SYNCHROTRON RADIATION AND PAIR CREATION OF MASSLESS CHARGES IN MAGNETIC FIELD

D. V. GAL’TSOV
Faculty of Physics, Moscow State University, 119899, Moscow, Russia.
galtsov@phys.msu.ru

We study the massless limit in synchrotron radiation and one-photon pair creation in magnetic field. In this limit Schwinger critical field $H_0 = m^2c^3/(e\hbar)$ tends to zero, so two characteristic quantum parameters $\eta = H/H_0$, $\chi = \eta E/mc^2$ are infinite, and the standard approximations used in analytical calculations fail. Applying Schwinger’s proper time methods we derive simple expressions for synchrotron radiation spectra emitted by massless charges of spins $s = 0, 1/2$ and the pair creation probability distribution in the quasiclassical (high Landau levels) regime exhibiting simple scaling properties and possessing universal spectral shapes.

Keywords: Strong magnetic fields, massless particles, synchrotron radiation, one-photon pair creation.

1. Introduction

It could be naively thought that the ultrarelativistic limit in the radiation problem is identical with the limit of zero mass of the radiating charge. That this is not so simple is clear already looking at the classical Lienard formula for radiation power of the relativistic massive charge moving with the transverse proper acceleration $a$ and the velocity $v$ (in the units $c = 1$): $P = 2/3e^2a^2(1 - v^2)^{-2}$, which diverges in the massless limit $v \rightarrow 1$. At the same time, it was argued that massless charges would not radiate at all, an assertion, which could partly explain their experimental non-observation. But an infinitely growing radiation power is unacceptable since the particle can not emit the energy greater than its own energy, so the quantum nature of radiation must be invoked. On the other hand, the absence of radiation contradicts to Bohr quantum principles, if the charge occupies the excited energy states. So, in some sense, taking the ultrarelativistic limit does not commute with quantization.

Passing to the most interesting case of radiation in magnetic field, one finds that the ultrarelativistic limit $E \gg m$ may be essentially different depending on whether the dimensionless quantum recoil parameter $\chi = eHE/m^3$ is small or large (we will use units $\hbar = c = 1$ unless stated explicitly). In the first case the second dimensionless parameter $\eta = eH/m^2$, defining the ratio of the Landau spacing to the mass in the energy spectrum $E = \sqrt{m^2 + eH(2n + 1)}$ (for zero spin) is also small, and the ultrarelativistic limit is mostly classical. This is typical for the laboratory situation. The ultra-quantum case $\chi \gg 1$ and $\eta \ll 1$ is also well-known: the corresponding conditions are commonly realized in pulsars with strong magnetic fields. But in the strictly massless case both parameters are infinite, so special consideration is required. In the existing literature one can find the double Taylor expansions of the radiation power in terms of $\chi$, $\eta$, and some complicated
formulas for arbitrary $\eta$. Here we derive quite simple expressions relevant to the case of both parameters infinite which have simple scaling features. In this case most of radiation from the highly excited Landau levels $1 \ll n \leq 10^7$ (the upper restriction ensuring the charge energy spectrum to be quasi-discrete) is emitted in the transitions $n \rightarrow n'$ with large $n'$. Since the standard approximations used in the massive case fail, one needs to redo the derivation. But after all, it turns out that that the main contribution can still be extracted from the massive ultra-quantum limit $\chi \gg 1$ and $\eta \ll 1$, provided the above restrictions on $n$ holds.

Although our primary goal is to clarify the double limit $\chi \rightarrow \infty$, $\eta \rightarrow \infty$ of the quantum synchrotron radiation spectrum of the massive charges, it is worth saying a few words about validity of QED for truly massless ones. Attempts to find theoretical arguments explaining non-existence of such charges have long history\textsuperscript{4,5}. In perturbation theory they cause collinear, or massless, singularities\textsuperscript{6,7} of the Feynmann diagrams occurring when the photons are emitted from the massless external legs with the momentum parallel to the charge momentum\textsuperscript{8}. This problem, however, can be cured just ramifying the calculations\textsuperscript{6,7}. Also, the arguments were presented that free massless charges would be screened by vacuum polarization\textsuperscript{9–11}, the phenomenon which was interpreted as charge confinement in the massless QED. The corresponding length, however, is quite large, so the smaller scale processes, like pair creation, are not forbidden. Anyway, independently of the consistency problem of perturbative QED, the situation changes once classical magnetic field is added non-perturbatively. Such magnetized massless QED looks to be a consistent theory leading to interesting predictions is the low-energy region\textsuperscript{12}. In\textsuperscript{13} we have considered synchrotron radiation from massless charges in magnetized scalar QED showing that radiation do exist and has the universal spectral shape depending on neither the magnetic field nor the energy. Here we briefly review this calculation and give similar formulas for the spin 1/2 charge and for the one-photon massless pair creation.

2. Universality of radiation spectra from massless charges

Quantum theory of synchrotron radiation has different formulations. Historically the first one was based on exact solutions of the Klein-Gordon and Dirac equations in magnetic field\textsuperscript{14} (for more recent review see\textsuperscript{15}). Later on, Schwinger and Tsai applied the “proper time” method to calculate the one-loop mass operator of massive charges with $s = 0, 1/2$ in the constant magnetic field $H$, its imaginary part gives the total rate of synchrotron radiation\textsuperscript{16,17}. We have repeated similar calculations for charged particles of strictly zero mass. For $s = 0$ the main steps go as follows\textsuperscript{13}.

The mass operator for the complex scalar field in Schwinger’s operator notation reads:

$$M = i e^2 \int \left[ (\Pi - k)^\mu \frac{1}{k^2} \frac{1}{(\Pi - k)^2} (\Pi - k)_\mu \right] \frac{dk}{(2\pi)^4} - M_0,$$  \hspace{1cm} (1)
where \( \Pi_\mu = -i\partial_\mu - eA_\mu \), \( A_\mu \) stands for the constant magnetic field, and \( M_0 \) is the subtraction term needed to ensure vanishing of \( M \) and its first derivative with respect to \( \Pi^2 \) at \( \Pi^2 = 0 \). Exponentiating the propagators,

\[
\frac{1}{k^2 (\Pi - k)^2} = -\int_0^\infty ds \int_0^1 e^{-is\Gamma} , \quad \mathcal{H} = (k - u\Pi)^2 - u(1 - u)\Pi^2 \ ,
\]

and replacing the \( k \)-integration by averaging over the eigenstates of the coordinate \( \xi^\mu \) of the fictitious particle, canonically conjugate to \( k_\mu \), one obtains:

\[
M = i e^2 \int_0^\infty ds \int_0^1 du \langle \xi | (\Pi - k)^2 e^{-is\Gamma} (\Pi - k)_\mu | \xi \rangle - M_0 \ .
\]

The quantity \( \mathcal{H} \) is then treated as the Hamiltonian, and the averaging is performed in the Heisenberg picture passing to \( s \)-dependent operators \( \kappa(s), \xi(s), \Pi(s) \), which can be found exactly in terms of \( H \). Performing calculations in the massless case one arrives at

\[
M = \frac{e^2}{4\pi} \int_0^1 du \int_0^\infty \frac{ds}{s} \left[ e^{i\psi} \Delta^{-1/2} \left( E^2 \Phi_1 + 4i eH \Phi_2 + i \Phi_3 / s \right) - 2i / s \right] ,
\]

where \( \Phi_{1,2,3} \) and \( \Delta \) are some functions of \( u \) and \( s \). The phase is \( \psi = (2n + 1)(\beta - (1 - u)x) \), \( x = eHsu \) and \( \beta = \arctan \left( \cot \left( x + u / [x(1 - u)] \right) \right)^{-1} \). This expression is valid for all Landau levels \( n \). In the case \( n \gg 1 \) the integrals can be evaluated expanding all \( x \)-dependent quantities in power series. Indeed, the main contribution to the integral over \( x \) comes from the region where the phase \( \psi(x, u) \lesssim 1 \), in which \( \beta \) for \( x \ll 1 \) can be approximated as \( \beta \approx (1 - u) x + u(1 - u)^2 x^3 / 3 \), so that

\[
\psi \approx 3 s^3 (eH E)^2 u^4 (1 - u)^2 / 3 .
\]

For large \( n \), apart from the narrow regions around the limiting points of \( u \), \( u > n^{-1} \), \( 1 - u > n^{-1/2} \), which give negligible contributions under conditions specified later on, the essential domain of \( x \) is \( x \lesssim n^{-3/2} \). Therefore we use (5) in the exponent, expanding the rest of the integrand in powers of \( x \):

\[
\Delta^{-1} \approx 1 + u(4 - 3 x^2) / 3 , \quad \Phi_1 \approx (8 - 32 u / 3 + 13 u^2 / 3 - u^3) (1 - u) x^2 ,
\]

\[
\Phi_2 \approx 2(1 - u + u^2) x , \quad \Phi_3 \approx -2(1 - 10 u / 3 + u^2) x^2 .
\]

The leading contribution comes from \( \Phi_1 \), while in \( \Phi_3 \) one has to keep only the zero order term. Introducing the decay rate \( \Gamma = -\left( E^2 \Phi_1 \sin \psi + 2 eH u / x (1 - \cos \psi) \right) \),

\[
\Gamma = \frac{4 e^2}{9 E} (2/3) (3 eH E)^{2/3} .
\]

With account for radiative decay, the energy levels are no longer stationary, but quasi-stationary, provided the level spacing \( \Delta E = E_n - E_{n-1} \approx (eH / 2n)^{1/2} \).
is larger than $\Gamma/2$. This leads to an upper bound for the Landau level $n$: $2n < 81/[2 \Gamma (2/3)][\alpha]^3 \approx 10^7$, where $\alpha = 1/137$. Thus our calculation is consistent under the following conditions for the energy: $eH \ll E^2 < 10^7 eH$.

The real part of the mass operator gives rise to magnetically induced mass square $\delta m^2$. This quantity is finite, while the corresponding linear correction $\delta m$ diverges for $m = 0$ in view of the variation formula $\delta m = \delta m^2/(2m)$. Keeping the leading terms in the real part of (4), we find:

$$\delta m^2 = \frac{e^2}{4\pi} \int_0^1 du \int_0^\infty \frac{dx}{x} \left( \cos \psi \Phi_1 + 2 \sin \psi \frac{eHu}{x} \right) = \frac{4e^2 \Gamma(2/3)}{9\sqrt{3}} (3eHE)^{2/3}. \tag{10}$$

This expression is essentially non-perturbative in $\alpha$.

To get the spectral power of radiation $P(\omega)$ one has to perform spectral decomposition in the mass operator. Denoting $v = \omega/E$, one obtains in the same approximation:

$$P(\omega) = \frac{e^2 v}{4\pi E} \int_0^\infty \left( E^2 (8 - v^2)/(1 - v)^2 x \sin \psi + \frac{eHu}{x^2} (1 - \cos \psi) \right) dx, \tag{11}$$

Evaluating the integral over $x$ and restoring $\hbar$, we get

$$P(\omega) = \frac{2e^2 \Gamma(2/3)}{27E} (3\hbar eHE)^{2/3} P_0 \left( \hbar \omega/E \right), \tag{12}$$

where the normalized spectral function is introduced

$$P_0(v) = \frac{27}{2\pi \sqrt{3}} v^{1/3}(1 - v)^{-1/3}, \quad \int_0^1 P_0(v) dv = 1, \tag{13}$$

which does not depend on any parameter. This spectrum, shown in Fig. 1, has maximum at $\hbar \omega_{\text{max}} = E/3$, the average photon energy being $\langle \hbar \omega \rangle = 4E/9$. Thus, radiation of the massless charge is essentially quantum, whose power diverges in the formal limit $\hbar \to 0$. Universality of the spectrum means that even in a weak magnetic field the massless charge converts its energy into quanta of the same order of energy.

In the case of spin $1/2$ calculations are essentially similar and lead to the following expression for the spectral power:

$$P(\omega) = \frac{e^2 Ev}{\pi} \int_0^\infty \frac{(v^2 - 2v + 2)(1 - v) x \sin \psi dx}{x} = \frac{e^2 \Gamma(2/3)2^5 (3eHE)^{2/3}}{3^5} P_{1/2}, \tag{14}$$

where the normalized universal spectral function reads

$$P_{1/2}(v) = \frac{81\sqrt{3}}{64\pi} v^{1/3}(1 - v)^{-1/3}(v^2 - 2v + 2). \tag{15}$$

At the upper limit $v \to 1$ the spin $1/2$ spectral power has an integrable divergence. Both $s = 0, 1/2$ spectra are plotted in Fig.1. The low-frequency limits are identical for both spins and coincide with the classical spectrum ($\hbar$ disappears):

$$P_{cl}(\omega) = \frac{e^2 3^{1/6}}{\pi} \Gamma(2/3) \left( \frac{\omega}{\omega_H} \right)^{1/3} \omega_H, \quad \omega_H = \frac{eH}{E}. \tag{16}$$
This power-law dependence exhibits ultraviolet catastrophe (no high-frequency cut-off), which is cured in quantum theory.

One can also investigate the case of vector massless particles, $s = 1$, but then the result is infinite: magnetized vector QED fails to describe radiation from massless vector charges. This could be expected in view of the results of Case and Gasiorovich, who gave the arguments that electromagnetic interaction of massless charged particles with spin one and higher is controversial.

3. One-photon pair creation

Consider now the one-photon pair creation $\gamma(k) \rightarrow e_-(p) + e_+(p')$ where the states of the massless pair in magnetic field are specified by two quantum numbers $p_z, n$. This process is possible due to non-conservation of the transverse momentum — only the energy and the longitudinal momentum are conserved in magnetic field:

$$\omega = E_-(p_z, n) + E_+(p'_z, n'), \quad p_z + p'_z = k_z. \tag{17}$$

Ignoring refraction due to vacuum polarization, we consider the on-shell $k^2 = 0$ photon propagating orthogonally to the magnetic field, $k_z = 0$. The probability of the pair creation was calculated using the solutions of the Dirac equation in magnetic field by Klepikov and as the imaginary part of the photon polarization operator in. Using the last method in the massless case, we obtain for the differential probability a simple formula, shown in Fig. 2,

$$\frac{dW}{dv} = \frac{e^{22/3} \Gamma(2/3)(3eHE)^{2/3}}{8\sqrt{3}\pi \omega} \frac{1 + v^2}{(1 - v^2)^{1/3}}, \tag{18}$$

where $v = (E_- - E_+)/\omega$, which exhibits similar scaling and universality features. The corresponding total probability reads

$$W = \frac{15e^{22/3} \Gamma^2(2/3)\Gamma(5/6)(3eHE)^{2/3}}{28\sqrt{3}\pi \omega}. \tag{19}$$

4. Conclusion

To summarize: massless magnetized QED for spins $s = 0, 1/2$ consistently describes synchrotron radiation and one-photon pair creation of massless charges. In the quasiclassical regime (high Landau levels of initial and final states) these processes exhibit common scaling law $(eHE)^{2/3}$ and have universal spectral distributions not depending on any parameters.

It is amusing to note that just during the MG-14 conference the announcement came about discovery of massless charged quasi-particles in semi-metals.

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Fig. 1. Universal synchrotron radiation spectra $\mathcal{P}_s(v) = \frac{1}{\mathcal{E}}$, for massless charges of spin $s = 0$ (regular curve) and $s = 1/2$ (divergent as $v \to 1$).

Fig. 2. Differential probability of the one-photon massless $s = 1/2$ pair creation as function of $v = (E_- - E_+)/\hbar \omega$.

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