(E, s^{(r)}) \in \check{H}^{2q}_{\text{ext}}(X, \mathcal{V}, V_Z)$, the localizations outside $Z$ with respect to $s^{(r)}$ of $c^q_{\text{ext}}(E)$, are called localization at $S$ with respect to $s^{(r)}$ of $c^q_{\text{ext}}(E)$.

5.2. The homomorphisms $P^*_k$ and $A^*_S k$. Let $X$ be a compact irreducible complex analytic variety of complex dimension $n$ and $[X]$ the fundamental class of $X$ (cp. Remark [12]). For each $k \in \{0, ..., 2n\}$ the cap product with $[X]$ induces a homomorphism $P^*_k : H^k(X) \to H_{2n-k}(X)$ called the $k$-Poincaré homomorphism. Namely, $P^*_k([c]) = [c] \cap [X]$ for any $[c] \in H^k(X)$.

Let $X$ be a compact irreducible complex analytic variety of complex dimension $n$. If $T$ is a finite triangulation of $X$ compatible with $\text{Sing}(X)$ and if an orientation of $(X, T)$ is already given, then it can be proved that $P^*_k$ is induced by

$$P_k : C^*_k(X) \cap c \mapsto C^*_{2n-k}(X) \cap \Delta = \sum_{\Delta \in T'_{2n-k}} c^*(\hat{\Delta}) \Delta,$$

where $T'$ is the first barycentric subdivision of $T$, $T^*$ is the dual block decomposition of $(X, T)$ and $\Delta$ denotes the dual block of $\Delta \in T_{2n-k}$ (cp. [Mu], Ch. 8, Sec. 64).

Let $X$ be an irreducible complex analytic variety of complex dimension $n$, $S$ a closed compact analytic subvariety of $X$ and $T$ a finite triangulation of $X$ compatible with $\text{Sing}(X)$ and $S$. If $c \in C^*_k(X, X \setminus S)$ is a $k$-cochain relative to $(X, X \setminus S)$, then in a sum as in (5.5) only appear the simplices in $S$. Namely,

$$\sum_{\Delta \in T_{2n-k}, \Delta \subseteq S} c^*(\hat{\Delta}) \Delta$$

Such a finite sum induces a homomorphism $A^*_S k : H^k(X, X \setminus S) \to H_{2n-k}(S)$ called the $(S, k)$-Alexander-Lefschetz homomorphism.

The following proposition is a direct consequence of the constructions of $P_k$ and $A^*_S k$ (cp. [Mu]).

Proposition 8. Let $X$ be a compact irreducible complex analytic variety of dimension $n$ and $S$ a compact analytic subvariety of $X$. Let $k \in \{0, ..., 2n\}$. Then the
following diagram is commutative

\[
\begin{array}{ccc}
H^k(X, X \setminus S) & \xrightarrow{j_*} & H^k(X) \\
\downarrow \alpha^*_{S,k} & & \downarrow p^*_k \\
H_{2n-k}(S) & \xrightarrow{i_*} & H_{2n-k}(X)
\end{array}
\]

5.3. Topological Chern classes. We briefly recall the construction of topological Chern classes for continuous complex vector bundles (cp. [St]).

Take \( r \in \mathbb{N} \), let \( r \in \mathbb{N} \) be such that \( r \leq e \) and denote by \( W_r(\mathbb{C}^r) \) the complex Stiefel manifold of \( r \)-frames. Recall that \( W_r(\mathbb{C}^r) \) is \((2e-2r+1)\)-connected and that \( \pi_{2e-2r+1}(W_r(\mathbb{C}^r)) \simeq \mathbb{Z} \) has a canonical generator \( \varsigma \) (cfr. [St], 25.7).

**Definition 8.** Let \( X \) be a finite dimensional complex analytic variety, \( Y \subseteq X \) a subset of \( X \) and \( E \to X \) a continuous complex vector bundle of rank \( e \).

An \( r \)-section of \( E \to X \) on \( Y \) is an ordered family \( s^{(r)} = (s_1, \ldots, s_r) \) of \( r \) continuous sections of \( E \to X \) over \( Y \). A singular point of \( s^{(r)} = (s_1, \ldots, s_r) \) is a point where the sections \( s_1, \ldots, s_r \) fail to be linearly independent over \( \mathbb{C} \). An \( r \)-frame (frame) of \( E \to X \) on \( Y \) is a non-singular \( r \)-section (\( e \)-section) over \( Y \).

Let \( X \) be a finite dimensional complex analytic variety, \( E \to X \) a continuous vector bundle of rank \( e \) and \( W_r(E) \) the continuous bundle of complex \( r \)-frames of \( E \) over \( X \). Recall that \( W_r(E) \) is a bundle associated with \( E \) whose fibre \( W_r(E_x) \) over a point \( x \in X \) is homeomorphic to \( W_r(\mathbb{C}^e) \).

Set \( q = e - r + 1 \) and note that \( 2e - 2r + 1 = 2q - 1 \). The primary obstruction to constructing a section of \( W_r(E) \) is the \( q^{\text{th}} \) topological Chern class \( c^q(E) \) of \( E \). A priori this class is defined in the cohomology with local coefficients system formed by \( \pi_{2q-1}(W_r(E_x)) \). But, since the group of transformations of \( E \to X \) is the unitary group, this is in fact trivial (cp. [St], 30.4). Then the obstruction class \( c^q(E) \) lies in the integral cohomology group \( H^{2q}(X) \).

Next, we introduce the notion of index of an \( r \)-section at its singular point.

Let \( U \) be an open set in \( X \) such that \( E|_U \psi \) is trivial and \( \kappa : U \times W_r(\mathbb{C}^e) \to W_r(\mathbb{C}^e) \) the projection on the second factor. Let \( B^{2q} \cup U \) be a suitably oriented ball of dimension \( 2q \) in \( U \), \( p \in B^{2q} \) a point in the interior of \( B^{2q} \) and \( S^{2q-1} \) the boundary of \( B^{2q} \). As a note, \( S^{2q-1} \) is an oriented \((2q-1)\)-sphere. If \( s^{(r)} \) is an \( r \)-section of \( E \) on a neighborhood of \( B^{2q} \) in \( U \) and \( p \) is an isolated singularity of \( s^{(r)} \), then the map

\[
\psi = \psi_p : S^{2q-1} \xrightarrow{s^{(r)}} W_r(E)|_U \simeq U \times W_r(\mathbb{C}^e) \xrightarrow{\kappa} W_r(\mathbb{C}^e)
\]

given by \( \psi = \kappa \circ s^{(r)} \) is well defined.

By the very definition of homotopy groups, the map \( \psi \) determines an element \( \psi \) in \( \pi_{2e-2r+1}(W_r(\mathbb{C}^e)) \). Furthermore, since \( \pi_{2e-2r+1}(W_r(\mathbb{C}^e)) \simeq \mathbb{Z} \), there exists an integer \( I(E|_{B^{2q}}, s^{(r)}, p) \in \mathbb{Z} \) such that

\[
(5.7) \quad \psi = I(E|_{B^{2q}}, s^{(r)}, p) \varsigma.
\]

Such an integer is the index of the \( r \)-section \( s^{(r)} \) of \( E \to X \) at the point \( p \).

Let \( X \) be a finite dimensional complex analytic variety, \( E \to X \) a continuous complex vector bundle and \( \mathbb{T} \) a triangulation of \( X \). We try to construct an \( r \)-frame of \( E \), that is a section of \( W_r(E) \), on each skeleton of \( \mathbb{T} \) inductively from the skeleton of dimension 0.
First of all, it is always possible to construct a section $s^{(r)}$ of $W_r(E)$ over $\text{Skel}^0(X, \mathbb{T})$.

Next, if $\Delta_h \in \mathbb{T}_h$ is contained in an open set $U$ of $X$ trivializing $W_r(E)$ and an $r$-section $s^{(r)}$ is given on $\partial \Delta_h$, then there is a well defined map $\kappa \circ s^{(r)} : S^{2h-1} \rightarrow W_r(E)|_U \cong U \times W_r(\mathbb{C}^e) \rightarrow W_r(\mathbb{C}^e)$ that determines an element in $\pi_{h-1}(W_r(\mathbb{C}^e))$, because $\partial \Delta_h$ is homeomorphic to a $(2h - 1)$-sphere $S^{2h-1}$. Then, by the very definition of homotopy groups, if $h \leq 2q - 1 = 2e - 2r + 1$, the section $s^{(r)}$ can be extended to a continuous $r$-section defined on the interior of $\Delta_h$, because for any $h \in \mathbb{N}$ such that $h \leq 2e - 2r + 1$ it results $\pi_{h-1}(W_r(\mathbb{C}^e)) = 0$.

Iterating this process, we reach an obstruction for $h = 2q$. Namely, the $r$-frame $s^{(r)}$ can be extended to an $r$-section on a suitably small $\Delta_{2q} \in \mathbb{T}_{2q}$ with at most a singularity at an interior point $p$ of $\Delta$ and the measure of such an obstruction is given by the index $I(E|_{\Delta}, s^{(r)}, p)$ of $s^{(r)}$ at $p$. So, up to choose a $p_\Delta$ in the interior of any $\Delta \in \mathbb{T}_{2q}$, we may define a cochain $\Gamma_{\Delta}^{2q} \in C_{2q}^{\mathbb{T}}(X)$ associated with $s^{(r)}$. Namely, $\Gamma_{\Delta}^{2q}$ is the cochain whose value at $\Delta \in \mathbb{T}_{2q}$ is

$$
(5.8) \quad \Gamma_{\Delta}^{2q} = I(E|_{\Delta}, s^{(r)}, p_\Delta)
$$

and then it is extended by linearity. It can be proved that $\Gamma_{\Delta}^{2q}$ is a cocycle and that the cohomology class represented by $\Gamma_{\Delta}^{2q}$ is independent of all the choices made in the definition (cp. [St]).

The class identified by $\Gamma_{\Delta}^{2q}$ is denoted by $c_{\text{top}}^q(E) \in H^{2q}(X)$ and it is called the $q^{th}$ topological Chern class of $E$ to $X$. The class $c_{\text{top}}^q(E) = \sum_{q=0}^{e} c_{\text{top}}^q(E)$, with $c_{\text{top}}^0(E) = 1$, is the total topological Chern class $c_{\text{top}}(E)$ of $E$ to $X$. As a note, $c_{\text{top}}(E)$ is an invertible element of the cohomology ring $\oplus_{r \in \mathbb{N}} H^r(X)$.

**Remark 19.** Let $X$ be a compact irreducible complex analytic variety of complex dimension $n$, $\mathbb{T}$ a finite triangulation of $X$ and $q \in \{0, ..., n\}$. Then, by [5.3], the class $P_{2q}^q(c_{\text{top}}^q(E)) \in H_{2n-2q}(X)$ is represented by the cycle

$$
(5.9) \quad \sum_{\Delta \in \mathbb{T}_{2n-2q}} I(E|_{\Delta}, s^{(r)}, p_\Delta) \Delta.
$$

We need observations about the localization of topological Chern classes.

Let $X$ be a complex analytic variety of complex dimension $n$, $Z$ a polyhedron of $X$ that is the closure of a non empty open set and $E \rightarrow X$ a continuous complex vector bundle of rank $e$. Let $\mathbb{T}$ be a triangulation of $X$ compatible with $Z$ and $\text{Sing}(X)$ and consider $Z$ as a closed subcomplex of $(X, \mathbb{T})$.

Let $q \in \{0, ..., n\}$ and set $r = e - q + 1$. If $s^{(r)}$ is a continuous $r$-section of $E \rightarrow X$ whose restriction at $Z$ is an $r$-frame, then $I(E|_{\Delta}, s^{(r)}, p_\Delta) = 0$ for any $\Delta \in \mathbb{T}_{2q}$ contained in $Z$. So, in this case, the cocycle $\Gamma_{\mathbb{T}}^{2q} \in C_{2q}^{\mathbb{T}}(X)$ representing $c_{\text{top}}^q(E)$ lies in $C_{2q}^{\mathbb{T}}(X, Z)$ and it represents a class $c_{\text{top}}^q(E, s^{(r)}) \in H^{2q}(X, Z)$. The class $c_{\text{top}}^q(E, s^{(r)})$ is the localization outside $Z$ with respect to $s^{(r)}$ of $c_{\text{top}}^q(E)$. Indeed, the image of $c_{\text{top}}^q(E, s^{(r)})$ via the map $j^* : H^{2q}(X, Z) \rightarrow H^{2q}(X)$ induced by the inclusion $j : (X, \emptyset) \rightarrow (X, Z)$ is $c_{\text{top}}^q(E)$. Note that, as a relative class, $c_{\text{top}}^q(E, s^{(r)})$ depends on the frame $s^{(r)}$ (cp. [St]).

Next, we study the following case (see the end of Subsection 5.1).

Let $X$ be a complex analytic variety of complex dimension $n$ and $E \rightarrow X$ be a continuous complex vector bundle of rank $e$. Take $q \in \{0, ..., n\}$ and set $r = e - q + 1$. Let be $s^{(r)}$ a continuous $r$-section of $E$ and denote by $S$ its singular locus. Suppose
that $S$ is a closed complex analytic subvariety of $X$ and note that, in this case, $S$ is a polyhedron of $X$.

Now, suppose that $S$ is compact. Let $U_S$ be an open neighborhood of $S$ enjoying (1) and (2) of Remark 5 and such that its closure $\overline{U_S}$ is also compact. Set $Z = X \setminus U_S$ and note that the restriction of $s^{(r)}$ at $Z$ is an $r$-frame, because $Z \subset X \setminus S$. In the above situation, $c^{q}_{\text{top}}(E, s^{(r)}) \in H^{2q}(X, Z)$, the localization outside $Z$ with respect to $s^{(r)}$ of $c^{q}_{\text{top}}(E)$, is called localization at $S$ with respect to $s^{(r)}$ of $c^{q}_{\text{top}}(E)$.

Furthermore, since $(X, Z) \sim (X, X \setminus S)$, $c^{q}_{\text{top}}(E, s^{(r)})$ corresponds to a class in $H^{2q}(X, X \setminus S)$, still denoted by $c^{q}_{\text{top}}(E, s^{(r)})$.

**Definition 9.** Let $X$ be an irreducible complex analytic variety of complex dimension $n$ and $E \to X$ a continuous complex vector bundle of rank $e$. Take $q \in \{0, \ldots, n\}$ with $q \leq e$ and set $r = e - q + 1$. Let $s^{(r)}$ be a continuous $r$-section of $E \to X$ and $S$ its singular locus. Suppose that $S$ is a closed compact complex analytic subvariety of $X$. The topological residue of $s^{(r)}$ for $c^{q}_{\text{top}}(E)$ at $S$ is the homology class $\text{TopRes}_{c^{q}_{\text{top}}}(E, s^{(r)}, S) \in H_{2n-2q}(S)$ defined by

\[
(5.10) \quad \text{TopRes}_{c^{q}_{\text{top}}}(E, s^{(r)}, S) = A^{*}_{S,2q}(c^{q}_{\text{top}}(E, s^{(r)}))
\]

We have the following remark (cp. Proposition 8).

**Remark 20.** Let $X$ be a compact irreducible complex analytic variety of complex dimension $n$ and $E \to X$ a continuous complex vector bundle of rank $e$. Take $q \in \{0, \ldots, n\}$ with $q \leq e$ and set $r = e - q + 1$. Let $s^{(r)}$ be a continuous $r$-section of $E \to X$ and $S$ its singular locus. Suppose that $S$ is a closed compact complex analytic subvariety of $X$. Then, by Proposition 8

\[
(5.11) \quad i_{*}(\text{TopRes}_{c^{q}_{\text{top}}}(E, s^{(r)}, S)) = P^{*}_{2q}(c^{q}_{\text{top}}(E))
\]

By (5.9) and (5.10), $\text{TopRes}_{c^{q}_{\text{top}}}(E, s^{(r)}, S)$ is represented by the cycle

\[
(5.12) \quad \sum_{\Delta \in \mathbb{T}_{2n-2q}} I(E|\Delta, s^{(r)}, S) \Delta
\]

If $S$ has a finite number of connected components $\{S_{\nu}\}_{\nu \in \{1, \ldots, N\}}$, then, correspondingly to the decomposition $H_{2n-2q}(S) = \oplus_{\nu \in \{1, \ldots, N\}} H_{2n-2q}(S_{\nu})$, for any $\nu \in \{1, \ldots, N\}$ we get the residue $\text{TopRes}_{c^{q}_{\text{top}}}(E, s^{(r)}, S_{\nu})$.

5.4. **Localization of differentiable Chern classes.** We consider the localization of differentiable Chern classes of a smooth complex vector bundle defined over a manifold, assuming that the construction of such classes is already familiar to the reader. For the necessary background of differential geometry, refer to [Su 1].

Let $M$ be an oriented differentiable manifold of real dimension $m$. Then $M$ is a triangulable space. From now on, suppose that a triangulation $\mathcal{T}$ of $M$ is already given and still denote by $M$ the simplicial complex associated with $M$ and $\mathcal{T}$. As a matter of notations, let $S$ be a subcomplex of $M$ and $k \in \{0, \ldots, m\}$. The $k^{th}$ de Rham cohomology group of $M$ and the $k^{th}$ de Rham cohomology group relative to $(M, M \setminus S)$ are denoted by $H^{k}_{dR}(M)$ and, respectively, $H^{k}_{dR}(M, M \setminus S)$. Recall that $H^{2q}_{dR}(M) \simeq H^{2q}(M)$ and $H^{2q}_{dR}(M, M \setminus S) \simeq H^{2q}(M, M \setminus S)$.

Let $E \to M$ be a differentiable complex vector bundle of rank $e$ over $M$ and for each $q \in \{0, \ldots, [\frac{m}{2}]\}$ denote by $c^{q}_{\text{dif}}(E) \in H^{2q}_{dR}(M)$ the $q^{th}$ differentiable Chern class of $E$. 
Next, take \( r \in \{0, \ldots, e\} \) and let \( q \in \{0, \ldots, \left[ \frac{m}{2} \right]\} \) be such that \( q \geq e - r + 1 \). Let \( S \) be a subcomplex of \( M \). If \( s^{(r)} \) is a differentiable \( r \)-section of \( E \to M \) whose restriction at \( M \setminus S \) is an \( r \)-frame, then there exists a cohomology class \( c^q_{\text{dif}}(E, s^{(r)}) \in H^2_{\text{dif}}(M, M \setminus S) \) called the localization at \( S \) with respect to \( s^{(r)} \) of \( c^q_{\text{dif}}(E) \).

To prove this, set \( D_0 = M \setminus S \) and let \( D_1 \) be an open neighborhood of \( S \) such that \( D_1 \sim S \). Then \( D = \{D_0, D_1\} \) is an open covering of \( M \). Let \( \nabla_0, \nabla_1 \) be differentiable linear connections for \( E|_{D_0} \to D_0 \) and, respectively, \( E|_{D_1} \to D_1 \) and consider the Čech-de Rham cocycle \( c^q_{\text{dif}}(\nabla_*) = (c^q_{\text{dif}}(\nabla_0), c^q_{\text{dif}}(\nabla_1), c^q_{\text{dif}}(\nabla_0 \nabla_1)) \) representing \( c^q_{\text{dif}}(E) \). Now, if \( \nabla_0 \) is \( s^{(r)} \)-trivial, then \( c^q_{\text{dif}}(\nabla_0) = 0 \) and \( c^q_{\text{dif}}(\nabla_*) \) represents a cohomology class \([c^q_{\text{dif}}(\nabla_*, s^{(r)}))] \in H^2_{\text{dif}}(M, M \setminus S) \). It can be proved that \([c^q_{\text{dif}}(\nabla_*, s^{(r)}))] \) does not depend on the choices of both \( \nabla_1 \) and \( \nabla_0 \) that is \( s^{(r)} \)-trivial (cp. [Su 1], Ch. III, Sec. 3). The class \( c^q_{\text{dif}}(E, s^{(r)}) \in H^2_{\text{dif}}(M, M \setminus S) \) is defined by \( c^q_{\text{dif}}(E, s^{(r)}) = [c^q_{\text{dif}}(\nabla_*, s^{(r)}))] \). As a note, \( j^*(c^q_{\text{dif}}(E, s^{(r)})) = c^q_{\text{dif}}(E) \), with \( j^*: H^2_{\text{dif}}(M, M \setminus S) \to H^2_{\text{dif}}(M) \) induced by the inclusion \( j : (M, \emptyset) \to (M, M \setminus S) \). Notice that, as a relative class, \( c^q_{\text{dif}}(E, s^{(r)}) \) depends on the frame \( s^{(r)} \) (cp. [Su 1], Ch. III, Sec. 3).

Now, suppose that \( S \) is also compact. Let \( \{S_\nu\}_{\nu \in \{1, \ldots, N\}} \) be the finite set of connected components of \( S \) and consider the \((S, 2q)\)-Alexander-Lefschetz duality \( A^*_S, 2q : H^{2q}(M, M \setminus S) \xrightarrow{\sim} \oplus_{\nu \in \{1, \ldots, N\}} H_{m-2q}(S_\nu) \). For each \( \nu \in \{1, \ldots, N\} \) the differentiable geometric residue of \( s^{(r)} \) for \( c^q_{\text{dif}} \) at \( S_\nu \) is the homology class \( \text{DiffRes}_{c^q_{\text{dif}}}(s^{(r)}, E, S_\nu) \) defined by

\[
\text{DiffRes}_{c^q_{\text{dif}}}(s^{(r)}, E, S_\nu) = A^*_S, 2q(c^q_{\text{dif}}(E, s^{(r)})).
\]

In order to give a description of such a residue, let \( D_0, D_1 \) be as above. For any \( \nu \in \{1, \ldots, N\} \) let \( D_\nu \) be an open neighborhood of \( S_\nu \) in \( D_1 \) and \( R_\nu \) an \( m \)-dimensional manifold with differentiable boundary such that \( S_\nu \subset R_\nu \subset R_\nu \subset D_\nu \). Assume that for any \( \nu_1, \nu_2 \in \{1, \ldots, N\} \) such that \( \nu_1 \neq \nu_2 \) it holds \( D_{\nu_1} \cap D_{\nu_2} = \emptyset \). Then the residue \( \text{DiffRes}_{c^q_{\text{dif}}}(s^{(r)}, E, S_\nu) \) is represented by an \((m - 2q)\)-cycle \( C_\nu \) in \( S_\nu \) such that for any closed differentiable \((m - 2q)\)-form \( \tau \in \Gamma(\Lambda^m T^* \nabla_\nu \nabla_\nu) \) it holds

\[
\int_{C_\nu} \tau = \int_{R_\nu} c^m_{\text{dif}}(\nabla_1) \wedge \tau + \int_{-\partial R_\nu} c^m_{\text{dif}}(\nabla_0, \nabla_1) \wedge \tau)
\]

If \( 2q = m \), then the differential geometric residue can be identified with the complex number given by

\[
\text{DiffRes}_{c^m_{\text{dif}}}(s^{(r)}, E, S_\nu) = \int_{R_\nu} c^m_{\text{dif}}(\nabla_1) + \int_{-\partial R_\nu} c^m_{\text{dif}}(\nabla_0, \nabla_1)
\]

Next, let \( M \) and \( E \to M \) be as above. We give a topological interpretation of the residue in case the compact subset \( S \) of \( M \) is just a point \( p \). Namely, \( S = \{p\} \). In order to proceed, we need some definitions. Let \( z_1, \ldots, z_m \) denote the complex coordinates on \( \mathbb{C}^m \). Write \( \Theta(z) = dz_1 \wedge \ldots \wedge dz_m \) and for each \( h \in \{1, \ldots, m\} \) set \( \Theta_h(z) = (-1)^h z_h (dz_1 \wedge \ldots \wedge dz_h \wedge \ldots \wedge dz_m) \). The Bochner-Martinelli kernel on \( \mathbb{C}^m \) is \((2m - 1)\)-form \( \beta_m : \mathbb{C}^m \to \Lambda^{2m-1} T^* \mathbb{C}^m \) defined by \( \beta_m(z) = \frac{(m-1)!}{(2\pi i)^m} \sum_{h=1}^m \Theta_h(z) \wedge \Theta(z) \).
Let $B^{2m} \subset \mathbb{R}^{2m} \simeq \mathbb{C}^m$ be a ball of real dimension $2m$ and $E \to D$ a differentiable complex vector bundle of rank $m$ defined on a open neighborhood $D$ of $B^{2m}$. Suppose there is a non vanishing differentiable section $s$ of $E \to D$ defined on an open neighborhood of the boundary $S^{2m-1}$ of $B^{2m}$. Suppose also that the $s$ is defined on the whole of $D$ with at most an isolated singularity at a point $p$ in the interior of $B^{2m}$. Indeed, this can be done, by dimensional reasons. Then, on one hand, we have the index $I(E|_{B^{2m}}, s, p) \in \mathbb{Z}$ (cp. (5.17)) and, on the other hand, we have $\operatorname{DifRes}^m_{\partial D^1}(s, E, p) \in H_0(p) \simeq \mathbb{C}$, the differential geometric residue of $s$ for $c^m_{\partial D^1}$ at $\{p\}$.

We claim that they are in fact the same number. To prove this, suppose that $E \to D$ is trivial and let $e^{(m)} = (e_1, \ldots, e_m)$ be a frame of $E \to D$ on $D$. Then $s = \sum_{h=1}^m f_h e_h$, with $f_h : D \to \mathbb{C}$ a differentiable function for any $h \in \{1, \ldots, m\}$. Then we have a differentiable map $f = (f_1, \ldots, f_m) : D \to \mathbb{C}^m$ that takes the value $(0, \ldots, 0) \in \mathbb{C}^m$ only at $p \in B^{2m}$. In the above situation it results $\operatorname{DifRes}^m_{\partial D^1}(s, E, p) = \int_{\partial B^{2m}} f^*(\beta_m)$. So, in particular

$$
(5.16) \quad \operatorname{DifRes}^m_{\partial D^1} = I(E|_{B^{2m}}, s, p)
$$

(cp. [Su 1], Ch. III, Sec. 4).

Now, we consider a more general case. Let $B^{2m} \subset \mathbb{R}^{2m} \simeq \mathbb{C}^m$ be a ball of real dimension $2m$ and $D$ a open neighborhood of $B^{2m}$. Denote by $S^{2m-1}$ the boundary of $B^{2m}$ and let $D'$ be an open neighborhood of $S^{2m-1}$. Let $E \to D$ be a differentiable complex vector bundle of rank $e$. Set $r = e - m + 1$ and let $s^{(r)} = (s_1, \ldots, s_r)$ be a differentiable $r$-section of $E \to D$ with at most an isolated singularity at a point $p$ in the interior of $B^{2m}$ and such that its restriction at $D'$ is an $r$-frame. In fact, as before, this can be done for dimensional reasons. Let $t \in \mathbb{N}$ be such that $t < r$, write $s^{(t)} = (s_1, \ldots, s_t)$ and set $s^{(r-t)} = (s_{t+1}, \ldots, s_r)$. Suppose that $s^{(t)}$ is non singular on the whole of $D$. So, $s^{(t)}$ generates a trivial complex vector bundle $E^t \to D$ of rank $t$ that is a subbundle of $E \to D$. Suppose that $E \to D$ is the trivial vector bundle. Then we have the following exact sequence

$$0 \to E^t \to E \to E^*, \to 0,$$

with $E^* \to D$ a complex vector bundle of rank $e - t$. Denote by $s^{(r-t)}_{(r-t)}$ the $(r-t)$-section of $E^* \to D$ induced by $s^{(r-t)}$. Then $s^{(r-t)}_{(r-t)}$ has at most an isolated singularity at $p \in B^{2m}$. In the above situation it results

$$
(5.17) \quad \operatorname{DifRes}^m_{\partial D^1}(s^{(r)}, E, p) = \operatorname{DifRes}^m_{\partial D^1}(s^{(r-t)}, E^*, p)
$$

and

$$
(5.18) \quad \operatorname{DifRes}^m_{\partial D^1}(s^{(r)}, E, p) = I(E|_{B^{2m}}, s^{(r)}, p)
$$

Indeed, if $t = r - 1$, then (5.18) follows from (5.16) and (5.17). For a proof of (5.17), refer to [Su 1], Ch. III, Sec. 4.

5.5. **Topological and extendable Chern classes.** We describe the topological Chern classes by means of the extendable Chern classes.

**Theorem 5.** Let $X$ be a complex analytic variety of complex dimension $n$ and $Z$ either the empty set or the closure of a non empty open set that is a polyhedron of $X$. Let $E \to X$ be a differentiable (holomorphic) complex vector bundle of rank $e$. Take $q \in \{0, \ldots, n\}$ with $q \leq e$ and set $r = e - q + 1$. 
Let $C = \{A_l\}_{l \in L}$ be an atlas of $X$ associated with $\Lambda^{p+1}\mathcal{C}_*X' \otimes E|_X$, for any $p \in \mathbb{N}$. Let $T$ be a $C$-small triangulation of $X$ compatible with $Sing(X)$ and $Z$. Let $Z_{2q-1}$ be a polyhedron of $X$ which is the closure of an open set so that $Z_{2q-1} \supset Skel^{2q-1}(X, T)$ and $Z_{2q-1} \sim Skel^{2q-1}(X, T)$. Set $Z_\bullet = Z_{2q-1} \cup Z$ and let $V_0$ be an open neighborhood of $Z_\bullet$ such that $V_0 \sim Z$ and such that for any $\Delta \in T_{2q}$ for which $\Delta \not\subseteq Z_\bullet$ it holds $\Delta \cap V_0 = \Delta \setminus \{p_\Delta\}$, where $p_\Delta$ is the barycentre of $\Delta$.

Let $s^{(r)}$ be a differentiable (holomorphic) $r$-section of $E$ whose restriction at $V_0$ is an $r$-frame. Consider the localizations $c_0^{q}(E, s^{(r)})$ and $c_{ext}^{q}(E, s^{(r)})$ outside $Z \subset V_0$ with respect to $s^{(r)}$ of $c_0^{q}(E)$ and, respectively, $c_{ext}^{q}(E)$. Then

$$c_0^{q}(E, s^{(r)}) = H_{Z}^{2q}(c_{ext}^{q}(E, s^{(r)})).$$

Proof. Let $V_1$ be an open set of $X$ such that $V_1 \cap Z_\bullet = \emptyset$ and $V_0 \cup V_1 = X$. Then the open covering $\mathcal{V} = \{V_0, V_1\}$ is adapted to $Z$, because $Z \subseteq Z_\bullet$.

Let $\nabla_0$ be an $s^{(r)}|_{V_0}$-trivial extendable linear connection for $E|_{V_0}$ and $\nabla_1$ an extendable linear connection for $E|_{V_1}$. Since $\nabla_0$ is $s^{(r)}$-trivial and $r = q + 1$, it results $c_0^{q}(\nabla_0) = 0$. Then, by $Z \subseteq Z_\bullet$, the cocycle $c_{ext}^{q}(\nabla_0, \nabla_1) = (c_{ext}^{q}(\nabla_0, \nabla_1), c_{ext}^{q}(\nabla_0, \nabla_1))$ represents both $c_{ext}^{q}(E, s^{(r)}) \in H_{ext}^{2q}(X, V_0)$ and $c_{ext}^{q}(E, s^{(r)}) \in H_{ext}^{2q}(X, Z)$.

Let $\{\rho_0, \rho_1 : X \to \mathbb{R}\}$ be an extendable partition of unity subordinate to $\mathcal{V}$. Then for any $\Delta \in T_{2q}$ we have $\rho_1|_{\partial\Delta} \equiv 0$, because $\partial\Delta \subseteq Skel^{2q-1}(X, T) \subseteq Z_{2q-1} \subseteq Z_\bullet$. Let $\mathcal{R} = \{R_0, R_1\}$ be a honeycomb cell system associated with $\mathcal{V}$ and suppose that $R_0 \supset Z_\bullet$, that $R_0 \sim Z_\bullet$ and that for any $\Delta \in T_{2q}$ it holds $R_1 \cap \Delta \sim \partial\Delta$. Then the inclusions $Z_\bullet \subset R_0 \subset V_0$ are homotopic equivalences.

Let $\Gamma^{2q}_{T}(Z)$ be the cocycle associated with $s^{(r)}$ and representing the class $c_0^{q}(E) \in H^{2q}(X)$. Such a cocycle is defined by assigning to each $\Delta \in T_{2q}$ the value $\Gamma^{2q}_{T}(\Delta) = I(E|_{\Delta}, s^{(r)}, p_\Delta)$ (cp. (5.18) and (5.19)). Furthermore, $\Gamma^{2q}_{T}$ belongs to $C_{T}^{2q}(X, Z)$ and it represents the class $c_0^{q}(E, s^{(r)}) \in H^{2q}(X, Z)$. Indeed, $s^{(r)}|_{Z}$ is an $r$-frame. So, $I(E|_{\Delta}, s^{(r)}, p_\Delta) = 0$ for any $\Delta \in T_{2q}$ such that $\Delta \subseteq Z$ (cp. Subsection 4.3).

As a matter of notations, for any $C \in C_{T}^{2q}(Z)$ set $C = C + C_{T}^{2q}(Z)$. Consider the operator of integration $\eta^{2q}_{Z} : Z_{T}^{2q}(X, V, V_{0}) \to Z_{T}^{2q}(X, Z)$ (cp. (4.18) and (4.19)). We claim that for any $\Delta \in T_{2q}$ it results $\eta^{2q}_{Z}(c_{ext}^{q}(\nabla_*))(\Delta) = \Gamma^{2q}_{T}(\Delta)$.

First of all, if $Y \in C_{T}^{2q}(Z)$, then $\eta^{2q}_{Z}(c_{ext}^{q}(\nabla_*))(Y) = 0$ (cp. Subsection 4.3). So, for any $Y \in C_{T}^{2q}(Z)$ it results

$$\eta^{2q}_{Z}(c_{ext}^{q}(\nabla_*))(\Delta) = \Gamma^{2q}_{T}(\Delta),$$

because $\Gamma^{2q}_{T}(\Delta) \equiv 0$.

Next, let $\Delta$ be any simplex in $T_{2q}$. We claim that

$$\int_{\Delta \cap R_1} c_{ext}^{q}(\nabla_1) - \int_{\Delta \cap R_1} c_{ext}^{q}(\nabla_0, \nabla_1) = I(E|_{\Delta}, s^{(r)}, p_\Delta)$$

and this will be enough to conclude, because of the hypothesis on $\mathcal{R}$ (see (5.24)) and, more generally, Subsection 4.3.

To prove (5.20), we proceed locally. Take $\Delta \in T_{2q}$ and let $l \in L$ be such that $A_l \supset \Delta$. Let $s_l^{(r)}$ be a differentiable $r$-section of $E_l \to U_l$ extending $s^{(r)}|_{A_l}$ on $A_l$. Let $\nabla_0$ be an $s_l^{(r)}$-trivial differentiable linear connection for $E_l$ extending $\nabla_0$ and $\nabla_1$ a differentiable linear connection for $E_l$ extending $\nabla_1$. 


As a matter of notations, write $\Lambda$, $\Lambda \cap R_1$, $\Lambda \cap R_{(1,0)}$ and $p_\Lambda$ instead of $F_i(\Delta)$, $F_i(\Delta \cap R_1)$, $F_i(\Delta \cap R_{(1,0)})$ and $F_i(\mu_\Delta)$. Then, by Proposition 5 and Remark 16 it suffices to prove that

\begin{equation}
\int_{\Lambda \cap R_1} c^0_{d1f}(\nabla_1) - \int_{\Lambda \cap R_{(1,0)}} c^0_{d1f}(\nabla_0, i\nabla_1) = I(E_i|\Lambda, S_i^{(r)}, p_\Lambda)
\end{equation}

Actually, (5.21) follows from (5.18) and (5.15), because \( \Lambda \) is a differentiable complex manifold. Then, by (5.19) and (5.20), the cocycles \( \hat{h}^q_Z(c^q_{ext}(\nabla_\ast)) \in C^q(X, Z) \) and \( \Gamma^q_T \in C^q_T(X, Z) \) coincide and we are done.

If \( Z \) is empty, then the following theorem holds.

**Theorem 6.** Let \( X \) be an abstract complex analytic variety of complex dimension \( n \) and \( E \to X \) a differentiable (holomorphic) complex vector bundle of rank \( e \). Take \( q \in \{1, \ldots, n\} \) with \( q \leq e \). Then

\[ c^q_{top}(E) = H^{2q}(c^q_{ext}(E)). \]

As an application of Theorem 5 we prove an abstract residue theorem. For the necessary background on topological Chern classes and their residues, see [Su 3], Ch. 1.

Let \( X \) be a complex analytic variety of complex dimension \( n \) and \( E \to X \) a holomorphic complex vector bundle of rank \( e \). Take \( q \in \{0, \ldots, n\} \) with \( q \leq e \) and set \( r = e - q + 1 \). Let \( s^{(r)} \) be a holomorphic \( r \)-section of \( E \) and denote by \( S \) its singular locus. Then \( S \) is a closed complex analytic subvariety of \( X \) that is also a polyhedron of \( X \).

Suppose that \( S \) is compact and let \( U_S \) be an open neighborhood of \( S \) in \( X \) enjoying (1) and (2) of Lemma 5 such that its closure \( U_S \) is also compact. Set \( Z = X \setminus U_S \), and let \( c^q_{top}(E, s^{(r)}) \in H^{2q}(X, Z) \) and \( c^q_{ext}(E, s^{(r)}) \in H^q_{ext}(X, Z) \) be the localization at \( S \) with respect to \( s^{(r)} \) of \( c^q_{top}(E) \) and, respectively, \( c^q_{ext}(E) \). Then, by Theorem 5 \( c^q_{top}(E, s^{(r)}) = H^q_Z(c^q_{ext}(E, s^{(r)})) \). Furthermore, since \( (X, Z) \sim (X, X \setminus S) \), \( c^q_{top}(E, s^{(r)}) \in H^q_{ext}(X, Z) \) corresponds to a class in \( H^q(X, X \setminus S) \) still denoted by \( c^q_{top}(E, s^{(r)}) \).

If \( X \) is compact and irreducible, then the following abstract residue theorem holds (see Proposition 8 and Remark 20).

**Theorem 7. (Residue theorem)** Let \( X \) be a compact and irreducible complex analytic variety of complex dimension \( n \) and \( E \to X \) a holomorphic complex vector bundle of rank \( e \). Take \( q \in \{0, \ldots, n\} \) with \( q \leq e \) and set \( r = e - q + 1 \). Let \( s^{(r)} \) be a holomorphic \( r \)-section of \( E \) and \( S \) the singular locus of \( s^{(r)} \). Then \( i_*(TopRes_{s^{(r)}}(E, s^{(r)}), S) = P^q_{2q} \circ H^{2q}(c^q_{ext}(E)) \). If \( q = n \), then

\begin{equation}
i_*(TopRes_{s^{(r)}}(E, s^{(r)}), S) = \int_{[X]} c^n_{ext}(E)
\end{equation}

As a note, under the hypotheses of Theorem 7, using the notations employed at the end of the proof of Theorem 5 the right hand side of (5.22) can be written as

\begin{equation}
\int_{[X]} c^n_{ext}(E) = \left[ \sum_{\Delta \in T_2} \left( \int_{\Lambda \cap R_1} c^0_{d1f}(\nabla_1) - \int_{\Lambda \cap R_{(1,0)}} c^0_{d1f}(\nabla_0, i\nabla_1) \right) p_\Lambda \right]
\end{equation}

In fact, (5.23) follows from (5.18), because of (5.12), (5.20) and (5.21) (cp. the proof of Theorem 5).
5.6. Generalized Camacho-Sad index theorem. As a matter of notations, the stalk at a of a sheaf $S \to A$ is denoted by $S_a$. Let $X$ be an abstract finite dimensional complex analytic variety. From now on, the sheaf of germs of holomorphic vector fields on $X$ is denoted by $TX$ instead of $\mathcal{O}_X(TX)$. For the necessary background about foliations, refer to [Br] and [Su 1], Ch. VI, Sec. 6.

Let $X$ be an abstract complex analytic variety of complex dimension $n$ and $Y$ a complex analytic subvariety of $X$ of complex dimension $m \leq n$ such that $Y \not\subseteq Sing(X)$. Set $Y' = Y \setminus ((Sing(X) \cap Y) \cup Sing(Y))$.

Let $\mathcal{F}$ be a holomorphic foliation of $X$ of rank $k \leq m$ and write $Sing(\mathcal{F}) = \{ x \in X^{Reg} : (TX/\mathcal{F})_x \text{ is locally free} \} \cup Sing(X)$. Then $\mathcal{F}|_{X \setminus Sing(\mathcal{F})}$ is the sheaf of holomorphic sections of a holomorphic vector bundle $F$ over $X \setminus Sing(\mathcal{F})$.

Suppose that $Y$ is $\mathcal{F}$-invariant. Then the image of the sheaf homomorphism $\mathcal{F} \otimes \mathcal{O}_Y \to TX \otimes \mathcal{O}_Y$, still denoted by $\mathcal{F} \otimes \mathcal{O}_Y$, is a holomorphic foliation of $Y$ of rank $k$ and $(\mathcal{F} \otimes \mathcal{O}_Y)|_{Y^{Reg}}$ is a possibly singular foliation of the manifold $Y^{Reg}$.

Consider the following exact sequence of sheaves $0 \to \mathcal{F} \otimes \mathcal{O}_X \otimes \mathcal{O}_Y \to TX \otimes \mathcal{O}_X \otimes \mathcal{O}_Y \to \mathcal{Q} \otimes \mathcal{O}_X \otimes \mathcal{O}_Y \to 0$, set $S = (Sing(\mathcal{F}) \cap Y) \cup Sing(Y)$ and write $Y'' = Y \setminus S$. Then $(\mathcal{F} \otimes \mathcal{O}_X \mathcal{O}_Y)|_{Y''}$ is the sheaf of holomorphic sections of a holomorphic vector bundle. By $Y'' = Y \setminus S \subseteq Y'$, we have the following diagram (5.24)

$$0 \to (\mathcal{F} \otimes \mathcal{O}_X \mathcal{O}_Y)|_{Y''} \to (TX \otimes \mathcal{O}_X \mathcal{O}_Y)|_{Y''} \to (\mathcal{Q} \otimes \mathcal{O}_X \mathcal{O}_Y)|_{Y''} \to 0$$

Remark 21. Let $M$ be a complex manifold. For definitions and general results concerning partial connections for a complex vector bundle $E \to M$, see [Ab-Br-To 2] and [Ba-Bo]. Let $H \subseteq TM$ be an involutive holomorphic bundle. For definitions and results about $H$-bundles and (flat) holomorphic actions of $H$ on a given holomorphic vector bundle over $M$, refer to [Ab-Br-To 2] and [Ba-Bo].

Denote by $N_{Y'} \to Y'$ the complex vector bundle associated with $N_{Y}|_{Y'}$. $N_{Y'}$ is the normal bundle of $Y'$. It is known that $N_{Y'}|_{Y''} \to Y''$ is an $(F|_{Y''})$-vector bundle with respect to the map

$$\tau : \Gamma(F|_{Y''}) \times \Gamma(N_{Y'}|_{Y''}) \to \Gamma(N_{Y'}|_{Y''})$$

$$(f, s) \mapsto \tau(f, s) = \pi(f, s|_{Y''})$$

where $f$ and $s$ are sections of $\Gamma((TX|_{X \setminus (Sing(\mathcal{F}) \cup Sing(Y))})$ such that $f|_{Y''} = f$ and, respectively, $s|_{Y''} = s$. Furthermore, $\tau$ is a flat holomorphic action of $F|_{Y''}$ on $N_{Y'}|_{Y''}$ (cp. [Ab-Br-To 2]).

Let $\nabla$ be a linear connection of type $(1, 0)$ for $N_{Y'}|_{Y''} \to Y''$ extending the partial connection $(F_{Y''} \oplus \mathcal{T}Y|_{Y''}, \nabla + \partial)$. Denote by $K$ the curvature of $\nabla$. Then for any symmetric homogeneous polynomial $\Phi \in \mathbb{C}[t_1, ..., t_n]$ of degree $q \in \{ m - k + 1, ..., m \}$ it results $\Phi(K) = 0$. In particular, $c^q_{ext}(\nabla) = c^q_{dif}(\nabla) = 0$ for any $q \in \{ m - k + 1, ..., m \}$ (cp. [Ab-Br-To 2], Theorem 6.1).

Suppose that $N_{Y'}|_{Y''} \to Y''$ is the restriction at $Y'' = Y \setminus S$ of a holomorphic vector bundle $N_Y \to Y$ defined over the whole of $Y$. In this case, $N_{Y'}|_{Y'} = N_{Y'} = (TX^{Reg}|_{Y'})/TY'$ and so $N_{Y'}|_{Y''} = N_{Y'}|_{Y''}$. Suppose that $Y$ is compact and globally irreducible. Then it is possible to localize some extendable Chern classes of $Y$ around $S$.

To prove this, let $V_0$ be the open subset of $Y$ defined as $V_0 = Y'' = Y \setminus S$. Let $V_1$ be a neighborhood of $S$ open in $Y$, homotopically equivalent to $S$ and such that...
$\mathbb{V}_1$ is compact. Set $Z = Y \setminus V_1$ and suppose that both $\mathbb{V}_1$ and $Z$ are polytopes with respect to a triangulation of $X$. Then $\mathcal{V} = \{V_0, V_1\}$ is an open covering of $Y$ adapted to $Z$. Let $V_0$ be a linear connection of type $(1,0)$ for $N_Y|V_0 \to V_0$ extending the partial connection $(F_{\mathcal{V}}^u \oplus \mathcal{T}_Y|_{\mathcal{V}}^u, \tau \oplus \bar{\partial})$. Let $\nabla_1$ be an extendable linear connection for $N_Y|V_1 \to V_1$. Take $q \in \{m-k+1, \ldots, m\}$. Then $c^q_{\text{ext}}(\nabla_*) = (c^q_{\text{ext}}(\nabla_0), c^q_{\text{ext}}(\nabla_1), c^q_{\text{ext}}(\nabla_0, \nabla_1)) = (0, c^q_{\text{ext}}(\nabla_1), c^q_{\text{ext}}(\nabla_0, \nabla_1))$, because $c^q_{\text{ext}}(\nabla_0) = c^q_{\text{dif}}(\nabla_0) = 0$ (cp. [Ab-Br-To 2]). So $[c^q_{\text{ext}}(\nabla_*)] \in H^q_{\text{ext}}(Y, \mathcal{V}, V_0)$.

Consider the following commutative diagram

\[
\begin{array}{cccccc}
H^q_{\text{ext}}(Y, Z) & \overset{H^q_{\text{ext}}(Y, Y \setminus S)}{\longrightarrow} & H^q_{\text{ext}}(Y, Y \setminus S) & \overset{A^q_{*, S, 2q}}{\longrightarrow} & H_{2n-2q}(S) \\
\downarrow & & \downarrow & & \downarrow \\
H^q(Z) & \overset{H^q(Y)}{\longrightarrow} & H^q(Y) & \overset{P^q_{2q}}{\longrightarrow} & H_{2n-2q}(Y)
\end{array}
\]

(5.26)

denote by $c^q_{\text{ext}}(N_Y, F, Y \setminus Z) \in H^q_{\text{ext}}(Y, Z)$ the cohomology class corresponding to $[c^q_{\text{ext}}(\nabla_*)] \in H^q_{\text{ext}}(Y, \mathcal{V}, V_0)$ and set $\text{Res}_{c^q_{\text{ext}}}(N_Y, F, S) = A_{S, 2q}^q \circ H^q_{\text{ext}}(c^q_{\text{ext}}(N_Y, F, Y \setminus Z))$.

**Theorem 8.** Let $X$ be an abstract complex analytic variety of complex dimension $n$ and $Y$ a compact and globally irreducible complex analytic subvariety of $X$ of complex dimension $m \leq n$ such that $Y \not\subseteq \text{Sing}(X)$. Let $F$ be a holomorphic foliation of $X$ of rank $k \leq m$ and suppose that $Y$ is $F$-invariant. Set $S = (\text{Sing}(F) \cap Y) \cup \text{Sing}(Y)$ and write $Y'' = Y \setminus S$. Let $N_Y \to Y$ be a vector bundle on $Y$ whose restriction at $Y''$ coincides with the normal bundle of $Y''$. Then $P^*_{2q} \circ H^q_{\text{ext}}(N_Y) = i_*(\text{Res}_{c^q_{\text{ext}}}(N_Y, F, S))$. If $q = m$, then

\[
\int[Y]^{c^m_{\text{ext}}}(N_Y) = i_*(\text{Res}_{c^m_{\text{ext}}}(N_Y, F, S))
\]

(5.27)

We have the following remark.

**Remark 22.** As an example of a complex vector bundle $N_Y \to Y$ enjoying the above hypotheses, consider the restriction at $Y$ of the line bundle $L_Y \to X$ canonically associated with a Cartier divisor $Y$ of the ambient variety $X$. Another class of examples is given by subvarieties $Y$ of $X$ which are defined as the zero locus of a section of a holomorphic vector bundle defined over the ambient variety $X$. The last example is similar to the construction for subvarieties of complex manifolds which are also strongly local complete intersection (cp. [Le-Su]).

Next, we give an explicit expression of $i_*(\text{Res}_{c^q_{\text{ext}}}(N_Y, F, S))$ in a simple but fundamental case. Let $X$ be an abstract complex analytic variety of complex dimension 2 and $Y$ be a compact and globally irreducible Cartier divisor of $X$ such that $Y \not\subseteq \text{Sing}(X)$. Consider be the line bundle $L_Y \to X$ canonically associated with $Y$ and let $N_Y \to Y$ be the restriction at $Y$ of $L_Y$.

Let $F$ be a holomorphic foliation of rank 1 of $X$ and suppose that $Y$ is $F$-invariant. Suppose that $S = (\text{Sing}(F) \cap Y) \cup \text{Sing}(Y)$ is an isolated singular point $p \in \text{Sing}(Y) \cap \text{Sing}(F) \cap \text{Sing}(X)$, and that the stalk $F_p$ is generated by $\mathcal{O}_{X,p}$ as a single element of $TX_p$. Write $Y'' = Y \setminus S$ and recall that $N_{Y''}|_{Y''} \to Y''$ is an $(F|\gamma''')$-bundle with respect to the action $\tau$ described in (5.25).

Let $W_1$ be a neighborhood of $p$ open in $X$ such that $W_1 \cap Y = \{x \in X : h(x) = 0\}$, where $h$ is a local holomorphic definition function for $Y$ defined on $W_1$. Denote by $h$ the non vanishing section of $L_Y|W_1 \to X|W_1$ associated with $h$. Let $V_1$ be the neighborhood of $p$ open in $Y$ defined by $V_1 = W_1 \cap Y$. Shrinking $W_1$, if necessary,
we can assume without loss of generality that $V_1$ is topologically contractible and that its closure $V_1^*$ is compact. Set $Z = Y \setminus V_1$ and suppose that both $V_1^*$ and $Z$ are polyhedra with respect to a triangulation $\mathcal{T}$ of $X$ compatible with $\text{Sing}(X) \cup Y$ and such that $p$ is in the interior of some 2-simplex of $\mathcal{T}$. Shrinking $V_1$, if necessary, we can also assume that on $W_1$ the foliation $\mathcal{F}$ is generated by one holomorphic vector field $\mathcal{F} \in TX$. Indeed, the sheaf $\mathcal{F}$ is coherent and $\mathcal{F}_p$ is generated by only one element of $\mathcal{T}X_p$.

Write $V_0 = Y \setminus \{p\}$, set $\mathcal{V} = \{V_0, V_1\}$ and note that $Z \subset V_0$. Let $\nabla_0$ be a linear connection of type $(1, 0)$ for $N_{Y^*}\mid_{V_0} \to V_0$ extending the partial connection $(F_0^\nu \otimes T Y\mid_{Y^*} \nu, \tau \otimes \partial)$. Let $\nabla_1$ be an $\mathfrak{h}\mid_{V_1}$-trivial extendable linear connection for $N_{Y^*}\mid_{V_1} \to V_1$. Then $c^1_{\text{ext}}(\nabla_*) = (c^1_{\text{ext}}(\nabla_0), c^1_{\text{ext}}(\nabla_1), c^1_{\text{ext}}(\nabla_0, \nabla_1)) = (0, 0, c^1_{\text{ext}}(\nabla_0, \nabla_1))$. Indeed, $c^1_{\text{ext}}(\nabla_0) = c^1_{\text{def}}(\nabla_0) = 0$, because of Theorem 6.1 of [Ab-Br-To 2], and $c^1_{\text{ext}}(\nabla_1) = 0$, because $\nabla_1$ is $\mathfrak{h}\mid_{V_1}$-trivial. Let $\{\rho_0, \rho_1 : Y \to \mathbb{R}\}$ be an extendable partition of unity subordinated to $\mathcal{V}$ and set $\nabla = \rho_0 \nabla_0 + \rho_1 \nabla_1$. Then, by (5.27), (4.17), (4.12) and (4.24), it results

\begin{equation}
(5.28) \quad i_*(\text{Res}_{\nabla, \mathcal{F}}(N_{Y^*}\mid_{\mathcal{F}}, p)) = \int_{\mathcal{V}} c^1_{\text{ext}}(N_{Y^*}) = \int_Y c^1_{\text{ext}}(\nabla) = -\int_{\text{Lk}(p)} c^1_{\text{ext}}(\nabla_0, \nabla_1),
\end{equation}

with $\text{Lk}(p) \subset V_0 \cap V_1$ the link of $p$ in $Y$ with respect to a triangulation $\mathcal{T}_*$ of $X$ compatible with $\text{Sing}(X) \cup Y \cup \{p\}$. As a note, $\mathcal{T}_* \neq \mathcal{T}$, because $p$ is not a vertex of $\mathcal{T}$.

So, we only have to explicitly compute the extendable Bott difference form $c^1_{\text{ext}}(\nabla_0, \nabla_1)$. To do this, observe that $c^1_{\text{ext}}(\nabla_0, \nabla_1)$ is defined on the differentiable complex manifold $V_{(0,1)} = V_0 \cap V_1$. Consider the differentiable vector bundle $E = N_{Y^*}\mid_{V_{(0,1)}} \times \mathbb{R} \to V_{(0,1)} \times \mathbb{R}$ and let $\nabla$ be the linear connection for $E$ defined by $\nabla = (1 - \varsigma)\nabla_0 + \varsigma \nabla_1$, with $\varsigma \in \mathbb{R}$. Let $\Xi_*$ denote the integration along the fibres of the projection $\Xi : V_{(0,1)} \times [0, 1] \to V_{(0,1)}$. Then, by its very definition, we have $c^1_{\text{ext}}(\nabla_0, \nabla_1) = \Xi_*(c^1_{\text{def}}(\nabla))$.

Let $\tilde{h}$ and $\tilde{\theta}$ be as above. By the parametrization theorem (cp. [Ca], Vol. II, Ch. D), we may find a holomorphic function $y : W_1 \to C$ defined on $W_1$ such that $(dh \wedge dy)|_Y$ does not vanish on a neighborhood $V$ of $Y^* \setminus \{p\}$ that, without loss of generality, we can assume to contain $V_{(0,1)}$. Then $(h, y)$ are local coordinates on $X^\text{reg}$ near $p$ and $y$ is a local coordinate on $Y^*$ near each point of $Y^* \setminus \{p\}$. In particular, $y$ is a local coordinate on $V_{(0,1)} \subset V_1 \setminus \{p\}$. Since $Y^*$ is $\mathcal{F}$-invariant, using the coordinates $(h, y)$ and with slight abuses of notation, we can write the holomorphic vector field $\mathcal{F} \in TX$ generating $\mathcal{F}$ on $W_1$ as $\mathcal{F} = a(h, y)\partial_{dh} + b(h, y)\partial_{dy}$, with $a$ and $b$ holomorphic functions defined on $W_1$ such that $b(0, y)$ is not identically equal to zero.

Let $\tilde{\theta}$ be the connection form of $\nabla$ and $\theta_0$ the connection form of $\nabla_0$. Since $\nabla_1$ is $\mathfrak{h}\mid_{V_1}$-trivial, the connection form $\theta_1$ with respect to $\mathfrak{h}\mid_{V_1}$ is zero. Then $\tilde{\theta} = (1 - \varsigma)\theta_0$ and

\begin{equation}
(5.29) \quad c^1_{\text{ext}}(\nabla_0, \nabla_1) = \frac{\sqrt{-1}}{2\pi} \Xi_*(\tilde{\theta}) = \frac{1}{2\pi\sqrt{-1}} \theta_0
\end{equation}

Now, to compute $\theta_0|_{Y^* \setminus \{p\}}$, look at the very definition of $\theta_0$ and $\tau$ (cp. (5.25)). In fact, since $\frac{\partial}{\partial \theta} \in \Gamma(V_{(0,1)}, TX|_{V_{(0,1)}})$ is an extension of $\mathfrak{h}\mid_{V_{(0,1)}} \in \Gamma(V_{(0,1)}, N_{Y^*}\mid_{V_{(0,1)}})$ on $V_{(0,1)}$, it results $\theta_0(\frac{\partial}{\partial \theta})|_{V_{(0,1)}} = (\nabla_0)\frac{\partial}{\partial \theta}(\mathfrak{h}|_{V_{(0,1)}}) = \tau(\frac{\partial}{\partial \theta})|_{Y^* \setminus \{p\}}$, $\mathfrak{h}|_{V_{(0,1)}} =$
\[ -\frac{a(0,y)}{b(0,y)} b|_{v(0,1)} \text{. Then } \theta_0 = \frac{a(0,y)}{b(0,y)} dy \text{. So, by (5.28) and (5.29), we get the following formula for the residue} \]

\[
\int_L^{k(p)} \frac{a(0,y)}{b(0,y)} dy
\]

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