Abstract  Modified teleparallel gravity theory with the torsion scalar has recently gained a lot of attention as a possible candidate of dark energy. We perform a thorough reconstruction analysis on the so-called $F(T)$ models, where $F(T)$ is some general function of the torsion term. We derive conditions for the equivalence between of $F(T)$ models with purely kinetic k-essence. We present a new class models of $F(T)$ gravity and k-essence.

Keywords  Cosmology · Dark energy · $F(T)$ gravity · k-essence

1 Introduction

The discovery of the accelerated expansion of the universe [1–3] has forced a profound shift in our cosmological paradigm. This discovery indicates that the universe is very nearly spatially flat and consists about 70% dark energy (DE) which drives the cosmic acceleration. The equation of state (EoS) parameter $w_{DE}$ for DE should be $w_{DE} < -1/3$ to maintain this acceleration. Modern constraints on the EoS parameter are around $w_{DE} = -1$. Here we can note that from the theoretical point of view there are three different cases: $w_{DE} > -1$ (quintessence), $w_{DE} = -1$ (cosmological constant), and $w_{DE} < -1$ (phantom). The cosmological constant can explain the present accelerated expansion of the universe, for which $w_{DE} = -1$. Although cosmological constant is the simplest candidate of DE, yet there are serious theoretical problems associated with it such as the fine-tuning problem, the coincidence problem and so on. To solve the cosmological constant problem, some scalar field and fermion-field...
models (phantom fields, k-essence, f-essence, g-essence and so on) are proposed. These models of DE arise by the modification of the energy momentum tensor in Einstein equations.

Another alternative approach dealing with the accelerated expansion of the universe is modifying the Einstein–Hilbert action: such as $F(R)$, $F(G)$ and $F(R, G)$ (see recent reviews [4–13]). Here the Lagrangian density $F$ of modified gravity theories is an arbitrary function of $R$, $G$ or both $R$ and $G$, which are respectively Ricci scalar and Gauss–Bonnet term. The field equations of these modified gravity theories are fourth order thereby making it difficult to obtain both exact and numerical solutions. A recent updated review of modified gravity is given in [14,15]. In fact, it is demonstrated that alternative gravity, especially $F(R)$, may give very realistic values of $w_{\text{DE}}$ and other cosmographic parameters being very close to values of Lambda cold dark matter (LCDM). Hence, alternative gravity may be viable candidate for effective LCDM or k-essence late-time acceleration.

Recently, however, some models with the field equations of second order [so-called $F(T)$ gravity] are proposed [16,17]. These models are based on the “teleparallel” equivalent of General Relativity (TEGR) [18–27], which, instead of using the curvature defined via the Levi-Civita connection, uses the Weitzenböck connection that has no curvature but only torsion. In [26,27], some models based on modified teleparallel gravity were presented as an alternative to inflationary models. The fact that the field equations of $F(T)$ gravity are always second order makes this theory simpler than other modified gravity theories like $F(R)$ or $F(G)$. More recently, some properties of $F(T)$ gravity were studied in [28–42]. For instance, it is demonstrated recently in [43] that $F(T)$ gravity may have very realistic cosmographic parameters, fitting it with luminosity distance and baryon acoustic oscillation (BAO). In fact, $F(T)$ cosmography [43] may give the regions overlapping with LCDM model. It is clear that $F(T)$ gravity presents a very rich behavior and deserves further investigation.

The purpose of the present paper is to investigate some models of $F(T)$ gravity as well as k-essence. Also we will study the equivalence of modified gravity theories with k-essence. This paper is organized as follows. In the following section we review $F(T)$ gravity and present some of its details. In Sect. 3 we investigate some models of k-essence. The relation between $F(T)$ gravity and k-essence is studied in Sect. 4. In Sect. 5 we will give some conclusions.

2 $F(T)$ gravity

2.1 Elements of $F(T)$ gravity

The action of $F(T)$ gravity reads as (see, e.g. [16,17,28])

$$S = \int d^4 x e \left[ \frac{1}{2\kappa^2} F(T) + L_m \right].$$

(2.1)

where $T$ is the torsion scalar, $e = \det (e^\mu_i) = \sqrt{-g}$ and $L_m$ stands for the matter Lagrangian. Here $e^\mu_i$ are the components of the vierbein vector field $e_A$ in a coordinate.
basis, that is $e_A = e^\mu_A \partial_\mu$. Note that in the teleparallel gravity, the dynamical variable is the vierbein field $e_A(x^\mu)$. The variation of the action with respect to this vierbein field leads to the following gravitational equations of motion

$$\left[ e^{-1} \partial_\mu \left( e S^\mu_\nu \right) - e^\lambda_i T^\rho_\mu S^\nu_\rho \right] F_T + S^\mu_\nu (\partial_\mu T) F_{TT} + \frac{1}{4} e^\nu_i F = \frac{1}{2} k^2 e^\rho_i T^\nu_\rho. \quad (2.2)$$

Here the torsion scalar $T$ is given by

$$T = S^\mu_\nu T^\rho_\mu \quad (2.3)$$

with

$$S^\mu_\nu T = \frac{1}{2} (K^\mu_\nu + \delta^\mu_\rho T^\rho_\nu - \delta^\nu_\rho T^\rho_\mu). \quad (2.4)$$

Here the contorsion tensor is defined as

$$K^\mu_\nu = -\frac{1}{2} (T^\mu_\nu - T^\nu_\mu - T^\nu_\rho) \quad (2.5)$$

and the torsion tensor looks like

$$T^\lambda_\mu_\nu = \Gamma w^\lambda_\nu_\mu - \Gamma w^\lambda_\mu_\nu = e^\lambda_i \left( \partial_\mu e^i_\nu - \partial_\nu e^i_\mu \right). \quad (2.6)$$

The vierbein vector fields are related with the metric through

$$g^\mu_\nu(x) = \eta^i_j e^i_\mu(x) e^j_\nu(x), \quad (2.7)$$

where $e_i \cdot e_j = \eta_{ij}$ and $\eta_{ij} = diag(1, -1, -1, -1)$. We will assume a flat homogeneous and isotropic Friedmann–Robertson–Walker (FRW) universe with the metric

$$ds^2 = -dt^2 + a(t)^2 \sum_{i=1}^3 (dx^i)^2, \quad (2.8)$$

where $t$ is the cosmic time. Then the modified Friedmann equations and the continuity equation read as (see, e.g. [16, 17, 28])

$$-2T F_T + F = 2k^2 \rho_m, \quad (2.9)$$

$$-8 \dot{H} T F_{TT} + (2T - 4\dot{H}) F_T - F = 2k^2 \rho_m, \quad (2.10)$$

$$\dot{\rho}_m + 3H(\rho_m + p_m) = 0. \quad (2.11)$$
This set of equations can be rewritten as

\[-T - 2T f_T + f = 2k^2 \rho_m, \tag{2.12}\]
\[-8\dot{H}T f_{TT} + (2T - 4\dot{H})(1 + f_T) - T - f = 2k^2 p_m, \tag{2.13}\]
\[\dot{\rho}_m + 3H(\rho_m + p_m) = 0, \tag{2.14}\]

with the action

\[S = \int d^4x e^{\left[\frac{1}{2k^2}(T + f(T)) + L_m\right]}, \tag{2.15}\]

where \(f = F - T\). Some properties of \(F(T)\) gravity were studied in [17–42]. Note that we can rewrite the gravitational Eqs. (2.9) and (2.10) as

\[\hat{M}_1 F = 2k^2 \rho_m, \tag{2.16}\]
\[\hat{M}_2 F = -\hat{M}_3 \hat{M}_1 F = 2k^2 p_m, \tag{2.17}\]
\[\hat{M}_3 \rho_m = -p_m, \tag{2.18}\]

where

\[\hat{M}_1 = -2T \partial_T + 1, \tag{2.19}\]
\[\hat{M}_2 = -8\dot{H}T \partial_T^2 + (2T - 4\dot{H})\partial_T - 1 = (4\dot{H}\partial_T - 1)\hat{M}_1 = -\hat{M}_3 \hat{M}_1, \tag{2.20}\]
\[\hat{M}_3 = \frac{1}{3H} \partial_t + 1. \tag{2.21}\]

Using these basic equations we can construct a hierarchy of \(F(T)\) gravity. For the case \(\rho_m = p_m = 0\) such hierarchy can be written as

\[\hat{M}_1^n F_n = 0, \tag{2.22}\]

where \(F_1 = F\). Some equations from this hierarchy for \(n = 1, 2, 3, \ldots\) are

\[-2T F_{1T} + F_1 = 0, \tag{2.23}\]
\[4T^2 F_{2TT} + F_2 = 0, \tag{2.24}\]
\[-8T^3 F_{3TTT} - 12T^2 F_{3TT} - 2T F_{3T} + F_3 = 0, \tag{2.25}\]

and so on. From the system (2.16)–(2.18) follows that any solution of the Eq. (2.16) automatically solves the Eqs. (2.17) and (2.18). It means that we need only to solve the Eq. (2.16), since it guarantees a solution to the Eqs. (2.17) and (2.18). Finally we present the effective EoS parameter

\[w_{\text{eff}} = -1 - 3^{-1}H^{-1}[\ln (\hat{M}_1 F)]_t = -1 - 3^{-1}[\ln (\hat{M}_1 F)]_N. \tag{2.26}\]
2.2 Particular models of $F(T)$ gravity

We note that some explicit models of $F(T)$ gravity appeared in the literature (see, e.g. [16,17,28,29,36,37,40]). Here we would like to present some new models of modified teleparallel gravity.

2.2.1 Example 1: The $M_{13}$-model

Let us consider the $M_{13}$-model. Its Lagrangian is

$$F(T) = \sum_{j=-m}^{n} v_j(t)T^j = v_{-m}(t)T^{-m} + \cdots + v_{-1}(t)T^{-1} + v_0(t) + v_1(t)T + \cdots + v_n(t)T^n. \quad (2.27)$$

Consider the particular example when $m = n = 1$ and $v_j = \text{consts.}$ Then

$$F = v_{-1}T^{-1} + v_0 + v_1T, \quad F_T = -v_{-1}T^{-2} + v_1, \quad F_{TT} = 2v_{-1}T^{-3}. \quad (2.28)$$

Substituting these expressions into (2.9) and (2.10) we obtain

$$3k^{-2}H^2 = \rho_{\text{eff}} + \rho_m, \quad (2.29)$$
$$-k^{-2}(2\dot{H} + 3H^2) = p_{\text{eff}} + p_m, \quad (2.30)$$

where

$$\rho_{\text{eff}} = k^{-2}[3H^2 - 1.5v_{-1}T^{-1} + 0.5v_1T - 0.5v_0], \quad (2.31)$$
$$p_{\text{eff}} = k^{-2}[6v_{-1}\dot{H}T^{-2} + 1.5v_{-1}T^{-1} - 0.5v_1T + 0.5v_0 + 2(v_1 - 1)\dot{H} - 3H^2]. \quad (2.32)$$

The effective EoS parameter is given by

$$w_{\text{eff}} = \frac{p_{\text{eff}}}{\rho_{\text{eff}}} = \frac{6v_{-1}\dot{H}T^{-2} + 1.5v_{-1}T^{-1} - 0.5v_1T + 0.5v_0 + 2(v_1 - 1)\dot{H} - 3H^2}{3H^2 - 1.5v_{-1}T^{-1} + 0.5v_1T - 0.5v_0}. \quad (2.33)$$

Let us set $v_1 = 1.$ Then

$$\rho_{\text{eff}} = k^{-2}[-1.5v_{-1}T^{-1} - 0.5v_0], \quad p_{\text{eff}} = k^{-2}[6v_{-1}\dot{H}T^{-2} + 1.5v_{-1}T^{-1} + 0.5v_0] \quad (2.34)$$
and

\[
 w_{\text{eff}} = \frac{p_{\text{eff}}}{\rho_{\text{eff}}} = \frac{6\nu_{-1}\dot{H}T^{-2} + 1.5\nu_{-1}T^{-1} + 0.5v_0}{-1.5\nu_{-1}T^{-1} - 0.5v_0} \\
= -1 - \frac{6\nu_{-1}\dot{H}T^{-2}}{1.5\nu_{-1}T^{-1} + 0.5v_0},
\]

(2.35)

respectively.

2.2.2 Example 2: The $M_{21}$-model

Our next example is the $M_{21}$-model

\[
 F = T + \alpha T^\delta \ln T.
\]

(2.36)

Then

\[
 F_T = 1 + \alpha \delta T^{\delta - 1} \ln T + \alpha T^{\delta - 1}, \quad F_{TT} = \alpha \delta(\delta - 1)T^{\delta - 2} \ln T + \alpha(2\delta - 1)T^{\delta - 2}.
\]

(2.37)

In this case, Eqs. (2.9) and (2.10) take the form

\[
 -T - 2\alpha T^\delta - \alpha(2\delta - 1)T^\delta \ln T = 2k^2\rho_m,
\]

(2.38)

\[
 \alpha(2\delta - 1)(T - 4\delta \dot{H})T^{\delta - 1} \ln T + T \\
-4\dot{H} + 2\alpha T^\delta - 4\alpha \dot{H}(4\delta - 1)T^{\delta - 1} = 2k^2p_m.
\]

(2.39)

So we have

\[
 \rho_{\text{eff}} = 0.5k^{-2}[2\alpha T^\delta + \alpha(2\delta - 1)T^\delta \ln T],
\]

(2.40)

\[
 p_{\text{eff}} = -0.5k^{-2}\alpha T^{\delta - 1}[(2\delta - 1)(T - 4\delta \dot{H}) \ln T + 2T - 4(4\delta - 1)\dot{H}].
\]

(2.41)

The special case $\delta = 0.5$ deserves separate consideration. In this case the above equations take a simpler form as

\[
 -T - 2\alpha T^{0.5} = 2k^2\rho_m, \quad T - 4\dot{H} + 2\alpha T^{0.5} - 4\alpha \dot{H}T^{-0.5} = 2k^2p_m.
\]

(2.42)

For the density of energy and pressure we get the following expressions

\[
 \rho_{\text{eff}} = k^{-2}\alpha T^{0.5}, \quad p_{\text{eff}} = -k^{-2}\alpha T^{-0.5}(T - 2\dot{H}).
\]

(2.43)
2.2.3 Example 3: The $M_{22}$-model

Now we consider the $M_{22}$-model

$$F = T + f(y), \quad y = \tanh[T]. \quad (2.44)$$

Then

$$F_T = 1 + f_y(1 - y^2), \quad F_{TT} = f_{yy}(1 - y^2)^2 - 2y(1 - y^2)f_y \quad (2.45)$$

so that Eqs. (2.9) and (2.10) take the form

$$-T - 2(1 - y^2)Tf_y + f = 2k^2 \rho_m, \quad (2.46)$$
$$T - 4\dot{H} - 8(1 - y^2)^2 T\dot{T}f_{yy} + (16\dot{y}^2T + 2T - 4\dot{H})(1 - y^2)f_y - f = 2k^2 \rho_m. \quad (2.47)$$

As a result we have

$$\rho_{\text{eff}} = 0.5k^{-2}[2(1 - y^2)Tf_y - f], \quad (2.48)$$
$$p_{\text{eff}} = 0.5k^{-2}[8(1 - y^2)^2 T\dot{H}f_{yy} - (16\dot{y}^2T + 2T - 4\dot{H})(1 - y^2)f_y + f]. \quad (2.49)$$

The corresponding EoS parameter takes the form

$$w_{\text{eff}} = \frac{8(1 - y^2)^2 T\dot{H}f_{yy} - (16\dot{y}^2T + 2T - 4\dot{H})(1 - y^2)f_y + f}{2(1 - y^2)Tf_y - f} \quad = -1 - \frac{8(1 - y^2)^2 T\dot{H}f_{yy} - (16\dot{y}^2T - 4\dot{H})(1 - y^2)f_y + f}{2(1 - y^2)Tf_y - f}. \quad (2.50)$$

2.2.4 Example 4: The $M_{25}$-model

In this part, we consider the $M_{25}$-model

$$F = \sum_{-m}^{n} v_j(t)\xi^j, \quad (2.51)$$

where $\xi = \ln T$. As an example we consider the case $m = n = 1$, $v_j = \text{const}$ that is

$$F = v_{-1}\xi^{-1} + v_0 + v_1\xi. \quad (2.52)$$

Then

$$F_{\xi} = -v_{-1}\xi^{-2} + v_1, \quad F_{\xi\xi} = 2v_{-1}\xi^{-3} \quad (2.53)$$
and

\[ F_T = (-v_{-1}\xi^{-2} + v_1)e^{-\xi}, \quad F_{TT} = (2v_{-1}\xi^{-3} + v_{-1}\xi^{-2} - v_1)e^{-2\xi}. \]  \tag{2.54}

For this case, Eqs. (2.9) and (2.10) read as

\[ 2v_{-1}\xi^{-2} + v_{-1}\xi^{-1} + v_0 - 2v_1 + v_1\xi = 2k^2\rho_m, \]  \tag{2.55}

\[ -4\dot{H}(4v_{-1}\xi^{-3} + v_{-1}\xi^{-2} - v_1)e^{-\xi} \\
-2v_{-1}\xi^{-2} - v_{-1}\xi^{-1} + 2v_1 - v_0 - v_1\xi = 2k^2\rho_m. \]  \tag{2.56}

3 k-essence

3.1 Elements of k-essence

The action of k-essence has the form [44–46]

\[ S = \int d^4x \sqrt{-g} \left[ \frac{1}{2\kappa^2} R + K(X, \phi) + L_m \right]. \]  \tag{3.1}

The corresponding closed set of equations for the FRW metric (2.8) reads as

\[ 3k^{-2}H^2 = 2XK_X - K + \rho_m, \]  \tag{3.2}

\[ -k^{-2}(2\dot{H} + 3H^2) = K + p_m, \]  \tag{3.3}

\[ (K_X + 2XK_X)\dot{X} + 6HXK_X - K\phi = 0, \]  \tag{3.4}

\[ \dot{\rho}_m + 3H(\rho_m + p_m) = 0, \]  \tag{3.5}

where \( X = -0.5\dot{\phi}^2 \). The equation of motion of the scalar field \( \phi \) is given as

\[ -(a^3\phi K_X)_t = a^3K\dot{\phi}, \]  \tag{3.6}

which is just another form of the Eq. (3.4). In the purely kinetic k-essence case we have \( K\phi = 0 \) and from the last equation we get (see, e.g. [47])

\[ a^3\phi K_X = a^3\sqrt{-2XK_X} = \sqrt{\kappa} = \text{const}. \]  \tag{3.7}

3.2 Particular models of k-essence

As examples, in this subsection we would like to present some new types of k-essence. We believe that all of them can give rise to cosmic acceleration.
3.2.1 Example 1: The $M_{12}$-model

Let us consider the $M_{12}$-model with the following Lagrangian

$$K = v_m(N)N^{-m} + \cdots + v_1(N)N^{-1} + v_0(N) + v_1(N)N + \cdots + v_n(N)N^n,$$

(3.8)

where in general $v_j = v_j(\phi) = v_j(N)$ and $N = \ln(a_0^{-1})$. As an example, we study the case $m = 0, n = 2, v_j = \text{const}$. In this case, the $M_{12}$-model becomes

$$K = v_0 + v_1 N + v_2 N^2.$$  

(3.9)

To find $v_j$ and $X$, we look for $H$ for instance as

$$H = \mu_0 + \mu_1 N,$$

(3.10)

where $\mu_j = \text{consts}$ [in general $\mu_j = \mu_j(t)$]. Of course

$$a = a_0 e^N.$$  

(3.11)

Finally, we obtain the following parametric form of the $M_{12}$-model (parametric purely kinetic k-essence)

$$K = -(2\mu_0 \mu_1 + 3\mu_0^2) - 2\mu_1 (\mu_1 + 3\mu_0)N - 3\mu_1^2 N^2,$$

(3.12)

$$X = k^{-1} a_0^6 \mu_1^2 (\mu_0 + \mu_1 N)^2 e^{6N}.$$  

(3.13)

3.2.2 Example 2: The $M_1$-model

Our next example is the $M_1$-model. Its Lagrangian looks like

$$K = v_m(t)t^{-m} + \cdots + v_1(t)t^{-1} + v_0(t) + v_1(t)t + \cdots + v_n(t)t^n,$$

(3.14)

where in general $v_j = v_j(\phi) = v_j(t)$. Let us explore this model for the case: $m = 0, n = 2$ and $v_j = \text{consts}$. In this case the $M_1$-model takes the form

$$K = v_0 + v_1 t + v_2 t^2.$$  

(3.15)

To find $v_j$ and $X$ we look for $H$, e.g. as

$$H = \mu_0 + \mu_1 t$$

(3.16)

so that

$$a = a_0 e^{\mu_0 t + 0.5 \mu_1 t^2},$$

(3.17)
where $\mu_j = \text{consts}$ [in general $\mu_j = \mu_j(t)$]. After some calculations we obtain the following explicit form of the k-essence Lagrangian

$$K = -(2\mu_1 + 3\mu_0^2) - 6\mu_0\mu_1 t - 3\mu_1^2 t^2. \quad (3.18)$$

At the same time, we have

$$2XK_X = 3H^2 + K = -2\dot{H} = -2\mu_1. \quad (3.19)$$

For $X$ we get the following expression

$$X = \gamma_2^{-1} e^{6\mu_0 t + 3\mu_1 t^2}, \quad \gamma_2^{-1} = \kappa^{-1} a_0^6 \mu_1^2. \quad (3.20)$$

Hence it follows that

$$t = \frac{1}{3\mu_1} \left[ -3\mu_0 \pm \sqrt{9\mu_0^2 + 3\mu_1 \ln(\gamma_2 X)} \right]. \quad (3.21)$$

Finally, we come to the following M$_{23}$-model

$$K = -2\mu_1 - 3\mu_0^2 - \mu_1 \ln[\gamma_2 X] = v_0 + v_1 \ln X. \quad (3.22)$$

[We recall that in general the M$_{23}$-model reads as

$$K = v_m(t)\xi^{-m} + \cdots + v_{-1}(t)\xi^{-1} + v_0(t) + v_1(t)\xi + \cdots + v_n(t)\xi^n, \quad (3.23)$$

where $\xi = \ln X.$]

### 3.2.3 Example 3: The M$_{24}$-model

Here we present the following M$_{24}$-model

$$K = \frac{2m\lambda\sigma^2(-2\beta v + \lambda v^2 + \lambda)(1 - v^2)}{(\beta - \lambda v)^2} - 3 \left[ n - \frac{m\lambda\sigma(1 - v^2)}{\beta - \lambda v} \right]^2, \quad (3.24)$$

$$X = \gamma_3(2\beta v - \lambda v^2 - \lambda)^2(1 - v^2)^2(\beta - \lambda v)^{6m-4}, \quad (3.25)$$

where $\gamma_3 = \kappa^{-1} a^6m^2\lambda^2\sigma^6$, $v = \tanh[\sigma t]$ and $\lambda, \sigma, \alpha, \beta, n, m$ are some constants.

Solving the Eq. (3.3) we obtain

$$H = n - \frac{m\lambda\sigma(1 - v^2)}{\beta - \lambda v} \quad (3.26)$$

and hence for the scale factor we get the following formula

$$a = \alpha[\beta - \lambda v]^m e^{nt}. \quad (3.27)$$
Note that
\[ \dot{H} = \frac{m\lambda \sigma^2 (2\beta v - \lambda v^2 - \lambda)(1 - v^2)}{(\beta - \lambda v)^2}. \] (3.28)

4 Equivalence between \(F(T)\) gravity and k-essence

In this section, our goal is to study the relation between modified teleparallel gravity and purely kinetic k-essence. In Appendix C, we will consider this relation in the context with the other modified gravity theories.

4.1 General case

4.1.1 Variant-I

Consider the transformation
\[
K = 8\dot{H}T f'' - 2(T - 2\dot{H}) f' + f, \tag{4.1}
\]
\[
X = \kappa^{-1} k^{-4} a^6 \dot{H} + 0.5k^2 (\rho_m + p_m))^2, \tag{4.2}
\]
where \(T = -6H^2\). Then Eqs. (2.12)–(2.14) take the form
\[
0 = -3k^{-2} H^2 + 2XK_X - K + \rho_m, \tag{4.3}
\]
\[
0 = k^{-2} (2\dot{H} + 3H^2) + K + p_m, \tag{4.4}
\]
\[
(K_X + 2XK_{XX})X + 6XK_X = 0, \tag{4.5}
\]
\[
\dot{\rho}_m + 3H (\rho_m + p_m) = 0. \tag{4.6}
\]

These are the equations of motion of purely kinetic k-essence. This result shows that modified teleparallel gravity and purely kinetic k-essence is equivalent to each other, at least in the equation’s level. This equivalence allows us to construct a new class of purely kinetic k-essence models starting from some models of modified teleparallel gravity. Let us demonstrate it for the following modified teleparallel gravity model: \(f(T) = \alpha T^n\) [16, 17]. In this case, we have
\[
f_T = \alpha n T^{n-1}, \quad f'' = \alpha n(n - 1) T^{n-2}. \tag{4.7}
\]

Substituting these expressions into the Eqs. (4.1) and (4.2) we get
\[
K = 8\alpha n T^{n-1} - 2\alpha n(T - 2\dot{H}) T^{n-1} + \alpha T^n, \tag{4.8}
\]
\[
X = \kappa^{-1} k^{-4} a^6 \dot{H} + 0.5k^2 (\rho_m + p_m))^2. \tag{4.9}
\]
(i) Let \( a = a_0 e^{g(t)} \) so that \( H = \dot{g}, \dot{H} = g \). In this case, \( K \) and \( X \) take the form

\[
K = 8\alpha n(n-1)\dot{g}(-6)^{n-1}\dot{g}^2(n-1) - 2\alpha n(-6\dot{g} - 2\ddot{g})(-6)^{n-1}\dot{g}^2(n-1) + \alpha(-6)^n\dot{g}^2.
\]
\[
X = \kappa^{-1}k^{-4}a^6\dot{g}^2.
\]

(4.10)

(4.11)

Now if we consider the simplest case \( g = t \) (that means \( \dot{g} = 1, \ddot{g} = 0 \)), then we get

\[
K = -2\alpha n(-6)^n + \alpha(-6)^n = (1 - 2n)\alpha(-6)^n,
\]
\[
X = 0.
\]

(4.12)

(4.13)

(ii) The more non-trivial model we get, if we consider the example \( a = a_0 t^m \). In this case \( H = mt^{-1}, \dot{H} = -mt^{-2}, T = -6m^2 t^{-2} \) so that \( K \) and \( X \) take the form

\[
K = 8\alpha n(n-1)\dot{H}\left(\frac{-6m^2}{t^2}\right)^{n-1} - 2\alpha n\left(\frac{-6m^2}{t^2} - 2\dot{H}\right)\left(\frac{-6m^2}{t^2}\right)^{n-1} + \alpha\left(\frac{-6m^2}{t^2}\right)^n,
\]
\[
X = \kappa^{-1}k^{-4}a_0^6m^{-2}t^{6m-4}
\]

(4.14)

(4.15)

or

\[
K = 2\alpha m(-6m^2)^{n-1}[-4n(n-1) + 2n(1 - 3m) + 3m]t^{-2n},
\]
\[
X = \kappa^{-1}k^{-4}a_0^6m^2t^{6m-4} = \gamma_5^{-1}t^{6m-4}.
\]

(4.16)

(4.17)

Since \( t = (\gamma_5 X)^{1/6m-4} \) finally we get the following purely kinetic k-essence model

\[
K = 2\alpha m(-6m^2)^{n-1}[-4n(n-1) + 2n(1 - 3m) + 3m](\gamma_5 X)^n_{6m-3m}.
\]

(4.18)

4.1.2 Variant-II

Let us rewrite Eqs. (2.12)–(2.14) as

\[
3k^{-2}H^2 = \rho_{\text{eff}} + \rho_m,
\]
\[
-\kappa^{-2}(2\dot{H} + 3H^2) = p_{\text{eff}} + p_m,
\]
\[
\dot{\rho}_m + 3H(\rho_m + p_m) = 0,
\]

(4.19)

(4.20)

(4.21)

where

\[
\rho_{\text{eff}} = 2Tf_T - f, \quad p_{\text{eff}} = 8\dot{H}f_{TT} - 2(T - 2\dot{H})f_T + f.
\]

(4.22)
We now introduce two functions $K$ and $X$:

\[
K = 8\dot{H}T f_{TT} - 2(T - 2\dot{H})f_T + f, \quad X = \frac{4\dot{H}^2(2T f_{TT} + f_T)^2}{\kappa a^{-6}}. \tag{4.23}
\]

Clearly these two functions $K$ and $X$ obey the system of Eqs. (4.3)–(4.6).

### 4.2 Special case: $\phi = \phi_0 + \ln a^{\pm \sqrt{12}}$

One of the interesting special cases is when:

\[
\phi = \phi_0 + \ln a^{\pm \sqrt{12}}. \tag{4.24}
\]

It deserves separate investigation. In fact for this case $\dot{\phi} = \pm \sqrt{12}H$ so that $X = -0.5\dot{\phi}^2 = -6H^2 = T$. Then the corresponding continuity equation is

\[
\ddot{\phi}(f_T - \dot{\phi}^2 f_{TT}) + 3H\dot{\phi} f_T = 0 \tag{4.25}
\]

or equivalently, in terms of $T$,

\[
(f_T + 2T f_{TT})\dot{T} + 6HT f_T = 0, \tag{4.26}
\]

where $\rho' = 2T f_T - f$, $p' = f$ and $\dot{\rho}' + 3H(\rho' + p') = 0$. Now let us split the Eq. (2.13) into two equations as

\[
4\dot{H}T f_{TT} - (T - 2\dot{H})f_T = 0 \tag{4.27}
\]

and

\[
-4\dot{H} + T - f = 2k^2 p_m. \tag{4.28}
\]

Equation (4.27) is satisfied automatically since it is just another form of the continuity Eq. (4.26). So finally the system of equations of $F(T)$ gravity takes the form

\[
-T - 2T f_T + f = 2k^2 \rho_m, \tag{4.29}
\]

\[
-4\dot{H} + T - f = 2k^2 p_m, \tag{4.30}
\]

\[
(f_T + 2T f_{TT})\dot{T} + 6HT f_T = 0, \tag{4.31}
\]

\[
\dot{\rho}_m + 3H(\rho_m + p_m) = 0. \tag{4.32}
\]

It transforms to Eqs. (4.3)–(4.6) after the identifications $T = X = -6H^2$ and $f = 2k^2 K$. So we can conclude that for the special case (4.24) both $F(T)$ gravity and purely kinetic k-essence are equivalent to each other at least at the equation’s level.
Some comments on the continuity Eq. (4.25) [= (4.26) = (4.27)]. It has two integrals of motion ($I_{jT} = 0$):

\[
I_1 = a_0^{-3} a^3 T^{0.5} f_T, \quad I_2 = f - a^3 T^{0.5} f_T \partial_T^{-1} (a^{-3} T^{-0.5}).
\]

(4.33)

Its general solution is given by

\[
f = C_2 + i C_1 a_0^{2} a_T^{-1} (a^{-3} T^{-0.5}), \quad C_j = \text{const.}
\]

(4.34)

Finally we would like to present the exact solution both $F(T)$ gravity and purely kinetic k-essence. As an example let us consider $\Lambda$CDM for which $a^{-3} = -\frac{1}{2\rho_0} (T + 2\Lambda) = -\frac{1}{2\rho_0} (X + 2\Lambda)$ so that

\[
f = f(X) = f(T) = C_2 - \frac{i C_1 a_0^{3}}{3\rho_0} (T^{1.5} + 6\Lambda T^{0.5})
\]

\[
= C_2 - \frac{i C_1 a_0^{3}}{3\rho_0} (X^{1.5} + 6\Lambda X^{0.5}),
\]

(4.35)

which is the M$_{32}$-model. It is the exact solution of the equations of motion of purely kinetic k-essence and $F(T)$ gravity simultaneously.

5 Conclusion

In this work we investigated the recently developed $F(T)$ gravity, which is a new modified gravity model capable of accounting for the present cosmic accelerating expansion with no need of DE. $F(T)$ gravity as the modified teleparallel gravity is the extension of the TEGR, which uses the zero curvature Weitzenböck connection instead of the torsionless Levi-Civita connection, in the same lines as $F(R)$ gravity is the extension of standard General Relativity. In particular, we presented some new models of $F(T)$ gravity. We analyze the relation between $F(T)$ gravity and k-essence. We also studied some new models of k-essence namely some parametric models of purely kinetic k-essence.

It is important to note that $F(T)$ gravity may be consistent with the observational data. For instance, the cosmographic parameters of $F(T)$ gravity found in [43] may lie in the region overlapping with those for LCDM. $F(T)$ EoS [30] or its cosmological perturbations [34,35] show very realistic behavior. $F(T)$ gravity may show LCDM-like, phantom-like or quintessence-like behavior at DE epoch as is shown in our work. Thus, $F(T)$ gravity may pretend to be viable and observationally friendly DE candidate. New precise observations may select most realistic candidate from existing number of DE models (for recent review, see [49]). Perhaps, cosmography may present such obserbational bounds.

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6 Appendix A: Multiple k-essence

For the multiple k-essence the action reads as

\[ S = \int d^4x \sqrt{-g} \left[ \frac{1}{2k^2} R + K(X_1, X_2, \ldots, X_n, \phi_1, \phi_2, \ldots, \phi_n) + L_m \right]. \]  

(6.1)

The corresponding closed set of equations reads as

\[ 3k^{-2}H^2 = 2 \sum_{j=1}^{n} X_j KX_j - K + \rho_m, \]  

(6.2)

\[ -k^{-2}(2\dot{H} + 3H^2) = K + p_m, \]  

(6.3)

\[ (KX_j + 2X_j KX_j)\ddot{X}_j + 6HX_j KX_j - K\phi_j = 0, \]  

(6.4)

\[ \dot{\rho}_m + 3H(\rho_m + p_m) = 0, \]  

(6.5)

where \( X_j = -0.5\phi_j^2 \). The k-essence energy density and pressure respectively, given by

\[ \rho_j = 2X_j KX_j - K, \quad p_j = w_j \rho_j. \]  

(6.6)

The equations of motion of the scalar fields \( \phi_j \) are given as

\[ -(a^3\dot{\phi}_j KX_j)_t = a^3K\phi_j. \]  

(6.7)

Note that the case \( K = \sum_{j=1}^{n} K_j (X_j, \phi_j) \) (mutually non-interacting scalar fields) was investigated in [48].

7 Appendix B: Some models of modified gravity theories and k-essence

In this appendix, we present some models of modified gravity theories and k-essence

\[ [N = \ln a, \quad R = 6(\dot{H} + 2H^2), \quad G = 24H^2(\dot{H} + H^2), \quad T = -6H^2, \quad \eta = \int a^{-1}dt = \int H^{-1}a^{-2}da, \quad \xi = \ln T, \quad \zeta = \ln X, \quad \zeta = \ln R, \quad \vartheta = \ln G]. \]  

In general the Lagrangian \( L_n \) of the \( M_n \)-model we write in the form

\[ L_n = F = K = \sum_j v_j(t)S_n^j, \]  

(7.1)

where \( S_n \) is some function of \( R, G, T, t, \eta, \xi, \zeta, \zeta, \vartheta \) and so on. For example, for the \( M_1 \)-model \( S_1 \) has the form \( S_1 = t \) that is \( L_1 = \sum_{j=-m}^{n} v_j(t)t^j \). Similarly, for the \( M_2 \)-model \( S_2 \) has the form \( S_2 = e^t \) so that its Lagrangian reads as \( L_2 = \sum_{j=-m}^{n} v_j(t)e^{jt} \).

Now let us we present the expressions for \( S_n \) of the \( M_n \)-model. We have:

\( S_4 = \tanh[t]; \quad S_5 = \cosh[t]; \quad S_6 = \tan[t]; \quad S_7 = \cos[t]; \quad S_8 = \cosh[t]; \quad S_9 = te^t; \quad S_{10} = H; \quad S_{11} = a; \quad S_{12} = N; \quad S_{13} = T; \quad S_{14} = G; \quad S_{15} = R; \quad S_{19} = e^N; \quad S_{20} = \eta; \)
$S_{26} = \zeta; S_{27} = \vartheta; S_{28} = \ln[\eta]; S_{29} = \tanh[\eta]; S_{30} = \ln[t]; S_{31} = \cosh[R]; S_{32} = X$ and so on.

8 Appendix C: Modified gravity theories as the particular reductions of purely kinetic k-essence

In this appendix, we show that some important modified gravity theories, namely, $F(G)$, $F(R)$ and $F(T)$ can written as particular reductions of purely kinetic k-essence.

8.1 $F(G)$ gravity

8.1.1 Variant-I

Let us consider following transformation (see, e.g. [5–13])

$$K = 8H^2\dddot{f}_G + 16H(\dot{H} + H^2)\dddot{f}_G + f - Gf_G,$$

$$X = k^{-1}k^{-4}a^6[\dot{H} + 0.5k^2(\rho_m + p_m)]^2.$$  (8.2)

Substituting these expressions e.g. into Eqs. (4.3)–(4.6) we get

$$0 = -3k^{-2}H^2 + Gf_G - f - 24\dot{H}H^3f_{GG} + \rho_m,$$  (8.3)

$$0 = 8H^2\dddot{f}_G + 16H(\dot{H} + H^2)\dddot{f}_G + k^{-2}(2\dot{H} + 3H^2) + f - Gf_G + p_m,$$  (8.4)

$$\dot{\rho}_m + 3H(\rho_m + p_m) = 0,$$  (8.5)

where

$$G = 24H^2(\dot{H} + H^2).$$  (8.6)

It is the system of equations of motion of $F(G)$ gravity with the action

$$S = \int d^4x\sqrt{-g}\left[\frac{1}{2k^2}R + f(G) + L_m\right].$$  (8.7)

Now let us consider the particular case when $K = f, \ X = G$. Then instead of Eqs. (8.3)–(8.5) we obtain the following system

$$0 = -3k^{-2}H^2 + 2Gf_G - f + \rho_m,$$  (8.8)

$$0 = k^{-2}(2\dot{H} + 3H^2) + f + p_m,$$  (8.9)

$$(f_G + 2Gf_{GG})\dot{G} + 6HGf_G = 0,$$  (8.10)

$$\dot{\rho}_m + 3H(\rho_m + p_m) = 0,$$  (8.11)

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and

\[ Gf_G + 24\dot{G}H^3 f_{GG} = 0, \] (8.12)
\[ 0 = 8H^2 \dot{f}_G + 16H(\dot{H} + H^2)\dot{f}_G - Gf_G, \] (8.13)
\[ \kappa^{-1}k^{-4}a^6[\dot{H} + 0.5k^2(\rho_m + p_m)]^2 = 24H^2(\dot{H} + H^2). \] (8.14)

Let’s make two steps back that is let’s simplify a problem: (1) we want to reduce the problem to the case \( \rho_m = p_m = 0; \) (2) we want illustrate our results on the pedagogical example: \( f(G) = \alpha G^n. \) As result, the \( k \)-fields take the more simple form

\[ K = \alpha(n - 1)G^{n-3}[8nH^2((n - 2)\dot{G} + \ddot{G}), \] (8.15)
\[ X = \kappa^{-1}k^{-4}a^6\dot{H}^2, \] (8.16)
\[ \phi = \pm i\sqrt{2}\kappa^{-1}k^{-2}d^{-1}(a^3\dot{H}). \] (8.17)

Here we want to construct two examples of induced purely kinetic \( k \)-essence models: one is in the standard “canonical” form that means in the form \( K = K(X) \) and another in the parametric form that means in the form \( K = K(t), X = X(t) \) (\( t \) plays the role of a parameter).

(i) Let \( a = \beta t^n. \) Then the corresponding purely kinetic \( k \)-essence reads as

\[ K = K(X) \]
\[ = 16an^9(n - 1)^3 \left[ (113 - 33n)n^3(n - 1) \left( \frac{X}{\gamma} \right)^{\frac{8}{3n}} - 8(n - 2) \left( \frac{X}{\gamma} \right)^{\frac{11}{3n-6n}} \right]. \] (8.18)

Such model we call the “canonical” \( k \)-essence model. Note that for this case

\[ X = \gamma t^{6n-4}, \quad \phi = \phi_0 + \frac{i\sqrt{2} \gamma}{3n - 1} t^{3n-1}, \quad \gamma = \kappa^{-1}k^{-4}\beta^n n^2. \] (8.19)

(ii) Now we want to present the parametric \( k \)-essence model. To do it, let us consider an example: \( H = \lambda t^m. \) In this case the purely kinetic \( k \)-essence equivalent counterpart of the corresponding \( F(G) \)-model is given by

\[ K = K(t) \]
\[ = 24^{n-3}\alpha\lambda^{3(n-3)}(n - 1)t^{(3m-1)(n-3)} \left[ m + \lambda t^{m+1} \right]^{n-3} [K_1 + K_2], \] (8.20)
\[ X = X(t) = \kappa^{-1}k^{-4}a_0^6\lambda^2 t^{2(m-1)} e^{\frac{6\lambda t^{m+1}}{m+1}} \] (8.21)
where

\[ K_1 = 192n^2m(n - 2)\lambda^5t^{5m-2}\left[3m - 1 + 4\lambda t^{m+1}\right] \]
\[ + 4608n^2m^2\lambda^8t^{8m-4}\left[m + \lambda t^{m+1}\right]\left[(3m-1)(3m-2)+4(4m-1)\lambda t^{m+1}\right]. \]  
(8.22)

\[ K_2 = 9216nm^2\lambda^8t^{8m-4}\left[m + t\right]\left[3m - 1 + 4\lambda t^{m+1}\right] \]
\[ - 13824\lambda^9t^{9m-3}\left[m + \lambda t^{m+1}\right]^3. \]  
(8.23)

To such a model we call the parametric k-essence model.

8.1.2 Variant-II

We now introduce two functions \( K \) and \( X \) as

\[ K = 8H^2 \dot{f}_G + 16H \dot{H} (\dot{H} + H^2) \dot{f}_G + f - Gf_G, \]
\[ X = \frac{8H^2 \dot{f}_G + 16H \dot{H} (\dot{H} + H^2) \dot{f}_G - 24\dot{G}H^3 f_{GG}}{4\kappa a^{-6}}. \]  
(8.24)

where \( f(G) \) obeys the system (8.3)–(8.5). Then these functions solve the system of the equations of motion of purely kinetic k-essence (4.3)–(4.6).

8.2 \( F(R) \) gravity

8.2.1 Variant-I

In this subsection we consider the following transformation (see, e.g. [5–13])

\[ K = 2[\ddot{f}_R + 2H \dot{f}_R + 0.5f - (\dot{H} + 3H^2)f_R], \]  
(8.25)

\[ X = \kappa^{-1}k^{-4}a^6[\dot{H} + 0.5k^2(\rho_m + p_m)]^2, \]  
(8.26)

where

\[ R = 6(\dot{H} + 2H^2). \]  
(8.27)

The substitution (8.25) and (8.26) into Eqs. (4.3)–(4.6) gives

\[ 0 = -3k^{-2}H^2 - 6H \ddot{H} f_{RR} + 6(\dot{H} + H^2)f_R - f + \rho_m, \]  
(8.28)

\[ 0 = 2[\ddot{f}_R + 2H \dot{f}_R + 0.5f - (\dot{H} + 3H^2)f_R] + k^{-2}(2\dot{H} + 3H^2) + p_m, \]  
(8.29)

\[ \dot{\rho}_m + 3H(\rho_m + p_m) = 0, \]  
(8.30)
which are equations of $F(R)$ gravity. The corresponding action is

$$S = \int d^4x \sqrt{-g} \left[ \frac{1}{2k^2} R + f(R) + L_m \right]. \quad (8.31)$$

Now let us consider the particular case when $K = f$, $X = R$. The corresponding continuity equation is $(f_R + 2Rf_{RR}) \dot{R} + 6HRf_R = 0$. Then instead Eqs. (8.28)–(8.30) we obtain the system

$$0 = -3k^{-2} H^2 + 2Rf_R - f + \rho_m, \quad (8.32)$$
$$0 = k^{-2}(2\dot{H} + 3H^2) + f + p_m, \quad (8.33)$$
$$(f_R + 2Rf_{RR}) \dot{R} + 6HRf_R = 0, \quad (8.34)$$
$$\rho_m' + 3H(\rho_m + p_m) = 0 \quad (8.35)$$

and

$$0 = -6H \dot{R} f_{RR} + 6(\dot{H} + H^2) f_R - 2Rf_R, \quad (8.36)$$
$$0 = \ddot{f}_R + 2H \dot{f}_R - (\dot{H} + 3H^2)f_R, \quad (8.37)$$
$$\kappa^{-1}k^{-4}a^6[\dot{H} + 0.5k^2(\rho_m + p_m)]^2 = 6(\dot{H} + 2H^2). \quad (8.38)$$

Let us construct an example of the purely kinetic k-essence model induced by $F(R)$ gravity. Let’s simplify a problem: we assume that $\rho_m = p_m = 0$ and $f(R) = \alpha R^n$. As a result, the k-essence Lagrangian takes the form

$$K = 2\alpha R^{n-3}[n(n-1)(n-2)\dddot{R}^2 + n(n-1)R\dddot{R} + 2n(n-1)HR\dddot{R}$$
$$+0.5R^3 - n(\dot{H} + 3H^2)R^2]. \quad (8.39)$$

Here

$$X = \kappa^{-1}k^{-4}a^6\dot{H}^2, \quad \phi = \pm i\sqrt{2k^{-1}k^{-2}\eta^{-1}(a^3\dot{H})}. \quad (8.40)$$

Now we construct a model for the case $a = \beta t^l$. Then the corresponding purely kinetic k-essence reads as

$$K = K(X) = \varsigma X^{\frac{n}{n-3}}, \quad (8.41)$$

where

$$\varsigma = 72\alpha l(2l - 1)^{n-1}(6l)^{n-3}[2\ln(n-1)(2n - 2l - 1)$$
$$+3(2l - 1) - nl^2(3l - 1)^2]\gamma^{\frac{n}{3l-2}}. \quad (8.42)$$

Note that $X$ and $\phi$ are given by

$$X = \gamma t^{6l-4}, \quad \phi = \phi_0 + \frac{i\sqrt{2\gamma}}{3l-1}t^{3l-1}, \quad \gamma = \kappa^{-1}k^{-4}\beta^6l^2. \quad (8.43)$$
Similarly we can construct a new class k-essence models induced by modified gravity theories. These new k-essence models give the equivalent descriptions of DE/matter.

8.2.2 Variant-II

If we introduce the following two functions $K$ and $X$

\[
K = 2[\ddot{f}_R + 2H \dot{f}_R + 0.5f - (\dot{H} + 3H^2)f_R],
\]
\[
X = \frac{2[\ddot{f}_R + 2H \dot{f}_R] - 6\dot{H} \dddot{f}_{RR} + 4\dot{H} f_R}{4\kappa a^{-6}},
\]

then they satisfy the Eqs. (4.3)–(4.6).

8.3 $F(R, G)$ gravity

The action of $F(R, G)$ gravity is given (see, e.g. [5–13])

\[
S = \int d^4x \sqrt{-g} \left[ \frac{1}{2k^2} F(R, G) + L_m \right].
\]

The corresponding system of equations is given by

\[
3k^{-2}H^2 = \rho_{\text{eff}},
\]
\[
-k^{-2}(2\dot{H} + 3H^2) = p_{\text{eff}},
\]
\[
\dot{\rho}_{\text{eff}} + 3H(\rho_{\text{eff}} + p_{\text{eff}}) = 0,
\]
\[
\dot{p}_m + 3H(\rho_m + p_m) = 0.
\]

Here

\[
\rho_{\text{eff}} = \frac{1}{F_R} \{\rho_m + 0.5k^{-2}[RF_R - F - 6H \dot{F}_R + GF_G - 24H^3 \dddot{F}_G]\},
\]
\[
p_{\text{eff}} = \frac{1}{F_R} \{p_m + 0.5k^{-2}[-RF_R + F + 4H \dot{F}_R + 2\dddot{F}_R
\]
\[
- GF_G + 16H(\dot{H} + H^2) \dot{F}_G + 8H^2 \dddot{F}_G]\}.
\]

Let us consider the following transformation

\[
K = \frac{1}{F_R} \{p_m + 0.5k^{-2}[-RF_R + F + 4H \dot{F}_R + 2\dddot{F}_R - GF_G
\]
\[
+ 16H(\dot{H} + H^2) \dot{F}_G + 8H^2 \dddot{F}_G]\},
\]
\[
X = 0.25k^{-1}a^{-6}F_R^{-1} \{\rho_m + p_m + k^{-2}[\dddot{F}_R - H \dddot{F}_R + 4H(2\dot{H} - H^2) \dot{F}_G + 4H^2 \dddot{F}_G]\}.
\]
After this transformation, Eqs. (8.46)–(8.49) convert to Eqs. (4.3)–(4.6). It is the system of equations of purely kinetic k-essence. So in this sense, both $F(R, G)$ gravity and purely kinetic k-essence is equivalent to each other. Hence there follows the results of previous two subsections. In fact $F(G)$ and $F(R)$ are particular reductions of $F(R, G)$ e.g. as: $F(R) = F(R, 0)$ and $F(G) = F(0, G)$.

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