Dimension Theory of Linear Solenoids

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Abstract

We develop the dimension theory for a class of linear solenoids, which have a "fractal" attractor. We will find the dimension of the attractor, proof formulas for the dimension of ergodic measures on this attractor and discuss the question whether there exists a measure of full dimension.

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1 Introduction

We consider in this article a class of dynamical systems given by piece-wise lineare maps acting on a cube. These dynamical system are very similar to the classical Smale-Williams Solenoids (see [6]), having a one dimensional unstable and a two dimensional stable manifold. As the solenoid the systems we study have a global attractor, which has a complicated "fractal" geometry. Thus we will discuss here the "fractal" dimension of this attractor. In Theorem 3.1 we determine the Hausdorff and Box-counting dimension, using results on the dimension of self-affine sets in the plane found in [9]. In the following we will apply symbolic dynamics and the theory of hyperbolic dynamics to the class of lineare solenoids. In section four we will find a coding of the dynamics through a shift on two symbols on a set of full measure. This allows us to to find a representation of all ergodic measures for the systems as images of shift ergodic measures under the coding map. In section five we will demonstrate the existence of Lyapunov exponents and Lyapunov charts for lineare solenoids with respect to any ergodic measure. This is the background we need to apply the general dimension theory of hyperbolic measures, see [7] and [2]. Using this theory we will show that ergodic measure for our systems are exact dimensional. Moreover we will find a formula for the dimension of ergodic measures in terms of entropy and Lyapunov exponents and the dimension of transversal measures (see Theorem 6.1). For Bernoulli measures this formula yield an explicit expression (see Corollary 6.2) for the dimension in terms of self-similar measure studied in [9]. In the last section of this article we will discuss the question whether there exists an ergodic measure of full dimension, which means that the dimension of the ergodic measure equals the dimension of the attractor. This question is widely open in the dimension theory of dynamical systems. It is of particular interest since ergodic measures of full dimension are of great geometrical signifcants, describing the long term behavior of orbits on the whole attractor in the dimensional theoretical sense. Results of Manning and McClusky [8] show that in the case of horseshoes diffeomorphisms there does not exist an ergodic measure of full dimension in general. One can not maximize the dimension in the stable and in the unstable direction at the same time. In [10] we demonstrate that for generalized Baker's transformations there exists parameter domains for which a measure of full
dimension exists and parameter domains where the dimension of the invariant set can not even be approximated by the dimension of ergodic measures. We observe the same phaenomenon in the case of lineare solenoids. We will show that there are manifolds in the parameter domain where there is a measure of full dimension and manifold where the variational principle of dimension does not hold (see Theorem 7.1). At the end of this paper the reader, who is not familiar with dimension theory, will find an appendix containing a short introduction to this field.

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2 Linear solenoids

Let $\mathbb{W} = [-1, 1]^3$. We consider the following class of piecewise affine maps $f_v : \mathbb{W} \mapsto \mathbb{W}$ given by

$$f_v(x, y, z) = \begin{cases} (2x - 1, \beta_1 y + (1 - \beta_1), \tau_1 z + (1 - \tau_1)) & \text{if } x \geq 0 \\
(2x + 1, \beta_2 y - (1 - \beta_2), \tau_2 z - (1 - \tau_2)) & \text{if } x < 0 \end{cases}$$

where we assume

$$v = (\beta_1, \beta_2, \tau_1, \tau_2) \in (0, 1)^4 \quad \text{and} \quad \tau_1 + \tau_2 < 1.$$ 

Figure 1: The action of $f_v$ on the cube $\mathbb{W}$.

Obvoiusly the maps $f_v$ are invertible and there is a global Attractor for the maps given by

$$\Lambda_v = \text{closure} \left( \bigcap_{n=0}^{\infty} f_v^n(\mathbb{W}) \right).$$

We call the system $(\Lambda_v, f_v)$ a linear Solenoid. We see that this system is quite similar to the classical Smale-Williams Solenoid $(\Delta_{\beta, \tau}, g_{\beta, \tau})$, see [6]. The Smale-Williams Solenoid is constructed by a family of maps $g_{\beta, \tau} : \mathbb{T}^2 \mapsto \mathbb{T}^2$ on the full torus $\mathbb{T}^2 = \mathbb{S}^1 \times \mathbb{D}^2$ defined by

$$g(\phi, y, z) = (2\phi \mod 2\pi, \beta y + 1/2 \cos(2\pi\phi), \tau z + 1/2 \sin(2\pi\phi))$$

with $\beta, \tau \in (0, 1/2)$.

$g_{\beta, \tau}$ has the global attractor

$$\Delta_{\beta, \tau} = \bigcap_{n=0}^{\infty} g_{\beta, \tau}^n(\mathbb{T}^2).$$
Figure 2 The action of $g_{\beta,\tau}$ on the full Torus $T^2$.

In fact the systems $(\Delta_{\beta,\tau}, g_{\beta,\tau})$ and $(\Lambda_v, f_v)$ have similar properties. Both systems are expanding in the first coordinate direction with expansion rate $\log 2$ and contracting in the two other coordinate direction. Moreover both maps are invertible and into with a global attractor. The classical Solenoid is hyperbolic and conjugated to the full Shift on two symbols, see [6]. In section four we will show that our linear Solenoids are up to a set of measure zero as well hyperbolic and conjugated to the full shift on two symbols. The last similarity is that both $\Lambda_v$ and $\Delta_{\beta,\tau}$ have a complicated non smooth geometry. Dimensional theoretical properties of the classical Solenoid where extensity studied, see [1], [16] or [12]. We will develop here the dimension theory for linear solenoids. In the next chapter we will present our results on the dimension of the attractor $\Lambda_v$.

3 Dimension of the attractor

We first give here an simple description of the attractor $\Lambda_v$ using iterated function systems, see [4].

Proposition 3.1 We have

$$\Lambda_v = [-1, 1] \times \Lambda_v^s$$

where $\Lambda_v^s$ is the unique compact set fulfilling

$$\Lambda_v^s = T_{\beta_1,\tau_1}(\Lambda_v^s) \cup T_{\beta_2,\tau_2}(\Lambda_v^s)$$

with $T_{\beta_1,\tau_1}, T_{\beta_2,\tau_2} : [-1, 1]^2 \mapsto [-1, 1]^2$ given by

$$T_{\beta_1,\tau_1}(y, z) = (\beta_1 y + (1 - \beta_1), \tau_1 z + (1 - \tau_1))$$

$$T_{\beta_2,\tau_2}(y, z) = (\beta_2 y + (1 - \beta_2), \tau_2 z + (1 - \tau_2)).$$

Proof. Let $T_1 := T_{\beta_1,\tau_1}$ and $T_2 = T_{\beta_2,\tau_2}$. We have

$$\text{closure}(f_v(W)) = [-1, 1] \times T_1(W) \cup [-1, 1] \times T_2(W)$$

and hence

$$\text{closure}(f_v^n(W)) = [-1, 1] \times \bigcup_{s_1,\ldots,s_n \in \{1,2\}} T_{s_1} \circ T_{s_2} \circ \ldots \circ T_{s_2}(W)$$

Now let

$$\Lambda_v^s = \bigcap_{n=1}^{\infty} \bigcup_{s_1,\ldots,s_n \in \{1,2\}} T_{s_1} \circ T_{s_2} \circ \ldots \circ T_{s_2}(W)$$

By this definition we get

$$\Lambda_v = [-1, 1] \times \Lambda_v^s.$$

Moreover $\Lambda_v^s$ is compact with

$$\Lambda_v^s = T_1(\Lambda_v^s) \cup T_2(\Lambda_v^s)$$
Uniqueness of $\Lambda^s_v$ with this property follows from \cite{5}.

Our results on the dimension of the attractor $\Lambda_v$ is now mainly a consequence of our results on the self-affine sets $\Lambda^s_v$ given in \cite{9} and and \cite{12}. In the following we denote by $\dim_B A$ the box-Counting dimension and by $\dim_H A$ the Hausdorff dimension of a set $A$; we refer to the appendix of this work for the definition of these quantities.

**Theorem 3.1** Let $v = (\beta_1, \beta_2, \tau_1, \tau_2) \in (0,1)^4$ with $\beta_1 + \beta_2 > \tau_1 + \tau_2$.

If $\beta_1 + \beta_2 < 1$ we have

$$\dim_B \Lambda_v = \dim_H \Lambda_v = d + 1$$

where $d$ is the solution of

$$\beta_1^d + \beta_2^d = 1.$$  

If $\beta_1 + \beta_2 \geq 1$ we have

$$\dim_B \Lambda_v = d + 2$$

where $d$ is the solution of

$$\beta_1 \tau_1^d + \beta_2 \tau_2^d = 1.$$  

Moreover for almost all $\beta_1, \beta_2 < 0.649$ we have

$$\dim_H \Lambda_v = \dim_B \Lambda_v.$$  

**Proof.** By proposition 8.1 of the appendix and proposition 3.1 we have

$$\dim_{H/B} \Lambda_v = \dim_{H/B} \Lambda^s_v + 1.$$  

If $\beta_1 + \beta_2 < 1$ we have by example 16.3 of \cite{12} $\dim_B \Lambda_v = \dim_H \Lambda_v = d$.

If $\beta_1 + \beta_2 \geq 1$ we get by theorem III of \cite{9} $\dim_B \Lambda^s_v = d + 2$ and generically $\dim_H \Lambda^s_v = \dim_B \Lambda^s_v$ under the assumption that $\beta_1, \beta_2 < 0.649$.

The condition $\beta_1, \beta_2 < 0.649$ in the last theorem is due to the technique we used in \cite{9}. We do not believe that this condition is essential, also we were not able to omit it. The identity of Box-Counting and Hausdorff dimension in the last statement does not hold in general. In \cite{10} we described numbertheoretical exceptions in the symmetric case $\beta_1 = \beta_2$.

## 4 Shift coding of the Dynamics

We need some notation to introduce a symbolic coding of the dynamics of system $(\Lambda_v, f_v)$.

Let $\Sigma = \{-1,1\}^\mathbb{Z}$ be the Shift space. With the product metric defined by

$$d(s, t) = \sum_{k=-\infty}^{\infty} |s_k - t_k|2^{-|k|}$$
Σ becomes a perfect, totally disconnected and compact metric space; see [3]. The forward shift map \( \sigma \) on \( \Sigma \) is given by \( \sigma((s_k)) = (s_{k+1}) \), the backward shift \( \sigma^{-1} \) is given by \( \sigma((s_k)) = (s_{k-1}) \).

For a sequence \( s \in \Sigma \) and \( \gamma_1, \gamma_2 \in (0, 1) \) we define a map

\[
\hat{\pi}_{\gamma_1, \gamma_2} : \Sigma^+ \rightarrow \left[ \frac{-\gamma_2}{1 - \gamma_1} \frac{\gamma_1}{1 - \gamma_2} \right]
\]

by

\[
\hat{\pi}_{\gamma_1, \gamma_2}(s) = \sum_{k=0}^{\infty} s_k \gamma_2 \bar{\gamma}(s,k) \gamma_1 \bar{\gamma}(s,k)
\]

where

\[
\bar{\gamma}(s,k) = \text{Cardinality}\{ s_i | s_i = -1 \quad i = 1, \ldots, k \}
\]

\[
\bar{\gamma}(s,k) = \text{Cardinality}\{ s_i | s_i = +1 \quad i = 1, \ldots, k \}.
\]

Let \( L_{\gamma_1, \gamma_2} \) be the monoton increasing linear map from \( \left[ \frac{-\gamma_2}{1 - \gamma_1} \frac{\gamma_1}{1 - \gamma_2} \right] \) onto \([-1, 1]\) and let \( \pi_{\gamma_1, \gamma_2} = L_{\gamma_1, \gamma_2} \circ \hat{\pi}_{\gamma_1, \gamma_2} \). Moreover we define the map of the signed dyadic expansion

\[
i : \Sigma \mapsto [-1, 1]
\]

by

\[
i(s) = \sum_{k=1}^{\infty} s_{-k}(1/2)^k.
\]

For \( v = (\beta_1, \beta_2, \tau_1, \tau_2) \in (0, 1)^4 \) we define the coding map

\[
\pi_v : \Sigma \mapsto \Lambda_v
\]

by

\[
\pi_v(s) = (i(s), \pi_{\beta_1, \beta_2}(s), \pi_{\tau_1, \tau_2}(s)).
\]

By this definitions we obviously have:

**Proposition 4.1** \( \pi_v \) is continuous and onto \( \Lambda_v \). Moreover the map is bijective from

\[
\tilde{\Sigma} := \Sigma \setminus \{(s_k)|\exists k_0 \forall k \leq k_0 \in \mathbb{Z} : s_k = 1\} \cup \{(1)\}
\]

onto \( \Lambda_v \) and we have

\[
\forall s \in \tilde{\Sigma} : \pi_v(\sigma^{-1}(s)) = f_v(\pi_v(s)).
\]

We can now represent all ergodic measures of the system \((\Lambda_v, f_v)\) using the coding map \( \pi \). Again we need some notations. Given a compact metric space \( X \) we denote the set of all Borel probability measures on \( X \) by \( M(X) \). With the weak* topology \( M(X) \) becomes a compact, convex and metricable space. If \( T \) is a Borel measurable transformation on \( X \) we call a measure \( \mu \) \( T \)-invariant if

\[
T(\mu) := \mu \circ T^{-1} = \mu.
\]
The set of all invariant measures forms a compact, convex and nonempty subset of \( M(X) \). An invariant measure \( \mu \) is called ergodic if

\[
T^{-1}(B) := B \Rightarrow \mu(B) \in \{0, 1\}
\]

hold for all Borel subsets \( B \) of the space \( X \). The set of all ergodic measures

\[
\mathcal{E}(X, T) := \{ \mu \in M(X) | \mu \text{-ergodic} \}
\]

is nonempty, convex and compact with respect to the weak* topology. It consists of the extreme points of the set of invariant measures. By \( b^p \) for \( p \in (0, 1) \) we denote the Bernoulli measure on \( \Sigma \), which is the product of the discrete measure giving 1 the probability \( p \) and \(-1\) the probability \((1-p)\). The Bernoulli measures are ergodic with respect to forward and backward shifts. Given \( b^p \) on \([-1, 1]^Z\) we define the corresponding Bernoulli measure \( \ell^p \) on \([-1, 1]\) by \( \ell^p = i(b^p) \). For the basic facts in ergodic theory mentioned here we refer to [3], [17] or [6]. Proposition 4.1 directly implies:

**Proposition 4.2** The map

\[
\mu \mapsto \mu_v := \mu \circ \pi_v^{-1}
\]

is an affine homeomorphism from \( M(\Sigma, \sigma) \) onto \( M(\Lambda_v, f_v) \). Moreover \( b^p \) is a product of the Bernoulli measure on \( \Lambda_v \) with \( \ell^p \).

## 5 Hyperbolicity

We will show here that there exists expansion and contraction rates (Lyapunov exponents) on the Solenoid \((\Lambda_v, f_v)\) for a set of full measure with respect to any ergodic measure \( \mu \in \mathcal{E}(\Lambda_v, f_v) \).

**Lemma 5.1** There is a subset \( \Omega_v \subseteq \Lambda_v \) which has full measure for all \( \mu_v \in \mathcal{E}(\Lambda_v, f_v) \) such that \( f_v \) is a bijection on \( \Omega_v \) and \( f_v \) is differentiable for all \( x \in \Omega_v \) with

\[
D_x f_v = \begin{pmatrix} 2 & 0 & 0 \\ 0 & \beta_1 & 0 \\ 0 & 0 & \tau_1 \end{pmatrix} \quad \text{if} \quad y > 0 \quad \text{and} \quad D_x f_v = \begin{pmatrix} 2 & 0 & 0 \\ 0 & \beta_2 & 0 \\ 0 & 0 & \tau_2 \end{pmatrix} \quad \text{if} \quad y < 0.
\]

**Proof.** Denote the singularity \( \{0\} \times [-1, 1]^2 \) of the map \( f_v \) by \( S \) and define the set \( \Omega_v \) by

\[
\Omega_v = \bigcap_{n=-\infty}^{\infty} f_v^n(\mathbb{W} \setminus S).
\]

By definition we have \( f_v(\Omega_v) = \Omega_v \) and since \( f_\theta \) is injective it is in fact a bijection on \( \Omega_v \). Moreover if \((x, y, z) \in \Omega_v \) then \((x, y, z) \not\in S \) and hence \( f_v \) is differentiable and has obviously the derivative that we stated in the lemma. It remains to show that \( \mu_v(\Omega_v) = 1 \).

By elemental calculations we see that

\[
\Omega_v = (\{(x, y, z) \in \Lambda_v | y \neq 1, \quad y \neq -1\} \cup \{(1, 1, 1), (-1, -1, -1)\}) \setminus \bigcup_{n=0}^{\infty} f_v^{-n}(S).
\]
Since $\mu_v$ is invariant and the union in the expression above is disjoint it has zero measure. It remains to show that $\mu_v(\{1\} \times [-1, 1] \times [-1, 1]) = \mu_v(\{(1, 1, 1)\})$ and $\mu_v(\{-1\} \times [-1, 1] \times [-1, 1]) = \mu_v(\{(-1, -1, -1)\})$. But this is obvious since $f_v$ is just a contraction with fixed point $(1, 1, 1)$ resp. $(-1, -1, -1)$ on the sets $\{1\} \times [-1, 1] \times [-1, 1]$ resp. $\{-1\} \times [-1, 1] \times [-1, 1]$.

We now define linear subspaces of $\mathbb{W}$ by

\[
\mathbb{E}^u = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \mathbb{E}^s = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \mathbb{E}^{ss} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \quad \mathbb{E}^{us} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.
\]

Given a Borel measure $\mu$ on $\Sigma$ and $\gamma_1, \gamma_2 \in (0, 1)$ we write

\[
\Xi_{\gamma_1, \gamma_2}^\mu = \mu(\{s \in \Sigma|s_0 = 1\}) \log \gamma_1 + \mu(\{s \in \Sigma|s_0 = -1\}) \log \gamma_2.
\]

**Proposition 5.1** Given $\mu \in \mathcal{E}(\Sigma, \sigma)$ we have for $\mu_v$-almost all $x \in \Lambda_v$

\[
\lim_{n \to \infty} \frac{1}{n} \log \|D_x f_v^n \nabla f_v\| = \log 2 \quad \forall \nabla f_v \in \mathbb{E}^u
\]

If $\Xi_{\beta_1, \beta_2}^\mu \geq \Xi_{\tau_1, \tau_2}^\mu$ then

\[
\lim_{n \to \infty} \frac{1}{n} \log \|D_x f_v^n \nabla f_v\| = \begin{cases} 
\Xi_{\beta_1, \beta_2}^\mu, & \nabla f_v \in \mathbb{E}^s \setminus \mathbb{E}^{ss} \\
\Xi_{\tau_1, \tau_2}^\mu, & \nabla f_v \in \mathbb{E}^{ss}
\end{cases}
\]

If $\Xi_{\beta_1, \beta_2}^\mu \leq \Xi_{\tau_1, \tau_2}^\mu$ then

\[
\lim_{n \to \infty} \frac{1}{n} \log \|D_x f_v^n \nabla f_v\| = \begin{cases} 
\Xi_{\beta_1, \beta_2}^\mu, & \nabla f_v \in \mathbb{E}^s \setminus \mathbb{E}^{us} \\
\Xi_{\tau_1, \tau_2}^\mu, & \nabla f_v \in \mathbb{E}^{us}
\end{cases}
\]

**Proof.** By lemma 5.1 we have for $\mu_v$-almost all $x \in \Lambda_v$ and all $n > 0$

\[
\log \|D_x f_v^n \left( \begin{array}{c} 0 \\ y \\ 0 \end{array} \right) \| = n \log 2 + \log y.
\]

This implies our claim about $\mathbb{E}^u$. Now consider $\mathbb{E}^s$. By lemma 5.1, proposition 4.1 and 4.2 we have for $\mu_v$-almost all $x \in \Lambda_v$ and all $n > 0$

\[
\log \|D_x f_v^n \left( \begin{array}{c} x \\ 0 \\ z \end{array} \right) \| = \log \sqrt{x^{\beta_1 n - \tilde{z}_n(s) + 1} \beta_2^{\tilde{z}_n(s)} + (z \tau_1^{n - \tilde{z}_n(s) + 1} \tau_2^{\tilde{z}_n(s)})^2}.
\]

where $s = (s_k) = \pi_v^{-1}(x)$ and $\tilde{z}_n(s)$ counts the number of entries in the set $\{s_0, s_{-1}, \ldots, s_{-n}\}$ that are $-1$. Applying Birkhoff’s ergodic theorem (see 4.1.2 of [3]) to the functions

\[
f_{ws}(s) = \begin{cases} 
\log \beta_1 & \text{if } s_0 = 1 \\
\log \beta_2 & \text{if } s_0 = -1
\end{cases}, \quad f_{ss}(s) = \begin{cases} 
\log \tau_1 & \text{if } s_0 = 1 \\
\log \tau_2 & \text{if } s_0 = -1
\end{cases}
\]

now yields the desired result.
Proposition 5.1 means that Lyapunov exponents exists almost everywhere for the ergodic systems $(\Lambda_v, f_v, \mu_\theta)$. $E^u$ is the unstable direction with Lyapunov exponent $\log 2$ and $E^s$ is the stable direction with exponent $\Xi_{\beta_1, \beta_2}$ or $\Xi_{\tau_1, \tau_2}$ depending on which quantity is bigger. Accordingly $E^{ss}$ (resp. $E^{ws}$) is the strong stable direction with Lyapunov exponent $\Xi_{\tau_1, \tau_2}$ (resp. $\Xi_{\beta_1, \beta_2}$). In order to guarantee the existence of Lyapunov charts associated with the Lyapunov exponents we have to show that the set of points that does not approach the singularity $S = \{0\} \times [-1, 1] \times [-1, 1]$ with exponential rate has full measure, see [14]. Precisely we have:

**Proposition 5.2** Given $\mu \in \mathcal{E}(\Lambda_v, f_v)$ we have for all $\epsilon > 0$

$$\mu_v(\{x \in \Lambda_v | \exists l > 0 \forall n > 0 : d(f^n(x), S) > (1/l)e^{-\epsilon n}\}) = 1.$$

**Proof.** Fix $\epsilon > 0$. First note that it is sufficient if we show

$$\mu_\theta(\{x \in \Lambda_\theta | \exists (n_k)_{k \in \mathbb{N}} \rightarrow \infty \forall k > 0 : d(f^{n_k}(x), S) \leq e^{-\epsilon n_k}\}) = 0$$

because if we have for a point $x$ that $\exists n_0 \forall n > n_0 \ d(f^n(x), S) > e^{-\epsilon n}$ then there exists $l > 0$ such that $d(f^n(x), S) > (1/l)e^{-\epsilon n} \forall n > 0$.

By proposition 4.2 and the definition of the measure $\mu_v$ this assertion is equivalent to $\mu(N) = 1$ where

$$N := \{s \in \hat{\Sigma} | \exists (n_k)_{k \in \mathbb{N}} \rightarrow \infty \forall k > 0 \ d(\sigma^{-n_k}(s), \tilde{S}) \leq e^{-\epsilon n_k}\}$$

and $\tilde{S}$ is the singularity in the symbolic coding, i.e.

$$\tilde{S} = \{s \in \Sigma | s_{-1} = 1 \quad s_k = -1 \forall k < -1\}.$$

We will prove this. If $s \in N$ we have

$$d(\sigma^{-n_k}(s), \tilde{S}) \leq e^{-\epsilon n_k} \quad \forall k > 0$$

By the definition of the metric $d$ this implies

$$(\sigma^i(s))_{-2} \neq 1 \quad \text{for} \quad i = n_k, \ldots, n_k + \lceil c\epsilon n_k \rceil - 1 \quad \forall k > 0.$$ 

where the constant $c$ is independent of $\epsilon$, $n_k$ and $\hat{S}$. Thus we have:

$$N \subseteq \{s | \exists (n_k)_{k \in \mathbb{N}} \rightarrow \infty \forall k > 0 : (\sigma^i(s))_{-2} \neq 1 \quad i = n_k, \ldots, n_k + \lceil c\epsilon n_k \rceil - 1\}.$$

Applying lemma 7.1. of [15] for the ergodic system $(\Sigma, \sigma, \mu)$ we obtain $\mu(N) = 0$. □
6 Dimension formulae for ergodic measures

Our results in the last section demonstrate that we may apply the general dimension theory for hyperbolic systems to the linear solenoids \((\Lambda_v, f_v, \mu_v)\). By proposition 5.1 and proposition 5.2 our systems fall into the class of generalized hyperbolic attractors in the sense of Schmeling and Troubetzkoy [14]. Usually the dimension theory of ergodic measures is stated in the context of \(C^2\)-diffeomorphisms in order to guarantee the existence of Lyapunov exponents and charts. But invertibility and the existence of Lyapunov exponents and charts almost everywhere is enough to apply this theory. We refer to section 4 of [14] for this fact.

First we define here stable partitions \(W^s\) and unstable partitions \(W^u\) of \(W\) by the partition elements

\[
W^s(x) = \{x\} \times [-1,1] \times [-1,1] \quad W^u(x) = [-1,1] \times \{y\} \times \{z\}
\]

where \(x = (x, y, z) \in \Lambda_v\). Given \(\mu_v \in \mathcal{E}(\Lambda_v, f_v)\) we have conditional measures \(\mu^s_v(x)\) on the partition \(W^s\) and conditional measures \(\mu^u_v(x)\) on the partition \(W^u\). These measures are unique \(\mu_v\)-almost everywhere fulfilling the relations:

\[
\mu_v(B) = \int \mu^s_v(x)(B \cap W^s(x))d\mu_v \quad \text{resp.} \quad \mu_v(B) = \int \mu^u_v(x)(B \cap W^u(x))d\mu_v
\]

for all Borel sets \(B\) in \(W\). We refer to [13] for information about conditional measures on measurable partitions.

To formulate our next theorem let us denote the entropy of an ergodic measure \(\mu\) by \(h(\mu)\). We recommend [17] for an introduction to theory of this invariant. Moreover we denote the dimension of a measure by \(\dim \mu\), so \(\mu\) is exact dimensional. In the end of the appendix the reader finds an introduction of this quantity.

Applying the dimension theory of hyperbolic systems by Barreira, Schmeling and Pesin [2] and Ledrappier Young [7] to the system \((\Lambda_v, f_v, \mu_v)\) we obtain the following theorem.

**Theorem 6.1** For all \(\mu \in \mathcal{E}(\Sigma, \sigma)\) the ergodic measures \(\mu_v \in \mathcal{E}(\Lambda_v, f_v)\) and the conditional measures \(\mu^s_v\) and \(\mu^u_v\) are exact dimensional with

\[
\dim \mu_v = \dim \mu^u_v + \dim \mu^s_v
\]

Moreover we have

\[
\dim \mu^u_v = \frac{h(\mu)}{\log 2}
\]

and

\[
\dim \mu^s_v = \frac{h(\mu)}{\Xi^{\mu}_{\beta_1, \beta_2}} + (1 - \frac{\Xi^{\mu}_{\beta_1, \beta_2}}{\Xi^{\mu}_{\gamma_1, \gamma_2}}) \dim \text{pr}_y(\mu_v)
\]

if \(\Xi^{\mu}_{\beta_1, \beta_2} \geq \Xi^{\mu}_{\gamma_1, \gamma_2}\), resp.

\[
\dim \mu^s_v = \frac{h(\mu)}{\Xi^{\mu}_{\beta_1, \beta_2}} + (1 - \frac{\Xi^{\mu}_{\beta_1, \beta_2}}{\Xi^{\mu}_{\beta_1, \beta_2}}) \dim \text{pr}_z(\mu_v)
\]

if \(\Xi^{\mu}_{\beta_1, \beta_2} < \Xi^{\mu}_{\gamma_1, \gamma_2}\). Here \(\text{pr}\) denotes the projection of the measure on second resp. third coordinate axis.
Proof. Exact dimensionality of the measures follows directly from [3] given our results in section five. The dimension formula for \( \mu^u \) follows directly from theorem \( C' \) of [7]. For the second formula we need an additional argument. If \( \Xi_{\beta_1, \beta_2} \geq \Xi_{\tau_1, \tau_2} \)

\[
\mathbb{W}^{ss}(x) = \{x\} \times \{y\} \times [-1, 1]
\]

forms the strong stable partition. We have conditional measures \( \mu^{ss}_v(x) \) on \( \mathbb{W}^{ss} \). These measures are unique \( \mu_v \)-almost everywhere fulfilling the relation:

\[
\mu_v(B) = \int \mu^{ss}_v(x)(B \cap \mathbb{W}^{ss}(x))d\mu_v
\]

for all Borel sets \( B \) in \( \mathbb{W} \). From the uniqueness of the conditional measures we have for \( \mu_v \)-almost all \( x \)

\[
\mu^s_v(x)(B) = \int \mu^{ss}_v(x)(B \cap \mathbb{W}^{ss}(x))d\mu_v
\]

for all Borel sets \( B \) in \( W^s(x) \). This statement means that the transversal measures in the sense of [7] of the nested partitions \( \mathbb{W}^s \) and \( \mathbb{W}^{ss} \) are given by \( p_{\mu_v} \mu_v \). Now the second formula follows again from theorem \( C' \) of [7]. The third formula is proved the same way just noticing that the strong stable partition is given by

\[
\mathbb{W}^{ss}(x) = \{x\} \times [-1, 1] \times \{z\}
\]

in this case. \( \square \)

The formula for the conditional measures \( \mu^s_v \) in theorem 6.1 is known in the dimension theory as Ledrappier-Young formula. For Bernoulli measures \( b^p \in M(\Sigma, \sigma) \) we get by theorem 6.1 the following explicit dimension formulas for the measures \( b^p_v \in M(\Lambda_v, f_v) \).

**Corollary 6.1**

\[
\dim b^p_v = \frac{p \log p + (1 - p) \log(1 - p)}{\log 2} + \frac{p \log p + (1 - p) \log(1 - p)}{p \log \tau_1 + (1 - p) \log \tau_2} + (1 - \frac{p \log \beta_1 + (1 - p) \log \beta_2}{p \log \tau_1 + (1 - p) \log \tau_2}) \dim \pi_{\beta_1, \beta_2}(b^p_v).
\]

if \( p \log \beta_1 + (1 - p) \log \beta_2 \geq p \log \tau_1 + (1 - p) \log \tau_2 \)

\[
\dim b^p_v = \frac{p \log p + (1 - p) \log(1 - p)}{\log 2} + \frac{p \log p + (1 - p) \log(1 - p)}{p \log \beta_1 + (1 - p) \log \beta_2} + (1 - \frac{p \log \tau_1 + (1 - p) \log \tau_2}{p \log \beta_1 + (1 - p) \log \beta_2}) \dim \pi_{\tau_1, \tau_2}(b^p_v).
\]

if \( p \log \beta_1 + (1 - p) \log \beta_2 < p \log \tau_1 + (1 - p) \log \tau_2 \).

**Proof.** It is well known in ergodic theory that

\[
h(b^p) = -(p \log p + (1 - p) \log(1 - p))
\]
see [3]. Furthermore we obviously have
\[ \Xi_{\gamma_1, \gamma_2}^{b^\nu} = (p \log \gamma_1 + (1 - p) \log \gamma_2) \]
Thus it remains to show \( pr_y(b^\nu) = \pi_{\beta_1, \beta_2}(b^\nu) \) (resp. \( pr_z(b^\nu) = \pi_{\tau_1, \tau_2}(b^\nu) \)) but this is immediate from the product property of Bernoulli measures and the definition of the coding map \( \pi_v \) in section four.

The self similar Bernoulli measures \( \pi_{\beta_1, \beta_2}(b^\nu) \) resp. \( \pi_{\tau_1, \tau_2}(b^\nu) \) where extensively studied in [9]. We have results on absolut continuity, singularity and the dimension of this measures.

7 Measures of full Dimension

In this section we will ask the question whether there exists ergodic measures of full dimension for linear solenoids. Our result is that in general such a measure does not exist, the dimension of an attractor can not even be approximated by the dimension of ergodic measures. On the other hand we will proof that under the assumption of certain symmetries of the system the equal weighted Bernoulli measure on the attractor has full dimension. Our theorem is a consequence of both our results on the dimension of the attractor in section three and our results on the dimension of ergodic measures in section 6.

**Theorem 7.1** For all \( v = (\beta_1, \beta_2, \tau_1, \tau_2) \in (0, 1) \) with \( \tau_1 + \tau_2 < \beta_1 + \beta_2 < 1 \) we have
\[ \dim b_v^{0.5} = \dim_B \Lambda_v = \dim_H \Lambda_v \]
if \( \beta_1 = \beta_2 \) and
\[ \sup \{ \dim \mu | \mu \in \mathcal{E}(\Lambda_v, f_v) \} < \dim_B \Lambda_v = \dim_H \Lambda_v \]
if \( \beta_1 \neq \beta_2 \).

For almost all \( \beta_1, \beta_2 \in (0, 0.649) \) with \( \beta_1 + \beta_2 \geq 1 \) and all \( \tau_1, \tau_2 \in (0, 1) \) with \( \tau_1 + \tau_2 < \beta_1 + \beta_2 \) we have
\[ \dim b_v^{0.5} = \dim_B \Lambda_v = \dim_H \Lambda_v \]
if \( \log \tau_2 2\beta_2 = \log \tau_1 2\beta_1 \) and
\[ \sup \{ \dim \mu | \mu \in \mathcal{E}(\Lambda_v, f_v) \} < \dim_B \Lambda_v = \dim_H \Lambda_v \]
if \( \log \tau_2 2\beta_2 \neq \log \tau_1 2\beta_1 \).

**Proof.** First assume \( \tau_1 + \tau_2 < \beta_1 + \beta_2 < 1 \) and let \( d \) be the solution of \( \beta_1^d + \beta_2^d = 1 \). In the case \( \beta_1 = \beta_2 = 1/2 \) we have \( \beta_1^d = \beta_2^d = 1/2 \). It is well know in dimension theory that \( \dim \pi_{\beta_1, \beta_2}(b^{0.5}) = d \), see for instance chapter 5 of [12] (the result follows from 15.4 of this work). Thus we get by Corollary 6.1
\[ \dim b_v^{0.5} = \frac{\beta_1^d \log \beta_1^d + \beta_2^d \log \beta_2^d}{\beta_1^d \log \tau_1 + \beta_2^d \log \tau_2} + (1 - \frac{\beta_1^d \log \beta_1 + \beta_2^d \log \beta_2}{\beta_1^d \log \tau_1 + \beta_2^d \log \tau_2}) d = d + 1 \]
But this is by theorem 3.1 the dimension of the attractor $\Lambda_v$. In the case $\beta_1 \neq \beta_2$ we know that

$$\dim \pi_{\beta_1, \beta_2}(b^{0.5}) = \frac{\log 2}{-(0.5 \log \beta_1 + 0.5 \log \beta_2)} < d,$$

see again 15.4 of [12]. Consider a sequence $\mu_n \in \mathcal{E}(\Sigma, \sigma)$ with $\mu_n \rightarrow b^{0.5}$. By theorem 6.1 we get

$$\lim_{n \rightarrow \infty} \dim_H(\mu_n)^s < d + \frac{-2 \log 2 - \log \beta_1^d - \log \beta_2^d}{\log \tau_1 + \log \tau_2} < d$$

We thus see that there is a weak* neighborhood $U$ of $b^{0.5}$ with

$$\sup \{ \dim \mu_v^s | \mu \in U \} < d$$

On the other hand it is well known in ergodic theory that

$$\sup \{ h(\mu) | \mu \in \mathcal{E}(\Sigma, \sigma) \setminus U \} < \log 2$$

and together we get by theorem 3.1

$$\sup \{ \dim \mu_v | \mu \in \mathcal{E}(\Sigma, \sigma) \} < d + 1$$

Our claim now follows from theorem 3.1 and proposition 4.2.

Now assume $\tau_1 + \tau_2 < \beta_1 + \beta_2$ and $\beta_1 + \beta_2 \geq 1$ and let $d$ be the solution of $\beta_1 \tau^d + \beta_2 \tau^d = 1$. In the case $\log_{\tau_1} 2 \beta_2 = \log_{\tau_2} 2 \beta_1$ we have $\beta_1 \tau_1^d = \beta_2 \tau_2^d = 1/2$. From [9] we know that for almost all $\beta_1, \beta_2 \in (0, 0.649)$ we have $\dim \pi_{\beta_1, \beta_2}(b^{0.5}) = 1$. Thus we get by Corollary 6.1

$$\dim_H b^{0.5} = 2 + \frac{\beta_1 \tau_1^d \log \beta_1 \tau_1^d + \beta_2 \tau_2^d \log \beta_2 \tau_2^d - (\beta_1 \tau_1^d \log \beta_1 + \beta_2 \tau_2^d \log \beta_2)}{\beta_1 \tau_1^d \log \tau_1 + \beta_2 \tau_2^d \log \tau_2} = 2 + \frac{\beta_1 \tau_1^d \log \tau_1^d + \beta_2 \tau_2^d \log \tau_2^d}{\beta_1 \tau_1^d \log \tau_1 + \beta_2 \tau_2^d \log \tau_2} = d + 2.$$

But this is by theorem 3.1 the dimension of the attractor $\Lambda_v$. Now let the $\log_{\tau_2} 2 \beta_2 \neq \log_{\tau_1} 2 \beta_1$. We get by theorem 6.1 the following upper estimate

$$\dim_H \mu_v^s \leq 1 - \frac{h_1(\mu) + \Xi_{\beta_1, \beta_2}}{\Xi_{\tau_1, \tau_2}}$$

for all $\mu \in \mathcal{E}(\Sigma, \sigma)$. If $\mu_n \rightarrow b^{0.5}$ this yield

$$\lim_{n \rightarrow \infty} \dim_H(\mu_n)^s \leq 1 - \frac{\log 2 + 0.5 \log \beta_1 + 0.5 \log \beta_2}{0.5 \log \tau_1 + 0.5 \log \tau_2} < d + 1.$$

We thus again see that there is a weak* neighborhood $U$ of $b^{0.5}$ with

$$\sup \{ \dim \mu_v^s | \mu \in U \} < d + 1$$

Now our claim follows in the same way as in the case $\beta_1 + \beta_2 < 1$. \hfill \square

Clearly the statement of theorem 7.1 holds as well if we interchange the role of $\beta$ and $\tau$. At the end of this work we again remark that it should be possible to replace the bound 0.649 in theorem 7.1 by 1 using new ideas on continuity of self-similar Bernoulli measures $\pi_{\beta_1, \beta_2}(b^{0.5})$ in the case $\beta_1 + \beta_2 \geq 1$. 

12
Appendix: General facts in dimension theory

We will here first define the most important quantities in dimension theory and collect some basic facts we need. We refer to the book of Falconer [4] and the book of Pesin [12] for a more detailed discussion of dimension theory.

Let $Z \subseteq \mathbb{R}^q$. We define the $s$-dimensional Hausdorff measure $H^s(Z)$ of $Z$ by

$$H^s(Z) = \lim_{\lambda \to 0} \inf \left\{ \sum_{i \in I} (\text{diam}U_i)^s | Z \subseteq \bigcup_{i \in I} U_i \text{ and } \text{diam}(U_i) \leq \lambda \right\}.$$

The Hausdorff dimension $\dim_H Z$ of $Z$ is given by

$$\dim_H Z = \sup \{ s | H^s(Z) = \infty \} = \inf \{ s | H^s(Z) = 0 \}.$$

Let $N_\varepsilon(Z)$ be the minimal number of balls of radius $\varepsilon$ that are needed to cover $Z$. We define the upper box-counting dimension $\overline{\dim}_B$ resp. lower box-counting dimension $\underline{\dim}_B$ of $Z$ by

$$\overline{\dim}_B Z = \lim_{\varepsilon \to 0} \frac{\log N_\varepsilon(Z)}{-\log \varepsilon} \quad \underline{\dim}_B Z = \lim_{\varepsilon \to 0} \frac{\log N_\varepsilon(Z)}{-\log \varepsilon}.$$

**Proposition 8.1** If $Z \subseteq \mathbb{R}^q$ and $I \subseteq \mathbb{R}$ is an interval then $\dim_H (Z \times I) = \dim_H + 1$ and $\dim_B (Z \times I) = \dim_B + 1$ holds for both upper and lower dimension.

The statement for the Hausdorff dimension follows from proposition 7.4. of [4] and the statement for the box-counting dimension is easy to see using 3.1. of [4].

Now let $\mu$ be a Borel probability measure on $\mathbb{R}^q$. We define the dimensional theoretical quantities for $\mu$ by

$$\dim_H \mu = \inf \{ \dim_H Z | \mu(Z) = 1 \}$$

and

$$\overline{\dim}_B \mu = \lim_{\rho \to 0} \inf \{ \overline{\dim}_B Z | \mu(Z) \geq 1 - \rho \}.$$

We introduce one more notion of dimension for a measure $\mu$. The upper local dimension $\overline{d}(x, \mu)$ resp. lower local dimension $\underline{d}(x, \mu)$ of the measure $\mu$ in a point $x$ is defined by

$$\overline{d}(x, \mu) = \lim_{\varepsilon \to 0} \frac{\mu(B_\varepsilon(x))}{\log \varepsilon} \quad \underline{d}(x, \mu) = \lim_{\varepsilon \to 0} \frac{\mu(B_\varepsilon(x))}{\log \varepsilon}.$$

The relations between the local dimension and the other notion of dimension of measures are described in the following theorem:

**Theorem 8.1** Let $\mu$ be a Borel probability measure on $\mathbb{R}^q$. If

$$\overline{d}(x, \mu) = \underline{d}(x, \mu) = c$$

almost for holds for almost all $x \in \mathbb{R}^q$ we have

$$\dim_H \mu = \dim_B \mu = c.$$ 

A proof of this theorem is contained in the work of Young [18]. If the conditional in this theorem holds, the measure $\mu$ is called **exact dimensional** and the common value of the dimensions is denoted by $\dim \mu$ and maybe called the fractal dimension of the measure.
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