Representations of a class of lattice type vertex algebras

Stephen Berman
Department of Mathematics and Statistics, University of Saskatchewan, Saskatoon, Saskatchewan, Canada S7N 5E6

Chongying Dong
Department of Mathematics, University of California at Santa Cruz, CA 95064

Shaobin Tan
Department of Mathematics, Xiamen University, Xiamen 361005, China

1 Introduction

The lattice vertex algebras, [B1], [FLM], form one of the most important and fundamental classes of vertex algebras. Beginning with a root lattice of simply laced type one constructs the fundamental representation for the corresponding affine Kac-Moody Lie algebra and this turns out to be one of the most basic examples of a lattice vertex algebra ([FK],[S]). The vertex algebra associated to the Leech lattice plays a fundamental role in the construction of the moonshine vertex operator algebra ([B1],[FLM]), while the vertex algebra associated to the rank 2 Lorentz lattice is used in constructing the Monster Lie algebra which in turn is used in a proof of the moonshine conjecture. Moreover lattice vertex algebras have been studied extensively from several points of view. Much work has been done on the representation

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theory ([FLM], [D], [DLM1], [DLM2]) and on fusion rules, [DL], for the lattice vertex algebras. Also, much is known about their automorphism groups, [DN], and there are some characterizations known [LX].

In this paper we study the representation theory for certain “half lattice” vertex algebras. Let $L$ be an even lattice and let $V_L$ be the associated lattice vertex algebra. Then, as a vector space, $V_L$ is the tensor product of a symmetric algebra $S(\mathfrak{h} \otimes \mathbb{C} t^{-1}[t^{-1}])$ with a group algebra $\mathbb{C}[L]$ where $\mathfrak{h} = \mathbb{C} \otimes \mathbb{Z} L$. Thus, $V_L = S(\mathfrak{h} \otimes \mathbb{Z} t^{-1}[t^{-1}]) \otimes \mathbb{C}[L]$. The lattice $L$ we consider in this paper is spanned by $c_i, d_i$ for $i = 1, ..., \nu$ with the $\mathbb{Z}$-bilinear form determined by $(c_i, c_j) = (d_i, d_j) = 0$ and $(c_i, d_j) = k\delta_{i,j}$. Here $k$ is an arbitrary non-zero integer. Our half lattice vertex algebra $V$ is then defined to be $S(\mathfrak{h} \otimes \mathbb{Z} t^{-1}[t^{-1}]) \otimes \mathbb{C}[L_C]$ where $L_C = \sum_{i=1}^{\nu} \mathbb{Z} c_i$. Notice that $V$ is not a lattice vertex algebra as originally defined in [B1] and [FLM], but it is a vertex subalgebra of $V_L$.

The motivation for studying $V$, as defined above, comes from the representation theory of certain toroidal Lie algebras and certain other Lie algebras related to them. In [T] certain vertex operator representations were given for a Lie algebra which is constructed from a natural Jordan algebra using the TKK construction. This algebra is of great interest in studying the structure theory and representation theory of extended affine Lie algebras in general because it has the smallest root system of any tame extended affine Lie algebra which is not of finite or affine type. It became clear that to study the representation theory of this Lie algebra one should use techniques offered by taking a vertex algebra point of view and that a certain natural toroidal Lie algebra entered into the picture here. The representation theory of the toroidal algebras have been studied for some time now and recently they too have been viewed from the vertex algebra perspective (see [BBS] and the references therein). In fact, the work [BBS] studies and introduces a vertex algebra which is the tensor product of three other basic vertex algebras, and one of these is nothing but our $V$ above. The other two are VOA’s associated to affine algebras and hence their representation theory is well developed. Thus, one needs the representation theory of the vertex algebra $V$ in order to understand that of the toroidal algebras in [BBS] as well as the algebras in [T]. This makes it natural to isolate $V$ and study it on its own.

Because of the motivation discussed above our main goal in this paper is to construct a large class of modules for $V$. Recall that in [FLM], for any even lattice $A$ and any element $\lambda$ in the dual lattice of $A$, an irreducible module $V_{A+\lambda}$ is constructed. The method in [FLM] for doing this uses the idea of coherent states and the authors make use of the fact that the lattice
In the work [D], employing the ideas of $\mathbb{Z}$-algebras developed in [LW], it is proved that all irreducible $V_A$-modules are of the form $V_{A+\lambda}$ as above. In our case, the sublattice $L_C$ does not span $\mathfrak{h}$, and hence we cannot use the coherent state argument to construct $V$-modules, but the $\mathbb{Z}$-algebra techniques still play an important role.

The main tools we use to construct $V$-modules is the theory of local systems, as developed in [L2]. It turns out that our proofs also apply to the “full lattice” vertex algebra. Thus we have obtained another proof that $V_{A+\lambda}$ is a $V_A$-module. More precisely, we first construct an associative algebra $A$ and a large class of $A$-modules in Section 3. We prove in Section 4 that for every $A$-module $W$ and $\lambda \in L_D$ where $L_D = \sum_{i=1}^{r} \mathbb{Z}d_i$ the space $V_{\lambda,W} = S(\mathfrak{h} \otimes_{\mathbb{Z}} t^{-1}\mathbb{C}[t^{-1}]) \otimes_{\mathbb{C}} W$ is a $V$-module. The definition of vertex operators $Y_{\lambda,W}(v,z)$ for $v \in V$ is essentially the same as in [FLM]. It follows from the theory of local system [L2] that these operators form such a local space. It is well known that the Jacobi identity for a module is equivalent to locality and associativity (see Section 4), so our main effort is to prove these operators satisfy associativity. Unfortunately, we cannot prove this directly. However, as we show in Section 4, associativity follows from the fact that this local space is closed under all $n$th products $u(x)_n v(x)$ for $n \in \mathbb{Z}$ and $u, v \in V$ where $u(x) = Y_{\lambda,W}(u,x)$. Our results then follow from this. Finally, in Section 5 we also discuss how to get $A$-modules from $V$-modules, and in Section 6 we study the Zhu algebra of $V$.

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## 2 Vertex Algebra and Module

In this paper, the sets of integers, positive integers and negative integers will be denoted respectively by $\mathbb{Z}$, $\mathbb{Z}^+$ and $\mathbb{Z}^-$. $z, z_1, z_2$ and $x$ will denote formal variables. The elementary properties of the $\delta$-function $\delta(z) = \sum_{n \in \mathbb{Z}} z^n$ can be found in [FLM] and [FHL]. First we give the definition of a vertex algebra and modules for them (see [B1], [FLM], [L1], and we also recall the theory of local system developed in [L2].

**Definition 2.1.** A vertex algebra is a quadruple $(V,Y,\mathbf{1},D)$ consisting of a vector space $V$, a vector $\mathbf{1} \in V$, a linear map $Y$ from $V$ to $(\text{End}V)[[z,z^{-1}]]$ and a linear map $D$ from $V$ to $V$ satisfying the following axioms:

1. For any $u, v \in V$, $u_n v = 0$ if $n$ is very large,
(2) $Y(1, z) = 1$,
(3) $Y(u, z)1 \in V[[z]]$ and $\lim_{z \to 0} Y(u, z)1 = u$ for $u \in V$,
(4) $[D, Y(u, z)] = \frac{d}{dz} Y(u, z) = Y(Du, z)$ for $u \in V$,
(5) The Jacobi identity holds for $u, v \in V$:

$$z_0^{-1}\delta\left(\frac{z_1 - z_2}{z_0}\right) Y(u, z_1)Y(v, z_2) - z_0^{-1}\delta\left(\frac{z_2 - z_1}{z_0}\right) Y(v, z_2)Y(u, z_1)
= z_2^{-1}\delta\left(\frac{z_1 - z_0}{z_2}\right) Y(Y(u, z_0)v, z_2). \quad (2.1)$$

We shall also use $V$ for the vertex algebra $(V, Y, D)$.

**Definition 2.2.** Let $V$ be a vertex algebra. A $V$-module is a vector space $W$ equipped with a linear map $Y_W$ from $V$ to $(\text{End } W)[[z, z^{-1}]]$ satisfying the following axioms:

1. For any $u \in V, w \in W$, $u_n w = 0$ if $n$ is very large,
2. $Y_W(1, z) = 1$,
3. The Jacobi identity holds for $u, v \in V$:

$$z_0^{-1}\delta\left(\frac{z_1 - z_2}{z_0}\right) Y_W(u, z_1)Y_W(v, z_2) - z_0^{-1}\delta\left(\frac{z_2 - z_1}{z_0}\right) Y_W(v, z_2)Y_W(u, z_1)
= z_2^{-1}\delta\left(\frac{z_1 - z_0}{z_2}\right) Y_W(Y(u, z_0)v, z_2). \quad (2.2)$$

One of the important consequences of the definition of module is the following $D$-derivative property:

$$Y_W(Du, z) = \frac{d}{dz}Y_W(u, z)$$

(see Lemma 2.2 of [DLM1]).

We now recall the theory of local system of vertex operators from [L2].

Let $M$ be a vector space. A vertex operator on $M$ is a formal series $a(z) = \sum_{n \in \mathbb{Z}} a_n z^{-n-1} \in (\text{End } M)[[z, z^{-1}]]$ such that for any $u \in M$, $a_n u = 0$ for sufficiently large $n$. All vertex operators on $M$ form a vector space (over $\mathbb{C}$), denoted by $VO(M)$. On $VO(M)$, we have a linear endomorphism $D = \frac{d}{dz}$, the formal differentiation.

Two vertex operators $a(z)$ and $b(z)$ on $M$ are said to be **mutually local** if there is a non-negative integer $k$ such that

$$(z_1 - z_2)^k a(z_1)b(z_2) = (z_1 - z_2)^k b(z_2)a(z_1). \quad (2.3)$$
A space $S$ of vertex operators is said to be local if any two vertex operators of $S$ are mutually local, and a maximal local space of vertex operators is called a local system.

Let $V$ be a local system on $M$. Then $V$ is closed under the formal differentiation $D = \frac{d}{dx}$. For $a(x), b(x) \in VO(M)$, we define

$$ Y(a(x), z)b(x) = \text{Res}_{z_1} \left( z^{-1} \delta \left( \frac{z_1 - x}{z} \right) a(z_1)b(x) - z^{-1} \delta \left( \frac{x - z_1}{-z} \right) b(x)a(z_1) \right) \quad (2.4) $$

Write $Y(a(x), z) = \sum_{n \in \mathbb{Z}} a(x)nz^{-n-1}$. Then (2.4) is equivalent to

$$ a(x)_n b(x) = \text{Res}_z ((z - x)^n a(z)b(x) - (-x + z)^n b(x)a(z)) \quad (2.5) $$

for $n \in \mathbb{Z}$. Denote by $I(x)$ the identity endomorphism of $M$.

**Theorem 2.3.** [L2] Let $M$ be a vector space and $V$ a local system on $M$. Then $(V, Y, D, I(x))$ is a vertex algebra with $M$ as a natural module such that $Y_M(a(x), z) = a(z)$ for $a(x) \in V$.

Let $A$ be any local space of vertex operators on $M$. Then there exists a local system $V$ that contains $A$. Let $\langle A \rangle$ be the vertex subalgebra of $V$ generated by $A$. Since the vertex operator “product” (2.4) does not depend on the choice of local system $V$, $\langle A \rangle$ is canonical. Then we have:

**Corollary 2.4.** [L2] Let $M$ be a vector space and $A$ any local space of vertex operators on $M$. Then $A$ generates a canonical vertex algebra $\langle A \rangle$ with $M$ as a natural module such that $Y_M(a(x), z) = a(z)$ for $a(x) \in A$.

Next we recall the well-known lattice vertex algebras from [B1] and [FLM]. We are working in the setting of [FLM]. In particular, $L$ is a lattice with nondegenerate symmetric $\mathbb{Z}$-bilinear $\mathbb{Z}$-valued form $\langle \cdot, \cdot \rangle$; $\mathfrak{h} = L \otimes_{\mathbb{Z}} \mathbb{C}$; $\mathfrak{h}_\mathbb{Z}$ is the corresponding Heisenberg algebra; $M(1)$ is the associated irreducible induced module for $\mathfrak{h}_\mathbb{Z}$ such that the canonical central element of $\mathfrak{h}_\mathbb{Z}$ acts as 1; $(\hat{L}, -)$ is a central extension of $L$ by group $\langle \kappa | \kappa^2 = 1 \rangle$ with commutator map $c(\alpha, \beta) = \kappa(\alpha, \beta)$ for $\alpha, \beta \in L$; $\chi$ is a faithful character of $\langle \kappa \rangle$ such that $\chi(\kappa) = -1$; $\mathbb{C}\{L\} = \text{Ind}_{\mathfrak{h}_\mathbb{Z}}^{\hat{L}} \mathbb{C}_\chi \simeq \mathbb{C}[L]$ (linearly), where $\mathbb{C}_\chi$ is the one-dimensional $\langle \kappa \rangle$-module defined by $\chi$; $\iota(a) = a \otimes 1 \in \mathbb{C}\{L\}$ for $a \in \hat{L}$; $V_L = M(1) \otimes \mathbb{C}\{L\}$; $1 = \iota(1)$; $\omega = \frac{1}{2} \sum_{r \geq 1} \beta_r (-1)^r$ where $\{\beta_1, \beta_2, \ldots\}$ is an orthonormal basis of $\mathfrak{h}$;

$$ V_L \rightarrow (\text{End} V_L)[[z, z^{-1}]] \quad v \rightarrow Y(v, z) = \sum_{n \in \mathbb{Z}} v_n z^{-n-1} \quad (v_n \in \text{End} V_L) \quad (2.6) $$
where the vertex operator $Y(v, z)$ is defined in detail in [FLM]; the space $V_L$ carries a natural $\mathbb{Z}$-grading determined by the conditions $\text{wt}(1 \otimes \iota(a)) = \frac{1}{2}(\bar{a}, \bar{a})$. Then $(V_L, Y, 1, L(-1))$ is a simple vertex algebra where $Y(\omega, z) = \sum_{n \in \mathbb{Z}} L(n) z^{-n-2}$ (see [B1] and [FLM]). Moreover, the component operators $L(n)$ satisfy the Virasoro algebra relation with central charge rank $L$ and $V_L = \oplus_{n \in \mathbb{Z}} (V_L)_n$ is $\mathbb{Z}$-graded where $(V_L)_n$ is the eigenspace of $L(0)$ with eigenvalue $n$ (see [B1] and [FLM]).

In this paper we consider certain even lattices below and study the modules for the “half lattice” vertex subalgebra of $V_L$.

Let $L_C = \oplus_{i=1}^{\nu} \mathbb{Z}c_i$, $L_D = \oplus_{i=1}^{\nu} \mathbb{Z}d_i$ and $L = L_C + L_D$. Define a symmetric bilinear form $(\cdot, \cdot)$ on $L$ such that $(c_i, d_j) = k \delta_{ij}$, $(c_i, c_j) = (d_i, d_j) = 0$ for $1 \leq i, j \leq \nu$, where $k \in \mathbb{Z} \setminus \{0\}$ is a constant. Then $L$ is an even lattice of rank $2\nu$, and $L$ is unimodular if $k = 1$.

For $\alpha \in (L_C \otimes \mathbb{C})/kL_C$ we define an automorphism $g_\alpha \in \text{Aut} V_L$ by

$$g_\alpha(u \otimes \iota(a)) = e^{2\pi i (\alpha, \bar{a})} u \otimes \iota(a)$$

for $u \in M(1)$ and $a \in \hat{L}$ (see [DM1]). Then $G = \{g_\alpha| \alpha \in (L_C \otimes \mathbb{C})/L_C\}$ is an abelian subgroup of $\text{Aut} V_L$. Let $V_L^G$ be the space of $G$-invariants. Then

$$V_L^G = M(1) \otimes \mathbb{C}\{L_C\}$$

where $\mathbb{C}\{L_C\}$ is spanned by $\iota(a)$ for $a \in \hat{L}$ such that $\bar{a} \in L_C$. It is clear that the algebra $\mathbb{C}\{L_C\}$ is isomorphic to the group algebra $\mathbb{C}[L_C]$. So we will use $e^a$ for basis elements of $\mathbb{C}[L_C]$ corresponding to $\alpha \in L_C$.

**Proposition 2.5.** $V_L^G = \oplus_{n=0}^{\infty} V_n$ is a $\mathbb{Z}$-graded simple vertex subalgebra of $V_L$, where

$$(V_L^G)_n = V_L^G \cap (V_L)_n = M(1)_n \otimes \mathbb{C}[L_C]$$

where

$$M(1)_n = \{\alpha_1(-n_1) \cdots \alpha_s(-n_s) | \alpha_i \in \mathfrak{h}, n_i > 0, \sum_{i=1}^{s} n_i = n\}.$$ 

**Proof.** The gradation is clear from the definition of the grading of $V_L$. In order to see that $V_L^G$ is simple we note that

$$V_L = \oplus_{\beta \in L_D} V_L^\beta$$

6
where $V_L^\beta = \{ v \in V_L | g_\alpha v = e^{2\pi i (\alpha, \beta)} v, \alpha \in (L_C \otimes Z) / kL_C \}$. Clearly, $V_L^0 = V_L^G$ and $u_n V_L^\beta \subset V_L^{\beta+\gamma}$ for $\beta, \gamma \in L_D, u \in V_L^\gamma$, $n \in \mathbb{Z}$. Since $V_L$ is simple, $V_L$ is spanned by $u_n v$ for $u \in V_L$ and $n \in \mathbb{Z}$ where $v$ is any nonzero vector in $V_L$ (see Corollary 4.2 of [DM2] and Proposition 4.1 of [L1]). It is immediate now that $V_L^G$ is simple and each $V_L^\beta$ is a simple $V_L^G$-module.

\[ \]

3 Associative Algebra and Modules

Our main goal in this paper is to study the representation theory for $V = V_L^G$. It turns out that this is closely related to the representation theory of an associative algebra $A$ which we will define in this section. It is convenient for us to begin by studying a larger algebra $B$ which then has $A$ as a homomorphic image. The idea to consider these algebras comes from considering $\mathbb{Z}$-algebras in our setting. Although our study was motivated by the idea of $\mathbb{Z}$-algebras we will see in Section 6 that $A$ is precisely the Zhu algebra $A(V)$, [Z]. The algebra $B$ is essentially a twisted tensor product of a group algebra on the lattice $L_C$ and the tensor algebra of the space $\mathfrak{h}_D = \mathbb{C} \otimes Z L_D$ where the elements $d_i$ act on $\mathbb{C}[L_C]$ as derivations. We also let $\mathfrak{h}_C = \mathbb{C} \otimes Z L_C$.

Formally, we let $B$ be an associative algebra generated by $e_\alpha$ and $d_i$ for $\alpha \in L_C$ and $1 \leq i \leq \nu$, subject to the following relation

$$
e_0 = 1, \quad e_\alpha + e_\beta = e_\alpha e_\beta, \quad d_i e_\alpha - e_\alpha d_i = (d_i, \alpha)e_\alpha$$

for $\alpha, \beta \in L_C, 1 \leq i \leq \nu$.

**Definition 3.1.** A $B$-module $W$ is called a weight module if $W = \bigoplus_{\lambda \in \mathfrak{h}_C} W_\lambda$, where

$$W_\lambda = \{ w \in W | d_i w = (\lambda, d_i) w, i = 1, ..., \nu \}.$$

Regarding $\mathfrak{h}$ as an abelian group, then the group algebra $\mathbb{C}[\mathfrak{h}]$ is a weight module for $B$ where $e_\alpha$ acts by addition and $d_i$ acts on $e_h$ for $h \in \mathfrak{h}_C$ as $(d_i, h)$. It is easy to see that $\mathbb{C}[L_C + \lambda]$ is a simple module for any $\lambda \in \mathfrak{h}_C$.

**Lemma 3.2.** Any weight $B$-module is semisimple. Moreover all the simple $B$-modules (up to isomorphism) are $\mathbb{C}[L_C + \lambda]$ for some $\lambda \in \mathfrak{h}_C$.

**Proof.** Let $w \in W_\lambda$ be nonzero for some $\lambda \in \mathfrak{h}_C$. Set $\langle w \rangle = \bigoplus_{\alpha \in L_C} C e_\alpha w$. Then $\langle w \rangle$ is a simple $B$-submodule of $W$ isomorphic to $\mathbb{C}[L_C + \lambda]$ and the lemma follows. \qed
Next we are going to construct a large class of simple $B$-modules which contains the weight modules as a special case. For simplicity we assume that $k = 1$ (the unimodular case). Let $1 \leq \mu \leq \nu + 1$ be an integer. Consider the Laurent polynomial algebras $\mathbb{C}[t_1^{\pm 1}, \ldots, t_{\mu-1}^{\pm 1}]$, and $\mathbb{C}[t_\mu, \cdots, t_\nu]$ with commuting variables. For any fixed Laurent polynomials $f_i = f_i(t_1, \cdots, t_{\mu-1}) \in \mathbb{C}[t_1^{\pm 1}, \ldots, t_{\mu-1}^{\pm 1}]$ for $i = 1, 2, \cdots, \mu - 1$, and nonzero complex numbers $a_i$ for $i = \mu, \cdots, \nu$ let $\omega = \omega(f_1, \cdots, f_{\mu-1}|a_\mu, \cdots, a_\nu)$ be a symbol. We will also use the symbols $\omega(a_1, \cdots, a_\nu)$ if $\mu = 1$, and $\omega(f_1, \cdots, f_\nu)$ if $\mu = \nu + 1$. Define

$$M_\omega = \mathbb{C}[t_1^{\pm 1}, \cdots, t_{\mu-1}^{\pm 1}, t_\mu, \cdots, t_\nu] \omega.$$

We define actions of $e_\alpha, d_j$ on $M_\omega$ for $\alpha = \sum_{i=1}^\nu m_ie_i$ and $1 \leq j \leq \nu$. For any $f \in \mathbb{C}[t_1^{\pm 1}, \cdots, t_{\mu-1}^{\pm 1}, t_\mu, \cdots, t_\nu]$,

$$e_\alpha.f \omega = \left(\prod_{i=1}^{\mu-1} t_i^{m_i} \right) \left(\prod_{i=\mu}^{\nu} (a_ie^{-\partial_i})^{m_i} \right) f \omega,$$

$$d_j.f \omega = (t_j \partial_j f + f_j f) \omega, \quad \text{for } 1 \leq j \leq \mu - 1,$$

$$d_j.f \omega = t_j f \omega, \quad \text{for } \mu \leq j \leq \nu.$$

where $\partial_i = \frac{\partial}{\partial t_i}$.

**Theorem 3.3.** $M_\omega$ is a simple $B$-module.

**Proof.** We first prove that $M_\omega$ is a $B$-module. For $1 \leq i \leq \mu - 1$, we have

$$(d_j e_\alpha - e_\alpha d_j).f \omega = d_j. \left[\prod_{i=1}^{\mu-1} t_i^{m_i} \left(\prod_{i=\mu}^{\nu} (a_i e^{-\partial_i})^{m_i} \right) f \omega - \prod_{i=\mu}^{\nu} (a_i e^{-\partial_i})^{m_i} \right] f \omega$$

$$= (t_j \partial_j f + f_j f) \left[\prod_{i=1}^{\mu-1} t_i^{m_i} \left(\prod_{i=\mu}^{\nu} (a_i e^{-\partial_i})^{m_i} \right) f \omega - \prod_{i=\mu}^{\nu} (a_i e^{-\partial_i})^{m_i} \right] f \omega$$

$$= m_j \left[\prod_{i=1}^{\mu-1} t_i^{m_i} \left(\prod_{i=\mu}^{\nu} (a_i e^{-\partial_i})^{m_i} \right) f \omega \right] (d_j, \alpha) e_\alpha. f \omega.$$
Moreover, for \( \mu \leq j \leq \nu \), we have

\[
(d_j e_\alpha - e_\alpha d_j).f \omega = d_j \left[ \prod_{i=1}^{\mu-1} t_i^{m_i} \prod_{i=\mu}^{\nu} (a_i e^{-\partial_i})^{m_i} f \right] \omega - e_\alpha (t_j f \omega)
\]

\[
= t_j \left[ \prod_{i=1}^{\mu-1} t_i^{m_i} \prod_{i=\mu}^{\nu} (a_i e^{-\partial_i})^{m_i} f \right] \omega
\]

\[
- \left[ \prod_{i=1}^{\mu-1} t_i^{m_i} \prod_{i=\mu}^{\nu} (a_i e^{-\partial_i})^{m_i} (t_j f) \right] \omega
\]

\[
= m_j \left[ \prod_{i=1}^{\mu-1} t_i^{m_i} \prod_{i=\mu}^{\nu} (a_i e^{-\partial_i})^{m_i} f \right] \omega = (d_j, \alpha)e_\alpha.f \omega
\]

as required, where we have used the fact

\[
(a_je^{\partial_j})^{m_j} (t_j f) = (t_j - m_j)(a_je^{\partial_j})^{m_j} f
\]

in the second to last identity. Other relations can be checked easily and so are omitted here.

Next we prove that the module is simple. Let \( M \) be a nonzero submodule of \( M_\omega \). One can take a nonzero element \( f \omega \in M \) so that \( f \in \mathbb{C}[t_1, \ldots, t_\nu] \). From the action \( d_j.f \omega = t_j (\partial_j f) \omega + f_j f \omega \), we see that \( (\partial_j f) \omega \in M \) for all \( 1 \leq j \leq \mu - 1 \). This implies that one can choose the nonzero polynomial \( f \in \mathbb{C}[t_\mu, \ldots, t_\nu] \), such that \( f \) has minimum degree. We apply \( e_\alpha \), for \( \alpha \in L_C \), on \( f \omega \) to get

\[
f(t_\mu - m_\mu, \ldots, t_\nu - m_\nu) \omega \in M
\]

for any \( m_\mu, \ldots, m_\nu \in \mathbb{Z} \). Therefore, if \( f \) depends on \( t_j \) for some \( \mu \leq j \leq \nu \), then

\[
(f(t_\mu, \ldots, t_j + 1, \ldots, t_\nu) - f(t_\mu, \ldots, t_\nu)) \omega \in M
\]

and \( f(t_\mu, \ldots, t_j + 1, \ldots, t_\nu) - f(t_\mu, \ldots, t_\nu) \) is nonzero with lower degree, which is a contradiction. Thus \( f \) is a nonzero constant, and so \( \omega \in M \). This finishes the proof. \( \square \)

**Theorem 3.4.** Let \( \omega_1 = \omega_1(f_1, \ldots, f_{\mu-1}|a_\mu, \ldots, a_\nu) \) and \( \omega_2 = \omega_2(g_1, \ldots, g_{\gamma-1}|b_\gamma, \ldots, b_\nu) \). Then, \( M_{\omega_1} \cong M_{\omega_2} \) as \( B \)-modules if and only if \( \mu = \gamma \), \( a_j = b_j \) for \( \mu \leq j \leq \nu \), and \( f_j - g_j \in \mathbb{Z} \) for \( 1 \leq j \leq \mu - 1 \).
Proof. Suppose $M_{\omega_1} \cong M_{\omega_2}$, and $\phi : M_{\omega_1} \to M_{\omega_2}$ is an isomorphism such that $\phi(h_{\omega_1}) = \omega_2$ for some $h \in \mathbb{C}[t_{1}^{\pm 1}, \cdots, t_{\mu-1}^{\pm 1}, t_{\mu}, \cdots, t_{\nu}]$. We first prove $\mu = \gamma$. Otherwise one may assume $\mu > \gamma$. We have

$$\phi(t_{\gamma}h_{\omega_1}) = \phi(e_{c_{\gamma}}h_{\omega_1}) = e_{c_{\gamma}}\phi(h_{\omega_1}) = e_{c_{\gamma}}\omega_2 = b_{\gamma}\omega_2 = \phi(b_{\gamma}h_{\omega_1})$$

which implies that $t_{\gamma}h_{\omega_1} = b_{\gamma}h_{\omega_1}$, and hence $h = 0$ a contradiction. Next we prove $a_i = b_i$ for $\mu \leq i \leq \nu$. Otherwise, we may assume $a_j \neq b_j$ for some $j$. Then

$$\phi(a_j h(t_1, \cdots, t_j - 1, \cdots, t_{\nu})\omega_1) = \phi(e_{c_j} h_{\omega_1}) = e_{c_j} \phi(h_{\omega_1})$$
$$= e_{c_j} \omega_2 = b_j \omega_2 = \phi(b_j h_{\omega_1})$$

which gives

$$a_j h(t_1, \cdots, t_j - 1, \cdots, t_{\nu})\omega_1 = b_j h_{\omega_1}.$$  

This implies that $h$ is independent of $t_j$. Moreover $a_j \neq b_j$ also forces $h = 0$, contradiction. Thus $a_i = b_i$ for all $i$. Finally we prove $f_j - g_j \in \mathbb{Z}$ for $1 \leq j \leq \mu - 1$. Set $\lambda_i = f_i - g_i$. We have

$$\phi(t_i(\partial_i h)\omega_1 + f_i h\omega_1) = \phi(d_i h\omega_1) = d_i \phi(h\omega_1)$$
$$= d_i \omega_2 = g_i \omega_2 = g_i \phi(h\omega_1).$$

Moreover it is easy to see that $g_i \phi(h\omega_1) = \phi(g_i h\omega_1)$, as $g_i \in \mathbb{C}[t_{1}^{\pm 1}, \cdots, t_{\mu-1}^{\pm 1}]$. Therefore we get

$$t_i \partial_i h = \lambda_i h$$
where $\lambda_i = f_i - g_i$, for $1 \leq i \leq \mu - 1$.

We claim, for $1 \leq i \leq \mu - 1$, that $\lambda_i$ is independent of $t_i$. To prove this statement, we first introduce some notation. For $p = \sum_{s=r}^{s} a_i t^i$ with $a_r, a_s \neq 0$, we define $\deg^+_{t_i} p = s$, $\deg^-_{t_i} p = r$. If $p = 0$, we define $\deg^+_{t_i} p = 0 = \deg^-_{t_i} p$.

If the statement is false, then $\deg^+_{t_i} \lambda_i = m^+ > 0$ or $\deg^-_{t_i} \lambda_i = m^- < 0$. Set $\deg^\pm_{t_i} h = n^\pm$. If $m^+ > 0$ then

$$n^+ \geq \deg^+_{t_i}(t_i \partial_i h) = \deg^+_{t_i} \lambda_i h = m^+ + n^+ > n^+$$

which is a contradiction. Similarly we will get contradiction for the case $m^- < 0$. Thus $\lambda_i$ is independent of $t_i$. Furthermore we claim that $\lambda_i$ is independent of $t_j$ for any $j = 1, \cdots, \mu - 1$. Indeed, let $1 \leq j \neq i \leq \mu - 1$ be fixed. We know that $\lambda_i$ and $\lambda_j$ are independent of $t_i$ and $t_j$ respectively, and also $\lambda_i, \lambda_j \in \mathbb{C}[t_{1}^{\pm 1}, \cdots, t_{\mu-1}^{\pm 1}]$, such that

$$\partial_i h = t_i^{-1} \lambda_i h,$$
\[ \partial_j h = t_j^{-1} \lambda_j h, \]

where \( h \in \mathbb{C}[t_1^{\pm 1}, \cdots, t_\mu^{\pm 1}, t_\mu, \cdots, t_\nu]. \)

Notice that
\[
\frac{\partial^2 h}{\partial t_j \partial t_i} = \frac{\partial}{\partial t_j} \left( t_i^{-1} \lambda_i h + \lambda_j \frac{\partial h}{\partial t_j} \right) = t_i^{-1} \left( \frac{\partial \lambda_i}{\partial t_j} h + \lambda_i t_i^{-1} \lambda_j h \right),
\]
and
\[
\frac{\partial^2 h}{\partial t_i \partial t_j} = \frac{\partial}{\partial t_i} \left( t_j^{-1} \lambda_j h + \lambda_i \frac{\partial h}{\partial t_i} \right) = t_j^{-1} \left( \frac{\partial \lambda_j}{\partial t_i} h + \lambda_j t_i^{-1} \lambda_j h \right),
\]
this gives us
\[
(t_i^{-1} \frac{\partial \lambda_i}{\partial t_j} - t_j^{-1} \frac{\partial \lambda_i}{\partial t_i}) h = 0
\]
or
\[
t_j \frac{\partial \lambda_i}{\partial t_j} - t_i \frac{\partial \lambda_j}{\partial t_i} = 0
\]
as \( h \neq 0. \) Set \( \Phi = t_i \frac{\partial \lambda_i}{\partial t_i} \in \mathbb{C}[t_1^{\pm 1}, \cdots, t_\mu^{\pm 1}], \) which is independent of \( t_j. \)
Then \( t_j \frac{\partial \lambda_i}{\partial t_j} = \Phi \) gives us \( \lambda_i = \Phi \ln t_j + C \in \mathbb{C}[t_1^{\pm 1}, \cdots, t_\mu^{\pm 1}]. \) Thus \( \Phi = 0. \)
That is \( t_j \frac{\partial \lambda_i}{\partial t_j} = 0, \) or \( \lambda_i \) is independent of \( t_j. \) This proves that \( \lambda_i \in \mathbb{C} \) for \( i = 1, \cdots, \mu - 1. \) Moreover the equation
\[
\frac{\partial h}{\partial t_i} = t_i^{-1} \lambda_i h
\]
gives \( h = C t_i^{\lambda_i}, \) but \( h \in \mathbb{C}[t_1^{\pm 1}, \cdots, t_\mu^{\pm 1}, t_\mu, \cdots, t_\nu] \) forces \( \lambda_i \in \mathbb{Z} \) as \( h \neq 0. \)

Now we suppose the conditions hold, and prove the two modules \( M_{\omega_1} \) and \( M_{\omega_2} \) are isomorphic, where
\[
\omega_1 = \omega_1(f_1, \cdots, f_{\mu-1}|a_\mu, \cdots, a_\nu)
\]
\[
\omega_2 = \omega_2(g_1, \cdots, g_{\mu-1}|a_\mu, \cdots, a_\nu)
\]
and \( g_i = f_i + N_i \) for some \( N_i \in \mathbb{Z}. \) We define \( \phi : M_{\omega_1} \rightarrow M_{\omega_2} \) by
\[
\phi(f_{\omega_1}) = t_1^{-N_1} \cdots t_{\mu-1}^{-N_{\mu-1}} f_{\omega_2}
\]
for \( f \in \mathbb{C}[t_1^{\pm 1}, \cdots, t_{\mu-1}^{\pm 1}, t_\mu, \cdots, t_\nu]. \) It is obvious that \( \phi \) defines a vector space isomorphism. We need to check that it also defines an algebra homomorphism.
Let $\alpha = \sum_{i=1}^{\nu} m_i c_i$. It is easy to see that

$$\phi(e_\alpha f \omega) = e_\alpha \phi(f \omega),$$

and $\phi(d_t f \omega) = d_t \phi(f \omega)$ for $\mu \leq l \leq \nu$. Therefore we only need to check the identity $\phi(d_t f \omega) = d_t \phi(f \omega)$ for $1 \leq l \leq \mu - 1$. In fact

$$d_t \phi(f \omega) = d_t \left( \prod_{i=1}^{\mu-1} t_i^{-N_i} \right) f \omega_2$$

$$= \left[ t_t \partial_t \left( \prod_{i=1}^{\mu-1} t_i^{-N_i} \right) f \right] + g_l \left( \prod_{i=1}^{\mu-1} t_i^{-N_i} \right) f \omega_2$$

$$= \left[ -N_l (\prod_{i=1}^{\mu-1} t_i^{-N_i}) f + t_l (\prod_{i=1}^{\mu-1} t_i^{-N_i}) \partial_t f + g_l (\prod_{i=1}^{\mu-1} t_i^{-N_i}) f \right] \omega_2$$

$$= \left[ \left( \prod_{i=1}^{\mu-1} t_i^{-N_i} \right) t_t \partial_t f + \left( \prod_{i=1}^{\mu-1} t_i^{-N_i} \right) f \partial_t f \right] \omega_2$$

$$= \phi((t_t \partial_t f + f \partial_t f) \omega) = \phi(d_t f \omega),$$

as required. This finishes the proof of the Theorem. \hfill \Box

Now we define the associative algebra $A$ to be the quotient of $B$ modulo relations $d_id_j = d_jd_i$ for all $i, j$.

**Theorem 3.5.** For $\omega = \omega(f_1, \ldots, f_{\mu-1}|a_\mu, \ldots, a_\nu)$, $M_\omega$ is an $A$-module if and only if $f_j = t_j \partial_j P + P_j(t_j)(j = 1, \ldots, \mu - 1)$ for some $P \in \mathbb{C}[t_1^{\pm 1}, \ldots, t_{\mu-1}^{\pm 1}]$ and $P_j(t_j) \in \mathbb{C}[t_j^{\pm 1}]$.

**Proof.** The proof of the "if" part is straightforward. To prove the "only if" part, we suppose $M_\omega$ is an $A$-module. From the relation $d_id_j = d_jd_i$ for $1 \leq i, j \leq \mu - 1$, one can easily obtain $D_if_j = D_jf_i$ for $1 \leq i, j \leq \mu - 1$, where $D_j = t_j \partial_j$ is the degree derivation of $\mathbb{C}[t_1^{\pm 1}, \ldots, t_{\mu-1}^{\pm 1}]$.

For this fixed $\omega = \omega(f_1, \ldots, f_{\mu-1}|a_\mu, \ldots, a_\nu)$, we can write $f_1, \ldots, f_{\mu-1}$ in the following form

$$f_i = \sum_{k_1, \ldots, k_{\mu-1} \in \mathbb{Z}} a_{k_1, \ldots, k_{\mu-1}} t_1^{k_1} \cdots t_{\mu-1}^{k_{\mu-1}} + P_i(t_i)$$

for $i = 1, 2, \ldots, \mu - 1$, where $a_{k_1, \ldots, k_{\mu-1}} \in \mathbb{C}$, and the homogeneous term $t_1^{k_1} \cdots t_{\mu-1}^{k_{\mu-1}} \not\in \mathbb{C}[t_1^{\pm 1}]$ if $a_{k_1, \ldots, k_{\mu-1}} \neq 0$. 

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Suppose \( a_{k_1, \ldots, k_{\mu - 1}}^{(i)} \neq 0 \) for a fixed \( i \), then, from \( D_j f_i = D_i f_j \), we have \( k_j a_{k_1, \ldots, k_{\mu - 1}}^{(i)} = k_i a_{k_1, \ldots, k_{\mu - 1}}^{(j)} \), which implies that \( k_i \neq 0 \) (as \( t_1^{k_1} \cdots t_{\mu - 1}^{k_{\mu - 1}} \notin \mathbb{C}[t_1^{\pm 1}] \), thus one can find \( j(\neq i) \) so that \( k_j \neq 0 \). Let \( c_{k_1, \ldots, k_{\mu - 1}} = \frac{1}{k_i} t_{k_1, \ldots, k_{\mu - 1}} \in \mathbb{C} \). It follows from \( k_j a_{k_1, \ldots, k_{\mu - 1}}^{(i)} = k_i a_{k_1, \ldots, k_{\mu - 1}}^{(j)} \) that

\[
a_{k_1, \ldots, k_{\mu - 1}}^{(j)} = c_{k_1, \ldots, k_{\mu - 1}} k_j
\]

for all \( 1 \leq j \leq \mu - 1 \). Therefore

\[
f_j = D_j \left( \sum_{k_1, \ldots, k_{\mu - 1} \in \mathbb{Z}} c_{k_1, \ldots, k_{\mu - 1}} t_1^{k_1} \cdots t_{\mu - 1}^{k_{\mu - 1}} \right) + P_j(t_j)
\]

for \( 1 \leq j \leq \mu - 1 \), as required. This also completes the proof. \( \square \)

**Corollary 3.6.** For \( \omega = \omega(f_1, \ldots, f_{\mu - 1}|a_\mu, \ldots, a_\nu) \), \( \omega \) is a simple \( A \)-module if and only if \( f_j = D_j P + P_j(t_j) \) \( (j = 1, \ldots, \mu - 1) \) for some Laurent polynomials \( P \in \mathbb{C}[t_1^{\pm 1}, \ldots, t_{\mu - 1}^{\pm 1}] \) and \( P_j(t_j) \in \mathbb{C}[t_j^{\pm 1}] \). Moreover, if \( M_{\omega_1} \) and \( M_{\omega_2} \) are \( A \)-modules with \( \omega_1 = \omega_1(f_1, \ldots, f_{\mu - 1}|a_\mu, \ldots, a_\nu) \) and \( \omega_2 = \omega_2(g_1, \ldots, g_{\gamma - 1}|b_\gamma, \ldots, b_\nu) \), then \( M_{\omega_1} \cong M_{\omega_2} \) as \( A \)-modules if and only if \( \mu = \gamma, a_j = b_j \) \( (1 \leq j \leq \nu) \), and \( f_j - g_j \in \mathbb{Z} \) \( (1 \leq j \leq \mu) \).

For \( \alpha = \sum_{i=1}^{\nu} m_i c_i \in L_C \), we set \( t^\alpha = t_1^{m_1} \cdots t_{\nu}^{m_\nu} \in \mathbb{C}[t_1^{\pm 1}, \ldots, t_{\nu}^{\pm 1}] \). Let \( \mu = \nu \) in the previous theorem, and take \( f_i = \lambda_i \in \mathbb{C} \) for \( 1 \leq i \leq \nu \). Then \( M_\omega = \bigoplus_{\alpha \in L_C} \mathbb{C} t^\alpha \), where \( \omega = \omega(\lambda_1, \ldots, \lambda_\nu) \). It is clear that

\[
d_i(t^\alpha \omega) = (d_i, \alpha + \lambda) t^\alpha \omega
\]

for \( 1 \leq i \leq \nu \), where \( \lambda = \sum_{i=1}^{\nu} \lambda_i c_i \in \mathfrak{h}_C \). Thus we have

**Corollary 3.7.** If \( \omega = \omega(\lambda_1, \ldots, \lambda_\nu) \) with \( \lambda = \sum_{i=1}^{\nu} \lambda_i c_i \in H_C \), then \( M_\omega \) is isomorphic to \( \mathbb{C}[L_C + \lambda] \) as \( A \)-modules. Moreover, \( \mathbb{C}[L_C + \lambda] \cong \mathbb{C}[L_C + \lambda'] \) if and only if \( \lambda - \lambda' \in L_C \).

The connection between \( A \)-modules and \( V \)-modules is studied in the next two sections.

### 4 Construction of \( V \)-Modules from \( A \)-Modules

Let \( W \) be an \( A \)-module. For \( \lambda \in \frac{1}{\ell} L_D \), set

\[
V_{\lambda, W} = M(1) \otimes W.
\]
Our objective is to make $V_{\lambda,W}$ a $V$-module.

Motivated by the representation theory for the lattice vertex algebras, we define an action of $\hat{h}$ on $V_{\lambda,W}$ as follows:

\[ \alpha(n) \mapsto \alpha(n) \otimes 1 \]
\[ \beta(0) \mapsto (\beta, \lambda) \]
\[ \gamma(0) \mapsto 1 \otimes \gamma \]
\[ c \mapsto 1 \]

for $n \in \mathbb{Z}$, $\alpha \in \mathfrak{h}$, $\beta \in \mathfrak{h}_C$, $\gamma \in \mathfrak{h}_D$. Then $V_{\lambda,W}$ is an $\hat{h}$-module. We also define operators $e^\alpha$, on $V_{\lambda,W}$ and $z^\alpha$ on $V_{\lambda,W}[[z, z^{-1}]]$ for $\alpha \in L_C$ as follows:

\[ e^\alpha \mapsto 1 \otimes e^\alpha, \quad z^\alpha \mapsto z^{(\alpha, \lambda)} \]

**Lemma 4.1.** For $\alpha, \beta \in L_C$, $\gamma \in \mathfrak{h}$, we have

\[ e^\alpha z^\beta = z^\beta e^\alpha, \quad \gamma(0)e^\alpha = e^\alpha \gamma(0) + (\gamma, \alpha)e^\alpha. \]

For $\alpha \in \mathfrak{h}$ we set

\[ \alpha(z) = \sum_{n \in \mathbb{Z}} \alpha(n)z^{-n-1}. \]

We also define

\[ Y_{\lambda,W}(e^\beta, z) = E^-(-\beta, z)E^+(-\beta, z)e^\beta z^\beta \]

for $\beta \in L_C$ where

\[ E^\pm(\beta, z) = \exp\left( \sum_{n \in \pm \mathbb{N}} \frac{\beta(n)z^{-n}}{n} \right). \]

Now we define a linear map

\[ Y_{\lambda,W} : \ V \to (\operatorname{End}V_{\lambda,W})[[z, z^{-1}]] \]
\[ v \mapsto Y_{\lambda,W}(v, z) = \sum_{n \in \mathbb{Z}} v_n z^{-n-1} \]

by

\[ Y_{\lambda,W}(\alpha_1(-n_1) \cdots \alpha_s(-n_s)e^\alpha, z) \]
\[ =: (\partial_{n_1-1}\alpha_1(z)) \cdots \partial_{n_s-1}\alpha_s(z))Y_{\lambda,W}(e^\alpha, z) : \]

where the normal ordering is defined in [FLM], $\partial_n = \frac{1}{n!} \left( \frac{\partial}{\partial z} \right)^n$. 

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Remark 4.2. The expression of the operator $Y_{\lambda,W}(v, z)$ here is the same as that in (2.7) except that we deal with abstract space $W$.

It is easy to see that $V_{\lambda,W}$ is a module for the Heisenberg vertex algebra $M(1)$. The rest of this section is devoted to the proof that $V_{\lambda,W}$ is a $V$-module. As we have already pointed out that we cannot prove the Jacobi identity for the operators $Y_{\lambda,W}(u, z)$ by using the coherent state argument given in Chapter 8 of [FLM] as our lattice $L_C$ does not span $\mathfrak{h}$. Instead we will apply the theory of local system developed in [L2] to our situation.

Recall that $\omega = \frac{1}{2} \sum_{i=1}^{2\nu} \beta_i (\frac{1}{2})^{i}$. It follows from the definition of $Y_{\lambda,W}(v, z)$ that

$$Y_{\lambda,W}(L(-1)v, z) = \frac{d}{dz} Y_{\lambda,W}(L(-1)v, z)$$

for all $v \in V$.

We need several lemmas.

Recall (2.7). The following lemma is straightforward.

**Lemma 4.3.** Let $\alpha_i \in \mathfrak{h}, 0 < n_i \in \mathbb{Z}$ for $i = 1, ..., s$ and $\alpha \in L_C$. Then

$$Y_{\lambda,W}(\alpha_1(-n_1) \cdots \alpha_s(-n_s) e^\alpha, x) = \alpha_1(x)^{-n_1} \cdots \alpha_s(x)^{-n_s} Y(e^\alpha, x).$$

Set $S_{\lambda,W} = \{Y_{\lambda,W}(v, x) | v \in V\}$. We have the following:

**Lemma 4.4.** $S_{\lambda,W}$ is a local space on $V_{\lambda,W}$.

**Proof.** Let $h, h' \in \mathfrak{h}$ and $\alpha, \beta \in L_C$. Since $V_{\lambda,W}$ is a module for the affine algebra $\hat{\mathfrak{h}}$ we immediately have

$$[h(x_1), h'(x_2)](x_1 - x_2)^2 = 0.$$

One can easily verify that

$$[h(x_1), Y_{\lambda,W}(e^\beta, x_2)](x_1 - x_2) = 0$$

$$[Y_{\lambda,W}(e^\alpha, x_1), Y_{\lambda,W}(e^\beta, x_2)] = 0$$

(cf. Chapter 7 of [FLM]). Thus $X = \langle \alpha(x), Y_{\lambda,W}(e^\beta, x) | \alpha \in \mathfrak{h}, \beta \in L_C \rangle$ is a local space on $V_{\lambda,W}$. By Lemma 4.3 $S_{\lambda,W}$ can be generated from $X$ by the operations $a(x)^n b(x)$ and by tacking derivations. Thus $S_{\lambda,W}$ is a local space on $V_{\lambda,W}$ (see [L2]).

**Remark 4.5.** Lemmas 4.3 and 4.4 also hold for vertex operators $Y(u, z)$ acting on $V$ (see [FLM], [DL]).
Now we set \( M = V \oplus V_{\Lambda, W} \) and define \( u(z) = Y_M(u, z) = Y(u, z) + Y_{\Lambda, W}(u, z) \) for \( u \in V \). We also set \( S = \langle Y_M(u, x) | u \in V \rangle \), a local space on \( M \). Let \( U \) be a vertex algebra generated by the local space \( S \). By Theorem 2.3, \( M \) is an \( U \)-module with \( Y_M(a(x), z) = a(z) \) for \( a(x) \in U \) and \( Y_M(u(x), z) = Y_M(u, z) \) if \( u \in V \).

By Lemma 2.2.5 of [L2], \( U \) is a module for the affine algebra \( \hat{\mathfrak{h}} \) with \( h(n) \) acting as \( h(x)_n \) for \( h \in \mathfrak{h}, n \in \mathbb{Z} \) and \( c \) acting as 1. Thus \( U \) contains a vertex subalgebra \( M(1) \) generated by \( h(x) \) for \( h \in \mathfrak{h} \). We should mention that the component operators \( \omega(x)_n \) for \( n \in \mathbb{Z} \) satisfy the Virasoro algebra relation

\[
[\omega(x)_{m+1}, \omega(x)_{n+1}] = (m - n)\omega(x)_{m+n+1} + \frac{m^3 - m}{6} \delta_{m+n,0} \nu
\]

which follows directly from the affine algebra relation in \( \hat{\mathfrak{h}} \).

Recall the component operators of \( Y(u(x), z) \) are given by \( Y(u(x), z) = \sum_{n \in \mathbb{Z}} u(x)_n z^{-n-1} \) where \( Y \) is the linear map from \( U \) to \( (\text{End} U)[[z, z^{-1}]] \) which is guaranteed since \( (U, Y, I(x), \frac{d}{dx}) \) is a vertex algebra.

**Lemma 4.6.** Let \( h \in \mathfrak{h}, m, n \in \mathbb{Z}, \alpha \in L_C \). Then

\[
[h(x)_n, e^\alpha(x)_m] = (h, \alpha)e^\alpha(x)_{m+n}. \quad (4.2)
\]

**Proof.** Since \( U \) is a vertex algebra we have the commutator formula which is a consequence of the Jacobi identity:

\[
[h(x)_n, e^\alpha(x)_m] = \sum_{i \geq 0} \binom{n}{i} (h(x)_i e^\alpha(x))_{m+n-i}.
\]

Note that

\[
h(x)_0 e^\alpha(x) = \text{Res}_z \{h(z)e^\alpha(x) - e^\alpha(x)h(z)\}
= [h(0), Y(e^\alpha, x)] = (h, \alpha) Y(e^\alpha, x).
\]

If \( i \geq 1 \) we know from the proof of Lemma 4.4 that

\[
h(x)_i e^\alpha(x) = \text{Res}_z \{ (z - x)^i h(z)e^\alpha(x) - (z - x)^i e^\alpha(x)h(z) \} = 0.
\]

The result follows immediately. \( \square \)

**Lemma 4.7.** Let \( \alpha, \beta \in L_C \), and \( m \in \mathbb{Z} \) Then \( e^\alpha(x)_m e^\beta(x) = 0 \) if \( m \geq 0 \) and \( e^\alpha(x)_m e^\beta(x) = Y(u^m \otimes e^{\alpha + \beta}, x) \) if \( m < 0 \) where \( u^m \in M(1) \) is determined by \( \frac{1}{(-m-1)!} L(-1)^{-m-1} e^\alpha = e^\alpha_m = u^m \otimes e^\alpha. \)
Proof. From the proof of Lemma 4.4 we know that if \( m \geq 0 \) then
\[
[e^\alpha(z), e^\beta(x)](z - x)^m = 0
\]
of nonnegative \( m \). It is immediate from the definition of \( e^\alpha(x)_m e^\beta(x) \) that
\( e^\alpha(x)_m e^\beta(x) = 0 \) in this case.

Now we deal with negative \( m \). If \( m = -1 \) then
\[
e^\alpha(x) - 1 e^\beta(x) = Y(e^\alpha, x) Y(e^\beta, x) + Y(e^\beta, x) Y(e^\alpha, x)^+
\]
where
\[
Y(e^\alpha, x)^+ = \sum_{s \geq 0} e^\alpha_s x^{-s-1}, Y(e^\alpha, x)^- = \sum_{s < 0} e^\alpha_s x^{-s-1}.
\]
Since
\[
[e^\alpha_s, e^\beta_t] = 0
\]
for any \( s, t \in \mathbb{Z} \) and \( (\alpha, \beta) = 0 \), we have
\[
e^\alpha(x) - 1 e^\beta(x) = Y(e^\alpha, x) Y(e^\beta, x) = Y(e^{\alpha+\beta}, x).
\]
Now let \( m = -1 - n \) for some nonnegative \( n \). A straightforward computation using (4.1) gives
\[
e^\alpha(x)_m e^\beta(x) = \frac{1}{n!} \left( \frac{d}{dx} \right)^n Y(e^\alpha, x) Y(e^\beta, x)
\]
\[
= \frac{1}{n!} Y(L(-1)^n e^\alpha, x) Y(e^\beta, x)
\]
\[
= Y(u^n \otimes e^{\alpha+\beta}, x),
\]
as required. \( \square \)

Proposition 4.8. We have \( S = U \). That is, \( S \) is a vertex algebra.

Proof. Since \( U \) is generated by
\[
X = \langle h(x), e^\alpha(x) | h \in \mathfrak{h}, \alpha \in L_C \rangle
\]
from the proof of Lemma 4.4 and Remark 4.5, it is enough to show that \( S \) is invariant under the operator \( u(x)_n \) for any \( u(x) \in X \) and \( n \in \mathbb{Z} \).

Note from Lemma 4.3 that \( S \) is spanned by
\[
h_1(x)_{-n_1} \cdots h_s(x)_{-n_s} e^\beta(x)
\]
for $h_i \in \mathfrak{h}$, $n_i > 0$ and $\beta \in L_C$. Also note that $h(x)_ne^\beta(x) = (h, \beta)\delta_{n,0}e^\beta(x)$ for nonnegative $n$ (see the proof of Lemma 4.6). Using the commutator relation $[h(x)_m, h'(x)_n] = m\delta_{m,-n}(h, h')$ we see that $S$ is invariant under $h(x)_n$ for all $h \in \mathfrak{h}, n \in \mathbb{Z}$.

Let $\alpha \in L_C$. By Lemma 4.7, $e^\alpha(x)_ne^\beta(x) \in S$. An induction by using Lemma 4.6 on $s$ then shows that $e^\alpha(x)_nh_1(x)_{-n_1} \cdots h_s(x)_{-n_s}e^\beta(x) \in S$. \hfill \Box

We are now in a position to prove the main result in this section.

**Theorem 4.9.** Let $\lambda, W$ be as before. Then $(V_{\lambda,W}, Y_{\lambda,W})$ is a $V$-module. Moreover, $V_{\lambda,W}$ is irreducible if and only if $W$ is simple.

**Proof.** By Theorem 2.3 and Proposition 4.8, $V_{\lambda,W}$ is a module for the vertex algebra $S$ under the action $Y_{V_{\lambda,W}}(u(x), z) = u(x)$ for any $u(x) \in S$. Note that $u(x)$ on $V_{\lambda,W}$ is exactly $Y_{\lambda,W}(u, x)$ for $u \in V$. So it is enough to prove that the map from $V$ to $S$ defined by sending $u$ to $Y(u, x) = u(x)$ is a vertex algebra isomorphism.

Let $\bar{V} = \{Y(u, x)|u \in V\}$ be the set of vertex operators on $V$. Then $\bar{V}$ is a vertex algebra under

$$Y(u, x)_nY(v, x) = \text{Res}_1((z-x)^nY(u, z)Y(v, x) - (-x+z)^nY(v, x)Y(u, z))$$

for $u, v \in V$ and both maps from $V$ and $S$ to $\bar{V}$ given by sending $v$ and $v(x)$ to $Y(v, x)$ for $v \in V$ are surjective vertex algebra homomorphisms (see [L2]). Note that $Y(v, z) = 0$ if and only if $v = 0$. This shows that both homomorphisms are isomorphisms. As a result, the map from $V$ to $S$ given by sending $v$ to $v(x)$ is a vertex algebra isomorphism.

Now we assume that $V_{\lambda,W}$ is an irreducible $V$-module. Let $M$ a nonzero $A$-submodule. It is clear from the definition of vertex operators $Y_{\lambda,W}(v, z)$ that $M(1) \otimes M$ is a $V$-submodule of $V_{\lambda,W}$. Thus $W$ is simple. It will be proved in the next section that if $W$ is a simple $A$-module then $V_{\lambda,W}$ is an irreducible $V$-module. \hfill \Box

## 5 Construction of $A$-Modules from $V$-Modules

Let $M = (M, Y_M)$ be a $V$-module. Set $Y_M(\alpha(-1), z) = \sum_{n \in \mathbb{Z}} \alpha(-n)z^{-n-1}$ for $\alpha \in \mathfrak{h}$. Then the operators $\alpha(m), \beta(n)$ satisfy the Heisenberg algebra relation

$$[\alpha(m), \beta(n)] = m(\alpha, \beta)\delta_{m+n,0}$$

(see [D] for the details). Thus $M$ is a module for the affine algebra $\hat{\mathfrak{h}}$. We now define the vacuum space $\Omega_M$ following [LW] (also see [D]):

$$\Omega_M = \{w \in M|\alpha(n)w = 0, \alpha \in \mathfrak{h}, n > 0\}.$$
In this section we prove that “weight space” $\Omega'_M$ of $\Omega_M$ is an $A$-module. We should point out that it is possible that $\Omega_M = 0$ without additional assumption.

We set $\bar{M} = M(1) \otimes \Omega_M$ then $\bar{M}$ is a subspace of $M$ and we will also show that $\bar{M}$ is a $V$-submodule of $M$.

For $\alpha \in L_C$, set

$$Z(\alpha, z) = \exp \left( \sum_{n \in \mathbb{Z}} \frac{\alpha(n)}{n} z^{-n} \right) Y_M(e^\alpha, z) \exp \left( \sum_{n \in \mathbb{Z}_+} \frac{\alpha(n)}{n} z^{-n} \right)$$

$$= \sum_{n \in \mathbb{Z}} Z(\alpha, n) z^{-n-1}.$$

**Lemma 5.1.** For $\alpha \in L_C$, $\beta \in H$ we have

1. $[\beta(0), Z(\alpha, z)] = (\beta, \alpha)Z(\alpha, z)$,
2. $[\beta(n), Z(\alpha, z)] = 0$, if $n \neq 0$,
3. $\frac{d}{dz} Z(\alpha, z) = Z(\alpha, z)\alpha(0) z^{-1}$,
4. $Z(\alpha, n)\Omega_M \subset \Omega_M$ for $n \in \mathbb{Z}$.
5. $[\alpha(0), Y_M(v, z)] = 0$, for $v \in V$.

**Proof.** The proofs follow from a similar argument as in [D]. We refer the reader to [D] for details.

**Lemma 5.2.** If $\Omega_M \neq 0$ there exists a nonzero $w \in \Omega_M$ and $\lambda \in \frac{1}{k}L_D$ such that $\alpha(0)w = (\lambda, \alpha)w$ for all $\alpha \in L_C$.

**Proof.** One of the important consequence of the Jacobi identity (2.2) is associativity (see [DL], [FLM], [L2]): Let $u, v \in V$ and $w \in M$, then there exists $n \geq 0$ such that

$$(z_2 + z_0)^n Y_M(u, z_0 + z_2)Y_M(v, z_2)w = Y_M(Y(u, z_0)v, z_2)w.$$ 

Using this associativity one can show (see the proof of Corollary 4.2 of [DM] or the proof of Proposition 4.1 of [L1]), that if $Y_M(u, z)w = 0$ for some nonzero $u \in V$ and $w \in M$ then $Y(v, z)w = 0$ for all $v \in V$. (One needs to use the fact that $V$ is simple.) Since $Y_M(1, z)$ is the identity operator, we conclude that for any nonzero $u \in V$ and nonzero $w \in M$, $Y_M(u, z)w \neq 0$. In particular, $Y_M(e^\alpha, z)w \neq 0$ for any $w \in M$.

From the definition of operator $Z(\alpha, z)$ we know that

$$Y_M(e^\alpha, z) = \exp \left( \sum_{n \in \mathbb{Z}_-} -\frac{\alpha(n)}{n} z^{-n} \right) \exp \left( \sum_{n \in \mathbb{Z}_+} -\frac{\alpha(n)}{n} z^{-n} \right) \otimes Z(\alpha, z)$$
on $M(1) \otimes \Omega_M$ by Lemma 5.1 (4). Thus $Z(\alpha, z)w \neq 0$ for any nonzero $w \in \Omega_M$. By Lemma 5.1 (3) we have the following
\[ \alpha(0)Z(\alpha, n) = (-n - 1)Z(\alpha, n) \]
for any $n \in \mathbb{Z}$. The exact same proof of Lemma 3.4 in [D] then works here. \[ \square \]

Motivated by Lemma 5.2 we define a weight subspace
\[ \Omega_M(\lambda) = \{ w \in \Omega_M | \alpha(0)w = (\alpha, \lambda)w \text{ for all } \alpha \in L_C \} \]
for any $\lambda \in L_D$. Set $\Omega'_M = \sum_{\lambda \in L_D} \Omega_M(\lambda)$. Now we define an operator $z^\alpha$ on $\Omega'_M$ for $\alpha \in L_C$ by saying it acts on $\Omega_M(\lambda)$ as $z^{(\alpha, \lambda)}$. Set
\[ T_\alpha = Z(\alpha, z)z^{-\alpha}. \]

Then by Lemma 5.1 (3),
\[ \frac{d}{dz} T_\alpha = 0. \]

That is, $T_\alpha$ is an operator on $\Omega'_M$ and is independent of the formal variable $z$.

**Lemma 5.3.** For $\alpha, \beta \in L_C, d \in \mathfrak{h}$, we have
\[ [d(0), T_\alpha] = (d, \alpha)T_\alpha, \quad T_\alpha z^\beta = z^\beta T_\alpha, \quad (5.2) \]
and
\[ T_\alpha T_\beta = T_{\alpha+\beta}. \quad (5.3) \]

**Proof.** (5.2) follows from Lemma 5.1 by noting that $d(0)z^\alpha = z^\alpha d(0)$.

For (5.3) we note that
\[ Y_M(e^\alpha, z) = E^\alpha(-\alpha, z)E^+(\alpha, z)T_\alpha z^\alpha. \]

From the proof of Lemma 4.7,
\[ Y(e^{\alpha+\beta}, z) = Y_M(e^\alpha, z)Y_M(e^\beta, z) = E^\alpha(-\alpha - \beta, z)E^+(\alpha - \beta, z)T_\alpha T_\beta z^{\alpha+\beta}. \]

Thus we obtain $T_\alpha T_\beta = T_{\alpha+\beta}$.

**Theorem 5.4.** Let $M$ be a $V$-module. Then

1. $\bar{M} = M(1) \otimes \Omega_M$ is a $V$-submodule of $M$.
2. $\Omega'_M$ is an $A$-module by sending $d$ to $d(0)$ and $e_\alpha$ to $T_\alpha$ for $d \in H_D$ and $\alpha \in L_C$.
3. If $M$ is irreducible and $\Omega_M \neq 0$ then $\Omega'_M = \Omega_M$ and $M$ is isomorphic to $V_{\lambda, \Omega_M}$ for some $\lambda \in L_D$. \[ \square \]
Proof. Clearly, $\bar{M}$ is invariant under the vertex operators $Y_M(d(-1), z)$ and $Y_M(e^\alpha, z)$ for $d \in \mathfrak{h}$ and $\alpha \in L_C$. We have already mentioned that $V$ is generated by $d(-1)$ and $e^\alpha$. It is clear now that $\bar{M}$ is a submodule. This proves (1).

(2) is a direct consequence of Lemma 5.3.

We now prove (3). By Lemma 5.2, $\Omega_M'^{\prime} \neq 0$. Set $M' = M(1) \otimes \Omega_M'^{\prime}$. Clearly $M'$ is invariant under the component operators of $Y_M(d(-1), z)$ and $Y_M(e^\alpha, z)$ for $d \in \mathfrak{h}$ and $\alpha \in L_C$. Again since $V$ is generated by $d(-1)$ and $e^\alpha$ for $d \in \mathfrak{h}$ and $\alpha \in L_C$, $M'$ is invariant under the component operators of $Y_M(v, z)$ for all $v \in V$. That is, $M'$ is a nonzero submodule of $M$. The irreducibility of $M$ then yields that $M = M'$. As a result we have $\Omega_M = \Omega_M'^{\prime}$.

We can now finish the proof of Theorem 4.9. That is, if $W$ is a simple $A$-module then $V_{\lambda, W}$ is an irreducible $V$-module. Note that $\Omega_{V_{\lambda, W}} = W$.

Recall from [Z] that $A(V) = V/O(V)$ where $O(V)$ is spanned by

$$u \circ v = \text{Res}_z (1 + z)^{wt_u} z^2 Y(u, z)v = \sum_{i=0}^{\infty} \left( \frac{wt(u)}{i} \right) u_{i-2}v$$

for homogeneous $u, v \in V$. The product is defined by

$$u \ast v = \text{Res}_z (1 + z)^{wt_u} z^{wt_v} Y(u, z)v = \sum_{i=0}^{\infty} \left( \frac{wt(u)}{i} \right) u_{i-1}v.$$

The following lemma can be found in [Z].

**Lemma 6.1.** Assume that $u \in V$ homogeneous, $v \in V$ and $n \geq 0$. Then

$$\text{Res}_z (1 + z)^{wt_u} z^{2+n} Y(u, z)v = \sum_{i=1}^{\infty} \left( \frac{wt(u)}{i} \right) u_{i-2}v \in O(V).$$
We will write $u \sim v$ if $u - v \in O(V)$ for $u, v \in V$. Note that $Y(h(-1), z) = \sum_{n \in \mathbb{Z}} h(-1) n z^{-n-1} = \sum_{n \in \mathbb{Z}} h(n) z^{-n-1}$ and the weight of $h(-1)$ is 1 for $h \in \mathfrak{h}$. By Lemma 6.1 we see that $h(-n - 1) w \sim -h(-n) w$ for any $w \in V$. Thus $A(V)$ is spanned by $\mathbb{C}[h(-1)|h \in \mathfrak{h}] \otimes \mathbb{C}[L_C]$.

For any $\alpha \in L_C$ the weight of $e^\alpha$ is 0. Thus
\[
e^\alpha \circ e^\beta = \text{Res}_{z} \left( \frac{1}{z^2} Y(e^\alpha, z)e^\beta \right) = \text{Res}_{z} \left( \frac{1}{z^2} E(-\alpha, z)e^{\alpha+\beta} \right) = \alpha(-1)e^{\alpha+\beta}
\]
for all $\alpha, \beta \in L_C$. Since $\alpha, \beta \in L_C$ are arbitrary we conclude that $\alpha(-1)e^{\beta} \in O(V)$. In particular, $\alpha(-1) \in O(V)$. Thus $\alpha(-1) \ast w = \alpha(-1) w$ lies in $O(V)$ for any $w \in V$. As a result we see that $A(V)$ is spanned by $\mathbb{C}[d_1(-1), ..., d_\nu(-1)] \otimes \mathbb{C}[L_C]$.

**Proposition 6.2.** The Zhu algebra $A(V)$ is isomorphic to $A$.

**Proof.** We define a linear map $I$ from $\tilde{A}$ to $A(V)$ by sending $d_1^{\alpha_1} \cdots d_{\nu}^{\alpha_{\nu}} e^\alpha$ to $d_1(-1)^{\alpha_1} \cdots d_{\nu}(-1)^{\alpha_{\nu}} e^\alpha + O(V)$. It is enough to prove that $I$ is an algebra isomorphism.

It is a straightforward verification that $d_i(-1), d_\nu(-1), e^\alpha, e^\beta$ satisfy the relations
\[
d_i(-1) \ast d_j(-1) = d_j(-1) \ast d_i(-1)
\]
\[
d_i(-1) \ast e^\alpha - e^\alpha \ast d_i(-1) = (d_i, \alpha) e^\alpha
\]
\[
e^\alpha \ast e^\beta = e^{\alpha+\beta}
\]
for $1 \leq i, j \leq \nu$ and $\alpha, \beta \in L_C$. So $I$ is an onto algebra homomorphism.

Proving that $I$ is injective is equivalent to proving that the intersection of $O(V)$ with the subspace $\mathbb{C}[d_1(-1), ..., d_\nu(-1)] \otimes \mathbb{C}[L_C]$ of $V$ is zero.

Recall that $V = \sum_{n \geq 0} V_n$ with $V_0 = \mathbb{C}[L_C]$. By the theory of $A(V)$ [Z], $\mathbb{C}[L_C]$ is a simple $A(V)$-module such that $o(u) = u_{\text{wt}u - 1}$ for a homogeneous $u \in V$. In particular, $o(e^\alpha)$ acts on $V_0$ as the multiplication by $e^\alpha$ for $\alpha \in L_C$ and $o(d_i(-1)) = d_i(0)$ acts on $e^\beta$ for $\beta \in L_C$ as scalar $(d_i, \alpha)$. If we identify $\mathbb{C}[L_C]$ with the ring $\mathbb{C}[t_1, t_1^{-1}, ..., t_\nu, t_\nu^{-1}]$ by identifying $e^{\sum_{i} n_i c_i}$ with $\prod_i t_i^{n_i}$ then $o(e^{\sum_{i} n_i c_i})$ acts on $\mathbb{C}[t_1, t_1^{-1}, ..., t_\nu, t_\nu^{-1}]$ as multiplication by $\prod_i t_i^{n_i}$.
and \( o(d_i(-1)) \) acts as the degree derivation \( t_i \frac{\partial}{\partial t_i} \). It is immediate then that if \( o(v) = 0 \) on \( V_0 \) for \( v \in \mathbb{C}[d_1(-1), ..., d_\nu(-1)] \otimes \mathbb{C}[L_C] \) then \( v = 0 \), as desired.

We next use \( A(V) \) to study the \( \mathbb{Z}_+ \)-graded modules for \( V \). A \( V \)-module \( M = (M, Y_M) \) is \( \mathbb{Z}_+ \)-graded if \( M = \oplus_{n \geq 0} M(n) \) such that \( u_n M(m) \subset M(wtu - n - 1 + m) \) for homogeneous \( u \in V \) and \( m, n \in \mathbb{Z} \). We may and will choose the gradation so that \( M(0) \neq 0 \). Then the theory of \( A(V) [Z] \) says that \( M(0) \) is a \( A(V) \)-module by sending \( v \) to \( o(v) \). Furthermore, \( M \mapsto M(0) \) gives a bijection between the equivalence classes of irreducible \( \mathbb{Z}_+ \)-graded \( V \)-modules and the equivalence classes of simple \( A(V) \)-modules.

Recall from Section 4 that given an \( A \)-module \( W \) and \( \lambda \in \frac{1}{k} L_D \) we have a \( V \)-module \( V_\lambda, W \). On the other hand, by the theory in [Z] there is an \( \mathbb{Z}_+ \)-graded \( V \)-module \( M = \oplus_{n \geq 0} M(n) \) such that \( M(0) \) is isomorphic to \( W \) as \( A(V) \)-modules. If \( W \) is simple then \( V_\lambda, W \) is irreducible \( V \)-module and we can choose \( M \) to be irreducible too. Here is the relation between \( V_\lambda, W \) and \( M \) if \( W \) is simple. In the construction of the module \( V_\lambda, W \) we let \( c_i(0) \) act on \( V_\lambda, W \) as \( (c_i, \lambda) \) \( (c_i \) is not an element of \( A \)). On the other hand, since \( c_i(-1) \in O(V) \) we see that \( c_i(0) \) acts on \( M = 0 \). Thus \( V_0, W \) and \( M \) are isomorphic.

Of course we could define the algebra \( A \) by including the central elements \( c_i \). Then \( A(V) \) would be a quotient of \( A \) modulo the ideal generated by \( c_i \) for \( i = 1, ..., \nu \).

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