Extendability of Simplicial Maps is Undecidable

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Abstract
We present a short proof of the Čadek–Krčál–Matoušek–Vokřínek–Wagner result from the title (in the following form due to Filakovský–Wagner–Zhechev). For any fixed even \( l \) there is no algorithm recognizing the extendability of the identity map of \( S^l \) to a PL map \( X \to S^l \) of given \( 2l \)-dimensional simplicial complex \( X \) containing a subdivision of \( S^l \) as a given subcomplex. We also exhibit a gap in the Filakovský–Wagner–Zhechev proof that embeddability of complexes is undecidable in codimension \( > 1 \).

Keywords
Extendability of continuous maps · Undecidability · Homotopy · Retraction · Wedge of spheres · Whitehead map

Mathematics Subject Classification
55-02 · 55P05 · 55S36 · 68-02 · 68U05

1 Extendability of Simplicial Maps is Undecidable

We present short proofs of recent topological undecidability results for simplicial complexes (hypergraphs): Theorems 1.1 and 1.2 [1, 2].

A complex \( K = (V, F) \) is a finite set \( V \) together with a collection \( F \) of subsets of \( V \) such that if a subset \( \sigma \) is in \( F \), then every subset of \( \sigma \) is in \( F \).\(^1\) In an equivalent geometric language, a complex is a collection of closed faces (= subsimplices) of some

\(^1\) We do not use longer name ‘abstract finite simplicial complex’. A \( k \)-hypergraph (more precisely, a \( (k + 1) \)-uniform hypergraph) \( (V, F) \) is a finite set \( V \) together with a collection \( F \) of \( (k + 1) \)-element subsets of \( V \). In topology it is more traditional (because often more convenient) to work with complexes not hypergraphs. The following results are stated for complexes, although some of them are correct for hypergraphs.

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A $k$-complex is a complex containing at most $(k + 1)$-element subsets, i.e., at most $k$-dimensional simplices. Elements of $V$ and of $F$ are called vertices and faces.

The complete $k$-complex on $n$ vertices (or the $k$-skeleton of the $(n - 1)$-simplex) is the $n$-element set $\binom{[n]}{k+1}$ of all at most $(k + 1)$-element subsets of $[n]$. For $n = k + 1$ we denote this complex by $D^k$ ($k$-simplex or $k$-disk), and for $n = k + 2$ by $S^k$ ($k$-sphere).

The subdivision of an edge operation is shown in Fig. 1. Exercise: represent the subdivision of a face operation shown in Fig. 1 as composition of several subdivisions of an edge and inverse operations. A subdivision of a complex $K$ is any complex obtained from $K$ by several subdivisions of edges.

A simplicial map $f : (V, F) \to (V', F')$ between complexes is a map $f : V \to V'$ (not necessarily injective) such that $f(\sigma) \in F'$ for each $\sigma \in F$. A piecewise-linear (PL) map $K \to K'$ between complexes is a simplicial map between certain their subdivisions. The body (or geometric realization) $|K|$ of a complex $K$ is the union of simplices of $K$. Below we often abbreviate $|K|$ to $K$; no confusion should arise. A simplicial or PL map between complexes induces a map between their bodies, which is called simplicial or PL, respectively.

The wedge $K_1 \vee \ldots \vee K_m$ of complexes $K_1 = (V_1, F_1), \ldots, K_m = (V_m, F_m)$ having disjoint vertices is the complex

- whose set of vertices is obtained from $V_1 \sqcup \ldots \sqcup V_m$ by choosing one vertex from each $V_j$, and identifying chosen vertices, and
- whose set of faces is obtained from $F_1 \sqcup \ldots \sqcup F_m$ by such identification.

The choice of vertices is important in general, but is immaterial in the examples below. Let $Y_l = S^l$ for $l$ even and $Y_l = S^l \vee S^l$ be the wedge of two copies of $S^l$ for $l$ odd.

**Theorem 1.1** (retractability is undecidable) For any fixed integer $l > 1$ there is no algorithm recognizing the extendability of the identity map of $Y_l$ to a PL map $X \to Y_l$ of given $2l$-complex $X$ containing a subdivision of $Y_l$ as a given subcomplex.

This result of [2] is implied by the following theorem and Proposition 1.8 (b). Let $V_m^d = S_1^d \vee \ldots \vee S_m^d$ be the wedge of $m$ copies of $S^d$.

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2 The related different notion of a continuous map between bodies of complexes is not required to state and prove the results of this text. In theorems below the existence of a continuous extension is equivalent to the existence of a PL extension (by the PL Approximation Theorem).
Theorem 1.2 (extendability is undecidable) For some fixed integer $m$ and any fixed integer $l > 1$ there is no algorithm recognizing extendability of given simplicial map $V_m^{2l-1} \to Y_l$ to a PL map $X \to Y_l$ of given $2l$-complex $X$ containing a subdivision of $V_m^{2l-1}$ as a given subcomplex.

This is a ‘concrete’ version of [1, Thm 1.1.a]. Remarks and examples below are formally not used later.

Remark 1.3 (a) Relation to earlier known results. For $l > 1$ any PL map $S^1 \to Y_l$ extends to $D^2$. The analogues of Theorems 1.1 and 1.2 for $Y_l$ replaced by a complex without this property (called simply-connectedness) were well known by mid-20th century. See more in [1, Sect. 1].

(b) Why this text might be interesting. Exposition of the proofs of Theorems 1.1 and 1.2 here is shorter and simpler than in [1]. The proof is structured here by explicitly stating the Brower–Hopf–Whitehead Theorems 1.5, 1.6, and Propositions 1.7 and 1.8. Theorems 1.5 and 1.6 relate homotopy classification to quadratic functions on integers. Thus they allow to prove the equivalence of extendability / retractability to homotopy of certain maps, and to solvability of certain Diophantine equations, see Propositions 1.7 and 1.8. Theorems 1.5, 1.6, and Proposition 1.8 are essentially known before [1, 2], and are essentially deduced in [1, 2] from other known results. (As far as I know, they were not explicitly stated earlier, not even in [1, 2]; cf. [1, Sect. 4.2] and [15, Rem. 2.1.b and 2.2.e].)

Also I present definitions in an economic way accessible to non-specialists (including computer scientists). In particular, I do not use cell complexes and simplicial sets. A reader might want to consider the proof below first for $l$ even. Then he/she can omit (b) and (b’) of Lemma 1.4, Theorem 1.5 (c), and (b) and (d) of Theorem 1.6.

Lemma 1.4 (a) For some fixed integers $m, s$ there is no algorithm that for given arrays $a = ((a_{i,j})_1, \ldots, (a_{i,j})_m), 1 \leq i < j \leq s$, and $b = (b_1, \ldots, b_m)$ of integers decides whether

\[ \sum_{1 \leq i < j \leq s} a_q^{i,j} x_i x_j = b_q \quad \text{for any} \quad 1 \leq q \leq m. \tag{SYM} \]

(b) Same as (a) for

\[ \sum_{1 \leq i < j \leq s} a_q^{i,j} (x_i y_j - x_j y_i) = b_q \quad \text{for any} \quad 1 \leq q \leq m. \tag{SKEW} \]

(b’) Same as (a) for the property (SKEW’) obtained from (SKEW) by replacing $a_q^{i,j}$ with $2a_q^{i,j}$.

See [1, Sect. 2] for deduction of (a) and (b) from insolvability of general Diophantine equations. Part (b’) follows by (b) because either all $b_q$ in (SKEW’) are even or the system (SKEW’) is unsolvable.
Denote by \( \simeq \) homotopy between maps. For \( n > 1 \) we use abelian group structure on the set \( \pi_n(X) \) of homotopy classes of PL maps \( S^n \to X \), see definition e.g. in [5] or [12, Sect. 14]. Let \( u, v : S^l \vee S^l \to S^l \) be the contractions of the second and the first sphere of \( S^l \vee S^l \).

**Theorem 1.5** For any integer \( l \) and simplicial map \( \varphi : P \to Q \) between subdivisions of \( S^l \) there is an effectively constructible integer \( \deg \varphi \) (called the degree of \( \varphi \)) such that

(a) for any integer \( k \) there is an effectively constructible PL map \( \hat{k} : S^l \to S^l \) of degree \( k \);
(b) for maps \( \varphi, \psi : S^l \to S^l \) if \( \deg \varphi = \deg \psi \), then \( \varphi \simeq \psi \);
(c) for \( l > 1 \) and maps \( \varphi, \psi : S^l \to S^l \vee S^l \) if \( \deg (u \circ \varphi) = \deg (u \circ \psi) \) and \( \deg (v \circ \varphi) = \deg (v \circ \psi) \), then \( \varphi \simeq \psi \).

**Sketch of a proof** Define \( \deg \varphi \) by to be the sum of signs of a finite number of points from \( \varphi^{-1} y \), where \( y \in S^l \) is a ‘random’ (i.e., regular) value of \( \varphi \). More precisely, \( y \) is any point outside the images of \((l-1)\)-simplices of \( P \). For the definition of sign and the proof of (b) see e.g. [5] or [12, Sect. 8]. Part (c) is a simple case of the Hilton Theorem.

Clearly, \( \deg \) defines a homomorphism \( \pi_l(S^l) \to \mathbb{Z} \). Let \( \hat{1} := \text{id} S^l \), let \( \hat{0} \) be the constant map, and let \( \hat{-1} \) be the reflection w.r.t. the equator \( S^{l-1} \subset S^l \). Then for \( k \neq 0 \) let \( \hat{k} \) be a representative of the sum of \( |k| \) summands \( \text{sgn} k \).

For a set \( x = (x_1, \ldots, x_s) \) of integers let \( \hat{x} : V^l_s \to S^l \) be the map whose restriction to \( S^l_j \) is \( \hat{x}_j \). Let \( \lambda, \mu : S^l \to S^l \vee S^l \) be the inclusions into the first and the second sphere of the wedge.

**Theorem 1.6** (proved in Sect. 2) For any integer \( a \) there exists an effectively constructible PL map \( W_2(a) : S^{2l-1} \to S^l_1 \vee S^l_2 \) such that for any \( l > 1 \) and the composition

\[
W(a) : S^{2l-1} \xrightarrow{W_2(a)} S^l_1 \vee S^l_2 \xrightarrow{\text{id} \vee \text{id}} S^l
\]

we have

(a) for \( l \) even \( W(a) \simeq W(a') \) only when \( a = a' \);
(b) \( W_2(a) \simeq W_2(a') \) only when \( a = a' \);
(c) \( (x_1, x_2) \circ W_2(a) \simeq W(ax_1, x_2) \);
(d) for \( l \) odd

\[
(\lambda \circ (x_1, x_2) + \mu \circ (y_1, y_2)) \circ W_2(2a) \simeq W(2a(x_1y_2 - x_2y_1)),
\]

where the map \( \lambda \circ \hat{x} + \mu \circ \hat{y} : S^l_1 \vee S^l_2 \to S^l \vee S^l \) is defined to be \( \lambda \circ \hat{x}_j + \mu \circ \hat{y}_j \) on \( S^l_j \).

**Proposition 1.7** Let \( a = ((a^{i,j})_1, \ldots, (a^{i,j})_m), 1 \leq i < j \leq s, \) and \( b = (b_1, \ldots, b_m) \) be arrays of integers. Let \( W(b) : V^{2l-1}_m \to S^l \) be the map whose restriction
to the $q$-th sphere is $W(b_q)$. Then there is an effectively constructible PL map $W_s(a): V_{m}^{l-1} \to V^l_i$ such that the property \((\text{SYM})\) for even $l$, and the property \((\text{SKEW})\) for odd $l > 1$ and all $a_{q,i,j}^{l,j}$ even, is equivalent to

$$\exists \text{ a PL map } \varnothing: V_{s}^l \to Y_l \text{ such that } \varnothing \circ W_{s}(a) \simeq \omega_l(b).$$  \hspace{1cm} (LD)

Here $\omega_l = W$ for $l$ even, and $\omega_l = W_2$ for $l$ odd.

Proposition 1.7 and Lemma 1.4, (a) and (b'), imply that homotopy left divisibility is undecidable. See the right triangle of the left diagram below for $P = V_{m}^{l-1}$, $Q = V_{s}^l$, $g = W_s(a)$, and $\omega = \omega_l(b)$.

\[
\begin{array}{c}
P \\ \xymatrix{ \subset \ar[r]^{g} & \omega \\ \subset Q \ar@{~>}[r] & \varnothing \\ Y \ar[ru] & }
\end{array}
\]

\[
\begin{array}{c}
P \subset X \subset X \cup P \text{ Cyl } g \\ \xymatrix{ g \ar[r] & X \ar[r] & X \cup P \ar[r] & \text{ Cyl } g \\ Y \ar[ru] & Q \ar[ru] & Q \ar[ru] & }
\end{array}
\]

**Proof of Proposition 1.7** Let

- $W_s^{i,j}(a_{q,i,j}^{l,j})$ be the composition $S^{2l-1} \xrightarrow{W_2(a_{q,i,j}^{l,j})} S_t^i \cup S_t^j \subset V^l_i$;
- $W_s(a_q): S^{2l-1} \to V^l_i$ be any PL map representing the sum of the maps $W_s^{i,j}(a_q)$;
- $W_s(a): V_{m}^{l-1} \to V_{s}^l$ be the map whose restriction to the $q$-th sphere is $W_s(a_q)$.

By Theorem 1.6 and using $(\alpha_1 + \alpha_2) \circ \beta = \alpha_1 \circ \beta + \alpha_2 \circ \beta$, for $l > 1$ we have

\[(\text{bs}) \quad W_s(a) \simeq W_s(a') \quad \text{only when } a = a';\]
\[(\text{cs}) \quad \widehat{\beta} \circ W_s(a) \simeq W(Q_x(a)), \quad \text{where } Q_x(a) := \sum_{1 \leq i < j \leq a} a^{i,j} x_i y_j;\]
\[(\text{ds}) \quad \text{for } l \text{ odd } (\lambda \circ \widehat{\beta} + \mu \circ \widehat{\gamma}) \circ W_s(2a) \simeq W_2(2R_{x,y}(a)), \text{ where } R_{x,y}(a) := \sum_{1 \leq i < j \leq s} a^{i,j} (x_i y_j - x_j y_i).\]

**Proof that (SYM) \(\Rightarrow\) (LD) for $l$ even.** Take an integer solution $x = (x_1, \ldots, x_s)$. Let $\varnothing := \widehat{\beta}$. Then by (cs), $\varnothing \circ W_s(a^q) \simeq W(Q_x(a^q)) = W(b_q)$ for each $q$. Thus $\varnothing \circ W_s(a) \simeq W(b)$.

**Proof that (LD) \(\Rightarrow\) (SYM) for $l$ even.** Take the PL map $\varnothing: V^l_s \to S^l$. Let $x_j := \deg(\varnothing|_{S^l_j})$. Then by Theorem 1.5 (b), $\varnothing \simeq \widehat{\beta}$. Take any $q$. Then by (cs)

$$W(Q_x(a^q)) \simeq \widehat{\beta} \circ W_s(a^q) \simeq \varnothing \circ W_s(a^q) \simeq W(b_q).$$

Hence by Theorem 1.6 (a), $Q_x(a^q) = b_q$.

**Proof that (SKEW) \(\Rightarrow\) (LD) for $l > 1$ odd and all $a_{q,i,j}^{l,j}$ even.** Take an integer solution $(x, y) = (x_1, \ldots, x_s, y_1, \ldots, y_s)$. Let $\varnothing := \lambda \circ \widehat{\beta} + \mu \circ \widehat{\gamma}$. Then by (ds), $\varnothing \circ W_s(a^q) \simeq W(R_{x,y}(a^q)) = W_2(b_q)$ for each $q$. Thus $\varnothing \circ W_s(a) \simeq W_2(b)$.
Proof that (LD) ⇒ (SKEW) for \( l > 0 \) and all \( a_{i,j}^q \) even. Let \( x_j := \deg (u \circ \varphi | s_j^l) \) and \( y_j := \deg (v \circ \varphi | s_j^l) \). Then by Theorem 1.5(c), \( \varphi \simeq \lambda \circ \widehat{x} + \mu \circ \widehat{y} \). Take any \( q \). Then by (ds)

\[
W_2(R_{x,y}(a^q)) \simeq (\lambda \circ \widehat{x} + \mu \circ \widehat{y}) \circ W_z(a^q) \simeq \varphi \circ W_z(a^q) \simeq W_2(b_q).
\]

Hence by (bs), \( R_{x,y}(a^q) = b_q \).

\[\square\]

**Proposition 1.8** For a simplicial map \( g : P \to Q \) between complexes there is an effectively constructible triple \((\text{Cyl} \: g; P, Q)\) of a complex \text{Cyl} \: g and its subcomplexes isomorphic to \( P \), \( Q \) such that

(a) for any complex \( Y \) a simplicial map \( \omega : P \to Y \) extends to \text{Cyl} \: g if and only if there is a PL map \( \varphi : Q \to Y \) such that \( \varphi \circ g \simeq \omega \);

(b) \( g \) extends to a complex \( X \supset P \) if and only if the identity map of \( Q \) extends to \( X \cup_P \text{Cyl} \: g \).

**Proof of the ‘extendability is undecidable’ Theorem 1.2** Take a simplicial subdivision of \( \omega_l(b) \) as a given map, and \( X = \text{Cyl} \: W_s(a_l) \). Apply Propositions 1.7 and 1.8(a) (in the latter take \( P = V_m^2 \), \( Q = V_s^l \), \( Y = Y_l \), \( g = W_s(a_l) \), and \( \omega = \omega_l(b) \)). We obtain that

- extendability of \( \omega_l(b) \) to \( X \) is equivalent to (SYM) for \( l \) even;

- when all \( a_{i,j}^l \) are even, extendability of \( \omega_l(b) \) to \( X \) is equivalent to (SKEW) for \( l \) odd.

The latter is undecidable by Lemma 1.4, (a) and (b').

\[\square\]

**Construction of Cyl \: g in Proposition 1.8.** For a map \( f : P \to Q \) between subsets \( P \subset \mathbb{R}^p \) and \( Q \subset \mathbb{R}^q \) define the mapping cylinder \( \text{Cyl} \: f \) to be the union of \( 0 \times Q \times 1 \subset \mathbb{R}^p \times \mathbb{R}^q \times \mathbb{R} = \mathbb{R}^{p+q+1} \) and segments joining points \((u, 0, 0) \in \mathbb{R}^{p+q+1}\) to \((0, f(u), 1) \in \mathbb{R}^{p+q+1} \), for all \( u \in P \). See [1, Figure, p.14]. We identify \( P \) with \( P \times 0 \times 0 \) and \( Q \) with \( 0 \times Q \times 1 \).

Define the map \( \text{ret} \: g : \text{Cyl} \: g \to Q \) by mapping \((0, v, 1) \) to \( v \), and mapping to \( g(u) \) the segment containing \((u, 0, 0) \). For a simplicial map \( g : P \to Q \) between complexes denote by \( |g| : |P| \to |Q| \) the corresponding PL map between their bodies. Then \( \text{Cyl} \: |g| \) is the body of certain complex \( \text{Cyl} \: g \)

- whose vertices are the vertices of \( P \) and the vertices of \( Q \);

- whose simplices are the simplices of \( P \), the simplices of \( Q \) and another simplices that are not hard to define.

**Example 1.9** (a) For the 2-winding \( \widehat{\varphi} : S^1 \to S^1 \) (i.e., for the quotient map \( S^1 \to \mathbb{R}P^1 \)) \( \text{Cyl} \: \widehat{\varphi} \) is the Möbius band (i.e., the complement to a 2-disk in \( \mathbb{R}P^2 \)).

(a') For the Hopf map \( \eta : S^3 \to S^2 \) (i.e., for the quotient map \( S^3 \to \mathbb{C}P^1 \)) \( \text{Cyl} \: \eta \) is the complement to a 4-disk in \( \mathbb{C}P^2 \) (i.e., the ‘complexified’ Möbius band).

(b) For the commutator map \( f : S^1 \to S^1 \) (i.e., \( f = aba^{-1}b^{-1} \)) \( \text{Cyl} \: f \) is the complement to a 2-disk in \( S^1 \times S^1 \).

(b') The cylinder of the map \( W_2(1) : S^{2l-1} \to S^l \) is the complement to a 2\( l \)-disk in \( S^l \times S^l \).
Proof of Proposition 1.8 (a), ‘only if’: Let \( \varphi \) be the restriction to \( Q \subset \text{Cyl} \, g \) of given extension.

(a), ‘if’: Let the required extension be \( \varphi \circ \text{ret} \, g \).

(b), ‘only if’: Let the required extension be \( \text{ret} \, g \) on \( \text{Cyl} \, g \) and the given extension on \( X \).

(b), ‘if’: Let \( r : X \cup_{P} \text{Cyl} \, g \to Q \) be given extension. The composition \( P \times [0, 1] \to \text{Cyl} \, g \to Q \) of the quotient map and \( r \) is a homotopy between \( r|_{P} \) and \( g \). Since \( r|_{P} \) extends to \( X \), by the Borsuk Homotopy Extension Theorem it follows that \( g \) extends to \( X \).

\[\square\]

2 Known Proof of Theorem 1.6

Construction of \( W_{2}(a) \). Decompose

\[ S^{2l-1} = \partial(D^{l} \times D^{l}) = S^{l-1} \times D^{l} \cup_{S^{l-1} \times S^{l-1}} D^{l} \times S^{l-1}. \]

Define the Whitehead map \( W_{2}(1) : S^{2l-1} \to S^{l} \vee S^{l} \) as the ‘union’ of the compositions

\[ S^{l-1} \times D^{l} \overset{pr_{2}}{\to} D^{l} \overset{c}{\to} S^{l} \overset{\lambda}{\to} S^{l} \vee S^{l} \quad \text{and} \quad D^{l} \times S^{l-1} \overset{pr_{1}}{\to} D^{l} \overset{c}{\to} S^{l} \overset{\mu}{\to} S^{l} \vee S^{l}. \]

Here \( pr_{j} \) is the projection onto the \( j \)-the factor, and \( c \) is contraction of the boundary to a point. It is easy to modify this ‘topological’ definition to obtain an effectively constructible PL map \( W_{2}(1) \). Define \( W_{2}(a) \) in the same way as \( W_{2}(1) \) except that \( \lambda \) is replaced with \( \lambda \circ \hat{a} \).

Sketch of a proof of (a) and (b) Denote by \( \text{lk} \) the linking coefficient of two collections of oriented closed polygonal lines in \( S^{3} \), or, more generally, of two integer \( l \)-cycles in \( S^{2l-1} \). See definition e.g. in [9], [10, Sect. 4].

For a PL map \( \psi : S^{2l-1} \to Y_{l} \) define \( H(\psi) := \text{lk}(\psi^{-1}y_{1}, \psi^{-1}y_{2}) \), where \( y_{1}, y_{2} \in Y_{l} \) are distinct ‘random’ (i.e., regular) values of \( \psi \), \( \psi^{-1} \) is ‘oriented’ preimage, and for \( l \) odd \( y_{1} \in S^{l} \vee *, y_{2} \in * \vee S^{l} \). More precisely, take subdivisions of \( S^{2l-1} \) and of \( Y_{l} \) for which \( \psi \) is simplicial. Then take \( y_{1}, y_{2} \) outside the image of any \((l - 1)\)-simplex of the subdivision of \( S^{2l-1} \). This is a well-defined homotopy invariant of \( \psi \) (for \( l \) even called Hopf invariant).

For \( l \) even \( HW(a) = \pm 2a \). Hence \( W(a) \simeq W(a') \) only when \( a = a' \). Clearly, \( HW_{2}(a) = \pm a \). Hence \( W_{2}(a) \simeq W_{2}(a') \) only when \( a = a' \).

\[\square\]

Sketch of a proof of (c) and (d) For a complex \( X \) and maps \( f, g : S^{l} \to X \) define a map \( \left[ f, g \right] : S^{2l-1} \to X \) in the same way as \( W_{2}(1) \) except that \( S^{l} \vee S^{l}, \lambda, \mu \) are replaced by \( X, f, g \). Then \( W_{2}(a) = \left[ \lambda \circ \hat{a}, \mu \right] \) and \( W(a) = \left[ \hat{a}, \hat{1} \right] \). This defines a map \( \left[ \cdot, \cdot \right] : \pi_{l}(X) \times \pi_{l}(X) \to \pi_{2l-1}(X) \) (called Whitehead product). For \( l > 1 \) we have

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3 This theorem states that if \((K, L)\) is a polyhedral pair, \( Q \subset \mathbb{R}^{d}, F : L \times I \to Q \) is a homotopy, and \( g : K \to Q \) is a map such that \( g|_{L} = F|_{L \times 0} \), then \( F \) extends to a homotopy \( G : K \times I \to Q \) such that \( g = G|_{K \times 0} \).

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\[ [\alpha_1 + \alpha_2, \beta] = [\alpha_1, \beta] + [\alpha_2, \beta] \quad \text{and} \quad [\alpha, \beta] = (-1)^l [\beta, \alpha]. \]

Then
\[ (\hat{x}_1 \lor \hat{x}_2) \circ W_2(a) \simeq [\hat{x}_1 \circ \hat{a}, \hat{x}_2] \simeq W(ax_1x_2). \]

For \( l \) odd denoting the homotopy class of a map by the same letter as the map we have
\[
(\lambda \circ \hat{x} + \mu \circ \hat{y}) \circ W_2(a) = (1) a[x_1\lambda + y_1\mu, x_2\lambda + y_2\mu] \\
= (2) a[x_1\lambda, y_2\mu] + a[y_1\mu, x_2\lambda] = (3) W_2(a(x_1y_2 - x_2y_1)).
\]

Here equalities (1), (2), (3) hold because \( (\lambda \circ \hat{x} + \mu \circ \hat{y})|_{S^l} = x_j\lambda + y_j\mu \), because \( [\lambda, \lambda] = -[\lambda, \lambda] \) and \( [\mu, \mu] = -[\mu, \mu] \), because \( [\mu, \lambda] = -[\lambda, \mu] \) and \( a \) is even, respectively.

\[\square\]

3 Appendix: Is Embeddability of Complexes Undecidable in Codimension \( > 1 \)?

Realizability of hypergraphs or complexes in the \( d \)-dimensional Euclidean space \( \mathbb{R}^d \) is defined similarly to the realizability of graphs in the plane. E.g. for 2-complex one ‘draws’ a triangle for every three-element subset. There are different formalizations of the idea of realizability.

A complex \((V, F)\) is simplicially (or linearly) embeddable in \( \mathbb{R}^d \) if there is a set \( V' \) of distinct points in \( \mathbb{R}^d \) corresponding to \( V \) such that for any subsets \( \sigma, \tau \subset V' \) corresponding to elements of \( F \) the convex hull \( \langle \sigma \rangle \) is a simplex of dimension \( |\sigma| - 1 \), and \( \langle \sigma \rangle \cap \langle \tau \rangle = \langle \sigma \cap \tau \rangle \). A complex is PL (piecewise linearly) embeddable in \( \mathbb{R}^d \) if some its subdivision is simplicially embeddable in \( \mathbb{R}^d \). For classical and modern results on embeddability and their discussion see e.g., surveys [10, Sect. 5], [11], [13, Sect. 3].

**Theorem 3.1** (embeddability is undecidable in codimension 1) *For every fixed \( d, k \) such that \( 5 \leq d \in \{k, k + 1\} \) there is no algorithm recognizing PL embeddability of \( k \)-complexes in \( \mathbb{R}^d \).*

This is deduced in [6, Thm. 1.1] from the Novikov theorem on unrecognizability of the \( d \)-sphere. Cf. [7, Rem. 3].

**Conjecture 3.2** (embeddability is undecidable in codimension \( > 1 \)) *For every fixed \( d, k \) such that \( 8 \leq d \leq (3k + 1)/2 \) there is no algorithm recognizing PL embeddability of \( k \)-complexes in \( \mathbb{R}^d \).*

Conjecture 3.2 is stated as a theorem in [2]. The proof in [2] contains a gap described below. Their idea is to elaborate the following remark to produce the reduction (described below) to the ‘retractability is undecidable’ Theorem 1.1.

**Remark 3.3** Homotopy classifications of maps \( S^{2l-1} \to S^l \) and \( S^{2l-1} \to S^l \lor S^l \) are related to isotopy classification of links of \( S^{2l-1} \lor S^{2l-1} \) and of \( S^{2l-1} \lor S^{2l-1} \lor S^{2l-1} \)
in \(\mathbb{R}^3\) [3] (including higher-dimensional Whitehead link and Borromean rings; see [11, Sect. 3]). E.g. the generalized linking coefficients of the Whitehead link and of the Borromean rings are (the homotopy classes) of the Whitehead maps \(W(1) : S^{2l-1} \to S^l\) and \(W_2(1) : S^{2l-1} \to S^l \vee S^l\) from Theorem 1.6. Analogous results for \(l = 1\) do illustrate some ideas, see a description accessible to non-specialists in [12, Sect. 3.2].

We use the notation of Sect. 1. Let \(a = ((a_1^{i,j})), \ldots, (a_s^{i,j}))_m, 1 \leq i < j \leq s,\) and \(b = (b_1, \ldots, b_m)\) be any arrays of integers. Define the double mapping cylinder \(X(a, b) = X_I(a, b)\) to be the union of \(\text{Cyl} W_s(a)\) and \(\text{Cyl} W_2(b) \supset Y_I\), in which \(V_m^{2l-1} \subseteq \text{Cyl} W_s(a)\) is identified with \(V_m^{2l-1} \subseteq \text{Cyl} W_2(b)\).

Embed \(Y_{2l+1}\) standardly into \(S^{3l+2}\). For \(l\) even take a small oriented \((l + 1)\)-disk \(D \subset S^{3l+2}\) whose intersections with \(Y_{2l+1} = S^{2l+1}\) is transversal and consist of exactly one point. Let \(\overline{Y}_I = \partial D \cong S^1\) be the meridian of \(Y_{2l+1}\). For \(l\) odd define analogously the meridian \(\overline{Y}_I := \partial D_+ \cup \partial D_- \cong S^1 \vee S^1\). Let \(\overline{D}_+ = \partial D_+ \cap S^1, \overline{D}_- = \partial D_- \cap S^1\). Let \(\overline{D}_+ \cap \overline{D}_-\) be a point in \(\partial D_+ \cup \partial D_-\).

**Conjecture 3.4** If either \(l\) is even or all \(a_{i,j}^l\) are even and \(l > 1\) is odd, then there is a \((2l + 1)\)-complex \(G \supset Y_I\) such that any of the following properties is equivalent to (SYM) for \(l\) even, and to (SKEW') for \(l\) odd:

\((\text{Ex})\) a PL homeomorphism \(Y_I \to \overline{Y}_I\) extends to a PL map \(X(a, b) \to S^{3l+2} - Y_{2l+1}\);

\((\text{Ex}')\) a PL homeomorphism of \(Y_I \to \overline{Y}_I\) extends to a PL embedding \(X(a, b) \to S^{3l+2} - Y_{2l+1}\);

\((\text{Em})\) \(X(a, b) \cup_{Y_I} \text{G embeds into } S^{3l+2}\).

All the implications except (Em) \(\Rightarrow (\text{Ex}')\) are correct results of [2]. The implication (Ex') \(\Rightarrow (\text{Ex})\) is clear. The equivalence of (Ex') and (SYM)/(SKEW) follows by Propositions 1.7 and 1.8, (a) and (b), because there is a strong deformation retraction \(S^{3l+2} - Y_{2l+1} \to \overline{Y}_I\). The implication (Ex) \(\Rightarrow (\text{Ex}')\) is implied by the following version of the Zeeman–Irwin Theorem [11, Thm. 2.9].

**Lemma 3.5** For any PL map \(f : X(a, b) \to S^{3l+2} - Y_{2l+1}\) there is a PL embedding \(f' : X(a, b) \to S^{3l+2} - Y_{2l+1}\) such that the restrictions of \(f\) and \(f'\) to \(Y_I \subset X(a, b)\) are homotopic.

Lemma 3.5 is essentially a restatement of [2, Thm. 10] accessible to non-specialists. See more detailed historical remark in [14, Rem. 3.9].

The idea of [2] to prove the implication (Em) \(\Rightarrow (\text{Ex}')\) is to construct the complex \(G\), and use a modification of the following lemma.

**Lemma 3.6** [8, Lem. 1.4] For any integers \(0 \leq l < k\) there is a \(k\)-complex \(F_\sim\) containing subcomplexes \(\Sigma^k \cong S^k\) and \(\Sigma^l \cong S^l\), PL embeddable into \(\mathbb{R}^{k+l+1}\) and such that for any PL embedding \(f : F_\sim \to \mathbb{R}^{k+l+1}\) the images \(f \Sigma^k\) and \(f \Sigma^l\) are linked modulo 2.

Lemma 30 of [2] is a modification of Lemma 3.6 with ‘linked modulo 2’ replaced by ‘linked with linking coefficient \(\pm 1\)’. The end of proof of Lemma 30 in [2], p. 778, used the following incorrect statement: If \(f : D^p \to \mathbb{R}^{p+q}\) and \(g : S^q \to \mathbb{R}^{p+q}\) are PL embeddings such that \(|f(D^p) \cap g(D^q)| = 1\), then the linking coefficient of \(f|_{S^{p-1}}\) and \(g\) is 1.
Example 3.7 For any integers $p, q \geq 2$ and $c$ there are PL embeddings $f : D^p \to \mathbb{R}^{p+q}$ and $g : S^q \to \mathbb{R}^{p+q}$ such that $|f(D^p) \cap g(S^q)| = 1$ and the linking coefficient of $f|_{S^{p-1}}$ and $g$ is $c$.

Proof Take PL embeddings $f_0 : S^{p-1} \to \mathbb{R}^{p+q-1}$ and $g_0 : S^{q-1} \to \mathbb{R}^{p+q-1}$ whose linking coefficient is $c$. Take points $A, B \in \mathbb{R}^{p+q} - \mathbb{R}^{p+q-1}$ on both sides of $\mathbb{R}^{p+q-1}$. Then $f = f_0 \ast A$ and $g = g_0 \ast \{A, B\}$ are the required embeddings. \qed

The modification [2, Lem. 30] of Lemma 3.6 is presumably incorrect, cf. [4, Thm. 1.6]. See more discussion and conjectures in [14, Sect. 3].

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