Thermal Casimir effect between random layered dielectrics

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We study the thermal Casimir effect between two thick slabs composed of plane-parallel layers of random dielectric materials interacting across an intervening homogeneous dielectric. It is found that the effective interaction at long distances is self averaging and is given by a description in terms of effective dielectric functions. The behavior at short distances becomes random (sample dependent) and is dominated by the local values of the dielectric function proximal to each other across the dielectrically homogeneous slab.

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Systems with spatially varying dielectric functions exhibit effective van der Waals interactions arising from the interaction between fluctuating dipoles in the system. These fluctuation interactions have two distinct components: (i) a classical or thermal component due to the zero frequency response of the dipoles and (ii) a quantum component due to the non-zero frequency/quantum response of the dipoles. Despite the clear physical differences in these contributions, the mathematical computation of the corresponding interaction is almost identical and boils down to the computation of an appropriate functional determinant. The full theory taking into account both of these component interactions is the celebrated Lifshitz theory of van der Waals interactions, based on boundary conditions imposed on the electromagnetic field at the bounding surfaces and the fluctuation-dissipation theorem for the electromagnetic potential operators. The original Casimir interaction is obtained in the limit of zero temperature and ideally polarizable bounding surfaces. At non-zero temperature the contribution of the zero frequency modes to the Lifshitz theory yields the classical thermal Casimir effect which is due to the non-retarded van der Waals interactions. The major mathematical problems in the computation of Casimir type interactions (setting aside the experimental and theoretical challenges to determine the correct dielectric behavior) are (i) the application of the Lifshitz approach to non-trivial geometries and (ii) taking into account local inhomogeneities in the dielectric properties of the media, always present in realistic systems. In this paper we will address the latter.

We consider the thermal Casimir interaction for the case where the local dielectric function is a random variable in the transverse direction. Specifically we will consider the interaction between two thick parallel dielectric slabs, separated by a homogenous dielectric medium, see Fig. 1. The thickness of both disordered dielectric slabs are \(L_1\) and \(L_2\) respectively and their separation is denoted by \(\ell\). In what follows we will study the limit of infinite slabs i.e. \(L_1, L_2 \to \infty\). The dielectric response within the two slabs is constant in the planes perpendicular to the slab normal, but varies in the direction of the surface normal. It is well known that this problem can be solved in the case where the dielectric constants of the slabs do not vary and the result can be tentatively applied to the case of fluctuating dielectric functions via an effective medium theory which consists of replacing the fluctuating dielectric functions by an effective (spatially constant within each of the slabs) dielectric tensor. The most commonly used approximation is that where the local dielectric tensor is replaced by the effective dielectric tensor, i.e.

\[
\langle \epsilon_{ij}(x) \rangle \rightarrow \epsilon_{ij}^{(e)},
\]

with the bulk dielectric tensor defined via \(\langle \epsilon_{ij}^{(e)} E_j \rangle = \langle \epsilon_{ij} E_j \rangle \). The use of the effective dielectric function is not easily justifiable mathematically as an approximation, although physically the effective dielectric function clearly does capture the bulk response to constant electric fields. We shall see that, for the random layered dielectric model studied here, the effective dielectric constant approximation of Eq. 1 does in fact give the correct value of the thermal Casimir interaction when the two slabs are widely separated. This can be expected on physical grounds since the fluctuating electromagnetic field modes with smallest wave-vectors (corresponding to variations on large scales) dominate the Casimir interaction for large inter-slab separation. The dielectric response of the material to a constant electric field is given by the effective dielectric constant and if the wave-vector depen-
The classical electromagnetic field energy in a dielectric medium is given by

\[ H[\phi] = \frac{1}{2} \int d\mathbf{x} \epsilon(\mathbf{x})(\nabla \phi(\mathbf{x}))^2 \]  

(2)

and the corresponding partition function is given by the functional integral

\[ Z = \int d[\phi] \exp(-\beta H[\phi]). \]

Differences in dielectric functions lead to the thermal Casimir effect. Here we will consider layered systems where the dielectric function \( \epsilon \) depends only on the \( z \) direction \( \epsilon(\mathbf{x}) = \epsilon(z) \).

If we express the field \( \phi \) in terms of its Fourier modes in the plane perpendicular to \( z \), and we take the area perpendicular to \( z \) as \( A \), with wave-vector \( \mathbf{k} = (k_x, k_y) \), then the Hamiltonian can be written as

\[ H_k = \frac{1}{2} \int dz \epsilon(z) \left( \left| \frac{d\phi(z, \mathbf{k})}{dz} \right|^2 + k^2 |\phi(z, \mathbf{k})|^2 \right). \]  

(3)

A direct consequence of this decomposition is that the partition function can be expressed as a sum over the partition functions of individual modes \( Z_k \) as

\[ \ln(Z) = \sum_k \ln(Z_k) \]

where

\[ Z_k = \int d[X] \exp \left( -\frac{1}{2} \int dz \epsilon(z) \left[ \left( \frac{dX}{dz} \right)^2 + k^2 X^2 \right] \right). \]  

(4)

Here \( k = |\mathbf{k}| \) and we have taken into account that the field \( \phi \) is real.

The problem of computing the interaction between slabs composed of layers of finite thickness can be studied using a transfer matrix method [5]. However we will use a method based on the Feynman path integral which is particularly well suited to the study of systems where the dielectric function can vary continuously [6]. If we specify the starting and ending points of the above path integral, we see that it has to be of a harmonic oscillator form defined by

\[ K(x, y; z) = \int d[X] \exp \left( -\frac{1}{2} \int dz M(z) \left[ \left( \frac{dX}{dz} \right)^2 + \omega^2 X^2 \right] \right). \]  

(5)

which can be computed using the generalized Pauli - van Vleck formula [6] [7] telling us that \( K \) must have the general form

\[ K(x, y; z) = \left( \frac{b}{2\pi} \right)^{\frac{1}{2}} \exp \left( -\frac{1}{2} a_z(z)x^2 - \frac{1}{2} a_f(z)y^2 + b(z)xy \right). \]  

(6)

We may now write down evolution equations for the coefficients \( a_z, a_f \) and \( b \) using the Markovian property of the path integral, \( K(x, y; z + z') = \int dw K(x, w; z)K(w, y; z') \) [6] [7], which can also be used to prove the generalized Pauli - van Vleck formula. We obtain the evolution equations

\[ \frac{da_z}{dz} = -\frac{b^2}{M}, \quad \frac{db}{dz} = -\frac{ba_f}{M}, \]  

(7)

\[ \frac{da_f}{dz} = M\omega^2 - \frac{a_f^2}{M}. \]  

(8)

We thus find that the \( \ell \)-dependent part of the free energy of the mode \( \mathbf{k} \) (up to a bulk term which can be subtracted...
off to get the interaction energy is given by
\[ F_k = \frac{k_B T}{2} \ln \left( 1 - \frac{(a_i^{(1)}(k) - \epsilon_0 k)(a_i^{(2)}(k) - \epsilon_0 k)}{(a_i^{(1)}(k) + \epsilon_0 k)(a_i^{(2)}(k) + \epsilon_0 k)} e^{-2\pi k} \right), \]
and the total \( \ell \) dependent free energy is \( F = \sum_k F_k \).
Here \( a_i^{(1,2)}(k) \) are the solutions to Eq. (3) evaluated at the opposing faces of each slab (1) and (2) respectively.

In order to evaluate the integrals of \( a_i^{(1,2)}(k) \), one first has to solve equations of motion Eqs. (8) to get the \( \ell \) dependence of \( a_i(k, z) \) and then proceed to the integrals that enter Eq. (9). The evolution equation for \( a_i(k) \) for either slab can be read off from Eq. (8) and is given by
\[ \frac{d a_i(k, z)}{d z} = \epsilon(z) k^2 - \frac{a_i^2}{\epsilon(z)}. \]

An appropriate Hopf-Cole transformation [7] shows this formalism to be equivalent to the transfer matrix method [2] or to the density functional method [8] for evaluating the van der Waals forces. This nonlinear formulation of an essentially linear problem simplifies evaluating the van der Waals forces. This nonlinear formalism to be equivalent to the transfer matrix method [2].

In the perpendicular direction the dielectric tensor is has the form \( \epsilon^{(c)} = \epsilon \) and \( \epsilon^{(e)} = \epsilon^* \), where \( \epsilon^* \) are defined via Eq. (15). The subscript \( i \) on the angled brackets signifies that we are averaging the dielectric function in the slab \( i \). The term \( H^* \) defines an effective disorder-dependent Hamaker coefficient. This therefore justifies physical arguments replacing the random layered material by an effective anisotropic medium where the dielectric tensor is has the form \( \epsilon^{(c)} = \epsilon \) and \( \epsilon^{(e)} = \epsilon^* \), and all other terms being zero by symmetry. The term \( \epsilon_{||} \) is the effective dielectric function in the \( \perp \) direction and \( \epsilon_{\perp} \) the perpendicular components are given by \( \epsilon_{\perp} = \langle \epsilon \rangle \). The expressions for \( \epsilon_{||} \) and \( \epsilon_{\perp} \) follow simply from the fact that in the perpendicular direction the dielectric function is obtained by analogy to capacitors in series and in the parallel direction by analogy to capacitors in parallel arrangement [10].

The effective value, \( e^* \), for dielectric constant of this system coincides with that of Eq. (15) above [7]. This result shows that for large separations (where \( \ell \) is much larger than the correlation length of the dielectric disorder) the thermal Casimir interaction free energy is self averaging and agrees with that given by physical reasoning.

One would imagine that as the distance between the slabs is reduced, the result will be increasingly dominated by the slab composition at the two opposite faces [2]. Indeed in the small \( \ell \) limit Eq. (11) is dominated by the large \( \ell \) behavior. The asymptotic behavior can be extracted if one assumes the ansatz
\[ \alpha(k, z) = \sum_{n=0}^{\infty} \frac{\alpha_n(z)}{k^n}. \]
Substituting this into Eq. (12) gives the following chain
of equations for $\alpha_n(z)$

$$
\frac{1}{k} \sum_{n=0}^{\infty} \frac{1}{k^n} \frac{d\alpha_n(z)}{dz} = \epsilon(z) - \frac{1}{\epsilon(z)} \sum_{n,m=0}^{\infty} \frac{\alpha_n(z)\alpha_m(z)}{k^{m+n}}. \quad (18)
$$

From here it is easy to see that to order $O(1)$ the leading asymptotic result of Eq. [23] is given by

$$
\alpha_0(z) = \epsilon(z). \quad (19)
$$

The equation for the corrections ($n \geq 1$) to this asymptotic limit is

$$
\frac{d\alpha_n(z)}{dz} = -\frac{1}{\epsilon(z)} \sum_{m=0}^{n} \alpha_m(z)\alpha_{n-m}(z), \quad (20)
$$

and the next two terms from this expansion yield

$$
\alpha_1(z) = \frac{1}{\epsilon(z)} \frac{d\epsilon(z)}{dz}, \quad (21)
$$

$$
\alpha_2(z) = \frac{1}{4} \frac{d^2\epsilon(z)}{dz^2} - \frac{1}{8\epsilon(z)} \left( \frac{d\epsilon(z)}{dz} \right)^2. \quad (22)
$$

It is straightforward to realize that these terms generate $O(1/\ell)$ corrections to the asymptotic result which are subdominant when $\ell$ is large. Thus to the leading order

$$
\alpha(z,k) \approx \alpha_0(z) = \epsilon(z) \quad (23)
$$

and from here it follows straightforwardly that

$$
\lim_{k \to \infty} \rho_i(y,k) = \rho_i(\epsilon) \quad (24)
$$

where $\rho_i$ is the probability density function of $\epsilon(z)$ in medium $i$. This result is easily understood from the physical discussion above. The average of the thermal Casimir interaction free energy Eq. (11) in the small separation limit is then given by

$$
\langle F \rangle (\ell \to 0) \sim \frac{k_B T A}{16\pi^2} \int \rho_1(\epsilon_1)\rho_2(\epsilon_2) d\epsilon_1 d\epsilon_2 \ln \left( 1 - \Delta_1 \Delta_2 e^{-u} \right), \quad (25)
$$

with $\Delta_i = (\epsilon_i - \epsilon_0)/(\epsilon_i + \epsilon_0)$. The forms of the thermal Casimir interaction free energy are thus given by Eqs. [25] and [10] in the small and large interslab separation limits respectively.

In the limit of large separation between the slabs we have obtained the limiting behavior of the thermal Casimir effect and shown that the free energy is given by self-averaging and that the distributions of $\alpha(k,z)$ are strongly peaked. It can be shown [7] that the attraction at large separation between two (statistically identical) homogeneous media (with $\epsilon = \langle \epsilon \rangle$) is stronger than that between the two fluctuating media if $1/\langle \epsilon \rangle^{-1} > \epsilon_0$. However it is always weaker if $\langle \epsilon \rangle < \epsilon_0$. So, depending on the details of the distribution of the fluctuating dielectric response in the two slabs and the dielectric response of the medium in-between, the effective interaction at large inter-slab separations can be stronger or weaker than that for a uniform medium with a dielectric constant equal to the mean dielectric function of the fluctuating media.

For small separations the interaction free energy is a random variable which has to be averaged over the probability density function of the dielectric functions in the media composing the two interacting slabs. The intermediate length scales can be analyzed via perturbation theory [7], and there may also exist models of disorder that can be treated exactly. The nonlinear formulation of the problem presented here should be equally useful to treat the case of deterministically varying dielectric functions and could open up a useful computational framework for designing materials where the effective interaction can be tuned to induce attractive or repulsive forces depending on the separation, for practical applications [11].

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