MOD 2 COHOMOLOGY OF 2-LOCAL FINITE GROUPS OF LOW RANK

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Abstract. We determine the mod 2 cohomology over the Steenrod algebra \( \mathcal{A}_2 \) of the classifying spaces of the free loop groups \( L_{G} \) for compact groups \( G = \text{Spin}(7), \text{Spin}(8), \text{Spin}(9), \) and \( F_4 \). Then, we show that they are isomorphic as algebras over \( \mathcal{A}_2 \) to the mod 2 cohomology of the corresponding Chevalley groups of type \( G_{2q} \), where \( q \) is an odd prime power. In a similar manner, we compute the cohomology of the free loop space over \( BDI(4) \) and show that it is isomorphic to that of \( BSol(q) \) as algebras over \( \mathcal{A}_2 \). This note is a revised version of [9].

1. Introduction

For a based self-map \( \phi : X \to X \) of a based topological space \( X \), the homotopy fixed point space (or the twisted loop space) \( L_\phi X \) of \( \phi \) is defined by the pullback

\[
\begin{array}{ccc}
\mathcal{L}_\phi X & \longrightarrow & X^{[0,1]} \\
\downarrow & & \downarrow (ev_0,ev_1) \\
X & \underset{(Id,\phi)}{\longrightarrow} & X \times X,
\end{array}
\]

where \( ev_i : X^{[0,1]} \to X \) (\( i = 0, 1 \)) are the maps assigning \( \gamma(i) \in X \) to a path \( \gamma \in X^{[0,1]} \). The free loop space \( LX = \{ \gamma : S^1 \to X \} \) over \( X \) is identified with the homotopy fixed point space of the identity map of \( X \). Kuribayashi [13] introduced a map from the cohomology of \( X \) to that of \( LX \) with degree \(-1\), called the module derivation, to compute the cohomology of \( LX \) by the Eilenberg-Moore spectral sequence of the above pullback diagram. Another interesting example of the homotopy fixed point space is that of an unstable Adams operation of the classifying space of a compact Lie group \( G \). Denote the Bousfield-Kan 2-completion [2] of \( X \) by \( X^2 \). Let \( q \) be an odd prime power throughout this note. There exists a self-map \( \psi^q \) of \( (BG)^2 \), called the unstable Adams operation of degree \( q \) ([18]), such that \( \psi^q \) induces multiplication by \( q^r \) on \( H^{2r}(BG)^2 ; \mathbb{Z}/2 \). Furthermore, we show that they are isomorphic to those of the corresponding Chevalley group of type \( G_{2q} \) as algebras over \( \mathcal{A}_2 \). In particular, \( H^*(G(q); \mathbb{Z}/2) \cong H^*(\mathcal{L}_{\psi^q}(BG)^2; \mathbb{Z}/2) \). Moreover, for the finite loop space \( BDI(4) \) at prime 2 constructed by Dwyer and Wilkerson [4], Notbohm [14] showed that there exists a self-map \( \psi^q \) of \( BDI(4) \), also called the unstable Adams operation of degree \( q \), which induces multiplication by \( q^r \) on \( H^{2r}(BDI(4); \mathbb{Z}/2) \otimes \mathbb{Q} \). Benson [11] defined the classifying space \( BSol(q) \) of an exotic 2-local finite group as \( L_{\psi^q}BDI(4) \), which can be regarded as the “classifying space” of Solomon’s non-existent finite group [16].

Kishimoto and Kono [9] generalized Kuribayashi’s work to calculate the cohomology of \( L_\phi X \). Their result provides an efficient computational tool when the cohomology of \( BG \) is a polynomial algebra. As an example, they computed \( H^*(LBG_2; \mathbb{Z}/2) \) for the compact simple exceptional Lie group \( G_2 \) and showed that it is isomorphic to \( H^*(BG_2(q); \mathbb{Z}/2) \) as algebras over the Steenrod algebra. In this note, we carry out the computation of the mod 2 cohomology of \( LBSpin(7), LBSpin(8), LBSpin(9), LBF_4, \) and \( LBDI(4) \) as algebras over the Steenrod algebra \( \mathcal{A}_2 \). Furthermore, we show that they are isomorphic as algebras over the Steenrod algebra \( \mathcal{A}_2 \) to those of the corresponding \( L_{\psi^q}(BG)^2 \). The main theorem is as follows:

\[
\text{2000 Mathematics Subject Classification.} \quad \text{Primary 55R35; Secondary 55S10.}
\]

Key words and phrases. mod 2 cohomology, free loop groups, 2-local finite groups.

The author was partially supported by the Grant-in-Aid for JSPS Fellows 182641.
Theorem 1.1. Let \( q \) be an odd prime power. The following are isomorphisms of algebras over the Steenrod algebra:

\[
\begin{align*}
H^* (\text{LBSpin}(n); \mathbb{Z}/2) &\cong H^* (\text{BSpin}_n(q); \mathbb{Z}/2) \quad (n = 7, 8, 9) \\
H^* (\text{LBF}_4; \mathbb{Z}/2) &\cong H^* (\text{BF}_4(q); \mathbb{Z}/2) \\
H^* (\text{LBDI}(4); \mathbb{Z}/2) &\cong H^* (\text{BSol}(q); \mathbb{Z}/2).
\end{align*}
\]

When \( G \) is a compact Lie group, by the homotopy equivalence \( B\text{LG} \cong \text{LBG} \) (see for example [13, §2]), the above theorem establishes cohomology isomorphisms between the classifying spaces of the loop groups \( LG \) and those of the corresponding finite groups \( G(q) \) for \( G = \text{Spin}(7), \text{Spin}(8), \text{Spin}(9), \) and \( F_4 \). The concrete presentations of the above algebras are given in Propositions 3.1, 3.2, 3.3, 4.1, and 5.1.

Acknowledgment. We would like to thank Professor Akira Kono for various suggestions. Also, we are very grateful to Anssi Lahtinen, who has corrected many errors in the previous version of this note. We have revised this note according to a detailed report kindly provided by him.

2. Main tool

Here, we summarize the results of Kishimoto and Kono [9] that are necessary for our purpose. Unless a coefficient ring is specified, \( H^*(X) \) always means the mod 2 cohomology of \( X \).

The twisted tube \( T_{\phi}X \) of \( X \) is defined by

\[
T_{\phi}X = \left[ [0,1] \times X \right] / (0,x) \sim (1, \phi(x))
\]

and there is a canonical inclusion \( \iota : X \hookrightarrow T_{\phi}X \) defined by \( \iota(x) = (0,x) \). For example, \( T_{\text{id}}X = S^1 \times X \) when \( \phi \) is the identity map \( \text{id} \). The cohomology of \( T_{\phi}X \) and \( X \) are related by the Wang exact sequence

\[
\cdots \rightarrow H^{n-1}(X) \xrightarrow{1-\phi^*} H^n(X) \xrightarrow{\delta} H^n(T_{\phi}X) \xrightarrow{\iota_*} H^n(X) \xrightarrow{1-\phi^*} H^n(X) \rightarrow \cdots,
\]

where \( \delta \) commutes with the action of \( \mathcal{A}_2 \). In particular, this exact sequence splits to short exact sequences when \( \phi^* \) is the identity map.

Let \( ev : S^1 \times LX \rightarrow X \) be the evaluation map defined by \( ev(t, \gamma) = \gamma(t) \). We define a map \( \sigma_X : H^*(X) \rightarrow H^{n-1}(LX) \) by the following equation:

\[
ev^*(x) = s \otimes \sigma_X(x) + 1 \otimes x, \quad (x \in H^*(X)),
\]

where \( s \in H^1(S^1) \) is the generator. The twisted cohomology suspension is defined by the following composition

\[
\delta_{\phi} : H^*(T_{\phi}X) \xrightarrow{\sigma r_{\phi}} H^*(L_{\phi}X) \xrightarrow{\iota^*} H^{n-1}(L_{\phi}X),
\]

where \( in : L_{\phi}X \rightarrow L_{\phi}X \) is defined by \( in(\gamma)(t) = (t, \gamma(t)) \). When there exists a ring homomorphism \( r : H^*(X) \rightarrow H^*(T_{\phi}X) \) such that \( \iota^* \circ r \) is the identity, we define the corresponding module derivation by \( \delta_{\phi} = \delta_{\phi} \circ r : H^*(X) \rightarrow H^{n-1}(L_{\phi}X) \). We call such a homomorphism \( r \) a section of \( \iota^* \).

Remark 1. When \( \phi^* \) is the identity map, we can take \( r \) to be \( \pi^* \), where \( \pi : T_{\text{id}}X = S^1 \times X \rightarrow X \) is the projection. In this case, \( \delta_{\phi} = \sigma_X \) coincides with Kuribayashi’s module derivation \( D_X \) [13].

The map \( \delta_{\phi} \) together with the Wang sequence above relates the cohomology of \( X \) to that of \( L_{\phi}X \). Consider the following conditions:

\[
\begin{align*}
\text{(i)} & \quad H^*(X) \text{ is a polynomial algebra } \mathbb{Z}[x_1, x_2, \ldots, x_t], \\
\text{(ii)} & \quad \phi^* \text{ is the identity map}.
\end{align*}
\]

Then, the result of [9] specializes to the following proposition.

Proposition 2.1 (Kishimoto-Kono). Assume that the conditions (i) and (ii) are satisfied and there is a section \( r \) of \( \iota^* \) which commutes with the action of \( \mathcal{A}_2 \). Denote by \( e : L_{\phi}X \rightarrow X \) the map defined by \( e(\gamma) = \gamma(0) \). Then, we have

\[
\begin{align*}
\text{(1)} & \quad \delta_{\phi} \text{ commutes with the action of } \mathcal{A}_2, \\
\text{(2)} & \quad \delta_{\phi}(xy) = \delta_{\phi}(x)e^r(y) + e^r(x)\delta_{\phi}(y) \text{ for } x, y \in H^*(X),
\end{align*}
\]
(3) the elements \( \{\tilde{\sigma}(x_1), \tilde{\sigma}(x_2), \ldots, \tilde{\sigma}(x_l)\} \) form a simple system of generators for \( H^\ast(L, X) \) as an algebra over \( \mathbb{Z}/2[e^\ast(x_1), \ldots, e^\ast(x_l)] \):

\[
H^\ast(L, X) \cong \mathbb{Z}/2[e^\ast(x_1), e^\ast(x_2), \ldots, e^\ast(x_l)] \otimes \Lambda(\tilde{\sigma}(x_1), \tilde{\sigma}(x_2), \ldots, \tilde{\sigma}(x_l)).
\]

This together with the action of \( A_2 \) on \( H^\ast(X) \) determines the ring structure of \( H^\ast(L, X) \) by \( \tilde{\sigma}(Sq^d(x)) \) with \( d = |x| - 1 \). In particular, we have an isomorphism \( H^\ast(L, X) \cong H^\ast(L, X) \) as algebras over \( A_2 \).

We consider the case when \( X \) is the 2-completion of either \( BSpin(7) \), \( BSpin(8) \), \( BSpin(9) \), \( BF_4 \), or \( BDI(4) \). When \( \psi \) is the identity map or an unstable Adams operation \( \psi^q \), the conditions (i) and (ii) can be verified by a case-by-case analysis. We construct a section \( r : H^\ast(BG) \to H^\ast(T_{\psi^q}BG) \) which commutes with the Steenrod operations, and use the above proposition to compute \( H^\ast(L, \psi^q BG) \).

3. Computations for \( G = Spin(7), Spin(8), Spin(9) \)

We recall the concrete presentations of the mod 2 cohomology of \( BSpin(7) \), \( BSpin(8) \) and \( BSpin(9) \) with the action of \( A_2 \) from [15, 10].

\( H^\ast(BSpin(7)) \cong \mathbb{Z}/2[w_4, w_6, w_7, w_8] \) and the action of \( A_2 \) is determined by

\[
\begin{array}{cccccc}
  w_4 & w_6 & w_7 & w_8 \\
  Sq^1 & 0 & w_7 & 0 & 0 \\
  Sq^2 & w_6 & 0 & 0 & 0 \\
  Sq^4 & w_4 w_6 & w_4 w_7 & w_4 w_8 & w_4 w_8.
\end{array}
\]

\( H^\ast(BSpin(8)) \cong \mathbb{Z}/2[w_4, w_6, w_7, w_8, e_8] \) and the action of \( A_2 \) is determined by

\[
\begin{array}{cccccc}
  w_4 & w_6 & w_7 & w_8 & e_8 \\
  Sq^1 & 0 & w_7 & 0 & 0 & 0 \\
  Sq^2 & w_6 & 0 & 0 & 0 & 0 \\
  Sq^4 & w_4 w_6 & w_4 w_7 & w_4 w_8 & w_4 w_8 & e_8.
\end{array}
\]

\( H^\ast(BSpin(9)) \cong \mathbb{Z}/2[w_4, w_6, w_7, w_8, e_{16}] \) and the action of \( A_2 \) is determined by

\[
\begin{array}{cccccc}
  w_4 & w_6 & w_7 & w_8 & e_{16} \\
  Sq^1 & 0 & w_7 & 0 & 0 & 0 \\
  Sq^2 & w_6 & 0 & 0 & 0 & 0 \\
  Sq^4 & w_4 w_6 & w_4 w_7 & w_4 w_8 & 0 \\
  Sq^8 & 0 & 0 & 0 & w_8 & 2 w_8 e_{16} + 2 w_2 e_{16}.
\end{array}
\]

Based on these results, we compute the mod 2 cohomology of \( LBG \) for \( G = Spin(7), Spin(8), \) and \( Spin(9) \). Note that the conditions (i) is seen to be satisfied from the above concrete presentation. The condition (ii) for \( \phi = \psi^q \) is verified as follows. The generators \( w_4, w_8, e_8 \), and \( e_{16} \) are seen to be the mod 2 reductions of torsion free integral classes by the Bockstein spectral sequence since the dimension of the \( \mathbb{Z}/2 \)-vector space \( H^\ast(BSpin(n), Sq^1) \) is same as the dimension of the \( \mathbb{Q} \)-vector space \( H^\ast(BSpin(n); \mathbb{Q}) \) when \( n = 7, 8, 9 \) and \( i = 4, 8, 16 \). (In fact, Kono [11] showed that this is true for any \( i \) and \( n \) and the torsion elements of \( H^\ast(BSpin(n); \mathbb{Z}) \) are of order \( 2 \).) Since \( (\psi^q)^q \) induces multiplication by a power of \( q \) on the rational cohomology, it acts as the identity on these generators in the mod 2 cohomology. By the compatibility of \( (\psi^q)^q \) with the action of \( A_2 \), the other generators \( w_6 = Sq^2(w_4) \) and \( w_7 = Sq^1 Sq^2 w_4 \) are also mapped identically by \( (\psi^q)^q \) on the mod 2 cohomology.

**Proposition 3.1.** \( H^\ast(LBSpin(7)) \cong \mathbb{Z}/2[v_4, v_6, v_7, v_8, y_3, y_5, y_7]/I \quad (|v| = i, |y| = i) \), where \( I \) is the ideal generated by

\[
|y_5 + y_2 v_4 + y_4 v_7, y_3 + y_2 v_6 + y_5 v_7, y_7 + y_2 v_8 + y_7 v_7|.
\]

The action of \( A_2 \) is determined by

\[
\begin{array}{cccccc}
  v_4 & v_6 & v_7 & v_8 & y_3 & y_5 & y_7 \\
  Sq^1 & 0 & v_7 & 0 & 0 & y_5 & 0 \\
  Sq^2 & v_6 & 0 & 0 & 0 & y_5 & 0 \\
  Sq^4 & v_4 v_6 & v_4 v_7 & v_4 v_8 & 0 & y_3 v_6 + y_5 v_4 & y_3 v_8 + y_7 v_4.
\end{array}
\]

Moreover, \( H^\ast(L_{\psi^q}(BSpin(7), \mathbb{Q})) \) is isomorphic to \( H^\ast(LBSpin(7)) \) as algebras over \( A_2 \).
Proof. When $\phi$ is the identity map, $\pi^*$ in Remark 1 serves as a section of $\iota^*$ which commutes with the action of $A_2$. We apply Proposition 2.1 to obtain $H^*(LBSpin(7)) \cong \mathbb{Z}/2[v_4, v_5, v_7, v_8] \otimes \Lambda[y_5, y_7, y_8, y_9]$, where $v_i = \iota^*(w_i)$ and $y_{i-1} = \delta_\phi(w_i)$ ($i = 4, 6, 7, 8$). The action of $A_2$ on $y_i$ ($i = 3, 5, 7$) are determined as follows:

$$
\begin{align*}
S^1 y_3 &= Sq^1 \delta_\phi(w_4) = \delta_\phi(Sq^1 w_4) = 0 \\
S^2 y_3 &= Sq^1 \delta_\phi(Sq^2 w_4) = \delta_\phi(w_6) = y_5 \\
S^3 y_3 &= \delta_\phi(Sq^1 w_5) = \delta_\phi(w_7) = y_6 \\
S^1 y_5 &= \delta_\phi(Sq^2 w_6) = 0 \\
S^2 y_5 &= \delta_\phi(Sq^2 w_6) = 0 \\
S^3 y_5 &= \delta_\phi(Sq^4 w_8) = \delta_\phi(w_8) = \delta_\phi(w_8) \iota^*(w_4) + \delta_\phi(w_6) \iota^*(w_4) = y_5 v_6 + y_5 v_4 \\
S^1 y_7 &= \delta_\phi(Sq^1 w_8) = 0 \\
S^2 y_7 &= \delta_\phi(Sq^2 w_8) = 0 \\
S^3 y_7 &= \delta_\phi(Sq^4 w_8) = \delta_\phi(w_8) y_3 v_8 + y_7 v_4.
\end{align*}
$$

With the aid of the Adem relations, we determine the ring structure as follows:

$$
\begin{align*}
y_2^2 &= Sq^3 y_3 = Sq^3 Sq^2 y_3 = Sq^1 y_5 = y_6 \\
y_2^3 &= Sq^3 y_5 = Sq^3 Sq^4 y_5 = Sq^1 y_5 (y_3 v_4 + y_5 v_4) = y_3 v_7 + y_6 v_4 = y_5 v_7 + y_7^2 v_4 \\
y_2^5 &= Sq^3 y_7 = Sq^3 Sq^2 Sq^2 y_7 = Sq^3 Sq^2 y_7 (y_3 v_8 + y_7 v_4) = Sq^3 y_5 v_8 + y_7 v_6 = y_5^2 v_8 + y_7 v_7 \\
y_4^2 &= y_2^2 y_6 = (Sq^2 Sq^4 + Sq^5 Sq^1) y_6 = \delta_\phi((Sq^2 Sq^4 + Sq^5 Sq^1) y_7) \\
&= \delta_\phi(Sq^2 w_4 v_7) = \delta_\phi(w_6 v_7) = y_5 v_7 + y_7^2 v_6.
\end{align*}
$$

To show that $H^*(LSpin(7))$ is isomorphic to $H^*(LBSpin(7))$ as algebras over $A_2$ by Proposition 2.1, we have to construct a section $r : H^*(LBSpin(7)) \to H^*(LSpin(7))$ of $\iota$ in (3) which commutes with the action of $A_2$. To do so, we carefully choose an element $u_i \in (\iota^{-1}(w_i) \subset H^*(LSpin(7))$ for each generator $w_i$ of $H^*(LSpin(7))$ so that the action of $A_2$ on $u_i$ is compatible with that on $w_i$.

The Wang sequences (3) give the following commutative diagram of exact sequences

$$
\begin{array}{cccc}
1-(\psi^\phi) & H^{n-1}(BSpin(7)_2^*; \mathbb{Z}/4) & \delta & H^n(T_{\psi^\phi}(BSpin(7))_2^*; \mathbb{Z}/4) \\
\rho & H^n(BSpin(7)_2^*; \mathbb{Z}/4) & \iota & H^n(BSpin(7)_2^*; \mathbb{Z}/4) \\
0 & H^{n-1}(BSpin(7)_2^*; \mathbb{Z}/2) & \delta & H^n(T_{\psi^\phi}(BSpin(7))_2^*; \mathbb{Z}/2) \\
\rho & H^n(BSpin(7)_2^*; \mathbb{Z}/2) & \iota & H^n(BSpin(7)_2^*; \mathbb{Z}/2) \\
\end{array}
$$

where $\rho$ is the map in the Bockstein exact sequence

$$
\begin{align*}
H^n(T_{\psi^\phi}(BSpin(7))_2^*; \mathbb{Z}/2) &\to H^n(T_{\psi^\phi}(BSpin(7))_2^*; \mathbb{Z}/4) \\
\delta &\to H^{n+1}(T_{\psi^\phi}(BSpin(7))_2^*; \mathbb{Z}/2)
\end{align*}
$$

Since $w_4$ and $w_8$ are the mod 2 reduction of torsion free integral classes, there are $\bar{w}_i$ ($i = 4, 8$) in $H^*(BSpin(7)_2^*; \mathbb{Z}/4)$ such that $\rho(\bar{w}_i) = w_i$. Furthermore, $H^*(\psi^\phi; \mathbb{Z}/4)$ acts as multiplication by one on $\bar{w}_i$ ($i = 4, 8$). Hence, we can take $\bar{w}_4, \bar{w}_8 \in H^*(T_{\psi^\phi}(BSpin(7))_2^*; \mathbb{Z}/4)$ such that $\rho \iota^*(\bar{w}_i) = \iota^* \rho(\bar{w}_i) = w_i$ ($i = 4, 8$). Define $u_4 = \rho(\bar{w}_4), u_8 = \rho(\bar{w}_8), u_6 = Sq^2 u_4$, and $u_7 = Sq^2 u_6$. Since $Sq^1 \rho = 0$, we have $Sq^1 (u_i) = 0$ for $i = 4, 8$. We compute $Sq^1 (u_4) = Sq^1 Sq^2 (u_4) = 0$. Since $\iota^* (Sq^4 (u_4)) = Sq^4 (u_6) = w_4 w_6 = \iota^* (u_4 u_6), Sq^4(u_6) + u_4 u_6$ is in the image of $\delta$. As $H^0(BSpin(7)_2^*) = 0$, we have $Sq^4 (u_6) = u_4 u_6$. Similarly, since $\iota^* (Sq^2 u_8) = Sq^2 \iota^* (u_8) = Sq^2 w_8 = 0$, we have $Sq^2 u_8 = 0$. Since $H^1(BSpin(7)_2^*) \cong \mathbb{Z}/2$ is generated by $w_3 w_7$ and $\iota^* (Sq^4 w_8) = Sq^4 w_8 = w_4 w_8 = \iota^* (u_4 u_8)$, we have $Sq^4 u_8 = u_4 u_8 + \epsilon \delta(w_4 w_7)$, where $\epsilon = 0$ or 1. To see $\epsilon = 0$, we compute

$$
Sq^4 Sq^4 u_8 = Sq^4 (u_4 u_8 + \epsilon \delta(w_4 w_7)) = Sq^4 (u_4 u_8) = u_4 \epsilon \delta(w_4 w_7).
$$

Since $Sq^4 Sq^4 u_8 = (Sq^0 Sq^4 + Sq^2 Sq^4) u_8 = 0$ by the Adem relation, we have $\epsilon = 0$.

Take $r$ to be the ring homomorphism defined by $r(w_i) = u_i$ ($i = 4, 6, 7, 8$), then $r$ is a section of $\iota$ which commutes with the action of $A_2$.

The computation for $G = Spin(8)$ is similarly conducted.
**Proposition 3.2.** \( H^*(\text{LBSpin}(8)) = \mathbb{Z}/2[v_4, v_6, v_7, v_8, f_8, y_3, y_5, y_7, z_7] / I \) (\( |v_i| = i, |y_i| = i, |f_8| = 8, |z_7| = 7 \)), where \( I \) is the ideal generated by\n\[ \{v_2^2 + y_3^2v_7 + y_3^3v_3 + y_3^2v_6 + y_5v_7, y_3^2 + y_3^2v_8 + y_7v_7, z_7^2 + y_3^2f_8 + z_7v_7 \}. \]

The action of \( \mathcal{A}_2 \) is determined by\n\[
\begin{array}{cccccccc}
v_4 & v_6 & v_7 & v_8 & f_8 & y_3 & y_5 & y_7 & z_7 \\
S^1 & 0 & v_7 & 0 & 0 & 0 & 0 & 0 & 0 \\
S^2 & v_6 & 0 & 0 & 0 & 0 & y_5 & 0 & 0 \\
S^4 & v_4 & v_4v_6 & v_4v_7 & v_4v_8 & v_4f_8 & 0 & y_3v_6 + y_5v_4 & y_3v_8 + y_7v_4 & y_3f_8 + z_7v_4.
\end{array}
\]

Moreover, \( H^*(\mathcal{L}_{\psi'}(\text{BSpin}(8)))_\phi \) is isomorphic to \( H^*(\text{LBSpin}(8)) \) as algebras over \( \mathcal{A}_2 \).

For the case when \( G = \text{Spin}(9) \), we make use of the following lemma.

**Lemma 3.1** (Lahtinen). Assume \( H^*(X) \cong \mathbb{Z}/2[x_1, \ldots, x_i] \) and \( \phi' \) is the identity. For \( 1 \leq i \leq l \), let \( u_i \in H^*(T_{\omega}X) \) be any element such that \( \iota^*(u_i) = x_i \). Denote by \( s \in H^1(T_{\omega}X) \) the pullback of the generator \( H^1(S^3) \) under the projection \( T_{\omega}X \to S^3 \). Then, we have\n\[ H^*(T_{\omega}X) \cong \mathbb{Z}/2[u_1, \ldots, u_s]/(s^2) \]
and \( \delta(p(x_1, \ldots, x_i)) = p(u_1, \ldots, u_s) \) for any polynomial \( p \in \mathbb{Z}/2[x_1, \ldots, x_i] \).

**Proposition 3.3.** \( H^*(\text{LBSpin}(9)) = \mathbb{Z}/2[v_4, v_6, v_7, v_8, f_1, f_6, y_3, y_5, y_7, z_{15}] / I \) (\( |v_i| = i, |y_i| = i, |f_1| = |f_6| = 16, |z_{15}| = 16 \)), where \( I \) is the ideal generated by\n\[ \{v_2^2 + y_3v_7 + v_4y_3, y_3^2 + y_3^2v_6 + y_5v_7, y_3^2 + y_3^2v_8 + y_7v_7, z_{15}^2 + y_3v_8z_{15} + y_7v_7f_1 + y_3^2v_8f_16 \} \]

The action of \( \mathcal{A}_2 \) is determined by\n\[
\begin{array}{cccccccc}
v_4 & v_6 & v_7 & v_8 & f_6 & y_3 & y_5 & y_7 & z_{15} \\
S^1 & 0 & v_7 & 0 & 0 & 0 & 0 & 0 & 0 \\
S^2 & v_6 & 0 & 0 & 0 & 0 & y_5 & 0 & 0 \\
S^4 & v_4 & v_4v_6 & v_4v_7 & v_4v_8 & v_4f_6 & 0 & y_3v_6 + y_5v_4 & y_3v_8 + y_7v_4 & 0 \\
S^8 & 0 & 0 & 0 & v_8^2 & v_8f_16 & 0 & 0 & y_7f_16 + v_8z_{15} + v_8^2z_{15}.
\end{array}
\]

Moreover, \( H^*(\mathcal{L}_{\psi'}(\text{BSpin}(9)))_\phi \) is isomorphic to \( H^*(\text{LBSpin}(9)) \) as algebras over \( \mathcal{A}_2 \).

**Proof.** In dimensions up to 8, the calculation of \( H^*(\text{LBSpin}(9)) \) is completely same as in the case of \( H^*(\text{LBSpin}(7)) \). Let \( \phi \) be the identity map and define \( f_{16} = e'_{16} \) and \( z_{15} = \delta_{\phi}(e_{16}) \). We have\n\[ S^8z_{15} = \delta_{\phi}(S^8e_{16}) = \delta_{\phi}(w_8e_{16} + w_6^2e_{16}) = y_7f_{16} + v_8z_{15} + v_8^2z_{15} \]
\[ z_{15}^2 = S^8z_{15} = \delta_{\phi}(S^8z_{15}) = \delta_{\phi}(S^8e_{16}) = \delta_{\phi}(S^8w_8e_{16}) = \delta_{\phi}(S^8w_8e_{16} + w_6^2e_{16}) = \delta_{\phi}(S^8w_8e_{16} + w_6^2e_{16}) \]
\[ = \delta_{\phi}(S^8w_8e_{16} = v_7v_8z_{15} + y_7v_7f_{16} + v_8^2f_{16}. \]

Now, we will construct a section \( \iota : H^*(\mathcal{L}_{\psi'}(\text{BSpin}(9)))_\phi \to H^*(T_{\omega}(\text{BSpin}(9)))_\phi \) of \( \iota \) which commutes with the action of \( \mathcal{A}_2 \). Define \( u_i \) (\( i = 4, 6, 7, 8 \)) as the case of \( \text{BSpin}(7) \). We can choose an element \( h_{16} \in H^{16}(T_{\omega}(\text{BSpin}(9)))_\phi \) such that \( \iota^*(h_{16}) = e_{16} \) and \( S^8h_{16} = 0 \) by the same argument for \( u_4 \) as in the case of \( \text{BSpin}(7) \). By the Wang sequence for \( \mathbb{Z}/2 \), we have \( S^8h_{16} = e_1d(w_4w_6w_7) \) since \( H^{17}(\text{BSpin}(9)) \cong \mathbb{Z}/2 \) is generated by \( w_4w_6w_7 \) and \( \iota^*(S^8h_{16}) = S^8z_{15} = 0 \). We see \( S^8h_{16} = e_1d(w_4w_6w_7) \) is \( e_1d(w_6^2w_7) \). Since \( S^4S^2h_{16} = S^3S^1h_{16} = 0 \) by the Adem relation, we have \( e_1 = 0 \).

Similarly, we have \( S^4h_{16} = e_2d(w_4^2w_7) + e_3d(w_6^2w_7) + e_4d(w_4w_7w_8) \) and \( S^4S^4h_{16} = (e_2 + e_3)d(w_4w_6w_7) + e_4d(w_6^2w_7) \). Since \( S^4S^4h_{16} = (S^4S^4) = S^6S^2h_{16} = 0 \), we have \( e_2 = e_3, e_4 = 0 \). Put \( h'_{16} = h_{16} + e_2d(w_6^2w_7) \), then we have \( h'_{16} = 0 \) since \( S^6(w_6^2w_7) = w_4w_7 + w_6^2w_7 \). Since \( S^4(w_6^2w_7) = 0 \) (\( i = 1, 2 \)), we have \( h'_{16} = 0 \) (\( i = 1, 2 \)).

Again by the Wang sequence, we have \( S^4h_{16} = u_8h_{16} + u_4h_{16} + e_2d(w_4w_7) + e_3d(w_6^2w_7) + e_4d(w_4w_7w_8) + e_5d(w_4w_6w_7) + e_6d(w_6^2w_7) + e_7d(w_7w_8) + e_8d(w_7e_{16}) \).
and by Lemma 3.1 we have
\[ Sq^8 Sq^8 h_1^8 = su_7 \left( e_5 u_4^6 + (e_5 + e_6) u_4^4 u_8 + e_7 u_4^2 u_6^2 + e_7 u_4 u_6 u_8 + (e_5 + e_7) u_6^4 + e_8 u_8^3 \right). \]

Since \( Sq^8 Sq^8 h_1^8 = (Sq^{12} Sq^4 + Sq^{14} Sq^2 + Sq^{15} Sq^1) h_1^8 = 0 \), we have \( e_5 = e_6 = e_7 = e_8 = 0 \). Put \( u_8' = u_8 + e_9 su_7 \).

We see
\[ Sq^8 h_1^8 = u_8 h_1^8 + u_4^2 h_1^6 + e_9 su_7 h_1^6 = u_8' h_1^8 + u_4^2 h_1^6. \]

Now, a section \( r' \) is obtained by setting \( r(w_4) = u_i \) (\( i = 4, 6, 7 \)), \( r(w_8) = u_8' \), and \( r(e_16) = h_1^8 \). By the proof of Proposition 3.1, we know that any lift \( u_8' \) of \( w_8 \) behaves similarly with respect to the action of \( \mathcal{A}_2 \), and hence, \( r \) commutes with the action of \( \mathcal{A}_2 \).

\[ \square \]

4. Computations for \( G = F_4 \)

Denote by \( i \) the classifying map of the canonical inclusion \( Spin(9) \hookrightarrow F_4 \), where \( F_4 \) is the compact exceptional Lie group of type \( F_4 \). Kono [10] showed that \( i \) is injective and the mod 2 cohomology of \( BF_4 \) over \( \mathcal{A}_2 \) was determined as follows:

\[ H^*(BF_4) = \mathbb{Z}/2[x_4, x_6, x_7, x_{16}, x_24], \]

where \( \bar{i}(x_4) = u_4, \bar{i}(x_6) = w_6, \bar{i}(x_7) = w_7, \bar{i}(x_{16}) = e_1 + w_6^2, \bar{i}(x_{24}) = w_8 e_{16} \), and the action of \( \mathcal{A}_2 \) is determined by

\[
\begin{array}{cccccc}
  x_4 & x_6 & x_7 & x_{16} & x_{24} \\
  Sq^1 & 0 & x_7 & 0 & 0 & 0 \\
  Sq^2 & x_6 & 0 & 0 & 0 & 0 \\
  Sq^4 & x_4 x_6 & x_4 x_7 & 0 & x_4 x_{24} & 0 \\
  Sq^8 & 0 & 0 & 0 & x_{24} + x_4^2 x_{16} & x_4^2 x_{24} \\
  Sq^{16} & 0 & 0 & 0 & x_{16}^2 x_{24} + x_4 x_{24} & x_{16}^2 x_{24} \\
\end{array}
\]

The condition (ii) for \( \phi = \psi^d \) is satisfied as \( \bar{i} \) is compatible with \( \psi^d \).

**Proposition 4.1.** \( H^*(LB_{F_4}) = \mathbb{Z}/2[v_4, v_6, v_7, v_{16}, v_{24}, y_3, y_5, y_{15}, y_{23}]/I \) \((\{y_i\} = i, |y_i| = i)\), where \( I \) is the ideal generated by

\[ \{y_5^2 + y_3 y_7 + v_4 y_5^2, y_3^4 + v_6 y_2^2 + y_5 y_7, y_2^2 + y_7 y_{23} + v_24 y_2^2, y_2^2 + y_3^2 v_16 v_{24} + v_7 v_{24} y_{15} + v_7 y_{16} y_{23}\}. \]

The action of \( \mathcal{A}_2 \) is determined by

\[
\begin{array}{cccccc}
  v_4 & v_6 & v_7 & v_{16} & v_{24} \\
  Sq^1 & 0 & v_7 & 0 & 0 & 0 \\
  Sq^2 & v_6 & 0 & 0 & 0 & 0 \\
  Sq^4 & v_4 & v_4 v_6 & v_4 v_7 & 0 & v_4 v_{24} \\
  Sq^8 & 0 & 0 & 0 & v_{24} + v_2^2 v_{16} & v_2^2 v_{24} \\
  Sq^{16} & 0 & 0 & 0 & v_{16}^2 v_{24} + v_4 v_2^2 v_{24} & 0 \\
\end{array}
\]

where \( I = v_{24} y_{15} + v_{16} y_{23} + y_3 v_6^2 v_{24} + v_4 v_6^2 y_{23} \). Moreover, \( H^*(L_{\psi^d}(BF_4)^2) \) is isomorphic to \( H^*(LB_{F_4}) \) as algebras over \( \mathcal{A}_2 \).

**Proof.** Let \( \phi \) be the identity map, and put \( v_i = e'(x_i) \) and \( y_{i-1} = \bar{\sigma}_\phi(x_i) \) \((i = 4, 6, 7, 16, 24)\). In dimensions up to 8, the calculation of \( H^*(LB_{F_4}) \) is completely parallel to the the case of \( H^*(L_{Spin(9)}) \). The rest is
computed as follows:

\[
\begin{align*}
Sq^1 y_{15} &= \delta_{\phi}(Sq^1 x_{16}) = 0 \\
Sq^2 y_{15} &= \delta_{\phi}(Sq^2 x_{16}) = 0 \\
Sq^4 y_{15} &= \delta_{\phi}(Sq^4 x_{16}) = 0 \\
Sq^8 y_{15} &= \delta_{\phi}(Sq^8 x_{16}) = \delta_{\phi}(x_{24} + x_4^2 x_{16}) = y_{23} + v_2^8 y_{15} \\
Sq^1 y_{23} &= \delta_{\phi}(Sq^1 x_{24}) = 0 \\
Sq^2 y_{23} &= \delta_{\phi}(Sq^2 x_{24}) = 0 \\
Sq^4 y_{23} &= \delta_{\phi}(Sq^4 x_{24}) = \delta_{\phi}(x_4 x_{24}) = y_3 v_{24} + v_4 y_{23} \\
Sq^8 y_{23} &= \delta_{\phi}(Sq^8 x_{24}) = \delta_{\phi}(x_4^2 x_{24}) = v_2^4 y_{23} \\
Sq^{16} y_{23} &= \delta_{\phi}(Sq^{16} x_{24}) = \delta_{\phi}(x_{16} x_{24} + x_4 x_2^2 x_{24}) = y_{15} v_{24} + v_{16} y_{23} + y_3 v_6 v_{24} + v_4 y_2^2 y_{23} \\
y_2^2 &= Sq^{15} y_{15} = \delta_{\phi}(Sq^{15} x_{16}) = \delta_{\phi}(Sq^7 Sq^8 x_{16}) = \delta_{\phi}(Sq^7 x_{24}) = \delta_{\phi}(x_7 x_{24}) = y_{15} y_2 v_{24} + y_2^7 v_{24} \\
y_2^2 &= Sq^{23} y_{23} = \delta_{\phi}(Sq^{23} x_{24}) = \delta_{\phi}(Sq^{16} x_{24}) = \delta_{\phi}(Sq^7 (x_{16} x_{24} + x_4 x_2^2 x_{24})) \\
\delta_{\phi}(x_{16} Sq^7 (x_{24}) + Sq^7 Sq^4 (x_2^2 x_{24})) &= \delta_{\phi}(x_7 x_{16} x_{24}) = y_2^3 v_{16} v_{24} + y_7 v_{15} v_{24} + y_7 v_{16} y_{23}.
\end{align*}
\]

Now, we will construct a section \( r : H^*(\mathcal{F}(\mathcal{F}_2^2)) \to H^*(\mathcal{T}_{\psi'}(\mathcal{F}_2^2)) \) of \( \iota' \) which commutes with the action of \( \mathcal{A}_2 \). By the uniqueness of the Adams operations [7], the following diagram is homotopy commutative:

\[
\begin{array}{ccc}
BSpin(9)^\wedge \mathcal{F}_2^2 & \xrightarrow{\psi_{BSpin(9)}^\wedge} & BSpin(9)^\wedge \\
\downarrow{\psi_{BSpin(9)}} & & \downarrow{\psi_{BSpin(9)}} \\
(BF_3)^\wedge & \xrightarrow{\psi_{BF_3}} & (BF_3)^\wedge
\end{array}
\]

By the naturality of the construction of the twisted tube, there is a map \( \mathcal{T}_{\psi'}(\iota_2^\wedge) : \mathcal{T}_{\psi'} BSpin(9)^\wedge \to \mathcal{T}_{\psi'} (BF_3)^\wedge \), which makes the following diagram commute:

\[
\begin{array}{cccccc}
0 & \xrightarrow{\delta_{BSpin(9)}^\wedge} & H^{n-1}(BSpin(9)^\wedge) & \xrightarrow{\delta_{BSpin(9)}^\wedge} & H^n(BSpin(9)^\wedge) & \xrightarrow{\delta_{BSpin(9)}^\wedge} & 0 \\
\downarrow{\iota'} & & \downarrow{\mathcal{T}_{\psi'}(\mathcal{F}_2^2)} & & \downarrow{\mathcal{T}_{\psi'}(\mathcal{F}_2^2)} & & \downarrow{\mathcal{T}_{\psi'}(\mathcal{F}_2^2)} \\
0 & \xrightarrow{\delta} & H^{n-1}((BF_3)^\wedge) & \xrightarrow{\delta} & H^n(BSpin(4)^\wedge) & \xrightarrow{\delta} & 0
\end{array}
\]

where the horizontal lines are the Wang sequences and the vertical arrows are injections. Let \( r_{BSpin(9)} \) be the section of \( \iota'_{BSpin(9)} \) constructed in the proof of Proposition 3.3. Let \( u_i = r_{BSpin(9)}(w_i) \) (\( i = 4, 6, 7, 8 \)) and \( h_{16} = r_{BSpin(9)}(e_{16}) \). Since \( r' \) is an isomorphism in degrees up to 8, so is \( (\mathcal{T}_{\psi'}(\iota_2^\wedge))^\wedge \). We define

\[
r(x_i) = ((\mathcal{T}_{\psi'}(\iota_2^\wedge))^\wedge)^{-1} u_i, \quad (i = 4, 6, 7).
\]

If \( r_{BSpin(9)}(\iota'(x_{16})) = r_{BSpin(9)}(e_{16} + w_8^2) = h_{16} + u_8^2 \) is in the image of \( (\mathcal{T}_{\psi'}(\iota_2^\wedge))^\wedge \), we define \( r(x_{16}) = ((\mathcal{T}_{\psi'}(\iota_2^\wedge))^\wedge)^{-1} (h_{16} + u_8^2) \). Otherwise, we replace \( h_{16} \) in the following manner. Since \( (r')^{-1}(e_{16} + w_8^2) = h_{16} + u_8^2 + \text{Im} \delta_{BSpin(9)}^\wedge \) and \( H^{15}(BSpin(9))/H^{15}(BF_3) \equiv Z/2[w_7 w_8] \), the element \( h_{16} + u_8^2 + \delta(w_7 w_8) \) should lie in the image of \( (\mathcal{T}_{\psi'}(\iota_2^\wedge))^\wedge \) in this case. We define \( r(x_{16}) = ((\mathcal{T}_{\psi'}(\iota_2^\wedge))^\wedge)^{-1} (h_{16} + u_8^2 + \delta(w_7 w_8)) \). Finally, noting \( x_{24} = Sq^8(x_{16}) + x_4^2 x_{16} \), we define \( r(x_{24}) = Sq^8 (r(x_{16})) + r(x_4^2 x_{16}) \). From the proof of Proposition 3.3 we see that \( r \) commutes with the action of \( \mathcal{A}_2 \).

\[\square\]

5. Computations for \( G = DI(4) \)

In [11], Dwyer and Wilkerson constructed a finite loop space \( DI(4) \), whose classifying space \( BDI(4) \) has the mod 2 cohomology isomorphic to the mod 2 Dickson invariant of rank 4, that is, \( H^*(BDI(4)) \equiv \)
$\mathbb{Z}/2[x_8, x_{12}, x_{14}, x_{15}]$, where $|x_j| = j$. The action of $A_2$ is determined by

| $x_8$ | $x_{12}$ | $x_{14}$ | $x_{15}$ |
|-------|-----------|-----------|-----------|
| $Sq^1$ | 0 | 0 | $x_{15}$ | 0 |
| $Sq^2$ | 0 | $x_{14}$ | 0 | 0 |
| $Sq^4$ | $x_{12}$ | 0 | 0 | 0 |
| $Sq^8$ | $x^2_8$ | $x_8 x_{12}$ | $x_8 x_{14}$ | $x_8 x_{15}$ |

Notbohm \cite{14} showed that there is an injection $H^*(BDI(4)) \to H^*(BSpin(7))$ compatible with the unstable Adams operation $\psi^i$, and hence, the condition (ii) is satisfied for $\phi = \psi^i$.

Grib \cite{6} calculated $H^*(BSol(q)) \cong H^*(\mathcal{L}_\psi BDI(4))$ over $A_2$ by the Eilenberg-Moore spectral sequence. Kuribayashi \cite{13} calculated the mod 2 cohomology of $H^*(LDI(4))$ over $A_2$. Here, we reproduce their results by the same method as in the previous sections.

**Proposition 5.1.** $H^*(LDI(4)) \cong \mathbb{Z}/2[v_8, v_{12}, v_{14}, v_{15}, y_7, y_{11}, y_{13}] / I (|v_i| = i, |y_i| = i)$, where $I$ is the ideal generated by

$$\{y_1^2 + y_7 y_{15} + v_8 y_7^2, y_{13}^2 + y_{11} y_{15} + v_{12} y_7^2, y_7^4 + y_{13} y_{15} + v_{14} y_7^2\}.$$

The action of $A_2$ is determined by

| $v_8$ | $v_{12}$ | $v_{14}$ | $v_{15}$ | $y_7$ | $y_{11}$ | $y_{13}$ |
|-------|-----------|-----------|-----------|-------|-----------|-----------|
| $Sq^1$ | 0 | 0 | $v_{15}$ | 0 | 0 | 0 | $y_7^2$ |
| $Sq^2$ | 0 | $v_{14}$ | 0 | 0 | 0 | $y_{13}$ | 0 |
| $Sq^4$ | $v_{12}$ | 0 | 0 | 0 | $y_{11}$ | 0 | 0 |
| $Sq^8$ | $v_8$ | $v_8 v_{12}$ | $v_8 v_{14}$ | $v_8 v_{15}$ | 0 | $v_8 y_{11} + y_7 v_{12}$ | $v_8 y_{13} + y_7 v_{14}$ |

Moreover, $H^*(\mathcal{L}_\psi BDI(4))$ is isomorphic to $H^*(LDI(4))$ as algebras over $A_2$.

**Proof.** Let $\phi$ be the identity map, and put $v_i = c'(x_i)$, $y_{i-1} = \delta_\phi(x_i)$ $(i = 8, 12, 14, 15)$. Just as in the previous sections, we have

$$y_7^2 = Sq^2 y_7 = \delta_\phi(Sq^2 x_8) = \delta_\phi(Sq^1 Sq^2 Sq^4 x_8) = \delta_\phi(x_{15}) = y_{14}$$

$$Sq^1 y_i = \delta_\phi(Sq^1 x_{i+1}) = 0 \quad (i = 7, 11)$$

$$Sq^1 y_{13} = \delta_\phi(Sq^1 x_{14}) = \delta_\phi(x_{15}) = y_{14} = y_7^2$$

$$Sq^2 y_i = \delta_\phi(Sq^2 x_{i+1}) = 0 \quad (i = 7, 13)$$

$$Sq^2 y_{11} = \delta_\phi(Sq^2 x_{12}) = \delta_\phi(x_{14}) = y_{13}$$

$$Sq^4 y_i = \delta_\phi(Sq^4 x_{i+1}) = 0 \quad (i = 11, 13)$$

$$Sq^4 y_7 = \delta_\phi(Sq^4 x_8) = \delta_\phi(x_{12}) = y_{11}$$

$$Sq^8 y_7 = \delta_\phi(Sq^8 x_8) = 0$$

$$Sq^8 y_{11} = \delta_\phi(Sq^8 x_{12}) = \delta_\phi(x_{8} x_{12}) = y_{7} v_{12} + v_8 y_{11}$$

$$Sq^8 y_{13} = \delta_\phi(Sq^8 x_{14}) = \delta_\phi(x_{8} x_{14}) = y_{7} v_{14} + v_8 y_{13}$$

$$y_{11}^2 = Sq^{11} y_{11} = \delta_\phi(Sq^{11} x_{12}) = \delta_\phi(Sq^1 Sq^2 Sq^8 x_{12}) = \delta_\phi(x_{8} x_{15}) = v_8 y_7^2 + y_7 v_{15}$$

$$y_{13}^2 = Sq^{13} y_{13} = \delta_\phi(Sq^{13} x_{14}) = \delta_\phi((Sq^1 Sq^4 Sq^8 + Sq^{11} Sq^2) x_{14})$$

$$= \delta_\phi(Sq^1 Sq^4 x_{8} x_{14}) = \delta_\phi(x_{12} x_{15}) = y_{11} v_{15} + v_{12} y_7^2$$

$$y_7^4 = Sq^{14} y_{14} = \delta_\phi(Sq^{14} x_{15}) = \delta_\phi((Sq^{11} Sq^3 + Sq^2 Sq^4 Sq^8) x_{15}) = \delta_\phi(x_{14} x_{15}) = y_{13} v_{15} + v_{14} y_7^2.$$

We can choose an element $u_8 \in H^{10}(\mathcal{T}_\psi BDI(4))$ such that $c'(u_8) = x_8$ and $Sq^1 u_8 = Sq^1 Sq^4 u_8 = 0$ by the same argument for $u_6$ as in the proof of Proposition 3.1. Put $u_{12} = Sq^4 u_{12}, u_{14} = Sq^2 u_{12},$ and $u_{15} = Sq^1 u_{14}$. Then, we have $Sq^1 u_i = 0 \quad (i = 8, 12, 15)$. Since $H^{10}(\mathcal{T}_\psi BDI(4)) = 0$, we have $Sq^2 u_8 = 0.$
Since $H^9(BDI(4)) = 0$ and $t^*(Sq^6u_{12}) = x_8x_{12}$, we have $Sq^6u_{12} = u_8u_{12}$. Moreover, we have

\[ Sq^i u_{12} = Sq^i Sq^4 u_8 = (Sq^7 Sq^i + Sq^6 Sq^6) u_8 = 0 \]
\[ Sq^2 u_{14} = Sq^2 Sq^2 u_{12} = 0 \]
\[ Sq^4 u_{14} = Sq^4 Sq^6 u_8 = (Sq^8 Sq^2 + Sq^6 u_{12}) u_8 = 0 \]
\[ Sq^6 u_{14} = Sq^6 Sq^2 u_{12} = (Sq^4 Sq^6 + Sq^2 Sq^6 + Sq^9 Sq^4) u_{12} = u_8 u_{14} \]
\[ Sq^2 u_{15} = Sq^2 Sq^7 u_8 = (Sq^8 Sq^1 + Sq^5) u_8 = 0 \]
\[ Sq^4 u_{15} = Sq^4 Sq^7 u_8 = (Sq^9 Sq^2 + Sq^{11}) u_8 = 0 \]
\[ Sq^8 u_{15} = Sq^8 Sq^1 u_{14} = (Sq^8 + Sq^2 Sq^7) u_{14} = Sq^1 Sq^8 u_{14} = u_8 u_{15}. \]

Therefore, we can construct a section $r : H^*(BDI(4)) \to H^*(T_{ψ_0} BDI(4))$ of $t^*$ by $r(x_i) = u_i$ ($i = 8, 12, 14, 15$) so that $r$ commutes with $A_2$. □

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