X-wave mediated instability of plane waves in Kerr media

Claudio Conti
NooEL, Nonlinear and Optoelectronics Laboratory, National Institute for the Physics of Matter, INFM - Roma Tre, Via della Vasca Navale 84, 00146 Rome - Italy
(Dated: March 31, 2022)

Plane waves in Kerr media spontaneously generate paraxial X-waves (i.e. non-dispersive and non-diffractive pulsed beams) that get amplified along propagation. This effect can be considered a form of conical emission (i.e. spatio-temporal modulational instability), and can be used as a key for the interpretation of the out of axis energy emission in the splitting process of focused pulses in normally dispersive materials. A new class of spatio-temporal localized wave patterns is identified. X-waves instability, and nonlinear X-waves, are also expected in periodical Bose condensed gases.

The dynamics of focused femtoseconds pulses (FFP) in optically nonlinear media has a fundamental importance and it is relevant in all of the applications of ultrafast optics. The basic mechanism, when dealing with the propagation in normally dispersive materials, is the splitting of the pulse, which has been originally predicted more than ten years ago [1, 2] and recently reconsidered, due to the development of the physics of FFP. [3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18]

If \( z \) is the direction of propagation, \( t \) is the time in the reference frame where the pulse is still, and \( r = \sqrt{x^2 + y^2} \) the radial cylindrical coordinate, this process can be roughly divided into a series of steps. They describe the propagation of a gaussian (in space and time) pulse, which travels in a focusing medium, and undergoes relevant reshaping, due to the interplay of diffraction, dispersion and the Kerr effect (when the peak power is sufficiently greater than the critical power \( P_c \) for self-focusing [1, 2]): 1) the out of axis energy is focused towards \( r = 0 \) around \( t = 0 \); 2) the pulse at \( r = 0 \) is compressed; 3) lobes appear in the on-axis temporal spectrum; 4) the pulse splits in the time domain.

Before the breakup, relevant out of axis energy emission and re-distribution occurs, as originally described by Rothenberg in Ref. 2. Looking at the spatio-temporal profile an X-shape (or hyperbola) is observed before the splitting (see, e.g., figures in 10, 12). This process has been theoretically described by different approaches 10, 12, 14, 15, and the onset of an X-shape can be ultimately related to the hyperbolic characteristics over which small perturbations evolve. 16

This can be checked, for example, by the hydrodynamical approach to the nonlinear Schrödinger equation 17, 18.

Recently, attention has been devoted to the existence of X-waves in optically nonlinear media. The latter are self-trapped (i.e. non-diffracting and non-dispersive) waves that are well known in the field of linear propagation in acoustics [14, 20, 21, 22] and in electromagnetism [23, 24, 27, 28, 29, 30]. The simplest X-wave has the shape of a double-cone, or clepsydra, that appears as an X when, e.g., a section is plotted in the plane \((x, t)\), and as V in the plane \((r, t)\), as shown in figure 11. Optical X-waves in nonlinear media have been theoretically predicted in [31] and experimental results, with a direct observation of the conical spatio-temporal shape, have confirmed their existence and generation in crystals for second-harmonic generation (SH-G) 32, 33, 34. In Ref. 35 it has been theoretically shown how, during non-depleted-pump SHG, the phase matched spatio-temporal harmonics let the SH beam become an X-wave. 38

Conical emission (CE), or spatio-temporal modulational instability (MI) [14, 36, 37, 38], has been addressed in Ref. 39 as a basic mechanism underlying the spontaneous formation of an X-wave and, as a foundation for the understanding of the splitting, in Ref. 10. In this Letter I consider the formation of X-waves in Kerr media (or in quadratic media, in the regime where they mimic cubic nonlinearity, see, e.g., the chapter after Torruellas, Kivshar and Stegeman in 40 and Ref. 41). A new form of instability of plane waves can be introduced by directly involving self-localized spatio-temporal wave patterns. The process strongly resembles CE, i.e. the amplification of plane waves from noise, but in this case X-waves, instead of periodical patterns, emerge from the breakup of an unstable pump beam. This mechanism is responsible of the first stage of the pulse-splitting, i.e. the out-of-axis energy redistribution, such as MI breaks a continuous wave signal into a periodical pattern of solitons. 12

The wave equation describing the propagation in nonlinear Kerr media, whose refractive \( n \) index obeys the law \( n = n_0 + n_2 I \), with \( I \) the optical intensity, in the framework of the paraxial and of the slowly varying envelope approximation, is written as

\[
i\partial_z A + ik'\partial_T A + \frac{1}{2k}\nabla^2_{XY} A - \frac{k''}{2}\partial_T T A + \frac{k n_2}{n_0}|A|^2 A = 0,
\]

where \( A \) is normalized such that \( |A|^2 = I \), and I have taken \( k'' > 0 \) (i.e. the medium is normally dispersive). \( X, Y, Z, T \) are the real world variables, \( \lambda \) the wavelength, \( n(\lambda) \) the refractive index and \( k(\lambda) = 2\pi n(\lambda)/\lambda \). Eq. 1 can be casted in the adimensional form:

\[
i\partial_\z u + \Delta_\z u - \partial_{tt} u + \chi |u|^2 u = 0,
\]

by defining \( z = Z/Z_{df} \) with \( Z_{df} = 2kW_0^2 \) and \( W_0^2 \) a reference beam waist; \( \Delta_\z = \partial_{xx} + \partial_{yy} \) the transverse Lapla-
cian with \((X, Y) = W_0(x, y)\); \(t = (T - Z/V_0)/T_0\) the retarded time in the frame travelling at the group velocity \(V_0 = 1/k'\) in units of \(T_0 = (k''Z_0^2)^{1/2}\). The optical field envelope \(A\) is given by \(A = A_0 u\), with \(A_0 = (n_0/k|n_2|Z_0^2)^{1/2}\). \(\chi > 0 (\chi < 0)\) identifies a focusing (de-focusing) medium being \(n_2 > 0 (n_2 < 0)\).

I start considering paraxial linear (i.e. \(a\) as MI, i.e. by perturbing the plane-wave solution of (2):

\[
\Delta \psi \equiv (\Delta - \partial_t)\psi = 0. \tag{3}
\]

Introducing the complex variable \(v = (\Delta - it)^2 + r^2\), with \(\Delta\) a real valued arbitrary coefficient, I have from (6), the equation

\[
6\partial_v \psi + 4v\partial_v \psi = 0, \tag{4}
\]

from which \(\psi = C_1/\sqrt{v} + C_2\). \(C_1\) and \(C_2\) are arbitrary complex coefficients. Note that both the real and the imaginary parts of this solution, are real-valued X-wave profiles. The former being the simplest X-wave, given by (the branch cut for the square root is along the negative real axis)

\[
\psi_X \equiv \Re \left( \frac{1}{\sqrt{v}} \right) = \Re \left( \frac{1}{\sqrt{(\Delta - it)^2 + r^2}} \right). \tag{5}
\]

Remarkably \(\psi_X\) still holds when referring to the Helmholtz equation, instead of the paraxial wave equation (see e.g. Ref. [11]). Its plot is given in figure 1.

X-wave instability can be introduced in the same way as MI, i.e. by perturbing the plane-wave solution of (2): \(a = a_P \equiv a_0 \exp(i\alpha_0 z)\), with \(a_0\) a real-valued constant. Letting \(a = (a_0 + \epsilon(x, y, t, z)) \exp(i\alpha_0 z)\) I have, at first order in \(\epsilon\):

\[
i\partial_\epsilon \epsilon + \epsilon + \chi a_0^2 (\epsilon^* + \epsilon). \tag{6}
\]

Writing \(\epsilon = \epsilon(r, t, z) = \psi_X(r, t)\mu(z)\) gives

\[
i\partial_\epsilon \mu + \chi a_0^2 (\mu + \mu^*) = 0. \tag{7}
\]

The perturbation is thus

\[
\epsilon = [\alpha + i(\beta + 2\chi a_0^2 \alpha z)]\psi_X(r, t) \tag{8}
\]

with \(\alpha\) and \(\beta\) arbitrary real-valued constants. Equation (8) represents an X-wave which grows linearly along propagation (independently on the sign of \(\chi\)), with amplification given by the intensity of the plane wave \(a_p\). This situation strongly resembles MI, where plane waves are exponentially amplified at the expense of the pump beam, with a gain determined by the pump intensity. For this reason, it is natural to refer to this process as X-wave instability. As for MI, the amplified wave can be artificially externally fed, or it can be generated by noise. Note also that X-wave instability can be triggered by conical emission. Indeed, as shown in [32], the latter generates the required spectrum to form an X-wave, which eventually gets amplified, as discussed above.

The previous treatment can be generalized in several ways. In particular it is possible to show that exponential amplification of X-wave-like beams can be attained. It is necessary to enlarge the definition of X-waves, i.e. Eq. (9), by introducing the equation:

\[
\psi = \kappa \psi, \tag{9}
\]

with \(\kappa\) a real constant \((\kappa \neq 0\) in the following). Letting \(\epsilon = \mu(z)\psi(r, t) + \nu(z)^* \psi(r, t)^*\) in (8), and setting to zero the coefficients of \(\psi^*\) and \(\psi^*\) the following linear system is obtained:

\[
\begin{align*}
i\partial_\mu + \kappa \mu + a_0^2 \psi (\mu + \nu) &= 0, \\
-i\partial_\nu + \kappa \mu + a_0^2 \psi (\mu + \nu) &= 0. \tag{10}
\end{align*}
\]

If \((\mu, \nu) = (\hat{\mu}, \hat{\nu}) \exp(\gamma z)\) I have

\[
\begin{bmatrix} i\gamma + \kappa + \chi a_0^2 & \chi a_0^2 \\
\chi a_0^2 & -i\gamma + \kappa + \chi a_0^2 \end{bmatrix} \begin{bmatrix} \hat{\mu} \\
\hat{\nu} \end{bmatrix} = 0. \tag{11}
\]

The solvability condition yields the allowed values for the gain \(\gamma = \pm \sqrt{-\kappa(\kappa + 2\chi a_0^2)}\). The perturbation grows along \(z\) if the following inequality is satisfied: \(-\kappa(\kappa + 2\chi a_0^2) > 0\). Thus, for a focusing (de-focusing) medium, generalized X-waves with \(-2a_0^2 < \kappa < 0\) \((0 < \kappa < 2a_0^2)\) are exponentially amplified. The gain \(\gamma\) is obtained: \(\gamma = \sqrt{-\kappa(\kappa + 2\chi a_0^2)}\), plotted in figure 2 has a maximum value which corresponds to the most exploding self-localized packet.

Now the question arises as to whether or not Eq. (9) admits solutions resembling X-waves. Closed forms can be found proceeding as before: In term of \(v\) Eq. (2) becomes:

\[
6\partial_v \psi + 4v\partial_v \psi = \kappa \psi, \tag{12}
\]

whose general solution is

\[
\psi = C_1 \frac{\exp(-\sqrt{\kappa v})}{\sqrt{v}} + C_2 \frac{\exp(\sqrt{\kappa v})}{\sqrt{v}}. \tag{13}
\]

FIG. 1: Tri-dimensional plot of \(\psi_X\), the simplest radial-symmetry X-wave \((\Delta = 1)\).
A real valued, localised, generalised X-wave is given by

\[ \psi_\kappa = \Re(\exp(-\sqrt{\kappa}) \sqrt{\kappa}). \]  

\[ (14) \]

depends on two parameters \( \Delta \) and \( \kappa \); while the first determines the decay constant as going far from the origin in the \((r,t)\) plane, the latter completely changes the shape of the wave. Examples for \( \Delta = 1 \) and \( \kappa = 1 \) are shown in figure 3 and in figure 4 for \( \Delta = 1 \) and \( \kappa = -1 \). Note that they are similar to the Bessel pulse beams described in [29]. To clarify the differences, I observe that the spatio-temporal spectrum develops around the curve

\[ \omega^2 = k_\perp^2 + \kappa, \]

with \( \omega \) the angular frequency corresponding to \( t \) and \( k_\perp \) the transversal wavenumber. In figure 5 I show the spectra for different \( \kappa \), with \( \kappa = 0 \) corresponding to the simplest X-wave \( \psi_X \). The appearance of these lines in the spatio-temporal far field is a clear signature of the X-waves instability, and can be directly observed in the experiments. Note that the spectra resemble the shape of the wave in the physical space as a consequence of their propagation invariance.

To show that X-wave instability can actually be observed in the experiments I numerically solved Eq. 2. I considered a gaussian input beam, \( A = A_0 \exp(-R^2/(2W_0^2) - T^2/(2T_0^2)) \), whose intensity profile FWHM spot and duration are 70\(\mu m\) and 100fs.

Eq. 2 is integrated with reference to fused silica, with \( \lambda = 800nm \) \((n_0 = 1.5, n_2 = 2.5 \times 10^{-20} k'' = 360^{-28}, \) SI units \), peak power \( P = 1.5P_c \), being \( P_c = (0.61\lambda)^2/(8n_0n_2) \approx 2.6MW \) the critical power for self-focusing. In figure 6 I show the spatio-temporal profile and the spectrum (in log scale) after 3 diffraction lengths \( L_{df} = \pi n_0 W_0^2/\lambda \). Clearly an X-like profile is formed and the spectrum shows the features in Fig. 5.
In conclusion I have shown that a plane wave in Kerr media gives rise to linear and exponential amplification of X-waves, thus leading to a significant beam reshaping. A new class of X-waves is involved in this nonlinear process. The reported analysis provides insights for the interpretation of the pulse-splitting of focused femtosecond beams, and related phenomena, in the same way as MI is relevant for solitons generation. Indeed, the spontaneous formation of an X-wave can be another explanation of the out of axis energy redistribution typically observed. X-wave instability can also be an effective approach for the controlled generation of non-dispersive and non-diffractive pulses.

These results appear to be susceptible of several generalizations, as considering quadratic nonlinearity or vectorial effects, and have implications in all the fields encompassing nonlinear wave propagation, as acoustics, hydrodynamics and plasma physics. For example, X-wave instability (as well as nonlinear X-waves) is also expected in periodical Bose-Einstein Condensates where Eq. (2) holds, being $t$ the direction of periodicity, in the presence of negative effective mass. [45, 46]

I thanks S. Trillo and E. Recami for fruitful discussions, and the Fondazione Tronchetti Provera for the financial support.

---

* Electronic address: c.conti@ele.uniroma3.it

URL: [http://optow.ele.uniroma3.it](http://optow.ele.uniroma3.it)

[1] P. Chernev and V. Petrov, Opt. Lett. 17, 172 (1992).
[2] J. Rothenberg, Opt. Lett. 17, 583 (1992).
[3] J. K. Ranka, R. W. Schirmer, and A. L. Gaeta, Phys. Rev. Lett. 77, 3783 (1996).
[4] L. Berge, J. J. Rasmussen, E. G. Shapiro, and S. K. Turitsyn, J. Opt. Soc. Am. B 13, 1879 (1996).
[5] T. Brabec and F. Krausz, Phys. Rev. Lett. 78, 3282 (1997).
[6] M. Trippenbach and Y. B. Band, Phys. Rev. A 56, 4242 (1997).
[7] J. K. Ranka and A. L. Gaeta, Opt. Lett. 23, 534 (1998).
[8] S. A. Diddams, H. K. Eaton, A. A. Zozulya, and T. S. Clement, Opt. Lett. 23, 379 (1998).
[9] A. A. Zozulya and S. A. Diddams, Opt. Expr. 4, 336 (1999).
[10] A. G. Litvak, V. A. Mironov, and E. M. Sher, Phys. Rev. E 61, 891 (2000).
[11] H. S. Eisenberg, R. Morandotti, Y. Silberberg, S. Bar-Ad, D. Ross, and J. S. Aitchison, Phys. Rev. Lett. 87, 43902 (2001).
[12] L. Berge, K. Germaschevski, R. Grauer, and J. J. Rasmussen, Phys. Rev. Lett. 89, 153902 (2002).
[13] H. Ward and L. Berge, Phys. Rev. Lett. 90, 053901 (2003).
[14] G. G. Luther, A. C. Newell, and J. V. Moloney, Physica D 74, 59 (1994).
[15] G. Fibich, V. M. Malkin, and G. C. Papanicolaou, Phys. Rev. Lett. 52, 4218 (1995).
[16] T. Levi-Civita, *Caratteristiche dei sistemi differenziali e propagazione ondosa* (Zanichelli, Bologna, Italy, 1931).
[17] K. Staliunas, Phys. Rev. A 48, 1573 (1993).
[18] S. Stringari, Phys. Rev. Lett. 77, 2360 (1996).
[19] R. W. Ziolkowski, D. K. Lewis, and B. D. Cook, Phys. Rev. Lett. 62, 147 (1989).
[20] J. Lu and J. F. Greenleaf, IEEE Trans. Ultrason. Ferroelec. Freq. contr. 39, 441 (1992).
[21] P. R. Stepanishen and J. Sun, J. Acous. Soc. Am. 102, 3308 (1997).
[22] J. Salo, J. Fagerholm, A. T. Friberg, and M. M. Salomaa, Phys. Rev. Lett. 83, 1171 (1999).
[23] J. Durbin, J. J. Miceli, and J. H. Eberly, Phys. Rev. Lett. 58, 1499 (1987).
[24] E. Recami, Physica A 252, 586 (1998).
[25] P. Saar and K. Reivelt, Phys. Rev. Lett. 79, 4135 (1997).
[26] H. Sonajalj, M. Rastep, and P. Saar, Opt. Lett. 22, 310 (1997).
[27] K. Reivelt and P. Saar, J. Opt. Soc. Am. A 17, 1785 (2000).
[28] D. Mugnai, A. Ranfagni, and R. Ruggeri, Phys. Rev. Lett. 84, 4830 (2000).
[29] C. R. Sheppard, J. Opt. Soc. Am. A 18, 2594 (2001).
[30] M. Zamboni-Rached, K. Z. Nobrega, H. E. Hernandez-Figueroa, and E. Recami, arXiv:physics/0209101 (2002).
[31] C. Conti, S. Trillo, P. D. Trapani, O. Jedrkiewicz, G. Valiulis, and J. Trull, arXiv:physics/0204066 (2002), Phys. Rev. Lett. submitted.
[32] G. Valiulis, J. Kilius, O. Jedrkiewicz, A. Bramati, S. Minardi, C. Conti, S. Trillo, A. Pikisarkas, and P. D. Trapani, in *Quantum Electronics and Laser Science Conference* (Optical Society of America, Washington, D.C., 2001), vol. 57 of *Trends in Optics and Photonics*, post-deadline Papers QPD10-1.
[33] C. Conti, P. D. Trapani, S. Trillo, G. Valiulis, S. Minardi, and O. Jedrkiewicz, in *Quantum Electronics and Laser Science Conference* (Optical Society of America, Washington, D.C., 2002), p. QTuJ6.
[34] P. D. Trapani et al., Nature (2003), submitted.
[35] C. Conti and S. Trillo, arXiv:physics/0208097 (2002), Opt. Lett. submitted.
[36] L. W. Liou, X. D. Cao, C. J. McKinstrie, and G. P. Agrawal, Phys. Rev. A 46, 4202 (1992).
[37] E. J. Fonseca, S. B. Cavalcanti, and J. M. Hickmann, Opt. Comm. 169, 199 (1999).
[38] A. Picozzi and M. Haelterman, Phys. Rev. Lett. 88, 083901 (2002).
[39] S. Trillo, C. Conti, P. D. Trapani, O. Jedrkiewicz, J. Trull, G. Valiulis, and G. Bellanca, Opt. Lett. 27, 1451 (2002).
[40] S. Trillo and Torreuellas, eds., *Spatial Solitons* (Springer-Verlag, Berlin, 2001).
[41] A. V. Buriak, P. D. Trapani, D. V. Skryabin, and S. Trillo, Phys. Rep. 370, 63 (2002).
[42] A. Hasegawa, Opt. Lett. 9, 288 (1984).
[43] M. A. Porras, S. Trillo, C. Conti, and P. D. Trapani (2002), Opt. Lett. submitted.
[44] P. Saari, arXiv:physics/0103054 (2001).
[45] V. V. Konotop and M. Salerno, Phys. Rev. A 65, 021602 (2002).
[46] M. Kramer, L. Pitaevskii, and S. Stringari, Phys. Rev. Lett. 88, 180404 (2002).
[47] J. Salo and M. M. Salomaa, J. Phys. A: Math. Gen. 34, 9319 (2001).
[48] For the sake of concreteness, I use the term “X-wave”, without making distinction among the many families of localized-wave patterns solutions to the linear Helmotz equation. See, e.g., Ref. 44.

[49] It is remarkable that both X-waves and MI plane waves, $\exp(i\omega t + ik_x x + ik_y y)$, fill all the space-time and have infinite energy. Furthermore X-waves may be enriched by orthogonality properties as described in Ref. 47.