Limiting effects in tori clusters

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We consider agglomerates of misaligned tori orbiting a supermassive black hole. The aggregate of tilted tori is modeled as a single orbiting configuration by introducing a leading function governing the distribution of toroids (and maximum pressure points inside the disks) around the black hole attractor. The orbiting clusters are composed by geometrically thick, pressure supported, perfect fluid tori. This analysis places constraints on the existence and properties of tilted tori and more general aggregates of orbiting disks. We study the constraints on the tori collision emergence and the instability of the agglomerates of tori with general relative inclination angles, the possible effects of the tori geometrical thickness and on the oscillatory phenomena. Some notes are discussed on the orbiting ringed structure in dependence of the dimensionless parameter $\xi$ representing the (total) BH rotational energy extracted versus the mass of the BH, associating $\xi$ to the characteristics of the accretion processes.

Keywords: Accretion disks; Accretion–jets; Black hole physics; Hydrodynamics

1. Introduction

We study agglomerates of tori orbiting one central Kerr super-massive black hole (SMBH), in two macro-configuration models: (i) as an equatorial-Ringed Accretion Disk (eRAD), where the toroids equatorial and symmetry planes coincide with the central Kerr BH equatorial plane \cite{11}; (ii) as Ringed Accretion Disk (RAD), where the agglomerate is composed by misaligned (tilted) tori \cite{12,13}.

The eRAD tori rotation orientation is a well defined quantity, and a couple of eRAD toroids can be both corotating or counter-rotating with respect to the central spinning BH, otherwise tori can have a relative alternate rotation orientation (with an inner corotating toroid and an outer counter-rotating toroid with respect to the central attractor or viceversa). Tori with different relative rotation orientation constitute an interesting case from the viewpoint of the constrains posed by the eRAD or RAD model, for example in the case of double tori accretion phase or also tori collision which depends generally on the spin-mass ratio of the central BH. It is clear that an orbiting tori aggregate model could be constructed considering a different disk model for each toroidal component. However in this analysis we mostly consider a Polish doughnut (PD) model. This is a well known geometrically thick disk model, widely used in literature in a variety of different applications \cite{13}. Polish doughnut shows a remarkably good fitting of main morphological characteristics of thick disks also in comparison with more refined dynamical GRHD or GRMHD models \cite{13}. RAD is a “constraining-models”, providing initial configurations for dy-
namical (GRMHD) situations. In general aggregates of toroidal structures orbiting one central BH attractor can result from different accreting phases of the SMBHs growing, where the infalling materials, having diverse angular momentum, may trace back the BH story.\textsuperscript{2,5,14–23}

This paper is structured in two parts: In the first part, Sec. (2), we discuss the eRAD model, focusing on the model parameters and distribution of pressure and density critical points. We introduce the concept of leading function for the agglomeration describing the tori distribution around the central attractor.

The second part, Sec. (3), focuses on aggregates of misaligned tori (RAD), discussing explicit solutions of its inner structure and an adapted parametrization for the toroidal components.

2. The equatorial Ringed Accretion Disks

2.1. Leading function and geodesic structure

A key step in the modelization of the orbiting agglomerate is to individuate an adapted “leading function”, representing the tori location around the central attractor. For large part of this analysis, we can identify the leading function with the definition of fluid specific angular momentum $\ell$. Different choices are the agglomerate leading functions also possibles as we see some examples in Sec. (2.4) and Sec. (3.1). In the Kerr spacetime there is

$$\ell \equiv \frac{L}{E} = -\frac{U\phi}{U_t} = \frac{-g_{\phi\phi} + g_{\phi t} U^t}{g_{tt} U^t + g_{\phi t} U^\phi} = -\frac{g_{t\phi} + g_{\phi\phi} \Omega}{g_{tt} + g_{t\phi} \Omega},$$

$$\Omega \equiv \frac{U^\phi}{U_t} = \frac{-E g_{\phi t} + g_{tt} L}{E g_{\phi\phi} + g_{\phi t} L} = -\frac{g_{t\phi} + g_{tt} \ell}{g_{\phi\phi} + g_{t\phi} \ell},$$

where $g_{\alpha\beta}$ are metric components in the Boyer-Lindquist coordinates, $U^\alpha$ is the fluid four velocity, $\Omega$ is the fluid relativistic angular velocity and $(E, L)$ are constants of motions—see for example.\textsuperscript{1,2} In general, we may interpret $E$, for timelike geodesics, as representing the total energy of the test particle coming from radial infinity, as measured by a static observer at infinity, and $L$ as the axial component of the angular momentum of the particle. (For the PD tori orbiting in a Kerr spacetime the set of results known as Von Zeipel theorem holds, therefore the fluid is barotropic and the surfaces of constant pressure coincide with the surfaces of constant density. In these spacetimes, the family of von Zeipel’s surfaces does not depend on the particular rotation law of the fluid, $\Omega = \Omega(\ell)$, but on the background spacetime only.\textsuperscript{24–28})

The leading function provides the distribution of the possible maximum points of pressure and density in the fluids surrounding the BH, which are identified as the RAD-“rings seeds”, coincident with torus centers $r_{\text{center}}$, and eventually the minimum points of pressure of the fluids orbiting around the central attractor. The minimum points of pressure are associated to the cusps $r_{\times}$ of the PD torus Roche lobe and are regulated by the geodesic structure of the background, composed by
the marginally stable circular orbit \((r_{mso})\), the marginally bounded circular orbit \((r_{mbo})\) and the marginally circular orbit \((r_\gamma)\) which is also a photon orbit. (The cusp location in the PD model, located in the range \([r_{mbo}, r_{mso}]\), could be related to the inner edge of an accreting torus.) In the RAD and eRAD models, the location of the maximum points of the pressure around the attractor are interpreted as “rings seeds”, and are regulated by a set of radii, associated to the geodesics structure, located in the “stability region” at \(r > r_{mso}\) and defined by

\[
r_{\pm}(mbo) : \ell_{\pm} = \ell_{\pm}(r_{\pm}(mbo)) \equiv \ell_{\pm}, \quad \text{with} \quad \ell_{\pm}(r_{\pm}) = \ell_{\pm}(r_{\pm}(\gamma)) = \ell_{\pm},
\]

where \(r_{\pm}(mbo) < r_{\pm}(mbo) < r_{\pm}(mbo) < r_{\pm}(\gamma)\) and \(r_{\pm}(M) : \ell_{\pm}(r_{\pm}(M)) = \ell_{\pm}(r_{\pm}(M)) = \ell_{\pm}\). (2)

In here and in the following with \(Q_{\pm}\) we indicate quantities \(Q\) associated to counter-rotating \((\ell_a < 0)\) or corotating \((\ell_a > 0)\) structures respectively, with respect to the central BH spin \(a/M\). In the following, for any quantity \(Q\) and radius \(r\), we adopt the notation \(Q_{\bullet} \equiv Q(r_{\bullet})\). Where more conveniently, we use dimensionless quantities where \(r \rightarrow r/M\) and \(a \rightarrow a/M\).

The leading function, solution of \(\partial_r^2 \ell_{\pm}(r, \theta; a) = 0\) (where \((r, \theta, \phi, t)\) are the Boyer-Lindquist coordinates of the Kerr metric), provides the point of maximum density of the rings seeds distribution around the BH.

The corotating and counter-rotating tori cusps orbital regions are shown in the Figs 1 for different BH attractors. In the Figs 1 the orbital regions for the ring seeds locations (tori centers) are also shown. The union of these regions provides the maximum rage of the location of the rings seeds and the maximum extension of the disks inner part (the region \([r_{inner}, r_{center}]\) where \(r_{inner} \leq r_{inner}\) is the torus cusp and \(r_{inner}\) is the inner edge of the quiescent (i.e. not cusped torus)). The orbital strips in Figs 1 relative to the corotating and counter-rotating fluids, cross in different points depending on the BH spin–mass ratio and particularly for slower spinning attractors.

2.2. Ideal GRMHD and GRHD

Polish doughnuts have been realized in different ideal GRMHD and GRHD setups, where for the ideal GRMHD (infinitely conductive plasma) case there is

\[
U_\alpha \nabla_\alpha \rho + (p + \rho) \nabla_\alpha U_\alpha = 0,
\]

\[
(p + \rho) U^\alpha \nabla_\alpha U^\gamma - \epsilon h^{\beta\gamma} \nabla_\beta p - \epsilon (\nabla^\alpha F_{\alpha}) F^\delta\beta\gamma = 0,
\]

\[
U^\alpha \nabla_\alpha s = 0.
\]

\((F^\beta_\alpha)\) is the Faraday tensor and \(\epsilon\) is a quantity related to the metric signature\(^{29,30}\). The electric field does not affect the continuity equation or the equation for the entropy. The entropy per particles \(s\) is conserved along the flow \(U^\alpha\), defined for each torus of the aggregate, and \(h_\alpha^\beta\) is the metric on the 3-sheet orthogonal to flow direction \(U^\alpha\), defining the projector tensor (there is \(\nabla_\alpha g_{\beta\gamma} = 0\)). The inner ringed
Fig. 1. Left: Fluid specific angular momentum $\ell^\pm$ of the eRAD tori, for corotating ($-$), and counterrotating ($+$), fluids, versus SMBH dimensionless spin $a/M$. Tori can form for $\ell^\pm > \ell^\pm_{mso}$ respectively. There is $\ell^\bullet \equiv \ell(r^*)$ for $r^* = \{r_{mso}, r_{mbo}, r_{\gamma}\}$; $r_{mso}$ is the marginally stable circular orbit, $r_{mbo}$ is the marginally bounded orbit and $r_{\gamma}$ the last circular (photon) orbit. Right: radii $r_{mbo}$ and $r_{mso}$ and the pair $r_{mbo}$ and $r_{mso}$ as functions of the dimensionless spin of the BH—see Eqs (2).

structure is defined by the boundary conditions determining the tori (edges)—see for more details. To simplify our discussion we consider in this analysis the GRHD scenario (see for example). With the barotropic equation of state, $p = p(\rho)$, the set of GRHD equations for the PD model, reduces to the only constrain equation for the pressure (Euler equation):

$$\nabla_{\mu}p_{,\mu} = -\nabla_{\mu}W + \frac{\Omega}{1 - \Omega^2}\nabla_{\mu}\ell.$$  

where $W = W(a; \ell, r, \theta)$ is an effective potential, function of $(r, \theta)$ for the Kerr metric (in the Boyer-Lindquist coordinates). The equation for the pressure critical points can be further simplified by considering $\ell$ as a model parameter, constant in each torus of the aggregate. In this setup the extreme points of pressure are the extremes of the potential $W$, and the leading function is given by the radial derivative of the effective potential. More precisely the tori parameters are the couple $(\ell, K)$ with $\ell = constant$ and $V_{eff} = K = constant$ for each torus, where $W = \ln V_{eff}$.

In this context, the eRAD leading function is $\ell(r) : \partial_r V_{eff} = 0$. The tori (Boyer) surfaces are equipressure surfaces (also surfaces of constant $(p, \rho, \ell, \Omega)$) and the fluids fill every equipressure surface.  

2.2.1. The energy function and tori energetics

The “energy function” $K(r) = V_{eff}(\ell(r))$ regulates, with the leading function $\ell(r)$, the RAD aggregate, where $K : K(r) = constant$ for each torus. More precisely, function $K(r, \rho; a) \equiv V_{eff}(\ell, r, \ell; a)|_{\ell(r, \rho, a)}$ determines the flow (and the torus) geometrical thickness, the tori extension on the equatorial plane, and it is uniquely identified by $\ell(r)$ in the case of cusped tori. The relation between the geometrical
maxima (defined by $K$) and the density maxima (fixed by $\ell$) is provided by the extreme of the leading function $\ell(r, \theta, a)$\cite{12}. We can relate $K(r)$ to certain features of the tori energetics, evaluating some characteristics related to the flow thickness, as mass accretion rate or cusp luminosity as listed in Table (1). It is clear that these quantities depend on the details of the different tori models, but this analysis can provide an estimation of these quantities with respect to the flow co-rotation or counter-rotation, the dimensionless BH rotational energy (or dimensionless spin) and the tori location in the aggregate\cite{13}.

Table 1. There is $\omega = n + 1$, with $\gamma = 1/n + 1$ being the polytropic index, $\kappa$ is the polytropic constant. $\Omega$ is the relativistic angular velocity. $W = \ln V_\text{eff}$ is the value of the equipotential surface, which is taken with respect to the asymptotic value, $W_x = \ln K_x$ is $W$ at the cusp $r_x$, while $W_x \geq W_x$ and $r_x$ is related to the accreting flow thickness. $\mathcal{L}$ representing the total luminosity, $\dot{M}$ the total accretion rate where, for a stationary flow, $\dot{M} = \dot{M}_{\text{Edd}}$, $\eta \equiv L/\dot{M}$ the efficiency, $D(n, \kappa), C(n, \kappa), A(n, \kappa), B(n, \kappa)$ are functions of the polytropic index and the polytropic constant. $L \times L$ is the fraction of energy produced inside the flow and not radiated through the surface but swallowed by central BH–see Figs 2.

### Quantities

| Quantities | $Q(r_x, n) \equiv q(n, \kappa)(W_x - W_x)d(n)$ | Quantities | $P \equiv \frac{Q(r_x, n)r_x}{H(r_x)}$ |
|-----------|---------------------------------|------------|----------------------------------|
| $R$-quantities: $R_x \equiv (W(r_x) - W_x)^{\omega}$ | $N$-quantities: $N_x \equiv \frac{r_x(W(r_x) - W_x)^{\omega}}{H(r_x)}$ |
| Enthalpy – flux : $D(n, \kappa)(W_x - W_x)^{n+3/2}$ | Mass – accretion – rate : $\dot{m} = \frac{M}{M_{\text{Edd}}}$ |
| Mass – Flux : $C(n, \kappa)(W_x - W_x)^{n+1/2}$ | Mass-accretion-rates: $\dot{M}_x = \frac{A(n, \kappa)r_x(W_x - W_x)^{n+1}}{H(r_x)}$ |
| $\mathcal{L}_x = \frac{B(n, \kappa)W_x - W_x}{A(n, \kappa)}\frac{n+1/2}{\kappa}$ | Cusp-luminosity: $\mathcal{L}_x = \frac{B(n, \kappa)r_x(W_x - W_x)^{n+2}}{H(r_x)}$ |

Evaluation of these quantities defined in Table (1) are in Figs 2 as functions of the BH dimensionless rotational energy for different models, fixed according to selected values of the fluid specific angular momentum $\ell$ (fixing the cusp location and the center of maximum density) and the $K_x \in [K_x, 1]$, fixing the flow thickness at the cusp\cite{14}.

### 2.3. The BH rotational energy

The BH rotational energy is related to the BH geometrical features through its irreducible mass $M_{\text{irr}}$. From the definition of irreducible mass $M_{\text{irr}} = (M^2 + \sqrt{M^2 - J^2})/2$, where $M$ is the BH total (ADM) mass and the $J = aM$, \cite{15}.
Fig. 2. Plots of $N^\pm \equiv r_x(W^\pm(r_x) - W^\pm_{r_{+}})\Omega(r_x)$ for $P$-quantities analysis and $R^\pm \equiv (W^\pm(r_s) - W^\pm_{r_{+}})\kappa$ for $O$-quantities analysis defined in Table 1 for corotating ((-)–continuum curves) and counterrotating ((+)–dashed curves) tori for different values of the cusps $r_s = r^\pm_x \in \{\bullet, \Box, \Diamond\}$ and radii $(r^\pm_x, r_s)$ and the associated angular momentum $\ell$ and $K$ parameters are shown with $\{\bullet, \Box, \Diamond, K_1, K_2, K_3\}$. $\Omega$ is the relativistic angular velocity. $\xi$ is the dimensionless BH rotational energy.

the dimensionless rotational energy $\xi$ is:

$$\xi \equiv \frac{M_{rot}}{M(0)} = 1 - \frac{\sqrt{1 + \frac{1 - J(0)}{M(0)^2} - \frac{J(0)^2}{M(0)^2}}}{2},$$

$$\xi^\pm = 1 \pm \sqrt{\frac{r^\pm_x}{2}},$$

$$\frac{\delta M_{irr}}{M_{irr}} = \frac{\delta M - \delta M_{H}^+}{\sqrt{M(0)^2 - \frac{J(0)^2}{M(0)^2}}}.$$
\[
\delta M_{\text{irr}} \geq 0 \text{ thus } (\delta M - \delta J \omega^2_H) \geq 0,
\]
where \( \xi = 1 - M_{\text{irr}}/M \), with quantities evaluated at an initial state \((0)\). It is \( \xi \in [0, \xi_l] \) where \( \xi_l \equiv \frac{1}{2} \left( 2 - \sqrt{2} \right) \), limiting the total rotational energy extracted to \( \approx 29\% \) of the total mass \( M \) for a process leading an extreme Kerr BH to the static Schwarzschild BH.

Fig. 3. Upper left panel: the BH outer horizon \( r_+ \) and the dimensionless BH spin function \( A(\xi) \), as functions of the dimensionless BH rotational energy \( \xi \). Extreme Kerr BH corresponds to \( \xi \approx 0.29 \). Upper right panel: marginally bounded orbit \( \text{mbo} \) (dashed curves), marginally stable orbit \( \text{mso} \) (plain curves), marginally circular orbit \( \text{mco} \) (dotted curves) for corotating motion (blue curves) and counter-rotating motion (red curves) as functions of the parameter \( \xi \). Center Bottom panels: the tori specific fluid angular momenta \( \ell^\pm = \text{constant} \) for corotating (center left panel) and counter-rotating (center right panel) tori, the RAD energy function \( K^\pm (r) = \text{constant} \) for the counter-rotating (bottom left panel) and corotating fluids (bottom right panel) as functions of \( \xi \), where \( r \gamma^\pm \) is the last circular circular orbit (photon orbit) for counter-rotating (+) and corotating orbits (-).

In Figs 2 and Figs 3 the tori aggregates energetic characteristics at the state 0, prior a possible process involving the BH and its environment, are related to the BH rotational energy. The energetic parameter \( \xi \) and the angular relativistic velocity of the BH determine the BH state prior the transition-34–36. This analysis turns particularly relevant for the RAD model which, because of its inner discrete
structure, can be characterized by phases of enhanced accretion rates. Furthermore, as each torus of the aggregate is a geometrically thick disk, each component can contribute with super-Eddington accretion rates, with the possibility of further effects as runaway instability\cite{37,38}, or in the case of RAD composed by misaligned tori, the Bardeen-Petterson effect\cite{39}.

2.4. Magnetized tori: alternative leading function

As mentioned in Sec. (2.1), the leading function is not necessary the fluid specific angular momentum. An example providing a different aggregate leading function is the case of orbiting agglomerates composed by magnetized tori with a “Komissarov” toroidal magnetic field\cite{40}. In this case the eRAD aggregate leading function is a function $S$, defined by the magnetic field parameters and proving the rings seeds locations, the minima of pressure (according to the conditions for the cusp formation), the maximum values of the $S$-parameter for the formation of a tori couple and constrains on the eRAD inner structure in dependence on the tori relative rotation orientation. More precisely, let us consider the toroidal magnetic field: $B^\phi = \sqrt{2} p_B / A$, where $A \equiv \ell^2 g_{tt} + 2 \ell g_{t\phi} + g_{\phi\phi}$, and $p_B = M (g_{t\phi} g_\phi - g_{tt} g_{\phi\phi}) q^{-1} \omega^q$ is the magnetic pressure, $\omega$ is the fluid enthalpy, $q$ and $M$ are constant; The Euler equation can be written as

$$\frac{\partial \rho p}{\rho + p} = G^{(f)} + G^{(em)}, \quad G^* = - \frac{\partial}{\partial \zeta} W^2; \quad \zeta = \{(em), (f)\}, \quad \partial \mu (W^{(f)} + W^{(em)}) = \partial \mu \ln (V_{eff} + \mathcal{G}) , \quad \mathcal{G}(r, \theta) = S \left( AV_{eff}^2 \right)^{q-1}, \quad S \equiv \frac{qM \omega^q}{q - 1}.
$$

The RAD leading $S$-function is

$$S_{crit} \equiv - \frac{\Delta - Q}{Q} f(a, \ell; r) \quad (Q \equiv q - 1),$$

where $f(a, \ell; r)$ is a function of the BH spin and fluid angular momentum\cite{41,42,43} (there is $\Delta \equiv r^2 - 2Mr + a^2$).

3. Misaligned tori

In this section we discuss the tori aggregates containing misaligned tori. To simply our discussion we consider misaligned tori orbiting a central static BH. In this case we can use different leading functions for the description of the aggregate introduced in Sec. (3.1). Tori geometrical thickness is discussed in Sec. (3.2). Frequency models in tori aggregates are considered in Sec. (3.2.1) and tori geometrical characteristics are deepened in Sec. (3.3) for quiescent (not cusped) tori in Sec. (3.3.1) and for cusped tori analyzed in Sec. (3.3.2).

3.1. Leading functions

In this section we introduce three definitions of leading functions for the tori aggregates–see\cite{44,45,46}. 

The critical $\bar{r}(r_i)$ and $r^c_x$ functions:

$$\bar{r}(r_i) = 2r_i \left[ \sqrt{2r_i - r_{+}} + 1 \right], \quad r^c_x \equiv \frac{r_{\text{center}} (\sqrt{2r_{\text{center}} - r_{+}} + 1)^2}{(r_{\text{center}} - r_{+})^2},$$  \(5\)

(solutions of $\ell(r) = \ell(r_p)$ for two orbits $(r, r_p)$). In this case the leading function is a relation between the extremes of pressure inside the tori $\bar{r}(r_i)$, or the cusp $r^c_x$ as function of the center of maximum pressure. The distance $r_{\text{center}} - r_x$ increases with the tori distance in the aggregates from the central BH attractor.

**Leading function** $\ell^o_{\text{crit}}(K)$:

There is

$$\ell^o_{\text{crit}}(K) \equiv \sqrt{\frac{27K^4 - K (9K^2 - 8)^{3/2} - 36K^2 + 8}{2K^2 (K^2 - 1)}},$$  \(6\)

$$\ell^i_{\text{crit}}(K) \equiv \sqrt{\frac{27K^4 + K (9K^2 - 8)^{3/2} - 36K^2 + 8}{2K^2 (K^2 - 1)}},$$  \(7\)

where $\ell^o_{\text{crit}}(K) > \ell^i_{\text{crit}}(K) > \ell_{\text{mso}}, \quad \ell^i_{\text{crit}}(K) \in [\ell_{\text{mso}}, \ell_i]$.

In this case, notably the leading function, relating the two tori parameters $\ell$ and $K$, has been split in function $\ell^o_{\text{crit}}(K)$, the leading function providing parameters $(\ell, K)$ at the tori centers (rings seeds), and function $\ell^i_{\text{crit}}(K)$ for the tori cusps.

Similarly we introduce the following alternative functions.

**Tori critical radius** $r^o_{\text{crit}}(K)$: There is

$$r^o_{\text{crit}}(K) \equiv -\frac{8}{K (\sqrt{9K^2 - 8} + 3K^2) - 4}, \quad r^i_{\text{crit}}(K) \equiv \frac{8}{K (\sqrt{9K^2 - 8} + 3K^2) + 4},$$

$$r^i_{\text{crit}}(K_{\times}) = r_x, \quad r^o_{\text{crit}}(K_{\text{center}}) = r_{\text{center}}$$  \(8\)

Similarly to $\ell^o_{\text{crit}}(K)$, the leading function is $r^o_{\text{crit}}(K)$, relating the center of maximum pressure (ring seed) and the $K$ parameter at the torus center.

### 3.2. Geometrical thickness

Disk geometrical thickness is an important characteristic for the RAD tori. The eRAD is a geometrical thin disk composed by geometrical thick tori with an inner articulated ringed structure, combining some features of geometrical thick disks, inherited by its components and features typical of the geometrical thin disks in its global structure. Geometrical thickness is a relevant parameter in the comparison with other disks model, in the assessment of the torus vertical structure and the influence of a possible poloidal magnetic field, for the accretion mechanism and the study of tori oscillations. The definition of geometrical thickness adopted here coincides with the thickness $S$ of the outer Roche lobe section of the PD torus. For large part of the $(\ell, K)$ range, cusped tori can be considered geometrically thin i.e. $S < 1$. There are classes of toroidal components with equal thickness. For example, in the “reference” case $S = 1$, distinguishing geometrically thin and geometrically
thick disks, where there are couples of toroids with equal energy parameter $K$, regulating also the flow thickness\cite{10,11}.

A further parameter for the evaluation of tori geometrical thickness is the dimensionless $\beta_{\text{crit}}$

$$
\beta_{\text{crit}} = \frac{(r_{\text{center}} - 2)(r_{\text{center}} - r_\times)\sqrt{r_{\text{center}}r_\times - 2(r_{\text{center}} + 2r_\times)} - 2}{r_{\text{center}}\sqrt{r_{\text{center}} - 3r_\times}\sqrt{r_\times - 2}},
$$

emerging from the analysis of cusped tori oscillation\cite{41,42,43}. Similarly to the leading function $\bar{r}(r_i)$ of Eq. (5), $\beta_{\text{crit}}$ depends on the distance between the maximum and minimum point of pressure in the tori, which increases with the distance from the central attractor. For small $\beta_{\text{crit}}$ ($\beta_{\text{crit}} \geq 0$), tori may be considered geometrically thin for radial and vertical oscillation, and can be described by the radial and vertical epicyclical frequencies from the hypothesis of thin (slender) tori, coincident therefore with the respective circular orbit frequencies\cite{42,44,45}. The conditions for geometrical thin components according to definition $S < 1$ and conditions for geometrical thin tori according to $\beta_{\text{crit}}$ coincide only for special conditions on $\ell$ and $K$ parameters (therefore depending on the tori location in the agglomerate and their dimension), having tori with combined characteristics typical of geometrical thin and thick disks\cite{10}.

### 3.2.1. Frequency models in tori aggregates

In the conditions where $\beta_{\text{crit}} \geq 0$, we can consider the circular orbit approximation for the oscillation frequencies. In\cite{10} different frequency models are applied to the RAD structure, interpreted as a frame for the high–frequency (HF) Quasi-Periodic Oscillations (QPOs), assuming the geodesic (nearly circular geodesic motion) frequencies

$$
\nu_r(r) = \nu_K(r) \sqrt{1 - \frac{r_{\text{mas}}}{r}}, \quad \nu_\theta(r) = \nu_K(r) \equiv \frac{1}{r^{3/2}},
$$

(9)
determined by the tori constraints. The frequencies (9) are combined for the fitting of resonance ratios, identifying the upper $\nu_U$ and lower $\nu_L$ frequencies from different oscillation models and assuming $(\nu_r(r), \nu_K(r))$ evaluated at different points $r$ of the tori surfaces. Therefore in\cite{10} we used the frequency models ($\text{TD,RP,RE,WD}$) listed Table (2), evaluated in different tori models. The twin peak quasi-periodic oscillations with resonant frequency ratios $\nu_U/\nu_L = \{3 : 2, 4 : 3, 5 : 4, 2 : 1, 3 : 1\}$ have been analyzed\cite{10}.

Different components of the aggregates fit different frequency models, according to tori location in the aggregate with respect to the central attractor distinguishing therefore the toroidal components and the different torus active part\cite{11}. In the models of Table (2), the torus inner edge has been considered the active part of
the emission process, the frequencies being evaluated at \( r_{\text{inner}} > r_x \) (for quiescent tori) or \( r_{\text{inner}} = r_x \) for cusped tori, and as \( (\beta_{\text{crit}} \gtrsim 0, S \ll 1) \), the maximum of pressure point, the outer edge and the geometrical maximum point have been also considered\(^{[36],[37]}\).

### 3.3. Tori geometrical characteristics

Frequencies models of Table 2 have been evaluated on the outer \( r_{\text{out}} \) and the inner \( r_x \) tori edges, the tori geometrical maximum \( r_{\text{max}} \) and tori center \( r_{\text{center}} \). The evaluation of the tori geometrical characteristics is relevant in the determination of inner ringed structure and tori collision. In this section we provide \( (r_x, r_{\text{out}}, r_{\text{center}}, r_{\text{max}}) \), the torus height \( h \) and the inner Roche lobe maximum high. These quantities are functions of tori parameters \((\ell, K)\) for quiescent (not cusped) tori, considered in Sec. 3.3.1 and \( \ell \) or \( K \) (or alternately the critical pressure points \( r_x \) and \( r_{\text{center}} \)) for cusped tori analyzed in Sec. (3.3.2).

#### 3.3.1. Roche lobes in quiescent and cusped tori

For quiescent and cusped tori, we provide below the outer and inner torus edge and the tori elongations \( \lambda \) on the tori symmetry plane–see (11).

| Torus outer edge: \( r_{\text{out}} \) | \( r_{\text{out}} \equiv \frac{2}{3K} \left[ 1 + K \tau \cos \left( \frac{1}{3} \cos^{-1}(\alpha) \right) \right] \) |
|-------------------------------------|-----------------------------------------------------------------------------------|
| Tori elongation: \( \lambda \)     | \( \lambda \equiv \frac{2 \tau \cos \left( \frac{1}{3} \left[ 2 \cos^{-1}(\alpha) + \pi \right] \right)}{\sqrt{3}} \) |
| Torus inner edge: \( r_{\text{inner}} \)| \( r_{\text{inner}} \equiv \frac{2}{3K} \left[ 1 - K \tau \sin \left( \frac{1}{3} \sin^{-1}(\alpha) \right) \right] \) |
where \( K \equiv \sqrt{1 - K^2} \), \( Q \equiv \ell^2 \) and \( (\alpha, K, \tau) \) are functions of \((\ell, K)\)–see \(^{11}\). (Note in the case of cusped tori \( r_{inner} = r_x \).)

For the inner Roche lobe, the inner edge an the elongation of the lobe on the symmetry plane are

\[
\begin{align*}
    r_{BH, inner}^{\ell} &\equiv 2 \left[ \frac{1}{K} - \tau \sin \left( \frac{1}{6} \left[ 2 \cos^{-1}(\alpha) + \pi \right] \right) \right], \\
    \lambda_{BH, inner}^{\ell} &\equiv 2 \frac{1}{K} \tau \left[ \sin \left( \frac{1}{6} \left[ 2 \cos^{-1}(\alpha) + \pi \right] \right) - \sin \left( \frac{1}{3} \sin^{-1}(\alpha) \right) \right].
\end{align*}
\tag{11}
\]

The geometrical maximum for the outer \( r_{max}^{o} \) and inner Roche lobes \( r_{max}^{i} \) are

\[
\begin{align*}
    \text{Outer lobe:} & \quad r_{max}^{o}(K, \ell) \equiv \sqrt{\frac{K^2 Q}{K^2 - 1} + 4 \sqrt{\frac{2}{3} \psi \cos \left( \frac{1}{3} \cos^{-1}(\psi_x) \right)}}, \\
    \text{Inner lobe:} & \quad r_{max}^{i}(K, \ell) \equiv \sqrt{\frac{K^2 Q}{K^2 - 1} - 4 \sqrt{\frac{2}{3} \psi \sin \left( \frac{1}{3} \sin^{-1}(\psi_x) \right)}},
\end{align*}
\]

and the torus height is

\[
\begin{align*}
    h_{max}^{o}(K, \ell) &\equiv \sqrt{\frac{K^2 Q}{1 - K^2} + Z - 4 \sqrt{\frac{2}{3} \psi \cos \left( \frac{1}{3} \cos^{-1}(\psi_x) \right)}},
\end{align*}
\]

where \((\psi, \psi_x, Z)\) are functions of \((\ell, K)^{11}\).

### 3.3.2. Cusped tori

In this section we specialize the analysis of Sec. \((3.3.1)\) for cusped misaligned tori, which are described by one only independent parameter \( K \) or \( \ell \) (or equivalently \( r_x \) or \( r_{center} \)).

We can express the critical points of pressure in the tori in terms of the parameter \( \ell \). The torus center and the point of minimum density (and hydrostatic pressure) are

\[
\begin{align*}
    r_{center}(\ell) &\equiv \frac{1}{3} \left[ Q + 2 L_\ell \cos \left( \frac{1}{3} u_a \right) \right], \\
    r_{x}(\ell) &\equiv \frac{1}{3} \left[ Q - 2 L_\ell \cos \left( \frac{1}{3} u_a + \pi \right) \right],
\end{align*}
\]

where \( L_\ell, L_{\ell} \) are functions of \( \ell \) and \( Q \equiv \ell^2^{11} \). The cusped torus outer edge is located at

\[
\begin{align*}
    r_{out,x}^\ell &\equiv \frac{2 \ell^2 \hat{\psi}_2}{3 \ell^2 \hat{\psi}_2 + \hat{\psi}_0 (6 - \hat{\psi}_0)} + 2 \hat{\psi}_4 \sqrt{\frac{\ell^2 \left[ \hat{\psi}_0^2 \left( 3 \ell^2 \hat{\psi}_2 (6 - \hat{\psi}_0) + \hat{\psi}_0 (\hat{\psi}_0^2 - 12 \psi_0 + 36) \right) + 12 \ell^2 \hat{\psi}_2^2 \right]}{3 \left( 3 \ell^2 \hat{\psi}_2 + \hat{\psi}_0 (6 - \hat{\psi}_0) \right)^2}},
\end{align*}
\]

(where \( \hat{\psi}_0, \hat{\psi}_2 \) are functions of \( \ell \)). Similarly to functions \((\ell^o, \ell^i)\) of Eqs \((6)\), the
leading function can be expressed in terms of the energy function $K$ as

$$K_{center}(r) \equiv \sqrt{\frac{[Q + 2L_x \cos \left( \frac{\psi}{\sigma} \right) - 6][Q + 2L_x \cos \left( \frac{\psi}{\sigma} \right)]^2}{3Q \left[ \frac{3}{2} \ell^4 + 2(2Q - 15) L_x \cos \left( \frac{\psi}{\sigma} \right) - 39Q + 2L_x^2 \cos \left( \frac{2\psi}{\sigma} \right) + 54 \right]}},$$

$$K_{\times}(r) \equiv \sqrt{\frac{[Q - 2L_x \sin \left( \frac{\psi}{\sigma} \right) - 6][Q - 2L_x \sin \left( \frac{\psi}{\sigma} \right)]^2}{3Q \left[ \frac{3}{2} \ell^4 + 2(15 - 2Q) L_x \sin \left( \frac{\psi}{\sigma} \right) - 39Q + 2L_x^2 \cos \left( \frac{2\psi}{\sigma} \right) + 54 \right]}},$$

$$i_a \equiv \cos^{-1}(L_{\perp + 1}), \quad i_b \equiv \sin^{-1}(L_{\perp + 2}).$$

(12)

(13)

(here $L_{\perp}$ is a function of the momentum $\ell$). Function $K_{center}(r)$ describes the rings seeds, while $K_{\times}(r)$ refers the tori cusps ($Q \equiv \ell^2$).

The cusped tori outer edge can be expressed as function of the cusp $r_x$ as follows:

$$r_{\text{out}}(r_x) = \frac{2}{3} \left[ \frac{(r_x - r_{\text{maxo}})^2 r_x^2}{(r_x - r_{\text{mbo}})^2} \cos \left( \frac{1}{3} \cos^{-1} X \right) + \frac{r_x}{r_x - r_{\text{mbo}}} + r_x \right],$$

where $X$ is a function of $r_x$, and the the geometrical maxima of cusped tori for the outer ($o$) and inner ($i$) Roche lobes are

$$r_{\text{max}}^o(r) = \sqrt{4} \left[ \frac{2}{3} \psi_\lambda \cos \left( \frac{1}{3} \cos^{-1} \left( \frac{3}{4} \sqrt{\frac{3}{2} \psi_\sigma^2} \right) \right) \right] + \frac{r^2}{(r - r_{\gamma}) \psi_{\sigma}},$$

$$r_{\text{max}}^i(r) = \sqrt{4} \left[ \frac{2}{3} \psi_\lambda \cos \left( \frac{1}{3} \cos^{-1} \left( -\frac{3}{4} \sqrt{\frac{3}{2} \psi_\sigma^2} \right) + \pi \right) \right],$$

(here $r$ is the fluid pressure critical point where $\psi_\lambda$, $\psi_\sigma$ are functions of $r$).

The cusped torus height is

$$h_{\text{max}}^o(r_x) = \left[ 2 \sqrt{6} \frac{(r_x - r_{\gamma})(r_x - r_+)^2 r_x^4}{(r_x - r_{\text{mbo}})^3} \sec \left( \frac{1}{3} \cos^{-1} (\psi_\rho) \right) + \frac{9(r_x - r_+)^2 r_+^2 \sec^2 \left( \frac{1}{4} \cos^{-1} (\psi_\rho) \right)}{8(r_x - r_{\text{mbo}})(r_x - r_{\gamma})} \left( \frac{r_x - r_+}{(r_x - r_{\text{mbo}})^2} \right)^{1/2} \right],$$

($\psi_\sigma$ is a function of $r_x$.) Finally we can express the cusped tori geometrical thickness $S_x = 2h_x / (\lambda_x)$ in terms of the pressure critical points where $\lambda_x$ is the cusped torus elongation on its symmetry plane and $h_x$ the cusped torus height.

4. Conclusion

We explored models of tori clusters orbiting around a central SMBH, detailing the morphological characteristics of the toroidal components. Configurations considered here can be used as initial data for dynamical (time-dependent, evolutive) GRMHD analysis. A “leading function” as been used to constraint the tori distribution around the central attractor, together with the energy function $K(r)$ regulating the
agglomerate stability (cusp emergence and tori collision), the flow thickness, mass accretion rate and cusp luminosity.

From the observational viewpoint the inner ringed structure offers several interesting scenarios arising from the unstable states associated to its inner activity, as the presence of multiple accretion points and inter disk shells of multiple jets. Eventually observational evidence of the RAD and the associated inter disk activity could be found in the obscuration of the X-ray emission spectrum, as a track of the agglomerate inner composition. An indication of the presence of multiple orbiting tori could be seen in an increasing BH accretion mass rate and the presence of interrupted phases of BH accretion, or in the emission associated to oscillation tori modes as in HF QPOs. The establishment of runaway instability and the tori self-gravity can be relevant further factors for eRAD tori agglomerate around SMBHs and the Bardeen–Petterson effect is main relevant in the misaligned tori case.

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