$\mathcal{N} = 2$ Supersymmetric $SO(N)/Sp(N)$ Gauge Theories from Matrix Model

Changhyun Ahn$^1$ and Soonkeon Nam$^2$

$^1$Department of Physics, Kyungpook National University, Taegu 702-701, Korea
$^2$Department of Physics and Research Institute for Basic Sciences, Kyung Hee University, Seoul 130-701, Korea

ahn@knu.ac.kr, nam@khu.ac.kr

Abstract

We use the matrix model to describe the $\mathcal{N} = 2$ $SO(N)/Sp(N)$ supersymmetric gauge theories with massive hypermultiplets in the fundamental representation. By taking the tree level superpotential perturbation made of a polynomial of a scalar chiral multiplet, the effective action for the eigenvalues of chiral multiplet can be obtained. By varying this action with respect to an eigenvalue, a loop equation is obtained. By analyzing this equation, we derive the Seiberg-Witten curve within the context of matrix model.
1 Introduction

Recently Dijkgraaf and Vafa [1] have made a conjecture, the exact superpotential and gauge couplings for a class of $\mathcal{N} = 1$ gauge theories can be obtained by calculating perturbative computations in a matrix model in which the superpotential of the gauge theory is interpreted as an ordinary potential. The earlier works [2, 3, 4] motivated this conjecture. Based on this observation, there are many works on this direction [5]-[32]. In particular, we restrict to the supersymmetric $SO(N)/Sp(N)$ gauge theories. The model with quartic tree level superpotential for adjoint chiral field was found [33] and the effective superpotential was computed in the context of matrix model and string theory on Calabi-Yau geometry with flux. The perturbative calculation for glueball superpotential was studied in [34]. For arbitrary tree level superpotentials, the planar and leading nonplanar contributions were derived by using higher genus loop equations and diagrammatics [35]. A field theoretic derivation of the superpotential was given in [36] based on the factorization property of Seiberg-Witten curve. An equivalence of $\mathcal{N} = 1$ gauge theories deformed from $\mathcal{N} = 2$ by the addition of superpotential terms was studied with flavors [37] and without flavors [38] based on the Cachazo-Vafa’s idea [2]: The low energy information is given by extremization of the effective superpotential and in the field theory analysis it is given by characterizing to the factorization locus of Seiberg-Witten curve and the equivalence of two description was given in [2].

In this paper, we compute the matrix path integral over tree level superpotential obtained from $\mathcal{N} = 2$ SQCD by taking arbitrary polynomial of a scalar chiral multiplet as a perturbation. The effective theory action can be expressed as a function of an eigenvalue of chiral multiplet. The saddle point equation implies an algebraic equation defined on a hyperelliptic Riemann surface. The presence of this curve allows us to study the relation of matrix model and the gauge theory result of perturbative calculation. By using the basic idea of matrix model, we calculate a partition function in terms of a glueball field, a distribution of eigenvalue of chiral multiplet, a perturbed superpotential and the mass of quarks. By reading off the two free energy contributions from a partition function, one obtains the final effective superpotential in terms of homology basis. By varying this effective superpotential with respect to the coefficient function appearing in the algebraic curve, one realizes the existence of a meromorphic function on Riemann surface with the appropriate structure of zeros and poles. In doing this, the correct counting of the number of physical D5-branes in the presence of orientifold planes (O5-planes) is very important because these values determine the structure of zeros and poles precisely. By identifying this function with the resolvent of matrix model, the Seiberg-Witten curves for $\mathcal{N} = 2$ $SO(N)/Sp(N)$ gauge theory with $N_f$ hypermultiplets are rederived. For $U(N)$ gauge theory with $N_f$ flavors of quarks in the fundamental representation, the derivation of Seiberg-Witten curve was found in [39]. It would be interesting to study the results in [40, 41, 42] to
2 \( SO(N) \) matrix model

We will derive the Seiberg-Witten curve for \( \mathcal{N} = 2 \) \( SO(N) \) gauge theory with \( N_f \) fundamental hypermultiplets by computing the matrix path integral using the saddle point method [43, 39]. Let us consider an \( \mathcal{N} = 2 \) supersymmetric \( SO(N) \) gauge theory with \( N_f \) flavors of quarks \( Q^i_a (i = 1, 2, \ldots, 2N_f, a = 1, 2, \ldots, N) \) in the vector (fundamental) representation [44, 45, 46, 47, 48, 49, 50, 51]. In terms of \( \mathcal{N} = 1 \) superfields, \( \mathcal{N} = 2 \) vector multiplet consists of a field strength chiral multiplet \( W_{ab}^i \) and a scalar chiral multiplet \( \Phi_{ab} \) both in the adjoint representation of the gauge group. The \( \mathcal{N} = 2 \) superpotential takes the form

\[
W_{\text{tree}}(\Phi, Q) = \sqrt{2}Q^i_a \Phi_{ab} Q^j_b J_{ij} + \sqrt{2}m_{ij} Q^i_a Q^j_a
\]  

(2.1)

where \( J_{ij} \) is the symplectic metric \( \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \otimes 1_{N_f \times N_f} \) used to raise and lower \( SO(N) \) flavor indices ( \( 1_{N_f \times N_f} \) is the \( N_f \times N_f \) identity matrix ) and \( m_{ij} \) is a quark mass matrix \( \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \otimes \text{diag}(m_1, \ldots, m_{N_f}) \). Classically, the global symmetries are the flavor symmetry \( Sp(2N_f) \) and \( U(1)_R \times SU(2)_R \) chiral R-symmetry. When \( N_f < N - 2 \), the theory is asymptotically free and generates dynamically a strong coupling scale \( \Lambda_{\mathcal{N}=2} \). The instanton factor is proportional to \( \Lambda_{\mathcal{N}=2}^{2N-4-2N_f} \). Then \( U(1)_R \) symmetry is anomalous and broken down to a discrete \( Z_{2N-2N_f-4} \) symmetry by instanton. By taking a tree level superpotential perturbation \( \Delta W \) made out of the adjoint field in the vector multiplet to the \( \mathcal{N} = 2 \) superpotential (2.1), the \( \mathcal{N} = 2 \) supersymmetry can be broken to \( \mathcal{N} = 1 \) supersymmetry. That is,

\[
W = W_{\text{tree}}(\Phi, Q) + \Delta W, \quad \Delta W \equiv \sum_{k=1}^{[\frac{N}{2}]} \frac{g_{2k}}{2k} \text{Tr} \Phi^{2k}.
\]

Then a microscopic \( \mathcal{N} = 1 \) \( SO(N) \) gauge theory is obtained from \( \mathcal{N} = 2 \) \( SO(N) \) Yang-Mills theory perturbed by \( \Delta W \).

By using the perturbed superpotential in addition to the tree level one, substituting the whole superpotential into the \( SO(N) \) matrix model at large \( N \) and replacing the gauge theory fields with matrices, we study the various contributions to the free energy. Then the partition function can be written as

\[
Z = \frac{1}{\text{vol}(SO(N))} \int [d\Phi] [dQ] \exp \left[ -\frac{1}{g_s} W(\Phi) - \sqrt{2}Q^i_a \Phi_{ab} Q^j_b J_{ij} - \sqrt{2}m_{ij} Q^i_a Q^j_a \right]
\]  

(2.2)

where for simplicity we change the notation

\[
W(\Phi) = \sum_{k=1}^{[\frac{N}{2}]+1} \frac{g_{2k}}{2k} \text{Tr} \Phi^{2k}.
\]
A superpotential $W$ of order $2\left(\left\lfloor \frac{N}{2} \right\rfloor + 1\right)$ breaks the gauge symmetry down to a direct product of $\left(\left\lfloor \frac{N}{2} \right\rfloor + 1\right)$ subgroup. One can write the derivative of $W$ with respect to the field

$$W'(x) = x^{2N} + \sum_{i=1}^{N} s_{2i} x^{2(N-i)} = \prod_{i=1}^{N} \left(x^2 - e_i^2\right)$$  \hspace{1cm} (2.3)

where $e_i$'s are the classical moduli and the symmetric polynomial $s_{2k}$ in $e_i^2$ is

$$s_{2k} = (-1)^k \sum_{i_1 < \cdots < i_k} e_{i_1}^2 \cdots e_{i_k}^2.$$  

The description for the addition of the mass term for the adjoint scalar only was studied in the context of matrix model [52]. The matrix description for pure flavors without any adjoint fields to check the Seiberg duality was observed in [53]. Let us study $SO(N)$ matrix model by considering even $N$ and odd $N$ case separately because the Jacobian has different form in each case and also the spectral curves are different. Now we first analyze $SO(2N)$ matrix model.

- **$SO(2N)$ matrix model**

  According to the procedure [3, 54, 39, 35] and by integrating over $Q$ in our case, the eigenvalue basis provides

$$Z \sim \int \prod_{a=1}^{N} [d\lambda] \prod_{a<b}^{N} \left(\lambda_a^2 - \lambda_b^2\right)^2 \exp \left[-\frac{1}{g_s} \sum_{a=1}^{N} 2W(\lambda_a) - \sum_{i=1}^{N_f} \log \left(\lambda_a^2 - m_i^2\right)\right]$$

where $\pm i\lambda_a$ are the eigenvalues of $\Phi = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \otimes \text{diag}(\lambda_1, \cdots, \lambda_N)$ and $m_i$ is a quark mass $\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \otimes \text{diag}(m_1, \cdots, m_{N_f})$. Note the factor 2 in the first term of the exponent. The second term comes from the determinant of $(\Phi + m)$. The new thing in our problem is the flavor part in the last term. For $SO(2N)$ theory without any flavors these terms are absent [35]. After exponentiating, the effective action for the eigenvalues is given by

$$S(\lambda) = -\sum_{a<b}^{N} \log \left(\lambda_a^2 - \lambda_b^2\right)^2 + \frac{1}{g_s} \sum_{a=1}^{N} 2W(\lambda_a) + \sum_{i=1}^{N_f} \log \left(\lambda_a^2 - m_i^2\right).$$

In this way, the potential $W(\lambda)$ contains a collection of $N$ variables $\lambda_1, \lambda_2, \cdots, \lambda_N$. Remember that due to the antisymmetric property of $\Phi$ and the trace of it vanishes, only even terms in the potential $W$ which is a polynomial of order $(2N + 2)$ contribute. The saddle point equations (classical equations of motion) coming from varying the action with respect to a single eigenvalue $\lambda_a$ are

$$\sum_{b \neq a}^{N} \frac{2\lambda_a}{\lambda_a^2 - \lambda_b^2} - \frac{1}{g_s} W'(\lambda_a) - \sum_{i=1}^{N_f} \frac{\lambda_a}{\lambda_a^2 - m_i^2} = 0.$$  \hspace{1cm} (2.4)

To solve this let us introduce the trace of the resolvent of the matrix $\Phi$ [43, 3, 54, 39, 35]

$$\omega(x) = \frac{1}{N} \text{Tr} \frac{1}{\Phi - x} = \frac{1}{N} \sum_{a=1}^{N} \frac{2x}{\lambda_a^2 - x^2}.$$  \hspace{1cm} (2.5)
Then multiplying (2.4) by $2\lambda_a/(x^2 - \lambda_a^2)$ and summing over an index $a$, one gets a loop equation for $\omega(x)$

$$\omega^2(x) + \frac{2}{S} \omega(x)W'(x) + \frac{f(x)}{S^2} = 0$$

where the $S$ is defined as $S \equiv g_s N$ being fixed in the large $N$ limit and the polynomial $f(x)$ is given by

$$f(x) \equiv 4g_s \sum_{a=1}^{N} \frac{\lambda_a W' (\lambda_a) - x W' (x)}{\lambda_a^2 - x^2}$$

which is a polynomial of order $(2N - 2)$ with even powers. Therefore the function $f(x)$ determines the solution of the matrix integral. Here we take the large $N$ limit and drop the terms like $\omega(x)/x$ and $\omega'(x)$ which will be important when we expand it with respect to $1/N$ in order to derive the Seiberg-Witten differential completely within the framework of the matrix model. The spectral curve reduces to

$$y^2 = W'(x)^2 - f(x), \quad f(x) = \sum_{n=0}^{N-1} b_{2n} x^{2n}$$

where we define

$$y(x) = W'(x) + S \omega(x).$$

This is nothing but a hyperelliptic curve in $(x, y)$ plane. We have to determine the $N$ unknown coefficients $b_{2n}$. As in [2], there exists two particular points denoted by $P$ and $Q$ located at the two pre images of $\infty$ of $x$. The force equation becomes

$$2y(\lambda) = -g_s \frac{\partial S}{\partial \lambda}.$$ 

The solution for resolvent is given by [43]

$$\omega(x) = \sqrt{W'(x)^2 - f(x) - W'(x)}.$$

which is expressed as an $N$ unknown coefficient function appearing in the polynomial $f(x)$. The resolvent has the branch cuts among which the eigenvalues of the matrix are distributed.

In the large $N$ limit, the distribution of eigenvalues can be written as

$$\rho(\lambda) = \frac{1}{N} \sum_{a=1}^{N} \delta(\lambda - \lambda_a), \quad \int \rho(\lambda) d\lambda = 1$$

and the resolvent becomes in this limit

$$\omega(x) = 2 \int_{0}^{\infty} \frac{x \rho(\lambda) d\lambda}{\lambda^2 - x^2} = \int_{0}^{\infty} \rho(\lambda) d\lambda \left( \frac{1}{\lambda - x} - \frac{1}{\lambda + x} \right) = \int_{-\infty}^{\infty} \frac{\rho(\lambda) d\lambda}{\lambda - x}$$
which implies that
\[ \rho(\lambda) = \frac{1}{2\pi i} \left[ \omega(\lambda + i\epsilon) - \omega(\lambda - i\epsilon) \right] = \frac{1}{2\pi i} \left[ y(\lambda + i\epsilon) - y(\lambda - i\epsilon) \right]. \]

The filling fractions are given by
\[ S_i = \frac{1}{2\pi i} \int_{A_i} y dx, \quad S_0 = \frac{1}{4\pi i} \int_{A_0} y dx \]
where we take the half of the cycle around \( A_0 \) due to the orientifold projection. In order to find out the functional behavior of \( f(x) \) within the matrix model, the saddle point computation of the partition function gives rise to up to \( 1/g_s \) term as follows:
\[ Z = \exp \left[ -2S \int d\lambda \rho(\lambda) W(\lambda) + S^2 \int d\lambda d\lambda' \rho(\lambda) \rho(\lambda') \log(\lambda^2 - \lambda'^2) \right] 
+ \exp \left[ -S \sum_{i=1}^{N_f} \int d\lambda \rho(\lambda) \log(\lambda^2 - m_i^2) \right] \equiv \exp \left[ -\frac{1}{g_s} F_2 - \frac{1}{g_s} F_1 \right]. \tag{2.6} \]

To get the effective superpotential, one should know both the variation of \( F_2 \) under a small change in \( S_i \) and \( F_1 \) that can be read off from (2.6). For the former, we take the following change in \( \rho(\lambda) \)
\[ \rho(\lambda) \rightarrow \rho(\lambda) + \frac{\delta S_i}{S} \delta(\lambda - e_i) \]
where \( e_i \) is an arbitrary point along the \( i \)-th cut on the hyperelliptic Riemann surface. From the explicit form of \( F_2 \) in (2.6) one considers
\[ \delta F_2 = \delta S_i \left( 2W(e_i) - 2S \int d\lambda \rho(\lambda) \log(\lambda^2 - e_i^2) \right) \]
up to \( S_i \) independent terms which are not relevant. Then the partial derivative of \( F_2 \) with respect to \( S_i \) becomes
\[ \frac{\partial F_2}{\partial S_i} = -2 \int_{e_i}^{P} dx W'(x) + 4S \int d\lambda \rho(\lambda) \int_{e_i}^{P} dx \frac{x}{\lambda^2 - x^2} = -2 \int_{e_i}^{P} y(x) dx - \left( \int_{e_i}^{P} + \int_{Q}^{\epsilon_i} \right) y(x) dx \tag{2.7} \]
up to an irrelevant constant of integration terms. Moreover the \( F_1 \) term can be written as
\[ F_1 = 2S \sum_{i=1}^{N_f} \int d\lambda \rho(\lambda) \int_{m_i}^{P} dx \frac{x}{\lambda^2 - x^2} = S \sum_{i=1}^{N_f} \int_{m_i}^{P} \omega(x) dx \]
\[ = \sum_{i=1}^{N_f} \int_{m_i}^{P} y(x) dx = \frac{1}{2} \sum_{i=1}^{N_f} \int_{m_i}^{P} y(x) dx + \frac{1}{2} \sum_{i=1}^{N_f} \int_{-m_i}^{P} y(x) dx. \tag{2.8} \]
up to the $S_i$ independent terms.

Combining the two contributions (2.7) and (2.8) one gets the effective superpotential

$$W = -\frac{1}{2}(2N - 2)\int_Q y dx + \frac{1}{2}\sum_{i=1}^{N_f} \int_{m_i}^P y dx + \frac{1}{2}\sum_{i=1}^{N_f} \int_{-m_i}^P y dx + \cdots \quad (2.9)$$

Here there are some remarks in order. In the type IIB string theory, the $U(N)$ gauge theory is realized by the worldvolume of $N$ D5-branes wrapped on $S^2$ and in the dual geometry D5-branes are replaced by RR fluxes generating the effective superpotential and the $S^2$ by $S^3$. In order to deal with the gauge groups $SO(N)$ and $Sp(N)$, one needs to introduce the orientifold plane into the geometry. This will change the contributions of RR fluxes below. The physical D5-brane charge of orientifold plane (O5-plane) is $-1$ and the total $2N_0$ D5-branes wrapping around the origin should be modified by $(2N_0 - 2)$. Moreover the branch cuts in figure 1 in [38] are symmetric, due to the $Z_2$ symmetry, with the one located in the center (We follow the notations given in [38]). This implies that the contribution from the compact cycle with $a_{-k}$ is exactly the same as the one with $a_k$. That is, by replacing the D5-branes with the fluxes, one gets [38]

$$\int_{a_k} h = \frac{1}{2} N_k = \int_{a_{-k}} h = \frac{1}{2} N_{-k}, \quad \int_{a_0} h = \frac{1}{2}(2N_0 - 2).$$

By summing over the all $\alpha_k$ contour,

$$\oint_{P} h = \left(2\sum_{k=1}^{N} N_k + 2N_0\right) - 2 = (2N - 2) - N_f, \quad \oint_{Q} h = -(2N - 2), \quad \oint_{m_i} h = -1$$

where $h$ is an one form on the Riemann surface. So $h$ should have a pole of order 1 at $P$ and $Q$ with residues $(2N - 2 - N_f)$ and $-(2N - 2)$ respectively [37]. By using the properties $C_{-k} = -\sum_{j=1}^{k} \beta_{-j} + C_0$ and $C_k = \sum_{j=1}^{k} \beta_j + C_0$ given in [38], one can divide into two parts: the cycle around $C_0$ and the one around $\beta_j, j = 1, 2, \cdots, k$. Then we have extra two contributions in (2.9) denoted by $\cdots$, Yang-Mills coupling term and the term with the cycle $\beta_j$. But if we take the variation of $W$, the first contribution will give rise to a trivial cycle and the second one gives an element of the period lattice. For $U(N)$ gauge theory, the discussion on this matter was considered in [2]. Since $S_i$ can be determined by $f(x)$ (and therefore $b_{2n}$), we have to compute the variation of $W$ with respect to $b_{2n}$.

$$\frac{\partial y}{\partial b_{2n}} dx = -\frac{x^{2n}}{2y} dx, \quad n = 0, 1, \cdots, N - 1$$

which are basis for the subspace of holomorphic differentials (one forms) which are odd under the $Z_2$ transformation $x \rightarrow -x$, on the Riemann surface. Note that the full space of holomorphic differentials has the dimension $(2N - 1)$ due to the genus $(2N - 1)$ of the Riemann surface.
By changing the bases to the homology basis, the extremum condition of $W$ will give rise to
\[
(2N - 2) \int_{p_0}^{Q} \zeta_k - (2N - 2N_f - 2) \int_{p_0}^{P} \zeta_k - \sum_{i=1}^{N_f} \left( \int_{p_0}^{m_i} + \int_{p_0}^{-m_i} \right) \zeta_k = 0
\]
modulo the period lattice and $p_0$ is an arbitrary generic point on the Riemann surface.

There exists a function on the Riemann surface with an $(2N - 2)$-th order pole at $Q$, an $(2N - 2N_f - 2)$-th order zero at $P$ and simple zeros at $x = \pm m_i$ for each $i = 1, 2, \cdots, N_f$, according to Abel’s theorem [2, 39]. The function is simply related to the resolvent divided by $x^2$ (We refer to [37] with flavors and [38] without flavors for the geometric picture)
\[
z(x) = \frac{y}{x^2} - \frac{W'(x)}{x^2} = \sqrt{\frac{W'(x)^2}{x^4} - \frac{f(x)}{x^4} - \frac{W'(x)}{x^2}}, \quad 2N_f < 2N - 2.
\]
This function has an $(2N - 2)$-th order pole at $Q$ and at least fourth order pole at $P$ since $f(x)$ is a polynomial of at most $(2N - 2)$-th order. Therefore $f(x)$ should contain a factor $(x^2 - m_i^2)$ for each $i$ in order for $z(x)$ to have a simple zero at $x = \pm m_i$ and should be at most $(2N_f + 4)$-th order in order for $z(x)$ to satisfy an $(2N - 2N_f - 2)$-th order zero at $P$. Combining these two conditions and putting the proportional constant as $A_{N=2}^{4N-2N_f-4}$, the spectral curve is
\[
y^2 = \prod_{i=1}^{N} (x^2 - e_i^2)^2 - A_{N=2}^{4N-2N_f-4} x^4 \prod_{i=1}^{N_f} (x^2 - m_i^2)
\]
which is exactly the Seiberg-Witten curve [55, 47, 48, 50, 51] in the field theory analysis where we used the relation (2.3). So far we assumed that the number of flavors are not too large. Next we consider the case of large flavors.

For $2N - 2 < 2N_f$, the function $z(x)$ is given by
\[
z(x) = \sqrt{\frac{A(x)^2}{x^4} - \frac{g(x)}{x^4} - \frac{A(x)}{x^2}}, \quad 2N - 2 < 2N_f < 4N - 4
\]
where $A(x)$ is an $2N$-th order polynomial and $g(x)$ is proportional to $x^4 \prod_{i=1}^{N_f} (x^2 - m_i^2)$. The presence of this new function comes from the fact that when the $2N_f$ is greater than $(2N - 2)$, it is not enough to have the function $f(x)$ only because the pole structure at $P$ needs to introduce a new function. Now we take the proportional constant as $A_{N=2}^{4N-2N_f-4}$. Then the function $z(x)$ vanishes at $x = \pm m_i$ for each $i = 1, 2, \cdots, N_f$ and has an $(2N - 2)$-th order pole at $Q$ and $(2N_f - 2N + 2)$-th order pole at $P$. The expression inside of square root in $z(x)$ should be proportional to $y(x)^2/x^4$
\[
\frac{A(x)^2}{x^4} - \frac{f(x)}{x^4} = \frac{W'(x)^2}{x^4} - \frac{W'(x)}{x^2}
\]
where \( f(x) \) is a polynomial of order at most \((2N - 2)\). The solution for this up to \( \mathcal{O}(\Lambda_{N=2}^{4N-2N_f-4}) \) is given by

\[
A(x) = \prod_{i=1}^{N}(x^2 - e_i^2) + \Lambda_{N=2}^{4N-2N_f-4} P(x),
\]

\[
f(x) = \Lambda_{N=2}^{4N-2N_f-4} \left( x^4 \prod_{i=1}^{N_f}(x^2 - m_i^2) - 2P(x) \prod_{i=1}^{N}(x^2 - e_i^2) \right)
\]

where \( P(x) \) is defined as a polynomial of degree \((2N_f - 4N + 4)\) in \( x \) and \( m_i \). Therefore the spectral curve and the function are given by

\[
y^2 = \prod_{i=1}^{N}(x^2 - e_i^2)^2 - f(x)
\]

\[
y = \prod_{i=1}^{N} \left[ (x^2 - e_i^2) + \Lambda_{N=2}^{4N-2N_f-4} P(x) \right]^2 - \Lambda_{N=2}^{4N-2N_f-4} x^4 \prod_{i=1}^{N_f}(x^2 - m_i^2),
\]

\[
z(x) = \frac{y}{x^2} - \frac{A(x)}{x^2}.
\]

Therefore in both regions of the number of flavors the spectral curve coming from the matrix model calculations coincides with precisely the known Seiberg-Witten curve.

- \( \text{SO}(2N+1) \) matrix model

Let us describe the odd case. Since the presentation looks similar to the one in previous discussion, we will present the main difference only. By integrating over \( Q \), the eigenvalue basis provides \([3, 54, 39, 35]\)

\[
Z \sim \int \prod_{a=1}^{N} [d\lambda] \prod_{a<b}^{N} \left( \lambda_a^2 - \lambda_b^2 \right)^2 \prod_{a=1}^{N} \lambda_a^2 \exp \left[ -\frac{1}{g_s} \sum_{a=1}^{N} 2W(\lambda_a) - \sum_{i=1}^{N_f} \log \left( \lambda_a^2 - m_i^2 \right) \right]
\]

where \( \pm i\lambda_a \) are the eigenvalues of \( \Phi = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \otimes \text{diag}(\lambda_1, \cdots, \lambda_N, 0) \) and \( m_i \) is a quark mass \( \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \otimes \text{diag}(m_1, \cdots, m_{N_f}) \). Note that there exists an extra factor \( \prod_{a=1}^{N} \lambda_a^2 \) in this case \([35]\). Then the effective action for the eigenvalues is given by

\[
S(\lambda) = -\sum_{a<b}^{N} \log \left( \lambda_a^2 - \lambda_b^2 \right)^2 - \sum_{a=1}^{N} \log \lambda_a^2 + \frac{1}{g_s} \sum_{a=1}^{N} 2W(\lambda_a) + \sum_{i=1}^{N_f} \log \left( \lambda_a^2 - m_i^2 \right).
\]

By varying the action with respect to an eigenvalue one gets

\[
\sum_{b \neq a} \frac{2\lambda_a}{\lambda_a^2 - \lambda_b^2} - \frac{1}{\lambda_a} - \frac{1}{g_s} W'(\lambda_a) - \sum_{i=1}^{N_f} \frac{\lambda_a}{\lambda_a^2 - m_i^2} = 0.
\]

In the large \( N \) limit, the second extra term comparing with the previous case \( \text{SO}(2N) \) matrix model does not contribute. Therefore all the arguments from (2.5) to (2.8) in even \( \text{SO}(2N) \)
gauge theory are valid for $SO(2N + 1)$ matrix model. By an appropriate counting the physical
D5-brane charge of the orientifold which is equal to $-\frac{1}{2}$, one gets

\[ \int_{a_k} h = \frac{1}{2} N_k = \int_{a_{-k}} h = \frac{1}{2} N_{-k}, \quad \int_{a_0} h = \frac{1}{2} (2N_0 - 1). \]

By summing over the all $\alpha_k$ contour,

\[ \oint_P h = \left( 2 \sum_{k=1}^N N_k + 2N_0 \right) - 1 = (2N - 1) - N_f, \quad \oint_Q h = -(2N - 1), \quad \oint_{\pm m_i} h = -1 \]

where $h$ is an one form on the Riemann surface.

The extremum condition of $W$ (2.9) where the coefficient $(2N - 2)$ is replaced with $(2N - 1)$
will give rise to

\[ (2N - 1) \int_{p_0}^Q \zeta_k - (2N - 2N_f - 1) \int_{p_0}^P \zeta_k - \sum_{i=1}^{N_f} \left( \int_{p_0}^{m_i} + \int_{p_0}^{-m_i} \right) \zeta_k = 0 \]

modulo the period lattice. Then the function is simply related to the resolvent divided by $x$

\[ z(x) = \frac{y}{x} - \frac{W'(x)}{x} = \sqrt{\frac{W'(x)^2}{x^2} - \frac{f(x)}{x^2}} - \frac{W'(x)}{x}, \quad 2N_f < 2N - 1. \quad (2.10) \]

This function has an $(2N - 1)$-th order pole at $Q$ and at least third order pole at $P$ since $f(x)$
is a polynomial of at most $(2N - 2)$-th order. Therefore $f(x)$ should contain a factor $(x^2 - m_i^2)$
for each $i$ in order for $z(x)$ to have a simple zero at $x = \pm m_i$ and should be at most $(2N_f + 2)$-th
order in order for $z(x)$ to satisfy an $(2N - 2N_f - 1)$-th order zero at $P$. Combining these two
conditions and putting the proportional constant as $\Lambda_4^{N-2N_f-2}$, the spectral curve is

\[ y^2 = \prod_{i=1}^{N} (x^2 - e_i^2)^2 - \Lambda_4^{N-2N_f-2} \prod_{i=1}^{N_f} (x^2 - m_i^2) \]

which is known as the Seiberg-Witten curve [56, 47, 48, 50, 51] in the perturbative calculation
in gauge theory side.

For $2N - 1 < 2N_f$, the function $z(x)$ is given by

\[ z(x) = \sqrt{\frac{A(x)^2}{x^2} - \frac{g(x)}{x^2} - \frac{A(x)}{x}}, \quad 2N - 1 < 2N_f < 4N - 2 \quad (2.11) \]

where $A(x)$ is an $2N$-th order polynomial and $g(x)$ is proportional to $x^2 \prod_{i=1}^{N_f} (x^2 - m_i^2)$. Now
we take the proportional constant as $\Lambda_4^{N-2N_f-2}$. Then the function $z(x)$ vanishes at $x = \pm m_i$
for each $i = 1, 2, \ldots, N_f$ and has an $(2N - 1)$-th order pole at $Q$ and $(2N_f - 2N + 1)$-th order
pole at $P$. The expression inside of square root in $z(x)$ should be proportional to $y(x)^2/x^2$

\[ \frac{A(x)^2}{x^2} - \Lambda_4^{N-2N_f-2} \prod_{i=1}^{N_f} (x^2 - m_i^2) = \frac{W'(x)^2}{x^2} - \frac{f(x)}{x^2} \]
where $f(x)$ is a polynomial of order at most $(2N - 2)$. It is straightforward to see the coincidence of the Seiberg-Witten curve. From the properties of $z(x)$ in (2.10) and (2.11), one can easily see that

$$h(x)dx = \frac{dz}{z}$$

is a meromorphic differential with simple poles at $P, Q$ and $x = \pm m_i$ with residues $(2N - 2N_f - 2), -(2N - 2)$ and 1 respectively for $SO(2N)$ gauge theory, for example.

### 3 $Sp(N)$ matrix model

In this section, we continue to study the matrix model for the symplectic group $Sp(N)$. Let us consider an $\mathcal{N} = 2$ supersymmetric $Sp(N)$ gauge theory with $N_f$ flavors of quarks $Q^i_i (i = 1, 2, \cdots, 2N_f, a = 1, 2, \cdots, 2N)$ in the fundamental representation. The tree level superpotential of the theory is obtained from

$$W_{\text{tree}}(\Phi, Q) = \sqrt{2}Q^i_i \Phi^i_a Q^i_a J^{bc} + \sqrt{2}m_{ij}Q^i_i Q^j_j (3.1)$$

where $J_{ab}$ is the symplectic metric ($0_{1 \times 1} - 1_{1 \times 1}$) $\otimes$ $\text{diag}(m_1, \cdots, m_{N_f})$. Classically, the global symmetries are the flavor symmetry $O(2N_f)$ and $U(1)_R \times SU(2)_R$ chiral R-symmetry. When $N_f < 2N + 2$, the theory is asymptotically free and generates dynamically a strong coupling scale $\Lambda_{N=2}$. The instanton factor is proportional to $\Lambda_{N=2}^{2N+2-N_f}$. Then $U(1)_R$ symmetry is anomalous and broken down to a discrete $Z_{2N-N_f+2}$ symmetry by instanton.

According to the procedure [3, 54, 39, 35] and by integrating over $Q$ in our case, the eigenvalue basis provides

$$Z \sim \int \prod_{a=1}^N [d\lambda] \prod_{a<b} (\lambda_a^2 - \lambda_b^2)^2 \prod_{a=1}^N \lambda_a^2 \exp \left[-\frac{1}{g_s} \sum_{a=1}^N 2W(\lambda_a) + \sum_{i=1}^{N_f} \log (\lambda_a^2 - m_i^2)\right]$$

where $\pm i\lambda_a$ are the eigenvalues of $\Phi = (1_{1 \times 1} 0_{1 \times 1}) \otimes \text{diag}(\lambda_1, \cdots, \lambda_N)$ and $m_i$ is a quark mass ($0_{1 \times 1} - 1_{1 \times 1}$) $\otimes$ $\text{diag}(m_1, \cdots, m_{N_f})$. By an appropriate counting the physical D5-brane charge of the orientifold which is equal to 1, one gets

$$\int_{a_k} h = \frac{1}{2}N_k = \int_{a_{-k}} h = \frac{1}{2}N_{-k}, \quad \int_{a_0} h = \frac{1}{2}(2N_0 + 2).$$

By summing over the all $\alpha_k$ contour,

$$\oint_P h = \left(2 \sum_{k=1}^N N_k + 2N_0\right) + 2 = (2N + 2) - N_f, \quad \oint_Q h = -(2N + 2), \quad \oint_{\pm m_i} h = -1$$
where \( h \) is an one form on the Riemann surface.

The extremum condition of \( W \) (2.9) where the coefficient \((2N - 2)\) is replaced with \((2N + 2)\) will give rise to

\[
(2N + 2) \int_{P_0}^Q \zeta_k - (2N - 2N_f + 2) \int_{P_0}^P \zeta_k - \sum_{i=1}^{N_f} \left( \int_{P_0}^{m_i} + \int_{P_0}^{-m_i} \right) \zeta_k = 0.
\]

There exists a function on the Riemann surface with an \((2N + 2)\)-th order pole at \( Q \), an \((2N - 2N_f + 2)\)-th order zero at \( P \) and simple zeros at \( x = \pm m_i \) for each \( i = 1, 2, \cdots, N_f \). The function is simply related to the resolvent [37, 38]

\[
z(x) = \sqrt{x^4W'(x)^2 - f(x) - x^2W'(x)}.
\]

This function has an \((2N + 2)\)-th order pole at \( Q \) and at least zero-th order pole at \( P \) since \( f(x) \) is a polynomial of at most \((2N - 2)\)-th order. Therefore \( f(x) \) should contain a factor \((x^2 - m_i^2)\) for each \( i \) in order for \( z(x) \) to have a simple zero at \( x = \pm m_i \) and should be at most \( 2N_f \)-th order in order for \( z(x) \) to satisfy an \((2N - 2N_f + 2)\)-th order zero at \( P \). Combining these two conditions and putting the proportional constant as \( \Lambda_{N=2}^{4N-2N_f+4} \), the spectral curve is

\[
y^2 = \left( x^2 \prod_{i=1}^{N} (x^2 - \epsilon_i^2) \right)^2 - \Lambda_{N=2}^{4N-2N_f+4} \prod_{i=1}^{N_f} (x^2 - m_i^2).
\]

In the gauge theory side, for the Seiberg-Witten curve [47, 50, 57] for \( Sp(N) \) case at least one hypermultiplet of exactly zero mass (for example, \( m_{N_f} = 0 \)), the above matrix model result is exactly the same the one in [47, 50, 57] because the extra piece which is peculiar to \( Sp(N) \) case, the product of quark mass term vanishes. For nonzero quark mass, one can consider the following resolvent \( z(x) = \sqrt{(x^2W'(x) + g)^2 - f(x) - (x^2W'(x) + g)} \) by including the constant term \((x \text{ independent term})\) \( g \) inside the square. By identifying this \( g \) with \( \Lambda_{N=2}^{2N+2-N_f} \prod_{i=1}^{N_f} m_i \), one obtains the Seiberg-Witten curve.

Acknowledgments

This research of CA was supported by Korea Research Foundation Grant(KRF-2002-015-CS0006). This research of SN was supported by Korea Research Foundation Grant KRF-2001-041-D00049. We thank Korea Institute for Advanced Study (KIAS) where part of this work was undertaken. CA thanks Y. Ookouchi for the correspondence on his paper.

References

[1] R. Dijkgraaf and C. Vafa, hep-th/0208048.
[2] F. Cachazo and C. Vafa, hep-th/0206017.
[3] R. Dijkgraaf and C. Vafa, Nucl.Phys. B644 (2002) 3, hep-th/0206255.
[4] R. Dijkgraaf and C. Vafa, Nucl.Phys. B644 (2002) 21, hep-th/0207106.
[5] M. Aganagic and C. Vafa, hep-th/0209138.
[6] F. Ferrari, Nucl.Phys. B648 (2003) 161, hep-th/0210135.
[7] D. Berenstein, Phys.Lett. B552 (2003) 255, hep-th/0210183.
[8] R. Dijkgraaf, S. Gukov, V.A. Kazakov and C. Vafa,
[9] R. Argurio, V.L. Campos, G. Ferretti and R. Heise, hep-th/0210291.
[10] R. Dijkgraaf, M.T. Grisaru, C.S. Lam, C. Vafa and D. Zanon, hep-th/0211017.
[11] H. Suzuki, hep-th/0211052.
[12] F. Ferrari, hep-th/0211069.
[13] I. Bena and R. Roiban, hep-th/0211075.
[14] Y. Demasure and R.A. Janik, Phys.Lett. B553 (2003) 105, hep-th/0211082.
[15] M. Aganagic, A. Klemm, M. Marino and C. Vafa, hep-th/0211098.
[16] R. Gopakumar, hep-th/0211100.
[17] S. Naculich, H. Schnitzer and N. Wyllard, hep-th/0211123.
[18] Y. Tachikawa, hep-th/0211189.
[19] R. Dijkgraaf, A. Neitzke and C. Vafa, hep-th/0211194.
[20] B. Feng, hep-th/0211202.
[21] B. Feng and Y.-H. He, hep-th/0211234.
[22] V.A. Kazakov and A. Marshakov, hep-th/0211236.
[23] R. Dijkgraaf, A. Sinkovics and M. Temurhan, hep-th/0211241.
[24] R. Argurio, V.L. Campos, G. Ferretti and R. Heise, Phys.Lett. B553 (2003) 332, hep-th/0211249.
[25] I. Bena, R. Roiban and R. Tatar, hep-th/0211271.
[26] Y. Tachikawa, hep-th/0211274.
[27] K. Ohta, hep-th/0212025.
[28] I. Bena, S. de Haro and R. Roiban, hep-th/0212083.
[29] C. Hofman, hep-th/0212095.
[30] H. Suzuki, hep-th/0212121.
[31] Y. Demasure and R.A. Janik, hep-th/0212212.
[32] B. Feng, hep-th/0212274.

[33] H. Fuji and Y. Ookouchi, JHEP 0212 (2002) 067, hep-th/0210148.

[34] H. Ita, H. Nieder and Y. Oz, JHEP 0301 (2003) 018, hep-th/0211261.

[35] S.K. Ashok, R. Corrado, N. Halmagyi, K.D. Kennaway and C. Romelsberger, hep-th/0211291.

[36] R.A. Janik and N.A. Obers, Phys.Lett. B553 (2003) 309, hep-th/0212069.

[37] Y. Ookouchi, hep-th/0211287.

[38] B. Feng, hep-th/0212010.

[39] S. Naculich, H. Schnitzer and N. Wyllard, JHEP 0301 (2003) 015, hep-th/0211254.

[40] F. Cachazo, M.R. Douglas, N. Seiberg and E. Witten, JHEP 0212 (2002) 071, hep-th/0211170.

[41] N. Seiberg, hep-th/0212225.

[42] F. Cachazo, N. Seiberg and E. Witten, hep-th/0301006.

[43] P.D. Francesco, P. Ginsparg and J. Zinn-Justin, Phys. Rept. 254 (1995) 1, hep-th/9306153.

[44] S. Terashima and S-K. Yang, Phys. Lett. B391 (1997) 107, hep-th/9607151.

[45] T. Kitao, S. Terashima and S-K. Yang, Phys. Lett. B399 (1997) 75, hep-th/9701009.

[46] P.C. Argyres, M.R. Plesser and A. D. Shapere, Nucl. Phys. B483 (1997) 172, hep-th/9608129.

[47] P.C. Argyres and A.D. Shapere, Nucl. Phys. B461 (1996) 437, hep-th/9509175.

[48] A. Hanany, Nucl. Phys. B466 (1996) 85, hep-th/9509176.

[49] T. Hirayama, N. Maekawa and S. Sugimoto, Prog. Theor. Phys. 99 (1998) 843, hep-th/9705069.

[50] E. D’Hoker, I.M. Krichever and D.H. Phong, Nucl. Phys. B489 (1997) 211, hep-th/9609145.

[51] C. Ahn, K. Oh and R. Tatar, J.Geom.Phys. 28 (1998) 163, hep-th/9712005.

[52] C. Ahn, hep-th/0301011.

[53] C. Ahn and S. Nam, hep-th/0212231.

[54] J. McGreevy, hep-th/0211009.

[55] A. Brandhuber and K. Landsteiner, Phys. Lett. B358 (1995) 73, hep-th/9507008.

[56] U.H. Danielsson and B. Sundborg, Phys. Lett. B358 (1995) 273, hep-th/9504102.

[57] C. Ahn, Phys. Lett. B426 (1998) 306, hep-th/9712149.