A pure subalgebra of a finitely generated algebra is finitely generated*

MITSUYASU HASHIMOTO

Graduate School of Mathematics, Nagoya University
Chikusa-ku, Nagoya 464–8602 JAPAN
hasimoto@math.nagoya-u.ac.jp

Abstract
We prove the following. Let $R$ be a Noetherian commutative ring, $B$ a finitely generated $R$-algebra, and $A$ a pure $R$-subalgebra of $B$. Then $A$ is finitely generated over $R$.

In this paper, all rings are commutative. Let $A$ be a ring and $B$ an $A$-algebra. We say that $A \to B$ is pure, or $A$ is a pure subring of $B$, if for any $A$-module $M$, the map $M = M \otimes_A A \to M \otimes_A B$ is injective. Considering the case $M = A/I$, where $I$ is an ideal of $A$, we immediately have that $IB \cap A = I$.

It has been shown that if $B$ has a good property and $A$ is a pure subring of $B$, then $A$ has a good property. If $B$ is a regular Noetherian ring containing a field, then $A$ is Cohen-Macaulay [3], [4]. If $k$ is a field of characteristic zero, $A$ and $B$ are essentially of finite type over $k$, and $B$ has at most rational singularities, then $A$ has at most rational singularities [1].

In this paper, we prove the following

**Theorem 1.** Let $R$ be a Noetherian ring, $B$ a finitely generated $R$-algebra, and $A$ a pure $R$-subalgebra of $B$. Then $A$ is finitely generated over $R$.

The case that $B$ is $A$-flat is proved in [3]. This theorem is on the same line as the finite generation results in [3].

To prove the theorem, we need the following, which is a special case of a theorem of Raynaud-Gruson [7], [8].

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Theorem 2. Let $A \rightarrow B$ be a homomorphism of Noetherian rings, and $\varphi: X \rightarrow Y$ the associated morphism of affine schemes. Let $U \subset Y$ be an open subset, and assume that $\varphi: \varphi^{-1}(U) \rightarrow U$ is flat. Then there exists some ideal $I$ of $A$ such that $V(I) \cap U = \emptyset$, and that the morphism $\Phi: \text{Proj} R_B(BI) \rightarrow \text{Proj} R_A(I)$, determined by the associated morphism of the Rees algebras $R_A(I) := A[tI] \rightarrow R_B(BI) := B[tBI]$, is flat.

The morphism $\Phi$ in the theorem is called a flattening of $\varphi$.

Proof of Theorem 2 Note that for any $A$-algebra $A'$, the homomorphism $A' \rightarrow B \otimes_A A'$ is pure.

Since $B$ is finitely generated over $R$, it is Noetherian. Since $A$ is a pure subring of $B$, $A$ is also Noetherian. So if $A_{\text{red}}$ is finitely generated, then so is $A$. Replacing $A$ by $A_{\text{red}}$ and $B$ by $B \otimes_A A_{\text{red}}$, we may assume that $A$ is reduced.

Since $A \rightarrow \prod_{P \in \text{Min}(A)} A/P$ is finite and injective, it suffices to prove that each $A/P$ is finitely generated for $P \in \text{Min}(A)$, where $\text{Min}(A)$ denotes the set of minimal primes of $A$. By the base change, we may assume that $A$ is a domain.

There exists some minimal prime $P$ of $B$ such that $P \cap A = 0$. Assume the contrary. Then take $a_P \in P \cap A \setminus \{0\}$ for each $P \in \text{Min}(B)$. Then $\prod_P a_P$ must be nilpotent, which contradicts to our assumption that $A$ is a domain.

So by [6, (2.11) and (2.20)], $A$ is a finitely generated $R$-algebra if and only if $A_p$ is a finitely generated $R_p$-algebra for each $p \in \text{Spec } R$. So we may assume that $R$ is a local ring.

By the descent argument [2, (2.7.1)], $\hat{R} \otimes_R A$ is a finitely generated $\hat{R}$-algebra if and only if $A$ is a finitely generated $\hat{R}$-algebra, where $\hat{R}$ is the completion of $R$. So we may assume that $R$ is a complete local ring. We may lose the assumption that $A$ is a domain (even if $A$ is a domain, $\hat{R} \otimes_R A$ may not be a domain). However, doing the same reduction argument as above if necessary, we may still assume that $A$ is a domain.

Let $\varphi: X \rightarrow Y$ be a morphism of affine schemes associated with the map $A \rightarrow B$. Note that $\varphi$ is a morphism of finite type between Noetherian schemes. We denote the flat locus of $\varphi$ by $\text{Flat}(\varphi)$. Then $\varphi(X \setminus \text{Flat}(\varphi))$ is a constructible set of $Y$ not containing the generic point. So $U = Y \setminus \varphi(X \setminus \text{Flat}(\varphi))$ is a dense open subset of $Y$, and $\varphi: \varphi^{-1}(U) \rightarrow U$ is flat. By Theorem 2 there exists some nonzero ideal $I$ of $A$ such that $\Phi: \text{Proj} R_B(BI) \rightarrow \text{Proj} R_A(I)$ is flat.
If $J$ is a homogeneous ideal of $R_A(I)$, then we have an expression $J = \bigoplus_{n \geq 0} J_n t^n$ ($J_n \subset I^n$). Since $A$ is a pure subalgebra of $B$, we have $J_n B \cap I^n = J_n$ for each $n$. Since $JR_B(BI) = \bigoplus_{n \geq 0} (J_n B)t^n$, we have that $JR_B(BI) \cap R_A(I) = J$. Namely, any homogeneous ideal of $R_A(I)$ is contracted from $R_B(BI)$.

Let $P$ be a homogeneous prime ideal of $R_A(I)$. Then there exists some minimal prime $Q$ of $PR_B(BI)$ such that $Q \cap R_A(AI) = P$. Assume the contrary. Then for each minimal prime $Q$ of $PR_B(BI)$, there exists some $a_Q \in (Q \cap R_A(AI)) \setminus P$. Then $\prod a_Q \in \sqrt{PR_B(BI) \cap R_A(AI)} \setminus P$. However, we have

$$\sqrt{PR_B(BI) \cap R_A(I)} = \sqrt{PR_B(BI) \cap R_A(I)} = \sqrt{P} = P,$$

and this is a contradiction. Hence $\Phi: \text{Proj } R_B(BI) \to \text{Proj } R_A(I)$ is faithfully flat.

Since $\text{Proj } R_B(BI)$ is of finite type over $R$ and $\Phi$ is faithfully flat, we have that $\text{Proj } R_A(I)$ is of finite type by [3, Corollary 2.6]. Note that the blow-up $\text{Proj } R_A(I) \to Y$ is proper surjective. Since $R$ is excellent, $Y$ is of finite type over $R$ by [3, Theorem 4.2]. Namely, $A$ is a finitely generated $R$-algebra. 

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