TWIST AUTOMORPHISMS ON QUANTUM UNIPOTENT CELLS AND DUAL CANONICAL BASES

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Abstract. In this paper, we construct twist automorphisms on quantum unipotent cells, which are quantum analogues of the Berenstein-Fomin-Zelevinsky twist automorphisms on unipotent cells. We show that those quantum twist automorphisms preserve the dual canonical bases of quantum unipotent cells.

Moreover we prove that quantum twist automorphisms are described by the syzygy functors for representations of preprojective algebras in the symmetric case. This is the quantum analogue of Geiß-Leclerc-Schröer’s description, and Geiß-Leclerc-Schröer’s results are essential in our proof. As a consequence, we show that quantum twist automorphisms are compatible with quantum cluster monomials. The 6-periodicity of specific quantum twist automorphisms is also verified.

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1. Introduction

1.1. Canonical bases and cluster algebras. Let $G$ be a connected, simply-connected, complex simple algebraic group with a fixed maximal torus $H$, a pair of Borel subgroups $B_\pm$ with $B_+ \cap B_- = H$, the Weyl group $W = \text{Norm}_G(H)/H$ and the maximal unipotent subgroups $N_\pm \subset B_\pm$ (In the main body of this paper, we deal with “the Kac-Moody groups”). Let $U_q(g)$ be the Drinfeld-Jimbo quantized enveloping algebra of the corresponding Lie algebra $g$, and $U_q^-(g)$ be its negative part which arises from the triangular decomposition of $g$. In [Lus90a], Lusztig constructed the canonical bases $B$ of $U_q^-(g)$ using perverse sheaves
on the varieties of quiver representations when $\mathfrak{g}$ is simply-laced. In [Kas91], Kashiwara constructed the lower global bases $\mathfrak{g}^{\text{low}} (\mathfrak{g})$ of $U_q^- (\mathfrak{g})$ in general. In simply-laced case, Lusztig [Lus90b] proved that the two bases of $U_q^- (\mathfrak{g})$ coincide. In this paper, we call the bases the \textit{canonical bases}. The canonical bases have interesting structures; one is positivity of structure constants of multiplications and \textit{(twisted) comultiplication}, and another is the combinatorial structure which is called Kashiwara crystal structure. Using the positivity of the canonical bases, Lusztig [Lus94] generalized the notion of the total positivity for reductive groups and related algebraic varieties.

Since $U_q^-$ has a natural non-degenerate Hopf pairing which makes it into a \textit{(twisted) self-dual} bialgebra, we can consider $U_q^-$ as a quantum analogue of the coordinate rings $\mathbb{C} [N_-]$. The combinatorial structure of $B^{\text{low}}$ and its dual basis $B^{\text{up}}$ \textit{(with respect to the non-degenerate Hopf pairing)}, called the \textit{dual canonical bases}, has been intensively studied by Lusztig [Lus10] Chapter 42 and Berenstein-Zelevinsky [BZ93, BZ96] \textit{(in the type A-case)} and it became one of the origins of cluster algebras introduced by Fomin-Zelevinsky [FZ02].

1.2. \textbf{Quantum unipotent subgroups and dual canonical bases}. For a Weyl group element $w \in W$ \textit{(and a lift $\hat{w} \in \text{Norm}_{\mathfrak{g}} (H)$)}, the unipotent root subgroups $N_- (w) := N_- \cap \hat{w} N \hat{w}^{-1}$ and the Schubert cells $B_- \hat{w} B_+ / B_+$ in the full flag varieties $G / B_+$ have attracted much attention in the development of the theory of total positivity for reductive groups. Geiß-Leclerc-Schröer [GLS11] introduced a cluster algebra structure on $\mathbb{C} [N_- (w)]$ \textit{using representation theory of preprojective algebras}, called an \textit{additive categorification}. They also proved that the dual semicanonical basis $S^*$ is compatible with $\mathbb{C} [N_- (w)]$, that is, $S^* \cap \mathbb{C} [N_- (w)]$ gives a basis of $\mathbb{C} [N_- (w)]$, and the set of cluster monomials is contained in the dual semicanonical basis $S^*$. Here we note that we identify the coordinate rings $\mathbb{C} [N_- (w)]$ of the unipotent subgroups $N_- (w)$ as invariant subalgebras $\mathbb{C} [N_-]^{N_- \cap \hat{w} N \hat{w}^{-1}}$ fixing a splitting $N_- \simeq (N_- \cap \hat{w} N \hat{w}^{-1}) \times N_-$ as varieties.

For a nilpotent Lie algebra $\mathfrak{n}_- (w)$ associated with the subgroup $N_- (w)$, a quantum analogue $U_q^- (\mathfrak{n}_- (w))$ of the universal enveloping algebra $U (\mathfrak{n}_- (w))$ has been introduced by De Concini-Kac-Processi [DKP93] and also by Lusztig [Lus10] \textit{as subalgebras of the quantized enveloping algebra} $U_q^-$. They are defined as subalgebras which are generated by quantum root vectors defined by Lusztig’s braid group symmetry on the quantized enveloping algebras $U_q (\mathfrak{g})$. Meanwhile they are the linear spans of their Poincaré-Birkhoff-Witt type orthogonal monomials with respect to the non-degenerate pairing on $U_q^-$. In [Kim12], the first author proved that the subalgebras $U_q^- (w)$ are compatible with the dual canonical bases, that is $B^{\text{up}} \cap U_q^- (w)$ is a base of $U_q^- (w)$ and the specialization of $U_q^- (w)$ \textit{(using the dual canonical basis)} at $q = 1$ is isomorphic to the coordinate ring $\mathbb{C} [N_- (w)]$, hence $U_q^- (w)$ is also considered as a quantum analogue of the coordinate ring $\mathbb{C} [N_- (w)]$ of the unipotent subgroup.

Geiß-Leclerc-Schröer [GLS13] proved that $U_q^- (w)$ admits a quantum cluster algebra structure in the sense of Berenstein-Zelevinsky if $\mathfrak{g}$ is symmetric via the additive categorification and Goodearl-Yakimov [GY14, GY17] proved the result using the framework of quantum nilpotent algebras in the symmetrizable case. Kang-Kashiwara-Kim-Oh [KKKO18] showed that the set of quantum cluster monomials is contained in the dual canonical bases via symmetric quiver Hecke algebras when $\mathfrak{g}$ is symmetric. See [KKKO18] \textit{Introduction} for the history of this topic.

1.3. \textbf{Unipotent cells and cluster structure}. For a pair $(w_+, w_-)$ of Weyl group elements, the intersections $G^{w_+, w_-} := B_+ \hat{w}_+ B_+ \cap B_- \hat{w}_- B_-$ are called double Bruhat cells and the
maximal torus $H$ acts $G^{w_+,w_-}$ by left (or right) multiplication. For a certain lift $\bar{w} \in G$ of $w_- \in W$, the intersection $B_+ \bar{w} B_+ \cap N_+ N_-$ is a section of the quotient $G^{w_+,w_-} \to H \setminus G^{w_+,w_-}$. The unipotent cells $N^w_+ := B_+ \bar{w} B_+ \cap N_-$ are special cases of reduced double Bruhat cells where $w_-$ is the unit of $W$. The (upper) cluster structure of the double Bruhat cells and unipotent cells have been studied in details, see Berenstein-Fomin-Zelevinsky [BFZ05] (see also Geiß-Leclerc-Schröer [GLS11] and Williams [Wil13]). In fact, in [GLS11], it is shown that the coordinate ring of the unipotent subgroup has a cluster algebra structure with unlocalized frozen variables, and that the coordinate ring of the unipotent cell has a cluster algebra structure with fully localized frozen variables.

For a Weyl group element $w \in W$, Berenstein-Fomin-Zelevinsky [BFZ96] (in the type A-case) and Berenstein-Zelevinsky [BZ97] (in general) introduced twist automorphisms which are automorphisms on unipotent cells $N^w_+$ for solving the factorization problems, called the Chamber Ansatz, which describe the inverse of the “toric chart” of the Schubert varieties.

In [GLS11, GLS12], Geiß-Leclerc-Schröer studied the additive categorification of the twist automorphism using representation theory of preprojective algebras, where it is given by the syzygy on the Frobenius subcategory associated with $w$. They treated the coordinate ring of the unipotent cells as the localization of the coordinate rings unipotent subgroups with respect to the (unipotent) minors associated with Weyl group elements. They also introduced the “dual semicanonical bases” of the coordinate ring of the unipotent cells, using the “multiplicative property” of dual semicanonical bases.

In this paper, we study the construction of a quantum analogue of the twist automorphisms on the quantum unipotent cells, which are the “quantized coordinate rings of the unipotent cells”, and its relation to the additive categorification.

1.4. Quantum unipotent cells. Quantum coordinate rings of double Bruhat cells, called quantum double Bruhat cells, are introduced by De Concini-Procesi [DP97] in the study of representation theory of quantum groups at root of unity and also intensively studied by Joseph [Jos95] in the study of prime spectra of quantized coordinate ring of $G$. Berenstein-Zelevinsky [BZ05] conjectured that quantum double Bruhat cells admit a structure of quantum cluster algebras via quantum minors. Goodearl-Yakimov [GY16] proved the conjecture using a quantum analogue of the Fomin-Zelevinsky twist of the double Bruhat cells.

In [DP97], De Concini-Procesi studied the relation between the quantum unipotent subgroups and the quantum unipotent cells in finite type case. In [Kim12], the injectivity result of De Concini-Procesi is generalized via the study of crystal bases.

Berenstein-Rupel [BR15] studied the quantum unipotent cells via the Hall algebra technique and they constructed quantum analogue of the twist maps under the conjecture concerning the quantum cluster algebra structure and they showed that the quantum twist automorphisms preserve the triangular bases (in the sense of Berenstein-Zelevinsky [BZ14]) of the quantum unipotent cells when the Weyl group element $w$ is the square of an acyclic Coxeter element $c$ with $\ell(w) = 2\ell(c)$. We note that Qin [Qin16] proved that the triangular bases (=localized dual canonical bases) in the sense of [Qin17] coincide with the triangular bases in the sense of Berenstein-Zelevinsky [BZ14] when $\mathfrak{g}$ is symmetric.

1.5. Quantum unipotent cells and the dual canonical bases. Our main results in this paper are the following:

(1) We prove the De Concini-Procesi isomorphisms between the localizations $A_q[N_-(w) \cap \bar{w} C^\text{min}_0]$ of the quantum unipotent subgroups $A_q[N_-(w)]$ and the quantum unipotent cells
A_q[N^w] for arbitrary symmetrizable Kac-Moody cases (Theorem 4.13). The quantum cluster structure on the quantum unipotent cells can be proved as a corollary of the existence of the De Concini-Procesi isomorphisms (Corollary 7.20).

We should remark that the original De Concini-Procesi isomorphisms [DP97, Theorem 3.2] were given under the assumption that g is of finite type. In [DP97], their existence was proved by downward induction on the length of elements of the Weyl group W from the longest element, which exists only in finite type cases.

(2) We introduce a quantum analogue \( \gamma_w \) of the twist isomorphism between the unipotent cells \( N^w \) and \( N_{-}(w) \cap \dot{w}G_{0}^{\text{min}} \) which is defined using the Gauss decomposition (Theorem 5.19).

(3) We introduce a quantum analogue of the twist automorphism of unipotent cells on the quantum coordinate ring \( A_q[N^w] \) of the unipotent cells (without referring the quantum cluster algebra structure) and show that the quantum twist preserves the dual canonical bases (Theorem 6.1). In fact, we introduce a quantum analogue of the twist automorphism as a composite of the De Concini-Procesi isomorphism and the quantum twist isomorphism. The result that the dual canonical bases are preserved under the twist automorphism from 2 is proved as a consequence of the properties of two isomorphisms and the dual canonical bases. We note that our construction is independent of the construction by Berenstein-Rupel [BR15].

(4) We relate the quantum twist automorphisms and the quantum cluster structure under the additive categorification (Theorem 7.25). We also prove the 6-periodicity of the twist automorphisms associated to the longest elements of the Weyl groups in finite type cases (Theorem 8.1).

1.6. Outline of the paper. The paper is organized as follows. In section 2 we prepare the notations for Kac-Moody Lie algebras, Kac-Moody groups, and flag schemes. Moreover, we give a description of the coordinate rings of unipotent cells, and express “classical twist maps”, which are defined by Berenstein-Zelevinsky [BZ97], in terms of matrix coefficients. In section 3 we give a brief review of quantum unipotent subgroups, quantum closed unipotent cells and canonical/dual canonical bases. The main result in this section is “a crystalized Kumar-Peterson identity” (Theorem 3.48). In section 4 we define the dual canonical bases of the localized quantum coordinate rings and prove the De Concini-Procesi isomorphisms under the arbitrary symmetrizable Kac-Moody setting. In section 5 a quantum analogue of the twist isomorphism is introduced. In section 6 we define a quantum analogue of the twist automorphism as a composite of the quantum twist isomorphism and the De Concini-Procesi isomorphism. In section 7 we relate the quantum twist automorphisms to the quantum cluster algebra structures via Geiß-Leclerc-Schröer’s additive categorification. In section 8 we study the periodicity of the twist automorphisms associated to the longest elements in finite type cases.

1.7. Further work. The comparison with the construction by Berenstein-Rupel [BR15] and a quantum analogue of the Chamber Ansatz will be discussed in another paper.

There is another type of “quantum twist map” which is not an automorphism, introduced by Lenagan-Yakimov [LY15]. This is a quantum analogue of the Fomin-Zelevinsky twist isomorphism [FZ99]. The authors showed that it also preserves the dual canonical basis of

\(^1\) After the submission of the present paper, the paper corresponding to these topics by the second author appeared as [Oya17].
1.8. Basic notation. (1) Let $k$ be a field. For a $k$-vector space $V$, set $V^* := \text{Hom}_k(V, k)$. Denote by $\langle \cdot, \cdot \rangle : V^* \times V \to k, (f, v) \mapsto (f, v)$ the canonical pairing.

(2) For a $k$-algebra $A$, we set $\{a_1, a_2\} := a_1a_2 - a_2a_1$ for $a_1, a_2 \in A$. An Ore set $\mathcal{M}$ of $A$ stands for a left and right Ore set consisting of non-zero divisors. Denote by $A[\mathcal{M}^{-1}]$ the algebra of fractions with respect to the Ore set $\mathcal{M}$. In this case, $A$ is naturally a subalgebra of $A[\mathcal{M}^{-1}]$.

(3) An $A$-module $V$ means a left $A$-module. The action of $A$ on $V$ is denoted by $a.v$ for $a \in A$ and $v \in V$. In this case, $V^*$ is regarded as a right $A$-module by $\langle f.a, v \rangle = \langle f, a.v \rangle$ for $f \in V^*, a \in A$ and $v \in V$.

(4) For two symbols $i, j$, the notation $\delta_{ij}$ stands for the Kronecker delta.

2. Preliminaries (1) : Kac-Moody Lie algebras and associated flag schemes

In this section, we fix the notation concerning (symmetrizable) Kac-Moody Lie algebras $g$ and associated Kac-Moody groups $G, G^{\text{min}}$ and (not necessarily a group) schemes $\mathcal{G}$. See Kashiwara [Kas89] (see also Kashiwara-Tanisaki [KT95]) for more details. In subsection 2.6, we describe the coordinate rings of unipotent cells explicitly, and review “classical twist maps”, which are defined by Berenstein-Zelevinsky [BZ97], in terms of matrix coefficients.

2.1. Kac-Moody Lie algebras and their representations.

Definition 2.1. A root datum $(I, h, P, \{\alpha_i\}_{i \in I}, \{h_i\}_{i \in I}, (\ , \))$ consists of the following data

1. $I$ : a finite index set,
2. $h$ : a finite dimensional $\mathbb{Q}$-vector space,
3. $P \subset h^*$ : a lattice, called the weight lattice,
4. $P^* = \{h \in h | \langle h, P \rangle \subset \mathbb{Z} \}$, called the coweight lattice, with the canonical pairing $\langle - , - \rangle : P^* \otimes \mathbb{Z} P \to \mathbb{Z}$,
5. $\{\alpha_i\}_{i \in I} \subset P$ : a subset, called the set of simple roots,
6. $\{h_i\}_{i \in I} \subset P^*$ : a subset, called the set of simple coroots,
7. $(\ , \) : P \times P \to \mathbb{Q} : a \mathbb{Q}$-valued symmetric $\mathbb{Z}$-bilinear form on $P,

satisfying the following conditions:

(a) $(\alpha_i, \alpha_i) \in 2\mathbb{Z}_{>0}$ for $i \in I$,
(b) $\langle h_i, \mu \rangle = 2(\alpha_i, \mu) / (\alpha_i, \alpha_i)$ for $\mu \in P$ and $i \in I$,
(c) $A = (\langle h_i, \alpha_j \rangle)_{i, j \in I}$ is a symmetrizable generalized Cartan matrix, that is $\langle h_i, \alpha_i \rangle = 2$,
(d) $\{\alpha_i\}_{i \in I} \subset h^*$, $\{h_i\}_{i \in I} \subset h$ are linearly independent subsets.

The $\mathbb{Z}$-submodule $Q = \sum_{i \in I} Z\alpha_i \subset P$ is called the root lattice, $Q' = \sum_{i \in I} Zh_i \subset P^*$ is called the coroot lattice. We set $Q_+ = \sum_{i \in I} Z_{\geq 0}\alpha_i \subset Q$ and $Q_- = -Q_+$. For $\xi = \sum_{i \in I} \xi_i$, we set $\text{ht} (\xi) = \sum_{i \in I} \xi_i \in \mathbb{Z}$. Let $P_+ := \{\lambda \in P \mid \langle h_i, \lambda \rangle \in \mathbb{Z}_{\geq 0} \text{ for all } i \in I\}$ and we assume that there exists $\{\varpi_i\}_{i \in I} \subset P_+$ such that $\langle h_i, \varpi_j \rangle = \delta_{ij}$. Set $p := \sum_{i \in I} \varpi_i \in P_+$.

The quadruple $(h, \{\alpha_i\}_{i \in I}, \{h_i\}_{i \in I}, (\ , \))$ is called a realization of $A$. Let $g$ be the associated Kac-Moody Lie algebra, that is, the Lie algebra $g$ over $\mathbb{C}$ which is generated by $\{\epsilon_i, f_i \mid i \in I\} \cup h$ with the following relations:

1. $h$ is a vector subspace of $g$,

$A_q[N_-(w)]$ [KO18]. However the authors do not know any explicit relations between this quantum twist map and the quantum twist automorphisms in this paper.
(2) \([h, h'] = 0\) for \(h, h' \in \mathfrak{h}\).
(3) \([e_i, f_j] = \langle h, \alpha_i \rangle e_i\) and \([h, f_i] = -\langle h, \alpha_i \rangle f_i\) for \(h \in \mathfrak{h}\) and \(i \in I\).
(4) \([e_i, f_j] = \delta_{ij} h_i\) for \(i, j \in I\).
(5) \(\text{ad}(e_i)^{1-a_{ij}}(e_j) = \text{ad}(f_i)^{1-a_{ij}}(f_j) = 0\) for \(i, j \in I\) with \(i \neq j\), where \(\text{ad}(x)(y) = [x, y]\).

Let \(\mathfrak{n}_+(\text{resp. } \mathfrak{n}_-)\) be the Lie subalgebra of \(\mathfrak{g}\) generated by \(\{e_i \mid i \in I\}\) (resp. \(\{f_i \mid i \in I\}\)). Then we have \(\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+\), and it is called a triangular decomposition of \(\mathfrak{g}\). Let \(\mathfrak{p}_i^+ = \mathfrak{n}_+ \oplus \mathfrak{h} \oplus \mathbb{C} f_i\), and \(\mathfrak{p}_i^- = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathbb{C} e_i\).

Let \(\mathfrak{g} = \bigoplus_{\alpha \in \mathfrak{h}^*} \mathfrak{g}_\alpha\) be its root space decomposition, \(\Delta = \{\alpha \in \mathfrak{h}^* \mid \mathfrak{g}_\alpha \neq 0\}\) \(\setminus\{0\}\) be the set of roots, and \(\Delta_\pm\) be the subsets of positive and negative roots. For a Lie algebra \(\mathfrak{s}\), its universal enveloping algebra is denoted by \(U(\mathfrak{s})\).

Let \(W\) be the Weyl group associated with the above root datum, that is the subgroup of \(GL(\mathfrak{h}^*)\) which is generated by simple reflections \(\{s_i\}_{i \in I}\), where

\[
s_i(\mu) = \mu - \langle h_i, \mu \rangle \alpha_i \quad (\mu \in \mathfrak{h}^*),
\]

and \(\ell : W \to \mathbb{Z}_{\geq 0}\) be the length function, that is \(\ell(w)\) is the smallest integer such that there exists \(i_1, \ldots, i_\ell \in I\) with \(w = s_{i_1} s_{i_2} \cdots s_{i_\ell}\). For \(w \in W\), set

\[
(2.1) \quad I(w) := \{i = (i_1, \ldots, i_{\ell(w)}) \in I^{\ell(w)} \mid w = s_{i_1} \cdots s_{i_{\ell(w)}}\}.
\]

An element of \(I(w)\) is called a reduced word of \(w\).

Let \(\Delta^{re} := W \{\alpha_i\}_{i \in I} \subset \Delta\) be the set of real roots and we set \(\Delta^{re}_\pm := \Delta_\pm \cap \Delta^{re}\).

**Definition 2.2.** (1) For \(\lambda \in P_+\), let \(V(\lambda)\) be the integrable highest weight \(\mathfrak{g}\)-module with highest weight \(\lambda\) of highest weight \(\lambda\).

(2) Let \(O(\mathfrak{g})\) be the category of integrable \(\mathfrak{g}\)-modules \(M\) satisfying the following condition:

(1) \(M = \bigoplus_{\mu \in P} M_{\mu}\) with \(M_{\mu} = \{m \in M \mid h.m = \langle h, \mu \rangle m\) for all \(h \in \mathfrak{h}\}\) and \(\dim M_{\mu} < \infty\) for \(\mu \in P\).

(2) there exists finitely many \(\lambda_1, \ldots, \lambda_k \in P_+\) such that \(P(M) := \{\mu \in P \mid M_{\mu} \neq 0\} \subset \bigcup_{1 \leq j \leq k} (\lambda_j + Q_-)\).

By definition, for a finitely generated (not necessarily integrable) \(\mathfrak{g}\)-module \(M\) satisfying the condition 1 above, the condition for \(M \in O(\mathfrak{g})\) is equivalent to \(\dim \mathbb{C} \bigoplus_U (P^+_\mathfrak{g}) m < \infty\) for all \(i \in I\) and \(m \in M\). It is well-known that \(O(\mathfrak{g})\) is semisimple with its simple object being isomorphic to the integrable highest weight modules \(\{V(\lambda) \mid \lambda \in P_+\}\).

Let \(\varphi : \mathfrak{g} \to \mathfrak{g}\) be the anti-involution defined by \(\varphi(e_i) = f_i, \varphi(f_i) = e_i, \varphi(h) = h\) for \(i \in I\) and \(h \in \mathfrak{h}\). For \(M \in O(\mathfrak{g})\), we denote by \(D_\varphi M\) the \(\mathfrak{g}\)-module \(\bigoplus_{\mu \in P} \text{Hom}(M_{\mu}, \mathbb{C})\) whose \(\mathfrak{g}\)-module structure is given by

\[
\langle x.f, m \rangle = \langle f, \varphi(x).m \rangle \quad \text{for } x \in \mathfrak{g} \text{ and } m \in M.
\]

We note that \(D_\varphi M \in O(\mathfrak{g})\). For a \(\mathfrak{g}\)-module \(M\), we denote by \(M^\vee\) the \(\mathfrak{g}^{op}\)-module \(\{m^\vee \mid m \in M\}\) whose \(\mathfrak{g}^{op}\)-module structure is given by

\[
x. (m^\vee) = (\varphi(x).m)^\vee \quad \text{for } x \in \mathfrak{g} \text{ and } m \in M.
\]

We denote by \(O^{op}(\mathfrak{g})\) be the category of integrable \(\mathfrak{g}^{op}\)-modules \(M^\vee\) such that \(M \in O(\mathfrak{g})\). We interpret the category of \(\mathfrak{g}^{op}\)-modules as the category of right \(U(\mathfrak{g})\)-modules.
2.2. (Pro-)unipotent subgroups. A subset $\Theta$ of $\Delta_{\pm}$ is called closed (resp. an ideal) if it satisfies $(\Theta + \Theta) \cap \Delta_{\pm} \subset \Theta$ (resp. $(\Theta + \Delta_{\pm}) \cap \Delta_{\pm} \subset \Theta$). For a closed subset (resp. an ideal) $\Theta \subset \Delta_{\pm}$, $n_{\pm}(\Theta) \colonequals \bigoplus_{\alpha \in \Theta} g_{\alpha}$ is a Lie subalgebra (resp. an ideal) of $n_{\pm}$.

Example 2.3. (1) For a Weyl group element $w \in W$, the subsets $\Delta_{\pm} \leq w \Delta_{\pm}$ and $\Delta_{\pm} \geq w \Delta_{\pm}$ are closed. Let $n_{\pm} \leq w \Delta_{\pm}$ and $n_{\pm} \geq w \Delta_{\pm}$ be the corresponding subalgebras. We have direct sum decompositions $n_{\pm} = n_{\pm} \leq w \Delta_{\pm} \oplus n_{\pm} \geq w \Delta_{\pm}$ for $w \in W$. For a simple reflection $s_{i}$, we have $\Delta_{\pm} \cap s_{i} \Delta_{\pm} \subset \alpha_{i}$ and $\Delta_{\pm} \cap s_{i} \Delta_{\pm} = \Delta_{\pm} \setminus \{\alpha_{i}\}$. Hence we have direct sum decompositions $n_{\pm} = g_{\pm} \alpha_{i} \oplus n_{\pm}^{\pm}$, where $n_{\pm}^{\pm} = n_{\pm} \Delta_{\pm} \setminus \{\alpha_{i}\}$. $\Theta$ is an ideal. Let $\Theta$ be the subgroup which is generated by $\pm \Theta$. Let $\Theta$ satisfy $(\Theta + \Theta)$.

(2) For $k \in \mathbb{Z}_{\geq 0}$, we set $\Delta_{\pm}^{\geq k} := \{\alpha \in \Delta_{\pm} \mid \lambda \geq k\}$ and $n_{\pm}^{\geq k} := n_{\pm} \Delta_{\pm}^{\geq k}$. Then we have $(\Delta_{\pm}^{\geq k} \cap \Delta_{\pm}) \subset \Delta_{\pm}^{\geq k}$. Hence $n_{\pm}^{\geq k}$ is an ideal of $n_{\pm}$.

It is clear that $n_{\pm}^{\geq k}$ is a finite dimensional nilpotent Lie algebra. We set

$$n_{\pm} = \lim_{\longleftarrow} n_{\pm}^{\geq k} = \prod_{\alpha \in \Delta_{\pm}} g_{\alpha}.$$

Let $N_{\pm}$ be the pro-unipotent group scheme whose pro-nilpotent pro-Lie algebra is $n_{\pm}$ that is defined by

$$N_{\pm} = \lim_{\longleftarrow} \exp \left( n_{\pm}^{\geq k} \right) = \text{Spec} \left( U (n_{\pm})^{*} \right),$$

where $\exp \left( n_{\pm}^{\geq k} \right)$ is an unipotent algebraic group whose Lie algebra is the nilpotent Lie algebra $n_{\pm}^{\geq k}$ and $U (n_{\pm})^{*}$ is the graded dual of $U(n_{\pm})$ with respect to the natural $Q_{\pm}$-grading on $U(n_{\pm})$ (the degrees of $e_{i}$ and $f_{i}$ are $\alpha_{i}$ and $-\alpha_{i}$, respectively). Note that the commutative algebra structure of $U (n_{\pm})^{*}$ is induced from the cocommutative usual coalgebra structure of $U(n_{\pm})$. Then we have $C[N_{\pm}] = U(n_{\pm})^{*}$. It is known that there exists an isomorphism of $C$-schemes $\text{Exp} : n_{\pm} \rightarrow N_{\pm}$.

For a subset $\Theta$ of $\Delta_{\pm}$ (resp. $\Delta_{-}$), we set

$$\hat{n}_{\pm}(\Theta) \colonequals \prod_{\alpha \in \Theta} g_{\alpha}, \quad N_{\pm}(\Theta) \colonequals \text{Exp}(\hat{n}_{\pm}(\Theta)).$$

Then $N_{\pm}(\Theta)$ is a closed subgroup of $N_{\pm}$ if $\Theta$ is closed and is a normal subgroup of $N_{\pm}$ if $\Theta$ is an ideal. Let $N_{\pm} \subset N_{\pm}^{*}$ be the subgroup which is generated by $\{N_{\pm}(\pm \alpha) \mid \alpha \in \Delta_{\pm}^{c}\}$, which has an ind-group scheme structure.

For a Weyl group element $w \in W$ and $i \in I$, let

$$N_{\pm}(w) := N_{\pm}(\Delta_{\pm} \leq w), \quad N'_{\pm}(w) := N_{\pm}(\Delta_{\pm} \geq w), \quad N_{\pm}^{s_{i}} := N'_{\pm}(\Delta_{\pm} > s_{i}).$$

Since $\Delta_{\pm} \cap w \Delta_{\pm} \subset \Delta_{\pm}^{c}$, we have $N_{\pm}(\Delta_{\pm} \leq w) \subset N_{\pm}$. In fact, $N_{\pm}(w)$ are unipotent subgroups of $N_{\pm}$ with $\dim (N_{\pm}(w)) = \ell(w)$.

We have the following isomorphisms

$$N_{\pm} \simeq N'_{\pm}(w) \times (N_{\pm}(w))$$

$$\simeq (N_{\pm}(w)) \times N'_{\pm}(w),$$

as schemes, see [Kum02] Lemma 6.1.2. We set $N'_{\pm}(w) := N_{\pm} \cap N'_{\pm}(w)$. We also have the decompositions $N_{\pm} \simeq N'_{\pm}(w) \times N_{\pm}(w) \simeq N_{\pm}(w) \times N'_{\pm}(w)$. 
2.3. Borel subgroups and minimal parabolic subgroups. Let us fix a root datum 
\((A, P, P^\vee, \{\alpha_i\}_{i \in I}, \{h_i\}_{i \in I})\) which gives a realization of \(A\). Set \(H := \text{Spec} (\mathbb{C} [P])\). Then 
\(H\) is the algebraic torus whose character lattice is \(P\) and whose \(\mathbb{C}\)-valued points are given by \(\text{Hom}_\mathbb{C} (P, \mathbb{C}^*)\). Since \(\mathbb{C} [N_\pm] = U(n_\pm)^*\) are \(Q(\subset P)\)-graded algebras, we have \(H\)-actions on \(N_\pm\). Moreover, since \(N_\pm(\pm \alpha)\), \(\alpha \in \Delta^\pm\) are preserved by these \(H\)-actions, the subgroups \(N_\pm\) are also preserved by these \(H\)-actions. Let \(B_\pm = H \ltimes N_\pm\), \(B_\pm = H \ltimes N_\pm\) be the semi-direct product groups.

For \(i \in I\), let \(G_i\) be the reductive group scheme whose Lie algebra is \(\mathfrak{h} \oplus \mathbb{C} e_i \oplus \mathbb{C} f_i\) with \(H\) a Cartan subgroup. Let \(\gamma : \text{SL}(2, \mathbb{C}) \rightarrow G_i\) be the morphism of algebraic groups which is induced by the homomorphism of Lie algebras given by \(e \mapsto e_i\) and \(f \mapsto f_i\). For a simple reflection \(s_i\), let \(\overline{s}_i \in G_i^\vee, \overline{s}_i \in G_i\) be the lift defined by

\[
\overline{s}_i = \gamma (\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} ) = \exp (-e_i) \exp (f_i) \exp (-e_i),
\]

\[
\overline{s}_i = \gamma (\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} ) = \exp (e_i) \exp (-f_i) \exp (e_i).
\]

Let \(G_i^+\) (resp. \(G_i^-\)) be the subgroup of \(G_i\) with \(\mathfrak{h} \oplus \mathbb{C} e_i\) (reps. \(\mathfrak{h} \oplus \mathbb{C} f_i\)) as its Lie algebra. We have \(G_i^+ = G_i \cap B_\pm\) and isomorphism \(B_\pm = G_i^\vee \ltimes N_i^\pm\) as schemes.

For \(i \in I\), let \((p_\pm^+, H)\)-mod (resp. \((p_\pm^+, H)\)^op-mod) be the category of left (resp. right) finite dimensional \(P\)-weighted \(\mathfrak{h}\)-semisimple \(U(p_\pm^+)\)-modules.

Let us consider the following \(\mathbb{C}\)-algebras:

\[
\mathbb{C} [P_i^\pm] := \left\{ f \in \text{Hom}_\mathbb{C} \left( U(p_\pm^+), \mathbb{C} \right) \left| U(p_\pm^+)/f \in (p_\pm^+, H)\text{-mod} \right\} \right.
\]

where we consider the \(U(p_\pm^+)-\)bimodule structure on \(\text{Hom}_\mathbb{C} \left( U(p_\pm^+), \mathbb{C} \right)\) defined by

\[
\langle f, y, u \rangle = \langle f, y, u \rangle \quad (x, y, u \in p_\pm^+).
\]

Then the coproduct \(U(p_\pm^+) \rightarrow U(p_\pm^+) \otimes U(p_\pm^+)\) induces a commutative algebra structure on \(\text{Hom}_\mathbb{C} \left( U(p_\pm^+), \mathbb{C} \right)\) and \(\mathbb{C} [P_i^\pm]\) is a subalgebra of \(\text{Hom}_\mathbb{C} \left( U(p_\pm^+), \mathbb{C} \right)\). We define a schemes \(P_i^\pm := \text{Spec} (\mathbb{C} [P_i^\pm])\) as spectrum. The product \(U(p_\pm^+) \otimes U(p_\pm^+) \rightarrow U(p_\pm^+)\) induces the morphism of schemes \(P_i^\pm \times P_i^\pm \rightarrow P_i^\pm\) and it gives the structure of group scheme on \(P_i^\pm\) and we have decomposition \(P_i^\pm \cong G_i \ltimes \mathbb{C} e_i\) and \(P_i^\pm \supset B_\pm\) for \(i \in I\). See [KT95] for more details.

2.4. Kac-Moody groups and flag schemes. Let \(G\) be the “maximal” Kac-Moody group over \(\mathbb{C}\) completed along the positive roots which is defined in Kumar [Kum02] 6.1.16] and let \(G_{\text{min}} \subset G\) be the “minimal” Kac-Moody group over \(\mathbb{C}\) defined in Kumar [Kum02] 7.4.1. They satisfy \(B_\pm \subset G\) and \(B_\pm \subset G_{\text{min}}\). See [Kum02] for details. We also introduce the scheme \(G_{\infty}\) and its open subscheme \(G\) following Kashiwara [Kas89] (see also Kashiwara-Tanisaki [KT95]).

We define the scheme \(G_{\infty} := \text{Spec} (\mathcal{R}_\mathbb{C} (\mathfrak{g}))\) as the spectrum of the ring of “strongly regular functions” introduced by Kac-Peterson [KP83], that is

\[
\mathcal{R}_\mathbb{C} (\mathfrak{g}) := \left\{ f \in \text{Hom}_\mathbb{C} \left( U(\mathfrak{g}), \mathbb{C} \right) \left| U(\mathfrak{g}) \cap U_{\text{int}}(\mathfrak{g}) \right\} \right.,
\]

where we consider the bimodule structure on \(\text{Hom}_\mathbb{C} \left( U(\mathfrak{g}), \mathbb{C} \right)\) defined by

\[
\langle f, y, u \rangle = \langle f, y, u \rangle \quad (x, y, u \in \mathfrak{g}, f \in \text{Hom}_\mathbb{C} \left( U(\mathfrak{g}), \mathbb{C} \right), u \in U(\mathfrak{g})).
\]
Let
\[ \Phi = \sum_{\lambda \in P_+} \Phi_{\lambda} : \bigoplus_{\lambda \in P_+} V_C(\lambda)^{\dagger} \otimes V_C(\lambda) \to R_C(g) \]
be the map defined by \((\Phi_{\lambda}(v_1 \otimes v_2), u) = (v_1, u.v_2)_\lambda\) for \(v_1, v_2 \in V_C(\lambda)\) and \(u \in U(g)\), where \((\ , \ )_\lambda : V_C(\lambda)^{\dagger} \otimes V_C(\lambda) \to \mathbb{C}\) is the symmetric bilinear form on \(V(\lambda)\) such that \((u_\lambda, u_\lambda)_\lambda = 1\) and \((x.v_1, v_2)_\lambda = (v_1, \phi(x).v_2)_\lambda\) for \(v_1, v_2 \in V_C(\lambda)\) and \(x \in g\). It is known [KP83, Theorem 1] that \(\Phi\) is an isomorphism of bimodules, called the Peter-Weyl isomorphism for symmetrizable Kac-Moody Lie algebras.

The multiplications \(U(p^-) \otimes U(g) \to U(g)\) and \(U(g) \otimes U(p^+) \to U(g)\) induce coaction morphisms \(R_C(g) \to C[p^-] \otimes R_C(g)\) and \(R_C(g) \to R_C(g) \otimes C[p^+]\). Hence we have the morphisms of schemes \(P^-_i \times G_\infty \to G_\infty\) and \(G_\infty \times P^+_i \to G_\infty\) which give rise to the left action of \(P^-_i\) and the right action of \(P^+_i\) on \(G_\infty\). The scheme \(G_\infty\) contains a canonical point \(e\).

**Definition 2.4.** Let \(G\) be the open subset of \(G_\infty\) which is given by the union of subsets \(P^-_{i_1} \cdots P^-_{i_m} e P^+_{j_1} \cdots P^+_{j_n} \subset G_\infty\), that is,
\[ G = \bigcup_{i_1, \ldots, i_m, j_1, \ldots, j_n \in I} \left( P^-_{i_1} \cdots P^-_{i_m} e P^+_{j_1} \cdots P^+_{j_n} \right). \]

**Proposition 2.5.** (1) The left \(P^-_i\) action on \(G\) is free and the right \(P^+_i\) action on \(G\) is free.

(2) The restricted left \(B_-\) actions from the left action of \(P^-_i\) on \(G\) and the restricted right \(B_+\) action on \(G\) from the right action of \(P^+_i\) on \(G\) are independent of \(i \in I\).

(3) For \(i \in I\) and \(g \in G_i\), we have \(g e = e g\), where the left action of \(G_i\) and the right action of \(G_i\) are defined via the left action of \(P^-_i\) and right action of \(P^+_i\) using the decomposition \(P_i^\pm = G_i \times N_i^\pm\).

Let \(N_- \times H \times N_+ \to G\) be the open immersion defined by the “multiplication” \((x, y, z) \mapsto x y z\) and denote its image by \(G_0\). Let
\[ \left[ \begin{array}{c} \cdot \\ H \end{array} \right] \times \left[ \begin{array}{c} \cdot \\ 0 \end{array} \right] \times \left[ \begin{array}{c} \cdot \\ \cdot \end{array} \right] : G_0 \to N_- \times H \times N_+ \]
be the inverse morphism of the “multiplication”. We note that we use only the left \(B_-\)-action and the right \(B_+\)-action on \(G\). For the minimal Kac-Moody group \(G^{\text{min}}\), it is known the same result holds, see Geiß-Leclerc-Schröer [GLST] Proposition 7.1.

For the Lie algebra anti-involution \(\varphi : g \to g\), let \(\varphi : U(g) \to U(g)\) be the induced anti-involution as a \(\mathbb{C}\)-algebra. We note that \(\varphi\) induces the anti-isomorphism of group schemes \(P_i^\pm \to P_i^\pm\) for \(i \in I\) and we have the following commutative diagram
\[ \begin{array}{ccc} U(p^-) \otimes U(g) & \longrightarrow & U(g) \\ \downarrow \text{flip}(\varphi \otimes \varphi) & & \downarrow \varphi \\ U(g) \otimes U(p^+) & \longrightarrow & U(g) \end{array} \]
where the horizontal homomorphisms are multiplications. Let \((\ )^T : G_\infty \to G_\infty\) be the induced morphism of schemes which intertwines the left \(P^-_i\)-action into the right \(P^+_i\)-action and vice versa. It is clear that \((\ )^T\) preserves \(G\) and \(G_0\) by its construction. We denote by \((\ )^T\) the restriction of \((\ )^T\) to \(G\) and \(G_0\) by abuse of notation. For each real root \(\alpha \in \Delta_+\), \((\ )^T\) maps \(N_+(\{\alpha\})\) to \(N_-((-\alpha))\), so \((\ )^T\) induces an involutive map on \(G^{\text{min}}\).
2.5. **Schubert cells and Schubert varieties.** For a Weyl group element \( w \in W \), we specify two lifts \( \overline{w}, \overline{w'} \in G \) of \( w \in W \). It is known that \( \{ \overline{w} \}_{i \in I} \) and \( \{ \overline{w'} \}_{i \in I} \) satisfy braid relations. It follows that the lifts \( \overline{w} \) and \( \overline{w'} \) can be uniquely defined by the condition
\[
\overline{w} \overline{w'} = \overline{w'} \overline{w},
\]
for \( w', w'' \in W \) with \( \ell (w'w'') = \ell (w') + \ell (w'') \).

**Proposition 2.6** ([FZ99 Proposition 2.1]). For \( w \in W \), we have the following properties:
\[
\overline{w}^{-1} = \overline{w'} = \overline{w''},
\]
\[
\overline{w}^{-1} = \overline{w'} = \overline{w''}.
\]

**Definition 2.7.** The flag scheme \( X \) is defined as a quotient scheme \( G/B_+ \).

It is known that \( X \) is an essentially smooth and separated (in general, not quasi-compact) scheme over \( \mathbb{C} \). Let \( e_X = B_+/B_+ \in X \) be the image of \( e \in G \).

**Notation 2.8.** For a set \( Y \) with a left (resp. right) \( H \)-action and \( w \in W \), we write \( \overline{w}Y \) as \( wY \) (resp. \( Y\overline{w} \) as \( Yw \)).

**Definition 2.9.**  (1) For \( w \in W \), we set \( \check{X}_w = wN_- (w^{-1}) e_X \subset X \) to be the locally closed subscheme of \( X \). Let \( X_w \) be the Zariski closure of \( \check{X}_w \) endowed with the reduced scheme structure. \( X_w \) (resp. \( \check{X}_w \)) are called (finite) Schubert varieties (resp. cells).

(2) For \( w \in W \), we set \( X^w := B_- w e X = N_- w e X \subset X \) to be the locally closed subscheme of \( X \). Let \( X^w \) be the Zariski closure of \( \check{X}^w \) endowed with the reduced scheme structure. \( X^w \) (resp. \( \check{X}^w \)) are called cofinite Schubert schemes (resp. cells).

(2) For \( w \in W \), we set \( U_w := w B_- e X = w N_- e X \subset X \).

**Proposition 2.10** ([KT95 Proposition 1.3.2]). (1) \( X_w \) is the smallest subscheme of \( X \) that is invariant by \( G_i^+ \)'s and contains \( w e_X \).

(2) There is an isomorphism \( N_+ (w) \to \check{X}_w \) given by \( x \mapsto x w e_X \). In particular, \( \check{X}_w \) is isomorphic to the affine space \( A^\ell(w) \).

(3) We have \( X_w = \bigsqcup_{y \leq w} \check{X}_y \), where \( \leq \) is the Bruhat order on \( W \).

We note that the morphism \( N_- (w) \times B_+ \to B_+ w B_+ \) given by \( (x, y) \mapsto x^T y \) is an isomorphism.

**Remark 2.11.** We note that the union \( X := \bigsqcup_{w \in W} X_w \subset X \) has a structure of an ind-scheme over \( \mathbb{C} \) and it is also called the flag variety. We have isomorphisms \( G_{min}/B_+ \cong G/B_+ = X \), see [Kum02 7.4.5 Proposition].

**Proposition 2.12** ([KT95 Proposition 1.3.1]). (1) There is an isomorphism \( N'_+ (w) \to \check{X}^w \) given by \( x \mapsto x w e_X \). In particular, \( \check{X}^w \) is affine space with \( \mathrm{codim} X^w = \ell (w) \).

(2) We have \( X = \bigsqcup_{w \in W} X^w \).

(3) We have \( X^w = \bigsqcup_{y \geq w} \check{X}^y \) for \( w \in W \).
Corollary 2.13 (Birkhoff decomposition). We have $G = \bigsqcup_{w \in W} B_+ w B_+$.

Proposition 2.14 ([Kas89]). $U_w$ is an affine open subset of $X$ and there is an isomorphism

$$(N'_i(w)) \times (N_+(w)) \simeq U_w$$

which is given by $(x, y) \mapsto xywX$. In particular, $N_- \to X$ given by $n_- \mapsto n_- e_X$ is an open immersion.

2.6. Unipotent cells and their automorphisms. In this subsection, we consider $N_-$ as an open subscheme of $X$ via the open immersion in Proposition 2.14.

Definition 2.15. For $w \in W$, we set

$$N^w_\cdot := N_- \cap \check{X}_w \subset X,$$

$N^w_\cdot$ is called the unipotent cell.

Since $wN_- (w^{-1}) B_+ \subset G$, we have

$$wN_- (w^{-1}) B_+ \cap N_+ B_+ = wN_- (w^{-1}) B_+ \cap N_+ B_+ \cap G = wN_- (w^{-1}) B_+ \cap N_+ B_+.$$ 

Therefore we have $N_- \cap \check{X}_w = N_- \cap \check{X}_w$. Similarly, it can be shown that $N_- \cap X_w = N_- \cap X_w$. Since $N_- \subset X$ is a Zariski open immersion, $N_- \cap X_w = N_- \cap X_w \subset X_w$ is an open immersion and $N_- \cap X_w$ is a closed (affine) subscheme of $N_-$. Moreover $N_- \cap X_w$ is reduced. It also coincides with the scheme-theoretic intersection.

We shall describe the coordinate ring of $N_- \cap X_w$ explicitly, that is, we describe the kernel of the quotient map $\mathbb{C}[N_-] \to \mathbb{C}[N_- \cap X_w]$, after preparing some notations. For a reduced expression $i = (i_1, \cdots, i_\ell) \in I(w)$ of a Weyl group element $w \in W$, we consider a morphism $y_i: \mathbb{A}^{\ell(w)} \to N_-$ defined by

$$y_i(z_1, \cdots, z_\ell) := \exp (z_1 f_{i_1}) \cdots \exp (z_\ell f_{i_\ell}).$$

We note that the associated ring homomorphism $y_i^*: \mathbb{C}[N_-] \to \mathbb{C}[\mathbb{A}^{\ell(w)}] = \mathbb{C}[z_1, \ldots, z_\ell]$ is nothing but the (classical) Feigin map or Geiß-Leclerc-Schröer’s $\varphi$-map (see Geiß-Leclerc-Schröer [GLS11] Section 6)]. Moreover, set

$$U_w := \sum_{a_1, \cdots, a_\ell \in \mathbb{Z}_{\geq 0}} \mathbb{C} f_{i_1}^{a_1} \cdots f_{i_\ell}^{a_\ell} \subset U(n_-).$$

Then $U_w$ is independent of the choice of $i \in I(w)$ (see also Proposition 3.34 and Remark 3.35).

Proposition 2.16. For $w \in W$, we have isomorphisms of $\mathbb{C}$-algebras:

$$\mathbb{C}[N_-] / (U_w)^\perp \simeq \mathbb{C}[N_- \cap X_w],$$

here $(U_w)^\perp := \{ f \in U(n_-)^{gr} \mid f(U_w) = 0 \}$ (recall that $\mathbb{C}[N_-] = U(n_-)^{gr}$).

Proof. Let us consider the morphism $y_i: \mathbb{A}^{\ell(w)} \to N_-$. It can be shown that the set-theoretic image of the morphism $y_i$ is included in $N_- \cap X_w$ and the set-theoretic image of $y_i |_{\mathbb{A}^{\ell(w)}}$ is dense in $N_- \cap X_w$ (cf. [Kum02] Proposition 7.1.15). Since $\mathbb{A}^{\ell(w)}$ is reduced and the Zariski closure of $N_- \cap \check{X}_w$ is $N_- \cap X_w$, the scheme-theoretic image of $y_i$ (into $N_-$) is $N_- \cap X_w$. 


The claim follows from the claim $\left(U_{w}^{w}\right)^{-1} = \{ f \in C[N_{-}] \mid y_{i}^{w}(f) = 0 \}$ which is clear from the definition of Geiß-Leclerc-Schröer’s $\varphi$-map.

We next describe the coordinate rings of $\mathbb{C}[N_{w}^{w}]$ and $\mathbb{C}[N_{-}(w) \cap wG_{0}^{\text{min}}]$. For $\lambda \in P_{+}$, we set $u_{w\lambda} := \overline{wu}_{\lambda}$. We set $\Delta_{w\lambda \lambda} := \Phi_{\lambda}(u_{w\lambda}^{T} \otimes u_{\lambda}) \in R_{C}(g)$. We regard $\Delta_{w\lambda \lambda}$ as a regular function on $G_{\infty}$ and its restriction to $G$. We have the following recognizing criterion of the point in the Schubert cells in the Schubert variety in terms of (unipotent) minors $\Delta_{w\lambda \lambda}$. It is proved by Williams [Wil13, Lemma 4.15] in the “minimal” Kac-Moody group setting.

**Lemma 2.17.** For $g \in G^{\text{min}}$ and a point $g e_{X}$ on the Schubert variety $X_{w}$ belongs to the Schubert cell $X_{w}$ if and only if $\Delta_{w\lambda \lambda}(ge_{X}) \neq 0$ for $\lambda \in P_{+}$, where $e_{X} := B_{+}/B_{+} \in G_{0}/B_{+}$.

Since $N_{w}^{w}$ is also reduced, the set-theoretic intersection $N_{-} \cap X_{w}$ coincides with the scheme-theoretic intersection, we obtain the following corollary.

**Corollary 2.18.** For $w \in W$, we have isomorphisms of $\mathbb{C}$-algebras:

$$
\left( \mathbb{C}[N_{-}] / U_{w} \right) \left[ \left\{ D_{w\lambda \lambda}^{C} \right\}^{-1} \mid \lambda \in P_{+} \right] \xrightarrow[\sim]{w} \mathbb{C}[N_{w}^{w}].
$$

where $\left[D_{w\lambda \lambda}^{C} \right]_{w} = \Delta_{w\lambda \lambda}|_{N_{-} \cap X_{w}} : N_{-} \cap X_{w} \to \mathbb{C}$ is the restriction of $\Delta_{w\lambda \lambda}$ to $N_{-} \cap X_{w}$.

By Corollary 2.13 we have

$$
wG_{0} = \{ g \in G \mid \Delta_{w\lambda \lambda}(g) \neq 0 \}.
$$

By [GLS11, Proposition 7.3] we have $G_{0}^{\text{min}} = G_{0} \cap G_{0}^{\text{min}}$, in particular, we obtain $N_{-}(w) \cap wG_{0}^{\text{min}} = N_{-}(w) \cap wG_{0}$. Hence we have

$$
\left( \mathbb{C}[N_{-}(w)] \left[ \left\{ D_{w\lambda \lambda}^{C} \right\}^{-1} \mid \lambda \in P_{+} \right] \xrightarrow[\sim]{w} \mathbb{C}[N_{-}(w) \cap wG_{0}^{\text{min}}].
$$

where $D_{w\lambda \lambda}^{C} := \Delta_{w\lambda \lambda}|_{N_{-}(w)}$.

Our next goal is to show Corollary 2.22. This is a classical counterpart of the De Concini-Procesi isomorphisms, which we prove in subsection 4.3. We first recall the (classical) twist isomorphism $\gamma_{w}$ and the (classical) twist automorphism $\eta_{w}$ following Berenstein-Zelevinsky and Geiß-Leclerc-Schröer.

**Definition 2.19.** For $w \in W$, let $O_{w} := N_{-} \cap wG_{0}$. We define a map $\tilde{\gamma}_{w} : O_{w} \to N_{-}$ by

$$
\tilde{\gamma}_{w}(z) = \left[z^{T} \varpi\right]_{-}.
$$

The following is proved by Berenstein-Zelevinsky [BZ97] (see also Geiß-Leclerc-Schröer [GLS11, Proposition 8.4, Proposition 8.5]).

**Proposition 2.20.** The following properties hold:

1. The map $\tilde{\gamma}_{w} : O_{w} \to N_{-}$ is a morphism between schemes.
2. The image of $\tilde{\gamma}_{w}$ is $N_{w}^{w}$.
3. The restriction $\gamma_{w} := \tilde{\gamma}_{w}|_{N_{-}(w) \cap wG_{0}^{\text{min}}} : N_{-}(w) \cap wG_{0}^{\text{min}} = N_{-}(w) \cap wG_{0} \to N_{w}^{w}$ is an isomorphism.
4. We have $N_{w}^{w} \subset wG_{0}^{\text{min}}(wG_{0})$ and $\eta_{w} := \gamma_{w}|_{N_{w}^{w}} : N_{w}^{w} \to N_{w}^{w}$ is an automorphism.
5. Let $\pi_{w} : N_{-} \to N_{-}(w)$ be the projection for the isomorphism $N_{-}(w) \times N_{-}(w) \xrightarrow[\sim]{\gamma} N_{-}$ given by multiplication (see subsection 2.2). Then $\pi_{w}$ restricts to $N_{w}^{w} \to N_{-}(w) \cap wG_{0}^{\text{min}}$, and $\eta_{w} = \gamma_{w} \circ \pi_{w}|_{N_{w}^{w}}$. 
Remark 2.21. In [GLS11], they define a twist isomorphism and a twist automorphism as restrictions of the morphism $\tilde{\gamma}_w: N_- \cap w\mathfrak{G}_0^{\min} \to N_- \cdot z \mapsto [z^T w]_-$ between ind-schemes. Eventually, it turns out that this twist isomorphism (resp. twist automorphism) coincides with our $\gamma_w$ (resp. $\eta_w$).

Let $\pi^*_w: \mathbb{C}[N_-(w)] \hookrightarrow \mathbb{C}[N_-]$ be the $\mathbb{C}$-algebra homomorphism induced from $\pi_w$. Then, by [GLS11] Proposition 8.2, the image of $\pi^*_w$ consists of the left $N'_{\cdot} (w)$-invariant functions in $\mathbb{C}[N_-]$. Note that our convention is the transpose of Geiß-Leclerc-Schröer's convention. Moreover, by the calculation in [GLS11] subsection 8.2, a function $\Phi_\lambda (u^r_{w\lambda} \otimes u)|_{N_-}$ is left $\mathbb{C}$ by Proposition 2.20. Hence we obtain the desired isomorphism with the universality of localization, we obtain the desired isomorphism with the universality of localization.

$$
\pi^*_w (\Phi_\lambda (u^r_{w\lambda} \otimes u)|_{N_-}) = \Phi_\lambda (u^r_{w\lambda} \otimes u)|_{N_-}.
$$

Corollary 2.22. For $w \in W$, we have an isomorphism of $\mathbb{C}$-algebras:

$$
\mathbb{C} [N_-(w) \cap w\mathfrak{G}_0^{\text{min}}] \xrightarrow{\sim} \mathbb{C} [N_-(w)],
$$

which is induced by localizing the homomorphism $\mathbb{C}[N_-(w)] \xrightarrow{\pi^*_w} \mathbb{C}[N_-] \to \mathbb{C}[N_- \cap X_w]$ with respect to $\{D^C_{w\lambda,\lambda} | \lambda \in P_+\}$.

Proof. By definition, the composite map $\iota: \mathbb{C}[N_-(w)] \xrightarrow{\pi^*_w} \mathbb{C}[N_-] \to \mathbb{C}[N_- \cap X_w]$ is induced from the morphism of schemes $\pi_w|_{N_- \cap X_w}: N_\cdot \cap X_w \to N_-(w)$. Moreover, by Corollary 2.18 and (2.2), the inclusions $N^w \to N_- \cap X_w$ and $N_-(w) \cap w\mathfrak{G}_0^{\text{min}} \to N_-(w)$ corresponds to the canonical $\mathbb{C}$-algebra homomorphisms

$$
\iota_1: \mathbb{C}[N_- \cap X_w] \to \mathbb{C}[N_- \cap X_w] \left[\left[D^C_{w\lambda,\lambda}\right]^{-1} | \lambda \in P_+\right] \simeq \mathbb{C} [N^w],
$$

$$
\iota_2: \mathbb{C}[N_-(w)] \to \mathbb{C}[N_-(w)] \left[\left(D^C_{w\lambda,\lambda}\right)^{-1} | \lambda \in P_+\right] \simeq \mathbb{C} [N_-(w) \cap w\mathfrak{G}_0^{\text{min}}].
$$

Therefore the composite map $\iota_1 \circ \iota: \mathbb{C}[N_-(w)] \to \mathbb{C} [N^w]$ is induced from $\pi_w|_{N^w}: N^w \to N_-(w)$. Moreover, by (2.3), $(\iota_1 \circ \iota)(D^C_{w\lambda,\lambda}) = \left[D^C_{w\lambda,\lambda}\right]_w$ for all $u \in V(\lambda), \lambda \in P_+$. Hence, by the universality of localization, $\iota_1 \circ \iota$ extends to $\mathbb{C}[N_-(w) \cap w\mathfrak{G}^{\text{min}}] \to \mathbb{C} [N^w]$. By construction this is induced from $\pi_w|_{N^w}: N^w \to N_-(w) \cap w\mathfrak{G}^{\text{min}}$, which is an isomorphism of schemes by Proposition 2.20. Hence we obtain the desired isomorphism $\mathbb{C} [N_-(w) \cap w\mathfrak{G}^{\text{min}}] \to \mathbb{C} [N^w]$. □

We conclude this subsection by describing the classical twist isomorphism $\gamma_w$ in terms of matrix coefficients.

Proposition 2.23. Let $\gamma^*_w: \mathbb{C} [N^w] \to \mathbb{C} [N_-(w) \cap w\mathfrak{G}^{\text{min}}]$ be the isomorphism of $\mathbb{C}$-algebras induced from $\gamma_w$. Then, for $w \in W$, $\lambda \in P_+$ and $u \in V(\lambda)$, we have

$$
\gamma^*_w \left(\left[D^C_{u^r,u\lambda}\right]_w\right) = \frac{D^C_{w\lambda,\lambda}}{D^C_{u^r,u\lambda}},
$$

here, $D^C_{u^r,u\lambda} := \Phi_\lambda (u^r \otimes u)|_{N^w}$ and $D^C_{w\lambda,\lambda} := \Phi_\lambda (u^r_{w\lambda} \otimes u)|_{N_-(w) \cap w\mathfrak{G}^{\text{min}}}$ (cf. Corollary 2.18 (2.2)).
Proof. We compute the value of functions. For \( z \in N_-(w) \cap O_w \), we have

\[
\left\langle \gamma_w^* \left( D_{u,u_\lambda}^C \right), z \right\rangle = (u, \gamma_w(z) u_\lambda)_\lambda = \left( u, [z^{-1} w]^{-1} u \right)_\lambda \\
= (u, z^{-1} w u_\lambda)_\lambda / (u_\lambda, [z^{-1} w]_0 u_\lambda)_\lambda \\
= (u, z^{-1} w u_\lambda)_\lambda / (u_\lambda, z^{-1} w u_\lambda)_\lambda \\
= (zu, u_\lambda)_\lambda / (zu_\lambda, u_\lambda)_\lambda.
\]

Hence we obtained the claim. \( \square \)

3. Preliminaries (2) : Quantized enveloping algebras and canonical bases

3.1. Quantized enveloping algebras. In this subsection, we present the definitions of quantized enveloping algebras. Let \( q \) be an indeterminate.

Notation 3.1. Set

\[
q_i := q^{(a_i, \alpha_i)}, \quad [n] := \frac{q^n - q^{-n}}{q - q^{-1}} \quad \text{for } n \in \mathbb{Z},
\]

\[
\left[ \begin{array}{c} n \\ k \end{array} \right] := \begin{cases} \frac{[n][n-1] \cdots [n-k+1]}{[k][k-1] \cdots [1]} & \text{if } n \in \mathbb{Z}, k \in \mathbb{Z}_{>0}, \\
1 & \text{if } n \in \mathbb{Z}, k = 0,
\end{cases}
\]

\[ [n]! := [n][n-1] \cdots [1] \quad \text{for } n \in \mathbb{Z}_{>0}, [0]! := 1. \]

For a rational function \( R \in \mathbb{Q}(q) \), we define \( R_i \) as the rational function obtained from \( R \) by substituting \( q \) by \( q_i \) \((i \in I)\).

Definition 3.2. The quantized enveloping algebra \( U_q \) associated with a root datum \((I, \mathfrak{h}, P, \{\alpha_i\}_{i \in I}, \{h_i\}_{i \in I})\), \((\ , \ )\) is the unital associative \( \mathbb{Q}(q) \)-algebra defined by the generators

\[ e_i, f_i \ (i \in I), q^h \ (h \in P^*), \]

and the relations (i)-(iv) below:

(i) \( q^0 = 1, \) \( q^h q^{h'} = q^{h+h'} \) for \( h, h' \in P^* \),

(ii) \( q^h e_i = q^{(h, \alpha_i)} e_i q^h, \) \( q^h f_i = q^{-(h,\alpha_i)} f_i q^h \) for \( h \in P^*, i \in I, \)

(iii) \( [e_i, f_j] = \delta_{ij} \frac{t_i - t_i^{-1}}{q_i - q_i^{-1}} \) for \( i, j \in I \) where \( t_i := q^{(a_i, \alpha_i)}/h_i, \)

(iv) \( \sum_{k=0}^{1-a_{ij}} (-1)^k \left[ \begin{array}{c} 1 - a_{ij} \\ k \end{array} \right] x_i^k x_j x_i^{1-a_{ij}-k} = 0 \) for \( i, j \in I \) with \( i \neq j, \) and \( x = e, f. \)

The \( \mathbb{Q}(q) \)-subalgebra of \( U_q \) generated by \( \{e_i\}_{i \in I} \) (resp. \( \{f_i\}_{i \in I}, \) \( \{q^h\}_{h \in P^*}, \) \( \{e_i, q^h\}_{i \in I, h \in P^*}, \)

\( \{f_i, q^h\}_{i \in I, h \in P^*} \) will be denoted by \( U_q^+ \) (resp. \( U_q^-, U_q^0, \) \( U_q^{\leq 0} \)).

For \( \alpha \in Q \), write \((U_q)_{\alpha} := \{x \in U_q \mid q^h x q^{-h} = q^{(h, \alpha)} x \) for all \( h \in P^*\} \). The elements of \((U_q)_{\alpha} \) are said to be homogeneous. For a homogeneous element \( x \in (U_q)_{\alpha} \), we set \( \text{wt } x = \alpha. \)

For any subset \( X \subset U_q \) and \( \alpha \in Q \), we set \( X_\alpha := X \cap (U_q)_{\alpha}. \)
Let \( \psi \) be the \( \mathbb{Q} \)-algebra involutions defined by
\[
\psi(e_i) = q_i^{-1}f_i^{-1}, \quad \psi(f_i) = q_i^{-1}t_i e_i, \quad \psi(q^h) = q^{-h}.
\]
Remark that \( \psi \) is also a \( \mathbb{Q}(q) \)-coalgebra homomorphism, and \( \varphi = \vee \circ \ast = \ast \circ \vee \).

In this paper, we also use the following variant \( \tilde{U}_q \) of the quantized enveloping algebra \( U_q \).

**Definition 3.4.** A variant \( \tilde{U}_q \) of the quantized enveloping algebra \( U_q \) is the unital associative \( \mathbb{Q}(q) \)-algebra defined by the generators
\[
e_i, f_i \ (i \in I), q^\mu \ (\mu \in P),
\]
and the relations (i)-(iv) below:
\[
(i) \quad q^0 = 1, \quad q^\mu q^{\mu'} = q^{\mu + \mu'} \quad \text{for} \quad \mu, \mu' \in P,
(ii) \quad q^\mu e_i = q^{(\mu, \alpha_i)} e_i q^\mu, \quad q^\mu f_i = q^{-(\mu, \alpha_i)} f_i q^\mu \quad \text{for} \quad \mu \in P, i \in I,
(iii) \quad [e_i, f_j] = \delta_{ij} t_i - t_i^{-1} q_i q_j^{-1} q_i^{-1} q_j \quad \text{for} \quad i, j \in I \text{ where } t_i := q^{\alpha_i} \text{ (abuse of notation)},
(iv) \quad \sum_{k=0}^{1-a_{ij}} (-1)^k \left[ \frac{1 - a_{ij}}{k} \right] x_i^k x_j x_i^{1-a_{ij}-k} = 0 \quad \text{for} \quad i, j \in I \text{ with } i \neq j, \text{ and } x = e, f.
\]

The \( \mathbb{Q}(q) \)-algebra \( \tilde{U}_q \) has a Hopf algebra structure given by the same formulae as \( U_q \). The notions, notations and maps defined in Definition 3.3 and 3.4 are immediately translated into those for \( \tilde{U}_q \). Note that \( \tilde{U}_q^\pm \) can be identified with \( U_q^\pm \) via \( e_i \mapsto e_i \) and \( f_i \mapsto f_i \), respectively.

### 3.2. Drinfeld pairings and Lusztig pairings

Some non-degenerate bilinear forms play a role of bridges between quantized enveloping algebras and their dual objects.

**Proposition 3.5** ([Dri87] [Tan92]). There uniquely exists a \( \mathbb{Q}(q) \)-bilinear map \( \langle , \rangle_D : \tilde{U}_q^{\geq 0} \times \tilde{U}_q^{\leq 0} \to \mathbb{Q}(q) \) such that
\[
(i) \quad \langle \Delta(x), y_1 \otimes y_2 \rangle_D = (x, y_1 y_2) \quad \text{for} \quad x \in \tilde{U}_q^{\geq 0}, y_1, y_2 \in \tilde{U}_q^{\leq 0},
(ii) \quad \langle x_2 \otimes x_1, \Delta(y) \rangle_D = (x_1 x_2, y) \quad \text{for} \quad x_1, x_2 \in \tilde{U}_q^{\geq 0}, y \in \tilde{U}_q^{\leq 0},
\]
(iii) \((e_i, q^h)_D = (q^h, e_i)_D = 0\) for \(i \in I\) and \(h \in P^*\), \(\mu \in P\),
(iv) \((q^\mu, q^h)_D = q^{-(\mu, \mu)}\) for \(\mu \in P, h \in P^*\),
(v) \((e_i, f_j)_D = -\delta_{ij} q_i^{-1} q_j^{-1}\) for \(i, j \in I\),

here the \(\mathbb{Q}(q)\)-bilinear map \((\ , \ )_D: \mathbb{U}_q^\geq \otimes \mathbb{U}_q^\geq \times \mathbb{U}_q^\leq \otimes \mathbb{U}_q^\leq \rightarrow \mathbb{Q}(q)\) is defined by \((x_1 \otimes x_2, y_1 \otimes y_2)_D = (x_1, y_1)_D(x_2, y_2)_D\) for \(x_1, x_2 \in \mathbb{U}_q^\geq, y_1, y_2 \in \mathbb{U}_q^\leq\).

The bilinear map \((\ , \ )_D\) is called the Drinfeld pairing. It has the following properties:

1. For \(\alpha, \beta \in \mathbb{Q}^+\), \((\ , \ )_D|_{(\mathbb{U}_q^\geq)_{\alpha} \times (\mathbb{U}_q^\geq)_{-\beta}} = 0\) unless \(\alpha = \beta\).
2. For \(\alpha \in \mathbb{Q}^+\), \((\ , \ )_D|_{(\mathbb{U}_q^+)^{\alpha} \times (\mathbb{U}_q^+)^{-\alpha}}\) is non-degenerate.
3. \((q^\mu x, q^h y)_D = q^{-(\mu, \mu)}(x, y)_D\) for \(\mu \in P, h \in P^*\) and \(x \in \mathbb{U}_q^+, y \in \mathbb{U}_q^\).

**Definition 3.6.** For \(i \in I\), define the \(\mathbb{Q}(q)\)-linear maps \(e'_i\) and \(\iota e': \mathbb{U}_q^- \rightarrow \mathbb{U}_q^-\) by

\[
\begin{align*}
e'_i(xy) & = e'_i(x)y + q_i^{-(\mathrm{wt}x)}xe'_i(y), & e'_i(f_j) & = \delta_{ij}, \\
\iota e'(xy) & = q_i^{-(\mathrm{wt}y)}y + xi e'(y), & \iota e'(f_j) & = \delta_{ij}
\end{align*}
\]

for homogeneous elements \(x, y \in \mathbb{U}_q^-\). For \(i \in I\), define the \(\mathbb{Q}(q)\)-linear maps \(f'_i\) and \(\iota f': \mathbb{U}_q^+ \rightarrow \mathbb{U}_q^+\) by

\[
\begin{align*}
f'_i(xy) & = f'_i(x)y + q_i^{-(\mathrm{wt}x)}xf'_i(y), & f'_i(e_j) & = \delta_{ij}, \\
\iota f'(xy) & = q_i^{-(\mathrm{wt}y)}y + xi f'(y), & \iota f'(e_j) & = \delta_{ij}
\end{align*}
\]

for homogeneous elements \(x, y \in \mathbb{U}_q^+\).

**Definition 3.7.** Define the \(\mathbb{Q}(q)\)-bilinear form \((\ , \ )_L: \mathbb{U}_q^- \times \mathbb{U}_q^- \rightarrow \mathbb{Q}(q)\) by \((x, y)_L := (\psi(x), y)_D\) for \(x, y \in \mathbb{U}_q^-\). Note that \(x\) is regarded as an element of \(\mathbb{U}_q^\leq\), while \(y\) is considered as an element of \(\mathbb{U}_q^\leq\). See Definition 3.3. Then this bilinear form satisfies

\[
(1, 1)_L = 1, \quad (f_i x, y)_L = \frac{1}{1 - q_i^2}(x, e'_i(y))_L, \quad (xf_i, y)_L = \frac{1}{1 - q_i^2}(x, i e'(y))_L.
\]

This is a symmetric bilinear form, called the Lusztig pairing. In fact, \((\ , \ )_L\) is the unique symmetric \(\mathbb{Q}(q)\)-bilinear form satisfying the properties above. Moreover, \((\ , \ )_L\) is non-degenerate and has the following property:

\[
(*)(x, *y)_L = (x, y)_L
\]

for all \(x, y \in \mathbb{U}_q^-\).

Similarly, define the \(\mathbb{Q}(q)\)-bilinear form \((\ , \ )_L^+: \mathbb{U}_q^+ \times \mathbb{U}_q^+ \rightarrow \mathbb{Q}(q)\) by \((x, y)_L^+ := (x, \psi(y))_D\) for \(x, y \in \mathbb{U}_q^+\). Then this bilinear form satisfies

\[
(1, 1)_L^+ = 1, \quad (e_i x, y)_L^+ = \frac{1}{1 - q_i^2}(x, f'_i(y))_L^+, \quad (xe_i, y)_L^+ = \frac{1}{1 - q_i^2}(x, i f'(y))_L^+.
\]

The forms \((\ , \ )_L\) and \((\ , \ )_L^+\) are related as follows:

\[
(3.2) \quad (x, y)_L = (x^\vee, y^\vee)_L^+
\]

for all \(x, y \in \mathbb{U}_q^-\). See [Lus10, Chapter 1] for more details.
The following Lemma can be proved easily from the definition, it is left as an exercise for readers.

**Lemma 3.8.** For \( \mu \in P, h \in P^*, y_1, y_2 \in U_q^- \) and \( x_1, x_2 \in U_q^+ \), we have
\[
(\psi(y_1q^\mu), y_2q^h)_D = q^{-(h,\mu)}(y_1, y_2)_L, \quad (x_1q^\mu, \psi(x_2q^h))_D = q^{-(h,\mu)}(x_1, x_2)_L^+.
\]

**Definition 3.9.** For a homogeneous \( x \in U_q^- \), we define \( \sigma(x) = \sigma_L(x) \in U_q^- \) by the property that
\[
(\sigma(x), y)_L = \overline{(x, y)}_L
\]
for an arbitrary \( y \in U_q^- \). By the non-degeneracy of \( ( , )_L \), the element \( \sigma(x) \) is well-defined. This map \( \sigma : U_q^- \to U_q^- \) is called the dual bar involution.

**Proposition 3.10 (Kim[12] Proposition 3.2, KO18 Proposition 2.6).** For a homogeneous element \( x \in U_q^- \), we have
\[
\sigma(x) = (-1)^{ht(x)}q^{(wt(x), wt(x))/2 - (wt(x), \rho)}(-\circ *) \sigma(x).
\]
In particular, for homogeneous elements \( x, y \in U_q^- \), we have
\[
\sigma(xy) = q^{(wt(x), wt(y))}\sigma(y)\sigma(x).
\]

**Definition 3.11.** Define a \( \mathbb{Q}(q) \)-linear isomorphism \( c_{tw} : U_q^- \to U_q^- \) by
\[
x \mapsto q^{(wt(x), wt(x))/2 - (wt(x), \rho)}x
\]
for every homogeneous element \( x \in U_q^- \). Set \( \sigma' := c_{tw}^{-1} \circ \sigma : U_q^- \to U_q^- \). We call \( \sigma' \) the twisted dual bar involution. By Proposition 3.10, \( \sigma'(x) = (-1)^{ht(x)}(-\circ *)(x) \) for every homogeneous element \( x \in U_q^- \). In particular, \( \sigma' \) is a \( \mathbb{Q} \)-algebra anti-involution.

**Remark 3.12.** Let \( x \in U_q^- \) be a homogeneous element. Then,
\[
\sigma(x) = x \text{ if and only if } \sigma'(x) = q^{-(wt(x), wt(x))/2 + (wt(x), \rho)}x.
\]

### 3.3. Canonical/Dual canonical bases.
In this subsection, we briefly review the properties of canonical/dual canonical bases of the quantized enveloping algebras and its integrable highest weight modules. See, for example, [Kas95] for the fundamental results on crystal bases and canonical bases.

**Definition 3.13.** For \( \lambda \in P_+ \), denote by \( V(\lambda) \) the integrable highest weight \( U_q^+ \)-module generated by a highest weight vector \( u_\lambda \) of weight \( \lambda \). Define the surjective \( U_q^- \)-module homomorphism \( \pi_\lambda : U_q^- \to V(\lambda) \) by
\[
\pi_\lambda(y) = y.u_\lambda.
\]
There exists a unique \( \mathbb{Q}(q) \)-bilinear form \( ( , )_\lambda^\varphi : V(\lambda) \times V(\lambda) \to \mathbb{Q}(q) \) such that
\[
(u_\lambda, u_\lambda)_\lambda^\varphi = 1 \quad (x.u_1, u_2)_\lambda^\varphi = (u_1, \varphi(x).u_2)_\lambda^\varphi
\]
for \( u_1, u_2 \in V(\lambda) \) and \( x \in U_q^- \). Then the form \( ( , )_\lambda^\varphi \) is non-degenerate and symmetric. See, for example, [GLS13] subsection 2.2, the equality (3.10)].

Set \( \mathcal{A} := \mathbb{Q}[q^{\pm 1}] \) and \( x_i^{[n]} := x_i^n/[n]_q \in U_q \) for \( i \in I, n \in \mathbb{Z}_{\geq 0}, x = e, f \). Denote by \( U^-_{\mathcal{A}} \) the \( \mathcal{A} \)-subalgebra of \( U_q^- \) generated by the elements \( \{ f_i^{[n]} \}_{i \in I, n \in \mathbb{Z}_{\geq 0}} \) and we set
\[
\mathcal{A}_{\mathbb{Q}[q^{\pm 1}]}[N_-] := \{ x \in U_q^- \mid (x, U^-_{\mathcal{A}})_L \subset \mathcal{A} \}.
\]
Lusztig [Lus90a, Lus91, Lus10] and Kashiwara [Kas91] have constructed the specific $\mathbb{Q}(q)$-basis $B^{\text{low}}$ (resp. $B^{\text{low}}(\lambda)$, $\lambda \in P_+$) of $U_q^{-}$ (resp. $V(\lambda)$), called the canonical basis (or the lower global basis), which is also an $A$-basis of $U_q^+$ (resp. $V(\lambda)_A := U_q^+ u_\lambda$). Moreover the elements of $B^{\text{low}}$ (resp. $B^{\text{low}}(\lambda)$) are parametrized by the Kashiwara crystal $B(\infty)$ (resp. $B(\lambda)$). We write

$$B^{\text{low}} = \{ G^{\text{low}}(b) \mid b \in B(\infty) \} \quad \text{and} \quad B^{\text{low}}(\lambda) = \{ G^{\text{low}}_\lambda(b) \mid b \in B(\lambda) \}.$$  

We follow the notation in [Kas93a] concerning the crystal $(B(\infty); \text{wt}, \{ e_i \}_{i \in I}, \{ f_i \}_{i \in I}, \{ \varepsilon_i \}_{i \in I}, \{ \varphi_i \}_{i \in I})$, $(B(\lambda); \text{wt}, \{ e_i \}_{i \in I}, \{ f_i \}_{i \in I}, \{ \varepsilon_i \}_{i \in I}, \{ \varphi_i \}_{i \in I})$. The unique element of $B(\infty)$ with weight 0 is denoted by $u_\infty$, and the unique element of $B(\lambda)$ with weight $w\lambda$ is denoted by $u_{w\lambda}$ for $\lambda \in P_+$ and $w \in W$ by abuse of notation.

Denote by $B^{\text{up}}$ (resp. $B^{\text{up}}(\lambda)$) the basis of $U_q^-$ (resp. $V(\lambda)$) dual to $B^{\text{low}}$ (resp. $B^{\text{low}}(\lambda)$) with respect to the bilinear form $(,)_L$ (resp. $(,)_\lambda$), that is, $B^{\text{up}} = \{ G^{\text{up}}(b) \}_{b \in B(\infty)}$ (resp. $B^{\text{up}}(\lambda) = \{ G^{\text{up}}_\lambda(b) \}_{b \in B(\lambda)}$) such that

$$\left( G^{\text{low}}(b), G^{\text{up}}(b') \right)_L = \delta_{b,b'} \quad \text{for } b, b' \in B(\infty) \quad \text{(resp. } b, b' \in B(\lambda) \text{).}$$

Proposition 3.14 ([Kas91, Theorem 5, Lemma 7.3.2], [Lus10, Theorem 14.4.11]). Let $\lambda \in P_+$. There exists a surjective map $\pi_\lambda : B(\infty) \to B(\lambda) \coprod \{ 0 \}$ such that

$$\pi_\lambda(G^{\text{low}}(b)) = G^{\text{low}}_\lambda(\pi_\lambda(b))$$

for $b \in B(\infty)$, here we set $G^{\text{low}}_\lambda(0) := 0$ as a convention. Moreover, $\pi_\lambda$ induces a bijection $\pi_\lambda^{-1}(B(\lambda)) \to B(\lambda)$.

Definition 3.15. Let $\lambda \in P_+$. Define $j_\lambda : V(\lambda) \hookrightarrow U_q^-$ as the dual homomorphism of $\pi_\lambda$ given by the non-degenerate bilinear forms $(,)_\lambda : V(\lambda) \times V(\lambda) \to \mathbb{Q}(q)$ and $(,)_L : U_q^- \times U_q^- \to \mathbb{Q}(q)$, that is

$$j_\lambda(v, y) = (v, \pi_\lambda(y))_\lambda = (v, y. u_\lambda)_\lambda.$$

The following proposition immediately follows from Proposition 3.14.

Proposition 3.16. There is an injective map $\mathcal{T}_\lambda : B(\lambda) \hookrightarrow B(\infty)$ such that

$$\left( G^{\text{up}}_\lambda(b), G^{\text{low}}(b') \cdot u_\lambda \right)_\lambda = \delta_{b', \mathcal{T}_\lambda(b)}$$

for any $b \in B(\lambda)$ and $b' \in B(\infty)$. That is, we have $j_\lambda \left( G^{\text{up}}_\lambda(b) \right) = G^{\text{up}}(\mathcal{T}_\lambda(b))$.

Remark 3.17. Let $\lambda \in P_+$. Then,

- $\text{wt} \mathcal{T}_\lambda(b) = \text{wt} b - \lambda$ for $b \in B(\lambda)$, and
- $\mathcal{T}_\lambda(u_\lambda) = b$ for $b \in \pi_\lambda^{-1}(B(\lambda))$.

Proposition 3.18 ([Kas93a, Lemma 7.3.4], [Lus10, 13.1.11]). For all $b \in B(\infty)$, we have $G^{\text{low}}(b) = G^{\text{low}}(\mathcal{T}_\lambda(b))$.

Note that Proposition 3.18 immediately implies $\sigma(G^{\text{up}}(b)) = G^{\text{up}}(b)$ for $b \in B(\infty)$.

Proposition 3.19 ([Kas93a, Theorem 2.1.1]). There exists an bijection $* : B(\infty) \to B(\infty)$ such that

$$* \left( G^{\text{low}}(b) \right) = G^{\text{low}}(\# b)$$

for $b \in B(\infty)$. 

Remark that Proposition 3.19 implies \( *(G^{\text{up}}(b)) = G^{\text{up}}(b) \) for \( b \in \mathcal{B}(\infty) \). See the equality \[3.1\].

**Definition 3.20.** The bijections \( * \) give new crystal structures on \( \mathcal{B}(\infty) \), defined by the maps\(^{1} \)

\[
\text{wt}^{*} := \text{wt} \circ * = \text{wt}, \quad e_{i}^{*} := e^{*} \circ \varphi, \quad f_{i}^{*} := f^{*} \circ \varphi, \quad \tilde{e}_{i}^{*} := e^{*} \circ \tilde{e}_{i} \circ * \quad \text{and} \quad \tilde{f}_{i}^{*} := f^{*} \circ \tilde{f}_{i} \circ *.
\]

**Proposition 3.21** (Kashiwara Theorem 7). Let \( \lambda \in P_{+} \). Then we have

\[
\mathcal{J}_{\lambda}(\mathcal{B}(\lambda)) = \{ \tilde{b} \in \mathcal{B}(\infty) \mid \varepsilon_{i}^{*}(\tilde{b}) \leq \langle h_{i}, \lambda \rangle \text{ for all } i \in I \}.
\]

**Proposition 3.22** (Kashiwara Lemma 5.1.1). For \( i \in I \), \( \lambda \in P \), \( b \in \mathcal{B}(\lambda) \) and \( \tilde{b} \in \mathcal{B}(\infty) \), we have

\[
e_{i}^{(e_{i}(b))}.G_{\lambda}^{\text{up}}(b) = G_{\lambda}^{\text{up}}(e_{i}^{*}(b)) G_{\lambda}^{\text{up}}(e_{i}(b)) = 0 \text{ if } k > \varepsilon_{i}(b),
\]

\[
f_{i}^{(e_{i}(b))}.G_{\lambda}^{\text{up}}(b) = G_{\lambda}^{\text{up}}(f_{i}^{*}(b)) G_{\lambda}^{\text{up}}(f_{i}(b)) = 0 \text{ if } k > \varphi_{i}(b),
\]

\[
(e_{i}^{(e_{i}(b))})^{G_{\lambda}^{\text{up}}}(\tilde{b}) = (1 - q_{i}^{2})e_{i}^{*}(b) G_{\lambda}^{\text{up}}(e_{i}^{*}(b)) \tilde{b} = 0 \text{ if } k > \varepsilon_{i}(\tilde{b}),
\]

\[
(e_{i}^{(e_{i}(b))})^{G_{\lambda}^{\text{up}}}(\tilde{b}) = (1 - q_{i}^{2})e_{i}^{*}(\tilde{b}) G_{\lambda}^{\text{up}}((e_{i}^{*})^{e_{i}^{*}(\tilde{b})}) \tilde{b} = 0 \text{ if } k > \varepsilon_{i}^{*}(\tilde{b}).
\]

Here \( (e_{i}^{(e_{i}(b))})^{n} := (e_{i}^{*})^{n}/[n]! \) and \( (e_{i}^{(e_{i}(b))})^{(n)} := (e_{i}^{(e_{i}(b))})^{n}/[n]! \) for \( n \in \mathbb{Z}_{\geq 0} \).

### 3.4. Quantum unipotent subgroups

In this subsection, we review the quantum unipotent subgroup \( A_{q}[\mathcal{N}_{-}(w)] \) which is a quantum analogue of the coordinate ring \( \mathbb{C}[\mathcal{N}_{-}(w)] \) of the unipotent subgroup \( \mathcal{N}_{-}(w) \) associated with \( w \in W \). See Theorem 3.29 below for the precise statement.

**Definition 3.23.** Following Lusztig [Lus10] Section 37.1.3, we define the \( \mathbb{Q} \langle q \rangle \)-algebra automorphism \( T_{i} : U_{q} \rightarrow U_{q} \) for \( i \in I \) by the following formulæ:

\[
T_{i}(q^{h}) = q^{s_{i}(h)},
\]

\[
T_{i}(e_{j}) = \begin{cases} 
-f_{i}t_{i} & \text{for } j = i, \\
\sum_{r+s = -(h_{i}, \alpha_{j})} (-1)^{r} q_{i}^{-r} e_{i}^{(s)} e_{i}^{(r)} & \text{for } j \neq i,
\end{cases}
\]

\[
T_{i}(f_{j}) = \begin{cases} 
-t_{i}^{-1} e_{i} & \text{for } j = i, \\
\sum_{r+s = -(h_{i}, \alpha_{j})} (-1)^{r} q_{i}^{r} f_{i}^{(r)} f_{j}^{(s)} & \text{for } j \neq i.
\end{cases}
\]

Its inverse map is given by

\[
T_{i}^{-1}(q^{h}) = q^{s_{i}(h)},
\]

\[
T_{i}^{-1}(e_{j}) = \begin{cases} 
-t_{i}^{-1} f_{i} & \text{for } j = i, \\
\sum_{r+s = -(h_{i}, \alpha_{j})} (-1)^{r} q_{i}^{-r} e_{i}^{(r)} e_{j}^{(s)} & \text{for } j \neq i,
\end{cases}
\]

\[
T_{i}^{-1}(f_{j}) = \begin{cases} 
-e_{i}t_{i} & \text{for } j = i, \\
\sum_{r+s = -(h_{i}, \alpha_{j})} (-1)^{r} q_{i}^{r} f_{i}^{(s)} f_{j}^{(r)} & \text{for } j \neq i.
\end{cases}
\]

The maps \( T_{i} \) and \( T_{i}^{-1} \) are denoted by \( T'_{i,1} \) and \( T'_{i,-1} \) respectively in [Lus10].
It is known that \( \{T_i\}_{i \in I} \) satisfies the braid relations, that is, for \( w \in W \), the \( \mathbb{Q}(q) \)-algebra automorphism \( T_w := T_{i_1} \cdots T_{i_\ell} : \mathbf{U}_q \to \mathbf{U}_q \) does not depend on the choice of \( (i_1, \ldots, i_\ell) \in I(w) \) (recall (2.1)). See [Lus10, Chapter 39].

**Definition 3.24.** (1) For \( w \in W \), we set \( \mathbf{U}_q^- (w) := \mathbf{U}_q^- \cap T_w (\mathbf{U}_q^\geq 0) \). These subalgebras of \( \mathbf{U}_q^- \) are called quantum nilpotent subalgebras.

(2) Let \( w \in W \) and \( i = (i_1, \ldots, i_\ell) \in I(w) \). For \( c = (c_1, \ldots, c_\ell) \in \mathbb{Z}_{\geq 0}^\ell \), we set

\[
F_{\text{low}} (c, i) := f_{i_1}^{(c_1)} T_{i_1} \left( f_{i_2}^{(c_2)} \cdots (T_{i_1} \cdots T_{i_{\ell-1}}) (f_{i_\ell}^{(c_\ell)}) \right),
\]

\[
F_{\text{up}} (c, i) := F_{\text{low}} (c, i) / (F_{\text{low}} (c, i), F_{\text{low}} (c, i))_L.
\]

**Proposition 3.25** ([DKP95 Proposition 2.2], [BCP99 Proposition 2.3], [Lus10 Proposition 38.2.3]). (1) \( F_{\text{low}} (c, i) \in \mathbf{U}_q^- (w) \) for \( c \in \mathbb{Z}_{\geq 0}^\ell \).

(2) \( \{F_{\text{low}} (c, i)\}_{c \in \mathbb{Z}_{\geq 0}^\ell} \) is an orthogonal basis of \( \mathbf{U}_q^- (w) \) with respect to the pairing \( (\ , \ )_L \), more precisely, we have

\[
( F_{\text{low}} (c, i), F_{\text{low}} (c', i) )_L = \delta_{c, c'} \prod_{k=1}^\ell \prod_{j=1}^{c_k} (1 - q_{ij}^{2j})^{-1},
\]

where \( i = (i_1, \ldots, i_\ell) \).

By Proposition 3.25, \( \{F_{\text{up}} (c, i)\}_{c \in \mathbb{Z}_{\geq 0}^\ell} \) is also an orthogonal basis of \( \mathbf{U}_q^- (w) \) with respect to the Lusztig pairing. The basis \( \{F_{\text{low}} (c, i)\}_{c \in \mathbb{Z}_{\geq 0}^\ell} \) is called the (lower) Poincaré-Birkhoff-Witt type basis associated with \( i \in I(w) \), and the basis \( \{F_{\text{up}} (c, i)\}_{c \in \mathbb{Z}_{\geq 0}^\ell} \) is called the dual (or upper) Poincaré-Birkhoff-Witt type basis.

**Definition 3.26.** For \( w \in W \), we set

\[
\mathbf{U}_q^+(w) := (\mathbf{U}_q^-(w))^\vee,
\]

\[
\mathbf{A}_q[N_-(w)] := * (\mathbf{U}_q^-(w)).
\]

We call \( \mathbf{A}_q[N_-(w)] \) a quantum unipotent subgroup. The quantum unipotent subgroup has a \( Q_- \)-graded algebra structure induced from that of \( \mathbf{U}_q^- \). Note that \( \varphi (\mathbf{A}_q[N_-(w)]) = \mathbf{U}_q^+(w) \).

**Proposition 3.27** ([Kim12 Theorem 4.25, Theorem 4.29]). Let \( w \in W \) and \( i \in I(w) \). Then the following hold:

(1) \( \mathbf{U}_q^-(w) \cap \mathbf{B}^\text{up} \) is a basis of \( \mathbf{U}_q^-(w) \).

(2) each element \( G^{\text{up}} (b) \) of \( \mathbf{U}_q^-(w) \cap \mathbf{B}^\text{up} \) is characterized by the following conditions:

\[
\sigma (G^{\text{up}} (b)) = G^{\text{up}} (b), \quad \text{(DCB1)}
\]

\[
G^{\text{up}} (b) = F_{\text{up}} (c, i) + \sum_{c' \prec c} d_{c, c'}^{i} F_{\text{up}} (c', i) \text{ with } d_{c, c'}^{i} \in q \mathbb{Z}[q] \text{ for some } c \in \mathbb{Z}_{\geq 0}^\ell.
\]

Here \( \prec \) denotes the left lexicographic order on \( \mathbb{Z}_{\geq 0}^\ell \), that is, we write \( (c_1, \ldots, c_\ell) < (c'_1, \ldots, c'_\ell) \) if and only if there exists \( k \in \{1, \ldots, \ell\} \) such that \( c_1 = c'_1, \ldots, c_{k-1} = c'_{k-1}, c_k < c'_k \).

**Definition 3.28.** Proposition 3.27 (2) says that each \( F_{\text{up}} (c, i) \) determines a unique dual canonical basis element \( G^{\text{up}} (b) \) in \( \mathbf{U}_q^-(w) \). We write the corresponding element of \( \mathcal{B}(\infty) \) as \( b_{\mathcal{B}} (c, i) \). Then

\[
\mathbf{U}_q^-(w) \cap \mathbf{B}^\text{up} = \{G^{\text{up}} (b_{\mathcal{B}} (c, i))\}_{c \in \mathbb{Z}_{\geq 0}^\ell}.
\]
Write \( \mathcal{B}(U_q(w)) := \{b(c, i)\} \in \mathbb{Z}_{\geq 0}^L \). Note that \( \mathcal{B}(U_q(w)) \) does not depend on the choice of \( i \in I(w) \). Set \( b_{-1}(c, i) := + (b(c, i)) \). Then \( A_q[N_-(w)] \cap B^{up} = \{G^{up}(b_{-1}(c, i))\} \in \mathbb{Z}_{\geq 0}^L \).

The following is the specialization result for the quantum unipotent subgroup which justifies the notation \( A_q[N_-(w)] \).

**Theorem 3.29 ([Kim12] Theorem 4.44).** For \( w \in W \), we set \( A_q[q^{\pm 1}][N_-(w)] := A_q[q^{\pm 1}][N_-(w)] \cap A_q[N_-(w)] \). Then we have

\[
A_q[q^{\pm 1}][N_-(w)] \otimes_A \mathbb{C} \simeq \mathbb{C}[N_-(w)],
\]

here we regard \( \mathbb{C} \) as an \( A \)-module via \( q^{\pm 1} \rightarrow 1 \).

**Remark 3.30.** In [Kim12], the \( A \)-form \( A_q[q^{\pm 1}][N_-(w)] \) is defined by the non-degenerate bilinear form \( ( , ) \) on \( U_q^-(g) \) with \( (f_i, f_i)_K = 1 \) for \( i \in I \). But this specialization result is not affected since the structure constants with respect to the dual canonical bases defined by \( ( , )_L \) and \( ( , )_K \) are the same. For more details, see [Kim12] Lemma 2.12.

### 3.5. Quantum closed unipotent cells

In this section, we review the definition of quantum closed unipotent cells. For more details, see [Kim12] Section 5.

**Definition 3.31.** Let \( M = \bigoplus_{\mu \in P} M_\mu \) be an integrable \( U_q \)-module (i.e., \( e_i \) and \( f_i \) act locally nilpotently on \( M \) for all \( i \in I \)) with weight space decomposition. For \( i \in I \), there exists a \( \mathbb{Q}(q) \)-linear automorphism \( T_i \) of \( M \) given by

\[
T_i(m) := \sum_{-a + b - c = (h_i, \mu)} (-1)^b \delta_i^{ac+b} e_i^{(a)} f_i^{(b)} e_i^{(c)} m,
\]

\[
T_i^{-1}(m) := \sum_{a - b + c = (h_i, \mu)} (-1)^b \delta_i^{ac-b} e_i^{(a)} f_i^{(b)} e_i^{(c)} m
\]

for \( m \in M_\mu, \mu \in P \). The maps \( T_i \) and \( T_i^{-1} \) are denoted by \( T_i = T_{i,1} \) and \( T_i = T_{i,-1} \) respectively in [Lus10] Chapter 5.

The following propositions are fundamental properties of \( T_i \). See, for example, [Lus10] Chapter 37, 39:

**Proposition 3.32.** Let \( M \) be an integrable \( U_q \)-module. (See Definition 3.31) (1) For \( x \in U_q \) and \( m \in M \), we have \( T_i(x.m) = T_i(x).T_i(m) \).

(2) For \( w \in W \), the composite map \( T_w := T_{i_1} \cdots T_{i_\ell} : M \rightarrow M \) does not depend on the choice of \( (i_1, \ldots, i_\ell) \in I(w) \).

(3) For \( \mu \in P \) and \( w \in W \), \( T_w \) maps \( M_\mu \) to \( M_{w\mu} \).

**Proposition 3.33.** Let \( \lambda \in P_+, \ w \in W \) and \( i = (i_1, \ldots, i_\ell) \in I(w) \). Recall that \( u_\lambda \) is a highest weight vector of \( V(\lambda) \) (Definition 3.13). Then we have

\[
u_{w\lambda} := (T_{w^{-1}})^{-1}(u_\lambda) = f_{i_1}^{(a_1)} f_{i_2}^{(a_2)} \ldots f_{i_\ell}^{(a_\ell)} u_\lambda,
\]

where \( a_1 = \langle h_{i_1}, \mu \rangle, \ldots, a_\ell = \langle h_{i_\ell}, \lambda \rangle \). Note that \( a_1, \ldots, a_\ell \in \mathbb{Z}_{\geq 0} \).

It is easy to show that \( (u_{w\lambda}, u_{w\lambda})_\lambda^e = 1 \) for \( \lambda \in P_+ \) and \( w \in W \). Actually, the vector \( u_{w\lambda} \) belongs to \( B^{low}(\lambda) \) and \( B^{up}(\lambda) \) [Kas93a] subsection 3.2.
Remark 3.36. We have
\[ \bigcup_{\lambda \in P_+} \mathfrak{f}_\lambda (\mathcal{B}_w (\lambda)) = \mathcal{B}_w (\infty). \]

See also Theorem 3.48.

Definition 3.37. Let \( w \in W \). Set
\[ (\mathcal{U}_w)_{\perp} = \{ x \in \mathcal{U}_w \mid (x, \mathcal{U}_w)_L = 0 \}. \]
Then, by \( \Delta (\mathcal{U}_w) \subset \mathcal{U}_w \mathcal{U}^0_q \mathcal{U}_w \) and Lemma 3.38, \( (\mathcal{U}_w)_{\perp} \) is a two-sided ideal of \( \mathcal{U}_w \).
Hence we obtain a \( \mathbb{Q}(q) \)-algebra
\[ A_q [N_- \cap X_w] := \mathcal{U}_w / (\mathcal{U}_w)_{\perp}, \]
called the quantum closed unipotent cell. The quantum closed unipotent cell has a \( q_- \)-graded algebra structure induced from that of \( \mathcal{U}_w \). Note that
\[ (\mathcal{U}_w)_{\perp} = \bigoplus_{b \in \mathcal{B}_w (\infty) \setminus \mathcal{B}_w (\infty) \setminus \mathcal{B}_w (\infty)} \mathbb{Q}(q) G^{up} (b). \]

Describe the canonical projection \( \mathcal{U}^-_q \rightarrow A_q[N_- \cap X_w] \) as \( x \mapsto [x] \). The element \([x]\) clearly depends on \( w \), however, we omit to write \( w \) because it will cause no confusion below.

Remark 3.38. In [Kim12 5.1.3], \( A_q[N_- \cap X_w] \) is denoted by \( \mathcal{O}_q [\overline{N_w}] \).
We set the $A$-form $A_q[N_- \cap X_w]$ of $A_q[N_- \cap X_w]$ by
$$A_q[N_- \cap X_w] := A_q/\left((U^-_w)\right)^{\perp} \cap A_q[N_-] .$$

Note that we have
$$\left(U^-_w\right)^{\perp} \cap A_q[N_-] = \bigoplus_{b \in B(\infty) \setminus B_u(\infty)} AG^{\text{unip}}(b) .

The following is the specialization result for quantum closed unipotent cell which justifies the notation $A_q[N_- \cap X_w]$.

**Theorem 3.39.** For $w \in W$, we have
$$A_q[N_- \cap X_w] \otimes_{\mathbb{C}} C \simeq C[N_- \cap X_w] .$$

**Proof.** We have an exact sequence of $A$-modules
$$0 \to \left(U^-_w\right)^{\perp} \cap A_q[N_-] \to A_q[N_-] \to A_q[N_-] \cap X_w \to 0 ,$$

here the second map is the inclusion and the third map is the projection. Moreover, $(U^-_w)^{\perp} \cap A_q[N_-]$ and $A_q[N_-]$ are free $A$-modules and an $A$-basis of the former can be chosen as the subset of that of the latter (see (3.6)). Therefore $A_q[N_- \cap X_w]$ is also a free $A$-module (more precisely, $A_q[N_- \cap X_w]$ admits the projected dual canonical basis), and we have
$$A_q[N_- \cap X_w] \otimes_{\mathbb{C}} C \simeq \frac{A_q[N_-] \otimes_{\mathbb{C}} C}{\left(U^-_w\right)^{\perp} \cap A_q[N_-] \otimes_{\mathbb{C}} C} = U_{(n_-)} / (U^-_w)^{\perp} \simeq C[N_- \cap X_w] .$$

3.6. Unipotent quantum matrix coefficients.

**Definition 3.40.** For $\lambda \in P_+$ and $u, u' \in V(\lambda)$, define the element $D_{u,u'} \in U_q$ by
$$(D_{u,u'}, x)_L = (u, u', x)_{\lambda}$$
for all $x \in U_q$. We call an element of this form a **unipotent quantum matrix coefficient**. Note that $\text{wt} \left((D_{u,u'})\right) = \text{wt} u - \text{wt} u'$ for weight vectors $u, u' \in V(\lambda)$. For $u, u' \in W$, we write
$$D_{u, u'} := D_{u, u'}_{\lambda} ,$$
which is called a **unipotent quantum minor**. See [Kim12, Section 6].

**Definition 3.41.** Let $\lambda \in P_+$. Define a surjective $\mathbb{Q}(q)$-linear map $\pi^\lambda_w \colon U_q / V_w(\lambda)$ by
$$\pi^\lambda_w(y) = (y)^{\perp} \cdot u_{\lambda} .$$

**Proposition 3.42** ([Lus10, Proposition 25.2.6], [Kas91, Section 8.2.2 (iii), (iv)]). Let $\lambda \in P_+$ and $w \in W$. Then there exists a surjective map $\pi^\lambda_w : B(\infty) \to B_w(\lambda) \prod \{0\}$ such that
$$\pi^\lambda_w(G^\text{low}(b)) = G^\text{low}_\lambda(\pi^\lambda_w(b))$$
for $b \in B(\infty)$, here $G^\text{low}_\lambda(0) = 0$. Moreover, $\pi^\lambda_w$ induces a bijection $(\pi^\lambda_w)^{-1}(B_w(\lambda)) \to B_w(\lambda)$. 

Definition 3.43. Let $\lambda \in P_+$ and $w \in W$. Set $V_w(\lambda)_{\perp} := \{ u \in V(\lambda) \mid (u, V_w(\lambda))_{\lambda} = 0 \}$. Define $j_w^\lambda: V(\lambda) / V_w(\lambda)_{\perp} \to U_q^-$ as the dual homomorphism of $\pi_w^\lambda$ given by the non-degenerate bilinear forms $(\cdot, \cdot)_{\lambda}: V(\lambda) \times V(\lambda) \to \mathbb{Q}(q)$ and $(\cdot, \cdot)_L: U_q^- \times U_q^- \to \mathbb{Q}(q)$, that is,

\[(j_w^\lambda(u), y)_L = (u, \pi_w^\lambda(y))_{\lambda} = (u, y^\vee, u_w^\lambda)_{\lambda} = (\varphi(y^\vee), u, u_w^\lambda)_{\lambda}.
\]

Proposition 3.44. Let $\lambda \in P_+$ and $w \in W$. Then there is an injective map $\mathfrak{T}_w^\lambda: \mathcal{B}_w(\lambda) \to \mathcal{B}(\infty)$ such that

\[\left( G_{\lambda}^{\sup}(b), G_{\lambda}^{\low}(b')^\vee, u_w^\lambda \right)_{\lambda} = \delta_{b'w} \mathfrak{T}_w^\lambda(b) \]

for any $b \in \mathcal{B}_w(\lambda)$ and $b' \in \mathcal{B}(\infty)$. That is, we have $j_w^\lambda \left( G_{\lambda}^{\sup}(b) \right) = G_{\lambda}^{\sup} \left( \mathfrak{T}_w^\lambda(b) \right)$.

Remark 3.45. Let $\lambda \in P_+$ and $w \in W$. Then,

- $\mathrm{wt} \mathfrak{T}_w^\lambda(b) = -\mathrm{wt} b + w \lambda$ for $b \in \mathcal{B}_w(\lambda)$, and
- $\mathfrak{T}_w^\lambda(\pi_w^\lambda(b)) = b$ for $b \in (\pi_w^\lambda)^{-1}(\mathcal{B}_w(\lambda))$.

Proposition 3.46. Let $\lambda \in P_+$ and $w \in W$. Then the following hold:

1. $D_{G_{\lambda}^{\sup}(b), u_w^\lambda} = G_{\lambda}^{\sup} \left( \mathfrak{T}_w^\lambda(b) \right)$ for all $b \in \mathcal{B}(\lambda)$,
2. $D_{u_w^\lambda, G_{\lambda}^{\sup}(b)} = G_{\lambda}^{\sup} \left( \ast \mathfrak{T}_w^\lambda(b) \right)$ for all $b \in \mathcal{B}_w(\lambda)$, and
3. $D_{u_w^\lambda, G_{\lambda}^{\sup}(b)} = 0$ for all $b \in \mathcal{B}(\lambda) \setminus \mathcal{B}_w(\lambda)$.

Proof. The equality (1) follows immediately by Proposition 3.16. For $y \in U_q^-$, we have

\[(D_{u_w^\lambda, G_{\lambda}^{\sup}(b)}(y))_L = (u_w^\lambda, y, G_{\lambda}^{\sup}(b))_{\lambda}^\varphi = (G_{\lambda}^{\sup}(b), (\ast(y))^\vee, u_w^\lambda)_{\lambda}^\varphi = (G_{\lambda}^{\sup} \left( \mathfrak{T}_w^\lambda(b) \right), \ast(y))_L = (G_{\lambda}^{\sup} \left( \ast \mathfrak{T}_w^\lambda(b) \right), y)_L.
\]

This completes the proof of (2). The assertion (3) follows from the similar calculation and Proposition 3.34. \[\square\]

Proposition 3.47 ([Kim12 Corollary 6.4]). Let $w \in W$ and $i = (i_1, \ldots, i_\ell) \in I(w)$. For $i \in I$, define $n^{(i)} = (n_1^{(i)}, \ldots, n_\ell^{(i)}) \in \mathbb{Z}_{\geq 0}^\ell$ by

\[n_k^{(i)} = \begin{cases} 1 & \text{if } i_k = i, \\ 0 & \text{otherwise}. \end{cases}
\]

For $\lambda \in P_+$, set $n^\lambda := \sum_{i \in I} (\lambda, h_i) n^{(i)}$. Then we have

\[D_{w, \lambda} = G_{\lambda}^{\sup}(b_{-1}(n^\lambda, i)).\]
3.7. **Kumar-Peterson identities.** We investigate the map $\mathcal{J}^\vee_{w\lambda}$ a little more. Kumar and Peterson studied the identity which expresses the $H$-characters of the coordinate ring $\mathbb{C}[X_w \cap U_v]$ of the intersection $X_w \cap U_v$ of Schubert varieties $X_w$ and $\nu$-translates of the open cell $U_v$ as the limit of a family of “twisted” characters of Demazure modules in general Kac-Moody Lie algebras, see Kumar [Kum02, Theorem 12.1.3]. In the special case with $v = w$, it reduces to the case of Schubert cells, that is, we have $X_w \cap U_w = X_w$ (see Kumar [Kum02, Lemma 7.3.10]) and the following equality can be considered as a crystalized Kumar-Peterson identity.

**Theorem 3.48.** We have

$$\bigcup_{\lambda \in P_+} \mathcal{J}^\vee_{w\lambda} (\mathcal{B}_w (\lambda)) = \mathcal{B} (U_q^{-} (w)).$$

The rest of this subsection is devoted to the proof of Theorem 3.48.

**Lemma 3.49 (KO18, Lemma 3.19).** For $w \in W$, let $U_q^{-} (w)^\perp$ be the orthogonal complement of $U_q^{-} (w)$ with respect to $(\ , \ )_L$. We have an isomorphism as $\mathbb{Q}(q)$-vector spaces:

$$U_q^{-} (w) \otimes (U_q^{-} \cap T_w U_q^{-} \cap \text{Ker} (\varepsilon)) \cong (U_q^{-} (w))^\perp \subset U_q^{-}$$

under the multiplication $U_q^{-} (w) \otimes (U_q^{-} \cap T_w U_q^{-}) \cong U_q^{-}$, here recall that $\varepsilon$ is the counit of $U_q$ (see Definition 3.3).

**Lemma 3.50.** For $y \in U_q^{-} (w)^\perp$, we have $y^\vee . u_{w\lambda} = 0$ for all $\lambda \in P_+$.

**Proof.** By Lemma 3.49 we write $y = \sum y_{(1)} y_{(2)}$ with $y_{(1)} \in U_q^{-} (w)$ and homogeneous elements $y_{(2)} \in U_q^{-} \cap T_w U_q^{-} \cap \text{Ker} (\varepsilon)$. Then we have

$$(y^\vee . u_{w\lambda} = (T_{w^{-1}})^{-1} \left( \sum T_{w^{-1}} (y^\vee_{(1)}) T_{w^{-1}} (y^\vee_{(2)}) . u_{\lambda} \right) = 0$$

because $\text{wt} (T_{w^{-1}} (y^\vee_{(2)})) \in \mathbb{Q}_+ \setminus \{0\}$. \qed

**Proposition 3.51.** We have

$$\bigcup_{\lambda \in P_+} \mathcal{J}^\vee_{w\lambda} (\mathcal{B}_w (\lambda)) \subset \mathcal{B} (U_q^{-} (w)).$$

**Proof.** Let $\pi (w) : U_q^{-} \rightarrow U_q^{-} (w)$ be the projection with respect to the decomposition $U_q^{-} = U_q^{-} (w) \oplus U_q^{-} (w)^\perp$. Since $U_q^{-} (w)^\perp \cap B^\text{low}$ is a basis of $U_q^{-} (w)^\perp$ by Proposition 3.27 we have $\pi (w) (G^\text{low} (b)) \neq 0$ if and only if $b \in \mathcal{B} (U_q^{-} (w))$ for $b \in \mathcal{B} (\infty)$. Let $b \in \bigcup_{\lambda \in P_+} \mathcal{J}^\vee_{w\lambda} (\mathcal{B}_w (\lambda))$. Then there exists $\lambda \in P_+$ such that $(G^\text{low} (b))^\vee . u_{w\lambda} \neq 0$. By Proposition 3.50 we have

$$(G^\text{low} (b))^\vee . u_{w\lambda} = (\pi (w) (G^\text{low} (b)))^\vee . u_{w\lambda}.$$ 

In particular, we have $\pi (w) (G^\text{low} (b)) \neq 0$. This completes the proof. \qed

We prove the opposite inclusion.

**Proposition 3.52.** We have

$$\mathcal{B} (U_q^{-} (w)) \subset \bigcup_{\lambda \in P_+} \mathcal{J}^\vee_{w\lambda} (\mathcal{B}_w (\lambda)).$$
Proof. Let \( b \in \mathcal{B}(\mathbb{U}_q^- (w)) \), that is \( 0 \neq \pi (w) (G^{\text{low}} (b)) \in \mathbb{U}_q^- (w) \). (See the proof of Proposition 3.41) By Proposition 3.42 and Remark 3.45 it suffices to show that \( G^{\text{low}} (b)^{\vee} .u_{w\lambda} = (\pi (w) (G^{\text{low}} (b)))^{\vee} .u_{w\lambda} \neq 0 \) for some \( \lambda \in P_+ \). Note that \( (\pi (w) (G^{\text{low}} (b)))^{\vee} .u_{w\lambda} \neq 0 \) is equivalent to \( (\pi (w) (G^{\text{low}} (b)))^{\vee} .u_{w\lambda} \neq 0 \).

By the way, we have
\[
\bar{y}^{\vee} .u_{w\lambda} = (T_{w^{-1}})^{-1} ((T_{w^{-1}} \circ \sigma) (y) .u_{\lambda}) = (T_{w^{-1}})^{-1} ((\sigma \circ \sigma T_{w^{-1}}) (y) .u_{\lambda}) .
\]

Since \( y_0 := \pi (w) (G^{\text{low}} (b)) \in \mathbb{U}_q^- \cap T_w \mathbb{U}_q^{>0} \), we have \( (\sigma \circ \sigma T_{w^{-1}}) (y_0) \in \mathbb{U}_q^{<0} \). It is well-known that, for \( \xi \in Q_- \), there exists an element \( \lambda \in P_+ \) such that the projection \( (\mathbb{U}_q^-)_\xi \to V (\lambda) \xi^{+\lambda} \) given by \( y \mapsto y .u_{\lambda} \) is an isomorphism of vector space. Hence it can be shown that there exists \( \lambda \in P_+ \) such that \( (\sigma \circ \sigma T_{w^{-1}}) (y_0) .u_{\lambda} \neq 0 \). \( \Box \)

4. Quantum unipotent cells and the De Concini-Procesi isomorphisms

In this section, we introduce quantum unipotent cells \( A_q[N^w] \) following De Concini-Procesi [DP97], and show that they are isomorphic to the quantum coordinate ring of \( N_-(w) \cap w G_0^{\text{min}} \). This isomorphism, called the De Concini-Procesi isomorphism, was proved in [DP97, Theorem 3.2] under the assumption that \( q \) is of finite type. We will prove it in the case of arbitrary symmetrizable Kac-Moody cases (Theorem 4.13). We also introduce the dual canonical bases of the quantum unipotent cells (Definition 4.6).

4.1. Quantum unipotent cells. To define the quantum unipotent cells, we use the localizations of \( A_q[N_-(w)] \) and \( A_q[N_- \cap X_w] \). We recall the Ore properties of the unipotent quantum minors. The following is the multiplicative property of the dual canonical bases with respect to the unipotent quantum minors.

Proposition 4.1 ([Kim12] Theorem 6.24, Theorem 6.25). Let \( w \in W \).

1. For \( \lambda \in P_+ \) and \( b \in \mathcal{B}_w (\infty) \), there exists \( b' \in \mathcal{B}_w (\infty) \) such that
\[
q^{-\langle \lambda, w b \rangle} [D_{w\lambda, \lambda}] [G^{\text{up}} (b)] = [G^{\text{up}} (b')].
\]

2. For \( \lambda \in P_+ \), \( i \in I(w) \) and \( c \in \mathbb{Z}_{>0}^{I(w)} \), we have
\[
q^{-\langle \lambda, w b_{-1}(c, i) \rangle} D_{w\lambda, \lambda} G^{\text{up}} (b_{-1}(c, i)) = G^{\text{up}} (b_{-1}(c + n^\lambda, i)),
\]
where \( n^\lambda \) is defined as in Proposition 3.47.

Proposition 4.1 together with Proposition 3.10 deduces the following (cf. Remark 5.21 below).

Proposition 4.2. Let \( w \in W \) and set \( D_w := \{ q^m D_{w\lambda, \lambda} \mid m \in \mathbb{Z}, \lambda \in P_+ \} \). Then the sets \( D_w \) and \([D_w]\) are Ore sets of \( A_q[N_-(w)] \) and \( A_q[N_- \cap X_w] \) respectively consisting of \( q \)-central elements. More explicitly, for \( \lambda, \lambda' \in P_+ \) and homogeneous elements \( x \in A_q[N_-(w)] \), \( y \in A_q[N_- \cap X_w] \), we have
\[
q^{-\langle \lambda, w \lambda'-\lambda' \rangle} D_{w\lambda, \lambda} D_{w\lambda', \lambda'} = D_{w(\lambda+\lambda'), \lambda+\lambda'} D_{w\lambda, \lambda} x = q^{\langle \lambda+\lambda, w x \rangle} D_{w\lambda, \lambda} \text{ in } A_q[N_-(w)] , \text{ and } [D_{w\lambda, \lambda}] [y] = q^{\langle \lambda+\lambda, w y \rangle} [D_{w\lambda, \lambda}] \text{ in } A_q[N_- \cap X_w].
\]

Using the Proposition 4.2 we obtain the definition of quantum unipotent cells.
Definition 4.3. For \( w \in W \), we set
\[
A_q[N_-(w) \cap wG_0^{min}] := A_q[N_-(w)] [D_w^{-1}], \\
A_q[N^w] := A_q[N_- \cap X_w] [D_w^{-1}].
\]
Those algebras have \( Q \)-graded algebra structures in an obvious way. The algebra \( A_q[N^w] \) is called a quantum unipotent cell.

Remark 4.4. We note that the notations \( A_q[N_-(w) \cap wG_0^{min}] \) and \( A_q[N^w] \) will be justified after proving the existence of the dual canonical bases of that.

4.2. Dual canonical bases of quantum unipotent cells. In this subsection, we define the dual canonical bases of quantum unipotent cells using localization and the “multiplicative property” of the dual canonical bases of \( A_q[N^w] \) and \( A_q[N_-(w) \cap wG_0^{min}] \).

Proposition 4.5. Let \( w \in W \) and \( i \in I(w) \). Then the following hold:
1. The subset
\[
\{ q^{(\lambda, wt b + \lambda - w \lambda)} [D_{w,\lambda,\lambda}]^{-1} [G^{up}(b)] \mid \lambda \in P_+, b \in B_w(\infty) \}
\]
of \( A_q[N^w] \) forms a \( Q(q) \)-basis of \( A_q[N^w] \).
2. The subset
\[
\{ q^{(\lambda, wt b - (c, i) + \lambda - w \lambda)} D_{w,\lambda,\lambda,0}^{-1} [G^{up}(b-1(c, i))] \mid \lambda \in P_+, c \in \mathbb{Z}_{\geq 0} \}
\]
of \( A_q[N_-(w) \cap wG_0^{min}] \) forms a \( Q(q) \)-basis of \( A_q[N_-(w) \cap wG_0^{min}] \).

Proof. We prove only (1). The assertion (2) is proved in the same manner. The given subset obviously spans the \( Q(q) \)-vector space \( A_q[N^w] \). Hence it remains to show that this set is a linearly independent set. For \((\lambda, b), (\lambda', b') \in P_+ \times B_w(\infty)\), write \((\lambda, b) \sim (\lambda', b')\) if and only if \(q^{(\lambda, wt b + \lambda - w \lambda)} [D_{w,\lambda,\lambda}]^{-1} [G^{up}(b)] = q^{(\lambda', wt b' + \lambda' - w \lambda')} [D_{w,\lambda',\lambda}^{-1} [G^{up}(b')]\). The relation \(\sim\) is clearly an equivalence relation, and we take a complete set \( F \) of coset representatives of \((P_+ \times B_w(\infty))/\sim\).

Suppose that there exists a finite subset \( F' \subset F \) and \( a_{\lambda,b} \in Q(q) \) \( ((\lambda, b) \in F')\) such that \( \sum_{(\lambda, b) \in F'} q^{(\lambda, wt b + \lambda - w \lambda)} a_{\lambda,b} [D_{w,\lambda,\lambda}]^{-1} [G^{up}(b)] = 0 \). There exists \( \lambda_0 \in P_+ \) such that \( \lambda_0 - \lambda \in P_+ \) for all \( \lambda \in \mathbb{P} \) such that \((\lambda, b) \in F' \) for some \( b \in B_w(\infty) \). Now the equality \( \sum_{(\lambda, b) \in F'} q^{(\lambda, wt b + \lambda - w \lambda)} a_{\lambda,b} [D_{w,\lambda,\lambda}]^{-1} [G^{up}(b)] = 0 \) is equivalent to the equality

\[
[D_{w,\lambda_0,\lambda_0}] \left( \sum_{(\lambda, b) \in F'} q^{(\lambda, wt b + \lambda - w \lambda)} a_{\lambda,b} [D_{w,\lambda,\lambda}]^{-1} [G^{up}(b)] \right) = 0.
\]

By Proposition 4.2 and Proposition 4.1, for \((\lambda, b) \in F'\), we have
\[
[D_{w,\lambda_0,\lambda_0}] \left( q^{(\lambda, wt b + \lambda - w \lambda)} [D_{w,\lambda,\lambda}]^{-1} [G^{up}(b)] \right) = q^{(-\lambda_0 - \lambda, w \lambda + \lambda)} [D_{w,\lambda_0 - \lambda, \lambda_0 - \lambda}] [G^{up}(b)] = q^{(\lambda_0, wt b + \lambda - w \lambda)} [G^{up}(b^{(\lambda_0 - \lambda)})] \]
for some \( b^{(\lambda_0 - \lambda)} \in B_w(\infty) \). Note that \( wt b + \lambda - w \lambda = wt b^{(\lambda_0 - \lambda)} - wt D_{w,\lambda_0,\lambda_0} \). Therefore if \( b^{(\lambda_0 - \lambda)} = (b')^{(\lambda_0 - \lambda)} \) for \((\lambda, b), (\lambda', b') \in F'\) then we have the equality
\[
[D_{w,\lambda_0,\lambda_0}] \left( q^{(\lambda, wt b + \lambda - w \lambda)} [D_{w,\lambda,\lambda}]^{-1} [G^{up}(b)] \right) = [D_{w,\lambda_0,\lambda_0}] \left( q^{(\lambda', wt b' + \lambda' - w \lambda')} [D_{w,\lambda',\lambda'}^{-1} [G^{up}(b')]] \right),
\]
hence $(\lambda, b) = (\lambda', b')$. Thus (4.1) implies $a_{\lambda, b} = 0$ for all $(\lambda, b) \in F'$. This completes the proof. 

**Definition 4.6.** Let $w \in W$. We call
\[ \tilde{B}_{u, w}^{\text{up}} := \left\{ q^{(\lambda, w \lambda + \lambda - w \lambda)} [D_{w, \lambda, \lambda}]^{-1}[G_{\text{up}}(b)] \mid \lambda \in P_+, b \in \mathcal{B}_w(\infty) \right\}, \]
and
\[ \tilde{B}_{u}^{\text{up}}(w) := \{ q^{(\lambda, w b_{-1}(c, i) + \lambda - w \lambda)} D_{w, \lambda, \lambda}^{-1} G_{\text{up}}(b_{-1}(c, i)) \mid \lambda \in P_+, c \in \mathbb{Z}_{\geq 0}^\ell(w) \} \]
the dual canonical bases of $A_q[N^w_\lambda]$ and $A_q[N_-(w) \cap wG^\text{min}_0]$, respectively.

For $\lambda \in P$, there exist $\lambda_1, \lambda_2 \in P_+$ such that $\lambda = -\lambda_1 + \lambda_2$. Set
\[ D_{w, \lambda} := q^{(\lambda_1, w \lambda - \lambda)} D_{w, \lambda_1, \lambda_1}^{-1} D_{w, \lambda_2, \lambda_2} \in \tilde{B}_{u, w}^{\text{up}}(w). \]
Then $D_{w, \lambda}$ does not depend on the choice of $\lambda_1, \lambda_2 \in P_+$ by Proposition 4.5. Note that $\text{wt} D_{w, \lambda} = w \lambda - \lambda$.

The following is straightforwardly proved by Proposition 4.2.

**Proposition 4.7.** Let $w \in W$ and $\lambda, \lambda' \in P_+$. Then the following hold:

1. $D_{w, \lambda} = q^{(\lambda, w \lambda - \lambda - 1)} D_{w, \lambda_1, \lambda_2} D_{w, \lambda_1, \lambda_1}^{-1}$ for $\lambda_1, \lambda_2 \in P_+$ with $\lambda = -\lambda_1 + \lambda_2$.
2. $D_{w, \lambda} D_{w, \lambda'} = q^{(\lambda, w \lambda - \lambda') D_{w, \lambda + \lambda'}}$. In particular, $D_{w, \lambda}^{-1} = q^{(\lambda, w \lambda - \lambda)} D_{w, -\lambda}$.
3. $D_{w, \lambda} x = q^{(\lambda_+ + w \lambda, w t x)} x D_{w, \lambda}$ for $\lambda \in P_+$ and a homogeneous element $x \in A_q[N_-(w) \cap wG^\text{min}_0]$.

**Remark 4.8.** By using Proposition 4.1 (2), we can parametrize explicitly the elements of $\tilde{B}_{u}^{\text{up}}(w)$. Fix $i = (i_1, \ldots, i_\ell) \in I(w)$. An element $c \in \mathbb{Z}^\ell_{\geq 0}$ is said to have gaps if $\min\{c_k \mid i_k = i\} = 0$ for all $i \in I$. Then, by Propositions 4.1 (2) and 4.3 (2), we obtain the non-overlapping parametrization of the elements of $\tilde{B}_{u}^{\text{up}}(w)$ as follows:
\[ \tilde{B}_{u}^{\text{up}}(w) = \{ q^{-(\lambda, w b_{-1}(c, i))} D_{w, \lambda} G_{\text{up}}(b_{-1}(c, i)) \mid \lambda \in P, c \in \mathbb{Z}^\ell_{\geq 0} \text{ has gaps} \} \]

We define the dual bar involutions on $A_q[N^w_\lambda]$ and $A_q[N_-(w) \cap wG^\text{min}_0]$, which are useful when we study the dual canonical bases.

**Proposition 4.9.** The following hold:

1. The twisted dual bar involution $\sigma'$ induces $Q$-algebra anti-involutions $A_q[N_-(w) \cap X_w] \rightarrow A_q[N_-(w) \cap X_w]$ and $A_q[N_-(w)] \rightarrow A_q[N_-(w)]$. See Definition 3.11 for the definition of $\sigma'$. Moreover these maps are extended to $Q$-algebra anti-involutions $\sigma'': A_q[N^w_\lambda] \rightarrow A_q[N^w_\lambda]$ and $\sigma'': A_q[N_-(w) \cap wG^\text{min}_0] \rightarrow A_q[N_-(w) \cap wG^\text{min}_0]$.

2. Define a $Q(q)$-linear isomorphism $c_{tw} : A_q[N^w_\lambda] \rightarrow A_q[N_-(w) \cap wG^\text{min}_0]$ (resp. $A_q[N_-(w) \cap wG^\text{min}_0] \rightarrow A_q[N^w_\lambda]$) by
\[ x \mapsto q^{(\text{wt } x, wt x)/2 - (\text{wt } x, \rho)} x \]
for every homogeneous element $x \in A_q[N^w_\lambda]$ (resp. $x \in A_q[N_-(w) \cap wG^\text{min}_0]$).

Set $\sigma := c_{tw} \circ \sigma'$. Then for homogeneous elements $x, y \in A_q[N^w_\lambda]$ (resp. $A_q[N_-(w) \cap wG^\text{min}_0]$), we have
\[ \sigma(x y) = q^{(\text{wt } x, wt y)} \sigma(y) \sigma(x). \]

Moreover the elements of the dual canonical bases $\tilde{B}_{u, w}^{\text{up}}$ and $\tilde{B}_{u}^{\text{up}}(w)$ are fixed by $\sigma$. 

Definition 4.10. The \( \mathbb{Q} \)-linear isomorphisms \( \sigma : A_q[N^w \cap wG^\text{min}_0] \to A_q[N^w \cap wG^\text{min}_0] \) defined in Proposition 4.9 will be also called the dual bar involution and the twisted dual bar involution, respectively.

Proof of Proposition 4.9. Recall that \( \sigma(G^\text{up}(b)) = q^{-(wt(b),wt(b))/2+(wt(b),\rho)}G^\text{up}(b) \) for all \( b \in B(\infty) \). See Remark 3.12. Hence (1) follows from the compatibility of the algebras \( A_q[N^- \cap X^w_0] \), \( A_q[N^- (w)] \) and the dual canonical basis (Definition 3.28 Definition 3.37), and the universality of localization [GW04 Proposition 6.3]. A direct calculation immediately shows the equality \( 4.2 \). For \( \lambda \in P_+ \), we have

\[
1 = \sigma(D_{w,\lambda} D^{-1}_{w,\lambda}) \\
= q^{-(\lambda - \lambda \otimes \lambda - \lambda)} \sigma(D^{-1}_{w,\lambda}) \sigma(D_{w,\lambda}) \\
= q^{2(\lambda \otimes \lambda - \lambda)} \sigma(D^{-1}_{w,\lambda}) D_{w,\lambda}
\]

in \( A_q[N^- (w) \cap wG^\text{min}_0] \). Hence

\[
\sigma(D^{-1}_{w,\lambda}) = q^{-2(\lambda \otimes \lambda - \lambda)} D^{-1}_{w,\lambda}.
\]

Let \( b \in B_0(\infty) \). Then, by Proposition 4.2 and the equality above, we have

\[
\sigma(q^{(\lambda, \lambda - \lambda \otimes \lambda - \lambda)} D_{w,\lambda}^{-1} G^\text{up}(b)) \\
= q^{-(\lambda, \lambda - \lambda \otimes \lambda - \lambda)} \sigma(G^\text{up}(b)) [D_{w,\lambda}]^{-1} \\
= q^{-(\lambda, \lambda - \lambda \otimes \lambda - \lambda)} \sigma([G^\text{up}(b)] [D_{w,\lambda}]^{-1}) \\
= q^{-(\lambda, \lambda - \lambda \otimes \lambda - \lambda)} [G^\text{up}(b)] [D_{w,\lambda}]^{-1} \\
= q^{-(\lambda, \lambda - \lambda \otimes \lambda - \lambda)} [G^\text{up}(b)].
\]

This proves the dual bar invariance property for \( B^\text{up,w} \). The assertion for \( B^\text{up,w} \) is proved in the same manner. \( \square \)

As a corollary of the existence of the dual canonical bases of \( A_q[N^- (w) \cap wG^\text{min}_0] \) and \( A_q[N^w] \), we have the following specialization theorem.

Corollary 4.11. Let \( w \in W \).

1. Set \( A_{\mathbb{Q}[q^\pm1]}[N^- (w) \cap wG^\text{min}_0] \) to be the free \( A \)-module spanned by \( B^\text{up,w} \). Then it is a \( A \)-subalgebra of \( A_q[N^- (w) \cap wG^\text{min}_0] \) and we have an isomorphism

\[
A_{\mathbb{Q}[q^\pm1]}[N^- (w) \cap wG^\text{min}_0] \otimes_A \mathbb{C} \simeq \mathbb{C}[N^- (w) \cap wG^\text{min}_0]
\]

as \( \mathbb{C} \)-algebras.

2. Set \( A_{\mathbb{Q}[q^\pm1]}[N^w] \) to be the free \( A \)-module spanned by \( B^\text{up,w} \). Then it is a \( A \)-subalgebra of \( A_q[N^w] \) and we have an isomorphism

\[
A_{\mathbb{Q}[q^\pm1]}[N^w] \otimes_A \mathbb{C} \simeq \mathbb{C}[N^w]
\]

as \( \mathbb{C} \)-algebras.
4.3. De Concini-Procesi isomorphisms. In this subsection, we give a proof of the De Concini-Procesi isomorphism between $A_q[N_-(w)]$ and $A_q[N_+ \cap X_w]$ for general symmetrizable Kac-Moody Lie algebras, by using theory of canonical bases and specialization. We should remark that the original proof in [DP97] uses the downward induction on the length of elements of the Weyl group $W$ from the longest element, which exists only in finite type cases.

**Proposition 4.12 ([Kim12, Theorem 5.13]).** Let $w \in W$. Define $\iota_w : A_q[N_-(w)] \rightarrow A_q[N_+ \cap X_w]$ as a $\mathbb{Q}(q)$-algebra homomorphism induced from the canonical projection $U_q^- \rightarrow A_q[N_+ \cap X_w]$. Recall Definition 3.26 and 3.37. Then $\iota_w$ is injective, or equivalently, $\ast(\mathcal{B}(U_q^-(w))) \subset \mathcal{B}_w(\infty)$.

**Theorem 4.13** (The De Concini-Procesi isomorphism). Let $w \in W$. Then $\iota_w$ induces an isomorphism:

$$\iota_w : A_q[N_-(w) \cap wC_G^{\min}] \sim A_q[N_w^+].$$

**Proof.** The map $\iota_w$ in Proposition 4.12 induces an injective algebra homomorphism $\iota_w : A_q[N_-(w)] \rightarrow A_q[N_w^+]$. Since this map sends $D_{w_{\lambda}, \lambda}$ to $[D_{w_{\lambda}, \lambda}]$ for $\lambda \in P_+$, it is extended to the injective algebra homomorphism

$$\iota_w : A_q[N_-(w) \cap wC_G^{\min}] = A_q[N_-(w)][D_w^{-1}] \rightarrow A_q[N_w^+]$$

by the universality of localization. It follows immediately from the definition of dual canonical bases and Proposition 4.12 that $\iota_w$ induces an injective map from $\hat{B}_{\up}(w)$ to $\hat{B}_{\up,w}$. Therefore the map (4.3) is an isomorphism if and only if the (well-defined) map

$$\iota_{w_{q=1}} : A_q[q_{\pm1}][N_-(w) \cap wG_0^{\min}] \otimes_A C \rightarrow A_q[q_{\pm1}][N_w^+] \otimes_A C$$

is an isomorphism. Through the isomorphisms in Corollary 4.11, the map $\iota_{w_{q=1}}$ coincides with the map in Corollary 2.22 by definition of $\iota_w$; hence it is an isomorphism. This completes the proof. \[\square\]

5. Quantum twist isomorphisms

In this section, we construct the quantum twist isomorphisms between $A_q[N_-(w) \cap wG_0^{\min}]$ and $A_q[N_w^+]$ (see Theorem 5.19) and define the quantum twist automorphisms on $A_q[N_w^+]$ as a composite of the quantum twist isomorphism and the De Concini-Procesi isomorphism.

5.1. Quantized coordinate algebras. In this subsection, we give a brief review on the quantized coordinate rings. For more details, see [Jos95, Chapter 9, 10].

**Definition 5.1.** Let $M$ be a $U_q$-module. For $f \in M^* := \text{Hom}_{\mathbb{Q}(q)}(M, \mathbb{Q}(q))$ and $u \in M$, define a $\mathbb{Q}(q)$-linear map $c_{f,u}^M \in U_q^*$ given by

$$x \mapsto \langle f, x, u \rangle$$

for $x \in U_q$. When $M = V(\lambda)$ ($\lambda \in P_+$), we abbreviate $c_{f,u}^{V(\lambda)}$ to $c_{f,u}^\lambda$. For $w, w' \in W$ and $\lambda \in P_+$, we write

$c_{w_{\lambda},w'_{\lambda}}^\lambda := c_{f_{w_{\lambda},w'_{\lambda}}}^\lambda$

here $f_{w_{\lambda}} \in V(\lambda)^*$ is defined by $u \mapsto (w_{\lambda}u)^\psi$. 


**Definition 5.2.** Let $M$ be a $\mathbf{U}_q$-module. For $\mu \in P$, we set

$$M_\mu := \{m \in M \mid q^h m = q^{(\mu, h)} m \text{ for all } h \in P^*\}.$$ 

For a $\mathbf{U}_q$-module $M = \bigoplus_{\mu \in P} M_\mu$ with weight space decomposition, we write its graded dual $\bigoplus_{\mu \in P} M^*_\mu$ as $M^*$. Note that $M^*$ is a right $\mathbf{U}_q$-module. For $\lambda \in P_+$, $V(\lambda)^*$ is an integrable highest weight right $\mathbf{U}_q$-module with highest weight $\lambda$. For $u \in V(\lambda)$, define $u^* \in V(\lambda)^*$ by $u' \mapsto (u, u')^\varphi_\lambda$. Then we have $V(\lambda)^* = \{u^* \mid u \in V(\lambda)\}$ since the bilinear form $(\ , \ )^\varphi_\lambda$ is non-degenerate. Note that $f_{w\lambda} = u^*_{w\lambda}$ for $w \in W$.

Let $\mathbf{R}_q$ be the $\mathbb{Q}(q)$-vector subspace of $\mathbf{U}_q^*$ spanned by the elements

$$\left\{c^\lambda_{f,u} \mid f \in V(\lambda)^*, u \in V(\lambda) \text{ and } \lambda \in P_+\right\}.$$

Henceforth, we consider the algebra structure of $\mathbf{U}_q^*$ induced from the coalgebra structure of $\mathbf{U}_q$.

**Proposition 5.3** ([Kas93b, Definition 7.2.1, Proposition 7.2.2]). The subspace $\mathbf{R}_q$ is a subalgebra of $\mathbf{U}_q^*$, which is isomorphic to $\bigoplus_{\lambda \in P_+} V(\lambda)^* \otimes V(\lambda)$ as a $\mathbf{U}_q$-bimodule.

The $\mathbb{Q}(q)$-algebra $\mathbf{R}_q$ is called the quantized coordinate algebra associated with $\mathbf{U}_q$.

**Definition 5.4.** Let $v, w \in W$ and $\lambda \in P_+$. Set

$$\mathbf{R}_q^{w(+)}(\lambda) := \{c^\lambda_{f,u\lambda} \mid f \in V(\lambda)^*, \mathbf{R}_q^{w(+)} := \sum_{\lambda' \in P_+} \mathbf{R}_q^{w(+)}(\lambda') \subset \mathbf{R}_q,$$

$$\mathbf{Q}_v^{w(+)}(\lambda) := \{c^\lambda_{f,u\lambda} \mid f \in V(\lambda)^*, \{f, \mathbf{U}_q^+ u_{v\lambda}\} = 0\}, \mathbf{Q}_v^{w(+)} := \sum_{\lambda' \in P_+} \mathbf{Q}_v^{w(+)}(\lambda') \subset \mathbf{R}_q.$$

When $w = e$, we write $\mathbf{R}_q^{e(+)}$ (resp. $\mathbf{Q}_v^{e(+)}$) as $\mathbf{R}_q^+$ (resp. $\mathbf{Q}_v^+$). It is easy to show that, for all $w \in W$, $\mathbf{R}_q^{w(+)}$ is a subalgebra of $\mathbf{R}_q$, and isomorphic to $\mathbf{R}_q^+$ as algebras via $c^\lambda_{f,u\lambda} \mapsto c^\lambda_{f,u\lambda}$.

See, for example, [Tan17 Chapter 3]. Moreover, for all $v, w \in W$, $\mathbf{Q}_v^{w(+)}$ is a two-sided ideal of $\mathbf{R}_q^{w(+)}$, and the above isomorphism induces an isomorphism from $\mathbf{R}_q^{w(+)}/\mathbf{Q}_v^{w(+)}$ to $\mathbf{R}_q^+/\mathbf{Q}_v^+$.

5.2. Other descriptions of quantum unipotent subgroups and quantum closed unipotent cells. In this subsection, we describe the algebras, quantum unipotent subgroups and quantum unipotent cells, by using the quantized coordinate algebra $\mathbf{R}_q$. The following descriptions are essentially shown in [Jos95 9.1.7], [Yak10 Theorem 3.7]. However, we restate them emphasizing the terms of dual canonical bases. Actually, we can now prove each statement immediately.

**Notation 5.5.** Let $v, w \in W$. By abuse of notation, we describe the canonical projection $\mathbf{R}_q^{w(+)} \to \mathbf{R}_q^{w(+)}/\mathbf{Q}_v^{w(+)}$ as $c \mapsto [c]$.

**Definition 5.6.** As a bridge between quantized enveloping algebras and quantized coordinate algebras, we consider the following two linear maps:

$$\Phi: \mathbf{U}_q^{\leq 0} \to (\mathbf{U}_q^{\leq 0})^*, \ y_1 \mapsto (y_2 \mapsto (\psi(y_1), y_2))_D,$$

$$\Phi^+: \mathbf{U}_q^{\geq 0} \to (\mathbf{U}_q^{\geq 0})^*, \ x_1 \mapsto (x_2 \mapsto (x_1, \psi(x_2)))_D.$$ 

By the properties of the Drinfeld pairing $(\ , \ )_D$, $\Phi$ is an injective algebra homomorphism and $\Phi^+$ is an injective algebra anti-homomorphism.
Definition 5.7. Let \( \lambda \in P_+ \). Set

\[ U_q^-(\lambda) := j_\lambda (V(\lambda)) = \sum_{b \in \mathcal{B}(\lambda)} Q(q) G^{up} (\mathcal{J}_\lambda(b)). \]

The following propositions follow from the non-degeneracy of the Drinfeld pairing, Lemma 3.8 and Proposition 3.46.

Proposition 5.8. The restriction map \( U_q^* \to (U_q^{<0})^* \) induces the injective algebra homomorphism \( r_{\leq 0} : R_q^+ \to (U_q^{<0})^* \), and \( \text{Im } r_{\leq 0} \subset \text{Im } \Phi \). Moreover the well-defined \( Q(q) \)-algebra homomorphism \( R_q^+ \to U_q^{<0}, c \mapsto (\Phi^{-1} \circ r_{\leq 0})(c) \) induces the \( Q(q) \)-algebra isomorphism \( \mathcal{I} : R_q^+ \to \sum_{\lambda \in P_+} U_q^-(\lambda) q^{-\lambda} \).

Proposition 5.9. For \( \lambda \in P_+ \) and \( b \in \mathcal{B}(\lambda) \), we have

\[ \mathcal{I} (c_{\lambda,u}^b) = G^{up} (\mathcal{J}_\lambda(b)) q^{-\lambda} = D_{G^{up}}(b,u) q^{-\lambda}. \]

In particular, we have

\[ \mathcal{I} (Q_q^+(\lambda)) = \sum_{b \in \mathcal{B}(\lambda) \setminus \mathcal{B}_w(\lambda)} Q(q) G^{up} (\mathcal{J}_\lambda(b)) q^{-\lambda}. \]

Definition 5.10. An element \( z \) of \( R_q^+ \) (resp. \( R_q^+/Q_w^+ \)) is said to be \( q \)-central if, for every weight vector \( f \in V(\lambda)^* \) and \( \lambda \in P_+ \), there exists \( l \in \mathbb{Z} \) such that

\[ zc_{\lambda,u}^f = q^l c_{\lambda,u}^f z \quad (\text{resp. } z[c_{\lambda,u}^f] = q^l [c_{\lambda,u}^f]) \]

Corollary 5.11. The set \( \{ c_{\lambda,u}^f \}_{\lambda \in P_+} \) is an Ore set in \( R_q^+ \) consisting of \( q \)-central elements. In particular, \( S := \{ c_{\lambda,u}^f \}_{\lambda \in P_+} \) is an Ore set in \( R_q^+/Q_w^+ \) consisting of \( q \)-central elements.

By Corollary 5.11, we can consider the algebra \( (R_q^+/Q_w^+) [S^{-1}] \). Proposition 5.8 and 5.9 together with Remark 3.36 immediately imply the following proposition. This gives the description of \( A_q[N_+ \cap X_w] \) in terms of the quantized coordinate algebra \( R_q \). This kind of description appears in [Jos95, 9.1.7].

Proposition 5.12. Let \( w \in W \). Set \( A_q[N_+ \cap X_w]^{ex} := U_q^{<0}/(U_w)^{-1} U_q^{>0} \). Note that \( (U_w)^{-1} U_q^{>0} \) is a two-sided ideal of \( U_q^{<0} \). Then the \( Q(q) \)-algebra isomorphism \( \mathcal{I}_w : (R_q^+/Q_w^+) [S^{-1}] \to A_q[N_+ \cap X_w]^{ex} \)

induces the \( Q(q) \)-algebra isomorphism

\[ \mathcal{I}_w : (R_q^+/Q_w^+) [S^{-1}] \to A_q[N_+ \cap X_w]^{ex}. \]

Moreover the \( Q(q) \)-algebra \( \sum_{\lambda \in P_+} (R_q^+(\lambda)/Q_w^+) [c_{\lambda,u}^f]^{-1} (\subset (R_q^+/Q_w^+) [S^{-1}]) \) is isomorphic to \( A_q[N_+ \cap X_w] \).

Next, we study the quantum unipotent subgroups via the quantized coordinate rings following Joseph and Yakimov. We consider the algebra \( R_q w^+(+)/Q_w w^+(+) \), which is isomorphic to \( R_q^+/Q_w^+ \). See Definition 5.4.

Definition 5.13. Let \( w \in W \) and \( \lambda \in P_+ \). Set

\[ U_q^+(w,\lambda) := \left( j_{\lambda,u}^w (V(\lambda)/V_w(\lambda)^-) \right)^\vee = \sum_{b \in \mathcal{B}_w(\lambda)} Q(q) G^{up} (\mathcal{J}_w^\vee(b))^\vee. \]

The following proposition follows again from the non-degeneracy of the Drinfeld pairing, the equality 3.22, Lemma 3.8 Proposition 3.19 and Proposition 3.46.
Proposition 5.14. Let \( w \in W \). The restriction map \( U_q^* \to (U_q^{>0})^* \) induces the algebra homomorphism \( r_w^w : R_q^{w(+)} \to (U_q^{>0})^* \), and it satisfies \( \text{Ker}(r_w^w) = Q_w^{w(+)} \) and \( \text{Im}(r_w^w) \subset \text{Im} \Phi^+ \). Hence \( r_w^w \) induces the \( (q,q) \)-algebra isomorphism \( \tau_{\geq 0}^w : R_q^{w(+)} / Q_w^{w(+)} \to R_q^{w(\geq 0)} \). Moreover we have a well-defined algebra anti-isomorphism \( T_w^+ : R_q^+/Q_w^+ \to R_q^+/Q_w^+ \) given by \([c_{f,u}] \mapsto (\Phi^+)^{-1} \circ \tau_{\geq 0}^w(\Phi^+(b))\) for \( f \in V(\lambda)^* \), \( \lambda \in P_+ \). We have

\[
T_w^+([c_{G_{\lambda}^b}^+,u_{\lambda}]) = G^+ \left( \tau_{\geq 0}^w(b) \right)^{q^{-w\lambda}} \varphi(D_{u_{w\lambda},G_{\lambda}^b}(b))q^{-w\lambda}
\]

for \( b \in B_w(\lambda) \).

Corollary 5.15. The set \( S_w := \{[c_{w\lambda}]\}_{\lambda \in P_+} \) is an Ore set in \( R_q^+/Q_w^+ \) consisting of \( q \)-central elements.

Remark 5.16. The description in Proposition 5.14 implies that the algebra \( R_q^+/Q_w^+ \) has no zero divisors.

By Corollary 5.15, we can consider the \( (q,q) \)-algebra \( (R_q^+/Q_w^+) | S_{w}^{-1} \). Proposition 5.14 immediately implies the following proposition. This gives the description of \( A_q[N_+(w)] \) in terms of the quantized coordinate algebra \( R_q \). This description appears in [Yak10, Theorem 3.7] modulo some difference of conventions.

Proposition 5.17. Let \( w \in W \). Then \( T_w^+ \) induces the algebra anti-isomorphism

\[
T_w^+ : \left( R_q^+/Q_w^+ \right) [S_{w}^{-1}] \to U_q^+(w)U_q^0.
\]

Moreover the \( (q,q) \)-algebra \( \sum_{\lambda \in P_+} \left( R_q^+/Q_w^+ \right) [c_{w\lambda}]^{-1} \subset \left( R_q^+/Q_w^+ \right) [S_{w}^{-1}] \) is anti-isomorphic to \( U_q^+(w) \), and is isomorphic to \( A_q[N_-(w)] \) via \( \varphi \).

Proof. It suffices to show that \( \sum_{\lambda \in P_+} U_q^+(w,\lambda) = U_q^+(w) \). This follows from Theorem 3.48.

5.3. Quantum twist isomorphisms and dual canonical bases. In this subsection, we prove the existence of quantum twist isomorphisms (Theorem 5.19). The following lemma easily follows from Corollary 5.11 and 5.15. See also [GW01, Proposition 6.3].

Lemma 5.18. Let \( w \in W \). Then the set \( \tilde{S}_w := \{q^m[c_{w\lambda}]c_{\lambda}^w \mid m \in \mathbb{Z}, \lambda, \lambda' \in P_+ \} \) is an Ore set in \( R_q^+/Q_w^+ \) consisting of \( q \)-central elements.

Moreover the maps \( \left( R_q^+/Q_w^+ \right) [S_{w}^{-1}] \to \left( R_q^+/Q_w^+ \right) [\tilde{S}_{w}^{-1}] \), \( [c_{w\lambda}]^{-1} \to [c_{f,u}]^{-1} \), \( [c_{\lambda}]^{-1} \to [c_{w\lambda}]^{-1} \) and \( \left( R_q^+/Q_w^+ \right) [S_{w}^{-1}] \to \left( R_q^+/Q_w^+ \right) [\tilde{S}_{w}^{-1}] \), \( [c_{w\lambda}]^{-1} \to [c_{f,u}]^{-1} \), \( [c_{\lambda}]^{-1} \to [c_{w\lambda}]^{-1} \) are injective \( (q,q) \)-algebra homomorphisms.

Theorem 5.19. There exists an isomorphism of the \( (q,q) \)-algebras

\[
\gamma_{w,q} : A_q[N_+^w] \to A_q[N_-(w) \cap wG_0^{min}]
\]

given by

\[
[D_{u,u}] \mapsto q^{-(\lambda,\omega u - \lambda)}D_{w\lambda,u}^{-1}D_{w,\lambda,u} \quad [D_{w\lambda}]^{-1} \mapsto q^{(\lambda,\omega u - \lambda)}D_{w,\lambda}
\]

for a weight vector \( u \in V(\lambda) \) and \( \lambda \in P_+ \).

Definition 5.20. We call \( \gamma_{w,q} \) a quantum twist isomorphism (cf. Proposition 2.23).
Proof of Theorem 5.19 By Proposition 5.12 (see also Proposition 5.9), we have the algebra isomorphism

\[ A_q[N_- \cap X_w] \xrightarrow{T_w^{-1}} \sum_{\lambda \in P_+} (R_q^+ \lambda / Q^+ w) [c_{\lambda,\lambda}]^{-1} \]

given by

\[ [D_{\lambda, u}] \mapsto [c_{\lambda,\lambda}]^{-1} \]

for \( \lambda \in P_+ \) and \( u \in V(\lambda) \). In particular, \( T_w^{-1}([D_{\lambda,\lambda}]) = [c_{\lambda,\lambda}]^{-1} \).

By Lemma 5.18 \( \sum_{\lambda \in P_+} (R_q^+ \lambda / Q^+ w) [c_{\lambda,\lambda}]^{-1} \) is naturally regarded as a subalgebra of \( (R_q^+ / Q^+ w)[S^{-1}_w] \), and in the latter algebra, the set \( \{ q^m [c_{\lambda,\lambda}]^{-1} \mid m \in \mathbb{Z}, \lambda \in P_+ \} \) is a multiplicative set consisting of invertible \( q \)-central elements. Hence the algebra isomorphism 5.11 is extended to the algebra isomorphism

\[ J_1: A_q[w^w] \to \sum_{\lambda, \lambda', \lambda'' \in P_+} (R_q^+ \lambda / Q^+ w) [c_{\lambda',\lambda''}]^{-1}. \]

On the other hand, by Proposition 5.17 (see also Proposition 5.14), we have an algebra isomorphism

\[ \sum_{\lambda \in P_+} (R_q^+ \lambda / Q^+ w) [c_{w,\lambda,\lambda}]^{-1} \xrightarrow{\psi \circ T_w^+} A_q[N_- w], \]

given by

\[ [c_{w,\lambda,\lambda}]^{-1} \mapsto D_{w,\lambda, u} \]

for \( \lambda \in P_+ \) and \( u \in V(\lambda) \). In particular, \( \phi \circ T_w^+([c_{w,\lambda,\lambda}]^{-1}) = D_{w,\lambda,\lambda} \).

As above, the set \( \{ q^m [c_{w,\lambda,\lambda}]^{-1} \mid m \in \mathbb{Z}, \lambda \in P_+ \} \) is a multiplicative set consisting of invertible \( q \)-central elements of \( (A_q^+ / Q^+ w)[S^{-1}_w] \). Hence the algebra isomorphism 5.4 is extended to the algebra isomorphism

\[ J_2: \sum_{\lambda, \lambda', \lambda'' \in P_+} (R_q^+ \lambda / Q^+ w) [c_{\lambda',\lambda''}]^{-1} \to A_q[N_- (w) \cap w \downarrow G_{\min}]. \]

By 5.3 and 5.6, we obtain the \( Q(q) \)-algebra isomorphism

\[ \gamma_{w, q} := J_2 \circ J_1: A_q[w^w] \to A_q[N_- (w) \cap w \downarrow G_{\min}]. \]

Moreover, for \( \lambda \in P_+ \) and a weight vector \( u \in V(\lambda) \), we have

\[ \gamma_{w, q}(D_{\lambda, u}) = J_2([c_{\lambda,\lambda}]^{-1}) \]

by 5.2,

\[ = J_2(q^{-\lambda, \lambda} \cdot D_{\lambda, u}) \]

by Proposition 5.9,

\[ = J_2(q^{-\lambda, \lambda} \cdot D_{\lambda, u}) \]

by \( q^{-\lambda, \lambda} D_{\lambda, u} \) by 5.5.

Moreover,

\[ 1 = \gamma_{w, q}(D_{\lambda, u}) \]

by 5.9,

\[ = q^{-\lambda, \lambda} D_{\lambda, u} \gamma_{w, q}(D_{\lambda, u}) \]

by 5.6.
Hence,
\[ \gamma_{w,q}([D_{w,\lambda},\lambda]^{-1}) = q^{(\lambda, w\lambda - \lambda)} D_{w,\lambda}. \]
This completes the proof of the theorem. \(\square\)

**Remark 5.21.** We can also deduce Proposition 5.12 from the descriptions
\[ [D_{w,\lambda},\lambda] = \mathcal{I}_w([c_{w,\lambda,\lambda}^{-1}], D_{w,\lambda} = (\varphi \circ \mathcal{I}_w^+)([c_{w,\lambda,\lambda}]) \]
appearing in the proof of Theorem 5.19 together with Propositions 5.9 and 5.14.

The quantum twist isomorphism \(\gamma_{w,q}\) is compatible with the dual canonical bases as follows:

**Theorem 5.22.** Let \( w \in \mathcal{W} \). Then the quantum twist isomorphism \( \gamma_{w,q} : A_q[N^w] \rightarrow A_q[N^w] \cap wG^\text{min} \) restricts to the bijection \( \hat{B}^w : \hat{B}^w \rightarrow \hat{B}^w \) given by
\[ q^{(\lambda, \text{wt}(\mathcal{I}_w(b))+\lambda'-w\lambda)}[D_{w,\lambda,\lambda}]^{-1}[G_{\text{up}}(\mathcal{I}_w(b))] \mapsto q^{-(\lambda'-w\lambda, \text{wt}(\mathcal{I}_w(b)))} D_{w,\lambda'-\lambda}G_{\text{up}}(\mathcal{I}_w(b)) \]
for \( \lambda, \lambda' \in P_+, b \in \mathcal{B}_w(\lambda') \). In particular, \( \gamma_{w,q}([D_{w,\lambda}]) = D_{w,-\lambda} \) for \( \lambda \in P \), and \( \gamma_{w,q} \circ \sigma = \sigma \circ \gamma_{w,q} \).

**Proof.** By Proposition 3.46, for \( \lambda, \lambda' \in P_+ \) and \( b \in \mathcal{B}_w(\lambda') \), we have
\[ \gamma_{w,q}(q^{(\lambda, \text{wt}(\mathcal{I}_w(b))+\lambda'-w\lambda)}[D_{w,\lambda,\lambda}]^{-1}[G_{\text{up}}(\mathcal{I}_w(b))]) = \gamma_{w,q}(q^{(\lambda, \text{wt}(\mathcal{I}_w(b))-w\lambda)}[D_{w,\lambda,\lambda}]^{-1}[G_{\text{up}}(\mathcal{I}_w(b), u_{\lambda'}))] \]
\[ = q^{(\lambda, \text{wt}(\mathcal{I}_w(b))+\lambda'-w\lambda)}(q^{(\lambda, w\lambda-\lambda)})^{-1} D_{w,\lambda'-\lambda}D_{w,\lambda'}G_{\text{up}}(\mathcal{I}_w(b)) \]
\[ = q^{-(\lambda'-w\lambda, \text{wt}(\mathcal{I}_w(b)))} D_{w,\lambda'-\lambda}G_{\text{up}}(\mathcal{I}_w(b)). \]
This completes the proof. \(\square\)

6. Twist Automorphisms on Quantum Unipotent Cells

We now obtain the twist automorphisms on quantum unipotent cells.

**Theorem 6.1.** Let \( w \in \mathcal{W} \). Then there exists a \( \mathbb{Q}(q) \)-algebra automorphism
\[ \eta_{w,q} := \iota_w \circ \gamma_{w,q} : A_q[N^w] \rightarrow A_q[N^w] \]
given by
\[ [D_{w,\lambda}, \lambda] \mapsto q^{-(\lambda, \text{wt}(u))-\lambda)}[D_{w,\lambda,\lambda}]^{-1}[D_{w,u}], \quad [D_{w,\lambda}, \lambda]^{-1} \mapsto q^{(\lambda, w\lambda-\lambda)}[D_{w,\lambda}] \]
for a weight vector \( u \in V(\lambda) \) and \( \lambda \in P_+ \). In particular, \( \text{wt}_{\eta_{w,q}}([x]) = -\text{wt}(x) \) for homogeneous elements \( [x] \in A_q[N^w] \). Moreover \( \eta_{w,q} \) restricts to a permutation on the dual canonical bases \( \hat{B}^w \). In particular, \( \eta_{w,q} \) commutes with the dual bar involution \( \sigma \), and \( \eta_{w,q}([D_{w,\lambda}]) = [D_{w,-\lambda}] \) for \( \lambda \in P_+ \).

The following follows from the theorem above and Proposition 2.22.

**Corollary 6.2.** Let \( w \in \mathcal{W} \). Then the \( \mathbb{Q}(q) \)-algebra automorphism \( \eta_{w,q} : A_q[N^w] \rightarrow A_q[N^w] \)
induces a \( A \)-algebra automorphism \( \eta_{w,A} : A_q[q^{\pm 1}][N^w] \rightarrow A_q[q^{\pm 1}][N^w] \) and a \( \mathbb{C} \)-algebra automorphism
\[ \eta_{w,q} |_{q=1} : A_q[q^{\pm 1}][N^w] \otimes_A \mathbb{C} \rightarrow A_q[q^{\pm 1}][N^w] \otimes_A \mathbb{C}. \]
Moreover, through the isomorphism in Corollary 4.11, the automorphism \( \eta_{w,q} |_{q=1} \) coincides with \( \eta^* \).
The based quantum torus $T$ for a $\ell \in \mathcal{B}(\infty)$, and has a $\tilde{\eta}$ of Appendix A. Note that $F$ is symmetric $Z$-skew-symmetric integer matrix. The skew-symmetric integer matrix $\Lambda$ determines a skew-

\section{7. Quantum twist automorphisms and quantum cluster algebras}

In this section, we consider an additive categorification of the twist automorphism $\eta_{w.g}$ on a quantum unipotent cell $A_q[N^w]$ in the sense of Geiß-Leclerc-Schröer. When $g$ is symmetric, Geiß-Leclerc-Schröer \cite{GLS12} obtained a categorification of the twist automorphism $\eta_{w}^*$ on the coordinate algebra of a unipotent cell $N^w$ (Proposition \ref{prop:7.24}). They used subcategories $C_w$, introduced by Buan-Iyama-Reiten-Scott \cite{BIRS09}, of the module category of the preprojective algebra $P$ corresponding to the Dynkin diagram for $g$. Geiß-Leclerc-Schröer \cite{GLS13} have also shown that the quantum unipotent subgroup $A_q[N^w(\omega)]$ is isomorphic to a certain quantum cluster algebra $A_q[N^w](C_w)$, which is determined by data of $C_w$ (Proposition \ref{prop:7.19}). Combining these results, we obtain a categorification of the twist automorphism $\eta_{w.g}$ (Theorem \ref{thm:7.25}). See also Corollary \ref{cor:7.26}.

In this section, we always consider the case that $g$ is symmetric. We assume that $(\alpha_i, \alpha_i) = 2$ for all $i \in I$, and thus $q_i = q$ for all $i \in I$.

\begin{notation}
For $m, m' \in \mathbb{Z}_{\geq 0}$ with $m \leq m'$, set $[m, m'] := \{ k \in \mathbb{Z} \mid m \leq k \leq m' \}$.
\end{notation}

\subsection{7.1. Quantum cluster algebras}

In this subsection, we briefly review quantum cluster algebras. The main references are \cite{BZ05} and \cite{GLS13}.

\begin{definition}
Let $n, \ell$ be positive integers such that $n \leq \ell$. Let $\Lambda = (\lambda_{ij})_{i,j \in [1,\ell]}$ be a skew-symmetric integer matrix. The skew-symmetric integer matrix $\Lambda$ determines a skew-

\begin{itemize}
\item $T$ is an associative algebra,
\item $X^a X^b = q^{\Lambda(a,b)} X^b X^a$ for $a, b \in \mathbb{Z}^\ell$,
\item $X^0 = 1$ and $(X^a)^{-1} = X^{-a}$ for $a \in \mathbb{Z}^\ell$.
\end{itemize}

The based quantum torus $T$ is contained in its skew-field of fractions $F(= F(\Lambda))$ \cite{BZ05}, Appendix A. Note that $F$ is a $\mathbb{Q}(q^{1/2})$-algebra. Write $X_i := X^{e_i}$ for $i \in [1, \ell]$.

Next we define an important operation, called mutation. Let $\tilde{B} = (b_{ij})_{i \in [1,\ell], j \in [1,\ell-n]}$ be an $\ell \times (\ell - n)$ integer matrix. The submatrix $B = (b_{ij})_{i,j \in [1,\ell-n]}$ of $\tilde{B}$ is called the principal part of $\tilde{B}$. The pair $(\Lambda, \tilde{B})$ is said to be compatible if, for $i \in [1, \ell]$ and $j \in [1, \ell - n]$, 

$$\sum_{k=1}^{\ell} b_{kj} \lambda_{ki} = \delta_{ij} d_j$$ 

for some $d_j \in \mathbb{Z}_{>0}$.
convention is slightly different from Geiß-Leclerc-Schröer’s one, see Remark 7.16.

In this subsection, we review the construction of the quantum unipotent cells.

7.2. 

Note that, when $(\Lambda, \tilde{B})$ is compatible, $\tilde{B}$ has full rank $\ell - n$ and its principal part $B = (b_{ij})_{i,j \in [1, \ell - n]}$ is skew-symmetrizable [BZ05 Proposition 3.3]. We will assume that $B$ is skew-symmetric.

For $k \in [1, \ell - n]$, define $E^{(k)} = (e_{ij})_{i,j \in [1, \ell]}$ and $F^{(k)} = (f_{ij})_{i,j \in [1, \ell - n]}$ as follows:

$$
e_{ij} = \begin{cases} 
\delta_{i,j} & \text{if } j \neq k, \\
-1 & \text{if } i = j = k, \\
\max(0, -b_{ik}) & \text{if } i \neq j = k.
\end{cases}$$

$$f_{ij} = \begin{cases} 
\delta_{i,j} & \text{if } i \neq k, \\
-1 & \text{if } i = j = k, \\
\max(0, b_{kj}) & \text{if } i = k \neq j.
\end{cases}$$

Set

$$\mu_k(\Lambda) = (E^{(k)})^T \Lambda E^{(k)} \\
\mu_k(\tilde{B}) = E^{(k)} \tilde{B} F^{(k)}.$$ 

Then $\mu_k(\Lambda, \tilde{B}) := (\mu_k(\tilde{B}), \mu_k(\Lambda))$ is again compatible [BZ05 Proposition 3.4]. It is said that $\mu_k(\Lambda, \tilde{B})$ is obtained from $(\Lambda, B)$ by the mutation in direction $k$. Note that $\mu_k(\mu_k(\Lambda, \tilde{B})) = (\Lambda, \tilde{B})$.

The pair $\mathcal{S} = \{(X_i)_{i \in [1, \ell]}, \tilde{B}, \Lambda\}$ is called a quantum seed in $F$, and $\{X_i\}_{i \in [1, \ell]}$ is called the quantum cluster of an arbitrary quantum seed obtained by iterated mutations of $\mathcal{S}$.

For $k \in [1, \ell - n]$, define $\mu_k(\{X_i\}_{i \in [1, \ell]}) = \{X'_i\}_{i \in [1, \ell]} \subset F \setminus \{0\}$ by

- $X'_i = X_i$ if $i \neq k$,
- $X'_k = X^{-e_k + \sum_{\{j, b_{jk} > 0\}} b_{jk} e_j} X^{-e_k - \sum_{\{j, b_{jk} > 0\}} b_{jk} e_j}$.

Then there is an injective $\mathbb{Q}[q^{\pm 1/2}]$-algebra homomorphism $\mathcal{T}(\mu_k(\Lambda)) \to F(\Lambda)$ given by $X_i^{\pm 1} \mapsto (X'_i)^{\pm 1}$ ($i \in [1, \ell]$). Moreover there exist a basis $\{c_i\}_{i \in [1, \ell]}$ of $\mathbb{Z}^\ell$ and a $\mathbb{Q}(q^{1/2})$-algebra automorphism $\tau : F(\Lambda) \to F(\Lambda)$ such that $\tau(X_i^{c_i}) = X'_i$ for $i \in [1, \ell]$ [BZ05 Proposition 4.7]. Hence the map above is extended to the isomorphism $F(\mu_k(\Lambda)) \to F(\Lambda)$. Through this isomorphism, we identify $F(\mu_k(\Lambda))$ with $F(\Lambda)$, and henceforth always write $F$ for this skew-field. Write

$$\mu_k(\mathcal{S}) := (\mu_k(\{X_i\}_{i \in [1, \ell]}), \mu_k(\tilde{B}), \mu_k(\Lambda))$$

and this is called a quantum seed obtained from the mutation of $\mathcal{S}$ in direction $k$. Note that $\mu_k(\mu_k(\mathcal{S}')) = \mathcal{S}'$ for any quantum seed $\mathcal{S}'$ and $k \in [1, \ell - n]$. By the argument above, we can consider the iterated mutations in arbitrary various directions $k \in [1, \ell - n]$. The subset $\{X_i \mid i \in [\ell - n + 1, \ell]\}$, called the set of frozen variables, is contained in the quantum cluster of an arbitrary seed obtained by iterated mutations of $\mathcal{S}$.

The quantum cluster algebra $A_{q^{\pm 1/2}}(\mathcal{S})$ is defined as the $\mathbb{Q}[q^{\pm 1/2}]$-subalgebra of $F$ generated by the union of the quantum clusters of all quantum seeds obtained by iterated mutations of $\mathcal{S}$. An element $M \in A_{q^{\pm 1/2}}(\mathcal{S})$ is called a quantum cluster monomial if there exists a quantum cluster $\{X'_i = (X'^{c_i})_{i \in [1, \ell]}\}$ of a quantum seed obtained by iterated mutations of $\mathcal{S}$ such that $M = (X')^{a}$ for some $a \in \mathbb{Z}_{\geq 0}$.

**Proposition 7.3** ([BZ05 Corollary 5.2]). The quantum cluster algebra $A_{q^{\pm 1/2}}(\mathcal{S})$ is contained in the based quantum torus generated by the quantum cluster of an arbitrary quantum seed obtained by iterated mutations of $\mathcal{S}$.

### 7.2. Quantum cluster algebra structures on quantum unipotent subgroups and quantum unipotent cells

In this subsection, we review the construction of the quantum cluster algebra structure on $A_q[N_{-\{w\}}]$ following [GLS11 GLS12 GLS13]. We note that our convention is slightly different from Geiß-Leclerc-Schröer’s one, see Remark 7.10.
**Definition 7.4.** A finite quiver $Q = (Q_0, Q_1, s, t)$ is a datum such that

1. $Q_0$ is a finite set, called the set of vertices,
2. $Q_1$ is a finite set, called the set of arrows,
3. $s, t: Q_1 \to Q_0$ are maps, and it is said that $a \in Q_1$ is an arrow from $s(a)$ to $t(a)$.

Here we take a finite quiver $Q$ such that $Q_0 = I$, $s(a) \neq t(a)$ for all $a \in Q_1$ and $a_{ij}(\triangleq \langle \alpha_i, \beta_j \rangle) = -\#\{a \in Q_1 | s(a) = i, t(a) = j\} - \#\{a \in Q_1 | s(a) = j, t(a) = i\}$. Such a quiver $Q$ is called a finite quiver without edge loops which corresponds to the symmetric generalized Cartan matrix $A$ [GLS11] Subsection 2.1 and 4.1. Let $\overline{Q} = (Q_0, \overline{Q}_1 := Q_1 \bigcup Q_1', s, t)$ be the double quiver of $Q$, which is obtained from $Q$ by adding to each arrow $a \in Q_1$ an arrow $a^* \in Q_1'$ such that $s(a^*) = t(a)$ and $t(a^*) = s(a)$. Set

$$\Pi := \mathbb{C}\overline{Q}/ \left( \sum_{a \in Q_1} (a^*a - aa^*) \right),$$

Here $\mathbb{C}\overline{Q}$ is a path algebra of $\overline{Q}$, and $\left(\sum_{a \in Q_1} (a^*a - aa^*)\right)$ is the two-sided ideal generated by $\sum_{a \in Q_1}(a^*a - aa^*)$. This is called the preprojective algebra associated with $Q$. Denote by $\epsilon_i$ the idempotent of $\Pi$ corresponding to $i \in I$. For a finite dimensional $\Pi$-module $X$, write $\dim X := -\sum_{i \in I}(\dim \mathbb{C}\epsilon_i X)\epsilon_i \in Q_+$. Remark that we do not regard $\dim X$ as an element of $Q_+$. A finite dimensional $\Pi$-module $X$ is said to be nilpotent if there exists $N \in \mathbb{Z}_{\geq 0}$ such that $a_1 \cdots a_N X = 0$ for any sequence $(a_1, \ldots, a_N) \in Q_1^N$.

**Proposition 7.5** ([CB00] Lemma 1). For any finite dimensional $\Pi$-module $X, Y$, we have

$$(\dim X, \dim Y) = \dim \mathbb{C} \hom_{\Pi}(X, Y) + \dim \mathbb{C} \hom_{\Pi}(Y, X) - \dim \mathbb{C} \ext^1_{\Pi}(X, Y).$$

**Remark 7.6.** For any finite dimensional $\Pi$-modules $X, Y$, we have

$$\dim \mathbb{C} \ext^1_{\Pi}(X, Y) = \dim \mathbb{C} \ext^1_{\Pi}(Y, X)$$

by Proposition 7.5.

A finite dimensional nilpotent $\Pi$-module $X$ determines an element $\varphi_X$ of $\mathbb{C} [N_+] = U(n_-)^*_{gr}$ through Lusztig’s construction of the universal enveloping algebra $U(n_-)$ as a space $\mathcal{M}$ consisting of certain constructible functions with convolution product [Lus00]. There is a short summary, for instance, in [GLS11] Subsection 2.2. However we remark that, in this paper, we consider the convolution product on $\mathcal{M}$ opposite to the one in [GLS11] Subsection 2.2. See also Remark 7.16. The following are important properties of $\varphi_X$.

**Proposition 7.7** ([Lus00] [GLS05] [GLS07b]). Let $X, Y$ be finite dimensional nilpotent $\Pi$-modules. The following hold:

1. $\wt(\varphi_X) = \dim X$.
2. $\varphi_X \varphi_Y = \varphi_{XY}$.
3. Suppose that $\dim \mathbb{C} \ext^1_{\Pi}(X, Y) = 1$. Write non-split short exact sequences as

$$0 \to X \to Z_1 \to Y \to 0 \quad 0 \to Y \to Z_2 \to X \to 0.$$

Then we have $\varphi_X \varphi_Y = \varphi_{Z_1} + \varphi_{Z_2}$. 


Notation 7.8. Let \( w \in W \) and fix \( i = (i_1, \ldots, i_\ell) \in I(w) \). Then, for \( k = 1, \ldots, \ell \), we set
\[
\begin{align*}
k^+ & := \min\{\ell + 1 \cup \{k + 1 \leq j \leq \ell \mid i_j = i_k\}\}, \\
k^- & := \max\{0 \cup \{1 \leq j \leq k - 1 \mid i_j = i_k\}\}, \\
k^\text{max} & := \max\{1 \leq j \leq \ell \mid i_j = i_k\}, \\
k^\text{min} & := \min\{1 \leq j \leq \ell \mid i_j = i_k\}.
\end{align*}
\]
Moreover, set \( I_w := \{i \in I \mid i = i_k \text{ for some } k = 1, \ldots, \ell\} \). Then we can easily check that \( I_w \) does not depend on the choice of \( i \).

Definition 7.9. Let \( S_i \) be the (simple) \( \Pi \)-module such that \( \dim S_i = -\alpha_i \) for \( i \in I \). For a \( \Pi \)-module \( X \) and \( i \in I \), define \( \text{soc}_i(X) \subset X \) by the sum of all submodules of \( X \) isomorphic to \( S_i \). For a sequence \( (i_1, \ldots, i_k) \in I^k \) \( (k \in \mathbb{Z}_{>0}) \), there exists a unique chain
\[
X \supset X_0 \supset X_1 \supset X_2 \supset \cdots \supset X_k = 0
\]
of submodules of \( X \) such that \( X_{j-1}/X_j \simeq \text{soc}_{i_j}(X/X_j) \) for \( j = 1, \ldots, k \). Set \( \text{soc}_{(i_1, \ldots, i_k)}(X) := X_0 \). For \( i \in I \), denote by \( \hat{I}_i \) the indecomposable injective \( \Pi \)-module with socle \( S_i \). Let \( w \in W \) and \( i = (i_1, \ldots, i_\ell) \in I(w) \). For \( k = 1, \ldots, \ell \), set
\[
V_{i,k} := \text{soc}_{(i_1, \ldots, i_k)}(\hat{I}_{i_k}).
\]
Set \( V_i := \bigoplus_{k=1,\ldots,\ell} V_{i,k} \). Define \( C_w \) as a full subcategory of the category of \( \Pi \)-modules consisting of all \( \Pi \)-modules \( X \) such that there exist \( t \in \mathbb{Z}_{>0} \) and a surjective homomorphism \( V_i^\oplus \to X \). Then it is known that \( C_w \) does not depend on the choice of \( i \in I(w) \). Note that all objects of \( C_w \) are nilpotent \( \Pi \)-modules. An object \( C \in C_w \) is called \( C_w \)-projective (resp. \( C_w \)-injective) if \( \text{Ext}_\Pi(C, X) = 0 \) (resp. \( \text{Ext}_\Pi(C, X) = 0 \)) for all \( X \in C_w \). The category \( C_w \) is closed under extension and is Frobenius. In particular, an object \( X \in C_w \) is \( \Pi \)-projective if and only if it is \( C_w \)-injective. An object \( T \) of \( C_w \) is called \( C_w \)-maximal rigid if \( \text{Ext}_\Pi^1(T \oplus X, X) = 0 \) for \( X \in C_w \) implies that \( X \) is isomorphic to a direct summand of a direct sum of copies of \( T \). A \( \Pi \)-module \( M \) is called basic if it is written as a direct sum of pairwise non-isomorphic indecomposable modules. Then, in fact, \( V_i \) is a basic \( C_w \)-maximal rigid module and \( V_{i,k} \) is the \( C_w \)-projective-injective module with socle \( S_i \) for \( k = 1, \ldots, \ell \). See [BIRS09] for more details, and [GLST1] Subsection 2.4 for more detailed summaries.

Let \( T \) be a basic \( C_w \)-maximal rigid module and \( T = T_1 \oplus \cdots \oplus T_\ell \) its indecomposable decomposition. From now on, we write \( I_w = [1, n] \) for simplicity, and always number indecomposable summands of \( T \) so that \( T_{\ell-n+i} \in I_w \) is the \( C_w \)-projective-injective module with socle \( S_i \). Note that this labelling is different from the labelling \( V_i = \bigoplus_{k \in [1, \ell]} V_{i,k} \). Let \( \Gamma_T \) be the Gabriel quiver of \( A_T := \text{End}_\Pi(T)^\text{op} \), that is, the vertex set of \( \Gamma_T \) is \([1, \ell]\) and \( d_{ij} := \dim C \text{Ext}_A^1(S_{T_i}, S_{T_j}) \) arrows from \( i \) to \( j \), where \( S_{T_i} \) is the head of a (projective) \( A_T \)-module \( \text{Hom}_\Pi(T, T_i) \). Define \( \tilde{B}_T \) by \( \tilde{B}_{ij} := (\tilde{d}_{ij} - d_{ij}) \). The following proposition is an essential result for the additive categorification of cluster algebras.

Proposition 7.10 ([BIRS09], [GLST2]). In the setting above, the following hold:
1. \( \ell = \ell(w) \).
2. For any \( k \in [1, \ell-n] \), there exists a unique indecomposable \( \Pi \)-module in \( C_w \) such that \( T_k \neq T_k \) and \( (T/T_k) \oplus T_k^* \) is a basic \( C_w \)-maximal rigid module. This basic \( C_w \)-maximal rigid module is denoted by \( \mu_T(T) \) and called the mutation of \( T \) in direction \( T_k \).
3. For any \( k \in [1, \ell-n] \), \( \mu_k(\tilde{B}_T) = \tilde{B}_{\mu_T(T)} \).


(4) For any $k \in [1, \ell - n]$, we have $\dim \mathbb{C} \mathrm{Ext}^1_H(T_k, T_k^*) = 1$, and there exists non-split exact sequences
\[
0 \to T_k \to T_- \to T_k^* \to 0 \quad \quad 0 \to T_k^* \to T_+ \to T_k \to 0
\]
such that $T_- \simeq \bigoplus_{j : b_{jk} < 0} T_j^\otimes (-b_{jk})$ and $T_+ \simeq \bigoplus_{j : b_{jk} > 0} T_j^\otimes b_{jk}$.

Note that, by Proposition 7.7 and 7.10 we have
\[
(7.1) \quad \varphi_{T_k} \varphi_T^* = \varphi_T + \varphi_{T_-} = \prod_{j : b_{jk} > 0} \varphi_{T_j}^{b_{jk}} + \prod_{j : b_{jk} < 0} \varphi_{T_j}^{-b_{jk}}.
\]

This is an additive categorification of mutation. See [GGLS11 Subsection 2.7] and references therein for more details. An object $T$ of $\mathcal{C}_w$ is said to be reachable (in $\mathcal{C}_w$) if $T$ is isomorphic to a direct summand of a direct sum of copies of a basic $\mathcal{C}_w$-maximal rigid module which is obtained from $V_i$ by iterated mutations. In fact, the notion of reachable does not depend on the choice of $i$ [BIRS09 Proposition III.4.3].

Remark 7.11. Let $T$ be a basic reachable $\mathcal{C}_w$-maximal rigid module, and $T = T_1 \oplus \cdots \oplus T_\ell$ its indecomposable decomposition. By Proposition 7.3, for any $i, j \in [1, \ell]$, we have
\[
(\dim T_i, \dim T_j) = \dim \mathbb{C} \mathrm{Hom}_H(T_i, T_j) + \dim \mathbb{C} \mathrm{Hom}_H(T_j, T_i).
\]

Definition 7.12. Let $i = (i_1, \ldots, i_\ell) \in I(w)$. For $1 \leq a \leq b \leq \ell$ with $i_a = i_b$, there exists a natural injective homomorphism $V_{i,a} \to V_{i,b}$ of $\Pi$-modules, and the cokernel of this homomorphism is denoted by $M_i(b, a)$. Here we set $V_{i,0} := 0$. In particular, $M_i(b, \min)$ is isomorphic to $V_{i,b}$. Geiß-Leclerc-Schröer shows that $M_i(b, a)$ is reachable for all $1 \leq a \leq b \leq \ell$ with $i_a = i_b$ [GGLS11 section 13].

We use the notation in Definition 7.9 Geiß-Leclerc-Schröer construct a quantum cluster algebra $\mathcal{A}_q(q)(\mathcal{C}_w)$ associated with $\mathcal{C}_w$ as we shall recall. Let $T$ be a basic $\mathcal{C}_w$-maximal rigid module and $T = T_1 \oplus \cdots \oplus T_\ell$ its indecomposable decomposition. Define $\Lambda_T := (\lambda_{ij})_{i,j \in [1,\ell]}$ by
\[
\lambda_{ij} := \dim \mathbb{C} \mathrm{Hom}_H(T_i, T_j) - \dim \mathbb{C} \mathrm{Hom}_H(T_j, T_i).
\]

Geiß-Leclerc-Schröer have shown the following properties:

Proposition 7.13 ([GGLS13 Proposition 10.1, Proposition 10.2]). (1) $(\widetilde{B}_T, \Lambda_T)$ is compatible in the sense of Definition 7.2.

(2) $\mu_k(\widetilde{B}_T, \Lambda_T) = (\widetilde{B}_{\mu_{T_k}(T)}, \Lambda_{\mu_{T_k}(T)})$ for $k \in [1, \ell - n]$.

Definition 7.14. The quantum cluster algebra $\mathcal{A}_{q^{\pm 1/2}}(\mathcal{C}_w)$ is defined as the quantum cluster algebra with the initial seed $((X_T)_i)_{i \in [1,\ell]}$, $(\widetilde{B}_T, \Lambda_T)$ for a basic reachable $\mathcal{C}_w$-maximal rigid module $T$. Note that this algebra $\mathcal{A}_{q^{\pm 1/2}}(\mathcal{C}_w)$ does not depend on the choice of $T$. By the properties above, we may write
\[
\mu_k((X_T)_i)_{i \in [1,\ell]}, (\widetilde{B}_T, \Lambda_T) = (((X_{\mu_{T_k}(T)})_i)_{i \in [1,\ell]}, (\widetilde{B}_{\mu_{T_k}(T)}, \Lambda_{\mu_{T_k}(T)})
\]
for $k \in [1, \ell - n]$. Moreover, for $a = (a_1, \ldots, a_\ell) \in \mathbb{Z}_{\geq 0}^\ell$, set $X_{\sum a_i}^{q^{\pm a_i}} := (X_T)^a$. Then the quantum cluster monomials of $\mathcal{A}_{q^{\pm 1/2}}(\mathcal{C}_w)$ are indexed by reachable $\Pi$-modules in $\mathcal{C}_w$.

Set
\[
Y_R := q^{\frac{\dim R \dim R}{4}} X_R.
\]
for every reachable Π-module \( R \) in \( \mathcal{C}_w \). Recall that \( \dim R \in \mathbb{Q}_- \). Define the \textit{rescaled quantum cluster algebra} \( \mathcal{A}_{q}^{±1}(C_w) \) as an \( \mathcal{A}(:= \mathbb{Q}[q^{±1}])\)-subalgebra of \( \mathcal{A}_{q}^{±1/2}(C_w) \) generated by \{\( Y_R \mid R \) is reachable in \( \mathcal{C}_w \)\}. For any basic reachable \( \mathcal{C}_w \)-maximal rigid module \( T = T_1 \oplus \cdots \oplus T_\ell \), the rescaled quantum cluster algebra \( \mathcal{A}_{q}^{±1}(C_w) \) is contained in the \textit{rescaled based quantum torus} \( T_{A,T} := A[Y_{T,k}^{±1}] \ k \in [1,\ell][\subset \mathcal{F}] \) \cite{GLS13} Lemma 10.4 and Proposition 10.5 (they are cited as \( \text{(7.2)} \)) and Proposition \( \text{(7.17)} \) below). Note that, for \( (a_1,\ldots,a_\ell) \in \mathbb{Z}_{≥0}^\ell \), we have

\[
Y_R = q^{α(R)}Y_{T_1}^{a_1} \cdots Y_{T_\ell}^{a_\ell},
\]

here we set \( R := \bigoplus_{i \in [1,\ell]} T_i^{a_i} \) and

\[
α(R) := \sum_{i \in [1,\ell]} a_i(a_i-1) \dim_{\mathbb{C}} \text{Hom}_T(T_i,T_i)/2 + \sum_{i<j; i,j \in [1,\ell]} a_ia_j \dim_{\mathbb{C}} \text{Hom}_T(T_j,T_i).
\]

Note that \( I := \{q^mY_i \bigoplus_{i \in [l-n+1,\ell]} T_i^{a_i} \mid (a_\ell-n+1,\ldots,a_\ell) \in \mathbb{Z}_{≥0}^\ell, m \in \mathbb{Z} \} \) is an Ore set in \( \mathcal{A}_{q}^{±1}(C_w) \).

Set \( \overline{\mathcal{A}}_{q}^{±1}(C_w) := \mathcal{A}_{q}^{±1}(C_w)[I^{-1}] \), and \( \mathcal{A}_{q}(C_w) := \mathbb{Q}(q) \otimes_{\mathcal{A}} \mathcal{A}_{q}^{±1}(C_w) \), \( \overline{\mathcal{A}}_{q}(C_w) := \mathbb{Q}(q) \otimes_{\mathcal{A}} \overline{\mathcal{A}}_{q}^{±1}(C_w) \).

For \( X \in \mathcal{C}_w \), denote by \( I(X) \) the injective hull of \( X \) in \( \mathcal{C}_w \), and by \( \Omega^{-1}_w(X) \) the cokernel of the corresponding injective homomorphism \( X \to I(X) \). Hence we have an exact sequence

\[
0 \to X \to I(X) \to \Omega^{-1}_w(X) \to 0.
\]

**Proposition 7.15** \cite{GLS11 Proposition 13.4]. Let \( w \in W \), \( T \) a basic reachable \( \mathcal{C}_w \)-maximal rigid module and \( T = T_1 \oplus \cdots \oplus T_\ell \) its indecomposable decomposition. Then \( T' := \Omega^{-1}_w(T) \oplus \bigoplus_{i \in I_w} T_i \oplus \cdots \oplus T_\ell \) is also a basic reachable \( \mathcal{C}_w \)-maximal rigid module; hence there exists a bijection \( [1,\ell-n] \to [1,\ell-n], k \mapsto k^\ast \) such that \( T_{k^\ast} = \Omega^{-1}_w(T_k) \).

Let \( k \in [1,\ell-n] \) and write \( µ_{T_k}(T) = (T/T_k) \oplus T_k^\ast \). Then we have

\[
µ_{T_k^\ast}(T') = (T'/T_k^\ast) \oplus T_k^\ast\cdot
\]

**Remark 7.16.** Let \( w \in W \). In this remark, we explain the difference between our convention and Geiß-Leclerc-Schröer’s one in \cite{GLS11 GLS12 GLS13}. An object \( \mathcal{X} \) in Geiß-Leclerc-Schröer’s papers is denoted by \( \mathcal{X}^{GLS} \) here.

The category \( \mathcal{C}_w \) is the same category as \( \mathcal{C}_w^{GLS} \). Moreover \( \mathcal{N}_w = (N(w-1)^{GLS})^T \) and \( \mathcal{N}_w^w = ((N(w-1))^{GLS})^T \). We omitted the definition of \( φ_X \) for a finite dimensional nilpotent \( Π\)-module \( X \), however the algebra \( \mathcal{M} \) used for its precise definition (see Definition \( \text{(7.3)} \)) is the same space as \( \mathcal{M}^{GLS} \) in \cite{GLS11} Subsection 2.2 equipped with the opposite convolution product.

Thus there exist algebra isomorphisms \( \mathbb{C}[N_w] \to \mathbb{C}[N(w-1)^{GLS}] \) and \( \mathbb{C}[N_w^w] \to \mathbb{C}[N(w-1,1,1)^{GLS}] \) given by \( f \to f \circ (−)^T \). Moreover \( φ_X = φ_X^{GLS} \circ (−)^T \) for all \( X \in \mathcal{C}_w = \mathcal{C}_w^{GLS} \). See also \cite{GLS11} Chapter 6). (This is the reason why we consider the opposite product on \( \mathcal{M} \).)

The quantum nilpotent subalgebra \( U_q(n(w-1))^{GLS} \) in \cite{GLS13} is equal to \( A_q[N(w-1)]^\vee \). Geiß-Leclerc-Schröer consider a \( (q)-\)algebra \( A_q(n(w-1))^{GLS} \), called the quantum coordinate ring, which is defined in \( (U_q^+)^* \) \cite{GLS13} (4.6), and define an algebra isomorphism \( ψ^{GLS} : U_q(n(w-1))^{GLS} \to A_q(n(w-1))^{GLS} \) by using a non-degenerate bilinear form \( ( , )^{GLS} \).
Proposition 4.1]. Actually, for \( x \in (U_q^+)_{\beta} \), \( y \in (U_q^+)_{\beta'} \) (\( \beta, \beta' \in Q_+ \)), we have
\[
(x, y)^{GLS} = \delta_{\beta, \beta'}(1 - q^{-2})\text{ht} \frac{x, y}{L}
\]
\[
= \delta_{\beta, \beta'}(1 - q^{-2})\text{ht} \frac{x', y'}{L}
\]
\[
= \delta_{\beta, \beta'}(1 - q^{-2})\text{ht} \left( x', \sigma(y') \right)_L
\]
\[
= q^{(\beta, \beta)/2}(q^{-1} - q)^{\text{ht}} \left( x', \varphi(y) \right)_L.
\]
The last equality follows from Proposition 3.10. By the way, there exists a \( \mathbb{Q}(q) \)-algebra automorphism \( m_{\text{norm}} : U_q^{-} \rightarrow U_q^{-} \) given by \( f_i \mapsto (q^{-1} - q)^{-1} f_i \) for \( i \in I \). We now have the following \( \mathbb{Q}(q) \)-algebra isomorphism;
\[
I_{\text{norm}} : A_q[N_-(w)] \xrightarrow{m_{\text{norm}}} A_q[N_-(w)] \xrightarrow{\psi_{GLS}} U_q(n(w^{-1}))^{GLS} \xrightarrow{\psi_{GLS}} A_q(n(w^{-1}))^{GLS},
\]
which maps \( x \in (U_q^{-})_{\beta} \) (\( \beta \in -Q_+ \)) to \( q^{(\beta, \beta)/2}(x, \varphi(-))^L \). By using this isomorphism, we describe their results. Note that \( I_{\text{norm}}(P_{w, \lambda, w'}) = q^{(w(-\lambda - w', \lambda - w)/2)}P_{GLS}^{GLS} \) for \( w, w' \in W \) and \( \lambda \in P_+ \). [GLS13 (5.5)].

The definitions of the quantum cluster algebra \( \mathcal{A}_{q^{1/2}}(C_w(GLS)) \) are the same. We have \( Y_R = q^{(\dim R, \dim R)/2}Y_{GLS}^{GLS} \) for every reachable II-module \( R \). [GLS13 (10.16)]. Note that \( (\dim R, \dim R)/2 \in \mathbb{Z} \). Therefore we have \( \mathcal{A}_{q^{1/2}}(C_w(GLS)) \).

The following propositions describe mutations of quantum clusters and twisted dual bar involutions in \( \mathcal{A}_{q^{1/2}}(C_w) \).

Proposition 7.17 ([GLS13 Proposition 10.5]). Let \( T \) be a basic reachable \( C_w \)-maximal rigid module, and \( T = T_1 \oplus \cdots \oplus T_k \) its indecomposable decomposition. Fix \( k \in [1, \ell - n] \). Write \( \tilde{B}_T = (b_{ij})_{i \in [1, \ell], j \in [1, \ell - n]} \) and \( \mu_{T_k}(T) = (T/T_k) \oplus T_k^e \). Set \( T_+ := \bigoplus_{j; b_{jk} > 0} T_j^{b_{jk}} \) and \( T_- := \bigoplus_{j; b_{jk} < 0} T_j^{b_{jk}} \). Then we have
\[
Y_{T_k}Y_{T_k} = q^{-\dim \text{Hom}_R(T_k, T_k^e)}(qY_{T_+} + Y_{T_-}).
\]

Proposition 7.18 ([GLS13 Lemma 10.6, Lemma 10.7]). Let \( T \) be a basic reachable \( C_w \)-maximal rigid module. Then there exists a unique \( \mathbb{Q} \)-algebra anti-involution \( \sigma' \) on \( T_{A,T} \) such that
\[
q \rightarrow q^{-1}, \ Y_R \rightarrow q^{-\dim R, \dim R/2 - \dim_C R}Y_R
\]
for every direct summand \( R \) of a direct sum of copies of \( T \). Moreover \( \sigma' \) induces \( \mathbb{Q} \)-algebra anti-involutions \( \sigma' \) on \( \mathcal{A}_{q^{1/2}}(C_w) \) and \( \mathcal{A}_{q^{1/2}}(C_w) \), and \( \sigma' \) does not depend on the choice of a basic reachable \( C_w \)-maximal rigid module \( T \).

Geiß-Leclerc-Schröer showed that a rescaled quantum cluster algebra \( \mathcal{A}_{Q(q)}(C_w) \) gives an additive categorification of the quantum unipotent subgroup \( A_q[N_-(w)] \) as follows.

Proposition 7.19 ([GLS13 Theorem 12.3]). Let \( w \in W \) and \( i = (i_1, \ldots, i_\ell) \in I(w) \). Then there is an isomorphism of \( \mathbb{Q}(q) \)-algebras \( \kappa : A_q[N_-(w)] \rightarrow \mathcal{A}_{Q(q)}(C_w) \) given by
\[
D_{s_{i_1} \cdots s_{i_\ell} \omega_{i_\ell} \cdots s_{i_1} \omega_{i_1}} \mapsto Y_{M[b,a]}
\]
for all \( 1 \leq a \leq b \leq \ell \) with \( i_a = i_b \). Moreover we have \( \sigma' \circ \kappa = \kappa \circ \sigma' \). Recall Definition 3.11.

By Theorem 4.13 this result also gives an additive categorification of the quantum unipotent cell \( A_q[N_+^w] \).
Corollary 7.20. Let $w \in W$ and $i = (i_1, \ldots, i_\ell) \in I(w)$. Then there is an isomorphism of \(\mathbb{Q}(q)\)-algebras \(\tilde{\kappa}: A_q[N_w^\omega] \to \tilde{A}_q(C_w)\) given by
\[
[D_{s_{i_1}^{-1} \cdots s_{i_\ell}^{-1}} w_{i_1} \cdots w_{i_\ell}] \mapsto Y_{M[i,a]}
\]
for all $1 \leq a \leq b \leq \ell$ with $i_a = i_b$. Moreover we have $\sigma' \circ \tilde{\kappa} = \tilde{\kappa} \circ \sigma'$. Recall Definition 4.10.

The following is the classical counterpart of the results above due to Geiß-Leclerc-Schröer. Note that we explain it as a “specialization” of the results above but it is actually the preceding result of them.

Proposition 7.21 ([GLS11] Theorem 3.1, Theorem 3.3). Let $w \in W$. For every reachable \(\Pi\)-module $R$ in $C_w$, we have $\varphi_R \in \mathbb{C}[N_-(w)]$, and the correspondence
\[
\varphi_R \quad \text{(resp. } \varphi_R) \mapsto 1 \otimes Y_R.
\]
gives the \(\mathbb{C}\)-algebra isomorphism from $\mathbb{C}[N_-(w)]$ (resp. $\mathbb{C}[N_w^\omega]$) to $\mathbb{C} \otimes_A A_q^{\pm 1} (C_w)$ (resp. $\mathbb{C} \otimes_A \tilde{A}_q^{\pm 1} (C_w)$).

Definition 7.22. Let $T$ be a basic reachable $C_w$-maximal rigid module and $T = T_1 \oplus \cdots \oplus T_\ell$ its indecomposable decomposition. Then the $Q$-grading on $\mathbb{Q}[q^\pm 1][Y_{T_k} \mid k = 1, \ldots, \ell] \subset T_{A,T}$ given by $\mathrm{wt} Y_{T_k} = \dim T_k$ is extended to the $Q$-grading on $T_{A,T}$. A homogeneous element $X \in T_{A,T}$ is said to be dual bar invariant if
\[
\sigma_T^*(X) = q^{-(\mathrm{wt} X, \mathrm{wt} X)/2 + (\mathrm{wt} X, \rho)} X,
\]
here recall that $\rho := \sum_{i \in I} w_i$ (Definition 2.1). When $X \in A_q[N_-(w)]$ (resp. $A_q[N_w^\omega]$), the $Q$-grading and the definition of dual bar invariance of homogeneous elements are compatible with the corresponding notions in $A_q[N_-(w)]$ (resp. $A_q[N_w^\omega]$) via $\kappa$ (resp. $\tilde{\kappa}$). See Remark 3.12. Note that $Y_R$ is dual bar invariant for any reachable $\Pi$-module $R$.

Remark 7.23. Through $\kappa$ (resp. $\tilde{\kappa}$), we can translate also the nontwisted dual bar involution $\sigma$ on $A_q[N_-(w)]$ (resp. $A_q[N_w^\omega]$) into the involution on $A_q[N_-(w)]$ (resp. $A_q[N_w^\omega]$). Then this involution coincides with the twisted bar involution in the sense of [BZ05] Section 6 if we take a grading datum $\Sigma = (\sigma_{ij})_{i,j \in [1, \ell]}$ associated with $T$ in Definition 7.22 as $\sigma_{ij} = -(\dim T_i, \dim T_j)$ for $i, j \in [1, \ell]$ (see [BZ05] Definition 6.5) for the definition of the notion of grading).

Geiß-Leclerc-Schröer also obtained an additive categorification of the twist automorphism $\eta^*_w$ on the coordinate algebra $\mathbb{C}[N_w^\omega]$ of a unipotent cell $N_w^\omega$ in non-quantum settings. Here the image of $\varphi_X \in \mathbb{C}[N_-]$ under the restriction map $\mathbb{C}[N_-] \to \mathbb{C}[N_w^\omega]$ is denoted by $[\varphi_X] \in \mathbb{C} \subset N_w^\omega$.

Proposition 7.24 ([GLS12] Theorem 6). Let $w \in W$. Then for every $X \in C_w$ we have
\[
\eta^*_w ([\varphi_X]) = \frac{[\varphi_{\Omega_w^{-1}}(X)]}{[\varphi_I(X)]}.
\]

7.3. Quantum twist automorphisms and the quantum algebra structures. Our main result in this subsection is the following quantum analogue of Proposition 7.24. Recall Proposition 7.15.
Theorem 7.25. Let $w \in W$, $T$ a basic reachable $C_w$-maximal rigid module, and $T = T_1 \oplus \cdots \oplus T_l$ its indecomposable decomposition. Through $\bar{\kappa}$ in Corollary 7.20 we regard the quantum twist map $\eta_{w,q}$ as an algebra automorphism on $\mathcal{Y}_{Q(q)}(C_w)$. Then, for every direct summand $R$ of a direct sum of copies of $T$ (that is, every reachable $\Pi$-module $R$ in $C_w$), we have

$$\eta_{w,q}(Y_R) = q^{\sum_{i \in I_w} \lambda_i \dim C_{\ell - n_i}} R_{I(R)}^{-1} Y_{\Omega_w^{-1}(R)}. \quad (7.3)$$

here we write $I(R) = \bigoplus_{i \in I_w} T_{\ell - n_i}^{\oplus \lambda_i}$.

As a corollary of the above result, we obtain the following.

Corollary 7.26. Let $R$ be a reachable $\Pi$-module in $C_w$. Then $\kappa^{-1}(Y_R) \in B_{up} \cap A_q[N_-(w)]$ if and only if $\kappa^{-1}(Y_{\Omega_w^{-1}(R)}) \in B_{up} \cap A_q[N_-(w)]$.

Before proving Theorem 7.25, we show its corollary.

Proof. By Theorem 6.1 and 7.25 $\kappa^{-1}(Y_R) \in B_{up}$ if and only if $\kappa^{-1}(q^{\sum_{i \in I_w} \lambda_i \dim C_{\ell - n_i}} R_{I(R)}^{-1} Y_{\Omega_w^{-1}(R)}) \in \widetilde{B}_{up}$. By Theorem 6.1 and the dual bar invariance of $Y_R$, the element $q^{\sum_{i \in I_w} \lambda_i \dim C_{\ell - n_i}} R_{I(R)}^{-1} Y_{\Omega_w^{-1}(R)}$ is also dual bar invariant. Combining this fact with the definition of $\tilde{B}_{up} = \iota_w(\tilde{B}_{up}(w))$ and the dual bar invariance of $Y_{\Omega_w^{-1}(R)}$, we have $\kappa^{-1}(q^{\sum_{i \in I_w} \lambda_i \dim C_{\ell - n_i}} R_{I(R)}^{-1} Y_{\Omega_w^{-1}(R)}) \in \tilde{B}_{up}$ if and only if $\kappa^{-1}(Y_{\Omega_w^{-1}(R)}) \in B_{up}$.

Remark 7.27. Kang-Kashiwara-Kim-Oh [KKKO18] have shown that all (rescaled) quantum cluster monomials belong to $B_{up}$ by using the categorification via representations of quiver Hecke algebras. (See [KKKO18] Introduction for several results of this direction before [KKKO18].) Hence we have already known that $Y_R$ is an element of $B_{up}$ for an arbitrary reachable $\Pi$-module in $C_w$. However there is no proof of this strong result through the additive categorification above. Therefore it would be interesting to determine the quantum monomials in $B_{up}$ which are obtained from Corollary 7.20 and, for example, $(Y_{\ell i}^a)$ for $a \in \mathbb{Z}_{>0}$ and $i \in I(w)$. Actually, it is easy to show that $(Y_{\ell i}^a) \in B_{up}$ by Proposition 4.1. Moreover it is unclear whether a quantum twist automorphism $\eta_{w,q}$ is categorified by using finite dimensional representations of quiver Hecke algebras. In particular, we do not know that quantum twist automorphisms preserve the basis coming from the simple modules of quiver Hecke algebras.

The rest of this subsection is devoted to the proof of Theorem 7.25. In this proof, we essentially use Geiß-Leclerc-Schröer’s theory.

Lemma 7.28. Let $T$ be a basic reachable $C_w$-maximal rigid module and $T = T_1 \oplus \cdots \oplus T_l$ its indecomposable decomposition. Take $(a_1, \ldots, a_l) \in \mathbb{Z}^l$. Then there exists a unique integer $m$ such that $q^m Y_{T_1}^{a_1} \cdots Y_{T_l}^{a_l}$ is dual bar invariant in $T_{A,T}$.

Proof. We have

$$\sigma_{T}(q^m Y_{T_1}^{a_1} \cdots Y_{T_l}^{a_l} = q^{-m} \sigma_T(q Y_{T_1})^{a_1} \cdots \sigma_T(Y_{T_l})^{a_l}$$

$$= q^{-m + \sum_{i \in [1, l]} a_i ((\dim_{T_i} \dim T_i)/2 + (\dim T_i, \rho))} Y_{T_1}^{a_1} \cdots Y_{T_l}^{a_l}$$

$$= q^{-m + \sum_{i \in [1, l]} a_i ((\dim_{T_i} \dim T_i)/2 + (\dim T_i, \rho)) - \sum_{i < j} a_i a_j \lambda_{ij}} Y_{T_1}^{a_1} \cdots Y_{T_l}^{a_l}.$$
Here we write $\Lambda_T = (\lambda_{ij})_{i,j \in [1, \ell]}$. Therefore $q^m Y_{T_1}^{a_1} \cdots Y_{T_\ell}^{a_\ell}$ is dual bar invariant if and only if

$$m - \sum_{i \in [1, \ell]} a_i^2 (\dim T_i, \dim T_i) / 2 - \sum_{i < j} a_i a_j (\dim T_i, \dim T_j) + \sum_{i \in [1, \ell]} a_i (\dim T_i, \rho)$$

$$= -m + \sum_{i \in [1, \ell]} a_i (- (\dim T_i, \dim T_i) / 2 + (\dim T_i, \rho)) - \sum_{i < j} a_i a_j \lambda_{ij}.$$  

By Remark 7.29 this is equivalent to

$$2m = \sum_{i \in [1, \ell]} a_i (a_i - 1) (\dim T_i, \dim T_i) / 2 + 2 \sum_{i < j} a_i a_j \dim \hom_{\mathbb{C}}(T_j, T_i).$$

The right-hand side is an element of $2\mathbb{Z}$. Therefore we can take an integer $m \in \mathbb{Z}$ uniquely which satisfies this equality.

**Remark 7.29.** For $(a_1, \ldots, a_\ell) \in \mathbb{Z}_{\geq 0}^\ell$, the unique dual bar invariant element in $\{q^m Y_{T_1}^{a_1} \cdots Y_{T_\ell}^{a_\ell} | m \in \mathbb{Z}\}$ is $Y_{T_1}^{a_1} \cdots T_{T_\ell}^{a_\ell}$.

**Lemma 7.30.** With the notation in Theorem 7.25 $q^{\sum_{i \in I_0} \lambda_i \dim \epsilon_i, R} Y_{I(R)}^{-1} Y_{\Omega_w^{-1}}(R)$ is dual bar invariant.

**Proof.** By Proposition 7.19

$$\kappa^{-1}(Y_{I(R)}) = D_{\omega, \lambda, \lambda},$$

here $\lambda := \sum_{i \in I_0} \lambda_i \omega_i$. Hence, by Proposition 7.12 we have

$$\kappa^{-1}(Y_{I(R)} Y_{\Omega_w^{-1}(R)}) = D_{\omega, \lambda, \lambda} \kappa^{-1}(Y_{\Omega_w^{-1}(R)})$$

$$= q^{(\lambda + w \lambda, \dim \Omega_w^{-1}(R))} \kappa^{-1}(Y_{\Omega_w^{-1}(R)}) D_{\omega, \lambda, \lambda}$$

$$= q^{(\lambda + w \lambda, \dim \Omega_w^{-1}(R))} \kappa^{-1}(Y_{\Omega_w^{-1}(R)}) Y_{I(R)}.$$  

By the way, $\dim \Omega_w^{-1}(R) = \dim I(R) - \dim R = w \lambda - \lambda - \dim R$. Hence $(\lambda + w \lambda, \dim \Omega_w^{-1}(R)) = - (\lambda + w \lambda, \dim R)$. Therefore

$$Y_{I(R)}^{-1} Y_{\Omega_w^{-1}(R)} = q^{(\lambda + w \lambda, \dim R)} Y_{\Omega_w^{-1}(R)} Y_{I(R)}^{-1}$$

Note that $\sum_{i \in I_0} \lambda_i \dim \epsilon_i, R = - (\lambda, \dim R)$. We have

$$q^{(\dim \Omega_w^{-1}(R) - \dim I(R)) (\dim \Omega_w^{-1}(R) - \dim I(R)) / 2 - (\dim \Omega_w^{-1}(R) - \dim I(R), \rho)}$$

$$= q^{(\dim R, \dim R) / 2 + (\dim R, \rho)} q^{(w \lambda, \dim R)} Y_{\Omega_w^{-1}(R)} Y_{I(R)}^{-1}$$

$$= q^{(\dim R, \dim R) / 2 + (\dim R, \rho) - (w \lambda, \dim R)} q^{(\dim I(R), \dim I(R)) / 2 + (\dim I(R), \dim I(R)) / 2 - (w \lambda, \dim R)} Y_{I(R)}^{-1} Y_{\Omega_w^{-1}(R)}$$

$$= q^{(\dim I(R) - w \lambda, \dim R)} Y_{I(R)}^{-1} Y_{\Omega_w^{-1}(R)}$$

This competes the proof.

**Lemma 7.31.** Let $T$ be a basic reachable $C_w$-maximal rigid module and $T = T_1 \oplus \cdots \oplus T_\ell$ its indecomposable decomposition. Then the equality (7.3) with $R = T_k$ holds for all $k = 1, \ldots, \ell$ if and only if the one with $R = T_1^{a_1} \oplus \cdots \oplus T_\ell^{a_\ell}$ holds for all $(a_1, \ldots, a_\ell) \in \mathbb{Z}_{\geq 0}^\ell$. 


Proof. The latter obviously implies the former. Suppose that the equality (7.3) holds for $R = T_k$, $k = 1, \ldots, \ell$. Write

$$\eta_{w,q}(Y_{T_k}) = q^{m_k}Y_{T_k}^{-1}Y_{\Omega_w^{-1}(T_k)}$$

for $k = 1, \ldots, \ell$. Set $R = T_k^{\oplus a_1} \oplus \cdots \oplus T_k^{\oplus a_\ell}$ for $(a_1, \ldots, a_\ell) \in \mathbb{Z}_0^\ell$. Note that $I(R) = I(T_k)^{\oplus a_1} \oplus \cdots \oplus I(T_\ell)^{\oplus a_\ell}$ and $\Omega_w^{-1}(R) = \omega_1^{-1}(T_1)^{\oplus a_1} \oplus \cdots \oplus \Omega_w^{-1}(T_\ell)^{\oplus a_\ell}$. (Actually $I(T_{\ell-n+i}) = T_{\ell-n+i}$ and $\Omega_w^{-1}(T_{\ell-n+i}) = 0$ for $i \in I_w$.) There exist unique $A_1, A_2, A_3 \in \mathbb{Z}$ such that the following hold:

$$\eta_{w,q}(Y_R) = q^{A_1} \eta_{w,q}(Y_{T_1}^{\oplus a_1} \cdots Y_{T_\ell}^{\oplus a_\ell})$$
$$= q^{A_1}(q^{m_1}Y_{I(T_1)}^{-1}Y_{\omega_1^{-1}(T_1)})^{a_1} \cdots (q^{m_\ell}Y_{I(T_\ell)}^{-1}Y_{\omega_\ell^{-1}(T_\ell)})^{a_\ell}$$
$$= q^{A_2}Y_{I(T_1)}^{-1}Y_{\omega_1^{-1}(T_1)}^{a_1} \cdots Y_{I(T_\ell)}^{-1}Y_{\omega_\ell^{-1}(T_\ell)}^{a_\ell}$$
$$= q^{A_3}Y_{I(R)}^{-1}Y_{\omega_1^{-1}(R)}^{a_1} \cdots Y_{I(R)}^{-1}Y_{\omega_\ell^{-1}(R)}^{a_\ell}.$$  

Moreover $\eta_{w,q}(Y_R)$ is dual bar invariant because of the dual bar invariance of $Y_R$ and Theorem 6.1. Hence, by Lemma 7.28 and Lemma 7.30 the equality (7.3) also holds for $R$.  

Proof of Theorem 7.25. Recall that we always assume that $T_{\ell-n+i}$ is a $C_w$-projective-injective module with socle $S_i$ for all $i \in I_w = [1, n]$, in particular, the isomorphism class of $T_{\ell-n+i}$ does not depend on the choice of $T$. From now on, we identify $\widetilde{I}(\mu_q)(C_w)$ with $A_q[N_w]$ via $\tilde{\kappa}$. First we consider the case that $R = A_q(N_w)$, where $\Omega_w^{-1}(A_q(N_w)) = 0$. Then the desired equality in this case since $I(T_{\ell-n+i}) = T_{\ell-n+i}$ and $\Omega_w^{-1}(T_{\ell-n+i}) = 0$.

Let $i \in I(w)$. Henceforth, we will prove the theorem by induction on the minimal length of sequences of mutations which we need to obtain $T$ from $V_i$. We begin with the case that $R = V_{i,k}$ for some $k \in [1, \ell]$ with $k^+ \neq \ell + 1$. Then $I(V_{i,k}) = V_{i,k}^{\text{max}}$ and $\Omega_w^{-1}(V_{i,k}) = M_i[k^{\text{max}}, k^+]$. Therefore we have

$$\eta_{w,q}(Y_{V_{i,k}}) = \eta_{w,q}(D_{s_i^1 \cdots s_i^k} \omega_{i,k} \omega_{i,k})$$
$$= q^{-\langle \omega_{i,k} - \omega_{i,k}\rangle}[D_{s_i^1 \cdots s_i^k} \omega_{i,k} \omega_{i,k}^{-1}][D_{u_{s_i^1 \cdots s_i^k} \omega_{i,k}^{-1} \omega_{i,k} \omega_{i,k}^{-1}}]$$
$$= q^{-\langle \omega_{i,k} \dim V_{i,k}\rangle}Y_{V_{i,k}^{\text{max}}}^{-1}Y_{M_i[k^{\text{max}}, k^+]^{-1}}$$
$$= q^{\dim C\epsilon_i Y_{V_{i,k}}^{-1}Y_{\Omega_w^{-1}(V_{i,k})}^{-1}}.$$  

Hence, by Lemma 7.31 the equality (7.3) holds when $R = V_i$. Next, suppose that the equality (7.3) holds for $R = T_1^{\oplus a_1} \oplus \cdots \oplus T_\ell^{\oplus a_\ell}$, where $T = T_1 \oplus \cdots \oplus T_\ell$ is a basic reachable $C_w$-maximal rigid module. Let $k \in [1, \ell - n]$ and write $\mu_{T_k}(T) = (T/T_k) \oplus T_k$, $I(T_k) = \bigoplus_{i \in I_w} T_{\ell-n+i}^{\oplus \lambda_i}$. Then, by Lemma 7.31 it remains to prove the following equality:

$$\eta_{w,q}(Y_{T_k}) = q^{\sum_{i \in I_w} \lambda_i \dim C\epsilon_i T_k Y_{I(T_k)}^{-1}Y_{\Omega_w^{-1}(T_k)}^{-1}}.$$
Write $\tilde{B}_T = (b_{ij})_{i \in [1, \ell], j \in [1, \ell-n]}$. Set $T_+ := \bigoplus_{j; b_{jk} > 0} T_j^{b_{jk}}$ and $T_- := \bigoplus_{j; b_{jk} < 0} T_j^{b_{jk}}$. By (7.1) and Proposition (7.10) (2) and Proposition (7.24), we have

$$\eta_w^*([\varphi_{T_+}]) = \eta_w^*([\varphi_{T_-}]) = \frac{[\varphi_{\Omega_w^{-1}(T_+)}]}{[\varphi_{I(T_+)}]} + \frac{[\varphi_{\Omega_w^{-1}(T_-)}]}{[\varphi_{I(T_-)}]}.$$

and

$$\eta_w^*([\varphi_{T_+}]) = \frac{[\varphi_{\Omega_w^{-1}(T_k)}]}{[\varphi_{I(T_k)}]} \cdot \frac{[\varphi_{\Omega_w^{-1}(T'_k)}]}{[\varphi_{I(T'_k)}]} = \frac{[\varphi_{\Omega_w^{-1}(T_k)} \varphi_{\Omega_w^{-1}(T'_k)}]}{[\varphi_{I(T_k \oplus T'_k)}]}.$$

Therefore,

$$(7.5) \quad [\varphi_{\Omega_w^{-1}(T_k)} \varphi_{\Omega_w^{-1}(T'_k)}] = [\varphi_{I(T_k \oplus T'_k)}] \left( \frac{[\varphi_{\Omega_w^{-1}(T_k)}]}{[\varphi_{I(T_k)}]} + \frac{[\varphi_{\Omega_w^{-1}(T'_k)}]}{[\varphi_{I(T'_k)}]} \right).$$

By Proposition (7.15) $T' := \Omega_w^{-1}(T) \oplus \bigoplus_{i \in I_w} T_{\ell-n+i}$ is a basic reachable $C_w$-maximal rigid module; hence there exists a bijection $[1, \ell-n] \to [1, \ell-n], j \mapsto j^*$ such that $T_{j^*} = \Omega_w^{-1}(T_j)$. Moreover we have

$$\mu_{T_{j^*}} (T') = \left( T'/T_{j^*} \right) \oplus \Omega_w^{-1}(T_{j^*}).$$

Write $\tilde{B}_{T'} = (b'_{ij})_{i \in [1, \ell], j \in [1, \ell-n]}$ and $(T_{j^*})^\ast := \Omega_w^{-1}(T_{j^*})$. Set $T'_+ := \bigoplus_{j; b'_{j^*k^*} > 0} (T_{j^*})^\ast \oplus b'_{j^*k^*}$ and $T'_- := \bigoplus_{j; b'_{j^*k^*} < 0} (T_{j^*})^\ast \oplus b'_{j^*k^*}$. Then, by (7.4) and (7.3), we have

$$(7.6) \quad \left[ \varphi_{I(T_k \oplus T'_k)} \right] \left( \frac{[\varphi_{\Omega_w^{-1}(T_k)}]}{[\varphi_{I(T_k)}]} + \frac{[\varphi_{\Omega_w^{-1}(T'_k)}]}{[\varphi_{I(T'_k)}]} \right) = [\varphi_{T'_k}] + [\varphi_{T'_k}].$$

Note that all $\Pi$-modules appearing in the equality (7.6) are direct summands of a direct sum of copies of $T'$. Therefore, by Proposition (7.17) (2), we have $I(T_k \oplus T'_k) = I(T+) \oplus I_- = I(T_-) \oplus I_+$ for some $C_w$-projective-injective modules $I_+, I_-$, and

$$(7.7) \quad \{ I_+ \oplus \Omega_w^{-1}(T_+), I_- \oplus \Omega_w^{-1}(T_-) \} = \{ T'_+, T'_- \}.$$

By the way, we recall our assumption that the equality (7.3) holds for $R = T_{j^*} \oplus T_{j^*} \oplus \cdots \oplus T_{j^*}$. By Proposition (7.17) and our assumption, there exist unique $A_1, A_1', A_2, A_2', A_3 \in \mathbb{Z}$ such that

$$\eta_{w,q}(Y_{T_k} Y_{T'_k}) = \eta_{w,q}(q^{A_1} Y_{T_+} + q^{A_2} Y_{T_-}) = q^{A_1} Y_{I(T_+)}^{-1} \Omega_w^{-1}(T_+) + q^{A_2} Y_{I(T_-)}^{-1} \Omega_w^{-1}(T_-);$$

and

$$\eta_{w,q}(Y_{T_k} Y_{T'_k}) = q^{A_1} Y_{I(T_k)}^{-1} Y_{T'_k} \eta_{w,q}(Y_{T'_k}).$$

These equalities together with (7.7) imply that there exist unique $A', A_1', A_2' \in \mathbb{Z}$ such that

$$\eta_{w,q}(Y_{T'_k}) = q^{-A'} Y_{T'_k}^{-1} Y_{I(T_k)} (q^{A_1'} Y_{I(T_+)}^{-1} \Omega_w^{-1}(T_+) + q^{A_2'} Y_{I(T_-)}^{-1} \Omega_w^{-1}(T_-))$$

$$= q^{A'} Y_{T'_k}^{-1} Y_{I(T_k \oplus T'_k)} (q^{A_1'} Y_{I(T_+)}^{-1} \Omega_w^{-1}(T_+) + q^{A_2'} Y_{I(T_-)}^{-1} \Omega_w^{-1}(T_-))$$

$$= Y_{I(T'_k)}^{-1} Y_{T'_k}^{-1} \left( q^{A_1'} Y_{T_+} + q^{A_2'} Y_{T_-} \right).$$

Note that all rescaled quantum cluster monomials appearing in the rightmost hand side of the equality above are monomials of the based quantum torus $T_{A,T'}$. Moreover, $\eta_{w,q}(Y_{T'_k})$
is dual bar invariant because, by Theorem 6.1, the quantum twist automorphism \( \eta_{w,q} \) preserves dual bar invariance property of elements of \( \mathcal{H}_q(C_w) \) (recall Definition 7.2.22). Hence \( q^{A^w_q} Y_{I(T^+_k)}^{-1} Y_{I(T^-_k)}^{-1} T^+_k \) and \( q^{A''_q} Y_{I(T^+_k)}^{-1} Y_{I(T^-_k)}^{-1} T^-_k \) are dual bar invariant elements of \( \mathcal{T}_{A,T} \). By Lemma 7.28, \( A^w_q \) and \( A''_q \) are uniquely determined by this property. On the other hand, by Proposition 7.17, \( q^{\sum_{\varepsilon \in w} \lambda_i \dim C \varepsilon_i T_k^+ Y_{I(T_k^+)}^{-1} \left( Y_{I(T_k^-)}^{-1} Y_{I(T_k^-)}^{-1} \right)^*} \) is of the following form as an element of \( \mathcal{T}_{q,\pm_T} \):

\[
Y_{I(T_k^+)}^{-1} Y_{I(T_k^-)}^{-1} \left( q^{M_1} Y_{I(T_k^+)}^{-1} + q^{M_2} Y_{I(T_k^-)} \right), \quad M_1, M_2 \in \mathbb{Z}.
\]

Moreover, by Lemma 7.30, \( q^{\sum_{\varepsilon \in w} \lambda_i \dim C \varepsilon_i T_k^+ Y_{I(T_k^+)}^{-1} \left( Y_{I(T_k^-)}^{-1} Y_{I(T_k^-)}^{-1} \right)^*} \) is dual bar invariant. Hence, by the argument above, \( M_1 = A^w_1 \) and \( M_2 = A''_q \). Therefore we obtain

\[
\eta_{w,q}(Y_k^+ \left( Y_{I(T_k^+)}^{-1} Y_{I(T_k^-)}^{-1} \right)^*) = q^{\sum_{\varepsilon \in w} \lambda_i \dim C \varepsilon_i T_k^+ Y_{I(T_k^+)}^{-1} \left( Y_{I(T_k^-)}^{-1} Y_{I(T_k^-)}^{-1} \right)^*},
\]

which completes the proof.

\[\square\]

8. Finite type cases : 6-periodicity

Since the map \( \eta_{q,w} \) is an automorphism, we can apply it repeatedly. In this section, we show the “6-periodicity” of specific quantum twist automorphisms. Assume that \( g \) is a finite dimensional Lie algebra, and let \( w_0 \) be the longest element of \( W \).

**Theorem 8.1.** For a homogeneous element \( x \in A_{q}[N_{w_0}^w] \), we have

\[
\eta_{w_0,q}^6(x) = q^{(w^{x+}w_0 w^{x} w^{x})} D_{w,\pm x}^{-w_0} w_0 x.
\]

**Remark 8.2.** When the action of \( w_0 \) on \( P \) is given by \( \mu \mapsto -\mu \), the theorem above states that \( \eta_{w_0,q}^6 = \text{id} \). Hence \( \eta_{w_0,q}^6 \) is “really” periodic. If \( g \) is simple, then this condition is satisfied in the case that \( g \) is of type \( B_n, C_n, D_2n, F_4, G_2 \). See [Hum90, Section 3.7].

When \( g \) is symmetric, such periodicity is also explained by Geiß-Leclerc-Schröer’s additive categorification of twist automorphisms (see Section 7). The periodicity corresponds to the well-known 6-periodicity of syzygy functors [AR96, ES98], that is, the property that \((\Omega_{w_0}^{-1})^6(M) \simeq M\) for an indecomposable non-projective-injective module \( M \) of \( \Pi \) in the notation of Section 7.

We can consider the similar periodicity problems for every \( w \in W \). It would be interesting to find the necessary and sufficient condition on \( w \in W \) for periodicity. Since quantum twist automorphisms restrict to permutations on dual canonical bases, the periodicity of a quantum twist automorphism \( \eta_{w,q} \) is equivalent to the periodicity of a (non-quantum) twist automorphism \( \eta_w \). See also Remark 8.3 below.

**Lemma 8.3.** Let \( \lambda \in P_+ \). Take \( u, u' \in V(\lambda) \) such that \( D_{u,u'} = G^{\text{up}}(\tilde{b}) \) for some \( \tilde{b} \in \mathcal{B}(\infty) \). Then, for \( i \in I \),

\[
\varepsilon_i(\tilde{b}) = \max\{ k \in \mathbb{Z}_{\geq 0} \mid D_{\varepsilon_i^t u, u'} \neq 0 \}, \quad \varepsilon^*_i(\tilde{b}) = \max\{ k \in \mathbb{Z}_{\geq 0} \mid D_{u, f_i^t u'} \neq 0 \}.
\]

In particular,

\[
\varepsilon_i(\overline{\lambda}(b)) = \varepsilon_i(b), \quad \varepsilon_i(\overline{\lambda}(b)) = \varphi_i(b) = \varepsilon_i(b) + (h_i, w t b).
\]
Proof. By Proposition 3.22
\[ (8.1) \quad \varepsilon_i(b) = \max\{ k \in \mathbb{Z}_{\geq 0} \mid (\varepsilon_i^k)(D_{u,u'}) \neq 0 \} \]
For \( k \in \mathbb{Z}_{\geq 0} \) and \( x \in U_q^- \), we have
\[
((\varepsilon_i^k)(D_{u,u'}), x)_L = (1 - q_i^2)^k(D_{u,u'}, f_i x)_L \\
= (1 - q_i^2)^k(u, (f_i^k x, u'))^\varepsilon_L \\
= (1 - q_i^2)^k((\varepsilon_i^k u, x, u'))^\varepsilon_L = (1 - q_i^2)^k(D_{e_i^k u, u'}, x)_L. 
\]
Hence \((\varepsilon_i^k)(D_{u,u'}) = (1 - q_i^2)^kD_{e_i^k u, u'}\). Combining this equality with (8.1), we obtain the first equality. The second equality is proved in the same manner. The last two equalities are deduced from Proposition 3.22 and 3.46. \( \square \)

Proof of Theorem 8.1 It is easily seen that we need only check the case that \( x \in U_q^- \). For \( i \in I \), we have \( D_{s_i \omega_i, \omega_i} = (1 - q_i^2) f_i \). We first consider the images of \( D_{s_i \omega_i, \omega_i} \), \( i \in I \) under iterated application of \( \eta_{\omega_0, q} \). If \( I = \{ i \} \), that is, \( \mathfrak{g} = \mathfrak{sl}_2 \), the quantum unipotent cell \( A_q[N^{\omega_0}] \) is generated by \( D_{\pm \omega_i, \omega_i} = D_{\pm \omega_i, \omega_i} \). In this case, \( \eta_{\omega_0, q}^2(D_{s_i \omega_i, \omega_i}) = D_{s_i \omega_i, \omega_i} \). Hence \( \eta_{\omega_0, q}^2 = \text{id} \), in particular, the theorem holds. Henceforth, we consider the case that \( \mathfrak{g} \) does not have ideals of Lie algebras which are isomorphic to \( \mathfrak{sl}_2 \). We have
\[
\eta_{\omega_0, q}(D_{s_i \omega_i, \omega_i}) \simeq D_{u_0 \omega_i, \omega_i}^{a_i} D_{u_0 \omega_i, s_i \omega_i}. 
\]
Here \( \simeq \) stands for the coincidence up to some powers of \( q \). Now, by Proposition 3.46
\[
D_{u_0 \omega_i, s_i \omega_i} = G_{u_0 \omega_i}^{a_i}(s_{u_0 \omega_i}(u_i \omega_i)). 
\]
By Lemma 8.3
\[
\varepsilon_j^*(s_{u_0 \omega_i}(u_i \omega_i)) = \varepsilon_j(s_{u_0 \omega_i}(u_i \omega_i)) = \varphi_j(u_i \omega_i) = \begin{cases} -a_ji & \text{if } j \neq i, \\ 0 & \text{if } j = i. \end{cases} 
\]
Therefore \( \sum_j \varepsilon_j^*(s_{u_0 \omega_i}(u_i \omega_i)) = \omega_i + s_i \omega_i \). Recall Remark 6.4. Then there exists \( b_1 \in B(\lambda_1) \) such that \( D_{u_0 \omega_i, s_i \omega_i} = D_{G_{u_0 \omega_i}^{a_i}(u_i \omega_i)} \), that is, \( \lambda_1(b_1) = s_{u_0 \omega_i}(u_i \omega_i) \). Then
\[
\eta_{\omega_0, q}^2(D_{s_i \omega_i, \omega_i}) \simeq D_{u_0 \omega_i, \omega_i} D_{u_0 \omega_i, s_i \omega_i} D_{u_0 \omega_i, \omega_i, \lambda_1} D_{u_0 \omega_i}. 
\]
As above, \( D_{u_0 \omega_i, \lambda_1}^{a_i b_1} = G_{u_0 \omega_i}^{a_i b_1}(b_1) \), and by Lemma 8.3
\[
\varepsilon_j^*(s_{u_0 \omega_i}^{a_i b_1}(b_1)) = \varepsilon_j(s_{u_0 \omega_i}^{a_i b_1}(b_1)) = \varepsilon_j(b_1) + \langle h_j, wt b_1 \rangle \\
= \varepsilon_j(\lambda_1(b_1)) + \langle h_j, u_0 \omega_i - s_i \omega_i + \lambda_1 \rangle \\
= \varepsilon_j(s_{u_0 \omega_i}^{a_i b_1}(u_i \omega_i)) + \langle h_j, u_0 \omega_i + \omega_i \rangle. 
\]
By Proposition 3.46 and Lemma 8.3
\[
\varepsilon_j(s_{u_0 \omega_i}^{a_i b_1}(u_i \omega_i)) = \max\{ k \in \mathbb{Z}_{\geq 0} \mid D_{e_i^k u_0 \omega_i, u_i \omega_i} \neq 0 \}. 
\]
By the way, there is an involution \( \theta \) on \( I \) defined by \( \omega_0 \omega_i = -\omega_i \omega_0 \). Then \( \omega_0 \omega_i = -\omega_i \omega_0 \) and \( s_\theta(i) u_0 \omega_i = u_0 s_i \omega_i \). When \( \mathfrak{g} \) does not have ideals of Lie algebras which are isomorphic to \( \mathfrak{sl}_2 \), we have \( D_{u_0 \omega_i, s_i \omega_i, \omega_i} \neq 0 \). Therefore \( \varepsilon_j(s_{u_0 \omega_i}^{a_i b_1}(u_i \omega_i)) = \delta_{j, \theta(i)}. \) Hence
\[
\varepsilon_j(s_{u_0 \omega_i}^{a_i b_1}(b_1)) = \delta_{j, \theta(i)} - \delta_{j, \theta(i)} + \delta_{ij} = \delta_{ij}. 
\]
Therefore \( \sum_{j \in I} \varepsilon_j^3(\mathfrak{w}_0^\lambda_1(b_1))\mathfrak{w}_j = \mathfrak{w}_i \). Then there exists \( b_2 \in \mathfrak{B}(\mathfrak{w}_i) \) such that \( D_{\mathfrak{w}_0\lambda_1, G^\uparrow_\mathfrak{w}_i}(b_2) = D_{G^\uparrow_\mathfrak{w}_i}(b_2, u_{\mathfrak{w}_i}) \). Then
\[
\eta^3_{\mathfrak{w}_0,q}(D_{\mathfrak{w}_i, \mathfrak{w}_i}) \simeq D_{\mathfrak{w}_0, \lambda_1}^{-1} D_{\mathfrak{w}_0, \lambda_1} D_{\mathfrak{w}_0, \lambda_1}^{-1} D_{u_{\mathfrak{w}_0, \mathfrak{w}_i}, G^\uparrow_\mathfrak{w}_i}(b_2) \simeq D_{u_{\mathfrak{w}_0, \mathfrak{w}_i}, G^\uparrow_\mathfrak{w}_i}(b_2).
\]

Here,
\[
\text{wt } D_{u_{\mathfrak{w}_0, \mathfrak{w}_i}, G^\uparrow_\mathfrak{w}_i}(b_2) = u_{0} \mathfrak{w}_i - \text{wt } b_2 = u_{0} \mathfrak{w}_i - (u_{0} \lambda_1 - \text{wt } b_1 + \mathfrak{w}_i) = u_{0} \mathfrak{w}_i - (u_{0} \lambda_1 - (u_{0} \mathfrak{w}_i - s_i \mathfrak{w}_i + \lambda_1) + \mathfrak{w}_i) = -\alpha_{\theta(i)}.
\]

Hence \( D_{u_{\mathfrak{w}_0, \mathfrak{w}_i}, G^\uparrow_\mathfrak{w}_i}(b_2) = D_{s_{\theta(i)} \mathfrak{w}_0, \theta(i), \mathfrak{w}_0, \theta(i)} \) because both sides are unique elements of the dual canonical basis of weight \(-\alpha_{\theta(i)}\). Therefore,
\[
\eta^6_{\mathfrak{w}_0,q}(D_{s_i, \mathfrak{w}_i}) \simeq D_{u_{\mathfrak{w}_0, \mathfrak{w}_i}, \mathfrak{w}_i}.
\]

Moreover, by Theorem 8.1 \( \eta^6_{\mathfrak{w}_0, q}(D_{s_i, \mathfrak{w}_i}) \) is an element of dual canonical basis, in particular, dual bar-invariant. Therefore,
\[
\eta^6_{\mathfrak{w}_0, q}(D_{s_i, \mathfrak{w}_i}) = q^{(\alpha_1 - \alpha_{\theta(i)}, \alpha_1)} D_{u_{\mathfrak{w}_0, \mathfrak{w}_i}, \theta(i)} D_{s_i, \mathfrak{w}_i}.
\]

By this result and Proposition 4.2 4.7 for \( i_1, \ldots, i_\ell \in I \), we have
\[
\eta^6_{\mathfrak{w}_0, q}(D_{s_{i_1} \mathfrak{w}_i, i_1} \cdots D_{s_{i_\ell} \mathfrak{w}_i, i_\ell}) = q^{\sum_{k=1}^\ell (\alpha_{i_k} - \alpha_{\theta(i_k)}, \alpha_{i_k})} D_{u_{\mathfrak{w}_0, \mathfrak{w}_i}, \theta(i_k)} D_{s_{i_1} \mathfrak{w}_i, i_1} \cdots D_{u_{\mathfrak{w}_0, \mathfrak{w}_i}, \theta(i_\ell)} D_{s_{i_\ell} \mathfrak{w}_i, i_\ell}
\]

Hence the desired equality in the theorem holds for all \( x \in A_{q, N_-} \) since the elements
\[
D_{s_i, \mathfrak{w}_i} = (1 - q_i^2)f_i, i \in I \text{ generate the quantum unipotent subgroup } A_{q, N_-}.
\]

Then we can easily extend this result to that for \( A_{q, N \uparrow} \) by straightforward calculation. The explicit calculation is left to the reader. \( \square \)

Remark 8.4. In the essential part of the proof of Theorem 8.1 we check the periodicity on generators of \( A_{q, N \uparrow} \). We should note that this set of generators is not the set of generators of \( C[N \uparrow] \) after specialization unless \( \mathfrak{g} = \mathfrak{sl}_2 \). Indeed, in general Kac-Moody cases, the quantum unipotent cell \( A_{q, N \uparrow} \) is generated by
\[
\{ [f_i] \mid i \in I \} \cup \{ [D_{w, \rho}]^{-1} \} \text{ (recall that } \rho := \sum_{i \in I} \mathfrak{w}_i \text{). In particular,}
\]
\[
(\text{the number of the generators of } A_{q, N \uparrow}) \leq \#I + 1.
\]

On the other hand,
\[
(\text{the number of the generators of } C[N \uparrow]) \geq \dim N \uparrow = \ell(w).
\]

Therefore the periodicity might be checked in the quantum setting more easily than in the classical setting by this decrease of numbers of generators.

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