Motion equations of a pendulum gear system

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Abstract. This work presents a dynamic system that is composed of gears which have an angular velocity \( \dot{\theta} \). Each of the gears was added a rod of known length and mass which can move freely. This system can move in such a way that the rods oscillate as the gears rotate. With the Lagrangiana formulation, the motion equations that govern the system were built, differential equations were solved with standardized numerical methods and graphs were obtained in different phase spaces.

1. Introduction
Within the field of physics, dynamic system study has been of great importance since we can understand in more detail the movement of a particle system. A very important case is the gear systems which are used to transmit power, between nearby, parallel, perpendicular or oblique axes [4], from one component to another within a machine, these are made up of two sprockets. The gears are used to transmit circular movement by contacting sprockets [1, 2]. We also know that there are large applications of these systems, however, little is said about mechanisms composed of gears. These systems are undoubtedly of great interest because their study involves employing the knowledge previously acquired in university life in physics courses, from the concept of potential and kinetic energy, simple and physical pendulum among other terms and above all what conditions will cause the system to go into chaos.

Our purpose in this work is, based on Lagrangian formalism, to obtain the equations describing the movement of the gear and rod system shown in Figure 1 and solve them with numerical methods using the standardized Runge-Kutta method of order 4.

2. Lagrangian equations
The system consists of two gears of \( R, r \) radius and mass \( M, m \), which rotate with an angular velocity \( \dot{\theta} \). In the center of each gear there is a thin rod of mass \( m_1 \) and length \( l \), which can rotate also due to the movement of the gear. It should be noted that the first rod is inverted i.e. forming an inverted physical pendulum, Figure 1.
The kinetic energy of the system can be written as:

\[ T = \frac{R^2}{4} (M + m + 4m_v) \dot{\theta}^2 + \frac{l^2}{8} m_v (\dot{\varphi}^2 + \dot{\beta}^2) - \frac{1}{2} RL \left[ \dot{\varphi} \sin(\theta + \varphi) + \dot{\beta} \sin\left(\frac{R}{r} \theta + \beta \right) \right], \]

and its potential energy as:

\[ V = m_v g \left( R \sin(\theta) + r \sin\left(\frac{R}{r} \theta \right) + \frac{l}{2} (\cos(\varphi) + \cos(\beta)) \right). \]

Knowing that in the Lagrangian formulation \( L = T - V \) the lagrangian of the whole system:

\[ L = \frac{R^2}{4} (M + m + 4m_v) \dot{\theta}^2 + \frac{l^2}{8} m_v (\dot{\varphi}^2 + \dot{\beta}^2) - \frac{1}{2} RL \left[ \dot{\varphi} \sin(\theta + \varphi) + \dot{\beta} \sin\left(\frac{R}{r} \theta + \beta \right) \right] - m_v g \left( R \sin(\theta) + r \sin\left(\frac{R}{r} \theta \right) + \frac{l}{2} (\cos(\varphi) + \cos(\beta)) \right), \]

and using the Euler-Lagrange equation \( \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}} \right) - \frac{\partial L}{\partial q} = 0 \) for each of the variables \( \theta, \beta \) and \( \varphi \) time-dependent, the three motion equations of the system are inferred.

For the variable \( \theta \) you get:

\[ (mR + MR + 4m_v) \ddot{\theta} + 2g m_v \cos(\theta) + 2g m_v \cos\left(\frac{R}{r} \theta \right) - l \left[ \cos\left(\beta + \frac{R}{r} \theta \right) + \dot{\beta}^2 \right] - l \left[ \cos(\theta + \varphi) + \dot{\varphi}^2 \right] = 0, \]

For the variable \( \varphi \) of the first pendulum:

\[ \ddot{\varphi} - \frac{2g}{l} \sin(\varphi) - \frac{2R}{lm_v} \cos(\theta + \varphi) \dot{\varphi}^2 - \frac{2R}{lm_v} \sin(\theta + \varphi) \dot{\varphi} = 0, \]

For the variable \( \beta \) of the second pendulum:

\[ \ddot{\beta} - \frac{2g \sin(\beta)}{l} - \frac{2R^2}{lm_v r} \cos\left(\beta + \frac{R}{r} \theta \right) \dot{\beta}^2 - \frac{2R}{lm_v} \sin\left(\beta + \frac{R}{r} \theta \right) \dot{\beta} = 0, \]

As you can see there is a coupling of differential equations (5) y (6).
3. Solving differential equations

The system formed by the two coupled differential equations was solved by computational numerical methods. The numerical solution was obtained using Runge-Kutta’s standardized and well-known method of order 4. The numerical results were obtained using the theoretical expressions (5) and (6) with the data presented in Table 1:

| Variables | Valor | Variables | Valor | Variables | Valor | Variables | Valor |
|-----------|-------|-----------|-------|-----------|-------|-----------|-------|
| \( \theta \) | 0     | \( \dot{\theta} \) | 0.9   | R         | 2     | M         | 4     |
| \( \varphi \) | 0     | \( \dot{\varphi} \) | 0.1   | r         | 0.5   | m         | 2     |
| \( \beta \) | 0     | \( \dot{\beta} \) | 0.3   | L         | 0.8   | m_v       | 2     |

N=10 loops were used for testing, with initial conditions for \( \theta \), \( \beta \) and \( \varphi \) within the integration, with a range of 0 to 100, dividing it into \( n = 90000 \) simulated data. Por so, the size of the step in the simulation is \( \delta h = 0.001 \).

4. Simulation Outcomes.

Below are the most representative results of the system. We show how is the movement behavior on the \( xy \) plane of the system(Figure 2a), blue the larger gear, in yellow the minor gear, in red the movement of the upper pendulum and in green that of the lower pendulum, considering that the system can be moved without any restrictions (Figure 2b).

![Figure 2](image)

Figure 2. Representation of the movement of the entire System in the \( xy \) plane. a) Simulation of movement on the \( xy \) plane. b) System at rest.

The phase planes are presented, as well as their behavior based on the time of each variable:

Case 1, for the angle \( \theta \):
Figure 3. a) Phase plane of gears b) Angle behavior $\theta$ depending on time.

Case 2, for the angle $\varphi$:

Figure 4. a) Phase plane of the upper pendulum b) Angle behavior $\varphi$ depending on time.

Case 3, for the angle $\beta$:

Figure 5. a) Phase plane of the lower pendulum. b) Angle behavior $\beta$ depending on time.

Below are the interactions between each of the variables, where you can see how each of these variables interacts and how they relate to each other:
As you can see the role that gears play in the system is of great importance because they give that ease of moving either linearly or chaotically to the pendulum system and above all how they can make the most of the energy it provides them.

5. Conclusions
We have presented the theoretical analysis of a system that consists of two gears of different mass masses and different radii which contain two physical pendulums of length l and mass m, with the intention of understanding the movement that occurs in this system. relation topics of great interest such as the conservation of energy, as well as the concept of the physical pendulum and the inverted pendulum. Numerical simulations were developed to obtain the solutions of the differential equations. From the numerical analysis it can be concluded that a chaotic system can occur in the system because the inverted pendulum and the normal pendulum can oscillate freely and considering the energy transfer provided by the gears, this system can be studied in different ways considering that now either the pendulums are in the same positions, that is, both inverted or both normal. In particular in this problem we can conclude that this system is already in chaos, this due to the inverted pendulum since when obtaining its phases planes, chaotic behaviors can be seen and the second or normal pendulum is also affected since all the energy is transferred it is observed to be chaotic; thus, raising the possibility of studying them through the methodology used to analyze chaos, such as: Poincaré sections and Lyapunov coefficients, for example.

References
[1] Albarrán J (s. f.) 2020 Engranajes e-REdING. Trabajos y protectos de la E.T.S.I. Recuperado 15 de junio de 2020, de http://bibing.us.es/proyectos/abreproy/4483/fichero/6.+Engranajes.pdf
[2] Marion J B 2010 Dinámica clásica de las partículas y sistemas, 2.ª ed., Vol. 1, Editorial Reverté
[3] Thomson W & Dahleh M D 1997 Theory of Vibrations with Applications, 5th Revised, ed. Pearson.
[4] Walter H 1968 Introducción a los Principios de la Mecánica (PRIMERA EDICIÓN EN ESPAÑOL ed., Vol. 1). UTEHA