SIMILARITY ISOMETRIES OF POINT PACKINGS

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Abstract. A linear isometry $R$ of $\mathbb{R}^d$ is called a similarity isometry of a lattice $\Gamma \subseteq \mathbb{R}^d$ if there exists a positive real number $\beta$ such that $\beta R \Gamma$ is a sublattice of (finite index in) $\Gamma$. The set $\beta R \Gamma$ is referred to as a similar sublattice of $\Gamma$. A (crystallographic) point packing generated by a lattice $\Gamma$ is a union of $\Gamma$ with finitely many shifted copies of $\Gamma$. In this study, the notion of similarity isometries is extended to point packings. We provide a characterization for the similarity isometries of point packings and identify the corresponding similar subpackings. Planar examples will be discussed, namely, the $1 \times 2$ rectangular lattice and the hexagonal packing (or honeycomb lattice). Finally, we also consider similarity isometries of point packings about points different from the origin. In particular, similarity isometries of a certain shifted hexagonal packing will be computed and compared with that of the hexagonal packing.

1. Introduction

A linear isometry $R$ of $\mathbb{R}^d$ is called a similarity isometry of a lattice $\Gamma \subseteq \mathbb{R}^d$ if there exists a positive real number $\beta$ such that $\beta R \Gamma$ is a sublattice of (finite index in) $\Gamma$. The set $\beta R \Gamma$ is referred to as a similar sublattice of $\Gamma$. Similarity isometries of lattices may be viewed as a generalization of coincidence isometries of lattices: in the latter, the lattice is only rotated while in the former, a uniform scaling factor is further applied to the lattice. Coincidence site lattices and coincidence isometries have been used by crystallographers to geometrically describe the interfaces where crystals and quasicrystals of different orientations meet (called grain boundaries) [16, 11]. On the other hand, similar sublattices and similarity isometries arise for instance in color symmetries of crystals and quasicrystals [4], and in structures that are self-similar (e.g.: fractals and tilings with singularities). Both similar and coincidence sublattices have been used in the design of multiple description lattice vector quantizers [1, 2, 24]. From a theoretical standpoint, the group of coincidence isometries and the group of similarity isometries are intimately related [17, 18, 25]. These topics belong to a wider class of combinatorial problems for lattices and $\mathbb{Z}$-modules (for a recent survey, see [10]).

A (crystallographic) point packing generated by a lattice $\Gamma$ is a union of $\Gamma$ with finitely many shifted copies of $\Gamma$. Point packings are precisely the locally finite point sets whose translation group forms a lattice of full rank [5]. In the context of the sphere packing problem, they appear as non-lattice periodic packings [13]. In crystallography, they serve as models of ideal crystals, that is, crystals having multiple atoms per primitive unit cell. Examples include the hexagonal packing or honeycomb lattice, diamond lattice (crystal structure of diamond, tin, silicon, and germanium), and hexagonal close packing (crystal structure of quartz).

Point packings appear in the literature under different names. For instance, Dolbilin et al. referred to point packings in [14] as ideal or perfect crystals, and gave minimal sufficient geometric conditions on a discrete subset of $\mathbb{R}^d$ to be an ideal crystal. Shchymura and Yuan used the term discrete lattice-periodic sets for point packings, and they considered the homogeneous and inhomogeneous problem for such sets in [23]. The term

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multilattice has also been used to pertain to a point packing, and arithmetic classification of multilattices have been studied in [22, 20].

In this study, we extend the notion of similarity isometries to point packings. Corollary 3.5 provides a characterization for the similarity isometries of point packings. We also establish that the set of similarity isometries of a point packing forms a monoid. Planar examples will be discussed, namely, the \(1 \times 2\) rectangular lattice and the hexagonal packing viewed as point packings. Finally, we also examine similarity isometries of point packings about points different from the origin. To this end, we consider (linear) similarity isometries of a translated copy of a point packing, or what we call a shifted point packing. This is because rotating a point packing about the point \(-x \in \mathbb{R}^d\) is equivalent to rotating the translate of the point packing by \(x\) about the origin. To illustrate this, we examine the similarity isometries of a certain shifted hexagonal packing and compare them with the similarity isometries of the hexagonal packing.

2. Preliminaries

A discrete subset \(\Gamma\) of \(\mathbb{R}^d\) is a lattice if it is the \(\mathbb{Z}\)-span of \(d\) linearly independent vectors in \(\mathbb{R}^d\) over \(\mathbb{R}\). As a group, \(\Gamma\) is isomorphic to the free abelian group of rank \(d\). A sublattice of \(\Gamma\) is a subgroup of \(\Gamma\) of full rank, that is, with finite index in \(\Gamma\). Geometrically, the index of \(\Gamma'\) in \(\Gamma\) may be viewed as the quotient of the volumes of the fundamental domains of \(\Gamma\) and \(\Gamma'\). The translation group of \(\Gamma\), denoted by \(\text{per}(\Gamma)\), is the set of all shift vectors under which \(\Gamma\) is invariant, that is, \(\text{per}(\Gamma) = \{t \in \mathbb{R}^d : t + \Gamma = \Gamma\}\).

A linear isometry \(R\) of \(\mathbb{R}^d\) is called a similarity isometry of \(\Gamma\) if there exists \(\beta \in \mathbb{R}^+\) for which \(\beta R\Gamma\) is a sublattice of \(\Gamma\). The sublattice \(\beta R\Gamma\) is referred to as a similar sublattice of \(\Gamma\). The set of similarity isometries of \(\Gamma\) forms a group and is denoted by \(\text{OS}(\Gamma)\) [6]. We write \(\text{OS}(\Gamma) = \{R \in \text{O}(d, \mathbb{R}) : \beta R\Gamma \subseteq \Gamma\text{ for some }\beta \in \mathbb{R}^+\}\). Similarly, the set of similarity rotations of \(\Gamma\) is denoted by \(\text{SOS}(\Gamma) = \text{OS}(\Gamma) \cap \text{SO}(d)\) and is a subgroup of \(\text{OS}(\Gamma)\). Existence of similar sublattices as well as properties of similarity isometries for particular lattices have been well-studied (see [12, 8, 19, 7, 9]).

Given a similarity isometry \(R\) of \(\Gamma\), the set of scaling factors \(\text{Scal}_R(\Gamma)\) of \(R\) with respect to \(\Gamma\) is given by \(\text{Scal}_R(\Gamma) = \{\beta \in \mathbb{R} : \beta R\Gamma \subseteq \Gamma\}\). Clearly, \(R \in \text{OS}(\Gamma)\) if and only if \(\text{Scal}_R(\Gamma) \neq \{0\}\). The smallest positive element of \(\text{Scal}_R(\Gamma)\) is called the denominator of \(R\) with respect to \(\Gamma\), denoted by \(\text{den}_R(\Gamma)\). From [10], we have \(\text{Scal}_R(\Gamma) = \{k \cdot \text{den}_R(\Gamma) : k \in \mathbb{Z}\}\) (see also [18] [17]).

We write an affine isometry of \(\mathbb{R}^d\) as \((v, R)\), where \(R \in \text{O}(d)\) and \(v \in \mathbb{R}^d\). The group of affine isometries of \(\mathbb{R}^d\) is denoted by \(\text{Isom}(\mathbb{R}^d)\).

Example 2.1. Let \(\Gamma = \mathbb{Z}[i] = \{a + bi : a, b \in \mathbb{Z}\}\) be the square lattice. Take \(\beta = \sqrt{5}\) and \(R\) to be the rotation about the origin by \(\tan^{-1}(2)\). Then \(R\) is a similarity isometry of \(\Gamma\) with \(\text{den}_R(\Gamma) = \sqrt{5}\) and \(\text{Scal}_R(\Gamma) = \sqrt{5}\mathbb{Z}\). Observe in Figure 1 that \(\beta R\Gamma \subseteq \Gamma\).

A subset \(L\) of \(\mathbb{R}^d\) is said to be a (crystallographic) point packing generated by the lattice \(\Gamma \subseteq \mathbb{R}^d\) if \(L\) is the union of \(\Gamma\) and a finite number of translated copies of \(\Gamma\), that is, \(L = \bigcup_{k \in \mathbb{N}}(x_k + \Gamma)\) where \(m \in \mathbb{N}, x_0 = 0, x_k \in \mathbb{R}^d\) for \(1 \leq k \leq m - 1\), and \(x_{k_1} - x_{k_2} \not\in \Gamma\) whenever \(k_1 \neq k_2\). We refer to the lattice \(\Gamma\) as the generating lattice of \(L\). The shifted lattices \(x_k + \Gamma\) comprising \(L\) are called components of \(L\), and the vector \(x_k\) is called a shift vector of \(L\). The set of shift vectors of \(L\) will be denoted by \(V_L := \{x_0, x_1, \ldots, x_{m-1}\}\). The translation group of \(L\), denoted by \(\text{per}(L)\), is the set of all shift vectors under which \(L\) is invariant, that is, \(\text{per}(L) = \{t \in \mathbb{R}^d : t + L = L\}\).

In general, point packings are not lattices. Note also that for a component \(x_k + \Gamma\) of \(L\), we can take \(x'_k = x_k + \ell\) with \(\ell \in \Gamma\) to be another shift vector for \(x_k + \Gamma\). Thus, without loss of generality, we will choose the shift vectors so that all of them lie within
Figure 1. Let $\Gamma$ be the square lattice $\mathbb{Z}[i]$, $\beta = \sqrt{5}$, and $R$ the rotation about the origin by $\tan^{-1}(2)$. The lattice $\beta R \Gamma$ (blue dots) is a similar sublattice of $\Gamma$ that is obtained by applying $R$ to $\Gamma$ (white dots), followed by a scaling factor of $\beta$. A fundamental domain of $\Gamma$. Nonetheless, even with such a choice of shift vectors, the decomposition of a point packing $L$ into a finite union of shifted lattices is not unique. This is illustrated by the following example.

Example 2.2. Let $L = \Gamma \cup (\frac{1+i}{2} + \Gamma)$ be the point packing generated by the square lattice $\Gamma = \mathbb{Z}[i]$, as shown in Figure 2. As a point packing generated by $\Gamma$, $L$ has two components, namely, $\Gamma$ and $\frac{1+i}{2} + \Gamma$. However, $L$ is also a square lattice obtained by rotating $\Gamma$ about the origin by $\frac{\pi}{4}$ followed by a scaling factor of $\frac{\sqrt{2}}{2}$. Writing $L = \Gamma'$, $L$ may now be viewed as a point packing generated by $\Gamma'$, having only one component. Note that $\Gamma \nsubseteq \Gamma'$ and $\text{per}(\Gamma) \subsetneq \text{per}(\Gamma')$, since the shift vector $\frac{1+i}{2}$ is contained in $\text{per}(\Gamma')$ but not in $\text{per}(\Gamma)$.

Figure 2. Let $L$ be the union of the square lattice $\Gamma = \mathbb{Z}[i]$ (black dots) and its shifted copy $\frac{1+i}{2} + \Gamma$ (gray dots). Here, $L$ is also the square lattice obtained by rotating $\Gamma$ about the origin by $\frac{\pi}{4}$ followed by a scaling factor of $\frac{\sqrt{2}}{2}$.

In general, given a point packing $L$ generated by a lattice $\Gamma$, there exists a generating lattice $\Gamma'$ containing $\Gamma$ such that $\text{per}(\Gamma') = \text{per}(L)$ ([24], an alternative proof is given in [15]). Consequently, when $L$ is generated by such a lattice $\Gamma'$, then $L$ is written with the least number of components. Hereafter, we assume that $L$ is written in this minimal form.
3. Similarity Isometries of Crystallographic Point Packings

We now proceed to generalize the notion of a similarity isometry to point packings. A point packing \( L' \) will be called a subpacking of \( L \) if \( L' \subseteq L \) such that \( L' \) has finite index in \( L \). By index of \( L' \) in \( L \), we mean the ratio of the density of points in \( L \) by the density of points in \( L' \). Here, we assume that the generating lattices of \( L \) and \( L' \) have the same rank, and hence the index of \( L' \) in \( L \) is finite.

A linear isometry \( R \) is a similarity isometry of \( L \) if there exists \( \beta \in \mathbb{R}^+ \) such that \( \beta R \) is a subpacking of \( L \). Here, \( \beta R \) is called a similar subpacking of \( L \). The set of similarity isometries of \( L \) is denoted by \( \text{OS}(L) = \{ R \in O(d, \mathbb{R}) : \beta R \subseteq L \text{ for some } \beta \in \mathbb{R}^+ \} \). The set of similarity rotations of \( L \) will be denoted by \( \text{SOS}(L) := \text{OS}(L) \cap SO(d) \). Given a similarity isometry \( R \) of a point packing \( L \), we define the set of scaling factors \( \text{Scal}_L(R) \) of \( R \) with respect to \( L \) by \( \text{Scal}_L(R) = \{ \beta \in \mathbb{R} : \beta RL \subseteq L \} \). The smallest positive element of \( \text{Scal}_L(R) \) will be called the denominator of \( R \) with respect to \( L \). Observe that 0 is always in \( \text{Scal}_L(R) \), and we have \( R \in \text{OS}(L) \) if and only if \( \text{Scal}_L(R) \neq \{0\} \).

We start by examining when a component of the similar subpacking \( \beta RL \) is a subset of at least one component of \( L \). The following theorem states that \( \beta RL \) is a similar subpacking of \( L \) if and only if every component of \( \beta RL \) is a subset of exactly one component of \( L \).

**Theorem 3.1.** Let \( L = \bigcup_{k=0}^{m-1} (x_k + \Gamma) \) be a point packing generated by a lattice \( \Gamma \subseteq \mathbb{R}^d \), \( R \in O(d, \mathbb{R}) \), and \( \beta \in \mathbb{R}^+ \). Then \( \beta RL \subseteq L \) if and only if for every \( k \in \{0, 1, \ldots, m-1\} \), there is a unique \( j \in \{0, 1, \ldots, m-1\} \) such that \( \beta R(x_k + \Gamma) \subseteq x_j + \Gamma \).

**Proof.** Suppose that for every \( k \in \{0, 1, \ldots, m-1\} \), there is a unique \( j \in \{0, 1, \ldots, m-1\} \) such that \( \beta R(x_k + \Gamma) \subseteq x_j + \Gamma \). Then \( \bigcup_{k=0}^{m-1} \beta R(x_k + \Gamma) \subseteq \bigcup_{j \in \Lambda} (x_j + \Gamma) \), for some \( \Lambda \subseteq \{0, 1, \ldots, m-1\} \). Thus, \( \beta RL \subseteq L \).

Conversely, suppose that \( \beta RL \subseteq L \), and let \( k \in \{0, 1, \ldots, m-1\} \). Since \( \beta RL \subseteq L \), \( \beta R(x_k + \Gamma) \) intersects at least one component of \( L \). Suppose \( \beta R(x_k + \Gamma) \) intersects two distinct components of \( L \), say, \( (x_{j_1} + \Gamma) \) and \( (x_{j_2} + \Gamma) \). Fix \( j \in \{j_1, j_2\} \). Since \( \beta R(x_k + \Gamma) \cap (x_j + \Gamma) = x_j + [(\beta Rx_k - x_j) R \Gamma] \cap \Gamma \) is non-empty, there exist \( \ell, \ell' \in \Gamma \) such that \( \beta Rx_k - x_j = \ell - \beta R\ell' \). Hence, we have \( (\beta Rx_k - x_j, \beta R) \Gamma \cap \Gamma = (\ell, \beta R) \Gamma \cap \Gamma = \ell + (\Gamma \cap \beta R) \).

Note that \( [\Gamma : \Gamma \cap \beta R] = [\Gamma : \ell + (\Gamma \cap \beta R)] \) since a translation does not change the volume of the fundamental domain of a lattice. Moreover, \( \beta RL \) is of finite index in \( L \), and so the index of \( (\beta Rx_k - x_j, \beta R) \Gamma \cap \Gamma \) in \( \Gamma \) is finite. Thus, the index \( [\Gamma : \Gamma \cap \beta R] \) is finite. We conclude that \( R \in \text{OS}(\Gamma) \) with \( \beta \in \text{Scal}_L(R) \), implying that \( \beta R \Gamma \subseteq \Gamma \). Consequently, \( \beta \in \text{Scal}_L(R) \) satisfies \( \beta Rx_k - x_j \in \Gamma + \beta R \Gamma = \Gamma \).

Since \( \beta Rx_k - x_j \in \Gamma \) for each \( j \in \{j_1, j_2\} \), it follows that \( x_{j_2} - x_{j_1} \in \Gamma \) which is a contradiction, since the components of \( L \) are distinct cosets of \( \Gamma \). Hence, \( \beta R(x_k + \Gamma) \) is a subset of exactly one component of \( L \). \( \Box \)

Theorem 3.1 states that each component of the similar subpacking \( \beta RL \) is contained in exactly one component of \( L \). However, a component of \( L \) may contain more than one component of \( \beta RL \). We see this in the following example.

**Example 3.2.** Let \( L = \Gamma \cup (\frac{1}{2} + \Gamma) \) be the \( 1 \times 2 \) rectangular lattice viewed as a point packing that is generated by the square lattice \( \Gamma = \mathbb{Z}[i] \). Take \( \beta = 2\sqrt{2} \) and \( R \) to be the rotation about the origin by \( \frac{\pi}{4} \) in the counterclockwise direction. Then \( R \) is a similarity isometry of \( L \) with \( \beta \in \text{Scal}_L(R) \), and so \( \beta RL \) is a similar subpacking of \( L \). Figure 3 shows that both components of \( \beta RL \), namely \( \beta R \Gamma \) and \( \beta R(\frac{1}{2} + \Gamma) \), are contained in \( \Gamma \).

We now investigate when a component of \( \beta RL \) is a subset of some component of \( L \). These conditions are given by Propositions 3.4. First, we have the following lemma.
Corollary 3.5. isometries of a point packing.β OS(Γ) and Γ ⊆ provides equivalent conditions for a component of the similar subpacking βRL

By Corollary 3.5, if in Proposition 3.4. The set of all such pairs (x) determined by identifying the pairs (R isometry L subset of some component of x have

Lemma 3.3. Let Γ and Γ' be lattices in R^d, and x_1, x_2 ∈ R^d. Then x_1 + Γ ⊆ x_2 + Γ' if and only if Γ ⊆ Γ' and x_1 - x_2 ∈ Γ'.

Proof. Let Γ ⊆ Γ' and x_1 - x_2 ∈ Γ'. Then x_1 = x_2 + ℓ' for some ℓ' ∈ Γ'. Given ℓ ∈ Γ, we have x_1 + ℓ = x_2 + (ℓ' + ℓ) ∈ x_2 + Γ'. Hence, x_1 + Γ ⊆ x_2 + Γ'. On the other hand, suppose that x_1 + Γ ⊆ x_2 + Γ'. It is easy to see that x_1 - x_2 ∈ Γ' since Γ contains 0. To show that Γ ⊆ Γ', let ℓ ∈ Γ. Since x_1 + ℓ ∈ x_2 + Γ', it follows that ℓ ∈ -(x_1 - x_2) + Γ' = Γ'.

The following proposition is an immediate consequence of Lemma 3.3. Proposition 3.4 provides equivalent conditions for a component of the similar subpacking βRL to be a subset of some component of L.

Proposition 3.4. Let R ∈ O(d, R), x_j, x_k ∈ R^d, and β ∈ R^+. Then βR(x_k + Γ) ⊆ x_j + Γ if and only if R ∈ OS(Γ) such that β ∈ Scal_Γ(R) and βRx_k - x_j ∈ Γ.

Combining Theorem 3.1 and Proposition 3.4 we obtain a characterization of similarity isometries of a point packing.

Corollary 3.5. Let L = ∪_{k=0}^{m-1}(x_k + Γ) be a point packing generated by a lattice Γ ⊆ R^d, R ∈ O(d, R), and β ∈ R^+. Then R ∈ OS(L) and β ∈ Scal_Γ(R) if and only if R ∈ OS(Γ) and β ∈ Scal_Γ(R) such that for every k ∈ {0, 1, ..., m - 1}, there is a unique j ∈ {0, 1, ..., m - 1} for which βRx_k - x_j ∈ Γ.

Corollary 3.5 implies that OS(L) ⊆ OS(Γ) and Scal_Γ(R) ⊆ Scal_Γ(R) for any similarity isometry R ∈ OS(L). Moreover, the components of the similar subpacking βRL are determined by identifying the pairs (x_k, x_j) of shift vectors of L satisfying the conditions in Proposition 3.4. The set of all such pairs (x_k, x_j) will be denoted by τ(βRL), and is given by

τ(βRL) = \{ (x_k, x_j) ∈ V_L × V_L : βR(x_k + Γ) ⊆ x_j + Γ \}

= \{ (x_k, x_j) ∈ V_L × V_L : βRx_k - x_j ∈ Γ \}.

By Corollary 3.5 if βRL = ∪_{k=0}^{m-1} βR(x_k + Γ) is a similar subpacking of L = ∪_{k=0}^{m-1} (x_k + Γ), then |τ(βRL)| = m.

Finally, let us examine the set of scaling factors Scal_Γ(R) of a similarity isometry R of L. Note that in general, negative scaling factors may exist, that is, it is possible that
\( \beta RL \) is a similar subpacking of \( L \) for some \( \beta < 0 \). In fact, for a lattice \( \Gamma \), if \( \beta \in \mathbb{R}^+ \) such that \( \beta \Gamma \subseteq \Gamma \), then \( -\beta \Gamma \subseteq \Gamma \). If a point packing \( L \) is invariant under inversion, that is, \( -L = L \), then for all \( R \in \text{OS}(L) \) such that \( 0 < \beta \in \text{Scal}_L(R) \), we have \( -\beta RL \subseteq L \).

Recall that for any similarity isometry \( R \) of a lattice \( \Gamma \), \( \text{Scal}_\Gamma(R) = \text{den}_\Gamma(R)\mathbb{Z} \), where \( \text{den}_\Gamma(R) \) is the smallest positive element of \( \text{Scal}_\Gamma(R) \). Hence, for a point packing \( L \) generated by a lattice \( \Gamma \subseteq \mathbb{R}^d \), we have

\[
\text{Scal}_L(R) \subseteq \text{Scal}_\Gamma(R) = \text{den}_\Gamma(R)\mathbb{Z}.
\]

In words, if \( R \) is a similarity isometry of a point packing \( L \), then any element of \( \text{Scal}_L(R) \) may be written as an integral multiple of \( \text{den}_\Gamma(R) \). This fact, together with Corollary 3.5, allows us to compute \( \text{Scal}_L(R) \) for a given similarity isometry \( R \in \text{OS}(L) \).

We now look at the algebraic structure of \( \text{OS}(L) \). The identity isometry \( R = 1 \) is clearly in \( \text{OS}(L) \). The following result shows that \( \text{OS}(L) \) is closed under composition.

**Proposition 3.6.** Let \( L = \cup_{k=0}^{m-1}(x_k + \Gamma) \) be a point packing generated by a lattice \( \Gamma \subseteq \mathbb{R}^d \). If \( R_1, R_2 \in \text{OS}(L) \), then \( R_2R_1 \in \text{OS}(L) \).

**Proof.** Take \( x_i \in V_L \). By Corollary 3.5, \( R_1 \in \text{OS}(\Gamma) \) and there exist \( \alpha \in \text{Scal}_\Gamma(R_1) \) and \( x_j \in V_L \) such that \( \alpha R_1x_i - x_j \in \Gamma \). Similarly, \( R_2 \in \text{OS}(\Gamma) \) and there exist \( \beta \in \text{Scal}_\Gamma(R_2) \) and \( x_k \in V_L \) such that \( \beta R_2x_j - x_k \in \Gamma \). Note that \( R_2R_1 \) is an element of the group \( \text{OS}(\Gamma) \), and \( \alpha \beta \in \text{Scal}_\Gamma(R_2R_1) \). In addition, \( \alpha \beta R_2R_1x_i - \beta R_2x_j \in \beta R_2\Gamma \subseteq \Gamma \), which implies that \( \alpha \beta R_2R_1x_i - x_k \in \Gamma \). It now follows from Corollary 3.5 that \( R_2R_1 \in \text{OS}(L) \). 

Proposition 3.6 assures us that \( \text{OS}(L) \) is at least a monoid. The closure of \( \text{OS}(L) \) under inverses, and hence whether \( \text{OS}(L) \) forms a group, remains an open problem. Nonetheless, there are many examples of planar point packings \( L \) for which \( \text{OS}(L) \) is a group, and we discuss them in the next section.

### 4. Planar Examples

We now apply the results in the previous section to planar point packings. In particular, we compute \( \text{SOS}(L) \) and \( \text{OS}(L) \) for certain point packings \( L \) generated by the square lattice \( \mathbb{Z}[i] \) or hexagonal lattice \( \mathbb{Z}[\omega] \) with \( \omega = e^{2\pi i/3} \), and show that they are groups. Afterwards, we investigate two concrete examples: the \( 1 \times 2 \) rectangular lattice viewed as a point packing generated by the square lattice, and the hexagonal packing viewed as a point packing generated by the hexagonal lattice.

It is known that any similarity rotation \( R \) of \( \mathbb{Z}[i] \) (or \( \mathbb{Z}[\omega] \)) corresponds to multiplication by a non-zero element of the form \( z/|z| \), where \( z \in \mathbb{Z}[i] \) (or \( \mathbb{Z}[\omega] \)). In addition, \( \text{den}_\Gamma(R) = |z| \). Meanwhile, any reflection \( T \) can be expressed as a product \( T = RT_r \), where \( R \) is a rotation, and \( T_r \) denotes the reflection about the real axis and corresponds to complex conjugation. Because of this, similarity rotations will be computed first in each example, followed by similarity reflections. Moreover, a reflection \( T \) will be represented in Tables 1, 3, and 5 by its rotational part \( R \) in the representation \( T = RT_r \).

Let us now proceed to compute for \( \text{SOS}(L) \) and \( \text{OS}(L) \) of point packings \( L \) that are generated by a square or a hexagonal lattice, all of whose shift vectors have rational components.

**Proposition 4.1.** Let \( L = \cup_{k=0}^{m-1}(x_k + \Gamma) \) be a point packing generated by the lattice \( \Gamma = \mathbb{Z}[i] \) (or \( \mathbb{Z}[\omega] \) with \( \omega = e^{2\pi i/3} \)). If \( x_k \in \mathbb{Q}(i) \) (or \( \mathbb{Q}(\omega) \)) for all \( k \in \{0, 1, \ldots, m-1\} \), then \( \text{SOS}(L) = \text{SOS}(\Gamma) \) and \( \text{OS}(L) = \text{OS}(\Gamma) \).

**Proof.** Let \( \Gamma = \mathbb{Z}[i] \) and \( R \in \text{SOS}(\Gamma) \) corresponding to multiplication by \( \frac{z}{|z|} \), where \( 0 \neq z \in \mathbb{Z}[i] \). For each \( x_k \in V_L \), write \( x_k = \frac{y_k}{t_k} \) for some \( y_k \in \mathbb{Z}[i] \) and \( 0 \neq t_k \in \mathbb{Z} \). Take
\(\beta = \text{lcm}(r_1, \ldots, r_{m-1}) \text{den}_T(R) \subseteq \text{Scal}_T(R)\). It follows that for all \(x_j \in V_L\), we have
\[
\beta R x_j - x_0 = \text{lcm}(r_1, \ldots, r_{m-1}) \frac{z}{|z|} \frac{y_j}{r_j} - 0 = q_j z y_j \in \Gamma,
\]
for some \(q_j \in \mathbb{Z}\). Hence, \(R \in \text{SOS}(L)\) and therefore, \(\text{SOS}(L) = \text{SOS}(\Gamma)\).

For reflections \(T = R T_r \in \text{OS}(L) \setminus \text{SOS}(L)\), note that \(\frac{\tau}{\tau} = \frac{\tau}{\tau}\). Using the same scaling factor \(\beta\), we obtain \(\beta T x_j - x_0 = \beta R T_j x_j - x_0 = q_j z \frac{y_j}{r_j} \in \Gamma\), for some \(q_j \in \mathbb{Z}\). This shows that \(T \in \text{OS}(L)\) and therefore, \(\text{OS}(L) = \text{OS}(\Gamma)\). The proof for \(\Gamma = \mathbb{Z} [\omega]\) is similar. \(\square\)

Hence, we obtain that \(\text{OS}(L)\) and \(\text{SOS}(L)\) are groups for a point packing \(L\) generated by a square or a hexagonal lattice, all of whose shift vectors have rational components.

We now look at two specific planar examples: the \(1 \times 2\) rectangular lattice and the hexagonal packing.

### 4.1. The \(1 \times 2\) rectangular lattice

Let \(L = \Gamma \cup (\frac{1}{2} + \Gamma)\) be the \(1 \times 2\) rectangular lattice viewed as a point packing that is generated by the square lattice \(\Gamma = \mathbb{Z} [i]\). Table 1 summarizes the results when Corollary 3.5 is applied. In Table 1, we give \(\text{Scal}_L(R)\) and \(\tau(\beta RL)\) for each \(R \in \text{OS}(\Gamma)\) corresponding to multiplication by \((a + bi)/|a + bi|\) and depending on the parity of \(a\) and \(b\). Since \(T_r L = L\), it follows that for any similarity reflection \(T \in \text{OS}(L) \setminus \text{SOS}(L)\) and \(\beta \in \text{Scal}_L(T)\), \(\beta TL = \beta RT_L L = \beta RL\). Hence, the results in Table 1 hold for any similarity isometry in \(\text{OS}(L)\), where a reflection \(T\) is represented in the table by its rotational part \(R\) in the representation \(T = R T_r\).

Table 1 implies that \(\text{OS}(L) = \text{OS}(\Gamma) = \{z/|z| : 0 \neq z \in \mathbb{Z} [i]\}\). This result also follows from Proposition 4.1 since the shift vectors 0 and \(\frac{1}{2}\) of \(L\) are rational.

The group \(\text{OS}(L)\) may also be obtained by applying \([17, \text{Corollary 3.7}]\) to \(L\), now viewed as an ordinary lattice. Nonetheless, viewing \(L\) here as a point packing gives us a more concrete approach that is easy to describe geometrically. It also allows us to extract exact information on the components of the similar subpacking \(\beta RL\), or in this case, the similar sublattice of the \(1 \times 2\) rectangular lattice \(L\).

| \(R : (a + bi)/|a + bi|\) | \(\text{Scal}_L(R)\) | \(\tau(\beta RL)\) |
|-----------------------------|-----------------|-----------------|
| \((a, b) \equiv (1, 0) (\text{mod } 2)\) | \(\text{den}_T(R) 2\mathbb{Z}\) | \((0, 0), (\frac{1}{2}, 0)\) |
| \((a, b) \equiv (0, 1), (1, 1) (\text{mod } 2)\) | \(\text{den}_T(R) (1 + 2\mathbb{Z})\) | \((0, 0), (\frac{1}{2}, \frac{1}{2})\) |

Table 1. Let \(L\) be the \(1 \times 2\) rectangular lattice viewed as a point packing generated by the square lattice \(\Gamma = \mathbb{Z} [i]\). The sets \(\text{Scal}_L(R)\) and \(\tau(\beta RL)\) are given for all similarity isometries \(R\) in \(\text{OS}(L)\). Here, reflections \(T = R T_r\) are represented by their rotational part \(R\).

**Example 4.2.** Take \(\beta = \sqrt{5}\) and \(R\) to be the rotation about the origin by \(\tan^{-1}(2)\) which corresponds to multiplication by \(\frac{1 + 2\sqrt{5}}{\sqrt{5}}\). Note that \(\text{den}_T(R) = \sqrt{5}\), and thus according to Table 1, \(R\) is a similarity isometry of \(L\) with \(\text{Scal}_L(R) = \sqrt{5}(2\mathbb{Z}) \cup \sqrt{5}(1 + 2\mathbb{Z}) = \sqrt{5}\mathbb{Z}\) and \(\text{den}_L(R) = \sqrt{5}\). Since \(\beta = \sqrt{5}\), the second row of Table 1 tells us that \(\beta R \Gamma \subseteq \Gamma\) and \(\beta R (\frac{1}{2} + \Gamma) \subseteq \frac{1}{2} + \Gamma\), as can be verified in Figure 4.

### 4.2. The hexagonal packing

Let \(L = \Gamma \cup (\frac{2}{3} \omega + \Gamma)\) be the hexagonal packing viewed as a point packing generated by the hexagonal lattice \(\Gamma = \mathbb{Z} [\omega]\) with \(\omega = e^{2\pi i/3}\). Note that \(L\) is not a lattice. Tables 2 and 3 summarize the similarity isometries of \(L\) obtained by applying Corollary 3.5. The results in Table 2 hold for any similarity rotation \(R \in \text{SOS}(L)\), while those in Table 3 hold for any similarity reflection \(T \in \text{OS}(L) \setminus \text{SOS}(L)\).
The two tables together imply that \( \text{OS}(L) = \text{OS}(\Gamma) = \{ z/|z| : 0 \neq z \in \mathbb{Z}[\omega] \} \), and thus, \( \text{OS}(L) \) is a group. This result is expected from Proposition 4.1, since the shift vectors 0 and \( \frac{2+\omega}{3} \) of \( L \) are both in \( \mathbb{Q}(\omega) \).

| \( R : (a+b\omega)/|a+b\omega| \) | \( \text{Scal}_L(R) \) | \( \tau(\beta RL) \) |
|----------------|----------------|----------------|
| \( a+b \equiv 1 (\text{mod } 3) \) | \( \text{den}_\Gamma(R)3\mathbb{Z} \) | \( \{(0,0),(\frac{2+\omega}{3},0)\} \) |
| | \( \text{den}_\Gamma(R)(1+3\mathbb{Z}) \) | \( \{(0,0),(\frac{2+\omega}{3},\frac{2+\omega}{3})\} \) |
| \( a+b \equiv 2 (\text{mod } 3) \) | \( \text{den}_\Gamma(R)3\mathbb{Z} \) | \( \{(0,0),(\frac{2+\omega}{3},0)\} \) |
| | \( \text{den}_\Gamma(R)(2+3\mathbb{Z}) \) | \( \{(0,0),(\frac{2+\omega}{3},\frac{2+\omega}{3})\} \) |
| \( a+b \equiv 0 (\text{mod } 3) \) | \( \text{den}_\Gamma(R)\mathbb{Z} \) | \( \{(0,0),(\frac{2+\omega}{3},0)\} \) |

Table 2. Let \( L \) be the hexagonal packing viewed as a point packing generated by the hexagonal lattice \( \Gamma = \mathbb{Z}[\omega] \), where \( \omega = e^{2\pi i/3} \). The sets \( \text{Scal}_L(R) \) and \( \tau(\beta RL) \) are given for all similarity rotations \( R \) in \( \text{SOS}(L) \).

| \( T = RT_r \) where \( R : (a+b\omega)/|a+b\omega| \) | \( \text{Scal}_L(T) \) | \( \tau(\beta TL) \) |
|----------------|----------------|----------------|
| \( a+b \equiv 1 (\text{mod } 3) \) | \( \text{den}_\Gamma(R)3\mathbb{Z} \) | \( \{(0,0),(\frac{2+\omega}{3},0)\} \) |
| | \( \text{den}_\Gamma(R)(2+3\mathbb{Z}) \) | \( \{(0,0),(\frac{2+\omega}{3},\frac{2+\omega}{3})\} \) |
| \( a+b \equiv 2 (\text{mod } 3) \) | \( \text{den}_\Gamma(R)3\mathbb{Z} \) | \( \{(0,0),(\frac{2+\omega}{3},0)\} \) |
| | \( \text{den}_\Gamma(R)(1+3\mathbb{Z}) \) | \( \{(0,0),(\frac{2+\omega}{3},\frac{2+\omega}{3})\} \) |
| \( a+b \equiv 0 (\text{mod } 3) \) | \( \text{den}_\Gamma(R)\mathbb{Z} \) | \( \{(0,0),(\frac{2+\omega}{3},0)\} \) |

Table 3. Let \( L \) be the hexagonal packing viewed as a point packing generated by the hexagonal lattice \( \Gamma = \mathbb{Z}[\omega] \), where \( \omega = e^{2\pi i/3} \). The sets \( \text{Scal}_L(T) \) and \( \tau(\beta TL) \) are given for all similarity reflections \( T \) in \( \text{OS}(L) \setminus \text{SOS}(L) \).

Example 4.3. Take \( \beta = 2 \) and \( R \) to be the rotation about the origin by \( \pi/3 \) which corresponds to multiplication by \( 1 + \omega \). Here, \( \text{den}_\Gamma(R) = 1 \), and thus according to Table 2 \( R \) is a similarity isometry of \( L \) with \( \text{Scal}_L(R) = 3\mathbb{Z} \cup (2+3\mathbb{Z}) \) and \( \text{den}_L(R) = 2 \).
Since $\beta = 2$, the fourth row of Table 2 states that $\beta R \Gamma \subseteq \Gamma$ and $\beta R (\frac{2+\omega}{3} + \Gamma) \subseteq \frac{2+\omega}{3} + \Gamma$, as can be verified in Figure 5.

Figure 5. Let $L$ be the hexagonal packing viewed as the union of the hexagonal lattice $\Gamma = \mathbb{Z}[\omega]$ (black dots), where $\omega = e^{2\pi i/3}$, and its shifted copy $\frac{2+\omega}{3} + \Gamma$ (gray dots). If $\beta = 2$ and $R$ corresponds to multiplication by $1 + \omega$ (rotation about the origin by $\frac{\pi}{3}$), then the set $\beta R L$, consisting of the union of $\beta R \Gamma$ (blue dots) and $\beta R (\frac{2+\omega}{3} + \Gamma)$ (yellow dots), is a similar subpacking of $L$.

5. Shifted Point Packings

Let us now consider rotations about points different from the origin. Note that rotating a point packing $L \subseteq \mathbb{R}^d$ about a point $-x \in \mathbb{R}^d$ is equivalent to rotating $x + L$ about the origin. For instance, consider the hexagonal packing $L = \Gamma \cup (x + \Gamma)$, where $\Gamma = \mathbb{Z}[\omega]$ is the hexagonal lattice and $x = \frac{2+\omega}{3}$. Observe that applying linear isometries to $L$ fixes the origin which is located at a vertex of a hexagon. To rotate about a center of a hexagon (a point of maximal symmetry of $L$), say, $-x$, one may first translate all points of $L$ by the vector $x$ and afterwards rotate about the origin. This observation leads us to investigate translates of point packings, which will be referred to as shifted point packings, and their similarity isometries. The same idea was employed in [3] to find the coincidence isometries of point packings about points different from the origin.

If $L = \bigcup_{k=0}^{m-1} (x_k + \Gamma)$ is a point packing generated by a lattice $\Gamma$, and $x \in \mathbb{R}^d$, then the shifted point packing $x + L$ is given by $x + L = \bigcup_{k=0}^{m-1} (x + x_k + \Gamma)$. Observe that $x + L$ is still a union of shifted copies of $\Gamma$ but having shift vectors $x + x_k$, that is, shift vectors $x_k$ of $L$ translated by $x$. Hence, the usual definitions and notations regarding similarity isometries of point packings will be applied to shifted point packings. Moreover, the results on similarity isometries of point packings in Section 3 are applicable to shifted point packings.

In particular, analogous results to Corollary 3.5 and (1) allow us to determine the set of similarity isometries $\text{OS}(x + L)$ and compute $\text{Scal}_{x+L}(R)$ for a given similarity isometry $R \in \text{OS}(x + L)$. In addition, components of the similar subpacking $\beta R (x + L)$ can be determined by identifying the pairs $(x + x_k, x + x_j)$ in

$$\tau(\beta R(x + L)) = \{(x + x_k, x + x_j) \in V_{x+L} \times V_{x+L} : \beta R(x + x_k + \Gamma) \subseteq x + x_j + \Gamma\}.$$

We illustrate these results by looking at the shifted hexagonal packing $x + L = (\frac{2+\omega}{3} + \Gamma) \cup (\frac{4+2\omega}{3} + \Gamma)$, where $L = \Gamma \cup (\frac{2+\omega}{3} + \Gamma)$ is the hexagonal packing, $\Gamma = \mathbb{Z}[\omega]$, with $\omega = e^{2\pi i/3}$, is the hexagonal lattice, and $x = \frac{2+\omega}{3}$. Tables 4 and 5 were obtained by applying analogous
results of Corollary 3.5 and 1. The results in Table 4 hold for any similarity rotation $R \in \text{SOS}(x + L)$, while those in Table 5 hold for any similarity reflection $T \in \text{OS}(x + L) \setminus \text{SOS}(x + L)$. The two tables together imply that $\text{OS}(x + L) \subseteq \text{OS}(L) = \text{OS}(\Gamma)$. One may verify that $\text{OS}(x + L)$ forms a group, and hence, is a proper subgroup of $\text{OS}(\Gamma)$.

| $R : (a + b\omega)/|a + b\omega|$ | $\text{Scal}_{x+L}(R)$ | $\tau(\beta R(x + L))$ |
|-----------------------------------|----------------------|--------------------------|
| $a + b \equiv 1 \pmod{3}$         | den$_{R}(R)(1 + 3\mathbb{Z})$ | $\{(\frac{2 + \omega}{3}, \frac{2 + 2\omega}{3}), (\frac{2 + 2\omega}{3}, \frac{2 + \omega}{3})\}$ |
|                                   | den$_{R}(R)(2 + 3\mathbb{Z})$ | $\{(\frac{2 + \omega}{3}, \frac{2 + 2\omega}{3}), (\frac{2 + 2\omega}{3}, \frac{2 + \omega}{3})\}$ |
| $a + b \equiv 2 \pmod{3}$         | den$_{R}(R)(1 + 3\mathbb{Z})$ | $\{(\frac{2 + \omega}{3}, \frac{2 + 2\omega}{3}), (\frac{2 + 2\omega}{3}, \frac{2 + \omega}{3})\}$ |
|                                   | den$_{R}(R)(2 + 3\mathbb{Z})$ | $\{(\frac{2 + \omega}{3}, \frac{2 + 2\omega}{3}), (\frac{2 + 2\omega}{3}, \frac{2 + \omega}{3})\}$ |

Table 4. Let $x + L$ be the shifted hexagonal packing where $L$ is the hexagonal packing generated by the hexagonal lattice $\Gamma = \mathbb{Z}[\omega]$ with $\omega = e^{2\pi i/3}$, and $x = \frac{2 + \omega}{3}$. The sets $\text{Scal}_{x+L}(R)$ and $\tau(\beta R(x + L))$ are given for all similarity rotations $R$ in $\text{SOS}(x + L)$.

| $T = RT_r$ with $R : (a + b\omega)/|a + b\omega|$ | $\text{Scal}_{x+L}(T)$ | $\tau(\beta T(x + L))$ |
|---------------------------------------------------|----------------------|--------------------------|
| $a + b \equiv 1 \pmod{3}$                          | den$_{R}(R)(1 + 3\mathbb{Z})$ | $\{(\frac{2 + \omega}{3}, \frac{2 + 2\omega}{3}), (\frac{2 + 2\omega}{3}, \frac{2 + \omega}{3})\}$ |
|                                                   | den$_{R}(R)(2 + 3\mathbb{Z})$ | $\{(\frac{2 + \omega}{3}, \frac{2 + 2\omega}{3}), (\frac{2 + 2\omega}{3}, \frac{2 + \omega}{3})\}$ |
| $a + b \equiv 2 \pmod{3}$                          | den$_{R}(R)(1 + 3\mathbb{Z})$ | $\{(\frac{2 + \omega}{3}, \frac{2 + 2\omega}{3}), (\frac{2 + 2\omega}{3}, \frac{2 + \omega}{3})\}$ |
|                                                   | den$_{R}(R)(2 + 3\mathbb{Z})$ | $\{(\frac{2 + \omega}{3}, \frac{2 + 2\omega}{3}), (\frac{2 + 2\omega}{3}, \frac{2 + \omega}{3})\}$ |

Table 5. Let $x + L$ be the shifted hexagonal packing where $L$ is the hexagonal packing generated by the hexagonal lattice $\Gamma = \mathbb{Z}[\omega]$ with $\omega = e^{2\pi i/3}$, and $x = \frac{2 + \omega}{3}$. The sets $\text{Scal}_{x+L}(T)$ and $\tau(\beta T(x + L))$ are given for all similarity reflections $T$ in $\text{OS}(x + L) \setminus \text{SOS}(x + L)$.

**Example 5.1.** Take $\beta = 1$ and $R$ to be the rotation about the origin by $\frac{\pi}{3}$ which corresponds to multiplication by $1 + \omega$. Note that den$_{R}(R) = 1$, and thus according to Table 4, $R$ is a similarity isometry of $x + L$ with $\text{Scal}_{x+L}(R) = (1 + 3\mathbb{Z}) \cup (2 + 3\mathbb{Z})$ and den$_{x+L}(R) = 1$. Since $\beta = 1$, the third row of Table 4 states that $\beta R(\frac{2 + \omega}{3} + \Gamma) \subseteq \text{SOS}(x + L)$ and $\beta R(\frac{2 + 2\omega}{3} + \Gamma) \subseteq \text{SOS}(x + L)$, as can be verified in Figure 6. In fact, Figure 6 shows that $\beta R(\frac{2 + \omega}{3} + L) = \frac{2 + \omega}{3} + L$, since $\beta R(\frac{2 + \omega}{3} + \Gamma) = \frac{2 + 2\omega}{3} + \Gamma$ and $\beta R(\frac{4 + 2\omega}{3} + \Gamma) = \frac{2 + \omega}{3} + \Gamma$. This is expected since the hexagonal packing is symmetric with respect to the $\frac{\pi}{3}$ rotation about the center of the hexagons.

We deduce from Tables 4 and 5 that $\text{OS}(x + L)$ is a proper subset of $\text{OS}(L) = \text{OS}(\Gamma)$. Consequently, there are more similarity isometries of $L$ when one rotates $L$ about a vertex of a hexagon than when one rotates $L$ about a center of a hexagon. This is a surprising result, since the hexagonal packing has more symmetries about the center of a hexagon than about a vertex of a hexagon.

Moreover, observe that if $R \in \text{OS}(x + L)$, then the sets of scaling factors $\text{Scal}_{x+L}(R)$ and $\text{Scal}_{L}(R)$ vary. For instance, consider Figures 5 and 6. Even if the rotation $R$ about the origin by $\frac{\pi}{3}$ is a similarity isometry of both $x + L$ and $L$, the sets of scaling factors obtained are different: $\text{Scal}_{L}(R) = 3\mathbb{Z} \cup (2 + 3\mathbb{Z})$ and $\text{Scal}_{x+L}(R) = (1 + 3\mathbb{Z}) \cup (2 + 3\mathbb{Z})$. Hence, changing the rotation point (from a vertex to a center of a hexagon) affects not only the set of similarity isometries, but also the set of scaling factors and the set $\tau$ that
Figure 6. Let $x + L$ be the shifted hexagonal packing given by the union of $\frac{2 + \omega}{3} + \Gamma$ (black outline) and $\frac{4 + 2\omega}{3} + \Gamma$ (gray outline), where $x = \frac{2 + \omega}{3}$, $L$ is the hexagonal packing $\Gamma \cup \left( \frac{2 + \omega}{3} + \Gamma \right)$, and $\Gamma$ is the hexagonal lattice $\mathbb{Z}[\omega]$ with $\omega = e^{2\pi i/3}$. If $\beta = 1$ and $R$ corresponds to multiplication by $1 + \omega$ (rotation about the origin by $\frac{\pi}{3}$), then the set $\beta R(x + L)$, consisting of the union of $\beta R(\frac{2 + \omega}{3} + \Gamma)$ (blue dots) and $\beta R(\frac{4 + 2\omega}{3} + \Gamma)$ (yellow dots), is a similar subpacking of $x + L$. Here, we see that $\beta R(x + L) = x + L$.

describes each component of the similar subpacking as a subset of some component of the point packing.

6. Outlook

The notions of a similarity isometry of a lattice and a similar sublattice were extended to (crystallographic) point packings. Equivalent conditions were obtained to characterize a similarity isometry of a point packing and its set of scaling factors. The approach employed here was to examine the components of the similar subpacking component-wise by associating each component of the similar subpacking to a unique component of the point packing that contains it. These results were applied to planar point packings whose generating lattice is the square lattice $\mathbb{Z}[i]$ or the hexagonal lattice $\mathbb{Z}[\omega]$.

The set $\text{OS}(L)$ of similarity isometries of a point packing $L$ is a subset of the group of similarity isometries of the generating lattice $\Gamma$ of $L$. Moreover, the set $\text{Scal}_L(R)$ of scaling factors with respect to $L$ is a subset of the set of scaling factors with respect to $\Gamma$. In general, $\text{OS}(L)$ is a monoid. Although the question of whether $\text{OS}(L)$ forms a group remains open, we have shown that $\text{OS}(L)$ is a group for point packings generated by a square or a hexagonal lattice, all of whose shift vectors have rational components. An example where $\text{OS}(L)$ is not a group may possibly be found in dimensions $d > 2$.

The framework and the methods presented in this study offer a more concrete approach in studying similarity isometries of ideal crystals that is easy to describe geometrically. Since point packings serve as models for crystals having multiple atoms per primitive unit cell, it would be interesting to investigate similarity isometries of other point packings, particularly three-dimensional ones that correspond to models of actual ideal crystals.

Finally, the notion of a shifted point packing was introduced in the interest of examining similarity isometries of a point packing about points different from the origin. This is because rotating a point packing about a point $-x$ is equivalent to rotating the translate of the point packing by $x$ about the origin. In particular, we investigate a shifted hexagonal packing and its similarity isometries that correspond to similarity isometries of the hexagonal packing about a center of a hexagon. The shifted hexagonal packing has less similarity isometries than the hexagonal packing. This is unexpected since the hexagonal packing has more symmetries about the center of a hexagon than about a vertex of a
hexagon. It is also notable that the set of similarity isometries of the shifted hexagonal packing forms a proper subgroup of the set of similarity isometries of the hexagonal packing.

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