Fractionally quantized Berry phases of magnetization plateaux in spin-$1/2$ Heisenberg multimer chains

Isao Maruyama$^1$ and Shin Miyahara$^2$

$^1$Department of Information and Systems Engineering, Fukuoka Institute of Technology, S-30-1 Wajiro-higashi, Higashi-ku, Fukuoka 811-0295, Japan
$^2$Department of Applied Physics, Fukuoka University, 8-19-1, Nanakuma, Jonan-ku, Fukuoka 814-0180, Japan

We study the fractionally quantized $Z_N$ Berry phase, $\gamma_N = 0, \frac{2\pi}{N}, \frac{4\pi}{N}, \ldots, \frac{2(N-1)\pi}{N}$, to characterize local $N$-mer spin structures at magnetization plateaux in spin-$1/2$ Heisenberg multimer ($N$-mer) models, i.e., highly frustrated $N$-leg ladder models, which are generalization of an orthogonal dimer chain and have exact ground states in the strong multimer coupling limit. We demonstrate that all $N$ kinds of Berry phases, characterizing magnetization-plateau phases, appear in a magnetic phase diagram when $N = 2$ and $4$. We show that magnetic-plateau with magnetization $\langle m \rangle$ and $D$-fold degenerated states has $\gamma_N = \pi((m - 1)D)$, except for the Haldane phase with $\gamma_N = 0$. In addition, we find a complementary $Z_N$ Berry phase becomes non-zero in the $S = N/2$ Haldane phase for $N = 2$ and $4$. Since the exact quantization of the $Z_N$ Berry phases protected by the translational (or rotational) symmetry along the rung direction, the $Z_N$ Berry phase has a possibility to be applied for wide class of magnetic plateaux in coupled multimer systems.

KEYWORDS: Orthogonal dimer model, magnetic plateau, Haldane phase, exact ground state

While the symmetry breaking has been successfully described by local order parameters, topological phases cannot be characterized by local order parameters. One of historical examples is a hidden $Z_2 \times Z_2$ symmetry breaking of the Haldane phase, which requires new quantities for characterization, such as a non-local string order parameter, twisted-order parameter, entanglement spectrum, and a quantized Berry phase. The quantized Berry phase defined through local flux is one of new topological invariants. A historical example of topological invariants is the Chern number for integer quantum Hall effects. As an advantage, the quantized Berry phase has a possibility to characterize various gapped phases, because various type of Berry phase can be defined depending on the choice of the local flux. In this sense, the Berry phase has more variety than anti-periodic boundary conditions, as also shown in a recent study using the $Z_2$ Berry phase. In addition, fractional $Z_N$ quantization of Berry phase can characterize $N$ types of phases. However, there is no demonstration that $N$ different phases, which are characterized by a $Z_N$ Berry phase defined in the same location, exist in a phase diagram.

In this letter, we show fractionally quantized $Z_N$ Berry phase successfully characterizes magnetization-plateau phases and the Haldane phase in a magnetic phase diagram of spin-$1/2$ Heisenberg $N = 2$ and $4$ multimer chains, that is, highly frustrated $N$-leg ladders. In a strong multimer coupling region of the model, the ground state is exactly obtained as a direct product state. Since the ground states are rigorously written, the model Hamiltonian is a useful reference to understand the role of the $Z_N$ Berry phase as well as the physics of magnetization, magnetization plateaux and jumps. The $N = 2$ model is equivalent to the edge-shared tetrahedral chain, which is an one-dimensional version of the Shastry-Sutherland model in a class of the orthogonal dimer models. Since the $Z_2$ Berry phase can detect the dimer-singlet in the two-dimensional orthogonal dimer model, we introduce the same $Z_2$ Berry phase in the dimer ($N = 2$) chain under the external magnetic fields. On the other hand, there is no previous study on the $Z_N$ Berry phase for multimer ($N$-mer) chains when $N \neq 2$ cases. Here, we mainly study the $Z_4$ Berry phase for the tetramer ($N = 4$) chain as a typical example of $N \neq 2$ cases.

Let us define the isolated spin-$1/2$ multimer ($N$-mer) Hamiltonian at the position $x$ along the leg:

$$h_x = \begin{cases} 0 & \text{for } N = 2 \\ J \sum_{y=1}^N \mathbf{s}_{x,y} \cdot \mathbf{s}_{x,y+1} & \text{for } N > 2 \end{cases}$$

We adopted the periodic boundary condition (PBC), $\mathbf{s}_{x,y} = \mathbf{s}_{x,y+L}$. Note that we use a one-dimensional ring structure for an $N$-mer for simplicity, which is slightly different from the definition in Ref.11 for $N \geq 4$. By using the total spin of each local multimer $\mathbf{S}_x = \sum_{y=1}^N \mathbf{s}_{x,y}$, the total Hamiltonian $\mathcal{H}$ of $N$-mer with the system size $L$ under the magnetic field $H$ is defined as

$$\mathcal{H} = \sum_{x=1}^L (h_x + J' \mathbf{S}_x \cdot \mathbf{S}_{x+1} - H \mathbf{S}_x^2).$$

Here, we adopted the PBC, $\mathbf{s}_{x,y} = \mathbf{s}_{x+L,y}$, along legs. We restricted ourselves for the case $J > 0$ and $J' > 0$ and used $J$ as a unit of energy. Schematic figures of the Hamiltonian (eq. (2)) for $N = 2, 3, 4$ are shown in Fig. 1. The second term of inter-multimer interaction $J'$ in eq. (2) has $N^2$ links. In the $N = 4$ case, next-nearest
neighbor interactions and third-nearest neighbor interactions have the identical strength $J'$ in eq. (2). In this sense, the inter-multimer interaction is artificial, but this is a key ingredient to obtain exact ground states under the magnetic field. In eq. (2), the local $S_y$ is a good quantum number due to the frustration.\(^\text{11}\) As a consequence, the model is equivalent to a spin chain model with different values of the magnitude of the spin at each site $x$.

To define the Berry phase for the local multimer at a site $x_0$, we introduce an one-parameter Hamiltonian $\mathcal{H}(\theta)$ by using the original Hamiltonian $\mathcal{H}$. Following Ref. 8, we define

$$\mathcal{H}(\theta) = \mathcal{H} + U(\theta)h_{x_0}U(\theta) - h_{x_0},$$

where $U(\theta)$ is a local gauge twist at the site $x_0$ defined as

$$U(\theta) = \prod_{y=1}^{N} e^{i\phi_y(\theta)(S_{y+1}-1/2)},$$

with

$$\phi_y(\theta) = \begin{cases} \frac{2\theta(y-1)}{N} + 2(\theta - \pi)\delta_{y,N}, & 0 \leq \theta < \pi, \\ 2\theta(y-1) - 2(\theta - \pi)\delta_{y,N}, & \pi \leq \theta \leq 2\pi. \end{cases}$$

Here we choose the spin-twist axis as the $z$-axis along the external magnetic field $H$. For $N = 2$, one can obtain $U(\theta) = e^{i\theta(S_{y+1}z-1/2)}$ and $U(\theta)h_{x_0}U(\theta) = (e^{-i\theta}s_{x_0}z + e^{i\theta}s_{x_0}z')/2 + s_{x_0}z = 1/2$, which corresponds to usual link-twist for the $Z_2$ Berry phase.\(^\text{5}\)

For general $N$, the local gauge twist $U(\theta)$ is equivalent to that in Ref. 8. A periodicity $\mathcal{H}(\theta) = \mathcal{H}(2\pi) \equiv \mathcal{H}$ is easily obtained from the relation $U(0) = U(2\pi) = 1$.

For any one-parameter Hamiltonian $\mathcal{H}(\theta)$ with the periodicity $\mathcal{H}(\theta) = \mathcal{H}(2\pi)$, the Berry phase $\gamma$ can be defined in modulo $2\pi$ for $D$-fold states under a energy gap: $\gamma = \frac{1}{2} \oint \text{tr} \text{A} \cdot D\text{D} \text{D}$ with the $D$-dimensional non-Abelian Berry connection $(A)_{ij} = \langle \psi_i(\theta)|i\psi_j(\theta)\rangle = \langle \psi_i(\theta) \rangle \frac{d}{d\theta} \langle \psi_j(\theta) \rangle d\theta$ for $D$-fold nearly degenerated ground states $|\psi_1(\theta)\rangle, \ldots, |\psi_D(\theta)\rangle$. The Berry phase $\gamma$ is numerically calculated\(^\text{16}\) as $\gamma = - \sum_{k=1}^{K} \arg \det C_k$ with using discretized integration variables $\theta_k = \frac{2\pi k}{N} + \theta_0 \mod 2\pi$ and an $D \times D$ matrix $(C_k)_{ij} = \langle \psi_i(\theta_k)|\psi_j(\theta_k)\rangle$. Numerical convergence of $\gamma$ is achieved for small $K$ when the Berry phase $\gamma$ is quantized.\(^\text{17}\)

For the model [eq. (2)], quantization of Berry phase comes from $y$-direction translational symmetry, that is, $\frac{2\pi}{N}$-rotational symmetry of all $N$-mers. As shown in Ref.8, $N\gamma = 0 \mod 2\pi$ is satisfied. Thus, the $Z_N$ Berry phase of an $N$-mer, $\gamma_N$, is quantized as $\gamma_N = 0, \frac{2\pi}{N}, \frac{4\pi}{N}, \ldots, \frac{2(N-1)\pi}{N}$.

Before calculating the $Z_N$ Berry phases of $N$-mer chains, let us consider a strong multimer coupling limit for general $N$ case. In the decoupled limit ($J' = 0$), the one-parameter Hamiltonian satisfies $H(\theta) = U(\theta)hU(\theta)^\dagger$ because of $U(\theta)S_{x_0}^zU(\theta) = S_{x_0}^z$. Then, one obtains the ground states $|\psi_i(\theta)\rangle = U(\theta)|\psi_0(\theta)\rangle$ and the Berry phase $\gamma_N = 2\pi \sum_{j=1}^{D} \langle \psi_j(\theta)|\psi_{j-N}(\theta)-1/2|\psi_0(\theta)\rangle$. Using the $x$- and $y$-direction translational symmetries of the $D$-fold degenerated ground states,\(^\text{18}\) the Berry phase can be written by the magnetization $\langle m \rangle = \frac{2\sum_{j} \langle S_j^z \rangle}{LN}$ as

$$\gamma_N = \pi(\langle m \rangle - 1)D.$$

This corresponds to the fact that $\gamma_N$ depends on the particle number in the decoupled limit of tight-binding models.\(^\text{8}\) In general, the inter-multimer coupling $J'$ makes $\gamma_N$ non-trivial.

To characterize the Haldane phase in large $J'$ region, one can define another quantized $Z_N$ Berry phase $\gamma_N$ by using another one-parameter Hamiltonian

$$\mathcal{H}(\theta) = U(\theta)i\mathcal{H}U(\theta)^\dagger = \mathcal{H} + U(\theta)\tilde{h}_{x_0}U(\theta) - \tilde{h}_{x_0},$$

where the local inter-multimer Hamiltonian $\tilde{h}_{x_0}$ is defined as $\tilde{h}_{x_0} = J'S_{x_0-1} \cdot S_{x_0} + J'S_{x_0} \cdot S_{x_0+1}$. In the decoupled limit $J' = 0$, $\gamma_N$ is zero because $H(\theta)$ does not depend on $\theta$. Since eigenvalues of $H(\theta)$ are the same with those of $\mathcal{H}(\theta)$, the complementary formula\(^\text{19}\) $\gamma_N = \gamma_N + \pi(\langle m \rangle - 1)D$ is valid in the small $J'/J$ region unless gap-closing occurs. While $\gamma_N$ is defined by local gauge twist of the local Hamiltonian $h_{x_0}$ in eq. (3) to characterize the local multimer, the complementary Berry phase $\gamma_N$ is defined by local gauge twist of the local inter-multimer Hamiltonian $\tilde{h}_{x_0}$ in eq. (5) to characterize a local quantum object on the inter-multimer coupling. It is natural to expect that $\gamma_N = 0$ in the large $J'/J$ region and we obtain non-zero $\gamma_N$. This naive expectation turns out to be true in the Haldane phases as shown in the following results.

For dimer case ($N = 2$), phase diagram and its magnetization plateaux have been studied.\(^\text{11}\) With using phase boundaries and magnetization curves in Ref.11, the phase diagram is reproduced in Fig. 2. In the small $J'/J$ region, the ground state of each magnetization plateaux is exactly written as the direct product state of the dimer singlet and triplet. On the other hand, in the large $J'/J$ region, the ground state of the model is equivalent to that of the $S = 1$ Heisenberg chain.

The phases are classified with $Z_2$ Berry phases $\gamma_2 = 0$ or $\pi$ of $D$-fold degenerated ground states except for the gapless phase, as shown in Fig. 2. It should be noted that $\gamma_2$ in each plateau phase requires numerical calculation because the ground states of $\mathcal{H}(\theta)$ are not simple direct product states when $\theta \neq 0$. $\gamma_2$ in dimer singlet phase ($\langle m \rangle = 0$) and singlet-triplet phase ($\langle m \rangle = 1/2$) are equivalent to those in the decoupled limit ($J' = 0$), i.e., $\gamma_2 = \pi(\langle m \rangle - 1)D$. When $J' = 0$, the local states for each total $S_z$ sector and its local Berry phase $\gamma_2$ are easily obtained; the spin-singlet $|s\rangle = (|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle)/\sqrt{2}$ with $\gamma_2 = \pi$.
Berry phase

for $\langle m \rangle = 0$, and the full-polarized $|t\rangle = |\uparrow\uparrow\rangle$ with $\gamma_2 = 0$ for $\langle m \rangle = 1$. Even for finite $J'$ case in Fig. 2, the exact ground state of the singlet phase is $|ss\cdots\rangle = |s\rangle|s\rangle\cdots$ with $\gamma_2 = \pi$ and that of the polarized phase is $|tt\cdots\rangle$ with $\gamma_2 = 0$. In short, $\gamma_2$ detects the singlet on a dimer. In the singlet-triplet phase, exact ground states are doubly-degenerated; $|stst\cdots\rangle$ and $|tsts\cdots\rangle$. The Berry phase of the doubly-degenerated ground states ($D = 2$) can be understood as the summation of $\gamma_2 = \pi$ for $|s\rangle$ and $\gamma_2 = 0$ for $|t\rangle$. Then, we obtain $\gamma_2 = \pi + 0 = \pi$ for the singlet-triplet phase. Finally, in the $S = 1$ Haldane phase, the ground state is not exact but the valence bond (VBS) type. Since the spin-singlets of the VBS state are existing on the inter-dimer links, it is natural that the ground state of the $S = 1$ Haldane phase has $\gamma_2 = 0$ and $\tilde{\gamma}_2 = \pi$.

As shown in Fig. 3, phase diagram of the tetramer chain ($N = 4$) can be classified by the $Z_4$ Berry phase; $\gamma_4 = 0$, $\frac{\pi}{2}$, $\pi$ and $\frac{3\pi}{2}$. The phase boundaries were determined by comparison of the energies between the exact direct product states, $S = 2$ chain with $L = 12$, and $S = 1$-2 chain with $L = 12$. All direct product states are described as the products of singlet $|s\rangle = \frac{1}{\sqrt{3}}(|\uparrow\uparrow\rangle - 2|\uparrow\downarrow\rangle + |\downarrow\downarrow\rangle)$, triplet $|t\rangle = \frac{1}{\sqrt{2}}(|\uparrow\uparrow\uparrow\rangle + |\downarrow\downarrow\rangle - |\uparrow\downarrow\rangle)$, and quintet $|q\rangle = |\uparrow\uparrow\uparrow\uparrow\rangle$ ($\langle h_q \rangle = -J$) states. Using the fact that the Berry phase is identical unless the energy-level crossing occurs, one can determine Berry phases continuously from the decouple limit to the small $J'/J$ region just by calculating the low energies by the Lanczos method.

As a result, the decoupled limit formula, $\gamma_4 = \pi(\langle m \rangle - \frac{1}{2})$, is satisfied for all plateau phases except for the $S = 2$ Haldane phase. For example, the singlet-quintet phase has doubly degenerated ground states; $|qsqs\cdots\rangle$ and $|qsqs\cdots\rangle$. After the summation of the two Berry phases at site $x_0$, $\gamma_4 = \pi$ for $|s\rangle$ and $\gamma_4 = 0$ for $|q\rangle$, we obtain $\gamma_4 = \pi$ for the doubly degenerated states in the singlet-quintet phase. In the other five phases except for the Haldane phase, the ground states of untwisted Hamiltonian $H$ and the Berry phase $\gamma_4$ of corresponding plateau phase are $|ss\cdots\rangle$ and $\gamma_4 = \pi$ for the singlet phase, and $|st\cdots\rangle$ and $|st\cdots\rangle$, $\gamma_4 = \pi/2$ for the singlet-triplet phase, $|tt\cdots\rangle$ and $\gamma_4 = 3\pi/2$ for the $S = 1$ polarized phase, $|tg\cdots\rangle$ and $|tg\cdots\rangle$, $\gamma_4 = 3\pi/2$ for the $S = 2 - 1$ polarized phase, $|qq\cdots\rangle$ and $\gamma_4 = 0$ for the $S = 2$ polarized phase. It should be emphasized that this simple understanding comes from simplicity of the present Hamiltonian with fully-connected inter-multimer interaction. Finally, in the $S = 2$ Haldane phase, the ground state is not exact but the VBS type. The Berry phases, $\gamma_4 = 0$ and $\tilde{\gamma}_4 = \pi$ in the $S = 2$ Haldane phase can be understood in the same way as $\gamma_2 = 0$ and $\tilde{\gamma}_2 = \pi$ in the $S = 1$ Haldane phase of Fig. 2; that is, the VBS state is disentangled on local-multimer sites and entangled on inter-multimer links. It will be an interesting problem if there are the $S = 1$ Haldane phase between the singlet phase and the $S = 2$ Haldane phase. However, comparison with numerical energies revealed that the $S = 1$ Haldane phase does not appear along an $H = 0$ line of phase diagram for the present model, nor for the model of Ref. 11.

A merit of the Berry phase is ability to identify phase transition even if gap-closing does not occur due to finite-size gap in small system-size, as demonstrated in bond-alternating models for $Z_N$ Berry phases. In the present model, since phase transition between plateau phases is
simple first-order transition,\(^{11}\) gap-closing at a transition point exists for small system-size and could survive for \(\theta > 0\) due to symmetry difference of ground states.\(^{15}\)

To check a detail of gap-closing line near a phase boundary, the excitation gap \(\Delta E\) is calculated by the exact diagonalization for \(|m\rangle = 1/2\) sector of 16 spins (\(L = 4\) and \(N = 4\)). In Fig. 4, the excitation gap \(\Delta E\) is shown as a function of \(\theta\) and \(J'/J\) around transition from the \(S = 1\) polarized phase through the singlet-quintet phase to the gapless-phase. Here, the double degeneracy of \(|qs\cdots\rangle\) and \(|qs\cdots\rangle\) in the singlet-quintet phase corresponds to \(\Delta E = 0\) at \(\theta = 0\) around \(0.5 \leq J'/J \leq 0.6\). For \(\theta > 0\), this double degeneracy is lifted because the local-quintet \(|q\rangle\) at the site \(x_0\) cannot be affected by \(\theta\) while the local-singlet stabilizes. This kind of gap-opening can be observed by the alternating strength of \(J\). With using this gap-opening, we can define each Berry phases for degenerated states, and we obtain \(\gamma_\pi = \pi\) for the local-singlet and \(\gamma_4 = 0\) for the local-quintet.

Finally, let us comment on the other \(N\) cases. For even \(N\), the similar analysis with the Berry phase \(\gamma_N\) can be applied. At least, in the weak coupling region, exact ground states can be obtained by the product state of the decoupled limits \((J' = 0)\) and the fractionally quantized Berry phase can be defined by \(\gamma_N = \pi((m) - 1)D\).

When \(N\) is odd, the plateau for \(|m\rangle = 0\) may vanish, since the model can be mapped to 1D Heisenberg model of half integer spins \(S = 1/2, 3/2, \cdots, N/2\). However, there is a room to obtain plateau phases for \(|m\rangle \neq 0\).

In conclusion, fractionally quantized \(Z_N\) Berry phases have classified magnetization plateaux in the spin-1/2 Heisenberg multimer (\(N\)-mer) chain as an extension of the orthogonal dimer (\(N = 2\)) chain. In addition to a transition from zero Berry phase to non-zero Berry phase, which is a usual case in the previous studies on the quantized Berry phase, the results in this letter show that one phase-diagram can be classified by all \(N\) kinds of the Berry phases \(\gamma_N = 0, \frac{2\pi}{N}, \frac{4\pi}{N}, \cdots, \frac{2(N-1)\pi}{N}\), which is defined for identification of the local multimer. Moreover, we have found the complementary Berry phase \(\tilde{\gamma}_N\) which becomes \(\tilde{\gamma}_N = \pi\) in the Haldane phases. This demonstration becomes a evidence that the \(Z_N\) Berry phase has a possibility to characterize various gapped phases.

Since the present model is a simple model to understand the physics of the magnetization plateaux and jumps, the simple interpretation of \(\gamma_N\) has been obtained from the decoupled limit formula, eq. (4), except for the Haldane phase. Such decoupled limit formula seems valid even when the ground state can not be represented by the direct product states unless the energy-gap closing occurs. We checked with an exact diagonalization for \(J_2 \neq J_3\) cases and \(N = 4\) in Fig. 1 (c) and show that the Berry phases in gapped states are quantized and given by the formula eq. (4). One of future problems is phase determination for a general case, e.g., \(J_2 \neq J_3\) for \(N = 4\).

Another future problem is to characterize general \(S > 2\) Haldane phase with the complementary Berry phase \(\tilde{\gamma}_{2S}\), which have successfully characterized the \(S = 2\) Haldane phase with non-trivial Berry phase \(\gamma_4 = \pi\). It should be emphasized that the conventional \(Z_2\) Berry phase defined by the link twist becomes zero in the even \(S\) Haldane phase, reflecting even number of spin singlets existing on a link. In this meaning, we have found a new topological invariant for the \(S = 2\) Haldane phase. These Berry phase, \(\gamma_N\) and \(\tilde{\gamma}_N\), can be applied to wide class of models because the quantization of the \(Z_N\) Berry phases holds for any Hamiltonian with the rotational symmetry of the local \(N\)-mer.

Acknowledgement The authors thank H. Ueda and T. Oka for useful discussions. The work is partially supported by a Grant-in-Aid for Scientific Research No. 17H02926. The computation was partially carried out using the computer facilities at Research Institute for Information Technology, Kyushu University.

---

1) T. Kennedy and H. Tasaki: Phys. Rev. B 45 (1992) 304.
2) M. den Nijs and K. Rommelse: Phys. Rev. B 40 (1989) 4709.
3) M. Nakamura and S. Todo: Phys. Rev. Lett. 89 (2002) 077204.
4) F. Pollmann, A. M. Turner, E. Berg, and M. Oshikawa: Phys. Rev. B 81 (2010) 064439.
5) Y. Hatsugai: J. Phys. Soc. Jpn. 75 (2006) 123601.
6) D. J. Thouless, M. Kohmoto, M. Nightingale, , and M. den Nijs: Phys. Rev. Lett. 49 (1982) 405.
7) N. Chepiga, I. Affleck, and F. Mila: Phys. Rev. B 94 (2016) 205112.
8) Y. Hatsugai and I. Maruyama: Europhys. Lett. 95 (2011) 20003.
9) T. Kariyado, T. Morimoto, and Y. Hatsugai: Phys. Rev. Lett. 120 (2018) 247202.
10) T. Kawarabayashi, K. Ishii, and Y. Hatsugai: ArXiv: 1806.10767 (2018).
11) A. Honecker, F. Mila, and M. Troyer: Eur. Phys. J. B 15 (2000) 227.
12) M. P. Gelfand: Phys. Rev. B 43 (1991) 8644.
13) B. S. Shastry and B. Sutherland: Physica B 108 (1981) 1069.
14) S. Miyahara and K. Ueda: J. Phys. cond. Matt. 15 (2003) 327.
15) I. Maruyama, S. Tanaya, M. Arikawa, and Y. Hatsugai: J. Phys.: Conf. Ser. 320 (2011) 012019.
16) R. D. King-Smith and D. Vanderbilt: Phys. Rev. B 47 (1993) 1651.
17) I. Maruyama and Y. Hatsugai: J. Phys. Soc. Jpn. 76 (2007) 113601.
18) M. Oshikawa, M. Yamanaka, and I. Affleck: Phys. Rev. Lett. 78 (1997) 1984.
19) T. Hirano, H. Katsura, and Y. Hatsugai: Phys. Rev. B 78 (2008) 054431.
20) K. Okunishi: Prog. Theor. Phys. Suppl. 145 (2002) 119.
21) S. Todo and K. Kato: Phys. Rev. Lett. 87 (2001) 047203.