Abstract: This manuscript is concerned with the oscillatory properties of 4th-order differential equations with variable coefficients. The main aim of this paper is the combination of the following three techniques used: the comparison method, Riccati technique and integral averaging technique. Two examples are given for applying the criteria.

Keywords: delay differential equations; oscillation; fourth-order

1. Introduction

Differential equations of fourth-order have applications in dynamical systems, optimization, and in the mathematical modeling of engineering problems [1]. The $p$-Laplace equations have some significant applications in elasticity theory and continuum mechanics, see, for example, [2,3]. Symmetry plays an important role in determining the right way to study these equations [4]. The main aim of this paper is the combination of the following three techniques used:

(a) The comparison method.
(b) Riccati technique.
(c) Integral averaging technique.

We consider the following fourth-order delay differential equations with $p$-Laplacian like operators

$$
\left( \frac{a(\zeta)}{u''(\zeta)} \right)' + \frac{q(\zeta)}{a(\eta(\zeta))} = 0,
$$

where $\zeta \geq \xi_0$. Throughout this work, we suppose that:

K1: $p > 1$ is a real number.
K2: $a \in C^1(\xi_0, \infty), a(\xi) > 0$, $a'(\xi) \geq 0$ and under the condition

$$
\int_{\xi_0}^{\infty} \frac{1}{a^{1/(p-1)}(s)} ds = \infty,
$$

K3: $q \in C(\xi_0, \infty), q(\xi) > 0$,
K4: $\eta \in C ([\xi_0, \infty), \mathbb{R})$, $\eta (\zeta) \leq \zeta$, $\lim_{\zeta \to \infty} \eta (\zeta) = \infty$,

K5: $g \in C (\mathbb{R}, \mathbb{R})$ such that $g (u) \geq m \|u\|^{p-2} u > 0$, for $u \neq 0$ and $m$ is a constant.

**Definition 1.** The function $u \in C^1 ([\xi_0, \infty), \mathbb{R})$, $\zeta \geq \xi_0$ is called a solution of (1), if $u (\zeta) \sim u'' (\zeta) \in C^1 ([\xi_0, \infty), \mathbb{R})$, and $u (\zeta)$ satisfies (1) on $[\xi_0, \infty)$. Moreover, the equation (1) is oscillatory if all its solutions oscillate.

In the last few decades, there have been a constant interest to investigate the asymptotic property for oscillations of differential equation, see [5–25]. Furthermore, there are some results that study the oscillatory behavior of 4th-order equations with $p$-Laplacian, we refer the reader to [26,27].

Now the following results are presented.

Grace and Lalli [28], Karpuz et al. [29] and Zafer [30] studied the even-order equation

$$u^{(\gamma)} (\zeta) + q (\zeta) u (\eta (\zeta)) = 0,$$

they used the Riccati substitution to find several oscillation criteria and established the following results, respectively:

$$\int_{\xi_0}^{\infty} \left( \delta (s) q (s) - \frac{(\gamma - 1)!}{2^{\gamma - 2} \eta^{\gamma - 2} (s) \eta' (s) \delta (s)} \right) ds = \infty, \quad (3)$$

where $\delta \in C^1 ([\xi_0, \infty), (0, \infty))$.

$$\liminf_{\zeta \to \infty} \int_{\eta (\zeta)}^{\zeta} q (s) \eta^{\gamma - 2} (s) ds > \frac{(\gamma - 1) 2^{(\gamma - 1)(\gamma - 2)}}{e} \quad (4)$$

and

$$\liminf_{\zeta \to \infty} \int_{\eta (\zeta)}^{\zeta} q (s) \eta^{\gamma - 2} (s) ds > \frac{(\gamma - 1)!}{e} \quad (5)$$

Zhang et al. [31,32] studied the even-order equation

$$\left( a (\zeta) \left( u^{(\gamma - 1)} (\zeta) \right)^\beta \right)' + q (\zeta) u^\beta (\eta (\zeta)) = 0, \quad (6)$$

where $\beta$ is a quotient of odd positive integers. They proved that it is oscillatory, if

$$\liminf_{\zeta \to \infty} \int_{\eta (\zeta)}^{\eta (\zeta)} \frac{q (s)}{a (\eta (s))} \left( \eta^{\gamma - 2} (s) \right)^\beta ds > \frac{(\gamma - 1)!}{e} \beta, \quad (7)$$

where $\gamma \geq 2$ is even and they used the compare with first order equations. If there exists a function $\delta \in C^1 ([\xi_0, \infty), (0, \infty))$ for all constants $M > 0$ such that

$$\liminf_{\zeta \to \infty} \int_{\xi_0}^{\infty} \delta (s) \left( \frac{q (s) - a (s) \left( \theta \eta M \eta^{\gamma - 2} (s) \eta' (s) \right)^{1-p} \left( \delta' (s) - \frac{a (s)}{\eta' (s)} \right)^p}{\delta (s)} \right) ds = \infty, \quad (8)$$

for some constant $\theta \in (0, 1)$.

Our aim in this work is to complement results in [28–32]. Two examples are given for applying the criteria.
Theorem 1. Let (2) holds. If the equations

\[ \left( \frac{2q^{p-1} (\xi')^{p-1}}{(\beta \xi^p)^{p-1}} (u' (\xi))^{p-1} \right)' + kq (\xi) \left( \frac{\eta^3 (\xi)}{\xi^3} \right)^{p-1} u^{p-1} (\xi) = 0 \]
where

\[ u''(\xi) + u(\xi) \int_{\xi}^{\infty} \left( \frac{1}{a(s)} \int_{\xi}^{s} q(s) \left( \frac{\eta(s)}{\xi} \right)^{p-1} ds \right)^{1/p-1} d\xi = 0 \quad (13) \]

are oscillatory, then every solution of (1) is oscillatory.

**Proof.** Assume, for the sake of contradiction, that \( u \) is a positive solution of (1). Then, we let \( u(\xi) > 0 \) and \( u(\eta(\xi)) > 0 \). By Lemma 3, we have \((S_1)\) and \((S_2)\).

Let case \((S_1)\) holds. Using [25], [Lemma 2.2.3], we find

\[ u'(\xi) \geq \frac{\theta}{2} \zeta^2 u'''(\xi), \]

for every \( \theta \in (0, 1) \).

From Lemma 2, we get

\[ \frac{u'(\xi)}{u(\xi)} \leq \frac{3}{\xi}. \]

Integrating from \( \eta(\xi) \) to \( \xi \), we find

\[ \frac{u(\eta(\xi))}{u(\xi)} \geq \frac{\eta^3(\xi)}{\xi^3}. \]

Defining

\[ \varphi(\xi) := \delta(\xi) \left( \frac{a(\xi) (u'''(\xi))^{p-1}}{u^{p-1}(\xi)} \right), \varphi(\xi) > 0, \]

where \( \delta \in C^1((\xi_0, \infty), (0, \infty)) \) and

\[ \varphi'(\xi) = \delta'(\xi) \frac{a(\xi) (u'''(\xi))^{p-1}}{u^{p-1}(\xi)} + \delta(\xi) \left( \frac{a(\xi) (u'''(\xi))^{p-1}}{u^{p-1}(\xi)} \right)'(\xi) - (p-1) \delta(\xi) \frac{u^{p-2}(\xi) u'(\xi) a(\xi) (u'''(\xi))^{p-1}}{u^{2(p-1)}(\xi)}. \]

Combining (14) and (16), we obtain

\[ \varphi'(\xi) \leq \frac{\delta'(\xi)}{\delta(\xi)} \varphi(\xi) + \delta(\xi) \left( \frac{a(\xi) (u'''(\xi))^{p-1}}{u^{p-1}(\xi)} \right)'(\xi) - (p-1) \delta(\xi) \frac{\theta \zeta^2 a(\xi) (u'''(\xi))^{p}}{u^{p}(\xi)} \]

\[ \leq \frac{\delta'(\xi)}{\delta(\xi)} \varphi(\xi) + \delta(\xi) \left( \frac{a(\xi) (u'''(\xi))^{\delta(\xi)}}{u^{\delta(\xi)}} \right)'(\xi) - \frac{(p-1) \beta \zeta^2}{2 \delta(\xi) a(\xi))^{\frac{p-1}{\delta(\xi)}}} \varphi^{\frac{p}{\delta(\xi)}}(\xi). \]

From (1) and (17), we find

\[ \varphi'(\xi) \leq \frac{\delta'(\xi)}{\delta(\xi)} \varphi(\xi) - m \delta(\xi) \frac{q(\xi) u^{p-1}(\eta(\xi))}{u^{p-1}(\xi)} - \frac{(p-1) \beta \zeta^2}{2 \delta(\xi) a(\xi))^{\frac{p-1}{\delta(\xi)}}} \varphi^{\frac{p}{\delta(\xi)}}(\xi). \]
From (15), we have

$$
\varphi' (\xi) \leq \delta' (\xi) \varphi (\xi) - m \delta (\xi) q (\xi) \left( \frac{\eta^3 (\xi)}{\xi^3} \right)^{p-1} - \frac{(p-1) \theta \xi^2}{2 (\delta (\xi) a (\xi))^{p-1}} \varphi^{p-1} (\xi). \tag{18}
$$

Let \( \delta (\xi) = m = 1 \) in (18), we have

$$
\varphi' (\xi) + \frac{(p-1) \theta \xi^2}{2 a^{p-1}} \varphi^{p-1} (\xi) + q (\xi) \left( \frac{\eta^3 (\xi)}{\xi^3} \right)^{p-1} \leq 0.
$$

Hence, the equation (12) is nonoscillatory, which is a contradiction.

Let case \( (S_2) \) holds. By Lemma 2, we find

$$
\frac{u' (\xi)}{u (\xi)} \leq \frac{1}{\xi}.
$$

Integrating again from \( \eta (\xi) \) to \( \xi \), we find

$$
\frac{u (\eta (\xi))}{u (\xi)} \geq \frac{\eta (\xi)}{\xi}. \tag{19}
$$

Defining

$$
\psi (\xi) := \vartheta (\xi) \frac{u' (\xi)}{u (\xi)} > 0,
$$

where \( \vartheta \in C^1 ([\xi_0, \infty), (0, \infty)) \) and

$$
\psi' (\xi) = \frac{\vartheta' (\xi)}{\vartheta (\xi)} \psi (\xi) + \vartheta (\xi) \frac{u'' (\xi)}{u (\xi)} - \frac{1}{\vartheta (\xi)} \psi (\xi)^2. \tag{20}
$$

Integrating (1) from \( \xi \) to \( x \) and using \( u' (\xi) > 0 \), we have

$$
a (x) (u'' (x))^{p-1} - a (\xi) (u'' (\xi))^{p-1} = - \int_{\xi}^{x} q (s) g (u (\eta (s))) ds.
$$

From (19), we get

$$
a (x) (u'' (x))^{p-1} - a (\xi) (u'' (\xi))^{p-1} \leq - ky^{p-1} (\xi) \int_{\xi}^{x} q (s) \left( \frac{\eta (s)}{s} \right)^{p-1} ds.
$$

Letting \( x \to \infty \), we have

$$
a (\xi) (u'' (\xi))^{p-1} \geq ky^{p-1} (\xi) \int_{\xi}^{\infty} q (s) \left( \frac{\eta (s)}{s} \right)^{p-1} ds
$$

and so

$$
u'' (\xi) \geq u (\xi) \left( \frac{m}{a (\xi)} \int_{\xi}^{\infty} q (s) \left( \frac{\eta (s)}{s} \right)^{p-1} ds \right)^{1/(p-1)}.
$$

Integrating again from \( \xi \) to \( \infty \), we get

$$
u'' (\xi) + u (\xi) \int_{\xi}^{\infty} \left( \frac{m}{a (\xi)} \int_{\xi}^{s} q (\mu) \left( \frac{\eta (\mu)}{\mu} \right)^{p-1} d\mu \right)^{1/(p-1)} d\xi \leq 0. \tag{21}
$$
Combining (20) and (21), we find
\[ \psi'(\zeta) \leq \frac{\partial'(\zeta)}{\partial(\zeta)} \Psi(\zeta) - \partial(\zeta) \int_{\zeta}^{\infty} \left( \frac{m}{a(\zeta)} \right) q(s) \left( \frac{\eta(s)}{s} \right)^{p-1} ds \right)^{1/(p-1)} \partial(\zeta) \Psi(\zeta)^2. \] (22)

If \( \partial(\zeta) = m = 1 \) in (22), we get
\[ \psi'(\zeta) + \psi^2(\zeta) + \int_{\zeta}^{\infty} \left( \frac{1}{a(\zeta)} \right) q(s) \left( \frac{\eta(s)}{s} \right)^{p-1} ds \right)^{1/(p-1)} \partial(\zeta) \Psi(\zeta)^2 \leq 0. \]

Thus, the Equation (13) is nonoscillatory, which is a contradiction. The proof of the theorem is complete. \( \square \)

Next, we obtain the following Hille and Nehari type oscillation criteria for (1) with \( p = 2 \).

**Theorem 2.** Let \( p = 2, m = 1 \). Assume that
\[ \int_{\zeta_0}^{\infty} \frac{\theta^2}{2a(\zeta)} d\zeta = \infty \]
and
\[ \lim \inf_{\zeta \to \infty} \left( \int_{\zeta_0}^{\zeta} \frac{\theta^2}{2a(s)} ds \right) \int_{\zeta}^{\infty} q(s) \left( \frac{\eta(s)}{s} \right)^{p-1} ds > \frac{1}{4}, \] (23)
for some constant \( \theta \in (0,1) \),
\[ \lim \inf_{\zeta \to \infty} \int_{\zeta_0}^{\zeta} \int_{\zeta}^{\infty} q(s) \left( \frac{\eta(s)}{s} \right)^{p-1} ds d\zeta dv > \frac{1}{4}. \] (24)
then all solutions of (1) is oscillatory.

In this theorem, we use the integral averaging technique:

**Theorem 3.** Let (2) holds. If there exist positive functions \( \delta, \partial \in C^1 ((\zeta_0, \infty), \mathbb{R}) \) such that
\[ \lim \sup_{\zeta \to \infty} \frac{1}{H_1(\zeta, \zeta_1)} \int_{\zeta_1}^{\zeta} \left( H_1(\zeta, s) m \delta(s) q(s) \left( \frac{\eta(s)}{s} \right)^{p-1} - \pi(s) \right) ds = \infty \] (25)
and
\[ \lim \sup_{\zeta \to \infty} \frac{1}{H_2(\zeta, \zeta_1)} \int_{\zeta_1}^{\zeta} \left( H_2(\zeta, s) \partial(s) a(s) - \frac{\partial(s) h^2(\zeta, s)}{4} \right) ds = \infty, \] (26)
where
\[ \pi(s) = \frac{h^p(\zeta, s) H_1^{p-1}(\zeta, s) 2^{p-1} \delta(s) a(s)}{(\theta \delta^2)^{p-1}}, \]
for all \( \theta \in (0,1) \), and
\[ \omega(s) = \left( \frac{1}{a(\zeta)} \right) \int_{\zeta}^{\infty} q(s) \left( \frac{\eta(s)}{s} \right)^{p-1} ds \right)^{1/(p-1)} d\zeta, \]
then (1) is oscillatory.
Proof. Proceeding as in the proof of Theorem 1. Assume that (S1) holds. From Theorem 1, we get that (18) holds. Multiplying (18) by $H_1 (\zeta, s)$ and integrating the resulting inequality from $\zeta_1$ to $\zeta$, we find that

$$\int^{\zeta}_{\zeta_1} H_1 (\zeta, s) m \delta (s) q (s) \left( \frac{\eta^3 (s)}{s^3} \right)^{p-1} ds \leq \varphi (\zeta_1) H_1 (\zeta, \zeta_1) + \int^{\zeta}_{\zeta_1} \left( \frac{\partial}{\partial s} H_1 (\zeta, s) + \frac{\delta' (s)}{\delta (s)} H_1 (\zeta, s) \right) \varphi (s) ds - \int^{\zeta}_{\zeta_1} \frac{(p-1) \theta s^2}{2 \left( \delta (s) a (s) \right)^{1-p} H_1 (\zeta, s) \varphi^{\frac{p}{p-1}} (s) ds. \quad (27)$$

From (10), we get

$$\int^{\zeta}_{\zeta_1} H_1 (\zeta, s) m \delta (s) q (s) \left( \frac{\eta^3 (s)}{s^3} \right)^{p-1} ds \leq \varphi (\zeta_1) H_1 (\zeta, \zeta_1) + \int^{\zeta}_{\zeta_1} h_1 (\zeta, s) H_1^{(p-1)/p} (\zeta, s) \varphi (s) ds - \int^{\zeta}_{\zeta_1} \frac{(p-1) \theta s^2}{2 \left( \delta (s) a (s) \right)^{1-p} H_1 (\zeta, s) \varphi^{\frac{p}{p-1}} (s) ds. \quad (27)$$

Using Lemma 1 with $V = (p-1) \theta s^2 / \left( 2 \left( \delta (s) a (s) \right)^{1-p} \right) H_1 (\zeta, s) \varphi (s)$ and $u = \varphi (s)$, we get

$$h_1 (\zeta, s) H_1^{(p-1)/p} (\zeta, s) \varphi (s) - \frac{(p-1) \theta s^2}{2 \left( \delta (s) a (s) \right)^{1-p} H_1 (\zeta, s) \varphi^{\frac{p}{p-1}} (s)} \leq \frac{h_1^p (\zeta, s) H_1^{p-1} (\zeta, s) 2^{p-1} \delta (s) a (s)}{(\theta s^2)^{p-1}},$$

which, with (27) gives

$$\frac{1}{H_1 (\zeta, \zeta_1)} \int^{\zeta}_{\zeta_1} \left( H_1 (\zeta, s) m \delta (s) q (s) \left( \frac{\eta^3 (s)}{s^3} \right)^{p-1} - \pi (s) \right) ds \leq \varphi (\zeta_1).$$

This contradicts (25).

Assume that (S2) holds. From Theorem 1, (22) holds. Multiplying (22) by $H_2 (\zeta, s)$ and integrating the resulting inequality from $\zeta_1$ to $\zeta$, we get

$$\int^{\zeta}_{\zeta_1} H_2 (\zeta, s) \theta (s) \omega (s) ds \leq \psi (\zeta_1) H_2 (\zeta, \zeta_1)$$

$$+ \int^{\zeta}_{\zeta_1} \left( \frac{\partial}{\partial s} H_2 (\zeta, s) + \frac{\delta' (s)}{\delta (s)} H_2 (\zeta, s) \right) \psi (s) ds$$

$$- \int^{\zeta}_{\zeta_1} \frac{1}{\theta (s)} H_2 (\zeta, s) \psi^2 (s) ds.$$

Thus, from (11), we get

$$\int^{\zeta}_{\zeta_1} H_2 (\zeta, s) \theta (s) \omega (s) ds \leq \psi (\zeta_1) H_2 (\zeta, \zeta_1) + \int^{\zeta}_{\zeta_1} h_2 (\zeta, s) \sqrt{H_2 (\zeta, s) \psi (s)} ds$$

$$- \int^{\zeta}_{\zeta_1} \frac{1}{\theta (s)} H_2 (\zeta, s) \psi^2 (s) ds$$

$$\leq \psi (\zeta_1) H_2 (\zeta, \zeta_1) + \int^{\zeta}_{\zeta_1} \frac{\theta (s) h_2^2 (\zeta, s)}{4} ds.$$
and so
\[
\frac{1}{H_2(\zeta, \xi_1)} \int_{\xi_1}^{\zeta} \left( H_2(\zeta, s) \vartheta(s) \omega(s) - \frac{\vartheta(s) h_2^2(\zeta, s)}{4} \right) ds \leq \psi(\xi_1),
\]
which contradicts (26). The proof of the theorem is complete. \( \square \)

**Example 1.** Consider the equation
\[
u^{(4)}(\zeta) + \frac{q_0}{\zeta^4} v \left( \frac{9\zeta}{10} \right) = 0, \quad \zeta \geq 1, \quad q_0 > 0. \tag{28}
\]

Let \( p = 2, \ a(\zeta) = 1, \ q(\zeta) = q_0/\zeta^4 \) and \( \eta(\zeta) = 9\zeta/10. \) If we set \( m = 1, \ H_1(\zeta, s) = (\zeta - s)^2 \) and \( \delta(s) = s^3, \) then \( h_1(\zeta, s) = (\zeta - s)(5 - 3\zeta s^{-1}), \) and conditions (23) becomes
\[
\limsup_{\zeta \to \infty} \frac{1}{H_1(\zeta, \xi_1)} \int_{\xi_1}^{\zeta} \left( H_1(\zeta, s) m \delta(s) q(s) \left( \frac{\eta^2(s)}{s^3} \right)^{p-1} - \pi(s) \right) ds
\]
\[
= \limsup_{\zeta \to \infty} \frac{1}{(\zeta - 1)^2} \int_{\xi_1}^{\zeta} \left( \frac{729q_0\zeta^2s^{-1}}{1000} + \frac{729q_0s}{1000} - \frac{729q_0s}{500} - \frac{s(25 + 9\zeta^2s^{-2} - 30\zeta s^{-1})}{2\theta} \right) ds
\]
if \( q_0 > 500/1810 \) for some \( \theta \in (0, 1), \) letting \( \theta = 81/82, \) then \( q_0 > 6.25. \)

Also, set \( H_2(\zeta, s) = (\zeta - s)^2 \) and \( \vartheta(s) = s, \) then \( h_2(\zeta, s) = (\zeta - s)(3 - \zeta s^{-1}), \varphi(s) = 3q_0/(20\zeta^2) \) and conditions (24) becomes
\[
\limsup_{\zeta \to \infty} \frac{1}{H_2(\zeta, \xi_1)} \int_{\xi_1}^{\zeta} \left( H_2(\zeta, s) \vartheta(s) \omega(s) - \frac{\vartheta(s) h_2^2(\zeta, s)}{4} \right) ds
\]
\[
= \limsup_{\zeta \to \infty} \frac{1}{(\zeta - 1)^2} \int_{\xi_1}^{\zeta} \left( \frac{3q_0\zeta^2s^{-1}}{20} + \frac{3q_0s}{20} - \frac{3q_0s}{40} - \frac{s(9 - 6\zeta s^{-1} + \zeta^2 s^{-2})}{4} \right) ds
\]
if \( q_0 > 5/3, \) From Theorem 3, all solutions of (28) are oscillatory, if \( q_0 > 6.25. \)

**Remark 1.** By comparing our results with previous results
1. By applying condition (3) in [28], we get
\[
q_0 > 1728,
\]
2. By applying condition (4) in [29], we get
\[
q_0 > 919.6,
\]
3. By applying condition (5) in [30], we get
\[
q_0 > 28.73,
\]
4. By applying condition (7) in [31], we get
\[
q_0 > 28.73,
\]
5. The condition (8) in [32] cannot be applied to Equation (28) due to the arbitrariness in the choice of \( \theta. \) Therefore, our result complement results [28–32].
Example 2. Let the equation

\[ u^{(4)}(\zeta) + \frac{q_0}{\zeta^4} u \left( \frac{1}{2} \zeta \right) = 0, \quad \zeta \geq 1, \quad q_0 > 0. \]  

(29)

Let \( a(\zeta) = 1 \), \( q(\zeta) = q_0 / \zeta^4 \) and \( \eta(\zeta) = \zeta / 2 \). If we set \( m = 1 \), then condition (23) becomes

\[ \lim \inf_{\zeta \to \infty} \left( \int_{\zeta_0}^\zeta \frac{q(s)}{2a(s)} ds \right) \int_{\zeta}^\infty \left( \frac{\eta^3(s)}{s^3} \right) ds = \lim \inf_{\zeta \to \infty} \left( \frac{\zeta^3}{3} \right) \int_{\zeta}^\infty \frac{q_0}{8s^4} ds = \frac{q_0}{72} > \frac{1}{4} \]

and condition (24) becomes

\[ \lim \inf_{\zeta \to \infty} \zeta \int_{\zeta_0}^\zeta \int_{\varphi(\zeta)}^\infty \left( \frac{1}{a(\zeta)} \int_{\zeta}^\infty q(s) \left( \frac{\eta(s)}{s} \right) ds \right) d\zeta d\varphi = \lim \inf_{\zeta \to \infty} \left( \frac{q_0}{12\zeta} \right) \]

\[ = \frac{q_0}{12} > \frac{1}{4}. \]

Hence, by Theorem 2, all solution equation (29) is oscillatory if \( q_0 > 18 \).

Remark 2. We point out that continuing this line of work, we can have oscillatory results for a fourth order equation of the type:

\[ \left( a(\zeta) |u'''(\zeta)|^{p-2} u'''(\zeta) \right)' + \sum_{i=1}^{m} q_i(\zeta) |u(\eta_i(\zeta))|^{p-2} u(\eta_i(\zeta)) = 0, \quad \zeta \geq \zeta_0, \quad m \geq 1, \]

under the condition

\[ \int_{\zeta_0}^\infty \frac{1}{a^{1/(p-1)}(s)} ds < \infty. \]

4. Conclusions

In this article, we studied some oscillation conditions for 4th-order differential equations by the comparison method, Riccati technique and integral averaging technique.

Further, in the future work we study Equation (1) under the condition \( \int_{\zeta_0}^\infty \frac{1}{a^{1/(p-1)}(s)} ds < \infty. \)

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