A PROOF OF THE PALAMODOV’S TOTAL INSTABILITY
CONJECTURE

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ABSTRACT. We give for the first time a detailed proof of the Palamodov’s total instability conjecture in Lagrangian dynamics. This proves an older related Lyapunov instability conjecture posed by Lyapunov and Arnold and reduces the Lagrange-Dirichlet converse problem in the class of real analytic potentials to the Lyapunov instability of non strict minimum critical points. It also proves the instability of charged rigid bodies under the presence of an external electrostatic field.

1. Introduction

In the context of the Lagrangian dynamics of a mechanical Lagrangian

\[ L(x, v) = Q_x(v) - U(x), \quad (x, v) \in TM \]

where \( M \) is a real analytic manifold, \( Q \) is a positive definite smooth quadratic form and the potential \( U \) is a real analytic function on \( M \), Palamodov announced the following instability Theorem:

**Theorem** (Palamodov [Pa], Theorem 2.1). For any point \( x_0 \) in \( M \), there exists a neighborhood \( W \) of \( x_0 \) and a positive function \( T \) defined on the interval \( (-\infty, U(x_0)) \) such that any motion with energy \( E < U(x_0) \) starting at \( W \) cannot spend a time lapse greater than \( T(E) \) inside \( W \).

However, although the proof is illuminating and has very inspiring ideas, unfortunately it is not a complete argument. It is believed that the argument can be completed [Pa2] but technically speaking, it does not constitute a proof leaving the announced Theorem so far as a conjecture.

In the recent paper [AR], Allaire and Rauch proved the Earnshaw’s Theorem asserting the absence of stable equilibrium configurations of conductors and dielectrics in an external electrostatic field. In that paper, referring to the weaker Lyapunov-Arnold’s conjecture (see below), we read ([AR], Section 1.7):

“In 1995 Palamodov [Pa] outlined a very interesting proof. A detailed proof has never been published. If this conjecture were proved it would imply our instability theorems and also the instability of charged rigid bodies that we do not prove.”

We believe that a detailed proof of the Palamodov’s conjecture would be of interest to the mathematical community, [BB], [GT], [GT2], [AKN], [FGT], [BGT], [AR] and [BMP], in chronological order. All of these cited references except for [AR], assume the conjecture as a valid Theorem.

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A proof of the Palamodov’s conjecture would be a breakthrough in the history of the long standing and more general open conjecture on the Lagrange-Dirichlet converse in the class of real analytic potentials for it reduces the problem to the study of the Lyapunov instability of non strict minimum points of a real analytic potential. In effect, the Theorem/Conjecture has the immediate corollary

**Corollary** (Palamodov [Pa], Corollary 2.2). If a critical point $x_0$ of the potential $U$ belongs to the closure of the set $\{ x \in M \mid U(x) < U(x_0) \}$, then the equilibrium position $x_0$ is unstable.

The Lagrange-Dirichlet converse for two degrees of freedom real analytic potentials follows from the results of Palamodov [Pa3] or Taliaferro [Ta] in conjunction with those of Kozlov [Ko1] and Laloy and Peiffer [LP].

Concerning the state of the art of the Lagrange-Dirichlet converse, we recommend the recent paper [Pa4] by Palamodov. In that paper, the hypothesis of the previous Palamodov’s Theorem/Conjecture is condensed in the notion of total instability ([Pa4], Section 3, Definition 1), although that paper does not mention the previous paper [Pa] where the Theorem was announced.

In view of the previous corollary, a proof of the Palamodov’s conjecture would also prove the related open Arnold’s conjecture in the context of Newtonian dynamics with a real analytic potential, posed in the form of a problem in [Ar]:

“1971-4. Prove the instability of the equilibrium 0 of the analytic system $\ddot{x} = -\partial U/\partial x$ in the case where the isolated critical point 0 of the potential $U$ is not a minimum.”

This problem/conjecture is a clear statement of a question previously posed by Lyapunov in ([Ly], Section 25) concerning the Lagrange-Dirichlet stability Theorem$^1$:

“But, in establishing that this condition is sufficient, the theorem in question does not allow any conclusion about the necessity of the same condition. That is why the question arises: will the position of equilibrium be unstable if the force-function is not maximum?”

In [Br], Brunella proved the Arnold’s conjecture for two degrees of freedom. Starting from Lyapunov [Ly] and continuing with [GT], [Ha], [Ko2], [Ko3], [Ku], [KP], [MN], [Ta2], many partial results have been given towards the solution of the Arnold’s conjecture and their common feature is that the Lyapunov instability criteria involves the lack of a local minimum at the origin of the first nonzero jet of the potential. In particular, these results are not sufficient to prove the Arnold’s conjecture hence, as the Palamodov’s conjecture, it remains open.

The applications of the Palamodov’s conjecture go far beyond the Lagrange-Dirichlet inverse problem and very interesting and important instability results would follow from the validity of it. As we quoted at the beginning of the introduction, in the recent paper [AR], Allaire and Rauch claimed that a proof of the Lyapunov-Arnold’s conjecture which follows from the Palamodov’s conjecture would imply the instability of charged rigid bodies under the presence of an external electrostatic field.

In this paper we give for the first time a detailed proof of the Palamodov’s conjecture. Although it is inspired in the Palamodov’s original argument, the most

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$^1$The force-function is the potential function, the opposite of the potential energy $U$. 
distinctive feature of the present proof is that it avoids the induction step on every blowup of a Hironaka’s monomialization [Hi] for the construction of an appropriate vector field for the problem. This is possible due to the key observation that, to study the instability of a point \( x_0 \), the mentioned vector field does not need to be defined on the locus of points with potential \( U(x_0) \), providing a proof of the conjecture.

2. Proof

By polarization, the quadratic form \( Q \) induces a Riemannian metric \( \rho \) on \( M \) such that \( Q_x(v) = \|v\|_x^2/2 \) for every point \( x \) in \( M \) and every vector \( v \) in the tangent space \( T_xM \). This Riemannian metric induce the Levi-Civita connection \( \nabla \) and the gradient of the potential \( \operatorname{grad}_\rho U \) on \( M \). A smooth curve \( \gamma \) verifies the Euler-Lagrange equation if

\[
\nabla \dot{\gamma} + \operatorname{grad}_\rho U = 0
\]

where the equation is evaluated at the curve \( \gamma \) for every instant where it is defined. A motion is a maximal solution of the Euler-Lagrange equation.

For every real analytic function \( f \) denote by \( \mathcal{V}(f) \) the real analytic variety of zeroes of \( f \). Without loss of generality, we suppose that \( x_0 \) is a zero potential point, that is to say, \( x_0 \) belongs to \( \mathcal{V}(U) \).

Denote by \( \mathcal{O}_M \) the structure sheaf of real analytic functions on \( M \) and define the sheaf of ideals \( \mathfrak{m}_{x_0} \) on \( M \) whose stalk at \( x_0 \) is the maximal ideal \( \mathfrak{m}_{M,x_0} \) of the local ring \( \mathcal{O}_{M,x_0} \) and \( \mathcal{O}_{M,y} \) at every other point \( y \) in \( M \). Consider the sheaf of ideals \( \mathcal{I}_U \) on \( M \) whose stalk at the point \( a \) is the localization of the ideal \( (U) \) at \( a \).

The following Lemma is due to Spivakovsky [Sp].

**Lemma 2.1.** Consider a Hironaka monomialization \( \sigma : \tilde{M} \to M \) of the sheaf of ideals \( \mathcal{I} = \mathfrak{m}_{x_0} \mathcal{I}_U \) with exceptional divisor \( E \). Then,

1. The inverse image of the sheaf of ideals \( \mathcal{I}_U \) by the monomialization \( \sigma \) is locally monomial.
2. The exceptional divisor \( E \) coincides with the preimage of the zero locus of \( U \) by \( \sigma \).
3. The preimage of the set \( \{x_0\} \) by \( \sigma \) is the union of a collection of irreducible components of \( E \).

**Proof.** By abuse of notation, we denote by \( \sigma^{-1} \) both the inverse image functor when it is evaluated on a sheaf and the preimage of a set when it is evaluated on a set.

By Hironaka’s monomialization, the inverse image of \( \mathcal{I} \) by \( \sigma \) is locally monomial and because

\[
\sigma^{-1}\mathcal{I} = \sigma^{-1}(\mathfrak{m}_{x_0}\mathcal{I}_U) = (\sigma^{-1}\mathfrak{m}_{x_0})(\sigma^{-1}\mathcal{I}_U)
\]

we conclude that the sheaves of ideals \( \sigma^{-1}\mathfrak{m}_{x_0} \) and \( \sigma^{-1}\mathcal{I}_U \) are locally monomial as well, proving in particular the first item.

Taking the support on the identity (3), we have

\[
\text{supp} (\sigma^{-1}\mathcal{I}) = \text{supp} (\sigma^{-1}\mathfrak{m}_{x_0}) \cup \text{supp} (\sigma^{-1}\mathcal{I}_U)
\]

where we have denoted by \( \text{supp} \) the support. Due to the fact that

\[
E = \text{supp} (\sigma^{-1}\mathcal{I}), \quad \sigma^{-1}(\{x_0\}) = \text{supp} (\sigma^{-1}\mathfrak{m}_{x_0})
\]

and \( \sigma^{-1}(\mathcal{V}(U)) = \text{supp} (\sigma^{-1}\mathcal{I}_U) \),
we have the identity
\[ E = \sigma^{-1}(\{x_0\}) \cup \sigma^{-1}(\mathcal{V}(U)). \]
Then, the second item follows from the previous identity and the inclusion
\[ \sigma^{-1}(\{x_0\}) \subset \sigma^{-1}(\mathcal{V}(U)) \] for \( x_0 \) belongs to \( \mathcal{V}(U) \).
Finally, because of the identity (3) and the fact that \( \sigma^{-1}\mathfrak{m}_{x_0} \) is locally monomial, its support is the union of a collection of irreducible components of the support of \( \sigma^{-1}\mathcal{I} \). Hence, the third item immediately follows from the identities (5) and this finishes the proof. \( \square \)

The following Lemma is the analog of Lemma 4.1 in [Pa].

**Lemma 2.2.** There is a neighborhood \( W \) of \( x_0 \) in \( M \), a smooth vector field \( V \) and a smooth real valued function \( P \) both defined on \( W - \mathcal{V}(U) \) such that
\[ P \geq 1, \quad V(U) = P U \quad and \quad (v, \nabla_v V) = (1 + o(1)) ||v||^2 \]
for every vector \( v \) in \( T_x M \) with \( x \) in \( W - \mathcal{V}(U) \) as the point \( x \) approaches to \( x_0 \).

**Proof.** By the previous Lemma and [Hi], there is a Hironaka’s monomialization
\[ \sigma : \tilde{M} \rightarrow M \]
of the sheaf of ideals \( \mathfrak{m}_{x_0} \mathcal{I} \) with exceptional divisor \( E \) such that \( \sigma^{-1}(x_0) \) is the union of a collection of irreducible components \( H_1, \ldots, H_m \) in \( E \). Denote by \( H \) the union of these distinguished irreducible components.

Because the exceptional divisor \( E \) has simple normal crossing, for every point \( p \) in \( H \) there is a coordinate neighborhood \( W_p \) of \( p \) in \( M \) with coordinates \( w_1, \ldots, w_n \) such that for every irreducible component \( E_i \) in \( E \) there is a coordinate \( w_i \) such that \( W_p \cap E_i = \mathcal{V}(w_i) \). Moreover, because of the first item in the previous Lemma, taking \( W_p \) small enough we also have
\[ \tilde{U}|_{W_p} = \pm w_1^{d_1} \ldots w_n^{d_n} \]
for some nonnegative integers \( d_1, \ldots, d_n \) where \( \tilde{U} = U \circ \sigma \). Note that, because \( \tilde{U} \) is zero on \( H \) for \( x_0 \) is a zero potential point, at least one of these nonnegative integers must be nonzero. On the neighborhood \( W_p \), define the real analytic vector field
\[ \tilde{V}_p = w_1 \partial_{w_1} + \ldots + w_n \partial_{w_n} \]
and note that it is is tangent to every irreducible component in \( E \) containing \( p \) and has the property that \( \tilde{V}_p(U) = c_p \tilde{U} \) on \( W_p \) for some positive integer \( c_p \).

The pushout \( V_p \) of \( \tilde{V}_p \) by \( \sigma \) is defined and real analytic on the open set \( \sigma(W_p - E) \) for \( \sigma \) is an isomorphism on \( \tilde{M} - E \) and it has the expression
\[ V_p = w'_1 \partial_{w'_1} + \ldots + w'_n \partial_{w'_n} \]
with respect to the coordinates \( w'_i = \sigma^{-1}(w_i) \) on \( \sigma(W_p - E) \).

Because of the tangency properties of \( \tilde{V}_p \), the field \( V_p \) continuously extends to the point \( x_0 \) just by defining it as the zero vector at this point. In particular, we have the following asymptotic property as the point approaches to \( x_0 \) on \( \sigma(W_p - E) \):
\[ (v, \nabla_v V_p) = g_{ab} v^a \left( v^c \partial_c V_p^b + \Gamma^b_{ij} v^j V_p^i \right) = g_{ab} v^a v^c \partial_c V_p^b + [ij, a] v^a v^j V_p^i \]
\[ = g_{ab} v^a v^b + \frac{1}{2} g_{ai,j} v^a v^j V_p^i = g_{ab} v^a v^b + \frac{1}{2} V_p(g_{ab}) v^a v^b = (1 + o(1)) ||v||^2. \]
We have used the Einstein’s sum convention on repeated indices and the calculation is with respect to the coordinates \( w'_1, \ldots, w'_n \) on \( \sigma(W_p - E) \).
The last equality follows because the second term is a smooth quadratic form on \( \sigma(W_p - E) \) for \( v \) and \( V_p(x) \rightarrow 0 \) as \( x \rightarrow x_0 \) on this region. It is clear that on this region, the field also verifies \( V_p(U) = c_p U \).

Define \( W' = \bigcup_{p \in H} \sigma(W_p - E) \) and consider a partition of unity \( \{ f_p \mid p \in H \} \) with the same index set and subordinate to the open cover \( \{ \sigma(W_p - E) \mid p \in H \} \) of \( W' \). Define the smooth vector field \( V \) on \( W' \) as
\[
V = \sum_{p \in H} f_p V_p.
\]

It has the property that \( V(U) = PU \) where \( P = \sum_{p \in H} f_p c_p \) is a smooth function on \( W' \) with \( P \geq 1 \) and verifying the following asymptotic property as the point approaches to \( x_0 \) for similar reasons as before:
\[
\langle v, \nabla_v V \rangle = \sum_{p \in H} (f_p \langle v, \nabla_v V_p \rangle + v(f_p) \langle v, V_p \rangle) = (1 + o(1)) \| v \|^2.
\]

Finally, there is a neighborhood \( W \) of \( x_0 \) in \( M \) contained in \( W' \cup \mathcal{V}(U) \) for otherwise there would be a sequence
\[
(x_n)_{n \in \mathbb{N}}, \quad x_n \notin W' \cup \mathcal{V}(U), \quad x_n \rightarrow x_0
\]
and a subsequence of \( (\sigma^{-1}(x_n)) \) converging to some point \( y \) in \( H \) for \( H \) is compact hence eventually contained in \( W_y - E \) which is absurd. In particular, \( W - \mathcal{V}(U) \) is contained in \( W' \). This finishes the proof.

\[
\square
\]

\textbf{Proof of the conjecture.} Let \( W \) be a small enough precompact neighborhood of \( x_0 \) such that \( \langle v, \nabla_v V \rangle \geq 0 \) for every vector \( v \) in \( T(W - \mathcal{V}(U)) \) and \( V \) is defined on \( \overline{W} - \mathcal{V}(U) \). There is a minimum \( m_U \) of \( U \) on \( \overline{W} \).

Suppose that the Theorem is false for \( W \). Then, there is a motion, that is to say a solution of (2), starting at time zero and defined for every instant in the future \( \gamma : [0, +\infty) \rightarrow W \) with energy \( E < 0 \). Note that necessarily \( 0 > E \geq m_U \), the motion lies in the region \( W \cap [U \leq E] \) and \( \| \dot{\gamma} \| \) is bounded from above by \( (2(E - m_U))^{1/2} \).

Define on \([0, +\infty)\) the smooth real valued function \( F \) by
\[
F(t) = \langle \dot{\gamma}(t), V(\gamma(t)) \rangle.
\]

By the Cauchy-Schwartz inequality, the function \( F \) is bounded for
\[
|F(t)| \leq \| \dot{\gamma}(t) \| \| V(\gamma(t)) \| \leq (2(E - m_U))^{1/2} M_V
\]
where \( M_V \) is the maximum of \( \| V \| \) on the compact set \( \overline{W} \cap [U \leq E] \).

However, because of the previous Lemma and the Euler-Lagrange equation (2)
\[
\dot{F} = \langle \nabla_x \dot{\gamma}, V \rangle + \langle \dot{\gamma}, \nabla_x V \rangle \geq -(\text{grad}_x U, V) = -V(U) = -PU \geq -E.
\]

In particular, \( F \) is not bounded from above which is absurd and we have the result.

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\square
\]
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References

[Ar] V. I. Arnold, Arnold’s Problems, Springer-Verlag, (2002).
[AKN] V. I. Arnold, V. V. Kozlov, A. I. Neishtadt, Mathematical aspects of classical and celestial mechanics, Springer, (2007).
[AR] G. Allaire, J. Rauch, Instability of Dielectrics and Conductors in Electrostatic Fields, Arch. Rational Mech. Anal., Springer, 224 (2017), 233–268.
[Br] M. Brunella, Instability of equilibria in dimension three, Ann I Fourier, 48 (1998).
[BB] M. L. Bertotti, S. V. Bolotin, On the influence of the kinetic energy on the stability of equilibria of natural Lagrangian systems, Arch. Rational Mech. Anal., 152 (2000), 65–79.
[BGT] R. B. Bortolatto, M. V. P. Garcia, F. A. Tal, Kinetic Energy and the Stable Set, Qualitative Theory of Dynamical Systems, 10 (2011).
[BMP] J. M. Burgos, E. Maderna, M. Paternain, On the Lyapunov instability in Newtonian dynamics, Nonlinearity, 34 (2021), 6719–6726.
[GT] M. V. P. Garcia, F. A. Tal, Stability of equilibrium of conservative systems with two degrees of freedom, J. of Differential Equations, 194 (2003), 364–81.
[GT2] M. V. P. Garcia, F. A. Tal, The influence of the kinetic energy in equilibrium of Hamiltonian systems, J. of Differential Equations, 213 (2005), 410–417.
[FGT] R. S. Freire, M. V. P. Garcia, F. A. Tal, Instability of equilibrium points of some Lagrangian systems, J. of Differential Equations, 245 (2008), 490–504.
[Ha] P. Hagedorn, Die Umkehrung der Stabilitätssätze von Lagrange-Dirichlet und Routh, Arch. Rational Mech. Anal., 42 (1971), 281–316.
[Hi] H. Hironaka, Resolution of singularities of an algebraic variety over a field of characteristic zero, Annals of Math., 79 (1964), 109–326.
[Ko1] V.V. Kozlov, Instability of equilibrium in a potential field, Russian Math. Surveys, 36 (1981), 238–239.
[Ko2] V. V. Kozlov, Asymptotic solutions of equations of classical mechanics, J. Appl. Math. Mech., 46 (1982), 454–7.
[Ko3] V. V. Kozlov, Asymptotic motions and the inversion of the Lagrange-Dirichlet theorem, J. Appl. Math. Mech., 50 (1987), 719–25.
[Ku] A. N. Kuznetsov, On existence of asymptotic solutions to a singular point of an autonomous system possessing a formal solution, Functional Anal. Appl., 23 (1989).
[KP] V. V. Kozlov, V. P. Palamodov, On asymptotic solutions of the equations of classical mechanics, Dokl. Akad. Nauk SSSR 263 285–9; English transl., Soviet Math. Dokl., 25 (1982), 335–9.
[Ly] A. M. Lyapunov, General problem of the stability of motion, (Kharkov Math. Soc, Kharkov, 1892; French transl.) Ann. of Math. Studies, 17 (1947).
[LP] M. Laloy, K. Peiffer, On the instability of equilibrium when the potential function has a non-strict local minimum, Arch. Rational Mech. Anal., 78 (1982), 213–22.
[MN] V. Maouro, P. Negrini, On the inversion of Lagrange-Dirichlet theorem, Differential Integral Equations, 2 (1989), 471–8.
[Pa1] V. P. Palamodov, Stability of motion and algebraic geometry, Transl. Amer. Math. Soc., 168 (1995), 5–20.
[Pa2] Communication with V. P. Palamodov.
[Pa3] V. P. Palamodov, Equilibrium stability in a potential field, Functional Anal. Appl., 11 (1977), 277–289.
[Pa4] V. P. Palamodov, On inversion of the Lagrange–Dirichlet theorem and instability of conservative systems, Russian Mathematical Surveys, 75 (2020), 107–122.
[Sp] Communication with M. Spivakovsky.
[Ta] S. D. Taliaferro, *Stability for two dimensional analytic potentials*, J. Differential Equations, 35 (1980), 248–265.
[Ta2] S. D. Taliaferro, *Instability of an equilibrium in a potential field*, Arch. Rational Mech. Anal., 109 (1990), 183–94.

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