Geometric realization of toroidal quadrangulations without hidden symmetries

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Abstract

It is shown that each quadrangulation of the 2-torus by the Cartesian product of two cycles can be geometrically realized in (Euclidean) 4-space without hidden symmetries—that is, so that each combinatorial cellular automorphism of the quadrangulation extends to a geometric symmetry of its Euclidean realization. Such realizations turn out to be new regular toroidal geometric 2-polyhedra which are inscribed in the Clifford 2-torus in 4-space, just as the five regular spherical 2-polyhedra are inscribed in the 2-sphere in 3-space. The following are two open problems: Realize geometrically (1) the regular triangulations and (2) the regular hexagonizations of the 2-torus without hidden symmetries in 4-space.

Keywords: quadrangulation, torus, Cartesian product of graphs, geometric realization, symmetry group, regular polyhedron.

MSC Classification: 51M20 (Primary); 52B15, 51F15, 20F65, 05C25, 57M20, 57M15 (Secondary).

1 Introduction

The concept of hidden symmetry was introduced by Hermann Weyl [9]. The basis of this concept is the understanding that if $P$ is a polyhedron in Euclidean $d$-space $\mathbb{R}^d$ with the group of obvious (that is, Euclidean) symmetries $\text{Sym}(P)$, $P$ may have hidden symmetries which are elements of a larger group—the combinatorial cellular automorphism group $\text{Aut}(P)$. Revealing hidden symmetries in polyhedra is an important problem along with visual geometric realization of those symmetries.
A fundamental theorem by Peter Mani [7] states that any polygonization of the 2-sphere with a 3-connected graph is realizable in \( \mathbb{R}^3 \) without hidden symmetries—that is, each combinatorial cellular automorphism of the polygonization extends to a geometric symmetry of its Euclidean realization. In this paper this result is generalized for quadrangulations of the 2-torus.

The (Cartesian) product of two graphs (that is, simplicial 1-complexes) \( G_1 \) and \( G_2 \) with disjoint vertex sets \( V(G_1) \) and \( V(G_2) \) is denoted by \( G_1 \times G_2 \) and is defined to be the graph whose vertex set is \( V(G_1 \times G_2) = V(G_1) \times V(G_2) \) and in which two vertices \((u_1, u_2)\) and \((v_1, v_2)\) are connected by an edge if either \( u_1 = v_1 \) and the vertices \( u_2, v_2 \) are connected by an edge in \( G_2 \) or \( u_2 = v_2 \) and the vertices \( u_1, v_1 \) are connected by an edge in \( G_1 \).

Let \( C_n \) denote the graph that is a simple cycle of length \( n \) (\( n \geq 3 \)). The Cartesian product \( C_n \times C_k \) naturally embeds in the 2-torus \( \mathbb{T}^2 \) as \( n \) parallels and \( k \) meridians (or vice versa) which collectively produce a quadrangulation \( C_n \times C_k \rightarrow \mathbb{T}^2 \) denoted by \( Q_{n,k} \). This quadrangulation has \( nk \) vertices, \( 2nk \) edges, and \( nk \) quadrilateral faces.

An analogue of Mani’s Theorem for toroidal quadrangulations \( Q_{n,k} \) is established in Section 3 as part of a more general result—Theorem 1. Geometric realization without hidden symmetries of polygonizations of 2-manifolds with transitive automorphism groups in \( \mathbb{R}^d \) produces regular 2-polyhedra in \( \mathbb{R}^d \). An infinite series of such 2-polyhedra is constructed in Section 4, in addition to the noble toroidal hexadecahedron constructed by the author in [6]. (A “noble polyhedron”, a term introduced by Branko Grünbaum [3], names a polyhedron whose full symmetry group is vertex- and face-transitive but not necessarily edge-transitive.) More precisely, it is shown in Section 4 that the quadrangulations \( Q_{n,k} \) are realizable as noble toroidal 2-polyhedra and, in the specific case \( n = k \), even regular toroidal 2-polyhedra which are inscribed in the Clifford 2-torus in \( \mathbb{R}^4 \), just as the five regular spherical 2-polyhedra are inscribed in the 2-sphere in \( \mathbb{R}^3 \). It is an open problem to realize geometrically the regular triangulations and hexagonizations without hidden symmetries in 4-space (Section 5).

2 Concepts, implications, and an example

Let \( G \) be a finite simplicial 1-complex, or in other words, a simple undirected graph, and let \( \mathbb{M}^2 \) be a 2-manifold. The faces of a topological embedding \( h : G \hookrightarrow \mathbb{M}^2 \) are the components of \( \mathbb{M}^2 - h(G) \). Such an embedding is called a polygonization of \( \mathbb{M}^2 \) with the graph \( G \) provided that the closure of each face is homeomorphic to a closed 2-disc. A polygonization in which each face is bounded by a cycle of \( G \) with length 4 is called a (topological) quadrangulation. An important family of quadrangulations \( Q_{n,k} \) was defined in the Introduction.

On the combinatorial side, a quadrangulation corresponds to an abstract 2-complex (in which each 2-cell corresponds to a quadrilateral face) provided that
the intersection of the closures of any two faces is either empty, a vertex or an edge (including its two vertices) of \( G \).

Let \( K^p \) and \( L^q \) be finite abstract CW-complexes of dimensions \( p \) and \( q \) (\( p \leq q \)), with vertex sets \( V(K^p) \) and \( V(L^q) \), respectively. A (combinatorial cellular) homomorphism \( K^p \rightarrow L^q \) is defined to be a cellular mapping \( \mu : V(K^p) \rightarrow V(L^q) \)—that is, if \( v_0, v_1, \ldots, v_r \) are the vertices of a cell of \( K^p \), then \( \mu(v_0), \mu(v_1), \ldots, \mu(v_r) \) are the vertices of a cell of \( L^q \). An injective homomorphism is called a monomorphism, and a surjective monomorphism is called an isomorphism. Especially, an isomorphism \( K^p \rightarrow K^p \) is called an automorphism of \( K^p \). The automorphism group of \( K^p \) is denoted \( \text{Aut}(K^p) \). The symbol \( \equiv \) designates identity of groups.

**Lemma 1.**

\[
|\text{Aut}(C_n \times C_k)| = \begin{cases} 
4nk & \text{if } n \neq k \\
8n^2 & \text{if } n = k \neq 4 \\
384 & \text{if } n = k = 4 
\end{cases} \tag{1}
\]

\[
|\text{Aut}(Q_{n,k})| = \begin{cases} 
4nk & \text{if } n \neq k \\
8n^2 & \text{if } n = k \end{cases} \tag{2}
\]

**Proof.** Firstly we prove Eq. (1) of the lemma by standard graph-theoretic techniques [4, 5]. Note that \( C_n \) can be extended to a nontrivial product of graphs if and only if \( n = 4 \), in which case \( C_4 = I \times J \), where \( I \) and \( J \) denote two disjoint 1-simplices. In this sense, for \( n \neq 4 \), \( C_n \) is a connected prime graph, and therefore (see [5])

\[
\text{Aut}(C_n \times C_n) \equiv \text{Aut}(C_n) \wr S_2 \equiv D_n \wr S_2
\]

which is the wreath product (called the composition in [4]) of the dihedral group \( D_n \) by the symmetric group \( S_2 \) and has order \( |D_n|^2 |S_2| = 8n^2 \). Furthermore, for \( n \neq k \), \( C_n \) and \( C_k \) are relatively prime graphs with respect to the graph product operation, and therefore (see [4])

\[
\text{Aut}(C_n \times C_k) \equiv \text{Aut}(C_n) \times \text{Aut}(C_k) \equiv D_n \times D_k
\]

which is the direct product of two dihedral groups and has order \( |D_n||D_k| = 4nk \). Finally, it is well known [4] that the automorphism group of the graph \( C_4 \times C_4 \) (that is, the 1-skeleton of the 4-cube) has order 384.

We now proceed to prove Eq. (2) of the lemma. Clearly, the group \( \text{Aut}(Q_{n,k}) \) is always vertex-transitive. If \( n = k \), the stabilizer of each vertex is isomorphic to the dihedral group \( D_4 \), and therefore \( |\text{Aut}(Q_{n,n})| = |V(Q_{n,n})| \times |D_4| = 8n^2 \). If \( n \neq k \), no cycle with length \( n \) can map onto a cycle with length \( k \), and therefore the stabilizer of each vertex \( v \) of \( Q_{n,k} \) is the group of order 4 generated by the permutations \( (\alpha, id_2) \) and \( (id_1, \beta) \) of the vertex set \( V(Q_{n,k}) = V(C_n \times C_k) \), where \( id_1 \) and \( id_2 \) are identical but \( \alpha \) and \( \beta \) are non-identical involutive automorphisms of
the factors $C_n$ and $C_k$ (respectively) that fix the vertex $v$. Therefore $|\text{Aut}(Q_{n,k})| = |V(Q_{n,k})| \times 4 = 4nk$.

A flag of a 2-complex is defined to be a triple of pairwise incident elements in the form of (vertex, edge, face).

**Corollary 1.** For any $n, k \geq 3$, $Q_{n,k}$ is a noble quadrangulation in the sense that $\text{Aut}(Q_{n,k})$ is vertex-transitive and face-transitive (but not edge-transitive when $n \neq k$). Furthermore, for any $n$, $Q_{n,n}$ is a regular quadrangulation in the sense that $\text{Aut}(Q_{n,n})$ is flag-transitive, which provides maximum possible order of the group.

For a finite abstract $p$-complex $K^p$, let $|K^p| : K^p \rightarrow \mathbb{R}^d$ denote a (geometric) realization of $K^p$—that is, a $p$-polyhedron whose cell structure is naturally inherited from $K^p$ and whose “corner points” correspond naturally and bijectively to the vertices of $K^p$. Euclidean motions leaving $|K^p|$ invariant form a finite subgroup of the group of Euclidean motions of $\mathbb{R}^d$. That subgroup is denoted by $\text{Sym}(|K^p|)$ and is called the (full) symmetry group of $|K^p|$. Acting on the vertex set $V(K^p)$, the group $\text{Sym}(|K^p|)$ corresponds to a subgroup of $\text{Aut}(K^p)$ and is often understood combinatorially in this paper—that is, as the group of the corresponding permutations of $V(K^p)$. It will be clear from the context whether we understand $\text{Sym}(|K^p|)$ combinatorially or geometrically.

A **polytope** is defined to be the convex hull of a finite set of points in $\mathbb{R}^d$. A **$d$-polytope** is a $d$-dimensional polytope. Let $\mu : K^p \rightarrow L^q$ be a combinatorial cellular monomorphism and let a realization $|L^q| : L^q \hookrightarrow \mathbb{R}^{q+1}$ be given by the boundary complex of some $(q+1)$-polytope in $\mathbb{R}^{q+1}$. Denote by $\hat{K}^p$ the image $\mu(K^p)$ and denote by $|\hat{K}^p|$ the realization $\hat{K}^p \hookrightarrow \mathbb{R}^{q+1}$ naturally induced by $|L^q|$. Therefore we obtain a realization $|\hat{K}^p| : K^p \hookrightarrow \mathbb{R}^{q+1}$ as

$$K^p \rightarrow \hat{K}^p \rightarrow |\hat{K}^p| \subseteq |L^q| \subset \mathbb{R}^{q+1}. \quad (3)$$

Realization (3) of $K^p$ by the polyhedron $|\hat{K}^p|$ in $\mathbb{R}^{q+1}$ is said to be without hidden symmetries provided that each automorphism of $K^p$ is induced by some Euclidean symmetry of $|\hat{K}^p|$.

On the algebraic side, assuming that the origin of $\mathbb{R}^{q+1}$ is fixed by the whole symmetry group of $|\hat{K}^p|$, if realization (3) of $K^p$ by $|\hat{K}^p|$ in $\mathbb{R}^{q+1}$ is without hidden symmetries, then the group $\text{Sym}(|\hat{K}^p|) \subset \text{O}(q+1)$ provides a faithful representation of the group $\text{Aut}(K^p)$ of degree $q+1$.

**Lemma 2.** For the absence of hidden symmetries in realization (3) it is sufficient that the following three conditions hold simultaneously:

(i) $\text{Sym}(|L^q|) \subseteq \text{Sym}(|\hat{K}^p|)$,

(ii) $|V(L^q)| = |V(K^p)|$,

(iii) $|\text{Sym}(|L^q|)| = |\text{Aut}(K^p)|$. 


Proof. By condition (i), each symmetry of $|L^q|$ is a symmetry of $|\hat{K}^p|$ and, therefore, induces some automorphism of $K^p$. By condition (ii), distinct symmetries of $|L^q|$ induce distinct automorphisms of $K^p$. By condition (iii), each automorphism of $K^p$ is induced by some symmetry of $|L^q|$ which is also a symmetry of $|\hat{K}^p|$ by (i). \hfill $\square$

For example, for $p = q = 1$ the cycle $K^1 = C_n$ is realized by the boundary complex $B(P^2_n)$ of a regular Euclidean $n$-gon $P^2_n$ in $\mathbb{R}^2$. Note that $P^2_n$ is a 2-polytope in $\mathbb{R}^2$, and also note that the groups $\mathrm{Sym}(B(P^2_n))$ and $\mathrm{Sym}(P^2_n)$ are identical as permutation groups on the set $V(P^2_n) = V(|\hat{K}^1|)$. The complex $B(P^2_n)$ corresponds to $|\hat{K}^1| = |L^1|$ in (3), and therefore condition (ii) of Lemma 2 holds. Since both groups $\mathrm{Sym}(P^2_n)$ and $\mathrm{Aut}(C_n)$ act on the set $V(C_n)$ as the $n$-gonal dihedral group $D_n$, condition (iii) also holds. Furthermore, since each element of the group $\mathrm{Sym}(P^2_n)$ leaves the 1-skeleton $C_n$ setwise invariant, condition (i) holds too, and by Lemma 2 the graph $C_n$ is realized by $B(P^2_n)$ without hidden symmetries.

## 3 Realization of toroidal quadrangulations

The construction of this section is a generalization of the example with the cycle $C_n$ at the end of the preceding section.

The Cartesian product of a regular $n$-gon $P^2_n$ and a regular $k$-gon $P^2_k$ $(n, k \geq 3)$ is known [8] as the $n, k$-duoprism, denoted by $P^4_{n,k}$ in this paper, and is a 4-polytope as the convex hull of the set of $nk$ points $M_{ij}(\cos \frac{2\pi i}{n}, \sin \frac{2\pi i}{n}, \cos \frac{2\pi j}{k}, \sin \frac{2\pi j}{k})$, where $i = 0, 1, \ldots, n - 1$ and $j = 0, 1, \ldots, k - 1$. Therefore the vertices $M_{ij}$ of $P^4_{n,k}$ lie on the Clifford 2-torus in $\mathbb{R}^4$ and the boundary complex $B(P^4_{n,k})$ is a 3-polyhedron [which plays the role of $|L^3|$ in realization (3)] inscribed in that 2-torus. It is not hard to observe that the 1-skeleton of $B(P^4_{n,k})$ is the graph $C_n \times C_k$ [which plays the role of $K^1$ in realization (3)] and that the 2-skeleton of $B(P^4_{n,k})$ contains a realization $Q_{n,k}$ of $Q_{n,k}$ plus $n$ more $k$-gons and $k$ more $n$-gons.

**Theorem 1.** For any $n, k \geq 3$, the graph $C_n \times C_k$ is geometrically realized in $\mathbb{R}^4$ without hidden symmetries by the 1-skeleton of the corresponding duoprism’s boundary complex $B(P^4_{n,k})$, and the toroidal quadrangulation $Q_{n,k}$ is geometrically realized without hidden symmetries in the 2-skeleton of $B(P^4_{n,k})$.

**Proof.** This proof is divided into three cases.

**Case 1.** $n \neq k$. We have

$$\mathrm{Sym}(P^4_{n,k}) \equiv \mathrm{Sym}(P^2_n \times P^2_k) \equiv \mathrm{Sym}(P^2_n) \times \mathrm{Sym}(P^2_k) \equiv D_n \times D_k,$$

The group obtained is the direct product of two dihedral groups and has order $4nk$. Condition (iii) of Lemma 2 holds by Lemma 1(1). Condition (ii) holds trivially. Furthermore, since each element of the group $\mathrm{Sym}(P^4_{n,k})$ leaves the 1-skeleton $C_n \times$
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$C_k$ setwise invariant, condition (i) holds as well. Therefore, by Lemma 2, the graph $C_n \times C_k$ is realized by the 1-skeleton of $B(P_{n,k}^4)$ without hidden symmetries. Then $Q_{n,k}$ is realized without hidden symmetries in the 2-skeleton of $B(P_{n,k}^4)$ because $\text{Aut}(Q_{n,k}) \subseteq \text{Aut}(C_n \times C_k)$.

Case 2. $n = k \neq 4$. Similarly, here we have $\text{Sym}(P_{4,n}^4) \equiv \text{Sym}(P_{n}^2 \times P_{n}^2) \equiv \text{Sym}(P_{n}^2) \wr S_2 \equiv D_n \wr S_2$ which is the wreath product of the dihedral group $D_n$ by the symmetric group $S_2$ and has order $|D_n|^2|S_2| = 8n^2$. By Lemma 1, $|\text{Sym}(P_{4,n}^4)| = |\text{Aut}(C_n \times C_n)|$, and the proof is completed as in Case 1.

Case 3. $n = k = 4$. In this case $P_{4,4}^4$ is the 4-cube and $\text{Sum}(P_{4,4}^4)$ is the hyper-octahedral group of order 384. Again by Lemma 1, $|\text{Sym}(P_{4,4}^4)| = |\text{Aut}(C_4 \times C_4)|$, and the proof is completed as above.

**Corollary 2.** $\text{Aut}(C_n \times C_k) \equiv \text{Sym}(P_{n,k}^4)$ $(n, k \geq 3)$, where $\text{Sym}(P_{n,k}^4)$ is regarded as a permutation group on the set $V(C_n \times C_k)$.

Notably, Case 3 is the only case where $|\text{Aut}(C_n \times C_k)| \neq |\text{Aut}(Q_{n,k})|$ (compare to Lemma 1), which means that there are at least two copies of $Q_{4,4}$ in the 2-skeleton of the 4-cube (all realized without hidden symmetries by Theorem 1). The exact number of such copies is equal to 3, which can be found by [1] formula (4) along with Lemma 1 as the ratio $|\text{Aut}(C_4 \times C_4)|/|\text{Aut}(Q_{4,4})| = 384/128 = 3$.

4 Regular 2-polyhedra

Any geometric realization of a polygonization of a closed 2-manifold in $\mathbb{R}^d$ without hidden symmetries is called a regular 2-polyhedron provided that the automorphism group of that polygonization is flag-transitive, and, following Branko Grünbaum [3], is called a noble 2-polyhedron provided that that group is vertex-transitive and face-transitive but not necessarily edge-transitive. A noble 2-polyhedron is both isogonal and isohedral—that is, all polyhedral angles at the vertices are congruent and all the faces are congruent. In addition to the above listed congruencies, a regular 2-polyhedron has all dihedral angles congruent. The following is a corollary of the combination of Corollary 1 and Theorem 1.

**Corollary 3.** For $n, k \geq 3$ the quadrangulation $Q_{n,k}$ is realizable as a noble toroidal 2-polyhedron in $\mathbb{R}^4$, and $Q_{n,n}$ is realizable as a regular toroidal 2-polyhedron in $\mathbb{R}^4$.

Note that all regular toroidal 2-polyhedra $\hat{Q}_{n,n}$ found in Section 3 are inscribed in the Clifford 2-torus in $\mathbb{R}^4$, just as the five regular spherical 2-polyhedra are inscribed in the 2-sphere in $\mathbb{R}^3$. 
5 Open problems

A classification of all regular toroidal polygonizations was given by Coxeter [2, pp. 25–27]. They split into three series: an infinite series of self-dual quadrangulations (with the degree of each vertex equal to $\delta = 4$), and two infinite (dual) series of triangulations ($\delta = 6$) and hexagonizations ($\delta = 3$). The quadrangulations are realized by regular toroidal 2-polyhedra in $\mathbb{R}^4$ by Corollary 3. The following are two open problems: Realize geometrically (1) the regular triangulations and (2) the regular hexagonizations without hidden symmetries in $\mathbb{R}^4$.

6 Concluding remarks

So, Mani’s Theorem doesn’t extend to polygonizations of 2-manifolds of higher genera unless we increase the dimension of the ambient Euclidean space. A rectangle subdivided by $n$ vertical and $k$ horizontal lines into $nk$ congruent subrectangles in $\mathbb{R}^2$ gives a realization of the corresponding quadrangulation of the 2-disc without hidden symmetries. Furthermore, we can isometrically bend the subdivided rectangle along the vertical lines and then indentify the left and right sides in $\mathbb{R}^3$ to obtain a realization of the corresponding quadrangulation of the 2-cylinder without hidden symmetries. Finally, we can isometrically bend the so obtained 2-cylinder along the horizontal circles (which progressed from the original horizontal lines) and then identify the upper and lower circles in $\mathbb{R}^4$ to obtain, by Theorem 1, a realization of the corresponding quadrangulation of the 2-torus without hidden symmetries. The first of the constructed quadrangulations is a prismatic 2-polytope, the second sits in the boundary complex of a prismatic 3-polytope, and the third sits in the 2-skeleton of the boundary complex of a prismatic 4-polytope.

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