On the Localization Methods of High Energy Transients for All-Sky Gamma-Ray Monitors

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ABSTRACT

Reliable localization of high-energy transients, such as Gamma-ray Bursts (GRBs) and Soft Gamma-ray Repeaters (SGRs), is the prerequisite for characterizing the burst properties (e.g. spectrum) and implementing the follow-up observations in the multi-wavelength and multi-messenger. Localization based on the relative counts of different detectors has been widely used for all-sky gamma-ray monitors, such as CGRO/BATSE, Fermi/GBM, POLAR, and GECAM. There are two major statistical frameworks for counts distribution localization methods: \( \chi^2 \) and Bayesian. Here, we studied and compared several localization methods based on these two statistical frameworks, by directly checking the localization probability map and credible region with comprehensive simulations. We find that the Bayesian method is generally more applicable for various bursts than \( \chi^2 \) method. We also proposed a location-spectrum iteration approach based on the Bayesian inference, which could not only take advantage of straightforward calculation but also alleviate the problems caused by the spectral difference between the burst and location templates.

Key words: methods: data analysis – methods: statistical – gamma-ray burst: general.

1 INTRODUCTION

High-energy transients, e.g. Gamma-ray Bursts (GRBs) and Soft Gamma-Ray Repeaters (SGRs), are usually first discovered in the gamma-ray band. Reliable localization of these bursts is critically important for joint observation in multi-wavelength and multi-messenger astronomy. For instance, in the case of the first gravitational wave electromagnetic counterpart event (GW 170817), the localization of GRB 170817A given by Fermi Gamma-ray Burst Monitor (GBM) and INTEGRAL/SPI-ACS (Abbott et al. 2017a,b; Goldstein et al. 2017; Savchenko et al. 2017; Li et al. 2018) helped to establish the association between GW 170817 and GRB 170817A, and guide the follow-up observations of this GW source in multi-wavelength.

Unlike soft X-rays, gamma-rays are very difficult to focus on for imaging. Thus various methods are proposed to localize gamma-ray transients with different kinds of instruments: (1) Counts distribution among detectors used by all-sky Gamma-ray monitors (Mazets & Golenetskii 1981; Pendleton et al. 1999; Briggs et al. 2009; Mee-
Table 1. Statistical frameworks comparison between the $\chi^2$ method (e.g. Fermi/GBM DoL algorithm) and Bayesian method (e.g. BALROG algorithm) for localization.

| Statistical Framework | $\chi^2$ method | Bayesian method |
|-----------------------|-----------------|----------------|
| $\chi^2$ Minimization  | Minimize $\chi^2$ | Maximize Poisson, Gaussian or PGSTAT² |
| with Approximate Solution | $\chi^2$ Minimization with Approximate Solution | Bayesian credible region with HPD4 |
| $\chi^2$ Minimization with Numerical Solution | $\chi^2$ Minimization with Numerical Solution | Bayesian credible region with HPD4 |

1 The Poisson data with Gaussian background (PGSTAT) statistic.
2 Highest Posterior Density (HPD).
3 The error estimation is for the localization error region.

Table 2. The localization methods described in Section 2.

| Abbreviation | Localization Method | Statistics | Error Region Estimation | Description |
|--------------|---------------------|------------|-------------------------|-------------|
| B⁴⁰⁸⁹      | Bayesian with Poisson Likelihood | Poisson Likelihood | Posterior + HPD Credible Region | Section 2.1.1 |
| B⁵⁴⁶⁶      | Bayesian with Simplified Gaussian Likelihood | Simplified Gaussian Likelihood | Posterior + HPD Credible Region | Section 2.1.2 |
| $\chi^2_{GBM}$ | $\chi^2$ Minimization with Approximate Solution | $\chi^2$ Minimization with Approximate Solution | Posterior + HPD Credible Region | Section 2.2.1 |
| $\chi^2_{MIN}$ | $\chi^2$ Minimization with Numerical Solution | $\chi^2$ Minimization with Numerical Solution | Posterior + HPD Credible Region | Section 2.2.2 |

Table 3. Characteristics of the burst used in the localization simulation. The incident angle is Zenith = 5.85°, Azimuth = 22.50° in GBM’s spacecraft coordinates which corresponds to RA = 184.65°, DEC = -67.72° (true position) at 2021-01-01T01:00:00 UTC. The fixed background level is set to 1000 counts/s for each detector. Fluence is calculated in 10-1000 keV.

| Source Intensity Type | Medium Bright Burst |
|----------------------|--------------------|
| Spectral Model       | Comptonized        |
| Spectral Index       | -1.50              |
| $E_{peak}$ (keV)     | 200                |
| Duration (s)         | 10.0               |
| Fluence (erg/cm²)    | $1.2 \times 10^{-5}$ |

et al. 2020). This approach fits the counts distribution in different detectors with several localization templates which are made from several fixed spectra. Obviously, the real spectrum of the burst could usually differ from that of the fixed templates, leading to systematic errors. (2) Localization with the Bayesian method and fitting the burst location and spectrum simultaneously, represented by the BAyesian Location Reconstruction Of GRBs (BALROG) algorithm (Burgess et al. 2017; Berlato et al. 2019) and an MCMC-based localization algorithm developed for Gravitational Wave High-energy Electromagnetic Counterpart All-sky Monitor (GECAM) (Luo et al. 2020).

For the Bayesian method, BALROG (Burgess et al. 2017; Berlato et al. 2019) showed a notable improvement over the DoL algorithm (i.e. $\chi^2$ minimization) both in accuracy and precision for localization. However, Goldstein et al. (2020) compared the accurateness and robustness of the BALROG algorithm and the updated GBM Team’s official automated system (RoboBA) based on the DoL algorithm and found that the updated RoboBA is more accurate for the selected GRBs and that some technical problems of BALROG algorithm such as convergence and sensitivity issues. Initiated by the debates between these two localization methods (Burgess et al. 2017; Berlato et al. 2019; Goldstein et al. 2020), here we investigate the difference between these two methods with a focus on their statistical frameworks, i.e. Bayesian method and $\chi^2$ minimization, with detailed formula derivation and simulations. In the second part of this paper, we propose a location-spectrum iteration approach based on the Bayesian inference, to mitigate the imperfection of the methods with fixed templates.

This paper is structured as follows: The localization principle of Bayesian methods and $\chi^2$ minimization methods are introduced in Section 2 and validated in Section 3. In Section 4, The location-spectrum iteration localization strategy is described. Finally, a summary is given in Section 5.

2 BAYESIAN METHOD AND $\chi^2$ MINIMIZATION METHOD

As mentioned above, localization methods of high-energy transients based on the counts distribution are generally represented by the GBM team’s DoL algorithm (Connaughton et al. 2015) and BALROG algorithm (Burgess et al. 2017; Berlato et al. 2019), as listed in Table 1. One of the critical differences between them is the statistical framework: $\chi^2$ for DoL and Bayesian for BALROG.

Here, for clarity, we just compare their statistical frameworks without getting the very detailed treatments used by these two methods. To make a fair comparison, we construct the Bayesian method with the same fixed spectral templates as the $\chi^2$ (DoL) method. Also, the treatment of background estimation is also simplified for both methods, that is, we assume the expectation of the background is precisely known. We note that this assumption is certainly invalid for real observation, however, it is appropriate and helpful for our study here which only focus on the investigation of the differences of these two statistical frameworks (i.e. $\chi^2$ and Bayesian).

2.1 Bayesian Methods

In analog to the DoL $\chi^2$ minimization method (see Section 2.2.1) with fixed spectral templates, we construct the Bayesian method using the fixed spectral templates with likelihood maximization for each incident direction and with the assumption of the known background. In the following, we present this Bayesian method invoking the Poisson likelihood and simplified Gaussian likelihood, respectively.

2.1.1 Bayesian Method with Poisson Likelihood

If the background is known precisely, the observational data (i.e. counts in a given detector) follow a simple Poisson distribution:
The cumulative probability

\[ P_{\text{Poisson}}(S | B, M) = \frac{(B + M)^S \exp(-(B + M))}{S!} \]  

(1)

where \( B \) is the expected background counts, \( M \) is the expected source counts, and \( S \) is the measured counts. The sum of the background \( B \) and source \( M \) is the expected value of the measured counts. Based on this Poisson distribution, the Poisson likelihood and its logarithmic form for fixed spectral templates localization method can be written as:

\[ \mathcal{L}(i) = \prod_{j=1}^{n} \frac{(b_j + f_i \cdot m_{j,i})^{s_j} \exp(-(b_j + f_i \cdot m_{j,i}))}{s_j!} \]  

(2)

\[ \ln \mathcal{L}(i) = \sum_{j=1}^{n} [s_j \cdot \ln(b_j + f_i \cdot m_{j,i}) - (b_j + f_i \cdot m_{j,i}) - \ln s_j!] \]  

(3)

where \( s_j \) is the total observed counts in detector \( j \), \( n \) is the total number of detectors and \( b_j \) is the expectation value of the background. Here we use \( f_i \cdot m_{j,i} \) as the expected source contribution, where \( m_{j,i} \) is the localization template of a specific spectrum, which is a matrix of counts of each detector \( j \) for each incident direction \( i \) (the whole sky is pixelized with HEALPix), and \( f_i \) is the normalization factor to account for the fluence ratio between the real burst and the preset fixed burst spectrum used to generate the template \( m_{j,i} \).

We note that, in the localization of real burst observations, the expected value of the background is unknown. We can obtain the estimated background \( \hat{B} \) and its uncertainty \( \sigma_B \) from background analysis (e.g., the polynomial fitting to the background intervals), and these background uncertainties should be considered in the likelihood of Poisson data. In this case, the Poisson data with Gaussian background (PGSTAT) statistic can be utilized, e.g., XSPEC (Arnaud et al. 2003) and the BALROG algorithm (see also Equation 9 to 10 in Burgess et al. 2017).

During the localization process with fixed templates, the \( f_i \) could be derived from the maximization for each direction \( i \), thus the burst position (i.e., direction \( i \)) is the only parameter of interest, whose prior could be assumed to be uniform all over the celestial sphere: \( P_{\text{prior}}(i) = \frac{1}{N} \), where \( N \) is the total number of the HEALPix pixels of all sky. With the parameter prior and likelihood as shown in Equation 3, the location results (location center, probability map, and credible region) could be derived through the Bayesian inference.

We summarize this Bayesian localization method based on Poisson likelihood (denoted as \( B_{\text{POIS}} \) hereafter) as follows:

**Step 1:** For each incident direction \( i \), maximize the likelihood \( \mathcal{L}(i) \) by adjusting the normalization factor \( f_i \). The maximization of likelihood (Equation 2) and logarithmic likelihood (Equation 3) are equivalent for this process.

![Figure 1](image)

**Figure 1.** Test results for medium bright bursts (Table 3) with simulations. The upper panels (a to d) show the statistical results (i.e., the fraction of bursts with the cumulative probability \( N\% \) of true position < the corresponding confidence level \( N\% \)) for 4 localization methods (see Table 2): (a) \( B_{\text{POIS}} \) (see Section 2.1.1), (b) \( B_{\text{GJ}} \) (see Section 2.1.2), (c) \( \chi^2_{\text{MIN}} \) (see Section 2.2.2), (d) \( \chi^2_{\text{GBM}} \) (see Section 2.2.1). The dashed line represents the one-one line. The confidence level is 10% to 90% step by 10% as well as 68.27%, 95.45% and 99.73%. The lower panels (e to h) show inspections of individual simulated bursts for the same localization method as the corresponding upper panels. The magenta and green lines mark 68.27% and 95.45% HPD credible regions, respectively. The purple cross and red star represent the location center and true position, respectively.

**Table 4.** Parameter evolution of the location-spectrum iteration for a simulated medium bright burst (Table 3), which is also shown in Figure 5 (b).

| Steps | Best Spectral Model | Index | \( E_{\text{peak}} \) (keV) | Offset (deg) |
|-------|---------------------|-------|-----------------------------|--------------|
| Step 1 | Comptonized | -1.50 | 200 | - |
| Step A | Comptonized | -1.15 | 350 | 5.68 |
| Step B | Comptonized | -1.15 ± 0.27 | 230 ± 38 | 3.71 |
| Step C | Comptonized | -1.33 ± 0.12 | 187 ± 21 | 2.00 |
Step 2: Calculate the posterior probability through Bayesian inference. Thus the posterior distribution, \( P(i|s) \), could be derived from the prior probability \( P_{\text{prior}}(i) \), conditional probability for a given direction \( i \) to obtain the observed counts \( s \) and evidence \( P(s) \):

\[
P(i|s) = \frac{P_{\text{prior}}(i) \cdot P(s|i)}{P(s)} = \frac{\frac{1}{N} \cdot P(s|i)}{\sum_{i'} \frac{1}{N} \cdot P(s|i')} = \frac{P(s|i)}{\sum_{i'} P(s|i')}
\]

By substituting the conditional probability \( (P(s|i)) \) with the likelihood \( (L(i)) \), one can get the posterior probability for each direction \( (i) \), which is also the localization probability map:

\[
P(i) = \frac{L(i)}{\sum_{i'} L(i')}
\]

Step 3: For simplicity, we take the direction with the maximum \( P(i) \) as the location center and the Bayesian credible region with \( N\% \) highest posterior density (HPD) as the \( N\% \) confidence interval of the burst position.
Figure 3. Test results for source-dominating weak bursts. The upper panels show the statistical results obtained with different localization methods (see Table 2) at ~570 total observed counts for 12 detectors (the brightest detector’s observed counts is ~80), including (a) \(B_{POIS}\) (see Section 2.1.1), (b) \(B_{SG}\) (see Section 2.1.2), (c) \(\chi^2_{MIN}\) (see Section 2.2.2), (d) \(\chi^2_{GBM}\) (see Section 2.2.1). The medium panels show the statistical results obtained with the different methods mentioned in the upper panels at ~100 total observed counts for 12 detectors (the brightest detector ~20 counts) and the lower panels show those results at ~20 total observed counts for 12 detectors (the brightest detector ~3 counts). The localization statistical error is reasonable for all methods at ~570 total observed counts, and all methods except for \(B_{POIS}\) start to deviate one-one line at ~100 total observed counts and then more severe at ~20 total observed counts. \(B_{SG}\) overestimates localization error regions while \(\chi^2_{MIN}\) and \(\chi^2_{GBM}\) underestimate localization error regions. However, \(B_{SG}\) and \(\chi^2_{MIN}\) are more closer to one-one line than original DoL method \(\chi^2_{GBM}\).

2.1.2 Bayesian Method with Simplified Gaussian Likelihood

In order to understand the GBM DoL’s \(\chi^2\) method (see Section 2.2.1) in the Bayesian framework, here we structure a simplified Gaussian likelihood. The original Gaussian distribution reads:

\[
P_{\text{Gaussian}}(x) = \frac{1}{\sqrt{2\pi}\cdot \sigma} \cdot \exp\left(-\frac{(x - (B + M))^2}{2\sigma^2}\right),
\]

where \(\sigma^2\) is the variance. Thus the Gaussian likelihood and its logarithmic form can be written as follows:

\[
\mathcal{L}_G(i) = \prod_{j=1}^{n} \frac{1}{\sqrt{2\pi}\cdot \sigma_j} \cdot \exp\left(-\frac{(y_j - (b_j + f_i \cdot m_{j,i}))^2}{2\sigma_j^2}\right)
\]

\[
\ln \mathcal{L}_G(i) = \sum_{j=1}^{n} \left[ \ln \frac{1}{\sqrt{2\pi}\cdot \sigma_j} - \frac{(y_j - (b_j + f_i \cdot m_{j,i}))^2}{2\sigma_j^2} \right]
\]

Generally, the variance \(\sigma_j^2\) could be either data-dependent or model-dependent. However, to approximate the mathematical form of DoL’s \(\chi^2\) (see Section 2.2.1), here the variance is chosen to be model-dependent:

\[
\sigma_j^2 = b_j + f_i \cdot m_{j,i}
\]

Parameters are defined the same as in the Poisson case mentioned above.

Note that the variance is equal to the expectation as the counts
Figure 4. Test results for localization setting of energy channels integral and divided with $B_{\text{POISS}}$ (see Section 2.1.1). The upper panels show the comparisons of (a) the statistical results, (b) the distribution of offset between location center and truth location, and (c) the distribution of 68.27% HPD error region for the medium bright source as shown in Table 3. The channels integral results are the same as Figure 1 (a). The lower panels show the comparisons of (d) the statistical results, (e) the distribution of the offset between location center and truth location, and (f) the distribution of the 68.27% HPD error region for the source-dominating weak bursts. The channels integral results are the same as Figure 3 (i). The results indicate that the energy channels divided localization setting reduces the error region and offset between location center and truth location on the premise that the error obeys the statistics for the medium bright source and the source-dominating weak bursts.

Figure 5. Test results of the location-spectrum iteration localization. (a) Results for fixed templates method (blue) and location-spectrum iteration (red). (b) Evolution of the location results given by the location-spectrum iteration localization for a simulated medium bright burst (see also Table 4). The blue[green][magenta] represents the location center and the 68.27% credible region of Step A[B][C], respectively. Other captions are the same as Figure 1.

follow the Poisson distribution. Using Gaussian distribution to approximate Poisson is generally valid when the number of counts is large (i.e. > 15-20).

Because the model-dependent variance term (Equation 9) of Equation 7 and 8 for each direction generally approaches $s_j$, this term could be dropped out through the maximizing process, Equation 8 thus could be written as a simplified Gaussian likelihood form:

$$
\ln L_{SG}(i) = -\sum_{j=1}^{n} \frac{(s_j - (b_j + f_i \cdot m_{j,i}))^2}{2\sigma_j^2}
$$

(10)

Now, it is clear that maximization of this simplified Gaussian logarithmic likelihood is equivalent to $\chi^2$ minimization used by DoL (see Section 2.2.1). From the framework of likelihood, it is explicit that the normalization factor $f_i$ used in DoL is an approximate solution that maximizes simplified Gaussian logarithmic likelihood (see
Equation 10 in Blackburn et al. 2015), resembling the approximate solution for minimizing $\chi^2$. Owing to the difficulty to obtain the analytical solution of $f_i$ in Equation 10, we employ the Powell algorithm (Powell 1964; Press et al. 2007) to numerically calculate the $f_i$ for the maximum of the likelihood. This numerical solution can also be used for the $\chi^2$ minimization.

Once the simplified Gaussian logarithmic likelihood of each incident direction is calculated, the posterior probability and credible region could be derived as Section 2.1.1. This Bayesian method with simplified Gaussian likelihood is denoted as $\mathcal{B}_SG$ hereafter.

### 2.2 $\chi^2$ Minimization Methods

#### 2.2.1 $\chi^2$ Minimization with Approximate Solution

The $\chi^2$ employed by GBM team's DoL algorithm is defined as (see Equation A1 and A2 in Connaughton et al. 2015):

$$\chi^2(i) = \sum_{j=1}^{n} \frac{(s_j - (b_j + f_i \cdot m_{j,i}))^2}{b_j + f_i \cdot m_{j,i}}$$

(11)

where $s_j$ and $b_j$ are the total observed and estimated background counts observed in detector $j$ (between 50 and 300 keV for Fermi/GBM NaI detectors), respectively, $m_{j,i}$ are the model counts (i.e. localization template) in the same energy range for detector $j$ in direction $i$. The normalization factor $f_i$ for direction $i$ is defined as:

$$f_i = \frac{\sum_{j=1}^{n} m_{j,i}(s_j - b_j)}{\sum_{j=1}^{n} s_j}$$

(12)

Once the $\chi^2$ for the whole sky map is calculated, the contour of $\Delta \chi^2 = C$ is regarded as the N% statistical error region, where $C$ is the Percent Point Function (PPF) of $\chi^2$ distribution with degree of freedom 2 for N%, i.e. $\Delta \chi^2 = 2.3$ represents the 68% statistical uncertainty.

From the comparison between this $\chi^2$ method and the above Bayesian method ($\mathcal{B}_SG$), the normalization factor $f_i$ used by DoL is just an approximate solution to minimize $\chi^2$ (see Section 2.1.2). Also, this $\chi^2$ does not consider the uncertainties of the estimated background. Furthermore, the large number of counts is implicitly assumed since Gaussian distribution is used.

This $\chi^2$ method used by the GBM DoL algorithm is denoted as $\chi^2_{GBM}$ hereafter.

#### 2.2.2 $\chi^2$ Minimization with Numerical Solution

As mentioned above, Equation 12 is an approximate solution to minimize $\chi^2$ and it could be accurately calculated by numerical solution. Thus we propose a $\chi^2$ statistic in which the normalization factor $f_i$ comes from a numerical solution and other calculations are the same as Section 2.2.1. This $\chi^2$ method is denoted as $\chi^2_{MIN}$ in this paper.

The main technical details of these 4 localization methods using different statistical frameworks mentioned in this chapter are summarized in Table 2.

### 3 COMPARISON AND VALIDATION

To quantitatively evaluate and compare the above 4 localization methods, we conduct a Monte Carlo (MC) simulation\(^1\) to make tests.

Again, since we focus on the comparison between the localization methods using two statistical frameworks, $\chi^2$ and Bayesian, therefore, several treatments are employed for simplicity and clarity: (1) Both methods are based on the fixed template with the burst spectrum which is also used to make simulation source counts, and (2) The expectation value of the background is assumed to be known which allows us to eliminate the influence of background uncertainties. Such an effect is trivial for the present comparison study. (3) The Fermi/GBM detector configuration (i.e. 12 NaI detectors) and instrumental response are adopted, however, these localization methods are applicable for any other all-sky monitors of a similar design, such as GECAM.

Key parameters (such as the position, and spectrum of the burst source) used in the simulation are listed in Table 3. The simulated counts in each detector are derived from the Poisson fluctuation of the total expected counts, which are the expected counts of source contribution (i.e. burst spectrum convolved with the detector response) plus the expected background.

The localization simulation results for the medium bright burst (see Table 3 for burst parameters) are shown in Figure 1. To validate the location probability map and credible region, we check the distribution of the real burst position’s cumulative probability in the location maps for simulated bursts. This distribution check and the detailed inspections of location maps for individual simulated bursts show that all these 4 localization methods based on Bayesian and $\chi^2$ minimization can give consistent and correct location results (especially the localization error region) for medium bright bursts. This finding is understandable because the medium bright bursts could give a large number of counts in detectors, the Gaussian distribution could well approximate the Poisson distribution, and the $\Delta \chi^2$ could be used to derive the confidence region.

However, we find that some localization methods would fail to give a correct probability map when the burst becomes weak to some extent. Here we explore the burst intensity threshold where these methods give generally reliable localization results for two different cases: background-dominant case and source-dominant case.

For the background-dominant case (i.e. the background counts are much more than burst source counts), we take the GRB 170817A as the input burst. The time-integrated spectrum of GRB 170817A (Goldstein et al. 2017) is adopted (the total source counts in the brightest 3 detectors ~ 900) and the background level is set to 600 counts/detector in our simulations. We verified the localization results as the input burst intensity decreases. Here we define the Signal-to-Noise Ratio (SNR) as $\frac{\text{NetCounts}}{\sqrt{\text{Background}}}$ for the 3 brightest detectors.

As shown in Figure 2, when the SNR decreases to 9, all localization results of $\mathcal{B}$ and $\chi^2$ are basically around one-one line. But when SNR is ~ $8$ (i.e. the summed source counts in the brightest 3 detectors ~ 430 and the input burst intensity decreases to ~ 38% of GRB 170817A), the two $\chi^2$ methods start to deviate one-one line, while the two Bayesian methods still could give a correct localization probability map.

When the burst intensity decreased to SNR=3 (the summed source counts in the brightest 3 detectors ~ 130 which means ~ 14% of GRB 170817A), all methods failed to give a correct localization probability

\(^1\) The real observation suffers from unknown systematic errors and is thus not suitable for this test.
For each detector or divided to 8 energy channels for each detector. Although it cannot give a correct probability map <90% confidence level (CL), we note that $B_{POIS}$ and $B_{SG}$ could give a correct credible region at >90% CL, i.e. 95.45% and 99.73% CL. As the detection horizon of gravitational detectors increases, the detected GW events would be further and the GW-associated GRBs might be weaker, say ~14% to ~30% of GRB 170817A, for which the $\chi^2$ methods will underestimate the location error, thus we suggest that a credible region >90% CL of $B_{POIS}$ and $B_{SG}$ should be used to estimate the location error of these weak bursts.

For the source-dominating case (i.e. the background counts are less than the source counts), the background level of 0.5 counts/detector is adopted for simulations. As shown in Figure 3, when the total observed counts is ~570 for 12 detectors (the brightest detector’s observed counts is ~80), these 4 localization methods can obtain the correct localization map basically. But when the total observed counts decreased to ~100 for 12 detectors (the brightest detector’s observed counts is ~20), all methods start to deviate from one-one line except for $B_{POIS}$. $B_{SG}$ and $\chi^2_{MIN}$ are obviously closer to the one-one line than $\chi^2_{GBM}$. When the total observed counts decreased to ~20 for 12 detectors (the brightest detector’s observed counts is ~3), a correct localization map could also be obtained for $B_{POIS}$. We note that some very short duration (~ms) bursts, e.g. the Terrestrial Gamma-ray Flashes (Fishman et al. 1994; Roberts et al. 2018), could be seen in Zhang et al. (2018); Ambrogi et al. (2019); Tang et al. (2021); Bray et al. (2021) could reach such low counts.

From these tests, one can note that the performance of Bayesian methods is better than $\chi^2$ methods for the same inputs and settings. And the Bayesian method with Poisson likelihood is more applicable than Gaussian-based methods. Therefore, $B_{POIS}$ is recommended for localization. It should be noted that, as mentioned above, the simple Poisson likelihood (Equation 3) used in this paper should be replaced by the PGSTAT likelihood in real data for considering the background uncertainties (Burgess et al. 2017).

For the original DoL method $\chi^2_{GBM}$, its normalization factor $f_i$ is an approximate solution of $\chi^2$ minimization. As improvements, $\chi^2_{MIN}$ conducts the minimization with a numerical solution, and $B_{SG}$ enhances it under the statistical framework of Bayesian. Thus in comparison with $\chi^2_{GBM}$, the improvements of $\chi^2_{MIN}$ and $B_{SG}$ could be seen in the tests as shown in the background-dominant and source-dominant weak bursts localization. Besides, due to the Gaussian assumption of $\chi^2$ localization methods, they could only be used for those cases with sufficient counts in detectors, which means the burst should not be too weak (say <30% of GRB 170817A) or too short (say ~ms). We note that the localization capabilities depend on the bursts’ properties (e.g. spectrum), detector configuration, and incident angle, thus the above threshold for the correct location may vary with the instrument setting and incident angle of bursts.

There are many settings and choices in the localization analysis that may potentially affect the final results. They are usually complex and coupled together, such as the selection of detectors, choice of spectral channels binning, spectral models, iteration termination criteria, etc. Some of them have been discussed in previous studies (e.g. Burgess et al. 2017; Berlato et al. 2019). Here we explore how to divide energy channels to optimize the localization results with simulations. To estimate the influence on the localization caused by the different data binning strategies in spectral channels, we did simulations for the medium bright burst with two kinds of binning data in the whole energy range (i.e. from 8 to 1000 keV); just one whole energy channel for each detector or divided to 8 energy channels for each detector.

For the divided energy channels, the mathematical Poisson likelihood and logarithmic likelihood are:

$$\mathcal{L}_{\text{Pois}}(i) = \prod_{j} \prod_{k} \frac{(b_{j,k} + f_{j,k} \cdot m_{j,k,i})^{s_{j,k} \cdot \exp(-(b_{j,k} + f_{j,k} \cdot m_{j,k,i}))}}{s_{j,k}!}$$

(13)

$$\ln \mathcal{L}_{\text{Pois}}(i) = \sum_{j} \sum_{k} [s_{j,k} \cdot \ln(b_{j,k} + f_{j,k} \cdot m_{j,k,i})$$

$$- (b_{j,k} + f_{j,k} \cdot m_{j,k,i}) - \ln s_{j,k}!]$$

where $s_{j,k}$ and $b_{j,k}$ are the total observed counts and background for energy channel $k$ in detector $j$. $f_{j,k} \cdot m_{j,k,i}$ is the expected source contribution in a single channel.

4 LOCATION-SPECTRUM ITERATION LOCALIZATION

In real observation, using the fixed spectral templates is imperfect. An inevitable problem of the fixed templates localization strategy is that the spectra of preset templates usually differ from those of bursts, which may introduce substantial systematic errors.

To illustrate the performance of the fixed templates localization strategy, i.e. the deviation of location induced by the difference of spectrum, a simulation using $B_{POIS}$ localization method with fixed templates for the medium bright burst is implemented. As shown in Figure 5, the statistical distribution of the location results significantly deviate from the expected value with the error region being underestimated.

Since the fixed template localization has advantages in the calculation, to alleviate the above issues of fixed templates localization, we propose a location-spectrum iteration approach for $B_{POIS}$ localization method, as described below:

Step A: First, derive the initial localization result with preset localization templates, which is similar to previous studies (Briggs et al. 2009; Connaughton et al. 2015; Goldstein et al. 2020), but implemented by $B_{POIS}$ (i.e. the Bayesian localization method with fixed spectral template).

Step B: With the initial location from Step A, spectral analysis of the burst is implemented with all detectors. Then redo the localization with the updated location template which is calculated based on the burst spectrum. Subsequently, iterate the spectral fitting and localization. The purpose of selecting all detectors for the spectral fitting here is to account for a large deviation of the location. To achieve an appropriate spectrum, 4 spectral models are employed to fit independently: the Band function, the Comptonized, power law, and power law + black body model. The best model is selected by the Bayesian information criterion (BIC) (Schwarz 1978a). This iteration will terminate if the observed counts are consistent with the expected

$^2$ Due to Gaussian priors and a Gaussian posterior are required, BIC is not a perfect model selection tool, however, we note the similar process could also be seen in Zhang et al. (2018); Ambrogi et al. (2019); Tang et al. (2021); Wang et al. (2022).
Step C: Based on the location obtained from Step B, a refined spectral analysis is executed with a sample of good detectors which are selected based on preset criteria, including incident angle <60° and significance > 5σ. With the refined spectrum, the template spectrum will be updated and the final localization result is obtained. The terminated condition is the same as Step B.

To quantitatively estimate the performance of this location-spectrum iteration localization strategy, a MC simulation has been implemented. As shown in Figure 5, the localization result of the iteration strategy is significantly improved (compared to that of the fixed templates) and thus quite consistent with the expectation. Taking a burst for example, we traced the evolution of the spectral and location parameters during the iteration, as shown in Table 4 and Figure 5 (b). Both the spectrum and location center tend to converge to the true value as iteration goes. Although the final spectrum (derived in Step C) is not exactly the same as the input one (which is not surprising if think of the spectral fitting error), the final location map given by this method is statistically reliable and correct according to the statistical validation in the left panel of Figure 5.

5 DISCUSSIONS AND CONCLUSIONS

In this paper, the two main localization statistical framework, the Bayesian methods and \( \chi^2 \) minimization localization methods, are studied and compared in details. This comparison was done with dedicated simulations which eliminate bias and impacts introduced by the inaccuracies in detector response, background estimation and knowledge of burst spectrum. Reliability and correctness of the location results are validated by directly checking the confidence regions of the localization probability map through comprehensive simulations.

All 4 localization methods studied in this paper give similar and reliable location results for medium bright bursts. For source-dominant weak bursts, we find that only \( \beta \text{PoIS} \) could give a correct localization probability map which is useful for some short-bursts, i.e. TGFs. For background-dominant weak bursts, \( \beta \text{PoIS} \) and \( \beta \text{SG} \) could give a correct localization probability map at >90% HPD credible region. Therefore, the Bayesian method with Poisson likelihood is recommended than the \( \chi^2 \) minimization method. In real observations, the more sophisticated PGSTAT statistic should be used instead of the simple Poisson likelihood.

We find that, compared to the original DoL method \( \chi^2_{\text{GBM}}, \chi^2_{\text{MIN}} \) and \( \beta \text{SG} \) could improve the location results by numerical solution during maximization and utilization of the Bayesian inference. We also demonstrate that the mismatch of the burst spectrum and template spectrum will cause location deviation, which may increase the systematic error of localization.

We also proposed a Bayesian-based location-spectrum iteration localization method to take the advantages but alleviate the issues of the fixed spectral template method. Compared to the existing methods for optimizing spectrum (i.e. fixed templates and location-spectrum simultaneously fitting), our location-spectrum iteration localization strategy features the following advantages: (1) the mismatch between the spectrum of fixed localization templates and burst spectrum could be fairly eliminated, and (2) the calculation of localization process is straightforward and fast. However, it may still have convergence problem during iteration for some cases. Besides, we find that, dividing the counts into several bins would improve the localization results than treating these counts as a single bin.

At last, there are some open questions in the localization, e.g., how to get a reliable localization probability map for background-dominant weak bursts. As mentioned in Section 3, the current methods studied in this paper will no longer be able to give the reliable location error regions <90% confidence level for weak bursts. On the other hand, these weak bursts might be very important as they could be associated with Gravitational Waves or Fast Radio Bursts, thus a joint time-delay localization with multiple all-sky instruments is highly required to provide reliable location results (Xiao et al. 2022).

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DATA AVAILABILITY

The data and codes underlying this article will be shared upon reasonable request to the corresponding author.

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