RESEARCH PAPER

Solving Linear Volterra Integro-Fractional Differential Equations in Caputo Sense with Constant Multi-Time Retarded Delay by Laplace Transform

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ABSTRACT:
In this paper presents Laplace transform methods for the first time to solve linear Volterra integro-differential equations of the fractional order in Caputo sense with constant multi-time Retarded delay. This method can be easily handling many linear Volterra problems and is capable of reducing computational analytical works where the kernel of difference and simple degenerate types. Analytical examples are presented to illustrate the efficiency and accuracy of the proposed methods.

KEY WORDS: Caputo fractional derivative, Integro-differential equation, Delay differential equations, Laplace transform, Difference and Simple Degenerate Kernels.
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1. INTRODUCTION:
The idea of this work is to solve linear Volterra Integro-Fractional Differential Equations (VIFDE’s) in Caputo sense with constant multi-time Retarded Delay (RD) in the general form:

\[
c_{0}^{a}D_{t}^{\alpha_{1}}u(t) + \sum_{i=1}^{m} P_{i}(t) \ c_{0}^{a}D_{t}^{\alpha_{i}}u(t) + P_{0}(t)u(t-\tau_{j}) = f(t) + \lambda \int_{0}^{t} \mathcal{K}_{j}(t,x) u(x-\tau_{j})dx,
\]

For \( \alpha_{n} > \alpha_{n-1} > \alpha_{n-2} > \alpha_{n-3} > \cdots > \alpha_{1} > \alpha_{0} = 0 \), with initial conditions which are given: \( u^{(k)}(0) = u_{k} \); \( k = 0,1,2,\ldots,\mu - 1 \) and historical \( \mu - th \) continuity differentiable functions \( u(t) = \varphi(t) \) for \( t \in [\bar{a}, 0] \), where \( \bar{a} = \max\{\tau_{i}; j = 1,2,\ldots,m\} \) and \( \mu = [\alpha_{n}] \). Connected, where \( u(t) \) is the unknown function which is the solution of equation (1) and \( \mathcal{K}_{j} \in C(\mathbb{R}^{n}, \mathbb{R}) \), and \( S = \{(t,x): 0 \leq x \leq t \leq b\} \) for all \( j = 1,2,\cdots,m \) given and \( f,P_{i} \in C([0,b], \mathbb{R}) \), for all \( i = 0,1,\ldots,n - 1 \) where \( u(t) \in \mathbb{R} \), \( c_{0}^{a}D_{t}^{\alpha_{i}}u(t) \) is the \( \alpha_{i}\)-fractional Caputo-derivative order of \( u \) on \( [0,b] \) and all \( \alpha_{i} \in \mathbb{R}^{+} \) for \( (i \neq 0) \), \( n_{\alpha_{i-1}} < \alpha_{i} \leq n_{\alpha_{i}}, n_{\alpha_{i}} = [\alpha_{i}], \) for all \( i = 1,2,\ldots,n \). Furthermore, the quantities \( \tau_{j} \in \mathbb{R}^{+} \) for all \( j = 1,2,\ldots,m \) are called the delay or time-lags, (Miran B. M. Amin, 2016).

Laplace transform is a very useful method for solving certain initial and boundary value problems associated with differential equations and partial differential with constant coefficients also this method is used in solving different types of equations such as integral equations, integro-
differential equations and fractional differential equations and Delay Differential equations (Abdul, 1985; Daniel, 1997; Dr. Muna, 2008; I. podlubny, 1999; Shazad, 2002; Shazad, 2009; Shokhan, 2011; Talhat, 2016; Rostam, 2016), here we discuss the transformation Laplace operator techniques that how can be used for solving the LVIFDE’s of RD that is expresses in equation (1). Before describing the Laplace transform technique, it is necessary to define and explain some important properties of Laplace method, and the way to drive Laplace transform for Delay functions and the Laplace transform of Caputo fractional derivative.

This paper is organized as follows: Section 2 presents Definition and some important property; section 3 Solve Linear Volterra Integro Differential Equation of Fractional Order with Constant multi-time Retarded Delay using Laplace Transforms Technique; our results illustrated throughout examples in section 4. Finally, section 5 includes a discussion for this method.

2. PRELIMINARIES AND PROPOSITIONS:

For this section we present the necessary information’s from fractional calculus and Laplace operator, this information’s are use in our suggested procedure to solve our problem, (1).

2.1 Fractional Calculus

For completeness, this part introduces the necessary definitions and important properties of fractional calculus theory, which are used throughout this paper. We begin by defining the function space which was used in development of the operational calculus for the differential operator. For more details, see (Anatoly et al., 2006; I. podlubny, 1999; Kelth, et al. 1974, Kenneth et al. 1993; SHAZAD, 2009):

Definition 1:
A real valued function u defined on [a, b] be in the space \( C_\delta \), -any real number, if there exists a real number \( \ell > \delta \), such that \( u(t) = (t-a)^\ell u_c(t) \), where \( u_c \in C[a,b] \), and it is said to be in the space \( C_\delta \), if and only if \( u^{(m)} \in C_\delta \), \( n \)-positive integer number with zero.

Definition 2:
Let \( u \in C_\delta \), \( \delta \geq -1 \) with any positive arbitrary real number \( \alpha \). Then the Riemann-Liouville fractional integral operator \( aI_t^\alpha \) of order \( \alpha \) of a function \( u \), is defined as:
\[
ad I_t^\alpha u(t) = \begin{cases} \int_a^t \frac{(t-\xi)^{\alpha-1}}{\Gamma(\alpha)} u(\xi) \, d\xi, & \alpha > 0 \\ u(t) \text{ whenever } \alpha = 0 \end{cases}
\]

Definition 3:
Let \( \alpha \geq 0 \), and \( m = [\alpha] \). the Riemann-Liouville fractional derivative operator \( R D_t^\alpha \), of order \( \alpha \) and \( u \in C_m^m[a,b] \) and defined as:
\[
R D_t^\alpha u(t) = \begin{cases} D_t^{m-\alpha} [aI_t^{m-\alpha} u(t)], & \alpha > 0 \\ u(t) \text{ whenever } \alpha = 0 \\ u^{(m)}(t), \text{if } \alpha = m \in \mathbb{N} \text{ and } u \in C^m[a,b] \end{cases}
\]

Definition 4:
The Caputo fractional derivative operator \( C D_t^\alpha \) of order \( \alpha \in \mathbb{R}^+ \) of a function \( u \in C^m[a,b] \) and \( m - 1 < \alpha \leq m \), \( m \in \mathbb{N} \) is defined as:
\[
C D_t^\alpha u(t) = \begin{cases} aI_t^{m-\alpha} [D_t^m u(t)], & \alpha > 0 \\ u(t) \text{ whenever } \alpha = 0 \\ u^{(m)}(t), \text{if } \alpha = m \in \mathbb{N} \text{ and } u \in C^m[a,b] \end{cases}
\]

Note that:
i. For \( \alpha \geq 0 \) and \( \beta > 0 \), then \( aI_t^\alpha (t-a)^{\beta-1} = \frac{\Gamma(\beta)}{\Gamma(\beta+\alpha)} (t-a)^{\beta+\alpha-1} \).

Note that:
i. For all \( \alpha \geq 0 \), \( \beta > 0 \) and \( u(t) \in C_\delta[a,b], \delta \geq -1 \), then:
\[
ad I_t^\alpha aI_t^\beta u(t) = aI_t^\alpha (t-a)^{\beta+\alpha-1} u(t)
\]

Note that:
i. Let \( u \in C^m[a,b] \), \( m \geq [\alpha] \), then:
\[
ad I_t^\alpha u(t) = aI_t^{\alpha+m} u(t) = aI_t^{\alpha+m} u(t)
\]

Note that:
i. Assume that \( u \in C_m^m[a,b] \) and \( C D_t^\alpha u(t) \), \( \alpha \geq 0 \), \( \alpha \in \mathbb{N} \) and \( m = [\alpha] \) then \( C D_t^\alpha u(t) \) is continuous on \([a,b]\), and \( C D_t^\alpha u(t) \) is any constant.

Note that:
i. Let \( \alpha \geq 0 \), \( m = [\alpha] \) and \( u \in C^m[a,b] \), then, the relation between the Caputo derivative and R-L integral are formed:
\[
C D_t^\alpha [aI_t^\alpha u(t)] = u(t)
\]

Note that:
i. For power function \( u(t) = (t-a)^k \), \( k! \) denotes the Taylor polynomial of degree \( m - 1 \) for the function \( u \), centered at \( a \).

Note that:
i. For power function \( u(t) = (t-a)^\beta \) for some \( \beta \geq 0 \). Then:
We adopt Caputo’s definition, which is a modification of the R-L definition and has the advantage of dealing properly with initial value problem, for the concept of the fractional derivative.

**Definition 5:** (RUDOLF et al, 2000)

The Laplace transforms of a function $u(t)$ of real variable $t \in \mathbb{R}^+$, denoted by $U(s)$, is defined by the equation

$$U(s) = \mathcal{L}\{u(t); s\} = \int_0^{\infty} e^{-st} u(t) \, dt \quad (2)$$

and its inverse is given for $t \in \mathbb{R}^+$ by the formula, symbolically written as: $\mathcal{L}^{-1}\{U(s); t\} = u(t)$

The Laplace transform has several properties which are important for our work, as explained below, from Lemmas (1-4) and lemma (5-i, 5-ii) are can be founding in references (Abdul, 1985; Mariwan, 2013; Murray, 1965; Peter, 1985; Shokhan, 2011) and (I. podlubny, 1999; Kenneth et al. 1993), respectively:

**Lemma 1:**

The Laplace transform is related to the transform of the $n$-th derivative of a function, where $U(s)$ is a Laplace of $u(t)$:

$$\mathcal{L}\left\{ \frac{d^n u(t)}{dt^n} \right\} = s^n U(s) - \sum_{k=0}^{n-1} s^{n-k-1} u^{(k)}(0)$$

where $u^{(k)}(0) = \frac{d^k u(t)}{dt^k} \bigg|_{t=0}$

**Lemma 2:**

The Laplace transform of the convolution of two functions is the product of their Laplace transforms. Thus $U(s)$ and $H(s)$ are the Laplace transforms of $u(t)$ and $h(t)$ respectively, then:

$$\mathcal{L}\{u(t)*h(t)\} = \mathcal{L}\left\{ \int_0^t u(t-x) h(x) \, dx \right\} = U(s)H(s) \quad (4)$$

If $U(s) = \mathcal{L}\{u(t)\}$, then:

$$\mathcal{L}\left\{ \int_0^t u(x) \, dx \right\} = \frac{1}{s} U(s) \quad (5)$$

**Lemma 3:**

If $U(s)$ is the Laplace of $u(t)$ and $t^n$ is a power function of order $n \in \mathbb{Z}^+$, then:

$$\mathcal{L}\{t^n u(t)\} = (-1)^n \frac{d^n}{ds^n} U(s) \quad (6)$$

**Lemma 4:**

Let $U(s)$ be the Laplace of $u(t)$ then:

$$\mathcal{L}\left\{ \int_0^t u(x) \, dx \right\} = -\frac{d}{ds} \left( \frac{1}{s} U(s) \right) \quad (7)$$

**Lemma 5:**

The Laplace transform of the R-L Fractional integral for order $\alpha \in \mathbb{R}^+$ using the convolution property (ii, 4), gives:

$$\mathcal{L}\{I^\alpha_t u(t)\} = \mathcal{L}\left\{ \int_0^{t^\alpha} * u(t) \right\} = \mathcal{L}\left\{ \int_0^{t^\alpha} \frac{tau^{\alpha-1}}{\Gamma(\alpha)} \right\} U(t) = s^{-\alpha} U(s) \quad (8)$$

**Lemma 6:** (New)

Let $u(t)$ be a continuous differentiable function on a closed bounded interval $[0, b]$, $b \in \mathbb{R}^+$ and let $\tau$ be a constant delay such that:

$$u(t) = \varphi(t), \quad for \quad -\tau \leq t < 0 \quad (10)$$

Then the Laplace transform of a $\tau-$ delay function is given by:

$$\mathcal{L}\{u(t-\tau)\} = e^{-st} [U(s) + Q(s, \tau)] \quad (11)$$

where

$$Q(s, \tau) = \int_{-\tau}^{0} e^{-sx} \varphi(t) \, dt$$

and $\mathcal{L}\{u(t)\} = U(s)$. If the historical function $\varphi(t)$ is defined by power function $t^{n}, (n \in \mathbb{Z}^+)$ we have:

$$\mathcal{L}\{u(t-\tau)\} = e^{-st} U(s) + \sum_{p=0}^{n} (-1)^{n-p} \frac{n!}{s^{p+1}} e^{-st} \quad (12)$$

**Proof:**
By taking Laplace transform of \( \tau \)-delay function \( u(t - \tau) \), as in definition (1), and applying the change of variable by \( t - \tau = x \) we obtain:

\[
L\{u(t - \tau)\} = \int_0^\infty e^{-st} u(t - \tau) \, dt = e^{-st} \int_0^\infty e^{-sx} u(x) \, dx = e^{-st} \left[ \int_{-\infty}^0 + \int_0^\infty \right] e^{-sx} u(x) \, dx \quad \ldots (13)
\]

Use by part integral method for solving first integral in (13) after instead \( u(x) \) by historical function \( \varphi(x) \) which is defined \( x^n, n \in \mathbb{Z}^+ \), we get:

\[
Q(s, \tau) = \int_0^\tau e^{-sx} \varphi(x) \, dx = \int_0^\tau e^{-sx} x^n \, dx = e^{st} \sum_{p=0}^{n} (-1)^{n-p} p! \left( \frac{n!}{s^{p+1}} \right) - \frac{n!}{s^{n+1}} \quad \ldots (14)
\]

And the second integral part in (13) is the Laplace transform of \( u(x) \), thus:

\[
\int_{-\infty}^0 e^{-sx} u(x) \, dx = U(s) \quad \ldots (15)
\]

Putting equations (14) and (15) into (13) we obtain:

\[
L\{u(t - \tau)\} = e^{-st} U(s) + \sum_{p=0}^{n} (-1)^{n-p} p! \left( \frac{n!}{s^{p+1}} \right) - \frac{n!}{s^{n+1}} e^{-st}
\]

which completes the proof. Note that, in a more general way, for H.F. which is defined

\[
\varphi(x) = \sum_{r=1}^{R} a_r x^{n_r}; \{ R \in \mathbb{Z}^+, n_r \in \mathbb{Z}^+, a_r \in \mathbb{Z}^+ \}
\]

Then the formula (12) becomes:

\[
L\{u(t - \tau)\} = e^{-st} U(s) + \sum_{r=1}^{R} a_r \left[ \sum_{p=0}^{n_r} (-1)^{n_r-p} p! \left( \frac{n_r!}{s^{p+1}} \right) - \frac{n_r!}{s^{n_r+1}} e^{-st} \right] \quad \ldots (16)
\]

3. ANALYSIS OF THE METHOD:

In this section we try to find general solution form of linear VIFDE's with multi-time RD by applying the Laplace transform method in two different types of kernel: difference and simple degenerate kernel.

3.1 Difference Kernel Type:

Recall equation (1) with difference kernels and initial point \( (a = 0) \). Moreover, take \( P_i(t) \) as a power function, say \( C_i t^{\ell_i}, \ell_i \in \mathbb{R} \) and \( \ell_i \) be any nonnegative integer numbers for all \( i \):

\[
\sum_{i=1}^{m} P_i(t) \frac{d^{\alpha - i}}{dt^{\alpha}} u(t) + P_0(t) u(t - \tau) = f(t) + \lambda \sum_{j=0}^{t} \int_{-\infty}^{t} \mathcal{K}_j(t - x) u(x - \tau_j) \, dx \quad \ldots (16)
\]

For all \( i \in I = [0, b] \) \( \alpha_1 > \alpha_{n-1} > \alpha_{n-2} > \alpha_{n-3} > \ldots > \alpha_1 > \alpha_0 = 0 \), with initial conditions which are given: \( u^{(k)}(0) = u_k, k = 0, 1, \ldots, \mu - 1 \) (\( \mu = [\alpha_n] \)) and historical \( \mu - \) th continuity differentiable functions \( u(t) = \varphi(t) \) for \( t \in [\bar{a}, 0] \), where \( \bar{a} = -\max\{\tau, \tau_j, j = 1: \frac{1}{m}\} \). Let \( U(s), F(s) \) and \( \mathcal{K}_j(t) \) be the Laplace transform of \( u(t), f(t) \) and \( \mathcal{K}_j(t) \), respectively. Take the Laplace transform of both sides of equation (16):

\[
L\{C_i \frac{d^{\alpha}}{dt^{\alpha}} u(t)\} + \sum_{i=1}^{m} L\{P_i(t) \frac{d^{\alpha - i}}{dt^{\alpha}} u(t)\} + L\{P_0(t) u(t - \tau)\} = L\{f(t)\} + \sum_{j=1}^{m} \lambda L\left\{ \int_{0}^{t} \mathcal{K}_j(t - x) u(x - \tau_j) \, dx \right\} \quad \ldots (17)
\]

Part A, using equation (9) and initial conditions, where \( m_{a_n-1} < a_n \leq m_{a_n} \), we obtain

\[
L\{C_i \frac{d^{\alpha}}{dt^{\alpha}} u(t)\} = s^{a_n} U(s) - \sum_{k=0}^{m_{a_n-1}} s^{a_n-k-1} u_k \quad \ldots (17, A)
\]

for part B, first using equation (6) and then applying equation (9), we get for all \( i = 1: n - \frac{1}{2} \):

\[
L\{P_i(t) \frac{d^{\alpha - i}}{dt^{\alpha}} u(t)\} = C_i (-1)^{\ell_i} \frac{d^{\ell_i}}{ds^{\ell_i}} \left( s^{a_n} U(s) \right) - C_i (-1)^{\ell_i} \frac{d^{\ell_i}}{ds^{\ell_i}} \sum_{k=0}^{m_{a_n-1}} \left[ s^{a_n-k-1} u_k \right] \quad \ldots (17, B)
\]

Where \( m_{a_n-1} < a_n-1 \leq m_{a_n-1} \), \( \ell_i \) is the order of \( P_i(t) \) for each \( i = 1: n - \frac{1}{2} \). Part C, first using equation (6) and then applying the Lemma (6), (using 11 and 12) respectively, we obtain:
As a special case, where the historical function \( \varphi(t) \) is \( t^q, q \in \mathbb{Z}^+ \), we have:

\[
\mathcal{L}\{P_0(t)u(t - \tau)\} = C_0(-1)^\ell_0 \frac{d^{\ell_0}}{ds^{\ell_0}}[e^{-st}(U(s) + Q(s, \tau))] \quad \ldots (17, C1)
\]

As a special case, where the historical function \( \varphi(t) \) is \( t^q, q \in \mathbb{Z}^+ \), we have:

\[
\mathcal{L}\{P_0(t)u(t - \tau)\} = C_0(-1)^\ell_0 \frac{d^{\ell_0}}{ds^{\ell_0}}[e^{-st}] \quad \ldots (17, C2)
\]

where \( \ell_0 \) is the order of \( P_0(t) \) and \( q \) is the order of historical polynomial function. For part \( D \), using the definition of Laplace transformation, we get:

\[
\mathcal{L}\{f(t)\} = F(s) \quad \ldots (17, D)
\]

Part \( E \), we apply equation (4) with Lemma (6), (11 and 12), respectively to obtain: for all \( j = 1, 2, \ldots, m \)

\[
\mathcal{L}\left\{ \int_0^t \mathcal{K}_j(t - x) u(x - \tau_j)dx \right\} = \mathcal{K}_j(s) e^{-stj} \left[ (U(s) + Q(s, \tau_j)) \right] \quad \ldots (17, E1)
\]

As a special case, where the historical function \( \varphi(t) \) is \( t^q, q \in \mathbb{Z}^+ \), we have:

\[
\mathcal{L}\left\{ \int_0^t \mathcal{K}_j(t - x) u(x - \tau_j)dx \right\} = \mathcal{K}_j(s) \left[ e^{-stj}U(s) + \sum_{p=0}^q (-1)^{q-p} p! \left( \frac{t_j^{q-p}}{s^{q+1}} \right) \right]
\]

\[- \frac{q!}{s^{q+1}} e^{-stj} \quad \ldots (17, E2)
\]

Putting equations (17: A, B, C1, D, E1) into the equation (17) and after some simple manipulations, we get the following equation:

\[
\sum_{i=1}^{n-1} C_i(-1)^\ell_i \frac{d^{\ell_i}}{ds^{\ell_i}}[s^{\alpha_n-i} U(s)]
\]

\[+ C_0(-1)^\ell_0 \frac{d^{\ell_0}}{ds^{\ell_0}}[e^{-st} U(s)]
\]

\[+ \left( s^{\alpha_n} - \lambda \sum_{j=1}^m \mathcal{K}_j(s) e^{-stj} \right) U(s)
\]

\[= F^*(s) \quad \ldots (18)
\]

where

\[
F^*(s) = F(s) + \lambda \sum_{j=1}^m \mathcal{K}_j(s) e^{-stj} Q(s, \tau_j)
\]

\[+ \sum_{k=0}^{m_{\alpha_n-1}} s^{\alpha_n-k-1} u_k
\]

\[+ \sum_{i=1}^{n-1} C_i(-1)^\ell_i \frac{d^{\ell_i}}{ds^{\ell_i}} \left[ \sum_{k=0}^{m_{\alpha_n-i-1}} s^{\alpha_n-i-k-1} u_k \right]
\]

\[- C_0(-1)^\ell_0 \frac{d^{\ell_0}}{ds^{\ell_0}}[e^{-st} Q(s, \tau)] \quad \ldots (19)
\]

If historical function is power function \( t^q, q \in \mathbb{Z}^+ \), putting equations (17: A, B, C2, D, E2) into equation (17), we obtain the following equations (20) instead of (18):

\[
F^*(s) = F(s)
\]

\[+ \lambda \sum_{j=1}^m \mathcal{K}_j(s) \sum_{p=0}^q (-1)^{q-p} p! \left( \frac{t_j^{q-p}}{s^{p+1}} \right) \]

\[- \frac{q!}{s^{q+1}} e^{-stj} + \sum_{k=0}^{m_{\alpha_n-1}} s^{\alpha_n-k-1} u_k
\]

\[+ \sum_{i=1}^{n-1} C_i(-1)^\ell_i \frac{d^{\ell_i}}{ds^{\ell_i}} \left[ \sum_{k=0}^{m_{\alpha_n-i-1}} s^{\alpha_n-i-k-1} u_k \right]
\]

\[- C_0(-1)^\ell_0 \frac{d^{\ell_0}}{ds^{\ell_0}} \sum_{p=0}^q (-1)^{q-p} p! \left( \frac{t_j^{q-p}}{s^{p+1}} \right)
\]

Consequently there is an ordinary differential equation in \( U(s) \); solve it to find \( U(s) \). Finally, use the inverse of Laplace transform on \( U(s) \) to obtain the solution \( u(s) \) of linear VIFDE’s for multi-time RD, (1). For a special case, if the \( P_i(t) \) and \( P_0(t) \) are only constants that is \( \ell_i \) and \( \ell_0 \) are equal to zero. Thus, equations (18), (19) and (20) become:

\[
H(s)U(s) = F^*(s)
\]

where

\[
F^*(s) = F(s)
\]
\[ + \frac{m}{\lambda} \sum_{j=1}^{m} \mathcal{K}_j(s) \left\{ \sum_{p=0}^{q} (-1)^{q-p} p! \left( \frac{q}{p} \right)^{\frac{q-p}{p+1}} - \frac{q!}{s^{q+1}} e^{-st_j} \right\} + \sum_{k=0}^{m_{an-1}} s^{\alpha_n-k-1} u_k + \sum_{i=1}^{n-1} C_i \left[ \sum_{k=0}^{m_{an_i-1}} s^{\alpha_n-i-k-1} u_k \right] \]

\[ = C_0 \left[ \sum_{p=0}^{q} (-1)^{q-p} p! \left( \frac{q}{p} \right)^{\frac{q-p}{p+1}} - \frac{q!}{s^{q+1}} e^{-st} \right] \]

\[ = -\lambda \sum_{j=1}^{m} \mathcal{K}_j(s) e^{-st_j} \] ... (23)

It has a unique solution providing that:

\[ s^\alpha + \sum_{i=1}^{n-1} C_i s^{\alpha_n-i} + C_0 e^{-st} \neq -\lambda \sum_{j=1}^{m} \mathcal{K}_j(s) e^{-st_j} \]

3.2 Simple Degenerate Kernel:

Laplace transform technique can be used to solve some kinds of Linear VIFDE’s of constant multi-time Retarded delays which the kernel is not necessarily difference kernel, here we take the same equation (16) with all conditions on that equation except the kernel which is a simple degenerate kernel. Thus:

\[ \frac{\partial}{\partial t} s^\alpha u(t) + \sum_{i=1}^{n-1} P_i(t) \frac{\partial}{\partial t} s^{\alpha_n-i} u(t) + P_0(t) u(t) = f(t) \]

\[ = f(t) + \lambda \sum_{j=1}^{m} \int_0^t \left[ c_j t^{k_j} + d_j x^{k_j} \right] u(x - t_j) dx \] ... (24)

Since \( c_j, d_j \in \mathbb{R} \) for all \( j = 1, 2, \ldots, m \) and \( k_j^1, k_j^2 \in \mathbb{Z}^+ \). Taking the Laplace transform of both sides of equation (24):

\[ \mathcal{L} \left[ \frac{\partial}{\partial t} s^\alpha u(t) \right] + \sum_{i=1}^{n-1} \mathcal{L} \left[ P_i(t) \frac{\partial}{\partial t} s^{\alpha_n-i} u(t) \right] + \mathcal{L} \left[ P_0(t) u(t - \tau) \right] = \mathcal{L} \left[ f(t) \right] \]

\[ + \sum_{i=1}^{n-1} C_i \left( \int_0^t \left[ c_j t^{k_j} + d_j x^{k_j} \right] u(x - t_j) dx \right) \] ... (25)

For parts A, B, C and D we obtain same equations as the section (3.1) respectively. Now for part E: we apply equation (7) with Lemma (5), (11 and 12) respectively, and using Leibniz’s formula for higher derivative of multiplication functions (Marc, 2005) then after some manipulating we obtain:

\[ \mathcal{L} \left\{ \int_0^t \left[ c_j t^{k_j} + d_j x^{k_j} \right] u(x - t_j) dx \right\} \]

\[ = \frac{e^{-st_j}}{s} \left\{ c_j \left( \sum_{r=0}^{k_j} r! \left( \frac{k_r^1}{r!} \right)^{\frac{1}{s^r} t_j^{k_r^1-r} \right) + d_j t_j^{k_j} \right\} \]

\[ + \left[ \frac{k_j^1-1}{s^r} \left( \sum_{r=0}^{k_j^1} \left( \sum_{r=0}^{k_j^1} \left( \frac{k_j^2}{r} \frac{d^{k_j^2-r}}{ds^{k_j^2-r}} \right) \right) \right) \}

\[ \left\{ c_j \sum_{r=0}^{k_j^1} \left( \frac{k_j^1}{r} \frac{d^{k_j^1-r}}{ds^{k_j^1-r}} \right) \right\} + \frac{1}{s^r} \left( \sum_{r=0}^{k_j^1} \left( \frac{k_j^2}{r} \frac{d^{k_j^2-r}}{ds^{k_j^2-r}} \right) \right) \]

\[ + \frac{1}{s} \left[ \int_0^t \left\{ \frac{k_j^1}{r} \frac{d^{k_j^1-r}}{ds^{k_j^1-r}} \right\} H_j^q(s) \right] \] ... (26)

for all \( j = 1, 2, \ldots, m \). Where,

\[ H_j^q(s) = e^{-st_j} Q(s,t_j) \]

\[ \text{if the (HP) any continuos differentiable function.} \]

\[ = \left\{ \sum_{r=0}^{q} (-1)^{q-r} p! \left( \frac{q}{p} \right)^{\frac{q-r}{p+1}} - \frac{q!}{s^{q+1}} e^{-st} \right\} \]

and \( Q(s,t_j) = \int_0^t e^{-st} \varphi(x) dx \)

After some simple manipulations, for par (17; A, B, C, D, E and 26) then putting equations into equation (25) we obtain the general solution for (24):

\[ \sum_{i=1}^{n-1} C_i \left( \int_0^t \left[ c_j t^{k_j} + d_j x^{k_j} \right] u(x - t_j) dx \right) \]

\[ + \frac{1}{s^r} \left[ \sum_{r=0}^{k_j^1} \left( \frac{k_j^1}{r} \frac{d^{k_j^1-r}}{ds^{k_j^1-r}} \right) \right] \]

\[ + C_0 \left( \int_0^t e^{-st} U(s) dx \right) \]

\[ \left\{ \sum_{j=1}^{m} \frac{e^{-st_j}}{s} \left[ c_j \left( \sum_{r=0}^{k_j} r! \left( \frac{k_j^1}{r!} \right)^{\frac{1}{s^r} t_j^{k_j^1-r}} \right) + d_j t_j^{k_j} \right] \right\} \]
If the historical function is any continuous differentiable function $\varphi(t)$, thus:

$$F^*(s) = F(s)$$

$$+ \sum_{j=1}^{m} \frac{1}{s} \left( \sum_{r=0}^{k_j^1} (-1)^{r+k_j^1} r! \left( \frac{k_j^1}{r} \right) \frac{d^{k_j^1-r}}{ds^{k_j^1-r}} \right) \left( \sum_{p=0}^{m_{\sigma_n}} \left( -1 \right)^{p} \tau_j^p \left( \frac{k_j^1 - r}{p} \right) \frac{d^{k_j^1-r-p}}{ds^{k_j^1-r-p}} \right)\right) \left( \sum_{k=0}^{\sigma_n-1-k-1} u_k \right)$$

$$+ \sum_{i=1}^{n} \sum_{k=0}^{m_{\sigma_n}} \left( -1 \right)^{i} \left( \frac{d^{i}}{ds^{i}} \right) \sum_{k=0}^{\sigma_n-1-k-1} u_k$$

$$- C_0 \left( -1 \right)^{s_0} \frac{d^{s_0}}{ds^{s_0}} \left[ e^{-s_0 Q(s, \tau)} \right] \quad \cdots \cdots (27 A)$$

If the historical function is $\varphi(t) = t^q, q \in \mathbb{Z}^+$ thus:

$$F^*(s) = F(s)$$

$$+ \sum_{j=1}^{m} \frac{1}{s} \left( \sum_{r=0}^{k_j^1} (-1)^{r+k_j^1} r! \left( \frac{k_j^1}{r} \right) \frac{d^{k_j^1-r}}{ds^{k_j^1-r}} \right) \left( \sum_{p=0}^{m_{\sigma_n}} \left( -1 \right)^{p} \tau_j^p \left( \frac{k_j^1 - r}{p} \right) \frac{d^{k_j^1-r-p}}{ds^{k_j^1-r-p}} \right)\right) \left( \sum_{k=0}^{\sigma_n-1-k-1} u_k \right)$$

$$+ \sum_{i=1}^{n} \sum_{k=0}^{m_{\sigma_n}} \left( -1 \right)^{i} \left( \frac{d^{i}}{ds^{i}} \right) \sum_{k=0}^{\sigma_n-1-k-1} u_k$$

$$- C_0 \left( -1 \right)^{s_0} \frac{d^{s_0}}{ds^{s_0}} \left[ e^{-s_0 Q(s, \tau)} \right] \quad \cdots \cdots (27 B)$$

Which is the Ordinary Differential Equation $U(s)$ with variable coefficients and solving it to find $U(s)$. Finally, use the inverse Laplace transform on $U(s)$ to obtain the solution $u(t)$ of Linear VIFDE’s with constant multi-time RD, (27).

4. ILLUSTRATIVE EXAMPLES:

In order to show the efficiency of our proposed method and investigate its accuracy how to solve linear VIFDE’s with constant multi-time Retarded delay (1), we present some examples.

**Example (1):** Consider linear VIFDE’s of constant multi-time Retarded delay with constant coefficients on $[0, 1]$:

$$c_0 \Delta^{0.9}_{0} u(t) - \frac{1}{2} c_0 \Delta^{0.5}_{0} u(t) + \frac{1}{2} u(t - 0.2) = f(t)$$

$$+ \int_{0}^{t} \left[ (t - x)u(x - 0.3) + (t - x)^2 u(x - 0.5) \right] dx$$

where

$$f(t) = \frac{2}{\Gamma(1.1)} t^{0.1} - \frac{1}{\Gamma(1.5)} t^{0.5} - \frac{1}{6} t^4 - \frac{1}{3} t^3$$

$$- \frac{1}{5} t^2 + t + \frac{3}{10}$$

with initial condition and historical function: $u(0) = 1$; with given historical function: $\varphi(t) = 2t + 1$.

Since here we notice that:

$$\mathcal{K}_1(t, x) = (t - x); \mathcal{K}_2(t, x) = (t - x)^2$$

$$\alpha_2 = 0.9; \alpha_1 = 0.5; m_{\alpha_2} = m_{\alpha_1} = 1;$$

$$P_1(t) = -1/2; P_0(t) = 1/2$$

$$P_2(t) = 1/2; P_3(t) = 1/2$$

$$P_4(t) = 1/2; P_5(t) = 1/2$$

$$P_6(t) = 1/2; P_7(t) = 1/2$$

$$P_8(t) = 1/2; P_9(t) = 1/2$$

$$P_{10}(t) = 1/2$$

$$P_{11}(t) = 1/2$$

$$P_{12}(t) = 1/2$$

$$P_{13}(t) = 1/2$$

$$P_{14}(t) = 1/2$$

$$P_{15}(t) = 1/2$$

$$P_{16}(t) = 1/2$$

$$P_{17}(t) = 1/2$$

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$$P_{22}(t) = 1/2$$

$$P_{23}(t) = 1/2$$

$$P_{24}(t) = 1/2$$

$$P_{25}(t) = 1/2$$

$$P_{26}(t) = 1/2$$

$$P_{27}(t) = 1/2$$

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$$P_{96}(t) = 1/2$$

$$P_{97}(t) = 1/2$$

$$P_{98}(t) = 1/2$$

$$P_{99}(t) = 1/2$$

$$P_{100}(t) = 1/2$$
and
\[ \tau = \tau_0 + 0.2, \tau_1 = 0.3, \tau_2 = 0.5 \]
which are constant different time delays. Taking Laplace transform to the equation mentioned above and finding \( F^*(s) \) with \( H(s) \) from equations (22) and (23), we have:
\[ L[K_1(t)] = \frac{1}{s^2} ; \quad L[K_2(t)] = \frac{2}{s^3} \]
and
\[ F^*(s) = \frac{2}{s^{1.1}} - \frac{1}{s^{1.5}} + s^{-0.1} - \frac{1}{2s^{-0.5}} - \frac{2}{s^4} e^{-0.3s} \]
\[ - \frac{1}{s^3} e^{-0.3s} - \frac{4}{s^5} e^{-0.5s} \]
\[ - \frac{2}{s^4} e^{-0.5s} + \frac{1}{s^2} e^{-0.2s} \]
\[ + \frac{1}{2s} e^{-0.2s} \]
with
\[ H(s) = s^{0.9} - \frac{1}{2} s^{0.5} + \frac{1}{2} e^{-0.2s} - \frac{1}{s^2} e^{-0.3s} \]
\[ - \frac{2}{s^3} e^{-0.5s} \]
Thus:
\[ U(s) = \frac{F^*(s)}{H(s)} = \frac{1}{s + \frac{2}{s^2}} \]
Then using inverse Laplace transform to obtain the exact solution \( u(t) \) for Linear VIFDDE’s with constant coefficient.
\[ u(t) = L^{-1} \left( \frac{1}{s + \frac{2}{s^2}} \right) = 2t + 1 \]
Which is the exact solution for our given problem.

**Example (2):** Consider linear VIFDE’s of constant multi-time Retarded delay with variable coefficients on \([0, 1]\):
\[ \frac{\partial}{\partial t} \int_0^t \frac{\partial^5}{\partial t^5} u(t) + t \frac{\partial^5}{\partial t^5} u(t) - 3t \ u(t - 1) = f(t) \]
\[ + \int_0^t [(t - x)u(x - 2) - e^{t-x}u(x - 1)] \ dx \]
where
\[ f(t) = \frac{2}{\Gamma(1.5)} t^{0.5} + \frac{2}{\Gamma(2.5)} t^{2.5} + e^{t} - \frac{1}{12} t^4 \]
\[ - \frac{7}{3} t^3 + 3t^2 - 3t - 1 \]
with initial condition and historical function:
\[ u(0) = u'(0) = 0; \quad \text{with historical function} \quad \phi(t) = t^2. \]
Since here we have:
\[ K_1(t, x) = (t - x) ; \quad K_2(t, x) = e^{t-x} \]
and
\[ \tau = \tau_0 = 1, \tau_1 = 2, \tau_2 = 1 \]
which are constant different time delays and \( P_1(t) = t; \quad P_0(t) = -3t \)
are variable coefficients. Taking Laplace transform to the above equation and using equations (18) and (20) to obtain:
\[ L[K_1(t)] = \frac{1}{s^2} ; \quad L[K_2(t)] = \frac{1}{s - 1} \]
Thus:
\[ s^{1.5} U(s) - s^{0.5} U'(s) - 0.5 s^{-0.5} U(s) \]
\[ + 3e^{-s} U'(s) - 3e^{-s} U(s) \]
\[ + \frac{1}{s^2} e^{-2s} U(s) + \frac{1}{s^2} e^{-s} U(s) \]
\[ = F^*(s) \]
where
\[ F^*(s) = \frac{2}{s^{1.5}} + \frac{5}{s^{3.5}} - \frac{2}{s^3} e^{-s} - \frac{6}{s^4} e^{-s} - \frac{2}{s^5} e^{-2s} \]
\[ + \frac{1}{s^2} e^{-2s} + \frac{1}{s^2} \frac{e^{-s}}{s - 1} \]
So the equation becomes:
\[ U'(s)[3e^{-s} - s^{0.5}] \]
\[ + U(s) \left[ s^{1.5} - 0.5 s^{-0.5} - 3e^{-s} \right] \]
\[ - \frac{1}{s^2} e^{-2s} + \frac{1}{s - 1} \frac{e^{-s}}{s} \]
\[ = F^*(s) \]
with \( U(\infty) = 0, (Rudolf, 2000) \), which is ODE of the first order, after solving it, the following is obtained:
\[ U(s) = \frac{2}{s^{3^{0.8}}} \]
By taking the inverse of Laplace transform of \( U(s) \) the exact solution \( u(t) \) is obtained
\[ u(t) = L^{-1} \left( \frac{2}{s^{3^{0.8}}} \right) = t^2 \]
Which is the exact solution for our given problem.

**Example (3):** Consider linear VIFDE’s of constant multi-time Retarded delay with variable coefficients with degenerate kernel on \([0, 1]\):
\[ \frac{\partial}{\partial t} \int_0^t \frac{\partial^3}{\partial t^3} u(t) + \frac{2}{3} \frac{\partial^4}{\partial t^4} u(t) - t \ u(t - 0.4) = f(t) \]
\[ + \int_0^t (2t + 3x^2)u(x - 0.2) \ dx \]
where
\[ f(t) = \frac{1}{\Gamma(1.7)} t^{0.7} + \frac{2}{3 \Gamma(2.6)} t^{1.6} - \frac{3}{10} t^5 \]
\[ - \frac{11}{60} t^4 - \frac{8}{25} t^3 + \frac{9}{25} t^2 - \frac{2}{25} t \]
with initial condition and historical function:
\[ u(0) = u'(0) = 0; \quad \text{with historical function} \quad \phi(t) = \frac{1}{2} t^2 \]
and constant different time delay
\[ \tau = \tau_0 = 0.4, \tau_1 = 0.2 \]

and

\[ P_1(t) = \frac{2}{3}, P_0(t) = -t \]

By Applying Laplace transform for solving our problem, using equations (27) and (27A) we obtain:

\[
s^{1.3} U(s) + \frac{2}{3} s^{0.4} U(s) + e^{-0.4s} U'(s) - 0.4e^{-0.4s} U(s) - \frac{2}{s^2} e^{-0.2s} U(s) - \frac{0.52}{s} e^{-0.2s} U(s) + \frac{3.2}{s} e^{-0.2s} U'(s) - \frac{3}{s} e^{-0.2s} U''(s)
\]

where

\[ F^*(s) = \frac{1}{s^{1.7}} + \frac{2}{3s^{2.6}} \frac{3}{s^4} e^{-0.4s} - \frac{0.4}{s^3} e^{-0.4s} - \frac{36}{s^6} e^{-0.2s} - \frac{11.6}{s^5} e^{-0.2s} - \frac{0.52}{s^4} e^{-0.2s} - \frac{0.1}{s^3} e^{-0.2s} \]

So the equation becomes:

\[
\left[ s^{1.3} + \frac{2}{3} s^{0.4} - 0.4e^{-0.4s} - \frac{2}{s^2} e^{-0.2s} - \frac{0.52}{s} e^{-0.2s} \right] U(s) + e^{-0.4s} + \frac{3.2}{s} e^{-0.2s} \right] U'(s) - \left[ \frac{3}{s} e^{-0.2s} \right] U''(s) = F^*(s)
\]

with \( U(\infty) = 0 \), which is ODE after solving it for \( U(s) \) we obtain:

\[ U(s) = \frac{1}{s^2} \]

By taking the inverse of Laplace transform of \( U(s) \) we get the exact solution

\[ u(t) = \mathcal{L}^{-1} \left[ \frac{1}{s^2} \right] = \frac{1}{2} t^2 \]

5. DISCUSSION:

In this paper, through the Laplace transform methods for solving linear Volterra integro-fractional differential equations of constant multi-time Retarded delay type with variable coefficients introduced with some illustrating examples for each cases, we pointed the following:

The Laplace transform method which was improved here provided good results and effectiveness for various problems.

The Laplace transforms was applied for difference kernel and simple degenerate kernel in general cases.

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