Loss of information in quantum guessing game

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Abstract

Incompatibility of certain measurements—impossibility of obtaining deterministic outcomes simultaneously—is a well known property of quantum mechanics. This feature can be utilized in many contexts, ranging from Bell inequalities to device dependent QKD protocols. Typically, in these applications the measurements are chosen from a predetermined set based on a classical random variable. One can naturally ask, whether the non-determinism of the outcomes is due to intrinsic hiding property of quantum mechanics, or rather by the fact that classical, incoherent information entered the system via the choice of the measurement. Authors Rozpedek \textit{et al} (2017 New J. Phys. 19 023038) examined this question for a specific case of two mutually unbiased measurements on systems of different dimensions. They have somewhat surprisingly shown that in case of qubits, if the measurements are chosen coherently with the use of a controlled unitary, outcomes of both measurements can be guessed deterministically. Here we extend their analysis and show that specifically for qubits, measurement result for any set of measurements with any \textit{a priori} probability distribution can be faithfully guessed by a suitable state preparation and measurement. We also show that up to a small set of specific cases, this is not possible for higher dimensions. This result manifests a deep difference in properties of qubits and higher dimensional systems and suggests that these systems might offer higher security in specific cryptographic protocols. More fundamentally, the results show that the impossibility of predicting a result of a measurement is not caused solely by a loss of coherence between the choice of the measurement and the guessing procedure.

1. Introduction

The impossibility of performing a general set of measurements on a fixed quantum state without disturbing it lies at the heart of quantum theory. This concept is often referred to as the measurement problem [2]. More concretely, one is not able to perform subsequently a series of incompatible measurements that would attain a well defined result, irrespectively on the measured state. A well known consequence of measurement incompatibility is the Heisenberg uncertainty principle [3, 4] which shows the impossibility to specify both position and momentum of a particle with arbitrary precision. Since this pioneering work, measurement incompatibility has been extensively studied in many contexts, e.g. uncertainty relations [5–19] or joint measurability [20–29].

The task can be slightly reformulated and instead of performing a series of subsequent measurements on the same system, the experimentalist performs only a single measurement on a state. The question then is, whether such measurement can produce a predictable outcome. The answer here is naturally affirmative, if the experimentalist is also allowed to choose the measured state, as for each projective measurement, its eigenstates produce a deterministic outcome. The experiment becomes equivalent to the subsequent measurement experiment, if instead of using a single measurement, the experimentalist is tasked to perform a randomly chosen measurement out of a predefined set and the goal is to prepare a state which produces deterministic outcomes for all measurements from this set.

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With this modification the task can be conveniently rephrased in terms of a guessing game [8]. The players, Alice and Bob, agree beforehand on a set of measurements that Alice can perform in her lab. Bob then prepares a probe state and sends it to Alice, who randomly performs one of the available measurements. The question of interest is to identify the conditions under which Bob is be able to predict the outcome of all Alice’s measurements with certainty.

Clearly, without any information about the choice of Alice, Bob only can guess correctly with probability 1, if there exists a state that is an eigenstate of all measurements in Alice’s set and additionally, all measurements associate the same outcome to this state. Only little changes when Bob, after the measurement, receives from Alice classical information about her measurement choice—here the set of measurements still needs to share a single common eigenstate, but it does not need to be associated with the same measurement outcome anymore. Bob’s strategy consists of preparing the common eigenstate as the probe state in the second step of the guessing game. By learning the specific measurement performed by Alice, Bob guesses the value associated to the probe eigenstate by Alice’s measurement.

One can ask whether the incapability of Bob to learn the measurement result of Alice in other non-degenerate cases, unique to quantum realm, is caused by intrinsic properties of the quantum theory or by the fact that Bob receives ‘only’ classical, rather than quantum, information from Alice.

To tackle with this question, one can re-design the guessing game to a fully quantum level as follows. Instead of letting Alice to choose the measurement randomly on her own, the choice of the measurement is made coherently with the help of a controlled unitary operation. After performing the measurement, Alice will send the control quantum system to Bob, sharing with him the information about her choice on quantum level. Now the natural question is to what extent this can help Bob determine Alice’s measurement outcomes.

Rozpedek et al in [11] examined this question for a specific choice of two distinct measurements performed by Alice in two Mutually Unbiased Bases (MUBs). They have shown that specifically for qubit, Bob can learn with certainty the outcome of the measurement by sending a specifically designed probe state. They have also shown that for higher dimensions and two fixed measurements in MUBs, perfect guessing is not possible anymore. This was intuitively explained by the fact that in the latter case, it is naturally impossible to perfectly determine a measurement on a higher dimensional system by measuring a qubit.

In this paper we extend the analysis of this game in two main directions. First, we fully characterize the game with qubit measurements. We show that Bob is always able to predict the measurement result of Alice, even if Alice is able to choose measurements from an arbitrarily large set, with the assumption that he knows the set of measurements Alice is using and the underlying probability distribution Alice uses to choose them in the experiment. This basically means that quantum regime with qubits is very similar to the classical scenario.

In the second part we analyze the guessing game with measurements on systems of higher dimensions. By parameter analysis we show that for larger quantum systems, Bob is not able to learn the measurement result of Alice with certainty, except for some degenerate cases. This is particularly also true in cases, where Alice uses the same or higher number of measurements compared to the dimension of Bob’s system. In other words, Bob cannot correctly guess the measurement outcome even in cases where he receives a quantum system of dimension higher than the number of possible measurement outcomes. This result therefore undermines the intuition that Bob’s incapability of guessing the correct outcome is caused by the deficient dimension of the system he receives from Alice.

The paper is organized as follows. In section 2 we rigorously define the guessing game. In section 3 we first analyze the qubit system under uniform selection of a measurement from an arbitrary set and subsequently we extend our analysis to non-uniform situations as well. In section 4 we use parameter counting analysis to show that perfect guessing is only possible for the set of measurements of measure zero. Particularly, we explicitly show that for a set of three mutually unbiased measurements on a qutrit the perfect guessing is not possible. We conclude in section 5.

2. The guessing game

The guessing game is schematically depicted in figure 1. Alice and Bob agree on a dimension $B$ and a set of $A$ projective measurements $M_i$ in this dimension. Bob prepares a state $|\phi_A\rangle$ of dimension $B$ and sends the state to Alice. Alice prepares a state

$$|\phi_A\rangle = \frac{1}{\sqrt{A}} \sum_{i=0}^{A-1} |i\rangle$$

(1)
and a set of unitary operations \( U_i^z \) acting on dimension \( B \) such that
\[
U_i|\tilde{j}\rangle = |M_i^j\rangle ,
\]
where \( |M_i^j\rangle \) is the state that yields an outcome \( j \), \( i \)th measurement is performed. Alice now performs a controlled operation on both states
\[
U_i|\phi_A\rangle |\psi_B\rangle = \frac{1}{\sqrt{A}} \sum_{i=0}^{A-1} |i\rangle U_i |\psi_B\rangle
\]
and performs a measurement in the computational basis on \( |\psi_B\rangle \), obtaining a result \( a \) of dimension \( B \). In the next step, Alice sends the state \( |\phi_A\rangle \) to Bob and asks him to guess her measurement outcome. In order to produce a correct guess, Bob measures the state \( |\phi_A\rangle \) and produces an outcome \( b \).

In full generality Bob is allowed to do any measurement on the state \( |\phi_A\rangle \) and can use any classical post-processing of the measurement outcome \( b \) to produce a guess of Alice’s outcome \( a \). In this work, however, we restrict ourselves to the situation when Bob is able to guess the outcome of Alice’s measurement with certainty. In such a case it is sufficient for Bob to use projective measurements, which yield a result \( b \) of dimension \( A \), and he succeeds if \( a = b \). For the case \( B > A \) this is naturally only possible if the probability of at least \( B - A \) outcomes of all Alice’s measurements is 0. Choosing a measurement that can distinguish states received by Bob is intimately connected to the choice of the probe state and Alice’s measurements, therefore it is an integral part of Bob’s strategy.

One can generalize the guessing game by allowing Alice to prepare a more general control state in the form
\[
|\phi_A^f\rangle = \sum_{i=0}^{A-1} \zeta_i |i\rangle.
\]

Now one can analyze different scenarios. In the first one, the amplitudes \( \zeta_i \) are fully known to Bob. On the other side of the spectrum, Bob does not have any information about \( \zeta_i \) and in the intermediate scenarios Bob can have some partial information about \( \zeta_i \). It is easy to see that complete ignorance of \( \zeta_i \) will not allow Bob to guess any better than in the classical scenario. In order to see this, consider a set of two incompatible measurements without a common eigenstate. If one of the measurements is chosen deterministically (say \( \zeta_0 = 1 \), Bob has to choose one of the eigenstates \( |M_0^0\rangle \), in order to guess Alice’s outcome with probability 1. On the other hand, for \( \zeta_i = 1 \) Bob has to choose one of the eigenstates \( |M_i^0\rangle \). Therefore, if two measurements do not share an eigenstate and Bob has no information about \( \zeta_i \), he cannot guess Alice’s outcome with certainty. In what follows, we analyze the scenarios in which Bob has at least some partial knowledge of \( \zeta_i \).

### 3. Guessing game with qubits

In this section we analyze the probability of correct guessing in scenarios where \( |\psi_B\rangle \) is a qubit. In this case \( |\psi_B\rangle \) can be parameterized as
\[
|\psi_B\rangle = \alpha |0\rangle + \beta |1\rangle,
\]
where we choose \( \alpha \) to be real and positive. Let us first consider a scenario, where all the measurements are chosen with uniform probability. In such a case, Alice has a general state \( (1) \) and the set of conditional operations \( U_i (2) \).
for each $i$ is defined in a following way:

$$
|0\rangle \rightarrow |\psi_i\rangle = \alpha_{i}|0\rangle + \beta_{i}|1\rangle,
$$

$$
|1\rangle \rightarrow |\psi_i^\dagger\rangle = \beta_{i}^\ast|0\rangle - \alpha_{i}|1\rangle,
$$

where again parameters $\alpha_i$ are chosen to be real and positive and $*$ denotes complex conjugation. It is straightforward to see that with suitable parameters $\alpha_i$ and $\beta_i$ one can encode any projective measurement on a qubit into a measurement in the computational basis.

Now we can write down the common state of Alice and Bob after the conditional operation (3) is performed

$$
|\Psi\rangle = \frac{1}{\sqrt{A}} \sum_{i=0}^{A-1} \left( (\alpha \alpha_i + \beta \beta_i^\ast) |i\rangle + (\alpha \beta_i - \beta \alpha_i) |i\rangle \right) |1\rangle.
$$

(6)

Then Alice performs a measurement in computational basis and depending on her outcome, the state that is sent to Bob is

$$
|\phi_\psi\rangle = \frac{1}{\sqrt{A}} \sum_{i=0}^{A-1} (\alpha \alpha_i + \beta \beta_i^\ast) |i\rangle
$$

(7)

or

$$
|\phi_\gamma\rangle = \frac{1}{\sqrt{A}} \sum_{i=0}^{A-1} (\alpha \beta_i - \beta \alpha_i) |i\rangle.
$$

(8)

Bob can learn the result of the measurement performed by Alice if and only if he can distinguish these two states, namely if

$$
\langle \phi_\gamma | \phi_\psi \rangle = 0.
$$

(9)

Here we remind that all the parameters in (7) and (8) are known to Bob, so he knows the full specification of the states he is trying to distinguish. This is why (9) is a sufficient condition for Bob’s perfect guessing.

Condition (9) can be expressed as a complex equation

$$
\frac{1}{A} \sum_{i=0}^{A-1} (\alpha \alpha_i + \beta \beta_i^\ast)(\alpha \beta_i - \beta \alpha_i) = 0,
$$

(10)

where $\alpha$ and $\beta$ are variables that can be chosen by Bob and parameters $\alpha_i$ and $\beta_i$ are chosen by Alice, but known to Bob. It remains to investigate under which conditions on $\alpha_i$ and $\beta_i$ the equation (10) has a solution.

In what follows, we rewrite (10) into a pair of equations involving only real parameters. To do so we define

$$
a = \alpha,
$$

$$
a_i = \alpha_i,
$$

$$
b = |\beta|,
$$

$$
b_i = |\beta_i|,
$$

$$
\varphi = \arg(\beta),
$$

$$
\varphi_i = \arg(\beta_i).
$$

(11)

Using this new set of parameters, equation (10) gets a form

$$
\sum_i [aa_i + bb_i(\cos \varphi \cos \varphi_i + \sin \varphi \sin \varphi_i) - jbb_i(\sin \varphi \cos \varphi_i - \cos \varphi \sin \varphi_i)]
$$

$$
\times [ab_i \cos \varphi_i - a_i b \cos \varphi + j(ab_i \sin \varphi_i - a_i b \sin \varphi)] = 0,
$$

(12)

where $j = \sqrt{-1}$. We can further separate it into two equations for real and imaginary part

$$
\sum_i [(aa_i + bb_i(\cos \varphi \cos \varphi_i + \sin \varphi \sin \varphi_i))(ab_i \cos \varphi_i - a_i b \cos \varphi)
$$

$$
+ bb_i(\sin \varphi \cos \varphi_i - \cos \varphi \sin \varphi_i)(ab_i \sin \varphi_i - a_i b \sin \varphi)] = 0,
$$

(13)

$$
\sum_i [(aa_i + bb_i(\cos \varphi \cos \varphi_i + \sin \varphi \sin \varphi_i))(ab_i \sin \varphi_i - a_i b \sin \varphi)
$$

$$
- bb_i(\sin \varphi \cos \varphi_i - \cos \varphi \sin \varphi_i)(ab_i \cos \varphi_i - a_i b \cos \varphi)] = 0.
$$

(14)

Let us now rewrite (13) and (14) into a form

$$
X_i b^2 + Y_i ab - X_i a^2 = 0,
$$

(15)

$$
X_i b^2 + Y_i ab - X_i a^2 = 0,
$$

(16)
where
\[ X_r = \sum_i a_i b_i (\cos^2 \varphi \cos \varphi_i - \cos \varphi \sin \varphi \sin \varphi_i - \sin^2 \varphi \cos \varphi_i + \sin \varphi \cos \varphi \sin \varphi_i) = -\sum_i a_i b_i \cos \varphi_i, \]
\[ Y_r = \sum_i [-a_i^2 \cos \varphi + b_i^2 \cos \varphi \cos^2 \varphi_i + 2b_i \sin \varphi \cos \varphi_i \sin \varphi_i - b_i^2 \cos \varphi \sin^2 \varphi_i], \]
\[ Z_r = \sum_i a_i b_i \cos \varphi_i = -X_r, \]
and
\[ X_I = \sum_i a_i b_i (\cos \varphi_i \cos \varphi \sin \varphi - \sin^2 \varphi \sin \varphi_i + \sin \varphi \cos \varphi \cos \varphi_i - \cos^2 \varphi \sin \varphi_i) = -\sum_i a_i b_i \sin \varphi_i, \]
\[ Y_I = \sum_i [-a_i^2 \sin \varphi + 2b_i^2 \cos \varphi \cos \varphi_i \sin \varphi_i + b_i^2 \sin \varphi \sin^2 \varphi_i - b_i^2 \sin \varphi \cos^2 \varphi_i], \]
\[ Z_I = \sum_i a_i b_i \sin \varphi_i = -X_I. \]

It is important to observe that \(X_r = -Z_r\) and \(X_I = Z_I\) depend only on parameters \(a_i, b_i\) and \(\varphi_i\) and not on variables \(\varphi\) and \(a\).

The set of equations (15), (16) can be joined by multiplying (15) by \(X_I\) and (16) by \(X\), and subtracting them:
\[ ba(Y_X - Y_I X) = 0. \tag{17} \]

This equation can have a solution in three cases. The obvious two are \(a = 0\) and \(b = 0\). These imply \(b = 1\) (or \(a = 1\), respectively) and thus \(X_r = X_I = 0\), which corresponds to a very specific set of parameters of the measurements. Naturally for \(a = 0\) or \(b = 0\) the value of the complex phase \(\varphi\) has no physical meaning.

The interesting case is \(Y_X Y_I - Y_I X = 0\), which translates to
\[ X_I \sum_i [-a_i^2 \cos \varphi + b_i^2 \cos \varphi \cos^2 \varphi_i + 2b_i \sin \varphi \cos \varphi_i \sin \varphi_i - b_i^2 \cos \varphi \sin^2 \varphi_i] \]
\[ = X_r \sum_i [-a_i^2 \sin \varphi + 2b_i^2 \cos \varphi \cos \varphi_i \sin \varphi_i + b_i^2 \sin \varphi \sin^2 \varphi_i - b_i^2 \sin \varphi \cos^2 \varphi_i]. \tag{18} \]

We can further restructure the equation to
\[ \cos \varphi \sum_i [-a_i^2 X_i + b_i^2 \cos^2 \varphi_i X_i \sin \varphi X_i - b_i^2 \sin^2 \varphi_i X_i - 2b_i^2 \cos \varphi_i \sin \varphi_i X_i] \]
\[ = \sin \varphi \sum_i [-a_i^2 X_i + b_i^2 \sin^2 \varphi_i X_i - b_i^2 \cos^2 \varphi_i X_i - 2b_i^2 \cos \varphi_i \sin \varphi_i X_i]. \tag{19} \]

and define the argument
\[ \varphi = \arctan \left( \frac{\sum_i [-a_i^2 X_i + b_i^2 \cos^2 \varphi_i X_i - b_i^2 \sin^2 \varphi_i X_i - 2b_i^2 \cos \varphi_i \sin \varphi_i X_i]}{\sum_i [-a_i^2 X_i + b_i^2 \sin^2 \varphi_i X_i - b_i^2 \cos^2 \varphi_i X_i - 2b_i^2 \cos \varphi_i \sin \varphi_i X_i]} \right). \tag{20} \]

For the specific case of the denominator of (20) being equal to 0 we define \(\varphi = \frac{\pi}{2}\).

Now the question is whether, substituting the result of (20) into (15) there exists a solution for \(b\). It is a quadratic equation with the discriminant
\[ D = a^2 Y_r^2 + 4a^2 X_r^2, \]
which is clearly non-negative. The solution for \(b\) will then be
\[ b = a \sqrt{\frac{Y_r^2 + 4X_r^2}{2X_r}}, \tag{21} \]
which still depends on \(a\). Imposing the normalization condition we get
\[ a = \frac{|2X_r|}{\sqrt{8X_r^2 + 2Y_r^2 - 2Y_r \sqrt{Y_r^2 + 4X_r^2}}}. \tag{22} \]

Thus, we can conclude that for every set of measurements that Alice chooses, Bob can prepare a test state specified by parameters \(\varphi\) (20) and \(a\) (22), such that with a suitable measurement in the basis specified by the states (7) and (8) he can predict with certainty the outcome of the measurement performed by Alice.

### 3.1. Non-uniform extension

Here we extend the analysis to a more general situation where Alice decides about the measurements based on a more general state (4). This includes the situations reachable classically, where Alice would simply favor some of
the measurements comparing to some others, but Alice could also do a more subtle change by just tweaking the phases between different measurements.

Two obvious questions arise here: is Bob able to retain his capability of perfect guessing even for this more complicated situation? And, to what extend does he need to adapt his strategy based on the knowledge of the state used by Alice?

If Alice uses the state \((4)\) instead of \((1)\), it causes the change of \((6)\) into

\[
|\Psi_i\rangle = \sum_{i=0}^{A-1} \left[ \zeta_i(\alpha\alpha_i + \beta\beta_i^*)|i\rangle|1\rangle + \zeta_i(\alpha\beta_i - \beta\alpha_i)|i\rangle|0\rangle \right]
\]

(23)

and the post-selected states depending on Alice’s measurement outcome to

\[
|\phi^0_i\rangle = \sum_{i=0}^{A-1} \zeta_i(\alpha\alpha_i + \beta\beta_i^*)|i\rangle
\]

(24)

and

\[
|\phi^1_i\rangle = \sum_{i=0}^{A-1} \zeta_i(\alpha\beta_i - \beta\alpha_i)|i\rangle.
\]

(25)

The orthogonality condition then reads

\[
\langle \phi^0_i|\phi^1_i\rangle = \sum_{i=0}^{A-1} |\zeta_i|^2 (\alpha\alpha_i + \beta\beta_i^*)^\dagger (\alpha\beta_i - \beta\alpha_i) = 0.
\]

(26)

It is straightforward to see that only the absolute values \(|\zeta_i|^2\) enter in further calculations for orthogonality, but the phases enter into the specification of the post-selected states \((24)\) and \((25)\). We can further rewrite \((26)\) to

\[
\sum_{i=0}^{A-1} (\alpha|\zeta_i|\alpha_i + \beta|\zeta_i||\beta_i|)(\alpha|\zeta_i|\beta_i - \beta|\zeta_i||\alpha_i) = 0
\]

(27)

and redefine the parameters used in the previous subsection \((11)\) into

\[
a = \alpha, \\
a_i = |\zeta_i|\alpha_i, \\
b = |\beta|, \\
b_i = |\zeta_i||\beta_i|, \\
\varphi = \text{arg} (\beta), \\
\varphi_i = \text{arg} (\beta). 
\]

(28)

Now it is straightforward to see that the specification of the optimal probe state defined by \((20)-(22)\) combined with \((28)\) is valid for the more general non-uniform case as well. Moreover, the definition of the probe state only depends on the absolute values of the parameters \(|\zeta_i|\), meaning Bob only needs the information about the frequencies of measurements beforehand. On the other hand, phases of \(|\zeta_i|\) enter into the specification of the two post-selected states that Bob needs to distinguish. So before performing his measurement, Bob needs to know the phases as well.

4. Measurements on a larger system

It was recently shown in [1] that for a specific case of two unbiased measurements, perfect guessing of their outcomes on a system larger than a qubit is not possible. From one point of view it sounds natural that with more possible measurement results the outcome will be harder to guess. Especially if \(B > A\), the amount of information extractable in principle from the state Bob gets from Alice is smaller than the amount Bob needs for perfect guessing. On the other hand, by increasing the number of possible measurements \(A\), Bob gets a larger system and, at least in principle, it could be easier for him to learn the outcome of the measurement. Let us now analyze in detail the simplest non-trivial configuration, namely the case of three unbiased measurements on a qutrit system.

4.1. Three unbiased measurements on a qutrit

Similarly as in the previous section, we characterize the state prepared by Bob by

\[
|\psi\rangle = \alpha|0\rangle + \beta|1\rangle + \gamma|2\rangle.
\]

(29)
Alice shall perform measurements in one of MUBs. These are generally defined for prime \( d \) by

\[
|\{M_k\} = \frac{1}{\sqrt{d}} | \sum_{j=0}^{d-1} \omega^{kj^2 + ij}|j\rangle,
\]

where \( d \) is the dimension of the system, \( k \) is the index of the base (starting from 0) and \( \omega = e^{i\pi/2} \) \([30]\). These bases can be obtained from the computational basis by applying a unitary transformation defined element-wise by

\[
U_k[i,j] = \frac{1}{\sqrt{d}} \omega^{kj^2 + ij}.
\]

We take (31) as the unitary operations associated with measurements (2) and build up the conditional operation (3).

For the specific case of a qutrit, the operations performed by Alice before measurement are

\[
U^0_1 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 1 & 1 \\ 1 & \omega & \omega^2 \\ 1 & \omega^2 & \omega \end{pmatrix},
U^1_1 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & \omega & \omega^2 \\ 1 & 1 & \omega \\ 1 & \omega & 1 \end{pmatrix},
U^2_1 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & \omega^2 & \omega \\ 1 & 1 & \omega \\ 1 & \omega & 1 \end{pmatrix},
\]

where \( \omega = e^{i\pi/3} \). The common state of Alice and Bob after performing conditional operation by Alice and before the measurement reads

\[
|\Psi\rangle = \frac{1}{\sqrt{3}} (|0\rangle (\alpha|0\rangle + \alpha|1\rangle + \alpha|2\rangle + \beta|0\rangle + \omega^2\beta|1\rangle + \gamma|0\rangle + \omega\gamma|1\rangle + \omega^2\gamma|2\rangle)
+ \frac{1}{\sqrt{3}} |1\rangle (\alpha|0\rangle + \alpha|1\rangle + \alpha|2\rangle + \omega^2\beta|0\rangle + \omega\beta|1\rangle + \beta|2\rangle + \omega^2\gamma|0\rangle + \gamma|1\rangle + \omega\gamma|2\rangle)
+ \frac{1}{\sqrt{3}} |2\rangle (\alpha|0\rangle + \alpha|1\rangle + \alpha|2\rangle + \omega\beta|0\rangle + \beta|1\rangle + \omega^2\beta|2\rangle + \omega\gamma|0\rangle + \omega^2\gamma|1\rangle + \gamma|2\rangle).
\]

Now the three post-selected states that are sent to Bob, based on the result obtained by Alice can be determined

\[
|\phi^0\rangle = \frac{\alpha + \beta + \gamma}{3} |0\rangle + \frac{\alpha + \omega^2\beta + \omega^2\gamma}{3} |1\rangle + \frac{\alpha + \omega\beta + \omega\gamma}{3} |2\rangle,
|\phi^1\rangle = \frac{\alpha + \omega^2\beta + \omega\gamma}{3} |0\rangle + \frac{\alpha + \omega\beta + \gamma}{3} |1\rangle + \frac{\alpha + \beta + \omega^2\gamma}{3} |2\rangle,
|\phi^2\rangle = \frac{\alpha + \omega\beta + \omega^2\gamma}{3} |0\rangle + \frac{\alpha + \beta + \omega\gamma}{3} |1\rangle + \frac{\alpha + \omega\beta + \gamma}{3} |2\rangle.
\]

To allow Bob to unambiguously guess the output of Alice, these three states need to be perpendicular to each other. To achieve that, they need to be pairwise perpendicular, which bring us to three equations (perpendicularity of pairs \(|\phi^0\rangle \perp |\phi^1\rangle, |\phi^0\rangle \perp |\phi^2\rangle\) and \(|\phi^1\rangle \perp |\phi^2\rangle\))

\[
(\alpha + \beta + \gamma) (\alpha + \omega^2\beta + \omega\gamma) + (\alpha + \omega\beta + \omega^2\gamma) (\alpha + \omega\beta + \gamma) + (\alpha + \beta + \omega\gamma) (\alpha + \omega\beta + \omega^2\gamma) = 0,
(\alpha + \beta + \gamma) (\alpha + \omega\beta + \omega^2\gamma) + (\alpha + \omega^2\beta + \omega\gamma) (\alpha + \beta + \omega\gamma) + (\alpha + \omega\beta + \gamma) (\alpha + \omega^2\beta + \gamma) = 0,
(\alpha + \omega^2\beta + \omega\gamma) (\alpha + \omega\beta + \gamma) + (\alpha + \omega\beta + \omega^2\gamma) (\alpha + \beta + \gamma) + (\alpha + \beta + \omega\gamma) (\alpha + \omega\beta + \omega^2\gamma) = 0.
\]

We can subtract the first two equations of (34) and get

\[
(\alpha + \beta + \gamma) (\omega^2 - \omega) (\beta - \gamma) + (\alpha + \omega^2\beta + \omega^2\gamma) (\omega - 1)(\beta - \gamma) + (\alpha + \omega\beta + \omega^2\gamma) (1 - \omega^2)(\beta - \gamma) = 0.
\]

This can be further decomposed to

\[
\omega(\omega - 1)^2(\beta - \gamma)(\beta + \gamma)^* = 0.
\]

Let us analyze the two possible solutions of (36).

**4.1.1. \( \beta = \gamma \)**

If \( \beta = \gamma \), from the third equation of (34) we get

\[
|\alpha + (\omega^2 + \omega)\beta|^2 + |\alpha + (1 + \omega)\beta|^2 + |\alpha + (\omega + 1)\beta|^2 = 0,
\]

which can obviously only be satisfied for \( \alpha = \beta = 0 \), leading to the trivial non-normalizable solution \( \alpha = \beta = \gamma = 0 \).
4.1.2. $\beta = -\gamma$

For the second solution of (36), we get from the first equation of (34)

$$
\alpha^4 (\alpha + \omega^2\beta - \omega\beta + \alpha + \omega\beta - \beta + \alpha + \beta - \omega^2\beta) = 3 |\alpha|^2 = 0.
$$

Plugging this into the third equation of (34) yields

$$(\omega - \omega^2)^2|\beta|^2 + (\omega^2 - 1)(1 - \omega^2)|\beta| + (1 - \omega)(\omega - 1) |\beta|^2 = 0,$$

which only has a trivial solution $\beta = 0$, leading again to the trivial non-normalizable solution $\alpha = \beta = \gamma = 0$. This concludes the proof there there does not exist a probe state for Bob that would allow him to learn with certainty the outcome of the measurement performed by Alice.

4.2. Parameter counting

In the previous paragraphs we have shown that unlike the qubit case, for three unbiased measurements on a qutrit Bob is not able to prepare a probe state that would allow him to guess the outcome of the measurement performed by Alice with probability 1. This however cannot be explained by the simple argument valid for two measurements on a qutrit, namely that the state obtained by Bob cannot contain enough information to guess the result. Also in a different perspective discussed in [1], three measurements on a qutrit allow to build up a maximally entangled state between Alice and Bob and yet it is not sufficient for reliable guessing. Here we argue by parameter counting that this is a general feature of the game for any system larger than a qubit, independently on the number of distinct measurements.

In the most general case, Bob designs a probe state by specifying $2B - 2$ real parameters. He then receives a state of a dimension $A$, which can be viewed as a mixture of post-selected states specified by the number of $B$ possible outcomes of the measurement performed by Alice. For successfully learning the measurement outcome, all these states have to be perpendicular to each other. This is naturally impossible for $B > A$, simply due to the dimension of the space, except for special cases discussed below.

Independently on $A$, perpendicularity of $B$ different states can be obtained by fulfilling $B(B - 1)$ simple equations for real parameters. Interestingly, for a qubit ($B = 2$) it holds

$$
B(B - 1) = 2B - 2
$$

and thus the number of free parameters is equal to the number of equations to fulfill. In the previous section we have shown that a solution always exist.

For any $B > 2$ it holds

$$
B(B - 1) > 2B - 2.
$$

Hence, for a general situation not all the equations can be fulfilled and thus Bob cannot guess Alice’s outcome with certainty. The exceptions are formed by a subset of measurements chosen by Alice that are of a measure zero within all possible set of measurements. The relative cardinality of the subset decreases with increasing $B$ due to the fact that Bob’s choice of the probe state can only lead to fulfillment of a linear portion from the quadratically increasing set of conditions and the rest needs to be guaranteed by the selection of measurements.

4.3. Special cases

Let us analyze here, what properties need to be fulfilled for the set of measurements selected by Alice to allow Bob, at least in principle, to guess with certainty the result of Alice’s measurement. Let us discuss first the case of $B > A$, which is easier to tackle.

Apparently, by measuring a state of dimension $A$, Bob only can learn up to $A$ different outcomes. Thus, he needs to have some a priori information about the possible outcomes of Alice’s measurement, which have $B > A$ possible outcomes. Moreover, as we are interested here in perfect guessing, this a priori information needs to be complete in a sense that Bob needs to know about some of the possible outcomes that they do not appear at all—the other option, knowing with certainty that a result will appear corresponds to the almost trivial case of all measurements sharing the same eigenstate.

To achieve this, there must exist a probe state for which at least $B - A$ measurement results (for each and every measurement) appear with zero probability. This can only be realized if all measurements share a common subspace of a dimension of at least $B - A$ by choosing the probe state perpendicular to this subspace. This naturally leads to a condition that this subspace must have a dimension smaller than $B$ (there still must exist at least one dimension outside the subspace in which the probe state can exist).

So we can summarize that the necessary condition for perfect guessing in the case $B > A$ is that there exist a decomposition of the $B$-dimensional Hilbert space into two non-trivial subspaces such that at least one of them is of a dimension of at least $B - A$ and for each measurement, each of its results falls strictly into one of these subspaces. However, this condition is not sufficient—even if it is fulfilled, the probe state needs to satisfy also the condition of mutual perpendicularity of the post-selected states in the form (34).
To check this condition, let us set the dimension of the Hilbert subspace containing the probe state to C. Then we arrive at \( 2(B - C) \) conditions on the probe state due to the perpendicularity to the rest of the Hilbert space and \( 2C(C - 1) \) conditions for the mutual perpendicularity of the post-selected states. This leads to in total \( 2B - 4C + 2C^2 \) conditions, which can be (potentially) satisfied using \( 2B - 2 \) parameters of the probe state only for \( C = 1 \). This however again leads to the semi-trivial solution of a single shared eigenstate, i.e. that the result of each measurement is perfectly predetermined.

For the case \( B < A \) the situation is a bit more complicated. Here Bob gets a state of a dimension that allows him, at least in principle, to learn enough information for perfect guessing. This is also the case for qubits, as investigated in the previous section, where Bob can guess in all cases. This result also can be used to show instances where perfect guessing is possible in higher dimensions.

If we choose \( B = n^2 \), we can view the probe state prepared by Bob as an \( n \)-qubit state and prepare it in the form of products of (5). We can restrict Alice to use only measurements factorable on these two-level subspaces, the same for all measurements. But even this is not enough, the set of measurements shall include all the combinations of different measurements on different qubits with the same weight (or appropriate relative weights for the non-uniform extension). For this, rather restricted, but still non-trivial set of measurements used by Alice, one can use procedures presented in the previous section to prepare a probe state that shall allow Bob’s perfect guessing. This example shows that the solution based on a single shared eigenstate presented above is not unique for systems larger than a qubit.

5. Conclusion

We have analyzed to what extent the impossibility of guessing an outcome of one of a prescribed set of measurements is connected with the inaccessibility of quantum information about the choice of the measurement. Interestingly, we have shown that specifically for two-dimensional systems (qubits), availability of coherent quantum information about the choice allows one to correctly guess the outcome of any measurement from any predefined set by choosing a proper probe state and final measurement. Both of these choices depend on the particular form of the measurements in the set, as well as on frequencies of their appearance in the more complex scenario.

Interestingly, the situation changes dramatically for larger systems. We have shown explicitly that for three measurements in MUBs performed on a qutrit, perfect guessing is not possible anymore. We have also shown that this is a general feature for all higher dimensional systems and sets of measurements for which guessing is possible form a subset of measure zero among all possible sets.

This result highlights the fundamental difference between the quantum properties of qubits and higher-dimensional systems, manifested e.g. by (non-)existence of bound entanglement [31], or (non-)existence of non-contextual hidden variable models [32]. Its direct manifestation in the proposed guessing game motivates further research of direct use of higher-dimensional states in secure quantum communication, potentially offering higher level of protection against adversaries.

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References

[1] Rozpedek F, Kaniewski J, Coles P J and Wehner S 2017 New J. Phys. 19 023038
[2] Nielsen M A and Chuang I L 2011 Quantum Computation and Quantum Information: 10th Anniversary Edition 10th edn (New York: Cambridge University Press)
[3] Heisenberg W 1927 Z. Phys. 43 172
[4] Kennard E H 1927 Z. Phys. 44 326
[5] Berta M, Coles P J and Wehner S 2014 Phys. Rev. A 90 062127
[6] Berta M, Fawzi O and Wehner S 2014 IEEE Trans. Inf. Theory 60 1168
[7] Christandl M and Winter A 2005 IEEE Trans. Inf. Theory 51 3159
[8] Berta M, Christandl M, Colbeck R, Renes J M and Renner R 2010 *Nat. Phys.* **6** 659
[9] Coles P J, Colbeck R, Yu L and Zoullak M 2012 *Phys. Rev. Lett.* **108** 210405
[10] Coles P J, Yu L, Gheorghiu V and Griffiths R B 2011 *Phys. Rev. A* **83** 062338
[11] Dupuis F, Fawzi O and Wehner S 2015 *IEEE Trans. Inf. Theory* **61** 1093
[12] Frank R L and Lieb E H 2013 *Commun. Math. Phys.* **323** 487
[13] Furrer F, Berta M, Tomamichel M, Scholz V B and Christandl M 2014 *J. Math. Phys.* **55** 122205
[14] Hall M J W 1995 *Phys. Rev. Lett.* **74** 3507
[15] Liu S, Mu L-Z and Fan H 2015 *Phys. Rev. A* **91** 042133
[16] Korzekwa K, Lostaglio M, Jennings D and Rudolph T 2014 *Phys. Rev. A* **89** 042122
[17] Luo S L 2005 *Theor. Math. Phys.* **143** 681
[18] Renes J M and Boileau J-C 2009 *Phys. Rev. Lett.* **103** 020402
[19] Sánchez-Ruiz J 1995 *Phys. Lett. A* **201** 125
[20] Ludwig G 1964 *Z. Phys.* **181** 233
[21] Busch P and Lahti P J 1984 *Phys. Rev. D* **29** 1634
[22] Busch P 1986 *Phys. Rev. D* **33** 2255
[23] de Muynck W M and Martens H 1990 *Phys. Rev. A* **42** 5079
[24] Lahti P 2003 *Int. J. Theor. Phys.* **42** 893
[25] Wolf M M, Perez-Garcia D and Fernandez C 2009 *Phys. Rev. Lett.* **103** 230402
[26] Reeb D, Reitzner D and Wolf M M 2013 *J. Phys. A: Math. Theor.* **46** 662002
[27] Uola R, Moroder T and Gühne O 2014 *Phys. Rev. Lett.* **113** 160403
[28] Heinosaari T, Miyadera T and Reitzner D 2014 *Found. Phys.* **44** 34
[29] Heinosaari T, Kiukas J and Reitzner D 2015 *Phys. Rev. A* **92** 022115
[30] Wootters W and Fields B D 1989 *Ann. Phys., NY* **191** 363
[31] Horodecki M, Horodecki P and Horodecki R 1998 *Phys. Rev. Lett.* **80** 5239
[32] Kochen S and Specker E P 1967 *J. Math. Mech.* **17** 59