CERTAIN CLASSES OF MEROMORPHIC $p$-VALENT FUNCTIONS
WITH POSITIVE COEFFICIENTS

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Abstract. We introduce the new class $L(\alpha, \beta, \lambda, p)$ of meromorphic $p$-valent functions. The aim of the paper is to obtain coefficient inequalities, growth and distortion, radii of convexity and starlikeness and the convex linear combinations for the class $L(\alpha, \beta, \lambda, p)$.

1. Introduction

Let $\Sigma_p$ denotes the class of the form

$$f(z) = \frac{1}{z^p} + \sum_{n=1}^{\infty} a_{p+n-1} z^{p+n-1} \quad (1.1)$$

which are analytic and $p$-valent in a punctured disc $D = \{ z : 0 < |z| < 1 \}$. Further let $\Sigma_p^\ast(\alpha)$ be the class of $\Sigma_p$ consisting of function $f$ which satisfies the inequality

$$\text{Re} \left( -\frac{zf'(z)}{f(z)} \right) > \alpha, \quad (0 \leq \alpha < p).$$

And let $\Sigma_p^C(\alpha)$ be the class of $\Sigma_p$ consisting of function $f$ which satisfies the inequality

$$\text{Re} \left( 1 - \frac{zf''(z)}{f'(z)} \right) > \alpha, \quad (0 \leq \alpha < p).$$

Clearly, we have

$$f \in \Sigma_p^C(\alpha) \iff zf'(z) \in \Sigma_p^\ast(\alpha), (0 \leq \alpha < p, p \in \mathbb{N}).$$

This condition is obviously analogous to the well-known Alexander equivalent (see for details [1]). Many important properties and characteristics of various interesting subclasses of the class $\Sigma_p$ of meromorphic $p$-valent functions, including the classes $\Sigma_p^\ast(\alpha)$

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and Σ_C^p(α) defined above, were studied rather extensively by (among others) Aouf et al. ([4], [5]), Kulkarni et al. [8], Mogra ([6], [7]) and Srivastava et al. ([2], [3]).

A function f given by (1.1) is said to be a member of the class L(α, β, λ, p) if it satisfies

\[ \frac{z^{p+1}f'(z) + p}{\alpha z^{p+1}f'(z) + [p + (\lambda - \alpha)(p - \alpha)]} < \beta \]

where \( 0 < \beta \leq 1, 0 < \lambda + p - \sqrt{(\lambda + p)^2 - 4p(1 + \lambda)} < \alpha < \lambda + p + \sqrt{(\lambda + p)^2 - 4p(1 + \lambda)} \leq p, (\lambda + p)^2 > 4p(1 + \lambda) \) and \( \lambda \geq \frac{p}{2} - 2 \) for all \( z \in D \).

In this paper sharp results concerning coefficients, distortion theorem and the radius of convexity for the class L(α, β, λ, p) are determined. Finally we prove that the class is closed under convex linear combination.

2. Coefficient Inequalities

Our first result for functions \( f \in L(\alpha, \beta, \lambda, p) \) is given as the following theorem.

**Theorem 2.1.** If \( f \in \sum_p \) given by (1.1) satisfies

\[ \sum_{n=1}^{\infty} (p + n - 1)(1 + \alpha \beta)|a_{p+n-1}| \leq \beta[p(1 - \alpha) + (\lambda - \alpha)(p - \alpha)]. \]

Then \( f \in L(\alpha, \beta, \lambda, p) \), where \( 0 < \beta \leq 1, 0 < \lambda + p - \sqrt{(\lambda + p)^2 - 4p(1 + \lambda)} < \alpha < \lambda + p + \sqrt{(\lambda + p)^2 - 4p(1 + \lambda)} \leq p, (\lambda + p)^2 > 4p(1 + \lambda) \) and \( \lambda \geq \frac{p}{2} - 2 \) for all \( z \in D \).

**Proof.** Let us suppose that

\[ \sum_{n=1}^{\infty} (p + n - 1)(1 + \alpha \beta)|a_{p+n-1}| \leq \beta[p(1 - \alpha) + (\lambda - \alpha)(p - \alpha)] \]

for \( f \in \sum_p \). Consider the expression

\[ M(f, f') = |z^{p+1}f'(z) + p| < \beta|\alpha z^{p+1}f'(z) + p + (\lambda - \alpha)(p - \alpha)|. \]

Then for \( 0 < |z| = r < 1 \) we have

\[ M(f, f') = \left| \sum_{n=1}^{\infty} (p + n - 1)a_{p+n-1}z^{2p+n-1} \right| - \beta \left| \sum_{n=1}^{\infty} (p + n - 1)a_{p+n-1}z^{2p+n-1} + p(1 - \alpha) + (\lambda - \alpha)(p - \alpha) \right|. \]
\[
\frac{M(f, f')}{r^p} \leq \left( \sum_{n=1}^{\infty} (p + n - 1)|a_{p+n-1}|r^{p+n-1} \right) - \beta \left( - \sum_{n=1}^{\infty} (p + n - 1)|a_{p+n-1}|r^{p+n-1} + \frac{p(1 - \alpha) + (\lambda - \alpha)(p - \alpha)}{r^p} \right) \\
\leq \sum_{n=1}^{\infty} (p + n - 1)(1 + \alpha \beta)|a_{p+n-1}|r^{p+n-1} - \beta[p(1 - \alpha) + (\lambda - \alpha)(p - \alpha)].
\]

The inequality above holds true for all \(r(0 < r < 1)\). Thus by letting \(r \to 1^-\) into the inequality above, we obtain \(M(f, f') \leq 0\). Hence it follows that (1.2) true, and so \(f \in L(\alpha, \beta, \lambda, p)\).

The result is sharp for functions \(f\) of the form

\[
f(z) = z^{-p} + \frac{\beta[p(1 - \alpha) + (\lambda - \alpha)(p - \alpha)]}{p(1 + \alpha \beta)}z^{p+n-1} \quad (n \geq 1).
\]

(2.4)

**Corollary 2.1.** Let the function be defined by (1.1). If \(f \in L(\alpha, \beta, \lambda, p)\), then

\[
|a_{p+n-1}| \leq \frac{\beta[p(1 - \alpha) + (\lambda - \alpha)(p - \alpha)]}{p(1 + \alpha \beta)}(p + n - 1)(1 + \alpha \beta), \quad (n \geq 1).
\]

The result is sharp for functions \(f\) given by (2.4).

**3. Distortion Theorem**

A distortion property for functions in the class \(f \in L(\alpha, \beta, \lambda, p)\), is given as follows:

**Theorem 3.1.** If the function \(f\) given by (1.1) is in the class \(f \in L(\alpha, \beta, \lambda, p)\), then for \(0 < |z| = r < 1\) we have

\[
\frac{1}{r^p} - \frac{\beta[p(1 - \alpha) + (\lambda - \alpha)(p - \alpha)]}{p(1 + \alpha \beta)}r^p \leq |f(z)| \leq \frac{1}{r^p} + \frac{\beta[p(1 - \alpha) + (\lambda - \alpha)(p - \alpha)]}{p(1 + \alpha \beta)}r^p
\]

with equality for

\[
f(z) = \frac{1}{z^p} + \frac{\beta[p(1 - \alpha) + (\lambda - \alpha)(p - \alpha)]}{p(1 + \alpha \beta)}z^p, \quad z = (ir, r)
\]

and

\[
\frac{p}{r^{p+1}} - \frac{\beta[p(1 - \alpha) + (\lambda - \alpha)(p - \alpha)]}{(1 + \alpha \beta)}r^{p-1} \leq |f'(z)| \leq \frac{p}{r^{p+1}} + \frac{\beta[p(1 - \alpha) + (\lambda - \alpha)(p - \alpha)]}{(1 + \alpha \beta)}r^{p-1}.
\]
With equality for,

\[ f(z) = \frac{1}{z^p} + \frac{\beta[p(1 - \alpha) + (\lambda - \alpha)(p - \alpha)]}{p(1 + \alpha\beta)} z^p, \quad z = (\pm ir, \pm r) \]

**Proof.** Since \( f \in L(\alpha, \beta, \lambda, p) \), Theorem 2.1 yields the inequality

\[ \sum_{n=1}^{\infty} \left| a_{p+n-1} \frac{\beta[p(1 - \alpha) + (\lambda - \alpha)(p - \alpha)]}{p + n - 1(1 + \alpha\beta)} \right| \leq 1 \tag{3.5} \]

Thus, for \( 0 < |z| = r < 1 \), and making use of (2.3) we have

\[
|f(z)| = \left| \frac{1}{z^p} + \sum_{n=1}^{\infty} a_{p+n-1} z^{p+n-1} \right|
\leq |z|^{-p} + \sum_{n=1}^{\infty} a_{p+n-1} |z|^{p+n-1}
\leq r^{-p} + \frac{\beta[p(1 - \alpha) + (\lambda - \alpha)(p - \alpha)]}{(p + n - 1)(1 + \alpha\beta)}, \quad \text{(we substitute in (3.5) when } n=1) \]

\[
\leq r^{-p} + \frac{\beta[p(1 - \alpha) + (\lambda - \alpha)(p - \alpha)]}{p(1 + \alpha\beta)}
= \frac{1}{r^p} + \frac{\beta[p(1 - \alpha) + (\lambda - \alpha)(p - \alpha)]}{p(1 + \alpha\beta)}.
\]

And

\[
|f(z)| = \left| \frac{1}{z^p} + \sum_{n=1}^{\infty} a_{p+n-1} z^{p+n-1} \right|
\geq |z|^{-p} - \sum_{n=1}^{\infty} a_{p+n-1} |z|^{p+n-1}
\geq r^{-p} - \frac{\beta[p(1 - \alpha) + (\lambda - \alpha)(p - \alpha)]}{(p + n - 1)(1 + \alpha\beta)}, \quad \text{(we substitute in (3.5) when } n=1) \]

\[
\geq r^{-p} - \frac{\beta[p(1 - \alpha) + (\lambda - \alpha)(p - \alpha)]}{p(1 + \alpha\beta)}
= \frac{1}{r^p} - \frac{\beta[p(1 - \alpha) + (\lambda - \alpha)(p - \alpha)]}{p(1 + \alpha\beta)}.
\]

Also from Theorem 2.1, it follows that

\[ \sum_{n=1}^{\infty} |a_{p+n-1}| \leq \frac{\beta[p(1 - \alpha) + (\lambda - \alpha)(p - \alpha)]}{(1 + \alpha\beta)}. \tag{3.6} \]
Thus

$$|f'(z)| = | -pz^{-p} + \sum_{n=1}^{\infty} (p+n-1)a_{p+n-1}z^{p+n-2}|,$$

$$\leq pr^{-p-1} + \beta[p(1-\alpha) + (\lambda-\alpha)(p-\alpha)]r^{p-1},$$

and

$$|f'(z)| = | -pz^{-p} + \sum_{n=1}^{\infty} (p+n-1)a_{p+n-1}z^{p+n-2}|,$$

$$\geq pr^{-p-1} + \beta[p(1-\alpha) + (\lambda-\alpha)(p-\alpha)]r^{p-1}.$$

Hence completes the proof of Theorem 3.1.

4. Radii of Starlikeness and Convexity

The radii of starlikeness and convexity for the class $L(\alpha, \beta, \lambda, p)$, is given by the following theorem:

**Theorem 4.1.** If the function $f$ defined by (1.1) is in the class $L(\alpha, \beta, \lambda, p)$, then $f$ is starlike of order $\rho (0 \leq \rho < p)$, in the disk $|z| < r_1(\alpha, \beta, \lambda, p, \rho)$, where $r_1(\alpha, \beta, \lambda, p, \rho)$ is the largest value for which

$$r_1 = r_1(\alpha, \beta, \lambda, p) = \inf_{n \geq 1} \left( \frac{(p-\rho)(p+n-1)(1+\alpha \beta)}{(3p+n-\rho-1)\beta[p(1-\alpha) + (\lambda-\alpha)(p-\alpha)]} \right)^{1/p+1-1}.$$

(4.7)

The result is sharp for functions $f$ given by (2.4).

**Proof.** It suffices to show that

$$\left| \frac{zf'(z)}{f(z)} + p \right| \leq (p-\rho).$$

(4.8)

For $|z| \leq r_1$, we have

$$\left| \frac{zf'(z)}{f(z)} + p \right| = \left| -pz^{-p} + \sum_{n=1}^{\infty} (p+n-1)a_{p+n-1}z^{p+n-1} \right| + p.$$

(4.9)
which gives
\[
\left| \sum_{n=1}^{\infty} \frac{(2p + n - 1) a_{p+n-1} z^{p+n-1}}{z^p + \sum_{n=1}^{\infty} a_{p+n-1} z^{p+n-1}} \right| = \frac{\sum_{n=1}^{\infty} (2p + n - 1) a_{p+n-1} z^{2p+n-1}}{1 - \sum_{n=1}^{\infty} a_{p+n-1} z^{2p+n-1}} \leq \frac{\sum_{n=1}^{\infty} (2p+n-1) \beta p(1-p) + (\lambda - \alpha)(p-\alpha))}{(p+n-1)(1+\alpha\beta)} |z|^{2p+n-1} \leq (p - \rho).
\]

The inequality above holds true if
\[
\sum_{n=1}^{\infty} \frac{(3p + n - \rho - 1) \beta p(1-\alpha) + (\lambda - \alpha)(p-\alpha))}{(p + n - 1)(1 + \alpha\beta)} |z|^{2p+n-1} \leq (p - \rho),
\]
and it follows that
\[
|z| \leq \left( \frac{(p - \rho)(p + n - 1)(1 + \alpha\beta)}{(3p + n - \rho - 1) \beta p(1-\alpha) + (\lambda - \alpha)(p-\alpha))} \right)^{\frac{1}{2p+n-1}}, \quad n \geq 1.
\]

Then we have
\[
r_1(\alpha, \beta, \lambda, p, \rho) = \inf_{n \geq 1} \left( \frac{(p - \rho)(p + n - 1)(1 + \alpha\beta)}{(3p + n - \rho - 1) \beta p(1-\alpha) + (\lambda - \alpha)(p-\alpha))} \right)^{\frac{1}{2p+n-1}}
\]
as required.

**Theorem 4.2.** If the function \( f \) defined by (1.1) is in the class \( L(\alpha, \beta, \lambda, p) \), then \( f \) is convex of order \( \rho(0 \leq \rho < p) \), in the disk \( |z| < r_2(\alpha, \beta, \lambda, p, \rho) \), where \( r_2(\alpha, \beta, \lambda, p, \rho) \) is the largest value for which
\[
r_2(\alpha, \beta, \lambda, p, \rho) = \inf_{n \geq 1} \left( \frac{p(p - \rho)(1 + \alpha\beta)}{(3p + n - \rho - 1) \beta p(1-\alpha) + (\lambda - \alpha)(p-\alpha))} \right)^{\frac{1}{2p+n-1}}.
\]

The result is sharp for functions \( f \) given by (2.4).

**Proof.** It suffices to show that
\[
\left| \frac{z f''(z)}{f'(z)} + (1 + p) \right| \leq (p - \rho).
\]

\[\text{(4.10)}\]
For $|z| \leq r_2$, we have

$$\left| \frac{zf''(z)}{f'(z)} + (1 + p) \right| = \frac{(p^2 + p)z^{-p-1} + \sum_{n=1}^{\infty} (p + n - 1)(p + n - 2)a_{p+n-1}z^{p+n-2}}{-pz^{-p-1} + \sum_{n=1}^{\infty} (p + n - 1)a_{p+n-1}z^{p+n-2}} + (p + 1)$$

$$= \frac{\sum_{n=1}^{\infty} (2p + n - 1)(p + n - 1)a_{p+n-1}z^{2p+n-1}}{-p + \sum_{n=1}^{\infty} (p + n - 1)a_{p+n-1}z^{2p+n-1}}$$

$$\leq \frac{\sum_{n=1}^{\infty} (2p + n - 1)(p + n - 1)a_{p+n-1}z^{2p+n-1}}{-p + \sum_{n=1}^{\infty} (p + n - 1)a_{p+n-1}z^{2p+n-1}} \leq (p - \rho).$$

The inequality above holds true if

$$\sum_{n=1}^{\infty} \frac{[3p + n - \rho - 1] \beta [p(1 - \alpha) + (\lambda - \alpha)(p - \alpha) + (1 + \alpha \beta)]}{(p + n - 1)(1 + \alpha \beta)} |z|^{2p+n-1} \leq p(p - \rho),$$

and it follows that

$$|z| \leq \left( \frac{p(p - \rho)(1 + \alpha \beta)}{[3p + n - \rho - 1] \beta [p(1 - \alpha) + (\lambda - \alpha)(p - \alpha)]} \right)^{\frac{1}{p + n - 1}}, \quad n \geq 1.$$

Then we have

$$r_2(\alpha, \beta, \lambda, p, \rho) = \inf_{n \geq 1} \left( \frac{p(p - \rho)(1 + \alpha \beta)}{(3p + n - \rho - 1) \beta [p(1 - \alpha) + (\lambda - \alpha)(p - \alpha)]} \right)^{\frac{1}{p + n - 1}},$$

as required.

5. Convex Linear Combination

Our next result involves a linear combination of function $f$ of the type (2.4).

**Theorem 5.1** Let

$$f_p(z) = z^{-p},$$

and

$$f_{p+n-1}(z) = z^{-p} + \frac{\beta [p(1 - \alpha) + (\lambda - \alpha)(p - \alpha)]}{(p + n - 1)(1 + \alpha \beta)} z^{p+n-1}, \quad (n \geq 1). \quad (5.12)$$
Then $f \in L(\alpha, \beta, \lambda, p)$ if and only if it can be expressed in the form

$$f(z) = \sum_{n=1}^{\infty} \lambda_{p+n-1} f_{p+n-1}(z),$$

(5.13)

where $\lambda_{p+n-1} \geq 0$ and $\sum_{n=1}^{\infty} \lambda_{p+n-1} = 1$.

(5.14)

**Proof.** From (5.12), (5.13) and (5.14), it is easily seen that

$$f(z) = \sum_{n=1}^{\infty} \lambda_{p+n-1} f_{p+n-1}(z),$$

$$= \sum_{n=1}^{\infty} \lambda_{p+n-1} \left( z^{-p} + \frac{\beta[p(1-\alpha) + (\lambda - \alpha)(p - \alpha)]_{p+n-1}}{(p+n-1)(1+\alpha\beta)} \right),$$

$$= z^{-p} + \sum_{n=1}^{\infty} \frac{\beta[p(1-\alpha) + (\lambda - \alpha)(p - \alpha)]_{p+n-1}}{(p+n-1)(1+\alpha\beta)}.$$ 

Then it follows that

$$\sum_{n=1}^{\infty} \frac{\beta[p(1-\alpha) + (\lambda - \alpha)(p - \alpha)]_{p+n-1}}{(p+n-1)(1+\alpha\beta)}$$

(5.15)

and

$$\sum_{n=2}^{\infty} \lambda_{p+n-1} = 1 - \lambda_{p} \leq 1.$$ 

(5.16)

So, by Theorem 2.1 we have $f \in L(\alpha, \beta, \lambda, p)$. Conversely, let us suppose that $f \in L(\alpha, \beta, \lambda, p)$. Then

$$a_{p+n-1} \leq \frac{\beta[p(1-\alpha) + (\lambda - \alpha)(p - \alpha)]}{(p+n-1)(1+\alpha\beta)}, \quad (n \geq 1).$$

(5.17)

Setting

$$\lambda_{p+n-1} = \frac{(p+n-1)(1+\alpha\beta)}{\beta[p(1-\alpha) + (\lambda - \alpha)(p - \alpha)]} a_{p+n-1} \quad (n \geq 1)$$

(5.18)

and $\lambda_{p} = 1 - \sum_{n=2}^{\infty} \lambda_{p+n-1}$. It follows that

$$f(z) = \sum_{n=1}^{\infty} \lambda_{p+n-1} f_{p+n-1}(z)$$

(5.19)

and the proof of the theorem is complete. Finally we prove

**Theorem 5.2.** The class $L(\alpha, \beta, \lambda, p)$ is closed under convex linear combinations.
**Proof.** Suppose that the functions $f_1$ and $f_2$ defined by,

\[ f_i(z) = z^{-p} + \sum_{n=1}^{\infty} a_{p+n-1,i} z^{p+n-1} \quad (i = 1, 2; z \in D) \quad (5.20) \]

are in the class \( L(\alpha, \beta, \lambda, p) \).

Setting \( h(z) = \mu f_1(z) + (1 - \mu) f_2(z) \) we want to show that \( f \in L(\alpha, \beta, \lambda, p) \).

For \( 0 \leq \mu \leq 1 \), we can write

\[
h(z) = z^{-p} + \mu \sum_{n=1}^{\infty} a_{p+n-1,1} + (1 - \mu) \sum_{n=1}^{\infty} a_{p+n-1,2} z^{p+n-1},
\]

\[
= z^{-p} + \sum_{n=1}^{\infty} \left\{ \mu a_{p+n-1,1} + (1 - \mu) a_{p+n-1,2} \right\} z^{p+n-1}, \quad (i = 1, 2; z \in D).
\]

In view of Theorem 2.1, we have

\[
\sum_{n=1}^{\infty} [(p + n - 1)(1 + \alpha \beta)] \left\{ \mu a_{p+n-1,1} + (1 - \mu) a_{p+n-1,2} \right\}
\]

\[
= \mu \sum_{n=1}^{\infty} [(p + n - 1)(1 + \alpha \beta)] a_{p+n-1,1} + (1 - \mu) \sum_{n=1}^{\infty} [(p + n - 1)(1 + \alpha \beta)] a_{p+n-1,2}
\]

\[
\leq \mu \beta [p(1 - \alpha) + (\lambda - \alpha)(p - \alpha)] + (1 - \mu) \beta [p(1 - \alpha) + (\lambda - \alpha)(p - \alpha)]
\]

\[
= \beta [p(1 - \alpha) + (\lambda - \alpha)(p - \alpha)]
\]

which shows that \( f \in L(\alpha, \beta, \lambda, p) \). Hence the theorem.

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