Application of double Laplace decomposition method to solve a singular one-dimensional pseudohyperbolic equation

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Abstract
In this work, the double Laplace decomposition method is applied to solve singular linear and nonlinear one-dimensional pseudohyperbolic equations. This method is based on double Laplace transform and decomposition methods. In addition, we prove the convergence of our method. This method is described and illustrated by some examples. These results show that the introduced method is highly accurate and easy to apply.

Keywords
Laplace transform, inverse double Laplace transform, singular pseudohyperbolic, single Laplace transform, decomposition methods and partial derivative

Introduction
The linear and nonlinear pseudohyperbolic equations are the important classes of evolution equations which have been developed in recent years, and there is an extensive application in chemistry, plasma physics, thermo-elasticity, and engineering. Many powerful methods have been developed to solve linear and nonlinear partial differential equations (PDEs), such as homotopy perturbation method,¹² combined Laplace transforms and decomposition method,³ the transformed rational function method which presents exact traveling wave solutions to nonlinear integro-differential equations has been studied in Ma and Lee,⁴ the bi-linear techniques⁵ which present multiple wave solutions to nonlinear differential equations, and the integral transform method.⁶⁻⁹ An auxiliary parameter method using Adomian polynomials and Laplace transformation have been powerfully combined¹⁰ to study the nonlinear differential equation. The one-dimensional nonlinear hyperbolic equation with Bessel operator is one of the fundamental nonlinear wave equations having many applications in science. The energy-integral method is used to handle nonlinear singular one-dimensional hyperbolic equation.¹¹ The convergence of Adomian’s method has been studied by several authors.¹²⁻¹⁸ In this article, we are concerned with the following problem

\[
\frac{\partial^2 u}{\partial t^2} - \frac{a}{x^m} \frac{\partial}{\partial x} \left( x^m \frac{\partial u}{\partial x} \right) - \frac{b}{x^m \partial t} \left( x^m \frac{\partial u}{\partial x} \right) - cx^2 \frac{\partial}{\partial x} \left( \frac{\partial u}{\partial x} \right) = f(x, t)
\]

subject to the initial conditions

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\[ u(x,0) = f_1(x), \quad \frac{\partial u(x,0)}{\partial t} = f_2(x) \quad (2) \]

where \(a, b, c\) are constants and \((1/x^m)(\partial / \partial x)(x^m(\partial u / \partial x))\) is called Bessel’s operator and \(f\) is a known function, where \(m = 1, 2, 3, \ldots\). Equation (1) is described by the following cases:

Case 1. At \(b = 0\), the equation is called singular one-dimensional like-wave equation;

Case 2. At \(c = 0\), the equation is called singular one-dimensional pseudohyperbolic equation;

Case 3. At \(a = 0\), the equation is called singular one-dimensional pseudohyperbolic-like-wave equation;

Case 4. At \(a = 0\) and \(b = 0\), the equation is called nonlinear one-dimensional like-wave equation;

Case 5. At \(b = 0\) and \(c = 0\), the equation is called singular one-dimensional wave equation;

Case 6. At \(a = 0\) and \(c = 0\), the equation is called singular one-dimensional pseudo wave equation.

In the general case when \(a \neq 0, b \neq 0,\) and \(c \neq 0,\) equation (1) is called singular one-dimensional pseudohyperbolic-like-wave equation.

The aim of this article is to use the double Laplace transform and domain decomposition method to obtain approximate solutions with high accuracy for a singular one-dimensional pseudohyperbolic equation and a singular one-dimensional pseudohyperbolic-like-wave equation. In addition, one of the main aims of this article is to provide a sufficient condition of convergence of the series.

Now, we recall the following definitions which are given by previous studies.\(^{19-22}\) The double Laplace transform is defined as

\[ L_x L_t [f(x,t)] = F(p,s) = \int_0^\infty \int_0^\infty e^{-pt} e^{-sx} f(x,t) dt dx \quad (3) \]

where \(x, t > 0\) and \(p, s\) are complex values, and further double Laplace transform of the first-order partial derivative is given by

\[ L_x L_t \left[ \frac{\partial u(x,t)}{\partial x} \right] = pU(p,s) - U(0,s). \quad (4) \]

Similarly, the double Laplace transform for second-order partial derivative with respect to \(x\) and \(t\) are defined as follows

\[ L_x L_t \left[ \frac{\partial^2 u(x,t)}{\partial^2 x} \right] = p^2 U(p,s) - pU(0,s) - \frac{\partial U(0,s)}{\partial x}, \]

\[ L_x L_t \left[ \frac{\partial^2 u(x,t)}{\partial t^2} \right] = s^2 U(p,s) - sU(0,s) - \frac{\partial U(0,s)}{\partial t}. \quad (5) \]

The following Lemma is used in this article.

**Lemma 1.** Double Laplace transform of the non-constant coefficient second-order partial derivative \(x^m(\partial^2 u / \partial t^2)\) and the function \(x^m f(x,t)\) are given by

\[ L_x L_t \left( x^m \frac{\partial^2 u}{\partial t^2} \right) = (-1)^m \frac{d^m}{dp^m} \left[ s^m L_x L_t (f(x,t)) \right] \]

and

\[ L_x L_t (x^m f(x,t)) = (-1)^m \frac{d^m}{dp^m} \left[ L_x L_t (f(x,t)) \right] \]

where \(r = 1, 2, 3, \ldots\)

One can prove this lemma using the definition of double Laplace transform in equations (3)–(5).

**Singular one-dimensional pseudohyperbolic equation**

To illustrate the basic idea of the modified double Laplace decomposition method, we assume that \(c = 0\) and \(m = 1\) in equation (1), we obtain the singular one-dimensional pseudohyperbolic equation

\[ \frac{\partial^2 u}{\partial t^2} - a \frac{\partial}{\partial x} \left( \frac{\partial u}{\partial x} \right) - b \frac{\partial^2}{\partial x \partial t} \left( \frac{\partial u}{\partial x} \right) = f(x,t) \quad (8) \]

subject to

\[ u(x,0) = f_1(x), \quad \frac{\partial u(x,0)}{\partial t} = f_2(x) \quad (9) \]

where the term \((1/x)(\partial / \partial x)(x(\partial u / \partial x))\) is Bessel operator. In the following theorem, we apply modified double Laplace decomposition methods.

**Theorem 1.** We claim that the solution of the singular one-dimensional pseudohyperbolic equation given in equation (8) is denoted by

\[ u(x,t) = L_p^{-1} L_s^{-1} \left( \frac{F_1(p)}{s} \right) + L_p^{-1} L_s^{-1} \left( \frac{F_2(p)}{s^2} \right) \]

\[ + \frac{1}{s^2} \left[ \int_0^p \left[ \frac{dF(p,s)}{dp} \right] dp \right] \]

\[ - L_p^{-1} L_s^{-1} \left[ \frac{1}{s^2} \left[ \int_0^p L_x L_t \left[ \frac{\partial}{\partial x} \left( \frac{\partial}{\partial x} \sum_{n=0}^{\infty} u_n \right) \right] dp \right] \right] \]

\[ - L_p^{-1} L_s^{-1} \left[ \frac{1}{s^2} \left[ \int_0^p L_x L_t \left[ \frac{\partial^2}{\partial x \partial t} \left( \frac{\partial}{\partial x} \sum_{n=0}^{\infty} u_n \right) \right] dp \right] \right] \quad (10) \]

where \(L_x L_t\) double Laplace transform with respect to \(x, t\) and \(L_p^{-1} L_s^{-1}\) double inverse Laplace transform with respect to \(p, s\), the function \(F(p,s), F_1(p),\) and \(F_2(p)\)
are Laplace transform of the functions $f(x,t)$, $f_1(x)$, and $f_2(x)$, respectively. Here, we provided double inverse Laplace transform with respect to $p$ and $s$ exist for each term in the right-hand side of equation (10).

**Proof.** By multiplying equation (8) by $x$ and using the definition of partial derivatives of the double Laplace transform, single Laplace transform, and the Lemma 1 for equation (8), respectively, we get

$$
\frac{dU(p,s)}{dp} = \frac{1}{s} \frac{dF_1(p)}{dp} + \frac{1}{s^2} \frac{dF_2(p)}{dp} - \frac{1}{s^2} L_x L_t \left[ \frac{dF(p,s)}{dp} \right] \, dp
$$

Applying the integral for both sides of equation (11) from 0 to $p$ with respect to $p$, we have

$$
U(p,s) = \frac{F_1(p)}{s} + \frac{F_2(p)}{s^2} + \frac{1}{s^2} \int_0^p \left[ \frac{dF(p,s)}{dp} \right] \, dp
$$

The next step in double Laplace decomposition method is representing the solution of singular one-dimensional pseudohyperbolic equation as $u(x,t)$ by the infinite series

$$
u(x,t) = \sum_{n=0}^{\infty} u_n(x,t)
$$

By applying double inverse Laplace transform for equation (12) and use equation (13), we obtain

$$
u(x,t) = L_p^{-1} L_s^{-1} \left( \frac{F_1(p)}{s} \right) + L_p^{-1} L_s^{-1} \left( \frac{F_2(p)}{s^2} \right) + L_p^{-1} L_s^{-1} \left[ \frac{1}{s^2} \int_0^p \left[ \frac{dF(p,s)}{dp} \right] \, dp \right]
$$

In particular, we have

$$
u_0 = - L_p^{-1} L_s^{-1} \left( \frac{F_1(p)}{s} \right) + L_p^{-1} L_s^{-1} \left( \frac{F_2(p)}{s^2} \right) + L_p^{-1} L_s^{-1} \left[ \frac{1}{s^2} \int_0^p \left[ \frac{dF(p,s)}{dp} \right] \, dp \right]
$$

and

$$
u_{n+1}(x,t) = - L_p^{-1} L_s^{-1} \left[ \frac{1}{s^2} \int_0^p \left[ \frac{dF(p,s)}{dp} \right] \, dp \right]
$$

By extending equation (8) as follows

$$
\frac{\partial^2 u}{\partial t^2} - a \frac{\partial}{\partial x} \left( \frac{\partial u}{\partial x} \right) - b \frac{\partial^2}{\partial x^2} \left( \frac{\partial u}{\partial x} \right) = f(x,t).
$$

On using the above theorem and Lemma 1, we have

$$
U(p,s) = \sum_{n=0}^{\infty} \left[ \frac{(-1)^n p^n}{s} \int_0^p \frac{d^n F_1(p)}{dp^n} \, dp \right] + \sum_{n=0}^{\infty} \left[ \frac{(-1)^n p^n}{s} \int_0^p \frac{d^n F_2(p)}{dp^n} \, dp \right] + \sum_{n=0}^{\infty} \left[ \frac{(-1)^n p^n}{s} \int_0^p \frac{d^n F(p,s)}{dp^n} \, dp \right] + \ldots
$$

By taking double inverse Laplace transform for equation (18), we get

$$
u(x,t) = L_p^{-1} L_s^{-1} \left( \frac{(-1)^n p^n}{s} \int_0^p \frac{d^n F_1(p)}{dp^n} \, dp \right) + L_p^{-1} L_s^{-1} \left( \frac{(-1)^n p^n}{s} \int_0^p \frac{d^n F_2(p)}{dp^n} \, dp \right) + L_p^{-1} L_s^{-1} \left( \frac{(-1)^n p^n}{s} \int_0^p \frac{d^n F(p,s)}{dp^n} \, dp \right) + \ldots
$$

Then, the solution of equation (17) is given by
with respect to $\partial_t^4 u$, we have

$$u(x, t) = x^2 t + x^2 \sin t - x^2 t + 4 \sin t - 4t + 4 \cos t - 4$$

$$-L_p^{-1} L_s^{-1} \left( \frac{1}{x^2} \right) \left( \frac{\partial}{\partial x} \left( x \frac{\partial u}{\partial x} \right) + \frac{\partial^2}{\partial x \partial t} \left( x \frac{\partial u}{\partial x} \right) \right) dp \right)

(25)

Using equations (15) and (16), we get

$$u_0 = x^2 t + x^2 \sin t - x^2 t + 4 \sin t - 4t + 4 \cos t - 4$$

$$u_0 = -L_p^{-1} L_s^{-1} \left( \frac{1}{x^2} \right) \left( \frac{\partial}{\partial x} \left( x \frac{\partial u}{\partial x} \right) + \frac{\partial^2}{\partial x \partial t} \left( x \frac{\partial u}{\partial x} \right) \right) dp \right)

(26)

The other components are given by

$$u(x, t) = u_0 + u_1 + \cdots$$

It is obvious that self-canceling some terms appear between various components and connected by coming terms, we have

$$u(x, t) = u_0 + u_1 + \cdots$$

therefore, the exact solution is given by

$$u(x, t) = x^2 \sin t$$

**Singular nonlinear one-dimensional pseudohyperbolic equation**

In this section, we discuss the use of modified double Laplace to solve the singular one-dimensional pseudohyperbolic equation

$$\frac{\partial^2 u}{\partial t^2} - \frac{1}{x} \frac{\partial}{\partial x} \left( x \frac{\partial u}{\partial x} \right) - \frac{1}{x} \frac{\partial^2}{\partial x \partial t} \left( x \frac{\partial u}{\partial x} \right)$$

$$-a(x) \frac{\partial u}{\partial x} + u^2 = f(x, t)$$

(27)

subject to

$$u(x, 0) = f_1(x), \quad \frac{\partial u(x, 0)}{\partial t} = f_2(x)$$

(28)

where $\frac{1}{x}(\partial / \partial x)(x(\partial u / \partial x))$ is Bessel operator, and $f(x, t)$ and $a(x)$ are known functions. To obtain the solution of singular one-dimensional pseudohyperbolic equation (27), we apply our method as follows. Using the definition of partial derivatives of the double Laplace transform, single Laplace transform for equations (27) and (28), respectively and Lemma 1, we have
\[
\frac{dU(p,s)}{dp} = \frac{1}{s} \frac{dF_1(p)}{dp} + \frac{1}{s^2} \frac{dF_2(p)}{dp} + \frac{1}{s^3} L_p L_s \left[ \frac{dF(p,s)}{dp} \right] \\
- \frac{1}{s^2} L_s L_t \left[ \frac{\partial}{\partial x} \left( x \frac{\partial U}{\partial x} \right) \right] - \frac{1}{s^2} L_s L_t \left[ \frac{\partial^2}{\partial x \partial t} \left( x \frac{\partial U}{\partial x} \right) \right] \\
- \frac{1}{s^3} L_s L_t \left[ x \frac{\partial u(x)}{\partial x} \right] + \frac{1}{s^2} L_s L_t \left[ xu^2 \right].
\]

By integrating both sides of equation (29) from 0 to \( p \) with respect to \( p \), we have

\[
U(p,s) = \frac{F_1(p)}{s} + \frac{F_2(p)}{s^2} + \frac{1}{s^3} \int_0^p \left( \frac{dF(p,s)}{dp} \right) dp \\
- \frac{1}{s^2} \int_0^p L_s L_t \left[ \frac{\partial}{\partial x} \left( x \frac{\partial u}{\partial x} \right) \right] dp \\
- \frac{1}{s^2} \int_0^p L_s L_t \left[ \frac{\partial^2}{\partial x \partial t} \left( x \frac{\partial u}{\partial x} \right) \right] dp \\
- \frac{1}{s^3} \int_0^p L_s L_t \left[ x \frac{\partial u(x)}{\partial x} \right] dp + \frac{1}{s^2} \int_0^p L_s L_t \left[ xu^2 \right] dp.
\]

The double Laplace Adomian decomposition method (DLADM) defines the solution of equation (27) as \( u(x, t) \) by the infinite series

\[
u(x, t) = \sum_{n=0}^{\infty} u_n(x, t).
\]

By applying double inverse Laplace transform for equation (30) and use equation (31) we get

\[
u(x, t) = L_p^{-1} L_s^{-1} \left[ \frac{F_1(p)}{s} + \frac{F_2(p)}{s^2} \right] \\
+ \frac{1}{s^2} \int_0^p \left( \frac{dF(p,s)}{dp} \right) dp \\
- \frac{1}{s^2} \int_0^p L_s L_t \left[ \frac{\partial}{\partial x} \left( x \frac{\partial u}{\partial x} \right) \right] dp \\
- \frac{1}{s^2} \int_0^p L_s L_t \left[ \frac{\partial^2}{\partial x \partial t} \left( x \frac{\partial u}{\partial x} \right) \right] dp \\
- \frac{1}{s^3} \int_0^p L_s L_t \left[ x \frac{\partial u(x)}{\partial x} \right] dp + \frac{1}{s^2} \int_0^p L_s L_t \left[ xu^2 \right] dp.
\]

The nonlinear operators can be defined as follows

\[
u_0(x, t) = L_p^{-1} L_s^{-1} \left[ \frac{F_1(p)}{s} + \frac{F_2(p)}{s^2} \right] \\
+ \frac{1}{s^2} \int_0^p \left( \frac{dF(p,s)}{dp} \right) dp.
\]
where $N_1$ and $N_2$ are defined in equation (33). By calculating the terms $u_0$, $u_1$, ... we obtain the solution as

$$u(x, t) = u_0 + u_1 + \cdots$$

To illustrate the modified double Laplace decomposition method for solving the singular nonlinear one-dimensional pseudohyperbolic equation, we let $g(x) = x^2$, $a = 8$, $h(x) = x/2$, and $f(u) = 0$ in equation (27), hence we have the following example.

**Example 2.** Consider the following nonlinear singular one-dimensional pseudohyperbolic equation

$$\frac{\partial^2 u}{\partial t^2} - \frac{1}{x} \frac{\partial}{\partial x} \left( x \frac{\partial u}{\partial x} \right) - \frac{1}{x} \frac{\partial^2}{\partial x \partial t} \left( x \frac{\partial u}{\partial x} \right) - \frac{x}{2} \frac{\partial u}{\partial x} + u^2 = x^2 e^{-t}$$  

subject to

$$u(x, 0) = x^2, \quad \frac{\partial u(x, 0)}{\partial t} = -x^2$$  

Using the modified double Laplace decomposition methods for equations (40) and (41) and applying equation (37), we have

$$u(x, t) = x^2 e^{-t} - L^{-1}_p L^{-1}_s$$

$$\left[ \frac{1}{x^2} \int_0^p L_s L_t \left[ (xu)_x + (xu)_xt + \frac{x}{2} u_x u - xu^2 \right] dp \right]$$  

By applying equations (38) and (39), we get

$$u_0 = x^2 e^{-t}$$

$$u_1 = -L^{-1}_p L^{-1}_s$$

$$\left[ \frac{1}{x^2} \int_0^p L_s L_t \left[ (xu)_x + (xu)_xt + \frac{x}{2} u_x u - xu^2 \right] dp \right]$$

$$u_1 = -L^{-1}_p L^{-1}_s$$

$$\left[ \frac{1}{x^2} \int_0^p L_s L_t \left[ 4xe^{-t} - 4xe^{-t} + x^2 e^{-2t} - x^2 e^{-2t} \right] dp \right] = 0$$

The other components are zeros

$$u(x, t) = u_0 + u_1 + \cdots$$

Therefore, the exact solution is given by

$$u(x, t) = x^2 e^{-t}.$$  

**Convergence analysis of the method**

In this section, we will discuss the convergence analysis of the modified double Laplace decomposition methods for the singular nonlinear one-dimensional pseudohyperbolic equation which is given by

$$\frac{\partial^2 u}{\partial t^2} = \frac{1}{x} \frac{\partial}{\partial x} \left( x \frac{\partial u}{\partial x} \right) + \frac{1}{x} \frac{\partial^2}{\partial x \partial t} \left( x \frac{\partial u}{\partial x} \right) + \frac{x}{2} \frac{\partial u}{\partial x} + f(u)$$

for all $u, v \in H$. We define $H$ as $H = L^2_p((a, b) \times [0, T])$, where $a < 0$ and

$$u : (a, b) \times [0, T] \to \mathbb{R} \times \mathbb{R}, \text{ with } ||u||_H^2 = \int_Q \left( xu^2(x, t) dxdt \right)$$

where $Q = (a, b) \times [0, T]$ and

$$H = \left\{ (u, v) : (a, b) \times 0, T \right\}, \text{ with }$$

$$L^{-1}_p L^{-1}_s \left[ \frac{1}{x^2} \int_0^p L_s L_t [u(x, t)](p, s) dp \right] (x, t) < \infty$$

Such that the solution satisfies the final condition $u(x, T) = 0$. Multiplying both sides of equation (43) by $x$ and writing the equation in the operator form

$$L(u) = \frac{\partial^2 u}{\partial t^2} = \frac{\partial u}{\partial x} + x \frac{\partial u}{\partial x} + \frac{\partial^2 u}{\partial x \partial t}$$

$$+ x \frac{\partial u}{\partial x} + xu \frac{\partial u}{\partial x} + xf(u)$$

For $L$ hemicontinuous operator, consider the following hypotheses:

1. (H1) $(L(u) - L(v), u - v) \geq k ||u - v||^2$; $k > 0, \forall u, v \in H.$

2. (H2) whatever may be $M > 0$, there exist a constant $C(M) > 0$ such that for $u, v \in H$ with $||u|| \leq M, ||v|| \leq M$ we have

$$(L(u) - L(v), w) \leq C(M) ||u - v|| ||w||$$

for every $w \in H$. In the next Theorem, we follow the literature.15,23–25
Theorem 2 (Sufficient condition of convergence). The Modified double Laplace decomposition methods applied to the nonlinear singular one-dimensional pseudohyperbolic equation (44) with homogeneous initial condition converges toward a solution.

Proof. To verify the convergence hypotheses (H1) for equation (44), we use the definition of our operator $L$, and we have the following form

\[
L(u) - L(v) = \left( \frac{\partial u}{\partial x} - \frac{\partial v}{\partial x} \right) + \left( \frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 v}{\partial x^2} \right) + \left( \frac{1}{2} \frac{\partial^2 u}{\partial x^2} - \frac{1}{2} \frac{\partial^2 v}{\partial x^2} \right) + x(f(u) - f(v))
\]

\[
= \frac{\partial}{\partial x}(u - v) + x \frac{\partial^2 u}{\partial x^2}(u - v) + \frac{\partial^2}{\partial x^2} \frac{\partial}{\partial x}(u - v) + x(f(u) - f(v))
\]

\[
= \frac{\partial}{\partial x}(u - v) + \frac{\partial^3}{\partial x^3}(u - v) + \frac{1}{2} x \frac{\partial}{\partial x} (u^2 - v^2) + x(f(u) - f(v))
\]

therefore

\[
(L(u) - L(v), u - v) = \left( \frac{\partial}{\partial x}(u - v), u - v \right)
\]

\[
+ \left( \frac{\partial^2}{\partial x^2}(u - v), u - v \right)
\]

\[
+ \left( \frac{\partial^3}{\partial x^3}(u - v), u - v \right)
\]

\[
+ \left( \frac{1}{2} x \frac{\partial}{\partial x} (u^2 - v^2), u - v \right)
\]

\[
+ \left( x(f(u) - f(v)), u - v \right)
\]

Since $\frac{\partial}{\partial x}$ and $\frac{\partial^2}{\partial x^2}$ are differential operators in $H$, there exists numbers $\alpha$, $\beta$, $\theta$, $\zeta$, and $\eta$ such that

\[
\left( \frac{\partial}{\partial x}(u - v), u - v \right) \geq \alpha \| u - v \|^2
\]  

(46)

where $\| x \| \leq a$, using Cauchy Schwartz inequality

\[
- \left( \frac{\partial^2}{\partial x^2}(u - v), u - v \right) \leq \| x \| \| \frac{\partial^2}{\partial x^2}(u - v) \| \| u - v \|
\]

\[
\leq a\beta \| u - v \|^2 \iff
\]

\[
\left( \frac{\partial^2}{\partial x^2}(u - v), u - v \right) \geq -a\beta \| u - v \|^2
\]  

(47)

\[
- \left( \frac{\partial}{\partial x}(u - v), u - v \right) \leq \| \frac{\partial}{\partial x}(u - v) \| \| u - v \|
\]

\[
\leq \eta \| u - v \|^2 \iff
\]

\[
\left( \frac{\partial}{\partial x}(u - v), u - v \right) \geq -\eta \| u - v \|^2
\]  

(48)

\[
- \left( \frac{\partial^3}{\partial x^3}(u - v), u - v \right) \leq \| x \| \| \frac{\partial^3}{\partial x^3}(u - v) \| \| u - v \|
\]

\[
\leq a\zeta \| u - v \|^2 \iff
\]

\[
\left( \frac{\partial^3}{\partial x^3}(u - v), u - v \right) \geq -a\zeta \| u - v \|^2
\]  

(49)

\[
- \left( \frac{1}{2} x \frac{\partial}{\partial x} (u^2 - v^2), u - v \right) \leq \frac{1}{2} \| x \| \| \frac{\partial}{\partial x} (u^2 - v^2) \| \| u - v \|
\]

\[
\leq \frac{1}{2} a\theta \| (u^2 - v^2) \| \| u - v \|
\]

\[
\leq \frac{1}{2} a\theta \| u + v \| \| u - v \| \| u - v \|^2 \iff
\]

\[
\left( \frac{1}{2} x \frac{\partial}{\partial x} (u^2 - v^2), u - v \right) \geq -a\theta M \| u - v \|^2
\]

(50)

According to Cauchy Schwartz inequality, where $\sigma > 0$ as $f$ is Lipschitzian function, we have

\[
-x(f(u) - f(v), u - v) \leq \| x \| \| f(u) - f(v) \| \| u - v \|
\]

\[
\leq a\| f(u) - f(v) \| \| u - v \|
\]

\[
\leq a\sigma \| u - v \|^2 \iff
\]

\[
(x(f(u) - f(v)), u - v) \geq -a\sigma \| u - v \|^2
\]

(51)

Substituting equations (46)-(51) into equation (45) gives

\[
(L(u) - L(v), u - v) \geq (\alpha - a\beta - \eta - a\zeta - a\theta M - a\sigma) \| u - v \|^2
\]

\[
\| u - v \|^2 (L(u) - L(v), u - v) \geq k \| u - v \|^2
\]

So, the hypothesis (H1) holds, where

\[
k = \alpha - a\beta - \eta - a\zeta - a\theta M - a\sigma > 0
\]

Now we verify the convergence hypotheses (H2) for the operator $L(u)$. For every $M > 0$, there exist a constant $C(M) > 0$ such that for $u, v \in H$ with $\| u \| \leq M$, $\| v \| \leq M$

\[
(L(u) - L(v), u - v) \leq C(M) \| u - v \| \| w \|
\]
for every $w \in H$. For that we have
\[
(L(u) - L(v), u - v) = \left( \frac{\partial}{\partial x} (u - v), w \right) + \left( x \frac{\partial^2}{\partial x^2} (u - v), w \right) + \left( \frac{\partial^2}{\partial x \partial t} (u - v), w \right) + \left( x \frac{\partial^3}{\partial x^3 \partial t} (u - v), w \right) + \frac{1}{2} \left( \frac{\partial}{\partial x} (u^2 - v^2), w \right) + (xf(u) - f(v), w)
\]
(52)

There exist numbers $\alpha_1, \alpha_2, \alpha_3, \beta_1, \beta_2$, and $\sigma_1$ such that, using the Schwartz inequality and the fact that $u$ and $v$ are bounded, we obtain
\[
\left( \frac{\partial}{\partial x} (u - v), w \right) \leq \alpha_1 \|u - v\| \|w\|,
\left( x \frac{\partial^2}{\partial x^2} (u - v), w \right) \leq a \alpha_2 \|u - v\| \|w\|,
\left( \frac{\partial^2}{\partial x \partial t} (u - v), w \right) \leq \beta_1 \|u - v\| \|w\|,
\left( x \frac{\partial^3}{\partial x^3 \partial t} (u - v), w \right) \leq a \beta_2 \|u - v\| \|w\|,
\frac{1}{2} \left( \frac{\partial}{\partial x} (u^2 - v^2), w \right) \leq \frac{1}{2} \alpha_3 \|u + v\| \|u - v\| \|w\| \leq a \sigma_1
\]
where $\|x\| \leq a$. We also have
\[
(L(u) - L(v), w) \leq (\alpha_1 + a \alpha_2 + \beta_1 + a \beta_2 + a \alpha_3 + a \sigma_1) \|u - v\| \|w\| = C(M) \|u - v\| \|w\|
\]
where
\[
C(M) = \alpha_1 + a \alpha_2 + \beta_1 + a \beta_2 + a \alpha_3 + a \sigma_1
\]
and therefore (H2) holds.

**Conclusion**

In this article, we proposed modified double Laplace decomposition methods to solve singular one-dimensional linear and nonlinear pseudohyperbolic equations. The efficiency and accuracy of the present scheme are validated through examples. This method can be applied to many complicated linear and nonlinear PDEs and also for system of PDEs and does not require linearization.

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