Relativistic field-theoretical approach to the inverse scattering problem.

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The inverse scattering problem for the relativistic three-dimensional equation
\[
(2E_{p'} - 2E_p) \langle p'|\Psi_p \rangle = - \int V(t) d^3p'' \langle p''|\Psi_p \rangle \text{ with } E_p = \sqrt{m^2 + p^2} \text{ and } t = (E_{p'} - E_{p''})^2 - (p' - p'')^2
\]
is considered. The field-theoretical potential \( V(t) \) of this equation is constructed in the framework of the old perturbation theory. It contains all contributions of diagrams with intermediate off-mass shell particles. In particular, this potential reproduces the OBE Bonn model of the \( NN \) potential exactly. For the \( \pi N \) scattering it is generated by \( \sigma, \rho, \omega \)-meson exchange diagrams. The inverse scattering problem is solved by reduction of these relativistic equations to the standard Schrödinger equations
\[
(\Delta_r + k^2) \langle r|\phi_p \rangle = -v(r) \langle r|\phi_k \rangle \text{ with } E_p = k^2/2m + m.
\]
The relation between the relativistic potential \( V(t) \) and its nonrelativistic representation \( v(r) \) is obtained.
The inversion method for the reconstruction of the potential in the Schrödinger equation is needed for the numerous problems in nuclear and particle physics \[1 - 4\]. However, the relativistic generalizations of the inverse scattering methods in the quantum scattering theory was carried out for the one particle relativistic equations such as the Dirac or Klein-Gordon equation only. \[1\]. The aim of this paper is to extend the inversion method for the determination of the potential in the field-theoretical equations. In particular we consider the scheme of the relativistic generalization of the Schrödinger equations for the NN scattering amplitudes\[5\].

The relativistic equation in the time-ordered three-dimensional field-theoretical formulation for the NN amplitude \(< p' | t(E(p)) | p > \)\[6,7\] has the form of the Lippmann-Schwinger equation

\[
< p' | t(E(p)) | p > = < p' | U(E(p)) | p > + \int dp'' < p' | U(E(p)) | p'' > \frac{1}{E(p) + i\omega - E(p'')} < p'' | t(E(p)) | p >,
\]

where \(p'\) and \(p\) denotes the nucleon relative momentum in the c.m. frame and \(E(p) = 2E_p = 2\sqrt{p^2 + m_N^2}\).

The relativistic equation (1) for the NN amplitude was derived in the framework of the standard S-matrix reduction technique by using the completeness condition for the asymptotic "in" states \[3,4]\. Moreover in Ref.\[7\] there were considered the explicit connections of the equation (1) with the quasipotential reduction of the Bethe-Salpeter equation and with the other three-dimensional time-ordered field-theoretical equations. The brief analysis of the Lippmann-Schwinger type equation (1) is given in the appendix.

The potential in the equation (1) consist of the on-mass shell meson exchange potential (A6c) (Fig.1) and of the NN potential \(Y(t)\) (A6a,b) which are generated by the equal-time commutators and which dependent on the square of the four-momentum transfer \(t\). The NN potential \(Y(t)\) contains the off-mass shell meson exchange part (Fig.2A) and the NN overlapping (contact) potential (Fig.2B).

The vertex functions \(< p'_N | J_{p_N}(0) | p_{\pi_1} p; in >\) and \(< 0 | J_{p_N}(0) | p_N p_{\pi_1}; in >\) of the transition \(N \rightarrow N\pi_1\) can be determined from the \(\pi N\) scattering phase shifts by using the dispersion relations \[9\]. The vertex functions \(< p'_N | J_{p_N}(0) | p_{\pi_1} p_{\pi_2}; in >\) and \(< 0 | J_{p_N}(0) | p_N p_{\pi_1} p_{\pi_2}; in >\) of the transition \(N \rightarrow N\pi_1\pi_2\) can be obtained by using the results of the calculation of the \(\pi N\) scattering amplitude with one crossed pion etc. Therefore we can assume that the nonlocal, on-mass shell meson exchange potential in eq.(A6c)-see (Fig.1) is defined “a priory”. Afterwards one can subtract the contributions of this “a priory” fixed potential from the complete NN phase shifts. Thus the inverse scattering problem for the relativistic Lippmann-Schwinger type equation (1) is reduced to the investigation of the relativistic equation

\[
< p' | T | p > = Y \left( (E_{p'} - E_p)^2 - (p' - p)^2 \right)
\]
Figure 1. Diagrammatic representation of the time-ordered NN interaction potential with on-mass shell intermediate pions. The full circles denote the vertex functions $<p_N' | j_{pN}(0) | p_{\pi_1}, p_{\pi_2} ... >$ in and $<0 | T_{pN}(0) | p_{pN} p_{\pi_1} p_{\pi_2} ... >$ in with the one off-mass shell nucleon.

Figure 2. The NN interaction with the off-mass shell $\pi, \sigma, \omega, ...$ meson exchange-diagram (A) and with the four nucleon overlapping or contact term diagram (B). The shaded circle corresponds to the vertex function $<p_N' | j_{pN}(0) | p_{\pi} >$ with off-mass shell $p_i$-meson and the other $\pi NN$ vertex functions in the diagram A are given in the tree approximation according to equation (4).

$$+ \int \mathcal{V} \left( (E_{p'} - E_{p''})^2 - (p' - p'')^2 \right) \frac{dp''}{2E_p + i\omega - 2E_{p''}} <p''|T|p>, \quad (2)$$

where $E_p = \sqrt{m_p^2 + p^2}$ and the NN potential $\mathcal{V}$ (A6b,c) is depicted on the Fig.2.

It is convenient to present equation (2) for the wave function $<p'|\Psi_p >$

$$\left(2E_{p'} - 2E_p\right) <p'|\Psi_p >= \int dp'' \mathcal{V} \left( (E_{p'} - E_{p''})^2 - (p' - p'')^2 \right) <p''|\Psi_p >, \quad (3)$$

where $<p'|T|p> = <p'|\mathcal{V}|\Psi_p >$.

In order to transform Eq.(3) to the nonrelativistic form we introduce the variables

$$k^2 = m_N(2E_p - 2m_N); \quad \frac{k}{k} = \frac{p}{m}$$

and

$$u = k' \sqrt{\frac{k''^2}{4m_N^2} + 1}; \quad v = k'' \sqrt{\frac{k''^2}{4m_N^2} + 1} \quad (5)$$
which satisfy the important condition [1]

\[ t = -(u - v)^2 = (E_{p'} - E_{p''})^2 - (p' - p'')^2. \]  

(6)

Then equation (3) takes the form

\[ (k'^2 - k^2) < k'|\psi_k >= \int dk'' Y\left(-(u - v)^2\right) < k''|\psi_k >, \]  

(7)

where \( J^{1/2}(k'') < k''|\psi_k >= < p''|\psi_p >; \) \( J^{1/2}(k') Y\left(-(u - v)^2\right) J^{1/2}(k'') = \mathcal{Y}(t) \) and \( J(k'') = p''^2 dp''/k''^2 dk''. \)

Now we present Eq. (7) in the coordinate space using the Fourier transform

\[ \left(\Delta_{r'} + k^2\right) < r'|\psi_k >= -\int e^{-ik'r'} \frac{dk'}{(2\pi)^3} Y\left(-(u - v)^2\right) e^{ik''r''} dr'' < r''|\psi_k >. \]  

(8)

Inserting

\[ Y\left(-(u - v)^2\right) = \int dz e^{-i(u-v)z} Y(z). \]  

(9)

into Eq. (8) we obtain

\[ \left(\Delta_{r'} + k^2\right) < r'|\psi_k >= -\left\{ \int e^{z\nabla_{r'}[\sqrt{\Delta_{r''}/4m_N^2} + 1]} e^{-z\nabla_{r''}[\sqrt{\Delta_{r''}/4m_N^2} + 1]} e^{-ik'r'} e^{ik''r''} e^{ik'z} e^{-ik''z} \right\} dr'' \frac{dk'}{(2\pi)^3} \frac{dk''}{(2\pi)^3} Y(z) dz < r''|\psi_k >, \]  

(10)

where the operators in the big curly brackets act on the function which are included in this brackets and \( \Delta_{r'} = \nabla_{r'}^2 \equiv \partial_{r'}^2 \).

After integration over \( k' \) and \( k'' \) we find

\[ \left(\Delta_{r'} + k^2\right) < r'|\psi_k >= -\left\{ \int e^{z\nabla_{r'}[\sqrt{\Delta_{r''}/4m_N^2} + 1]} e^{-z\nabla_{r''}[\sqrt{\Delta_{r''}/4m_N^2} + 1]} dr'' \delta(z - r') \delta(r'' - z) \right\} Y(z) dz < r''|\psi_k >. \]  

(11)

The operators \( e^{z\nabla_{r'}[\sqrt{\Delta_{r''}/4m_N^2} + 1]} \) and \( e^{-z\nabla_{r''}[\sqrt{\Delta_{r''}/4m_N^2} + 1]} \) compensate each other on the surface \( r' = r'' = z \). Therefore we obtain the standard Schrödinger equation

\[ \left(\Delta_{r'} + k^2\right) < r'|\psi_k >= -Y(r') < r'|\psi_k >. \]  

(12)

1 For the variables \( u \) and \( v \) there is a more transparent representation \( u = p' \sqrt{E_{p'} + m_N} \) and \( v = p'' \sqrt{E_{p''} + m_N} \).
Our main result is the explicit reduction of the relativistic Lippmann-Schwinger equation (3) with the potential $Y(t)$ to the standard Schrödinger equation (12) with the auxiliary variable $2m_N\sqrt{\mathbf{p}^2 + m_N^2} - 2m_N^2 = k^2$ (4). This result allows to determine the relativistic potential $Y(t)$ (or $Y(t')$) from the nonrelativistic potential $Y(r')$ using the standard Scarödinger equations (12) and the well known inverse scattering theory [1–3].

Summarizing the above formulation of the $NN$ scattering problem, we see that in order to apply the inverse scattering methods to the present field-theoretical equation (1) one must take into account the structure of the $NN$ potential. The complete $NN$ potential consists of the nonlocal (on-mass-shell pions exchange) part (Fig.1) and of the off-mass shell meson exchange part $Y(t)$ (Fig.2). The contribution of the potential shown in Fig.1 is constructed from the $\pi N$ vertex functions with one of the nucleons being off-mass shell. These vertex functions can be obtained from the $\pi N$ phase shifts by using the dispersion relations. Therefore in the present formulation the inverse scattering problem is reduced to the determination of the $Y(t)$ potential in the equation (2). This is because the other part of the $NN$ potential in the equation (1) can be constructed from the $\pi N$ scattering amplitudes. One can build the potential $Y$ from the corresponding $NN$ phase shifts, by using the inverse scattering methods for the nonrelativistic potential $v(r)$ of the Schrödinger equation (12).

The potential $Y(t)$ is generated by the equal-time anticommutation relation (A6a) and it is generally the function of the $t$-variable. For the simplest Lagrangian (A5) $Y(t)$ consist of the terms, describing the exchange by the off-mass shell mesons: $\pi, \sigma, \rho, \omega, ...$ (A6a) (Fig.2A) and of the contact terms (A6b) (Fig.2B). Therefore the determination of the potential $Y$ in the framework of the inverse scattering methods can help us to clarify the form of the general meson-nucleon Lagrangians. The formulation considered above can be extended easily to the $\pi N - \pi N$ and $\pi N - \gamma N$ scattering reactions.

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APPENDIX

The well known expression for the $NN$ scattering $S$-matrix with the local meson and the local nucleon field operators $\Phi(x)$ and $\Psi(x)$ has the form [14,15]

$$S_{N'N;\xi=NN} \equiv \langle \text{out}; \mathbf{p}'_1 s'_1 \mathbf{p}'_2 s'_2 | \mathbf{p}_1 s_1 \mathbf{p}_2 s_2 | \text{in} \rangle = \langle \text{in}; \mathbf{p}'_1 s'_1 \mathbf{p}'_2 s'_2 | \mathbf{p}_1 s_1 \mathbf{p}_2 s_2 | \text{out} \rangle,$$

\footnote{Unlike the nonrelativistic case, in the considered relativistic formulation $r$ does not have the physical meaning of the coordinate. This variable should be treated as the auxiliary variable which is conjugate to the momentum $\mathbf{p}$. In the Ref. [2] $x = i\sqrt{1 + p^2/m^2}\partial/\partial \mathbf{p}$ was treated as the generators of translation in the Euclidean $p$-space with the $x^2 - L^2/m^2$ Casimir operator of the Lorentz group. In this field-theoretical approach the relativistic analogue of the Fourier transformations (the Shapiro transformation [3] with the complete set of the functions $\xi(\mathbf{p}, \mathbf{x})$) was used. The final equation have the form $H_o(2E_p - H_o) < r|\phi_p > = -1/2mV(r, E_p) < r|\phi_p >$ with the free Hamiltonian $H_o = 2m \cosh(i/m\partial/\partial r) + 2i/r \sinh(i/m\partial/\partial r) + \Delta_{\theta,\phi}/mr^2 \exp(i/m\partial/\partial r)$.}
\[ A_{N'N''\equiv NN} = \mathcal{P}_{1'2'} < out; p'_{i_1} s'_{i_1} | J_{p_2' s_2'}(0) | p_1 s_1 p_2 s_1; in > = \text{equal time anticommutators} + \]
\[ \mathcal{P}_{1'2'} \mathcal{P}_{12} \int d^4x e^{(-ip_1x)} < p'_{i_1} s'_{i_1} | \left( J_{p_2' s_2'}(0) \theta(-x_o) \bar{J}_{p_1 s_1}(x) - \bar{J}_{p_1 s_1}(x) \theta(x_o) J_{p_2' s_2'}(0) \right) | p_{2'2} > \]

(A2)

where \( p'_{i},s_{i} \) and \( p_{i_1}s_{i_1} \) denote the three momentum and the spin of the nucleon, \( i = 1, 2 \) correspondingly in the final and in the initial states, \( \mathcal{P}_{12} = 1/2(1 - \hat{P}_{12}) \) stands for the antisymmetrization operator with the nucleon transposition operator \( \hat{P}_{12} \), \( J_{p_2' s_2'}(x) \equiv \bar{\nu}(p_{2'2'}) \gamma(x) \bar{\nu}(p_{2'2'}) \frac{1}{2m_N} \mu - m_N \nu(x) \) is the nucleon source operator which is determined by the Dirac equation of motion, \( u(p_{N1}) \) stands for the Dirac bispinor function and \( \theta(x_o) = 1 \) if \( x_o > 0 \) and \( \theta(x_o) = 0 \) if \( x_o < 0 \) is the well known step function.

Substituting the completeness condition \( \sum_n |n; in > < in; n| = 1 \) into eq. (A2) between the source operators, we obtain after integration over \( x \)

\[ A_{N'N''\equiv NN} = \sum_{n = d,NN,NN, \ldots} < p'_{i_1} s'_{i_1} | J_{p_2' s_2'}(0) | n; in > \frac{\delta(p_1 + p_2 - P_n)}{E_{p_1} + E_{p_2} - P_n + i0} < in; n | J_{p_1 s_1}(0) | p_{2} > 
- \sum_{m = \pi, \pi, \ldots} < p'_{i_1} s'_{i_1} | J_{p_1 s_1}(0) | m; in > \frac{\delta(p_1 - p_1 - P_m)}{E_{p_1} - E_{p_1} - P_m} < in; m | J_{p_2' s_2'}(0) | p_{2} > \],

(A3)

where \( p_N = (E_{p_N}, \nu_N) = \sqrt{p_N^2 + m_N^2}, p_N \) denotes the four momentum of the on-mass shell nucleon.

Comparing equation (A2) with equation (A3), we see, that the time-ordering procedure in eq. (A2) is replaced by the linear propagator which consists from the energies of the outside and inside particles. Besides in the equation (A2) only the sum of the three-momentums of the all intermediate particles is conserved. However, the main property of the present field-theoretical formulation is that it is the only one, where the one variable vertex functions like \( < p'_N | J_{\beta}(0) | p_\pi > \) or \( < p'_N | j_{\pi'}(0) p''_N > \) (in the equal time anticommutators) are required as input functions for the construction of the \( NN \) potential.

The equal time anticommutators in equations (A2) and (A3) have the form

\[ \text{equal time anticommutators} \equiv \mathcal{Y}(t) = \mathcal{P}_{1'2'} \mathcal{P}_{12} < p'_{i_1} s'_{i_1} | \left( J_{p_2' s_2'}(0), b_{p_1 s_1}^\dagger(0) \right) | p_{2} > , \]

(A4)

where the operator \( b_{p_1 s_1}^\dagger(y_o) = \int d^4y e^{-ip_1 y} \gamma_0 \bar{\nu}(y) u(p_1 s_1) \) tends to the nucleon creation (annihilation) operator in the asymptotic region \( \lim_{x \to \pm \infty} b_{p_1}^\dagger(x_o) \Rightarrow b_{p_1}(out \ or \ in) \) and satisfy the anticommutation relations \( \{ b_{p_1 s_1}(x_o), b_{p_2 s_2}(x_o) \} = E_p / m_N \delta_{s_1 s_2} \delta(p' - p) \).

The exact form of the equal-time anticommutators (A4) can be derived using the usual form of the meson-nucleon Lagrangians.
\[ L_{\text{int}} = g_\sigma \Psi \sigma + \frac{f_\pi}{m_\pi} \overline{\Psi} \gamma_5 \gamma_\mu \Psi \partial^\mu \Phi_\sigma + g_\nu \overline{\Psi} \gamma_\mu \Psi V^\mu + \frac{f_V}{4m_N} \overline{\Psi} \sigma_{\mu\nu} \Psi (\partial^\mu V^\nu - \partial^\nu V^\mu), \]  

where \( \Phi_\sigma, \Phi_\pi \) and \( \Phi_V \) denote the field operators of the \( \sigma, \pi \) and of the \( V = \rho, \omega \) mesons. Using the equal-time commutation relation between the Heisenberg field operators, we obtain

\[ \mathcal{Y}(t) \equiv \mathcal{P}_{1'} \mathcal{P}_{12} < \mathbf{p}'_1 s'_1 | \{ J_{p_x s'_2}(0), b^\dagger_{p_1 s_1}(0) \} | \mathbf{p}_2 s_2 > = \mathcal{P}_{1'} \mathcal{P}_{12} \left( \right. \\
\left. g_\sigma \overline{u}(\mathbf{p}'_2 s'_2) u(\mathbf{p}_1 s_1) \frac{< \mathbf{p}'_1 s'_1 | j_\sigma(0) | \mathbf{p}_2 s_2 >}{t - m_\sigma^2} + i \frac{f_\pi}{2m_N m_\pi} \overline{u}(\mathbf{p}'_2 s'_2) \gamma_5 u(\mathbf{p}_1 s_1) \frac{< \mathbf{p}'_1 s'_1 | j_\pi(0) | \mathbf{p}_2 s_2 >}{t - m_\pi^2} \\
+ g_V \overline{u}(\mathbf{p}'_2 s'_2) \gamma_\mu u(\mathbf{p}_1 s_1) \frac{< \mathbf{p}'_1 s'_1 | \gamma^\mu(0) | \mathbf{p}_2 s_2 >}{t - m_\pi^2} \right) + \text{contact terms} \quad (A6a) \]

\[ \text{contact terms} = \mathcal{P}_{1'} \mathcal{P}_{12} \left( \frac{E_{\mathbf{p}'_2} - E_{\mathbf{p}_1}}{2m_N} \overline{u}(\mathbf{p}'_2 s'_2) i \frac{f_\pi}{2m_N m_\pi} \gamma_5 u(\mathbf{p}_1 s_1) < \mathbf{p}'_1 s'_1 | j_\pi(0) | \mathbf{p}_2 s_2 > \\
+ \frac{f_V}{8m_N} \overline{u}(\mathbf{p}'_2 s'_2) \gamma_\mu u(\mathbf{p}_1 s_1) < \mathbf{p}'_1 s'_1 | \gamma^\mu(0) | \mathbf{p}_2 s_2 > + ... \right) \quad (A6b) \]

where \( t = (p_N - p_N)^2 \equiv (E_{\mathbf{p}'_2} - E_{\mathbf{p}_1})^2 - (\mathbf{p}'_2 - \mathbf{p}_1)^2 \) and due to the Lorentz-covariance of the scalar \( (\sigma) \), pseudoscalar \( (\pi) \) and vector \( (V = \rho, \omega) \) vertex functions the following simple expressions

\[ < \mathbf{p}'_N | j_\sigma(0) | \mathbf{p}_N > = g_\sigma G_\sigma(t) \overline{u}(\mathbf{p}'_N) u(\mathbf{p}_N); \quad < \mathbf{p}'_N | j_\pi(0) | \mathbf{p}_N > = i g_\pi G_\pi(t) \overline{u}(\mathbf{p}'_N) \gamma_5 u(\mathbf{p}_N) \]

\[ < \mathbf{p}'_N | j_\nu(0) | \mathbf{p}_N > = g_V G_V(t) \overline{u}(\mathbf{p}'_N) \gamma^\mu u(\mathbf{p}_N). \]

are valid. The diagrammatic representations of the three-dimensional equations (A6a) and (A6b) are given in the Fig. 2. The first three terms of the eq. (A6a) corresponds to the one off-mass shell meson exchange \( NN \) interaction potential and these terms exactly coincides with the OBE \( NN \) Bonn potential. However, in the original derivation of the Bonn OBE potential model from quasipotential equation, the dependence of meson-nucleon vertices on the off mass-shell variables \( p_N^2 \neq m_N^2 \) and \( p'_N^2 \neq m_N^2 \) has been neglected. In the present field-theoretical formulation this approximation is not needed. In this meaning the present derivation can be considered as an additional justification for the Bonn OBE potential model of the \( NN \) interaction. The contact or overlapping terms in eq. (A6b) are generated by the nonrenormalizable pseudoscalar and vector parts in the Lagrangian (A5). If we take the renormalizable pseudoscalar coupling \( L_{\rho \pi} = i g_\pi \overline{\Psi} \gamma_5 \Psi \) instead of the pseudovector coupling in the Lagrangian (A5), the first term in the right hand side of eq. (A6b) vanishes \( \Box \Box \). The other terms of the relation (A6b) are produced by the nonrenormalizable fourth term of the Lagrangian (A5). In Ref. \( \Box \Box \) the structure
of the contact (overlapping) terms of the $NN$ interaction potential with quark degrees of freedom was investigated. The contact terms of the $NN$ potential were shown to consist of quark-gluon exchange contributions. In addition it was obtained that due to the structure of the equal-time commutators the quark-gluon exchange terms do not violate the unitarity condition for the $NN$ scattering amplitude.

The considered field-theoretical equations are connected analytically with all the other field-theoretical equations, i.e. we can derive the Bethe-Salpeter equation in the framework of the $S$-matrix reduction technique. Therefore, all results obtained in the framework of this time-ordered three-dimensional equations remain valid in the other field-theoretical approaches as well. This formulation is free from the "three-dimensional ambiguities" which emerge during the reduction of the Bethe-Salpeter equation in the three dimensional form. The structure of the present field-theoretical equations does not depend on the choice of the form of the effective Lagrangian.

The second term of the eq. (A3) is depicted in the Fig.1a as the on-mass shell meson exchange diagram. After the cluster decomposition procedure i.e. after separation of connected and disconnected parts in the amplitudes $<\text{in}; m|\mathcal{J}_{p'_{1}s'_{1}}(0)|p's>$ and $<\text{in}; n|\mathcal{J}_{p_{s}}(0)|p's>$, one obtain the 8 skeleton diagrams for the connected parts of the transition amplitudes. We can omit the diagrams with the two and more on-mass shell meson exchange and we will neglect the contributions of the $\pi d$ and $\pi NN$ intermediate states, and of the anti-particle $d, N, ...$ intermediate states in the low energy region. Therefore, after cluster decomposition we obtain only other time time-ordered pi-meson exchange diagram which is depicted in the Fig.1b. Thus we can define the inhomogeneous term of the eq. (A3) as

$$<p'_{1}s'_{1}p'_{2}s'_{2}|W|p_{1}s_{1}p_{2}s_{2}> = \text{equal time anticommutators}$$

$$+(2\pi)^3\mathcal{P}_{12} \left( - \sum_{m=\pi,\pi\pi,...} <p'_{1}s'_{1}|\mathcal{J}_{p_{1}s_{1}}(0)|m;\text{in}> c \frac{\delta(p'_{1} - p_{1} - p_{m})}{E_{p'_{1}} - E_{p_{1}} - P_{m}} <\text{in}; m|\mathcal{J}_{p'_{2}s'_{2}}(0)|p_{2}s_{2}> c 
+ \sum_{m=\pi,\pi\pi,...} <0|\mathcal{J}_{p_{1}s_{1}}(0)|p_{2}s_{2}; m;\text{in}> c \frac{\delta(-p_{2} - p_{1} - p_{m})}{-E_{p_{2}} - E_{p_{1}} - P_{m}} <\text{in}; m|p'_{1}s'_{1}|\mathcal{J}_{p'_{2}s'_{2}}(0)|0> \right),$$

(A6c)

where the subscript 'c'" denotes the connected part of the corresponding transition amplitude.

Using the eq. (A6c) we can rewrite the equation (A3) in the c.m. frame as

$$<p'|T|p> = <p'|W|p> + \sum_{d} <p'|T|P_{d} > \frac{1}{E(p) - m_{d}} <P_{d}T|p>$$

$$+ \int <p'|T|p'' > \frac{dp''}{E(p) + io - E(p'')} <p''|T|p>$$

(A8)

where we have omitted the spin variables for the sake of simplicity, $E(p) = 2\sqrt{p^{2} + m_{N}^{2}}$ is the energy of the $NN$ state, $P_{d} = (m_{d}, 0)$ is the four momentum of deuteron in the c.m. frame.
\[ <p'|T|p> \equiv - <p'_1 s'_1 |J_{p'_2 s'_2}|0|p_1 s_1 p'_1 s'_2 p_2 s_2; in > \] (A9a)

\[ <p'|T|P_d> \equiv - <p'_1 s'_1 |J_{p'_2 s'_2}(0)|P_d S_d; in > \] (A9b)

The potential term \( V \) in eq. (A8) consist of the on-mass shell pion exchange diagrams (see the second term of eq. (A3) and the diagrams shown in Fig.1) and of the equal-time commutator (A4). In the general case the commutator is reduced to the off-mass shell \( \pi, \sigma, \rho, \omega \)-meson exchange diagrams and to the contact terms shown in Fig.2A,B. In the framework of the simplest Lagrangian (A5) the exact form of this commutators is given by eqs.(A6a,b).

\[ W = V(\text{on mass shell meson exchange}) + Y(t)(\text{off mass shell meson exchange}) \] (A10)

The procedure of the exact linearization of the quadratically nonlinear equations (A8) for \( \pi N \), \( NN \) and \( \gamma N - \pi N - \gamma \pi N ) - \pi \pi N \) scattering reactions is described in the ref.[6,7,10,11].

The on-mass shell meson exchange part of the \( NN \) potential \( V \) (Fig.1) is nonhermitian, but \( Y^\dagger(t) = Y(t) \). Nevertheless we can obtain the linear energy dependent potential \( <p'|U(E)|p> \) from \( <p'|W|p> \)

\[ <p'|U(E) = E(p')|p> \geq <p'|W|p> \] (A11)

so that off energy shell \( U(E) \) is hermitian \( <p'|U^\dagger(E)|p> = <p'|U(E)|p> \), but on the half-energy shell \( U(E) \) (as well as \( W \)) is not hermitian \( <p'|U^\dagger(E = E(p'))|p> \neq <p'|U(E)|p> \). This property allows us to derive the Lippmann-Schwinger type equation exactly, by using Eq. (A8) [6,7] exactly

\[ <p'|t(E)|p> = <p'|U(E)|p> + \int <p'|U(E)|p''> \frac{d\rho''}{E + i\rho'' - E(p'')} <p''|t(E)|p> \] (A12)

On the energy shell the solution of equations (A8) and (A12) coincide, i.e. \( <p'|t(E(p) = E(p'))|p> \geq <p'|T|p> \). Equation (A12) coincide with the equation (1) on the half-energy shell \( E = E(p) \).

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