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Abstract

Khan and Penrose obtained an exact solution for colliding plane impulsive gravitational waves with the remarkable feature that the spacetime before the collision is flat and after the collision is not only curved, but develops a future curvature singularity. The spacetime has been extensively investigated analytically by various authors using the Newman–Penrose equations. We feel that a separate insight is provided by using the method of Weber and Wheeler to calculate the momentum imparted to test particles by the colliding waves. In this paper we probe the curvature by considering the momentum imparted to test particles by the colliding gravitational waves so as to try to visualise the way the curvature increases in the chosen coordinates as the singularity is approached.

1. Introduction

Penrose [1] had developed a topological procedure to cut the Minkowski spacetime at a null cone and rejoin it with a jump. This discontinuity in the metric tensor of a flat space, leads to a delta function in its first derivative, corresponding to a delta-function gravitational wave with a ‘strength’ given by the ‘size’ of the jump. Subsequently Khan and Penrose wrote down a solution for colliding plane impulsive gravitational waves of equal (unit) strength [2], which we shall call KP waves. The spacetime can be divided into four regions: (I) where neither wave arrived at the given point; (II) where one has passed but the other still has not arrived; (III) where the other has passed and the first one has not arrived; and (IV) where both have passed. For concreteness, take the two waves to lie along the \( xy \)-plane and travel in the positive and negative \( z \)-directions, respectively. The first one is given by a delta function in the retarded time, \( \nu = (t - z)/T \), and the second in the advanced time, \( \nu = (t + z)/T \). For simplicity, since the delta function, \( \delta(x) \), can be taken to be the derivative of a step function, \( \theta(x) \), i.e. \( \delta(x) = \theta'(x) \), the metric for the first and second waves can be taken to have the relevant coefficients given by \( \theta(u) \) and \( \theta(v) \), respectively. Thus region (I) would be for \( u, v < 0 \); (II) for \( u, 0, v ) 0 \); (III) for \( u, 0, v ) 0 \); and (IV) for \( u, v > 0 \); see figure 1.

The colliding waves metric must be a solution of the vacuum Einstein field equations involving the product of the two step functions. It is given in double null coordinates by

\[
ds^2 = \frac{2t^3}{rw(pq + rw)^2} du dv - t^2 \left( \frac{r + q}{r - q} \right) \left( \frac{w + p}{w - p} \right) dx^2 - t^2 \left( \frac{r - q}{r + q} \right) \left( \frac{w - p}{w + p} \right) dy^2,
\]

where

\[
p = u\theta(u), \quad q = v\theta(v), \quad r = \sqrt{1 - p^2}, \quad w = \sqrt{1 - q^2}, \quad t = \sqrt{1 - p^2 - q^2}.
\]

It has a curvature singularity at \( u^2 + v^2 = 1 \). See figure 1.
In region IV, $u, v > 0$ and $u^2 + v^2 < 1$ and the metric becomes,

$$ds^2 = 2 \frac{(1 - u^2 - v^2)^{3/2}}{\sqrt{1 - u^2} \sqrt{1 - v^2} (uv + \sqrt{1 - u^2} \sqrt{1 - v^2})} du dv - (1 - u^2 - v^2) \left( \frac{1 - u^2 + v}{\sqrt{1 - u^2} - v} \right) \left( \frac{1 - v^2 + u}{\sqrt{1 - v^2} - u} \right) dx^2$$

$$- (1 - u^2 - v^2) \left( \frac{1 - u^2 - v}{\sqrt{1 - u^2} + v} \right) \left( \frac{1 - v^2 - u}{\sqrt{1 - v^2} + u} \right) dy^2.$$  \hspace{1cm} (3)

Its Weyl tensor, representing the gravitational field, has ten linearly independent components given in appendix A.

There seems to be a singularity at $v = 1$ in region II and $u = 1$ in region III, apart from the singularity in region IV. This may appear strange, as the disturbance is a shock wave and leaves a flat spacetime after passing. Indeed, they can be removed by transforming coordinates [3]. Region IV is curved and the singularity in it is a curvature singularity at $T$, where this (KP) Universe ends. As such we call $T$ doomsday. Thus, even though there is no intrinsic curvature singularity at $t = T$ in region II at $(v = 1, u < 0)$ and analogously in region III at $(u = 1, v < 0)$, there remains a topological singularity. Physically, doomsday comes with no warning in regions II and III but with a warning in region IV. In this paper, we intend to see how much of a warning of doomsday there is, in the coordinates chosen and as a fraction of the time to doomsday. Though there is warning, doomsday comes dramatically fast.

There had been questions about whether gravitational waves really exist [4]. This question was studied thoroughly by Weber and Wheeler [5] by considering the curvature invariant involving the Riemann tensor components for the Einstein–Rosen cylindrical gravitational waves [6] and by considering the momentum imparted to test particles by them. Their result is also given in a book by Weber [7]. Later Ehlers and Kundt [8] used this procedure for plane waves. Qadir and Sharif provided a general closed formula for the momentum imparted to test particles in time-varying spacetimes [9], which has been applied elsewhere [10–12]. This was called the $\psi N$ formalism. Though the Newman–Penrose formalism [13] has been used, we will use this formula for our purpose. As such, we will compute the momentum imparted to test particles by the colliding waves, after both have passed by. It may be argued that there is no need for more when the analytic behaviour has already been investigated. Our point of view is that the extra insight provided by using our method is analogous to the difference between giving algebraic expressions for curves and surfaces and of giving their graphs and 3-d plots. The geometric significance of the frame used in this formalism is explained in the next section and further discussed in the Conclusion.

The plan of the paper is as follows. In the next section we give a brief review of the $\psi N$ formalism. In section three we shall discuss the numerical and analytic results obtained and follow this up a brief discussion and conclusion in section four.

2. The pseudo-newtonian formalism

Einstein had replaced the concept of forces by the curvature of spacetime and the new description not only fitted all experimental evidence at the time but made predictions that have withstood all experimental tests so far.
However, only the gravitational field is incorporated into the curvature description and all other forces are described by other fields. This problem may be at the root of the difficulty of finding a theory that gives quantum mechanics and general relativity in appropriate limits (normally called the problem of ‘quantum gravity’). Since we are unable to incorporate the other forces satisfactorily into geometry, it seems useful to re-express the geometric results of general relativity in terms of forces, despite one’s inhibition against ‘rendering asunder’ that which Einstein had ‘joined together’. A formalism was developed for this purpose for static spacetimes by identifying the tidal acceleration with geodesic deviation and then asking for the vector whose directional derivative along the direction for which the maximum magnitude of the tidal acceleration is to be found, see [14] and references therein. This quantity would be the acceleration due to gravity. In the frame for which the vector is purely spacelike, this would give the general relativistic analogue of the Newtonian gravitational field. When multiplied by the mass of the test particle in the field, this was called the ψ/N force. The required frame is the ψ/N frame, which is a special case of Fermi normal coordinates which are attached to the ‘freely falling rest frame’, by which is meant the frame of reference of an observer falling freely from infinity. It has been shown that this is the frame in which an observer would see the space as locally flat, i.e. as Minkowski space [15]. Consequently, one would expect that the frame corresponds to that obtained from the foliation of the spacetime by flat spacelike hypersurfaces, i.e. those in which the 3-curvature tensor is flat, \( R^i_{jkl} = 0 \), \((i, j, k, l = 1, 2, 3)\). This, indeed, turns out to be the case. The general expression for this force in the appropriate frame turns out to be

\[
F_{\psi N} = \nabla \ln \sqrt{g_{00}},
\]

(4)

where the signature has been taken to be \((+, -, -, -)\).

Extending the formalism to time varying spacetimes [9, 16] required that we dispense with the assumption that the preferred frame exists for all time and replace it by the gauge requirement that the shift vector be zero and then satisfy the additional condition. With this modification the previous formalism can be used and the resultant acceleration, or ‘force’ becomes time dependent. This is reasonable as it has to account for the energy non-conservation in the time-varying spacetime. (It may be noted that this modification will not change the crucial geometric requirement, that the foliation is flat, is met.) Therefore, the expression for the extended pseudo-Newtonian (evN) force [17] is given as,

\[
F_i = m \left( \left( \ln \lambda A \right)_0 \frac{g_{00}}{4A} \right),
\]

(5)

\[
F_i = m \left( \ln \sqrt{g_{00}} \right)_i, \quad i = 1, 2, 3
\]

where \( A = (\ln \sqrt{g_{00}})_0, g = \det(g_{ij}) \) and \( f = 1/\sqrt{g_{00}} \). The quantity whose proper time derivative is \( F_{\mu} \), is called the momentum four-vector for the test particle. Thus the momentum four-vector, \( P_{\mu} \), is

\[
P_{\mu} = \int F_{\mu} dt \quad \mu = 0, 1, 2, 3
\]

(6)

Since the energy is anyhow re-scaled, the physical significance of \( P_0 \) is not clear and only the 3-momentum seen in the ψ/N-frame, which gives the momentum transfer, is relevant as defined in the preferred frame (in which \( g_{00} = 0 \)).

To illustrate the significance of this formalism, consider its application to cylindrical gravitational waves [17]. The line element for the cylindrical gravitational waves depends on two arbitrary functions of the time \( t \) and a radial coordinate \( \rho \),

\[
ds^2 = e^{2(\gamma - \psi)}(dt^2 - d\rho^2) - e^{-2\psi}d\phi^2 - e^{2\psi}dz^2.
\]

(7)

Einstein’s vacuum equations lead to the constraints on the two functions,

\[
\dot{\psi} + \frac{1}{\rho} \psi' - \ddot{\psi} = 0, \quad \gamma' = \rho(\dot{\psi}^2 + \ddot{\psi}^2), \quad \gamma = 2\rho \dot{\psi} \psi',
\]

(8)

where ‘ ’ and ‘ ′′ represents differentiation with respect to \( t \) and \( \rho \) respectively.

The evψN-force for the line element (4) is given by [17],

\[
F_i = m \left\{ A_{ij}(x) \cos(\omega t) + BY_{ij}(x) \sin(\omega t) - \frac{1}{2}[A^2 J_0(x) J_0'(x) - B^2 Y_0(x) Y_0'(x)] \right\} \cos(2\omega t)
\]

\[
+ \omega \rho \left( [(A^2 J_0(x) J_0'(x) - B^2 Y_0(x) Y_0'(x))] \right) \cos(2\omega t)
\]

\[
- \frac{1}{2} AB[2(J_0(x) Y_0''(x) + Y_0(x) J_0'(x)) - \omega \rho (J_0(x) Y_0''(x) + Y_0(x) J_0'(x))] \sin(\omega t)
\]

\[
- \frac{1}{2} AB[4(J_0(x) Y_0''(x) - Y_0(x) J_0'(x)) + 2\omega \rho (J_0(x) Y_0''(x) - Y_0(x) J_0'(x))] \omega t \right\},
\]

(9)
where $x = \omega \rho$, and $\omega$ is the angular frequency, $J_0(x)$ and $Y_0(x)$ are the Bessel and the Neumann functions of order zero respectively. Here $''$ represents differentiation with respect to $x$.

The corresponding momentum imparted to the test particle is given by [17],

$$P_i = \frac{m}{\omega} \left[ AJ_0'(x) \sin(\omega t) - BY_0'(x) \cos(\omega t) \right] - \frac{1}{4}[(A^2J_0(x)J_0' - B^2Y_0(x)Y_0')]
+ \omega^2((A^2J_0(x)J_0' - B^2Y_0(x)Y_0')\sin(2\omega t) - \frac{1}{2}AB(J_0'(x)Y_0'(x) + Y_0'(x)J_0'(x))
+ \rho(\omega^2J_0'(x)Y_0'(x) + Y_0'(x)J_0'(x))\cos(2\omega t) - \frac{\omega^2}{2}[(J_0'(x)Y_0'(x) - Y_0'(x)J_0'(x))]
- \frac{\rho\omega}{2}(J_0'(x)Y_0'(x) - Y_0'(x)J_0'(x)) \right] + f_i(\omega \rho),$$

(10)

where $f_1$ and $f_2$ are ‘functions of integration’.

The $(evN)$ formalism [17], yielded only the trivial result of a constant for plane impulsive waves, as the spacetime either side of the wave is flat. Since the KP metric in region IV is curved, it was thought that the method could give more meaningful results for this metric. As will be demonstrated shortly, this turns out to be the case. We will use it to probe the fascinating behaviour of test particles near the KP singularity.

3. Analysis of the KP spacetime

The $(evN)$ formalism needs the metric in block diagonal form. This is easily provided by converting from the double null coordinates $u$ and $v$ to a timelike $t = (u + v)/\sqrt{2}$ and a spacelike $z = (u - v)/\sqrt{2}$, to give

$$ds^2 = \frac{8}{\sqrt{2 - (t - z)^2}} \frac{(1 - t^2 - z^2)^{3/2}}{\sqrt{2 - (t + z)^2}} dt^2
- (1 - t^2 - z^2) \left[ \frac{1}{2} \frac{(2 - (t - z)^2 + (t + z)}{2 - (t + z)^2 - (t - z)} \right] dx^2
- (1 - t^2 - z^2) \left[ \frac{1}{2} \frac{(2 - (t - z)^2 - (t + z)}{2 - (t + z)^2 + (t - z)} \right] dy^2
- \frac{8}{\sqrt{2 - (t - z)^2}} \frac{(1 - t^2 - z^2)^{3/2}}{\sqrt{2 - (t + z)^2}} dz^2.$$ (11)

In these coordinates the singularity now occurs at $t^2 + z^2 = 1$ and the domain is $-1 < z < 1$, $0 < t < 1$, or equivalently $|z| < t$, $t^2 + z^2 < 1$.

The time component of the $(evN)$-force gives the re-scaling of the dial of the accelerometer, which shows the zero position of the needle in the absence of a central force. It corresponds to the rate of change of energy of a test particle of mass $m$. However, the expression, given in appendix $B$, does not convey much wisdom. The spatial components give the spatial rate of change of its potential energy. The momentum imparted to test particles is only given by the time integral of the spatial part of the $(evN)$-force. From equation (3) the $(evN)$-force for this metric has $F_x = 0 = F_y$ and

$$F_z = \frac{z(3t^2 + z^2 - 2)(t^2 + z^2)\sqrt{2 - (t + z)^2}(t^2 - z^2)}{2((t - z)^2 - 2)(t^2 + z^2 - 1)((t + z)^2 - 2)},$$

(12)

The relevant quantity is the magnitude of the spatial part of the force 4-vector, i.e. the above expression multiplied by $\sqrt{g^{zz}}$

$$|F| = \sqrt{-F_z E_z g^{zz}} = \left[ \sqrt{2 - (t - z)^2} \frac{z(3t^2 + z^2 - 2)}{2 - (t + z)^2} \sqrt{2 - (t + z)^2} (t^2 - z^2)
+ \sqrt{2 - (t - z)^2} \sqrt{2 - (t + z)^2} (t^2 - z^2) + 2\sqrt{2 - (t - z)^2} \sqrt{2 - (t + z)^2} \right]$$

$$/ \left[ 4\sqrt{2 - (t - z)^2} + 1 \right]^{1/4} (t^2 - 2z^2 + 2(t^2 - 2z^2)).$$

(13)

To better visualise the behavior of the force near the singularity $t^2 + z^2 = 1$, we express the spacetime coordinates in ‘polar’ form, with the ‘radial’ coordinate being timelike and the ‘angular’ one spacelike: $r = \sqrt{t^2 + z^2}$ and $\theta = \tan^{-1}(t/z)$. The wave front of one wave occurs at $\theta = \pi/4$ and of the other at $\theta = -\pi/4$ and the time domain is $0 < t < 1$. Obviously $|F|$ is a symmetric function of $z$ or $\pi/2 - \theta$. As such, we only consider $\pi/4 < \theta < \pi/2$ and reflect in the line $z = 0$ or $\theta = \pi/2$.

Taking $r = 0.989846$ and varying $\theta$ from $\pi/4$ to $\pi/2$, we find that $F$ rapidly increases, reaching a maximum at $\theta \approx 1.22134$ and then decreases to zero. In table $1$ we give $F(\theta)$ in tabular form in arbitrary units. It is found that the same behavior occurs for all constant $r$. In figure $2$, $F(\theta)$ is given graphically for $r = 0.989846$. 


we can construct starts increasing very rapidly after we use value is 7.341. The momentum imparted by the two waves in opposite directions would cancel out.

Now, fixing θ at the maximum value, we compute $F(r)$. It increases very slowly till $r$ approaches 1 and then starts increasing very rapidly after $r = 0.9$. This function is given in tabular form in table 2. For our convenience we use $\epsilon = 1 - r$, instead of $r$. Notice that $F$ is zero at $z = 0$ or $\theta = \pi/2$ as it should be since the momentum imparted by the two waves in opposite directions would cancel out.

Now keeping $F$ constant, we construct $\epsilon(\theta)$. Table 3 gives it for $F = 2.90957521$. The maximum of $\epsilon$ is $7.341 \times 10^{-2}$ occurs at $\theta = 1.26516$. For $F = 7.381660$ the function is given in table 4. Here the maximum value is $\epsilon = 5.111 \times 10^{-2}$ at $\theta = 1.25465$. These curves are plotted in figure 3, in terms of $r = 1 - \epsilon$. Similarly we can construct $\epsilon(\theta)$ for $F = 3.3780842$ and $F = 4.1597548$. The maximum of $\epsilon$ occurs at $\theta = 1.25886$ and

### Table 1. $|F(\theta)|$ for $r = 0.989846$ is maximum at $\theta \approx 1.22134$.

| $\theta$ | $|F(\theta)|$ |
|---------|-------------|
| 0.79972 | $1.747 \times 10^{6}$ |
| 0.814097 | $2.304 \times 10^{6}$ |
| 0.843697 | $1.364 \times 10^{7}$ |
| 0.882315 | $9.528 \times 10^{7}$ |
| 0.933585 | $6.951 \times 10^{7}$ |
| 0.970789 | $5.866 \times 10^{7}$ |
| 1.00707 | $1.208 \times 10^{8}$ |
| 1.04254 | $1.433 \times 10^{8}$ |
| 1.09099 | $1.732 \times 10^{8}$ |
| 1.156 | $1.922 \times 10^{8}$ |

### Table 2. $|F(\epsilon)|$ for $\theta = 1.22134$.

| $\epsilon = 1 - r$ | $|F(\epsilon)|$ |
|------------------|----------------|
| $9.346 \times 10^{-1}$ | $2.24573 \times 10^{-2}$ |
| $8.762 \times 10^{-1}$ | $4.28428 \times 10^{-2}$ |
| $7.886 \times 10^{-1}$ | $7.46654 \times 10^{-2}$ |
| $5.841 \times 10^{-1}$ | $1.60318 \times 10^{-3}$ |
| $4.381 \times 10^{-1}$ | $2.63274 \times 10^{-3}$ |
| $3.505 \times 10^{-1}$ | $2.84141 \times 10^{-3}$ |
| $2.357 \times 10^{-1}$ | $2.9669 \times 10^{-3}$ |
| $1.168 \times 10^{-1}$ | $4.88637 \times 10^{-4}$ |
| $3.505 \times 10^{-2}$ | $1.72862 \times 10^{1}$ |

$$
\int_{\theta}^{1.26516} F(\theta) \, d\theta = 1.25886. \text{ These curves are plotted in figure 3, in terms of } r = 1 - \epsilon. \text{ Similarly we can construct } \epsilon(\theta) \text{ for } F = 3.3780842 \text{ and } F = 4.1597548. \text{ The maximum of } \epsilon \text{ occurs at } \theta = 1.25886 \text{ and}$$
\[ \theta = 1.25664 \] respectively. Finally, we provide a 3-d plot of \( r, \theta, F \). The spacetime structure after the collision of KP-waves can be seen graphically in figure 4.

### 4. Discussion and conclusion

Penrose introduced a method for constructing an impulsive gravitational wave in which the spacetime is flat everywhere, except at a hypersurface where the Riemann tensor has a \( \delta \)-function discontinuity. The rigorous

### Table 3.
The function \( \epsilon(\theta) \) for \( F = 2.90957521 \). The maximum occurs at \( \theta = 1.26516 \).

| \( \theta \) | \( \epsilon = 1 - r \) | \( \theta \) | \( \epsilon = 1 - r \) |
|---|---|---|---|
| 0.79972 | 5.255 \times 10^{-3} | 1.25886 | 7.339 \times 10^{-2} |
| 0.814097 | 9.756 \times 10^{-3} | 1.26516 | 7.341 \times 10^{-2} |
| 0.858873 | 2.204 \times 10^{-2} | 1.26711 | 7.342 \times 10^{-2} |
| 0.939235 | 4.018 \times 10^{-2} | 1.27563 | 7.337 \times 10^{-2} |
| 1.01052 | 5.288 \times 10^{-2} | 1.28085 | 7.332 \times 10^{-2} |
| 1.0698 | 6.116 \times 10^{-2} | 1.35641 | 7.029 \times 10^{-2} |
| 1.13636 | 6.798 \times 10^{-2} | 1.40732 | 6.543 \times 10^{-2} |
| 1.21333 | 7.25 \times 10^{-2} | 1.49471 | 4.915 \times 10^{-2} |
| 1.23248 | 7.304 \times 10^{-2} | 1.56566 | 1.296 \times 10^{-2} |
| 1.25465 | 7.337 \times 10^{-2} | 1.5708 | 9.304 \times 10^{-2} |

### Table 4.
The function \( \epsilon(\theta) \) for \( F = 7.381660 \). The maximum occurs at \( \theta = 1.25465 \).

| \( \theta \) | \( \epsilon = 1 - r \) | \( \theta \) | \( \epsilon = 1 - r \) |
|---|---|---|---|
| 0.79972 | 4.04 \times 10^{-3} | 1.25465 | 5.111 \times 10^{-2} |
| 0.814097 | 7.303 \times 10^{-3} | 1.25664 | 5.11 \times 10^{-2} |
| 0.858873 | 1.609 \times 10^{-2} | 1.25886 | 5.109 \times 10^{-2} |
| 0.890055 | 2.147 \times 10^{-2} | 1.26711 | 5.102 \times 10^{-2} |
| 0.939235 | 2.898 \times 10^{-2} | 1.27563 | 5.092 \times 10^{-2} |
| 1.01052 | 3.791 \times 10^{-2} | 1.35641 | 4.793 \times 10^{-2} |
| 1.0698 | 4.361 \times 10^{-2} | 1.40732 | 4.397 \times 10^{-2} |
| 1.13636 | 4.814 \times 10^{-2} | 1.49471 | 3.183 \times 10^{-2} |
| 1.21333 | 5.084 \times 10^{-2} | 1.56566 | 7.81 \times 10^{-3} |
| 1.23248 | 5.107 \times 10^{-2} | 1.5708 | 5.465 \times 10^{-2} |

Figure 3. The curves \( r(\theta) \) for constant \( F \).
mathematical study of the problem of impulsive gravitational waves spacetime is addressed in [18, 19], in detail. The curvature singularity and some generalizations have been thoroughly investigated using invariant methods [20, 21]. More recently the solution has been extended to include a cosmological constant [22]. As stated earlier, we feel that additional insight is provided by considering the momentum imparted to test particles in a frame of special significance. This is a Fermi-Walker frame corresponding to an observer falling freely from infinity and has the remarkable feature that it also corresponds to a foliation of the spacetime by flat spacelike hypersurfaces, i.e. those with a zero Riemann tensor. We feel that this method enhances the visualization of the curvature after both waves have passed and before doomsday.

It is physically obvious that there is no momentum imparted to test particles at $z = 0$ as the two impulsive waves exactly cancel each other out. Hence the curvature must be zero at $z = 0$. However, the curvature must be infinite at the singularity. This is seen graphically in the numerical results. What this amounts to is that at $z = 0$ we have a delta function in the gravitational field that has its infinity at doomsday, $t = T$.

The other interesting feature is seen in terms of the variables $r^2 = t^2 + z^2$ and $\theta = \tan^{-1}(t/z)$. One might have expected that for any $r < T$ the maximum of the force would be at $\theta = \pi/4$. This would be what one would expect on one’s ‘linear’ intuition. It turns out that the nonlinear reality is very different. The maximum occurs around $3\pi/4$ but not exactly there and is not at a constant $\theta$! The values are seen in table 1 and figure 2. It would be of interest to understand why this occurs.

What would happen if the impulsive waves were not taken to be of equal strength? Qualitatively we know that the argument for no force at $z = 0$ would fail there. However, our expectation that it would only be shifted towards the weaker wave may not be correct. From the above behaviour of the maximum, it might be expected that it would not be at a constant $\theta$. For that matter, it is not clear how to formulate the problem of different strength waves. The normal way of writing the KP-metric does not allow for different strengths. Since the ‘strength’ is related to the value of $T$ by an inverse behaviour, it would, presumably, be necessary to introduce two time scales, $T_1$ and $T_2$, instead of one $T$.

A solution for colliding ‘sandwich’ plane waves was published by Szekres [23]. It would be interesting to see how the curvature of the spacetime in the fourth region behaves for them in comparison with the KP waves. Of course, if the formulation for different strengths is available, it would be worthwhile to compute the results for the sandwich waves of different strengths and then see how the results approach the limits for the special cases. It would also be worthwhile to extend our analysis to colliding gravitational waves in a cosmological context, using the work of [22].

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Appendix A. The curvature tensor components

The curvature tensor can be uniquely decomposed into three parts namely the Weyl tensor, the Ricci tensor and the Ricci scalar. Since Khan-Penrose spacetime is the vacuum solution therefore the Ricci tensor and its trace would be zero. Thus we have only the trace-free part of the curvature tensor, $C^a_{b c d}$, called the Weyl tensor. It has...
ten linearly independent components given below,

\[
C_{332}^2 = \frac{2(\sqrt{1 - u^2} - \sqrt{1 - v^2} + uv)(u^2(2v^2 - 1) + \sqrt{1 - u^2} uv\sqrt{1 - v^2} - v^2 + 1)}{(\sqrt{1 - u^2} + v)^2 - u^2 - v^2 + 1 (u + \sqrt{1 - v^2})^2} \tag{A1}
\]

\[
C_{220}^0 = (\sqrt{1 - u^2} - \sqrt{1 - v^2} + uv)(u^2v(3v^2 - 2) - u(1 - v^2)^{1/2}(2\sqrt{1 - u^2} v^2
+ \sqrt{1 - u^2} - 3v) + v^2 - 1)^2(\sqrt{1 - u^2} - v^2 + 2u^2v^2 - 1)(3\sqrt{1 - u^2} v^2 + \sqrt{1 - u^2} - 2v^3)
+ \sqrt{1 - u^2} - 2v^3) - u^2\sqrt{1 - v^2}(-3\sqrt{1 - u^2} v^2 + \sqrt{1 - u^2} + 6v^3 - 3v)
+ u^2(6\sqrt{1 - u^2} v^2 - 6\sqrt{1 - u^2} v^2 + \sqrt{1 - u^2} - 3v^3 - v^3 + 3v^3)
+ u^2\sqrt{1 - v^2}(-3\sqrt{1 - u^2} v^4 - 2\sqrt{1 - u^2} v^2 + 2\sqrt{1 - u^2} + 9v^3
- 6v^3)/(\sqrt{1 - u^2} - (\sqrt{1 - u^2})^2 - u^2 - v^2 + 1 (\sqrt{1 - v^2} - u)^3) \tag{A2}
\]

\[
C_{330}^0 = (\sqrt{1 - u^2} - \sqrt{1 - v^2} + uv)(u^2v(3v^2 - 2) - u(1 - v^2)^{1/2}(2\sqrt{1 - u^2} v^2 + \sqrt{1 - u^2} + 3v)
- (v^2 - 1)^2(\sqrt{1 - u^2} + v)^2 + 2u^2v^2 - 1)(3\sqrt{1 - u^2} v^2 - \sqrt{1 - u^2} + 2v^3)
+ u^2\sqrt{1 - v^2}(3\sqrt{1 - u^2} v^2 - \sqrt{1 - u^2} + 6v^3 - 3v) - u^2(6\sqrt{1 - u^2} v^4
- 6\sqrt{1 - u^2} v^2 + \sqrt{1 - u^2} + 3v^3 + v^3 - 3v^3)
- 2\sqrt{1 - u^2} v^2 + 2\sqrt{1 - u^2} + 9v^3 + 6v^3)//(\sqrt{1 - u^2} - (\sqrt{1 - u^2})^2 - u^2 - v^2 + 1 (u + \sqrt{1 - v^2})^3) \tag{A3}
\]

\[
C_{331}^0 = -3u(\sqrt{1 - u^2} - \sqrt{1 - v^2} + uv)(u^2v(3v^2 - 2) - u(v^2 - 1)(3\sqrt{1 - u^2} v^2 + \sqrt{1 - u^2} + 3v)
+ 2v^2 + 1) - (1 - v^2)^{1/2}(\sqrt{1 - u^2} v^2 + 1) - u^2\sqrt{1 - v^2}(2\sqrt{1 - u^2} v^2 - \sqrt{1 - u^2} + 2v^3)
+ \sqrt{1 - u^2} + 5v^2 - 2) + u^2\sqrt{1 - v^2}(2\sqrt{1 - u^2} v^2 + 4v^2 - 1)
- u^2(4\sqrt{1 - u^2} v^2 - 3\sqrt{1 - u^2} v + 2v^2 + v^2 - 2))//(\sqrt{1 - v^2}(\sqrt{1 - u^2} + v)^2 - (\sqrt{1 - v^2} - u^2 + v^2 + 1 (u + \sqrt{1 - v^2})^3) \tag{A4}
\]

\[
C_{220}^1 = -3uv(\sqrt{1 - u^2} - \sqrt{1 - v^2} + uv)(u^2v(3v^2 - 2) + u(v^2 - 1)(3\sqrt{1 - u^2} v^2
+ \sqrt{1 - u^2} - 3v) - (v^2 - 1)^2(\sqrt{1 - u^2} - v^2) - u^2\sqrt{1 - v^2}(-4\sqrt{1 - u^2} v^2
+ \sqrt{1 - u^2} + 3v^3 + v) - u^2\sqrt{1 - v^2}(-2\sqrt{1 - u^2} v^2 + \sqrt{1 - u^2} + 4v^3 - 3v))//(\sqrt{1 - u^2} v^2 - \sqrt{1 - u^2} + v^2 + 1 (\sqrt{1 - u^2} - v^3) \tag{A5}
\]

\[
C_{221}^0 = -(\sqrt{1 - u^2} - \sqrt{1 - v^2} + uv)(u^2v(3v^2 - 4v^2 + 1) - u(v^2 - 1)^2(3\sqrt{1 - u^2} v^2 - 2v^2 - 1)
+ (1 - v^2)^{1/2}(\sqrt{1 - u^2} v^2 - 1) + u^2(1 - v^2)^3/2(3\sqrt{1 - u^2} v^2 + \sqrt{1 - u^2} v^2
- 6v^2 + 2) - u^2\sqrt{1 - v^2}(-3\sqrt{1 - u^2} v^3 + 2\sqrt{1 - u^2} v^2 + 6v^4 - 6v^2 + 1)
+ u^2(3v^2 - 1)(4\sqrt{1 - u^2} v^2 - 3\sqrt{1 - u^2} v^3 + 3v^4 - 2v^2 - 2))//(\sqrt{1 - u^2} v^2 - \sqrt{1 - u^2} + v^2 + 1 (\sqrt{1 - u^2} - v^3) \tag{A6}
\]

\[
C_{330}^1 = 3uv(\sqrt{1 - u^2} - \sqrt{1 - v^2} + uv)(u^2v(3v^4 - 4v^2 + 1) - u(v^2 - 1)^2(3\sqrt{1 - u^2} v^2
+ \sqrt{1 - u^2} + 3v) + (v^2 - 1)^2(\sqrt{1 - u^2} + v) - u^2(v^2 - 1)(4\sqrt{1 - u^2} v^2
- \sqrt{1 - u^2} + 2v^3 + v) + u^2\sqrt{1 - v^2}(-2\sqrt{1 - u^2} v^2 - \sqrt{1 - u^2} + 4v^3
- 3v))//(\sqrt{1 - u^2}(-\sqrt{1 - u^2} + v)^3 + u^2 - v^2 + 1 (u + \sqrt{1 - v^2})^3) \tag{A7}
\]

\[
C_{010}^0 = 2(3uv(2v^2 - 1)v^2 - \sqrt{1 - v^2} + 3u^2v\sqrt{1 - v^2} + uv)(3v^2 - 2) + \sqrt{1 - u^2}(v^2 - 1)^2
- 2\sqrt{1 - u^2} u^2(3v^4 - 4v^2 + 1) + \sqrt{1 - u^2} u^2(6v^4 - 6v^2 + 1)
+ 3uv(1 - v^2)^{1/2})/(u^2 - 1)\sqrt{1 - v^2}(u^2 + v^2 - 1)^2(\sqrt{1 - u^2}\sqrt{1 - v^2} + uv)^2) \tag{A9}
\]
\[
C^{1}_{331} = -(\sqrt{1 - u^2} \sqrt{1 - v^2} + uv)(u^2(3v^4 - 4v^2 + 1) + u(v^2 - 1)^2(3\sqrt{1 - u^2}v + 2v^2 + 1)
+ (1 - v^2)^2(3\sqrt{1 - u^2}v + 1) + u^2(1 - v^2)^2(3\sqrt{1 - u^2}v^3 + \sqrt{1 - u^2}v^3 + \sqrt{1 - u^2}v^3 + 1 - v^2 - 2) + u^4(1 - v^2)(3\sqrt{1 - u^2}v^3 - 2\sqrt{1 - u^2}v + 6v^4 - 6v^2 + 1)
- u^4(v^4 - 1)(6\sqrt{1 - u^2}v^3 - 3\sqrt{1 - u^2}v + 3v^4 + 2v^2 - 2))
/ (\sqrt{1 - u^2} (\sqrt{1 - u^2} + v)^2 - u^2 - v^2 + 1 (u + \sqrt{1 - v^2})^2)
\]

(A10)

Appendix B. The time component of \(e\)/\(\gamma\)/\(N\)-force

The time component of \(e\)/\(\gamma\)/\(N\)-force is,

\[
F_t = (15t^{10} + 3(-7z^2 - 47)t^{14} - 3((51z^4 + 2(M - 42)z^2 + 37M - 187))t^{12}
+ (1564 - 532M)z^2
- 540M + 1692)z^0
+ [[885z^8 - 24(17M - 35)z^6 + 2(169M - 703)z^2 (1564 - 532M)z^2
- 540M + 1692)]z^8
+ (723M - 1771)z^8 + (452M - 672)z^6
+ 4(334M + 315)z^4 - 16(76M + 77)z^2 - 48(M - 22)z^4
+ (932z^8 - 3(27M + 289)z^12 + (673M + 3223)z^{10}
- (1885M + 6348)z^8 + (4(694M + 1731)z^6
- 6(619M + 1012)z^8 + 64(17M + 22)z^2 - 96(M + 4))z^2
- 4(z - 1)z^0(z + 1)(z^2 - 2)(3z^4 - 5z^2 + 5)(z^2 - M - 2)z^2
- 2M + 8))/(t^4(t - z)^2 - 2(t^2 - t z + 1)(t^2 + z^2 - 1)(t^2 - z^2 - M)^3
\times t^4 - 2(3z^2 + M + 3)z^2 + 5z^4 - 2z^2(3M + 5) + 4M + 8],
\]

(B1)

where \(M = \sqrt{2 - (t - z)^2}(\sqrt{2 - (t + z)^2})^2\).

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