The $\rho$ meson decay constant using a tadpole-improved action

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The $\rho$ meson decay constant and the associated renormalization factor are computed in the quenched approximation on coarse lattices using a tadpole-improved action which is corrected at the classical level to $\mathcal{O}(a^2)$. The improvement is displayed by comparing to Wilson action calculations.

1. INTRODUCTION

By improving a lattice action, one hopes that calculations at a fixed lattice spacing will be closer to their continuum values than with the original action, and similarly that calculations of a fixed accuracy can be performed on coarser lattices than those required, for example, by the Wilson action. For the present work, we consider an action which has classical errors beginning only at $\mathcal{O}(a^3)$, and which is also tadpole-improved\textsuperscript{6}. This action has been used to compute the spectrum of light hadron masses\textsuperscript{2,3}, and it was found that hadron mass ratios are considerably improved compared to Wilson action results at comparable lattice spacings. In this work we address the question of whether matrix elements of the vector current renormalization factor and vector meson decay constant.

2. IMPROVED ACTION

We use an improved gauge field action which involves a sum over 1×2 plaquettes as well as 1×1 plaquettes\textsuperscript{4}, and an improved fermion action which contains next-nearest-neighbour interactions\textsuperscript{5},

\begin{align*}
S_F(\bar{\psi}, \psi; U) &= -\sum_x \bar{\psi}(x)\psi(x) \\
&\quad + \frac{4}{3}\kappa \sum_{x, \mu} \left[ \bar{\psi}(x)(1 - \gamma_\mu)U_\mu(x)\psi(x + \mu) \\
&\quad \quad + \bar{\psi}(x + \mu)(1 + \gamma_\mu)U_\mu^\dagger(x)\psi(x) \right]
\end{align*}

\begin{align*}
-\frac{\kappa}{6U_0} \sum_{x, \mu} \left[ \bar{\psi}(x)(2 - \gamma_\mu)U_\mu(x)U_\mu(x + \mu)\psi(x + 2\mu) \\
+ \bar{\psi}(x + 2\mu)(2 + \gamma_\mu)U_\mu^\dagger(x + \mu)U_\mu^\dagger(x)\psi(x) \right].
\end{align*}

The tadpole factor\textsuperscript{1} is $U_0 = (\frac{1}{2}\text{Re}\text{Tr}U_\mu)^{1/4}$.

The presence of three timesteps in a single term of the action introduces artifacts\textsuperscript{6}, such as oscillations in meson mass functions near the source. However on our lattices, the plateau itself is not altered.

3. METHOD

In the continuum, the $\rho$ meson decay constant $f_\rho$ is defined as follows:

\begin{equation}
\langle 0|V_\mu|\rho\rangle\text{cont} = f_\rho^{-1}m_\rho^2\epsilon_\mu.
\end{equation}

On a lattice, matrix elements of the vector current get renormalized by a factor $Z_V$,

\begin{equation}
\langle f|V_\mu|i\rangle\text{cont} = Z_V(g^2)\langle f|V_\mu^C|i\rangle + \ldots ,
\end{equation}

where the dots represent finite lattice spacing effects. For the improved action, the $\mathcal{O}(a, a^2)$ terms vanish identically and terms proportional to the coupling are kept small by the tadpole factor. By considering the ratio of the local vector current $V_\mu^L(x) = \bar{\psi}(x)\gamma_\mu\psi(x)$ to the conserved vector current $V_\mu^C(x)$, we can compute the renormalization factor,

\begin{equation}
Z_V(g^2) = \langle f|V_\mu^C|i\rangle / \langle f|V_\mu^L|i\rangle + \ldots .
\end{equation}

We will consider a variety of choices for the initial and final states in this ratio, and the differences in the resulting values for $Z_V$ will reflect the finite lattice spacing effects.
4. CALCULATION

We have performed calculations at two values of $\beta$ with the improved action which correspond to lattice spacings of 0.4fm and 0.27fm as derived from the string tension. For comparison, we have also used the Wilson action to calculate at the same lattice spacings. For the gauge fields we used pseudo-heatbath updating with periodic boundary conditions; the first 4000 sweeps were discarded and then 250 sweeps (200 sweeps) were discarded between each pair of improved (Wilson) configurations that was kept.

A stabilized biconjugate gradient algorithm was used for the fermion matrix inversion at three choices of the hopping parameter. Due to the modest number of lattice sites, Dirichlet boundary conditions were used for fermions propagating in the time direction but periodic boundaries were used in all spacial directions. The source was placed two timesteps away from the boundary.

| Lattice | $N_U$ | $\kappa$ | $a_{st}$[fm] | $\beta$ |
|---------|-------|---------|---------------|--------|
| Improved Action |       |         |               |        |
| $6^3 \times 12$ | 100   | 0.625   | 0.162, 0.168, 0.174 | 6.25   |
| $8^3 \times 14$ | 50    | 6.8     | 0.150, 0.154, 0.158  | 0.27   |
| Wilson Action    |       |         |               |        |
| $6^3 \times 12$ | 75    | 4.5     | 0.189, 0.201, 0.213  | 0.4    |
| $8^3 \times 14$ | 30    | 5.5     | 0.164, 0.172, 0.180  | 0.27   |

5. RESULTS

Figure 1 shows our determination of $Z_V$ from four separate choices of the states in Eq. (4), $\langle \pi|V|\pi \rangle, \langle \rho|V|\rho \rangle, \langle N|V|N \rangle, \langle 0|V|\rho \rangle$. The extrapolation to the chiral limit was done linearly in $m_\pi^2$. The results of Wilson calculations by other groups are also shown. Notice that for both actions, $Z_V$ at fixed $M_\rho a$ is approximately independent of the amplitude used, except for the decay amplitude which predicts a smaller value for $Z_V$. For fixed lattice spacing, the improved action does give improved values of $Z_V$; in fact the improved action matches the Wilson predictions for $Z_V$ when the lattice spacing is larger by a factor of about three. Figure 2 shows that the dependence on $m_\pi^2$ is also quite similar for the two actions.

![Figure 1](image1.png)

Figure 1. The renormalization constant in the chiral limit. Solid (open) symbols are from the improved (Wilson) action. The lowest lying point at each $M_\rho a$ is from the decay amplitude.

![Figure 2](image2.png)

Figure 2. The renormalization constant as a function of $M_\rho^2$. Solid symbols are from the improved action at $\beta = 6.8$; open stars and diamonds are Wilson at $\beta = 6.0$; open squares are Wilson at $\beta = 5.85$. 
Figure 3 shows the lattice decay constant (i.e. without $Z_V$). Again, we find that the improved action does give improved results when compared to Wilson calculations at the same lattice spacing, and that the improved prediction is similar to the Wilson prediction at 1/3 of the lattice spacing.

Our best determination of the decay constant comes from the product of $Z_V$ and $1/(Z_V f_\rho)$, and this is shown in Figure 4. The cluster of data points which lie above the experimental result rely on tadpole-improved perturbation theory to estimate $Z_V$. Notice that the completely non-perturbative determinations of $1/f_\rho$ are quite flat over a substantial range of lattice spacing, and that for both the Wilson and improved actions the result is near experiment.

In summary, we have found that the $O(a^2)$ classical and tadpole improvements of our action manifest themselves in vector current matrix elements in essentially the same way as in masses.$^2$ Improved action results for $1/f_\rho$ and $Z_V$ are consistent with Wilson action computations done at a lattice spacing about 1/3 the size.

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