STABILITY INEQUALITIES FOR PROJECTIONS OF CONVEX BODIES

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ABSTRACT. The projection function $P_K$ of an origin-symmetric convex body $K$ in $\mathbb{R}^n$ is defined by $P_K(\xi) = |K|\xi \perp$, $\xi \in S^{n-1}$, where $K|\xi \perp$ is the projection of $K$ to the central hyperplane $\xi \perp$ perpendicular to $\xi$, and $|K|$ stands for volume of proper dimension.

We prove several stability and separation results for the projection function. For example, if $D$ is a projection body in $\mathbb{R}^n$ which is in isotropic position up to a dilation, and $K$ is any origin-symmetric convex body in $\mathbb{R}^n$ such that there exists $\xi \in S^{n-1}$ with $P_K(\xi) > P_D(\xi)$, then

$$\max_{\xi \in S^{n-1}} (P_K(\xi) - P_D(\xi)) \geq \frac{c}{\log^2 n}(|K|^{\frac{n-1}{n}} - |D|^{\frac{n-1}{n}}),$$

where $c$ is an absolute constant.

As a consequence, we prove a hyperplane inequality

$$S(D) \leq C \log^2 n \max_{\xi \in S^{n-1}} S(D|\xi \perp) |D|^{\frac{1}{n}},$$

where $D$ is a projection body in isotropic position, up to a dilation, $S(D)$ is the surface area of $D$, and $C$ is an absolute constant. The proofs are based on the Fourier analytic approach to projections developed in [KRZ].

1. INTRODUCTION

Let $K$ be an origin-symmetric convex body in $\mathbb{R}^n$. The projection function $P_K: S^{n-1} \to (0, \infty)$ of $K$ is defined for every $\xi \in S^{n-1}$ as the $(n-1)$-dimensional volume of the orthogonal projection of $K$ to the central hyperplane $\xi \perp$ perpendicular to $\xi$. We write

$$P_K(\xi) = |K|\xi \perp, \quad \forall \xi \in S^{n-1},$$

where $K|\xi \perp$ is the projection of $K$ to $\xi \perp$, and $|K|$ stands for volume of proper dimension.

The classical uniqueness theorem of Aleksandrov [A] states that every origin-symmetric convex body is uniquely determined by the function $P_K$; see also [G, Theorem 3.3.6]. The corresponding volume comparison question was posed by Shephard [Sh] in 1964. Suppose that $K, D$
are origin-symmetric convex bodies in $\mathbb{R}^n$, and $P_K(\xi) \leq P_D(\xi)$ for every $\xi \in S^{n-1}$. Does it necessarily follow that $|K| \leq |D|$? Shephard’s problem was solved by Petty [Pe] and Schneider [S1], independently, and the answer is affirmative only in dimension $n = 2$. Both solutions were based on a connection with projection bodies (see definition in Section 2), as follows. If $D$ is a projection body, the answer to Shephard’s question is affirmative for every $K$. On the other hand, if $K$ is not a projection body, one can construct $D$ giving together with $K$ a counterexample. The final answer to the problem follows from the fact that only in dimension $n = 2$ all origin-symmetric convex bodies are projection bodies.

For the affirmative cases in volume comparison problems, stability and separation problems were proposed in [K2]. Suppose that $\varepsilon > 0$, $K, L$ are origin-symmetric convex bodies in $\mathbb{R}^n$, and $L$ is a projection body. The stability problem asks whether there exists a constant $c > 0$ not dependent on $K, L$ or $\varepsilon$ and such that the inequalities $P_K(\xi) \leq P_L(\xi) + \varepsilon$, for all $\xi \in S^{n-1}$, imply $|K|^{\frac{n-1}{n}} \leq |L|^{\frac{n-1}{n}} + c\varepsilon$. The separation problem asks whether inequalities $P_K(\xi) \leq P_L(\xi) - \varepsilon$, for all $\xi \in S^{n-1}$, imply $|K|^{\frac{n-1}{n}} \leq |L|^{\frac{n-1}{n}} - c\varepsilon$, again with a constant $c > 0$ not dependent on $K, L, \varepsilon$.

Separation was proved in [K2] (see also [K3], where the constant is written precisely) with the best possible constant

$$c = c_n = \frac{|B_2^n|^{\frac{n-1}{n}}}{|B_2^{n-1}|}, \quad c_n \in \left(\frac{1}{\sqrt{e}}, 1\right),$$

where $B_2^n$ is the unit Euclidean ball in $\mathbb{R}^n$. Stability was proved in [K2] with constants dependent on the bodies, using rough estimates for $M$ and $M^*$ parameters of convex bodies. In this article we prove stability with constants not dependent on the bodies, but under an additional assumption that $L$ is in isotropic position, up to a dilation; see Proposition 1. To do this, we use a recent $M^*$-estimate of Milman [M]. We also prove that, if the projection body condition is dropped, one can get results going in the direction opposite to stability and separation. Namely, we give examples of origin-symmetric convex bodies $K, L$ such that $P_K \leq P_L + \varepsilon$, but $|K|^{\frac{n-1}{n}} \geq |L|^{\frac{n-1}{n}} + c\varepsilon$, and also examples of $K, L$ where $P_K \leq P_L - \varepsilon$, but $|K|^{\frac{n-1}{n}} \geq |L|^{\frac{n-1}{n}} - c\varepsilon$, with $c$ not dependent on the bodies or small enough $\varepsilon$. In some sense, these results provide a quantitative version of the solution to Shephard’s problem.

Stability and separation immediately imply what we call volume difference inequalities. In fact, if stability holds and there exists $\xi \in S^{n-1}$ such that $P_K(\xi) > P_L(\xi)$, then we put $\varepsilon = \max_{\xi \in S^{n-1}} (P_K(\xi) - P_L(\xi))$, 

$$\varepsilon = \max_{\xi \in S^{n-1}} (P_K(\xi) - P_L(\xi)),$$
and get
\[ |K|^\frac{n-1}{n} - |L|^\frac{n-1}{n} \leq c \max_{\xi \in S^{n-1}} (P_K(\xi) - P_L(\xi)). \]
Similarly, if separation holds, we get
\[ |L|^\frac{n-1}{n} - |K|^\frac{n-1}{n} \geq c \min_{\xi \in S^{n-1}} (P_L(\xi) - P_K(\xi)). \]

We provide such an inequality for the stability result mentioned above; see Theorem 1.

Volume difference inequalities lead to hyperplane inequalities for surface area of projection bodies. It was proved in [K3] that if \(D\) is a projection body in \(\mathbb{R}^n\), then
\[ S(D) \geq c \min_{\xi \in S^{n-1}} S(D|\xi^\perp) |D|^\frac{1}{n}, \tag{1} \]
where \(S(D)\) is the surface area of \(D\), and \(c\) is an absolute constant.

In this paper we prove an inequality that complements (1). If \(D\) is a projection body in isotropic position, up to a dilation, then
\[ S(D) \leq C \log^2 n \max_{\xi \in S^{n-1}} S(D|\xi^\perp) |D|^\frac{1}{n}, \]
where \(C\) is an absolute constant; see Theorem 2 below.

Finally, we show that if the condition that \(D\) is a projection body is removed, volume difference inequalities can go in the opposite direction; see Theorems 3 and 4.

We extensively use the Fourier analytic approach to projections of convex bodies developed in [KRZ]; see also [K1, Chapter 8].

2. Stability theorems

We need several definitions from convex geometry. We refer the reader to [S2] for details.

The support function of a convex body \(K\) in \(\mathbb{R}^n\) is defined by
\[ h_K(x) = \max_{\{\xi \in \mathbb{R}^n : \|\xi\|_K = 1\}} (x, \xi), \quad x \in \mathbb{R}^n. \]
If \(K\) is origin-symmetric, then \(h_K\) is a norm on \(\mathbb{R}^n\).

The surface area measure \(S(K, \cdot)\) of a convex body \(K\) in \(\mathbb{R}^n\) is a measure on \(S^{n-1}\) defined as follows. For every Borel set \(E \subset S^{n-1}\), \(S(K, E)\) is equal to Lebesgue measure of the part of the boundary of \(K\) where normal vectors belong to \(E\). We usually consider bodies with absolutely continuous surface area measures. A convex body \(K\) is said to have the curvature function \(f_K : S^{n-1} \rightarrow \mathbb{R}\) if its surface
area measure $S(K, \cdot)$ is absolutely continuous with respect to Lebesgue measure $\sigma_{n-1}$ on $S^{n-1}$, and
\[
\frac{dS(K, \cdot)}{d\sigma_{n-1}} = f_K \in L_1(S^{n-1}),
\]
so $f_K$ is the density of $S(K, \cdot)$.

The volume of a body can be expressed in terms of its support function and curvature function:
\[
|K| = \frac{1}{n} \int_{S^{n-1}} h_K(x) dS(K, x) = \frac{1}{n} \int_{S^{n-1}} h_K(x) f_K(x) \, dx,
\]
with the latter equality if $f_K$ exists.

If $K$ and $L$ are two convex bodies in $\mathbb{R}^n$, the mixed volume $V_1(K, L)$ is equal to
\[
V_1(K, L) = \frac{1}{n} \lim_{\varepsilon \to 0^+} \frac{|K + \varepsilon L| - |K|}{\varepsilon}.
\]
The first Minkowski inequality (see for example [K1, p.23]) asserts that for any convex bodies $K, L$ in $\mathbb{R}^n$,
\[
V_1(K, L) \geq |K|^{\frac{n-1}{n}} |L|^{\frac{1}{n}}.
\]
Mixed volume can also be expressed in terms of the support and curvature functions:
\[
V_1(K, L) = \frac{1}{n} \int_{S^{n-1}} h_L(x) dS(K, x) = \frac{1}{n} \int_{S^{n-1}} h_L(x) f_K(x) \, dx.
\]

Let $K$ be an origin-symmetric convex body in $\mathbb{R}^n$. The projection body $\Pi K$ of $K$ is defined by
\[
h_{\Pi K}(\theta) = |K|^{\frac{1}{n}} |\theta^\perp|, \quad \forall \theta \in S^{n-1}.
\]

If $L$ is the projection body of some convex body, we simply say that $L$ is a projection body. We refer the reader to [K1, Chapter 8] for necessary information about projection bodies. We just mention here that, by a result of Bolker [Bl], an origin-symmetric convex body is a projection body if and only if its polar body is the unit ball of a subspace of $L_1$. The unit balls of the spaces $\ell_p^n$, $p \geq 2$ are projection bodies, while the unit balls of $\ell_p^n$, $p < 2, n \geq 3$ are not. There are other examples of bodies in $\mathbb{R}^n$, $n \geq 3$ that are not projection bodies; see [S2] or [K1].

The classes of projection bodies $K$ for which the functions $h_K$ and $f_K$ are infinitely smooth are dense in the class of all projection bodies in the Hausdorff metric; see [S2, p.151] and [GZ]. The class of projection
bodies for which the curvature function is strictly positive is also dense in the class of all projection bodies; see [GZ] or [K1, p.158, p. 161].

We say that a body $K$ is in isotropic position if $|K| = 1$ and there exists a constant $L_K > 0$ such that

$$\int_K (x, \xi)^2 dx = L_K^2, \quad \forall \xi \in S^{n-1}. $$

For every origin-symmetric convex body $K$ in $\mathbb{R}^n$ there exists $T \in \text{GL}_n$ such that $TK$ is in isotropic position. The constant $L_K$ is called the isotropic constant of $K$. It is known that $L_K \geq L_{B^n_2}$ for every symmetric convex body $K$ in $\mathbb{R}^n$; see [MP, p.93]. The question of whether $L_K$ is bounded from above by an absolute constant is the matter of the well-known and still open slicing problem. The best known estimate $L_K \leq O(n^{1/4})$ is due to Klartag [Kl], who improved an earlier estimate of Bourgain [Bo]. We refer the reader to [BGVV] for these results and more about the isotropic position and the slicing problem.

We use the Fourier approach to projections of convex bodies developed in [KRZ]; see also [K1, Chapter 8]. We consider Schwartz distributions, i.e. continuous functionals on the space $S(\mathbb{R}^n)$ of rapidly decreasing infinitely differentiable functions on $\mathbb{R}^n$. The Fourier transform of a distribution $f$ is defined by $\langle \hat{f}, \phi \rangle = \langle f, \hat{\phi} \rangle$ for every test function $\phi \in S(\mathbb{R}^n)$. For any even distribution $f$, we have $(\hat{f})^\wedge = (2\pi)^n f$.

For $f \in C^\infty(S^{n-1})$ and $p = 1$ or $p = -n - 1$, we denote by

$$(f \cdot r^p)(x) = f(x/|x|_2)|x|_2^p$$

the extension of $f$ to a homogeneous function of degree $p$ on $\mathbb{R}^n$. By [K1, Lemma 3.16], there exists $g \in C^\infty(S^{n-1})$ such that

$$(f \cdot r^p)^\wedge = g \cdot r^{-n-p}. \quad (6)$$

In particular, if the support function $h_K$ is infinitely smooth on $S^{n-1}$, then the Fourier transform of $h_K \cdot r$ is an infinitely smooth function $g$ on the sphere extended to a homogeneous function of degree $-n - 1$ on $\mathbb{R}^n \setminus \{0\}$. In this case we simply write $(h_K \cdot r)^\wedge(\theta)$ for $\theta \in S^{n-1}$, meaning the function $g$. It was proved in [KRZ] (see also [K1, Theorem 8.6]) that an origin-symmetric convex body $K$, for which $h_K$ is infinitely smooth, is a projection body if and only if

$$(h_K \cdot r)^\wedge(\theta) \leq 0, \quad \forall \theta \in S^{n-1}. \quad (7)$$

It was also proved in [KRZ] (see also [K1, Theorem 8.2]) that if an origin-symmetric body $K$ has curvature function $f_K$, then

$$(f_K \cdot r^{-n-1})^\wedge(\theta) = -\pi |K|_{\theta \perp} = -\pi P_K(\theta), \quad \forall \theta \in S^{n-1}. \quad (8)$$
The following version of Parseval’s formula was proved in [KRZ] (see also [K1, Lemma 8.8]). If \( K, L \) are origin-symmetric convex bodies, \( K \) has infinitely smooth support function and \( L \) has infinitely smooth curvature function, then
\[
\int_{S^{n-1}} (h_K \cdot r)^\wedge(\xi)(f_L \cdot r^{-n-1})^\wedge(\xi) \\ d\xi = (2\pi)^n \int_{S^{n-1}} h_K(x)f_L(x) \\ dx.
\]
(9)

**Lemma 1.** Let \( K \) be an origin-symmetric convex body in \( \mathbb{R}^n \) such that the support function \( h_K \) is infinitely smooth. Then
\[
\int_{S^{n-1}} (h_K \cdot r)^\wedge(\xi) \\ d\xi \leq -\frac{(2\pi)^n}{\pi} c_n |K|^{1/n}.
\]

Recall that \( c_n = |B_2^n|^{1/n}/|B_2^{n-1}| \in (\sqrt{\frac{1}{e}}, 1) \).

**Proof :** The curvature function of the unit Euclidean ball \( B_2^n \) is constant, \( f_2 \equiv 1 \). By (8),
\[
(f_2 \cdot r^{-n-1})^\wedge(\xi) = -\pi |B_2^{n-1}|, \quad \forall \xi \in S^{n-1}.
\]

By (4), (3) and (9),
\[
\int_{S^{n-1}} (h_K \cdot r)^\wedge(\xi) \\ d\xi = -\frac{1}{\pi|B_2^{n-1}|} \int_{S^{n-1}} (h_K \cdot r)^\wedge(\xi)(f_2 \cdot r^{-n-1})^\wedge(\xi) \\ d\xi
\]
\[
= -\frac{(2\pi)^n}{\pi|B_2^{n-1}|} \int_{S^{n-1}} h_K(x)f_2(x) \\ dx \leq -\frac{(2\pi)^n |K|^{1/n}}{\pi|B_2^{n-1}|} |B_2^n|^{(n-1)/n}. \quad \Box
\]

**Lemma 2.** Let \( K \) be an origin-symmetric convex body in \( \mathbb{R}^n \) such that the support function \( h_K \) is infinitely smooth and \( K \) is a dilate of an isotropic body. Then
\[
\int_{S^{n-1}} (h_K \cdot r)^\wedge(\xi) \\ d\xi \geq -C(2\pi)^n n \log^2(1 + n) L_K |K|^{1/n},
\]
where \( C \) is an absolute constant.

**Proof :** By the same argument as in Lemma 1 (recall that \( f_2 \equiv 1 \)) we have
\[
\int_{S^{n-1}} (h_K \cdot r)^\wedge(\xi) \\ d\xi = -\frac{(2\pi)^n}{\pi|B_2^{n-1}|} \int_{S^{n-1}} h_K(x) \\ dx.
\]

Now use the following estimate of E.Milman [M, Theorem 1.1]
\[
\frac{1}{|S^{n-1}|} \int_{S^{n-1}} h_K(x) \\ dx \leq C_1 \sqrt{n} \log^2(1 + n) L_K |K|^{1/n},
\]
where \( C_1 \) is an absolute constant, and note that \( |S^{n-1}|/|B_2^{n-1}| \sim \sqrt{n} \) to get the result. \( \Box \)
We now prove stability in the affirmative direction of Shephard’s problem under the additional condition that the body $D$ is a dilate of an isotropic body.

**Proposition 1.** Suppose that $\varepsilon > 0$, $K$ and $D$ are origin-symmetric convex bodies in $\mathbb{R}^n$, and $D$ is a projection body which is a dilate of an isotropic body. If for every $\xi \in S^{n-1}$

$$P_K(\xi) \leq P_D(\xi) + \varepsilon,$$

then

$$|K|^{\frac{n-1}{n}} \leq |D|^{\frac{n-1}{n}} + C\varepsilon \log (1 + n) L_D,$$

where $C$ is an absolute constant.

**Proof:** By approximation ([S2, Th. 3.3.1] and [GZ, Section 5]), we can assume that $D$ has infinitely smooth support function and both $D$ and $K$ have infinitely smooth curvature functions. By (8), the condition (10) can be written as

$$-\frac{1}{\pi} (f_K \cdot r^{-1})^\wedge(\xi) \leq -\frac{1}{\pi} (f_D \cdot r^{-1})^\wedge(\xi) + \varepsilon, \quad \forall \xi \in S^{n-1}.$$

(11)

By (7), $(h_D \cdot r)^\wedge \leq 0$ on the sphere $S^{n-1}$. Therefore, integrating (11) with respect to a negative density, we get

$$\int_{S^{n-1}} (h_D \cdot r)^\wedge(\xi) (f_D \cdot r^{-1})^\wedge(\xi) d\xi$$

$$\geq \int_{S^{n-1}} (h_D \cdot r)^\wedge(\xi) (f_K \cdot r^{-1})^\wedge(\xi) d\xi + \pi \varepsilon \int_{S^{n-1}} (h_D \cdot r)^\wedge(\xi) d\xi.$$

Using this, (2), (4), Parseval’s formula (9) and the first Minkowski inequality (3),

$$(2\pi)^n n |D| = (2\pi)^n \int_{S^{n-1}} h_D(x) f_D(x) \, dx$$

$$\geq (2\pi)^n \int_{S^{n-1}} h_D(x) f_K(x) \, dx + \pi \varepsilon \int_{S^{n-1}} (h_D \cdot r)^\wedge(\xi) d\xi.$$ 

$$\geq (2\pi)^n |D|^\frac{1}{2} |K^{\frac{n-1}{n}} + \pi \varepsilon \int_{S^{n-1}} (h_D \cdot r)^\wedge(\xi) d\xi.$$

The result follows from Lemma 2. \qed

Now we show that if the projection body condition is dropped, the result may go in the opposite direction.
Proposition 2. Suppose that $K$ is an origin-symmetric convex body in $\mathbb{R}^n$, which is not a projection body. Suppose also that $h_K, f_K \in C^\infty(S^{n-1})$, and $f_K$ is strictly positive on $S^{n-1}$. Then for small enough $\varepsilon > 0$ there exists an origin-symmetric convex body $D$ in $\mathbb{R}^n$ so that
\[
P_D(\theta) \leq P_K(\theta) \leq P_D(\xi) + \varepsilon, \quad \forall \theta \in S^{n-1},
\]
but
\[
|K|^{\frac{n-1}{n}} \geq |D|^{\frac{n-1}{n}} + c_n \varepsilon.
\]

Proof: By (7), since $K$ is not a projection body, there exists a symmetric open set $\Omega \subset S^{n-1}$, where $(h_K \cdot r)^\wedge > 0$. Let $v$ be an even infinitely smooth non-negative function supported in $\Omega$. Extend $v$ to a homogeneous function $v \cdot r$ of degree 1 on $\mathbb{R}^n$. The Fourier transform of $v \cdot r$ is a homogeneous of degree $-n-1$ function $g \cdot r^{-n-1}$, where $g$ is an infinitely smooth function on the sphere; recall (6).

Define an even function $h$ on the sphere $S^{n-1}$ by
\[
f_K = h + \delta g + \frac{\varepsilon}{|B_2^{n-1}|}.
\]
Choose $\varepsilon, \delta > 0$ small enough so that $h > 0$ everywhere on $S^{n-1}$ (recall that $f_K > 0$). By the Minkowski existence theorem (see, for example, [S2, Section 7.1]), there exists an origin-symmetric convex body $D$ in $\mathbb{R}^n$, whose curvature function $f_D = h$. Extend the functions in the definition of $h$ to even homogeneous of degree $-n-1$ functions on $\mathbb{R}^n$:
\[
f_K \cdot r^{-n-1} = f_D \cdot r^{-n-1} + \delta g \cdot r^{-n-1} + \frac{\varepsilon}{|B_2^{n-1}|} \cdot r^{-n-1}. \tag{12}
\]
By (8), $(r^{-n-1})^\wedge(\theta) = (f_2 \cdot r^{-n-1})^\wedge(\theta) = -\pi |B_2^{n-1}|$ for every $\theta \in S^{n-1}$ (here $f_2 \equiv 1$ is the curvature function of the unit Euclidean ball).

Taking the Fourier transform of both sides of (12) and again using (8), we get
\[
(f_K \cdot r^{-n-1})^\wedge(\theta) = (f_D \cdot r^{-n-1})^\wedge(\theta) + (2\pi)^n \delta v(\theta) - \pi \varepsilon, \quad \forall \theta \in S^{n-1}, \tag{13}
\]
and
\[
-\pi P_K(\theta) = -\pi P_D(\theta) + (2\pi)^n \delta v(\theta) - \pi \varepsilon, \quad \forall \theta \in S^{n-1}.
\]
Since $v \geq 0$, the latter implies
\[
P_K(\theta) \leq P_D(\theta) + \varepsilon, \quad \forall \theta \in S^{n-1}.
\]
Also, choosing $\delta$ small enough, we can assure that $P_K(\theta) \geq P_D(\theta)$ for every $\theta \in S^{n-1}$.
On the other hand, multiplying (13) by \((h_K \cdot r)^{\wedge}(\theta)\) integrating over the sphere, and using the fact that \(v \geq 0\) is supported in \(\Omega\), where \((h_K \cdot r)^{\wedge} > 0\), we get

\[
\int_{S^{n-1}} (h_K \cdot r)^{\wedge}(\theta)(f_K \cdot r^{-n-1})^{\wedge}(\theta)d\theta
\]

\[
= \int_{S^{n-1}} (h_K \cdot r)^{\wedge}(\theta)(f_D \cdot r^{-n-1})^{\wedge}(\theta)d\theta
\]

\[
+(2\pi)^n \delta \int_{S^{n-1}} v(\theta)(h_K \cdot r)^{\wedge}(\theta)d\theta - \pi \varepsilon \int_{S^{n-1}} (h_K \cdot r)^{\wedge}(\theta)d\theta
\]

\[
\geq \int_{S^{n-1}} (h_K \cdot r)^{\wedge}(\theta)(f_D \cdot r^{-n-1})^{\wedge}(\theta)d\theta - \pi \varepsilon \int_{S^{n-1}} \hat{h}(\theta)d\theta.
\]

Now, by Parseval’s formula (9), (2), (4) and the first Minkowski inequality,

\[
(2\pi)^n n |K| \geq (2\pi)^n \int_{S^{n-1}} h_K(\theta)f_D(\theta)d\theta - \pi \varepsilon \int_{S^{n-1}} (h_K \cdot r)^{\wedge}(\theta)d\theta
\]

\[
\geq (2\pi)^n n |K|^\frac{1}{n} |D|^\frac{n-1}{n} - \pi \varepsilon \int_{S^{n-1}} (h_K \cdot r)^{\wedge}(\theta)d\theta.
\]

The result follows from Lemma 1. \(\square\)

The following separation result will be used to prove Theorem 4.

**Proposition 3.** Suppose that \(K\) is an origin-symmetric convex body in \(\mathbb{R}^n\) with strictly positive curvature which is not a projection body and is a dilate of an isotropic body. Also suppose that the support and curvature functions of \(K\) are infinitely smooth. Then for small enough \(\varepsilon > 0\) there exists an origin-symmetric convex body \(D\) in \(\mathbb{R}^n\) so that

\[
P_K(\theta) \leq P_D(\theta) - \varepsilon, \quad \forall \theta \in S^{n-1},
\]

but

\[
|D|^\frac{n-1}{n} \leq |K|^\frac{n-1}{n} + C \log^2(1 + n)L_K\varepsilon,
\]

where \(C\) is an absolute constant.

**Proof:** The proof follows the steps of the proof of Theorem 2. Define the functions \(v\) and \(g\) in the same way. Then define a body \(D\) by

\[
f_D = f_K - \delta g + \frac{\varepsilon}{|B_2^{n-1}|}
\]

At the very end use Lemma 2 instead of Lemma 1. \(\square\)
3. VOLUME DIFFERENCE AND HYPERPLANE INEQUALITIES

In this section we apply stability theorems to prove our main results. We start with the volume difference inequality of Theorem 1.

**Theorem 1.** Let $D$ be a projection body in $\mathbb{R}^n$ in isotropic position up to a dilation, and let $K$ be any origin-symmetric convex body in $\mathbb{R}^n$. Suppose that there exists $\xi \in S^{n-1}$ so that $P_K(\xi) > P_D(\xi)$. Then

$$\max_{\xi \in S^{n-1}} (P_K(\xi) - P_D(\xi)) \geq \frac{c}{\log^2 n} (|K|^{\frac{n-1}{n}} - |D|^{\frac{n-1}{n}}),$$

(14)

where $c$ is an absolute constant.

**Proof:** Let $\varepsilon = \max_{\xi \in S^{n-1}} (P_K(\xi) - P_D(\xi))$. By the condition of Theorem 1, $\varepsilon > 0$. Now we can apply Proposition 1 to $K, D, \varepsilon$. We get

$$|K|^{\frac{n-1}{n}} \leq |D|^{\frac{n-1}{n}} + C \log^2 (1 + n) L_D \max_{\xi \in S^{n-1}} (P_K(\xi) - P_D(\xi)),$$

where $C$ is an absolute constant. The result follows from the fact that isotropic constants of projection bodies (zonoids) are uniformly bounded by an absolute constant; see [MP, p.96]. □

“Differentiating” the inequality of Theorem 1, we prove a hyperplane inequality for the surface area of projection bodies.

**Theorem 2.** Suppose that $D$ is a projection body in $\mathbb{R}^n$ in isotropic position up to a dilation (see definition in Section 2). Then

$$S(D) \leq C \log^2 n \max_{\xi \in S^{n-1}} S(D|\xi^\perp) |D|^{\frac{1}{n}},$$

(15)

where $C$ is an absolute constant.

**Proof:** The surface area of $D$ can be computed as

$$S(D) = \lim_{\varepsilon \to +0} \frac{|D + \varepsilon B^n_2| - |D|}{\varepsilon}.$$

The inequality (14) with the bodies $K = D + \varepsilon B^n_2$ and $D$ implies

$$\frac{|D + \varepsilon B^n_2|^{\frac{n-1}{n}} - |D|^{\frac{n-1}{n}}}{\varepsilon} \leq C \log^2 n \max_{\xi \in S^{n-1}} \frac{|(D|\xi^\perp) + \varepsilon B^{n-1}_2| - |D|\xi^\perp|}{\varepsilon},$$

(16)

where $C$ is an absolute constant.

By the Minkowski theorem on mixed volumes ([S2, Theorem 5.1.6] or [G, Theorem A.3.1]),

$$\frac{|(D|\xi^\perp) + \varepsilon B^{n-1}_2| - |D|\xi^\perp|}{\varepsilon} = \sum_{i=1}^{n-1} \binom{n-1}{i} W_i(D|\xi^\perp) \varepsilon^{i-1},$$

(17)
where \( W_i \) are quermassintegrals. The function \( \xi \mapsto D|\xi^\perp \) is continuous from \( S^{n-1} \) to the class of origin-symmetric convex sets equipped with the Hausdorff metric, and \( W_i \)'s are also continuous with respect to this metric (see [S2, p.275]), so the functions \( \xi \mapsto W_i(D|\xi^\perp \) are continuous and, hence, bounded on the sphere. This implies that the left-hand side of (17) converges to \( S(D|\xi^\perp \) as \( \varepsilon \to 0 \), uniformly with respect to \( \xi \). The latter allows to switch the limit and maximum in the right-hand side of (16), as \( \varepsilon \to 0 \). Sending \( \varepsilon \) to zero in (16), we get

\[
\frac{n - 1}{n} |D|^{-1/n} S(D) \leq C \log^2 n \max_{\xi \in S^{n-1}} S(D|\xi^\perp) .
\]

\( \Box \)

Theorems 3 and 4 follow from Propositions 2 and 3 by putting \( \varepsilon = \max_{\xi \in S^{n-1}} (P_K(\xi) - P_D(\xi)) \) and \( \varepsilon = \min_{\xi \in S^{n-1}} (P_D(\xi) - P_K(\xi)) \), correspondingly.

**Theorem 3.** Suppose that \( K \) is an origin-symmetric convex body in \( \mathbb{R}^n \) with strictly positive curvature that is not a projection body. Then there exists an origin-symmetric convex body \( D \) in \( \mathbb{R}^n \) so that \( P_K(\xi) \geq P_D(\xi) \) for all \( \xi \in S^{n-1} \) and

\[
\max_{\xi \in S^{n-1}} (P_K(\xi) - P_D(\xi)) \leq \frac{1}{c_n} (|K|^{\frac{n-1}{n}} - |D|^{\frac{n-1}{n}}).
\]

**Theorem 4.** Suppose that \( K \) is an origin-symmetric convex body in \( \mathbb{R}^n \) that is not a projection body and is in isotropic position up to a dilation, with isotropic constant \( L_K \). Then there exists an origin-symmetric convex body \( D \) in \( \mathbb{R}^n \) so that \( P_D(\xi) \geq P_K(\xi) \) for all \( \xi \in S^{n-1} \) and

\[
\min_{\xi \in S^{n-1}} (P_D(\xi) - P_K(\xi)) \geq \frac{c}{L_K \log^2 n} (|D|^{\frac{n-1}{n}} - |K|^{\frac{n-1}{n}}),
\]

where \( c \) is an absolute constant.

**Remark.** Putting \( D = \beta B_2^n \) in (14) and sending \( \beta \to 0 \), we get a hyperplane inequality for volume

\[
\max_{\xi \in S^{n-1}} P_K(\xi) \geq \frac{c}{\log^2 n} |K|^{\frac{n-1}{n}}.
\]

However, a stronger inequality

\[
|L|^{\frac{n-1}{n}} \leq c_n \max_{\xi \in S^{n-1}} |L|\xi^\perp |
\]

holds for all origin-symmetric convex bodies and follows from the Cauchy projection formula for the surface area and the classical isoperimetric inequality; see [G, p. 363].
It can be deduced directly from the solution to Shephard’s problem (see [G, Corollary 9.3.4]) that, if $L$ is a projection body in $\mathbb{R}^n$, then

$$|L|^{\frac{n-1}{n}} \geq c_n \min_{\xi \in S^{n-1}} |L|_{\xi^\perp}.$$

(18)

Recall that $c_n > 1/\sqrt{e}$. For general symmetric convex bodies, Ball [Ba] proved that $c_n$ may and has to be replaced in (18) by $c/\sqrt{n}$, where $c$ is an absolute constant.

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