ASYMPTOTIC ANALYSIS OF MHD SYSTEMS

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Abstract. In this paper, we study the convergence of strong solutions of a Magneto-Hydro-Dynamic system. On the torus $\mathbb{T}^3$, the proof is based on Schochet’s methods, whereas in the case of the whole space $\mathbb{R}^3$, we use Strichartz’s type estimates and a product law’s $2D \times 3D$.

1. Introduction

In this paper we study unique local existence and asymptotic behaviour of solutions for the MHD system, see [6],

$$
\begin{align*}
\partial_t u + u.\nabla u - \frac{E}{\varepsilon}\Delta u + \frac{e \times u}{\varepsilon} - \frac{\Lambda}{\varepsilon}\text{curl}(b) \times e' - \frac{\Lambda\theta}{\varepsilon}\text{curl} b \times b &= -\frac{\nabla p}{\varepsilon}, \\
\partial_t b - \frac{\Delta b}{\theta} + u.\nabla b - b.\nabla u - \frac{u \times e'}{\theta} &= 0,
\end{align*}
$$

in $\mathbb{R}_+^T \times \Omega_x$, where $\Omega$ is the whole space $\mathbb{R}^3$ or the torus $\mathbb{T}^3 := \mathbb{R}^3/\mathbb{Z}^3$; $u$ is the velocity field, $b$ is the magnetic field and $e, e'$ are two fixed vectors. The parameters $E, \varepsilon, \Lambda$ and $\theta$ represent consecutively the Ekman number, the Rossby number, the Elsasser number and the magnetic Reynolds number. We notice that these parameters satisfy, according to [6],

$$
\varepsilon \to 0, \quad \Lambda = O(1), \quad \varepsilon\theta \to 0 \quad \text{and} \quad E \sim \varepsilon^2.
$$

These equations modalize the magneto-hydro-dynamic flow in the Earth’s core which is believed to support a self-excited dynamo process generating the Earth’s magnetic field. Here, we present the analytical study of a simplified problem where we choose $E = \varepsilon^2$, $\Lambda = \varepsilon^{3/2}$, $\theta = \varepsilon^{-1/2}$ and $e = e' = -e_3$. Then, we can see that our system take the following form

$$
(MHD^\varepsilon)
\begin{align*}
\partial_t u - \varepsilon\Delta u + u.\nabla u - \text{curl} (b) \times b + \sqrt{\varepsilon}\text{curl} (b) \times e_3 + \frac{u \times e_3}{\varepsilon} &= -\nabla p, \\
\partial_t b - \sqrt{\varepsilon}\Delta b + u.\nabla b - b.\nabla u + \sqrt{\varepsilon}\text{curl} (u \times e_3) &= 0,
\end{align*}
$$

in $\mathbb{R}_+^T \times \Omega_x$, where $\Omega$ is the whole space $\mathbb{R}^3$ or the torus $\mathbb{T}^3 := \mathbb{R}^3/\mathbb{Z}^3$; $u$ is the velocity field, $b$ is the magnetic field and $e, e'$ are two fixed vectors. The parameters $E, \varepsilon, \Lambda$ and $\theta$ represent consecutively the Ekman number, the Rossby number, the Elsasser number and the magnetic Reynolds number. We notice that these parameters satisfy, according to [6],

$$
\varepsilon \to 0, \quad \Lambda = O(1), \quad \varepsilon\theta \to 0 \quad \text{and} \quad E \sim \varepsilon^2.
$$

The goal of this study is to find the "limit system" when $\varepsilon$ goes to zero. We denote by $\mathbb{P}$ the $L^2$ orthogonal projection on divergence-free vector fields. Applying $\mathbb{P}$ to the first equation of $(MHD^\varepsilon)$, then one can see that $U^\varepsilon = (u^\varepsilon, b^\varepsilon)$ is a solution of the following abstract form

$$
(S^\varepsilon)
\begin{align*}
\partial_t U + Q(U, U) + a_5^\varepsilon(D)U + L^\varepsilon(U) &= 0 \quad \text{in} \quad \mathbb{R}_+^T \times \Omega_x, \\
\text{div} u &= 0, \quad \text{div} b = 0,
\end{align*}
$$

where the quadratic term $Q$ is defined by

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Q(U, U) = \left( \mathbb{P}(u, \nabla u) - \mathbb{P}(b, \nabla b), u, \nabla b - b, \nabla u \right),

the viscous term is

\begin{equation*}
a^\varepsilon(D)U = \left( -\varepsilon \Delta u, -\sqrt{\varepsilon} \Delta b \right),
\end{equation*}

and the linear perturbation \( L^\varepsilon \) is given by

\begin{equation*}
L^\varepsilon(U) := \left( \mathbb{P}(\frac{u + e_3}{\varepsilon}) + \sqrt{\varepsilon} \partial_3 b, \sqrt{\varepsilon} \partial_3 u \right).
\end{equation*}

In the literature, \( a^\varepsilon(D) \) is elliptic and, in the case of a rotating fluid \( L^\varepsilon = \frac{1}{\varepsilon}L \) with \( L \) a skew-symmetric linear operator.

Singular limits in system such as \( (S^\varepsilon) \) have been studied by several authors. In the hyperbolic case, namely \( a^\varepsilon(D)U = 0 \), A. Babin, A. Mahalov and B. Nicolaenko [1] studied the incompressible rotating Euler equation on the torus. Using the method introduced by S. Schochet (see [14] and [15]), I. Gallagher studied in [8] this problem in its abstract hyperbolic form. In the case of the incompressible rotating Navier-Stokes equation on the torus, it is shown (see [1] and [9]) that the solutions converge to a solution of a certain diffusion equation. Moreover, for a special initial condition, there exists a sequence of solutions convergent to a solution of a two dimensional Navier-Stokes equation. Motivated by this case, J. Y. Chemin, B. Desjardin, I. Gallagher and E. Grenier studied in [5] the incompressible fluids with anisotropic viscosity on the whole space, the key of their proof is an anisotropic version of Strichartz estimates. We refer to I. Gallagher [7] for the study of the abstract parabolic form. Among others, we also refer to the basic results of [5], [7], [9], [10], [11], [12], [13].

Notice that the existence results follow directly from the Friedrichs’s method and the energy estimates. Following the approximation scheme of Friedrichs, we shall prove in section two global existence of “Leray’s solutions” and local existence of strong solutions on uniform time; namely solutions defined by the following results.

**Theorem 1.1.** Let \( \varepsilon > 0 \) and \( U_0 = (u_0, b_0) \in L^2(\Omega) \) such that

\begin{equation*}
\text{div } u_0 = 0, \quad \text{div } b_0 = 0.
\end{equation*}

There exists \( U^\varepsilon := (u^\varepsilon, b^\varepsilon) \) a solution of \((\text{MHD}^\varepsilon)\) with \( U^\varepsilon \in L^\infty(\mathbb{R}^+, L^2) \cap L^2(\mathbb{R}^+, \dot{H}^1) \).

Moreover, \( U^\varepsilon \) satisfies the following energy estimate. For all \( t \geq 0 \),

\begin{equation}
\|U^\varepsilon(t)\|^2_{L^2(\Omega)} + 2\varepsilon \int_0^t \|\nabla u^\varepsilon\|^2_{L^2} + 2\sqrt{\varepsilon} \int_0^t \|\nabla b^\varepsilon\|^2_{L^2} \leq \|U_0\|^2_{L^2}.
\end{equation}

**Theorem 1.2.** Let \( U_0 = (u_0, b_0) \in H^s(\Omega) \) with \( s > \frac{3}{2} + 2 \) an integer, such that

\begin{equation*}
\text{div } u_0 = 0, \quad \text{div } b_0 = 0.
\end{equation*}

Then there exists \( T > 0 \), and a constant \( C > 0 \) such that for all \( \varepsilon \in ]0,1[ \), there exists a unique solution \( U^\varepsilon := (u^\varepsilon, b^\varepsilon) \in C^0_T(H^s) \cap L^2_T(H^{s+1}) \) of system \((\text{MHD}^\varepsilon)\) satisfying; for all \( t \in [0, T] \)

\begin{equation}
\|U^\varepsilon(t)\|^2_{H^s(\Omega)} + 2\varepsilon \int_0^t \|\nabla u^\varepsilon\|^2_{H^s(\Omega)} + 2\sqrt{\varepsilon} \int_0^t \|\nabla b^\varepsilon\|^2_{H^s(\Omega)} \leq 2\|U_0\|^2_{H^s(\Omega)}.
\end{equation}

Moreover, if \( \|U_0\|_{H^s} \leq c\varepsilon \) \( (c := \frac{1}{4}) \), then the solution is global.

We are interested now to seeing the limit of strong solutions of the \((\text{MHD})\) system when \( \varepsilon \) goes to zero. Since \( \partial_t u^\varepsilon \) is not bounded in \( \varepsilon \), one cannot take the limit directly in the system, and then the classical proofs (see for example [10], [17]) no longer work.
• In the case $\Omega = \mathbb{T}^3$, a method to get round this difficulty is to use the group $\mathcal{L}(t)$ associated with the operator $L(u) := \mathbb{P}(u \times e_3)$. We consider the filtered solution $v^\varepsilon := \mathcal{L}(\frac{t}{\varepsilon})u^\varepsilon$ and we look for the limit system in $\mathcal{D}'$ satisfied by the "eventuel" limit of $(v^\varepsilon, b^\varepsilon)$,

\[
\begin{align*}
\partial_t v + q^0(v, v) - \mathbb{P}(b, \nabla b) &= 0 \\
\partial_t b + \bar{v} \cdot \nabla b - b \cdot \nabla \bar{v} &= 0 \quad \text{in} \quad \mathbb{R}^+_t \times \mathbb{T}^3 \\
(v, b)_{/t=0} &= (u_0, b_0) \\
\text{div } v &= \text{div } b = 0,
\end{align*}
\]

(\text{LS})

where

\[
q^0(v, v) := \lim_{\frac{t}{\varepsilon} \to 0} \mathcal{L}
\left(\frac{t}{\varepsilon}\right) \left(\mathbb{P}(\mathcal{L}(\frac{t}{\varepsilon})v).\nabla (\mathcal{L}(\frac{t}{\varepsilon})v)\right)
\]

and here, we have denoted $\bar{f}(x_1, x_2) = \int f(x_1, x_2, x_3) dx_3$ and $f_{\text{osc}} = f - \bar{f}$.

The system (LS) is taking in the sobolev space, precisely we have the following result.

\textbf{Theorem 1.3.} Let $s > \frac{3}{2} + 2$ be an integer and $U_0 = (u_0, b_0) \in H^s(\mathbb{T}^3)$ such that $\text{div } u_0 = 0$, $\text{div } b_0 = 0$. Then there exists $T > 0$, and a unique solution $(v, b) \in C^0_T(H^s)$ of system (LS).

More, precisely we have the following convergence result.

\textbf{Theorem 1.4.} Let $s > \frac{3}{2} + 2$ be an integer and $U_0 = (u_0, b_0) \in H^s(\mathbb{T}^3)$ such that $\text{div } u_0 = 0$, $\text{div } b_0 = 0$. We denote by $U^\varepsilon = (u^\varepsilon, b^\varepsilon)$ the family of solutions of (MHD) given by Theorem \ref{Theorem 1.3}. Then, for all $s' < s$,

\[
\begin{align*}
\text{div } v - \mathcal{L}(\frac{t}{\varepsilon}) v &= o(1) \quad \text{in} \quad L^\infty_T(H^{s'}(\mathbb{T}^3)) \\
b^\varepsilon - b &= o(1) \quad \text{in} \quad L^\infty_T(H^{s'}(\mathbb{T}^3)),
\end{align*}
\]

where $(v, b)$ is the solution of (LS).

• In the case $\Omega = \mathbb{R}^3$, we have the following result

\textbf{Theorem 1.5.} Let $s > \frac{3}{2} + 2$ be an integer and $U_0 = (u_0, b_0) \in H^s(\mathbb{R}^3)$ such that $\text{div } u_0 = 0$, $\text{div } b_0 = 0$. We denote by $U^\varepsilon = (u^\varepsilon, b^\varepsilon)$ the family of solutions of (MHD) given by Theorem \ref{Theorem 1.3}. Then, for all $s' < s$

\[
\begin{align*}
u^\varepsilon &= o(1) \quad \text{in} \quad L^\infty_T(C^{s'-\frac{3}{2}}(\mathbb{R}^3)) \\
b^\varepsilon - b_0 &= o(1) \quad \text{in} \quad L^\infty_T(H^{s'}(\mathbb{R}^3)).
\end{align*}
\]

Since the limit of the system (MHD) is the bidimensional Navier-Stokes equations, it is natural to consider initial data of the type $(u_0, b_0) = (\bar{u}_0 + u_0, b_0)$, where $\bar{u}_0 = \bar{u}_0(x_h)$ and $u_0 = u_0(x_h, x_3), b_0 = b_0(x_h, x_3)$ (see \cite{3}.

Before starting the results, it will be useful to consider the following system

\[
\begin{align*}
\partial_t \bar{u} - \varepsilon \Delta \bar{u} + \bar{u} \cdot \nabla \bar{h} \bar{u} &= - (\nabla_h p, 0) \quad \text{in} \quad \mathbb{R}^+_t \times \mathbb{R}^2_h \\
div_h \bar{u} &= 0 \\
\bar{u}_{/t=0} &= \bar{u}_0.
\end{align*}
\]

(\text{NS2D}^\varepsilon)

Using the classical Friedrich’s scheme, we can prove the existence of strong solutions on uniform time for the system (\text{NS2D}^\varepsilon). Precisely, we have the classical result

\textbf{Theorem 1.6.} Let $\bar{u}_0 \in \left(H^\sigma(\mathbb{R}^2)\right)^3$ be a divergence-free vector field with $\sigma > \frac{3}{2} + 2$ an integer. Then there exists $T_0 := \frac{1}{C(\sigma) \|\bar{u}_0\|_{H^\sigma}}$ such that for all $\varepsilon > 0$, there exists a
unique solution $\bar{u}^\varepsilon \in C^0_{T_0}(H^s(\mathbb{R}^2)) \cap L^2_{T_0}(H^{s+1}(\mathbb{R}^2))$ of system (NS2D$^\varepsilon$) satisfying; for all $t \in [0,T_0]$ 
\[ \|\bar{u}^\varepsilon(t)\|_{H^s}^2 + 2\varepsilon \int_0^t \|\nabla \bar{u}^\varepsilon\|^2_{L^2} \leq 2\|\bar{u}_0\|_{H^s}^2. \]

We pose
\[
(2D \times 3D) \left\{ \begin{array}{l}
(u_0,b_0) = (\bar{u}_0 + w_0,b_0) \\
\text{div}_h \bar{u}_0 = \text{div} w_0 = \text{div} b_0 = 0 \\
\bar{u}_0 = \bar{u}_0(x_h) \in H^{s+1}(\mathbb{R}^2) \\
w_0 = w_0(x_h,x_3) \in H^s(\mathbb{R}^3) \\
b_0 = b_0(x_h,x_3) \in H^s(\mathbb{R}^3)
\end{array} \right. \\
\text{with } s > \frac{3}{2} + 2 \text{ an integer.}
\]

We suppose the condition is satisfy and we denoted $(\bar{u}^\varepsilon)$ the family of solutions of (NS2D). Now we considered the following system

(MHD3D$^\varepsilon$) \[
\left\{ \begin{array}{l}
\partial_t w - \varepsilon \Delta w + \frac{1}{\varepsilon} w \times e_3 + \sqrt{\varepsilon} \partial_3 B - B \nabla B = -\nabla p_L \\
\partial_t B - \sqrt{\varepsilon} \Delta B + \sqrt{\varepsilon} \partial_3 w + (\bar{u}^\varepsilon + w) \nabla B = 0 \\
\text{in } [0,T_0] \times \mathbb{R}^3 \\
(w,B)(0) = (w_0,b_0) \\
(\text{div } w, \text{div } B) = (0,0).
\end{array} \right.
\]

Theorem 1.7. Suppose that the condition (2D × 3D) is satisfied. There exists $0 < T_1 \leq T_0$ such that for all $\varepsilon > 0$, there exists a unique solution $(w^\varepsilon,B^\varepsilon) \in L^\infty_T(H^s(\mathbb{R}^3)) \cap L^2_T(H^{s+1}(\mathbb{R}^2))$ of system (MHD3D$^\varepsilon$) satisfying; for all $t \in [0,T_1]$ 
\[ \|w^\varepsilon(t)\|_{H^s}^2 + \|B^\varepsilon(t)\|_{H^s}^2 + 2\varepsilon \int_0^t \|\nabla w^\varepsilon\|^2_{H^s} + 2\varepsilon \int_0^t \|\nabla B^\varepsilon\|^2_{H^s} \leq 2\|(w_0,b_0)\|^2_{H^s}. \]

Moreover 
\[ w^\varepsilon \to 0 \text{ in } L^4([0,T_1],C^{s-rac{3}{2}}(\mathbb{R}^3)) \text{ ; all } s'>s. \]

Theorem 1.8. Suppose that the condition (2D × 3D) is satisfy.

Then, for all $\varepsilon > 0$, there exists a unique solution $(u^\varepsilon,b^\varepsilon)$ of the system (MHD$^\varepsilon$) such that 
\[ u^\varepsilon = \bar{u}^\varepsilon, \quad b^\varepsilon \in L^\infty_T(H^s(\mathbb{R}^3)) \cap L^2_T(H^{s+1}(\mathbb{R}^3)). \]

Moreover, we have for all $t \in [0,T_1]$, 
\[ \|u^\varepsilon(t) - \bar{u}^\varepsilon(t)\|^2_{H^s} + \|b^\varepsilon(t)\|^2_{H^s} + 2\varepsilon \int_0^t \|\nabla (u^\varepsilon - \bar{u}^\varepsilon)(\tau)\|^2_{H^s} + 2\varepsilon \int_0^t \|\nabla b^\varepsilon\|^2_{H^s} d\tau \leq 2\|(w_0,b_0)\|^2_{H^s}. \]

Now we are ready to state the main convergence result in the case of the whole space $\mathbb{R}^3$.

Theorem 1.9. We keep the same hypothesis as in Theorem 1.8 above and we suppose $s > \frac{3}{2} + 4$ an integer. Then, for all $s'>s$ 
\[ u^\varepsilon - \bar{u}^\varepsilon - w^\varepsilon = o(1) \text{ in } L^\infty_T(H^{s'}(\mathbb{R}^3)) \]
\[ b^\varepsilon - B^\varepsilon = o(1) \text{ in } L^\infty_T(H^{s'}(\mathbb{R}^3)). \]

The structure of this paper is as follows. In the next section, we present the proofs of the existence theorems (Theorem 1.1, 1.2). The third section is devoted to the proof of the convergence result in the case $\Omega = \mathbb{T}^3$ (Theorem 1.4) and the study of the system (LS) (Theorem 1.3). In the final section, we consider the case of the whole space $\mathbb{R}^3$; We give the proof of the Theorems 1.5, 1.7, 1.8, 1.9.
2. Existence results

2.1. Proofs of Theorems 1.1 and 1.2

In this section we shall prove Theorems 1.1 and 1.2. We begin by observing that using the energy methods, one can prove global existence of so-called “Leray’s solutions” for the system (MHD). The crucial fact is the following $L^2$-energy estimate

$$\|U^\varepsilon(t)\|_{L^2}^2 + 2\varepsilon \int_0^t \|\nabla u^\varepsilon(\tau)\|_{L^2}^2 d\tau + 2\sqrt{\varepsilon} \int_0^t \|\nabla b^\varepsilon(\tau)\|_{L^2}^2 d\tau \leq \|U_0\|_{L^2}^2. \quad (2.3)$$

We now turn to the case of strong solutions. Let us introduce, for a strictly positive integer $n$, the Friedrich’s operator $J_n$ defined by:

$$J_n u = \mathcal{F}^{-1}\left(1_{B(0,n)}\mathcal{F}(u)\right).$$

After this definition, we consider the following approximate Magneto-Hydro-Dynamic system (MHD$_n$)

$$\begin{cases}
\partial_t u_n - \varepsilon \Delta J_n u_n + J_n \text{div} (J_n u_n \otimes J_n u_n) - J_n \text{div} (J_n b_n \otimes J_n b_n) + \sqrt{\varepsilon} \partial_3 (J_n b_n) + \frac{J_n u_n \times e}{\varepsilon} = \nabla \Delta^{-1} \text{div} \left( \varepsilon \nabla J_n u_n \otimes J_n u_n \right) - J_n \text{div} (J_n b_n \otimes J_n b_n) + \frac{J_n u_n \times e}{\varepsilon}, \\
\partial_t b_n - \sqrt{\varepsilon} \Delta J_n b_n + J_n \text{div} (J_n u_n \otimes J_n b_n) - J_n \text{div} (J_n b_n \otimes J_n u_n) + \sqrt{\varepsilon} \partial_3 (J_n u_n) = 0, \\
(u_n|_{t=0}, b_n|_{t=0}) = (J_n u_0, J_n b_0).
\end{cases} \quad (MHD_n)$$

By the theory of ordinary differential equations in $H^s$ we know that the system (MHD$_n$) has a unique maximal solution $U_n := (u_n, b_n)$ in the space $C^1([0, T^*_n(\varepsilon)], H^s)$. Using uniqueness and the fact that $\text{div} u_n = \text{div} b_n = 0$ and $J_n^2 = J_n$, we can re-write the system

$$\begin{cases}
\partial_t u_n - \varepsilon \Delta u_n + J_n (u_n \nabla u_n) - J_n (b_n \nabla b_n) + \sqrt{\varepsilon} \partial_3 b_n + \frac{u_n \times e}{\varepsilon} = \\
\nabla \Delta^{-1} \text{div} \left( \varepsilon \nabla u_n \otimes u_n \right) - J_n \text{div} (b_n \otimes b_n) + \frac{u_n \times e}{\varepsilon}, \\
\partial_t b_n - \sqrt{\varepsilon} \Delta b_n + J_n (u_n \nabla b_n) - J_n (b_n \nabla u_n) + \sqrt{\varepsilon} \partial_3 u_n = 0, \\
(u_n|_{t=0}, b_n|_{t=0}) = (J_n u_0, J_n b_0).
\end{cases} \quad (MHD_n)$$

To continue the proof, we recall without proof the following product law.

**Lemma 2.1.** Let $\sigma > \frac{3}{2} + 2$ be an integer and $a$, $b$ and $c$ three vectors field in $H^\sigma(\Omega)$ such that $\text{div} a = 0$. Then, a constant $C$ exists such that

$$\|a \cdot b\|_{H^\sigma} \leq C \|\nabla a\|_{H^{\sigma-1}} \|\nabla b\|_{H^{\sigma-1}}, \quad \|a \cdot c\|_{H^\sigma} \leq C \|\nabla a\|_{H^{\sigma-1}} \|\nabla b\|_{H^{\sigma-1}} \|\nabla c\|_{H^{\sigma-1}}.$$

(See the Appendix for the proof of the lemma.)

We take the scalar product in $H^s$ and we use the lemma above, we obtain for all $t \in [0, T^*_n(\varepsilon)]$,

$$\frac{1}{2} \frac{d}{dt} \|U_n(t)\|_{H^s}^2 + \varepsilon \|\nabla u_n(t)\|_{H^s}^2 + \sqrt{\varepsilon} \|\nabla b_n(t)\|_{H^s}^2 \leq C \|\nabla U_n(t)\|_{H^{s-1}}^2. \quad (2.4)$$

Then

$$\|U_n(t)\|_{H^s}^2 + 2\varepsilon \int_0^t \|\nabla u_n(\tau)\|_{H^s}^2 d\tau + 2\sqrt{\varepsilon} \int_0^t \|\nabla b_n(\tau)\|_{H^s}^2 d\tau \leq \|U_0\|_{H^s}^2 + C \int_0^t \|\nabla U_n(\tau)\|_{H^{s-1}}^2 d\tau. \quad (2.5)$$

We set $T(n, \varepsilon) := \text{Sup} \{0 \leq t < T^*_n(\varepsilon); \forall \tau \in [0, t], \|U_n(\tau)\|_{H^s} \leq 2 \|U_0\|_{H^s}\}$. Using (2.5) and Gronwall lemma we obtain, for all $t \in [0, T(n, \varepsilon)]$,

$$\|U_n(t)\|_{H^s}^2 \leq \|U_0\|_{H^s}^2 \exp(2Ct\|U_0\|_{H^s}).$$
Thus,

\[ T(n, \varepsilon) > T : = \frac{1}{C'\|U_0\|_{H^s}} > 0. \]

Moreover, for all \( t \in [0, T] \),

\[
\|U_n(t)\|_{H^s} + 2\varepsilon \int_0^t \|\nabla u_n(\tau)\|_{H^s}^2 d\tau + 2\sqrt{\varepsilon} \int_0^t \|\nabla b_n(\tau)\|_{H^s}^2 d\tau \leq 2\|U_0\|_{H^s}^2.
\]

Now, the problem is to pass to the limit. Using Ascoli’s theorem, the Cantor’s diagonal process as in Navier-Stokes equations (see [14]) and the estimate (2.6), we obtain a solution satisfying, for all \( t \in [0, T] \),

\[
\|U^n(t)\|_{H^s} + 2\varepsilon \int_0^t \|\nabla u^n(\tau)\|_{H^s}^2 d\tau + 2\sqrt{\varepsilon} \int_0^t \|\nabla b^n(\tau)\|_{H^s}^2 d\tau \leq 2\|U_0\|_{H^s}^2.
\]

This regularity implies in a standard way the uniqueness. It remains to prove the global existence when the initial data is small enough. We assume now that \( \|U_0\|_{H^s} \leq c\varepsilon \), \((c = \frac{1}{2})\), and we set

\[ T_n(\varepsilon) := \text{Sup}\{0 \leq t < T^*_n(\varepsilon); \forall \tau \in [0, t], \|U_n(\tau)\|_{H^s} \leq c\varepsilon\}. \]

Using (2.5), it suffices to show that \( T_n(\varepsilon) = T^*_n(\varepsilon) \). By (2.4) we have \( \frac{d}{dt}\|U_n\|_{H^s}(0) < 0 \), then there exists \( t_n > 0 \) such that \( \|U_n(t_n)\|_{H^s} < c\varepsilon \). Since the quantity \( \|U_n(t)\|_{H^s} \) is decreasing on \([t_n, T^*_n(\varepsilon)]\), then \( T_n(\varepsilon) = T^*_n(\varepsilon) \). This achieves the proof.

3. The case \( \Omega = T^3 \)

Let \((U^\varepsilon)\) be a family of strong solutions of the system \((S^\varepsilon)\) with initial data \(U_0\). To take the limit when \( \varepsilon \to 0 \), the classical proofs no longer work because \( (\partial_t u^\varepsilon) \) is not uniformly bounded. An idea (as in [15] for instance) is to “filter” the system by the group \( L(t) \) associated to \( L \).

In what follows, we recall some properties of the Coriolis force \( L(u) \). We consider, as in [9], the ”wave equation”

\[
\begin{aligned}
\frac{\partial}{\partial t} u + L(u) &= 0 \quad \text{in} \quad \mathbb{R}_t \times T^3, \\
u(0) &= u_0 \quad \text{with} \quad \text{div} u_0 = 0.
\end{aligned}
\]

**Lemma 3.1.** The above system has a global solution denoted by \( u(t) = L(t)u_0 \), such that for all \( s \in \mathbb{R} \) and for all \( u_0 \in H^s(T^3) \),

\[
\|L(t)u_0\|_{H^s(T^3)} = \|u_0\|_{H^s(T^3)} \quad \text{and} \quad \|L(t)u_0\|_{H^s(T^3)} = \|u_0\|_{H^s(T^3)}.
\]

Moreover, if we denote by \( k = (k_1, k_2, k_3) \) the Fourier coordinates, then \( u \) is explicitly given by

\[
F u(t, k) = \exp(i\omega(k)t)(F u(0, k), \nu_k^+) + \exp(-i\omega(k)t)(F u(0, k), \nu_k^-),
\]

where \( \omega(k) = \frac{k_3}{|k|}, \nu_k^\pm \) are given unit vectors and \((\cdot, \cdot)\) denotes the usual scalar product.

Now we define

\[
v^\varepsilon(t) = L\left(\frac{-t}{\varepsilon}\right)u^\varepsilon(t)
\]

then, \( V^\varepsilon := (v^\varepsilon, b^\varepsilon) \), satisfies the following system

\[
\begin{aligned}
\frac{\partial}{\partial t} v^\varepsilon - \varepsilon \Delta v^\varepsilon + L\left(\frac{-t}{\varepsilon}\right)\mathbb{P}(L\left(\frac{-t}{\varepsilon}\right)v^\varepsilon, \nabla L\left(\frac{-t}{\varepsilon}\right)v^\varepsilon)\mathbb{P}(\text{curl} b^\varepsilon \times b^\varepsilon) + \sqrt{\varepsilon} L\left(\frac{-t}{\varepsilon}\right)\mathbb{P}(\text{curl} b^\varepsilon \times e_3) &= 0 \\
\frac{\partial}{\partial t} b^\varepsilon - \sqrt{\varepsilon} \Delta b^\varepsilon + (L\left(\frac{-t}{\varepsilon}\right)v^\varepsilon). \nabla b^\varepsilon - b^\varepsilon. \nabla (L\left(\frac{-t}{\varepsilon}\right)v^\varepsilon) + \sqrt{\varepsilon} \text{curl}((L\left(\frac{-t}{\varepsilon}\right)v^\varepsilon) \times e_3) &= 0 \\
div v^\varepsilon = div b^\varepsilon &= 0 \\
(v^\varepsilon, b^\varepsilon)(0) &= (u_0, b_0).
\end{aligned}
\]
This system can be re-written in the following way

\[
\begin{align*}
(\tilde{\mathcal{S}}) \quad \left\{ \begin{array}{l}
\partial_t V^\varepsilon + Q^\varepsilon(V^\varepsilon, V^\varepsilon) + a_2^\varepsilon(D)V^\varepsilon + \tilde{L}^\varepsilon(V^\varepsilon) = 0 & \text{in } \mathbb{R}_t^+ \times T_x^3 \\
\text{div } V^\varepsilon = 0 & \text{in } \mathbb{R}_t^+ \times T_x^3 \\
V^\varepsilon(0) = U_0 = (u_0, b_0),
\end{array} \right.
\end{align*}
\]

where the “filtered” quadratic form \(Q^\varepsilon\) is given by

\[
Q^\varepsilon(V, V) = \left( \mathcal{L} \left( \frac{-t}{\varepsilon} \right) \mathcal{P}(\mathcal{L} \left( \frac{t}{\varepsilon} \right) v, \nabla \mathcal{L} \left( \frac{t}{\varepsilon} \right) v \right) - \mathcal{L} \left( \frac{-t}{\varepsilon} \right) \mathcal{P}(b, \nabla b), (\mathcal{L} \left( \frac{t}{\varepsilon} \right) v, \nabla b - b, \nabla \mathcal{L} \left( \frac{t}{\varepsilon} \right) v) \right),
\]

and,

\[
a_2^\varepsilon(D)V = \left( -\varepsilon \Delta v, -\sqrt{\varepsilon} \Delta b \right),
\]

\[
\tilde{L}^\varepsilon(V) = \sqrt{\varepsilon} \left( \mathcal{L} \left( \frac{-t}{\varepsilon} \right) \partial b, \partial L \left( \frac{t}{\varepsilon} \right) v \right).
\]

When \(\varepsilon\) goes to 0, we obtain formally the following limit system

\[
(\text{LS}) \quad \left\{ \begin{array}{l}
\partial_t V + Q^0(V, V) = 0 & \text{in } \mathbb{R}_t^+ \times T_x^3 \\
\text{div } v = \text{div } b = 0 & \text{in } \mathbb{R}_t^+ \times T_x^3 \\
V(0) = U_0 = (u_0, b_0),
\end{array} \right.
\]

where \(Q^0(V, V)\) is the limit in \(\mathcal{D}'\) of \(Q^\varepsilon(V, V)\).

### 3.1. Proof of Theorem 1.3

The proof is similar to the one of Theorem 1.2. We have just to estimate the term

\[
\left| \int_0^t < q^0(v_n, v_n), v_n >_{H^s} \right|
\]

where \((v_n, b_n)\) is the solution of the approximate limit system. Observe that

\[
\lim_{\varepsilon \to 0} \int_0^t < q^\varepsilon(v_n, v_n), v_n >_{H^s} = \int_0^t < q^0(v_n, v_n), v_n >_{H^s}
\]

and using the product law given by Lemma 2.1 we obtain

\[
\left| \int_0^t < q^0(v_n, v_n), v_n >_{H^s} \right| \leq C \int_0^t \| \nabla v_n(\tau) \|_{H^{s-1}}^3 d\tau
\]

which completes the proof.

### 3.2. Proof of Theorem 1.4

The proof of this theorem is based on a method used in [7], [9] for instance.

Let \(W^\varepsilon = V^\varepsilon - V = (v^\varepsilon - v, b^\varepsilon - b) = (W_1^\varepsilon, W_2^\varepsilon)\), then \(W^\varepsilon\) satisfies

\[
\left\{ \begin{array}{l}
\partial_t W^\varepsilon + Q^\varepsilon(W^\varepsilon, W^\varepsilon + 2V) + a_2^\varepsilon(D)W^\varepsilon + \tilde{L}^\varepsilon(W^\varepsilon) = F^\varepsilon + R_{\text{osc}}^\varepsilon & \text{in } \mathbb{R}_t^+ \times T_x^3 \\
\text{div } W_1^\varepsilon = \text{div } W_2^\varepsilon = 0 & \text{in } \mathbb{R}_t^+ \times T_x^3 \\
W^\varepsilon(0) = (0, 0),
\end{array} \right.
\]

where

\[
F^\varepsilon = -a_2^\varepsilon(D)V - \tilde{L}^\varepsilon V
\]

\[
= (\varepsilon \Delta v - \sqrt{\varepsilon} \mathcal{L}(-\frac{t}{\varepsilon})b; \sqrt{\varepsilon} \Delta b - \sqrt{\varepsilon} \mathcal{L}(\frac{t}{\varepsilon})v)
\]

and \(R_{\text{osc}}^\varepsilon = Q^0(V, V) - Q^\varepsilon(V, V)\).

We recall that \(v\) and \(b\) are in the space \(C_T^0(H^s)\), then

\[
F^\varepsilon \longrightarrow 0 \quad \text{in } C_T^0(H^{s-2})
\]
We begin by studying the term \( A_0^\epsilon \).

\[
A_0^\epsilon = \mathcal{L}(-\frac{t}{\epsilon})\mathcal{P}(\mathcal{L}(\frac{t}{\epsilon})v,\nabla \mathcal{L}(\frac{t}{\epsilon})v) - q^0(v,v)
\]

where

\[
q^0(v,v) = \lim_{\epsilon \to 0} \mathcal{L}(-\frac{t}{\epsilon})\mathcal{P}(\mathcal{L}(\frac{t}{\epsilon})v,\nabla \mathcal{L}(\frac{t}{\epsilon})v) \quad \text{in} \quad \mathcal{D}'.
\]

In the sequel, for any three-vector field \( X \), we shall note,

\[
X^\pm(n) = (X,\nu^\pm(n))\nu^\pm(n).
\]

We can write

\[
R_{osc}^\epsilon = \sum_{k=0}^3 A_k^\epsilon.
\]

By the non-stationary phase's theorem we have

\[
\mathcal{F}(A_0^\epsilon)(t,n) = \sum_{\sigma \in \{\pm\}^3} \sum_{k+m+n=0} e^{-i\frac{\epsilon}{\omega_\sigma} (n,k,m)}((\mathcal{F}v)^\sigma_1(t,k), (\mathcal{F}\nabla v)^\sigma_2(t,m)^\sigma_3(n),
\]

where \( \sigma = (\sigma_1, \sigma_2, \sigma_3) \) and \( \omega_\sigma(n,k,m) = \frac{\sigma_1 n_3}{|n|} - \frac{\sigma_2 k_3}{|k|} - \frac{\sigma_3 m_3}{|m|}. \)

Hence we can write

\[
A_0^\epsilon = \mathcal{F}^{-1}\left( \sum_{\sigma \in \{\pm\}^3} \sum_{k+m+n=0, \omega_\sigma(n,k,m) \neq 0} e^{-i\frac{\epsilon}{\omega_\sigma} (n,k,m)}(e_{\sigma}(n,k,m)f_{\sigma}(t,k)g_{\sigma}(t,m)) \right).
\]

Similarly, we have

\[
A_1^\epsilon = \mathcal{F}^{-1}\left( 1_{\{n_3 \neq 0\}}(n) e^{-i\frac{\epsilon}{\omega_\sigma} (n,k,m)}(\mathcal{F}(b,\nabla b))^+(t,n) + 1_{\{n_3 \neq 0\}}(n) e^{i\frac{\epsilon}{\omega_\sigma} (n,k,m)}(\mathcal{F}(b,\nabla b))^-(t,n) \right),
\]

\[
A_2^\epsilon = \mathcal{F}^{-1}\left( \sum_{k_3 \neq 0} e^{i\frac{\epsilon}{|k_3|} (\mathcal{F}v)^+(t,k),(\mathcal{F}\nabla v)(t,n-k)} + \sum_{k_3 \neq 0} e^{-i\frac{\epsilon}{|k_3|} (\mathcal{F}v)^-(t,k),(\mathcal{F}\nabla v)(t,n-k)} \right)
\]

and finally,

\[
A_3^\epsilon = \mathcal{F}^{-1}\left( \sum_{k_3 \neq 0} e^{i\frac{\epsilon}{|k_3|} (\mathcal{F}b)(t,n-k),(\mathcal{F}v)^+(t,k)} + \sum_{k_3 \neq 0} e^{-i\frac{\epsilon}{|k_3|} (\mathcal{F}b)(t,n-k),(\mathcal{F}v)^-(t,k)} \right).
\]

Since \( R_{osc}^\epsilon \) converges weakly to 0 (but not “strongly”), we shall divide it (as in [7] for instance) into high and low frequencies terms. Precisely, for any integer \( N > 1 \), we define

\[
A_{0,N}^\epsilon = \mathcal{F}^{-1}\left( 1_{\{|n| \leq N\}}\sum_{\sigma \in \{\pm\}^3} \sum_{k+m+n=0, \omega_\sigma(n,k,m) \neq 0, |k| |m| \leq N} e^{-i\frac{\epsilon}{\omega_\sigma} (n,k,m)}(e_{\sigma}(n,k,m)f_{\sigma}(t,k)g_{\sigma}(t,m)) \right).
\]
\[ A^\varepsilon_{1,N} = \mathcal{F}^{-1}\left(1_{\{n_3 \neq 0,|n| \leq N\}}(n) e^{-\frac{i t \varepsilon b_n}{|n|}}(\mathcal{F}(b, \nabla b))^{+}(t,n) + 1_{\{n_3 \neq 0,|n| \leq N\}}(n) e^{\frac{i t \varepsilon b_n}{|n|}}(\mathcal{F}(b, \nabla b))^{-}(t,n)\right), \]
\[ A^\varepsilon_{2,N} = \mathcal{F}^{-1}\left(1_{\{|n| \leq N\}} \sum_{k_3 \neq 0} e^{\frac{i \varepsilon b_n}{|n|}}(\mathcal{F}v)^{+}(t,k), (\mathcal{F} \nabla b)(t,n-k) + e^{-\frac{i \varepsilon b_n}{|n|}}(\mathcal{F}v)^{-}(t,k), (\mathcal{F} \nabla b)(t,n-k)\right) \]
and
\[ A^\varepsilon_{3,N} = \mathcal{F}^{-1}\left(1_{\{|n| \leq N\}} \sum_{k_3 \neq 0} e^{\frac{i \varepsilon b_n}{|n|}}(\mathcal{F}b)(t,n-k), (\mathcal{F} \nabla v)^{+}(t,k) + e^{-\frac{i \varepsilon b_n}{|n|}}(\mathcal{F}b)(t,n-k), (\mathcal{F} \nabla v)^{-}(t,k)\right). \]

Now, the idea is to absorb the low frequency terms. For that, we set
\[ \tilde{A}^\varepsilon_{0,N} = \mathcal{F}^{-1}\left(1_{\{|n| \leq N\}} \sum_{\sigma \in \{\pm\}^3} \sum_{k_3 \neq 0} e^{-\frac{i \varepsilon \omega_\sigma(n,k,m)}{|n|}}(\mathcal{F} \nabla \sigma(n,k,m) f_\sigma(t,k) g_\sigma(t,m))\right), \]
\[ \tilde{A}^\varepsilon_{1,N} = \mathcal{F}^{-1}\left(1_{\{n_3 \neq 0,|n| \leq N\}}(n) e^{-\frac{i t \varepsilon b_n}{|n|}}(\mathcal{F}(b, \nabla b))^{+}(t,n) + 1_{\{n_3 \neq 0,|n| \leq N\}}(n) e^{\frac{i t \varepsilon b_n}{|n|}}(\mathcal{F}(b, \nabla b))^{-}(t,n)\right), \]
\[ \tilde{A}^\varepsilon_{2,N} = \mathcal{F}^{-1}\left(1_{\{|n| \leq N\}} \sum_{k_3 \neq 0} e^{\frac{i \varepsilon b_n}{|n|}}(\mathcal{F}v)^{+}(t,k), (\mathcal{F} \nabla b)(t,n-k) + e^{-\frac{i \varepsilon b_n}{|n|}}(\mathcal{F}v)^{-}(t,k), (\mathcal{F} \nabla b)(t,n-k)\right) \]
and
\[ \tilde{A}^\varepsilon_{3,N} = \mathcal{F}^{-1}\left(1_{\{|n| \leq N\}} \sum_{k_3 \neq 0} e^{\frac{i \varepsilon b_n}{|n|}}(\mathcal{F}b)(t,n-k), (\mathcal{F} \nabla v)^{+}(t,k) + e^{-\frac{i \varepsilon b_n}{|n|}}(\mathcal{F}b)(t,n-k), (\mathcal{F} \nabla v)^{-}(t,k)\right) \]
and we define
\[ R^\varepsilon_{osc,N} = \sum_{k=0}^{3} \tilde{A}^\varepsilon_{k,N}, \]
\[ R^\varepsilon_{osc} = R^\varepsilon_{osc} - R^\varepsilon_{osc,N}, \quad \tilde{R}^\varepsilon_{osc,N} = \sum_{k=0}^{3} \tilde{A}^\varepsilon_{k,N}. \]

Considering \( \varphi^\varepsilon_N = W^\varepsilon + \varepsilon \tilde{R}^\varepsilon_{osc,N} = (\varphi^\varepsilon_{N,1}, \varphi^\varepsilon_{N,2}) \), then \( \varphi^\varepsilon_{N} \) satisfies the following equation
\[ \partial_t \varphi^\varepsilon_{N} + Q^\varepsilon(\varphi^\varepsilon_{N}, \varphi^\varepsilon_{N}) + 2 \varepsilon \tilde{R}^\varepsilon_{osc,N} + 2V + a^2_\varepsilon(D) \varphi^\varepsilon_{N} + \tilde{L}^\varepsilon(\varphi^\varepsilon_{N}) = F^\varepsilon + R^\varepsilon_{osc} + \varepsilon v^\varepsilon_{osc,N}, \]
where
\[ \varepsilon v^\varepsilon_{osc,N} = \varepsilon \left(Q^\varepsilon(\tilde{R}^\varepsilon_{osc,N}, \varepsilon \tilde{R}^\varepsilon_{osc,N} + 2V) + a^2_\varepsilon(D) \tilde{R}^\varepsilon_{osc,N} + \tilde{L}^\varepsilon(\tilde{R}^\varepsilon_{osc,N})\right) + \left(R^\varepsilon_{osc,N} + \varepsilon \partial_t \tilde{R}^\varepsilon_{osc,N}\right). \]

Now, we have to studying the low frequencies terms. This study is easy. In fact, we have the following result.

**Lemma 3.2.** A constant \( C_N(T) \) exists, depending only \( T \) and \( N \) such that
\[ \| \tilde{R}^\varepsilon_{osc,N} \|_{L^2(T)} \leq C_N(T), \]
\[ \| v^\varepsilon_{osc,N} \|_{L^2(T)} \leq C_N(T). \]

**Proof.** Let us recall that all the functions considered here are truncated in low frequencies. Hence the result is simply due to the fact that \( v, b \in C^0_T(H^s) \), \( \partial_t v, \partial_t b \in C^0_T(H^{s-2}) \) and the following product law.
Proposition 3.1. Let $s$ be an integer, $\sigma > \frac{3}{2}$. A constant $C$ exists such that for all $f \in H^\sigma$ and $g \in H^{\sigma+1}$ with $\text{div}(f) = \text{div}(g) = 0$, we have

$$ |< Q^\varepsilon(f, g), f >_{H^\sigma} | \leq C \| f \|_{H^\sigma}^2 \| g \|_{H^{\sigma+1}} $$

and

$$ |< Q^\varepsilon(f, f), f >_{H^\sigma} | \leq C \| f \|_{H^\sigma}^3. $$

(The proof of the proposition used the fact $L(t)$ is a isometri in the Sobolev space and lemma 2.1.)

Lemma 3.3. For any function $f \in C^0_T(H^s)$ with $s \in \mathbb{R}$, the high frequency term

$$ f^N = \mathcal{F}^{-1} \left( 1_{[N, +\infty]} \mathcal{F}(f) \right) $$

goes to zero when $N$ goes to infinity in $C^0_T(H^s)$.

Proof. Let us recall that

$$ \| f^N(t) \|_{H^s}^2 = \sum_{|k| \geq N} |k|^{2s} |\mathcal{F}(f)(t, k)|^2. $$

So, the desired result is simply due to Dini's theorem which implies that $\| f^N(t) \|_{H^s}^2$ goes to zero uniformly in $t$.

This lemma implies in a straightforward way the following result.

Lemma 3.4. The high frequency term $R_{\text{osc}}^\varepsilon,N$ goes to zero in $C^0_T(H^{s-2}) \cap L^2_T(H^{s-1})$ when $N$ goes to infinity, uniformly in $\varepsilon$; precisely $\| R_{\text{osc}}^\varepsilon,N \|_{C^0_T(H^{s-2}) \cap L^2_T(H^{s-1})} \leq \eta_N$ with $\eta_N \to 0$.

Now, we can end the proof of the theorem. By the energy estimate in $H^{s-2}(\mathbb{T}^3)$ we obtain

$$ \frac{1}{2} \frac{d}{dt} \| \varphi^\varepsilon_N \|_{H^{s-2}}^2 + \varepsilon \| \nabla \varphi^\varepsilon_{N,1} \|_{H^{s-2}}^2 + \sqrt{v} \| \nabla \varphi^\varepsilon_{N,2} \|_{H^{s-2}}^2 \leq |< f^\varepsilon + R_{\text{osc}}^\varepsilon,N + \varepsilon r_{\text{osc}}^\varepsilon,N, \varphi^\varepsilon_N >_{H^{s-2}} | $$

$$ + 2 |< Q^\varepsilon(\varphi^\varepsilon_N, \varepsilon R_{\text{osc}}^\varepsilon,N + V), \varphi^\varepsilon_N >_{H^{s-2}} | $$

$$ + |< Q^\varepsilon(\varphi^\varepsilon_N, \varepsilon \varphi^\varepsilon_N), \varphi^\varepsilon_N >_{H^{s-2}} | $$

Which leads, using the product law (Proposition 3.1), to

$$ \frac{1}{2} \frac{d}{dt} \| \varphi^\varepsilon_N \|_{H^{s-2}}^2 \leq C \left[ \| \varphi^\varepsilon_N \|_{H^{s-2}}^2 (\| \varphi^\varepsilon_N \|_{H^{s-2}} + \| R_{\text{osc}}^\varepsilon,N + V \|_{H^{s-1}} + 1) \right] $$

Integrating this inequality and using Lemma 3.2, we obtain

$$ \| \varphi^\varepsilon_N(t) \|_{H^{s-2}}^2 \leq \varepsilon C_N(T)^2 + 4 \varepsilon^2 \| u \|_{L^2_T(H^{s-2})}^2 + 4 \| R_{\text{osc}}^\varepsilon,N \|_{L^2_T(H^{s-2})}^2 $$

$$ + 4 \varepsilon^2 C_N(T)^2 + C \int_0^t \| \varphi^\varepsilon_N(\tau) \|_{H^{s-2}}^2 \left( B(T) + \varepsilon C_N(T) + \| \varphi^\varepsilon_N(\tau) \|_{H^{s-2}} \right) d\tau, $$

where $B(T) = 1 + \| V \|_{L^2_T(H^{s-2})}$.

We set

$$ T^* = \sup \left\{ 0 \leq t < T \ / \ \| \varphi^\varepsilon_N \|_{L^\infty_T(H^{s-2})} \leq B(T) \right\}. $$

Then, for all $0 \leq t < T^*$, we can write

$$ \| \varphi^\varepsilon_N(t) \|_{H^{s-2}}^2 \leq \varepsilon C_N(T)^2 + 4 \varepsilon^2 \| u \|_{L^2_T(H^s)}^2 + 4 \eta_N^2 + 4 \varepsilon^2 C_N(T)^2 $$

$$ + \left( c B(T) + \varepsilon C_N(T) \right) \int_0^t \| \varphi^\varepsilon_N(\tau) \|_{H^{s-2}}^2 d\tau. $$
A classical Gronwall estimate gives
\[ \| \varphi^\varepsilon_N(t) \|_{H^{s-2}}^2 \leq \left( \varepsilon C_N(T)^2 + 4 \varepsilon^2 \| u \|_{L^2_t(H^s)}^2 + 4 \eta_N^2 + 4 \varepsilon^2 C_N(T)^2 \right) \exp \left( cTB(T) + cT \varepsilon C_N(T) \right). \]

For \( N \) large and \( \varepsilon \) small enough we obtain, thanks to Lemma 3.3
\[ \| \varphi^\varepsilon_N(t) \|_{H^{s-2}} \leq \frac{B(T)}{2}, \]
which easily implies that \( T^* = T \).

Using Lemma 3.2 and letting \( \varepsilon \to 0, N \to +\infty \), we obtain
\[ W^\varepsilon \to 0 \quad \text{in} \quad C_0^1(H^{s-2}). \]
An interpolation argument concludes the proof of Theorem 1.4.

4. THE CASE \( \Omega = \mathbb{R}^3 \)

This section is devoted to study dispersion phenomena in the \((\text{MHD}^\varepsilon)\) system in the case of the space \( \mathbb{R}^3 \). Let us introduce the “linearized” equation in \( u^\varepsilon \) of the first equation of \((\text{MHD}^\varepsilon)\).

\[
\begin{cases}
\partial_t u^\varepsilon + \frac{1}{\varepsilon} Lu^\varepsilon = -\nabla p & \text{in} \quad \mathbb{R}_t \times \mathbb{R}_x^3 \\
\text{div} u^\varepsilon = 0 \\
u^\varepsilon(0) = u_0.
\end{cases}
\]

In Fourier variables \( \xi \in \mathbb{R}^3 \), we obtain
\[ \partial_t \mathcal{F}(u^\varepsilon) + \frac{\xi_3}{\varepsilon |\xi|^2} \xi \times \mathcal{F}(u^\varepsilon) = 0. \]

Hence, we are led to study the following family of operators
\[
\mathcal{G}^\varepsilon : f \mapsto \int_{\mathbb{R}_t^3} \mathcal{F}(f)(\xi) \exp \left( \mp i t \frac{\xi_3}{\varepsilon |\xi|} + ix.\xi \right) d\xi
\]
\[ = \int_{\mathbb{R}_y^3 \times \mathbb{R}_t^3} f(y) \exp \left( \mp i t \frac{\xi_3}{\varepsilon |\xi|} + i(x-y).\xi \right) dy d\xi. \]

We notice that the phase function \( \frac{\xi_3}{\varepsilon |\xi|} \) is almost stationary when \( \xi_3 \) is almost equal to 0 as well as when \( |\xi_3| \) is much larger then \( |\xi_h| \). So, for some \( 0 < r < R \), let us define the domain \( \mathcal{C}_{r,R} \) by
\[ \mathcal{C}_{r,R} = \{ \xi \in \mathbb{R}^3 ; \ |\xi_3| > r \quad \text{and} \quad |\xi| \leq R \}. \]
We consider \( \psi \) a cut-off function, which is radial with respect to horizontal variable \( \xi_h = (\xi_1, \xi_2) \) and whose value is 1 near \( \mathcal{C}_{r,R} \).

First, we study the case when \( \mathcal{F}(f) \) is supported in \( \mathcal{C}_{r,R} \). We can write
\[ \mathcal{G}^\varepsilon f(t,x) = \left( K(\frac{t}{\varepsilon},.) * f \right)(x), \]
where the kernel \( K \) is defined by
\[ K(t,z) = \int_{\mathbb{R}^3} \psi(\xi) e^{it\frac{\xi_3}{\varepsilon |\xi|} + iz.\xi} d\xi. \]
As in [5], we recall the following property of \( K \).
Lemma 4.1. For all \( r, R \) such that \( 0 < r < R \), there exists a constant \( C_{r,R} \) such that
\[
\| K(t, \cdot) \|_{L^\infty(\mathbb{R}^3)} \leq C_{r,R} \min\{1, t^{-\frac{1}{2}}\}.
\]

Let us denote by \( \varepsilon \) the solution of
\[
(PLF_{\varepsilon}) \quad \left\{ \begin{array}{l}
\partial_t w^\varepsilon + \frac{1}{\varepsilon} L w^\varepsilon = f \quad \text{in } \mathbb{R}_t \times \mathbb{R}^3_x \\
w^\varepsilon(0) = w_0.
\end{array} \right.
\]

Lemma 4.1 yields, in a standard way, the following Strichartz-estimate (see [5]).

Corollary 4.1. For all constants \( r < R \) such that \( 0 < r < R \), let \( C_{r,R} \) be the domain defined above. Then a constant \( C_{r,R} \) exists such that if
\[
\operatorname{supp} \mathcal{F}(w_0) \cup \operatorname{supp} \mathcal{F}(f) \subset C_{r,R}
\]
then the solution \( w^\varepsilon \) of \( (PLF_{\varepsilon}) \) with the forcing term \( f \) and initial data \( w_0 \) satisfies
\[
\| w^\varepsilon \|_{L^4(\mathbb{R}^3 \times \mathbb{R}^3)} \leq C_{r,R} \varepsilon^2 \left( \| w_0 \|_{L^2} + \| f \|_{L^1(\mathbb{R}^3 \times \mathbb{R}^3)} \right).
\]

We notice that the constant \( C_{r,R} \) does not depend on \( \varepsilon \).

Using the above estimate, we are able to prove the convergence result.

4.1. Proof of Theorem 1.5. We start by proving a convergence result for the Leray’s solutions. We shall apply the method used in [5] for the first equation of our system. Let \( R > 0 \) and \( \chi \) be a cut-off function in \( \mathcal{D}(\mathbb{R}) \) taking the value 1 near the origin. We set \( u^\varepsilon_R := \chi(\frac{|D|}{R}) u^\varepsilon = \mathcal{F}^{-1} \left( \chi(\frac{|\xi|}{R}) \mathcal{F}(u^\varepsilon)(\xi) \right) \). Then \( u^\varepsilon_R \) satisfies
\[
\partial_t u^\varepsilon_R + \frac{1}{\varepsilon} \mathbb{P}(u^\varepsilon_R \times e_3) = f^\varepsilon_R,
\]
with
\[
f^\varepsilon_R = -\chi(\frac{|D|}{R}) \left( \varepsilon \Delta u^\varepsilon + \mathbb{P}(u^\varepsilon \nabla u^\varepsilon) - \mathbb{P}(b^\varepsilon \nabla b^\varepsilon) - \sqrt{\varepsilon} \partial_3 b^\varepsilon \right).
\]

By Duhamel’s formula, we can write
\[
u^\varepsilon_R = e^{t \sigma^\varepsilon(D)} u_{0,R} + \int_0^t e^{(t-t') \sigma^\varepsilon(D)} f^\varepsilon_R(t') dt',
\]
where \( \sigma^\varepsilon(\xi) u = \frac{\xi_3}{\varepsilon R} u \times \xi. \)

So we have
\[
\| u^\varepsilon_R \|_{L^2_t(L^\infty(\mathbb{R}^3))} \leq \| e^{t \sigma^\varepsilon(D)} u_{0,R} \|_{L^2_t(L^\infty(\mathbb{R}^3))} + \| \int_0^t e^{(t-t') \sigma^\varepsilon(D)} f^\varepsilon_R(t') dt' \|_{L^2_t(L^\infty(\mathbb{R}^3))}.
\]

For the first term of the right side of the above inequality, we localize on \( C_{r,R} \). Hence, for all \( 0 < r < R \), we write
\[
\| e^{t \sigma^\varepsilon(D)} u_{0,R} \|_{L^2_t(L^\infty(\mathbb{R}^3))} \leq \| e^{t \sigma^\varepsilon(D)} \chi(\frac{D_3}{r}) u_{0,R} \|_{L^2_t(L^\infty(\mathbb{R}^3))} + \| e^{t \sigma^\varepsilon(D)} (Id - \chi(\frac{D_3}{r}) u_{0,R} \|_{L^2_t(L^\infty(\mathbb{R}^3))},
\]

Using Corollary 4.1 we obtain
\[
(4.9) \quad \| e^{t \sigma^\varepsilon(D)} u_{0,R} \|_{L^2_t(L^\infty(\mathbb{R}^3))} \leq C T^{\frac{1}{4}}(R^2 r)^{\frac{1}{2}} \| u_0 \|_{L^2} + \varepsilon^{\frac{1}{4}} C_{r,R} \| u_0 \|_{L^2}.
\]
For the other term we have
\[
\| \int_0^t e^{(t-t') \sigma^c(D)} f_R^e(t') dt' \|_{L^1_t(L^\infty)} \leq \| \int_0^t e^{(t-t') \sigma^c(D)} \chi \left( \frac{D_3}{r} \right) f_R^e(t') dt' \|_{L^1_t(L^\infty)} + \\| \int_0^t e^{(t-t') \sigma^c(D)} (Id - \chi \left( \frac{D_3}{r} \right)) f_R^e(t') dt' \|_{L^1_t(L^\infty)}.
\]

Corollary 4.1 and Bernstein's lemma imply the following
\[
\| \int_0^t e^{(t-t') \sigma^c(D)} f_R^e(t') dt' \|_{L^1_t(L^\infty)} \leq C T^2 R^2 \| \chi \left( \frac{D_3}{r} \right) \chi \left( \frac{|D|}{R} \right) f^e \|_{L^\infty_t(L^2)}
\]
\[
+ C_{r,R} \varepsilon \| (Id - \chi \left( \frac{D_3}{r} \right)) f^e \|_{L^1_t(L^\infty)} \leq C T^2 R^2 \| \chi \left( \frac{D_3}{r} \right) \chi \left( \frac{|D|}{R} \right) f^e \|_{L^\infty_t(L^2)}
\]
\[
+ C_{r,R} \varepsilon \| \chi \left( \frac{|D|}{R} \right) f^e \|_{L^\infty_t(L^2)}.
\]

Now, we focus our attention to the first right term of the last inequality.
\[
\chi \left( \frac{D_3}{r} \right) \chi \left( \frac{|D|}{R} \right) f^e = \varepsilon \chi \left( \frac{D_3}{r} \right) \chi \left( \frac{|D|}{R} \right) \Delta u^e - \varepsilon \chi \left( \frac{D_3}{r} \right) \chi \left( \frac{|D|}{R} \right) \mathbb{P}(u^e, \nabla u^e)
\]
\[
+ \chi \left( \frac{D_3}{r} \right) \chi \left( \frac{|D|}{R} \right) \mathbb{P}(b^e, \nabla b^e) + \sqrt{\varepsilon} \chi \left( \frac{D_3}{r} \right) \chi \left( \frac{|D|}{R} \right) \partial_3 b^e.
\]
First, notice that we have
\[
\| \chi \left( \frac{D_3}{r} \right) \chi \left( \frac{|D|}{R} \right) \mathbb{P}(u^e, \nabla u^e) \|_{L^\infty_t(L^2)} \leq R \| \chi \left( \frac{D_3}{r} \right) \chi \left( \frac{|D|}{R} \right) (u^e \otimes u^e) \|_{L^\infty_t(L^2)}
\]
\[
\leq R \| \mathcal{F}^{-1} \chi \left( \frac{\xi_3}{r} \right) \chi \left( \frac{|\xi|}{R} \right) \ast (u^e \otimes u^e) \|_{L^\infty_t(L^2)}
\]
\[
\leq R \| \mathcal{F}^{-1} \chi \left( \frac{\xi_3}{r} \right) \chi \left( \frac{|\xi|}{R} \right) \|_{L^\infty_t(L^2)} \| u^e \otimes u^e \|_{L^\infty_t(L^1)}
\]
\[
\leq R (R^2 r)^{\frac{3}{2}} \| U_0 \|_{L^2}^2.
\]

Then, in a same manner as above, we obtain
\[
\| \chi \left( \frac{D_3}{r} \right) \chi \left( \frac{|D|}{R} \right) \mathbb{P}(b^e, \nabla b^e) \|_{L^\infty_t(L^2)} \leq R (R^2 r)^{\frac{3}{2}} \| U_0 \|_{L^2}^2
\]
and finally
\[
\| \chi \left( \frac{D_3}{r} \right) \chi \left( \frac{|D|}{R} \right) \partial_3 b^e \|_{L^\infty_t(L^2)} \leq r \| \chi \left( \frac{D_3}{r} \right) \chi \left( \frac{|D|}{R} \right) b^e \|_{L^\infty_t(L^2)} \leq r \| b^e \|_{L^\infty_t(L^2)} \leq r \| U_0 \|_{L^2}^2.
\]

For the second right term we use
\[
\| \chi \left( \frac{|D|}{R} \right) b^e \|_{L^\infty_t(L^2)} \leq \| \chi \left( \frac{|D|}{R} \right) \mathbb{P}(u^e, \nabla u^e) \|_{L^\infty_t(L^2)} + \| \chi \left( \frac{|D|}{R} \right) \mathbb{P}(b^e, \nabla b^e) \|_{L^\infty_t(L^2)}
\]
\[
+ \sqrt{\varepsilon} \| \chi \left( \frac{|D|}{R} \right) \partial_3 b^e \|_{L^\infty_t(L^2)} \leq CR^2 \| U_0 \|_{L^2}^2 + CR^2 \| U_0 \|_{L^2}^2 + R \| b^e \|_{L^\infty_t(L^2)}
\]
\[
\leq CR^2 \| U_0 \|_{L^2}^2 + R \| U_0 \|_{L^2}.
\]
Finally, we have
\[ \| \int_0^t e^{(t-t')(\sigma^\varepsilon(D) - \sigma^\varepsilon(b))} f_R^\varepsilon(t') dt' \|_{L^2(L^\infty)} \leq C T \frac{\bar{\varepsilon}}{\varepsilon} R^\frac{3}{2} T \left( \| U_0 \|_{L^2}^2 R^2 \bar{\varepsilon} \varepsilon + \| U_0 \|_{L^2} \right) + C_{\bar{r}, R} \frac{\bar{\varepsilon}}{\varepsilon} T \left( \frac{\bar{\varepsilon}}{\varepsilon} R^\frac{3}{2} T \| U_0 \|_{L^2} + R \| U_0 \|_{L^2} \right). \] (4.10)
Inequalities (4.9) and (4.10) imply in a simple manner that
\[ u^\varepsilon_R \rightarrow 0 \quad \text{as} \quad \varepsilon \rightarrow 0 \quad \text{in} \quad L^4_T(L^\infty). \] (4.11)
Using the embedding \( L^\infty \hookrightarrow C^{-\frac{\sigma}{4}} \), we deduce that
\[ u^\varepsilon_R \rightarrow 0 \quad \text{as} \quad \varepsilon \rightarrow 0 \quad \text{in} \quad L^4_T(C^{-\frac{\sigma}{4}}). \]
On the other hand, we have
\[ \| u^\varepsilon - u^\varepsilon_R \|_{L^2_\varepsilon(L^{-\frac{\sigma}{4}})} \leq C \| u^\varepsilon - u^\varepsilon_R \|_{L^2_\varepsilon(H^{-\frac{\sigma}{4}})} \leq C R^{-\frac{\sigma}{4}} \| u^\varepsilon \|_{L^2_\varepsilon(L^2(\mathbb{R}^3))} \]
and then by energy estimate \( 1.2 \) we conclude that
\[ \| u^\varepsilon - u^\varepsilon_R \|_{L^2_\varepsilon(L^{-\frac{\sigma}{4}})} \leq C R^{-\frac{\sigma}{4}} T \frac{\bar{\varepsilon}}{\varepsilon} \| U_0 \|_{L^2(\mathbb{R}^3)}. \] (4.12)
Hence, inequalities (4.11) and (4.12) and an interpolation argument, we can deduce the first part of Theorem 1.5.

Now, we come to the proof of the second part of Theorem 1.5. We recall that we have
\[ \partial_t b^\varepsilon = \varepsilon \Delta b^\varepsilon + b^\varepsilon \nabla u^\varepsilon - u^\varepsilon \nabla b^\varepsilon - \sqrt{\varepsilon} \partial_3 u^\varepsilon. \]
We take the scalar produce in \( H^{s-3} \) with \( b^\varepsilon - b_0 \), we obtain
\[ \| b^\varepsilon(t) - b_0 \|_{H^{s-3}}^2 \leq \int_0^t |< b^\varepsilon(\tau), \nabla u^\varepsilon(\tau), b^\varepsilon(\tau) - b_0 >_{H^{s-3}}| d\tau \]
\[ + \varepsilon \int_0^t |< u^\varepsilon(\tau), \nabla b^\varepsilon(\tau), b^\varepsilon(\tau) - b_0 >_{H^{s-3}}| d\tau \]
\[ + \sqrt{\varepsilon} \int_0^t |\partial_3 b^\varepsilon(\tau) - b^\varepsilon(\tau) - b_0 >_{H^{s-3}}| d\tau, \]
using the energy estimate \( 1.2 \) and the following product law.

**Lemma 4.2.** Let \( \sigma > 4 \) an integer and \( a, b \) and \( c \) three vectors field such that \( a \in H^\sigma(\mathbb{R}^3) \cap C^{\sigma + \frac{1}{2}}(\mathbb{R}^3) \), \( b \in H^{\sigma + 1}(\mathbb{R}^3) \cap C^{\sigma + \frac{3}{2}}(\mathbb{R}^3) \), \( c \in H^\sigma(\mathbb{R}^3) \) and \( \text{div} \ a = 0 \). Then, a constant \( C(\sigma) \) exists such that
\[ |< a, \nabla b, c >_{H^\sigma}| \leq C(\sigma) \| c \|_{H^\sigma} \min \left( \| a \|_{C^{\sigma + \frac{1}{2}}}, \| \nabla b \|_{H^\sigma}, \| a \|_{H^\sigma}, \| b \|_{C^{\sigma + \frac{3}{2}}} \right). \]
We obtain
\[ \| b^\varepsilon(t) - b_0 \|_{H^{s-3}}^2 \leq C \sqrt{T} \| (w(0, b_0))_0 \|_{H^s}^2 \| u^\varepsilon \|_{L^2_\varepsilon(H^{s-\frac{s}{4}})} \]
\[ + C \sqrt{\varepsilon} \| (w(0, b_0))_0 \|_{H^s}^2, \]
the fact
\[ u^\varepsilon \rightarrow 0 \quad \text{in} \quad L^4_T(C^{s-\frac{s}{4}}) \]
and an interpolation argument, we can deduce the desired result. \( \blacksquare \)
4.2. Proof of Theorem 4.7. We used the following lemma

**Lemma 4.3.** For $\bar{a} \in \left( H^{\sigma}(\mathbb{R}^3) \right)^3$ and $b, c \in \left( H^{\sigma}(\mathbb{R}^3) \right)^3$ ($\sigma > \text{eq4 an integer}$) such that $\text{div}_h \bar{a} = 0$, then

$$\left\| < \bar{a} \cdot \nabla b, c >_{H^\sigma} + < \bar{a} \cdot \nabla c, b >_{H^\sigma} \right\| \leq C \| \bar{a} \|_{H^\sigma} \| b \|_{H^\sigma} \| c \|_{H^\sigma}$$

Moreover, if $\bar{a} \in \left( H^{\sigma+1}(\mathbb{R}^3) \right)^3$, then

$$\left\| < b \cdot \nabla \bar{a}, c >_{H^\sigma} \right\| \leq C \| \bar{a} \|_{H^\sigma+1} \| b \|_{H^\sigma} \| c \|_{H^\sigma}$$

and the approximate system

**(MHD3D)\text{D}_n**

$$\partial_t w - \varepsilon \Delta J_n w + \frac{1}{\varepsilon} J_n w \times e_3 + \sqrt{\varepsilon} \partial_3 J_n B - J_n (J_n B \cdot \nabla J_n B)$$

$$= \nabla \Delta^{-1} \text{div} \left( J_n (J_n B \cdot \nabla J_n B) - \frac{1}{\varepsilon} J_n w \times e_3 \right)$$

$$\partial_t B^\varepsilon - \sqrt{\varepsilon} \Delta J_n B + \sqrt{\varepsilon} \partial_3 J_n w + J_n ((J_n \bar{u}^\varepsilon + J_n w) \cdot \nabla J_n B) - J_n (J_n B \cdot \nabla (J_n \bar{u}^\varepsilon + J_n w) ) = 0$$

$$(w, B)(0) = (J_n w_0, J_n b_0)$$

$$(\text{div} w, \text{div} B)(0) = (0, 0).$$

The Lemmas 2.1 and 3.1 implies that (MHD3D\text{D}_n) have a ordinary differential equation form in $H^s(\mathbb{R}^3)$, then there exists $W_n := (w_n, B_n) \in C^1([0, T_n \varepsilon), H^s(\mathbb{R}^3))$ a maximal solution of (MHD3D\text{D}_n). The uniqueness and the fact $\bar{J}_n = J_n$ implies that $J_n W_n = W_n$.

Moreover $T_n \varepsilon > T_1 := \frac{1}{C(s)(\| \bar{u}_0 \|_{H^{\sigma+1}} + \| (w_0, b_0) \|_{H^\sigma})}$ and for all $t \in [0, T_1]$

$$\| W_n(t) \|_{H^\sigma}^2 < 2 \varepsilon \int_0^t \| \nabla w_n \|_{H^\sigma}^2 + 2 \varepsilon \int_0^t \| \nabla B_n \|_{H^\sigma}^2 \leq 2 \| (w_0, b_0) \|_{H^\sigma}^2.$$  

(4.13)

In the end we used the proof of theorem 4.5, we obtain existence of a $W^\varepsilon := (w^\varepsilon, B^\varepsilon) \in L^\infty_T(\mathbb{R}^3) \cap L^2_T(\mathbb{H}^{\sigma+1})$ solution of (MHD3D\text{D}^\varepsilon), satisfy for all $t \in [0, T_1]$

$$\| W^\varepsilon(t) \|_{H^\sigma}^2 < 2 \varepsilon \int_0^t \| \nabla w^\varepsilon \|_{H^\sigma}^2 + 2 \varepsilon \int_0^t \| \nabla B^\varepsilon \|_{H^\sigma}^2 \leq 2 \| (w_0, b_0) \|_{H^\sigma}^2.$$  

(4.14)

The proof of the uniqueness is easy.

4.3. Proof of Theorem 4.8. We posed $v^\varepsilon := u^\varepsilon - \bar{u}^\varepsilon$, then $(v^\varepsilon, b^\varepsilon)$ is solution of the following system

**\text{(MHD}^\varepsilon)\text{F}**

$$\partial_t v^\varepsilon - \varepsilon \Delta v^\varepsilon + v^\varepsilon \cdot \nabla v^\varepsilon + v^\varepsilon \cdot \nabla \bar{u}^\varepsilon + \bar{u}^\varepsilon \cdot \nabla v^\varepsilon$$

$$- \text{curl} b^\varepsilon \times b^\varepsilon + \sqrt{\varepsilon} \text{curl} b^\varepsilon \times e_3 + \frac{v^\varepsilon \times e_3}{\varepsilon} = - \nabla p^\varepsilon$$

$$\partial_t b^\varepsilon - \sqrt{\varepsilon} \Delta b^\varepsilon + \bar{u}^\varepsilon \cdot \nabla b^\varepsilon + v^\varepsilon \cdot \nabla b^\varepsilon - b^\varepsilon \cdot \nabla v^\varepsilon$$

$$- b^\varepsilon \cdot \nabla \bar{u}^\varepsilon + \sqrt{\varepsilon} \text{curl} (v^\varepsilon \times e_3) = 0$$

$$\text{div} v^\varepsilon = 0$$

$$\text{div} b^\varepsilon = 0$$

$$(v^\varepsilon, b^\varepsilon)_{t=0} = (w_0, b_0)$$
Now we consider the approximate system

\[
\begin{align*}
(MHDF_n^\varepsilon) \\
\frac{\partial v}{\partial t} &= -\varepsilon \Delta J_n v + J_n(J_n v.\nabla J_n v) + J_n(J_n v.\nabla J_n \tilde{u}^\varepsilon) + J_n(J_n \tilde{u}^\varepsilon.\nabla J_n v) \\
&+ J_n(J_n \tilde{u}^\varepsilon.\nabla J_n v) - J_n(\text{curl} \, J_n b \times J_n b) + \sqrt{\varepsilon} \text{curl} \, J_n b \times e_3 + \\
J_n v \times e_3 &= -\nabla \Delta^{-1} \text{div} \left( J_n(J_n v.\nabla J_n v) + J_n(J_n v.\nabla J_n \tilde{u}^\varepsilon) + J_n(J_n \tilde{u}^\varepsilon.\nabla J_n v) \right) \\
&+ \frac{J_n^\varepsilon \nu}{\varepsilon} \times e_3 \\
\frac{\partial b}{\partial t} &= -\sqrt{\varepsilon} \Delta J_n b + J_n(J_n \tilde{u}^\varepsilon.\nabla J_n b) + J_n(J_n v.\nabla J_n b) \quad \text{in} \quad (0, T_0) \times \mathbb{R}^3 \\
&- J_n(J_n b.\nabla J_n v) - J_n(J_n b.\nabla J_n \tilde{u}^\varepsilon) \\
&+ \sqrt{\varepsilon} \text{curl} \, (J_n v \times e_3) = 0 \\
div v &= div b = 0 \\
(v, b)(0) &= (J_n w_0, J_n b_0)
\end{align*}
\]

We use Lemmas 2.1, 3.2 to prove that \((MHDF_n^\varepsilon)\) have an ordinary differential equation form in \(H^s(\mathbb{R}^3)\), then there exists \(V_n := (v_n, b_n) \in C^1((0, T_n^\varepsilon), H^s(\mathbb{R}^3))\) a maximal solution of \((MHDF_n^\varepsilon)\). The uniqueness and the fact \(J_n^\varepsilon = J_n\) implies that \(J_n V_n = V_n\). The Lemmas 2.1 and 3.1 implies \(T_n^\varepsilon > T_1\) and for all \(t \in [0, T_1]\)

\[
\|V_n(t)\|^2_{H^s} + 2\varepsilon \int_0^t \|\nabla v_n\|^2_{H^s} + 2\sqrt{\varepsilon} \int_0^t \|\nabla b_n\|^2_{H^s} \leq 2\|(w_0, b_0)\|^2_{H^s}.
\]

The end is similar of the proof for Theorem 1.5, then we obtain existence of a \(V^\varepsilon := (v^\varepsilon, b^\varepsilon) \in L^\infty_{T_1}(H^s) \cap L^2_{T_1}(H^{s+1})\) solution of \((MHDF^\varepsilon)\), satisfy for all \(t \in [0, T_1]\)

\[
\|V^\varepsilon(t)\|^2_{H^s} + 2\varepsilon \int_0^t \|\nabla v^\varepsilon\|^2_{H^s} + 2\sqrt{\varepsilon} \int_0^t \|\nabla b^\varepsilon\|^2_{H^s} \leq 2\|(w_0, b_0)\|^2_{H^s}.
\]

The proof of the uniqueness is easy.

\[\blacksquare\]

4.4. Proof of Theorem 1.9. We posed \(\psi^\varepsilon := (u^\varepsilon - \tilde{u}^\varepsilon - u^\varepsilon, b^\varepsilon - B^\varepsilon) = (\psi_1^\varepsilon, \psi_2^\varepsilon)\), \(\psi^\varepsilon\) satisfy the followings equations

\[
\begin{align*}
\frac{\partial \psi_1^\varepsilon}{\partial t} - \varepsilon \Delta \psi_1^\varepsilon + \sqrt{\varepsilon} \partial_3 \psi_2^\varepsilon &= f_1^\varepsilon \\
\frac{\partial \psi_2^\varepsilon}{\partial t} - \sqrt{\varepsilon} \Delta \psi_2^\varepsilon + \sqrt{\varepsilon} \partial_3 \psi_1^\varepsilon &= f_2^\varepsilon,
\end{align*}
\]

where

\[
\begin{align*}
f_1^\varepsilon &= -\nabla p^\varepsilon - \psi_1^\varepsilon.\nabla \psi_1^\varepsilon - \psi_1^\varepsilon.\nabla \tilde{u}^\varepsilon - \tilde{u}^\varepsilon.\nabla \psi_1^\varepsilon + \psi_2^\varepsilon.\nabla \psi_2^\varepsilon + \psi_2^\varepsilon.\nabla B^\varepsilon + B^\varepsilon.\nabla \psi_1^\varepsilon - w^\varepsilon.\nabla u^\varepsilon - \tilde{u}^\varepsilon.\nabla w^\varepsilon \\
f_2^\varepsilon &= -\psi_2^\varepsilon.\nabla \psi_2^\varepsilon - w^\varepsilon.\nabla \psi_2^\varepsilon - \tilde{u}^\varepsilon.\nabla \psi_2^\varepsilon - \psi_1^\varepsilon.\nabla B^\varepsilon + B^\varepsilon.\nabla \psi_1^\varepsilon + \psi_2^\varepsilon.\nabla \tilde{u}^\varepsilon + \psi_2^\varepsilon.\nabla \psi_1^\varepsilon + \psi_2^\varepsilon.\nabla \psi_2^\varepsilon.
\end{align*}
\]

We take the scalar produce in \(H^{s-3}(\mathbb{R}^3)\) we obtain

\[
\frac{1}{2} \frac{d}{dt} \|\psi^\varepsilon\|^2_{H^{s-3}} + \varepsilon \|\nabla \psi_1^\varepsilon\|^2_{H^{s-3}} + \sqrt{\varepsilon} \|\nabla \psi_2^\varepsilon\|^2_{H^{s-3}} \leq \sum_{k=1}^{14} I_k,
\]
where

\[ I_1 := |\psi_1^e \cdot \nabla \psi_1^e, \psi_1^e >_{H^{s-3}}|, \quad I_2 := |\tilde{u}^e \cdot \nabla \psi_1^e, \psi_1^e >_{H^{s-3}}| \]
\[ I_3 := |\psi_1^e \cdot \nabla \tilde{u}^e, \psi_1^e >_{H^{s-3}}|, \quad I_4 := |\psi_2^e \cdot B^e, \psi_1^e >_{H^{s-3}}| \]
\[ I_5 := |\psi_2^e \cdot \nabla \psi_2^e, \psi_1^e >_{H^{s-3}} + |\psi_2^e \cdot \nabla \psi_1^e, \psi_2^e >_{H^{s-3}}| \]
\[ I_6 := |B^e \cdot \nabla \psi_2^e, \psi_1^e >_{H^{s-3}} + |B^e \cdot \nabla \psi_1^e, \psi_2^e >_{H^{s-3}}| \]
\[ I_7 := |\psi_1^e \cdot \nabla \psi_2^e, \psi_2^e >_{H^{s-3}}|, \quad I_8 := |\psi_1^e \cdot B^e, \psi_2^e >_{H^{s-3}}| \]
\[ I_9 := |\tilde{u}^e \cdot \nabla \psi_2^e, \psi_2^e >_{H^{s-3}}|, \quad I_{10} := |w^e \cdot \nabla \psi_2^e, \psi_2^e >_{H^{s-3}}| \]
\[ I_{11} := |\psi_2^e \cdot \nabla \tilde{u}^e, \psi_2^e >_{H^{s-3}}|, \quad I_{12} := |\psi_2^e \cdot w^e, \psi_1^e >_{H^{s-3}}| \]
\[ I_{13} := |\tilde{u}^e \cdot \nabla w^e, \psi_1^e >_{H^{s-3}}|, \quad I_{14} := |w^e \cdot \nabla \tilde{u}^e, \psi_1^e >_{H^{s-3}}|. \]

Using Cauchy-Schwarz inequality and the fact \( H^{s-3} \) is an algebra we obtain

\[ I_4 \leq C \| B^e \|_{H^{s-2}} \| \psi_2^e \|_{H^{s-3}} \| \psi_1^e \|_{H^{s-3}} \leq C \| \psi^e \|_{H^{s-3}}^2 \]
\[ I_8 \leq C \| B^e \|_{H^{s-2}} \| \psi_1^e \|_{H^{s-3}} \| \psi_2^e \|_{H^{s-3}} \leq C \| \psi^e \|_{H^{s-3}}^2 \]
\[ I_{12} \leq C \| w^e \|_{H^{s-2}} \| \psi_1^e \|_{H^{s-3}} \| \psi_2^e \|_{H^{s-3}} \leq C \| \psi^e \|_{H^{s-3}}^2. \]

Using lemma 4.3 we obtain

\[ I_1 \leq C \| \psi_1^e \|_{H^{s-3}}^3 \leq C \| \psi^e \|_{H^{s-3}}^2 \]
\[ I_2 \leq C \| \tilde{u}^e \|_{H^{s-2}} \| \psi_1^e \|_{H^{s-3}}^2 \leq C \| \psi^e \|_{H^{s-3}}^2 \]
\[ I_3 \leq C \| \tilde{u}^e \|_{H^{s-2}} \| \psi_1^e \|_{H^{s-3}}^2 \leq C \| \psi^e \|_{H^{s-3}}^2 \]
\[ I_5 \leq C \| \psi_1^e \|_{H^{s-2}} \| \psi_1^e \|_{H^{s-3}}^2 \leq C \| \psi^e \|_{H^{s-3}}^2 \]
\[ I_6 \leq C \| B^e \|_{H^{s-3}} \| \psi_1^e \|_{H^{s-3}} \| \psi_1^e \|_{H^{s-3}} \leq C \| \psi^e \|_{H^{s-3}}^2 \]
\[ I_7 \leq C \| \psi_1^e \|_{H^{s-2}} \| \psi_2^e \|_{H^{s-3}} \leq C \| \psi^e \|_{H^{s-3}}^2 \]
\[ I_9 \leq C \| \tilde{u}^e \|_{H^{s-3}} \| \psi_2^e \|_{H^{s-3}} \leq C \| \psi^e \|_{H^{s-3}}^2 \]
\[ I_{10} \leq C \| w^e \|_{H^{s-2}} \| \psi_2^e \|_{H^{s-3}} \leq C \| \psi^e \|_{H^{s-3}}^2 \]
\[ I_{11} \leq C \| \tilde{u}^e \|_{H^{s-3}} \| \psi_2^e \|_{H^{s-3}} \leq C \| \psi^e \|_{H^{s-3}}^2. \]

The difficult is the terms \( I_{13} \) and \( I_{14} \), for exemple study the term \( I_{14} \).
Set \( \chi \in D(\mathbb{R}) \) valu 1 near the origin. For \( R > 0 \) we pose

\[ w_R^e := \mathcal{F}^{-1} \left( \frac{|\xi|}{R} \mathcal{F}(w^e) \right), \quad \tilde{w}_R^e := w^e - w_R^e. \]

We have

\[ I_{14} \leq J + \tilde{J}, \]
where $J := \langle w^\varepsilon_R \cdot \nabla \bar{u}^\varepsilon; \psi_1^\varepsilon \rangle_{H^{s-3}}$, $\tilde{J} := \langle \tilde{w}^\varepsilon_R \cdot \nabla \tilde{u}^\varepsilon; \psi_1^\varepsilon \rangle_{H^{s-3}}$.

- Study of $J$

\[
J = \left| \sum_{|\alpha| \leq s-3} \left( \int w^\varepsilon_R \cdot \nabla \bar{u}^\varepsilon \partial^\alpha \psi_1^\varepsilon \right) \left( \int \partial^\alpha (w^\varepsilon_R \cdot \nabla \bar{u}^\varepsilon) \partial^\alpha \psi_1^\varepsilon \right) \right|
\]
\[
= \left| \sum_{|\alpha| \leq s-3} \sum_{\beta \leq \alpha} C_{\alpha}^\beta \left( \int \partial^\beta \bar{u}^\varepsilon \nabla \partial^{\alpha-\beta} \bar{u}^\varepsilon \partial^\alpha \psi_1^\varepsilon \right) \right|
\]
\[
\leq \sum_{|\alpha| \leq s-3} \sum_{\beta \leq \alpha} C_{\alpha}^\beta \left( \int \partial^\beta \bar{u}^\varepsilon \nabla \partial^{\alpha-\beta} \bar{u}^\varepsilon \partial^\alpha \psi_1^\varepsilon \right).
\]

For any $\alpha, \beta$ we have

\[
| \int \partial^\beta \bar{u}^\varepsilon \nabla \partial^{\alpha-\beta} \bar{u}^\varepsilon \partial^\alpha \psi_1^\varepsilon | = | \int \mathcal{F}(\partial^\beta \bar{u}^\varepsilon \nabla \partial^{\alpha-\beta} \bar{u}^\varepsilon \partial^\alpha \psi_1^\varepsilon) | = | \int \mathcal{F}(\partial^\beta \bar{u}^\varepsilon \nabla \partial^{\alpha-\beta} \bar{u}^\varepsilon \partial^\alpha \psi_1^\varepsilon) | \leq \| \partial^\beta \bar{u}^\varepsilon \|_{L^\infty} \| \mathcal{F}^{-1}(\mathcal{F}(\partial^\beta \bar{u}^\varepsilon \nabla \partial^{\alpha-\beta} \bar{u}^\varepsilon \partial^\alpha \psi_1^\varepsilon)) \|_{L^1}
\]

Young inequality and the fact

\[
(*) \quad C_{\frac{1}{4}}^\varepsilon \hookrightarrow L^\infty
\]

we have

\[
| \int \partial^\beta \bar{u}^\varepsilon \nabla \partial^{\alpha-\beta} \bar{u}^\varepsilon \partial^\alpha \psi_1^\varepsilon | \leq C \| \partial^\beta \bar{u}^\varepsilon \|_{C^{s,3+\frac{1}{4}}} \| \mathcal{F}^{-1}(\mathcal{F}(\partial^\beta \bar{u}^\varepsilon \nabla \partial^{\alpha-\beta} \bar{u}^\varepsilon \partial^\alpha \psi_1^\varepsilon)) \|_{L^1} \| \nabla \partial^{\alpha-\beta} \bar{u}^\varepsilon \partial^\alpha \psi_1^\varepsilon |_{L^2} \]
\[
\leq C \| \partial^\beta \bar{u}^\varepsilon \|_{C^{s,3+\frac{1}{4}}} \| \nabla \partial^{\alpha-\beta} \bar{u}^\varepsilon \|_{L^\infty} \| \partial^\alpha \psi_1^\varepsilon \|_{L^2}
\]

the property $(*)$ imply

\[
| \int \partial^\beta \bar{u}^\varepsilon \nabla \partial^{\alpha-\beta} \bar{u}^\varepsilon \partial^\alpha \psi_1^\varepsilon | \leq C \| \partial^\beta \bar{u}^\varepsilon \|_{C^{s,3+\frac{1}{4}}} \| \bar{u}^\varepsilon \|_{C^{s-2+\frac{1}{4}}} \| \psi_1^\varepsilon \|_{H^{s-3}}
\]

the Sobolev injection

\[
H^\sigma(\mathbb{R}^2) \hookrightarrow C^{\sigma-1}(\mathbb{R}^2)
\]

imply

\[
| \int \partial^\beta \bar{u}^\varepsilon \nabla \partial^{\alpha-\beta} \bar{u}^\varepsilon \partial^\alpha \psi_1^\varepsilon | \leq C \| \partial^\beta \bar{u}^\varepsilon \|_{C^{s,3+\frac{1}{4}}} \| \bar{u}^\varepsilon \|_{H^{s-\frac{3}{2}+\frac{1}{4}}} \| \psi_1^\varepsilon \|_{H^{s-3}}
\]
\[
\leq C \| \partial^\beta \bar{u}^\varepsilon \|_{C^{s,3+\frac{1}{4}}} \| \psi_1^\varepsilon \|_{H^{s-3}}
\]
\[
\leq C \| \bar{u}^\varepsilon \|_{C^{s,3+\frac{1}{4}}}^2 + \| \psi_1^\varepsilon \|_{H^{s-3}}^2.
\]

(4.30)
• Study of \( \tilde{J} \)

\[
\tilde{J} = |(\tilde{w}_R^\varepsilon, \nabla \tilde{u}^\varepsilon; \psi_1^\varepsilon)_{H^{s-3}}|
\]

\[
= |\sum_{|\alpha| \leq s-3} \int \partial^\alpha (\tilde{w}_R^\varepsilon, \nabla \tilde{u}^\varepsilon) \partial^\alpha \psi_1^\varepsilon|
\]

\[
\leq \sum_{|\alpha| \leq s-3} \sum_{\beta \leq \alpha} C_{\alpha}^\beta \int \partial^\beta \tilde{w}_R^\varepsilon, \nabla \partial^\alpha - \beta \tilde{u}^\varepsilon \partial^\beta \psi_1^\varepsilon|
\]

\[
\leq C \|\tilde{w}_R^\varepsilon\|_{H^{s-3}} \|\psi_1^\varepsilon\|_{H^{s-3}} \sum_{|\alpha| \leq s-3} \sum_{\beta \leq \alpha} C_{\alpha}^\beta \|\nabla \partial^\alpha - \beta \tilde{u}^\varepsilon\|_{L^\infty}
\]

\[
\leq C \|\tilde{w}_R^\varepsilon\|_{H^{s-3}} \|\psi_1^\varepsilon\|_{H^{s-3}} \|\tilde{u}^\varepsilon\|_{H^{s-2+\frac{1}{4}}}
\]

\[
\leq C \|\tilde{w}_R^\varepsilon\|_{H^{s-3}} \|\psi_1^\varepsilon\|_{H^{s-3}}.
\]

The fact \( \mathcal{F}(\tilde{w}_R^\varepsilon) \equiv 0 \) in \( B(0, R) \), imply

\[
\|\tilde{w}_R^\varepsilon\|_{H^{s-3}} \leq R^{-3} \|w^\varepsilon\|_{H^s}
\]

\[
\leq C.R^{-3}
\]

then

\[
\tilde{J} \leq C.R^{-3} \|\psi_1^\varepsilon\|_{H^{s-3}}
\]

\[
\leq C.R^{-2} \|\psi_1^\varepsilon\|_{H^{s-3}}
\]

\[
\leq C.R^{-4} + \|\psi_1^\varepsilon\|_{H^{s-3}}^2.
\]

(4.31)

Using (4.17)...(4.31) we obtain

\[
\|\psi_1^\varepsilon(t)\|_{H^{s-3}}^2 \leq C.R^{-4} + C_R \int_0^{T_1} \|w_R^\varepsilon(\tau)\|_{C^{s-3+\frac{1}{4}}^4}^2 d\tau + C \int_0^t \|\psi_1^\varepsilon(\tau)\|_{H^{s-3}} d\tau,
\]

Gronwall lemma imply

\[
\|\psi_1^\varepsilon(t)\|_{H^{s-3}}^2 \leq \left( C.R^{-4} + C.R \int_0^{T_1} \|w_R^\varepsilon(\tau)\|^2_{C^{s-3+\frac{1}{4}}^4} d\tau \right) \exp(CT_1),
\]

The proofs of Theorem 1.5 and 1.7 imply that

\[
w_R^\varepsilon \to 0 \quad \text{in} \quad L^4([0, T_1], C^{s-\frac{2}{s'}}), \quad \forall s' < s;
\]

we can deduce the desired result.

5. Appendix

In this section we shall prove the product laws stated in Lemmas 2.3, 4.2 and 4.3.
5.1. Proof of Lemma 2.1. We suppose that \( a, b, c \) are three vectors fields in \( H^\infty(\mathbb{R}^3) \) such that \( \text{div} \ a = 0 \).

For the first point:
We write
\[
|<a . \nabla b, b>_{H^\sigma(\mathbb{R}^3)}| = \sum_{|\alpha| \leq \sigma} \left| \int \partial^\alpha (a . \nabla b) \partial^\alpha b \right|,
\]
then
\[
|<a . \nabla b, b>_{H^\sigma(\mathbb{R}^3)}| \leq \sum_{|\alpha| \leq \sigma} \left| \int \partial^\alpha (a . \nabla b) \partial^\alpha b \right| \leq \sum_{|\alpha| \leq \sigma} \sum_{\beta \leq \alpha} C^{\beta}_{\alpha} A_{\alpha,\beta},
\]
where
\[
A_{\alpha,\beta} := \left| \int (\partial^{\alpha-\beta} a . \nabla \partial^{\beta} b) \partial^\alpha b \right|.
\]
The most important term is for \( |\alpha| = \sigma \).
- \( \text{div} \ a = 0 \) imply that \( A_{\alpha,\alpha} = 0 \).
- For \( 0 < |\beta| < \sigma \), we are

\[
A_{\alpha,\alpha} \leq \int |\partial^{\alpha-\beta} a| \| \nabla \partial^{\beta} b \| \| \partial^\alpha b \|
\leq |\partial^{\alpha-\beta} a| \| \nabla \partial^{\beta} b \|_L^2 \| \partial^\alpha b \|_L^2
\leq |\partial^{\alpha-\beta} a| \| \nabla \partial^{\beta} b \|_H^{\sigma-1} \| \partial^\alpha b \|_H^{\sigma-1}
\leq |\nabla a| \| \nabla \partial^{\beta} b \|_H^{\sigma-1} \| \partial^\alpha b \|_H^{\sigma-1}
\leq C |\nabla a| \| \nabla \partial^{\beta} b \|_H^{\sigma-1} \| \partial^\alpha b \|_H^{\sigma-1}
\]
- For \( \beta = 0 \),

\[
A_{\alpha,\beta} \leq \int |\partial^{\alpha} a| \| \nabla b \| \| \partial^\alpha b \|
\leq |\partial^{\alpha} a| \| \nabla b \|_L^\infty \| \partial^\alpha b \|_L^2
\leq |\nabla a| \| \nabla \partial^{\alpha} b \|_C^{1} \| \nabla \partial^{\alpha} b \|_H^{\sigma-1}
\leq C |\nabla a| \| \nabla \partial^{\alpha} b \|_H^{\sigma-1} \| \nabla \partial^{\alpha} b \|_H^{\sigma-1}
\leq C |\nabla a| \| \nabla \partial^{\alpha} b \|_H^{\sigma-1} \| \nabla \partial^{\alpha} b \|_H^{\sigma-1}
\]

For the second point, we write
\[
|<a \nabla c, b>_{H^\sigma} + <a \nabla b, c>_{H^\sigma}| = \sum_{\alpha \in \mathbb{N}^3} (\sum_{|\alpha| \leq \sigma} \left| \int \partial^\alpha (a . \nabla c) \partial^\alpha c + \int \partial^\alpha (a . \nabla b) \partial^\alpha b \right|)
\leq \sum_{\alpha \in \mathbb{N}^3} \sum_{\beta \leq \alpha} C^{\beta}_{\alpha} B_{\alpha,\beta},
\]
where
\[
B_{\alpha,\beta} := \left| \int (\partial^{\alpha-\beta} a . \nabla \partial^{\beta} b) \partial^\alpha c + \int (\partial^{\alpha-\beta} a . \nabla \partial^{\beta} c) \partial^\alpha b \right|.
\]
The most important term is for \( |\alpha| = \sigma \).
- \( \text{div} \ a = 0 \) imply that \( B_{\alpha,\alpha} = 0 \).
- For \( \beta \neq \alpha \) we apply the first step.
5.2. Proof of Lemma 4.2. We write

\[ \langle a \nabla b, c \rangle_{H^s(\mathbb{R}^3)} = \left| \sum_{|\alpha| \leq \sigma} \int \partial^n (a \nabla b) \partial^\alpha c \right| \]
\[ = \left| \sum_{|\alpha| \leq \sigma} \int \partial^n (a \nabla b) \partial^\alpha c \right| \]
\[ = \left| \sum_{|\alpha| \leq \sigma} \sum_{\beta \leq \alpha} C^\beta_\alpha \int (\partial^{\alpha - \beta} a \nabla \partial^\beta b) \partial^\alpha c \right| \]
\[ \leq \sum_{|\alpha| \leq \sigma} \sum_{\beta \leq \alpha} C^\beta_\alpha D_{\alpha, \beta}, \]

where

\[ D_{\alpha, \beta} := \left| \int (\partial^{\alpha - \beta} a \nabla \partial^\beta b) \partial^\alpha c \right| \]

the Cauchy-Schwarz inequality imply

\[ D_{\alpha, \beta} \leq \left\| \partial^{\alpha - \beta} a \nabla \partial^\beta b \right\|_{L^2} \left\| \partial^\alpha c \right\|_{L^2} \]
\[ \leq \left\| \partial^\alpha c \right\|_{L^2} \min \left( \left\| \partial^{\alpha - \beta} a \right\|_{L^2}, \left\| \partial^{\alpha - \beta} a \right\|_{L^\infty}, \left\| \partial^{\alpha - \beta} a \right\|_{L^\infty} \left\| \nabla \partial^\beta b \right\|_{L^2} \right) \]
\[ \leq \left\| \partial^\alpha c \right\|_{L^2} \min \left( \left\| a \right\|_{H^s}, \left\| \nabla b \right\|_{H^\sigma}, \left\| \partial^\beta a \right\|_{C^{\epsilon}}, \left\| \partial^\beta \partial^\beta b \right\|_{L^2} \right) \]

This achieves the proof.  

\[ \square \]

5.3. Proof of Lemma 4.3. We write

\[ \langle \bar{a} \nabla b, c \rangle_{H^s(\mathbb{R}^3)} + \langle \bar{a} \nabla c, b \rangle_{H^s(\mathbb{R}^3)} = \left| \sum_{|\alpha| \leq \sigma} \int \partial^n (\bar{a} \nabla b) \partial^\alpha c + \int \partial^n (\bar{a} \nabla c) \partial^\alpha b \right| \]
\[ = \left| \sum_{|\alpha| \leq \sigma} \sum_{\beta \leq \alpha} C^\beta_\alpha \left( \int (\partial^{\alpha - \beta} \bar{a} \nabla \partial^\beta b) \partial^\alpha c \right. 
\[ + \left. \int (\partial^{\alpha - \beta} \bar{a} \nabla \partial^\beta c) \partial^\alpha b \right) \right| \]
\[ \leq \sum_{|\alpha| \leq \sigma} \sum_{\beta \leq \alpha} C^\beta_\alpha E_{\alpha, \beta}, \]

where

\[ E_{\alpha, \beta} := \left| \int (\partial^{\alpha - \beta} \bar{a} \nabla \partial^\beta b) \partial^\alpha c + \int (\partial^{\alpha - \beta} \bar{a} \nabla \partial^\beta c) \partial^\alpha b \right|. \]

The most important term is for \( |\alpha| = \sigma. \)

\bullet \ \text{div}_h \ \bar{a} = 0 \ \text{imply that} \ E_{\alpha, \alpha} = 0.

For \( \beta \neq \alpha \) we are \( E_{\alpha, \beta} \leq E_{\alpha, \beta}^{(1)} + E_{\alpha, \beta}^{(2)} \),

where \( E_{\alpha, \beta}^{(1)} = \left| \int (\partial^{\alpha - \beta} \bar{a} \nabla \partial^\beta b) \partial^\alpha c \right| \text{ and } E_{\alpha, \beta}^{(2)} = \left| \int (\partial^{\alpha - \beta} \bar{a} \nabla \partial^\beta c) \partial^\alpha b \right|.\)

\bullet \ \text{0} \leq |\beta| \leq \sigma - 2.

In this case we have

\[ E_{\alpha, \beta}^{(1)} \leq \left\| \partial^{\beta} \bar{a} \right\|_{L^\infty(\mathbb{R}^3)} \left\| \nabla \partial^{\alpha - \beta} b \right\|_{L^2(\mathbb{R}^3)} \left\| \partial^\alpha c \right\|_{L^2(\mathbb{R}^3)}. \]

Using the injections

\[ H^{1+\frac{3}{4}}(\mathbb{R}^2) \leftrightarrow C^{\frac{3}{4}}(\mathbb{R}^2) \leftrightarrow L^\infty(\mathbb{R}^2) \]

we deduce

\[ E_{\alpha, \beta}^{(1)} \leq C \left\| \bar{a} \right\|_{H^s(\mathbb{R}^2)} \left\| \nabla b \right\|_{H^{s-1}(\mathbb{R}^3)} \left\| \nabla c \right\|_{H^{s-1}(\mathbb{R}^3)}. \]
\[ E_{\alpha,\beta}^{(1)} \leq \int_{x_3} \left( \int_{x_h} |\partial^\beta \alpha(x_h)| \|\nabla \partial^{\alpha-\beta} b(x_h, x_3)||\partial^\alpha c(x_h, x_3)| \right) \]
\[ \leq \|\partial^\beta \alpha\|_{L^2(\mathbb{R}^2)} \left( \int_{x_3} \|\nabla \partial^{\alpha-\beta} b(x_3)\|_{L^\infty(\mathbb{R}^2)}^2 dx_3 \right)^{\frac{1}{2}} \left( \int_{x_3} \|\partial^\alpha c(x_3)\|_{L^2(\mathbb{R}^2)}^2 dx_3 \right)^{\frac{1}{2}}. \]

But
\[ \|\nabla \partial^{\alpha-\beta} b(x_3)\|_{L^\infty(\mathbb{R}^2)} \leq C \|\nabla \partial^{\alpha-\beta} b(x_3)\|_{H^{\frac{1}{2}}(\mathbb{R}^2)} \]
\[ \leq C \|\nabla \partial^{\alpha-\beta} b(x_3)\|_{H^2(\mathbb{R}^2)}, \]
\[ \left( \int_{x_3} \|\partial^\alpha c(x_3)\|_{L^2(\mathbb{R}^2)}^2 dx_3 \right)^{\frac{1}{2}} = \|\partial^\alpha c\|_{L^2(\mathbb{R}^3)}, \]
and
\[ \left( \int_{x_3} \|\nabla \partial^{\alpha-\beta} b(x_3)\|_{H^2(\mathbb{R}^2)}^2 dx_3 \right)^{\frac{1}{2}} \leq \|\nabla \partial^{\alpha-\beta} b\|_{H^2(\mathbb{R}^3)} \]
so
\[ E_{\alpha,\beta} \leq \|\bar{a}\|_{H^\alpha(\mathbb{R}^2)} \|\nabla b\|_{H^{\alpha-1}(\mathbb{R}^3)} ||c||_{H^\alpha(\mathbb{R}^3)}. \]

For the second estimate we remark that, if \( c = b, \)
\[ \text{div}_h \bar{a} = 0 \text{ imply } E_{\alpha,\alpha} = 0 \text{ for all } |\alpha| \leq \sigma. \]

For the third point we write
\[ |< b, \nabla \bar{\alpha}, c |_{H^\alpha(\mathbb{R}^3)} | = \left| \sum_{|\alpha| \leq \sigma} \int \partial^\alpha (b, \nabla \bar{\alpha}) \partial^\alpha c \right|, \]
then
\[ |< b, \nabla \bar{\alpha}, c |_{H^\alpha(\mathbb{R}^3)} | \leq \sum_{|\alpha| \leq \sigma} \sum_{\beta \leq \alpha} C^\beta \alpha F_{\alpha,\beta}, \]
where
\[ F_{\alpha,\beta} := \left| \int (\partial^\beta b, \nabla \partial^{\alpha-\beta} \bar{\alpha}) \partial^\alpha c \right|. \]

The most important case is when \( |\alpha| = \sigma. \)

\textbf{For }3 \leq |\beta| \leq \sigma, \text{ we can write }
\[ F_{\alpha,\beta} \leq \|\nabla \partial^{\alpha-\beta} \bar{\alpha}\|_{L^\infty(\mathbb{R}^2)} \|\partial^\beta b\|_{L^2(\mathbb{R}^3)} \|\partial^\alpha c\|_{L^2(\mathbb{R}^3)}. \]

But
\[ \|\nabla \partial^{\alpha-\beta} \bar{\alpha}\|_{L^\infty(\mathbb{R}^2)} \leq C \|\bar{\alpha}\|_{H^\alpha(\mathbb{R}^2)}. \]
So
\[ F_{\alpha,\beta} \leq C \|\bar{\alpha}\|_{H^\alpha(\mathbb{R}^2)} \|b\|_{H^\alpha(\mathbb{R}^3)} ||c||_{H^\alpha(\mathbb{R}^3)}. \]

\textbf{For }|\beta| \leq 2, \text{ we have by the Cauchy-Schwarz inequality }
\[ F_{\alpha,\beta} \leq \int_{x_3} \left( \int_{x_h} \|\nabla \partial^{\alpha-\beta} \bar{\alpha}(x_h)||\partial^\beta b(x_h, x_3)||\partial^\alpha c(x_h, x_3)\right) \]
\[ \leq \|\nabla \partial^{\alpha-\beta} \bar{\alpha}\|_{L^2(\mathbb{R}^2)} \left( \int_{x_3} \|\partial^\beta b(x_3)\|_{L^\infty(\mathbb{R}^2)}^2 dx_3 \right)^{\frac{1}{2}} \left( \int_{x_3} \|\partial^\alpha c(x_3)\|_{L^2(\mathbb{R}^2)}^2 dx_3 \right)^{\frac{1}{2}} \]
\[ \leq C \|\bar{\alpha}\|_{H^{\sigma+1}(\mathbb{R}^2)} \|b\|_{H^\sigma(\mathbb{R}^3)} ||c||_{H^\sigma(\mathbb{R}^3)}. \]
This end the proof of lemma [L3].
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