CHEMOTAXIS EFFECT VS LOGISTIC DAMPING ON
BOUNDEDNESS IN THE 2-D MINIMAL KELLER-SEGEL MODEL

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Abstract. We study chemotaxis effect vs logistic damping on boundedness for the two-dimensional minimal Keller-Segel model with logistic source:

\[
\begin{aligned}
u_t &= \nabla \cdot (\nabla u - \chi \nabla v) + u - \mu u^2, \quad x \in \Omega, t > 0, \\
v_t &= \Delta v - v + u, \quad x \in \Omega, t > 0
\end{aligned}
\]

in a smooth bounded domain $\Omega \subset \mathbb{R}^2$ with $\chi, \mu > 0$, nonnegative initial data $u_0, v_0$ and homogeneous Neumann boundary data. It is well-known that this model allows only for global and uniform-in-time bounded solutions for any $\chi, \mu > 0$. Here, we carefully employ a simple and new method to regain its boundedness and, with particular attention to how upper bounds of solutions qualitatively depend on $\chi$ and $\mu$. More precisely, it is shown there exists $C = C(u_0, v_0, \Omega) > 0$ such that

\[
\|u(\cdot, t)\|_{L^\infty(\Omega)} \leq C \left[ 1 + \frac{1}{\mu} + \chi K(\chi, \mu) N(\chi, \mu) \right]
\]

and

\[
\|v(\cdot, t)\|_{W^{1, \infty}(\Omega)} \leq C \left[ 1 + \frac{1}{\mu} + \frac{4}{\mu} K^2(\chi, \mu) \right] = CN(\chi, \mu)
\]

uniformly on $[0, \infty)$, where

\[
K(\chi, \mu) = M(\chi, \mu) E(\chi, \mu), \quad M(\chi, \mu) = 1 + \frac{1}{\mu} + \sqrt{1 + \frac{1}{\mu^2}}
\]

and

\[
E(\chi, \mu) = \exp \left[ \frac{\chi C^2 \min\{1, \frac{1}{2}\}}{2 \min\{1, \frac{1}{4}\}} \left( \frac{1}{\mu} \|u_0\|_{L^1(\Omega)} + \frac{13}{2 \mu^2} |\Omega| + \|\nabla v_0\|_{L^2(\Omega)}^2 \right) \right].
\]

We notice that these upper bounds are increasing in $\chi$, decreasing in $\mu$ and have only one singularity at $\mu = 0$, where the corresponding minimal model (removing the term $+u - \mu u^2$ in the first equation) is widely known to possess blow-ups for large initial data.

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\]

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1. Introduction and main results

In this work, we are concerned with the well-known and extensively explored Keller-Segel minimal chemotaxis model with logistic source:

\[
\begin{align*}
    u_t &= \nabla \cdot (\nabla u - \chi uv) + ru - \mu u^2, & x \in \Omega, t > 0, \\
    v_t &= \Delta v - v + u, & x \in \Omega, t > 0, \\
    \frac{\partial u}{\partial \nu} &= \frac{\partial v}{\partial \nu} = 0, & x \in \partial \Omega, t > 0, \\
    u(x, 0) &= u_0(x) \geq 0, v(x, 0) &= v_0(x) \geq 0, & x \in \Omega,
\end{align*}
\]

(1.1)

where $\Omega \subset \mathbb{R}^n (n \geq 1)$ is a bounded smooth domain, $r \geq 0, \chi, \mu > 0$ and $u$ and $v$ respectively denote the density of cells and the concentration of the chemical signal. The chemotactic flux $-\chi uv$ (defining term in chemotaxis models) models the directed movement that $u$ moves towards the higher concentration of $v$. This is commonly termed as chemotactic movement, a biological phenomenon whereby biological individuals orient their movement in response to some external signaling substances which attract cells to aggregate.

Without logistic source, i.e., $r = 0, \mu = 0$, the system (1.1) is known as the classical Keller-Segel minimal model [14], which and whose variants have been widely investigated since 1970. The striking feature of KS type models is the possibility of blow-up of solutions in a finite/infinite time, which strongly depends on the space dimension. A finite/infinite time blow-up never occurs in 1-D [7, 24, 37], a critical mass blow-up occurs in 2-D: when the initial mass $\|u_0\|_{L^1} < 4\pi/\chi$, solutions exist globally and converge to a single equilibrium in large time, whereas, when $\|u_0\|_{L^1} > 4\pi/\chi$, there exist solutions blowing up in finite time, cf. [10, 8, 21, 20, 25], and even small initial mass can result in blow-ups in $\geq 3$-D [31, 33]. See [1, 11] for more surveys on the classical KS model and its variants.

The logistic source was introduced by Mimura and Tsujikawa [19], where they study aggregating patterns based on the chemotaxis, diffusion and growth of bacteria. First, this additional logistic term apparently destroys the conservation law of mass of the classical KS model. On the other hand, it exerts a certain growth-inhibiting influence on the global existence and boundedness of solutions to the corresponding Keller-Segel models. Indeed, in the case $n = 1, 2$, even arbitrarily small $\mu > 0$ will be enough to prevent blow-ups by guaranteeing all solutions to (1.1) are global-in-time and uniformly bounded [24, 23, 38]. This is even true for a 2-D simplified version of parabolic-elliptic (the second PDE in (1.1) is replaced with $0 = \Delta v - v + u$) chemotaxis system with singular sensitivity [5]. Whereas, in the case $n \geq 3$, the global existence and boundedness were first obtained for a parabolic-elliptic simplification of (1.1) under $\mu > \frac{(n-2)}{n^2} \chi$ [30]. Nowadays, this result has been improved to the borderline case $\mu \geq \frac{n-2}{n^2} \chi$ [13, 29]. Moreover, with a very slow self-diffusion of cells, the $u$ component can exceed the carrying capacity $\frac{r}{\mu}$ to an arbitrary extent at some intermediate time [17, 35]. Coming back to our parabolic mode (1.1), for $\Omega$ being convex, Winkler first derived the boundeness and global existence provided that $\mu$ is beyond a certain number $\mu_0$ not explicitly known [32]. A further progress in this regard was derived as long as $\mu > \theta_0 \chi$ for some implicit positive constant $\theta_0$ in [10]. An explicit lower bound for a 3-D chemotaxis-fluid system with logistic source, when applied to (1.1) with $\chi = 1$, which states that $\mu \geq 23$ is enough to ensure boundedness [27]. This bound $\mu_0$
was further improved by Lin and Mu [13] in 3-D, wherein they replaced the logistic source in (1.1) by the damping term $u - \mu u^r$ with $r \geq 2$ to derive the boundedness under $\mu^{\frac{9}{\chi_1^2}} > 20\chi$. Very recently, for a full-parameter version of (1.1), we calculate out the explicit formula for $\mu_0$ in terms of the involving parameters, which states that $\mu > \frac{9}{\chi_1^2} \chi = (7.743416 \cdots) \chi$ ensures boundedness and global existence for (1.1) in 3-D [39]. Yet, it is a big open challenging problem whether or not blow-up occurs in (1.1) for small $\mu > 0$, even though the existence of global weak solutions is available in convex 3-D domains for $\mu > 0$ [16]. Under further conditions on $\chi, \mu$ or $r$, convergence of bounded solutions to the constant equilibrium $(\frac{x}{\mu}, \frac{\mu}{r})$ as well as its convergence rates are available [6] [13] [33] [59]. It also needs to be mentioned that for certain choices of the parameters, the solutions of (1.1) even may oscillate drastically in time, as numerically illustrated in [8], and that the solutions may undergo transient growth phenomena, as demonstrated in [17] [35] [56].

In contrast to the rich knowledge on boundedness, convergence and other dynamical properties for (1.1) and its variants, understanding the qualitative or quantitative properties even of bounded solutions to chemotaxis problems seems much less developed. In this direction, a work was considered by Tao and Winkler in [28] to show the mass persistence phenomenon for (1.1), i.e., for any supposedly given global classical and bounded nontrivial solution $(u, v)$ of (1.1), there is $m_0 > 0$ such that $\|u(t)\|_{L^1} \geq m_0$ for all $t > 0$. To our best knowledge, there seems no work on how boundedness or upper bounds of solutions of (1.1) depends on the system parameters, say, $\chi, \mu$ or $r$. In this paper, we aim as a first step to study chemotaxis effect vs logistic damping on boundedness for the minimal chemotaxis-logistic model (1.1) in 2-D. We do so partially because all solutions in 2-D are global and bounded by [23] [35]. We are particularly interested in the dependence of upper bounds of solutions to (1.1) on the most interesting parameters $\chi$ and $\mu$. We hope that this qualitative boundedness would stimulate new research directions, especially, the same problem in higher dimensions. Since the constant $r$ doesn’t bother us much in our derivation, we include it here. With this goal in mind, our main qualitative boundedness result reads as follows:

**Theorem 1.1.** Let $\chi, \mu > 0, r > 0, \Omega \subset \mathbb{R}^2$ be a bounded domain with a smooth boundary and let the initial data $u_0 \in C(\Omega)$ and $v_0 \in W^{1,\infty}(\Omega)$ be nonnegative. Then the Keller-Segel chemotaxis-logistic model (1.1) has a unique global classical nonnegative solution $(u, v)$ on $\Omega \times [0, \infty)$ for which

$$\|u(t)\|_{L^\infty(\Omega)} \leq C \left[1 + \frac{1}{\mu} + \chi K(\chi, \mu)N(\chi, \mu)\right] =: CL(\chi, \mu)$$  \hspace{1cm} (1.2)

and

$$\|v(t)\|_{W^{1,\infty}(\Omega)} \leq C \left[1 + \frac{1}{\mu} + \frac{\chi^2}{\mu} K^2(\chi, \mu)\right] =: CN(\chi, \mu)$$  \hspace{1cm} (1.3)

uniformly on $[0, \infty)$ and for some $C$ depending on $u_0, v_0, r$ and $|\Omega|$, where

$$K(\chi, \mu) = M(\chi, \mu)E(\chi, \mu), \quad M(\chi, \mu) = 1 + \frac{1}{\mu} + \sqrt{\chi(1 + \frac{1}{\mu^2})}$$  \hspace{1cm} (1.4)

and

$$E(\chi, \mu) = e^\frac{\chi^2}{2 \min\{1, \frac{1}{\mu^2}\}} \left[\frac{(\chi+3)}{\mu^2} \|u_0\|_{L^1(\Omega)}^2 + \frac{(\chi+3)^2}{\mu^2} |\Omega| + \|\nabla v_0\|_{L^2(\Omega)}^2 + \frac{(\chi+3)^2}{2\mu^2} |\Omega|\right].$$  \hspace{1cm} (1.5)

Up to a scaling constant, Theorem 1.1 provides explicit upper bounds for $\|u(t)\|_{L^\infty}$ and $\|v(t)\|_{W^{1,\infty}}$ in terms of the most interesting parameters $\chi$ and $\mu$ in 2-D.
The crucial point of the proof of Theorem 1.1 consists in deriving a uniform-in-time estimate for \( \|u(t)\|_{L^2} \) rather than \( \|(u+1) \ln(u+1)\|_{L^1} \) as in [1, 38], indeed, we obtain an explicit uniform-in-time bound for \( \|u(t)\|_{L^2}^2 \) as follows:

\[
\|u(t)\|_{L^2}^2 \leq E(\chi, \mu) \left\{ \|u_0\|_{L^2}^2 + \frac{8 \min\{1, \frac{2}{\mu}\}}{C_{GN}^2} + \frac{(r+1)}{\mu} \|u_0\|_{L^1} \right. \\
+ \left. \frac{3\chi C_{GN}^2}{4} \left[ \|u_0\|_{L^1}^2 + \frac{(r+1)^2}{4\mu} |\Omega| \right]^4 + \frac{(r+1)^3}{4\mu^2} |\Omega| + \frac{8r^3}{9\mu^2} |\Omega| \right\},
\]

where \( E \) is defined by (1.5) and \( C_{GN} \) is the Gagliardo-Nirenberg constant. After obtaining precise bounds on \( \|u\|_{L^1}, \|\nabla v\|_{L^2} \) and space-time integrals on \( u^2 \) and \( |\Delta v|^2 \), cf. Lemmas 2.4 and 3.1, we can use the 2-D Gagliardo-Nirenberg interpolation inequality to derive a differential inequality for \( y(t) = \|u(t)\|_{L^2}^2 + a \) of the form:

\[
y'(t) \leq ky(t)z(t) + b, \quad z(t) = \|\Delta v(t)\|_{L^2}^2
\]

for some \( a, k, b > 0 \) and then solving this ODE successively and using the gained space-time bounds, we achieve the desired estimate (1.6). This is inspired by the ideas presented in [26, Lemma 3.4]. Thanks to the \( L^p \)-boundedness criterion in [1, 38], the uniform-in-time bound for \( \|u(t)\|_{L^2} \) indeed implies the global existence and boundedness. While, to dig out the dependence of boundedness on \( \chi \) and \( \mu \), we first use the established \( L^2 \)-estimate of \( u \) together with a widely used 'reciprocal' lemma obtained from the \( v \)-equation, cf. Lemma 3.3 to bound \( \|\nabla v\|_{L^q} \) for any \( q \in (1, \infty) \), and then, we test the \( u \)-equation in (1.1) by \( u^2 \) to derive the \( L^q \)-estimate of \( u \), and finally, we apply the variation-of-constants formula for \( u \) and \( v \) and use the well-known smoothing \( L^p-L^q \) type estimates for the Neumann heat semigroup in \( \Omega \), cf. [2, 31] to conclude the respective bounds for \( (u, v) \) in (1.2) and (1.3).

**Remark 1.2.** From (1.2), (1.3), (1.4) and (1.5), one can see that \( L, M, N, K, E \) defined on \( [0, \infty) \times (0, \infty) \) are decreasing in \( \mu \) and are increasing in \( \chi \) have only one singularity at \( \mu = 0 \). Therefore, our obtained bounds for \( \|u(t)\|_{L^\infty} \) and \( \|v(t)\|_{W^{1, \infty}} \) enjoy these properties. It is worthwhile to observe that, when \( \mu = 0 \), even \( r = 0 \), the corresponding KS minimal model possesses blow-ups for large initial data [11, 21, 29, 25], illustrating the reasonableness of adding a logistic source to the KS minimal model to prevent blow-up.

2. Preliminaries

For convenience, we start with the well-known Young’s inequality with \( \epsilon \):

**Lemma 2.1.** (Young’s inequality with \( \epsilon \)) Let \( p \) and \( q \) be two given positive numbers with \( \frac{1}{p} + \frac{1}{q} = 1 \). Then, for any \( \epsilon > 0 \), it holds

\[
ab \leq ca^p + \frac{b^q}{(c^p)^{\frac{q}{p}}} + \epsilon a, \quad \forall a, b \geq 0.
\]

**Lemma 2.2.** (Gagliardo-Nirenberg interpolation inequality [1, 22]) Let \( p \geq 1 \) and \( q \in (0, p) \). Then there exist a positive constant \( C_{GN} \) depending on \( p \) and \( q \) such that

\[
\|w\|_{L^p} \leq C_{GN} \left( \|\nabla w\|_{L^2}^\delta \|w\|_{L^s}^{(1-\delta)} + \|w\|_{L^q} \right), \quad \forall w \in H^1 \cap L^q,
\]

where \( s > 0 \) is arbitrary and \( \delta \) is given by

\[
\frac{1}{p} = \delta \left( \frac{1}{2} - \frac{1}{n} \right) + \frac{1-\delta}{q} \iff \delta = \frac{n}{2} - \frac{n}{p} + \frac{n}{q} \in (0, 1).
\]
The basic result on local existence, uniqueness and extendibility of classical solutions for the minimal KS system \((1.1)\) can be found in [22, Lemma 1.1].

**Lemma 2.3.** Let \(\chi, \mu > 0, r \geq 0, \Omega \subset \mathbb{R}^n(n \geq 1)\) be a bounded smooth domain and let the initial data \(u_0 \in C(\overline{\Omega})\) and \(v_0 \in W^{1,\infty}(\Omega)\) be nonnegative. Then there is a unique, nonnegative and classical maximal solution \((u, v)\) of the IBVP \((1.1)\) on some maximal interval \((0, T_m)\) with \(0 < T_m \leq \infty\) such that

\[
\begin{align*}
    u &\in C(\overline{\Omega} \times [0, T_m)) \cap C^{2,1}(\Omega \times (0, T_m)), \\
v &\in C(\overline{\Omega} \times [0, T_m)) \cap C^{2,1}(\Omega \times (0, T_m)) \cap L^\infty_{loc}([0, T_m); W^{1,s}(\Omega))
\end{align*}
\]

for any \(s > n\). In particular, if \(T_m < \infty\), then

\[
\|u(t)\|_{L^\infty} + \|v(t)\|_{W^{1,s}} \to \infty \quad \text{as } t \to T_m^{-}.
\]

**Lemma 2.4.** For any \(t \in [0, T_m)\), the nonnegative solution \((u, v)\) of \((1.1)\) satisfies

\[
\|u\|_{L^1} \leq \|u_0\|_{L^1} + \frac{(r + 1)^2}{4\mu}|\Omega| := k_1
\]

and

\[
\|
\nabla v\|_{L^2}^2 \leq \frac{2}{\mu} \left[ \|u_0\|_{L^1} + \frac{\mu}{2}\|
\nabla v_0\|_{L^2}^2 + \frac{(r + 2)^2}{4\mu}|\Omega| \right] := k_2.
\]

**Proof.** The nonnegativity of \(u, v\) follows from the maximum principle. Then integrating the \(u\)-equation and using the homogeneous Neumann boundary condition, we derive

\[
\frac{d}{dt}\int_\Omega u = r\int_\Omega u - \mu \int_\Omega u^2 \leq -\int_\Omega u + \frac{(r + 1)^2}{4\mu}|\Omega|,
\]

which yields the \(L^1\)-bound for \(u\) in \((2.1)\).

Then testing the \(v\)-equation in \((1.1)\) against \(-\Delta v\) and integrating by parts and using Young’s inequality with epsilon as stated in Lemma 2.1, we obtain

\[
\frac{1}{2}\frac{d}{dt}\int_\Omega |\nabla v|^2 + \frac{1}{2}\int_\Omega |\Delta v|^2 \leq \int_\Omega |\nabla v|^2 + \frac{1}{2}\int_\Omega u^2,
\]

which together with the reasoning leading to \((2.3)\) gives us

\[
\frac{d}{dt}\int_\Omega (u + \frac{\mu}{2}|\nabla v|^2) + 2\int_\Omega (u + \frac{\mu}{2}|\nabla v|^2) \leq \frac{(r + 2)^2}{2\mu}|\Omega|.
\]

Solving this standard Gronwall inequality shows

\[
\|u\|_{L^1} + \frac{\mu}{2}\|
\nabla v\|_{L^2}^2 \leq \|u_0\|_{L^1} + \frac{\mu}{2}\|
\nabla v_0\|_{L^2}^2 + \frac{(r + 2)^2}{4\mu}|\Omega|,
\]

which directly leads to \((2.2)\).

\[\square\]

3. **Chemotaxis vs logistic on Boundedness in 2-D**

In 2-D, it is well-known that any presence of logistic source will be sufficient to suppress blow-up by ensuring all solutions to \((1.1)\) are global-in-time and uniformly bounded \([23, 38]\). In this section, we carefully scrutinize a different method motivated from \([26, \text{Lemma 3.4}]\) to regain its boundness and, with particular focus on the qualitative dependence of upper bounds of solutions to \((1.1)\) on \(\chi\) and \(\mu\), and thus accomplish the proof of Theorem 1.3.
Lemma 3.1. Given $\tau \in (0, T_m)$, then, for any $t \in [0, T_m - \tau)$, the solution $(u, v)$ of the KS model \eqref{eq:KS_model} fulfills
\[ \int_t^{t+\tau} \int_{\Omega} u^2 \leq \frac{(r+1)k_1}{\mu} \max\{\tau, 1\} =: k_3 \max\{\tau, 1\}, \quad (3.1) \]
\[ \int_t^{t+\tau} \int_{\Omega} |\nabla v|^2 \leq k_2 \max\{\tau, 1\} \quad (3.2) \]
and
\[ \int_t^{t+\tau} \int_{\Omega} |\Delta v|^2 \leq (k_3 + k_2) \max\{\tau, 1\} =: k_4 \max\{\tau, 1\}. \quad (3.3) \]

Proof. For any $t \in [0, T_m - \tau)$, integrating the $u$-equation in \eqref{eq:KS_model} over $\Omega \times (t, t+\tau)$ and using Lemma 2.4, we deduce that
\[ \mu \int_t^{t+\tau} \int_{\Omega} u^2 \leq r \int_t^{t+\tau} \int_{\Omega} u + \int_{\Omega} u \leq (r+1)k_1 \max\{\tau, 1\}, \]
yielding the desired inequality \eqref{eq:3.1}.

The estimate \eqref{eq:3.2} follows directly from \eqref{eq:2.2}. Next, an integration of \eqref{eq:2.4} over $(t, t+\tau)$ and the use of \eqref{eq:2.2} and \eqref{eq:3.1} telescope
\[ \int_t^{t+\tau} \int_{\Omega} |\nabla v|^2(s) \leq \int_t^{t+\tau} \int_{\Omega} u^2(s) + \int_{\Omega} |\nabla v|^2(t) \leq (k_3 + k_2) \max\{\tau, 1\}, \]
which is exactly \eqref{eq:3.3}. \hfill \Box

Here, with Lemma 3.1 at hand, in 2-D setting, we can make use of the Gagliardo-Nirenberg interpolation inequality in Lemma 2.2 to derive an ODE satisfied by $\Box u$, which enables us to deduce an estimate for $\|u\|_{L^2}$. This is the key point for us to derive qualitative bounds for $\|u\|_{L^\infty}$ and $\|\nabla v\|_{W^{1, \infty}}$ later on.

Lemma 3.2. Given $\tau \in (0, T_m)$, then the $u$-component of the solution $(u, v)$ of the KS minimal model \eqref{eq:KS_model} satisfies the explicit uniform-in-time bound:
\[ \|u(t)\|_{L^2} \leq C \left[ \frac{8 \min\{1, \frac{2}{3}\}}{C_{GN}^2} + \frac{3 \chi C_{GN}^2}{4} \left( \frac{\|u_0\|_{L^1} + \frac{(r+1)^2}{4\mu} |\Omega|}{\max\{1, \frac{1}{\tau}\}} \right)^4 \right. \]
\[ \left. + \frac{(r+1)}{\mu} \frac{\|u_0\|_{L^1} + \frac{(r+1)^3}{4\mu^2} |\Omega| + \frac{8r^3}{9\mu^2} |\Omega|}{\max\{1, \frac{1}{\tau}\}} \right] \max\{1, \tau, 1\} \]
\[ \times e^{-\frac{\chi C_{GN}^2}{2\mu} \int_{1}^{t+\tau} \left( \frac{(r+1)^3}{4\mu^2} |\Omega| + \|\nabla u_0\|^2_{L^2} + \frac{\chi C_{GN}^2}{4\mu} |\Omega| \right) \max\{1, \tau\}}, \quad (3.4) \]
and so a uniform estimate for $\|u\|_{L^2}$ in terms of $\chi$ and $\mu$ follows:
\[ \|u(t)\|_{L^2} \leq C \left[ 1 + \frac{1}{\mu} + \sqrt{\chi} \left( 1 + \frac{1}{\mu^2} \right) \right] \max\{\sqrt{\tau}, \frac{1}{\sqrt{\tau}}\} E^{\max\{1, \tau\}}(\chi, \mu) \]
\[ (3.5) \]
for all $t \in (0, T_m)$ and for some $C = C(u_0, r, |\Omega|)$, where $E$ is defined by \eqref{eq:energy}. \hfill \Box
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Proof. We test the \( u \)-equation in (1.1) by \( u \) and integrate by parts to deduce from Hölder’s inequality that

\[
\frac{1}{2} \frac{d}{dt} \int_{\Omega} u^2 + \int_{\Omega} |\nabla u|^2 = \frac{\chi}{2} \int_{\Omega} \nabla (u^2) \nabla v + \int_{\Omega} u^2 (r - \mu u) \\
= -\frac{\chi}{2} \int_{\Omega} u^2 \Delta v + \int_{\Omega} u^2 (r - \mu u) \\
\leq \frac{\chi}{2} \left( \int_{\Omega} u^4 \right)^{\frac{1}{4}} \left( \int_{\Omega} |\Delta v|^2 \right)^{\frac{3}{4}} + \int_{\Omega} u^2 (r - \mu u). \tag{3.6}
\]

Applying the GN interpolation inequality in Lemma 2.2 with \( n = 2 \) and the boundedness of \( \|u\|_{L^1} \) in (2.1), we estimate

\[
\left( \int_{\Omega} u^4 \right)^{\frac{1}{4}} \left( \int_{\Omega} |\Delta v|^2 \right)^{\frac{3}{4}} \leq C_{GN} \left( \|\nabla u\|_{L^2} \|u\|_{L^2} + \|u\|_{L^1}^2 \right) \leq C_{GN} \left( \|\nabla u\|_{L^2} \|u\|_{L^2} + k_1^2 \right).
\]

Hence, upon twice uses of Young’s inequality with epsilon, cf. Lemma 2.1, for any \( \epsilon > 0 \), it follows that

\[
\left( \int_{\Omega} u^4 \right)^{\frac{1}{4}} \left( \int_{\Omega} |\Delta v|^2 \right)^{\frac{3}{4}} \leq C_{GN} \|\nabla u\|_{L^2} \|\Delta v\|_{L^2} + \frac{k_1^2 C_{GN}}{4} \|\Delta v\|_{L^2} \leq \epsilon \|\nabla u\|_{L^2}^2 + \frac{C_{GN}^2}{4\epsilon} \|u\|_{L^2}^2 \|\Delta v\|_{L^2}^2 + \left( \frac{4C_{GN}^2}{\epsilon} + \frac{k_1^4 C_{GN}}{4} \right) \|\Delta v\|_{L^2}^2 + \frac{8r^3}{27\mu^2} \|\Omega\|.
\]

Inserting (3.7) into (3.9), we conclude that

\[
\frac{1}{2} \frac{d}{dt} \int_{\Omega} u^2 + \int_{\Omega} |\nabla u|^2 \leq \frac{\chi}{2} \left( \epsilon \|\nabla u\|_{L^2}^2 + \frac{C_{GN}^2}{4\epsilon} \|u\|_{L^2}^2 \|\Delta v\|_{L^2}^2 + \frac{k_1^4 C_{GN}}{4} \|\Delta v\|_{L^2}^2 + \frac{8r^3}{27\mu^2} \|\Omega\| \right)
\]

from which, upon setting

\[
\epsilon = \min \left\{ 1, \frac{\chi}{2} \right\}, \tag{3.8}
\]

we deduce

\[
\frac{d}{dt} \int_{\Omega} u^2 \leq \frac{\chi C_{GN}^2}{4\epsilon} \left( \|u\|_{L^2}^2 + \frac{4\epsilon}{C_{GN}^2} \|\Delta v\|_{L^2}^2 \right) + \frac{8r^3}{27\mu^2} \|\Omega\| + k_5 y(t) z(t) + k_6, \tag{3.9}
\]

where \( k_5 = \frac{\chi C_{GN}^2}{4\epsilon} \), \( k_6 = \frac{k_1^4 C_{GN}}{4} + \frac{8r^3}{27\mu^2} \|\Omega\| \) and

\[
y(t) = \|u\|_{L^2}^2 + \frac{4\epsilon}{C_{GN}^2}, \quad z(t) = \|\Delta v\|_{L^2}^2. \tag{3.10}
\]

For any \( s \geq 0 \) and any \( t \geq s \), multiplying the integrating factor \( \exp(-k_5 \int_s^t z(\lambda)d\lambda) \) on both sides of (3.9), we deduce that

\[
y(t) \leq y(s) e^{k_5 \int_s^t z(\sigma)d\sigma} + k_6 \int_s^t e^{k_5 \int_{\xi}^{\tau} z(\sigma)d\sigma} d\xi, \quad \forall t \in [s, \infty) \cap [0, T_m). \tag{3.11}
\]
In view of (3.1) and Lemma 3.1 and the mean value theorem, one infers from the definitions of $y$ and $z$ in (3.10) that

$$y(s_i) = \frac{1}{\tau} \int_{s_i}^{(i+1)\tau} y(s) ds \leq (k_3 + \frac{4\epsilon}{C_{GN}^2}) \max\{1, \frac{1}{\tau}\} =: k_4 \max\{1, \frac{1}{\tau}\}$$

(3.12)

and

$$\int_{s_i}^{(i+1)\tau} z(s) ds \leq k_4 \max\{\tau, 1\}$$

(3.13)

for some $s_i \in [i\tau, (i+1)\tau]$ and any nonnegative integers $i < \frac{T_m}{\tau} - 1$.

First, for $t \in [0, \tau]$, we set $s = 0$ in (3.11) and $i = 0$ in (3.13) to infer

$$y(t) \leq y(0)e^{k_5 \int_0^\tau z(\sigma)d\sigma} + k_6 \int_0^\tau e^{k_5 \int_0^\tau z(\sigma)d\sigma} d\xi$$

$$\leq (y(0) + k_6) \max\{1, \frac{1}{\tau}\} e^{k_5 k_4 \max\{\tau, 1\}}.$$  

(3.14)

Next, for $t \in [\tau, 2\tau]$, we will always assume that $t < T_m$, we put $s = s_0 \in [0, \tau]$ in (3.11), and then infers from (3.12) and (3.13) that

$$y(t) \leq y(s_0)e^{k_5 \int_0^\tau z(\sigma)d\sigma} + k_6 \int_{s_0}^t e^{k_5 \int_0^\tau z(\sigma)d\sigma} d\xi$$

$$\leq y(s_0)e^{k_5 \int_0^{(i+1)\tau} z(\sigma)d\sigma} + k_6 \int_{s_0}^t e^{k_5 \int_0^{(i+1)\tau} z(\sigma)d\sigma} d\xi$$

(3.15)

$$\leq (k_7 + 2k_6) \max\{1, \frac{1}{\tau}\} e^{2k_5 k_4 \max\{\tau, 1\}}.$$  

In general, for any $t \in (\tau, T_m)$, one first chooses $i \geq 0$ such that $t \in [(i+1)\tau, (i+2)\tau]$ and set $s = s_i \in [i\tau, (i+1)\tau]$ in (3.11), and then infers from (3.12) and (3.13) that

$$y(t) \leq y(s_i)e^{k_5 \int_{s_i}^{(i+1)\tau} z(\sigma)d\sigma} + k_6 \int_{s_i}^t e^{k_5 \int_{s_i}^{(i+1)\tau} z(\sigma)d\sigma} d\xi$$

$$\leq y(s_i)e^{k_5 \int_{s_i}^{(i+2)\tau} z(\sigma)d\sigma} + k_6 \int_t^{(i+1)\tau} e^{k_5 \int_{s_i}^{(i+2)\tau} z(\sigma)d\sigma} d\xi$$

(3.16)

Recalling from the definition of $y(t)$ in (3.10), we then conclude from (3.14), (3.15) and (3.16) the uniform $L^2$-estimate of $u$:

$$\|u\|_{L^2}^2 + \frac{4\epsilon}{C_{GN}^2} \leq (y(0) + k_7 + 3k_6) \max\{1, \frac{1}{\tau}\} e^{2k_5 k_4 \max\{\tau, 1\}}$$

$$= \left(\|u_0\|_{L^2}^2 + \frac{8\min\{1, \frac{\tau}{\Omega}\}}{C_{GN}^2} + k_3 \frac{3\chi k_4^3 C_{GN}^2}{4} + \frac{8\tau^3}{9\mu_\epsilon|\Omega|}\right)$$

$$\times \max\{1, \frac{1}{\tau}\} e^{\frac{8\min\{1, \frac{\tau}{\Omega}\}}{k_3 + k_2} \max\{1, \tau\}},$$

where we have substituted the definitions of $k_4, k_5, k_6, k_7$ and $\epsilon$ in (3.3), (3.4), (3.12) and (3.8). In the above inequality, a further substitution of $k_3, k_2, k_3$ as defined in (2.1), (2.2) and (5.1) yields the desired $L^2$-estimate of $u$ in (3.7). □

**Remark 3.3.** Another way to view the uniform $L^2$-norm of $u$ could be arguing as follows: Assume $T_m < \infty$. Then, for any given large natural number $N \gg 1$, we set $\tau = \frac{T_m}{N}$ so that $N\tau = T_m$. Then as arguing above we can obtain that $\|u(t)\|_{L^2}$ is
uniformly bounded in \((0, T_m)\), which violates the \(L_{\chi, \mu}^p\)-criterion in \((1.1)\) with \(n = 2\). Hence, \(T_m = \infty\) and \(\|u(t)\|_{L_{\chi, \mu}^p} \) is uniformly bounded on \((0, \infty)\). Furthermore, this energy method offers a simple proof for global-in-time boundedness in 2-D setting compared to existing literature, c.f. \([23, 38]\).

In the sequel, we shall seek how the \((L^\infty, W^{1,\infty})\)-bound of \((u, v)\) depends on \(\chi\) and \(\mu\). Since the solution \((u, v)\) is global in time by Remark \((3.3)\) we will set \(\tau = 1\) to simplify our calculations. To get higher order regularity of \(L\), known smoothing bounds of \(v\), and \(\mu\leq cf. \([2, 31]\), for \(1\)

\[
L \text{ -equation in (1.1), we have the following widely known 'reciprocal' lemma, cf. [12, Lemma 4.1], [15, Lemma 1], [38, Lemma 3.5] for instance. Applying these heat Neumann semigroup estimates to the (1.1) with } \(\nu = 1\), \(\mu = 2\). Since the solution \((u, v)\) is globally bounded on \((0, \infty)\), we will set \(\nu = 1\) to simplify our calculations. To get higher order regularity of \(L\), known smoothing bounds of \(v\), and \(\mu\leq cf. \([2, 31]\), for \(1\)
Multiplying the \( u \)-equation in (1.1) by \( u^2 \), integrating by parts and using Young’s inequality with epsilon, we arrive at
\[
\frac{1}{3} \frac{d}{dt} \int_\Omega u^3 + 2 \int_\Omega u |\nabla u|^2 \\
= 2 \chi \int_\Omega u^2 \nabla u \nabla v + \int_\Omega (ru^3 - \mu u^4) \\
\leq 2 \int_\Omega u |\nabla u|^2 + \frac{\lambda^2}{2} \int_\Omega u^3 |\nabla v|^2 + \int_\Omega (ru^3 - \mu u^4) \\
\leq 2 \int_\Omega u |\nabla u|^2 + \frac{\mu}{2} \int_\Omega u^4 + \frac{3^3 \chi^8}{2 \cdot 4^4 \mu^3} \int_\Omega |\nabla v|^2 + \int_\Omega (ru^3 - \mu u^4),
\]
which along with the algebraic fact \( ru^3 - \frac{\mu}{2} u^4 \leq -\frac{1}{4} u^3 + \frac{3^3 (r + \frac{1}{4})^4}{2^5 \mu^3} \) shows that
\[
\frac{d}{dt} \int_\Omega u^3 + \int_\Omega u^3 \leq \frac{3^3 \chi^8}{2 \cdot 4^4 \mu^3} \|\nabla v\|_{L^8}^8 + \frac{3^4 (r + \frac{1}{4})^4}{2^5 \mu^3} |\Omega|.
\]
Solving this standard Gronwall differential inequality, we directly have
\[
\|u\|_{L^3}^3 \leq \|u_0\|_{L^3}^3 + \frac{3^3 \chi^8}{2 \cdot 4^4 \mu^3} \sup_{t \in (0, \infty)} \|\nabla v(t)\|_{L^8}^8 + \frac{3^4 (r + \frac{1}{4})^4}{2^5 \mu^3} |\Omega|,
\]
which together with (3.24) with \( q = 8 \) yields the desired estimate (3.23). \( \square \)

**Proof of Theorem 1.1.** The \( W^{1, \infty} \)-bound of \( v \) in (1.5) follows directly from the uniform \( L^3 \)-estimate of \( u \) in (3.23) and Lemma 3.4 with \( (n, p, q) = (2, 3, \infty) \).

For the \( L^\infty \)-bound of \( u \), we first apply the variation-of-constants formula to the \( u \)-equation in (1.1) to represent \( u \) as
\[
u(t) = e^{(\Delta - 1) t} u_0 - \chi \int_0^t e^{(t-s)(\Delta - 1)} \nabla \cdot ((u \nabla v)(s)) \, ds \\
+ \int_0^t e^{(t-s)(\Delta - 1)} [(r + 1) u(s) - \mu u^2(s)] \, ds
\]
\[
= u_1(t) + u_2(t) + u_3(t).\tag{3.25}
\]
Because \( u \) is nonnegative and smooth, we thus have
\[
\|u(t)\|_{L^\infty} = \sup_{x \in \Omega} u(x, t) \leq \sup_{x \in \Omega} u_1(x, t) + \sup_{x \in \Omega} u_2(x, t) + \sup_{x \in \Omega} u_3(x, t).
\]
Thanks to the the maximum principle, the Neumann heat semigroup \( e^{t \Delta} \) is order preserving. This allows us to control \( u_1 \) and \( u_3 \) as follows:
\[
\|u_1(t)\|_{L^\infty} = \|e^{(\Delta - 1) t} u_0\|_{L^\infty} \leq e^{-t} \|u_0\|_{L^\infty} \leq \|u_0\|_{L^\infty}.
\]
as well as
\[
u_3(t) = \int_0^t e^{-(t-s)} e^{(t-s) \Delta} [(r + 1) u(s) - \mu u^2(s)] \, ds \\
\leq \int_0^t e^{-(t-s)} e^{(t-s) \Delta} \frac{(r + 1)^2}{4 \mu} \, ds \leq \frac{(r + 1)^2}{4 \mu} \tag{3.27}
\]
To estimate \( u_2 \), we recall one more property of the Neumann heat semigroup \( e^{t \Delta} \), cf. (3.21) for any \( 1 \leq q \leq p \leq \infty \), there exists \( k_{11} > 0 \) such that
\[
\|e^{t \Delta} \nabla \cdot w\|_{L^p} \leq k_{11} \left( 1 + t^{\frac{n}{2} - 1} \frac{(r + \frac{1}{4})^4}{2^5 \mu^3} \right) e^{-\lambda_1 t} \|w\|_{L^q}, \forall t > 0, w \in (W^{1, p})^n. \tag{3.28}
\]
Using the definition of $u_2$ in (3.25), (3.28) with $n = 2$ and Hölder interpolation inequality, we deduce that

$$\|u_2(t)\|_{L^\infty} \leq \chi \int_0^t \|e^{(t-s)(1-\frac{1}{2})\Delta} \nabla \cdot (u(s)\nabla v(s))\|_{L^\infty} ds$$

$$\leq k_{111} \chi \int_0^t (1 + (t-s)^{-\frac{1}{2}+\frac{q}{2}}) e^{-(\lambda_1+1)(t-s)} \|u(s)\nabla v(s)\|_{L^2} ds$$

$$\leq k_{111} \chi \int_0^t (1 + (t-s)^{-\frac{1}{2}+\frac{q}{2}}) e^{-(\lambda_1+1)(t-s)} \|u(s)\|_{L^3} \|\nabla v(s)\|_{L^{15}} ds$$

$$\leq k_{111} \chi \sup_{s \in (0,\infty)} \|u(s)\|_{L^3} \|\nabla v(s)\|_{L^{15}} \int_0^t (1 + \sigma^{1+\frac{q}{2}}) e^{-(\lambda_1+1)\sigma} ds$$

$$=: k_{122} \chi \sup_{s \in (0,\infty)} \|u(s)\|_{L^3} \sup_{s \in (0,\infty)} \|\nabla v(s)\|_{L^{15}}.$$

This in conjunction with (3.23) and (3.24) with $q = 15$ gives the estimate of $u_2$:

$$\|u_2(t)\|_{L^\infty} \leq C \chi M(\chi,\mu) E(\chi,\mu) \left[ 1 + \frac{1}{\mu} + \frac{\chi^{\frac{q}{2}}}{\mu} M \hat{\chi}(\chi,\mu) E \hat{\chi}(\chi,\mu) \right].$$

(3.29)

A substitution of (3.24), (3.27) and (3.29) into (3.25) yields the desired uniform bound for $\|u(t)\|_{L^\infty}$ as stated in (1.2).

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References

[1] N. Bellomo, A. Bellouquid, Y. Tao, M. Winkler Toward a mathematical theory of Keller-Segel models of pattern formation in biological tissues, Math. Models Methods Appl. Sci. 25 (2015), 1663-1763.

[2] X. Cao, Global bounded solutions of the higher-dimensional Keller-Segel system under smallness conditions in optimal spaces, Discrete Contin. Dyn. Syst. 35 (2015), 1891–1904.

[3] E. Feireisl, P. Laurençot and H. Petzeltova, On convergence to equilibria for the Keller-Segel chemotaxis model, J. Differential Equations, 236 (2007), 551–569.

[4] A. Friedman, Partial differential equations. Holt, Rinehart and Winston, New York-Montreal, Que.-London, 1969.

[5] K. Fujie, M. Winkler and T. Yokota, Blow-up prevention by logistic sources in a parabolic-elliptic Keller-Segel system with singular sensitivity, Nonlinear Anal. 109 (2014), 56–71.

[6] X. He and S. Zheng, Convergence rate estimates of solutions in a higher dimensional chemotaxis system with logistic source, J. Math. Anal. Appl. 436 (2016), 970–982.

[7] T. Hillen and A. Potapov, The one-dimensional chemotaxis model: global existence and asymptotic profile, Math. Methods Appl. Sci. 27 (2004), 1783–1801.

[8] T. Hillen and K. Painter, Spatio-temporal chaos in a chemotaxis model, Phys. D 240 (2011), 363–375.

[9] B. Hu and Y. Tao, Boundedness in a parabolic-elliptic chemotaxis-growth system under a critical parameter condition, Appl. Math. Lett. 64 (2017), 1–7.

[10] D. Horstmann and G. Wang, Blow-up in a chemotaxis model without symmetry assumptions, European J. Appl. Math. 12 (2001), 159–177.

[11] D. Horstmann, From 1970 until now: the Keller-Segel model in chemotaxis and its consequence I, Jahresber DMV, 105 (2003), 103–165.

[12] D. Horstmann and M. Winkler, Boundedness vs. blow-up in a chemotaxis system, J. Differential Equations 215 (2005), 52–107.

[13] K. Kang and A. Stevens, Blowup and global solutions in a chemotaxis-growth system, Nonlinear Anal. 135 (2016), 57–72.

[14] E. Keller and L. Segel, Initiation of slime mold aggregation viewed as an instability, J. Theoret Biol., 26 (1970), 399–415.
[15] R. Kowalczyk and Z. Szymańska, On the global existence of solutions to an aggregation model, J. Math. Anal. Appl. 343 (2008), 379–398.
[16] J. Lankeit, Eventual smoothness and asymptotics in a three-dimensional chemotaxis system with logistic source, J. Differential Equations 258 (2015), 1158–1191.
[17] J. Lankeit, Chemotaxis can prevent thresholds on population density, Discrete Contin. Dyn. Syst. Ser. B 20 (2015), 1499–1527.
[18] K. Lin and C. Mu, Global dynamics in a fully parabolic chemotaxis system with logistic source. Discrete Contin. Dyn. Syst. 36 (2016), 5025–5046.
[19] M. Mimura and T. Tsujikawa, Aggregating pattern dynamics in a chemotaxis model including growth, Physica A 230 (1996), 449–543.
[20] T. Nagai, Blowup of nonradial solutions to parabolic-elliptic systems modeling chemotaxis in two-dimensional domains, J. Inequal. Appl. 6 (2001), 37–55.
[21] T. Nagai, T. Senba and K. Yoshida, Application of the Trudinger-Moser inequality to a parabolic system of chemotaxis, Funkcial. Ekvac. 40 (1997), 411–433.
[22] L. Nirenberg, An extended interpolation inequality, Ann. Scuola Norm. Sup. Pisa. (3) 20 1966, 733-737.
[23] K. Osaki, T. Tsujikawa, A. Yagi, M. Mimura, Exponential attractor for a chemotaxis-growth system of equations, Nonlinear Anal. 51, 119-144 (2002).
[24] K. Osaki and A. Yagi, Finite dimensional attractor for one-dimensional Keller-Segel equations, Funkcial. Ekvac. 44 (2001), 441–469.
[25] T. Senba and T. Suzuki, Parabolic system of chemotaxis: blowup in a finite and the infinite time, Methods Appl. Anal. 8 (2001), 349–367.
[26] C. Stinner, C. Surulescu and M. Winkler, Global weak solutions in a PDE-ODE system modeling multiscale cancer cell invasion, SIAM J. Math. Anal. 46 (2014), 1969–2007.
[27] Y. Tao and M. Winkler, Boundedness and decay enforced by quadratic degradation in a three-dimensional chemotaxis-fluid system, Z. Angew. Math. Phys. 66 (2015), 2555–2573.
[28] Y. Tao and M. Winkler, Persistence of mass in a chemotaxis system with logistic source, J. Differential Equations, 259 (2015), 6142–6161.
[29] Z. Wang and T. Xiang, A class of chemotaxis systems with growth source and nonlinear secretion, arXiv:1510.07204, 2015.
[30] J. Tello and M. Winkler, A chemotaxis system with logistic source, Comm. Partial Differential Equations, 32 (2007), 849–877.
[31] M. Winkler, Aggregation vs. global diffusive behavior in the higher-dimensional Keller-Segel model, J. Differential Equations, 248 (2010), 2889–2905.
[32] M. Winkler, Boundedness in the higher-dimensional parabolic-parabolic chemotaxis system with logistic source, Comm. Partial Differential Equations, 35 (2010), 1516–1537.
[33] M. Winkler, Finite-time blow-up in the higher-dimensional parabolic-parabolic Keller-Segel system, J. Math. Pures Appl. 100 (2013), 748–767.
[34] M. Winkler, Global asymptotic stability of constant equilibria in a fully parabolic chemotaxis system with strong logistic dampening, J. Differential Equations, 257 (2014), 1056–1077.
[35] M. Winkler, How far can chemotactic cross-diffusion enforce exceeding carrying capacities? J. Nonlinear Sci. 24 (2014), 809–855.
[36] M. Winkler, Emergence of large population densities despite logistic growth restrictions in fully parabolic chemotaxis systems, Discrete Contin. Dyn. Syst. Ser. B 22 (2017), 2777–279.
[37] T. Xiang, On effects of sampling radius for the nonlocal Patlak-Keller-Segel chemotaxis model, Discrete Contin. Dyn. Syst. 34 (2014), 4911–4946.
[38] T. Xiang, Boundedness and global existence in the higher-dimensional parabolic-parabolic chemotaxis system with/without growth source, J. Differential Equations, 258 (2015), 4275–4323.
[39] T. Xiang, How strong a logistic damping can prevent blow-up for the minimal Keller-Segel chemotaxis system? J. Math. Anal. Appl. 459 (2018), 1172–1200.
[40] C. Yang, X. Cao, Z. Jiang and S. Zheng, Boundedness in a quasilinear fully parabolic Keller-Segel system of higher dimension with logistic source, J. Math. Anal. Appl. 430 (2015), 585–591.
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