Two-dimensional turbulence in magnetized plasmas

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Abstract
In an inhomogeneous magnetized plasma the transport of energy and particles perpendicular to the magnetic field is in general mainly caused by quasi two-dimensional turbulent fluid mixing. The physics of turbulence and structure formation is of ubiquitous importance to every magnetically confined laboratory plasma for experimental or industrial application. Specifically, high-temperature plasmas for fusion energy research are also dominated by the properties of this turbulent transport. Self-organization of turbulent vortices to mesoscopic structures like zonal flows is related to the formation of transport barriers that can significantly enhance the confinement of a fusion plasma. This subject, of great importance in research, is rarely touched on in introductory plasma physics or continuum dynamics courses. Here a brief tutorial on 2D plasma turbulence is presented as an introduction to the field, appropriate for inclusion in undergraduate and graduate courses.

1. Introduction

Turbulence is a state of spatio-temporal chaotic flow generically attainable for fluids with access to a sufficient source of free energy. A result of turbulence is enhanced mixing of the fluid which is directed towards a reduction of the free energy. Mixing typically occurs by the formation of vortex structures on a large range of spatial and temporal scales that span between system, energy injection and dissipation scales [1–3].

Fluids comprise the states of matter of liquids, gases and plasmas [4]. A common free energy source that can drive turbulence in neutral (or more precisely: non-conducting) fluids is a strong enough gradient (or ‘shear’) in flow velocity, which can lead to vortex formation by Kelvin–Helmholtz instability. Examples for turbulence occurring from this type of instability are forced pipe flows, where a velocity shear layer is developing at the wall boundary, or a fast jet streaming into a stationary fluid. Another source of free energy is a thermal gradient in connection with an aligned restoring force (as in liquids heated from below in a gravity field) that leads to Rayleigh–Benard convection [5].
Several routes for the transition from laminar flow to turbulence in fluids have been proposed. For example, in some specific cases the Ruelle–Takens scenario occurs, where by linear instability through a series of a few period doubling bifurcations a final nonlinear transition to flow chaos is observed when a control parameter (such as the gradient of velocity or temperature) is increased [6]. For other scenarios, as in pipe flow, a sudden direct transition by subcritical instability to fully developed turbulence or an intermittent transition are possible [7, 8].

The complexity of the flow dynamics is considerably enhanced in a plasma compared to a non-conducting fluid. A plasma is a macroscopically neutral gas composed of many electrically charged particles that is essentially determined by collective degrees of freedom [9, 10]. Space and laboratory plasmas are usually composed of positively charged ions and negatively charged electrons that are dynamically coupled by electromagnetic forces. Thermodynamic properties are governed by collisional equilibration and conservation laws as in non-conducting fluids. The additional long-range collective interaction by spatially and temporally varying electric and magnetic fields allows for rich dynamical behaviour of plasmas with the possibility for complex flows and structure formation in the presence of various additional free energy sources [11].

The basic physics of plasmas in space, laboratory and fusion experiments is introduced in detail in a variety of textbooks (e.g. [12–14]).

Although the dynamical equations for fluid and plasma flows can be conceptually simple, they are highly nonlinear and involve an infinite number of degrees of freedom [15]. Analytical solutions are therefore in general impossible. The description of fluid and plasma dynamics mainly relies on statistical and numerical methods.

2. Continuum dynamical theory of fluids and plasmas

Computational models for fluid and plasma dynamics may be broadly classified into three major categories:

(1) Microscopic models: many-body dynamical description by ordinary differential equations and laws of motion;
(2) mesoscopic models: statistical physics description (usually by integro-differential equations) based on probability theory and stochastic processes;
(3) macroscopic models: continuum description by partial differential equations based on conservation laws for the distribution function or its fluid moments.

Examples of microscopic models are molecular dynamics (MD) methods for neutral fluids that model motion of many particles connected by short-range interactions [16], or particle-in-cell (PIC) methods for plasmas including electromagnetic forces [17, 18]. Such methods become important when relevant microscopic effects are not covered by the averaging procedures used to obtain meso- or macroscopic models, but they usually are intrinsically and computationally expensive.

Complications result from multi-particle or multi-scale interactions. Mesoscopic modelling treats such effects on the dynamical evolution of particles (or modes) by statistical assumptions on the interactions [19]. These may be implemented either on the macroscopic as spectral fluid closure schemes as, for example, in the direct interaction approximation (DIA), or on the microscale as advanced collision operators as in Fokker–Planck models. An example of a mesoscopic computational model for fluid flow is the lattice Boltzmann method (LBM) that combines free streaming particle motion by a minimalistic discretization in velocity space with suitable models for collision operators in such a way that fluid motion is recovered on
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Figure 1. Computation of decaying two-dimensional fluid turbulence, showing contours of vorticity $\omega = \nabla \times \mathbf{u}$.

Macroscopic models are based on the continuum description of the kinetic distribution function of particles in a fluid, or of its hydrodynamic moments. The continuum modelling of fluids and plasmas is introduced in more detail below.

Computational methods for turbulence simulations have been developed within the framework of all particle, mesoscopic or continuum models. Each of the models has both advantages and disadvantages in their practical numerical application. The continuum approach can be used in situations where discrete particle effects on turbulent convection processes are negligible. This is to some approximation also the case for many situations and regimes of interest in fusion plasma experiments that are dominated by turbulent convective transport, in particular at the (more collisional) plasma edge.

Within the field of computational fluid dynamics the longest experience and broadest applications have been obtained with continuum methods [20]. Many numerical continuum methods that were originally developed for neutral fluid simulation have been straightforwardly applied to plasma physics problems [21].

In continuum kinetics, the time evolution of the single-particle probability distribution function $f(x, v, t)$ for particles of each species (e.g. electrons and ions in a plasma) in the presence of a mean force field $F(x, t)$ and within the binary collision approximation (modelled by an operator $C$) is described by the Boltzmann equation [22]

$$\left(\partial_t + v \cdot \nabla_x + F \cdot \nabla_v\right)f = Cf. \quad (1)$$

In a plasma the force field has to be self-consistently determined by solution of the Maxwell equations. Usually, kinetic theory and computation for gas and plasma dynamics make use of further simplifying approximations that considerably reduce the complexity: in the Vlasov equation binary collisions are neglected ($C = 0$), and in the drift-kinetic or gyro-kinetic plasma equations further reducing assumptions are taken about the time and space scales under consideration.
The continuum description is further simplified when the fluid can be assumed to be in local thermodynamic equilibrium. Then a hierarchical set of hydrodynamic conservation equations is obtained by construction of moments over velocity space \[23, 24\]. In lowest orders of the infinite hierarchy, the conservation equations for mass density \(n(x, t)\), momentum \(n u(x, t)\) and energy density \(E(x, t)\) are obtained. Any truncation of the hierarchy of moments requires the use of a closure scheme that relates quantities depending on higher order moments by a constitutive relation to the lower order field quantities.

An example of a continuum model for neutral fluid flow are the Navier–Stokes equations. In their most widely used form (in particular for technical and engineering applications) the assumptions of incompressible divergence free flow (i.e., \(n\) is constant on particle paths) and of an isothermal equation of state are taken \[25\]. Then the description of fluid flow can be reduced to the solution of the (momentum) Navier–Stokes equation

\[
(\partial_t + u \cdot \nabla) u = -\nabla P + \nu \Delta u
\]

under the constraints given by

\[
\nabla \cdot u = 0
\]

and by boundary conditions. Most numerical schemes for the Navier–Stokes equation require solution of a Poisson-type equation for the normalized scalar pressure \(P = p/\rho_0\) in order to guarantee divergence free flow. The character of solutions for the Navier–Stokes equation intrinsically depends on the ratio between the dissipation time scale (determined by the kinematic viscosity \(\nu\)) and the mean flow time scale (determined by the system size \(L\) and mean velocity \(U\)), specified by the Reynolds number

\[
Re = LU/\nu.
\]

For small values of \(Re\) the viscosity will dominate the time evolution of \(u(x)\) in the Navier–Stokes equation, and the flow is laminar. For higher \(Re\) the advective nonlinearity is dominant and the flow can become turbulent. The Rayleigh number has a similar role for the onset of thermal convective turbulence \[6\].

3. Drift-reduced two-fluid equations for plasma dynamics

Flow instabilities as a cause for turbulence, such as those driven by flow shear or thermal convection, do in principle also exist in plasmas similar to neutral fluids \[26\], but are in general found to be less dominant in strongly magnetized plasmas. The most important mechanism which results in turbulent transport and enhanced mixing relevant to confinement in magnetized plasmas \[27–29\] is an unstable growth of coupled wave-like perturbations in plasma pressure \(\tilde{p}\) and electric fields \(\tilde{E}\). The electric field forces a flow with the ExB (‘E-cross-B’) drift velocity

\[
v_{\text{ExB}} = \frac{1}{B^2} \tilde{E} \times B
\]

of the plasma perpendicular to the magnetic field \(B\). A phase shift, caused by any inhibition of a fast parallel Boltzmann response of the electrons, between pressure and electric field perturbation in the presence of a pressure gradient can lead to an effective transport of plasma across the magnetic field and to an unstable growth of the perturbation amplitude. Nonlinear self-advection of the ExB flow and coupling between perturbation modes (‘drift waves’) can finally lead to a fully developed turbulent state with strongly enhanced mixing.

A generic source of free energy for magnetized laboratory plasma turbulence resides in the pressure gradient: in the core of a magnetic confinement region both plasma density and temperature are usually much larger than near the bounding material wall, resulting in a
pressure gradient directed inwards to the plasma centre. Instabilities which tap this free energy
tend to lead to enhanced mixing and transport of energy and particles down the gradient [30].
For magnetically confined fusion plasmas, this turbulent convection by \( E \times B \) drift waves often
dominate collisional diffusive transport mechanisms by orders of magnitude, and essentially
determines energy and particle confinement properties [31, 32]. The drift wave turbulence is
regulated by formation of mesoscopic streamers and zonal structures out of the turbulent flows
[33].

Continuum models for drift wave turbulence have to capture the different dynamics of
electrons and ions parallel and perpendicular to the magnetic field and the coupling between
both species by electric and magnetic interactions [34, 35]. Therefore, a single-fluid magneto-
hydrodynamic (MHD) model cannot appropriately describe drift wave dynamics: one has to
refer to a set of two-fluid equations, treating electrons and ions as separate species, although
the plasma on macroscopic scales remains quasi-neutral with nearly identical ion and electron
density, \( n_i \approx n_e \equiv n \).

The two-fluid equations require quantities such as collisional momentum exchange rate,
pressure tensor and heat flux to be expressed by hydrodynamic moments based on solution of
a kinetic (Fokker–Planck) model. The most widely used set of such fluid equations has been
derived by Braginskii [36] and is, e.g., presented in brief in [37].

The most general continuum descriptions for the plasma species, based either on the kinetic
Boltzmann equation or on the hydrodynamic moment approach as in the Braginskii equations,
are covering all time and space scales, including detailed gyro-motion of particles around the
magnetic field lines, and the fast plasma frequency oscillations. From experimental observation
it is on the other hand evident [38–42], that the dominant contributions to turbulence and
transport in magnetized plasmas originate from time and space scales that are associated with
frequencies in the order the drift frequency \( \omega \sim \left( \rho_s / L_\perp \right) \Omega_i \), that are much lower than the ion
gyro-frequency \( \Omega_i = q_i B / M_i \) by the ratio between drift scale \( \rho_s = \sqrt{T_e M_i / (eB)} \) to gradient
length \( L_\perp \),

\[
\omega \sim \Omega_i \ll \Omega_e. \tag{6}
\]

Under these assumptions one can apply a ‘drift ordering’ based on the smallness of the order
parameter \( \delta = \omega / \Omega_i \ll 1 \). This can be introduced either on the kinetic level, resulting
in the drift-kinetic model, or on the level of two-fluid moment equations for the plasma,
resulting in ‘drift-reduced two-fluid equations’, or simply called ‘drift wave equations’
[29, 43, 44]: neglect of terms scaling with \( \delta \) in higher powers than 2 considerably simplifies
both the numerical and analytical treatment of the dynamics, while retaining all effects of the
perpendicular drift motion of guiding centres and nonlinear couplings that are necessary to
describe drift wave turbulence.

For finite ion temperature, the ion gyro-radius \( \rho_i = \sqrt{T_i M_i / (eB)} \) can be of the same
magnitude as typical fluctuation scales, with wave numbers found around \( k_\perp \rho_s \sim 0.3 \) in the
order of the drift scale

\[
\rho_s = \frac{\sqrt{T_e M_e}}{eB}. \tag{7}
\]

Although the gyro-motion is still fast compared to turbulent time scales, the ion orbit then is
of similar size as spatial variations of the fluctuating electrostatic potential. Finite gyro-radius
(or ‘finite Larmor radius’, FLR) effects are captured by appropriate averaging procedures
over the gyrating particle trajectory and modification of the polarization equation, resulting in
‘gyrokinetic’ or ‘gyrofluid models’ for the plasma.
4. Turbulent vortices and mean flows

The prevalent picture of drift wave turbulence is that of small-scale, low-frequency ExB vortices in the size of several gyro-radii, that determine mixing and transport of the plasma perpendicular to the magnetic field across these scales.

Beyond that, turbulence in magnetized plasmas exhibits large-scale structure formation that is linked to this small-scale eddy motion: the genesis of mean zonal flow structures out of an initially nearly homogeneous isotropic vortex field and the resulting shear-flow suppression of the driving turbulence is a particular example of a self-organizing regulation process in a dynamical system [27, 45–51]. The scale of these macroscopic turbulent zonal flows is that of the system size, setting up a radially localized differential ExB rotation of the whole plasma on an entire toroidal flux surface (see figure 2).

Moreover, the process of self-organization to zonal flow structures is thought to be a main ingredient in the still unresolved nature of the L–H transition in magnetically confined toroidal plasmas for fusion research [52]. The L–H transition is experimentally found to be a sudden improvement in the global energy content of the fusion plasma from a low to high (L–H) confinement state when the central heat source power is increased above a certain threshold [53–56]. The prospect of operation in a high confinement H-mode is one of the main requirements for successful performance of fusion experiments like ITER.

The mechanism for spin-up of zonal flows in drift wave turbulence is a result of the quasi two-dimensional nature of the nonlinear ExB dynamics, in connection with the double periodicity and parallel coupling in a toroidal magnetized plasma.

Basic concepts and terminology for the interaction between vortex turbulence and mean flows have been first developed in the context of neutral fluids. It is therefore instructive to briefly review these relations in the framework of the Navier–Stokes equation (2) before applying them to plasma dynamics.

Small (space and/or time) scale vortices and mean flows may be separated formally by an ansatz known as Reynolds decomposition,

$$\mathbf{u} = \bar{\mathbf{u}} + \tilde{\mathbf{u}},$$

splitting the flow velocity into a mean part $\bar{\mathbf{u}} = \langle \mathbf{u} \rangle$, averaged over the separation scale, and small-scale fluctuations $\tilde{\mathbf{u}}$ with $\langle \tilde{\mathbf{u}} \rangle = 0$. While the averaging procedure, $\langle \cdots \rangle$, is mathematically most unambiguous for the ensemble average, the physical interpretation
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Figure 3. Example for a fluid simulation of 2D grid turbulence with a lattice Boltzmann code [65]: a high Reynolds number flow with \( R_e = 5000 \) is entering from the left of the domain and passes around a grid of obstacles. The shading depicts vorticity. In the near field directly behind the grid the particular vortex streets can be distinguished. In the middle of the domain neighbouring eddies are strongly coupled to a quasi homogeneous (statistically in the perpendicular direction) vortex field. On the far right side eddies decay into larger structures in the characteristic way of 2D turbulence. The simulation agrees well with flowing soap film experiments [66].

in fluid dynamics makes a time or space decomposition more appropriate. Applying this averaging on the Navier–Stokes equation (2), one obtains the Reynolds equation (or Reynolds averaged Navier–Stokes equation (RANS))

\[
(\partial_t + \bar{U} \cdot \nabla)\bar{U} = -\nabla \bar{P} + \nabla \mathbf{R} + \nu \Delta \bar{U}.
\]

This mean flow equation has the same structure as the original Navier–Stokes equation with one additional term including the Reynolds stress tensor \( \mathbf{R}_{ij} = \langle \tilde{u}_i \tilde{u}_j \rangle \). Momentum transport between turbulence and mean flows can thus be caused by a mean pressure gradient, viscous forces and Reynolds stress. A practical application of the RANS is in large eddy simulation (LES) of fluid turbulence, which efficiently reduces the time and space scales necessary for computation by modelling the Reynolds stress tensor for the smaller scales as a local function of the large scale flow. LES is however not applicable for drift wave turbulence computations, as here in any case all scales down to the effective gyro-radius (or drift scale \( \rho_s \)) have to be resolved in direct numerical simulation (DNS).

5. Two-dimensional fluid turbulence

Turbulent flows are generically three dimensional. In some particular situations the dependence of the convective flow dynamics on one of the Cartesian directions can be negligible compared to the others and the turbulence becomes quasi two dimensional [57, 58]. Examples for such 2D fluid systems are thin films (e.g. soap films; see simulation in figure 3), rapidly rotating stratified fluids, or geophysical flows of ocean currents and the (thin) planetary atmosphere. In particular, also the perpendicular ExB dynamics in magnetized plasmas behaves similarly to a 2D fluid [59]. The two-dimensional approximation of fluid dynamics not only simplifies the treatment, but moreover introduces distinctly different behaviour.

The major difference can be discerned by introducing the vorticity

\[
\omega = \nabla \times \mathbf{u}
\]
and taking the curl of the Navier–Stokes equation (2) to get the vorticity equation

\[
(\partial_t + u \cdot \nabla) \omega = (\omega \cdot \nabla) u + \nu \Delta \omega. \tag{11}
\]

In a two-dimensional fluid with \( v = v_x e_x + v_y e_y \) and \( v_z = 0 \) the vorticity reduces to \( \omega = w e_z \) with \( w = \partial_x v_y - \partial_y v_x \). The vortex stretching and twisting term \( (\omega \cdot \nabla) u \) is zero in 2D, thus eliminating a characteristic feature of 3D turbulence. For unforced inviscid 2D fluids then due to \( (\partial_t + u \cdot \nabla) \omega = 0 \) the vorticity \( w \) is constant in flows along the fluid element. This implies conservation of total enstrophy \( W = \int (1/2) |\omega|^2 \, dx \) in addition to the conservation of kinetic flow energy \( E = \int (1/2) |u|^2 \, dx \).

The 2D vorticity equation can be further rewritten in terms of a scalar stream function \( \phi \) that is defined by \((v_x, v_y) = (\partial_y, -\partial_x) \phi \) so that \( w = \nabla^2 \phi \), to obtain

\[
\partial_t w + [\phi, w] = \nu \Delta w. \tag{12}
\]

Here the Poisson bracket \([a, b] = \partial_a a \partial_b b - \partial_b a \partial_a b\) is introduced. For force-driven flows a term given by the curl of the force adds to the right-hand side of equation (12). Although the pressure is effectively eliminated from that equation, it is still necessary to similarly solve a (nonlocal) Poisson equation for the stream function. For ExB flows in magnetized plasmas the stream function \( \phi \) is actually represented by the fluctuating electrostatic potential.

The energetics of homogeneous 3D turbulence is usually understood in terms of a direct cascade of energy from the injection scale down to small (molecular) dissipation scales [1]: large vortices break up into smaller ones due to mutual stretching and shearing. In terms of the Reynolds equation (9) this means that the Reynolds stress transfer is usually negative, taking energy out of mean flows into small-scale vortices. The interaction between scales takes place basically by three-mode coupling maintained by the convective quadratic nonlinearity. This leads to the generic Kolmogorov \( k^{-5/3} \) power spectrum of Fourier components \( E(k) = \int dx (1/2) |u(x)|^2 \exp(-ik \cdot x) \) in the cascade range of 3D turbulence when energy injection and dissipation scales are well separated by a high Reynolds number [60].

In two dimensions the behaviour is somewhat different: Kraichnan, Leith and Batchelor [61–63] have conjectured that the energy has an inverse cascade property to scales larger than the injection. Smaller vortex structures self-organize to merge into bigger ones as a result of the absence of vortex stretching. For unforced turbulence the Reynolds stress transfer is on the average positive and into the mean flows.

The classical theory of 2D fluid turbulence by Kraichnan et al predicts a \( k^{-3} \) energy spectrum and a \( k^{-1} \) enstrophy spectrum in the inertial range. Numerical simulations of 2D Navier–Stokes turbulence however rather find, for example, a \( k^{-5/3} \) inverse cascade for energy on large scales and a \( k^{-3} \) direct cascade for enstrophy on small scales [64], modifying the classical predictions due to the existence of intermittency and coherent structures, although the extent of the modification is still under discussion. The (limiting or periodic) domain boundary in 2D simulations has also been found to have stronger influence than for the 3D case.

Periodicity in one dimension of the 2D system can lead to the spin up of sustained zonal structures of the mean flow out of the turbulence by inverse cascade. Prominent examples of zonal flows in planetary atmospheric dynamics are the well visible structures spanning around the planet Jupiter approximately along constant latitude, and jet streams in the earth’s atmosphere. Zonal flows are also observed in fluids rotating in a circular basin.

Drift wave turbulence in magnetized plasmas also has basically a 2D character and exhibits zonal structure formation in the poloidally and toroidally periodic domain on magnetic flux surfaces of a torus. These zonal plasma flows have finite radial extension and constitute a differential, sheared rotation of the whole plasma on flux surfaces.
6. Turbulence in magnetized plasmas

Drift wave turbulence is nonlinear, non-periodic motion involving disturbances on a background thermal gradient of a magnetized plasma and eddies of fluid-like motion in which the advecting velocity of all charged species is the ExB velocity. The disturbances in the electric field $E$ implied by the presence of these eddies are caused by the tendency of the electron dynamics to establish a force balance along the magnetic field $B$.

Pressure disturbances have their parallel gradients balanced by a parallel electric field, whose static part is given by the parallel gradient of the electrostatic potential. This potential in turn is the stream function for the ExB velocity in drift planes, which are locally perpendicular to the magnetic field. The turbulence is driven by the background gradient, and the electron pressure and electrostatic potential are coupled together through parallel currents. Departures from the static force balance are mediated primarily through electromagnetic induction and resistive friction, but also the electron inertia, which is not negligible.

The dynamical character of cross-field ExB drift wave turbulence in the edge region of a tokamak plasma is governed by the electromagnetic and dissipative effects in the parallel response.

The most basic drift-Alfvén (DALF) model to capture the drift wave dynamics includes nonlinear evolution equations of three fluctuating fields: the electrostatic potential $\tilde{\phi}$, electromagnetic potential $\tilde{A}_\parallel$ and density $\tilde{n}$. The tokamak edge usually features a more or less pronounced density pedestal (see figure 4), and the dominant contribution to the free energy drive to the turbulence by the inhomogeneous pressure background is thus due to the density gradient.

On the other hand, a steep enough ion temperature gradient (ITG) does not only change the turbulent transport quantitatively, but adds new interchange physics into the dynamics. In addition, more field quantities have to be treated: parallel and perpendicular temperatures $\tilde{T}_\parallel$ and $\tilde{T}_\perp$ and the associated parallel heat fluxes, for a total of six moment variables for each species. Finite Larmor radius effects introduced by warm ions require a gyrofluid description of the turbulence equations.

Both the resistive DALF and the ITG models can be covered by using the six-moment electromagnetic gyrofluid model GEM by Scott [67], but for basic studies it is also widely used in its more economical two-moment version for scenarios where the DALF model is applicable [68]. The gyrofluid model is based upon a moment approximation of the underlying gyrokinetic equation.
Figure 5. Basic drift waves mechanism: (1) an initial pressure perturbation $\tilde{p}$ leads to an ambipolar loss of electrons along the magnetic field $B$, whereas ions remain more immobile. (2) The resulting electric field $\tilde{E} = -\nabla \tilde{\phi}$ convects the whole plasma with $v_{ExB}$ around the perturbation in the plane perpendicular to $B$. (3) In the presence of a pressure gradient, $\tilde{p}$ propagates in electron diamagnetic drift direction with $v_e \sim \nabla p \times B$. This ‘drift wave’ is stable if the electrons establish $\tilde{\phi}$ according to the Boltzmann relation without delay (‘adiabatic response’). A non-adiabatic response due to collisions, magnetic flutter or wave-kinetic effects causes a phase shift between $\tilde{p}$ and $\tilde{\phi}$. The $ExB$ velocity is then effectively transporting plasma down the gradient, enhances the principal perturbation and leads to an unstable growth of the wave amplitude.

The first complete six-moment gyrofluid formulation was given for slab geometry by Dorland et al [69], and later extended by Beer et al to incorporate toroidal effects [70] using a ballooning-based form of flux surface geometry [71].

Electromagnetic induction and electron collisionality were then included to form a more general gyrofluid for edge turbulence by Scott [72], with the geometry correspondingly replaced by the version from the edge turbulence work, which does not make ballooning assumptions and in particular represents slab and toroidal mode types equally well and does not require radial periodicity [73]. Energy conservation considerations were solidified first for the two-moment version [68], and recently for the six-moment version in [67].

7. Basic drift wave instability

Destabilization of the $ExB$ drift waves occurs when the parallel electron dynamics deviates from a fast ‘adiabatic’ response to potential perturbations, resulting in a phase shift between the density and potential fluctuations. In this section the basic linear instability mechanism is discussed in the most basic electrostatic, cold ion limit for a straight magnetic field.

Figure 5 schematically shows a localized perturbation of plasma pressure $\tilde{p}$ (left) that results in a positive potential perturbation $\tilde{\phi} > 0$ (middle) due to ambipolar diffusion. For typical tokamak parameters it is found that the perturbation scale $\Delta \gg \lambda_D = \sqrt{\epsilon_0 T_e/(n e^2)}$ is much larger than the Debye length $\lambda_D$, so that quasi neutrality $n_i \approx n_e \equiv n$ can be assumed.

In accordance with the stationary parallel electron momentum balance equation

$$-e n_0 \tilde{k} \tilde{E}_1 - \nabla \parallel \tilde{p}_e = 0$$

with $\tilde{p}_e = \tilde{n}_e T_e$, the isothermal electrons try to locally establish along the field line a Boltzmann relation $n_e = n_0(r) \exp(e \tilde{\phi}/T_e)$. Under quasi neutrality $n_i = n_e = n_0(r) + \tilde{n}_e$, where $n_0(r)$ is the (in general radially varying) background density.

Without restrictions on the parallel electron dynamics (e.g., due to collisions, Alfvén waves or kinetic effects such as Landau damping and particles trapped in magnetic field inhomogeneities) this balance is established instantaneously on the drift time scale and is usually termed an ‘adiabatic response’.
Already at homogeneous background density the perturbation convects the plasma with the ExB drift velocity \( v_\perp = v_{\text{ExB}} = (B^{-2}) \mathbf{E} \times \mathbf{B} \) equal for electrons and ions. When a perpendicular background pressure gradient \( \nabla p \) is present, the perturbed structure propagates in the electron diamagnetic drift direction \( \sim \nabla p \times \mathbf{B} \).

In the continuity equation
\[
\partial_t n + \nabla \cdot (n \mathbf{v}) = 0 \quad \text{(14)}
\]
for cold ions in a homogeneous magnetic field and by neglecting ion inertia the only contribution to the velocity is the perpendicular ExB drift velocity \( v_{\text{ExB}} = -B^{-2}(\nabla \tilde{\phi} \times \mathbf{B}) \).

Using the Boltzmann relation in equation (14) one gets
\[
\partial_t n_0 \exp \left( \frac{e\tilde{\phi}}{T_e} \right) - \nabla \cdot \left[ \frac{1}{B^2} (\nabla \tilde{\phi} \times \mathbf{B}) n_0 \exp \left( \frac{e\tilde{\phi}}{T_e} \right) \right] = 0, \quad \text{(15)}
\]
and due to the straight \( \mathbf{B} = B \mathbf{e}_j \) it is obtained
\[
\partial_t \tilde{\phi} - \left( \frac{T_e}{eB} \right) (\partial_r \ln n_0) \partial_\theta \tilde{\phi} = 0. \quad \text{(16)}
\]

Assuming a perturbation periodical in the electron diamagnetic drift coordinate \( \theta \) with \( \tilde{\phi} = \Phi \exp[-i\omega t + i\xi_0 \theta] \), the electron drift wave frequency is found to be
\[
\omega_{ce} = \frac{T_e}{eB} k_\theta = \frac{c_s}{L_n} [\rho_s k_\theta] = \frac{\rho_s}{L_n} \Omega_i [\rho_s k_\theta]. \quad \text{(17)}
\]
Here the density gradient length \( L_n = (\partial_r \ln n_0)^{-1} \) and the drift scale \( \rho_s = \sqrt{m_i T_e/(eB)} \), representing an ion radius at electron temperature, have been introduced.

The motion of the perturbed structure perpendicular to the magnetic field and the pressure gradient in the electron diamagnetic drift direction \( k_\theta \sim \nabla p \times \mathbf{B} \) is in this approximation still stable and does so far not cause any transport down the gradient.

The drift wave is destabilized only when a phase shift \( \delta_k \) between potential and density perturbation is introduced by ‘non-adiabatic’ electron dynamics
\[
\tilde{n}_e = n_0 (1 - \tilde{\delta}_k) \frac{c_s}{T_e} \tilde{\phi}. \quad \text{(18)}
\]
The imaginary term \( \tilde{\delta}_k \) in general is an anti-Hermitian operator and describes dissipation of the electrons that causes the density perturbations to proceed the potential perturbations in \( \theta \) by slowing down the parallel equilibration. This leads to an exponential growth of the perturbation amplitude by \( \exp(\gamma_k t) \) with linear growth rate \( \gamma_k \sim \delta_k \omega_k \).

Parallel electron motion also couples drift waves to shear Alfvén waves, which are parallel propagating perpendicular magnetic field perturbations. With the vector potential \( A_\parallel \) as a further dynamic variable, the parallel electric field \( E_\parallel \), parallel electron motion and nonlinearly the parallel gradient are modified. The resulting nonlinear drift-Alfvén equations are discussed in the following section.

The stability and characteristics of drift waves and resulting plasma turbulence are further influenced by inhomogeneities in the magnetic field, in particular by field line curvature and shear. The normal and geodesic components of field line curvature have different roles for drift wave turbulence instabilities and saturation. The field gradient force associated with the normal curvature, if aligned with the plasma pressure gradient, can either act to restore or amplify pressure gradient driven instabilities by compression of the fluid drifts, depending on the sign of alignment. The geodesic curvature describes the compression of the field strength in perpendicular direction on a flux surface and is consequently related to the compression of large-scale (zonal) ExB flows.
Transition from stable drift waves to turbulence has been studied experimentally in linear and simple toroidal magnetic field configurations, and by direct numerical simulation. Experimental investigations in a magnetized low-beta plasma with cylindrical geometry by Klinger et al. have demonstrated that the spatiotemporal dynamics of interacting destabilized travelling drift wave follows a bifurcation sequence towards weakly developed turbulence according to the Ruelle–Takens–Newhouse scenario [74]. The relationship between observations made in linear magnetic geometry, purely toroidal geometry and magnetic confinement is discussed in [75], where the role of large-scale fluctuation structures has been highlighted. The role of parallel electron dynamics and Alfvén waves for coherent drift modes and drift wave turbulence have been studied in a collisionality dominated high-density helicon plasma [76]. Measurements of the phase coupling between spectral components of interchange unstable drift waves at different frequencies in a basic toroidal magnetic field configuration have indicated that the transition from a coherent to a turbulent spectrum is mainly due to three-wave interaction processes [77].

The competition between drift wave and interchange physics in ExB drift turbulence has been studied computationally in tokamak geometry with respect to the linear and nonlinear mode structure by Scott [78]. A quite remarkable aspect of fully developed drift wave turbulence in a sheared magnetic field lying in closed surfaces is its strong nonlinear character, which can be self-sustaining even in the absence of linear instabilities [79]. This situation of self-sustained plasma turbulence does not have any analogy in neutral fluid dynamics and, as shown in numerical simulations by Scott, is mostly applicable to tokamak edge turbulence, where linear forcing is low enough so that the nonlinear physics can efficiently operate [80].

8. Drift-Alfvén turbulence simulations for fusion plasmas

The model DALF3 by Scott [80], in the cold ion approximation without gyrofluid FLR corrections, represents the four field version of the dissipative drift-Alfvén equations, with disturbances (denoted by the tilde) in the ExB vorticity \( \tilde{\Omega} \), electron pressure \( \tilde{p}_e \), parallel current \( \tilde{j}_|| \) and parallel ion velocity \( \tilde{u}_|| \) as dependent variables. The equations are derived under gyro-drift ordering, in a three-dimensional globally consistent flux tube geometry [73, 81], and appear (in cgs units as used in the references) as

\[
\frac{nM_i c^2}{B^2} (\partial_t + \mathbf{v}_E \cdot \nabla) \tilde{\phi} = \nabla_|| \tilde{j}_|| - K(\tilde{\phi}),
\]

\[
\frac{1}{c} \partial_t \tilde{A}_|| + \frac{m_e}{n_e e^2} (\partial_t + \mathbf{v}_E \cdot \nabla) \tilde{j}_|| = \frac{1}{n_e} \nabla_|| (p_e + \tilde{p}_e) - \nabla_|| \tilde{\phi} - \eta_|| \tilde{j}_||, \tag{20}
\]

\[
(\partial_t + \mathbf{v}_E \cdot \nabla) (p_e + \tilde{p}_e) = \frac{T_e}{e} \nabla_|| \tilde{j}_|| - p_e \nabla_|| \tilde{u}_|| + p_e K(\tilde{\phi}) - \frac{T_e}{e} K(\tilde{p}_e), \tag{21}
\]

\[
nM_e (\partial_t + \mathbf{v}_E \cdot \nabla) \tilde{u}_|| = -\nabla_|| (p_e + \tilde{p}_e), \tag{22}
\]

with the parallel magnetic potential \( \tilde{A}_|| \) given by \( \tilde{j}_|| = -(c/4\pi) \nabla_\perp^2 \tilde{A}_|| \) through Ampère’s law, and the vorticity \( \tilde{\Omega} = \nabla_\perp^2 \tilde{\phi} \). Here, \( \eta_|| \) is the Braginskii parallel resistivity, \( m_e \) and \( M_i \) are the electron and ion masses, \( n \) is the electron (and ion) density and \( T_e \) is the electron temperature with pressure \( p_e = nT_e \). The dynamical character of the system is further determined by a set of parameters characterizing the relative role of dissipative, inertial and electromagnetic effects in addition to the driving by gradients of density and temperature.
The flux surface geometry of a tokamak enters into the fluid and gyrofluid equations via the curvilinear generalization of differentiation operators and via inhomogeneity of the magnetic field strength $B$. The different scales of equilibrium and fluctuations parallel and perpendicular to the magnetic field motivate the use of field aligned flux coordinates. The differential operators in the field aligned frame are the parallel gradient
\[
\nabla_\parallel = \left(\frac{1}{B}\right)(B + \tilde{B}_\perp) \cdot \nabla,
\]
with magnetic field disturbances $\tilde{B}_\perp = (-1/B)B \times \nabla \tilde{A}_\parallel$ as additional nonlinearities, the perpendicular Laplacian
\[
\nabla_\perp^2 = \nabla \cdot \left[(-1/B^2)B \times (B \times \nabla)\right],
\]
and the curvature operator
\[
\mathcal{K} = \nabla \cdot \left[(e/B^2)B \times (B \times \nabla)\right].
\]

The DALF equations constitute the most basic model containing the principal interactions of dissipative drift wave physics in a general closed magnetic flux surface geometry. The drift wave coupling effect is described by $\nabla_\parallel$ acting upon $\tilde{\rho}_e/p_e - e\tilde{\phi}/T_e$ and $\tilde{j}_\parallel$, while interchange forcing is described by $\mathcal{K}$ acting upon $\tilde{\rho}_e$ and $\tilde{\phi}$ [82]. In the case of tokamak edge turbulence, the drift wave effect is qualitatively more important [80], while the most important role for $\mathcal{K}$ is to regulate the zonal flows [83]. Detailed accounts on the role of magnetic field geometry shape in tokamaks and stellarators on plasma edge turbulence can be found in [84–87], in particular with respect to effects of magnetic field line shear [88] and curvature [89].

An example for typical experimental parameters are those of the ASDEX Upgrade (AUG) edge pedestal plasmas in L mode near to the L–H transition for deuterium ions with $M_i = M_D$; electron density $n_e = 3 \times 10^{13}$ cm$^{-3}$, temperatures $T_e = T_i = 70$ eV, magnetic field strength $B = 2.5$ T, major radius $R = 165$ cm, perpendicular gradient length $L_\perp = 4.25$ cm and safety factor $q = 3.5$.

The dynamical character of the DALF/GEM system is determined by a set of parameters characterizing the relative role of dissipative, inertial and electromagnetic effects in addition to the driving by gradients of density and temperature. In particular, for the above experimental values, these are collisionality $C = 0.51\tilde{\epsilon}(v_eL_\perp/c_e)(m_e/M_i) = 5$, magnetic induction $\tilde{B} = \tilde{\epsilon}(4\pi\rho_e/B^2) = 1$, electron inertia $\tilde{\mu} = \tilde{\epsilon}(m_e/M_i) = 5$ and ion inertia $\tilde{\epsilon} = (qR/L_\perp)^2 = 18350$.

The normalized values are similar in edge plasmas of other large tokamaks like JET. The parameters can be partially obtained even by smaller devices such as the torsatron TJ-K at the University of Stuttgart [90, 91], which therefore provides ideal test situations for comparison between simulations and the experiment [92].

A review of and introduction to drift wave theory in inhomogeneous magnetized plasmas has been presented by Horton in [93], although its main emphasis is placed on linear dynamics. An excellent introduction to and overview of turbulence in magnetized plasma and its nonlinear properties by Scott can be found in [94], and a very detailed survey on drift wave theory with emphasis on the plasma edge is given by Scott in [95, 96].

However, no tokamak edge turbulence simulation has yet reproduced the important threshold transition to the high confinement mode known from experimental fusion plasma operation. The possibility to obtain a confinement transition within first principle computations of edge turbulence will have to be studied with models that at least include full temperature dynamics, realistic flux surface geometry, global profile evolution including equilibrium, and
a coupling of edge and SOL regions with realistic sheath boundary conditions. In addition the model still has to maintain sufficient grid resolution, grid deformation mitigation and energy plus enstrophy conservation in the vortex/flow system.

Such ‘integrated’ fusion plasma turbulence simulation codes are currently under development. The necessary computing power to simulate the extended physics models and computation domains is going to be available within the next few years. This may facilitate international activities (for example, within the European Task Force on integrated tokamak modelling) towards a ‘computational’ tokamak plasma with a first-principles treatment of both transport and equilibrium across the whole cross-section. The objective of this extensive project in computational plasma physics is to provide the means for a direct comparison between our theoretical understanding and the emerging burning-plasma physics of the next large international fusion experiment ITER.

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