$L^p$–Variational Solution of Backward Stochastic Differential Equation driven by subdifferential operators on a deterministic interval time

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Abstract

Our aim is to study the existence and uniqueness of the $L^p$–variational solution, with $p > 1$, of the following multivalued backward stochastic differential equation with $p$–integrable data:

$$\begin{cases}
-dY_t + \partial_y \Psi (t, Y_t) dQ_t \ni H (t, Y_t, Z_t) dQ_t - Z_t dB_t, \quad t \in [0, T], \\
Y_T = \eta,
\end{cases}$$

where $Q$ is a progressively measurable increasing continuous stochastic process and $\partial_y \Psi$ is the subdifferential of the convex lower semicontinuous function $y \mapsto \Psi (t, y)$.

In the framework of [12] (the case $p \geq 2$), the strong solution found it there is the unique variational solution, via the uniqueness property proved in the present article.

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1 Introduction

The study of the standard backward stochastic differential equations (BSDEs) was initiated by E. Pardoux and S. Peng in [16]. The authors have proved the existence and the uniqueness of the solution for the BSDE on fixed time interval, under the assumption of Lipschitz continuity of the generator $F$ with respect to $y$ and $z$ and square integrability of $\eta$ and $F (t, 0, 0)$. The case of BSDEs on random time interval have been treated by R.W.R. Darling and E. Pardoux in [4], where it is obtained, as application, the existence of a continuous viscosity solution to the elliptic partial differential equations (PDEs) with Dirichlet boundary conditions. The more general case of reflected BSDEs was considered for the first time by N. El Karoui et al. in [6].

In the present paper, we prove the existence and uniqueness of a new type of solution, called $L^p$–variational solution, in the case $p > 1$, of the generalized backward stochastic variational inequality (BSVI for short) with $p$–integrable data:

$$\begin{cases}
Y_t + \int_t^T dK_s = \eta + \int_t^T [F (s, Y_s, Z_s) ds + G (s, Y_s) dA_s] - \int_t^T Z_s dB_s, \quad t \in [0, T], \\
dK_t \in \partial \phi (Y_t) dt + \partial \psi (Y_t) dA_t, \quad \text{on } [0, T],
\end{cases}$$

(1)
where $\partial \varphi$ and $\partial \psi$ are the subdifferentials of two convex lower semicontinuous functions $\varphi$ and $\psi$ and $\{A_t: t \geq 0\}$ is a progressively measurable increasing continuous stochastic process.

We prove the uniqueness property of the solution on a random time interval $[0, \tau]$; the existence is obtained only in the case of a deterministic time interval, i.e. $\tau = T > 0$, and it is made using the Moreau–Yosida regularization of $\varphi$ and $\psi$ and the mollifier approximations of the generators $F$ and $G$. The proof of the existence in the case of a random time interval and the existence and the uniqueness problem for the same equation in the case $p = 1$ are, for the moment, in work and there will be the subjects of a future article.

In fact, we will define and prove the existence and uniqueness of the $L^p$ variational solution for an equivalent form of (1):

$$
\begin{align*}
\begin{cases}
Y_t + \int_t^T dK_s = \eta + \int_t^T H(s, Y_s, Z_s) dQ_s - \int_t^T Z_s dB_s , & t \in [0, T] \\
\int_0^T dK_t \in \partial \Psi(t, Y_t) dQ_t , & \text{on } [0, T],
\end{cases}
\end{align*}
$$

with $Q$, $H$ and $\Psi$ adequately defined.

The second condition in (1) says, among others, that the first component $Y$ of the solution is forced to stay in the set $\text{Dom}(\partial \varphi) \cap \text{Dom}(\partial \psi)$. The role of $K$ is to act in the evolution of the process $Y$ and also to keep $Y$ in these domains.

We mention that the presence of the process $A$ is justified by the possible applications of equation (1) in proving probabilistic proofs for the existence of a solution of PDEs with Neumann boundary conditions on a domain from $\mathbb{R}^n$. The stochastic approach of the existence problem for multivalued parabolic PDEs, was considered by L. Maticiuc and A. Răşcanu in [11] and [13]. We emphasize that if the obstacles are fixed, the reflected BSDEs becomes a particular case of the BSVI of type (1), by taking $\varphi$ as convex indicator of the interval defined by obstacles. In this case the solution of the BSVI belongs to the domain of the multivalued operator $\partial \varphi$ and it is reflected at the boundary of this domain.

The standard work on BSVI in the finite dimensional case is that of E. Pardoux and A. Răşcanu [17], where it is proved the existence and uniqueness of the solution $(Y, Z, K)$ for the BSVI (1) with $A \equiv 0$, under the following assumptions on $F$: continuity with respect to $y$, monotonicity with respect to $y$ (in the sense that $(y' - y, F(t, y', z) - F(t, y, z)) \leq \alpha |y' - y|^2$, lipschitzianity with respect to $z$ and a sublinear growth for $F(t, y, 0)$). Moreover, it was shown that, unlike the forward case, the process $K$ is absolute continuous with respect to $dt$. In [18] the same authors extend these results to the Hilbert spaces framework.

We mention that assumptions of Lipschitz continuity of the generator $F$ with respect to $y$ and $z$ and the square integrability of the final condition and $F(t, 0, 0)$ (as in articles El Karoui et al. [6] and E. Pardoux and S. Peng [16]) are sometimes too strong for applications (see, e.g., D. Duffie and L. Epstein [5] and El Karoui et al. [7] for the applications in mathematical finance and P. Briand et al. [2] and A. Rozkosz and L. Slominski [22] for the applications to PDEs). A possibility is to weaken the integrability conditions imposed on $\eta$ and $F$ or to weaken the assumption which concerns the Lipschitz continuity of the generators. In P. Briand and R. Carmona [2] or E. Pardoux [15] it is considered the case where the generators are Lipschitz continuous with respect to $z$, continuous with respect to $y$ and satisfy a monotonicity condition and a growth condition of the type $|F(t, y, z)| \leq |F(t, 0, z)| \phi(|y|)$, where $\phi$ is a polynomial or even an arbitrary positive increasing continuous function.

We recall that the previous assumption was used in [15] in order to prove the existence of a solution in $L^2$. This result was generalized by P. Briand et al. in [3], where it is proved the existence and uniqueness of $L^p$ solutions, with $p \in [1, 2]$, for BSDEs considered with a random terminal time $T$: in the case $p \in (1, 2)$, if $\eta \in L^p$, $\int_0^T |F(s, 0, 0)| ds \in L^p$, for any $r > 0$, $\int_0^T \sup_{|y| \leq r} |F(s, y, 0) - F(s, 0, 0)| ds \in L^p$, for
$L^1$ and $F$ is Lipschitz continuous with respect to $z$, continuous with respect to $y$ and satisfy a monotonicity condition, then there exists a unique $L^p$ solution; in the case $p = 1$ similar result is proved if $T$ is a fixed deterministic terminal time and under additional assumptions.

We also note that the study of the reflected BSDEs was the subject, e.g., of the papers: J.P. Lepeltier et al. [9] (in the case of the general growth condition with respect to $y$ and for $p = 2$), S. Hamadène and A. Popier [8] (in the case of Lipschitz continuity with respect to $y$ the and for $p \in (1, 2)$). Studies made, roughly speaking, under the assumptions of [3] are, e.g.: A. Aman [1] (in the case of a generalized reflected BSDE and for $p \in (1, 2)$), A. Rozkosz and L. Słomiński [21] (for $p \in [1, 2]$) and T. Klimsiak [10] (in the case of BSDE with two irregular reflecting barriers and for $p \in [1, 2]$).

Our paper generalizes the existence and uniqueness results from [17] by considering the $L^p$ solutions in the case $p \in (1, 2)$, the Lebesgue–Stieltjes integral terms, and by assuming a weaker boundedness condition for the generator $F$ (instead of the sublinear growth):

$$
\mathbb{E}\left(\int_0^T F_p^\#(s)ds\right)^p < \infty, \quad \text{where } F_p^\#(t) := \sup_{|y| \leq p} |F(t, y, 0)|.
$$

We remark that article [12] concerns the same type of backward equation as in our study (and under the similar assumptions), but considered in the infinite dimensional framework and in the case $p \geq 2$. In addition, it is worth pointing out that in the case $p \geq 2$, if we are in the framework of [12], our variational solution is a strong one since we have proved the uniqueness property of the variational solution.

In this paper we use the following notation: $(\Omega, \mathcal{F}, \mathbb{P})$ is a complete probability space, the set $\mathcal{N}_P := \{A \in \mathcal{F} : \mathbb{P}(A) = 0\}$, $\{\mathcal{F}_t\}_{t \geq 0}$ is a right continuous and complete filtration generated by a standard $k$–dimensional Brownian motion $(B_t)_{t \geq 0}$.

$S^p_m[0, T]$ is the space of (equivalent classes of) continuous progressively measurable stochastic processes (p.m.s.p. for short) $X : \Omega \times [0, T] \rightarrow \mathbb{R}^m$ such that $\mathbb{E}\sup_{t \in [0, T]} |X_t|^p < +\infty$, if $p > 0$.

$\Lambda^p_m(0, T)$ is the space of p.m.s.p. $X : \Omega \times (0, T) \rightarrow \mathbb{R}^m$ such that such that $\int_0^T |X_t|^2 dt < +\infty$, $\mathbb{P}$-a.s. if $p = 0$ and $\mathbb{E}\left(\int_0^T |X_t|^2 dt\right)^{p/2} < +\infty$, if $p > 0$.

The article is organized as follows: next section is dedicated to the presentation of the assumptions needed in our study. In the third section we present firstly a intuitive introduction for the notion of $L^p$-variational solution and the we prove the uniqueness property. The fourth section is devoted to the proof of the existence of our type of solution in the case of a deterministic time interval, i.e. $\tau = T > 0$. The Appendix contains, following [19], some results useful throughout the paper.

### 2 Assumptions and definitions

In the beginning of this subsection we introduce the assumptions regarding equation (1). Since we provide a definition of the variational solution on a random time interval $[0, \tau]$ and we prove the uniqueness and a continuity property of the solution with respect to this definition, we will give here also the framework corresponding to $[0, \tau]$.

We consider throughout this paper that $p > 1$.

(A1) The random variable $\tau : \Omega \rightarrow [0, \infty]$ is a stopping time;

(A2) The random variable $\eta : \Omega \rightarrow \mathbb{R}^m$ is $\mathcal{F}_\tau$-measurable such that $\mathbb{E}|\eta|^p < \infty$ and $(\xi, \zeta) \in S^p_m \times \Lambda^p_{m \times k}(0, \infty)$ is the unique pair associated to $\eta$ given by the martingale representation formula (see [19, Corollary 2.44])

\[
\begin{align*}
\xi_t &= \eta - \int_t^\infty \zeta_s dB_s, \quad t \geq 0, \; \text{a.s.,} \\
\xi_t &= \mathbb{E}^{\mathcal{F}_t}\eta \quad \text{and} \quad \zeta_t = [0, \tau](t) \zeta_t
\end{align*}
\]
(or equivalently, \(\xi_t = \eta - \int_{t\wedge T}^T \zeta_s dB_s, \ t \geq 0, \ \text{a.s.}\))

(A3) The process \(\{A_t : t \geq 0\}\) is a increasing and continuous p.m.s.p. such that \(A_0 = 0\) and 
\[
\mathbb{E} \left( e^{\alpha A_T} \right) < \infty, \quad \text{for any } \alpha, T > 0;
\]

(A4) \(\varphi, \psi : \mathbb{R}^m \rightarrow [0, +\infty]\) are proper convex lower semicontinuous (l.s.c. for short) functions, \(\partial \varphi\) and \(\partial \psi\) denote their subdifferentials and we suppose that \(0 \in \partial \varphi (0) \cap \partial \psi (0)\) (or equivalently \(0 = \varphi (0) \leq \varphi (y)\) and \(0 = \psi (0) \leq \psi (y)\) for all \(y \in \mathbb{R}^m\)).

(A5) The functions \(F : \Omega \times \mathbb{R}_+ \times \mathbb{R}^m \times \mathbb{R}^m \times k \rightarrow \mathbb{R}^m\) and \(G : \Omega \times \mathbb{R}_+ \times \mathbb{R}^m \rightarrow \mathbb{R}^m\) are such that \(F (\cdot, \cdot, y, z), G (\cdot, \cdot, y)\) are p.m.s.p., for all \((y, z) \in \mathbb{R}^m \times \mathbb{R}^m \times k\), \(F (\omega, t, \cdot, \cdot), G (\omega, t, \cdot)\) are continuous functions, \(d\mathbb{P} \otimes dt\)-a.e. and, \(\mathbb{P}\)-a.s.,
\[
\int_0^T F^\#_{\rho} (s) ds + \int_0^T G^\#_{\rho} (s) dA_s < \infty, \quad \text{for all } \rho, T \geq 0,
\]
where
\[
F^\#_{\rho} (s) := \sup_{|y| \leq \rho} |F (\omega, s, y, 0)|, \quad G^\#_{\rho} (s) := \sup_{|y| \leq \rho} |G (\omega, s, y)|;
\]

(A6) Let
\[
n_{\rho} := 1 \wedge (p - 1) \quad \text{and} \quad \lambda \in (0, 1),
\]
Assume there exist three p.m.s.p. \(\mu, \nu : \Omega \times \mathbb{R}_+ \rightarrow \mathbb{R}, \ell : \Omega \times \mathbb{R}_+ \rightarrow \mathbb{R}_+\), such that
\[
\mathbb{E} \exp \left( p \int_0^T (|\mu_s| + \frac{1}{2n_{\rho}} \lambda^2 s) ds + p \int_0^T |\nu_s| dA_s \right) < \infty, \quad \text{for all } T > 0,
\]
and for all \(t \geq 0, y, y' \in \mathbb{R}^m, z, z' \in \mathbb{R}^m\), \(\mathbb{P}\)-a.s.
\[
\langle y' - y, F (t, t, z) - F (t, y, z) \rangle \leq \mu_t |y' - y|^2,
\]
\[
\langle y' - y, G (t, y') - G (t, y) \rangle \leq \nu_t |y' - y|^2,
\]
\[
|F (t, t, z) - F (t, y, z)| \leq \ell_t |z' - z|.
\]

We define
\[
Q_t (\omega) = t + A_t (\omega),
\]
and let \(\{\alpha_t : t \geq 0\}\) be the real positive p.m.s.p. such that \(\alpha \in [0, 1] \) and \(dt = \alpha_t dQ_t + dA_t = (1 - \alpha_t) dQ_t\).

Let us introduce the functions
\[
\begin{align*}
H (t, y, z) & := \mathbb{1}_{[0, \tau]} (t) [\alpha_t F (t, y, z) + (1 - \alpha_t) G (t, y)], \\
\Psi (\omega, t, y) & := \mathbb{1}_{[0, \tau] (\omega)} (t) [\alpha_t (\omega) \varphi (y) + (1 - \alpha_t (\omega)) \psi (y)].
\end{align*}
\]

Obviously, from (9) we see that
\[
\begin{align*}
\langle y' - y, H (t, y', z) - H (t, y, z) \rangle & \leq \mathbb{1}_{[0, \tau]} (t) [\mu_t \alpha_t + \nu_t (1 - \alpha_t)] |y' - y|^2, \\
|H (t, y, z') - H (t, y, z)| & \leq \mathbb{1}_{[0, \tau]} \alpha_t \ell_t |z' - z|.
\end{align*}
\]

Here and subsequently, \(\lambda \in (0, 1)\)
\[
V_t \overset{\text{def}}{=} \int_0^t \mathbb{1}_{[0, \tau]} (r) \left( \mu_r + \frac{1}{2n_{\rho}} \lambda^2 r \right) dr + \int_0^t \mathbb{1}_{[0, \tau]} (r) \nu_r dA_r.
\]
By the assumption \((8)\) we clearly have, for all \(T > 0\),

\[
\mathbb{E} \exp (p V_T) \leq \mathbb{E} \exp \left( p V_T^{(+) \left(0\right)} \right) \leq \mathbb{E} \exp \left( p \int_0^T \left| \mu_s \right| + \frac{1}{2n^2 \lambda} \right) \left| \nu_s \right| dA_s + p \int_0^T \left| \nu_s \right| dA_s < \infty.
\]  

**Definition 1** The notation \(dK_t \in \partial_y \Psi (t, Y_t) dQ_t\) means that \(K\) is \(\mathbb{R}^m\)-valued locally bounded variation stochastic process, \(Q\) is a real increasing stochastic process, \(Y\) is \(\mathbb{R}^m\)-valued continuous stochastic process such that \(\int_0^T \Psi (t, Y_t) dQ_t < \infty\), a.s. for all \(T \geq 0\) and, \(\mathbb{P}\)-a.s., for any \(0 \leq t \leq s\)

\[
\int_t^s \langle y (r) - Y_r, dK_r \rangle + \int_t^s \Psi (r, Y_r) dQ_r \leq \int_t^s \Psi (r, y (r)) dQ_r, \quad \text{for any } y \in C (\mathbb{R}_+; \mathbb{R}^m).
\]

**Remark 2** The condition \(0 \in \partial \varphi (0) \cap \partial \psi (0)\) does not restrict the generality of the problem, since from \(\text{Dom} \ (\partial \varphi) \cap \text{Dom} \ (\partial \psi) \neq \emptyset\) it follows that there exists \(u_0 \in \text{Dom} \ (\partial \varphi) \cap \text{Dom} \ (\partial \psi)\) and \(u_{01} \in \partial \varphi (u_0), u_{02} \in \partial \psi (u_0). \) In this case equation \((1)\) is equivalent to

\[
\begin{aligned}
\dot{Y}_t &= Y_t - u_0, \quad \dot{Z}_t := Z_t, \quad \eta := \eta - u_0,
\end{aligned}
\]

where

\[
\dot{Z}_t := dK_t - \dot{u}_0 dt - \dot{u}_2 dt A_t.
\]

Let \(\varepsilon > 0\) and the Moreau–Yosida regularization of \(\varphi:\)

\[
\varphi_{\varepsilon} (y) := \inf \left\{ \frac{1}{2 \varepsilon} |y - v|^2 + \varphi (v) : v \in \mathbb{R}^m \right\}, \quad (14)
\]

which is a \(C^1\)-convex function.

The gradient \(\nabla \varphi_{\varepsilon} (x) = \partial \varphi_{\varepsilon} (x) = \partial \varphi (J_{\varepsilon} (x))\), where \(J_{\varepsilon} (x) := x - \varepsilon \nabla \varphi_{\varepsilon} (x)\) and satisfies

\[
\begin{aligned}
(a) & \quad |J_{\varepsilon} (x) - J_{\varepsilon} (y)| \leq |x - y|, \\
(b) & \quad |\nabla \varphi_{\varepsilon} (x) - \nabla \varphi_{\varepsilon} (y)| \leq \frac{1}{\varepsilon} |x - y|, \\
(c) & \quad \varphi_{\varepsilon} (y) = \frac{|y - J_{\varepsilon} (y)|^2}{2 \varepsilon} + \varphi (J_{\varepsilon} (y))
\end{aligned}
\]

and

\[
- \langle u - v, \nabla \varphi_{\varepsilon} (u) - \nabla \varphi_{\varepsilon} (v) \rangle \leq (\varepsilon + \delta) \langle \nabla \varphi_{\varepsilon} (u), \nabla \varphi_{\varepsilon} (v) \rangle \leq \frac{\varepsilon + \delta}{2} \left[ |\nabla \varphi_{\varepsilon} (u)|^2 + |\nabla \varphi_{\varepsilon} (v)|^2 \right]
\]

\[
(16)
\]
Also it holds that \( \varphi (J_\varepsilon u) \leq \varphi (u) \), for any \( u \in \mathbb{R}^m \),
\[
J_\varepsilon (0) = 0, \quad \nabla \varphi_\varepsilon (0) = 0, \quad \text{and} \quad \varphi_\varepsilon (0) = 0.
\] 

Also it holds that \( \varphi (J_\varepsilon u) \leq \varphi (u) \), for any \( u \).

We introduce the compatibility conditions between \( \varphi, \psi \) and \( F, G \).

(A7) For all \( \varepsilon > 0, t \geq 0, y \in \mathbb{R}^m, z \in \mathbb{R}^{m \times k} \)

(i) \( \langle \nabla \varphi_\varepsilon (y), \nabla \psi_\varepsilon (y) \rangle \geq 0 \),

(ii) \( \langle \nabla \varphi_\varepsilon (y), G (t, y) \rangle \leq |\nabla \psi_\varepsilon (y)| |G (t, y)|, \quad \mathbb{P} \text{-a.s.}, \)

(iii) \( \langle \nabla \psi_\varepsilon (y), F (t, y, z) \rangle \leq |\nabla \varphi_\varepsilon (y)| |F (t, y, z)|, \quad \mathbb{P} \text{-a.s.} \) \hspace{1cm} (18)

Example 3

(a) If \( \varphi = \psi \) then the compatibility assumptions (18) are clearly satisfied.

(b) Let \( m = 1 \). Since \( \nabla \varphi_\varepsilon \) and \( \nabla \psi_\varepsilon \) are increasing monotone functions on \( \mathbb{R} \), we see that, if \( y \cdot G (t, y) \leq 0 \) and \( y \cdot F (t, y, z) \leq 0 \), for all \( t, y, z \), then the compatibility assumptions (18) are satisfied.

(c) Let \( m = 1 \). If \( \varphi, \psi : \mathbb{R} \to (-\infty, +\infty] \) are the convexity indicator functions \( \varphi (y) = \begin{cases} 0, & \text{if } y \in [a, b], \\ +\infty, & \text{if } y \notin [a, b], \end{cases} \) and \( \psi (y) = \begin{cases} 0, & \text{if } y \in [c, d], \\ +\infty, & \text{if } y \notin [c, d], \end{cases} \) where \( -\infty \leq a \leq b \leq +\infty \) and \( -\infty \leq c \leq d \leq +\infty \) are such that \( 0 \in [a, b] \cap [c, d] \) (see (A6)), then \( \nabla \varphi_\varepsilon (y) = \frac{1}{\varepsilon} [(y - b)^+ - (a - y)^+] \), and \( \nabla \psi_\varepsilon (y) = \frac{1}{\varepsilon} [(y - d)^+ - (c - y)^+] \).

Assumption (A7 - i) is clearly fulfilled; the remaining compatibility assumptions are satisfied if, for example, \( G (t, y) \geq 0 \), \( y \leq a \), \( G (t, y) \leq 0 \), \( y \geq b \), and, respectively, \( F (t, y, z) \geq 0 \), \( y \leq c \), \( F (t, y, z) \leq 0 \), for \( y \geq d \).

3 \hspace{1cm} \mathbb{L}^p - variational solutions

3.1 \hspace{1cm} Intuitive introduction

At the beginning of this section, until further notice, we will consider \( p \geq 0 \) and \( n_p = (p - 1)^+ \land 1 \) and let be fixed an arbitrary \( \lambda \in (0, 1) \). We recall definition (12) of \( V \) and we extend the definition of \( V \) to the case \( p \in [0, 1] \) by considering
\[
V_t \overset{\text{def}}{=} \int_0^t \mathbf{1}_{[0, \tau]} (r) \mu_r dr + \int_0^t \mathbf{1}_{[0, \tau]} (r) \nu_r dA_r, \quad \text{if} \ p \in [0, 1]
\]
(in the case \( p \in [0, 1] \) we will consider \( \ell = 0 \) (i.e. \( H \) is independent of \( z \)) and \( \ell^2 / \nu_p = 0 \)).

By the assumption (8) we clearly have
\[
\mathbb{E} \left( \sup_{r \in [0, T]} e^{p \nu_r} \right) < \infty, \quad \text{for all} \ T > 0.
\]
Let us define the space $S^p_m(\gamma, N, R; V)$, $p \geq 0$, of the continuous stochastic process $M$ of the form

$$M_t = \gamma - \int_0^t N_r dQ_r + \int_0^t R_r dB_r,$$

or equivalent

$$M_t = M_T + \int_t^T N_r dQ_r - \int_t^T R_r dB_r, \quad M_0 = \gamma$$

where $\gamma \in \mathbb{R}^m$ and $N : \Omega \times \mathbb{R}^+ \to \mathbb{R}^m$, $R : \Omega \times \mathbb{R}^+ \to \mathbb{R}^{m \times k}$ are p.m.s.p. such that for all $T > 0$:

$$\mathbb{E} \left( \int_0^T e^{\gamma r} N_r dr \right)^p + \mathbb{E} \left( \int_0^T e^{2\gamma r} |R_r|^2 dr \right)^{p/2} < \infty, \quad \text{if } p > 0$$

and

$$\int_0^T e^{\gamma r} N_r dr + \int_0^T e^{2\gamma r} |R_r|^2 dr < \infty, \quad \mathbb{P} - \text{a.s., if } p = 0.$$

Clearly $M$ is a continuous p.m.s.p. and for all $p > 0$

$$\mathbb{E} \left( \sup_{t \in [0,T]} e^{p\gamma} |M_t|^p \right) < \infty, \quad \text{for all } T > 0.$$

For a intuitive introduction let $p \geq 1$ and $(Y, Z, U)$ be a strong a solution of (1) or (2) that is $Y, Z$, and $U$ are p.m.s.p., $Y$ has continuous trajectories,

$$\int_0^T e^{2\gamma r} |Z_r|^2 dr + \int_0^T e^{2\gamma r} |U_r|^2 dr < \infty, \quad \mathbb{P} - \text{a.s., for all } T \geq 0;$$

the following equation is satisfied for all $T \geq 0$

$$\begin{cases}
Y_t + \int_t^T dK_r = Y_T + \int_t^T H(r,Y_r,Z_r) dQ_r - \int_t^T Z_r dB_r, \quad \text{a.s. for all } t \in [0,T], \\
K_r = U_r dQ_r \in \mathcal{E}_\gamma Y, dQ_r;
\end{cases}$$

and

$$e^{\gamma r} |Y_t - \xi_t| + \int_0^\infty e^{2\gamma r} |Z_r - \zeta_r|^2 dr \overset{\mathbb{P} \text{ a.s.}}{\to} 0, \quad \text{as } t \to \infty.$$

For $\delta \in (0, 1]$ we define

$$\delta_q := \delta 1_{[1,2]}(q) = \begin{cases}
\delta, & \text{if } 1 \leq q < 2, \\
0, & \text{if } q \geq 2.
\end{cases} \quad (19)$$

Let $q \in [1, 2]$ and $M \in S^q_m(\gamma, N, R; V)$. By Itô’s formula for $(\Gamma_t)^q$, where

$$\Gamma_t := \left( |M_t - Y_t|^2 + \delta_q \right)^{1/2}$$

(see (109) from Proposition 23) with

$$M_t = M_T + \int_t^T N_r dQ_r - \int_t^T R_r dB_r,$$

we deduce by inequality (110) from Remark 24 that for all $0 \leq t \leq s$ and for all $\delta \in (0, 1]$

$$\begin{align*}
&(\Gamma_t)^q + \frac{q}{2} n_q \int_t^s (\Gamma_r)^q |R_r - Z_r|^2 dr + q \int_t^s (\Gamma_r)^q |M_r - Y_r|^2 dQ_r \\
&\leq (\Gamma_s)^q + q \int_t^s (\Gamma_r)^q (M_r - Y_r, N_r - H(r, Y_r, Z_r))dQ_r \\
&\quad - q \int_t^s (\Gamma_r)^q (M_r - Y_r, (R_r - Z_r) dB_r),
\end{align*} \quad (20)$$

7
where \( U_t dQ_t \in \partial_q \Psi (t, Y_t) dQ_t \) and \( n_q := (q - 1) \land 1 = q - 1. \)

Using the subdifferential inequality
\[
\langle M_r - Y_r, U_t dQ_t \rangle + \Psi (r, Y_r) dQ_r \leq \Psi (r, M_r) dQ_r
\]
we get, from (20),
\[
(\Gamma_t)^q + \frac{q}{2} \int_t^s (\Gamma_r)^{q-2} |R_r - Z_r|^2 \, dr + q \int_t^s (\Gamma_r)^{q-2} \Psi (r, Y_r) \, dQ_r
\]
\[
= (\Gamma_s)^q + q \int_t^s (\Gamma_r)^{q-2} \Psi (r, M_r) \, dQ_r + q \int_t^s (\Gamma_r)^{q-2} (M_r - Y_r, N_r - H (r, Y_r, Z_r)) \, dQ_r
\]
\[-q \int_t^s (\Gamma_r)^{q-2} \langle M_r - Y_r, (R_r - Z_r) \rangle dB_r.\] (21)

### 3.2 Definition and preliminary estimates

Following the approach for the forward stochastic variational inequalities from article [20], we propose, stating from (21), the next variational formulation for a solution of the multivalued BSDE (2).

**Definition 4** Let \( V \) be given by definition (12). We say that \((Y_t, Z_t)_{t \geq 0}\) is a \( L^p \)–variational solution of (2) if:

- \( Y : \Omega \times \mathbb{R}_+ \rightarrow \mathbb{R}^m \) and \( Z : \Omega \times \mathbb{R}_+ \rightarrow \mathbb{R}^{m \times k} \) are two \( p \)-m.s.p., \( Y \) has continuous trajectories
  \[
  \mathbb{E} \left( \sup_{r \in [0, T]} e^{pV_r} |Y_r|^p \right) < \infty, \quad \text{for all } T > 0
  \] (22)

  and
  \[
  \mathbb{E} \left( \int_0^T e^{2V_r} |Z_r|^2 \, dr \right)^{p/2} + \mathbb{E} \left( \int_0^T e^{2V_r} \Psi (r, Y_r) \, dQ_r \right)^{p/2} < \infty, \quad \text{for all } T > 0;
  \] (23)

- \( (Y_t, Z_t) = (\xi_t, \zeta_t) = (\eta, 0), \) for \( t > \tau \) and
  \[
  \left( e^{pV_T} |Y_T - \xi_T|^p \right) + \mathbb{E} \left( \int_T^\infty e^{2V_s} |Z_s - \zeta_s|^2 \, ds \right)^{p/2} \xrightarrow{\text{prob.}} 0. \] (24)

- if \( \Gamma_t := \left( |M_t - Y_t|^2 + \delta_q \right)^{1/2}, \) where \( \delta_q \) is defined by (19), it holds
  \[
  (\Gamma_t)^q + \frac{q(q - 1)}{2} \int_t^s (\Gamma_r)^{q-2} |R_r - Z_r|^2 \, dr + q \int_t^s (\Gamma_r)^{q-2} \Psi (r, Y_r) \, dQ_r
  \]
  \[
  \leq (\Gamma_s)^q + q \int_t^s (\Gamma_r)^{q-2} \Psi (r, M_r) \, dQ_r + q \int_t^s (\Gamma_r)^{q-2} (M_r - Y_r, N_r - H (r, Y_r, Z_r)) \, dQ_r
  \]
  \[-q \int_t^s (\Gamma_r)^{q-2} \langle M_r - Y_r, (R_r - Z_r) \rangle dB_r, \] (25)

for any \( q \in \{2, p \land 2\}, \delta \in (0, 1], 0 \leq t \leq s < \infty, \) and \( M \in \mathcal{S}^0_m (\gamma, N, R; V). \)

**Remark 5** For \( q = 2 \) inequality (25) becomes
\[
|M_t - Y_t|^2 + \int_t^s |R_r - Z_r|^2 \, dr + 2 \int_t^s \Psi (r, Y_r) \, dQ_r
\]
\[
\leq |M_s - Y_s|^2 + 2 \int_t^s \Psi (r, M_r) \, dQ_r + 2 \int_t^s \langle M_r - Y_r, (R_r - Z_r) \rangle \, dB_r, \quad \mathbb{P} \text{-a.s.}
\] (26)
which was in [12] the definition of the variational solution in the case \( p \geq 2 \).

**Remark 6** Let

\[
\Lambda_t = \frac{q}{2} (q - 1) \int_0^t (\Gamma_r)^{q-2} |R_r - Z_r|^2 \, dr + q \int_0^t (\Gamma_r)^{q-2} \Psi (r, Y_r) \, dQ_r \\
- q \int_0^t (\Gamma_r)^{q-2} \Psi (r, M_r) \, dQ_r - q \int_0^t (\Gamma_r)^{q-2} \langle M_r - Y_r, N_r - H (r, Y_r, Z_r) \rangle \, dQ_r \\
+ q \int_0^t (\Gamma_r)^{q-2} \langle M_r - Y_r, (R_r - Z_r) dB_r \rangle,
\]

Since from (25) it follows that

\[ t \mapsto (\Gamma_t)^q - \Lambda_t \]

is a nondecreasing stochastic process then \( t \mapsto \Gamma_t^q = [(\Gamma_t)^q - \Lambda_t] + \Lambda_t \) is a semimartingale and consequently for all \( 0 \leq t \leq s \)

\[
e^{\Psi V_r} (\Gamma_s)^q - e^{\Psi V_r} (\Gamma_t)^q = \int_t^s d [e^{\Psi V_r} (\Gamma_r)^q] \\
= q \int_t^s e^{\Psi V_r} (\Gamma_r)^q \, dV_r + \int_t^s e^{\Psi V_r} d [(\Gamma_r)^q - \Lambda_t] + \int_t^s e^{\Psi V_r} d \Lambda_t \\
\geq q \int_t^s e^{\Psi V_r} (\Gamma_r)^q \, dV_r + \int_t^s e^{\Psi V_r} d \Lambda_t
\]

which yields

\[
e^{\Psi V_r} (\Gamma_t)^q + q \int_t^s e^{\Psi V_r} (\Gamma_r)^q \, dV_r + \frac{q}{2} (q - 1) \int_t^s e^{\Psi V_r} (\Gamma_r)^{q-2} |R_r - Z_r|^2 \, dr \\
+ q \int_t^s e^{\Psi V_r} (\Gamma_r)^{q-2} \Psi (r, Y_r) \, dQ_r \\
\leq e^{\Psi V_r} (\Gamma_s)^q + q \int_t^s e^{\Psi V_r} (\Gamma_r)^{q-2} \Psi (r, M_r) \, dQ_r + q \int_t^s e^{\Psi V_r} (\Gamma_r)^{q-2} \langle M_r - Y_r, N_r - H (r, Y_r, Z_r) \rangle \, dQ_r \\
- q \int_t^s e^{\Psi V_r} (\Gamma_r)^{q-2} \langle M_r - Y_r, (R_r - Z_r) dB_r \rangle.
\]

(27)

for any \( q \in \{2, p \wedge \delta\} \), \( \delta \in (0, 1] \), \( 0 \leq t \leq s < \infty \), and \( M \in S_m^q (\gamma, N, R; V) \).

Following the previous calculus, we see that, in fact, inequality (27) holds true for any arbitrary continuous bounded variation p.m.s.p. \( \{V_t : t \geq 0\} \).

**Remark 7** Let \( \{V_t : t \geq 0\} \) be defined by (12) and \( M \in S_m^q (\gamma, N, R; V) \), \( q \in \{2, p \wedge \delta\} \). Since by assumption (8) we have

\[
\mathbb{E} \left( \delta_T \sup_{r \in [0,T]} e^{\Psi V_r} \right) < \infty, \text{ for all } T > 0,
\]

\[ 9 \]
we deduce

\[
\mathbb{E} \left[ \int_0^T e^{2qV_r} (\Gamma_r)^{2q-4} |M_r - Y_r|^2 |R_r - Z_r|^2 \, dr \right]^{1/2} \\
\leq \mathbb{E} \left[ \int_0^T e^{2qV_r} (\Gamma_r)^{2q-2} |R_r - Z_r|^2 \, dr \right]^{1/2} \\
\leq \mathbb{E} \left[ \sup_{r \in [0,T]} e^{(q-1)V_r} (|M_r - Y_r|^2 + \delta_q)^{(q-1)/2} \left( \int_0^T e^{2V_r} |R_r - Z_r|^2 \, dr \right)^{1/2} \right]^{(q-1)/q} \\
\leq \mathbb{E} \left( \sup_{r \in [0,T]} e^{V_r} (|M_r - Y_r|^2 + \delta_q)^{q/2} \right)^{1/q} \mathbb{E} \left( \int_0^T e^{2V_r} |R_r - Z_r|^2 \, dr \right)^{q/2}. 
\]

In the case \( q = p \wedge 2 \), we infer that

\[
\mathbb{E} \left[ \int_0^T e^{2qV_r} (\Gamma_r)^{2q-4} |M_r - Y_r|^2 |R_r - Z_r|^2 \, dr \right]^{1/2} < \infty
\]

and the stochastic integral \( J_t = \int_0^t e^{qV_r} (\Gamma_r)^{q-2} \langle M_r - Y_r, (R_r - Z_r) dB_r \rangle \) is a continuous martingale; therefore for all stopping times \( 0 \leq \sigma \leq \theta \leq T \):

\[
\mathbb{E}^{\mathcal{F}_\sigma} \int_\sigma^T e^{qV_r} (\Gamma_r)^{q-2} \langle M_r - Y_r, (R_r - Z_r) dB_r \rangle = 0.
\]

We also have

\[
\mathbb{E} \int_0^T e^{qV_r} (\Gamma_r)^{q-2} |\langle M_r - Y_r, N_r - H (r, Y_r, Z_r) \rangle| dQ_r \\
\leq \mathbb{E} \int_0^T e^{qV_r} (\Gamma_r)^{q-1} |\langle N_r + |H (r, Y_r, Z_r)\rangle| dQ_r \\
\leq \mathbb{E} \left( \sup_{r \in [0,T]} e^{(q-1)V_r} (|M_r - Y_r|^2 + \delta_q)^{(q-1)/2} \left( \int_0^T e^{V_r} [|N_r + |H (r, Y_r, Z_r)|] dQ_r \right) \right)^{q/q} \\
\leq \left[ \mathbb{E} \left( \sup_{r \in [0,T]} e^{V_r} (|M_r - Y_r|^2 + \delta_q)^{q/2} \right)^{1/q} \mathbb{E} \left( \int_0^T e^{V_r} [|N_r + |H (r, Y_r, Z_r)|] dQ_r, dr \right)^{q/2} \right]^{1/q}. 
\]

Hence if \((Y_t, Z_t)_{t \geq 0}\) is an \(L^p\)—variational solution of (2) then for all \( T \geq 0 \) such that

\[
\mathbb{E} \left( \int_0^T e^{V_r} |H (r, Y_r, Z_r)| dQ_r \right)^{p/2} < \infty
\]
the following inequality is satisfied \( \mathbb{P} - \text{a.s.} \)

\[
e^{qV_r \Gamma^q} \Gamma^q_0 + q \mathbb{E} \int_0^T e^{qV_r \Gamma^q} dV_r + \frac{q}{2} n_q \mathbb{E} \int_0^T e^{qV_r \Gamma^q - 2} \delta |R_r - Z_r|^2 \, dr \\
+ q \mathbb{E} \int_\sigma^T e^{qV_r \Gamma^q - 2} \Psi (r, Y_r, Z_r) \, dQ_r,
\]

(28)

for \( q = p \wedge 2 \) and all \( M \in \mathcal{S}_m^q (\gamma, N, R; V) \) and for all stopping times \( 0 \leq \sigma \leq \theta \leq T \).

**Remark 8** It is obviously that a strong solution \((Y, Z) \in \mathcal{S}_m^q \times \Lambda_{m \times k}^q \) for (2) such that (22), (23) and (24) are satisfied is also an \( L^p \)-variational solution (see the intuitive introduction for inequality (21)).

Conversely, if \((Y, Z)\) is an \( L^p \)-variational solution of the BSDE (2) with \( \varphi = \psi = 0 \), \( V \) is a nondecreasing stochastic process and

\[
\mathbb{E} \left( \int_0^T e^{qV_r} |H (r, Y_r, Z_r)| \, dQ_r \right)^q < \infty, \quad \text{for all } T > 0,
\]

then \((Y, Z)\) is a strong solution of BSDE (2).

Indeed, by [19, Corollary 2.45] there exists a unique pair \((M, R) \in \mathcal{S}_m^0 [0, T] \times \Lambda_{m \times k}^0 (0, T) \) such that

\[ M_t = Y_T + \int_t^T H (r, Y_r, Z_r) \, dQ_r - \int_t^T R_r \, dB_r \]

and

\[
\mathbb{E} \sup_{t \in [0, T]} |e^{qV_t} M_t|^q + \mathbb{E} \left( \int_0^T e^{qV_r} |R_r|^2 \, dr \right)^{q/2} < \infty.
\]

With this \( M \) the inequality (27) becomes, \( \mathbb{P} - \text{a.s.} \)

\[
e^{qV_t (\Gamma_t)^q} + q \int_t^s e^{qV_r (\Gamma_r)^q} \, dV_r + \frac{q}{2} (q - 1) \int_t^s e^{qV_r (\Gamma_r)^q - 2} |R_r - Z_r|^2 \, dr \\
\leq e^{qV_t (\Gamma_t)^q} - q \int_t^s e^{qV_r (\Gamma_r)^q - 2} (M_r - Y_r, (R_r - Z_r) \, dB_r).
\]

for any \( q \in \left\{ 2, p \wedge 2 \right\}, \delta \in (0, 1], 0 \leq t \leq s < \infty.

By Remark 7 for \( q = p \wedge 2 \) the stochastic integral is a martingale and therefore since \( 0 < \delta \leq 1 \), we obtain \( \mathbb{P} - \text{a.s.} \)

\[
e^{qV_t (\Gamma_t)^q} + \frac{q}{2} (q - 1) \mathbb{E} \int_t^T e^{qV_r} \frac{|R_r - Z_r|^2}{(|M_r - Y_r|^2 + 1)^{(2-q)/2}} \, dr \leq (\delta_q)^q \mathbb{E} \mathcal{F}_t e^{qV_T} \quad \text{for all } 0 \leq t \leq T. \quad (29)
\]

Passing to limit as \( \delta \to 0_+ \), by Fatou’s Lemma we obtain, \( \mathbb{P} - \text{a.s.} \)

\[
e^{qV_t} |M_t - Y_t|^q + \frac{q}{2} (q - 1) \mathbb{E} \int_t^T e^{qV_r} \frac{|R_r - Z_r|^2}{(|M_r - Y_r|^2 + 1)^{(2-q)/2}} \, dr = 0, \quad \forall t \in [0, T].
\]
that clearly yields \((M, R) = (Y, Z)\) in \(S_m^0 [0, T] \times \Lambda_{m \times k}^T (0, T)\). Consequently
\[
Y_t = Y_T + \int_t^T H (r, Y_r, Z_r) \, dQ_r - \int_t^T Z_r \, dB_r.
\]

**Proposition 9** Let \(M \in S_m^0 (\gamma, N, R; V)\). Let \(Y : \Omega \times \mathbb{R}_+ \rightarrow \mathbb{R}^m\) and \(Z : \Omega \times \mathbb{R}_+ \rightarrow \mathbb{R}^{m \times k}\) be two p.m.s.p. with \(Y\) having continuous trajectories and
\[
\begin{align*}
(\ i & \ \int_0^T e^{2V_r} |R_r - Z_r|^2 \, dr + \int_0^T e^{2V_r} \Psi (r, Y_r) \, dQ_r < \infty, \quad \text{for all } T > 0, \ \mathbb{P} \ - \ a.s., \\
(ii) & \ \Psi (r, M_r) \leq 1_{q \geq 2} \Psi (r, M_r) \\
(iii) & \ \langle M_r - Y_r, N_r \rangle \, dQ_r \leq |M_r - Y_r| \, dL_r
\end{align*}
\]
with \(L\) an increasing and continuous p.m.s.p. \(L_0 = 0\).

**I. If inequality (25) holds for \(q = 2\), then for all \(k > 0\) and for any stopping times \(0 \leq \sigma \leq \theta < \infty\)
\[
\begin{align*}
& \mathbb{E}^{\mathcal{F}_\sigma} \left( \int_0^\theta e^{2V_r} |R_r - Z_r|^2 \, dr \right)^{k/2} + \mathbb{E}^{\mathcal{F}_\sigma} \left( \int_0^\theta e^{2V_r} \Psi (r, Y_r) \, dQ_r \right)^{k/2} \\
& \leq C_{k, \lambda} \left[ \mathbb{E}^{\mathcal{F}_\sigma} \sup_{r \in [\sigma, \theta]} e^{kV_r} |M_r - Y_r|^k + \mathbb{E}^{\mathcal{F}_\sigma} \left( \int_\sigma^\theta e^{V_r} \Psi (r, M_r) \, dQ_r \right)^{k/2} \right] \\
& + \mathbb{E}^{\mathcal{F}_\sigma} \left( \int_\sigma^\theta e^{V_r} |M_r - Y_r| \, dL_r + |H (r, M_r, R_r)| \, dQ_r \right)^{k/2} \\
& \leq 2C_{k, \lambda} \left[ \mathbb{E}^{\mathcal{F}_\sigma} \sup_{r \in [\sigma, \theta]} e^{kV_r} |M_r - Y_r|^k + \mathbb{E}^{\mathcal{F}_\sigma} \left( \int_\sigma^\theta e^{V_r} \Psi (r, M_r) \, dQ_r \right)^{k/2} \right] \\
& + \mathbb{E}^{\mathcal{F}_\sigma} \left( \int_\sigma^\theta e^{V_r} |dL_r + |H (r, M_r, R_r)| \, dQ_r| \right)^{k/2}, \ \mathbb{P} \ - \ a.s.
\end{align*}
\]
In particular for \(\gamma = 0, N = 0, R = 0, L = 0, M = 0, \Psi (r, M) = \Psi (r, 0) = 0\) it follows
\[
\begin{align*}
& \mathbb{E}^{\mathcal{F}_\sigma} \left( \int_\sigma^\theta e^{2V_r} |Z_r|^2 \, dr \right)^{k/2} + \mathbb{E}^{\mathcal{F}_\sigma} \left( \int_\sigma^\theta e^{2V_r} \Psi (r, Y_r) \, dQ_r \right)^{k/2} \\
& \leq C_{k, \lambda} \left[ \mathbb{E}^{\mathcal{F}_\sigma} \sup_{r \in [\sigma, \theta]} e^{kV_r} |Y_r|^k + \mathbb{E}^{\mathcal{F}_\sigma} \left( \int_\sigma^\theta e^{V_r} |Y_r| \, |H (r, 0, 0)| \, dQ_r \right)^{k/2} \right] \\
& \leq 2C_{k, \lambda} \left[ \mathbb{E}^{\mathcal{F}_\sigma} \sup_{r \in [\sigma, \theta]} e^{kV_r} |Y_r|^k + \mathbb{E}^{\mathcal{F}_\sigma} \left( \int_\sigma^\theta e^{V_r} |H (r, 0, 0)| \, dQ_r \right)^{k} \right], \ \mathbb{P} \ - \ a.s.
\end{align*}
\]

**II. If inequality (25) holds and for some fixed stopping times \(0 \leq \sigma \leq \theta < \infty, \ 1 < q \leq k\)
\[
\mathbb{E} \left( \sup_{r \in [\sigma, \theta]} e^{kV_r} |M_r - Y_r|^k \right) < \infty,
\]

\(12\)
then
\[\mathbb{E}^{F_r} \sup_{r \in [\sigma, \theta]} e^{kV_r} |M_r - Y_r|^k \leq C_{\lambda, q, k} \left( \mathbb{E}^{F_r} e^{kV_\theta} |M_\theta - Y_\theta|^k + \mathbb{E}^{F_r} \left( \int_0^\theta e^{qV_r} |M_r - Y_r|^{q-2} |Y_r| dQ_r \right)^{k/q} \right) \]

\[\mathbb{E}^{F_r} \left( \sup_{r \in [\sigma, \theta]} e^{kV_r} |M_r - Y_r|^k \right) + \mathbb{E}^{F_r} \left( \int_0^\theta e^{qV_r} |M_r - Y_r|^{q-2} |Y_r| dQ_r \right)^{k/q} \]

\[\leq C_{\lambda, q, k} \left( \mathbb{E}^{F_r} e^{kV_\theta} |M_\theta - Y_\theta|^k + \mathbb{E}^{F_r} \left( \int_0^\theta e^{qV_r} |Y_r|^{q-2} |Z_r|^2 dQ_r \right)^{k/q} \right) \leq C_{\lambda, q, k} \mathbb{E}^{F_r} \left( e^{kV_\theta} |Y_\theta|^k + \left( \int_0^\theta e^{qV_r} |H (r, 0, 0)| dQ_r \right)^{k/q} \right) \]

**Proof.** Using the monotonicity of \( H \):

\[\langle M_r - Y_r, -H (r, Y_r, Z_r) dQ_r \rangle\]

\[= \langle M_r - Y_r, -H (r, M_r, R_r) dQ_r \rangle + \langle M_r - Y_r, H (r, M_r, R_r) - H (r, Y_r, Z_r) dQ_r \rangle\]

\[\leq |M_r - Y_r| |H (r, M_r, R_r)| dQ_r + |M_r - Y_r|^2 dV_r + \frac{\alpha p \lambda}{2} |R_r - Z_r|^2 ds\]
we obtain from (25):
\[
(\Gamma_t)^q + \frac{q}{2} (q - 1 - n_p \lambda) \int_t^s (\Gamma_r)^{q-2} |R_r - Z_r|^2 \, dr + q \delta q \int_t^s (\Gamma_r)^{q-2} \, dV_r
+ q \int_t^s (\Gamma_r)^{q-2} \Psi (r, Y_r) \, dQ_r
\leq (\Gamma_s)^q + q \int_t^s (\Gamma_r)^q \, dV_r + q \int_t^s (\Gamma_r)^{q-2} 1_{q \geq 2} \Psi (r, M_r) \, dQ_r
+ q \int_t^s (\Gamma_r)^{q-2} |M_r - Y_r| \, dL_r + |H (r, M_r, R_r)| \, dQ_r
- q \int_t^s (\Gamma_r)^{q-2} (M_r - Y_r, (R_r - Z_r) \, dB_r).
\]

for all \(0 \leq t \leq s < \infty\).
Since \(p > 1\) and \(q \in \{2, p \wedge 2\}\), then \(n_p = (p - 1) \wedge 1 \leq q - 1\) and
\[(q - 1)(1 - \lambda) \leq (q - 1 - n_p \lambda).
\]

Applying a Gronwall’s type stochastic inequality (see Lemma 12 from the Appendix of [14]) we conclude that
\[
e^{qV_t} (\Gamma_t)^q + \frac{q}{2} (q - 1) (1 - \lambda) \int_t^s e^{qV_r} (\Gamma_r)^{q-2} |R_r - Z_r|^2 \, dr + q \delta q \int_t^s e^{qV_r} (\Gamma_r)^{q-2} \, dV_r
+ q \int_t^s e^{qV_r} (\Gamma_r)^{q-2} \Psi (r, Y_r) \, dQ_r
\leq e^{qV_t} (\Gamma_s)^q + q \int_t^s (\Gamma_r)^{q-2} 1_{q \geq 2} \Psi (r, M_r) \, dQ_r
+ q \int_t^s e^{qV_r} (\Gamma_r)^{q-2} |M_r - Y_r| \, dL_r + |H (r, M_r, R_r)| \, dQ_r
- q \int_t^s e^{qV_r} (\Gamma_r)^{q-2} (M_r - Y_r, (R_r - Z_r) \, dB_r).
\]

I. Writing (37) for \(q = 2\) we get
\[
e^{2V_t} |M_t - Y_t|^2 + (1 - \lambda) \int_t^s e^{2V_r} |R_r - Z_r|^2 \, dr + 2 \delta q \int_t^s e^{2V_r} \Psi (r, Y_r) \, dQ_r
\leq e^{2V_t} |M_s - Y_s|^2 + 2 \int_t^s \Psi (r, M_r) \, dQ_r + 2 \int_t^s e^{2V_r} |M_r - Y_r| \, dL_r + |H (r, M_r, R_r)| \, dQ_r
- 2 \int_t^s e^{2V_r} (M_r - Y_r, (R_r - Z_r) \, dB_r), \ \mathbb{P}\text{-a.s.}
\]

for all \(0 \leq t \leq s < \infty\), that yields (30) by Proposition 17 from Appendix.

II. Using Fatou’s Lemma, Lebesgue dominated convergence theorem and the continuity in probabil-
ity of the stochastic integral we clearly deduce from (37), as $\delta \to 0+$, that:

$$e^{qV_t} |M_t - Y_t|^q + \frac{q}{2} (q - 1) (1 - \lambda) \int_t^s e^{qV_r} |M_r - Y_r|^{q-2} |R_r - Z_r|^2 dr$$

$$+ q \int_t^s e^{qV_r} |M_r - Y_r|^{q-2} \Psi (r, Y_r) dQ_r$$

$$\leq e^{qV_s} |M_s - Y_s|^q + q \int_t^s |M_r - Y_r|^{q-2} 1_{q \geq 2} \Psi (r, M_r) dQ_r$$

$$+ q \int_t^s e^{qV_r} |M_r - Y_r|^{q-1} [dL_r + |H (r, M_r, R_r)|] dQ_r$$

$$- q \int_t^s e^{qV_r} (|M_r - Y_r|^{q-2} (M_r - Y_r), (R_r - Z_r) dB_r).$$

Using Proposition 18 from inequality (38) we get (33) and (34).

### 3.3 Uniqueness and Continuity

**Theorem 10** Let $p > 1$, $q = p \wedge 2$ and the assumptions $(A_1 - A_6)$ be satisfied. Then the backward stochastic variational inequality (2) has at most solution $(Y, Z)$ in the sense of Definition 4.

Moreover, if $(\tilde{Y}, \tilde{Z})$ and $(\tilde{Y}, \tilde{Z})$ are two $L^p-$variational solutions of (2) corresponding to $(\tilde{\eta}, \tilde{H})$ and $(\tilde{\eta}, \tilde{H})$ respectively, where $\tilde{H}$ and $\tilde{H}$ have the same coefficients $\mu, \nu, \ell$ (there are constants functions), then for any stopping time $\sigma$, $0 \leq \sigma \leq \tau$, it holds, $\mathbb{F}$–a.s.,

$$qV_\sigma |\tilde{Y}_\sigma - \tilde{Y}_\sigma|^q + c_{q,\lambda} \times E^{F_\sigma} \int_\sigma^\tau e^{qV_r} \frac{|\tilde{Z}_r - Z_r|^2}{(|\tilde{Y}_r - \tilde{Y}_r| + 1)^{2-q}} dr$$

$$\leq E^{F_\sigma} e^{qV_\sigma} |\tilde{\eta} - \tilde{\eta}|^q + q E^{F_\sigma} \int_\sigma^\tau e^{qV_r} |\tilde{Y}_r - \tilde{Y}_r|^{q-1} |H (r, \tilde{Y}_r, \tilde{Z}_r) - \tilde{H} (r, \tilde{Y}_r, \tilde{Z}_r)| dQ_r$$

$$\leq E^{F_\sigma} e^{qV_\sigma} |\tilde{\eta} - \tilde{\eta}|^q + M_{\sigma, \tau} \left[ E^{F_\sigma} \left( \int_\sigma^\tau e^{V_r} |H (r, \tilde{Y}_r, \tilde{Z}_r) - \tilde{H} (r, \tilde{Y}_r, \tilde{Z}_r)| dQ_r \right)^q \right]^{1/q},$$

where

$$M_{\sigma, \tau} = C_{q, \lambda} \left[ E^{F_\sigma} \left( e^{qV_\sigma} |\tilde{\eta}|^q + \left( \int_\sigma^\tau e^{V_r} |H (r, 0, 0)| dQ_r \right)^q \right) \right]^{(q-1)/q}$$

and $c_{q, \lambda}, C_{q, \lambda}$ are positive constants depending only $q$ and $\lambda$.

Moreover, for all $0 < \alpha < 1$

$$E \sup_{t \in [0, \tau]} e^{\alpha q V_t} |\tilde{Y}_t - \tilde{Y}_t|^\alpha q + \left( E \int_0^\tau \frac{1}{e^{V_r} |\tilde{Y}_r - \tilde{Y}_r| + 1} e^{2V_r} |\tilde{Z}_r - Z_r|^2 dr \right)^\alpha$$

$$\leq C_{\alpha, q, \lambda} \left[ E e^{q V_\sigma} |\tilde{\eta} - \tilde{\eta}|^q + K \left( E \left( \int_0^\tau e^{V_r} |H (r, \tilde{Y}_r, \tilde{Z}_r) - \tilde{H} (r, \tilde{Y}_r, \tilde{Z}_r)| dQ_r \right)^q \right)^{1/q} \right]^\alpha.$$
where
\[
K = \mathbb{E} \left( e^{q V_r |\hat{\eta}|^q} + \left( \int_0^T e^{q V_r |\hat{H}(r, 0, 0)| dQ_r} \right)^q \right) \\
+ \mathbb{E} \left( e^{q V_r |\hat{\eta}|^q} + \left( \int_0^T e^{q V_r |\tilde{H}(r, 0, 0)| dQ_r} \right)^q \right)^{(q-1)/q}.
\]
and \(C_{\alpha,q,\lambda}\) is a positive constant depending only on \((\alpha, q, \lambda)\).

**Proof.** Let \((\hat{Y}, \hat{Z})\) and \((\tilde{Y}, \tilde{Z})\) be two \(L^p\)-variational solutions of (2) corresponding to \((\hat{\eta}, \hat{H})\) and \((\tilde{\eta}, \tilde{H})\) respectively, where \(\hat{H}\) and \(\tilde{H}\) have the same constants functions \(\mu, \nu, \ell\) as monotonicity and Lipschitz coefficients. Let \(M \in S_\eta^n (\gamma, N, R; V)\) and
\[
\hat{\Gamma}_t = \left( |M_t - \hat{Y}_t|^2 + \delta_q \right)^{1/2} \text{ and } \tilde{\Gamma}_t = \left( |M_t - \tilde{Y}_t|^2 + \delta_q \right)^{1/2}
\]
Let \(T > 0\) and the stopping times \(0 \leq \sigma \leq \theta \leq T\) such that
\[
\mathbb{E} \left( \int_\sigma^\theta e^{q V_r} \left[ |\hat{H}(r, \hat{Y}_r, \hat{Z}_r)| + |\tilde{H}(r, \tilde{Y}_r, \tilde{Z}_r)| \right] dQ_r \right)^{q < \infty}
\]
From (28) we deduce that \(P - a.s.
\[
[q \mathbb{E}_{\mathcal{F}_\sigma} \int_\sigma^\theta e^{q V_r} (\hat{\Gamma}_r)^q + (\hat{\Gamma}_r)^q] dV_r \\
+ \frac{q(q-1)}{2} \mathbb{E}_{\mathcal{F}_\sigma} \int_\sigma^\theta e^{q V_r} (\hat{\Gamma}_r)^q \left[ |R_r - \hat{Z}_r|^2 + (\hat{\Gamma}_r)^{-2} |R_r - \hat{Z}_r|^2 \right] dr \\
+ q \mathbb{E}_{\mathcal{F}_\sigma} \int_\sigma^\theta e^{q V_r} [\hat{\Gamma}_r]^q \Psi \left( r, \hat{Y}_r \right) + (\hat{\Gamma}_r)^{-2} \Psi \left( r, \hat{Y}_r \right)] dQ_r
\leq q \mathbb{E}_{\mathcal{F}_\sigma} \int_\sigma^\theta e^{q V_r} \left[ (\hat{\Gamma}_r)^{-2} + \left( \hat{\Gamma}_r \right)^q \right] \Psi (r, M_r) dQ_r \\
+ q \mathbb{E}_{\mathcal{F}_\sigma} \int_\sigma^\theta e^{q V_r} \left( \hat{\Gamma}_r \right)^q \Psi \left( r, \hat{Y}_r, \hat{Y}_r, \hat{Z}_r, \hat{Z}_r \right) dQ_r
\]
Let \(Y_r = \frac{1}{2} \left( \hat{Y}_r + \tilde{Y}_r \right)\). We have for all \(\beta > 0\)
\[
|M - \hat{Y}|^2 \leq \frac{1 + \beta}{\beta} |M - Y|^2 + \frac{1 + \beta}{4} |\hat{Y} - \tilde{Y}|^2, \hspace{1cm} \text{and}
|M - \tilde{Y}|^2 \leq \frac{1 + \beta}{\beta} |M - Y|^2 + \frac{1 + \beta}{4} |\hat{Y} - \tilde{Y}|^2
\]
and taking in account that \(1 \leq q \leq 2\) then
\[
(\hat{\Gamma})^q |R - \hat{Z}|^2 + (\hat{\Gamma})^q |R - \hat{Z}|^2
\leq \left( |M - \hat{Y}|^2 + \delta_q \right)^{(q-2)/2} |R - \hat{Z}|^2 + \left( |M - \tilde{Y}|^2 + \delta_q \right)^{(q-2)/2} |R - \hat{Z}|^2
\geq \left[ \frac{1 + \beta}{\beta} |M - Y|^2 + \frac{1 + \beta}{4} |\hat{Y} - \tilde{Y}|^2 + \delta_q \right]^{(q-2)/2} \left[ |R - \hat{Z}|^2 + |R - \hat{Z}|^2 \right]
\geq \frac{1}{2} \left[ \frac{1 + \beta}{\beta} |M - Y|^2 + \frac{1 + \beta}{4} |\hat{Y} - \tilde{Y}|^2 + \delta_q \right]^{(q-2)/2} |\hat{Z} - \tilde{Z}|^2.
\]
Hence

$$\hat{\Gamma}^{q-2}|R - \hat{Z}|^2 + (\hat{\Gamma})^{q-2}|R - \hat{Z}|^2 \geq \frac{1}{2} \left[ \frac{1 + \beta}{\beta} M - Y |^2 + \frac{1 + \beta}{4} |\hat{Y} - \hat{Y}|^2 + \delta_q \right] \frac{(q-2)/2}{|\hat{Z} - \hat{Z}|^2}. \quad (42)$$

Let $0 < \varepsilon \leq 1$ and

$$M^\varepsilon_t = \mathbb{E}^\mathbb{F}_t \int_{t \wedge \varepsilon}^\infty \frac{1}{Q_\varepsilon} e^{-\frac{Q_{r - Q_{r \wedge \varepsilon}}}{Q_\varepsilon}} Y_r dQ_r, \quad t \geq 0.$$ 

with $Y_r = \frac{\hat{Y}_r + \hat{Y}_{r \wedge \varepsilon}}{2}$. 

Then by Proposition 28 $(M^\varepsilon, R^\varepsilon) \in S^p_m \times \Lambda^p_{m \times k}$ is the unique solution of the BSDE:

$$\begin{cases} 
M^\varepsilon_t = M^\varepsilon_T + \int_t^T 1_{[\varepsilon, \infty)} (r) \frac{1}{Q_\varepsilon} (Y_r - M^\varepsilon_r) dQ_r - \int_t^T R^\varepsilon_s dB_s, \quad \text{for any } T > 0, \quad t \in [0, T], \\
\lim_{T \to \infty} \mathbb{E} |M^\varepsilon_T - \xi_T|^p = 0.
\end{cases}$$

and

(a) $|M^\varepsilon_t| \leq \mathbb{E}^\mathbb{F}_t \sup_{r \geq 0} |Y_r|$, a.s., for all $t \geq 0$,

(b) $\lim_{\varepsilon \to 0} M^\varepsilon_t = Y_t$, $\mathbb{P}$-a.s., for all $t \geq 0$.

(c) $\lim_{\varepsilon \to 0} \sup_{t \in [0, T]} |M^\varepsilon_t - Y_t|^p = 0$, for all $T > 0$.

We replace $M$ by $M^\varepsilon$ in (41) and remark that

$$\langle M^\varepsilon_{r \wedge \varepsilon} - \hat{Y}_{r \wedge \varepsilon}, N^\varepsilon_{r \wedge \varepsilon} \rangle = \langle M^\varepsilon_{r \wedge \varepsilon} - \hat{Y}_{r \wedge \varepsilon}, \frac{1}{Q_\varepsilon} (Y_r - M^\varepsilon_r) \rangle = \frac{1}{2Q_\varepsilon} \langle M^\varepsilon_{r \wedge \varepsilon} - \hat{Y}_{r \wedge \varepsilon}, (\hat{Y}_{r \wedge \varepsilon} - M^\varepsilon_{r \wedge \varepsilon}) + (\hat{Y}_{r \wedge \varepsilon} - M^\varepsilon_{r \wedge \varepsilon}) \rangle$$

$$\leq \frac{1}{2Q_\varepsilon} \left[ |M^\varepsilon_{r \wedge \varepsilon} - \hat{Y}_{r \wedge \varepsilon}|^2 + |M^\varepsilon_{r \wedge \varepsilon} - \hat{Y}_{r \wedge \varepsilon}|M^\varepsilon_{r \wedge \varepsilon} - \hat{Y}_{r \wedge \varepsilon}| \right] = \frac{1}{2Q_\varepsilon} \left[ |M^\varepsilon_{r \wedge \varepsilon} - \hat{Y}_{r \wedge \varepsilon}|^2 + |M^\varepsilon_{r \wedge \varepsilon} - \hat{Y}_{r \wedge \varepsilon}|M^\varepsilon_{r \wedge \varepsilon} - \hat{Y}_{r \wedge \varepsilon}| \right]$$

and similar

$$\langle M^\varepsilon_{r \wedge \varepsilon} - \hat{Y}_{r \wedge \varepsilon}, N^\varepsilon_{r \wedge \varepsilon} \rangle \leq \frac{1}{2Q_\varepsilon} \left[ |M^\varepsilon_{r \wedge \varepsilon} - \hat{Y}_{r \wedge \varepsilon}| - |M^\varepsilon_{r \wedge \varepsilon} - \hat{Y}_{r \wedge \varepsilon}| \right] |M^\varepsilon_{r \wedge \varepsilon} - \hat{Y}_{r \wedge \varepsilon}|.$$

Hence

$$\left[ (\hat{\Gamma})^{q-2} \langle M^\varepsilon_{r \wedge \varepsilon} - \hat{Y}_{r \wedge \varepsilon}, N^\varepsilon_{r \wedge \varepsilon} \rangle + (\hat{\Gamma})^{q-2} \langle M^\varepsilon_{r \wedge \varepsilon} - \hat{Y}_{r \wedge \varepsilon}, N^\varepsilon_{r \wedge \varepsilon} \rangle \right]$$

$$= \left( |M^\varepsilon_{r \wedge \varepsilon} - \hat{Y}_{r \wedge \varepsilon}|^2 + \delta_q \right)^{(q-2)/2} \langle M^\varepsilon_{r \wedge \varepsilon} - \hat{Y}_{r \wedge \varepsilon}, N^\varepsilon_{r \wedge \varepsilon} \rangle + \left( |M^\varepsilon_{r \wedge \varepsilon} - \hat{Y}_{r \wedge \varepsilon}|^2 + \delta_q \right)^{(q-2)/2} \langle M^\varepsilon_{r \wedge \varepsilon} - \hat{Y}_{r \wedge \varepsilon}, N^\varepsilon_{r \wedge \varepsilon} \rangle$$

$$\leq \frac{1}{2Q_\varepsilon} \left[ \left( |M^\varepsilon_{r \wedge \varepsilon} - \hat{Y}_{r \wedge \varepsilon}|^2 + \delta_q \right)^{(q-2)/2} |M^\varepsilon_{r \wedge \varepsilon} - \hat{Y}_{r \wedge \varepsilon}| - \left( |M^\varepsilon_{r \wedge \varepsilon} - \hat{Y}_{r \wedge \varepsilon}|^2 + \delta_q \right)^{(q-2)/2} |M^\varepsilon_{r \wedge \varepsilon} - \hat{Y}_{r \wedge \varepsilon}| \right]$$

$$\cdot \left[ |M^\varepsilon_{r \wedge \varepsilon} - \hat{Y}_{r \wedge \varepsilon}| - |M^\varepsilon_{r \wedge \varepsilon} - \hat{Y}_{r \wedge \varepsilon}| \right] \leq 0 \quad (43)$$

since for all $a, b \geq 0, \delta \geq 0$ and $\beta \geq -1/2$ we have

$$\left[ (a^2 + \delta)^{\beta} a - (b^2 + \delta)^{\beta} b \right] (a - b) \geq 0.$$
We use inequalities (42) and (43) in (41) for $M = M^*$ and it follows:

$$
\begin{align*}
&\left[e^{\varphi_{V_k}} (\hat{\Gamma}_r)^q + e^{\varphi_{V_k}} (\hat{\Gamma}_r)^q\right] + q \mathbb{E}^{F}_\sigma \int_0^\sigma e^{\varphi_{V_k}} \left[ (\hat{\Gamma}_r)^q + (\hat{\Gamma}_r)^q \right] \, dV_r \\
&+ q \left(\frac{q}{2}\right) \mathbb{E}^{F}_\sigma \int_0^\sigma e^{\varphi_{V_k}} \left[ 1 + \frac{\beta}{\beta} |M^* - Y|^2 + \frac{4}{\beta} |\bar{Y}_r - \bar{Y}_r|^2 + \delta_q \right] \frac{(q-2)^2}{2} |\bar{Z}_r - \bar{Z}_r|^2 \, dr \\
&+ q \mathbb{E}^{F}_\sigma \int_0^\sigma e^{\varphi_{V_k}} \left[ (\hat{\Gamma}_r)^q - 2 \Psi (r, \bar{Y}_r) + (\hat{\Gamma}_r)^q \right] \, dQ_r \\
&\leq \mathbb{E}^{F}_\sigma \left[ e^{\varphi_{V_k}} (\hat{\Gamma}_r)^q + e^{\varphi_{V_k}} (\hat{\Gamma}_r)^q\right] + q \mathbb{E}^{F}_\sigma \int_0^\sigma e^{\varphi_{V_k}} \left[ (\hat{\Gamma}_r)^q + (\hat{\Gamma}_r)^q \right] \Psi (r, M^*_r) \, dQ_r \\
&+ q \mathbb{E}^{F}_\sigma \int_0^\sigma e^{\varphi_{V_k}} \left[ (\hat{\Gamma}_r)^q - 2 \Psi (r, \bar{Y}_r) + (\hat{\Gamma}_r)^q \right] \, dQ_r.
\end{align*}
$$

(44)

Let $0 \leq t \leq s \leq T$, $0 < u \leq r \leq v$ and the stopping times $v^* = Q^{-1}_v$, $u^* = Q^{-1}_u$, $r^* = Q^{-1}_r$, where $Q^{-1}(\omega)$ is the inverse mapping of the function $r \mapsto Q_r(\omega) : [0, \infty) \to [0, \infty)$ and, for each $k, i \in \mathbb{N}^*$, the stopping times

$$
\alpha_k = \inf \left\{ u \geq 0 : \mathbb{1} V_{\alpha_k} + \sup_{r \in [0, u]} |e^{V_r} \bar{Y}_r - \bar{Y}_r| + \sup_{r \in [0, u]} |e^{V_r} \bar{Y}_r - \bar{Y}_r| + \int_0^u e^{2V_r} |\bar{Z}_r|^2 \, dr \\
+ \int_0^u e^{2V_r} Z_r \, dr + \int_0^u e^{V_r} H (r, \bar{Y}_r, \bar{Z}_r) \, dQ_r + \int_0^u e^{V_r} H (r, \bar{Y}_r, \bar{Z}_r) \, dQ_r \\
+ \int_0^u e^{2V_r} \Psi (r, \bar{Y}_r) \, dQ_r + \int_0^u e^{2V_r} \Psi (r, \bar{Y}_r) \, dQ_r \geq k \right\}
$$

and

$$
u^*_{k+i} := t \wedge u^* \wedge \alpha_k \quad \text{and} \quad u^*_{k+i} := s \wedge v^* \wedge \alpha_{k+i}.
$$

We put in (44)

$$
\sigma = u^*_k \quad \text{and} \quad \theta = v^*_{k+i}.
$$

Now passing to $\lim \inf_{\varepsilon \searrow 0}$ in (44) we obtain (using Proposition 28-(3), Fatou’s Lemma and Lebesgue dominated convergence theorem):

$$
\begin{align*}
2 e^{\varphi_{V_k}} \left( \frac{1}{4} |\bar{Y}_r - \bar{Y}_r|^2 + \delta_q \right)^{q/2} + 2q \mathbb{E}^{F}_{u^*_k} \int_{u^*_k}^{v^*_{k+i}} e^{\varphi_{V_k}} \left( \frac{1}{4} |\bar{Y}_r - \bar{Y}_r|^2 + \delta_q \right)^{q/2} \, dV_r \\
+ q \left(\frac{q}{2}\right) \mathbb{E}^{F}_{u^*_k} \int_{u^*_k}^{v^*_{k+i}} e^{\varphi_{V_k}} \left[ 1 + \frac{\beta}{\beta} |\bar{Y}_r - \bar{Y}_r|^2 + \delta_q \right] \frac{(q-2)^2}{2} |\bar{Z}_r - \bar{Z}_r|^2 \, dr \\
+ q \mathbb{E}^{F}_{u^*_k} \int_{u^*_k}^{v^*_{k+i}} e^{\varphi_{V_k}} \left[ (\hat{\Gamma}_r)^q - 2 \Psi (r, \bar{Y}_r) + (\hat{\Gamma}_r)^q \right] \, dQ_r \\
\leq 2 \mathbb{E}^{F}_{u^*_k} e^{\varphi_{V_k}} \left( \frac{1}{4} |\bar{Y}_r - \bar{Y}_r|^2 + \delta_q \right)^{q/2} \\
+ q \mathbb{E}^{F}_{u^*_k} \int_{u^*_k}^{v^*_{k+i}} e^{\varphi_{V_k}} \left[ 1 + \frac{\beta}{\beta} |\bar{Y}_r - \bar{Y}_r|^2 + \delta_q \right] \frac{(q-2)^2}{2} |\bar{Z}_r - \bar{Z}_r|^2 \, dr \\
+ q \mathbb{E}^{F}_{u^*_k} \int_{u^*_k}^{v^*_{k+i}} e^{\varphi_{V_k}} \left[ (\hat{\Gamma}_r)^q - 2 \Psi (r, \bar{Y}_r) + (\hat{\Gamma}_r)^q \right] \, dQ_r \\
+ \frac{q}{2} \mathbb{E}^{F}_{u^*_k} \int_{u^*_k}^{v^*_{k+i}} e^{\varphi_{V_k}} \left[ (\hat{\Gamma}_r)^q - 2 \Psi (r, \bar{Y}_r) + (\hat{\Gamma}_r)^q \right] \, dQ_r.
\end{align*}
$$

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By Fatou’s Lemma passing to limit as \( \beta \to 0 \) this last inequality is true for \( \beta = 0 \).

We remark now that
\[
2 \Psi (r, Y_r) = 2 \Psi \left(r, \frac{1}{2} \tilde{Y}_r + \frac{1}{2} \hat{Y}_r \right) \leq \Psi(r, \tilde{Y}_r) + \Psi(r, \hat{Y}_r).
\]

and
\[
\langle \tilde{Y}_r - \hat{Y}_r, \tilde{H}(r, \tilde{Y}_r, \tilde{Z}_r) - \hat{H}(r, \tilde{Y}_r, \hat{Z}_r) \rangle + \alpha \langle \tilde{Y}_r - \hat{Y}_r, \beta \tilde{Z}_r - \hat{Z}_r \rangle = 0 \text{ dr}.
\]

Hence
\[
2 \mathbb{E} e^{q V_{\tilde{Z}_r}} \left( \frac{1}{4} |\tilde{Y}_{u_k} - \hat{Y}_{u_k}|^2 + \delta_q \right)^{q/2} + 2 q \delta_q \mathbb{E} F_{\tilde{Z}_r} \int_{u_k}^{v_{k+1}} e^{q V_r} \left( \frac{1}{4} |\tilde{Y}_r - \tilde{Y}_r|^2 + \delta_q \right)^{(q-2)/2} \text{ d}V_r
\]
\[
\quad + \frac{q}{2} \left( q - 1 - \frac{n_p \lambda}{2} \right) \mathbb{E} F_{\tilde{Z}_r} \int_{u_k}^{v_{k+1}} e^{q V_r} \left( \frac{1}{4} |\tilde{Y}_r - \tilde{Y}_r|^2 + \delta_q \right)^{(q-2)/2} |\tilde{Z}_r - \hat{Z}_r|^2 \text{ dr}
\]
\[
\quad \leq 2 \mathbb{E} F_{\tilde{Z}_r} e^{q V_{\tilde{Z}_r}} \left( \frac{1}{4} |\tilde{Y}_{v_{k+1}} - \tilde{Y}_{v_{k+1}}|^2 + \delta_q \right)^{q/2}
\]
\[
\quad + \frac{q}{2} \mathbb{E} F_{\tilde{Z}_r} \int_{u_k}^{v_{k+1}} e^{q V_r} \left( \frac{1}{4} |\tilde{Y}_r - \tilde{Y}_r|^2 + \delta_q \right)^{(q-2)/2} (\tilde{Y}_r - \hat{Y}_r, \tilde{H}(r, \tilde{Y}_r, \tilde{Z}_r) - \hat{H}(r, \tilde{Y}_r, \hat{Z}_r)) dQ_r,
\]

for all \( \delta > 0 \) with \( \delta_q = \delta 1_{1,2} (q) \).

Passing to limit as \( \delta \to 0 \) and taking in account that \( q - 1 = n_p \) for \( q = p \wedge 2 \) we get
\[
e^{q V_{\tilde{Z}_r}} |\tilde{Y}_{u_k} - \hat{Y}_{u_k}|^q + \frac{1}{2} q \left(q - 1 \right) (2 - \lambda) \mathbb{E} F_{\tilde{Z}_r} \int_{u_k}^{v_{k+1}} \left| \tilde{Y}_r - \hat{Y}_r \right|^q |\tilde{Z}_r - \hat{Z}_r|^2 \text{ dr}
\]
\[
\quad \leq \mathbb{E} F_{\tilde{Z}_r} e^{q V_{\tilde{Z}_r}} |\tilde{Y}_{v_{k+1}} - \tilde{Y}_{v_{k+1}}|^q + q \mathbb{E} F_{\tilde{Z}_r} \int_{u_k}^{v_{k+1}} e^{q V_r} \left| \tilde{Y}_r - \hat{Y}_r \right|^q |\tilde{H}(r, \tilde{Y}_r, \tilde{Z}_r) - \hat{H}(r, \tilde{Y}_r, \hat{Z}_r)| dQ_r.
\]

We remark that for any nonnegative measurable function \( K \) we have
\[
\int_{u_k}^{v_{k+1}} e^{V_r} K_r dQ_r = \int_{s \wedge n^+ \wedge \alpha_k}^{s \wedge n^+ \wedge \alpha_k} K_r dQ_r \leq \int_{s \wedge n^+ \wedge \alpha_k}^{s \wedge n^+ \wedge \alpha_k} K_r dQ_r.
\]

Passing first to \( \lim_{i \to \infty} \) and then \( \lim_{k \to \infty} \) in (46) we infer (using Fatou’s Lemma and Lebesgue dominated convergence theorem via the condition (22)):
\[
e^{q V_{\tilde{Z}_r}} |\tilde{Y}_{u_k} - \hat{Y}_{u_k}|^q + \frac{1}{2} q \left(q - 1 \right) (2 - \lambda) \mathbb{E} F_{\tilde{Z}_r} \int_{s \wedge n^+}^{s \wedge n^+} \frac{1}{e^{V_r} \left| \tilde{Y}_r - \hat{Y}_r \right| + 1} 2^{-q} e^{2 V_r} |\tilde{Z}_r - \hat{Z}_r|^2 \text{ dr}
\]
\[
\quad \leq \mathbb{E} F_{\tilde{Z}_r} e^{q V_{\tilde{Z}_r}} \left| \tilde{Y}_{s \wedge n^+} - \tilde{Y}_{s \wedge n^+} \right|^q
\]
\[
\quad + q \mathbb{E} F_{\tilde{Z}_r} \int_{s \wedge n^+}^{s \wedge n^+} e^{q V_r} \left| \tilde{Y}_r - \hat{Y}_r \right|^{q-1} |\tilde{H}(r, \tilde{Y}_r, \tilde{Z}_r) - \hat{H}(r, \tilde{Y}_r, \hat{Z}_r)| dQ_r, \quad \text{P-a.s.}
\]
Moreover, by the Lebesgue dominated convergence theorem (based on (22)) passing to $\lim_{n \to \infty}$ it follows that for all $0 \leq t \leq s \leq T < \infty$, $\mathbb{P}$-a.s.,

$$
e^{qV_t} |\hat{Y}_t - \hat{Y}_s|^q + \frac{1}{2} q (q - 1) (2 - \lambda) \mathbb{E}^{F_t} \int_t^s \frac{1}{\left( e^{qV_r} |\hat{Y}_r - \hat{Y}_s| + 1 \right)^2} e^{2V_r} |\hat{Z}_r - \hat{Z}_s|^2 \, dr
\leq \mathbb{E}^{F_t} e^{qV_t} |\hat{Y}_s - \hat{Y}_s|^q + q \mathbb{E}^{F_t} \int_t^s e^{qV_r} |\hat{Y}_r - \hat{Y}_s|^{q-1} |\hat{H}(r, \hat{Y}_r, \hat{Z}_r) - \hat{H}(r, \hat{Y}_s, \hat{Z}_s)| \, dQ_r.
$$

Using here Proposition 19 and inequality (36) for $|\hat{Y} - \hat{Y}|^q \leq 2^{(q-1)} \left( |\hat{Y}|^q + |\hat{Y}|^q \right)$ we easily obtain (40).

The uniqueness property follows, since, if $\hat{\eta} = \hat{\eta}$ and $\hat{H} = \hat{H}$ we have from (40), $\hat{Y} = \hat{Y}$ in $S^0_m$ and $\hat{Z} = \hat{Z}$ in $\Lambda^0_m \times k$. ■

### 3.4 Existence of the solution on a deterministic interval time $[0, T]$

The existence will be proved only in the case of a deterministic time interval, i.e. $\tau = T > 0$.

**Lemma 11 (Strong solution)** We suppose that assumptions $(A_1 - A_7)$ are satisfied. Let $0 < \lambda < 1 < p$, $n_p = (p - 1) \wedge 1$ and

$$V_t^{(+)} \overset{\text{def}}{=} \int_0^t \left[ \mu_r + \frac{1}{2 n_p \lambda} \ell_r^2 + \nu_r^+ \, dA_r \right].$$

Moreover, we assume:

(i) there exists $L > 0$ such that

$$|\eta| + \ell_t + F^{#}_1 (t) + G^{#}_1 (t) \leq L, \quad a.e. \ t \in [0, T], \ a.s.;$$

(ii) there exists $\delta > 0$ such that

$$\mathbb{E} \exp \left[ (2 + \delta) V_T^{(+)} \right] < \infty;$$

(iii) there exists $\tilde{L} > 0$ such that

$$|e^{V_T^{(+)} \eta}|^2 + \left( \int_0^T e^{V_r^{(+)} \left( F^{#}_1 (r) \, dr + G^{#}_1 (r) \, dA_r \right) } \right)^2 \leq \tilde{L}, \quad \mathbb{P} - a.s.;$$

(iv) for $\rho_0 = (C_\lambda \hat{L})^{1/2} > 0$ it holds

$$\mathbb{E} \int_0^T e^{2V_r^{(+)} \left( F^{#}_{1+\rho_0} (r) \right)^2 + \left( G^{#}_{1+\rho_0} (r) \right)^2 \, dA_r} < \infty.$$

Then the multivalued BSDE

$$
\begin{cases}
Y_t + \int_t^T dK_r = Y_T + \int_t^T H (r, Y_r, Z_r) \, dQ_r - \int_t^T Z_r \, dB_r, \ a.s., \text{for all } t \in [0, T], \\
dK_r = U^{(1)}_r \, dr + U^{(2)}_r \, dA_r, \\
U^{(1)}_r \, dr \in \partial \varphi (Y_r) \, dr \quad \text{and} \quad U^{(2)}_r \, dA_r \in \partial \psi (Y_r) \, dA_r
\end{cases}
$$

\footnote{The constant $\hat{L}$ is given by (52) and the constant $C_\lambda = C_{p, \lambda}$ is given by (106) with $p = 2.$}
has a strong solution \((Y, Z, U^{(1)}, U^{(2)}) \in S^0_m \times \Lambda^0_m \times \Lambda^0_m \times \Lambda^0_m\) such that

\[
\mathbb{E} \sup_{t \in [0, T]} e^{2V_r^{(+)} |Y|_t^2} + \mathbb{E} \int_0^T e^{2V_r^{(+)} |Z|_r^2} \, dr + \mathbb{E} \int_0^T e^{2V_r^{(+)} |U_r^{(1)}|_r^2} \, dr + \mathbb{E} \int_0^T e^{2V_r^{(+)} |U_r^{(2)}|_r^2} \, dA_r < \infty.
\]

**Proof.** Let \(0 < \varepsilon \leq 1\). We consider the approximating backward stochastic equation

\[
Y_t^\varepsilon + \int_t^T \nabla_y \Psi^\varepsilon (r, Y_r^\varepsilon) \, dQ_r = \eta + \int_t^T H_\varepsilon (r, Y_r^\varepsilon, Z_r^\varepsilon) \, dQ_r - \int_t^T Z_r^\varepsilon \, dB_r, \quad \mathbb{P}\text{-a.s., } t \in [0, T],
\]  

(54)

where

\[
\Psi^\varepsilon (\omega, r, y) := \alpha_r (\omega) \varphi_\varepsilon (y) + (1 - \alpha_r (\omega)) \psi_\varepsilon (y)
\]

\[
\nabla_y \Psi^\varepsilon (\omega, r, y) = [\alpha_r (\omega) \nabla_y \varphi_\varepsilon (y) + (1 - \alpha_r (\omega)) \nabla_y \psi_\varepsilon (y)] \, 1_{[0, 1]} (A_r),
\]

(55)

\[
H_\varepsilon (\omega, r, y, z) := [\alpha_r (\omega) F_\varepsilon (\omega, r, y, z) + (1 - \alpha_r (\omega)) G_\varepsilon (\omega, r, y)] \, 1_{[0, 1]} (A_r),
\]

with \(\varphi_\varepsilon\) and \(\psi_\varepsilon\) being the Maurey-Yosida’s regularization given by (14) and \(F_\varepsilon, G_\varepsilon\) being the mollifier approximations introduced in the Appendix Section 4.4.

By (123) and (15), the function

\[
\Phi_\varepsilon (\omega, r, y, z) := H_\varepsilon (\omega, r, y, z) - \nabla_y \Psi^\varepsilon (\omega, r, y)
\]

is a Lipschitz function:

\[
|\Phi_\varepsilon (\omega, r, y, z) - \Phi_\varepsilon (\omega, r, \hat{y}, \hat{z})|
\]

\[
\leq \alpha_r (\omega) \left( \ell_t |y - \hat{y}| + \frac{\kappa (1 + \ell_t)}{\varepsilon^2} |y - \hat{y}| \right) + (1 - \alpha_r (\omega)) \frac{\kappa}{\varepsilon^2} |y - \hat{y}|
\]

\[
+ \frac{1}{\varepsilon} \alpha_r (\omega) |y - \hat{y}| + \frac{1}{\varepsilon} (1 - \alpha_r (\omega)) |y - \hat{y}| \, 1_{[0, 1]} (A_r)
\]

\[
\leq \alpha_r (\omega) \frac{\kappa L + \kappa + 1}{\varepsilon^2} |y - \hat{y}| + (1 - \alpha_r (\omega)) \frac{\kappa}{\varepsilon^2} |y - \hat{y}| + \alpha_r (\omega) L |z - \hat{z}| \, 1_{[0, 1]} (A_r)
\]

\[
\leq \frac{\kappa L + \kappa + 1}{\varepsilon^2} |y - \hat{y}| + \alpha_r (\omega) L |z - \hat{z}| \, 1_{[0, 1]} (A_r).
\]

The assumptions of [19, Lemma 5.20] are satisfied for all \(p' \geq 2\). Therefore equation (54) has a unique solution \((Y^\varepsilon, Z^\varepsilon) \in S^0_m [0, T] \times \Lambda^p_{m \times k} (0, T)\) and consequently, for all \(p' \geq 2\),

\[
\mathbb{E} \sup_{t \in [0, T]} |Y_t^\varepsilon|^{p'} < \infty.
\]

Remark that, by (124),

\[
(Y_t^\varepsilon, \Phi_\varepsilon (t, Y_t^\varepsilon, Z_t^\varepsilon)) \, dQ_t
\]

\[
= \left(Y_t^\varepsilon, F_\varepsilon (t, Y_t^\varepsilon, Z_t^\varepsilon) \right) \, 1_{[0, 1]} (A_r) \, dt + \left(Y_t^\varepsilon, G_\varepsilon (t, Y_t^\varepsilon) \right) \, 1_{[0, 1]} (A_r) \, dA_t
\]

\[
- \left(Y_t^\varepsilon, \nabla \varphi_\varepsilon (t, Y_t^\varepsilon) \right) \, 1_{[0, 1]} (A_r) \, dt - \left(Y_t^\varepsilon, \nabla \psi_\varepsilon (t, Y_t^\varepsilon) \right) \, 1_{[0, 1]} (A_r) \, dA_t
\]

\[
\leq \left[Y_t^\varepsilon \right] F^\# (t) + \left[\frac{\mu_t + 1}{2n_p \lambda} \ell_t^2 \right] + \left[Y_t^\varepsilon \right]^2 + \left[\frac{\lambda}{2} |Z_t^\varepsilon|^2 \right] \, dt + \left[\frac{\lambda}{2} |Y_t^\varepsilon |^{p'} + \nu_t^{+} |Y_t^\varepsilon |^{2} \right] \, dA_t
\]

\[
\leq |Y_t^\varepsilon| \tilde{H}^\# (t) \, dQ_t + |Y_t^\varepsilon|^2 \, dV_r^{(+)} + \frac{\lambda}{2} |Z_t^\varepsilon|^2 \, dt,
\]
Remark that compatibility assumptions (and similar inequality for $\psi$) where

Since by (51)

$$
\mathbb{E} \sup_{t \in [0,T]} e^{2V_{1T}(\cdot)} |Y_t|^2 \leq \mathbb{E} \left( \exp 2V_{1T}(\cdot) \right) \sup_{t \in [0,T]} |Y_t|^2 \\
\leq \left[ \frac{2}{2 + \delta} \mathbb{E} \exp (2 + \delta) V_{1T}(\cdot) + \frac{\delta}{2 + \delta} \mathbb{E} \sup_{t \in [0,T]} |Y_t|^{(4 + 2\delta)/\delta} \right] < \infty,
$$

by Proposition 21 we have

$$
\mathbb{E} \mathcal{F}_t \left( \sup_{r \in [t,T]} e^{V_{rT}(\cdot)} Y_r^2 \right) + \mathbb{E} \mathcal{F}_t \left( \int_t^T e^{V_{rT}(\cdot)} |Z_r|^2 \, dr \right) \\
\leq C \mathbb{E} \left[ e^{V_{Tt}(\cdot)} \eta^2 + \left( \int_t^T e^{V_{rT}(\cdot)} H_1^\# (r) dQ_r \right)^2 \right]
$$

($C = C_{p, \lambda}$ is the constant given by (106) with $p = 2$).

By assumption (52) we get

$$
|Y_t|^2 \leq e^{V_{rT}(\cdot)} Y_r^2 \\
\leq \left[ \mathbb{E} \mathcal{F}_t \left( \sup_{r \in [t,T]} e^{V_{rT}(\cdot)} Y_r^2 \right) \right]^{1/2} \leq (C \tilde{L})^{1/2} = \rho_0, \text{ a.s., for all } t \in [0, T],
$$

$$
\mathbb{E} \left( \int_0^T e^{2V_{rT}(\cdot)} |Z_r|^2 \, dr \right) \leq \rho_0^2,
$$

(56)

$$
|F_\varepsilon (t, Y_t^\varepsilon, Z_t^\varepsilon)| \leq \ell \varepsilon |Z_t^\varepsilon| + F^\#_{1+\rho_0} (r), \quad |G_\varepsilon (t, Y_t^\varepsilon)| \leq G^\#_{1+\rho_0} (r)
$$

$$
|H_\varepsilon (r, Y_r^\varepsilon, Z_r^\varepsilon)| \leq \left[ \alpha_r \left( \ell \varepsilon |Z_r^\varepsilon| + F^\#_{1+\rho_0} (r) \right) + (1 - \alpha_r) G^\#_{1+\rho_0} (r) \right] 1_{[0, 1]} (A_r).
$$

Following the ideas from [11], [12] and Section 5.6.2 from [19] and using the stochastic subdifferential inequality from Lemma 2.38 and Remark 2.39 from [19], for all $0 \leq t \leq s \leq T$

$$
e^{2V_{1T}(\cdot)} \varphi_\varepsilon (Y_t^\varepsilon) \leq e^{2V_{1T}(\cdot)} \varphi_\varepsilon (Y_s^\varepsilon) + \int_t^s e^{2V_{1T}(\cdot)} \langle \nabla \varphi_\varepsilon (Y_r^\varepsilon), \Phi_\varepsilon (r, Y_r^\varepsilon, Z_r^\varepsilon) \rangle \, dQ_r \\
- \int_t^s e^{2V_{1T}(\cdot)} \langle \nabla \varphi_\varepsilon (Y_r^\varepsilon), Z_r^\varepsilon \, dB_r \rangle
$$

(and similar inequality for $\psi_\varepsilon$) we deduce that:

$$
e^{2V_{1T}(\cdot)} [\varphi_\varepsilon (Y_t^\varepsilon) + \psi_\varepsilon (Y_t^\varepsilon)] \\
+ \int_t^s e^{2V_{1T}(\cdot)} \left[ \alpha_r |\nabla \varphi_\varepsilon (Y_r^\varepsilon)|^2 + \langle \nabla \varphi_\varepsilon (Y_r^\varepsilon), \nabla \psi_\varepsilon (Y_r^\varepsilon) \rangle + (1 - \alpha_r) |\nabla \psi_\varepsilon (Y_r^\varepsilon)|^2 \right] dQ_r \\
\leq e^{2V_{1T}(\cdot)} [\varphi_\varepsilon (Y_t^\varepsilon) + \psi_\varepsilon (Y_s^\varepsilon)] + \int_t^s e^{2V_{1T}(\cdot)} \langle \nabla \varphi_\varepsilon (Y_r^\varepsilon) + \nabla \psi_\varepsilon (Y_r^\varepsilon), H_\varepsilon (r, Y_r^\varepsilon, Z_r^\varepsilon) \rangle \, dQ_r \\
- \int_t^s e^{2V_{1T}(\cdot)} \langle \nabla \varphi_\varepsilon (Y_r^\varepsilon) + \nabla \psi_\varepsilon (Y_r^\varepsilon), Z_r^\varepsilon \, dB_r \rangle.
$$

(57)

Remark that compatibility assumptions (18) yield for $|y| \leq \rho_0$:
\[
\langle \nabla \psi_\varepsilon(y), F_\varepsilon(t, y, z) \rangle \\
= \int_{B(0,1)} \langle \nabla \psi_\varepsilon(y) - \nabla \psi_\varepsilon(y - \varepsilon u), F(t, y - \varepsilon u, \beta_\varepsilon(z)) \rangle \mathbf{1}_{[0,1]}(\varepsilon |F(t, y - \varepsilon u), 0|) \rho(u) du \\
+ \int_{B(0,1)} \langle \nabla \psi_\varepsilon(y - \varepsilon u), F(t, y - \varepsilon u, \beta_\varepsilon(z)) \rangle \mathbf{1}_{[0,1]}(\varepsilon |F(t, y - \varepsilon u), 0|) \rho(u) du \\
\leq \int_{B(0,1)} \frac{1}{\varepsilon} |\varepsilon u| |F(t, y - \varepsilon u, \beta_\varepsilon(z))| \mathbf{1}_{[0,1]}(\varepsilon |F(t, y - \varepsilon u), 0|) \rho(u) du \\
+ \int_{B(0,1)} |\nabla \varphi_\varepsilon(y - \varepsilon u)||F(t, y - \varepsilon u, \beta_\varepsilon(z))| \mathbf{1}_{[0,1]}(\varepsilon |F(t, y - \varepsilon u), 0|) \rho(u) du \\
\leq |F|_\varepsilon(t, y, z) + \int_{B(0,1)} [ |\nabla \varphi_\varepsilon(y - \varepsilon u)| + |\nabla \varphi_\varepsilon(y)|] \\
\times |F(t, y - \varepsilon u, \beta_\varepsilon(z))| \mathbf{1}_{[0,1]}(\varepsilon |F(t, y - \varepsilon u), 0|) \rho(u) du \\
\leq |F|_\varepsilon(t, y, z) + (1 + |\nabla \varphi_\varepsilon(y)|) |F|_\varepsilon(t, y, z) \\
= (2 + |\nabla \varphi_\varepsilon(y)|) |F|_\varepsilon(t, y, z) \\
\leq \ell_\varepsilon |z| + |F|_\varepsilon(t, y, 0) + (1 + |\nabla \varphi_\varepsilon(y)|) \ell_\varepsilon |z| + |F|_\varepsilon(t, y, 0) \\
= (2 + |\nabla \varphi_\varepsilon(y)|) (\ell_\varepsilon |z| + |F|_\varepsilon(t, y, 0)) \\
\leq 2L |z| + 2F_{1+\rho_0}^\#(t) + |\nabla \varphi_\varepsilon(y)| \left( L |z| + F_{1+\rho_0}^\#(t) \right)
\]

and similarly

\[
\langle \nabla \varphi_\varepsilon(y), G_\varepsilon(t, y) \rangle \leq (2 + |\nabla \psi_\varepsilon(y)|) |G|_\varepsilon(t, y) \\
\leq 2G_{1+\rho_0}^\#(t) + |\nabla \psi_\varepsilon(y)| G_{1+\rho_0}^\#(t) .
\]

Hence using definition of the function \(H_\varepsilon(t, y, z)\) we have for \(|y| \leq \rho_0\) :

\[
\langle \nabla \varphi_\varepsilon(y) + \nabla \psi_\varepsilon(y), H_\varepsilon(s, y, z) \rangle \\
= \langle \nabla \varphi_\varepsilon(y) + \nabla \psi_\varepsilon(y), \alpha_\varepsilon F_\varepsilon(s, y, z) + (1 - \alpha_\varepsilon) G_\varepsilon(s, y) \rangle \mathbf{1}_{[0,1]}(A_r) \\
\leq \alpha_\varepsilon \left[ 2 + 2 |\nabla \varphi_\varepsilon(y)| \right] |F|_\varepsilon(t, y, z) + (1 - \alpha_\varepsilon) \left[ 2 + 2 |\nabla \psi_\varepsilon(y)| \right] |G|_\varepsilon(t, y) \\
\leq \alpha_\varepsilon \left[ \frac{1}{2} |\nabla \varphi_\varepsilon(y)|^2 + 1 + 3 |F|_\varepsilon(t, y, z)^2 \right] + (1 - \alpha_\varepsilon) \left[ \frac{1}{2} |\nabla \psi_\varepsilon(y)| + 1 + 3 |G|_\varepsilon(t, y)|^2 \right] .
\]

Consequently, using inequality

\[
E \left[ e^{2V_t^{\varepsilon,+}} (\varphi_\varepsilon(Y^\varepsilon_t) + \psi_\varepsilon(Y^\varepsilon_t)) \right] \leq E \left[ e^{2V_t^{(\varepsilon,+)}} (\varphi(\eta) + \psi(\eta)) \right] 
\]

and the fact that

\[
M_\varepsilon^t = \int_0^t e^{2V_r^{\varepsilon,+}} \langle \nabla \varphi_\varepsilon(Y^\varepsilon_r) + \nabla \psi_\varepsilon(Y^\varepsilon_r), Z^\varepsilon_r dB_r \rangle 
\]

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is a martingale we obtain that for all \(0 \leq t \leq s \leq T\):

\[
\begin{align*}
\mathbb{E} e^{2V^\varepsilon_t} \varphi_\varepsilon(Y_t^\varepsilon) + \mathbb{E} e^{2V^\varepsilon_t} \psi_\varepsilon(Y_t^\varepsilon) + \frac{1}{2} \mathbb{E} \int_t^T e^{2V^\varepsilon_r} \left[ |\nabla \varphi_\varepsilon(Y_r^\varepsilon)|^2 dr + |\nabla \psi_\varepsilon(Y_r^\varepsilon)|^2 dA_r \right] + \mathbb{E} \int_t^T e^{2V^\varepsilon_r} \left( 1 + 3 [G^\varepsilon_r (r, Y_r^\varepsilon)]^2 \right) dA_r
\leq \mathbb{E} e^{2V^\varepsilon_t} (\varphi(\eta) + \psi(\eta))
+ \mathbb{E} \int_t^T e^{2V^\varepsilon_r} \left( 1 + 3 [H_r (r, Y_r^\varepsilon, Z_r^\varepsilon)]^2 \right) dr + \mathbb{E} \int_t^T e^{2V^\varepsilon_r} \left( 1 + 3 [G^\varepsilon_r (r, Y_r^\varepsilon)]^2 \right) dA_r
\end{align*}
\]

which clearly yields

\[
(\text{a}) \quad \sup_{t \in [0, T]} \mathbb{E} e^{2V^\varepsilon_t} \varphi_\varepsilon(Y_t^\varepsilon) + \mathbb{E} e^{2V^\varepsilon_t} \psi_\varepsilon(Y_t^\varepsilon) \leq C = C_{\rho_0, \varepsilon, T, \lambda}
\]

\[
(\text{b}) \quad \mathbb{E} \int_0^T e^{2V^\varepsilon_t} \left[ |\nabla \varphi_\varepsilon(Y_t^\varepsilon)|^2 dr + |\nabla \psi_\varepsilon(Y_t^\varepsilon)|^2 dA_r \right] \leq C
\]

(with \(C\) a constant independent of \(\varepsilon\).

Let \(\varepsilon, \delta \in (0, 1]\). We have

\[
Y_t^\varepsilon - Y_t^\delta = \int_t^T dK_r^\varepsilon,\delta - \int_t^T (Z_r^\varepsilon - Z_r^\delta) dB_r , \quad \text{a.s.,}
\]

with

\[
dK_r^\varepsilon,\delta = \left[ H_\varepsilon(r, Y_r^\varepsilon, Z_r^\varepsilon) - H_\delta(r, Y_r^\delta, Z_r^\delta) - \left[ \nabla \varphi_\varepsilon(r, Y_r^\varepsilon) - \nabla \varphi_\delta(r, Y_r^\delta) \right] \right] dQ_r
\]

\[
= \alpha_r \left[ F_\varepsilon(r, Y_r^\varepsilon, Z_r^\varepsilon) - F_\delta(r, Y_r^\delta, Z_r^\delta) \right] 1_{[0, \varepsilon]} (A_r) dr
+ (1 - \alpha_r) \left[ G_\varepsilon(r, Y_r^\varepsilon) - G_\delta(r, Y_r^\delta) \right] 1_{[\varepsilon, \frac{1}{2}]} (A_r) dA_r,
\]

By (16) we have

\[
\langle Y_t^\varepsilon - Y_t^\delta, dK_r^\varepsilon,\delta \rangle \leq dR_r^\varepsilon,\delta + |Y_r^\varepsilon - Y_r^\delta| dN_r^\varepsilon,\delta + |Y_r^\varepsilon - Y_r^\delta|^2 dV_r^{(+)} + \frac{1}{2} |Z_r^\varepsilon - Z_r^\delta|^2 dr
\]

where

\[
dR_r^\varepsilon,\delta = |\varepsilon - \delta| \left[ \mu_r^+ |\varepsilon - \delta| + 2F^\#_{1+r, \rho_0} (r) + 2\ell_r |Z_r^\varepsilon| \right] dr + \frac{\varepsilon + \delta}{2} \left[ |\nabla \varphi_\varepsilon(Y_r^\varepsilon)|^2 + |\nabla \varphi_\delta(Y_r^\delta)|^2 \right] dr
\]

\[
+ \frac{\varepsilon + \delta}{2} \left( |\nabla \varphi_\varepsilon(Y_r^\varepsilon)|^2 + |\nabla \varphi_\delta(Y_r^\delta)|^2 \right) dA_r.
\]

and

\[
\begin{align*}
\text{a.r.}
\end{align*}
\]

\[
\text{a.r.}
\]
By Proposition 21 we get
\[
\mathbb{E} \sup_{r \in [0,T]} e^{2V_r(\varepsilon)} |Y_r^\varepsilon - Y^\delta_r|^2 + \mathbb{E} \int_0^T e^{2V_r(\varepsilon)} |Z_r^\varepsilon - Z^\delta_r|^2 \, dr \\
\leq C_\lambda \left[ \mathbb{E} \int_0^T e^{2V_r(\varepsilon)} dR_r^\delta + \mathbb{E} \left( \int_0^T e^{V_r(\varepsilon)} dN_r^\varepsilon \right)^2 \right].
\]

Boundedness assumptions (50), (51), (52), (53) and boundedness result (60) yield that
\[
\lim_{\varepsilon, \delta \to 0} \mathbb{E} \int_0^T e^{2V_r(\varepsilon)} dR_r^\delta + \mathbb{E} \left( \int_0^T e^{V_r(\varepsilon)} dN_r^\varepsilon \right)^2 = 0.
\]

Consequently there exists \((Y, Z) \in S^0_m \times \Lambda^0_{m \times k}\) such that
\[
\mathbb{E} \sup_{r \in [0,T]} e^{2V_r(\varepsilon)} |Y_r^\varepsilon - Y_r|^2 + \mathbb{E} \int_0^T e^{2V_r(\varepsilon)} |Z_r^\varepsilon - Z_r|^2 \, dr \to 0, \text{ as } \varepsilon \to 0.
\]

From (60) there exist two p.m.s.p. \(U^{(1)}\) and \(U^{(2)}\), such that along a sequence \(\varepsilon_n \to 0\), we have
\[
e^{V_r(\varepsilon)} \nabla \varphi_{\varepsilon}(Y_r^{\varepsilon_n}) \to e^{V_r(\varepsilon)} U^{(1)} \quad \text{weakly in } L^2(\Omega \times [0, T], dP \otimes dt; \mathbb{R}^m),
\]
\[
e^{V_r(\varepsilon)} \nabla \psi_{\varepsilon}(Y_r^{\varepsilon_n}) \to e^{V_r(\varepsilon)} U^{(2)} \quad \text{weakly in } L^2(\Omega \times [0, T], dP \otimes dA; \mathbb{R}^m).
\]

Passing to limit in the approximating equation for \(\varepsilon = \varepsilon_n \to 0\) we infer
\[
Y_t + \int_t^T U_r \, dQ_r = \eta + \int_t^T H(r, Y_r, Z_r) \, dQ_r - \int_t^T Z_r \, dB_r, \text{ a.s.}
\]

where
\[
U_r = [\alpha_r U_r^1 + (1 - \alpha_r) U_r^2], \text{ for } r \in [0, T].
\]

Since \(\nabla \varphi_{\varepsilon}(y) \in \partial \varphi(y - \varepsilon \nabla \varphi_{\varepsilon}(y))\) then for all \(E \in \mathcal{F}, 0 \leq t \leq s \leq T\) and \(X \in S^2_m [0, T]\)
\[
\mathbb{E} \int_t^s \left( e^{2V_r(\varepsilon)} \nabla \varphi_{\varepsilon_n}(Y_r^{\varepsilon_n}), X_r - Y_r^{\varepsilon_n} \right) 1_E \, dr + \mathbb{E} \int_t^s e^{2V_r(\varepsilon)} \varphi(Y_r^{\varepsilon_n} - \varepsilon \nabla \varphi_{\varepsilon_n}(Y_r^{\varepsilon_n})) 1_E \, dr \leq \mathbb{E} \int_t^s e^{2V_r(\varepsilon)} \varphi(X_r) 1_E \, dr.
\]

Passing to \(\liminf_{n \to +\infty}\) in the above inequality we obtain \(U^{(1)}_s \in \partial \varphi(Y_s), dP \otimes ds\text{-a.e. and, with similar arguments, } U^{(2)}_s \in \partial \psi(Y_s), dP \otimes dA_s\text{-a.e.}\) Summarizing the above conclusions we conclude that \((Y, Z, U) \in S^0_m [0, T] \times \Lambda^0_{m \times k} [0, T] \times \Lambda^0_{m \times k} [0, T]\) is strong solution of
\[
\begin{cases}
Y_t + \int_t^T \left( U^{(1)}_s ds + U^{(2)}_s dA_s \right) = \eta + \int_t^T \left[ F(s, Y_s, Z_s) ds + G(s, Y_s) dA_s \right] \\
- \int_t^T Z_s dB_s, \quad t \in [0, T],
\end{cases}
\]

\(U^{(1)}_s \in \partial \varphi(Y_s), dP \otimes ds\text{-a.e. and } U^{(2)}_s \in \partial \psi(Y_s), dP \otimes dA_s\text{-a.e. on } [0, T].\)

Moreover, from (56),
\[
(a) \quad |Y_t| \leq e^{V^+_r} |Y_t| \leq \left[ \mathbb{E} \sup_{r \in [0,T]} \left| e^{V^+_r} Y_r \right|^2 \right]^{1/2} \leq (C_\lambda \mathbb{L})^{1/2} = \rho_0 \quad \text{and}
\]
\[
(b) \quad \mathbb{E} \left( \int_0^T e^{2V^+_r} |Z_r|^2 \, dr \right) \leq \rho_0^2.
\]
Since
\[
0 \leq |F_\varepsilon (r, Y_r^\varepsilon, Z_r^\varepsilon)|^2 \leq 6L^2 |Z_r|^2 + 6 \left(\frac{F_{1+\rho_0}}{t} \right)^2
\]
\[
0 \leq |G_\varepsilon (r, Y_r^\varepsilon)|^2 \leq \left(\frac{G_{1+\rho_0}}{t} \right)^2,
\]
passing to \(\lim \inf_{\varepsilon \to 0}\) in \((59)\), we have by Fatou’s Lemma and by the Lebesgue dominated convergence theorem that
\[
\frac{1}{2} \mathbb{E} \int_0^T e^{2V_r^+} \left[|U_r^{(1)}|^2 dr + |U_r^{(2)}|^2 dA_r\right]
\leq \mathbb{E} \left[e^{2V_r^+} (\varphi (\eta) + \psi (\eta))\right] + \mathbb{E} \int_0^T e^{2V_r^+} \left(1 + 6 |Z_r|^2 + 6 |F (r, Y_r, 0)|^2\right) dr
\]
\[
+ \mathbb{E} \int_0^T e^{2V_r^+} \left(1 + 3 |G (r, Y_r)|^2\right) dA_r
\]
(63)

Proposition 12 (\(L^p\)-variational solution) We suppose that assumptions \((A_1 - A_7)\) are satisfied. Let \(0 < \lambda < 1 < p, n_p = (p - 1) \land 1\) and \(V^+\) be given by \((49)\).
Moreover, we assume:

(i) there exists \(\hat{L} > 0\) such that
\[
|e^{V_r^+} \eta|^2 + \left(\int_0^T e^{V_r^+} (|F (r, 0, 0)| dr + |G (r, 0)| dA_r)\right)^2 \leq \hat{L};
\]
(64)

(ii) there exists \(a \in (1 + n_p \lambda, p \land 2)\) such that
\[
(a) \quad \mathbb{E} \left(\int_0^T e^{2V_r^+} d\ell(s)\right)^{\frac{a}{2}} < \infty,
\]
\[
(b) \quad \mathbb{E} \left[\int_0^T e^{2V_r^+} \left(F_{1+\hat{\rho}}^+ (s) ds + G_{1+\hat{\rho}}^+ (s) dA_s\right)\right]^a < \infty,
\]
where \(\hat{\rho} = (C_\lambda \hat{L})^{1/2}\);

(iii) there exists a positive p.m.s.p. \((\Theta_t)_{t \geq 0}\) and, for each \(\rho \geq 0\), there exist a non-decreasing function \(K_\rho : \mathbb{R}_+ \to \mathbb{R}_+\) such that
\[
F_{\rho}^+ (t) + G_{\rho}^+ (t) \leq K_\rho (\Theta_t), \quad \text{a.e. } t \in [0, T], \quad \mathbb{P} - \text{a.s.}
\]
(66)

Then the multivalued BSDE
\[
\begin{aligned}
Y_t + \int_t^T dK_r &= \eta + \int_t^T H (r, Y_r, Z_r) dQ_r - \int_t^T Z_r dB_r, \text{ a.s., for all } t \in [0, T], \\
dK_r &= U_r dQ_r \in \partial_y \Psi (r, Y_r) dQ_r
\end{aligned}
\]
has a unique \(L^p\)-variational solution, in the sense of Definition 4.

\(\text{The constant } \hat{L} \text{ is given by } (64) \text{ and the constant } C_\lambda = C_{\rho, \lambda} \text{ is given by } (106) \text{ with } p = 2.\)
Moreover, for all $t \in [0, T]$, $\mathbb{P}$-a.s.,
\[
\mathbb{E}^F_t \left( \sup_{s \in [t, T]} |e^{V_s^{(+)}} Y_s|^p \right) + \mathbb{E}^F_t \left( \int_t^T e^{2V_s^{(+)}} \left( \varphi (Y_s) \, ds + \psi (Y_s) \, dA_s \right) \right)^{p/2} + \mathbb{E}^F_t \left( \int_0^T e^{2V_s^{(+)}} |Z_s|^2 \, ds \right)^{p/2} \leq C_{p, \lambda} \mathbb{E}^F_t \left[ e^{pV_t^{(+)}} |\eta|^p + \left( \int_t^T e^{V_s^{(+)}} (|F (r, 0, 0)| \, dr + |G (t, 0)| \, dA_r) \right)^p \right].
\]

(67)

**Proof.** Let $t \in [0, T]$, $n \in \mathbb{N}^*$ and
\[
\beta_t = t + A_t + |\mu_t| + |\nu_t| + \ell_t + V_t^{(+) + } + F_{1+\beta}^\# (t) + G_{1+\beta}^\# (t) + \Theta_t.
\]
Consider the BSDE
\[
\begin{cases}
Y_t^{(n)} + \int_t^T U_s^{(n)} \, dQ_s = \eta^{(n)} + \int_t^T H^{(n)} \left( s, Y_s^{(n)}, Z_s^{(n)} \right) \, dQ_s - \int_t^T Z_s^{(n)} \, dB_s, & t \in [0, T], \\
U_s^{(n)} = \alpha_t U_s^{(2,n)} + (1 - \alpha_t) U_s^{(1,n)}
\end{cases}
\]
and $U_s^{(2,n)} \in \partial \varphi (Y_s^{(n)})$, $dP \otimes ds$ - a.e. and $U_s^{(2,n)} \in \partial \psi (Y_s^{(n)})$, $dP \otimes dA_s$ - a.e. on $[0, T]$, (68)

where
\[
\begin{align*}
\eta^{(n)} &:= \eta 1_{[0, n]} (|\eta| + V_T^{(+) + }), \\
F^{(n)} (t, y, z) &:= F (t, y, z) 1_{[0, n]} (\beta_t), \\
G^{(n)} (t, y, z) &:= G (t, y) 1_{[0, n]} (\beta_t), \\
H^{(n)} (s, y, z) &:= \alpha_s F^{(n)} (s, y, z) + (1 - \alpha_s) G^{(n)} (s, y).
\end{align*}
\]
We have
\[
\left\langle y - \hat{y}, H^{(n)} (t, y, z) - H^{(n)} (t, \hat{y}, z) \right\rangle \leq (\alpha_t \mu_t + (1 - \alpha_t) \nu_t) 1_{[0, n]} (\beta_t) |y - \hat{y}|^2
\]
and
\[
\left| H^{(n)} (t, y, z) - H^{(n)} (t, \hat{y}, \hat{z}) \right| \leq 1_{[0, n]} (\beta_t) \alpha_t \ell_t |z - \hat{z}|.
\]
Remark that
\[
\mu_t^{(n)} = \mu_t 1_{[0, n]} (\beta_t), \quad \nu_t^{(n)} = \nu_t 1_{[0, n]} (\beta_t), \quad \ell_t^{(n)} = \ell_t 1_{[0, n]} (\beta_t)
\]
and
\[
F_{1+\beta}^{(n+) \#} (t) \leq \sup_{|u| \leq 1} |F^{(n)} (t, u, 0)| \leq n 1_{[0, n]} (\beta_t), \quad G_{1+\beta}^{(n+) \#} (t) = \sup_{|u| \leq 1} |G^{(n)} (t, u)| \leq n 1_{[0, n]} (\beta_t).
\]
Let $\theta_n = \inf \left\{ r \geq 0 : r + A_r + V_r^{(+) + } > n \right\}$. Then $1_{[0, n]} (\beta_r) \leq 1_{[0, \theta_n]} (r)$ and
\[
V_t^{(n, +)} = \int_0^t \left[ \mu_t^{(n)} + \frac{1}{2 n \rho \lambda} (\ell_t^{(n)})^2 \right]^+ \, dr + \nu_t^{(n)} \, dA_r
\]
\[
\leq \int_0^t 1_{[0, n]} (\beta_r) \left[ \left( \mu_r + \frac{1}{2 n \rho \lambda} (\ell_r)^2 \right)^+ \right] \, dr + \nu_r^+ \, dA_r
\]
\[
= V_{t \wedge \theta_n}^{(n, +)} \leq V_{\theta_n}^{(n, +)} \leq n
\]

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and

\[
\left|e^{Y^{(n,+)}_t} \eta^{(n)}_t\right|^2 + \left(\int_0^T e^{Y^{(n,+)}_t} \left(F^{(n)}_1 (r) \, dr + G^{(n)}_1 (r) \, dA_r \right)\right)^2 \\
\leq n^2 e^{2n} + e^{2n} n^2 \left(\int_0^{T \wedge \theta_n} (dr + dA_r)\right)^2 \leq e^{2n} n^2 \left(1 + n^2\right) = \hat{L}^{(n)}
\]

for every \( \rho \geq 0 \)

\[
F^{(n)}_p (t) + G^{(n)}_p (t) \leq \left[F^{#}_p (t) + G^{#}_p (t)\right] 1_{[0,n]} (\beta_t) \\
\leq K_\rho (\Theta_t) 1_{[0,n]} (\beta_t) \\
\leq K_\rho \rho \cdot n.
\]

Therefore assumptions (50), (51), (52) and (53) are satisfied.

Hence by Lemma 11 there exists a unique (strong) solution \((Y^{(n)}, Z^{(n)}, U^{(n)}) \in S^0_m [0,T] \times \Lambda^0_m \times k (0,T) \times \Lambda^0_m (0,T)\) of BSDE (68).

We have

\[
\left\langle Y^{(n)}_t, H^{(n)} (t, Y^{(n)}_t, Z^{(n)}_t) - U^{(n)}_t \right\rangle dQ_t \\
\leq \left[\left(\alpha_t \mu_t + (1 - \alpha_t) \nu_t + \alpha_t \frac{1}{2n_p \lambda} \ell^2_t\right) 1_{[0,n]} (\beta_t) \left|Y^{(n)}_t\right|^2 \\
+ \alpha_t 1_{[0,n]} (\beta_t) \frac{n \lambda}{2} \left|Z^{(n)}_t\right|^2 + \left|H^{(n)} (t, 0, 0) \right| \left|Y^{(n)}_t\right|\right] dQ_t \\
\leq \left|Y^{(n)}_t\right| dN_t + \left|Y^{(n)}_t\right|^2 dV^{(+)\_t} + \frac{\lambda}{2} \left|Z^{(n)}_t\right|^2 dr,
\]

where

\[N_t = \int_0^t \left[|F (r, 0, 0)| \, dr + |G (t, 0)| \, dA_r\right] \quad \text{and} \quad V^{(+)\_t} = \int_0^t \left[\left(\mu_r + \frac{1}{2n_p \lambda} \ell^2_r\right) \, dr + \nu_r^+ \, dA_r\right].
\]

Since by (62)

\[
\left|Y^{(n)}_t\right|^2 \leq \left|e^{Y^{(n,+)}_t} Y^{(n)}_t\right|^2 \leq \mathbb{E}^{\mathbb{F}_t} \left(\sup_{r \in [t,T]} \left|e^{V^{(+)\_r}_r} Y^{(n)}_r\right|^2\right) \\
\leq C_\lambda \hat{L}^{(n)} =: (\rho_n)^2, \quad \text{for all} \ t \in [0,T], \mathbb{P} - a.s.
\]

and \(|\eta^{(n)}| \leq |\eta|\), we deduce from (69), by Proposition 21, that for all \( t \in [0,T] \):

\[
\mathbb{E}^{\mathbb{F}_t} \left(\sup_{r \in [t,T]} \left|e^{V^{(+)\_r}_r} Y^{(n)}_r\right|^2\right) \leq C_\lambda \mathbb{E}^{\mathbb{F}_t} \left[\left|e^{V^{(+)\_t}_t} \eta\right|^2 + \left(\int_t^T e^{V^{(+)\_r}} \, dN_r\right)^2\right]
\]

\((C_\lambda = C_{p, \lambda} \text{ is the constant defined by (106) for } p = 2)\).

By assumption (64) we have

\[
\left|Y^{(n)}_t\right| \leq \left|e^{Y^{(n,+)}_t} Y^{(n)}_t\right| \leq \left[\mathbb{E}^{\mathbb{F}_t} \left(\sup_{r \in [t,T]} \left|e^{V^{(+)\_r}_r} Y^{(n)}_r\right|^2\right)\right]^{1/2} \\
\leq (C_\lambda \hat{L})^{1/2} =: \hat{\rho}, \quad \text{for all} \ t \in [0,T], \mathbb{P} - a.s.,
\]

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and
\[ \mathbb{E} \left( \int_0^T e^{2V_s^{(i)}(t)} |Z_s^{(i)}|^2 dt \right) \leq \rho^2. \]

Let \( n, i \in \mathbb{N}^* \). Then \( Y^{(n+i)} - Y^{(n)} \) satisfies the following BSDE
\[
Y_t^{(n+i)} - Y_t^{(n)} + \int_t^T \left[ U_s^{(n+i)} - U_s^{(n)} \right] dQ_s
= \eta^{(n+i)} - \eta^{(n)} + \int_t^T \left[ H^{(n+i)}(s, Y_s^{(n+i)}, Z_s^{(n+i)}), H^{(n)}(s, Y_s^{(n)}, Z_s^{(n)}) \right] dQ_s
- \int_t^T (Z_s^{(n+i)} - Z_s^{(n)}) dB_s.
\]

Since
\[
\left\langle Y_t^{(n+i)} - Y_t^{(n)}, (U_s^{(n+i)} - U_s^{(n)}) dQ_s \right\rangle \geq 0
\]
and
\[
\left\langle Y_t^{(n+i)} - Y_t^{(n)}, \left[ H^{(n+i)}(s, Y_s^{(n+i)}, Z_s^{(n+i)}), H^{(n)}(s, Y_s^{(n)}, Z_s^{(n)}) \right] dQ_s \right\rangle
= \left\langle Y_t^{(n+i)} - Y_t^{(n)}, \left[ H^{(n+i)}(s, Y_s^{(n+i)}, Z_s^{(n+i)}), H^{(n)}(s, Y_s^{(n)}, Z_s^{(n)}) \right] dQ_s \right\rangle
+ \left\langle Y_t^{(n+i)} - Y_t^{(n)}, \left[ H^{(n+i)}(s, Y_s^{(n)}, Z_s^{(n)}), H^{(n)}(s, Y_s^{(n)}, Z_s^{(n)}) \right] dQ_s \right\rangle
\leq 1_{[0,n+i]}(\beta_s) \left( \mu_s ds + \nu_s dA_s + \frac{1}{2n_p\lambda} \right) |Y_s^{(n+i)} - Y_s^{(n)}|^2 + \frac{n_p\lambda}{2} |Z_s^{(n+i)} - Z_s^{(n)}|^2 ds
+ |Y_s^{(n+i)} - Y_s^{(n)}| \left( 1_{[0,n]}(\beta_s) - 1_{[n,\infty]}(\beta_s) \right) \left[ 2\ell_s |Z_s^{(n)}| + F^\#(s) dA_s + G^\#(s) dA_s \right]
\leq |Y_s^{(n+i)} - Y_s^{(n)}| \left( 1_{[0,n]}(\beta_s) \right) \left[ 2\ell_s |Z_s^{(n)}| + F^\#(s) dA_s + G^\#(s) dA_s \right]
+ |Y_s^{(n+i)} - Y_s^{(n)}|^2 dV_s^{(i)} + \frac{n_p\lambda}{2} |Z_s^{(n+i)} - Z_s^{(n)}|^2 ds
\]

where using the assumption (ii) of our Proposition we have \( 1 + n_p\lambda < a < p \land 2, n_a = (a-1) \land 1 = a - 1 \) and
\[
n_p\lambda \leq (a-1) \frac{n_p\lambda + a - 1}{2(a-1)} = n_a \lambda \quad \text{with} \quad \lambda = \frac{n_p\lambda + a - 1}{2(a-1)} \in (0, 1)
\]

Now by Proposition 21, we obtain:
\[
\mathbb{E} \left( \sup_{s \in [0,T]} e^{aV_s^{(i)}(t)} Y_s^{(n+i)} - Y_s^{(n)} \right) + \mathbb{E} \left( \int_0^T e^{aV_s^{(i)}(t)} |Z_s^{(n+i)} - Z_s^{(n)}|^2 dt \right)^{1/2}
\leq C_{a,\lambda} \mathbb{E} \left[ e^{aV_T^{(i)}(t)} |\eta|^a 1_{(n,\infty)}(\eta) + V_T^{(i)}(t) \right]^a
+ C_{a,\lambda} \mathbb{E} \left( \int_0^T e^{V_s^{(i)}(t)} 1_{(n,\infty)}(\beta_s) \left[ 2\ell_s |Z_s^{(n)}| + F^\#(s) dA_s + G^\#(s) dA_s \right]^a \right)
\leq C_{a,\lambda} \mathbb{E} \left[ e^{V_T^{(i)}(t)} |\eta|^a 1_{(n,\infty)}(\eta) + V_T^{(i)}(t) \right]^a + 2C_{a,\lambda} \mathbb{E} \left( \int_0^T e^{V_s^{(i)}(t)} 1_{(n,\infty)}(\beta_s) 2\ell_s |Z_s^{(n)}| ds \right)^a
+ 2C_{a,\lambda} \mathbb{E} \left( \int_0^T e^{V_s^{(i)}(t)} 1_{(n,\infty)}(\beta_s) \left( F^\#(s) dA_s + G^\#(s) dA_s \right) \right)^a
\]
\[
C_{a, \lambda} \mathbb{E} \left[ e^{a \nu(T)} | \eta | \right] + C'_{a, \lambda} \mathbb{E} \left[ \left( \int_0^T \ell^2 \mathbf{1}_{[0, \alpha]} (\beta_s) ds \right)^{\alpha / 2} \right] + C''_{a, \lambda} \mathbb{E} \left[ \left( \int_0^T \ell^2 \mathbf{1}_{[0, \alpha]} (\beta_s) ds \right)^{\alpha / 2} \right] \\
+ 2C_{a, \lambda} \mathbb{E} \left( \int_0^T e^{V_s} \mathbf{1}_{[0, \alpha]} (\beta_s) \left( F_{\rho} (s) ds + G_{\rho} (s) dA_s \right) \right) \\
\leq C_{a, \lambda} \lambda^{n / 2} \mathbb{E} \left[ \mathbf{1}_{[0, \alpha]} (|\eta| + V_T) \right] + C'_{a, \lambda} \mathbb{E} \left[ \left( \int_0^T \ell^2 \mathbf{1}_{[0, \alpha]} (\beta_s) ds \right)^{\alpha / 2} \right] + \mathbb{E} \left[ \left( \int_0^T e^{V_s} \mathbf{1}_{[0, \alpha]} (\beta_s) \right) \right] \\
+ 2C_{a, \lambda} \mathbb{E} \left( \int_0^T e^{V_s} \mathbf{1}_{[0, \alpha]} (\beta_s) \left( F_{\rho} (s) ds + G_{\rho} (s) dA_s \right) \right).
\]

Hence there exists \((Y, Z) \in S_{m}^{0} [0, T] \times \Lambda_{m \times k}^{0} (0, T)\) such that

\[(j) \quad |Y_t| \leq e^{V_t} |Y_t| \leq (C_{a, \lambda} \lambda)^{1 / 2} = \rho, \quad \text{for all } t \in [0, T], \mathbb{P}\text{-a.s.,}
\]

\[(jj) \quad \mathbb{E} \int_0^T e^{2V_t} |\ell^2| dt \leq 2 \rho^2,
\]

\[(jjj) \quad \lim_{n \to \infty} \mathbb{E} \sup_{s \in [0,T]} e^{aV_s} |Y_s^N - Y_s|^a + \mathbb{E} \left( \int_0^T e^{aV_t} |Z_s^N - Z_s|^2 ds \right)^{a / 2} = 0,
\]

\[(jv) \quad (Y_t, Z_t) = (\eta, 0), \quad \text{for all } t > T.
\]

We remark that

\[
\varphi \left( Y_t^{(n)} \right) dt + \psi \left( Y_t^{(n)} \right) dA_t \leq \left< Y_t^{(n)}, U_{t}^{(1,n)} \right> dt + \left< Y_t^{(n)}, V_t^{(2,n)} \right> dA_t
\]

and

\[
\varphi \left( Y_t^{(n)} \right) dt + \psi \left( Y_t^{(n)} \right) dA_t + \left< Y_t^{(n)}, H(t, Y_t^{(n)}, Z_t^{(n)}) - U_t^{(n)} \right> dQ_t \\
\leq \left< Y_t^{(n)}, H(t, Y_t^{(n)}, Z_t^{(n)}) \right> dQ_t \\
\leq \left[ \left( \alpha_t \mu_t + (1 - \alpha_t) \nu_t + \alpha_t \frac{1}{2 \pi p \lambda} \ell^2 \right) \mathbf{1}_{[0, \alpha]} (\beta_t) |Y_t^{(n)}| \right]^2 \\
+ \alpha_t \mathbf{1}_{[0, \alpha]} (\beta_t) \frac{np \lambda}{2} |Z_t^{(n)}|^2 + \left| H^{(n)} (t, 0, 0) \right| |Y_t^{(n)}| \right) dQ_t \\
\leq |Y_t^{(n)}| |N_t| + |Y_t^{(n)}|^2 |V_t^{(n)}| + \frac{np \lambda}{2} |Z_t^{(n)}|^2 dt \text{,}
\]

where

\[
N_t = \int_0^t \left[ |F (r, 0, 0)| dr + |G (t, 0)| dA_r \right].
\]

Also by (70-j) and the assumption (8) we have

\[
\mathbb{E} \sup_{t \in [0,T]} e^{pV_t} |Y_t^{(n)}|^p \leq \rho^p \mathbb{E} \exp \left( p \int_0^T \left( \mu_s + \frac{1}{2 \pi p \lambda} \ell^2 \right) ds + p \int_0^T |\nu_s| dA_s \right) < \infty.
\]
Hence by Proposition 21 we deduce for all \( t \in [0, T] \)
\[
\mathbb{E}^F_t \left( \sup_{s \in [t, T]} |e^{V_s^{(t)}} Y_s^{(t)}|^p \right) + \mathbb{E}^F_t \left( \int_t^T e^{2V_s^{(t)}} \left( \varphi \left(Y_s^{(t)}\right) ds + \psi \left(Y_s^{(t)}\right) dA_s \right) \right)^{p/2} \\
+ \mathbb{E}^F_t \left( \int_t^T e^{2V_s^{(t)}} |Z_s^{(t)}|^2 ds \right)^{p/2} \\
\leq C_{p, \lambda} \mathbb{E}^F_t \left[ e^{pV_t^{(t)}} |\eta|^p + \left( \int_t^T e^{V_s^{(t)}} dN_s \right)^p \right], \ a.s.
\] (71)

By Remark 8 and
\[
V_t = \int_0^t \left( \mu_r + \frac{1}{2n_p \lambda} \right) dr + \nu_t dA_t \leq V_s^{(t)},
\] (72)
\((Y^{(n)}, Z^{(n)})\) as strong solution of (68) is also an \( L^p \)-variational solution on \([0, T]\) for (68).

Hence for \( q \in \{2, p \land 2\} \), \( \delta_q = \delta(1,12) \) and \( \Gamma_t^{(n)} = (|M_t - Y_t^{(n)}|^2 + \delta_q)^{1/2} \) it holds
\[
(\Gamma_t^{(n)})^q + \frac{q(q-1)}{2} \int_t^s \left( \Gamma_r^{(n)} \right)^{q-2} |R_r - Z_r^{(n)}|^2 dr + q \int_t^s \left( \Gamma_r^{(n)} \right)^{q-2} \Psi(r, Y_r^{(n)}) dQ_r \\
\leq (\Gamma_t^{(n)})^q + q \int_t^s \left( \Gamma_r^{(n)} \right)^{q-2} \Psi(r, M_r) dQ_r \\
+ q \int_t^s \left( \Gamma_r^{(n)} \right)^{q-2} (M_r - Y_r^{(n)}, N_r - H(r, Y_r^{(n)}, Z_r^{(n)})) dQ_r \\
- q \int_t^s \left( \Gamma_r^{(n)} \right)^{q-2} (M_r - Y_r^{(n)}, (R_r - Z_r^{(n)}) dB_r);
\] (73)
for all \( \delta \in (0, 1] \), for all \( 0 \leq t \leq s < \infty \), for all \( M \in S_{\infty}^q (\gamma, N, R; V) \).

By convergence result (70-\(jjj\)) and the assumptions \((A_4 - A_6)\) we can pass to \( \lim \inf_{n \to \infty} \) (on a subsequence) in (71) and (73) to conclude that \((Y, Z)\) is also an \( L^p \)-variational solution on \([0, T]\) and the inequality (67) holds.

**Corollary 13** Let the assumptions of Proposition 12 be satisfied. If, moreover, \( \varphi = \psi = 0 \), then BSDE
\[
Y_t = \eta + \int_t^T H \left( t, Y_r, Z_r \right) dQ_r - \int_t^T Z_r dB_r, \ a.s., \ for \ all \ t \in [0, T],
\] (74)
has a unique solution \((Y, Z) \in S^p_m [0, T] \times \Lambda^p_{m \times k} (0, T) \).

**Proof.** Based on the results from (70) and the assumptions \((A_4 - A_6)\) we can pass to \( \lim \inf_{n \to \infty} \) in the approximating equation (68) with \( \varphi = \psi = 0 \) and \( U = U^{(1)} = U^{(2)} = 0 \) to infer that \((Y, Z)\) satisfies (74). From (67), (72) and the assumption (64) we get \((Y, Z) \in S^p_m [0, T] \times \Lambda^p_{m \times k} (0, T) \). Moreover by (70-\(j\))
\[
|Y_t| \leq \hat{\rho}, \ for \ all \ t \in [0, T], \ \mathbb{P}-a.s.
\]

**Corollary 14** Let the assumptions of Proposition 12 be satisfied. If, moreover,
\[
(i) \ \mathbb{E} \left[ e^{2V_T^{(t)}} (\varphi(\eta) + \psi(\eta)) \right] < \infty,
\]
\[
(ii) \ \mathbb{E} \int_0^T e^{2V_s^{(t)}} dQ_r < \infty,
\]
\[
(iii) \ \mathbb{E} \int_0^T e^{2V_r^{(t)}} \left( \left| F_r (r) \right|^2 + \left| G_r (r) \right|^2 dA_r \right) < \infty
\] (75)
where \( \hat{\rho} = (C_\lambda \hat{L})^{1/2} \), then the BSDE

\[
\begin{align*}
Y_t + \int_t^T dK_r &= Y_T + \int_t^T H (r, Y_r, Z_r) \, dQ_r - \int_t^T Z_r \, dB_r, \text{ a.s., for all } t \in [0, T], \\
K_r &= U_r^{(1)} \, dr + U_r^{(2)} \, dA_r, \\
U_r^{(1)} \, dr &= \partial \varphi (Y_r) \, dr \quad \text{and} \quad U_r^{(2)} \, dA_r = \partial \psi (Y_r) \, dA_r
\end{align*}
\]

has a unique strong solution \((Y, Z, U^{(1)}, U^{(2)}) \in S^0_{m} \times L^0_{m \times k} \times L^0_{m} \times L^0_{m} \) such that

\[
\mathbb{E} \sup_{t \in [0, T]} e^{2V_t} |Y_t|^2 + \mathbb{E} \left( \int_0^T e^{2V_r} |Z_r|^2 \, dr \right) + \mathbb{E} \left( \int_0^T e^{2V_r} \left| U_r^{(1)} \right|^2 \, dr \right) + \mathbb{E} \left( \int_0^T e^{2V_r} \left| U_r^{(2)} \right|^2 \, dA_r \right) < \infty.
\]

Moreover

\[
|Y_t| \leq e^{V(t)} |Y_t| \leq (C_\lambda \hat{L})^{1/2} = \hat{\rho}, \quad \text{for all } t \in [0, T], \quad \mathbb{P}-a.s.
\]

**Proof.** We expand the proof of Proposition 12. By (63)

\[
\begin{align*}
\frac{1}{2} \mathbb{E} \int_0^T e^{2V_r} \left| U_r^{(1)} \right|^2 \, dr &+ \mathbb{E} \left( \int_0^T e^{2V_r} \left| U_r^{(2)} \right|^2 \, dA_r \right) \\
\leq \mathbb{E} \left[ e^{2V_r} (\varphi (\eta (\cdot)) + \psi (\eta (\cdot))) \right] + \mathbb{E} \int_0^T e^{2V_r} \left( 1 + 6 |Z_r|^2 + 6 |F (r, Y_r, 0)|^2 \right) \, dr \\
&+ \mathbb{E} \int_0^T e^{2V_r} \left( 1 + 3 |G (r, 0)|^2 \right) \, dA_r \\
&\leq \mathbb{E} \left[ e^{2V_r} (\varphi (\eta) + \psi (\eta)) \right] + \mathbb{E} \int_0^T e^{2V_r} \, (dr + dA_r) + 6 \hat{\rho}^2 \\
&+ 6 \mathbb{E} \int_0^T e^{2V_r} |F^\# (r)|^2 \, dr + 3 \mathbb{E} \int_0^T e^{2V_r} |G^\# (r)|^2 \, dA_r.
\end{align*}
\]

Hence there exists \((U^{(1)}, U^{(2)}) = \left( e^{-V^{(+)}} \hat{U}^{(1)}, e^{-V^{(+)}} \hat{U}^{(2)} \right) \in \Lambda^0_m (0, T) \times \Lambda^0_m (0, T) \) such that on a subsequence also denoted \( \{U^{(1, n)}, U^{(2, n)}; \ n \in \mathbb{N}^* \} \)

\[
e^{V^{(+)} U^{(1, n)}} \rightarrow e^{V^{(+)} U^{(1)}}, \quad \text{weakly in } L^2 (\Omega \times [0, T], d\mathbb{P} \otimes dt; \mathbb{R}^m),
\]

\[
e^{V^{(+)} U^{(2, n)}} \rightarrow e^{V^{(+)} U^{(2)}}, \quad \text{weakly in } L^2 (\Omega \times [0, T], d\mathbb{P} \otimes dA_t; \mathbb{R}^m).
\]

and

\[
\begin{align*}
\frac{1}{2} \mathbb{E} \int_0^T e^{2V_r} \left| U_r^{(1)} \right|^2 \, dr &+ \mathbb{E} \left( \int_0^T e^{2V_r} \left| U_r^{(2)} \right|^2 \, dA_r \right) \\
&\leq \mathbb{E} \left[ e^{2V_r} (\varphi (\eta) + \psi (\eta)) \right] + \mathbb{E} \int_0^T e^{2V_r} \left( 1 + 6 |Z_r|^2 + 6 |F (r, Y_r, 0)|^2 \right) \, dr \\
&\quad + \mathbb{E} \int_0^T e^{2V_r} \left( 1 + 3 |G (r, Y_r)|^2 \right) \, dA_r.
\end{align*}
\]

Passing to \( \lim_{n \to \infty} \) in (68) using the results from the proof of Proposition 12 we infer

\[
\begin{align*}
Y_t + \int_t^T U_s \, dQ_s &= \eta + \int_t^T H (s, Y_s, Z_s) \, dQ_s - \int_t^T Z_s \, dB_s, \quad t \in [0, T], \\
U_s &= \alpha_r U_s^{(1)} + (1 - \alpha_r) U_s^{(2)}, \\
U_s^{(1)} &\in \partial \varphi (Y_s), \quad d\mathbb{P} \otimes ds \text{ a.e. and } \quad U_s^{(2)} \in \partial \psi (Y_s), \quad d\mathbb{P} \otimes dA_s \text{ a.e. on } [0, T],
\end{align*}
\]

\( ^3 \)The constant \( \hat{L} \) is given by (64) and the constant \( C_\lambda = C_{p, \lambda} \) is given by (106) with \( p = 2 \).
and the conclusion follows.

**Theorem 15 (L^p– variational solution)** Let \( 0 < \lambda < 1 < p, n_p = (p - 1) \land 1 \) and \( q \in \{2, p \land 2\} \). We suppose that assumptions \((A_1 – A_7)\) are satisfied and

\[
E \left[ e^{p V_T} |\eta|^p + \left( \int_0^T e^{V_r} (|F (r, 0, 0)| \, dr + |G (t, 0)| \, dA_r) \right)^p \right] < \infty, \tag{78}
\]

where \( V \) is defined by \((12)\). We also assume

(i) there exists \( a \in (1 + n_p \lambda, p \land 2) \) such that

\[
\begin{align*}
(a) & \quad E \left( \int_0^T \ell^2_r ds \right)^{\frac{2}{2-a}} < \infty, \\
(b) & \quad E \left[ \int_0^T e^{V_r^+} \left( F^\#_r (s) \, ds + G^\#_r (s) \, dA_s \right) \right]^a < \infty, \quad \text{for all } \rho > 0,
\end{align*}
\]

where \( V^+ \) be given by \((49)\) and \( F^\#, G^\# \) are defined by \((6)\).

(ii) there exists a p.m.s.p. \((\Theta_t)_{t \geq 0}\) and for each \( \rho \geq 0 \) there exist an non-decreasing function \( K_\rho : \mathbb{R} \to \mathbb{R} \) such that

\[
F^\#_\rho (t) + G^\#_\rho (t) \leq K_\rho (\Theta_t), \quad \text{a.e. } t \in [0, T]. \tag{80}
\]

Then the multi-valued BSDE

\[
\begin{cases}
Y_t + \int_t^T dK_r = \eta + \int_t^T H (r, Y_r, Z_r) \, dQ_r - \int_t^T Z_r dB_r, \quad \text{a.s., for all } t \in [0, T], \\
dK_r = U_r \, dQ_r \in \partial_\eta \Psi (r, Y_r) \, dQ_r
\end{cases}
\]

has a unique \( L^p \)–variational solution, in the sense of Definition 4. Moreover this solution satisfies

\[
E \left( \sup_{t \in [0,T]} e^{p V_t} |Y_t|^p \right) + E \left( \int_0^T e^{2 V_r} |Z_r|^2 \, dr \right)^{p/2} + E \left( \int_0^T e^{2 V_r} \Psi (r, Y_r) \, dQ_r \right)^{p/2} \\
+ E \left( \int_0^T e^q V_r |\eta|^p \, dr \right)^{p/q} + E \left( \int_0^T e^{q V_r} |Y_r|^{q-2} \Psi (r, Y_r) \, dQ_r \right)^{p/q} \\
\leq C_{p, \lambda, q} E \left[ e^{p V_T} |\eta|^p + \left( \int_0^T e^{V_r} (|F (r, 0, 0)| \, dA_r) \right)^p \right].
\]

**Proof.** Let \( t \in [0, T] \) and

\[
\beta_t = t + A_t + |\mu_t| + |\nu_t| + \ell_t + V^+ (t) + |F (t, 0, 0)| + |G (t, 0)| + \Theta_t,
\]

Define, for \( n \in \mathbb{N}^* \),

\[
\eta^{(n)} = \eta \mathbf{1}_{[0,n]} (|\eta| + V^+ (t)) , \quad F^{(n)} (t, y, z) = F (t, y, z) - F (t, 0, 0) \mathbf{1}_{[0,n]} (\beta_t) , \quad G^{(n)} (t, y) = G (t, y) - G (t, 0) \mathbf{1}_{[0,n]} (\beta_t) , \quad H^{(n)} (t, y, z) = \alpha_t F^{(n)} (t, y, z) + (1 - \alpha_t) G^{(n)} (t, y) .
\]

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We highlight the following properties of the function \( H^{(n)} \)

\[
\begin{align*}
(j) & \quad \langle y', y, H^{(n)}(t, y', z) - H^{(n)}(t, y, z) \rangle \leq \mu_t \langle \alpha_t + \nu_t (1 - \alpha_t) \rangle \| y' - y \|^2, \\
(jj) & \quad |H^{(n)}(t, y, z') - H^{(n)}(t, y, z)| \leq \alpha_t \ell_t \| z' - z \|, \\
(jjj) & \quad |H^{(n+1)}(t, y, z) - H^{(n)}(t, y, z)| \leq [\alpha_t |F(t, 0, 0)| + (1 - \alpha_t) |G(t, 0)|] 1_{(n, \infty)}(\beta_t).
\end{align*}
\]  

(81)

and the monotonicity properties

\[
\begin{align*}
\langle y, H^{(n)}(t, y, z) \rangle \leq & |y| [\alpha_t |F(t, 0, 0)| + (1 - \alpha_t) |G(t, 0)|] 1_{(0, \infty)}(\beta_t) + |y|^2 dV_s + \alpha_t \frac{n_p \lambda}{2} |z|^2 \\
\leq & |y| [\alpha_t |F(t, 0, 0)| + (1 - \alpha_t) |G(t, 0)|] 1_{(0, \infty)}(\beta_t) + |y|^2 dV_s + \alpha_t \frac{n_p \lambda}{2} |z|^2
\end{align*}
\]

and

\[
\begin{align*}
\langle Y'_t - Y_t, H^{(n)}(t, Y'_t, Z'_t) - H^{(n)}(t, Y_t, Z_t) \rangle dQ_t \leq & |Y'_t - Y_t|^2 dV_t + \frac{n_p \lambda}{2} |Z'_t - Z_t|^2 dt \\
& \leq |Y'_t - Y_t|^2 dV_s + \frac{n_p \lambda}{2} |Z'_t - Z_t|^2 dt.
\end{align*}
\]

Clearly, the assumptions of Proposition 12 are satisfied for the approximating BSDE

\[
\begin{align*}
Y^{(n)}_t + & \int_t^T dK_s = \eta^{(n)} + \int_t^T H^{(n)}(s, Y^{(n)}_s, Z^{(n)}_s) dQ_s - \int_t^T Z^{(n)}_s dB_s, \quad t \in [0, T], \\
dK_s \in & \partial_y \Psi (r, Y^{(n)}_r) dQ_r = \alpha_r \partial_y (Y^{(n)}_r) dr + (1 - \alpha_r) \partial_y (Y^{(n)}_r) dA_r
\end{align*}
\]

(82)

(with \( \eta := \eta^{(n)}, F := F^{(n)}, G := G^{(n)}, H := H^{(n)}. \))

Hence by Proposition 12 the approximating BSDE (82) has a unique \( L^p \)-variational solution \( (Y^{(n)}, Z^{(n)}). \) Therefore

\[
\mathbb{E} \left( \sup_{r \in [0, T]} e^{pV_r} |Y^{(n)}_r|^p \right) + \mathbb{E} \left( \int_0^T e^{2V_t} |Z^{(n)}_t|^2 dt \right)^{p/2} + \mathbb{E} \left( \int_0^T e^{2V_r} \Psi (r, Y^{(n)}_r) dQ_r \right)^{p/2} < \infty
\]

and for \( q \in \{2, p \wedge 2\}, \delta_q = \delta 1_{[1, 2]}(q) \) and \( \Gamma^{(n)}_t = \left( |M_t - Y^{(n)}_t|^2 + \delta_q \right)^{1/2} \) it holds

\[
\begin{align*}
(\Gamma^{(n)}_t)^q + & \frac{q (q - 1)}{2} \int_t^s (\Gamma^{(n)}_r)^q - 2 |R_r - Z^{(n)}_r|^2 dr + q \int_t^s (\Gamma^{(n)}_r)^q - 2 \Psi (r, Y^{(n)}_r) dQ_r \\
\leq & \left( \Gamma^{(n)}_t \right)^q + q \int_t^s (\Gamma^{(n)}_r)^q - 2 \Psi (r, M_r) dQ_r \\
& + q \int_t^s (\Gamma^{(n)}_r)^q - 2 \langle M_r - Y^{(n)}_r, N_r - H^{(n)}(r, Y^{(n)}_r, Z^{(n)}_r) \rangle dQ_r \\
& - q \int_t^s (\Gamma^{(n)}_r)^q - 2 \langle M_r - Y^{(n)}_r, (R_r - Z^{(n)}_r) dB_r \rangle
\end{align*}
\]

(83)

for all \( \delta \in (0, 1], \) for all \( 0 \leq t \leq s < \infty, \) for all \( M \in S^0_{\mathcal{M}}(\gamma, N, R; V). \)

Since \( \mathbb{E} \left( \sup_{r \in [0, T]} e^{pV_r} |Y^{(n)}_r|^p \right) < \infty \) and inequality (83) holds for \( 1 < q = p \wedge 2 \leq p, \) inequalities
(36) and (31) yield
\[
\mathbb{E}\left( \sup_{t \in [0,T]} e^{\eta_Y(t)} |Y_t^{(n)}|^p \right) + \mathbb{E}\left( \int_0^T e^{2\eta_Y(r)} |Z_r^{(n)}|^2 \, dr \right)^{p/2} + \mathbb{E}\left( \int_0^T e^{2\eta_Y(r)} \Psi(r, Y_r^{(n)}) \, dQ_r \right)^{p/2}
\]
\[
+ \mathbb{E}\left( \int_0^T e^{\eta_Y(r)} |Y_r^{(n)}|^{q-2} |Z_r^{(n)}|^2 \, dr \right)^{p/q} + \mathbb{E}\left( \int_0^T e^{\eta_Y(r)} |Y_r^{(n)}|^{q-2} \Psi(r, Y_r^{(n)}) \, dQ_r \right)^{p/q} \quad \text{(84)}
\]
\[
\leq C_{p,\lambda,q} \mathbb{E}\left[ e^{\eta_Y(r)} |\eta^{(n+1)} - \eta^{(n)}| \left( \int_0^T e^{\eta_Y(r)} \left| H(r, Y_r^{(n)} - H^n(r) \neq 0 \right) \, dQ_r \right)^{\eta} \right]^{1/q} 
\]
\[
+ K \left( \mathbb{E}\left( \int_0^T e^{\eta_Y(r)} |H^n(t, Y_t^{(n)}, Z_t^{(n)}) - H^n(t, Y_t^{(n)}, Z_t^{(n)})| \, dQ_r \right)^{\eta} \right) \right]^{1/q}
\]
\[
\leq 2^{(q-1)/q} \left[ \mathbb{E}\left( e^{\eta_Y(r)} \left| \eta^{(n+1)} - \eta^{(n)} \right| + \left( \int_0^T e^{\eta_Y(r)} \left| H^n(t, r, 0) \right| \, dQ_r \right)^{\eta} \right) \right]^{(q-1)/q}
\]
and \(C_{\alpha,\eta,\lambda}\) is a positive constant depending only \(\alpha, q\) and \(\lambda\).

First we remark
\[
\mathbb{E}\left( e^{\eta_Y(r)} \left| \eta^{(n+1)} - \eta^{(n)} \right| \right) \leq \mathbb{E}\left( e^{\eta_Y(r)} \left| \eta \right| \right) \left( \left| \eta \right| + Y_T^{(n)} \left( \left| \eta \right| + Y_T^{(n)} \right) \right) \rightarrow 0, \quad \mathbb{P} - \text{a.s., for } n \rightarrow \infty,
\]

since by \(1 < q \leq p\) and assumption (78) we have
\[
\mathbb{E}\left( e^{\eta_Y(r)} \left| \eta \right| \right) \leq \left( \mathbb{E}\left( e^{\eta_Y(r)} \left| \eta \right| \right)^p \right)^{q/p} < \infty.
\]

Secondly, we remark that under assumption (78)
\[
\mathbb{E}\left( \int_0^T e^{\eta_Y(r)} \left| H^n(t, Y_t^{(n)}, Z_t^{(n)}) - H^n(t, Y_t^{(n)}, Z_t^{(n)}) \right| \, dQ_r \right)^{\eta} \]
\[
\leq \mathbb{E}\left( \int_0^T e^{\eta_Y(r)} \left[ |F(r, 0, 0) \, 1_{(n, \infty)}(\beta_r) \, dr + |G(r, 0) \, 1_{(n, \infty)}(\beta_r) \, dA_r \right) \right)^{\eta} \]
\[
\leq 2^{p^{-1}} \left[ \mathbb{E}\left( \int_0^T e^{\eta_Y(r)} \left| F(r, 0, 0) \, 1_{(n, \infty)}(\beta_r) \right| \, dr \right)^{\eta} + \mathbb{E}\left( \int_0^T e^{\eta_Y(r)} \left| G(r, 0) \, 1_{(n, \infty)}(\beta_r) \, dA_r \right) \right)^{\eta} \right]
\]
\[
\rightarrow 0, \quad \text{a.s. for } n \rightarrow \infty.
\]
By (85) we conclude that there exists \((Y, Z) \in S_m^0 \times \Lambda_{m \times k}^0\) such that (on a subsequence)

\[
\sup_{t \in [0, T]} |Y_t^{(n)} - Y_t| + \int_0^T |Z_t^{(n)} - Z_t|^2 \, dr \to 0, \quad P \text{-a.s., for } n \to \infty.
\]

Passing to \(\lim \inf_{n \to \infty}\) in (84) and (83) we infer that \((Y, Z)\) is an \(L^p\)-variational solution. 

\[\square\]

4 Appendix

In this section we recall from [19] some results frequently used in our paper. These results concern inequalities for backward stochastic differential equations and are interesting by themselves. For more details the interested readers are referred to the monograph of Pardoux and Răşcanu [19].

Let \(\{B_t : t \geq 0\}\) be a \(k\)-dimensional Brownian motion with respect to a given stochastic basis \((\Omega, \mathcal{F}, \mathbb{P}, \{\mathcal{F}_t\}_{t \geq 0})\), where \(\mathcal{F}_t\) is the natural filtration associated to \(\{B_t : t \geq 0\}\).

**Notation 16** If \(p \geq 1\) we denote \(n_p := 1 \wedge (p - 1)\).

4.1 Backward stochastic inequalities

Based on [19, Proposition 6.80] and its proof we adapt here the Pardoux–Răşcanu’s inequalities (6.92) and (6.94) from [19] to the case of BSVI.

**Proposition 17** Let \((Y, Z) \in S_m^0 \times \Lambda_{m \times k}^0\) and \(a \geq 0, \gamma \in \mathbb{R}\) such that for all \(0 \leq t \leq s < \infty\)

\[
\int_t^s |Z_r|^2 \, dr + \int_t^s dD_r \leq a |Y_s|^2 + a \int_t^s (dR_r + |Y_r| \, dN_r) + \gamma \int_t^s \langle Y_r, Z_r dB_r \rangle, \quad \mathbb{P}\text{-a.s.},
\]

where \(R, N\) and \(D\) are increasing and continuous p.m.s.p. \(R_0 = N_0 = D_0 = 0\). Then for all \(q > 0\) and for all stopping times \(0 \leq \sigma \leq \theta < \infty\), the following inequality hold:

\[
\mathbb{E}^{\mathcal{F}_\sigma} \left( \int_\sigma^\theta |Z_r|^2 \, dr \right)^{q/2} + \mathbb{E}^{\mathcal{F}_\sigma} \left( \int_\sigma^\theta dD_r \right)^{q/2} \leq C_{a, \gamma, q} \left[ \sup_{r \in [\sigma, \theta]} |Y_r|^q + \left( \int_\sigma^\theta dR_r \right)^{q/2} + \left( \int_\sigma^\theta |Y_r| \, dN_r \right)^{q/2} \right] \leq 2C_{a, \gamma, q} \mathbb{E}^{\mathcal{F}_\sigma} \left[ \sup_{r \in [\sigma, \theta]} |Y_r|^q + \left( \int_\sigma^\theta dR_r \right)^{q/2} + \left( \int_\sigma^\theta |Y_r| \, dN_r \right)^{q/2} \right], \quad \mathbb{P}\text{-a.s.}
\]

where \(C_{a, \gamma, q}\) is a positive constant depending only on \(a, \gamma\) and \(q\).

**Proof.** We follow the first part of the proof of [19, Proposition 6.80]. Let the sequence of stopping times

\[
\theta_n = \theta \wedge \inf \left\{ s \geq \sigma : \sup_{r \in [\sigma, \sigma \vee s]} |Y_r - Y_\sigma| + \int_\sigma^{\sigma \vee s} |Z_r|^2 \, dr + \int_\sigma^{\sigma \vee s} d(D_r + R_r + N_r) \geq n \right\}.
\]

We have for \(q > 0\)

\[
\mathbb{E}^{\mathcal{F}_\sigma} \left( \int_\sigma^{\theta_n} |Z_r|^2 \, dr \right)^{q/2} + \mathbb{E}^{\mathcal{F}_\sigma} \left( \int_\sigma^{\theta_n} dD_r \right)^{q/2} \leq 2 \mathbb{E}^{\mathcal{F}_\sigma} \left( \int_\sigma^{\theta_n} |Z_r|^2 \, dr \right)^{q/2} + \int_\sigma^{\theta_n} dD_r \right)^{q/2} \leq C_{a, \gamma, q} \mathbb{E}^{\mathcal{F}_\sigma} \left[ |Y_{\theta_n}|^q + \left( \int_\sigma^{\theta_n} dR_r \right)^{q/2} + \left( \int_\sigma^{\theta_n} |Y_r| \, dN_r \right)^{q/2} + \left( \int_\sigma^{\theta_n} \langle Y_r, Z_r dB_r \rangle \right)^{q/2} \right].
\]
By Burkholder–Davis–Gundy inequality we get
\[ C'_{a,\gamma,q} \mathbb{E}^{F_r} \left[ \int_0^{\theta_n} \langle Y_r, Z_r dD_r \rangle \right]^{q/2} \leq C''_{a,\gamma,q} \mathbb{E}^{F_r} \left( \int_0^{\theta_n} |Y_r|^2 |Z_r|^2 \, dr \right)^{q/4} \]
\[ \leq C''_{a,\gamma,q} \mathbb{E}^{F_r} \sup_{r \in [\sigma, \theta_n]} |Y_r|^{q/2} \left( \int_0^{\theta_n} |Z_r|^2 \, dr \right)^{q/4} \]
\[ \leq \frac{1}{2} \left( C''_{a,\gamma,q} \right)^2 \mathbb{E}^{F_r} \sup_{r \in [\sigma, \theta_n]} |Y_r|^q + \frac{1}{2} \mathbb{E}^{F_r} \left( \int_0^{\theta_n} |Z_r|^2 \, dr \right)^{q/2} \]
and consequently from (88) the following inequality holds
\[ \mathbb{E}^{F_r} \left( \int_0^{\theta_n} |Z_r|^2 \, dr \right)^{q/2} + \mathbb{E}^{F_r} \left( \int_0^{\theta_n} dD_r \right)^{q/2} \]
\[ \leq C_{a,\gamma,q} \left[ \mathbb{E}^{F_r} \sup_{r \in [\sigma, \theta]} |Y_r|^q + \left( \int_0^\theta dR_r \right)^{q/2} + \left( \int_\sigma^\theta |Y_r| dN_r \right)^{q/2} \right] \]
(89)
Since
\[ \left( \int_\sigma^\theta |Y_r| dN_r \right)^{q/2} \leq \sup_{r \in [\sigma, \theta]} |Y_r|^q + \left( \int_\sigma^\theta dN_r \right)^q \]
then from (89) we infer
\[ \mathbb{E}^{F_r} \left( \int_0^{\theta_n} |Z_r|^2 \, dr \right)^{q/2} + \mathbb{E}^{F_r} \left( \int_0^{\theta_n} dD_r \right)^{q/2} \]
\[ \leq C_{a,\gamma,q} \left[ \mathbb{E}^{F_r} \sup_{r \in [\sigma, \theta]} |Y_r|^q + \left( \int_\sigma^\theta dR_r \right)^{q/2} + \left( \int_\sigma^\theta dN_r \right)^q \right] \]
(90)
Consequently by Fatou’s Lemma, as \( n \to \infty \), inequality (86) follows.

**Proposition 18** Let \((Y, Z) \in S_0^0 \times \Lambda_{m \times k}^0, a \geq 0, \gamma \in \mathbb{R} \) and \( 1 < q \leq p \) satisfying for all \( 0 \leq t \leq s < \infty \):
\[ |Y_t|^q + \int_t^s |Y_r|^{q-2} 1_{Y_r \neq 0} |Z_r|^2 \, dr + \int_t^s |Y_r|^{q-2} 1_{Y_r \neq 0} dD_r \]
\[ \leq a |Y_s|^q + a \int_t^s \left[ |Y_r|^{q-2} 1_{Y_r \neq 0} q_{\geq 2} dR_r + |Y_r|^{q-1} dN_r \right] + \gamma \int_t^s \langle |Y_r|^{q-2} Y_r, Z_r dB_r \rangle, \quad \mathbb{P} \text{-a.s.,} \]
where \( R, N \) and \( D \) are increasing and continuous p.m.s.p. \( R_0 = N_0 = D_0 = 0 \). If \( \sigma \) and \( \theta \) are two stopping times such that \( 0 \leq \sigma \leq \theta < \infty \) and
\[ \mathbb{E} \sup_{r \in [\sigma, \theta]} |Y_r|^p < \infty \]
then, \( \mathbb{P} \text{-a.s.} \)
\[ \mathbb{E}^{F_r} \sup_{r \in [\sigma, \theta]} |Y_r|^p \leq C_{p,q,a,\gamma} \mathbb{E}^{F_r} \left[ |Y_0|^p + \left( \int_\sigma^\theta |Y_r|^{q-2} 1_{Y_r \neq 0} q_{\geq 2} dR_r \right)^{p/q} + \left( \int_\sigma^\theta |Y_r|^{q-1} dN_r \right)^{p/q} \right] \]
(91)
and
\[ \mathbb{E}^{F_r} \left( \sup_{r \in [\sigma, \theta]} |Y_r|^p \right) + \mathbb{E}^{F_r} \left( \int_\sigma^\theta |Y_r|^{q-2} 1_{Y_r \neq 0} |Z_r|^2 \, dr \right)^{p/q} + \mathbb{E}^{F_r} \left( \int_\sigma^\theta |Y_r|^{q-2} 1_{Y_r \neq 0} dD_r \right)^{p/q} \]
\[ \leq C_{p,q,a,\gamma} \mathbb{E}^{F_r} \left[ |Y_0|^p + \left( \int_\sigma^\theta 1_{q_{\geq 2} dR_r} \right)^{p/2} + \left( \int_\sigma^\theta dN_r \right)^p \right]. \]
with \( C_{p,q,a,\gamma} \) a positive constant depending only \((p, q, a, \gamma)\).
Proof. We follow the proof of [19, Proposition 6.80]. Let the stopping time $\theta_n$ be defined by

$$\theta_n = \theta \land \inf \left\{ s \geq \sigma : \sup_{r \in [\sigma, \sigma \lor s]} |Y_r - Y_\sigma| + \int_{\sigma}^{\sigma \lor s} |Z_r|^2 \, dr + \int_{\sigma}^{\sigma \lor s} d(D_r + R_r + N_r) \geq n \right\}$$

For any stopping time $\tau \in [\sigma, \theta_n]$ we have

$$|Y_\tau|^q + \int_{\tau}^{\theta_n} |Y_r|^{q-2} 1_{Y_r \neq 0} |Z_r|^2 \, dr + \int_{\tau}^{\theta_n} |Y_r|^{q-2} 1_{Y_r \neq 0} dD_r$$

$$\leq a |Y_{\theta_n}|^q + a \int_{\tau}^{\theta_n} \left( |Y_r|^{q-2} 1_{Y_r \neq 0} 1_{q \geq 2} dR_r + |Y_r|^{q-1} dN_r \right) + \gamma \int_{\tau}^{\theta_n} (|Y_r|^{q-2} Y_r, Z_r dR_r).$$

Remark that

$$M_s = \int_0^s 1_{[\sigma, \theta_n]} (r) (|Y_r|^{q-2} Y_r, Z_r dR_r), \quad s \geq 0$$

is a martingale, since for all $T > 0$

$$\mathbb{E} \left( \int_0^T 1_{[\sigma, \theta_n]} (r) (|Y_r|^{2q-2} |Z_r|^2 \, dr) \right)^{1/2} \leq \mathbb{E} \sup_{r \in [\sigma, \theta_n]} |Y_r|^{q-1} \left( \int_{\sigma}^{\theta_n} |Z_r|^2 \, dr \right)^{1/2}$$

$$\leq \left[ \frac{q-1}{q} \mathbb{E} \sup_{r \in [\sigma, \theta_n]} |Y_r|^q + \frac{1}{q} \mathbb{E} \left( \int_{\sigma}^{\theta_n} |Z_r|^q \, dr \right)^{q/2} \right]$$

$$\leq \frac{q-1}{q} \mathbb{E} (|Y_\sigma| + n)^q + \frac{1}{q} n^{q/2} < \infty.$$

Therefore from (93)

$$\mathbb{E}^F_\tau \left[ \left( \int_{\tau}^{\theta_n} |Y_r|^{q-2} |Z_r|^2 \, dr \right)^{p/q} + \left( \int_{\tau}^{\theta_n} |Y_r|^{q-2} dD_r \right)^{p/q} \right]$$

$$\leq C_{p,q,a} \mathbb{E}^F_\tau \left[ |Y_{\theta_n}|^p + \left( \int_{\sigma}^{\theta_n} \left( |Y_r|^{q-2} 1_{Y_r \neq 0} 1_{q \geq 2} dR_r \right)^{p/q} + \left( \int_{\sigma}^{\theta_n} |Y_r|^{q-1} dN_r \right)^{p/q} \right) \right]$$

(94)

and

$$\mathbb{E}^F_\tau \sup_{\tau \in [\sigma, \theta_n]} |Y_\tau|^p \leq C_{p,q,a} \gamma \left[ \mathbb{E}^F_\tau |Y_{\theta_n}|^p + \mathbb{E}^F_\tau \left( \int_{\sigma}^{\theta_n} |Y_r|^{q-2} 1_{Y_r \neq 0} 1_{q \geq 2} dR_r \right)^{p/q} \right. \right.$$

$$+ \mathbb{E}^F_\tau \left( \int_{\sigma}^{\theta_n} |Y_r|^{q-1} dN_r \right)^{p/q} + \mathbb{E}^F_\tau \sup_{\tau \in [\sigma, \theta_n]} |M_{\theta_n} - M_{\tau}|^{p/q} \left. \right]$$

$$\leq C_{p,q,a} \gamma \left[ \mathbb{E}^F_\tau |Y_{\theta_n}|^p + \mathbb{E}^F_\tau \left( \int_{\sigma}^{\theta_n} |Y_r|^{q-2} 1_{Y_r \neq 0} 1_{q \geq 2} dR_r \right)^{p/q} \right.$$

$$+ \mathbb{E}^F_\tau \left( \int_{\sigma}^{\theta_n} |Y_r|^{q-1} dN_r \right)^{p/q} + \mathbb{E}^F_\tau \left( \int_{\sigma}^{\theta_n} |Y_r|^{2q-2} |Z_r|^2 \, dr \right)^{p/(2q)} \right].$$

(95)
But

\[ C_{p,q,a,\gamma}^{m} \mathbb{E}^{F_{e}} \left( \int_{\sigma}^{\theta_{n}} |Y_{r}|^{2q-2} |Z_{r}|^{2} dr \right)^{p/(2q)} \]

\[ \leq C_{p,q,a,\gamma}^{m} \mathbb{E}^{F_{e}} \sup_{r \in [\sigma, \theta_{n}]} |Y_{r}|^{p/2} \left( \int_{\sigma}^{\theta_{n}} |Y_{r}|^{q} |Z_{r}|^{2} dr \right)^{p/(2q)} \]

\[ \leq \frac{1}{2} \mathbb{E}^{F_{e}} \sup_{r \in [\sigma, \theta_{n}]} |Y_{r}|^{p} + \frac{(C_{p,q,a,\gamma}^{m})^{2}}{2} \mathbb{E}^{F_{e}} \left( \int_{\sigma}^{\theta_{n}} |Y_{r}|^{q} |Z_{r}|^{2} dr \right)^{p/q} \]

\[ \leq \frac{1}{2} \mathbb{E}^{F_{e}} \sup_{r \in [\sigma, \theta_{n}]} |Y_{r}|^{p} + \left( \int_{\sigma}^{\theta_{n}} (|Y_{r}|^{q-2} \mathbf{1}_{Y_{r} \neq 0} \mathbf{1}_{q \geq 2} dR_{r}) \right)^{p/q} + \left( \int_{\sigma}^{\theta_{n}} |Y_{r}|^{q-1} dN_{r} \right)^{p/q} \]

Using this last inequality in (95) we obtain

\[ \mathbb{E}^{F_{e}} \sup_{r \in [\sigma, \theta_{n}]} |Y_{r}|^{p} \leq C_{p,q,a,\gamma} \mathbb{E}^{F_{e}} \left[ |Y_{\theta_{n}}|^{p} + \left( \int_{\sigma}^{\theta_{n}} (|Y_{r}|^{q-2} \mathbf{1}_{Y_{r} \neq 0} \mathbf{1}_{q \geq 2} dR_{r}) \right)^{p/q} + \left( \int_{\sigma}^{\theta_{n}} |Y_{r}|^{q-1} dN_{r} \right)^{p/q} \right] \]

(96)

Now by Hölder’s inequality

\[ C_{p,q,a,\gamma} \mathbb{E}^{F_{e}} \left[ \left( \int_{\sigma}^{\theta_{n}} |Y_{r}|^{q-2} \mathbf{1}_{Y_{r} \neq 0} \mathbf{1}_{q \geq 2} dR_{r} \right)^{p/q} + \left( \int_{\sigma}^{\theta_{n}} |Y_{r}|^{q-1} dN_{r} \right)^{p/q} \right] \]

\[ \leq C_{a,\gamma} \mathbb{E}^{F_{e}} \left[ \left( \sup_{r \in [\sigma, \theta_{n}]} (|Y_{r}|^{q-2} \mathbf{1}_{Y_{r} \neq 0} \mathbf{1}_{q \geq 2}) \int_{\sigma}^{\theta_{n}} \mathbf{1}_{q \geq 2} dR_{r} \right)^{p/q} + \left( \sup_{r \in [\sigma, \theta_{n}]} |Y_{r}|^{q-1} \int_{\sigma}^{\theta_{n}} dN_{r} \right)^{p/q} \right] \]

\[ \leq \frac{1}{2} \mathbb{E}^{F_{e}} \sup_{r \in [\sigma, \theta_{n}]} |Y_{r}|^{p} + \hat{C}_{p,q,a,\gamma} \mathbb{E}^{F_{e}} \left( \int_{\sigma}^{\theta_{n}} \mathbf{1}_{q \geq 2} dR_{r} \right)^{p/2} + \hat{C}_{p,q,a,\gamma} \mathbb{E}^{F_{e}} \left( \int_{\sigma}^{\theta_{n}} dN_{r} \right)^{p} \]

that yields via (96):

\[ \mathbb{E}^{F_{e}} \sup_{r \in [\sigma, \theta_{n}]} |Y_{r}|^{p} \leq C_{p,q,a,\gamma} \mathbb{E}^{F_{e}} \left[ |Y_{\theta_{n}}|^{p} + \left( \int_{\sigma}^{\theta_{n}} \mathbf{1}_{q \geq 2} dR_{r} \right)^{p/2} + \left( \int_{\sigma}^{\theta_{n}} dN_{r} \right)^{p} \right] \]

(97)

Hence form this last two inequalities we have

\[ \mathbb{E}^{F_{e}} \left[ \left( \int_{\sigma}^{\theta_{n}} |Y_{r}|^{q-2} \mathbf{1}_{Y_{r} \neq 0} \mathbf{1}_{q \geq 2} dR_{r} \right)^{p/q} + \left( \int_{\sigma}^{\theta_{n}} |Y_{r}|^{q-1} dN_{r} \right)^{p/q} \right] \]

\[ \leq \hat{C}_{p,q,a,\gamma} \mathbb{E}^{F_{e}} \left[ |Y_{\theta_{n}}|^{p} + \left( \int_{\sigma}^{\theta_{n}} \mathbf{1}_{q \geq 2} dR_{r} \right)^{p/2} + \left( \int_{\sigma}^{\theta_{n}} dN_{r} \right)^{p} \right] \]

(98)

By Beppo Levi monotone convergence theorem and by Lebesgue dominated convergence theorem, as \( n \to \infty \), we deduce, from (98), (97), (96) and (94), inequalities (91) and (92).

**Proposition 19** Let \((Y, Z) \in S_{m}^{0} \times \Lambda_{m \times k}^{0}, q > 1 \text{ and } b, L > 0 \text{ satisfying} \)

\[ \mathbb{E} \sup_{r \in [0, T]} |Y_{r}|^{q} \leq L \]

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where $R$, $N$ and $D$ are increasing and continuous p.m.s.p. $R_0 = N_0 = D_0 = 0$. Then for all $0 < \alpha < 1$

$$\mathbb{E}\sup_{t \in [0, T]} |Y_t|^\alpha + \left(\mathbb{E}\int_0^T dD_r\right)^\alpha$$

$$\leq \frac{2b^\alpha}{1 - \alpha} \left[ |Y_T|^\alpha + \mathbb{E}\left(\int_0^T |Y_r|^q 1_{Y_r \neq 0} 1_{q \geq 2} dR_r + \int_0^T |Y_r|^q dN_r\right)\right]$$

where $\alpha = \frac{q}{4}$.

Proof. By [19, Proposition 1.56] we obtain for all $0 < \alpha < 1$:

$$\mathbb{E}\sup_{t \in [0, T]} |Y_t|^q + \mathbb{E}\int_0^T dD_r$$

$$\leq \frac{b^\alpha}{1 - \alpha} \left[ |Y_T|^q + \mathbb{E}\left(\int_0^T |Y_r|^q 1_{Y_r \neq 0} 1_{q \geq 2} dR_r + \int_0^T |Y_r|^q dN_r\right)\right]$$

and inequality (99) follows since

$$\sup_{t \in [0, T]} |Y_t|^\alpha \leq \sup_{t \in [0, T]} \left( |Y_t|^q + \mathbb{E}\int_0^T dD_r\right)^\alpha$$

and

$$\left(\mathbb{E}\int_0^T dD_r\right)^\alpha \leq \sup_{t \in [0, T]} \left( |Y_t|^q + \mathbb{E}\int_0^T |Y_r|^q 1_{Y_r \neq 0} dD_r\right)^\alpha$$

Proposition 20 Let:

- $(Y, Z) \in \mathcal{S}_m^0 \times \Lambda_m^0$ ;
- $K \in S_m^0$ and $K(\cdot) \in BV_{loc}(\mathbb{R}_+; \mathbb{R}^m)$, $\mathbb{P}$–a.s.;
- $D, R, N, \tilde{R}$ be some increasing continuous p.m.s.p. with $D_0 = R_0 = N_0 = 0$;
• $V$ be a bounded variation p.m.s.p. with $V_0 = 0$;

• $\sigma$ and $\theta$ be two stopping times such that $0 \leq \sigma \leq \theta < \infty$.

I. If for all $0 \leq t \leq s < \infty$, $\mathbb{P}$-a.s.

$$|Y_t|^2 + \int_t^s |Z_s|^2 \, dr + \int_t^s dD_r \leq |Y_s|^2 + 2 \int_t^s \langle Y_r, dK_r \rangle - 2 \int_t^s \langle Y_r, Z_r dB_r \rangle, \quad (100)$$

and for some $\lambda < 1$

$$\int_t^s \langle Y_r, dK_r \rangle \leq \int_t^s (dR_r + |Y_r| dN_r + |Y_r|^2 dV_r) + \frac{\lambda}{2} \int_t^s |Z_r|^2 \, dr, \quad (101)$$

then, for any $q > 0$, there exists a positive constant $C_{q, \lambda}$ such that $\mathbb{P}$-a.s.

$$\mathbb{E}^F_r \left( \int_{\sigma}^\theta \right) e^{2V_r} |Z_r|^2 \, ds \right)^{q/2} + \mathbb{E}^F_r \left( \int_{\sigma}^\theta \right) e^{2V_r} dD_r \right)^{q/2} \leq C_{q, \lambda} \mathbb{E}^F_r \left[ \sup_{r \in [\sigma, \theta]} e^{V_r} |Y_r| + \left( \int_{\sigma}^\theta e^{2V_r} dR_r \right)^{q/2} + \left( \int_{\sigma}^\theta e^{2V_r} dN_r \right)^{q/2} \right] \leq 2C_{q, \lambda} \mathbb{E}^F_r \left[ \sup_{r \in [\sigma, \theta]} e^{V_r} |Y_r| + \left( \int_{\sigma}^\theta e^{2V_r} dR_r \right)^{q/2} + \left( \int_{\sigma}^\theta e^{V_r} dN_r \right)^q \right]. \quad (102)$$

II. If $q > 1$, $n_q = (q - 1) \wedge 1$,

$$(i) \quad |Y_t|^q + \frac{q}{2} n_q \int_t^s |Y_r|^{q-2} 1_{Y_r \neq 0} |Z_r|^2 \, dr + \int_t^s |Y_r|^{q-2} 1_{Y_r \neq 0} dD_r

\leq |Y_s|^q + q \int_t^s |Y_r|^{q-2} 1_{Y_r \neq 0} \left[ dR_r + \langle Y_r, dK_r \rangle \right] - q \int_t^s |Y_r|^{q-2} 1_{Y_r \neq 0} \langle Y_r, Z_r dB_r \rangle, \quad (103)$$

and for some $\lambda < 1$

$$dR_r + \langle Y_r, dK_r \rangle \leq (1_{q \geq 2} dR_r + |Y_r| dN_r + |Y_r|^2 dV_r) + \frac{n_q}{2} \lambda |Z_r|^2 \, dt, \quad (104)$$

then there exists some positive constant $C_{q, \lambda}, C'_{q, \lambda}$ such that $\mathbb{P}$-a.s.,

$$\mathbb{E}^F_r \left( \sup_{r \in [\sigma, \theta]} e^{V_r} |Y_r|^q \right) + \mathbb{E}^F_r \left( \int_{\sigma}^\theta e^{V_r} |Y_r|^{q-2} 1_{Y_r \neq 0} |Z_r|^2 \, ds \right) + \mathbb{E}^F_r \left( \int_{\sigma}^\theta e^{V_r} |Y_r|^{q-2} 1_{Y_r \neq 0} dD_r \right)

\leq C_{q, \lambda} \mathbb{E}^F_r \left[ e^{V_0} |Y_\theta|^q + \left( \int_{\sigma}^\theta e^{V_r} |Y_r|^{q-2} 1_{Y_r \neq 0} 1_{q \geq 2} dR_r \right) + \left( \int_{\sigma}^\theta e^{V_r} |Y_r|^{q-1} dN_r \right) \right] \leq C'_{q, \lambda} \mathbb{E}^F_r \left[ e^{V_0} |Y_\theta|^q + \left( \int_{\sigma}^\theta e^{2V_r} 1_{q \geq 2} dR_r \right)^{q/2} + \left( \int_{\sigma}^\theta e^{V_r} dN_r \right)^q \right]. \quad (105)$$

**Proof.** Using inequality (101) and (100) we obtain for all $0 \leq t \leq s < \infty$

$$|Y_t|^2 + (1 - \lambda) \int_t^s |Z_r|^2 \, dr + \int_t^s dD_r \leq |Y_s|^2 + \int_t^s \left[ (2dR_r + 2|Y_r| dN_r) + |Y_r|^2 d(2V_r) \right] - 2 \int_t^s \langle Y_r, Z_r dB_r \rangle$$
which yields, applying \([19, \text{Proposition 6.69}] \) (or \([14, \text{Lemma 12}] \)),
\[
|e^{\nu} Y_t|^2 + (1 - \lambda) \int_t^\sigma |e^{\nu} Z_r|^2 \, dr + \int_t^\sigma e^{2\nu} \, dD_r
\]
\[
\leq |e^{\nu} Y_s|^2 + 2 \int_t^\sigma [e^{2\nu} \, dR_r + |e^{\nu} Y_r| \, dN_r] - 2 \langle e^{\nu} Y_r, e^{\nu} Z_r \, dB_r \rangle.
\]
Now inequality \((102)\) clearly follows by Proposition 17.
In the same manner, using \((104)\) and \((103)\), \([19, \text{Proposition 6.69}] \) and Proposition 18 we infer
\[
|e^{\nu} Y_t|^q + \frac{q}{2} n_q (1 - \lambda) \int_t^\sigma |e^{\nu} Y_r|^q - 2 1_{Y_r \neq 0} \, e^{2\nu} \, dD_r
\]
\[
\leq |e^{\nu} Y_s|^q + q \int_t^\sigma |e^{\nu} Y_r|^q - 2 1_{Y_r \neq 0} q e^{2\nu} \, dR_r + |e^{\nu} Y_r| e^{\nu} \, dN_r
\]
\[
- q \int_t^\sigma 1_{Y_r \neq 0} \langle e^{\nu} Y_r, e^{\nu} Z_r \, dB_r \rangle,
\]
which yields \((105)\).
With a similar proof we deduce (see also \([19, \text{Corollary 6.81}] \)) the next results.

**Proposition 21 (see \([19, \text{Proposition 6.80}] \))** Let \((Y, Z) \in S_m^0 \times \Lambda_m^0 \times K_s^0 \) satisfying
\[
Y_t = Y_T + \int_t^T K_s - \int_t^T Z_s dB_s, \quad 0 \leq t \leq T, \quad \mathbb{P}-\text{a.s.},
\]
where \(K \in S_m^0 \) and \(K, (\omega) \in \text{BV}_{loc}(\mathbb{R}_+; \mathbb{R}^m), \mathbb{P}-\text{a.s.} \).
Let \(\tau \) and \(\sigma \) be two stopping times such that \(0 \leq \tau \leq \sigma < \infty \). Assume that there exists three increasing and continuous p.m.s.p. \(D, R, N \) with \(D_0 = R_0 = N_0 = 0 \) and a bounded variation p.m.s.p. \(V \) with \(V_0 = 0 \) such that for \(\lambda < 1 \),
\[
dD_t + \langle Y_t, dK_t \rangle \leq dR_t + |Y_t| dN_t + |Y_t|^2 dV_t + \frac{\lambda}{2} |Z_t|^2 \, dt.
\]
Then, for any \(q > 0 \), there exists a positive constant \(C_{q, \lambda} \) such that \(\mathbb{P}-\text{a.s.} \).
\[
\mathbb{E}^F_T \left( \int_\tau^\sigma e^{2\nu} \, dD_s \right)^{q/2} + \mathbb{E}^F_T \left( \int_\tau^\sigma e^{2\nu} \, |Z_s|^2 \, ds \right)^{q/2}
\]
\[
\leq C_{q, \lambda} \mathbb{E}^F_T \left[ \sup_{s \in [\tau, \sigma]} \right] e^{\nu} Y_s \Bigg|^{q} + \left( \int_\tau^\sigma e^{2\nu} \, dR_s \right)^{q/2} + \left( \int_\tau^\sigma e^{\nu} \, dN_s \right)^{q}.
\]
Moreover, if \(p > 1 \) and
\[
dD_t + \langle Y_t, dK_t \rangle \leq (1 + 2dR_t + |Y_t| dN_t + |Y_t|^2 dV_t) + \frac{n_p}{2} \lambda |Z_t|^2 \, dt,
\]
\[
\mathbb{E} \sup_{s \in [\tau, \sigma]} e^{\nu} Y_s \Bigg| \Bigg|^{p} < \infty,
\]
then there exists a positive constant \(C_{p, \lambda} \) such that \(\mathbb{P}-\text{a.s.} \).
\[
\mathbb{E}^F_T \left( \sup_{s \in [\tau, \sigma]} |e^{\nu} Y_s| \right)^p + \mathbb{E}^F_T \left( \int_\tau^\sigma e^{2\nu} \, dD_s \right)^{p/2} + \mathbb{E}^F_T \left( \int_\tau^\sigma e^{2\nu} \, |Z_s|^2 \, ds \right)^{p/2}
\]
\[
\leq C_{p, \lambda} \mathbb{E}^F_T \left[ e^{\nu} Y_s \right|^p + \left( \int_\tau^\sigma e^{2\nu} \, dR_s \right)^{p/2} + \left( \int_\tau^\sigma e^{\nu} \, dN_s \right)^p \right].
\]
Based mainly on this previous result one can prove:

**Proposition 22** (see [19, Corollary 6.81]) Let \((Y, Z) \in S_m^0 \times \Lambda_m^0\) satisfying

\[
Y_t = Y_T + \int_t^T dK_s - \int_t^T Z_s dB_s, \quad 0 \leq t \leq T, \quad \mathbb{P} - a.s.,
\]

where \(K \in S_m^0\) and \(K_t \in BV_{loc}([\tau] : \mathbb{R}^n), \quad \mathbb{P} - a.s.

Let \(\tau\) and \(\sigma\) be two stopping times such that \(0 \leq \tau \leq \sigma < \infty\). Assume that there exists two increasing and continuous p.m.s.p. \(D, N\) with \(N_0 = 0\) and a bounded variation p.m.s.p. \(V\) with \(V_0 = 0\) such that for \(\lambda < 1,\)

\[
dD_t + \langle Y_t, dK_t \rangle \leq |Y_t|dN_t + |Y_t|^2dV_t,
\]

\[
\mathbb{E}\sup_{s \in [\tau, \sigma]} |e^{V_s}Y_s| < \infty.
\]

Then

\[
e^{V_\tau} |Y_\tau| \leq \mathbb{E}^{F_\tau} e^{V_\tau} |Y_\sigma| + \mathbb{E}^{F_\tau} \int_\tau^\sigma e^{V_s} dN_s
\]

and for all \(0 < a < 1\)

\[
\sup_{s \in [\tau, \sigma]} \left[ \mathbb{E} \left( e^{V_s} |Y_s| \right) \right]^a + \mathbb{E} \left( \sup_{s \in [\tau, \sigma]} |e^{V_s}Y_s|^a \right) + \mathbb{E} \left( \int_{\tau}^{\sigma} e^{2V_s} |Z_s|^2 \, ds \right)^{a/2} + \mathbb{E} \left( \int_{\tau}^{\sigma} e^{2V_s} \, dD_s \right)^{a/2}
\]

\[
\leq C_a \left[ \mathbb{E} \left( e^{V_\tau} |Y_\tau| \right) \right]^a + C_a \left[ \mathbb{E} \left( \int_\tau^\sigma e^{V_s} dN_s \right) \right]^a
\]

### 4.2 An Itô's formula and some backward stochastic inequalities

**Proposition 23** (see [19, Section 2.3.1]) Let \(p \in \mathbb{R}, \rho \geq 0\) and \(\delta \geq 0\) if \(p \geq 2\) and \(\delta > 0\) if \(p < 2\). Let \(Y \in S_m^0\) be a local semimartingale of the form

\[
Y_t = Y_0 - \int_0^t F_r \, dr + \int_0^t R_r dB_r, \quad t \geq 0 \quad \text{or equivalently}
\]

\[
Y_t = Y_T + \int_t^T F_r \, dr - \int_t^T R_r dB_r, \quad \text{for all } 0 \leq t \leq T,
\]

where \(R \in \Lambda_m^0\), \(F \in S_m^0\). Let \(\varphi = \varphi_{p, \rho, \delta} : \mathbb{R}^d \to [0, \infty]\)

\[
\varphi(x) = \varphi_{p, \rho, \delta}(x) = \left( \frac{|x|^2}{1 + \rho |x|^p} + \delta \right)^{1/2}.
\]

By Itô's formula for \(\varphi_{p, \rho, \delta}(Y_t), p \in \mathbb{R}\) we have for all \(0 \leq t \leq s \leq T:\)

\[
\begin{align*}
\varphi_{p, \rho, \delta}^p(Y_t) + \frac{p}{2} \int_t^s R_r^{p, \rho, \delta} \, dr + \frac{p}{2} \left[ L_{r}^{p, \rho, \delta} - L_{t}^{p, \rho, \delta} \right] \\
= \varphi_{p, \rho, \delta}^p(Y_s) + \frac{p}{2} \int_t^s Q_r^{p, \rho, \delta} \, dr + p \int_t^s \left( U_r^{p, \rho, \delta}, F_r \right) - p \int_t^s \left( U_r^{p, \rho, \delta}, R_r dB_r \right), \quad a.s.
\end{align*}
\]

where

\[
U_r^{p, \rho, \delta} = \varphi_{p, \rho, \delta}^{p-2}(Y_r) \frac{1}{(1 + \rho |Y_r|^2)^2} Y_r,
\]

\[
R_r^{p, \rho, \delta} = \varphi_{p, \rho, \delta}^{p-4}(Y_r) \frac{1}{(1 + \rho |Y_r|^2)^3} \left[ \frac{p - 1}{1 + \rho |Y_r|^2} |R_r Y_r|^2 + \left( |R_r|^2 |Y_r|^2 - |R_r Y_r|^2 \right) \right]
\]

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4.3 Smoothing approximations

Let $G : \Omega \times \mathbb{R}_+ \to \mathbb{R}^m$ be a measurable stochastic process such that $\sup_{t \in \mathbb{R}_+} |G_t(\omega)| < \infty$, $\mathbb{P}$-a.s.. Define

$$G_t^\varepsilon = \int_{t \vee \varepsilon}^{\infty} \frac{1}{Q_t} e^{-\frac{Q_t - Q_s}{Q_t}} G_s dQ_s.$$

Then $G_t^\varepsilon : \Omega \times \mathbb{R}_+ \to \mathbb{R}^m$ are continuous stochastic processes and $\mathbb{P}$-a.s.

(a) $|G_t^\varepsilon(\omega)| \leq \sup_{r \geq 0} |G_r(\omega)|$, for all $t \geq 0$;

(b) $\lim_{\varepsilon \to 0} G_t^\varepsilon(\omega) = G_t(\omega)$, a.e. $t \geq 0$;

(c) $|G_t^\varepsilon - G_t| \leq \sup_{r \geq 0} |G_r| \exp \left(2 - \frac{1}{Q_{t}}\right)$

$$+ \sup_{r \geq 0} \left\{|G_r - G_t| : 0 \leq Q_r - Q_t \leq \sqrt{Q_r}\right\}, \text{ for all } t \geq 0.$$

If, moreover, $G$ is a continuous stochastic process then for all $T > 0$

$$\lim_{\varepsilon \to 0} \left(\sup_{\omega \in [0,T]} |G_t^\varepsilon(\omega) - G_s(\omega)|\right) = 0, \mathbb{P}$-a.s. (12)
Proof. \((b)\) Let \(n \in \mathbb{N}^+\). We can assume \(0 < \varepsilon < t\).

\[
|G_t^n - G_t| \leq \int_t^\infty e^{-\frac{Q_t - Q_s}{\varepsilon}} \frac{1}{Q_s} |G_r - G_t| dQ_r \\
\leq \int_t^\infty e^{-s} |G_{Q^{-1}(Q_s + \varepsilon)} - G_{Q^{-1}(Q_t)}| ds \\
\leq \int_0^n |G_{Q^{-1}(Q_{t_s(s)} + \varepsilon)} - G_{Q^{-1}(Q_t)}| ds + 2 \sup_{r \geq 0} |G_r| \int_n^\infty e^{-s} ds.
\]

Since
\[
\lim_{\varepsilon \to 0} \int_0^n |G_{Q^{-1}(Q_{t_s(s)} + \varepsilon)} - G_{Q^{-1}(Q_t)}| ds = 0, \text{ a.e. } t \in [0, n],
\]
we have for all \(n \in \mathbb{N}^+\)

\[
\limsup_{\varepsilon \to 0} |G_t^n - G_t| \leq 2e^{-n} \sup_{r \geq 0} |G_r|, \text{ a.e. } t \in (0, T).
\]

which yields \((b)\).

\((c)\) Let \(t_\varepsilon = Q^{-1}(Q_t + \sqrt{Q_\varepsilon})\). We have

\[
|G_t^n - G_t| \leq \int_{t_\varepsilon}^\infty e^{-\frac{Q_t - Q_s}{\varepsilon}} \frac{1}{Q_s} |G_r - G_t| dQ_r \\
\leq \sup_{r \in [t_\varepsilon, t_\varepsilon]} |G_r - G_t| \int_{t_\varepsilon}^\infty e^{-\frac{Q_t - Q_s}{\varepsilon}} \frac{1}{Q_s} dQ_r + 2 \sup_{s \geq 0} |G_s| \int_{t_\varepsilon}^\infty e^{-\frac{Q_t - Q_s}{\varepsilon}} \frac{1}{Q_s} dQ_r \\
\leq \sup_{r \in [t_\varepsilon, t_\varepsilon]} |G_r - G_t| + 2 e^{-\frac{Q_t - Q_s}{\varepsilon}} \sup_{s \geq 0} |G_s|.
\]

Since
\[
\frac{Q_{t_\varepsilon} - Q_{t_\varepsilon}}{Q_s} = \frac{\sqrt{Q_\varepsilon}}{Q_s} + \frac{Q_s - Q_{t_\varepsilon}}{Q_s} \geq \frac{1}{\sqrt{Q_\varepsilon}} - 1,
\]
we obtain \((111)\-d)\).

Clearly, \((112)\) follows from \((111)\).

\[\square\]

Remark 26 \(\varepsilon > 0\). Let \(Q : \Omega \times \mathbb{R} \to \mathbb{R}, \ Q_0 = 0\), be a strictly increasing continuous stochastic process, \(\lim_{t \to \infty} Q_t = \infty, \lim_{t \to -\infty} Q_t = -\infty\) and \(G : \Omega \times \mathbb{R} \to \mathbb{R}^m\) be bounded measurable stochastic processes then similar boundedness and convergence results hold for \(G^{1,\varepsilon} : \Omega \times \mathbb{R} \to \mathbb{R}^m\) defined by

\[
G_t^{1,\varepsilon} = \int_t^\infty G_r \frac{1}{Q_s} e^{-\frac{Q_s - Q_t}{\varepsilon}} dQ_r, \quad t \in \mathbb{R},
\]
\[
G_t^{2,\varepsilon} = \int_{-\infty}^t G_r \frac{1}{Q_s} e^{-\frac{Q_s - Q_t}{\varepsilon}} dQ_r, \quad t \in \mathbb{R},
\]
\[
G_t^{3,\varepsilon} = e^{-\frac{Q_t}{Q_s}} G_0 + \int_0^t G_r \frac{1}{Q_s} e^{-\frac{Q_s - Q_t}{\varepsilon}} dQ_r
\]
\[
= \int_{-\infty}^t [1_{(-\infty, 0)}(r) G_0 + 1_{[0, \infty)}(r) G_r] \frac{1}{Q_s} e^{-\frac{Q_s - Q_t}{\varepsilon}} dQ_r, \quad t \geq 0,
\]
\[
G_t^{4,\varepsilon} = 1_{[0, \varepsilon)}(t) G_0 + 1_{[\varepsilon, \infty)}(t) \int_0^t G_r \frac{1}{Q_s} e^{-\frac{Q_s - Q_t}{\varepsilon}} dQ_r, \quad t \geq 0.
\]
Corollary 27 Let the assumptions of Lemma 25 be satisfied and \( \varphi : \mathbb{R}^m \to (-\infty, +\infty] \) be a proper convex lower semicontinuous function such that \( \int_0^\infty |\varphi(G_u)| \, dQ_u < \infty \). Then for all \( 0 \leq \alpha \leq \beta \):

\[
\lim_{\varepsilon \to 0} \int_\alpha^\beta \varphi(G_r^\varepsilon) \, dQ_r = \int_\alpha^\beta \varphi(G_r) \, dQ_r.
\]

Moreover, if \( \mathbb{E} \int_0^\infty |\varphi(G_u)| \, dQ_u < \infty \). Then for any stopping times \( 0 \leq \sigma \leq \theta \)

\[
\lim_{\varepsilon \to 0} \mathbb{E} \int_\sigma^\theta \varphi(G_r^\varepsilon) \, dQ_r = \mathbb{E} \int_\sigma^\theta \varphi(G_r) \, dQ_r.
\]

Proof. We have

\[
\int_\alpha^\beta \varphi(G_r^\varepsilon) \, dQ_r \leq \int_\alpha^\beta \left( \int_{r \leq \varepsilon}^\infty 1_{Q_r < Q_u} e^{-\frac{Q_u - Q_r}{Q_r}} \varphi(G_u) \right) \, dQ_r
\]

\[
= \int_0^\infty \varphi(G_u) \left( \int_0^\infty 1_{[\alpha, \beta]}(r) \left( \int_{r \leq \varepsilon}^\infty 1_{[r, \infty)}(u) \frac{1}{Q_r} e^{-\frac{Q_u - Q_r}{Q_r}} \right) \right) \, dQ_u
\]

\[
= \int_0^\infty \varphi(G_u) 1_{[\tau, \infty)}(u) \left( \int_0^u 1_{[\alpha, \beta]}(r) \frac{1}{Q_r} e^{-\frac{Q_u - Q_r}{Q_r}} \right) \, dQ_u,
\]

since \( 1_{[r \leq \varepsilon, \infty)}(u) = 1_{[0,u]}(r) 1_{[\tau, \infty)}(u) \).

We remark that

\[
\int_0^u 1_{[\alpha, \beta]}(r) \frac{1}{Q_r} e^{-\frac{Q_u - Q_r}{Q_r}} dQ_r \leq \int_0^u 1_{[\alpha, \beta]}(r) \frac{1}{Q_r} e^{-\frac{Q_u - Q_r}{Q_r}} dQ_r
\]

\[
= \int_0^u 1_{[\alpha, \beta]}(r) \left( 1_{[0,\varepsilon)}(r) \frac{1}{Q_r} e^{-\frac{Q_u - Q_r}{Q_r}} + 1_{[\varepsilon, \infty)}(r) \frac{1}{Q_r} e^{-\frac{Q_u - Q_r}{Q_r}} \right) \, dQ_r
\]

\[
\leq \frac{Q_u \wedge \varepsilon}{Q_r} e^{-\frac{Q_u - Q_r}{Q_r}} + \int_0^u 1_{[\alpha, \beta]}(r) \frac{1}{Q_r} e^{-\frac{Q_u - Q_r}{Q_r}} \, dQ_r
\]

and by Remark 26 (with the extension \( Q_r = r \) for \( r < 0 \))

\[
\lim_{\varepsilon \to 0} \int_0^u 1_{[\alpha, \beta]}(r) \frac{1}{Q_r} e^{-\frac{Q_u - Q_r}{Q_r}} \, dQ_r = \lim_{\varepsilon \to 0} \int_{-\infty}^u 1_{[\alpha, \beta]}(r) \frac{1}{Q_r} e^{-\frac{Q_u - Q_r}{Q_r}} \, dQ_r = 1_{[\alpha, \beta]}(u), \quad \text{a.e. } u \geq 0.
\]

Hence

\[
\lim_{\varepsilon \to 0} \int_0^u 1_{[\alpha, \beta]}(r) \frac{1}{Q_r} e^{-\frac{Q_u - Q_r}{Q_r}} \, dQ_r = 1_{[\alpha, \beta]}(u), \quad \text{a.e. } u \geq 0.
\]

Moreover,

\[
0 \leq \int_0^u 1_{[\alpha, \beta]}(r) \frac{1}{Q_r} e^{-\frac{Q_u - Q_r}{Q_r}} \, dQ_r \leq e + 1.
\]

Then, by Fatou’s Lemma and by the Lebesgue dominated convergence theorem, we infer

\[
\int_\alpha^\beta \varphi(G_r) \, dQ_r \leq \liminf_{\varepsilon \to 0} \int_\alpha^\beta \varphi(G_r^\varepsilon) \, dQ_r \leq \limsup_{\varepsilon \to 0} \int_\alpha^\beta \varphi(G_r^\varepsilon) \, dQ_r = \int_\alpha^\beta \varphi(G_r) \, dQ_r.
\]

The second assertion of this corollary follows in the same manner.

Proposition 28 Let \( Q : \Omega \times [0, T] \to \mathbb{R}_+ \), \( Q_0 = 0 \), be a strictly increasing continuous stochastic process. Let \( \tau : \Omega \to [0, \infty] \) be a stopping time, \( \eta : \Omega \to \mathbb{R}^m \) is \( \mathcal{F}_\tau \)-measurable random variable such that \( \mathbb{E} |\eta|^p < \infty \),
$p > 1$, and $(\xi, \zeta) \in S^p_m \times \Lambda^p_{m \times k} (0, \infty)$ is the unique pair associated to $\eta$ given by the martingale representation formula (see [19, Corollary 2.44])

\[
\begin{align*}
\left\{ \begin{array}{l}
\xi_t = \eta - \int_t^\infty \zeta_s dB_s, \ t \geq 0, \ a.s., \\
\xi_t = \mathbb{E}^{\mathcal{F}_t} \eta \quad \text{and} \quad \zeta_t = [0,\tau] (t) \zeta_t
\end{array} \right.
\end{align*}
\]

(or equivalently, $\xi_t = \eta - \int_t^\tau \zeta_s dB_s, \ t \geq 0, \ a.s.$).

Let $U \in S^p_m$, $p > 1$ be such that

\[(a) \quad \mathbb{E} \sup_{t \geq 0} |U_t|^p < \infty, \]
\[(b) \quad \lim_{t \to \infty} \mathbb{E} |U_t - \xi_t|^p = 0. \]

Define

\[M^\epsilon_t = \mathbb{E}^{\mathcal{F}_t} \int_{t \vee \epsilon}^\infty \frac{1}{Q_s} e^{-\frac{Q_s - Q_t}{\epsilon}} U_s dQ_s, \ t \geq 0, \quad (113)\]

Then:

I. \quad $\lim_{t \to \infty} M^\epsilon_t = U_t$, $\mathbb{P} - a.s.,$ for all $t \geq 0$;
\[(jj) \quad \lim_{t \to \infty} \mathbb{E} \sup_{r \geq 0} |M^\epsilon_t|^p = 0, \quad $for all $T > 0. \quad (116)\]

II. $M^\epsilon$ is the unique solution of the BSDE:

\[
\begin{align*}
\left\{ \begin{array}{l}
M^\epsilon_t = M^\epsilon_T + \int_t^T 1_{[t,\infty)} (r) \frac{1}{Q_r} \left( U_r - M^\epsilon_r \right) dQ_r - \int_t^T R_r dB_r, \quad \text{for all } T > 0, \ t \in [0, T], \\
\lim_{t \to \infty} \mathbb{E} |M^\epsilon_t - \xi_t|^p = 0.
\end{array} \right.
\end{align*}
\]

Moreover, we also have

\[
\lim_{t \to \infty} \mathbb{E} \sup_{s \geq t} |U_t - \xi_t|^p = 0 \quad \implies \quad \lim_{t \to \infty} \mathbb{E} \left( \sup_{s \geq t} |M^\epsilon_s - \xi_s|^p \right) = 0. \quad (118)\]

III. Let $\varphi: \mathbb{R}^m \to (-\infty, +\infty]$ be a proper convex lower semicontinuous function such that

\[
\mathbb{E} \int_0^\infty |\varphi (U_t)| dQ_t < \infty.
\]

Let $0 \leq s \leq r \leq t$ and the stopping times $s^* = Q_s^{-1}$, $t^* = Q_t^{-1}$, $r^* = Q_r^{-1}$, where $Q^{-1} (\omega)$ is the inverse mapping of the function $r \mapsto Q_r (\omega) : [0, \infty) \to [0, \infty)$. Then

\[
\lim_{\epsilon \to 0} \mathbb{E} \int_{s^*}^{t^*} \varphi (M^\epsilon_r) dQ_r = \mathbb{E} \int_{s^*}^{t^*} \varphi (U_r) dQ_r.
\]
Moreover, if \( g : \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R}_+ \) is a continuous function, \( D : \Omega \times \mathbb{R}_+ \to \mathbb{R}^n \) is a continuous stochastic process such that for all \( R > 0 \)

\[
\mathbb{E} \int_0^R |\varphi (U_r)| \sup_{\theta \in [0, r]} g (U_\theta, D_\theta) dQ_r + \mathbb{E} \int_0^R |\varphi (U_r)| \sup_{\theta \in [0, r]} \sup_{0 \leq e \leq 1} g (M^e_\theta, D_\theta) dQ_r < \infty,
\]

then for all \( 0 \leq T \leq \infty \)

\[
(c_1) \quad \mathbb{E} \int_{T \land s^*} T \land s^* g (M^\varepsilon_T, D_r) \varphi (M^\varepsilon_T) dQ_r \leq \mathbb{E} \int_{T \land s^*} g (M^\varepsilon_T, D_r) \varphi (U_r) dQ_r,
\]

\[
(c_2) \quad \lim_{\varepsilon \to 0} \mathbb{E} \int_{T \land s^*} g (M^\varepsilon_T, D_r) \varphi (M^\varepsilon_T) dQ_r = \mathbb{E} \int_{T \land s^*} g (U_r, D_r) \varphi (U_r) dQ_r,
\]

where

\[
U^\varepsilon_t = \int_{t \vee \varepsilon}^{\infty} \frac{1}{Q^{\varepsilon}} e^{-\frac{Q^{\varepsilon} \varepsilon}{Q^{\varepsilon} T}} U_r dQ_r.
\]

**Proof.** Remark that

\[
M^\varepsilon_t = \mathbb{E}^F_t (U^\varepsilon_t), \quad \text{for all } t \in [0, T],
\]

that yields \((114)-j\). By Doob’s inequality (see [19, Theorem 1.60]) from \((114)-j\) we get the estimate \((114)-j\).

Clearly \(|M^\varepsilon_t - U_t| \leq \mathbb{E}^F_t \sup_{t \in [0, T]} |U^\varepsilon_t - U_t|\) and the conclusions \((115)\) and \((116)\) hold by Lemma 25 and Doob’s inequality.

Let us to prove \((117)\).

By representation theorem we have

\[
\int_{t \vee \varepsilon}^{\infty} \frac{1}{Q^{\varepsilon}} e^{-\frac{Q^{\varepsilon} \varepsilon}{Q^{\varepsilon} T}} U_r dQ_r = \mathbb{E}^F_t \int_{t \vee \varepsilon}^{\infty} \frac{1}{Q^{\varepsilon}} e^{-\frac{Q^{\varepsilon} \varepsilon}{Q^{\varepsilon} T}} U_r dQ_r + \mathbb{E}^F_t \int_{t \vee \varepsilon}^{\infty} \frac{1}{Q^{\varepsilon}} \tilde{R}_r dB_r
\]

\[
= e^{-\frac{Q^{\varepsilon} \varepsilon}{Q^{\varepsilon} T}} M^\varepsilon_t + \mathbb{E}^F_t \int_{t \vee \varepsilon}^{\infty} \frac{1}{Q^{\varepsilon}} e^{-\frac{Q^{\varepsilon} \varepsilon}{Q^{\varepsilon} T}} U_r dQ_r + \mathbb{E}^F_t \int_{t \vee \varepsilon}^{\infty} \tilde{R}_r dB_r
\]

that yields

\[
e^{-\frac{Q^{\varepsilon} \varepsilon}{Q^{\varepsilon} T}} M^\varepsilon_t = \int_{t \vee \varepsilon}^{\infty} 1_{[\varepsilon, \infty)} (r) \frac{1}{Q^{\varepsilon}} e^{-\frac{Q^{\varepsilon} \varepsilon}{Q^{\varepsilon} T}} U_r dQ_r - \int_{t}^{\infty} \tilde{R}_r dB_r.
\]

Now by Itô’s formula

\[
M^\varepsilon_t = M^\varepsilon_T - \int_{t}^{T} d \left[ e^{\frac{Q^{\varepsilon} \varepsilon}{Q^{\varepsilon} T}} \left( e^{-\frac{Q^{\varepsilon} \varepsilon}{Q^{\varepsilon} T}} M^\varepsilon_T \right) \right]
\]

\[
= M^\varepsilon_T - \int_{t}^{T} 1_{[\varepsilon, \infty)} (r) \frac{1}{Q^{\varepsilon}} e^{\frac{Q^{\varepsilon} \varepsilon}{Q^{\varepsilon} T}} \left( e^{-\frac{Q^{\varepsilon} \varepsilon}{Q^{\varepsilon} T}} M^\varepsilon_T \right) dQ_r - \int_{t}^{T} e^{\frac{Q^{\varepsilon} \varepsilon}{Q^{\varepsilon} T}} d \left( e^{-\frac{Q^{\varepsilon} \varepsilon}{Q^{\varepsilon} T}} M^\varepsilon_T \right)
\]

\[
= M^\varepsilon_T - \int_{t}^{T} 1_{[\varepsilon, \infty)} (r) \frac{1}{Q^{\varepsilon}} M^\varepsilon_r dQ_r + \int_{t}^{T} e^{\frac{Q^{\varepsilon} \varepsilon}{Q^{\varepsilon} T}} 1_{[\varepsilon, \infty)} (r) \frac{1}{Q^{\varepsilon}} e^{-\frac{Q^{\varepsilon} \varepsilon}{Q^{\varepsilon} T}} U_r dQ_r - \int_{t}^{T} e^{\frac{Q^{\varepsilon} \varepsilon}{Q^{\varepsilon} T}} \tilde{R}_r dB_r
\]

\[
= M^\varepsilon_T + \int_{t}^{T} 1_{[\varepsilon, \infty)} (r) \frac{1}{Q^{\varepsilon}} (U_r - M^\varepsilon_r) dQ_r - \int_{t}^{T} \tilde{R}_r dB_r,
\]

where \(R^\varepsilon_r = e^{\frac{Q^{\varepsilon} \varepsilon}{Q^{\varepsilon} T}} \tilde{R}_r\).

The convergence result from \((117)\) is obtained as follows:

\[
M^\varepsilon_t - \xi_t = \mathbb{E}^F_t \int_{t \vee \varepsilon}^{\infty} \frac{1}{Q^{\varepsilon}} e^{-\frac{Q^{\varepsilon} \varepsilon}{Q^{\varepsilon} T}} (U_r - \xi_r) dQ_r + \mathbb{E}^F_t \int_{t \vee \varepsilon}^{\infty} \frac{1}{Q^{\varepsilon}} e^{-\frac{Q^{\varepsilon} \varepsilon}{Q^{\varepsilon} T}} (\xi_r - \xi_t) dQ_r
\]

\[
= \mathbb{E}^F_t \int_{0}^{T} e^{-s} (U_{Q^{-1}(s \varepsilon)} + Q_{s \varepsilon} - \xi_{Q^{-1}(s \varepsilon)} + Q_{s \varepsilon}) ds + \mathbb{E}^F_t \sup_{r \geq t} |\xi_r - \xi_t|
\]

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Here first by Jensen’s inequality and then by Burkholder–Davis–Gundy inequality (see [19, Corollary 2.9]) we have
\[
\left( \mathbb{E}^{F_t} \sup_{r \geq t} |\xi_r - \xi_t| \right)^p \leq \left( \mathbb{E}^{F_t} \sup_{r \geq t} \left| \int_t^r \zeta_s dB_s \right| \right)^p \leq C_p \mathbb{E}^{F_t} \left( \int_t^\infty |\zeta_s|^2 \, ds \right)^{p/2}.
\]
Hence
\[
\mathbb{E} |M_t^x - \xi_t|^p \leq 2^{p-1} \mathbb{E} \left( \mathbb{E}^{F_t} \int_0^\infty e^{-s} |U_{Q^{-1}(sQ_+ + Q_+)} - \xi_{Q^{-1}(sQ_+ + Q_+)}| \, ds \right)^p + 2^{p-1} \mathbb{E} \left( \mathbb{E}^{F_t} \sup_{r \geq t} |\xi_r - \xi_t| \right)^p \leq 2^{p-1} \int_0^\infty e^{-s} \mathbb{E} |U_{Q^{-1}(sQ_+ + Q_+)} - \xi_{Q^{-1}(sQ_+ + Q_+)}|^p \, ds + C_p \mathbb{E} \left( \int_t^\infty |\zeta_r|^2 \, ds \right)^{p/2}
\]
and using the Lebesgue dominated convergence theorem we get
\[
\lim_{t \to \infty} \mathbb{E} |M_t^x - \xi_t|^p = 0.
\]
To prove (118) we have for \( t \geq T > \varepsilon \) and \( 1 < q < p \):
\[
|M_t^x - \xi_t|^{p/q} \leq 2^{p-1} \mathbb{E}^{F_t} \sup_{r \geq T} |U_r - \xi_t|^p + 2^{p-1} \left( \mathbb{E}^{F_t} \sup_{r \geq t} |\xi_r - \xi_t| \right)^{p/q}
\]
and consequently by Doob’s inequality,
\[
\mathbb{E} \sup_{t \geq T} |M_t^x - \xi_t|^p \leq 2^{p-1} \sup_{r \geq T} |U_r - \xi_t|^p + C_{p,q} \mathbb{E} \sup_{t \geq T} \left[ \mathbb{E}^{F_t} \left( \int_T^\infty |\zeta_s|^2 \, ds \right)^{\eta/2} \right]^{p/q} \leq C_p \mathbb{E} \sup_{r \geq T} |U_r - \xi_t|^p + C'_{p,q} \left( \int_T^\infty |\zeta_s|^2 \, ds \right)^{p/2}
\]
that yields (118).

Finally
\[
\mathbb{E} \int_{T \wedge \tau}^\pi g (M_r^x, D_r) \varphi (M_r^x) \, dQ_r = \mathbb{E} \int_{s^*}^\tau \mathbb{1}_{[0,T]} (r) g (M_r^x, D_r) \varphi (\mathbb{E}^{F_r} (U_r^x)) \, dQ_r
\]
\[
\leq \mathbb{E} \int_{s^*}^\tau \mathbb{E}^{F_r} \left[ \mathbb{1}_{[0,T]} (r) g (M_r^x, D_r) \varphi (U_r^x) \right] \, dQ_r = \mathbb{E} \int_{s}^\tau \mathbb{E}^{F_r} \left[ \mathbb{1}_{[0,T]} (r^*) g (M_r^x, D_r) \varphi (U_r^x) \right] \, dr
\]
\[
= \mathbb{E} \int_{s}^\tau \mathbb{1}_{[0,T]} (r^*) g (M_r^x, D_r) \varphi (U_r^x) \, dr = \mathbb{E} \int_{s}^\tau \mathbb{1}_{[0,T]} (r^*) g (M_r^x, D_r) \varphi (U_r^x) \, dr
\]
\[
= \mathbb{E} \int_{s}^\tau \mathbb{1}_{[0,T]} (r) g (M_r^x, D_r) \varphi (U_r^x) \, dQ_r = \mathbb{E} \int_{T \wedge \tau}^{\pi} g (M_r^x, D_r) \varphi (U_r^x) \, dQ_r
\]

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and as in the proof of Corollary 27 we have

\[
\mathbb{E} \int_{T^{\wedge \tau^*}} g(M^*_r, D_r) \varphi(U^*_r) \, dQ_r \\
= \mathbb{E} \int_0^\infty \varphi(U_0) 1_{[\epsilon, \infty)}(\theta) \left( \int_0^\theta 1_{[T^{\wedge \tau^*}, T^{\wedge \tau^*}]}(r) g(M^*_r, D_r) \frac{1}{Q_\epsilon} e^{-\frac{Q_\epsilon - Q_{\epsilon^*}}{Q_\epsilon}} \, dQ_r \right) \, dQ_\theta \\
\leq \mathbb{E} \int_0^\infty \varphi(U_0) 1_{[\epsilon, \infty)}(\theta) \sup_{r \in [0, \theta]} |g(M^*_r, D_r) - g(U_r, D_r)| \, dQ_\theta \\
+ \mathbb{E} \int_0^\infty \varphi(U_0) 1_{[\epsilon, \infty)}(\theta) \left( \int_0^\theta 1_{[T^{\wedge \tau^*}, T^{\wedge \tau^*}]}(r) g(U_r, D_r) \frac{1}{Q_\epsilon} e^{-\frac{Q_\epsilon - Q_{\epsilon^*}}{Q_\epsilon}} \, dQ_r \right) \, dQ_\theta.
\]

Now by Fatou’s Lemma we have

\[
\mathbb{E} \int_{T^{\wedge \tau^*}} g(U_0, D_0) \varphi(U_0) \, dQ_0 \leq \lim \inf_{\varepsilon \rightarrow 0^+} \mathbb{E} \int_{T^{\wedge \tau^*}} g(M^*_r, D_r) \varphi(M^*_r) \, dQ_r \\
\leq \lim \inf_{\varepsilon \rightarrow 0^+} \mathbb{E} \int_{T^{\wedge \tau^*}} g(M^*_r, D_r) \varphi(U^*_r) \, dQ_r \leq \lim \sup_{\varepsilon \rightarrow 0^+} \mathbb{E} \int_{T^{\wedge \tau^*}} g(M^*_r, D_r) \varphi(U^*_r) \, dQ_r \\
\leq \lim \sup_{\varepsilon \rightarrow 0^+} \mathbb{E} \int_0^\infty \varphi(U_0) 1_{[\epsilon, \infty)}(\theta) \left( \int_0^\theta 1_{[T^{\wedge \tau^*}, T^{\wedge \tau^*}]}(r) g(U_r, D_r) \frac{1}{Q_\epsilon} e^{-\frac{Q_\epsilon - Q_{\epsilon^*}}{Q_\epsilon}} \, dQ_r \right) \, dQ_\theta \\
\leq \mathbb{E} \int_{T^{\wedge \tau^*}} g(U_0, D_0) \varphi(U_0) \, dQ_0
\]

and the convergence result follows. \hfill \blacksquare

### 4.4 Mollifier approximation

Let \( F : \Omega \times \mathbb{R}_+ \times \mathbb{R}^m \times \mathbb{R}^{m \times k} \rightarrow \mathbb{R}^m \) and \( G : \Omega \times \mathbb{R}_+ \times \mathbb{R}^m \rightarrow \mathbb{R}^m \) be such that assumptions (A5) and (A6) are satisfied.

Let \( \rho \in C_0^\infty(\mathbb{R}^m; \mathbb{R}_+) \) such that \( \rho(y) = 0 \) if \( |y| \geq 1 \) and \( \int_{\mathbb{R}^d} \rho(y) \, dy = 1 \).

Let \( \kappa 1_{B(0,1)}(y) \geq |\nabla y \rho(y)| \), for all \( y \in \mathbb{R}^m \).

Define, for \( 0 < \varepsilon \leq 1 \),

\[
F_\varepsilon(t, y, z) = \int_{B(0, 1)} F(t, y - \varepsilon u, \beta_\varepsilon(z)) 1_{[0, 1]}(\varepsilon |F(t, y - \varepsilon u)|) \rho(u) \, du \\
= \frac{1}{\varepsilon^{m+1}} \int_{\mathbb{R}^m} \varepsilon F(t, u, \beta_\varepsilon(z)) 1_{[0, 1]}(\varepsilon |F(t, u)|) \rho \left( \frac{y - u}{\varepsilon} \right) \, du,
\]

where

\[
\beta_\varepsilon(z) = \frac{z}{1 \wedge (\varepsilon |z|)} = \text{Pr}_{B(0, 1/\varepsilon)}(z).
\]

Clearly for all \( y, u \in \mathbb{R}^m, |u| \leq 1 \) and \( z \in \mathbb{R}^{m \times k} \),

\[
|F(t, y - \varepsilon u, \beta_\varepsilon(z))| \leq \ell_t |z| + F_\varepsilon^\#(y+1) + t
\]

and consequently

\[
|F_\varepsilon(t, y, z)| \leq \ell_t |z| + F_\varepsilon^\#(y+1) \quad \text{and} \quad |F_\varepsilon(t, 0, 0)| \leq F_1^\#(t).
\]
It is easy to prove that this mollifier approximation of $F$ satisfies the following properties:

\[
\begin{align*}
(a) \quad |F_\varepsilon (t, y, z)| & \leq \ell_t \beta_\varepsilon (z) + \frac{1}{\varepsilon} \leq \frac{1}{\varepsilon} (1 + \ell_t), \\
(b) \quad |F_\varepsilon (t, y, z) - F_\varepsilon (t, y, \hat{z})| & \leq \ell_t |z - \hat{z}| \\
(c) \quad |F_\varepsilon (t, y, z) - F_\varepsilon (t, \hat{y}, z)| & \leq \frac{\kappa}{\varepsilon} |y - \hat{y}||\ell_t |\beta_\varepsilon (z)| + \frac{1}{\varepsilon} \leq \frac{\kappa (1 + \ell_t)}{\varepsilon^2} |y - \hat{y}|.
\end{align*}
\]

(123)

Also we have for all $y, \hat{y} \in \mathbb{R}^m$, $|\hat{y}| \leq \rho$:

\[
\begin{align*}
(y - \hat{y}, F_\varepsilon (t, y, z)) & \leq \mu_t^+ |y - \hat{y}|^2 + |y - \hat{y}| \left[ F_{\rho + 1}^\# (t) + \ell_t |z| \right] \\
& \leq |y - \hat{y}| F_{\rho + 1}^\# (t) + \left( \mu_t + \frac{1}{2n_p \lambda} \ell_t^2 \mathbf{1}_{z \neq 0} \right) |y - \hat{y}|^2 + \frac{n_p \lambda}{2} |z|^2, \quad \text{for all } \lambda > 0.
\end{align*}
\]

(124)

where $p > 1$, $n_p = (p - 1) \wedge 1.$

Indeed, by taking

\[
\alpha_\varepsilon (t, y) = \int_{\mathbb{B}(0,1)} \mathbf{1}_{[0,1]} (\varepsilon |F(t, y - \varepsilon u), 0|) \rho(u) \, du,
\]

we have $0 \leq \alpha_\varepsilon (t, y) \leq 1$ and

\[
\begin{align*}
(y - \hat{y}, F_\varepsilon (t, y, z)) &= \int_{\mathbb{B}(0,1)} (y - \hat{y}, F(t, y - \varepsilon u, \beta_\varepsilon (z))) \mathbf{1}_{[0,1]} (\varepsilon |F(t, y - \varepsilon u), 0|) \rho(u) \, du \\
& \quad + \int_{\mathbb{B}(0,1)} (y - \hat{y}, F(t, y - \varepsilon u, \beta_\varepsilon (z))) \mathbf{1}_{[0,1]} (\varepsilon |F(t, y - \varepsilon u), 0|) \rho(u) \, du \\
& \quad + \int_{\mathbb{B}(0,1)} (y - \hat{y}, F(t, \hat{y} - \varepsilon u, 0)) \mathbf{1}_{[0,1]} (\varepsilon |F(t, y - \varepsilon u), 0|) \rho(u) \, du \\
& \leq \left[ \mu_t |y - \hat{y}|^2 + |y - \hat{y}| \ell_t |\beta_\varepsilon (z)| \right] \alpha_\varepsilon (t, y) + |y - \hat{y}| F_{\rho + 1}^\# (t).
\end{align*}
\]

Moreover, for all $y, \hat{y} \in \mathbb{R}^m$, $|y| \leq \rho$, $|\hat{y}| \leq \rho$:

\[
\begin{align*}
(a) \quad (y - \hat{y}, F_\varepsilon (t, y, z) - F_\varepsilon (t, \hat{y}, z)) & \leq \mu_t^+ |y - \hat{y}|^2 \\
& \quad + |y - \hat{y}| \left[ F_{\rho + 1}^\# (t) + \ell_t |z| \right] \mathbf{1}_{[0,\infty)} (F_{\rho + 1}^\# (t)) \\
(b) \quad (y - \hat{y}, F_\varepsilon (t, y, z) - F_\varepsilon (t, \hat{y}, \hat{z})) & \leq |y - \hat{y}| \left[ F_{\rho + 1}^\# (t) + \ell_t |\hat{z}| \right] \mathbf{1}_{[0,\infty)} (F_{\rho + 1}^\# (t)) \\
& \quad + \left( \mu_t + \frac{1}{2n_p \lambda} \ell_t^2 \mathbf{1}_{z \neq 0} \right) |y - \hat{y}|^2 + \frac{n_p \lambda}{2} |z - \hat{z}|^2, \quad \text{for all } \lambda > 0.
\end{align*}
\]

(125)

\[
\begin{align*}
(c) \quad (y - \hat{y}, F_\varepsilon (t, y, z) - F_\varepsilon (t, \hat{y}, \hat{z})) & \leq |\varepsilon - \delta| \left[ \mu_t^+ |\varepsilon - \delta| + 2F_{\rho + 1}^\# (t) + 2\ell_t |z| \right] \\
& \quad + |y - \hat{y}| \left[ 2 |\mu_t| |\varepsilon - \delta| + \ell_t |\hat{z}| \mathbf{1}_{[0,\infty)} (|\hat{z}|) \mathbf{1}_{z \neq \delta} \right] \\
& \quad + (F_{\rho + 1}^\# (t) + \ell_t |\hat{z}|) \mathbf{1}_{[0,\infty)} (F_{\rho + 1}^\# (t)) \\
& \quad + \left( \mu_t + \frac{1}{2n_p \lambda} \ell_t^2 \mathbf{1}_{z \neq \hat{z}} \right) |y - \hat{y}|^2 + \frac{n_p \lambda}{2} |z - \hat{z}|^2.
\end{align*}
\]
It is sufficient to prove (125-c); inequalities (125-a,b) are obtained by particularization of (125-c). We have

\[
\langle y - \hat{y}, F_\varepsilon (t, y, z) - F_\delta (t, \hat{y}, \hat{z}) \rangle \\
\leq \int_{B(0,1)} (y - \varepsilon u - (\hat{y} - \delta u) + (\varepsilon - \delta) u, F (t, y - \varepsilon u, \beta_\varepsilon (z)) - F (t, \hat{y} - \delta u, \beta_\delta (z))) \cdot \\
\cdot 1_{[0,1]} (\varepsilon |F (t, y - \varepsilon u), 0|) \rho (u) \, du \\
+ \int_{B(0,1)} (y - \hat{y}, F (t, \hat{y} - \delta u, \beta_\varepsilon (z)) - F (t, \hat{y} - \delta u, \beta_\delta (z))) \cdot 1_{[0,1]} (\varepsilon |F (t, y - \varepsilon u), 0|) \rho (u) \, du \\
+ \int_{B(0,1)} (y - \hat{y}, F (t, \hat{y} - \delta u, \beta_\delta (\hat{z}))) \cdot \\
\cdot 1_{[0,1]} (\varepsilon |F (t, y - \varepsilon u), 0|) - 1_{[0,1]} (\delta |F (t, \hat{y} - \delta u), 0|) \rho (u) \, du \\
\leq \mu_\varepsilon |y - \varepsilon u - (\hat{y} - \delta u)|^2 \alpha_\varepsilon (t, y) + 2 |\varepsilon - \delta| \left[ F^#_{\rho + 1} (t) + \ell_t |\beta_\varepsilon (z)| \right] \\
+ |y - \hat{y}| \left[ F^#_{\rho + 1} (t) + \ell_t |\beta_\delta (\hat{z})| \right] 1_{[\frac{1}{\lambda}, \lambda, \infty]} (t) 1_{\varepsilon \neq \delta}.
\]

But

(a) \( \mu_\varepsilon |y - \varepsilon u - (\hat{y} - \delta u)|^2 \alpha_\varepsilon (t, y) \leq \mu_\varepsilon |y - \hat{y}|^2 \alpha_\varepsilon (t, y) + 2 |\mu_\varepsilon| |y - \hat{y}| |\varepsilon - \delta| + \mu_\varepsilon^+ |\varepsilon - \delta|^2 \)

(b) \( |\beta_\varepsilon (z)| \leq |z| \wedge \frac{1}{\varepsilon} \leq |z| \)

(c) \( |\beta_\varepsilon (z) - \beta_\delta (\hat{z})| \leq |\beta_\varepsilon (z) - \beta_\varepsilon (\hat{z})| + |\beta_\varepsilon (\hat{z}) - \beta_\delta (\hat{z})| \leq |z - \hat{z}| + |\hat{z}| 1_{[\frac{1}{\lambda}, \lambda, \infty]} (|\hat{z}|) 1_{\varepsilon \neq \delta}. \)

Hence

\[
\langle y - \hat{y}, F_\varepsilon (t, y, z) - F_\delta (t, \hat{y}, \hat{z}) \rangle \leq \left( \mu_\varepsilon + \frac{1}{2n_\rho \lambda} \ell_t^2 1_{\varepsilon \neq \delta} \right)^+ |y - \hat{y}|^2 + \frac{n_\rho \lambda}{2} |z - \hat{z}|^2 \\
+ |y - \hat{y}| \left[ 2 |\mu_\varepsilon| |\varepsilon - \delta| + \ell_t |\hat{z}| 1_{[\frac{1}{\lambda}, \lambda, \infty]} (|\hat{z}|) 1_{\varepsilon \neq \delta} + (F^#_{\rho + 1} (t) + \ell_t |\hat{z}|) 1_{[\frac{1}{\lambda}, \lambda, \infty]} (F^#_{\rho + 1} (t)) \right] \\
+ \mu_\varepsilon^+ |\varepsilon - \delta|^2 + 2 |\varepsilon - \delta| \left[ F^#_{\rho + 1} (t) + \ell_t |z| \right].
\]

Remark 29 The function \( G \) will be approximate in the same manner. For \( 0 < \varepsilon \leq 1 \):

\[
G_\varepsilon (t, y) = \int_{B(0,1)} G (t, y - \varepsilon u) 1_{[0,1]} (\varepsilon |G (t, y - \varepsilon u)|) \rho (u) \, du.
\]

Similar properties (123) and (125) are satisfied with \( z = \hat{z} = 0 \) and \( \ell = 0 \).

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