Mass-dependence of pseudocritical temperature in mean field approximation

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We implement the Dyson-Schwinger equations approach to study the mass dependence of pseudocritical temperature of QCD phase transition at zero chemical potential. We restrict our computation in the mean field approximation which could lead to a clear critical behavior. We analyze the scaling behavior with different shape of interaction kernel by considering different dressed-gluon models. The critical exponent we obtained is consistent with that in the 3D O(4) universality class. The size of critical region is up to $m_0 \lesssim 2 \sim 4$ MeV in this mean field approximation which sets naturally an upper bound of the critical region since the fluctuations beyond mean-field usually diminish the critical region. Besides, we analyze the possible percentage of the maximum chiral susceptibility and pion mass range at which the chiral phase transition temperature is independent of the current quark mass. The results show that the percentage and the pion mass range depend on the details of interaction kernel, which differs in gluon models.

I. INTRODUCTION

The basic degrees of freedom in Quantum Chromodynamics (QCD) – quarks and gluons – do not exist as asymptotic states, i.e., these partonic excitations do not propagate with integrity over the length-scales that exceed some modest fraction of the proton’s radius. The forces responsible for this phenomenon also appear to generate more than 98% of the mass of visible matter. These features are known as confinement and dynamical chiral symmetry breaking (DCSB). Theoretically, these features could be described via the appearance of momentum dependent mass-functions for quarks and gluons even in the absence of any Higgs-like mechanism [1–50]. While including the medium effect at finite temperature and chemical potential, these features will lead the QCD matter to have a rich phase structure as the thermal fluctuation dramatically affect the momentum dependence of mass kernel. Considering the chiral symmetry property, as the temperature and chemical potential increases, owing to the asymptotic freedom of QCD [51, 52], QCD matter transits from a DCSB phase to a chiral symmetry preserving (CS) phase. Such features cover a vast array of empirical aspects, from gluon and quark interactions at the highest energies achievable with the heavy ion collision, to the nature of nuclear matter in the inner part of a compact star, which then deliver a sketch of QCD phases at finite temperature and chemical potential [1, 28, 53–61].

It has been shown that if one sets the up and down quarks in chiral limit, the crossover behavior of the matter at zero chemical potential steepens into a second order chiral phase transition which belongs to the 3-dimensional (3D) O(4) universality class, if the $U_A(1)$ anomaly still exists in the phase transition region [51, 62–66]. If $U_A(1)$ symmetry has been restored sufficiently, it is expected to be a larger 3D universality class. At physical mass, the critical end point (CEP) is also expected to be 3D Ising class [64, 66]. Based on these critical behavior analysis, the phase structure at vanishing density could be helpful for determining the CEP at the quark mass, for instance, the phase transition temperature at chiral limit could be considered as the upper bound for the temperature of CEP.

However, the reconstruction with a certain universality class is subtle owing to the uncertainty of the size of the critical region. Generally, the fluctuations drive deviation of the scaling behavior, and a detailed analysis reveals that the actual size of the scaling behavior could be only up to $m_\pi \sim 1$ MeV [67].

Since the Dyson-Schwinger equations (DSEs) approach has been shown to be a sophisticated QCD approach [1, 68, 70], we study the scaling behavior of chiral susceptibility in the DSEs approach in this paper. We will focus on the computation in rainbow approximation which is generally working at mean-field approximation level [71]. Under the mean field approximation, it would have clearer signal of critical behavior and broader critical region. Besides, we take three different dressed-gluon models to explore the effect of the interaction feature on the critical behavior.

The remainders of this article are organized as follows. In Sec. II, we reiterate briefly the DSEs approach at finite temperature including the models and the chiral phase transition criterion. Sec. III represents our results of the chiral susceptibility and the pseudocritical temperature,
as well as their current mass dependence. Finally, we summarize in Sec. IV.

II. QUARK GAP EQUATION AT FINITE TEMPERATURE

The quark propagator at finite temperature can be determined by the gap equation

\[ S(\vec{p}, \Omega_n) = \frac{Z_2 i \gamma \cdot \vec{p} + Z_2 C i \gamma \cdot \Omega_n + Z_4 m_0 + Z_1 \Sigma(\vec{p}, \Omega_n)}{\Omega_n}, \]

(1)

\[ \Sigma(\vec{p}, \Omega_n) = T \sum_{l=-\infty}^{\infty} \int \frac{d^3q}{(2\pi)^3} \frac{g^2 D_{\mu\nu}(\vec{p} - \vec{q}, \Omega_n; T, \mu)}{2 \gamma^\mu S(\vec{q}, \Omega_l) \gamma^\nu} \frac{\lambda^a}{2} \Gamma_\nu(\vec{q}, \Omega_l, \vec{p}, \Omega_n), \]

(2)

where \(Z_1, Z_2, Z_2C, Z_4\) are the renormalization constants, \(m_0\) is the current quark mass, \(\Omega_n = (2n + 1)\pi T\) being the quark Matsubara frequency, \(\Omega_{nl} = \Omega_n - \omega_l; D_{\mu\nu}\) is the dressed-gluon propagator; and \(\Gamma_\nu\) is the dressed-quark-gluon interaction vertex.

According to the Lorentz structure analysis, the gap equation’s solution can be decomposed as

\[ S(\vec{p}, \Omega_n) = i \gamma \cdot \vec{p} A(p^2, \omega_n^2) + i\gamma \cdot \Omega_n C(p^2, \omega_n^2) + B(p^2, \omega_n^2). \]

(3)

In our calculation, we take the approximation \(\Gamma_\nu(\vec{q}, \Omega_l, \vec{p}, \Omega_n) = \gamma_\nu\) for the quark-gluon interaction vertex which is referred as the rainbow approximation. The rainbow approximation is the leading order approximation. It is essentially a mean-field approximation, which will then lead to a clear analysis of the universality class.

To solve the gap equation, we should also have the information of the dressed-gluon propagator. The dressed-gluon propagator is usually approximated as

\[ g^2 D_{\mu\nu}(\vec{k}, \Omega_{nl}) = G(k^2) D^{\text{free}}_{\mu\nu}(k), \]

(4)

where \(D^{\text{free}}_{\mu\nu}\) is the free gluon propagator

\[ D^{\text{free}}_{\mu\nu}(k) = \left( \delta_{\mu\nu} - \frac{k_\mu k_\nu}{k^2} \right) \frac{1}{k^2}, \]

(5)

and \(k_\mu = (\vec{k}, \Omega_{nl})\). The \(G(k^2)\) is the effective interaction which depends on models.

Recent studies of QCD’s gauge sector confirmed a massive gluon propagator on the domain at \(q^2 = 0\) [12, 17, 24, 38]. The behavior could be well described by the infrared constant model, known as the Qin-Chang (QC) model [38]. The QC gluon model is a QCD based model with the correct momentum dependence of QCD, which reads

\[ \frac{G(k^2)}{k^2} = 8\pi^2 D \frac{1}{\omega^4} e^{-k^2/\omega^2} \]

\[ + \frac{8\pi^2 m^2}{\ln[\tau + (1 + k^2/\Lambda^2_{\text{QCD}})^2]} F(k^2), \]

(6)

with \(F(k^2) = (1 - \exp(-k^2/4m_0^2))/k^2, \tau = e^2 - 1, m_\tau = 0.5 \text{ GeV}, m_{\gamma} = 12/25, \) and \(\Lambda_{\text{QCD}} = 0.234 \text{ GeV}.\)

There are two parameters, \(D \) and \(\omega\) in the QC model. In vacuum, we can take \((D\omega)^{1/3} = 0.82 \text{ GeV}\) and \(\omega = 0.5 \text{ GeV}\) together with light quark mass \(m_\pi = 6.6 \text{ MeV}\), which reproduces pion properties \(m_\pi = 0.14 \text{ GeV}\) and \(f_\pi = 0.092 \text{ GeV}\) [38].

Noticing that the function \(G(k^2)\) in above equation is served as a momentum distribution function, this model could cover various types of the interaction via varying the \(\omega\) which stands for the interaction width of QCD. Except for the perturbative correction, as \(\omega\) goes to 0, the distribution becomes a \(\delta\)-function, which is just the so called Munczek-Nemirovsky (MN) model [87].

For very large \(\omega\), it becomes approximately the contact model which is then a representative of NJL model in DSEs approach [38].

At finite temperature, the MN model could be generalized as

\[ g^2 D_{\mu\nu}(\vec{p}, \Omega_k) = \frac{\delta_{\mu\nu} - \frac{p_\mu p_\nu}{p^4 + \Omega_k^2}}{2\pi^3 T} \delta(\vec{p}), \]

(7)

where \(\Omega_k = 2kT\) is the boson Matsubara frequency and \((p_\mu) = (\vec{p}, \Omega_k)\). \(\eta\) is a mass-scale parameter. Ref. [87] shows that with parameters \(\eta = 1.37 \text{ GeV}\) and \(m = 30 \text{ MeV}\), one can get the pion mass \(m_\pi = 140 \text{MeV}\).

The contact model reads

\[ g^2 D_{\mu\nu} = \delta_{\mu\nu} \frac{1}{m_G^2}, \]

(8)

where \(m_G\) is an interaction strength parameter and it can be taken \(m_G = 0.132 \text{ GeV}\) in vacuum [38].

At finite temperature, however, the interaction strength should be modified. For the QC model, we multiply a damping factor to the coupling constant \(D\):

\[ D(T) = D \times \frac{1}{(1 + \alpha T/m_\pi^2)^2}. \]

(9)

And for the MN model and contact model, the same damping factor \(1/(1 + \alpha T/m_\pi^2)^2\) can be directly multiplied to the dressed-gluon propagator.

The dimensionless damping parameter \(\alpha\) is calibrated to reproduce the critical temperature \(T_c = 156.5 \text{ MeV}\) at physical pion mass [38]. One has then \(\alpha = 0.1731, 0.5274\) and 0.1477 for the QC model, MN model and contact model, respectively.

The chiral symmetry property manifests by the quark propagator straightforwardly. The order parameter of chiral symmetry is usually defined as the quark condensate which is an integral of the quark propagator. This definition needs an ultraviolet subtraction at finite current quark mass (see, e.g. Ref. [38] and the reference therein). Besides, noticing that the dominant contribution in the ultraviolet region is perturbative due to the asymptotic free behavior, and the DCSB effect is generated mainly by the infrared domain, the order parameter could also be interpreted as the running mass of the
The pseudocritical temperature is then just the temperature at which the chiral susceptibility becomes divergent which means it becomes the second order phase transition [8]. The critical temperature at chiral limit is $T_c = 143.6\text{MeV}$. Similar behaviors could also be found in the other two models, and we obtain $T_c = 142.9\text{MeV}$ for the contact model, and $T_c = 142.0\text{MeV}$ for the MN model in chiral limit.

In Fig. 2 we present the calculated result of the pseudocritical temperature as a function of current quark mass. The current quark mass corresponding to the physical pion mass $m_\pi = 140 \text{MeV}$. The different symbols with different colors correspond to the result calculated using different model, The dashed lines correspond to the fit of the calculated data using the formula $T = a \times m_0^{2/3} + T^{\text{chiral}}_c$.

In order to further study the behavior of the phase transition, we take a formula $T = a \times m_0^{2/3} + T^{\text{chiral}}_c$ to fit the data. We fix $c = T^{\text{chiral}}_c$ with $T^{\text{chiral}}_c$ the critical temperature at $m_0 = 0$ since there’s no ambiguity in our chiral limit results as the susceptibility becomes divergent. For the power index $b$, we could make use of the value of the critical exponent as $b = 1/\beta \delta = 2/3$ in our mean field approximation, which is consistent with the $3D$ $O(4)$ universality class [60, 71]. As can be seen from the figure, the calculated result can be fitted quite well with the mean field critical exponent. Such a result is also consistent with that given by analyzing the heat capability in the DSE approach of QCD [64].

Moreover, we compute the scaling behavior of chiral susceptibility with respect to the current quark mass in the QC model. The universality class analysis gives

$$\chi = \frac{\partial B(\omega_0, \vec{p} = 0)}{\partial m_0}. \quad (10)$$

The pseudocritical temperature is then just the temperature for the $\chi$ to reach its maximum. Such a definition of the chiral susceptibility is not exactly the same as that used in lattice QCD and functional renormalization group calculations, however the difference between the extracted pseudocritical temperature should be quite limited [89]. Therefore the exact value of chiral susceptibility in this paper is different from those in Refs. [51, 63], but the obtained pseudocritical temperatures are comparable (see next Section).

### III. RESULTS AND DISCUSSIONS

In this section, we present our numerical results calculated by the DSE with the three models mentioned above. In Fig. 1 we show the chiral susceptibility as a function of temperature for several values of the current quark mass $m_0$ in the QC model. It is evident that the peak gets narrower and narrower as the current quark mass becomes very small. At chiral limit, the chiral susceptibility becomes divergent which means it becomes the second order phase transition [8]. The critical temperature of temperature for several values of the current quark mass $m_0$ in the QC model.

![FIG. 1. (color online) Calculated chiral susceptibility as a function of temperature for several current quark masses.](image1)

![FIG. 2. (color online) Calculated critical temperature as a function of current quark mass rescaled by $m_{\text{phys}}$ which is the quark mass corresponding to the physical pion mass $m_\pi = 140 \text{MeV}$. The different symbols with different colors correspond to the result calculated using different model, The dashed lines correspond to the fit of the calculated data using the formula $T = a \times m_0^{2/3} + T^{\text{chiral}}_c$.](image2)
FIG. 3. (color online) Calculated chiral susceptibility as a function of current quark mass. The dashed lines represent the fitted results of the calculated data using the formula \( \chi = Am^{1-1/\delta} \) with \( \delta = 3 \).

In Fig. 3 we illustrate the calculated data and the fitted result. It is apparent that we can assign the parameter as \( \delta = 3 \), which is consistent with 3D O(4) universality class. In some details, we can extract that the critical region is up to \( m_q \sim 4 \text{ MeV} \) for the QC model, and for the contact and the MN model, the critical region is \( m_0 \lesssim 2 \text{ MeV} \). This means that the MN model has a smaller critical region than the contact model, since the physical current quark mass is larger in the MN model. After analyzing the scaling behavior of the chiral susceptibility with respect to temperature, we get another critical exponent as \( \beta = 1/2 \).

The scaling behavior could also be demonstrated in terms of the mass of pion since the mass of pion is an experimental quantity, therefore, it is better than the current quark mass which is dependent on the renormalization schemes. It has been well known that the Gell-Mann–Oakes–Renner (GOR) relation indicates a simple connection between the pion mass and the current quark mass near the chiral limit, which reads

\[ f_\pi^2m_\pi^2 = -2m(\zeta)(\bar{q}q), \]  

with \( f_\pi \) the pion decay constant, \( m(\zeta) \) is the current quark mass at renormalization scale \( \zeta \). In this paper, we can then take the leading order of this relation and have

\[ m_\pi = C\sqrt{m_0}, \text{ in the unit of MeV}, \]  

where \( C \) is a coefficient depending on the model. The value of \( C \) is determined by fitting the results from Bethe-Salpeter equation calculation, and is 54.49 for QC model \( [85] \), 52.76 for contact model \( [88] \), and 25.56 for MN model \( [87] \). It is clear that the pion mass obtained via the DSE with the RL truncation and the QC gluon model is quite the same with that given in the contact model, while differs from that in the MN model since the \( m_0 \) in the MN model is much larger than that in the other models.

The calculated result of the \( T_c \), as a function of \( m_\pi \), is shown in Fig. 4. For comparison we plot also the result obtained with the functional renormalization group approach of QCD (fQCD) \( [65] \) and that given in lattice QCD (HotQCD) \( [50, 65] \). One can notice from Fig. 4 that, in contrast to the linear dependence from fQCD and HotQCD result, our calculated \( T_c \) has a \( \sim 4/3 \) power dependence on the pion mass. Beyond the critical region, we can infer from the figure that the relation between \( T_c \) and \( m_\pi \) goes gradually to a simple linear relation. Recalling that the critical exponent of 3D O(4) class beyond mean-field is around 1 \( [51, 65] \), therefore, the full QCD computation could find a linear relation between \( T_c \) and \( m_\pi \) without the limitation of critical region.

In the lattice QCD simulation, it has been argued that the temperature at which the chiral susceptibility is 60%
of its maximum remains nearly the same for a large range of quark mass, so that the critical temperature for the chiral symmetry to be restored in chiral limit is obtained by extrapolation \cite{50}. In order to investigate such an extrapolation scheme, we present the calculated critical temperature with a certain percentage of the maximal susceptibility via the pion mass. The solid lines in Fig. 5 correspond to a percentage of 100\%, i.e., they correspond to the typical pseudocritical temperature, \( T_c \). The horizontal dashed lines stand for the critical temperature in the chiral limit, \( T_{\text{chiral}} \). The symbols \( \bullet \), \( \Delta \) and \( \blacktriangle \) represent the temperature corresponding to a percentage of the maximal chiral susceptibility, \( T_{\text{percent}} \), in the three models respectively.

As we can see from the Fig. 5 for all the three models, when the pion mass (and the current quark mass) is large enough, the \( T_{\text{percent}} \) deviates from the horizontal dashed line, \( T_{\text{chiral}} \), in the corresponding model. This means that when we take a fixed percentage of the \( \chi_{\text{max}} \) as a criterion, the pseudocritical temperature is not invariable for a large range of pion mass.

However, if we survey the obtained results with various percentages carefully, we observe that there exist a pion mass range where \( T_{\text{percent}} \) remains almost invariable. And the percentage for the pion mass range to reach its maximum is 78\%, 79\% and 80\% for the QC model, the contact model and the MN model, respectively. As we can see from the figure, for the contact model and the MN model, the \( T_{\text{percent}} \) remains unchanged up to physical pion mass, but the \( T_{\text{percent}} \) begins to increase at \( m_\pi \sim 100\text{MeV} \) in the QC model.

IV. SUMMARY

It has been argued that the phase transition of QCD at chiral limit belongs to the 3D \( O(4) \) universality class. The universality class analysis gives the critical behavior at the chiral limit as that relation between the pseudocritical temperature and the current quark mass behaves as \( T_c \sim m_0^{1/\beta_3} \). We restrict our computation in the mean field approximation of QCD which could lead to a clear critical behavior, and give \( 1/\beta_3 = 2/3 \), which is consistent with the universality class result. We analyze the scaling behavior with different features of interaction by considering different dressed-gluon models. The obtained critical exponent is consistent with the 3D \( O(4) \) universality class. The size of critical region is up to \( m_0 \sim 2 - 4 \text{ MeV} \) in this mean field approximation which sets naturally an upper bound of the critical region since the fluctuations beyond mean-field rely usually on a more subtle universal classification of the system and blur the regime of critical behaviors.

We have implemented three different types of dressed-gluon models which represent different momentum dependent behaviors of the interaction kernel. Our obtained results lead us to a conclusion that the momentum dependence of the interaction kernel does not change the mean field approximation. To go beyond the mean field approximation, people need to either introduce the complicated tensor structures of the quark-gluon interaction vertex of which the coefficients involve explicitly the quark mass dependence \cite{90} or employ the effective pion exchange which also contains the mass dependence via the pion mass.

Another interesting quantity has been marked by lattice QCD simulation analysis is the pseudocritical temperature where the chiral susceptibility is the 60\% of the maximum. At this point, the temperature is found to be independent of current quark mass. However, our results indicate that this is only true when the current quark mass and pion mass is not too large. For the QC model, this temperature is invariant only for \( m_\pi \leq 100 \text{ MeV} \), while for the contact and the MN model it can be invariant up to the physical pion mass. The fixed percentage of the maximal chiral susceptibility is also different for different models, which reads 78\%, 79\% and 80\% for the QC, the contact and the MN model.

Despite of the mean field approximation, here we obtain the chiral phase transition temperature at chiral limit as \( T_c^0 = 142.8 \pm 0.8 \text{MeV} \) with the error from the different gluon models.

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