INTEGRAL FORMULAS FOR QUANTUM ISOMONODROMIC SYSTEMS

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ABSTRACT. We consider time-dependent Schrödinger systems, which are quantizations of the Hamiltonian systems obtained from a similarity reduction of the Drinfeld-Sokolov hierarchy by K. Fuji and T. Suzuki, and a similarity reduction of the UC hierarchy by T. Tsuda, independently. These Hamiltonian systems describe isomonodromic deformations for certain Fuchsian systems. Thus, our Schrödinger systems can be regarded as quantum isomonodromic systems. Y. Yamada conjectured that our quantum isomonodromic systems determine instanton partition functions in \( N = 2 \) \( SU(L) \) gauge theory.

The main purpose of this paper is to present integral formulas as particular solutions to our quantum isomonodromic systems. These integral formulas are generalizations of the generalized hypergeometric function \( {}_L F_{L-1} \).

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1. Introduction

Fix integers \( L \geq 2 \) and \( N \geq 1 \). We consider following time-dependent Schrödinger system

\[
\kappa \frac{\partial}{\partial z_i} \Psi(q, z) = H_i \left( q, \frac{\partial}{\partial q}, z \right) \Psi(q, z) \quad (1 \leq i \leq N)
\]

where \( \kappa \in \mathbb{C} \) and \( \Psi(q, z) \) is an unknown function of

\[
q = (q^{(1)}_1, \ldots, q^{(1)}_{L-1}, q^{(2)}_1, \ldots, q^{(2)}_{L-1}, \ldots, q^{(N)}_1, \ldots, q^{(N)}_{L-1})
\]

and \( z = (z_1, \ldots, z_N) \). The Hamiltonians \( H_i \) are defined in Definition 2.1.

The Schrödinger system (1.1) is a quantization of the classical Hamiltonian system \( \mathcal{H}_{L,N} \) obtained from a similarity reduction of the Drinfeld-Sokolov hierarchy by K. Fuji and T. Suzuki \( (L = 3, N = 1) \) [3], T. Suzuki \( (L \geq 2, N = 1) \) [10], and a similarity reduction of the UC hierarchy by T. Tsuda \( (L \geq 2, N \geq 1) \) [14], independently. In [14], T. Tsuda showed that the classical Hamiltonian system \( \mathcal{H}_{L,N} \) is equivalent to a Schlesinger system governing isomonodromic deformation for a certain Fuchsian system.

On the other hand, Y. Yamada conjectured in the context of the so-called AGT relation that the instanton partition function, in the presence of the full surface operator in \( N = 2 \) \( SU(L) \) gauge theory, is determined by the Schrödinger system (1.1) for \( N = 1 \) [15]. In the case of \( L = 2 \), the Schrödinger system (1.1) is a quantization of the Garnier systems [4], [6], which has been appeared in the conformal field theory [12].
In this paper, we present a family of hypergeometric integrals as particular solutions to the Schrödinger system (1.1). These solutions are polynomials in \( q \) with the degree \( M \in \mathbb{Z}_{\geq 1} \) and the coefficients are integral representations of hypergeometric type.

A key to find special solutions to quantum isomonodromic systems is to observe special solutions to the corresponding classical isomonodromic systems. For example, both the classical and quantum sixth Painlevé equation has a particular solution expressed in terms of the Gauss hypergeometric function [7].

It is known that the classical Hamiltonian system \( \mathcal{H}_{L,N} \) has a particular solution expressed in terms of a generalization of the Gauss hypergeometric function by T. Suzuki (\( L \geq 2, N = 1 \)) [11], T. Tsuda (\( L \geq 2, N \geq 1 \)) [13]. Observing the linear Pfaffian system derived from this generalization of the Gauss hypergeometric function, we see indeed that hypergeometric integrals given in [13] yield a particular solution to the Schrödinger system (1.1):

**Theorem 1.1.** The integral formula

\[
\int_{\Delta} \prod_{n=1}^{L-1} t_n^{a_n/k} \prod_{i=1}^{N} \left( 1 - z t_{iL-1} \right)^{-\beta_i/k} \prod_{n=1}^{L-1} (t_{n-1} - t_n)^{-\gamma_n/k} \left( \varphi_0(t) - \sum_{i=1}^{N} \sum_{n=1}^{L-1} \varphi_n^{(i)}(t) q_n^{(i)} \right), \quad (1.2)
\]

which is a polynomial in \( q \) with the degree 1, is a particular solution to the Schrödinger system (1.1). Here \( \Delta \) is a twist cycle and \( \varphi_0(t) \), \( \varphi_n^{(i)}(t) \) are certain rational \( (L-1) \)-forms defined in (4.1).

(see Theorem [4.1])

In order to generalize the integral formula as particular solution to the case of polynomials in \( q \) with the degree \( M \in \mathbb{Z}_{\geq 2} \), let us recall equivalence between the Knizhnik-Zamolodchikov equation of the conformal field theory and a quantization of a Schlesinger system [5], [8]. The KZ equations for the simple Lie algebra \( \mathfrak{g} \), have integral representations as solutions taking values in tensor products of Verma modules of \( \mathfrak{g} \) (see, for example, [1], [9]). From the point of view that the integral formula (1.2) may be a solution to the Knizhnik-Zamolodchikov equation, it should be viewed that the integral variables are corresponding to the simple roots of \( \mathfrak{sl}_L \). While, for the case of \( L = 2 \) and \( N = 1 \), it is known that the Schrödinger system (1.1), the quantum sixth Painlevé equation, has hypergeometric solutions [7]:

\[
\int_{\Delta} \prod_{1 \leq a < b \leq M} (t^{(a)} - t^{(b)})^{2/k} \prod_{a=1}^{M} (t^{(a)})^{a/k} (1 - z t^{(a)})^{-\beta_a/k} (1 - t^{(a)})^{-\gamma_a/k} \left( \varphi_0(t^{(a)}) - \varphi_1^{(1)}(t^{(a)}) q_1^{(1)} \right). \]

Note that the integrand above consists of \( M \)-copies of the integrand of (1.2) multiplied by the coupled term \( \prod_{1 \leq a < b \leq M} (t^{(a)} - t^{(b)})^{2/k} \).

Considering upon these, we arrive at

**Theorem 1.2.** The integral formula

\[
\int_{\Delta} \prod_{1 \leq a \leq M, \ 1 \leq n \leq L-1} (t_n^{(a)} - t_n^{(b)})^{2/k} \prod_{1 \leq a \leq M, \ 1 \leq n \leq L-2} (t_n^{(a)} - t_{n+1}^{(b)})^{-1/k}
\]
\[
\times \prod_{a=1}^{M} \left\{ \prod_{n=1}^{L-1} \left( \frac{t^{(a)}}{a^n} \right)^{\alpha_n/k} \prod_{i=1}^{N} \left( 1 - z_i t^{(a)} \right)^{-\beta_i/k} \left( 1 - t^{(a)} \right)^{-\gamma/k} \left( \varphi_0(t^{(a)}) - \sum_{i=1}^{N} \sum_{n=1}^{L-1} \varphi_i^{(i)}(t^{(a)}) q_i^{(i)} \right) \right\},
\]
which is a polynomial in \( q \) with the degree \( M \), is a particular solution to the Schrödinger system \( (1.1) \). Here \( \Delta \) is a skew-symmetric twist cycle.

(see Theorem 4.3)

The remainder of this paper is organized as follows. In section 2, we introduce quantizations of the classical Hamiltonians of \( \mathcal{H}_{L,N} \) and show that those quantum Hamiltonians are mutually commutative. In section 3, we introduce our Schrödinger systems and discuss properties of them. In section 4, we give integral formulas for solutions.

**Remark 1.3.** As mentioned above, the classical Hamiltonian system \( \mathcal{H}_{L,N} \) describes isomonodromic deformation for an \( L \times L \) Fuchsian system

\[
\frac{\partial}{\partial u} \Phi(u) = \sum_{i=0}^{N+1} A_i \frac{1}{u - u_i} \Phi(u),
\]

where \( u_0 = 1, u_i = 1/z_i \ (1 \leq i \leq N) \), and \( u_{N+1} = 0 \), whose spectral type is given by the \( (N+3) \)-tuple

\[
(1, 1, \ldots, 1), (1, 1, \ldots, 1), (L - 1, 1), \ldots, (L - 1, 1)
\]

of partitions of \( L \). A spectral type defines multiplicities of the eigenvalues of each residue matrix \( A_i \). Consequently, \( L - 1 \) parameters are associated with singular points \( 0 \) and \( \infty \), and one parameter is associated with each singular point \( u_i \) for \( i = 0, \ldots, N \). Notice that in the integrand given in the Theorems above, \( L - 1 \) parameters are associated with the singular point \( 0 \), and one parameter is associated with each singular point \( 1, 1/z_i \ (1 \leq i \leq N) \).

2. **Hamiltonian**

Let us define a non-commutative associative algebra \( W_{L,N} \) over \( \mathbb{C} \) with generators

\[
q_m^{(i)}, p_m^{(i)} \quad (1 \leq m \leq L - 1, \ 1 \leq i \leq N),
\]

\[
e_n, \kappa_n, \theta_j, \hbar \quad (0 \leq n \leq L - 1, \ 0 \leq j \leq N)
\]

and commutation relations

\[
\left[ p_m^{(i)}, q_n^{(i)} \right] = \delta_{n,m} \delta_{i,j} \hbar \quad (1 \leq n, m \leq L - 1, \ 1 \leq i, j \leq N),
\]

(2.1)

where \( \delta_{i,j} \) is Kronecker’s delta, and the other commutation relations are zero, and relations

\[
\sum_{m=0}^{L-1} e_m = \frac{L - 1}{2}, \quad \sum_{m=0}^{L-1} \kappa_m = \sum_{i=0}^{N} \theta_i.
\]

The non-commutative associative algebra \( W_{L,N} \) is an Ore domain, so that we can define its skew field \( \mathcal{K}_{L,N} \) (see, for example, [2], Chapter 1, Section 8).
**Definition 2.1.** We introduce Hamiltonians $H_i$ ($i = 1, \ldots, N$) in the rational function field $\mathbb{W}_{L,N}(z_1, \ldots, z_N)$ in variables $z_1, \ldots, z_N$ by

$$z_i H_i = \sum_{n=0}^{L-1} e_n q_n^{(i)} p_n^{(i)} + \sum_{j=0}^{N} \sum_{0 \leq m < n \leq L-1} q_m^{(i)} p_m^{(j)} q_n^{(j)} p_n^{(i)} + \frac{1}{z_i - 1} \sum_{m,n=0}^{L-1} q_m^{(i)} p_m^{(0)} q_n^{(0)} p_n^{(i)}$$

$$+ \sum_{j=1 \neq i}^{N} \sum_{m,n=0}^{L-1} q_m^{(i)} p_n^{(j)} q_n^{(j)} p_m^{(i)} + \theta_i \left( e_0 + \kappa_0 - \sum_{j=1}^{N} \theta_j - \sum_{j=1 \neq i}^{N} \theta_j z_j \right),$$

(2.2)

where

$$q_0^{(i)} = \theta_i + \sum_{m=1}^{L-1} q_m^{(i)} p_m^{(i)}, \quad p_0^{(i)} = -1 \quad (1 \leq i \leq N),$$

$$q_m^{(0)} = -1, \quad p_m^{(0)} = \kappa_m + \sum_{i=1}^{N} q_m^{(i)} p_m^{(i)}, \quad (1 \leq m \leq L - 1),$$

$$q_0^{(0)} = \kappa_0 - \sum_{i=1}^{N} \theta_i - \sum_{i=1}^{N} \sum_{m=1}^{L-1} q_m^{(i)} p_m^{(i)}, \quad p_0^{(0)} = -1.$$

The Hamiltonians $H_i$ ($i = 1, \ldots, N$) are canonical quantization of the polynomial Hamiltonians in [14], Appendix A. What we mean by canonical quantization is, to replace the Poisson bracket with the commutator.

Since the canonical variables in the classical Hamiltonians are not separated, quantization of the Hamiltonians is not unique. In the following, we show that the Hamiltonians $H_i$ are mutually commutative and the Schrödinger equations associated with the Hamiltonians $H_i$ have integral formulas.

**Example 2.2.** We give an example of the Hamiltonians $H_i$ in the case of $L = 2$. Set $(q_i, p_i) = (q_i^{(0)} , p_i^{(0)})$. The Hamiltonian $H_i$ is expressed as follows:

$$z_i (z_i - 1) H_i = q_i \left( \kappa_1 - \theta_0 + \sum_{j=1}^{N} q_j p_j \right) \left( \kappa_1 + \sum_{j=1}^{N} q_j p_j \right) + z_i (\theta_i + q_i p_i) p_i$$

$$- \sum_{j=1 \neq i}^{N} \sum_{j=1}^{N} \frac{z_j}{z_i - z_j} (\theta_j + q_j p_j) q_i p_j - \sum_{j=1 \neq i}^{N} \frac{z_i}{z_i - z_j} (\theta_i + q_i p_i) q_j p_j$$

$$- \sum_{j=1 \neq i}^{N} \frac{z_i(z_j - 1)}{z_j - z_i} (\theta_i + q_i p_i) q_j p_j - \sum_{j=1 \neq i}^{N} \frac{z_i(z_j - 1)}{z_j - z_i} (\theta_j + q_j p_j) q_i p_i$$

$$- (z_i + 1)(\theta_i + q_i p_i) q_i p_i - (e_1 - e_0) z_i + e_0 - e_1 - \hbar + \kappa_1 - \kappa_0) q_i p_i$$

plus some function in only $(z_1, \ldots, z_N)$. These Hamiltonians are quantizations of the polynomial Hamiltonians for the Garnier system [6].
Example 2.3. We give an example of the Hamiltonians $H_1$ in the case of $N = 1$. Set $(q_m, p_m) = (q_m^{(1)}, p_m^{(1)})$, $H = H_1$, and $z = z_1$. The Hamiltonian $H$ is written in a coupled form as follows:

$$z(z - 1)H = \sum_{m=1}^{L-1} H_{V1} \left( \sum_{n=0}^{L-1} \alpha_{2n+1} - \alpha_{2m-1} - \eta, \sum_{n=0}^{L-1} \alpha_{2n}, \sum_{n=m}^{L-1} \alpha_{2n+1} \eta \right) q_m, p_m \right)$$

$$+ \frac{1}{4} \sum_{1 \leq m < n \leq L-1} (((q_m - 1) p_m q_m + q_m p_m (q_m - 1) + 2 \alpha_{2m-1} (q_m - 1)) (p_n (q_n - z) + (q_n - z) p_n)
$$

plus some function in only $(z_1, \ldots, z_N)$, where

$$H_{V1} (a_0, a_1, a_z; a, q, p) = \frac{1}{6} (qp(q - 1)p(q - z) + (q - 1)p(q - z)pq + (q - z)pqp(q - 1) +
$$

$$(q - z)p(q - 1)pq + (q - 1)pqp(q - z) + qp(q - z)p(q - 1))
$$

$$- \frac{1}{2} (a_0((q - 1)p(q - z) + (q - z)p(q - 1)) + a_1(qp(q - z) + (q - z)pq)
$$

$$+(a_2 - 1)(qp(q - 1) + (q - 1)pq)) + aq.$$

Here, we let

$$\alpha_{2m-1} = \kappa_m - \hbar \quad (1 \leq m \leq L - 1), \quad \alpha_{2m} = e_m - e_{m+1} - \kappa_m + \hbar \quad (1 \leq m \leq L - 2),
$$

$$\alpha_0 = e_0 - e_1, \quad \alpha_{2L-1} = -\kappa_0 + (L - 2)\hbar, \quad \sum_{m=0}^{2L-1} \alpha_m = \kappa, \quad \eta = -\kappa_0 + \theta_1 - \frac{L - 2}{2} \hbar.$$

The Hamiltonian $H$ is a quantization of the Hamiltonian obtained by Fuji-Suzuki ($L = 3$) [3] and Suzuki ($L \geq 3$) [10]. The Hamiltonian $H_{V1}$ is of the sixth quantum Painlevé equation with the affine Weyl group symmetry of type $D_4^{(1)}$ introduced in [?].

2.1. Commutativity. The Hamiltonians $H_i (i = 1, \ldots, N)$ are expressed in the following forms

$$-z_i^2 H_i = \sum_{j \neq i}^{N+1} \frac{\Omega_{i,j}}{u_i - u_j},$$

where $\Omega_{i,j}$ are elements in $W_{L,N}$ and $u_0 = 1$, $u_i = 1/z_i$ $(i = 1, \ldots, n)$ and $u_{N+1} = 0$.

For $i, j = 1, \ldots, N$, the forms $\Omega_{i,j}$ read as

$$\Omega_{i,j} = \frac{1}{2} \text{tr} \left( \hat{A}^{(i)} \hat{A}^{(j)} \right),$$

where $\hat{A}^{(i)}$ is a $L \times L$ matrix defined as

$$\left( \hat{A}^{(i)} \right)_{m,n} = q_m^{(i)} p_n^{(i)}$$

for $m, n = 0, 1, \ldots, L - 1$, where $\left( \hat{A}^{(i)} \right)_{m,n}$ is the $(m, n)$ entry of the matrix $\hat{A}^{(i)}$. The entries of $\hat{A}^{(i)}$ satisfy the following commutation relations.
Lemma 2.4. It holds that
\[ \frac{1}{\hbar} \left[ \hat{A}_m, \hat{A}_{m'} \right] = \delta_{m,m'} \left( \delta_{n,n'} \hat{A}_m - \delta_{n',m} \hat{A}_{m'} \right) \] (2.5)
for \( 0 \leq m, n, m', n' \leq L - 1 \) and \( 1 \leq i, j \leq N \).

A proof is given by a straightforward calculation.

Recall the definition of the Gaudin Hamiltonians (see, for example, [5], Section 2). The Gaudin Hamiltonians \( G_i \) \( (i = 1, \ldots, N) \) for \( \mathfrak{gl}_L \) are defined as
\[ G_i = \frac{1}{2} \sum_{j \neq i}^N \frac{\text{tr} \left( B^{(i)} B^{(j)} \right)}{u_i - u_j}, \]
where \( B^{(i)} \) \( (i = 1, \ldots, N) \) are \( L \times L \) matrices whose entries satisfy the commutation relations (2.5). Since the commutativity of Gaudin Hamiltonians equals the so-called infinitesimal braid relations, Lemma 2.4 yields:
\[ \left[ \Omega_{i,j}, \Omega_{k,l} \right] = 0 \quad (i, j, k, l \text{ are distinct}), \]
\[ \left[ \Omega_{i,j}, \Omega_{i,k} + \Omega_{k,j} \right] = 0 \quad (i, j, k \text{ are distinct}) \] (2.6) (2.7)
for \( i, j, k, l = 1, \ldots, N \).

The other elements \( \Omega_{i,0} \) and \( \Omega_{i,N+1} \) \( (i = 1, \ldots, N) \) are not expressed in a similar way as (2.3) and (2.4). However, we can check by a straightforward calculation that the infinitesimal braid relations above hold even if \( i, j, k, l = 0, 1, \ldots, N + 1 \). Therefore, we have

**Proposition 2.5.** Hamiltonians \( H_i \) \( (i = 1, \ldots, N) \) are mutually commutative.

3. Schrödinger system

Denote by
\[ H_i \left( \mathbf{q}, \frac{\partial}{\partial \mathbf{q}}, \mathbf{z} \right) \] (3.1)
for \( i = 1, \ldots, N \), the Hamiltonians obtained by substituting \( q^{(i)}_m \) and \( \partial/\partial q^{(i)}_m \) into \( q^{(i)}_m \) and \( p^{(i)}_m \), respectively, of the Hamiltonians \( H_i \) defined in Definition 2.1.

We consider the following Schrödinger system:
\[ \kappa \frac{\partial}{\partial z_i} \Psi(\mathbf{q}, \mathbf{z}) = H_i \left( \mathbf{q}, \frac{\partial}{\partial \mathbf{q}}, \mathbf{z} \right) \Psi(\mathbf{q}, \mathbf{z}) \] (3.2)
where \( \kappa \in \mathbb{C} \), \( \Psi(\mathbf{q}, \mathbf{z}) \) is an unknown function of
\[ \mathbf{q} = (q^{(1)}_1, \ldots, q^{(1)}_{L-1}, q^{(2)}_1, \ldots, q^{(2)}_{L-1}, \ldots, q^{(N)}_1, \ldots, q^{(N)}_{L-1}) \]
and \( \mathbf{z} = (z_1, \ldots, z_N) \). Here, we regard \( e_n, \kappa_n, \theta_i \) as complex parameters.
Proposition 3.1. The Schrödinger system (3.2) is completely integrable in the sense of Frobenius, that is, it holds

$$\left[ \kappa \frac{\partial}{\partial z_i} - H_i \left( \mathbf{q}, \frac{\partial}{\partial \mathbf{q}}, \mathbf{z} \right), \kappa \frac{\partial}{\partial z_j} - H_j \left( \mathbf{q}, \frac{\partial}{\partial \mathbf{q}}, \mathbf{z} \right) \right] = 0,$$

for $i, j = 1, \ldots, N$.

Proof. Thanks to Proposition 2.5, we have only to show

$$\frac{\partial}{\partial z_i} H_j \left( \mathbf{q}, \frac{\partial}{\partial \mathbf{q}}, \mathbf{z} \right) = - \frac{\partial}{\partial z_j} H_i \left( \mathbf{q}, \frac{\partial}{\partial \mathbf{q}}, \mathbf{z} \right).$$

It is easily calculated as follows. For $i \neq j$, we have

$$\frac{\partial}{\partial z_i} H_j \left( \mathbf{q}, \frac{\partial}{\partial \mathbf{q}}, \mathbf{z} \right) = \frac{1}{(z_i - z_j)^2} \left( \sum_{m, n=0}^{N-1} q_m^{(j)} p_n^{(j)} q_n^{(0)} p_m^{(0)} - \theta_j \right).$$

From Lemma 2.4, the last line is symmetrical with respect to $i$ and $j$. Thus, we finished the proof. \(\square\)

In the most simplest case, namely, the case of $L = 2$ and $N = 1$, the Schrödinger system (3.2) is the quantum sixth Painlevé equation. In the previous work [7], we showed that the quantum sixth Painlevé equation has polynomial solutions in terms of $q$.

In the general case, the Schrödinger system (3.2) also has polynomial solutions in terms of $\mathbf{q}$ due to the following proposition.

For a $L - 1 \times N$ matrix $A$ whose entries are non-negative integers, let $q^A$ be the monomial defined by

$$q^A = \prod_{m=1}^{L-1} \prod_{i=1}^{N} \left( q_m^{(i)} \right)^{A_{m,i}},$$

where $A_{m,i}$ is the $(m, i)$ entry of the matrix $A$. Set $d(A) = \sum_{m=1}^{L-1} \sum_{i=1}^{N} A_{m,i}$.

Let $V(M) \ (M \in \mathbb{Z}_{\geq 0})$ be the subspace of the polynomial ring $\mathbb{C}[\mathbf{q}]$ such that the degree of elements in $V(M)$ is less than $M$, namely, $V(M) = \bigoplus_A \mathbb{C} q^A$, where the summation is taken over all $L - 1 \times N$ matrices $A$ such that $d(A) \leq M$.

Proposition 3.2. For each $i = 1, \ldots, N$, the Hamiltonian $H_i(\mathbf{q}, \partial/\partial \mathbf{q}, \mathbf{z})$ acts on $V(M)$ if $\kappa_0 - \sum_{i=1}^{N} \theta_i = M$.

Proof. We compute the action of the Hamiltonian $H_i(\mathbf{q}, \partial/\partial \mathbf{q}, \mathbf{z})$ on $q^A$ such that $d(A) = M$ as follows.

$$z_i(z_i - 1) H_i \left( \mathbf{q}, \frac{\partial}{\partial \mathbf{q}}, \mathbf{z} \right) q^A = - \sum_{n=1}^{L-1} q_n^{(i)} q_0^{(0)} p_0^{(0)} q^A + f(\mathbf{q})$$
\[ - \sum_{n=1}^{L-1} \left( \kappa_0 - \sum_{i=1}^{N} \theta_i - M \right) \left( \kappa_n + \sum_{i=1}^{N} A_{n,i} \right) q^{(i)} q^A + f(q). \]  

(3.3)

Here \( f(q) \) is a polynomial whose degree is equal to or less than \( M \). Hence, if \( \kappa_0 - \sum_{i=1}^{N} \theta_i = M \), then the first term of (3.3) vanishes, which finishes the proof. \( \square \)

By virtue of Proposition 3.2 for the Schrödinger equation (3.2), we can consider polynomial solutions

\[ \Psi(q, z) = \sum_{A \in A_M} c_A(z) q^A, \]

where

\[ A \in \mathcal{A}_M = \left\{ A = (A_{m,i}) \mid A_{m,i} \in \mathbb{Z}_{\geq 0}, d(A) \leq M \right\} \]

and \( c_A(z) \) is a function of \( z \). In the next section, we present integral formulas taking values in \( V(M) \) and show that they are solutions to the Schrödinger system (3.2).

The Hamiltonians \( H_i \) act on another subspaces of the polynomial ring \( \mathbb{C}[q] \). Let \( F(T_1, \ldots, T_{L-1}) \) \((T_1, \ldots, T_{L-1} \in \mathbb{Z}_{\geq 0})\) be the subspace of the polynomial ring \( \mathbb{C}[q] \) defined as \( F(T_1, \ldots, T_{L-1}) = \bigoplus A \mathbb{C} q^A \), where the summation is taken over all \( L-1 \times N \) matrices \( A \) such that the entries of \( A \) are non-negative integers and \( \sum_{i=1}^{N} A_{m,i} \leq T_m \) \((m = 1, \ldots, L-1)\). Set \( d_m(A) = \sum_{i=1}^{N} A_{m,i} \).

**Proposition 3.3.** For each \( i = 1, \ldots, N \), the Hamiltonian \( H_i(q, \partial / \partial q, z) \) acts on \( F(T_1, \ldots, T_{L-1}) \) if \( \kappa_m = -T_m \).

**Proof.** Take a \( L-1 \times N \) matrix \( A \) such that the entries of \( A \) are non-negative integers and \( d_m(A) = T_m \) for any \( m \in \{1, \ldots, L-1\} \). The Hamiltonian \( H_i(q, \partial / \partial q, z) \) acts on \( q^A \) as follows:

\[ z_i(z_i - 1) H_i \left( q, \frac{\partial}{\partial q}, z \right) q^A = - \left( \sum_{m=1}^{L-1} q^{(i)} m_0 \right) P_m^{(i)} + \sum_{n=m+1}^{L-1} q^{(i)} m_0 P_n^{(i)} + \sum_{n=m+1}^{L-1} q^{(i)} m_0 P_n^{(i)} \left( q^A + f(q) \right) \]

\[ = - (\kappa_m + T_m) \left( \kappa_0 - \sum_{j=1}^{N} \theta_j - d(A) + \sum_{n=m+1}^{L-1} P_n^{(i)} + \sum_{n=m+1}^{L-1} P_n^{(i)} \right) q^{(i)} q^A + f(q). \]

(3.5)

Here, \( f(q) \) is a polynomial such that as a polynomial in terms of \( q^{(1)}_m, \ldots, q^{(N)}_m \), the degree of \( f(q) \) is equal to or less than \( d_m(A) \). Thus, if \( \kappa_m = -T_m \), then the first term of (3.5) vanishes, which finishes the proof. \( \square \)

Consequently, we can also consider polynomial solutions taking values in \( F(T_1, \ldots, T_{L-1}) \).

4. **Integral Formula**

In this section, we construct integral formulas for the Schrödinger systems (3.2), as particular solutions.

Recall that the Gauss hypergeometric function is a particular solution to both the classical and quantum sixth Painlevé equation [7]. Hypergeometric solutions to the classical Hamiltonian
systems $\mathcal{H}_{L,N}$ were given by T. Suzuki ($L \geq 2$, $N = 1$) \[11\] and T. Tsuda ($L \geq 2$, $N \geq 1$) \[13\] independently, under the condition $\kappa_0 - \sum_{i=1}^{N} \theta_i = 0$.

These hypergeometric solutions are the generalized hypergeometric functions (Thomae’s hypergeometric function) $F_{L-1}$ in the case of $(L \geq 2$, $N = 1)$ and their generalizations in the case of $(L \geq 2$, $N \geq 1)$.

We expect that these generalized hypergeometric functions are also solutions to a quantization of the classical Hamiltonian systems $\mathcal{H}_{L,N}$, the Schrödinger systems \[3,2\]. Indeed, this is true if we consider polynomial solutions to the Schrödinger systems \[3,2\] with $\kappa_0 - \sum_{i=1}^{N} \theta_i = 1$.

Set $\kappa_0 - \sum_{i=1}^{N} \theta_i = M \in \mathbb{Z}_{\geq 0}$. We begin with the case $M = 1$ and later we deal with general case.

4.1. The case of $M = 1$. Consider a multivalued function

$$U(t) = \prod_{n=1}^{L-1} \prod_{i=1}^{N} (1 - z_i t_{L-1})^{-\beta_i/k} \prod_{n=1}^{L-1} (t_{n-1} - t_n)^{-\gamma_n/k},$$

with $t_0 = 1$ defined on the complement $T \subset \mathbb{C}^{(L-1)}$ of singular locus $D$ given by

$$D = \bigcup_{1 \leq n \leq L-1} \{t_{n-1} = t_n\} \cup \bigcup_{1 \leq n \leq L-1} \{t_n = 0\} \cup \bigcup_{1 \leq n \leq N} \{t_{L-1} = 1/z_i\}.$$

Let $S$ be the rank one local system determined by $U(t)$ and $S^*$, the dual local system of $S$. The hypergeometric paring between the twisted homology group and twisted de Rham cohomology group is

$$H_{L-1}(T, S^*) \times H^{L-1}(T, \nabla) \longrightarrow \mathbb{C}$$

$$(\Delta, \varphi) \longmapsto \int_{\Delta} U(t) \varphi,$$

where $\varphi$ is a rational $(L - 1)$-form holomorphic outside $D$ and $\nabla$ is the covariant differential operator given by $\nabla = d + d \log(U(t)) \wedge$.

According to \[14\], the following rational $(L - 1)$-forms

$$\varphi_0(t) = \frac{dt_1 \wedge \cdots \wedge dt_{L-1}}{t_{L-1}} \prod_{n=1}^{L-1} \frac{1}{t_{n-1} - t_n},$$

$$\varphi_n^{(i)}(t) = \frac{dt_1 \wedge \cdots \wedge dt_{L-1}}{(1 - z_i t_{L-1}) t_{L-1}} \prod_{m=1, m \neq n}^{L-1} \frac{1}{t_{m-1} - t_m}$$

(4.1)

represent a basis of $H^{L-1}(T, \nabla)$.

Define the integral formula $\Psi_1(\mathbf{q}, \mathbf{z})$ by

$$\Psi_1(\mathbf{q}, \mathbf{z}) = \int_{\Delta} U(t) \left( \varphi_0(t) - \sum_{i=1}^{N} \sum_{n=1}^{L-1} \varphi_n^{(i)}(t) q_n^{(i)} \right)$$

with $\Delta \in H_{L-1}(T, S^*)$. From Proposition \[3,2\] when $\kappa_0 - \sum_{i=1}^{N} \theta_i = 1$, the action of Hamiltonian $H_i$ ($i = 1, \ldots, N$) on the integral formula, $H_i \Psi_1(\mathbf{q}, \mathbf{z})$, is also a polynomial of degree equal to or less than 1 and then the constant term and the coefficient of $q_n^{(i)}$ ($1 \leq n \leq L - 1$, $1 \leq j \leq N$) of $H_i \Psi_1(\mathbf{q}, \mathbf{z})$ are linear combinations of $\int_{\Delta} U(t) \varphi_0(t)$ and $\int_{\Delta} U(t) \varphi_n^{(i)}(t)$ ($1 \leq n \leq L - 1$, $1 \leq j \leq N$).
Remarkably they coincide with $\kappa \partial \varphi_0(t)/\partial z_i$ and $\kappa \partial \varphi_n^{(i)}(t)/\partial z_i$ with appropriate correspondence between parameters. Namely, we have

**Theorem 4.1.** If $\kappa_0 - \sum_{i=1}^N \theta_i = 1$, then the integral formula $\Psi_1(q, z)$ is a solution to the Schrödinger system (3.2), with

$$\alpha_n = e_{n+1} - e_n + \kappa_{n+1}, \quad \beta_n = -\theta_i, \quad \gamma_n = \kappa_n,$$

for $1 \leq n \leq L - 1$ and $1 \leq i \leq N$, where $e_L = e_0$ and $\kappa_L = 1$.

### 4.2. The case of $M \geq 2$

Fix $M \in \mathbb{Z}_{\geq 2}$. We consider a multivalued function

$$U(t) = \prod_{1 \leq a \leq M} \left( t_n^{(a)} - t_n^{(b)} \right)^{2/\kappa} \prod_{1 \leq a \leq M} \left( t_n^{(a)} - t_{n+1}^{(b)} \right)^{-1/\kappa}$$

$$\times \prod_{a=1}^M \left\{ \prod_{n=1}^{L-1} \left( t_n^{(a)} \right)^{a_n/\kappa} \prod_{n=1}^N \left( 1 - z_i t_{L-1}^{(a)} \right)^{-\beta_n/\kappa} \left( 1 - t_1^{(a)} \right)^{-\gamma_n/\kappa} \right\}$$

defined on the complement $T \in \mathbb{C}^{(L-1)M}$ of singular locus $D$ given by

$$D = \bigcup_{1 \leq a \leq M} \left\{ t_n^{(a)} = t_n^{(b)} \right\} \bigcup \bigcup_{1 \leq a \leq M} \left\{ t_n^{(a)} = t_{n+1}^{(b)} \right\} \bigcup \bigcup_{1 \leq a \leq M} \left\{ t_n^{(a)} = 0 \right\} \bigcup \bigcup_{1 \leq a \leq M} \left\{ t_{L-1}^{(a)} = 1/z_i \right\} \bigcup \bigcup_{1 \leq a \leq M} \left\{ t_1^{(a)} = 1 \right\}.$$

Let $S$ be the rank one local system determined by $U(t)$ and $S^*$, the dual local system of $S$. The hypergeometric pairing between the twisted homology group and twisted de Rham cohomology group is

$$H_{(L-1)M}(T, S^*) \times H^{(L-1)M}(T, \nabla) \longrightarrow \mathbb{C}$$

$$(\Delta, \varphi) \longmapsto \int_\Delta U(t) \varphi,$$

where $\varphi$ is a rational $(L-1)M$-form holomorphic outside $D$ and $\nabla$ is the covariant differential operator given by $\nabla = d + d \log(U(t)) \wedge$.

Denote by $\mathcal{E}_M^{L-1}$, $(L-1)$-th products of the symmetric group with the degree $M$. Let the action of $\mathcal{E}_M^{L-1}$ on a rational function $f(t)$ of variables $t = (t_1^{(1)}, \ldots, t_1^{(M)}, \ldots, t_{L-1}^{(M)})$ be defined by

$$\sigma(f(t)) = f(t_1^{(\sigma_1(1))}, \ldots, t_{L-1}^{(\sigma_{L-1}(1))}, \ldots, t_1^{(\sigma_1(M))}, \ldots, t_{L-1}^{(\sigma_{L-1}(M))})$$

for $\sigma = (\sigma_1, \ldots, \sigma_{L-1}) \in \mathcal{E}_M^{L-1}$. Let $\text{Sym}[f(t)]$ be the symmetrization of $f(t)$, given by $\text{Sym}[f(t)] = \sum_{\sigma \in \mathcal{E}_M^{L-1}} \sigma(f(t))$.

**Definition 4.2.** For $M \in \mathbb{Z}_{\geq 2}$, we define an integral formula by

$$\Psi_M(q, z) = \int_\Delta U(t) \cdot \text{Sym} \left[ \prod_{a=1}^M \left( f_0(t_n^{(a)}) - \sum_{i=1}^N \sum_{n=1}^{L-1} f_n^{(i)}(t_n^{(a)}) t_n^{(i)} \right) \right] dt,$$

where $\Delta \in H_{(L-1)M}(T, S^*)$ and

$$f_0(t_0^{(a)}) = \prod_{m=1}^{L-1} \frac{1}{t_0^{(a)} - t_m^{(a)}}, \quad f_n^{(i)}(t_n^{(a)}) = \frac{1}{z_i t_n^{(a)} - t_m^{(a)}}, \quad t_0^{(a)} = 1.$$
\[ dt = dt_1^{(1)} \wedge \cdots \wedge dt_L^{(1)} \wedge dt_1^{(2)} \wedge \cdots \wedge dt_L^{(2)} \wedge \cdots \wedge dt_1^{(m)} \wedge \cdots \wedge dt_L^{(m)}. \]

**Theorem 4.3.** If \( \kappa_0 - \sum_{i=1}^N \theta_i = M \) and \( \kappa_n = 1 \) \( (2 \leq n \leq L - 1) \), then the integral formula \( \Psi_M(q, z) \) is a solution to the Schrödinger system (3.2), with

\[ \alpha_n = e_{n+1} - e_n + 1, \quad \beta_i = -\theta_i, \quad \gamma = \kappa_1 + M - 1, \]

for \( 1 \leq n \leq L - 1 \) and \( 1 \leq i \leq N \), where \( e_L = e_0. \)

For \( A \in \mathcal{A}_M \), let \( \varphi_A(t) \) be the rational \((L - 1)M\)-form holomorphic outside \( D \) defined by

\[ \varphi_A(t) = \text{Sym} \left[ (-1)^{M-A_0} \begin{pmatrix} M \\ A \end{pmatrix} \prod_{i=1}^N \prod_{n=1}^{L-1} S_n^{(i)} \prod_{a=M-A_0+1}^M f_n^{(i)}(t^{(a)}) \prod_{a=M-A_0+1}^M f_0^{(i)}(t^{(a)}) \right] dt, \]

where

\[ A_0 = M - \sum_{i \leq N, \ 1 \leq n \leq L - 1} A_{n,i}, \quad \begin{pmatrix} M \\ A \end{pmatrix} = \frac{M!}{A_0!} \prod_{1 \leq n \leq L - 1} A_{n,j}!, \quad S_n^{(i)} = \sum_{j=1}^L \sum_{m=1}^n A_{m,j} + \sum_{m=1}^n A_{m,i}. \]

Then, the integral formula is expressed as

\[ \Psi_M(q, z) = \sum_{A \in \mathcal{A}_M} q^A \int_{\Delta} U(t) \varphi_A(t). \]

Since, in general, it holds for an \((L - 1)M\)-form \( \varphi \) that

\[ \frac{\partial}{\partial z_i} \int_{\Delta} U(t) \varphi = \int_{\Delta} U \left( \frac{1}{U} \frac{\partial U}{\partial z_i} \varphi + \frac{\partial \varphi}{\partial z_i} \right), \]

let a linear operator \( \nabla_i \) \( (i = 1, \ldots, N) \) acting on \( \varphi \) be defined as

\[ \nabla_i \varphi = \frac{1}{U} \frac{\partial U}{\partial z_i} \varphi + \frac{\partial \varphi}{\partial z_i}. \]

Let us explain our proof of Theorem 4.3 briefly. We compute \( \kappa \nabla \varphi_A(t) \) and obtain the linear Pfaffian system for \( \{ \int_{\Delta} U(t) \varphi_A(t) | A \in \mathcal{A}_M \} \). While we compute the action of the Hamiltonians \( H_i \) on \( q^A \) and obtain the coefficient of \( q^A \) of \( H_i \Psi_M(q, z) \) as a linear combination of elements of \( \{ \int_{\Delta} U(t) \varphi_A(t) | A \in \mathcal{A}_M \} \). Finally, comparing both results, we obtain Theorem 4.3.

**A proof of Theorem 4.3**

Fix \( i \in \{1, \ldots, N\} \) and \( A \in \mathcal{A}_M \). We compute \( \nabla_i \varphi_A(t) \) as follows. First, we have

\[ \kappa \nabla \varphi_A(t) = \text{Sym} \left[ \left( \begin{array}{c} \beta_i \sum_{j \neq i}^{n_{L-1}} \sum_{n=1}^{L-1} A_{n,j} \frac{S_n^{(i)}}{1 - z_i S_n^{(i)}} + \beta_i \frac{A_{n,i} (M-A_0+1)}{1 - z_i (M-A_0+1)} + (\beta_i + \kappa) \sum_{m=1}^{L-1} A_{n,m} \frac{S_n^{(i)}}{1 - z_i S_n^{(i)}} \end{array} \right) \varphi_A(t) \right] dt, \]

where \( \varphi_A(t) \) is defined by

\[ \varphi_A(t) = (-1)^{M-A_0} \begin{pmatrix} M \\ A \end{pmatrix} \prod_{i=1}^N \prod_{n=1}^{L-1} S_n^{(i)} \prod_{a=M-A_0+1}^M f_n^{(i)}(t^{(a)}) \prod_{a=M-A_0+1}^M f_0^{(i)}(t^{(a)}) \]
Using a relation
\[
\frac{t}{1 - z_i t} = \frac{1 - z_j t}{z_i - z_j} \left( \frac{1}{1 - z_i t} - \frac{1}{1 - z_j t} \right),
\]
we get

the first term of (4.3) \( = \beta_i \sum_{j=1}^{N} \sum_{n=1}^{L-1} \frac{1}{z_i - z_j} \left( (A_{n,i} + 1) \varphi_{(A_{n-1,i},A_{n,i+1})}(t) - A_{n,j} \varphi_A(t) \right) \),

where \( \varphi_{(A_{n-1,i},A_{n,i+1})}(t) \) is the rational \((L - 1)M\)-form defined for the matrix in \( \mathcal{A}_M \) whose \((n, j)\) entry is \( A_{n,j} - 1 \) and \((n, i)\) entry is \( A_{n,i} + 1 \), and the other \((m, k)\) entries are \( A_{m,k} \).

As for the second term of (4.3), using a relation
\[
\frac{t_{L-1}}{1 - z_i t_{L-1}} = \frac{1}{(z_i - 1) f_0(t)} \left( -f_0(t) + \sum_{n=1}^{L-1} f_n^{(i)}(t) \right)
\]
we obtain

the second term of (4.3) \( = -\beta_i \frac{z_i}{z_i - 1} \left( A_0 \varphi_A(t) + \sum_{n=1}^{L-1} (A_{n,i} + 1) \varphi_{(A_{n,i+1})}(t) \right) \),

where \( \varphi_{(A_{n,i+1})}(t) \) is the rational \((L - 1)M\)-form defined for the matrix in \( \mathcal{A}_M \) whose \((n, i)\) entry is \( A_{n,i} + 1 \), and the other \((m, k)\) entries are \( A_{m,k} \).

In order to calculate the third term of (4.3), we compute coboundaries \( X_n^{(i)} \) \((n = 1, \ldots, L - 1)\) defined by
\[
X_n^{(i)} = \kappa \sum_{m=n}^{L-1} \sum_{\sigma \in \mathcal{S}_m^{(i)}} \left( \sum_{\gamma \in \mathcal{B}_m^{(i)}} \varphi_{(A_{n,i})} \right) \ast dt_m^{(\sigma n(S^{(i)}_m))},
\]
where \( \ast dt_m^{(a)} \) is defined by
\[
\ast dt_m^{(a)} = ( -1 )^{(L-1)(a-1)+m-1} dt_1^{(1)} \wedge \cdots \wedge dt_m^{(a)} \wedge \cdots \wedge t_{L-1}^{(M)}
\]
so that \( dt_m^{(a)} \wedge \ast dt_m^{(a)} = dt \).

For \( m \neq n \), denote by \( \varphi_{(A_{n-1,i},A_{n,i+1})}(t) \), the rational \((L - 1)M\)-form defined for the matrix in \( \mathcal{A}_M \) whose \((n, i)\) entry is \( A_{n,i} - 1 \) and \((m, i)\) entry is \( A_{m,i} + 1 \), and the other \((l, k)\) entries are \( A_{l,k} \), and denote by \( \varphi_{(A_{n-1,i})}(t) \), the rational \((L - 1)M\)-form defined for the matrix in \( \mathcal{A}_M \) whose \((n, i)\) entry is \( A_{n,i} - 1 \), and the other \((l, k)\) entries are \( A_{l,k} \).

Using the relations (4.4) and (4.5), we obtain by straightforward calculations
\[
X_n^{(i)} = \text{Sym} \left[ \left( \beta_i + \kappa \right) \frac{t_{L-1}^{(S^{(i)}_m)}}{1 - z_i t_{L-1}} \varphi_A(t) \right] dt
\]
\[+ \left( \sum_{m=n}^{L-1} \alpha_m - (L - 1 - n) \right) \varphi_A(t) + (1 + \delta_{n,1}(\gamma - 1)) \sum_{m=n+1}^{L-1} \frac{A_{m,i} + 1}{A_{n,i}} \varphi_{(A_{n,i+1},A_{n,i-1})}(t) \]

where \( \delta_{n,1} \) is the Kronecker delta.
where

\[
Y_n^{(i)} = \text{Sym} \left[ \sum_{m=n}^{L-1} \left( S_n^{(i)} W_{n,m} \varphi_A(t) \right) dt, (4.7) \right.
\]

with

\[
W_{n,m}^{(i)} = \sum_{a=1, a \neq i}^{M} \left( \frac{-1}{t_m^{(a)}} - \frac{2}{t_m^{(a)}} - \frac{1}{t_m^{(a)}} + \frac{1}{t_m^{(a)}} - 1 \right).
\]

We compute \( Y_n^{(i)} \) in Lemmas 4.4, 4.5, 4.6 and 4.7. Owing to those lemmas, we obtain

\[
\kappa \bar{z} \nabla \varphi_A(t) - \sum_{n=1}^{L-1} A_n X_n^{(i)} = \left\{ - \sum_{n=1}^{L-1} A_{n,i} \left( \sum_{m=1}^{L-1} \alpha_m + L - n - \beta_i + \sum_{m=1}^{n} A_{m,i} \right) \\
+ \frac{1}{\kappa \bar{z}_i - 1} \left[ A_0 \left( \sum_{n=1}^{L-1} A_{n,i} - \beta_i \right) - \sum_{j=1}^{N} \sum_{n=1}^{L-1} A_{n,j} A_{n,j} + A_{1,j}(M - \gamma) \right] \\
+ \sum_{j=1}^{N} \frac{z_j}{\kappa \bar{z}_i - 1} \sum_{n=1}^{L-1} \left( A_{n,i} \left( \sum_{m=1}^{L-1} A_{m,j} + A_{n,j} - \beta_j \right) - \beta_i A_{n,j} \right) \right\} \varphi_A(t) \\
- \frac{A_0 + 1}{\kappa \bar{z}_i - 1} \sum_{n=1}^{L-1} \left( \sum_{j=1}^{N} A_{n,j} + \delta_n \gamma - M \right) \varphi_{(A_{n,j}-1)}(t) + \frac{z_j}{\kappa \bar{z}_i - 1} \sum_{n=1}^{L-1} (A_{m,i} - \beta_i) \sum_{m=1}^{n} (A_{m,i} + 1) \varphi_{(A_{m,i}+1)}(t) \\
- \frac{1}{\kappa \bar{z}_i - 1} \sum_{n=1}^{L-1} \left( \sum_{j=1}^{N} A_{n,j} + \delta_n \gamma - M \right) \left( \sum_{m=1}^{n} (A_{m,i} + 1) \varphi_{(A_{m,i}+1,A_{n,j}-1)}(t) + z_i \sum_{m=n+1}^{L-1} (A_{m,i} + 1) \varphi_{(A_{m,i}+1,A_{n,j}-1)}(t) \right) \\
+ \sum_{j=1}^{N} A_{n,j} + 1 \left( z_j \sum_{m=1}^{n} (A_{m,i} + 1) \varphi_{(A_{m,i}+1,A_{n,j}+1)}(t) + z_i \sum_{m=n+1}^{L-1} (A_{m,i} + 1) \varphi_{(A_{m,i}+1,A_{n,j}+1)}(t) \right) \\
+ \sum_{j=1}^{N} \frac{z_j}{\kappa \bar{z}_i - 1} \left( \beta_i - \sum_{m=1}^{L-1} A_{m,j} \right) \sum_{n=1}^{L-1} (A_{n,j} + 1) \varphi_{(A_{n,j}+1)}(t) \\
+ \sum_{j=1}^{N} \frac{z_j}{\kappa \bar{z}_i - 1} \left( \beta_i - \sum_{m=1}^{L-1} A_{m,j} \right) \sum_{n=1}^{L-1} (A_{n,j} + 1) \varphi_{(A_{n,j}+1)}(t), \quad (4.8)
\]

where the rational \((L - 1)M\)-form \( \varphi_{(A_{m,j}-1,A_{n,i}+1,A_{n,i}-1,A_{n,j}+1)}(t) \) be defined for the matrix in \( \mathcal{A}_M \) whose \((m, j), (m, i), (n, i), (n, j)\) entries are \( A_{m,j} - 1, A_{m,i} + 1, A_{n,i} - 1, \) and \( A_{n,j} + 1, \) respectively,
and the other \((l,k)\) entries are \(A_{l,k}\). Hence, for \(A \in \mathcal{A}_M\), as an element in the twisted de Rham cohomology group \(H^{(L-1)M}(T, \nabla)\), \(\kappa \nabla_{i} \varphi_A(t)\) is expressed in terms of elements of \(\{\varphi_B(t)\}B \in \mathcal{A}_M\).

On the other hand, computations of the action of the Hamiltonian \(H_i\) on \(q^A\) for \(A \in \mathcal{A}_M\) is straightforward and it is easy to see that the coefficient of \(q^A\) of \(H_i \psi_M(q, z)\) is equal to the hypergeometric pairing between the cycle \(\Delta \in H^{(L-1)M}(T, \mathcal{S}^\ast)\) and the right hand side of (4.8). Therefore, we complete our proof. \(\square\)

4.3. **Lemmas.** Through lemmas below, fix \(1 \leq n \leq L - 1, 1 \leq i \leq N\) and \(A \in \mathcal{A}_M\). For a triple \((n, i, A)\), the coboundary \(X^{(1)}_n\) is defined by (4.6) and expressed as a linear combination of elements in \(\{\varphi_B(t)\}B \in \mathcal{A}_M\), and \(Y^{(1)}_n\). In this subsection, we compute \(Y^{(1)}_n\), so that we show that they are also expressed as a linear combination of elements in \(\{\varphi_B(t)\}B \in \mathcal{A}_M\).

We divide \(Y^{(1)}_n\) as

\[
Y^{(1)}_n = \sum_{1 \leq j \leq N} \left( Y^{(1)}_{n,j} + Y^{(1)}_{n,0} \right)
\]

and we compute \(Y^{(1)}_{n,j}\) and \(Y^{(1)}_{n,0}\), where for \(l \neq n\) or \(j \neq i\),

\[
\begin{align*}
Y^{(1)}_{n,j} &= \text{Sym} \left[ A_{l,j} C \left( n, S^{(1)}_n, S^{(1)}_l \right) \bar{\varphi}_A(t) \right] dt, \\
Y^{(1)}_{n,j} &= \text{Sym} \left[ (A_{n,j} - 1) C \left( n, S^{(1)}_n, S^{(1)}_n - 1 \right) \bar{\varphi}_A(t) \right] dt, \\
Y^{(1)}_{n,0} &= \text{Sym} \left[ A_0 C \left( n, S^{(1)}_n, M - A_0 + 1 \right) \bar{\varphi}_A(t) \right] dt.
\end{align*}
\]

Here, for \(1 \leq a \neq b \leq M\),

\[
C(n, a, b) = \sum_{m=n}^{L-1} t^{(a)}_m \left( \frac{-1}{t^{(a)}_m - t^{(b)}_{m-1}} + \frac{2}{t^{(a)}_m - t^{(b)}{m+1}} - \frac{-1}{t^{(a)}_m - t^{(b)}_{m+1}} \right) + \delta_{n,1} \frac{j^{(a)}_1}{f^{(a)}_1 - 1}.
\]

Let the rational functions \(f^{(a)}_{l,m}(t^{(a)})\) be defined by

\[
f^{(a)}_{l,m}(t^{(a)}) = \frac{1}{1 - z^{(a)}_l} \sum_{k=m}^{L-1} \frac{1}{t^{(a)}_k - t^{(a)}_m}.
\]

**Lemma 4.4.** When \(1 \leq l < n\), for \(1 \leq j \neq i \leq N\), we have

\[
\begin{align*}
\left( Y^{(1)}_{n,j} \right)_{n,j} &= \left( A_{l,j} + 1 \right) \varphi \left( A_{l,j-1}, A_{l,j-1} \right) (t) + \frac{z_j}{z_i - z_j} \left( \left( A_{l,j} + 1 \right) \varphi \left( A_{l,j-1}, A_{l,j-1} \right) (t) \right. \\
&\quad - A_{l,j} \varphi_A(t) - A_{l,j} \left( A_{n,j} + 1 \right) \varphi \left( A_{n,j-1}, A_{n,j-1} \right) (t) + \frac{A_{n,j} + 1}{A_{n,j}} \frac{A_{l,j}}{A_{n,j}} \varphi \left( A_{n,j}, A_{n,j} \right) (t),
\end{align*}
\]

and for \(j = i\), we have

\[
\left( Y^{(1)}_{n,i} \right)_{n,i} = A_{l,i} \varphi_A(t).
\]

**Proof.** It suffices to show that

\[
\text{Sym} \left[ C(n, 1, 2) \frac{1}{f^{(1)}_{L-1}} f^{(2)}_{L-1} f^{(3)}(t^{(1)}) \frac{1}{f^{(3)}_{L-1}} \right]
\]
where the symmetrization \( \text{Sym} \{ f(t) \} \) stands for \( \sum_{i \in \mathbb{Z}^+} \sigma(f(t)) \) (see (4.2)), the rational functions \( f_n^{(i)}(t^{(a)}) \) are defined in Definition [4.2] and if \( j = i \), then we understand that the second line of the right hand side of (4.9) is vanished.

Firstly, we claim that for \( n \leq k \leq L - 2 \), we have

\[
\text{Sym} \left[ \sum_{m=n}^k \frac{t_m^{(1)}}{t_m^{(1)} - t_m^{(2)}} \left( \frac{-1}{t_m^{(1)} - t_m^{(2)}} + \frac{2}{t_m^{(1)} - t_m^{(2)}} + \frac{-1}{t_m^{(1)} - t_m^{(2)}} \right) \frac{1}{t_{L-1}} f_n^{(i)}(t^{(1)}) \frac{1}{t_{L-1}} f_l^{(j)}(t^{(2)}) \right]
\]

(4.10)

We show (4.10) by induction. Let \( k = n \), then, we have

\[
\text{Sym} \left[ \frac{1}{t_n^{(1)} - t_n^{(2)}} + \frac{1}{t_n^{(1)} - t_n^{(2)}} \frac{t_n^{(1)}}{t_n^{(1)} - t_n^{(2)}} f_n^{(i)}(t^{(1)}) \frac{1}{t_{L-1}} f_l^{(j)}(t^{(2)}) \right]
\]

(4.11)

where in the last line, we interchange \( t_n^{(1)} \) with \( t_n^{(2)} \) and

\[
\text{Sym} \left[ \frac{1}{t_n^{(1)} - t_n^{(2)}} + \frac{1}{t_n^{(1)} - t_n^{(2)}} \frac{t_n^{(1)}}{t_n^{(1)} - t_n^{(2)}} f_n^{(i)}(t^{(1)}) \frac{1}{t_{L-1}} f_l^{(j)}(t^{(2)}) \right]
\]

(4.12)

Thus, the left hand side of (4.10) for \( k = n \), that is, (4.11) plus (4.12), becomes the right hand side of (4.10) for \( k = n \).

Suppose (4.10) holds for \( k - 1 \), then

\[
\text{Sym} \left[ \left( \frac{-1}{t_{k-1}^{(1)} - t_{k-1}^{(2)}} + \frac{1}{t_{k-1}^{(1)} - t_{k-1}^{(2)}} \right) + \sum_{m=n}^{k-1} \frac{t_m^{(1)}}{t_m^{(1)} - t_m^{(2)}} \left( \frac{-1}{t_m^{(1)} - t_m^{(2)}} + \frac{2}{t_m^{(1)} - t_m^{(2)}} + \frac{-1}{t_m^{(1)} - t_m^{(2)}} \right) \frac{1}{t_{L-1}} f_n^{(i)}(t^{(1)}) \frac{1}{t_{L-1}} f_l^{(j)}(t^{(2)}) \right]
\]

(4.13)

\[
\text{Sym} \left[ \frac{1}{t_k^{(1)} - t_k^{(2)}} \frac{t_k^{(1)}}{t_k^{(1)} - t_k^{(2)}} f_n^{(i)}(t^{(1)}) \frac{1}{t_{L-1}} f_l^{(j)}(t^{(2)}) \right]
\]

(4.14)

\[
\text{Sym} \left[ \frac{1}{t_k^{(1)} - t_k^{(2)}} \frac{1}{t_{k+1}^{(1)} - t_{k+1}^{(2)}} f_n^{(i)}(t^{(1)}) \frac{1}{t_{L-1}} f_l^{(j)}(t^{(2)}) \right]
\]

(4.15)
where in the last line, we interchange \( t_i^{(1)} \) with \( t_k^{(2)} \). Since

\[
\text{Sym} \left[ \left( \frac{1}{t_k^{(1)} - t_k^{(2)}} + \frac{-1}{t_k^{(1)} - t_k^{(2)}} \right) \frac{t_i^{(1)}}{t_i^{(1)} - t_{k+1}^{(2)}} f_n^{(i)}(t^{(1)}) \frac{1}{t_i^{(1)} - t_{k+1}^{(2)}} f_i^{(j)}(t^{(2)}) \right]
= \text{Sym} \left[ \frac{1}{t_i^{(1)} - t_i^{(2)}} \frac{1}{t_i^{(1)} - t_i^{(2)}} f_n^{(i)}(t^{(1)}) \frac{1}{t_i^{(1)} - t_i^{(2)}} f_i^{(j)}(t^{(2)}) \right],
\]

the left hand side of (4.10) for \( k \) becomes the right hand side of (4.10) for \( k \).

Secondly, using (4.10) for \( k = L - 2 \), we have

\[
\text{Sym} \left[ \left( \frac{-1}{t_L^{(1)} - t_L^{(2)}} + \frac{1}{t_L^{(1)} - t_L^{(2)}} \right) + \sum_{m=n}^{L-2} \left( \frac{-1}{t_m^{(1)} - t_m^{(2)}} + \frac{2}{t_m^{(1)} - t_m^{(2)}} + \frac{-1}{t_m^{(1)} - t_m^{(2)}} \right) \frac{1}{t_L^{(1)} - t_L^{(2)}} f_n^{(i)}(t^{(1)}) \frac{1}{t_L^{(1)} - t_L^{(2)}} f_i^{(j)}(t^{(2)}) \right]
= \text{Sym} \left[ \frac{-1}{t_L^{(1)} - t_L^{(2)}} f_n^{(i)}(t^{(1)}) \frac{1}{t_L^{(1)} - t_L^{(2)}} f_i^{(j)}(t^{(2)}) \right],
\]

where in the last line, we interchange \( t_L^{(1)} \) with \( t_i^{(2)} \). Hence, the left hand side of (4.9) is equal to

\[
\text{Sym} \left[ \frac{-1}{t_L^{(1)} - t_L^{(2)}} f_n^{(i)}(t^{(1)}) \frac{1}{t_L^{(1)} - t_L^{(2)}} f_i^{(j)}(t^{(2)}) + \frac{1}{t_L^{(1)} - t_L^{(2)}} f_n^{(i)}(t^{(1)}) \frac{1}{t_L^{(1)} - t_L^{(2)}} f_i^{(j)}(t^{(2)}) \right]
= \text{Sym} \left[ \frac{1}{t_L^{(1)} - t_L^{(2)}} f_n^{(i)}(t^{(1)}) f_i^{(j)}(t^{(2)}) \frac{1-z(t_L^{(1)} + t_i^{(2)}) + z_i t_i^{(2)}}{(1-z t_L^{(1)})(1-z t_i^{(2)})} \right].
\]

Therefore, the relation (4.9) holds.

\[ \square \]

**Lemma 4.5.** When \( n < l \leq L - 1 \), for \( 1 \leq j \neq i \leq N \), we have

\[
\left( Y_{n}^{(i)} \right)_{l,j} = -A_{l,j} \left( \frac{\delta_{n,1}}{z_i - 1} + \frac{z_j}{z_i - z_j} \right) \varphi_A(t) + (A_{l,j} + 1) \frac{z_j}{z_i - z_j} \varphi(A_{l,j-1},A_{l,j+1}) (t) - \left( A_{l,j} + 1 \right) \frac{z_i}{A_{n,i}} \varphi(A_{n,j-1}) (t) + \left( A_{n,j} + 1 \right) \frac{z_j}{z_i - z_j} \varphi(A_{n,j+1},A_{n,j}) (t) - \frac{A_{l,j}}{A_{n,i}} \left( \left( A_0 + 1 \right) \varphi(A_{n,j-1}) (t) + z_i \sum_{m=2}^{L-1} \left( A_{m,j} + 1 \right) \varphi(A_{n,j+1},A_{m,j-1}) (t) \right),
\]

and for \( j = i \), we have

\[
\left( Y_{n}^{(i)} \right)_{l,i} = -\frac{\delta_{n,1} A_{l,i}}{z_i - 1} \left( \varphi_A(t) + \left( A_0 + 1 \right) \frac{z_i}{A_{1,i}} \varphi(A_{l,i-1}) (t) + z_i \sum_{m=2}^{L-1} \left( A_{m,i} + 1 \right) \frac{z_j}{A_{1,i}} \varphi(A_{m,i+1},A_{m,i-1}) (t) \right).
\]

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Proof. It suffices to show that for \( n \geq 2 \),

\[
\text{Sym} \left[ C(n, 1, 2) \frac{1}{t_{L-1}^{(1)}} f_n^{(0)}(t^{(1)}) \frac{1}{t_{L-1}^{(2)}} f_l^{(0)}(t^{(2)}) \right]
\]

\[
= \frac{1}{z_l - z_j} \text{Sym} \left[ \frac{1}{t_{L-1}^{(1)}} \frac{1}{t_{L-1}^{(2)}} \left( f_n^{(0)}(t^{(1)}) - f_l^{(0)}(t^{(1)}) \right) \left( z_l f_n^{(0)}(t^{(2)}) - z_j f_l^{(0)}(t^{(2)}) \right) \right],
\]

and for \( n = 1 \),

\[
\text{Sym} \left[ C(1, 1, 2) \frac{1}{t_{L-1}^{(1)}} f_1^{(0)}(t^{(1)}) \frac{1}{t_{L-1}^{(2)}} f_l^{(0)}(t^{(2)}) \right]
\]

\[
= \frac{1}{z_l - z_j} \text{Sym} \left[ \frac{1}{t_{L-1}^{(1)}} \frac{1}{t_{L-1}^{(2)}} \left( f_1^{(0)}(t^{(1)}) - f_l^{(0)}(t^{(1)}) \right) \left( z_l f_1^{(0)}(t^{(2)}) - z_j f_l^{(0)}(t^{(2)}) \right) \right],
\]

where the symmetrization \( \text{Sym}[f(t)] \) stands for \( \sum_{\sigma \in \mathbb{Z}_L^*} \sigma(f(t)) \) (see (4.2)), and if \( j = i \), then we understand that the right hand side of (4.13) and the first line of the right hand side of (4.14) is zero.

We shall show (4.13). Firstly, using (4.10) for \( k = l - 2 \), we have

\[
\text{Sym} \left[ \sum_{m=1}^{L-2} t_m \left( \frac{-1}{t_m^{(1)} - t_m^{(2)}} + \frac{2}{t_m^{(1)} - t_m^{(2)}} + \frac{-1}{t_m^{(1)} - t_m^{(2)}} \right) + t_{L-1} \left( \frac{-1}{t_{L-1}^{(1)} - t_{L-1}^{(2)}} + \frac{1}{t_{L-1}^{(1)} - t_{L-1}^{(2)}} \right) \right] \frac{1}{t_{L-1}^{(1)}} f_n^{(0)}(t^{(1)}) \frac{1}{t_{L-1}^{(2)}} f_l^{(0)}(t^{(2)})
\]

\[
= \text{Sym} \left[ \frac{1}{t_{l-1}^{(1)} - t_{l-1}^{(2)}} \frac{1}{t_{l-1}^{(2)} - t_{l-1}^{(2)}} \frac{1}{t_{L-1}^{(1)} - t_{L-1}^{(2)}} f_n^{(0)}(t^{(1)}) \frac{1}{t_{L-1}^{(2)}} f_l^{(0)}(t^{(2)}) \right],
\]

where in the last line, we interchange \( t_{l-1}^{(1)} \) with \( t_{l-1}^{(2)} \). Thus, (4.15) is equal to

\[
-\text{Sym} \left[ \frac{t_{l-1}^{(1)}}{t_{l-1}^{(1)} - t_{l-1}^{(2)}} \frac{-t_{l-1}^{(1)}}{t_{l-1}^{(1)} - t_{l-1}^{(2)}} \frac{1}{t_{L-1}^{(1)}} f_n^{(0)}(t^{(1)}) \frac{1}{t_{L-1}^{(2)}} f_l^{(0)}(t^{(2)}) \right].
\]

Secondly, we claim that for \( l \leq k \leq L - 2 \), we have

\[
\text{Sym} \left[ \sum_{m=1}^{k} \left( \frac{-t_{m+1}^{(1)}}{t_{m+1}^{(1)} - t_m^{(2)}} + \frac{2t_m^{(1)}}{t_m^{(1)} - t_m^{(2)}} + \frac{-t_{m-1}^{(1)}}{t_{m-1}^{(1)} - t_m^{(2)}} \right) \frac{1}{t_{L-1}^{(1)}} f_n^{(0)}(t^{(1)}) \frac{1}{t_{L-1}^{(2)}} f_l^{(0)}(t^{(2)}) \right]
\]

\[
= \text{Sym} \left[ \frac{1}{(t_k^{(1)} - k_{k+1})} \frac{1}{(t_k^{(2)} - t_{k+1})} \frac{1}{t_{L-1}^{(1)}} f_{n,k+1}^{(0)}(t^{(1)}) \frac{1}{t_{L-1}^{(2)}} f_{l,k+1}^{(0)}(t^{(2)}) \right].
\]

We can prove (4.16) by induction and omit the proof of this claim.
Using (4.16) for $k = L - 2$, we have
\begin{align*}
\text{Sym} & \left[ \frac{t_{L-1}^{(1)}}{t_{L-1}^{(1)} - t_{L-1}^{(2)}} + \frac{-t_{L-2}^{(1)}}{t_{L-2}^{(1)} - t_{L-1}^{(2)}} + \sum_{m=1}^{L-2} \left( \frac{-t_{m+1}^{(1)}}{t_{m+1}^{(1)} - t_{m}^{(2)}} + \frac{2t_{m}^{(1)}}{t_{m}^{(1)} - t_{m}^{(2)}} + \frac{-t_{m-1}^{(1)}}{t_{m-1}^{(1)} - t_{m}^{(2)}} \right) \right] \frac{1}{t_{L-1}^{(1)}} f_n^{(1)}(t^{(1)}) \frac{1}{t_{L-1}^{(2)}} f_n^{(1)}(t^{(2)}) \\
&= \text{Sym} \left[ \frac{1}{t_{L-1}^{(1)} - t_{L-1}^{(2)}} f_n^{(1)}(t^{(1)}) f_n^{(1)}(t^{(2)}) \right] \\
&= \text{Sym} \left[ \frac{-1}{t_{L-1}^{(1)} - t_{L-1}^{(2)}} f_n^{(1)}(t^{(1)}) f_n^{(1)}(t^{(2)}) \right],
\end{align*}
where in the last line, we interchange $t_{L-1}^{(1)}$ with $t_{L-1}^{(2)}$. Hence, the left hand side of (4.13) is equal to
\begin{align*}
\text{Sym} & \left[ \frac{-1}{t_{L-1}^{(1)} - t_{L-1}^{(2)}} f_n^{(1)}(t^{(1)}) f_n^{(1)}(t^{(2)}) \right] \\
&= (z_i - z_j) \text{Sym} \left[ \frac{1}{1 - z_j t_{L-1}^{(1)}} f_n^{(1)}(t^{(1)}) \frac{1}{t_{L-1}^{(2)}(1 - z_i t_{L-1}^{(2)})} f_n^{(1)}(t^{(2)}) \right].
\end{align*}
Therefore, the relation (4.13) holds.

We shall show (4.14). We compute the left hand side of (4.14) as follows.
\begin{align*}
\text{L.H.S. of (4.14)} = & \text{Sym} \left[ \left( \frac{-t_{L-1}^{(1)}}{t_{L-1}^{(1)} - 1} + C(1, 1, 2) \right) \frac{1}{t_{L-1}^{(1)}} f_n^{(1)}(t^{(1)}) \frac{1}{t_{L-1}^{(2)}} f_n^{(1)}(t^{(2)}) \right] \\
&+ \text{Sym} \left[ \frac{t_{L-1}^{(1)}}{t_{L-1}^{(1)} - 1} f_n^{(1)}(t^{(1)}) \frac{1}{t_{L-1}^{(2)}} f_n^{(1)}(t^{(2)}) \right].
\end{align*}
The first line of the right hand side of the relation above becomes the first line of the right hand side of (4.14) in the same way of the proof of (4.13). While, we have
\begin{align*}
\text{Sym} & \left[ \frac{1}{t_{L-1}^{(1)} - 1} f_n^{(1)}(t^{(1)}) \frac{1}{t_{L-1}^{(2)}} f_n^{(1)}(t^{(2)}) \right] \\
&= \text{Sym} \left[ \frac{t_{L-1}^{(1)}}{t_{L-1}^{(1)} + \sum_{m=2}^{L-1} t_{m-1}^{(1)} - t_{m}^{(1)}} \frac{1}{t_{L-1}^{(1)}} f_n^{(1)}(t^{(1)}) \frac{1}{t_{L-1}^{(2)}} f_n^{(1)}(t^{(2)}) \right] \\
&= \text{Sym} \left[ \frac{1}{z_i - 1} \frac{1}{t_{L-1}^{(1)}} \left( f_0(t^{(1)}) - f_1(t^{(1)}) - z_i \sum_{m=2}^{L-1} f_n^{(1)}(t^{(1)}) \right) \frac{1}{t_{L-1}^{(2)}} f_n^{(1)}(t^{(2)}) \right].
\end{align*}
Therefore, the relation (4.14) holds. \hfill \Box

**Lemma 4.6.** For $1 \leq j \neq i \leq L - 1$, we have
\begin{align*}
\left( T_n^{(i)} \right)_{n,j} = & A_{n,j} \left( \frac{1 - \delta_{n,1}}{z_i - 1} - \frac{2z_j}{z_i - z_j} \right) \varphi_A(t) + \frac{1 - \delta_{n,1}}{z_i - 1} A_{n,i} (A_0 + 1) A_{n,j} \varphi (A_{n-1})(t) \\
&+ \frac{1 - \delta_{n,1}}{z_i - 1} A_{n,j} \sum_{m=1}^{n-1} (A_{m,i} + 1) \varphi (A_{m+1}, A_{n-1})(t) + z_i \sum_{m=n+1}^{L-1} (A_{m,i} + 1) \varphi (A_{m+1}, A_{n-1})(t)
\end{align*}
\[ + (A_{n,j} + 1) \frac{z_i}{z_i - z_j} \varphi(A_{n,j-1,A_{n,j}+1})(t) + \frac{(A_{n,j} + 1)A_{n,j}}{z_i - z_j} \varphi(A_{n,j+1,A_{n,j}-1})(t), \]

and for \( j = i \), we have

\[ (y^{(i)}_n)_{n,i} = (A_{n,i} - 1) \left( \frac{z_i - \delta_{n,1}}{z_i - 1} \right) \varphi_A(t) + \frac{1 - \delta_{n,1} (A_0 + 1)(A_{n,i} - 1)}{z_i - 1} \varphi_{A_{n,i}}(t) \]

\[ + \frac{1 - \delta_{n,1} A_{n,i} - 1}{A_{n,i}} \sum_{m=1}^{n-1} (A_{m,i} + 1) \varphi_{A_{m,i+1,A_{m,i}-1}}(t) + z_i \sum_{m=n+1}^{L-1} (A_{m,i} + 1) \varphi_{A_{m,i+1,A_{m,i}-1}}(t). \]

Proof. It suffices to show that

\[ \text{Sym} \left[ C(n, 1, 2) \frac{1}{t_{L-1}^{(1)}} f_n^{(0)}(t^{(1)}) \frac{1}{t_{L-1}^{(2)}} f_n^{(0)}(t^{(2)}) \right] \]

\[ = \frac{1 - \delta_{n,1}}{z_i - 1} \text{Sym} \left[ \frac{1}{t_{L-1}^{(1)}} \left( -f_0(t^{(1)}) + \sum_{m=1}^{n} f_m^{(0)}(t^{(1)}) + z_i \sum_{m=n+1}^{L-1} f_m^{(0)}(t^{(1)}) \right) \frac{1}{t_{L-1}^{(2)}} f_n^{(0)}(t^{(2)}) \right] \]

\[ + \frac{1}{z_i} \text{Sym} \left[ \frac{1}{t_{L-1}^{(1)}} \frac{1}{t_{L-1}^{(2)}} \left( f_m^{(0)}(t^{(1)}) - f_n^{(0)}(t^{(1)}) \right) \left( f_m^{(0)}(t^{(2)}) - f_n^{(0)}(t^{(2)}) \right) \right]. \]

(4.17)

where the symmetrization \( \text{Sym}[f(t)] \) stands for \( \sum_{\sigma \in S_{L-1}} \sigma(f(t)) \) (see (4.2)), and if \( j = i \), then we understand that the third line of the right hand side of (4.17) is vanished.

Firstly, we have

\[ \text{Sym} \left[ \frac{-f_n^{(1)}}{t_{L-1}^{(1)} - t_{L-1}^{(2)}} f_n^{(0)}(t^{(1)}) \frac{1}{t_{L-1}^{(2)}} f_n^{(0)}(t^{(2)}) \right] = \text{Sym} \left[ \left( \frac{1}{t_{L-1}^{(1)} - t_{L-1}^{(1)}} + \sum_{m=1}^{L-1} \frac{1}{t_{L-1}^{(1)} - t_{L-1}^{(1)}} \right) \frac{1}{t_{L-1}^{(2)}} f_n^{(0)}(t^{(1)}) \frac{1}{t_{L-1}^{(2)}} f_n^{(0)}(t^{(2)}) \right] = \frac{1}{z_i - 1} \text{Sym} \left[ \frac{1}{t_{L-1}^{(1)}} \left( -f_0(t^{(1)}) + \sum_{m=1}^{n} f_m^{(0)}(t^{(1)}) + z_i \sum_{m=n+1}^{L-1} f_m^{(0)}(t^{(1)}) \right) \frac{1}{t_{L-1}^{(2)}} f_n^{(0)}(t^{(2)}) \right]. \]

Secondly, we notice that for \( n \leq m \leq L - 2 \), we have

\[ \text{Sym} \left[ \left( \frac{-f_{m+1}^{(1)}}{t_{m+1}^{(1)} - t_{m+1}^{(2)}} + \frac{2t_m^{(1)}}{t_{m-1}^{(1)} - t_m^{(2)}} + \frac{-t_m^{(1)}}{t_{m-1}^{(1)} - t_m^{(2)}} \right) \frac{1}{t_{L-1}^{(1)}} f_n^{(0)}(t^{(1)}) \frac{1}{t_{L-1}^{(2)}} f_n^{(0)}(t^{(2)}) \right] = 0. \]

Thirdly, we compute the remaining term as follows.

\[ \text{Sym} \left[ \frac{2}{t_{L-1}^{(1)} - t_{L-1}^{(2)}} f_n^{(0)}(t^{(1)}) \frac{1}{t_{L-1}^{(2)}} f_n^{(0)}(t^{(2)}) \right] \]

\[ = \text{Sym} \left[ \frac{1}{t_{L-1}^{(1)} - t_{L-1}^{(2)}} f_n^{(0)}(t^{(1)}) \frac{1}{t_{L-1}^{(2)}} f_n^{(0)}(t^{(2)}) + \frac{1}{t_{L-1}^{(1)} - t_{L-1}^{(2)}} f_n^{(0)}(t^{(1)}) \frac{1}{t_{L-1}^{(2)}} f_n^{(0)}(t^{(2)}) \right] \]

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Lemma 4.7. Therefore, the relation (4.17) holds.

Lemma 4.7. We have

\[
\left( Y_n^{(i)} \right)_0 = - \frac{1 + \delta_{n,1}}{z_i - 1} A_0 \varphi_A(t) + \frac{z_i}{z_i - 1} \sum_{m=1}^{L-1} (A_{m,i} + 1) \varphi(A_{m,i+1})(t) - \frac{1}{(z_i - 1)} A_0 \left( A_0 + 1 \right) \varphi(A_{1,i-1})(t) - \delta_{n,1} \left( \varphi_A(t) - z_i \sum_{m=2}^{L-1} A_0 \left( A_m + 1 \right) \varphi(A_{m,i+1,A_{i-1}})(t) \right).
\]

Proof. It suffices to show that

\[
\text{Sym} \left[ C(n, 1, 2) \frac{1}{t_{L-1}^{(1)}} f_n^{(i)}(t^{(1)}) \frac{1}{t_{L-1}^{(2)}} f_0(t^{(2)}) \right]
\]

\[
= \text{Sym} \left[ \frac{1}{(z_i - 1) t_{L-1}^{(1)} t_{L-1}^{(2)}} f_n^{(i)}(t^{(1)}) \left( - (1 + \delta_{n,1}) f_0(t^{(2)}) + z_i \sum_{m=1}^{L-1} f_m^{(i)}(t^{(2)}) \right) \right]
\]

\[
+ \delta_{n,1} \text{Sym} \left[ \frac{1}{(z_i - 1) t_{L-1}^{(1)} t_{L-1}^{(2)}} f_0(t^{(2)}) \left( f_0(t^{(1)}) - z_i \sum_{m=2}^{L-1} f_m^{(i)}(t^{(1)}) \right) \right],
\]

(4.18)

where the symmetrization \text{Sym}[f(t)] stands for \( \sum_{\sigma \in \mathbb{Z}_2} \sigma(f(t)) \) and the rational functions \( f(t^{(a)}) \) are defined in Definition 4.2.

Firstly, using (4.10) for \( l = 0 \) and \( k = L - 2 \), we have

\[
\text{Sym} \left[ \left( C(n, 1, 2) + \delta_{n,1} \frac{1}{t_{L-1}^{(1)} - 1} \frac{1}{t_{L-1}^{(1)}} f_n^{(i)}(t^{(1)}) \frac{1}{t_{L-1}^{(2)}} f_0(t^{(2)}) \right) \right]
\]

\[
= \text{Sym} \left[ \frac{-1}{t_{L-1}^{(1)} - t_{L-1}^{(2)}} \frac{1}{1 - z_i t_{L-1}^{(2)}} f_n^{(i)}(t^{(1)}) \frac{1}{t_{L-1}^{(1)}} f_0(t^{(2)}) + \frac{1}{t_{L-1}^{(1)} - t_{L-1}^{(2)}} f_n^{(i)}(t^{(1)}) \frac{1}{t_{L-1}^{(2)}} f_0(t^{(2)}) \right]
\]

\[
= \text{Sym} \left[ \frac{1}{t_{L-1}^{(1)}} f_n^{(i)}(t^{(1)}) \frac{1}{t_{L-1}^{(2)}} f_0(t^{(2)}) \right].
\]

Secondly, we have

\[
\text{Sym} \left[ \frac{1}{t_{L-1}^{(1)} - t_{L-1}^{(2)}} f_1^{(i)}(t^{(1)}) \frac{1}{t_{L-1}^{(2)}} f_0(t^{(2)}) \right]
\]

\[
= \text{Sym} \left[ \frac{1}{t_{L-1}^{(1)} - t_{L-1}^{(2)}} f_1^{(i)}(t^{(1)}) \frac{1}{t_{L-1}^{(2)}} f_0(t^{(2)}) \right]
\]

\[
= \text{Sym} \left[ \frac{1}{z_i - 1} \left( f_0(t^{(1)}) - f_1(t^{(1)}) - z_i \sum_{m=2}^{L-1} f_m^{(i)}(t^{(1)}) \right) \frac{1}{t_{L-1}^{(2)}} f_0(t^{(2)}) \right].
\]

Therefore, the relation (4.18) holds. □
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