Abstract. In this paper, we define a new subclass of bi-univalent functions involving $q$-difference operator in the open unit disk. For functions belonging to this class, we obtain estimates on the first two Taylor-Maclaurin coefficients $|a_2|$ and $|a_3|$.

1. Introduction and Preliminaries

Let $A$ denote the class of functions of the form

\[ f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \]

which are analytic in the open unit disk

\[ \Delta = \{ z : z \in \mathbb{C} \text{ and } |z| < 1 \}. \]

The convolution or Hadamard product of two functions $f, h \in A$ is denoted by $f * h$ and is defined as $(f * h)(z) = z + \sum_{n=2}^{\infty} a_n b_n z^n$, where $f(z)$ is given by (1) and $h(z) = z + \sum_{n=2}^{\infty} b_n z^n$.

An analytic function $f$ is subordinate to an analytic function $h$, written $f(z) \prec h(z) \ (z \in \Delta)$, provided there is an analytic function $w$ defined on $\Delta$ with $w(0) = 0$ and $|w(z)| < 1$ satisfying $f(z) = h(w(z))$.

By $S$ we shall denote the class of all functions in $A$ which are univalent in $\Delta$. Some of the important and well-investigated subclasses of the univalent function

AMS (2010) Subject Classification: 30C45.

Keywords and phrases: Univalent function, Bi-Starlike function, Bi-Convex function, Hadamard product, $q$-derivative operator.

* Corresponding author.
class $S$ include (for example) the class $S^\alpha(\alpha)$ of starlike functions of order $\alpha$ in $\Delta$ and the class $K(\alpha)$ of convex functions of order $\alpha$ in $\Delta$.

Ma and Minda [13] unified various subclasses of starlike functions and convex functions which consist of functions $f \in A$ satisfying the subordinations

$$\frac{zf'(z)}{f(z)} < \phi(z) \text{ and } 1 + \frac{zf''(z)}{f'(z)} < \phi(z),$$

respectively, here (and throughout this paper) $\phi$ with positive real part in the unit disk $\Delta$, $\phi(0) = 1$, $\phi'(0) > 0$ and $\phi$ maps $\Delta$ onto a region starlike with respect to 1 and symmetric with respect to the real axis. Such a function has the form

$$\phi(z) = 1 + B_1 z + B_2 z^2 + B_3 z^3 + \ldots, \quad B_1 > 0.$$

It is well known that every function $f \in S$ has an inverse $f^{-1}$, defined by

$$f^{-1}(f(z)) = z, \quad z \in \Delta$$

and

$$f(f^{-1}(w)) = w, \quad |w| < r_0(f), \quad r_0(f) \geq \frac{1}{4}.$$

where

$$g(w) = f^{-1}(w) = w - a_2 w^2 + (2a_2^2 - a_3)w^3 - (5a_2^3 - 5a_2a_3 + a_4)w^4 + \ldots. \quad (3)$$

A function $f \in A$ is said to be bi-univalent in $\Delta$ if both $f$ and $f^{-1}$ are univalent in $\Delta$, in the sense that $f^{-1}$ has a univalent analytic continuation to $\Delta$. Let $\Sigma$ denote the class of bi-univalent functions in $\Delta$ given by (1).

A function $f$ is bi-starlike of Ma-Minda type or bi-convex of Ma-Minda type if both $f$ and $f^{-1}$ are respectively Ma-Minda starlike or convex. These classes are denoted respectively by $S^*_{\Sigma}(\phi)$ and $K_{\Sigma}(\phi)$.

Now we recall here the notion of $q$-operator i.e., $q$-difference operator that play vital role in the theory of hypergeometric series, quantum physics and in the operator theory. The application of $q$-calculus was initiated by Jackson [8], recently Kanas and Râducanu [11] have used the fractional $q$-calculus operators in investigations of certain classes of functions which are analytic in $U$.

Let $0 < q < 1$. For any non-negative integer $n$, the $q$-integer number $n$ is defined by

$$[n]_q = \frac{1 - q^n}{1 - q} = 1 + q + \ldots + q^{n-1}, \quad [0]_q = 0. \quad (4)$$

In general, we will denote

$$[x]_q = \frac{1 - q^x}{1 - q}$$

for a non-integer number $x$. Also the $q$-number shifted factorial is defined by

$$[n]_q! = [n]_q[n - 1]_q \ldots [2]_q[1]_q, \quad [0]_q! = 1.$$

Clearly,

$$\lim_{q \to 1^-} [n]_q = n \quad \text{and} \quad \lim_{q \to 1^-} [n]_q! = n!.$$
For $0 < q < 1$, the Jackson’s $q$-derivative operator (or $q$-difference operator) of a function $f \in \mathcal{A}$ given by (1) defined as follows [8]

$$D_q f(z) = \begin{cases} f(z) - f(qz) & \text{for } z \neq 0, \\ f'(0) & \text{for } z = 0, \end{cases}$$

$$D_0^q f(z) = f(z),$$

$$D_m^q f(z) = D_q(D_{m-1}^q f(z)) \quad \text{for } m \in \mathbb{N} = \{1, 2, \ldots\}.$$

From (5), we have

$$D_q f(z) = 1 + \sum_{n=2}^{\infty} [n]_q a_n z^n - 1, \quad z \in \Delta,$$

where $[n]_q$ is given by (4).

For a function $h(z) = z^n$ we obtain

$$D_q h(z) = D_q z^n = \frac{1 - q^n}{1 - q} z^{n-1} = [n]_q z^{n-1}$$

and

$$\lim_{q \to 1^-} D_q h(z) = \lim_{q \to 1^-} ([n]_q z^{n-1}) = nz^{n-1} = h'(z),$$

where $h'$ is the ordinary derivative.

Let $t \in \mathbb{R}$ and $n \in \mathbb{N}$. The $q$-generalized Pochhammer symbol is defined by

$$[t; n]_q = [t]_q [t+1]_q [t+2]_q \ldots [t+n-1]_q$$

and for $t > 0$ the $q$-gamma function is defined by

$$\Gamma_q(t+1) = [t]_q \Gamma_q(t) \quad \text{and} \quad \Gamma_q(1) = 1.$$

Using the $q$-difference operator, Kannas and Raducanu [11] defined the Ruscheweyh $q$-differential operator as below. For $f \in \mathcal{A}$,

$$R_q^\delta f(z) = f(z) * F_{q,\delta+1}(z), \quad \delta > -1, \quad z \in \Delta,$$

where

$$F_{q,\delta+1}(z) = z + \sum_{n=2}^{\infty} \frac{\Gamma_q(n + \delta)}{[n-1]_q! \Gamma_q(1 + \delta)} z^n = z + \sum_{n=2}^{\infty} \frac{[\delta + 1; n]_q}{[n-1]_q!} z^n.$$  

We note that

$$\lim_{q \to 1^-} F_{q,\delta+1}(z) = \frac{z}{(1 - z)^{\delta+1}}, \quad \lim_{q \to 1^-} R_q^\delta f(z) = f(z) * \frac{z}{(1 - z)^{\delta+1}}.$$

Making use of (6) and (7), we have

$$R_q^\delta f(z) = z + \sum_{n=2}^{\infty} \frac{\Gamma_q(n + \delta)}{[n-1]_q! \Gamma_q(1 + \delta)} a_n z^n, \quad z \in \Delta.$$
From \( (8) \), we note that
\[
\mathfrak{R}_0^q f(z) = f(z), \quad \mathfrak{R}_1^q f(z) = z D_q f(z), \quad \mathfrak{R}_m^q f(z) = \frac{z D_q^m(z^{m-1} f(z))}{[m]_q!} \text{ for } m \in \mathbb{N}.
\]
Also we have
\[
D_q(\mathfrak{R}_q^q f(z)) = 1 + \sum_{n=2}^{\infty} \Theta_n(q, \delta) a_n z^{n-1}, \quad (9)
\]
where
\[
\Theta_n := \Theta_n(q, \delta) = \frac{[n]_q \Gamma_q(n + \delta)}{[n-1]_q \Gamma_q(1 + \delta)}. \quad (10)
\]

For our study, we will use the short presentation
\[
\Theta_2 = \Theta_2(q, \delta) = \frac{[2]_q \Gamma_q(2 + \delta)}{\Gamma_q(1 + \delta)}, \quad \Theta_3 = \Theta_3(q, \delta) = \frac{[3]_q \Gamma_q(3 + \delta)}{[2]_q \Gamma_q(1 + \delta)}.
\]

Recently there has been triggering interest to study bi-univalent function class \( \Sigma \) and obtained non-sharp coefficient estimates on the first two coefficients \( |a_2| \) and \( |a_3| \) of \( (1) \). But the coefficient problem for each of the following Taylor-Maclaurin coefficients
\[
|a_n|, \quad n \in \mathbb{N} \setminus \{1, 2, 3\}
\]
is still an open problem (see \[2, 3, 4, 12, 14, 18\]). Many researchers (see \[1, 7, 9, 17\]) have recently introduced and investigated several interesting subclasses of the bi-univalent function class \( \Sigma \).

Motivated by the earlier work of Bulut \[5\], Deniz \[6\], Inayat Noor \[10\] and Srivastava et al. \[16\], in the present paper we introduce new families of Bazilevič functions of complex order of the function class \( \Sigma \), involving the operator \( D_q(\mathfrak{R}_q^q f(z)) \), and find estimates on the coefficients \( |a_2| \) and \( |a_3| \) for functions in the new subclass of function class \( \Sigma \). Several related classes are also considered, and connection to earlier known results are made.

**Definition 1.1**
A function \( f \in \Sigma \) given by \( (1) \) is said to be in the class \( S^q_2(\gamma, \lambda, \delta; \phi) \) if the following conditions are satisfied:
\[
1 + \frac{1}{\gamma} \left( z^{1-\lambda} D_q(\mathfrak{R}_q^q f(z)) - 1 \right) \prec \phi(z)
\]
and
\[
1 + \frac{1}{\gamma} \left( w^{1-\lambda} D_q(\mathfrak{R}_q^q g(w)) - 1 \right) \prec \phi(w),
\]
where \( z, w \in \Delta, \gamma \in \mathbb{C} \setminus \{0\}, \delta > -1, \lambda \geq 0 \) and the function \( g = f^{-1} \) is given by \( (3) \).
Remark 1.1
The following special cases of Definition 1.1 are worthy of note:

(i) A function $f \in \Sigma$ given by (1) is said to be in the class $S_q^\gamma(\gamma, 0, \delta; \phi) \equiv S_q^\gamma(\gamma, \delta; \phi)$ if the following conditions are satisfied:

$$1 + \frac{1}{\gamma} \left( \frac{z^D_q(R_\delta^q f(z))}{R_\delta^q f(z)} - 1 \right) \prec \phi(z)$$

and

$$1 + \frac{1}{\gamma} \left( \frac{w^D_q(R_\delta^q g(w))}{R_\delta^q g(w)} - 1 \right) \prec \phi(w),$$

where $z, w \in \triangle, \gamma \in \mathbb{C} \setminus \{0\}, \delta > -1$ and the function $g = f^{-1}$ is given by (3).

(ii) A function $f \in \Sigma$ given by (1) is said to be in the class $S_q^\gamma(\gamma, 1, \delta; \phi) \equiv H_q\Sigma^\gamma(\gamma, \delta; \phi)$ if the following conditions are satisfied:

$$1 + \frac{1}{\gamma} (D_q(R_\delta^q f(z)) - 1) \prec \phi(z)$$

and

$$1 + \frac{1}{\gamma} (D_q(R_\delta^q g(w)) - 1) \prec \phi(w),$$

where $z, w \in \triangle, \gamma \in \mathbb{C} \setminus \{0\}, \delta > -1$ and the function $g = f^{-1}$ is given by (3).

(iii) If we set $\phi(z) = \frac{1+Az}{1+Bz}, -1 \leq B < A \leq 1$, then the class $S_q^\gamma(\gamma, \lambda, \delta; \phi) \equiv S_q^\gamma(\gamma, \lambda, \delta; A, B)$ which is defined as $f \in \Sigma$,

$$1 + \frac{1}{\gamma} \left( z^{1-\lambda} D_q(R_\delta^q f(z)) \right) \left( R_\delta^q f(z) \right)^{1-\lambda} - 1 \prec \frac{1 + Az}{1 + Bz}$$

and

$$1 + \frac{1}{\gamma} \left( w^{1-\lambda} D_q(R_\delta^q g(w)) \right) \left( R_\delta^q g(w) \right)^{1-\lambda} - 1 \prec \frac{1 + Aw}{1 + Bw},$$

where $z, w \in \triangle, \gamma \in \mathbb{C} \setminus \{0\}, \delta > -1, \lambda \geq 0$ and the function $g = f^{-1}$ is given by (3).

(iv) If we set $\phi(z) = \frac{1+1-2\beta z}{1-z}, 0 \leq \beta < 1$, then the class $S_q^\gamma(\gamma, \lambda, \delta; \phi) \equiv S_q^\gamma(\gamma, \lambda, \delta; \beta)$ which is defined as $f \in \Sigma$,

$$\Re \left[ 1 + \frac{1}{\gamma} \left( z^{1-\lambda} D_q(R_\delta^q f(z)) \right) \left( R_\delta^q f(z) \right)^{1-\lambda} - 1 \right] > \beta$$

and

$$\Re \left[ 1 + \frac{1}{\gamma} \left( w^{1-\lambda} D_q(R_\delta^q g(w)) \right) \left( R_\delta^q g(w) \right)^{1-\lambda} - 1 \right] > \beta,$$

where $z, w \in \triangle, \gamma \in \mathbb{C} \setminus \{0\}, \delta > -1, \lambda \geq 0$ and the function $g = f^{-1}$ is given by (3).
On specializing the parameters $\lambda$ and $\delta$, one can state the various new subclasses of $\Sigma$.

2. Coefficient Bounds for the class $S^q_{\Sigma}(\gamma, \lambda, \delta; \phi)$

We begin by finding the estimates on the coefficients $|a_2|$ and $|a_3|$ for functions in the class $S^q_{\Sigma}(\gamma, \lambda, \delta; \phi)$.

In order to derive our main results, we shall need the following lemma.

**Lemma 2.1** (see [15])
If $p \in \mathcal{P}$, then $|p_k| \leq 2$ for each $k$, where $\mathcal{P}$ is the family of all functions $p$ analytic in $\Delta$ for which $\Re(p(z)) > 0$, where $p(z) = 1 + p_1 z + p_2 z^2 + \ldots$ for $z \in \Delta$.

**Theorem 2.1**
Let the function $f(z)$ given by (1) be in the class $S^q_{\Sigma}(\gamma, \lambda, \delta; \phi)$. Then

$$|a_2| \leq \sqrt{\frac{N}{D}},$$

where

$$N = 2|\gamma|^2 B_1^2 (1 + q)^2 (1 + q + q^2),$$

$$D = |\gamma B_1^2 [2(1 + q)^2 (\lambda + q + q^2) \Theta_3 + (\lambda - 1)(\lambda + 2q)(1 + q + q^2) \Theta_3^2] - 2(B_2 - B_1)(\lambda + q)^2 (1 + q + q^2) \Theta_3^2|$$

and

$$|a_3| \leq \left(\frac{|\gamma| B_1 (1 + q)}{(\lambda + q) \Theta_2} \right)^2 + \left(\frac{|\gamma| B_1 (1 + q + q^2)}{(\lambda + q + q^2) \Theta_3} \right).$$

**Proof.** Let $f \in S^q_{\Sigma}(\gamma, \lambda, \delta; \phi)$ and $g = f^{-1}$ be given by (3). Then there are analytic functions $u, v: \Delta \rightarrow \Delta$ with $u(0) = 0 = v(0)$, satisfying

$$1 + \frac{1}{\gamma} \left(\frac{z^{1-\lambda} \Re_d(\Re_d f(z))}{(\Re_d f(z))^{1-\lambda}} - 1\right) = \phi(u(z))$$

and

$$1 + \frac{1}{\gamma} \left(\frac{z^{1-\lambda} \Re_d(\Re_d g(w))}{(\Re_d g(w))^{1-\lambda}} - 1\right) = \phi(v(w)).$$

Define the functions $p(z)$ and $q(z)$ by

$$p(z) := \frac{1 + u(z)}{1 - u(z)} = 1 + p_1 z + p_2 z^2 + \ldots$$

and

$$q(z) := \frac{1 + v(z)}{1 - v(z)} = 1 + q_1 z + q_2 z^2 + \ldots$$

or, equivalently,

$$u(z) := \frac{p(z) - 1}{p(z) + 1} = \frac{1}{2} \left[p_1 z + \left(p_2 - \frac{p_2^2}{2}\right) z^2 + \ldots\right].$$
From (19) and (21), it follows that
\[ v(z) := \frac{q(z) - 1}{q(z) + 1} = \frac{1}{2} \left[ q_1 z + \left( q_2 - \frac{q_1^2}{2} \right) z^2 + \ldots \right]. \] (16)

Then \( p(z) \) and \( q(z) \) are analytic in \( \Delta \) with \( p(0) = q(0) = 1 \). Since \( u, v: \Delta \to \Delta \), the functions \( p(z) \) and \( q(z) \) have a positive real part in \( \Delta \), and \( |p_k| \leq 2 \) and \( |q_k| \leq 2 \) for each \( k \). Using (15) and (16) in (13) and (14) respectively, we have
\[ 1 + \frac{1}{\gamma} \left( \frac{z^{1-\lambda} D_q \left( D_q f(z) \right)}{(R_q g(w))^{1-\lambda}} - 1 \right) = \phi \left( \frac{1}{2} \left[ p_1 z + \left( p_2 - \frac{p_1^2}{2} \right) z^2 + \ldots \right] \right) \] (17)
and
\[ 1 + \frac{1}{\gamma} \left( \frac{z^{1-\lambda} D_q \left( R_q g(w) \right)}{(R_q g(w))^{1-\lambda}} - 1 \right) = \phi \left( \frac{1}{2} \left[ q_1 w + \left( q_2 - \frac{q_1^2}{2} \right) w^2 + \ldots \right] \right). \] (18)

In light of (3)–(10), from (17) and (18), it is evident that
\[ \begin{aligned} 1 + \frac{1}{\gamma} \left( \frac{z^{1-\lambda} D_q \left( R_q g(w) \right)}{(R_q g(w))^{1-\lambda}} - 1 \right) &= \phi \left( \frac{1}{2} \left[ q_1 w + \left( q_2 - \frac{q_1^2}{2} \right) w^2 + \ldots \right] \right), \\
&= 1 + \frac{1}{2} B_1 q_1 w + \left[ \frac{1}{2} B_1 \left( p_2 - \frac{p_1^2}{2} \right) + \frac{1}{4} B_2 p_1^2 \right] w^2 + \ldots \end{aligned} \]
and
\[ \begin{aligned} 1 - \frac{1}{\gamma} \left( \frac{z^{1-\lambda} D_q \left( R_q g(w) \right)}{(R_q g(w))^{1-\lambda}} - 1 \right) &= \phi \left( \frac{1}{2} \left[ - \frac{\lambda + q}{1 + q} \Theta_2 a_2 + \frac{\lambda + q + q^2}{1 + q + q^2} \Theta_3 a_3 \right. \\
&\left. + \left( \frac{2 \lambda + q + q^2}{1 + q + q^2} \Theta_3 + \frac{(\lambda - 1)(\lambda + 2q)}{2(1 + q)^2} \Theta_2^2 \right) a_2 \right] w^2 + \ldots \right), \\
&= 1 + \frac{1}{2} B_1 q_1 w + \left[ \frac{1}{2} B_1 \left( - \frac{q_1}{2} \right) + \frac{1}{4} B_2 q_1^2 \right] w^2 + \ldots \end{aligned} \]
which yields the following relations:
\[ \frac{\lambda + q}{1 + q} \Theta_2 a_2 = \frac{\gamma}{2} B_1 p_1, \] (19)
\[ \frac{\lambda + q + q^2}{1 + q + q^2} \Theta_3 a_3 + \frac{(\lambda - 1)(\lambda + 2q)}{2(1 + q)^2} \Theta_2^2 a_2 = \frac{\gamma}{2} B_1 \left( p_2 - \frac{p_1^2}{2} \right) + \frac{\gamma}{4} B_2 p_1^2, \] (20)
\[ - \frac{\lambda + q}{1 + q} \Theta_2 a_2 = \frac{\gamma}{2} B_1 q_1, \] (21)
and
\[ \begin{aligned} \frac{\lambda + q + q^2}{1 + q + q^2} \Theta_3 a_3 + \left( \frac{2 \lambda + q + q^2}{1 + q + q^2} \Theta_3 + \frac{(\lambda - 1)(\lambda + 2q)}{2(1 + q)^2} \Theta_2^2 \right) a_2 \\
&= \frac{\gamma}{2} B_1 \left( - \frac{q_1}{2} \right) + \frac{\gamma}{4} B_2 q_1^2. \end{aligned} \] (22)

From (19) and (21), it follows that
\[ p_1 = -q_1 \] (23)
Adding (20) and (22), we obtain
\[
\left(2\lambda + q + q^2\right)\Theta_3 + \frac{(\lambda - 1)(\lambda + 2q)}{(1 + q)^2}\Theta_2\hat{a}_2^2
\]
\[
= \frac{\gamma B_1}{2}(p_2 + q_2) + \frac{\gamma}{4}(B_2 - B_1)(p_1^2 + q_1^2).
\]

Using (24) in (25), we get
\[
\hat{a}_2^2 = \frac{N_0}{D_0},
\]
where
\[
N_0 = \gamma^2 B_1^3(1 + q)^2(1 + q + q^2)(p_2 + q_2),
\]
\[
D_0 = 2\gamma B_1^3[2(1 + q)^2(\lambda + q + q^2)\Theta_3 + (\lambda - 1)(\lambda + 2q)(1 + q + q^2)\Theta_2^2]
\]
\[- 4(B_2 - B_1)(\lambda + q)^2(1 + q + q^2)\Theta_3^2].
\]

Applying Lemma 2.1 for the coefficients \(p_2\) and \(q_2\), we immediately have
\[
|\hat{a}_2|^2 \leq \frac{N}{D},
\]
where
\[
N = 2|\gamma|^2 B_1^3(1 + q)^2(1 + q + q^2),
\]
\[
D = |\gamma B_1^3[2(1 + q)^2(\lambda + q + q^2)\Theta_3 + (\lambda - 1)(\lambda + 2q)(1 + q + q^2)\Theta_2^2]
\[- 2(B_2 - B_1)(\lambda + q)^2(1 + q + q^2)\Theta_3^2]].
\]

This gives the bound on \(|\hat{a}_2|\) as asserted in (11).

Next, in order to find the bound on \(|\hat{a}_3|\), by subtracting (22) from (20), we get
\[
2\lambda + q + q^2\Theta_3\hat{a}_3 - 2\lambda + q + q^2\Theta_3\hat{a}_2^2
\]
\[
= \frac{\gamma B_1}{2}\left[(p_2 - q_2) - \frac{1}{2}(p_1^2 - q_1^2)\right] + \frac{\gamma B_2}{4}(p_1^2 - q_1^2).\]

Using (23) and (24) in (26), we get
\[
\hat{a}_3 = \frac{\gamma^2 B_1^2(1 + q)^2(p_1^2 + q_1^2)}{8(\lambda + q)^2\Theta_3^2} + \frac{\gamma B_1(1 + q + q^2)(p_2 - q_2)}{4(\lambda + q + q^2)\Theta_3}.
\]

Applying Lemma 2.1 once again for the coefficients \(p_1, q_1, p_2\) and \(q_2\), we readily get (12). This completes the proof of Theorem 2.1.
3. Corollaries and Consequences

By setting \( \lambda = 0 \) in Theorem 2.1, we have the following Theorem.

**Theorem 3.1**

*Let the function \( f(z) \) given by (1) be in the class \( S^q_{\Sigma}(\gamma, \delta; \phi) \). Then*

\[
|a_2| \leq \frac{|\gamma|^2 B_1^2 (1 + q)^2 (1 + q + q^2)}{q|\gamma| B_1^2 [(1 + q)^3 \Theta_3 - (1 + q + q^2) \Theta_1^2] - (B_2 - B_1)q(1 + q + q^2) \Theta_2^2}
\]

and

\[
|a_3| \leq \left( \frac{|\gamma| |B_1(1 + q)|}{q \Theta_2} \right)^2 + \frac{|\gamma| |B_1(1 + q + q^2)|}{q(1 + q) \Theta_3}.
\]

By setting \( \lambda = 1 \) in Theorem 2.1, we have the following result.

**Theorem 3.2**

*Let the function \( f(z) \) given by (1) be in the class \( H^q_{\Sigma}(\gamma, \delta; \phi) \). Then*

\[
|a_2| \leq \sqrt{\frac{2|\gamma|^2 B_1^2 (1 + q)^2}{|\gamma| B_2^2 [2(1 + q)^2 \Theta_3 + (1 + 2q) \Theta_1^2] - 2(B_2 - B_1)(1 + q + q^2) \Theta_2^2}}
\]

and

\[
|a_3| \leq \left( \frac{|\gamma| B_1}{\Theta_2} \right)^2 + \frac{|\gamma| B_1}{\Theta_3}.
\]

By setting \( \phi(z) = \frac{1 + Az}{1 + Bz} \), \(-1 \leq B < A \leq 1\), in Theorem 2.1 we state the following Theorem.

**Theorem 3.3**

*Let the function \( f(z) \) given by (1) be in the class \( S^q_{\Sigma}(\gamma, \delta; A, B) \). Then*

\[
|a_2| \leq \sqrt{\frac{N}{D}},
\]

where

\[
N = 2|\gamma|^2 (A - B)^2 (1 + q)^2 (1 + q + q^2),
\]

\[
D = |\gamma| (A - B)[2(1 + q)^2 (\lambda + q + q^2) \Theta_3 + (\lambda - 1)(\lambda + 2q)(1 + q + q^2) \Theta_1^2]
+ 2(B + 1)(\lambda + q)^2 (1 + q + q^2) \Theta_2^2]
\]

and

\[
|a_3| \leq \left( \frac{|\gamma|(A - B)(1 + q)}{(\lambda + q) \Theta_2} \right)^2 + \frac{|\gamma|(A - B)(1 + q + q^2)}{(\lambda + q + q^2) \Theta_3}.
\]

Further, by setting \( \phi(z) = \frac{1 + (1 - 2\beta)z}{1 - z} \), \( 0 \leq \beta < 1 \) in Theorem 2.1 we get the following result.
Theorem 3.4

Let the function \( f(z) \) given by (1) be in the class \( S_q^\gamma(\gamma, \lambda, \delta; \beta) \). Then

\[
|a_2| \leq \sqrt{\frac{4|\gamma|(1-\beta)(1+q)^2(1+q+q^2)}{[2(1+q)^2(\lambda+q+q^2)\Theta_3 + (\lambda-1)(\lambda+2q)(1+q+q^2)\Theta_3^2]}}
\]

and

\[
|a_3| \leq \frac{(2|\gamma|(1-\beta)(1+q))}{(\lambda+q)\Theta_2} \left( \frac{2\gamma(1-\beta)(1+q+q^2)}{(\lambda+q+q^2)\Theta_3} \right)^2 + \frac{2\gamma(1-\beta)(1+q+q^2)}{(\lambda+q+q^2)\Theta_3}.
\]

Concluding Remarks. By taking \( \delta = 0 \) and specializing the parameters \( \lambda \) and \( \gamma \), various other interesting corollaries and consequences of our main results (which are asserted by Theorem 2.1 above) can be derived easily hence we omit the details.

References

[1] Ali, Rosihan M. et all. "Coefficient estimates for bi-univalent Ma-Minda starlike and convex functions." Appl. Math. Lett. 25, no. 3 (2012): 344–351. Cited on [8]

[2] Brannan, David A., and James Clunie, and William E. Kirwan. "Coefficient estimates for a class of star-like functions." Canad. J. Math. 22 (1970): 476–485. Cited on [8]

[3] Aspects of Contemporary Complex Analysis. Edited by David A. Brannan and James G. Clunie. London: Academic Press, 1980. Cited on [5]

[4] Brannan, David A., and T.S. Taha. "On some classes of bi-univalent functions." Studia Univ. Babeş-Bolyai Math. 31, no. 2 (1986): 70–77. Cited on [5]

[5] Bulut, Serap. "Certain subclasses of analytic and bi-univalent functions involving the q-derivative operator." Commun. Fac. Sci. Univ. Ank. Sér. A1 Math. Stat. 66, no. 1 (2017): 108–114. Cited on [8]

[6] Deniz, Erhan. "Certain subclasses of bi-univalent functions satisfying subordinate conditions." J. Class. Anal. 2, no. 1 (2013): 49–60. Cited on [9]

[7] Frasin, Basem A., and Mohamed K. Aouf. "New subclasses of bi-univalent functions." Appl. Math. Lett. 24, no. 9 (2011): 1569–1573. Cited on [8]

[8] Jackson, F.H. "On q-functions and a certain difference operator." Trans. Royal Soc. Edinburgh 46, no. 2 (1909): 253–281. Cited on [6] and [7]

[9] Hayami, Toshio, and Shigeyoshi Owa. "Coefficient bounds for bi-univalent functions." PanAmer. Math. J. 22, no. 4 (2012): 15–26. Cited on [8]

[10] Inayat Noor, Khalida. "On Bazilevic functions of complex order." Nihonkai Math. J. 3, no. 2 (1992): 115–124. Cited on [8]

[11] Kanas, Stanisława, and Dorina Răducanu. "Some class of analytic functions related to conic domains." Math. Slovaca 64, no. 5 (2014): 1183–1196. Cited on [6] and [7]

[12] Lewin, Mordechai. "On a coefficient problem for bi-univalent functions." Proc. Amer. Math. Soc. 18 (1967): 63–68. Cited on [9]

[13] Ma, Wan Cang, and David Minda. "A unified treatment of some special classes of univalent functions." In Proceedings of the Conference on Complex Analysis (Tianjin, 1992), 157–169. Vol I of Conf. Proc. Lecture Notes Anal. Cambridge, MA: Int. Press, 1994. Cited on [9]
Bi-Bazilevič functions of complex order involving Ruscheweyh type operator

[14] Netanyahu, Elisha. "The minimal distance of the image boundary from the origin and the second coefficient of a univalent function in \(z < 1\)." *Arch. Rational Mech. Anal.* 32 (1969): 100–112. Cited on 8

[15] Pommerenke, Christian. *Univalent functions.* Vol. 25 of Mathematische Lehrbücher. Göttingen: Vandenhoeck & Ruprecht, 1975. Cited on 10

[16] Srivastava, H. M., and G. Murugusundaramoorthy, and K. Vijaya. "Coefficient estimates for some families of bi-Bazilevič functions of the Ma-Minda type involving the Hohlov operator." *J. Class. Anal.* 2, no. 2, (2013): 167–181. Cited on 8

[17] Srivastava, Hari M., and Akshaya K. Mishra, and Priyabrat Gochhayat. "Certain subclasses of analytic and bi-univalent functions." *Appl. Math. Lett.* 23, no. 10, (2010): 1188–1192. Cited on 8

[18] Taha, T.S. *Topics in Univalent Function Theory*, Ph.D. Thesis. London: University of London, 1981. Cited on 8

Gangadharan Murugusundaramoorthy
School of Advanced Sciences
VIT University
Vellore, India - 632 014
India
E-mail: gmsmoorthy@yahoo.com

Serap Bulut
Faculty of Aviation and Space Sciences
Arslanbey Campus
Kocaeli University
41285 Kartepe-Kocaeli
Turkey
E-mail: serap.bulut@kocaeli.edu.tr

Received: November 16, 2017; final version: February 6, 2018; available online: April 3, 2018.