Wave fields with a periodic orbital angular momentum gradient along a single axis: a chain of vortices

A O Santillán¹, K Volke-Sepúlveda²,³ and A Flores-Pérez¹,⁴

¹ Centro de Ciencias Aplicadas y Desarrollo Tecnológico, Universidad Nacional Autónoma de México, Apdo. Postal 70-186, 04510 México DF, México
² Instituto de Física, Universidad Nacional Autónoma de México, Apdo. Postal 20-364, 01000 Mexico DF, Mexico
E-mail: karen@fisica.unam.mx

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Abstract. We present a theoretical analysis and an experimental demonstration of a wave field constituted by an axial array of vortices consecutively inverting the topological charge over distances of one-quarter of a wavelength. It exhibits a periodic gradient of the orbital angular momentum density along a single axis, although the field is created by the superposition of counter-propagating waves possessing no angular momentum themselves. The experiment is carried out with acoustic waves, whose wavelength is sufficiently large to allow a three-dimensional mapping of the field in amplitude and phase.

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³ Author to whom any correspondence should be addressed.
⁴ Scholar in a research assistance program of professional practices (Facultad de Ciencias, UNAM).
1. Introduction

The existence of different kinds of wave front dislocations or singularities was established by Nye and Berry in 1974 [1]. The term ‘dislocation’ was coined in connection with crystal physics, due to the topological similitude between wave front singularities and dislocation defects in crystals [1]. Currently, wave field topology is recognized as a major subject in different scientific areas [1]–[6], including geology and oceanography, quantum mechanics, acoustics and, in general, all the areas involving wave phenomena. Particularly, in optics this has given birth to a wide area known at present as singular optics [7, 8].

A wave vortex is a screw dislocation, which lies in a plane perpendicular to the wave vector and whose phase changes by an integer multiple of $2\pi$ along a closed loop around the vortex core, where the amplitude vanishes [1]. This integer number is known as the topological charge or singularity strength. In three dimensions, wave vortices correspond to lines [1, 8]. They might interact with one another [9] or even form links and knots [10]–[13], some of which have been experimentally observed [14, 15]. Furthermore, vortex arrays can be created by the interference of three or more plane waves [16, 17] or spherical waves [17, 18]. These kinds of superpositions are the basis for modeling the speckle patterns arising in coherent optical waves [19]–[21]. An especially relevant feature of screw type singularities is its potential indication of the presence of orbital angular momentum (OAM) in a wave field.

In fact, light beams with OAM are examples of optical vortices. Since the calculation of the OAM of the Laguerre–Gaussian (LG) laser modes [22], these beams and other optical vortices have been used to optically trap and rotate microscopic particles [23]–[25], cold atom clouds [26] and Bose–Einstein condensates [27, 28]. Interestingly, it has been demonstrated that modes with phase singularities in Bose–Einstein condensates are robust to decoherence effects, opening the possibility of quantum information storage in atomic vapors [28].

On the other hand, wave front dislocations have been studied in the realm of acoustics since the earliest stage of development of the wave field topology area [1, 29]. It is relatively recently, however, that acoustic vortices have been experimentally generated and studied in both the linear [30]–[33] and nonlinear [32]–[34] regimes. Moreover, the torque exerted by sound vortices on matter has been quantified for the case of a disk [35]–[37]. In terms of topological studies of scalar wave fields, sound waves might represent a more versatile alternative than light in some cases, due to the larger wavelength and the possibility of performing a direct scan of both amplitude and phase of the field.

In this paper, we present a general theoretical analysis of wave fields with a periodic array of oppositely charged vortices along a single axis, with a consequent gradient of OAM. An experimental demonstration of this chain of vortices is carried out with acoustical waves. Nevertheless, this kind of field is realizable with other type of waves, such as electromagnetic and matter waves, which may lead to a number of potential applications.

2. Theoretical analysis of wave fields with an OAM gradient (OAMG)

Our analysis is based on a scalar description of monochromatic wave fields, appropriated for longitudinal sound waves and for paraxial light fields. We also assume propagation through a linear medium, where the waves have a wavelength $\lambda$, an angular frequency $\omega$ and a harmonic time dependence of the form $\exp(i\omega t)$ (set by the experimental apparatus).
In optics, an analogy was established between the circularly polarized light and single-ringed LG beams with topological charge \( l = \pm 1 \), and between the linearly polarized light and Hermite–Gauss (HG) beams (TEM\(_{nm} \), transverse laser modes) of the type TEM\(_{01} \), TEM\(_{10} \) [38, 39]. In this context, a Poincaré sphere was proposed to describe the relations between the LG modes and the HG modes. However, it is not possible to extend this formalism to higher order Gaussian modes or nonparaxial beams.

Alternatively, it is known that any wave vortex propagating along the \( z \)-axis can be decomposed into two equally weighted orthogonal standing modes with a relative phase difference of \( \pm \pi / 2 \) as \( (x \pm iy)^l = \rho^l(\cos l\varphi \pm i \sin l\varphi) \), where \( \rho = \sqrt{x^2 + y^2} \) and \( \tan \varphi = y/x \). This is valid for any rotating beam in a region close to the vortex core, regardless of its topological charge and the nature of the wave field. Reciprocally, the standing modes \( \cos l\varphi \) and \( \sin l\varphi \) can be produced by the superposition of two counter-rotating vortices of the same amplitude and frequency [36], for instance, \( \cos(l\varphi) = \frac{1}{2}(\exp(il\varphi) + \exp(-il\varphi)) \). In this sense, we consider the superposition of these two orthogonal standing modes, \( \cos l\varphi \) and \( \sin l\varphi \), but propagating in opposite directions along the same axis (\( z \)) with an arbitrary phase difference \( \theta_0 \). We wonder if this superposition could give rise to planes where the interference is a vortex. The corresponding wave function in cylindrical coordinates is described by

\[
\Psi(\rho, \varphi, z) = \frac{1}{\sqrt{2}} u(\rho, z) \left[ \cos l\varphi \exp \left\{ i \left( kz - \frac{\theta_0}{2} \right) \right\} + \sin l\varphi \exp \left\{ -i \left( kz - \frac{\theta_0}{2} \right) \right\} \right] \\
= u(\rho, z) \left[ \cos \left( k\rho - \frac{\theta_0}{2} \right) \cos \left( l\varphi - \frac{\pi}{4} \right) - i \sin \left( k\rho - \frac{\theta_0}{2} \right) \sin \left( l\varphi - \frac{\pi}{4} \right) \right], \quad (1)
\]

where \( k = 2\pi/\lambda \). The function \( u(\rho, z) \) satisfies either the Helmholtz equation or the paraxial wave equation, and \( u(\rho, z) \propto \rho^l \) when \( \rho \) is small. For the rest of the analysis, we will express the wave function in (1) as \( \Psi(\rho, \varphi, z) = u(\rho, z)\psi(\varphi, z) \), and we will focus only on the function \( \psi(\varphi, z) \). In terms of its modulus and phase, \( \psi(\varphi, z) \) can be rewritten as

\[
\psi(\varphi, z) = \left( \sqrt{\frac{1 + \sin 2l\varphi \cos (2k\rho - \theta_0)}{2}} \right) \exp \left\{ -i \arctan \left[ \tan \left( k\rho - \frac{\theta_0}{2} \right) \tan \left( l\varphi - \frac{\pi}{4} \right) \right] \right\}. \quad (2)
\]

From expression (2) it can be seen that for transverse planes such that \( z = (\zeta + \theta_0/2\pi)\lambda/2 \), the field takes the form \( \psi = (1/\sqrt{2}) \exp[\mp i(l\varphi - \pi/4)] \), where the minus and plus signs in the phase correspond to \( \zeta = n + 1/4 \) and \( \zeta = n + 3/4 \), respectively, with \( n \) an integer. Therefore, the wave field exhibits vortices of opposite strengths in those planes, separated by a distance equal to \( \lambda/4 \) along the \( z \)-axis. On the other hand, when \( \zeta = n \) the field is given by \( \psi = (-1)^n \cos (l\varphi - \pi/4) \), and when \( \zeta = n + 1/2 \) it takes the form \( \psi = i (-1)^{n+1} \sin(l\varphi - \pi/4) \). In both cases, the wave field exhibits edge type dislocations [1] along the nodal lines in the respective transverse planes. The function \( \psi(\varphi, z) \) is periodic along the \( z \)-axis with a period of \( \lambda \), but the topological features repeat themselves every \( \lambda/2 \). Figure 1 illustrates the rms amplitude (top row) and the phase (bottom row) for an ideal field described by \( \Psi(\vec{r}) = (k\rho)^l\psi(\varphi, z) \) over a distance of \( \lambda/2 \) along the \( z \)-axis, with \( \theta_0 = 0 \) and \( l = 1 \).

The transverse field distribution occurring in a fixed \( z \)-plane can be controlled by means of the parameter \( \theta_0 \). Hence, in terms of interaction with matter, a continuous variation of \( \theta_0 \) in a fixed \( z \)-plane would be equivalent to a displacement through the wave pattern along the
Figure 1. Wave field with a gradient of OAM density along the $z$-axis and a topological charge $l = 1$. The length unit is the wavelength. Top row: snapshots of the normalized rms amplitude. Bottom row: snapshots of the phase in units of $\pi$ rad. Notice that $2\pi$—phase cycles are represented with a color map going from red to green, blue and finally red again.

$z$-axis. In both cases the field alternates between edge and screw dislocations with periodic inversions of the topological charge. It is worth recalling at this point a question raised by Dennis [13]: ‘one may ask whether other nodal configurations, such as chains, are possible in beam superpositions’. We believe this is a good example of a chain with interesting topological features, namely, a chain of vortices.

A single inversion of the topological charge along a propagation axis has been previously observed for an optical vortex due to the presence of an astigmatic element [40]. In our case, we have a standing wave configuration and there are no external elements to achieve the multiple inversions of the topological charge. On the other hand, the wave field studied here also differs from the straight row of unit-strength optical vortices produced by the breakup of a single optical vortex of high topological charge under elliptic perturbations [41].

As an example of the general character of the wave field studied here, figure 2 illustrates chains of vortices with topological charges (a) $l = 2$ and (b) $l = 3$, with a radial dependence given by a Bessel function of order $l$, $u(\rho, z) = J_l(k_\rho \rho)$. This is in contrast with the previously established analogy between HG beams of order $N = n + m = 1$ ($n, m \in [0, 1, 2, 3, \ldots]$) only and LG beams of the same order with linear and circular polarization states, respectively [38, 39]. In addition, we want to remark that our analysis can be applied to any kind of wave fields.

In order to analyze the dynamical properties of the chain of vortices, we will start from a general expression for the time-averaged energy flux density vector of a monochromatic scalar
Figure 2. Wave fields with a gradient of OAM density along the $z$-axis with topological charges: (a) $l = 2$ and (b) $l = 3$. In this case, the radial dependence of the fields is governed by a Bessel function of order $l$: $J_l(k \rho \rho)$. The length unit is the wavelength. Top row: snapshots of the normalized rms amplitude in different planes. Bottom row: snapshots of the phase in units of $\pi$ rad at the same planes.

wave field [42], $\langle \vec{F}(\vec{r}) \rangle = (i \omega / 4) \alpha (\Psi^* \nabla \Psi - \Psi \nabla \Psi^*)$, which in our case becomes

$$\langle \vec{F}(\vec{r}) \rangle = \frac{i}{4} (i \omega) \alpha |\psi|^2 (u^* \nabla u - u \nabla u^*) + |u|^2 (\psi^* \nabla \psi - \psi \nabla \psi^*).$$

(3)

Here $\alpha$ represents a constant that is associated with the particular wave field of interest. For example, $\alpha = \varepsilon_0$ for electromagnetic waves propagating in vacuum, where $\varepsilon_0$ denotes the electric permittivity. For sound waves within the linear acoustic regime $\alpha = \rho_0$, with $\rho_0$ being the mean density of the medium. Moreover, expression (3) may also represent the pseudomomentum flux for electromagnetic or acoustic waves in a medium [32] when $\alpha = \varepsilon_0 / v^2$ or $\alpha = \rho_0 / v^2$, respectively, where $v$ is either the speed of light or the speed of sound in the medium. We will focus our attention on the second term in parenthesis on the right-hand side of (3), since it is the only one that can contribute to the angular momentum of the field along the $z$-axis, via a tangential component of the energy flux vector. By substituting (1) into (3), we obtain for the $z$ component of the time-averaged angular momentum density

$$\langle J_z(\vec{r}) \rangle = \frac{k^2 \alpha}{2} |u(\rho, z)|^2 \left( \frac{l}{\omega} \right) \sin(2kz - \theta_0),$$

(4)

where we have also used the fact that $\langle J_z(\vec{r}) \rangle = \vec{r} \times \langle \vec{F}_\phi(\vec{r}) \rangle / v^2$, with $v = \omega / k$. This is a general result, regardless of the explicit form of $u(\rho, z)$. We can see that $\langle J_z(\vec{r}) \rangle$ smoothly alternates between positive and negative values of $(l / \omega) W(\rho, z)$ at the planes $z = (n + 1/4 + \theta_0 / 2 \pi) \lambda / 2$ and $z = (n + 3/4 + \theta_0 / 2 \pi) \lambda / 2$, respectively, where $n$ is an integer and the factor

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Figure 3. Experimental set-up. Four sound drivers were used to produce two orthogonal acoustic dipoles separated by a distance \( L \); each dipole was placed in a corner of a table. A global phase \( \theta_0 \) was added to one of the dipoles. The signals were reproduced by using a sound card controlled with a computer and passed through independent channels of two audio amplifiers connected to the drivers. The sound pressure field was recorded point by point with an omnidirectional microphone, following a square grid.

\[
W(\rho, z) = (k^2 \alpha/2)|u(\rho, z)|^2
\]

is proportional to the energy density of the field. Therefore, the scalar wave field described by (1) exhibits an OAMG along the \( z \)-axis.

3. Experiment

We carried out an experimental demonstration of a wave field with an OAMG using sound waves with \( \lambda \approx 21.5 \) cm and for the case \( l = 1 \), as described below. The experimental set-up is illustrated in figure 3.

In our experiment, sound waves with a frequency of 1.6 kHz were produced by using drivers from horn loudspeakers. As the wavelength was more than eight times larger than the inner diameter of the mouth of the driver (equal to 2.5 cm) from which the sound was radiated, this represents a good approximation to a simple sound source [43].

Consider a pair of identical simple sound sources at the points \((\pm a, 0, z_0)\) radiating with a relative phase of \( \pi \) with respect to each other. The sound pressure is given by

\[
\Psi(\vec{r}, t) = A \left\{ \frac{\exp[i(\omega t - kr_1)]}{r_1} - \frac{\exp[i(\omega t - kr_2)]}{r_2} \right\},
\]

(5)

where \( A \) is a constant and \( r_{1,2} = \sqrt{(x \mp a)^2 + y^2 + (z - z_0)^2} \).

By making a parabolic wave approximation assuming \( \rho = \sqrt{x^2 + y^2} \ll z \), equation (5) can be rewritten as

\[
\Psi(\vec{r}, t) = \frac{4A\pi i \alpha \rho}{\lambda(z - z_0)^2} \cos \varphi \times \exp \left[ -i \left( k \frac{\rho^2 + a^2}{2|z - z_0|} \right) \right] \exp[i(\omega t - k|z - z_0|)].
\]

(6)

The two simple sound sources form an acoustic dipole with its axis oriented along the \( x \)-axis, with an azimuthal dependence of the form \( \cos \varphi \), as described by (6). Notice that a dipole with its
axis oriented along the $y$-axis will exhibit an azimuthal dependence of the form $\sin \varphi$. We used two of these dipoles placed in the planes $z_0 = \pm L/2$, with $L = 60$ cm, and a separation between the radiation centers of $2a = 13.4$ cm (see figure 3). The axes of the dipoles were perpendicular to each other and to the $z$-axis, so as to approximately reproduce the wave field described by (1) with $l = 1$. The two pairs of loudspeakers were placed on the edge of two separated tables.

The experiments were carried out inside a rectangular room of approximately $4.5$ m × $6.0$ m, and $5.0$ m high, which was well isolated from exterior noise. Acoustic absorptive material was used on the floor underneath the experimental set-up in order to minimize reflections.

We generated burst signals consisting of 23 periods using Matlab. These were reproduced by means of a sound card controlled with a computer. Each of the four signals from the sound card was passed through an independent channel of an audio amplifier; we used two amplifiers, two channels per amplifier. The outputs of the amplifiers were in turn connected to the drivers. The sound pressure field was recorded by means of a half-inch omnidirectional microphone connected to the same sound card, and the data were stored in the computer. The recording process was synchronized with the reproduction of the signals. The sampling frequency was 48 kHz.

For processing the recorded signals, we considered only the part of the signal contained in the time interval after the transient response of the system, due mainly to the drivers, and before the arrival of the first reflection from the surfaces of the room. In this way, the analysis was carried out during steady state conditions of the sound field and under free field propagation. Hence, the first $28.125$ ms of each recorded signal were eliminated, and the next $3.125$ ms (exactly five periods at the reproduced frequency) were used to calculate the amplitude and the phase as described next. Since we used an integer number of periods and the sampling frequency was a multiple of the frequency of the generated sound waves, we applied the FFT to the signal in the $3.125$ ms interval and one of the resulting discrete complex numbers corresponded to the frequency of $1.6$ kHz. From that complex number we determined both the rms amplitude of the sound pressure and its phase.

Each of the four reproduction chains was initially characterized. For this purpose, we determined the amplitude and the phase of the sound field produced at the origin by each driver at a time, all of them fed with the same input signal. The amplitudes and phases were then compensated for undesirable deviations in order to generate the same values at the origin with each of the four sound sources. Finally, the desired phase shift was added in each of the four signals.

4. Results and discussion

The resulting field was an approximation to the ideal case of figure 1. The differences become larger as we move away from the plane $z = 0$ (see figure 3) and also as we move radially outwards. For this reason, we measured the experimentally generated sound field at the plane $z = 0$ in most of the cases, and we varied $\theta_0$ in order to control the complex amplitude distribution of the total acoustic field.

The produced sound field was sampled over the transverse plane of interest at discrete points $(x_j, y_j)$ distributed on a square reference grid with a spatial separation of $1$ cm from one another, where $-6$ cm $\leq x_j \leq 6$ cm and $-6$ cm $\leq y_j \leq 6$ cm. The experimental results are shown in figure 4. The top row corresponds to the rms output signal of the microphone in volts, which is proportional to the amplitude of the sound pressure, and the bottom row is
Figure 4. Measurements of the rms sound pressure (top) and of the phase in degrees (bottom) of the generated acoustic field. The columns correspond to: (a) $z = 0, \theta_0 = -\pi/2$; (b) $z = 0, \theta_0 = 0$; (c) $z = 0, \theta_0 = \pi/2$; (d) $z = 0, \theta_0 = \pi$ and (e) $z = -\lambda/8, \theta_0 = \pi$. The grid in the figures represents the spatial resolution of our experiment.

Figure 5. Numerical simulations of the normalized rms sound pressure (top) and of the phase in degrees (bottom). As in figure 4, the columns correspond to: (a) $z = 0, \theta_0 = -\pi/2$; (b) $z = 0, \theta_0 = 0$; (c) $z = 0, \theta_0 = \pi/2$; (d) $z = 0, \theta_0 = \pi$ and (e) $z = -\lambda/8, \theta_0 = \pi$. For comparison, we present theoretical simulations of the experiment in figure 5 for the same values of $z$ and $\theta_0$. The agreement between the experiments and the simulations is, in general, very good. The deviations were mainly due to an uncertainty in the position of the microphone and to the warming of the voice coils of the drivers. The distinctive characteristics of a vortex field were correctly reproduced as can be seen in figures 4(a), (c) and (e). The inversion of the topological...
charge is also observed between figures 4(a) and (c). On the other hand, figures 4(b) and 4(d) exhibit mutually orthogonal edge phase dislocations. According to equation (3) and the results presented in figure 4, the OAM of the wave field in a fixed $z$-plane can be controlled by means of the global phase $\theta_0$.

It is worth noticing that this kind of wave field with an OAMG can be generated with laser light, for example, by means of spatial light modulators technology [14]. However, an experimental analysis of the field variations produced over distances of $\lambda/8$ represents a big challenge at optical frequencies. Hence, the best way to investigate this optical field would be perhaps through its interaction with atoms.

5. Conclusions

We have shown that there is a kind of wave field that exhibits a periodic gradient of OAM along a single axis ($z$), with variations of the angular momentum density that are proportional to $2|l|$, where $l$ is the topological charge. The wave field corresponds to a standing wave configuration that alternates between edge and screw dislocations with periodic inversions of the topological charge, representing by itself an interesting subject of study from the topological viewpoint. This chain of vortices can be generated by means of a co-lineal superposition of counter-propagating modes having angular dependences of the form $\cos l\phi$ and $\sin l\phi$, and an arbitrary phase difference $\theta_0$ between them. We performed a probe-of-principle experimental demonstration for the case of a topological charge $l = 1$ using sound waves, which allowed us to realize a mapping of both the amplitude and the phase of the sound field. We verified that the topology of the wave field in a given $z$-plane can be controlled by setting a global phase $\theta_0$ in one of the interfering modes. Our analysis is not restricted to sound waves, but can be applied to any kind of wave field, including light and matter waves, which may lead to interesting potential applications. For example, in acoustics, it could be useful for the rotational control of acoustically levitated objects. Optical fields with a gradient of OAM could be applied in matter wave physics to rotate Bose–Einstein condensates [27] or even to the storage of quantum information in atomic vapors [28]. In fact, we notice that the OAMG proposed here is analogous to the polarization (spin) gradients used in atom cooling by means of the Sisyphus effect [44].

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