FUNCTIONAL EQUATIONS FOR ZETA FUNCTIONS OF $F_1$-SCHEMES

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ABSTRACT. For a scheme $X$ whose $\mathbb{F}_q$-rational points are counted by a polynomial $N(q) = \sum a_i q^i$, the $\mathbb{F}_1$-zeta function is defined as $\zeta_X(s) = \prod (s - i)^{-a_i}$. Define $\chi = N(1)$. In this paper we show that if $X$ is a smooth projective scheme, then its $\mathbb{F}_1$-zeta function satisfies the functional equation $\zeta_X(n-s) = (-1)^\chi \zeta_X(s)$. We further show that the $\mathbb{F}_1$-zeta function $\zeta_G(s)$ of a split reductive group scheme $G$ of rank $r$ with $N$ positive roots satisfies the functional equation $\zeta_G(r + N - s) = (-1)^\chi (\zeta_G(s))^r$.

1. INTRODUCTION

In recent years around a dozen different suggestion of what a scheme over $\mathbb{F}_1$ should be appeared in literature (cf. [6]). The common motivation for all these approaches is to provide a framework in which Deligne’s proof of the Weyl conjectures can be transfered to characteristic 0 in order to proof the Riemann hypothesis. Roughly speaking, $\mathbb{F}_1$ should be thought of as a field of coefficients for $\mathbb{Z}$, and $\mathbb{F}_1$-schemes $X$ should have a base extension $X_{\mathbb{Z}}$ to $\mathbb{Z}$ which is a scheme in the usual sense.

Though it is not clear yet whether one of the existing $\mathbb{F}_1$-geometries comes close to this goal, and thus in particular it is not clear what the appropriate notion of an $\mathbb{F}_1$-scheme should be, the zeta function $\zeta_X(s)$ of such an elusive $\mathbb{F}_1$-scheme $X$ is determined by the scheme $X = X_{\mathbb{Z}}$.

Namely, let $X$ be a variety of dimension $n$ over $\mathbb{Z}$, i.e. a scheme such that $X_k$ is an $\mathbb{F}$-variety of dimension $n$ for any field $k$. Assume further that $X$ has a counting polynomial $N(q)$, i.e. the number of $\mathbb{F}_q$-rational points is counted by $\#X(\mathbb{F}_q) = N(q)$ for every prime power $q$. If $X$ descents to an $\mathbb{F}_1$-scheme $\mathcal{X}$, i.e. $\mathcal{X}_{\mathbb{Z}} \simeq X$, then $\mathcal{X}$ has the zeta function $\zeta_{\mathcal{X}}(s) = \lim_{q \to 1} \left( q - 1 \right)^N \zeta_X(q, s)$ where $\zeta_X(q, s) = \exp \left( \sum_{r \geq 1} N(q^r) q^{-sr}/r \right)$ is the zeta function of $X \otimes \mathbb{F}_q$ if $q$ is a prime power and $\chi = N(1)$ is the order the pole of $\zeta_X(q, s)$ in $q = 1$ (cf. [9]). This expression comes down to

$$\zeta_{\mathcal{X}}(s) = \prod_{i=0}^n (s - i)^{-a_i}.$$  \[9, Lemme 1\]

From this it is clear that $\zeta_{\mathcal{X}}(s)$ is a rational function in $s$ and that its zeros (resp. poles) are at $s = i$ of order $-a_i$ for $i = 0, \ldots, n$. The only statement from the Weyl conjectures which is not obvious for zeta functions of $\mathbb{F}_1$-schemes is the functional equation.

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I like to thank Takashi Ono for drawing my attention to the symmetries occuring in the counting polynomials of split reductive group schemes, and I like to thank Markus Reineke for his explanations on the comparision theorem for liftable smooth varieties.
2. The functional equation for smooth projective \( \mathbb{F}_1 \)-schemes

Let \( X \) be an (irreducible) smooth projective variety of dimension \( n \) with a counting polynomial \( N(q) \). Let \( b_0, \ldots, b_{2n} \) be the Betti numbers of \( X \), i.e. the dimensions of the singular homology groups \( H_0(X_{\mathbb{C}}), \ldots, H_{2n}(X_{\mathbb{C}}) \). By Poincaré duality, we know that \( b_{2n-i} = b_i \).

As a consequence of the comparision theorem for smooth liftable varieties and Deligne’s proof of the Weil conjectures, we know that the counting polynomial is of the form

\[
N(q) = \sum_{i=0}^{n} b_{2i} q^i
\]

and that \( b_i = 0 \) if \( i \) is odd (cf. \cite{2} and \cite{8}). Thus \( \chi = \sum_{i=0}^{n} b_{2i} \) is the Euler characteristic of \( X_{\mathbb{C}} \) in this case (cf. \cite{4}).

Suppose \( X \) has an elusive model \( X \) over \( \mathbb{F}_1 \). Then \( X \) has the zeta function \( \zeta_X(s) = \prod_{i=0}^{n} (s - i)^{-b_{2i}} \).

**Theorem 1.** The zeta function \( \zeta_X(s) \) satisfies the functional equation

\[
\zeta_X(n - s) = (-1)^{\chi} \zeta_X(s)
\]

and the factor equals \(-1\) if and only if \( n \) is even and \( b_n \) is odd.

**Proof.** We calculate

\[
\zeta_X(n - s) = \prod_{i=0}^{n} ((n - s) - i)^{-b_{2i}}
\]

\[
= \prod_{i=0}^{n} (-1)^{b_{2i}} (s - (n - i))^{-b_{2i}}
\]

\[
= (-1)^{\chi} \prod_{i=0}^{n} (s - (n - i))^{-b_{2n-2i}}
\]

where we used \( b_{2n-2i} = b_{2i} \) in the last equation. If we now substitute \( i \) by \( n - i \) in this expression, we obtain

\[
\zeta_X(n - s) = (-1)^{\chi} \prod_{i=0}^{n} (s - i)^{-b_{2i}} = (-1)^{\chi} \zeta_X(s).
\]

If \( n \) is odd, then there is an even number of non-trivial Betti numbers and \( \chi = 2b_0 + 2b_2 + \cdots + 2b_{n-1} \) is even. If \( n \) is odd, then \( \chi = 2b_0 + 2b_2 + \cdots + 2b_{n-2} + b_n \) has the same parity as \( b_n \). Thus the additional statement.

**Remark.** Note the similarity with the functional equation for motivic zeta functions as in \cite{3, Thm. 1}. Amongst other factors, also \((-1)^{\chi(M)}\) appears in the functional equation of the zeta function of a motive \( M \) where \( \chi(M) \) is the (positive part of the) Euler characteristic of \( M \).

3. The functional equation for reductive groups over \( \mathbb{F}_1 \)

The above observations imply further a functional equation for reductive group schemes over \( \mathbb{F}_1 \). Note that Soule’s and Connes and Consani’s approaches towards \( \mathbb{F}_1 \)-geometry indeed succeeded in descending split reductive group schemes from \( \mathbb{Z} \) to \( \mathbb{F}_1 \) (cf. \cite{11}, \cite{5}, \cite{7}).
Let $G$ be a split reductive group scheme of rank $r$ with Borel group $B$ and maximal split torus $T \subset B$. Let $N$ be the normalizer of $T$ in $G$ and $W = N(\mathbb{Z})/T(\mathbb{Z})$ be the Weyl group. The Bruhat decomposition of $\lambda$ where $\lambda$ is the number of induced by the subscheme inclusions $BwB \to G$, which has the property that it induces a bijection between the $k$-rational points for every field $k$. We have $B \simeq G^r_{\mathbb{A}} \times \mathbb{A}^N$ as schemes where $N$ is the number of positive roots of $G$, and $BwB \simeq G^r_{\mathbb{A}} \times \mathbb{A}^{N+\lambda(w)}$ where $\lambda(w)$ is the length of $w \in W$. With this we can calculate the counting polynomial of $G$ as

$$N(q) = \# \prod_{w \in W} BwB(\mathbb{F}_q) = (q - 1)^r q^N \sum_{w \in W} q^{\lambda(w)}.$$  

The quotient variety $G/B$ is a smooth projective scheme of dimension $N$ with counting function $N_{G/B}(q) = ((q - 1)^r q^N)^{-1} N(q) = \sum_{w \in W} q^{\lambda(w)}$. Let $b_0, \ldots, b_{2N}$ be the Betti numbers of $G/B$, then we know from the previous section that $N_{G/B}(q) = \sum_{l=0}^{2N} b_{2l} q^l$ and that $b_{2N-2l} = b_{2l}$.

Thus we obtain for the counting polynomial of $G$ that

$$N(q) = q^N \left( \sum_{k=0}^{r} (-1)^{r-k} \binom{r}{k} q^k \right) \cdot \left( \sum_{l=0}^{N} b_{2l} q^l \right)$$

$$= \sum_{i=0}^{d} \left( \sum_{k+l=i-N} (-1)^{r-k} \binom{r}{k} b_{2l} \right) q^i$$

where $d = r + 2N$ is the dimension of $G$ and with the convention that $\binom{i}{k} = 0$ if $k < 0$ or $k > r$. Denote by $a_i = \sum_{k+l=i-N} (-1)^{r-k} \binom{r}{k} b_{2l}$ the coefficients of $N(q)$.

Lemma. We have $a_0 = \cdots = a_{N-1} = 0$ and $a_{d-1} = (-1)^r a_{i+N}$.

Proof. The first statement follows from the fact that $N(q)$ is divisible by $q^N$. For the second statement we use the symmetries $\binom{i}{k} = \binom{r}{r-k}$ and $b_{2N-2l} = b_{2l}$ to calculate

$$a_{d-i} = \sum_{k+l=d-i-N} (-1)^{r-k} \binom{r}{k} b_{2l}$$

$$= \sum_{k+l=d-i-N} (-1)^r (-1)^k \binom{r}{r-k} b_{2N-2l}.$$ 

When we substitute $k$ by $r - k$ and $l$ by $N - l$ in this equation and use $d = r + 2N$, then we obtain

$$a_{d-i} = (-1)^r \sum_{k+l=(i+N)-N} (-1)^{r-k} \binom{r}{k} b_{2l},$$

which is the same as $(-1)^r a_{i+N}$. \hfill $\square$

Suppose $G$ has an elusive model $\mathcal{G}$ over $\mathbb{F}_1$. Then $\mathcal{G}$ has the zeta function $\zeta_{\mathcal{G}}(s) = \prod_{i=0}^{d} (s - i)^{-a_i}$. Let $\chi = N(1) = \sum_{i=0}^{d} a_i$.

Theorem 2. The zeta function $\zeta_{\mathcal{G}}(s)$ satisfies the functional equation

$$\zeta_{\mathcal{G}}(r + N - s) = (-1)^{\chi} \left( \zeta_{\mathcal{G}}(s) \right)^{(-1)^r}.$$
Proof. We use of the previous lemma and \( r + N = d - N \) to calculate that

\[
\zeta_G(r + N - s) = \prod_{i=0}^{n} (r + N - s - i)^{-a_i}
\]

\[
= \prod_{i=0}^{n} (d - N - s - i)^{(-1)^r a_d - N - i}.
\]

After substituting \( i \) by \( d - N - i \), we find that

\[
\zeta_G(d - N - s) = \prod_{i=0}^{n} (i - s)^{(-1)^r a_i}
\]

\[
= (-1)^{\chi} \left( \sum_{i=0}^{n} \frac{n!}{\prod_{i=0}^{n} (s - i)^{-a_i}} \right)^{-1}.
\]

\[\square\]

Remark. Kurokawa calculates the \( \mathbb{F}_1 \)-zeta functions of \( \mathbb{P}^n \), \( \text{GL}(n) \) and \( \text{SL}(n) \) in [4]. One can verify the functional equation for these examples immediately.

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