Article

Consistency and General Solutions to Some Sylvester-like Quaternion Matrix Equations

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Abstract: This article makes use of simultaneous decomposition of four quaternion matricies to investigate some Sylvester-like quaternion matrix equation systems. We present some useful necessary and sufficient conditions for the consistency of the system of quaternion matrix equations in terms of the equivalence form and block matrices. We also derive the general solution to the system according to the partition of the coefficient matrices. As an application of the system, we present some practical necessary and sufficient conditions for the consistency and general solution to a constrained system of quaternion matrix equations in terms of the equivalence form and block matrices. We also provide the general \( \phi \)-Hermitian solution to the system when the equation system is consistent. Moreover, we present some numerical examples to illustrate the availability of the results of this paper.

Keywords: quaternion matrix equation; matrix decomposition; consistency; general solution

1. Introduction

It is well known that quaternion and quaternion matrixes have a great range of applications in color image processing ([1,2]), quantum physics ([3,4]) and signal processing ([5]), etc. Quaternion matrix equations play an important role in mathematics and other domains, such as theoretical mechanics, optics, digital image processing, aerospace technology, etc. There are a great deal of papers from different aspects investigating quaternion matrix equations, such as solvability conditions, general solutions, extreme rank of solutions, minimum norm least squares solutions and their applications (e.g., [1,2,4,5,6–31]).

Kyrchei investigated two-sided generalized Sylvester matrix equation \( A_1X_1B_1 + A_2X_2B_2 = C \) over quaternion refer to Cramer’s Rules and Moore–Penrose inverse [13]. Liu et al. [17] considered the existence and uniqueness of solutions of some quaternion matrix equations with three new defined real representations. Kyrchei deduced the minimum norm least squares solutions of some quaternion matrix equations [11]. Futorny et al. [32] utilized Roth’s solvability criteria derived some solvability conditions for a few quaternion equations in terms of the equivalence of quaternion matrix. Xu et al. [28] provided some useful necessary and sufficient consitions and general solution to a constrained system of Sylvester-like matrix equations over the quaternion in terms of ranks and Moore–Penrose inverse of the coefficient matrixes. Mehany et al. [18] showed some solvaility conditions and general solution for three symmetrical coupled Sylvester-type matrix equations systems over quaternion from the aspects of ranks and generalized inverse. Liu et al. [15] gave solvability conditions and general solution to a Sylvester-like quaternion matrix equation \( A_1X + YB_1 + A_2Z_2B_2 = A_3Z_3B_3 + A_4Z_4B_4 = C \) with five unknown matrices. Wang et al. [27] formed some practical solvability conditions for a Sylvester-like matrix equation system and gave an application involving an \( \eta \)-Hermitian solution. Dmytryshyn et al. [33] showed some solvability conditions for a class of quaternion matrix equations. He [10] gave some useful necessary and sufficient conditions for the existence of

\[
A_iX_i + Y_iB_i + C_iZ_iD_i + F_iWG_i = E_i, i = 1, 2,
\]
in terms of ranks and Moore–Penrose inverses. Wang et al. [25] considered a system of constrained two-sided coupled Sylvester-type quaternion matrix equations

\[
\begin{align*}
A_1 X &= C_1, & A_2 Y &= C_2, & A_3 Z &= C_3, \\
X B_1 &= D_1, & Y B_2 &= D_2, & Z B_3 &= D_3, \\
A_4 X B_4 + C_4 Y D_4 &= P, & A_5 Z B_5 + C_5 Y D_5 &= Q,
\end{align*}
\]

and gave some solvability conditions and general solution to the system. As far as we know, there has been little information on the solvability conditions to the system

\[
\begin{align*}
AX E + CY F + DZ + WG &= \Omega, \\
BX H &= \Phi,
\end{align*}
\]

in terms of quaternion matrix decomposition. One goal of this paper is to give some practical equivalent conditions for the consistency of the system (1) in terms of block matrixes and ranks.

\(\phi\)-Hermitian quaternion matrix was put forward by Rodman in [34] at first, and some properties about it were given. In addition to general solutions, \(\phi\)-Hermitian solutions of quaternion matrix equations play an important role in signal processing, engineering, systems and cybernetics, etc. With the need of practical application, the \(\phi\)-Hermitian solutions of quaternion matrix equations have been paid more and more attention. He et al. [8] derived some practical necessary and sufficient conditions for the existence of the \(\phi\)-Hermitian solution to the system of quaternion matrix equations

\[
\begin{align*}
AX A_{\phi} + CY C_{\phi} + DZ D_{\phi} &= \Omega, \\
BX B_{\phi} &= \Theta,
\end{align*}
\]

where \(A, B, C, D, \Omega,\) and \(\Theta\) are given quaternion matrixes. He et al. [7] considered the following systems of quaternion matrix equations involving \(\phi\)-Hermiticity

\[
\begin{align*}
A_1 X - Y B_1 &= C_1, & A_2 Z - Y B_2 &= C_2, & Z &= Z_{\phi},
\end{align*}
\]

and

\[
\begin{align*}
A_1 X - Y B_1 &= C_1, & A_2 Y - Z B_2 &= C_2, & Z &= Z_{\phi}.
\end{align*}
\]

He [6] derived necessary and sufficient conditions for the existence of the solution to quaternion matrix equations

\[
BX B_{\phi} + CY C_{\phi} + DZ D_{\phi} = A, \quad X = X_{\phi}, \quad Y = Y_{\phi}, \quad Z = Z_{\phi}
\]

and

\[
BXC + (BXC)_{\phi} + D Y D_{\phi} = A, \quad Y = Y_{\phi}
\]

involving \(\phi\)-hermiticity in terms ranks of the given quaternion matrixes. To the best of our knowledge, there is little information on the general \(\phi\)-Hermitian solution to the system of quaternion matrix equations

\[
\begin{align*}
AX A_{\phi} + CY C_{\phi} + DZ + (DZ)_{\phi} &= \Omega, \\
BX B_{\phi} &= \Phi.
\end{align*}
\]

Another goal of this article is to give some solvability conditions and the expressions of the general \(\phi\)-Hermitian solution to the system (2) via using the similar method with the system (1).

Let \(\mathbb{R}\) stand for the real number field and \(\mathbb{H}\) represent the real quaternion number field which is a four-dimensional linear space over \(\mathbb{R}\) with a pattern that

\[i^2 = j^2 = k^2 = ijk = -1.\]

Let \(\mathbb{H}^{m \times n}\) stand for all \(m \times n\) matrixes over the real quaternion algebra, and if \(A \in \mathbb{H}^{m \times n}\), then sign \(A^T\) the transpose matrix of \(A\). \(A = A_{\phi}\) is \(\phi\)-Hermitian, if \(A_{\phi}\) is a \(n \times m\) quaternion matrix obtained through applying a nonstandard involution transform \(\phi\) with transformation matrix \(
\begin{pmatrix}
1 & 0 \\
0 & Q
\end{pmatrix}
\)

based on basis \(\{1, i, j, k\}\) on \(A^T\), where \(Q\) is a \(3 \times 3\) real orthogonal symmetric matrix with eigenvalues
1, 1, −1 [34]. The rank of a quaternion matrix $A$ is defined to be the right linear independent columns maximum number of $A$ and is represented by symbol $r(A)$. Note that $A$ and $PAQ$ have the same rank for any invertible matrices $P$ and $Q$ with appropriate sizes. The identity matrix and zero matrix with appropriate size are labeled by $I$ and $0$, respectively.

2. Solvability Conditions to the System (1)

In this section, we deduce the consistency conditions of the system of quaternion matrix Equation (1),

$$\begin{align*}
AXE + CYF + DZ + WG &= \Omega, \\
BXH &= \Phi,
\end{align*}$$

where $A$, $B$, $C$, $D$, $E$, $F$, $G$, $H$, $\Omega$, and $\Phi$ are given quaternion matrixes, through utilizing the equivalence canonical form of four matrixes.

According to the matrix product order principle, we can observe that coefficient matrixes $A$, $B$ have the same number of columns, $A$, $C$, $D$ have the same number of rows, $E$, $F$, $G$ have the same number of columns, and $E$, $H$ have the same number of rows, so they can be formed in the following two quaternion matrix arrays

$$\begin{bmatrix}
A & C & D \\
B & & \\
& & \\
& & \\
& & \\
& & \\
\end{bmatrix}, \quad \begin{bmatrix}
p & q & s \\
m & & \\
& & \\
& & \\
& & \\
& & \\
\end{bmatrix},$$

where $A \in \mathbb{H}^{m \times p}, B \in \mathbb{H}^{t \times p}, C \in \mathbb{H}^{m \times q}, D \in \mathbb{H}^{m \times t}, E \in \mathbb{H}^{p_1 \times m_1}, H \in \mathbb{H}^{p_1 \times t_1}, F \in \mathbb{H}^{p_1 \times m_1}$, and $G \in \mathbb{H}^{p_1 \times m_1}$.

To solve the equation system, we first give the simultaneous decomposition and equivalence canonical form of four quaternion matrixes in the following lemma.

**Lemma 1.** ([9, 26]) Given $A \in \mathbb{H}^{m \times p}, B \in \mathbb{H}^{t \times p}, C \in \mathbb{H}^{m \times q}$, and $D \in \mathbb{H}^{m \times t}$. Then, there exist nonsingular matrices $M$, $P$, $T$, $Q$, and $S$ such that

$$\begin{bmatrix}
MAP & MCQ & MDS & TBP \\
\end{bmatrix} = \begin{bmatrix}
S_a & S_c & S_d & S_b
\end{bmatrix},$$

where

$$\begin{bmatrix}
p & q & s \\
m & & \\
& & \\
& & \\
& & \\
& & \\
\end{bmatrix} = \begin{bmatrix}
\begin{array}{cccc}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
\end{array}
\end{bmatrix}.$$  

The order of identity matrices in (4) are shown in [9] directly.

We utilize Lemma 1 to transform the matrix arrays (3) into two simple forms such as (4) and
where \(S_a, S_b, S_c, S_d, S_e, S_f, S_g, S_h\) satisfy

\[
MAP = S_a, \quad MCQ = S_c, \quad MDS = S_d, \quad TBP = S_b,
\]

and

\[
P_1EM_1 = S_c, \quad P_1HT_1 = S_h, \quad Q_1FM_1 = S_f, \quad S_1GM_1 = S_g,
\]

and \(M, P, T, Q, S, M_1, P_1, T_1, Q_1, S_1\) are nonsingular.

Hence, the system (1) is equivalent to

\[
\begin{align*}
S_h(P^{-1}XP_1^{-1})S_c &+ S_c(Q^{-1}YQ_1^{-1})S_f + S_d(S^{-1}ZM_1) + (MWS_1^{-1})S_g = M\Omega M_1, \\
S_h(P^{-1}XP_1^{-1})S_h &+ T\Phi T_1.
\end{align*}
\]

Set

\[
\hat{X} = P^{-1}XP_1^{-1} = (X_{ij})_{14 \times 14}, \quad \hat{Y} = Q^{-1}YQ_1^{-1} = (Y_{ij})_{10 \times 10},
\]

\[
\hat{Z} = S^{-1}ZM_1 = (Z_{ij})_{18 \times 18}, \quad \hat{W} = MWS_1^{-1} = (W_{ij})_{18 \times 18}.
\]

\[
M\Omega M_1 = (\Omega_{ij})_{18 \times 18}, \quad T\Phi T_1 = (\Phi_{ij})_{8 \times 8},
\]

where \((\Omega_{ij})_{18 \times 18}\) and \((\Phi_{ij})_{8 \times 8}\) have the same block rows as \(S_a\) and \(S_b\), the same block columns as \(S_c\) and \(S_h\), respectively. Then, substituting (9)–(11) into the system (8) becomes

\[
\begin{align*}
S_h\hat{X}S_c &+ S_c\hat{Y}S_f + S_d\hat{Z} + \hat{W}S_g = (\Omega_{ij})_{18 \times 18}, \\
S_h\hat{X}S_h &+ (\Phi_{ij})_{8 \times 8}.
\end{align*}
\]

A result can be conclude that the system (1) is equivalent with the system (12). Substituting (9)–(11) into the system (12) yields

\[
S_a\hat{X}S_c + S_c\hat{Y}S_f + S_d\hat{Z} + \hat{W}S_g = (\Omega_{ij})_{18 \times 18},
\]

and

\[
S_h\hat{X}S_h = (\Phi_{ij})_{8 \times 8}.
\]
where

\[
(\mathcal{Q}_I)_{18 \times 18} =
\begin{bmatrix}
X_{10} + Y_{10} + Z_{10} + W_{10} & X_{20} + Y_{20} + Z_{20} + W_{20} & \cdots & X_{18} + Y_{18} + Z_{18} + W_{18} \\
X_{11} + Y_{11} + Z_{11} + W_{11} & X_{21} + Y_{21} + Z_{21} + W_{21} & \cdots & X_{12} + Y_{12} + Z_{12} + W_{12} \\
\vdots & \vdots & \ddots & \vdots \\
X_{18} + Y_{18} + Z_{18} + W_{18} & X_{28} + Y_{28} + Z_{28} + W_{28} & \cdots & X_{36} + Y_{36} + Z_{36} + W_{36}
\end{bmatrix}
\]

\[
(\Phi_I)_{8 \times 8} =
\begin{bmatrix}
X_{10} X_{10} X_{10} X_{10} X_{10} X_{10} X_{10} X_{10} \\
X_{11} X_{11} X_{11} X_{11} X_{11} X_{11} X_{11} X_{11} \\
\vdots & \vdots & \ddots & \vdots \\
X_{36} X_{36} X_{36} X_{36} X_{36} X_{36} X_{36} X_{36}
\end{bmatrix}
\]

The following theorem takes into account the solvability conditions to the system (1) from the level of the partion of the coefficient matrices of equivalent matrix equation system (12) and deduces solvability condition in terms of ranks which is equivalent to block matrices condition.

**Theorem 1.** Consider the system of quaternion matrix Equation (1). Then the following statements are equivalent:

(1) The system (1) is consistent.

(2) [10] The ranks satisfy:
\[ r(B \Phi) = r(B), \quad r(H) = r(H), \] (15)

\[ r\begin{pmatrix} A & C & D \\ 0 & 0 & 0 \\ G & 0 & 0 \end{pmatrix} = r(A) + r(G), \] (16)

\[ r\begin{pmatrix} E & 0 \\ F & 0 \\ G & 0 \end{pmatrix} = r(E) + r(D), \] (17)

\[ r\begin{pmatrix} A & D \\ 0 & 0 \\ G & 0 \end{pmatrix} = r(A) + r(D), \] (18)

\[ r\begin{pmatrix} C & D \\ 0 & 0 \\ G & 0 \end{pmatrix} = r(C) + r(D), \] (19)

\[ r\begin{pmatrix} A & D \\ 0 & 0 \\ G & 0 \end{pmatrix} = r(A) + r(D), \] (20)

\[ r\begin{pmatrix} A & C & D \\ 0 & 0 & 0 \\ G & 0 & 0 \end{pmatrix} = r(A) + r(D), \] (21)

(3) The block matrices satisfy:

\[ (\Phi_{81} \Phi_{82} \ldots \Phi_{88}) = 0, \] (22)

\[ \begin{pmatrix} \Phi_{18} \\ \Phi_{28} \\ \vdots \\ \Phi_{78} \end{pmatrix} = 0, \] (23)

\[ \left\{ \begin{array}{l} \Omega_{18,4} = \Omega_{18,7}, \quad \Omega_{18,11} = \Omega_{18,13}, \quad \Omega_{18,5} = \Omega_{18,14}, \quad \Omega_{18,18} = 0, \\
\Omega_{18,2} = 0, \quad \Omega_{18,6} = 0, \quad \Omega_{18,9} = 0, \quad \Omega_{18,12} = 0, \quad \Omega_{18,16} = 0,
\end{array} \right. \] (24)

\[ \left\{ \begin{array}{l} \Omega_{4,18} = \Omega_{7,18}, \quad \Omega_{5,18} = \Omega_{14,18}, \quad \Omega_{11,18} = \Omega_{13,18}, \\
\Omega_{2,18} = 0, \quad \Omega_{6,18} = 0, \quad \Omega_{9,18} = 0, \quad \Omega_{12,18} = 0, \quad \Omega_{16,18} = 0,
\end{array} \right. \] (25)

\[ \Omega_{16,6} = 0, \quad \Omega_{16,12} = 0, \] (26)

\[ \Omega_{6,16} = 0, \quad \Omega_{12,16} = 0, \] (27)
In this case, the general solution to the system (1) can be written as

\[ X = P \tilde{X} P_1, \quad Y = Q \tilde{Y} Q_1, \quad Z = S \tilde{Z} M_1^{-1}, \quad W = M^{-1} \tilde{W} S_1, \]

(30)

where

\[ \tilde{X} = \begin{pmatrix}
\Phi_{11} & \Phi_{12} & \Phi_{13} & \Phi_{14} & \Phi_{15} & \Phi_{16} & \Phi_{17} \\
\Phi_{21} & \Phi_{22} & \Phi_{23} & \Phi_{24} & \Phi_{25} & \Phi_{26} & \Phi_{27} \\
\Phi_{31} & \Phi_{32} & \Phi_{33} & \Phi_{34} & \Phi_{35} & \Phi_{36} & \Phi_{37} \\
\Phi_{41} & \Phi_{42} & \Phi_{43} & \Phi_{44} & \Phi_{45} & \Phi_{46} & \Phi_{47} \\
\Phi_{51} & \Phi_{52} & \Phi_{53} & \Phi_{54} & \Phi_{55} & \Phi_{56} & \Phi_{57} \\
\Phi_{61} & \Phi_{62} & \Phi_{63} & \Phi_{64} & \Phi_{65} & \Phi_{66} & \Phi_{67} \\
\Phi_{71} & \Phi_{72} & \Phi_{73} & \Phi_{74} & \Phi_{75} & \Phi_{76} & \Phi_{77}
\end{pmatrix},
\]

\[ \tilde{Y} = \begin{pmatrix}
Y_{11} & Y_{12} & Y_{13} & Y_{14} & Y_{15} & Y_{16} & Y_{17} \\
Y_{21} & Y_{22} & Y_{23} & Y_{24} & Y_{25} & Y_{26} & Y_{27} \\
Y_{31} & Y_{32} & Y_{33} & Y_{34} & Y_{35} & Y_{36} & Y_{37} \\
Y_{41} & Y_{42} & Y_{43} & Y_{44} & Y_{45} & Y_{46} & Y_{47} \\
Y_{51} & Y_{52} & Y_{53} & Y_{54} & Y_{55} & Y_{56} & Y_{57} \\
Y_{61} & Y_{62} & Y_{63} & Y_{64} & Y_{65} & Y_{66} & Y_{67} \\
Y_{71} & Y_{72} & Y_{73} & Y_{74} & Y_{75} & Y_{76} & Y_{77}
\end{pmatrix},
\]

\[ \tilde{Z} = \begin{pmatrix}
Z_{11} & Z_{12} & Z_{13} & Z_{14} & Z_{15} & Z_{16} & Z_{17} \\
Z_{21} & Z_{22} & Z_{23} & Z_{24} & Z_{25} & Z_{26} & Z_{27} \\
Z_{31} & Z_{32} & Z_{33} & Z_{34} & Z_{35} & Z_{36} & Z_{37} \\
Z_{41} & Z_{42} & Z_{43} & Z_{44} & Z_{45} & Z_{46} & Z_{47} \\
Z_{51} & Z_{52} & Z_{53} & Z_{54} & Z_{55} & Z_{56} & Z_{57} \\
Z_{61} & Z_{62} & Z_{63} & Z_{64} & Z_{65} & Z_{66} & Z_{67} \\
Z_{71} & Z_{72} & Z_{73} & Z_{74} & Z_{75} & Z_{76} & Z_{77}
\end{pmatrix},
\]

\[ \tilde{W} = \begin{pmatrix}
W_{11} & W_{12} & W_{13} & W_{14} & W_{15} & W_{16} & W_{17} \\
W_{21} & W_{22} & W_{23} & W_{24} & W_{25} & W_{26} & W_{27} \\
W_{31} & W_{32} & W_{33} & W_{34} & W_{35} & W_{36} & W_{37} \\
W_{41} & W_{42} & W_{43} & W_{44} & W_{45} & W_{46} & W_{47} \\
W_{51} & W_{52} & W_{53} & W_{54} & W_{55} & W_{56} & W_{57} \\
W_{61} & W_{62} & W_{63} & W_{64} & W_{65} & W_{66} & W_{67} \\
W_{71} & W_{72} & W_{73} & W_{74} & W_{75} & W_{76} & W_{77}
\end{pmatrix},
\]

\[ \tilde{S} = \begin{pmatrix}
S_{11} & S_{12} & S_{13} & S_{14} & S_{15} & S_{16} & S_{17} \\
S_{21} & S_{22} & S_{23} & S_{24} & S_{25} & S_{26} & S_{27} \\
S_{31} & S_{32} & S_{33} & S_{34} & S_{35} & S_{36} & S_{37} \\
S_{41} & S_{42} & S_{43} & S_{44} & S_{45} & S_{46} & S_{47} \\
S_{51} & S_{52} & S_{53} & S_{54} & S_{55} & S_{56} & S_{57} \\
S_{61} & S_{62} & S_{63} & S_{64} & S_{65} & S_{66} & S_{67} \\
S_{71} & S_{72} & S_{73} & S_{74} & S_{75} & S_{76} & S_{77}
\end{pmatrix}.
\]
\[ Z = \begin{pmatrix}
Z_{11} & Z_{12} & Z_{13} & Z_{14} & Z_{15} & Z_{16} & Z_{17} \\
Z_{21} & Z_{22} & Z_{23} & Z_{24} & Z_{25} & Z_{26} & Z_{27} \\
Z_{31} & Z_{32} & Z_{33} & Z_{34} & Z_{35} & Z_{36} & Z_{37} \\
Z_{41} & Z_{42} & Z_{43} & Z_{44} & Z_{45} & Z_{46} & Z_{47} \\
Z_{51} & Z_{52} & \Omega_{13,1} - W_{12} & \Omega_{34,1} - W_{13} & \Omega_{13,1} - W_{14} & \Omega_{13,1} - W_{15} & \Omega_{13,1} - W_{16} \\
Z_{61} & Z_{62} & \Omega_{13,1} - W_{17} & \Omega_{13,1} - W_{18} & \Omega_{13,1} - W_{19} & \Omega_{13,1} - W_{20} & \Omega_{13,1} - W_{21} \\
Z_{71} & Z_{72} & \Omega_{13,1} - W_{22} & \Omega_{13,1} - W_{23} & \Omega_{13,1} - W_{24} & \Omega_{13,1} - W_{25} & \Omega_{13,1} - W_{26} \\
Z_{81} & Z_{82} & \Omega_{13,1} - W_{27} & \Omega_{13,1} - W_{28} & \Omega_{13,1} - W_{29} & \Omega_{13,1} - W_{30} & \Omega_{13,1} - W_{31}
\end{pmatrix}
\]

\[ W = \begin{pmatrix}
W_{11} & W_{12} & W_{13} & W_{14} & W_{15} & W_{16} & W_{17} & W_{18} & W_{19} & W_{20} \\
W_{21} & W_{22} & W_{23} & W_{24} & W_{25} & W_{26} & W_{27} & W_{28} & W_{29} & W_{30} \\
W_{31} & W_{32} & W_{33} & W_{34} & W_{35} & W_{36} & W_{37} & W_{38} & W_{39} & W_{40} \\
W_{41} & W_{42} & W_{43} & W_{44} & W_{45} & W_{46} & W_{47} & W_{48} & W_{49} & W_{50} \\
W_{51} & W_{52} & W_{53} & W_{54} & W_{55} & W_{56} & W_{57} & W_{58} & W_{59} & W_{60} \\
W_{61} & W_{62} & W_{63} & W_{64} & \Omega_{6,13} & \Omega_{6,14} & \Omega_{6,15} & \Omega_{6,16} & \Omega_{6,17} & \Omega_{6,18} \\
W_{71} & W_{72} & W_{73} & W_{74} & W_{75} & W_{76} & W_{77} & W_{78} & W_{79} & W_{80} \\
W_{81} & W_{82} & W_{83} & W_{84} & W_{85} & W_{86} & W_{87} & W_{88} & W_{89} & W_{90} \\
W_{91} & W_{92} & W_{93} & W_{94} & \Omega_{9,13} & \Phi_{9,14} & \Phi_{9,15} & \Phi_{9,16} & \Phi_{9,17} & \Phi_{9,18} \\
W_{101} & W_{102} & W_{103} & W_{104} & W_{105} & W_{106} & W_{107} & W_{108} & W_{109} & W_{110} \\
W_{111} & W_{112} & W_{113} & W_{114} & W_{115} & W_{116} & W_{117} & W_{118} & W_{119} & W_{120} \\
W_{121} & W_{122} & W_{123} & W_{124} & W_{125} & W_{126} & W_{127} & W_{128} & W_{129} & W_{130} \\
W_{131} & W_{132} & W_{133} & W_{134} & W_{135} & W_{136} & W_{137} & W_{138} & W_{139} & W_{140} \\
W_{141} & W_{142} & W_{143} & W_{144} & W_{145} & W_{146} & W_{147} & W_{148} & W_{149} & W_{150} \\
W_{151} & W_{152} & W_{153} & W_{154} & W_{155} & W_{156} & W_{157} & W_{158} & W_{159} & W_{160} \\
W_{161} & W_{162} & W_{163} & W_{164} & W_{165} & W_{166} & W_{167} & W_{168} & W_{169} & W_{170} \\
W_{171} & W_{172} & W_{173} & W_{174} & W_{175} & W_{176} & W_{177} & W_{178} & W_{179} & W_{180} \\
W_{181} & W_{182} & W_{183} & W_{184} & W_{185} & W_{186} & W_{187} & W_{188} & W_{189} & W_{190}
\end{pmatrix}
\]

\[ P, Q, S, M, P_1, Q_1, S_1, M_3 \text{ are defined in (6) and (7), } \Omega_{ij}, \Phi_{ij} \text{ are defined in (13), (14), respectively, } X_{ij}, Y_{ij}, Z_{ij} \text{ and } W_{ij} \text{ are arbitrary matrices over } \mathbb{H} \text{ with appropriate sizes.}

\textbf{Proof.} (1) \Rightarrow (2): Suppose \( (X_0, Y_0, Z_0, W_0) \) is a solution to the system (1), that is

\[ AX_0E + CY_0F + DZ_0 + W_0G = \Omega, \]

\[ BX_0H = \Phi, \]

we can employ elementary matrix operations to show that the rank equalities (15)–(21) hold.

(2) \iff (3): From \( S_d, S_b, S_c, S_d, S_c, S_f, S_g, S_h \) in (4) and (5), we can infer that

\[ r(B \Phi) = r(B) \iff r(S_b (\Phi_{ij})_{8 \times 8}) = r(S_b) \iff (\Phi_{81} \Phi_{82} \ldots \Phi_{88}) = 0, \]

\[ r(H) = r(H) \iff r((\Phi_{ij})_{8 \times 8}) = r(S_t) \iff (\Phi_{18} \Phi_{28} \ldots \Phi_{88}) = 0. \]

\[ P_{ij}, Q_{ij}, S_{ij}, M_{ij} \text{ are defined in (6) and (7), } \Omega_{ij}, \Phi_{ij} \text{ are defined in (13), (14), respectively, } X_{ij}, Y_{ij}, Z_{ij} \text{ and } W_{ij} \text{ are arbitrary matrices over } \mathbb{H} \text{ with appropriate sizes.} \]
Similarly, we have
\[ r \begin{pmatrix} Ω & A & C & D \\ G & 0 & 0 & 0 \end{pmatrix} = r \begin{pmatrix} A & C & D \end{pmatrix} + r(G) \]
\[ ⇔ \{ \begin{align*} Ω_{18,4} &= Ω_{18,7}, \quad Ω_{18,11} = Ω_{18,13}, \quad Ω_{18,15} = Ω_{18,14}, \quad Ω_{18,18} = 0, \\ Ω_{18,2} &= 0, \quad Ω_{18,6} = 0, \quad Ω_{18,9} = 0, \quad Ω_{18,12} = 0, \quad Ω_{18,16} = 0, \end{align*} \}
\]
\[ r \begin{pmatrix} Ω & D \\ E & 0 \\ F & 0 \\ G & 0 \end{pmatrix} = r \begin{pmatrix} E \\ F \\ G \end{pmatrix} + r(D) \]
\[ ⇔ \{ \begin{align*} Ω_{18,4} &= Ω_{18,7}, \quad Ω_{18,11} = Ω_{18,13}, \quad Ω_{18,15} = Ω_{18,14}, \quad Ω_{18,18} = 0, \\ Ω_{18,2} &= 0, \quad Ω_{18,6} = 0, \quad Ω_{18,9} = 0, \quad Ω_{18,12} = 0, \quad Ω_{18,16} = 0, \end{align*} \}
\]
\[ r \begin{pmatrix} Ω & A \\ F & 0 \\ G & 0 \end{pmatrix} = r(A) + r(F) \]
\[ ⇔ \{ \begin{align*} Ω_{16,6} &= 0, \quad Ω_{16,12} = 0, \quad Ω_{16,18} = 0, \quad Ω_{18,6} = 0, \quad Ω_{18,12} = 0, \quad Ω_{18,18} = 0, \end{align*} \}
\]
\[ r \begin{pmatrix} Ω & C \\ E & 0 \\ G & 0 \end{pmatrix} = r(E) + r(C) \]
\[ ⇔ \{ \begin{align*} Ω_{16,6} &= 0, \quad Ω_{16,12} = 0, \quad Ω_{18,6} = 0, \quad Ω_{18,12} = 0, \quad Ω_{18,18} = 0, \end{align*} \}
\]
\[ r \begin{pmatrix} A & D \\ Ω & 0 \\ B & 0 & 0 & -Φ \\ 0 & 0 & E & H \\ 0 & 0 & F & 0 \\ 0 & 0 & G & 0 \end{pmatrix} = r \begin{pmatrix} A & D \\ B & 0 \end{pmatrix} + r \begin{pmatrix} E & H \\ F & 0 \\ G & 0 \end{pmatrix} \]
\[ ⇔ \{ \begin{align*} Ω_{9,12} &= Φ_{36}, \quad Ω_{12,12} = 0, \quad Ω_{18,18} = 0, \quad Ω_{16,12} = 0, \\ Ω_{12,12} &= Φ_{66}, \quad Ω_{16,18} = 0, \quad Ω_{12,18} = 0, \\ Φ_{56} &= Ω_{11,12} - Ω_{13,12}, \quad Ω_{11,18} = Ω_{13,18}, \end{align*} \}
\]
\[ r \begin{pmatrix} A & C \\ D & 0 \\ B & 0 & 0 & 0 & -Φ \\ 0 & 0 & E & H \\ 0 & 0 & G & 0 \end{pmatrix} = r \begin{pmatrix} A & C & D \\ B & 0 & 0 \end{pmatrix} + r \begin{pmatrix} E & H \\ G & 0 \end{pmatrix} \]
\[ ⇔ \{ \begin{align*} Ω_{12,9} &= Φ_{33}, \quad Ω_{18,9} = 0, \quad Ω_{18,12} = 0, \quad Ω_{18,16} = 0, \\ Ω_{12,12} &= Φ_{66}, \quad Ω_{18,18} = 0, \quad Ω_{12,16} = 0, \quad Ω_{12,18} = 0, \\ Φ_{65} &= Ω_{12,11} - Ω_{12,13}, \quad Ω_{18,11} = Ω_{18,13} = 0. \end{align*} \}

(3) ⇒ (1): If (22)–(29) are established, then (30) is the general solution of the system (1).

Remark 1. We use a new method which differs from the one presented in [10] to obtain our result.
3. A Numerical Example of the System (1)

In this section, we give an numerical example to demonstrate the availability of Theorem 1.

Example 1. Let

\[
A = \begin{pmatrix} j & k & i \\ 0 & 1 & k \\ 1 & 0 & j \end{pmatrix}, \quad B = \begin{pmatrix} j & k & i \\ 3j & 3k & 3i \\ -2j & -2k & -2i \end{pmatrix}, \quad C = \begin{pmatrix} j - k & 2i + k & 0 \\ 1 & 0 & i \\ 1 + j - k & 2i + k & i \end{pmatrix},
\]

\[
D = \begin{pmatrix} i & j & k \\ i + j + k & 0 & 1 \\ j + k & -j & 1 - k \end{pmatrix}, \quad E = \begin{pmatrix} j + k & j - k & i \\ 1 + j + k & 0 & 1 \\ i & j & k \end{pmatrix}, \quad F = \begin{pmatrix} k & j & 0 \\ i & 1 & k \\ 1 + k & i + j & k \end{pmatrix},
\]

\[
G = \begin{pmatrix} 0 & j & 1 \\ i & 0 & k \\ i & 1 & j + k \end{pmatrix}, \quad H = \begin{pmatrix} -3 & -k & -i \\ -3j & -3i & k \\ -3 - 3j & -k - i & k \end{pmatrix},
\]

\[
\Omega = \begin{pmatrix} -29 - 13i + 2j + 14k & 12 - 2i + 14j - 8k & -18 + 6i + 15j - 4k \\ -5 + 4i + 10j + 6k & -8 + 3i + 6j - 5k & -6 + 5i - 2j + 6k \\ -19 - 1 + 13j + 16j & 5 - 14i + 18j - 10k & -20 + 3i + 5j + 6k \end{pmatrix},
\]

\[
\Phi = \begin{pmatrix} 9 - 6i + 3j - 3k & 1 + i + 2j + 3k & 2 + 3i - j - k \\ 27 - 18i + 9j - 9k & 3 + 3i + 6j + 9k & 6 + 9i - 3j - 3k \\ -18 + 12i - 6j + 6k & -2 - 2i - 4j - 6k & -4 - 6i + 2j + 2k \end{pmatrix}.
\]

Consider the system (1). By a straightforward calculation, we have

\[
r(B \ \Phi) = r(B) = 1, \quad r(\Phi) = r(H) = 1,
\]

\[
r(\begin{pmatrix} \Omega & A & C & D \\ G & 0 & 0 & 0 \end{pmatrix}) = r(A \ C \ D) + r(G) = 3 + 2 = 5,
\]

\[
r(\begin{pmatrix} \Omega & A & C & D \\ E & 0 & F & 0 \end{pmatrix}) = r(E) + r(D) = 3 + 2 = 5,
\]

\[
r(\begin{pmatrix} \Omega & A & D \\ F & 0 & 0 \end{pmatrix}) = r(A \ D) + r(F) = 3 + 3 = 6,
\]

\[
r(\begin{pmatrix} \Omega & C & D \\ E & 0 & 0 \end{pmatrix}) = r(C \ D) + r(E) = 3 + 3 = 6,
\]

\[
r(\begin{pmatrix} A & D & \Omega & 0 \\ B & 0 & 0 & -\Phi \\ 0 & 0 & E & H \end{pmatrix}) = r(A \ D) + r(E \ H) = 4 + 4 = 8,
\]

\[
r(\begin{pmatrix} A & D & \Omega & 0 \\ B & 0 & 0 & -\Phi \\ 0 & 0 & F & 0 \end{pmatrix}) = r(A) + r(F) = 2 + 2 = 4,
\]

\[
r(\begin{pmatrix} A & D & \Omega & 0 \\ B & 0 & 0 & -\Phi \\ 0 & 0 & G & 0 \end{pmatrix}) = r(A) + r(G) = 2 + 2 = 4.
\]
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The system (1) is consistent since all the rank equalities in (15)–(21) hold. Moreover, it is easy to show that

\[
X = \begin{pmatrix} 1 & j & i+2k \\ 2 & i+3j & k \\ i-k & i & j \end{pmatrix}, \quad Y = \begin{pmatrix} k & 3+2j & i+j \\ 0 & -1+2j & 8j \\ j & 2+3j & i+k \end{pmatrix},
\]

and

\[
Z = \begin{pmatrix} i+k & 2k & 2+j \\ 2j & -1+k & 2 \\ 2+j & 2j & -1-k \end{pmatrix}
\]

is a solution that satisfies the system (1).

4. Solvability Conditions to the System (2) Involving \(\phi\)-Hermicity

In this section, we provide some consistency conditions to the system of quaternion matrix Equation (2),

\[
\left\{ \begin{array}{l}
AXA_\phi + CYC_\phi + DZ + (DZ)_\phi = \Omega, \\
BXB_\phi = \Theta,
\end{array} \right.
\]

to obtain a \(\phi\)-Hermitian solution as an application of the system (1), where \(A, B, C, D, \Omega = \Omega_\phi\) and \(\Theta = \Theta_\phi\) are given quaternion matrices, \(X\) and \(Y\) are \(\phi\)-Hermitian unknowns.

According to system (8), system (2) is equivalent to the following system

\[
\left\{ \begin{array}{l}
S_a(P^{-1}X(P_\phi)^{-1})(S_a)_\phi + S_c(Q^{-1}Y(Q_\phi)^{-1})(S_c)_\phi + S_d(S^{-1}ZM_\phi) + (S_d(S^{-1}ZM_\phi))_\phi = M\Omega M_\phi, \\
S_b(P^{-1}X(P_\phi)^{-1})(S_b)_\phi = T\Theta T_\phi,
\end{array} \right.
\]

(32)

where \(S_a, S_b, S_c\) and \(S_d\) are given in (4), \(M, P, Q, S\) and \(T\) are defined in (6) are nonsingular. Put

\[
\tilde{X} = P^{-1}X(P_\phi)^{-1} = (X_{ij})_{14\times14}, \quad \tilde{Y} = Q^{-1}Y(Q_\phi)^{-1} = (Y_{ij})_{10\times10}, \quad \tilde{Z} = S^{-1}Z(M_\phi) = (Z_{ij})_{10\times18},
\]

(33)

\[
M\Omega M_\phi = (\Omega_{ij})_{18\times18}, \quad T\Theta T_\phi = (\Theta_{ij})_{8\times8},
\]

(34)

where \((\Omega_{ij})_{18\times18}\) and \((\Theta_{ij})_{8\times8}\) have the same block rows as \(S_a\) and \(S_b\) and the same block columns as \((S_a)_\phi\) and \((S_b)_\phi\), respectively. Then, substituting (33) and (34) into the system (32) becomes

\[
\left\{ \begin{array}{l}
S_a\tilde{X}(S_a)_\phi + S_c\tilde{Y}(S_c)_\phi + S_d\tilde{Z} + (S_d\tilde{Z})_\phi = (\Omega_{ij})_{18\times18}, \\
S_b\tilde{X}(S_b)_\phi = (\Theta_{ij})_{8\times8}.
\end{array} \right.
\]

(35)

Hence, the system (2) is equivalent to the system (35). Substituting (33) and (34) into the system (35) yields

\[
S_a\tilde{X}(S_a)_\phi + S_c\tilde{Y}(S_c)_\phi + S_d\tilde{Z} + (S_d\tilde{Z})_\phi = (\Omega_{ij})_{18\times18},
\]

(36)

and

\[
S_b\tilde{X}(S_b)_\phi = (\Theta_{ij})_{8\times8},
\]

(37)
where

\[
(\Omega_i)_{18 \times 18} = \begin{pmatrix}
X_{9} + Y_{11} + Z_{11} + (Z_{11})_p & X_{8} + Y_{12} + Z_{12} & X_{4} + Z_{12} + (Z_{12})_p & Y_{11} + Z_{12} + (Z_{12})_p & X_{12} + Z_{12} + (Z_{12})_p & X_{11} + Z_{12} + (Z_{12})_p & X_{12} + Z_{12} + (Z_{12})_p & X_{12} + Z_{12} + (Z_{12})_p & \\
X_{8} + Z_{13} + (Z_{13})_p & X_{8} + Z_{14} + (Z_{14})_p & X_{8} + Z_{15} + (Z_{15})_p & X_{8} + Z_{16} + (Z_{16})_p & X_{8} + Z_{17} + (Z_{17})_p & X_{8} + Z_{18} + (Z_{18})_p & X_{8} + Z_{19} + (Z_{19})_p & X_{8} + Z_{20} + (Z_{20})_p & \\
X_{9} + Z_{21} + (Z_{21})_p & X_{10} + Z_{21} + (Z_{21})_p & X_{11} + Z_{21} + (Z_{21})_p & X_{12} + Z_{21} + (Z_{21})_p & X_{13} + Z_{21} + (Z_{21})_p & X_{14} + Z_{21} + (Z_{21})_p & X_{15} + Z_{21} + (Z_{21})_p & X_{16} + Z_{21} + (Z_{21})_p & \\
X_{8} + Z_{22} + (Z_{22})_p & X_{8} + Z_{23} + (Z_{23})_p & X_{8} + Z_{24} + (Z_{24})_p & X_{8} + Z_{25} + (Z_{25})_p & X_{8} + Z_{26} + (Z_{26})_p & X_{8} + Z_{27} + (Z_{27})_p & X_{8} + Z_{28} + (Z_{28})_p & X_{8} + Z_{29} + (Z_{29})_p & \\
Y_{1} + Z_{1} + (Z_{1})_p & Y_{2} + Z_{1} + (Z_{1})_p & Y_{3} + Z_{1} + (Z_{1})_p & Y_{4} + Z_{1} + (Z_{1})_p & Y_{5} + Z_{1} + (Z_{1})_p & Y_{6} + Z_{1} + (Z_{1})_p & Y_{7} + Z_{1} + (Z_{1})_p & Y_{8} + Z_{1} + (Z_{1})_p & \\
X_{8} + Z_{1} + (Z_{1})_p & X_{8} + Z_{2} + (Z_{2})_p & X_{8} + Z_{3} + (Z_{3})_p & X_{8} + Z_{4} + (Z_{4})_p & X_{8} + Z_{5} + (Z_{5})_p & X_{8} + Z_{6} + (Z_{6})_p & X_{8} + Z_{7} + (Z_{7})_p & X_{8} + Z_{8} + (Z_{8})_p & \\
X_{8} + Z_{9} + (Z_{9})_p & X_{8} + Z_{10} + (Z_{10})_p & X_{8} + Z_{11} + (Z_{11})_p & X_{8} + Z_{12} + (Z_{12})_p & X_{8} + Z_{13} + (Z_{13})_p & X_{8} + Z_{14} + (Z_{14})_p & X_{8} + Z_{15} + (Z_{15})_p & X_{8} + Z_{16} + (Z_{16})_p & \\
(X_{12}, Y_{5}, Z_{16}) & (X_{12}, Y_{5}, Z_{17}) & (X_{12}, Y_{5}, Z_{18}) & (X_{12}, Y_{5}, Z_{19}) & (X_{12}, Y_{5}, Z_{20}) & (X_{12}, Y_{5}, Z_{21}) & (X_{12}, Y_{5}, Z_{22}) & (X_{12}, Y_{5}, Z_{23}) & \\
(X_{12}, Y_{6}, Z_{14}) & (X_{12}, Y_{6}, Z_{15}) & (X_{12}, Y_{6}, Z_{16}) & (X_{12}, Y_{6}, Z_{17}) & (X_{12}, Y_{6}, Z_{18}) & (X_{12}, Y_{6}, Z_{19}) & (X_{12}, Y_{6}, Z_{20}) & (X_{12}, Y_{6}, Z_{21}) & \\
(X_{12}, Y_{7}, Z_{12}) & (X_{12}, Y_{7}, Z_{13}) & (X_{12}, Y_{7}, Z_{14}) & (X_{12}, Y_{7}, Z_{15}) & (X_{12}, Y_{7}, Z_{16}) & (X_{12}, Y_{7}, Z_{17}) & (X_{12}, Y_{7}, Z_{18}) & (X_{12}, Y_{7}, Z_{19}) & \\
(X_{12}, Y_{8}, Z_{10}) & (X_{12}, Y_{8}, Z_{11}) & (X_{12}, Y_{8}, Z_{12}) & (X_{12}, Y_{8}, Z_{13}) & (X_{12}, Y_{8}, Z_{14}) & (X_{12}, Y_{8}, Z_{15}) & (X_{12}, Y_{8}, Z_{16}) & (X_{12}, Y_{8}, Z_{17}) & \\
\end{pmatrix}
\]
The following theorem takes into consideration the solvability conditions to the system (2) from the aspect of the partition of the coefficient matrixes of equivalent matrix equation system (35) and derives the solvability condition in terms of ranks which have an equivalence to the block matrixes condition.

**Theorem 2.** Consider the system of quaternion matrix Equation (2). Then the following statements are equivalent:

1. The system (2) is consistent.
2. The ranks satisfy:
   \[ r \begin{pmatrix} \Theta \\ B \end{pmatrix} = r(B), \]  
   \[ r \begin{pmatrix} \Omega & D \\ A \end{pmatrix} = r(A \ C \ D) + r(D), \]  
   \[ r \begin{pmatrix} \Omega & C \\ A \phi \end{pmatrix} = r(C \ D) + r(A \ D), \]
   \[ r \begin{pmatrix} A & D & \Omega & 0 \\ B & 0 & 0 & -\Theta \\ 0 & 0 & A \phi & B \phi \\ 0 & 0 & C \phi & 0 \end{pmatrix} = r \begin{pmatrix} A & D \\ B & 0 \end{pmatrix} + r \begin{pmatrix} A & C \\ B & 0 & 0 \end{pmatrix}. \]
3. The block matrixes satisfy:
   \[ \begin{pmatrix} \Theta_{18} \\ \Theta_{28} \\ \vdots \\ \Theta_{88} \end{pmatrix} = 0, \]
   \[ \left\{ \begin{array}{l} \Omega_{2,18} = 0, \Omega_{6,18} = 0, \Omega_{9,18} = 0, \Omega_{12,18} = 0, \Omega_{16,18} = 0, \Omega_{18,18} = 0, \\
   \Omega_{4,18} = 0, \Omega_{7,18} = 0, \Omega_{11,18} = 0, \Omega_{14,18} = 0, \Omega_{15,18} = 0, \Omega_{16,18} = 0, \end{array} \right. \]
   \[ \Omega_{6,16} = 0, \Omega_{12,16} = 0, \]
   \[ \left\{ \begin{array}{l} \Omega_{12,12} = \Theta_{66}, \\
   \Omega_{11,12} - \Omega_{12,13} \phi = \Theta_{56}, \\
   \Omega_{9,12} = \Theta_{36}. \end{array} \right. \]

In this case, the general solution to the system (2) can be written as
   \[ X = P \bar{X} P, \quad Y = Q \bar{Y} Q, \quad Z = S Z M_{\phi}^{-1}, \]
We utilize the method similar with solving the system (1) to solve the system (2). Matrixes over $\mathbb{Q}$, $\mathbb{W}$, and $\mathbb{S}$ are defined in (6), $S_{ij}$ and $\Theta_{ij}$ are defined in (34), $X_{ij}$, $Y_{ij}$, and $Z_{ij}$ are arbitrary matrixes over $\mathbb{H}$ with appropriate sizes.

Proof. We utilize the method similar with solving the system (1) to solve the system (2).

(1) $\Rightarrow$ (2): Suppose $(X_0, Y_0, Z_0)$ is a solution to the system (2), that is

$$\begin{align*}
AX_0A^* + CY_0C^* + DZ_0 + (DZ_0)\lambda &= \Omega, \\
BX_0B^* &= \Theta.
\end{align*}$$

(47)

We can make advantage of elementary matrix operations to show that the rank equalities (38)–(41) hold.

(2) $\Leftrightarrow$ (3): From $S_{ij}, S_{ij}, \Theta_{ij}$ in (4), we can have that

$$r \left( \begin{array}{c} \Theta \\ B^* \end{array} \right) = r(B) = r(S_{ij}) = r(S_{ij})\lambda = r(B^*)$$

$\Leftrightarrow r \left( \begin{array}{c} \Theta \\ B^* \end{array} \right) = r(B^*)$ $\Leftrightarrow r \left( \begin{array}{c} \Theta \\ S_{ij} \end{array} \right) = r(S_{ij})\lambda$ $\Leftrightarrow r \left( \begin{array}{c} \Theta \\ S_{ij} \end{array} \right) = r(S_{ij})\lambda$

$\Leftrightarrow \cdot = 0.$

Other equivalent relation between block matrixes and ranks can be obtain through similar method as shown in above, and then we only give out the results for the remaining three correspondences:

$$\begin{align*}
r \left( \begin{array}{c} \Omega \\ A^* \\ D^* \end{array} \right) &= r(A, C, D) + r(D), \\
\Omega_{18} = 0, \Omega_{4,18} = 0, \Omega_{9,18} = 0, \Omega_{12,18} = 0, \Omega_{16,18} = 0, \Omega_{18,18} = 0, \\
\Omega_{4,18} = \Omega_{7,18}, \Omega_{5,18} = \Omega_{14,18}, \Omega_{11,18} = \Omega_{13,18}.
\end{align*}$$
\[ r \begin{pmatrix} \Omega & C & D \\ A_\phi & 0 & 0 \\ D_\phi & 0 & 0 \end{pmatrix} = r(C, D) + r(A, D) \]
\[ \Leftrightarrow \Omega_{6,16} = 0, \Omega_{6,18} = 0, \Omega_{12,16} = 0, \Omega_{12,18} = 0, \Omega_{18,16} = 0, \Omega_{18,18} = 0, \]
\[ r \begin{pmatrix} A & D & 0 \\ B & 0 & -\Theta \\ 0 & C_\phi & 0 \\ 0 & 0 & D_\phi \end{pmatrix} = r \begin{pmatrix} A & D \\ B & 0 \end{pmatrix} + r \begin{pmatrix} A & C & D \\ B & 0 & 0 \end{pmatrix} \]
\[ \Leftrightarrow \begin{cases} \Omega_{9,12} = \Theta_{36}, \Omega_{18,18} = 0, (\Omega_{12,16})_{\phi} = 0, \\ \Omega_{12,12} = \Theta_{66}, \Omega_{16,18} = 0, \Omega_{12,18} = 0, \\ \Theta_{56} = \Omega_{11,12} - (\Omega_{13,12})_{\phi}, \Omega_{11,18} = \Omega_{13,18}. \end{cases} \]

(3) \Rightarrow (1): If (42)–(45) have been set up, then (46) is the general solution of the system (2).

**Remark 2.** We utilize a new method which differs from the one in [10] to obtain our result.

5. A Numerical Example to System (2)

The goal of this section is to present a numerical example to the system (2) to illustrate the availability of the Theorem 2.

**Example 2.** Let

\[ A = \begin{pmatrix} i & 1 & -1 \\ j & j & i \\ -i & k & k \end{pmatrix}, \quad B = \begin{pmatrix} i - k & 2 & 0 \\ j & i & k \\ 0 & j & 1 \end{pmatrix}, \]

\[ C = \begin{pmatrix} k & 3 & j \\ 0 & i & 0 \\ i - k & 2 & 1 + k \end{pmatrix}, \quad D = \begin{pmatrix} k & 1 & 2i \\ i & j & -j \\ -k & 3k & i \end{pmatrix}, \]

\[ \Omega = \Omega_\phi = \begin{pmatrix} 5 + 6i + 20j & -2 + 4i - j + 2k & 4 - 2i + 4j - 4k \\ -2 + 4i - j - 2k & -2 + 3i + 3j & -1 + 2j - 8k \\ 4 - 2i + 4j + 4k & -1 + 2j + 8k & 5 + 3i + 3j \end{pmatrix}, \]

\[ \Theta = \Theta_\phi = \begin{pmatrix} -8 - 2i + 4j & 2 - 2i - 2k & i - 4j + k \\ 2 - 2i + 2k & 1 - i + j & -1 \\ i - 4j - k & -1 & 3 + j \end{pmatrix}, \]

where \(\phi(a) = a^{k*} = -ka^*k\) for \(a \in \mathbb{H}\). We consider the \(\phi\)-Hermitian solution to the system (2). Check that

\[ r \begin{pmatrix} \Theta \\ B_\phi \end{pmatrix} = r(B) = 3, \]

\[ r \begin{pmatrix} \Omega & D \\ A_\phi & 0 \\ C_\phi & 0 \\ D_\phi & 0 \end{pmatrix} = r(A, C, D) + r(D) = 3 + 3 = 6, \]

\[ r \begin{pmatrix} \Omega & C & D \\ A_\phi & 0 & 0 \\ C_\phi & 0 & 0 \\ D_\phi & 0 & 0 \end{pmatrix} = r(C, D) + r(A, D) = 3 + 3 = 6. \]
\[
\begin{pmatrix}
A & D & \Omega & 0 \\
B & 0 & 0 & -\Theta \\
0 & 0 & A_\phi & B_\phi \\
0 & 0 & C_\phi & 0 \\
0 & 0 & D_\phi & 0 \\
\end{pmatrix}
= r \begin{pmatrix}
A & D \\
B & 0 \\
\end{pmatrix}
+ r \begin{pmatrix}
A & C & D \\
B & 0 & 0 \\
\end{pmatrix}
= 6 + 6 = 12.
\]

The system (2) has a \(\phi\)-Hermitian solution since all the rank equalities in (38)–(41) hold. Note that

\[
X = X_\phi = \begin{pmatrix}
1 & i & -k & j \\
i + k & j & 1 - j & 1 \\
j & 1 - j & 1 & 1 \\
\end{pmatrix},
Y = Y_\phi = \begin{pmatrix}
j & i & k \\
i & 1 & 1 - k & 1 \\
-k & 1 + k & 1 + i & 1 \\
\end{pmatrix},
\]

and

\[
Z = \begin{pmatrix}
1 & i & k \\
-j & 2 & i & 0 \\
\end{pmatrix}
\]

is a solution that satisfies the system (2).

6. Conclusions

We have investigated a Sylvester-like quaternion matrix equation system (1) making use of simultaneous decomposition of four quaternion matrixes to deduce some useful equivalent conditions of equation system consistency in terms of the partition of the coefficient matrixes of equivalent matrix equation system, also derived ranks condition according to the partition condition. Therefore, we have provided the general solution to the system (1). Based on the result of the system (1), we have infered the consistency conditions and general \(\phi\)-Hermitian solution to the system (2). We also give several numerical examples to illustrate the main outcome.

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References

1. Yu, S.W.; He, Z.H.; Qi, T.C.; Wang, X.X. The equivalence canonical form of five quaternion matrices with applications to imaging and Sylvester-type equations. *J. Comput. Appl. Math.* **2021**, *393*, 113494. [CrossRef]
2. Yuan, S.F.; Wang, Q.W. On solutions of the quaternion matrix equation AX=B and their applications in color image restoration. *Appl. Math. Comput.* **2013**, *221*, 10–20.
3. Adler, S.L. Scattering and decay theory for quaternionic quantum mechanics, and the structure of induced Tnonconservation. *Phys. Rev. D* **1988**, *37*, 3654–3662. [CrossRef] [PubMed]
4. Marek, D.; Lucjan, S. Foundations of the Quaternion Quantum Mechanics. *Entropy* **2020**, *22*, 1424.
5. Bihan, N.L.; Mars, J. Singular value decomposition of quaternion matrices: A new tool for vector-sensor signal processing. *Signal Process.* **2004**, *84*, 1177–1199. [CrossRef]
6. He, Z.H. Some quaternion matrix equations involving \(\phi\)-Hermicity. *Filomat* **2019**, *33*, 5097–5112. [CrossRef]
7. He, Z.H.; Liu, J.; Tam, T.Y. The general \(\phi\)-Hermitian solution to mixed pairs of quaternion matrix Sylvester equations. *Electron. J. Linear Algebra* **2017**, *32*, 475–499. [CrossRef]
8. He, Z.H.; Wang, M. Solvability conditions and general solutions to some quaternion matrix equations. *Math. Methods Appl. Sci.* **2021**, *44*, 14274–14291. [CrossRef]
9. He, Z.H.; Wang, Q.W.; Zhang, Y. The complete equivalence canonical form of four matrices over an arbitrary division ring. *Linear Multilinear Algebra* **2018**, *66*, 74–95. [CrossRef]
10. He, Z.H. A system of coupled quaternion matrix equations with seven unknowns and its applications. *Adv. Appl. Clifford Algebr. 2019*, 29, 38. [CrossRef]
11. Kyrchei, I. Explicit representation formulas for the minimum norm least squares solutions of some quaternion matrix equations. *Linear Algebra Appl. 2018*, 438, 136–152. [CrossRef]
12. Kyrchei, I. Determinantal representations of solutions to systems of quaternion matrix equations. *Adv. Appl. Clifford Algebras 2018*, 28, 23. [CrossRef]
13. Kyrchei, I. Cramer’s rules for Sylvester quaternion matrix equation and its special cases. *Adv. Appl. Clifford Algebras 2018*, 28, 90. [CrossRef]
14. Kyrchei, I. Cramer’s Rules of η-(skew-) Hermitian solutions to the quaternion Sylvester-type matrix equations. *Adv. Appl. Clifford Algebras 2019*, 29, 56.
15. Liu, L.S.; Wang, Q.W.; Mehany, M.S. A Sylvester-Type Matrix Equation over the Hamilton Quaternions with an Application. *Mathematics 2022*, 10, 1758.
16. Liu, X.; Zhang, Y. Consistency of split quaternion matrix equations $AX^* - XB = CY + D$ and $X - AX^*B = CY + D$. *Adv. Appl. Clifford Algebras 2019*, 29, 64. [CrossRef]
17. Liu, X.; Wang, Q.W.; Zhang, Y. Consistency of quaternion matrix equations $AX^* - XB = C$ and $X - AX^*B = C$. *Electron. J. Linear Algebra 2019*, 35, 394–407. [CrossRef]
18. Mehany, M.S.; Wang, Q.W. Three symmetrical systems of coupled Sylvester-like quaternion matrix equations. *Symmetry 2022*, 14, 550. [CrossRef]
19. Rehman, A.; Wang, Q.W. A system of matrix equations with five variables. *Appl. Math. Comput. 2015*, 271, 805–819. [CrossRef]
20. Song, C.; Chen, G.; Liu, Q. Explicit solutions to the quaternion matrix equations $X - AXF = C$ and $X - A^TF = C$. *Int. J. Comput. Math. 2012*, 89, 890–900. [CrossRef]
21. Song, G.J.; Wang, Q.W.; Yu, S.W. Cramer’s rule for a system of quaternion matrix equations with applications. *Appl. Math. Comput. 2018*, 336, 490–499. [CrossRef]
22. Wang, Q.W. Bisymmetric and centrosymmetric solutions to system of real quaternion matrix equations. *Comput. Math. Appl. 2005*, 49, 641–650. [CrossRef]
23. Wang, Q.W. The general solution to a system of real quaternion matrix equations. *Comput. Math. Appl. 2005*, 49, 665–675. [CrossRef]
24. Wang, Q.W.; van der Woude, J.W.; Chang, H.X. A system of real quaternion matrix equations with applications. *Linear Algebra Appl. 2009*, 431, 2291–2303. [CrossRef]
25. Wang, Q.W.; He, Z.H. Constrained two-sided coupled Sylvester-type quaternion matrix equations. *Automatica 2019*, 101, 207–213. [CrossRef]
26. Wang, Q.W.; Zhang, X.; van der Woude, J.W. A new simultaneous decomposition of a matrix quaternity over an arbitrary division ring with applications. *Comm. Algebra 2012*, 40, 2309–2342. [CrossRef]
27. Wang, R.N.; Wang, Q.W.; Zhang, X.; Liu, L.S. Solving a System of Sylvester-like Quaternion Matrix Equations. *Symmetry 2022*, 14, 1056. [CrossRef]
28. Xu, Y.F.; Wang, Q.W.; Liu, L.S.; Mehany, M.S. A constrained system of matrix equations. *Comput. Appl. Math. 2022*, 41, 166. [CrossRef]
29. Yuan, S.F.; Tian, Y.; Li, M.Z. On Hermitian solutions of the reduced biquaternion matrix equation $(AXB, CXD) = (E, G)$. *Linear Multilinear Algebra 2020*, 68, 1355–1373. [CrossRef]
30. Zhang, F.X.; Wei, M.S.; Li, Y.; Zhao, J.L. Special least squares solutions of the quaternion matrix equation $AX = B$ with applications. *Appl. Math. Comput. 2015*, 270, 425–433.
31. Zhang, Y.; Wang, R.H. The exact solution of a system of quaternion matrix equations involving η-Hermicity. *Appl. Math. Comput. 2013*, 222, 201–209. [CrossRef]
32. Futorny, V.; Klymchuk, T. Roth’s solvability criteria for the matrix equations $AX - \hat{X}B = C$ and $X - AX\hat{B} = C$ over the skew field of quaternions with an involutive automorphism $q \to \hat{q}$. *Linear Algebra Appl. 2016*, 510, 246–258. [CrossRef]
33. Dmytryshyn, A.; Futorny, V.; Klymchuk, T.; Sergeichuk, V.V. Generalization of Roth’s solvability criteria to systems of matrix equations. *Linear Algebra Appl. 2017*, 527, 294–302. [CrossRef]
34. Rodman, L. *Topics in Quaternion Linear Algebra*; Princeton University Press: Princeton, NJ, USA, 2014.