Three component fermion pairing in two dimensions

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We study pairing of an interacting three component Fermi gas in two dimensions. By using a mean field theory to decouple the interactions between different pairs of Fermi components, we study the free energy landscapes as a function of various system parameters including chemical potentials, binding energies, and temperature. We find that the s-wave pairing channel is determined by both chemical potentials and the interaction strengths between the three available channels. We find a second order thermal phase transition and a series of first order quantum phase transitions for a homogenous system as we change the parameters. In particular, for symmetric parameters, we find the simultaneous existence of three superfluid orders as well as re-entrant quantum phase transitions as we tune the parameters.

I. INTRODUCTION

Recent experimental progress achieved in ultra-cold atomic gases allows one to set up test beds for controlled study of many body physics. By tuning three dimensional two-body scattering length between atoms in different hyperfine spin states of a dilute system at low temperatures, interaction strength between atoms can be controlled very precisely [1]. This can be done by using magnetically tuned Feshbach resonance [2]. Moreover, interaction and spatial dimensionality can be effectively controlled by applying an optical lattice. An effective two-dimensional system can be created by applying a relatively strong one-dimensional optical lattice to an ordinary three-dimensional system. As there are always two body bound states exist for attractive two body potentials in two dimensions [3], different pairs of Fermi atoms can undergo Bose Einstein condensation and form superfluidity at low temperatures. The two dimensional bound state energies can be controlled by tuning either three dimensional scattering length or the laser intensity which used to create one dimensional lattice to accommodate 2D layers.

In this paper we study three component Fermi gases in two dimensions. A mixture of $^6$Li atoms which has favorable collisional properties among its lowest three hyperfine spin states will be an ideal system to explore novel three component superfluidity. Three component $^6$Li mixture in three dimensions have already been trapped and manipulated experimentally [4] [5]. In Ref. [4], using radio frequency spectroscopic data and a quantum scattering model, scattering lengths and the Feshbach resonance positions in the lowest three channels of $^6$Li atoms have been determined. As there are three broad s-wave Feshbach resonances, one can prepare the system at various interaction strengths between each pair. In Ref. [5], collisional stability of the lowest three channels of $^6$Li atoms has been studied. As the spin relaxation time is large compared to the other time scales in the experiments, experimentalists were able to maintain a fixed spin population throughout the experiments. As a physically accessible system, a three-component ultracold atomic system can be used to study the physics of nuclear matter. Three-component Fermi pairing is believed to occur in the interior of neutron stars and in heavy-ion collisions [6]. Nevertheless, this system can be used to understand the competition between quantum phases and re-entrant phase transitions.

Properties of three-component Fermi systems have been extensively studied in recent past [7] [8] [9] [10] [11] [12]. However, all these studies are carried out in a three dimensional or one dimensional environments. Further, authors in all these references except Ref. [8] have restricted their parameter space either by assuming equal interaction strengths or equal chemical potentials or by neglecting the interaction between some hyperfine spin components. In ref. [5], the authors have studied the properties of a harmonically trapped three component gas in three dimensions.

In this paper, we neglect the possibility of three body bound states in two dimensions and consider only two body pairing states. This is reasonable, because of the system we are considering is dilute and the atomic interactions are short range in nature. Therefore, it is unlikely to have many atoms interacting in the same region of space. Further, we neglect the harmonic confinement and consider the system as spatially homogenous. Two-component Fermi gases in two dimensions have already been studied in theory [13] [14]. The purpose of this paper is to investigate how pairing will occur when a third spin component is added to such a two-component gas. More precisely, we study the competition of individual components to form Bose condensed pairs by investigating the landscapes of the free energy as a function of various parameters which include the temperature, chemical potentials of the hyperfine spin components, and the interaction strengths between different pairs of fermions components. For appropriately chosen parameters, we find that the system undergoes a second order thermal phase transition from normal state to a superfluid state as one lowers the temperature. At low temperatures, we find a series of first order quantum phase transitions...
as we change the chemical potentials or the interactions between hyperfine spin components. At low temperatures, we find simultaneous existence of three types of superfluid phases (at symmetric parameters corresponding to different pairing channels) and normal phases in this novel three-component Fermi system. Interestingly, we find that the system can undergo re-entrant phase transitions as we simultaneously tune the chemical potentials and interactions.

The paper is organized as follows. In the following section, we introduce the theoretical model and use a mean field approximation to decouple the interaction terms. Then using a canonical transformation, we diagonalize the Hamiltonian to derive the free energy of the system. In section III, we present our results with a discussion. Finally, our summary and conclusions are given in section IV.

II. FORMALISM

We consider an interacting three-component Fermi atomic gas trapped in two dimensions. We take the model Hamiltonian of the system as

\[
H = \int d^2 r \left\{ \sum_n \psi_n^\dagger(r) \left[ -\frac{\hbar^2 \nabla_{2D}^2}{2m} - \mu_n \right] \psi_n(r) + \frac{1}{2} \sum_{n \neq n'} U_{nn'} \psi_n^\dagger(r) \psi_n^\dagger(r) \psi_n(r) \psi_{n'}(r) \right\}
\]

where \( r^2 = x^2 + y^2 \), \( \nabla_{2D} \) is the 2D gradient operator and \( U_{nn'} \) is the 2D interaction strength between component \( n \) and \( n' \). The operator \( \psi_n^\dagger(r) \) creates a fermion of mass \( m \) with hyperfine spin \( n = 1, 2, 3 \) at position \( r = (x, y) \). The chemical potential of the \( n \)th component is \( \mu_n \). Notice that we have neglected the interaction between the same components. This is reasonable as we are considering a dilute atomic system, and the interactions are short-range in nature, s-wave scattering channel is dominated over the other scattering channels. By using a mean field decoupling of the interacting terms, the mean field Hamiltonian in the momentum space can be written as,

\[
H_{MF} = \sum_{ij} \psi_i^\dagger A_{ij} \psi_j + \frac{1}{2} \sum_{ij} (\psi_i^\dagger B_{ij} \psi_j + h.c) - \sum_{i \neq j} \left| \Delta_{ij} \right|^2 \frac{U_{ij}}{U_{ij}}
\]

where \( i = k, n \) and \( j = -k, n \). Here we defined the superfluid order parameters \( U_{ij} \) as \( \langle \psi_i \psi_j \rangle = \Delta_{ij} \) and two matrices \( A \) and \( B \),

\[
A = \begin{pmatrix}
\epsilon_1 & 0 & 0 \\
0 & \epsilon_2 & 0 \\
0 & 0 & \epsilon_3
\end{pmatrix}
\]

\[
B = \begin{pmatrix}
0 & \Delta_3 & -\Delta_1 \\
-\Delta_3 & 0 & \Delta_1 \\
\Delta_2 & -\Delta_1 & 0
\end{pmatrix}
\]

where \( \epsilon_n = \hbar^2 k^2 / (2m) - \mu_n \) and \( \Delta_{ij} = \epsilon_{ijk} \Delta_k \). As the mean field Hamiltonian is quadratic in Fermi operators, it can be diagonalized with a canonical transformation to get

\[
H_{MF} = \sum_{n,k} \Lambda_n n_n^\dagger \eta_n + \frac{1}{2} \sum_{n,k} (\epsilon_n - \Lambda_n) - \sum_n \left| \Delta_n \right|^2 U_n
\]

where we use the notation \( U_{ij} = |\epsilon_{ijk}| U_k \). The Fermi operators \( \eta_n \) represent the quasi particles in the system with \( n = 1, 2, 3 \). The quasi particle energies \( \Lambda_n = \sqrt{\lambda_n} \) are given by \( \lambda_n = 2\sqrt{Q} \cos[(\theta + (n - 1)2\pi)/3] - A_k/3 \). The parameter \( \theta = \text{arccos}[\sqrt{\text{RQ}}] \) with \( Q = (3B_k - A_k^2)/9 \) and \( R = (9A_k B_k - 27C_k - 2A_k^3)/54 \). Here we defined \( A_k = -\sum_n \epsilon_n^2 - 2\sum_n \Delta_n^2, B_k = \sum_n \Delta_n^4 + 2\sum_n \Delta_n^2 \epsilon_n + \sum_{n \neq m} \Delta_n \Delta_m + 1/2 \sum_{n \neq m} \epsilon_{nm}^2 + \sum_{n \neq m \neq l} \Delta_n^2 \epsilon_{nl} \) and...
FIG. 2: Superfluid order parameters $\Delta_3$ as a function of temperature. We fixed the binding energies $E_{B1} = 0.1\mu_1$, $E_{B2} = 0.5\mu_1$, $E_{B3} = \mu_1$ and chemical potentials $\mu_2 = 0.8\mu_1$ and $\mu_3 = 0.75\mu_1$.

$C_k = -\langle\sum_n \Delta_n^2 \epsilon_n + \epsilon_1 \epsilon_2 \epsilon_3\rangle^2$. The grand potential of the system $\Omega = -1/(\beta) \ln[Z_G]$ with $Z_G = \text{tr}\{e^{-\beta H_M}\}$ is then given by

$$\Omega = -1/(\beta) \sum_{n,k} \ln[1 + e^{-\beta \Lambda_n}] + \frac{1}{2} \sum_{n,k} (\epsilon_n - \Lambda_n) - \sum_n \frac{|\Delta_n|}{U_n} \quad (6)$$

where $\beta = 1/(k_B T)$ is the inverse temperature and $k_B$ is the Boltzmann constant. As the short range nature of the interaction, the grand potential is diverging so that regularization must be done in standards way by writing $-|\Delta_n|^2/U_n = \sum_k |\Delta_n|^2/(\hbar^2 k^2/m + E_{Bn})$. Here $E_{Bij} = |\epsilon_{ijk}| E_{Bk}$ is the binding energy between two hyperfine spin components $i$ and $j$. Notice that we use the same notation for $E_{Bn}$ as we used for $\Delta_{ij} = \epsilon_{ijk} \Delta_k$ and $U_{ij} = |\epsilon_{ijk}| U_k$. For two dimensions, converting the $\sum_k$ into integral $\int d^2k/(2\pi)^2$ and then by changing the variable by $k^2 = z$, the grand potential can be converted into an one dimensional integral. We numerically perform this integral and numerically minimize the grand potential for the seven parameter space (three chemical potentials, three binding energies and the temperature).

III. RESULTS AND DISCUSSION

In Fig. 2 we plot the free energy landscapes for different temperatures at a selected set of parameters. We choose the binding energy in the $2 - 3$ channel to be small ($E_{B1} = 0.1\mu_1$) so that pairing is not possible in this channel. As a result, we find a second order thermal phase transition as we lower the temperature. As can be seen, at high temperature [Fig. 2(a)], free energy is minimum when both pairing order parameters ($\Delta_2$ and $\Delta_3$) in channels $1 - 3$ and $1 - 2$ are zero. As we lower the temperature, channel $1 - 2$ undergoes pairing and form Bose condensation. This is because the binding energy and average chemical potential in this channel is larger than those of channel $1 - 3$ and $2 - 3$. Further lowering the temperature results more atom pairing and condensation in channel $1 - 2$ giving larger superfluid order parameter $\Delta_3$. In principle, it is possible to have a sequence of second order thermal and first order quantum phase transitions at three different critical temperatures, if one change the interactions or the chemical potentials together with temperature. The reason for this sequence of phase transition is that there are many ways of pairing when various favorable channels are available.

In Fig. 3 we plot the temperature dependence of the superfluid order parameter ($\Delta_3$) in channel $1 - 2$ for chosen values of parameters. We choose the parameters such that pairing is possible only in channel $1 - 2$ so that a single minimum is available in the free energy. As can be seen, the superfluid order parameter continuously increases as one lower the temperature, showing a second order thermal phase transition.

By varying the average chemical potentials and binding energies of the pairing channels at low temperatures, one can control the first order quantum phase transitions
from one superfluid phase to another. As a demonstration, we plot the free energy landscapes in Fig. 3 for a selected set of parameters. Again, we chose the parameters such that the pairing in channel 2–3 is very weak and the corresponding superfluid order parameter \( \Delta_1 \) is zero. When the binding energy in channel 1–2 is larger than that of the channel 1–3, the free energy minimum is at a non-zero value of \( \Delta_3 \) but zero value of \( \Delta_2 \) at the same chemical potentials. However, when the binding energies are equal at equal average chemical potentials, both superfluid order parameters are non-zero and free energy gives many stable stationary points as seen in Fig. 3(b) (now the global minimum is not a point, but a quarter of a circle). The reason for this line of global minimum is the symmetry of the parameter space. By increasing the binding energy in channel 1–3 over the channel 1–2, the minimum free energy pass to the non-zero \( \Delta_2 \) but zero \( \Delta_3 \). Similar first order quantum phase transitions can be seen by controlling the average chemical potentials at fixed and equal binding energies. More generally, by controlling the chemical potentials and binding energies, one can observe not only a series of quantum phase transition, but a phase with multi-component superfluid order (simultaneous existence of three superfluid order parameters). Notice that one can have a re-entrant quantum phase transition by changing both chemical potentials and binding energies simultaneously.

In Fig. 4 we plot superfluid order parameters as a function of the chemical potential of the third component. The binding energies are fixed to be the same for all three channels. As can be seen in figure, fermions pairing occurs in channel 1–2 at smaller \( \mu_3 \). This is because the average chemical potential in this channel is the largest for \( \mu_3 < E_B \). Further, as the average chemical potential is constant, superfluid order parameter \( \Delta_3 \) is constant. For \( \mu_3 > E_B \), average chemical potential in channel 2–3 is the largest and increasing with increasing \( \mu_3 \). As a result, superfluid order parameter \( \Delta_1 \) increases with \( \mu_3 \). In the entire range of \( \mu_3 \), average chemical potential in channel 1–3 is smaller than that of the other channels so that the pairing in this channel is not favorable. As seen in Fig. 4, the superfluid order parameters have large discontinuity which represents a sharp first order quantum phase transition.

At the same chemical potentials and the same interaction strengths of the channels, free energy gives many stable stationary points at which the condition \( \Delta_1^2 + \Delta_2^2 + \Delta_3^2 = C \) is satisfied. The constant \( C \) depends on both the chemical potentials and the interaction strengths (binding energies). As shown in Fig. 5 when the free energy has a minimum, the order parameters represent a surface in order parameter space.

Within our mean field description, we were able to handle only two-body correlations. One needs to go beyond mean field theory to understand the role of three-body correlations in a three-component system. If three atoms can overlap in the same region of space, then the three-body correlations can play a role giving Thomas effect [17] and Efimov effect [18]. Thomas effect is the collapse of a three-body system due to the overlap of atoms. Atom loss in a trap is undoubtedly related to the Thomas effect. In a three component atomic system, collision between a condensed pair and a third species atom can support the Thomas effect. Efimov effect is the accumulation of three-body bound states at strongly interacting limit. These effects are forbidden in two component gases. For sufficiently low densities, these effects are forbidden even in three component systems so that our results are applicable to dilute ultra-cold atomic gases. We discussed the pure 2D limit in this paper, however one can generalize the theory to include the weak atom tunneling between layers as done in Ref. [14] for two component gases.

Our results in two dimensions look qualitatively similar to the ones obtained in three dimensions in Ref. [7]. However, we find that the superfluidity is more sensitive to the parameters in two dimensions than three dimensional systems. As we have seen above, superfluidity is very sensitive to both chemical potentials and the interactions between different pairs. In current experimental setups, Feshbach resonance allows one to control the interactions to different values by tuning the scattering lengths. However, the scattering lengths between different pairs of fermions cannot be controlled independently. Therefore, chemical potential is the suitable parameter to drive the quantum phase transitions. Typically, chemical potentials can be controlled by changing the atomic population in both two dimensions and three dimensions. However, in order to change the chemical potentials this way, one has to start the experiment all over with a different atomic sample. The advantage of using quasi two dimensional system is that one can change the effective chemical potential by controlling the tunneling between layers. This tunneling can be controlled by the laser in-

![FIG. 4: Superfluid order parameters \( \Delta_1/E_B \) (black line) and \( \Delta_3/E_B \) (gray line) as a function of \( \mu_3/E_B \). We fixed the binding energies \( E_{B1} = E_{B2} = E_{B3} = E_B \) and the temperature \( k_B T = E_B/50 \). The chemical potentials are \( \mu_1 = E_B \) and \( \mu_2 = 1.2 E_B \). In the entire range of \( \mu_3, \Delta_2 \) is zero.](image-url)
quantum phase transition can be induced by increasing the average chemical potential or the interaction strength of one channel over the other. If the pairing strengths are equal in all three channels, then it is possible to have three superfluid phases simultaneously. As we have not considered the interaction between condensed pairs, we do not expect the phase separation of superfluid phases in spatially homogenous environments [9]. However in trapped systems, it has been shown that the phase separation of superfluid phases is possible in three dimensions [8].

We speculate that the simultaneous existence of three types of superfluid phases in trapped systems can be detected by standard experimental methods. For example, superfluidity can be demonstrated by the creation of vortices [19] and then distinguished them by probes coupling to each atom types. Alternatively, one can measure the energy gap using radio frequency spectroscopy [20] or measure condensate fraction using the pair projection method [21].

![Image](image_url)

**FIG. 5:** Superfluid order parameters for symmetric parameters. We use the same chemical potentials for the three species $\mu_1 = \mu_2 = \mu_3 = \mu$ and same interaction strengths for the three channels $E_{B1} = E_{B2} = E_{B3} = E_B$. The three surfaces ($\Delta_1^2 + \Delta_2^2 + \Delta_3^2 = \text{constant}$) shown in the order parameter space are for $E_B = 0.5\mu$, 1.2$\mu$, and 2.0$\mu$. We fix the temperature to be $\kappa_0 T = \mu/50$.

**IV. SUMMARY AND CONCLUSIONS**

We studied Fermi superfluidity of an interacting three component system in two dimensions. We used a mean field theory to investigate the behavior of free energy as a function of chemical potentials, binding energies, and the temperature. Depending on the chemical potentials and binding energies, we find a second order thermal phase transition as we lower the temperature. At low temperatures, by controlling the parameters, one can have a series of first order quantum phase transitions and re-entrant phase transitions. These series of phase transitions are associated with pairing between different hyperfine spin components of fermions.

The possible pairing is determined by both average chemical potentials and the interaction strengths of the paring channels. The channel which has largest paring strength forms the superfluid, while the unpaired component form a Fermi sea. At low temperature, first order
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