The theory of $G$-structures provides us with a unified framework for a large class of geometric structures, including symplectic, complex and Riemannian structures, as well as foliations and many others. Surprisingly, contact geometry – the “odd-dimensional counterpart” of symplectic geometry – does not fit naturally into this picture. In this paper, we introduce the notion of a homogeneous $G$-structure, which encompasses contact structures, as well as some other interesting examples that appear in the literature.

The theory of $G$-structures places a variety of geometric structures on equal footing, the idea being to encode a structure on a manifold $M$ by its set of compatible frames, which (in many interesting examples) forms a reduction of the frame bundle of the manifold to a structure group $G \subset \text{GL}_n(\mathbb{R})$ (with $n = \dim M$). The group $G$ plays a key role in the theory, namely that of the linear model for the geometric structure. For example, a symplectic manifold induces a reduction of its frame bundle to the symplectic group, complex manifolds are modeled by the complex general linear group, Riemannian manifolds by the orthogonal group, volume forms by the special linear group, and so forth (see [5, 9, 2] for introductions to the theory of $G$-structures).

The pattern that repeats itself in each example is as follows: every structure, say one modeled by the group $G$, has a corresponding almost structure where the integrability axiom is removed. The instances of the almost structure that a given manifold admits are in one-to-one correspondence with reductions of the frame bundle of the manifold to $G$, and, of those, the instances of the structure correspond to so-called integrable reductions, which means that the manifold admits an atlas of coordinate charts that are compatible with the reduction (see Section 4 for more details). For example, almost symplectic structures (i.e. non-degenerate 2-forms) on a given manifold are in one-to-one correspondence with reductions of the frame bundle of the manifold to the symplectic group, and the symplectic structures (i.e. closed non-degenerate 2-forms) correspond to the integrable reductions. In this case, integrability is equivalent to the existence of an atlas consisting of Darboux charts.

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Contact structures, albeit being so similar to symplectic structures (most notably, due to the contact version of Darboux’s theorem [3]), do not fit into this picture. While the frame bundle of a contact manifold can be reduced to the group $U(n) \times 1$ (see [12, Ch. 1, Prop. 1.3]), the reduction is not canonical, and, more problematically, the integrability axiom of the structure does not translate to the condition of the reduction being integrable as a $G$-structure. In this paper, we provide a solution to this anomaly by introducing the notion of a *homogeneous $G$-structure*. Let us illustrate the general idea by explaining what happens in the special case of contact structures.

**Contact Structures as Homogeneous Symplectic Structures.** A contact structure on a manifold $M$ is a corank-one distribution $H \subset TM$ which is maximally non-integrable (i.e. the curvature 2-form of $H$, rather than vanishing as in the integrable case, is non-degenerate). Let us write $L := TM/H$ for the line bundle associated with $H$ and $\tilde{L} := L^* \setminus \{0\}$ for the complement of the zero section of the dual. The latter has the structure of a principal bundle when equipped with the obvious projection map $p : \tilde{L} \to M$ and the action

$$h : \mathbb{R}^\times \times \tilde{L} \to \tilde{L}, \quad (r, e) \mapsto h_\ast(e) = re$$

of the multiplicative group $\mathbb{R}^\times := (\mathbb{R} \setminus \{0\}, \cdot)$. A contact structure $H$ on $M$ induces a symplectic structure on $\tilde{L}$ via a construction known as the “symplectization trick”. The symplectic form, which we denote by $\omega_H \in \Omega^2(\tilde{L})$, is obtained by pulling back the quotient map $TM \to TM/H = L$, viewed as an $L$-valued 1-form on $M$, to a usual 1-form on $\tilde{L}$, and then applying the de Rham differential. Apart from being closed and non-degenerate, this 2-form also satisfies the homogeneity property

$$h_\ast^\ast r \omega_H = r \omega_H, \quad \forall r \in \mathbb{R}^\times.$$

Accordingly, we say that $\omega_H$ is homogeneous of degree 1, since $r$ appears to the first power on the right hand side.

Conversely, given a line bundle $L$ over $M$, any homogeneous of degree 1 symplectic form $\omega \in \Omega^2(\tilde{L})$ induces a contact structure on $M$ by contraction with the infinitesimal generator of the action $h$ (known as the Euler vector field). Indeed, by the homogeneity property, the resulting 1-form descends to an $L$-valued 1-form on $M$ whose kernel is a contact structure. Two such pairs $(L, \omega)$ and $(L', \omega')$ may induce the same contact structure on $M$, but, when they do, they are related by an equivalence, namely a vector bundle isomorphism $L \cong L'$ covering the identity map under which $\omega$ corresponds to $\omega'$ (auto equivalences are sometimes called conformal transformations).

The above constructions are inverse to one another, and, for a fixed manifold $M$, they define a one-to-one correspondence between contact structures $H$ on $M$, on the one hand, and pairs $(L, \omega)$ consisting of a line bundle $L$ over $M$ and a homogeneous of degree 1 symplectic structure $\omega$ on $\tilde{L}$ modulo equivalence, on the other.

**Homogeneous $G$-Structures.** The “symplectization trick” hints at the idea of encoding a contact structure $H$ on $M$, with an associated line bundle $L = TM/H$, as a reduction of the frame bundle of $\tilde{L} = L^* \setminus \{0\}$ to the symplectic group. However, to obtain a one-to-one correspondence as in the above examples of $G$-structures, we must be able to identify those reductions that are “homogeneous of degree 1”, i.e. that come from a homogeneous of degree 1 symplectic structure. The approach we propose in this paper is to encode the homogeneity property of the symplectic form as the invariance of the corresponding reduction under a twisted action of $\mathbb{R}^\times$ on the frame bundle of $\tilde{L}$. The key, of course, is in the choice of the twisting.

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1For more details, we recommend the mathoverflow discussion [https://mathoverflow.net/questions/281256/do-contact-and-cr-structures-have-corresponding-g-structures](https://mathoverflow.net/questions/281256/do-contact-and-cr-structures-have-corresponding-g-structures).
As we will see, the twisting is characterized by a map we call the \( \text{degree} \), a Lie group homomorphism of the form

\[ \alpha : \mathbb{R}^X \to N(G)/G, \]

with \( G \) the structure group, in this case the symplectic group, and \( N(G) \) its normalizer inside the general linear group. In short – the symplectic structure being homogeneous of degree 1 translates into the reduction being \( \alpha \)-homogeneous, for an appropriate choice of \( \alpha \).

An advantage of this approach is that it can be generalized to other structure groups \( G \) and other degree maps \( \alpha \). Given any line bundle \( L \) over a manifold \( M \), any Lie subgroup \( G \subset \text{GL}_{n+1}(\mathbb{R}) \) (with \( n = \dim M \)) and any map \( \alpha \) as above, we will define the notion of an \( \alpha \)-homogeneous \( G \)-structure on \( L \) (Definition 3.4). We will show that in addition to contact structures, the “odd-dimensional counterparts” of symplectic structures, our framework also encompasses the “odd-dimensional counterparts” of complex structures, a “contact analogue” of Riemannian metrics, and an example coming from Poisson geometry, or, more specifically, from structures known as \( b \)-symplectic manifolds (or also as log-symplectic manifolds).

**Outline of the Paper.** The paper is organized as follows: in Section 2 we collect some facts about line bundles. In Section 3, we introduce the notion of a homogeneous \( G \)-structure, and in Section 4, the notion of homogeneous integrability. In Section 5, we prove that contact structures are in one-to-one correspondence with homogeneous \( \text{Sp} \)-structures of an appropriate degree \( \alpha \) that are homogeneous integrable, where \( \text{Sp} \) denotes the symplectic group, and we conclude in Section 6 by proving analogous theorems for the three other examples mentioned above.

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2. **Line Bundles**

Let \( M \) be a manifold and let \( L \) be a line bundle over \( M \). We use the standard notation that \( \mathcal{C}^\infty(M) \) denotes the ring of functions on \( M \), \( \mathfrak{X}(M) \) its \( \mathcal{C}^\infty(M) \)-module of vector fields, and \( \Gamma(L) \) the \( \mathcal{C}^\infty(M) \)-module of sections of \( L \), all in the smooth category. When working with contact structures and other examples of homogeneous \( G \)-structures, we will need to pass from “usual” geometry on \( M \) to geometry on the line bundle \( L \). The picture to keep in mind is the following:

\[ \text{Objects on } M \quad \text{----} \quad \text{Atiyah objects on } L \quad \begin{array}{c} 1 \mapsto \rightarrow \end{array} \quad \text{Homogeneous objects on } \tilde{L} = L^* \setminus \{0\}, \]

where “objects” refers to the basic building blocks – functions, vector fields, differential forms, etc. Let us explain this in slightly more detail (and we refer the reader to Section 2 of [11] for further details).

The role that functions on \( M \) have in usual geometry is played by sections of \( L \) (“Atiyah functions”). These, in turn, are in one-to-one correspondence with homogeneous of degree 1 functions on \( \tilde{L} \) (i.e. functions \( f \in \mathcal{C}^\infty(\tilde{L}) \) satisfying \( h^*_r f = rf \) for all \( r \in \mathbb{R}^X \)) via the correspondence \( \lambda \in \Gamma(L) \mapsto \tilde{\lambda} \in \mathcal{C}^\infty(\tilde{L}), \) where \( \tilde{\lambda}(e) := e(\lambda(p(e))) \).

Vector fields on \( M \) are replaced by derivations of \( \Gamma(L) \) (“Atiyah vector fields”), i.e. linear maps

\[ \Delta : \Gamma(L) \to \Gamma(L) \]

for which there exists a (necessarily unique) vector field \( X_\Delta \in \mathfrak{X}(M) \) (the symbol of \( \Delta \)) such that

\[ \Delta(f \lambda) = f \Delta(\lambda) + X_\Delta(f) \lambda, \quad \forall \lambda \in \Gamma(L), \ f \in \mathcal{C}^\infty(M). \]
These are in one-to-one correspondence with homogeneous of degree 0 vector fields on \( \tilde{L} \) (i.e. vector fields \( X \in \mathfrak{x}(\tilde{L}) \) satisfying \( (h_r)_*X = X \) for all \( r \in \mathbb{R}^\times \)) via the correspondence \( \Delta \mapsto \tilde{\Delta} \in \mathfrak{x}(\tilde{L}), \) where \( \tilde{\Delta}(\lambda) := \tilde{\Delta}(\lambda) \), for all \( \lambda \in \Gamma(L) \). Under this correspondence, the identity operator \( 1 \in \Gamma(DL) \) corresponds to the infinitesimal generator \( \mathfrak{e} \) of the action \( h \), namely the restriction to \( L \) of the Euler vector field on \( L^\ast \).

A useful point of view to take is that derivations of \( L \) can be realized as the sections of a Lie algebroid \( DL \) over \( M \), called the Atiyah algebroid of \( L \) (see Example 3.3.4 in [6] or Section 2 of [10]). This is the Lie algebroid whose fiber at \( x \in M \) consists of all pointwise derivations at \( x \), i.e. linear maps \( \Delta_x : \Gamma(L) \to L_x \) for which there exists a (necessarily unique) vector \( X_{\Delta_x} \in T_xM \) such that \( \Delta_x(f \lambda) = f(x)\Delta_x(\lambda) + X_{\Delta_x}(f)\lambda_x \) for all \( \lambda \in \Gamma(L) \) and \( f \in C^\infty(M) \). Its bracket is the commutator bracket (of derivations) and its anchor is the symbol map \( DL \to TM, \Delta_x \mapsto X_{\Delta_x} \).

Going back to the picture above, one should think that the tangent bundle of \( M \) is replaced by the Atiyah algebroid of \( DL \). This allows us to complete the picture by replacing differential forms \( \Omega^\bullet(M) \) on \( M \) by differential forms on the Atiyah algebroid \( DL \) with values in \( L \) ("Atiyah forms"):

\[
\Omega^\bullet(DL; L) := \Gamma(\Lambda^\bullet(DL^\ast) \otimes L).
\]

These, in turn, are in one-to-one correspondence with homogeneous of degree 1 differential forms on \( \tilde{L} \) (i.e. differential forms \( \Omega \in \Omega^\bullet(\tilde{L}) \) satisfying \( (h_r)_\ast \Omega = r \Omega \) for all \( r \in \mathbb{R}^\times \)) via the correspondence \( \omega \in \Omega^k(DL; L) \mapsto \tilde{\omega} \in \Omega^k(\tilde{L}), \) where \( \tilde{\omega}(\tilde{\Delta}_1, \ldots, \tilde{\Delta}_k) = \omega(\Delta_1, \ldots, \Delta_k) \), for all \( \Delta_1, \ldots, \Delta_k \in \Gamma(DL) \). Moreover, the de Rham differential \( d : \Omega^\bullet(M) \to \Omega^{\bullet+1}(M) \) is replaced by the Lie algebroid differential \( d_D : \Omega^\bullet(DL; L) \to \Omega^{\bullet+1}(DL; L) \) with values in the tautological representation \( \Gamma(DL) \times \Gamma(L) \to \Gamma(L), (\Delta, \lambda) \mapsto \Delta(\lambda) \), under the one-to-one correspondence, is mapped to the usual de Rham differential \( d : \Omega^\bullet(L) \to \Omega^{\bullet+1}(L) \).

Remark 2.1. In addition to homogeneous of degree 1 functions on \( \tilde{L} \), we could also consider homogeneous functions of different degrees. In fact, for any Lie group homomorphism \( \phi : \mathbb{R}^\times \to \mathbb{R}^\times \), we can consider \( \phi \)-homogeneous functions on \( \tilde{L} \), i.e. functions \( f \in C^\infty(\tilde{L}) \) such that \( h_r^\ast f = \phi(r)f \) for all \( r \in \mathbb{R}^\times \). Writing \( p : \phi(L) \to M \) for the associated line bundle constructed out of the principal bundle \( L \) and the Lie group homomorphism \( \phi \) (seen as a representation of the structure group \( \mathbb{R}^\times \) on \( \mathbb{R} \)), sections of \( \phi(L) \) are in one-to-one correspondence with \( \phi \)-homogeneous functions via the correspondence \( \lambda \in \Gamma(\phi(L)) \mapsto \tilde{\lambda} \in C^\infty(\tilde{L}), \) with \( \tilde{\lambda}(\epsilon) := s \), where \( s \in \mathbb{R} \) the unique real number such that \( \lambda(p(\epsilon)) = [\{\epsilon, s\}] \).

We also note that when \( \phi : \mathbb{R}^\times \to \mathbb{R}^\times \) is a Lie group homomorphism with a non-trivial associated Lie algebra homomorphism (i.e. \( \phi \) is not locally constant), derivations of \( \phi(L) \) are again in one-to-one correspondence with homogeneous of degree 0 vector fields on \( \tilde{L} \) via the correspondence \( \Delta \mapsto \tilde{\Delta} \) given by the same formula as above, \( \tilde{\Delta}(\lambda) = \tilde{\Delta}(\lambda) \) for all \( \lambda \in \Gamma(\phi(L)) \). It follows that derivations of \( \phi(L) \) are also in one-to-one correspondence with derivations of \( L \), and hence this correspondence establishes a canonical Lie algebroid isomorphism \( D\phi(L) \cong DL \) (be aware that this works only when \( \phi \) is not locally constant).
3. Homogeneous G-structures

In this section, we introduce the notion of a homogeneous G-structure. Recall first that a G-structure on an n-dimensional manifold \( M \), with \( G \subset \text{GL}_n(\mathbb{R}) \) a Lie subgroup, is a reduction of the frame bundle of \( M \),

\[
\text{Fr}(M) = \{ \psi : \mathbb{R}^n \xrightarrow{\psi} T_x M \mid x \in M \},
\]
to the group \( G \). Spelled out, it is a submanifold \( S \subset \text{Fr}(M) \) that: 1) is invariant under the restriction of the right action of \( \text{GL}_n(\mathbb{R}) \),

\[
\text{Fr}(M) \times \text{GL}_n(\mathbb{R}) \to \text{Fr}(M), \quad (\psi, g) \mapsto \psi \circ g,
\]
to the subgroup \( G \), and 2) has the structure of a principal \( G \)-bundle over \( M \) when equipped with the restrictions of the action and the projection.

Now, let \( L \) be a line bundle over an \( n \)-dimensional manifold \( M \), and recall that \( \widetilde{L} := L^* \setminus \{0\} \) and that \( \rho : \widetilde{L} \to M \) denotes the projection. Given a section of the frame bundle of \( \widetilde{L} \),

\[
\text{Fr}(\widetilde{L}) \xrightarrow{\sigma} \widetilde{L},
\]
or, in short, a frame of \( \widetilde{L} \), there exists a (necessarily unique and smooth) map

\[
A_{\sigma} : \mathbb{R}^n \times \widetilde{L} \to \text{GL}_{n+1}(\mathbb{R}), \quad (r, \epsilon) \mapsto A_{\sigma}(r, \epsilon),
\]
that satisfies \( \sigma(\epsilon) = (h_\epsilon)_* \circ \sigma(\epsilon) \circ A_{\sigma}(r, \epsilon)^{-1} \), for all \( \epsilon \in \widetilde{L} \) and \( r \in \mathbb{R}^n \). Thus, \( A_{\sigma} \) measures how \( \sigma \) varies along the orbits of the action of \( \mathbb{R}^n \) on \( \widetilde{L} \).

**Definition 3.1.** A frame \( \sigma \) of \( \widetilde{L} \) is **homogeneous** if \( A_{\sigma}(r, \epsilon) \) is independent of \( \epsilon \), i.e. if \( A_{\sigma} \) descends to a map

\[
A_{\sigma} : \mathbb{R}^n \to \text{GL}_{n+1}(\mathbb{R}). \quad (3.1)
\]

**Lemma 3.2.** If a frame \( \sigma \) of \( \widetilde{L} \) is homogeneous, then \( A_{\sigma} \) is a Lie group homomorphism, and it induces a left action

\[
\mathbb{R}^n \times \text{Fr}(\widetilde{L}) \to \text{Fr}(\widetilde{L}), \quad (r, \psi) \mapsto r \cdot A_{\sigma} \psi := (h_r)_* \circ \psi \circ A_{\sigma}(r)^{-1}. \quad (3.2)
\]

**Proof.** Let \( r, s \in \mathbb{R}^n \) and \( \epsilon \in \widetilde{L} \). Since \( \sigma(\epsilon r) = (h_\epsilon)_* \circ \sigma(\epsilon) \circ A_{\sigma}(r)^{-1} \), then

\[
\sigma(\epsilon rs) = (h_{\epsilon s})_* \circ \sigma(\epsilon s) \circ A_{\sigma}(rs)^{-1},
\]

and thus \( A_{\sigma}(rs) = A_{\sigma}(r)A_{\sigma}(s) \). The second assertion is now straightforward. \( \square \)

**Remark 3.3.** Lie group homomorphisms of the type \( A : \mathbb{R}^n \to \text{GL}_{n+1}(\mathbb{R}) \) are in one-to-one correspondence with pairs \((B, C) \in \text{gl}_{n+1}(\mathbb{R}) \times \text{gl}_{n+1}(\mathbb{R})\) that satisfy

\[
C^2 = I \quad \text{and} \quad C \exp(Bt) = \exp(Bt)C, \quad \text{for all} \ t \in \mathbb{R}. \quad (3.3)
\]

In one direction, one sets \( B := \text{Lie}(A)(1) \in \text{gl}_{n+1}(\mathbb{R}) \), where \( \text{Lie}(A) : \mathbb{R} \to \text{gl}_{n+1}(\mathbb{R}) \) is the induced Lie algebra homomorphism, and \( C := A(-1) \in \text{GL}_{n+1}(\mathbb{R}) \). Conversely, we recover \( A \) by

\[
A(r) = \begin{cases} 
\exp(B \log(r)) & \text{if} \ r > 0, \\
C \exp(B \log(|r|)) & \text{if} \ r < 0.
\end{cases}
\]

While homogeneous frames (and, in general, frames) may fail to exist globally, they always exist locally on saturated open subsets of \( \widetilde{L} \), i.e. open subsets of the type

\[
\widetilde{L}_U := \rho^{-1}(U) \subset \widetilde{L}, \quad \text{with} \ U \subset M \text{ open},
\]
assuming that \( U \) is sufficiently small. Indeed, for any \( x \in M \), we may construct an open neighborhood \( U \subset M \) such that there exists a section \( \eta \) of \( \tilde{L}_U \rightarrow U \) and a local section \( \sigma_0 \) of \( \text{Fr}(\tilde{L}_U) \rightarrow \tilde{L}_U \) defined on a neighborhood of \( \eta(U) \). Then, given any Lie group homomorphism \( A : \mathbb{R}^\times \rightarrow GL_{n+1}(\mathbb{R}) \), we define a homogeneous frame \( \sigma \) of \( \tilde{L}_U \) with \( A_\sigma = A \) by imposing invariance under the action (3.2), i.e. by setting \( \sigma(e) := (h_{\psi(e)}), \circ \sigma_0(\psi(p(e)))\circ A(r(e))^{-1} \) for all \( e \in \tilde{L}_U \), where \( r(e) \in \mathbb{R}^\times \) is determined by \( e = r(e)\psi(p(e)) \). We will use the term semi-local homogeneous frame of \( \tilde{L} \) around \( e \) (or semi-local homogeneous section of \( \text{Fr}(\tilde{L}) \) around \( e \)) for a homogeneous frame defined on a saturated open neighborhood of \( e \in \tilde{L} \).

**Definition 3.4.** Let \( G \subset GL_{n+1}(\mathbb{R}) \) be a Lie subgroup. A homogeneous \( G \)-structure on \( \tilde{L} \) (with \( \dim \tilde{L} = n + 1 \)) is a \( G \)-structure \( S \subset \text{Fr}(\tilde{L}) \) such that, for any \( e \in \tilde{L} \),

1. there exists a semi-local homogeneous section \( \sigma \) of \( S \) around \( e \), say with domain \( \tilde{L}_U \subset \tilde{L} \),
2. \( S|_{\tilde{L}_U} \) is preserved by the action (3.2) induced by \( A_\sigma \), i.e. \( r \cdot A_\sigma S|_{\tilde{L}_U} = S|_{\tilde{L}_U} \) for all \( r \in \mathbb{R}^\times \).

Note that homogeneous \( G \)-structures can be restricted to saturated open subsets, namely if \( S \) is a homogeneous \( G \)-structure on \( \tilde{L} \), then \( S|_{\tilde{L}_U} \) is a homogeneous \( G \)-structure on \( \tilde{L}_U \).

The second condition in the above Definition has the following useful characterization:

**Lemma 3.5.** Let \( S \) be a \( G \)-structure on \( \tilde{L} \) and let \( \sigma \) be a homogeneous section of \( S \) (i.e. a homogeneous frame of \( \tilde{L} \) with values in \( S \)). Then \( r \cdot A_\sigma S = S \) for all \( r \in \mathbb{R}^\times \) if and only if \( A_\sigma \) takes values in the normalizer \( N(G) \) of \( G \).

**Proof.** Assume that \( r \cdot A_\sigma S = S \) for all \( r \in \mathbb{R}^\times \). For all \( e \in \tilde{L} \), \( g \in G \) and \( r \in \mathbb{R}^\times \), the left hand sides of

\[
\begin{align*}
\sigma(re) \circ g' &= (h_r), \circ \sigma(e) \circ gA_\sigma(r)^{-1} \\
\sigma(re) \circ g' &= (h_r), \circ \sigma(e) \circ A_\sigma(r)^{-1}g',
\end{align*}
\]

are equal for some \( g' \in G \), and the equality on the right hand sides then implies that \( A_\sigma(r)gA_\sigma(r)^{-1} = g' \), and hence that \( A_\sigma \) takes values in \( N(G) \). Conversely, assume that \( A_\sigma \) takes values in \( N(G) \). Any \( \psi \in S \) can be written as \( \sigma(e) \cdot g \) for some \( e \in \tilde{L} \) and \( g \in G \). Given \( r \in \mathbb{R}^\times \), the two right hand sides above are equal for some \( g' \in G \) due to the assumption, and so the equality on the left hand sides shows that \( r \cdot A_\sigma \psi \in S \). \( \square \)

Of course, the above definition of a homogeneous \( G \)-structure should not depend on the choices of semi-local homogeneous sections:

**Proposition 3.6.** Let \( S \) be a homogeneous \( G \)-structure on \( \tilde{L} \). If \( \sigma \) is a homogeneous section of \( S \) such that \( A_\sigma \) takes values in \( N(G) \), then, given any other homogeneous section \( \sigma' \) of \( S \), \( A_{\sigma'} \) takes values in \( N(G) \). Furthermore,

\[
A_\sigma \equiv A_{\sigma'} \quad \text{mod} \ G,
\]

in the sense that the compositions of \( A_\sigma \) and \( A_{\sigma'} \) with the projection \( N(G) \rightarrow N(G)/G \) are equal.

**Proof.** For all \( e \in \tilde{L} \),

\[
\sigma'(e) = \sigma(e) \circ g(e)
\]

for some smooth function \( g : \tilde{L} \rightarrow G \). Since \( \sigma \) and \( \sigma' \) are both homogeneous, then for all \( r \in \mathbb{R}^\times \),

\[
\begin{align*}
\sigma'(re) &= (h_r), \circ \sigma'(e) \circ A_{\sigma'}(r)^{-1}, \\
\sigma(re) \circ g(re) &= (h_r), \circ \sigma(e) \circ A_\sigma(r)^{-1}g(re) = (h_r), \circ \sigma'(e) \circ A_{\sigma'}(r)^{-1}g(re).
\end{align*}
\]

Since the two left hand sides are equal, it follows that

\[
A_{\sigma'}(r) = g(re)^{-1}A_\sigma(r)g(e),
\]

and hence \( A_{\sigma'}(r) \) takes values in \( N(G) \) (since \( G \subset N(G) \), and \( N(G) \) is a group), and \( A_\sigma(r) \) and \( A_{\sigma'}(r) \) belong to the same coset of \( G \) in \( N(G) \) (since \( G \) is normal in \( N(G) \), and left and right cosets coincide). \( \square \)
A consequence of this proposition is that, given a homogeneous $G$-structure $S$ over $\tilde{L}$, there is a canonical Lie group homomorphism
\[
\alpha : \mathbb{R}^n \to N(G)/G, \quad r \mapsto A_\alpha(r)G,
\]
associated with every connected component $M_0$ of the base manifold $M$. Here, $\sigma$ is any choice of a homogeneous section of $S|_{\tilde{L}_U}$, with $U$ a sufficiently small, non-empty open subset in $M_0$. When this map is the same for all connected components, we call $\alpha$ the degree of $S$, and we say that $S$ is an $\alpha$-homogeneous $G$-structure. A lift of $\alpha$ is any Lie group homomorphism $A : \mathbb{R}^n \to N(G)$ such that $\alpha$ is equal to the composition of $A$ with the projection $N(G) \to N(G)/G$.

**Proposition 3.7.** Let $S$ be an $\alpha$-homogeneous $G$-structure on $\tilde{L}$. Given $\epsilon \in \tilde{L}$ and a lift $A : \mathbb{R}^n \to N(G)$ of $\alpha$ (if one exists), there exists a semi-local homogeneous section $\sigma$ of $S$ around $\epsilon$ such that $A_\sigma = A$.

**Proof.** Start with any homogeneous section $\sigma_0$ of $S|_{\tilde{L}_U}$, with $\tilde{L}_U$ a sufficiently small saturated open neighborhood of $\epsilon$ such that there exists a section $\eta$ of the projection $\tilde{L}_U \to U$. Then set
\[
\sigma(\epsilon) := \sigma_0(\epsilon) \circ A_{\sigma_0}(r(\epsilon))A(r(\epsilon))^{-1},
\]
where $r(\epsilon) \in \mathbb{R}^n$ is determined by $\epsilon = r(\epsilon) \cdot \eta(p(\epsilon))$. \qed

**Remark 3.8.** Proposition 3.7 gives an obstruction for the existence of a homogeneous $G$-structure with a prescribed degree $\alpha : \mathbb{R}^n \to N(G)/G$, namely that $\alpha$ must admit a lift to a Lie group homomorphism $A : \mathbb{R}^n \to N(G)$. This is not the case, for example, when $M$ is a point (and so $n = 0$ and $\text{GL}_{n+1}(\mathbb{R}) = \mathbb{R}^n$), $G$ is the (closed) subgroup generated by 2 (in which case $N(G) = \mathbb{R}^n$), and $\alpha : \mathbb{R}^n \to \mathbb{R}^n/G$ is given by
\[
\alpha(r) = \begin{cases} 
 rG & \text{if } r > 0 \\
 \sqrt{2}|r|G & \text{if } r < 0.
\end{cases}
\]
Here, $\alpha(-1) = \sqrt{2}G$, and, since there is no order two element in the lateral $\sqrt{2}G$, $\alpha$ cannot be lifted to a homomorphism $\mathbb{R}^n \to \mathbb{R}^n$. This counter-example can be easily generalized to higher dimensions. When $\alpha$ does admit a lift $A$, while there is still no guarantee for the global existence of an $\alpha$-homogeneous $G$-structure on a given $\tilde{L}$, it is, however, sufficient for local existence, since we can construct a homogeneous frame $\sigma$ with $A_\sigma = A$ on a saturated open subset $\tilde{L}_U$ (as explained above), and extend it to the unique homogeneous $G$-structure on $\tilde{L}_U$ that contains the image of $\sigma$.

4. Homogeneous Integrability

We now move on to discuss integrability in the context of homogeneous $G$-structures. Recall that a $G$-structure $S \subset \text{Fr}(M)$ on a manifold $M$ is said to be integrable if around every point in $M$ there exists a coordinate chart $(U, \chi)$ such that the induced frame
\[
\sigma_\chi := \left( \frac{\partial}{\partial \chi^1}, \ldots, \frac{\partial}{\partial \chi^n} \right),
\]
viewed as a local section of $\text{Fr}(M)$, takes values in $S$. In the case of homogeneous $G$-structures, motivated by the examples that will be presented in the following two sections, we are interested in homogeneous coordinate charts. As always, $L \to M$ is a line bundle and $\tilde{L} = L^* \setminus \{0\}$.

**Definition 4.1.** A coordinate chart $(V, \chi)$ of $\tilde{L}$ is homogeneous if, locally around every point of $V$, the induced frame $\sigma_\chi$ is the restriction of a semi-local homogeneous frame of $\tilde{L}$. A homogeneous $G$-structure $S$ on $\tilde{L}$ is homogeneous integrable if around every point of $\tilde{L}$ there exists a homogeneous coordinate chart such that $\sigma_\chi$ takes values in $S$. 


Remark 4.2. In all of the examples that we have of homogeneous $G$-structures, homogeneous integrability is equivalent to integrability (see Theorems 5.2, 6.1, 6.3 and 6.6). However, the proof in each example is rather different and particular to that case, and we do not know if this fact is true in general.

The condition for a coordinate chart to be homogeneous can also be rephrased in the following more intrinsic way:

Proposition 4.3. A coordinate chart $(V, \chi)$ of $\mathcal{L}$ is homogeneous if for every point $\epsilon_0 \in V$, there exist

1. an open neighborhood $V_0$ of $\epsilon_0$ in $V$,
2. an open neighborhood $I_0$ of 1 in $\mathbb{R}^\times$,
3. a Lie group homomorphism $(A, b): \mathbb{R}^\times \to \text{Aff}_{n+1}(\mathbb{R}) = \text{GL}_{n+1}(\mathbb{R}) \ltimes \mathbb{R}^{n+1}$, such that $h_\epsilon(V_0) \subset V$, and

$$h_\epsilon^* \chi = A(r) \chi + b(r) \quad \text{on } V_0, \tag{4.1}$$

for all $r \in I_0$. Furthermore, if $\sigma$ is a semi-local homogeneous frame of $\mathcal{L}$ whose restriction to $V$ is $\sigma_X$, then $A_\sigma = A$ on the connected component of the identity $\mathbb{R}_+^\times \subset \mathbb{R}^\times$.

Proof. Begin with a homogeneous coordinate chart $(V, \chi)$ on $\mathcal{L}$, let $\epsilon_0 \in V$, and let $V_0$ be a connected open neighborhood of $\epsilon_0$ in $V$ such that $\sigma_\chi$ agrees with a local homogeneous frame, say $\sigma$, in $V_0$. Shrinking $V_0$ if necessary, we can assume that $h_\epsilon(V_0) \subset V$ for all $r$ in a sufficiently small interval $I_0 \subset \mathbb{R}$ containing 1. It is easy to see that $A_\sigma(r)$ maps the $i$-th element in the canonical frame of $\mathbb{R}^{n+1}$ to

$$\left( \frac{\partial h_\epsilon^*(\chi^i)}{\partial \chi^j}(\epsilon), \ldots, \frac{\partial h_\epsilon^*(\chi^{n+1})}{\partial \chi^j}(\epsilon) \right)$$

for all $\epsilon \in V_0$, and $r \in I_0$. As $A_\sigma(r)$ is independent of $\epsilon$, we have

$$\frac{\partial^2 h_\epsilon^*(\chi^k)}{\partial \chi^i \partial \chi^j} = 0,$$

for all $i, j, k = 1, \ldots, n + 1$, hence

$$h_\epsilon^* \chi = A_\sigma(r) \chi + b(r),$$

for some $b(r) \in \mathbb{R}^{n+1}$. From the group property of $h_r$, $(A_\sigma, b)$ is a local Lie group homomorphism. As such, it can be uniquely extended to a Lie group homomorphism from the connected component of the identity $\mathbb{R}_+^\times$ of $\mathbb{R}^\times$ to $\text{Aff}_{n+1}(\mathbb{R})$. Finally, extend $(A_\sigma, b)$ arbitrarily to a Lie group homomorphism

$$(A, b): \mathbb{R}^\times \to \text{Aff}_{n+1}(\mathbb{R}).$$

Conversely, let $(V, \chi)$ be a chart on $\mathcal{L}$ as in the statement. Put $U_0 = \rho(V_0)$, and $x_0 = \rho(\epsilon_0)$. Shrinking $V_0$ if necessary, we can assume that $\rho : V_0 \to U_0$ is a trivial fiber bundle. Let $\eta$ be a section of $V_0 \to U_0$ such that $\eta(x_0) = \epsilon_0$. As already remarked before, we can construct a semi-local homogeneous frame $\sigma$ on $\mathcal{L}_{U_0}$ by setting $\sigma(\epsilon) := (h_\epsilon(x_0)) \circ \sigma_0(\eta(\rho(\epsilon))) \circ A(r(\epsilon))^{-1}$ for all $\epsilon \in \mathcal{L}_{U_0}$, where $r(\epsilon) \in \mathbb{R}$ is determined by $\epsilon = r(\epsilon) \eta(\rho(\epsilon))$. By construction, $A_\sigma = A$. Finally, it easily follows from (4.1), that $\sigma$ and $\sigma_X$ agree on $V_0$. \hfill $\Box$

Note that we do not require that the domain of a homogeneous coordinate chart be a saturated open subset as we do for semi-local homogeneous frames. The reason for this is that charts of this type cannot always be extended to saturated domains, as the following example shows:
Example 4.4. Let $M = \mathbb{R}^n$ with the standard coordinates $(x^1, \ldots, x^n)$, and let $L$ be the trivial line bundle $\mathbb{R}_M := M \times \mathbb{R} \to M$, so that $\tilde{L} = \mathbb{R}^n \times \mathbb{R}_\xi$. Let $\mu$ be the standard coordinate on $\mathbb{R}_\xi$, and consider the coordinate chart 

$$(V = \mathbb{R}^n \times \mathbb{R}_\xi, \chi = (x^1, \ldots, x^n, \log \mu))$$

on $\tilde{L}$. The induced coordinate frame is

$$\sigma_\chi = \left( \frac{\partial}{\partial x^1}, \ldots, \frac{\partial}{\partial x^n}, \frac{\partial}{\partial \mu} \right).$$

This frame extends (via the same formula) to a commuting homogeneous section of $\text{Fr}(\tilde{L})$, with $A_\sigma$ the trivial homomorphism, while $(V, \chi)$ cannot be extended to the whole of $\tilde{L}$ since $\chi$ is already surjective onto $\mathbb{R}^{n+1}$.

5. Contact Structures as Homogeneous G-Structures

Our main and motivating example of a homogeneous G-structure is a contact structure. As explained in the introduction, the linear model for a contact structure on an odd dimensional manifold $M$, with $n = \dim M$, is the symplectic group $\text{Sp}_k \subset \text{GL}_{2k}(\mathbb{R})$, with $k = (n+1)/2$, consisting of $2k \times 2k$ matrices $A$ satisfying $A^t J A = J$. Here,

$$J = \begin{pmatrix} 0 & I_k \\ -I_k & 0 \end{pmatrix},$$

with $I_k$ the $k \times k$ unit matrix.

Lemma 5.1 ([8, Theorem 1.10]). The normalizer $N(\text{Sp}_k)$ of the symplectic group $\text{Sp}_k \subset \text{GL}_{2k}(\mathbb{R})$ fits in the split short exact sequence of Lie group homomorphisms

$$1 \to \text{Sp}_k \to N(\text{Sp}_k) \xrightarrow{p} \mathbb{R}^\times \to 1,$$  

with $p : N(\text{Sp}_k) \to \mathbb{R}^\times$ defined by $p(B)I_{n+1} = J^t B J^t B$. Furthermore, a splitting $A$ is given by

$$A : \mathbb{R}^\times \to N(\text{Sp}_k), \quad r \mapsto \begin{pmatrix} 1 & 0 \\ 0 & r1 \end{pmatrix},$$

and therefore $N(\text{Sp}_k)$ decomposes as the semidirect product of $\text{Sp}_k$ and the 1-dimensional subgroup consisting of matrices of the form $A(r)$, with $r \in \mathbb{R}^\times$.

As a preparation for the theorem below, let us explain how a contact structure is constructed out of an $\alpha$-homogeneous $\text{Sp}_k$-structure $\Sigma$ on $\tilde{L}$, where the relevant degree in this case is simply the identity map $\alpha : \mathbb{R}^\times \to N(\text{Sp}_k)/\text{Sp}_k \cong \mathbb{R}^\times$, where the last isomorphism is the one induced by (5.2). By Proposition 3.7, around any point in $\tilde{L}$, there is a saturated open neighborhood $\tilde{L}_U$ and a homogeneous section $\sigma$ of $S|_{\tilde{L}_U}$ such that $A_\sigma = A$, where $A$ is the lift of $\alpha$ given by (5.3). Due to the specific form of $A$, the components $X_1, \ldots, X_k, Y^1, \ldots, Y^k \in \tilde{\omega}(\tilde{L}_U)$ of $\sigma$ satisfy the homogeneity conditions

$$(h_r)_*(X_i) = X_i \quad \text{and} \quad (h_r)_*(Y^i) = r Y^i \quad \forall \; r \in \mathbb{R}^\times, \; i = 1, \ldots, k.$$  

Denoting the components of the dual coframe by $\xi^1, \ldots, \xi^k, \eta_1, \ldots, \eta_k \in \Omega^1(\tilde{L}_U)$, we define a symplectic form $\tilde{\omega} \in \Omega^2(\tilde{L})$ on $\tilde{L}$ by setting

$$\tilde{\omega}|_{\tilde{L}_U} = \xi^1 \wedge \eta_1 + \cdots + \xi^k \wedge \eta_k$$

on the saturated open neighborhood $\tilde{L}_U$. Due to (5.4), $\tilde{\omega}$ satisfies the homogeneity condition

$$(h_r)_* \tilde{\omega} = r \tilde{\omega}, \quad \forall \; r \in \mathbb{R}^\times,$$
which, as explained in Section 2, implies that \( \tilde{\omega} \) uniquely determines a non-degenerate Atiyah 2-form \( \omega : \wedge^2 DL \to L \). Setting
\[
\Theta := i_\omega, \quad \Upsilon := i_d d_\omega,
\]
where \( \mathbb{I} \in \Gamma(DL) \) is the identity operator, we have that \( i \Theta = i \Upsilon = 0 \), which implies that \( \Theta \) and \( \Upsilon \) descend to an \( L \)-valued 1-form \( \vartheta \in \Omega^1(M; L) \) and an \( L \)-valued 2-form \( \omega \in \Omega^2(M; L) \) uniquely defined by
\[
\vartheta(X_\Delta) = \Theta(\Delta), \quad \omega(X_\Delta, X_{\Delta'}) = \Upsilon(\Delta, \Delta'), \quad \forall \Delta, \Delta' \in \Gamma(DL).
\]
(5.5)

Recall from Section 2 that \( X_\Delta \) denotes the symbol of the derivation \( \Delta \).

**Theorem 5.2.** Let \( L \to M \) be a line bundle, with \( n = \dim M \) odd, and set \( k = (n + 1)/2 \). The assignment \( S \mapsto (\vartheta, \omega) \) described above defines a one-to-one correspondence between
- (i) \( \alpha \)-homogeneous \( Sp_k \)-structures \( S \) on \( \tilde{L} \), with \( \alpha : \mathbb{R}^+ \to \text{N}(Sp_k)/Sp_k \cong \mathbb{R}^+ \) the identity map,
- (ii) pairs \( (\vartheta, \omega) \) consisting of an \( L \)-valued one-form \( \vartheta \in \Omega^1(M; L) \), and an \( L \)-valued 2-form \( \omega \in \Omega^2(M; L) \)
such that
  - \( \vartheta \) is nowhere zero,
  - \( \vartheta|_H - R_H \) is a non-degenerate 2-form on \( H := \ker \vartheta \), where \( R_H \) is the curvature of \( H \).

Furthermore, the following conditions are equivalent:
- (1) \( S \) is homogeneous integrable,
- (2) \( S \) is integrable,
- (3) \( H = \ker \vartheta \) is a contact structure and \( \omega = 0 \).

**Proof.** We first show that the pair \( (\vartheta, \omega) \) associated with an \( \alpha \)-homogeneous \( Sp_k \)-structure \( S \) satisfies the two properties in item (ii). The graded commutator \([d_D, i] = \text{ker} \vartheta \) acts like the identity on \( \Omega^*(DL, L) \) and it follows that
\[
\omega = d_D i_\omega + i_d d_\omega = d_\omega + \Upsilon.
\]
Consequently,
\[
\omega(\Delta, \Delta') = \Delta(\vartheta(X_\Delta')) - \Delta'(\vartheta(X_\Delta)) - \vartheta([X_\Delta, X_{\Delta'}]) + \omega(X_\Delta, X_{\Delta'}), \quad \forall \Delta, \Delta' \in \Gamma(DL).
\]
(5.6)

Since \( \omega \) is non-degenerate and \( \mathbb{I} \) is nowhere zero, then \( \Theta = i_\omega \), and hence \( \vartheta \), is nowhere zero. Let \( H := \ker \vartheta \subset TM \) be the induced hyperplane distribution. We want to show that \( \vartheta|_H - R_H \) is a non-degenerate, \( L \)-valued 2-form on \( H \). Recall that the curvature \( R_H \) is a distribution \( H \) is the \( L \)-valued 2-form on \( H \), defined by \( R_H(X, Y) = [X, Y] \mod \Gamma(H) \in \Gamma(L) \) for all \( X, Y \in \Gamma(H) \). Now, pick a connection \( \nabla \) on \( L \), and note that
\[
R_H = -d_\vartheta \vartheta|_H
\]
where \( d_\vartheta \) is the associated connection-differential. So, it is enough to show that the intersection
\[
H \cap \ker (\vartheta + d_\vartheta \vartheta)
\]
is trivial. The claim will then follow from the fact that, in this case, \( \vartheta + d_\vartheta \vartheta \) can only have rank 1 kernel transversal to \( H \), hence \( (\vartheta + d_\vartheta \vartheta)|_H = \vartheta|_H - R_H \) must be non-degenerate. So, let \( X \in \Gamma(H) \) be such that
\[
\vartheta(X, Y) + d_\vartheta \vartheta(X, Y) = 0
\]
for all \( Y \in \mathfrak{X}(M) \). This means that
\[
0 = \vartheta(X, Y) + \nabla_X(\vartheta(Y)) - \nabla_Y(\vartheta(X)) - \vartheta([X, Y]) = \omega(\nabla_X, \nabla_Y)
\]
for all \( Y \), where we used (5.6). But
\[
\omega(\nabla_X, \mathbb{I}) = -\Theta(\nabla_X) = -\vartheta(X)
\]
vanishes as well. Hence
\[ \omega(\nabla_X, \Delta) = 0 \]
for all \( \Delta \in \Gamma(DL) \). As \( \omega \) is non-degenerate, we conclude that \( X = 0 \).

Conversely, let \( (\partial, \nu) \) be as in (ii), define \( \Theta, \Upsilon \) via (5.5) (so that \( i_1 \Theta = i_1 \Upsilon = 0 \)) and finally put
\[ \omega = d_\partial \Theta + \Upsilon \]
(so that \( \Theta = i_1 \omega \), and \( \Upsilon = i_1 d_\partial \omega \)). We want to show that \( \omega \) is non-degenerate. To do this let \( \Delta \in \Gamma(DL) \) be such that
\[ 0 = \omega(\Delta, \Delta') = \Delta(\partial(X_{\Delta'})) - \Delta'(\partial(X_{\Delta})) - \partial([X_{\Delta}, X_{\Delta'}]) + \nu(X_{\Delta}, X_{\Delta'}) \]
for all \( \Delta' \in \Gamma(DL) \). In particular,
\[ 0 = \omega(\Delta, \partial) = -\partial(X_{\Delta}), \]
showing that \( X_{\Delta} \in \Gamma(H) \). More generally, let us assume that \( X_{\Delta'} \in \Gamma(H) \). Then we get
\[ 0 = \omega(\Delta, \Delta') = -\partial([X_{\Delta}, X_{\Delta'}]) + \nu(X_{\Delta}, X_{\Delta'}) = -R_H(X_{\Delta}, X_{\Delta'}) + \nu(X_{\Delta}, X_{\Delta'}). \]
As \( X_{\Delta'} \in \Gamma(H) \) is otherwise arbitrary and \( \nu|_H - R_H \) is non-degenerate, we conclude that \( X_{\Delta} = 0 \), so that \( \Delta = f\partial \) for some function \( f \in C^\infty(M) \), and
\[ 0 = i_{\Delta} \omega = f i_1 \omega = f \Theta. \]
But, from \( \partial \neq 0 \), it follows that \( \Theta \neq 0 \) everywhere, hence \( f = 0 \), i.e. \( \Delta = 0 \), showing that \( \omega \) is non-degenerate as claimed. This means that, locally, around every point of \( M \), we can choose

(1) a basis \( \lambda \) of \( \Gamma(L) \), and

(2) a symplectic frame with components
\[ \delta_1, \ldots, \delta_k, \eta^1, \ldots, \eta^k \]
for the fiber-wise symplectic structure
\[ \lambda^* \circ \omega : \lambda^* DL \rightarrow \mathbb{R}. \]

where \( \lambda^* \) is the dual basis of \( \lambda \) in \( \Gamma(L^*) \): \( \lambda^*(\lambda) = 1 \). It is easy to see that the vector fields
\[ \delta_1, \ldots, \delta_k, \lambda^{-1} \cdot \eta^1, \ldots, \lambda^{-1} \cdot \eta^k \]
are the components of a semi-local homogeneous frame \( \sigma \) of \( \tilde{L} \) with the following homogeneity property
\[ \sigma(re) = (h_r)_{*} \circ \sigma(e) \circ A(r)^{-1} \]
where \( A \) is given in (5.3). All such frames span an \( \alpha \)-homogeneous \( Sp_k \)-structure \( S \) on \( \tilde{L} \) with \( \alpha \) being the identity, and this construction inverts the assignment \( S \mapsto (\partial, \nu) \).

For the second part of the statement, that (1) implies (2) is obvious. Let us show that (2) implies (3). So, let \( S \) be integrable. Then the associated almost symplectic structure \( \tilde{\omega} \) is actually a symplectic structure. As the de Rham differential of \( \tilde{\omega} \) is equal to \( d_H \omega \), we conclude that \( \omega \) is \( d_H \)-closed. Then \( \Upsilon = i_1 d_H \omega = 0 \), hence \( \nu = 0 \) as well, and \( R_H \) is non-degenerate, so that \( H \) is a contact structure.

It remains to show that (3) implies (1). To do this, assume that \( S \) is such that \( H = \ker \partial \) is a contact structure, and \( \nu = 0 \). Choose Darboux coordinates \( (x^1, u, p_1) \) on \( M \), so that
\[ \partial = (du - p_1 dx^1) \otimes \lambda \]
where \( \lambda = \partial \left( \frac{\partial}{\partial u} \right) \neq 0 \) everywhere. It is then easy to see that
\[ \tilde{\omega} = d \left( \lambda (du - p_1 dx^1) \right) = -du \wedge d\lambda + dx^1 \wedge d(\lambda p_1). \]

This shows that \( \chi = (u, x^1, \lambda, \lambda p_1) \) are homogeneous coordinates such that \( \sigma_{\chi} \) takes values in \( S \). \( \square \)
Theorem 5.2 shows that, given an integrable (hence homogeneous integrable) $\alpha$-homogeneous $\text{Sp}_k$-structure on $\widetilde{L}$, with $\alpha$ the identity map, we get a contact structure $H$ on $M$ together with an isomorphism $L \cong TM/H$, and vice versa. Thus, contact structures indeed fit in the framework of integrable homogeneous $G$-structures, and this also suggests that (at least from the point of view of $G$-structures) the correct notion of an almost structure in contact geometry is a pair $(\partial, \nu)$ as in the statement of the theorem.

6. Other Examples

6.1. The symplectic group again. Let $n > 0$ be an odd integer, and set $k = (n + 1)/2$. Let us consider homogeneous $\text{Sp}_k$-structures $S$ whose degree $\alpha_S$ is trivial, i.e. $\alpha_S : \mathbb{R}^\chi \to N(\text{Sp}_k)/\text{Sp}_k \cong \mathbb{R}^\chi$, with $\alpha_S(r) = 1$ for all $r \in \mathbb{R}^\chi$. As we will see, these type of $G$-structures arise naturally in $b$-symplectic geometry.

Recall that a $b$-manifold is a pair $(N, M)$ consisting of a manifold $N$ and a closed hypersurface $M \subset N$ (see, e.g., [4]). The $b$-tangent bundle of $(N, M)$ is the vector bundle $T^b N$ over $N$ whose sections are vector fields on $N$ that are tangent to $M$. The $b$-tangent bundle has the structure of a Lie algebroid, where the Lie bracket is given by the commutator of vector fields (tangent to $M$) and the anchor map is the identity map at the level of sections. The (point-wise) restriction $T^b N|_M \to M$ is a subalgebroid that fits in the following short exact sequence of vector bundles over $M$:

$$0 \to K \to T^b N|_M \to TM \to 0. \tag{6.1}$$

The projection $T^b N|_M \to TM$ maps (the point-wise restriction to $M$ of) a section of $T^b N$ to its restriction to $M$ as a vector field, and it is well-defined by the definition of $T^b N$. The kernel $K$ admits a canonical nowhere-zero section $I$, which, in a coordinate chart $(t, z^a)$ of $N$ adapted to $M$ (i.e. for which $M$ is the zero set of $t$), is given by $I = \frac{t}{\partial t}$.

Now, let $\nu := TN|_M/TM \to M$ be the normal bundle to $M$, and let $L := \nu^* \to M$ be the conormal bundle. In particular, $L$ is a line bundle, and it is not hard to see that there is an isomorphism of Lie algebroids $T^b N|_M \cong DL$ which maps the point-wise restriction to $M$ of a section $X$ of $T^b N$ to the derivation $\Delta_X$ of $L$ defined as follows: any $\lambda \in \Gamma(L)$ is the point-wise restriction to $M$ of a 1-form on $N$ whose pull-back to $M$ vanishes, and

$$\Delta_X \lambda := (L_X \eta)|_M.$$ 

Under the isomorphism $T^b N|_M \cong DL$, $I$ becomes the identity derivation $\hat{I}$, and, hence, the short exact sequence (6.1) becomes

$$0 \to \mathbb{R}_M \to DL \to TM \to 0.$$

A $b$-symplectic structure on a $b$-manifold $(N, M)$ is a symplectic structure on the $b$-tangent bundle. $b$-symplectic structures are important in Poisson geometry since they provide particularly nice instances of Poisson manifolds, namely Poisson manifolds $(N, \pi)$ whose Poisson tensor $\pi$ is everywhere non-degenerate except for a hypersurface $M \subset N$, where $\pi$ satisfies an suitable transversality condition. Given a $b$-symplectic structure on $(N, M)$, the restriction of the symplectic form to $T^b N|_M$ can be seen as a symplectic structure on the Atiyah algebroid $DL$ under the isomorphism $T^b N|_M \cong DL$:

$$\omega : \wedge^2 DL \to \mathbb{R}_M.$$ 

Such symplectic structures, as Theorem 6.1 below shows, are examples of $\alpha$-homogeneous $\text{Sp}_k$-structures with $\alpha = 1$. Let us explain how the symplectic form is constructed from such a structure.

Let $L \to M$ be a line bundle with $\dim M = n$ odd, and set $k = (n + 1)/2$. Let $S$ be an $\alpha$-homogeneous $\text{Sp}_k$-structure on $\widetilde{L}$, with $\alpha = 1$. By Proposition 3.7, around any point in $\widetilde{L}$, there is a saturated open neighborhood $L_U$ and a homogeneous section $\sigma$ of $S|_{L_U}$ such that $A_{\sigma} = 1$. The components $X_1, \ldots, X_k, Y^1, \ldots, Y^k$ of $\sigma$ satisfy the homogeneity conditions

$$(h_r)_*(X_i) = X_i \quad \text{and} \quad (h_r)_*(Y^i) = Y^i \quad \forall \ r \in \mathbb{R}^\chi, \ i = 1, \ldots, k.$$
Denoting the components of the dual coframe by $\xi^1, \ldots, \xi^k, \eta_1, \ldots, \eta_k \in \Omega^1(\tilde{L}_U)$, we define an almost symplectic structure $\tilde{\omega}$ on $\tilde{L}$ by setting

$$\tilde{\omega}|_{\tilde{L}_U} = \xi^1 \wedge \eta_1 + \cdots + \xi^k \wedge \eta_k,$$

for any saturated open neighborhood $\tilde{L}_U$ as above. It follows that $\tilde{\omega}$ satisfies the homogeneity property

$$h_r^*(\tilde{\omega}) = \tilde{\omega}, \quad \forall r \in \mathbb{R}^\times.$$

Equivalently, $\tilde{\omega}$ maps homogeneous vector fields of degree 0 to homogeneous functions of degree 0, i.e. fiber-wise constant functions, and hence it defines a non-degenerate 2-form $\omega : \wedge^2 DL \to \mathbb{R}_M$.

**Theorem 6.1.** Let $L \to M$ be a line bundle, with $n = \dim M$ odd, and set $k = (n+1)/2$. The assignment $S \mapsto \omega$ described above defines a one-to-one correspondence between $\alpha$-homogeneous $Sp_k$-structures $S$ on $\tilde{L}$, with $\alpha = 1$, and non-degenerate 2-forms

$$\omega : \wedge^2 DL \to \mathbb{R}_M.$$

**Furthermore,** the following conditions are equivalent:

1. $S$ is homogeneous integrable,
2. $S$ is integrable,
3. $\omega$ is a cocycle in the de Rham complex of $DL$ (with trivial coefficients).

**Proof.** Begin with a non-degenerate 2-form $\omega : \wedge^2 DL \to \mathbb{R}_M$. Locally, around every point of $M$, we can choose a symplectic frame of $DL$, with components

$$\delta_1, \ldots, \delta_k, \xi^1, \ldots, \xi^k$$

and

$$\tilde{\delta}_1, \ldots, \tilde{\delta}_k, \tilde{\xi}^1, \ldots, \tilde{\xi}^k \in \mathfrak{x}(\tilde{L})$$

are the components of a semi-local homogeneous frame $\sigma$ on $\tilde{L}$ with the following homogeneity property: $\sigma(re) = (h_r)_* \circ \sigma(e)$. All such frames span an $\alpha$-homogeneous $Sp_k$-structure $S \subset Fr(\tilde{L})$ with $\alpha = 1$, and this construction inverts the correspondence $S \mapsto \omega$.

For the second part of the statement, that (1) implies (2) is obvious. Let us show that (2) implies (3). So, let $S$ be integrable. Then the associated almost symplectic structure $\tilde{\omega}$ is actually a symplectic structure. Similarly as in the previous section the de Rham differential of $\tilde{\omega}$ is equal to $\tilde{D} \omega$ (where $\tilde{D}$ is the de Rham differential of the Atiyah algebroid $DL$). We conclude that $d_D \omega = 0$. It remains to show that (3) implies (1). To do this, assume that $S$ is such that $d_D \omega = 0$. Then $d \tilde{\omega} = 0$, i.e. $\tilde{\omega}$ is a symplectic structure. Additionally, it follows from the homogeneity condition $h_r^*(\tilde{\omega}) = \tilde{\omega}$, that the Euler vector field $E \in \mathfrak{x}(\tilde{L})$ is an infinitesimal symplectomorphism. The Carathéodory Theorem then states that, around every point in $\tilde{L}$, there is a Darboux chart $(V, \chi)$ for $\tilde{\omega}$ such that $\frac{\partial}{\partial \chi^i} = E$. Integrating the commutation relations

$$\left[ E, \frac{\partial}{\partial \chi^i} \right] = \left[ \frac{\partial}{\partial \chi^i}, \frac{\partial}{\partial \chi^j} \right] = 0, \quad \forall i = 2, \ldots, n + 1,$$

we easily see that $(V, \chi)$ is a homogeneous chart (with $A : \mathbb{R}^\times \to GL_{n+1}(\mathbb{R})$ being the trivial homomorphism). \hfill \Box

### 6.2. The complex group

Let $n > 0$ be an odd integer, and set $k = (n+1)/2$. Let us now consider homogeneous $GL_k(\mathbb{C})$-structures, where $GL_k(\mathbb{C})$ is the group of invertible $k \times k$ complex matrices embedded as the subgroup of $GL_{n+1}(\mathbb{R})$ consisting of matrices $A$ satisfying $A^{-1}JA = J$, with $J$ given by (5.1).
Lemma 6.2. The normalizer $N(GL_k(\mathbb{C}))$ of the complex general linear group $GL_k(\mathbb{C})$ in $GL_{n+1}(\mathbb{R})$ fits in the split short exact sequence of Lie group homomorphisms

$$1 \longrightarrow GL_k(\mathbb{C}) \longrightarrow N(GL_k(\mathbb{C})) \overset{p}{\longrightarrow} Z_2 \longrightarrow 1,$$

with $p : N(GL_k(\mathbb{C})) \to Z_2$ defined by $(-)^p(B)I_{n+1} = J^{-1}B^{-1}JB$. Furthermore, a splitting $\Sigma$ is given by $\Sigma(\mathbb{I}) = I_{n+1}$ and

$$\Sigma(\mathbb{I}) = V := \left( \begin{array}{cc} O & I \\ I & O \end{array} \right),$$

and therefore $N(GL_k(\mathbb{C}))$ decomposes as the semidirect product of $GL_k(\mathbb{C})$ and the two element subgroup $\{I_{n+1}, V\}$.

Proof. We use a similar strategy as that of [8]. Begin noticing that the matrix $V$ in the statement is indeed in the normalizer. Now let $B \in GL_{n+1}(\mathbb{R})$. It is easy to see that $B$ is in $N(GL_k(\mathbb{C}))$ iff $J^{-1}B^{-1}JB$ is in the centralizer of $GL_k(\mathbb{C})$. In its turn, the centralizer consists of the scalar multiplications of vectors in $\mathbb{R}^{n+1} = \mathbb{C}^k$ by invertible complex numbers, i.e. matrices $C$ of the form

$$C = aI_{n+1} + bJ, \quad a + ib \in \mathbb{C} \setminus \{0\},$$

as one can easily show using that elements of the centralizer commute with matrices of the form

$$\left( \begin{array}{cc} U & O \\ O & U \end{array} \right) \text{ and } \left( \begin{array}{cc} O & U \\ -U & O \end{array} \right)$$

with $U \in GL_k(\mathbb{R})$. A direct computation then reveals that $J^{-1}B^{-1}JB$ is $\pm I_{n+1}$. As $J^{-1}V^{-1}JV = -I_{n+1}$ this concludes the proof. \qed

The surjective homomorphism $p$ in Lemma 6.2 induces an isomorphism of groups $N(GL_k(\mathbb{C}))/GL_k(\mathbb{C}) \cong Z_2$. There are, therefore, only two Lie group homomorphisms $\alpha : \mathbb{R}^X \to N(GL_k(\mathbb{C}))/GL_k(\mathbb{C})$, the trivial one and the sign. We restrict our attention to the trivial case. The other case is similar and is left to the reader.

Let $L \to M$ be a line bundle with dim $M = n$, and let $S$ be an $\alpha$-homogeneous $GL_k(\mathbb{C})$-structures on $\tilde{L}$ with $\alpha = 1$ (the trivial map). As in the previous example, around any point in $\tilde{L}$ there is a saturated open neighborhood $\tilde{L}_U$ and a homogeneous section of $S|\tilde{L}_U$ such that $A_\sigma = 1$. The components $X_1, \ldots, X_k, Y_1, \ldots, Y_k$ of $\sigma$ satisfy

$$(h_r)_\ast(X_i) = X_i \quad \text{and} \quad (h_r)_\ast(Y^i) = Y^i \quad \forall \ r \in \mathbb{R}^X, \ i = 1, \ldots, k.$$

Denote by $\xi^1, \ldots, \xi^k, \eta^1, \ldots, \eta^k$ the components of the dual coframe, and define a complex structure $\tilde{j}$ by setting

$$\tilde{j}_{\mid \tilde{L}_U} = \xi^1 \otimes Y_1 + \cdots + \xi^k \otimes Y_k - \eta^1 \otimes X_1 - \cdots - \eta^k \otimes X_k.$$

Clearly,

$$h_r^\ast(\tilde{j}) = \tilde{j}, \quad \forall \ r \in \mathbb{R}^X.$$

Equivalently, $\tilde{j}$ maps homogeneous vector fields of degree 0 to themselves, and hence it defines a fiber-wise complex structure $K : DL \to DL$.

Theorem 6.3. Let $L \to M$ be a line bundle, with $n = \dim M$ odd, and set $k = (n + 1)/2$. The assignment $S \mapsto K$ defines a one-to-one correspondence between

(i) $\alpha$-homogeneous $GL_k(\mathbb{C})$-structures on $\tilde{L}$, with $\alpha = 1$,

(ii) fiber-wise complex structures $K$ on $DL$.

Furthermore, the following conditions are equivalent:
(1) $S$ is homogeneous integrable,
(2) $K$ is integrable,
(3) $K$ is a complex structure, in the sense that the (Lie-algebroid) Nijenhuis torsion of $K$ vanishes identically.

Proof. Begin with a fiber-wise complex structure $K : DL \to DL$. Locally, around every point of $M$, we can choose a complex frame of $DL$, with components $\delta_1, \ldots, \delta_k, \zeta_1, \ldots, \zeta_k$, and $\overline{\delta}_1, \ldots, \overline{\delta}_k, \overline{\zeta}_1, \ldots, \overline{\zeta}_k$ are the components of a semi-local homogeneous frame $\sigma$ on $\overline{L}$ such that: $\sigma(\epsilon r) = (h_r) \circ \sigma(\epsilon)$. All such frames span an $\alpha$-homogeneous $GL_k(\mathbb{C})$-structure $S \subset Fr(\overline{L})$ with $\alpha = 1$, and this construction inverts the correspondence $S \mapsto K$.

For the second part of the statement, that (1) implies (2) is obvious. Let us show that (2) implies (3). So, let $S$ be integrable. Then the associated almost complex structure $\tilde{j}$ is actually a complex structure. As the Nijenhuis torsion of $\tilde{j}$ vanishes iff so does the Nijenhuis torsion of $K$ (see [11, Example 2.3.4]), we conclude that $K$ is a complex structure on the Atiyah algebroid $DL$. It remains to show that (3) implies (1), but this is essentially contained in the proof of [7, Theorem A.1.1].

□

Remark 6.4. A fiber-wise complex structure $K$ on $DL$ is essentially the same as an almost contact structure on $M$ (see [7, Appendix]), and $K$ is integrable iff the associated almost contact structure is normal (see [1]). Thus, (normal) almost contact structures fit well in our setting.

6.3. The orthogonal group. We move on to homogeneous $O_{n+1}$-structures, where $O_{n+1} \subset GL_{n+1}(\mathbb{R})$ is the orthogonal group.

Lemma 6.5 ([8, Theorems 1.10 and 2.9]). The normalizer $N(O_{n+1})$ of the orthogonal group $O_{n+1}$ in $GL_{n+1}(\mathbb{R})$ fits in the split short exact sequence of Lie group homomorphisms

\[
1 \longrightarrow O_{n+1} \longrightarrow N(O_{n+1}) \overset{p}{\longrightarrow} \mathbb{R}^*_+ \longrightarrow 1,
\]

with $\mathbb{R}^*_+$ the multiplicative group of positive reals, and $p : N(O_{n+1}) \to \mathbb{R}^*_+$ defined by $p(B) = B^{-1}B$. Furthermore, a splitting $\Sigma$ is given by by $\Sigma(r) = r^{1/2}I_{n+1}$, and therefore $N(O_{n+1})$ decomposes as the semidirect product of $O_{n+1}$ and the 1-dimensional subgroup consisting of positive scalar matrices.

The surjective homomorphism $p$ in Lemma 6.5 induces an isomorphism $N(O_{n+1})/O_{n+1} \cong \mathbb{R}^*_+$. In this last example, we will look at $\alpha$-homogeneous $O_{n+1}$-structures with $\alpha : \mathbb{R}^*_+ \to N(O_{n+1})/O_{n+1} \cong \mathbb{R}^*_+$ the square root of the absolute value, i.e. $\alpha(r) = |r|^{1/2}$. As before, the other cases are similar and are left to the reader.

Let $L \to M$ be a line bundle with dim $M = n$, and let $S$ be an $\alpha$-homogeneous $O_{n+1}$-structure with $\alpha$ as above. Around any point in $\overline{L}$ there is a saturated open neighborhood $\overline{L}_U$ and a homogeneous section $\sigma$ of $S|_{\overline{L}_U}$ such that $A_\sigma(r) = |r|^{1/2}I_{n+1}$ for all $r \in \mathbb{R}^*_+$. The components $X_1, \ldots, X_{n+1}$ of $\sigma$ satisfy

\[
(h_r)_\alpha(X_i) = |r|^{1/2}X_i, \quad \forall r \in \mathbb{R}^*_+, \quad i = 1, \ldots, k.
\]

Denote by $\xi^1, \ldots, \xi^{n+1}$ the components of the dual coframe and define a Riemannian metric $\tilde{g}$ by setting

\[
\tilde{g}|_{\overline{L}_U} = \xi^1 \odot \xi^1 + \cdots + \xi^{n+1} \odot \xi^{n+1}.
\]

We have

\[
h^*_\alpha(\tilde{g}) = |r|\tilde{g}, \quad \forall r \in \mathbb{R}^*_+.
\]

Equivalently, $\tilde{g}$ maps a pair of homogeneous vector fields of degree 0, say $X, Y$, to a function $\tilde{g}(X, Y)$ such that $h^*_\alpha(\tilde{g}(X, Y)) = |r|\tilde{g}(X, Y)$. Hence $\tilde{g}$ defines a definite, symmetric bilinear form (see Remark 2.1)

\[
G : DL \odot DL \to |L|.
\]
Since there is a canonical isomorphism of Lie algebroids $D[L] \cong DL$ (see again remark 2.1), $G$ is the same as a definite, symmetric $|L|$-valued bilinear form on $D[L]$, which we also denote by $G$. In particular, $G(\mathbb{I}, \mathbb{I})$ is a non-zero section of $|L|$ and it induces a trivialization $\phi : |L| \cong \mathbb{R}_M$. In the following, we will identify $|L|$ with the trivial line bundle $\mathbb{R}_M$ using $\phi$. The $G$-orthogonal bundle $\mathbb{I}^\perp \subset D[L]$ is the image of a unique linear connection $\nabla^{[L]} : TM \to D[L]$ in $|L|$. Since $|L|$ is a trivial line bundle, the connection $\nabla^{[L]}$ defines a connection 1-form $\eta$ on $M$ via $\eta(X) = (\phi \circ \nabla^{|L|}_X \circ \phi^{-1})(1)$, for all $X \in \mathfrak{X}(M)$. Finally, we can also use $\nabla^{[L]}$ to identify $TM$ and $\mathbb{I}^\perp$, which allows us to regard the restriction of $G$ to $\mathbb{I}^\perp$ as the Riemannian metric $g$ on $M$ defined by

$$g(X, Y) = (\phi \circ G)(\nabla^{|L|}_X, \nabla^{|L|}_Y).$$

**Theorem 6.6.** Let $L \to M$ be a line bundle, with $n = \dim M$. The assignment $S \mapsto (\phi, g, \eta)$ establishes a one-to-one correspondence between

(i) $\alpha$-homogeneous $O_{n+1}$-structures on $\tilde{L}$, with $\alpha_S$ being the square root of the absolute value,

(ii) triples $(\phi, g, \eta)$ consisting of a trivialization $\phi : |L| \cong \mathbb{R}_M$ of the line bundle $|L|$, a Riemannian metric $g$ on $M$, and a 1-form $\eta \in \Omega^1(M)$.

Furthermore, the following conditions are equivalent:

1. $S$ is homogeneous integrable,
2. $S$ is integrable,
3. $\eta = 0$, and $g$ has constant curvature equal to $1/4$.

**Proof.** Begin with a triple $(\phi, g, \eta)$ as in the statement. We can use $\phi$ to identify $|L|$ and $\mathbb{R}_M$, and $\eta$ with a linear connection $\nabla^{[L]} : TM \to D[L]$. As $\nabla^{[L]}$, is an isomorphism on its image $H = \nabla^{[L]}(TM)$, $g$ defines a fiber-wise scalar product on $H$, and can be uniquely extended to a fiber-wise scalar product $G$ on $D[L]$ such that $G(\mathbb{I}, \mathbb{I}) = 1$ and $H = \mathbb{I}^\perp$. Identifying $|L|$ with the trivial line bundle again, and $D[L]$ with $DL$, we can regard $G$ as a definite, symmetric bilinear form $G : DL \otimes DL \to |L|$. Locally, around every point of $M$, we can choose a basis $\lambda$ of $|L|$, an orthonormal frame with components $\delta_1, \ldots, \delta_{n+1}$ for the fiber-wise scalar product $\lambda^* \circ G : DL \otimes DL \to \mathbb{R}_M$, where $\lambda^*$ is the dual basis of $\lambda$ in $\Gamma(|L|^{-1})$. It is easy to see that the vector fields $[\hat{\lambda}]^{-1/2} \cdot \tilde{\delta}_1, \ldots, [\hat{\lambda}]^{-1/2} \cdot \tilde{\delta}_{n+1}$ are well-defined and that they are the components of a semi-local homogeneous frame $\sigma$ of $\tilde{L}$ such that $\sigma(\epsilon r) = |r|^{-1/2}(h_\epsilon) \circ \sigma(\epsilon)$. All such frames span an $\alpha$-homogeneous $O_{n+1}$-structure $S$ on $\tilde{L}$ with $\alpha$ being the square-root of the absolute value. This construction inverts the assignment $S \mapsto (\phi, g, \eta)$.

For the second part of the statement, that (1) implies (2) is obvious. Let us prove that (2) implies (3). So, take an integrable $\alpha$-homogeneous $O_{n+1}$-structure $S$ on $\tilde{L}$ with $\alpha$ being the square-root of the absolute value. As above, $S$ determines a Riemannian metric $\tilde{g}$ on $\tilde{L}$. From integrability, $\tilde{g}$ is a flat metric and we want to unveil how does this translate in terms of the data $(\phi, g, \eta)$. To do this, first of all denote by $u$ the non-zero section of $|L|$ determining the trivialization $\phi$. If we use the trivialization to identify $|L|$ with the trivial line bundle, then $u$ identifies with the constant function $1$. Now, recall that $\tilde{g}$ determines a definite, symmetric bilinear form $G : DL \otimes DL \to |L|$, and a (Lie algebroid) version of the Fundamental Theorem of Riemannian Geometry says that there exists a unique $D[L]$-connection $\nabla^D$ in $D[L]$ such that

1. $\nabla^D_{\Delta_1} \Delta_2 - \nabla^D_{\Delta_2} \Delta_1 = [\Delta_1, \Delta_2]$ (i.e. $\nabla^D$ is a symmetric connection), and
2. $\Delta_1(G(\Delta_2, \Delta_3)) = G(\nabla^D_{\Delta_1} \Delta_2, \Delta_3) + G(\Delta_2, \nabla^D_{\Delta_1} \Delta_3)$ (i.e. $\nabla^D$ is a metric connection),

for all $\Delta_1, \Delta_2, \Delta_3 \in \Gamma(D[L])$. The flatness of $\tilde{g}$ is equivalent to the vanishing of the curvature

$$R^D : \wedge^2 D[L] \to \text{End} D[L], \quad (\Lambda_1, \Lambda_2) \mapsto [\nabla^D_{\Lambda_1}, \nabla^D_{\Lambda_2}] - \nabla^D_{[\Lambda_1, \Lambda_2]}.$$ 

Obviously, $R^D$ can be expressed in terms of $u, g$ and $\eta$. The formulas, however, are rather complicated and we will not provide a full account of them. Our main concern is proving that $R^D = 0$ iff $\eta = 0$, and $g$ has
constant curvature equal to 1/4, as claimed. Before doing this, we need to express the connection $\nabla^D$ in terms of $u, g$ and $\eta$. We begin noticing that every derivation $\Delta \in \Gamma(D|L|)$ can be uniquely written in the form

$$\Delta = \nabla^{|L|}_\xi + f\|$$

for some function $f \in C^\infty(M)$, where $\Phi = X_\Delta$ is the symbol of $\Delta$, and $\nabla^{|L|}$ is the connection in $|L|$ with connection 1-form $\eta$, induced by the trivialization $\{L| \cong \mathbb{R}_M$, i.e.:

$$\nabla^{|L|}_\xi(gu) = X(g)u + g\eta(X)u, \quad \forall g \in C^\infty(M), \ X \in \mathfrak{x}(M).$$

In particular, there exist a vector field $Z \in \mathfrak{x}(M)$ and a smooth function $h \in C^\infty(M)$ such that

$$\nabla^{|L|}_Z = \nabla^{|L|}_Z + h\|$$. \hfill (6.5)

Additionally, for every $X, Y \in \mathfrak{x}(M)$, there exist vector fields $A(X), B(X), C(X, Y)$, and smooth functions $E(X), F(X), H(X, Y)$ such that

$$\nabla^{|L|}_X \nabla^{|L|}_X = \nabla^{|L|}_{\Phi(X)} + E(X)\|,$$

$$\nabla^{|L|}_X \nabla^{|L|}_Y = \nabla^{|L|}_{\Phi(X)} + F(X)\|,$$

$$\nabla^{|L|}_X \nabla^{|L|}_Y = \nabla^{|L|}_{\Phi(X, Y)} + H(X, Y)\|.$$ \hfill (6.6)

It is easy to see, using the axioms of a $D|L|$-connection, that $A, B$ are $(1, 1)$-tensors, $E, F$ are 1-forms, $H$ is a $(0, 2)$-tensor, and

$$C(X, Y) = \nabla_X Y.$$ \hfill (6.7)

for some standard (a priori non-necessarily torsion-free) connection $\nabla$ in $TM$. Using the properties of $\nabla^D$, it is possible to compute $A, B, E, F, H, \nabla$ in terms of $u, g, \eta$. To do this, first we use that $\nabla^D$ is a symmetric connection. So, from

$$\nabla^{|L|}_X \nabla^{|L|}_X - \nabla^{|L|}_X \nabla^{|L|}_Y = [\nabla^{|L|}_X, \ nabla^{|L|}_Y] = 0$$

it follows that

$$A = B \quad \text{and} \quad E = F.$$ \hfill (6.8)

Similarly, from

$$\nabla^{|L|}_X \nabla^{|L|}_Y - \nabla^{|L|}_Y \nabla^{|L|}_X = [\nabla^{|L|}_X, \ nabla^{|L|}_Y] = \nabla^{|L|}_{[X, Y]} + d\eta(X, Y)\|$$

it follows that $\nabla$ is actually a symmetric connection, and

$$H(X, Y) - H(Y, X) = d\eta(X, Y).$$ \hfill (6.9)

Next, we use that $\nabla^D$ is a metric connection. for a 1-form $\rho \in \Omega^1(M)$, denote by $\rho^\sharp \in \mathfrak{x}(M)$ the vector field uniquely determined by $g(\rho^\sharp, X) = \rho(X)$, for all $X \in \mathfrak{x}(M)$. Then, from

$$u = \mathbb{I}(\mathbb{I}, \mathbb{I}) = G(\nabla^D\mathbb{I}, \mathbb{I}) + G(\mathbb{I}, \nabla^D\mathbb{I}),$$

$$\eta(X)u = \nabla^{|L|}_X(\mathbb{I}, \mathbb{I}) = G(\nabla^D\mathbb{I}, \nabla^{|L|}_X\mathbb{I}) + G(\mathbb{I}, \nabla^D\mathbb{I}, \nabla^{|L|}_X\mathbb{I}),$$

$$0 = \mathbb{I}(\mathbb{I}, \nabla^{|L|}_X\mathbb{I}) = G(\nabla^D\mathbb{I}, \mathbb{I}) + G(\mathbb{I}, \nabla^D\mathbb{I}, \nabla^{|L|}_X\mathbb{I}),$$

it follows that

$$h = 1/2, \quad E = F = \eta/2, \quad Z = -\eta^\sharp/2,$$ \hfill (6.10)

respectively. Next, for a $(1, 1)$-tensor $K : TM \to TM$, we denote by $K^+ : TM \to TM$ its $g$-adjoint $(1, 1)$-tensor. Then, from

$$g(X, Y)u = \mathbb{I}(G(\nabla^{|L|}_X\mathbb{I}, \nabla^{|L|}_Y\mathbb{I})) = G(\nabla^{|L|}_X\nabla^{|L|}_Y\mathbb{I}, \mathbb{I}) + G(\nabla^{|L|}_X\mathbb{I}, \nabla^{|L|}_Y\mathbb{I})$$
it follows that
\[(A + A^\dagger)(X) = (B + B^\dagger)(X) = X, \quad (A - A^\dagger)(X) = (B - B^\dagger)(X) = (i_X d\eta)^\# \],
respectively. Finally, let $W \in \mathfrak{X}(M)$ be another vector field. Then from
\[\nabla_X^{|L|}(G(\nabla^{|L|}_Y, \nabla^{|L|}_W)) = G(\nabla^D_{\nabla^{|L|}_X} Y, \nabla^{|L|}_W) + G(\nabla^D_{\nabla^{|L|}_X} W, \nabla^{|L|}_W)\]
it follows, after some computations, that
\[\nabla_X Y - \nabla^g_X Y = \frac{1}{2} \left( g(X, Y)\eta^\# - \eta(X) Y - \eta(Y) X \right), \tag{6.12}\]
where $\nabla^g$ is the Levi-Civita connection of $g$.

Formulas (6.5)–(6.7), together with (6.8)–(6.12), can be used to compute $R^D$ in terms of $u, g, \eta$. Namely, let $X \in \mathfrak{X}(M)$. A direct computation exploiting Formulas (6.5)–(6.12) shows that
\[R^D(\|, \nabla^{|L|}_X)\| = \nabla^{|L|}_X J(X)\| \tag{6.13}\]
where $K$ is the $(1, 1)$-tensor given by
\[K(X) = \frac{1}{2} \nabla^g_X \eta^\# + \frac{1}{4} \left( g(\eta^\#, \eta^\#) X - \eta(X) \eta^\# + (i_X i_X d\eta)^\# \right) \tag{6.14}\]
and $J$ is the 1-form
\[J = \frac{1}{4} \left( \eta - i_\eta d\eta \right). \tag{6.15}\]
Using (6.14), it can be easily checked that the skew-adjoint part of $K$ can be obtained from $3d\eta/2$ by raising one index with $g$. Hence, if $R^D = 0$, necessarily $d\eta = 0$, and, from (6.15), $\eta = 0$. In other words, $\nabla^{|L|}$ is a flat connection, and $u$ is a flat section. Now on, we assume $\eta = 0$. In this case, for all vector fields $X, Y$,
\[R^D(\|, \nabla^{|L|}_X)\| = R^D(\nabla^{|L|}_X, \nabla^{|L|}_Y)\| = 0\]
identically. Finally, let $X, Y, W$ be vector fields. Then, in the hypothesis $\eta = 0$, we have
\[R^D(\nabla^{|L|}_X, \nabla^{|L|}_Y)\nabla^{|L|}_W = \nabla^{|L|}_U, \]
where $U$ is the vector field
\[U = R^g(X, Y) W + \frac{1}{4} \left( g(Y, W) X - g(X, W) Y \right), \]
$R^g$ being the Riemann tensor of $g$. When $R^D = \eta = 0$, $U$ must vanish for all $X, Y, W \in \mathfrak{X}(M)$, and this happens exactly when $g$ has constant curvature equal to $1/4$.

It remains to show that (3) implies (1). So, assume that $\eta = 0$ and that $g$ has constant curvature equal to $1/4$. Then, locally around every point, $(M, g)$ is isometric to the radius $2$ $n$-sphere. So, if we work locally, we can assume $g = 4g(S^n)$ where $g(S^n)$ is the metric of the unit sphere. And we can also assume that $g(S^n)$ is written, for instance, in spherical coordinates, say $(z^1, \ldots, z^n)$. Now, recall that
\begin{enumerate}
\item $\tilde{g}$ is equivalent to a definite, symmetric bilinear form $G : DL \otimes DL \equiv D|L| \otimes D|L| \rightarrow |L|$;
\item we are encoding $G$ in the triple $(\phi, g, \eta)$, where $\phi : |L| \equiv \mathbb{R}_M$ is the trivialization identifying $u = G(\|, \|)$ with the constant function $1$, and $\eta$ is the connection 1-form of the unique connection $\nabla^{|L|}$ in $|L|$ whose image is the $G$-orthogonal complement $L^\perp$;
\item if $\eta = 0$, then $\nabla^{|L|}$ is a flat connection and $u$ is a flat section.
\end{enumerate}
\begin{enumerate}
\item being a section of $|L|$, $u$ corresponds to a non-negative smooth function $\tilde{u} \in C^\infty(\tilde{L})$ such that $h^\ast_r(\tilde{u}) = |r|\tilde{u}$, for all $r \in \mathbb{R}_X$.
\end{enumerate}
From these facts, and the concrete relationship between $\tilde{g}$ and $G$, it is easy to see that, when $\eta = 0$, and $g = 4g(S^n)$, we have, locally

$$\tilde{g} = \tilde{u}(\tilde{u}^{-2} d\tilde{u} \otimes d\tilde{u} + 4g(S^n))$$

Hence, setting $R = 2\tilde{u}^{1/2}$, we find

$$\tilde{g} = dR \otimes dR + R^2 g(S^n),$$

which is a flat metric. More importantly, passing from spherical coordinates $(R, z^1, \ldots, z^n)$ to Cartesian coordinates $(\chi^1, \ldots, \chi^{n+1})$, we can put $\tilde{g}$ in the normal form

$$\tilde{g} = d\chi^1 \otimes d\chi^1 + \cdots + d\chi^{n+1} \otimes d\chi^{n+1}.$$

The Cartesian coordinates are of the form $R \cdot Y^i(z^1, \ldots, z^n)$ for some smooth functions $Y^i$ of the variables $z^i$. Hence, they are homogeneous:

$$h^*_r (\chi^i) = h^*_r (R) \cdot Y^i(z^1, \ldots, z^n) = |r|^{1/2} \chi^i, \quad \forall \ r \in \mathbb{R}^\times, \ i = 1, \ldots, n + 1.$$

We conclude that the $O_{n+1}$-structure consisting of orthonormal frames of $\tilde{g}$ is homogeneous integrable, as claimed. □

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