An algebraic theory of infinite classical lattices II: Axiomatic theory

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Abstract

We apply the algebraic theory of infinite classical lattices from Part I to write an axiomatic theory of measurements, based on Mackey’s axioms for quantum mechanics. The axioms give a complete theory of measurements in the sense of Haag and Kastler, taking the traditional form of a logic of propositions provided with a classical spectral theorem. The results are expressed in terms of probability distributions of individual measurements. As applications, we give a separation theorem for states by the set of observables and discuss its relationship to the equivalence of ensembles in the thermodynamic-limit program. We also introduce a weak equivalence of states based on the theory.

MSC 46A13 (primary), 46M40 (secondary)
I. Introduction

There are two standard approaches to the study of infinite lattice systems, the algebraic approach from quantum field theory (QFT) ([3], [4], [5], [18]) and the thermodynamic limit (TL) ([10], [17]). In Part I of this series [15], we presented an algebraic theory of infinite classical lattices, constructed using the axioms of Haag and Kastler [7] from the QFT. We showed that the two approaches may be regarded as two aspects of a single theory, linked by a unique relation between their states based on expectation values. The kinds of questions they can ask are different, however. TL theory is designed to study states, especially the equilibrium states, and the expectation values they assign to observables. We shall find that with the algebraic theory, one may study the statistical properties of the individual measurement. This is therefore the setting for a theory of measurement.

In this paper, we show that the abstract system of algebraic observables of the theory satisfies the Mackey axioms I-VI from quantum mechanics. This will permit us to base the axiomatization on a logic of propositions provided with a classical spectral theorem. Following Birkhoff and von Neumann, the theory is then centered on the question, “If I measure a certain quantity on a lattice prepared in a given state, what is the probability the outcome will lie in a fixed interval \((a, b)\)?” ([2], [9], [12]).

II. Classical measurements

A. The Haag-Kastler frame.

In agreement with Haag and Kastler, we should treat measurements such that their “state and operation are defined in terms of laboratory procedures” [7, p.850]. For this purpose, we view the lattice as representing a finite system immersed homogeneously in an (infinite) surround which acts as a generalized temperature bath. We denote by \(\mathcal{P}\) the set of all possible systems, indexed by \(J\).

According to the Haag-Kastler axioms, the algebraic structure is derived from the local texture, i.e., the pairing of each system with the set of functions representing measurements on that system. We denote the configuration space of the lattice by \(\Omega\), written as the Cartesian product of the single-site configurations, so that for any system \(\Lambda_t\), \(t \in J\), we may write \(\Omega = \Omega_{\Lambda_t} \times \Omega_{\Lambda^\prime_t}\). To each system \(\Lambda_t\), we assign the set \(\mathfrak{W}(\mathfrak{A}^t)\) of functions on \(\Omega\) representing measurements on \(\Lambda_t\) and the compact set \(E_t\) of states on \(\mathfrak{W}(\mathfrak{A}^t)\).

The axioms then direct formation of the algebraic theory from the texture. We showed that for any compact convex set \(K\) of algebraic states, we may construct the triple \(\{X, \mathcal{C}(X), K\mathcal{C}(X)\}\), dependent on \(K\), where the Segal algebra
$\mathcal{C}(X)$ is the set of continuous functions on a compact space $X$, and $\mathcal{K}\mathcal{C}(X)$ is the set of states on $\mathcal{C}(X)$. The triple has the following structure:

1. $\mathcal{K}\mathcal{C}(X)$ is isomorphic with $K$.
2. All of the local functions ($\mathfrak{M}(\mathfrak{A})$) representing measurements on finite systems of the lattice map to unique points in $\mathcal{C}(X)$. Functions measuring the same physical quantity on different systems map to the same point in $\mathcal{C}(X)$.
3. $X$ is homeomorphic with the set $\partial e\mathcal{K}\mathcal{C}(X)$ of extremal points of $\mathcal{K}\mathcal{C}(X)$.
4. By the Riesz representation theorem, for every state $\zeta \in \mathcal{K}\mathcal{C}(X)$, there exists a unique Radon probability measure $\sigma$ on $X$ such that

$$\zeta(f) = \int_X f(x)d\sigma(x) \quad \forall f \in \mathcal{C}(X) \tag{2.1}$$

This is formally the expectation value of a point $f \in \mathcal{C}(X)$ for a lattice in state $\zeta$. Physically, it is the expectation value of any local measurement that maps to $f$. Note especially that this is a decomposition theorem, i.e. it decomposes any state $\zeta \in \mathcal{K}\mathcal{C}(X)$ into an integral over the pure states of $\mathcal{K}\mathcal{C}(X)$.

In this paper we shall only be concerned with a particular choice of $K$, the compact convex set of all stationary states of the lattice. For this case, we showed the crucial additional fact that

5. the space $X$ is a Stonean (compact extremely disconnected) topological space (Theorem V.3).

The point is that this triple is an algebraic theory well-defined by these five properties without any reference to an underlying structure. We apply Mackey’s axioms to this structure and derive our theory of measurements in terms of it.

The similarity of eq. (2.1) to the integration over phase space in ordinary CSM to obtain expectation values might lead to the question of whether the Mackey axioms could be applied directly to the classical problem with its configurational phase space. However, usually the set of continuous functions is not large enough to represent the observables of a classical problem. For example, the ($\mathfrak{M}(\mathfrak{A})$) are the sets of all bounded measurable functions compatible with the preparation of systems for measurement. Furthermore, a phase space with a Stonean topology, which will be essential in the following, excludes most interesting mechanical problems.

We shall have occasion to use a term “microcanonical state” for the lattice. This term clearly refers to local states. There are two ways of describing states of the infinite system. The one is in terms of $\mathcal{K}\mathcal{C}(X)$, the states (positive linear
functionals of norm 1) on the algebra \( \mathcal{C}(X) \). Part I gives another way, namely, in terms of an inverse limit object of the \( (\mathcal{E}_t)_{t \in J} \), denoted there by \( E_\infty \). The elements of \( E_\infty \) are threads \((\mu_t)_{t \in J}\) giving the local state of each finite system in the lattice. It is shown in Part I that the two sets \( \mathcal{K}\mathcal{C}(X) \) and \( E_\infty \) are isomorphic. The “microcanonical state” refers to a state on \( \mathcal{C}(X) \) identified with a thread \((\mu_t) \in E_\infty \) in which all local states are microcanonical.

**B. Measurements**

We adopt Segal’s interpretation of the algebraic observables. In Segal’s terminology, the elements of \( \mathcal{C}(X) \) are the observables, and the values \( f(x) \), \( x \in X \), the spectral values of \( f \in \mathcal{C}(X) \). They are the only possible values of any measurement \( f^t \in \mathcal{M}(\mathfrak{A}) \) representable by \( f \). The mathematical states \( \zeta_\mu \in \mathcal{K}\mathcal{C}(X) \) define ensembles or distributions of the pure states \( X \), so that the expectation values of measurements are the quantities \( \zeta_\mu(f) \).

A description based on this terminology requires something conceptually close to the following algebraic picture of the classical measurement. In the preparation for measurement, the lattice is brought into a given state \( \zeta \in \mathcal{K}\mathcal{C}(X) \). The measurement begins with an instantaneous isolation that leaves it in a MC state \( x_\mu \in X \) randomly chosen from the ensemble defined by \( \zeta \). The outcome of the measurement \( f \in \mathcal{C}(X) \) is the MC average \( f(x_\mu) \), the result of time averaging, say. Its expectation value is \( \zeta(f) \), the integral over the possible outcomes \( x \in X \).

**III Mackey’s axioms**

This description of a measurement is readily turned into an axiomatic theory based on Mackey’s axioms for a quantum theory [12]. It will have the traditional form of a logic of propositions introduced by Birkhoff and von Neumann (2). For the commutative case, a Mackey system is defined by six axioms. In the following, we construct such a system from our triple \( \{X, \mathcal{C}(X), \mathcal{K}\mathcal{C}(X)\} \).

A theorem in Mackey gives sufficient conditions for a lattice of observables and its states to display his six axioms [12, p.68]. The next two propositions satisfy these conditions. The first pertains to observables.

**Proposition III.1.** The mapping \( \phi : \mathfrak{P} \rightarrow \mathcal{B}(X) \) by \( \phi(\chi_F^{(X)}) = F \) is a lattice-isomorphism from the class \( \mathfrak{P} \) of idempotents of \( \mathcal{C}(X) \) onto the topology \( \mathcal{B}(X) \) of \( X \). Hence, \( \mathfrak{P} \) is a complete Boolean algebra.

**Proof.** Observe first that \( \chi_F^{(X)} \in \mathcal{C}(X) \) iff \( F \) is clopen (=closed-and-open). It was shown in Part I that all open sets are clopen (Theorem VI.3). Hence, the idempotents are exactly the characteristic functions \( \chi_F^{(X)} \), \( F \in \mathcal{B}(X) \). Then \( \phi \) is clearly 1:1 and onto, i.e., \( \mathfrak{P} = \mathcal{B}(X) \). Also, \( \forall E, F \in \mathcal{B}(X), \phi(\chi_E^{(X)} \lor \chi_F^{(X)}) = \phi(\chi_{E \cup F}^{(X)}) = E \cup F = \phi(\chi_E^{(X)}) \lor \phi(\chi_F^{(X)}) \), and \( \phi(\chi_E^{(X)} \land \chi_F^{(X)}) = \phi(\chi_{E \cap F}^{(X)}) = F \).
\[ E \cap F = \phi(\chi_E^{(X)}) \cap \phi(\chi_F^{(X)}). \] The complementation is defined by \((\chi_F^{(X)})' = 1 - \chi_F^{(X)} = \chi_F^{(X)}\) and hence \(\phi((\chi_F^{(X)})') = F'\). Hence, \(\phi\) is a lattice isomorphism. For completeness, note simply that for an arbitrary net \((\chi_{F_i}^{(X)}) \cup F_i \in \mathcal{B}(X)\) (clopen), so that \(\bigvee \chi_{F_i}^{(X)} = \chi_{\bigcup F_i}^{(X)} \in \mathcal{P}\). One shows similarly that the lattice \(\mathcal{P}\) is distributive. The distributive property for infinite operations is given by Semadeni [19, Proposition 16.6.3]

The completeness of the lattice \(\mathcal{P}\) is equivalent to having \(X\) Stonean [14, 6.2.4], as pointed out in Part I. It is analogous to the completeness of the lattice of projections of the von Neumann algebra in algebraic QFT [20, Proposition V.1.1]. The distributive lattices are exactly those with a set representation [14] Birkhoff-Stone theorems, p. 104. Since the distributive property assures the pairwise compatibility of measurements, this Proposition assures that we are dealing with classical theory.

The second condition pertains to states.

**Proposition III.2.** Denote by \(\mathcal{S}\) the set of all restrictions \(\{\zeta_\mu^{(X)}, \mu \in E_\infty\}\). Then \(\mathcal{S}\) is a full and strongly convex set of states on \(\mathcal{P}\).

**Proof.** The state \(\zeta_\mu \in \mathcal{S}\) is a state on \(\mathcal{P}\) in Mackey’s sense if, in addition to \(\zeta_\mu(\chi_0^{(X)}) = 0\) and \(\zeta_\mu(\chi_X^{(X)}) = 1\), one has that for all sets of questions \((\chi_{F_i}^{(X)}) \in \mathcal{P}\) with \(F_i \cap F_j = \emptyset \forall i \neq j\), \(\zeta_\mu(\bigvee\chi_{F_i}^{(X)}) = \zeta_\mu(\chi_{\bigcup F_i}^{(X)}) = \sum \zeta_\mu(\chi_{F_i}^{(X)})\).

Certainly for all finite subsets of \((\chi_{F_i}^{(X)}), \zeta_\mu(\bigvee_{i=1}^k \chi_{F_i}^{(X)}) = \sum_{i=1}^k \zeta_\mu(\chi_{F_i}^{(X)}).\) The result then follows by continuity. Now note that for any pair \(\chi_E^{(X)}, \chi_F^{(X)},\) if \(\chi_E^{(X)}\) is not \(\leq \chi_F^{(X)},\) then there exists \(x_\mu \in X\) such that \(\chi_E^{(X)}(x_\mu) = 1\) and \(\chi_F^{(X)}(x_\mu) = 0\).

But \(\delta_{x_\mu} \in \mathcal{K}(X)\), while \(\delta_{x_\mu}(\chi_E^{(X)}) = 1\) and \(\delta_{x_\mu}(\chi_F^{(X)}) = 0\). Hence, \(\mathcal{S}\) is a full set of states, i.e., if \(\zeta_\mu(\chi_E^{(X)}) \leq \zeta_\mu(\chi_F^{(X)})\) for all \(\mu \in \mathcal{S}\), then \(\chi_E^{(X)} \leq \chi_F^{(X)}\). Finally, the set of states \(\mathcal{S}\) is strongly convex in Mackey’s sense if for any sequence \((t_n) \in [0, 1]\) such that \(\sum_{i=1}^\infty t_n = 1\) and any set \((\zeta_{\mu_n}) \in \mathcal{S}, \sum_{i=1}^\infty t_n \zeta_{\mu_n} \in \mathcal{S}.\) Certainly \(\sum_{i=1}^\infty t_n \zeta_{\mu_n}\) is a positive linear functional on \(\mathcal{K}(X)\) by continuity. Furthermore, \(\|\sum_{i=1}^\infty t_n \zeta_{\mu_n}\| = \sup_{\|f\| \leq 1} \sum_{i=1}^\infty t_n \zeta_{\mu_n}(f) = \sum_{i=1}^\infty t_n \zeta_{\mu_n}(\chi_X) = \sum_{i=1}^\infty t_n = 1.\) Hence, \(\sum_{i=1}^\infty t_n \zeta_{\mu_n} \in \mathcal{K}(X).\)

The Mackey axioms are in terms of a class of functions of the following form.

**Definition III.3.** Denote by \(\mathcal{B}\) the Borel sets of the real line \(\mathbb{R}\). The function \(Q : \mathcal{B} \to \mathcal{P}\) is called a \(\mathcal{P}\)-valued measure on \(\mathbb{R}\) iff the following obtain:

(a) \(Q(\emptyset) = 0, Q(\mathbb{R}) = 1;\)
(b) If \((B_n)\) is any family in \(\mathcal{B}\), and \(B_i \cap B_j = \emptyset \forall i \neq j\), then \(Q(\bigcup B_n) = \bigvee Q(B_n).\)
Note that $\bigvee Q(B_n) \in \mathfrak{P}$ because $\mathfrak{P}$ is complete.

Let $\mathcal{O}$ be the set of all $\mathfrak{P}$-valued measures on $(\mathbb{R}, \mathcal{B})$. $\mathcal{O}$ is the set of observables of the Mackey system $(\mathcal{O}, \mathcal{S}, \mathcal{B})$. With Propositions III.1 and III.2, we have proven the following.

**Theorem III.4.** The triple $\{\mathcal{O}, \mathcal{S}, \mathcal{B}\}$ is a Mackey system, satisfying Axioms I - VI. ■

It is noteworthy that with $\mathfrak{P}$ a complete lattice, the system $\{X, \mathcal{C}(X), \mathcal{K}\mathcal{C}(X)\}$ likewise satisfies the axioms of Piron from QFT [13].

### IV The theory of measurement

We divide discussion into two sections, dealing respectively with observables and states.

**A. Observables**

The role of the quasilocal observables depends on their identification with the elements of $\mathcal{O}$. Observe first that $\mathcal{O}$ is a large set. In fact, if $f \in \mathcal{C}(X)$ is any observable, and $B \in \mathcal{B}$, define $Q^f : \mathcal{B} \to \mathbb{Q}$ by $Q^f(B) \equiv Q^f_B = \chi_{\mathbb{Q}} \circ f = \chi_{\{f \in B\}}^{(X)}$. Recall that $\bigvee \chi^{(X)}_{B_i} = \chi^{(X)}_{\bigcup B_i}$. Hence $Q^f \in \mathcal{O}$. Axiom VI says that all of $\mathcal{O}$ is of this form:

**Proposition IV.1.** For any $f \in \mathcal{C}(X)$, define $Q^f : \mathcal{B} \to \mathcal{C}(X)$ by $Q^f_B = \chi^{(X)}_{\{f \in B\}}$. Then $Q^f \in \mathcal{O}$. Conversely, if $Q \in \mathcal{O}$ is any $\mathfrak{P}$-valued measure, then there exists a function $f \in \mathcal{C}(X)$ such that $Q = Q^f$. Thus, $\mathcal{O} = \mathcal{C}(X)$. ■

This gives a classical spectral theorem for $\mathcal{C}(X)$ as follows:

**Proposition IV.2.** For any $f \in \mathcal{C}(X)$, define $Q^f(\lambda) = \chi^{(X)}_{\{f \leq \lambda\}} \forall \lambda \in \mathbb{R}$. Then one may write any $f \in \mathcal{C}(X)$ in the following integral form:

$$f = \int_{-\infty}^{\infty} \lambda dQ^f(\lambda)$$  \hspace{1cm} (4.1)

Furthermore, for any continuous Borel function of $f$,

$$g \circ f = \int_{-\infty}^{\infty} g(\lambda) dQ^f(\lambda).$$  \hspace{1cm} (4.2)

**Proof.** Eq.(4.1) follows from the fact that for all $x \in X$, $Q^f(\cdot)(x)$ is a nondecreasing function on $\mathbb{R}$ [3, Theorem III.8.7]). Eq.(4.2) is by Mackey’s axiom III. ■
Using the language from Hilbert spaces, we call the $\mathfrak{P}$-valued measure $Q^f$ the spectral measure corresponding to the observable $f$, and eq. (4.1) the spectral decomposition of $f$.

Birkhoff and von Neumann motivated their logic of quantum mechanics with the epistemological judgment that “Before a phase-space can become imbued with reality, its elements and subsets must be correlated in some way with experimental propositions”, i.e., with the Borel sets of the real line $\mathbb{R}$ and its products $\mathbb{R}^n$ [2] p.825. The designation of the space $X$ as the algebraic theory’s “phase space” is their terminology. Each spectral measure $Q^f \in \mathcal{O}$ defines a correlation of the Borel sets in $\mathbb{R}$ with sets in the algebraic phase space $X$ as follows:

**Proposition IV.3.** For any $f \in \mathcal{C}(X)$, the measure $Q^f \in \mathcal{O}$ is a lattice homomorphism on $\mathcal{B}$ into the lattice $\mathfrak{P}$, transforming the operations $(\subseteq, \cup, \cap, \cap')$ to $(\leq, \vee, \wedge, \wedge')$ and preserving set inclusion. Hence, for any $f \in \mathcal{C}(X)$, the composite $\phi \circ Q^f : \mathcal{B} \to \mathcal{B}(X)$ is a lattice homomorphism on the Borel sets of $\mathbb{R}$ into $\mathcal{B}(X)$, where $\phi$ is the isomorphism defined in Proposition III.1.

**Proof.** Recall that $\mathfrak{P}$ is a complete lattice. One has $Q^f_{B_1 \cap B_2} = \chi_{[f \in B_1] \cap [f \in B_2]}^{(X)} = \chi_{[f \in B_1]}^{(X)} \wedge \chi_{[f \in B_2]}^{(X)}$ and $Q^f_{B_1 \cup B_2} = \chi_{[f \in B_1 \cup B_2]}^{(X)} = \chi_{[f \in B_1]}^{(X)} \vee \chi_{[f \in B_2]}^{(X)}$ for the meet and join, and $Q^f_{B^c} = 1 - \chi_{[f \in B]}^{(X)}$ for complementation. Furthermore, $E, F \in \mathcal{B}$, $E \subseteq F$, goes to $\chi_E^{(X)} \leq \chi_F^{(X)}$. 

This establishes the role of the algebraic observables in the theory.

**B. States**

The probability of the set $[f \in B]$ in the initial state $\zeta_\mu \in \mathcal{K}\mathcal{C}(X)$ is just the expectation value with respect to the probability measure $\sigma_\mu$ of the random variable $\chi_{[f \in B]}^{(X)}$:

$$\zeta_\mu(\chi_{[f \in B]}^{(X)}) = \int_X \chi_{[f \in B]}^{(X)} d\sigma_\mu = \int_{[f \in B]} d\sigma_\mu = \sigma_\mu([f \in B]) \quad (4.3)$$

This is the probability the measurement finds the system in a MC state $x$ belonging to the set $[f \in B] \subseteq X$ when the lattice is in state $\zeta_\mu$. It is given in terms of the spectral measures $Q^f$ by Mackey’s Axiom I as follows:

**Proposition IV.4.** For any observable $f \in \mathcal{C}(X)$ and any state $\zeta_\mu \in \mathcal{K}\mathcal{C}(X)$, the function $p$ defined by

$$p(Q^f, \zeta_\mu, B) = \zeta_\mu(\chi_{[f \in B]}^{(X)}) \quad (4.4)$$
is a probability measure on \((\mathbb{R}, \mathcal{B})\).

Following Haag and Kastler we call an algebraic theory a complete theory of measurement if for all Borel sets \(F \in \mathcal{B}(X)\), and for all algebraic states \(\zeta_\mu \in \mathcal{K}C(X)\) one can write the probability of finding the system in a MC state \(x \in F \subseteq X\), given that it is initially in the state \(\zeta_\mu\). To show that we have a complete theory in this sense, set \(f = \chi_F^{(X)}\). Then \(f \in \mathcal{C}(X)\), and from eq. (4.3), \(\zeta_\mu(\chi_{\{f \in (1/2,3/2)\}}) = \sigma_\mu(\{x \in F\})\). We have treated the measurements themselves as represented by local observables \(W_\Lambda\) and their states \(E_\Lambda\) referred to a particular system \(\Lambda\). Haag and Kastler regard operations of the form \(f = \chi_F^{(X)}\) as filters, passing the MC states in \(F\) and blocking the rest. Correspondingly, they call the probabilities \(\sigma_\mu(\{x \in F\})\) transmission probabilities.

V Applications

We conclude with two applications of the axiomatic theory. They pertain especially to results on the equivalence of ensembles in the thermodynamic-limit program. The first is a very basic question for the axiomatic theory itself: is the theory’s set of observables \(\mathcal{C}(X)\) large enough? Given any two states \(\zeta_\mu, \zeta_\nu \in \mathcal{K}C(X)\), is there an observable \(f \in \mathcal{C}(X)\) such that \(\zeta_\mu(f) \neq \zeta_\nu(f)\)? Since the mapping in Proposition IV.3 is into, not onto, the correlation provided by a measure \(Q^f \in \mathcal{O}\) does not in general define a state \(\zeta_\mu\) on all of \(\mathcal{B}(X)\). Nevertheless, it has the ability to distinguish two states \(\zeta_\mu, \zeta_\nu\) by measurements, as follows.

Proposition V.1. If \(p(Q^f, \zeta_\mu, B) = p(Q^g, \zeta_\mu, B)\) for all \(\mu \in E_\infty, B \in \mathcal{B}\), then \(f = g\). Conversely, if \(p(Q^f, \zeta_\mu, B) = p(Q^f, \zeta_\nu, B)\) for all \(f \in \mathcal{C}(X), B \in \mathcal{B}\), then the states \(\zeta_\mu = \zeta_\nu\).

Proof. Axiom III.

That is, states separate observables, and observables separate states. \(\mathcal{C}(X)\) contains points that do not represent measurements because \(\mathcal{W}\) is the completion of \(\mathcal{W}^\infty\). Nevertheless, the set \(W = \psi_K \circ \Delta_K(\mathcal{W}^\infty)\) is (strongly) dense in \(\mathcal{C}(X)\) (Part A, Corollary II.12, Theorem II.15), so that if any \(f \in C\) separates the states \(\zeta_\mu, \zeta_\nu\), we may construct a convergent sequence of functions \((g_n) \in W\) such that \(g_n \to f\), and it does contain points that separate these states.

It is noteworthy that this separation property does not conflict with results on the equivalence of ensembles. These theorems have to do with the convergence of sequences of local observables when the lattice is in one of the standard
ensembles (MC, canonical, grand canonical). They show that the sequences
converge in probability to the same limit functions for all three ensembles [11]
Theorem A5.8], i.e., as the sizes of systems get larger and larger, measurements
of any quantity give the *same values* in the three ensembles except possibly on
sets of configurations of decreasing probability. But in general, \( f^t \xrightarrow{P} f \) does
not assure that \( \int f^t d\mu_t \to \int f d\mu \) unless there exists an integrable dominating
function \( g \) such that \( |f^t| \leq |g| \) for all \( t \in J \) (Lebesgue Dominated Convergence
Theorem [11 Theorem 7.2.C]). Thus, agreement of the limit functions does not
assure agreement of the limits of their expectation values. In physical terms,
the dominating function has the effect of excluding large fluctuations from the
limiting value of an observable.

For the second application, we show that there is a weak equivalence of states
if one allows some experimental error in measurements. Specifically, suppose
the expectation value of a particular observable \( f \in C(X) \) in a given initial
state \( \zeta_\mu \) is only determined (or estimated) to within an accuracy of \( \zeta_\mu(f) \pm \varepsilon \).
Then the measurement cannot be used to separate \( \zeta_\mu \) from any state \( \zeta_\nu \) in
the \( w^* \)-neighborhood of \( \zeta_\mu \) defined by the basic open set \( N(\zeta_\mu; f, \varepsilon) = \{ \zeta_\nu : |\zeta_\mu(f) - \zeta_\nu(f)| < \varepsilon \} \). With repeated measurements, one can estimate the
relative frequency (or probability) of a set \( [f \in \mathcal{B}] \) to any degree of precision.
However, this cannot exclude these considerations with any finite number of
measurements. States close together in this sense are essentially physically
equivalent.

ACKNOWLEDGEMENT.

The author wishes to express his gratitude to Rudolf Haag for his many
suggestions during the writing of this manuscript.

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