Multiple orthogonal polynomial ensembles

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Dedicated to Guillermo López Lagomasino, on the occasion of his 60th birthday

ABSTRACT. Multiple orthogonal polynomials are traditionally studied because of their connections to number theory and approximation theory. In recent years they were found to be connected to certain models in random matrix theory. In this paper we introduce the notion of a multiple orthogonal polynomial ensemble (MOP ensemble) and derive some of their basic properties. It is shown that Angelesco and Nikishin systems give rise to MOP ensembles and that the equilibrium problems that are associated with these systems have a natural interpretation in the context of MOP ensembles.

1. Introduction

Multiple Orthogonal Polynomials (MOPs) were introduced and studied for problems in analytic number theory (irrationality and transcendence proofs). Later they appeared in approximation theory, most notably in the theory of Hermite-Padé approximation and in this context they are also called Hermite-Padé polynomials \[2, 4, 20, 33, 34, 40, 41, 42, 52, 60\]. MOPs were also studied from the point of view of new special functions \[6, 19, 22, 49, 64, 67\]. See the books \[44, 59\] and the survey papers \[3, 8, 65, 66\] for these aspects of MOPs. Further developments in these directions are reported in e.g. \[7, 10, 16, 21, 23, 24, 37, 38, 50, 54\].

Recently MOPs also appeared in a natural way in probability theory and mathematical physics in certain models coming from random matrix theory and non-intersecting paths. The connection was first observed in \[14\] where MOPs were used in in a random matrix model with external source. In the Gaussian case, the external source model has an equivalent interpretation in terms of non-intersecting Brownian motions. The external source model was further analyzed with the use of multiple Hermite and multiple Laguerre polynomials in \[5, 9, 15, 17, 31, 43, 61, 53, 55, 56, 69\], see also \[6, 9, 16, 30, 51\]. A related non-intersecting path model was studied in \[46\] using MOPs for modified Bessel weights that were introduced earlier in \[22\]. The biorthogonal polynomials arising in the two matrix model were identified as MOPs in \[47\]. For a special case

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they were asymptotically analyzed in [36, 57]. The Cauchy two matrix model and
their associated Cauchy biorthogonal polynomials have a number of similar features
[12, 13]. MOPs were generalized to MOPs of mixed type in [1, 26, 27, 28].

Asymptotic results were mainly obtained from an analysis of the Riemann-
Hilbert problem for MOPs, formulated by Van Assche et al. [68] as an extension of
the Riemann-Hilbert problem for orthogonal polynomials [39]. The application of
the Deift/Zhou steepest descent analysis [29] to the Riemann-Hilbert problem for
MOPs presents several interesting new features that however we will not discuss
here.

It is the aim of this paper to give an introductory account of MOPs from the
point of view of determinantal point processes. After discussing the definition and
some of the basic properties of MOPs we discuss a multiple integral representa-
tions for the type II MOPs, which is essentially taken from [14]. Under a suitable constant
sign condition the formula can be interpreted as the expectation value of the random
polynomial \( \prod_{j=1}^{n} (z - x_j) \) with roots \( x_1, \ldots, x_n \) from a determinantal point process
(called a MOP ensemble) on the real line.

The constant sign condition holds in particular for Angelesco and Nikishin
systems. For both of these systems we show that the joint p.d.f. of the associated
MOP ensemble takes on a particular nice form. In the large \( n \) limit it allows for
a natural probabilistic interpretation of the vector equilibrium problems that are
associated with Angelesco and Nikishin systems.

2. Multiple orthogonal polynomials

2.1. Definitions. Given weight functions \( w_1, \ldots, w_p \) on \( \mathbb{R} \) and a multi-index
\( \vec{n} = (n_1, \ldots, n_p) \in \mathbb{N}^p \), the type II MOP is a monic polynomial \( P_{\vec{n}} \) of degree
\( |\vec{n}| = n_1 + \cdots + n_p \) such that

\[
\int_{-\infty}^{\infty} P_{\vec{n}}(x) x^k w_j(x) dx = 0, \quad k = 0, \ldots, n_j - 1, \quad j = 1, \ldots, p.
\]

Throughout we will write
\( n = |\vec{n}| = n_1 + \cdots + n_p. \)

The conditions (2.1) give a system of \( n \) linear equations for the \( n \) free coefficients
of the polynomial \( P_{\vec{n}} \) (recall that \( P_{\vec{n}} \) is monic). If the system has a unique solution
we say that the multi-index \( \vec{n} \) is normal (with respect to the weights \( w_1, \ldots, w_p \)).

In this paper we mainly deal with the type II MOP, but at times it is useful
to consider the dual notion of type I MOPs as well. These are polynomials \( A_{\vec{n}}^{(j)} \),
\( j = 1, \ldots, p \), of degrees \( \deg A_{\vec{n}}^{(j)} = n_j - 1 \), such that the linear form

\[
Q_{\vec{n}}(x) = \sum_{j=1}^{p} A_{\vec{n}}^{(j)}(x) w_j(x)
\]

satisfies

\[
\int_{-\infty}^{\infty} x^k Q_{\vec{n}}(x) dx = 0, \quad k = 0, 1, \ldots, n - 2.
\]

If we supplement this with the normalizing condition

\[
\int_{-\infty}^{\infty} x^{n-1} Q_{\vec{n}}(x) dx = 1,
\]
then again we have a system of \( n = |\vec{r}| \) linear equations for the in total \( n \) coefficients of the polynomials \( A_{\vec{r}}^{(j)}, j = 1, \ldots, p \).

### 2.2. Determinantal expressions

Let

\[
c_k^{(j)} = \int_{-\infty}^{\infty} x^k w_j(x) dx
\]

denote the \( k \)th moment of the weight \( w_j \), and let

\[
H^{(j)}_{m,n} = \left( c_k^{(j)} \right)_{k=0, \ldots, m, t=0, \ldots, n}
\]

be the \((m+1) \times (n+1)\) Hankel matrix with the moments of \( w_j \). The conditions (2.3) and (2.4) give rise to a linear system whose matrix has the block Hankel structure

\[
M_{\vec{r}} = \begin{bmatrix}
H_{n-1,n_1-1}^{(1)} & H_{n-1,n_2-1}^{(2)} & \cdots & H_{n-1,n_p-1}^{(p)} \\
\end{bmatrix}.
\]

Therefore the type I MOPs uniquely exist if and only if

\[
D_{\vec{r}} := \det M_{\vec{r}} = \left| H_{n-1,n_1-1}^{(1)} \ H_{n-1,n_2-1}^{(2)} \ \cdots \ H_{n-1,n_p-1}^{(p)} \right| \neq 0.
\]

The linear system arising from the type II conditions (2.1) has a matrix which is the transpose of (2.5). Therefore the non-vanishing of the determinant (2.6) also guarantees the existence and uniqueness of the type II MOP.

Suppose \( D_{\vec{r}} \neq 0 \). Then it is easy to see that the type II MOP has the determinantal formula

\[
P_{\vec{r}}(x) = \frac{1}{D_{\vec{r}}} \left| H_{n,n_1-1}^{(1)} \ H_{n,n_2-1}^{(2)} \ \cdots \ H_{n,n_p-1}^{(p)} \right| \begin{bmatrix}
1 \\
x \\
x^2 \\
\vdots \\
x^n
\end{bmatrix}.
\]

Indeed, the right-hand side of (2.7) is a monic polynomial of degree \( n \). If we multiply the right-hand side of (2.7) by \( x^k w_j(x) \) and integrate with respect to \( x \), we can perform these operations in the last column to obtain a determinant with two equal columns if \( k \leq n_j - 1 \). This proves the type II orthogonality conditions (2.1).

The type I MOPs have a similar determinantal expression. For \( j = 1, \ldots, p \) we have

\[
A_{\vec{r}}^{(j)}(x) = \frac{1}{D_{\vec{r}}} \times \begin{bmatrix}
H_{n-2,n_1-1}^{(1)} & \cdots & H_{n-2,n_j-1}^{(j-1)} & H_{n-2,n_j-1}^{(j)} & H_{n-2,n_j+1-1}^{(j+1)} & \cdots & H_{n-2,n_p-1}^{(p)} \\
0 & \cdots & 0 & 1 & x & \cdots & x^{n_j-1} & 0 & \cdots & 0
\end{bmatrix}.
\]

These and similar determinantal formulas have recently been considered from the point of view of integrable systems in [11].
2.3. Multiple integral representation. For what follows it is convenient to write
\[ N_j = \sum_{i=1}^{j} n_i, \quad N_0 = 0 \]
and to introduce two sequences of functions \( f_1, \ldots, f_n \) and \( g_1, \ldots, g_n \) by
\[
(2.9) \quad f_j(x) = x^{j-1}, \quad j = 1, \ldots, n
\]
and
\[
(2.10) \quad g_{i+N_j-1}(x) = x^{i-1} w_j(x), \quad i = 1, \ldots, n_j, \quad j = 1, \ldots, p.
\]
Then the block Hankel matrix (2.5) can be written as
\[
(2.11) \quad M_{\vec{n}} = \left[ m_{j,k} \right]_{j,k=1}^{n}, \quad m_{j,k} = \int_{-\infty}^{\infty} f_j(x) g_k(x) \, dx
\]
and
\[
D_{\vec{n}} = \det M_{\vec{n}} = \det \left[ \int_{-\infty}^{\infty} f_j(x) g_k(x) \, dx \right]_{j,k=1}^{n}.
\]
For general \( m \) and \( n = |\vec{n}| \) we also write
\[
(2.12) \quad M_{m,n} = \left[ m_{j,k} \right]_{j=1}^{m}, k=1,\ldots,n,
\]
so that we have by (2.7)
\[
(2.13) \quad P_{\vec{n}}(x) = \frac{1}{D_{\vec{n}}} \left| \begin{array}{c} 1 \\ x \\ \vdots \\ x^n \end{array} \right| M_{n+1,n}
\]
and by (2.9) and (2.10)
\[
(2.14) \quad Q_{\vec{n}}(x) = \sum_{j=1}^{n} A_{\vec{n}}^{(j)}(x) \cdot w_j(x) = \frac{1}{D_{\vec{n}}} \left| \begin{array}{c} 1 \\ g_1(x) \\ g_2(x) \cdot \cdots \cdot g_n(x) \end{array} \right|.
\]
The following lemma is standard, see e.g. [45] Proposition 2.10] where it is called a generalized Cauchy-Binet identity.

**Lemma 2.1.** We have
\[
(2.15) \quad D_{\vec{n}} = \frac{1}{n!} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \det \left[ f_j(x_k) \right]_{j,k=1}^{n} - \det \left[ g_j(x_k) \right]_{j,k=1}^{n} \prod_{k=1}^{n} dx_k.
\]

**Proof.** Expanding the two determinants on the right-hand side of (2.15) we get
\[
\det \left[ f_j(x_k) \right]_{j,k} - \det \left[ g_j(x_k) \right]_{j,k} = \sum_{\sigma} \sum_{\tau} (-1)^{\text{sgn } \sigma + \text{sgn } \tau} \prod_{k=1}^{n} f_{\sigma(k)}(x_k) g_{\tau(k)}(x_k)
\]
where the sums are for \( \sigma \) and \( \tau \) over the symmetric group \( S_n \). By (2.11) the right-hand side of (2.15) is equal to
\[
(2.16) \quad \frac{1}{n!} \sum_{\sigma} \sum_{\tau} (-1)^{\text{sgn } \sigma + \text{sgn } \tau} \prod_{k=1}^{n} m_{\sigma(k),\tau(k)} = \frac{1}{n!} \sum_{\sigma} \sum_{\tau} (-1)^{\text{sgn } (\sigma \tau^{-1})} \prod_{k=1}^{n} m_{\sigma \tau^{-1}(k),k}.
\]
For any fixed \( \sigma \), we have that \( \sigma \circ \tau^{-1} \) runs through \( S_n \) as \( \tau \) runs through \( S_n \). Hence
\[
(2.17) \quad \sum_{\tau} (-1)^{\text{sgn}(\sigma \circ \tau^{-1})} \prod_{k=1}^{\tilde{n}} m_{\sigma \circ \tau^{-1} \circ (k), k} = \det M_{\tilde{\tau}} = D_{\tilde{\tau}}.
\]

The equality (2.15) follows from (2.16) and (2.17).

There is a similar multiple integral representation for the type II MOPs, which was stated for a special case in [14], see also [32]. We emphasize that it is important here that \( f_j(x) = x^j \).

**Proposition 2.2.** Assume \( D_{\tilde{\tau}} \neq 0 \). Then the type II MOP has the multiple integral representation
\[
(2.18) \quad P_{\tilde{\tau}}(z) = \frac{1}{D_{\tilde{\tau}} \cdot n!} \times \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \prod_{k=1}^{n} (z - x_k) \cdot \det [f_j(x_k)]_{j,k=1,\ldots,n} \cdot \det [g_j(x_k)]_{j,k=1,\ldots,n} \prod_{k=1}^{n} dx_k.
\]

**Proof.** Since \( f_j(x) = x^j \) we have that \( \det [f_j(x_k)] \) is a Vandermonde determinant, and therefore
\[
\prod_{k=1}^{\tilde{n}} (z - x_k) \cdot \det [f_j(x_k)]_{j,k=1,\ldots,n} = \det [f_j(x_k)]_{j,k=1,\ldots,n+1}
\]
where we have put
\[
f_{n+1}(x) = x^n, \quad \text{and} \quad x_{n+1} = z.
\]
Thus, by expanding the determinant we have
\[
\prod_{k=1}^{n} (z - x_k) \cdot \det [f_j(x_k)]_{j,k=1,\ldots,n} = \sum_{\sigma \in S_{n+1}} (-1)^{\text{sgn} \sigma} \prod_{k=1}^{n} f_{\sigma(k)}(x_k) \cdot f_{\sigma(n+1)}(z)
\]
and similarly
\[
\det [g_j(x_k)]_{j,k=1,\ldots,n} = \sum_{\tau \in S_n} (-1)^{\text{sgn} \tau} \prod_{k=1}^{n} g_{\tau(k)}(x_k).
\]

Integrating the product of the two above expressions with respect to \( x_1, \ldots, x_n \) we obtain
\[
(2.19) \quad \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \prod_{k=1}^{n} (z - x_k) \cdot \det [f_j(x_k)]_{j,k=1,\ldots,n} \cdot \det [g_j(x_k)]_{j,k=1,\ldots,n} \prod_{k=1}^{n} dx_k
\]

\[
= \sum_{\sigma \in S_{n+1}} \sum_{\tau \in S_n} (-1)^{\text{sgn} \sigma + \text{sgn} \tau} \prod_{k=1}^{n} m_{\sigma(k), \tau(k)} \cdot f_{\sigma(n+1)}(z)
\]

\[
= \sum_{\tau \in S_n} \sum_{\sigma \in S_{n+1}} (-1)^{\text{sgn}(\sigma \circ \tau^{-1})} \prod_{k=1}^{n} m_{\sigma \circ \tau^{-1} \circ (k), k} \cdot z^{\sigma(n+1)-1},
\]
where we used the definition of \( m_{j,k} \) as given in (2.11) also for \( j = n + 1 \).

For each fixed \( \tau \in S_n \) we have that the sum over \( \sigma \) in (2.19) is equal to the determinant in the right-hand side of (2.13) and the proposition follows. \( \square \)
In an analogous way we find the following multiple integral representation for
the linear form of type I MOPs, which is due to Desrosiers and Forrester \cite{32}.

**Proposition 2.3.** Assume $\vec{D}_n \neq 0$. Then the linear form of type I MOPs
satisfies

\[
\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \prod_{k=1}^{n} (z - x_k)^{-1} \cdot \det [f_j(x_k)]_{j,k=1,\ldots,n} \cdot \det [g_j(x_k)]_{j,k=1,\ldots,n} \prod_{k=1}^{n} dx_k.
\]

**Proof.** Here we use the property

\[
\prod_{k=1}^{n} (z - x_k)^{-1} \cdot \det [f_j(x_k)]_{j,k=1,\ldots,n} = \det [f_j(x_k)]_{j,k=1,\ldots,n+1}
\]

where now we put

\[
f_{\vec{D}_n+1}(x) = \frac{1}{z - x}, \quad \text{and} \quad x_{n+1} = z.
\]

The rest of the proof follows along the same lines as the proof of Proposition 2.2.

We omit the details, see also \cite{32}. \hfill \Box

3. MOP ensembles

**3.1. Probabilistic interpretation.** The multiple integral representations \cite{21,22,23} have a natural probabilistic interpretation in case the product of
determinants

\[
\det [f_j(x_k)]_{j,k=1,\ldots,n} \cdot \det [g_j(x_k)]_{j,k=1,\ldots,n}
\]
is of a fixed sign for $(x_1,\ldots,x_n) \in \mathbb{R}^n$. That is, if it is always $\geq 0$ or always $\leq 0$.

Indeed, in that case it follows by \cite{21} that

\[
\mathcal{P}(x_1,\ldots,x_n) = \frac{1}{Z_n} \det [f_j(x_k)]_{j,k=1,\ldots,n} \cdot \det [g_j(x_k)]_{j,k=1,\ldots,n}
\]
is a probability density function on $\mathbb{R}^n$, where

\[
Z_n = D_n n!
\]
is the normalizing constant (also called partition function in statistical mechanics
literature), so that $\int \cdots \int \mathcal{P}(x_1,\ldots,x_n) dx_1 \cdots dx_n = 1$.

The multiple integral representations \cite{21,22} then show that

\[
\mathcal{P}(z) = \mathbb{E} \left[ \prod_{k=1}^{n} (z - x_k) \right], \quad z \in \mathbb{C},
\]

\[
Q(z) = \mathbb{E} \left[ \prod_{k=1}^{n} (z - x_k)^{-1} \right], \quad z \in \mathbb{C} \setminus \mathbb{R},
\]

where the mathematical expectation is taken with respect to the p.d.f. \cite{31}.

Thus $\mathcal{P}(z)$ is the average of the polynomials $\prod_{k=1}^{n} (z - x_k)$ where the roots
$x_1,\ldots,x_n$ are distributed according to \cite{31}. In cases where the distribution \cite{31}
can be interpreted as the eigenvalue distribution of a random matrix ensembles,
one would call $\mathcal{P}$ the average characteristic polynomial.
3.2. Biorthogonal ensembles. A biorthogonal ensemble, see [18], is a probability density function on $\mathbb{R}^n$ of the form (3.1) with certain given functions $f_1, \ldots, f_n$ and $g_1, \ldots, g_n$, not necessarily of the form (2.9) and (2.10). The p.d.f. is invariant under permutations of variables. We think of the ensemble as giving us $n$ random points or particles $x_j$ on the real line, and so it is a random point process.

A biorthogonal ensemble is a special case of a determinantal point process, see e.g. [45, 63]. This means that there is a correlation kernel $K_n(x, y)$ so that

$$P(x_1, \ldots, x_n) = \frac{1}{n!} \det [K_n(x_j, x_k)]_{j,k=1,\ldots,n}$$

and so that marginal densities ($m$ point correlation functions) are determinants

$$\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} P(x_1, \ldots, x_n) dx_{m+1} \cdots dx_n = \frac{(n-m)!}{n!} \det [K_n(x_j, x_k)]_{j,k=1,\ldots,m}.$$

Taking for example $m = 1$ we have that $\frac{1}{n}K_n(x, x)$ is the mean density of points, that is

$$\frac{1}{n} \int_a^b K_n(x, x) dx$$

is the expected fraction of points lying in the interval $[a, b]$.

In a biorthogonal ensemble, the correlation kernel can be written as a bordered determinant

$$K_n(x, y) = -\frac{1}{\det M_n} \begin{vmatrix} f_1(x) & f_2(x) & \cdots & f_n(x) \\ g_1(y) & g_2(y) & \cdots & g_n(y) \end{vmatrix}$$

where $M_n$ is the matrix

$$M_n = [m_{j,k}]_{j,k=1,\ldots,n}, \quad m_{j,k} = \int_{-\infty}^{\infty} f_j(x)g_k(x) dx$$

In the formulation of the biorthogonal ensemble (3.1), we have some freedom in choosing the functions $f_1, \ldots, f_n$ and $g_1, \ldots, g_n$. Indeed, if $\phi_1, \ldots, \phi_n$ and $\psi_1, \ldots, \psi_n$ are functions with the same linear span as the $f_j$’s and $g_j$’s, respectively, then we could use these functions instead. A particular nice form appears if the functions $\phi_j$ and $\psi_k$ are biorthogonal, i.e.,

$$\int_{-\infty}^{\infty} \phi_j(x)\psi_k(x) dx = \delta_{j,k}.$$

Then the representation (3.5) reduces to

$$K_n(x, y) = -\begin{vmatrix} I_n & \phi_1(x) \\ \psi_1(y) & \cdots & \phi_n(x) \end{vmatrix} = \sum_{j=1}^{n} \phi_j(x)\psi_j(y).$$
3.3. OP ensembles. If \( f_j(x) = g_j(x) = x^{j-1}\sqrt{w(x)} \), \( j = 1, \ldots, n \) for some non-negative weight function \( w \) on \( \mathbb{R} \), then

\[
\frac{1}{Z_n} \det[f_j(x_k)] \cdot \det[g_j(x_k)] = \frac{1}{Z_n} \prod_{1 \leq j < k \leq n} (x_k - x_j)^2 \cdot \prod_{k=1}^n w(x_k)
\]

which is indeed of constant sign. This is the form of the joint p.d.f. for the eigenvalues of a unitary random matrix ensemble

\[
\frac{1}{Z_n} \exp(-\text{Tr} V(H)) dH, \quad w(x) = e^{-V(x)},
\]

defined on \( n \times n \) Hermitian matrices \( H \), see [29].

In this case the biorthogonal functions take the form

\[
\phi_j(x) = \psi_j(x) = p_{j-1}(x)\sqrt{w(x)}
\]

where \( p_{j-1} \) is the orthonormal polynomial of degree \( j - 1 \) with respect to the weight \( w \) on \( \mathbb{R} \), and by (3.6),

\[
K_n(x, y) = \sqrt{w(x)} \sqrt{w(y)} \sum_{j=0}^{n-1} p_j(x)p_j(y)
\]

is the correlation kernel, which in this situation is also called the OP kernel or the Christoffel-Darboux kernel.

The OPs are characterized by a \( 2 \times 2 \) matrix valued Riemann-Hilbert problem due to Fokas, Its, and Kitaev [39],

- \( Y : \mathbb{C} \setminus \mathbb{R} \to \mathbb{C}^{2 \times 2} \) is analytic,
- \( Y_+(x) = Y_-(x) \begin{pmatrix} 1 & w(x) \\ 0 & 1 \end{pmatrix}, \quad x \in \mathbb{R}, \)
- \( Y(z) = (I_2 + O(1/z)) \text{diag}(z^n \quad z^{-n}) \) as \( z \to \infty \).

The correlation kernel for the OP ensemble can be given directly in terms of the solution \( Y \) of the RH problem

\[
K_n(x, y) = \frac{1}{2\pi i(x - y)} \sqrt{w(x)} \sqrt{w(y)} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} Y_+(y)^{-1}Y_+(x) \begin{pmatrix} 1 \\ 0 \end{pmatrix}.
\]

This follows from an explicit formula for \( Y \) in terms of the orthogonal polynomials \( p_n \) and \( p_{n-1} \), and the Christoffel-Darboux formula for orthogonal polynomials.

3.4. MOP ensembles. We have a MOP ensemble if \( f_1, \ldots, f_n \) and \( g_1, \ldots, g_n \) are given by (2.9) and (2.10), and if

\[
\det[f_j(x_k)]_{j,k=1,\ldots,n} = \prod_{1 \leq j < k \leq n} (x_k - x_j)^2 \cdot \prod_{k=1}^n w(x_k)
\]

has constant sign. The case \( p = 1 \) reduces essentially to the OP case.

For a MOP ensemble we have that the correlation kernel \( K_n \) given by the determinant (3.3) has another expression in terms of the RH problem for multiple orthogonal polynomials. MOPs (with \( p \) weights) satisfy a \( (p+1) \times (p+1) \) matrix valued RH problem [68]

- \( Y : \mathbb{C} \setminus \mathbb{R} \to \mathbb{C}^{(p+1)\times(p+1)} \) is analytic,
The correlation kernel for the MOP ensemble is given as follows in terms of the solution $Y$ of the RH problem

$$K_n(x,y) = \frac{1}{2\pi i(x-y)} \begin{pmatrix} 0 & w_1(y) & \cdots & w_p(y) \end{pmatrix} Y_+(y)^{-1} Y_+(x) \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}.$$  

The proof is based on a Christoffel-Darboux formula for MOPs, see [14] for the case $p = 2$ and [25] for general $p$. An extension to MOP of mixed type is given in [26].

4. Special classes

The condition that (3.10) is of constant sign is automatically satisfied in the OP case, but it becomes relevant in the MOP case. We call it the MOP ensemble condition. It is of interest to identify classes for which the MOP ensemble condition holds. In the literature on Hermite-Padé approximation a number of special classes of MOPs were identified for which rather complete convergence results could be established. These are in particular the Angelesco systems [2, 40] and the Nikishin systems, see e.g. [4, 20, 34, 42, 50]. It turns out that for these special classes the MOP ensemble condition holds. Before we turn to that, we make some preliminary observations.

4.1. Preliminary observations. The first observation is that the product of determinants (3.10) is invariant under permutations of the $x_k$’s. It is also clear that (3.10) is zero in case two or more of the $x_k$’s coincide. Therefore we may restrict ourselves to strictly ordered sets of points

$$x_1 < x_2 < \cdots < x_n.$$  

The second observation is that the first factor in (3.10) is a Vandermonde determinant (due to the fact that $f_j(x) = x^{j-1}$)

$$\det [f_j(x_k)]_{j,k=1,\ldots,n} = \prod_{j<k} (x_k - x_j),$$

which is positive for ordered points (4.1). Therefore the MOP ensemble condition comes down to the condition stated in the following lemma.

**Lemma 4.1.** The MOP ensemble condition is satisfied if and only if either

$$\det [g_j(x_k)]_{j,k=1,\ldots,n} \geq 0$$

whenever $x_1 < x_2 < \cdots < x_n$, or

$$\det [g_j(x_k)]_{j,k=1,\ldots,n} \leq 0$$

whenever $x_1 < x_2 < \cdots < x_n$. 


Since Vandermonde-like products will appear frequently in what follows we use the abbreviations
\begin{equation}
\Delta(X) = \prod_{1 \leq j < k \leq n} (x_k - x_j)
\end{equation}
if \(X = (x_1, \ldots, x_n)\), and
\begin{equation}
\Delta(X, Y) = \prod_{k=1}^{n} \prod_{j=1}^{m} (x_k - y_j)
\end{equation}
if \(X = (x_1, \ldots, x_n)\) and \(Y = (y_1, \ldots, y_m)\).

4.2. Angelesco ensemble. The weights \(w_1, \ldots, w_p\) are an Angelesco system if there are disjoint intervals \(\Gamma_1, \ldots, \Gamma_p \subseteq \mathbb{R}\), such that
\[
\text{supp}(w_j) \subseteq \Gamma_j, \quad j = 1, \ldots, p.
\]
We write \(\Gamma_j = [\alpha_j, \beta_j]\) and without loss of generality we assume that
\begin{equation}
\beta_j < \alpha_{j+1}, \quad \text{for } j = 1, \ldots, p-1.
\end{equation}
We extend \(w_j\) to all of \(\mathbb{R}\) by defining \(w_j(x) = 0\) if \(x \in \mathbb{R} \setminus \Gamma_j\).

An Angelesco system always gives rise to a MOP ensemble. Indeed it is easy to see that in the Angelesco case \(\det [g_j(x_k)]\) is of block form, and it can only be non-zero if \(n_j\) of the points \(x_k\) belong to \(\Gamma_j\) for \(j = 1, \ldots, p\), and so this is what we will assume. Then writing \(N_j = \sum_{i=1}^{n_i} n_i\), \(N_0 = 0\), we have that
\[
x_k^{(i)} := x_{N_{j-1} + k} \in \Gamma_j, \quad k = 1, \ldots, n_j, \quad j = 1, \ldots, p.
\]
Because of the orderings (4.1) and (4.6) the determinant then has a block diagonal form where the \(i\)th block is
\[
\begin{pmatrix}
w_i(x_1^{(i)}) & \cdots & w_i(x_{n_i}^{(i)}) \\
x_1^{(i)} w_i(x_1^{(i)}) & \cdots & x_{n_i}^{(i)} w_i(x_{n_i}^{(i)}) \\
\vdots & & \vdots \\
x_1^{(i)} \prod_{j=1}^{n_i} w_i(x_1^{(i)}) & \cdots & x_{n_i}^{(i)} \prod_{j=1}^{n_i} w_i(x_{n_i}^{(i)})
\end{pmatrix}
\]
whose determinant is
\[
\prod_{1 \leq j < k \leq n_i} (x_k^{(i)} - x_j^{(i)}) \cdot \prod_{k=1}^{n_i} w_i(x_k^{(i)}) = \Delta(X^{(i)}) \prod_{k=1}^{n_i} w_i(x_k^{(i)})
\]
where \(X^{(i)} = (x_1^{(i)}, \ldots, x_{n_i}^{(i)})\). The result is that
\begin{equation}
\det [g_j(x_k)] = \prod_{i=1}^{p} \left( \Delta(X^{(i)}) \cdot \prod_{k=1}^{n_i} w_i(x_k^{(i)}) \right)
\end{equation}
and this is \(\geq 0\) for every choice of \(x_1 < \cdots < x_n\).

Thus, by Lemma (4.4), an Angelesco system gives rise to a MOP ensemble and we call it an Angelesco ensemble. We see from the above calculation that the joint
The p.d.f. in an Angelesco ensemble is

\[\frac{1}{Z_n} \det[f_j(x_k)] \det[g_j(x_k)] = \frac{1}{Z_n} \prod_{i=1}^{p} \Delta(X^{(i)})^2 \cdot \prod_{1 \leq i < j \leq p} \Delta(X^{(i)}, X^{(j)}) \cdot \prod_{i=1}^{p} \prod_{k=1}^{n_i} w_i(x_k^{(i)}).\]

**4.3. AT ensemble.** Assume that \(w_1, \ldots, w_p\) are weights defined on a fixed interval \(\Gamma \subset \mathbb{R}\). Then \(w_1, \ldots, w_p\) are an AT system on \(\Gamma\) if the functions \(g_j\) are an algebraic Chebyshev system on \(\Gamma\). This means that every non-trivial linear combination

\[\sum_{j=1}^{n} \lambda_j g_j\]

has at most \(n - 1\) zeros in \(\Gamma\). Equivalently, an algebraic Chebyshev system means that

\[\det [g_j(x_k)] \neq 0\]

for every choice of distinct points \(x_k\) in \(\Gamma\). The property of being an AT system also depends on the multi-indices \(n_1, \ldots, n_p\).

If the weights \(w_j\) in an AT system are continuous functions on \(\Gamma\), then it clearly follows from (4.9) by continuity that \(\det [g_j(x_k)]\) has constant sign (either > 0 or < 0) whenever the \(x_k\) are strictly ordered points in \(\Gamma\). Therefore, in that case, we have a MOP ensemble by Lemma 4.1 which we will call an AT ensemble.

**4.4. Nikishin ensemble.**

4.4.1. **Definition of a Nikishin system.** Certain AT systems with special properties were first described by Nikishin [58] and are therefore called Nikishin systems. We state it first for \(p = 2\) continuous weight functions \(w_1, w_2\) defined on an interval \(\Gamma_1 \subset \mathbb{R}\).

The Nikishin assumption is that the ratio \(w_2/w_1\) can be written as a Markov function for a non-negative weight function (or more generally a measure) supported on an interval \(\Gamma_2\), disjoint from \(\Gamma_1\), that is, if

\[\frac{w_2(x)}{w_1(x)} = \pm \int_{\Gamma_2} \frac{v(y)}{x-y} dy, \quad x \in \Gamma_1,\]

where \(v(s)\) is a non-negative weight function with

\[\text{supp}(v) = \Gamma_2, \quad \Gamma_2 \cap \Gamma_1 = \emptyset.\]

We choose the + sign in (4.10) if \(\Gamma_2\) lies to the left of \(\Gamma_1\); otherwise we choose the − sign. Then we call \(w_1, w_2\) a Nikishin system on \(\Gamma_1\) for the intervals \(\Gamma_1, \Gamma_2\). A Nikishin system with \(p \geq 3\) weights is defined inductively. Suppose \(\text{supp}(w_j) = \Gamma_1\) for all \(j = 1, \ldots, p\), where \(\Gamma_1\) is an interval. Suppose

\[\frac{w_j(x)}{w_1(x)} = \pm \int_{\Gamma_2} \frac{v_j(y)}{x-y} dy, \quad x \in \Gamma_1, \quad j = 2, \ldots, p,\]

where \(\Gamma_2 \cap \Gamma_1 = \emptyset\) and where \(v_2, \ldots, v_p\) is a Nikishin system on \(\Gamma_2\) for the intervals \(\Gamma_2, \ldots, \Gamma_p\). Then we call \(w_1, \ldots, w_p\) a Nikishin system on \(\Gamma_1\) for the intervals \(\Gamma_1, \ldots, \Gamma_p\).
Note that in a Nikishin system two consecutive intervals \( \Gamma_j \) and \( \Gamma_{j+1} \) are disjoint. However, if \( |j - k| \geq 2 \), then \( \Gamma_j \) and \( \Gamma_k \) may very well have a non-empty intersection.

This construction might not seem very natural at first sight, but it is actually a very beautiful structure. A main result is that for multi-indices \( \vec{n} = (n_1, \ldots, n_p) \) such that

\[
n_j \geq n_{j+1} - 1, \quad j = 1, \ldots, p - 1
\]
a Nikishin system is an AT system, see [59], and therefore the type I and type II MOPs exist. As we have seen in the previous subsection, there is also an associated MOP ensemble, which we call a Nikishin ensemble.

4.4.2. Nikishin ensemble with 2 weights. Here we show that a Nikishin ensemble has a natural interpretation as the marginal distribution of an extended ensemble. The following calculations are due to Coussement and Van Assche [22].

For reasons of clarity we take \( p = 2 \) and we assume that \( \Gamma_2 \) is to the left of \( \Gamma_1 \). Then for \( x_1, \ldots, x_n \) in \( \Gamma_1 \) we have

\[
\det [g_j(x_k)] = \prod_{k=1}^n \frac{w_1(x_k)}{w_1(x_1)} \begin{vmatrix}
 w_1(x_1) & w_1(x_2) & \cdots & w_1(x_n) \\
 x_1 w_1(x_1) & x_2 w_1(x_2) & \cdots & x_n w_1(x_n) \\
 \vdots & \vdots & & \vdots \\
 x_1^{n_1-1} w_1(x_1) & x_2^{n_1-1} w_1(x_2) & \cdots & x_n^{n_1-1} w_1(x_n) \\
 x_1^{n_2-1} w_2(x_1) & x_2^{n_2-1} w_2(x_2) & \cdots & x_n^{n_2-1} w_2(x_n)
\end{vmatrix}
\]

Now we replace each ratio \( \frac{w_2(x_n)}{w_1(x_n)} \) by the integral (4.10), we use \( y_j \) as the integration variable in row \( n_1 + j \), and we take the integrals as well as the factors \( v(y_j) \) out of the determinant, to obtain

\[
\prod_{k=1}^n \frac{w_1(x_k)}{w_1(x_1)} \int_{\Gamma_2} \cdots \int_{\Gamma_2} \prod_{j=1}^{n_2} v(y_j) \begin{vmatrix}
 1 & 1 & \cdots & 1 \\
 x_1 & x_2 & \cdots & x_n \\
 \vdots & \vdots & & \vdots \\
 x_1^{n_1-1} & x_2^{n_1-1} & \cdots & x_n^{n_1-1} \\
 \frac{\frac{x_1^{n_2-1} w_2(x_1)}{w_1(x_1)}}{\frac{x_1^{n_2-1} w_2(x_n)}{w_1(x_n)}} & \frac{\frac{x_2^{n_2-1} w_2(x_2)}{w_1(x_2)}}{\frac{x_2^{n_2-1} w_2(x_n)}{w_1(x_n)}} & \cdots & \frac{\frac{x_n^{n_2-1} w_2(x_n)}{w_1(x_n)}}{\frac{x_n^{n_2-1} w_2(x_n)}{w_1(x_n)}}
\end{vmatrix} \prod_{j=1}^{n_2} dy_j.
\]

Since

\[
n_1 \geq n_2 - 1
\]
we can perform elementary row operations to reduce the remaining determinant to

\[
\begin{bmatrix}
1 & 1 & \cdots & 1 \\
\vdots & \vdots & \ddots & \vdots \\
\frac{x_1}{x_1-y_1} & \frac{1}{x_2-y_1} & \cdots & \frac{1}{x_n-y_1} \\
\frac{y_1^{n_1-1}}{x_1-y_2} & \frac{1}{x_2-y_2} & \cdots & \frac{1}{x_n-y_2} \\
\vdots & \vdots & \ddots & \vdots \\
\frac{y_1^{n_1-1}}{x_1-y_{n_2}} & \frac{1}{x_2-y_{n_2}} & \cdots & \frac{1}{x_n-y_{n_2}}
\end{bmatrix}
\]

which is a mixture of a Vandermonde and a Cauchy determinant. It can be evaluated to give

\[
\prod_{j=1}^{n_2} y_j^{j-1} \frac{\Delta(X) \Delta(Y)}{\Delta(X,Y)}.
\]

where \(X = (x_1, \ldots, x_n)\) and \(Y = (y_1, \ldots, y_{n_2})\). Thus

\[
(4.12) \quad \det [g_j(x_k)] = \prod_{k=1}^{n} w_1(x_k) \Delta(X) \int_{\Gamma_2} \cdots \int_{\Gamma_2} \prod_{j=1}^{n_2} v(y_j) \prod_{j=1}^{n_2} y_j^{j-1} \frac{\Delta(Y)}{\Delta(X,Y)} \prod_{j=1}^{n_2} dy_j.
\]

Now we symmetrize the multiple integral with respect to the integration variables \(y_j\) (which is a standard trick in determinantal point processes). That is, for any permutation \(\sigma \in S_{n_2}\) we make the change of variables \(y_j \mapsto y_{\sigma(j)}\) and we average over all permutations \(\sigma \in S_{n_2}\). Using the fact that

\[
\sum_{\sigma \in S_{n_2}} (-1)^{sgn \sigma} \prod_{j=1}^{n_2} y_j^{j-1} = \det [y_j^{-1}]_{j,k=1,\ldots,n_2} = \Delta(Y)
\]

we then obtain that (4.12) is equal to

\[
(4.13) \quad \det [g_j(x_k)] = \frac{1}{n_2} \prod_{k=1}^{n} w_1(x_k) \Delta(X) \int_{\Gamma_2} \cdots \int_{\Gamma_2} \prod_{j=1}^{n_2} v(y_j) \frac{\Delta(Y)^2}{\Delta(X,Y)^{n_2}} \prod_{j=1}^{n_2} dy_j.
\]

The joint p.d.f. for the Nikishin ensemble is therefore (since \(\det [f_j(x_k)] = \Delta(X)\))

\[
(4.14) \quad \frac{1}{Z_n} \det [f_j(x_k)] \det [g_j(x_k)] = \frac{1}{Z_n n_2!} \prod_{k=1}^{n} w_1(x_k) \Delta(X)^2 \int_{\Gamma_2} \cdots \int_{\Gamma_2} \prod_{j=1}^{n_2} v(y_j) \frac{\Delta(Y)^2}{\Delta(X,Y)^{n_2}} \prod_{j=1}^{n_2} dy_j.
\]
By dropping the integrals over the $y_j$ variables, we can view (4.14) as a marginal density of an extended ensemble defined by the joint p.d.f.

$$P_{\text{ext}}(x_1, \ldots, x_n, y_1, \ldots, y_{n_2}) = \frac{1}{Z_n n_2!} \prod_{k=1}^{n} w_1(x_k) \prod_{j=1}^{n_2} v(y_j) \cdot \frac{\Delta(X)^2 \cdot \Delta(Y)^2}{\Delta(X, Y)}$$

defined for $x_1, \ldots, x_n \in \Gamma_1$ and $y_1, \ldots, y_{n_2} \in \Gamma_2$. Note that the factor $\Delta(X, Y)$ in (4.15) is positive, since $\Gamma_2$ lies to the left of $\Gamma_1$ so that $x_k > y_j$ for every $k = 1, \ldots, n$ and $j = 1, \ldots, n_2$.

4.4.3. Nikishin ensemble with $p \geq 2$ weights. The above considerations can be extended to general $p \geq 2$. Let $w_1, \ldots, w_p$ be a Nikishin system with $p$ weights for the intervals $\Gamma_1, \ldots, \Gamma_p$. Assume that

$$n_j \geq n_{j+1} - 1, \quad \text{for } j = 1, \ldots, p - 1.$$ 

Then the joint p.d.f. for the Nikishin ensemble is a marginal density of an extended ensemble defined on

$$\Gamma_1^{N_1} \times \Gamma_2^{N_2} \times \cdots \times \Gamma_p^{N_p}, \quad \text{where } N_j = \sum_{i=j}^{p} n_i,$$

with joint p.d.f. of the form

$$\frac{1}{\tilde{Z}_n} \prod_{j=1}^{p} \prod_{k=1}^{N_j} w^{(j)}(x^{(j)}_k) \cdot \prod_{j=1}^{p} \frac{\Delta(X^{(j)})^2}{\prod_{j=1}^{p-1} \Delta(X^{(j)}, X^{(j+1)})}$$

where $w^{(j)}$ is a certain weight function on $\Gamma_j$ for $j = 1, \ldots, p$, with $w^{(1)} = w_1$. Here

$$X^{(j)} = (x^{(j)}_1, x^{(j)}_2, \ldots, x^{(j)}_{N_j}) \in \Gamma_j^{N_j}$$

and $\tilde{Z}_n$ is a normalizing constant.

5. Weak asymptotics

An important question about a sequence of polynomials with increasing degrees, is about the asymptotic behavior as the degree tends to $\infty$.

5.1. Vector equilibrium problems. To describe the weak asymptotics of the type II MOPs, as well as the convergence for the Hermite-Padé rational approximation problems, vector equilibrium problems were identified that are relevant for the Angelesco and Nikishin systems. Here we show that these equilibrium problems have a natural interpretation in terms of the joint p.d.f.’s of the Angelesco and Nikishin ensembles.

We assume that we are considering MOPs $P_{\bar{n}}$ for a sequence of multi-indices $\bar{n}$ such that $n = |\bar{n}| \to \infty$ and $n_j \to \infty$ for every $j = 1, \ldots, p$ in such a way that

$$\frac{n_j}{n} \to r_j \quad \text{for } j = 1, \ldots, p.$$ 

The limiting ratios $r_j$ should satisfy

$$0 < r_j < 1, \quad \sum_{j=1}^{p} r_j.$$
Here and in the following we use
\[
I(\mu) = \int \int \log \frac{1}{|x - y|} d\mu(x) d\mu(y)
\]
to denote the logarithmic energy of the measure \(\mu\), and
\[
I(\mu, \nu) = \int \int \log \frac{1}{|x - y|} d\mu(x) d\nu(y),
\]
which is the mutual logarithmic energy of the two measures \(\mu\) and \(\nu\).

For a discrete measure we introduce the reduced logarithmic energy
\[
I^*(\mu) = \int \int_{x \neq y} \log \frac{1}{|x - y|} d\mu(x) d\mu(y)
\]
Note that, if
\[
\nu_X = \sum_{k=1}^{n} \delta_{x_k}
\]
is the point counting measure of \(X = (x_1, \ldots, x_n) \in \mathbb{R}^n\), then
\[
\log \Delta(X)^2 = -I^*(\nu_X).
\]

### 5.2. Angelesco System

The Angelesco ensemble has the joint p.d.f. (4.8).

A configuration of points \(X\) in an Angelesco ensemble is of the form
\[
X = (X^{(1)}, \ldots, X^{(p)}), \quad \text{where} \quad X^{(i)} = (x_1^{(i)}, \ldots, x_n^{(i)}) \in \Gamma_i^n.
\]
The most likely configuration minimizes
\[
-\sum_{j=1}^{p} \log \Delta(X^{(j)})^2 - \sum_{i=1}^{p-1} \sum_{j=i+1}^{p} \log \Delta(X^{(i)}, X^{(j)}) + \sum_{j=1}^{p} \sum_{k=1}^{n_j} Q_j(x_k^{(j)})
\]
where \(w_j = e^{-Q_j}\), among all \(X\) of the form (5.2). Introducing the normalized point counting measures
\[
\nu_j = \frac{1}{n} \nu_X^{(j)}, \quad j = 1, \ldots, p,
\]
we can rewrite (5.3), after dividing by \(n^2\), as
\[
\sum_{j=1}^{p} I^*(\nu_j) + \sum_{i=1}^{p-1} \sum_{j=i+1}^{p} I(\nu_i, \nu_j) + \frac{1}{n} \sum_{j=1}^{p} \int Q_j(x) d\nu_j(x).
\]

In the limit (5.1) we forget about the discreteness of the measures \(\nu_j\). Then instead of minimizing (5.3) among all vectors of measures \((\nu_1, \ldots, \nu_p)\) with \(\nu_j\) a measure of total mass \(n_j/n\) on \(\Gamma_j\) of the form (5.4), we come to minimize the energy functional
\[
E(\mu_1, \mu_2, \ldots, \mu_p) = \sum_{j=1}^{p} I(\mu_j) + \sum_{j=1}^{p-1} \sum_{k=j+1}^{p} I(\mu_j, \mu_k)
\]
among all vectors of measures \((\mu_1, \ldots, \mu_p)\) with
\[
supp(\mu_j) \subset \Gamma_j \quad \text{and} \quad \int d\mu_j = r_j.
\]

Under the assumption that each \(\Gamma_j\) is compact, and that \(w_j(x) \geq 0\) almost everywhere on \(\Gamma_j\), Gonchar and Rakhmanov [40] showed that the zeros of the
type II MOP $P_\vec{n}$ are distributed according to the minimizer $(\mu_1, \mu_2, \ldots, \mu_r)$ for the vector equilibrium problem. More precisely, for every $j = 1, \ldots, p$, there are $n_j$ simple zeros of $P_\vec{n}$ in $\Gamma_j$, say $x_1^{(j)}, \ldots, x_{n_j}^{(j)}$, and the normalized zero counting measure

$$\nu_j = \frac{1}{n} \sum_{k=1}^{n_j} \delta_{x_k^{(j)}}$$

converges in the limit (5.1) weakly to $\mu_j$.

Under the same conditions, it seems likely that the vector of normalized counting measures $(\nu_1, \ldots, \nu_p)$ (as in (5.4)) of a random point (5.2) from an Angelesco ensemble tends to the vector of nonrandom measures $(\mu_1, \ldots, \mu_p)$, almost surely, but this has not been established rigorously.

In a situation of varying weights in an Angelesco ensemble, such as for example

$$w_j(x) = e^{-nV_j(x)}, \quad x \in \Gamma_j, \quad j = 1, \ldots, p,$$

we have to add external field terms to (5.6) and the relevant energy functional becomes

$$(5.8) \quad \sum_{j=1}^{p} I(\mu_j) + \sum_{j=1}^{p-1} \sum_{k=j+1}^{p} I(\mu_j, \mu_k) + \sum_{j=1}^{p} \int V_j(x) d\mu_j(x).$$

This concept is well known in the orthogonal polynomial case, see [62] for the standard reference on logarithmic potential theory with external fields.

### 5.3. Nikishin system.

Similar considerations apply to the joint p.d.f. (4.16) in the extended Nikishin ensemble. Here the most likely configuration

$$X = (X^{(1)}, X^{(2)}, \ldots, X^{(p)})$$

where

$$X^{(j)} = (x_{1}^{(j)}, x_{2}^{(j)}, \ldots, x_{N_j}^{(j)}) \in \Gamma_j^{N_j}$$

is the one that minimizes

$$(5.9) \quad - \sum_{j=1}^{p} \log \Delta(X^{(j)})^2 + \sum_{j=1}^{p-1} \log \Delta(X^{(j)}, X^{(j+1)}) + \sum_{j=1}^{p} \sum_{k=1}^{N_j} Q_j(x_k^{(j)})$$

where $w^{(j)} = e^{-Q_j}$. In terms of the normalized point counting measures

$$\nu_j = \frac{1}{n} \nu_{X^{(j)}}, \quad j = 1, \ldots, p,$$

the expression (5.9) is, after dividing by $n^2$,

$$(5.10) \quad \sum_{j=1}^{p} I^*(\nu_j) - \sum_{j=1}^{p-1} I(\nu_j, \nu_{j+1}) + \frac{1}{n} \sum_{j=1}^{p} \int Q_j(x) d\nu_j(x).$$

The measure $\nu_j$ has total mass

$$\int d\nu_j = \frac{N_j}{n} = \frac{1}{n} \sum_{i=j}^{p} n_i,$$

and in the limit (5.11) we have

$$\frac{N_j}{n} \to \sum_{i=j}^{p} r_i.$$
So in the limit (5.1) we are led to the following energy functional

\[
E(\mu_1, \mu_2, \ldots, \mu_p) = \sum_{j=1}^{p} I(\mu_j) - \sum_{j=1}^{p-1} I(\mu_j, \mu_{j+1})
\]

among all vectors of measures \((\mu_1, \ldots, \mu_p)\) with

\[
supp(\mu_j) \subset \Gamma_j \quad \text{and} \quad \int d\mu_j = \frac{p}{r_j}, \quad j = 1, \ldots, p.
\]

In the special case that \(n_1 = n_2 = \cdots = n_p\) we have that all \(r_j\) are equal to \(1/p\), and then the normalizations are

\[
\int d\mu_j = 1 - \frac{j-1}{p}, \quad j = 1, \ldots, p.
\]

5.4. Interaction matrix. In both an Angelesco system and a Nikishin system we minimize

\[
E(\mu_1, \mu_2, \ldots, \mu_p) = \sum_{j=1}^{p} \sum_{k=1}^{p} c_{jk} I(\mu_j, \mu_k)
\]

with a certain positive definite interaction matrix \(C = (c_{jk})\).

The Angelesco interaction matrix is a full matrix

\[
C = \begin{pmatrix}
1 & \frac{1}{2} & \frac{1}{2} & \cdots & \frac{1}{2} & \frac{1}{2} \\
\frac{1}{2} & 1 & \frac{1}{2} & \cdots & \frac{1}{2} & \frac{1}{2} \\
\frac{1}{2} & \frac{1}{2} & 1 & \cdots & \frac{1}{2} & \frac{1}{2} \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
\frac{1}{2} & \frac{1}{2} & \cdots & \frac{1}{2} & 1 & \frac{1}{2} \\
\frac{1}{2} & \cdots & \cdots & \cdots & \frac{1}{2} & 1
\end{pmatrix}
\]

and the Nikishin interaction matrix is a tridiagonal matrix

\[
C = \begin{pmatrix}
1 & -\frac{1}{2} & 0 & \cdots & \cdots & 0 \\
-\frac{1}{2} & 1 & -\frac{1}{2} & \cdots & \cdots & \vdots \\
0 & -\frac{1}{2} & 1 & \cdots & \cdots & \vdots \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
\vdots & \vdots & \vdots & \cdots & -\frac{1}{2} & 0 \\
0 & \cdots & \cdots & 0 & -\frac{1}{2} & 1
\end{pmatrix}
\]

The Angelesco interaction matrix (5.14) shows the repulsion that exists between \(\mu_j\) and \(\mu_k\) when \(j \neq k\). However, the repulsion is only half as strong as the inner repulsion between each measure \(\mu_j\) itself.

Since the non-zero off-diagonal entries in (5.15) are negative, there is an attraction in a Nikishin ensemble between two consecutive measures \(\mu_j\) and \(\mu_{j+1}\) for \(j = 1, \ldots, p-1\). The measures \(\mu_j\) and \(\mu_k\) with \(|j-k| \geq 2\) do not interact. The Nikishin interaction also arises in the asymptotic analysis of eigenvalues of banded Toeplitz matrices \([35]\) as well as in the two-matrix model \([36]\) where it appears with both an external field and an upper constraint.
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