The Scalar Curvature Problem on the Four Dimensional Half Sphere

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Abstract. In this paper, we consider the problem of prescribing the scalar curvature under minimal boundary conditions on the standard four dimensional half sphere. We provide an Euler-Hopf type criterion for a given function to be a scalar curvature to a metric conformal to the standard one. Our proof involves the study of critical points at infinity of the associated variational problem.

MSC classification: 35J60, 35J20, 58J05.

Key words: Variational problems, Lack of compactness, Scalar curvature, Conformal invariance, Critical points at infinity.

1 Introduction and the Main Results

This paper is devoted to some nonlinear problem arising from conformal geometry. Precisely, Let \((M^n, g)\) be a \(n\)-dimensional Riemannian manifold with boundary, \(n \geq 3\), and let \(\tilde{g} = u^{4/(n-2)}g\), be a conformal metric to \(g\), where \(u\) is a smooth positive function, then the scalar curvatures \(R_g\), \(R_{\tilde{g}}\) and the mean curvatures \(h_g\), \(h_{\tilde{g}}\), with respect to \(g\) and \(\tilde{g}\) respectively, are related by the following equations.

\[
(P_1) \begin{cases}
-c_n \Delta_g u + R_g u &= R_{\tilde{g}} u^{\frac{n+2}{n-2}} \quad \text{in } M \\
\frac{2}{n-2} \frac{\partial u}{\partial \nu} + h_g u &= h_{\tilde{g}} u^{\frac{n}{n-2}} \quad \text{on } \partial M
\end{cases}
\]

where \(c_n = 4(n-1)/(n-2)\) and \(\nu\) denotes the outward normal vector with respect to the metric \(g\).

In view of \((P_1)\), the following problem naturally arises: given two functions \(K : M \to \mathbb{R}\) and \(H : \partial M \to \mathbb{R}\), does exist a metric \(\tilde{g}\) conformally equivalent to \(g\) such that \(R_{\tilde{g}} = K\) and \(h_{\tilde{g}} = H\)?

From equations \((P_1)\), the problem is equivalent to finding a smooth positive solution \(u\) of the following equation

\[
(P_2) \begin{cases}
-c_n \Delta_g u + R_g u &= K u^{\frac{n+2}{n-2}} \quad \text{in } M \\
\frac{2}{n-2} \frac{\partial u}{\partial \nu} + h_g u &= H u^{\frac{n}{n-2}} \quad \text{on } \partial M
\end{cases}
\]

This problem has been studied in earlier works (see [2], [12], [18], [19], [20], [21], [23], [24], [25], [27], [30] and the references therein).
In this paper we consider the case of the standard four dimensional half sphere under minimal boundary conditions. We are thus reduced to look for positive solutions of the following problem

\[
\begin{align*}
L_g u := -\Delta_g u + \frac{n(n-2)}{4} u &= K u^{\frac{n+2}{n-2}} \quad \text{in} \quad S^n_+ \\
\frac{\partial u}{\partial \nu} &= 0 \quad \text{on} \quad \partial S^n_+ 
\end{align*}
\]

where \( n = 4 \) and where \( g \) is the standard metric of \( S^n_+ = \{ x \in \mathbb{R}^{n+1} / |x| = 1, x_{n+1} > 0 \} \).

This problem has been studied by Yanyan Li [30], and Djadli-Malchiodi-Ould Ahmedou [19], on the three dimensional standard half sphere. Their method involves a fine blow up analysis of some subcritical approximations and the use of the topological degree tools. In a previous work [12], we gave some topological conditions on \( K \) to prescribe the scalar curvature under minimal boundary conditions on half spheres of dimension bigger than or equal 4 . In this paper, which is a continuation of [12], we single out the four dimensional case to give more existence results, in particular a Hopf type formula reminiscent to Bahri-Coron [9] formula for the scalar curvature problem on \( S^3 \), see also [11] [15].

Notice that, Problem (1) is in a natural way related to the well-known Scalar Curvature Problem on \( S^n \)

\[
\begin{align*}
-\Delta_g u + \frac{n(n-2)}{4} u &= K u^{(n+2)/(n-2)} \quad \text{in} \quad S^n 
\end{align*}
\]

to which much works have been devoted (see [1, 4, 5, 6, 7, 11, 17, 15, 16, 18, 22, 26, 29, 33 and the references therein ). As for Problem (2), also for problem (1) there are topological obstructions for existence of solutions, based on Kazdan-Warner type conditions, see [13]. Hence it is not expectable to solve problem (1) for all the functions \( K \), and it is natural to impose some conditions on them.

Regarding Problem (2), Bahri [7] observed that a new phenomenon appears in dimension \( n \geq 5 \) due to the fact that the self interaction of the functions failing the Palais-Smale condition dominates the interaction of two of those functions. In the dimensional three, the reverse happens(see [3]). In dimension 4, we have a balance phenomenon, that is, the self interaction and the interaction are of the same size (see [11]).

For Problem (1), Djadli-Malchiodi-Ould Ahmedou [19] showed that such a balance phenomenon appears in \( S^4_+ \). Such a result suggests that there is a dimension break between Problem (1) and Problem (2). In this paper, we prove that a dimension’s break is not always true. Precisely, we show that \( S^4_+ \) behaves like \( S^4 \) under some conditions on the behavior of the function \( K \) on the boundary \( \partial S^4_+ \).

In order to state our results, we need to introduce some notations and the assumptions that we are using in our results.

Let \( G \) be the Green’s function of \( L_g \) on \( S^4_+ \) and \( H \) its regular part defined by

\[
\begin{align*}
G(x, y) &= (1 - \cos(d(x, y)))^{-1} + H(x, y), \\
\Delta H &= 0 \quad \text{in} \quad S^4_+, \quad \partial G/\partial \nu = 0 \quad \text{on} \quad \partial S^4_+
\end{align*}
\]
Let $K$ be a $C^3$ positive Morse function on $\overline{S^4_+}$. We say that $K$ satisfies the condition (C) if

$$
\begin{align*}
\text{If } a \in S^4_+, \nabla K(a) &= 0, \quad \text{then } -\frac{\Delta K(a)}{3K(a)} - 4H(a, a) \neq 0, \\
\text{If } a \in \partial S^4_+, \nabla_T K(a) &= 0, \quad \text{then } \frac{\partial K}{\partial \nu}(a) < 0,
\end{align*}
$$

where $\nabla_T K$ denotes the tangential part of the gradient of $K$.

For sake of simplicity, we assume that $y_1, ..., y_l$ ($l \leq N$) are all the critical points of $K$ for which $(-\Delta K(y_i)/(3K(y_i))) - 4H(y_i, y_i) > 0$, for $i = 1, ..., l$. For $s \in \{1, ..., l\}$ and for any $s$-tuple $\tau_s = (i_1, ..., i_s) \in \{1, ..., l\}^s$ such that $i_p \neq i_q$ if $p \neq q$, we define a Matrix $M(\tau_s) = (M_{pq})_{1 \leq p, q \leq s}$ by

$$
M_{pp} = -\frac{\Delta K(y_i_p)}{3K(y_i_p)^2} - 4 \frac{H(y_{i_p}, y_{i_p})}{K(y_{i_p})}, \\
M_{pq} = -\frac{4G(y_{i_p}, y_{i_q})}{(K(y_{i_p})K(y_{i_q}))^{1/2}} \quad \text{for } p \neq q,
$$

and we denote by $\rho(\tau_s)$ the least eigenvalue of $M(\tau_s)$.

We then have the following results.

**Theorem 1.1** Assume that $K$ satisfies condition (C), and assume further that for any $s \in \{1, ..., l\}$, $M(\tau_s)$ is nondegenerate.

If

$$
1 \neq \sum_{s=1}^{l} \sum_{\tau_s = (i_1, ..., i_s) / \rho(\tau_s) > 0} (-1)^{s-1-\sum_{j=1}^{s} k_{ij}},
$$

where $k_{ij} = \text{index } (K, y_{i_j})$, denotes the Morse Index of $K$ at $y_{i_j}$, then Problem (1) has a solution.

**Theorem 1.2** Assume that $K$ satisfies condition (C), and assume further that:

If $a \in S^4_+, \nabla K(a) = 0$, we have

$$
-\frac{\Delta K(a)}{3K(a)} \leq \frac{1}{d^2(a, \partial S^4_+)},
$$

then Problem (1) has a solution.

The proof of the above results involves the construction of a special pseudogradient for the associated variational problem for which the Palais Smale condition is satisfied along the decreasing flow lines, as long as these flow lines do not enter the neighbourhood of a finite numbers of critical points of $K$ such that the related matrix $M(\tau)$ is positive definite. Moreover along the flow lines of such a pseudogradient there can be only finitely many blow up points. Furthermore if some blow up points are close and the interactions between them is large, then the flow lines starting from there will enter the zone with at least one less blow up points. Using such a pseudogradient, a Morse Lemma at infinity is performed and an Euler Poincaré characteristic argument allows us to derive the existence
of solution for this problem. Our proof goes along the method initiated by Bahri-Coron [9], see also [11], however in our case the presence of the boundary makes the analysis more involved. Moreover it turns out that the interaction of the bubbles and the boundary creates a phenomenon of new type which is not present in the sphere’s case.

We organize the remainder of the present paper as follows. In section 2, we set up the variational structure and give some careful expansion of the Euler functional $J$ associated to (1). In section 3, we perform the construction of a pseudogradient of $J$ whose zeros are the critical points at infinity of $J$. The last section is devoted to the proofs of our results.

Acknowledgment. The authors would like to thank Professor Abbas Bahri for his encouragement and constant support over the years. The third Author (M.O.A) is grateful to Professor Reiner Schätzle for his support and interest in this work.

2 General Framework and Expansion of the functional

Problem (1) has a variational structure, the functional being

$$J(u) = \frac{\int_{S_+^4} |\nabla u|^2 + 2 \int_{S_+^4} u^2}{\left(\int_{S_+^4} Ku^4\right)^{\frac{1}{2}}} ,$$

defined on the unite sphere of $H^1(S_+^4)$ equipped with the norm

$$||u||^2 = \int_{S_+^4} |\nabla u|^2 + 2 \int_{S_+^4} u^2 .$$

Problem (1) is equivalent to finding the critical points of $J$ to the constraint $u \in \Sigma^+$, where

$$\Sigma^+ = \{u \in \Sigma / u \geq 0\}, \quad \Sigma = \{u \in H^1(S_+^4) / ||u|| = 1\}$$

The Palais-Smale condition fails to be satisfied for $J$ on $\Sigma^+$. To describe the sequences failing the Palais-Smale condition, we need to introduce some notations. For $a \in S_+^4$ and $l > 0$, let

$$\delta_{a,l}(x) = \frac{l}{l^2 + 1 + (1 - l^2) \cos d(a, x)} ,$$

where $d$ is the geodesic distance on $(S_+^4, g)$. This function satisfies the following equation

$$-\Delta \delta_{a,l} + 2 \delta_{a,l} = 8 \delta_{a,l}^3, \quad \text{in } S_+^4$$
Let $\varphi_{(a,l)}$ be the function defined on $S_4^+$ and satisfying

$$-\Delta \varphi_{(a,l)} + 2 \varphi_{(a,l)} = -\Delta \delta_{a,l} + 2 \delta_{a,l} \text{ in } S_4^+, \quad \frac{\partial \varphi_{(a,l)}}{\partial \nu} = 0 \text{ on } \partial S_4^+$$

Regarding $\varphi_{(a,l)}$ we prove the following lemma

**Lemma 2.1** For $a \in \partial S_4^+$, we have $(\partial \delta_{a,l})/(\partial \nu) = 0$ and therefore $\varphi_{(a,l)} = \delta_{a,l}$. For $a \notin \partial S_4^+$, we have

$$\varphi_{(a,l)} = \delta_{a,l} + \frac{H(a,.)}{l} + f_{(a,l)},$$

where $f_{(a,l)}$ satisfies

$$|f_{(a,l)}|_{L^\infty} \leq \frac{c}{l^3d^4}, \quad l \frac{\partial f}{\partial l} = O\left( \frac{1}{l^3d} \right)$$

and where $d = d(a, \partial S_4^+)$.  

**Proof.** Using a stereographic projection, we are led to prove the corresponding estimates on $\mathbb{R}_4^+$. We still denote by $G$ and $H$ the Green’s function and its regular part of Laplacian on $\mathbb{R}_4^+$ under Neumann boundary conditions. In this case, we have

$$\delta_{a,l} = \frac{l}{1 + l^2|a-x|^2} \quad \text{and} \quad H(a,x) = \frac{1}{|a-x|^2},$$

where $\bar{a}$ is the symmetric of $a$ with respect to $\partial \mathbb{R}_4^+$. Observe that, for $\theta = \delta_{a,l} + \varphi_{(a,l)} - H(a,.)/l$, we have

$$\Delta \theta = 0 \text{ in } \mathbb{R}_4^+, \quad \frac{\partial \theta}{\partial \nu} = \frac{\partial \delta}{\partial \nu} + \frac{1}{l} \frac{\partial H}{\partial \nu} = O\left( \frac{1}{l^3d^3} \right)$$

Thus, using the Green’s formula, we derive

$$\theta(y) = c \int_{\partial \mathbb{R}_4^+} G \left( \frac{\partial \delta}{\partial \nu} + \frac{1}{l} \frac{\partial H}{\partial \nu} \right) \leq \frac{c}{l^3d^2} \int_{\partial \mathbb{R}_4^+} G \frac{1}{|a-x|^3},$$

where $d_a$ is the distance of $a$ to the boundary. But $G$ satisfies

$$\int_{\partial \mathbb{R}_4^+} G(x,y) \frac{1}{|a-x|^3} = O\left( \frac{1}{d_a^2} \right).$$

Thus, the first estimate follows. The second estimate can be proved by the same way. $\square$  

Now, for $\varepsilon > 0$ and $p \in \mathbb{N}^*$, let us define

$$V(p, \varepsilon) = \{ u \in \Sigma / \exists \alpha_1, \ldots, a_p \in S_4^+, \exists l_1, \ldots, l_p > 0, \exists \alpha_1, \ldots, \alpha_p > 0 \text{ s.t. } ||u - \sum_{i=1}^p \alpha_i \delta_i|| < \varepsilon,$$

$$|\frac{\alpha_i^2K(a_i)}{\alpha_j^2K(a_j)} - 1| < \varepsilon, l_i > \varepsilon^{-1}, \varepsilon_{ij} < \varepsilon \text{ and } l_i d_i < \varepsilon \text{ or } l_i d_i > \varepsilon^{-1} \},$$
where \( \delta_i = \delta_{a_i,i}, d_i = d(a_i, \partial S^4_+) \) and \( \varepsilon^{-1} = l_i/l_j + l_j/l_i + \ldots + l(1 - \cos d(a_i, a_j))/2 \).

The failure of Palais-Smale condition can be described, following the ideas introduced in [14], [31], [34] as follows:

**Proposition 2.2** Assume that \( J \) has no critical point in \( \Sigma^+ \) and let \( (u_k) \in \Sigma^+ \) be a sequence such that \( J(u_k) \) is bounded and \( \nabla J(u_k) \to 0 \). Then, there exist an integer \( p \in \mathbb{N}^* \), a sequence \( \varepsilon_k > 0 \) (\( \varepsilon_k \to 0 \)) and an extracted subsequence of \( u_k \), again denoted \( (u_k) \), such that \( u_k \in V(p, \varepsilon_k) \).

If a function \( u \) belongs to \( V(p, \varepsilon) \), we assume, for the sake of simplicity, that \( l_d_i < \varepsilon \) for \( i \leq q \) and \( l_d_i > \varepsilon^{-1} \) for \( i > q \). We consider the following minimization problem for \( u \in V(p, \varepsilon) \) with \( \varepsilon \) small

\[
\min \{ \|u - \sum_{i=1}^{q} \alpha_i \delta_{a_i,i} - \sum_{i=q+1}^{p} \alpha_i \varphi_{b_i,i} \|, \alpha_i > 0, l_i > 0, a_i \in \partial S^4_+ \text{ and } b_i \in S^4_+ \} \tag{2.1}
\]

We then have the following proposition which defines a parametrization of the set \( V(p, \varepsilon) \). It follows from corresponding statements in [7], [8], [32].

**Proposition 2.3** For any \( p \in \mathbb{N}^* \), there is \( \varepsilon_p > 0 \) such that if \( \varepsilon < \varepsilon_p \) and \( u \in V(p, \varepsilon) \), the minimization problem \( 2.1 \) has a unique solution (up to permutation). In particular, we can write \( u \in V(p, \varepsilon) \) as follows

\[
u = \sum_{i=1}^{q} \bar{\alpha}_i \delta_{a_i,i} + \sum_{i=q+1}^{p} \bar{\alpha}_i \varphi_{a_i,i} + v,
\]

where \( (\bar{\alpha}_1, ..., \bar{\alpha}_p, \bar{\alpha}_1, ..., \bar{\alpha}_p, \bar{l}_1, ..., \bar{l}_p) \) is the solution of \( 2.1 \) and \( v \in H^1(S^4_+) \) such that

\[
\|v\| \leq \varepsilon, \quad (v, \psi) = 0 \quad \text{for} \quad \psi \in \{ \delta_i, \partial \delta_i/\partial l_i, \delta_i/\partial a_i, \varphi_j, \partial \varphi_j/\partial a_i, \partial \varphi_j/\partial a_j/ i \leq q, j > q \}
\]

We also have the following proposition whose proof is similar, up to minor modification to corresponding statements in [6] (see also [32]).

**Proposition 2.4** There exists a \( C^1 \) map which, to each \( (\alpha_1, ..., \alpha_p, a_1, ..., a_p, l_1, ..., l_p) \) such that \( \sum_{i=1}^{p} \alpha_i \delta_i \in V(p, \varepsilon) \) with small \( \varepsilon \), associates \( \bar{\nu} = \bar{\nu}(\alpha_i, a_i, l_i) \) satisfying

\[
J \left( \sum_{i=1}^{q} \alpha_i \delta_i + \sum_{i=q+1}^{p} \alpha_i \varphi_i + \bar{\nu} \right) = \min \left\{ J \left( \sum_{i=1}^{q} \alpha_i \delta_i + \sum_{i=q+1}^{p} \alpha_i \varphi_i + v \right), \ v \text{ satisfies } (V_0) \right\}.
\]

Moreover, there exists \( c > 0 \) such that the following holds

\[
\|\bar{\nu}\| \leq c \left( \sum_{i \leq q} \frac{1}{l_i} + \sum_{i > q} \frac{\|\nabla K(a_i)\|}{l_i} + \sum_{i > q} \frac{1}{(l_id_i)^2} + \sum_{k \neq r} \varepsilon_{kr} (\log(\varepsilon^{-1}))^{1/2} \right).
\]
Next, we are going to give an useful expansion of functional $J$ and its gradient in $V(p, \varepsilon)$.

**Proposition 2.5** For $\varepsilon > 0$ small enough and $u = \sum_{i=1}^{p} \alpha_i \varphi(a_i, l_i) \in V(p, \varepsilon)$, we have the following expansion

$$
J(u) = \frac{8S^{1/2}}{(\sum \alpha_i^4 K(a_i))^{1/2}} \left( 1 + \frac{\varphi_3}{8S} \left( \sum K(a_i)^{-1} \right)^{-1} \left( \sum \left( -\frac{\Delta K(a_i)}{3l_i^2 K(a_i)^2} - \frac{4H(a_i, a_i)}{l_i^2 K(a_i)} \right) \right) - \frac{2}{(K(a_i) K(a_j))^{1/2}} \left( \sum_{i \neq j} \varphi_{ij} + \frac{2H(a_i, a_j)}{l_i l_j} \right) \right) + o \left( \sum \varepsilon_{kr} + \frac{1}{(l_k d_k)^2} \right),
$$

where

$$
S = \int_{\mathbb{R}^4} \frac{dx}{(1 + |x|^2)^4}.
$$

**Proof.** We need to estimate

$$
N = ||u||^2 \quad \text{and} \quad D^2 = \int_{S^4_+} K u^4
$$

Observe that

$$
N = \sum \alpha_i^2 ||\varphi_i||^2 + \sum_{i \neq j} \alpha_i \alpha_j (\varphi_i, \varphi_j) \quad (2.2)
$$

As in [12], we have

$$
||\varphi_i||^2 = 8S + 8\omega \frac{H(a_i, a_i)}{2l_i^2} + o \left( \frac{1}{(l_i d_i)^2} \right) \quad (2.3)
$$

We also have, for $i \neq j$

$$
(\varphi_i, \varphi_j) = 2\varphi_3 \left( \varepsilon_{ij} + \frac{2H(a_i, a_j)}{l_i l_j} \right) + o \left( \sum \frac{1}{(l_k d_k)^2} \right) \quad (2.4)
$$

For the denominator, we have

$$
D^2 = \int_{S^4_+} K \left( \sum \alpha_i \varphi_i \right)^4 = \sum \alpha_i^4 \int_{S^4_+} K \varphi_i^4 + 4 \sum_{i \neq j} \alpha_i^3 \alpha_j \int_{S^4_+} K \varphi_i^3 \varphi_j + o(\sum \varepsilon_{kr}) \quad (2.5)
$$

Observe that

$$
\int_{S^4_+} K \varphi_i^4 = K(a_i) S + \frac{\varphi_3 \Delta K(a_i)}{12l_i} + 2\varphi_3 K(a_i) \frac{H(a_i, a_i)}{l_i^2} + o \left( \frac{1}{(l_i d_i)^2} \right) \quad (2.6)
$$

We also have, for $i \neq j$

$$
\int_{S^4_+} K \varphi_i^3 \varphi_j = \frac{\varphi_3}{4} K(a_i) \left( \varepsilon_{ij} + \frac{2H(a_i, a_j)}{l_i l_j} \right) + o \left( \frac{1}{(l_k d_k)^2} \right) \quad (2.7)
$$

Using (2.2), ..., (2.7), the result follows. \qed
Proposition 2.6 (see [12]) For $u = \sum_{i \leq q} \alpha_i \delta_i + \sum_{j=q+1}^p \alpha_j \varphi_j \in V(p, \varepsilon)$, we have the following expansions

$$
(\nabla J(u), l_i \partial \delta_i / \partial l_i) = c_1 J(u) \sum_{i \leq q, j \neq i} \alpha_k l_i \frac{\partial \varepsilon_{ik}}{\partial l_i} + c_2 J(u)^3 \alpha_i^3 \frac{\partial K}{\partial \nu}(a_i) \\
+ O \left( \sum_{k \leq q} \frac{1}{l_k^2} + \sum_{k=q} \varepsilon_{kr} \right) + o(\sum_{k,r \leq q} \varepsilon_{kr})
$$

$$
(\nabla J(u), l_i^{-1} \partial \delta_i / \partial a_i) = c_3 \alpha_i J(u) e_4 \left( c_4 (1 - J(u)^2 \alpha_i^2 K(a_i)) + J(u)^3 \alpha_i^3 \frac{c_5}{l_i} \frac{\partial K(a_i)}{\partial \nu} \right) \\
- c_5 J(u) \sum_{k \leq q} \alpha_i \frac{1}{l_i} \frac{\partial \varepsilon_{ik}}{\partial a_i} (1 + o(1)) - 4 J(u)^3 \alpha_i^3 c_6 \nabla_T K(a_i) \\
+ o(\sum_{k,r \leq q} \varepsilon_{kr}) + O \left( \sum_{k \leq q} \frac{1}{l_k^2} + \sum_{k \leq q} \varepsilon_{kr} \right)
$$

$$
(\nabla J(u), \delta_i) = c_7 J(u) \alpha_i (1 - J(u)^2 \alpha_i^2 K(a_i)) + O \left( \frac{1}{l_i} + \sum_{ij} \varepsilon_{ij} \right)
$$

where $c_1, \ldots, c_6$ are positive constants, and $(e_1, \ldots, e_4)$ denotes an orthonormal basis of $T_a S_+^4$.

Proposition 2.7 For $u = \sum_{i \leq q} \alpha_i \delta_i + \sum_{j=q+1}^p \alpha_j \varphi_j \in V(p, \varepsilon)$, we have the following expansions

$$
(\nabla J(u), l_j \partial \varphi_j / \partial l_j) = 2 \phi_3 J(u) \left( -2 \sum_{k \neq j} \alpha_k l_j \frac{\partial \varepsilon_{jk}}{\partial l_j} + 4 \sum_{k=q+1, k \neq j}^{p} \alpha_k \frac{H(a_j, a_k)}{l_j l_k} \right) \\
+ \alpha_j \frac{\Delta K(a_j)}{3 l_j^2 K(a_j)} + 4 \alpha_j \frac{H(a_j, a_j)}{l_j} + o \left( \sum_{k=1}^{p} \frac{1}{(l_k d_k)^2} + \sum_{k \neq r} \varepsilon_{kr} \right)
$$

$$
(\nabla J(u), l_j^{-1} \partial \varphi_j / \partial a_j). \nabla K(a_j) \geq c \frac{|\nabla K(a_j)|^2}{l_j} + O \left( \frac{1}{(l_j d_j)^2} + \sum_{k \neq j} \varepsilon_{kj} \right)
$$

$$
3 \int_{S_+^4} K \varphi_j \varphi_j^2 l_j \frac{\partial \varphi_j}{\partial l_j} = \frac{\phi_4}{4} K(a_j) \left( \frac{\partial \varepsilon_{ij}}{\partial l_j} - 2 \frac{H(a_i, a_j)}{l_i l_j} \right) + o \left( \varepsilon_{ij} + \sum_{k \neq j} \frac{1}{(l_k d_k)^2} \right)
$$

Proof. First observe that easy computations show the following estimates:
trivial in scalar curvature problems whose functional’s levels on $V$.

Some levels of $V$ blow up points. Such a flow is defined by combining two basic facts. On the one hand, the first one comes from the Morse Lemma at infinity which moves points and concentrations they are boundary points, concentrations move so as to decrease the Functional $\sum \partial \varphi_i$.

On the other hand, there is another pseudogradient when the points are very close and the flows to keep the pseudogradient property, to avoid the creation of new asymptotes and follow using similar arguments as in [11].

\[ \int S^4 K \delta_i^3 l_j \frac{\partial \varphi_i}{\partial l_j} = \frac{\phi_3}{4} K(a_i) \frac{\partial \varepsilon_{ij}}{\partial l_j} + o(\varepsilon_{ij}) \]
\[ \int S^4 K \varphi_i^3 l_j \frac{\partial \varphi_i}{\partial l_j} = \frac{\phi_3}{4} \Delta K(a_i) \left( l_j \frac{\partial \varepsilon_{ij}}{\partial l_j} - 2 \frac{H(a_i, a_i)}{l_j} \right) + o(\varepsilon_{ij} + \sum \frac{1}{(l_j d_j)^2} ) \]
\[ 3 \int S^4 K \varphi_i \varphi_j^2 l_j \frac{\partial \varphi_i}{\partial l_j} = \frac{\phi_3}{4} K(a_j) \frac{\partial \varepsilon_{ij}}{\partial l_j} + o(\varepsilon_{ij} \) \]
\[ 3 \int S^4 K \varphi_i \varphi_j^2 l_j \frac{\partial \varphi_i}{\partial l_j} = \frac{\phi_3}{4} K(a_j) \left( l_j \frac{\partial \varepsilon_{ij}}{\partial l_j} - 2 \frac{H(a_i, a_i)}{l_j} \right) + o(\varepsilon_{ij} + \sum \frac{1}{(l_j d_j)^2} ) \]

Using the above estimates and the fact that $J(u)^2 \alpha_i^2 K(a_i) = 8 + o(1)$, the Proposition follows using similar arguments as in [11].

\[ \square \]

3 Construction of a pseudogradient flow

In this section we are going to construct a global pseudogradient flow for the Functional $J$ under assumption $(C)$ on $K$. Along its flow lines there can be only finitely many isolated blow up points. Such a flow is defined by combining two basic facts. On the one hand, the first one comes from the Morse Lemma at infinity which moves points and concentrations as follows: points move according to $-\nabla K$ if they are interior points and along $\partial \nu$ $K$ if they are boundary points, concentrations move so as to decrease the Functional $J$. On the other hand, there is another pseudogradient when the points are very close and the total interaction $\sum \varepsilon_{ij}$ is large with respect to $\sum \frac{1}{p}$. We need to convex-combine both flows to keep the pseudogradient property, to avoid the creation of new asymptotes and to ensure the property that the flow lines when they leave some $V(p, \varepsilon)$ will loose at least one bubble, that is the flow will never come back to $V(q, \varepsilon)$ for $q \leq p$, a fact which is not trivial in scalar curvature problems whose functional’s levels on $V(p, \varepsilon)$ are not constant. Some levels of $V(p, \varepsilon)$ might be below some other levels of $V(q, \varepsilon)$ for some $q < p$. 

\[ (\delta_i, l_j \frac{\partial \varphi_j}{\partial l_j}) = 2 \phi_3 l_j \frac{\partial \varepsilon_{ij}}{\partial l_j} + o(\varepsilon_{ij} + \frac{1}{(l_j d_j)^2}) \]
\[ (\varphi_i, l_j \frac{\partial \varphi_j}{\partial l_j}) = 2 \phi_3 l_j \frac{\partial \varepsilon_{ij}}{\partial l_j} - 4 \phi_3 H(a_i, a_j) \frac{l_i l_j}{l_j} + o(\varepsilon_{ij} + \sum \frac{1}{(l_j d_j)^2}) \]
\[ (\varphi_j, l_j \frac{\partial \varphi_j}{\partial l_j}) = -4 \phi_3 \frac{H(a_j, a_j)}{l_j} + o(\frac{1}{(l_j d_j)^2}) \]

Scalar Curvature Problem
As a by product of the construction of our pseudogradient, we able to identify the critical points at infinity of our problem. We recall that the critical point at infinity are the orbits of the gradient flow of \( J \) which remain in \( V(p, \varepsilon(s)) \), where \( \varepsilon(s) \), a given function, tends to zero when \( s \) tends to \( +\infty \) (see [6]).

**Proposition 3.1** For \( p \geq 1 \), there exists a pseudo-gradient \( W \) so that the following holds: There is a constant \( c > 0 \) independent of \( u = \sum_{i=1}^{q} \alpha_{i} \xi_{i} + \sum_{j=q+1}^{p} \alpha_{j} \varphi_{j} \in V(p, \varepsilon) \) so that

\[
(i) \quad (-\nabla J(u), W) \geq c \left( \sum_{k \neq r} \varepsilon_{kr} + \sum_{i \leq q} \frac{1}{l_{i}} + \frac{\abs{\nabla K(a_{i})}}{l_{i}} + \sum_{j=q+1}^{p} \frac{1}{(l_{j}d_{j})^{2}} \right)
\]

\[
(ii) \quad (-\nabla J(u + \varpi), W + \frac{\partial W}{\partial(\alpha_{i}, a_{i}, l_{i})}) \geq c \left( \sum_{k \neq r} \varepsilon_{kr} + \sum_{i \leq q} \frac{1}{l_{i}} + \frac{\abs{\nabla K(a_{i})}}{l_{i}} + \sum_{j=q+1}^{p} \frac{1}{(l_{j}d_{j})^{2}} \right)
\]

\((iii) |W| \) is bounded. Furthermore, the only case where the maximum of the \( l_{i} \)'s is not bounded is when each point \( a_{i} \) is close to a critical point \( y_{j} \) of \( K \) with \( j_{i} \neq j_{k} \) for \( i \neq k \) and \( \rho(y_{i}, ..., y_{p}) > 0 \), where \( \rho(y_{i}, ..., y_{p}) \) denotes the least eigenvalue of \( M(y_{i}, ..., y_{p}) \).

**Proof.** Without loss of generality, we can assume that

\[
l_{1} \leq ... \leq l_{q}, \quad \text{and} \quad l_{q+1} \leq ... \leq l_{p}.
\]

Let

\[
u = \sum_{i \leq q} \alpha_{i} \xi_{i} + \sum_{i > q} \alpha_{i} \varphi_{i} \in V(p, \varepsilon)
\]

Since \((\partial K(z))/(\partial \nu) < 0\) for any critical point \( z \) of \( K_{1} = K_{1}/\partial S_{+}^{1} \), there exist \( \mu > 0 \) and \( c > 0 \) such that \((\partial K(a))/(\partial \nu) < -c < 0\) for any \( a \in \partial B(z_{k}, 2\mu) \). In the first step, we will build a vector field \( Y_{b} \) using the indices \( i \in \{1, ..., q\} \). For this purpose, we introduce the following sets

\[
P = \{ i \leq q/ \exists \text{ a sequence } i_{1} = i, ..., i_{j} \text{ s.t. } a_{i_{j}} \in \partial B(z_{k}, \mu) \text{ and } d(a_{i_{k}}, a_{i_{k-1}}) < \frac{1}{p} \mu \forall k \leq j \}
\]

\[
I = \{ i \leq q/ l_{i} \leq M l_{1} \}, \quad \text{where } M \text{ is a large constant.}
\]

We observe that, if \( i \in P \) then there exists \( r \) such that \( d(a_{i}, z_{r}) < 2\mu \) and therefore \((\partial K(a_{i}))/(\partial \nu) < -c < 0\). But if \( i \notin P \) then, for any \( r \) we have \( d(a_{i}, z_{r}) \geq \mu \). Furthermore if \( j \notin P \) and \( i \in P \) then \( d(a_{i}, a_{j}) \geq \frac{\mu}{p} \). We also have for \( l_{i} \geq l_{j} \)

\[
-2l_{i} \frac{\partial \varepsilon_{ij}}{\partial l_{i}} - l_{j} \frac{\partial \varepsilon_{ij}}{\partial l_{j}} = 2\varepsilon_{ij}(1 - 2\frac{l_{i}}{l_{j}}\varepsilon_{ij}) + \varepsilon_{ij}(1 - 2\frac{l_{i}}{l_{j}}\varepsilon_{ij}) \geq \varepsilon_{ij}(1 + o(1)) \quad (3.1)
\]
Now, we define the following vector fields

\[ Z_1 = \sum_{i \in P} l_i \frac{\partial \delta_i}{\partial a_i} \quad \text{and} \quad Z_2 = \sum_{i \in I \setminus P} \frac{1}{l_i} \frac{\partial \delta_i}{\partial a_i} \frac{\nabla T K(a_i)}{|\nabla T K(a_i)|} \]

Using (3.1) and Proposition 2.6, we derive

\[ \left( -\nabla J(u), Z_1 \right) \geq c \sum_{i \in P, j \leq q} \varepsilon_{ij} + O \left( \sum_{i \in P, j > q} \varepsilon_{ij} \right) + \sum_{i \in P} c \frac{1}{l_i} + o \left( \sum_{k \leq q} \varepsilon_{kr} \right) + O \left( \sum_{k \in P} \frac{1}{l_k^2} \right) \quad (3.2) \]

and

\[ \left( -\nabla J(u), Z_2 \right) \geq \sum_{i \in I \setminus P} \left( \frac{c}{l_i} + O \left( \sum_{j \in I \setminus P} \frac{1}{l_i} \left| \frac{\partial \varepsilon_{ij}}{\partial a_i} \right| \right) \right) + O \left( \sum_{j \notin I \cup P} \varepsilon_{ij} + \sum_{j > q} \varepsilon_{ij} \right) \]

\[ + o \left( \sum_{k, r \leq q} \varepsilon_{kr} \right) + O \left( \sum_{k \leq q} \frac{1}{l_k} \right) \quad (3.3) \]

Notice that for \( i, j \in I \), \( l_i d(a_i, a_j) \) is very large and thus

\[ \frac{1}{l_i} \left| \frac{\partial \varepsilon_{ij}}{\partial a_i} \right| \leq c l_j d(a_i, a_j) \leq \frac{\varepsilon_{ij}}{l_i d(a_i, a_j)} = o(\varepsilon_{ij}) \quad (3.4) \]

We also notice that if \( i \notin P \) and \( j \in P \), thus \( d(a_i, a_j) > \frac{4}{p} \) and therefore

\[ \frac{1}{l_i} \left| \frac{\partial \varepsilon_{ij}}{\partial a_i} \right| = O \left( \frac{1}{l_i} + \frac{1}{l_j^2} \right) \quad (3.5) \]

Let us define now the following vector field

\[ Z_3 = -\sqrt{M} \sum_{i \notin I \cup P} l_i \frac{\partial \delta_i}{\partial a_i}^2 - m \sum_{i \in I} l_i \frac{\partial \delta_i}{\partial l_i}, \]

where \( m \) is a small positive constant.

Using (3.1) and Proposition 2.6, we derive

\[ \left( -\nabla J(u), Z_3 \right) \geq c \sqrt{M} \sum_{i \notin I \cup P} \left( \sum_{j \leq q, j \notin P} \varepsilon_{ij} + O \left( \sum_{j \in P} \varepsilon_{ij} + \sum_{j > q} \varepsilon_{ij} + \sum_{k \leq q} \frac{1}{l_k^2} + \frac{1}{l_i} \right) \right) \]

\[ + cm \sum_{i \in I} \left( \sum_{j \in I} \varepsilon_{ij} + O \left( \sum_{j \notin I} \varepsilon_{ij} + \frac{1}{l_i} \right) \right) \quad (3.6) \]
Observe that, if \( i \notin I \) and \( i \leq q \), we have \( \sqrt{M}/l_i \leq (l_1 \sqrt{M})^{-1} = o(l_1^{-1}) \), for \( M \) large enough.

We also define the following vector field
\[
Z_4 = \sum_{i \leq q} \psi \left( l_i (1 - J(u)^2 \alpha_i^2 K(a_i)) \right) \delta_i,
\]
where \( \psi \) is a \( C^\infty \) function which satisfies
\[
\psi(t) = -1 \text{ if } t > 2, \quad \psi(t) = 0 \text{ if } |t| \leq 1 \quad \text{and} \quad \psi(t) = 1 \text{ if } t < -2
\]
Using Proposition 2.6, we derive
\[
\langle -\nabla J(u), Z_4 \rangle \geq \sum_{i \leq q} |\psi(l_i (1 - J(u)^2 \alpha_i^2 K(a_i)))| \left| 1 - J(u)^2 \alpha_i^2 K(a_i) \right| + O \left( \frac{1}{l_i} + \sum_{j \neq i} \varepsilon_{ij} \right) \quad (3.7)
\]
Now, we introduce the following vector field
\[
Y_b = M_1 Z_1 + \sqrt{M_1} Z_2 + Z_3 + Z_4
\]
Using (3.2),...,(3.7), we derive
\[
\langle -\nabla J(u), Y_b \rangle \geq c \sum_{i \leq q} \left( \frac{1}{l_i} + |1 - J(u)^2 \alpha_i^2 K(a_i)| \sum_{j \leq q} \varepsilon_{ij} \right) + O \left( \sum_{i \leq q, j > q} \varepsilon_{ij} \right) \quad (3.8)
\]
Secondly, we need to construct a vector field using the indices \( i > q \). We claim that, for \( i \leq q \) and \( j > q \), we have
\[
- l_j \frac{\partial \varepsilon_{ij}}{\partial l_i} = \varepsilon_{ij} \left( 1 - 2 \frac{l_i}{l_j} \varepsilon_{ij} \right) = \varepsilon_{ij} (1 + o(1)) \quad (3.9)
\]
Indeed, if \( l_j \geq l_i \), our claim is easy, if \( l_j \leq l_i \), we obtain \( l_j d(a_i, a_j) \geq l_j d(a_i, a_j) > l_j d(a_j, \partial S_4^+) > \varepsilon^{-1} \), thus \( \frac{l_i}{l_j} \varepsilon_{ij} \leq \frac{1}{l_j d(a_i, a_j)} = o(1) \), then our claim follows.
Now, we see that there exists \( d_0 > 0 \) small enough such that for any \( a \) satisfying \( d_a = d(a, \partial S_4^+) \leq d_0 \), we have
\[
H(a, a) \geq M'|\Delta K(a)| \quad \text{and} \quad H(a, a) \sim \frac{c}{d_a^2},
\]
where \( M' \) is a large constant.
We need to introduce the subset of points which are close to the boundary, for that purpose, let us introduce the following set
\[
F = \{ i > q / \exists i_1 = i, ..., i_l \text{ s.t. } d(a_{i_1}, \partial S_4^+) < \frac{d_0}{p} \text{ and } d(a_{i_k}, a_{i_{k-1}}) < \frac{d_0}{p} \forall k \leq l \}
\]
It is easy to see that the following claims hold:
- if \( i \in F, j \notin F, j \geq q + 1 \), we have \( d(a_i, a_j) \geq d_0/p \)
- for any \( i \in F \), we have \( H(a_i, a_i) \geq M' |\Delta K(a_i)| \) and \( H(a_i, a_i) \sim c/d_0^2 \)
- for any \( j \notin F \) and \( j \geq q + 1 \), we have \( d_j > d_0/p \). Furthermore, if \( i \leq q \), we have \( \varepsilon_{ij} = o(l_i^{-1}) \)

Thus, (3.8) becomes

\[
< -\nabla J(u), Y_b > \geq c \sum_{i \leq q} \left( \frac{1}{l_i} + |1 - J(u)^2 \alpha_i^2 K(a_i)| + \sum_{k \leq q} \varepsilon_{ki} \right) \\
+ c \sum_{i \leq q, j \notin F} \varepsilon_{ij} + O \left( \sum_{k \leq q, r \in F} \varepsilon_{kr} \right) 
\]

(3.10)

Now, we introduce

\[
Z_5 = -\sum_{i \in F} l_i \frac{\partial \varphi_i}{\partial l_i} 2^i 
\]

Using Proposition 2.6, we derive

\[
< -\nabla J(u), Z_5 > \geq c \sum_{k \in F} \varepsilon_{kr} + c \sum_{k \in F} \frac{1}{(l_k d_k)^2} + o(\sum \varepsilon_{kr}) 
\]

(3.11)

Next, we deal with the points which are far away from the boundary. Let \( \bar{l} = \max(l_1, \min\{l_i, i \in F\}) \) (recall that \( l_1 = \min\{l_i, i \leq q\} \)).

We introduce the following sets

\[
L = \{ i > q / i \notin F \text{ and } l_i \leq \bar{l}/2 \}, \quad L' = \{ i > q / i \notin F \text{ and } i \notin L \} 
\]

(3.12)

We denote by \( Z_6 \) the following vector field

\[
Z_6 = -\sum_{i \in L'} l_i \frac{\partial \varphi_i}{\partial l_i} 2^i 
\]

Thus, we have

\[
< -\nabla J(u), Z_6 > \geq c \sum_{k \in L'} \varepsilon_{kr} + \sum_{k \in L'} O \left( \frac{1}{l_k^2} \right) + o(\sum \varepsilon_{kr}) \\
+ \sum_{k \in L'} \varepsilon_{kr} + O \left( \frac{1}{l_k^2} \right) + o(\sum \varepsilon_{kr}) 
\]

(3.13)

Now, we define

\[
Z_7 = M_2 Z_5 + Y_b + Z_6, 
\]
where $M_2$ is a positive constant large enough.

Using (3.10), (3.11) and (3.13), we derive
\[
< -\nabla J(u), Z_7 > \geq c \sum_{k,r \in L} \varepsilon_{kr} + c \sum_{i \leq q} \left( |1 - J(u)^2 \alpha_i^2 K(a_i)| + \frac{1}{l_i} \right) \\
+ c \sum_{k \in F \cup L'} \frac{1}{(l_i d_i)^2} + o(\sum \varepsilon_{kr})
\]
(3.14)

Now, we observe that if $L \neq \{1, ..., p\}$, then
\[
\max \{l_i, i \in L \} \leq (1/2) \max \{l_i, i = 1, ..., p\},
\]
(3.15)

where $L$ is defined in (3.12). Notice that, if $i \in L$, we have $d_i \geq d_0$, thus the function $H$ and its gradient are bounded. In this case, we can use the vector field (denoted here $Z_8$) defined in Lemma 3.3 of [11]. We will apply $Z_8$ only to $u_1 = \sum_{i \in L} \alpha_i \phi_i$ forgetting the indices $i \notin L$. Thus, we have
\[
< -\nabla J(u), Z_8(u_1) > \geq c \left( \sum_{k,r \in L} \varepsilon_{kr} + \sum_{i \in L} \frac{1}{l_i^2} \right) + O \left( \sum_{k \in L, r \notin L} \varepsilon_{kr} \right) + o \left( \sum \frac{1}{l_i^2} \right)
\]
(3.16)

Now, we define
\[
Z_9 = Z_8 + M_3 Z_7,
\]

where $M_3$ is a positive constant large enough.

Using (3.14) and (3.16), we derive
\[
< -\nabla J(u), Z_9 > \geq c \left( \sum_{k,r \in L} \varepsilon_{kr} + \sum_{i \leq q} |1 - J(u)^2 \alpha_i^2 K(a_i)| \frac{1}{l_i} + \sum_{i > q} \frac{1}{(l_i d_i)^2} \right)
\]
(3.17)

To obtain the estimate (i), we need to introduce the following vector field
\[
Z_{10} = \sum_{i > q} \frac{1}{l_i d_i} \frac{\partial \phi_i}{\partial a_i} \frac{\nabla K(a_i)}{|\nabla K(a_i)|} \psi(l_i |\nabla K(a_i)|),
\]

where $\psi$ is a $C^\infty$ function satisfying $\psi(t) = 1$ if $t \geq 2$ and $\psi(t) = 0$ if $t \leq 1$.

Our vector field $W$ will be the following
\[
W = M_4 Z_9 + Z_{10},
\]

where $M_4$ is a positive constant large enough.

Thus, using (3.17) and Proposition 2.6, the estimate (i) follows.

Regarding the estimate (ii), it can be obtained once we have (i), using the estimates of $||\nabla J(u + \tilde{v})||$ and $||\tilde{v}||$, arguing as in Appendix B of [11].

Now, we observe that if the set $L$ defined in (3.12) is equal to $\{1, ..., p\}$, thus using (3.15)
and the fact that $Z_7$ only decreases the $l_i$’s, we derive that the maximum of the $l_i$’s is a decreasing function in this case. In other case, that is $L = \{1, \ldots, p\}$, Claim (ii) follows from the definition of $Z_8$ (see Lemma 3.3 of [11]). Thus, the proof of our proposition is completed. \hfill $\square$

**Corollary 3.2** Assume that $J$ has no critical points in $\Sigma^+$. Then the only critical points at infinity of $J$ correspond to

$$\sum_{j=1}^p K(y_{ij})^{-1/2} \varphi(y_{ij}, \infty), \quad \text{with } p \in \mathbb{N}^* \text{ and } \rho(y_{i1}, \ldots, y_{ip}) > 0.$$  

Furthermore, such a critical point at infinity has a Morse index equal to 

$$5p - 1 - \sum_{j=1}^p \text{index}(K, y_{ij})$$

where $\text{index}(K, y_{ij})$ is the Morse index of $K$ at $y_{ij}$.

**Proof.** From Proposition 3.1 we know that the only region where the $l_i$’s are unbounded is when each point $a_i$ is close to a critical point $y_{ij}$, with $j_i \neq j_k$ for $i \neq k$ and $\rho(y_{i1}, \ldots, y_{ip}) > 0$. In this region, arguing as in [7] and [11], we can find a change of variable $(a_1, \ldots, a_p, l_1, \ldots, l_p) \rightarrow (\tilde{a}_1, \ldots, \tilde{a}_p, \tilde{l}_1, \ldots, \tilde{l}_p) := (\tilde{a}, \tilde{l})$

such that

$$J\left(\sum_{i=1}^p \alpha_i \varphi_i + \tau\right) = \psi(\alpha, \tilde{a}, \tilde{l}) := \frac{8 S_4^{1/2} \sum_{i=1}^p \alpha_i^2}{(\sum_{i=1}^p \alpha_i^4 K(a_i))^{1/2}} \left(1 + \eta \right) \left(\sum_{i=1}^p \frac{1}{K(y_{ij})}\right)^{-1} \left(\sum_{i=1}^p \frac{1}{K(y_{ij})}\right)^{-1} L M(\tau_p L)$$

where $\alpha = (\alpha_1, \ldots, \alpha_p)$, $c$ is a positive constant, $\eta$ is a small positive constant, $L = (\tilde{l}_1, \ldots, \tilde{l}_p)$, $\tau_p = (y_{j_1}, \ldots, y_{j_p})$ and $S_4 = \int_{\mathbb{R}^4} \delta_{(0,1)}^4$.

This yields a split of variables $\tilde{a}$ and $\tilde{l}$, thus it is easy to see that if $\tilde{a}$ is equal to $(y_{j_1}, \ldots, y_{j_p})$, only $\tilde{l}$ can move. Since $\rho(y_{i1}, \ldots, y_{ip}) > 0$, in order to decrease the functional $J$, we have to increase $\tilde{l}$, and we obtain a critical point at infinity only in this case.

It remains to compute the Morse index of such a critical point at infinity. In order to compute such a Morse index, we observe that $M(\tau_p)$ is definite positive and the function $\psi$ possesses, with respect to the variables $\alpha_i$’s, an absolute degenerate maximum with one dimensional nullity space. Then the Morse index of such a critical point at infinity is equal to $(5p - 1 - \sum_{i=1}^p (4 - \text{index}(K, y_{ij})))$. Thus our result follows. \hfill $\square$

4 Proof of Theorems

In this section we give the proof of Theorems 1.1 and 1.2.

**Proof of Theorem 1.1** For $\eta > 0$ small enough, we introduce the following neighborhood of $\Sigma^+$

$$V_\eta(\Sigma^+) = \{u \in \Sigma/ e^{2J(u)} J(u)^3 |u|_{L^4}^2 < \eta\},$$
where $u^- = \max(0, -u)$. Recall that, we already built in Proposition 3.1 a vector field $W$ defined in $V(p, \varepsilon)$ for $p \geq 1$. Outside $\cup_{p \geq 1} V(p, \varepsilon/2)$, we will use $-\nabla J$ and our global vector field $Z$ will be built using a convex combination of $W$ and $-\nabla J$. $V_\eta(\Sigma^+)$ is invariant under the flow line generated by $Z$ (see [11]). Since $V_\eta(\Sigma^+)$ is contractible, we have $\chi(V_\eta(\Sigma^+)) = 1$, where $\chi$ is the Euler-Poincare characteristic. Arguing by contradiction, we assume that $J$ has no critical points in $V_\eta(\Sigma^+)$. It follows from Corollary 3.2 that the only critical points at infinity of $J$ in $V_\eta(\Sigma^+)$ correspond to

$$\sum_{j=1}^p K(y_{i_j})^{-1/2} \varphi(y_{i_j}, \infty), \quad \text{with } p \in \mathbb{N}^* \text{ and } \rho(y_{i_1}, ..., y_{i_p}) > 0,$$

and such a critical point at infinity has a Morse index equal to $(5p - 1 - \sum_{j=1}^p \text{index}(K, y_{i_j}))$. Using the vector field $Z$, we have $V_\eta(\Sigma^+)$ retracts by deformation on $\cup W_u(w_\infty)$ (see section 7 and 8 of [10]), where $W_u(w_\infty)$ is the unstable manifold at infinity of a critical point at infinity $w_\infty$. Then, we have

$$1 = \chi(V_\eta(\Sigma^+)) = \sum_{p=1}^l \sum_{\tau_p = (i_1, ..., i_p) / \rho(\tau_p) > 0} (-1)^{(5p - 1 - \sum_{j=1}^p \text{index}(K, y_{i_j}))},$$

which is in contradiction with the assumption of our theorem. Thus there exists a critical point of $J$ in $V_\eta(\Sigma^+)$. Arguing as in [11], we prove that this critical point is positive and hence our result follows. 

\textbf{Proof of Theorem 1.2} Again, we argue by contradiction. We assume that $J$ has no critical points in $V_\eta(\Sigma^+)$. We observe that

$$H(y, y) = \frac{1}{4} d_y^{-2}, \quad \text{for any } y \in S^4_+,$$

where $d_y = d(y, \partial S^4_+)$. Thus, under the assumption of our theorem, we derive that

$$(-\Delta K(y)/(3K(y))) - H(y, y) < 0, \quad \text{for any critical point } y \text{ of } K.$$

Using Corollary 3.2, we deduce that there is no critical points at infinity of $J$. Let $Z$ be the vector field defined in the proof of Theorem 11. Let $u_0 \in \Sigma^+$ and let $\eta(s, u_0)$ be the one parameter group generated by $Z$. It is known that $|\nabla J|$ is lower bounded outside $V(p, \varepsilon/2)$, for any $p \in \mathbb{N}^*$ and for $\varepsilon$ small enough, by a fixed constant which depends only on $\varepsilon$. Thus, the flow line $\eta(s, u_0)$ cannot remain outside of the set $V(p, \varepsilon/2)$. Furthermore, if the flow line travels from $V(p, \varepsilon/2)$ to the boundary of $V(p, \varepsilon)$, $J(\eta(s, u_0))$ will decrease by a fixed constant which depends on $\varepsilon$. Then, this travel cannot be repeated in an infinite
time. Thus, there exist \( p_0 \) and \( s_0 \) such that the flow line enters into \( V(p_0, \varepsilon/2) \) and it does not exit from \( V(p_0, \varepsilon) \). But in \( V(p_0, \varepsilon) \), by Proposition 3.1 we know that the maximum \( l_{max} \) of the \( l_i \)'s is bounded by \( l_{max}(s_0) \) and therefore \( |\nabla J| \) is lower bounded. Then when \( s \) goes to \( +\infty \), \( J(u(s)) \) goes to \( -\infty \) and this yields a contradiction, hence our result follows. 

\[ \square \]

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