Large N Expansion From Fuzzy AdS$_2$

Pei-Ming Ho$^1$, Miao Li$^{2,1}$

$^1$ Department of Physics, National Taiwan University, Taipei 106, Taiwan

$^2$ Institute of Theoretical Physics, Academia Sinica, Beijing 100080

pmho@phys.ntu.edu.tw
mli@phys.ntu.edu.tw

Abstract

We study the quantum analogue of primary fields and their descendants on fuzzy AdS$_2$, proposed in [hep-th/0004072]. Three-point vertices are calculated and shown to exhibit the conventional $1/N$ expansion as well as nonperturbative effects in large $N$, thus providing a strong consistency check of the fuzzy AdS$_2$ model. A few new physical motivations for this model are also presented.
1 Introduction

Quantum mechanics and Einstein’s general relativity are two major achievements in physics in twentieth century. In the last thirty years, string theory has become the most promising theory unifying these two otherwise highly irreconcilable theories. Nevertheless in string theory, until quite recently, spacetime has been treated as a continuous platform upon which one builds up a perturbation theory. It was suspected by many that when gravitational interactions become strong, a continuous spacetime is doomed, and one has to replace it by something else. In other words, when nonperturbative effects are properly taken into account, spacetime in string theory will become fundamentally fuzzy.

A concrete model in which space and time are quantized was proposed very recently [1]. The classical counter-part is the two dimensional anti-de Sitter space which one can obtain as the near horizon limit of an extremal black hole in 4 dimensions. Quantization of spacetime has a long history, it was first proposed in [4, 5]. The motivation at that time is to cure the UV divergences in QED, and this is rendered unnecessary by the modern quantum field theory. However, very general arguments show that spacetime are fundamentally noncommutative in string theory [6]. And that some unusual physics required to explain quantum black holes also indicates that space and time are quantized simultaneously [3]. One piece of such unusual physics is teleology which was beautifully demonstrated in a recent paper [6] in one of the simplest models of noncommutative spacetime. Another manifestation of noncommutativity of spacetime is the UV/IR relation in the AdS/CFT correspondence [7], indeed one of our motivations to propose the fuzzy $AdS_2$ model is to derive holography from spacetime noncommutativity.

Note that quantum spheres in the AdS/CFT correspondence were first proposed in [8, 9] to explain the stringy exclusion principle [10]. A remarkable physical mechanism for this phenomenon is proposed in [11], and subsequently it was shown in [12] that this mechanism is compatible with the stringy spacetime uncertainty relation, providing another indication of quantization of space and time. For even dimensional spheres, it is shown in [13] that the quantum geometry of the sphere is really a fuzzy sphere. In particular, for $S^4$, the model already made its appearance in [13, 14]. The fuzzy $S^2$ is determined using the dipole mechanism and the fuzzy $AdS_2$ is proposed based on analogy.

The fuzzy $AdS_2$ model resembles much the noncommutative horizon of a 4 dimensional Schwarzschild black hole proposed by ’t Hooft [15]. Its study and clarification of many physics issues in this context will undoubtedly shed much light on the quantum
physics of black holes which have been source of riddles as well as inspirations.

We shall make a first step in studying physics in the fuzzy $AdS_2$ in this paper. The uncertainty relation results from this model is unusual in that it is much larger than the one showing up for instance in \cite{1}. This is not surprising to us, since this model is a descendant of the 4 dimensional extremal black hole whose quantum properties are unusual too. The most convincing check that our model is correct is the right pattern of $1/N$ expansion for the effective interaction vertices. We will in the next section review some discussion of \cite{15} on the solution to the Laplace equation in $AdS_2$, and in sect.3 review our definition of fuzzy $AdS_2$. We fix the geometric data of the fuzzy $AdS_2$ in sect.4 and solve the quantum analogue of the Laplace equation. Sect.5 computes the three-point vertices, and demonstrates $1/N$ expansion in these vertices. Remarkably, there are also instanton effects such as terms $\exp(-2\pi nN)$, similar to \cite{16}. In the discussion section we present a shock-wave argument for fuzzy $AdS_2$.

We leave many interesting questions, such as how to understand holography, to future work.

## 2 $AdS_2/CFT_1$ Duality

In this section we review what has been discussed about the $AdS_2/CFT_1$ duality \cite{15}.

For a gravitational theory on $AdS_2$, the isometry group $SL(2,\mathbb{R})$ is enhanced to half of the 1+1 dimensional conformal group. We denote the Killing vectors for the isometry group by $L_0, L_1, L_{-1}$.

For a scalar field $\Phi$ of mass $M$ propagating in the bulk of $AdS_2$, its wave equation is

$$\nabla^2 \Phi = M^2 \Phi. \quad (1)$$

The Laplacian $\nabla^2$ is just $4/R^2$ times the Casimir of $SL(2,\mathbb{R})$. To get a primary solution of the wave equation, we impose the constraints

$$L_1 \Phi_h = 0, \quad L_0 \Phi_h = \hbar \Phi_h \quad (2)$$
on $\Phi$. Then the solution is easily found to be

$$\Phi_h = (e^{-2iu^+} - e^{-2iu^-})^h, \quad (3)$$

where we used the global coordinates

$$ds^2 = \frac{4R^2}{\sin^2(u^+ - u^-)} du^+ du^- . \quad (4)$$
This solution is normalizable and has

\[ M^2 = 4h(h - 1)/R^2. \]  

The primary solution in the bulk corresponds to a primary operator on the boundary CFT.

Other solutions of the wave equation with which \( \Phi_h \) forms an irreducible representation of \( SL(2, \mathbb{R}) \) can be obtained by acting \( L_{-1} \) on \( \Phi_h \). Denote the normalized solutions obtained this way by \( \tilde{\Psi}_{hn} \), where \( n \) is the number of times \( L_{-1} \) acts on \( \Phi_h \). This is a highest weight representation. Its complex conjugation gives a lowest weight representation. All together they form a complete basis with which one can expand a scalar field of mass \( M = 2\sqrt{h(h - 1)/R} \) as

\[ \Phi = \sum_{n=0}^{\infty} a_n^{\dagger} \tilde{\Psi}_{hn}^{\dagger} + a_n \tilde{\Psi}_{hn}. \]  

The canonical commutation relation is

\[ [a_m, a_n^{\dagger}] = \delta_{mn}. \]  

As \( h \) is roughly proportional to the mass of a scalar field, it was shown that there should exist a cutoff of \( h \) around the value

\[ h \sim 1/G_N, \]  

where \( G_N \) is the Newton constant in \( AdS_2 \).

### 3 Fuzzy \( AdS_2 \times S^2 \)

Consider the near horizon limit of a 4 dimensional charged black hole in string theory [18]. For instance, by wrapping two sets of membranes and two sets of M5-branes in \( T^7 \), one obtains a 4D charged, extremal black hole [19]. The brane configuration is as follows. Denote the coordinates of \( T^7 \) by \( x_i, i = 1, \ldots, 7 \). A set of membranes are wrapped on \( (x_1, x_2) \), another set are wrapped on \( (x_3, x_4) \). A set of M5-branes are wrapped on \( (x_1, x_3, x_5, x_6, x_7) \), the second set are wrapped on \( (x_2, x_4, x_5, x_6, x_7) \). By setting all charges to be \( N \), one finds the metric of \( AdS_2 \times S^2 \) for \( (x_0, x_8, x_9, x_{10}) \):

\[ ds^2 = l_p^2 \left( -\frac{r^2}{N^2} dt^2 + \frac{N^2}{r^2} dr^2 + N^2 d\Omega_2 \right), \]

\[ F = -Nd\Omega_{1+1} - Nd\Omega_2, \]  

3
where $l_p$ is the 4 dimensional Planck length, $d\Omega_{1+1}$ and $d\Omega_2$ are the volume forms on $AdS_2$ and $S^2$, respectively. The field $F$ is just the linear combination of all anti-symmetric tensor fields involved. Note that here for simplicity, we consider the most symmetric case in which all the charges appearing in the harmonics $1 + Q_i l_p/r$ are just $N$ which in turn is equal to the number of corresponding branes used to generate this potential. As a consequence, the tension of the branes compensates the volume of the complementary torus. This means that the size of each circle of $T^7$ is at the scale of the M theory Planck length.

The same space $AdS_2 \times S^2$ can also be obtained by taking the near horizon limit of the 4 dimensional extremal Reissner-Nordtström solution.

### 3.1 Fuzzy $S^2$

In [1] we proposed that the $S^2$ part of the $AdS_2 \times S^2$ space is a fuzzy $S^2$ [21] defined by

$$[Y^a, Y^b] = i\epsilon^{abc}Y^c,$$  

(11)

where $Y_a$’s are the Cartesian coordinates of $S^2$. (We use the unit system in which $l_p = 1$.) This algebra respects the classical $SO(3)$ invariance.

The commutation relations (11) are the same as the $SU(2)$ Lie algebra. For the $(2N + 1)$ dimensional irreducible representation of $SU(2)$, the spectrum of $Y_a$ is $\{-N, -(N-1), \cdots, (N-1), N\}$. and its second Casimir is

$$\sum_{a=1}^{3}(Y_a)^2 = N(N+1).$$  

(12)

Since the radius of the $S^2$ is $Nl_p$ (in the leading power of $N$), we should realize the $Y_a$’s as $N \times N$ matrices on this irreducible representation of $SU(2)$.

An evidence for this proposal is the following. For a fractional membrane wrapped on $(x_1, x_3)$ or $(x_2, x_4)$, it is charged under the $F$ field generated by a set of M5-branes. Denote the polar and azimuthal angles of $S^2$ by $(\theta, \phi)$. The stable trajectories of the membrane with angular momentum $M$ are discrete, and

$$\cos \theta = \frac{M}{N}.$$  

(13)

Furthermore, they all have the same energy of $1/N$. It follows that, since $M$ is conjugate to $\phi$, $\cos \theta$ and $\phi$ do not commute with each other in the quantized theory. The resulting Poisson structure on $S^2$ is precisely that of the fuzzy sphere.
In [17], it was proposed that in 2 + 1 dimensions the spacetime coordinates are quantized according to

\[ [x, y] = \frac{i}{\cos^2 \mu} L, \]  

where \( \cos \mu \) is related to the mass of the particle, and \( L \) is the angular momentum on the 2 dimensional space. To complete the algebra we also have

\[ [L, x] = iy, \quad [L, y] = -ix. \]  

This algebra is Lorentz invariant, and its 3+1 dimensional generalization was given by Yang [3] much earlier. Note that this algebra (14) was proposed based on general grounds for a gravitational theory in 2+1 dimensions, and we are content with the fact that it is actually a consequence of the fuzzy \( S^2 \) for massless particles \((\cos \mu = 1)\), where \( Y_3 \) acts on \( Y_1 \) and \( Y_2 \) as the angular momentum operator.

### 3.2 Fuzzy \( AdS_2 \)

In [1] we further proposed that the \( AdS_2 \) part is also quantized. Let \( X^{-1}, X^0, X^1 \) be the Cartesian coordinates of \( AdS_2 \). The algebra of fuzzy \( AdS_2 \) is

\[ [X^{-1}, X^0] = -iX^1, \]  
\[ [X^0, X^1] = iX^{-1}, \]  
\[ [X^1, X^{-1}] = iX^0, \]

which is obtained from the fuzzy \( S^2 \) by a “Wick rotation” of the time directions \( X^0, X^{-1} \). The “radius” of \( AdS_2 \) is \( R = Nl_p \), so

\[ \eta_{\mu\nu} X^\mu X^\nu = (X^{-1})^2 + (X^0)^2 - (X^1)^2 = R^2, \]  

where \( \eta = \text{diag}(1, 1, -1) \). The isometry group \( SL(2, \mathbb{R}) \) of the classical \( AdS_2 \) is a symmetry of this algebra, and thus is also the isometry group of the fuzzy \( AdS_2 \).

For later use, we define the raising and lowering operators

\[ X_\pm \equiv X^{-1} \pm iX^0, \]  

which satisfy

\[ [X^1, X_\pm] = \pm X_\pm, \quad [X_+, X_-] = -2X^1, \]  

according to (16)-(18).
The radial coordinate \( r \) and the boundary time coordinate \( t \) are defined in terms of the \( X \)'s as
\[
  r = X^{-1} + X^1, \quad t = \frac{R}{2} (r^{-1} X^0 + X^0 r^{-1}),
\]  \tag{22}
where we symmetrized the products of \( r^{-1} \) and \( X^0 \) so that \( t \) is a Hermitian operator. The metric in terms of these coordinates assumes the form (I). It follows that the commutation relation for \( r \) and \( t \) is
\[
  [r, t] = -i R l_p. \tag{23}
\]

The following simple heuristic argument also suggests this commutation relation. Consider a closed string in \( AdS_2 \). (Since the space is one dimensional, the closed string actually looks like an open string with twice the tension.) Take the Nambu-Goto action for a fractional string of tension \( 1/N \) and take the static gauge \( t = p_0 \tau \). It follows that the action is
\[
  S = \frac{1}{2\pi N \alpha'} \int_{-\infty}^{\infty} dt \int_0^{2\pi} d\sigma \sqrt{(p_0 \dot{r}^2)}
  = \frac{1}{\pi N \alpha'} \int dt \int_0^{\pi} p_0 |\dot{r}|
  = \frac{1}{N \alpha'} \int d\tau \dot{r} \frac{\partial t}{\partial \tau}, \tag{24}
\]
where we have assumed that \( \dot{r} > 0 \) for \( 0 < \sigma < \pi \) and \( \dot{r} < 0 \) for \( \pi < \sigma < 2\pi \). The last line above shows clearly that \( r \) is the conjugate variable to \( t \) up to a factor of \( N \alpha' = R l_p \), and so the quantization of this fundamental string is (23). We will set \( l_p = 1 \) in the rest of the paper.

4 Properties of Fuzzy \( AdS_2 \)

One can realize the algebra (16)-(18), which is the same as the Lie algebra of \( SL(2, \mathbb{R}) \), on a unitary irreducible representation of \( SL(2, \mathbb{R}) \). The question is which representation is the correct choice. As we show in appendix A, since the range of \( X_1 \) goes from \( -\infty \) to \( \infty \) for \( AdS_2 \); when \( R > 1/2 \), the proper choice is the principal continuous series, and when \( R < 1/2 \), it is the complementary series. Since we have \( R = N > 1 \) for our physical system, we should consider the principal continuous series only. A representation in this series is labeled by two parameters \( j = 1/2 + is \) and \( \alpha \), where \( s, \alpha \) are real numbers, and \( 0 \leq \alpha < 1 \). The label \( j \) determines the second Casimir as
\[
  c_2 = \eta_{\mu\nu} X^\mu X^\nu. \tag{25}
\]
It follows from (19) and $R = N$ that one should take

$$j = 1/2 + iN.$$  \hspace{1cm} (26)

However, strictly speaking, we can only be sure of these relations in the leading power of $N$.

What is the physical interpretation of the other label $\alpha$ of these representations? We will argue in sect.5 that $\alpha$ corresponds to the VEV of axion when viewing this system as IIB string theory via duality. However, for most of the discussions below, we will focus on the representation with $(j = 1/2 + iN, \alpha = 0)$, to be denoted by $D_N$. This is the case for which the reflection symmetry $X^1 \rightarrow -X^1$ is not broken.

Functions on the fuzzy $AdS_2$ are functions of the $X$’s. They form representations of the isometry group $SL(2, \mathbb{R})$. The three generators $L_{\mu\nu}$ of the isometry group act on $X$ as

$$[L_{\mu\nu}, X^\kappa] = i(\delta^\kappa_\nu X_\mu - \delta^\kappa_\mu X_\nu),$$ \hspace{1cm} (27)

where $X_\mu = \eta_{\mu\nu}X^\nu$. A very interesting property of the algebra of fuzzy $AdS_2$ is that the action of the generators $L_{\mu\nu}$ is the same as the adjoint action of the $X_\kappa$. That is,

$$[L_{\mu\nu}, f(X)] = \epsilon_{\mu\nu\kappa}[X^\kappa, f(X)]$$ \hspace{1cm} (28)

for an arbitrary function $f(X)$. The operators $L_{\mu\nu}$ act on the functions as differential operators.

The integration over the fuzzy $AdS_2$ is just the trace over the representation $D_N$

$$\int f(X) \equiv c\text{Tr}(f(X)) = c\sum_{n \in \mathbb{Z}} \langle n|f(X)|n\rangle,$$ \hspace{1cm} (29)

where $c$ is a real number. This integration is invariant under $SL(2, \mathbb{R})$ transformations. In the large $N$ limit, $c_2 \gg 1$, the trace can be calculated and its comparison with an ordinary integration on the classical $AdS_2$ with metric (9) shows that

$$c = 2\pi N$$ \hspace{1cm} (30)

in the leading power of $N$. The inner product of two functions, as well as the norm of a function are defined by integration over the fuzzy $AdS_2$ in the usual way:

$$\langle f(X)|g(X)\rangle = \int f^\dagger(X)g(X), \quad \|f(X)\|_2 = \int f(X)^\dagger f(X).$$ \hspace{1cm} (31)

In view of organizing the functions into representations of $SL(2, \mathbb{R})$, in order to describe the boundary CFT dual to the bulk theory on fuzzy $AdS_2$ via holography,
we derived all functions corresponding to the lowest and highest weight states in the principal discrete series. The information about the underlying fuzzy $AdS_2$, i.e., the value of $N$, is encoded in the precise expressions of these functions.

Denote the lowest weight state by $\Psi_{jj}$, or just $\Psi_j$. The states of higher weights $\Psi_{jm} (m > j)$ in the same irreducible representation are obtained as

$$[X_+, [X_+, \cdots [X_+, \Psi_j] \cdots]],$$

where $X_+$ appears $(m-j)$ times. In appendix B, we find the explicit expressions for the functions $\Psi_j$ as

$$\Psi_j = \left( \frac{1}{X^1(X^1-1)+c_2X^+} \right)^j.$$  

(32)

In the large $N$ limit, using $c_2 = R^2$ and the following coordinate transformation

$$X^1 = R \cot(u^+ - u^-), \quad X_\pm = \frac{R}{\sin(u^+ - u^-)} e^{\pm i(u^+ + u^-)},$$

(34)

one finds

$$\Psi_j \rightarrow \left( \frac{e^{-2iu^+} - e^{-2iu^-}}{-2iR} \right)^j,$$

(35)

where $u^+, u^-$ are the coordinates appearing in (34), in agreement with (3).

Let

$$I_{jm} \equiv \text{Tr}(\Psi_{jm}^\dagger \Psi_{jm}),$$

(36)

then the normalized states are

$$\tilde{\Psi}_{jm} \equiv \frac{1}{\sqrt{cI_{jm}}} \Psi_{jm}.$$  

(37)

As a normalized basis of an $SL(2, \mathbb{R})$ representation, they satisfy

$$[X_+, \tilde{\Psi}_{jm}] = a_{jm+1} \tilde{\Psi}_{jm+1},$$

(38)

$$[X_-, \tilde{\Psi}_{jm}] = a_{jm} \tilde{\Psi}_{jm-1},$$

(39)

where

$$a_{jm} = \sqrt{m(m-1) - j(j-1)}.$$  

(40)

The explicit expressions of $I_m \equiv I_{mm}$ are given in appendix C. They will be used later when we try to extract physical information from the fuzziness of the $AdS_2$.

For a field $\Phi$ in the bulk of the fuzzy $AdS_2$, one can decompose it into the basis functions $\tilde{\Psi}_{jm}$ as

$$\Phi(X) = \sum_{jm} \phi_{jm} \tilde{\Psi}_{jm}(X),$$

(41)
where $\phi_{jm}$ are the creation/annihilation operators of the physical state with the wave function $\tilde{\Psi}(X)$. By holography, these states are identified with those in the boundary theory, which is a one-dimensional theory.

An interesting question for a wave function on noncommutative theory is how to define expectation values. For instance, should the expectation value of $X_1$ for a wave function $\Psi$ be

$$\text{(1)} \int \tilde{\Psi}^{\dagger} X_1 \tilde{\Psi}, \quad \text{(2)} \int \Psi X_1^{\dagger} \Psi, \quad \text{or (3)} \frac{1}{2} \int (X_1^{\dagger} \Psi \Psi + \Psi^{\dagger} \Psi X_1)? \quad \text{(42)}$$

The answer is that it depends on how one measures it. If one measures the $X_1$ location of the wave function according to its interaction with another field $\Phi$ under control in the experiment, and if the interaction is described in the action by a term like

$$\int \tilde{\Psi}^{\dagger} \Phi \Psi, \quad \text{(43)}$$

then we expect that the choice (1) is correct. But if the interaction is written differently, the definition of expectation value should be modified accordingly.

Fixing a definition of the expectation value of $X_1$, such as case (1) in (42), one finds that for the same $j$, the larger $m$ is, which means larger energy at the boundary, the larger is the expectation value of $X_1$ for the wave function $\Psi_{jm}$. This can also be viewed as a manifestation of the UV/IR relation in $AdS$ space.

**5 Interaction in Fuzzy $AdS_2$**

To see how the noncommutativity of the fuzzy $AdS_2$ incorporate physical data, presumably including the effect of string quantization on $AdS_2$, we consider an interaction term in the action of the bulk theory of the form

$$S_I = \lambda \int \Phi_1^{\dagger} \Phi_2 \Phi_3, \quad \text{(44)}$$

where $\lambda$ is the coupling constant for this three point interaction.

Expanding all three $\Phi_i$'s as $\text{(11)}$ in the action, one obtains the vertex

$$c\lambda \text{Tr}(\tilde{\Psi}_{1m_1}^{\dagger} \tilde{\Psi}_{2m_2} \tilde{\Psi}_{3m_3}) \quad \text{(45)}$$

for the three states $(\phi_1)_{1m_1}$, $(\phi_2)_{2m_2}$ and $(\phi_3)_{3m_3}$. Obviously, due to the isometry, the vertex vanishes unless $m_1 = m_2 + m_3$. 9
For simplicity, consider the special case where all three states participating the interaction are lowest weight states. Then the vertex \( \lambda V_{m_1m_2m_3} \) in question is \( \lambda V_{m_1m_2m_3} = \frac{1}{c I_{m_1} I_{m_2} I_{m_3}} \). (46)

In appendix C, we find that

\[
I_m = \frac{(2m - 2)!}{((m - 1)!)^2} \left[ \prod_{k=1}^{m-1} \frac{1}{k^2 - 1 + 4c_2} \right] I_1,
\]

where

\[
I_1 = \frac{\pi}{\sqrt{c_2 - 1/4}} \tanh \left( \pi \sqrt{c_2 - 1/4} \right).
\]

We therefore have the large \( N \) expansion of the vertex (46). Using (26), one finds

\[
V_{m_1m_2m_3}^2 = \mathcal{N}_{m_2m_3} \frac{\left[ \prod_{j=1}^{m_2} (1 + j_2^2/4N^2) \right] \left[ \prod_{j=1}^{m_3} (1 + j_3^2/4N^2) \right]}{8\pi^2 N^2 \left[ \prod_{j=1}^{m_1} (1 + j_1^2/4N^2) \right]} \left( 1 + 2 \sum_{n=1}^{\infty} e^{-2\pi nN} \right),
\]

where

\[
\mathcal{N}_{m_2m_3} = \frac{[2(m_1 - 1)]![(m_2 - 1)]![(m_3 - 1)]!^2}{[(m_1 - 1)!]^2[2(m_2 - 1)]![2(m_3 - 1)]!}.
\]

With the possibility of corrections to (26) of order \( 1/N^0 \sim 1 \), the \( 1/N \) expansion of the vertex is of the form

\[
V_{m_1m_2m_3} \sim \frac{K N}{N} \left( 1 + \sum_{n=1}^{\infty} \frac{a_n}{N^{2n}} + \sum_{n=1}^{\infty} e^{-2\pi nN} \sum_{k=0}^{\infty} \frac{b_{nk}}{N^{2k}} \right).
\]

This expression is reminiscent of a correlation function in the case of type IIB strings on \( AdS_5 \times S^5 \). It calls for an analogous interpretation.

The \( 1/N^2 \) expansion in (51) suggests that the coupling constant in \( AdS_2 \) is of order \( 1/N^2 \). This is indeed the case. The 11 dimensional Newton constant is just 1 in Planck units. Compactifying on \( S^2 \times T^7 \) of size \( 4\pi N^2 \) results in a dimensionless effective Newton constant of order \( 1/N^2 \) in \( AdS_2 \).

The overall factor of \( 1/N \) in (51) is what one expects for a three-point correlation function. The Newton constant is \( 1/N^2 \) in \( AdS_2 \) as we just mentioned, on the other hand it is also \( g_s^2 \) from the viewpoint of a string theory on \( AdS_2 \). This means that \( g_s \sim 1/N \). While the coupling constant \( \lambda \) in the bulk theory should not depend on \( N \) since it should not depend on the geometry of the space it lives on, the vertex (13) should have the leading dependence on \( N \) of a three-point function for closed string
states, which is of order \( g_s \sim 1/N \). This is exactly the dependence we find for the vertex (51).

Finally, we identify the terms in (51) proportional to \( \exp(-2\pi nN) \) as contributions from instantons. This implies that the action of a single instanton equals \( 2\pi N \). We have just argued that the string coupling constant \( g_s \) is of order \( 1/N \). Since the D-instanton action is \( 2\pi/g_s \), it is precisely \( 2\pi N \) as we wish.

Similarly, we can consider \( n \)-point interaction vertex in the bulk theory on \( AdS_2 \):

\[
S_n = \lambda \int \Phi_1^\dagger \Phi_2 \cdots \Phi_n.
\]  

(52)

The leading dependence of the vertex will be \( 1/N^{n-2} \), which is exactly what it should be for an \( n \)-point function in string theory with coupling constant \( g_s \sim 1/N \).

We conjecture that for M theory compactified on \( AdS_2 \times S^2 \), the perturbative as well as nonperturbative effects of string quantization are encoded in the noncommutativity of the fuzzy \( AdS_2 \times S^2 \), in the sense that the low energy effective theory is most economically written as a field theory on this noncommutative space.

Finally we make a comment on the other label \( \alpha \) for a representation in the principal continuous series. We have set \( \alpha = 0 \) in the above. When \( \alpha \) is turned on, the expression of \( a_j(m+\alpha) \) becomes

\[
a_j(m+\alpha) = \sqrt{(m - 1/2 + \tau)(m - 1/2 + \bar{\tau})}
\]

(53)

for \( m \in \mathbb{Z} \), where

\[
\tau = \alpha + iN \simeq \alpha + \frac{i}{g_s}.
\]

(54)

The appearance of this combination of \( \alpha \) with \( g_s \) suggests the identification of \( \alpha \) with \( \chi/2\pi \), where \( \chi \) is the VEV of the axion field in IIB theory. While \( \chi \) is only defined up to \( 2\pi \), \( \alpha \) is also defined only up to an integer.

6 Discussion

Although we already argued for the spacetime uncertainty in the fuzzy \( AdS_2 \) from the general stringy uncertainty, it should be interesting to compare the way ’t Hooft introduces spacetime noncommutativity in his S-matrix ansatz [4, 22]. The basic method is to study the effect of a shock-wave on a particle moving in the opposite direction.

Use the global coordinates (4). The metric induced by a left-moving shock-wave assumes the form

\[
ds^2 = -e^{2\phi}dudv + h(du)^2,
\]

(55)
where
\[ e^{2\phi} = \frac{4R^2}{\sin^2(u^+ - u^-)}. \]
The scalar curvature is perturbed by a term
\[ \frac{1}{2} e^{-2\phi} \partial_- (e^{-2\phi} \partial_+ h), \] (56)
and the Einstein equation with a constant negative curvature is solved provided
\[ h = \cot(u^+ - u^-) g(u^+) + f(u^+). \] (57)

In the full 4 dimensions, we expect another Einstein equation of the form \( G_{++} = 8\pi G_4 T_{++} \), where \( G_4 \) is the 4D Newton constant. Now \( G_{++} \sim h\mathcal{R}, \mathcal{R} \) is the scalar curvature. For a stress tensor \( T_{++} \) proportional to \( \delta(u^+ - x^+) \), the only solution is \( g(u^+) = 0 \) and
\[ h \sim l_p^2 \delta(u^+ - x^+). \] (58)

The proportionality constant is determined by how the stress tensor is normalized. For a S-wave shock-wave smeared over \( S^2 \), it is \( p_+ / R^2 \). However, the dipole mechanism of [1] seems to indicate that a shock-wave must be localized on a strip on \( S^2 \) whose area is proportional to \( Nl_p^2 \). If true, we expect
\[ \int du^+ T_{++} \sim \frac{p_+}{R l_p}, \] (59)
and this leads to
\[ h \sim R l_p p_+ \delta(u^+ - x^+). \] (60)

Now the shift on \( u^- \) induced on a right-moving particle by the shock-wave is
\[ \Delta u^- \sim \frac{p_+}{N} \sin^2(x^+ - u^-) \] (61)
as can be computed using (53). This shift suggests a commutator
\[ [u^+, u^-] \sim \frac{i}{N} \sin^2(u^+ - u^-) \] (62)
the one that is compatible with our fuzzy AdS_2 model.

**Acknowledgment**

This work is supported in part by the National Science Council, Taiwan, and the Center for Theoretical Physics at National Taiwan University. The work of M.L. is also supported by a “Hundred People Project” grant.
A Representations of $SL(2, \mathbb{R})$

The algebra of fuzzy $AdS_2$ can be realized on an irreducible representation of $SL(2, \mathbb{R})$ ($SU(1,1)$). Similar to the case of $SU(2)$, a representation of $SL(2, \mathbb{R})$ can be easily constructed using the raising and lowering operators (20). The $X$’s act on the basis $\{|m\rangle\}$ as

\begin{align*}
X^1|m\rangle & = m|m\rangle, \\
X_+|m\rangle & = a_{m+1}|m+1\rangle, \\
X_-|m\rangle & = a_m|m-1\rangle,
\end{align*}

where

\[ a_m = \sqrt{m(m-1) + c_2}. \]

The second Casimir is

\[ c_2 = (X^{-1})^2 + (X^0)^2 - (X_1)^2 = -j(j-1), \]

which should be identified with $R^2 = N^2$.

However, not all such representations are unitary representations. They are unitary only in the following four situations.

1. Principal discrete representations (lowest weight):
   \[ j \in \mathbb{R}, j > 0, \alpha = 0 \text{ and } m = j, j+1, j+2, \cdots. \]

2. Principal discrete representations (highest weight):
   \[ j \in \mathbb{R}, j > 0, \alpha = 0 \text{ and } m = -j, -j-1, -j-2, \cdots. \]

3. Principal continuous representations:
   \[ j = 1/2 + is, \text{ where } s \in \mathbb{R}, \text{ and without loss of generality, one can choose } \alpha \text{ such that } 0 \leq \alpha < 1. \text{ The eigenvalues of } X^1 \text{ are } m = \alpha, \alpha \pm 1, \alpha \pm 2, \cdots. \]

4. Complementary representations:
   \[ 1/2 < j < 1 \text{ and } (j - 1/2) < |\alpha - 1/2| \text{ (} j \in \mathbb{R} \text{)}, \text{ where again we assumed that } 0 \leq \alpha < 1. \text{ The eigenvalues of } X^1 \text{ are again } m = \alpha, \alpha \pm 1, \alpha \pm 2, \cdots. \]

All unitary irreducible representations of $SL(2, \mathbb{R})$ fit in these four cases, except the trivial identity representation. The two principal discrete representations correspond to a fuzzy version of the de Sitter space, which can be obtained as one of the two hypersurfaces in 2+1 dimensional Minkowski space by requiring that $(X^{-1})^2 + (X^0)^2 -$
\((X^1)^2 = -R^2\). This can be seen by examining the spectrum of \(X^1\). On the other hand, the spectrum of \(X^1\) for the other two series are both extending from \(-\infty\) to \(\infty\). Furthermore, from \((67)\) we see that For \(R > 1/2\) we have to choose the principal continuous series, and for \(R < 1/2\) the complementary series.

\section{Functions on Fuzzy AdS\(_2\) as Rep.s of \(SL(2,R)\)}

A function \(\Psi\) on the fuzzy AdS\(_2\) corresponds to a lowest weight state if

\[ [X_-, \Psi]|n\rangle = 0 \quad \text{for all } n \in \mathbb{Z}. \]  

(68)

Take the ansatz for the lowest weight state in a principal discrete representation with \(j = m \in \mathbb{Z}\) as

\[ \Psi_m = f_m(X^1)X^m_+. \]  

(69)

One can solve the function \(f_m(X^1)\) up to an overall normalization by imposing (68). The result is

\[ f_m(X^1) = \frac{1}{\prod_{k=1}^{m}(X^1 - k + 1)(X^1 - k) + c_2} = \frac{\Gamma(X^1 - m + (1 - \sigma)/2)\Gamma(X^1 - m + (1 + \sigma)/2)}{\Gamma(X^1 + (1 - \sigma)/2)\Gamma(X^1 + (1 + \sigma)/2)}, \]  

(70)

where \(\sigma = \sqrt{1 - 4c_2}\). Inserting this back to (69), we find

\[ \Psi_m = \left(\frac{1}{X^1(X^1 - 1) + c_2}X^1_+\right)^m. \]  

(71)

States in the same representation of higher weights can be obtained by taking commutators of \(X_+\) with \(\Psi_m\) as in (62).

The highest weight representations in the principal discrete series can be obtained in a similar way, or by simply taking Hermitian conjugation of the functions in the lowest weight representations.

\section{Calculation of \(I_m\)}

From (71) and (62), we find

\[ I_m(x) = \sum_{n = -\infty}^{\infty} \langle n | \Psi^1_m \Psi_m | n \rangle = \sum_{n = -\infty}^{\infty} \prod_{k=1}^{m} \frac{1}{a_{m+k}^2(x)}, \]  

(72)
where
\[ a_m^2(x) = m(m-1) + x. \] (73)

In the expressions above we view \( I_m \) as a function of the Casimir \( c_2 \) and we have denoted \( c_2 \) by \( x \).

The expression of \( I_m(x) \) can be reduced algebraically to an expression involving only \( I_1 \) times a function of \( x \). While generic \( I_m \) can be calculated similarly, we show here only the case of \( I_2 \):

\[
I_2(x) = \sum_{n=-\infty}^{\infty} \frac{1}{(n(n-1)+x)(n(n+1)+x)}
\]
\[
= \sum_{n} \frac{1}{2n} \left( \frac{1}{n(n-1)+x} - \frac{1}{n(n+1)+x} \right)
\]
\[
= \sum_{n} \frac{1}{2} \left( \frac{1}{n(n-1)} - \frac{1}{n(n+1)} \right) \frac{1}{n(n-1)+x}
\]
\[
= \sum_{n} \frac{-1}{2} \frac{1}{n(n-1)(n(n+1)+x)}
\]
\[
= \sum_{n} \frac{-1}{2x} \left( \frac{1}{n(n-1)} - \frac{1}{n(n-1)+x} \right)
\]
\[
= \frac{-1}{2x} \left( \sum_{n} \left( \frac{1}{n} - \frac{1}{n} \right) - \sum_{n} \frac{1}{n(n-1)+x} \right)
\]
\[
= \frac{1}{2x} I_1. \] (74)

This derivation is not rigorous because terms involving \( 1/n \) appear and it diverges at \( n = 0 \). However it suffices to show the idea. It is easy to modify it to get a well-defined, rigorous derivation which leads to exactly the same final result. The general expression for \( I_m \) is

\[
I_m(x) = \frac{(2m-2)!}{[(m-1)!!]^2} \left[ \prod_{k=1}^{m-1} \frac{1}{k^2 - 1 + 4c_2} \right] I_1(x). \] (75)

It is just \( I_1 \) divided by a polynomial of \( x \) of order \((m-1)\). So our task is reduced to calculate \( I_1(x) \).

Viewing \( I_1(x) \) as a holomorphic function of \( x \in \mathbb{C} \), one can convince him/herself of the following equality

\[
I_1(x) = \frac{\pi}{\sqrt{x-1/4}} \tanh \left( \pi \sqrt{x-1/4} \right) \] (76)

by checking that the two sides of the equal sign share the following properties:

1. All the poles they have are at \( x = -n(n-1) \) with the residue 2 for all integers
$n \in \mathbb{Z}$.

(2) They have zeros at $x = -n(n - 1)$ for any half-integer $n$ except $1/2$.

(3) They equal $\pi^2$ at $x = 1/4$.

(4) They approach the function $\pi/\sqrt{x}$ as $x \to \infty$. 

References

[1] P.-M. Ho, M. Li, “Fuzzy Spheres in AdS/CFT Correspondence and Holography from Noncommutativity”, hep-th/0004072.

[2] H. S. Snyder, “Quantized Space-Time”, Phys. Rev. 71 (1946) 38; “The Electromagnetic Field in Quantized Space-Time”, Phys. Rev. 72 (1947) 68.

[3] C. N. Yang, “On Quantized Space-Time”, Phys. Rev. 72 (1947) 874.

[4] T. Yoneya, p. 419 in “Wandering in the Fields”, eds. K. Kawarabayashi and A. Ukawa (World Scientific, 1987); see also p. 23 in “Quantum String Theory”, eds. N. Kawamoto and T. Kugo (Springer, 1988); M. Li and T. Yoneya, hep-th/9611072, Phys. Rev. Lett. 78 (1997) 1219; M. Li and T. Yoneya, “Short-distance Space-time Structure and Black Holes in String Theory: A Short Review of the Present Status”, hep-th/9806240, Jour. Chaos, Solitons and Fractals 10 (1999) 423.

[5] G. ’t Hooft, Physica Scripta, Vol. T15 (1987) 143; Nucl. Phys. B335 (1990) 138.

[6] N. Seiberg, L. Susskind and N. Toumbas, “The Teleological Behavior of Rigid Regge Rods”, hep-th/0005015.

[7] L. Susskind and E. Witten, “The Holographic Bound in Anti-de Sitter Space”, hep-th/9805113; A. Peet and J. Polchinski, “UV/IR Relations in AdS Dynamics”, hep-th/9809022.

[8] A. Jevicki and S. Ramgoolam, “Noncommutative gravity from AdS/CFT correspondence”, hep-th/9902059, JHEP 9904 (1999) 032.

[9] P.-M. Ho, S. Ramgoolam and R. Tatar, “Quantum Spacetimes and Finite N Effects in 4D Yang-Mills Theories”, hep-th/9907143.

[10] J. Maldacena and A. Strominger, “AdS3 Black Holes and a Stringy Exclusion Principle”, hep-th/9804083, JHEP 9812 (1998) 005.
[11] J. McGreevy, L. Susskind and N. Toumbas, “Invasion of the Giant Gravitons from Anti-de Sitter Space”, hep-th/0003075.

[12] M. Li, “Fuzzy Gravitons From Uncertain Spacetime”, hep-th/0003173.

[13] J. Castelino, S. Lee and W. Taylor, “Longitudinal 5-branes as 4-sphere in Matrix Theory”, hep-th/9712105, Nucl. Phys. B526 (1998) 334.

[14] M. Berkooz and H. Verlinde, “Matrix Theory, AdS/CFT and Higgs-Coulomb Equivalence”, hep-th/9907100, JHEP 9911 (1999) 037.

[15] A. Strominger, “AdS2 Quantum Gravity and String Theory”, hep-th/9809027, JHEP 9901 (1999) 007;
J. Maldacena, J. Michelson and A. Strominger, “Anti-de Sitter Fragmentation”, hep-th/9812073, JHEP 9902 (1999) 011.

[16] T. Banks, M. Green, “Nonperturbative Effects in $AdS_5 \times S^5$ String Theory and $d=4$ SUSY Yang-Mills”, hep-th/9804170, JHEP 9805 (1998) 002.

[17] G. ’t Hooft, “Quantization of Point Particles in 2 + 1 Dimensional Gravity and Space-Time Discreteness”, gr-qc/9601014.

[18] J. Maldacena and A. Strominger, “Statistical Entropy of Four-Dimensional Black Holes”, hep-th/9603069, Phys. Rev. Lett. 77 (1996) 428.

[19] I. R. Klebanov and A. A. Tseytlin, “Intersecting M-branes as Four-Dimensional Black Holes”, hep-th/9604166.

[20] S. Gukov, I. R. Klebanov, A. M. Polyakov, “Dynamics of $(n,1)$ Strings”, hep-th/9711112.

[21] F. Berezin, “General Concept of Quantization”, Comm. Math. Phys. 40, 153 (1975);
J. Madore, “Introduction to Noncommutative Differential Geometry and Its Physical Applications”, Cambridge U. Press, 2nd (1999).
[22] G. ’t Hooft, “Transplanckian Particles and the Quantization of Time”, gr-qc/9805079.