On the uniqueness result for the BSDE with continuous coefficient

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Abstract

In this paper, we study one-dimensional backward stochastic differential equation (BSDE, for short), whose coefficient $f$ is Lipschitz in $y$ but only continuous in $z$. In addition, if the terminal condition $\xi$ has bounded Malliavin derivative, we prove some uniqueness results for the BSDE with quadratic and linear growth in $z$, respectively.

Keywords: Backward stochastic differential equation, uniqueness result, Malliavin calculus.

1. Introduction

We study backward stochastic differential equations (BSDEs) of the following type:

$$Y_t = \xi + \int_t^T f(s, Y_s, Z_s) \, ds - \int_t^T Z_s \, dW_s, \quad 0 \leq t \leq T,$$

where $(W_t)_{0 \leq t \leq T}$ be a standard $d$-dimensional Brownian motion on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, with $\mathbb{F} = (\mathcal{F}_t)_{0 \leq t \leq T}$ is the augmented natural filtration generated by $W$. Fixed $T > 0$ the time horizon of the study is called the terminal time. The terminal condition $\xi$ is a $\mathcal{F}_T$-measurable random variable. The coefficient (or generator) $f$ of the BSDE (1.1) is assumed to be a function from $[0, T] \times \Omega \times \mathbb{R} \times \mathbb{R}^d$ to $\mathbb{R}$ that is measurable with respect to $\text{Prog} \otimes \mathcal{B}(\mathbb{R}) \otimes \mathcal{B}(\mathbb{R}^d)$, where $\text{Prog}$ is the progressive sigma-algebra on $[0, T] \times \Omega$. The triple $(\xi, T, f)$ is the parameters of the BSDE (1.1).

Nonlinear BSDEs were first introduced by Pardoux and Peng (1990). They proved that when $\xi$ is square integrable and $f$ is uniformly Lipschitz in $(y, z)$ there exists a unique adapted solution with square integrability properties. Since then, BSDEs have been studied with great interest, due to their wide range of applications in mathematical finance, stochastic optimal control, and partial differential equations. In particular, many efforts have been made to relax the assumptions on the parameters $\xi$ and $f$ of the BSDE (1.1). The full list of contributions is too long to give and we will only quote results in our framework. A class of BSDEs with
the generator of quadratic growth concerning the variable $z$, has received a lot of attention in recent years. For convenience, hereafter, by a quadratic BSDEs, we mean that in BSDE (1.1), the map $z \mapsto f(t, y, z)$ grows no more than quadratically. Concerning the scalar case, and restricting to the bounded terminal condition $\xi$, Kobylanski (2000) proved the existence of a solution $(Y, Z)$ such that $Y$ is a bounded scalar process. Since then, a large number of papers have dealt with extensions and applications.

In this paper, we will focus on the uniqueness results for the BSDEs in the one-dimensional case. Our main purpose is to strengthen the uniqueness result for the quadratic BSDEs with generators $f(t, y, z)$ that are only continuous in $z$. This is different from the previous works, which assume that the generators $f$ are convex functions with respect to $z$ (see e.g. Briand and Hu (2008), Delbaen et al. (2011)) or other additional conditions (see e.g. Briand and Richou (2019), Fan et al. (2020), Richou (2012)). On the other hand, we require the terminal conditions $\xi$ that have bounded Malliavin derivative. Let us recall our strategy in detail. To obtain the main result of this paper, we first use the sup-convolution approximation techniques (see Lemma 3.7) to get the approximate generators with regularity properties. Then the solution $(Y^n, Z^n)$ of the BSDE (3.3) with parameters $(\xi, f_n)$ converges to the maximal solution $(\overline{Y}, \overline{Z})$ of the BSDE (1.1) by the monotone stability proposition (see Proposition 3.9). In addition, if the terminal condition $\xi$ is Malliavin differentiable with a bounded Malliavin derivative, then $Z^n$ is bounded uniformly with respect to $n$. Thus, we can get that $\overline{Z}$ is bounded. Moreover, applying similar arguments (see Lemma 3.10), the minimal solution $(\underline{Y}, \underline{Z})$ of the BSDE (1.1) is also bounded (see Proposition 3.11). Finally, noting that the $\underline{Z}$ and $\overline{Z}$ are bounded, we localize the generator $f$ by $\overline{f}$, then under Assumptions (A1)-(A2) the BSDE with parameters $(\overline{f}, \xi)$ has a unique solution. This implies that Theorem 3.12 holds. Let us emphasize that, applying the above-mentioned proof strategy, we can get a uniqueness result for the BSDEs with the generators that have linear growth in $z$ (see Theorem 4.1).

The paper is organized as follows. In Section 2, we introduce some notations and definitions that will be used in the sequel. The uniqueness of the solution to quadratic BSDE with the terminal condition that has bounded Malliavin derivative is obtained in Section 3. In Section 4, we prove the uniqueness of the solution for BSDE when the generator is only continuous with linear growth in $z$.

2. Preliminaries

Let us introduce some notations and spaces that will be used below. For a positive integer $d$, any element $x \in \mathbb{R}^d$ will be identified to a column vector with $i$-th component $x^i$ and Euclidean norm denoted by $|x|$. For $k \in \mathbb{N}$ and Euclidean spaces $\mathbb{H}$ and $\mathbb{G}$, denote by $C^k_b(\mathbb{H}, \mathbb{G})$ the set of functions of class $C^k$ from $\mathbb{H}$ to $\mathbb{G}$ whose partial derivations of order less
than or equal to \( k \) are bounded. For any real \( p > 1 \), define
\[
L_p^\infty(\Omega; \mathbb{H}) = \left\{ \xi : \Omega \to \mathbb{H} \mid \xi \text{ is } \mathcal{F}_T\text{-measurable, } \|\xi\|_{L_p} \triangleq (\mathbb{E} |\xi|^p)^{\frac{1}{p}} < \infty \right\},
\]
and for any \( t \in [0, T) \), define
\[
L_p^\infty(t, T; \mathbb{H}) = \left\{ \varphi : [t, T] \times \Omega \to \mathbb{H} \mid \varphi \text{ is } \mathcal{F}_\tau\text{-progressively measurable, } \|\varphi\|_{L_p^\infty(t, T; \mathbb{H})} \triangleq \text{esssup}_{(s, \omega) \in [t, T] \times \Omega} |\varphi_s(\omega)| < \infty \right\},
\]
and for any \( t \in [0, T) \), define
\[
S^p(\Omega; \mathbb{H}) = \left\{ \xi : \Omega \to \mathbb{H} \mid \xi \text{ is } \mathcal{F}_T\text{-measurable, } \|\xi\|_\infty \triangleq \text{esssup}_\omega |\xi(\omega)| < \infty \right\},
\]
and for any \( t \in [0, T) \), define
\[
S^p(t, T; \mathbb{H}) = \left\{ \varphi : [t, T] \times \Omega \to \mathbb{H} \mid \varphi \text{ is } \mathcal{F}_\tau\text{-progressively measurable, } \|\varphi\|_{S^p(t, T; \mathbb{H})} \triangleq \text{esssup}_{(s, \omega) \in [t, T] \times \Omega} |\varphi_s(\omega)| < \infty \right\},
\]
and for any \( t \in [0, T) \), define
\[
S^p(t, T; \mathbb{H}) = \left\{ \varphi : [t, T] \times \Omega \to \mathbb{H} \mid \varphi \text{ is } \mathcal{F}_\tau\text{-progressively measurable, continuous, } \|\varphi\|_{S^p(t, T; \mathbb{H})} \triangleq \text{esssup}_{(s, \omega) \in [t, T] \times \Omega} |\varphi_s(\omega)| < \infty \right\},
\]
and for any \( t \in [0, T) \), define
\[
\mathcal{H}^2_{BMO}(t, T) = \left\{ \varphi \in L^2(t, T; \mathbb{H}) \mid \|\varphi\|_{\mathcal{H}^2_{BMO}(t, T)} \triangleq \sup_{t \leq \tau \leq T} \mathbb{E}_\tau \left[ \int_\tau^T |\varphi_s|^2 ds \right]^{\frac{1}{2}} < \infty \right\}.
\]
where \( \tau \) is the stopping time and \( \mathbb{E}_\tau \) is the conditional expectation given \( \mathcal{F}_\tau \).

Next, let us recall the notion of derivation on Wiener space, and for more information about Malliavin calculus we refer the readers to Nualart (2006). Denote by \( \mathbb{S} \) the set of random variable \( \xi \) of the form \( \xi = \varphi(W(h^1), ..., W(h^n)) \), where \( \varphi \in C^\infty_c(\mathbb{R}^n, \mathbb{R}) \), \( h^i \in L^2([0, T]; \mathbb{R}^d) \) and \( W(h^i) = \int_0^T h^i_t dW_t \) is the Wiener integral for \( i = 1, ..., n \). To such a random variable \( \xi \), we associate a “derivative process” \( \{D_i \xi; t \in [0, T]\} \) defined as
\[
D_i \xi \triangleq \sum_{i=1}^n \partial_{x_i} \varphi(W(h^1), ..., W(h^n)) h^i_t, \quad t \in [0, T],
\]
whose components we denote by \( D_i \xi \), for \( i = 1, ..., d \). For \( \xi \in \mathbb{S} \), we define a kind of Sobolev norm by
\[
\|\xi\|_{\mathcal{H}^1_{BMO}(t, T)} \triangleq \mathbb{E}_\tau \left[ |\xi|^2 + \int_0^T |D_i \xi|^2 dt \right].
\]
It can be shown that the operator \( D \) has a closed extension to the space \( \mathbb{D}^{1,2} \), the closure of \( \mathbb{S} \) with respect to the norm \( \| \cdot \|_{1,2}^2 \).

Throughout this paper, to avoid the additional Malliavin regularization technicalities (see Cheridito and Nam (2014) for more detail), we only consider the deterministic generator \( f \) case. We work with the following

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Definition 2.1. A solution of the BSDE (1.1) is a pair \((Y_t, Z_t)_{0 \leq t \leq T}\) of adapted processes taking values in \(\mathbb{R} \times \mathbb{R}^d\) such that \(\int_0^T (|f(t, Y_t, Z_t)| + |Z_t|^2) \, dt < \infty\) and (1.1) holds for all \(0 \leq t \leq T\), \(\mathbb{P}\)-a.s.

This minimal definition will be completed later on by some further integrability assumptions. In order to simplify the notations, we sometimes write \((Y, Z)\) for the processes \((Y_t, Z_t)_{0 \leq t \leq T}\).

3. The unique solution to the quadratic BSDE

In this section, we study a unique solution for the quadratic BSDE (1.1) with the continuous coefficient and the terminal condition that has bounded Malliavin derivative.

We work with the following assumptions:

(A1) The bounded terminal condition \(\xi\) is in \(D^{1,2}\) and there exists a positive constant \(L\) such that \(||\xi||_\infty \leq L\) and \(|D_t^i \xi| \leq L, d\mathbb{P} \otimes dt\text{-a.e., for all } i = 1, \ldots, d\).

(A2) The generator \(f\) is Borel measurable and continuous with respect to \(y, z\), and there exists a positive constant \(L\) such that for all \(t \in [0, T], y, y' \in \mathbb{R}\) and \(z \in \mathbb{R}^d\)

\[|f(t, y, z)| \leq L(1 + |y| + |z|^2)\text{ and }|f(t, y, z) - f(t, y', z)| \leq L|y - y'|.\]

(A3) There is a positive constant \(K\) such that for all \(t \in [0, T], y \in \mathbb{R}\) and \(z, z' \in \mathbb{R}^d\),

\[|f(t, y, z) - f(t, y, z')| \leq K(1 + |z| + |z'|)|z - z'|.\]

The following proposition was first obtained by Theorem 2.3 of Kobylanski (2000), and some extended results can refer to El Karoui et al. (2009), etc.

Proposition 3.1. If Assumptions (A1)-(A2) hold, then the BSDE (1.1) has the maximal solution \((\overline{Y}, \overline{Z})\) (resp. the minimal solution \((\underline{Y}, \underline{Z})\)) in \(S^\infty_\mathbb{F}(0, T; \mathbb{R}) \times L^2_\mathbb{F}(0, T; \mathbb{R}^d)\), i.e. for any solution \((Y, Z)\) of the BSDE (1.1), \(\overline{Y} \geq Y\) (resp. \(\underline{Y} \leq Y\), \(\mathbb{P}\)-a.s.).

Remark 3.2. Notice that if \((\overline{Y}, \overline{Z})\) and \((\tilde{Y}, \tilde{Z})\) are both maximal solutions of BSDE (1.1), then necessarily \(\overline{Y} \equiv \tilde{Y}\) and \(\overline{Z} \equiv \tilde{Z}\).

Let us recall the comparison principle which is one of the most important properties of BSDEs. The following proposition which is obtained under quite general conditions is an extension of Theorem 7.3.1 of Zhang (2017).
**Proposition 3.3.** Assume that $\xi$ and $f$ satisfy Assumptions (A1)-(A3), let $(Y,Z) \in S^\infty_F(0,T;\mathbb{R}) \times L^2_F(0,T;\mathbb{R}^d)$ be the unique solution of the BSDE (1.1), and $(Y',Z') \in S^\infty_F(0,T;\mathbb{R}) \times L^2_F(0,T;\mathbb{R}^d)$ be a solution of the BSDE with parameters $(\xi', f')$ satisfying Assumptions (A1)-(A2). If $\xi \geq \xi'$, $\mathbb{P}$-a.s., and $f(t, Y'_t, Z'_t) \geq f'(t, Y'_t, Z'_t)$, $d\mathbb{P} \otimes dt$-a.e., then $Y_t \geq Y'_t$, $0 \leq t \leq T$, $\mathbb{P}$-a.s.

**Proof.** First, under Assumptions (A1)-(A3), since the solution $(Y,Z)$ of the BSDE (1.1) in $S^\infty_F(0,T;\mathbb{R}) \times L^2_F(0,T;\mathbb{R}^d)$, by Proposition 2.1 of Briand and Elie (2013), we have $Z \in H^2_B[0,T]$. If the parameters $(\xi', f')$ satisfy Assumptions (A1)-(A2) and $(Y', Z') \in S^\infty_F(0,T;\mathbb{R}) \times L^2_F(0,T;\mathbb{R}^d)$, by Proposition 1.1 of Barrieu et al. (2008), we can have $Z' \in H^2_B[0,T]$. Next, to simply notation we restrict ourselves to case $d = 1$. Denote

$$\Delta Y \triangleq Y - Y', \quad \Delta Z \triangleq Z - Z', \quad \Delta \xi \triangleq \xi - \xi', \quad \Delta f \triangleq f - f'.$$

Then, by the classical linearization argument, we have

$$\Delta Y_t = \Delta \xi + \int_t^T [a_r \Delta Y_r + b_r \Delta Z_r + \Delta f(r, Y'_r, Z'_r)] dr - \int_t^T \Delta Z_r dW_r, \quad 0 \leq t \leq T, \quad (3.1)$$

where the processes $a_r$ and $b_r$ are defined by

$$a_r \triangleq \begin{cases} \frac{f(r, Y'_r, Z'_r) - f(r, Y'_r, Z'_r)}{\Delta Y_r}, & \text{if } \Delta Y_r \neq 0, \\ 0, & \text{if } \Delta Y_r = 0, \end{cases} \quad \text{and} \quad b_r \triangleq \begin{cases} \frac{f(r, Y'_r, Z'_r) - f(r, Y'_r, Z'_r)}{\Delta Z_r}, & \text{if } \Delta Z_r \neq 0, \\ 0, & \text{if } \Delta Z_r = 0. \end{cases}$$

Now since the generator $f$ satisfies Assumptions (A2)-(A3), it implies that

$$|a_r| \leq L \quad \text{and} \quad |b_r| \leq K(1 + |Z_r| + |Z'_r|).$$

Solving the linear BSDE (3.1) leads to

$$\Gamma_t \Delta Y_t = \mathbb{E} \left[ \Delta \xi \Gamma_T + \int_t^T \Gamma_r \Delta f(r, Y'_r, Z'_r) dr \mid \mathcal{F}_t \right],$$

where $\Gamma$ is the adjoint (positive) process of the linear BSDE (3.1), defined by the following SDE

$$d\Gamma_t = \Gamma_t (a_t dt + b_t dW_t), \quad \Gamma_0 = 1.$$

For $a$ is a bounded process and $b \in H^2_B[0,T]$, then the $\Gamma$ is well-defined (see Kazamaki (2006)).

Besides, $\Delta \xi$ and $\Delta f(r, Y'_r, Z'_r)$ are nonnegative, then $\Delta Y$ is nonnegative. Thus Proposition 3.3 is proved.

**Remark 3.4.** The above proof method comes from Theorem 2.2 of El Karoui et al. (1997) and Theorem 7.3.1 of Zhang (2017). The terminal condition $\xi$ that has bounded Malliavin derivative in Assumption (A1) can be dropped. We can also get the strict comparison theorem similar to El Karoui et al. (1997).
The following result gives the regular properties of the solutions of the BSDEs, which comes from Cheridito and Nam (2014). For the convenience of readers, we give a brief proof.

**Proposition 3.5.** If Assumptions (A1)-(A3) hold, then the BSDE (1.1) has a unique solution \((Y, Z)\) in \(S^\infty_f(0, T; \mathbb{R}) \times L^\infty_f (0, T; \mathbb{R}^d)\), and \(|Y_t| \leq (L + 1)e^{L(T-t)} - 1, |Z^i_t| \leq Le^{L(T-t)}, d\mathbb{P} \otimes dt\text{-a.e. for all } i = 1, \ldots, d.\)

**Proof.** Under Assumptions (A1)-(A3), we set \(\rho(x) \triangleq K (1 + 2x)\) which is a nondecreasing function. Since \(\rho(|z| \vee |z'|) = K [1 + 2 (|z| \vee |z'|)] \geq K (1 + |z| + |z'|)\), we have

\[
|f(t, y, z) - f(t, y, z')| \leq K (1 + |z| + |z'|) |z - z'|
\leq \rho(|z| \vee |z'|) |z - z'|.
\]

Then the terminal condition \(\xi\) and the deterministic generator \(f\) satisfy the assumptions of the corollary 2.8 of Cheridito and Nam (2014), so we can get that there exists a unique solution \((Y, Z)\) in \(S^\infty_f(0, T; \mathbb{R}) \times L^\infty_f (0, T; \mathbb{R}^d)\) to the BSDE (1.1) and \(|Y_t| \leq (L + 1)e^{L(T-t)} - 1, |Z^i_t| \leq Le^{L(T-t)}, d\mathbb{P} \otimes dt\text{-a.e. for all } i = 1, \ldots, d.\)

\[\square\]

**Remark 3.6.** It should be pointed out that the bounds of the solution \((Y, Z)\) are not depend on the constant \(K\) of (A3). This is important for us to obtain the uniformly bounded solutions of the BSDEs with the approximation coefficients in Proposition 3.9.

Now we present the regularization of the generator of BSDE (1.1) through sup-convolution in the following lemma, which is useful for us to study the quadratic BSDEs later.

**Lemma 3.7.** Let the function \(f\) satisfies Assumption (A2), then the sequence of functions

\[
f_n(t, y, z) \triangleq \sup_{v \in \mathbb{R}^d} \{f(t, y, v) - n|z - v|^2\},
\]

is well defined for \(n \geq 2L\). Moreover, it satisfies

1. For any \(t \in [0, T], y \in \mathbb{R}\) and \(z \in \mathbb{R}^d, |f_n(t, y, z)| \leq L (1 + |y| + 2|z|^2).\)
2. For any \(t \in [0, T], y \in \mathbb{R}\) and \(z \in \mathbb{R}^d, f_n(t, y, z)\) is decreasing in \(n.\)
3. If \(z_n \to z\) as \(n \to \infty\), then \(f_n(t, y, z_n) \to f(t, y, z).\)
4. For any \(t \in [0, T], y_1, y_2 \in \mathbb{R}\) and \(z_1, z_2 \in \mathbb{R}^d\) and for sufficiently large \(n,\)

\[
|f_n(t, y_1, z_1) - f_n(t, y_2, z_2)| \leq L |y_1 - y_2| + n (1 + |z_1| + |z_2|) |z_1 - z_2|.
\]
**Proof.** We consider the case of the generator \( f \) independent of \( t \) and \( y \), i.e., \( f(\cdot) = f(z) \), and the case of \( f(t, y, z) \) could be proved similar without substantial difference. Let \( h: \mathbb{R}^d \to \mathbb{R} \) be a continuous function and \( |h(z)| \leq L(1 + |z|^2) \), then for any \( n \geq 2L \), we define

\[
h_n(z) \triangleq \sup_{v \in \mathbb{R}^d} \{ h(v) - n|z - v|^2 \}.
\]

From the definition, we see that \( h_n(z) \geq h(z) \geq -L (1 + |z|^2) \) and

\[
h_n(z) \leq \sup_{v \in \mathbb{R}^d} \{ L + L|v|^2 - n|z - v|^2 \}
= \sup_{v \in \mathbb{R}^d} \{ L + L|v|^2 - 2L|z - v|^2 - (n - 2L)|z - v|^2 \}
\leq \sup_{v \in \mathbb{R}^d} \{ L + 2L|z|^2 - (n - 2L)|z - v|^2 \}
= L + 2L|z|^2.
\]

So \( |h_n(z)| \leq L (1 + 2|z|^2) \), from which item (1) holds. Moreover, item (2) comes from the definition of \( h_n(\cdot) \) directly.

Now we prove the item (3), and consider \( z_n \to z \), then for every \( n \), there exists \( v_n \in \mathbb{R}^d \) such that

\[
h(z_n) \leq h_n(z_n) \leq h(v_n) - n|z_n - v_n|^2 + \frac{1}{n}
\leq L + L|v_n|^2 - n|z_n - v_n|^2 + \frac{1}{n}
\leq L + 2L|z_n|^2 - (n - 2L)|z_n - v_n|^2 + \frac{1}{n}.
\]

Since \( h(z_n) \) is bounded, we deduce that \( \lim_{n \to \infty} \sup (n - 2L)|z_n - v_n|^2 < \infty \). In particular, when \( v_n \to z \), we have

\[
\lim_{n \to \infty} \sup (n - 2L)|z_n - z|^2 < \infty.
\]

Moreover, we have that

\[
h(z_n) \leq h_n(z_n) \leq h(v_n) + \frac{1}{n}
\]
from which \( h_n(z_n) \to h(z) \) when \( z_n \to z \), this implies that item (3) holds.

In order to prove item (4), for any \( z \in \mathbb{R}^d \), we take \( \epsilon > 0 \) and consider \( v_\epsilon \in \mathbb{R}^d \) (in fact for sufficiently large \( n \), we can assumed that \( |z - v_\epsilon| \leq \frac{1}{2} \)) such that

\[
h_n(z) \leq h(v_\epsilon) - n|z - v_\epsilon|^2 + \epsilon
= h(v_\epsilon) - n|v - v_\epsilon|^2 + n|v - v_\epsilon|^2 - n|z - v_\epsilon|^2 + \epsilon
\leq h_n(v) + n|v - v_\epsilon|^2 - n|z - v_\epsilon|^2 + \epsilon
\leq h_n(v) + n|z - v|(|v - v_\epsilon| + |z - v_\epsilon|) + \epsilon
\leq h_n(v) + n|z - v|(1 + |z| + |v|) + \epsilon.
\]
Therefore, interchanging the role of \( z \) and \( v \), and since \( \epsilon \) is arbitrary we can get
\[
|h_n(z) - h_n(v)| \leq n(1 + |z| + |v|)|z - v|.
\]
Finally, for any \( y_1, y_2 \in \mathbb{R}, z \in \mathbb{R}^d \) and \( t \in [0, T] \), by the inequality
\[
\left| \sup_{v \in \mathbb{R}^d} f(v) - \sup_{v \in \mathbb{R}^d} g(v) \right| \leq \sup_{v \in \mathbb{R}^d} |f(v) - g(v)|
\]
and the Lipschitz condition of \( f \) with respect to the spatial variable \( y \), we get that
\[
|f_n(t, y_1, z) - f_n(t, y_2, z)| \leq L |y_1 - y_2|
\]
from which the desired result follows.

\[\square\]

**Remark 3.8.** The idea of the proof of the above lemma is inspired by Lemma 1 in Lepeltier and San Martin (1997).

In the following, for positive number \( n \geq 2L \), we consider the following BSDE:
\[
Y_t^n = \xi + \int_t^T f_n(r, Y_r^n, Z_r^n)dr - \int_t^T Z_r^n dW_r, \quad 0 \leq t \leq T, \tag{3.3}
\]
where \( f_n(t, y, z) \) is defined by (3.2). For the above BSDE (3.3), we have the following properties concerning the solutions.

**Proposition 3.9.** Under Assumptions (A1)-(A2), then for any sufficiently large \( n \in \mathbb{N} \), the BSDE (3.3) admits a unique solution \((Y^n, Z^n) \in S^\infty_\mathbb{F}(0, T; \mathbb{R}) \times L^\infty_\mathbb{F}(0, T; \mathbb{R}^d)\), and \( Y^n \) is decreasing in \( n \). Moreover, \((Y^n, Z^n) \to (\overline{Y}, \overline{Z})\) in the space \( S^\infty_\mathbb{F}(0, T; \mathbb{R}) \times L^\infty_\mathbb{F}(0, T; \mathbb{R}^d)\) as \( n \to \infty \), where \((\overline{Y}, \overline{Z})\) in \( S^\infty_\mathbb{F}(0, T; \mathbb{R}) \times L^\infty_\mathbb{F}(0, T; \mathbb{R}^d)\) is the maximal solution of BSDE (1.1).

**Proof.** Under Assumptions (A1)-(A2), by Lemma 3.7, we know that for any sufficiently large \( n \in \mathbb{N} \), the function \( f_n(r, y, z) \) and terminal condition \( \xi \) satisfy the conditions of Proposition 3.5, then the BSDE (3.3) has a unique solution \((Y^n, Z^n)\) and
\[
|Y^n_t| \leq (L + 1)e^{L(T-t)} - 1, \text{ for all } t \in [0, T], \quad |Z_t^{n,i}| \leq Le^{L(T-t)}, \quad d\mathbb{P} \otimes dt\text{-a.e. for all } i = 1, \ldots, d.
\]

Note that the sequence \( \{f_n(r, y, z)\}_n \) is decreasing, so from the comparison principle (see Theorem 7.3.1 of Zhang (2017)), we have \( Y^n \geq Y^{n+1}, \mathbb{P}\text{-a.s.} \).

Let us recapitulate what we have obtained. The sequence \( \{f_n(t, y, z)\}_n \) converges locally uniformly in \((y, z)\) to the generator \( f(t, y, z) \), and \( |f_n(t, y, z)| \leq L (1 + |y| + 2|z|^2) \). The BSDE (3.3) with parameters \((f_n, \xi)\) has a unique bounded solution \((Y^n, Z^n)\) such that the sequence \( \{Y^n\}_n \) is decreasing. Therefore, applying the monotone stability proposition (see Proposition 2.4 of Kobyanski (2000) or Lemma 3 of Briand and Hu (2006)), we have \((Y^n, Z^n) \to (\overline{Y}, \overline{Z})\)
in the space $S^2_F(0, T; \mathbb{R}) \times L^2_F(0, T; \mathbb{R}^d)$ as $n \to \infty$, and $(\overline{Y}, \overline{Z})$ is a solution of the BSDE (1.1) with parameters $(f, \xi)$. The solution $(\overline{Y}, \overline{Z})$ is the maximal solution of the BSDE (1.1) was followed by $f_n(t, y, z) \leq f(t, y, z)$ and the comparison principle (see Proposition 3.3). Finally, since the bounds of the solution $(Y^n, Z^n)$ does not depend on $n$, we have $(\overline{Y}, \overline{Z}) \in S^\infty_F(0, T; \mathbb{R}) \times L^\infty_F(0, T; \mathbb{R}^d)$, and $|\overline{Y}_t| \leq (L + 1)e^{L(T-t)} - 1$, $|\overline{Z}_t| \leq L e^{L(T-t)}$, $d\mathbb{P} \otimes dt$-a.e. for all $i = 1, \ldots, d$.

Furthermore, to study the minimal solution of BSDE (1.1), we apply the similar above arguments. Firstly, we give the following lemma involving the regularization of the generator $f$ by inf-convolution techniques. The proof of Lemma 3.10 is similar to Lemma 3.7, so we omit the proof.

**Lemma 3.10.** Let the function $f$ satisfies Assumption (A2), then the sequence of functions

$$
\hat{f}_n(t, y, z) \triangleq \inf_{v \in \mathbb{R}^d} \{ f(t, y, v) + n|z - v|^2 \},
$$

is well defined for $n \geq 2L$. Moreover, it satisfies

1. For any $t \in [0, T]$, $y \in \mathbb{R}$ and $z \in \mathbb{R}^d$, $|\hat{f}_n(t, y, z)| \leq L(1 + |y| + 2|z|^2)$.

2. For any $t \in [0, T]$, $y \in \mathbb{R}$ and $z \in \mathbb{R}^d$, $\hat{f}_n(t, y, z)$ is increasing in $n$.

3. If $z_n \to z$ as $n \to \infty$, then $\hat{f}_n(t, y, z_n) \to f(t, y, z)$.

4. For any $t \in [0, T]$, $y_1, y_2 \in \mathbb{R}$ and $z_1, z_2 \in \mathbb{R}^d$ and for sufficiently large $n$,

$$
|\hat{f}_n(t, y_1, z_1) - \hat{f}_n(t, y_2, z_2)| \leq L|y_1 - y_2| + n(1 + |z_1| + |z_2|)|z_1 - z_2|.
$$

In the following, for positive number $n \geq 2L$, we consider the following BSDE:

$$
Y^n_t = \xi + \int_t^T \hat{f}_n(r, Y^n_r, Z^n_r)dr - \int_t^T Z^n_r dW_r, \quad 0 \leq t \leq T,
$$

where $\hat{f}_n(t, y, z)$ is defined by (3.4). For the above BSDE (3.5), by the similar arguments in Proposition 3.9, we have the following proposition.

**Proposition 3.11.** Under Assumptions (A1)-(A2), then for any sufficiently large $n \in \mathbb{N}$, the BSDE (3.5) admits a unique solution $(Y^n, Z^n) \in S^\infty_F(0, T; \mathbb{R}) \times L^\infty_F(0, T; \mathbb{R}^d)$, and $Y^n$ is increasing in $n$. Moreover, $(Y^n, Z^n) \to (\overline{Y}, \overline{Z})$ in the space $S^2_F(0, T; \mathbb{R}) \times L^2_F(0, T; \mathbb{R}^d)$ as $n \to \infty$, where $(\overline{Y}, \overline{Z})$ in $S^\infty_F(0, T; \mathbb{R}) \times L^\infty_F(0, T; \mathbb{R}^d)$ is the minimal solution of BSDE (1.1).

Now, we give the main result in this section.


Theorem 3.12. Under Assumptions (A1)-(A2), the BSDE (1.1) has a unique adapted solution \((Y, Z)\) in \(S^\infty_F(0, T; \mathbb{R}) \times L^\infty_F(0, T; \mathbb{R}^d)\).

Proof. First of all, by Proposition 3.9 and Proposition 3.11, the BSDE (1.1) has the maximal and minimal solution \((\bar{Y}, \bar{Z})\) and \((\underline{Y}, \underline{Z})\) in \(S^\infty_F(0, T; \mathbb{R}) \times L^\infty_F(0, T; \mathbb{R}^d)\), respectively. The bound of \(Z\) and \(\bar{Z}\) is denoted by the constant \(M\).

Next, we define
\[
\bar{f}(t, y, z) \triangleq \begin{cases} 
 f(t, y, z), & \text{if } |z| \leq M; \\
 f(t, y, \frac{Mz}{|z|}), & \text{if } |z| > M.
\end{cases}
\]

Then under Assumption (A2), the generator \(\bar{f}(t, y, z)\) is Lipschitz continuous in \(y\) and uniformly continuous in \(z\). Furthermore, by Theorem 2 of Fan and Jiang (2011), the BSDE with parameters \((\bar{f}, \xi)\) has a unique solution. Now we can prove that the maximal solution \((\bar{Y}, \bar{Z})\) of the BSDE (1.1) must be equal to the minimal solution \((\underline{Y}, \underline{Z})\), since solving the BSDE (1.1) with the generator \(\bar{f}\) that coincides with the generator \(f\) for \(|z| \leq M\), where \(M\) is the bound on \(\bar{Z}\) and \(\underline{Z}\).

Remark 3.13. In the above theorem, we see that if we assume that the terminal condition \(\xi\) has bounded Malliavin derivative, even though relaxing some assumptions on the generator \(f\), we can also get the uniqueness result for the BSDE.

Remark 3.14. For the unbounded terminal condition case, similar results will be in our coming paper. In fact, we only need the comparison theorem (see Proposition 3.3) holds for the solution \((Y', Z')\) in the appropriate space.

4. The uniqueness solution to the BSDEs with linear growth

In this section we state and prove a uniqueness result for the BSDE (1.1), whose generator \(f(t, y, z)\) is Lipschitz in \(y\) and only continuous with linear growth in \(z\). The following theorem is the main result of this section.

Theorem 4.1. Assume that the following conditions hold:

(B1) The terminal condition \(\xi\) is in \(D^{1,2}\) and there exists a positive constant \(L\) such that \(|D^i_\xi| \leq L, d\mathbb{P} \otimes dt\text{-a.e.}, \) for all \(i = 1, \ldots, d;\)

(B2) The generator \(f\) is Borel measurable and continuous with respect to \(y, z\), and there exists a positive constant \(L\) such that for all \(t \in [0, T], y, y' \in \mathbb{R}\) and \(z \in \mathbb{R}^d\)
\[
|f(t, y, z)| \leq L(1 + |y| + |z|) \text{ and } |f(t, y, z) - f(t, y', z)| \leq L|y - y'|.
\]

Then the BSDE (1.1) has a unique adapted solution \((Y, Z)\) in \(S^2_F(0, T; \mathbb{R}) \times L^\infty_F(0, T; \mathbb{R}^d)\).
Proof. Since the only novelty is the gathering of known results and applying the proof strategy in Section 3, we will only sketch the proof.

1. First of all, since the terminal condition $\xi$ satisfies Assumption (B1), by Lemma 2.5 of Cheridito and Nam (2014), we have $\xi \in L^2_{\mathcal{F}_T} (\Omega; \mathbb{R})$. Moreover, since the generator $f$ satisfies Assumption (B2), we regularize $f$ by

\[ \tilde{f}_n(t, y, z) \triangleq \inf_{v \in \mathbb{R}^d} \{ f(t, y, v) + n|z - v| \}, \]

and for $n \geq L$, the sequence of functions $\tilde{f}_n(t, y, z)$ satisfies the following properties (see Lemma 1 in Lepeltier and San Martin (1997))

(i) For any $t \in [0, T]$, $y \in \mathbb{R}$ and $z \in \mathbb{R}^d$, $|\tilde{f}_n(t, y, z)| \leq L(1 + |y| + |z|);

(ii) For any $t \in [0, T]$, $y \in \mathbb{R}$ and $z \in \mathbb{R}^d$, $\tilde{f}_n(t, y, z)$ is increasing in $n$;

(iii) If $z_n \to z$ as $n \to \infty$, then $\tilde{f}_n(t, y, z_n) \to f(t, y, z)$;

(iv) For any $t \in [0, T]$, $y_1, y_2 \in \mathbb{R}$ and $z_1, z_2 \in \mathbb{R}^d$, $|\tilde{f}_n(t, y_1, z_1) - \tilde{f}_n(t, y_2, z_2)| \leq L|y_1 - y_2| + n|z_1 - z_2|.

2. For any $n \geq L$, we consider the following BSDE

\[ \hat{Y}_t^n = \xi + \int_t^T \tilde{f}_n(r, \hat{Y}_r^n, \hat{Z}_r^n)dr - \int_t^T \hat{Z}_r^n dW_r, \quad 0 \leq t \leq T, \]  

(4.1)

with the generator $\tilde{f}_n$. Now $(\xi, \tilde{f}_n)$ satisfy the assumptions of Theorem 1 of Lepeltier and San Martin (1997) (or Theorem 8.4 of El Karoui et al. (2009)), then the unique solution $(\hat{Y}^n, \hat{Z}^n)$ of the BSDE (4.1) converges to the minimal solution $(\bar{Y}, \bar{Z})$ of the BSDE (1.1), in the space $L^2_{\mathcal{F}_T} (0, T; \mathbb{R}) \times L^2_{\mathcal{F}_T} (0, T; \mathbb{R}^d)$ as $n \to \infty$. In addition, by Theorem 2.2 and Remark 2.3 of Cheridito and Nam (2014), we have that the solution $(\hat{Y}^n, \hat{Z}^n) \in S^2_{\mathcal{F}_T} (0, T; \mathbb{R}) \times L^2_{\mathcal{F}_T} (0, T; \mathbb{R}^d)$, and $|\hat{Z}^n_i| \leq L e^{L(T-t)}$, $d\mathbb{P} \otimes dt$-a.e. for all $i = 1, \ldots, d$. So we have $|\hat{Z}_i| \leq L e^{L(T-t)}$, $d\mathbb{P} \otimes dt$-a.e. for all $i = 1, \ldots, d$.

If we regularize $f$ by $\tilde{f}_n(t, y, z) \triangleq \sup_{v \in \mathbb{R}^d} \{ f(t, y, v) - n|z - v| \}$, and applying the similar arguments, then under Assumptions (B1)-(B2) the BSDE (1.1) admits the maximal solution $(\bar{Y}, \bar{Z}) \in S^2_{\mathcal{F}_T} (0, T; \mathbb{R}) \times L^\infty_{\mathcal{F}_T} (0, T; \mathbb{R}^d)$, and $|\bar{Z}_i| \leq Le^{L(T-t)}$, $d\mathbb{P} \otimes dt$-a.e. for all $i = 1, \ldots, d$.

3. Now let us summarize the conclusions we have obtained. The BSDE (1.1) admits the maximal $(\bar{Y}, \bar{Z})$ and minimal solution $(\hat{Y}, \hat{Z})$ in $S^2_{\mathcal{F}_T} (0, T; \mathbb{R}) \times L^\infty_{\mathcal{F}_T} (0, T; \mathbb{R}^d)$, respectively. Moreover, for all $i = 1, \ldots, d$, $|\bar{Z}_i|$ and $|\hat{Z}_i|$ are bounded by constant $C \triangleq Le^L$. Since $f$ satisfies Assumption (B2), then $f$ is Lipschitz continuous in $y$ and uniformly continuous in $z$ for $z \in [-C, C]^d$. Furthermore, by Theorem 2 of Fan and Jiang (2011), we have $(\bar{Y}, \bar{Z}) \equiv (\hat{Y}, \hat{Z})$.\[\square\]
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