On Eigenvalues Problem for Self-adjoint Operators with Singular Perturbations

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Abstract

We investigate the eigenvalues problem for self-adjoint operators with the singular perturbations. The general results presented here include weakly as well as strongly singular cases. We illustrate these results on two models which correspond to so-called additive strongly singular perturbations.

1 INTRODUCTION

Ever since Kr¨ onig, Penney [18] and Bethe, Peierls [9] used potential supported by isolated points the Hamiltonians with perturbations on null sets attracted physicists and mathematicians. Special progress in understanding of mathematical aspects started in 80’s. Most of the researches has been devoted to study of the potentials supported by null sets but it turns out that the perturbations by the dynamics of the systems living on null sets are also interested (see for example [2, 6]). In this paper we are concerned with method of perturbations applicable in both cases.

Let \( A \) represents Hamiltonian of free system \( S \) and \( V \) corresponds to a potential supported by a null set or to a Hamiltonian of system located on a null set. There are many papers devoted to problem of the construction of Hamiltonian of composite system \( S \). This problem can be formulated as follows. Give a meaning of self-adjoint operator to the following formal sum

\[
A + V.
\] (1)

In this paper we investigate the eigenvalues problem for the self-adjoint operators with the singular perturbations.

Let us describe the idea of singular perturbations more precisely. The main concept is based on the theory of the extensions of symmetric operators developed by von Neuman and Krein.
Let $A$ be a self-adjoint, strictly positive operator in the Hilbert space $H$ and $D(A)$ denote its domain. We say that a self-adjoint, invertible operator $\tilde{A}$ in $H$ is a singular perturbation of $A$ if $\tilde{A}$ coincides with $A$ on a set dense in $H$. The class of all singular perturbation of $A$ we denote by $\mathcal{A}_s(A)$. In other words, any operator $\tilde{A} \in \mathcal{A}_s(A)$ is the self-adjoint extension of closed, symmetric operator $\hat{A} \equiv A|D$ where $D \equiv D(A) \cap D(\hat{A})$ and $|D$ stands for the restriction to $D$.

In accordance with the standard notations, we put $N_0$ for the deficiency space of $\hat{A}$ which coincides with $\ker \hat{A}^*$ where $\hat{A}^*$ stands for the adjoint operator to $\hat{A}$. It is known [3, 12] that any self-adjoint extension $\tilde{A}$ of $\hat{A}$, can be represented by its inverse in the following way

$$\tilde{A}^{-1} = A^{-1} + \hat{B}^{-1}$$

where $\hat{B}^{-1} : H \to N_0$ is self-adjoint. Note that (2) is just the Krein formula at point zero.

The class $\mathcal{A}_s(A)$ can be decomposed into so-called weakly and strongly singular classes i.e. $\mathcal{A}_s(A) = \mathcal{A}_{ws}(A) \cup \mathcal{A}_{ss}(A)$. The precise definitions of $\mathcal{A}_{ws}(A)$ and $\mathcal{A}_{ss}(A)$ we will be formulated later.

Some of operators from $\mathcal{A}_s(A)$ can be interpreted as the self-adjoint realizations of (1). But we would like to emphasize that the sum (1) only in some remote sense corresponds to addition and operator $V$ does not act in $H$. In fact, we usually write

$$\tilde{A} = A + V$$ for $\tilde{A} \in \mathcal{A}_{ws}(A)$

and

$$\tilde{A} = A + V$$ for $\tilde{A} \in \mathcal{A}_{ss}(A)$. (4)

The constructions of (5) and (6) were described in [4] [10] [16] [15] respectively.

The aim of this paper is to describe some of the spectral properties of operators from $\mathcal{A}_s(A)$. Precisely, we are interested in the problem of eigenvalues

$$\tilde{A}f = Ef \quad E \in \mathbb{R} \setminus \{0\}, \quad f \in D(\tilde{A}).$$

Our first goal is to characterize the solutions of (5) in the terms of $\hat{B}^{-1} = \tilde{A}^{-1} - A^{-1}$. The eigenvalues problem for operator given by $A + \alpha V \in \mathcal{A}_{ws}(A)$ where $\alpha \in \mathbb{R}$ was studied in [4]. Here we start from the general situation and in further discussion focus on some subfamily of $\mathcal{A}_s(A)$ which we denote by $\mathcal{A}'_s(A)$. However $\mathcal{A}'_s(A) \cap \mathcal{A}_{ws}(A) \neq \{0\}$ as well as $\mathcal{A}'_s(A) \cap \mathcal{A}_{ss}(A) \neq \{0\}$ holds.

In the second part of this paper we discuss (5) for $\tilde{A}$ given by $A + \alpha V \in \mathcal{A}_{ss}(A)$. We consider two models in which $A$ corresponds to the Laplace operators in $L^2(0, \pi) \times \mathbb{R}^2, dx)$ and $L^2(\mathbb{R}^3, dx)$ and $V$ refers to self-adjoint operators in $L^2(I, dx_1)$ and $L^2(C, d\phi)$ where $I$, $C$ denote an interval and circle respectively. In particular we investigate the asymptotic behaviour of the solutions of (5) as $\alpha \to \infty$. 

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1.1 Definitions and Notations

Let $A$ be a self-adjoint, strictly positive operator in separable Hilbert space $\mathcal{H}$ with an inner product $(\cdot, \cdot)$ and norm $\|\cdot\|$. By $D(A)$ and $\text{Ran}(A)$ we denote the domain and range of $A$.

Put $\rho(A)$ and $\sigma(A)$ for the resolvent set and spectrum of $A$ respectively. In this paper we assume for simplicity that the spectrum of $A$ is purely absolutely continuous, $\sigma(A) = \sigma_{ac}(A)$.

Let $q$ be integer with $|q| \leq 2$. Define the inner product in $\mathcal{D}(A)$ by $$(u, v)_q = (A^{q/2}u, A^{q/2}v).$$

Putting $\mathcal{H}_q$ for the completion of $\mathcal{D}(A)$ in the norm $\|\cdot\|_q$ we get the chain of the Hilbert spaces

$$\mathcal{H}_{-2} \supset \mathcal{H}_{-1} \supset \mathcal{H}_0 \equiv \mathcal{H} \supset \mathcal{H}_1 \supset \mathcal{H}_2.$$  (6)

Clearly, for $q=1, 2$ the space $\mathcal{H}_q$ coincides with $\mathcal{D}(A^{q/2})$.

From the construction of (6) follows that $\mathcal{H}_q$ and $\mathcal{H}_{-q}$ are mutually conjugate with respect to $\mathcal{H}_0$. Set $\langle \cdot, \cdot \rangle$ for the duality between $\mathcal{H}_q$ and $\mathcal{H}_{-q}$.

In chain (6) the operator $A$ is unitary as a map from $\mathcal{H}_2$ to $\mathcal{H}_0$ and it acts isometrically from $\mathcal{H}_1$ to $\mathcal{H}_{-1}$ and from $\mathcal{H}_0$ to $\mathcal{H}_{-2}$. Putting $\tilde{A}$ for the closure of $A : \mathcal{H}_0 \to \mathcal{H}_{-2}$ we obtain unitary operator.

By definition, a self-adjoint, invertible operator $\tilde{A}$ in $\mathcal{H}$ is called a singular perturbation of $A$ if the set

$$D = \{ \varphi \in D(A) \cap D(\tilde{A}) : A\varphi = \tilde{A}\varphi \}$$  (7)

is dense in $\mathcal{H}$. The set of all singular perturbations of $A$ will be denoted by $\tilde{A} \in \mathcal{A}_{sa}(A)$.

So, we see that any $\tilde{A} \in \mathcal{A}_{sa}(A)$ is a self-adjoint extension of symmetric operator $\dot{A} = A|D$ which is automatically closed. Then $D$ is the proper subspace of $\mathcal{H}_2$. Denote $M_2 = D$ and $N_2 = \mathcal{H}_2 \ominus M_2$. Conversely, if a linear space $M_2 \subset \mathcal{H}_2$ is closed in $\mathcal{H}_2$ and dense in $\mathcal{H}_0$ then $A = A|M_2$ is symmetric and its self-adjoint extension belongs to $\mathcal{A}_{sa}(A)$.

By unitarity of $A : \mathcal{H}_2 \to \mathcal{H}_0$ we can ”shift” the above decomposition of $\mathcal{H}_2$ onto $\mathcal{H}_0$ i.e.

$$\mathcal{H}_0 = M_0 \oplus N_0$$

where $M_0 = AM_2$, $N_0 = AN_2$. Note that $N_0$ coincides with $\ker \dot{A}^*$ and is the defect space for $A$.  

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We shall say that operator \( \tilde{B}^{-1} \) belongs to \( \tilde{B}(A) \) if there exists a proper subspace \( N \) of \( \mathcal{H}_0 \) so that \( M_+ \equiv \mathcal{H}_2 \oplus A^{-1}N \) is dense in \( \mathcal{H}_0 \) and
\[
\tilde{B}^{-1} = 0 \oplus B^{-1} : M \oplus N \to \mathcal{H}_0, \quad M \equiv AM_+
\]
where \( B^{-1} \) is self-adjoint, invertible in \( N \). The following theorem states the relation between \( \tilde{B}(A) \) and \( A_s(A) \).

**Theorem 1** ([3, 12]) If \( \tilde{B}^{-1} \in \tilde{B}(A) \) then
\[
\tilde{A}^{-1} = A^{-1} + \tilde{B}^{-1}
\]
is invertible and \( \tilde{A} \in A_s(A) \). Moreover \( \tilde{A}|_{\ker A^{-1}\tilde{B}^{-1}} = \tilde{A} \). Conversely, if \( \tilde{A} \in A_s(A) \) then there exists exactly one \( \tilde{B}^{-1} \in \tilde{B}(A) \) so that (8) holds and \( \ker A^{-1}\tilde{B}^{-1} = D(\tilde{A}) \).

In \( A_s(A) \) we can select weakly singular class defined by
\[
A_{ws}(A) := \{ \tilde{A} \in A_s(A) : D(\tilde{A}) \subset \mathcal{H}_1 \}
\]
and strongly singular class
\[
A_{ss}(A) := A_s(A) \setminus A_{ws}(A).
\]

## 2 Eigenvalues problem for singular perturbation

We keep notations introduced in the previous section. Now, our goal will be to investigate eigenvalues problem for \( \tilde{A} \in A_s(A) \). Precisely, we would like to formulate conditions ensuring the existence of solutions
\[
\tilde{A}f = Ef, \quad f \in D(\tilde{A}), \quad E \in \mathbb{R} \setminus \{0\}. \tag{9}
\]
The pure point spectrum of \( \tilde{A} \) i.e. the set of all \( E \) satisfying (9) we denote by \( \sigma_p(\tilde{A}) \).

Due to theorem 1 for any \( \tilde{A} \in A_s(A) \) there exists uniquely defined \( \tilde{B}^{-1} \in \tilde{B}(A) \) so that \( \tilde{A}^{-1} = A^{-1} + \tilde{B}^{-1} \). Thus (9) can be rewritten in the form
\[
(A^{-1} + \tilde{B}^{-1})g = E^{-1}g, \quad g = \tilde{A}f, \quad E \in \mathbb{R} \setminus \{0\}. \tag{10}
\]

For all \( E \in \mathbb{R} \setminus \{0\} \) we define \( U_{0E} = (A - E)A^{-1} : \mathcal{H} \to \mathcal{H}, \mathcal{H} \equiv \mathcal{H}_0 \). With this notation we can formulate the following theorem.

**Theorem 2** A pair \( E \in \mathbb{R} \setminus \{0\}, f \in \mathcal{H} \) solves eq. (10) iff \( f \) has the form
\[
f = \tilde{A}^{-1}g
\]
where \( g \in \ker(U_{0E} - E\tilde{B}^{-1}) \).
Proof. Note that (10) is equivalent to
\[(I - EA^{-1})g - E\tilde{B}^{-1}g = 0, \quad g = \tilde{A}f.\] (11)
In turn, (11) can be written as
\[U_0Eg - E\tilde{B}^{-1}g = 0, \quad g = \tilde{A}f.\]
So the theorem is proven. ■

It is not hard to see that theorem 2 is a generalization of theorem 5 [4] but the proof presented above is less complicated.

To discuss eigenvalues problem (11) in more details we shall consider cases \(E \in \rho(A) \cap \mathbb{R}\) and \(E \in \sigma(A)\) separately.

First, assume \(E \in \rho(A) \cap \mathbb{R}\). Define operator-valued function on \(\rho(A) \cap \mathbb{R}\) by \(R_E := (A - E)^{-1}: \mathcal{H} \to \mathcal{H}\). Then \(U_{E0} \equiv (U_0E)^{-1} = AR_E : \mathcal{H} \to \mathcal{H}\). Let \(K_E\) stand for the set of eigenvectors of \(U_{E0}\).

Given \(E \in \rho(A) \cap \mathbb{R}\) and \(h \in K_E\) the expression
\[a_h(E') \equiv a_{h,E}(E') := \|h\|^{-2} (U_{E0}\tilde{B}^{-1}h, h)\]
defines a real, continous function on \(\rho(A) \cap \mathbb{R}\).

Equivalently
\[a_h(E') = \|h\|^{-2} \left< (A - E')^{-1}\tilde{B}^{-1}h, Ah \right> = \|h\|^{-2} \int_{\lambda}^{\infty} \frac{1}{t - E'} d \left< \mathbb{E}_t \tilde{B}^{-1}h, Ah \right>\]
where \(\lambda > 0\) is the lower bound for \(A\) and \(\mathbb{E}_t\) is the spectral resolution for \(A\).

The problem at hand is to express solutions of
\[\tilde{A}f = Ef, \quad f \in D(\tilde{A}), \quad E \in \rho(A) \cap \mathbb{R}\]
by eigenvectors of \(U_{E0}\). This is given by the following theorem.

**Theorem 3** A pair \(f \in \mathcal{H}, \quad E \in \rho(A) \cap \mathbb{R}\) solves (4) if and only if there exists \(g \in K_E\) so that
\[f = \tilde{A}^{-1}g\] (12)
and condition
\[a_g(E) = E^{-1}.\] (13)
holds.
Proof. Observe that \( g \in \ker(U_0E - E\tilde{B}^{-1}) \iff g \in K_E \) and \( a_g(E) = E^{-1} \).

Then, by theorem 2 we get that \( f \in H, E \in \rho(A) \cap \mathbb{R} \) solve (12) \iff \( f \) is given by (13) where \( g \in K_E \) and condition (13) is satisfied. 

Now, we shall be interested in eigenvalues problem for \( E \in \sigma(A) \). Recall that we assume \( \sigma(A) = \sigma_{ac}(A) \).

For further discussion we shall need the spectral representation of \( A \) supplied by the following theorem.

Put \( I \) for a countable set.

Theorem 4 (Spectral Theorem) Let \( A \) be self-adjoint in separable Hilbert space \( \mathcal{H} \). Then there exist measures \( \{\mu_n\}_{n \in I} \) on \( \sigma(A) \) and unitary operator \( U: \mathcal{H} \to \bigoplus_{n \in I} L^2(\mathbb{R}, d\mu_n) \) so that

\[
(UA^{-1}\psi)_n(t) = t\psi'_n(t)
\]

where \( \psi \) is a vector from \( \bigoplus_{n \in I} L^2(\mathbb{R}, d\mu_n) \) with coordinates \( \psi_n \in L^2(\mathbb{R}, d\mu_n) \) for each \( n \in I \).

By the unitarity of \( U \) the direct sum decomposition \( \bigoplus_{n \in I} L^2(\mathbb{R}, d\mu_n) \) can be ”transmitted” to \( \mathcal{H} \). Precisely, we have

\[
\mathcal{H} = \bigoplus_{n \in I} \mathcal{H}^n \text{ where } \mathcal{H}^n = U^{-1}L^2(\mathbb{R}, d\mu_n).
\]

(14)

For all \( n \in I \) introduce

\[
A^n = A|\mathcal{H}^n \text{ and } \mathcal{H}^n = \mathcal{A}|\mathcal{H}^n.
\]

(15)

Clearly \( \sigma(A^n) = \text{supp}\mu_n \).

For further discussion we need the following definition.

Definition Let \( \tilde{A} \in \mathcal{A}_s(A) \) and \( \tilde{B}^{-1} = \tilde{A}^{-1} - A^{-1} \). We will say that operator \( \tilde{A} \) satisfies condition \( \tilde{\sigma} \) if for all \( n \in I \) we have

\[
\text{Ran}\tilde{B}^{-1} \cap \text{Ran}(A^n - E) = \{0\} \text{ where } E \text{ runs over } \sigma(A^n).
\]

(16)

In the next section we will discuss condition (16) in more details.

Let \( \tau \) be a subset of \( I \). Define

\[
\mathcal{H}^\tau := \bigoplus_{j=1}^\tau \mathcal{H}^j
\]

(17)

and

\[
A^\tau = A|\mathcal{H}^\tau.
\]

(18)

For \( E \in \rho(A) \cap \mathbb{R} \) put \( U_{E0}^\tau = A^\tau(A^\tau - E)^{-1}, U_{0E}^\tau = (U_{E0}^\tau)^{-1} \) and \( K_E^\tau = \{g \in \mathcal{H}^\tau \text{ and } g \text{ is an eigenvector of } U_{E0}^\tau \tilde{B}^{-1}\} \).

Now we are in position to formulate the following theorem.
**Theorem 5** Assume that \( \tilde{A} \in \mathcal{A}_s(A) \) satisfies \( \hat{\sigma} \) and \( g \in \mathcal{H}^r \). A pair \( f = \tilde{A}^{-1}g, E \in \sigma(A) \) solves

\[
\tilde{A}f = Ef
\]  

iff \( E \in \rho(A^*) \cap \sigma(A), g \in \mathcal{K}_E \) and condition

\[
a_g(E) \equiv \|g\|^{-2} (U_{E_0}^\tau \tilde{B}^{-1}g, g) = E^{-1}
\]  

holds.

**Proof.** Let \( \tilde{A} \in \mathcal{A}_s(A) \) satisfies \( \hat{\sigma} \) and \( g \in \mathcal{H}^r \). Assume that a pair \( f = \tilde{A}^{-1}g, E \in \sigma(A) \) solve (19). Using \( \tilde{A}^{-1} = A^{-1} + \tilde{B}^{-1} \) we can rewrite (19) in the form

\[
E^{-1}g = (A^{-1} + \tilde{B}^{-1})g.
\]

A direct calculation yields

\[
(A - E)A^{-1}g = E\tilde{B}^{-1}g.
\]  

Then \( (A - E)A^{-1}g \in \text{Ran}(A^* - E) \cap \text{Ran}\tilde{B}^{-1} \). Due to condition (16) we come to the conclusion that \( E \in \rho(A^*) \). Moreover from (21) follows \( g \in \mathcal{K}_E \) and \( a_g(E) = E^{-1} \).

Conversely, assume \( E \in \rho(A^*) \cap \sigma(A) \) and \( g \in \mathcal{H}^r \). Besides, let \( g \in \mathcal{K}_E \) and condition (20) holds. Then we have

\[
(\mathbb{I} - EU_{E_0}^\tau \tilde{B}^{-1})g = 0
\]

or equivalently

\[
(U_{0E}^\tau - E\tilde{B}^{-1})g = 0.
\]  

In turn (22) can be written as

\[
A^{-1}g + \tilde{B}^{-1}g = E^{-1}g.
\]

Putting \( f = A^{-1}g + \tilde{B}^{-1}g = \tilde{A}^{-1}g \) we come to (19).

Theorems 3 and 5 solve the eigenvalues problem for operator \( \tilde{A} \in \mathcal{A}_s(A) \) satisfying condition \( \hat{\sigma} \). Moreover, from above mentioned theorems we immediately get.

**Corollary 6** Let \( \tilde{A} \in \mathcal{A}_s(A) \) and satisfy \( \hat{\sigma} \). If \( E \in \sigma_p(\tilde{A}) \) then \( E \in \rho(A) \cap \mathbb{R} \).

Particularly, if \( I = \{1\} \) and \( E \in \sigma_p(\tilde{A}) \) then \( E \in \rho(A) \cap \mathbb{R} \).

We will formulate our next results for the following case. Let again \( A \) be a self-adjoint, strictly positive operator in \( \mathcal{H} \) with lower bound \( \lambda \). Additionally we assume that \( A \) has the following orthogonal sum decomposition

\[
A = \bigoplus_{n \in I} A^n
\]
where $A^n$ are operators defined by (15) with spectrum sets

$$\sigma(A^n) = [m_n, \infty), \quad m_n \geq \lambda.$$  \hfill (23)

In fact, it is not difficult to generalize further results for the case when $A^n$ have absolutely continuous spectrums with gaps. However, for simplicity we make assumption (23).

Define operator $\tilde{B}^{-1} \in \mathcal{B}(A)$ by

$$\tilde{B}^{-1} = \sum_{k \in S} b^{-1}_k(\cdot, e_k)e_k$$ \hfill (24)

where $b_k$ are real constants, $b_k \neq 0$, $\{e_k\}_{k \in S}$ is an orthogonal basis in $N_0$ and for each $k \in S$ there exists $\tau_k \subset \mathbb{I}$, so that $e_k \in \mathcal{H}^{\tau_k}$ (see [7]) and $\bigcap_{k \in S} \tau_k = \{0\}$.

Now, we will study the eigenvalues problem for $\tilde{A} \in A_s(A)$ where $\tilde{B}^{-1} = \tilde{A}^{-1} - A^{-1}$ has the form (24). In particular, our aim will be to characterize the following functions

$$N_-(\tilde{A}) = \#\{E < 0 : E \text{solves (19)}\}$$

and

$$N_+(\tilde{A}) = \#\{E > 0 : E \text{solves (19)}\}$$

where $\#$ denotes the number of elements of the set $\{\cdot\}$ including its multiplicity.

One can easily note that

$$\sigma(A^{\tau_k}) = [M_k, \infty) \quad \text{where} \quad M_k \equiv \inf_{n \in \tau_k} m_n.$$  \hfill (15)

Let $E_k \in \mathbb{R}\setminus[M_k, \infty)$. Henceforth, we will omit the subscript $k$ putting $E_k \equiv E$. We also abbreviate $K^k_{E_k} \equiv k^{\tau_k}, U^{k}_{E_k} \equiv U^{\tau_k}_{E_0}$ and $U^k_{0E} \equiv U^0_{0E}$.

Assume $g \in K^k_{E_k}$. Then we have

$$U^{k}_{E_0} \tilde{B}^{-1}g = b^{-1}_k(g, e_k)U^{k}_{E_0}e_k.$$  \hfill (25)

Since $g$ is an eigenvalue of $U^{k}_{E_0} \tilde{B}^{-1}$ we conclude that $g$ belongs to the subspace spanned by $U^{k}_{E_0}e_k$. For simplicity we put $g = U^{k}_{E_0}e_k$. With this observation theorems 3 and 5 read.

**Theorem 7** Let $\tilde{A}$ satisfy $\tilde{\sigma}$ and $\tilde{B}^{-1}$ be defined by (24). Assume that given $k \in S$ there exists number $E \in \mathbb{R}\setminus\{0\}$ so that condition

$$a_k(E) \equiv b^{-1}_k(U^{k}_{E_0}e_k, e_k) = E^{-1}$$ 

fulfills. Then $E \in \mathbb{R}\setminus[M_k, \infty)$, and the pair $E = \tilde{A}^{-1}g_k$ where $g_k = U^{k}_{E_0}e_k$ solves (15). Moreover $\{k\}_{k \in S'}$ where $S' = \{k \in S : (25) \text{ holds}\}$ is the complete system of eigenvectors for $\tilde{A}$.
From above theorem follows that \( E \in \sigma_p(\tilde{A}) \) iff \( E \) solves (25).

For each \( k \in S \) introduce functions on \( \mathbb{R} \setminus [M_k, \infty) \) defined by

\[
s_k(E) \equiv E(U^k_{E_0}e_k, e_k) = E \int_{(M_k, \infty)} \frac{t}{t-E} d\nu_k(t) \tag{26}
\]

where \( \nu_k(t) \) stands for the spectral measure associated to \( e_k \) i.e. \( \nu_k(t) = (E_t e_k, e_k) \). With this notation (25) can be rewritten as

\[
s_k(E) = b_k. \tag{27}
\]

Obviously, by the construction \( s_k(E) \) are continuous. Moreover the fact that \( \frac{E^k_{E_k}}{E} \) grows monotonically as the function of \( E \) implies that \( s_k(E) \) grow monotonically too. Besides, \( E < 0 \) iff \( s_k(E) < 0 \). Then as follows from (27) \( E \in \sigma_p(\tilde{A}) \) iff

\[
b_k \in (s_k(-\infty), s_k(M_k))
\]

where \( s_k(-\infty) \equiv \lim_{E \to \infty} s_k(E) \) and \( s_k(M_k) \equiv \lim_{E \to M_k} s_k(E) \). So, we have to find \( s_k(-\infty) \) and \( s_k(M_k) \). The first result is given in the following statement.

**Proposition 8** For \( e_k \in \mathcal{H}_1 \) the expression \( s_k(-\infty) \) is finite and given by

\[
s_k(-\infty) = -\|e_k\|_1^2.
\]

Otherwise, i.e. for \( e_k \in \mathcal{H}_0 \setminus \mathcal{H}_1 \) we have \( s_k(-\infty) = -\infty \).

**Proof.** From (26) we immediately get that \( s_k(-\infty) = -\langle e_k, e_k \rangle = -\|e_k\|_1^2 \).

Clearly, this expression is finite iff \( e_k \in \mathcal{H}_1 \). ■

To present the next result we need some preparation. For \( i = 1, 2 \) define space \( \mathcal{H}'_q \) as the completion of \( D(A) \) in the norm

\[
\|u\|_i^q = \left\| \sum_{k \in S} (A^k - M_k)^{i/2} u \right\|_0. \tag{28}
\]

Let \( \mathcal{H}'_{-i} \) stand for the dual spaces to \( \mathcal{H}'_i \) with the respect to \( \mathcal{H} \). We keep \( q \) for the integer with \( |q| \leq 2 \). Note, that \( \mathcal{H}'_q \) in some sense generalize the notation of a homogenous Sobolev space \( [19] \). In further discussion we shall use only \( \mathcal{H}'_1 \) and \( \mathcal{H}'_{-1} \). Obviously, we have \( \mathcal{H}_1 \subset \mathcal{H}'_1 \) and \( \mathcal{H}'_{-1} \subset \mathcal{H}_{-1} \).

**Proposition 9** For \( A^{1/2} e_k \in \mathcal{H}'_{-1} \) the expression \( s_k(M_k) \) is finite and given by

\[
s_k(M_k) = M_k \left\| A^{1/2} e_k \right\|_{-1}^2. \tag{29}
\]

Otherwise, i.e. for \( A^{1/2} e_k \in \mathcal{H}_{-1} \setminus \mathcal{H}'_{-1} \) we have \( s_k(M_k) = \infty \).
Proof. Note that \( s_k(E) = E(U_{E0}^k e_k, e_k) \) can be equivalently written as 
\[ s_k(E) = E \| A^{1/2} (A - E)^{-1/2} e_k \|^2. \]
Then we have \( s_k(M_k) = M_k \| A^{1/2} e_k \|_{-1}^2 \) which is finite iff \( A^{1/2} e_k \in H'_{-1}. \]

To summarize above discussion let us select four different cases.

Case 1. Let \( e_k \in H_0 \setminus H_1 \) and \( A^{1/2} e_k \in H_{-1} \setminus H'_{-1} \) for all \( k \in \mathbb{S} \). Then
\[ N_\pm = \# \{ b_k : b_k < 0 \}, \quad N_\pm = \# \{ b_k : b_k > 0 \}. \]

Case 2. Let \( e_k \in H_0 \setminus H_1 \) and \( A^{1/2} e_k \in H'_{-1} \) for all \( k \in \mathbb{S} \). Then
\[ N_\pm = \# \{ b_k : b_k < 0 \}, \quad N_\pm = \# \{ b_k : 0 < b_k \leq M_k \| A^{1/2} e_k \|_{-1} \}. \]

Case 3. Let \( e_k \in H_1 \) and \( A^{1/2} e_k \in H_{-1} \setminus H'_{-1} \) for all \( k \in \mathbb{S} \). Then
\[ N_\pm = \# \{ b_k : -\| e_k \|_1^2 \leq b_k < 0 \}, \quad N_\pm = \# \{ b_k : b_k > 0 \}. \]

Case 4. Let \( e_k \in H_1 \) and \( A^{1/2} e_k \in H'_{-1} \) for all \( k \in \mathbb{S} \). Then
\[ N_\pm = \# \{ b_k : -\| e_k \|_1^2 \leq b_k < 0 \}, \quad N_\pm = \# \{ b_k : 0 < b_k \leq M_k \| A^{1/2} e_k \|_{-1} \}. \]

Observe that for cases 1, 2 operator \( \tilde{A} \) belongs to strongly singular class \( A_{ss}(A) \). In turn, for cases 3, 4 \( \tilde{A} \) belongs to weakly singular class \( A_{ws}(A) \).

We also remark that for \( e_k \in H_1 \) condition \( A^{1/2} e_k \in H'_{-1} \) is equivalent to \( \tilde{A} e_k \in H'_{-1} \). Indeed, one can show
\[ M_k \| A^{1/2} e_k \|_{-1}^2 = \| \tilde{A} e_k \|_{-1}^2 - \| e_k \|_1^2. \]
So, the pure point spectrum \( \sigma_p(\tilde{A}) \) can be characterized in the terms of \( \tilde{A} e_k \) as was done in \( \text{[8]} \) for the case \( \tilde{A} \in A_{ws}(A) \).

Let us make short digression about the absolutely continuous spectrum of \( \tilde{A} \) with \( B^{-1} = A^{-1} - A^{-1} \) of type \( \text{[24]} \). Note that we have
\[ \sigma_{ac}(\tilde{A}) = \bigcup_{k \in \mathbb{S}} \sigma(\tilde{A}^{\tau_k}) \]
where the inverse of \( \tilde{A}^{\tau_k} \) has the form
\[ (\tilde{A}^{\tau_k})^{-1} = (A^{\tau_k})^{-1} + b_k^{-1} (-, e_k) e_k. \]
So, we see that \( \sigma_{ac}(\tilde{A}^{\tau_k}) = \sigma_{ac}(A^{\tau_k}) = [M_k, \infty) \). Moreover, observing that \( \bigcup_{k \in \mathbb{S}} [M_k, \infty) = [\lambda, \infty) \) we get
\[ \sigma_{ac}(\tilde{A}) = \sigma(A) = [\lambda, \infty). \] (30)
3 EIGENVALUES PROBLEM FOR ADDITIVE STRONGLY SINGULAR PERTURBATION OF LAPLACE OPERATOR.

In this section we put $H = L^2(\mathbb{R}^3, dx) \equiv L^2$ and $A = -\Delta + \lambda : D(A) \to L^2$ where $\Delta$ stands for the self-adjoint Laplace operator in $L^2$ and $\lambda > 0$. Clearly, we have $\sigma(A) = [\lambda, \infty)$. For our convenience we put $G := A^{-1}$. Then $G$ is an integral operator with kernel $G(x-y)$ given by

$$G(x) = \frac{1}{4\pi} \frac{\exp(-\sqrt{\lambda}|x|)}{|x|}. \quad (31)$$

As in general discussion we construct the chain of Hilbert spaces $\mathcal{H}_q$ defined as the completions of $D(A)$ in norms

$$\|f\|_q^2 = \int_{\mathbb{R}^3} \left|(-\Delta + \lambda)^{q/2} f(x)\right|^2 dx \quad (32)$$

coincides with the Sobolev spaces $W^{2,q}(\mathbb{R}^3) \equiv W^{2,q}$. As before, we put $\mathcal{A}$ for the extension of $A : D(A) \subset L^2 \to W^{2,-2}$ and $\mathcal{G}$ for its inverse.

Consider operator $\mathcal{V} : D(\mathcal{V}) \subset C(\mathbb{R}^3) \equiv C^0 \to W^{2,-2}$ satisfying two conditions

- $K)$ $\ker \mathcal{V} \cap W^{2,2}$ is dense in $L^2$.
- $R)$ $\text{Ran}\mathcal{V} \cap W^{2,-1} = \{0\}$.

The technics proposed in [13] allows to construct the operator belonging to $\mathcal{A}_s(A)$ which is, in some sense, a sum of $\mathcal{A}$ and $\mathcal{V}$. The concept is based on the analogy with the generalized sum (see for example [3, 8, 10, 14, 17]). Now, we present the main results of [13].

Let us introduce $G_r, G_s$ for the integral operator with kernels given by

$$G_s(x) := \frac{1}{4\pi} \frac{1}{|x|}, \quad G_r(x) := G(x) - G_s(x).$$

Define

$$C_r := I + G_r \mathcal{V} : D(C_r) = \{g \in D(\mathcal{V}) : C_r g \in W^{2,2}\} \to W^{2,2}$$

and

$$C_s := I - G_s \mathcal{V} : D(C_s) = D(C_r) \to L^2.$$
Let $f \in D(C_{s}^{-1})$ and $f_{r} = C_{s}^{-1}f$. Define the set
\[ D(\mathcal{A}+\mathcal{V}) = \{ f \in D(C_{s}^{-1}) : \mathcal{A}f + \mathcal{V}f_{r} \in L^{2} \} \] (33)
and operator $\mathcal{A}+\mathcal{V}$ which acts as
\[ (\mathcal{A}+\mathcal{V})f = \mathcal{A}f + \mathcal{V}f_{r}, \quad f \in D(\mathcal{A}+\mathcal{V}). \] (34)

To explain (33) and (34) we assume $f \in D(\mathcal{A}+\mathcal{V})$. Then $\mathcal{A}f + \mathcal{V}f_{r} = g \in L^{2}$ implies $f = Gg - \mathcal{G}\mathcal{V}f_{r}$. Since $Gg \in W^{2,2}$ by the Sobolev theorem we have $Gg \in C^{0}$. However $f$ possess singularity induced by $\mathcal{G}\mathcal{V}f_{r}$. Thus, to regularize $f$ consider $f + G_{s}\mathcal{V}f_{r}$. On the other hand $f_{r} = C_{s}^{-1}f$ yields $f_{r} = f + G_{s}\mathcal{V}f_{r}$.

So we see that $f_{r}$ is just a regularization of $f$.

With this notation we have the following theorem.

**Theorem 10** Let us assume that $C_{r}$ is invertible and operator
\[ \hat{B}^{-1} = -\mathcal{G}\mathcal{V}C_{r}^{-1}G : D(\hat{B}^{-1}) = AD(C_{r}^{-1}) \to L^{2} \] (35)
is self-adjoint. Then $\mathcal{A}+\mathcal{V} \in \mathcal{A}_{s}(A)$ and its inverse is given by
\[ (\mathcal{A}+\mathcal{V})^{-1} = A^{-1} + \hat{B}^{-1} : D(\hat{B}^{-1}) \to L^{2}. \] (36)

**Proof.** Let $C_{r}$ be invertible. First we shall show that $A+\mathcal{V}$ is invertible also and its inverse has the form (36). Let $f \in \ker(A+\mathcal{V})$ and $f_{r} = C_{s}^{-1}f$. Then
\[ \mathcal{A}f + \mathcal{V}f_{r} = 0 \]
i.e.
\[ f = -\mathcal{G}\mathcal{V}f_{r} = -\mathcal{G}\mathcal{V}C_{s}^{-1}f. \]

Then we get
\[ C_{r}C_{s}^{-1}f = 0. \]
Since $\ker C_{r} = \ker C_{s}^{-1} = \{ 0 \}$ we obtain $f = 0$. Now let $f \in D(\mathcal{A}+\mathcal{V})$, $f_{r} = C_{s}^{-1}f$. Then
\[ g = (\mathcal{A}+\mathcal{V})f = \mathcal{A}f + \mathcal{V}f_{r} \in L^{2}. \]
So, we have
\[ f = A^{-1}g - \mathcal{G}\mathcal{V}C_{s}^{-1}f. \] (37)
From (37) follows
\[ C_{r}C_{s}^{-1}f = Gg, \] (38)
i.e. $Gg \in D(C_{r}^{-1})$. Finally after inserting (38) to (37) we get

$$f = A^{-1}g - G\mathbb{V}C_{r}^{-1}Gg = A^{-1}g + \tilde{B}^{-1}g.$$  

This means that operator $(A\tilde{+}\mathbb{V})^{-1}$ is given by (36). Now we shall show that $B^{-1} \in B(A)$. For this aim let us note that by a self-adjointness of $B^{-1}$ we have $G \ker \tilde{B}^{-1} = \{ GRan \tilde{B}^{-1} \}$, where $\perp$ denotes orthogonal completion in $W^{2,2}$ topology. Further let us note that $[G^{2}Ra\tilde{n}] \subseteq [GRan \tilde{B}^{-1}]$. In turn, since operator $\mathbb{V}$ has property $R$ we get $GRan \mathbb{V} \perp \subseteq GRan \tilde{B}^{-1}[GRan \tilde{B}^{-1}]$. Hence we can conclude that $GRan \tilde{B}^{-1} \perp$ is dense in $L^{2}$. Hence we get the density of $GRan \tilde{B}^{-1} \perp$ and $GRan \tilde{B}^{-1} \perp$ in $L^{2}$. Operator $\tilde{B}^{-1}$ belongs to $B(A)$.

In fact the above construction of $A\tilde{+}\mathbb{V}$ can be repeated for an arbitrary $d \geq 3$. However in general case we define $G_{s}$ as the integral operator with the kernel $G_{s}(x) = c_{1}|x|^{d-2}$ where $c_{1}$ is apropriated constance.

To investigate eigenvalues problem for $A\tilde{+}\mathbb{V}$ let us recall spectral representation of $A := -\Delta + \lambda : D(A) \rightarrow L^{2}$. Put $r, \theta, \phi$ for the spherical coordinates in $\mathbb{R}^{3}$. It is known that $L^{2}([0, \infty))$ can be written as

$$L^{2}([0, \infty)) = L^{2}([0, \infty), r^{2}dr) \otimes L^{2}(S^{1})$$

where $S^{1}$ is unit sphere. This leads to the following direct sum decomposition

$$L^{2}(\mathbb{R}^{3}) = \bigoplus_{l=0}^{\infty} \bigoplus_{m=-l}^{l} \mathcal{H}^{lk}, \quad \mathcal{H}^{lk} = L^{2}([0, \infty), r^{2}dr) \otimes Y_{lk}(\theta, \phi) \quad (39)$$

where $Y_{lk}(\theta, \phi)$ are spherical harmonics. Then $f \in L^{2}$ can be represented in the following form

$$f(x) = \sum_{l=0}^{\infty} \sum_{k=-l}^{l} \int_{0}^{\infty} dp p^{2} \hat{f}_{lk}(p) j_{l}(pr) Y_{lk}(\theta, \phi)$$

where $j_{l}(pr)$ are spherical Bessel’s functions. Keeping consistency with (39) define unitary operator

$$U : L^{2}(\mathbb{R}^{3}) \rightarrow \bigoplus_{l=0}^{\infty} \bigoplus_{k=-l}^{l} L^{2}([\lambda, \infty), d\mu_{lk}), \quad d\mu_{lk}(t) = \frac{1}{2} \sqrt{t-\lambda} dt \quad (40)$$

by

$$(Uf)_{lk} = \tilde{f}_{lk}(\sqrt{p} - \lambda).$$
Then we have
\[ U(\Delta + \lambda)^{-1}\psi)_{lk} = t\psi_{lk}(t), \quad \psi \in \bigoplus_{l=0}^{\infty} \bigoplus_{k=-l}^{l} L^2([\lambda, \infty), d\mu_{lk}). \]

In accordance with (39) operator \( A \) can be decomposed as
\[ A = \bigoplus_{l=0}^{\infty} \bigoplus_{k=-l}^{l} A^{lk}; \quad A^{lk} = A\mathcal{H}^{lk} \]
and
\[ \sigma(A^{lk}) = [\lambda, \infty) \quad \text{for all } l, k. \quad (41) \]

Assume that \( \tilde{A} \in A_s(A) \) and its inverse is given by \( \tilde{A}^{-1} = A^{-1} + \tilde{B}^{-1} \) where \( A = -\Delta + \lambda \). Observe that by (41) \( \tilde{A} \) satisfies \( \hat{\sigma}(16) \) iff
\[ \text{Ran} \tilde{B}^{-1} \cap \text{Ran}(A - E) = \{0\} \quad \text{for all } E \in [\lambda, \infty). \quad (42) \]

The problem at hand is to select class of operators \( V \) so that \( A \hat{+} V \) satisfy \( \hat{\sigma} \).

Let \( N \) be a compact with boundary of \( C^1 \) class. We shall say that operator \( V \) satisfying \( K, R \) has the property \( \hat{N} \) if any \( \mu \in \text{Ran} V \) is a distribution from \( W^{2,2} \) supported by the compact set \( N \).

**Theorem 11** Let \( V \) have property \( \hat{N} \) and \( A \hat{+} V \in A_s(A) \). Then \( A \hat{+} V \) satisfies \( \hat{\sigma} \).

**Proof.** Let \( V \) have property \( \hat{N} \). Assume that \( g \in \text{Ran} \tilde{B}^{-1} \cap \text{Ran}(A - E) \) where \( E \in [\lambda, \infty) \). Since \( \text{Ran} \tilde{B}^{-1} \subseteq \text{Ran} G V \) (see (35)) there exists \( \mu \in W^{2,2} \) supported by the compact set \( N \)) so that \( g = G\mu \). Clearly, \( \mu \in \text{Ran}(A - E) \).

Let \( f \in \mathcal{H} \) satisfies
\[ \mu(x) = (A - E)f(x). \quad (43) \]
Expressing \( G(x - y) \) (31) in the spherical system of coordinates one can show that \( |G\mu| \) behaves like \( \frac{|\mu|}{|x|} \) as \( |x| \to \infty \). Thus \( g \in L^{2,s} \equiv L^2(\mathbb{R}^3, (|x|^2 + \lambda)^s dx), \)
\( s > 1/2 \) and \( \mu \in W^{2,-2,s} \equiv W^{2,-2}((\mathbb{R}^3, (|x|^2 + \lambda)^s dx) \). Relying on the results of [1] we get that for all \( E \in [\lambda, \infty), s > 1/2 \) the limits
\[ K^\pm \equiv \lim_{\varepsilon \to 0} (A - E \pm i\varepsilon)^{-1} : W^{2,-2,s} \to L^{2,-s} \]
exist in the uniform operator norm. Besides
\[ K^\pm \mu = f. \]

It known that \( K^\pm \) can be represented as the integral operator with the kernel given by
\[ K^\pm(x - y) = (4\pi)^{-1} \exp(\pm i\sqrt{E - \lambda} |x - y|) |x - y|^{-1}. \]
Therefore, \( f \) can be written as

\[
f(x) = \int_{\mathbb{R}^3} K^\pm(x - y) \mu(y) dy = \int_{\mathcal{N}_\mu} K^\pm(x - y) \mu(y) dy.
\]

Similarly as before, expressing \( K^\pm(x - y) \) in the spherical system of coordinates, we get the following asymptotics:

\[
|f(x)| \sim |x|^{-1} \quad \text{for} \quad |x| \to \infty.
\]

Then the fact that \( f \in L^2 \) implies \( f = 0 \).

Thus, \( \mu = 0 \) as well as \( g = 0 \) and (42) is satisfied.

Above theorem shows that if \( \mathbf{V} \) has property \( \hat{\mathbf{N}} \) then operator \( \hat{A} + \mathbf{V} \) satisfies \( \hat{\sigma} \). This result and the fact \( \sigma(A^k) = \sigma(A) \) implies by corollary 6 the following statement.

**Proposition 12** Let \( \mathbf{V} \) have the property \( \hat{\mathbf{N}} \) and \( \hat{A} + \mathbf{V} \in \mathcal{A}_s(A) \). Then \( \sigma_p(\hat{A} + \mathbf{V}) \subset (-\infty, \lambda) \).

Now, we shall consider two particular cases of \( \hat{A} + \mathbf{V} \) and investigate eigenvalues problem relying on abstract results of section 2 and 3.

### 4 EXAMPLES

#### 4.1 STRONGLY SINGULAR PERTURBATION OF LAPLACE OPERATOR BY THE DYNAMICS LIVING ON INTERVAL

This example is not strictly in the scheme of our previous discussion because we take \( \mathcal{H} = L^2(\Omega, dx) \equiv L^2(\Omega) \), \( \Omega = (0, \pi) \times \mathbb{R}^2 \) instead of \( L^2(\mathbb{R}^3, dx) \).

Let \( A \equiv -\Delta \) stand for the Laplace operator with the Dirichlet boundary condition in \( L^2(\Omega) \)

\[
D(A) = \left\{ \frac{1}{2} \pi^3 \sum_{k=1}^{\infty} \int_{\mathbb{R}^2} dp \tilde{u}_k(p) e^{ipx} \sin kx_1; \quad 2 \sum_{k=1}^{\infty} \int_{\mathbb{R}^2} dp \left| (p^2 + k^2) \tilde{u}_k(p) \right|^2 < \infty \right\} \to L^2(\Omega).
\]

Then we have \( \sigma_{ac}(A) = \sigma(A) = [1, \infty) \). Setting \( G \) for the inverse of \( A \) we obtain an integral operator with the kernel

\[
G(x, y) = \frac{1}{2} \pi^3 \sum_{k=1}^{\infty} \int_{\mathbb{R}^2} dp \frac{e^{ip(x-y)}}{p^2 + k^2} \sin kx_1 \sin ky_1
\]

where \( p = (p_2, p_3) \), \( x = (x_2, x_3) \) and \( p^2 = p_2^2 + p_3^2 \). According to the general discussion (see (6)) we construct spaces \( \mathcal{H}_q \) which are given by

\[
\mathcal{H}_q = \left\{ u(x) = \frac{1}{2} \pi^3 \sum_{k=1}^{\infty} \int_{\mathbb{R}^2} dp \tilde{u}_k(p) e^{ipx} \sin kx_1; \quad \sum_{k=1}^{\infty} \int_{\mathbb{R}^2} dp (p^2 + k^2)^q \left| \tilde{u}_k(p) \right|^2 < \infty \right\}.
\]
Now, our aim is to construct an operator from $A_s(A)$ corresponding to the formal expression

$$-\Delta + \alpha (-\frac{\partial^2}{\partial x_1^2})\delta(x).$$

where $\alpha \in \mathbb{R}$. Let

$$\mathcal{V}_\alpha \equiv -\alpha \Delta_1 \delta(x) : D(\mathcal{V}_\alpha) \to W^{2,-2}$$

be given by

$$\mathcal{V}_\alpha f = \sum_{k=1}^{\infty} \alpha k^2 c_k(f) \sin kx_1 \delta(x), \quad D(\mathcal{V}_\alpha) = \{ f \in C(\Omega) : \mathcal{V}_\alpha f \in W^{2,-2} \}$$

where $c_k(f) = \int_0^\pi dx_1 f(x_1,0,0) \sin kx_1$. It is not hard to see that $\text{Ran} \mathcal{V}_\alpha \subset \mathcal{H}_{-2} \setminus \mathcal{H}_{-1}$.

Similarly as in the previous discussion we can construct operator $-\Delta + \alpha (-\Delta_1) \delta(x)$ which acts $( -\Delta + \alpha (-\Delta_1) \delta(x) ) f = -\Delta f + \alpha (-\Delta_1) \delta(x) f_r$ where $f_r$ is some regularization of $f$. This problem is discussed in detail in [13]. Now we give only the final result.

For each $k \in \mathbb{N}$ we $s_k = \pi^2 (-\ln \frac{k}{2} + C)$ where $C$ is the Euler constant. We also define

$$e_k(x) := \frac{1}{2} \pi^{-3} \int_\Omega dy G(x,y) \sin ky_1 \delta(y) = \frac{1}{2} \pi^{-3} \int_{\mathbb{R}^2} \frac{e^{ipy}}{(p^2 + k^2)} \sin ky_1 \quad (44)$$

One can show by a direct calculation that $\{e_k\}_{k \in \mathbb{N}}$ is the orthogonal system.

**Theorem 13 ([13])** Operator $-\Delta + (-\Delta_1) \delta(x) \in A_{ss}(A)$ and its inverse is given by

$$(-\Delta + (-\Delta_1) \delta(x))^{-1} = G + \tilde{B}^{-1} : D(\tilde{B}^{-1}) = \{ f \in L^2(\Omega) : \tilde{B}^{-1} f \in L^2(\Omega) \}$$

where

$$\tilde{B}^{-1} f = \sum_{k=1}^{\infty} b_k^{-1}(f,e_k)e_k, \quad b_k^{-1} = b_{\alpha,k}^{-1} = -\alpha k^2 (1 + \alpha k^2 s_k)^{-1}.$$

To solve the eigenvalues problem for operator $(-\Delta + \alpha (-\Delta_1) \delta(x))$ we will apply the results of section 2. First, let us note that $L^2(\Omega)$ possess the following orthogonal decomposition

$$L^2(\Omega) = \bigoplus_{k=1}^{\infty} \mathcal{H}^k, \quad \text{where} \quad \mathcal{H}^k = L^2(\mathbb{R}^2) \otimes \sin kx_1.$$

Further for all $k \in \mathbb{N}$ we have

$$\mathcal{H}^k = \bigoplus_{l=-\infty}^{\infty} L^2([0,\infty), r dr) \otimes e^{ik\phi} \otimes \sin kx_1.$$
Then any function \( f \in L^2(\Omega) \) can be written in the following form

\[
f(x) = \sum_{k=1}^{\infty} \sum_{l=-\infty}^{\infty} \int_{0}^{\infty} pdp \hat{f}_{lk}(p) J_{l}(pr)e^{ik\varphi} \sin kx_1
\]

where \( J_l(pr) \) are cylindrical Bessel’s functions. Define unitary operator

\[
U : L^2(\Omega) \to L^2([k^2, \infty), dp_{lk}), \quad dp_{lk}(t) = 1/2tdt
\]

by

\[
(Uf)_{lk} = \hat{f}_{lk}(\sqrt{t - \lambda}).
\]

The spectral representation of \(-\Delta\) has the form

\[
(U(-\Delta)U^{-1}\psi)_{lk}(t) = t\psi_{lk}(t), \quad \psi \in \bigoplus_{k=1}^{\infty} L^2([k^2, \infty), dp_{lk}).
\]

For all \( k \in \mathbb{N} \), put \( -\Delta^k \equiv -\Delta|D(A) \cap \mathcal{H}^k \). Then \( \sigma(-\Delta^k) = [k^2, \infty) \).

Let us note that operator \(-\Delta + \alpha V\) satisfies \( \Theta \) iff for each \( k \in \mathbb{N} \) we have

\[
e_k \notin \text{Ran}(-\Delta^k - E) \quad \text{for all } E \in [k^2, \infty).
\]

(46)

Proceeding analogously as in the proof of theorem 12 one can show that (48) is fulfilled. Next, observing that \( e_k \in \mathcal{H}^k \) for all \( k \in \mathbb{N} \) we get that \( \tilde{B}^{-1} \) has the form of (24). This allows to use theorem 7.

Let us mention that this model corresponds to case 1 described at the end of section 2. However instead of showing this fact we solve this example explicitely.

Given \( k \in \mathbb{N} \) assume \( E_k \in (-\infty, k^2) \) and define \( U_{E_k,0} \equiv U_{E_k} = -\Delta^k(-\Delta^k - E_k)^{-1} : \mathcal{H}^k \to \mathcal{H}^k \). As follows from theorem 7 number \( E_k \) belongs to \( \sigma_p(-\Delta + \alpha(-\Delta_1)\delta(x)) \) iff condition

\[
b_k^{-1}(U_{E_k,0}, e_k) = E_k^{-1}
\]

(47a)

holds. A direct calculation yields

\[
(U_{E_k,0}, e_k, e_k) = \frac{1}{8\pi^4 E_k} \frac{1}{\alpha k^2 - E_k} \ln \frac{\alpha k^2}{\alpha k^2 - E_k}.
\]

Then, from (47a) we get \( E_k = k^2(1 - e^{-b_k}) \) where \( b_k = \frac{1}{8\pi^4 E_k} \ln \frac{\alpha k^2}{\alpha k^2 - E_k} \). By theorem 7 we obtain the following result.

**Corollary 14** The discrete spectrum of \(-\Delta + \alpha(-\Delta_1)\delta(x)\) is given by

\[
E_k = k^2(1 - e^{-2b_k})
\]

(48)

where \( b_k = -\frac{1}{8\pi^4} (\alpha k^2)^{-1} (1 + s_k \alpha k^2), s_k = \pi^{-2}(-\ln k/2 + C) \). The corresponding eigenvectors have the form

\[
f_k = ((-\Delta + (-\Delta_1)\delta(x))^{-1} U_{E_k,0}, e_k = \frac{1}{2} \pi^{-3} \int_{\mathbb{R}^2} dp \frac{e^{ipx}}{\sqrt{p^2 + k^2}} \frac{1}{p^2 + k^2 - E_k} \sin kx_1
\]

\[
+ \frac{1}{2E_k} \ln \frac{k^2}{k^2 + E_k} \int_{\mathbb{R}^2} dp \frac{e^{ipx}}{p^2 + k^2} \sin kx_1).
\]
Let us close this example by a short discussion of (48). One can check that 
\( s_k < 0 \) iff \( k < 3 \). So, to describe positive and negative pure point spectrum we select two cases.

Let \( k \leq 3 \). Then \( E_k > 0 \) iff \( \alpha \in (0, -k^{-2}s_k^{-1}) \).

Let \( k > 3 \). Then \( E_k > 0 \) iff \( \alpha \in (-\infty, k^{-2}s_k^{-1}) \cup (0, \infty) \).

From (48) we obtain the following asymptotic behaviour of \( E_k = E_k(\alpha) \) as \( \alpha \to -\infty \)

\[
\lim_{\alpha \to -\infty} E_k(\alpha) = k^2(1 - e^{C(\frac{2}{R})^2s^2}).
\]

Finally, we observe that

\[
\lim_{k \to \infty} E_k = \infty.
\]

Since \( \sigma_{ac}(-\Delta) = \sigma_{ac}((-\Delta + \alpha(-\Delta_1)\delta(x)) = [1, \infty) \) (see 30) we get an interesting result that \( E_k \in \sigma_{ac}((-\Delta + \alpha(-\Delta_1)\delta(x)) \) for sufficiently large \( k \) i.e.

\[
\sigma_p((-\Delta + \alpha(-\Delta_1)\delta(x)) \cap \sigma_{ac}((-\Delta + \alpha(-\Delta_1)\delta(x)) \neq \{0\}.
\]

4.2 STRONGLY SINGULAR PERTURBATION OF LAPLACE OPERATOR BY THE DYNAMICS LIVING ON CIRCLE

Let \( \mathcal{H} \) and \( A \) be as in section 3 i.e. \( \mathcal{H} = L^2(\mathbb{R}^3, dx) \equiv L^2 \), \( A = -\Delta + \lambda \), \( \lambda > 0 \).

Then \( \mathcal{H}_q \) coincides with the Sobolev spaces \( W^{2,q}(\mathbb{R}^3) \equiv W^{2,q} \) (see (32)).

Let us recall that operator \( G = A^{-1} \) can be presented by the integral kernel

\[
G(x) = \frac{1}{4\pi} \exp(-\sqrt{\lambda} |x|) = \frac{1}{(2\pi)^3} \int_{\mathbb{R}^3} dp \frac{e^{ipx}}{p^2 + \lambda}.
\]

We also introduced notation \( G_r \) for operator with kernel \( G_r(x - y) \) given by

\[
G_r(x) = G(x) - \frac{1}{4\pi} \frac{1}{|x|}.
\]

As was shown the spectral representation of \( A \) determinates the following decomposition \( A = \bigoplus_{l=0}^{\infty} \bigoplus_{k=-l} A^{lk} \) where \( \sigma(A^{lk}) = [\lambda, \infty) \) for each \( k,l \). Similarly, as in the general discussion (28) we define spaces \( \mathcal{H}'_k \) as the completions of \( D(A) \) in norms

\[
\|u\|_k' = \left\| (A - \lambda)^{k/2} u \right\|_0
\]

i.e. we have

\[
\|u\|_k = \left\| (-\Delta)^{k/2} u \right\|_0 = \left\{ \int_{\mathbb{R}^3} dx \left| (-\Delta)^{k/2} u(x) \right|^2 \right\}^{1/2}.
\]
We again put \( r, \theta, \phi \) for the spherical coordinates in \( \mathbb{R}^3 \). Given real function \( V \in C(\mathbb{R}) \) define self-adjoint operator \( V(\partial^2/\partial^2 \phi) \) in \( L^2((0, 2\pi), d\phi) \) where \( \partial^2/\partial^2 \phi \) is the Laplacian with periodic boundary condition. For each \( k \in \mathbb{Z} \) put \( v_k = V(k^2) \).

In this example we will investigate the eigenvalues problem for operator \( \tilde{A} \in \mathcal{A}_\alpha(A) \) which formally corresponds to

\[
(-\Delta + \lambda) + V(\partial^2/\partial^2 \phi) \delta(r-1)\delta(\cos \theta).
\]

Let us put for abbreviation \( N = \{ r = 1, \theta = \frac{\pi}{2}, \phi \in (0, 2\pi) \} \) and \( \delta_c(r, \theta) \equiv \delta(r-1)\delta(\cos \theta) \) where the subscript \( c \) suggests that the support of \( \delta_c \) coincides with the circle in \( \mathbb{R}^3 \).

Define operator \( V_\alpha = -\alpha V(\Delta \phi) \delta_c : D(V_\alpha) \rightarrow W^{2,-2} \) by

\[
V_\alpha f = \sum_{k \in \mathbb{Z}} \alpha v_k c_k(f)e^{ik\phi} \delta_c(r, \theta); \quad D(V_\alpha) = \{ f \in C(\mathbb{R}^3) : V_\alpha f \in W^{2,-2} \}
\]

where \( c_k(f) = \int_0^{2\pi} d\phi f(r = 1, \theta = \frac{\pi}{2}, \phi)e^{-ik\phi} \) and \( \alpha \in \mathbb{R}\setminus\{0\} \).

The following facts \( \text{Ran} V_\alpha \subset W^{2,-2}\setminus W^{2,-1} \) and \( C_\alpha(\mathbb{R}^3\setminus N) \subset \ker V_\alpha \) ensure that conditions K) and R) are satisfied.

According to general discussion presented in the previous section we construct operator

\[
\tilde{A}_V = (-\Delta + \lambda) + (-\alpha V(\Delta \phi) \delta_c).
\]

This construction is described in \([11]\). Below we give the final result.

Let us abbreviate \( (p, \phi) = p_1 \cos \phi + p_2 \sin \phi + p_3 \) and put for each \( k \in \mathbb{Z} \)

\[
e_k = Ge^{ik(\cdot)} \delta_c = \frac{1}{(2\pi)^3} \int_{\mathbb{R}^3} dp \int_0^{2\pi} d\phi \frac{e^{i(px-(p,\phi))}}{p^2 + \lambda} e^{ik\phi}
\]

and

\[
q_k \equiv c_k(G e^{ik(\cdot)}).
\]

One can check that \( \{e_k\}_{k\in\mathbb{Z}} \) is the orthogonal system.

**Theorem 15 \([11]\)** Let us assume that \( \alpha v_k q_k \neq -1 \) for all \( k \in \mathbb{Z} \). Then operator \( \tilde{A}_V \in \mathcal{A}_{ss}(A) \) and its inverse is given by

\[
\tilde{A}_V^{-1} = G + \tilde{B}^{-1} : D(\tilde{B}^{-1}) = \{ f \in L^2(\Omega) : \tilde{B}^{-1} f \in L^2(\Omega) \}
\]

where

\[
\tilde{B}^{-1} f = \sum_{k=1}^{\infty} b_k^{-1}(f, e_k) e_k, \quad b_k^{-1} \equiv b_{\alpha,k}^{-1} = -\alpha v_k (1 + \alpha v_k q_k)^{-1}.
\]
Since this model is more complicated from the technical point of view we restrict ourselves to some estimations.

First, we will show that this example represents case 2 described at the end of section 2. Clearly $\tilde{A}_V$ satisfies $\tilde{\sigma}$. Further, let us note that decomposition (39) can be equivalently written

$$L^2(\mathbb{R}^3) = \bigoplus_{l=0}^{\infty} \mathcal{H}^l = \bigoplus_{k=\infty}^{\infty} \mathcal{H}^k$$

where $\mathcal{H}^k = L^2((0, \infty), r^2dr) \otimes Y_k(\theta, \phi)$. A direct computation shows

$$e_k \in \bigoplus_{l=0}^{\infty} \mathcal{H}^l$$

for each $k \in \mathbb{Z}$.

Therefore $\tilde{B}^{-1}$ has a form of (24). Now, it suffices to check that $\tilde{A}_V^{1/2}e_k \in \mathcal{H}'_{-1}$ for all $k \in \mathbb{Z}$. Indeed, we have

$$\|\tilde{A}_V^{1/2}e_k\|_{-1}^2 = \int_{\mathbb{R}^3} dp \int_0^{2\pi} d\phi \int_0^{2\pi} d\phi' e^{i(p,\phi)e^{i(p,\phi')}} p^2 (p^2 + \lambda) \leq 4\pi^2 \int_0^{\infty} dp \frac{1}{p^2 + \lambda} = 2\pi^3 \lambda^{-1/2}.$$ 

So, we get $E_k \in \sigma_p\left(\tilde{A}_V\right)$ iff

$$b_k \in (-\infty, \lambda \|\tilde{A}_V^{1/2}e_k\|_{-1}^2).$$

Moreover, we have

$$N_+ = \#\{b_k : 0 < b_k < \lambda \|\tilde{A}_V^{1/2}e_k\|_{-1}^2\}, \quad N_- = \#\{b_k : b_k < 0\}.$$ 

Note that (50) is equivalent to

$$-\frac{1 + \alpha v_k q_k}{\alpha v_k} \leq \lambda \|\tilde{A}_V^{1/2}e_k\|_{-1}^2.$$ 

A direct calculation using (49) we get

$$q_k = -\alpha v_k \|\tilde{A}_V^{1/2}e_k\|_{-1}^2.$$ 

So (51) is satisfied iff $\alpha v_k > 0$. On the other hand for $\alpha v_k > 0$ we have $b_k < 0$. Hence

$$N_+ = 0, \quad N_- = \#\{v_k : \alpha v_k > 0\}.$$ 

Then in this model we have

$$\sigma_p(\tilde{A}_V) \cap \sigma_{ac}(\tilde{A}_V) = \{0\}.$$ 

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