GLOBAL CLASSICAL SOLUTION OF THE CAUCHY PROBLEM TO 1D COMPRESSIBLE NAVIER-STOKES EQUATIONS WITH LARGE INITIAL DATA

QUANSENI JIU

School of Mathematical Sciences, Capital Normal University
Beijing 100048, P. R. China
Email: jiuqs@mail.cnu.edu.cn

MINGJIE LI

College of Science, Minzu University of China
Beijing 100081, P. R. China
Email: lmjmath@gmail.com

YULIN YE

School of Mathematical Sciences, Capital Normal University
Beijing 100048, P. R. China
Email: nkyelin@163.com

Abstract: In this paper, we prove that the 1D Cauchy problem of the compressible Navier-Stokes equations admits a unique global classical solution \((\rho, u)\) if the viscosity \(\mu(\rho) = 1 + \rho^\beta\) with \(\beta \geq 0\). The initial data can be arbitrarily large and may contain vacuum. Some new weighted estimates of the density and velocity are obtained when deriving higher order estimates of the solution.

Keywords: compressible Navier-Stokes equations; density-dependent viscosity; global classical solution; vacuum; weighted estimates.

1. Introduction and Main Results

In this paper, we consider the following compressible Navier-Stokes equations with density-dependent viscosity coefficients:

\[
\begin{cases}
\rho_t + (\rho u)_x = 0, \\
(\rho u)_t + (\rho u^2)_x + [p(\rho)]_x = [\mu(\rho)u_x]_x,
\end{cases}
\]

where \(t \geq 0, x \in \mathbb{R}, \rho = \rho(x,t)\) and \(u = u(x,t)\) represent the fluid density and velocity respectively and the pressure \(P\) is given by

\[
P(\rho) = R\rho^\gamma, \, \gamma > 1.
\]

For simplicity, we assume that

\[
\mu(\rho) = 1 + b\rho^\beta, \, \beta \geq 0.
\]
In the sequel, we set $R = b = 1$ without loss of generality. We consider the Cauchy problem for (1.1) with $(\rho, u)$ vanishing at infinity. The initial data is imposed as

$$(\rho, u)|_{t=0} = (\rho_0(x), u_0(x)), \quad x \in \mathbb{R}. \quad (1.4)$$

It is known that in the presence of vacuum, the solution of the compressible Navier-Stokes equations with constant viscosity will behave singularly (see [26], [27], [9]) in general. By some physical considerations, Liu, Xin and Yang in [21] introduced the modified compressible Navier-Stokes equations with density-dependent viscosity coefficients for isentropic fluids. In fact, while deriving the compressible Navier-Stokes equations from the Boltzmann equations by the Chapman-Enskog expansions, the viscosity depends on the temperature and correspondingly depends on the density for isentropic cases. Moreover, the viscous Saint-Venant system for the shallow water equations, derived from the incompressible Navier-Stokes equations with a moving free surface, corresponds a kind of compressible Navier-Stokes equations with density-dependent viscosity (see [6] and references therein).

The one-dimensional compressible Navier-Stokes equations with density-dependent viscosity have been widely studied (see [4, 7, 12, 19, 14, 15, 20, 23, 24, 28, 29] and references therein). However, the global well-posedness of classical solutions with large initial data in multi-dimensional case is completely open. Even the global existence of weak solutions in multi-dimensional case remains open except under spherically symmetric assumptions [6]. In [25], Vaigant-Kazhikhov first proposed and studied the following two-dimensional Navier-Stokes equations

$$\begin{cases}
\rho_t + \text{div}(\rho U) = 0, \\
(\rho U)_t + \text{div}(\rho U \otimes U) + \nabla P(\rho) = \mu \triangle U + \nabla \left( (\mu + \lambda(\rho))\text{div}U \right).
\end{cases} \quad (1.5)$$

Here $\rho(x, t)$ and $U = (u_1(x, t), u_2(x, t))$ represent the density and velocity of the flow respectively. It is assumed in [25] that the shear viscosity $\mu > 0$ is a positive constant and the bulk viscosity satisfies $\lambda(\rho) = \rho^\beta$ with $\beta > 0$ in general. For the periodic problem on the torus $\mathbb{T}^2$ and under assumptions that the initial density is uniformly away from vacuum and $\beta > 3$, Vaigant-Kazhikhov established the global well-posedness of the classical solution to (1.5) in [25]. Jiu-Wang-Xin [17] improved the result and obtained the global well-posedness of the classical solution with large initial data permitting vacuum. Later on, Huang-Li relaxed the index $\beta$ to be $\beta > 4/3$ and studied the large time behavior of the solutions in [10]. For the 2D Cauchy problems with vacuum states at far fields, Jiu-Wang-Xin [16] and Huang-Li [11] independently considered the global well-posedness of classical solution in different weighted spaces. Recently, Jiu-Wang-Xin in [18] studied the global well-posedness to the Cauchy problem with absence of vacuum at far fields and proved that if there is no vacuum initially then there will not appear vacuum in any finite time.

In this paper, we will study the global existence and uniqueness of classical solution to the one-dimensional Cauchy problem for the isentropic compressible Navier-Stokes equations (1.1)-(1.4). The initial data is assumed to be large and may contain vacuum. The index $\beta \geq 0$ is much more general in comparison with that in [25], [16], [11] and [18] such that the constant viscosity is permitted in our result. Note
Theorem 1.1. Suppose that the initial values \((\rho_0, u_0)(x)\) satisfy

\[0 \leq (\rho_0, \rho_0^\beta, \rho_0^\gamma) \in L^1(\mathbb{R}) \cap H^2(\mathbb{R}), \quad u_0 \in D^1(\mathbb{R}) \cap D^2(\mathbb{R}),\]

\[\sqrt{\rho_0}u_0(1 + |x|^\frac{\gamma}{2}) \in L^2(\mathbb{R}), \quad \partial_x u_0|x|^\frac{\gamma}{2} \in L^2(\mathbb{R}), \quad (|x|^\frac{\beta}{2} \rho_0^\beta, |x|^\frac{\gamma}{2} \rho_0^\gamma) \in L^2(\mathbb{R}),\]

for \(\beta \geq 0\) and \(2 < \alpha < 1 + \frac{2}{1+\frac{\alpha}{4}},\) and the compatibility condition

\[\left[\mu(\rho_0)u_0x\right]_x - [p(\rho_0)]_x(x) = \sqrt{\rho_0}g(x), \quad x \in \mathbb{R},\]

with some \(g\) satisfying \(\sqrt{\rho_0}g(1 + |x|\frac{\gamma}{2}) \in L^2(\mathbb{R}).\) Then for any \(T > 0,\) there exists a unique global classical solution \((\rho, u)(t, x)\) to the Cauchy problem \((1.1)-(1.4),\) satisfying

\[0 \leq \rho \leq C, \quad (\rho, \rho^\gamma, \rho^\beta) \in C([0, T]; H^2(\mathbb{R})), \quad (\rho_t, (\rho^\gamma)_t, (\rho^\beta)_t) \in C([0, T]; H^1(\mathbb{R})),\]

\[\rho_0 \in C([0, T]; L^2(\mathbb{R})), \quad ((\rho^\gamma)_t, (\rho^\beta)_t) \in L^\infty([0, T]; L^2(\mathbb{R})), \quad (\rho u)_t \in C([0, T]; H^1(\mathbb{R})),\]

\[\sqrt{\rho u}(1 + |x|^\frac{\gamma}{2}), \sqrt{\rho u}(1 + |x|^\frac{\gamma}{2}), \rho u_0|x|^\frac{\gamma}{2}, (|x|^\frac{\beta}{2} \rho_0^\beta, |x|^\frac{\gamma}{2} \rho_0^\gamma) \in C([0, T]; L^2(\mathbb{R}))\]

\[u \in C([0, T]; L^{\frac{2}{1+\frac{\alpha}{4}}}(\mathbb{R} \cap D^2(\mathbb{R})) \cap L^2(0, T; L^{\frac{2}{1+\frac{\alpha}{4}}}(\mathbb{R} \cap D^3(\mathbb{R}))), \sqrt{tu}_u \in L^\infty(0, T; D^3),\]

\[u_t \in L^\infty(0, T; L^{\frac{2}{1+\frac{\alpha}{4}}}(\mathbb{R} \cap D^1(\mathbb{R}))), \sqrt{t}u_t \in L^2(0, T; D^2(\mathbb{R})) \cap L^\infty(0, T; L^{\frac{4}{1+\frac{\alpha}{4}}}(\mathbb{R} \cap D^1(\mathbb{R}))),\]

\[t \rho u_{tt} \in L^\infty(0, T; L^2(\mathbb{R})), \sqrt{t}t \rho u_{tt} \in L^2(0, T; L^2(\mathbb{R})), t \sqrt{t} \rho u_{tt} \in L^\infty(0, T; L^2(\mathbb{R})),\]

\[t \partial_x u_{tt} \in L^2(0, T; L^2(\mathbb{R})),\]

(1.8)
where $\dot{u}$ is the material derivative of $u$ defined as $\dot{u} = (\partial_t + u \cdot \partial_x)u$.

**Remark 1.1.** In our result, the index $\beta \geq 0$ is more general in comparison with the results obtained in [4] for the initial-boundary problem and in [16] for the two-dimensional Cauchy problem of (1.5). Much more general viscosity $\mu(\rho)$ can be treated in a similar way.

The rest of the paper is organized as follows: In Section 2, we collect some elementary facts and derive the a priori estimates of the solution which are needed to extend the local solution to a global one. In Section 3, we give the proof of the main result.

2. **A priori estimates**

In this section, we will establish various a priori estimates and weighted estimates on classical solution $(\rho, u)$ on the interval $[0, T]$ for any $T > 0$. Before that, we give the Caffarelli-Kohn-Nirenberg weighted inequalities, which will be used in the a priori estimates of the higher order derivatives of the solution.

**Lemma 2.1 (Caffarelli-Kohn-Nirenberg weighted inequality [1], [2]).**

(1) $\forall h \in C_0^\infty(\mathbb{R})$, it holds that

$$\| |x|^\kappa h\|_r \leq C \| |x|^{\alpha} |\partial_x h|\|_p \| |x|^\beta h\|_q^{1-\theta}$$

(2.1)

where $1 \leq p, q < \infty, 0 < r < \infty, 0 \leq \theta \leq 1, \frac{1}{p} + \alpha > 0, \frac{1}{q} + \beta > 0, \frac{1}{r} + \kappa > 0$ and satisfying

$$\frac{1}{r} + \kappa = \theta \left(\frac{1}{p} + \alpha - 1\right) + (1 - \theta)\left(\frac{1}{q} + \beta\right)$$

(2.2)

and

$$\kappa = \theta \sigma + (1 - \theta)\beta$$

with $0 \leq \alpha - \sigma$ if $\theta > 0$ and $0 \leq \alpha - \sigma \leq 1$ if $\theta > 0$ and $\frac{1}{p} + \alpha - 1 = \frac{1}{r} + \kappa$.

(2) (Best constant for Caffarelli-Kohn-Nirenberg weighted inequality) $\forall h \in C_0^\infty(\mathbb{R})$, it holds that

$$\| |x|^b h\|_p \leq C_{a,b} \| |x|^a \partial_x h\|_2$$

(2.3)

where $a > \frac{1}{2}, a - 1 \leq b \leq a - \frac{1}{2}$ and $p = \frac{2}{2(a-b)-1}$. If $b = a - 1$, then $p = 2$ and the best constant in the inequality (2.3) is

$$C_{a,b} = C_{a,a-1} = \left|\frac{2a - 1}{2}\right|$$

The proof of (1) can be found in [1] and the proof of (2) can be found in [2].
2.1. Uniform upper bound of the density.

In this subsection, we will present a new approach to obtain the upper bound of the density. The following is the usual energy estimate.

**Lemma 2.2.** Let \((\rho, u)\) be a smooth solution to (1.1) - (1.4). Then for any \(T > 0\), it holds

\[
\int_{\mathbb{R}} (\rho u^2 + \rho^\gamma) \, dt + \int_{0}^{T} \int_{\mathbb{R}} (1 + \rho^\beta) (u_x)^2 \, dx \leq C.
\]

The upper bound of the density is stated as follows.

**Lemma 2.3.** Suppose that \((\rho, u)\) is a smooth solution to (1.1) - (1.4). Then for any \(T > 0\), there exists an absolute constant \(C > 0\) which depends on the initial data and \(\beta \geq 0\) such that

\[
\rho(x, t) \leq C, \quad (x, t) \in \mathbb{R} \times (0, T].
\]

**Proof.** Let

\[
\xi = \int_{-\infty}^{x} \rho u(y) \, dy.
\]

Using the momentum equation (1.12), we have

\[
\xi_{tx} + (\rho u^2)_x = (\mu(\rho) u_x)_x - p_x.
\]

Integrating with respect to \(x\) over \((-\infty, x)\) yields

\[
\xi_t + \rho u^2 - \mu(\rho) u_x + p = 0,
\]

(2.4)

Using the mass equation (1.11), we rewrite (2.4) as

\[
\xi_t + \rho u^2 + \mu(\rho) \frac{\rho_x + u \rho_x}{\rho} + p = 0.
\]

(2.5)

Let \(X(t, x)\) be the particle trajectory defined by

\[
\begin{cases}
\frac{dX(t, x)}{dt} = u(X(t, x), t), \\
X(0, x) = x.
\end{cases}
\]

Then

\[
\frac{d\xi}{dt}(X(t, x), t) = \xi_t + u \xi_x = \xi_t + \rho u^2.
\]

(2.6)

Denote

\[
\eta(\rho) = \int_{1}^{\rho} \frac{\mu(s)}{s} \, ds = \begin{cases} 
\ln \rho + \frac{1}{\beta} (\rho^\beta - 1), & \text{if } \beta > 0, \\
2 \ln \rho, & \text{if } \beta = 0.
\end{cases}
\]

It follows from (2.5) and (2.6) that

\[
\frac{d}{dt}(\xi + \eta)(X(t, x), t) \leq \frac{d}{dt}(\xi + \eta)(X(t, x), t) + p(X(t, x), t) = 0.
\]

(2.7)

Integrating (2.7) over \((0, t)\), we have

\[
(\xi + \eta)(X(t, x), t) \leq \xi(X(0, x), 0) + \eta(X(0, x), 0).
\]
Since
\[ \xi(X(0, x), 0) = \int_{-\infty}^{X((0, x), 0)} \rho_0 u_0(y)dy \leq \int_{\mathbb{R}} \rho_0 u_0 dy \leq \| \sqrt{\rho_0 u_0} \|_{L^2} \| \rho_0 \|_{L^1}^{1/2} \leq C, \]
\[ \eta(X(0, x), 0) = \int_{1}^{\rho_0} \frac{\mu(s)}{s} ds = \begin{cases} \ln \rho_0 + \frac{1}{\beta}(\rho_0^\beta - 1), & \text{if } \beta > 0 \\ 2 \ln \rho_0, & \text{if } \beta = 0 \\ \rho_0 + \frac{1}{\beta}(\rho_0^\beta - 1), & \text{if } \beta > 0 \\ 2 \rho_0, & \text{if } \beta = 0 \end{cases}, \]
we have
\[ \xi(x, t) + \eta(x, t) \leq C, \]
which implies
\[ \ln \rho + \frac{1}{\beta}(\rho^\beta - 1) \leq C - \int_{-\infty}^{x} \rho u dx \leq C + \int_{\mathbb{R}} |\rho u| dx \leq C + \| \sqrt{\rho u} \|_{L^2} \| \rho \|_{L^1} \leq C, \beta > 0, \]
or
\[ 2 \ln \rho \leq C - \int_{-\infty}^{x} \rho u dx \leq C + \int_{\mathbb{R}} |\rho u| dx \leq C + \| \sqrt{\rho u} \|_{L^2} \| \rho \|_{L^1} \leq C, \beta = 0. \]
Then we have
\[ \ln \rho \leq \begin{cases} C + \frac{1}{\beta}, & \text{if } \beta > 0 \\ C, & \text{if } \beta = 0 \end{cases}. \]
Consequently
\[ \rho(x, t) \leq C, \beta \geq 0. \]
The proof of the lemma is completed. \qed

2.2. The estimates of the first derivatives.

The first derivative estimates of the velocity is as follows.

**Lemma 2.4.** Suppose that \((\rho, u)\) is a smooth solution to (1.1) – (1.4). Then for any \(T > 0\), it holds
\[ \int_{\mathbb{R}} (u_x)^2 dx + \int_{0}^{T} \int_{\mathbb{R}} \rho u_x^2 dx dt + \int_{0}^{T} \| u_x \|_{L^\infty}^2 dt \leq C(T). \]

**Proof.** Using (1.1)1, we rewrite (1.1)2 as
\[ \rho u_t + \rho u u_x + (\rho^\gamma)_x = (\mu(\rho) u_x)_x. \] (2.8)
Multiplying on both sides of (2.8) by \( u_t \), integrating over \( \mathbb{R} \), we have
\[
\int_{\mathbb{R}} \rho u_t^2 \, dx + \frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}} \mu(\rho) u_x^2 \, dx = \frac{1}{2} \int_{\mathbb{R}} (\rho^\beta)_t u_x^2 \, dx - \int_{\mathbb{R}} \rho u_t u u_x \, dx - \int_{\mathbb{R}} (\rho^\gamma)_x u_t \, dx \\
\leq \frac{d}{dt} \int_{\mathbb{R}} \rho^\gamma u_x \, dx + \gamma \int_{\mathbb{R}} \rho^\gamma - 1 (\rho u_x + u \rho_x) u_x \, dx - \frac{1}{2} \int_{\mathbb{R}} (u(\rho^\beta)_x + \beta \rho^\beta u_x) u_x^2 \, dx \\
+ \frac{1}{2} \int_{\mathbb{R}} \rho u_t^2 \, dx + C \int_{\mathbb{R}} \rho u^2 u_x^2 \, dx.
\]
It yields
\[
\frac{1}{2} \int_{\mathbb{R}} \rho u_t^2 \, dx + \frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}} \mu(\rho) u_x^2 \, dx \leq \frac{d}{dt} \int_{\mathbb{R}} \rho^\gamma u_x \, dx + \gamma \int_{\mathbb{R}} \rho^\gamma - 1 (\rho u_x + u \rho_x) u_x \, dx \\
- \frac{1}{2} \int_{\mathbb{R}} (u(\rho^\beta)_x + \beta \rho^\beta u_x) u_x^2 \, dx + C \int_{\mathbb{R}} \rho u^2 u_x^2 \, dx \quad (2.9)
\]
where we used the following equation
\[
(\rho^\beta)_t + u(\rho^\beta)_x + \beta \rho^\beta u_x = 0, \quad \beta \geq 0.
\]
Then we estimate the terms \( I_1 - I_2 \) as follows.
\[
I_1 = \gamma \int_{\mathbb{R}} \rho^\gamma u_x^2 \, dx + \gamma \int_{\mathbb{R}} \rho^\gamma - 1 \rho_x u u_x \, dx \\
\leq C \|\rho\|_{L^\infty}^\gamma \int_{\mathbb{R}} u_x^2 \, dx + \gamma \int_{0}^{\rho} \frac{s^{\gamma - 1}}{\mu(s)} \mu(\rho) u_x - \rho^\gamma \, ds \, dx + \gamma \int_{\mathbb{R}} \left( \int_{0}^{\rho} \frac{s^{\gamma - 1}}{\mu(s)} ds \right) u(\rho u_t + \rho u u_x) \, dx \\
\leq C \int_{\mathbb{R}} \mu(\rho) u_x^2 \, dx + C(T) - \gamma \int_{0}^{\rho} \frac{s^{\gamma - 1}}{\mu(s)} ds u(\rho u_t + \rho u u_x) \, dx \\
+ C \int_{0}^{\rho} (s^{2\gamma - 1} ds) |u|_x \, dx \\
\leq C \int_{\mathbb{R}} \mu(\rho) u_x^2 \, dx + \epsilon \int_{\mathbb{R}} \rho u_t^2 \, dx + C \int_{\mathbb{R}} \rho u^2 u_x^2 \, dx + C(T). \quad (2.10)
\]
\[
I_2 = -\frac{1}{2} \int_{\mathbb{R}} (u(\rho^\beta)_x + \beta \rho^\beta u_x) u_x^2 \, dx = -\frac{1}{2} \int_{\mathbb{R}} \beta \rho^\beta u_x^2 \, dx - \frac{1}{2} \int_{\mathbb{R}} \beta \rho^\beta - 1 \rho_x u u_x^2 \, dx \\
= I_{21} + I_{22}. \quad (2.11)
\]
Direct estimates give
\[
I_{21} = -\frac{1}{2} \int_{\mathbb{R}} \beta \rho^\beta u_x^2 \, dx \leq C \|u_x\|_{L^\infty} \int_{\mathbb{R}} \mu(\rho) u_x^2 \, dx \quad (2.12)
\]
\[ I_{22} = -\frac{1}{2} \int_{\mathbb{R}} \beta \rho^{\beta - 1} \rho_x u_x^2 dx \]
\[ = -\frac{1}{2} \int_{\mathbb{R}} \frac{\beta \rho^{\beta - 1} \rho_x u(\mu(\rho) u_x - \rho)}{(\mu(\rho))^2} dx - \int_{\mathbb{R}} \frac{\beta \rho^{\beta - 1} \rho_x u(\mu(\rho)) u_x \rho \gamma dx}{(\mu(\rho))^2} \]
\[ + \frac{1}{2} \int_{\mathbb{R}} \beta \rho^{2\gamma + \beta - 1} \rho_x u dx \]
\[ \leq -\frac{1}{2} \int_{\mathbb{R}} (\int_{0}^{\rho} \frac{\beta s^{\beta - 1}}{(\mu(s))^2} ds \rho_x u(\mu(s) u_x - \rho) dx + 2 u(\mu(s) u_x - \rho)(\mu(s) u_x - \rho) \gamma dx) \]
\[ + \int_{\mathbb{R}} (\int_{0}^{\rho} \frac{\beta s^{\gamma + \beta - 1}}{(\mu(s))^2} ds \rho_x u(\mu(s) u_x - \rho) + u(\mu(s) u_x - \rho) \gamma dx) \]
\[ + \frac{1}{2} \int_{\mathbb{R}} (\int_{0}^{\rho} \frac{\beta s^{2\gamma + \beta - 1}}{(\mu(s))^2} ds \rho_x u dx \]
\[ \leq C(T) + C \int_{\mathbb{R}} \mu(s) u_x^2 dx - (\int_{\mathbb{R}} (\int_{0}^{\rho} \frac{\beta s^{\beta - 1}}{(\mu(s))^2} ds \rho_x u(\mu(s) u_x - \rho) \rho u_t + \rho u u_x) dx \]
\[ - \int_{\mathbb{R}} (\int_{0}^{\rho} \frac{\beta s^{\gamma + \beta - 1}}{(\mu(s))^2} ds \rho_x u dx + \frac{1}{2} \int_{\mathbb{R}} \frac{1}{2\gamma + \beta} \rho^{2\gamma + \beta} u x dx \]
\[ \leq C(T) + C \int_{\mathbb{R}} \mu(s) u_x^2 dx + \varepsilon \int_{\mathbb{R}} \rho u_t^2 dx + C \int_{\mathbb{R}} \rho u^2 dx + C \int_{\mathbb{R}} \rho^2 u_x^2 dx. \]

Using the Gagliardo-Nirenberg inequality, we get
\[ \|u_x\|_{L^\infty} \leq \|\mu(s) u_x - \rho\|_{L^\infty} + \|\rho\|_{L^\infty} \leq C\|\mu(s) u_x - \rho\|_{L^\infty} + C \]
\[ \leq \|\mu(s) u_x - \rho\|_{L^2}^{\frac{1}{2}} \|\mu(s) u_x - \rho\|_{L^\infty}^{\frac{1}{2}} \|ho\|_{L^2} + C \]
\[ \leq C(\|\sqrt{\mu(s)} u_x\|_{L^2} + 1)^{\frac{1}{2}} \|\sqrt{\rho u_t}\|_{L^2} + \|u_x\|_{L^\infty} \|\sqrt{\rho u}\|_{L^2} + C \]
\[ \leq C(\|\sqrt{\mu(s)} u_x\|_{L^2} + 1)^{\frac{1}{2}} \|\sqrt{\rho u_t}\|_{L^2} + C(\|\sqrt{\mu(s)} u_x\|_{L^2} + 1)^{\frac{1}{2}} \|u_x\|_{L^\infty} + C \]
\[ \leq \frac{1}{2} \|u_x\|_{L^\infty} + C \|\sqrt{\mu(s)} u_x\|_{L^2} + \varepsilon \|\sqrt{\rho u_t}\|_{L^2}. \]

It concludes that
\[ \|u_x\|_{L^\infty} \leq C(\|\sqrt{\mu(s)} u_x\|_{L^2} + \varepsilon \|\sqrt{\rho u_t}\|_{L^2}) + C. \]

It follows that
\[ \int_{\mathbb{R}} \rho u^2 u_x dx \leq C \|u_x\|_{L^\infty} \int_{\mathbb{R}} \rho u^2 dx \leq C \|u_x\|_{L^\infty} \leq C(\|\sqrt{\mu(s)} u_x\|_{L^2} + \varepsilon \|\sqrt{\rho u_t}\|_{L^2} + C(T), \]

\[ (2.15) \]
and
\[ \int_{\mathbb{R}} \rho u_x^2 \, dx \leq \| u_x \|_{L^\infty}^2 \int_{\mathbb{R}} \rho u^2 \, dx \leq C \| \sqrt{\mu(\rho)} u_x \|_{L^2}^2 + \| \rho u_t \|_{L^2} + C(T). \quad (2.16) \]

Substituting (2.10) − (2.16) into (2.9), we get
\[ \frac{d}{dt} \int_{\mathbb{R}} \mu(\rho) u_x^2 \, dx + \int_{\mathbb{R}} \rho u_x u_t \, dx \leq \frac{d}{dt} \int_{\mathbb{R}} \rho^\gamma u_x \, dx + C(\int_{\mathbb{R}} \mu(\rho) u_x^2 \, dx)^2 + C(T). \quad (2.17) \]

Integrating (2.17) over \([0, T]\) and using the Cauchy inequality and Gronwall inequality, we have
\[ \int_{\mathbb{R}} \mu(\rho) u_x^2 \, dx + \int_0^T \int_{\mathbb{R}} \rho u_x^2 \, dx \, dt \leq C(T). \]

Using (2.14) again yields
\[ \int_0^T \| u_{xx} \|_{L^\infty} \, dt \leq C(T). \]

The proof of the lemma is completed. \( \square \)

Next we show the first derivative estimates of the density.

**Lemma 2.5.** Let \((\rho, u)\) be the smooth solution to (1.1) − (1.4). Then for any \(T > 0\), it holds
\[ \int_{\mathbb{R}} \left( \rho_x^2 + (\rho^\gamma)_x^2 + (\rho^\beta)_x^2 \right) \, dx \leq C(T). \]

**Proof.** Differentiating (1.1) with respect to \(x\), multiplying the resulting equation by \(\rho_x\) and integrating over \(\mathbb{R}\), we have
\[ \frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}} \rho_x^2 \, dx = \frac{3}{2} \int_{\mathbb{R}} \rho_x u_x \, dx - \int_{\mathbb{R}} \rho u_{xx} \rho_x \, dx \leq C \| u_x \|_{L^\infty} \int_{\mathbb{R}} \rho_x^2 + C \| \rho \|_{L^\infty} \| u_{xx} \|_{L^2} \| \rho_x \|_{L^2}. \quad (2.18) \]

Since
\[ \rho u_t + \rho u u_x + (\rho^\gamma)_x = (\mu(\rho) u_x)_x, \]
combining (2.14) and Lemma 2.3 we have
\[ \| u_{xx} \|_{L^2} \leq C(\| \rho u_t \|_{L^2} + \| \rho u u_x \|_{L^2} + \| (\rho^\gamma)_x \|_{L^2} + \| (\rho^\beta)_x u_x \|_{L^2}) \leq C(\| \sqrt{\rho} u_t \|_{L^2} + \| u_x \|_{L^\infty} + \| (\rho^\gamma)_x \|_{L^2} + \| u_x \|_{L^\infty} \| (\rho^\beta)_x \|_{L^2}) \leq C(\| \sqrt{\rho} u_t \|_{L^2} + 1 + \| (\rho^\gamma)_x \|_{L^2} + (\| \sqrt{\rho} u_t \|_{L^2} + 1) \| (\rho^\beta)_x \|_{L^2}). \quad (2.19) \]
Putting (2.14), (2.19) into (2.18), we get
\[
\frac{d}{dt} \int_{\mathbb{R}} \rho_x^2 dx \leq C(\|\sqrt{\rho}u_t\|_{L^2}^2 + 1) \int_{\mathbb{R}} \rho_x^2 dx + C \left[ \|\sqrt{\rho}u_t\|_{L^2} + 1 + \|(\rho^\gamma)_x\|_{L^2} \right] \|\rho_x\|_{L^2} \\
\leq C(\|\sqrt{\rho}u_t\|_{L^2}^2 + 1) \int_{\mathbb{R}} \rho_x^2 dx + C(\|(\rho^\gamma)_x\|_{L^2}^2 + \|(\rho^\beta)_x\|_{L^2}^2 + 1) \\
\leq C(\|\sqrt{\rho}u_t\|_{L^2}^2 + 1) \int_{\mathbb{R}} \rho_x^2 + (\rho^\gamma)_x^2 + (\rho^\beta)_x^2 dx + C(\|\sqrt{\rho}u_t\|_{L^2}^2 + 1).
\]
(2.20)

Note that
\[
(\rho^\gamma)_t + u(\rho^\gamma)_x + \gamma(\rho^\gamma)u_x = 0
\]
and
\[
(\rho^\beta)_t + u(\rho^\beta)_x + \beta(\rho^\beta)u_x = 0,
\]
for any $\gamma > 1, \ \beta \geq 0$. We can obtain in a similar way that
\[
\frac{d}{dt} \int_{\mathbb{R}} (\rho^\gamma)_x^2 + (\rho^\beta)_x^2 dx \\
\leq C(\|\sqrt{\rho}u_t\|_{L^2}^2 + 1) \int_{\mathbb{R}} \rho_x^2 + (\rho^\gamma)_x^2 + (\rho^\beta)_x^2 dx + C(\|\sqrt{\rho}u_t\|_{L^2}^2 + 1). 
\]
(2.21)

Substituting (2.21) into (2.20), using Lemma 2.4 and the Gronwall inequality, we have
\[
\int_{\mathbb{R}} \left( \rho_x^2 + (\rho^\gamma)_x^2 + (\rho^\beta)_x^2 \right) dx \leq C(T).
\]
The proof of the lemma is completed. \(\square\)

2.3. Weighted energy estimates.
In this subsection, we will establish the weighted energy estimates. These will be used in estimates of the higher derivatives of the velocity.

**Lemma 2.6.** Let $(\rho, u)$ be the smooth solution to (1.11) – (1.14). Then for any $T > 0$ and $\alpha > 0$, it holds
\[
\int_{\mathbb{R}} \rho |u|^\alpha dx + \int_0^T \int_{\mathbb{R}} \mu(\rho)u_x^2 |u|^{\alpha} dx dt \leq C(T).
\]
Proof. For any \(\alpha > 0\), multiplying \((1.1)_2\) by \((\alpha + 2)|u|^\alpha u\) and integrating with respect to \(x\) over \(\mathbb{R}\) yields that

\[
\frac{d}{dt} \int_{\mathbb{R}} \rho|u|^\alpha u^2 \, dx + (\alpha + 2)(\alpha + 1) \int_{\mathbb{R}} \mu(\rho)|u|^\alpha u_x^2 \, dx \\
= \int_{\mathbb{R}} \rho_t |u|^\alpha u^2 \, dx - (\alpha + 2) \int_{\mathbb{R}} \rho |u|^\alpha u_x^2 \, dx - \int_{\mathbb{R}} (\rho^\gamma)_x (\alpha + 2) |u|^\alpha u^2 \, dx \\
= (\alpha + 2)(\alpha + 1) \int_{\mathbb{R}} \rho^\gamma |u|^\alpha u_x^2 \, dx \\
= (\alpha + 2)(\alpha + 1) \int_{\mathbb{R}} (|u|^2 u_x)(\rho |u|^2) \frac{\alpha}{2(\alpha + 2)} \rho^{- \frac{\alpha}{2(\alpha + 2)}} \, dx \\
\leq \varepsilon \int_{\mathbb{R}} |u|^\alpha u_x^2 \, dx + C \int_{\mathbb{R}} \rho |u|^\alpha u^2 \, dx + C.
\]

Use the Gronwall inequality to get

\[
\int_{\mathbb{R}} \rho|u|^\alpha u^2 \, dx + \int_0^T \int_{\mathbb{R}} \mu(\rho) u_x^2 |u|^\alpha dx \, dt \leq C(T).
\]

The proof of the lemma is completed. \(\Box\)

**Lemma 2.7.** Let \((\rho, u)\) be the smooth solution to \((1.1) - (1.4)\). Then for any \(T > 0\) and \(2 < \alpha < 1 + \frac{2}{3 + \sqrt{3}}\), it holds

\[
\int_{\mathbb{R}} |x|^\alpha (\rho u^2 + \rho^\gamma + \rho^\beta) \, dx + \int_0^T \int_{\mathbb{R}} |x|^\alpha \mu(\rho) u_x^2 \, dx \, dt \leq C(T).
\]

*Proof.* Multiplying \((1.1)_2\) by \(|x|^\alpha u\) and integrating with respect to \(x\) over \(\mathbb{R}\) yields that

\[
\frac{d}{dt} \int_{\mathbb{R}} |x|\alpha (\frac{1}{2} \rho u^2 + \frac{1}{\gamma - 1} \rho^\gamma + \rho^\beta) \, dx + \int_{\mathbb{R}} \mu(\rho)|x|\alpha u_x^2 \, dx \\
= \frac{1}{2} \int_{\mathbb{R}} \alpha \rho u^3 |x|\alpha - 2 u^2 \, dx + \frac{\gamma \alpha}{\gamma - 1} \int_{\mathbb{R}} \rho^\gamma |x|\alpha - 2 u^2 \, dx - \alpha \int_{\mathbb{R}} \mu(\rho)|x|\alpha - 2 u u_x \, dx \\
- (\beta - 1) \int_{\mathbb{R}} |x|^\alpha \rho^\beta u_x \, dx + \alpha \int_{\mathbb{R}} \rho^\beta |x|\alpha - 2 u x \, dx \\
\equiv \sum_{i=1}^5 J_i.
\]

The terms \(J_i (i = 1, 2 \cdots 5)\) on the right side of \((2.22)\) are estimated as follows. By the Hölder inequality, Caffarelli-Kohn-Nirenberg weighted inequality and Young
inequality, it holds that
\[
J_1 \leq C \int_{\mathbb{R}} \rho u^3 |x|^\alpha dx = C \int_{\mathbb{R}} (\rho u^2 |x|^\alpha)^{\frac{\alpha - 1}{\alpha}} \rho^{\frac{1}{\alpha}} u^{3 - 2(\frac{\alpha - 1}{\alpha})} dx
\]
\[
= C \int_{\mathbb{R}} (\rho u^2 |x|^\alpha)^{\frac{\alpha - 1}{\alpha}} (\rho u^{\alpha + 2})^{\frac{1}{\alpha}} dx
\]
\[
\leq ||x|^\frac{\alpha}{2} \sqrt{\rho u}||_{L^2}^{2(\frac{\alpha - 1}{\alpha})} ||\sqrt{\rho} u^{\frac{\alpha}{2} + 1}||_{L^2}^{\frac{2}{\alpha}}
\]
\[
\leq C \int_{\mathbb{R}} |x|^\alpha \rho u^2 dx + C(T),
\]
(2.23)

\[J_2 \] can be estimated as
\[
J_2 \leq C \int_{\mathbb{R}} \rho^2 |u||x|^\alpha dx = C \int_{\mathbb{R}} (\sqrt{\rho^2} |x|^{\frac{\alpha}{2}})(|u||x|^{\frac{\alpha}{2} - 1}) \sqrt{\rho^2} dx
\]
\[
\leq C \left( \int_{\mathbb{R}} \rho^\gamma |x|^\alpha dx \right)^{\frac{1}{2}} \rho ||x|^{\frac{\alpha}{2} - 1} u ||_{L^2}
\]
\[
\leq C \left( \int_{\mathbb{R}} \rho^\gamma |x|^\alpha dx \right)^{\frac{1}{2}} ||u_x||_{L^2} ||x|^{\frac{\alpha}{2} - 1} ||_{L^2}
\]
\[
\leq \varepsilon \int_{\mathbb{R}} \mu(\rho)|x|^\alpha u_x^2 dx + C \int_{\mathbb{R}} \rho^\gamma |x|^\alpha dx,
\]
(2.24)

where the index \(\alpha\) satisfies
\[
\frac{1}{2} + \frac{\frac{\alpha}{2} - 1}{1} > 0 \implies \alpha > 1.
\]

\[J_3 \] can be rewritten as
\[
J_3 = -\alpha \int_{\mathbb{R}} \mu(\rho)|x|^\alpha - 2 x uu_x dx
\]
\[
= -\alpha \int_{\mathbb{R}} |x|^\alpha - 2 x uu_x dx - \alpha \int_{\mathbb{R}} \rho^\beta |x|^\alpha - 2 x uu_x dx
\]
\[
\equiv J_{31} + J_{32}.
\]
(2.25)

Direct estimates give
\[
J_{31} = -\alpha \int_{\mathbb{R}} |x|^\alpha - 2 x uu_x dx = -\alpha \int_{\mathbb{R}} \frac{1}{2} |x|^\alpha - 2 x (u^2)_x dx
\]
\[
= \frac{\alpha}{2} \int_{\mathbb{R}} u^2 \left( (\alpha - 2)|x|^\alpha - 2 \frac{x}{|x|} x + |x|^{\alpha - 2} \right) dx
\]
\[
= \frac{\alpha}{2} \int_{\mathbb{R}} u^2 \left( (\alpha - 1)|x|^{\alpha - 2} \right) dx \leq \frac{\alpha(\alpha - 1)}{2} ||x|^{\frac{\alpha}{2} - 1} u ||_{L^2}^2
\]
\[
\leq \frac{\alpha(\alpha - 1)^3}{8} ||x|^{\frac{\alpha}{2}} u_x ||_{L^2}^2.
\]
(2.26)

The weight index \(\alpha > 0\) will be chosen to satisfy
\[
\frac{\alpha(\alpha - 1)^3}{8} < 1.
\]
(2.27)
$J_{32}$ is estimated as

$$J_{32} \leq C \int \rho^3 |x|^\alpha - 1 |u| |u_x| dx = C \int \rho^3 \frac{|x|^{\frac{\alpha}{\beta} - 1} |u| |x|^{\frac{\alpha}{\beta}} |u_x|}{|x|^{\alpha - 1} |u|} \rho^\beta \frac{\hat{\omega}_n}{\hat{\omega}_n} dx$$

$$\leq C \| |x|^{\frac{\alpha}{\beta} - 1} u \|_{L^3} \| |x|^{\frac{\alpha}{\beta}} u_x \|_{L^2} \| \rho^\beta |x|^\alpha \|_{L^1}^\frac{1}{\beta}$$

$$\leq C \| u_x \|_{L^2}^{1-\theta} \| |x|^{\frac{\alpha}{\beta}} u_x \|_{L^2}^\theta \| |x|^{\frac{\alpha}{\beta}} u_x \|_{L^2} \| \rho^\beta |x|^\alpha \|_{L^1}^\frac{1}{\beta}$$

$$\leq C \| |x|^{\frac{\alpha}{\beta}} u_x \|_{L^2}^{1+\theta} \| \rho^\beta |x|^\alpha \|_{L^1}^\frac{1}{\beta}$$

$$\leq \varepsilon \| |x|^{\frac{\alpha}{\beta}} u_x \|_{L^2}^2 + C \| \rho^\beta |x|^\alpha \|_{L^1} + C(T),$$

where $\theta \in (0, 1)$ and $\alpha > 1$ are chosen to satisfy

$$\frac{1}{3} + \frac{\alpha}{3} - 1 = (\frac{1}{2} - 1)(1 - \theta) + (\frac{1}{2} + \frac{\alpha}{2} - 1)\theta, \quad \frac{1}{3} + \frac{\alpha}{3} - 1 > 0,$$

which implies

$$\theta = \frac{2\alpha - 1}{3\alpha} < \frac{2}{3}, \quad \alpha > 2. \quad (2.29)$$

By (2.27) and (2.29), we first choose $\alpha$ as

$$(\alpha - 1)^3 < \frac{8}{\alpha} < 4 \implies \alpha < 1 + \sqrt[3]{4}.$$  

Then, to guarantee (2.27), we impose

$$\frac{\alpha(\alpha - 1)^3}{8} < \frac{(1 + \sqrt[3]{4})(\alpha - 1)^3}{8} < 1,$$

which implies that

$$(\alpha - 1)^3 < \frac{8}{1 + \sqrt[3]{4}} \implies \alpha < 1 + \frac{2}{\sqrt[3]{1 + \sqrt[3]{4}}}. \quad (2.30)$$

Combining (2.29) and (2.30), the index $\alpha$ is chosen to satisfy

$$2 < \alpha < 1 + \frac{2}{\sqrt[3]{1 + \sqrt[3]{4}}}. \quad (2.31)$$

Concerning $J_4$ and $J_5$, we have

$$J_4 + J_5 \leq C \int |x|^{\alpha} \rho^3 |u_x| dx + C \int \rho^3 |x|^{\alpha - 1} |u| dx$$

$$\leq C \left( \int |x|^{\alpha} \rho^3 dx \right)^{\frac{1}{2}} |x|^{\frac{\alpha}{\beta}} u_x \|_{L^2} + C \left( \int |x|^{\alpha} \rho^3 dx \right)^{\frac{1}{2}} |x|^{\frac{\alpha}{\beta} - 1} \|_{L^2}$$

$$\leq \varepsilon \| |x|^{\frac{\alpha}{\beta}} u_x \|_{L^2}^2 + C \int \rho^\beta |x|^\alpha dx.$$  

Substituting (2.23) – (2.32) into (2.22) and applying the Gronwall inequality lead to

$$\int |x|^\alpha (\rho u^2 + \rho^\gamma + \rho^3) dx + \int_0^T \int |x|^\alpha \mu(\rho) u_x^2 dx dt \leq C(T).$$

The proof of the lemma is completed. \square
2.4. Estimates of the higher order derivatives.

**Lemma 2.8.** Let \((\rho, u)\) be the smooth solution to (1.1) – (1.4). Then for any \(T > 0\), it holds
\[
\int_{\mathbb{R}} \rho(u_t)^2 dx + \int_0^T \int_{\mathbb{R}} \mu(\rho)(u_{tx})^2 dx dt + \int_0^T \int_{\mathbb{R}} \left( \rho_i^2 + (\rho^\gamma)_t^2 + (\rho^\beta)_t^2 \right) dx dt \leq C(T).
\]

**Proof.** Differentiating (1.1) with respect to \(t\), we have
\[
\rho u_{tt} + \rho_t u_t + \rho_t \rho_{xx} + \rho u_{tx} + \rho u_{xt} + (\rho^\gamma)_{xx} = (\mu(\rho)u_{xt} + (\rho^\beta)_t u_x)_x.
\]
Multiplying on both sides of (2.33) by \(u_t\), integrating over \(\mathbb{R}\) and using (1.1), we have
\[
\frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}} \rho u_t^2 dx + \int_{\mathbb{R}} \mu(\rho)u_t^2 dx = -\int_{\mathbb{R}} \rho_l uu_x u_t dx - \int_{\mathbb{R}} \rho u_t^2 u_x dx - \int_{\mathbb{R}} (\rho^\gamma)_t u_t dx
\]
\[
- \int_{\mathbb{R}} (\rho^\beta)_t u_x u_{xt} dx - 2 \int_{\mathbb{R}} \rho uu_{tx} dx \equiv K_1 + K_2 + K_3 + K_4 + K_5.
\]
Now we estimate the terms \(K_1 - K_5\) as follows
\[
K_1 + K_2 = -\int_{\mathbb{R}} \rho uu_x u_t dx - \int_{\mathbb{R}} \rho u_t^2 u_x dx = \int_{\mathbb{R}} (\rho u)_x uu_x u_t dx - \int_{\mathbb{R}} \rho u_t^2 u_x dx
\]
\[
= -\int_{\mathbb{R}} \rho (u_x^2 u_t + uu_{xx} u_t + uu_x u_{xt}) dx - \int_{\mathbb{R}} \rho u_t^2 u_x dx
\]
\[
\leq \int_{\mathbb{R}} (\sqrt[\rho]{|u_t|})(\sqrt{\rho}|u|)|u_x|^2 dx + \int_{\mathbb{R}} (\sqrt[\rho]{|u_t|})(\sqrt{\rho}u^2)|u_{xx}| dx + \int_{\mathbb{R}} (\sqrt{\rho}u)|u_x| (\sqrt{\rho}u_{xx}) dx
\]
\[
\leq C\|u_x\|_{L^\infty}\|\sqrt{\rho}u_t\|_{L^2}\|\sqrt{\rho}u\|_{L^2} + C\|u\|_{L^\infty}\|\sqrt{\rho}u_t\|_{L^2}\|u_{xx}\|_{L^2}
\]
\[
+ C\|u\|_{L^\infty}\|u_x\|_{L^\infty}\|u_{tx}\|_{L^2}\|\sqrt{\rho}u\|_{L^2} + C\|u_x\|_{L^\infty}\int_{\mathbb{R}} \rho u_t^2 dx
\]
\[
\leq C\|\sqrt{\rho}u_t\|_{L^2}^2 + C\|\sqrt{\rho}u_t\|_{L^2} + C\|u\|_{L^\infty}\|\sqrt{\rho}u_t\|_{L^2}\|\sqrt{\rho}u_t\|_{L^2} + 1)
\]
\[
+ C\|\sqrt{\rho}u_t\|_{L^2} + 1\|\sqrt{\rho}u_t\|_{L^2} + 1)\|u_{xt}\|_{L^2} + C\|\sqrt{\rho}u_t\|_{L^2} + 1)\int_{\mathbb{R}} \rho u_t^2 dx
\]
\[
\leq \varepsilon \int_{\mathbb{R}} \mu(\rho)u_{tx}^2 dx + C(\|u_x\|_{L^2}^2 + \|\sqrt{\rho}u_t\|_{L^2}^2 + \int_{\mathbb{R}} \rho u_t^2 dx + 1)\int_{\mathbb{R}} \rho u_t^2 dx + C(\|u_x\|_{L^2}^2 + 1),
\]
where we used the fact
\[
\|u_x\|_{L^\infty} + \|u_{xx}\|_{L^2} \leq C(\|\sqrt{\rho}u_t\|_{L^2} + 1),
\]
which follows from (2.14) and (2.19).

Note that
\[
\|u\|_{L^\infty} \leq \|u\|_{L^{\infty, \frac{2}{\gamma + 1}}} u_x \|_{L^\infty}^{1 - \frac{\gamma}{\gamma + 1}} \leq C(\|u\|_{L^{\infty, \frac{2}{\gamma + 1}}} + \|u_x\|_{L^2})
\]
\[
\leq C(\|u_x\|_{L^{2, \frac{2}{\gamma + 1}}}),
\]
(2.36)
Lemma 2.9. Let $(\rho, u)$ be the smooth solution to \((\text{1.1}) - (\text{1.4})\). Then for any $T > 0$, it holds
\[
\|u_x\|_{L^\infty}(t) + \int_0^T (u_{xx})^2 dx \leq C(T).
\]

Proof. Applying \((\text{2.19})\), Lemma 2.5 and Lemma 2.8, we have
\[
\|u_{xx}\|_{L^2} \leq C \left[ \|\sqrt{\rho u_t}\|_{L^2} + 1 + \|\rho^\gamma\|_{L^2} + \|\sqrt{\rho u_t}\|_{L^2} + (\|\sqrt{\rho u_t}\|_{L^2} + 1)\|\rho^\beta\|_{L^2} \right] \leq C(T),
\]
and

\[ \| u_x \|_{L^\infty} \leq C(\| u_x \|_{L^2} + \| u_{xx} \|_{L^2}) \leq C(T). \]

Then we complete the proof of the lemma.

\[ \square \]

Inspired by Hoff [8], we estimate the weighted material derivative \( \dot{u} = (\partial_t + u \partial_x) u \).

**Lemma 2.10.** Let \((\rho, u)\) be the smooth solution to (1.1) – (1.4). Then for any \( T > 0 \) and \( 2 < \alpha < 1 + \frac{2}{\sqrt{1 + \sqrt{4}}} \), it holds

\[ \int R \rho \dot{u} \alpha |x|^\alpha dx + \int_0^T \int R (1 + |x|^\alpha) \mu(\rho) \dot{u}_x^2 dx dt + \int_0^T \| \dot{u} \|_{L^\frac{2}{\alpha-1}}^2 dt \leq C(T). \]

**Proof.** Firstly, applying \( \partial_t + \partial_x (u \cdot) \) to equation (1.1) gives

\[ \rho \dot{u}_t + \rho \dot{u} u_x - (\mu(\rho) \dot{u}_x)_x = (\gamma p u - \mu(\rho) u^2_x - \beta \rho^2 u^2 x)_x. \quad (2.41) \]

Multiplying on both sides of (2.41) by \( |x|^\alpha \dot{u} \) and integrating over \( \mathbb{R} \), we have

\[ \frac{1}{2} \frac{d}{dt} \int R \rho |x|^\alpha \dot{u} dx + \int R \mu(\rho) \dot{u}_x^2 |x|^\alpha dx \]

\[ = \frac{1}{2} \alpha \int R \rho |x|^{\alpha-2} x \dot{u}^2 dx - \int R \mu(\rho) \dot{u}_x \dot{u} \alpha |x|^\alpha dx \]

\[ - \int R (\gamma p u - \mu(\rho) u^2_x - \beta \rho^2 u^2 x)(|x|^\alpha \dot{u}_x + \alpha |x|^{\alpha-2} x \dot{u}) dx \]

\[ \equiv L_1 + L_2 + L_3. \quad (2.42) \]

Direct estimates give

\[ L_1 \leq \frac{1}{2} \alpha \int R \rho |x|^{\alpha-2} x \ddot{u}^2 dx \leq \frac{\alpha}{2} \int R \rho |u| |x|^{\alpha-1} \dddot{u}^2 dx = \frac{\alpha}{2} \int R (\rho \dddot{u}^2 |x|^{\alpha})^{\frac{\alpha-1}{\alpha}} (\rho^{\frac{1}{\alpha}})^{\frac{1}{\alpha}} |u| \alpha dx \]

\[ \leq C \| u \|_{L^\infty} \int R \rho \dddot{u}^2 |x|^\alpha dx + C \| u \|_{L^\infty} \int R \rho \dddot{u}^2 dx \]

\[ \leq C(\| \dddot{u} \|_{L^2} + 1) \int R \rho \dddot{u}^2 |x|^\alpha dx + C(\| \dddot{u} \|_{L^2} + 1)(\int R \rho \dddot{u}^2 dx + \int R \rho \dddot{u}_x^2 dx) \]

\[ \leq C(\| \dddot{u} \|_{L^2} + 1) \int R \rho \dddot{u}^2 |x|^\alpha dx + C(\| \dddot{u} \|_{L^2} + 1), \quad (2.43) \]
Moreover, the restriction (2.31) guarantees that

\[ L_2 \leq | - \int_{\mathbb{R}} \mu(x)|\dot{u}_x \dot{u}_x x^{|\alpha^2}dx| \leq | \int_{\mathbb{R}} \mu(x)| \dot{u}_x \dot{u}_x x^{|\alpha^2}dx + \int_{\mathbb{R}} \rho^\beta \dot{u}_x \dot{u}_x x^{|\alpha^2}dx | \]

\[ \leq \frac{\alpha(\alpha - 1)}{2} \int_{\mathbb{R}} u^2 x^{|\alpha^2}dx \leq C(\int_{\mathbb{R}} (|x|^\frac{\alpha}{2} \dot{u}_x)(|x|^\frac{\alpha}{2} - 1 ) \dot{u}_x x^{|\alpha^2}dx) \]

\[ \leq \frac{\alpha(\alpha - 1)^3}{8} \| |x|^\frac{\alpha}{2} \dot{u}_x \|_{L^2}^2 + C \| |x|^\frac{\alpha}{2} - 1 \dot{u}_x \|_{L^2} \| |x|^\frac{\alpha}{2} \dot{u}_x \|_{L^2} \| \rho^\beta \|_{L^1} \]

\[ \leq \frac{\alpha(\alpha - 1)^3}{8} \| |x|^\frac{\alpha}{2} \dot{u}_x \|_{L^2}^2 + C \| \dot{u}_x \|_{L^2}^{\frac{1}{\theta}} \| |x|^\frac{\alpha}{2} \dot{u}_x \|_{L^2}^{\frac{1}{\theta}} \]

\[ \leq \frac{\alpha(\alpha - 1)^3}{8} \| |x|^\frac{\alpha}{2} \dot{u}_x \|_{L^2}^2 + \varepsilon \| |x|^\frac{\alpha}{2} \dot{u}_x \|_{L^2}^2 + C \| \dot{u}_x \|_{L^2}^2. \]

(2.44)

The index \( \alpha \) and \( \theta \) in (2.44) are chosen to satisfy

\[ \frac{1}{3} + \frac{\alpha}{3} - 1 = \left( \frac{1}{2} + \frac{-1}{1} \right)(1 - \theta) + \left( \frac{1}{3} + \frac{\alpha}{3} - 1 \right) \theta, \]

which implies

\[ \theta = \frac{2\alpha - 1}{3\alpha} < \frac{2}{3}. \]

Moreover, the restriction (2.31) guarantees that

\[ \frac{1}{2} + \frac{\alpha}{3} - 1 > 0, \quad \frac{1}{3} + \frac{\alpha}{3} - 1 > 0, \quad \frac{\alpha(\alpha - 1)^3}{8} < 1. \]

Concerning \( L_3 \), we have

\[ L_3 = - \int_{\mathbb{R}} (\gamma p u_x - \mu(x)u^2_x - \beta \rho^\beta u^2_x)(|x|^\alpha \dot{u}_x + \alpha \dot{x}^{|\alpha^2}dx) \]

\[ \leq C \int_{\mathbb{R}} (\gamma p + \mu(x)|u_x| + \beta \rho^\beta |u_x|)(|x|^\frac{\alpha}{2} |u_x|)(|\dot{u}_x| |x|^\frac{\alpha}{2} )dx, \]

\[ \leq C \| |x|^\frac{\alpha}{2} u_x \|_{L^2} \| |x|^\frac{\alpha}{2} \dot{u}_x \|_{L^2} \]

\[ \leq \varepsilon \| |x|^\frac{\alpha}{2} u_x \|_{L^2}^2 + C \| |x|^\frac{\alpha}{2} u_x \|_{L^2}^2 \]

(2.45)

Since

\[ \| \dot{u}_x \|_{L^2} \leq C \| (u_t + uu_x) x \|_{L^2} \leq C \left( \| u_x t \|_{L^2} + \| u^2_x \|_{L^2} + \| uu_{xx} \|_{L^2} \right) \]

\[ \leq C \left( \| u_x t \|_{L^2} + \| u \|_{L^\infty} + 1 \right) \leq C \left( \| u_x t \|_{L^2} + \| |x|^\frac{\alpha}{2} u_x \|_{L^2} + 1 \right), \]

by Lemma 2.7 and Lemma 2.8 we get

\[ \int_{0}^{T} \int_{\mathbb{R}} \dot{u}_x^2 dx dt \leq C(T). \]

(2.46)
Substituting (2.43)-(2.45) into (2.42) and using (2.46), Lemma 2.11 and the Gronwall inequality, we derive
\[
\int_{\mathbb{R}} \rho u^2 |x|^\alpha dx + \int_0^T \int_{\mathbb{R}} \mu(\rho) \dot{u}_x^2 |x|^\alpha dxdt \leq C(T).
\]

It follows from the Caffarelli-Kohn-Nirenberg weighted inequality that
\[
\int_0^T \|u_t\|_{L^{\frac{n+2}{n-2}}}^2 dt \leq \int_0^T \|\dot{u} - uu_x\|_{L^{\frac{n+2}{n-2}}}^2 dt \leq \int_0^T \|\dot{u}\|_{L^{\frac{n+2}{n-2}}}^2 dt + \int_0^T \|u_x\|_{L^\infty}^2 \|u\|_{L^{\frac{n+2}{n-2}}}^2 dt \leq \int_0^T \|\dot{u}\|_{L^\infty}^2 \|u\|_{L^2}^2 dt + C \int_0^T \|\dot{u}\|_{L^2}^2 \|u\|_{L^2}^2 dt \leq C(T).
\]

The proof of the lemma is completed. \(\square\)

Lemma 2.11. Let \((\rho, u)\) be the smooth solution to (1.1) - (1.4). Then for any \(T > 0\), it holds
\[
\int_{\mathbb{R}} (\rho_{xx}^2 + (\rho_x)^2 + (\rho_\gamma)^2_{xx}) dx + \int_0^T \int_{\mathbb{R}} \rho_{xxt}^2 dxdt \leq C(T).
\]

Proof. Differentiating (1.11) with respect to \(x\) twice, we get
\[
(\rho_{xx})_t + 3\rho_{xx} u_x + 3\rho_x u_{xx} + \rho u_{xxx} + u\rho_{xxx} = 0. \tag{2.47}
\]

Multiplying (2.47) by \(\rho_{xx}\), integrating over \(\mathbb{R}\) and using (1.11), we have
\[
\frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}} \rho_{xx}^2 dx = -3 \int_{\mathbb{R}} \rho_{xx}^2 u_x dx - 3 \int_{\mathbb{R}} \rho_x u_{xx} \rho_{xx} dx - \int_{\mathbb{R}} \rho u_{xxx} \rho_{xx} dx - \int_{\mathbb{R}} u\rho_{xxx} \rho_{xx} dx \leq C \|u_x\|_{L^\infty} \int_{\mathbb{R}} \rho_{xx}^2 dx + C \|\rho_x\|_{L^2} \|u_{xx}\|_{L^2} \|\rho_{xx}\|_{L^2}
\]
\[
+ C \|\rho_{xx}\|_{L^2} \|u_{xxx}\|_{L^2} + C \|u_x\|_{L^\infty} \int_{\mathbb{R}} \rho_{xx}^2 dx \leq C \int_{\mathbb{R}} \rho_{xx}^2 dx + C(\|\rho_{xx}\|_{L^2} + 1) \|\rho_{xx}\|_{L^2} + C \|u_{xxx}\|_{L^2} \|\rho_{xx}\|_{L^2}
\]
\[
\leq C \int_{\mathbb{R}} \rho_{xx}^2 dx + C(\|u_{xxx}\|_{L^2}^2 + 1). \tag{2.48}
\]

Similarly, we have
\[
\frac{d}{dt} \int_{\mathbb{R}} ((\rho_\gamma)^2_{xx} + (\rho_\beta)^2_{xx}) dx \leq C \int_{\mathbb{R}} ((\rho_\gamma)^2_{xx} + (\rho_\beta)^2_{xx}) dx + C(\|u_{xxx}\|_{L^2}^2 + 1). \tag{2.49}
\]

Combining (2.48) with (2.49), we obtain
\[
\frac{d}{dt} \int_{\mathbb{R}} (\rho_{xx}^2 + (\rho_\gamma)^2_{xx} + (\rho_\beta)^2_{xx}) dx \leq C \int_{\mathbb{R}} (\rho_{xx}^2 + (\rho_\gamma)^2_{xx} + (\rho_\beta)^2_{xx}) dx + C(\|u_{xxx}\|_{L^2}^2 + 1). \tag{2.50}
\]

Differentiating (1.1) with respect to \(x\), we have
\[
\mu(\rho) u_{xxx} = -2(\rho_\beta)_x u_{xx} - (\rho_\beta)_{xx} u_x + \rho_x u_t + \rho u_{xt} + \rho_x u u_x + \rho u_x^2 + \rho u u_{xx} + (\rho_\gamma)_{xx}, \tag{2.51}
\]
which yields
\[ \|u_{xxx}\|_{L^2} \leq C \left( \left\| (\rho^\beta)_x u_{xx} \right\|_{L^2} + \left\| (\rho^\beta)_x u_x \right\|_{L^2} + \|\rho_x u_t\|_{L^2} + \|\rho u_{tx}\|_{L^2} \right. \\
\left. + \|\rho_x u_{tx}\|_{L^2} + \|\rho u_{ xx}\|_{L^2} + \|\rho u_{ xx}\|_{L^2} + \left\| (\rho^\gamma)_{xx} \right\|_{L^2} \right) \]
\[ \leq \left( \left\| (\rho^\beta)_x \right\|_{L^\infty} \|u_{xx}\|_{L^2} + \|u_x\|_{L^\infty} \left\| (\rho^\beta)_x \right\|_{L^2} + \|u_t\|_{L^\infty} \|\rho_x\|_{L^2} \right. \\
\left. + \|\rho\|_{L^\infty} \|u_{xt}\|_{L^2} + \|u\|_{L^\infty} \|u_x\|_{L^2} \right. \\\n\left. + \|\rho\|_{L^\infty} \|u_{xx}\|_{L^2} + \left\| (\rho^\gamma)_{xx} \right\|_{L^2} \right) \]
\[ \leq C \left( \|\rho_{xx}\|_{L^2} + \left\| (\rho^\beta)_{xx} \right\|_{L^2} + \|x\|^\frac{2}{\alpha} \left\| \frac{\partial }{\partial x} u_{xx} \right\|_{L^2} + \|u_{xt}\|_{L^2} + \left\| x \right\|^\frac{2}{\alpha} \left\| u_{xx} \right\|_{L^2} \right) \]
\[ + 1 + \left\| (\rho^\gamma)_{xx} \right\|_{L^2} \right). \]

Combining (2.50) with (2.52), we get
\[ \frac{d}{dt} \int_\mathbb{R} (\rho^2_{xx} + (\rho^\gamma)^2_{xx} + (\rho^\beta)^2_{xx}) dx \]
\[ \leq C \int_\mathbb{R} (\rho^2_{xx} + (\rho^\gamma)^2_{xx} + (\rho^\beta)^2_{xx}) dx + C \left( \left\| x \right\|_{L^2} \left\| \frac{\partial }{\partial x} u_{xx} \right\|_{L^2} + \|u_{xt}\|_{L^2} + \left\| x \right\|_{L^2} \left\| u_{xx} \right\|_{L^2} + 1 \right) \]

Using Lemmas 2.7, 2.8, 2.10 and the Gronwall inequality, we get
\[ \int_\mathbb{R} (\rho^2_{xx} + (\rho^\gamma)^2_{xx} + (\rho^\beta)^2_{xx}) dx \leq C(T). \]

Furthermore, differentiating (1.11) with respect to \( x \), we have
\[ \rho_{xt} + 2\rho_x u_x + \rho u_{xx} + u_{xx} = 0 \]
which implies
\[ \|\rho_{xx}\|_{L^2} \leq C \left( \|\rho_x u_x\|_{L^2} + \|\rho u_{xx}\|_{L^2} + \|u_{xx}\|_{L^2} \right) \]
\[ \leq C (1 + \|u\|_{L^\infty}) \]
\[ \leq C (1 + \left\| x \right\|_{L^2} \left\| u_{xx} \right\|_{L^2}) \]
\[ (2.53) \]
and
\[ \int_0^T \int_\mathbb{R} \rho^2_{xx} dx dt \leq C(T). \]

The proof of the lemma is completed. \( \square \)

**Lemma 2.12.** Let \((\rho, u)\) be the smooth solution to \((1.1) - (1.4)\). Then for any \( T > 0 \) and \( 2 < \alpha < 1 + \frac{2}{\sqrt{1+\sqrt{4}}} \), it holds
\[ \int_\mathbb{R} \left( \rho^2_t + (\rho^\gamma)^2_t + (\rho^\beta)^2_t + \rho^2_{xt} + (\rho^\gamma)^2_{xt} + (\rho^\beta)^2_{xt} + \mu(\rho)|x|^\alpha u^2_x \right) dx + \int_0^T \int_\mathbb{R} \rho u^2_t |x|^\alpha dx dt \leq C(T). \]
Proof. Rewriting the equation (2.54) as
\[ \rho \dot{u} + (\rho^\gamma)_x = (\mu(\rho)u_x)_x. \] (2.54)
Multiplying on both sides of (2.54) by \(|x|^\alpha \dot{u}\) and integrating on \(\mathbb{R}\) with respect to \(x\), we have
\[
\frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}} \mu(\rho)|x|^\alpha u_x^2 dx + \int_{\mathbb{R}} \rho \dot{u}^2 |x|^\alpha dx
= \frac{1}{2} \int_{\mathbb{R}} (\rho^\beta)_t |x|^\alpha u_x^2 dx - \int_{\mathbb{R}} \mu(\rho)u_x |x|^\alpha (u u_x)_x dx
- \int_{\mathbb{R}} \mu(\rho)u_x \alpha |x|^\alpha u dx - \int_{\mathbb{R}} (\rho^\gamma)_x |x|^\alpha \dot{u} dx
\equiv M_1 + M_2 + M_3 + M_4. \] (2.55)
Next we estimate the terms \(M_1 - M_4\),
\[
M_1 = \frac{1}{2} \int_{\mathbb{R}} (\rho^\beta)_t |x|^\alpha u_x^2 dx \leq C \|(\rho^\beta)_t\|_{L^\infty} \int_{\mathbb{R}} |x|^\alpha u_x^2 dx \leq C(1 + \|\|\| x^{\frac{3}{2}} u_x\|_{L^2}) \int_{\mathbb{R}} |x|^\alpha u_x^2 dx, \] (2.56)
where
\[
\|\rho_t\|_{L^\infty} \leq C(\|\rho_t\|_{L^2} + \|\rho_{xt}\|_{L^2}) \leq C(\|\rho u_x\|_{L^2} + \|\mu u_x\|_{L^2} + \|\rho_{xu}\|_{L^2} + \|\rho_{ux}\|_{L^2} + \|\mu_{ux}\|_{L^2}) \leq C(1 + \|\|\| x^{\frac{3}{2}} u_x\|_{L^2}). \] (2.57)
Similar to (2.57), we get
\[
\|(\rho^\gamma)_t\|_{L^\infty} + \|(\rho^\beta)_t\|_{L^\infty} \leq C(1 + \|\|\| x^{\frac{3}{2}} u_x\|_{L^2}). \] (2.58)
Integration by parts yields
\[
M_2 = -\int_{\mathbb{R}} u_x |x|^\alpha (u u_x)_x dx - \int_{\mathbb{R}} \rho^\beta u_x |x|^\alpha (u u_x)_x dx
= \frac{1}{2} \int_{\mathbb{R}} \alpha |x|^{-2} x u u_x^2 dx - \frac{1}{2} \int_{\mathbb{R}} |x|^\alpha u_x^2 dx - \frac{1}{2} \int_{\mathbb{R}} \rho^\beta |x|^\alpha u_x^2 dx
+ \frac{1}{2} \int_{\mathbb{R}} (\rho^\beta)_x u_x^2 |x|^\alpha u dx + \frac{1}{2} \int_{\mathbb{R}} \rho^\beta u_x^2 \alpha |x|^{-2} x dx
\leq C \|u_x\|_{L^\infty} \||\| x^{\frac{3}{2}} u_x\|_{L^2} \|\| x^{\frac{3}{2}} u^{\alpha - 1}\|_{L^2} + C \|u_x\|_{L^\infty} \int_{\mathbb{R}} |x|^\alpha u_x^2 dx
+ \|(\rho^\beta)_x\|_{L^\infty} \|u\|_{L^\infty} \int_{\mathbb{R}} |x|^\alpha u_x^2 dx + C \|u_x\|_{L^\infty} \||\| x^{\frac{3}{2}} u_x\|_{L^2} \||\| x^{\frac{3}{2}} u^{\alpha - 1}\|_{L^2}
\leq C(1 + \|\|\| x^{\frac{3}{2}} u_x\|_{L^2}) \int_{\mathbb{R}} |x|^\alpha u_x^2 dx, \] (2.59)
\[
M_3 = -\int_{\mathbb{R}} \mu(\rho)u_x \alpha |x|^{-2} x \dot{u} \leq C \int_{\mathbb{R}} |u_x| \|\| x^{\alpha - 1}\| \int_{\mathbb{R}} (\|x^{\alpha - 1}\|u_x)(\|x^{\alpha - 1}\|\dot{u}) \leq C \||\| x^{\frac{3}{2}} u_x\|_{L^2} \||\| x^{\frac{3}{2}} u^{\alpha - 1}\|_{L^2} \leq C \||\| x^{\frac{3}{2}} u_x\|_{L^2} \||\| x^{\frac{3}{2}} \dot{u}_x\|_{L^2} \leq C \||\| x^{\frac{3}{2}} u_x\|_{L^2}^2 + C \||\| x^{\frac{3}{2}} \dot{u}_x\|_{L^2}^2. \] (2.60)
Similarly, we have
\[
M_4 = -\int_\mathbb{R} (\rho^\gamma) x |x|^{\alpha} \hat{u} dx = \int_\mathbb{R} \rho^\gamma (|x|^{\alpha} u_x + \alpha |x|^{\alpha-2} x \hat{u}) dx
\]
\[
\leq C \| \rho \|_{L^\infty} \| x \|_{L^\infty} \| \hat{u} \|_{L^2} + C \| \rho \|_{L^\infty} \| x \|_{L^\infty} \| \hat{u} \|_{L^2} \tag{2.61}
\]
\[
\leq C \| x \|_{L^\infty} \| \hat{u} \|_{L^2}.
\]
Substituting (2.56), (2.59), (2.60), (2.61) into (2.55) yields
\[
\int_\mathbb{R} \mu(\rho) |x|^{\alpha} u_x^2 dx + \int_0^T \int_\mathbb{R} \rho \hat{u}^2 |x|^{\alpha} dx dt \leq C(T).
\]
It follows from (2.37) and (2.53) that
\[
\| \rho_t \|_{L^2} + \| \rho_{xt} \|_{L^2} \leq C(1 + \| x \|_{L^\infty} \| \hat{u} \|_{L^2}) \leq C(T).
\]
Similarly, we have
\[
\| (\rho^\gamma)_t \|_{L^2} + \| (\rho^\beta)_t \|_{L^2} + \| (\rho^\gamma)_x \|_{L^2} + \| (\rho^\beta)_x \|_{L^2} \leq C(T).
\]
The proof of the lemma is completed. \(\square\)

**Lemma 2.13.** Let \((\rho, u)\) be the smooth solution to \((1.1) - (1.4)\). Then for any \(0 \leq t \leq T\), it holds
\[
t \int_\mathbb{R} \mu(\rho) \hat{u}_x^2 dx + \int_0^T t \| \sqrt{\rho} \hat{u}_t \|_{L^2}^2 + \| \hat{u}_{xx} \|_{L^2}^2 dt \leq C(T).
\]

**Proof.** Multiplying on both sides of (2.41) by \(\hat{u}_t\) and integrating the resulted equation with respect to \(x\) over \(\mathbb{R}\), we have
\[
\frac{1}{2} \frac{d}{dt} \int_\mathbb{R} \mu(\rho) \hat{u}_x^2 dx + \int_\mathbb{R} \rho \hat{u}_t^2 dx = \frac{d}{dt} \int_\mathbb{R} (\mu(\rho) u_x^2 + \beta \rho^\beta u_x^2 - \gamma \rho^\gamma u_x) \hat{u}_x dx
\]
\[
+ \frac{1}{2} \int_\mathbb{R} (\rho^\beta)_t \hat{u}_x^2 dx - \int_\mathbb{R} \rho u_x \hat{u}_t dx - \int_\mathbb{R} (\rho^\gamma)_t u_x^2 dx
\]
\[
- \int_\mathbb{R} (2 \mu(\rho) u_x + 2 \beta \rho^\beta u_x - \gamma \rho^\gamma) (\hat{u} - uu_x) \hat{u}_x dx
\]
\[
\leq C \| (\rho^\beta)_t \|_{L^\infty} \int_\mathbb{R} \hat{u}_x^2 dx + \| \sqrt{\rho} \|_{L^\infty} \| u \|_{L^\infty} \| \sqrt{\rho} \hat{u}_t \|_{L^2} \| \hat{u}_x \|_{L^2}
\]
\[
\leq C(1 + \| x \|_{L^\infty} \| \hat{u} \|_{L^2}) \int_\mathbb{R} \hat{u}_x^2 dx + C(1 + \| x \|_{L^\infty} \| \hat{u} \|_{L^2}) \| \sqrt{\rho} \hat{u}_t \|_{L^2} \| \hat{u}_x \|_{L^2}
\]
\[
\leq \varepsilon \int_\mathbb{R} \rho \hat{u}_x^2 dx + C \int_\mathbb{R} \hat{u}_x^2 dx.
\]

Cauchy problem of the 1-D compressible N-S equations
where (2.58) has been used. Due to Lemma 2.12, we obtain

\[-\int_{\mathbb{R}} \left( (\rho^2)_{\tau} u_{\tau x}^2 + \beta (\rho^2)_{\tau} u_{\tau x}^2 - \gamma (\rho^\gamma)_{\tau} u_{\tau x} \right) \dot{u} x \dot{x} dx \]

\[\leq C \| (\rho^2)_{\tau} \| L^\infty \| u_x \|_{L^2} \| \dot{u}_x \|_{L^2} + \| (\rho^\gamma)_{\tau} \| L^\infty \| u_x \|_{L^2} \| \dot{u}_x \|_{L^2} \]

\[\leq C (1 + \| |x|^\beta u_x \|_{L^2}) \| \dot{u}_x \|_{L^2} \]

\[\leq C \| \dot{u}_x \|_{L^2}. \quad (2.64)\]

Similarly,

\[-\int_{\mathbb{R}} (2 \mu (\rho) u_x + 2 \beta \rho^\gamma u_x - \gamma (\rho^\gamma) \dot{u} u \dot{x} u \dot{x} \dot{x} dx \]

\[= - \int_{\mathbb{R}} (2 \mu (\rho) u_x + 2 \beta \rho^\gamma u_x - \gamma (\rho^\gamma) (\ddot{u}_x - u_x^2 \dot{u}_x - uu_{xx} \ddot{u}_x) dx \]

\[\leq C \int_{\mathbb{R}} \ddot{u}_x^2 dx + C \| u_x \|_{L^2} \| \ddot{u}_x \|_{L^2} + C \| u \|_{L^\infty} \| u_{xx} \|_{L^2} \| \ddot{u}_x \|_{L^2} \]

\[\leq C \int_{\mathbb{R}} \ddot{u}_x^2 dx + C (T). \quad (2.65)\]

Substituting (2.63), (2.64), (2.65) into (2.62), multiplying the resulted equation by \( t \) and integrating respect to \( t \) over \([\tau, t_1] \) with \( \tau, t_1 \in [0, T] \) give that

\[t_1 \| \sqrt{\mu (\rho)} \ddot{u}_x \|_{L^2}^2 (t_1) + \int_{\tau}^{t_1} t \| \sqrt{\rho} \dot{u}_t \|_{L^2}^2 dt \]

\[\leq \tau \| \sqrt{\mu (\rho)} \ddot{u}_x \|_{L^2}^2 (\tau) + t_1 G(t_1) - \tau G(\tau) + \int_{\tau}^{t_1} \| \sqrt{\mu (\rho)} \ddot{u}_x \|_{L^2}^2 dt \]

\[- \int_{\tau}^{t_1} G(t) dt + C \int_{\tau}^{t_1} t \| \ddot{u}_x \|_{L^2}^2 dt \]

\[\leq \tau \| \sqrt{\mu (\rho)} \ddot{u}_x \|_{L^2}^2 (\tau) - \tau G(\tau) + C \int_{\tau}^{t_1} t \| \ddot{u}_x \|_{L^2}^2 dt + C (T) \]

\[(2.66)\]

where

\[G(t) = \int_{\mathbb{R}} \left( (\mu (\rho) u_x^2 + \beta \rho^\gamma u_x^2 - \gamma (\rho^\gamma) u_x \right) \dot{u} x \dot{x} dx.\]

It follows from Lemma 2.10 that \( G(t) \in L^1(0, T), \| \sqrt{\mu (\rho)} \ddot{u}_x \|_{L^2} \in L^2(0, T) \). Thus, there exists a subsequence \( \tau_k \) such that

\[\tau_k \to 0, \quad \tau_k G(\tau_k) \to 0, \quad \tau_k \| \sqrt{\mu (\rho)} \ddot{u}_x \|_{L^2}^2 (\tau_k) \to 0, \quad \text{as} \quad k \to +\infty.\]

Taking \( \tau = \tau_k \) in (2.66), then letting \( k \to +\infty \) and using the Gronwall inequality, one gets that

\[t \| \sqrt{\mu (\rho)} \ddot{u}_x \|_{L^2}^2 + \int_{0}^{T} t \| \sqrt{\rho} \dot{u}_t \|_{L^2}^2 dt \leq C (T).\]

It follows from (2.41) that

\[\mu (\rho) \dddot{u}_{xx} = \rho \ddot{u}_t + \rho u \ddot{u}_x - (\gamma pu_x - \mu (\rho) u_x^2 - \beta \rho^\gamma u_x^2) \dot{u} x - (\rho^\gamma) \ddot{u}_x.\]
Which yields

\[ \|\dddot{u}\|_{L^2} \leq C \left( \|\dot{\rho}\|_{L^2} + \|\rho \dot{u}\|_{L^2} + \| (\gamma \rho u - \mu (\rho) u^2 - \beta \dot{\rho} u_x) \|_{L^2} + \| (\rho^3)_x \ddot{u} \|_{L^2} \right) \]

\[ \leq C(\sqrt{\rho \dot{u}} \|_{L^2} + \| \dddot{u} \|_{L^2} + 1) \]

and

\[ \int_0^T t \|\dddot{u}\|_{L^2} dt \leq C(T) \]

The proof of the lemma is completed. \( \square \)

**Lemma 2.14.** Let \((\rho, u)\) be the smooth solution to (1.1) – (1.4). Then for any \(0 \leq t \leq T\), it holds

\[ t^2 \int_{\mathbb{R}} \rho \dddot{u}^2 dx + \int_{0}^{T} t^2 \int_{\mathbb{R}} \mu(\rho) \dddot{u}^2_{xt} dx dt \leq C(T). \]

**Proof.** Differentiating (2.41) with respect to \(t\), we have

\[ \rho \dddot{u} + \rho \dddot{u} + \rho \dddot{u} + \rho \dddot{u} - (\mu(\rho) \dddot{u})_x - ((\rho^3)_t \dddot{u})_x \]

\[ = (\gamma (\rho^3)_t u_x + \gamma (\rho^3) u_{xt} - (\rho^3)_t u_x^2 - 2 \mu(\rho) u_x u_{xt} - \beta (\rho^3)_t u_x^2 - 2 \beta \rho^3 u_x u_{xt})_x. \]  

(2.67)

Multiplying on both sides of (2.67) by \(\ddot{u}_t\), integrating the resulted equation over \(\mathbb{R}\), we have

\[ \frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}} \rho \dddot{u}^2 dx + \int_{\mathbb{R}} \mu(\rho) \dddot{u}^2_{xt} dx \]

\[ \leq - \int_{\mathbb{R}} \rho \dddot{u} \dddot{u}_t dx - \int_{\mathbb{R}} \rho \dddot{u} \dddot{u}_t dx - \int_{\mathbb{R}} (\rho^3)_t \dddot{u}_x u_{xt} dx - 2 \int_{\mathbb{R}} \rho u_{xt} \dddot{u}_t dx \]

\[ + C \|\ddot{u}_x\|_{L^2} (\|\rho^3\|_{L^\infty} \|u_x\|_{L^2} + \|\dot{u}_x\|_{L^2} + \|\rho^3\|_{L^\infty} \|u_x\|_{L^2}^2) \]

\[ \leq - \int_{\mathbb{R}} \rho \dddot{u} \dddot{u}_t dx - \int_{\mathbb{R}} \rho \dddot{u} \dddot{u}_t dx - \int_{\mathbb{R}} (\rho^3)_t \dddot{u}_x u_{xt} dx - 2 \int_{\mathbb{R}} \rho u_{xt} \dddot{u}_t dx + C \|\ddot{u}_t\|_{L^2} \]

(2.68)

It follows that

\[ - \int_{\mathbb{R}} \rho \dddot{u} \dddot{u}_t dx = \int_{\mathbb{R}} (\rho u)_x u \dddot{u}_t dx \]

\[ = - \int_{\mathbb{R}} \rho u (u_x \dddot{u}_t + u \dddot{u}_x \ddot{u}_t + u \dddot{u}_x \ddot{u}_t dx \]

\[ \leq C(\sqrt{\rho \dot{u}} \|_{L^2} \ddot{u}_x \|_{L^2}^2 + \| \dddot{u}_x \|_{L^2}^2 + \| \dddot{u}_t \|_{L^2}^2) \]

\[ \leq \varepsilon \|\ddot{u}_t\|_{L^2}^2 + C(\|\sqrt{\rho \dot{u}} \|_{L^2}^2 + \| \dddot{u}_x \|_{L^2}^2 + \| \dddot{u}_x \|_{L^2}^2) \]

(2.69)
Thus there exists a subsequence

\[ \tau = \sqrt{\frac{2}{\beta}} \]

Similarly, we have

\[ \tau \]

Taking

\[ t \]

Substituting (2.69), (2.70), (2.71), (2.72) into (2.68), multiplying the resulted equation by \( t^2 \) and integrating with respect to \( t \) over \([r, t_1]\) with \( r, t_1 \in [0, T]\) give that

\[
t_1^2 \| \sqrt{\rho \dot{u}_t} \|_{L^2}^2 + \int_{\tau}^{t_1} t^2 \| \sqrt{\mu (\rho) \ddot{u}_t} \|_{L^2}^2 dt \\
\leq \tau^2 \| \sqrt{\rho \dot{u}_t} \|_{L^2}^2 + 2 \int_{\tau}^{t_1} t \| \sqrt{\rho \dot{u}_t} \|_{L^2}^2 dt + C \int_{\tau}^{t_1} t^2 \| \sqrt{\rho \dot{u}_t} \|_{L^2}^2 dt \\
+ C \int_{\tau}^{t_1} t^2 (\| \ddot{u}_t \|_{L^2}^2 + \| \ddot{u}_x \|_{L^2}^2 + \| |x| \ddot{u}_x \|_{L^2}^2 + 1) dt.
\]

Since

\[ t \| \sqrt{\rho \dot{u}_t} \|_{L^2} \in L^2(0, T). \]

Thus there exists a subsequence \( \tau_k \) such that

\[ \tau_k \to 0, \quad \tau_k^2 \| \sqrt{\rho \dot{u}_t} (\tau_k) \|_{L^2}^2 \to 0, \quad as \quad k \to +\infty. \]

Taking \( \tau = \tau_k \) in (2.73), then letting \( k \to +\infty \) and using the Gronwall inequality, one gets that

\[
t^2 \int_{\tau}^{t} \rho \dot{u}_t^2 dx + \int_{0}^{T} t^2 \int_{\tau}^{T} \mu (\rho) \ddot{u}_x dx dt \leq C(T).
\]

The proof of the lemma is completed. \( \square \)

**Lemma 2.15.** Let \((\rho, u)\) be the smooth solution to (1.1) – (1.4). Then for any \( 0 \leq t \leq T, \ 2 \alpha < 1 + \frac{2}{\sqrt{1 + \sqrt{\alpha}}}, \) it holds

\[
t \| \sqrt{\mu (\rho)} |x| |\ddot{u}_x| \|_{L^2}^2 + \int_{0}^{T} t \| \sqrt{\rho \dot{u}_t} |x| ^{\frac{\alpha}{2}} \|_{L^2}^2 dt \leq C(T).
\]
Proof. Multiplying on both sides of (2.41) by $|x|^\frac{\alpha}{2}\dot{u}_t$, integrating the resulted equation over $\mathbb{R}$, we have

$$\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}} \mu(\rho)\dot{u}_x^2 |x|^\frac{\alpha}{2}dx + \int_{\mathbb{R}} \rho \dot{u}_t^2 |x|^\frac{\alpha}{2}dx \leq \frac{d}{dt}\int_{\mathbb{R}} (\mu(\rho)u_x^2 + \beta \rho^\gamma u_x^2 - \gamma \rho^\gamma u_x)(|x|^\frac{\alpha}{2}\dot{u}_x + \frac{\alpha}{2}|x|^\frac{\alpha}{2}x \dot{u})dx$$

$$+ \frac{1}{2} \int_{\mathbb{R}} (\rho^\gamma t \dot{u}_x^2 |x|^\frac{\alpha}{2} dx - \frac{\alpha}{2} \int_{\mathbb{R}} \mu(\rho)\dot{u}_x |x|^\frac{\alpha}{2}x \dot{u}_tdx - \int_{\mathbb{R}} \rho u \dot{u}_x |x|^\frac{\alpha}{2}x \dot{u}_tdx \tag{2.74}$$

$$- \int_{\mathbb{R}} (\rho^\gamma 2 \mu(\rho)u_x u_{xt} + \beta (\rho^\gamma t u_x^2 + 2 \beta \rho^\gamma u_x u_{xt} - \gamma (\rho^\gamma t u_x - \gamma \rho^\gamma u_{xt})$$

$$\times (|x|^\frac{\alpha}{2}\dot{u}_x + \frac{\alpha}{2}|x|^\frac{\alpha}{2}x \dot{u})dx$$

It follows from (2.58) that

$$\frac{1}{2} \int_{\mathbb{R}} (\rho^\gamma t \dot{u}_x^2 |x|^\frac{\alpha}{2} dx \leq C \|(\rho^\gamma)_t\|_{L^\infty} \int_{\mathbb{R}} |x|^\frac{\alpha}{2}\dot{u}_x^2 dx \leq C \int_{\mathbb{R}} |x|^\frac{\alpha}{2}\dot{u}_x^2 dx. \tag{2.75}$$

Direct estimates lead to

$$- \frac{\alpha}{2} \int_{\mathbb{R}} \mu(\rho)\dot{u}_x |x|^\frac{\alpha}{2}x \dot{u}_tdx$$

$$= - \frac{\alpha}{2} \frac{d}{dt} \int_{\mathbb{R}} \mu(\rho)\dot{u}_x |x|^\frac{\alpha}{2}x \dot{u}_tdx + \frac{\alpha}{2} \int_{\mathbb{R}} (\rho^\gamma t \dot{u}_x |x|^\frac{\alpha}{2}x \dot{u}_tdx + \frac{\alpha}{2} \int_{\mathbb{R}} \mu(\rho)\dot{u}_{xt} |x|^\frac{\alpha}{2}x \dot{u}_tdx$$

$$\leq - \frac{\alpha}{2} \frac{d}{dt} \int_{\mathbb{R}} \mu(\rho)\dot{u}_x |x|^\frac{\alpha}{2}x \dot{u}_tdx + C \|(\rho^\gamma)_t\|_{L^\infty} \|\dot{u}_x\|_{L^2} \||x|^\frac{\alpha}{2} - 1\ddot{u}\|_{L^2} + C \|\dot{u}_{xt}\|_{L^2} \||x|^\frac{\alpha}{2} - 1\ddot{u}\|_{L^2}$$

$$\leq - \frac{\alpha}{2} \frac{d}{dt} \int_{\mathbb{R}} \mu(\rho)\dot{u}_x |x|^\frac{\alpha}{2}x \dot{u}_tdx + C \|\dot{u}_x\|_{L^2} \|\dot{u}_{xt}\|_{L^2} \tag{2.76}$$

$$- \int_{\mathbb{R}} \rho u \dot{u}_x |x|^\frac{\alpha}{2}x \dot{u}_tdx \leq C \int_{\mathbb{R}} (\sqrt{\rho} \dot{u}_x |x|^\frac{\alpha}{2})(\dot{u}_x |x|^\frac{\alpha}{2}) \sqrt{\rho} u dx$$

$$\leq C \sqrt{\rho} u \|L^\infty \|L^\infty \|L^2 \|\dot{u}_x\|_{L^2} \|\dot{u}_x\|_{L^2} \tag{2.77}$$

$$\leq C \int_{\mathbb{R}} \rho \dot{u}_t^2 |x|^\frac{\alpha}{2} dx + C \|\dot{u}_x\|_{L^2}^2,$$
and
\[-\int_{\mathbb{R}} \left((\rho^\beta)_t u_x^2 + 2\mu(\rho)u_ux_{xt} + \beta(\rho^\beta)_t u_x^2 + 2\beta\rho^\beta u_x u_{xt} - \gamma(\rho^\gamma)_t u_x - \gamma\rho^\gamma u_{xt}\right)\]
\[\left(|x|^{\frac{\beta}{2}} u_x + \frac{\alpha}{2} |x|^{\frac{\beta}{2}} x u_x\right) dx\]
\[\leq C\left(|||x|^{\frac{\beta}{2}} u_x||_{L^2}^2 + |||x|^{\frac{\beta}{2}} - 1 u_x||_{L^2}^2\right) (\gamma(\rho^\gamma)_t ||u_x||_{L^2}^2 + ||u_x||_{L^\infty} ||u_{xt}||_{L^2})\]
\[+ ||(\rho^\beta)_t ||_{L^\infty} ||u_x||_{L^2}^2 + ||u_x||_{L^\infty} ||u_{xt}||_{L^2} + ||(\rho^\gamma)_t ||_{L^\infty} ||u_x||_{L^2} + ||\rho^\gamma||_{L^\infty} ||u_{xt}||_{L^2}\]
\[\leq C\left|||x|^{\frac{\beta}{2}} u_x||_{L^2}\right||, (2.78)\]

Substituting (2.75) - (2.78) into (2.74), we get
\[\frac{d}{dt} \int_{\mathbb{R}} \mu(\rho)u_x^2 + \frac{d}{dt} \int_{\mathbb{R}} \rho u_x^2 + ||u_x||_{L^2}^2 dx\]
\[\leq \frac{d}{dt} \int_{\mathbb{R}} \mu(\rho)u_x^2 + \beta\rho^\beta u_x^2 - \gamma\rho^\gamma u_x) \left(|x|^{\frac{\beta}{2}} u_x + \frac{\alpha}{2} |x|^{\frac{\beta}{2}} x u_x\right) dx\]
\[\leq \frac{d}{dt} F(t) - \frac{d}{dt} H(t) + \gamma(\rho^\gamma) ||u_x||_{L^2}^2 + ||u_x||_{L^\infty} ||u_{xt}||_{L^2}\]
where
\[F(t) = \int_{\mathbb{R}} \mu(\rho)u_x^2 + \beta\rho^\beta u_x^2 - \gamma\rho^\gamma u_x) \left(|x|^{\frac{\beta}{2}} u_x + \frac{\alpha}{2} |x|^{\frac{\beta}{2}} x u_x\right) dx,\]
\[H(t) = \frac{\alpha}{2} \int_{\mathbb{R}} \mu(\rho)u_x^2 + \beta\rho^\beta u_x^2 - \gamma\rho^\gamma u_x) \left(|x|^{\frac{\beta}{2}} u_x + \frac{\alpha}{2} |x|^{\frac{\beta}{2}} x u_x\right) dx.\]

Multiplying the inequality (2.79) by t and then integrating the resulted inequality with respect to t over [\tau, t_1] with both \tau, t_1 \in [0, T] give
\[\int_{\tau}^{t_1} \sqrt{\mu(\rho)}|x|^\frac{\beta}{2} u_x ||_{L^2}^2 dt + \int_{\tau}^{t_1} t \sqrt{\rho u_t} |x|^\frac{\beta}{2} ||_{L^2}^2 dt\]
\[\leq \tau \sqrt{\mu(\rho)}|x|^\frac{\beta}{2} u_x ||_{L^2}^2 (\tau) + \int_{\tau}^{t_1} ||\mu(\rho)|||x|^\frac{\beta}{2} u_x ||_{L^2}^2 dt + t_1 F(t_1) - \tau F(\tau)\]
\[+ \int_{\tau}^{t_1} \frac{d}{dt} F(t) dt - t_1 H(t_1) + \tau H(\tau) + \int_{\tau}^{t_1} H(t) dt\]
\[+ C \int_{\tau}^{t_1} t(1 + ||u_{xt}||_{L^2}^2 + ||x|^\frac{\beta}{2} u_x ||_{L^2}^2) dt.\]

Note that
\[\int_{0}^{T} \sqrt{\mu(\rho)}|x|^\frac{\beta}{2} ||_{L^2}^2 dt \leq \int_{0}^{T} \sqrt{\mu(\rho)}|x|(1 + \sqrt{\beta} u_x ||_{L^2}^2 dt \leq C(T),\]
\[
\int_0^T F(t) dt = \int_0^T \int_\mathbb{R} (\mu(\rho)u^2_x + \beta \rho^\beta u^2_x - \gamma \rho^\gamma u_x)(|x|^\nu \dot{u}_x + \frac{\alpha}{2}|x|^{\frac{n}{2}-2} x \dot{u}) dx dt \\
\leq C \int_0^T \|x|^{\frac{n}{2}} \dot{u}_x\|^2_{L^2} dt \leq C(T),
\]

and
\[
\int_0^T H(t) dt = \int_0^T \mu(\rho) \dot{u}_x |x|^{\frac{n}{2}-2} x \dot{u} dx dt \\
\leq C \int_0^T \|u_x\|_{L^2} \|x|^{\frac{n}{2}-1} \dot{u}\|_{L^2} dt \\
\leq C \int_0^T (\|u_x\|^2_{L^2} + \|x|^{\frac{n}{2}} \dot{u}_x\|^2_{L^2}) dt \leq C(T).
\]

There exists a subsequence \(\tau_k\) such that as \(k \to +\infty\),
\[
\tau_k \to 0, \quad \tau_k \left(\int_\mathbb{R} (\mu(\rho)u^2_x + \beta \rho^\beta u^2_x - \gamma \rho^\gamma u_x)(|x|^\nu \dot{u}_x + \frac{\alpha}{2}|x|^\frac{n}{2} x \dot{u}) dx\right)(\tau_k) \to 0,
\]
and
\[
\tau_k \left(\int_\mathbb{R} (\mu(\rho) u^2_x + \beta \rho^\beta u^2_x - \gamma \rho^\gamma u_x)(|x|^\nu \dot{u}_x + \frac{\alpha}{2}|x|^\frac{n}{2} x \dot{u}) dx\right)(\tau_k) \to 0.
\]

Taking \(\tau = \tau_k\) in (2.80), then letting \(k \to +\infty\) and using the Cauchy inequality and Gronwall inequality, one can obtain
\[
t \|\sqrt{\mu(\rho)} |x|^{\frac{n}{2}} \dot{u}_x\|^2_{L^2} + \int_0^T t \|\sqrt{\rho} |x|^{\frac{n}{2}} \dot{u}\|^2_{L^2} dt \leq C(T).
\]

The proof of the lemma is completed. \(\square\)

**Lemma 2.16.** Let \((\rho, u)\) be the smooth solution to (1.1) - (1.4). Then for any \(0 \leq t \leq T\), it holds
\[
t \int_\mathbb{R} u^2_{xxx} dx \leq C(T).
\]

**Proof.** Rewriting the equation (1.1) as
\[
\dot{\rho} \dot{u} + (\rho^\gamma)_x = \mu(\rho) u_{xx} + (\rho^\beta)_x u_x.
\]

Differentiating the above equation with respect to \(x\), we have
\[
\mu(\rho) u_{xxx} = \rho_{x} \dot{u} + \dot{\rho} u_{x} + (\rho^\gamma)_{xx} - (\rho^\beta)_{xx} u_x - 2(\rho^\beta) u_{xx}.
\]

This implies that
\[
\|u_{xxx}\|_{L^2} \leq C \|\rho_{x} \dot{u} + \dot{\rho} u_{x} + (\rho^\gamma)_{xx} - (\rho^\beta)_{xx} u_x - 2(\rho^\beta) u_{xx}\|_{L^2} \\
\leq C(\|\dot{u}\|_{L^\infty} \|\rho_{x}\|_{L^2} + 1) \leq C(\|\dot{u}\|_{L^\infty} + 1) \\
\leq C(\|\dot{u}\|^{\frac{1}{2}}_{L^{\frac{1}{2}}} + \|\dot{u}_{x}\|_{L^2} + 1) \\
\leq C(\|x|^{\frac{n}{2}} \|\dot{u}_{x}\|_{L^2} + 1).
\]
Thanks to Lemma 2.13 it deduces

\[ t \int_{\mathbb{R}} u_{xx}^2 \, dx \leq C(T). \]

The proof of the lemma is completed. \( \Box \)

3. PROOF OF THEOREM 1.1

In this section, we give the proof of our main result.

We first state the local existence and uniqueness of classical solution when the initial data may contain vacuum, the proof is referred to \[22\], \[3\].

Lemma 3.1. Under assumptions of Theorem 1.1, there exists a \( T_* > 0 \) and an
unique classical solution \((\rho, u)\) to the Cauchy problem (1.1) – (1.4) satisfying (1.8)
with \( T \) replaced by \( T_* \).

With all the a priori estimates in Section 2 at hand, we are ready to prove the
main results of this paper in this section.

Proof of Theorem 1.1 We first show that \((\rho, u)\) is a classical solution to (1.1) – (1.4)
if \((\rho, u)\) satisfies (1.8). Since \( u \in L^2(0, T; L_{\frac{3}{2}}^{\frac{2}{3}} \cap D^3(\mathbb{R})) \) and \( u_t \in L^2(0, T; L_{\frac{3}{2}}^{\frac{2}{3}} \cap D^1(\mathbb{R})) \), the Sobolev’s embedding theorem implies that

\[ u \in C([0, T]; L_{\frac{3}{2}}^{\frac{2}{3}} \cap D^2(\mathbb{R})) \hookrightarrow C([0, T] \times \mathbb{R}) \]

It follows from \((\rho, p(\rho) ) \in L^\infty (0, T; H^2(\mathbb{R})), \) and \((\rho, P(\rho))_t \in L^\infty (0, T; H^1(\mathbb{R})) \) that \((\rho, P(\rho)) \in C([0, T]; W^{1,q}(\mathbb{R})) \cap C([0, T]; H^2(\mathbb{R}) \text{ weak}) \) with \( 2 < q < +\infty \). This, together with (1.1) and [3], implies that

\[ (\rho, P(\rho)) \in C([0, T]; H^2(\mathbb{R})) \]

Moreover, since for any \( \tau \in (0, T) \)

\[ u \in L^\infty (\tau, T; L_{\frac{3}{2}}^{\frac{2}{3}} \cap D^3(\mathbb{R})), \quad u_t \in L^\infty (\tau, T; L_{\frac{3}{2}}^{\frac{2}{3}}(\mathbb{R})) , \]

we have

\[ u \in C([\tau, T]; W^{3,q}(\mathbb{R})) \hookrightarrow C([\tau, T]; C^{2,\zeta}(\mathbb{R})), \quad 0 < \zeta < \frac{1}{2} \text{ for } 1 < q < 2. \]

Note that

\[ (\rho_x, (P(\rho))_x) \in C([0, T]; H^1(\mathbb{R})) \hookrightarrow C([0, T] \times \mathbb{R}). \]

Using the continuity equation (1.4)1, one has

\[ \rho_t = -(\rho u_x + u\rho_x) \in C([\tau, T] \times \mathbb{R}). \]

Using the momentum equation (1.4)2, one has

\begin{align*}
(\rho u)_t &= (\mu(\rho) u_x)_x - (\rho u^2)_x - p_x \\
&= \mu(\rho) u_{xx} + (\rho^3)_x u_x - \rho_x u^2 - 2\rho uu_x - (\rho^\gamma)_x \\
&\in C([\tau, T] \times \mathbb{R})
\end{align*}

Theorem 1.1 follows from Lemma 3.1 which is about the local well-posedness of
the classical solution and global (in time) a priori estimates in Section 2. In fact, by
Lemma 3.1, there exists a local classical solution \((\rho, u)\) on the time interval \((0, T_\ast]\) with \(T_\ast > 0\). Now let \(T^\ast\) be the maximal existing time of the classical solution \((\rho, u)\) in Lemma 3.1. Then obviously one has \(T^\ast \geq T_\ast\). Now we claim that \(T^\ast \geq T\) for any \(T > 0\) being any fixed positive constant given in Theorem 1.1. Otherwise, if \(T^\ast < T\), then all the a priori estimates in Section 2 hold with \(T\) being replaced by \(T^\ast\). In particular, from the Lemma 2.8, it holds that

\[(1 + |x|^{\frac{n}{2}})\sqrt{\rho} u \in C([0, T^\ast]; L^2(\mathbb{R})).\]

Therefore, it follows from a priori estimates in Section 2 that \((\rho, u)(x, T^\ast)\) satisfy (1.8) and the compatibility condition (1.7) at time \(t = T^\ast\) with \(g(x) = \sqrt{\rho} u(x, T^\ast)\). By using lemma 3.1 again, there exists a \(T_1^\ast > 0\) such that the classical solution \((\rho, u)\) in Lemma 3.1 exists on \((0, T^\ast + T_1^\ast]\), which contradicts with \(T^\ast\) being the maximal existing time of the classical solution \((\rho, u)\). Thus it holds that \(T^\ast \geq T\). The proof of Theorem 1.1 is completed. \(\square\)

References

[1] L. Caffarelli, R. Kohn, L. Nirenberg, First order interpolation inequality with weights, *Compositio Mathematica*, 53 (1984) 259-275.
[2] F. Catrina, Z.Q. Wang, On the Caffarelli-Kohn-Nirenberg inequalities: Sharp constants, existence (and nonexistence) and symmetry of extremal functions, *Comm.Pure Appl.Math.*, LIV, (2001) 229-258.
[3] Y. Cho, H. Kim, On classical solutions of the compressible Navier-Stokes equations with nonnegative initial densities, *Manuscripta Math.*, 120 (2006) 91-129.
[4] S.J. Ding, H.Y. Wen, C.J. Zhu, Global classical large solutions to 1D compressible Navier-Stokes equations with density-dependent viscosity and vacuum, *J.Differential.Equations*. 251 (2011) 1696-1725.
[5] E. Feireisl, A. Novotný, H. Petzeltová, On the existence of globally defined weak solutions to the Navier-Stokes equations, *J.Math.Fluid Mech.*, 3 (2001) 358-392.
[6] Z.H. Guo, Q.S. Jiu, Z.P. Xin, Spherically symmetric isentropic compressible flows with density-dependent viscosity coefficients, *SIAM J. Math. Anal.* 39(5) (2008) 1402-1427.
[7] D. Hoff, Global existence for 1D compressible isentropic Navier-Stokes equations with large initial data, *Trans. Amer. Math. Soc.*, 303, no. 1, (1987) 169-181.
[8] D. Hoff, Global solutions of the Navier-Stokes equations for multidimensional compressible flow with discontinuous initial data, *J. Differ. Eqs.*, 120, no. 1,(1995) , 215-254.
[9] D. Hoff, D. Serre, The Failure of Continuous Dependence on Initial data for the Navier-Stokes equations of Compressible Flow, *SIAM J. Math. Anal.* 51 (1991) 887-898.
[10] X. D. Huang, J. Li, Existence and blowup behavior of global strong solutions to the two-dimensional baratropic compressible Navier-Stokes system with vacuum and large initial data, *preprint*, arxiv: 1205. 5342.
[11] X.D. Huang, J. Li, Global well-posedness of classical solutions to the Cauchy problem of two-dimensional baratropic compressible Navier-Stokes system with vacuum and large initial data, *preprint*, arXiv: 1207. 3746 (2012).
[12] S. Jiang, Global smooth solutions of the equations of a viscous, heat-conducting one-dimensional gas with density-dependent viscosity, *Math. Nachr.* 190 (1998) 169-183.
[13] S. Jiang, Z.P. Xin, P. Zhang, Global weak solutions to 1D compressible isentropic Navier-Stokes equations with density-dependent viscosity, *Methods Appl. Anal.* 12 (2005) 239-251.
Q. S. Jiu, Y. Wang, Z. P. Xin, Stability of rarefaction waves to the 1D compressible Navier-Stokes equations with density-dependent viscosity, Comm. Part. Diff. Equ., 36, (2011) 602-634.

Q. S. Jiu, Y. Wang, Z. P. Xin, Vacuum behaviors around the rarefaction waves to the 1D compressible Navier-Stokes equations with density-dependent viscosity, arxiv.org/abs/1109.0871

Q. S. Jiu, Y. Wang, Z. P. Xin, Global well-posedness of the Cauchy problem of two-dimensional compressible Navier-Stokes equations in weighted spaces, J. Differential. Equations. 255 no 3 (2013) 351-404

Q. S. Jiu, Y. Wang, Z. P. Xin, Global well-posedness of 2D compressible Navier-Stokes equations with large data and vacuum, arxiv:1202.1382 (2012)

Q. S. Jiu, Y. Wang, Z. P. Xin, Global classical solutions to the two-dimensional compressible Navier-Stokes equations in $R^2$, arxiv:1209.0157 (2012)

Q. S. Jiu, Z. P. Xin, The Cauchy problem for 1D compressible flows with density-dependent viscosity coefficients, Kinet. Relat. Models 1(2) (2008) 313-330

A. V. Kazhikhov, V. V. Shelukhin, Unique global solution with respect to time of initial-boundary value problems for one-dimensional equations of a viscous gas, J. Appl. Math. Mech 41 (1977) 273-282

T. P. Liu, Z. P. Xin, T. Yang, Vacuum states of compressible flow, Discrete Contin. Dyn. Syst., 4, 1-32 (1998).

Z. Luo, Local existence of classical solutions to the two-dimensional viscous compressible flows with vacuum, Commun. Math. Sci, Vol.10, (2012) 527-554

A. Mellet, A. Vasseur, Existence and uniqueness of global strong solutions for one-dimensional compressible Navier-Stokes equations, SIAM J. Math. Anal. 39(4) (2008) 1344-1365

I. Straskraba, A. Zlotnik, Global properties of solutions to 1D viscous compressible barotropic fluid equations with density dependent viscosity, Z. Angew. Math. Phys. 54(4) (2003) 593-607

V. A. Vaigant, A. V. Kazhikhov, On the existence of global solutions of two-dimensional Navier-Stokes equations of a compressible viscous fluid, Sibirsk. Mat. Zh. 36(6) (1995) 1283-1316

Z. P. Xin, Blow-up of smooth solution to the compressible Navier-Stokes equations with compact density, Comm. Pure Appl. Math. 51 (1998) 229-240

Z. P. Xin, W. Yan, On blowup of classical solutions to the compressible Navier-Stokes equations, Comm. Math. Phys. (2013)

T. Yang, Z. A. Yao, C. J. Zhu, Compressible Navier-Stokes equations with density-dependent viscosity and vacuum, Comm. Partial Differential Equations 26(5-6) (2001) 965-981

T. Yang, C. J. Zhu, Compressible Navier-Stokes equations with degenerate viscosity coefficient and vacuum, Comm. Math. Phys. 230(2) (2002) 329-363