Time-reversal waves and super resolution

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Abstract. Time-reversal mirrors (TRMs) refocus an incident acoustic field to the position of the original source regardless of the complexity of the propagation medium. TRM’s have now been implemented in a variety of physical scenarios from MHz ultrasonics with order centimeter aperture size to hundreds/thousands of Hz in ocean acoustics with order hundred meter aperture size. Common to this broad range of scales is a remarkable robustness exemplified by observations at all scales that the more complex the medium between the probe source and the TRM, the sharper the focus. The relation between the medium complexity and the size of the focal spot is studied in this paper. It is certainly the most exciting property of TRM compared to standard focusing devices. A TRM acts as an antenna that uses complex environments to appears wider than it is, resulting for a broadband pulse in a refocusing quality that does not depend of the TRM aperture.

In this paper, we investigate the time-reversal approach in various media of increasing complexity. We will also demonstrated that time-reversal focusing opens completely new approaches to super-resolution. We will show that in medium made of random distribution of sub-wavelength scatterers, a time-reversed wave field interacts with the random medium to regenerate not only the propagating but also the evanescent waves required to refocus below the diffraction limit. Finally, we will discuss the link existing between time-reversal approaches and new imaging methods recently developed where Green’s functions of complex media can be extracted from diffusive noise by cross-correlating the recordings of a diffuse random wave field.

1. Introduction

It is well known that the acoustic wave equation in a non-dissipative heterogeneous medium is invariant under a time reversal operation. Indeed, it contains only a second-order time-derivative operator. Therefore, for every burst of sound $\varphi(r, t)$ diverging from a source- and possibly reflected, refracted or scattered by any heterogeneous media- there exists in theory a set of waves $\varphi(r, -t)$ that precisely retraces all of these complex paths and converges in synchrony, at the original source, as if time were going backwards. This idea gives the basis of time reversal acoustics.

Taking advantage of these two properties the concept of time-reversal mirror has been developed and several devices have been built which illustrated the efficiency of this concept [1, 2, 3]. In such a device, an acoustic source, located inside a lossless medium, radiates a brief transient pulse that propagates and is distorted by the medium. If the acoustic field can be measured on every point of a closed surface surrounding the medium (acoustic retina), and retransmitted through the medium in a time-reversed chronology, then the wave will travel back to its source. Note that it requires both time reversal invariance and spatial reciprocity.
[4] to reconstruct the time-reversed wave in the whole volume by means of a two-dimensional time-reversal operation. From an experimental point of view a closed TRM consists of a two-dimensional piezoelectric transducer array that samples the wavefield over a closed surface. An array pitch of the order of $\lambda/2$ where $\lambda$ is the smallest wavelength of the pressure field is needed to insure the recording of all the information on the wavefield. Each transducer is connected to its own electronic circuitry that consists of a receiving amplifier, an A/D converter, a storage memory and a programmable transmitter able to synthesize a time-reversed version of the stored signal. In practice, closed TRMs are difficult to realize and the TR operation is usually performed on a limited angular area, thus apparently limiting focusing quality. A TRM consists typically of a small number of elements or time-reversal channels. The major interest of TRM, compared to classical focusing devices (lenses and beam forming) is certainly the relation between the medium complexity and the size of the focal spot. A TRM acts as an antenna that uses complex environments to appear wider than it is, resulting in a refocusing quality that does not depend of the TRM aperture.

It must be noticed that the research on time-reversal acoustics was initially focused on two main applications: ultrasound therapy (tumor or kidney stone destruction) and acoustic communications in the ocean. In both applications, self focusing on the source with a TRM is accomplished without any knowledge of the medium between the source and the TRM. Active or passive sources can be used.

This paper focus mainly on the behavior of time reversal mirrors in complex media. It will be shown that in presence of multiple reflections or multiple scattering, a small size time reversal mirror manages to focus a pulse back to the source with a spatial resolution that beats the diffraction limit. The resolution is no more dependent on the mirror aperture size but it is only limited by the spatial correlation of the wave field. In these media, due to a sort of kaleidoscopic effect that creates virtual transducers, the TRM appears to have an effective aperture that is much larger that it’s physical size.

Both the duration of the time reversal window and the bandwidth of the time reversed signals play a crucial role in this process and these points will be emphasized. Optimal duration of the time reversal window will be discussed for various scenarios from waveguides, to closed cavities and to open random media. Moreover, a successful time reversal focusing with only a one channel TRM is possible, provided that the time reversal operation is conducted with broadband signals. Despite the apparent equivalence between time reversal operation and phase conjugation operating on monochromatic signal, focusing with a phase conjugated mirror of punctual size is not possible. This point will also be emphasized to show the fundamental difference between ‘true’ time reversal and phase conjugation.

Finally, the link between time reversal and diffusive noise interferometry (local helioseismology) will also be discussed and a discussion on super-resolution with TRMs will be presented.

2. Basic limitations of time reversal mirrors

The basic theory employs a scalar wave formulation $\varphi(\mathbf{r}, t)$ and, hence, is strictly applicable to acoustic or ultrasound propagations in fluid. However, the basic ingredients and conclusions apply equally well to elastic waves in solid and to electromagnetic fields.

In any propagation experiment, the acoustic sources and the boundary conditions determine a unique solution $\varphi(\mathbf{r}, t)$ in the fluid. The goal, in time-reversal experiments, is to modify the initial conditions in order to generate the dual solution $\varphi(\mathbf{r}, T - t)$ where $T$ is a delay due to causality requirements. Cassereau and Fink [4] and Jackson and Dowling [5] have studied theoretically the conditions necessary to insure the generation of $\varphi(\mathbf{r}, T - t)$ in the entire volume of interest.
2.1. An ideal time reversal experiment

Although reversible acoustic retinas usually consist of discrete elements, it is convenient to examine the behavior of idealized continuous retinas, defined by two-dimensional surfaces. In the case of a time-reversal cavity, we assume that the retina completely surrounds the source. The analysis in this paragraph borrows from the back-propagation formalism [6] developed for the inverse source problem. It is important to appreciate the distinction between the back-propagation problem and the time reversal cavity problem. In both, one deals with propagation of a time-reversed field, but the propagation is real in the time reversal cavity problem and simulated in the back-propagation problem. The most important distinction is that source localization methods such as back-propagation demand knowledge of the propagating medium, while the time-reversal cavity method does not.

![Image of time reversal experiment](image)

Figure 1.

(a) Recording step: A closed surface is filled with transducer elements. A point like source generates a wave front which is distorted by heterogeneities. The distorted pressure field is recorded on the cavity elements.

(b) Time-reversed or reconstruction step: The recorded signals are time-reversed and reemitted by the cavity elements. The time-reversed pressure field back-propagates and refocuses exactly on the initial source.

The time reversed field is related to another concept: the adjoint field approach introduced by Tarantola [7]. Although the adjoint field is generally nonphysical, it can be computed using numerical algorithms essentially identical to those used to solve the forward problem. The main difference is that in adjoint methods, one tries to model the elastic structure of the medium while in time reversal experiments one retransmits true time-reversed acoustic signals to focus on its source. In adjoint methods [8, 9], one compares the data recorded by an array of receivers to synthetic waveforms for a given reference model. To localize the origin of the discrepancies between the recorded data and the synthetics, one uses time-reversed residuals as radiating sources that focus on those parts of the model that are inadequate. Time-reversal focusing is also related to the concept of reversed time migration that was introduced in the mid 1980’s
The basic time-reversal experiment can be described in the following way:

In a first step, let us consider a point-like source located at $r_0$ inside a volume $V$ surrounded by the retina surface $S$, emitting a time modulation $s(t)$. The wave equation in a medium of density $\rho(r)$ and compressibility $\kappa(r)$ is given by

$$\left( L_r + L_t \right) \phi(r, t) = -\delta(r - r_0)s(t)$$

with the differential operators

$$L_r = \nabla \cdot \left( \frac{1}{\rho(r)} \nabla \right) \quad \text{and} \quad L_t = -\kappa(r) \partial_t$$

Considering an impulsive source $s(t) = \delta(t)$ at time 0, the causal solution to equation (1) reduces to the retarded Green’s function $G_{ret}(r, r_0; t)$ that takes into account the heterogeneities and the boundaries of the medium. Note that to respect causality in physics, the causal Green’s function (retarded) is only selected while the advanced Green’s function (the anti causal) is neglected. The goal of a time-reversed experiment is to generate this advanced Green’s function by modifying the initial conditions. In the following, we will use the notation $G(r, r_0; t)$ instead of $G_{ret}(r, r_0; t)$.

We assume that we are able to measure the pressure field and its normal derivative at any point $r'$ on the surface $S$ during the interval $[t_{min}, t_{min} + \Delta T]$ where $t_{min}$ is the time of the first arrivals on $S$. As time-reversal experiments are based on a two-step process, the measurement step must be limited in time by the duration of the time reversal window $\Delta T$. In the following paragraph, we suppose that the contribution of multiple scattering decreases with time, and that $\Delta T$ is chosen such that the information loss can be considered as negligible inside the volume $V$. However, in the next chapter that deals with time reversal in complex media we will discuss more carefully the optimal time duration $\Delta T$ for various situations.

During the second step of the time-reversal process, the initial source at $r_0$ is removed (this key point will be discussed later in the paragraph concerning the acoustic sink) and we create on the surface of the cavity monopole and dipole sources that correspond to the time-reversal of $S$ key point will be discussed later in the paragraph concerning the acoustic sink) and we create more carefully the optimal time duration $\Delta V$. However, in the next chapter that deals with time reversal in complex media we will discuss more carefully the optimal time duration $\Delta T$ for various situations.

In this equation, $\partial_n$ is the normal derivative operator with respect to the normal direction $\vec{n}$ to $S$, oriented outward. Due to these secondary sources on $S$, a time-reversed pressure field $\varphi_tr(r, t)$ propagates inside the cavity. It can be calculated using a modified version of the Helmholtz-Kirchhoff integral

$$\varphi_tr(r, t) = \int_{-\infty}^{+\infty} dt' \int_S \left[ G(r, r'; t - t') \partial_n \varphi_s(r', t') - \varphi_s(r', t') \partial_n G(r, r'; t - t') \right] \frac{d^2 r'}{\rho(r')}$$

Following a classical argument on the self-adjointness of the operator $L_r$, the integral over the surface $S$ can be changed in an integral over the volume $V$ surrounded by $S$

$$\varphi_tr(r, t) = \int_{-\infty}^{+\infty} dt' \int_V \left[ G(r, r'; t - t') L_r \varphi_s(r', t') - \varphi_s(r', t') L_r' G(r, r'; t - t') \right] d^3 r'$$

Replacing $\varphi_s(r', t')$ by $G(r', r_0; T - t')$, we obtain

$$\varphi_tr(r, t) = \int_{-\infty}^{+\infty} dt' \int_V \left[ G(r, r'; t - t') L_r G(r', r_0; T - t') - G(r', r_0; T - t') L_r' G(r, r'; t - t') \right] d^3 r'$$
Taking into account Green’s function definition, spatial reciprocity and time reversal invariance ($L_t = L_{T-t}$) yield the following expression [4, 12]:

$$
\varphi_{tr}(\mathbf{r}, t) = G(\mathbf{r}, \mathbf{r}_0; T - t) - G(\mathbf{r}, \mathbf{r}_0; t - T) \tag{7}
$$

Note that this expression can be written, neglecting the causal time delay $T$, as

$$
\varphi_{tr}(\mathbf{r}, t) \propto G_{adv}(\mathbf{r}, \mathbf{r}_0; t) - G_{ret}(\mathbf{r}, \mathbf{r}_0; t) = K(\mathbf{r}, \mathbf{r}_0; t) \tag{8}
$$

where $K(\mathbf{r}, \mathbf{r}_0; t)$ is the propagator. This equation can be interpreted as the difference of advanced and retarded waves, centered on the initial source position. The converging wave (advanced) collapses at the origin and is always followed by a diverging (retarded) wave. Thus the time-reversed field observed as a function of time, from any location in the cavity, shows two wavefronts of opposite sign.

In the case of a homogeneous medium, assuming that the retina does not perturb the field propagation (free-space assumption), the free-space retarded Green’s function $G_0$ reduces to a diverging spherical impulse wave that depends only on $\mathbf{r} - \mathbf{r}_0$ and propagates with a sound speed $c$. Thus the propagator reads:

$$
K_0(\mathbf{r}, t) = \frac{1}{4\pi r} \delta \left( t + \frac{\mathbf{r}}{c} \right) - \frac{1}{4\pi r} \delta \left( t - \frac{\mathbf{r}}{c} \right) \tag{9}
$$

Note that the propagator $K_0(\mathbf{r}, t)$ is odd in $t$ and is the difference between the advanced and the retarded Green’s functions. It results therefore that the self-focusing process with a closed cavity leads to the emission of a field that is not exactly a time-reversed version of the initial field.

Although both converging and diverging spherical waves show a singularity at the origin, it is crucial to note that the time-reversed field remains finite for all time throughout the cavity. This result can be explained in the frame of the general theory for wave propagation. Indeed, during the second or reconstruction step, the initial source is removed or remains passive. As a consequence, there is no more spatial discontinuity with respect to the acoustic field that propagates inside the cavity, and the pressure field resulting from this back-propagation cannot be discontinuous inside volume $V$. More explicitly, the time-reversed field, observed as a function of time, shows two different wavefronts with opposite sign. For an observer located at $\mathbf{r}$, the arrival-time difference between the two wavefronts is $\frac{2|\mathbf{r} - \mathbf{r}_0|}{c}$, this difference increases with the distance separating the observation point and the initial source position.

Experimentally, due to the finite bandwidth in any experiment, we have to take into account the duration of the initial source signal $s(t)$. In this case, the time-reversed field is given by:

$$
\varphi_{tr}(\mathbf{r}, t) = s(-t) \otimes K(\mathbf{r}, \mathbf{r}_0; t) \tag{10}
$$

If the arrival-time difference is greater that the duration of $s(t)$, the two wavefronts can be separated in time. Otherwise (for an observation point located near the origin), the two wavefronts overlap, therefore resulting in a distortion of the temporal variations of the time-reversed pressure field. This distortion effect leads to the time derivative of the excitation function in the neighbourhood of the origin that reads:

$$
\varphi_{tr}(\mathbf{r} = \mathbf{r}_0, t) = -\frac{1}{2\pi c} s'(T - t) \tag{11}
$$

where $s'(t)$ is the time derivative of the source modulation.

In the case of a narrow band excitation (monochromatic excitation of pulsation $\omega$), the interference between the converging and the diverging fields leads to the classical diffraction
limits. Indeed by calculating the Fourier transform of equation (7) over the time variable $t$, we obtain
\[ \Phi_{tr}(r, \omega) = \frac{\exp(-jk|r - r_0|)}{4\pi|r - r_0|} - \frac{\exp(jk|r - r_0|)}{4\pi|r - r_0|} = -2j \sin(k|r - r_0|) = -2j \mathcal{I} G(r - r_0, \omega) \tag{12} \]
where $j^2 = -1$. Note that the time-reversed field at initial point is finite because it is the difference between a converging and a diverging wave and not the addition (otherwise it will give a discontinuity at the focus).

As a consequence, the time-reversed field is focused on the initial source position, with a focal spot size limited to one half-wavelength that corresponds to the standard formulation for the Green’s function. Note that the point spread function is proportional to the imaginary part of the monochromatic Green’s function. Note that the point spread function is proportional to the imaginary part of the monochromatic Green’s function.

In the case of any loss less arbitrary inhomogeneous medium, a similar interpretation can be given, but the retarded Green’s function $G$ is no more dependent on $r - r_0$ but is now a function separately of both $r$ and $r_0$ taking into account the geometry of the heterogeneities.

\[ \hat{\Phi}_{tr}(r, \omega) = \iint_S \left[ \partial_n \hat{G}^*(r', r_0; \omega) \hat{G}(r, r'; \omega) - \hat{G}^*(r', r_0; \omega) \partial_n \hat{G}(r, r'; \omega) \right] d^2r' = -2j \mathcal{I} \hat{G}(r, r_0; \omega) \tag{13} \]

Note that the choice of the integration boundary $S$ is arbitrary as long as it encloses $r$ and $r_0$. This formula (the so called time reversal formula) has many fundamental implications in wave physics. Note that equation (13) requires that the time reversal cavity is made of monopole and dipole sources. In fact, if the time reversal cavity is located in the far field of both source and observation points, and if the medium closed to the time reversal cavity is homogeneous this expression can be simplified. In this case we may assume that

\[ \partial_n \hat{G}(r, r'; \omega) \simeq jk \hat{G}(r, r'; \omega) \tag{14} \]

Thus, the time reversed field can be written as
\[ \hat{\Phi}_{tr}(r, \omega) \simeq 2j \frac{\omega}{\rho c} \iint_S \hat{G}^*(r', r_0; \omega) \hat{G}(r, r'; \omega) d^2r' \tag{15} \]

Note that if we come back in the time domain, equation (15) can be written as
\[ \varphi_{tr}(r, t) \simeq 2 \frac{\partial}{\rho c \partial t} \iint_S G(r', r_0; -t) \otimes G(r, r'; t) d^2r' \tag{16} \]

From an experimental point of view, it is not easy to measure and re-emit the field at any point of a surface $S$: experiments are carried out with transducer arrays that spatially sample the receiving and emitting surface Assuming that the time reversal retina consists of discrete elements located at position $r_i$, this allows to replace the integration over $S$ in equation (16) by a summation over $N$ surface element positions
\[ \varphi_{tr}(r, t) = C \frac{\partial}{\partial t} \sum_{i=1}^{N} G(r_i, r_0; -t) \otimes G(r, r_i; t) \tag{17} \]
where $C$ is a scaling factor. Taking into account spatial reciprocity this expression can be written a summation of cross-correlation functions
\[ \varphi_{tr}(r, t) \simeq C \frac{\partial}{\partial t} \sum_{i=1}^{N} G(r_0, r_i; -t) \otimes G(r, r_i; t) \tag{18} \]

The spatial sampling of surface $S$ by a set of elements may introduce grating lobes. These lobes can be avoided by using an array pitch smaller than $\lambda_{\text{min}}/2$, where $\lambda_{\text{min}}$ is the smallest wavelength of the incident pressure field.
3. TRM in complex media

It is generally difficult to use acoustic arrays that surround completely the area of interest, and the closed cavity is usually replaced by a TRM of finite angular aperture. This yields an increase of the point spread function dimension that is related to the limited angular size of the mirror observed from the source. In the standard approach of diffraction in homogeneous free space, the point spread function is related to the source angular-spectrum captured by the aperture. For a closed time reversal mirror, the $\vec{k}$ vectors of the radiated field span the whole $4\pi$ solid angle and the focal spot dimension is minimal ($\lambda/2$). As the TRM covers a limited solid angle, the spatial diversity of the $\vec{k}$ vectors that interact with the TRM is reduced. Therefore the focal spot size is increased.

One part of the transducers is replaced by reflecting boundaries. In the first step (receive mode) the wave radiated by the source is recorded by a set of transducers through the reverberation inside the cavity. In the second step, the recorded signals are time-reversed and reemitted by the transducers.

The main interest of focusing with TRM is that in propagating media of complex structure the spatial diversity of the $\vec{k}$ vectors captured by a small size TRM can be significantly increased. Wave propagation in medium with complex boundaries or through random scattering medium can strongly increase the apparent aperture of the TRM, resulting in a focal spot size much smaller than the one predicted by classical formulas.

The basic idea is to replace one part of the transducers needed to sample a closed time reversal surface by reflecting boundaries that redirect one part of the incident wave towards the TRM aperture. (see Figure 2) When a source radiates a wave field inside a closed cavity or in a waveguide, multiple reflections along the medium boundaries can significantly increase the apparent aperture of the TRM. Thus spatial information on the $\vec{k}$ vectors that is usually lost with a finite aperture TRM is converted into the time domain and the reversal quality depends crucially on the duration of the time-reversal window, i.e., the length of the recording to be reversed.
Such a concept is strongly related to a kaleidoscopic effect that appears thanks to the multiple reverberations on the waveguide boundaries, waves emitted by each transducer are multiply reflected, creating at each reflection ‘virtual’ transducers that can be observed from the desired focal point. Thus, we create a large virtual array from a limited number of transducers. The result of such an operation is that a small number of transducers is multiplied to create a “kaleidoscopic” transducer array. Three different examples will be presented (waveguide, chaotic cavity and multiply scattering medium).

3.1. Time reversal in acoustic waveguide

The simplest boundaries that can give rise to such a kaleidoscopic effect are plane boundaries as in rectangular waveguides or cavities. A first experiment conducted in the ultrasonic regime by P. Roux et al [13, 14] showed clearly this effect with a TRM made of a 1D transducer array located in a rectangular ultrasonic waveguide (see Figure 3a). For an observer, located in the waveguide, the TRM seems to be escorted by a periodic set of virtual images related to multipath propagation and effective aperture 10 times larger than the real aperture was observed.

The experiment was conducted in a waveguide whose interfaces (water-air or water-steel interfaces) are plane and parallel. The length of the guide is $L \sim 800$ mm, which is on the order of 20 times the water depth $H \sim 40$ mm. A subwavelength ultrasonic source is located on one side of the waveguide. On the other side, a 1D time-reversal mirror, made of a 96 elements array fills exactly the waveguide aperture. All the transducers work around a center frequency of 3.5 MHz with a 50% bandwidth. Due to experimental imitations, the array pitch is equal to 0.7λ. Then, a time-reversal experiment is performed in the following way: (1) the point source emits a pulsed wave (1 μs duration), (2) the TRM receives, selects a time reversal window, time-reverses and re-transmits the field which has propagated from the source through the waveguide, (3) after back propagation, the time-reversed field is scanned in the plane of the source.

Figure 3b shows the incident field recorded by the array after forward propagation through the channel. After the arrival of the first wavefront corresponding to the direct path we observe a set of signals, due to multiple reflections of the incident wave between the interfaces that spread over 100 μs. Figure 3c represents the signal received on one transducer of the TRM.

After the time-reversal of all the signals recorded by the array during 100 μs, we observe a remarkable temporal compression at the source location (see Figure 4). This means that multipath effects are fully compensated. Figure 4 shows that the time-reversed signal observed at the source is nearly identical to the one received in a time-reversed experiment conducted in free space.

In this experiment, the transfer function of the waveguide has been completely compensated by the time-reversal process. The time-reversal process enables the compensation of the waveguide transfer function. The analysis of Figure 4 shows that the ratio between the peak signal and the side lobe level is on the order of 45 dB.

Not only a remarkable time recompression is observed but the spatial focusing of the time reversed field is of interest. Figure 5 shows the directivity pattern of the time-reversed field observed in the source plane. The time-reversed field is focused on a spot which is much smaller than the one obtained with the same TRM working in free space. In our experiment, the 6 dB lateral resolution is improved by a factor of 9. This can be easily interpreted by the images theorem in a medium bounded by two mirrors. For an observer, located at the source point, the 40-mm TRM appears to be accompanied by a set of virtual images related to multipath reverberation. The effective TRM is then a set of TRM’s as shown on Figure 6. This is the kaleidoscopic effect. When taking into account the first ten arrivals, the theoretical effective aperture of the mirror array is ten times larger than the real aperture. However, in practice as the replica arrive later, their amplitudes decrease. The angular directivity of the transducers
Figure 3.

(a) Schematic of the acoustic waveguide: the guide length ranges from 40 cm to 80 cm and the water depth from 1 cm to 5 cm. The central acoustic wavelength (\(\lambda\)) is 0.5 mm. The array element spacing is 0.42 mm. The TRM is always centered at the middle of the water depth.

(b) Spatial-temporal representation of the incident acoustic field received by the TRM; the amplitude of the field is in dB.

(c) Temporal evolution of the signal measured on one transducer of the array.

leads to an apodization (smoothness of the aperture function) of the effective aperture of the TRM. Figure 7 shows the effect of the time-reversed window duration \(\Delta T\) on the width of the focal spot. The size of the focal spot decreases when the number of replica selected by the window increases. This clearly shows that the effective aperture of the TRM is directly related to the time-reversal window duration \(\Delta T\). It emphasized the way the \(\vec{k}\) vector information is translated in the time domain through the interaction with the boundaries. In the set of replica that are recorded in figure 3a, the last arrival corresponds to the highest values of the source angular spectrum. Thus, the \(-6\) dB beamwidth of the focal spot is given by

\[
\Delta = \frac{\lambda f}{D'}
\]

where \(D'\) is an effective aperture dimension of the TRM that depends on the duration \(\Delta T\):
Figure 4. Time-reversed signal measured at the point source. Figure 5. Directivity pattern of the time-reversed field in the plane of source: dotted line corresponds to free space, full line to the waveguide.

Figure 6. The principle of mirror images (kaleidoscope) applied to the waveguide.

when all the replica are selected ($\Delta T = 100\mu s$), we measure $D' = 9D$ where $D$ is the array aperture. Note that the optimal time duration of the time reversal window is directly related to the waveguide dispersion and depends on the source-TRM distance.

3.1.1. Time-reversal: matched-filter or inverse filter

One important aspect of this experiment is the impressive time recompression that is observed in a waveguide. It can be interpreted in terms of matched filter. Note that in all the following paragraphs, to comment time-reversed experiments, we will compute time-reversed fields without taking care of the time derivative that appears in equation (15), (16) and (17). Indeed this time derivative is needed in a perfect experiment where both the field and the normal derivative are recorded and where both monopoles and dipoles sources are spread along the array. In real experiment things are simpler and for many applications we can consider that we use only monopole sources that play back
Figure 7. Directivity patterns of the time-reversed field versus the number of echoes selected in the time-reversed window.

the recorded pressure signals, thus the experimental time-reversed field at the source location is

\[ \varphi_{\text{tr}}(r = r_0, t) \propto \sum_{i=1}^{N} G(r_0, r_i; -t) \otimes G(r_0, r_i; t) \]  

(20)

It is a sum of autocorrelation functions. In terms of signal processing, wave propagation through a waveguide may be described as a linear system with different impulse responses (Green’s functions). Each term in the summation corresponds to a matched filter. Given a signal as input, a matched filter is a linear filter whose output is optimal in some sense. Whatever the impulse response \( h_i(t) = G(r_0, r_i; t) \), the convolution \( h_i(-t) \otimes h_i(t) \) is maximum at time \( t = 0 \). This maximum is always positive and equals \( \int h_i^2(t) dt \), i.e. the energy of the signal \( h_i(t) \). This has an important consequence. Indeed, with an \( N \)-elements array, the time-reversed signal recreated on the source depends on the sum

\[ \sum_{i=1}^{N} h_i(-t) \otimes h_i(t) \]  

(21)

Even if each of the \( h_i(t) \) have different behavior, each term in this sum reaches its maximum at time \( t = 0 \). So all contributions add constructively around \( t = 0 \), whereas at earlier or later times uncorrelated contributions tend to destroy one another. Thus the re-creation of a sharp peak after time-reversal on an \( N \)-elements array can be viewed as an interference process between the \( N \) outputs of \( N \) matched filters (see Figure 8).

When the number \( N \) of channels is sufficient, that is to say when the TRM aperture fills exactly the whole waveguide aperture and when the sampling pitch is small enough, we observed that the time reversal process is not only a spatio-temporal matched filter but it is also a good approximation of an inverse filter of the propagation operator [15, 16]. 96 elements were needed to reach this goal, and grating lobes around the focal spot were avoided because the array pitch was close to \( \lambda/2 \) and the 96 elements array fills the whole waveguide aperture.

However, grating lobes appear when smaller TRM are used that does not fill completely the water depth \( H \). Indeed, in this case the kaleidoscopic effect introduces a clear periodicity in the
Each individual contribution of the time-reversed field is a symmetrical signal with a maximum at the same time $T$ for each transducer. When all the transducers of the TRM work together the summation of all the signals give a perfect time recompression at the origin.

Virtual array. The virtual array is made of a small periodic TRM's with spacing $H$. According to diffraction laws, this periodicity generates high grating lobes on each side of the main lobe at distance $m\lambda L/H$, where $m$ is an integer. Experimentally, $\lambda L/H = 8$ mm, which corresponds to the position of the side lobes observed on Figure 9.

Acoustic waveguides are currently found in underwater acoustic, especially in shallow water, and TRMs can compensate for the multipath propagation in oceans that limits the capacity of underwater communication systems. The problem arises because acoustic transmission in shallow water bounce off the ocean surface and floor, so that a transmitted pulse gives rise to multiple copies of itself that arrive at the receiver. Underwater acoustic experiments have been conducted by W. Kuperman and his group from San Diego University in a sea water channel of 120 m depth, with a 24 elements TRM working at 500 Hz and 3.5 kHz. They observed focusing with super resolution and multipath compensation at a distance up to 30 kms [17]. Such properties open the field of new discrete communications systems in underwater applications as it was experimentally demonstrated by different groups [18, 19].

As a conclusion of these various experiments, two points must be emphasized. Boundaries help to obtain a better resolution by redirecting $\vec{k}$ vectors, associated to high spatial frequency, toward the TRM aperture. However a small aperture TRM made of a reduced number of transducers does not work perfectly, because the symmetries of the waveguide introduced grating lobes. There are two principal ways of reducing the grating lobes. One is to work with broadband signals. Thus for each spectral component, the grating lobe position is shifted, so an averaging effect reduces their amplitude compared to the main lobe. The other alternative is to work with boundaries that are no more symmetric and this is the case of the so called chaotic cavities that we will explore now.
3.2. Time reversal in chaotic cavities

In this paragraph, we are interested in another aspect of multiply reflected waves: waves confined in closed reflecting cavities with non symmetrical geometry. With closed boundary conditions, no information can escape from the system and a reverberant acoustic field is created. If, moreover, the geometry of the cavity shows ergodic and mixing properties, one may hope to collect all information at only one point. Ergodicity means that, due to the boundary geometry, any acoustic ray radiated by a point source and multiply reflected would pass every location in the cavity. Therefore, all the information about the source can be redirected towards a single time reversal transducer (we will see later, using an eigenmode decomposition of the wavefield, that it is not exactly true). This is the regime of fully diffuse wave fields that can be also defined as in room acoustics as an uncorrelated and isotropic mix of plane waves of all propagation directions [20, 21]. Draeger and Fink [22, 23, 24] shown experimentally and theoretically that in this particular case a time-reversal focusing with $\lambda/2$ spot can be obtained using only one TR channel operating in a closed cavity.

The first experiments were made with elastic waves propagating in cavity with negligible absorption. The experiment is 2-dimensional and has been carried out by using elastic surface waves propagating along monocrystalline silicon wafer whose shape is a $D$-shaped stadium. This geometry is chosen to avoid quasi-periodic orbits that introduced grating lobes. $D$-shaped cavities, as many others shapes, are not completely symmetrical objects and therefore ergodicity is guaranteed and the eigenmodes are not degenerated [25, 26]. The role of this important point will be emphasized in the next paragraph.

Silicon was also selected for its weak absorption. The elastic waves which propagate in such a plate are Lamb waves. An aluminum cone coupled to a longitudinal transducer generates these waves at one point of the cavity. A second transducer is used as a receiver. The central frequency of the transducers is 1 MHz and its bandwidth is 100%. At this frequency, only three Lamb modes are possible (one flexural, two extensional). The source is isotropic and considered

Figure 9.

Directivity patterns obtained after time-reversal in a wave guide with a limited number of transducers in the TRM aperture. This clearly show the presence of side lobes due to spatial aliasing for 5, 10 and 20 transducers.
point-like because the cone tip is much smaller than the central wavelength. A heterodyne laser interferometer measures the displacement field as a function of time at different points on the cavity. Assuming that there is nearly no mode conversion between the flexural mode and other modes at the boundaries, we have only to deal with one field, the flexural-scalar field.

The experiment is a “two step-process” as described above: In the first step, one of the transducers, located at point A, transmits a short omnidirectional signal of duration $0.5 \, \mu s$ into the wafer. Another transducer, located at B, observes a very long chaotic signal that results from multiple reflections of the incident pulse along the boundaries of the cavity, and which continue for more than 50 milliseconds corresponding to some hundred reflections along the boundaries. Then, a portion $\Delta T$ of the signal is selected, time-reversed and re-emitted by point B. As the time reversed wave is a flexural wave that induces vertical displacements of the silicon surface, it can be observed using the optical interferometer that scan the surface around point A (see Figure 10).

![Figure 10.](image)

Time reversal experiment conducted in a chaotic cavity with flexural waves. In a first step, a point transducer located at point A transmits a 1 ms long signal. The signal is recorded at point B by a second transducer. The signal spreads on more than 30 ms due to reverberation. In the second step the experiment, a 1 ms portion of the recorded signal is time-reversed and retransmitted back in the cavity.

For time reversal windows of sufficient long duration $\Delta T$, one observes both an impressive time recompression at point A and a refocusing of the time-reversed wave around the origin (see Figures 11a and 11b for $\Delta T = 1 \, ms$), with a focal spot whose radial dimension is equal to half the wavelength of the flexural wave. Using reflections at the boundaries, the time-reversed wave field converges towards the origin from all directions and gives a circular spot, like the one
that could be obtained with a closed time reversal cavity covered with transducers. A complete study of the dependence of the spatio-temporal side lobes around the origin shows a major result (Draeger et al. 1999): a time duration $\Delta T$ of nearly 1 ms is enough to obtain a good focusing. For values of $\Delta T$ larger than 1 ms, the sidelobes’ shape and the signal-to-noise ratio (focal peak/ sidelobes) do not change any more. There is a saturation regime. Once the saturation regime is reached, point $B$ will receive redundant information on the various eigenmodes of the cavity. The saturation regime is reached after a time $\tau_{\text{Heisenberg}}$ that is called the Heisenberg time (in quantum chaos theory). It is the minimum time duration needed to resolve each of the eigenmodes in the cavity. It can also be interpreted as the time it takes, for all the rays radiated by a point source, to reach the vicinity of any point in the cavity in a wavelength. This guarantees enough interference between all the multiply reflected waves to build each of the eigenmodes in the cavity. As it was shown by Berry, in a chaotic cavity, due to the lack of symmetry, the eigenmodes are not degenerated and therefore it exists a mean distance $\Delta \omega$ between the eigenfrequencies that is related to the Heisenberg time $\tau_{\text{Heisenberg}}$ of the cavity by $\tau_{\text{Heisenberg}} = \frac{1}{\Delta \omega}$.

The success of this time-reversal experiment in closed chaotic cavity is particularly interesting with respect to two aspects. Firstly, it proves the feasibility of acoustic time-reversal in cavities of complex geometry that give rise to chaotic ray dynamics. Paradoxically, in the case of one-channel time-reversal, chaotic dynamics is not only harmless but even useful, as it guarantees ergodicity and mixing. Secondly, using a source of vanishing aperture, there is an almost perfect focusing quality. The procedure approaches the performance of a closed TRM, which has an aperture of 360°. Hence, a one-point time-reversal in a chaotic cavity produces better results than a TRM in an open system. Using reflections at the edge, focusing quality is not aperture limited, and in addition, the time-reversed collapsing wavefront approaches the focal spot from all directions.

Although one obtains excellent focusing, a one-channel time-reversal is not perfect, as a weak noise level throughout the system can be observed. There is a saturation regime beyond the Heisenberg time. Residual temporal and spatial sidelobes persist even for time-reversal windows of duration larger than the Heisenberg time. They are due to multiple reflections passing over the locations of the TR transducer and they have been expressed in closed form by Draeger and Fink. Using an eigenmode analysis of the wavefield, they explain that, for long time reversal windows, there is a saturation regime that limits the signal-to-noise ratio (SNR). This corresponds to the “Cavity Formula” that is explains now.

3.2.1. The cavity formula  As it was stated before, time reversal focusing is a matched filter. It does not usually behave as an inverse filter, except if the TRM array is made of many elements that span a large aperture. Here we want to understand this relation for a one channel TRM. Therefore we have to compute $h_{AB}(-t) \otimes h_{BA}(t)$ where use the notation $h_{AB}(t) = G(A, B, t)$.

Taking into account the modal decomposition of the Green’s function $G(A, B, t)$ on each of the eigenmodes $u_j(r)$ of the cavity with eigenfrequency $\omega_j$, we get

$$h_{AB}(t) = \sum_j u_j(A)u_j(B)\frac{\sin(\omega_j t)}{\omega_j} \quad (t > 0)$$

This signal is recorded in $B$ and a part $\Delta T = [t_1; t_2]$ is time-reversed and reemitted as

$$h_{AB}^{\Delta T}(-t) = \begin{cases} h_{AB}(-t) & t \in [-t_2, -t_1] \\ 0 & \text{elsewhere} \end{cases}$$
(a) time-reversed signal observed at point A. The observed signal is 210 \( \mu s \) long.
(b) time-reversed wavefield observed at different times around point A on a square of 15 mm \( \times \) 15 mm.

So equation (16) gives for the time-reversed signal at the source location A:

\[
\varphi_{tr}^{\Delta T}(A, t) \propto \int_{t_1}^{t_2} d\tau h_{AB}(t + \tau)h_{AB}(\tau) \\
\propto \sum_j \frac{1}{\omega_j} u_j(A)u_j(B) \sum_i \frac{1}{\omega_i} u_i(A)u_i(B) I_{ij}
\]

(24)

with \( I_{ij} \) equal to:

\[
I_{ij} = \int_{t_1}^{t_2} d\tau \sin(\omega_i \tau) \sin \omega_j(\tau + t)
\]
\[ I_{ij} = \begin{cases} 
\frac{1}{2} \sin(\omega_i t) \int_{t_1}^{t_2} d\tau \left[ \sin \left( (\omega_i - \omega_j)\tau \right) + \sin \left( (\omega_i + \omega_j)\tau \right) \right] + \\
\frac{1}{2} \cos(\omega_i t) \int_{t_1}^{t_2} d\tau \left[ \cos \left( (\omega_i - \omega_j)\tau \right) - \cos \left( (\omega_i + \omega_j)\tau \right) \right] 
\end{cases} \] (25)

The second term of each integral gives a contribution of order \(1/(\omega_i + \omega_j)\ll\Delta T\) which can be neglected. Thus we obtain

\[ I_{ij} \equiv \begin{cases} 
\frac{1}{2} \Delta T \cos(\omega_j t) & \text{if } \omega_i = \omega_j \\
\frac{1}{2} \sin(\omega_j t) \left[ - \cos \left( (\omega_i - \omega_j)\tau_2 \right) + \cos \left( (\omega_i - \omega_j)\tau_1 \right) \right] + \\
\frac{1}{2} \cos(\omega_j t) \left[ \sin \left( (\omega_i - \omega_j)\tau_2 \right) - \sin \left( (\omega_i - \omega_j)\tau_1 \right) \right] & \text{if } \omega_i \neq \omega_j 
\end{cases} \] (26)

Under the assumptions that the eigenmodes are not degenerated, then \(\omega_i = \omega_j \iff i = j\), and the second term represents the diagonal elements of the sum over \(i\) and \(j\). The first term is only important if the difference \(\omega_i - \omega_j\) is small, i.e., for neighboring eigenfrequencies. In the case of a chaotic cavity, next neighbors tend to repel each other and if the characteristic distance \(\Delta \omega\) is sufficiently large so that \(\Delta T >> \tau_{\text{Heisenberg}} = 1/\Delta \omega\), the non-diagonal terms of \(I_{ij}\) are negligible compared to the diagonal contributions and one obtains

\[ I_{ij} = \frac{1}{2} \delta_{ij} \Delta T \cos(\omega_j t) + O\left(\frac{1}{\Delta \omega}\right) \] (27)

In the limit \(\Delta T \to \infty\), the time-reversed signal observed in \(A\) by a reversal in \(B\) is given by

\[ \varphi_{\text{tr}}^{\Delta T}(A, t) \propto \frac{1}{2} \Delta T \sum_i \frac{1}{\omega_i^2} u_i^2(A) u_j^2(B) \cos(\omega_i t) \] (28)

This expression gives a simple interpretation of the residual temporal lobes which are observed experimentally. The time-reversed signal observed at the origin cannot be simply reduced to a Dirac distribution \(\delta t\), but is proportional, even for \(\Delta T >> \tau_{\text{Heisenberg}}\), to the cross-correlation product.

\[ C_{AB}(t) = h_{AA}(-t) * h_{BB}(t) \] (ΔT)

\[ = \int_{t_1}^{t_2} d\tau h_{BB}(t + \tau) h_{AA}(\tau) \]

\[ = \sum_i \frac{1}{\omega_i} u_i^2(B) \sum_j \frac{1}{\omega_j} u_j^2(A) I_{ij} \] (29)

where the impulse responses \(h_{AA}\) and \(h_{BB}\) describe the backscattering properties of \(A\) and \(B\) due to the boundaries of the cavity. Each impulse response is composed of a first peak at \(t = 0\) followed by multiple reflections that pass over the source point even after the excitation has ended. Hence, the signal observed in \(A\), after a Dirac excitation, can be described by \(h_{AA}(t)\). Therefore, a perfect time-reversal operation (i.e., we simply count time backwards) would give in \(A\) the signal \(h_{AA}(-t)\), i.e. some multiple reflections with a final peak at \(t = 0\). For the same reason, the reversed point \(B\) cannot exactly transmit any waveform into the cavity. Due to the boundaries, a Dirac excitation at \(B\) will also give rise to a transmitted signal \(h_{BB}(t)\). So, in the
limit of very long time-reversal window we get for a one channel time-reversal experiment the cavity formula.

\[ h_{AB}(-t) \otimes h_{BA}(t) = h_{AA}(-t) \otimes h_{BB}(t) \]  

(30)

This result shows clearly that even in a chaotic cavity, the time reversal process is not an exact inverse filter. Equation (29) proves that, contrary to the assumption (asymptotic regime for ray) that in an ergodic cavity all the information about the source can be redirected towards a single time reversal transducer, a more rigorous analysis using a modal decomposition of the field is needed. Point \( A \) and point \( B \) cannot exchange completely all the information. Indeed, points \( A \) and \( B \) are always located at the nodes of some eigenfrequencies of the signal spectrum and therefore \( A \) and \( B \) cannot exchange any information at these frequencies.

3.2.2. The focal spot as the spatial correlation of the field  
Spatial focusing properties of the time-reversed wave can also be calculated by computing the field at another point \( C \) different or identical to the observation point \( A \). The time reversed field at point \( C \) is equal to

\[ \varphi_{tr}^{\Delta \tau}(C, t) \propto \frac{1}{2} \Delta \tau \sum_{i} \frac{1}{\omega_{i}^{2}} u_i(A) u_i(C) u_i^2(B) \cos(\omega_i t) \]  

(31)

As the cavity formula can be simply generalized in the form:

\[ h_{AB}(-t) \otimes h_{BC}(t) = h_{AC}(-t) \otimes h_{BB}(t) \]  

(32)

Note that at the focal time \( t = 0 \) (collapse) the directivity pattern of the time-reversed wavefield is

\[ \varphi_{tr}(C, 0) \propto \sum_{i} \frac{1}{\omega_{i}^{2}} u_i(A) u_i(C) u_i^2(B) \]  

(33)

Note that in a real experiment one has to take into account the limited bandwidth of the transducers, so a spectral function \( F(\omega) \) centered on center frequency \( \omega_c \), with bandwidth \( \Delta \omega \), must be introduced and we can write equation (33) in the form

\[ \varphi_{tr}(C, 0) = \sum_{i} \frac{1}{\omega_{i}^{2}} u_i(A) u_i(C) u_i^2(B) F(\omega_i) \]  

(34)

Thus the summation is limited to a finite number of modes, which is typical in our experiment of the order of some hundreds. As we do not know the exact eigenmode distribution for each chaotic cavity, we cannot evaluate this expression directly. However, due to the ergodic properties of the cavity one may use a statistical approach and consider the average over different realizations, which consist in summing over different cavity realizations. So we replace in equation (34) the eigenmodes product by their expectation values \( \langle \cdot \cdot \cdot \rangle \). We use also a qualitative argument proposed by Berry to characterize irregular modes in chaotic system. If chaotic rays support an irregular mode, it can be considered as a superposition of a large number of plane waves with random direction and phase. This implies that the amplitude of an eigenmode has a Gaussian distribution with \( \langle u_i^2 \rangle = \sigma^2 \) and a short-range isotropic correlation function given by a Bessel function that reads:

\[ \langle u_i(A) u_i(C) \rangle = J_0 \left( \frac{2\pi |A-C|}{\lambda_i} \right) \]  

(35)

with \( \lambda_i \) is the wavelength corresponding to \( \omega_i \). If \( A \) and \( C \) are sufficiently far apart from \( B \) not to be correlated, then

\[ \langle u_i(A) u_i(C) u_i^2(B) \rangle = \langle u_i(A) u_i(C) \rangle \langle u_i^2(B) \rangle \]  

(36)
One obtains finally:

$$\langle \varphi_{tr}(\mathbf{C}, 0) \rangle = \sum_{i} \frac{1}{\omega_{i}^{2}} J_{0}(\frac{2\pi |\mathbf{C} - \mathbf{A}|}{\lambda_{i}}) \sigma^{2} F(\omega_{i})$$

(37)

The experimental results obtained on Fig. 11 agree with this prediction and shows that in a chaotic cavity the spatial resolution is independent from the time reversal mirror aperture. Indeed, with a one-channel time-reversal mirror, the directivity patterns at $t = 0$ is closed to the Bessel function $J_{0}(2\pi |\mathbf{C} - \mathbf{A}|/\lambda_{c})$ corresponding to the central frequency of the transducers.

One can also observe, on Figure. 11b, a very good estimate of the eigenmode correlation function experimentally obtained with only one realization. A one-channel omnidirectional transducer is able to refocus a wave in a chaotic cavity, and if the bandwidth is very large, we have not have to average the data over different cavities or over different positions of the transducer $\mathbf{B}$.

### 3.2.3. Phase conjugation versus time reversal: self averaging in the time domain

This interesting result emphasizes the great interest of time-reversal experiments, compared to phase conjugated experiments. In phase conjugation, one only works with monochromatic waves. Time reversal of $\varphi(\mathbf{r}, t)$ is equivalent, for each of the spectral components $\tilde{\Phi}(\mathbf{r}, \omega)$, to complex conjugation. For single-frequency signal, time reversal is equivalent to complex conjugation of amplitude. For example, if one works only at a mode frequency $\omega_{i}$ so that there is only one term in Eq. (33), one cannot refocus a wave on point $\mathbf{A}$. Indeed, an omnidirectional transducer, located at any position $\mathbf{B}$, working in monochromatic mode, sends a diverging wave in the cavity that have no any reason to refocus on point $\mathbf{A}$: the directivity pattern is nothing else than the eigenmode pattern corresponding to the selected frequency. The refocusing process works only with broadband pulses, with a large number of eigenmodes in the transducer bandwidth. Here, the averaging process that gives a good focusing is obtained by a sum over the different modes in the cavity by assuming that in a chaotic cavity, we have a statistical decorrelation of the different eigenmodes, the time reversed field can be computed by adding the various frequency components (each individual mode) and it can be represented as a sum of Fresnel vectors (Figure 12).

At the source position, all these phase conjugated fields have a zero phase (this comes from the phase conjugation operation that exactly compensates for the forward phase) and even if there is no amplitude focusing for each spectral contributions, there is a constructive interference between all this fields at the focusing time as

$$\sum_{i} |H_{AB}(\omega_{i})|^{2}$$

where $H_{AB}(\omega)$ is the Fourier transform of $h_{AB}(t)$. Thus, the total field at the focusing time increases proportionally to the number $I$ of modes (or arrows). Outside the source position, at point $\mathbf{C}$, we observe $\sum_{i} H_{AB}(\omega_{i})H_{AB}^{*}(\omega_{i})$, the contributions of each individual mode are decorrelated because there is no more coherent phase compensation and therefore the total length only increases as $\sqrt{I}$. On the whole, the focusing peak emerges at the focusing time from the noise when the bandwidth is large enough to contain many different modes. Ideally, if we could indefinitely expand the bandwidth, the background level on the directivity patterns should decrease as $1/\sqrt{I}$. As the number of eigenmodes available in the transducer bandwidth increases, the refocusing quality becomes better and the focal spot pattern becomes closed to the ideal Bessel function.

As a conclusion, it must be emphasizes that in a closed cavity a one channel time reversal mirror can focused with $\lambda/2$ resolution if the duration of time reversal window is equivalent to the Heisenberg time of the cavity. Larger time windows do not improve the focusing quality. However, larger bandwidth $\Delta \omega$ reduces the side lobe levels as $1/\sqrt{\Delta \omega}$.

Transposition in the audible frequency range of this concept has been shown to be a very efficient localizing technique in solid objects. The idea consists in detecting the acoustic waves in solid objects generated by a slight finger knock (for example a table or a glass plate). As in a
reverberating object, a one channel TRM has the memory of its source location, the information related to the source location is extracted from a simulated time reversal experiment in the computer. Then, any action, turn on the light or a compact disk player, for example is associated with each location. Thus, the whole system transforms solid objects into interactive interfaces. Compared to the existing acoustic techniques, it presents the great advantage of being simple and easily applicable to inhomogeneous objects whatever their shapes. The number of possible touch locations at the surface of objects is directly related to the number of independent time-reversed focal spots that can be obtained. For example, a virtual keyboard can be drawn on the surface of an object; the sound made by fingers when a text is captured, is used to localize impacts. Then, the corresponding letters are displayed on a computer screen [27].

3.3. Time reversal in open systems : random medium

The ability to focus with a one channel time reversal mirror is not only limited to experiments conducted inside closed cavity. Similar results have also been observed in time-reversal experiments conducted in open random medium with multiple scattering [28, 29, 30]. A. Derode carried out the first experimental demonstration of the reversibility of an acoustic wave propagating through a random collection of scatterers with strong multiple scattering contributions. A multiple scattering sample is immersed between the source and an TRM array made of 128 elements. The scattering medium consists of a set of 2000 parallel steel rods (diameter 0.8 mm) randomly distributed. The sample thickness is \( L = 40 \) mm, and the average distance between rods is 2.3 mm. The elastic mean free path in this sample was found to be \( 4 \) mm (see Figure 13), The source is 30 cm away from the TRM and transmits a short (1\( \mu \)s) ultrasonic pulse (3 cycles of a 3.5 MHz).

Figure 14a shows one part of the waveform received on the TRM by one of the element. It spread over more than 200 \( \mu \)s, i.e \( \sim 200 \) times the initial pulse duration. After the arrival of a first wavefront corresponding to the ballistic wave, a long incoherent wave is observed, which results from the multiply scattered contribution. In the second step of the experiment, any number of signals (between 1 and 128) is time-reversed and transmitted and an hydrophone measures the time reversed wave around the source location. For a TRM made of 128 elements, with a time reversal window of 300 \( \mu \)s, the time-reversed signal received on the source is represented on Fig.
Figure 13.

Time-reversal focusing through a random medium. In the first step the source (A) transmits a short pulse that propagates through the rods. The scattered waves are recorded on a 128-element array (B). In the second step, \( N \) elements of the array (0 < \( N < 128 \)) retransmit the time-reversed signals through the rods. The piezoelectric element (A) is now used as a detector, and measures the signal reconstructed at the source position. It can also be translated along the x-axis while the same time-reversed signals are transmitted by B, in order to measure the directivity pattern.

14b : an impressive compression is observed, since the received signal lasts about 1 \( \mu s \), against over 300 \( \mu s \) for the scattered signals. The directivity pattern of the TR field is also plotted on Fig. 15. It shows that the resolution (i.e. the beam width around the source) is significantly finer than in the absence of any scattering medium: the resolution is 30 times finer, and the background level is always below −20 dB. Moreover Fig 16 shows that the resolution is independent of the array aperture: even with only one transducer doing the time-reversal operation, the quality of focusing is quite good and the resolution remains approximately the same than with an aperture 128 times larger. This is clearly the same effect that the one observed with closed cavity where a one channel time reversal mirror was able to focus. High spatial frequencies that would have been lost otherwise in an homogeneous medium are redirected towards the array, due to the presence of the random scatterers distribution. Once again this result illustrates the major difference between phase conjugation and time reversal. If the experiment had been quasi-monochromatic and the array element had simply phase conjugated one frequency component, the conjugated wave field would never have focused on the source position. Indeed, whatever its phase, there is no reason for a monochromatic wave emanating from a point source to be focused on a particular point on the other side of a random sample. The phase conjugated field at one frequency in
the source plane is perfectly random and verifies the classical speckle distribution. Here again, for a broadband signal, a similar analysis than in the last paragraph can be conducted in order to predict the side lobes level around the focal peak. As we are no more in a closed cavity a modal decomposition of the field is no more adequate. However, if we keep in mind, that the focusing with one channel occurs only for a broadband transducer, we have to found the number of uncorrelated speckle fields that are transmitted in the transducer bandwidth $\Delta \omega$. For this, we have to define the spectral correlation length $\delta \omega$ of the scattered waves. Two monochromatic wave fields produced by a point source separated by a frequency shift of $\delta \omega$ are decorrelated. Then there are $\Delta \omega/\delta \omega$ uncorrelated spectral information in the frequency bandwidth, and the signal-to-noise is expected to depends of $\sqrt{\Delta \omega/\delta \omega}$.

![Figure 14.](image)

Experimental results. (a) Signal transmitted through the sample ($L = 40$ mm) and recorded by the array element $n^64$, and (b) signal recreated at the source after time reversal.

The same analysis can also be made in the time domain. Indeed, second-order moments in time and frequency are related by the Wiener-Kinchin theorem $\int \langle H(\omega)H^*(\omega + \delta \omega) \rangle d\omega = \int \langle |h(t)|^2 \rangle e^{j\delta \omega t} dt$. In other words, the spectral correlation function (averaged over the frequency bandwidth) is the Fourier transform of the ‘time of flight’ distribution $\langle |h(t)|^2 \rangle$. Let $\delta t$ be the duration of the pulse obtained after time reversal (i.e. the correlation time of the transmitted signal) and $\Delta T$ the typical duration of the transmitted intensity $|h(t)|^2$. In a multiply scattering medium, in the diffusive approximation, it is well known that the typical spreading time (the so called Thouless time) is equal to $\tau_{\text{thouless}} = L^2/D$ where $D$ is a diffusion coefficient related to the mean free path). Therefore $\Delta T$ grows proportionally to $L^2$. We have $\Delta T = 1/\delta \omega$ and
Figure 15. Directivity pattern of the time-reversed waves around the source position, in water (thick line) and through the rods (thin line), with a 16-element aperture. The sample thickness is \( L = 40 \text{ mm} \). The \(-6\) dB widths are 0.8 and 22 mm, respectively.

Figure 16. Directivity pattern of the time-reversed waves around the source position through \( L = 40 \text{ mm} \), with \( N = 128 \) transducers (thin line) and \( N = 1 \) transducer (thick line). The \(-6\) dB resolutions are 0.84 and 0.9 mm respectively.

\[ \Delta \omega = 1/\delta t, \text{ so the number } I \text{ of uncorrelated frequencies grows with } L \text{ and may be expressed in the time or in the frequency domain as } I = \Delta T/\delta t = \Delta \omega/\delta \omega \propto L^2. \]

Ideally, if we could indefinitely expand the bandwidth or decrease the correlation frequency \( \delta \omega \), the background level of the directivity patterns should decrease as \( 1/\sqrt{I} \). But dissipation effects limit the performance of a time-reversal mirror. When the sample thickness becomes too large, the cumulative effect of dissipation can break the time reversal invariance. Indeed, acoustic losses are taken into account in the wave equation by a first order time derivative that breaks the reversal invariance. In each experiment, we have to compare a dissipative time \( \tau_{\text{dissipat}} \) that measures the dissipative loss to the Thouless time. Ideally \( \tau_{\text{dissipat}} \geq \tau_{\text{thouless}} \). However, we have also extended this study to various media. It was shown for example, that the porosity of the skull bone produces a strong dissipation that breaks the TR symmetry of the wave equation for ultrasound in the MHz frequency range. Therefore TR focusing is no longer appropriate to compensate for the skull properties and new focusing technique have been studied to correct these effects. In a dissipative medium, the time reversal process remains a spatio-temporal matched filter, that is focused on the source, but it no more an inverse filter of the propagation. Therefore strong spatio-temporal side lobes appear. An iterative time-reversal approach has been developed to compensate for these effects [31].

4. Focusing below the diffraction limit

It is important to understand the apparent failure of the time-reversed operation that leads to diffraction limitation. We have to understand the very different roles played respectively by the sources and by the fields. A time reversal operation applied on the wave field do not necessary involves the time reversal of the source. Reversing the field using a TRM without reversing the source (i.e., without creating a local sink) does not correspond to a complete time reversal of the field. Indeed, the basic TR experiment can be interpreted in the following way. The time reversal step described above is not strictly the time-reversal of the first step. During the second step of an ideal time-reversed experiment, the initial active source (that injects some energy into the system) must be replaced by a sink (the time-reversal of a source). An acoustic sink is a device
that absorbs all arriving energy without reflecting it. Taking into account the source term in the
wave equation, reversing time leads to the transformation of source into sink. For an initial
point source transmitting a waveform \( s(t) \), the wave field obeys equation (1) (inhomogeneous
wave equation) with a source term that can be transformed, in the time reversal operation in
\[
(L_r + L_t)\varphi(r, -t) = -\delta(r - r_0)s(-t)
\] (38)
To achieve a perfect time reversal experimentally both the field on the surface of the cavity has
to be time reversed, and the source has to be transformed into a sink. Therefore one may achieve
time-reversed focusing below the diffraction limit. The role of the new source term \( \delta(r - r_0)s(-t) \)
is to transmit a diverging wave that exactly cancels the usual outgoing spherical wave.
In a monochromatic approach, taking into account the evanescent waves concept, the necessity
of replacing a source by a sink in the complete time-reversed operation can be interpret as follows.
In the first step a point like source of size quite smaller than the transmitted wavelengths radiates
a field that can be described as a superposition of homogeneous plane waves propagating in
the various directions \( \vec{k} \) and of decaying, nonpropagating, evanescent plane waves [32]. The
amplitude of the evanescent waves (that contains details of the source smaller than half the
wavelength) decays exponentially with the distance to the source, so that it excludes evanescent
wave contribution in the far field. If the closed cavity is located in the far field of the source, the
time-reversed field retransmitted by the surface of the cavity does not contain these evanescent
components. The role of the sink is to radiate exactly, with the good timing, the evanescent waves
that have been lost during the first step. Therefore the resulting field contains the evanescent
part that is needed to focus below diffraction limits.

Time reversal below the diffraction limit has been experimentally demonstrated in acoustics,
using an acoustic sink placed at the focal point and focal spots with dimension \( \lambda/14 \) have been
observed [33].

One severe drawback for applications of the acoustic sink is the need to use an active source at
the focusing point to exactly cancel the usual diverging wave created during the focusing process.
Since the focusing spot is also proportional to the imaginary part of the Green function in the case
of any heterogeneous non-dissipative medium, another more interesting and general approach
consists to surround the focusing point by a microstructured medium with typical length scales
well below the wavelength, e.g., putting some strong scatterers in the near-field of the source.
In this case, the microstructured medium completely modifies the spatial dependence of the
imaginary part of the Green function that now oscillates on scales smaller than the wavelength.
This is exactly the idea we exploit in the field of time-reversal with microwaves to create focal
spots much thinner than the wavelength. In a recent experiment [34] we consider 8 possible
focusing points placed in a strong reverberating chamber (Fig.17.a). Eight electromagnetic
sources are placed at these 8 locations to be used in the learning step of the TR process. These
sources consist of wire antennas used at a central frequency of 2.45 GHz (i.e., \( \lambda = 12\text{cm} \)). The
pitch between them is \( \lambda/30 \) ! These eight antennas form an array which will be referred to as the
receiving array. Each antenna in this array is surrounded by a microstructure consisting here
of a random distribution of thin copper wires (Fig. 17.b). A TRM made of eight commercial
dipolar antennas is placed in the far-field, ten wavelengths apart from the receiving array. The
set “reverberant chamber/TRM” acts as a virtual far-field time-reversal cavity. When antenna
marked #3 in Fig. 1 sends a short electromagnetic pulse (10 ns), the 8 signals received at
the TRM are much longer than the initial pulse due to strong reverberation in the chamber
(typically 500 ns). As an example the signal received at one of the antennas of the TRM is
shown in Fig. 18.a. When antenna marked #4 is in its turn used as a source, it is remarkable
to point out that now the signal received at the same antenna in the TRM (shown in Fig. 18.b)
looks significantly different although sources #3 and #4 were \( \lambda/30 \) apart from each other. When
these signals are time-reversed and transmitted back, the resulting waves converge respectively
to antenna #3 and #4 where they recreate pulses as short as the initial ones (Fig. 18.c and 18.d). Measuring the signal received at the other antennas of the receiving array gives access to the spatial focusing around antennas #3 and #4 (Fig. 17.e). The remarkable result is that the two antennas can be addressed independently since the focusing spots created around them have a size much less than the wavelength (here typically $\lambda/30$): the diffraction limit is overcome although the focusing points are in the far-field of the TRM!

Figure 17.

(a) A Time Reversal Mirror (TRM) made of eight commercial dipolar antennas operating at 2.45 GHz (i.e., $\lambda = 12$ cm) is placed in a 1-m$^3$ reverberating chamber. Ten wavelengths away from the TRM is placed a subwavelength receiving array, consisting of eight microstructured antennas $\lambda/30$ apart from one another.
(b) Details of one microstructured antenna. It consists of the core of a coaxial line which comes out 2 mm from an insulating layer and is surrounded by a microstructure consisting of a random distribution of thin copper wires.
(c) Photo of the 8-element subwavelength array surrounded by the random distribution of copper wires. Antennas #3 and #4 are indicated by the red and blue arrows.

Contrary to the acoustic sink experiment, in the microwave time reversal experiment, the
In (a) (respectively (b)) is shown the signal received at one antenna of the TRM when a 10-ns pulse is sent from antenna #3 (resp. #4) of the subwavelength array. The signals in (a) and (b) look significantly different although antenna #3 and antenna #4 are distant from $\lambda/30$.

In (c) (resp. (d)) is shown the time compression obtained at antenna #3 (resp. #4) obtained when the eight signals coming from antenna #3 (resp. antenna #4) are time-reversed and sent back from the TRM.

In (e) are shown the focusing spots obtained around antenna #3 and #4. Their width is $\lambda/30$. Thus antenna #3 and #4 can be addressed independently.

Source remains passive and high spatial frequency components of the field are created upon scattering at the disordered structure. Reciprocity ensures that the time-reversed scattering process creates a sub wavelength focus around the source location [35]. The initial evanescent waves created around the initial wire are converted into propagating waves by the random distribution of wires. In the time-reversed step, these propagating waves are playback, from the far field, with reverse $\vec{k}$. Spatial reciprocity ensures that each propagating waves with a reverse $\vec{k}$ interact with the random distribution of wires to exactly recreate the initial evanescent waves around the focus.
5. Link between the time-reversed field and spatial correlation of random wavefield

It has been experimentally demonstrated by various authors that the Green’s function of a complex medium or an irregular finite body can be obtained by cross-correlating the recordings of a diffuse random wave field at two receiver positions [36, 37, 38, 39, 40]. The cross-correlation approach has been applied successfully to helioseismic data [41, 42, 43], ultrasonic measurements [44, 45, 46] and more recently to seismic surface waves [47, 48] and to ocean acoustics [49, 50]. These random wavefields are excited by spatially distributed sources of noise such as turbulence inside the sun, thermal noise in ultrasound, random distribution of seismic events (large distributions of numerous earthquakes during a long period, or seismic noise produced by the Sea-Earth interaction), and acoustic noise in the ocean.

The theoretical aspect of this research is strongly related to the theory of the time reversal cavity and a simple demonstration will be given in the following.

Let us assume that we have a random distribution of noise sources distributed along a surface $S$ surrounding two observations points located respectively at positions $r_0$ and $r$. For example if the diffusive field is radiated by an ensemble of individual incoherent sources $n(r_i, t)$ located at different position $r_i$ (like in expression (16) and (17)) the random field recorded at both points $r$ and $r_0$ can be computed as

$$\varphi_{\text{diff}}(r, t) = \sum_i G(r, r_i; t) \otimes n(r_i, t)$$

Taking into account the noise source cross correlation $\langle n(r_i, t)n(r_j, t') \rangle = \delta(t - t')\delta_{ij}$, it is easy to see that the field-field correlation at $r_0$ and $r$ reads:

$$C_{r,r_0}(t) = \langle \varphi_{\text{diff}}(r, \tau)\varphi_{\text{diff}}(r_0, t + \tau) \rangle = \sum_i G(r_0, r_i; -t) \otimes G(r, r_i; t)$$

If the source of noise is distributed along all the directions with a sufficient sampling, mimicking a perfect time-reversal cavity, then combining equations (8) and (18) gives:

$$\frac{\partial}{\partial t} C_{r,r_0}(t) = G(r, r_0; -t) - G(r, r_0; t)$$

The TR analogy preliminarily indicates that to retrieve the exact details of the Green’s function from the noise correlations, the sources should be placed so that they completely surround the medium and the two sensors. Fortunately in medium with multiple scattering, even if the source of noise is confined in a limited portion of space, a small aperture time-reversal mirror may have a larger virtual aperture. As we have seen, using a large frequency bandwidth with multiple scattering is a way of improving the self averaging process. However, enlarging the frequency band cannot solve all the problems; in particular, it cannot replace source averaging. If we want to retrieve all the details of the exact Green’s function, the only solution is to have sources surrounding the medium. But if we just need a simple estimation of the first arrival of the Green’s function (the ballistic contribution), then enlarging the frequency band helps.

6. Conclusion

In this paper, we have shown that in presence of multiple reflections or multiple scattering, a small size time reversal mirror manages to focus a pulse back to the source with a spatial resolution that beats the diffraction limit. The resolution is no more dependent on the mirror aperture size but it is only limited by the spatial correlation of the wave field. In these media, due to a sort of kaleidoscopic effect that creates virtual transducers, the TRM appears to have an effective aperture that is much larger than its physical size. Resolution can be improved in reverberating media using this concept. Time reversal focusing opens also completely new
approaches to super-resolution. We have show that in medium made of random distribution of sub-wavelength scatterers, a time-reversed wave field interacts with the random medium to regenerate not only the propagating but also the evanescent waves required to refocus below the diffraction limit. Focal spots as small as $\lambda/30$ have been demonstrated with microwaves. This results in a large increase of the information transfer rate by time reversal in such disordered media.

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