Radial perturbations of the scalarized black holes in Einstein-Maxwell-conformally coupled scalar theory

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Abstract

We perform the stability analysis for the scalarized charged black holes obtained from Einstein-Maxwell-conformally coupled scalar theory by employing the radial perturbations. The targeting black holes include a single branch of scalarized charged black hole with $\alpha > 0$ inspired by the constant scalar hairy black hole as well as infinite branches of $n = 0(\alpha \geq 8.019), 1(\alpha \geq 40.84), 2(\alpha \geq 99.89), \cdots$ scalarized charged black holes found through the spontaneous scalarization on the Reissner-Nordström black hole. It turns out that all scalarized charged black holes are unstable against the $l = 0(s$-mode$)$ scalar perturbation.

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1 Introduction

No-hair theorem implies that a black hole is completely described by mass, electric charge, and angular momentum \[1\]. In this connection, we know well that Maxwell and gravitational fields satisfy the Gauss-law outside the horizon. It is interesting to note that a minimally coupled scalar does not obey the Gauss-law and thus, a black hole could not have a scalar hair in the Einstein-scalar theory \[2\]. On the other hand, introducing the Einstein-conformally coupled scalar theory leads to a secondary scalar hair around the BBMB (Bocharova-Bronnikov-Melnikov-Bekenstein) black hole \[3, 4\]. This corresponds to the first counterexample to the no-hair theorem for black holes.

Including the Maxwell theory with scalar coupling term into the Einstein-conformally coupled scalar theory leads to the EMCS theory. The EMCS theory without scalar coupling has admitted the charged BBMB black hole \[4\] and the constant scalar hairy black hole \[5\]. It is emphasized that the former implies the secondary scalar hair which blows up on the horizon, while the latter has a constant scalar hair. In this sense, we would like to mention that both of these black holes could not have a truly scalar hair apart from the fact that the former is unstable against the radial perturbation, while the latter is stable against the full perturbation.

Scalarized charged black holes were obtained, through spontaneous scalarization \[6\], from the instability of Reissner-Norström (RN) black hole in the Einstein-Maxwell scalar theory \[7\]. Recently, we have obtained a single branch of scalarized charged black holes inspired by the constant scalar hairy black hole as well as infinite branches of \(n = 0(\alpha \geq 8.019), 1(\alpha \geq 40.84), 2(\alpha \geq 99.89), \cdots\) scalarized charged black holes found through spontaneous scalarization in the EMCS theory \[8\]. These solutions are regarded really as charged black holes with scalar hair because they all have a primary scalar which takes a finite value on the horizon.

Therefore, it is very important to investigate their stability analysis by considering radial perturbations around scalarized charged black holes. In this work, we would be better to choose the radial perturbations because the full perturbations around scalarized charged black holes (numerical solutions) would encounter some difficulty in achieving the stability analysis for numerical black holes.
2 EMCS theory

The action for Einstein-Maxwell-conformally coupled scalar (EMCS) theory takes the form

\[ S_{EMCS} = \frac{1}{16\pi G} \int d^4x \sqrt{-g} \left[ R - (1 + \alpha \phi^2)F_{\mu\nu}F^{\mu\nu} - \beta \left( \phi^2 R + 6 \partial_\mu \phi \partial^\mu \phi \right) \right] , \]  

(1)

where \( \alpha \) denotes a coupling parameter and the last term corresponds to a conformally coupled scalar action with parameter \( \beta \). In this work, we choose \( \beta = 1/3 \) and \( G = 1 \) for simplicity. In the decoupling limit of \( \alpha \to 0 \), the above action reduces to the \( \alpha = 0 \) EMCS theory which allowed the constant scalar hairy black hole and charged BBMB black hole.

The Einstein equation is derived from (1) as

\[ G_{\mu\nu} = 2(1 + \alpha \phi^2)T^M_{\mu\nu} + T^\phi_{\mu\nu} , \]  

(2)

where the energy-momentum tensors for Maxwell theory and conformally coupled scalar theory are given, respectively, by

\[ T^M_{\mu\nu} = F_{\mu\rho}F_{\nu}^{\rho} - \frac{F^2}{4}g_{\mu\nu} , \]  

(3)

\[ T^\phi_{\mu\nu} = \beta \left[ \phi^2 G_{\mu\nu} + g_{\mu\nu} \nabla^2 (\phi^2) - \nabla_\mu \nabla_\nu (\phi^2) + 6 \nabla_\mu \phi \nabla_\nu \phi - 3 (\nabla \phi)^2 g_{\mu\nu} \right] . \]

Here, we observe the traceless condition of \( T^M_{\mu\mu} = 0 \) for Maxwell field. The Maxwell equation is given by

\[ \nabla^\mu F_{\mu\nu} = 2\alpha \phi \nabla^\mu (\phi) F^2 . \]  

(4)

On the other hand, the scalar equation is given by

\[ \nabla^2 \phi - \frac{1}{6} R \phi - \frac{\alpha}{6\beta} F^2 \phi = 0 . \]  

(5)

Considering the trace of the Einstein equation \( [2] \) together with \( [5] \) implies a non-vanishing Ricci scalar given by

\[ R = -\alpha \phi^2 F^2 . \]  

(6)

Finally, we obtain a non-minimally coupled scalar equation

\[ \nabla^2 \phi + \frac{\alpha}{6} \left[ \phi^2 - \frac{1}{\beta} \right] F^2 \phi = 0 . \]  

(7)

In case of \( \alpha = 0 \), one finds a minimally coupled scalar equation \( (\nabla^2 \phi = 0) \) which admitted the charged BBMB black hole \( [4] \) and the constant scalar hairy black hole \( [5] \).
3 Scalarized charged black holes

Before we proceed, we would like to mention an analytical solution of the RN black hole without scalar hair found in the EMCS theory

\[
ds_{\text{RN}}^2 = -f(r)dt^2 + \frac{dr^2}{f(r)} + r^2d\Omega_2^2,
\]

\[
f(r) = 1 - \frac{2m}{r} + \frac{q^2}{r^2}, \quad \bar{\phi} = 0, \quad \bar{A}_t = \frac{q}{r} - \frac{q}{r_+},
\]

and the constant scalar hairy black hole obtained from the \(\alpha = 0\) EMCS theory \cite{5}

\[
ds_{\text{csbh}}^2 = -\tilde{f}(r)dt^2 + \frac{dr^2}{\tilde{f}(r)} + r^2d\Omega_2^2,
\]

\[
\tilde{f}(r) = 1 - \frac{2m}{r} + \frac{Q^2 + q^2}{r^2}, \quad \bar{\phi}_c = \sqrt{\frac{3q_s^2}{q_s^2 + Q^2}}, \quad \bar{A}_t = \frac{Q}{r} - \frac{Q}{r_+},
\]

where \(q_s\) does not represent a truly scalar charge \(Q_s\) existing in the \(\alpha \neq 0\) EMCS theory.

Let us assume the metric and fields to find scalarized charged black holes

\[
ds_{\text{scbh}}^2 = -N(r)e^{-2\delta(r)}dt^2 + \frac{dr^2}{N(r)} + r^2d\Omega_2^2,
\]

\[
N(r) = 1 - \frac{2m(r)}{r}, \quad \bar{\phi} = \phi(r), \quad \bar{A}_t = v(r).
\]

Substituting (10) into Eqs.(2), (4), and (7), one finds four equations for \(m(r), \delta(r), v(r),\) and \(\phi(r)\) as

\[
3e^{2\delta}r^2\alpha \phi(\phi^2 - 3)v^2 - 18(m - m'r)\phi - r(r - 2m)(9 + \phi^2)\phi'' - (r - 2m)\left[\phi'(\phi^2 - 3)\delta + \left(18 + r(\phi^2 - 9)\delta'\right)\phi' - 2r\phi\phi'\right] = 0,
\]

\[
3e^{2\delta}r^2(2 + r\delta' + \frac{2r\alpha\phi}{1 + \alpha\phi^2})v' + rv'' = 0,
\]

\[
r\phi'' - 2r\phi'^2 + \delta'(\phi^2 - 3 + r\phi\phi') = 0,
\]

where the prime (’) denotes differentiation with respect to \(r\).
3.1 Scalarized charged black holes in the single branch

In this section, we briefly review how to derive the scalarized charged black holes inspired by the constant scalar hairy black hole (9). Implementing an outer horizon located at \( r = r_+ \), one may introduce an approximate solution to (11)-(14) in the near-horizon region

\[
m(r) = \frac{r_+}{2} + m_1(r - r_+) + \cdots, \tag{15}
\]

\[
\delta(r) = \delta_0 + \delta_1(r - r_+) + \cdots, \tag{16}
\]

\[
\phi(r) = \phi_0 + \phi_1(r - r_+) + \cdots, \tag{17}
\]

\[
v(r) = v_1(r - r_+) + \cdots, \tag{18}
\]

where the coefficients are determined by

\[
m_1 = \frac{[(\alpha \phi_0^2 (\phi_0^2 - 12) - 9)Q^2}{6r_+^2(\phi_0^2 - 3)(1 + \alpha \phi_0^2)^2},
\]

\[
\delta_1 = \frac{\alpha \phi_0^2 Q^2 (\phi_0^2 - 3)}{2r_+(1 + \alpha \phi_0^2) \left[Q^2(9 - \alpha \phi_0^2 (\phi_0^2 - 12)) + 3r_+^2(\phi_0^2 - 3)(1 + \alpha \phi_0^2)^2\right]^2 \times \left[12r_+^3(\phi_0^2 - 3)(1 + \alpha \phi_0^2)^3 + Q^2 \{18 + \alpha(27 + (48 + 63\alpha)\phi_0^2 + (6\alpha - 7)\phi_0^4 - 3\alpha \phi_0^6)\}\right]},
\]

\[
\phi_1 = \frac{\alpha \phi_0 Q^2 (\phi_0^2 - 3)^2}{r_+ Q^2 (9 - \alpha \phi_0^2 (\phi_0^2 - 12)) + 3r_+^3(\phi_0^2 - 3)(1 + \alpha \phi_0^2)^2},
\]

\[
v_1 = -e^{-\delta_0} Q \frac{r_+^2}{r_+^2(1 + \alpha \phi_0^2)^2}. \tag{20}
\]

Here, we note that \( \delta_1 \) takes a complicated form because of a conformal coupling term \( \phi^2 R \).

In case of \( \alpha = 0 \), the above coefficients reduce to those for the constant scalar hairy black hole exactly

\[
m_1 = -\frac{9Q^2}{6r_+^2(\phi_0^2 - 3)} = \frac{Q^2 + q_s^2}{2r_+^2}, \quad \delta_0 = \delta_1 = 0,
\]

\[
\phi_0 = \bar{\phi}_c = \sqrt{\frac{3q_s^2}{Q^2 + q_s^2}}, \quad \phi_1 = 0, \quad v_1 = -\frac{Q}{r_+^2}. \tag{21}
\]

The near-horizon solution (19) involves two essential parameters of \( \phi_0 = \phi(r_+, \alpha) \) and \( \delta_0 = \delta(r_+, \alpha) \), which can be determined by matching (15)-(18) with an asymptotic solution
Figure 1: (Left) Plot of a scalarized black hole with $\alpha = 63.75$, compared to the constant scalar hairy black hole $[\tilde{f}(r), \tilde{\phi}_c = 0.7746]$. The horizon is located at $\ln r = \ln r_+ = -0.0542$. The right picture indicates that $\delta(r)$ is negative [$\delta(r) = 0$ for the constant scalar hairy black hole] and $v(r)$ is a negative function.

in the far-region as

$$m(r) = M - \frac{3Q_s^2(1 + \phi_\infty^2)}{2(\phi_\infty^2 - 3)^2r} - \frac{Q^2(2\alpha\phi_\infty^4 - 15\alpha\phi_\infty^2 - 9)}{6(\phi_\infty^2 - 3)(1 + \alpha\phi_\infty^2)^2r} - \frac{MQ_s\phi_\infty}{(\phi_\infty^2 - 3)r} \cdots,$$

$$\delta(r) = \frac{2Q_s\phi_\infty}{(\phi_\infty^2 - 3)r} + \cdots,$$

$$\phi(r) = \phi_\infty + \frac{Q_s}{r} + \cdots,$$

$$v(r) = \Psi + \frac{Q}{(1 + \alpha\phi_\infty^2)r} + \cdots,$$

where $\Psi$ is an electrostatic potential. In case of $\alpha = 0$ and $Q_s = 0$, the above reduces to those for the constant scalar hairy black hole exactly

$$m(r) = M + \frac{9Q_s^2}{6(\phi_\infty^2 - 3)r} = M - \frac{Q^2 + q_e^2}{2r}, \quad \delta(r) = 0, \quad \phi_\infty = \tilde{\phi}_c, \quad v(r) = \Psi + \frac{Q}{r}. \quad (22)$$

We obtain a single branch of scalarized charged black holes with a positive $\alpha$, implying no restriction on $\alpha$. Explicitly, we wish to show a (numerical) scalarized charged black hole with $\alpha = 63.75$ in Fig. 1. $N(r)$ and $\tilde{f}(r)$ represent metric function for scalarized charged black hole and constant scalar hairy black hole, respectively. The magnification in the left picture indicates an enlarged decrease of scalar hair $\phi(r)$, showing a clear difference from a constant hair $\tilde{\phi}_c = 0.7746$ in the constant scalar hairy black hole. It is worth noting that this scalar hair does not blow up on the horizon and thus, it is surely a primary one.
\[ \phi(r) \quad f(r) \quad \bar{A}_t(r) \]

Figure 2: Graphs of a scalarized charged black hole with \( \alpha = 65.25 \) in the \( n = 0 \) branch. Here \( f(r) \) and \( \bar{A}_t \) represent the metric function and vector potential for the RN black hole with \( \delta(r) = 0 \). We plot all figures in terms of ‘ln \( r \)’ and thus, the horizon is located at \( \ln r = \ln r_+ = -0.154 \).

### 3.2 \( n = 0, 1, 2, \cdots \) scalarized charged black holes

In this case, an approach to finding infinite black hole solutions through the spontaneous scalarization is the nearly same in the previous one except that the the asymptotic solution in the far-region is given by

\[
\begin{align*}
m(r) &= M - \frac{3Q^2 + Q_s^2}{6r} + \cdots, \quad \phi(r) = \frac{Q_s}{r} + \cdots, \\
\delta(r) &= \frac{Q_s^2[2Q_s^2 - 6M^2 + 3Q^2(2 + \alpha)]}{108r^4} + \cdots, \\
v(r) &= \Psi + \frac{Q}{r} + \cdots,
\end{align*}
\]

which could be obtained when imposing \( \phi_\infty = 0 \) in (22). Thus, any scalar hairs are absent in the far-region, differing from the constant scalar hair \( \phi_\infty \neq 0 \) found in the single branch of scalarized charged black holes. We have obtained the infinite branches of solutions labeled by \( n = 0(\alpha \geq 8.019), n = 1(\alpha \geq 40.84), n = 2(\alpha \geq 99.89), \cdots \) scalarized charged black holes. At this stage, we wish to note that the appearance of these black holes with scalar hair is closely connected to the instability of RN black holes determined by the linearized scalar equation of \( \bar{\nabla}^2 \delta \phi + (\alpha \bar{F}^2/2)\delta \phi = 0 \) with \( q = 0.7 \).

Explicitly, we choose the horizon radius \( r_+ = 0.857 \) and electric charge \( Q = 0.35 \) to construct the \( n = 0 \) scalarized charged black hole with \( \alpha = 65.25 \) shown in Fig. 2.

### 4 Stability of scalarized charged black holes

Before we proceed, we have to mention that it is not an easy task to carry out the stability of scalarized charged black holes because these black holes come out as not analytic solutions
but numerical solutions. In order to develop the stability analysis, one needs to obtain hundreds of numerical solutions depending on the coupling constant $\alpha$ in each branch. Also, the full (axial+polar) perturbations require a complicated decoupling process because the linearized EMCS theory contains five physically propagating modes on these black hole background. In addition, we note that the $l = 0$ (s-mode) scalar propagation determines mainly the stability of these black holes. In the conformal coupling theory (EMCS theory), it is would be better to choose the radial (spherically symmetric) perturbations starting with two metric and vector perturbations which is regarded as a simpler version of the polar perturbation as far as the scalar perturbation is concerned.

Let us introduce the radial perturbations around the scalarized black holes as

$$ds^2_{\text{rad}} = -N(r)e^{-2\delta(r)}(1 + \epsilon H_0)dt^2 + \frac{dr^2}{N(r)(1 + \epsilon H_1)} + r^2 (d\theta^2 + \sin^2 \theta d\varphi^2),$$

where $N(r)$, $\delta(r)$, $\phi(r)$, and $v(r)$ represent a scalarized charged black hole background, while $H_0(t,r)$, $H_1(t,r)$, $\Phi(t,r)$, and $\delta v(t,r)$ denote the perturbed fields around the scalarized black hole background. From now on, we confine ourselves to analyzing the $l = 0$ (s-mode) propagation, implying that higher angular momentum modes ($l \neq 0$) are excluded. In this case, all perturbed fields except the perturbed scalar $\Phi$ may belong to redundant fields.

Expanding equations up to the first order $\epsilon (\epsilon \ll 1)$, we obtain the seven-coupled linearized equations. Four equations of $(t, t)$, $(t, r)$, $(r, r)$, and $(\theta, \theta)$-component are given by

$$t : \quad A_0 H_1 + A_1 \Phi + A_2 \Phi' = 0,$$
$$t : \quad -\frac{A_2}{g^2} \Phi'' + A_3 \Phi' + A_4 \Phi + A_5 H_1 - \frac{A_0}{g^2} H_0' + A_6 H_0 + A_7 \delta v = 0,$$
$$r : \quad -A_7 g^2 \delta v - A_2 g^2 \dot{\Phi} + A_8 \Phi' + A_9 \Phi + A_{10} H_0 H_1 + A_0 H_0' + A_{11} H_0 = 0,$$
$$\theta, \theta : \quad B_0 \delta v + B_1 H_0 + B_2 H_1 + B_3 \Phi + B_4 H_0' + B_5 H_1' + B_6 \Phi' + B_7 \Phi'' + B_8 H_0'' + B_9 \dot{H}_1 + B_{10} \dot{\Phi} = 0$$

with $g^2(r) = \frac{e^{2\delta(r)}}{N(r)}$.

Two linearized Maxwell equations take the forms

$$t : \quad A_{12} \delta v + A_{13} \delta v' + A_{14} \Phi' + A_{15} \Phi + A_{16} (H_0' - H_1') = 0,$$
$$r : \quad A_{13} \delta \Phi + A_{14} \dot{\Phi} + A_{16} (\dot{H}_0 - \dot{H}_1) = 0.$$
Finally, one has a linearized scalar equation

\[
B_{11} \delta v + B_{12} H_0 + B_{13} \Phi + B_{14} H_1 + B_{15} H'_0 + B_{16} H'_1 + B_{17} \Phi' + B_{18} \Phi'' \\
+ B_{19} H''_0 + B_{20} \ddot{H}_1 + B_{21} \dddot{\Phi} = 0.
\]

(31)

Here, the overdot (\(\dot{}\)) denotes derivative with respect to time \(t\), and \(A_i (i = 0, 1..., 16)\) and \(B_i (i = 0, 1..., 21)\) are functions of \(r\) appeared in Appendix A.

Now, our main task is to diagonalize the linearized scalar equation (31) by exploiting the remaining six equations. From (25), we have

\[
H_1 = -\frac{A_1}{A_0} \Phi - \frac{A_2}{A_0} \Phi' \tag{32}
\]

which implies that \(H_1\) is a redundant field. Integrating (30) with respect to \(t\) leads to \(\delta v\)

\[
\delta v = -\frac{A_{14}}{A_{13}} \Phi - \frac{A_{16}}{A_{13}} (H_0 - H_1) \tag{33}
\]

which means that \(\delta v\) is also a redundant field. Using (28), we transform (31) to a reduced scalar equation without \(H''_0\) and \(\dddot{H}_1\)

\[
A_{17} (H'_0 + H'_1) + A_{18} \Phi'' - A_{19} g^2 \ddot{\Phi} + A_{19} \delta v + A_{20} \Phi' + A_{21} \Phi + A_{22} H_0 + A_{23} H_1 = 0. \tag{34}
\]

Making use of (33), (32), \(\delta v'\), and \(H'_1\), we could express (34) in terms of \(H'_0\), \(\Phi\), \(\Phi'\) and \(\Phi''\); called the reduced (34). Combining (27) with reduced (34) to eliminate \(H'_0\) arrives at the master scalar equation for testing the stability of scalarized charged black holes as

\[
\left[ g^2 (r) \frac{\partial^2 \Phi}{\partial t^2} \right] - \frac{\partial^2 \Phi}{\partial r^2} + C_1 (r) \frac{\partial \Phi}{\partial r} + U (r) \Phi = 0, \tag{35}
\]

where \(C_1 (r)\) and \(U (r)\) are expressed in Appendix B. At this stage, we wish to mention that two equations (26) and (29) are redundant.

Introducing a further separation of \(\Phi(t, r) = \delta \phi (r) e^{-i \omega t}\), we obtain the Schrödinger-like equation from (35)

\[
\frac{d^2 Z(r)}{dr_*^2} + [\omega^2 - V (r, \alpha)] Z(r) = 0, \tag{36}
\]

where \(r_*\) is a tortoise coordinate to extend from \(r \in [r_+, \infty]\) to \(r_* \in [-\infty, \infty]\) and \(Z(r)\) is a redefined scalar, expressed by

\[
r_* = \int_{r_+}^{\infty} g(r) dr, \quad Z(r) = \frac{\delta \phi (r)}{C_0 (r)}. \tag{37}
\]
\[ \alpha = 0.0142 \]

\[
\begin{align*}
\alpha & = 63.7571 \\
\alpha & = 359
\end{align*}
\]

The potential seems to be independent of the coupling parameter \( \alpha > 0 \).

Figure 3: Scalar potential \( V(r, \alpha) \) around scalarized charged black holes in the single branch.

Here the potential takes the form

\[
V(r, \alpha) = \frac{U(r) - C'_1(r)}{g^2(r)} + \frac{C_1 g'(r) + g''(r)}{g^3(r)} - \left[ \frac{2g'(r)}{g^3(r)} \right]^2. \tag{38}
\]

We point out that \( C_0(r) \) is the solution to the differential equation

\[
[\ln C_0(r)]' = C_1(r) - [\ln g(r)]'. \tag{39}
\]

Table 1: Results for numerical integration (NI) of \( \int_{r_+}^{\infty} dr[g(r)V(r, \alpha)] \) for single branch (SB) and \( n = 0, 1, 2 \) branches. Bold figures represent cases appeared in potentials in Figs. 3-4.
stability of the black hole. It is suggested from Fig. 3 that the potential $V(r, \alpha)$ around the black hole in the single branch indicates negative regions for large $r$, suggesting an instability. We inform from Fig. 4 (Left) that the potential around the $n = 0$ black hole indicates negative region for large $r$, implying an instability. On the other hand, from Fig. 4 (Middle, Right), the potentials around the $n = 1, 2$ black holes show large negative regions outside the horizon, showing the strong instability.

In addition, a sufficient condition for instability is given by $\int_{r_*}^{\infty} dr [g(r)V] < 0$ in accordance with the existence of unstable modes [18]. However, a potential with negative region whose integral ($\int_{r_*}^{\infty} dr [g(r)V] > 0$) is positive does not exclude the existence of unstable modes. We observe from Table 1 that all $\int_{r_*}^{\infty} dr [g(r)V]$ for single branch are positively small and those are also positive for $n = 0$ branch with $\alpha > 16.2$. It is clear from Table 1 that all $\int_{r_*}^{\infty} dr [g(r)V]$ are negative for $n = 1, 2$ branches, showing the instability.

Importantly, to determine the (in)stability of the black hole, we have to solve (36) numerically by imposing an appropriate boundary condition that $Z(r)$ has an outgoing wave at infinity and an ingoing wave on the horizon: $Z(r) \sim e^{-i\omega(t-r_*)}$ at $r_* \to \infty$ and $Z(r) \sim e^{i\omega(t+r_*)}$ at $r_* \to -\infty$. If one finds an exponentially growing mode of $e^{\Omega t}$ ($\omega = i\Omega$), the corresponding black hole is unstable against the scalar perturbation. The linearized scalar equation (36) around the scalarized charged black hole in the single branch and the $n = 0, 1, 2$ scalarized charged black holes may allow either a stable (decaying) mode with $\Omega < 0$ or an unstable (growing) mode with $\Omega > 0$. In case of unstable modes, we may solve (36) numerically with a boundary condition that $Z(r) = 0$ at $r_* = \infty$ and $Z(r) = 0$ at $r_* = -\infty$. We find that the black hole in the single branch is unstable against the $l = 0$-scalar mode (Fig. 5) as well as the $n = 0, 1, 2$ black holes are unstable against the $l = 0$-scalar mode (Fig. 6).

5 Discussions

First of all, we would like to mention that all scalarized charged black holes found from the EMCS theory are unstable against the $l = 0$-scalar mode perturbation.

We summarize the stability issues for scalarized black holes obtained from three theories in Table 2.

We compare the stability of scalarized charged black holes obtained from the EMCS
Figure 5: The positive \( \Omega \) as function \( \alpha \) for \( l = 0 \)-scalar mode around the scalarized black holes in the single branch.

Figure 6: (Left) The positive \( \Omega \) as functions of \( \alpha \) for \( l = 0 \)-scalar mode around the \( n = 0(\alpha \geq 8.019) \), \( 1(\alpha \geq 40.84) \), \( 2(\alpha \geq 99.89) \) black holes. A red curve with \( q = 0.7 \) denotes the positive \( \Omega \) of \( l = 0 \)-scalar mode as function of \( \alpha \) around the RN black hole, indicating the unstable RN black holes for \( \alpha > 8.019 \). (Right) The enlarged picture shows the small positive \( \Omega \) for \( l = 0 \)-scalar mode around the \( n = 0 \) black hole.

theory with quadratic coupling to that from the EMS theory with quadratic coupling. The former includes all unstable branches, whereas the latter has a stable \( n = 0 \) branch. Actually, there is no difference between radial and full perturbations as far as the \( s \)-mode scalar perturbation is concerned. At this stage, it would be better to ask why all scalarized charged black holes are unstable. To answer to this question, we remind the reader that the BBMB (extremal) black hole whose scalar hair is nontrivial is unstable, while the constant scalar hairy black holes whose scalar is trivial are stable against full perturbations. It implies that the inclusion of a conformally coupled scalar may make the scalarized charged black holes unstable in the EMCS theory, in compared to non-minimally coupling in the EMS theory. However, the spontaneous scalarization scheme seems to persist in obtaining
scalarized black holes from both theories.

Since all scalarized charged black holes obtained from the EMCS theory are unstable, these are not considered as final remnants. It suggests that the scalarized charged black hole in the single branch radiates to yield the constant scalar hairy black hole (endpoint of scalarized charged black hole in the single branch is the constant scalar hairy black hole), while scalarized charged black holes in the infinite branches might not radiate to the RN black hole because they all are unstable.

| Theory         | $n = 0$ | $n = 1$ | $n = 2$ | Single Branch | Perturbations | Reference |
|----------------|---------|---------|---------|---------------|---------------|-----------|
| EGBS with EC   | NA      | NA      | NA      | U(SBH)        | Radial        | [9, 10, 11]|
| EGBS with QC   | U       | U       | U       | NA            | Radial        | [12]      |
| EGBS with qC   | S       | U       | U       | NA            | Radial        | [13, 14]  |
| EGBS with EC   | S       | -       | -       | NA            | Axial         | [15]      |
| EMS with EC    | NA      | NA      | NA      | U(RNBH)       | Radial        | [7]       |
| EMS with QC    | S       | U       | U       | NA            | Full          | [16]      |
| EMCS with QC   | U       | U       | U       | U             | Radial        | This work |

Table 2: Summary of stability analysis for scalarized black holes obtained from three theories. EGBS stands for Einstein-Gauss-Bonnet-Scalar and EMS denotes Einstein-Maxwell-Scalar. EC, QC, and qC represent exponential coupling, quadratic coupling, quartic coupling. U(S) indicates unstable (stable) black hole. Full implies axial+polar. SBH (NRBH) mean Schwarzschild (Reissner-Norström) black holes without scalar hair. Finally, NA represents not available and ‘·’ implies not computed.

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Appendix A

In Appendix A, we display the coefficients in seven Eqs. (25)-(34).

The coefficients in (25) are given by

\[
A_0 = -\frac{\phi^2 - 3 + r\phi\phi'}{3r}, \quad A_1 = \frac{1}{3} \left( \frac{\phi N'}{N} - 2\phi\delta' + 4\phi' \right), \quad A_2 = -\frac{2\phi}{3}.
\]

Those in (26) have

\[
A_3 = e^{-2\delta N} \left( \frac{\phi N'}{3} + \frac{4N\phi}{3r} - \frac{2N\phi'}{3} \right),
\]
\[
A_4 = e^{-2\delta N} \left( 2\phi(N - 1 + rN' - 3e^{-2\delta}r^2v^2) + r(rN'\phi' + 2N(2\phi' + r\phi'')) \right),
\]
\[
A_5 = e^{-2\delta N} \left( r(-3e^{2\delta}r(1 + \alpha\phi^2)v^2 + N'(\phi^2 - 3 + r\phi\phi')) + N(-3 + \phi^2 - r^2\phi^2 + 2r\phi(2\phi' + r\phi'')) \right),
\]
\[
A_6 = e^{-2\delta N} \left( 3 - 3rN' + \phi^2(rN' - 1) + r^2\phi N'\phi' + N(\phi^2 - 3 - r^2\phi^2 + 2r\phi(2\phi' + r\phi'')) \right),
\]
\[
A_7 = -2N(1 + \alpha\phi^2)v'.
\]

The coefficients in (27) are given by

\[
A_8 = \frac{1}{3} \left( \phi(2\delta' - 4 - \frac{N'}{N}) - 6\phi' \right),
\]
\[
A_9 = \frac{1}{3r^2N} \left( \phi(2 - 2rN' + 6\alpha e^{-2\delta}r^2v^2 + N(4r\delta' - 2)) + r(-rN' + 2N(r\delta' - 2))\phi' \right),
\]
\[
A_{10} = \frac{\phi^2 - 3}{3r^2N}, \quad A_{11} = -\frac{e^{2\delta}(1 + \alpha\phi^2)v^2}{N}.
\]
The coefficients in (28) possess

\[ B_0 = -2e^{2\delta}r^2(1 + \alpha \phi'^2)v', \quad B_1 = e^{2\delta}r^2(1 + \alpha \phi^2)v', \]
\[ B_2 = \frac{1}{3\phi^2}(\phi'^4(N - 1 + rN' - 4rN\delta') - 9r\phi\phi'(rN' + N(2 - 2r\delta')) - 9rN(3\delta' + 2r\phi'^2) + \phi^2(-3rN' - 1 + e^{2\delta}r^2(1 + 3\alpha)v'^2) + N(-3 + 21r\delta' + 7r^2\phi'^2) + r\phi^3(rN'\phi' - 2N(2(-1 + r\delta')\phi' + r\phi''))), \]
\[ B_3 = -\frac{2}{3}\phi'(-\phi(1 + N + rN' - 4rN\delta') - 3rN(3\delta' + 2r\phi'^2) + r\phi(-2rN'\phi' + N(5(r\delta' - 1)\phi' + r\phi''))), \]
\[ B_4 = \frac{r}{12}(3r(3 + \phi^2)N' + 2N(3 - 6r\delta' + \phi^2(2r\delta' - 1) - 2r\phi')), \]
\[ B_5 = \frac{r}{12}(-r(3 + \phi^2)N' + 2N(3 - 3r\delta' + \phi^2(r\delta' - 1) - 2r\phi')), \]
\[ B_6 = \frac{2r}{3}(-r\phi N' + N(\phi(r\delta' - 1) + r\phi')), \quad B_7 = \frac{2r^2N\phi}{3}, \]
\[ B_8 = \frac{r^2(\phi'^2 - 3)N}{6}, \quad B_9 = \frac{r^2(\phi'^2 - 3)e^{2\delta}}{6N}, \quad B_{10} = \frac{2e^{2\delta}r^2\phi}{3N}. \]

The coefficients in (29) and (30) take the forms

\[ A_{12} = e^{2\delta}\left((1 + \alpha \phi^2)\left(\frac{2}{r} + \delta'\right) + 2\alpha \phi\phi'\right), \quad A_{13} = e^{2\delta}r^2(1 + \alpha \phi^2), \quad A_{14} = 2\alpha e^{2\delta}\phi v', \]
\[ A_{15} = -\frac{2\alpha e^{2\delta}(-1 + \alpha \phi^2)\phi'v'}{1 + \alpha \phi^2}, \quad A_{16} = -\frac{1}{2}e^{2\delta}r^2(1 + \alpha \phi^2)v'. \]

The coefficients in (31) are given by

\[ B_{11} = 2\alpha e^{2\delta}\phi v', \quad B_{12} = -\alpha e^{2\delta}\phi v'^2, \]
\[ B_{13} = \alpha e^{2\delta}v'^2 + \frac{\phi(-rN'\phi' + N(r\delta' - 2)\phi' - r\phi'')}{r(\phi^2 - 3)}, \]
\[ B_{14} = \alpha e^{2\delta}\phi v'^2 + \frac{\phi}{3r^2} + \frac{3(-rN'\phi' + N(r\delta' - 2)\phi' - r\phi'')}{r(\phi^2 - 3)}, \]
\[ B_{15} = \frac{N\phi'}{3r} + \frac{\phi}{12}(3N' - 4N\delta') + \frac{N\phi'}{2}, \]
\[ B_{16} = \frac{N\phi'}{3r} + \frac{\phi}{12}(N' - 2N\delta') + \frac{N\phi'}{2}, \]
\[ B_{17} = N' + N\left(\frac{2}{r} - \delta'\right), \quad B_{18} = N, \quad B_{19} = \frac{N\phi}{6}, \]
\[ B_{20} = \frac{e^{2\delta}\phi}{6N}, \quad B_{21} = \frac{2e^{2\delta}r^2\phi}{3N}. \]
Finally, those for (34) lead to

\[ A_{17} = \frac{r N^2}{36} \left( (\phi^2 - 3)\phi + (\phi^2 - 9)r\phi' \right), \quad A_{18} = \frac{r^2 N^2}{18} (\phi^2 - 9), \quad A_{19} = -e^{2\delta} r^2 \left( \frac{1}{3} + \alpha \right) N\phi v', \]

\[ A_{20} = \frac{r N}{18} \left( r(\phi^2 - 9)N' + N(\phi^2(4 - r\delta') + 9(r\delta' - 2) + 2r\phi\phi') \right), \]

\[ A_{21} = -\frac{N}{18\phi^2} \left( \phi^4(2 - 2r N' + N(5r\delta' - 2) - 9r\phi\phi'(rN' + 2N(1 - r\delta'))) - 6r^2 N\phi^2\phi'^2 - 9rN(3\delta' + 2r\phi'^2) + r\phi^3(-rN'\phi' + 2N(2(r\delta' - 1)\phi' + r\phi''))) \right), \]

\[ A_{22} = \frac{r N N'}{18} \left( (\phi^2 - 3)\phi + (\phi^2 - 9)r\phi' \right) - e^{2\delta} r^2 \left( \frac{1}{6} + \frac{\alpha}{2} \right) N\phi v'^2 + \frac{r^2 N^2}{18} (\phi^2 - 9)\phi'' - \frac{N^2}{18} \left( \phi^3(r\delta' - 1) + r\phi^2\phi'(r\delta' - 4) + 9r\phi'(2 - r\delta') + \phi(3r\delta' + r^2\phi'^2 - 3) \right), \]

\[ A_{23} = e^{2\delta} r^2 \left( \frac{1}{6} + \frac{\alpha}{2} \right) N\phi v'^2. \]

**Appendix B**

For the master scalar equation (35), \( C_1 \) is given by

\[ C_1 = C_2 + C_3, \]

\[ C_2 = -\frac{3e^{2\delta} r^2 (1 + \alpha \phi^2) v'^2 + \phi^2 - 3}{N r(\phi^2 - 3 + r\phi\phi')}, \]

\[ C_3 = \frac{1}{r(\phi^2 - 3)(\phi^2 - 3 + r\phi\phi')} \left[ 9(1 + 3r\delta' + 3r^2\phi'^2) - 9r\phi\phi'(r\delta' - 2) - \phi^2(6 + 18r\delta' + 7r^2\phi'^2) - 3r\phi^3\phi'(2 - r\delta') + \phi^4(1 + 3r\delta') \right]. \]

On the other hand, \( U(r) \) takes a complicated form

\[ U = U_1 + U_2 + \frac{U_3}{U_4}, \]

\[ U_1 = -\frac{e^{2\delta}(1 + \alpha \phi^2)v'^2(\phi N' - 2\phi N\delta' + 4N\phi')(\phi^2 + \phi^3 - 9r\phi' + r\phi^2\phi')}{6N^2(\phi^2 - 3)(-3 + \phi^2 + r\phi\phi')}, \]

\[ U_2 = -\frac{2e^{2\delta} \alpha \phi v'^2(-9r\phi' + 3(1 + 6\alpha)\phi - (1 + 15\alpha)r\phi^2\phi' - (1 + 9\alpha)\phi^3 + \alpha r\phi^4\phi' + \alpha \phi^5)}{3N(\phi^2 - 3)(1 + \alpha \phi^2)}, \]

\[ U_3 = 18r N^2 \phi(\phi^2 - 3)^2(-3 + \phi^2 + r\phi\phi'), \]

\[ U_4 = D_0 + \phi D_1 + \phi^2 D_2 + \phi^3 D_3 + \phi^4 D_4 + \phi^5 D_5 + \phi^6 D_6 + \phi^7 D_7 + \phi^8 D_8 + \phi^9 D_9 \]
with

\[ D_0 = -972rN^2\phi'(3\phi' + 2r\phi'^2), \]
\[ D_1 = 162N \left( 3e^{2\delta} + 3\phi'(N - rN' + 2rN\phi') + 2r(1 - 3rN' + N(9r\phi' - 3))\phi'^2 + 2r^3N\phi'^4 \right), \]
\[ D_2 = -27\phi' \left( 3rN'(rN' - 1) + 2N^2(12r^2\delta'^2 + r\phi'(9r^2\phi'^2 - 40) - 18r^2\phi'^2 - 10) \right) + N(8 + 30ae^{2\delta}r^2v^2 + 6r\phi' - rN'(5 + 18r\phi' + 9r^2\phi'^2)) , \]
\[ D_3 = 9 \left( 2N(2r(5N - 4)\phi'^2 + 6rN\delta'^2(r^2\phi'^2 - 10) + 3ae^{2\delta}rv^2(4r^2\phi'^2 - 9) - \delta'(3 + N(24 + 71r^2\phi'^2))) + 3rN^2(r^2\phi'^2 - 1) + N'(3 + N(-3 + 41r^2\phi'^2 + \delta'(66r - 12r^3\phi'^2))) \right), \]
\[ D_4 = 3\phi' \left( 3rN'(8rN' - 7) + 2N^2(84r^2\phi'^2 - 54 - 8r^2\phi'^2 + 5r\phi'(2r^2\phi'^2 - 21)) + N(6(6 + 30ae^{2\delta}rv^2 + 7r\phi') - rN'(3 + 132r\delta' + 10r^2\phi'^2)) \right), \]
\[ D_5 = -3 \left( 2N(8rN\delta'^2(r^2\phi'^2 - 9) + 3ae^{2\delta}rv^2(4r^2\phi'^2 - 9) + 2r\phi'^2(N - 2 - r^2N\phi'^2) - 9\phi' \right) - 2N\phi'(9 + 16r^2\phi'^2)) + N'(9 + N(-9 + 20r^2\phi'^2 + \delta'(90r - 16r^2\phi'^2)) + rN'^2(4r^2\phi'^2 - 9)), \]
\[ D_6 = -\phi' \left( 3rN'(7rN - 5) + 2N^2 + (60r^2\delta'^2 + r\phi'(7r^2\phi'^2 - 66) - 2(9 + 7r^2\phi'^2)) \right) - rN(-30(3ae^{2\delta}rv^2 + \delta') + N'(-39 + 102r\delta' + 7r^2\phi'^2))). \]
\[ D_7 = rN^2(r^2\phi'^2 - 9) - 2N \left( 9ae^{2\delta}rv^2 + 2r(1 - 5N)\phi'^2 - 2rN\delta'^2(r^2\phi'^2 - 20) + 3\phi'(3 + 5r^2N\phi'^2) \right), + N'(9 + N(-9 + 15r^2\phi'^2 + \delta'(54r - 4r^3\phi'^2))). \]
\[ D_8 = \phi' \left( rN'(2rN' - 1) + 2N^2(2 - 9r\phi' + 4r^2\phi'^2) + N(-4 + 2r\phi' + rN'(9 - 8r\phi')) \right), \]
\[ D_9 = (N' - 2N\phi')(N - 1 + rN' - 2rN\phi'). \]
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