Jamming in granular hopper flow

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Large-scale three dimensional molecular dynamics simulations of hopper flow are presented. The flow rate of the system is controlled by the width of the aperture at the bottom. As the steady-state flow rate is reduced, the force distribution $P(f)$ changes only slightly, while there is a large change in the impulse distribution $P(i)$. In both cases, the distributions show an increase in small forces or impulses as the systems approach jamming, the opposite of that seen in previous Lennard-Jones simulations. This occurs dynamically as well for a hopper that transitions from a flowing to a jammed state over time. The final jammed $P(f)$ is quite distinct from a poured packing $P(f)$ in the same geometry. The change in $P(i)$ is a much stronger indicator of the approach to jamming. The formation of a peak or plateau in $P(f)$ at the average force is not a general feature of the approach to jamming.

I. INTRODUCTION

The vertical pressure of grains in a silo becomes independent of depth for sufficiently tall silos $[1,2,3]$. The weight of sand in the column is ultimately supported by the lateral containing walls. Experiments $[4,5,6,7]$ and simulations $[8,9,10,11]$ all show that these stresses are transmitted in an inhomogeneous manner. One measure of the inhomogeneity in a granular pile is expressed through the distribution of normal forces, $P$mal forces, or "jammed" state by analyzing the change in inhomogeneities as one moves from one to the other. In the context of a general conception of jamming $[17]$, it has been conjectured $[18]$ that the $P(f)$ of a flowing system exhibits a decrease at small forces ($f < 1$) as the system moves from a flowing to a jammed state.

An ideal system to probe this transition is the hopper $[19]$. Its jammed state (a silo) has been extensively studied. For a granular packing of sufficient height, the pressure in the depth of the packing is independent of the height of the granular matter above it $[1]$. A granular silo is also constrained and well-controlled - there is no free surface aside from the top. A silo transitions to a hopper when an aperture at the base is opened. Numerous experiments and simulations have examined the transition to jamming with the goal of predicting its onset $[20,21,22]$. Here we focus specifically on the distribution of forces and impulses in the approach to jamming.

In experiments the forces between particles are difficult to measure directly $[23]$, so Longhi et al $[24]$ measured the distribution of impulses $P(i)$ (where the normalized impulse is $i = I/I$, $I$ is the impulse, and $⟨I⟩$ is the average impulse) between particles and the wall and related this distribution to the macroscopic behavior of the system as the system approached jamming. This quasi-2D experiment found that the impulse has an exponential tail at all flow rates and that at small flow rates, the distribution of $P(i)$ increases with decreasing flow rate. The distribution of times between collisions tends to a power law $P(τ_c) \approx τ_c^{-3/2}$, that is, the mean time interval tends to diverge just as in the glass transition $[24]$. Ferguson et al found using 2D event-driven frictionless simulations $[25]$ that $P(i)$ exhibits an increase in small impulses ($i < 1$) and argue that this arises from the contribution of “frequently-colliding” particles that are spatially correlated into 1D collapse strings.

Here we present fully 3D molecular dynamics simulations of hopper flow. We measure the distributions of normal forces $P(f)$ and impulses $P(i)$ for the system as the system approaches jamming and relate our results to previous experiments and 2D simulations. We find that $P(f)$ is not a strong indicator of jamming, and that it exhibits an increase at small forces as the system approaches jamming, exactly the opposite of the predicted behavior from earlier simulations $[18]$. $P(f)$ of a jammed system is markedly different from a poured packing in a similar geometry, showing that $P(f)$ can depend strongly on the history of the system. We also find that $P(i)$ exhibits an increase at small impulses, in agreement with previous experiments $[24]$ and simulations $[25]$. 

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II. SIMULATION METHOD

Our geometry was inspired by that used in the quasi-2D experiments: a conical hopper [20]. The system is periodic in the z direction: particles that fall out of the bottom of the system reappear at the top and refill the container, creating a steady-state flowing system. The system is shown in Figure 1(a). The top of the hopper is a cylindrical container with radius \( R = 10d \), where \( d \) is the particle diameter. At the bottom of this cylindrical region, a cone is joined to the cylinder, with radius varying smoothly from \( R \) to \( r_f \) over a distance \( z_f = 50d \). The angle \( \theta \) between vertical and the funnel is \( \theta = \tan^{-1} \left( \frac{R}{r_f} \right) \). This system tends to jam for \( r_f < 1.5d \). We deliberately make the cylindrical region very deep to assure that in the static case the pressure is independent of depth. Those particles that leave the hopper opening and then rain down on the top of the pile quickly lose their kinetic energy and do not affect the flow through the opening. The system is prepared by pouring particles into the top of the container with aperture radius \( r_f = d \) and a plate at the bottom to prevent the exit of particles. After the packing is formed, the plate is removed and the aperture widened to the desired radius. All measurements are taken after the system has cycled particles through at least once and is in a steady state unless specifically noted otherwise. We focus our attention on the funnel region, and unless otherwise stated, all measurements are only of particles in that region.

We use a molecular dynamics code with a Hertzian force law that is described in detail elsewhere [4, 27]. All results will be presented in dimensionless units based on the mass of a particle \( m \), the diameter of a particle \( d \), and the force of gravity \( g \). In this case, the simulations used \( N = 40000 \) monodisperse particles of diameter \( d \). The unit of time is \( \tau = \sqrt{d/g} \), the time it takes a particle to fall its radius from rest under gravity. Most of the simulations were carried out with a spring constant \( k_n = 2 \times 10^5 mg/d \), for which the simulation time step \( \delta t = 10^{-4} \tau \). The particle-particle friction and particle-wall friction are the same: \( \mu = \mu_w = 0.5 \).

The flow profile is dependent on the geometry of the system, as shown in Fig. 2. The flow is essentially plug-like and constant in the cylindrical region of the hopper for particles more than one particle away from the wall. A one-particle thick layer near the wall flows more slowly due to friction with the wall. This plug breaks up in the funnel region, with particles in the center of the flow flowing more quickly and accelerating as they approach the hopper opening.

Experimental measurements have shown that hoppers produce a mass flow rate dependence of \( M = (r_f - wd)^{5/2} \), where \( w \) is a constant related to the shape of the particles [28]. We see a similar dependence in our simulations, with \( w = 1.2 \), consistent with the experimental measurement of \( w \approx 1.5 \) [28].

III. FORCES AND IMPULSES

The distribution of particle-particle normal forces \( P(f) \) for the funnel region is shown in Fig. 3 for a variety of aperture radii \( r_f \). \( P(f) \) is exponential at large forces, just as observed in static piles [3, 4, 12, 13, 14]. All the flowing states of the system have similar distributions. In Fig. 3 (inset) we show a closeup of \( P(f) \) for a range of flowing states. There is a slight change in \( P(f) \) as the system approaches jamming — it displays a slight increase at small forces. The peak of the distribution moves to smaller forces as the system approaches jamming. This behavior is the opposite of the behavior seen in 2D Lennard-Jones simulations [18]. This change in \( P(f) \) is robust and suggests that the behavior of \( P(f) \) in a jammed system is dependent on the specific physics of the interactions between particles.

We also analyzed the dynamical nature of \( P(f) \) for \( r_f = 1.4d \). In this case, the system flows freely for a short period and then jams. In Fig. 4 \( P(f) \) is shown for this system as a function of time. As the jam forms, the distribution abruptly increases at small forces [20]. This behavior is compared to that of a poured system with \( r_f = d \) that never flows after the aperture is opened. In that case, \( P(f) \) is lower at small forces than the flowing state. The formation of a jam is history-dependent, and the static packing that results from a jam is fundamentally distinct from a poured packing. \( P(f) \) in a jammed packing is strongly dependent on the dynamic processes used to create the jam and is not equivalent to the \( P(f) \) of a granular packing created with an alternate method. There is no general \( P(f) \) that one should expect for all jammed packings.

This hysteretic sudden change in \( P(f) \) has also been seen in reverse in shearing experiments in cylinders. In that case, \( P(f) \) is essentially unchanged until yield stress is attained, at which point \( P(f) \) changes discontinuously to a new form [30].

We can also probe the dependence of \( P(f) \) on the particle interactions. We carried out additional simulations with a much softer normal spring constant, \( k_n = 2 \times 10^5 mg/d \). Experiments have shown that very soft particle interactions can have a strong effect on \( P(f) \) [12]. We compare the behavior of \( P(f) \) for the two systems in Figure 4. The trend of increasing the \( P(f) \) at small forces is the same in both systems, but the increase is more pronounced and more localized at small forces for the stiffer springs. \( P(f) \) is thus sensitive to the compressibility of the particles and softness of the particle interactions.

These analyses provide a clue to the discrepancy between our simulations and those of O’Hern et al [18] and others. Two possible effects separate our simulations and theirs. The first and more important effect arises from the ahistorical nature of the O’Hern simulations, in which the packings are produced by conjugate gradient methods, which have no information on the dynamics of the transition from flowing state to jammed state. As demonstrated above, in granular materials the history of the
FIG. 1: (color online) (a) Geometry of the system. The total height of the system is $160d$, with the freefall region roughly $20d$ in height, the cylindrical region $90d$, and the funnel region $50d$. The bottom aperture radius $r_f$ is varied to control the flow. In this case, $r_f = 3d$. All particles with $z$ positions $120d < z < 130d$ are colored gray (green). (b) Only the tracer particles after time $t = 300\tau$. (c) Tracer particles only at $t = 580\tau$. (d) Tracer particles only at $t = 680\tau$.

FIG. 2: Velocity profiles for $v_z$ at a number of heights $z$ in a flowing hopper with $r_f = 3d$. The profile with the largest magnitude was taken at $z = 10d$, and at every $5d$ afterward, up to a maximum of $130d$. $|v_z|$ decreases as the funnel widens with increasing height, until one reaches the cylindrical region, where it is roughly constant.

FIG. 3: Distribution of normal forces $P(f)$ for flowing hopper systems with various apertures $r_f$. $P(f)$ increases slightly at small forces as the system approaches jamming. Inset: closeup of $P(f)$ at small forces. The lines are guides to the eye.

system cannot be ignored, and the $P(f)$ of a jammed state is quite distinct from that of a poured state. The $P(f)$ obtained for the repulsive interactions by O’Hern et al more closely resemble those for static granular packings and are in fact produced by the quasi-static method of conjugate gradient. In our simulations a poured packing does exhibit a deficit at small forces in relation to flowing systems, but all “jammed” systems clearly do not exhibit the same characteristics. A poured packing
In summary, our simulations show that for a granular hopper system with history-dependent interactions and friction, \( P(f) \) exhibits an increase in small forces \((f < 1)\) as the system approaches jamming. This increase is robust and hysteretic; the \( P(f) \) of a jammed packing is distinct from the \( P(f) \) of a poured packing. The behavior of \( P(f) \) is somewhat dependent on the interaction: softer
particle interactions tend to diminish this effect. The history dependence of granular materials distinguishes our simulations from many 2D simulations [18], which have no memory of dynamics, and thus predict a general behavior of $P(f)$ that is exactly the opposite of that observed in our simulations. The trend is the same for the distribution of impulses $P(i)$, which see an increase in small impulses ($i < 1$) as the system approaches jamming. This trend is also observed in experiments and quasi-2D simulations. The behavior of $P(f)$ does not appear to be a general feature of jamming, but instead depends on the particle interactions and the hysteretic quality of the system.

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[1] H. A. Janssen, Z. Ver. Dt. Ing 39, 1045 (1895).
[2] H. M. Jaeger, S. R. Nagel, and R. P. Behringer, Rev. Mod. Phys. 68, 1250 (1996).
[3] L. Vanel, P. Claudin, J.-P. Bouchaud, M. Cates, E. Clément, and J. P. Wittmer, Phys. Rev. Lett. 80, 1430 (2000).
[4] D. Howell, R. P. Behringer, and C. Veje, Phys. Rev. Lett. 82, 5241 (1999).
[5] C. h Liu, S. R. Nagel, D. A. Schecter, S. N. Coppersmith, S. Majumdar, O. Narayan, and T. A. Witten, Science 269, 513 (1995).
[6] D. L. Blair, N. W. Mueggenburg, A. H. Marshall, H. M. Jaeger, and S. R. Nagel, Phys. Rev. E 63, 041304 (2001).
[7] J. Geng, D. Howell, E. Longhi, R. P. Behringer, G. Rey-dellet, L. Vanel, E. Clément, and S. Luding, Phys. Rev. Lett. 87, 035506 (2001).
[8] S. N. Coppersmith, C. h Liu, S. Majumdar, O. Narayan, and T. A. Witten, Phys. Rev. E 53, 4673 (1996).
[9] J. W. Landry, G. S. Grest, L. E. Silbert, and S. J. Plimpton, Phys. Rev. E 67, 041303 (2003).
[10] H. A. Makse, D. L. Johnson, and L. M. Schwartz, Phys. Rev. Lett. 84, 4160 (2000).
[11] L. E. Silbert, G. S. Grest, and J. W. Landry, Phys. Rev. E 66, 061303 (2002), cond-mat/0208462.
[12] J. M. Eriksson, N. W. Mueggenburg, H. M. Jaeger, and S. R. Nagel, Phys. Rev. E 66, 040301 (2002).
[13] G. Løvoll, K. J. Måløy, and E. G. Flekkøy, Phys. Rev. E 60, 5872 (1999).
[14] J. Brujić, S. F. Edwards, D. V. Grinev, I. Hopkinson, D. Brujić, and H. A. Makse, Faraday Discussions 123, 1 (2002).
[15] J. H. Snoeijer, M. van Hecke, E. Somfei, and W. van Saarloos, pre 70, 011301 (2004).
[16] P. A. Metzger, Phys. Rev. E 70, 051303 (2004).
[17] A. J. Liu and S. R. Nagel, Nature (London) 396, 21 (1998).
[18] C. S. O’Hern, S. A. Langer, A. J. Liu, and S. R. Nagel, Phys. Rev. Lett. 86, 111 (2001).
[19] H. P. Zhu and A. B. Yu, Granular Matter (2005).
[20] K. To and P.-Y. Lai, pre 66, 011308 (2002).
[21] K. To, P.-Y. Lai, and H. K. Pak, prl 86, 71 (2001).
[22] I. Zuriguel, Luis A Pugnaloni, A. Garcimartín, and D. Maza, pre 68, 030301 (2003).
[23] In some cases, forces can be obtained through careful use of the impulse and timing data. Work is underway on this problem.
[24] E. Longhi, N. Easwar, and N. Menon, Phys. Rev. Lett. 89, 045501 (2002).
[25] A. Ferguson, B. Fisher, and B. Chakraborty, Europhysics Letters 66, 277 (2004).
[26] R. M. Nedderman, Statics and Kinematics of Granular Materials (Cambridge University Press, 1992).
[27] L. E. Silbert, D. Ertaç, G. S. Grest, T. C. Halsey, D. Levine, and S. J. Plimpton, Phys. Rev. E 64, 051302 (2001).
[28] R. M. Nedderman, U. Tüziün, S. B. Savage, and G. T. Houlsby, Chemical Engineering Science 37, 1597 (1982).
[29] Similar behavior was observed for other flowing apertures slightly less than $r_f = 1.5d$.
[30] E. Corwin, private communication.
[31] C. Denniston and H. Li, Phys. Rev. E 59, 3289 (1999).
[32] The distribution of impulses for an essentially static system is difficult to interpret.
[33] There is some evidence to suggest that the $P(f)$ distribution for flowing 3D experimental systems is also slightly non-exponential.
[34] N. Easwar, private communication.