CATALAND: WHY THE FUSS?

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Abstract. The main objects of noncrossing Catalan combinatorics associated
to a finite Coxeter system are noncrossing partitions, sortable elements, and
cluster complexes. The first and the third of these have known Fuss–Catalan
generalizations, for which we provide new viewpoints. We introduce a corre-
sponding generalization of sortable elements as elements in the positive Artin
monoid, and show how this perspective ties together all three generalizations.

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1. Introduction

Fix a finite Coxeter system \((W, S)\) and a (standard) Coxeter element \(c \in W\)—
that is, a product of the simple generators \(S\) in any order. There are three non-
crossing Catalan objects associated to this data [Rea07a, Rea07b, BRT12, PS11]:

- the \(c\)-noncrossing partitions \(NC(W, c)\);
- the \(c\)-sortable elements \(Sort(W, c)\); and
- the \(c\)-cluster complexes \(Asso(W, c)\).

We use the term “noncrossing” here for historical reasons—noncrossing partitions
generalize noncrossing set partitions to all finite Coxeter systems and all standard
Coxeter elements. We also attach this term to the other families, since they are
in uniform bijection with the noncrossing partitions, and to distinguish them from
the nonnesting Catalan objects, which we do not consider in this paper. We denote
the cluster complex by \(Asso(W, c)\) in order to emphasize that it generalizes the
well-studied dual associahedron.

Although no uniform proof is currently known, these noncrossing families are
counted by the Catalan number of type \(W\)

\[
\text{Cat}(W) := \prod_{i=1}^{n} \frac{h + d_i}{d_i},
\]

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where \( d_1 \leq d_2 \leq \ldots \leq d_n \) are the degrees and \( h := d_n = \text{ord}(c) \) is the Coxeter number of \( W \). These noncrossing Catalan objects have two natural lattice structures, each of which has an associated cyclic group action:

- The \( c \)-noncrossing partition lattice \( \text{NCL}(W,c) \) and its anti-automorphism of order \( 2h \) given by the Kreweras complement \( \text{Krew}_c \); and
- the \( c \)-Cambrian lattice \( \text{Camb}(W,c) \) and its underlying graph automorphism of order \( h + 2 \) given by the Cambrian rotation \( \text{Camb}_c \).

The noncrossing partitions, their lattices and Kreweras complements, and the cluster complex and Cambrian rotation have all been generalized in the literature by introducing a nonnegative integral parameter \( m \). Since many generalizations exist, and since many names in the theory already carry the term “generalized”, we refer to this particular generalization as an \( m \)-eralization. The objects associated with these \( m \)-eralizations are counted by the \textit{Fuß-Catalan number of type} \( W \)

\[
\text{Cat}^{(m)}(W) := \prod_{i=1}^{n} \frac{m h + d_i}{d_i}.
\]

More precisely,

- S. Fomin and N. Reading defined and studied the \( m \)-eralized \( c \)-cluster complex \( \text{Asso}^{(m)}(W,c) \) for a bipartite Coxeter element \( c \) in \cite{FR05}; and
- D. Armstrong defined and studied the \( m \)-eralized \( c \)-noncrossing partitions \( \text{NC}^{(m)}(W,c) \) for any Coxeter element \( c \) in \cite{Arn06}, generalizing P. Dehornoy’s construction from \cite{Edel80} to all finite Coxeter groups.

There are several components missing from this story. The aim of this paper is to complete the \( m \)-eralization of noncrossing Catalan objects using the spherical Artin group corresponding to the finite Coxeter system \((W; S)\).

**Coxeter-sortable elements.** The most glaring omission is that no \( m \)-eralization of \( W \), Reading’s \( c \)-sortable elements \cite{Rea07b} has appeared in the literature. It is straightforward to extract a definition involving chains of sortable elements by combining D. Armstrong’s definition of \( m \)-eralized \( c \)-noncrossing partitions and N. Reading’s shard order (see Section 4.7), but such a definition is unsatisfactory without a corresponding \( m \)-eralization of the weak order. At the heart of this paper is the observation that such an \( m \)-eralization is given by the interval \([c, w_0^m] \) in the weak order of the positive Artin monoid \( B^+ = B^+(W) \). These intervals have been previously studied by P. Dehornoy in an enumerative context \cite{Deh07}, and we study some of their properties in Section 2.8.

**Definition 1.1.** \( W^{(m)} := \{ w \in B^+(W) : w \leq w_0^m \in B^+(W) \} \).

There is a natural bijection between the elements of \( W \) and \( W^{(1)} \). Furthermore, \( W^{(m)} \) is a rank-symmetric lattice under the weak order, and we recover the usual weak order when \( m = 1 \). In Section 4.3, we provide the missing \( m \)-eralization of the \( c \)-sortable elements as a lift of N. Reading’s definition from \( W \) to \( W^{(m)} \).

**Definition 1.2.** Let \( c \in W \) be a Coxeter element. An element \( w \in W^{(m)} \) is \( c \)-\textit{sortable} if the \( c \)-sorting word \( w(c) \) for \( w \) defines a decreasing sequence of subsets of positions in \( c \). Denote the set of all such \( m \)-eralized \( c \)-sortable elements by \( \text{Sort}^{(m)}(W,c) \).

We refer to Section 2.1 for the definition of the \( c \)-sorting word \( w(c) \). We provide three characterizations of \( \text{Sort}^{(m)}(W,c) \) in Definition 4.5, Proposition 4.9, and in Definition 4.13, extending N. Reading’s characterization of \( c \)-sortable elements when \( m = 1 \), as given in \cite[Sections 2 and 4]{Rea07a}. The relationship between chains of \( c \)-sortable elements in shard order and \( \text{Sort}^{(m)}(W,c) \) turns out to be analogous.
to the relationship between chains of noncrossing partitions and delta sequences, as explained in Theorem 4.56.

Cluster complexes. The second missing component is that there is no simple combinatorial definition of the \( m \)-eralized \( c \)-cluster complex when the Coxeter element \( c \) is not bipartite. We supply this definition in Section 5.4, giving an \( m \)-eralized \( c \)-compatibility relation. We prove the existence and uniqueness of this relation using an \( m \)-eralization of the subword complex approach to \( c \)-cluster complexes given in [CLS14, PS11].

Definition 1.3. The simplicial complex \( \textnormal{Asso}^{(m)}(W, c) \) is the set of subwords (as indices of positions) of the word \( w^m_0(c) \) whose complements contain a word for \( w^m_0 \in W \) of length \( mN \), where \( N = \ell(w_0) \).

Using this definition, we recover in Proposition 5.25 that \( \textnormal{Asso}^{(m)}(W, c) \) is vertex-decomposable, and hence shellable. With the \( m \)-eralizations of noncrossing partitions, sortable elements, and subword complexes in hand, we prove the following theorem and several refinements in Theorems 4.34, 4.56, 5.33 and 5.37.

Theorem 1.4. There are explicit and uniform bijections between the three families

- the \( m \)-eralized \( c \)-noncrossing partitions \( \text{NC}^{(m)}(W, c) \);
- the \( m \)-eralized \( c \)-sortable elements \( \text{Sort}^{(m)}(W, c) \); and
- the \( m \)-eralized \( c \)-cluster complexes \( \text{Asso}^{(m)}(W, c) \).

These bijections are equivariant under \( m \)-eralized Cambrian rotation.

By “explicit,” we mean that we provide a bijection, rather than just proving existence of a bijection by a counting argument. The term “uniform” specifies that the description of the bijection does not use the classification theorem of finite Coxeter systems.

Cambrian lattices. Although the exchange graph of S. Fomin and N. Reading’s \( m \)-eralized \( c \)-cluster complex can be used to define a Cambrian graph for bipartite Coxeter elements, no corresponding poset has been considered in the literature for \( m > 1 \). In particular, no orientation of the exchange graph was known to be a lattice. In Sections 3.4, 4.6 and 5.7, we close this third gap with definitions of the \( m \)-eralized \( c \)-Cambrian lattice on each of \( \text{NC}^{(m)}(W, c) \), \( \text{Sort}^{(m)}(W, c) \), and \( \text{Asso}^{(m)}(W, c) \). Our construction \( m \)-eralizes N. Reading’s Cambrian lattices, which are themselves generalizations of the classical Tamari lattices.

For \( \text{NC}^{(m)}(W, c) \) and \( \text{Asso}^{(m)}(W, c) \), we construct these posets as the transitive closures of the objects under certain flips. Exactly as in the case \( m = 1 \), the structure on \( \text{Sort}^{(m)}(W, c) \) is inherited from the weak order on \( W^{(m)} \). This yields the following theorem which is proven in Theorems 4.40, 4.42 and 5.42.

Theorem 1.5. The restriction of the weak order to \( \text{Sort}^{(m)}(W, c) \) is a lattice. It is isomorphic to the increasing flip posets of \( \text{NC}^{(m)}(W, c) \) and of \( \text{Asso}^{(m)}(W, c) \).

We call this common lattice the \( m \)-eralized \( c \)-Cambrian lattice. The intuition behind this construction on \( \text{NC}^{(m)}(W, c) \)—which appears to be new even when \( m = 1 \)—is given in Section 3.4, where it naturally arises from transporting a construction of weak order via descent and ascent sets to the noncrossing partitions. Another intuition comes from the isomorphism between shard order restricted to \( c \)-sortable elements and the noncrossing partition lattice: just as D. Armstrong considered component-wise absolute order on chains in \( \text{NCL}(W, c) \) [Arm06], our \( m \)-eralized \( c \)-Cambrian lattices can be seen to be component-wise weak order on the corresponding chains in \( \text{Shard}(W) \).
In summary, we place the program of \( m \)-eralizing noncrossing Coxeter–Catalan combinatorics in the context of the corresponding positive Artin monoid. For the reader’s convenience, Figure 1 displays the Fuß-Catalan objects we consider, and the bijections between the various objects.

At least in the classical types, our definitions naturally extend to Coxeter-theoretic constructions of noncrossing objects in the framework of D. Armstrong’s program of “rational Catalan combinatorics” [ARW13, ALW14], which we plan to explore in the future. In a subsequent paper, we will study local versions of the Kreweras complement and of Cambrian rotation, and we will relate these local actions to the study of nonnesting partitions.

2. \( m \)-ERALIZED WEAK ORDER

In this section, we first recall some background on finite Coxeter groups and their Artin groups. Further background can be found in [BS72, DDG+13, Hum90]. We will then consider inversion sequences, introduce subword complexes and dual subword complexes, and recall N. Reading’s definition of Coxeter-sorting words. Finally, we define an \( m \)-eralized version of the weak order and study some of its properties.

2.1. Coxeter and Artin groups. A finite Coxeter system \((W, S)\) of rank \( n = |S|\) is a finite group \(W\) together with a distinguished subset \(S \subseteq W\) of generators and a presentation

\[ W = \langle S : s^2 = e, (st)^{m(s,t)/2} = (ts)^{m(t,s)/2} \text{ for } s, t \in S \text{ with } s \neq t \rangle, \]

for integers \(m(s,t) = m(t,s) \geq 2\), and where we set

\[ (st)^{m(s,t)/2} := \begin{cases} 
(st)^{\frac{m(s,t)-1}{2}} & \text{if } m(s,t) \text{ even} \\
(st)^{\frac{m(s,t)-1}{2}} & \text{if } m(s,t) \text{ odd} 
\end{cases} \]

The elements in the set \(S\) are called simple generators or simple reflections. The relations \(sts \cdots = tst \cdots\) are called braid relations, and replacing \(sts \cdots\) by \(tst \cdots\) in a word in \(S\) is called a braid move. Observe that for \(s, t \in S\), we have \(m(s,t) = 2\) if and only if \(s\) and \(t\) commute in \(W\).

It follows from the classification of finite Coxeter systems that we have the decomposition \(S = S_L \sqcup S_R\), where all reflections in \(S_L\) pairwise commute, as do those in \(S_R\). To not have to distinguish various cases later, we fix such a decomposition throughout the paper.
The set of reflections in $W$ is defined to be
$$\mathcal{R} := \{s^w : s \in \mathcal{S}, w \in W\},$$
where we write $u^w := uwu^{-1}$.

The spherical Artin system $(B(W), \mathcal{S})$ corresponding to the finite Coxeter system $(W, \mathcal{S})$ is the group $B(W)$ given by (a formal copy of) the generators $\mathcal{S}$ and the presentation
$$B(W) = \langle \mathcal{S} : s^m(s,s')^2 = (s's)^m(s',s)^2 \text{ for } s, s' \in \mathcal{S} \text{ with } s \neq s' \rangle.$$

When $W$ is a symmetric group, $B(W)$ is called a braid group. For ease of notation, we will not explicitly distinguish between the simple generators of $W$ and of $B(W)$. Rather than studying $B(W)$, we will restrict ourselves to the positive Artin monoid $B^+(W)$ (or simply $B^+$ if there is no confusion), which is the submonoid of $B(W)$ generated by $\mathcal{S}$.

**Example 2.1.** To illustrate our definitions and results, we will use the running example of the symmetric group on three letters. This group is the Coxeter group $A_2$ with simple generators $\mathcal{S} = \{s, t\}$ that correspond to the simple transpositions $s = (12)$ and $t = (23)$. The third reflection is denoted by $u = (13)$, giving
$$\mathcal{R} = \{s, u, t\} = \{(12), (13), (23)\}.$$ Beside these reflections and the identity element $e$, the group $A_2$ also contains the two “rotations”
$$(123) = st = tu = us \quad \text{and} \quad (321) = ts = ut = su.$$ The positive Artin monoid $B^+(A_2)$ consists of all words in $s$ and $t$, subject to the relation $sts = tst$.

2.2. Coxeter length, support, and absolute length. The (Coxeter) length of an element $w$ in the group $W$ or in the positive Artin monoid $B^+$ is the length $\ell(w)$ of a shortest expression for $w$ as a product of the generators in $\mathcal{S}$. An $\mathcal{S}$-word $s_1 \cdots s_p$ with $s_1, \ldots, s_p \in \mathcal{S}$ is reduced if $w = s_1 \cdots s_p$ and $p = \ell(w)$. Figures 2 and 3 illustrate several examples. We use Roman letters to indicate that we consider words in $\mathcal{S}$ rather than elements in $W$ or in $B^+$.

It is classical that any reduced word of an element in $W$ may be transformed to any other by a sequence of braid moves [BB05, Theorem 3.3.1]. A reduced word for an element $w \in W$ gives rise to a corresponding element in $B(W)$; since $B(W)$ contains the braid relations, any two reduced words for $w \in W$ specify the same element in $B(W)$. The unique longest element in $W$ is denoted by $w_0$, and we will also use the notation $w_0$ to refer to the corresponding element in $B^+(W)$. This element is called the fundamental element in Garside theory. We denote its length by $N = \ell(w_0) = |\mathcal{R}|$.

We define the support $\text{supp}(w)$ for $w \in W$ to be the set $\{s_1, \ldots, s_p\} \subseteq \mathcal{S}$ of simple reflections contained in any reduced word $s_1 \cdots s_p$ for $w$. Analogously, we define the support $\text{supp}(w)$ for $w \in B^+(W)$, since all words for $w \in B^+(W)$ are—by definition—related by a sequence of braid moves. We again refer to Figures 2 and 3 for examples.

Instead of considering words in simple reflections for elements in $W$, it is natural to also consider words in general reflections. The absolute length of an element $w$ in $W$ is the length $\ell_{\mathcal{R}}(w)$ of a shortest expression for $w$ as a product of reflections. An $\mathcal{R}$-word $r_1 \cdots r_p$ with $r_1, \ldots, r_p \in \mathcal{R}$ is reduced if $w = r_1 \cdots r_p$ and $p = \ell_{\mathcal{R}}(w)$. We now only refer to Figure 2 for examples. See [BKL98] for a construction of the braid group and [Bes03] for constructions of spherical Artin groups using a related construction.
2.3. Finite reflection groups. We briefly review the geometry of Coxeter systems. Let $V$ be an $n$-dimensional Euclidean vector space. For $v \in V \setminus \{0\}$, we denote by $s_v$ the reflection interchanging $v$ and $-v$ while fixing the orthogonal hyperplane pointwise. We consider a finite (real) reflection group $W$ acting on $V$, i.e., a finite group generated by the reflections $R$ it contains. We assume without loss of generality that the intersection of all the corresponding reflecting hyperplanes is $\{0\}$. Fix a connected component—the fundamental chamber—of the complement of the reflecting hyperplanes. The simple reflections of $W$ with respect to this choice of a fundamental chamber then are the $n$ reflections orthogonal to its facet-defining hyperplanes. The set $S \subseteq R$ of simple reflections generates $W$, and the pair $(W, S)$ then forms a Coxeter system. Moreover, every Coxeter system may be obtained in this way.

2.4. Root systems. Let $W$ be a finite reflection group with a fixed choice of a fundamental chamber. For simple reflections $s, t \in S$, denote by $m_{st}$ the order of the product $st \in W$. Fix a generalized Cartan matrix for $(W, S)$, i.e., a matrix $(a_{st})_{s,t\in S}$ such that $a_{ss} = 2$, $a_{st} \leq 0$, $a_{st}a_{ts} = 4\cos^2\left(\frac{\pi}{m_{st}}\right)$ and $a_{st} = 0 \Leftrightarrow a_{ts} = 0$ for all $s \neq t \in S$. We can associate to each simple reflection $s$ a simple root $\alpha_s \in V$, orthogonal to the reflecting hyperplane of $s$ and pointing toward the half-space containing the fundamental chamber, so that $s(\alpha_s) = \alpha_t - a_{st}\alpha_s$ for all $s, t \in S$. The set of all simple roots is denoted by $\Delta := \{\alpha_s : s \in S\}$. The orbit $\Phi := \{w(\alpha_s) : w \in W, s \in S\}$ of $\Delta$ under $W$ is a root system for $W$. It is invariant under the action of $W$ and contains precisely two opposite roots orthogonal to each reflecting hyperplane of $W$.

When discussing finite Coxeter systems, we always think of the Coxeter system together with a fixed root system associated to it, since there is not necessarily a unique root system associated to a finite reflection group $W$.

The set $\Delta$ of simple roots forms a linear basis of $V$ (since we assumed $W$ to act essentially on $V$). The root system $\Phi$ is the disjoint union of the positive roots $\Phi^+ := \Phi \cap \mathbb{R}_{\geq 0}[\Delta]$ (non-negative linear combinations of simple roots) and the negative roots $\Phi^- := -\Phi^+$. We write $\beta > 0$ or $\beta < 0$ to indicate if $\beta$ is a positive or a negative root, and we denote by $|\beta|$ the positive root in $\{\pm \beta\}$. Each reflecting hyperplane is orthogonal to one positive and one negative root. For a reflection $r \in R$, we set $\alpha_r$ to be the unique positive root orthogonal to the reflecting hyperplane of $r$, so that $r = s_{\alpha_r}$.

**Example 2.2.** The Coxeter group $A_2$ considered above can be realized as the dihedral group of isometries of a regular triangle. Its root system consists of the simple and positive roots

$$\Delta = \{\alpha, \beta\} \quad \text{and} \quad \Phi^+ = \{\alpha, \gamma, \beta\}$$

with $\alpha = e_2 - e_1, \beta = e_3 - e_2$, and $\gamma = e_3 - e_1 = \alpha + \beta$ and such that

$$s = s_\alpha, \quad t = s_\beta, \quad u = s_\gamma = s_\alpha s_\beta s_\alpha = s_\beta s_\alpha s_\beta.$$

The Coxeter group $W$ is said to be crystallographic if it stabilizes a lattice in $V$ when considered as a finite reflection group. This can only happen if all entries of the Cartan matrix are integers, or, equivalently, if $m(s, t) \in \{2, 3, 4, 6\}$ for all $s, t \in S$ with $s \neq t$. Reciprocally, if all entries of the Cartan matrix are integers, then the lattice generated by the simple roots $\Delta$ is fixed by the Coxeter group $W$.

2.5. Inversion sets and inversion sequences. The Coxeter length of an element $w \in W$ is equal to the cardinality of the inversion set of $w$, defined as the set $\text{inv}(w) := \Phi^+ \cap w(\Phi^-)$ of positive roots sent to negative roots by $w^{-1}$. Indeed,

$$\text{inv}(w) = \{\alpha_s, s_1(\alpha_s), \ldots, s_1 \cdots s_{l-1}(\alpha_s)\} \subseteq \Phi^+.$$
for any reduced expression $w = s_1 \cdots s_\ell$. We write

$$\text{inv}_R(w) := \{s_1, s_2 \cdots, s_\ell, s_\ell^{-1} \cdots s_1^{-1}\}$$

for the set of corresponding reflections.

The verbatim generalization of inversion sets to inversion multisets (or colored inversion sets, see below) for the positive Artin monoid $B^+$ does not generally distinguish positive braids. For example, $ssst \neq tstt$ in $B^+(A_2)$, even though both have inversion multiset $\{\pm \alpha, \pm \beta\}$.

A particular choice of a reduced expression $s_1 \cdots s_\ell$ for $w \in W$ induces a total order $\text{inv}(s_1 \cdots s_\ell)$ on the positive roots $\text{inv}(w)$, and a reduced expression of the longest element $w_0$ induces an order $\text{inv}(w_0)$ on $\Phi^+ = \text{inv}(w_0)$ such orders on $\Phi^+$ are called root orders. For simplicity of notation, we also refer with this term to the analogous orders on $R$ induced by $\text{inv}_R(w_0)$. We refer to [Dye93] for further details on root orders.

Given any $S$-word $Q = s_1 \cdots s_p$, one can still consider the corresponding inversion sequence $(\alpha_{s_1}, s_1(\alpha_{s_2}), \ldots, s_1 \cdots s_{p-1}(\alpha_{s_p}))$. Since such sequences contain negative roots, and possibly even repeated roots, if the word Q is not reduced, we now consider colored inversion sequences on colored positive roots, which are simply positive roots together with a color given by a nonnegative integer. A simple reflection $s \in S$ acts on a colored positive root as $s(\beta^{(k)}) = [s(\beta)]^{(k)}$ if $\beta \neq \alpha_s$ and $s(\beta^{(k)}) = \beta^{(k+1)}$ if $\beta = \alpha_s$. An element $w \in W$ therefore acts on colored positive roots as

$$w(\beta^{(k)}) = \begin{cases} [w(\beta)]^{(k)} & \text{if } w(\beta) > 0 \\ [-w(\beta)]^{(k+1)} & \text{if } w(\beta) < 0 \end{cases}.$$  

The colored inversion sequence of $Q$ is given by

$$\text{inv}(Q) := (\beta_1^{(m_1)}, \ldots, \beta_p^{(m_p)}),$$

where $\beta_i^{(m_i)}$ is given by $s_1 \cdots s_{i-1}(\alpha_{s_i}^{(0)})$. In other words, $\beta_i = [s_1 \cdots s_{i-1}(\alpha_{s_i})] \in \Phi^+$ and $m_i$ is given by the number of times $s_j \cdots s_{i-1}(\alpha_{s_i}) \in \{\pm \alpha_{s_i}\}$ for $j = 1, 2, \ldots, i - 1$, or, equivalently, the number of times $s_j \cdots s_{i-1}(\alpha_{s_i})$ and $s_{j+1} \cdots s_{i-1}(\alpha_{s_i})$ have different signs. For ease of notation and agreement with the above definition for reduced $S$-words, we may suppress the superscripts if $s_1 \cdots s_p$ is reduced.

It will be useful to also define the corresponding (uncolored) reflection sequence by

$$\text{inv}_R(Q) := (s_{\beta_1}, s_{\beta_2}, \ldots, s_{\beta_p}) = (s_1, s_2 \cdots, s_p^{s_1^{-1} \cdots s_{p-1}^{-1}}).$$

**Example 2.3.** The inversion sequence in $A_2$ for the reduced $S$-word $sts$ of the element $u = sts$ is $\text{inv}(sts) = \alpha, \gamma, \beta$, and the colored inversion sequence of its square $stssts \in B^+(A_2)$ is

$$\text{inv}(stssts) = (\alpha^{(0)}, \gamma^{(0)}, \beta^{(1)}, \gamma^{(1)}, \alpha^{(1)}),$$

with reflection sequence $\text{inv}_R(ststs) = (s, u, t, t, u, s)$.

### 2.6. Subword complexes and dual subword complexes

We next consider certain natural simplicial complexes associated to words in $S$ or in $R$ and elements in $W$.

**Definition 2.4.** Let $Q$ be a word in $S$, let $w \in W$, and let $a = \ell(w) + 2g$ for some $g \geq 0$. The subword complex $\text{SUB}_S(Q, w, a)$ is the simplicial complex with facets being all subsets of (positions of) letters in $Q$ whose complements yield an $S$-word for $w$ of length $a$. 
Definition 2.5. Let $Q$ be a word in $\mathcal{R}$, let $w \in W$, and let $a = \ell_R(w) + 2g$ for some $g \geq 0$. The dual subword complex $\text{Sub}_\mathcal{R}(Q, w, a)$ is the simplicial complex with facets being all subsets of (positions of) letters in $Q$ that yield an $\mathcal{R}$-word for $w$ of length $a$.

The definition of subword complexes for $g = 0$ was given by A. Knutson and E. Miller in [KM04], where they provided various properties of such subword complexes. In this case, we drop the parameter $a$ from the notation, and we write $\text{Sub}_\mathcal{R}(Q, w)$ and $\text{Sub}_\mathcal{R}(Q, w, a)$, respectively. For $w = w_\gamma$, such subword complexes were studied in [CLS14, PS11]. After discussing the duality between the two notions of subword complexes, we will extend some properties from these two references to the case of general parameter $g \geq 0$.

Subword and dual subword complexes are indeed dual to each other in the following sense.

Proposition 2.6. Let $Q = s_1 \cdots s_p$ be a word in $\mathcal{S}$ and let $w \in W$. Let $a = \ell(w) + 2g$ for some $g \geq 0$, $b = p - a$, and $w' = (ws_p \cdots s_1)^{-1}$. Then the two complexes $\text{Sub}_\mathcal{S}(Q, w, a)$ and $\text{Sub}_\mathcal{R}(\text{inv}_\mathcal{R}(Q), w', b)$ coincide.

Proof. The proof follows the same lines as the proof of [IS10, Lemma 3.2] and of [CLS14, Proposition 2.8]. Let $\text{inv}_\mathcal{R}(Q) = r_{i_1} \cdots r_{i_p}$ and let $1 \leq i_1 \leq \cdots \leq i_b \leq p$ be an increasing sequence of positions. This is a facet of $\text{Sub}_\mathcal{S}(Q, w, a)$ if and only if

$$s_1 \cdots \hat{s}_{i_1} \cdots \hat{s}_{i_b} \cdots s_p = w.$$  

Since the left-hand side equals $r_{i_b} \cdots r_{i_1} s_1 \cdots s_p$, the set $\{i_1, \ldots, i_b\}$ is a facet of $\text{Sub}_\mathcal{S}(Q, w, a)$ if and only if

$$r_{i_b} \cdots r_{i_1} = ws_p \cdots s_1 = (w')^{-1},$$

which is equivalent to saying that $\{i_1, \ldots, i_b\}$ is a facet of $\text{Sub}_\mathcal{R}(\text{inv}_\mathcal{R}(Q), w', b)$. $\Box$

Example 2.7. Let $Q = \text{statstst}$ be the word in simple reflections in $A_2$, let $w = e = w_2^\circ$, and let $a = 6$. The reflection sequence corresponding to $Q$ is then

$$\text{inv}_\mathcal{R}(Q) = (s, u, t, s, u, t, s, u).$$

The 12 facets of the subword complex $\text{Sub}_\mathcal{S}(Q, w, a)$ can be found in Figure 12 on page 46.

For $Q = s_1 \cdots s_p$, $w \in W$, and $a = \ell(w) + 2g$, the facets of the subword complex $\text{Sub}_\mathcal{S}(Q, w, a)$ are specified by those sequences $i_1 < \cdots < i_{p-a}$ such that

$$s_1 \cdots \hat{s}_{i_1} \cdots \hat{s}_{i_{p-a}} \cdots s_p = w.$$  

We now consider a slight variation of the colored inversion sequence, where the simple reflections $s_{i_1}, \ldots, s_{i_{p-a}}$ are omitted when computing $\beta_i^{(m)}$ in Equation (1) (see also [CLS14, Definition 3.2] and [PS11, Definition 3.1]). The root configuration of the facet $I$ is the set

$$\text{R}(I) = \{r_I(i) : i \in I\}$$  

of all colored positive roots $r_I(i) = \beta_i^{(m)}$ associated to the elements of $I$ in the above procedure. We call the colored positive root $r_I(i)$ the root vector of $I$ at position $i$. Observe that the root configuration has a natural order induced by the order of the (positions of) letters in $I$. We thus consider the root configuration throughout the paper as an ordered set, and whenever we write a union of colored roots in a root configuration, we mean the disjoint union ordered by concatenation of the individual orderings.
Example 2.8. The root configurations of the facets of the subword complex in Example 2.7 can be found in Figure 12 on page 46.

The role of the root configuration for subword complexes is the same in the present generality as it is for \( g = 0 \). The original proofs of the following lemmas do not rely on the assumption that the word for \( w \) inside \( Q \) is reduced, and therefore generalize verbatim.

Lemma 2.9 ([PS11, Lemma 3.3(2)]). Let \( I \) be any facet of the subword complex \( \text{Sub}_S(Q, w, a) \), and let \( r_I(i) = \beta(k) \). If \( J \) is a facet of \( \text{Sub}_S(Q, w, a) \) adjacent to \( I \) (i.e., \( I \setminus i = J \setminus j \)), the position \( j \) is one of the positions in the complement of \( I \) for which \( r_I(j) = \beta(k) \) for \( k \neq \ell \). Moreover, \( i < j \) if and only if \( k < \ell \).

In the situation of this lemma, we say that the facets \( I \) and \( J \) are connected by a flip. Such a flip from \( I \) to \( J \) is increasing if \( i < j \) and decreasing otherwise. Finally, the direction of this flip from \( I \) to \( J \) is given by \( r_I(i) \).

Lemma 2.10 ([PS11, Lemma 3.4 and Remark 3.5]). A facet \( I \) of the subword complex \( \text{Sub}_S(Q, w, a) \) is uniquely determined by its root configuration \( R(I) \), and can be reconstructed from it.

Definition 2.11. Let \( \text{Sub}_S(Q, w, a) \) be a subword complex. The increasing flip graph \( G(Q, w, m) \) of \( \text{Sub}_S(Q, w, a) \) is the directed graph of increasing flips. Its vertices are the facets of \( \text{Sub}_S(Q, w, a) \) and two facets \( I \) and \( J \) are connected by a directed edge if \( I \setminus i = J \setminus j \) with \( i < j \). Finally, the increasing flip poset \( \Gamma(Q, w, m) \) is the transitive closure of the increasing flip graph.

Lemma 2.9 states that an increasing flip is also characterized by \( I \setminus i = J \setminus j \) with \( r_I(i) = \beta(k) \) and \( r_J(j) = \beta(\ell) \) with \( k < \ell \). We conclude that the Hasse diagram of the increasing flip poset \( \Gamma(Q, w, m) \) is given by those edges \( I \setminus i = J \setminus j \) in the increasing flip graph \( G(Q, w, m) \) for which \( r_I(i) = \beta(k) \) and \( r_J(j) = \beta(k+1) \) for some \( k \).

Example 2.12. Figure 13 on page 53 shows the increasing flip poset \( \Gamma(Q, w, m) \) for the \( S \)-word \( Q = \text{ststst} \) in \( A_2 \), the element \( w = e = w_2^2 \), and \( a = 6 \) with increasing flips labeled by their direction.

Remark 2.13. There is an alternative generalization of subword complexes which computations suggest also works for our applications, although we do not pursue its development here. Let \( Q = s_1 \cdots s_p \) and let \( w \in B^+(W) \). The Artin subword complex \( \text{Sub}_S^0(Q, w) \) is the simplicial complex with facets being all subsets of (positions of) letters in \( Q \) that yield an \( S \)-word for \( w \). Then there is an injection from the Artin subword complex \( \text{Sub}_S^0(Q, w) \) to the subword complex \( \text{Sub}_S(Q, \overline{w}, \ell(w)) \), where \( \overline{w} \) is the projection of \( w \) into \( W \). We have chosen the above approach for the duality in Proposition 2.6.

It was shown in [CLS14, Section 3.3] that certain subword complexes with \( g = 0 \) are canonically isomorphic. The analogous statements hold in this more general context with the same proofs. We call two \( S \)-words \( Q \) and \( Q' \) commutation equivalent if or equal up to commutation if they are obtained from each other by a sequence of interchanging consecutive commuting simple reflections. Two consecutive letters commute if or only if the corresponding reflections in \( \text{inv}_R(Q) \) and \( \text{inv}_R(Q') \) commute. Accordingly, we call two words \( Q \) and \( Q' \) in \( R \) commutation equivalent or equal up to commutation if they are obtained from each other by a sequence of interchanging consecutive commuting reflections.

Proposition 2.14. Let \( Q, Q' \) be commutation equivalent \( S \)-words (resp. \( R \)-words). Then the two complexes \( \text{Sub}_S(Q, w, a) \) and \( \text{Sub}_S(Q', w, a) \) (resp. \( \text{Sub}_R(Q, w, b) \) and \( \text{Sub}_R(Q', w, b) \)) are canonically isomorphic.
In the case of \( w = w_o^m \in \{ e, w_o \} \subseteq W \) for some integer \( m \), there is the following further rotational symmetry. Observe that \( s(w_o^m) = w_o^m s w_o^m \in W \) is again a simple reflection for any \( s \in S \). The word \( Q' \) in the following proposition is therefore again a word in \( S \).

**Proposition 2.14.** Let \( Q = s_1 \cdots s_p \) be an \( S \)-word and let \( Q' = s_2 \cdots s_p s_1^{(w_o)} \). Then \( \text{SUB}_S(Q, w_o^n, a) \) and \( \text{SUB}_S(Q', w_o^n, a) \) are canonically isomorphic.

**Proofs of Propositions 2.14 and 2.15.** The proofs given in [CLS14, Propositions 3.8 and 3.9] extend directly to these more general cases. \( \square \)

This more general construction of subword complexes does not guarantee vertex-decomposability for \( g > 0 \), as seen in the following example.

**Example 2.16.** Let \( Q = s_1 s_2 s_3 \) in type \( A_1 \times A_1 \). Then the facets of \( \text{SUB}_S(Q, e, 2) \) are given by \{12, 34\}. Since this complex is not connected, it is neither shellable nor vertex-decomposable.

We remark that the Artin subword complexes of Remark 2.13 are also not vertex-decomposable in general, although Proposition 2.15 still holds since for \( w_o^m \in B^+ \) and \( s \in S \), we still have \( s(w_o^m) \in S \).

**Example 2.17.** Let \( Q = tsstst \), and let \( w = ssts = stst = tsts \in B^+(W) \). Then, as an Artin subword complex,
\[
\text{SUB}_S^F(Q, w) = \{12, 13, 16, 25, 35\}
\]
is not vertex-decomposable. In particular,
\[
\text{del}_1(\text{SUB}_S^F(Q, w)) = \{25, 35, 6\}
\]
is not an Artin subword complex. Using a slightly more complicated example, one can also show that vertex-decomposability does not even hold in the special case \( w = w_o^m \).

### 2.7 Coxeter elements and sorting words

A (standard) Coxeter element \( c \) for \((W, S)\) and for \((B(W), S)\) is defined to be the product of the simple reflections \( S \) in any order. Given the fixed bipartition \( S = S_L \sqcup S_R \) discussed in Section 2.1, a bipartite Coxeter element is the product of the reflections in \( S_L \) followed by the product of the reflections in \( S_R \). As the Dynkin diagram for a finite Coxeter group \( W \) is a tree, we have that all Coxeter elements in \( W \) are conjugate. We denote their common order by the Coxeter number \( h \). The rank \( n \) of \( W \), the number \( N = \ell(w_o) = |\Phi^+| \), and the Coxeter number are related by \( N = nh/2 \).

Let \( c = s_1 \cdots s_n \) be a particular reduced word for a Coxeter element \( c \), and let \( w \) be an element in \( W \) (or in \( B^+ \)). The \textit{c-sorting word} \( w(c) \) for \( w \) is then defined to be the lexicographically first subword of \( c^\infty = (s_1 \cdots s_n)^\infty \) which is a reduced expression for \( w \). To emphasize the different copies of \( s_1 \cdots s_n \), we may separate them by vertical bars, as in Figure 2 on page 12. Observe that although the \( c \)-sorting word is attached to a particular reduced word for \( c \)—rather than to \( c \) itself—the property that all reduced words for \( c \) are related by commutation relations implies that the corresponding sorting words are also equal up to commutations. Finally, we write \( \leq_c \) for the root order \( \text{inv}(w_o(c)) \) on \( \Phi^+ \) and analogously for the root order \( \text{inv}_R(w_o(c)) \) on \( R \).

The following lemma collects a few properties of sorting words and their reflection sequences.

**Lemma 2.18** ([CLS14, Proposition 4.3], [RS11, Lemma 3.8]). Let \( c = s_1 \cdots s_n \) be a Coxeter element for a finite Coxeter system \((W, S)\) with \( s = s_1 \) initial in \( c \), let \( c' = s_2 \cdots s_n s_1 \), and let \( c'' = s_n^{w_o} \cdots s_1^{w_o} \). For a word \( Q \) in \( S \) let \( \bar{Q} \) denote the word obtained from \( Q \) by replacing every letter \( s \) in \( Q \) by \( s^{w_o} \) (which is again in \( S \)). Then
We refer to Proposition 2.19. Let $s \in S$. For $w \in W$ with $\ell(sw) > \ell(w)$ or for $w \in B^+(W)$, we have
$$\text{des}_L(sw) \subseteq \{s\} \cup \text{des}_L(w).$$

\textbf{Proof.} Recall that if $s, t \in \text{des}_L(u)$ for $u \in W$ or for $u \in B^+(W)$, then $s \vee t = \frac{st \cdot \cdots}{\ell(s \vee t) \text{ factors}}$ is a left factor of $u$ [BST2]. Applying this to the element $\ell(s \vee t)$ factors $\ell(s \vee t)$ factors
Figure 2. All 6 elements in $A_2$ with their $st$-sorting word, length, reflection length, left descents, right descents, covered reflections, and covering reflections.

| $w$ | $w(st)$ | $\ell$ | $\ell_R$ | $\text{supp}$ | $\text{des}_L$ | $\text{des}_R$ | $\text{cov}_1$ | $\text{cov}^\dagger$ |
|-----|----------|-------|--------|-----------|---------|---------|---------|---------|
| $e$  | $\cdot\cdot\cdot s\cdot\cdot\cdot s$ | 0     | 0      | $-$       | $-$     | $-$     | $-$     | $s,t$    |
| $s$  | $s\cdot\cdot\cdot s\cdot\cdot\cdot s$ | 1     | 1      | $s$      | $s$     | $s$     | $u$     |          |
| $t$  | $t\cdot\cdot\cdot s\cdot\cdot\cdot t$ | 1     | 1      | $t$      | $t$     | $t$     | $u$     |          |
| $st$ | $st\cdot\cdot\cdot s\cdot\cdot\cdot t$ | 2     | 2      | $s,t$    | $s$     | $t$     | $u$     | $t$      |
| $ts$ | $st\cdot\cdot\cdot s\cdot\cdot\cdot t$ | 2     | 2      | $s,t$    | $s$     | $u$     | $s$     |          |
| $sts$| $st\cdot\cdot\cdot s\cdot\cdot\cdot t$ | 3     | 1      | $s,t$    | $s,t$   | $s,t$   | $-$     |          |

Figure 3. The 13 elements in $B^+(A_2)$ having exactly two Garside factors with their $st$-sorting word and their descent sets.

$u = sw$ yields for $t \in \text{des}_L(sw)$, that $s \lor t$ is a left factor of $sw$. If $t \neq s$ (and thus $\ell(s \lor t) > 1$), we conclude that $\ell_{(s \lor t)}$ is a left factor of $u$. □

\begin{theorem}[(BB05, DDG+13)]\label{thm:weakorder}
Weak(W) and Weak(B+) are both lattices.
\end{theorem}

The meet of $w_1, w_2, \ldots, w_q$ is denoted $\gcd(w_1, w_2, \ldots, w_q)$, while their join is $\text{lcm}(w_1, w_2, \ldots, w_q)$. Observe that the Coxeter group $W$ injects into $B^+(W)$ as the interval $[e, w_0]_{\text{Weak}(B^+)}$ with $w_0 = \text{lcm}(S)$. This injection preserves the weak order, \[ \text{Weak}(W) = [e, w_0]_{\text{Weak}(B^+)} \]

The embedding of the weak order on $W$ into the weak order on $B^+$ suggests the following generalization, which plays a crucial role in the development of the $m$-eralized theory. It was previously considered in [Deh07].

\begin{definition}
The (right) $m$-weak order is defined as the interval \[ \text{Weak}^{(m)}(W) := [e, w_0^m]_{\text{Weak}(B^+)} \]

We denote the set of elements simply by $W^{(m)} := \{ w : w \in \text{Weak}^{(m)}(W) \}$.
\end{definition}

\begin{example}
Figure 4 illustrates the Hasse diagram of $\text{Weak}^{(2)}(A_2)$.

each element $w \in B^+$ has a normal form called Garside factorization $\text{garside}(w)$, which we will use to give an alternative description of the $m$-weak order. To this end, set $w_1 = w$ and $v_1 = \gcd(w_1, w_0)$. For $i = 2, 3, \ldots$, as long as $w_{i-1} \neq e$, let
\[ w_i = v_{i-1}^{-1} w_{i-1}, \quad v_i = \gcd(w_i, w_0). \]
Then
\[ \text{garside}(w) = w^{(1)} \cdot w^{(2)} \cdot \ldots \cdot w^{(k)}, \]

where \( w^{(i)} := v_i \), and the degree \( \text{deg}(w) \) is defined to be \( k \). By construction, every factor sits inside the interval \([e, w_o]\) and so can be treated as an element of \( W \). A factorization \( v_1 \cdot v_2 \cdot \cdots \cdot v_k \) with \( v_i \in W \) is the Garside factorization of the element \( w = v_1 \cdots v_k \in B^+ \) if and only if
\[
\text{des}_R(v_i - 1) \supseteq \text{des}_L(v_i).
\]

For further details, see [DDG+13, Proposition 2.4, Chapter I and Proposition 1.30, Chapter IX]. To emphasize different Garside factors, we may separate them by a centered dot, as done for example in Figure 2.

**Lemma 2.23.** [EM94, Lemma 2.10] Let \( uw \geq w_o \) in \( B^+ \), and let \( w^{(1)} \) be the first Garside factor of \( w \). Then
\[
uw^{(1)} \geq w_o.
\]

**Proof.** We argue by induction on \( \ell(u) \), the base case when \( u = e \) being trivial. Otherwise, let \( u = u's \) and \( w' = sw \) for some \( s \in S \), so that \( u'w' = uw \geq w_o \). By induction, \( u'(w')^{(1)} \geq w_o \). Since \( s \in \text{des}_L(w') \), the first Garside factor \( (w')^{(1)} = \gcd(w', w_o) \) has \( s \in \text{des}_L((w')^{(1)}) \). We may therefore write \( (w')^{(1)} = sv \) for some \( v \leq w_o \) with \( v \leq w \). Since \( w^{(1)} = \gcd(w, w_o) \), \( v \leq w_o \), and \( v \leq w \), we have that \( v \leq w^{(1)} \). Then \( uw^{(1)} \geq uw = u'(w')^{(1)} \geq w_o \).

We have the following proposition, generalizing [EM94, Theorem 2.11], where it was stated for the positive braid monoid, of type \( A_n \).

**Proposition 2.24.** [EM94, Theorem 2.11] Let \( u \in B^+(W) \). Then \( u \in W^{(m)} \) if and only if \( u \) has degree at most \( m \).

**Proof.** Suppose that \( w \leq w_o^r \) with \( r \) minimal and that \( w \) has degree \( k \), so that garside\((w) = w^{(1)} \cdot w^{(2)} \cdot \cdots \cdot w^{(k)} \). We will show that \( r = k \) by induction on the number of factors of \( w \).

The base case, when \( k = r = 0 \), is trivial. Suppose then that \( k \geq 1, r \geq 1 \). Since \( w \leq w_o^r \), there exists \( u \in B^+ \) with \( uw = w_o^r \). Then \( uw \geq w_o^r \), so that \( uw^{(1)} = w_o^ru' \) by Lemma 2.23. We can therefore write \( uw = w_o^ru'(w)^{(2)} \cdots (w)^{(k)} \),
from which it follows that \( w^{(2)} \cdots w^{(k)} \leq w^{r-1}_0 \). By induction, \( w^{(2)} \cdots w^{(k)} \leq w^{k-1}_0 \) with \( k \) minimal, so that \( k \leq r \).

On the other hand, it is easy to show that \( k \geq r \) by showing that \( w \leq w^{k}_0 \): let \( u_1 \) be the element such that \( u_1 w^{(1)} = w_0 \). Push this copy of \( w_0 \) to the right to obtain the factorization \( (w^{(2)})^{u_0} \cdot (w^{(3)})^{u_0} \cdots (w^{(k)})^{u_0} \). Iterating, we obtain the desired conclusion.

Therefore, if \( w \) has degree \( k \), then \( w \leq w^{k}_0 \) with \( k \) minimal, from which we conclude the theorem. \( \square \)

**Remark 2.25.** Both \( \text{Weak}(W) = \text{Weak}^{(1)}(W) \) and \( \text{Weak}(B^+) = \text{Weak}^{(\infty)}(W) \) are known to have the beautiful product rank-generating functions [SD08, Lemma 2.1]

\[
\text{Weak}(W; q) := \prod_{i=1}^{n} [d_i]_q, \quad \text{and} \quad \text{Weak}^{(\infty)}(W; q) := \left( \sum_{J \subseteq \mathbb{S}} (-1)^{|J|} q^{\ell(w_0(J))} \right)^{-1}.
\]

Unfortunately, we do not know of any similar expression for \( \text{Weak}^{(m)}(W) \) when \( 1 < m < \infty \)—for example, \( \text{Weak}^{(2)}(A_3) \) has 211 elements and the rank generating function is an irreducible polynomial over \( \mathbb{R} \).

However, \( \text{Weak}^{(m)}(W) \) inherits the lattice property from \( \text{Weak}(B^+) \).

**Theorem 2.26.** \( \text{Weak}^{(m)}(W) \) is a rank-symmetric, self-dual lattice.

To prove this theorem, we recall some standard results from the weak order of Coxeter theory in the context of \( \text{Weak}^{(m)}(W) \). Just as \( \text{Weak}(W) \) has an anti-automorphism given by acting by the longest element \( w_0 \), we show that \( \text{Weak}^{(m)}(W) \) has an anti-automorphism given by acting by \( w_0^m \).

**Lemma 2.27.** If \( u \) is a left divisor of \( w_0^m \in B^+ \), then it is also a right divisor.

*Proof.* For \( s \in S \), \( w_0^m s = s' w_0^m \in B^+ \) for \( s' = \psi^m(s) \) with \( \psi(s) = s^{w_0} = w_0 s w_0 \in S \subseteq W \). Given a reduced expression \( u s_1 \cdots s_k = w_0^m \in B^+ \), \( \psi^m(s_1) \cdots \psi^m(s_k) u \) is again a reduced expression for \( w_0^m \). \( \square \)

**Lemma 2.28.** The map \( \phi(w) := w_0^m w_0^{-1} \) is an anti-isomorphism from right to left weak order when restricted to \( \text{Weak}^{(m)}(W) \).

*Proof.* We have seen in **Lemma 2.27** that \( w \in W^{(m)} \) is both a left and right divisor of \( w_0^m \). We also have

\[
\ell(w_0^m w_0^{-1}) = \ell(w_0^{-1} w_0^m) = \ell(w_0^m) - \ell(w) = m N - \ell(w).
\]

To show that the map reverses the order, it suffices to consider the case for \( w \leq w_0^m \). Then \( \ell(\psi(w)) = \ell(w_0^m) - \ell(w) \) and \( \ell(\phi(w)) = \ell(w_0^m) - \ell(w) = \ell(w_0^m) - \ell(w - 1) \), so that \( \ell(\phi(ws)) + 1 = \ell(\phi(w)) \). Furthermore,

\[
\phi(ws) = w_0^m s^{-1} w_0^{-1} = \psi(s)^{-1} w_0^m w_0^{-1} = \psi(w_0)^{-1} \phi(w),
\]

so that \( \phi(w) = \psi(s)^{-1} \phi(w) \). \( \square \)

To convert back from left weak order to right weak order, we introduce the reverse map \( \text{rev} \). If \( u = s_i \cdots s_p \in B^+(W) \), then \( \text{rev}(u) \) is given by \( s_p \cdots s_i \in B^+(W) \).

**Lemma 2.29.** The map \( \text{rev} \) is an isomorphism from left weak order to right weak order that preserves the interval \( \text{Weak}^{(m)}(W) \).

*Proof.* Since \( \text{rev}(sw) = \text{rev}(ws) \), \( \text{rev} \) is an isomorphism from left weak order to right weak order. If \( w \leq w_0^m \), then there exists \( u \in B^+(W) \) such that \( uw = w_0^m \) so that \( \text{rev}(w) \text{rev}(u) = \text{rev}(uw) = \text{rev}(w_0^m) = w_0^m \) and \( \text{rev}(w) \leq w_0^m \). \( \square \)
Proposition 2.30. The composition rev ◦ φ is an anti-isomorphism of the m-weak order \text{Weak}^m(W).

Proof. This follows from Lemmas 2.28 and 2.29. □

Proof of Theorem 2.26. Any interval in a lattice is again a lattice, thus \text{Weak}^m(W) is a lattice by Theorem 2.20. The self-duality is a direct consequence of the existence of the anti-isomorphism rev ◦ φ. Finally, it is rank-symmetric since it is graded by the Coxeter length. □

2.9. Parabolic subgroups and parabolic quotients. For a Coxeter system \((W, S)\) and a subset \(J \subseteq S\), the parabolic subgroup \(W_J\) is the subgroup of \(W\) generated by \(J\), and the corresponding parabolic root subsystem is given by \(\Phi_J = \Phi \cap R\{ \alpha_s : s \in J \}\). Likewise, for an Artin system \((B, S)\), the parabolic subgroup \(B_J\) is the subgroup of \(B\) generated by \(J\). The positive parabolic submonoid is denoted \(B_J^+\). Let

\[
W_J^m := W^{(m)} \cap B_J^+
\]

with induced weak order \(\text{Weak}^m(W_J)\), and write \(w_J^m(J)\) for the longest element in \(W_J^m\). A parabolic subsystem is called maximal if it is generated by all but one generator. We use the notation \((s) := S \setminus \{ s \}\) for such situations.

The corresponding parabolic quotients in \(W\) and in \(B^+\) are given as

\[
W_J := \{ w \in W : w \neq su \text{ for any } s \in J, u \in W \} \\
= \{ w \in W : \text{des}_L(w) \cap J = \emptyset \}
\]

and

\[
B_J^+ := \{ w \in B^+ : w \neq su \text{ for any } s \in J, u \in B^+ \} \\
= \{ w \in W : \text{des}_L(w) \cap J = \emptyset \}.
\]

The parabolic quotient of the \(m\)-weak order is

\[
W_J^m := W^{(m)} \cap B_J^+
\]

with induced weak order denoted \(\text{Weak}^m(W_J)\).

We continue our development of Coxeter-theoretic results by extending the well-known decomposition \(w = w_J w_J^J\) for \(w \in W\), \(w_J^J \in W_J\), and \(w_J \in W_J\) to the \(m\)-weak order. See also [DMR07, Lemma-Definition 2.1.5].

Proposition 2.31. Fix a subset \(J \subseteq S\). Every \(w \in W^{(m)}\) has a unique factorization \(w = w_J w_J^J\), where \(w_J := w \wedge w_J^m(J) \in W_J^m\), \(w_J^J := w_J^{-1} w \in W_J^J\), and \(\ell(w) = \ell(w_J) + \ell(w_J^J)\).

Proof. By construction, \(w_J \in W_J^m\). We need to show that \(w_J^J\) has no initial \(s\), for any \(s \in J\). Suppose it did, so that \(w_J \leq w_J^m(J)^m\) but \(w_J s \leq w_J^m(J)^m\). Since \(s \in J\), \(w_J s \leq w_J^m(J)^{m+1}\), so that by Proposition 2.24, \(w_J s\) has degree greater than \(m\) in \(W_J\), and therefore also in \(W\). But then \(w_J s \leq w_J^m\), so that \(w_J w_J^J = w \leq w_J^m\), contradicting the assumption that \(w \in W^{(m)}\). □

Let \(w_J\) be the element in \(W_J^m\) arising from the factorization \(w = w_J w_J^J\) of Proposition 2.31, and observe that \(w_J \leq w\) in \(\text{Weak}^m(W)\).

Proposition 2.32 ([God02, Theorem 2.10]). The map \(w \mapsto w_J\) is a lattice homomorphism. If

\[
garside(w) = w^{(1)} \cdot w^{(2)} \cdot \ldots \cdot w^{(k)},
\]

then

\[
garside(w_J) = w^{(1)}_J \cdot w^{(2)}_J \cdot \ldots \cdot w^{(k)}_J.
\]
Remark 2.33. Since any element $w \in B^+$ lies in some $W^{(m)}$, for $J \subseteq S$, we remark that there is a well-defined parabolic decomposition according to $J$ in $B^+$. The map $w \mapsto w_J$ is again a lattice homomorphism.

Remark 2.34. Parabolic quotients are not well-behaved in $W^{(m)}$. Although the minimal coset representatives have a natural (left) order, there is generally not a unique maximal element, unlike in the situation of $\text{Weak}(W)$. Moreover, because of the restriction that elements in $\text{Weak}^{(m)}(W)$ have only $m$ Garside factors, not all cosets have the same size.

3. $m$-eralized noncrossing partitions

We first recall the definition of noncrossing partitions and their $m$-eralization in Sections 3.1 and 3.2. In Section 3.3 we then define the $m$-eralized Kreweras complement, the $m$-eralized Cambrian recurrence, and the $m$-eralized Cambrian rotation. We conclude our discussion in Section 3.4 by constructing the $m$-eralized Cambrian poset on noncrossing partitions.

3.1. Noncrossing partitions. Similarly to the (right) weak order, the absolute order on $W$ is defined by considering the absolute length rather than the usual Coxeter length.

Definition 3.1. The absolute order $\text{Abs}(W) = (W, \leq_R)$ is defined by

$$w \leq_R u \iff \ell_R(w) = \ell_R(w) + \ell_R(u).$$

The $c$-noncrossing partitions $\text{NC}(W,c) \subseteq W$ are all elements $w \in W$ such that $w \leq_R c$ in absolute order.

The $c$-noncrossing partitions under the induced order

$$\text{NC}(W,c) = [c,c]_{\text{Abs}(W)} \subseteq \text{Abs}(W)$$

is a lattice, which we denote by $\text{NCL}(W,c)$ [BW08, Theorem 7.8]. $\text{NCL}(A_2, st)$ is drawn in Figure 10 on page 37.

The following proposition is a direct consequence of the definition.

Proposition 3.2. Conjugation by $c$ is an automorphism of $\text{NCL}(W,c)$.

In [ABW07], C. Athanasiadis, T. Brady, and C. Watt used, for a bipartite Coxeter element $c$, the natural edge labelling of the noncrossing lattice $\text{NCL}(W,c)$ by reflections to prove that $\text{NCL}(W,c)$ is EL-shellable. This shelling gives a unique factorization of each element $w \in \text{NC}(W,c)$ into reflections that increase with respect to the root order $\text{inv}_R(w,c)$ [ABW07, Definition 3.1]. This factorization was extended to all (standard) Coxeter elements in [Rea07a].

Proposition 3.3 ([Rea07a, Remark 6.8]). Each $w \in \text{NC}(W,c)$ has a unique reduced $\mathcal{R}$-word $w_{\mathcal{C}} = r_1r_2\cdots r_p$ with $r_i \in \mathcal{R}$ and $r_1 <_c r_2 <_c \cdots <_c r_p$. We write $r \in w$ for $r \in \{r_1, \ldots, r_p\}$ in this case.

Note the pleasant similarity between Proposition 3.3 and the definition of $c$-sorting word in Section 2.7.

The Kreweras complement $\text{Krew}_c : \text{NC}(W,c) \to \text{NC}(W,c)$ is defined by

$$\text{Krew}_c(w) := cw^{-1}.$$
Proposition 3.3. \(
m = \) are the \(m\)-delta sequences, and the \(m\)-cluster complex.

Definition 3.4. \((\delta_0,\delta_1,\ldots,\delta_m) \in NC(W,c)\) is an \(m\)-delta sequence if

- \(\delta_0\delta_1\cdots\delta_m = c\); and
- \(\sum_{i=0}^m \ell_R(\delta_i) = \ell_R(c)\).

We denote the set of all such sequences by \(NC^{(m)}_\delta(W,c)\), and write \(r \in \delta = (\delta_0,\delta_1,\ldots,\delta_m)\) if \(r \in \delta_i\) for some \(0 \leq i \leq m\).

Proposition 3.7. The map \(NC^{(m)}(W,c) \rightarrow NC^{(m)}_\delta(W,c)\), given by

\[
\mathbf{w} = (w_1 \geq_R \cdots \geq_R w_m) \mapsto \mathbf{w}_\delta = (c_1w_1^{-1}, w_1w_2^{-1}, \ldots, w_{m-1}w_m^{-1}, w_m),
\]

is a bijection. Its inverse is

\[
\delta_{\mathbf{w}_\delta} = (\delta_0,\ldots,\delta_m) \mapsto \mathbf{w} = (\delta_1 \cdots \delta_m \geq_R \cdots \geq_R \delta_{m-1}\delta_m \geq_R \delta_m).
\]

Proof. Immediate from the definitions. \(\square\)

There is more than one way to define the map \(\mathbf{w} \leftrightarrow \delta_{\mathbf{w}_\delta}\). Our convention is chosen to ease the connection between \(m\)-eralized \(c\)-noncrossing partitions, \(m\)-eralized \(c\)-sortable elements, and the \(m\)-eralized \(c\)-Kreweras complex.

Using \(m\)-delta sequences, one can \(m\)-eralize the Kreweras complement as follows [Arm06, Chapter 3].

Definition 3.8. Let \(\delta = (\delta_0,\delta_1,\ldots,\delta_m) \in NC^{(m)}_\delta(W,c)\). The \(m\)-eralized \(c\)-Kreweras complement \(\text{Krew}_c : NC^{(m)}_\delta(W,c) \rightarrow NC^{(m)}(W,c)\) is defined to be

\[
\text{Krew}_c(\delta) := (\delta_1,\delta_2,\ldots,\delta_m, c^{-1}\delta_0c).
\]
The two equalities $\ell_R(c\delta_0c^{-1}) = \ell_R(\delta_0)$, and
$$\delta_1\delta_2\cdots\delta_m \cdot (c^{-1}\delta_0c) = \delta_0^{-1}c \cdot c^{-1}\delta_0c = c,$$
together with Proposition 3.2, imply that $Krew_c(\delta)$ is again an $m$-delta sequence. It is well-known that the square of the Kreweras complement may be interpreted as a rotation on the combinatorial realizations of $m$-eralized noncrossing partitions in the classical types [Arm06]. In general, we have the following proposition.

**Proposition 3.9.** The order of the $m$-eralized c-Kreweras complement is $(m+1)h$.

**Proof.** This follows from the definition of the Coxeter number $h$ as the order of $c$. \(\square\)

Analogously to Proposition 3.4, one can describe $m$-eralized $c$-noncrossing partitions. By Lemma 2.18, the two words $\text{inv}_R(w_c(c))^{m+1}$ and $\text{inv}_R(\delta^{m+1})$ with
$$e^{\frac{m+1}{2}} := \begin{cases} e^{\frac{m+1}{2}} & \text{if } m+1 \text{ even} \\ c^{\delta}w_c(c) & \text{if } m+1 \text{ odd} \end{cases}$$
are equal up to commutations.

**Definition 3.10.** Define $\text{NC}_{\Delta}^{(m)}(W,c)$ as the dual subword complex
$$\text{NC}_{\Delta}^{(m)}(W,c) := \text{Sub}_R(\text{inv}_R(\delta^{m+1}),c).$$

We obtain the following proposition.

**Proposition 3.11.** There is an explicit bijection between $m$-eralized $c$-noncrossing partitions $\text{NC}^{(m)}(W,c)$ and $\text{NC}_{\Delta}^{(m)}(W,c)$.

**Proof.** This is nearly identical to the proof of Proposition 3.4, except that our $m$-delta sequence has $m+1$ components (rather than just two). As before, the proposition follows by applying the factorization of Proposition 3.3 separately to each component of the $m$-delta sequence. \(\square\)

We may denote a facet $I$ of $\text{NC}^{(m)}(W,c)$ as the tuple $(r_1^{(i_1)}, \ldots, r_n^{(i_n)})$, such that $r_1\cdots r_n = c$ and where each reflection $r_j$ is colored by which copy of $\text{inv}_R(w_c(c))$ it lies in, so that $0 \leq i_1 \leq \cdots \leq i_n \leq m$. We write $r \in I$ for $r \in \{r_1, \ldots, r_n\}$.

As we have now simple, explicit bijections between the three sets $\text{NC}^{(m)}(W,c)$, $\text{NC}_{\delta}^{(m)}(W,c)$, and $\text{NC}_{\Delta}^{(m)}(W,c)$, we will freely move between the three descriptions.

**Example 3.12.** Figure 5 shows all 12 elements of
$$\text{NC}^{(m)}(A_2, st) \cong \text{NC}_{\delta}^{(m)}(A_2, st) \cong \text{NC}_{\Delta}^{(m)}(A_2, st).$$

We close this section with the definition of the support of $m$-eralized $c$-noncrossing partitions.

**Definition 3.13.** Let $\underline{w} = (w_1 \geq_R \cdots \geq_R w_m) \in \text{NC}^{(m)}(W,c)$ and let $\delta_{\underline{w}} = (\delta_0, \ldots, \delta_m)$ be the corresponding element in $\text{NC}_{\delta}^{(m)}(W,c)$. The support $\text{supp}(\underline{w}) = \text{supp}(\delta_{\underline{w}})$ is defined to be $\text{supp}(w_1) = \text{supp}(\delta_1\cdots\delta_m) \subseteq S$.

Figure 5 shows the support of the 12 elements in $\text{NC}^{(m)}(A_2, st)$. 

The composition of these two bijections is the map

$$\text{Shift}_r : \text{NC}_\Delta^{(m)}(W, c) \rightarrow \text{NC}_\Delta^{(m)}(W, s^{-1}cs)$$

We will see analogous shift operations again in Section 4.3 and Section 5.5.
Example 3.14. As before, consider $\NC^{(2)}(A_2, sl)$. Then one orbit of the shift operation is given by

\[
\begin{array}{cccc}
\text{Shift}_r & \text{Shift}_l & \text{Shift}_r & \text{Shift}_l \\
\text{sut, sut, sut} & \text{tuts, tut, tuts} & \text{shifts, shift, shifts} & \text{sut, sut, sut}
\end{array}
\]

We may now compose the shift operations in the order specified by any reduced $S$-word for the Coxeter element $c$. This composition does not depend on the chosen reduced word since two shifts $\text{Shift}_r$ and $\text{Shift}_l$ commute for commuting $s, t \in S$.

Definition 3.15. Let $c$ be a Coxeter element. The \textit{m-eralized c-Cambrian rotation} $\text{Camb}_c : \NC^{(m)}(W, c) \to \NC^{(m)}(W, c)$ is given by

\[
\text{Camb}_c = \text{Shift}_{s_n} \circ \cdots \circ \text{Shift}_{s_1},
\]

for any reduced $S$-word $s_1s_2 \cdots s_n$ for $c$.

As an example, observe that the elements of $\NC^{(m)}(A_2, sl)$ in Figure 5 are grouped into their orbits under Cambrian rotation, the first one being the orbit discussed in Example 3.14.

Rather than append a final $s$ to $\delta_m$ in the first case of the definition of the shift operator, we could send $\delta_m$ to the delta sequence $(s^{-1}\delta_0, \delta_1, \ldots, \delta_{m-1}, \delta_m) \in \NC^{(m)}_\delta(W_s, s^{-1}c)$, performing a descent into a parabolic subgroup. This modification of Cambrian rotation yields the \textit{m-eralized c-Cambrian recurrence} on noncrossing partitions, which we phrase as the following characterization of $\NC^{(m)}_\delta(W, c)$.

Proposition 3.16. Let $s$ be initial in a Coxeter element $c$. Then

\[
\delta_m \in \NC^{(m)}_\delta(W, c) \iff \begin{cases} (s^{-1}\delta_0, \delta_1, \ldots, \delta_m) \in \NC^{(m)}_\delta(W_s, s^{-1}c) & \text{if } s \in \delta_0; \\ \text{Shift}_s(\delta_{m}) \in \NC^{(m)}_\delta(W, s^{-1}cs) & \text{if } s \notin \delta_0. \end{cases}
\]

Proof. The recurrence follows directly from Lemma 2.18(6) and (7). \qed

Remark 3.17. Some care must be taken to correctly run this recurrence in reverse. For any $(\delta_0, \delta_1, \ldots, \delta_m) \in \NC^{(m)}_\delta(W_s, s^{-1}c)$, we can certainly produce the element $(s\delta_0, \delta_1, \ldots, \delta_m) \in \NC^{(m)}_\delta(W, c)$. A point of confusion can now occur when running the recurrence backwards, starting with an element of $\NC^{(m)}_\delta(W, s^{-1}cs)$. Those elements of $\NC^{(m)}_\delta(W, s^{-1}cs)$ containing $s \in \delta_m$ have already been accounted for by $\NC^{(m)}_\delta(W_s, s^{-1}c)$. When running the recurrence backwards, it is therefore crucial to not begin with an element of $\NC^{(m)}_\delta(W, s^{-1}cs)$ containing $s \in \delta_m$.

We emphasize once and for all here that in order to reverse any of the Cambrian recurrences in Propositions 4.2, 4.9, 4.14, 5.31 and 4.32, we will always require that the elements on the right hand side have the form specified by the recurrence.

Just as with Cambrian rotation, Proposition 3.16 can be refined to a statement on $\NC_\delta(W, c)$ under the canonical factorization of Proposition 3.3.

Example 3.18. Consider the 3-delta sequence $(e, s, t) \in \NC^{(2)}_\delta(A_2, sl)$ Then the m-eralized c-Cambrian recurrence is computed as

\[
\begin{array}{cccc}
(e, s, t) & \to & (s, e, u) & \to & (u, e, s) & \to & (t, s, e) & \to & (e, s, t) & \to & (s, e, c) & \to & (e, c, e) & .
\end{array}
\]

$\NC^{(2)}_\delta(A_2, st)$ $\NC^{(2)}_\delta(A_2, et)$ $\NC^{(2)}_\delta(A_2, se)$ $\NC^{(2)}_\delta(A_2, es)$ $\NC^{(2)}_\delta(A_1, st)$ $\NC^{(2)}_\delta(A_1, et)$ $\NC^{(2)}_\delta(A_1, se)$ $\NC^{(2)}_\delta(A_1, es)$
3.4. m-eralized Cambrian lattices. To place the definition of our m-eralized c-Cambrian lattice in context, we first consider the situation for m = 1.

The weak order \text{Weak}(W) can be described using the covering and covered reflections of the elements in \(W\). Given that \(\text{cov}^\uparrow(e) = S\) and \(\text{cov}_\downarrow(e) = \emptyset\), one can reconstruct the covered and covering reflections of all elements as follows. Supposing that \(\text{cov}^\uparrow(w)\) and \(\text{cov}_\downarrow(w)\) are known, choose some \(r \in \text{cov}^\uparrow(w)\) and let \(\alpha_r\) be the associated positive root. The sets \(\text{cov}^\uparrow(rw)\) and \(\text{cov}_\downarrow(rw)\) are then given by

\[
\text{cov}^\uparrow(rw) = \{ u^r : u \in \text{cov}^\uparrow(w), r(\alpha_u) > 0 \} \cup \{ u^r : r \in \text{cov}_\downarrow(w), r(\alpha_u) < 0 \}
\]

and

\[
\text{cov}_\downarrow(rw) = \{ u^r : u \in \text{cov}^\uparrow(w), r(\alpha_u) > 0 \} \cup \{ u^r : u \in \text{cov}^\uparrow(w), r(\alpha_u) < 0 \}.
\]

An element \(w \in W\) can be reconstructed from its covered and covering reflections—these tell us exactly which hyperplanes bound the corresponding Weyl chamber, which determines \(w\) up to multiplication by \(w_\circ\). Since we have distinguished covering and covered reflections, we have therefore uniquely specified \(w\).

**Remark 3.19.** One naive m-eralization of the weak order would be to allow \(m + 1\) sets of reflections \((\text{cov}_0, \text{cov}_1, \ldots, \text{cov}_m)\), generalizing the two sets \(\text{cov}^\uparrow\) and \(\text{cov}_\downarrow\). Setting the minimal element to be \(\{(s_1, s_2, \ldots, s_n), \emptyset, \ldots, \emptyset\}\), we were unable to find a \(m\)-eralization of the \(m = 1\) action by (simple) reflections for which there was a unique maximal element in this framework.

We will next mimic the characterization of \text{Weak}(W) given above the remark. When \(m = 1\), delta sequences in \(\text{NC}_{c}(m)(W, c)\) have two components, \(\delta_w = (\delta_0, \delta_1)\). The analogy we wish to draw is that the factorization of \(\delta_0\) of Proposition 3.33 should be thought of as a set \(\text{cov}^\uparrow(w)\), while the factorization of \(\delta_1\) behaves like \(\text{cov}_\downarrow(w)\). Jumping immediately to the definition for general \(m\) does not introduce any additional complication.

Let \(I\) be a facet of \(\text{NC}_{c}(m)(W, c)\), and consider \(I\) as a reduced \(\mathcal{R}\)-word \(r_1 \cdots r_n\) for \(c\) as a subword of

\[
Q = \text{inv}_\mathcal{R}(w_\circ(c))^{m+1} = \text{inv}_\mathcal{R}(w_\circ^{m+1}(c)).
\]

Observe that for any reflection \(r \in \mathcal{R}\), every other reflection appears exactly once between two consecutive copies of \(r\) inside \(Q\). If \(r = r_j\) does not appear inside the final copy of \(w_\circ(c)\), the *increasing flip* \(\text{Flip}_\uparrow^c(I)\) is given by

\[
(5) \quad r_1 \cdots r_{j-1} r_{j+1}^c \cdots r_k^c r_j r_{k+1} \cdots r_n,
\]

where \(k\) is chosen maximally such that \(r_k\) still appears before the next consecutive copy of \(r\) inside \(Q\), and the reflections chosen between those two consecutive copies of \(r\) are \(r_{j+1}^c, \ldots, r_k^c\). By Proposition 3.20 below, this is again a subword of \(Q\). We define the *decreasing flip* \(\text{Flip}_\downarrow^c(I)\) for \(r\) not appearing in the initial copy of \(w_\circ(c)\) analogously.

Just as for covered and covering reflections in \text{Weak}(W), individual reflections are shuttled between the two copies of \(r\) inside \(Q\) to maintain the structure imposed by the absolute order on \(\mathcal{R}\) induced by \(c\). Examples of increasing and decreasing flips can be found in Figure 6.

**Proposition 3.20.** We have \(\text{Flip}_\uparrow^c(I) \in \text{NC}_{c}(m)(W, c)\).

**Proof.** We must check that the reflections \(r_{j+1}^c \cdots r_k^c\) indeed appear in this order between two copies of \(r\) inside \(Q\). We have already mentioned that every reflection except \(r\) appears exactly once between these two copies of \(r\). Denote by \(Q_r\) this sequence, together with an initial \(r\). Let \(w_\circ(c) = s_1 \cdots s_N\) and let the element \(w \in W\) be given by the prefix \(s_1 \cdots s_{i-1}\) such that \(r = s_i^w\). Applying Lemma 2.18(6)
to $w_0(c)$ along the sequence $s_1, \ldots, s_{i-1}$ yields that $s_i$ is initial in $c'$ with $c = (c')^w$, where $Q_r$ is the sequence given by $t^w$ for $t \in \text{inv}_R(w_0(c'))$. The sequence $r_j \cdots r_k$ is contained in $\text{NC}(W, c')$ starting with $r_j = s_i$. It follows (see e.g. [BDSW14, Theorem 1.4]) that each of $r_j \cdots r_k$ lives in the parabolic subgroup $W(s_i)$. By Lemma 2.18(7), the ordering of the reflections in $W(s_i)$ agrees in each of $w_0(c')$, $w_0(s_i'c')$, and $w_0(s_i'c's_i)$, so that conjugating the reflections $r_j \cdots r_k$ by $s_i$ does not change their order. Conjugating this statement by $w$, we conclude that the word $r_j' \cdots r_k'$ appears in the same order in $Q_r$ as the word $r_j+1 \cdots r_k$.

In analogy to the weak order, flips can be used to define an $m$-eralized $c$-Cambrian poset on $\text{NC}^{(m)}(W, c)$.

**Definition 3.21.** The $m$-eralized $c$-Cambrian poset $\text{Camb}_{\text{NC}}^{(m)}(W, c)$ is the partial order on the elements of $\text{NC}^{(m)}(W, c)$ with minimal element $(c, e, \ldots, e)$ and with covering relations $\delta_w \triangleleft \text{Flip}^r_w(\delta_w)$ for $r \in I$ contained in the $i$-th copy of $w_0(c)$ with $i < m$. These edges are labeled by the colored reflection $r(i)$.

We may also define a Cambrian graph, where we allow longer flips that send a reflection from the $i$-th copy of $w_0(c)$ to the $j$-th copy for $i < j$.

**Definition 3.22.** The $m$-eralized $c$-Cambrian graph $\mathcal{G}_{\text{Camb}}^{(m)}(W, c)$ is the graph on the elements of $\text{NC}^{(m)}(W, c)$ with minimal element $(c, e, \ldots, e)$ and with covering relations $\delta_w \triangleleft (\text{Flip}^r_w)^k(\delta_w)$ for $r \in I$ contained in the $i$-th copy of $w_0(c)$ with $i + k < m$ and $k > 0$.

Figure 6 illustrates an $m$-eralized $c$-Cambrian poset in type $A_2$ with $m = 2$. The following proposition collects a few first consequences of the definition.
Proposition 3.23. We have for \(c = s_1 \cdots s_n\) and \(c' = s_n^{w_n} \cdots s_1^{w_1}\) that

(i) the posets \(\text{Camb}_{\text{NC}}^{(m)}(W, c)\) and \(\text{Camb}_{\text{NC}}^{(m)}(W, c')\) are dual;
(ii) the generating function counting elements of \(\text{Camb}_{\text{NC}}^{(m)}(W, c)\) by the number of their outgoing edges and the corresponding generating function for incoming edges are given by \(\sum q^{\ell(w)}\), where the sum ranges over all \((w_1, \ldots, w_m) \in \text{NC}^{(m)}(W, c)\) and \(w = w_1, \text{ or, equivalently, over all } (\delta_0, \ldots, \delta_m) \in \text{NC}^{(m)}_\delta(W, c)\) and \(w = \delta_1 \cdots \delta_m\).

Proof. The first item (i) follows from Lemma 2.18(5). To obtain (ii) observe that the definition of \(\text{Camb}_{\text{NC}}^{(m)}\) yields

\[
\begin{align*}
\sum q^{\ell(w')} &= \{F \in \text{Camb}_{\text{NC}}^{(m)}(W, c) : F \text{ has exactly } i \text{ outgoing edges} \} \\
\sum q^{\ell(w)} &= \{F \in \text{Camb}_{\text{NC}}^{(m)}(W, c) : F \text{ has exactly } i \text{ incoming edges} \}
\end{align*}
\]

where both sums range over all \((\delta_0, \ldots, \delta_m) \in \text{NC}^{(m)}_\delta(W, c)\) and \(w' = \delta_0 \cdots \delta_{m-1}, \ w = \delta_1 \cdots \delta_m\).

The description in terms of \(m\)-delta sequences follows from the observation that the absolute lengths \(\ell_{\mathcal{R}}(\delta_0 \cdots \delta_{m-1})\) and \(\ell_{\mathcal{R}}(\delta_1 \cdots \delta_m)\) are equidistributed on all \(m\)-delta sequences in \(\text{NC}^{(m)}_\delta(W, c)\). The description in terms of \(\text{NC}^{(m)}(W, c)\) is obtained by applying the isomorphism \(\text{NC}^{(m)}(W, c) \xrightarrow{\sim} \text{NC}^{(m)}_\delta(W, c)\).

The shift operation \(\text{Shift}_s\) and the flip operation \(\text{Flip}_s\) are indeed closely related as explained in the following proposition.

Proposition 3.24. Let \(s\) be initial in \(c\). If \(\delta_{m} \prec \text{Flip}_s^r(\delta_{m})\), then

\[
\begin{align*}
\text{Shift}_s(\delta_m) &= \left(\text{Flip}_s\right)^r(\text{Shift}_s(\text{Flip}_s^r(\delta_m))) \quad \text{if } s \in \delta_0 \text{ and } r = s; \\
\text{Flip}_s(\text{Shift}_s(\delta_m)) &= \text{Shift}_s(\text{Flip}_s^r(\delta_m)) \quad \text{if } s \in \delta_0 \text{ and } r \neq s. \\
\text{Flip}_s(\text{Shift}_s(\delta_m)) &= \text{Shift}_s(\text{Flip}_s^r(\delta_m)) \quad \text{if } s \not\in \delta_0.
\end{align*}
\]

Proof. This follows immediately from the definition of \(\text{Shift}_s\) in Section 3.3 and the definition of \(\text{Flip}_s\).

Corollary 3.25. Let \(c, c'\) be two Coxeter elements. Then there is an undirected graph isomorphism \(\mathcal{G}\text{Camb}_{\text{NC}}^{(m)}(W, c) \xrightarrow{\sim} \mathcal{G}\text{Camb}_{\text{NC}}^{(m)}(W, c')\).

Proof. Since any two Coxeter elements \(c, c'\) are conjugate in \(W\), we have \(c' = c^{s_{p-1} \cdots s_1}\); this undirected graph isomorphism is induced by the composition of the shift operations for the sequence \(s_1, \ldots, s_p\) of initial letters.

4. \(m\)-eralized Coxeter-sortable elements

In this section, we define an \(m\)-eralization of \(c\)-sortable elements as a certain subset \(\text{Sort}^{(m)}(W, c)\) of \(\text{Sort}^{(m)}(W)\). We start by reviewing N. Reading’s theory of \(c\)-sortable elements [Rea06, Rea07b, Rea07a], recalling two characterizations of these elements in Definition 4.1 and Proposition 4.2. We \(m\)-eralize \(c\)-sortable elements in Section 4.2, providing the analogous characterizations as Definition 4.5 and Proposition 4.9. We give a characterization of \(\text{Sort}^{(m)}(W, c)\) on Garside factors in Definition 4.13, and we give a bijection between \(\text{Sort}^{(m)}(W, c)\) and noncrossing partitions in Section 4.5. Generalizing the case for \(m = 1\), we prove in Section 4.6 that \(\text{Sort}^{(m)}(W, c)\) is a sublattice of \(\text{Weak}^{(m)}(W)\). Finally, in Section 4.7, we explain the connection between \(\text{Sort}^{(m)}(W, c)\) and \(\text{NC}^{(m)}(W, c)\) by giving a bijection between \(\text{Sort}^{(m)}(W, c)\) and chains in N. Reading’s shard intersection order.
4.1. Coxeter-sortable elements. N. Reading introduced and studied $c$-sortable elements in [Rea06, Rea07a]. The $c$-sortable elements have three different characterizations, each of which is useful in different ways.

**Definition 4.1** (N. Reading [Rea06]). An element $w \in W$ is $c$-sortable if the $c$-sorting word $w(c)$ for $w$ defines a decreasing sequence of subsets of positions in $c$. We denote the set of $c$-sortable elements by $\text{Sort}(W, c)$. The $c$-Cambrian lattice $\text{Camb}_{\text{Sort}}(W, c)$ is the restriction of $\text{Weak}(W)$ to $\text{Sort}(W, c)$.

Although the definition of being $c$-sortable depends on a particular choice of a reduced word $c$ for the Coxeter element $c$, we have seen in Section 2.7 that all $c$-sorting words $w(c)$ are commutation equivalent. Therefore, the property of being $c$-sortable does not depend on a particular chosen word.

The second characterization is the $c$-Cambrian recurrence, which is immediate from Definition 4.1.

**Proposition 4.2** (N. Reading [Rea06, Lemma 2.1 and 2.2]). Let $s$ be initial in $c$. Then
\[
 w \in \text{Sort}(W, c) \iff \left\{ \begin{array}{ll}
 w \in \text{Sort}(W_s, s^{-1}c) & \text{if } s \in \text{asc}_L(w) \\
 s^{-1}w \in \text{Sort}(W, s^{-1}cs) & \text{if } s \in \text{des}_L(w)
\end{array} \right..
\]

N. Reading’s third characterization describes $c$-sortable elements by their inversion sets as the $c$-aligned elements. We do not $m$-eralize this definition, and so omit further discussion, except to summarize the following properties of $c$-sortable elements.

**Lemma 4.3** ([Rea07a, Corollary 4.5], [Rea07b, Theorem 1.2]). Let $w, u, v$ be elements in $\text{Sort}(W, c)$. Then
\begin{enumerate}
  
  
  \item $w, j \in \text{Sort}(W_j, c_J)$, where $J \subseteq S$ and $c_J$ is obtained from $c$ by deleting the letters $S - J$ from any reduced word for $c$;

  \item $\text{inv}(u \land v) = \text{inv}(u) \cap \text{inv}(v)$. Furthermore, $u \land v \in \text{Sort}(W, c)$.
\end{enumerate}

Finally, $\text{Camb}_{\text{Sort}}(W, c)$ is a sublattice of $\text{Weak}(W)$.

**Lemma 4.4** (N. Reading [Rea07b, Lemma 2.8 and 2.9]). Let $s$ be initial in $c$ and let $w \in \text{Sort}(W_s, sc)$. Then $w \lor s$ is both $c$- and $(s^{-1}cs)$-sortable with
\[
 \text{cov}_{\downarrow}(w \lor s) = \text{cov}_{\downarrow}(w) \cup \{s\}.
\]

4.2. $m$-eralized Coxeter-sortable elements. We first define $\text{Sort}^{(m)}(W, c)$ as elements in $W^{(m)}$ in analogy with Definition 4.1.

**Definition 4.5**. An element $w \in W^{(m)}$ is $c$-sortable if the $c$-sorting word $w(c)$ for $w$ defines a decreasing sequence of subsets of positions in $c$. We denote the set of $c$-sortable elements by $\text{Sort}^{(m)}(W, c)$.

**Example 4.6**. Definition 4.5 is illustrated with the following non-example. In $\text{Weak}^{(2)}(A_3)$ with $c = s_1s_2s_3$, the element $w = s_1s_2s_3s_1s_2 \cdot s_3s_2s_1$ (where the dot, as usual, denotes the separation of Garside factors), has $c$-sorting word
\[
 w(c) = \left( \begin{array}{cccc}
 1 & 2 & 3 \\
 a & s_2 & s_3 \\
 s_1 & s_2 & s_3 \end{array} \right).
\]

As in the situation of $m$-eralized $c$-noncrossing partitions, we conclude this section by discussing the support of $m$-eralized $c$-sortable elements. Since these are elements in the positive Artin monoid $B^{\downarrow}(W)$, they inherit the notion of support from Section 2.2. Figure 7 shows the support of the 12 elements in $\text{Sort}^{(m)}(A_2, st)$.
4.3. \textit{m-eralized Cambrian rotation and recurrence}. The Cambrian rotation and the Cambrian recurrence both depend on an operation \textsf{Shift}_{s} for an initial simple reflection \( s \) in a Coxeter element \( c \). The map

\[
\text{Shift}_{s} : \text{Sort}^{(m)}(W, c) \to \text{Sort}^{(m)}(W, s^{-1}cs)
\]

is defined for \( w \in \text{Sort}^{(m)}(W, c) \) by

\[
\text{Shift}_{s}(w) = \begin{cases} 
  w \lor s^{m} & \text{if } s \in \text{asc}_{L}(w) \\
  s^{-1}w & \text{if } s \in \text{des}_{L}(w).
\end{cases}
\]

This operation does not a priori yield an element in \( \text{Sort}^{(m)}(W, s^{-1}cs) \). The proof that \( w \lor s^{m} \) is indeed \((s^{-1}cs)\)-sortable is will be given in Proposition 4.25.

\textbf{Example 4.7.} Parallel to Example 3.14, we consider \( \text{Sort}^{(2)}(A_{2}, st) \). One orbit of the shift operation is given by

\[
\begin{align*}
& e \\
\xrightarrow{\text{Shift}_{s}} & s \cdot s \xrightarrow{\text{Shift}_{s}} sts \cdot sts \\
\xrightarrow{\text{Shift}_{t}} & tst \cdot st \xrightarrow{\text{Shift}_{s}} sts \cdot t \\
\xrightarrow{\text{Shift}_{t}} & tst \xrightarrow{\text{Shift}_{s}} st \\
\xrightarrow{\text{Shift}_{t}} & t \xrightarrow{\text{Shift}_{s}} e
\end{align*}
\]

After removing the initial \( e \), which is thought of as an element of \( \text{Sort}^{(2)}(A_{2}, st) \), the right column consists of elements of \( \text{Sort}^{(2)}(A_{2}, st) \), while the left column contains elements of \( \text{Sort}^{(2)}(A_{2}, ts) \).

We may now compose the shift operations in the order specified by any reduced \( \mathcal{S} \)-word for the Coxeter element \( c \). This composition does not depend on the chosen reduced word, since two shifts \( \text{Shift}_{s} \) and \( \text{Shift}_{t} \) commute for commuting \( s, t \in \mathcal{S} \).

\textbf{Definition 4.8.} Let \( c \) be a Coxeter element. The \textit{m-eralized c-Cambrian rotation} \( \text{Camb}_{c} : \text{Sort}^{(m)}(W, c) \to \text{Sort}^{(m)}(W, c) \) is given by

\[
\text{Camb}_{c} = \text{Shift}_{s_{n}} \circ \cdots \circ \text{Shift}_{s_{1}}
\]

for any reduced \( \mathcal{S} \)-word \( s_{1}s_{2} \cdots s_{n} \) for \( c \).

Rather than take the join with \( s^{m} \) in the first case of the definition of the shift operator, we could have sent \( w \) to itself, viewed as an element of a parabolic subgroup. We call this process, as given in the following proposition, the \textit{m-eralized c-Cambrian recurrence} on Coxeter-sortable elements.

\textbf{Proposition 4.9.} Let \( s \) be initial in a Coxeter element \( c \). Then

\[
w \in \text{Sort}^{(m)}(W, c) \iff \begin{cases} 
  w \in \text{Sort}^{(m)}(W, s^{-1}c) & \text{if } s \in \text{asc}_{L}(w) \\
  s^{-1}w \in \text{Sort}^{(m)}(W, s^{-1}cs) & \text{if } s \in \text{des}_{L}(w).\end{cases}
\]

\textbf{Example 4.10.} Consider \( sts \cdot s \in \text{Sort}^{(2)}(A_{2}, st) \). Then the \( m \)-eralized \( c \)-Cambrian recurrence is computed as

\[
\begin{align*}
\text{Sort}^{(2)}(A_{2}, st) & \xrightarrow{sts \cdot s} \text{Sort}^{(2)}(A_{2}, ts) \xrightarrow{\text{Sort}^{(2)}(A_{2}, ts)} \text{Sort}^{(2)}(A_{1}, e) \xrightarrow{\text{Sort}^{(2)}(A_{1}, e)} \text{Sort}^{(2)}(A_{0}, e).\end{align*}
\]

\textbf{Example 4.11.} Parallel to Figure 5, Figure 7 shows all 12 elements of \( \text{Sort}^{(2)}(A_{2}, st) \cong \text{Sort}^{(2)}_{\text{shard}}(A_{2}, st) \), with their support. The notion \( \text{Sort}^{(2)}_{\text{shard}}(A_{2}, st) \) will be defined and studied in Section 4.7.
4.4. Factorwise Coxeter-sortable elements. We now give an alternative description of \( m \)-eralized Coxeter-sortable elements using Garside factors.

Let \( w \in W \), let \( c = s_1 s_2 \cdots s_n \) be a Coxeter element, and let \( \text{des}_R(w) = \{s_i, s_{i+1}, \ldots, s_k\} \), where \( s_i \leq_c s_{i+1} \leq_c \cdots \leq_c s_k \) in the root order associated to \( c \). Define the restriction of \( c \) with respect to the element \( w \) to be the Coxeter element \( c|_w := s_i s_{i+1} \cdots s_k \) of the Coxeter system given by the standard parabolic subgroup \( W_{\text{des}_R(w)} \) generated by the simple reflections in \( \text{des}_R(w) \). Observe that \( c|_w \) contains the same simple reflections as the restriction of \( c \) to this standard parabolic subgroup, but that the order in which the simple reflections appear is not necessarily the order in which they appear in \( c \)—rather, it is the order coming from the root order associated to \( c \).

Example 4.12. Consider type \( A_3 \) with \( c = s_1 s_2 s_3 \) and let \( w = s_1 s_2 s_3 s_2 = s_1 s_3 s_2 s_3 \), so that \( \text{des}(w) = \{s_2, s_3\} \). Since \( s_2^{(1)} = (13) \leq_c (34) = s_3^{(1)} \), we then have \( c|_w = s_3 s_2 \).

Definition 4.13. Let \( w \) be an element in \( W^{(m)} \) with \( \text{garside}(w) = w^{(1)}, \ldots, w^{(m)} \). We say that \( w \) is \textbf{factorwise c-sortable} if \( w^{(i)} \in W_{\text{des}_R(w^{(i-1)})} \) is \( c^{(i)} \)-sortable for all \( 1 \leq i \leq m \), where \( w^{(0)} := w, c^{(0)} := c, c^{(i)} := c^{(i-1)}|_{w^{(i-1)}} \). We denote the set of factorwise c-sortable elements by \( \text{Sort}_{\text{fact}}^{(m)}(W, c) \).

We next show that \( \text{Sort}_{\text{fact}}^{(m)}(W, c) = \text{Sort}^{(m)}(W, c) \) by proving that factorwise c-sortable elements satisfy the Cambrian recurrence.

Proposition 4.14. Let \( s \) be initial in \( c \). Then

\[
W \in \text{Sort}_{\text{fact}}^{(m)}(W, c) \iff \begin{cases} 
W \in \text{Sort}_{\text{fact}}^{(m)}(W_{s^{-1}} c) & \text{if } s \in \text{asc}_{L}(w) \\
W \in \text{Sort}_{\text{fact}}^{(m)}(W, s^{(i)} c) & \text{if } s \in \text{des}_{L}(w) 
\end{cases}
\]

Proof. Let \( w \in \text{Sort}_{\text{fact}}^{(m)}(W, c) \) and suppose \( \text{garside}(w) = w^{(1)}, w^{(2)}, \ldots, w^{(m)} \).

If \( s \in \text{asc}_{L}(w) = \text{asc}_{L}(w^{(1)}) \), then \( w^{(1)} \) is \( c \)-sortable by assumption, and we obtain by Proposition 4.9 that \( s \notin \text{supp}(w^{(1)}) \) and thus that \( s \notin \text{des}_{R}(w^{(1)}) \). We conclude that \( s \notin \text{supp}(w) \) since all further Garside factors \( w^{(2)}, \ldots, w^{(m)} \) live inside
\(W_{\text{des}}(w(1))\), implying that \(w\) is \((s^{-1}c)\)-sortable as an element of \(W^{(m)}\). The converse direction, supposing \(w \in \text{Sort}^{(m)}_{\text{fact}}(W, s^{-1}c)\), is immediate.

Otherwise, \(s \in \text{des}^{(m)}_{L}(w)\) and \(c' = s^{-1}cs\). Then \(w = su\) and we set \(\text{garside}(u) = u(1) \cdot u(2) \cdot \ldots \cdot u(m)\). We need to show that

\[
w \in \text{Sort}^{(m)}_{\text{fact}}(W, c) \iff u \in \text{Sort}^{(m)}_{\text{fact}}(W, c').
\]

If \(su^{(1)} \leq w_{o}\), then \(\text{garside}(w) = su^{(1)} \cdot u^{(2)} \cdot \ldots \cdot u^{(m)}\) by the characterization of Garside factorizations in Section 2.8, since \(\text{des}_{R}(su^{(1)}) \supseteq \text{des}_{R}(u^{(1)})\). As \(w^{(1)}\) is \(c\)-sortable if and only if \(s^{-1}w^{(1)} = u^{(1)}\) is \(c'\)-sortable (by Proposition 4.9), we conclude this case.

We consider both implications individually in the final case \(su^{(1)} \not\leq w_{o}\). Note that in this case, \(s \in \text{des}^{(m)}_{L}(u^{(1)})\).

Suppose \(w \in \text{Sort}^{(m)}_{\text{fact}}(W, c')\). Since \(s\) is the final reflection in the reflection order associated to \(c' = s^{-1}cs\), the simple reflection \(s\) is the final reflection in the inversion sequence \(\text{inv}(u^{(1)}(c'))\) (since \(u^{(1)}\) is \(c'\)-sortable), and so corresponds to the final simple reflection \(r\) in the \(c'\)-sorting word \(u^{(1)}(c')\). More succinctly, we can write

\[
w^{(1)}r = su^{(1)}\]

so that \(w^{(1)} = su^{(1)}r^{-1}\). In this case, set \(w^{(1)} = u^{(1)} = su^{(1)}r^{-1}\), which is \(c\)-sortable by Definition 4.5.

We drop the first Garside factor and repeat the preceding argument on the element \(ru^{(2)} \cdot \ldots \cdot u^{(m)}\), obtaining an element of \(\text{Sort}^{(m)}_{\text{fact}}[m - 1](W, c)\) with Garside factorization \(w^{(2)} \cdot \ldots \cdot w^{(m)}\).

We claim that \(w^{(1)} \cdot w^{(2)} \cdot \ldots \cdot w^{(m)}\) is the Garside factorization of \(w\). Since we did not change the first Garside factor \(w^{(1)} = u^{(1)}\), since \(r \in \text{des}_{R}(u^{(1)})\) and \(\text{des}_{L}(ru^{(2)}) \subseteq \{r\} \cup \text{des}_{L}(u^{(2)})\) (by Proposition 2.19), \(u^{(2)}\) lies in \(W^{(1)}\). We conclude that \(w^{(1)}\) was indeed the first Garside factor of \(w\). The result follows by induction on the number of Garside factors of \(w\).

Now suppose \(w \in \text{Sort}^{(m)}_{\text{fact}}(W, c)\). Running the argument above in reverse, we obtain the candidate Garside factorization of \(u\), \(u^{(1)} = w^{(1)}\) and \(u^{(2)} \cdot \ldots \cdot u^{(m)} = r^{-1}w^{(2)} \cdot \ldots \cdot w^{(m)}\), where \(w^{(1)} = su^{(1)}\). Since \(\text{des}_{R}(w^{(1)}) = \text{des}_{R}(u^{(1)})\), and since \(w^{(2)}\) only uses simple reflections in \(\text{des}_{R}(w^{(1)})\), \(u^{(1)}\) is indeed the first Garside factor. The result again follows by induction on the number of Garside factors of \(w\).

**Example 4.15.** Continuing with Example 4.12, consider the Garside factorization

\[
w = w^{(1)} \cdot w^{(2)} = s_{1}s_{2}s_{3}s_{2} \cdot s_{3}s_{2} \quad \text{of} \quad w \in \text{Sort}^{(2)}(A_{3}, c)\]

with \(c = c(1) = s_{1}s_{2}s_{3}\) and \(c(2) = c(1)\left|_{w(1)} = s_{3}s_{2}\right.\). Then \(w^{(1)}\) is \(c(1)\)-sortable in \((W, S), s_{3}s_{2}\) is \(c(2)\)-sortable, and \(s_{1}s_{2}s_{3}s_{2}s_{3}s_{2}\) is \(c\)-sortable by Definition 4.5.

**Corollary 4.16.** An element \(w \in W^{(m)}\) is \(c\)-sortable if and only if it is factorwise \(c\)-sortable, i.e.

\[
\text{Sort}^{(m)}_{c}(W, c) = \text{Sort}^{(m)}_{\text{fact}}(W, c).
\]

**Proof.** Both \(\text{Sort}^{(m)}(W, c)\) and \(\text{Sort}^{(m)}_{\text{fact}}(W, c)\) satisfy the same recurrence and initial conditions. \(\square\)

Having established the two different descriptions of an element \(w\) in \(\text{Sort}^{(m)}(W, c)\) and in \(\text{Sort}^{(m)}_{\text{fact}}(W, c)\), we now prove the equivalence of the sorting word coming from the description in \(\text{Sort}^{(m)}(W, c)\) and the sorting words coming from the description in \(\text{Sort}^{(m)}_{\text{fact}}(W, c)\).

**Proposition 4.17.** Let \(w \in \text{Sort}^{(m)}_{\text{fact}}(W, c)\) with sorting word \(w(c)\), and let

\[
\text{garside}(w)(c) := w^{(1)}(c^{(1)}) \cdot w^{(2)}(c^{(2)}) \cdot \ldots \cdot w^{(m)}(c^{(m)})
\]
be the concatenation of the $c(i)$-sorting words coming from the Garside factors of $w$. Then $w(c)$ and $\text{garside}(w)(c)$ are commutation equivalent. In particular, the Garside factorization of $w$ can be obtained from the sorting word $w(c)$ using only commutations.

Proof. Let $w \in \text{Sort}^m(W,c)$ with $s$ initial in $c$ and $s \in \text{des}_R(w)$. If we let $w = su$, then the proof of Proposition 4.14 shows that the Garside factorization of $u$ is obtained by removing the initial $s$ from the Garside factorization of $w$ and performing commutations. The same relationship is trivially true for the sorting words of $w$ and $u$. The result now follows from the Cambrian recurrence. □

Example 4.18. In type $A_3$ with $c = s_1s_2s_3$,

$$\text{garside}(w_3^2)(c) = w_6(c) \cdot w_6(\bar{c}) = (s_1s_2s_3s_1s_2s_1) \cdot (s_3s_2s_1s_3s_2s_3),$$

which is commutation equivalent to

$$w_3^2(c) = c^4 = s_1s_2s_3s_1s_2s_3s_1s_2s_3.$$

As indicated in Example 4.18, Proposition 4.17 generalizes Lemma 2.18(4) to general sorting words of sortable elements in $B^+$. As a corollary, we obtain that the colors in the colored inversion sequence of the $c$-sorting word $w(c)$ are ordered according to the Garside factors.

Corollary 4.19. Let $w \in \text{Sort}^m(W,c)$ with $\text{garside}(w)(c)$ as in Proposition 4.17. The colors in the inversion sequence $\text{inv}(w(c)) = (\beta_1^{(m_1)}, \ldots, \beta_p^{(m_p)})$ are then given by $m_i = j$ if $s_i$ appears in the $(j + 1)^{\text{st}}$ Garside factor of $w$.

Proof. Let $w = w^{(1)} \cdots w^{(m)}$ be the Garside factorization of $w$. The factor $w^{(i)}$ lives, by Definition 4.13, inside the parabolic generated by $\text{des}_R(w^{(i-1)})$. Note that the longest element of the parabolic subgroup generated by $\text{des}_R(w^{(i-1)})$ is a right factor of $w^{(i-1)}$. The statement then follows by Proposition 4.17 and induction. □

The next example shows that this property does not hold in general for non-sortable elements.

Example 4.20. In $A_2^{(2)}$, let $w = st$ with Garside factorization $s \cdot s$.

$$\text{inv}(s \cdot s) = \left(\alpha^{(0)}, \alpha^{(0)}, \beta^{(0)}\right).$$

We next generalize Lemma 4.3(1) as follows.

Proposition 4.21. For $w \in \text{Sort}^m(W,c)$, $\text{inv}(w_J) = \text{inv}(w) \cap \Phi^+(W_J)$ and $w_J \in \text{Sort}^m(W_J, c_J)$.

Proof. The inversion set $\text{inv}(w_J)$ is contained in the restriction of $\text{inv}(w)$, since it is an initial segment of $w$. We now show that there are exactly the right number of inversions in $\text{inv}(w) \cap \Phi^+(W_J)$. Consider the roots in $W_J$ coming from the $i$th Garside factor of $w$; by restricting to $w_J$, we compute from Proposition 2.32 that the roots in the $i$th Garside factor of $w_J$ are $(w^{(1)}_J \cdots w^{(i-1)}_J) \text{inv}(w^{(i)}_J) = (w^{(1)}_J \cdots w^{(i-1)}_J) \text{inv}(w^{(i)}_J)$. But $(w^{(1)}_J \cdots w^{(i-1)}_J)_J$ is a bijection on $\Phi(W_J)$, so

$$(w^{(1)}_J \cdots w^{(i-1)}_J) \text{inv}(w^{(i)}_J) = (w^{(1)}_J \cdots w^{(i-1)}_J) \text{inv}(w^{(i)}_J) \cap \Phi(W_J).$$

In particular, there are the same number of roots in $W_J$ in the $i$th Garside factor of $w$ as there are in $w_J$, and so $\text{inv}(w_J) = \text{inv}(w) \cap \Phi(W_J)$.

The second statement follows directly from Proposition 2.32 and the statement for $m = 1$, given in Lemma 4.3(1). □

The next theorem generalizes Lemma 4.3(2).
**Theorem 4.22.** For \( w, u \in \text{Sort}^{(m)}(W, c) \),

1. \( w \leq u \) if and only if \( \text{inv}(w) \subseteq \text{inv}(u) \); and
2. \( \text{inv}(w \wedge u) = \text{inv}(w) \cap \text{inv}(u) \).

**Proof.** We prove \( w \leq u \) if and only if \( \text{inv}(w) \subseteq \text{inv}(u) \) first. If \( w \leq u \) then it is clear that \( \text{inv}(w) \) is contained in \( \text{inv}(u) \), since \( w \) is a left factor of \( u \). We now argue the converse. Suppose \( \text{inv}(w) \subseteq \text{inv}(u) \).

- If \( s \in \text{asc}_{L}(w) \) and \( s \in \text{asc}_{L}(u) \), then we are done by restriction to \( W(s) \).
- We cannot have \( s \) initial in \( w \) but not in \( u \), so suppose that \( s \in \text{asc}_{L}(w) \) and \( s \in \text{des}_{L}(u) \). It is clear that \( u_{(s)} \leq u \). Since \( w \in \text{Sort}^{(m)}(W(s), s^{-1}c) \), we have that \( \text{inv}(w) \subseteq \text{inv}(u_{(s)}) \) by **Proposition 4.21**, so that by induction on rank (since both \( w, u_{(s)} \in \text{Sort}^{(m)}(W(s), s^{-1}c) \)), \( w \leq u_{(s)} \). Since \( u_{(s)} \leq u \), we conclude that \( w \leq u \).
- Finally, if \( s \in \text{des}_{L}(w) \) and \( s \in \text{des}_{L}(u) \), then we get the statement for \( s^{-1}u = w' \) and \( s^{-1}w = w' \) by induction on length. Multiplying by \( s \) then does not change containment of inversion sets (since multiplication by \( s \) just conjugates all reflections by \( s \), and then adds \( s \)).

We now show \( \text{inv}(w \wedge u) = \text{inv}(w) \cap \text{inv}(u) \).

- If \( s \in \text{asc}_{L}(w) \) and \( s \in \text{asc}_{L}(u) \), then we are done by restriction to \( W(s) \).
- If \( s \in \text{des}_{L}(w) \) and \( s \in \text{asc}_{L}(u) \), then \( u \in \text{Sort}^{(m)}(W(s), s^{-1}c) \) so that \( w \wedge u = (w \wedge u)_{(s)} = w_{(s)} \wedge u \). We conclude the result by induction on rank.
- The case \( s \in \text{des}_{L}(w) \) and \( s \in \text{asc}_{L}(w) \) follows by symmetry.
- Finally, if \( s \in \text{des}_{L}(w) \) and \( s \in \text{des}_{L}(u) \), then we obtain \( \text{inv}(s^{-1}w \wedge s^{-1}u) = \text{inv}(s^{-1}w) \cap \text{inv}(s^{-1}u) \) by induction on length. Then \( \text{inv}(w \wedge u) = \{\alpha_{s}^{(0)}\} \cup s(\text{inv}(s^{-1}w) \cap \text{inv}(s^{-1}u)) = \{\alpha_{s}^{(0)}\} \cup s(\text{inv}(s^{-1}w) \cap \text{inv}(s^{-1}u)) = \text{inv}(w) \cap \text{inv}(u) \).

\( \square \)

**Remark 4.23.** The most naive guess for how weak order might be characterized on \( \text{Sort}^{(m)}(W, c) \) in a componentwise fashion would be to compare individual Garside factors in weak order—this naive guess is wrong. For example, \( s \cdot s \) and \( sts \cdot s \) are not comparable, even though individually \( s \leq sts \) and \( s \leq s \).

**Remark 4.24.** We emphasize that the first part of **Theorem 4.22** does not hold in general for non-sortable elements, although it is true when \( m = 1 \). The second part doesn’t even hold when \( m = 1 \) for non-sortable elements.

The remainder of this section is devoted to the following technical proposition, which we used previously to establish that the shift operator \( \text{Shift}_{s} : \text{Sort}^{(m)}(W, c) \to \text{Sort}^{(m)}(W, s^{-1}cs) \) is well-defined.

**Proposition 4.25.** Let \( c \) be a Coxeter element, let \( s \) be initial in \( c \), and let \( w \in \text{Sort}^{(m)}(W(s), cs) \). For \( 0 \leq k \leq m \), \( w \vee sk \) is simultaneously c-sortable and \((s^{-1}cs)\)-sortable.

Let \( w(0) := w_{0}^{(1)} \cdots w_{0}^{(m)} \) be the Garside factorization of the element \( w \) in the proposition, and then inductively set

\[
\begin{align*}
\text{for } 1 \leq k \leq m \text{ where } s_{k} \text{ and } v_{k} \text{ are the following elements of } W: \\
& s_{k} = s(w_{1}^{(2)} \cdots w_{k-1}^{(k-1)})^{-1} = s_{k-1}^{(k-1)} \circ s_{k}^{-1}, \\
v_{k} = (w_{k}^{(k)})^{-1}w_{k-1}^{(k-1)} = (w_{k-1}^{(k-1)} \circ s_{k})^{-1}w_{k-1}^{(k-1)}.
\end{align*}
\]
We will show that the decomposition \( w(k) = w_k^{(1)} \cdot \cdots \cdot w_k^{(m)} \) is the Garside factorization of \( w \vee s^k \), and that this factorization is factorwise \( c \)- and \((s^{-1}cs)\)- sortable.

**Lemma 4.26.** The element \( w(k) \) is \( c \)- and \((s^{-1}cs)\)- sortable with Garside factorization \( w_k^{(1)} \cdot \cdots \cdot w_k^{(m)} \).

As the proof of this lemma is a slightly involved induction on \( k \), we extract the base case \( k = 1 \) into a separate lemma for readability.

**Lemma 4.27.** The element \( w(1) \) is \( c \)- and \((s^{-1}cs)\)- sortable with Garside factorization
\[
  w(1) = (w_0^{(1)} \vee s) \cdot (w_0^{(2)})^v \cdot \cdots \cdot (w_0^{(m)})^v
\]
where \( v = (w_1^{(1)})^{-1} w_0^{(1)} = (w_0^{(1)} \vee s)^{-1} w_0^{(1)} \in W \). Furthermore, \( \text{cov}_i(w_0^{(1)} \vee s) = \text{cov}_i(w_0^{(1)}) \cup \{s\} \).

**Proof.** Since \( w(0) \) is \( c \)- sortable (and thus factorwise \( c \)- sortable) by assumption, its first factor \( w_0^{(1)} \) is also \( c \)- sortable. This element \( w_0^{(1)} \) satisfies the assumption of Lemma 4.4, and we obtain that \( w_0^{(1)} \vee s \) is both \( c \)- and \((s^{-1}cs)\)- sortable, and that
\[
  \text{cov}_i(w_0^{(1)} \vee s) = \text{cov}_i(w_0^{(1)}) \cup \{s\}.
\]

The factorwise \( c \)- sortability also implies that the Garside factors \( w_0^{(2)}, \ldots, w_0^{(m)} \) all live in the parabolic subgroup \( W_{\text{des}}(w_0^{(1)}) \). Now, conjugating all these Garside factors by \( v \) simply takes those right descents of \( w_0^{(1)} \), maps them to the cover reflection \( \text{cov}_i(w_0^{(1)}) \) by conjugating with \( w_0^{(1)} \), and then turns these cover reflections back to the corresponding right descents of \( w_1^{(1)} = w_0^{(1)} \vee s \) using Equation (6). Since this rearrangement of the right descents is clearly compatible with the defining property of factorwise \( c \)- and \((s^{-1}cs)\)- sortability, the statements follow. \( \square \)

**Proof of Lemma 4.26.** We write \( c_i^{(i)} \) for the Coxeter element \( c_i^{(i)} \) for \( w(j) \), as defined at the beginning of Section 4.4. We will prove the statement of Lemma 4.26 along with
\[
  \bullet \quad \text{cov}_i(w_k^{(k)}) = \text{cov}_i(w_k^{(k)} \vee s_k) = \text{cov}_i(w_{k-1}^{(k)}) \cup \{s_k\}; \quad \text{and}
\]
\[
  \bullet \quad s_k \text{ is initial in } c_{k-1}^{(k-1)}.
\]
by induction on \( k \). These are established for \( k = 1 \) by Lemma 4.27. It remains to conclude the statements for \( k \), assuming that they hold for \( k - 1 \).

The first \( k - 1 \) Garside factors have not changed, and so are still Garside factors, and each of them is sortable in its corresponding parabolic subgroup given by the definition of factorwise sortability.

We next show that \( s_k \in \text{des}_R(w_{k-1}^{(k-1)}) \) (which, in particular, shows that it is a simple reflection) and \( s_k \notin \text{supp}(w_{k-1}^{(k)}) \). We can assume by induction that
\[
  \text{cov}_i(w_{k-1}^{(k-1)}) = \text{cov}_i(w_{k-2}^{(k-1)} \vee s_{k-1}) = \text{cov}_i(w_{k-2}^{(k-1)}) \cup \{s_{k-1}\}.
\]
Therefore, \( s_k = s_{k-1}^{(k-1)} \) is the right descent of \( w_{k-1}^{(k-1)} \) corresponding to its cover reflection \( s_{k-1} \), implying the first property \( s_k \in \text{des}_R(w_{k-1}^{(k-1)}) \). Moreover, \( w_{k-1}^{(k)} = (w_{k-2}^{(k)}) \vee s_{k-1} \) sits inside the right descents of \( w_{k-1}^{(k-1)} \) in the same way as \( w_{k-2}^{(k)} \) sits in the right descents of \( w_{k-2}^{(k-1)} \). Since \( s_{k-1} \) was not a covered reflection of \( w_{k-2}^{(k-1)} \), the right descent \( s_k \) of \( w_{k-1}^{(k-1)} \) corresponding to this covered reflection cannot be contained in the support of \( w_{k-1}^{(k-1)} \), yielding the second property \( s_k \notin \text{supp}(w_{k-1}^{(k)}) \).
The induction hypothesis gives us that \( s_{k-1} \) is initial in \( c_{k-2}^{(k-2)} \). Therefore \( s_k \) is initial in \( c_{k-1}^{(k-1)} \) by the definition \( c_{k-1}^{(k-1)} \) since \( s_{k-1} \in \text{cov}_k(w_{k-1}^{(k-1)}) \) is the cover reflection corresponding to \( s_k \in \text{des}_k(w_{k-1}^{(k-1)}) \) and \( s_{k-1} \).

We can therefore apply Lemma 4.4 to the \( c_{k-1}^{(k-1)} \)-sortable element \( w_{k-1}^{(k)} \) to obtain that \( w_{k}^{(k)} = w_{k-1}^{(k)} \lor s_k \) is again \( c_{k-1}^{(k-1)} \) - and \( (s_k^{-1}c_k^{-1} s_k) \)-sortable with

\[
\text{cov}_k(w_{k}^{(k)}) = \text{cov}_k(w_{k-1}^{(k)} \lor s_k) = \text{cov}_k(w_{k-1}^{(k-1)}) \cup \{s_k\}.
\]

Therefore \( w_{k-1}^{(k)} \) lives in the parabolic subgroup generated by \( \text{cov}_k(w_{k-1}^{(k-1)}) = \text{cov}_k(w_{k-1}^{(k-1)}) \).

The final part of the proof is to conjugate the remaining Garside factors \( w_{k}^{(k+1)} \) through \( w_{k-1}^{(k+1)} \) by \( w_k \). This part is completely analogous to the argument given in the proof of Lemma 4.27.

**Proof of Proposition 4.25.** We show that \( w(k) = w \lor s^k \), and again first consider the case \( k = 1 \).

Clearly, \( s \leq w(1) \) since the Garside factorization begins with \( w_0^{(1)} \lor s \) which is above \( s \) in \( W \) and therefore has a reduced word starting with \( s \). Also \( w \leq w(1) \) since

\[
w(1) = w_1^{(1)} \cdot w_1^{(2)} \cdot \cdots \cdot w_1^{(m)} = (w_0^{(1)} w_1^{(1)}) \cdot (w_1^{(2)} v_1^{(1)}) \cdot \cdots \cdot (w_1^{(m)} v_1^{(1)})
\]

where we write \( v_1 = (w_0^{(1)})^{-1} w_0^{(1)} = (w_0^{(1)} \lor s_k)^{-1} w_0^{(1)} \) as before, and write \( v_1^{-1} = (w_0^{(1)})^{-1} (w_0^{(1)} \lor s_k) \in B^+ \). The first equality is given by the definition of \( w(1) \) in terms of \( w(0) \). Then Lemma 4.27 implies that \( w_1^{(1)} \cdot \cdots \cdot w_1^{(m)} \) is indeed the Garside factorization \( w(1) \).

It remains to show that \( w(1) \) is minimal among all elements above \( s \) and \( w \). Although the colored inversion set of an element of \( B^+ \) is not necessarily unique to that element, the number of inversions still tell us its length. Any element above \( w \) must contain all inversions of \( w \), and any element above \( w_0^{(1)} \) and \( s \) must contain the inversions of \( w_0^{(1)} \lor s \). The inversion set of \( w(1) \) contains all these inversions and no others, and therefore has the minimal desired length; we conclude that \( w(1) = w \lor s \).

For the case of general \( k \), we first check that \( w(k-1) \leq w(k) \) and that \( s^k \leq w(k) \).

For \( w(k-1) \), we have

\[
w(k) = w_k^{(1)} \cdot w_k^{(2)} \cdot \cdots \cdot w_k^{(k-1)} \cdot (w_k^{(k)} \lor sk) \cdot (w_{k-1}^{(k+1)}) v_k \cdot \cdots \cdot (w_{k-1}^{(m)}) v_k
\]

\[= w_k^{(1)} w_k^{(2)} \cdots w_k^{(k-1)} (w_k^{(k)} v_k) = (w_{k-1}^{(k+1)}) v_k \cdots (w_{k-1}^{(m)}) v_k
\]

\[= w_k^{(1)} w_k^{(2)} \cdots w_k^{(k-1)} (w_k^{(k)} \lor sk)\]

For \( s^k \), we have

\[
w(k) = w_k^{(1)} w_k^{(2)} \cdots w_k^{(k-1)} (w_k^{(k)} \lor sk)(w_{k-1}^{(k+1)}) v_k \cdots (w_{k-1}^{(m)}) v_k
\]

\[= w_k^{(1)} w_k^{(2)} \cdots w_k^{(k-1)} (w_k^{(k)} \lor sk) \cdots w_k^{(1)} w_k^{(2)} \cdots w_k^{(k-1)} sk \cdots
\]

\[= s(w_k^{(1)} w_k^{(2)} \cdots w_k^{(k-1)}) \cdots s s^{k-1} \cdots s^k \cdots
\]

We now show that \( w(k) \) is the minimal element above \( s^k \) and \( w(k-1) \). Let \( u = (w_k^{(1)} w_k^{(2)} \cdots w_k^{(k-1)} \lor sk) \). We claim that \( s^k \lor u = (su) \lor w(k-1) \). We first show that \( s^k \lor u = su \). The element \( s^k \lor u \) is divisible by \( s^k \), and since \( u \) is
divisible by $s^{k-1}$, the length of $s^k \vee u$ is at least one more than the length of $u$. As the element $su = us_k$ is divisible by both $s^k$ and $u$, we conclude that $su = s^k \vee u$. Therefore, since $u$ is a left factor of $w(k-1)$, we conclude that

$$su \vee w(k-1) = s^k \vee u \vee w(k-1) = s^k \vee w(k-1).$$

Now $su = us_k$, so that $us_k$ and $w(k-1)$ share their first $k-1$ Garside factors. Therefore,

$$su \vee w(k-1) = us_k \vee w(k-1) = u(s_k \vee (w_{k-1}^k \cdot \cdots \cdot w_{k-1}^m)).$$

By the inversion set argument used above for $k = 1$, we may now conclude that the final $m - k + 1$ Garside factors $s_k \vee (w_{k-1}^k \cdot \cdots \cdot w_{k-1}^m)$ are of specified form, so that $w(k) = s^k \vee w(k-1)$. □

**Example 4.28.** Proposition 4.25 does not hold for non-sortable elements. For example, in type $A_3$, if $c = s_1s_2s_3$ and $u = s_3s_2s_1$, then

$$u \vee s_1 = s_3s_3s_2s_1s_2 = s_3s_3s_1s_2s_1 = s_1s_3s_3s_2s_1 = s_1s_3 \cdot s_3s_2s_1.$$

4.5. *m*-eralized Coxeter-sortable elements and noncrossing partitions. We begin this section with a direct construction of a bijection between *m*-eralized c-sortable elements an *m*-eralized c-noncrossing partitions.

Generalizing the situation in [RS11] and in [PS11, Proposition 6.20], we define the skip set for c-sortable elements as follows. Given $w \in \text{Sort}^{(m)}(W; c)$, let its c-sorting word be $w(c) = s_1 \cdots s_p$ and let $s \in S$. We say that $w$ *skips* $s$ in position $k + 1$ if the leftmost instance of $s$ in $c^\infty$ not used in $w(c)$ occurs between $s_k$ and $s_{k+1}$. The *skip set* (of colored positive roots) is then defined by

$$C_c(w) = \{ \beta^{(c^\infty)}_s : s \in S \}$$

where $\beta^{(c^\infty)}_s$ is the colored positive root $s_1 \cdots s_k(\alpha_s^{(0)})$ and $k$ is chosen such that $w$ skips $t$ in position $k + 1$. A skip is *a-forced* if it occurs with color $a$; a skip is *forced* if it is *m*-forced and *unforced* otherwise. As in the definition of root configurations of facets of subword complexes, we consider skip sets as naturally ordered induced by the order of $S$ induced by $c$. (As usual, this ordering is only defined up to commutations.) Again, whenever we write a union of colored roots in a skip set, we mean the disjoint union ordered by concatenation of the individual orderings.

**Remark 4.29.** This terminology comes from the following observation: for $w \in \text{Sort}^{(m)}(W; c)$, if $w$ is modified to postpone an *m*-forced skip, then the resulting element is no longer less than or equal to $w^m_0$.

**Example 4.30.** Figure 7 shows the skip sets of all elements in $\text{Sort}^{(2)}(A_2, st)$.

**Remark 4.31.** It is easy to recover the c-sorting word $w(c)$ for $w \in \text{Sort}^{(m)}(W; c)$ from its skip set $C_c(w)$ as follows. Begin reading the word $c^\infty = s_1s_2\ldots$ from left to right; we will specify a procedure for deleting letters. If there is no next letter, the letters remaining spell $w(c)$. Otherwise, if the next letter in $c^\infty$ is a skip—in the sense that the corresponding colored positive root is contained in $C_c(u)$, where $u$ is the product of the undeleted letters strictly left of the current position—then it and all of its occurrences to the right in $c^\infty$ are deleted from $c^\infty$.

The skip set $C_c(w)$ is closely linked to $\text{NC}^{(m)}_\Delta(W; c)$, and has the following inductive structure, proven for $m = 1$ by N. Reading and D. Speyer in [RS11].
Proposition 4.32. Let $s$ be initial in $c$ and let $w \in \text{Sort}^{(m)}(W, c)$. The skip set satisfies

$$C_c(w) = \begin{cases} \{a_s^{(0)}\} \cup C_{s^{-1}c}(w) & \text{if } s \in \text{asc}_L(w) \\ sC_{s^{-1}c}(s^{-1}w) & \text{if } s \in \text{des}_L(w) \end{cases}$$

Proof. If $s \in \text{asc}_L(w)$ then $w \in \text{Sort}^{(m)}(W, s^{-1}c)$ with skip set $C_{s^{-1}c}(w)$. If we treat $w$ as an element of $\text{Sort}^{(m)}(W, c)$, this does not change the positions of the skips $t \neq s$. But $w$ now skips $s$ in position 1, so that the new colored positive root $\{a_s^{(0)}\}$ is added to the skip set. The second case for $s \in \text{des}_L(w)$ is similar—no simple reflection is skipped in position 1, and so each colored positive root $s_1 \cdots s_k(\alpha_t) \in C_c(w)$ corresponds to a colored positive root $s_2 \cdots s_k(\alpha_t) \in sC_{s^{-1}c}(s^{-1}w)$.

Example 4.33. Considering $sts \cdot s \in \text{Sort}^{(2)}(A_2, st)$ as in Example 4.10, the sequence of skip sets is

$$\begin{array}{cccccccc}
\{1\} & \{2\} & \{0\} & \{2\} & \{0\} & \{1\} & \{0\} & \{1\} \\
\pi^{(2)}(A_2, st) & \pi^{(2)}(A_2, st) & \pi^{(2)}(A_2, st) & \pi^{(2)}(A_2, st) & \pi^{(2)}(A_2, st) & \pi^{(2)}(A_2, st) & \pi^{(2)}(A_2, st) & \pi^{(2)}(A_2, st)
\end{array}$$

Let $w \in \text{Sort}^{(m)}(W, c)$ with skip set $C_c(w) = \{\beta_1^{(t)}(\alpha), \ldots, \beta_n^{(t)}(\alpha)\}$. Since $w$ divides $w^m \in B^+(W)$, the maximal possible color in $C_c(w)$ is indeed $m$. We conclude the following theorem.

Theorem 4.34. The map $C_c$ induces a bijection

$$\text{Sort}^{(m)}(W, c) \rightarrow \text{NC}^{(m)}_\Delta(W, c)$$

that respects Cambrian rotation, the Cambrian recurrence, and support. The individual factors of color $i$ in $C_c(w)$ give the corresponding factors as an element of $\text{NC}^{(m)}_\Delta(W, c)$.

Proof. By Proposition 4.32 and the discussion after Proposition 3.16, the inductive structure on $\text{Sort}^{(m)}(W, c)$ is sent by $C_c$ to the inductive structure on $\text{NC}^{(m)}_\Delta(W, c)$, and they have the same initial conditions.

Example 4.35. The $m$-eralized $c$-sortable elements in Figure 7 are mapped to the $m$-eralized $c$-noncrossing partitions in Figure 5.

Remark 4.36. For $m = 1$, the bijection $\text{Sort}^{(m)}(W, c) \rightarrow \text{NC}^{(m)}_\Delta(W, c)$ reduces to the bijection between $\text{Sort}(W, c)$ and $\text{NC}^{(m)}_\Delta(W, c)$ given by N. Reading in [Rea07a]. By [RS11, Proposition 5.2], since covered reflections of a $c$-sortable element coincide with their forced skips, our generalized skip set may be thought of as an $m$-eralized version of covered reflections for $c$-sortable elements. Compare with our failed construction in Remark 3.19.

4.6. $m$-eralized Cambrian lattices. N. Reading’s $c$-sortable elements are the key to understand certain order congruences on the weak order that respect the lattice structure of the weak order (and are therefore lattice congruences). We briefly summarize some results of [Rea07b]. N. Reading defined an order-preserving preserving projection $\pi^+_c : \text{Weak}(W) \rightarrow \text{Sort}(W, c)$ sending an element $w$ to the largest $c$-sortable element less than or equal to $w$. Likewise, there is a related order-preserving preserving map $\pi^+_c$ that maps $w$ to the smallest $c$-sortable greater than or equal to $w$. N. Reading showed that the fibers of $\pi^+_c$ and $\pi^+_c$ are equal and that the fiber containing $w$ is the interval $[\pi^+_c(w), \pi^+_c(w)]_{\text{Weak}(W)}$. This turns out to be enough to conclude that the $c$-sortable elements form a lattice quotient of the weak order. Each of these congruences defines an associated dodecahedron corresponding to $c$; the 1-skeletons of these $c$-associahedra are the Hasse diagrams of the $c$-Cambrian lattices.
In contrast, the \(m\)-eralized \(c\)-sortable elements no longer form a lattice quotient of \(\text{Weak}^{(m)}(W)\), as indicated in the following example.

**Example 4.37.** Let \(\pi^c_1: \text{Weak}^{(m)}(W) \to \text{Sort}^{(m)}(W,c)\) be defined as the largest \(c\)-sortable element less than or equal to \(w\). In type \(A_2\) with \(m = 3\), this does not define a lattice quotient, as we now illustrate. The element \(s_2s_1 \cdot s_1 s_2\) is maximal in the fiber above \(s_2\) because its one cover is the element \(w_0 \cdot s_1 s_2 \cdot s_2\), which is above \(w_0 \cdot s_1\). On the other hand, the element \(s_2s_1 \cdot s_1 s_2 \cdot s_2 s_1\) is also maximal in the fiber above \(s_2\) because its single cover is the element \(w_0 \cdot s_1 s_2 \cdot s_2 s_1\), which is also above \(w_0 \cdot s_1\). Therefore, there is more than one maximal element in the fiber above \(s_2\), the fiber is not an interval, and so \(\pi^c_1\) cannot define a lattice quotient.

One might hope that the failure arose because we were “missing” the full fiber above \(s_2\) by restricting to only those elements below \(w^3_0\). It would make sense to therefore take the entire monoid \(B^+\) to obtain the fiber. In fact, this does not resolve the problem, since

\[
(s_2s_1 \cdot s_1 s_2 \lor s_2 s_1 \cdot s_1 s_2 \cdot s_2 s_1) = w_0 \cdot s_1 s_2 \cdot s_2 s_1 \leq w^3_0.
\]

**Definition 4.38.** The \(m\)-eralized \(c\)-Cambrian poset \(\text{Camb}^{(m)}_{\text{Sort}}(W, c)\) is the restriction of \(\text{Weak}^{(m)}(W)\) to \(\text{Sort}^{(m)}(W, c)\).

**Remark 4.39.** F. Bergeron defined an \(m\)-Tamari lattice on \(m\)-Dyck paths. When \(m = 1\), this lattice is isomorphic to \(\text{Camb}_{\text{Sort}}(A_n, s_1 \cdots s_n)\), but this is no longer the case for \(m > 1\). The \(m\)-Tamari lattice is shown for \(m = 2\) in Figure 8; compare with Figure 9. We refer to [Ber12] for definitions and further details. For conjectural similarities between the two lattices, see Remark 5.44.

Although we no longer have a lattice quotient, the restriction of \(\text{Weak}^{(m)}(W)\) to \(\text{Sort}^{(m)}(W, c)\) still yields a lattice.

**Theorem 4.40.** \(\text{Camb}^{(m)}_{\text{Sort}}(W, c)\) is a sublattice of \(\text{Weak}^{(m)}(W)\).
Proof. The proof is analogous to the proof of [Rea07b, Theorem 1.2], although—as illustrated in Example 4.37—we do not have a projection map \( \pi^c_i \), and so cannot rely on its properties to compute the join.

We first show that \( u \wedge v \in \text{Sort}^{(m)}(W,c) \) for \( u, v \in \text{Sort}^{(m)}(W,c) \). Let \( s \) be initial in \( c \).

- If \( s \in \text{asc}_L(u) \) and \( s \in \text{asc}_L(v) \), then \( u, v \in \text{Sort}^{(m)}(W,(s)) \) and we conclude the result by induction on rank.
- If \( s \in \text{des}_L(u) \) and \( s \in \text{asc}_L(v) \), then \( v \in \text{Sort}^{(m)}(W,(s),s^{-1}c) \). Since \( w \mapsto w_{(s)} \) is a meet-semilattice homomorphism by Proposition 2.32, \( u \wedge v = (u \wedge v)_{(s)} = u_{(s)} \wedge v_{(s)} \). By Proposition 4.21, \( u_{(s)} \) is \((s^{-1}c)\)-sortable, so that \( u \wedge v \) is \( c \)-sortable by the previous case.
- The case \( s \in \text{des}_L(v) \) and \( s \in \text{asc}_L(u) \) follows by symmetry.
- Finally, suppose \( s \in \text{des}_L(u) \) and \( s \in \text{des}_L(v) \). Let \( c' = s^{-1}cs \) so that \( s^{-1}u \) and \( s^{-1}v \) are both \( c' \)-sortable. By induction on length, \((s^{-1}u) \wedge (s^{-1}v) \) is \( c' \)-sortable. Then \( u \wedge v = s(s^{-1}u \wedge s^{-1}v) \), which is \( c \)-sortable by Proposition 4.9.

We now show that \( u \vee v \in \text{Sort}^{(m)}(W,c) \) for \( u, v \in \text{Sort}^{(m)}(W,c) \). Again, let \( s \) initial in \( c \).

- If \( s \in \text{des}_L(u) \) and \( s \in \text{des}_L(v) \), then \( s \in \text{des}_L(u \vee v) \). Let \( c' = s^{-1}cs \) so that \( s^{-1}u \) and \( s^{-1}v \) are both \( c' \)-sortable. By induction on length, \((s^{-1}u) \vee (s^{-1}v) \) is \( c' \)-sortable. Then \( u \vee v = s(s^{-1}u \vee s^{-1}v) \), which is \( c \)-sortable by Proposition 4.9.
- If \( s \in \text{des}_L(v) \) and \( s \in \text{asc}_L(u) \), then \( u \in \text{Sort}^{(m)}(W,(s),s^{-1}c) \) and \( s \vee u \) is \( c \)-sortable by Proposition 4.25. We compute that \( u \vee v = s(v(u \vee v)) = (s \vee u) \vee v \), so that \( u \vee v \) is \( c \)-sortable by the previous case.
- The case \( s \in \text{des}_L(u) \) and \( s \in \text{asc}_L(v) \) follows by symmetry.
- Finally, suppose \( s \in \text{asc}_L(u) \) and \( s \in \text{asc}_L(v) \). Then \( u, v \in \text{Sort}^{(m)}(W,(s)) \) and we conclude the result by induction on rank.

\( \square \)

Example 4.41. Figure 9 shows all 12 \( st \)-sorting elements in \( \text{Camb}_{(2)}^{\text{Sort}}(A_2, st) \).

In the remainder of this section, we prove that the two posets \( \text{Camb}^{(m)}_{\text{Sort}}(W,c) \) and \( \text{Camb}_{\text{NC}}^{(m)}(W,c) \) are isomorphic.

Theorem 4.42. We have \( \text{Camb}_{\text{Sort}}^{(m)}(W,c) \cong \text{Camb}_{\text{NC}}^{(m)}(W,c) \). In particular, the poset \( \text{Camb}_{\text{NC}}^{(m)}(W,c) \) is a lattice.

We will prove this theorem by induction, using the Cambrian recurrence. As a preliminary result, we analyze the situation of the skip set in Proposition 4.25. Let \( c \) be a Coxeter element, let \( s \) be initial in \( c \), and let \( w \in \text{Sort}^{(m)}(W,(s),sc) \). For \( 0 \leq k \leq m \), we have seen that \( w \wedge s^k \) is \( c \)-sortable and \((s^{-1}c)s\)-sortable.

Lemma 4.43. The skip set \( C_c(w \wedge s^k) \) is obtained from the skip set \( C_c(w) \) by replacing \( \alpha_1^{(0)} \) by \( \alpha_1^{(k)} \), replacing all other roots of the form \( \beta^{(i)} \) by \([s(\beta)]^{(i)} \) for \( 0 \leq i < k \), and leaving all roots of the form \( \beta^{(i)} \) for \( i \geq k \) unchanged.

Proof. The proof of Proposition 4.25 describes exactly how the Garside factorization of \( w \wedge s^{k-1} \) is obtained from the Garside factorization of \( w \). This shows that the colored inversion sequence of the \( c \)-sorting word only changes within the \((k-1)^{st}\) Garside factor. Since we know from the case \( m = 1 \) that the roots in this Garside factor change exactly in the described way [RS11, Proposition 5.4], we conclude that the skip set \( C_c(w \wedge s^k) \) is obtained from the skip set \( C_c(w \wedge s^{k-1}) \) by
Figure 9. Camb\textsuperscript{(2)}\textsubscript{Sort}(A\textsubscript{2}, st).

- replacing $\alpha_s^{(k-1)}$ by $\alpha_s^{(k)}$;
- replacing all other $(k-1)$-colored roots $\beta^{(k-1)}$ by $[s(\beta)]^{(k-1)}$; and
- leaving all other colored roots in the skip set invariant.

The lemma then follows by applying this procedure $k$ times, starting with $w = w \lor s^0$.

Proof of Theorem 4.42. We show that a cover relation $u \leq v$ in Camb\textsubscript{Sort}(W, c) corresponds to a cover relation $I \leq J$ in Camb\textsubscript{NC}(W, c) under the map given in Theorem 4.34. To this end, let $\delta = (\delta_0, \ldots, \delta_m)$ and $\delta' = (\delta'_0, \ldots, \delta'_m)$ be the delta sequences corresponding to $I$ and to $J$, respectively.

- If $s \in \text{asc}_L(u)$ and $s \in \text{asc}_L(v)$ (equivalently, $s \in \delta_0$ and $s \in \delta'_0$), the statement follows by the Cambrian recurrences given in Propositions 4.2 and 3.16 and their relation in Theorem 4.34.
- The case $s \in \text{des}_L(u)$ and $s \in \text{asc}_L(v)$ (equivalently, $s \not\in \delta_0$ and $s \in \delta'_0$) is impossible since $u \leq v$ in weak order.
- If $s \in \text{des}_L(u)$ and $s \in \text{des}_L(v)$ (equivalently, $s \not\in \delta_0$ and $s \not\in \delta'_0$), the statement follows again by the Cambrian recurrences.
- Finally consider the case $s \in \text{asc}_L(u), s \in \text{des}_L(v)$ (equivalently, $s \in \delta_0, s \not\in \delta'_0$). We first analyze $\text{Flip}_{L}^1(I) = J$. By Equation (5), the set of $(m + 1)$-colored positive roots in $J$ is obtained from the set of $(m + 1)$-colored positive roots in $I$ by replacing $\alpha_s^{(0)}$ by $\alpha_s^{(1)}$, conjugating all other roots of the form $\beta^{(0)}$ by $[s(\beta)]^{(0)}$, and leaving all roots of the form $\beta^{(\ell)}$ for $\ell > 1$ unchanged. On the other hand, we have that $u \leq v$ in weak order and, since $s \in \text{supp}(v)$, $s \leq v$ in weak order, implying that $u \lor s \leq v$ in weak order. Since $s \not\in \nu s \not\in \text{supp}(u)$, we can apply Lemma 4.43 with $k = 1$ and obtain that $u \lor s$ is $c$-sortable. Therefore, $v = u \lor s$ and the skip set of $v$ is obtained from the skip set of $u$ in the expected way.

□

□
4.7. m-eralized Coxeter-sortable elements and the shard intersection order. In this last section on Coxeter-sortable elements, we provide an alternative route to describe the bijection in Theorem 4.34 which—although a concatenation of other results—has not been considered in the literature before. This approach reconciles our m-eralization of c-sortable elements with D. Armstrong’s m-eralization of noncrossing partitions.

D. Armstrong’s construction of NC\((m)\)(\(W,c\)) as \(m\)-multichains in NCL\((W,c)\) was given in Definition 3.5. On the other hand, N. Reading defined an order Shard\((W)\) on the elements of \(W\), with the property that the restriction of Shard\((W)\) to Sort\((W,c)\) is isomorphic to the noncrossing partition lattice NCL\((W,c)\) [Rea11]. It is natural to use this isomorphism to m-eralize Coxeter-sortable elements as chains of c-sortable elements in Shard\((W)\).

In Theorem 4.52, we give a bijection between this seemingly artificial m-eralization of c-sortable elements as chains and Sort\((m)\)(\(W,c\)). As a corollary, we obtain a second bijection between Sort\((m)\)(\(W,c\)) and NC\((m)\)(\(W,c\)); we show in Theorem 4.56 that this recovers the bijection previously given in Theorem 4.34.

In [Rea11], N. Reading defined a delicate slicing procedure on simplicial hyperplane arrangements that cuts hyperplanes into several pieces called shards. The shard intersection order Shard\((W)\) is the set of all intersections of these hyperplane pieces, ordered by reverse inclusion. N. Reading proved that the intersection of the lower shards of an element \(w \in W\) is a bijection between \(W\) and the set of shard intersections. The longest element \(w_0\) is mapped to the maximal element in the shard intersection order under this bijection. It is indeed possible to define the shard intersection order directly on \(W\).

Definition 4.44. Let \(u, v \in W\). The shard intersection order Shard\((W)\) is given by \(u \preceq v\) if and only if \(\langle \text{cov}_1(u) \rangle \subseteq \langle \text{cov}_1(v) \rangle\) and \(\text{inv}(u) \subseteq \text{inv}(v)\).

Example 4.45. Figure 10 shows the two lattices NCL\((A_2, st)\) and Shard\((A_2)\) restricted to Sort\((A_2, st)\). Figure 11 shows Shard\((A_3)\) restricted to Sort\((A_3, s_1s_2s_3)\). In both figures, the sortable elements are given by their inversion sets in the positive roots. Covered reflections are circled in grey, and further inversions in the parabolic subgroup generated by the covered reflections are circled in white.

A (non-stuttering) gallery, or simply gallery, is a walk on the connected regions of the complement of the hyperplane arrangement of \(W\), where two regions are connected if they share a bounding hyperplane, and any hyperplane is crossed at most once.

Lemma 4.46. Any gallery from \(e\) to \(v\) necessarily crosses every lower shard of \(v\).
Proof. Let $H$ be the hyperplane corresponding to a lower shard $t$ of $v$ and consider a different shard $t'$ in $H$. It is not hard to see from the definition of shard [Rea11] that there is a hyperplane $K$ which cuts $H$ such that $t$ and $e$ are on one side of $K$, and $t'$ is on the other side. Then it is clear that a gallery from $e$ to $v$ will not cross $K$ and so will not cross $t'$. □

**Proposition 4.47.** Definition 4.44 agrees with the definition given in [Rea11, Section 4].

Proof. We first show that $\langle \text{cov}_1(u) \rangle \subseteq \langle \text{cov}_1(v) \rangle$ and $\text{inv}(u) \subseteq \text{inv}(v)$ implies $u \preceq v$. By the correspondence between the lattice of (conjugates of) parabolic subgroups and the intersection lattice of the hyperplane arrangement of $W$ [BI99], $\langle \text{cov}_1(u) \rangle \subseteq \langle \text{cov}_1(v) \rangle$ implies that

$$\bigcap_{H \in \text{cov}_1(u)} H \subseteq \bigcap_{H \in \text{cov}_1(v)} H.$$  

If we in addition have $\text{inv}(u) \subseteq \text{inv}(v)$, then there exists a gallery from $e$ to $u$ to $v$. Let $\Upsilon$ be the union of the set of shards that this gallery crosses, and note that because the gallery is non-stuttering, there is exactly one shard for each hyperplane in $\text{inv}(v)$. For any region $r$ with lower hyperplane $H$, the entire facet of $r$ corresponding to $H$ is part of the same shard. By Lemma 4.46, each hyperplane in $\text{cov}_1(r)$ intersected with the shards arising from a gallery gives the corresponding lower shard of $r$. Now taking the intersection of both sides of Equation (7) with $\Upsilon$ implies that the intersection of the lower shards of $u$ is contained in the intersection of the lower shards of $v$.

We now show that $u \preceq v$ implies that $\langle \text{cov}_1(u) \rangle \subseteq \langle \text{cov}_1(v) \rangle$ and $\text{inv}(u) \subseteq \text{inv}(v)$. By [Rea11, Proposition 5.5], we know that $\langle \text{cov}_1(u) \rangle \subseteq \langle \text{cov}_1(v) \rangle$, since sending shards to their containing hyperplane induces an order-preserving map between the shard intersection order and the intersection lattice of $W$. Finally, by [Rea11, Proposition 4.7 (ii)], sending regions to the intersection of their lower shards is an order-preserving map between the shard intersection order and the weak order. □
Proposition 4.48 ([Rea11, Proposition 1.2]). The interval $[c, w]_{\text{Shard}(W)}$ is isomorphic to $\text{Shard}(W_{\text{cov}}(w)) \cong \text{Shard}(W_{\text{des}}(u)).$

The isomorphism $[c, w]_{\text{Shard}(W)} \cong \text{Shard}(W_{\text{cov}}(w))$ is given by sending $u \preceq w$ to $u_{\text{cov}}(w),$ the element in $W_{\text{cov}}(w)$ with inversion set $\text{inv}(u) \cap \Phi^+(W_{\text{cov}}(w)).$ Likewise, the isomorphism $[c, w]_{\text{Shard}(W)} \cong \text{Shard}(W_{\text{des}}(u))$ is given by sending $u \preceq w$ to $u_{\text{des}}.$

N. Reading showed that the shard intersection order provides an alternative way of thinking about the noncrossing partition lattice $\text{NCL}(W_c):$ it is the restriction of $\text{Shard}(W)$ to the sortable elements $\text{Sort}(W, c)$.

Theorem 4.49 ([Rea11, Theorem 8.5]). The restriction of $\text{Shard}(W)$ to $\text{Sort}(W, c)$ is isomorphic to $\text{NCL}(W_c).$ The isomorphism is given by sending $w \in \text{Sort}(W, c)$ to the product of its covered reflections in the order they appear in $\text{inv}(c(w)).$

Combining these observations with Definition 3.5—D. Armstrong’s definition of $m$-noncrossing partitions as $m$-multichains in the noncrossing partition lattice—allows us to immediately extract a definition of $m$-sortable elements as $m$-multichains of sortable elements in $\text{Shard}(W)$.

Definition 4.50. Let $\text{Sort}_{\text{shard}}^{(m)}(W, c)$ be the set of $m$-multichains of $c$-sortable elements in $\text{Shard}(W),$

$$\text{Sort}_{\text{shard}}^{(m)}(W, c) := \{ (w_1 \preceq w_2 \preceq \cdots \preceq w_m) : w_i \in \text{Sort}(W, c) \}.$$ 

By construction, the elements of $\text{Sort}_{\text{shard}}^{(m)}(W, c)$ are in bijection with the noncrossing partitions $\text{NC}^{(m)}(W, c)$.

Theorem 4.51. There is an explicit bijection $\text{Sort}_{\text{shard}}^{(m)}(W, c)$ and $\text{NC}^{(m)}(W, c).$

We next relate $\text{Sort}_{\text{shard}}^{(m)}(W, c)$ to $\text{Sort}^{(m)}(W, c)$ using the description of $m$-eralized $c$-sortable elements given in Corollary 4.16.

Theorem 4.52. There is an explicit bijection $\text{Sort}_{\text{shard}}^{(m)}(W, c) \longrightarrow \text{Sort}^{(m)}(W, c).$

Remark 4.53. The most naive guess for such a bijection would be to multiply the individual factors of $\text{Sort}_{\text{shard}}^{(m)}(W, c)$ as an element of $B^+(w)$—this naive guess is wrong. For example, in type $A_3$ with $c = s_1 s_2 s_3,$ this map would send the chain of sortable elements in shard order

$$(s_1 s_2 s_3 s_1 \preceq s_1 s_2 s_3)$$ to the element $s_1 s_2 s_3 | s_1 \cdot s_1 s_2 s_3 \in B^+(A_3).$$

This element is evidently not $c$-sortable.

Proof of Theorem 4.52. Given a multichain $(w_1 \preceq w_2 \preceq \cdots \preceq w_m) \in \text{Sort}_{\text{shard}}^{(m)}(W, c),$ we produce an element in $\text{Sort}^{(m)}(W, c)$ as follows. Recall that Proposition 4.48 gives a bijection between the interval $[c, w]_{\text{Shard}(W)}$ and $\text{Shard}(W_{\text{des}}(w)).$ It follows from the bijection between $c$-sortable elements and noncrossing partitions in [Rea11] that if $w \preceq u$ with $w, u \in \text{Sort}(W, c),$ then $u$ is sent to a $u$-sort element in $\text{Sort}(W_{\text{des}}(w)).$ This process is bijective, since every step is bijective.

We now iterate this procedure to obtain elements $w^{(1)}, w^{(2)}, \ldots, w^{(m)}$ such that $w^{(1)} := w_1 \in \text{Sort}(W, c),$ and for all $1 < i \leq m,$ $w^{(i)} \in \text{Sort}(W_{\text{des}}(w^{(i-1)})), c^{(i)} \in W_{\text{des}}(w^{(i-1)}),$ where $c^{(i)} := c^{(i-1)}|_{W_{\text{des}}(w^{(i-1)})}.$ Then $w^{(1)}, w^{(2)}, \ldots, w^{(m)}$ satisfies the condition in Equation (4) to be a Garside factorization, it satisfies the factorwise conditions of Definition 4.13, and so it is an element of $\text{Sort}^{(m)}(W, c)$ by Corollary 4.16.
The inverse of this bijection is given by explicitly describing the inverse of Proposition 4.48 on c-sortable elements: given \( w \in W_{\text{des}}(w) \), we conjugate it by \( w \) to an element of \( W_{\text{cov},(w)} \). Since c-sortable elements are uniquely defined by their cover reflections, there is now a unique way to complete the inversion set in \( W_{\text{cov},(w)} \) to the inversion set of a c-sortable element in \( W \) by filling in initial segments of dihedral subgroups of \( W \) according to their c-orientation.

\[ \square \]

**Remark 4.54.** In analogy to componentwise absolute order on \( NC_4^{(m)}(W,c) \)—as considered by D. Armstrong in [Arm06]—by Theorems 4.22 and 4.52, componentwise weak order on \( \text{Sort}_{\text{shard}}^{(m)}(W,c) \) defines the \( m \)-eralized c-Cambrian lattice.

**Example 4.55.** For type \( A_3 \) with \( m = 2 \) and \( c = s_1s_2s_3 \), consider the delta sequence

\( (23), (34), (14), \)

which maps under Proposition 3.7 to the chain of noncrossing partitions in absolute order

\( (13)(34) \geq_R (14). \)

By Theorem 4.51, this chain corresponds to the chain of sortable elements in shard order

\( (w_1 \geq_R w_2) = (s_1s_2s_3s_2s_1s_1s_2s_3) \in \text{Sort}_{\text{shard}}^{(m)}(A_3, c). \)

We now compute the bijection of Theorem 4.52.

\[ \text{inv}_R(w_1) = \{(12), (13), (14), (34)\} \supseteq \{(12), (13), (14)\} = \text{inv}_R(w_2). \]

Note that \( w_1 \) has covered reflections \( \{(13), (34)\} \) and associated simple reflections \( \{s_3, s_2\} \), since \( s_3^{w_1} = (34)^{w_1} = (13) \) and \( s_2^{w_1} = (23)^{w_1} = (34) \). The covered reflections generate the nonstandard parabolic subgroup \( W_{\text{cov},(w_1)} \) containing the reflections \( \{(13), (14), (34)\} \). Therefore, \( w_2 \) gets restricted by the first isomorphism of Proposition 4.48 to the element in \( W_{\text{cov},(w_1)} \) with reflection inversion set \( \{(13), (14)\} \). (Note that we can uniquely recover \( \text{inv}_R(w_2) \) by adding in missing initial segments of oriented dihedral subgroups—in this case, the dihedral generated by \( \{s_1, s_2\} \) forces \( (1, 2) \) to be added back to the inversion set).

Using the second isomorphism in Proposition 4.48, we conjugate these inversions by \( w_1^{-1} \) to pass to the corresponding standard parabolic subgroup \( W_{\text{des}}(w_1) \), obtaining the inversion set \( \{(34), (24)\} \). This inversion set corresponds to an element with reduced word \( s_2s_3 \). Therefore, \( (s_1s_2s_3s_2 \geq_R s_1s_2s_3) \) is mapped to the Garside factorization \( s_1s_2s_3s_2 \cdot s_3s_2 \).

On the other hand, we use Theorem 4.34 to compute the skip set of this element to be

\[
\begin{array}{ccc|ccc}
    s_1 & s_2 & s_3 & (23)^{(0)} & s_2 & s_3 & (34)^{(1)} \\hline
    & & & & s_2 & & (14)^{(2)}
\end{array}
\]

recovering our initial delta sequence.

**Theorem 4.56.** Starting at any corner of the square below and traveling around the square via the bijections results in the identity map.

\[ \begin{array}{c|c|c}
    NC_5^{(m)}(W,c) & \text{Theorem 4.34} & \text{Sort}^{(m)}(W,c) \\
    \text{Proposition 3.7} & \text{Theorem 4.52} & \text{Sort}_{\text{shard}}^{(m)}(W,c)
\end{array} \]
Proof. It suffices to check that the composition of the bijections described in Proposition 3.7, Theorem 4.51, and Theorem 4.52 is the inverse of the bijection $C_w$ given in Theorem 4.34.

Choose $(u_1 \geq_R u_2 \geq_R \cdots \geq_R u_m) \in NC(m)(W,c)$, denote the corresponding chain of sortable elements by $(w_1 \geq w_2 \geq \cdots \geq w_m) \in Sort_m(W,c)$, and denote the corresponding factorwise $c$-sortable element by $w = w_1 w_2 u_2^{-1} \cdots u_m^{-1} \in Sort_m(W,c)$. Let $C_w(w) = (\delta_0, \delta_1, \delta_2, \ldots, \delta_m) \in NC_{\delta}(m)(W,c)$. We must show that $(\delta_0, \delta_1, \delta_2, \ldots, \delta_m) = (cu_1^{-1}, u_1 u_2^{-1}, u_2 u_3^{-1}, \cdots, u_m)$. We argue by induction on $m$, the base case following from the $m = 1$ theory.

We note that any 0-forced skips that could be attributed to the $w_1 w_2 \cdots w_m$ piece of $C_w(w)$ are already 0-forced skips from the $w_1$ piece of the product, since the support of $w_2 \cdots w_m$ is contained in the support of $w_1$. By Proposition 4.17, these 0-forced skips are unaffected by the addition of the piece $w_2 \cdots w_m$, and therefore account for the first piece of the delta sequence, $cu_1^{-1}$ by the $m = 1$ bijection between sortable elements and delta sequences.

Since the covering reflections of $w_1$ correspond to a parabolic Coxeter element, $(u_2 \geq_R \cdots \geq_R u_m) \in NC(m)\left(W_{\text{cov}}(w_1), (c|_{w_1}) w_1\right)$ corresponds to $w_2 \cdots w_m \in Sort(m-1)(W_{\text{des}}(w_1), c|_{w_1})$, so that by conjugating by $w_1$ to map $W_{\text{des}}(w_1)$ to $W_{\text{cov}}(w_1)$, we have by induction that the skip set of $w_2 \cdots w_m$ is indeed $(u_1 u_2^{-1}, u_2 u_3^{-1}, \cdots, u_m)$.

Splicing these two pieces together concludes the proof. \hfill \Box

5. m-eralized cluster complexes

In this section, we study an $m$-eralization of the $c$-cluster complex. We begin in Section 5.1 with the definition of $c$-cluster complexes, as introduced by N. Reading in [Rea07b], and its description in terms of subword complexes. We then recall S. Fomin and N. Reading’s $m$-eralization of cluster complexes from [FR05], and generalize their definition to general Coxeter elements in Definitions 5.8 and 5.13. In Section 5.4, we show existence of our generalized cluster complexes by explicitly constructing them as subword complexes. We prove that they are vertex-decomposable and that they are a wedge of spheres. After defining Cambrian rotation and the Cambrian recurrence, in Section 5.6 we give bijections to noncrossing partitions and sortable elements. We conclude by constructing the $m$-eralized Cambrian lattices on our subword complexes.

5.1. Cluster complexes. Let $(W,S)$ be a finite Coxeter system with root system $\Phi$, let $c$ be a Coxeter element, and let $\Phi_{\geq 1} = \Phi^+ \cup -\Delta$ be the set of almost positive roots. Following N. Reading, the $c$-compatibility relations is the unique family of relations $\|_{c}$ on $\Phi_{\geq 1}$ characterized by the following two properties [RS11, Section 5]:

(i) for $\alpha \in \Delta$ and $\beta \in \Phi_{\geq 1}$,
\[-\alpha \|_{c} \beta \iff \beta \in (\Phi(\alpha))_{\geq 1},\]

(ii) for $\beta_1, \beta_2 \in \Phi_{\geq 1}$ and $s$ initial in $c$,
\[\beta_1 \|_{c} \beta_2 \iff \tau_s(\beta_1) \|_{s^{-1} \alpha_s} \tau_s(\beta_2),\]

where $\tau_s$ is defined as
\[\tau_s(\beta) = \begin{cases} 
\beta & \text{if } \beta \in (-\Delta \setminus \alpha_s) \\
\delta' & \text{otherwise}
\end{cases}.
\]

The $c$-cluster complex $Asso(W,c)$ is the simplicial complex given by all collections of pairwise $c$-compatible almost positive roots; in other words, it is the clique complex...
complex of the graph on $\Phi_{\geq -1}$ whose edges are given by $\|_{\alpha}$. A $c$-\textit{cluster} is a facet of the $c$-cluster complex $\text{Asso}(W, c)$—that is, a maximal subset of almost positive roots which are pairwise $c$-compatible.

For a fixed Coxeter group $W$ and any two Coxeter elements $c$ and $c'$, the $c$- and $c'$-cluster complexes are isomorphic [Read07a, Proposition 7.2], [BMR+06, Proposition 4.10]. In crystallographic type, the $c$-cluster complexes are isomorphic to the cluster complex defined in [FZ02].

5.2. Cluster complexes as subword complexes. We may encode clusters using E. Tzanaki’s explicit decreasing factorizations of Coxeter elements [Tza08, Theorem 1.1], see also [IS10]. In our notation, this encoding has the following form.

\textbf{Theorem 5.1} (E. Tzanaki [Tza08], K. Igusa and R. Schiffler [IS10]). The $c$-\textit{cluster complex can be described as the dual subword complex}

\[
\text{Asso}(W, c) \cong \text{Asso}_{\tau}(W, c) \\
:= \text{SUB}_{c}(\text{inv}_{c}(cw_{s}(c)), c^{-1}).
\]

More recently, C. Ceballos, J.-P. Labbé, and C. Stump [CLS14] defined the dual notion to the decreasing $c$-factorizations using subword complexes.

\textbf{Theorem 5.2} ([CLS14, Theorem 2.2]). The $c$-\textit{cluster complex can be described as the subword complex}

\[
\text{Asso}(W, c) \cong \text{Asso}_{\Delta}(W, c) \\
:= \text{SUB}_{c}(cw_{s}(c), w_{0}).
\]

The isomorphism $L_{c}$ between (positions of) letters in $cw_{s}(c)$ and almost positive roots is given by sending the letter $s_{i}$ of $c = s_{1} \cdots s_{i}$ to the simple negative root $-\alpha_{s_{i}}$, and the letter $w_{i}$ of $w_{s}(c)$ to the $i$-th root in $\text{inv}(w_{s}(c))$.

\textbf{Remark 5.3.} By Proposition 2.6, these two theorems are equivalent. It follows from [KM04, Theorem 2.5] that the $c$-cluster complex is vertex-decomposable.

This alternative description of the $c$-cluster complex provides a simple way to prove that all $c$-cluster complexes are isomorphic. We say that a sequence $s_{1}, \ldots, s_{p}$ of simple reflections in a \textit{sequence of initial letters} for a Coxeter element $c$ if $s_{i}$ is initial in $c^{s_{i} \cdots s_{1}}$.

\textbf{Proposition 5.4} ([CLS14, Theorem 2.6]). For any two Coxeter elements $c, c'$, the two complexes $\text{Asso}_{\Delta}(W, c)$ and $\text{Asso}_{\Delta}(W, c')$ are isomorphic.

\textbf{Proof.} It is well-known that for any two Coxeter elements $c, c'$, there is a sequence of initial letters $s_{1}, \ldots, s_{p}$ such that $c' = c^{s_{1} \cdots s_{p}}$. The statement then follows from Proposition 2.14 and Lemma 2.18(1) and (6). \hfill $\square$

5.3. \textit{m-eralized cluster complexes}. In [FR05], S. Fomin and N. Reading gave an $m$-eralization of the definitions in Section 5.1. Let

\[
\Phi_{\geq -1}^{(m)} = \{ \beta^{(k)} : \beta \in \Phi^{+}, 0 \leq k < m \} \cup \{ \alpha^{(m)} : \alpha \in \Delta \}
\]

be the set of \textit{m-colored almost positive roots}. Rather than use the negative simple roots in $\Phi_{\geq -1}$, as we did for $\Phi_{\geq -1}$ in Section 5.1, we instead use the color $m$ to denote this extra copy of the simple roots. We embed $\Phi_{\geq -1}$ in $\Phi_{\geq -1}^{(m)}$ by sending the simple negative roots of $\Phi_{\geq -1}$ to the simple $m$-colored roots and the positive roots of $\Phi_{\geq -1}$ to the positive roots of color $m - 1$.

The \textit{Fomin-Reading map} $\text{HR}^{(m)} : \Phi_{\geq -1}^{(m)} \rightarrow \Phi_{\geq -1}^{(m)}$ is defined by

\[
\text{HR}^{(m)}(\beta^{(k)}) = \begin{cases} 
\beta^{(k+1)} & \text{if } 0 \leq k < m - 1, \\
\text{HR}(\beta)^{(0)} & \text{otherwise}
\end{cases}
\]

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\beta^{(k+1)} & \text{if } 0 \leq k < m - 1, \\
\text{HR}(\beta)^{(0)} & \text{otherwise}
\end{cases}
\]
where

\[
\begin{cases}
  (-\beta)^{(m)} & \text{if } \beta \in -\Delta, \\
  \beta^{(0)} & \text{otherwise,}
\end{cases}
\]

with \(HR(\beta) = \tau_S \tau_R = \prod_{s \in S_L} \tau_s \prod_{t \in S_R} \tau_t\), and \(S = S_L \cup S_R\) is the bipartition of the simple reflections given in Section 2.1.

The \textit{m-eralized compatibility relation} \(\parallel\) is the unique relation \(\parallel\) on \(\Phi_{\geq -1}^{(m)}\) characterized by the following two properties [FR05, Theorem 3.4]:

(i) for \(\alpha \in \Delta\) and \(\beta^{(k)} \in \Phi_{\geq -1}^{(m)}\),

\[
a^{(m)}_c \parallel \beta^{(k)} \iff \beta \in (\Phi_{(s_c)})_{\geq -1}^{(m)},
\]

(ii) for \(\beta^{(k)}, \gamma^{(t)} \in \Phi_{\geq -1}^{(m)}\),

\[
\beta^{(k)} \parallel \gamma^{(t)} \iff HR^{(m)}(\beta^{(k)}) \parallel HR^{(m)}(\gamma^{(t)}).
\]

The \textit{existence} of a binary relation on \(\Phi_{\geq -1}^{(m)}\) satisfying these two properties is not immediate [FR05, Section 8]. The \textit{m-eralized cluster complex} \(\text{Asso}^{(m)}(W)\) is the simplicial complex given by all collections of pairwise compatible \(m\)-colored almost positive roots. A \textit{generalized cluster} is a facet of the cluster complex \(\text{Asso}^{(m)}(W)\).

It is unnecessary to treat bipartite \(c\) as a special case. We present an approach to the Fomin-Reading map that puts all Coxeter elements on equal footing, and which we then use to define \(\text{Asso}(W, c)\) for any Coxeter element \(c\). For \(s \in S\), we define the \textit{m-eralization} of the map \(\tau_s\) from Section 5.1 to be the bijection

\[
\tau^{(m)}_s : \Phi_{\geq -1}^{(m)} \rightarrow \Phi_{\geq -1}^{(m)}
\]

is given by

\[
\tau^{(m)}_s(\beta^{(k)}) = \begin{cases} 
\beta^{(k-1)} & \text{if } \beta = \alpha_s, \\
\gamma^{(k)} & \text{if } \beta \neq \alpha_s, k = m \\
[s(\beta)]^{(k)} & \text{otherwise,}
\end{cases}
\]

where we consider colors modulo \((m + 1)\). It follows from the definition that the order of \(\tau^{(m)}_s\) is given by \(\text{lcm}\{m + 1, 2\}\), so that it is an involution only for \(m = 1\).

The \textit{m-eralized c-compatibility relation} \(\parallel_c\) on \(m\)-colored almost positive roots is the unique family of relations \(\parallel_c\), on \(\Phi_{\geq -1}^{(m)}\) characterized by the following two properties, analogous to the defining properties of the \(c\)-compatibility relation given in Section 5.1:

(i) for \(\alpha \in \Delta\) and \(\beta^{(k)} \in \Phi_{\geq -1}^{(m)}\),

\[
a^{(m)}_c \parallel \beta^{(k)} \iff \beta \in (\Phi_{(s_c)})_{\geq -1}^{(m)},
\]

(ii) for \(\beta^{(k)}, \gamma^{(t)} \in \Phi_{\geq -1}^{(m)}\) and \(s\) initial in \(c\),

\[
\beta^{(k)} \parallel_c \gamma^{(t)} \iff \tau_s^{(m)}(\beta^{(k)}) \parallel s^{-1}cs \tau_s^{(m)}(\gamma^{(t)}).
\]

The existence of an \textit{m-eralized c-compatibility relation} satisfying the above properties is again not immediate. To prove existence, one would need to show that the two properties above uniquely determine the binary relation \(\parallel_c\) and that \(\parallel_c\) is symmetric. We remark that the proofs given in [Rea07a] combined with the ideas in [FR05, Section 6] could probably be adapted to provide a proof of the existence. We prefer the viewpoint in Section 5.4, which gives a construction that directly implies the existence of \(\parallel_c\).
We begin by showing that a binary relation on $\Phi^{(m)}_{\geq 1}$ satisfying these two properties is unique.

**Lemma 5.5.** Let $\alpha$ be a Coxeter element and let $\beta \in \Phi^+$ be a positive root. Then there exists a sequence $s_1, \ldots, s_p$ of initial letters in $c$ such that
\[
\tau_{s_p}^{(m)} \circ \cdots \circ \tau_{s_1}^{(m)}(\beta^{(k)}) = \alpha^{(k)}
\]
for a simple root $\alpha = \alpha_s$, such that $s$ is initial in $c^{e_{\rho_{\alpha}}}$.

**Proof.** Since $\tau_{s}^{(m)}$ coincides with $\tau_s$ on nonsimple roots of a given color, this proof reduces to [Rea07a, Lemma 1.10]. See also [Rea07a, Proof of Lemma 7.1]. □

**Proposition 5.6.** Let $c$ be a Coxeter element and let $\beta \in \Phi^+$ be a positive root. Then there exists a sequence $s_1, \ldots, s_p$ of initial letters in $c$ such that
\[
\tau_{s_p}^{(m)} \circ \cdots \circ \tau_{s_1}^{(m)}(\beta^{(k)}) = \alpha^{(m)}
\]
for a simple root $\alpha = \alpha_s$. In particular, the $m$-eralized $c$-compatibility relation is uniquely determined by its two defining properties.

**Proof.** After moving the root $\beta^{(k)}$ to a simple root $\alpha^{(k)}$ by the sequence provided by Lemma 5.5, we apply $\tau_{s}^{(m)}$ to obtain $\alpha^{(k-1)}$. If $k = 0$, we have sent $\beta^{(k)}$ to $\alpha^{(1)}$, and are finished. Otherwise, we again apply such a sequence provided by the lemma to get an initial simple $(\alpha')^{(k-2)}$. Repeating $k+1$ times therefore yields the desired sequence. □

**Remark 5.7.** We note that, in the proof of Proposition 5.6, we cannot simply apply $\tau_{s_\alpha}$ as many times as needed after using Lemma 5.5 once. The simple reflection $s_\alpha$ becomes final after applying $\tau_{s_\alpha}^{(m)}$, and we therefore cannot apply $\tau_{s_\alpha}^{(m)}$ again immediately after.

Having proven uniqueness of the $m$-eralized $c$-compatibility relation, we postpone the proof of existence to instead define the $m$-eralized $c$-cluster complex. The following definition assumes the existence of $c$, which will be established in Section 5.4.

**Definition 5.8.** Let $(W,S)$ be a finite Coxeter system and let $c$ be a Coxeter element. The $m$-eralized $c$-cluster complex $\text{Asso}^{(m)}(W,c)$ is the flag simplicial complex defined as the clique complex of the graph with
- vertices given by $\Phi^{(m)}_{\geq -1}$; and
- edges given by all pairs $\{\beta^{(k)}, \gamma^{(l)}\}$ such that $\beta^{(k)} \parallel \gamma^{(l)}$.

**Proposition 5.9.** $\text{Asso}^{(1)}(W,c) = \text{Asso}(W,c)$.

**Proof.** This follows from the observation that, after identifying $\Phi^{(1)}_{\geq -1}$ and $\Phi_{\geq -1}$ as above, $\tau_{s}^{(m)}$ for $m = 1$ specializes to the involution $\tau_s$. □

When $c = c_{LR}$ is bipartite, the $m$-eralized $c$-cluster complex recovers the Fomin-Reading $m$-eralaized cluster complex.

**Proposition 5.10.** The Fomin-Reading map $\mathcal{R}^{(m)}$ satisfies
\[
(\mathcal{R}^{(m)})^{-1} = \psi \circ \tau_{s_N}^{(m)} \circ \cdots \circ \tau_{s_1}^{(m)},
\]
where $s_1 \ldots s_N = c_{c_{LR}} \cdots c_{L/R} = w_\alpha$ is the $c_{LR}$-sorting word for $w_\alpha$, and where $\psi : S \to S$ is the involution on $S$-words given by $\psi(s) = s^{w_\alpha}$ for $s \in S$.

**Proof.** It is a straightforward check that this composition satisfies the two cases in the definition of the Fomin-Reading map $\mathcal{R}^{(m)}$. □
Corollary 5.11. Let \( c = c_{L,C} \) be a bipartite Coxeter element. Then
\[
\text{Asso}^{(m)}(W, c) = \text{Asso}^{(m)}(W).
\]

Proof. It follows from [FR05, Lemma 6.2] that the \( m \)-eralized compatibility relation is invariant under the application of \(-w_o, \) i.e.,
\[
\beta^{(k)} \parallel \gamma^{(l)} \Leftrightarrow [-w_o(\beta)]^{(k)} \parallel [-w_o(\gamma)]^{(l)},
\]
and is therefore the \( m \)-eralized compatibility relation induced by \( \tau_{s_1}^{(m)} \cdots \tau_{s_n}^{(m)} \) in the previous proposition. We conclude that \( m \)-eralized \( c \)-compatibility relation \(|c|\) for \( c = c_{L,C} \) coincides with the \( m \)-eralized compatibility relation \(|c|\). \( \square \)

Remark 5.12. As the situation for general \( m \) is similar, we give a more structured viewpoint on the relation given in Proposition 5.10 for \( m = 1 \). The operation \( \tau_s \) corresponds to the rotation operation described in Proposition 2.15 on the subword complex of Theorem 5.2. In this subword complex, the rotation of the initial \( n \) letters induced an automorphism on \( \text{Asso}(W, c) \) as does the inverse rotation of the final \( N = \ell(w_o) \) letters, and these two automorphisms are obtained from each other by twisting with \( w_o \).

The equality of the two operations in Proposition 5.10 should be thought of an \( m \)-eralization of this observation. The automorphism on \( \text{Asso}^{(m)}(W) \) induced by \( \mathcal{H}_{\mathcal{R}}^{(m)} \) is, by Proposition 5.10, given by \( \tau_{s_1}^{(m)} \cdots \tau_{s_n}^{(m)} \), after twisting with \( w_o \).

5.4. \( m \)-eralized cluster complexes as subword complexes. In this section, we define the \( m \)-eralized \( c \)-cluster complex in two different ways without using the \( c \)-compatibility relation. We then use these definitions to conclude the existence of the symmetric \( m \)-eralized \( c \)-compatibility relation \(|c|\).

Define the map \( \text{Lr}^{(m)}_{c} \) between (positions of) letters in \( cw_0^m(c) \) and \( m \)-colored almost positive roots by sending the letter \( s_i \) of the initial \( c = s_1 \cdots s_n \) in \( cw_0^m(c) \) to the \( m \)-colored simple root \( \alpha_i^{(m)} \), and the letter \( w_i \) of \( w_o^m(c) \) to the \( i \)-th colored positive root in the colored inversion sequence \( \text{inv}(w_o^m(c)) \). Note that the definition of \( \text{Lr}^{(m)}_{c} \) \( m \)-eralizes the definition of \( \text{Lr}_{c} \) in Theorem 5.2.

Definition 5.13. The \( m \)-eralized \( c \)-cluster complex has the following three equivalent manifestations.
\[
\text{Asso}^{(m)}_\Delta(W, c) := \text{Sub}_{\mathcal{S}}(cw_0^m(c), w_o^m, mN)
\]
\[
\text{Asso}^{(m)}_{\mathcal{C}}(W, c) := \text{Sub}_{\mathcal{R}}(\text{inv}_{\mathcal{R}}(cw_0^m(c)), c^{-1}),
\]
where \( N := \ell(w_o) \). Lastly, we define \( \text{Asso}^{(m)}_\Delta(W, c) \) to be the image of the faces of \( \text{Asso}^{(m)}_\Delta(W, c) \) under the map \( \text{Lr}^{(m)}_{c} \).

The three definitions of the \( m \)-eralized \( c \)-cluster complex are clearly equivalent. On the other hand, we have already given a definition of \( \text{Asso}^{(m)}(W, c) \) in Definition 5.8, though that definition is yet to be proven to be valid. We thus aim to show that the definition we give here indeed matches the definition given in Definition 5.8.

Remark 5.14. We conjecture that, in the case of this definition, the injection mentioned in Remark 2.13 of the Artin subword complex into corresponding the Coxeter subword complex is actually an isomorphism. That is, we conjecture that \( \text{Asso}^{(m)}_\Delta(W, c) = \text{Sub}_{\mathcal{S}}(cw_0^m(c), w_o^m, mN) \equiv \text{Sub}_{\mathcal{B}}(cw_0^m(c), w_o^m). \)

Example 5.15. Parallel to Figures 5 and 7, Figure 12 shows all 12 facets of \( \text{Asso}^{(2)}(A_2, st) \equiv \text{Asso}^{(2)}_{\mathcal{C}}(A_2, st) \equiv \text{Asso}^{(2)}(A_2, st) \).
| \(\text{Asso}^{(m)}_\Delta(A_2, st)\) | \(\text{Asso}^{(m)}_\Delta(A_2, st)\) | \(\text{Asso}^{(m)}(A_2, st)\) | \(\mathcal{R}(I)\) | \(\text{supp}(I)\) |
|----------------|----------------|----------------|--------|---------|
| \(st,st,st\)  | \(su,su,su\) | \(\alpha^{(2)}, \beta^{(2)}\) | \(\alpha^{(0)}, \beta^{(0)}\) | – |
| \(st,st,st\)  | \(su,su,su\) | \(\gamma^{(1)}, \beta^{(1)}\) | \(\alpha^{(2)}, \beta^{(2)}\) | \(s, t\) |
| \(st,st,st\)  | \(su,su,su\) | \(\beta^{(0)}, \alpha^{(1)}\) | \(\gamma^{(1)}, \alpha^{(2)}\) | \(s, t\) |
| \(st,st,st\)  | \(su,su,su\) | \(\alpha^{(0)}, \gamma^{(0)}\) | \(\beta^{(0)}, \gamma^{(1)}\) | \(s, t\) |
| \(st,st,st\)  | \(su,su,su\) | \(\beta^{(2)}, \alpha^{(0)}\) | \(\gamma^{(0)}, \alpha^{(1)}\) | \(s\) |
| \(st,st,st\)  | \(su,su,su\) | \(\alpha^{(2)}, \beta^{(1)}\) | \(\alpha^{(0)}, \beta^{(2)}\) | \(t\) |
| \(st,st,st\)  | \(su,su,su\) | \(\beta^{(0)}, \alpha^{(1)}\) | \(\beta^{(1)}, \gamma^{(2)}\) | \(s, t\) |
| \(st,st,st\)  | \(su,su,su\) | \(\alpha^{(1)}, \beta^{(1)}\) | \(\alpha^{(1)}, \beta^{(1)}\) | \(s, t\) |
| \(st,st,st\)  | \(su,su,su\) | \(\beta^{(2)}, \alpha^{(1)}\) | \(\alpha^{(0)}, \beta^{(1)}\) | \(t\) |
| \(st,st,st\)  | \(su,su,su\) | \(\alpha^{(0)}, \gamma^{(1)}\) | \(\beta^{(0)}, \gamma^{(2)}\) | \(s, t\) |
| \(st,st,st\)  | \(su,su,su\) | \(\gamma^{(0)}, \beta^{(1)}\) | \(\alpha^{(0)}, \beta^{(1)}\) | \(s\) |
| \(st,st,st\)  | \(su,su,su\) | \(\alpha^{(1)}, \beta^{(2)}\) | \(\alpha^{(1)}, \beta^{(2)}\) | \(s, t\) |

Figure 12. The 12 facets of the \(m\)-eralized \(c\)-cluster complex for \(A_2\) with \(m = 2\) and \(c = st\) in its three instances, together with their root configurations and support. Facets are grouped according to their orbits under Cambrian rotation.

Before showing that the definition of compatibility arising from \(\text{Asso}^{(m)}(W, c)\) satisfies the defining properties of an \(m\)-eralized \(c\)-compatibility relation, we show that Definition 5.13 yields isomorphic complexes for the various Coxeter elements, \(m\)-eralizing the situation in Proposition 5.4. To this end, let \(c, c' = s^{-1}cs\) be Coxeter elements and denote by \(\text{Shift}_s\), the identification between letters in \(cw^\circ_r(c)\) and in \(c'w^\circ_r(c')\), as in Lemma 2.18(6). This defines an isomorphism

\[
\text{Shift}_s : \text{Asso}^{(m)}_\Delta(W, c) \cong \text{Asso}^{(m)}_\Delta(W, s^{-1}cs).
\]

Example 5.16. Parallel to Examples 4.7 and 3.14, we consider \(\text{Asso}^{(2)}_\Delta(A_2, st)\). One orbit of the shift operation is given by

\[
\begin{align*}
\text{ststst} & \xrightarrow{\text{Shift}_s} \text{tststs} \xrightarrow{\text{Shift}_s} \text{tststst} \\
\text{tststs} & \xrightarrow{\text{Shift}_s} \text{tststst} \xrightarrow{\text{Shift}_s} \text{tststst} \\
\text{tststst} & \xrightarrow{\text{Shift}_s} \text{tststst} \xrightarrow{\text{Shift}_s} \text{tststst} \\
\end{align*}
\]

Proposition 5.17. For any two Coxeter elements \(c, c'\), the two \(m\)-eralized cluster complexes \(\text{Asso}^{(m)}_\Delta(W, c)\) and \(\text{Asso}^{(m)}_\Delta(W, c')\) are isomorphic.

Proof. Let \(s_1, \ldots, s_p\) be a sequence of initial letters of \(c\) such that \(c' = c^{s_1 \cdots s_1}\). Then,

\[
\text{Shift}_{s_p} \circ \cdots \circ \text{Shift}_{s_1} : \text{Asso}^{(m)}_\Delta(W, c) \cong \text{Asso}^{(m)}_\Delta(W, c')
\]

We next show that the \(m\)-eralized \(c\)-cluster complex is flag.

Proposition 5.18. The \(m\)-eralized \(c\)-cluster complex in Definition 5.13 is the clique complex of its edges.

Proof. We show that all of the minimal nonfaces of \(\text{Asso}^{(m)}_\Delta(W, c)\) have cardinality 2. An ordered nonface \(\{i_1, \ldots, i_k\}\) of \(\text{Asso}^{(m)}_\Delta(W, c)\) has \(r_k \cdots r_1 \not\subset \mathcal{R} c\), where \(r_i\) is the \(i\)th reflection in \(\text{inv}_\mathcal{R}(cw^\circ_r(c))\). It follows from [ABMW06, Lemma 2.1(iv)] and the lattice property of \(\mathcal{N}(W, c)\) that for a reduced \(\mathcal{R}\)-word \(r_1 \cdots r_k\),

\[
r_k \cdots r_1 \not\subset \mathcal{R} c \iff r_i r_j \not\subset \mathcal{R} c \text{ for all } i > j.
\]
Thus, there are indices $k \geq a > b \geq 1$ such that $r_ar_b \not\in R c$, implying that $\{i_a, i_b\}$ is also a nonface of $\text{Asso}_\Delta^{(m)}(W, c)$.  \hfill $\square$

**Remark 5.19.** This statement was inadvertently omitted in [CLS14], where it was only proven that $\text{Asso}_\Delta^{(1)}(W, c)$ is a subcomplex of $\text{Asso}_\Delta^{(1)}(W, c)$. It was implicit in [CLS14] that equality follows, since $\text{Asso}_\Delta^{(1)}(W, c)$ was previously proven to be a simplicial sphere of the same dimension as $\text{Asso}_\Delta^{(1)}(W, c)$. The proof given here implies all known geometric properties of $\text{Asso}_\Delta^{(m)}(W, c)$ from combinatorial properties of $\text{Asso}_\Delta^{(m)}(W, c)$ and $\text{Asso}_\Delta^{(m)}(W, c)$, avoiding the use of any previous results on cluster complexes along the way.

We now show that the construction of $\text{Asso}^{(m)}(W, c)$ given in Definition 5.13 implies the existence of $\parallel_c$, completing the proof that Definition 5.8 is valid.

**Theorem 5.20.** The $m$-eralized $c$-compatibility relation $\parallel_c$ is given by $\beta(k) \parallel_c \gamma(\ell)$ if and only if $\{\beta(k), \gamma(\ell)\}$ is a face of $\text{Asso}_\Delta^{(m)}(W, c)$.

For clarity, we denote by $\parallel'_c$ the relation given by $\beta(k) \parallel'_c \gamma(\ell)$ if and only if $\{\beta(k), \gamma(\ell)\}$ is a face of $\text{Asso}_\Delta^{(m)}(W, c)$ as defined in Definition 5.13. The remainder of this section is devoted to the proof that $\parallel'_c$ is indeed the unique $m$-eralized $c$-compatibility relation $\parallel_c$.

The proof of Theorem 5.20 is analogous to [CLS14, Proof of Theorem 2.2]. We begin by relating the $m$-eralized $c$-compatibility relation with the $m$-eralized $s^{-1}cs$-compatibility relation for $s$ initial in $c$, $m$-eralizing [CLS14, Theorem 5.7].

**Proposition 5.21.** Let $\beta(k), \gamma(\ell) \in \Phi_\Delta^{(m)}$ and let $s$ initial be in $c$. Then

$$\beta(k) \parallel'_c \gamma(\ell) \Leftrightarrow r_s^{(m)}(\beta(k)) \parallel_{s^{-1}cs} r_s^{(m)}(\gamma(\ell)).$$

**Proof.** Let $c' = s^{-1}cs$ and consider the isomorphism

$$\text{Shift}_s : \text{Asso}_\Delta^{(m)}(W, c) \rightarrowtail \text{Asso}_\Delta^{(m)}(W, c').$$

It can be checked directly on the two colored inversion sequences $\text{inv}(w^m_s(c))$ and $\text{inv}(w^m_s(c'))$ that the shift operation $\text{Shift}_s$, sending letters in $w^m_s(c)$ to letters in $c'w^m_s(c')$, is mapped by $\text{Lr}_c^{(m)}$ to the operation $r_s^{(m)}$ on $m$-colored almost positive roots. The statement follows. \hfill $\square$

We now explain how to descend into a parabolic subgroup, $m$-eralizing [CLS14, Theorem 5.1].

**Proposition 5.22.** Let $a \in \Delta$ and $\beta(k) \in \Phi_\Delta^{(m)}$. Then

$$a^{(m)} \parallel'_c \beta(k) \Leftrightarrow \beta \in (\Phi_{(s)}^{(m)})_{\Delta_{\geq -1}}.$$
The three embeddings $\text{inv}_R(c_i w^n(c)) \leftrightarrow \text{inv}_R(c w^n(c))$ for $1 \leq i \leq 3$ are then given by

\[
\begin{align*}
\text{inv}_R(c w^n(c)) & = (12)(13)(14)(23)(24)(12)(34)(13)(14) \\
\text{inv}_R(c_1 w^n(c_1)) & \rightarrow (13)(14) \\
\text{inv}_R(c_2 w^n(c_2)) & \rightarrow (12)(14)(23) \\
\text{inv}_R(c_3 w^n(c_3)) & \rightarrow (12)(13)(23)(24)(34)
\end{align*}
\]

Given this embedding, we can now deduce the following lemma.

Lemma 5.24. Let $c = s_1 \cdots s_n$ be a Coxeter element and let $\hat{c} = s_1 \cdots s_k \cdots s_{n}$ be obtained from $c$ by removing the letter $s_k$. Let $Q$ be an initial segment of $w^n(c)$ and let $\hat{Q}$ be the corresponding initial segment of $w^n(\hat{c})$ given by the above embedding $\text{inv}_R(\hat{c} w^n(\hat{c})) \leftrightarrow \text{inv}_R(c w^n(c))$. Then

$$\text{SUB}_R(\text{inv}_R(\hat{c}Q), \hat{c}^{-1}) \cong \text{SUB}_R(\text{inv}_R(cQ), \hat{c}^{-1}).$$

When $Q = w^n(c)$, we have $\hat{Q} = w^n(\hat{c})$, and

$$\text{SUB}_R(\text{inv}_R(\hat{c} w^n(\hat{c})), \hat{c}^{-1}) \cong \text{SUB}_R(\text{inv}_R(c w^n(c)), \hat{c}^{-1}).$$

Proof. The above embedding directly extends to an embedding of dual subword complexes $\text{SUB}_R(\text{inv}_R(\hat{c}Q), \hat{c}^{-1})$ into $\text{SUB}_R(\text{inv}_R(cQ), \hat{c}^{-1})$. This embedding is an isomorphism since a reduced $R$-word for $\hat{c}^{-1}$ must necessarily be contained in $W_{(s_k)}$ (see, for example, [BDSW14, Theorem 1.3]).

As we have seen in Example 2.16, we cannot use the usual theory of subword complexes to conclude that the generalized subword complex of Definition 5.13 is vertex-decomposable. Nevertheless, the $m$-eralized cluster complex $\text{Asso}^{(m)}(W)$ is vertex-decomposable, as shown by C. Athanasiadis and E. Tzanaki in [AT08]. Using the previous lemma, we give a short proof in Proposition 5.25. We emphasize that we will later need vertex-decomposability to prove Theorem 5.20, and we therefore may not simply deduce this property from previous work.

Proposition 5.25. Let $c = s_1 \cdots s_n$, let $Q = s_1 \cdots s_p$ be an initial segment of the word $w^n(c)$, and let $w = s_1 \cdots s_p \in W$ be the corresponding element in $W$. Then $\text{SUB}_R(cQ, w, p)$ is vertex-decomposable. In particular, $\text{Asso}^{(m)}(W, c)$ is vertex-decomposable.

Proof. We show that both the link and the deletion of the first vertex are vertex-decomposable subword complexes by simultaneous induction on the rank of $W$ and on the length of $Q$.

The facets of the link of the first vertex in $\text{SUB}_R(cQ, w, p)$ are given by

$$\{ I \setminus 1 : I \text{ a facet of } \text{SUB}_R(cQ, w, p) \text{ with } 1 \in I \}.$$  

When $\hat{c} = s_2 \cdots s_n$, 

$$\text{SUB}_R(\hat{c}Q, \hat{c}^{-1}) \cong \text{SUB}_R(\hat{Q}, \hat{c}^{-1})$$

for $\hat{Q}$ as given in Lemma 5.24. This complex is vertex-decomposable by induction on the rank of $W$.

We now consider the deletion. Let $I$ be a facet of $\text{SUB}_R(cQ, w, p)$ with $1 \in I$, so that $I \setminus 1$ is a face of the deletion of this letter. To set up an inductive argument on the length of $Q$, we show that there is a facet $J$ of $\text{SUB}_R(cQ, w, p)$ with $J \setminus j = I \setminus 1$ and $j > 1$. Letting $c' = s_j^{-1} s_1$, the facet $J$ is naturally a facet of $\text{SUB}_R(c' s_2 \cdots s_n, w, p)$. We first extend $Q$ to $w^n(c)$ and observe that every facet of $\text{SUB}_R(cQ, w, p)$ is also a facet of $\text{Asso}^{(m)}(W, c)$. Treating $I$ as a facet of $\text{Asso}^{(m)}(W, c)$, we may then flip $1 \in I$ to the position $j$ for which $r_j(j) = \alpha_j^{(0)}$. 

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After performing this flip, if there is a position \( k \in I \) with \( j < k \), then this flip did not cause the facet to leave the initial segment \( cQ \). Otherwise, we claim that the facet \( I \) was equal to the first \( n \) letters of \( cQ \), and that the position \( j \) was the position of the first letter of \( Q \). By assumption, no position of a letter to the right of \( j \) is inside \( I \). Since the product of the root configuration corresponding to the letters of \( I \) is \( c^{-1} \), we can use Lemma 2.18(5) to read the letters of \( w_n^c(c) \) from right to left until we pass \( j \). From the form of \( w_n^c(c) \), it is clear that the right-most possible location of the colored positive root \( \alpha_{s_1}^{(0)} \) is the left-most position of \( Q \). Therefore, \( I \) is indeed the initial copy of \( c \). We can now replace \( c \) by \( s_1^{-1}cs_1 \), and deduce the statement by induction on the length of \( Q \).

Finally, we note that the two base cases of \( W \) of rank 1 and the empty word \( Q \) are trivially vertex-decomposable.

**Corollary 5.26.** The lexicographic order of the facets of \( \text{Asso}^{(m)}(W,c) \) is a shelling order.

**Proof.** The order on the vertices in the vertex-decomposition in Proposition 5.25 is the usual order. The lexicographic order on \( \text{Asso}^{(m)}(W,c) \) is therefore a shelling order. □

We now deduce the following theorem from Corollary 5.26. It was first proven by S. Fomin and N. Reading in [FR05, Proposition 11.1].

**Theorem 5.27.** The \( m \)-eralized \( c \)-cluster complex \( \text{Asso}^{(m)}(W,c) \) has the homotopy type of a wedge of \( \text{Cat}^{(m-1)}(W) \) spheres of dimension \( n-1 \).

**Proof.** Because the complex is vertex-decomposable, it clearly has the homotopy type of a wedge of \( (n-1) \)-dimensional spheres. We count the number of such spheres using a technique of A. Björner and M. Wachs in [BW96] (see also [Wac07, Theorem 3.1.3]). The number of spheres is given by the number of **homology facets** of \( \text{Asso}^{(m)}(W,c) \)—that is, those facets whose entire boundary is contained in the union of the earlier facets—in the lexicographic shelling order of Corollary 5.26. Lemma 2.18(4) implies that the word \( cw_n^c(c) \) starts, up to commutations, with an initial copy of \( w_\circ(c) \). We will conclude the theorem by showing that the homology facets of \( \text{Asso}^{(m)}(W,c) \) are exactly those facets that do not contain a position corresponding to one of the letters in this initial copy of \( w_\circ(c) \).

We first consider the case when \( I \) is a facet of \( \text{Asso}^{(m)}(W,c) \) that does not contain a position corresponding to a letter of the initial \( w_\circ(c) \). Then, by definition, \( k > 0 \) for \( \beta^{(k)} \in R(I) \). Since the root vectors in this initial copy of \( w_\circ(c) \) are the positive zero-colored roots, we can use Lemma 2.9 to flip any position \( i \in I \) into a letter in this initial copy of \( w_\circ(c) \) to obtain another facet \( J \) with \( I \setminus i = J \setminus j \) for some position \( j \). Since \( j < i \) by construction, the obtained facet \( J \) is lexicographically smaller and contains the boundary face \( I \setminus i \) of \( I \). By Corollary 5.26, the facet \( I \) is therefore a homology facet.

Suppose now that \( I \) is a facet containing a position \( i \) of a letter in this initial \( w_\circ(c) \). The root vector \( \beta^{(k)} = r_I(i) \) is now colored by \( k = 0 \), and the position \( i \) in \( I \) can not be flipped to the left. The boundary face \( I \setminus i \) is therefore not contained in any lexicographically smaller facet, and \( I \) is thus not a homology facet.

The subcomplex of \( \text{Asso}^{(m)}(W,c) \) not containing positions corresponding to a letter in the initial copy of \( cw_n^c(c) \) is equal to the cluster complex \( \text{Asso}^{(m-1)}(W,c) \). By Proposition 5.17 and Corollary 5.11, \( \text{Asso}^{(m-1)}(W,c) \) is isomorphic to the \( m \)-eralized cluster complex \( \text{Asso}^{(m-1)}(W) \) of S. Fomin and N. Reading. Since the latter is known to be counted by \( \text{Cat}^{(m-1)}(W) \), we conclude the theorem. □
We now prove Proposition 5.22.

**Proof of Proposition 5.22.** Using the root configuration as defined in Equation (2) together with the fact that $\text{Ass}^{(m)}(W,c)$ is vertex-decomposable—and therefore shellable—we obtain that

$$a^{(m)} |_{c} \beta^{(k)} \Rightarrow \beta \in (\Phi_{(s_{c})})^{(m)}_{\geq 1},$$

using the same proof as [CLS14, Proposition 5.3]. We require shellability to ensure that all facets of $\text{Ass}^{(m)}(W,c)$ containing the first letter are connected by flips not using this first letter.

The other direction

$$\beta \in (\Phi_{(s_{c})})^{(m)}_{\geq 1} \Rightarrow a^{(m)} |_{c} \beta^{(k)}$$
can then be deduced from Lemma 5.24 analogously to the proof of [CLS14, Proposition 5.6]. □

**Proof of Theorem 5.20.** Proposition 5.6 showed that the two defining properties uniquely determine $|_{c}$. Propositions 5.21 and 5.22 show that $|_{c}$ satisfies the defining properties of an $m$-eralized c-compatibility relation, so that $|_{c}$ and $|_{c}$ coincide. □

We conclude this section by defining the support of a facet of the $m$-eralized c-cluster complex.

**Definition 5.28.** The support of a facet $I$ of $\text{Ass}^{(m)}_{\Delta}(W,c)$ is defined to be the set of simple reflections corresponding to the positions in the initial $c$ of the word $cw^{(m)}_{c}(c)$ that are not contained in $F$. Analogously, the support of a facet $I$ of $\text{Ass}^{(m)}(W,c)$ is defined as

$$\text{supp}(I) = S \setminus \{\alpha : a^{(m)} \in I\}.$$

Figure 12 shows the support of the 12 facets of $\text{Ass}^{(m)}(A_{2}, st)$.

5.5. $m$-eralized Cambrian rotation and recurrence.

**Definition 5.29.** Let $s_{1}s_{2}\cdots s_{n}$ be a reduced $S$-word for the Coxeter element $c$. The $m$-eralized c-Cambrian rotation Camb$_{c} : \text{Ass}^{(m)}(W,c) \rightarrow \text{Ass}^{(m)}(W,c)$ is given by

$$\text{Camb}_{c} := \text{Shift}_{s_{n}} \circ \cdots \circ \text{Shift}_{s_{1}}.$$

**Proposition 5.30.** The order of the automorphism

$$\text{Camb}_{c} : \text{Ass}^{(m)}(W,c) \rightarrow \text{Ass}^{(m)}(W,c)$$
is

$$\text{ord}(\text{Camb}_{c}) = \begin{cases} mh + 2 & \text{if } \psi \equiv \mathbb{1} \text{ and } m \text{ odd} \\ (mh + 2)/2 & \text{otherwise} \end{cases}.$$

**Proof.** The statement follows from the facts that $w_{c}$ is an involution on $S$, $N = nh/2$ and the length of the word $cw^{(m)}_{c}(c)$ is $n + mN$. □

We modify the shift operator $\text{Shift}_{s}$ to produce the $m$-eralized c-Cambrian recurrence on facets of $\text{Ass}^{(m)}_{\Delta}(W,c)$, depending on if the initial $s$ in the word $cw^{(m)}_{c}(c)$ is in a given facet $F$.

**Proposition 5.31.** Let $s$ be initial in a Coxeter element $c$. Then

$$I \text{ a facet of } \text{Ass}^{(m)}_{\Delta}(W,c) \Leftrightarrow \begin{cases} I \setminus 1 \text{ a facet of } \text{Ass}^{(m)}_{\Delta}(W,s^{-1}c) \quad & \text{if } 1 \in I; \\ \text{Shift}_{s}(I) \text{ a facet of } \text{Ass}^{(m)}_{\Delta}(W,s^{-1}cs) \quad & \text{otherwise.} \end{cases}$$

**Proof.** This follows from Proposition 5.17 and Lemma 5.24. □
Example 5.32. Parallel to Examples 4.10 and 3.18, we consider the facet $sts \in Asso_\Delta^{(2)}(A_2, st)$ Then the $m$-eralized $c$-Cambrian recurrence is computed as
\[
sts \mapsto \Delta(A_2, st) \mapsto \Delta(A_2, st) \mapsto \Delta(A_2, st) \mapsto \Delta(A_2, st) \mapsto \Delta(A_2, st) \mapsto \Delta(A_2, st) \mapsto \Delta(A_2, st) \mapsto \Delta(A_2, st) \mapsto \Delta(A_2, st).
\]

5.6. $m$-eralized cluster complexes, noncrossing partitions, sortable elements. The connection between $Asso_\Delta^{(m)}(W, c)$ and $NC_\Delta^{(m)}(W, c)$ is based on the $m$-eralization of [PS11, Proposition 6.20].

Theorem 5.33. Mapping the root configuration of a facet of an $m$-eralized $c$-cluster complex to the corresponding reflections in $\text{inv}_R(c^h_{\frac{m+1}{2}})$ is a bijection
\[
Asso_\Delta^{(m)}(W, c) \rightarrow NC_\Delta^{(m)}(W, c)
\]
that respects Cambrian rotation, the Cambrian recurrence, and support.

In order to prove this proposition, we establish the Cambrian recurrence for the root configuration.

Proposition 5.34. Let $s$ be initial in $c$ and let $I$ be a facet of $Asso_\Delta^{(m)}(W, c)$. The root configuration satisfies
\[
R(I) = \begin{cases} 
\{a_{x(0)}^{(1)}\} \cup R(I \setminus 1) & \text{if } 1 \in I \\
R(\text{Shift}_s(I)) & \text{if } 1 \notin I
\end{cases}
\]
where $I \setminus 1$ is considered as a facet of $Asso_\Delta^{(m)}(W_s, s^{-1}c)$, and $\text{Shift}_s(I)$ as a facet of $Asso^{(m)}(W, s^{-1}cs)$.

Proof. Both cases are clear from the definition of the root configuration and the previously-established properties of the embedding given before Lemma 5.24. □

Proof of Theorem 5.33. This follows from the inductive description in Proposition 3.16 and the corresponding description of the root configuration in Proposition 5.34. □

We record the following corollary of Proposition 5.34, generalizing the last part of [PS11, Proposition 6.18].

Corollary 5.35. The colors of the colored positive roots in the root configuration $R(I)$ are weakly increasing, up to commutations of consecutive colored roots for which the corresponding reflections commute.

Proof. This follows from Proposition 5.34 and the observation that applying $s$ to $R(\text{Shift}_s(I))$ does not decrease the color of those colored positive roots. □

Similarly, we obtain a bijection between the $m$-eralized $c$-cluster complex and $m$-eralized $c$-sortable elements, by identifying the root configuration and the skip set.

Corollary 5.36. Identifying the root configuration of a facet of $Asso_\Delta^{(m)}(W, c)$ with the skip set of an element of $\text{Sort}^{(m)}(W, c)$ yields a bijection
\[
Asso_\Delta^{(m)}(W, c) \rightarrow \text{Sort}^{(m)}(W, c)
\]
that respects Cambrian rotation, the Cambrian recurrence, and support.

Proof. This is a direct consequence of Propositions 4.32 and 5.34. □
Generalizing [Rea07a, Theorem 8.1], we conclude this section with a direct bijection between \( \text{Sort}^{(m)}(W, c) \) and \( \text{Asso}^{(m)}(W, c) \).

For \( w \in \text{Sort}^{(m)}(W, c) \), consider the \( c \)-sorting \( w(c) = s_1 \cdots s_p \). For \( s \in \supp(w) \), let \( \beta_s^{(m)} \) be the colored positive root \( s_1 \cdots s_{k-1}(\alpha_s^{(0)}) \), where \( s_k \) is the last occurrence of the letter \( s \) in \( w(c) \).

**Theorem 5.37.** The composition

\[
\text{Sort}^{(m)}(W, c) \xrightarrow{w(c)} \text{Asso}^{(m)}(W, c) \xrightarrow{c} \text{Asso}^{(m)}(W, c)
\]

given by sending \( w \in \text{Sort}^{(m)}(W, c) \) to the set

\[
C^c_s(w) := \{ \beta_s^{(m)} : s \in \supp(w) \} \cup \{ \alpha_s^{(m)} : s \not\in \supp(w) \}
\]

of \( m \)-colored positive roots is a bijection that respects Cambrian rotation, the Cambrian recurrence, and support.

We prove this theorem by showing that both satisfy the Cambrian recurrence. The notational similarity to the skip set is chosen to emphasize its very similar recurrence. For simplicity, denote by \( \nu_s^{(m)} \) the inverse of the map \( \tau_s^{(m)} \) on \( m \)-colored almost positive roots.

**Proposition 5.38.** Let \( s \) be initial in \( c \) and let \( w \in \text{Sort}^{(m)}(W, c) \). Then

\[
C^c_s(w) = \begin{cases} 
C^c_{s-1,c}(w) \cup \{ \alpha_s^{(m)} \} & \text{if } s \in \text{asc}_L(w) \\
\nu_s^{(m)}C^c_{s-1,c}(s^{-1}w) & \text{if } s \in \text{des}_L(w).
\end{cases}
\]

**Proof.** The first case \( s \in \text{asc}_L(w) \) is clear, since we have \( s \not\in \supp(w) \) in this case. For the second case \( s \in \text{des}_L(w) \), observe that \( \nu_s^{(m)} \) acts as the simple reflection \( s \) on \( \{ \beta_s^{(m)} : s \in \supp(w) \} \), while fixing \( \{ \alpha_s^{(m)} : s \not\in \supp(w) \} \). This is exactly the modification to \( C^c_s \) when going from \( s^{-1}w \) to \( w \) in the second case. \( \square \)

**Proof of Theorem 5.37.** We show that clusters also satisfy the recurrence given in Proposition 5.38. For a facet \( I \) of \( \text{Asso}^{(m)}(W, c) \), recall the definition of \( \text{Lr}^{(m)}(I) \) from the beginning of Section 5.4. Let \( s \) be initial in \( c \) and let \( I \) be a facet of \( \text{Asso}^{(m)}(W, c) \). From Proposition 5.31, we see that

\[
\text{Lr}^{(m)}(I) = \begin{cases} 
\text{Lr}^{(m)}_s(I \setminus 1) \cup \{ \alpha_s^{(m)} \} & \text{if } 1 \in I \\
\nu_s^{(m)}(\text{Lr}^{(m)}_{s^{-1}c}(\text{Shift}^c_s(I))) & \text{if } 1 \not\in I,
\end{cases}
\]

where \( I \setminus 1 \) is considered as a facet of \( \text{Asso}^{(m)}(W_{s^{-1}c}) \), and \( \text{Shift}^c_s(I) \) as a facet of \( \text{Asso}^{(m)}(W_{s^{-1}c}) \). Note as in the proof of Proposition 5.38 that \( \nu_s^{(m)} \) acts as the simple reflection \( s \) on \( \text{Lr}^{(m)}_s(I) \cap \{ \beta^{(k)} : \beta \in \Phi^+, 0 \leq k < m \} \) while fixing \( \text{Lr}^{(m)}_c(I) \cap \{ \alpha^{(m)} : \alpha \in \Delta_s \} \). \( \square \)

### 5.7. \( m \)-eralized Cambrian Lattices

In this section, we give a description of the \( m \)-eralized \( c \)-Cambrian poset using \( \text{Asso}(W, c) \).

**Definition 5.39.** The **\( m \)-eralized \( c \)-Cambrian poset** \( \text{Camb}^{(m)}_{\text{Asso}}(W, c) \) is the increasing flip poset of \( \text{Asso}^{(m)}(W, c) \).

**Example 5.40.** Figure 13 shows \( \text{Camb}^{(2)}_{\text{Asso}}(A_2, st) \).
**Figure 13.** The flip poset $\text{Camb}^{(2)}_{\text{Asso}}(A_2, st)$ on the 12 facets of $\text{Asso}^{(2)}(A_2, st)$. The positive roots labelling the edges correspond to the direction of the flips, in the two colors.

**Proposition 5.41.** The $h$-polynomial $h(q) = h_0 + h_1 q + \cdots + h_d q^d$ of the $m$-eralized $c$-cluster complex $\text{Asso}^{(m)}(W, c)$ given by the in-degree or by the out-degree generating function of $\text{Camb}^{(m)}_{\text{Asso}}(W, c)$. This is to say that

$$h_i = \{ F \in \text{Camb}^{(m)}_{\text{Asso}}(W, c) : F \text{ has exactly } i \text{ incoming edges} \}$$

$$= \{ F \in \text{Camb}^{(m)}_{\text{Asso}}(W, c) : F \text{ has exactly } i \text{ outgoing edges} \}$$

**Proof.** Since the lexicographic order is a linear extension of $\text{Camb}^{(m)}_{\text{Asso}}(W, c)$, the statement follows from Corollary 5.26 together with the fact that the reverse of a shelling order is again a shelling order. □

**Theorem 5.42.** The bijection $\text{Asso}^{(m)}(W, c) \rightarrow \text{NC}^{(m)}_{\Delta}(W, c)$ in Corollary 5.36 is a poset isomorphism $\text{Camb}^{(m)}_{\text{Asso}}(W, c) \cong \text{Camb}^{(m)}_{\text{NC}}(W, c)$.

**Proof.** We show that flips in $\text{Asso}^{(m)}(W, c)$ are sent to flips in $\text{NC}^{(m)}_{\Delta}(W, c)$ under the bijection of Corollary 5.36. By construction (see the discussion after Definition 5.13 for $\text{Asso}^{(m)}(W, c)$ and Proposition 3.24 for $\text{NC}^{(m)}_{\Delta}(W, c)$), flips in $\text{Asso}^{(m)}(W, c)$ and $\text{NC}^{(m)}_{\Delta}(W, c)$ not involving the initial letter $s$ are preserved under Cambrian rotation. By symmetry and by applying $\text{Shift}$ a suitable number of times, we may therefore assume that the given flip is positive and uses the initial letter $s^{(0)}$ in the search word for $\text{Asso}^{(m)}(W, c)$, where $s$ is initial in $c$.

We now choose to work in $\text{Asso}^{(m)}_c(W, c)$, whose search word $P := p_1 \cdots p_{mN+n}$ is related to the search word $Q := q_1 \cdots q_{mN+n}$ of $\text{Asso}^{(m)}_{\Delta}(W, c)$ by $p_i = q_i^1 q_i^2 \cdots q_i^{i-1}$. Since the bijection of Corollary 5.36 sends a facet $x_{\Delta}$ of $\text{Asso}^{(m)}_{\Delta}(W, c)$ to a facet $x_{\text{NC}}$ of $\text{NC}^{(m)}_{\Delta}(W, c)$ by taking the root configuration of $x_{\Delta}$, if we write $x_{\Delta} = \{q_1, \ldots, q_n\}$ with $i_1 < i_2 < \cdots < i_n$ in $\text{Asso}^{(m)}_{\Delta}(W, c)$ as $x_c = \{p_1, \ldots, p_n\}$, then
the bijection is given by \( x_{\mathcal{C}} \mapsto x_{NC} \) using
\[
(8) \quad p_{ij} \mapsto r_j := p_{ij}^{p_{ij} \cdots p_{ij}^{-1}},
\]
where colors are assigned as in Corollary 5.36.

Since two facets of \( \text{Asso}_{\mathcal{C}}^{(m)}(W, c) \) are connected by a flip if and only if they differ in a single position, and since the letters of a facet of \( \text{Asso}_{\mathcal{C}}^{(m)}(W, c) \) multiply to \( c^{-1} \), we can explicitly describe what increasing flips involving the initial \( s^{(0)} \) must look like.

There are two types of increasing flips that could involve \( s^{(0)} \). The first is given by pushing \( s^{(0)} \) to its right past the letters \( p_2, p_3, \ldots \) (where we have sorted so that elements of the same color appear consecutively and in \( c \)-root order), conjugating by those letters that it passes and increasing the color by one when \( p_1 \cdots p_d(a_s) < 0 \), until its first legal position, should one exist. The second type of flip occurs if \( s^{(0)} \) commutes with all other letters of its color. In this case, we may simply increase its color by one.

With this description of the flips, we see that a flip in \( \text{Asso}_{\mathcal{C}}^{(m)}(W, c) \) involving \( s^{(0)} \) may be written
\[
\text{Flip}_s(x_{\mathcal{C}}) = \text{Flip}_s \left( \left\{ s^{(0)}, p_{i_2}, \ldots, p_{i_n} \right\} \right) = \left\{ p_{i_2}, \ldots, p_{i_j}, (s^{p_{i_2} \cdots p_{i_j}})^{(k)}, \ldots, p_{i_n} \right\},
\]
where the color \( k \) is either 0 or 1—depending on which of the two types of increasing flips it is involved in—is sent by Equation (8) to the pair of facets of \( \text{NC}_{\Delta}^{(m)}(W, c) \)
\[
\left\{ s^{(0)}, r_2, \ldots, r_n \right\} \text{ and } \left\{ r_2^s, \ldots, r_j^s, s^{(k)}, \ldots, r_n \right\}.
\]
Since these are both legal facets of \( \text{NC}_{\Delta}^{(m)}(W, c) \) by Corollary 5.36, we conclude from Section 3.4 that they are related by a flip in \( \text{NC}_{\Delta}^{(m)}(W, c) \). The converse follows identically. \( \square \)

Corollary 5.43. The \( h \)-polynomial \( h_t(q) = h_0 + h_1q + \cdots + h_dq^d \) of \( \text{Asso}^{(m)}(W, c) \) is given by \( \sum q^{\pi(w)} \) where the sum ranges over all \( (w_1, \ldots, w_m) \in \text{NC}^{(m)}(W, c) \) and \( w = w_m \), or, equivalently, over all \( (\delta_0, \ldots, \delta_m) \in \text{NC}_{\delta}^{(m)}(W, c) \) and \( w = \delta_1 \cdots \delta_m \).

Proof. This follows from Proposition 5.41, from Theorem 5.42, and from Proposition 3.23(ii). \( \square \)

Remark 5.44. Figure 8 and Remark 4.39 show that our \( m \)-eralized \( (12 \cdots n) \)-Cambrian lattice in type \( A_{n-1} \) is not isomorphic to F. Bergeron’s \( m \)-Tamari lattice for \( m \geq 2 \). For example, we proved in Proposition 3.23(ii) that the poset \( \text{Camb}^{(m)}_{\text{Asso}}(A_{n-1}, (12 \cdots n)) \) is self-dual, while the \( m \)-Tamari lattice is not, when \( m \geq 2 \). Proposition 5.41 and Corollary 5.43 nevertheless yield
\[
\sum_{I} q^{\# \text{incoming edges of } I} = \sum_{D} q^{\# \text{incoming edges of } D},
\]
where the first sum ranges over all elements in \( \text{Camb}^{(m)}_{\text{Asso}}(A_{n-1}, (12 \cdots n)) \), and the second sum ranges over all elements in the \( m \)-Tamari lattice. This can be deduced as follows. The out-degree of an \( m \)-Dyck path in \( m \)-Tamari lattice is given by its number of valleys. The generating function of all \( m \)-Dyck paths by their number of valleys is well-known to be the \( m \)-Narayana polynomial. The \( m \)-Narayana polynomial is also equal to the \( h \)-polynomial of \( \text{Asso}^{(m)}(A_{n-1}, (12 \cdots n)) \) [FR05], which, by Proposition 5.41, is the in-degree generating function on \( \text{Camb}^{(m)}_{\text{Asso}}(A_{n-1}, (12 \cdots n)) \). This also follows from Corollary 5.43 and [Arm06, Section 3.5].
Remark 5.45. For any two Coxeter elements \( c \) and \( c' \), the underlying unoriented flip graphs of the complexes \( \text{Asso}^{(m)}(W,c) \) and \( \text{Asso}^{(m)}(W,c') \) are isomorphic. The isomorphism is induced by the shift operation, and is given by the isomorphism in Proposition 5.17. In particular, the map \( \text{Camb}_{c} \) is a graph automorphism. For \( m = 1 \), the increasing flip graph coincides with the Hasse diagram of the Cambrian poset, while for \( m \geq 2 \) the increasing flip graph is no longer transitively reduced. Therefore, the shift operation does not induce a isomorphism between the unoriented Hasse diagrams of \( \text{Camb}^{(m)}_{\text{Asso}}(W,c) \) and \( \text{Camb}^{(m)}_{\text{Asso}}(W,c') \).

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