The existence and linear stability of periodic solution for a free boundary problem modeling tumor growth with a periodic supply of external nutrients

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Abstract

We study a free boundary problem modeling tumor growth with a T-periodic supply of external nutrients. The model contains two parameters \(\mu\) and \(\bar{\sigma}\). We first show that (i) zero radially symmetric solution is globally stable if and only if \(\bar{\sigma} \geq \frac{1}{T} \int_0^T \Phi(t) dt\); (ii) If \(\bar{\sigma} < \frac{1}{T} \int_0^T \Phi(t) dt\), then there exists a unique radially symmetric positive solution \((\sigma_*(r,t), p_*(r,t), R_*(t))\) with period \(T\) and it is a global attractor of all positive radially symmetric solutions for all \(\mu > 0\). These results are a perfect answer to open problems in Bai and Xu [Pac. J. Appl. Math. 2013(5), 217-223]. Then, considering non-radially symmetric perturbations, we prove that there exists a constant \(\mu_* > 0\) such that \((\sigma_*(r,t), p_*(r,t), R_*(t))\) is linearly stable for \(\mu < \mu_*\) and linearly unstable for \(\mu > \mu_*\).

Keywords: Tumor growth, Free boundary problem, Periodic solution, Linear stability.

1. Introduction

Consider a free boundary problem modeling tumor growth with a periodic supply of external nutrients:

\[
\begin{align*}
\Delta \sigma &= \sigma \quad x \in \Omega(t), t > 0, \\
-\Delta p &= \mu(\sigma - \bar{\sigma}) \quad x \in \Omega(t), t > 0, \\
V_n &= -\frac{\partial p}{\partial n} \quad x \in \partial\Omega(t), t > 0, \\
\sigma &= \Phi(t) \quad x \in \partial\Omega(t), t > 0, \\
p &= \gamma \kappa \quad x \in \partial\Omega(t), t > 0, \\
\Omega(0) &= \Omega_0,
\end{align*}
\]

where \(\Omega(t) \subseteq \mathbb{R}^3\) is the domain occupied by tumor at time \(t\), \(\sigma\) denotes the concentration of nutrients, \(p\) is the pressure in the tumor, \(\bar{\sigma}\) denotes a threshold concentration for proliferation, \(\mu\) is proportional coefficient in Darcy law, \(V_n\) denotes the velocity of the free boundary in the unit outward normal direction \(n\), \(\gamma\) is cell adhesiveness coefficient, \(\kappa\) is the mean curvature and \(\Phi(t)\) is concentration of external nutrients which is a continuous, positive periodic function satisfying \(\Phi(t+T) = \Phi(t)\). The detailed introduction of this model is referred to [1, 2].

At first, we study radially symmetric solutions. For radially symmetric solutions, problem (1.1)–(1.6) is reduced to

\[
\begin{align*}
\frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \frac{\partial \sigma}{\partial r} \right) &= \sigma \quad r \in (0, R(t)), t > 0, \\
-\frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \frac{\partial p}{\partial r} \right) &= \mu(\sigma - \bar{\sigma}) \quad r \in (0, R(t)), t > 0, \\
\frac{dR}{dt}(t) &= -\frac{\partial p}{\partial r} \quad r = R(t), t > 0, \\
\sigma &= \Phi(t) \quad r = R(t), t > 0, \\
p &= \frac{\gamma}{R(t)} \quad r = R(t), t > 0,
\end{align*}
\]

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Corollary 1.4. If $\sigma > \Phi$, then the solution $R(t) \equiv 0$ of (1.15) is globally stable. If the solution $R(t) \equiv 0$ of (1.15) is globally stable, then $\overline{\Phi} \geq \Phi$;

(ii) If $\overline{\Phi} > \Phi_*$, then (1.15) admits a unique T-periodic solution $R_*(t)$. Moreover, any positive solution $R(t)$ of (1.15) converges to $R_*(t)$ as $t \to +\infty$.

Notice that when $\Phi(t)$ is not a constant function, then $\Phi_* < \overline{\Phi}$ and there is a gap $\overline{\Phi} \in (\Phi_*, \overline{\Phi})$ between the results of (i) and (ii). So they proposed two open problems:

(1) Is $R_*(t) \equiv 0$ globally stable as $\overline{\Phi} = \Phi$?

(2) Does there exist a unique periodic solution $R_*(t)$ of (1.15) as $\overline{\Phi} < \Phi$? Is it a global attractor of all positive solutions?

We give an affirmative answer to the two open problems. The following theorems are our main results.

Theorem 1.1. When $\overline{\Phi} = \overline{\Phi}$, the solution $R(t) \equiv 0$ of (1.15) is globally stable.

Together with [1, Theorem 2.2 and 2.3] or the above result (i), it implies the following result.

Theorem 1.2. The solution $R(t) \equiv 0$ of (1.15) is globally stable if and only if $\overline{\Phi} \geq \overline{\Phi}$.

Theorem 1.3. If $\overline{\Phi} < \overline{\Phi}$, then

(i) (1.15) admits a unique T-periodic positive solution $R_*(t)$.

(ii) For any the positive solution $R(t)$, there exist $\delta > 0$ and $C > 0$ such that

$$|R(t) - R_*(t)| \leq Ce^{-\delta t} \quad \text{for } t > 0,$$

i.e., $R(t) - R_*(t)$ decreases exponentially fast to 0.

Theorem 1.2 and 1.3 give a complete classification for the parameter $\overline{\Phi}$ and show us that if the average of the supply $\Phi(t)$ of external nutrients isn’t larger than the threshold concentration $\overline{\Phi}$ for proliferation, then all spherical tumors will disappear while if the average of the supply $\Phi(t)$ of external nutrients is larger than the threshold concentration $\overline{\Phi}$ for proliferation, then there exists a unique spherical tumor with periodic change and all the other spherical tumors don’t disappear and they evolve to this periodical tumor.

As a direct corollary of Theorem 1.3, we have the following result.

Corollary 1.4. If $\overline{\Phi} < \overline{\Phi}$, then there exists a unique T-periodic solution $(\sigma_*(r,t), p_*(r,t), R_*(t))$ of (1.7)–(1.12) given by

$$\sigma_*(r,t) = \Phi(t) \frac{R_*(t)}{\sinh R_*(t)} \sinh r,$$

$$p_*(r,t) = \frac{1}{6} \mu \overline{\Phi} r^2 - \mu \sigma_*(r,t) + \frac{\gamma}{R_*(t)} - \frac{1}{6} \mu \overline{\Phi} \overline{\Phi}^2(t) + \mu \Phi(t),$$

where $R_*(t)$ is the unique T-periodic positive solution of (1.15).
Recently, for two-space dimensional problem of (1.1)–(1.6), Huang, Zhang and Hu ([2]) have studied the linear stability of the periodic solution under all non-radially symmetric perturbations.

In this paper, we also extend the linear stability ([2, Theorem 1.1]) of the periodic solution under non-radially symmetric perturbations from two-space dimensional case to three-space dimensional case. Precisely, considering non-radially symmetric perturbations, we prove that there exists a constant $\mu_0 > 0$ such that the periodic solution $(\sigma_\varepsilon(r,t), p_\varepsilon(r,t), R_\varepsilon(t))$ (given in Corollary 1.4) is linearly stable for $\mu < \mu_\ast$ and linearly unstable for $\mu > \mu_\ast$ (see Theorem 4.6).

In recent years, many research works have been done on various tumor models (see e.g., [3–26] and the references therein). If $\Phi(t)$ is a constant and (1.1) is replaced by $\varepsilon \Phi_t - \Delta \sigma + \sigma = 0$, many interesting results about the existence and stability of the stationary solution have been established (see [7–9, 22]). The tumor model with the general consumption rate of the nutrients and the general tumor cell proliferation rate has been studied by Cui and Escher (see [10]). If the boundary (1.4) is replaced by the boundary condition $\partial_\sigma \sigma + a(\sigma - \sigma^*) = 0 (a > 0)$, this model and the general case have been considered by Huang, Zhang and Hu ([24]) and Cui and Zhuang ([27–29]). The results about the existence and stability of the stationary solution have been extended in [21, 30–32] to the case involving Gibbs-Thomson relation on the boundary. The tumor model with ECM models in fluid-like tissue.

In this paper, we also extend the linear stability ([2, Theorem 1.1]) of the periodic solution under non-radially symmetric perturbations in Section 4. Precisely, considering non-radially symmetric perturbations from two-space dimensional case to three-space dimensional case. The tumor model with the general consumption rate of the nutrients and the general tumor cell proliferation rate has been studied by Cui and Escher (see [10]). If the boundary (1.4) is replaced by the boundary condition $\partial_\sigma \sigma + a(\sigma - \sigma^*) = 0 (a > 0)$, this model and the general case have been considered by Huang, Zhang and Hu ([24]) and Cui and Zhuang ([27–29]). The results about the existence and stability of the stationary solution have been extended in [21, 30–32] to the case involving Gibbs-Thomson relation on the boundary. The tumor model with ECM and MDE interactions was analyzed in [26, 36]. Friedman et al. ([12, 37–39]) have studied the various tumor models in fluid-like tissue.

The paper is organized as follows. In Section 2, we establish Theorem 1.1 about the global stability of zero solution of (1.15). In Section 3, we derive Theorem 1.3 about the existence and asymptotically stable of the periodic solution under radially symmetric perturbations. We discuss the linear stability of the periodic solution of three-space dimensional problem under non-radially symmetric perturbations in Section 4.

### 2. Stability of zero Equilibrium

In this section, we shall prove Theorem 1.1.

**Proof of Theorem 1.1.** Assume $\sigma = \bar{\sigma}$ and $R(t)$ is a solution of (1.15) with the initial value $R_0 > 0$. Then $R(t) > 0$. We split the proof into three steps.

**Step 1:** At first, we claim

\[
R(t + T) \leq R(t) \quad \text{for } t > 0, \tag{2.1}
\]

\[
R(t) \leq R(a)e^{\mu \frac{\Phi^* - \bar{\sigma}}{3} T} \quad \text{for } t \in [a, a + T], \tag{2.2}
\]

where $a \geq 0$.

Since $0 < P_0(x) < \frac{1}{3}$ ([8]), (1.15) implies

\[
\frac{dR}{dt} \leq \mu R(t) \left[ \frac{\Phi(t)}{3} - \frac{\bar{\sigma}}{3} \right]. \tag{2.3}
\]

Hence

\[
R(t + T) \leq R(t)e^{\int^t_0 \mu \left[ \frac{\Phi(t)}{3} - \frac{\bar{\sigma}}{3} \right] dt} = R(t)e^{\mu \left[ \frac{\Phi^*}{3} T - \frac{\bar{\sigma}}{3} T \right]} = R(t).
\]

Then (2.1) holds.

From (2.3), we have

\[
\frac{dR}{dt} \leq \mu R(t) \left[ \frac{\Phi^*}{3} - \frac{\bar{\sigma}}{3} \right],
\]

which implies

\[
R(t) \leq R(a)e^{\mu \frac{\Phi^* - \bar{\sigma}}{3} (t - a)} \leq R(a)e^{\mu \frac{\Phi^* - \bar{\sigma}}{3} T} \quad \text{for } t \in [a, a + T].
\]

Then (2.2) is true.

**Step 2:** We claim

\[
\liminf_{t \to +\infty} R(t) = 0. \tag{2.4}
\]
Assume on the contrary
\[
\liminf_{t \to +\infty} R(t) = \alpha > 0.
\]
For every \(\varepsilon > 0\), there exists \(M > 0\) such that
\[
R(t) > \alpha - \varepsilon \quad \text{for } t > M.
\] (2.5)

From (1.15), the fact that \(P_0\) is strictly decreasing ([8]) implies
\[
\frac{dR}{dt} = \mu R(t) \left[ \Phi(t) P_0(R(t)) - \frac{\bar{\sigma}}{3} \right] \leq \mu R(t) \left[ \Phi(t) P_0(\alpha - \varepsilon) - \frac{\bar{\sigma}}{3} \right] \quad \text{for } t > M.
\]
Then
\[
R(t^* + nT) \leq R(t^*) e^{\int_{t^*}^{t^* + nT} \mu \left[ \Phi(t) P_0(\alpha - \varepsilon) - \frac{\bar{\sigma}}{3} \right] dt} = R(t^*) e^{\mu nT P_0(\alpha - \varepsilon) \Phi - \frac{\bar{\sigma}}{3}},
\]
where \(n \geq 1\) is an integer and \(t^* > M\). Notice
\[
P_0(\alpha - \varepsilon) \Phi - \frac{\bar{\sigma}}{3} < \frac{1}{3} \Phi - \frac{\bar{\sigma}}{3} = 0.
\]

Letting \(n \to \infty\) in (2.6), we have
\[
R(t^* + nT) \to 0.
\]

It contracts with (2.5). Hence (2.4) holds.

**Step 3:** We shall prove
\[
\lim_{t \to +\infty} R(t) = 0.
\] (2.7)

From (2.4), for all \(\varepsilon > 0\), there exist a sequence \(t_n \to \infty\) and \(M > 0\) such that
\[
R(t_n) < \varepsilon \quad \text{for } t_n > M.
\]
(2.1) implies
\[
R(t_N + kT) \leq R(t_N) < \varepsilon,
\]
where \(t_N > M\) and \(k \geq 1\) is an integer. For \(t > t_N\), there exists \(k_0\) such that \(t \in t_N + k_0T, t_N + (k_0 + 1)T\). Together with (2.2), we obtain
\[
R(t) \leq R(t_N + k_0T) e^{\mu \frac{\Phi' - \bar{\sigma}}{3} T} \leq \varepsilon e^{\mu \frac{\Phi' - \bar{\sigma}}{3} T}.
\]

Then (2.7) is true, which completes the proof. \(\square\)

**Remarks 2.1.** If \(\bar{\sigma} > \Phi\), the key step ([1, (2.4)]) tells
\[
R(\bar{\xi} + nT) \leq R(\bar{\xi}) e^{\frac{\mu}{3} (\Phi - \bar{\sigma})} \to 0, \quad n \to \infty.
\]

Then \(R(t) \equiv 0\) is globally stable. If \(\bar{\sigma} = \Phi\), (2.4) in [1] only shows
\[
R(\bar{\xi} + nT) \leq R(\bar{\xi}).
\]

By this method, one can not get \(R(t)\) converges to 0 as \(t \to \infty\).

When \(\bar{\sigma} = \Phi\), we find good properties (2.1) and (2.2), i.e., \(R(t + nT)\) is a decreasing function in \(n\) and in one period, \(R(t)(t \in [a, a + T])\) can be controlled by \(CR(a)\). The two properties and \(\liminf_{t \to +\infty} R(t) = 0\) (2.4) in Step 2) ensure Theorem 1.1 holds.
3. Existence, Uniqueness and Stability of the Periodic Solution

In this section, we shall prove Theorem 1.3.

**Proof of Theorem 1.3.** The facts that $\bar{\sigma} < \Phi \leq \Phi^*$, $0 < P_0(x) < \frac{1}{3}$ and $P_0(x)$ is strictly decreasing imply that $x_2 = P_0^{-1}(x)$ are well defined. Since $P_0(x)$ is strictly decreasing, it follows

$$\Phi < x_2.$$ 

For each $R_0 \in [\Phi, x_2]$, we let $R(t)$ be the solution of (1.5) with the initial value $R(0) = R_0$. Define the map $F: [\Phi, x_2] \rightarrow \mathbb{R}$ by

$$F(R_0) = R(T).$$ 

At first, we show that $F$ maps $[\Phi, x_2]$ into itself.

Since $x_2$ is a upper solution of the (1.5) and $R(0) \leq x_2$, the comparison theorem implies

$$R(t) \leq x_2 \quad \text{for } t > 0.$$ 

Then

$$R(T) \leq x_2. \quad (3.1)$$ 

On the other hand, we define $\Phi$ by

$$\begin{cases}
\frac{d\Phi}{dt} = \mu\Phi(t) \left[ \Phi(t)P_0(\Phi(t)) - \frac{\bar{\sigma}}{3} \right], \\
\Phi(0) = \Phi.
\end{cases} \quad (3.2)$$ 

By comparison theorem, we obtain

$$R(t) \geq \Phi(t) \quad \text{for } t > 0. \quad (3.3)$$ 

The fact that $0 < P_0(x) < \frac{1}{3}$ implies

$$\frac{d\Phi}{dt} = \mu\Phi(t) \left[ \Phi(t)P_0(\Phi(t)) - \frac{\bar{\sigma}}{3} \right] \leq \mu\Phi(t) \left[ \Phi^* - \frac{\bar{\sigma}}{3} \right].$$ 

Then

$$\Phi(t) \leq \Phi(t) = P_0^{-1} \left( \frac{\bar{\sigma}}{3\Phi} \right) \quad \text{for } t \in [0, T].$$ 

Since $P_0(x)$ is strictly decreasing, it follows

$$P_0(\Phi(t)) \geq \frac{\bar{\sigma}}{3\Phi} \quad \text{for } t \in [0, T].$$ 

Together with the first equation of (3.2), we get

$$\frac{d\Phi}{dt} \geq \mu\Phi(t) \left[ \Phi(t) \frac{\bar{\sigma}}{3\Phi} - \frac{\bar{\sigma}}{3} \right] \quad \text{for } t \in (0, T).$$ 

Hence

$$\Phi(T) \geq \Phi(0)e^{\int_0^T \mu \left[ \Phi(t) \frac{\bar{\sigma}}{3\Phi} - \frac{\bar{\sigma}}{3} \right] dt} = \Phi(0) = \Phi.$$ 

From (3.1), (3.3) and (3.4), we get

$$R(T) \in [\Phi, x_2].$$ 

Then $F$ maps $[\Phi, x_2]$ into itself. Since the solution $R$ depends continuously on the initial value $R_0$, it follows that $F$ is continuous. Brouwer’s fixed point theorem implies that $F$ has a fixed point $R_*(0)$. Then the solution $R_*(t)$ of (1.5) with the initial value $R_*(0)$ is a positive $T$-periodic solution. So far, we have shown the
existence of the periodic solution. The uniqueness of the periodic solution will be given at the end of the proof.

Let

\[ R_{\min} = \min_{t > 0} \{ R_s(t) \} \quad \text{and} \quad R_{\max} = \max_{t > 0} \{ R_s(t) \}. \tag{3.5} \]

The uniqueness of the solution to the initial value problem implies that \( R_{\min} > 0 \) and \( R_{\max} > 0 \).

Next we turn to prove (ii). Assume that \( R(t) \) is the solution of (1.15) with the initial value \( R(0) > 0 \).

Let

\[ R(t) = R_s(t)e^{y(t)}. \]

Then \( y \) satisfies the following equation

\[ y'(t) = \mu \Phi(t) [P_0(R_s(t)e^{y(t)}) - P_0(R_s(t))]. \tag{3.6} \]

To prove (1.17), it is sufficient to show that there exist \( \delta > 0 \) and \( C > 0 \) such that

\[ |e^{y(t)} - 1| \leq Ce^{-\delta t} \quad \text{for} \ t > 0. \]

The uniqueness of the solution to the initial value problem implies that if \( R(0) > R_s(0) \), then \( R(t) > R_s(t) \) and if \( R(0) < R_s(0) \), then \( R(t) < R_s(t) \). Then \( y(t) > 0 \) if \( y(0) > 0 \) and \( y(t) < 0 \) if \( y(0) < 0 \). Hence the arguments are divided into two cases according to the sign of \( y(t) \).

Case A: \( y(t) > 0 \).

From (3.6) and the mean value theorem, we get

\[ y'(t)e^{y(t)} = \mu \Phi(t) P_0'(\xi(t)) R_s(t)(e^{y(t)} - 1)e^{y(t)} \leq -\mu \Phi_s M_{\min} R_{\min}(e^{y(t)} - 1), \]

where \( \xi(t) \in [R_s(t), R_s(t)e^{y(t)}] \subseteq [R_{\min}, R_{\max}e^{\Phi(0)}] \) and \( M_{\min} = \min_{x \in [R_{\min}, R_{\max}e^{\Phi(0)}]} \{ -P_0'(x) \} > 0 \).

Hence

\[ \frac{(e^{y(t)} - 1)}{(e^{y(t)} - 1)} \leq -\mu \Phi_s M_{\min} R_{\min}. \tag{3.7} \]

Integrating (3.7) over \([0, t]\), we have

\[ e^{y(t)} - 1 \leq (e^{y(0)} - 1)e^{-\mu \Phi_s M_{\min} t} \quad \text{for} \ t > 0. \tag{3.8} \]

Case B: \( y(t) < 0 \).

From (3.6) and the fact that \( P_0 \) is strictly decreasing, we get that \( y'(t) > 0 \). Combining the mean value theorem, we have

\[ -y'(t)e^{y(t)} = -\mu \Phi(t) P_0'(\eta(t)) R_s(t)(e^{y(t)} - 1)e^{y(t)} \leq -\mu \Phi_s M_{\min} R_{\min}(1 - e^{y(t)})e^{y(0)}, \]

where \( \eta(t) \in [R_s(t)e^{y(t)}, R_s(t)] \subseteq [R_{\min}e^{y(0)}, R_{\max}] \).

Hence

\[ \frac{(1 - e^{y(t)})}{(1 - e^{y(t)})} \leq -\mu \Phi_s M_{\min} e^{y(0)}. \tag{3.9} \]

Integrating (3.9) over \([0, t]\), we have

\[ 1 - e^{y(t)} \leq (1 - e^{y(0)})e^{-\mu \Phi_s M_{\min} e^{y(0)} t} \quad \text{for} \ t > 0. \tag{3.10} \]

Taking \( \delta = \min \{ \mu \Phi_s M_{\min} R_{\min}, \mu \Phi_s M_{\min} R_{\min} e^{y(0)} \} \) and \( C = |1 - e^{y(0)}| \). (3.8) and (3.10) imply (1.17).

Finally, we show that the solution \( R_s(t) \) is unique. Otherwise, by (1.17), we obtain

\[ |R_1(t) - R_2(t)| = |R(t) - R_1(t)| + |R(t) - R_2(t)| \to 0 \quad t \to \infty. \]

Hence \( R_1(t) = R_2(t) \), which completes the proof. \( \square \)
Remarks 3.1. Bai and Xu ([1]) applied Brouwer’s fixed point theorem to show the existence of periodic solution. They need the condition \( \bar{c} < \Phi_* \) to construct sub-solution \( x_1 = P_0^{-1} \left( \frac{\bar{c}}{\Phi_*} \right) \). To get rid of the condition \( \bar{c} < \Phi_* \), we use a sub-solution \( \mathcal{K}(t) \) (given in (3.2)) to replace \( x_1 \).

Remarks 3.2. The proof of Theorem 1.3 is still valid for two-space dimensional problem. The existence scope \( \bar{c} \in (0, \Phi_*) \) of periodic solution in [2, Theorem 2.1] can be extended to \( \bar{c} \in (0, \Phi) \).

4. Linear stability of the periodic solution under non-radially symmetric perturbations

In this section, we consider linear stability of the unique radially symmetric T-periodic positive solution \( (\sigma_s(r,t), \rho_s(r,t), 0) \) obtained in Corollary 1.4 under non-radially symmetric perturbations.

Substituting
\[
\begin{align*}
\frac{\partial \Omega(t)}{\partial r} = \sigma_r(r,t) + \epsilon \rho \frac{\partial \sigma_r}{\partial r}(R_s(t), t) \rho(\theta, \phi, t) + O(\epsilon^2), \\
\sigma(r, \theta, \phi, t) &= \sigma_r(r, t) + \epsilon w(r, \theta, \phi, t) + O(\epsilon^2), \\
p(r, \theta, \phi, t) &= p_r(r, t) + \epsilon q(r, \theta, \phi, t) + O(\epsilon^2)
\end{align*}
\]

into (1.1)–(1.6) and collecting the \( \epsilon \)-order terms, we can get the linearized system of (1.1)–(1.6) at the radially symmetric T-periodic solution \( (\sigma_s(r,t), \rho_s(r,t), 0) \).

(1.3) and [7, 8, 22] imply
\[
V_n = R_n^\ast(t) + \epsilon \rho t + O(\epsilon^2) \quad \text{and} \quad \kappa = \frac{1}{R_n^\ast(t)} - \frac{\epsilon}{R_n^\ast(t)} \left( \rho + \frac{1}{2} \Delta \omega \rho \right) + O(\epsilon^2).
\]

Then the linearized system has the following form:
\[
\begin{align*}
\Delta w &= w, & r \in (0, R_s(t)), t > 0, \\
w(R_s(t), \theta, \phi, t) &= -\frac{\partial \sigma_r}{\partial r}(R_s(t), t) \rho(\theta, \phi, t), & t > 0, \\
-\Delta q &= \mu w, & r \in (0, R_s(t)), t > 0, \\
q(R_s(t), \theta, \phi, t) &= -\frac{\partial p_r}{\partial r}(R_s(t), t) \rho(\theta, \phi, t) - \frac{\gamma}{R_n^\ast(t)} \left( \rho + \frac{1}{2} \Delta \omega \rho \right), & t > 0, \\
\frac{d \rho}{dt} &= -\frac{\partial^2 p_r}{\partial r^2}(R_s(t), t) \rho(\theta, \phi, t) - \frac{\partial q}{\partial r}(R_s(t), t), & t > 0.
\end{align*}
\]

At first, we give some preliminaries.

The Bessel function \( (40) \) is given by
\[
I_n(r) = \left( \frac{r}{2} \right)^n \sum_{k=0}^{\infty} \frac{\Gamma(n+k+1)}{k!} \left( \frac{r}{2} \right)^{2k},
\]
and has the following properties:
\[
\begin{align*}
\frac{\partial^2}{\partial r^2} + \frac{2}{r} \frac{\partial}{\partial r} - \frac{n(n+1)}{r^2} I_{n-1/2}(r) &= \frac{I_{n+1/2}(r)}{r^{1/2}}, \\
\frac{d}{dr} \left( \frac{n}{r} \right) \frac{I_{n+1/2}(r)}{r^{1/2}} &= \frac{I_{n+3/2}(r)}{r^{1/2}}.
\end{align*}
\]

A useful function \( P_n ([8]) \) is given by
\[
P_n(r) = \frac{I_{n+3/2}(r)}{I_{n+1/2}(r)}, \quad n = 0, 1, 2, 3, \ldots,
\]
and has the following properties:
\[
\begin{align*}
P_0(r) &= r^{-1} \coth r - r^{-2}, \\
\frac{dP_n}{dr}(r) &= 0 \quad r > 0, \\
0 < P_n(r) &\leq \frac{1}{2n+3} \quad r \geq 0, \\
P_0(r) &= \frac{1}{r^2 P_1(r) + 3}, \\
P_n(r) &> P_{n+1}(r) \quad \forall n \geq 0, r > 0.
\end{align*}
\]
Lemma 4.1. The following relations hold:

\[
\begin{align*}
\frac{\partial \sigma_s}{\partial r}(R_s(t), t) &= \Phi(t) R_s(t) P_0(R_s(t)), \\
\frac{\partial^2 \sigma_s}{\partial r^2}(R_s(t), t) &= \Phi(t) \{1 - 2P_0(R_s(t))\}, \\
\frac{\partial p_s}{\partial r}(R_s(t), t) &= -\frac{dR_s(t)}{dt}, \\
\frac{\partial^2 p_s}{\partial r^2}(R_s(t), t) &= -\frac{1}{R_s(t)} \frac{dR_s(t)}{dt} - \mu \Phi(t) R_s^2(t) P_0(R_s(t)) P_1(R_s(t)).
\end{align*}
\]

Proof. The proofs of (4.5)–(4.6) are similar to [22, Lemma 2.1], we omit them. It remains to show that (4.7)–(4.8) hold.

(1.9) implies (4.7). From (1.18) and (4.6), we obtain

\[
\frac{\partial^2 p_s}{\partial r^2}(R_s(t), t) = \frac{1}{3} \bar{\sigma} - \mu \Phi(t) [1 - 2P_0(R_s(t))].
\]

Together with (1.15) and (4.3), we have

\[
\frac{\partial^2 p_s}{\partial r^2}(R_s(t), t) = -\frac{1}{R_s(t)} \frac{dR_s(t)}{dt} - \mu \Phi(t) [1 - 3P_0(R_s(t))] = -\frac{1}{R_s(t)} \frac{dR_s(t)}{dt} - \mu \Phi(t) R_s^2(t) P_0(R_s(t)) P_1(R_s(t)).
\]

Then (4.8) is true.

Let

\[
\begin{align*}
\rho(\theta, \phi, t) &= \sum_{n=0}^{+\infty} \sum_{m=-n}^{n} \rho_{n,m}(t) Y_{n,m}(\theta, \phi), \\
w(r, \theta, \phi, t) &= \sum_{n=0}^{+\infty} \sum_{m=-n}^{n} w_{n,m}(r, t) Y_{n,m}(\theta, \phi), \\
q(r, \theta, \phi, t) &= \sum_{n=0}^{+\infty} \sum_{m=-n}^{n} q_{n,m}(r, t) Y_{n,m}(\theta, \phi),
\end{align*}
\]

where the spherical harmonic function

\[
Y_{n,m}(\theta, \phi) = (-1)^m \sqrt{\frac{(2n+1)(n-m)!}{2(n+m)!}} P_n^m(\cos \theta) e^{im\phi} \quad (m = -n, \ldots, n)
\]

in \( \mathbb{R}^3 \), where

\[
P_n^m(z) = \frac{1}{2^n n!} (1 - z^2)^{n/2} d_{n+m}(z^2 - 1)^n
\]

is the Legendre polynomial. \( \{Y_{n,m}\} \) is a complete orthonormal basis for \( L^2(S^2) \).

Applying the relation (4.8)

\[
\Delta_{\omega} Y_{n,m}(\theta, \phi) + n(n+1) Y_{n,m}(\theta, \phi) = 0,
\]

we have

\[
\begin{align*}
\frac{\partial^2 w_{n,m}}{\partial r^2}(r, t) + \frac{2}{r} \frac{\partial w_{n,m}}{\partial r}(r, t) - \left( \frac{n(n+1)}{r^2} + 1 \right) w_{n,m}(r, t) &= 0 & r \in (0, R_s(t)), t > 0, \\
\omega_{n,m}(R_s(t), \theta, \phi, t) &= -\frac{\partial \sigma_s}{\partial r}(R_s(t), t) \rho_{n,m}(t) & t > 0, \\
\frac{\partial^2 q_{n,m}}{\partial r^2}(r, t) + \frac{2}{r} \frac{\partial q_{n,m}}{\partial r}(r, t) - \left( \frac{n(n+1)}{r^2} + 1 \right) q_{n,m}(r, t) &= -\mu \omega_{n,m}(r, t) & r \in (0, R_s(t)), t > 0, \\
q_{n,m}(R_s(t), t) &= \left( \frac{n(n+1)}{2} - 1 \right) \frac{\gamma \rho_{n,m}(t)}{R_s^2(t)} - \frac{\partial p_s}{\partial r}(R_s(t), t) \rho_{n,m}(t) & t > 0,
\end{align*}
\]
\[
\frac{d\rho_{n,m}(t)}{dt} = -\frac{\partial^2 p_s}{\partial r^2}(R_s(t), t) \rho_{n,m}(t) - \frac{\partial q_{n,m}}{\partial r}(R_s(t), t) \quad t > 0. (4.13)
\]

The solution \(w_{n,m}\) of (4.9)–(4.10) is given by
\[
w_{n,m}(r, t) = -\frac{\partial \sigma_s}{\partial r}(R_s(t), t) \frac{R_s^{1/2}(t)}{I_{n+1/2}(R_s(t))} \frac{l_{n+1/2}(r)}{r^{1/2}} \rho_{n,m}(t).
\]

Define
\[
\psi_{n,m} = q_{n,m} + \mu w_{n,m}.
\]

From (4.9)–(4.12), we obtain
\[
\frac{\partial^2 \psi_{n,m}(r, t)}{\partial r^2} + \frac{2}{r} \frac{\partial \psi_{n,m}(r, t)}{\partial r} - \frac{n(n + 1)}{r^2} \psi_{n,m}(r, t) = 0 \quad r \in (0, R_s(t)),
\]
\[
\psi_{n,m}(R_s(t), t) = \left( \frac{n(n + 1)}{2} - 1 \right) \gamma \rho_{n,m}(t) \frac{R_s(t)}{R_s^\gamma(t)} - \frac{\partial p_s}{\partial r}(R_s(t), t) \rho_{n,m}(t) - \mu \frac{\partial \sigma_s}{\partial r}(R_s(t), t) \rho_{n,m}(t).
\]

The solution of the above problem is
\[
\psi_{n,m}(r, t) = \frac{r^n}{R_s^n(t)} \left[ \left( \frac{n(n + 1)}{2} - 1 \right) \gamma \rho_{n,m}(t) \frac{R_s(t)}{R_s^\gamma(t)} - \frac{\partial p_s}{\partial r}(R_s(t), t) \rho_{n,m}(t) - \mu \frac{\partial \sigma_s}{\partial r}(R_s(t), t) \rho_{n,m}(t) \right].
\]

Then
\[
q_{n,m}(r, t) = \frac{r^n}{R_s^n(t)} \left[ \left( \frac{n(n + 1)}{2} - 1 \right) \gamma \rho_{n,m}(t) \frac{R_s(t)}{R_s^\gamma(t)} - \frac{\partial p_s}{\partial r}(R_s(t), t) \rho_{n,m}(t) - \mu \frac{\partial \sigma_s}{\partial r}(R_s(t), t) \rho_{n,m}(t) \right]
\]
\[+ \mu \frac{\partial \sigma_s}{\partial r}(R_s(t), t) \frac{R_s^{1/2}(t)}{I_{n+1/2}(R_s(t))} \frac{l_{n+1/2}(r)}{r^{1/2}} \rho_{n,m}(t).
\]

Differentiating the above equation in \(r\), using (4.1) and taking \(r = R_s(t)\), we obtain
\[
\frac{\partial q_{n,m}}{\partial r}(R_s(t), t) = \frac{n}{R_s(t)} \left[ \left( \frac{n(n + 1)}{2} - 1 \right) \gamma \rho_{n,m}(t) \frac{R_s(t)}{R_s^\gamma(t)} - \frac{\partial p_s}{\partial r}(R_s(t), t) \rho_{n,m}(t) - \mu \frac{\partial \sigma_s}{\partial r}(R_s(t), t) \rho_{n,m}(t) \right]
\]
\[+ \mu \frac{\partial \sigma_s}{\partial r}(R_s(t), t) \frac{n}{R_s(t)} \left[ \frac{n}{R_s(t)} + \frac{l_{n+3/2}(R_s(t))}{l_{n+1/2}(R_s(t))} \right] \rho_{n,m}(t)
\]
\[= \frac{n}{R_s(t)} \left[ \left( \frac{n(n + 1)}{2} - 1 \right) \gamma \rho_{n,m}(t) \frac{R_s(t)}{R_s^\gamma(t)} - \frac{\partial p_s}{\partial r}(R_s(t), t) \rho_{n,m}(t) \right]
\]
\[+ \mu \frac{\partial \sigma_s}{\partial r}(R_s(t), t) R_s(t) P_n(R_s(t)) \rho_{n,m}(t).
\]

Plugging (4.5) and (4.7) into the above equation, we obtain
\[
\frac{\partial q_{n,m}}{\partial r}(R_s(t), t)
\]
\[= \left\{ \frac{n}{R_s(t)} \left[ \gamma \rho_{n,m}(t) \frac{R_s(t)}{R_s^\gamma(t)} \left( \frac{n(n + 1)}{2} - 1 \right) \frac{dR_s}{dt} \right] + \mu \Phi(t) R_s^{2/2}(t) P_0(R_s(t)) P_n(R_s(t)) \right\} \rho_{n,m}(t). (4.14)
\]

Substituting (4.8) and (4.14) into (4.13), we have
\[
\frac{d\rho_{n,m}}{dt} = -\left\{ \frac{dR_s}{dt} \frac{n-1}{R_s(t)} + \gamma \rho_{n,m}(t) \frac{n(n + 1)}{2} - 1 \right\} - \mu \Phi(t) R_s^{2/2}(t) P_0(R_s(t)) [P_1(R_s(t)) - P_n(R_s(t))]
\]
\[\rho_{n,m}(t).
\]

Hence,
\[
\rho_{n,m}(t) = \rho_{n,m}(0) \exp \left\{ -\int_0^t \left( \frac{dR_s}{dt} \frac{n-1}{R_s(t)} + \gamma \rho_{n,m}(t) \frac{n(n + 1)}{2} - 1 \right) - \mu \Phi(t) R_s^{2/2}(t) P_0(R_s(t)) [P_1(R_s(t)) - P_n(R_s(t))] \right\} dt.
\]

At first, we give an estimate for \(\rho_0(t)\). 

Lemma 4.2. For any \( \mu > 0 \), there exist \( \delta > 0 \) and \( M \) such that
\[
|\rho_0(t)| \leq |\rho_0(0)| e^{-\delta t} \quad \text{for } t > M.
\]

Proof. Plugging \( n = 0 \) into (4.15), we have
\[
\rho_0(t) = \rho_0(0) \exp \left\{ - \int_0^t \frac{dR_s(t)}{R_s(t)} + \mu \Phi(t) R_2^s(t) P_0(R_s(t)) [P_0(R_s(t)) - P_1(R_s(t))] dt \right\} \\
= \rho_0(0) \frac{R_s(t)}{R_s(0)} \exp \left\{ - \int_0^t \mu \Phi(t) R_2^s(t) P_0(R_s(t)) [P_0(R_s(t)) - P_1(R_s(t))] dt \right\}.
\]
From (4.4), we get
\[
\mu \Phi(t) R_2^s(t) P_0(R_s(t)) [P_0(R_s(t)) - P_1(R_s(t))] \geq 0. \quad (4.16)
\]
For any \( t > T \), there exist a positive integer \( m \) and \( \tau \in [0, T) \) such that \( t = mT + \tau \). The fact that \( R_s(t) \) and \( \Phi(t) \) are \( T \)-periodic and (4.16) imply
\[
|\rho_0(t)| = |\rho_0(0)| \frac{R_s(t)}{R_s(0)} \exp \left\{ - \int_0^t \left( \int_0^{mT} \mu \Phi(t) R_2^s(t) P_0(R_s(t)) [P_0(R_s(t)) - P_1(R_s(t))] dt \right) \right\} \\
\leq |\rho_0(0)| \frac{R_s(\tau)}{R_s(0)} \exp \left\{ - m \int_0^T \mu \Phi(t) R_2^s(t) P_0(R_s(t)) [P_0(R_s(t)) - P_1(R_s(t))] dt \right\} \\
\leq |\rho_0(0)| \frac{R_{\max} (\tau)}{R_{\min} (0)} \exp \left\{ - \mu \Phi \bar{\delta} mT \right\} \\
\leq |\rho_0(0)| \frac{R_{\max}}{R_{\min}} \exp \left\{ - \mu \Phi \bar{\delta} (t - T) \right\} \\
= |\rho_0(0)| \frac{R_{\max}}{R_{\min}} e^{\mu \Phi \bar{\delta} (t - T)}.
\]
where \( R_{\min} = \min_{t > 0} \{ R_s(t) \} \), \( R_{\max} = \max_{t > 0} \{ R_s(t) \} \) and \( \bar{\delta} = \min_{x \in [R_{\min}, R_{\max}]} \{ x^2 P_0(x) [P_0(x) - P_1(x)] \} > 0 \).

Then there exists \( M > 0 \) such that
\[
|\rho_0(t)| \leq |\rho_0(0)| e^{-\frac{\mu \Phi \bar{\delta}}{2} t} \quad \text{for } t > M.
\]
Taking \( \delta = \frac{\mu \Phi \bar{\delta}}{2} \), we complete the proof.

Next, we give an estimate for \( \rho_1(t) \).

Lemma 4.3. For any \( \mu > 0 \), we have
\[
\rho_{1,n}(t) = \rho_{1,n}(0). \quad (4.17)
\]

Proof. Substituting \( n = 1 \) into (4.15), we get that (4.17) holds.

At last, we give an estimate for \( \rho_n(t) \) \((n \geq 2)\).

Define
\[
\vartheta_n = \frac{\int_0^T \frac{\gamma n}{R_2^s(t)} \left( \frac{n(n+1)}{2} - 1 \right) dt}{\int_0^T \Phi(t) R_2^s(t) P_0(R_s(t)) [P_1(R_s(t)) - P_n(R_s(t))] dt} \quad n \geq 2. \quad (4.18)
\]

Lemma 4.4. For \( n \geq 2 \), \( \vartheta_n < \vartheta_{n+1} \).

Proof. From (4.18), we have
\[
\vartheta_n = \frac{\int_0^T \frac{\gamma n}{R_2^s(t)} dt}{\int_0^T \Phi(t) R_2^s(t) P_0(R_s(t)) \frac{P_1(R_s(t)) - P_n(R_s(t))}{n \left( \frac{n(n+1)}{2} - 1 \right)} dt} \quad n \geq 2.
\]
Since the sequence \( \frac{n(n+1)/2 - 1}{P_1(R) - P_2(R)} \) is strictly increasing in \( n \) as \( n \geq 2 \) ([5]), we get that this lemma is true.

□
Lemma 4.2 and 4.3 hold for all $\mu > 0$, so we define $\vartheta_0 = \vartheta_1 = \infty$. Set

$$
\mu_* = \min \{ \vartheta_0, \vartheta_1, \vartheta_2, \vartheta_3, \vartheta_4, \ldots \}.
$$

(4.19)

From Lemma 4.4, we obtain that $\mu_* = \vartheta_2$.

**Lemma 4.5.** For $n \geq 2$ and $0 < \mu < \vartheta_2$, there exist $\delta > 0$ and $M > 0$ such that

$$
|\rho_{n,m}(t)| \leq |\rho_{n,m}(0)| e^{-\delta(n^3+1)t} \quad t > M,
$$

where $\delta$ and $M$ are independent of $n$ and depend on $R_s(t), \Phi(t), T, \mu$ and $\gamma$.

**Proof.** For any $t > T$, there exist a positive integer $m$ and $\tau \in [0, T)$ such that $t = mT + \tau$. From (4.2), we have

$$
-\int_0^T \frac{\gamma n}{R^2_s(t)} \left( \frac{n(n+1)}{2} - 1 \right) - \mu \Phi(t) R^2_s(t) P_0(R_s(t)) \{ P_1(R_s(t)) - \rho_n(R_s(t)) \} dt
\leq \int_0^T \mu \Phi(t) R^2_s(t) P_0(R_s(t)) P_1(R_s(t)) dt
\leq \frac{1}{15} \vartheta_2 \Phi^* R^2_{\text{max}} T,
$$

(4.20)

where $R_{\text{max}} = \max_{t>0} \{ R_s(t) \}$. From (4.18), we obtain

$$
\int_0^T \frac{\gamma n}{R^2_s(t)} \left( \frac{n(n+1)}{2} - 1 \right) - \mu \Phi(t) R^2_s(t) P_0(R_s(t)) \{ P_1(R_s(t)) - \rho_n(R_s(t)) \} dt
\leq \mu \vartheta_2 \int_0^T \frac{\vartheta_2 n}{R^2_s(t)} \left( \frac{n(n+1)}{2} - 1 \right) - \vartheta_2 \Phi(t) R^2_s(t) P_0(R_s(t)) \{ P_1(R_s(t)) - \rho_n(R_s(t)) \} dt
\geq \mu \vartheta_2 \left( \frac{\vartheta_2 n}{R^2_{\text{max}}} - 1 \right) \int_0^T \frac{\gamma n}{R^2_s(t)} \left( \frac{n(n+1)}{2} - 1 \right) dt
\geq \mu \vartheta_2 \left( \frac{\vartheta_2 n}{R^2_{\text{max}}} - 1 \right) \frac{n^3+1}{4} T = M_1(n^3+1) T,
$$

(4.21)

where $M_1 = \frac{\mu}{\vartheta_2} \left( \frac{\vartheta_2 n}{R^2_{\text{max}}} - 1 \right) > 0$.

The fact that $R_s(t)$ and $\Phi(t)$ are $T$-periodic, (4.15), (4.20) and (4.21) imply

$$
|\rho_n(t)|
= |\rho_n(0)| \frac{R^{n-1}_s(0)}{R^{n-1}_s(\tau)} \exp \left\{ -\int_0^{mT+\tau} \frac{\gamma n}{R^2_s(t)} \left( \frac{n(n+1)}{2} - 1 \right) - \mu \Phi(t) R^2_s(t) P_0(R_s(t)) \{ P_1(R_s(t)) - \rho_n(R_s(t)) \} dt \right\}
\leq |\rho_n(0)| \frac{R^{n-1}_s(0)}{R^{n-1}_s(\tau)} \exp \left\{ -\int_0^{mT+\tau} \frac{\gamma n}{R^2_s(t)} \left( \frac{n(n+1)}{2} - 1 \right) - \mu \Phi(t) R^2_s(t) P_0(R_s(t)) \{ P_1(R_s(t)) - \rho_n(R_s(t)) \} dt \right\}
\exp \left\{ -\int_0^{mT+\tau} \frac{\gamma n}{R^2_s(t)} \left( \frac{n(n+1)}{2} - 1 \right) - \mu \Phi(t) R^2_s(t) P_0(R_s(t)) \{ P_1(R_s(t)) - \rho_n(R_s(t)) \} dt \right\}
\leq |\rho_n(0)| \frac{R^{n-1}_s}{R^{n-1}_m} \exp \left\{ \frac{1}{15} \vartheta_2 \Phi^* R^2_{\text{max}} T \right\} \exp \left\{ - M_1(n^3+1) mT \right\}
\leq |\rho_n(0)| \frac{R^{n-1}_s}{R^{n-1}_m} \exp \left\{ \frac{1}{15} \vartheta_2 \Phi^* R^2_{\text{max}} T \right\} \exp \left\{ - M_1(n^3+1)(t - T) \right\}.
$$

Then there exists $M > 0$ such that

$$
|\rho_n(t)| \leq |\rho_n(0)| e^{-\frac{M_1}{T}(n^3+1)t} \quad t > M.
$$

Taking $\delta = \frac{M_1}{2}$, we complete the proof. \qed

Our main result for three-space dimensional problem is the following theorem.
Theorem 4.6. Assume $\rho_0 \in L^2(\partial B_{R_s}(0))$. Then
(i) If $\mu \in (0, \theta_1)$, then the radially symmetric $T$-periodic solution $(\sigma_s(r,t), p_s(r,t), R_s(t))$ (obtained in Corollary 1.4) is linearly stable, i.e., for any positive integer $k$, there exist $\delta > 0$, $C > 0$ and $t_0 > 0$ such that
$$
\|\hat{\rho}(\theta, \phi, t) - \sum_{m=-1}^{\infty} \rho_{1,m}(0) Y_{1,m}\|_{\mathcal{H}^{k+\frac{1}{2}}(\partial B_{R_s}(t))} \leq Ce^{-\delta t} \quad \text{for } t > t_0.
$$

(ii) If $\mu > \theta_2$, then the radially symmetric $T$-periodic solution $(\sigma_s(r,t), p_s(r,t), R_s(t))$ is linearly unstable.

Proof. For any positive integer $k$, Lemma 4.3, [6, Lemma 8.2], Lemma 4.2 and Lemma 4.5 imply that there exist $\delta > 0$, $C > 0$ and $M > 0$ such that
$$
\bigg\|\rho(\theta, \phi, t) - \sum_{m=-1}^{\infty} \rho_{1,m}(0) Y_{1,m}\bigg\|_{\mathcal{H}^{k+\frac{1}{2}}(\partial B_{R_s}(t))} \leq Ce^{-\delta t} \quad \text{for } t > M.
$$

We next turn to linear instability for $\mu > \theta_2$. For any $t > T$, there exist a positive integer $m$ and $\tau \in [0, T)$ such that $t = mT + \tau$. By (4.4), we derive
$$
\int_0^T -\frac{4\gamma}{R_s^2(t)} + \mu \Phi(t) R_s^2(t) P_0(R_s(t)) [P_1(R_s(t)) - P_2(R_s(t))] dt \\
\geq -\int_0^T \frac{4\gamma}{R_s^2(t)} dt \geq -\frac{4\gamma}{R_{min}^3} T,
$$

(4.22)

where $R_{min} = \min_{t \geq 0} \{R_s(t)\}$. From (4.18), we obtain
$$
\int_0^T -\frac{4\gamma}{R_s^2(t)} + \mu \Phi(t) R_s^2(t) P_0(R_s(t)) [P_1(R_s(t)) - P_2(R_s(t))] dt \\
= \frac{\mu}{\theta_2} \int_0^T -\frac{4\gamma}{R_s^2(t)} + \theta_2 \Phi(t) R_s^2(t) P_0(R_s(t)) [P_1(R_s(t)) - P_2(R_s(t))] dt \\
= \frac{\mu}{\theta_2} \int_0^T -\frac{4\gamma}{R_s^2(t)} dt \\
\geq \frac{\mu}{\theta_2} \int_0^T -\frac{4\gamma}{R_s^2(t)} dt \\
(4.23)

where $R_{max} = \max_{t \geq 0} \{R_s(t)\}$.

The fact that $R_s(t)$ and $\Phi(t)$ are T-periodic, (4.15), (4.22) and (4.23) imply
$$
|\rho_{2,m}(t)| = |\rho_{2,m}(0)| \exp \left\{ -\int_0^t \frac{dR_s}{R_s(t)} + \frac{4\gamma}{R_s^3(t)} - \mu \Phi(t) R_s^2(t) P_0(R_s(t)) [P_1(R_s(t)) - P_2(R_s(t))] dt \right\} \\
= |\rho_{2,m}(0)| \frac{R_s(0)}{R_s(t)} \exp \left\{ -\int_0^t -\frac{4\gamma}{R_s^2(t)} dt + \mu \Phi(t) R_s^2(t) P_0(R_s(t)) [P_1(R_s(t)) - P_2(R_s(t))] dt \right\} \\
= |\rho_{2,m}(0)| \frac{R_s(0)}{R_s(t)} \exp \left\{ -\int_0^t -\frac{4\gamma}{R_s^2(t)} dt + \mu \Phi(t) R_s^2(t) P_0(R_s(t)) [P_1(R_s(t)) - P_2(R_s(t))] dt \right\} \\
\geq |\rho_{2,m}(0)| \frac{R_{min}}{R_{max}} \exp \left\{ -\frac{4\gamma}{R_{min}^3} T \right\} \exp \left\{ \frac{\mu}{\theta_2} (1 - \frac{\theta_2}{\mu}) \frac{4\gamma}{R_{max}^3} mT \right\} \\
\geq |\rho_{2,m}(0)| \frac{R_{min}}{R_{max}} \exp \left\{ -\frac{4\gamma}{R_{min}^3} T \right\} \exp \left\{ \frac{\mu}{\theta_2} (1 - \frac{\theta_2}{\mu}) \frac{4\gamma}{R_{max}^3} (t - T) \right\}.
$$
Then \[ |\rho_{2,m}(t)| \to \infty \quad t \to \infty. \]

Therefore, the proof is completed. □

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