A Formula for the S-Class Number of an Algebraic Torus

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Abstract

We obtain a formula for the S-class number of an algebraic torus defined over a number field in terms of the étale and Galois cohomology groups of its character module. As applications, we give different proofs of some classical class number formulas of Shyr, Ono, Katayama and Morishita.

1 Introduction

Let $T$ be an algebraic torus defined over a number field $K$ with character group $\hat{T}$. Let $S_\infty$ be the set of all archimedean places of $K$ and $S$ be a finite set of places of $K$ containing $S_\infty$. Let $O_{K,S}$ be the ring of $S$-unit integers and $U = \text{Spec}(O_{K,S})$. We write $j : \text{Spec}(K) \to U$ for the inclusion of the generic point.

For each finite prime $v$ of $K$, let $I_v$ be the inertia group of $v$. For $n \geq 1$, the Tate-Shafarevich group $\text{III}^n(T)$ is defined as the kernel of the restriction map

$$\Delta^n : H^n(K,T) \to \prod_{v \in S} H^n(K_v,T).$$

We also write $[M]$ for the order of a finite group $M$. Our main result is the following formula for the $S$-class number $h_{T,S}$ of $T$.

**Main Theorem.** Let $T$ be an algebraic torus over a number field $K$ with character group $\hat{T}$. Let $\Psi^n(j_\ast \hat{T})$ be the map $H^n_{\text{et}}(U,j_\ast \hat{T}) \to \prod_{v \in S} H^n(K_v,\hat{T})$. Then

$$h_{T,S} = \frac{[\text{Ext}^1_U(j_\ast \hat{T},\mathbb{G}_m)][H^1(K,\hat{T})]}{[\ker\Psi^1(j_\ast T)][\text{III}^1(T)] \prod_{v \in S} [H^1(K_v,T)] \prod_{v \notin S} [H^0(\hat{Z},H^1(I_v,\hat{T}))]}.$$  \hspace{1cm} (1.1)

The class number is an important invariant of an algebraic torus. For example, it appears in the formulas for the special values of the L-function of the torus [Ono61]. Shyr proved a formula relating two class numbers of two isogenous tori in [Shy77]. Using Shyr’s result, Ono [Ono87] obtained a formula for the class number of a norm torus of a Galois extension. Using a different method, Katayama [Kat91] found formulas for the class numbers of a norm torus and its dual for an arbitrary extension. Morishita used the techniques from Nisnevich cohomology to generalize Ono’s formula to $S$-class number [Mor91]. More recently, Gonzalez-Aviles used the Nisnevich cohomology and the result of Xarles [Xar93] on the group of components of Néron-Raynauld models of tori to give a generalization of Chevalley’s ambiguous class number formula and the Capitulation problem [GA08,GA10]. Even though the Nisnevich cohomology seems more natural for the questions concerning the class numbers of tori, we work with the étale cohomology as it is more elementary and has more machinery. The proof of our Main Theorem uses homological algebra techniques together with results in Galois and étale cohomology, for example the Poitou-Tate exact sequence.
and the Artin-Verdier Duality. As applications, we will give different proofs of the formulas of Shyr, Ono, Katayama and Morishita mentioned above. Although the following is not discussed in this paper, we cannot resist mentioning that (1.1) will be used in a future paper to prove the following theorem.

**Theorem 1.1.** Let $T$ be an algebraic torus defined over a number field $K$ with character group $\hat{T}$. Suppose $S$ is a finite set of places of $K$ containing $S_\infty$. We denote $h_{T,S}$, $R_{T,S}$ and $\omega_T$ for the $S$-class number, the $S$-regulator and the number of roots of unity of $T$ respectively. Let $L_S(\hat{T}, s)$ be the partial Artin $L$-function associated with the $G_K$-representation $\hat{T} \otimes_{\mathbb{Z}} \mathbb{C}$ modulo the local factors at $S$. Then $\operatorname{ord}_{s=0} L_S(\hat{T}, s) = \operatorname{rank}_\mathbb{Z} T(O_{K,S})$ and

$$L_S^*(\hat{T}, 0) = \pm \frac{h_{T,S} R_{T,S} \left[ \prod \left( \begin{array}{c} 1 \\ \hat{T} \end{array} \right) \right]}{\omega_T} \left[ H^1(K, \hat{T}) \right] \prod_{v \in S} \left[ H^1(K_v, \hat{T}) \right] \prod_{v \notin S} \left[ H^0(\hat{\mathbb{Z}}, H^1(I_v, \hat{T})) \right]. \quad (1.2)$$

This paper is organized as follows. In section 2, we recall some basic facts about algebraic tori and the Artin-Verdier Duality. In section 3, we obtain an exact sequence for Galois modules over local fields which is necessary for the proof of (1.1) given in section 4. The rest of the paper is for applications of (1.1). More precisely, we give different proofs of the formulas of Ono, Katayama and Morishita in section 5 and generalize the formula of Shyr in section 6.

This paper is part of my PhD thesis written at Brown University. I would like to express my gratitude to my advisor Professor Stephen Lichtenbaum for his guidance and encouragement. It is my pleasure to acknowledge the support from the Deutsche Forschungsgesellschaft through the SFB 1085 Higher Invariant Research Group at University of Regensburg. Finally, I would like to thank Professor Guido Kings for his support and my friend Yigeng Zhao for many helpful conversations.

## 2 Preliminaries

### 2.1 Algebraic Tori

Let $T$ be an algebraic torus defined over a number field $K$ with character group $\hat{T}$. For each place $v$ of $K$, let $T_v$ be the base extension of $T$ to $K_v$ and let $\hat{T}_v$ be the character group of $T_v$. We define $T_v^c$ to be

$$T_v^c := \{ x \in T_v(K_v) \text{ such that for all } \chi \in H^0(K_v, \hat{T}_v), |\chi(x)|_v = 1 \}.$$ 

Then $T_v^c$ is the unique maximal compact subgroup of $T_v(K_v)$. If $v$ is a finite place then $T_v^c \simeq \text{Hom}_{K_v}(\hat{T}_v, \hat{O}_v)$ [Ono61] page 115-116]. The following definitions are taken from [Ono61].

**Definition 2.1.** Let $S$ be a finite set of places of $K$ which contains $S_\infty$. Define

$$T_{h,S} := \prod_{v \in S} T_v(K_v) \times \prod_{v \notin S} T_v^c \quad \& \quad T_h := \lim_{S \text{ finite}} T_{h,S}$$

**Definition 2.2.** Let $S$ be a finite set of places of $K$ containing $S_\infty$. Define

1. $T(O_{K,S}) := T_{h,S} \cap T(K)$ - the $S$-units of $T$.
2. $\text{Cl}_S(T) := T_h/T(K)T_{h,S}$ - the $S$-class group of $T$.

Then $T(O_{K,S})$ is finitely generated and $\text{Cl}_S(T)$ is a finite abelian group. We denote $h_{T,S}$ for the order of $\text{Cl}_S(T)$, it is also called the $S$-class number of $T$.  

Proposition 2.3. We have the exact sequences

\[ 0 \to T(O_{K, S}) \to T(K) \to \prod_{v \not\in S} T(K_v)/T_v^c \to Cl_S(T) \to 0, \tag{2.1} \]

\[ 0 \to T(O_K) \to T(O_{K, S}) \to \prod_{v \in S - S_\infty} T(K_v)/T_v^c \to Cl(T) \to Cl_S(T) \to 0. \tag{2.2} \]

In particular, \( T(O_{K, S})_{tor} \simeq T(O_K)_{tor} \).

Proof. Note that \( T_h/T_h,S \simeq \prod_{v \not\in S} T(K_v)/T_v^c \). Then (2.1) follows immediately. To prove (2.2), we only need to apply the Snake Lemma to the following commutative diagram

\[ \begin{CD}
0 @>>> 0 @>>> T(K) @>>> T(K) @>>> 0
\end{CD} \]

\[ \begin{CD}
0 @>>> \prod_{v \in S - S_\infty} T(K_v)/T_v^c @>>> \prod_{v \not\in S_\infty} T(K_v)/T_v^c @>>> \prod_{v \not\in S} T(K_v)/T_v^c @>>> 0
\end{CD} \]

and invoke (2.1) for \( S \) and \( S_\infty \). The last statement follows from (2.2) and the fact that \( T(K_v)/T_v^c \) is torsion free.

2.2 Artin-Verdier Duality

Let \( K \) be a number field with Galois group \( G_K \) and \( X = \text{Spec}(O_K) \). Let \( U \) be an open subscheme of \( X \). For any sheaf \( \mathcal{F} \) on \( U \), let \( \mathcal{F}_v \) be the unique discrete \( G_K_v \)-module corresponding to the pull-back of \( \mathcal{F} \) to \( \text{Spec}(K_v) \). In \cite{Milne2006} (page 165), Milne constructed the cohomology groups with compact support \( H^r_c(U, \mathcal{F}) \) which satisfies the following long exact sequence

\[ \ldots \to H^r_c(U, \mathcal{F}) \to H^r_{et}(U, \mathcal{F}) \to \bigoplus_{v \not\in U} H^r(K_v, \mathcal{F}_v) \to \ldots \tag{2.3} \]

where \( H^r(K_v, \mathcal{F}_v) \) is the usual Galois cohomology if \( v \) is a finite prime and is the Tate cohomology if \( v \) is an archimedean prime. For an abelian group \( A \), let \( \hat{A} \) be the profinite completion of \( A \) and \( A^D := \text{Hom}_\mathbb{Z}(A, \mathbb{Q}/\mathbb{Z}) \). The following theorem will be used many times in this paper.

Theorem 2.4 (Artin-Verdier Duality). There is a canonical isomorphism \( H^3_c(U, \mathbb{G}_m) \simeq \mathbb{Q}/\mathbb{Z} \).

Furthermore, for any \( \mathbb{Z} \)-constructible sheaf \( \mathcal{F} \) on \( U \), the pairing

\[ H^2_{c-r}(U, \mathcal{F}) \times \text{Ext}_U^r(\mathcal{F}, \mathbb{G}_m) \to H^3_c(U, \mathbb{G}_m) \]

induces a map \( \alpha^r(\mathcal{F}) : \text{Ext}_U^r(\mathcal{F}, \mathbb{G}_m) \to H^3_{c-r}(U, \mathcal{F})^D \) which satisfies

1. For \( r = 0, 1 \), \( \text{Ext}_U^r(\mathcal{F}, \mathbb{G}_m) \) is finitely generated and \( \alpha^r(\mathcal{F}) \) induces an isomorphism of profinite groups

\[ \alpha^r(\mathcal{F}) : \text{Ext}_U^r(\mathcal{F}, \mathbb{G}_m) \to H^3_{c-r}(U, \mathcal{F})^D. \]

2. For \( r = 2, 3 \), \( \alpha^r(\mathcal{F}) \) is an isomorphism between groups of cofinite type.

Furthermore, if \( \mathcal{F} \) is constructible then \( \alpha^r(\mathcal{F}) \) is an isomorphism of finite groups for all \( r \).

Proof. See \cite{Milne2006} II.3.1.\]
Definition 2.5. We say $\mathcal{F}$ is a negligible sheaf on $U$ if $\mathcal{F}$ has finite support and its stalks are finite everywhere. Note that negligible sheaves are constructible.

Lemma 2.6. Let $\mathcal{F}$ be a negligible sheaf on $U$. Then

1. $H^0_{\text{et}}(U, \mathcal{F}) \simeq H^0_{\text{et}}(U, \mathcal{F})$ for all $n$.

2. $H^n_{\text{et}}(U, \mathcal{F}) = 0$ for $n > 1$ and $\text{Ext}^n_{\text{et}}(\mathcal{F}, \mathbb{G}_m) = 0$ for $n = 0, 1$.

3. $[H^0_{\text{et}}(U, \mathcal{F})] = [H^1_{\text{et}}(U, \mathcal{F})]$ and $[\text{Ext}^1_{\text{et}}(\mathcal{F}, \mathbb{G}_m)] = [\text{Ext}^2_{\text{et}}(\mathcal{F}, \mathbb{G}_m)]$.

Proof. It suffices to assume $\mathcal{F} = i_* M$ where $i : p \to U$ is the immersion of a closed point $p$ of $U$ and $M$ is a finite discrete $\hat{Z}$-module. As the generic stalk of $i_* M$ vanishes, $H^0_{\text{et}}(U, i_* M) \simeq H^0_{\text{et}}(U, i_* M)$ by (2.3). Since $H^n_{\text{et}}(U, i_* M) \simeq H^n(\hat{Z}, M)$ and $\hat{Z}$ has cohomological dimension 2, $H^n_{\text{et}}(U, i_* M) = 0$ for $n > 1$. Hence, by Theorem 2.4 $\text{Ext}^n_{\text{et}}(i_* M, \mathbb{G}_m) = 0$ for $n = 0, 1$. Finally, $[H^0(\hat{Z}, M)] = [H^1(\hat{Z}, M)]$ by [Mil06 page 32].

Example 2.7. Let $N$ be a discrete $G_K$-module, torsion free as an abelian group. Then any subsheaf of $R^1 j_* N$ is negligible. Indeed, the generic stalk of $R^1 j_* N$ vanishes so it only has finite support. Moreover, if $\bar{p}$ is a geometric point over a closed point $p$ of $U$ then $(R^1 j_* N)_{\bar{p}} \simeq H^1(I_p, N)$ which is finite.

Proposition 2.8. Let $M$ be a discrete $G_K$-module, torsion free and finitely generated as an abelian group. Then

1. $H^n_{\text{et}}(U, j_* M), H^n_{\text{et}}(U, j_* M), H^1_{\text{et}}(U, j_* M)$ and $\text{Hom}_{\text{et}}(j_* M, \mathbb{G}_m)$ are finitely generated.

2. $H^1_{\text{et}}(U, j_* M), H^2_{\text{et}}(X, j_* M)$ and $\text{Ext}^1_{\text{et}}(j_* M, \mathbb{G}_m)$ are finite abelian groups.

3. $\text{Ext}^n_{\text{et}}(j_* M, \mathbb{G}_m)$ are of cofinite type for $n = 2, 3$.

Proof. By [Ono61, 1.5.1], there exist finitely many Galois extensions $\{K_\mu\}_\mu, \{K_\lambda\}_\lambda$ of $K$, a positive integers $n$, and a finite $G_K$-module $N$ such that we have the exact sequence

$$0 \to M^n \oplus \prod_\mu (\pi_\mu)_* \mathbb{Z} \to \prod_\lambda (\pi_\lambda)_* \mathbb{Z} \to N \to 0,$$

where $\pi_\mu$ is the map $\text{Spec}(K_\mu) \to \text{Spec}(K)$. Let $V_\mu$ be the normalization of $U$ in $K_\mu$ and $\pi'_\mu$ be the map $V_\mu \to U$. Applying $j_*$ to (2.4) we obtain

$$0 \to (j_* M)^n \oplus \prod_\mu (\pi'_\mu)_* \mathbb{Z} \to \prod_\lambda (\pi'_\lambda)_* \mathbb{Z} \to \mathcal{Q} \to 0$$

where $\mathcal{Q}$ is a constructible sheaf. From [Mil06 II.2.1], $\text{Ext}^n_{\text{et}}((\pi'_\mu)_* \mathbb{Z}, \mathbb{G}_m) \simeq H^n_{\text{et}}(V_\mu, \mathbb{G}_m)$ is finitely generated, finite, cofinite type for $n = 0, 1$ and $2, 3$ respectively. Therefore, by the long exact sequence of the $\text{Ext}$-groups associated to (2.5), we obtain the similar statement for $\text{Ext}^n_{\text{et}}(j_* M, \mathbb{G}_m)$.

As $H^n_{\text{et}}(U, j_* M) \simeq H^n(K, M)$, it is finitely generated. By the long exact sequence of $H^n_{\text{et}}(U, j_* M)$ associated with (2.5) and the fact that $H^1_{\text{et}}(U, \mathbb{Z}) = 0$, we deduce that $H^1_{\text{et}}(U, j_* M)$ is finite. Finally, we have the exact sequence from (2.3)

$$0 \to \prod_{v \in S} H^{-1}(K_v, M) \to H^0_{\text{et}}(U, j_* M) \to H^0(K, M) \to \prod_{v \in S} H^0(K_v, M) \to H^1_{\text{et}}(U, j_* M) \to H^1_{\text{et}}(U, j_* M)$$

Since $\prod_{v \in S} H^n(K_v, M)$ is finitely generated for all $n$ (note that for $v \in S_{\infty}, H^n(K_v, M)$ means the Tate cohomology group), $H^1_{\text{et}}(U, j_* M)$ is finitely generated for $n = 0, 1$. □
3 Local Galois Cohomology

Let $K$ be a number field with Galois group $G_K$. Let $U$ be an open subscheme of $\text{Spec}(O_K)$ and $j: \text{Spec}(K) \to U$. For each finite place $v$ of $K$, we write $K_v^{ur}$ for the maximal unramified extension of the completion $K_v$ of $K$. We denote by $I_v = G(K_v/K_v^{ur})$ the inertia group of $v$. Let $O_v$, $O_v^{ur}$ and $\hat{O}_v$ be the valuation rings of $K_v$, $K_v^{ur}$ and $K_v$ respectively. For the rest of this paper, by a discrete $G_K$-module, we mean a finitely generated abelian group with continuous $G_K$-action.

**Lemma 3.1.** Let $\mathcal{F}$ be a sheaf on $U$ and $\mathcal{F}_K$ be the $G_K$-module corresponding to the sheaf $j^* \mathcal{F}$ on $\text{Spec}(K)$. Then $\mathcal{F}_K = \text{Hom}_\mathbb{Z}(\mathcal{F}_K, \mathbb{K}^*)$.

1. Let $\text{Ext}^0_U(\mathcal{F}, j_* \mathcal{G}_m) \simeq H^n(K, \hat{\mathcal{F}}_K)$. In particular, $\text{Ext}^0_U(j_* \hat{T}, j_* \mathcal{G}_m) \simeq H^n(K, T)$.

2. Let $i: v \to U$ be a closed immersion. Then $\text{Ext}^0_U(\mathcal{F}, i_* \mathbb{Z}) \simeq \text{Ext}^0_Z(\mathcal{F}_K^I, \mathbb{Z})$. In particular, $\text{Ext}^0_U(j_* \hat{T}, i_* \mathbb{Z}) \simeq \text{Ext}^0_Z(T^I, \mathbb{Z})$.

**Proof.** 1. From [Mil06 II.1.4], $R^q j_* \mathcal{G}_m = 0$ for $q > 0$. Thus the spectral sequence $\text{Ext}^p_U(\mathcal{F}, R^q j_* \mathcal{G}_m) \Rightarrow \text{Ext}^{p+q}_{G_K}(\mathcal{F}_K, \mathbb{K}^*)$ collapses and yields $\text{Ext}^p_U(\mathcal{F}, j_* \mathcal{G}_m) \simeq \text{Ext}^p_{G_K}(\mathcal{F}_K, \mathbb{K}^*)$. From [Mil06 I.0.8], there is a spectral sequence $H^p(G_K, \text{Ext}^q_Z(\mathcal{F}_K, \mathbb{K}^*)) \Rightarrow \text{Ext}^{p+q}_{G_K}(\mathcal{F}_K, \mathbb{K}^*)$.

Since $\hat{K}^*$ is divisible, $\text{Ext}^q_Z(N, \hat{K}^*) = 0$ for $q > 0$. Thus, $H^0(G_K, \text{Hom}_\mathbb{Z}(\mathcal{F}_K, \hat{K}^*)) \simeq \text{Ext}^0_{G_K}(\mathcal{F}_K, \hat{K}^*)$. Hence, $\text{Ext}^0_U(\mathcal{F}_K, j_* \mathcal{G}_m) \simeq H^n(K, \hat{\mathcal{F}}_K)$. Finally, if $\mathcal{F} = j_* \hat{T}$ then $\hat{\mathcal{F}}_K = \text{Hom}_\mathbb{Z}(\hat{T}, \hat{K}^*) \simeq T(\hat{K})$. Therefore, $\text{Ext}^0_U(j_* \hat{T}, j_* \mathcal{G}_m) \simeq H^n(K, T)$.

2. Since $i_*$ is exact, the spectral sequence $\text{Ext}^p_U(\mathcal{F}, R^m i_* \mathbb{Z}) \Rightarrow \text{Ext}^{p+m}_{G_K}(i^* \mathcal{F}, \mathbb{Z}) \simeq \text{Ext}^{p+m}_Z(T^I, \mathbb{Z})$ implies that $\text{Ext}^p_U(\mathcal{F}, i_* \mathbb{Z}) \simeq \text{Ext}^p_Z(T^I, \mathbb{Z})$.

**Proposition 3.2.** Let $\mathbb{N}$ be a discrete $G_{K_v}$-module. Let $\hat{\mathbb{N}} = \text{Hom}_\mathbb{Z}(N, \hat{K}_v^*)$ and $\hat{\mathbb{N}}^c = \text{Hom}_\mathbb{Z}(N, \hat{O}_v^*)$.

Then

1. $H^0(K_v, \hat{\mathbb{N}}^c) = \{ f \in \text{Hom}_{G_{K_v}}(N, \hat{K}_v^*) : \text{ for all } x \in H^0(K_v, N), \text{ we have } f(x) \in O_v^* \}$.

2. $\text{Ext}^2_Z(N^I_v, \mathbb{Z}) \simeq H^2(K_v, \hat{\mathbb{N}})$.

3. The following sequence is exact

$$0 \to H^0(K_v, \hat{\mathbb{N}}^c) \to H^0(K_v, \hat{\mathbb{N}}) \to \text{Hom}_\mathbb{Z}(N^I_v, \mathbb{Z}) \to H^0(\hat{Z}, H^1(I_v, N))^D \to$$

$$\to H^1(K_v, \hat{\mathbb{N}}) \to \text{Ext}^1_Z(N^I_v, \mathbb{Z}) \to 0.$$ (3.1)

**Proof.** 1. Let $f \in H^0(K_v, \hat{\mathbb{N}}^c)$. For any $x \in H^0(K_v, N)$, $f(x) \in O_v^*$ by definition. As $f$ is $G_{K_v}$-invariant, $f(x) \in O_v^*$. Thus, $H^0(K_v, \hat{\mathbb{N}}^c)$ is a subset of the right hand side.

Conversely, let $f$ be an element of the right hand side. Let $L_v$ be a finite Galois extension of $K_v$ such that the Galois group $G_{L_v}$ acts trivially on $N$. For $x \in N$, $f(x) \in L_v^*$ as $N = H^0(L_v, N)$. We have $N_{L_v/K_v}(f(x)) = f(T_{L_v/K_v}(x))$. As $T_{L_v/K_v}(x) \in H^0(K_v, N)$, $f(T_{L_v/K_v}(x)) \in O_v^*$. Hence, $N_{L_v/K_v}(f(x)) \in O_v^*$. We deduce that $f(x) \in O_v^* \subset O_v^*$. As a result, the right hand side is a subset of $H^0(K_v, \hat{\mathbb{N}}^c)$. 

2. By [Mil06 I.1.10], \( \text{Ext}^2_{\hat{\mathbb{Z}}}(N^t, \mathbb{Z}) \simeq H^0(\hat{\mathbb{Z}}, N^t)^D \simeq H^0(K_v, N)^D \). From [Mil06 I.2.1], we have \( H^2(K_v, \hat{\mathbb{N}}) \simeq H^0(K_v, N)^D \). Thus, \( H^2(K_v, \hat{\mathbb{N}}) \simeq \text{Ext}^2_{\hat{\mathbb{Z}}}(N^t, \mathbb{Z}) \).

3. From the spectral sequence \( H^r(\hat{\mathbb{Z}}, H^s(I_v, N)) \Rightarrow H^{r+s}(K_v, N) \), we obtain

\[
0 \to H^1(\hat{\mathbb{Z}}, N^t) \to H^1(K_v, N) \to H^0(\hat{\mathbb{Z}}, H^1(I_v, N)) \to H^2(\hat{\mathbb{Z}}, N^t) \to H^2(K_v, N)
\]

Taking Pontryagin dual and use Tate’s local duality Theorem, we have \( v \) is the normalized valuation of \( K_v \). Therefore, \( \hat{\mathbb{G}} \). As \( H^0(\hat{\mathbb{Z}}, H^1(I_v, N)) \) is finite, so is \( W \). To complete the proof, we shall show the following sequence is exact.

\[
0 \to H^0(K_v, \hat{\mathbb{N}}^c) \to H^0(K_v, \hat{\mathbb{N}}) \xrightarrow{\Psi} H^0_{\hat{\mathbb{Z}}}(N^t, \mathbb{Z}) \to W \to 0.
\]

The map \( \Psi \) is defined as follows: for \( f \in H^0(K_v, \hat{\mathbb{N}}) \) and \( x \in N^t \), \( \Psi(f) := v(f(x)) \) where \( v \) is the normalized valuation of \( K_v \). Then \( \Psi \) is a continuous map and

\[
\ker \Psi = \{ f \in H_{G_{K_v}}^0(N, K_v^c) : \text{for all } x \in N^t, \text{we have } f(x) \in (O_v^{ur})^* \}.
\]

Claim: \( \ker \Psi = H^0(K_v, \hat{\mathbb{N}}^c) \).

Proof of claim:

- Let \( f \in H^0(K_v, \hat{\mathbb{N}}^c) \) and \( x \in N^t \). Then \( f(x) \in H^0(I_v, K^*) \cap \hat{O}_v^* = (O_v^{ur})^* \). Therefore, \( H^0(K_v, \hat{\mathbb{N}}^c) \subset \ker \Psi \).
- To prove the other inclusion, we use the description of \( H^0(K_v, \hat{\mathbb{N}}^c) \) from part 1. Let \( f \in \ker \Psi \) and \( x \in H^0(K, N) \). Then \( f(x) \in (O_v^{ur})^* \) by definition. Since \( f(x) \in K_v^* \), \( f(x) \in (O_v^{ur})^* \cap K_v^* = O_v^* \). Hence, \( \ker \Psi \subset H^0(K_v, \hat{\mathbb{N}}^c) \).

Let \( W' = \text{cok}(\Psi) \). We have the following exact sequence where all the maps are strict morphisms [Mil06 page13],

\[
0 \to H^0(K_v, \hat{\mathbb{N}}^c) \to H^0(K_v, \hat{\mathbb{N}}) \xrightarrow{\Psi} H^0_{\hat{\mathbb{Z}}}(N^t, \mathbb{Z}) \to W' \to 0.
\]

As profinite completion is exact for sequences with strict morphisms [Mil06 page14],

\[
0 \to H^0(K_v, \hat{\mathbb{N}}^c) \to H^0(K_v, \hat{\mathbb{N}}) \xrightarrow{\Psi} H^0_{\hat{\mathbb{Z}}}(N^t, \mathbb{Z}) \to W' \to 0.
\]

As \( H^0(K_v, \hat{\mathbb{N}}^c) \) is compact and totally disconnected (topologically it is a product of finitely many copies of \( O_v^* \)), it is a profinite group. Therefore, \( H^0(K_v, \hat{\mathbb{N}}^c) = H^0(K_v, \hat{\mathbb{N}}^c) \). Moreover, \( \hat{W}' = W \) which is a finite group. Hence \( W' = W \). That completes the proof of the proposition. \( \square \)

Example 3.3. 1. Let \( N = \mathbb{Z} \). Then \( \hat{N} = \hat{K}_v^* \) and \( \hat{N}^c = \hat{O}_v^* \). We have \( H^1(I_v, \mathbb{Z}) = 0 \), \( H^1(K_v, \mathbb{G}_m) = 0 \) and \( \text{Ext}^1_{\hat{\mathbb{Z}}}(\mathbb{Z}, \mathbb{Z}) = 0 \). In this case, (3.1) is no other than

\[
0 \to O_v^* \to K_v^* \to \mathbb{Z} \to 0.
\]
2. Let \( N = \mathbb{Z}/n \). We have \( \widehat{\mathbb{Z}}/n \cong \mu_n \). Therefore, (3.1) becomes
\[
0 \to H^0(\widehat{\mathbb{Z}}, H^1(I_v, \mathbb{Z}/n)) \to H^1(K_v, \mu_n) \to \text{Ext}^1_{\mathbb{Z}}(\mathbb{Z}/n, \mathbb{Z}) \to 0. \quad (3.3)
\]
We have \( H^1(K_v, \mu_n) \cong K_v^*/(K_v^*)^n \) and \( \text{Ext}^1_{\mathbb{Z}}(\mathbb{Z}/n, \mathbb{Z}) \cong \mathbb{Z}/n \). By [CF10 page 144], \( I_v^{ab} \cong O_v^* \). Thus, \( H^2(I_v, \mathbb{Z}) \cong H^1(I_v, \mathbb{Q}/\mathbb{Z}) \cong I_v^D \cong (O_v^*)^D \). Moreover, \( H^1(I_v, \mathbb{Z}) = 0 \). Hence, \( H^1(I_v, \mathbb{Z}/n) \cong (O_v^*/(O_v^*)^n)^D \). Therefore, (3.3) reduces to
\[
0 \to O_v^*/(O_v^*)^n \to K_v^*/(K_v^*)^n \to \mathbb{Z}/n \to 0.
\]

3. Let \( N = \hat{T} \) for some torus \( T \) over \( K_v \). Then \( H^0(K_v, \hat{N}) = T(K_v), H^0(K_v, \hat{N}^c) = T_v^c \), the maximal compact subgroup of \( T(K_v) \) and
\[
0 \to \frac{T(K_v)}{T_v^c} \to \text{Hom}_{\mathbb{Z}}(\hat{T}^I_v, \mathbb{Z}) \to H^0(\widehat{\mathbb{Z}}, H^1(I_v, \hat{T})) \to H^1(K_v, T) \to \text{Ext}^1_{\mathbb{Z}}(\hat{T}^I_v, \mathbb{Z}) \to 0.
\]
(3.4)

4 Proof of the Main Theorem

Let \( T \) be an algebraic torus over a number field \( K \) with character group \( \hat{T} \). Let \( S \) be a finite set of places of \( K \) containing \( S_\infty \). For \( n \geq 1 \), let \( \Pi^n_S(T) \) be the kernel of the restriction map
\[
\Delta^n_S : H^n(K, T) \to \prod_{v \notin S} H^n(K_v, T).
\]

Lemma 4.1. We have the formula
\[
\frac{[\text{cok} \Delta^n_S]}{[\Pi^n_S(T)]} = \frac{[H^1(K, \hat{T})]}{[\Pi^n(T)][\Pi^2(\hat{T})][H^1(K_v, T)]}. \quad (4.1)
\]

Proof. We have the following commutative diagram
\[
\begin{array}{ccc}
0 & \to & 0 & \to & H^n(K, T) & \to & H^n(K, T) & \to & 0 \\
& & \downarrow & & \Delta^n & & \Delta^n_S & & \\
0 & \to & \prod_{v \in S} H^n(K_v, T) & \to & \prod_v H^n(K_v, T) & \to & \prod_{v \notin S} H^n(K_v, T) & \to & 0.
\end{array}
\]
Applying the Snake lemma to the above diagram yields the exact sequence
\[
0 \to \Pi^n(T) \to \Pi^n_S(T) \to \prod_{v \in S} H^n(K_v, T) \to \text{cok} \Delta^n \to \text{cok} \Delta^n_S \to 0. \quad (4.2)
\]

Now we recall the Poitou-Tate exact sequence [Mil06 I.4.20],
\[
0 \to \Pi^1(T) \to H^1(K, T) \xrightarrow{\Delta^1} \prod_v H^1(K_v, T) \to \\
\to H^1(K, \hat{T})^D \to H^2(K, T) \xrightarrow{\Delta^2} \prod_v H^2(K_v, T) \to H^0(K, \hat{T})^D \to 0. \quad (4.3)
\]
From (4.3), we deduce that \( \text{cok} \Delta^2 \cong H^0(K, \hat{T})^D \) and \( [H^1(K, \hat{T})] = [\text{cok} \Delta^1][\Pi^2(T)] \). Note that for \( n = 1 \), (4.2) is an exact sequence of finite groups. Therefore, (4.1) follows from (4.2) and the fact that \( [H^1(K, \hat{T})] = [\text{cok} \Delta^1][\Pi^2(T)] \).
Theorem 4.2. Let $T$ be an algebraic torus over a number field $K$ with character group $\hat{T}$. Let $S$ be a finite set of places of $K$ containing $S_\infty$. Let $j : Spec(K) \to U = Spec(O_{K,S})$ be the inclusion of the generic point. Let $\Psi^n(j_s\hat{T})$ be the map $H^i_{et}(U, j_*\hat{T}) \to \prod_{v \in S} H^n(K_v, \hat{T})$. Then

$$h_{T,S} = \frac{[\text{Ext}_1^n(j_s\hat{T}, G_m)] [H^1(K, \hat{T})]}{[\ker \Psi^1(j_s\hat{T})][\prod_{v \in S} H^1(K_v, T)] \prod_{v \in S} [H^0(\hat{Z}, H^1(I_v, \hat{T}))]}.$$

(4.4)

Proof. From the short exact sequence of étale sheaves on $U$

$$0 \to G_m \to j_* G_m \to \prod_{\nu \notin S} i_* Z \to 0$$

we obtain the long exact sequence

$$\ldots \to \text{Ext}^0_U(j_s\hat{T}, G_m) \to \text{Ext}^0_U(j_s\hat{T}, j_* G_m) \to \prod_{\nu \in S} \text{Ext}^0_U(j_s\hat{T}, i_* Z) \to \ldots$$

(4.5)

By Lemma 3.1 and Proposition 3.2, (4.5) can be rewritten as

$$\ldots \to \text{Ext}^0_U(j_s\hat{T}, G_m) \to H^n(K, T) \xrightarrow{\Theta_S} \prod_{\nu \in S} \text{Ext}^0_Z(\hat{T}^\nu, \mathbb{Z}) \to \ldots$$

(4.6)

From (4.6), we note that $\ker(\Theta^0_S) \simeq Hom_U(j_s\hat{T}, G_m)$ and

$$0 \to \text{cok}(\Theta^0_S) \to \text{Ext}^1_U(j_s\hat{T}, G_m) \to \ker(\Theta^1_S) \to 0,$$

(4.7)

$$0 \to \text{cok}(\Theta^1_S) \to \text{Ext}^1_U(j_s\hat{T}, G_m) \to \prod_{\nu \in S} \text{Ext}^1_Z(\hat{T}^\nu, \mathbb{Z}) \to 0.$$  

(4.8)

Note that $\text{cok}(\Theta^0_S), \ker(\Theta^1_S)$ are finite groups as $\text{Ext}^1_U(j_s\hat{T}, G_m)$ is finite. For each finite prime $v$ of $K$, we split the sequence (3.4) into

$$0 \to T(K_v)/T_v^c \to Hom_{\hat{Z}}(\hat{T}^\nu, \mathbb{Z}) \to S_v \to 0,$$

(4.9)

$$0 \to S_v \to H^0(\hat{Z}, H^1(I_v, \hat{T}))D \to H^1(K_v, T) \to \text{Ext}^1_Z(\hat{T}^\nu, \mathbb{Z}) \to 0.$$  

(4.10)

From (4.9), we obtain the following commutative diagram

$$\begin{array}{ccccccccc}
0 & \xrightarrow{} & T(K) & \xrightarrow{} & T(K) & \xrightarrow{} & 0 & \xrightarrow{} & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \xrightarrow{} & \prod_{\nu \notin S} T(K_v)/T_v^c & \xrightarrow{} & \prod_{\nu \notin S} Hom_{\hat{Z}}(\hat{T}^\nu, \mathbb{Z}) & \xrightarrow{} & \prod_{\nu \notin S} S_v & \xrightarrow{} & 0
\end{array}$$

The Snake Lemma combining with (2.1) yield $T(O_{K,S}) \simeq Hom_U(j_s\hat{T}, G_m)$ and

$$0 \to Cl_S(T) \xrightarrow{} \text{cok}(\Theta^0_S) \xrightarrow{} \prod_{\nu \notin S} S_v \to 0.$$  

(4.11)

As $\text{cok}(\Theta^0_S)$ is finite, (4.11) implies $\prod_{\nu \notin S} S_v$ is finite and

$$h_{T,S} = \frac{[\text{cok}(\Theta^0_S)]}{\prod_{\nu \notin S} [S_v]} = \frac{[\text{Ext}^1_U(j_s\hat{T}, G_m)]}{[\ker(\Theta^1_S)] \prod_{\nu \notin S} [S_v]}.$$  

(4.12)
From (4.10), we have

\[
\begin{array}{ccccccc}
0 & \rightarrow & 0 & \rightarrow & H^1(K, T) & \rightarrow & H^1(K, T) & \rightarrow & 0 \\
\downarrow & & & & \downarrow & & \Theta_S & & \\
0 & \rightarrow & \prod_{v \notin S} \frac{H^0(\hat{Z}, H^1(I_v, \hat{T}))}{S_v} & \rightarrow & \prod_{v \notin S} H^1(K_v, T) & \rightarrow & \prod_{v \notin S} Ext^1_{\hat{Z}}(\hat{T}^v, \hat{Z}) & \rightarrow & 0
\end{array}
\]

By the Snake Lemma, we obtain

\[
0 \rightarrow \mathbb{III}^1(T) \rightarrow \ker(\Theta_S^1) \rightarrow \prod_{v \notin S} \frac{H^0(\hat{Z}, H^1(I_v, \hat{T}))}{S_v} \rightarrow \cok(\Delta^1) \rightarrow \cok(\Theta_S^1) \rightarrow 0. \tag{4.13}
\]

As \( \cok(\Delta^1) \) and \( \prod_{v \notin S} S_v \) are finite, so are \( \cok(\Theta_S^1) \) and \( \prod_{v \notin S} H^0(\hat{Z}, H^1(I_v, \hat{T})) \). Therefore from (4.13) and Lemma 4.1, we obtain

\[
[\ker(\Theta_S^1)] = \frac{[\cok(\Theta_S^1)][\mathbb{III}^1(T)][\mathbb{III}^2(T)] \prod_{v \in S} [H^1(K_v, T)] \prod_{v \notin S} [H^0(\hat{Z}, H^1(I_v, \hat{T}))]}{[H^1(K, T)]}. \tag{4.14}
\]

Putting together (4.12) and (4.14), we obtain

\[
h_{T,S} = \frac{[Ext^1_j(j^*_T, \mathbb{G}_{m})][H^1(K, \hat{T})]}{[\cok(\Theta_S^1)][\mathbb{III}^1(T)][\mathbb{III}^2(T)] \prod_{v \in S} [H^1(K_v, T)] \prod_{v \notin S} [H^0(\hat{Z}, H^1(I_v, \hat{T}))]]. \tag{4.15}
\]

The theorem will follow from the next lemma.

\[\square\]

**Lemma 4.3.** Notations as in Theorem 4.2. Then

\[
[\mathbb{III}^2(T)][\cok(\Theta_S^1)] = [\ker\Psi^1(j^*_T)]. \tag{4.16}
\]

**Proof.** From (2.3), we have an exact sequence

\[
H^0(K, \hat{T}) \rightarrow \prod_{v \in S} H^0(K_v, \hat{T}) \rightarrow H^1_c(U, j^*_T) \rightarrow \ker\Psi^1(j^*_T) \rightarrow 0. \tag{4.17}
\]

Let \( R_1 \) be the cokernel of the map \( H^0(K, \hat{T}) \rightarrow \prod_{v \in S} H^0(K_v, \hat{T}) \). Then \( R^D_1 \) can be identified with the kernel of the map

\[
\prod_{v \in S} H^2(K_v, T) \rightarrow H^0(K, \hat{T})^D
\]

from (4.13). As \( Ext^2_U(j^*_T, \mathbb{G}_{m}) \simeq H^1_c(U, j^*_T)^D \), taking dual of (4.17) yields

\[
0 \rightarrow (\ker\Psi^1(j^*_T))^D \rightarrow Ext^2_U(j^*_T, \mathbb{G}_{m}) \rightarrow R^D_1 \rightarrow 0.
\]

From (4.2) for \( n = 2 \), we have the commutative diagram

\[
\begin{array}{ccccccc}
0 & \rightarrow & (\ker\Psi^1(j^*_T))^D & \rightarrow & Ext^2_U(j^*_T, \mathbb{G}_{m}) & \rightarrow & R^D_1 & \rightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow \simeq & & \downarrow & & \\
0 & \rightarrow & \mathbb{III}^2(T) & \rightarrow & \mathbb{III}^2(T) & \rightarrow & R^D_1 & \rightarrow & 0
\end{array}
\]

The lemma follows from (4.8) and diagram chasing. As a result, Theorem 4.2 is proved.

\[\square\]

**Corollary 4.4.** \( T(O_{K,S}) \simeq Hom_U(j^*_T, \mathbb{G}_{m}) \).
5 Class Number Formulas for Norm Tori and Their Duals

Let $L/K$ be a Galois extension of number fields with Galois group $G$. For each prime $v$ of $K$, choose a prime $w$ of $L$ lying over $v$. If $v$ is finite, let $D_w$ and $I_w$ be the decomposition group and the inertia group of $w$. Let $e_v(L/K)$ be the ramification index of $v$ in $L$. For any finite Galois extension $L/K$, let $L'/K$ be the maximal abelian subextension of $L/K$. Let $S$ be a finite set of places of $K$ containing $S_\infty$ and $S'$ be the set of places of $L$ which lie over a place in $S$. Let $\pi : \text{Spec}(L) \to \text{Spec}(K)$ and $\pi' : \text{Spec}(O_{L,S'}) \to \text{Spec}(O_{K,S})$ be the induced maps. If $G$ is a finite group and $M$ is a discrete $G$-module, we write $H^m_T(G, M)$ for the Tate cohomology.

Let $T = R_{L/K}^{(1)}(\mathbb{G}_m)$ be the norm torus corresponding to $L/K$ and $T' = R_{L/K}(\mathbb{G}_m)/\mathbb{G}_m$ be the dual torus of $T$. In this section, we shall compute $h_{T,S}$ and $h_{T',S}$ using (5.1).

**Lemma 5.1.** Let $T = R_{L/K}^{(1)}(\mathbb{G}_m)$. Then

1. $H^0(K, \hat{T}) = 0$ and $[H^1(K, \hat{T})] = [L : K]$.
2. For any place $v$, $[H^1(K_v, T)] = [H^1(K_v, \hat{T})] = [I'_w : K_v]$.
3. For $v \notin S_\infty$, if $w$ is a place of $L$ lying above $v$ then $[H^0(\hat{Z}, H^1(I_v, \hat{T}))] = e_v(L'/K)$.
4. Let $I_L$ be the group of ideles of $L$. Then $\prod I^n(T) \simeq \ker(H^0_T(G, L^*) \to H^0_T(G, I_L))$

**Proof.**

1. We have the exact sequence

\[ 0 \to \mathbb{Z} \to \mathbb{Z}[G] \to \hat{T} \to 0. \]  \tag{5.1}

From (5.1) and the fact that $H^n(G, \mathbb{Z}[G]) = 0$ for $n > 0$, we deduce that

\begin{align*}
H^0(K, \hat{T}) &\simeq H^0(G, \hat{T}) \simeq H^1(G, \mathbb{Z}) = 0, \\
H^1(K, \hat{T}) &\simeq H^1(G, \hat{T}) \simeq H^2(G, \mathbb{Z}) \simeq H^1(G, \mathbb{Q}/\mathbb{Z}) \simeq \text{Hom}_{\mathbb{Z}}(G^{ab}, \mathbb{Q}/\mathbb{Z}).
\end{align*}

2. From Tate’s local duality, $[H^1(K_v, T)] = [H^1(K_v, \hat{T})] = [H^1(D_w, \hat{T})]$. Since $\mathbb{Z}[G]$ is also an induced $D_w$-module, part 2 can be obtained from (5.1) as in part 1.

3. We have $H^0(\hat{Z}, H^1(I_v, \hat{T})) \simeq H^0(D_w/I_w, H^1(I_w, \hat{T}))$. From sequence (5.4) and the fact that $\mathbb{Z}[G]$ is an induced $I_w$-module, $H^1(I_w, \hat{T}) \simeq H^2(I_w, \mathbb{Z})$. Consider the spectral sequence

\[ E_2^{m,n} = H^m(D_w/I_w, H^n(I_w, \mathbb{Z})) \Rightarrow E^{m+n} = H^m(D_w, \mathbb{Z}). \]

Note that $E_2^{m,1} = 0$ for all $m$ and $E_2^{m,0} = 0$ for $m$ odd. Therefore, we obtain the exact sequence

\[ 0 \to (D_w/I_w)^D \to D_w^D \to H^0(D_w/I_w, H^2(I_w, \mathbb{Z})) \to 0. \]

Hence, $H^0(D_w/I_w, H^2(I_w, \mathbb{Z})) \simeq I_w^D$. Therefore, $[H^0(\hat{Z}, H^1(I_v, \hat{T}))] = e_v(L'/K)$.

4. This is proved in [PR92, page 307].

**Theorem 5.2 ([Ono87], [Mor91]).** Let $T = R_{L/K}^{(1)}(\mathbb{G}_m)$. Then

\[ h_{T,S} = \frac{h_{L,S'}[L' : K][H^0_T(G, O_{L,S'})]}{h_{K,S}[\ker(H^0_T(G, L^*) \to H^0_T(G, I_L))] \prod_{v \in S}[L'_v : K_v] \prod_{v \in S} e_v(L'/K)}. \]  \tag{5.2}
Proof. We have the exact sequence of $G_K$-modules

$$0 \to \mathbb{Z} \to \pi_* \mathbb{Z} \to \hat{T} \to 0.$$  \hfill (5.3)

Since $R^1 j_* \mathbb{Z} = 0$, (5.3) induces the exact sequence of étale sheaves on $U = \text{Spec}(O_K, S)$

$$0 \to \mathbb{Z} \to \pi'_* \mathbb{Z} \to j_* \hat{T} \to 0.$$  

The long exact sequence of Ext-groups yields

$$0 \to H^0_U(G, O'_{L, K}) \to \text{Ext}^1_U(j_* \hat{T}, \mathbb{G}_m) \to \text{Pic}(O_{L, S'}) \to \text{Pic}(O_K, S) \to R \to 0.$$  \hfill (5.4)

By Artin-Verdier Duality, $R^0$ is the kernel of the map $H^2_c(U, \mathbb{Z}) \to H^2_c(U, \pi'_* \mathbb{Z})$. From [Mil06 II.2.11] and the fact that $H^1_{\text{ét}}(U, \mathbb{Z}) = 0$ and $H^1(K, \mathbb{Z}) = 0$, there is a commutative diagram

\[
\begin{array}{c}
\begin{array}{ccc}
H^1_{\text{ét}}(U, j_* \hat{T}) & \xrightarrow{\Psi^1(j_* \hat{T})} & \prod_{v \in S} H^1(K_v, \hat{T}) \\
H^2_c(U, \mathbb{Z}) & \xrightarrow{\text{Ext}^1_U(j_* \hat{T}, \mathbb{G}_m)} & \prod_{v \in S} H^2(K_v, \mathbb{Z}) \\
H^2_c(U, \pi'_* \mathbb{Z}) & \xrightarrow{\text{Pic}(O_{L, S'})} & \prod_{v \in S} H^2(K_v, \pi'_* \mathbb{Z})
\end{array}
\end{array}
\]

By diagram chasing, $R^0 \simeq \ker \Psi^1(j_* \hat{T})$. By (5.4),

$$\frac{[\text{Ext}^1_U(j_* \hat{T}, \mathbb{G}_m)]}{[\ker \Psi^1(j_* \hat{T})]} = \frac{h_{L, S'}[H^0_U(G, O'_{L, S'})]}{h_{K, S}}.$$  \hfill (5.5)

Now (5.2) follows from (5.5), Theorem 4.2 and Lemma 5.1.

Remark 5.3. We note a simple but useful fact: if $\text{Pic}(O_K) = 0$ then $\ker \Psi^1(j_* \hat{T}) = 0$.

Lemma 5.4. Let $T' = R_{L/K}(\mathbb{G}_m)/\mathbb{G}_m$. Then

1. $H^0(K, \hat{T}') = 0$ and $H^1(K, \hat{T}') \simeq \mathbb{Z}/[G]\mathbb{Z}$.

2. For $v \notin S_{\infty}$, $[H^0(\hat{Z}, H^1(I_v, \hat{T}'))] = e_v(L/K)$.

3. $\text{Pic}(T') = 0$.

Proof. The character group of $T'$ satisfies the following exact sequence

$$0 \to \hat{T}' \to \mathbb{Z}[G] \xrightarrow{\cdot G} \mathbb{Z} \to 0.$$  \hfill (5.6)

Consider the long exact sequence of (5.8)

$$0 \to H^0(G, \hat{T}') \to H^0(G, \mathbb{Z}[G]) \xrightarrow{\cdot G} H^0(G, \mathbb{Z}) \to H^1(G, \hat{T}') \to 0.$$
The augmentation map $\epsilon$ in this case is just the multiplication-by-$[G]$ map. Hence, part 1) follows. Part 3) is proved in [Kat91 page 685]. Let us prove part 2). Let $w$ be a prime of $L$ dividing $v$.

\[ H^0(\mathcal{Z}, H^1(I_w, \hat{T}')) \simeq H^0(D_w/I_w, H^1(I_w, \hat{T}')). \]

By (5.6), $H^1(I_w, \hat{T}) \simeq H^0(I_w, \mathbb{Z})$. As $D_w/I_w$ acts trivially on $H^0(I_w, \mathbb{Z})$,

\[ [H^0(D_w/I_w, H^1(I_w, \hat{T}') \asymprod [H^0(I_w, \mathbb{Z})] = [I_w] = e_v(L/K). \]

\[ \square \]

**Theorem 5.5 (Kat91).** Let $T' = R_{L/K}(\mathbb{G}_m)/\mathbb{G}_m$. Then

\[ h_{T', S} = \frac{h_{L,S'}[H^1(G, O_{L,S'})]}{h_{K,S} \prod_{v \in S} e_v(L/K)}. \]  

(5.7)

**Proof.** We have the following exact sequence of étale sheaves on $\text{Spec}(K)$

\[ 0 \to \hat{T}' \to \pi_* \mathbb{Z} \to \mathbb{Z} \to 0. \]

(5.8)

As $R^1j_*(\pi_* \mathbb{Z}) = 0$, (5.8) induces

\[ 0 \to j_* \hat{T}' \to \pi'^* \mathbb{Z} \to \mathbb{Z} \to R^1j_* \hat{T}' \to 0. \]

(5.9)

To ease notation, let $\mathcal{R} = R^1j_* \hat{T}'$. We split (5.9) into

\[ 0 \to j_* \hat{T}' \to \pi'^* \mathbb{Z} \to \mathbb{Q} \to 0. \]

(5.10)

\[ 0 \to \mathbb{Q} \to \mathbb{Z} \to \mathcal{R} \to 0. \]

(5.11)

Let $\beta^n$ be the map $H^n_\varepsilon(X, \mathbb{Q}) \to H^n_\varepsilon(X, \mathbb{Z})$. From the long exact sequences of $\text{Ext}$-groups of (5.11), we obtain $\text{Hom}_U(\mathbb{Q}, \mathbb{G}_m) \simeq O_{K,S}$ and

\[ \text{[Ext}_{U}[\mathbb{Q}, \mathbb{G}_m)] = \frac{h_{K,S}[\text{Ext}_{U}[\mathcal{R}, \mathbb{G}_m)]}{\text{cok}(\beta^n_1)}. \]

(5.12)

Let $\alpha^n_\varepsilon$ be the map $H^n_\varepsilon(X, \pi'^* \mathbb{Z}) \to H^n_\varepsilon(X, \mathbb{Q})$. Similar argument applied to sequence (5.10) yields

\[ \text{[Ext}_{U}[j_* \hat{T}', \mathbb{G}_m)] = \frac{h_{L,S'}[\text{cok}(\alpha^n_1)]\text{[S]}\text{[Ext}_{U}[\mathbb{Q}, \mathbb{G}_m)]}{h_{K,S}[\text{Ext}_{U}[\mathcal{R}, \mathbb{G}_m)]}. \]

(5.13)

where $S$ satisfies the exact sequence

\[ 0 \to O_K \to O_* \to \text{Hom}_U(j_* \hat{T}', \mathbb{G}_m) \to S \to 0. \]

(5.14)

We claim that $S \simeq H^1(G, O_{L,S'})$. Indeed, from Corollary 4.4, $\text{Hom}_U(j_* \hat{T}', \mathbb{G}_m) \simeq T'(O_{K,S})$. Therefore, (5.13) can be identified with the sequence

\[ 0 \to \mathbb{G}_m(O_{K,S}) \to R_{L/K}(\mathbb{G}_m)(O_{K,S}) \to T'(O_{K,S}) \to S \to 0. \]

(5.15)

which is part of the long exact sequence of cohomology associated with

\[ 0 \to \mathbb{G}_m(O_{L,S'}) \to R_{L/K}(\mathbb{G}_m)(O_{L,S'}) \to T'(O_{L,S'}) \to 0. \]
Note that \( R_{L/K}(G_m)(O_{L,S'}) \) is an induced \( G \)-module thus \( H^1(G, R_{L/K}(G_m)(O_{L,S'})) = 0 \). Consider the long exact sequence of cohomology associated with (5.15) and compare with (5.14), we obtain \( S \cong H^1(G, O_{L,S'}) \). From Theorem 5.2 and Lemma 5.4 we have

\[
h_{T', S} = \frac{h_{L,S'}[H^1(G, O_{L,S'})][L : K][\text{cok}(\alpha_c^1)][\text{cok}(\beta_c^1)]}{h_{K,S} \prod_{v \in S} e_v(L/K) \prod_{v \in S} [H^1(K_v, \hat{T'})][\ker \Psi^1(j_*\hat{T'})][\text{Ext}^2_U(R, G_m)]}.
\]

Theorem 5.5 will follow from the next lemma.

**Lemma 5.6.** Notations as in Theorem 5.5 Then

\[
\frac{[\text{cok}(\alpha_c^1)][\text{cok}(\beta_c^1)][L : K]}{[\text{Ext}^2_U(R, G_m)][\ker \Psi^1(j_*\hat{T'})]} = \prod_{v \in S} [H^1(K_v, \hat{T'})].
\]

**Proof.** Consider the following commutative diagram

\[
\begin{array}{cccccccc}
0 & \rightarrow & H^0(U, \mathbb{Q}) & \rightarrow & H^0(U, \mathbb{Q}) & \rightarrow & H^0(U, \mathbb{Q}) & \rightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \rightarrow & H^0(U, \mathbb{Z}) & \rightarrow & H^0(U, \mathbb{Z}) & \rightarrow & H^0(U, \mathbb{Z}) & \rightarrow & 0 \\
\end{array}
\]

As \( R \) is negligible, \( R_v = 0, Q_v \cong \mathbb{Z} \) and \( \beta_S^0 \) is an isomorphism. Hence \( \text{cok}(\beta_c^1) = 0 \). Next we consider the diagram.

\[
\begin{array}{cccccccc}
0 & \rightarrow & H^0(U, \mathbb{Q}) & \rightarrow & H^0(U, \mathbb{Q}) & \rightarrow & H^0(U, \mathbb{Q}) & \rightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \rightarrow & H^0(U, \mathbb{Z}) & \rightarrow & H^0(U, \mathbb{Z}) & \rightarrow & H^0(U, \mathbb{Z}) & \rightarrow & 0 \\
\end{array}
\]

By diagram chasing, we have

\[
[\text{cok}(\alpha_c^1)] = [H^1(U, \mathbb{Q})][\text{cok}\Psi^1(j_*\hat{T'})] = \frac{[H^1(U, \mathbb{Q})][\ker \Psi^1(j_*\hat{T'})]\prod_{v \in S} [H^1(K_v, \hat{T'})]}{[\text{Ext}^2_U(R, \mathbb{Q})]}. \quad (5.17)
\]

It is not hard to show that \([\text{cok}(\alpha_c^0)] = [H^1(U, j_*\hat{T'})], [\text{cok}(\beta_c^0)] = [H^0(U, \mathbb{Q})]/[H^1(U, \mathbb{Q})]. \) Let \( \varepsilon_c^0 : H^0(U, \mathbb{Z}) \rightarrow H^0(U, \mathbb{Z}) \) be the composition \( \beta_c^0 \circ \alpha_c^0 \). Note that \( \varepsilon_c^0 \) can be identified with the augmentation map \( H^0(G, \mathbb{Z}, \mathbb{G}_m) \rightarrow H^0(G, \mathbb{Z}) \). Hence, \( \text{cok}(\varepsilon_c^0) \cong \mathbb{Z}/[L : K] \mathbb{Z} \). As \( \beta_c^0 \) is injective, \([\text{cok}(\alpha_c^0)][\text{cok}(\beta_c^0)] = [\text{cok}(\varepsilon_c^0)]. \) Therefore,

\[
[L : K] = \frac{[H^1(U, j_*\hat{T'})][H^0(U, \mathbb{Q})]}{[\text{Ext}^2(U, \mathbb{Q})]}. \quad (5.18)
\]

That completes the proofs of the lemma and Theorem 5.5. □
Corollary 5.7 ([Neu99] VI.3.5). Suppose \( L/K \) is a cyclic extension. Then the Herbrand quotient of \( O_{L,S}^* \) is given by

\[
h(G, O_{L,S}^*) = \prod_{v \in S} [L_v : K_v] / [L : K].
\]

Proof. Since \( L/K \) is cyclic, by Hasse’s theorem \( \ker(H^n_{\ell}(G, L^*) \to H^n_{\ell}(G, I_L)) = 0 \). In addition, \( T \cong T' \). Thus, the corollary follows by comparing the two formulas in Theorems 5.2 and 5.5. \( \square \)

6 A Relative Class Number Formula for Isogenous Tori

Let \( \lambda : T \to T' \) be an isogeny between two algebraic tori defined over a number field \( K \). In other words, we have exact sequences

\[
0 \to T'' \to T \to T' \to 0 \quad \& \quad 0 \to \hat{T}'' \to \hat{T} \to \hat{T}' \to 0 \tag{6.1}
\]

where \( T'' \) is a finite algebraic group over \( K \) and \( \hat{T}'' \) is a finite discrete \( G_K \)-module. Then \( \lambda \) induces the following maps all of which have finite kernels and cokernels.

- \( \lambda(O_{K,S}) : T(O_{K,S}) \to T'(O_{K,S}) \).
- \( \hat{\lambda}(H^n_{\alpha})(\hat{T}'') : H^n_{\alpha}(X, j_\ast \hat{T}'') \to H^n_{\alpha}(X, j_\ast \hat{T}) \) and \( \hat{\lambda}(H^n_{\alpha})(\hat{T}'') : H^n_{\alpha}(X, j_\ast \hat{T}') \to H^n_{\alpha}(X, j_\ast \hat{T}) \).
- Let \( \alpha(T) = \ker(\Psi^1(j_\ast \hat{T})) \) and similarly for \( \alpha(T') \). Then there is a map \( \hat{\lambda}(\alpha) : \alpha(T') \to \alpha(T) \).
- For finite prime \( v \), let \( c_v(T) = H^0(\hat{T}, H^1(I_v, \hat{T}))^D \) and similarly for \( c_v(T') \). We have \( \lambda(O_v) : T_v^c \to T_v^c \) and \( \hat{\lambda}(c_v) : c_v(T) \to c_v(T') \).
- For any prime \( v \), \( \lambda(H^n(K_v)) : H^n(K_v, T) \to H^n(K_v, T') \) and \( \hat{\lambda}(H^n(K_v)) : H^n(K_v, \hat{T}') \to H^n(K_v, \hat{T}) \).
- For infinite prime \( v \), \( \lambda(H^n_{\ell}(K_v)) : H^n_{\ell}(K_v, T) \to H^n_{\ell}(K_v, T') \) and \( \hat{\lambda}(H^n_{\ell}(K_v)) : H^n_{\ell}(K_v, \hat{T}') \to H^n_{\ell}(K_v, \hat{T}) \).

Let \( \alpha \) be a group homomorphism with finite kernel and cokernel. We define \( q(\alpha) := [\cok(\alpha)]/[\ker(\alpha)] \). Then \( q(\alpha) \) is multiplicative with respect to exact sequences [Ono61] 0.3.1. The purpose of this section is to generalize a formula of Shyr for \( h_{T,S}/h_{T',S} \) using \( (1.4) \).

Lemma 6.1.

\[
q(\hat{\lambda}(H^0(K_v))) = \begin{cases} \frac{[\cok(\hat{\lambda}(H^0(K_v)))]}{[\ker(\hat{\lambda}(H^{-1}(K_v)))[\cok(\hat{\lambda}(H^0(K_v)))]]} & v \notin S_\infty, \\ \frac{[\ker(\hat{\lambda}(H^{-1}(K_v)))[\cok(\hat{\lambda}(H^0(K_v)))]}{[H^0(K_v, T'')]} & v \in S_\infty. \end{cases}
\]

Proof. If \( v \notin S_\infty \) then this is clear. If \( v \in S_\infty \) then this follows from diagram chasing and the fact that \( [H^{-1}_T(K_v, T'')] = [H^0_{\ell}(K_v, \hat{T}'')] \). \( \square \)

Lemma 6.2. For any prime \( v \),

\[
q(\lambda(H^1(K_v))) = \frac{[\hat{T}'']_v}{[\cok(\lambda(H^0(K_v)))]}. \]
Proof. From the exact sequence

$$0 \to H^0(K_v, T'') \to H^0(K_v, T) \to H^0(K_v, T') \to H^1(K_v, T'') \to \ker(\lambda(H^1(K_v))) \to 0$$

and the facts that $\ker(\lambda(H^1(K_v))) = \cok(\hat{\lambda}(H^1(K_v)))$ and $[H^n(K_v, T'')] = [H^{2-n}(K_v, \hat{T}'')]$, we obtain

$$\frac{q(\lambda(H^1(K_v)))}{q(\lambda(H^0(K_v)))} = \frac{[H^2(K_v, \hat{T}'')] [\ker(\hat{\lambda}(H^1(K_v)))]}{[H^1(K_v, \hat{T}'')]} = \frac{[\hat{T}']_v [\ker(\hat{\lambda}(H^1(K_v)))]}{[H^0(K_v, \hat{T}'')] [\cok(\hat{\lambda}(H^0(K_v)))]},$$

where the second equality follows from [Mil06 I.2.8]. \qed

**Lemma 6.3.** Let $v$ be a finite prime of $K$. Then

$$q(\lambda(c_v)) = q(\lambda(O_v)) [\hat{T}']_v.$$  \hspace{1cm} (6.2)

**Proof.** From Corollary 3.3 we have the following commutative diagram

$$
\begin{array}{ccccccc}
T'_v & \longrightarrow & T(K_v) & \longrightarrow & Hom_{\hat{Z}}(\hat{T}'_v, \hat{Z}) & \longrightarrow & H^0(\hat{Z}, H^1(I_v, \hat{T}^D)) & \longrightarrow & H^1(K_v, T) & \longrightarrow & \text{Ext}_{\hat{Z}}^1(\hat{T}'_v, \hat{Z}) \\
\downarrow \lambda(O_v) & & \downarrow \lambda(H^0(K_v)) & & \downarrow \lambda(Hom) & & \downarrow \hat{\lambda}(c_v) & & \downarrow \lambda(H^1(K_v)) & & \downarrow \lambda(Ext) \\
T'_v & \longrightarrow & T(K_v) & \longrightarrow & Hom_{\hat{Z}}(\hat{T}'_v, \hat{Z}) & \longrightarrow & H^0(\hat{Z}, H^1(I_v, \hat{T}^D)) & \longrightarrow & H^1(K_v, T') & \longrightarrow & \text{Ext}_{\hat{Z}}^1(\hat{T}'_v, \hat{Z})
\end{array}
$$

(6.3)

Note that all the vertical maps have finite kernels and cokernels. We have

$$\frac{q(\hat{\lambda}(c_v))}{q(\lambda(O_v))} = \frac{q(\lambda(Hom))q(\lambda(H^1(K_v)))}{q(\lambda(Ext))q(\lambda(H^0(K_v)))}. \hspace{1cm} (6.4)$$

From (6.1), we have an exact sequence of $\hat{Z}$-modules

$$0 \to \hat{T}'_v \to \hat{T}^D_v \to \hat{T}_v \to 0$$

where $P$ is finite. As $P$ is finite, $[\text{Ext}^1_{\hat{Z}}(P, Z)] = [H^2-P(\hat{Z}, P)]$. In particular, $[\text{Ext}^1_{\hat{Z}}(P, Z)] = [\text{Ext}^2_{\hat{Z}}(P, Z)]$. Moreover, $\text{Ext}^1_{\hat{Z}}(T'_v, Z) \simeq H^0(\hat{Z}, \hat{T}'_v)^D \simeq H^0(K_v, \hat{T})$. Thus, $[\ker(\lambda(\text{Ext}^2))] = [\cok(\hat{\lambda}(H^0(K_v)))].$ By direct calculation,

$$\frac{q(\lambda(Hom))}{q(\lambda(Ext))} = \frac{[\text{Ext}^1_{\hat{Z}}(P, Z)[\ker(\lambda(\text{Ext}^2))]}{[\text{Ext}^2_{\hat{Z}}(P, Z)]} = [\cok(\hat{\lambda}(H^0(K_v)))]. \hspace{1cm} (6.5)$$

Now put together (6.4), (6.5) and Lemma 6.2 we obtain (6.2). \qed

**Theorem 6.4 ([Shy77]).** Let $\lambda : T \to T'$ be an isogeny between two algebraic tori defined over a number field $K$. Let $\tau(T)$ and $\tau(T')$ be the Tamagawa numbers of $T$ and $T'$ respectively. Then

$$h_{T,S} / h_{T',S} = \frac{\tau(T) \prod_{v \in S} q(\lambda(H^0(K_v))) \prod_{v \notin S} q(\lambda(O_v))}{\tau(T') q(\lambda(O_{K,S})) q(\lambda(H^0_{et}))}.$$
Proof. By Theorem 4.2 and the fact that \( \tau(T) = [H^1(K, \hat{T})]/[\mathbb{H}^1(T)] \) (see [Ono63]),

\[
\frac{h_{T,S}}{h_{T',S}} = \frac{\tau(T)}{\tau(T')} \frac{[\text{Ext}^1_U(j_*\hat{T}, \mathbb{G}_m)]}{[\text{Ext}^1_U(j_*\hat{T}', \mathbb{G}_m)]} \prod_{v \in S} q(\hat{\lambda}(c_v)) \prod_{v \in S} q(\lambda(H^1(K_v))) q(\hat{\lambda}(\alpha)).
\] (6.6)

Combining Lemmas 6.2 and 6.3

\[
\prod_{v \in S} q(\hat{\lambda}(c_v)) \prod_{v \in S} q(\lambda(H^1(K_v))) = \prod_{v \notin S} q(\lambda(O_v)) \prod_{v \in S} \frac{q(\lambda(H^0(K_v)))}{[\text{cok}(\lambda(H^0(K_v)))]}.
\] (6.7)

Sequence (6.1) induces an exact sequence of sheaves

\[
0 \to j_*\hat{T}' \to j_*\hat{T} \to j_*\hat{T}'' \to \mathcal{R} \to 0
\]

where \( \mathcal{R} \) is negligible as it is a subsheaf of \( R^1j_*\hat{T}' \). We can split this sequence into

\[
0 \to j_*\hat{T}' \to j_*\hat{T} \to Q \to 0,
\] (6.8)

\[
0 \to Q \to j_*\hat{T}'' \to \mathcal{R} \to 0
\] (6.9)

where \( Q \) is a constructible sheaf. The long exact sequence of Ext-groups associated to (6.8) can be split into

\[
0 \to \text{Hom}_U(Q, \mathbb{G}_m) \to \text{Hom}_U(j_*\hat{T}, \mathbb{G}_m) \to \text{Hom}_U(j_*\hat{T}', \mathbb{G}_m) \to S_1 \to 0,
\] (6.10)

\[
0 \to S_1 \to \text{Ext}^1_U(Q, \mathbb{G}_m) \to \text{Ext}^1_U(j_*\hat{T}, \mathbb{G}_m) \to \text{Ext}^1_U(j_*\hat{T}', \mathbb{G}_m) \to \text{Ext}^2_U(Q, \mathbb{G}_m) \to S_2 \to 0.
\] (6.11)

The Artin-Verdier duality implies \( S_2^D \cong \text{cok}(\hat{\lambda}(H^1_1)) \). As \( \text{Hom}_U(j_*\hat{T}, \mathbb{G}_m) \cong T(O_{K,S}) \), (6.10) yields

\[
q(\lambda(O_{K,S})) = \frac{[S_1]}{[\text{Hom}_U(Q, \mathbb{G}_m)]}.
\] (6.12)

Combining (6.11) and (6.12) gives us

\[
\frac{[\text{Ext}^1_U(j_*\hat{T}, \mathbb{G}_m)]}{[\text{Ext}^1_U(j_*\hat{T}', \mathbb{G}_m)]} = \frac{[\text{Ext}^1_U(Q, \mathbb{G}_m)][\text{cok}(\hat{\lambda}(H^1_1))]}{[\text{Ext}^2_U(Q, \mathbb{G}_m)][\text{Hom}_U(Q, \mathbb{G}_m)]q(\lambda(O_{K,S}))}.
\] (6.13)

From [Mil06 II.2.13] and Artin-Verdier Duality, we deduce

\[
\frac{[\text{Ext}^1_U(Q, \mathbb{G}_m)][H^0(U, Q)]}{[\text{Ext}^2_U(Q, \mathbb{G}_m)][\text{Hom}_U(Q, \mathbb{G}_m)]} = \prod_{v \in S_{\infty}} [H^0(K_v, Q_v)] = \prod_{v \in S_{\infty}} [H^0(K_v, \hat{T}''_v)].
\]

Therefore,

\[
\frac{[\text{Ext}^1_U(j_*\hat{T}, \mathbb{G}_m)]}{[\text{Ext}^1_U(j_*\hat{T}', \mathbb{G}_m)]} = \prod_{v \in S_{\infty}} [H^0(K_v, \hat{T}''_v)][\text{cok}(\hat{\lambda}(H^1_1))]/[H^0(U, Q)]q(\lambda(O_{K,S})).
\] (6.14)

By diagram chasing, we can see that

\[
\frac{q(\hat{\lambda}(H^1_1))}{q(\lambda(H^0_1))} = \frac{[\text{cok}(\hat{\lambda}(H^1_1))][\text{ker}(\lambda(H^0_1))]}{[H^0(U, Q)]}.
\]
Thus, (6.14) can be rewritten as

\[
\frac{[\text{Ext}^1_U(j_*\hat{T}, \mathbb{G}_m)]}{[\text{Ext}^1_U(j_*\hat{T}', \mathbb{G}_m)]} = \frac{\prod_{v \in S} \text{Ext}^0(K_v, \hat{T}')}{|\ker(\text{Ext}^0(H^0_0))| q(\text{Ext}^0(H^0_0))} (6.15)
\]

We have the following commutative diagram

\[
\begin{array}{ccccccccc}
\prod_{v \in S} H^{-1}_T(K_v, \hat{T}') & \longrightarrow & H^0_c(U, j_*\hat{T}') & \longrightarrow & H^0_{et}(U, j_*\hat{T}') & \longrightarrow & \prod_{v \in S} H^0(K_v, \hat{T}') & \longrightarrow & H^1_c(U, j_*\hat{T}') & \longrightarrow & \alpha(T') \\
\lambda(H^{-1}_S) & \downarrow & \lambda(H^0_0) & \downarrow & \lambda(H^0_0) & \downarrow & \lambda(H^0_0) & \downarrow & \lambda(H^0_0) & \downarrow & \lambda(H^0_0) \\
\prod_{v \in S} H^{-1}_T(K_v, \hat{T}) & \longrightarrow & H^0_c(U, j_*\hat{T}) & \longrightarrow & H^0_{et}(U, j_*\hat{T}) & \longrightarrow & \prod_{v \in S} H^0(K_v, \hat{T}) & \longrightarrow & H^1_c(U, j_*\hat{T}) & \longrightarrow & \alpha(T)
\end{array}
\]

As \(\lambda(H^0_0)\) is injective, \(\ker(\lambda(H^0_0)) \simeq \ker(\lambda(H^{-1}_S)) \simeq \prod_{v \in S} \ker(\lambda(H^{-1}_S))\). Furthermore,

\[
\frac{q(\lambda(H^1_0))}{q(\lambda(H^0_0))} = \frac{q(\lambda(\alpha))q(\lambda(H^0_0))}{q(\lambda(H^0_0))q(\lambda(H^{-1}_S))} = \frac{q(\lambda(\alpha))}{q(\lambda(\alpha))} \prod_{v \in S} \frac{q(\lambda(H^0_0))}{q(\lambda(H^0_0))} \prod_{v \in S} \frac{q(\lambda(H^0_0))}{q(\lambda(H^0_0))} (6.16)
\]

Combining (6.15), (6.16) with Lemma 6.4 yields

\[
\frac{[\text{Ext}^1_U(j_*\hat{T}, \mathbb{G}_m)]}{[\text{Ext}^1_U(j_*\hat{T}', \mathbb{G}_m)]} = \frac{q(\lambda(\alpha))}{q(\lambda(\alpha))} \prod_{v \in S} \frac{\text{cok}(\lambda(H^0_0))}{\text{cok}(\lambda(H^0_0))} (6.17)
\]

Now Theorem 6.4 follows from (6.6), (6.7) and (6.17).

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