Turán numbers of hypergraph trees

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Abstract

An \( r \)-graph is an \( r \)-uniform hypergraph tree (or \( r \)-tree) if its edges can be ordered as \( E_1, \ldots, E_m \) such that \( \forall i > 1 \exists \alpha(i) < i \text{ such that } E_i \cap \left( \bigcup_{j=1}^{i-1} E_j \right) \subseteq E_{\alpha(i)} \). The Turán number \( \text{ex}(n, \mathcal{H}) \) of an \( r \)-graph \( \mathcal{H} \) is the largest size of an \( n \)-vertex \( r \)-graph that does not contain \( \mathcal{H} \). A cross-cut of \( \mathcal{H} \) is a set of vertices in \( \mathcal{H} \) that contains exactly one vertex of each edge of \( \mathcal{H} \). The cross-cut number \( \sigma(\mathcal{H}) \) of \( \mathcal{H} \) is the minimum size of a cross-cut of \( \mathcal{H} \). We show that for a large family of \( r \)-graphs (largest within a certain scope) that are embeddable in \( r \)-trees, \( \text{ex}(n, \mathcal{H}) = (\sigma - 1)n^r - 1 + o(n^r - 1) \) holds, and we establish structural stability of near extremal graphs. From stability, we establish exact results for some subfamilies.

1 Introduction, reducible and embeddable hypertrees

In this paper, \( r \)-graphs refer to \( r \)-uniform hypergraphs. We will use \( \mathcal{F} \subseteq \binom{V}{r} \) to indicate that \( \mathcal{F} \) is an \( r \)-graph on the set \( V \), usually we take \( V = [n] := \{1, \ldots, n\} \). Given an \( r \)-graph \( \mathcal{H} \) and a positive integer \( n \), the Turán number \( \text{ex}(n, \mathcal{H}) \) is the largest size of an \( r \)-graph on \( n \) vertices not containing \( \mathcal{H} \) as a subgraph. While the study of hypergraph Turán numbers is a notoriously difficult area of extremal combinatorics there have been active developments in recent years concerning “tree-like” hypergraphs.

The classic Erdős-Ko-Rado theorem \cite{EK} determines the maximum size of an \( r \)-graph not containing two disjoint edges (i.e., a matching of size 2). See \cite{G, Gr, Ho, M, Pf, S} for recent work. Note that a matching is an \( r \)-tree. There are many theorems and conjectures inspired by and/or generalize the Erdős-Ko-Rado theorem, including results on set systems not containing \( d \)-clusters and their generalizations \cite{CM, Ho, Lo, M, S} and on set systems not containing \( d \)-simplex or strong \( d \)-simplex \cite{C, Ho, Lo}.

An \( r \)-tree \( \mathcal{H} \) is a tight \( r \)-tree if its edges can be ordered as \( E_1, \ldots, E_m \) such that \( \forall i > 1 \exists \alpha(i) < i \) one has \( E_i \cap \left( \bigcup_{j<i} E_j \right) \subseteq E_{\alpha(i)} \) and \( |E_i \cap E_{\alpha(i)}| = r - 1 \) (hence \( |E_i \setminus \left( \bigcup_{j<i} E_j \right)| = 1 \)).

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Conjecture 1.1 (Erdős-Sós for graphs and Kalai 1984 for \( r \geq 3 \)) Let \( r \geq 2 \) and \( \mathcal{H} \) a tight \( r \)-tree on \( v \) vertices. Then
\[
\text{ex}(n, \mathcal{H}) = \frac{v - r}{r} \binom{n}{r - 1} + o(n^{r-1}).
\]

To this date the conjecture was verified only for star-shaped tight \( r \)-trees by Frankl and Füredi [11]. While Kalai’s conjecture is wide open, asymptotically tight or even exact results have been obtained concerning some families of reducible hypergraph trees and hypergraphs that are embeddable in them [2, 16, 17, 18, 27, 28]. We say that a hypergraph \( \mathcal{H} \) is \( k \)-reducible if each edge of \( \mathcal{H} \) contains at least \( k \) degree 1 vertices.

A hypergraph \((V(\mathcal{H}), \mathcal{H})\) is said to be embeddable in a hypergraph \((V(\mathcal{G}), \mathcal{G})\) if it is a subgraph of it, more precisely there is a mapping \( f : V(\mathcal{H}) \rightarrow V(\mathcal{G}) \) such that \( f(E) \in \mathcal{G} \) for all \( E \in \mathcal{H} \).

Example 1.2 (An \( r \)-graph embeddable in an \( r \)-tree may not be an \( r \)-tree itself)
The 3-uniform linear cycle of length \( m \) is a hypergraph \( \mathcal{H} := \{a_ia_{i+1}b_i : 0 \leq i \leq m - 1\} \) (subscripts taken mod \( m \)). It is not a 3-tree though it is embeddable in a 3-tree \( \{a_0a_1a_2 : 1 \leq i \leq m - 2\} \cup \mathcal{H} \).

It is easy to see that \( \text{ex}(n, \mathcal{H}) \leq (p - r)(\binom{n}{r - 1}) \) for any \( r \)-tree on \( p \) vertices (see Proposition 5.4) and so the same holds for any hypergraph \( \mathcal{H} \) embeddable in an \( r \)-tree. On the other hand, \( \binom{n - 1}{r - 1} \leq \text{ex}(n, \mathcal{H}) \) for any \( r \)-graph with \( \bigcap_{F \in \mathcal{H}} F = \emptyset \).

In this paper, we are interested to determine Turán numbers asymptotically in this Erdős-Ko-Rado zone, i.e., when \( \text{ex}(n, \mathcal{H}) = \Theta(n^{r-1}) \). We substantially extend and generalize most of the recent results by asymptotically determining \( \text{ex}(n, \mathcal{H}) \) for all graphs \( \mathcal{H} \) that are embeddable in a 2-reducible \( r \)-tree, when \( r \geq 4 \). We also obtain structural stability of near extremal graphs and use this to determine the exact value of \( \text{ex}(n, \mathcal{H}) \) for certain graphs. We describe these results in details in Section 4.

Let us note that simple \( r \)-graphs that are embeddable in \( r \)-trees are 1-degenerate (i.e. every subgraph has a vertex of degree at most 1). Answering a question of Erdős, Füredi [15] showed that \( \text{ex}(n, \mathcal{G}) = \Theta(n^2) \) for the hypergraph \( \mathcal{G} := \{123, 124, 356, 456\} \). Note that \( \mathcal{G} \) is not 1-degenerate. Furthermore, \( \mathcal{G} \) is 2-regular. See [32, 33, 34], and [22] for some related work. In general, there are many \( r \)-graphs \( \mathcal{H} \) with \( \text{ex}(n, \mathcal{H}) = \Theta(n^{r-1}) \) that are not embeddable in \( r \)-trees. There is a lot more to be done concerning Turán type problems for such \( r \)-graphs \( \mathcal{H} \).

2 Definitions and Notation
A hypergraph \( \mathcal{H} = (V, \mathcal{E}) \) is a finite set \( V \) (called vertices) and a collection \( \mathcal{E} \) of subsets of \( V \) (the edge-set of \( \mathcal{H} \)). We do not allow multiple edges (we call these simple hypergraphs) unless otherwise stated. Many times we identify a simple hypergraph by its edge-set, and write about hypergraph \( \mathcal{E} \).

A hypergraph \( \mathcal{G} \) (with multiple copies of the same edge allowed) is a hypergraph tree if its edges can be ordered as \( E_1, E_2, \ldots, E_m \) such that \( \forall i > 1, \) there exists \( \alpha(i) < i \) such that \( E_i \cap (\bigcup_{j \leq \alpha(i)} E_j) \subseteq E_{\alpha(i)} \). Even though there may exist more than one edge that could serve as \( E_{\alpha(i)} \), we will always implicitly fix a choice in advance so that \( \alpha \) is a function. We call \( E_{\alpha(i)} \) the parent of \( E_i \). We call the ordering \( E_1, \ldots, E_m \) a tree-defining ordering of \( \mathcal{G} \). The relation \( \alpha(i) < i \) defines a partial order on \([m]\). It is not hard to verify that any linear extension of this order (i.e., a permutation \( \pi : [m] \rightarrow [m] \) with \( \pi(\alpha(i)) < \pi(i) \) for all \( i \geq 2 \)) \( E_{\pi(1)}, \ldots, E_{\pi(m)} \) is also a tree-defining ordering of \( \mathcal{G} \).
Suppose that $G$ is hypergraph tree defined by the sequence $E_1, \ldots, E_m$. Let $G'$ be the corresponding simple hypergraph. Then it is a hypergraph tree as well, as it can be seen from the list $E'_1, E'_2, \ldots, E'_p$ obtained by keeping only one copy, namely the first appearance, of each edge. Due to this, we do not always explicitly distinguish between a hypergraph tree in which duplicated edges are allowed and one in which there is no duplicated edge. An $r$-uniform hypergraph tree is also called an $r$-tree. An $r$-tree is tight if $\forall i > 1, |E_i \cap E_{\alpha(i)}| = r - 1$ or equivalently $|E_i \setminus (\bigcup_{j<i} E_j)| = 1$.

If a hypergraph $H$ is a subgraph of another hypergraph $G$ then we say that $H$ is embedded/embeddable in $G$. As we have seen in Example 1.2 a hypergraph embeddable in a hypergraph tree may not be a hypergraph tree itself.

An $r$-graph $H$ is $r$-partite if its vertex set can be partitioned into $r$ sets $X_1, \ldots, X_r$ such that each edge of $H$ contains exactly one vertex from each $X_i$. We call such a partition compatible with $H$ and the parts $X_i$’s are the color classes of the $r$-partition. The following can be easily verified using induction.

**Proposition 2.1 (r-trees are r-partite)** Every $r$-tree is $r$-partite. Every tight $r$-tree has a unique compatible $r$-partition up to the permutation of color classes. \hfill $\square$

Given a hypergraph $H$, a set $S$ of vertices is a vertex cover of $H$ if $S$ contains at least one vertex of each edge of $H$. A vertex cover $S$ of $H$ is called a cross-cut of $H$ if it contains exactly one vertex of each edge of $H$. Every hypergraph has a vertex cover (if $\emptyset \notin H$). Not every hypergraph has a cross-cut. However, every $r$-partite $r$-graph $H$ has at least one cross-cut. Namely, every color class of an $r$-partition of $H$ is a cross-cut of $H$. We let $\tau(H)$ denote the minimum size of a vertex cover of $H$ and call it the vertex cover number of $H$. If $H$ has cross-cuts, then we let $\sigma(H)$ denote the minimum size of a cross-cut of $H$ and call it the cross-cut number of $H$.

Let $F$ be a hypergraph on $V = V(F)$. We define the $p$-shadow of $F$ to be

$$\partial_p(F) := \{D : |D| = p, \exists F \in F, D \subseteq F\}.$$ 

The Lovász’ [29] version of the Kruskal-Katona theorem states that if $F$ is an $r$-graph of size $|F| = \binom{x}{r}$, where $x \geq r - 1$ is a real number, then for all $p$ with $1 \leq p \leq r - 1$ one has

$$|\partial_p(F)| \geq \binom{x}{p}. \tag{1}$$

Given $D \subseteq V(F)$, the degree $\deg_F(D)$ of $D$ in $F$ is defined as

$$\deg_F(D) := |\{F : F \in F, D \subseteq F\}|.$$

Given an $r$-graph $F$ and integer $i$ with $1 \leq i \leq r - 1$ let

$$\delta_i(F) := \min\{\deg_F(D) : D \in \partial_i(F)\}.$$

A family of sets $F_1, \ldots, F_s$ is said to form an $s$-star, a sunflower, or $\Delta$-system of size $s$ with kernel $D$ if $F_i \cap F_j = D$ for all $1 \leq i < j \leq s$ and $\forall i \in [s]$, $F_i \setminus D \neq \emptyset$. The sets $F_1, \ldots, F_s$ are called the petals of this $s$-star. Note that $D = \emptyset$ is allowed. Let $\mathcal{L}_p^r$ denote the $r$-uniform sunflower with a single vertex in the kernel and $p$ petals. We call $\mathcal{L}_p^r$ an $r$-uniform linear star with $p$ edges.
The kernel degree \( \deg_F^s(D) \) of \( D \) in \( F \) is defined as

\[
\deg_F^s(D) := \max\{s : F \text{ contains an } s\text{-star with kernel } D\}.
\]

Given a positive integer \( s \), we define the kernel graph \( \text{Ker}_s(F) \) of \( F \) with threshold \( s \) to be

\[
\text{Ker}_s(F) := \{D \subseteq V(F) : \deg_F^s(D) \geq s\}.
\]

For each \( 1 \leq p \leq r - 1 \), the \( p \)-kernel graph \( \text{Ker}_s^{(p)}(F) \) of \( F \) with threshold \( s \) is defined to be

\[
\text{Ker}_s^{(p)}(F) := \{D \subseteq V(F) : |D| = p, \deg_F^s(D) \geq s\}.
\]

The following fact follows easily from the definition of \( \deg_F^s(D) \) and will be used frequently.

Given a set \( Y \). If \( \deg_F^s(D) > |Y| \) then \( \exists F \in F \) such that \( D \subseteq F \) and \( (F \setminus D) \cap Y = \emptyset \). \hspace{1cm} (2)

Given a hypergraph \( F \) and a vertex \( x \in V(F) \), let \( L_F(x) := \{F \setminus \{x\} : x \in F \in F\} \). We call \( L_F(x) \) the link graph of \( x \) in \( F \). Given a set \( A \subseteq V(F) \), let \( L_F(A) := \bigcap_{a \in A} L_F(x) \). In other words,

\[
L_F(A) = \{D \subseteq V(F) \setminus A : \forall a \in A, D \cup a \in F\}.
\]

We call \( L_F(A) \) the common link graph of \( A \) in \( F \).

Given two hypergraphs \( A \) and \( B \), the product of \( A \) and \( B \) is defined as

\[
A \times B := \{A \cup B : A \in A, B \in B\}.
\]

If \( G \) is an \( r \)-graph with \( V = V(G) \), then its complement \( \overline{G} \) is the \( r \)-graph on \( V \) with edge set \((V) \setminus G\).

Given an \( r \)-graph \( G \), and set \( S \subseteq V(G) \), the trace of \( G \) on \( S \), denoted by \( G|_S \) is the hypergraph with edge set \( \{E \cap S : E \in G\} \) (after eliminating resulting duplicated edges). Let \( G - S := G|_{V(G) - S} \).

3 Lower bounds, \( \sigma \)-tight families, and \( \tau \)-perfect families

In this section, we present two lower bound constructions on \( \text{ex}(n, \mathcal{H}) \); one based on the vertex cover number of \( \mathcal{H} \) and the other based on the cross-cut number of \( \mathcal{H} \).

Let \( n, r, t \) be positive integers. Define

\[
\mathcal{S}^r_{n,t} := \{F : F \in \binom{[n]}{r}, F \cap [t] \neq \emptyset\}
\]

and

\[
\mathcal{C}^r_{n,t} := \{F : F \in \binom{[n]}{r}, |F \cap [t]| = 1\} = \binom{[t]}{1} \times \binom{[n] \setminus [t]}{r - 1}.
\]

So, \( \mathcal{S}^r_{n,t} \) consists of all the \( r \)-sets in \([n]\) intersecting a given \( t \)-set and \( \mathcal{C}^r_{n,t} \) consists of all the \( r \)-sets in \([n]\) intersecting a given \( t \)-set in exactly one vertex, respectively. We have \(|\mathcal{S}^r_{n,t}| = \binom{n}{r} - \binom{n-t}{r} = \binom{n-t}{r-1} \).
\[ (\binom{n-1}{r-1} + \binom{n-2}{r-1} + \ldots + \binom{n-t}{r-1}) \text{ and } |C_{n,t}^r| = t(\binom{n-t}{r-1}). \] In particular,

\[ |S_{n,t}^r|, |C_{n,t}^r| \sim t\left(\frac{n}{r-1}\right) \text{ as } n \to \infty \text{ and } t \text{ and } r \text{ are fixed.} \]

For any given \( r \)-graph \( H \), observe that \( H \not\subseteq S_{n,\tau-1}^r \), where \( \tau = \tau(H) \) and if \( \sigma(H) < \infty \) then \( H \not\subseteq C_{n,\sigma-1}^r \), where \( \sigma = \sigma(H) \). Hence

**Proposition 3.1** Let \( r \geq 2 \). Let \( H \) be an \( r \)-graph with \( \tau(H) = \tau \). Then

\[ \text{ex}(n, H) \geq |S_{n,\tau-1}^r| = \left(\binom{n-1}{r-1}\right) + \left(\binom{n-2}{r-1}\right) + \ldots + \left(\binom{n-\tau+1}{r-1}\right). \]

**Proposition 3.2** Let \( r \geq 2 \). Let \( H \) be an \( r \)-graph. Suppose \( \sigma(H) = \sigma < \infty \). Then

\[ \text{ex}(n, H) \geq |C_{n,\sigma-1}^r| = (\sigma - 1)\left(\binom{n-\sigma+1}{r-1}\right). \]

Recent works on hypergraph forests have identified many \( r \)-trees for which Proposition 3.2 is asymptotically tight, i.e., \( \text{ex}(n, H) = (\sigma(H) - 1)\binom{n}{r-1} + o(n^{r-1}) \). We call such graphs \( \sigma \)-tight. There have been a few \( r \)-trees for which equality holds in Proposition 3.1 for sufficiently large \( n \), i.e., \( \text{ex}(n, H) = \binom{n}{r} - \binom{n-\tau(H)+1}{r} \) for all sufficiently large \( n \). We call such \( r \)-graphs \( \tau \)-perfect. For example, \( r \)-uniform matchings are both \( \sigma \)-tight and \( \tau \)-perfect (Erdős [5]). See [2, [16], [17], [18], [27], [28] for more recent works on \( \sigma \)-tight and \( \tau \)-perfect families. Our main results in this paper generalize most of these recent results. Cross-cuts were introduced by Frankl and Füredi [11]. But they focused only on cases where \( \sigma \) was small. [16] was the first paper dealing with cases \( \sigma \geq r - 1 \), followed shortly by [18], [17], [27], and [28]. Our work in this paper builds on [16], [17], [18], unifies and substantially extends these results as well as others. In particular, we obtain exact results for many graphs.

## 4 Main results: asymptotic and stability

Let \( r, k \) be positive integers where \( r \geq k + 1 \). An \( r \)-graph \( H \) is \( k \)-reducible if each edge of \( H \) contains at least \( k \) vertices of degree 1. If \( H \) is \( k \)-reducible, we call the unique \((r-k)\)-graph \( G \) obtained by \( H \) by deleting \( k \) degree 1 vertices from each edge of \( H \) the \( k \)-reduction of \( H \). In general, the \( k \)-reduction \( G \) of \( H \) may be a multi-hypergraph. (For instance, the \((r-p)\)-reduction of an \( r \)-uniform \( s \)-petal sunflower with kernel size \( p \) consists of \( s \) copies of one edge of size \( p \)). We denote the underlying simple hypergraph of \( G \) by \( G' \). We may view \( H \) as being obtained from \( G' \) by enlarging each edge \( E \) to a given positive number \( \mu(E) \) of new edges of size \( r \) by adding \( k \) new vertices (called expansion vertices) per new edge in such a way that different new edges use disjoint sets of expansion vertices. If \( \mu(E) = 1 \) for all \( E \in G' \), i.e. if \( G = G' \), then we call \( H \) a simple expansion of an \((r-k)\)-graph; otherwise we call \( H \) a multi-expansion of an \((r-k)\)-graph.

**Theorem 4.1 (Asymptotic)** Let \( H \) be an \( r \)-graph that is embeddable in a \((r-k)\)-reducible \( r \)-tree, where \( r \geq 4 \). Let \( \sigma = \sigma(H) \), define \( \beta := 1/(\tau - 2)(\sigma + 1) + 1 \). Then

\[
(\sigma - 1)\left(\frac{n}{r-1}\right) + O(n^{r-2}) \leq \text{ex}(n, H) \leq (\sigma - 1)\left(\frac{n}{r-1}\right) + O(n^{r-1-\beta}).
\]
Theorem 4.1 is best possible in this sense. To construct a non-$\sigma$-tight $r$-uniform multi-expansion of an $(r - 1)$-tree, one can take an $r$-graph $S$ of size $s$ such that $|\bigcap_{F \in \mathcal{H}} F| = r - 1$ (a sunflower of size $s$). Obviously $\sigma(S) = 1$. On the other hand, it is known (Rödl [36] and Keevash [25]) that whenever $s$ and $r$ are fixed and $n \to \infty$ there are $S$-free families (called $P_{s-1}(n, r, r - 1)$ packings) of size $\frac{n-1}{r} \binom{n}{r-1} + O(n^{r-2})$. So $S$ is not $\sigma$-tight. For all $r \geq 4$, Irwin and Jiang [21] also constructed infinitely many $r$-uniform simple expansions $\mathcal{G}$ of $(r - 1)$-trees that are not $\sigma$-tight. In fact, in their construction $\mathcal{G}$, all but one edge of $\mathcal{G}$ have two degree 1 vertices. By contrast, Kostochka, Mubayi, and Verstraëte [28] had earlier showed that every 3-uniform simple expansion of a 2-tree is $\sigma$-tight.

Theorem 4.2 (Structural Stability) Let $\mathcal{H}$ be an $r$-graph that is embeddable in a 2-reducible $r$-tree, where $r \geq 4$. Let $\sigma = \sigma(\mathcal{H})$, suppose $\sigma \geq 2$ and define $\beta := 1/(r - 2)(\sigma + 1) + 1)$. Suppose that $\mathcal{F} \subseteq \binom{[n]}{r}$ is $\mathcal{H}$-free and $n \geq n(r, s)$, (where $n(r, s)$ is a function of $r$ and $s$ only). If

$$|\mathcal{F}| \geq (\sigma - 1)\binom{n}{r-1} - Kn^{r-1-\beta}$$

for $K \geq 0$, then there exists a set $A$ of $\sigma - 1$ vertices such that

$$|\mathcal{L}_\mathcal{F}(A)| \geq \binom{n}{r-1} - (r - 1)(K + 2s^2)n^{r-1-\beta}.$$

Furthermore, all but at most

$$((\sigma - 1)(r - 1)(K + 2s^2) + 2s^2)n^{r-1-\beta}$$

members of $\mathcal{F}$ meet $A$ in exactly one element.

Using the structural stability we obtain exact results for certain critical graphs.

Theorem 4.3 (Critical graphs) Let $\mathcal{H}$ be an $r$-graph that is embeddable in a 2-reducible $r$-tree, where $r \geq 4$. Let $\sigma = \sigma(\mathcal{H}) > 1$. Suppose $\mathcal{H}$ contains an edge $F_0$ such that $\sigma(\mathcal{H} \setminus F_0) = \sigma - 1$. Then there is a positive integer $n_0$ such that $\text{ex}(n, \mathcal{H}) \leq \binom{n}{r} - \binom{n-\sigma+1}{r}$ for all $n \geq n_0$. If, in addition, $\tau(\mathcal{H}) = \sigma(\mathcal{H})$, then for all $n \geq n_0$ we have

$$\text{ex}(n, \mathcal{H}) = \binom{n}{r} - \binom{n-\sigma+1}{r}.$$

For $r \geq 5$, Theorem 4.3 implies the exact results for $r$-uniform linear paths and cycles of odd length [17] [18] and also the exact results [2] on the disjoint union of linear paths and cycles at least one of which has odd length. For general 2-reducible $r$-trees, we can sharpen the error term in the upper bound of Theorem 4.1 to $O(n^{r-2})$.

Theorem 4.4 (Sharper estimates for trees) Let $\mathcal{H}$ be a 2-reducible $r$-tree, where $r \geq 4$. Let $\sigma = \sigma(\mathcal{H})$. Then $\text{ex}(n, \mathcal{H}) \leq (\sigma - 1)\binom{n}{r-1} + O(n^{r-2}).$
For \((r - 2)\)-reducible \(r\)-trees, i.e., \(r\)-uniform multi-expansions of 2-trees, we can sharpen our estimates even further, which sometimes yields exact results.

**Theorem 4.5 (Sharper results on multi-expansions of 2-trees)** Let \(r \geq 4\) and \(H\) an \((r - 2)\)-reducible \(r\)-tree with \(\sigma(H) = t + 1\). Let \(\pi\) be a tree-defining ordering of \(H\) and \(S\) be a minimum cross-cut of \(H\). Let \(w\) be the last vertex in \(S\) that is included in \(\pi\) and \(H_w\) the subgraph of \(H\) consisting of all the edges containing \(w\). Suppose that \(H_w\) is a linear star \(L^r_p\) (i.e., a sunflower with a single vertex in the kernel and \(p\) petals). Then there exists a positive integer \(n_1\) such that for all \(n \geq n_1\) we have

\[
\text{ex}(n, H) \leq \binom{n}{r} - \binom{n - t}{r} + \text{ex}(n - t, L^r_p).
\]

In many cases, Theorem 4.5 reduces the determination of the Turán number of an \((r - 2)\)-reducible \(r\)-tree to the determination of the Turán number of a linear star \(L^r_p\). For all \(r \geq 5\) and \(p \geq 2\), Frankl and Füredi [11] determined \(\text{ex}(n, L^r_p)\) asymptotically, showing that \(\text{ex}(n, L^r_p) = (\varphi(2, p) + o(1))(\frac{n - 2}{r - 2})^t\), where \(\varphi(2, p)\) is the maximum size of a 2-graph not containing a star of size \(p\) or a matching of size \(p\). As determined by Abbott et al. [1], \(\varphi(2, p) = p(p - 1)\) if \(p\) is odd and that \(\varphi(2, p) = (p - 1)^2 + \frac{1}{2}(p - 2)\) if \(p\) is even. Using the asymptotic result of Frankl and Füredi [11] and the stability method used in this paper, Irwin and Jiang [21] were able to determine the exact value of \(\text{ex}(n, L^r_p)\) for all \(r \geq 5\), \(p \geq 2\) when \(n\) is large. Based on this, Theorem 4.5 can then be used to obtain the exact value of \(\text{ex}(n, H)\) for many \((r - 2)\)-reducible \(r\)-trees \(H\).

For instance, for \(r \geq 4\) and large \(n\), as was already obtained by Frankl [8], \(\text{ex}(n, L^r_2) = \binom{n - 2}{r - 2}\). If \(H\) is a linear path of even length or if \(H\) is the disjoint union of linear paths all of which have even length, then \(H = L^2_2\) and for large \(n\) Theorem 4.5 shows that \(\text{ex}(n, H) \leq \binom{n}{r} - \binom{n - t}{r} + \binom{n - t - 2}{r - 2}\), where \(t = \sigma(H) - 1\). On the other hand, a trivial construction shows that \(\text{ex}(n, H) \geq \binom{n}{r} - \binom{n - t}{r} + \binom{n - t - 2}{r - 2}\).

So for \(r \geq 4\), we immediately retrieve the even case of the exact results from [18] and [2].

5 Lemmas on \(r\)-trees, partial \(r\)-trees, and cross-cuts

In this section, we develop a series of lemmas. The following (easy) lemma was given in [16].

**Proposition 5.1** Every \(r\)-tree \(H\) is contained in a tight \(r\)-tree \(G\) with \(V(G) = V(H)\). Furthermore, a starting edge of \(H\) can be used as a starting edge in \(G\). \(\square\)

**Lemma 5.2 (Tree embedding)** Let \(H\) be an \(r\)-tree, where \(r \geq 2\). Let \(E_1\) be a starting edge of \(H\). Let \(F\) be an \(r\)-graph with \(\delta_{r-1}(F) \geq |V(H)| \geq r + 1\). Then any mapping \(f : V(E_1) \rightarrow V(F)\) such that \(f(E_1) \subseteq F\) can be extended to an embedding of \(H\) in \(F\).

**Proof.** By Proposition 5.1 we may assume that \(H\) is a tight \(r\)-tree. Let \(E_1, \ldots, E_m\) be an ordering of the edges of \(H\) defining \(H\) as a tight \(r\)-tree. For each \(i \in [m]\) let \(H_i = \{E_j : j \leq i\}\) be the initial segment of \(H\). Then \(H_i\) is a tight \(r\)-tree. We use induction on \(i\) to show that \(f\) can be extended to an embedding \(f_i\) of \(H_i\) in \(F\). For the basis step, let \(f_1 = f\). In general, let \(2 \leq i \leq |E(H)|\) and suppose \(f\) can be extended to an embedding \(f_{i-1}\) of \(H_{i-1}\) in \(F\). Let \(E_{\alpha(i)}\) be a parent of \(E_i\) and \(D = E_i \cap E_{\alpha(i)}\). By definition, \(|D| = r - 1\). Let \(D' = f_{i-1}(D)\). Since \(\delta_{r-1}(F) \geq |V(H)| - r + 1\) we have \(\deg_F(D') > |f_{i-1}(\bigcup H_{i-1})| - |D'|\). So one can find an edge \(F\) in \(F\) containing \(D'\) such that
Proposition 5.3 (Embedding an expansion) Let $r \geq 2$ be an integer. Suppose that $G$ is an $r$-graph with $s$ vertices and $S$ is a set of degree $1$ vertices in $G$. Let $F$ be an $r$-graph. If $G - S \subseteq \text{Ker}_s(F)$, then $G \subseteq F$.

Proof. Easy from definitions. Let $E_1, \ldots, E_m$ be the edges of $G$. For each $i$, let $D_i = E_i \setminus S$. Note that $\{D_1, D_2, \ldots, D_m\}$ may be a multi-set. Let $f$ be an embedding of $G - S$ into $\text{Ker}_s(F)$. Let $W = \bigcup_{i=1}^m f(D_i)$. To obtain a copy of $G$ in $F$ it suffices to extend each $f(D_i)$, $1 \leq i \leq m$, to some edge $F_i$ of $F$ containing $f(D_i)$ such that for $i = 1, \ldots, m$, $F_i \setminus f(D_i)$ is pairwise disjoint and that each $F_i \setminus f(D_i)$ is disjoint from $W$. On can define the appropriate $F_i$’s one by one using (2).

Proposition 5.4 (Trees and shadows) Let $H$ be an $r$-tree with $p$ vertices. Let $F$ be an $r$-graph on $[n]$ not containing $H$. Then $|F| \leq (p-r)|\partial_{r-1}(F)|$. In particular, we have $\text{ex}(n,H) \leq (p-r){n \choose r-1}$.

Proof. The second statement follows from the first since $|\partial_{r-1}(F)| \leq {n \choose r-1}$. Suppose $|F| > (p-r)|\partial_{r-1}(F)|$. We successively remove edges from $F$ that contain an $(r-1)$-set $D$ whose degree becomes at most $p-r$ until no such edge remains. Denote the remaining graph by $F'$. Since at most $(p-r)$ edges were removed for each such $D$ and there are at most $|\partial_{r-1}(F)|$ such $D$, $F'$ is nonempty. By definition, $\delta_{r-1}(F') \geq p-r+1$. By Lemma 5.2, $H \subseteq F' \subseteq F$, a contradiction.

Lemmas similar to Proposition 5.4 were given in [16] and in [27]. For specific $r$-trees, more is known. Katona [24] showed that for an intersecting family of $r$-sets $F$ (i.e. $F$ avoids a matching of size 2) one has $|F| \leq |\partial_{r-1}(F)|$. This was recently extended by Frankl [29] who showed that if an $r$-graph $F$ avoids $M_s$ (a matching of size $s$) then $|F| \leq (s-1)|\partial_{r-1}(F)|$.

Lemma 5.5 (The first edge containing a vertex) Let $E_1, \ldots, E_m$ be an ordering of edges that defines a hypergraph tree $H$. Fix an $i$, where $2 \leq i \leq m$. Let $x \in E_i \setminus E_{\alpha(i)}$. Then $E_i$ is the first edge in the ordering that contains $x$. Also, if $y \in E_{\alpha(i)} \setminus E_i$ then no edge of $H$ contains both $x$ and $y$.

Proof. By definition, $E_i \cap (\bigcup_{j<i} E_j) \subseteq E_{\alpha(i)}$. Since $x \notin E_{\alpha(i)}$, $x \notin (\bigcup_{j<i} E_j)$. So $E_i$ is the first edge in the ordering that contains $x$. Let $y \in E_{\alpha(i)} \setminus E_i$. At the moment of $E_i$’s addition, $x, y$ are both in $H$ but there is no edge containing both $x, y$. Suppose some later edge contains both $x$ and $y$. Let $E_j$ be an earliest such edge, where $j > i$. Then $E_{\alpha(j)}$ must already contain both $x, y$, contradicting our $E_j$. So no edge of $H$ contains both $x$ and $y$.

Lemma 5.6 (Compression of trees) Let $H$ be a hypergraph tree with a defining ordering $E_1, \ldots, E_m$. Fix one $i$, where $2 \leq i \leq m$. Suppose $x \in E_i \setminus E_{\alpha(i)}$ and $y \in E_{\alpha(i)} \setminus E_i$. For each $j = 1, \ldots, m$, let $E'_j = E_j \setminus \{x\} \cup \{y\}$ if $x \in E_j$ and let $E'_j = E_j$ if $x \notin E_j$. Then the list $E'_1, \ldots, E'_m$ defines a hypergraph tree $H'$.

Proof. By Lemma 5.5, $E_i$ is the first edge in the ordering that contains $x$ and that no edge of $H$ contains both $x$ and $y$. Next we show that the multi-list $E'_1, \ldots, E'_m$ defines a hypergraph tree. It suffices to check that

$$E'_1 \cap \left( \bigcup_{1 \leq k < j} E'_k \right) \subseteq E'_{\alpha(j)}$$

(4)
holds for each $2 \leq j \leq m$. By the definition of $\alpha(j)$, we have

$$E_j \cap \left( \bigcup_{1 \leq k < j} E_k \right) \subseteq E_{\alpha(j)}. \quad (5)$$

First suppose $j < i$. Since $E_i$ is the first edge in the ordering that contains $x$, we have $E_{\ell} = E_{\ell}$ for all $\ell \leq j$. So (1) is the same as (5), which holds. Next, suppose $j = i$. Then (1) holds since $E_i \cap \left( \bigcup_{k < i} E_k \right) \subseteq (E_i \cap \left( \bigcup_{k < i} E_k \right)) \cup \{y\} \subseteq E_{\alpha(i)} \cup \{y\} = E_{\alpha(i)}$, where the last equality follows from the fact that $E_{\alpha(i)}$ contains $y$ but not $x$. Finally, suppose $j > i$. Observe that the edges are unchanged outside $\{x, y\}$ and that the new edges do not contain $x$. Suppose (1) does not hold. Then we must have $y \in E_j$ and $y \notin E_{\alpha(j)}$. The latter implies $x \notin E_{\alpha(j)}$ and so $E_{\alpha(j)} = E_{\alpha(j)}$. Thus, $y \notin E_{\alpha(j)}$. There are two subcases to check.

First, suppose $y \in E_j$. Then $y \in E_j \setminus E_{\alpha(j)}$. By Lemma 5.5, $E_j$ is the first edge in the ordering that contains $y$. Since $y \notin E_{\alpha(i)}$, we have $j \leq \alpha(i) < i$, contradicting $j > i$. Next, suppose $y \notin E_j$. Since $y \notin E_j$, we have $x \in E_j$. But $x \notin E_{\alpha(j)}$. So $x \in E_j \setminus E_{\alpha(j)}$. By Lemma 5.5, $E_j$ is the first edge in the ordering that contains $x$, contradicting $E_i$ being the first edge that contains $x$.

We have shown that the multi-list $E_1', \ldots, E_m'$ satisfies (1). \hfill \square

**Corollary 5.7 (Smallest hosting tree)** Let $H$ be an $r$-graph that is embeddable in an $r$-tree $T$ and let $V_1, \ldots, V_r$ be a good $r$-coloring of $V(T)$. For each $i = 1, \ldots, r$, define $X_i := V_i \cap V(H)$. Then there exists an $r$-tree $G$ containing $H$ satisfying that $V(G) = V(H)$ and that $X_1, \ldots, X_r$ is an $r$-coloring of $G$.

**Proof.** Let $E_1, \ldots, E_m$ be an ordering of the edges of $T$ that defines $T$ as a hypergraph tree. If $V(T) = V(H)$ then we let $G = T$. Otherwise, let $x \in V(T) \setminus V(H)$. Suppose $x \in V_i$. Let $E_i$ be the first edge in the ordering that contains $x$. Then $x \in E_i \setminus E_{\alpha(i)}$. Let $y$ be the unique vertex in $E_{\alpha(i)} \cap V_r$. Then $y \in E_{\alpha(i)} \setminus E_i$. Let $T'$ denote the hypergraph tree obtained from $T$ by applying the compression procedure in Lemma 5.6. All the edges in $T$ that do not contain $x$ remain unchanged. Hence $H \subseteq T'$. We repeat this procedure until we obtain $G$. \hfill \square

For the next proposition, the reader should recall the definition of a trace, given in Section 2. The following can be immediately verified using definitions.

**Proposition 5.8 (The trace of a tree)** Let $G$ be a hypergraph tree and $S \subseteq V(G)$. Then $G|_S$ and $G - S$ are also hypergraph trees. \hfill \square

**Proposition 5.9 (Subtrees through one vertex)** Let $H = \{E_1, \ldots, E_m\}$ be an $r$-tree with this ordering, $r \geq 2$. For each vertex $x$, let $H_x$ denote the subgraph consisting of the edges containing $x$. Then $H_x$ is an $r$-tree.

**Proof.** As usual, let $E_{\alpha(i)}$ the parent of $E_i$ in $H$ (for $i \geq 2$). Order the edges of $H_x$ in the same way as they were in $H$. Let $E_i$ be an edge in $H_x$ that is not the first edge of $H_x$. Then $E_i$ is not the first edge in $H$ that contains $x$. Hence its parent $E_{\alpha(i)}$ must already contain $x$. So $E_{\alpha(i)}$ is also in $H_x$ and appears before $E_i$. Let $E_j$ be any edge in $H_x$ appearing before $E_i$. Then it also appears before $E_i$ in $H$ and hence $E_j \cap E_i \subseteq E_{\alpha(i)}$. This shows that $E_{\alpha(i)}$ still serves as a parent of $E_i$ in $H_x$. \hfill \square
Proposition 5.10 (Deleting a cross-cut) Let \( r \geq 3 \). Let \( \mathcal{H} \) be an \( r \)-graph embeddable in a 1-reducible \( r \)-tree \( T \). Let \( S \) be a cross-cut of \( \mathcal{H} \). Then \( \mathcal{H} - S \) is embeddable in an \((r-1)\)-tree on the same vertex set as \( \mathcal{H} - S \).

**Proof.** Starting with \( S \), from each edge of \( T \) that does not intersect \( S \) we select a vertex of degree 1 and add it to \( S \). Call the resulting set \( S' \). Then \( \mathcal{H} - S \subseteq T - S' \). By Lemma 5.18 \( T - S' \) is a hypergraph tree. Each edge in \( T - S' \) has size at most \( r - 1 \). We can round out those edges of \( T - S' \) of size smaller than \( r - 1 \) to \((r-1)\)-sets by adding new expansion vertices. Call the resulting \((r-1)\)-graph \( T' \). Then \( T' \) is an \((r-1)\)-tree that contains \( \mathcal{H} - S \). By Corollary 5.7 there exists an \((r-1)\)-tree \( G \) containing \( \mathcal{H} - S \) on the same vertex set as \( \mathcal{H} - S \). \( \square \)

Let us mention a potential difficulty in extending results on \( r \)-trees to those embeddable in \( r \)-trees. Namely, if \( \mathcal{H} \) is an \( r \)-graph embeddable in an \( r \)-tree, then \( \mathcal{H} \) may not always have a minimum cross-cut that can be extended to a cross-cut of some \( r \)-tree \( T \) that contains \( \mathcal{H} \).

Example 5.11 (Cross-cuts of an \( r \)-graph embeddable in an \( r \)-tree might not extend)

Define the 4-graph \( \mathcal{H}_1 := \{1abx_{a,b}, 1bcx_{b,c}, 1cdx_{c,d}, 2aby_{a,b}, 2bcy_{b,c}, 2cdy_{c,d}\} \), where 1, 2, a, b, c, d, \( x_{a,b}, x_{b,c}, x_{c,d}, y_{a,b}, y_{b,c}, y_{c,d} \) are 12 different vertices. Then \( \mathcal{H} \) is embeddable in the 4-tree \( \{12ab, 12bc, 12cd\} \cup \mathcal{H}_1 \), and \( S = \{1, 2\} \) is the unique minimum cross-cut of \( \mathcal{H} \). But every 4-tree \( T \) that contains \( \mathcal{H} \) must have an edge containing both 1 and 2. So there is no 4-tree \( T' \) with a crosscut \( S' \) such that \( \mathcal{H} \subseteq T' \) and \( S \subseteq S' \).

By comparison, the case \( \sigma = 1 \) is simpler.

Proposition 5.12 Let \( \mathcal{H} \) be an \( r \)-graph embeddable in an \( r \)-tree. Suppose \( \sigma(\mathcal{H}) = 1 \) and \( \{x\} \) is a cross-cut of \( \mathcal{H} \). Then there is an \( r \)-tree \( G \) with \( V(G) = V(\mathcal{H}) \) such that \( \{x\} \) is a cross-cut of \( G \). Moreover, if \( T \) is \( k \)-reducible then \( G \) can be \( k \)-reducible, too.

**Proof.** Let \( X_1, \ldots, X_r \) be an \( r \)-partition of \( \mathcal{H} \). Note that \( \{x\} \) must by itself be one of the \( X_i \)'s. The claim then follows immediately from Corollary 5.7. \( \square \)

Lemma 5.13 (Subtrees and detachable limbs) Let \( \mathcal{H} = \{E_1, \ldots, E_m\} \) be an \( r \)-tree with this ordering, \( r \geq 2 \). For each \( x \), let \( H_x \) denote the subtree consisting of edges containing \( x \). Suppose \( S \) is a cross-cut of \( \mathcal{H} \), \( |S| \geq 2 \). Then \( \exists w \in S \) such that \( H' = H \setminus H_w \) is an \( r \)-tree. Furthermore, there exist an \( E \in H_w \) and \( F \in H' \) such that \( E \) is a starting edge of \( H_w \) and \( V(H_w) \cap V(H') = E \cap F \).

**Proof.** Let \( E_{\alpha(j)} \) denote a fixed parent of \( E_i \) in \( \mathcal{H} \). Let \( w \) be the last vertex in \( S \) that is included as we add edges of \( \mathcal{H} \) in the order of \( \pi := \{1, 2, 3, \ldots\} \). We now verify that \( H' = H \setminus H_w \) is an \( r \)-tree. Let \( \pi' \) be obtained from \( \pi \) by deleting the edges of \( H_w \) and keeping the relative order of the remaining edges. Let \( E_j \) be an edge in \( \pi' \) that is not the first edge. Since \( S \) is a cross-cut of \( \mathcal{H} \) and \( E_j \notin H_w \), \( E_j \) contains exactly one vertex \( x \) of \( S \) and \( x \neq w \). If \( E_{\alpha(j)} \) contains \( w \) then \( x \in E_j \setminus E_{\alpha(j)} \). By Lemma 5.5, \( E_j \) is the first edge in \( \pi \) that contains \( x \). But \( E_{\alpha(j)} \) appears earlier than \( E_j \). So \( w \) is included earlier than \( x \), contradicting our choice of \( w \). So \( w \notin E_{\alpha(j)} \), which means \( E_{\alpha(j)} \) is also in \( H' \) and appears earlier than \( E_j \). For any \( E_\ell \) in \( \pi' \) that appears earlier than \( E_j \), it also appears earlier than \( E_j \) in \( \pi \) and we have \( E_\ell \cap E_j \subseteq E_{\alpha(j)} \) since \( E_{\alpha(j)} \) is a parent of \( E_j \) in \( \pi \). This shows that \( E_{\alpha(j)} \) is still a parent of \( E_j \) in \( \pi' \). So \( \pi' \) defines \( H' \) as an \( r \)-tree.
Let $E_k$ be the first edge in $\pi_w$. Then $E_{\alpha(k)} \in H'$. We show that if $A \in H_w$ and $B \in H'$ then $A \cap B \subseteq E_k \cap E_{\alpha(k)}$. For convenience for each edge $D$ in $H$ we let $\alpha(D)$ denote a fixed parent of it in $\pi$. Observe that we have (a) $A \cap B \subseteq A \cap \alpha(B)$ if $A$ appears before $B$ in $\pi$, and (b) $A \cap B \subseteq \alpha(A) \cap B$ if $B$ appears before $A$ in $\pi$. Starting with $A \cap B$, we may obtain a superset by either replacing $B$ with $\alpha(B)$ or by replacing $A$ with $\alpha(A)$ depending on which of (a), (b) applies. If $A \neq E_k$ then by earlier discussion, $\alpha(A)$ is still in $H_w$. Also, since $B \in H'$, $\alpha(B) \in H'$ by earlier discussion. In particular, $\alpha(B) \notin H_w$. So we may repeatedly apply (a) or (b) in a way until $A = E_k$ and $B$ is an edge appearing before $E_k$ in $\pi$. Then $A \cap B \subseteq E_k \cap E_{\alpha(k)}$ holds. This proves the second part. \hfill \Box

One of the subtleties in this paper is the distinction between an $r$-graph embeddable in an $r$-tree (like linear cycles) and an $r$-tree itself. One of the difficulties in extending results on $r$-trees to those embeddable in $r$-trees is that the latter class is not known to possess the nice decomposition property described in the above Lemma. 

6 Reduction to centralized families

In this section, we use the delta system method to reduce the problem of embedding 2-reducible hypergraph trees and their subgraphs into a host graph $F$ to one where $F$ belongs to a so-called centralized family. The following lemma was developed using the delta system method, and was used in earlier works. See [19,18,16,17] for some recent applications. In particular, [18] and [17] contain some detailed discussions that are most relevant to what is needed in this paper.

Let $F$ be an $r$-partite $r$-graph with an $r$-partition $(X_1, \ldots, X_r)$. So, each edge of $F$ contains exactly one element of each $X_i$. Given $F \in F$ and $I \subseteq [r]$, let $F[I] = F \cap (\bigcup_{i \in I} X_i)$. In other words, $F[I]$ is the projection of $F$ onto those parts indexed by $I$. If $I = \{i\}$, we write $F[i]$ for $F[I]$. Let $F[I] = \{F[I] : F \in F\}$.

Lemma 6.1 (The homogeneous subfamily lemma, see [14]) For any positive integers $s$ and $r$, there is a positive constant $c(r,s)$ such that for every family $F \subseteq \binom{[n]}{r}$ there exist $F^* \subseteq F$ with $|F^*| \geq c(r,s) |F|$ and some $J \subseteq 2^r \setminus [r]$ (called the intersection pattern) such that

1. $F^*$ is $r$-partite, together with an $r$-partition $(X_1, \ldots, X_r)$.

2. $\forall F \in F^*, \forall I \in J, \deg_{\pi}(F[I]) \geq s$ and $\forall I \notin J \forall F' \in F^*$ satisfying $F \cap F' = F[I]$.

3. $J$ is closed under intersection, i.e., for all $I, I' \in J$ we have $I \cap I' \in J$ as well. \hfill \Box

We will call $F^*$ (with the corresponding $J$) $(r,s)$-homogeneous with intersection pattern $J$.

Lemma 6.2 [11] Let $n \geq r \geq 3$. Let $F^* \subseteq \binom{[n]}{r}$ be an $(r,s)$-homogeneous family with a corresponding $r$-partition $(X_1, \ldots, X_r)$ and intersection pattern $J \subseteq 2^r$. Then one of the following holds:

1. $|F^*| \leq \binom{r-2}{2}$, or
2. $\exists a, b \in [r]$ such that $2^r \setminus \{a,b\} \subseteq J$, or
3. $\exists i \in [r]$, such that $\forall F \in F^*, \deg_{\pi}(F \setminus F[i]) = 1$ but $\forall I \subseteq [r]$, with $i \in I$ we have $\deg_{\pi}(F[I]) \geq s$.

Proof. If $J$ contains all the $(r-1)$-subsets of $[r]$ then since $J$ is closed under intersection, we have $J = 2^r \setminus [r]$, in which case (2) holds trivially. Hence, we may assume that there is at least
one \((r-1)\)-subset of \([r]\) not in \(J\). This implies that there are proper subsets of \([r]\) that are not contained in any member of \(J\). Among them let \(D\) be one with minimum size. Suppose there exist \(F,F' \in F^*, F \neq F'\), such that \(F[D] = F'[D]\). Then \(F[D] \subseteq F \cap F' = F[B]\) for some \(D \subseteq B \subseteq [r]\).

By Lemma 6.1 item 2, \(B \in J\), which contradicts our assumption about \(D\). So \(F[D]\) are different from for each \(F \in F^*\). Hence \(|F^*| \leq \binom{n}{|D|-1}\). If \(|D| \leq r-2\) then (1) holds and we are done. So assume \(|D| = r-1\).

Without loss of generality, suppose \([r] \setminus \{i\} \not\in J\) for \(i = 1, \ldots, t\) and \([r] \setminus \{i\} \in J\) for \(i = t+1, \ldots, r\). By our assumption \(t \geq 1\). Suppose first that \(t \geq 2\). For any \(i, j \in [t], i \neq j\), observe that \([r] \setminus \{i, j\} \in J\), since otherwise \(r \setminus \{i, j\}\) is a \((r-2)\)-set not contained in any member of \(J\), contradicting our assumption about \(D\). Let \(I\) be any subset of \([r] \setminus \{1, 2\}\). Then \(I\) can be written as the intersection of sets of the form \([r] \setminus \{i\}\) for \(i \in \{t+1, \ldots, r\}\) and \([r] \setminus \{i, j\}\), for \(i, j \in [t], i \neq j\). Since each set of one of these forms are in \(J\) and \(J\) is closed under intersection, \(I \in J\). So we have \(2^{[t]} \setminus \{1, 2\} \subseteq J\), and (2) holds. Finally, assume \(t = 1\). Let \(I\) be any proper subset of \([r]\) containing 1. Then since \([r] \setminus \{i\} \in J\) for \(i = 2, \ldots, r\) and \(J\) is closed under intersection, we have \(I \in J\). By Lemma 6.1 item 2, \(\forall F \in F^*\) we have \(\deg_F^s(F[I]) \geq s\). By our assumption, \([r] \setminus \{1\} \not\in J\). By Lemma 6.1 item 2, \(\forall F \in F^*\) \(F'[r] \setminus \{1\}\) is contained only in \(F\) and not in any other member of \(F^*\). So (3) holds.

**Definition 6.3** If \(F^* \subseteq \binom{[n]}{r}\) is an \((r,s)\)-homogeneous family with a corresponding \(r\)-partition \((X_1, \ldots, X_r)\) and intersection pattern \(J\) for which Lemma 6.2 item 3 holds, then we say that \(F^*\) is **homogeneously centralized** with threshold \(s\). We call \(i \in [r]\) the **central element** of \(F^*\). For each \(F \in F^*\), we let \(c(F) = F[i]\) and call it the **central element** of \(F\). More generally, \(F \subseteq \binom{[n]}{r}\) is a **centralized family** with threshold \(s\) if each \(F \in F\) contains an element \(c(F)\) such that \(\forall c(F) \in D \subseteq F\) we have \(\deg_F^s(D) \geq s\). (The choice of \(c(F)\) may not be unique, but we fix one.)

**Remark 6.4** Note the following distinction between homogeneously centralized families and centralized families: If \(F\) is homogeneously centralized then \(\forall F \in F\), \(\deg_F(F \setminus c(F)) = 1\) (see Lemma 6.2 item 3). However, if \(F\) is simply centralized, then this condition need not hold.

**Lemma 6.5** Let \(r \geq 4\). Let \(\mathcal{H}\) be an \(r\)-graph that is embeddable in a 2-reducible \(r\)-tree. Suppose \(\mathcal{H}\) has \(s\) vertices. Let \(F \subseteq \binom{[n]}{r}\) be an \((r,s)\)-homogeneous family with a corresponding \(r\)-partition \((X_1, \ldots, X_r)\) and intersection pattern \(J\). If \(\mathcal{H} \nsubseteq F\), then either \(|F| \leq \binom{n}{r-2}\) or \(F\) is homogeneously centralized.

**Proof.** By Lemma 6.2 it suffices to rule out item 2. Suppose otherwise that item 2 holds for \(F\) and \(J\). So there exist \(M \subseteq [r]\) with \(|M| = r-2\) such that \(2^M \subseteq J\). By our assumption, for all \(I \subseteq M\) and all \(F \in F\), \(F[I] \in \text{Ker}_s(F)\). In particular, we have \(F[M] \subseteq \text{Ker}_s(F)\). Let \(D \in \partial_{r-3}(F[M])\). Then \(D = F[I]\) for some \(F \in F\) and \(I \subseteq M\) with \(|I| = r-3\). By our assumption, there is an \(s\)-star in \(F\) with kernel \(D\). The restriction of the members of this \(s\)-star to \(\bigcup_{i \in M} X_i\) are \(s\) edges in \(F[M]\) containing \(D\). This shows that \(\partial_{r-3}(F[M]) \geq s\).

Let \(\mathcal{T}\) be a 2-reducible \(r\)-tree that contains \(\mathcal{H}\). Let \(\mathcal{H}^*\) be obtained from \(\mathcal{H}\) by removing two degree 1 vertices from each edge of \(\mathcal{H}\) and eliminating duplicated edges and let \(\mathcal{T}^*\) be obtained from \(\mathcal{T}\) by removing two degree 1 vertices from each edge of \(\mathcal{T}\) and eliminating duplicated edges. Then clearly \(\mathcal{H}^*\) and \(\mathcal{T}^*\) are both \((r-2)\)-uniform and \(\mathcal{H}^* \subseteq \mathcal{T}^*\). By Lemma 5.8 \(\mathcal{T}^*\) is an \((r-2)\)-tree. So
$\mathcal{H}^*$ is embeddable in an $(r-2)$-tree. By Lemma 5.1 and Lemma 5.7 there exists a tight $(r-2)$-tree $G$ containing $\mathcal{H}^*$ with $V(G) = V(\mathcal{H}^*)$. In particular, $\mathcal{G}$, has at most $s$ vertices. Since $\delta_{r-3}(\mathcal{F}[M]) \geq s$, by Lemma 5.2, $\mathcal{F}[M] \supseteq \mathcal{G}$ and thus $\mathcal{F}[M]$ contains a copy $\mathcal{H}'$ of $\mathcal{H}^*$. Since $\mathcal{F}[M] \subseteq \text{Ker}_s(\mathcal{F})$, each edge of $\mathcal{H}'$ has kernel degree at least $s$ in $\mathcal{F}$. By Lemma 5.3, $\mathcal{H} \subseteq \mathcal{F}$, contradicting $\mathcal{H} \not\subseteq \mathcal{F}$.

**Theorem 6.6 (The reduction theorem)** Let $r \geq 4$. Let $\mathcal{H}$ be an $r$-graph that is embeddable in a 2-reducible $r$-tree. Suppose $\mathcal{H}$ has $s$ vertices. If $\mathcal{H} \not\subseteq \mathcal{F}$, then $\mathcal{F}$ can be split into subfamilies $\mathcal{F}^*$ and $\mathcal{F}_0$ such that $\mathcal{F}^*$ is centralized with threshold $s$ and $|\mathcal{F}_0| \leq \frac{1}{c(r,s)}(\binom{n}{r-2})$. Further, if $\sigma(\mathcal{H}) = 1$ then $\mathcal{F}^* = \emptyset$ and thus $|\mathcal{F}| \leq \frac{1}{c(r,s)}(\binom{n}{r-2})$.

**Proof.** First we apply Lemma 6.1 to $\mathcal{F}$ to get an $(r,s)$-homogeneous subfamily $\mathcal{F}_1$ with intersection pattern $\mathcal{J}_1$ such that $|\mathcal{F}_1| \geq c(r,s)|\mathcal{F}|$. By Lemma 6.3 either $|\mathcal{F}_1| \leq \binom{n}{r-2}$ or $\mathcal{F}_1$ is centralized. If $|\mathcal{F}_1| \leq \binom{n}{r-2}$ then we stop. Otherwise we apply Lemma 6.1 again to $\mathcal{F} \setminus \mathcal{F}_1$ to get an $(r,s)$-homogeneous subfamily $\mathcal{F}_2$ with intersection pattern $\mathcal{J}_2$ such that $|\mathcal{F}_2| \geq c(r,s)(|\mathcal{F}| - |\mathcal{F}_1|)$. We continue like this until $|\mathcal{F}_i| \leq \binom{n}{r-2}$. Let $m$ be the smallest index $i$ such that $|\mathcal{F}_i| \leq \binom{n}{r-2}$. By our assumption, for each $i \in [m-1]$, $\mathcal{F}_i$ is homogeneously centralized. Let $\mathcal{F}^* = \bigcup_{i=1}^{m-1} \mathcal{F}_i$ and let $\mathcal{F}_0 = \mathcal{F} \setminus \mathcal{F}^*$. Then clearly $\mathcal{F}^*$ is centralized with threshold $s$. Also, by the algorithm, $|\mathcal{F}_m| \geq c(r,s)|\mathcal{F}_0|$ and hence $|\mathcal{F}_0| \leq \frac{1}{c(r,s)}|\mathcal{F}_m| \leq \frac{1}{c(r,s)}(\binom{n}{r-2})$.

Next, suppose $\sigma(\mathcal{H}) = 1$ with $\{x\}$ being a cross-cut of $\mathcal{H}$. By Proposition 5.12 there exists an $r$-tree $\mathcal{G}$ containing $\mathcal{H}$ such that $\{x\}$ is a cross-cut of $\mathcal{G}$. Suppose $\mathcal{F}^* \neq \emptyset$. Then $\mathcal{F}_1 \neq \emptyset$. By our assumption, $\mathcal{F}_1$ is homogeneously centralized with threshold $s$. So there exists an $r$-partition $X_1, \ldots, X_r$ of $\mathcal{F}_1$ together with a central element $c \in \{1, \ldots, r\}$, such that $\forall F \in \mathcal{F}_1$ and $c \in I \subseteq [r]$, $\deg_{\mathcal{F}_1}(F[I]) \geq s$. This allows us to greedily embed $\mathcal{G}$ into $\mathcal{F}_1$, contradicting $\mathcal{F}_1$ being $\mathcal{H}$-free.

**7 Proof of the asymptotic in Theorem 4.1**

Suppose that $\mathcal{H}$ has $s$ vertices. Since $\mathcal{H} \not\subseteq \mathcal{F}$, by Theorem 6.6, $\mathcal{F}$ can be split into subfamilies $\mathcal{F}^*$ and $\mathcal{F}_0$ such that $\mathcal{F}^*$ is centralized with threshold $s$ and

$$|\mathcal{F}_0| \leq \frac{1}{c(r,s)}(\binom{n}{r-2}).$$

If $\sigma = 1$, then by Theorem 6.6 $|\mathcal{F}| = |\mathcal{F}_0| \leq \frac{1}{c(r,s)}(\binom{n}{r-2})$, which implies the upper bound in Theorem 4.1. Hence, for the rest of this proof, we suppose that $\sigma > 1$.

By the definition of a centralized family, for each $F \in \mathcal{F}^*$, there is a central element $c(F) \in F$ such that for all proper subsets $D$ of $F$ containing $c(F)$ we have $\deg_{\mathcal{F}^*}(D) \geq s$. Let

$$\varepsilon := \frac{r-2}{(\sigma + 1)(r-2) + 1} \quad \text{and} \quad h := \lceil n^\varepsilon \rceil.$$

We accomplish the proof of Theorem 4.1 in three steps.

**Step 1.** $\exists W \subseteq [n]$ and a subfamily $\mathcal{F}_1^* \subseteq \mathcal{F}^*$ such that $|W| = \lceil n^\varepsilon \rceil$, $F \cap W = \{c(F)\}$ for $\forall F \in \mathcal{F}_1^*$, and

$$|\mathcal{F}^* \setminus \mathcal{F}_1^*| \leq s^2n^{r-1-(\varepsilon/(r-2))} + n^{r-2+2}\varepsilon.$$
Proof of Step 1. We partition $\mathcal{F}^*$ according to $c(F)$. For each $i \in [n]$, let

$$\mathcal{A}_i = \{F \in \mathcal{F}^* : c(F) = i\}, \quad \text{and} \quad \mathcal{A}_i' = \{F \setminus \{i\} : F \in \mathcal{A}_i\}.$$  

By Lemma 6.2 (2) we have that $D \cup \{i\} \in \text{Ker}^{(r-1)}_s(\mathcal{F}^*)$ for all $D \in \partial_{r-2}(\mathcal{A}_i')$. Hence

$$|\text{Ker}^{(r-1)}_s(\mathcal{F}^*)| \geq \frac{1}{r-1} \sum_{i=1}^n |\partial_{r-2}(\mathcal{A}_i')|.$$  

(9)

Let $\mathcal{T}$ be a 2-reducible $r$-tree that contains $\mathcal{H}$. Let $\mathcal{T}^*$ be obtained from $\mathcal{T}$ by deleting a degree 1 vertex from each edge and $\mathcal{H}^*$ be obtained from $\mathcal{H}$ by deleting a degree 1 vertex from each edge. Then $\mathcal{H}^* \subseteq \mathcal{T}^*$ and by Lemma 5.8 $\mathcal{T}^*$ is an $(r-1)$-tree. So $\mathcal{H}^*$ is embeddable in an $(r-1)$-tree. By Lemma 5.7 there exists an $(r-1)$-tree $\mathcal{G}$ containing $\mathcal{H}^*$ such that $V(\mathcal{G}) = V(\mathcal{H}^*)$. If $\text{Ker}^{(r-1)}_s(\mathcal{F}^*)$ contains a copy of $\mathcal{G}$, then it contains a copy of $\mathcal{H}^*$ by Lemma 5.3. Since $\mathcal{H} \subseteq \mathcal{F}^* \subseteq \mathcal{F}$, this is a contradiction. So, $\text{Ker}^{(r-1)}_s(\mathcal{F}^*)$ does not contain $\mathcal{G}$. By Lemma 5.4 we have

$$|\text{Ker}^{(r-1)}_s(\mathcal{F}^*)| \leq s \left(\frac{n}{r-2}\right).$$  

(10)

For each $i \in [n]$, let $x_i \geq r - 2$ be the real such that $|\partial_{r-2}(\mathcal{A}_i')| = (x_i)$. Hence, $x_1 \geq \ldots \geq x_n$. By (9) and (10), we have

$$\sum_{i=1}^n \left(\frac{x_i}{r-2}\right) \leq s(r-1)\left(\frac{n}{r-2}\right).$$  

(11)

Since $x_1 \geq \ldots \geq x_n$, (11) gives $\left(\frac{x_h}{r-2}\right) \leq \frac{x_1}{n}$. Hence, $x_h - r + 3 \leq \left(\frac{sr}{h}\right)^{(r-2)n}$. Kruskal-Katona theorem [1] implies that $|\mathcal{A}_i'| \leq \left(\frac{x_i}{r-1}\right)$ holds $\forall i \in [n]$, since $|\partial_{r-2}(\mathcal{A}_i')| = (x_i)$. Note that $|\mathcal{A}_i| = |\mathcal{A}_i'|$. We obtain

$$\sum_{i=h+1}^n |\mathcal{A}_i| \leq \sum_{i=h+1}^n \left(\frac{x_i}{r-1}\right) \leq \frac{x_h - r + 3}{r-1} \sum_{i=h+1}^n \left(\frac{x_i}{r-2}\right) \leq \left(\frac{sr}{h}\right)^{(r-2)n} \left(\frac{n}{r-2}\right) < s^2 n^{r-1 - (\varepsilon/(r-2))}.$$  

Define $W := [h]$. Let $\mathcal{F}_1 = \{F \in \mathcal{F}^* : c(F) \not\in W\}$. We have

$$|\mathcal{F}_1| \leq s^2 n^{r-1 - (\varepsilon/(r-2))}.$$  

Let $\mathcal{F}_2 = \{F : F \in \mathcal{F}^* \setminus \mathcal{F}_1, |F \cap W| \geq 2\}$. Then

$$|\mathcal{F}_2| \leq \binom{|W|}{2} \left(\frac{n-|W|}{r-2}\right) \leq n^{r-2+2\varepsilon}.$$  

Let $\mathcal{F}_1^* = \mathcal{F}^* \setminus (\mathcal{F}_1 \cup \mathcal{F}_2)$. By definition, $\forall F \in \mathcal{F}_1^*$, we have $F \cap W = \{c(F)\}$. The above two bounds imply

$$|\mathcal{F}^* \setminus \mathcal{F}_1^*| = |\mathcal{F}_1 \cup \mathcal{F}_2| \leq s^2 n^{r-1 - (\varepsilon/(r-2))} + n^{r-2+2\varepsilon}.$$  

This completes Step 1. \hfill \Box
Let $S$ be a cross-cut of $H$ with $|S| = \sigma = \sigma(H)$. For the next claim, the reader should recall the definition of a common link graph from Section 2.

**Step 2.** For every $A \in \binom{[n]}{\sigma}$, we have $|\mathcal{L}_{F_1^r}(A)| \leq s\left(\frac{n}{r-3}\right)$ and

$$|\bigcap_{x \in A} \partial_{r-2}(\mathcal{L}_{F_1^r}(x))| \leq s\left(\frac{n}{r-3}\right). \tag{12}$$

*Proof of Step 2.* By Lemma 5.10 there exists an $(r-1)$-tree $G$ containing $H - S$ with $V(G) = V(H - S)$. In particular, $G$ has at most $s$ vertices and by Lemma 5.1 ex($n, G) \leq s\left(\frac{n}{r-2}\right)$. Suppose there exists a $\sigma$-set $A$ in $W$ with $\mathcal{L}_{F_1^r}(A) > s\left(\frac{n}{r-2}\right)$, then $\mathcal{L}_{F_1^r}(A)$ contains a copy of $H'$ of $H - S$. We then obtain a copy of $H$ in $F_1^r$ be mapping $S$ to $A$, a contradiction.

Let us now select a degree 1 vertex outside $S$ from each edge of $H$ and denote the resulting set $S'$. The set $S'$ is well-defined since each edge of $H$ contains at least two degree 1 vertices at most one of which is in $S$. Observe that $S$ and $S'$ are two disjoint cross-cuts of $H$. Let $H^* = H - S'$. By Lemma 5.10 $H^*$ is embeddable in an $(r-1)$-tree on the same vertex set as $H^*$. Clearly, $S$ is still a cross-cut of $H^*$. Applying Lemma 5.10 again, there exists an $(r-2)$-tree $G'$ containing $H^* - S$ on the same vertex set as $H^* - S$. In particular, $G'$ has at most $s$ vertices and hence $\text{ex}(n, G') \leq s\left(\frac{n}{r-3}\right)$.

Suppose there exists a $\sigma$-set $A$ in $W$ such that $|\bigcap_{x \in A} \partial_{r-2}(\mathcal{L}_{F_1^r}(x))| > s\left(\frac{n}{r-3}\right)$. Then there exists a copy $H'$ of $H^* - S$ in $Z = \bigcap_{x \in A} \partial_{r-2}(\mathcal{L}_{F_1^r}(x))$. By embedding $S$ to $A$, we see that $A \times Z$ contains a copy of $H^*$. By the definitions of $F_1^r$, $A$, and $Z$, we have $A \times Z \subseteq \text{Ker}_{s\left(r-1\right)}(F^*)$. Hence $H^* \subseteq \text{Ker}_{s\left(r-1\right)}(F^*)$. Since $H^* = H - S'$, by Lemma 5.3 we get $H \subseteq F^* \subseteq F$, a contradiction. \qed

**Step 3.** \exists $F_2^* \subseteq F_1^r$ such that

$$\text{deg}_{F_2^*}(F \setminus W) \leq \sigma - 1 \tag{13}$$

holds for $\forall F \in F_2^*$ and

$$|F_1^r \setminus F_2^*| \leq sn^{r-2+\sigma+\varepsilon}. \tag{14}$$

Furthermore, for each $(r-2)$-set $D \subseteq [n] \setminus W$,

$$|\left(\bigcup\{F \in F_2^* : D \subseteq F\}\right) \cap W| \leq \sigma - 1. \tag{15}$$

Note that (13) gives

$$|F_2^*| \leq (\sigma - 1)\left(\frac{n - |W|}{r - 1}\right). \tag{16}$$

*Proof of Step 3.* Note that $W$ is a cross-cut of $F_1^r$. First, we clean out edges in $F_1^r$ that contain $(r-2)$-sets in $[n] \setminus W$ that lie in $\partial_{r-2}(\mathcal{L}_{F_1^r}(x))$ for at least $\sigma$ different $x$ in $W$. Formally, let

$$B := \bigcup_{A \in \binom{[n]}{\sigma}} \left(\bigcap_{x \in A} \partial_{r-2}(\mathcal{L}_{F_1^r}(x))\right).$$

By (12), we have

$$|B| \leq \left(\frac{|W|}{\sigma}\right)s\left(\frac{n}{r-3}\right) < sn^{r-3+\sigma+\varepsilon}.$$
Let

\[ \mathcal{F}_3 := \{ F \in \mathcal{F}_1^* : \exists D \in \mathcal{B}, D \subseteq F \} \]

Then

\[ |\mathcal{F}_3| \leq |\mathcal{B}| |W| n < sn^{r-2+(\sigma+1)\epsilon}. \]

Let \( \mathcal{F}_2^* := \mathcal{F}_1^* \setminus \mathcal{F}_3 \). Then \( |\mathcal{F}_1^* \setminus \mathcal{F}_2^*| = |\mathcal{F}_3| \leq sn^{r-2+(\sigma+1)\epsilon} \). For each \((r-2)\)-set \( D \in [n] \setminus W \) that is contained in an edge of \( \mathcal{F}_2^* \), we have \( D \notin \mathcal{B} \). So \( |\bigcup \{ F \in \mathcal{F}_2^* : D \subseteq F \}| \leq n - 1 \). This also implies that \( \deg_{\mathcal{F}_2^*}(F \setminus W) \leq \sigma - 1 \). For each \((r-2)\)-set \( D \) in \([n] \setminus W\) that is contained in an edge of \( \mathcal{F}_2^* \), we have \( D \notin \mathcal{B} \). So

\[ |\left( \bigcup \{ F \in \mathcal{F}_2^* : D \subseteq F \} \right) \cap W| \leq \sigma - 1. \]

This also implies that \( \deg_{\mathcal{F}_2}(F \setminus W) \leq \sigma - 1 \) holds for \( \forall F \in \mathcal{F}_2^* \). This completes Step 3.

The obvious identity

\[ |\mathcal{F}| = |\mathcal{F}_0| + |\mathcal{F}^* \setminus \mathcal{F}_1^*| + |\mathcal{F}_1^* \setminus \mathcal{F}_2^*| + |\mathcal{F}_2^*| \]

together with (6), (8), (14), and (16) yield

\[ |\mathcal{F}| \leq \frac{1}{c(r, s)} \left( \frac{n}{r-2} \right) + (s^2 n^{r-1-(\epsilon/(r-2))} + n^{r-2+2\epsilon}) + sn^{r-2+(\sigma+1)\epsilon} + |\mathcal{F}_2^*| \]

\[ \leq |\mathcal{F}_2^*| + (s^2 + s + 2)n^{r-2+(\sigma+1)\epsilon} \leq (\sigma - 1) \left( \frac{n}{r-1} \right) + 2s^2 n^{r-1-(\sigma+1)(r-2)\epsilon+1}, \quad (17) \]

for \( n \geq n(r, s) \), where \( n(r, s) \) is some function of \( r \) and \( s \). This completes the proof of Theorem 4.1.

\[ \square \]

8 Proof of the stability in Theorem 4.2

This is a continuation of the previous section. Recall that we now assume \( |\mathcal{F}| \geq (\sigma - 1) \left( \frac{n}{r-1} \right) - Kn^{r-1-\beta}. \) We already have from (17) and from the lower bound constraint for \( |\mathcal{F}| \) that for \( n > n(r, s) \)

\[ |\mathcal{F}_2^*| \geq |\mathcal{F}| - 2s^2 n^{r-1-\beta} \geq (\sigma - 1) \left( \frac{n}{r-1} \right) - (K + 2s^2)n^{r-1-\beta}. \quad (18) \]

Let\n
\[ \mathcal{F}_3^* = \{ F \in \mathcal{F}_2^* : \deg_{\mathcal{F}_2^*}(F \setminus W) = \sigma - 1 \}. \]

Then \( \forall F \in \mathcal{F}_2^* \setminus \mathcal{F}_3^* \) we have \( \deg_{\mathcal{F}_2^*}(F \setminus W) \leq \sigma - 2 \). Counting the degrees of the \((r-1)\)-sets of \([n] \setminus W\) in \( \mathcal{F}_2^* \) we obtain that

\[ \frac{|\mathcal{F}_3^*|}{\sigma - 1} + (\sigma - 2) \left( \frac{n - |W|}{r-1} \right) \geq |\mathcal{F}_3^*|. \]

This and (15) give

\[ \frac{|\mathcal{F}_3^*|}{\sigma - 1} \geq \left( \frac{n}{r-1} \right) - (K + 2s^2)n^{r-1-\beta}. \quad (19) \]

Let \( A_1, A_2, \ldots, A_m \) be all the subsets of \( W \) of size \( \sigma - 1 \) with \( \mathcal{L}_{\mathcal{F}_3^*}(A_i) \neq \emptyset \). Then

\[ \mathcal{F}_3^* = \bigcup_{1 \leq i \leq m} (A_i \times \mathcal{L}_{\mathcal{F}_3^*}(A_i)). \]
By (15) and that fact that $\mathcal{F}_3^* \subseteq \mathcal{F}_2^*$,

$$\forall i, j \in [m], i \neq j, \partial_{r-2}(\mathcal{L}_{\mathcal{F}_3^*}(A_i)) \cap \partial_{r-2}(\mathcal{L}_{\mathcal{F}_3^*}(A_j)) = \emptyset.$$

(20)

For each $i \in [m]$, let $y_i \geq r - 1$ denote the real such that $|\mathcal{L}_{\mathcal{F}_3^*}(A_i)| = \binom{y_i}{r-1}$. Without loss of generality, we may assume that $y_1 \geq y_2 \geq \cdots \geq y_m$. For each $i \in [m]$, by the Kruskal-Katona theorem (1), we have $|\partial_{r-2}(\mathcal{L}_{\mathcal{F}_3^*}(A_i))| \geq \binom{y_i}{r-2}$. By (20), we have

$$\sum_{1 \leq i \leq m} \binom{y_i}{r-2} \leq \sum_{i=1}^m |\partial_{r-2}(\mathcal{L}_{\mathcal{F}_3^*}(A_i))| \leq \binom{n}{r-2}.$$

The disjointness of the $\mathcal{L}_{\mathcal{F}_3^*}(A_i)$’s imply $|\mathcal{F}_3^*| = (\sigma - 1) \sum_{i=1}^m |\mathcal{L}_{\mathcal{F}_3^*}(A_i)|$. Use this, then the fact that $y_i \leq y_1$ for all $i$ and then the last displayed inequality. We obtain

$$\frac{|\mathcal{F}_3^*|}{\sigma - 1} = \sum_{i=1}^m |\mathcal{L}_{\mathcal{F}_3^*}(A_i)| = \sum_{i=1}^m \binom{y_i}{r-1} \leq \frac{y_1 - r + 2}{r - 1} \sum_{i=1}^m \binom{y_i}{r-2} \leq \frac{y_1 - r + 2}{r - 1} \binom{n}{r - 2}.$$

Compare this to the lower bound (19). We get

$$\frac{y_1 - r + 2}{r - 1} \binom{n}{r - 2} \geq \frac{|\mathcal{F}_3^*|}{\sigma - 1} \geq \frac{n - r + 2}{r - 1} \binom{n}{r - 2} - (K + 2s^2)n^{r-1-\beta}.$$

Hence

$$(r - 1)(K + 2s^2)n^{r-1-\beta} \geq (n - y_1) \binom{n}{r - 2}.$$

Take $A = A_1$. We have

$$|\mathcal{L}_F(A)| \geq |\mathcal{L}_{\mathcal{F}_3^*}(A_1)| = \binom{y_1}{r-1} \geq \binom{n}{r - 1} - (n - y_1) \binom{n}{r - 2} \geq \binom{n}{r - 1} - (r - 1)(K + 2s^2)n^{r-1-\beta}.$$

This, together with (17), also yields

$$|\mathcal{F} \setminus (A \times \mathcal{L}_F(A))| \leq ((\sigma - 1)(r - 1)(K + 2s^2) + 2s^2)n^{r-1-\beta}. \quad \square$$

9 Structures of near extremal families

To prove Theorems 4.3 4.4 4.5 we analyze the structure of near extremal families.

Lemma 9.1 (Missing edges vs. non-\(M\) edges) Let $\mathcal{M}$ be an $r$-graph with $m$ edges, $m \geq 2$. Let $\mathcal{G}$ be an $r$-graph on $[n]$, $\overline{\mathcal{G}}$ its complement. Let $\mathcal{G}_0$ be the subgraph of $\mathcal{G}$ consisting of the edges of $\mathcal{G}$ that do not lie in any copy of $\mathcal{M}$. Then $|\mathcal{G}_0| \leq (m - 1)|\overline{\mathcal{G}}|$.

Proof. Let $\mathcal{M}_1, \ldots, \mathcal{M}_h$ be all the labelled copies of $\mathcal{M}$ on $[n]$. By symmetry, each $r$-set in $[n]$ lies in the same number $t$ of these copies. If some $\mathcal{M}_i$ contains an edge of $\mathcal{G}_0$ then not all of its
edges are in $G$ and so it contains an edge of $\overline{G}$. Let $\mu$ be the number of triples $(e, M, f)$, where $e \in G_0$, $M \in \{M_1, \ldots, M_h\}$, $f \in \overline{G}$, and $e, f \in M$. Then $t|G_0| \leq \mu$ and $\mu \leq t(m-1)|\overline{G}|$. □

Let $K^p_0(s)$ denote the complete $p$-partite $p$-graphs with $s$ vertices in each part. By a well-known result of Erdős [4], $ex(n, K^p_0(s)) \leq c_1(p, s)n^{p-(1/s^p)-1}$ for all $n$ where $c_1$ depends only on $p$ and $s$.

Lemma 9.2 Let $s, p \geq 2$ be fixed. Let $n \geq n_2(p, s)$ be sufficiently large. Let $G, D \subseteq \binom{[n]}{p}$, $D \neq \emptyset$. Suppose that

$$|G| \geq \binom{n}{p} - n^{p-(1/s^p)-1}. \tag{21}$$

Let $G^* \subseteq G$ consist of all edges of $G$ that lie in copies of $K^p_0(s)$. Suppose $\partial_{p-1}(G^*) \cap \partial_{p-1}(D) = \emptyset$. Then $|D| \leq c_2n^{-1/((p-1)s^{p-1})|\overline{G}|}$, for some positive constant $c_2 := c_2(p, s)$.

Proof. Let $G_0 = G \setminus G^*$. By Erdős’ theorem, $|G_0| < c_1n^{p-(1/s^p)-1}$. By Lemma 9.1, $|G_0| \leq (s^p-1)|\overline{G}|$. Hence $|G^*| \leq s^p|G|$. This and (21) gives

$$|G^*| \geq \binom{n}{p} - c_3n^{p-(1/s^p)-1}, \tag{22}$$

where the positive constant $c_3 := c_3(p, s)$ depends only on $p$ and $s$.

Let $x, y \geq p$ be positive reals such that $|\partial_{p-1}(G^*)| = \binom{x}{p-1}$ and $|\partial_{p-1}(D)| = \binom{y}{p-1}$. The Kruskal-Katona theorem [1] implies that

$$|D| \leq \binom{y}{p} \text{ and } |G^*| \leq \binom{x}{p}. \tag{23}$$

The inequality (22) gives

$$x \geq n - c_4n^{1-(1/s^p)-1} \tag{24}$$

for some positive constant $c_4 = c_4(p, s)$.

Since $\partial_{p-1}(D) \cap \partial_{p-1}(G^*) = \emptyset$, we have

$$\binom{y}{p-1} + \binom{x}{p-1} \leq \binom{n}{p-1}. \tag{25}$$

This implies $\binom{y}{p-1} \leq \binom{n}{p-1} - \binom{x}{p-1} \leq (n-x)\binom{n}{p-2}$. Using this and (24) we get

$$y \leq c_5n^{1-1/((p-1)s^{p-1})}, \tag{26}$$

where $c_5 := c_5(p, s)$ and $n$ is large enough ($n > n_5(p, s)$).

Rewrite $\binom{n}{p-1}$ as $\frac{p}{n-p+1}\binom{n}{p}$ and multiply (25) by $(x-p+1)/p$. We obtain

$$\frac{x-p+1}{y-p+1}\binom{y}{p} + \binom{x}{p} \leq \frac{x-p+1}{n-p+1}\binom{n}{p} \leq \binom{n}{p}.$$

By (23), we have

$$\frac{x-p+1}{y-p+1}|D| + |G^*| \leq \binom{n}{p}.$$

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Hence, by (24) and (26), and using $n$ being large enough, we have

$$|D| \leq \frac{y-p+1}{x-p+1}|G| \leq \frac{y-p+1}{x-p+1}s^p|G| \leq c_2 n^{1-(p-1)s^{p-1}}|G|,$$

where $c_2 := c_2(p,s)$ depends only on $p$ and $s$. \qed

Lemma 9.3 Let $H$ be an $r$-graph embeddable in a 2-reducible $r$-tree. Suppose $H$ has $s$ vertices and $\sigma(H) = \sigma \geq 2$. Let $n \geq n_3(r,s) \geq n(r,s)$ be sufficiently large. Let $F \subseteq \binom{[n]}{r}$ such that $H \not\subseteq F$ and that $|F| \geq (\sigma-1)\binom{n-1}{r} - n^{r-1-\beta}$. Let $A$ be a $(\sigma-1)$-set guaranteed by Theorem 4.2 that satisfies

$$L_F(A) \geq \binom{n}{r-1} - (r-1)(1+2s^2)n^{r-1-\beta}, \text{ where } \beta = ((r-2)(\sigma+1)+1)^{-1}. \quad (27)$$

Let $L^* \subseteq L_F(A)$ consist of all edges of $L_F(A)$ that lie in copies of $K_{r-1}^{(\sigma-1)}(s)$. Let

$$F_A := \{F \in F : F \cap A \neq \emptyset\}, \quad S_A := \{F \in \binom{[n]}{r} : F \cap A \neq \emptyset\}, \text{ and } B \subseteq F \setminus F_A.$$

Suppose either $B = \emptyset$ or there is an $(r-1)$-graph $D$ on $[n] \setminus A$ satisfying

$$\partial_{r-2}(D) \cap \partial_{r-2}(L^*) = \emptyset \text{ and } |D| \geq \gamma |B|. \quad (28)$$

for some $\gamma > 0$. Then $|F_A \cup B| \leq \binom{n}{r} - \binom{n-\sigma+1}{r}$ holds for $n > n_4 := n_4(r,s,\gamma)$.

Proof. Note that $L_F(A)$ is an $(r-1)$-graph on $[n] \setminus A$ and for sufficiently large $n \geq n_3$, we have

$$L_F(A) \geq \binom{n-\sigma+1}{r-1} - (n-\sigma+1)^{(r-1-1/s^{r-2})}.$$

Let $\overline{A} = \binom{[n]}{r-1} \setminus L_F(A)$, i.e. $\overline{A}$ is the complement of $L_F(A)$ on $[n] \setminus A$. Since $D$ is an $(r-1)$-graph on $[n] \setminus A$ satisfying $\partial_{r-2}(D) \cap \partial_{r-2}(L^*) = \emptyset$, we can apply Lemma 9.2 with $G := L_F(A)$. We obtain

$$|D| \leq c_2(n-\sigma+1)^{-1/(r-2)s^{r-2}}|\overline{A}|.$$

If $B = \emptyset$ then the lemma holds trivially. So assume $B \neq \emptyset$. By our assumption about $B$ and $D$,

$$|B| \leq \frac{1}{\gamma} |D| \leq \frac{c_2}{\gamma}(n-\sigma+1)^{-1/(r-2)s^{r-2}}|\overline{A}|. \quad (29)$$

Each $(r-1)$-set in $\overline{A}$ contributes at least one edge to $S_A \setminus F_A$. So,

$$|S_A \setminus F_A| \geq |\overline{A}|.$$

Now,

$$|F_A \cup B| \leq |S_A| - |\overline{A}| + |B|.$$

By (29), we have $|B| \leq |\overline{A}|$ for sufficiently large $n$. Hence $|F_A \cup B| \leq |S_A| = \binom{n}{r} - \binom{n-\sigma+1}{r}$ for sufficiently large $n$. \qed
10 Proof of Theorem 4.3 on critical edges

Let \( \mathcal{H} \) be the \( r \)-graph on \( s \) vertices embeddable in a 2-reducible \( r \)-tree in Theorem 4.3 and let \( \mathcal{F} \subseteq \binom{[n]}{r} \) such that \( \mathcal{H} \not\subseteq \mathcal{F} \) and such that \( n \geq n_4(r, s, \gamma) \), where \( n_4(r, s, \gamma) \) is specified in Lemma 9.3 with \( \gamma = 1/s \). We may assume that \( |\mathcal{F}| \geq \binom{n}{r} - \binom{n-\sigma+1}{r} \), since otherwise we are done. By Theorem 4.2, there exists a \((\sigma - 1)\)-set \( A \) that satisfies (27). Define \( \mathcal{L}^*, \mathcal{F}_A, S_A \) as in Lemma 9.3, and define \( \mathcal{B} := \mathcal{F} \setminus \mathcal{F}_A \). If \( \mathcal{B} = \emptyset \), then we are done. Suppose \( \mathcal{B} \neq \emptyset \). If we can show that there exists an \((r-1)\)-graph \( \mathcal{D} \) on \([n] \setminus A \) satisfying (28), i.e. \( \partial_{r-2}(\mathcal{D}) \cap \partial_{r-2}(\mathcal{L}^*) = \emptyset \) and \( |\mathcal{D}| \geq \gamma|\mathcal{B}| \), then by Lemma 9.3, \( |\mathcal{F}| = |\mathcal{F}_A \cup \mathcal{B}| \leq \binom{n}{r} - \binom{n-\sigma+1}{r} \), and we are done.

Towards that goal, let \( \mathcal{D} := \partial_{r-1}(\mathcal{B}) \) and \( \gamma := 1/s \). Since \( \mathcal{H} \not\subseteq \mathcal{B} \), by Lemma 5.4 \(|\mathcal{B}| \leq s|\partial_{r-1}(\mathcal{B})| = s|\mathcal{D}| \), and hence \(|\mathcal{D}| \geq (1/s)|\mathcal{B}| \), as desired. By our assumption about \( \mathcal{H} \), \( \mathcal{H} \) contains an edge \( F_0 \) such that \( \sigma(\mathcal{H} \setminus F_0) = \sigma - 1 \). Let \( W = F_0 \cap V(\mathcal{H} \setminus F_0) \). Since \( F_0 \) contains at least two degree 1 vertices, \(|W| \leq r - 2 \). By our assumption, \( \mathcal{H} \setminus F_0 \) has a cross-cut \( S' \) of size \( \sigma - 1 \). Let \( x \) be a degree 1 vertex in \( F_0 \setminus W \). Then \( S = S' \cup \{x\} \) is a cross-cut of \( \mathcal{H} \). By Proposition 5.10 \( \mathcal{H} - S \) is embeddable in an \((r-1)\)-tree and hence it is \((r-1)\)-partite. Since \( \mathcal{H}' - S' \subseteq \mathcal{H} - S \), \( \mathcal{H}' - S' \) is an \((r-1)\)-partite \((r-1)\)-graph.

Suppose for contradiction that \( \exists U \subseteq \partial_{r-2}(\mathcal{L}^*) \cap \partial_{r-2}(\mathcal{B}) \). Let \( E \) be an edge of \( \mathcal{B} \) that contains \( U \). Let \( M \) be an edge of \( \mathcal{L}^* \) that contains \( U \). By our assumption, there exists a copy \( \mathcal{K} \) of \( K_{(r-1)}^{(r-1)}(s) \) in \( \mathcal{L}^* \) containing \( M \). Let \( U' \) be a subset of \( U \) of size exactly \(|W'| \). Since \( \mathcal{H}' - S' \) is an \((r-1)\)-partite \((r-1)\)-graph on fewer than \( s \) vertices, we can easily find a mapping \( f \) of \( \mathcal{H}' - S' \) into \( \mathcal{K} \) such that \( W \) is mapped onto \( U' \) and such that \( f(\mathcal{H}' - S') \) does not contain any vertex of \( E \setminus U' \). Now \( (A \times f(\mathcal{H}' - S')) \cup E \subseteq \mathcal{F} \) contains a copy of \( \mathcal{H} \), a contradiction. Hence \( \partial_{r-2}(\mathcal{D}) \cap \partial_{r-2}(\mathcal{L}^*) = \emptyset \). This completes the proof.

\[ \Box \]

11 Proof of Theorem 4.4, sharper error term for \( r \)-trees

Let \( \mathcal{H} \) be the 2-reducible \( r \)-tree on \( s \) vertices in Theorem 4.4, \( \sigma = \sigma(\mathcal{H}) \), and let \( \mathcal{F} \subseteq \binom{[n]}{r} \) such that \( \mathcal{H} \not\subseteq \mathcal{F} \). We will show that

\[ |\mathcal{F}| \leq \binom{n}{r} - \binom{n-\sigma+1}{r} + \frac{1}{c(r,s)} \binom{n}{r-2} \]  

(30)

for sufficiently large \( n \), where \( c(r,s) \) is the constant in Theorem 6.1. We may assume that \( |\mathcal{F}| \geq \binom{n}{r} - \binom{n-\sigma+1}{r} + \frac{1}{c(r,s)} \binom{n}{r-2} \), since otherwise there is nothing to prove.

By Lemma 9.3 there exists a \((\sigma - 1)\)-set \( A \) that satisfies (27). Define \( \mathcal{L}^*, \mathcal{F}_A, S_A \) as in Lemma 9.3 and define \( \mathcal{B} \subseteq \mathcal{F} \setminus \mathcal{F}_A \). By Lemma 6.1 there exists an \((r, s)\)-homogeneous subfamily \( \mathcal{B}^* \) of \( \mathcal{B} \) with \(|\mathcal{B}^*| \geq c(r,s)|\mathcal{B}| \). By Lemma 6.5 either \(|\mathcal{B}^*| \) is homogeneously centralized with threshold \( s \) or \(|\mathcal{B}^*| \leq \binom{n}{r-2} \). If \(|\mathcal{B}| \leq \frac{1}{c(r,s)} \binom{n}{r-2} \), then since \( \mathcal{F} = \mathcal{F}_A \cup \mathcal{B} \subseteq S_A \cup \mathcal{B} \), (30) already holds. Hence, we may assume that \( \mathcal{B}^* \) is homogeneously centralized with threshold \( s \).

For each \( F \in \mathcal{B}^* \) as usual let \( c(F) \) denote the central element of \( F \). By the remarks before Lemma 6.5, the kernel degree of \( D \) is at least \( s \) in \( \mathcal{B}^* \) for all proper subsets \( D \) of \( F \) containing \( c(F) \). Furthermore, \( F \setminus c(F) \) is contained in precisely one edge, namely \( F \), of \( \mathcal{B}^* \). Let \( X_1, \ldots, X_r \) be an associated partition of \( \mathcal{B}^* \). Without loss of generality we may assume that \( \forall F \in \mathcal{B}^*, F \cap X_1 = \{c(F)\} \). For each \( x \in X_1 \), let \( \mathcal{B}_x = \{F \in \mathcal{B}^* : c(F) = x\} \) and \( \mathcal{D}_x = \{F \setminus x : F \in \mathcal{B}^*\} \). The families \( \mathcal{B}_x \)'s
partition \( B^* \). Define \( D := \bigcup_{x \in X_1} D_x \).

Claim 11.1. \( \partial_{r-2}(\mathcal{L}^*) \cap \partial_{r-2}(D) = \emptyset \).

Proof of Claim 11.1. Let \( S \) be a minimum cross-cut. By Lemma 5.13, there exists a \( w \in S \) such that \( \mathcal{H}_w \) and \( \mathcal{H}' = \mathcal{H}' \setminus \mathcal{H}_w \) are \( r \)-trees. Furthermore, there are \( E \in \mathcal{H}_w \) and \( F \in \mathcal{H}' \) such that \( E \) is a starting edge of \( \mathcal{H}_w \) and \( V(\mathcal{H}') \cap V(\mathcal{H}_w) = E \cap F \). Define \( D := E \cap F \). Since \( E \) has at least two degree 1 vertices, \( |D| \leq r - 2 \).

It suffices to show that \( \forall x \in X_1, \partial_{r-2}(\mathcal{L}^*) \cap \partial_{r-2}(D_x) = \emptyset \). Suppose, on the contrary, that for some \( x \in X_1, \exists M \in \partial_{r-2}(\mathcal{L}^*) \cap \partial_{r-2}(D_x) \). Let \( M' \) be a subset of \( M \) of size \( |D| \). Let \( E' \) be an edge of \( D_x \) that contains \( M \) and thus contains \( M' \). Let \( f \) be a mapping that maps \( D \) onto \( M' \). Since \( \mathcal{H}_w - \{w\} \) is an \((r - 1)\)-tree on fewer than \( s \) vertices, by Lemma 5.13, \( f \) can be extended to an embedding of \( \mathcal{H}_w - \{w\} \) into \( D_x \). Let \( E'' \) be any edge of \( \mathcal{L}^* \) that contains \( M' \). By the definition of \( \mathcal{L}^* \), there exists a copy \( \mathcal{K} \) of \( \mathcal{K}^{(r-1)}_r \) in \( \mathcal{L}^* \) that contains \( E'' \). Let \( S' = S \setminus \{w\} \). Since \( \mathcal{H}' - \mathcal{H}' \) is \((r - 1)\)-partite and \( \mathcal{H} \) has \( s \) vertices, we can find a mapping \( g \) of \( \mathcal{H}' - \mathcal{H}' \) into \( \mathcal{K} \) that agrees with \( f \) on vertices in \( D \) and such that \( g(\mathcal{H}' - \mathcal{H}') \cap M' \) is disjoint from \( f(\mathcal{H}_w - \{w\}) \). Now since \( g(\mathcal{H}' - \mathcal{H}') \subseteq \mathcal{K} \) is in the common link graph of \( A \) and \( f(\mathcal{H}_w - \{w\}) \) is in the link of \( x \), we can obtain a copy of \( \mathcal{H} \) in \( \mathcal{F} \) by mapping \( S' \) to \( A \) and \( w \) to \( x \), a contradiction. \( \square \)

Since for each \( F \), \( \deg_{\mathcal{G}_r}(F \setminus c(F)) = 1 \), we have \( |\mathcal{B}^*| = |D| \geq c(r, s)|\mathcal{B}| \). Hence \( D \) satisfies both conditions in (28) with \( \gamma := c(r, s) \). By Lemma 9.3, \( |\mathcal{F}| \leq \binom{n}{r} - \binom{n - \sigma + 1}{r} \) for sufficiently large \( n \), which contradicts our earlier assumption about \( |\mathcal{F}| \). This completes the proof of Theorem 4.3. \( \square \)

12 Proof of Theorem 4.5 on 2-tree expansions

Let \( \mathcal{H} \) be an \((r - 2)\)-reducible \( r \)-tree on \( s \) vertices with \( \sigma(\mathcal{H}) = t + 1 \) and \( S \) a minimum cross-cut, \( \pi \) a tree-defining ordering of \( \mathcal{H} \) and \( w \) the last vertex in \( S \) that is included in \( \pi \). Let \( \mathcal{H}_w \) be the subgraph of \( \mathcal{H} \) consisting of all the edges containing \( w \). Since \( \mathcal{H} \) can be obtained from a 2-uniform forest by expanding each edge into a number of \( r \)-sets through expansion vertices we have that every two edges of \( \mathcal{H} \) intersect in at most two vertices.

By Lemma 5.13, \( \mathcal{H}_w \) and \( \mathcal{H}' = \mathcal{H} \setminus \mathcal{H}_w \) are \( r \)-trees and that \( \exists E \in \mathcal{H}_w \) and \( F \in \mathcal{H}' \) such that \( E \) is a starting edge of \( \mathcal{H}_w \) and \( V(\mathcal{H}_w) \cap V(\mathcal{H}') = E \cap F \). If \( \mathcal{H}_w \) only has one edge, then Theorem 4.3 applies and we get \( \exp(n, \mathcal{H}) \leq \binom{n}{r} - \binom{n - \sigma + 1}{r} \) and we are done. Hence, we may assume that \( \mathcal{H}_w \) contains at least two edges. So \( w \) has degree at least two in \( \mathcal{H} \). Since \( \mathcal{H} \) is \((r - 2)\)-reducible, \( E \) has at most two vertices of degree two or higher, one of which is \( w \), \( w \notin V(\mathcal{H}') \). So \( E \cap F \) contains at most one vertex. Also, since \( \mathcal{H}_w \) is a linear hypergraph, if \( E \cap F \) contains a vertex \( y \) then no edge in \( \mathcal{H}_w \) other than \( E \) contains \( y \).

Let \( \mathcal{F} \subseteq \binom{\mathcal{H}}{2} \) such that \( \mathcal{H} \not\subseteq \mathcal{F} \). We may assume that \( |\mathcal{F}| \geq \binom{n}{r} - \binom{n - \sigma + 1}{r} \), since otherwise we are done. By Lemma 9.3, there exists a \((\sigma - 1)\)-set \( A \) that satisfies (27). Define \( \mathcal{L}^*, \mathcal{F}_A \), and \( \mathcal{S}_A \) as in Lemma 9.3. Define

\[
\mathcal{B} := \{F \in \mathcal{F} \setminus \mathcal{F}_A : |F \cap V(\mathcal{L}^*)| \leq 1\}
\]

\[
\mathcal{C} := \{F \in \mathcal{F} \setminus \mathcal{F}_A : |F \cap V(\mathcal{L}^*)| \geq 2\}.
\]
We use Lemma 9.3 to show that $|\mathcal{F}_A \cup \mathcal{B}| \leq \binom{n}{r} - \binom{n-\sigma+1}{r}$. This holds trivially if $\mathcal{B} = \emptyset$. So assume $\mathcal{B} \neq \emptyset$. Let $D := \partial_{r-1}(\mathcal{B})$. Then $D$ is an $(r-1)$-graph on $[n] \setminus A$. Also, $\partial_{r-2}(D) = \partial_{r-2}(\mathcal{B})$. Since $r \geq 4$, by the definition of $\mathcal{B}$, $\partial_{r-2}(D) \cap \partial_{r-2}(\mathcal{L}^*) = \emptyset$. Since $\mathcal{H} \not\subseteq \mathcal{B}$, by Proposition 5.4, $|D| \geq (1/s)|\mathcal{B}|$. Hence $D$ satisfies both conditions in (28) with $\gamma := 1/s$. By Lemma 9.3
\[ |\mathcal{F}_A \cup \mathcal{B}| \leq \binom{n}{r} - \binom{n-\sigma+1}{r}. \] (31)

**Claim 12.1.** We have $\mathcal{H}_w \not\subseteq \mathcal{C}$.

**Proof of Claim 12.1.** Suppose $\mathcal{C}$ contains a copy $\tilde{H}$ of $\mathcal{H}_w$, we derive a contradiction. $\mathcal{H}_w$ is a linear star centered at $w$. By earlier discussion, either $\mathcal{H}_w$ is vertex disjoint from $\mathcal{H}'$ or one of its edges $E$ intersects $V(\mathcal{H}')$ at one vertex $y \neq w$ and no other edge of $\mathcal{H}_w$ contains any vertex of $\mathcal{H}'$. In the former case, we can take a copy $\mathcal{K}$ of $\mathcal{K}_w^{(r-1)}(s)$ in $\mathcal{L}^*$ and find a copy of $\mathcal{H}'$ in $A \times \mathcal{K}$ that avoids $\tilde{H}$, which then gives us a copy of $\mathcal{H}$ in $\mathcal{F}$, a contradiction. Consider now the latter case. Since $\mathcal{H}_w$ is a linear star centered at $w$, any of its edges can play the role of $E$ and any of the vertex in $E \setminus \{w\}$ can play the role of $y$. Let $\tilde{w}$ denote the image of $w$ in $\tilde{H}$. Let $\tilde{E}$ be any edge in $\tilde{H}$. Since $\tilde{E} \in \mathcal{C}$, by definition, $|\tilde{E} \cap V(\mathcal{L}^*)| \geq 2$. Let $\tilde{y} \neq \tilde{w}$ be a vertex in $\tilde{E} \cap V(\mathcal{L}^*)$. Since $\tilde{y} \in V(\mathcal{L}^*)$, there exists a copy $\mathcal{K}$ of $\mathcal{K}_w^{(r-1)}(s)$ in $\mathcal{L}^*$ that contains $\tilde{y}$. Now we can easily find a copy $\mathcal{H}^*$ of $\mathcal{H}'$ in $A \times \mathcal{K}$ such that $y$ is mapped to $\tilde{y}$ and such that $V(\mathcal{H}^*) \cap V(\tilde{H}) = \{\tilde{y}\}$. This gives us a copy of $\mathcal{H}$ in $\mathcal{F}$, a contradiction.\]

By Claim 12.1, we have
\[ |\mathcal{C}| \leq \text{ex}(n - \sigma + 1, \mathcal{H}_w). \] (32)

Now, since $\mathcal{F} = \mathcal{F}_A \cup \mathcal{B} \cup \mathcal{C}$, by (31) and (32), we have
\[ |\mathcal{F}| = |\mathcal{F} \cup \mathcal{B}| + |\mathcal{C}| \leq \binom{n}{r} - \binom{n-\sigma+1}{r} + \text{ex}(n - \sigma + 1, \mathcal{H}_w). \]

This completes the proof of Theorem 4.5.\]

13 Concluding remarks

We have identified a large class of $r$-trees $\mathcal{H}$ (i.e. 2-reducible ones) with $\text{ex}(n, \mathcal{H}) \sim (\sigma(\mathcal{H}) - 1)\binom{n}{r-1}$. By contrast, Kalai’s conjecture states that for a tight $r$-trees $\mathcal{H}$ on $n$ vertices $\text{ex}(n, \mathcal{H}) \sim \frac{\overline{\omega}_{\mathcal{H}}}{r} \binom{n}{r-1}$. Already, the family of 1-reducible $r$-trees lie somewhere in-between. There are 1-reducible $r$-trees whose Turán number is more dependent on its cross-cut number and there are 1-reducible $r$-trees whose Turán number is more dependent on its number of vertices. The situation with general $r$-trees is likely even more complex, providing many intriguing questions.

References

[1] H. Abbott, D. Hanson, N. Sauer: Intersection theorems for systems of sets, *J. Combin. Theory Ser. A* 12 (1972), 381–389.

[2] N. Bushaw, N. Kettle: Turán numbers for forests of paths in hypergraphs, *SIAM J. Discree Mathematics*, 28 (2014), 711–721.
[1] V. Chvátal: An extremal set-intersection theorem, *J. London Math. Soc.* 9 (1974/1975), 355–359.

[2] P. Erdős: On extremal problems of graphs and generalized graphs, *Israel Journal of Mathematics* 2 (1964), 183-190.

[3] P. Erdős: A problem on independent r-tuples, *Ann. Univ. Sci. Budapest* 8 (1965), 93–95.

[4] P. Erdős, T. Gallai: On maximal paths and circuits of graphs, *Acta Math. Acad. Sci. Hungar.* 10 (1959), 337–356.

[5] P. Erdős, C. Ko, R. Rado: Intersection theorems for systems of finite sets, *Quart. J. Math. Oxford Ser. (2)* 12 (1961), 313–320.

[6] P. Frankl: On families of finite sets no two of which intersect in a singleton, *Bull. Austral. Math. Soc.* 17 (1977), 125–134.

[7] P. Frankl: Improved bounds for Erdős’ matching conjecture, *J. Combin. Th. Ser. A* 120 (2013), 1068–1072.

[8] P. Frankl, Z. Füredi: A new generalization of the Erdős-Ko-Rado theorem, *Combinatorica* 3 (1983), 341–349.

[9] P. Frankl, Z. Füredi: Exact solution of some Turán-type problems, *J. Combin. Th. Ser. A* 45 (1987), 226–262.

[10] P. Frankl, T. Łuczak, K. Mieczkowska: On matchings in hypergraphs, *Electronic J. Combin.* 19 (2012), Paper 42, 5 pp.

[11] P. Frankl, V. Rödl, A. Ruciński: On the maximum number of edges in a triple system not containing a disjoint family of a given size, *Combinatorics, Probability and Computing* 21 (2012), 141–148.

[12] Z. Füredi: On finite set-systems whose every intersection is a kernel of a star, *Discrete Math.* 47 (1983), 129–132.

[13] Z. Füredi: Hypergraphs in which all disjoint pairs have distinct unions, *Combinatorica* 4 (1984), 161–168.

[14] Z. Füredi: Linear trees in uniform hypergraphs, *European J. Combinatorics,* 35 (2014), 264–272.

[15] Z. Füredi, T. Jiang: Hypergraph Turán numbers of linear cycles, *J. Combin. Th. Ser. A* 123 (2014), 252–270.

[16] Z. Füredi, T. Jiang, R. Seiver: Exact Solution of the hypergraph Turán problem for k-uniform linear paths, *Combinatorica,* 34 (2014), 299–322.

[17] Z. Füredi, L. Özkahya: Unavoidable subhypergraphs: a-clusters, *J. Combin. Th. Ser. A* 118 (2011), 2246–2256.

[18] H. Huang, P. Loh, B. Sudakov: The size of a hypergraph and its matching number, *Combinatorics, Probability and Computing* 21 (2012), 442–450.

[19] D. Irwin, T. Jiang: Turán numbers of clusters, in preparation.

[20] T. Jiang, X. Liu: Turán numbers of a class of r-uniform 2-regular graphs, in preparation.

[21] T. Jiang, O. Pikhurko, Z. Yilma: Set-systems without a strong simplex, *SIAM J. Discrete Math.* 24 (2010), 1038–1045.
[24] G. Katona, Intersection theorems for systems of finite sets, *Acta Math. Acad. Sci. Hungar.* 15 (1964), 329–337.

[25] P. Keevash: On the existence of designs, submitted. (see arXiv:1401.3665)

[26] P. Keevash, D. Mubayi: Set systems without a simplex or a cluster, *Combinatorica* 30 (2010), 175–200.

[27] A. Kostochka, D. Mubayi, J. Verstraëte: Turán problems and shadows I: Paths and Cycles, *Journal of Combin. Th. Ser. A*, 129 (2015), 57–79.

[28] A. Kostochka, D. Mubayi, J. Verstraëte: Turán problems and shadows II: trees, submitted. (See arXiv:1402.0544)

[29] L. Lovász: *Combinatorial Problems and Exercises*, Problem 13.31. Akadémiai Kiadó, Budapest and North Holland, Amsterdam, 1979.

[30] T. Łuczak, K. Mieczkowska: On Erdős’ extremal problem on matchings in hypergraphs, *J. Combin. Theory Ser. A* 124 (2014), 178-194.

[31] D. Mubayi: Erdős-Ko-Rado for three sets, *J. Combinatorial Theory Ser. A* 113 (2006), 547–550.

[32] D. Mubayi, R. Ramadurai: Set systems with union and intersection constraints, *J. Combinatorial Theory Ser. B* 99 (2009), 639–642.

[33] D. Mubayi, J. Verstraëte: A hypergraph extension of the bipartite Turán problem, *J. Combinatorial Theory Ser. A* 106 (2004), 237-253.

[34] D. Mubayi, J. Verstraëte: Proof of a conjecture of Erdős on triangles in set systems, *Combinatorica* 25 (2005), 599–614.

[35] D. Mubayi, J. Verstraëte: Minimal paths and cycles in set systems, *European J. Combin.* 28 (2007), 1681–1693.

[36] V. Rödl: On a packing and covering problem, *European J. Combin.* 6 (1985), 69–78.