Color Superconductivity in $\mathcal{N} = 2$ Supersymmetric Gauge Theories

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Abstract

We study vacuum structure of $\mathcal{N} = 2$ supersymmetric (SUSY) QCD, based on the gauge group $SU(2)$ with $N_f = 2$ flavors of massive hypermultiplet quarks, in the presence of non-zero baryon chemical potential ($\mu$). The theory has a classical vacuum preserving baryon number symmetry, when a mass term, which breaks $\mathcal{N} = 2$ SUSY but preserves $\mathcal{N} = 1$ SUSY, for the adjoint gauge chiral multiplet ($m_{ad}$) is introduced. By using the exact result of $\mathcal{N} = 2$ SUSY QCD, we analyze low energy effective potential at the leading order of perturbation with respect to small SUSY breaking parameters, $\mu$ and $m_{ad}$. We find that the baryon number is broken as a consequence of the $SU(2)$ strong gauge dynamics, so that color superconductivity dynamically takes place at the non-SUSY vacuum.

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1 Introduction

Recently phase structure of quark matter under non-zero baryon density has been intensively studied. In particular, it has been emphasized that quark matter is expected to be in a BCS-type superconducting phase at high baryon densities [1]. In the case with high baryon densities the Fermi surface lies at a high energy regime, and perturbative description of QCD is applicable. There exists an attractive interaction between quarks through the exchange of one gluon in a color anti-symmetric $\bar{3}$ state. Therefore, QCD at high densities is expected to behave as a color superconductor. One of important consequences of color superconductivity is spontaneous breaking of baryon number symmetry. In fact, spontaneous breaking of the baryon number symmetry is observed in high density QCD for three or more quark flavors with breaking of color and flavor symmetries (color flavor locking) [2]. However, in the case of low baryon densities, where QCD is in strong coupling regime, perturbative calculations is no longer applicable. It is an open question whether the color superconducting phase appears even at low baryon densities.

If some QCD-like theories exist which is calculable beyond perturbation, it would be interesting to investigate vacuum structure of such theories in the presence of non-zero baryon chemical potential. There have been prominent developments of SUSY gauge theory in the past decade. In $\mathcal{N} = 1$ SUSY gauge theory, the exact low energy effective superpotential was derived by using holomorphy properties of the superpotential and the gauge kinetic function (for review, see Ref. [3]). Seiberg and Witten derived the exact low energy Wilsonian effective action for $\mathcal{N} = 2$ SUSY SU(2) Yang-Mills theory and generalized it to the case up to four massive hypermultiplets by using holomorphy and duality arguments [4]. By using the exact superpotential, Harnik et al. [5] studied vacuum structure of $\mathcal{N} = 1$ SUSY QCD with non-zero baryon chemical potential. They found that the baryon number symmetry is spontaneously broken due to the effect of the strong gauge dynamics and also examined the breaking pattern of flavor symmetries. However, in $\mathcal{N} = 1$ SUSY framework, the region that we can investigate is limited, because effective Kähler potential, which is not a holomorphic variable, can not be determined beyond perturbations, while effective superpotential is exact.

In this paper, we study vacuum structure of $\mathcal{N} = 2$ SUSY QCD in the presence of non-zero baryon chemical potential. Remarkable point is that in the theory we can derive exact low energy effective theory including effective Kähler potential. The theory we will investigate is based on the gauge group $SU(2)$ with $N_f = 2$ flavors of hypermultiplet quarks with the same masses. In order to provide definite classical vacua, we also introduce $\mathcal{N} = 1$ SUSY preserving mass term for the gauge adjoint chiral multiplet. At the classical level, the theory has two discrete vacua in some parameter region for the baryon chemical potential $\mu$, the common masses of hypermultiplet quarks $m$ and the adjoint mass $m_{ad}$. One is a local minimum where baryon number symmetry is preserved, the other is a global one breaking the baryon number. Our main interest is whether the baryon number symmetry is spontaneously broken or not after taking all quantum corrections into account, and so we concentrate on the classical vacuum preserving the baryon number symmetry. By using the exact results of $\mathcal{N} = 2$ SUSY QCD, we derive low energy effective theory and investigate how the classical vacuum is deformed due to the $SU(2)$ strong gauge dynamics.

As the same as in the analysis of $\mathcal{N} = 1$ SUSY theory [5], there is a clear limitation on
our analysis, namely SUSY breaking parameters in our theory, the chemical potential $\mu$ and the adjoint mass $m_{ad}$, should be much smaller than the dynamical scale of the $SU(2)$ gauge interaction $\Lambda$. Keeping them in such a region, we use the exact results of $\mathcal{N} = 2$ SUSY QCD and derive low energy effective theory at the leading order of perturbation with respect to such small SUSY breaking parameters. Note that the parameter $\mu$ we consider is small, where gauge coupling is in strong coupling regime. Thus, the results of our analysis would give some insights for color superconducting phase in strong coupling regime, complementary to the results obtained in perturbative analysis of non-SUSY QCD with large baryon chemical potential. Unfortunately, in our analysis, diquark fields do not appear as dynamical variables and we cannot examine their condensations $\langle qq \rangle$ (for a related work, see Ref. [6]).

The organization of this paper is as follows. In section 2, we define our classical Lagrangian and find classical vacua. In Section 3, we derive the low energy effective action by using exact results of $\mathcal{N} = 2$ SUSY QCD. Potential analysis is performed numerically in section 4. Finally we summarize our results.

## 2 Vacuum structure of classical theory

In this section, we first define our classical Lagrangian and analyze its classical vacuum. We will find that the theory has at least one vacuum preserving the baryon number symmetry.

Our theory is based on the $\mathcal{N} = 2$ $SU(2)$ gauge theory with $N_f = 2$ quark hypermultiplets having the same masses $m$. In terms of $\mathcal{N} = 1$ superfield language, this theory is described by two pairs of chiral quark and anti-quark superfields $Q_i$ and $\tilde{Q}_i$ ($i = 1, 2$ denotes the flavor index) in the (anti) fundamental representation of the $SU(2)$ gauge group, and the $SU(2)$ vector multiplet consisting of vector and chiral superfields $(V, A)$ in the adjoint representation. There is a global symmetry $SO(4) \times U(1)_B$, where $SO(4)$ is a quark flavor symmetry and $U(1)_B$ is a baryon number symmetry. According the prescription in Ref. [5], we incorporate a baryon chemical potential $\mu$ into this theory by introducing a background fictitious $U(1)_B$ super gauge field $V_B$ with an appropriate vacuum expectation values (see Eq. (2.5)). A chiral superfield $A_B$ which is a superpartner of $V_B$ is also introduced because of $\mathcal{N} = 2$ SUSY. The chemical potential leads to a tachyonic mass term for scalar quarks so that they immediately undergoes Bose-Einstein condensations and the baryon number symmetry is broken at the classical level. Then, to realize a classical vacuum preserving the baryon number symmetry, which we are interested in, we further introduce $\mathcal{N} = 1$ SUSY preserving soft mass term for the adjoint chiral gauge multiplet. Putting all together, the resulting classical Lagrangian of our theory is defined as

\[
\mathcal{L} = \mathcal{L}_{\text{HM}} + \mathcal{L}_{\text{VM}} + \mathcal{L}_{\text{soft}},
\]

\[
\mathcal{L}_{\text{HM}} = \sum_{i=1}^{2} \left[ \int d^4 \theta \left( Q_i^\dagger e^{2V + 2S V_B} Q_i + \tilde{Q}_i e^{-2V - 2S V_B} \tilde{Q}_i^\dagger \right) 
+ \sqrt{2} \left( \int d^2 \theta \tilde{Q}_i \left( A + S \frac{m}{\sqrt{2}} \right) Q_i + \text{h.c.} \right) \right],
\]

\[
\mathcal{L}_{\text{VM}} = \frac{2}{4\pi} \text{Im} \left[ \text{tr} \left\{ \tau \left( \int d^4 \theta A^\dagger e^{2V} A e^{-2V} + \frac{1}{2} \int d^2 \theta W^2 \right) \right\} \right],
\]
\[ \mathcal{L}_{\text{soft}} = m_{\text{ad}} \int d^2 \theta \mathrm{tr} A^2 + \text{h.c.}, \] (2.4)

where \( \tau = \frac{4\pi i}{g^2} + \frac{\theta}{2\pi} \), \( W \) is a \( SU(2) \) gauge superfield strength, \( S = 1 \) is a baryon number quark superfield carries, and \( m_{\text{ad}} \) is the adjoint mass. Here, we took the normalization \( T(R)\delta^{ab} = \mathrm{tr}(T^a T^b) = \frac{1}{2} \delta^{ab} \). The vacuum expectation values for vector and chiral superfields \( (V_B, A_B) \) are taken to be \( [5] \)

\[ \langle V_B \rangle = \theta \sigma^\mu \bar{\theta} \langle A_\mu \rangle \quad \text{with} \quad \langle A_\mu \rangle = (\mu, 0, 0, 0), \quad \langle A_B \rangle = 0. \] (2.5)

In general one can take the non-zero vacuum expectation value for \( A_B \) and introduce it as a soft SUSY breaking term. We fix it to be zero, for simplicity.

The scalar potential is written down as

\[
V = q_i^\dagger \left( 2|A + Sm/\sqrt{2}|^2 - \mu^2 \right) q_i + \bar{q}_i \left( 2|A + Sm/\sqrt{2}|^2 - \mu^2 \right) \bar{q}_i^\dagger \\
+ \frac{1}{2b} \sum_{a=1}^{3} (q_i^\dagger T^a q_i - \bar{q}_i T^a \bar{q}_i^\dagger)^2 + b \mathrm{tr}[A, A^\dagger]^2 + q_i^\dagger [A^\dagger, A] q_i - \bar{q}_i [A^\dagger, A] \bar{q}_i^\dagger \\
+ \frac{1}{b} \sum_{a=1}^{3} |\sqrt{2} q_i T^a \bar{q}_i + m_{\text{ad}} A^a|^2, \tag{2.6}
\]

where the quantity \( b \) is the coupling constant defined by \( b \equiv \frac{1}{4\pi} \mathrm{Im} \tau \). The fields \( A, q_i \) and \( \bar{q}_i \) are the scalar components of the corresponding chiral superfields, respectively. The index \( i \) for the flavor symmetry is implicitly summed.

Solving stationary conditions with respect to the hypermultiplet scalars \( q_i \) and \( \bar{q}_i \) yields \(^4\)

\[
q_i = \begin{pmatrix} v_i \\ 0 \end{pmatrix}, \quad \bar{q}_i = \begin{pmatrix} \bar{v}_i \\ 0 \end{pmatrix}, \quad v_i = \rho_i e^{ia}, \quad \bar{v}_i = \rho_i, \quad A = \frac{1}{2} \begin{pmatrix} a & 0 \\ 0 & -a \end{pmatrix}, \quad a \in \mathbb{C}, \tag{2.7}
\]

from which we find

\[
V = -\frac{\rho_i^4}{2b} + \frac{|m_{\text{ad}} a|^2}{b}, \tag{2.9}
\]

with the quark condensation

\[
\rho_i^2 = \begin{cases} 
2b \left( \mu^2 - 2 \left| \frac{1}{2} a + m_{\text{ad}}/\sqrt{2} \right|^2 \right) + \sqrt{2}|m_{\text{ad}} a| & \text{for } \rho_i^2 > 0, \\
0 & \text{otherwise}.
\end{cases} \tag{2.10}
\]

\(^4\)In the case with \( \mu > m \), we also have another solution \( |v_i|^2 = 2b(\mu^2 - m^2) \left( 1 + \sqrt{1 - \frac{4m^2}{m_{\text{ad}}}} \right) \) and \( |\bar{v}_i|^2 = 2b(\mu^2 - m^2) \left( 1 - \sqrt{1 - \frac{4m^2}{m_{\text{ad}}}} \right) \). This solution breaks the baryon number and out of our interests. In addition, we will consider the opposite case \( \mu < m \) in the following.
The vacuum expectation value of the condensation $\langle \rho \rangle$ is an order parameter for the baryon number symmetry breaking.

We found that the theory has a vacuum (a local minimum) where the baryon number symmetry is preserved. Figs. 1 and 2 show an example realizing such a situation. A 3D plot of the effective potential for the complex variable $a$ is depicted in Fig. 1. In the parameter choice shown in the caption in Fig. 1, one can see that the theory has two discrete vacua along the real $a$ axis. The plots in Fig. 2 show the potential (left) and the quark condensation (right) along real $a$ axis. At a vacuum with $\langle a \rangle \neq 0$, the $SU(2)$ gauge symmetry is broken down to $U(1)$ and quarks become massless (quark singular point). Since the quarks condensate there, the baryon number symmetry is broken. On the other hand, we find a local minimum at the origin, where there is no quark condensation and the baryon number symmetry is preserved. This is the classical vacuum which we are interested in.

We also investigate the behavior of the scalar potential with respect to other choices of the parameters. First of all, the quark singular point depends only on $m$. As $\mu$ decreases with $m$ and $m_{ad}$ fixed, the quark condensation becomes small and the vacuum energy at the quark singular point increases. For vanishing $\mu$, $\mathcal{N} = 1$ SUSY is restored and two equivalent discrete vacua
appear. As $\mu$ increases, the quark condensation grows and its width becomes wider and reaches the origin. As a result, the minimum at the origin is smeared away and the baryon number preserving vacuum disappears (see Fig. 3). As $m$ decreases with $\mu$ and $m_{\text{ad}}$ fixed, the quark singular point approaches to the origin and, eventually, the quark condensation covers the origin and eliminates the minimum at the origin. As $m_{\text{ad}}$ decreases with $\mu$ and $m$ fixed, the potential becomes flatter with the quark condensation being small. In the limit of vanishing $m_{\text{ad}}$ the flat direction appears except around quark singular point at which vacuum energy is negative due to the quark condensation caused by non-zero $\mu$.

As discussed above, the vacuum preserving the baryon number symmetry is smeared away for small $m$ and large $\mu$. From Eq. (2.10), it is easy to find that a vacuum preserving the baryon symmetry exist if the condition $\mu < m$ is satisfied. Here, we assume real $m$, for simplicity. This situation is similar to the case of $\mathcal{N} = 1$ SUSY QCD investigated in Ref. [5], where the authors introduced soft SUSY breaking masses, corresponding to $m$ in our analysis, to provide a classical vacuum preserving the baryon number symmetry. In our $\mathcal{N} = 2$ gauge theory, we need to introduce the adjoint mass term as understood from the above discussions.

### 3 Quantum theory

In this section, we describe low energy Wilsonian effective Lagrangian of our theory. Firstly we recall the result without the chemical potential. In this case, effective action is just of $\mathcal{N} = 2$ SUSY QCD perturbed by adjoint mass term preserving $\mathcal{N} = 1$ SUSY. For the small SUSY breaking parameters $m_{\text{ad}} \ll \Lambda$, the effective action is described as [4]

\[
\mathcal{L}_{\text{eff}} = \mathcal{L}_{\text{SQCD}} + \mathcal{L}_{\text{soft}},
\]

\[
\mathcal{L}_{\text{soft}} = m_{\text{ad}} \int d^2 \theta U(A) + h.c.,
\]

where $\mathcal{L}_{\text{SQCD}}$ is the effective Lagrangian containing full SUSY quantum corrections and $U(A)$ is a superfield whose scalar component is $u(a) = \text{tr} A^2$. The first term consists of vector multiplet and
hypermultiplet parts as will be seen below. The vector multiplet part is described by prepotential being a function of a chiral superfield $A$, vector superfield $V$ and their dual fields $A_D$ and $V_D$. The hypermultiplet part is described by a light hypermultiplet with appropriate quantum numbers $(n_e, n_m)_S$, where $n_e$ is electric charge, $n_m$ is magnetic charge, and $S$ is $U(1)_B$ charge. They couple to vector multiplet superfields $V(V_D)$ and $A(A_D)$ according to their quantum numbers. In the effective Lagrangian around a singular point, the hypermultiplet is expected to be light and enjoys correct degrees of freedom in the effective theory.

Next we incorporate the baryon chemical potential into this system. As in the same manner in the classical theory, the fictitious $U(1)_B$ gauge multiplet is introduced and vacuum expectation values for $V_B$ and $A_B$ are taken to be the same as in the classical theory (see Eq. (2.5)). The light hypermultiplet couples to the gauge multiplet according to its $U(1)_B$ charge. One might consider to introduce a dual field of $U(1)_B$ gauge multiplet in the strong coupling region of the moduli space. However, this is not the case, because the $U(1)_B$ gauge multiplet is not a dynamical field and does not affect the $SU(2)$ gauge dynamics. Resulting effective action is described by

$$L_{\text{eff}} = L_{\text{HM}} + L_{\text{VM}} + L_{\text{soft}},$$

$$L_{\text{HM}} = \sum_{i=2}^{N_f} \left[ \int d^4 \theta \left( H_i^\dagger e^{2n_m V_D + 2n_e V + 2S V_B} H_i + \tilde{H}_i e^{-2n_m V_D - 2n_e V - 2S V_B} \tilde{H}_i^\dagger \right) + \left( \int d^2 \theta \tilde{H}_i (n_m A_D + n_e A + Sm/\sqrt{2}) H_i + h.c. \right) \right],$$

$$L_{\text{VM}} = \frac{1}{4\pi} \text{Im} \left( \int d^4 \theta \frac{\partial F(A)}{\partial A} A_i^i + \int d^2 \theta \frac{1}{2} W^2 \right).$$

Here $H_i$ and $\tilde{H}_i$ denote light quarks, light monopoles or light dyons, and the function $F(A)$ is a prepotential. The effective coupling $\tau$ is written by the prepotential as

$$\tau = \frac{\partial^2 F(a)}{\partial a^2}. \tag{3.16}$$

The scalar potential is easily read off from the Lagrangian (3.13) as

$$V = \left( 2|a + Sm/\sqrt{2}|^2 - \mu^2 \right) (|h_i|^2 + |\tilde{h}_i|^2) + \frac{1}{2b} (|h_i|^2 - |\tilde{h}_i|^2)^2$$

$$+ \frac{1}{b} (2|h_i\tilde{h}_i| + |m_{ad}\kappa|^2) + \frac{2\sqrt{2}}{b} \text{Re}(h_i\tilde{h}_i m_{ad}\kappa), \tag{3.17}$$

where $a$, $h_i$ and $\tilde{h}_i$ are scalar fields of corresponding superfields, and $\kappa = \frac{\partial u}{\partial a}$. Solving stationary conditions with respect to light hypermultiplets, we find $^5$

$$V = -\frac{2\rho_i^4}{b} + \frac{|m_{ad}\kappa|^2}{b}, \tag{3.18}$$

$^5$Similarly to the classical case, we also have another solution which does not exist in the case $\mu > m$. In what follows, we will not consider this solution, since we consider the solution of $\mathcal{N} = 2$ SUSY QCD perturbed by a small $\mu$. 

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where $\rho_i$ is the vacuum expectation value of the condensation of light matters

$$\rho_i^2 = \begin{cases} \frac{b}{2} \left( \mu^2 - 2 |a + S \frac{m}{\sqrt{2}}|^2 \right) + \frac{1}{\sqrt{2}} |m_{ad} \kappa| \quad (\text{for } \rho_i^2 > 0), \\ 0 \quad (\text{otherwise}). \end{cases} \quad (3.19)$$

Here, we took the phase $m_{ad} \kappa = -e^{i\alpha} |m_{ad} \kappa|$ so that the potential is minimized, for simplicity.

We supposed that the potential is described by the adequate variables associated with the light BPS states. For instance, the variable $a$ is understood implicitly as $-a_D$, when we consider the effective potential for the monopole. The same assumption is implicitly included in the following analysis.

4 Numerical analysis on effective potential

In the following, we investigate the potential minimum on the $u$-plane numerically. The scalar potential is a function of $a(u)$, $a_D(u)$, gauge coupling $b = \frac{1}{4\pi} \text{Im} \tau$ and its dual coupling. All their explicit forms needed for our analysis are given in Ref. [9]. Here, we briefly summarize the results for readers’ convenience.

The scalar fields $a$ and $a_D$ are obtained as periods of the elliptic curves. The elliptic curves of $\mathcal{N} = 2$ SUSY QCD with $N_f = 2$ hypermultiplets having the same mass $m$ were found to be

$$y^2 = x^2(x - u) + P_{N_f}(x, u, m, \Lambda), \quad (4.20)$$

where $\Lambda$ is a dynamical scale of the $SU(2)$ gauge interaction. In this case, the polynomials $P_{N_f}$ is given by

$$P_{N_f=2} = -\frac{\Lambda^4}{64} (x - u) + \frac{\Lambda^2}{4} m^2 x - \frac{\Lambda^4}{32} m^2. \quad (4.21)$$

The mass formula of the BPS state with the quantum number $(n_e, n_m)_{S}$ is given by $M_{BPS} = \sqrt{2}|n_m a_D - n_e a + S m/\sqrt{2}|$. If $\lambda$ is a meromorphic differential on the curve Eq. (4.20) such that

$$\frac{\partial \lambda}{\partial u} = \frac{\sqrt{2}}{8\pi} \frac{dx}{y}, \quad (4.22)$$

the periods are given by the contour integrals

$$a_D = \oint_{\alpha_1} \lambda, \ a = \oint_{\alpha_2} \lambda, \quad (4.23)$$

where the cycles $\alpha_1$ and $\alpha_2$ are defined so as to encircle $e_2$ and $e_3$, and $e_1$ and $e_3$, respectively, where they are roots of the algebraic curve Eq. (4.20) in Weierstrass normal form (see Eqs. (4.26) and (4.32)). Meromorphic differentials are given by

$$\lambda_{SW}^{(N_f=2)} = -\frac{\sqrt{2}}{4\pi} \frac{y dx}{x^2 - \frac{\Lambda^4}{64}} = -\frac{\sqrt{2}}{4\pi} \frac{dx}{y} \left[ x - u + \frac{m^2 \Lambda^2}{4 \left( x + \frac{\Lambda^2}{8} \right)} \right]. \quad (4.24)$$
They have a single pole at $x = -\frac{\Lambda}{8}$ and the residue is given by

$$\text{Res}_{\text{SW}}^{(N_f)} = \frac{1}{2\pi i}(-1)\frac{m}{\sqrt{2}}.$$  \hfill (4.25)

We calculate the periods by using the Weierstrass normal form for later convenience. In this form, the algebraic curve is rewritten by new variables $x = 4X + \frac{u}{3}$ and $y = 4Y$, such that

$$Y^2 = 4X^3 - g_2X - g_3 = 4(X - e_1)(X - e_2)(X - e_3), \quad \sum_{i=1}^{3} e_i = 0,$$

where $g_2$ and $g_3$ are explicitly written by

$$g_2 = \frac{1}{16} \left( \frac{4}{3} u^2 + \frac{\Lambda^4}{16} - m^2 \Lambda^2 \right),$$

$$g_3 = \frac{1}{16} \left( \frac{m^2 \Lambda^4}{32} - \frac{u}{12} m^2 \Lambda^2 - \frac{u \Lambda^4}{96} + \frac{2 u^3}{27} \right).$$ \hfill (4.26)

Converting the Seiberg-Witten differentials of Eq. (4.24) into the Weierstrass normal form and substituting it into Eq. (4.23), we obtain the integral representations of the periods as follows ($a_D$ and $a$ are denoted by $a_1$ and $a_2$, respectively):

$$a_i = -\frac{\sqrt{2}}{4\pi} \left( -\frac{4}{3} I_1^{(i)} + 8 I_2^{(i)} + \frac{m^2 \Lambda^2}{8} I_3^{(i)}(c) \right),$$ \hfill (4.29)

where $c = -\frac{u}{12} - \frac{\Lambda^2}{32}$ is the pole of the differentials. Integrals $I_1^{(i)}$, $I_2^{(i)}$ and $I_3^{(i)}$ are defined as

$$I_1^{(i)} = \frac{1}{2} \oint_{a_i} \frac{dX}{Y}, \quad I_2^{(i)} = \frac{1}{2} \oint_{a_i} \frac{X dX}{Y}, \quad I_3^{(i)}(c) = \frac{1}{2} \oint_{a_i} \frac{dX}{Y(X - c)}.$$

The roots $e_i$ of the polynomial defining the cubic are chosen so as to lead to the correct asymptotic behavior for large $|u|$,

$$a_D(u) \sim i \frac{4 - N_f}{2\pi} \sqrt{2u \log \frac{u}{\Lambda^2}}, \quad a(u) \sim \sqrt{2u} \frac{2}{2}.$$ \hfill (4.31)

A correct choice is the following:

$$e_1 = \frac{u}{24} - \frac{\Lambda^2}{64} - \frac{1}{8} \sqrt{u + \frac{\Lambda^2}{8} + \Lambda m \sqrt{u + \frac{\Lambda^2}{8} - \Lambda m}},$$

$$e_2 = \frac{u}{24} - \frac{\Lambda^2}{64} + \frac{1}{8} \sqrt{u + \frac{\Lambda^2}{8} + \Lambda m \sqrt{u + \frac{\Lambda^2}{8} - \Lambda m}},$$

$$e_3 = -\frac{u}{12} + \frac{\Lambda^2}{32}.\quad (4.32)$$

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Fixing the contours of the cycles relative to the positions of the poles, which is equivalent to fix the $U(1)$ baryon numbers for the BPS states, the final formulae are given by

$$a_i = -\frac{\sqrt{3}}{4\pi} \left(-\frac{4}{3}u I_1^{(i)} + 8I_2^{(i)} + \frac{m^2\Lambda^2}{8}f_3^{(i)} \left(-\frac{u}{12} - \frac{\Lambda^2}{32}\right)\right) - \frac{m}{\sqrt{2}}\delta_{ij}, \quad (4.33)$$

with the integral $I_j^{(1)} (j = 1, 2, 3)$ explicitly given by

$$I_1^{(1)} = \int_{e_2}^{e_3} \frac{dX}{Y} = \frac{iK(k')}{\sqrt{e_2 - e_1}}, \quad (4.34)$$

$$I_2^{(1)} = \int_{e_2}^{e_3} \frac{XdX}{Y} = \frac{ie_1}{\sqrt{e_2 - e_1}}K(k') + i\sqrt{e_2 - e_1}E(k'), \quad (4.35)$$

$$I_3^{(1)} = \int_{e_2}^{e_3} \frac{dX}{Y(X - c)} = -\frac{i}{(e_2 - e_1)^{3/2}} \left\{ \frac{1}{k + \tilde{c}}K(k') + \frac{4k}{1 + k} - \frac{1}{c^2 - k^2} \Pi_1 \left(\nu, \frac{1 - k}{1 + k}\right) \right\}, \quad (4.36)$$

where $k^2 = \frac{e_3 - e_1}{e_2 - e_1}$, $k'^2 = 1 - k^2 = \frac{e_3 - e_3}{e_2 - e_1}$, $\tilde{c} = \frac{e_2 - e_1}{e_2 - e_1}$, and $\nu = -\left(\frac{k + \tilde{c}}{k - \tilde{c}}\right)^2 \left(\frac{1 - k}{1 + k}\right)^2$. The formulae for $I_j^{(2)}$ are obtained from $I_j^{(1)}$ by exchanging the roots, $e_1$ and $e_2$. In Eqs. (4.34)-(4.36), $K$, $E$, and $\Pi_1$ are the complete elliptic integrals \[7\] given by

$$K(k) = \int_0^1 \frac{dx}{\sqrt{(1 - x^2)(1 - k^2x^2)}}, \quad (4.37)$$

$$E(k) = \int_0^1 dx \left(\frac{1 - k^2x^2}{1 - x^2}\right)^{1/2},$$

$$\Pi_1(\nu, k) = \int_0^1 \frac{dx}{\sqrt{(1 - x^2)(1 - k^2x^2)}(1 + \nu x^2)}.$$

The effective coupling $\tau$ and $\kappa$ can be calculated by using above results.

$$\tau = \frac{\partial a_D}{\partial a} = \frac{\omega_1}{\omega_2}, \quad (4.38)$$

$$\kappa = \frac{\partial u}{\partial a} = \frac{1}{\omega_1}, \quad (4.39)$$

where $\omega_i$ is the period of the Abelian differential,

$$\omega_i = \int_{\alpha_i} \frac{dX}{Y} = 2I_1^{(i)} \quad (i = 1, 2). \quad (4.40)$$

Before going to numerical analysis of the effective potential, let us see behaviors of the singular point with respect to the mass of hypermultiplets. The effective potential is expected to have its minimum around the singular points, because the singular point is energetically favored due to the non-zero condensation (see Eq. (3.18)) of the light BPS state such as a quark, monopole or dyon with appropriate quantum number $(n_e, n_m)$. The solution of the cubic polynomial determines the position of the singular number on the $u$-plane \[4\]. In $N_f = 2$ case with the same masses, the solution is given by

$$u_1 = -m\Lambda - \frac{\Lambda^2}{8}, \quad u_2 = m\Lambda - \frac{\Lambda^2}{8}, \quad u_3 = m^2 + \frac{\Lambda^2}{8}. \quad (4.41)$$
In the following, we fix the dynamical scale $\Lambda$ to be $\Lambda = 2\sqrt{2}$. The flow of the singular points with respect to the hypermultiplet mass is sketched in Fig. 4. For $m = 0$, the singular points appear at $u_1 = u_2 = -1$ and $u_3 = 1$. Here, at $u = -1$, two singular points degenerate. For non-zero $m > 0$, this singular point splits into two singular points $u_1$ and $u_2$, which correspond to the BPS states with quantum numbers $(1, 1)_{-1}$ and $(1, 1)_1$, respectively. As $m$ is increasing, these singular points, $u_1$ and $u_2$, are moving to the left and the right on real $u$-axis, respectively. Two singular points, $u_2$ and $u_3$, collide and degenerate at the Argyres-Douglas point $\mathbf{8}$ ($u = \frac{3\Lambda^2}{8}$) for $m = \frac{\Lambda}{2}$, where it is believed that the theory becomes superconformal one. As $m$ is increasing further, there appear two singular points $u_2$ and $u_3$ again, and quantum numbers of the corresponding BPS states, $(1, 1)_1$ and $(0, 1)_0$, change into $(1, 0)_1$ and $(1, -1)_1$, respectively. The singular point $u_2$ is moving to the right faster than $u_3$.

In the following, to see the effect of the dynamics on breaking of the baryon number symmetry, we analyze only the case with $m > \Lambda/2$. In this parameter region, the quantum theory has the quark singular point at weak coupling region, which corresponds to the one in the classical theory. Therefore, other potential minima, if we find them, correspond to the vacuum preserving

\footnote{We consider only the case $m > 0$, since the result for $m < 0$ can be obtained by exchanging $u_1 \leftrightarrow u_2$, as can be seen from the first two equations in Eq. (4.11).}
Figure 6: Plots of the potential and the condensations for two dyons and quark along real $u$ axis for $\mu = 0.06$, $m_{ad} = 0.01$ and $m = 5$.

the baryon number symmetry at the classical level. On the other hand, for $m < \Lambda/2$, the quantum theory has no quark singular point, while the classical theory has it. The quark cannot appear as a light state and, instead, other solitonic state such as dyon and monopole appear due to the strong gauge dynamics at the region $u < \Lambda^2$.

Now we are ready for numerical analysis. The effective potential and the condensations for light matters along real $u$ axis are shown in Figs. 5 and 6. When $m$ is real, the singular point is on the real $u$ axis and the effective potential has its minimum there. In the left figure in Fig. 6, the right minimum corresponds to the quark singularity. The quark condensates around this singular point is depicted in the right figure in Fig. 6. Since the quark has the $U(1)_B$ baryon number $S = 1$, the baryon number symmetry is broken. This potential minimum corresponds to the global minimum at the classical theory and the baryon number symmetry is already broken there. The left and middle minima in the left figure in Fig. 6 correspond to the dyon singular points where they have non-zero vacuum expectation values. Since these dyons have $U(1)_B$ baryon number $S = -1$ and $S = 1$ for the left and middle singular point, respectively, the baryon number symmetry is broken at these points. Recall that these minima corresponds to the classical minimum at the origin. Therefore, we see that the classical minimum splits into two dyon singular points and the baryon number is spontaneously broken due to the strong gauge dynamics.

Note that these dyon singular points are not global minimum and the global one is located at quark singular point. This situation is similar to the classical theory where the global minimum appears at the quark singular point while the minimum at the origin is the local one. Fig. 7 shows the vacuum energy for each singular points as a function of the baryon chemical potential $\mu$.

The effective potential around the left dyon point is magnified in Fig. 8. We can see the cusp at the minimum, which originates from the non-zero baryon chemical potential. The same things happen at other potential minima. The existence of the cusps would imply that, for precise description of the effective potential around the potential minima, the periods ($a$ and $a_D$) and the dyon hypermultiplets are not the complete set of dynamical variables and introduction of new variables is necessary, by which the cusps are smoothed away. Unfortunately, we could not find
Figure 7: The evolutions of the potential energies at quark (solid line), the left dyon (dashed line) and the middle dyon (dotted line) singular points as a function of $\mu$ for $m_{ad} = 0.01$ and $m = 5$.

Figure 8: Plot around the left dyon singular point.

such variables. However, considering that the cusps disappear in the limit $\mu \to 0$, we can expect that the role of such variables is just to smooth the cusps away. Thus, the vacuum structure of our system would not be drastically changed, and the baryon number is still broken at the minima after such new variables are taken into account.

We have found that, once non-perturbative effects are taken into account, the baryon number is always broken for any baryon chemical potential. Fig. 9 shows the effect of the baryon chemical potential on the effective potential. We can see that each potential well becomes deeper and wider as the chemical potential is raised (see also Fig. 7). When we raise the chemical potential further, potential wells very much overlap with. In this case, description of our effective theory by using each suitable dynamical variable around each potential minimum breaks down. This corresponds to the breakdown of our effective theory approach for the case with $\mu > \Lambda$. 
Figure 9: Plots of the scalar potential along real $a$ axis for $m_{ad} = 0.01$ and $m = 5$ with $\mu = 0.06$(left), 0.15(middle), 0.3(right).

5 Conclusion

We investigate vacuum structure of $\mathcal{N} = 2$ SUSY QCD in the presence of the baryon chemical potential. Our theory is based on the gauge group $SU(2)$ massive $N_f = 2$ quark hypermultiplets. The baryon chemical potential is introduced in the theory through the fictitious $U(1)_B$ gauge field. When the mass term for the adjoint chiral gauge multiplet, which breaks $\mathcal{N} = 2$ SUSY into $\mathcal{N} = 1$, is introduced, the classical theory has two discrete vacua in some parameter region for the chemical potential, the common mass of the quark hypermultiplets and the adjoint mass. One is the global minimum where the baryon number is broken, while the other is the local one preserving the baryon number symmetry. We concentrated on the local minimum and examined how the baryon number preserving minimum is deformed when all (supersymmetric) quantum corrections are taken into account. In our analysis, the SUSY breaking parameters, namely the baryon chemical potential and the adjoint mass, were taken to be much smaller than the dynamical scale of the $SU(2)$ gauge interaction and the exact result on $\mathcal{N} = 2$ SUSY QCD was used to describe the effective potential at the leading order of perturbation with respect to the small SUSY breaking parameters. We found that the classical minimum was deformed due to the strong $SU(2)$ gauge dynamics and the baryon number was spontaneously broken by the dyon condensations. Therefore, color superconductivity takes place due to the strong gauge dynamics. One the other hand, the global minimum at the classical level lies in the perturbative regime and thus remains at the quantum level.

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References

[1] D. Bailin and A. Love, Phys. Rept. 107 (1984) 325; M. Iwasaki and T. Iwado, Phys. Lett. B 350 (1995) 163; M. G. Alford, K. Rajagopal and F. Wilczek, Phys. Lett. B 422 (1998) 247 [arXiv:hep-ph/9711395]; R. Rapp, T. Schafer, E. V. Shuryak and M. Velkovsky, Phys. Rev. Lett. 81 (1998) 53 [arXiv:hep-ph/9711396].

[2] M. G. Alford, K. Rajagopal and F. Wilczek, Nucl. Phys. B 537 (1999) 443 [arXiv:hep-ph/9804403].

[3] K. A. Intriligator and N. Seiberg, Nucl. Phys. Proc. Suppl. 45BC, 1 (1996) [arXiv:hep-th/9509066].

[4] N. Seiberg and E. Witten, Nucl. Phys. B 426, 19 (1994) [Erratum-ibid. B 430, 485 (1994)] [arXiv:hep-th/9407087]; Nucl. Phys. B 431, 484 (1994) [arXiv:hep-th/9408099].

[5] R. Harnik, D. T. Larson and H. Murayama, JHEP 0403, 049 (2004) [arXiv:hep-ph/0309224].

[6] N. Maru and M. Tachibana, [arXiv:hep-ph/0411079].

[7] A. Erdelyi et al., *Higher Transcendental Functions*, Vol. 1, McGraw-Hill, New York (1953).

[8] P. C. Argyres and M. R. Douglas, Nucl. Phys. B 448, 93 (1995) [arXiv:hep-th/9505062].

[9] M. Arai and N. Okada, Phys. Rev. D 64, 025024 (2001) [arXiv:hep-th/0103157].