Decoherence dynamics of qubits coupled to systems at quantum transitions

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We study the decoherence properties of a two-level (qubit) system homogeneously coupled to an environmental many-body system $S$ at a quantum transition (QT), considering both continuous and first-order quantum transitions (CQTs and FOQTs respectively). In particular, we consider a $d$-dimensional quantum Ising model as environment system $S$, and study the qubit decoherence (QD) dynamics along the global quantum evolution starting from pure states of the qubit and the ground state of $S$. This issue is discussed within dynamic finite-size scaling (DFSS) frameworks. We analyze the DFSS of appropriate QD functions. At CQTs they develop power laws of the size of $S$, with a substantial enhancement of the growth rate of the QD with respect to the case $S$ is in normal noncritical conditions. The enhancement of the QD growth rate appears much larger at FOQTs, leading to exponential laws when increasing the size of $S$.

I. INTRODUCTION

Decoherence generally arises when a quantum system interacts with an environmental many-body system $S$. This issue is crucially related to the emergence of classical behaviors in quantum systems [1–2], quantum effects such as interference and entanglement [3–4], and it is particularly relevant for the efficiency of quantum information protocols [5]. The decoherence dynamics has been investigated in some paradigmatic models, such as two-level (qubit) systems interacting with many-body systems, in particular the so-called central spin models, see, e.g., Refs. [6–17], where the qubit is coupled to all components of the environmental system $S$.

A typical problem concerns the coherence loss of the qubit during the entangled quantum evolution of the global system, starting from pure states of the qubit and the ground state of $S$. The decoherence rate may significantly depend on the quantum phase of $S$, and, in particular, whether $S$ develops critical behaviors arising from quantum transitions (QTs). Indeed the response of many-body systems at QTs is generally amplified by critical quantum fluctuations. At QTs, small variations of the driving parameter give rise to significant changes of the ground state and low-excitation properties of many-body systems [18]. At first-order QTs (FOQTs) the ground-state properties are discontinuous, generally arising from level crossings in the infinite-volume limit. Continuous QTs (CQTs) show continuous change of the ground state at the transition point, and correlation functions develop a divergent length scale.

Environmental systems at QTs may significantly drive the qubit decoherence (QD) dynamics. An enhanced QD has been put forward [10] in the case of CQTs. In this paper we return to this issue, providing a quantitative scaling framework to support the enhancement of the growth rate of the QD, and extending the analysis to the case the environmental system is at a FOQT.

We consider a qubit homogeneously coupled to a $d$-dimensional many-body system $S$ of size $L$ (or equivalently with $N \sim L^d$ degrees of freedom). In particular, as environmental systems we consider the paradigmatic $d$-dimensional quantum Ising models, whose quantum phase diagrams present both CQTs and FOQTs [18]. The two-level qubit system is equally coupled to all $L^d$ spin states of $S$. We consider the standard out-of-equilibrium protocol in which the initial global state is a product of pure states of the qubit and $S$. We study the QD dynamics during the quantum evolution of the global system, as measured by its density matrix, obtained tracing out the $S$-states.

We investigate the QD dynamics when the environmental Ising system experiences a QT. The QD properties are analyzed within dynamic finite-size scaling (DFSS) frameworks. At both CQTs and FOQTs, DFSS behaviors arise from the interplay among the coupling of the qubit with $S$, the Hamiltonian parameters of $S$ close to the QT, and the size $L$ of $S$. We show that the critical conditions of the environmental system at QTs give rise to a substantial enhancement of the growth rate of the QD dynamics with respect to systems outside criticality. In particular, the QD growth rate (QDGR) at CQTs shows power laws $L^\zeta$ of the size $L$, with exponents $\zeta$ that are larger than that of the volume $L^d$ law expected for systems in normal conditions. The rate enhancement of the qubit coherence loss is even more substantial at FOQTs, whose DFSS is generally characterized by exponentially laws, predicting exponentially large QDGR $\sim \exp(bL^d)$.

The paper is organized as follows. In Sec. II we present the general setting of the out-of-equilibrium problem that we consider. In Sec. III we discuss the QD properties when the environmental system is critical at a CQT, within a DFSS framework, and show the enhanced growth rate of decoherence with respect to normal conditions. Sec. IV extends this analysis to FOQTs, showing that the QDGR enhancement is even more pronounced. Finally in Sec. V we summarize and draw our conclusions.
where the trace of the square density matrix
\[ \text{Tr} P_v = 0 \] and 0 \leq F_D(t) \leq 1. The function \( F_D \) measures the QD, quantifying the departure from a pure state. Indeed \( F_D(t) = 0 \) implies that the qubit is in a pure state, while \( F_D(t) = 1 \) indicates that the qubit is maximally entangled. Of course, the time evolution of the QD function depends on the parameters of the global system, i.e., \( v \) that measures the distance of the many-body system from the QT, the coupling \( w \) between the qubit and the system, and the size \( L \) of the system.

Note that the overlap \( L_D \equiv \left| \langle \phi_{v-w}(t) | \phi_{v+w}(t) \rangle \right| \) entering the definition of \( F_D \) can be interpreted as the fidelity or Loschmidt echo, see, e.g., Ref. [10], of the S-states associated with two different quench protocols involving the isolated system \( S \). For both of them the system \( S \) starts from the ground state of the Hamiltonian \( H_S(v) \) as \( t = 0 \); then one considers, and compares using \( L_D \), the quantum evolutions at the same \( t \), arising from the sudden change of the Hamiltonian parameter \( v \) to \( v - w \) and to \( v + w \).

Noting that
\[ \langle \phi_{v-w}(t) | \phi_{v+w}(t) \rangle = \langle G_v | e^{i H_S(v-w)t} e^{-i H_S(v+w)t} | G_v \rangle , \]
one can easily show that \( F_D \) is an even function of \( w \). Therefore, since \( F_D(0, v, L, t) = 0 \), and assuming an analytical behavior around \( \mu = 0 \) (at finite \( L \) and \( t \)), we can write
\[ F_D(w, v, L, t) = \frac{w^2}{2} C_D(v, L, t) + O(w^4) . \]
Therefore the growth rate of the QD in the limit of small qubit-S coupling \( w \) is described by the QDGR function
\[ C_D(v, L, t) = \partial^2 F_D / \partial w^2 \bigg|_{w=0} , \]
for a given value \( v \) of \( H_S \). It measures the sensitivity of the system to the qubit-S coupling.

The above setting can be straightforwardly extended to \( n \)-level systems coupled to an environmental many-body system. The scaling arguments we will report in the paper can be extended as well.

As concrete examples of environmental systems \( S \), we consider the paradigmatic \( d \)-dimensional quantum Ising models defined on a \( L^d \) lattice
\[ H_I = -J \sum_{\langle x, y \rangle} \sigma_x^{(3)} \sigma_y^{(3)} - g \sum_x \sigma_x^{(1)} , \]
where \( \sigma^{(k)} \) are the Pauli matrices, the first sum is over all bonds connecting nearest-neighbor sites \( \langle x, y \rangle \), while the other sums are over the sites. We assume \( h = 1 \), \( J = 1 \) and \( g > 0 \). At \( g = g_c \) (for one-dimensional quantum Ising systems \( g_c = 1 \)), the model undergoes a CQT belonging to the \((d+1)\)-dimensional Ising universality class [15–20], separating a disordered phase \( (g > g_c) \) from an ordered \((g < g_c) \) one. For any \( g < g_c \), the presence of a longitudinal external field \( v \) coupled to
\[ P_t = - \sum_x \sigma_x^{(3)} \]
drives FOQTs along the \( v = 0 \) line.

Then we consider a two-level qubit system, described by the Pauli operator \( \Sigma^{(3)} \) globally coupled to the Ising system by the Hamiltonian term

\[
H_q = w \Sigma^{(3)} P_t .
\]  

We are interested in the coherence properties of the qubit when the system \( S \) is a \( d \)-dimensional Ising model with Hamiltonian

\[
H_S(v) = H_I + v P_t ,
\]

cf. Eqs. (16) and (17), and the qubit coupling is described by \( H_q \) given in Eq. (18).

In the following sections we show that, at both CQTs and FOQTs of the environmental Ising systems \( S \), the interplay among the coupling with the qubit, the Hamiltonian parameters, the size \( L \), gives rise to dynamic scaling behaviors of the QD function \( F_D(w, v, L, t) \), and correspondingly of QDGR function \( C_D(v, L, t) \). For this purpose we consider DFSS frameworks, which allows us to characterize the QD dynamics at both CQTs and FOQTs. We derive the general features of the DFSS of \( F_D \) and \( C_D \), evidencing the differences between CQTs and FOQTs.

### III. THE DECOHERENCE DYNAMICS WITH A CRITICAL ENVIRONMENTAL SYSTEM

The theory of finite-size scaling (FSS) at quantum transitions is well established, see, e.g., Refs. 21–24 and references therein. The CQT of the Ising model (16) is characterized by two relevant parameters, \( r = g - g_c \) and \( v \) (such that they vanish at the critical point), with RG dimension \( y_r \) and \( y_v \), respectively. The relevant FSS variables are

\[
\kappa_r = L^{y_r} r, \quad \kappa_v = L^{y_v} v.
\]

The FSS limit is obtained by taking \( L \to \infty \) keeping \( \kappa_r \) and \( \kappa_v \) fixed.

The equilibrium critical exponents \( y_r \) and \( y_v \) are those of the \((d + 1)\)-dimensional Ising universality class [18–20]. Therefore, for one-dimensional systems they are \( y_r = 1/\nu = 1 \) and \( y_h = (d + 3 - \eta)/2 = (4 - \eta)/2 \) with \( \eta = 1/4 \). For two-dimensional models the critical exponents are not known exactly, but there are very accurate estimates, see, e.g., Refs. 25–29; in particular 28, \( y_r = 1/\nu \) with \( \nu = 0.629971(4) \) and \( y_h = (5 - \eta)/2 \) with \( \eta = 0.036298(2) \). For three-dimensional systems they assume mean-field values, \( y_r = 2 \) and \( y_h = 3 \), apart from logarithms. The temperature \( T \) gives rise to a relevant perturbation at CQTs, associated with the scaling variable \( \tau = L^z T \) where \( z = 1 \) (for any spatial dimension) is the dynamic exponent characterizing the behavior of the energy differences of the lowest-energy states and, in particular, the gap \( \Delta \sim L^{-z} \). In the following we assume \( T = 0 \).

A generic observable \( O \) in the FSS limit behaves as

\[
O(r, v, L) \approx L^{y_o} \mathcal{O}(\kappa_r, \kappa_v) ,
\]

where the exponent \( y_o \) is the renormalization-group (RG) dimension associated with \( O \), and \( \mathcal{O}(x, y) \) is a universal FSS function. The approach to such an asymptotic behavior is characterized by power-law corrections, typically controlled by irrelevant perturbations at the corresponding fixed point [21]. The equilibrium FSS at quantum transitions has been also extended to quantum-information concepts [2, 30–32], such as the fidelity and its susceptibility, which measure the change of the ground state when varying the Hamiltonian parameters around a QT [33].

Out-of-equilibrium time-dependent processes require also an appropriate rescaling of the time \( t \), encoded by the scaling variable

\[
\theta = L^{-z} t \sim \Delta(L)t .
\]

For example, we may consider the dynamic behavior of an isolated system after a quench associated with a sudden change of the parameter \( v \), from \( v \) to \( v + w \) at \( t = 0 \) (keeping \( g \) fixed), starting from the ground state \( |G_v\rangle \). The resulting quantum evolution is described by the state

\[
|\phi(t)\rangle = e^{-iH_S(v+w)t}|G_v\rangle .
\]

This problem can be studied within a DFSS framework [34]. The DFSS limit is defined as the infinite-volume \( L \to \infty \) limit, keeping the scaling variables \( \theta, \kappa_r, \kappa_v \), and \( \kappa_w = L^{y_w} w \) fixed. Then a generic observable \( O \) in the DFSS limit is expected to behave as

\[
O(r, v, w, L, t) \approx L^{y_o} \mathcal{O}(\kappa_r, \kappa_v, \kappa_w, \theta) ,
\]

where again \( y_o \) is the RG dimension of \( O \), and \( \mathcal{O}(x, y, z) \) is a DFSS function. The equilibrium (ground-state) FSS behavior is recovered in the limit \( w \to 0 \).

An analogous DFSS is also shown by the Loschmidt echo that quantifies the deviation of the post-quench state at time \( t > 0 \) from the initial \( t = 0 \) state. For example, at \( r = 0 \), the Loschmidt echo, defined as

\[
L_e(w, v, L, t) = |\langle G_v|e^{-iH_S(v+w)t}|G_v\rangle| ,
\]

shows the asymptotic DFSS [34]

\[
L_e(w, v, L, t) \approx L^{\kappa_{w}}(\kappa_w, \kappa_v, \theta) .
\]

The above DFSS behaviors have been confirmed by numerical calculations within the one-dimensional Ising model around its CQT at \( g_c = 1 \) [34].

In order to derive the DFSS behavior of the QD function \( F_D \), cf. Eq. (11), we recall that \( F_D \) is related to the Loschmidt echo \( L_D \) of quantum states of \( S \), along
the quantum evolutions arising from two different quench protocols involving the isolated system $S$. Indeed $\mathcal{L}_D$ measures the overlap of the evolving states at the same $t$, arising from the quantum evolutions with Hamiltonians $H_S(v \pm w)$ when starting from the same state $|G_v\rangle$. Therefore, we expect that $F_D$ develops a DFSS analogous to that of the Loschmidt echo (25), i.e.

$$ F_D(w, v, L, t) \approx F_D(\kappa_w, \kappa_v, \theta), $$

with

$$ F_D(\kappa_v = 0, \kappa_v, \theta) = 0, \quad F_D(\kappa_w, \kappa_v, \theta = 0) = 0. \quad (29) $$

Note that the DFSS Eq. (28) holds when also the coupling $w$ between the qubit and $S$ is sufficiently small, indeed the DFSS limit requires that $\kappa_w = L^{\nu_w} w$ must be kept constant in the large-$L$ limit. We do not expect particular scaling behaviors without such a rescaling, i.e. for generic finite values of $w$.

Moreover, Eq. (28) implies that the QDGR function $C_D$, cf. Eqs. (14) and (15), behaves as

$$ C_D(v, L, t) \approx L^{2 \nu_w} C_D(\kappa_w, \kappa_v, \theta). \quad (30) $$

This scaling equation characterizes the amplified $O(L^{\nu_w})$ rate of departure from coherence of the qubit when the environment system $S$ is at a CQT. Indeed, in the case of systems out of criticality one generally expects $C_D \sim L^d$, and

$$ 2 \nu_w = d + 3 - \eta > d. \quad (31) $$

The DFSS at CQTs is expected to be universal, i.e., independent of the microscopic features of the system $S$. Its main features only depend on the universality class of the CQT of $S$ and the general properties of the coupling between the qubit and the system $S$. In the case at hand the qubit is coupled to the order parameter of the magnetic transition. Note that the DFSS functions generally depend on the boundary conditions and the geometry of the system, while the power laws of the observables and the scaling variables remain unchanged.

We may also consider the case in which the qubit is homogeneously coupled to the transverse spin operators, i.e., we replace $P_t$, cf. Eq. (17), with $P_t = -\sum_\sigma \sigma^{(1)}_x$, and the qubit-$S$ coupling (18) with $H_{q,t} = u \Sigma^{(1)} P_t$. For simplicity we assume that $S$ is initially prepared in the ground state for $v = 0$ and a given $r = g - g_c$. Using scaling arguments analogous to those leading to Eq. (28), we arrive at the following DFSS

$$ F_D(u, r, L, t) \approx F_D(\kappa_v, \kappa_v, \theta), \quad (32) $$

with $\kappa_v$ defined in Eq. (20), and $\kappa_u = L^{\nu_u} u$. This also implies

$$ C_D(r, L, t) \approx L^{2 \nu_u} C_D(\kappa_v, \theta). \quad (33) $$

Note again that $2 \nu_r > d$. The decoherence dynamics of this central spin model, with the qubit homogeneously coupled to the transverse spin variables of a one-dimensional Ising model, was also considered in Ref. (10); the scaling behavior of its numerical results appears consistent with the DFSS prediction (22).

IV. THE QUBIT DECOHERENCE WITH ENVIRONMENTAL SYSTEMS AT FOQTS

In this section we extend the DFSS of the QD dynamics to the case of the environmental system $S$ at a FOQT, i.e. along the line $g < g_c$, of the phase diagram of the $d$-dimensional Ising models. We again consider the quantum evolution of the global system starting from pure states of both the qubit and the environmental Ising system $S$.

As shown by earlier works (22, 35–37), the FSS behaviors of isolated many-body systems at FOQTs significantly depend on the type of boundary conditions, in particular whether they favor one of the phases or they are neutral, giving rise to FSS characterized by exponential or power-law behaviors. In the following we consider Ising systems with boundary conditions which do not favor any of the two magnetized phases, such as periodic and open boundary conditions (PBC and OBC respectively).

The FOQT line for $g < g_c$ are related to the level crossing of the two lowest states $|\uparrow\rangle$ and $|\downarrow\rangle$ for $v = 0$, such that $|\uparrow\rangle |\sigma_x^{(3)}| |\uparrow\rangle = m_0$ and $|\downarrow\rangle |\sigma_x^{(3)}| |\downarrow\rangle = -m_0$ (independently of $\mathbf{x}$) with $m_0 > 0$. The degeneracy of these states at $v = 0$ is lifted by the longitudinal field $v$. Therefore, $v = 0$ is a FOQT point, where the longitudinal magnetization $M = L^{-d} \sum_\mathbf{x} M_\mathbf{x}$, with $M_\mathbf{x} \equiv \langle \sigma_\mathbf{x}^{(3)} \rangle$, becomes discontinuous in the infinite-volume limit. The FOQT separates two different phases characterized by opposite values of the magnetization $m_0$, i.e.

$$ \lim_{v \to 0^+} \lim_{L \to \infty} M = \pm m_0. \quad (34) $$

For one-dimensional systems, $m_0 = (1 - g^2)^{1/8}$.

In a finite system of size $L$, the two lowest states are superpositions of two magnetized states $|\pm\rangle$ and $|\mp\rangle$ such that $\langle \pm |\sigma_\mathbf{x}^{(3)}| \pm \rangle = \pm m_0$ for all sites $\mathbf{x}$. Due to tunneling effects, the energy gap $\Delta$ vanishes exponentially as $L$ increases, (22, 39)

$$ \Delta(L) \sim e^{-cL^d}, \quad (35) $$

apart from powers of $L$. In particular, the energy gap $\Delta(L)$ of one-dimensional Ising systems for $g < 1$ is exponentially suppressed as $\Delta_0(L) \approx 2 (1 - g^2) g^L [1 + O(g^{2L})]$ for OBC, (36)

$$ \Delta_0(L) \approx 2 (\pi L)^{1/2} (1 - g^2) g^L \quad \text{for PBC, (37)} $$

while the differences $\Delta_i = E_i - E_0$ for the higher excited states $(i > 1)$ are finite for $L \to \infty$.

The emergence of a DFSS after a quench protocol is also expected along the FOQT line for $g < g_c$, (34), associated with a sudden change of the parameter $v$, from $v$ to $v + w$ at $t = 0$, starting from the ground state $|G_w\rangle$. Extending to generic dimensions the arguments of Refs. (34, 41), we identify the following scaling variables

$$ \kappa = \frac{2 m_0 \nu L^d}{\Delta(L)}, \quad \kappa_w = \frac{2 m_0 w L^d}{\Delta(L)}, \quad \theta = t \Delta(L). \quad (38) $$
In particular, the scaling variables $\kappa_v$ and $\kappa_w$ are the ratios between the energy associated with the corresponding longitudinal-field perturbations, which are approximately given by $2m_0vL^d$ and $2m_0wL^d$ respectively, and the energy difference $\Delta(L)$ of the two lowest states at $v = 0$. Then, the expected DFSS of the magnetization is

$$M(w, v, L, t) = m_q \mathcal{M}(\kappa_w, \kappa_v, \theta).$$

This DFSS is expected to hold for any $g < g_c$. The scaling function $\mathcal{M}$ is independent of $g$, apart from trivial normalizations of the arguments. This DFSS at FOQTs has been numerically confirmed in the case of the one-dimensional Ising model [34]. The approach to the asymptotic DFSS is expected to be exponential in the size of the system. An analogous DFSS applies to the Loschmidt echo defined as in Eq. (26), we expect $L_c(w, v, L, t) \approx \mathcal{L}_c(\kappa_w, \kappa_v, \theta)$, which is formally identical to Eq. (27).

Then, using the same arguments of the previous section, i.e., noting that the QD function $F_D$ can be written in terms of quench-like amplitudes related to the environmental system only, we conjecture an analogous DFSS for the QD function

$$F_D(w, v, L, t) \approx \mathcal{F}_D(\kappa_w, \kappa_v, \theta).$$

Correspondingly, matching the expansion of the $F_D$ in powers of $w$ and that of $\mathcal{F}_D$ in powers of $\kappa_w$, we obtain the QDGR function

$$C_D(v, L, t) \approx \frac{4m_0^2L^{2d}}{\Delta(L)^2} \mathcal{C}_D(\kappa_v, \theta).$$

Therefore, when the environment system $S$ is at a FOQT the QDGR gets significantly enhanced, increasing exponentially with $L$. Indeed the prefactor of Eq. (41) behaves as

$$\frac{4m_0^2L^{2d}}{\Delta(L)^2} \sim \exp(bL^d),$$

apart from powers of $L$.

In the case of the quantum Ising systems with PBC or OBC, the DFSS functions can be exactly computed, exploiting a two-level truncation of the spectrum [22, 41]. As shown in Ref. [41], in the long-time limit and for large systems, the scaling properties in a small interval around $v = 0$, more precisely for $m_0|v| \ll \Delta_0 = O(1)$, are captured by a two-level truncation, which only takes into account the two nearly-degenerate lowest-energy states. The effective evolution is determined by the Schrödinger equation

$$i\frac{\partial}{\partial t} \Psi(t) = H_2(v) \Psi(t),$$

where $\Psi(t)$ is a two-component wave function, whose components correspond to the states $|+\rangle$ and $|-\rangle$, and

$$H_2(v) = -\beta \sigma^{(3)} + \delta \sigma^{(1)},$$

$$\beta = m_0vL^d, \quad \delta = \frac{\Delta(L)}{2}, \quad \kappa_v = \frac{\beta}{\delta},$$

where $\sigma^{(k)}$ are the Pauli matrices. The initial condition is given by the ground state of $H_2(v)$, i.e., by

$$\Psi(w, v, L, t = 0) = \sin(\alpha_w/2) |+\rangle + \cos(\alpha_w/2) |-\rangle,$$

with $\tan \alpha_v = \kappa_v^{-1}$ and $\alpha_v \in (0, \pi)$. The quantum evolution after quenching from $v$ to $v+w$ can be easily obtained by diagonalizing $H_2(v + w)$, whose eigenstates are

$$|0\rangle = \sin(\alpha_v + w/2) |+\rangle + \cos(\alpha_v + w/2) |-\rangle,$$

$$|1\rangle = \cos(\alpha_v + w/2) |+\rangle - \sin(\alpha_v + w/2) |-\rangle,$$

where $\tan \alpha_v + w = (\kappa_v + \kappa_w)^{-1}$ with $\alpha_v + w \in (0, \pi)$. Their eigenvalue difference is given by

$$E_1 - E_0 = \Delta(L) \sqrt{1 + (\kappa_v + \kappa_w)^2}.$$ 

Then, apart from an irrelevant phase, the time-dependent state evolves as

$$\Psi(w, v, L, t) = \cos \left(\frac{\alpha_v - \alpha_v + w}{2}\right) |0\rangle + e^{-i\theta} \sqrt{1 + (\kappa_v + \kappa_w)^2} \sin \left(\frac{\alpha_v - \alpha_v + w}{2}\right) |1\rangle.$$

Note that the time-dependent wave function is written in terms of scaling variables only. The DFSS of the magnetization can be easily obtained [34] by computing the expectation value of the operator $\sigma^{(3)}$ over the state $|\Psi(w, v, L, t)\rangle$.

The QD function $F_D$ can be straightforwardly obtained by computing

$$F_D(w, v, L, t) = 1 - |\langle \Psi(-w, v, L, t) | \Psi(w, v, L, t) \rangle|^2.$$ 

Using Eq. (49), one can immediately see that $F_D(w, v, L, t)$ is a function of $\kappa_w$, $\kappa_v$ and $\theta$ only, confirming the DFSS Eq. (10). The resulting expression is quite cumbersome, some plots are shown in Fig. 1. The curves are also characterized by revivals, typical of two-level systems.

The DFSS of QDGR function $C_D$ is obtained by computing

$$C_D(v, L, t) = \frac{\partial^2 F_D}{\partial w^2} |_{w=0} = \left(\frac{\partial \kappa_w}{\partial w}\right)^2 C_D(\kappa_w, \theta),$$

which leads to the analytical result

$$C_D(\kappa_v, \theta) = \frac{2(1 - \cos(\theta \sqrt{1 + \kappa_v^2}))}{(1 + \kappa_v^2)^2}.$$ 

Note the simple result for $\kappa_v = 0$,

$$C_D(0, \theta) = 2(1 - \cos \theta),$$

and that $C_D(\kappa_v, \theta)$ vanishes for $\kappa_v \to \infty$. We stress that the above DFSS functions are expected to be independent of $g < g_c$ along the FOQT line, apart from trivial $g$-dependent normalizations of the scaling variables.
FIG. 1: Some plots of the scaling function $F_D(\kappa, \psi)$ associated with the QD at FOQTs, cf. Eq. (10). The top figure shows plots versus $\theta$ for $\kappa_w = 1$ and some values of $\kappa_v$; the middle figure shows plots versus $\kappa_w$ for $\theta = 1$ and some values of $\kappa_v$; the bottom figure shows plots versus $\kappa_w$ for $\kappa_v = 0$ and some values of $\theta$.

We finally mention that a notable feature of one-dimensional quantum Ising systems at FOQTs with neutral boundary conditions is their rigidity with respect to external perturbations [22, 41], i.e., their response to global or local longitudinal perturbations is analogous. Therefore, an analogous QD dynamics at the FOQT line is expected in the case of a local coupling between the longitudinal parameter $v$, the qubit and the Ising chain, for example when replacing $P_\ell$, cf. Eqs. (17) and (18), with

$$p_\ell = -\sigma^{(3)}_{x_\ell}, \quad h_q = w \Sigma^{(3)} p_\ell$$

respectively, where $x_\ell$ is one of the sites of the chain (sufficiently far from the boundaries). The only difference is that the relevant scaling variables turn into $\kappa_w = 2m_yv/\Delta(L)$ and $\kappa_w = 2m_yw/\Delta(L)$, instead of those reported in Eq. (28). They give rise to a two-level scenario as well, with the same DFSS functions.

V. CONCLUSIONS

We have investigated the decoherence dynamics of a two-level qubit system globally and homogeneously coupled to a many-body spin system $S$, such as a $d$-dimensional quantum Ising system, at a QT. In particular, we have considered the out-of-equilibrium quantum evolution of the global system starting from pure states of both the qubit and $S$. The decoherence dynamics of the qubit is described by the time evolution of its density matrix, obtained tracing out the states of $S$. Its behavior can be characterized by the QD function $F_D$ defined in Eq. (11), which quantifies the departure of the qubit from a pure state, independently of its initial pure state. The sensitivity to the qubit-$S$ coupling $w$ is measured by the QDGR function $C_D = \partial^2 F_D/\partial w^2|_{w=0}$, cf. Eq. (14).

We have shown that the rate of the QD gets enhanced when the environmental system $S$ experiences a QT. At both CQTs and FOQTs of $S$, the interplay among the coupling between the qubit and $S$, the Hamiltonian parameters and the size of $S$, during the quantum evolution gives rise to scaling behaviors of the QD function $F_D(w, v, L, t)$, cf. Eq. (12), and the corresponding QDGR function $C_D(v, L, t)$, cf. Eq. (15), in the limit of large size $L$ of $S$. This is shown within DFSS frameworks, which allow us to determine the behaviors of the QD functions at both CQTs and FOQTs of the environmental system $S$, in appropriate DFSS limits.

We derive the general properties of the DFSS of the QD functions $F_D$ and $C_D$, evidencing the differences between CQTs and FOQTs. We show that they are characterized by power laws of the size $L$ at CQTs, while exponential laws generally emerge at FOQTs. These behaviors represent a substantial enhancement of the rate of the QD dynamics. For example, at CQTs, when the qubit couples longitudinally to the Ising model, the rate function turns out to increase as $C_D \sim L^\zeta$ where $\zeta = 2y_h = 15/8$ for $d = 1$, $\zeta = 2y_h \approx 4.96$ for $d = 2$, and $\zeta = 2y_h = 6$ for $d = 3$ (apart from logarithms). Therefore they show a significant enhancement of the QDGR, when compared with the general volume $L^d$ law expected for systems in normal conditions. The QDGR enhancement appears even more substantial at FOQTs, where $C_D \sim \exp(bL^d)$ increases exponentially.

Note that the main features of the DFSS, such as the general size dependence and the scaling functions, are ex-
expected to be universal, i.e., they are expected not to depend on the microscopic details of the models. Therefore, their predictions can be extended to all CQTs belonging to the same Ising universality class with analogous coupling between qubit and system. An analogous statement holds for the DFSS with environmental systems at FOQTs. In particular the DFSS with environmental Ising systems is expected to be the same, apart from normalizations, along the FOQT line for $g < g_c$, and in any system sharing the same global properties, such as FOQTs arising from an avoided two-level crossing phenomenon in the large-$L$ limit.

Finally we would like to stress that the DFSS framework, exploited to study the decoherence dynamics of qubit coupled longitudinally and transversally to Ising systems at QTs, can be straightforwardly extended to general CQTs and FOQTs, and generic couplings of the qubit to its environmental system.

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