Zero Net Flux MRI Turbulence in Disks: Sustenance Scheme and Magnetic Prandtl Number Dependence

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Abstract

We investigate sustenance and dependence on magnetic Prandtl number (Pm) for magnetorotational instability (MRI)-driven turbulence in Keplerian disks with zero net magnetic flux using standard shearing box simulations. We focus on the turbulence dynamics in Fourier space, capturing specific/noncanonical anisotropy of nonlinear processes due to disk flow shear. This is a new type of nonlinear redistribution of modes over wavevector orientations in Fourier space—the nonlinear transverse cascade—which is generic to shear flows and fundamentally different from the usual direct/inverse cascade. The zero flux MRI has no exponentially growing modes, so its growth is transient, or nonmodal. Turbulence self-sustenance is governed by constructive cooperation of the transient growth of MRI and the nonlinear transverse cascade. This cooperation takes place at small wavenumbers (on the flow size scales) referred to as the vital area in Fourier space. The direct cascade transfers mode energy from the vital area to larger wavenumbers. At large Pm, the transverse cascade prevails over the direct one, keeping most of modes’ energy contained in small wavenumbers. With decreasing Pm, however, the action of the transverse cascade weakens and can no longer oppose the action of the direct cascade, which more efficiently transfers energy to higher wavenumbers, leading to increased resistive dissipation. This undermines the sustenance scheme, resulting in the turbulence decay. Thus, the decay of zero net flux MRI turbulence with decreasing Pm is attributed to the topological rearrangement of the nonlinear processes when the direct cascade begins to prevail over the transverse cascade.

Unified Astronomy Thesaurus concepts: Stellar accretion disks (1579); Magnetohydrodynamics (1964); Plasma astrophysics (1261); High energy astrophysics (739); Protoplanetary disks (1300); Magnetic fields (994); Interplanetary turbulence (830)

1. Introduction

The challenge of understanding accretion processes and the associated angular momentum transport in astrophysical disks requires a comprehensive study of the nonlinear dynamics of perturbations in this kind of flow. In recent decades, this has prompted a vast number of theoretical and numerical analyses. Keplerian differential rotation plays a significant role in the linear stability and nonlinear dynamics of the disk flow. In fact, in the purely hydrodynamic case, Keplerian flows are both linearly and nonlinearly stable, ruling out self-sustained turbulence in them (e.g., Hawley et al. 1999; Lesur & Longaretti 2005; Shen et al. 2006; Rincon et al. 2007). Recent developments revealed, however, other physical processes, such as vertical shear instability, convection, and vortices capable of driving weak turbulence and transport in nonmagnetic regions of disks (e.g., Lyra & Umurhan 2019; Pfeil & Klahr 2019).

The situation is fundamentally different in magnetized regions of disks. The magnetic field imposed on Keplerian rotation of conducting disk matter gives rise to dynamic activity, which is primarily manifested in the linear magnetorotational instability (MRI) and its nonlinear outcome, magnetohydrodynamic (MHD) turbulence (Balbus & Hawley 1991; Hawley et al. 1995; Balbus & Hawley 1998). The self-sustaining mechanism, statistical characteristics, and transport properties of this MRI-driven turbulence strongly depend on the configuration of the background magnetic field itself.

Disks threaded by a weak net vertical magnetic field are unstable to MRI; axisymmetric perturbations can grow exponentially on a fast orbital time at a linear stage that greatly facilitates the onset of turbulence (e.g., Hawley et al. 1995; Sano & Inutsuka 2001; Lesur & Longaretti 2007; Pessah & Goodman 2009; Pessah 2010; Longaretti & Lesur 2010). In addition to these exponentially growing axisymmetric (channel) modes, the system contains transiently, or nonmodally, growing nonaxisymmetric MRI modes due to shear flow nonnormality (Balbus & Hawley 1992; Pessah & Chan 2012; Mamatsashvili et al. 2013; Squire & Bhattacharjee 2014a, 2014b; Zhuravlev & Razdoburdin 2014; Razdoburdin & Zhuravlev 2017), which, despite being transient, strongly affect the statistical properties of MRI turbulence (Bodo et al. 2008; Longaretti & Lesur 2010; Murphy & Pessah 2015; Gogichaishvili et al. 2018, hereafter Paper II).

Disks threaded by a net azimuthal magnetic field lack linear exponentially growing axisymmetric modes; hence, the turbulence can be energetically fueled only by the linear process of transient/nonmodal growth of nonaxisymmetric MRI (e.g., Hawley et al. 1995; Brandenburg & Dröndorp 2001; Simon & Hawley 2009; Pessah & Chan 2012; Squire & Bhattacharjee 2014a; Mamatsashvili & Stefani 2017). In this case, the role of a specific, or noncanonical, nonlinearity—the so-called nonlinear transverse cascade (see below)—is crucial, since it ensures positive feedback for nonmodally growing MRI modes (Gogichaishvili et al. 2017, hereafter Paper I).
1.1. An Overview of Zero Net Flux MRI Turbulence

Disks with zero net magnetic flux occupy a special place in MRI turbulence research and have been very actively studied since the seminal paper by Hawley et al. (1996), which demonstrated for the first time a link between MRI and self-sustaining magnetic dynamo action in disks. The zero net flux MRI has also spurred much debate in the last decade because of the nonconvergence issue in numerical simulations (Bodo et al. 2011, 2014; Davis et al. 2010; Fromang & Papaloizou 2007; Guan et al. 2009; Pessah et al. 2007; Ryan et al. 2017). The chief appeal of the zero net flux case is that it offers the prospect of a universal state of MRI turbulence in which the disk generates and maintains its own magnetic field via the joint action of the dynamo mechanism and MRI, not requiring any imposed net field. The resulting transport of angular momentum would then depend on the disk properties but not on the magnitude and direction of the magnetic field in the disk.

Unlike the net vertical field case but similar to the net azimuthal field case, there are no purely exponentially growing large-scale linear MRI modes for the zero net flux configuration. The growth of MRI is then of a transient type; hence, the resulting turbulence and associated dynamo action are subcritical by nature (Rincon et al. 2008), being energetically powered by this linear nonmodal MRI growth mechanism. But the self-sustaining schemes and statistical characteristics of the subcritical MRI turbulence generally differ for these two configurations of the magnetic field. Analyzing the self-sustaining schemes of such subcritical MHD turbulence in disks, one is tempted to appeal to mathematical concepts and tools that lie at the basis of self-sustenance of subcritical turbulence in hydrodynamic (Couette, Poiseuille, pipe, etc.) shear flows. Yet such an approach can be misleading because these flows are bounded and the corresponding self-sustaining schemes (elaborated by the hydrodynamic community over the past three decades; see, e.g., Hamilton et al. 1995; Waleffe 1995; Schmid & Henningson 2001) assign a key role to the effect of boundaries. However, in astrophysical disks, especially in local shearing box considerations, there are effectively no rigid boundaries. Therefore, the self-sustaining scheme of the subcritical turbulence in the disks should be built on the interplay between linear nonmodal MRI and nonlinear processes.

Over the past decade, a large number of numerical studies have extensively explored the statistical properties of unstratified zero net flux MRI turbulence in disks. Using the spherical shell-averaging technique in Fourier space (e.g., Verma 2004; Alexakis et al. 2007; Alexakis & Biferale 2018), the spectra of magnetic and kinetic energies, as well as Maxwell stress, were analyzed first without explicit dissipation, where the infamous issue of numerical nonconvergence arises (e.g., Pessah et al. 2007; Fromang & Papaloizou 2007; Simon et al. 2009; Guan et al. 2009; Bodo et al. 2011; Shi et al. 2016). This prompted further studies, where different configurations and physical ingredients were also considered, for example, explicit/physical viscous and resistive dissipation, different boundary conditions, and box aspect ratios (e.g., Fromang et al. 2007; Liljeström et al. 2009; Fromang 2010; Käpylä & Korpi 2011; Herault et al. 2011; Riols et al. 2013, 2015; Bhat et al. 2016; Shi et al. 2016; Walker et al. 2016; Riols et al. 2017; Potter & Balbus 2017; Walker & Boldyrev 2017; Nauman & Pessah 2016, 2018). These studies revealed quite complex behavior, where convergence may depend on several factors. It was also demonstrated that at sufficiently high Reynolds and magnetic Reynolds numbers, the saturated value of the total turbulent stress appears to depend only on the magnetic Prandtl number (Pm) scaling as its power law at Pm $\gtrsim$ 1, whereas the turbulence is generally not sustained at Pm $\lesssim$ 1 in standard boxes (i.e., with radial/azimuthal sizes larger than the vertical one). Recently, Guseva et al. (2017) reported sustained zero net flux MRI turbulence and dynamo at large Pm in a magnetized cylindrical Taylor–Couette flow, going beyond the Cartesian shearing box.

Despite the above numerous theoretical efforts that have gone into the analysis of the zero net flux MRI turbulence phenomenon, relatively less attention has been devoted to properly understanding an underlying self-sustaining mechanism and consequently elucidating the physics behind the Pm dependence of the turbulence. Clearly, one could not do the latter without first doing the former. Perhaps this is because most of these studies have concentrated in physical space, hence offering a limited insight into mode dynamics, while a much richer dynamical picture unfolds in Fourier space, which is the basis of the turbulence self-sustenance. Only a few papers looked into the dynamics of zero net flux MRI turbulence in Fourier (k-) space (e.g., Fromang & Papaloizou 2007; Fromang et al. 2007; Simon et al. 2009; Davis et al. 2010). However, they used various types of averaging techniques (over spherical shells of constant wavevector magnitude $k = |k|$ or slices with different directions of $k$, etc.), which in the end led to overlooking a key ingredient of the dynamics—specific spectral anisotropy of nonlinear transfers due to flow shear, i.e., the transverse cascade—and thus somewhat hindered uncovering the full dynamical story. With a similar approach, Lesur & Longaretti (2011) considered the spectral dynamics of MRI turbulence in Fourier space but with a nonzero net vertical field. They emphasized the nonlocality of nonlinear transfers and focused only on explaining the dependence on Pm, rather than clarifying the self-sustaining mechanism. In fact, in this case, there are exponentially growing MRI channel modes in the flow and hence no shortage of energy supply to the turbulence. So, the role of nonlinearity in the self-sustenance of perturbations is not as crucial as in the net azimuthal or zero net flux cases.

In this quest for self-sustenance in the zero net flux case, Herault et al. (2011) and Riols et al. (2013, 2015, 2017) carried out a detailed study of the spectral dynamics of nonlinear MRI dynamo states in Fourier space in the shearing box. They also interpreted the dependence of the nonlinear state and associated transport on Pm as the effect of turbulent, or nonlinear, magnetic diffusion, which increases with decreasing Pm. However, these studies adopted relatively small Reynolds and magnetic Reynolds numbers and specially designed azimuthally or vertically elongated boxes. Due to this, their models comprise only a relatively low number of active modes (degrees of freedom) with small wavenumbers and are therefore nonturbulent, low-order models of zero net flux MRI. So, the self-sustenance scheme of the MRI dynamo states elaborated by these authors is based on the interplay of the dominant large-scale axisymmetric mode responsible for the dynamo azimuthal field and the next large-scale nonaxisymmetric modes. On the other hand, the number of active modes generally involved in the fully developed MRI turbulence is orders of magnitude higher, spanning a broader range of wavenumbers (see also Papers I and II). Consequently, the overall self-sustenance scheme of the MRI turbulence is more complex, being determined by the
1.2. Goals of This Study

For a deeper understanding of the self-sustaining process of the zero net flux MRI turbulence, a detailed analysis of scale-to-scale nonlinear interactions (transfers), linear processes, and their interplay in Fourier $k$-space is necessary without resorting to low-order models. Based on this, one can then explain the dependence of the turbulence level on $Pm$. This is exactly the program we set as the main goal in this paper. We reveal and analyze the structural type of nonlinear action in disks with zero net magnetic flux. It ensures the replenishment of modes capable of drawing shear flow energy through the linear nonmodal MRI mechanism and, in this way, a continual energy supply to the turbulence. Spectral analysis performed here indicates that the main nonlinear process in this case is angular, that is, over wavevector orientation redistribution of modes in Fourier space. As mentioned above, we call this anisotropic type of nonlinear action in shear flows the nonlinear transverse cascade. Such an angular redistribution in Fourier space due to nonlinearity was first found for a simplified two-dimensional (2D) model problem in constant shear flows by Chagelishvili et al. (2002). Later, analyzing the interplay of linear transient growth and nonlinear processes, Horton et al. (2010) and Mamatsashvili et al. (2014) revealed and distinctly described the action of the nonlinear transverse cascade in spectrally stable 2D hydrodynamic and MHD plane shear flows, which is mainly responsible for the turbulence sustenance therein. A similar analysis, showing the importance of the transverse cascade for the nonzero net flux MRI turbulence in Keplerian shear flows, is carried out in Papers I and II. It should be emphasized that the nonlinear transverse cascade, being induced by flow shear, essentially differs from the canonical direct/inverse cascades in classical (i.e., without mean shear flow), Kolmogorov, Iroshnikov–Kraichnan, or Goldreich–Sridhar theories of turbulence (e.g., Biskamp 2003; Alexakis & Biferale 2018).

In this paper, we first demonstrate the significance of the nonlinear transverse cascade for the sustenance of zero net flux MRI turbulence at $Pm > 1$ (specifically, we consider the $Pm = 4$ case). Then, we show that with decreasing $Pm$, the action of the transverse cascade weakens, so that it can no longer interfere with the action of the direct cascade, which transfers energy from small wavenumber modes (from the so-called vital area; see, e.g., Papers I and II) toward higher wavenumber ones. Such a course of events ultimately leads to the increased resistive dissipation of the mode energy and, consequently, the decay of the turbulence. In other words, we attribute the decay of the zero net flux MRI turbulence with decreasing $Pm$ to structural/topological changes in the nonlinear processes when the direct cascade begins to dominate the nonlinear transverse cascade, which is crucial for the turbulence sustenance.

The paper is organized as follows. The physical model and main equations in Fourier space are given in Section 2. Simulations of zero net flux MRI turbulence are described in Section 3. The spectral dynamics of the turbulence in Fourier space and the proposed sustenance scheme are presented in Section 4, and the dependence of the turbulence dynamics on magnetic $Pm$ is analyzed in Section 5. A summary and relation to other works are given in Section 6.

2. Physical Model and Basic Equations

We adopt the shearing box model of an accretion disk (Goldreich & Lynden-Bell 1965), where a local Cartesian coordinate system $(x, y, z)$ centered at a radius $r_0$ orbits with angular velocity $\Omega$ of the disk at this radius. This reference frame has the unit vectors $(e_x, e_y, e_z)$, respectively, in the radial $(x)$, azimuthal $(y)$, and vertical $(z)$ directions. The main equations of incompressible nonideal MHD in the shearing box are

$$\frac{\partial U}{\partial t} + (U \cdot \nabla) U = -\frac{1}{\rho} \nabla P + \frac{(B \cdot \nabla) B}{4\pi \rho} - 2\Omega e_x \times U + 2q\Omega^2 x e_e + \nu \nabla^2 U,$$

$$\frac{\partial B}{\partial t} = \nabla \times (U \times B) + \eta \nabla^2 B,$$

$$\nabla \cdot U = 0, \quad \nabla \cdot B = 0,$$

where $\rho$ is the fluid density, $U$ is the velocity in the rotating frame, $B$ is the magnetic field, and $P$ is the sum of the thermal and magnetic pressures. The explicit kinematic viscosity, $\nu$, and ohmic resistivity, $\eta$, of the fluid are constant. In the shearing box, Keplerian rotation of the disk is represented as an azimuthal flow $U_0 = -q\Omega x e_x$ with constant radial shear parameter $q = 3/2$, spatially uniform pressure $P_0$, and density $\rho_0$. The flow domain is a rectangle with sizes $(L_x, L_y, L_z)$.

We normalize time by $\Omega^{-1}$, length by vertical box size $L_z$, velocities by $\Omega L_z$, magnetic field by $\Omega L_z \sqrt{4\pi \rho_0}$, and the pressure by $\rho_0 \Omega^2 L_z^2$. The relevant parameters of the problem are the Reynolds number, $Re = \Omega L_z^2 / \nu$; the magnetic Reynolds number, $Rm = \Omega L_z^2 / \eta$; and the magnetic Pm, $Pm = \nu / \eta = Rm / Re$, which plays a very important role in MRI turbulence.

An initial field configuration represents a purely vertical magnetic field, which varies only along $x$ as

$$B = B_0 \sin \left( \frac{2\pi x}{L_x} \right) e_z,$$

and hence has zero net flux within the domain. The field amplitude, $B_0$, is specified by the parameter $\beta = 2\Omega L_z \sqrt{\nu_\Lambda^2 v_\Lambda^2}$, where $\nu_\Lambda = B_0 / (4\pi \rho_0)^{1/2}$ is the amplitude of the associated Alfvén speed. In the following, we divide the total velocity $U$ into the stationary background Keplerian flow $U_0$ and perturbation $u$ on top of it, $U = U_0 + u$, and work with $u$ and $B$ (Appendix A gives the equations for these variables as derived from Equations (1)–(3)). Our primary goal is to deepen our understanding of the self-sustaining dynamics of zero net flux MRI turbulence and its dependence on viscous and resistive dissipation. To this end, we focus on the turbulence dynamics in Fourier space, applying similar tools as used in Paper I.

2.1. Equations in Fourier Space

The second part of our study starts with the derivation of spectral dynamical equations for the velocity and magnetic field components in Fourier space. All variables are decomposed into spatial Fourier modes,

$$f(r, t) = \int \hat{f}(k, t) \exp(i k \cdot r) d^3k,$$
where $f$ stands for $(u, P, B)$ and $\tilde{f}$ stands for the corresponding Fourier transforms $(\tilde{u}, \tilde{P}, \tilde{B})$. Substituting Equation (4) into Equations (A1)–(A8), we obtain Equations (A22)–(A24) for the spectral velocity and Equations (A12)–(A14) for the magnetic field components. Below, we give the final equations for the quadratic forms of these quantities, while their derivation is given in Appendix A.

Multiplying Equations (A22)–(A24), respectively, by $\tilde{u}_x^* \tilde{u}_y^*$, $\tilde{u}_x^* \tilde{u}_z^*$ and summing with their conjugates, we have

$$\frac{\partial}{\partial t} \frac{|\tilde{u}_x|^2}{2} = -q_{xy} \frac{\partial}{\partial k_x} \frac{|\tilde{u}_y|^2}{2} + \mathcal{H}_x + \mathcal{D}^{(u)} + \mathcal{N}^{(u)}, \quad (5)$$

$$\frac{\partial}{\partial t} \frac{|\tilde{u}_y|^2}{2} = -q_{yx} \frac{\partial}{\partial k_y} \frac{|\tilde{u}_x|^2}{2} + \mathcal{H}_y + \mathcal{D}^{(u)} + \mathcal{N}^{(u)}, \quad (6)$$

$$\frac{\partial}{\partial t} \frac{|\tilde{u}_z|^2}{2} = -q_{yz} \frac{\partial}{\partial k_z} \frac{|\tilde{u}_z|^2}{2} + \mathcal{H}_z + \mathcal{D}^{(u)} + \mathcal{N}^{(u)}, \quad (7)$$

where the linear terms are\(^8\)

$$\mathcal{H}_x = \left(1 - \frac{k_x^2}{k^2}\right)(\tilde{u}_x \tilde{u}_y^* + \tilde{u}_y \tilde{u}_x^*) + 2(1 - q) k_x k_y \frac{k_x}{k^2} |\tilde{u}_x|^2,$$

$$\mathcal{H}_y = \frac{1}{2} \left[q - 2 - (\bar{q} - 1) \frac{k_x^2}{k^2}\right](\tilde{u}_y \tilde{u}_z^* + \tilde{u}_z \tilde{u}_y^*) - 2 \frac{k_y}{k^2} |\tilde{u}_y|^2,$$

$$\mathcal{H}_z = (1 - \bar{q}) \frac{k_x k_z}{k^2}(\tilde{u}_x \tilde{u}_y^* + \tilde{u}_y \tilde{u}_x^*) - \frac{k_y k_z}{k^2} (\tilde{u}_z \tilde{u}_x^* + \tilde{u}_x \tilde{u}_z^*),$$

and the negative terms of viscous dissipation are

$$\mathcal{D}^{(u)} = -\frac{k^2}{Re} |\tilde{u}|^2,$$

while the modified nonlinear transfer functions in these spectral equations are

$$\mathcal{N}^{(u)} = \frac{1}{2}(\tilde{u}_x q_{y}^* + \tilde{u}_y q_{x}^*),$$

with the index $i = x, y, z$ here and everywhere below. The quantity $Q_i$ (given by Equation (A25)) describes the nonlinear redistributions via triadic interactions for the spectral velocities $\tilde{u}_i$, in Equations (A22)–(A24). The sum of $\mathcal{H}_i$ is equal to the Reynolds stress spectrum multiplied by $q$, $\mathcal{H} = \mathcal{H}_x + \mathcal{H}_y + \mathcal{H}_z = q(\tilde{u}_x \tilde{u}_y^* + \tilde{u}_y \tilde{u}_x^* + \tilde{u}_z \tilde{u}_z^*) / 2$.

Next, multiplying Equations (A12)–(A14), respectively, by $\tilde{B}_x^* \tilde{B}_y^*$, $\tilde{B}_x^* \tilde{B}_z^*$ and summing with their conjugates, we obtain

$$\frac{\partial}{\partial t} \frac{|\tilde{B}_x|^2}{2} = -q_{xy} \frac{\partial}{\partial k_x} \frac{|\tilde{B}_y|^2}{2} + \mathcal{D}^{(b)} + \mathcal{N}^{(b)}, \quad (8)$$

$$\frac{\partial}{\partial t} \frac{|\tilde{B}_y|^2}{2} = -q_{yx} \frac{\partial}{\partial k_y} \frac{|\tilde{B}_x|^2}{2} + \mathcal{M} + \mathcal{D}^{(b)} + \mathcal{N}^{(b)}, \quad (9)$$

$$\frac{\partial}{\partial t} \frac{|\tilde{B}_z|^2}{2} = -q_{yz} \frac{\partial}{\partial k_z} \frac{|\tilde{B}_z|^2}{2} + \mathcal{D}^{(b)} + \mathcal{N}^{(b)}, \quad (10)$$

where the linear terms are the Maxwell stress spectrum $\mathcal{M}$ multiplied by $q$,

$$\mathcal{M} = -\frac{q}{2}(\tilde{B}_x \tilde{B}_y^* + \tilde{B}_y \tilde{B}_x^*),$$

and the negative terms of resistive dissipation are

$$\mathcal{D}^{(b)} = -\frac{k^2}{Rm} |\tilde{B}_t|^2,$$

while the modified nonlinear terms in these equations are

$$\mathcal{N}^{(b)} = \frac{1}{2} \tilde{B}_x^*[k_x \tilde{F}_z - k_z \tilde{F}_x] + c.c.,$$

$$\mathcal{N}^{(b)} = \frac{1}{2} \tilde{B}_y^*[k_z \tilde{F}_x - k_x \tilde{F}_y] + c.c.,$$

$$\mathcal{N}^{(b)} = \frac{1}{2} \tilde{B}_z^*[k_x \tilde{F}_y - k_y \tilde{F}_x] + c.c.$$
then, because the integral of the drift terms over the $k_r$-axis is zero, $-\int qk\partial(\ldots)/\partial k_r dk_r = 0$. As a result, the Eulerian radial wavenumber of each nonaxisymmetric mode (shearing wave) varies linearly with time, $k_r(t) = k_r(0) + qk_r t$, in the shearing box. So, the effect of this drift is straightforward—it only makes the growth of individual modes transient as it sweeps them in $k$-space. For this reason, we do not show this term in the spectral plots below.

For the finite simulation box, the grid in Fourier space is determined by the box sizes $L_i$ and numerical resolution $N_i$, $i = x, y, z$, so that the cell sizes are given by $\Delta k_i = 2\pi/L_i$, and hence the wavenumbers run through values $k_i = 2\pi n_i/L_i$, where $n_i = 0, \pm 1, \pm 2\ldots, \pm N_i/2$. So, everywhere below, we use these wavenumbers normalized by the corresponding grid cell sizes, after which they become all integers, $k_i/\Delta k_i = k_i = n_i$. However, the radial wavenumber of nonaxisymmetric modes, varying with time as $k_r(t) = 2\pi n_r/L_r + qt(2\pi n_r/L_r)$, becomes an integer at discrete time moments $t_n = nL_r/(q|n_r|L_z)$, where $n$ is an integer multiple of $n_y$ (Hawley et al. 1995).

### 3. General Properties of the Simulations

We solve Equations (1)–(3) using the pseudospectral code SNOOPY (Lesur & Longaretti 2007). The computational domain has sizes $L_\alpha, L_r, L_z = (3, 3, 1)$ and numerical resolution $(N_\alpha, N_r, N_z) = (512, 512, 128)$. Note that in contrast to previous studies on zero net flux MRI turbulence (e.g., Fromang & Papaloizou 2007; Lesur & Ogilvie 2008; Herault et al. 2011; Shi et al. 2016; Riols et al. 2017), we adopt a box with equal radial and azimuthal sizes, $L_r = L_z$, since it is itself isotropic in the $(x, y)$ slice and does not cause “numerical deformation” of the inherent anisotropic dynamics of the MRI turbulence (a more detailed analysis of the effect of the aspect ratio $L_z/L_r$ on the numerical deformation due to box asymmetry is presented in Section 5.3 of Paper I). The standard shearing box boundary conditions (Hawley et al. 1995) are used in the code that keeps the net magnetic flux in the box at zero at all times.

We initialize our simulations by imposing random velocity perturbations on the Keplerian flow and trace their subsequent evolution. First, we ran the case with $Rm = 1.2 \times 10^4$, $Re = 3 \times 10^4$, having $Pm = 4$. After it has settled down into a quasi-steady turbulent state, from the flow snapshot of this run at $t = 400$, we started three additional runs with lower $Pm = 1$, 2, 3 and the same $Rm$. These new runs thus have higher Reynolds numbers, $Re = [1.2, 0.6, 0.4] \times 10^4$, respectively. Table 1 summarizes the volume- and time-averaged values of the kinetic, $E_K = \rho u^2/2$, and magnetic, $E_M = B^2/8\pi$, energy densities, as well as the Reynolds, $u_\alpha u_\alpha$, and Maxwell, $-B_\alpha B_\alpha$, stresses in all four runs.

Figure 1 shows the evolution of the volume-averaged magnetic and Maxwell stress for various $Pm = 1$, 2, 3 and 4, which recovers the classical results of other simulations of zero net flux MRI turbulence in the literature (e.g., Fromang et al. 2007; Nauman & Pessah 2016; Shi et al. 2016; Walker et al. 2016; Walker & Boldyrev 2017; Nauman & Pessah 2018). Turbulence is sustained at $Pm = 3$ and 4, but its level decreases with $Pm$. It no longer persists at lower $Pm = 1$ and 2, decaying faster the smaller $Pm$ is. Thus, for our box, the critical value separating the sustained and decaying cases is between $Pm = 2$ and 3 consistent with Riols et al. (2017). But Nauman & Pessah (2016) did find sustained turbulence for $Pm < 1$ provided the vertical size of a box, $L_z$, is sufficiently large.

The structure of the MRI turbulence is different in the sustained and decaying cases. As $Pm$ decreases, the characteristic length scale (correlation length) of the magnetic field structures decreases as well (Riols et al. 2017; Walker & Boldyrev 2017). Performing a spectral analysis of the dynamical processes at different $Pm$ below (Section 5), we show that this behavior in physical space is a consequence of the fact that the energy of large wavenumber modes relative to that of small wavenumber ones increases with decreasing $Pm$.

We also carried out simulations including thermal stratification along the vertical direction in the Boussinesq approximation. Its role, however, turned out to be negligible in the sustaining dynamics of MRI turbulence. Thus, here we do not take the stratification into account, although it plays an important role in the disk dynamo (e.g., Davis et al. 2010; Bodo et al. 2012; Gressel 2013).

| $(L_\alpha, L_r, L_z)$ | $(N_\alpha, N_r, N_z)$ | Re | Rm | Pm | $\langle E_K \rangle$ | $\langle E_M \rangle$ | $\langle (u_\alpha u_\alpha) \rangle$ | $\langle (-B_\alpha B_\alpha) \rangle$ |
|------------------------|------------------------|----|----|----|---------------------|---------------------|-----------------------|-----------------------|
| $(3, 3, 1)$            | $(512, 512, 128)$      | $1.2 \times 10^4$ | $1.2 \times 10^4$ | 1   | ...                | ...                | ...                   | ...                   |
| $(3, 3, 1)$            | $(512, 512, 128)$      | $0.6 \times 10^4$ | $1.2 \times 10^4$ | 2   | ...                | ...                | ...                   | ...                   |
| $(3, 3, 1)$            | $(512, 512, 128)$      | $0.4 \times 10^4$ | $1.2 \times 10^4$ | 3   | $4.2 \times 10^{-3}$ | $1.31 \times 10^{-2}$ | $8.62 \times 10^{-4}$ | $5.8 \times 10^{-3}$ |
| $(3, 3, 1)$            | $(512, 512, 128)$      | $0.3 \times 10^4$ | $1.2 \times 10^4$ | 4   | $7 \times 10^{-3}$  | $2.3 \times 10^{-2}$ | $1.5 \times 10^{-3}$ | $1.04 \times 10^{-2}$ |

Figure 1. Evolution of the volume-averaged magnetic energy (top) and Maxwell stress (bottom) at different $Pm = 1$ (black), 2 (red), 3 (green), and 4 (blue). The runs at $Pm = 1, 2, 3$ are started from the flow snapshot of the $Pm = 4$ run at $t = 400$. The runs for $Pm = 3$ and 4 are self-sustained, with the turbulence level decreasing with $Pm$, whereas the runs at $Pm = 1$ and 2 decay slowly.
4. Turbulence Dynamics in Fourier Space

In this section, adopting the approach of our previous studies (Mamatsashvili et al. 2014; Papers I and II), we investigate the spectral dynamics and self-sustenance of the zero flux MRI turbulence by computing and visualizing the individual linear and nonlinear terms in spectral Equations (5)–(10) from the simulation data. Below, we focus on the dynamics of the radial and azimuthal components of the spectral velocity, \( \bar{u}_r \), \( \bar{u}_\phi \), and magnetic field, \( \bar{B}_r \), \( \bar{B}_\phi \), because these are the most important ones for the sustenance process, making up the Reynolds and Maxwell stresses that extract energy from the flow. Here we consider the Pm = 4 case, while the dependence on Pm is explored in the next section. Since we deal with the quasi-steady turbulence in this section, we time-average the spectra of the velocity, magnetic field, and dynamical terms from \( t = 80 \) to \( t = 1000 \).

4.1. \( k_z \) Spectra of the Magnetic Field and Velocity

We begin the analysis in Fourier space by first looking at the distribution of the spectral quantities as a function of the vertical wavenumber. Figure 2 shows the integrated in \( (k_x, k_y) \) slice and time-averaged spectra of the radial and azimuthal velocity and magnetic field components, \( [\bar{u}_{r(z)}]_v = \int [\bar{u}_{r(z)}]^2 dk_x dk_y \), \( [\bar{B}_{r(z)}]_v = \int [\bar{B}_{r(z)}]^2 dk_x dk_y \), versus \( k_z \) for Pm = 4. Both the magnetic field and the velocities’ spectra reach their highest values at small \( |k_z| \) and decrease as \( |k_z| \) increases. The maximum of \( [\bar{B}_{r(z)}]_v \) and \( [\bar{B}_{\phi}]_v \) comes, respectively, at \( |k_z| = 2 \) and \( |k_z| = 1 \), while that of the velocity spectra, \( [\bar{u}_{r(z)}]_v \) and \( [\bar{u}_{\phi}]_v \), comes, respectively, at \( |k_z| = 1 \) and \( k_z = 0 \). Out of these four spectra, the spectrum of the azimuthal velocity \( \bar{u}_{\phi} \) noticeably differs from the remaining three in that the peak at \( k_z = 0 \) is much more pronounced. We show below that at \( k_z = 0 \), most of the contribution in the azimuthal velocity spectrum comes from the largest-scale axisymmetric \( k_z = 0 \) mode with the smallest nonzero radial wavenumber in the box, \( k_r = \pm 1 (\pm 2 \pi / L_r \text{ in dimensional units}) \). So this peak, belonging to the mode \( \mathbf{k}_x = (\pm 1, 0, 0) \), corresponds to the zonal flow accompanying MRI turbulence (Johansen et al. 2009; Walker & Boldyrev 2017; Papers I and II). Its formation is a consequence of the action of the nonlinearity on the perturbations. Thus, most of the power for the velocities and magnetic field is contained in small \( k_z \), which thus plays a main, dynamically important role in the sustainment process of the turbulence.

4.2. Active Modes and the Vital Area

Having analyzed the spectra of the velocity and magnetic field along vertical wavenumbers, we move now to the analysis of the spectral dynamics in the horizontal \( (k_x, k_y) \) slices at a given \( k_z \). We choose the first few small vertical wavenumbers \( |k_z| = 0, 1, 2 \), which carry most of the kinetic and magnetic energy and hence play a central role in the turbulence dynamics. Below, we first identify those active modes that participate in and shape the self-sustaining dynamics of the turbulence.

Figure 3 shows the energy-carrying, or active, modes (colored dots) in \( (k_x, k_y) \) slices at \( k_z = 0, 1, 2 \) for Pm = 4 and 1 placed next to each other for the sake of comparison. Here the active modes are defined as those modes whose spectral magnetic energy, \( \mathcal{E}_M = (|\bar{B}_r|^2 + |\bar{B}_\phi|^2 + |\bar{B}_z|^2)/2 \), becomes larger than 50% of the maximum spectral magnetic energy, \( \mathcal{E}_{M, \text{max}} \), at least once during the evolution. Although this definition is somewhat arbitrary, it provides information on the location and number of the dynamically important modes in Fourier space. Following Paper I, these modes are identified by tracing the temporal evolution of all modes in the box during the entire simulation and recording at each instant only those whose magnetic energy is higher than the above given fraction of the maximum spectral magnetic energy at that moment. At the end of the simulation, the fraction of time during which a given mode from this set retains this high a magnetic energy relative to the total simulation time is calculated. The resulting ratio is represented on the color bars.

The active modes are distributed quite anisotropically in the \( (k_x, k_y) \) slice, being mostly concentrated on the \( k_y/k_x > 0 \) side due to the shear. For Pm = 4, these modes occupy a broader range of radial wavenumbers, \( |k_r| \leq 11 \), but a narrower range of azimuthal ones, \( |k_\phi| \leq 3 \). As noted above, these active modes have small vertical wavenumbers, \( |k_z| \lesssim 2 \). Since this area of \( k \)-space encompasses the active modes that most contribute to the turbulence dynamics, we refer to it as the vital area of the turbulence. For Pm = 1, the radial and azimuthal extents of the vital area are somewhat smaller, \( |k_r| \leq 10 \),...
\( |k_x| \leq 2 \). We will see below (Section 5) that this is because, in the decaying turbulence at \( P_m = 1 \), in contrast to the sustained one at \( P_m = 4 \), the active modes gain less energy due to MRI and the nonlinear transverse cascade, while their energy is rapidly lost to higher wavenumber modes primarily via direct cascade.

The mode \( k_d = (0, 0, \pm 1) \) \( (0, 0, \pm 2\pi/L_x) \) in dimensional units) in Figure 3 bears the large-scale dynamo field (\( d \) stands for “dynamo”; Lesur & Ogilvie 2008; Herault et al. 2011; Shi et al. 2016). It somewhat stands out among other active modes in that it has a magnetic energy larger than the 50\% threshold during the longest period of time. However, it is also seen in these plots that other nearby active modes retain similar energies for comparable time intervals. We demonstrate in Appendix B that the nonlinear interaction between the \( k_d \) mode and these active modes with comparable amplitudes governs the dynamics of the large-scale field. For this reason and the sake of generality, as distinct from the self-sustaining schemes of Herault et al. (2011) and Riols et al. (2015, 2017), we prefer not to separate out the dynamo \( k_d \) mode in the dynamics but rather treat it on an equal footing with other modes in Fourier space.

Finally, note that the number of active modes in Figure 3 is fairly large: 112 for \( P_m = 4 \) and 59 for \( P_m = 1 \) (double these, if we also take into account negative \( k_x \)). This indicates that the self-sustaining process of zero net flux MRI turbulence is quite complex, involving a broad range of scales, and, therefore, it should not be treated within simplified low-order models of the self-sustaining processes. Those models, involving a much smaller number of modes, do not account for all of the essential nonlinear mode interactions that lie at the heart of the self-sustaining process of MRI turbulence.

### 4.3. Dynamical Balances

We now investigate the spectral dynamics of the turbulence in \((k_x, k_y)\) slices for the same first few smallest vertical wavenumbers at \( P_m = 4 \). Herewith, we characterize the dynamical balances individually for the radial and azimuthal velocity and magnetic field components. Understanding these dynamical balances allows us to formulate a main self-sustaining scheme for the zero net flux MRI turbulence. Figures 4–7 present the time-averaged spectra of \( |\vec{\alpha}_x|, |\vec{\alpha}_y|, |\vec{\beta}_x|, \) and \( |\vec{\beta}_y| \), as well as the corresponding main governing linear \( M, H_x, H_y \) and nonlinear \( N_x^{(b)}, N_y^{(b)}, N_x^{(u)}, N_y^{(u)} \) dynamical terms in \((k_x, k_y)\) slices at \( k_z = 0, 1, 2 \). These \((k_x, k_y)\) sections of the spectral dynamics contain the full information on the linear and nonlinear dynamical processes and their interplay, ensuring the turbulence sustenance. The ranges of the radial and azimuthal wavenumbers are taken as \( |k_x| \leq 16, |k_y| \leq 8 \), so as to fully encompass the vital area defined above—the location of the core of the sustaining process. We first describe the dynamical balances depicted in each figure. Here we do not show the negative definite viscous, \( D_x^{(a)}, \) and resistive, \( D_y^{(b)}, \) terms, since they always oppose the sustenance. Moreover, these terms are negligible in the vital area compared to the above dynamical terms.

As is typical of shear flows, a key common feature of all of these spectra of the magnetic field and velocity, as well as the dynamical terms depicted in Figures 4–7, is that they all display a common type of anisotropy in Fourier space due to the shear, i.e., a strong variation over polar angle in \((k_x, k_y)\) slices with inclination preferably on the \( k_x/k_y > 0 \) side (e.g., Lesur & Longaretti 2011; Mamatsashvili et al. 2014; Murphy & Pessah 2015; Paper I). The anisotropic linear terms \( M, H_x, H_y \), as discussed above, characterize the nonmodal growth of zero net flux MRI. These terms act as a source when they are positive (red and yellow areas) and as a sink when they are negative (blue areas). For the nonlinear processes described by \( N_x^{(b)} \) and \( N_x^{(u)} \), this spectral anisotropy evidently cannot be fitted into the framework of classical (spherically symmetric) forms of nonlinear—direct and inverse—cascades.

We now describe the spectra of the magnetic field and velocity components, as well as the dynamical balances for them. We start with the magnetic field spectra.

1. The radial field spectrum, \( |\vec{\beta}_x| \), is plotted in Figure 4, together with the governing nonlinear transfer term \( N_x^{(b)} \), which is mainly responsible for its generation and amplification. The power of the anisotropic spectrum of the radial field is mainly concentrated in nonaxisymmetric \((k_x \neq 0)\) modes with \( k_x/k_y < 0 \). The mode with \( |k_x| = 3, |k_y| = k_z = 1 \) has the largest amplitude, although the nearby nonaxisymmetric modes at \( k_z = 0 \) and 2 have comparable amplitudes. This anisotropy of
[\vec{B}_i] is a consequence of the anisotropic structure of $N^{(b)}$, whose main notable effect is the transverse (i.e., over wavevector polar angle $\varphi = \sin^{-1}(k_y/(k_x^2 + k_y^2)^{1/2}))$ redistribution/transfer of power in the $(k_x, k_y)$ slice for all considered $k_z = 0, 1, and 2$. As seen in the bottom panels of Figure 4, this term transfers the spectral energy of the radial field, $|\vec{B}_i|^2/2$, from the area of “giver” wavenumbers, where it is negative, $N^{(b)}(r) < 0$ (blue), into the area of “receiver” wavenumbers, where it is positive, $N^{(b)}(r) > 0$ (red and yellow), as well as among different components of the velocities and magnetic field. The structure of these two areas strongly depends on the polar angle in $(k_x, k_y)$ slices. The yellow and red areas are essential for the sustenance process, since they comprise the wavenumbers at which $N^{(b)}$ continually regenerates the radial field as a result of nonlinear interactions between other pairs of wavenumbers forming a triad with these. These modes, being thereby replenished in the red and yellow areas, are nonaxisymmetric with $k_x/k_y > 0$. In other words, one can call these red and yellow areas together the growth area. The linear drift (the first right-hand side term of Equation (8)), on the other hand, advects $[\vec{B}_i]$ of an individual nonaxisymmetric mode (shearing wave with given $k_y$ and $k_z$) along the $k_y$-axis from the growth (red and yellow) area back into the blue area, where $N^{(b)}(r) < 0$ and, therefore, $|\vec{B}_i|$ decreases while still remaining significant in this blue area. This interplay between the nonlinear transfer and linear drift establishes the specific anisotropic spectrum of $[\vec{B}_i]$ that overlaps with both—red/yellow and blue—areas, as is illustrated by the contours drawn on $N^{(b)}_x$ in Figure 4. In the quasi-steady state, the drift and the transverse cascade balance each other in the vital area, forming a closed cycle. Thus, the linear drift in Fourier space gives the dynamical processes a transient feature, and therefore the permanent regeneration due to the nonlinear transverse cascade (exhibited by $N^{(b)}_x$) plays a key role in the self-sustaining dynamics of the turbulence.

2. The azimuthal field spectrum, $[\vec{B}_i]$, is shown in Figure 5, together with the governing Maxwell stress, $\mathcal{M}$, and the nonlinear transfer term, $N^{(b)}$. We have seen already in Figure 3 that the dynamo mode $k_d$ stands out among other active modes. Here we also observe that, on average in time, it carries the largest azimuthal field energy among the other modes. Nearby energetic nonaxisymmetric active modes have $|k_d| = 1$ (see also Herault et al. 2011; Riols et al. 2015, 2017). The Maxwell stress, $\mathcal{M}$, is positive in the $(k_x, k_y)$ slices at $k_x/k_y > 0$ and dominant in the vital area (red and yellow), where it supplies (injects) energy in the azimuthal field (including that of the $k_d$ mode; see also Appendix B). As a result, $|\vec{B}_i|$ undergoes nonmodal MRI growth. By contrast, the nonlinear transfer term is mainly negative, $N^{(b)}(r) < 0$, in this area (blue), draining the azimuthal field energy there and transferring it to large wavenumbers, where this term is positive but small (light green area), as well as to different components. Thus, the azimuthal field of larger wavenumber modes is supplied as a result of nonlinear transfers from the smaller wavenumber modes in the vital area but not from the Maxwell stress directly, which is negligible there.7 Note also that $\mathcal{M}$ and $N^{(b)}$ have nearly similar absolute values and shapes in Fourier space, implying that in the quasi-steady state, the energy injection into and its nonlinear “removal” from the active modes in the vital area are approximately in balance; the

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7 These larger wavenumber modes are what Riols et al. (2017) referred to as “slaved modes.”
action of the linear drift for $|\vec{B}|$ (the first right-hand side term of Equation (9)) plays only a minor role in this balance. However, because of this drift, the structure of the $|\vec{B}|$ spectrum in the quasi-steady state, mainly resulting from the action of the Maxwell stress and the nonlinear transfer term, is a bit more inclined toward the $k_z$-axis than $M$, as seen from the contours of $|\vec{B}|$ drawn on this term in Figure 5.

3. The radial velocity spectrum, $|\vec{u}_r|$, is shown in Figure 6, together with its governing linear, $\mathcal{H}_r$, and nonlinear, $\mathcal{N}^{(u)}_r$, transfer terms. Here $|\vec{u}_r|$ reaches higher values at $k_z = 0$, with a maximum at $k_z = 0$, $|\vec{u}_r| = 1$. At $k_z > 1$, it is maintained by mostly positive $\mathcal{H}_r > 0$ at the expense of the mean flow and drained by the negative $\mathcal{N}^{(u)}_r < 0$. At $k_z = 0$, $\mathcal{H}_r$ can be positive (red areas at $k_z/k_x < 0$) and negative (blue areas at $k_z/k_x > 0$) in the $(k_x, k_z)$ slice, similar to that in the 2D hydrodynamic case (Chagelishvili et al. 2003; Johnson & Gammie 2005; Horton et al. 2010). This implies that the injected $|\vec{u}_r|$ with $k_z/k_x < 0$, undergoing the linear drift along the $k_z$-axis, achieves a maximum at $k_z = 0$ and then enters the area $k_z/k_x > 0$, where it falls off due to the negative values of $\mathcal{H}_r$ and $\mathcal{N}^{(u)}_r$.

4. The azimuthal velocity spectrum, $|\vec{u}_\varphi|$, is shown in Figure 7, together with the governing linear, $\mathcal{H}_\varphi$, and nonlinear, $\mathcal{N}^{(u)}_\varphi$, transfer terms. Note that the $|\vec{u}_\varphi|$ spectrum is markedly dominated by the $k_{cf} = (\pm 1, 0, 0)$ mode, which, as discussed above, corresponds to the zonal flow excited in MRI turbulence. Just this mode gives rise to the sharp peak of $|\vec{u}_\varphi|^2$ at $k_z = 0$ (Figure 2). Here $\mathcal{N}^{(u)}_\varphi$ is always positive at $k_z \geq 1$, supplying the azimuthal velocity, while at $k_z = 0$, this term can be positive or negative. At $k_z = 0$, the linear term $\mathcal{H}_\varphi$ also can be positive or negative, while it is always negative for $k_z = 1$ and 2, draining the azimuthal velocity. In any case, for the dominant $k_{cf}$ mode, $\mathcal{H}_\varphi(k_{cf}) = 0$, $\mathcal{N}^{(u)}_\varphi(k_{cf}) > 0$, implying that the generation and maintenance of the zonal flow is exclusively due to nonlinear transfers. Note also that the radial velocity at $k_z = 0$ is several times smaller than the azimuthal velocity of the zonal flow, $|\vec{u}_\varphi(0, 1, 0)| < |\vec{u}_\varphi(k_{cf})|$, as follows from the comparison of Figures 6 and 7.

4.4. The Core Subcycle of the Self-sustaining Process

Having described the dynamical balances in Fourier space in the sustained case with $Pm = 4$, we are now in a position to construct on their basis the main self-sustaining scheme/cycle of the zero net flux MRI turbulence. Since the magnetic energy and Maxwell stress are much larger and hence more important than the kinetic energy and Reynolds stress, the sustenance should be primarily magnetically driven. Specifically, the core of this process should involve the most energy-carrying radial and azimuthal field components at those wavenumbers where they are appreciable, i.e., in the vital area. In the presence of
both shear and rotation, these two components undergo both shear and rotation, these two components undergo amplification due to the nonmodal MRI process, mediated by the Maxwell stress. The nonlinear transfers, also involving the rest of the velocity and magnetic field components, ensure continual regeneration of the radial field, which initiates the MRI growth. Thus, the interplay of the linear nonmodal MRI growth and the constructive nonlinear feedback due to the transverse cascade in fact determines the self-sustaining process of the MRI turbulence in the zero net flux case. The situation here is analogous to that of the MRI turbulence in the zero net azimuthal field case studied in Paper I, except that here there are no linear kinetic–magnetic exchange terms contributing to MRI. The detailed nonlinear interactions between velocity and magnetic field components at many different wavenumbers are described in general by rather complex terms under integrals in expressions (A17) and (A18)–(A19). Since the number of active modes is quite large (Figure 3), this makes the entire self-sustaining process not amenable to a vivid schematization. Nevertheless, assuming that all details of the nonlinear mode-to-mode interactions are encapsulated in the nonlinear magnetic transfer terms $N_{v}^{(n)}$ and $N_{v}^{(b)}$ (Figures 4 and 5), we can clearly disentangle the core subcycle of the turbulence sustenance from the overall dynamical picture.

This core subcycle is sketched in Figure 8 (solid arrows inside the rectangle) and can be characterized as follows. One can start each such cycle with the regeneration/seeding of the radial field $B_{r}$ by the nonlinear term $N_{v}^{(b)}$ via the transverse cascade process at those wavenumbers where this term is positive (red and yellow areas in Figure 4). These modes with a newly regenerated radial field, in turn, initiate the linear nonmodal MRI process due to the shear (via the shear-proportional linear term in Equation (A13)), which produces positive Maxwell stress, $M > 0$, that, in turn, amplifies the azimuthal field energy $|\tilde{B}_{\gamma}|^2/2$. This is in agreement with the noticeable correlation between the distributions of $|\tilde{B}_{\gamma}|$ and $M$ in $(k_{x}, k_{y})$ slices seen by comparing Figures 4 and 5. This growth is higher for nonaxisymmetric modes and thus is of a transient type because of their drift along the $k_{y}$-axis. Because of this transient nature of the MRI growth (and hence energy extraction from the flow), the continual seeding of the radial field due to the nonlinear transverse cascade is crucial for the turbulence self-sustenance. As these modes drift, they also cross the area in Fourier space where the nonlinear term $N_{v}^{(b)}$ is negative (blue areas in Figure 5) and therefore drains the azimuthal field energy of these modes. Eventually, due to the linear drift process, the nonaxisymmetric modes move away from the amplification red area, where the Maxwell stress $M$ is appreciable (Figure 5), and decay due to resistivity. Since this azimuthal field is the dominant field component, it gives a main contribution—positive feedback—to $N_{v}^{(b)}$, which, in turn, produces a new seed radial field, thereby closing the cycle. This self-sustenance scheme of MRI turbulence, underlying the dynamics of active modes, naturally determines the behavior of the axisymmetric $k_{d}$ dynamo mode too (see Appendix B).

We have outlined above a central part of the full self-sustaining scheme at $Pm = 4$, in which, in principle, the velocity components and vertical field also participate. However, they contribute through the nonlinear term, $N_{v}^{(b)}$, which is key to the turbulence sustenance, but it is still determined primarily by the dominant $B_{\gamma}$. In Figure 8, the...
dashed arrows denote the contributions from these quantities extrinsic to the core subcycle. The proposed self-sustaining scheme shares some similarities with the nonlinear 3D MRI dynamo cyclic solutions reported by Herault et al. (2011) and Riols et al. (2015, 2017), despite the fact that these dynamo states are more regular in space and time (not fully turbulent), with a much-reduced number of active modes. A detailed comparison of our self-sustaining scheme with that proposed in those papers is discussed in Section 6.1.

5. Dependence of the Turbulence Sustenance and Dynamics on \( \text{Pm} \)

In this section, we explore the effects of reducing \( \text{Pm} \) on the nonlinear transfers, and hence on the magnetic field spectrum, which eventually lead to the decay of the turbulence (Figure 1). We do a comparative analysis by juxtaposing spectral quantities at \( \text{Pm} = 4 \), when turbulence is sustained, and at \( \text{Pm} = 1 \), when it decays fastest, and characterizing the differences between the turbulence dynamics in Fourier space in these two cases. Such an analysis is at first glance analogous to that carried out by Lesur & Longaretti (2011) for a nonzero net vertical flux case at different \( \text{Pm} \), who, however, used a spherical shell-averaging technique, overlooking the spectral anisotropy and hence the transverse cascade, whose weakening with \( \text{Pm} \), as shown below, is a main cause for the decay of the turbulence. As seen in Figure 1, the turbulence slowly decays from the moment \( (t = 400) \) at which \( \text{Pm} \) is abruptly lowered from 4 to 1. Due to this, it is not possible to do time-averaging of the spectra at \( \text{Pm} = 1 \) over an entire evolution, as done for the quasi-steady state at \( \text{Pm} = 4 \) in the previous section. In this case, we instead choose two time moments, \( t = 440 \) in about the middle and \( t = 490 \) near the end of the decay (see Figure 1), and average the spectral quantities over a short time.
interval around these moments in order to filter noise and make spectral quantities smoother. Since the radial field and its regeneration process are the most important ingredients in the turbulence sustenance, in this section, we focus mainly on the behavior of its spectrum and nonlinear term $N^{(b)}$ at different $P_m$, though we also consider the behavior of the azimuthal field and the Maxwell stress.

Finally, in the decaying turbulence, all of the spectra of the magnetic field and dynamical terms decrease in time. This makes it somewhat difficult to see how the shape of these spectral quantities changes with time at $P_m = 1$, as well as with respect to the quasi-steady ones at $P_m = 4$. This is important because changes in the form of the spectra with time can give clues to the transfer directions in Fourier space when $P_m$ is varied. To circumvent this, we compare below normalized spectra that better highlight the differences in the magnetic field and dynamical terms’ spectra in the sustained and decaying cases. So, below, we describe how the spectra of the main quantities change with decreasing $P_m$ and clarify its physics by examining the behavior of the nonlinear transfers.

5.1. Spectra at Different $P_m$

We start with comparing the spectra of the radial field, azimuthal field, and Maxwell stress integrated in the $(k_x, k_z)$ slice, which are shown in Figure 9 at $P_m = 4$ and for the above two moments at $P_m = 1$. These $k_z$ spectra are normalized by the corresponding values integrated over $k_z$, i.e., by $\langle B_r^2 \rangle$, $\langle B_z^2 \rangle$, and $q(-B_z, B_r)$, respectively. These plots show how the power is redistributed along $k_z$ in the process of the turbulence decay with time at $P_m = 1$ compared to the sustained case at $P_m = 4$. The relative power in the radial and azimuthal components and hence the Maxwell stress mainly decreases with time at small $k_z \leq 3$, with a slight increase at larger $k_z$. Thus, with decreasing $P_m$, the magnetic energy is taken from the modes of the vital area by the nonlinear terms and distributed among higher $k_z$ modes.

Let us now see how the spectra of these quantities are redistributed in $(k_x, k_z)$ slices at different $k_z$. First, we consider the normalized spectra of these quantities averaged over rings of constant wavenumber magnitude $k \equiv (k_x^2 + k_z^2)^{1/2}$ in $(k_x, k_z)$ slices, $\hat{N}_{x, k}^{(b)} = \int_{0}^{2\pi} N_{x, k}^{(b)} d\varphi$, and normalized by its integrated value for each $k_z = 0, 1, 2$, and 6, i.e., $\hat{N}_{x, k}^{(b)} = \int N_{x, k}^{(b)} dk$, at the same times $t = 440$ and 490 at $P_m = 1$ and its time-averaged value at $P_m = 4$. The ring averaging conceals the dependence of the spectrum on the polar angle in the $(k_x, k_z)$ slice; hence, $\hat{N}_{x, k}^{(b)}$ describes the transfer of the radial field only along wavevector $k = (k_x, k_z)$, i.e., the direct/inverse cascade.

Figure 11 shows that at $P_m = 1$ and small $k_z = 0, 1, 2$, the normalized $N_{x, k}^{(b)} / \hat{N}_{x, k}^{(b)}$ decreases with time at $k \leq k_0$ in the vital area but increases at $k \geq k_0$, that is, it supplies more power to modes with $k$ higher than $k_0$ and less power to modes with $k$ smaller than $k_0$. As a result, the ring-averaged spectrum of the radial field exhibits a similar behavior with time, as we have seen above in Figure 10. We interpret this as the transfer of power from the vital area to larger $k$ due to the intensified direct cascade caused by the decrease of $P_m$, that is, the increase of Re (at fixed $R_m$), as if the latter opens up a “channel” of magnetic energy flux toward small-scale modes. For the Maxwell stress spectrum, this transfer implies that the energy injection into the turbulence also shifts to higher $k$. By contrast, at $P_m = 4$, the nonlinear term, and hence the spectral magnetic field and Maxwell stress, are concentrated within the vital area, implying that the role of the energy transfer to higher wavenumber modes outside the vital area is not so important.

Figure 9. Spectra of the radial and azimuthal magnetic field, as well as the Maxwell stress integrated in $(k_x, k_z)$ slices, vs. $k$, and normalized by their respective total values, $\langle B_r^2 \rangle$, $\langle B_z^2 \rangle$, and $q(-B_z, B_r)$. They are averaged in time over the quasi-steady state for $P_m = 4$ and in the vicinity of two time moments, $t = 440$ and 490, in the decaying case for $P_m = 1$ (see text). In the latter case, all of these normalized spectra decrease with time, mostly at small $|k_z| \leq 3$, but slightly increase at larger $|k_z|$. few $k_z = 0, 1, 2$ and larger $k_z = 6$. Here each spectral quantity is normalized by its integrated values over $(k_x, k_z)$ slices at a given $k_z$, i.e., by $\langle B_r^2 \rangle$, $\langle B_z^2 \rangle$, and $\mathcal{M}$, respectively. At $P_m = 1$ and small $k_z$, the normalized spectra of these quantities decrease with time for $k \lesssim k_0 = 10$ (to $k_0$ is approximately the radial extent of the vital area in Figure 3) but increase for larger $k \gtrsim k_0$, with the maximum also decreasing and moving toward higher $k$. We demonstrate below that this behavior is a consequence of the enhancement of the role of the direct cascade and decrease of the role of the transverse cascade when $P_m$ is reduced from 4 to 1. It is also seen in the azimuthal field spectrum at $P_m = 1$ that there is, additionally, some temporal accumulation of power in the dynamo mode $k = 0, k_z = 1$ with time.

This tendency of the enhanced transfer of the spectral magnetic energy to larger wavenumbers with lowering $P_m$ is in fact related to the specific action of its governing nonlinear term $N^{(b)}$. Figure 11 depicts this nonlinear term ring-averaged in $(k_x, k_z)$ slices,

$$N_{x, k}^{(b)} = \int_{0}^{2\pi} N_{x, k}^{(b)} d\varphi,$$

and normalized by its integrated value for each $k_z = 0, 1, 2$, and 6, i.e., $\hat{N}_{x, k}^{(b)} = \int N_{x, k}^{(b)} dk$, at the same times $t = 440$ and 490 at $P_m = 1$ and its time-averaged value at $P_m = 4$. The ring averaging conceals the dependence of the spectrum on the polar angle in the $(k_x, k_z)$ slice; hence, $\hat{N}_{x, k}^{(b)}$ describes the transfer of the radial field only along wavevector $k = (k_x, k_z)$, i.e., the direct/inverse cascade.
respectively integrated in these slices, the values \( |\hat{B}|^2, |\hat{B}_\perp|^2, \hat{M} \) at different \( k_z = 0, 1, 2 \) and larger \( k_z = 6 \) for \( Pm = 4 \) and 1 (time averages are done in the same manner as in Figure 9). At \( Pm = 1 \), all of these spectra decrease with time in the vital area \( k \lesssim k_0 = 10 \) but increase at larger \( k \gtrsim k_0 \) mainly for \( k_z = 0, 1, 2 \), so there is a flux (direct cascade) of these quantities out of the vital area toward larger wavenumbers. However, at \( k_z = 1 \), there is additionally some accumulation of power in the large-scale azimuthal field at \( k = 0 \). At higher \( k_z = 6 \), however, the shape of these normalized spectra does not change with time or \( Pm \).

On the other hand, at higher \( k_z \), say, \( k_z = 6 \), the normalized spectra of \( N^{(b)}_x \), as well as the spectral radial and azimuthal fields and the Maxwell stress, no longer vary with either time or \( Pm \), as seen in Figures 10 and 11. This is also consistent with Figure 9, indicating that most of the changes in the \( k_z \) spectra of these quantities with time and \( Pm \) occur at lower \( k_z \).

Figures 10 and 11 give a first indication of increasing transfers of the ring-averaged spectral energy and shift of Maxwell stress toward larger wavenumbers with lowering \( Pm \). Figures 12 and 13 give a fuller picture of how these transfers proceed with time in \( (k_x, k_y) \) slices as a result of the variation of the nonlinear term. At \( Pm = 1 \), the action of \( N^{(b)}_x \) shifts from small to large \( k_x \) and \( k_y \) (on the \( k_x/k_y > 0 \) side) as the turbulence decays, spreading over a broader range of wavenumbers, while its normalized value, \( N^{(b)}_x / \tilde{N}^{(b)}_x \), gradually decreases (Figure 12). The intensity (color saturation) of positive (red and yellow) and negative (blue) areas of \( N^{(b)}_x \) fades away in the vital area, indicating the weakening efficiency of the nonlinear transverse cascade with time there relative to the direct cascade. By contrast, in the sustained turbulence at \( Pm = 4 \), the transverse cascade is more pronounced and efficient, being concentrated mainly at small wavenumbers in the vital area.

This behavior of the nonlinear transfer terms with time at \( Pm = 1 \) induces a similar redistribution of the spectra of the radial field and Maxwell stress in \( (k_x, k_y) \) slices, as seen in
panels in order to better highlight the dynamics of other modes in the Pm of the red, yellow, and blue areas also fades away, implying the reduction of the transverse cascade efficiency. The large-scale mode nonlinear term is less concentrated in the vital area normalized spectra of the magnetic field and Maxwell stress also spread to larger wavenumbers with time, as well as decrease in magnitude. Physically, the transverse cascade is characterized by its positive magnitudes \( N_x^{(b)} \) that lie outside the vital area and \( \tilde{N}_x^{(b)} \) at a given \( k_x \).

\[ [N_x^{(b)}]_{k < k_0} = \frac{1}{N_x^{(b)}} \sum_{k > k_0} |N_x^{(b)}|. \]

Note that in this expression, we take the absolute value of \( N_x^{(b)} \), because the transverse cascade is characterized by its positive and negative areas. These would otherwise nearly cancel each other had we simply taken the sum of \( \tilde{N}_x^{(b)} \) over the wavenumbers. Physically, \( [N_x^{(b)}]_{k < k_0} \) describes the role of the regeneration process of the radial field by the transverse cascade inside \( k < k_0 \) with respect to the total value \( \tilde{N}_x^{(b)} \) at the same \( k_x \). From the top panel of Figure 14, it is seen that this ratio is largest at \( Pm = 4 \) and decreases with time at \( Pm = 1 \), indicating a drop in the intensity of the transverse cascade for active mode wavenumbers in the vital area.

The second process competing with the transverse cascade is the direct cascade of power from the active modes inside the vital area to larger wavenumber modes outside it. Following Lesur & Longaretti (2011), we can characterize this by the ratio of the total energy gain for all modes with wavenumber magnitudes \( k > k_0 \) that lie outside the vital area and \( \tilde{N}_x^{(b)} \) at a given \( k_x \).

\[ [N_x^{(b)}]_{k > k_0} = \frac{1}{N_x^{(b)}} \sum_{k > k_0} N_x^{(b)}. \]

This measure of the direct cascade is plotted in the bottom panel of Figure 14. Its time-averaged value is smallest in the quasi-steady turbulence for \( Pm = 4 \) and increases with time at

Figure 12. Normalized spectrum of the nonlinear term for the radial field, \( N_x^{(b)} \), in the \((k_x, k_y)\) slice at \( k_x = 1 \) for \( Pm = 4 \) (left) and \( Pm = 1 \) for two moments \( t = 440 \) (middle) and 490 (right). The contours of the corresponding azimuthal field spectrum, \( \tilde{N}_x \), are drawn on this spectrum. In contrast to the sustained \( Pm = 4 \) case, at \( Pm = 1 \) this nonlinear term is less concentrated in the vital area \( |k_x| \lesssim k_0 = 10, |k_y| \lesssim 3 \) and spreads out with time to larger \( k_x \) and \( k_y \), decreasing in amplitude. The color intensity of the red, yellow, and blue areas also fades away, implying the reduction of the transverse cascade efficiency.

Figure 13. Normalized spectra of the radial field, \( \tilde{N}_x \), and Maxwell stress, \( \tilde{M}/N_x \), in the \((k_x, k_y)\) slice at \( k_x = 1 \) for \( Pm = 4 \) (top row) and \( Pm = 1 \) at two moments \( t = 440 \) (middle) and 490 (right). The contours of the corresponding azimuthal field spectrum, \( \tilde{N}_x \), are drawn on the Maxwell stress spectrum in the bottom row. The large-scale mode \( k = (\pm 1, 0, 1) \) giving a large contribution to the Maxwell stress at \( Pm = 1 \) has been artificially faded in the middle and right panels in order to better highlight the dynamics of other modes in the \((k_x, k_y)\) slice. As a consequence of the action of the nonlinear term (Figure 12), at \( Pm = 1 \), these normalized spectra of the magnetic field and Maxwell stress also spread to larger wavenumbers with time, as well as decrease in magnitude.

5.2. MRI Turbulence Decay at Low Pm

To better characterize the above-discussed nonlinear transfers and, especially, the relative role of the transverse and direct cascades with decreasing Pm, we introduce a measure of the intensity of the transverse cascade in the vital area. It is defined as the ratio of the sum of the absolute values of the nonlinear transfer term within the vital area \( k < k_0 \) to the total value \( \tilde{N}_x^{(b)} \) in \((k_x, k_y)\) slices (for a given \( k_x \)).

\[ [N_x^{(b)}]_{k < k_0} = \frac{1}{N_x^{(b)}} \sum_{k > k_0} |N_x^{(b)}|. \]
Pm = 1, indicating an increasing intensity of the direct cascade for all \( k_z \).

From the above analysis, we can interpret the decay of zero net flux MRI turbulence with decreasing Pm as follows. At Pm = 4, the transverse cascade concentrated in the vital area is strong enough to supply the active modes there and counteract the effect of the direct cascade, which acts to transfer magnetic energy to higher wavenumber modes outside it. However, as seen from Figure 14, the decrease of Pm initiates the topological rearrangement of the nonlinear cascade processes:

1. it weakens the transverse cascade in the vital area, that is, the angular transfer of modes in Fourier space, and thus the replenishment of the MRI growing modes, which is essential for the turbulence sustenance; and

2. it intensifies the direct cascade, that is, the transfer of modes from small to high wavenumbers that acts as a sort of nonlinear (turbulent) diffusion for the active modes (see also Riols et al. 2015, 2017). This transfer (leakage) of the magnetic energy from the vital area toward large wavenumber modes, in turn, leads to enhanced resistive dissipation of these modes.

Thus, the role of the nonlinear cascades in the turbulence decay is indirect; the topological rearrangement of the nonlinear cascade processes modifies the spectral distribution of the magnetic field, thereby intensifying the resistive dissipation at large wavenumbers and weakening the regeneration of MRI growing modes (mediated by the transverse cascade) in the vital area, i.e., weakening the energy supply to the turbulence. The diminishing of the transverse cascade and intensification of the direct cascade for the magnetic field ultimately result in the drop of the turbulence level (at Pm = 3) or its decay (at Pm = 1 and 2). In the decaying case, the reduced efficiency of the regeneration of MRI growing modes cannot counteract the increased resistive dissipation, making the self-sustaining scheme incapable of maintaining the turbulence.

6. Summary and Discussion

In this paper, we investigated the nature of MRI turbulence—dynamical balances, self-sustenance, and dependence on magnetic Pm—in the unstratified shearing box approximation of Keplerian disks with zero net magnetic flux. New insights into these processes were gained by first performing simulations of the turbulence at different Pm = 1–4 and then carrying out a detailed analysis of the individual linear and nonlinear dynamical terms in the main MHD equations and their interplay in Fourier (\( k \)-)space. We found, in agreement with other related studies, that in standard boxes at Pm \( \gtrsim 1 \), MRI turbulence is sustained, while at Pm \( \lesssim 1 \), it decays at a timescale of 100 orbits.

As distinct from previous studies, we did the spectral analysis in a general way, without doing spherical shell averaging in Fourier space and thereby capturing the spectral anisotropy of nonlinear processes—the transverse cascade—due to the flow shear. In the net zero flux case, there are no large-scale purely exponentially growing MRI modes, so MRI is of transient, or nonmodal, type. This transient growth itself is “flawed” in the sense that it lasts for a limited (dynamical/orbital) time and therefore needs to be supported by nonlinear positive feedback for the long-term maintenance of the turbulence, which is provided just by the transverse cascade. In other words, the nonlinear transverse cascade lies at the heart of the zero net flux MRI turbulence, ensuring its self-sustenance. This type of new cascade generic to shear flows is fundamentally (topologically) different from the usual classical (inverse/direct) nonlinear cascades. So, the conventional characterization of nonlinear MHD cascade processes in strongly sheared Keplerian flows in terms of direct and inverse cascades only (e.g., using shell averaging; Fromang & Papaloizou 2007; Simon et al. 2009; Lesur & Longaretti 2011), which ignores the shear-induced spectral anisotropy and the resulting transverse cascade, is incomplete.

Specifically, the main results of this paper are as follows.

1. Self-sustenance scheme. In the sustained case Pm = 4, we established that the zero net flux MRI turbulence is sustained by the interplay of the linear nonmodal growth of MRI and the nonlinear transverse cascade in Fourier space. Although this interplay is generally quite intricate, we were able to isolate a core subcycle of the turbulence sustenance from it (Figure 8). The radial and azimuthal components of the magnetic field play the main role in this process, which proceeds as follows. The radial field is generated/seeded and maintained by the nonlinear term \( N_x^{(b)} \) through the transverse cascade process. The azimuthal field is then produced from the radial field through the linear nonmodal MRI process. The azimuthal field, in turn, largely contributes to the production of the nonlinear source (feedback) \( N_x^{(b)} \) for the radial field, thereby closing the self-sustaining cycle. This self-sustaining dynamics of the turbulence mainly occupies a small wavenumber area of Fourier space, the so-called vital area, and involves a sizable number (\( \gtrsim 100 \)) of large-scale active modes. For a relatively large Pm = 4, the transverse cascade is able to replenish the radial field in the vital area sufficiently faster than the direct cascade transfers it to larger wavenumber modes; hence, the sustenance is possible. As is typical of MRI turbulence,
large-scale zonal flow is generated with the dominant azimuthal velocity.

2. Dependence on \( P_m \). We also performed a comparative analysis of the sustained (at \( P_m = 4 \)) and decaying (at \( P_m = 1 \)) cases, exploring the specific differences between the nonlinear transfers and, consequently, of the magnetic field and Maxwell stress spectra in Fourier space. We showed that the essence of these differences is that decreasing \( P_m \) leads to the structural/topological rearrangement of the nonlinear transfers, such that, in contrast to the sustained case, the direct cascade now prevails over the transverse one and shifts the magnetic field spectrum to higher wavenumbers. In other words, at \( P_m = 1 \), the action of the nonlinear transverse cascade, particularly the associated regeneration of MRI growing modes, weakens in the vital area (top panel of Figure 14). It can no longer oppose the increased action of the direct cascade, which more rapidly transfers magnetic energy from small wavenumber modes in the vital area to higher wavenumber ones outside it (bottom panel of Figure 14).

This intensified transfer of the magnetic field to the small-scale modes, in turn, enhances resistive dissipation of these modes relative to the regeneration of MRI growing modes due to the transverse cascade. This enhancement of resistive dissipation in conjunction with the reduction of MRI growth ultimately makes the self-sustaining scheme ineffective for the turbulence sustenance and leads to its decay. Thus, the decrease of \( P_m \) results in weakening the transverse cascade at small wavenumbers (large scales) and a simultaneous intensification of the direct cascade to large wavenumbers, which, in turn, leads to enhanced resistive dissipation of the magnetic field at small scales. However, the reason this course of events takes place still needs to be better understood.

6.1. Connection to Related Studies

The self-sustenance and dynamics of 3D nonlinear periodic MRI dynamo solutions in the presence of zero net magnetic flux and their dependence on viscous and resistive dissipation were studied previously by Herault et al. (2011) and Riols et al. (2015, 2017) in Fourier space. Therefore, making connections with their works is in order. We note from the beginning that these papers were focused on smaller Reynolds numbers, smaller numerical resolution, and larger box aspect ratios, \( L_x/L_y, L_x/L_z \), etc. than used here. For comparison, we took a much higher numerical resolution \( (N_x, N_y, N_z) = (512, 512, 128) \) and \( Rm = 1.2 \times 10^4 \) in the standard box \( (L_x, L_y, L_z) = (3, 3, 1) \). These limiting factors in the above studies likely resulted in the resistive dissipation penetrating into the vital area, as we call it, and hence reducing the number of active modes participating in the self-sustaining dynamics to only the first few large-scale nonaxisymmetric and two axisymmetric ones. These two axisymmetric modes are the \( k_d = (0, 0, \pm 2\pi/L_z) \) dynamo mode, which carries a slowly varying in time, large-scale azimuthal magnetic field, and \( k_m = (\pm 2\pi/L_x, 0, \pm 2\pi/L_z) \) mode, which carries a radially modulated magnetic field. Overall, the nonlinear MRI dynamo state considered in those papers is not fully turbulent but instead spatially and temporally more regular with a scarce spectrum of active modes contributing to the dynamics, in contrast to our case of the fully developed MRI turbulence.

A self-sustaining scheme of this zero net flux MRI dynamo state was worked out in Herault et al. (2011). In this scheme, the role of the nonlinearity is twofold.

1. To maintain the large-scale axisymmetric radial field of the \( k_d \) dynamo mode via nonlinear scattering of non-axisymmetric MRI-amplified modes (shearing waves). The azimuthal dynamo field, in turn, is produced by shear-induced stretching of this radial field.

2. To regenerate (seed) leading nonaxisymmetric modes due to the nonlinear scattering of trailing nonaxisymmetric modes with the \( k_m \) mode. These leading modes are then capable of undergoing transient MRI growth against the background of the large-scale azimuthal axisymmetric dynamo field.

The next step of zero net magnetic flux MRI study was performed in Riols et al. (2015), again considering a low-order model (an azimuthally very elongated box \( (L_x, L_y, L_z) = (0.7, 20, 2) \), low numerical resolution \( (N_x, N_y, N_z) = (48, 48, 72) \) and \( (128, 128, 96) \), and moderate \( Re, Rm = 100–3000 \) and focusing on the mechanism of the decrease/disappearance of the MRI dynamo states with decreasing \( P_m \). To explain this behavior, Riols et al. (2015) split the nonlinear terms in the magnetic field equation into positive induction and negative advection parts. It was shown that there is an enhanced nonlinear magnetic diffusion due to the transfer of energy from small to large wavenumbers (direct cascade) with decreasing \( P_m \) that results in turbulence decay.

At the last step, Riols et al. (2017) increased \( Re \) and \( Rm \) and took moderate aspect ratios, \( (L_x, L_y, L_z) = (0.7, 6, 2), (0.5, 2, 1) \). In this case, although the dynamical picture is more complex (so-called “chimera” MRI dynamo states) than that in the previous low-order cases, it is still not completely turbulent; the vital area contains only a few largest-scale dominant active modes with \( |k_x| \leq 4, |k_y|, |k_z| \leq 1 \). In that study, using the same approach, the roles/interplay of the induction and advection terms were analyzed in the decay of magnetic perturbations with decreasing \( P_m \) given this more complex nature of the MRI dynamo states.

As distinct from the works of Herault et al. (2011) and Riols et al. (2015, 2017), we deal with a much larger number of active modes, i.e., with fully developed MRI turbulence, and propose a general scheme of the turbulence self-sustenance. Our simulations indicate that in this turbulent state, not only the large-scale axisymmetric azimuthal magnetic field but also a certain quite extensive set of nonaxisymmetric modes undergo appreciable nonmodal/transient MRI growth without mediation of this axisymmetric field. The existence of nonlinear feedback provided by transversal/angular redistribution of the nonaxisymmetric modes in Fourier space is a key factor for the turbulence self-sustenance, as it replenishes those new nonaxisymmetric modes experiencing nonmodal amplification. This angular redistribution of the multitude of nonaxisymmetric modes can be structurally/topologically classified as a nonlinear transverse cascade. In conclusion, the nonlinear transverse cascade is naturally inherent in the turbulence dynamics (bottom panels of Figure 4) in shear flows and therefore is an integral part of our scheme.

9 It should be mentioned that Riols et al. (2015, 2017) summed the spectra of the dynamical terms along the radial wavenumber \( k_r \), thereby missing out on essential shear-induced anisotropy of the nonlinear processes in Fourier space and hence the transverse cascade.
Due to its general nature, the transverse cascade also comprises a particular nonlinear feedback process central to the self-sustenance scheme of Herault et al. (2011): replenishment of new leading nonaxisymmetric modes via scattering of trailing ones by the $k_m$ mode. Our study indicates that, in contrast to those low-order models, this particular nonlinear process is not the only and only process resupplying new active modes in zero net flux MRI turbulence. A broad class of other such “constructive” nonlinear triad interactions is collectively accounted for in the transverse cascade process. Besides, our approach gives us insight into the topological changes of the nonlinear cascades in Fourier space with lowering $P_m$, which approach gives us insight into the topological changes of the turbulence in standard ($L_c/L_\alpha \lesssim 1$) boxes.

The representation of the nonlinear magnetic term in the induction and advection parts, as done in Riol et al. (2015, 2017), is useful and interesting in its own right. However, dealing with fully developed turbulence (i.e., a much larger number of active modes), instead of the explicit decomposition of the nonlinear term into induction and advection parts, we use another approach—structural/topological analysis of the linear and nonlinear dynamical terms of the induction equation in Fourier space—which we previously vindicated for hydrodynamic (Horton et al. 2010) and MHD (Mamatsashvili et al. 2014) 2D shear flows. In our opinion, such a topological analysis is more general/universal, as it has confirmed viability with regard to hydrodynamic shear flows too (Horton et al. 2010; Mamatsashvili et al. 2016) when these induction and advection nonlinear terms of magnetic origin are absent by definition. In these papers investigating the nonlinear terms of fluid equations in Fourier space, we elucidated, for example, the self-sustenance scheme of coherent vortical perturbations and homogeneous shear turbulence in hydrodynamic 2D and 3D constant shear flows, respectively.

In related papers, Nauman & Pessah (2016, 2018) carried out a series of zero net flux MRI turbulence simulations for taller boxes with vertical aspect ratios $L_c/L_\alpha = 4–32$ and a wide range of $P_m = 0.1–10$. They found sustained turbulence even at $P_m \lesssim 1$ for large enough $L_c/L_\alpha \gtrsim 4$ in contrast to standard boxes with $L_c/L_\alpha \sim 1$, where there is no turbulence. The main difference between the tall and standard boxes is that the large-scale azimuthal dynamo field is much stronger and varies more slowly in tall boxes (see also Lesur & Ogilvie 2008; Shi et al. 2016). In this case, this mean azimuthal field contributes to and enhances the nonmodal growth of nonaxisymmetric MRI modes by introducing linear kinetic–magnetic exchange terms in the momentum and induction equations. In Fourier space, this linear exchange term can be comparable to the action of the nonlinear term, i.e., transverse cascade (Paper I). As a result, the turbulence dynamics becomes more similar to that in the presence of a nonzero net azimuthal field, which is known to persist at lower $P_m \lesssim 1$, but with a level decreasing with $P_m$ (e.g., Simon & Hawley 2009; Meheut et al. 2015). Apparently, this mean large-scale azimuthal field helps the MRI turbulence to persist at small $P_m$ by enhancing its energetic supply, despite the reduction of the transverse cascade efficiency with decreasing $P_m$. However, this is only a qualitative interpretation, and for a proper understanding of the sustenance of zero net flux MRI turbulence in tall boxes, one should do a similar analysis of the turbulence dynamics in Fourier space at large $L_c/L_\alpha \gtrsim 4$ and different $P_m$.

### 6.2. Alternative Instability Mechanisms at Small $P_m$

It is well known by now that the existence of MRI turbulence and associated dynamo action is quite problematic, nearly impossible at small $P_m \lesssim 1$, both for nonzero and zero net magnetic fluxes, as also confirmed in the present study. On the other hand, from global magnetized Taylor–Couette studies—laboratory analogs of accretion disks—it has long been shown that at small $P_m \ll 1$, the helical (with both azimuthal and vertical fields) and azimuthal (with a purely azimuthal field) versions of MRI, or, for short, HMRI and AMRI, can operate (Hollerbach & Rüdiger 2005; Stefani et al. 2009; Hollerbach et al. 2010; Kirillov et al. 2014; Seilmayer et al. 2014; Mamatsashvili & Stefani 2016, e.g.), even down to very low $P_m \sim 10^{-6}–10^{-5}$ relevant to cold interiors of protoplanetary disks. These instabilities, relatives of MRI, have been overlooked in shearing box studies because they rely on the curvature term proportional to the background azimuthal magnetic field in the linearized induction equation, which is present only in the global cylindrical geometry (Pessah & Psaltis 2005). Another reason was that they require a radial shear of the rotational velocity larger than the Keplerian if the azimuthal field is current-free (Liu et al. 2006). The last constraint may not be realized in protoplanetary disks, and the dominant azimuthal field can, in principle, deviate from the current-free profile. In this case, as demonstrated by Kirillov & Stefani (2013), HMRI and AMRI can in fact extend to Keplerian rotation. This opens up a new possibility that these instabilities can be alternatives of standard MRI in the cold and dense parts (“dead zones”) of protoplanetary disks with small $P_m \ll 1$. They can play a central role in driving dynamical processes and magnetic dynamo action there, through which, in turn, they may also sustain themselves. However, the viability of this proposition should be further explored in global disk simulations.

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### Appendix A

#### Perturbation Equations in Physical and Fourier Space

Substituting velocity and pressure perturbations, $u = U - U_0$, $p = P - P_0$, about the background Keplerian shear flow $U_0 = -q_0 \Omega \epsilon_x$, into Equations (1)–(3), we get

$$
\frac{Du_x}{Dt} = 2\Omega u_y - \frac{1}{\rho_0} \frac{\partial p}{\partial x} + \frac{\partial}{\partial x} \left( \frac{B_z^2}{4\pi\rho_0} - u_z^2 \right) + \frac{\partial}{\partial y} \left( \frac{B_x B_y}{4\pi\rho_0} - u_x u_y \right) + \frac{\partial}{\partial z} \left( \frac{B_z^2}{4\pi\rho_0} - u_z u_x \right) + \nu \nabla^2 u_x,
$$

(A1)
\[
\frac{Du_x}{Dt} = (q - 2)\Omega u_x - \frac{1}{\rho_0} \frac{\partial}{\partial y} + \frac{\partial}{\partial z} \left( \frac{B_x B_z}{4\pi\rho_0} - u_x u_z \right) \\
+ \frac{\partial}{\partial y} \left( \frac{B_y^2}{4\pi\rho_0} - u_y^2 \right) + \frac{\partial}{\partial z} \left( \frac{B_z^2}{4\pi\rho_0} - u_z^2 \right) + \nu \nabla^2 u_y,
\]

(A2)

\[
\frac{Du_y}{Dt} = -\frac{1}{\rho_0} \frac{\partial}{\partial z} + \frac{\partial}{\partial x} \left( \frac{B_x B_z}{4\pi\rho_0} - u_x u_z \right) \\
+ \frac{\partial}{\partial y} \left( \frac{B_y^2}{4\pi\rho_0} - u_y^2 \right) + \frac{\partial}{\partial z} \left( \frac{B_z^2}{4\pi\rho_0} - u_z^2 \right) + \nu \nabla^2 u_y,
\]

(A3)

\[
\frac{Du_z}{Dt} = \frac{\partial}{\partial y} (u_x B_y - u_y B_x) - \frac{\partial}{\partial z} (u_x B_z - u_z B_x) + \eta \nabla^2 B_x,
\]

(A4)

\[
\frac{DB_y}{Dt} = -\frac{\partial}{\partial z} (u_x B_y - u_y B_x) - \frac{\partial}{\partial y} (u_x B_z - u_z B_x) + \eta \nabla^2 B_y,
\]

(A5)

\[
\frac{DB_z}{Dt} = \frac{\partial}{\partial x} (u_x B_x - u_y B_y) - \frac{\partial}{\partial y} (u_x B_y - u_z B_z) + \eta \nabla^2 B_z,
\]

(A6)

\[
\frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} + \frac{\partial u_z}{\partial z} = 0,
\]

(A7)

\[
\frac{\partial B_x}{\partial x} + \frac{\partial B_y}{\partial y} + \frac{\partial B_z}{\partial z} = 0,
\]

(A8)

where \(D/Dt = \partial/\partial t - q\Omega x\partial/\partial y\) is the total derivative along the Keplerian shear flow.

Substituting decomposition (Equation 4) into Equations (A1)--(A8) and taking into account the normalization, we obtain the evolutionary equations for the Fourier transforms of the velocity and magnetic field components:

\[
\left( \frac{\partial}{\partial t} + q k_y \frac{\partial}{\partial k_x} \right) \tilde{u}_x = 2\tilde{u}_x - ik_y \tilde{p} - \frac{k^2}{Re} \tilde{u}_x \\
+ ik_x N^{(a)}_{xx} + ik_y N^{(a)}_{xy} + ik_z N^{(a)}_{xz},
\]

(A9)

\[
\left( \frac{\partial}{\partial t} + q k_y \frac{\partial}{\partial k_x} \right) \tilde{u}_y = (q - 2)\tilde{u}_y - ik_x \tilde{p} - \frac{k^2}{Re} \tilde{u}_y \\
+ ik_x N^{(a)}_{xy} + ik_y N^{(a)}_{yy} + ik_z N^{(a)}_{yz},
\]

(A10)

\[
\left( \frac{\partial}{\partial t} + q k_y \frac{\partial}{\partial k_x} \right) \tilde{u}_z = -\tilde{u}_z - ik_x \tilde{p} - \frac{k^2}{Re} \tilde{u}_z \\
+ ik_x N^{(a)}_{xz} + ik_y N^{(a)}_{yz} + ik_z N^{(a)}_{zz},
\]

(A11)

\[
\left( \frac{\partial}{\partial t} + q k_y \frac{\partial}{\partial k_x} \right) \tilde{B}_x = ik_x \tilde{F}_y - ik_y \tilde{F}_z - \frac{k^2}{Rm} \tilde{B}_x,
\]

(A12)

\[
\left( \frac{\partial}{\partial t} + q k_y \frac{\partial}{\partial k_x} \right) \tilde{B}_y = -q \tilde{B}_z + ik_z \tilde{F}_x - ik_x \tilde{F}_z - \frac{k^2}{Rm} \tilde{B}_y,
\]

(A13)

\[
\left( \frac{\partial}{\partial t} + q k_y \frac{\partial}{\partial k_x} \right) \tilde{B}_z = ik_x \tilde{F}_y - ik_y \tilde{F}_z - \frac{k^2}{Rm} \tilde{B}_z,
\]

(A14)

\[
k_x \tilde{a}_x + k_y \tilde{a}_y + k_z \tilde{a}_z = 0,
\]

(A15)

\[
k_x \tilde{B}_x + k_y \tilde{B}_y + k_z \tilde{B}_z = 0,
\]

(A16)

where \(k^2 = k_x^2 + k_y^2 + k_z^2\). These spectral equations involve the linear and nonlinear \((N^{(a)}_{ij}(k, t), \tilde{F}_i(k, t))\), where \(i, j = x, y, z\) terms that are the Fourier transforms of the corresponding linear and nonlinear terms of Equations (A1)–(A8) in physical space. These spectral nonlinear terms are given by convolutions

\[
N^{(a)}_{ij}(k, t) = \int d^3k'[\tilde{B}_i(k', t) \tilde{F}_j(k - k', t)] \\
- \tilde{u}_i(k', t) \tilde{u}_j(k - k', t)
\]

(A17)

and \(\tilde{F}_x, \tilde{F}_y, \tilde{F}_z\), which are the Fourier transforms of the respective components of the perturbed electromotive force \(\mathbf{F} = \mathbf{u} \times \mathbf{B}\), where

\[
\tilde{F}_x(k, t) = \int d^3k'[\tilde{u}_x(k', t) \tilde{B}_y(k - k', t)] \\
- \tilde{u}_x(k', t) \tilde{B}_y(k - k', t),
\]

(A18)

\[
\tilde{F}_y(k, t) = \int d^3k'[\tilde{u}_y(k', t) \tilde{B}_x(k - k', t)] \\
- \tilde{u}_y(k', t) \tilde{B}_x(k - k', t),
\]

(A19)

\[
\tilde{F}_z(k, t) = \int d^3k'[\tilde{u}_z(k', t) \tilde{B}_y(k - k', t)] \\
- \tilde{u}_z(k', t) \tilde{B}_y(k - k', t)
\]

(A20)

describe the effect of nonlinearity on the magnetic field perturbations. From Equations (A9)–(A11) and the divergence-free conditions (A15) and (A16), we can express pressure

\[
\rho = 2i(1 - q) \frac{k_y}{k^2} \tilde{a}_x - 2i \frac{k_z}{k^2} \tilde{a}_y + \frac{k_x}{k^2} + \sum_{(i,j)=(x,y,z)} \frac{k_i k_j}{k^2} N^{(a)}_{ij}.
\]

(A21)

Inserting it back into Equations (A9)–(A11), we have

\[
\left( \frac{\partial}{\partial t} + q k_y \frac{\partial}{\partial k_x} \right) \tilde{u}_x = 2 \left( 1 - \frac{k_x^2}{k^2} \right) \tilde{u}_y + 2(1 - q) \frac{k_x k_y}{k^2} \tilde{a}_x \\
- \frac{k^2}{Re} \tilde{u}_x + Q_x,
\]

(A22)

\[
\left( \frac{\partial}{\partial t} + q k_y \frac{\partial}{\partial k_x} \right) \tilde{u}_y = \left[ q - 2 - 2(q - 1) \frac{k_y^2}{k^2} \right] \tilde{u}_y \\
- 2 \frac{k_x k_y}{k^2} \tilde{a}_x + \frac{k_x k_z}{k^2} \tilde{a}_y + \frac{k^2}{Re} \tilde{u}_y + Q_z,
\]

(A23)

\[
\left( \frac{\partial}{\partial t} + q k_y \frac{\partial}{\partial k_x} \right) \tilde{u}_z = 2(1 - q) \frac{k_y k_z}{k^2} \tilde{a}_x - 2 \frac{k_x k_z}{k^2} \tilde{a}_y \\
- \frac{k^2}{Re} \tilde{a}_z + Q_z,
\]

(A24)
where

\[ Q_i = i \sum_j k_j N_j^{(w)} - i k_j \sum_{m,n} \frac{k_m k_n}{k^2} N_m^{(w)} N_n^{(w)}, \quad i, j, m, n = x, y, z. \]

(A25)

Appendix B
Dynamics of the Large-scale Field in the Turbulent State

In our setup, the MRI turbulence does support the large-scale dynamo action producing axisymmetric radial and azimuthal magnetic fields belonging to the \( \vec{k}_d = (0, 0, \pm 1) \) mode with only large-scale vertical variation. This dynamo action and the dynamics of this mode were previously studied in unstratified zero net flux MRI turbulence (Lesur & Ogilvie 2008; Herault et al. 2011; Shi et al. 2016; Riols et al. 2017; Walker & Boldyrev 2017), where it was shown to exhibit either regular/ quasi-periodic or more chaotic spatiotemporal behavior, depending on the vertical aspect ratio of a box. In the taller and azimuthally more elongated boxes with aspect ratios \( L_y/L_x \geq 4 \) and \( L_z/L_x \geq 2 \) mostly considered in those papers, this large-scale dynamo mode exhibits variations on a timescale of several tens of orbits or more. By contrast, in the present setup with the standard box where \( L_y/L_x = 1/3 \), this mode and associated large-scale dynamo field do not display any remarkable long-time quasi-periodic behavior and vary instead on shorter timescales, of the order of dynamical/orbital time.

This behavior is seen in Figure 15, depicting the evolution of the radial \( \vec{B}_r(k_d) \) and dominant azimuthal \( \vec{B}_\theta(k_d) \) fields associated with the \( k_d \) mode, together with driving Maxwell stress \( \mathcal{M}(k_d) \) and the nonlinear terms \( \mathcal{N}_{\theta}(k_d) \), \( \mathcal{N}_{\phi}(k_d) \) at \( P_m = 4 \). As it is often done in MRI turbulence studies, in Figure 16, we also plot the corresponding spacetime diagrams of the radial and azimuthal fields in physical space averaged horizontally in the \( (x, y) \) slice, which are dominated by the \( k_d \) mode. In these figures, we have chosen a similar total time interval, 334 (in units of \( \Omega^{-1} \)), as used by Lesur & Ogilvie (2008) and Shi et al. (2016) in order to facilitate a comparison with the related plots of the horizontally averaged fields in those papers. It is seen that the variation of \( \vec{B}_r(k_d) \) and \( \vec{B}_\theta(k_d) \) with time is even more irregular and nonperiodic than those for taller \( (L_x/L_z \geq 2) \) boxes, occurring over a shorter timescale \( \lesssim 10 \), nearly as fast as the characteristic time of nonmodal MRI growth of nonaxisymmetric active modes. This timescale of the mean field is consistent with the estimates of Shi et al. (2016) for standard boxes. Note that the radial field exhibits variations on even shorter timescales compared to the azimuthal field. Since the \( k_d \) mode makes the largest contribution to the horizontally (over \( x \) and \( y \)) averaged fields, \( \langle B_r \rangle \) and \( \langle B_\theta \rangle \), the latter also exhibit temporal variations on a similar timescale (Figure 16). As a result, the overall pattern of the large-scale field as a function of \( t \) and \( z \) appears to be less organized than that in taller boxes considered in Lesur & Ogilvie (2008), Shi et al. (2016), and Walker & Boldyrev (2017).

The evolution of the large-scale radial and azimuthal fields is a consequence of the specific temporal behavior of the Maxwell stress and the nonlinear terms also shown in Figure 15. Here \( \mathcal{N}_{\phi}(k_d) \) oscillates irregularly and rapidly, increasing the radial field when positive and draining it when negative. The Maxwell stress \( \mathcal{M}(k_d) \) is always positive, acting as a main driver of the azimuthal field, and varies with time slower than \( \mathcal{N}_{\phi}(k_d) \) does. On the other hand, \( \mathcal{N}_{\theta}(k_d) \) is always negative, draining the azimuthal field. It nearly balances the positive Maxwell term, closely following its peaks.

The above analysis shows that in the standard box \((L_x, L_y, L_z) = (3, 3, 1)\) used here, there is no temporal scale separation between the \( k_d \) mode and other nonaxisymmetric smaller-scale modes. Besides, as we have also checked, for our setup, the contribution of the \( k_d \) mode’s interaction with other nonaxisymmetric modes is small in the nonlinear transfer terms. For these reasons, unlike Herault et al. (2011) and Riols et al. (2015, 2017), we have preferred not to separate out the \( k_d \) dynamo mode, which carries the large-scale axisymmetric azimuthal field, and consider the dynamics of nearby nonaxisymmetric modes against its background; instead, we treat it from a general standpoint of the dynamics and nonlinear interactions of modes in Fourier space. In fact, the nonlinear

Figure 15. Evolution of the real and imaginary parts of the radial, \( \vec{B}_r(k_d) \), and azimuthal, \( \vec{B}_\theta(k_d) \), magnetic fields of the \( \vec{k}_d = (0, 0, \pm 1) \) mode at \( P_m = 4 \), together with the corresponding governing terms, \( \mathcal{N}_{\theta}(k_d) \), \( \mathcal{N}_{\phi}(k_d) \), as well as the Maxwell stress, \( \mathcal{M}(k_d) \). The azimuthal field is about 10 times larger than the radial one. Both of these field components and the dynamical terms exhibit quite irregular variations on a relatively short timescale of \( \lesssim 10 \). The nonlinear term \( \mathcal{N}_{\phi}(k_d) \) alternates signs and causes corresponding dips and rises of the radial field. The Maxwell stress \( \mathcal{M}(k_d) \) is always positive, maintaining the azimuthal field, whereas the nonlinear term \( \mathcal{N}_{\theta}(k_d) \) is always negative, draining the azimuthal field, i.e., acting as a turbulent magnetic dissipation for it.
interaction of the $k_d$ mode with other modes, whatever effect it has, is already taken care of in the nonlinear terms $\mathcal{N}^{(l)}$, while its energy supply from the background shear flow is described by the Maxwell stress, which are all computed in Section 4. This is one of the advantages of our general analysis of the spectral dynamics of the turbulence, that it gives a good overview of the active modes, as well as the underlying linear and nonlinear processes in Fourier space.

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**Figure 16.** Spacetime diagrams of the radial and azimuthal magnetic fields in physical space averaged horizontally in the $(x, y)$ slice at $P_m = 4$. Their temporal behavior appears to be less regular, varying on a similar timescale as the $k_d$ mode in Figure 15 (perhaps even on a shorter timescale due to the contribution of higher $k_z \geq 2$ axisymmetric modes). Consequently, the patches of positive (yellow and red) and negative (blue) values of the fields are smaller, contrary to those in taller boxes.
