Semiclassical limits for the QCD Dirac operator

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Abstract

We identify three semiclassical parameters in the QCD Dirac operator. Mutual coupling of the different types of degrees of freedom (translational, colour and spin) depends on how the semiclassical limit is taken. We discuss various semiclassical limits and their potential to describe spectrum and spectral statistics of the QCD Dirac operator close to zero virtuality.

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1 Introduction

Quantum chromo dynamics (QCD) is generally believed to be the correct theory for describing the strong force between quarks and gluons. The property of asymptotic freedom makes it possible to use perturbation theory at large momentum transfer and allows for a precise description of many scattering experiments as carried out in the big accelerator facilities.

As the fundamental theory for the interaction of quarks and gluons it also has to be able to describe bound states of quarks and anti-quarks, i.e. hadronic matter such as the proton, the neutron, pions etc. However, in this energy regime ordinary perturbation theory (expansion in the coupling constant) is bound to fail and QCD becomes extremely hard to solve. So far there are no promising analytical approaches at hand which would allow for a calculation of hadronic masses from QCD. It is the main goal of lattice gauge theory to numerically calculate hadron masses from first principles.

Within lattice gauge theory QCD is not formulated in the continuum but on a discrete and finite space or space-time lattice. Hadronic masses can then be extracted from the decay of fermionic correlation functions. In Euclidean lattice gauge theory these correlation functions are given by Euclidean path integrals which in turn can be evaluated numerically by Monte Carlo methods. The fermionic degrees of freedom are formally integrated out and the Monte Carlo integration is carried out for a bosonic path integral only, which, however, contains the spectral determinant of the Dirac operator in the integration measure.

The evaluation of fermionic determinants, which now has to be carried out for each update of the gauge field configuration, is computationally intensive, in particular for realistic, i.e. small, quark masses. Thus, in the past many studies have been performed in the so-called quenched approximation in which the fermionic determinant is neglected completely. This is equivalent to giving the quarks infinite mass or setting the number of flavours to zero. Large scale unquenched lattice calculations have only become available in recent years, and calculations with realistic quark masses will only be possible with the next generation of specialised super computers.

Therefore, any other way of obtaining independent information on the spectrum of the Dirac operator, and thus the fermionic determinant, is of great interest.

In the early nineties it turned out that chiral random matrix theory (RMT) describes spectral correlations of the QCD Dirac operator extremely well \[1, 2\] and can even predict the microscopic spectral density, i.e. the distribution of small eigenvalues of the Dirac operator, see \[3\] for an overview. However, the somewhat surprising information that the spectra of lattice QCD are, up to a certain scale, indistinguishable from the spectra of random matrices cannot be exploited directly in order to facilitate lattice calculations. The scale mentioned above is the equivalent of the Thouless energy in disordered systems. It was theoretically derived in \[4, 5\] and identified in lattice QCD data in \[6, 7\].

This situation is reminiscent of the situation in low dimensional quantum chaos. There short range spectral correlations of individual quantum systems can be described by RMT if the corresponding classical system is chaotic. In this context a two-fold role is played by semiclassical methods, in particular by the Gutzwiller trace formula \[8\].
they provide an explanation for the correspondence of classical chaoticity and quantum spectral correlations being described by RMT. On the other hand they also predict and describe deviations from RMT in long-range correlations, linking them to non-universal features of short periodic orbits\(^9\). Up to now, such a scale could not be identified in spectra of the QCD Dirac operator for a frozen, i.e. fixed, configuration of the gauge fields. We notice that the equivalent of the Thouless energy mentioned above is an effect due to the fluctuation of the gauge fields and can thus only be seen after averaging over all configurations, see the discussion in \([10, 11]\).

Inspired by this analogy one may ask: Are semiclassical contributions the missing ingredient which would make it possible to constructively use the RMT information when calculating fermionic determinants? As a first step towards an answer we develop semiclassical approaches to the Euclidean QCD Dirac operator and in particular discuss qualitative features of the classical dynamics arising in this context. Notice that the word “semiclassical” in this context always refers to asymptotic statements about the spectrum of the Dirac operator – technically a problem in single particle quantum mechanics rather than in quantum field theory – which is not the same as loop expansions which are also called “semiclassical” in quantum field theory. Thus, our approach is in a similar spirit as works relating the spectral analysis of the QCD Dirac operator to the theory of disordered systems\(^4, 5\).

This article is organised as follows. In section 2 we review some basic formulae and discuss the semiclassical structure of the QCD Dirac operator. In section 3 we briefly sketch a strategy for deriving trace formulae which we will follow in the subsequent sections. The discussion of semiclassical approaches to the Dirac operator in Abelian gauge fields presented in section 4 serves as prerequisite for our semiclassical analysis for the QCD Dirac operator which follows in section 5. The latter contains the main results of this article identifying three semiclassical parameters and discussing the classical dynamics arising in different (combined) semiclassical limits. Some details left open in sections 4 and 5 are solved by our study of the squared Dirac operator in section 6. In section 7 we discuss whether and how our theory can be used for describing features of Dirac spectra close to zero virtuality. Section 8 illustrates our theory for an explicit example. We conclude with a discussion of our findings and by indicating possible future directions of research in section 9.

### 2 The QCD Dirac operator

The free Euclidean Dirac operator describing massless spin 1/2-particles reads

\[
\hat{D} = \frac{\hbar}{i} \gamma_{\mu} \partial_{\mu}.
\]  

(2.1)

We adopt the summation convention over repeated Greek indices from 1 to \(d\), the number of space-time dimensions. The \(\gamma\)-matrices satisfy

\[
\{ \gamma_{\mu}, \gamma_{\nu} \} = 2 \delta_{\mu\nu}.
\]  

(2.2)
Describing massive particles simply amounts to adding $-im$ to (2.1). Also note that in the context of lattice gauge theory often the anti-Hermitean operator $i \hat{D}$ is called Dirac operator. In dimension $d = 4$ we will later explicitly use the chiral representation,\[ \gamma = \begin{pmatrix} 0 & -i\sigma \\ i\sigma & 0 \end{pmatrix}, \quad \gamma_4 = \begin{pmatrix} 0 & 1_2 \\ 1_2 & 0 \end{pmatrix}, \quad (2.3) \]

where $1_n$ denotes the $n \times n$ unit matrix and $\sigma$ is the three-vector of Pauli matrices,\[ \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \quad (2.4) \]

In this representation $\gamma_5 = \gamma_1\gamma_2\gamma_3\gamma_4$ reads\[ \gamma_5 = \begin{pmatrix} 1_2 & 0 \\ 0 & -1_2 \end{pmatrix}. \quad (2.5) \]

When now introducing a non-Abelian gauge field we put special emphasis on the appearance of $\hbar$ and its consequences for semiclassics. In this way we will identify the most natural asymptotic treatment from the perspective of semiclassical physics. However, by introducing fields with an $\hbar$-dependent magnitude alternative options are also possible and we will remark on those in the appropriate places. Moreover, we will explain which situations in standard QCD language correspond to the scenarios discussed.

A non-Abelian gauge field $A_\mu(x)$ is introduced by minimal coupling,\[ \hat{D} = \gamma_\mu \left( \frac{\hbar}{i} \partial_\mu - \hbar g A_\mu(x) \right). \quad (2.6) \]

Notice the appearance of $\hbar$, together with the coupling constant $g$ which turns the covariant derivative into\[ D_\mu = \partial_\mu - ig A_\mu(x), \quad (2.7) \]

$\hat{D} = -i\hbar \gamma_\mu D_\mu$. This is different from the Abelian case, i.e. quantum electro dynamics (QED), where the minimal coupling prescription reads\[ \frac{\hbar}{i} \partial_\mu \quad \mapsto \quad \frac{\hbar}{i} \partial_\mu - e A_\mu(x). \quad (2.8) \]

Here we denote the coupling constant, i.e. the electric charge of the fermion, by $e$ (we set $c = 1$) and thus the covariant derivative reads\[ D_\mu = \partial_\mu - i\frac{e}{\hbar} A_\mu. \quad (2.9) \]

The reason for the different appearance of $\hbar$ in these two cases is that non-Abelian fields couple to themselves. More precisely, when writing down the QCD-Lagrangian which upon variation yields both, the Dirac equation and the classical Yang-Mills equations for
\( A_\mu \), the latter would contain a self-interaction term which would explicitly depend on \( \hbar \) if the covariant derivative \( (2.7) \) had the same \( \hbar \)-dependence as \( (2.9) \). Since this cannot be true for a classical equation, formula \( (2.7) \) is the correct choice for non-Abelian fields. To illustrate this consider the field strength tensor deriving from \( (2.7) \),

\[
F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - ig[A_\mu, A_\nu] \tag{2.10}
\]

which does not contain \( \hbar \). On the other hand an Abelian field does not couple to itself, the last term in \( (2.10) \) vanishes, and therefore in QED a covariant derivative of the form \( (2.9) \) is allowed, because it does not lead to an \( \hbar \)-dependence of the field strength. The very same mechanism is responsible for the well-known fact that one can have elementary particles with different electric charges but that all particles which couple to the non-Abelian colour field do so with the same coupling constant, i.e. they all have the same colour charge.

Notice that the observation described between \( (2.6) \) and here holds true as long as all \( \hbar \)-dependencies in the formulae are displayed explicitly, i.e. quantities such as \( g \), \( A_\mu \) or \( F_{\mu\nu} \) do not depend on \( \hbar \). In particular, the argumentation as laid out above is independent of the scaling properties of the QCD-action. If one, e.g., rescales the fields according to \( A_\mu = A'_\mu / \sqrt{\hbar} \) then eqs. \( (2.6) \) and \( (2.10) \) will read \( \hat{D}_\mu = \gamma_\mu (\frac{\hbar}{4} \partial_\mu - \hbar^{3/2} g A'_\mu) \) and \( F_{\mu\nu} = \sqrt{\hbar} \partial_\mu A'_\nu - \sqrt{\hbar} \partial_\nu A'_\mu - i\hbar g [A'_\mu, A'_\nu] \), respectively. Thus, we have formally produced powers of \( \hbar \) in unfamiliar places. However, as long as the original fields \( A_\mu \) are of order 1 then the rescaled fields \( A'_\mu \) are of order \( \hbar^{-1/2} \), and therefore the \( \hbar \)-dependence of the couplings between fermion and gauge field and of the gauge field to itself are as before.

The situation changes if we, instead of just rescaling the fields, consider fields whose order of magnitude is \( \hbar \)-dependent. For instance, a gauge field of order \( 1/\hbar \) gives rise to a Dirac operator in which the coupling of fermion and colour field has the same \( \hbar \)-dependence as in QED. In QCD such a strong field is called an external colour field.

In the situation, however, which was described between \( (2.6) \) and \( (2.10) \), the electromagnetic fields are external fields whereas the colour fields are microscopic or dynamical fields. Since the \( \hbar \)-dependence of the latter is chosen such that the classical field equations are \( \hbar \)-independent it is also common in QCD to speak of a “classical gauge field” in this context. From the point of view of high energy physics it may appear slightly inconsistent to discuss external electro-magnetic fields and microscopic colour fields in the same context. From the point of view of semiclassical physics, however, we have treated both types of fields on the same footing.

In the following we will concentrate on the situation with microscopic gauge fields, i.e. on the Dirac operator \( (2.6) \) where all \( \hbar \)-dependencies are displayed explicitly. A treatment of external colour fields would lead to different semiclassical asymptotics.

In order to shed some more light on the physics behind the \( \hbar \)-dependence discussed above, decompose the non-Abelian field in terms of the generators \( X^a \) of the gauge group \( G \), say SU\((N)\),

\[
A_\mu = \frac{1}{2} A^a_\mu X^a, \tag{2.11}
\]

where summation over the repeated Latin index \( a \) is from 1 to the dimension of the Lie algebra. The \( X^a \) are traceless, Hermitean \( N \times N \)-matrices satisfying the Lie algebra relations...
\[ [X^a, X^b] = f^{abc} X^c, \]  
with structure constants \( f^{abc} \), and are normalised according to
\[ \text{tr}(X^a X^b) = 2\delta_{ab}. \]  
(2.13)

If we now view
\[ \hat{C}^a := \frac{\hbar}{2} X^a \]  
(2.14)
as the quantum observable describing the colour degrees of freedom (of the fermion) the Dirac operator (2.6) takes the form
\[ \hat{D} = \gamma_\mu \left( \hat{p}_\mu - g \hat{C}^a A^a_\mu(x) \right), \]  
(2.15)
with the colour and momentum operators \( \hat{C}^a \) and \( \hat{p}_\mu \), representing the quantisation of some classical observables \( C^a \) and \( p_\mu \).

The point of view adopted in (2.14) is typical for internal, i.e. microscopic, degrees of freedom, a familiar example being the non-relativistic spin operator \( \hat{s} = \hbar \sigma / 2 \), which has the same structure as (2.14) with \( X^a \) replaced by the Pauli matrices, the generators of \( SU(2) \).

For later reference let us also introduce the matrix valued function on classical phase space,
\[ D(p, x) = \gamma_\mu \left( p_\mu - \frac{\hbar g}{2} X^a A^a_\mu(x) \right), \]  
(2.16)
from which the QCD Dirac operator can be obtained by replacing \( p_\mu \) with \(-i\hbar \delta_\mu\). In microlocal analysis or Wigner-Weyl calculus (2.16) is referred to as the Weyl symbol of the Dirac operator (2.6), which in turn can be obtained from its symbol by Weyl quantisation,
\[ (\hat{D}\Psi)(x) = \frac{1}{(2\pi\hbar)^d} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} D\left(p, \frac{x+y}{2}\right) e^{i\pi p_{\mu}(x_\mu - y_\mu)} \Psi(y) \, dp \, dq \, dy. \]  
(2.17)
Wigner-Weyl calculus is a particularly useful tool when studying semiclassical asymptotics. In a setting where the semiclassical limit is identified with \( \hbar \to 0 \) one would classify the terms in (2.16) according to their \( \hbar \)-dependence as the principal symbol
\[ D_0(p, x) = \gamma_\mu p_\mu \]  
(2.18)
and the sub-principal symbol
\[ D_1(p, x) = \frac{g}{2} \gamma_\mu X^a A^a_\mu(x), \]  
(2.19)
respectively. Eventually we will also use the notation
\[ \text{symb}[\hat{D}](p, x) \equiv D(p, x) \]  
(2.20)
for the Weyl symbol of an operator.
3 Semiclassical trace formulae

Before we go into details about the semiclassics for the QCD Dirac operator let us say a few words about semiclassical trace formulae in general and briefly sketch one method for deriving them; for details, however, we refer to the cited literature.

We are interested in the spectrum of the Hermitean operator $\hat{D}$. For simplicity assume that the spectrum is pure point, i.e. we have a set of eigenvalues $\lambda_n$ and a complete orthonormal set of corresponding eigenstates $\Psi_n$,

$$\hat{D}\Psi_n = \lambda_n \Psi_n.$$  \hfill (3.1)

Our main focus lies on the spectral density

$$\rho(\lambda) = \sum_n \delta(\lambda - \lambda_n),$$  \hfill (3.2)

which is gauge invariant. In order to derive a semiclassical expression for $\rho(\lambda)$ consider the evolution equation

$$i\hbar \frac{\partial}{\partial t} \Psi(x, t) = \hat{D}\Psi(x, t).$$  \hfill (3.3)

Note that the time parameter $t$ is not the physical time but an auxiliary variable. The physical time is already included in the components of $x$ and we are dealing with the Euclidean Dirac operator. Likewise the spectral parameter $\lambda$ is not an energy but referred to as virtuality. Now define the evolution kernel $K(x, y, t)$ by

$$\Psi(x, t) = \int_{\mathbb{R}^d} K(x, y, t) \Psi(y, 0) \, d^d y,$$  \hfill (3.4)

which has the spectral representation

$$K(x, y, t) = \sum_n \Psi_n(x) \Psi_n^\dagger(y) e^{-i\frac{\hbar}{\pi} \lambda_n t}.$$  \hfill (3.5)

Obviously, $K(x, y, t)$ also has to solve (3.3) with initial condition

$$K(x, y, 0) = \delta(x - y).$$  \hfill (3.6)

By Fourier transforming the evolution kernel and taking the trace on $L^2(\mathbb{R}^d) \otimes \mathbb{C}^J \otimes \mathbb{C}^4$, where $J$ denotes the dimension of the representation of the gauge group, we obtain the spectral density,

$$\text{tr} \, \frac{1}{2\pi\hbar} \int_{\mathbb{R}^d} \int_{-\infty}^{\infty} K(x, x, t) e^{i\pi \hbar t} \, dt \, d^d x = \rho(\lambda)$$  \hfill (3.7)

Here $\text{tr}$ denotes the trace over the matrix degrees of freedom.

In order to obtain a semiclassical approximation for the spectral density, one can begin with a WKB-type ansatz for the time evolution kernel,

$$K(x, y, t) = \frac{1}{(2\pi\hbar)^d} \int_{\mathbb{R}^d} [a_0(x, \xi, t) + \hbar a_1(x, \xi, t) + \ldots] e^{i\hat{S}(x, \xi, t) - \xi y} \, d^d \xi.$$  \hfill (3.8)
Inserting into (3.3) and sorting by powers of $\hbar$ one finds a sequence of equations which can be solved order by order yielding $S, a_0, a_1, \ldots$. In leading order one always finds a Hamilton-Jacobi equation for the phase $S$,

$$\Lambda(\nabla_x S, x) + \frac{\partial S}{\partial t} = 0,$$

with a classical Hamiltonian $\Lambda$ given by an eigenvalue of the principal symbol of $\hat{D}$. Classical Hamilton-Jacobi theory now tells us that the solution $S$ of (3.9) generates classical dynamics from the phase space point $(\xi, \nabla_\xi S)$ to $(\nabla_x S, x)$ in time $t$, showing that the integration parameter $\xi$ of the ansatz (3.8) plays the role of an initial momentum for the classical system.

In order to derive a semiclassical approximation to $\rho(\lambda)$, one also needs to determine the leading order amplitude $a_0$ which is fixed by the next-to-leading order equation. The latter, usually referred to as transport equation, has the following structure,

$$\left( \frac{\partial}{\partial t} + (\nabla_p \Lambda) \nabla_x \right) a_0 + \frac{1}{2} \left( \frac{\partial^2 \Lambda}{\partial x_\mu \partial p_\mu} + \frac{\partial^2 \Lambda}{\partial p_\mu \partial p_\nu} \frac{\partial^2 S}{\partial x_\mu \partial x_\nu} \right) a_0 + \ldots = 0$$  \hspace{1cm} (3.10)

The reader easily verifies this structure by explicitly doing the calculation for an operator of his choice. A derivation of the general result can, e.g., be found in appendix E of [12].

The first bracket on the l.h.s. of (3.10) is a derivative along the trajectory generated by the $\Lambda$-dynamics, whereas the second term, roughly speaking, measures the behaviour of neighbouring phase space points. Without additional terms (3.10) is solved by $\sqrt{\det \frac{\partial^2 S}{\partial x \partial \xi}}$, see, e.g., appendix B of [12] for a compact derivation. If, besides the terms displayed explicitly in (3.10), further contributions show up in the transport equation then they represent the transport of internal degrees of freedom (such as spin or colour as we will see below) along the trajectory of the flow with Hamiltonian $\Lambda$.

Therefore, in order to find the full classical system corresponding to the quantum Hamiltonian $\hat{D}$ one has to (i) determine the Hamiltonian(s) $\Lambda$ and (ii) carefully analyse all contributions to the transport equation.

Having determined $S(x, \xi, t)$ and $a(x, \xi, t)$, i.e. having obtained a semiclassical approximation to the kernel $K(x, y, t)$, a trace formula can be derived in a straightforward manner by inserting everything into (3.7) and evaluating all integrals in leading order with the method of stationary phase.

The stationarity conditions for the $x$- and $\xi$-integrals read

$$\nabla_x S(x, \xi, t) = \xi \quad \text{and} \quad \nabla_\xi S(x, \xi, t) = x.$$  \hspace{1cm} (3.11)

According to classical Hamilton-Jacobi theory this means that both initial and final momentum as well as initial and final position of the trajectory generated by $S$ have to be identical. Thus, only phase space points that lie on periodic orbits contribute to the semiclassical expression for $\rho(\lambda)$.

A special role is played by the periodic points of period zero which are given by the whole hypersurface of constant virtuality $\lambda$,

$$\Omega_\lambda := \{(p, x) \mid \Lambda(p, x) = \lambda\}.$$  \hspace{1cm} (3.12)
Since their action is also zero they yield the only non-oscillating contribution to the spectral density and thus constitute the mean density, often called Weyl term,

$$\bar{\rho}(\lambda) = \frac{|\Omega_\lambda|}{(2\pi \hbar)^d} = \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \delta(\Lambda(p, x) - \lambda) \frac{d^dp \, d^dx}{(2\pi \hbar)^d}, \quad (3.13)$$

with a possible multiplicity factor deriving from the internal degrees of freedom such as spin or colour.

Finally we obtain the following general structure for a semiclassical trace formula,

$$\rho(\lambda) \sim \bar{\rho}(\lambda) + \sum_{\gamma} A_{\gamma}(\lambda) e^{i\frac{\hbar}{\pi} S_{\gamma}(\lambda)}. \quad (3.14)$$

Here $\gamma$ labels both, isolated periodic orbits and larger families of periodic points, like, e.g., Liouville-Arnold tori in integrable systems. The amplitudes $A_{\gamma}(\lambda)$ are derived by keeping track of all contributions in the various stationary phase approximations involved.

If one is interested in the precise mathematical meaning of this distributional identity and in an absolutely convergent version of the trace formula, which can be used for numerical calculations, it is convenient to multiply the expressions with a test function in $t$ before taking the Fourier transforms, see e.g. [13, 14, 15].

## 4 Semiclassical parameters in the Abelian case

In this section we discuss semiclassical approximations to the Dirac operator in Abelian gauge fields. We will keep the presentation short and closely follow similar studies for the Dirac Hamiltonian which were carried out in [16, 15], however, pointing out small differences which are due to the fact that we are dealing with the Euclidean Dirac operator instead. The results obtained here will also be needed for our discussion of the non-Abelian case in the following section.

Consider the equation of motion (3.3) for the time evolution kernel,

$$i\hbar \frac{\partial}{\partial t} K(x, y, t) = \hat{D} K(x, y, t), \quad (4.1)$$

where the derivatives in the Dirac Hamiltonian,

$$\hat{D} = \gamma_\mu \left( \frac{\hbar}{i} \partial_\mu - e A_\mu(x) \right), \quad (4.2)$$

with Abelian $A_\mu$, act on the first argument of $K$. Inserting an ansatz of type (3.8) with scalar phase $S$ and matrix-valued amplitudes $a_k$ into the evolution equation we find in leading orders

$$\left[ \frac{\partial S}{\partial t} + D(\nabla_x S, x) \right] a_0 = 0, \quad (4.3)$$

$$\left[ \frac{\partial S}{\partial t} + D(\nabla_x S, x) \right] a_1 + \left( \frac{\partial}{\partial t} + \gamma_\mu \partial_\mu \right) a_0 = 0, \quad (4.4)$$
where

\[ D(p, x) = \begin{pmatrix} 0 & \pi_4 - i\sigma_\pi \\ \pi_4 + i\sigma_\pi & 0 \end{pmatrix} \] (4.5)

is the (principal) symbol of \( \hat{D} \) and \( \pi_\mu := p_\mu - eA_\mu(x) \) denotes the kinetic momenta. For (4.3) to have non-trivial solutions the term in square brackets must have an eigenvalue zero. The eigenvalues of \( D(p, x) \) are given by

\[ \Lambda^\pm(p, x) = \pm \sqrt{\pi_\mu \pi_\mu} =: \pm \Lambda, \] (4.6)

both having multiplicity two. We collect the corresponding eigenvectors columnwise in the \( 2 \times 4 \)-matrices

\[ V_+(p, x) = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ \frac{\pi_4 - i\sigma_\pi}{\Lambda} \end{pmatrix}, \quad V_-(p, x) = \frac{1}{\sqrt{2}} \begin{pmatrix} \frac{\pi_4 + i\sigma_\pi}{\Lambda} \\ -1 \end{pmatrix}. \] (4.7)

Thus, the solvability condition for (4.3) yields the Hamilton-Jacobi equations

\[ \Lambda^\pm(\nabla_x S^\pm, x) + \frac{\partial S^\pm}{\partial t} = 0, \] (4.8)

and the general solution of (4.8) is a superposition of terms with positive (+) and negative (−) virtuality.

Equation (4.8) alone does not solve (4.3), but in addition the leading order amplitude has to satisfy \( D a_0^\pm = \Lambda^\pm a_0^\pm \). This is guaranteed by the following ansatz,

\[ a_0^\pm(x, \xi, t) = V_\pm(\nabla_x S^\pm, x) b_\pm(x, \xi, t) V_\pm^\dagger(\xi, y) \] (4.9)

where the \( 2 \times 2 \)-matrices \( b_\pm \) have to be determined by (4.4). To this end we multiply (4.4) with \( V_\pm^\dagger(\nabla_x S^\pm, x) \) from the left and \( V_\pm(\xi, y) \) from the right, yielding

\[ V_\pm^\dagger(\nabla_x S^\pm, x) \left( \gamma_\mu \partial_\mu + \frac{\partial}{\partial t} \right) V_\pm(\nabla_x S^\pm, x) b_\pm = 0, \] (4.10)

since

\[ V_\pm^\dagger(\nabla_x S^\pm, x) \left[ \frac{\partial S}{\partial t} + D(\nabla_x S, x) \right] = 0. \] (4.11)

After a lengthy calculation, which is sketched in appendix A, one finds

\[ V_\pm^\dagger V_\pm \frac{\partial b_\pm}{\partial t} + V_\pm^\dagger V_\pm(\partial_\mu b_\pm) = \left( \frac{\partial}{\partial t} + \frac{\partial \Lambda^\pm}{\partial p_\mu} \partial_\mu \right) b_\pm =: \dot{b}_\pm, \] (4.12)

\[ V_\pm^\dagger \frac{\partial}{\partial t} V_\pm + V_\pm^\dagger \gamma_\mu \partial_\mu V_\pm = \frac{1}{2} \left( \frac{\partial^2 \Lambda^\pm}{\partial x_\mu \partial p_\mu} + \frac{\partial^2 \Lambda^\pm}{\partial p_\mu \partial x_\mu} \partial^2 S^\pm \partial x_\mu \partial x_\mu \right) - \frac{ie}{2\Lambda^\pm} \sigma(E \pm B), \] (4.13)
where the dot in \(4.12\) denotes a derivative along the trajectory generated by \(S^\pm\). In addition we have introduced the electric and magnetic components, \(E\) and \(B\), of the field strength \(F_{\mu\nu}\), according to

\[
F = \begin{pmatrix}
0 & B_3 & -B_2 & -E_1 \\
-B_3 & 0 & B_1 & -E_2 \\
B_2 & -B_1 & 0 & -E_3 \\
E_1 & E_2 & E_3 & 0
\end{pmatrix}.
\] (4.14)

Since we already know how to solve a transport equation of type \(3.10\) the product ansatz

\[
b_\pm = \sqrt{\det \frac{\partial^2 S_\pm}{\partial x \partial \xi} d_\pm}
\] (4.15)

with a \(2 \times 2\) matrix \(d\) lends itself to simplify the transport equation to

\[
\dot{d}_\pm - \frac{ie}{2\Lambda_\pm} \sigma (E \pm B) d_\pm = 0.
\] (4.16)

This equation describes the transport of the spin degrees of freedom along the trajectory determined by the Hamiltonian \(\Lambda^\pm\). Obviously \(d_\pm\) takes values in \(\text{SU}(2)\) and in the trace formula the contribution of each periodic orbit is weighted with the trace of the corresponding \(d_\pm\), i.e. with a character.

Equation \(4.16\) can be mapped from \(\text{SU}(2)\) to \(S^2\) by looking at the time evolution of the expectation value \(s\) of the spin operator \(\hbar \frac{\sigma}{2}\) in an arbitrary state \(u \in \mathbb{C}^2\) – i.e. \(s_\pm = u^\dagger d^\dagger_\pm \hbar \sigma d_\pm u\) – as induced by \(4.16\),

\[
\dot{s}_\pm = s_\pm \times \frac{e}{\Lambda_\pm} (E \pm B).
\] (4.17)

This equation describes classical spin precession, i.e. it is a Euclidean analogue of the Thomas- or Bargman-Michel-Telegdi(BMT)-equation [17, 18]. Although \(4.17\) looks like an equation for the three-vector \(s\) classical spin precession actually takes place on the sphere \(S^2\) since total spin, i.e. \(|s|^2\), is conserved. The two-sphere in turn is a symplectic manifold and \(4.17\) defines a volume-preserving flow on it. These facts together justify the notion of “classical spin dynamics” in this context. Mathematically speaking, we map the equation from (the representation of) the group to its coadjoint orbit, see e.g. [19, 20].

Had we dealt with a particle with higher spin from the beginning we would have obtained a similar spin transport equation as \(4.10\) with only \(\sigma\) replaced by generators of a higher dimensional representation of \(\text{SU}(2)\) and \(d_\pm\) now taking values in that representation. The weight factor \(\text{tr} d_\pm\) in the trace formula would still be a character and the analogous mapping to the sphere would lead to exactly the same classical spin precession equation. The character entering the trace formula is always completely determined by classical spin precession [12].

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We have thus identified the total classical dynamics arising from a semiclassical analysis of the Euclidean Dirac operator as a combination of the Hamiltonian flows with Hamiltonians $\Lambda^\pm(p, x)$ accompanied by spin precession along the orbits. Since there is no back-reaction of spin dynamics on the Hamiltonian part the total dynamics can be formulated as a skew product flow, either on $\mathbb{R}^{2d} \times SU(2)$ or $\mathbb{R}^{2d} \times S^2$ [21, 22].

So far we have discussed what happens in the single semiclassical limit $\hbar \to 0$. However, also the limit of large spin can be considered as a semiclassical limit, cf. the so-called kicked top [23]. If one simultaneously lets $\hbar \to 0$ and $s \to \infty$, where $2s + 1$ denotes the dimension of the representation of $SU(2)$, such that the product $\hbar s$ is kept non-zero and finite, also the back-reaction of spin on the translational degrees of freedom shows up in the classical picture, see e.g. [21, 25, 26]. We emphasise that, although claimed otherwise in [25], even for Hamiltonians linear in the spin degrees of freedom semiclassical asymptotics can only then display both, spin dynamics and back reaction, simultaneously in leading semiclassical order if one considers the large spin limit. This says, however, nothing about the possible practical use of this type of approximation even when the actual value of $s$ is rather small.

The situation here is somehow reminiscent of the limit of large colour, $N \to \infty$, [27], which yields valuable insights although we are mostly interested in $N = 3$.

We can almost write down the Hamiltonians $\Lambda^\pm(p, x, s)$ for the combined dynamics already with the information gathered so far. Omitting the spin-dependent terms it has to reduce to $\Lambda^\pm(p, x)$, i.e., formally, $\Lambda^\pm(p, x, 0) = \Lambda^\pm(p, x)$, and it has to give rise to the spin precession (4.17). The relativistic Pauli Hamiltonian,

$$\Lambda_{\text{Pauli}}^\pm(p, x, s) = \Lambda^\pm(p, x) - \frac{e}{\Lambda^\pm(p, x)} s(E(x) \pm B(x)), \quad (4.18)$$

fulfils these requirements, but so does, e.g., the alternative square root type Hamiltonian

$$\Lambda_{\text{sqrt}}^\pm(p, x, s) = \pm \sqrt{(p_\mu - eA_\mu(x))(p_\mu - eA_\mu(x)) - 2es(E(x) \pm B(x))} \quad (4.19)$$

Both types of Hamiltonians agree in the limit of small spin contribution (for illustration one may formally consider the limit $s \to 0$) but in general they lead to different back-reactions of spin on the translational degrees of freedom. This difference becomes particularly important for small virtualities $\lambda$. In section 6 we will discuss a simple method for deciding which Hamiltonians to use, without explicitly developing a full symbol calculus for the combined limits.

5 Semiclassical parameters in the non-Abelian case

With a discussion of semiclassical parameters and limits of the QCD Dirac operator and the different classical dynamics arising in this context the present section contains the central results of this work. We perform our analysis along the same lines as laid out in sections 3 and 4 and build on the results obtained in section 4.

Consider the Dirac operator (2.6) with Weyl symbol (2.16). The time evolution kernel $K(x, y, t)$ is now a $4J \times 4J$ matrix, where $J$ denotes the dimension of the representation...
of the gauge group, i.e. \( J = 3 \) for QCD with SU(3) gauge fields in the fundamental representation. Inserting an ansatz of type (3.8) into the equation of motion (4.1) with Dirac operator (2.6) yields in leading orders

\[
\left[ \frac{\partial S}{\partial t} + D_0(\nabla_x S, x) \right] a_0 = 0 ,
\]

(5.1)

\[
\left[ \frac{\partial S}{\partial t} + D_0(\nabla_x S, x) \right] a_1 + \left( \frac{\partial}{\partial t} + \gamma_\mu \partial_\mu + D_1(\nabla_x S, x) \right) a_0 = 0 ,
\]

(5.2)

where we have used the notation for the principal and sub-principal symbol which was introduced in eqs. (2.18) and (2.19). The principal symbol has eigenvalues

\[
\Lambda^\pm(p, x) = \pm \sqrt{p_\mu p^\mu} =: \pm \Lambda
\]

(5.3)

with corresponding eigenvectors collected in the matrices

\[
V_+ = \frac{1}{\sqrt{2}} \left( \begin{array}{c} 1 \\ \frac{1}{2} \pm \frac{i}{2} \end{array} \right) , \quad V_- = \frac{1}{\sqrt{2}} \left( \begin{array}{c} \frac{1}{2} \pm \frac{i}{2} \\ -1 \end{array} \right).
\]

(5.4)

As in section 4, eq. (5.1) demands that the phase of the ansatz (3.8) solves a Hamilton-Jacobi equation,

\[
\Lambda^\pm(\nabla_x S^\pm, x) + \frac{\partial S^\pm}{\partial t} = 0 ,
\]

(5.5)

and suggests the following ansatz for the leading order amplitude,

\[
a_0^\pm(x, \xi, t) = V_\pm(\nabla_x S^\pm, x) b_\pm(x, \xi, t) V_\pm^\dagger(\xi, y) .
\]

(5.6)

The projected transport equation for the \( 2J \times 2J \) matrix \( b_\pm \) reads

\[
V_\pm^\dagger \left( \gamma_\mu \partial_\mu + \frac{\partial}{\partial t} - \frac{ig}{2} \gamma_\mu A_\mu^a X^a \right) V_\pm b_\pm = 0 ,
\]

(5.7)

which, using eqs. (4.12) and (4.13) with the substitution \( \pi_\mu \to p_\mu \), cf. (5.3), simplifies to,

\[
\dot{b}_\pm + \frac{1}{2} \left( \frac{\partial^2 \Lambda^\pm}{\partial x_\mu \partial p_\mu} + \frac{\partial^2 \Lambda^\pm}{\partial p_\mu \partial p_\nu} \frac{\partial^2 S^\pm}{\partial x_\mu \partial x_\nu} \right) b_\pm - \frac{ig}{2} \partial_\mu \frac{\partial \Lambda^\pm}{\partial p_\mu} A_\mu^a X^a b_\pm = 0 .
\]

(5.8)

As in the Abelian case we separate the translational part according to (4.15) and obtain the following equation for the \( 2J \times 2J \) matrix \( d_\pm \),

\[
\dot{d}_\pm - \frac{ig}{2} \partial_\mu A_\mu^a X^a d_\pm = 0 .
\]

(5.9)

In contrast to the Abelian case this equation does not involve the spin but the colour degrees of freedom. Accordingly we will refer to it as colour transport equation.
As in the case of spin transport we obtain classical dynamics by looking at the equation of motion satisfied by the expectation value $C^a$ of $\hat{C}^a = \frac{\hbar}{2} X^a$,

$$\dot{C}^a = -\frac{g}{2} \frac{\partial A^\pm}{\partial \rho_\mu} f^{abc} A^b_\mu C^c,$$

which we call colour precession. Equation (5.10) is the colour part of the Wong equations [28] to which we will come back later.

As in the case of spin precession discussed in the preceding section there are certain conditions restricting the possible values which the variables $C^a$ can assume, thus confining the dynamics (5.10) to a compact manifold: The (representations of the) Casimir operators of the gauge algebra are constants of motion for (5.9) and from those derive constants of motion for the precession equation (5.10). In the case of SU(2) there is only the quadratic Casimir operator (total spin in the preceding section) which confines the precession to the sphere $S^2 = SU(2)/U(1)$. If the gauge group is SU(3) then we have two Casimir operators, one quadratic and one cubic in the generators (or components $C^a$ of classical colour). The dynamics of the $C^a$ is thus reduced from $R^8$ (8 generators) to a six dimensional manifold, the flag manifold $F^3 = SU(3)/U(1) \times U(1)$, see e.g. [29]. For the gauge group SU($N$) we would have $F^N = SU(N)/U(1)^{n-1}$, instead. In all cases these are maximal coadjoint orbits [19], which are not only even dimensional but naturally endowed with a symplectic structure, thus constituting the classical phase space for internal degrees of freedom such as spin or colour.

Having understood (5.9) as transport equation for the colour degrees of freedom and characterised the underlying classical phase space and dynamics we can now ask ourselves why the spin degrees of freedom do not show up at this level of the semiclassical treatment, neither in the Hamiltonians (5.3) nor in the transport equation. The answer is that spin and translational degrees of freedom are coupled by the gauge fields and thus only via the internal colour degrees of freedom. With both, spin and colour, being internal degrees of freedom a coupling between them, which has to involve the product of $\hat{s} = \frac{\hbar}{2} \sigma$ and $\hat{C}^a = \frac{\hbar}{2} X^a$, is automatically at least of order $\hbar^2$. Therefore, it does not enter the leading order phases and amplitudes of $\hbar \to 0$ asymptotics.

Comparing with the results of the preceding section we should expect a spin precession equation like (4.17) with the electric and magnetic fields replaced by their non-Abelian analogues. At this point we can thus guess the following semiclassical hierarchy (which we will confirm to be correct in the following section): In pure $\hbar \to 0$ asymptotics the phase of semiclassical approximations is determined by free translational dynamics alone. The leading order amplitude is affected by the colour transport along particle orbits, whereas there is no back-reaction of colour onto the translational degrees of freedom. Spin shows up only as an $\hbar$-correction to the amplitude. While spin precession is driven by both translational and colour dynamics it does not act back on either of them. Thus, we have a double skew product structure.

Back reaction can be forced to show up explicitly in the semiclassical approximations by considering combined limits. To this end choose a $J$ dimensional unitary irreducible
representation of the gauge group and consider the combined limits $\hbar \to 0$ and $J \to \infty$ with the product $\hbar J$ kept constant. This will lead to colour entering on the same level as the translational degrees of freedom, i.e. we will have to deal with the minimally coupled classical Hamiltonians

$$\Lambda^\pm(p,x,C) = \pm \sqrt{(p_\mu - gA_\mu(x)C^a)(p_\mu - gA_\mu(x)C^b)}.$$  \hspace{1cm} (5.11)

The coupled classical dynamics arising from these Hamiltonians are known as the Wong equations \[28, 30\]. Spin will enter on the level of the transport equation as for pure $\hbar \to 0$ asymptotics in the Abelian case. Thus we have moved from a double skew product structure to an ordinary skew product. We mention in passing that by performing the additional limit $J \to \infty$ with $\hbar J$ fixed we have changed the order of magnitude of the term $g\hbar A_\mu$, appearing in the Dirac operator (2.6), from $\hbar$ to 1. Thus, this situation may be physically related to that of an external colour field in the language of QCD, cf. the discussion following (2.10). Mathematically, however, the scenario introduced here is different.

If we go even one step further by taking the triple limit $\hbar \to 0$, $J \to \infty$ and $s \to \infty$ with the products $\hbar J$ and $\hbar s$ fixed we will find fully coupled Hamiltonian dynamics on the total phase space $\mathbb{R}^{2d} \times \mathbb{R}^3 \times S^2$ (for SU(3)-gauge fields). As for the relevant Hamiltonians we have to solve the same problem as at the end of section 4. With the knowledge obtained so far it could be either of Pauli type (4.18) or of square root type (4.19), with $A_\mu$, $E$ and $B$ replaced by their non-Abelian analogues.

6 The squared Dirac operator:

Confirming the semiclassical hierarchy

The following study of the squared Dirac operator serves two purposes. On the one hand we prove that the semiclassical hierarchy conjectured in the preceding section is correct and on the other hand we determine the functional form of the classical Hamiltonians corresponding to the Dirac operator in simultaneous semiclassical limits.

We calculate the square of the Dirac operator (2.6) and determine the Weyl symbol of $\hat{D}^2$ for different symbol calculi. The principle underlying this approach is that in a symbol calculus there exists always a so-called Moyal product which expresses the symbol of the product of two operators as an asymptotic expansion in the semiclassical parameter(s) in terms of the symbols of the individual operators. The leading term in this expansion, i.e. the principle symbol of the product, is always given by the product of the principal symbols. Thus, from the appearance (or absence) of certain dynamical variables at given order in the symbol of the squared Dirac operator we can conclude at which order these variables appear in the symbol of the operator itself. Moreover, the principal symbol of the squared operator allows us to draw conclusions about the functional form of (the eigenvalues of) the principal symbol of the operator itself.

The square of $\hat{D}$ is most conveniently calculated by decomposing the products $\gamma_\mu \gamma_\nu$
and $D_\mu D_\nu$ into their symmetric and antisymmetric parts,

$$
\hat{D}^2 = -\hbar^2 \gamma_\mu \gamma_\nu D_\mu D_\nu = -\frac{\hbar^2}{4} \left( \{ \gamma_\mu, \gamma_\nu \} + [\gamma_\mu, [\gamma_\nu]] \right) \left( \{ D_\mu, D_\nu \} + [D_\mu, D_\nu] \right)
$$

$$
= -\frac{\hbar^2}{4} \left( \{ \gamma_\mu, \gamma_\nu \} \{ D_\mu, D_\nu \} + [\gamma_\mu, [\gamma_\nu]] [D_\mu, D_\nu] \right)
$$

$$
= -\hbar^2 \left( D_\mu D_\mu - ig \frac{\gamma_\mu, \gamma_\nu}{4} F_{\mu\nu} \right),
$$

(6.1)

where on the last line we have used (2.2) and the definition (2.10) of the non-Abelian field strength tensor, $F_{\mu\nu} = \frac{1}{g} [D_\mu, D_\nu]$. With the representation (2.3) we have the commutators

$$
[\gamma_4, \gamma] = 2i \begin{pmatrix} \sigma & 0 \\ 0 & -\sigma \end{pmatrix}
$$

and

$$
[\gamma_j, \gamma_k] = 2i \epsilon_{jkl} \begin{pmatrix} \sigma_l & 0 \\ 0 & \sigma_l \end{pmatrix}, \quad j, k, l = 1, 2, 3.
$$

(6.2)

Introducing colour-electric and colour-magnetic fields as in (4.14) we finally obtain

$$
\hat{D}^2 = -\hbar^2 D_\mu D_\nu - \frac{\hbar^2}{4} \left( \sigma (B + E) \begin{pmatrix} 0 & 0 \\ 0 & \sigma (B - E) \end{pmatrix} \right).
$$

(6.3)

Thus, the matrix-valued Weyl-symbol of $\hat{D}^2$ reads

$$
symb[\hat{D}^2](p, x) = (p_\mu - hg A_\mu)(p_\mu - hg A_\mu) - \hbar^2 g \left( \begin{pmatrix} \sigma (B + E) & 0 \\ 0 & \sigma (B - E) \end{pmatrix} \right).
$$

(6.4)

From this we can easily read off at which order in $\hbar$ the different degrees of freedom will enter a semiclassical approximation. To this end recall that $A_\mu = \frac{1}{2} A_\mu^a X^a$, $E = \frac{1}{2} E^a X^a$ and $B = \frac{1}{2} B^a X^a$. At order $\hbar^0$ only the translational degrees of freedom show up in symb[\hat{D}^2]. The colour degrees of freedom, $X^a$, appear for the first time at order $\hbar^1$, whereas the spin degrees of freedom are absent unless we proceed up to order $\hbar^2$. The first two observations are in agreement with the semiclassical analysis of the non-Abelian Dirac operator, and the last one provides the missing element in order to prove the semiclassical hierarchy anticipated at the end of section 5.

If, instead of Wigner-Weyl calculus for the translational degrees of freedom only, we used a symbol calculus which also maps the internal matrix degrees of freedom, spin and colour, to classical variables, i.e. symb[$\frac{1}{2} \sigma$] = $s$ and symb[$\frac{1}{2} X^a$] = $C^a$, then the symbol of $\hat{D}^2$ reads

$$
symb[\hat{D}^2](p, x) = (p_\mu - g A_\mu^a C^a)(p_\mu - g A_\mu^b C^b) - 2g \left( B^a + E^a \right) C^a \begin{pmatrix} 0 & 0 \\ 0 & s (B^a - E^a) C^a \end{pmatrix}.
$$

(6.5)

In the simultaneous limit $\hbar \to 0$, $J \to \infty$ and $s \to \infty$ with $\hbar J$ and $\hbar s$ fixed this total symbol consists of a (diagonal $2 \times 2$) principal symbol only, i.e. there are no higher order terms in any of the three semiclassical parameters. The linearity of (6.3) in the spin degrees of freedom shows that the eigenvalues of the symbol of the Dirac operator $\hat{D}$ itself have to be of square root type (4.18) rather than of Pauli type (4.19).
We remark that in the Abelian case \(^{(6.4)}\) reads

\[
\text{symb}[\hat{D}^2](p, x) = (p_\mu - eA_\mu)(p_\mu - eA_\mu) - \hbar e \begin{pmatrix} \sigma(B + E) & 0 \\ 0 & \sigma(B - E) \end{pmatrix},
\]

which is consistent with spin appearing in the leading order transport equation in pure \(\hbar \to 0\) asymptotics.

7 Mean density in stochastic fields

We have motivated this study with the question whether and how semiclassics can be of use for the understanding of spectral properties of the QCD Dirac operator. Of interest are here in particular spectral functions averaged over an ensemble of gauge fields, as they appear in the calculation of path integrals in (lattice) quantum field theory. An important example is the averaged spectral density,

\[
\langle \rho(\lambda) \rangle := \int \rho(\lambda) e^{S[A]} DA,
\]

where the action \(S[A]\), and thus the integration measure, can, e.g., be just the Yang-Mills action (quenched approximation) or the bosonic part of the full QCD action, including fermionic determinants.

Prominent features of \(\langle \rho(\lambda) \rangle\) are the so-called chiral condensate, a non-zero value at virtuality \(\lambda = 0\), and a universal functional form for small values of the virtuality. Due to the Banks-Casher relation \([31]\), see also \([32, 33]\), \(\langle \rho(0) \rangle\) is proportional to the expectation value \(\langle \bar{\psi}\psi \rangle\) of the quantised quark fields \(\psi\) in the ground state. Thus, a non-zero value indicates the spontaneous breaking of chiral symmetry. Moreover, after suitably rescaling the virtuality with the chiral condensate, the microscopic density becomes universal and can be calculated in chiral RMT. Notice that the chiral condensate is not determined by the number of exact zero modes, which is a topological invariant of the Dirac operator, but it arises, in a suitable limit, from the accumulation of small but non-zero eigenvalues.

In this section we investigate whether and to what extent the different semiclassical approaches characterised in section \(\textit{5}\) are able to explain the formation of a chiral condensate on the level of the Weyl term \((3.13)\), which is the semiclassical description of the mean spectral density. For these considerations we restrict the \(x\)-integration to a subset \(V \subset \mathbb{R}^d\) with finite volume \(V\) and since the density is symmetric about zero it is sufficient to consider only positive \(\lambda\).

7.1 \(\hbar \to 0\)

In pure \(\hbar \to 0\) asymptotics the periodic orbit structure of the trace formula and the hypersurface \(\Omega_\lambda\) \((3.12)\) determining the Weyl term are derived from the translational degrees
of freedom only. Moreover the translational dynamics are extremely simple, namely free. Spin and colour enter only as multiplicity pre-factors,

$$\rho(\lambda) = \frac{J(2s+1)}{(2\pi \hbar)^d} \int_{\mathbb{R}^d} \delta(\sqrt{p_\mu p_\mu} - \lambda) \, dp \, d^d x$$

$$= \frac{J(2s+1)}{(2\pi \hbar)^d} \int_{V} \int_{R} \delta(\sqrt{p_\mu p_\mu} - \lambda) \, dp \, d^d x$$

$$= \frac{2\pi^{d/2} J(2s+1) V}{(2\pi \hbar)^d \Gamma(d/2)} \lambda^{d-1}.$$  \hfill (7.2)

Thus, the Weyl term reduces to the free mean spectral density and is in particular independent of the gauge field configuration. For $\hbar = 1$, dimension $d = 4$, $J = N = 3$, the number of colours, and spin $s = 1/2$ the mean density reads

$$\rho(\lambda) = \frac{3V}{4\pi^2} \lambda^3.$$ \hfill (7.3)

### 7.2 $\hbar \to 0$, $J \to \infty$

If we take this combined limit then colour precession \[^{5.10}\] appears on the same level as translational dynamics. The Hamiltonians are not only functions of $p$ and $x$ but also of the classical colour degrees of freedom $C$. In order to be able to integrate over this larger phase space we need a parameterisation of the colour part. If the gauge group is $SU(N)$ (in a faithful representation) then $C$ has $N^2 - 1$ components. Colour dynamics, however, lives on the $N(N - 1)$-dimensional manifold $\mathbb{F}^N$. Let $C(\xi)$ be a parameterisation of $\mathbb{F}^N$, then the correctly normalised integration measure, which complements $d^d p \, d^d x / (2\pi \hbar)^d$ stemming from the translational degrees of freedom, is

$$\frac{J}{|\mathbb{F}^N|} d^{N(N-1)} \xi$$ \hfill (7.4)

with $|\mathbb{F}^N| = \int_{\mathbb{F}^N} d^{N(N-1)} \xi$ being the volume of the flag manifold. Thus the Weyl term now reads

$$\rho(\lambda) = \frac{J}{|\mathbb{F}^N|} \frac{(2s+1)}{(2\pi \hbar)^d} \int_{\mathbb{F}^N} \int_{\mathbb{V}} \int_{\mathbb{R}^d} \delta(\Lambda^+(p, x, C(\xi)) - \lambda) \, dp \, d^d x \, d^{N(N-1)} \xi.$$ \hfill (7.5)

However, with $\Lambda^+ = \sqrt{(p_\mu - gA_\mu^a(x)C^a)(p_\mu - gA_\mu^b(x)C^b)}$, see \[^{5.11}\], after a simple shift of variables in the $p$-integrals for fixed $x$ and $C$, $p_\mu \mapsto p_\mu + gA_\mu^a(x)C^a$, integration over the colour degrees of freedom becomes trivial and (7.5) reduces to (7.2) and thus once more to the free result.
7.3 $\hbar \to 0$, $J \to \infty$, $s \to \infty$

In this triple limit all degrees of freedom – translational, colour and spin – appear on the same footing in the Hamiltonian,

$$\Lambda^+ = \sqrt{(p_\mu - g A^a_\mu(x) C^a)(p_\mu - g A^b_\mu(x) C^b) - 2gs(E^a(x) + B^a(x))C^a}.$$  \hspace{1cm} (7.6)

In order to calculate the Weyl term we also need to parametrise the phase space of spin, $S^2$, which we do in spherical coordinates denoting the solid angle by $\omega$. The correctly normalised measure is $\frac{2s+1}{2}\frac{1}{|S^2|} d^2\omega$, with volume $|S^2| = 4\pi$. The mean density thus reads

$$\bar{\rho}(\lambda) = \frac{(2s + 1)J}{4\pi|B^N|}(2\pi \hbar)^d \int_{S^2} \int_{B^N} \int_\mathbb{V} \int_{\mathbb{R}^d} \delta(\Lambda^+(p, x, C(\xi), s(\omega)) - \lambda) d^d p d^d x d^{N(N-1)} \xi d^2 \omega.$$  \hspace{1cm} (7.7)

As before we can shift the integration variable in the $p$-integrals in order to remove the explicit appearance of the gauge potentials $A^a_\mu$. However, through the field strengths $E$ and $B$ the expression still depends on the gauge fields and the integration over the internal degrees of freedom does not become trivial. Yet we are able to calculate the mean density if we consider the average over an ensemble of gauge fields, which is the function we are interested in anyway,

$$\langle \bar{\rho}(\lambda) \rangle = \int \bar{\rho}(\lambda) \mathcal{D}A.$$  \hspace{1cm} (7.8)

Using the following property,

$$\delta(\Lambda^+ - \lambda) = 2\lambda \delta(\Lambda^+ - \lambda^2),$$  \hspace{1cm} (7.9)

and employing the Fourier representation of the $\delta$-function we have to calculate

$$\langle \bar{\rho}(\lambda) \rangle = \frac{(2s + 1)J \lambda}{4\pi^2|B^N|(2\pi \hbar)^d} \int_{\mathbb{R}} \int_{S^2} \int_{B^N} \int_{\mathbb{V}} \int_{\mathbb{R}^d} e^{i(\Lambda^+ - \lambda^2) t} d^d p d^d x d^{N(N-1)} \xi d^2 \omega d t \mathcal{D}A.$$  \hspace{1cm} (7.10)

For simplicity we calculate this expression using stochastic fields. More precisely, we assume locally independent Gaussian fluctuations with the same variance $\sigma$ for all components of $E$ and $B$. According to a relation derived in appendix B for Weyl terms this is equivalent to averaging over constant random fields, i.e.

$$\int \ldots \mathcal{D}A \mapsto \frac{1}{(2\pi\sigma^2)^{3N(N-1)}} \int_{\mathbb{R}^{6N(N-1)}} \ldots e^{-(E^a E^a + B^a B^a)/(2\sigma^2)} d^{3N(N-1)} E d^{3N(N-1)} B.$$  \hspace{1cm} (7.11)

Now the total exponent is quadratic in $E$ and $B$ and an average over the fields yields,

$$\frac{1}{(2\pi\sigma^2)^{3N(N-1)}} \int_{\mathbb{R}^{6N(N-1)}} e^{-i2gs(E^a + B^a)} C^a t e^{-(E^a E^a + B^a B^a)/(2\sigma^2)} d^{3N(N-1)} E d^{3N(N-1)} B \quad \exp \left( -4\sigma^2 g^2 s^2 C^a C^a t^2 \right).$$  \hspace{1cm} (7.12)
Since $s^2$ and $C^aC^a$ correspond to the quadratic Casimir operators of SU(2) and SU($N$), respectively, they are constants, i.e. they depend only on $s$ and $J$ but not on $\omega$ and $\xi$. Hence,

\[
\langle \rho(\lambda) \rangle = \frac{VJ(2s + 1)}{\pi(2\pi \hbar)^d} \lambda \int_{\mathbb{R}^d} e^{(p_\mu p_\mu - \lambda^2)t - 4\sigma^2 g^2 s^2 C^a C^a t^2} dt d^d p,
\]

\[
= \frac{2VJ(2s + 1)}{(2\pi \hbar)^d\sqrt{2\pi v^2}} \lambda \int_{\mathbb{R}^d} e^{-\frac{1}{2\sigma^2}(p_\mu p_\mu - \lambda^2)^2} d^d p,
\]

\[
= \frac{4\pi^{d/2}VJ(2s + 1)}{(2\pi \hbar)^d \Gamma(\frac{d}{2})} \frac{\lambda}{\sqrt{2\pi v^2}} \int_0^\infty e^{-\frac{1}{\pi v^2}(p^2 - \lambda^2)^2} p^{d-1} dp,
\]

where we have introduced the abbreviation $v := \sqrt{8\sigma^2 |s| C^a C^a}$. This parameter, being proportional to the variance $\sigma$ and the coupling constant $g$, is a measure for the strength of the fields. The field free situation corresponds to $v = 0$ and one easily confirms that in this case (7.13) reduces to (7.2). Similarly, for large $\lambda$ the integral expression grows proportional to $\lambda^{d-2}$ and we once more obtain the free density (7.2). For arbitrary $\lambda$ the remaining integral in (7.13) can be expressed in terms of generalised Laguerre functions.

The most important observation is that for $\lambda \to 0$ the integral converges to a constant,

\[
\frac{1}{\sqrt{2\pi v^2}} \int_0^\infty e^{-\frac{p^4}{4\pi^2} v^{d-1}} dp = \frac{(2v^2)^{d/4-1/2}}{4\sqrt{\pi}} \Gamma(\frac{d}{2}).
\]

Thus, for small virtualities the mean density now grows linearly instead of being proportional to $\lambda^{d-1}$ as in the previous cases. For instance, for $d = 4$, where the behaviour changes from cubic to linear, this means a dramatic increase in the number of small eigenvalues. If the triple limit discussed here was related to the scenario of a strong external field then the linear density for small $\lambda$ could be interpreted as the density within the first Landau band, cf. [34]. In any case, the behaviour of the density resulting from our semiclassical approach shows some remarkable features which we discuss in the following.

### 7.4 Discussion of Weyl terms

We have calculated the (averaged) Weyl terms for the spectral density of the QCD Dirac operator in all 3 different semiclassical limits introduced in section 5. In two cases the resulting leading order mean density is just the mean density for the free Dirac operator. Only in the triple limit, $\hbar \to 0$, $J \to \infty$ and $s \to \infty$, have we observed a dependence on gauge fields and internal degrees of freedom. In particular we have derived an increase in the number of small eigenvalues.

In this last case we have replaced the QCD or Yang-Mills action in the path integral over the colour fields by a Gaussian measure, thus neglecting details of the gauge dynamics. The success of random matrix models and in particular related work on stochastic field theories [II, III] makes us believe that our results are nevertheless relevant for QCD.
To further study the accumulation of small eigenvalues as borne out by (7.13) we have to examine the integral expression

$$
\Phi_d(\lambda) := \frac{1}{\sqrt{2\pi v^2}} \int_0^\infty e^{-\frac{1}{2\pi^2}(p^2-\lambda^2)^2} p^{d-1} \, dp.
$$

(7.15)

Its value in dimension \( d = 4 \) is given by

$$
\Phi_4(\lambda) = \frac{1}{\sqrt{2\pi v^2}} \int_0^\infty e^{-\frac{1}{2\pi^2}(p^2-\lambda^2)^2} p^3 \, dp
= \frac{\sqrt{2v^2}}{4\sqrt{\pi}} e^{\frac{\lambda^4}{2v^2}} + \frac{\lambda^2}{4} \left( 1 + \text{erf} \left( \frac{\lambda}{\sqrt{2v^2}} \right) \right).
$$

(7.16)

For large \( \lambda \) the integral is dominated by the last term, i.e. it grows like \( \lambda^2/2 \), which restores the \( \lambda^3 \)-behaviour of the free mean density (7.3). On the other hand for small \( \lambda \) the integral is determined by the first term, giving rise to a linear spectral density, i.e. (for \( d = 4, J = 3, s = \frac{1}{2} \) and \( \hbar = 1 \))

$$
\langle \rho(\lambda) \rangle \approx \frac{3V}{(2\pi)^{5/2}} \lambda v e^{-\frac{\lambda^4}{2v^2}}.
$$

(7.17)

It is interesting to see how this expression changes under the variation of external parameters. If one wants to consider finite temperatures in a field theoretical setting one has to choose an asymmetric subset \( V \subset \mathbb{R}^4 \), say a box with lengths \( L_\mu \) with fixed \( L_4 \ll L_1, L_2, L_3 \). Then the inverse of \( L_4 \) is essentially the temperature, see e.g. [35]. Having one smaller dimension sets a natural scale for the small eigenvalues, namely \( \lambda \approx 2\pi/L_4 \) (cf. the eigenvalues of the free Dirac operator in a box with lengths \( L_4 \ll L_1, L_2, L_3 \)). Since it is this accumulation of small but non-zero eigenvalues which we want to investigate further, it is instructive to look at the averaged spectral density on this scale,

$$
\langle \rho \left( \frac{2\pi}{L_4} \right) \rangle \approx \frac{3V}{(2\pi)^{3/2}} \frac{v}{L_4} \exp \left( -\frac{(2\pi)^4}{2v^2L_4^4} \right).
$$

(7.18)

In fig. \( \Box \) we plot the scaled density

$$
r = \frac{L_4^3}{3V \sqrt{2\pi}} \langle \rho \rangle
$$

(7.19)

as a function of the scaled variance

$$
\zeta = \frac{vL_4^2}{(2\pi)^2},
$$

(7.20)

observing a curve which is reminiscent of a critical phenomenon with the spectral density itself playing the rôle of the order parameter. The behaviour for \( \zeta > \zeta_c \) would be interpreted as an indication for a non-zero density for small \( \lambda \), which, on the other hand, vanishes (exponentially) for \( \zeta < \zeta_c \). Thus, we are tempted to view \( \zeta_c \) as a value indicating a phase
transition. In terms of the original quantities, the variance $v$, measuring the (coupling) strength of the gauge fields, and the inverse temperature $L_4$, this implies the following. For fixed temperature, on the one hand, the phase transition would occur at a critical strength of the gauge fields with a vanishing order parameter for weak fields. For fixed $v$, on the other hand, we would observe the phase transition for a critical temperature with a non-vanishing order parameter at low temperatures only.

We find it remarkable that our simple semi-classical argument is capable of showing a behaviour which seems to hint at a critical phenomenon. In view of this we are tempted to put forward the following speculation. According to the Bank-Casher relation [31] a non-zero averaged density at $\lambda = 0$ results from the formation of a chiral condensate. However, in the derivation of the Banks-Casher relation the limit $\lambda \to 0$ may only be considered after one has first taken the infinite volume limit $V \to \infty$, for a suitably normalised expression, and then the chiral limit $m \to 0$ of vanishing sea quark mass(es). If one interchanges the latter two limits, i.e. if one performs $m \to 0$ before $V \to \infty$ then the chiral condensate vanishes, see e.g. [36]. In our semi-classical calculation we do not have a mass parameter, which we could vary accordingly. Nevertheless, we find an accumulation of small eigenvalues hinting at a critical phenomenon. Could it be that this relates to the formation of a chiral condensate? – However, we do not want to conceal that our discussion only takes into account the leading order Weyl term. Higher order semiclassical corrections to the mean density may contribute where the leading order term vanishes, as we will demonstrate for an example in section 8 and the above discussion also ignores the periodic orbit contributions to the spectral density. Recall that a trace formula provides a decomposition of the density of states $\rho$ into a mean term $\overline{\rho}$ and a periodic orbit sum $\rho_{osc}$, cf. (3.14), of which only the latter oscillates as a function of the virtuality $\lambda$. It is now natural to expect that after averaging over the gauge fields only the non-oscillating Weyl-term contributes to $\langle \rho \rangle$. The above discussion is based on this tacit assumption. In general we have

$$\langle \rho(\lambda) \rangle \sim \langle \overline{\rho}(\lambda) \rangle + \langle \rho_{osc}(\lambda) \rangle$$

(7.21)

and it is not guaranteed that $\langle \rho_{osc} \rangle = 0$ holds in a short virtuality interval close to $\lambda = 0$.

If besides the formation of the chiral condensate one also wants to explain the universal microscopic density, characteristic of the chiral ensembles of random matrix theory and observed in lattice calculations, cf. 3, a theory involving (correlations in) the periodic orbit contributions will be required. The analysis in 37, 38, 39, where the emergence of universal microscopic densities is discussed within a graph model should be viewed as a
guideline which, combined with our semiclassical approximations, would put the semiclassical understanding of spectral correlations in QCD on a similar level as in (low dimensional) quantum chaos.

8 Example: Fermions in SU(2)-fields on $\mathbb{T}^2$

As an illustration for the structure of semiclassical trace formulae and for the calculation of some of the contributions we discuss the example of the Dirac operator on a two-dimensional torus $\mathbb{T}^2$ with constant SU(2) gauge fields. With “constant field” we actually mean constant potentials $A^n_\mu$, which, due to the non-Abelian character, can give rise to a non-vanishing field strength, cf. (2.10). For this scenario we can analytically calculate the eigenvalues and derive an exact trace formula. The contributions to this trace formula are then compared to the corresponding semiclassical expressions. In order to keep the presentation simple we only discuss the case with fixed representations for the internal degrees of freedom, i.e. pure $\hbar \to 0$ asymptotics.

Consider a two-dimensional Euclidean Dirac operator, in external SU(2) fields,

$$\hat{D} = \left( \frac{\hbar}{i} \partial_\mu - \frac{\hbar}{2} g A_\mu \sigma \right) \gamma_\mu.$$  

(8.1)

In two dimensions the $\gamma$-matrices can be chosen of type $2 \times 2$, e.g

$$\gamma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \gamma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}.$$  

(8.2)

More precisely, we should write

$$\gamma_\mu = 1_2 \otimes \sigma^\mu$$  

(8.3)

and replace $\sigma$ in (8.1) by $\sigma \otimes 1_2$. As configuration space we choose a two dimensional box with lengths $L_\mu$ and periodic boundary conditions,

$$\Psi(x_1 + L_1, x_2) = \Psi(x_1, x_2), \quad \Psi(x_1, x_2 + L_2) = \Psi(x_1, x_2),$$  

(8.4)

i.e. we put the system on a torus $\mathbb{T}^2$. With the ansatz $\Psi(x) = u \exp(\frac{i}{\hbar} p_\mu x_\mu)$ the boundary conditions require

$$p_\mu = \frac{2\pi \hbar}{L_\mu} n_\mu, \quad n_\mu \in \mathbb{Z} \quad \text{(no summation convention!)}$$  

(8.5)

and the Dirac operator reduces to an ordinary $4 \times 4$ matrix, which has to be diagonalised. The eigenvalues are most conveniently determined via the square $\hat{D}^2$ of the Dirac operator. By a calculation similar to that in section 4 one finds

$$\lambda_\pm = \sqrt{p_\mu p_\mu + \frac{\hbar^2}{4} g^2 A_\mu A_\mu \pm \sqrt{\hbar^2 g^2 p_\mu p_\nu A_\mu A_\nu + \frac{\hbar^4}{4} g^4 (A_1 \times A_2)^2}}$$  

(8.6)
with $p_\mu$ as in (8.6). The spectrum is once more symmetric about $\lambda = 0$ and we only show the positive eigenvalues.

For the following we concentrate on the special case with

$\begin{align*}A_1 &= \begin{pmatrix} A_0 & 0 \\ 0 & 0 \end{pmatrix} , \\
A_2 &= \begin{pmatrix} 0 & A_0 \\ A_0 & 0 \end{pmatrix} \end{align*}$ \hfill (8.7)

in which yet none of the four contributions in (8.6) vanishes. Introducing the abbreviation $a := \hbar g A$ \hfill (8.8)

we have

$\lambda^\pm_n = \sqrt{p_\mu p_\mu + \frac{a^2}{2}} \pm \sqrt{a^2 p_\mu p_\mu + \frac{a^4}{4}} = \sqrt{p_\mu p_\mu + \frac{a^2}{4}} \pm \frac{a}{2}$. \hfill (8.9)

The spectral density (for positive virtuality) thus reads

$\rho(\lambda) = \sum_{n \in \mathbb{Z}^2} \left[ \delta(\lambda - \lambda^+_n) + \delta(\lambda - \lambda^-_n) \right]$. \hfill (8.10)

When the spectrum is already known exactly a trace formula can usually be derived by employing the Poisson summation formula which expresses a sum over integers by a sum over the Fourier transformed addends,

$\sum_{n \in \mathbb{Z}^d} f(n) = \sum_{k \in \mathbb{Z}^d} \int_{\mathbb{R}^d} f(n) e^{2\pi i k_\mu n_\mu} \, d^n n$. \hfill (8.11)

Doing this for the spectral density (8.10) and changing variables from $n_\mu$ to $p_\mu = L_\mu n_\mu/(2\pi \hbar)$ (no summation convention) we have

$\rho(\lambda) = \frac{L_1 L_2}{(2\pi \hbar)^2} \sum_{k \in \mathbb{Z}^2} \int_{\mathbb{R}^2} \left[ \delta(\lambda - \lambda^+_n) + \delta(\lambda - \lambda^-_n) \right] e^{\frac{\pi q_\mu p_\mu}{\hbar}} \, d^2 p$, \hfill (8.12)

where we have introduced

$q_\mu := k_\mu L_\mu$ (no summation convention). \hfill (8.13)

Introducing radial coordinates, $p = \sqrt{p_1^2 + p_2^2}$, $q = \sqrt{k_1^2 L_1^2 + k_2^2 L_2^2}$, we obtain

$\rho(\lambda) = \frac{L_1 L_2}{(2\pi \hbar)^2} \sum_{k \in \mathbb{Z}^2} \int_0^{2\pi} \int_0^\infty \left[ \delta(\lambda - \lambda^+(p)) + \delta(\lambda - \lambda^-(p)) \right] e^{\frac{\pi q p \cos \phi}{\hbar}} \, p \, dp \, d\phi$ \hfill (8.14)

with

$\lambda^\pm(p) := \sqrt{p^2 + \frac{a^2}{4}} \pm \frac{a}{2}$. \hfill (8.15)
The $\delta$-functions select $p = \sqrt{\lambda(\lambda - a)}$ and $p = \sqrt{\lambda(\lambda + a)}$ in the first and second term, respectively, and the $\phi$-integral yields a Bessel function. Hence,

$$\rho(\lambda) = \frac{L_1 L_2}{2\pi \hbar^2} \left[ \Theta(\lambda - a) \left( \lambda - \frac{a}{2} \right) + \left( \lambda + \frac{a}{2} \right) \right]$$

$$+ \frac{L_1 L_2}{2\pi \hbar^2} \sum_{k \in \mathbb{Z}^2, \ k_{\mu} \neq 0} \left[ \Theta(\lambda - a) \left( \lambda - \frac{a}{2} \right) J_0 \left( \frac{q}{\hbar} \sqrt{\lambda(\lambda - a)} \right) + \left( \lambda + \frac{a}{2} \right) J_0 \left( \frac{q}{\hbar} \sqrt{\lambda(\lambda + a)} \right) \right],$$

(8.16)

where we have separated the mean density,

$$\overline{\rho}(\lambda) = \frac{L_1 L_2}{2\pi \hbar^2} \left[ \Theta(\lambda - a) \left( \lambda - \frac{a}{2} \right) + \left( \lambda + \frac{a}{2} \right) \right],$$

(8.17)

which derives from $k_1 = k_2 = 0$. Notice that – since $a$ is of order $\hbar$, cf. (8.8) – in leading semiclassical order the mean density is given by

$$\overline{\rho}(\lambda) \sim \frac{L_1 L_2}{\hbar^2} \lambda$$

(8.18)

which agrees with (7.2) with $d = 2$, $J = 2$ and, formally, $s = 0$, since there is no dynamical spin in $1 + 1$ dimensions. For small $\lambda$, however, the exact mean density reads

$$\overline{\rho}(\lambda) = \frac{L_1 L_2}{2\pi \hbar^2} \left( \lambda + \frac{a}{2} \right), \quad \lambda < \frac{a}{2},$$

(8.19)

giving rise to a non-zero value at $\lambda = 0$ which cannot be seen by leading order semiclassical asymptotics.

Before we can compare the periodic orbit sum with semiclassical theories we have to expand the result asymptotically for $\hbar \rightarrow 0$. To this end recall that $a$ is of order $\hbar$. Using the asymptotic behaviour of the Bessel function we obtain

$$J_0 \left( \frac{q}{\hbar} \sqrt{\lambda(\lambda + a)} \right) \sim \sqrt{\frac{2}{\pi q\sqrt{\lambda(\lambda + a)}}} \cos \left( \frac{q}{\hbar} \sqrt{\lambda(\lambda + a) - \frac{\pi}{4}} \right)$$

$$\sim \sqrt{\frac{2}{\pi q\lambda}} \cos \left( \frac{gA}{2} \right) \cos \left( \frac{q}{\hbar} \lambda - \frac{\pi}{4} \right),$$

(8.20)

and thus the semiclassical periodic orbit sum reads

$$\rho_\text{osc}(\lambda) \sim \frac{L_1 L_2}{(\pi \hbar)^{3/2}} \sum_{k \in \mathbb{Z}^2, \ k_{\mu} \neq 0} \sqrt{\frac{2\lambda}{q}} \cos \left( \frac{gA}{2} \right) \cos \left( \frac{q}{\hbar} \lambda - \frac{\pi}{4} \right),$$

(8.21)

Let us analyse some contributions. The rapidly oscillating term, the last cosine, contains the argument $q\lambda/\hbar$. The geometric length of a periodic orbit on the torus with winding
numbers \( k_1 \) and \( k_2 \) is given by \( q = \sqrt{k_1^2 L_1^2 + k_2^2 L_2^2} \) and with the Hamiltonian (5.3) the action of this orbit reads
\[
\oint p_\mu \, dx_\mu = \lambda q .
\] (8.22)
The colour field shows up only in the argument of the other cosine, which has to derive from colour precession. For SU(2) fields the colour transport equation (5.9) reads
\[
\dot{d} - \frac{g p_\mu}{2 \lambda} A_\mu \sigma \, d = 0.
\] (8.23)
With the choice (8.7) this becomes
\[
\dot{d} - \frac{gA}{2 \lambda} p_\mu \sigma^{\mu} \, d = 0
\] (8.24)
which we have to integrate with initial condition \( d(0) = 1 \) up to the period of a periodic orbit on the torus. Due to the Hamiltonian (5.3) the period equals the geometric length of the orbit and thus the solution is given by
\[
d = \exp \left( -\frac{i gA}{2 \lambda} p_\mu \sigma^{\mu} q \right).
\] (8.25)
The trace of this expression,
\[
\text{tr} \, d = 2 \cos \left( \frac{gA \sqrt{p_\mu p_\mu}}{2 \lambda} q \right) = 2 \cos \left( \frac{gAq}{2} \right),
\] (8.26)
enters as a weight factor in the periodic orbit sum, and indeed gives rise to the aforesaid cosine factor. The remaining factors can be calculated with standard methods, see e.g. [40, 41] for the general case or [12, section 3.6.2] for a related example.

9 Conclusions and outlook

We have discussed the semiclassical structure of the QCD Dirac operator, and in particular the interplay of three semiclassical parameters, namely Planck’s constant \( \hbar \), and the spin and colour quantum numbers \( s \) and \( J \), respectively. This situation allows for various semiclassical scenarios, with combined semiclassical asymptotics considered. We have encountered a rich family of classical dynamics of translational, colour and spin degrees of freedom, whose mutual coupling depends on how the semiclassical limit is taken.

The influence of these different types of dynamics in semiclassical trace formulae has been discussed and, in particular, we have analysed the behaviour of the Weyl term, the mean density of states, in different semiclassical scenarios. Based on this analysis we have critically evaluated which of the semiclassical scenarios has the potential of describing the spectrum of the QCD Dirac operator near zero virtuality, leading us to a speculative discussion of the mechanism behind the chiral phase transition. We certainly do not want
to overstate this speculation, a definite statement requires further work, as indicated in section 7.4. There are various directions of research which would naturally continue the present analysis.

So far we have mainly discussed the Weyl term, which in a trace formula gives rise to the mean density of states, but not the periodic orbit sum which is responsible for spectral correlations. An analysis based on periodic orbits should, e.g., lead to a semiclassical theory for the universal microscopic spectral density of the QCD Dirac operator as described by chiral RMT. Moreover, such an approach would also describe deviations from RMT behaviour on large spectral scales, cf. saturation effects as described in [9], and thus potentially provide the missing link asked for in the introduction, which would make it possible to directly use RMT information when calculating fermionic determinants. Here, one should keep in mind that the equivalent of the Thouless energy [1, 5] sets another scale. It is not present in Dirac spectra for frozen gauge fields, but it is an ensemble effect resulting from the propagation of the gauge fields.

Besides the various semiclassical scenarios which we have described in this article there is an additional strategy for taking the semiclassical limit of multi-component wave equations. In this approach one does not treat the matrix degrees of freedom, i.e. colour and spin, dynamically but rather considers polarised Hamiltonians, which describe a particle with the spin or colour projection locked to the “direction” of the external gauge field, see e.g. [2, 3, 13, 14]. Such an approach may also prove useful in the case of the QCD Dirac operator.

In lattice gauge theory, which we have referred to in various places, the Dirac operator is implemented as a difference instead of a differential operator. This has consequences which could also be analysed within the semiclassical picture. On the one hand discretisation leads to a modified dispersion relation, i.e. to different classical Hamiltonians $\Lambda$. Roughly speaking, the momenta are replaced by suitably normalised sines of momenta, which in the semiclassical picture changes both, Weyl terms and the periodic orbit structure. On the other hand the discretised theory lives in a finite dimensional Hilbert space. In a semiclassical context the dimension of this Hilbert space also becomes a semiclassical parameter (cf. the theory of quantised maps, [15], see also [16] for an overview), which would allow for the continuum limit to be discussed on a semiclassical footing.

Acknowledgement

We thank Tilo Wettig for numerous stimulating discussions and helpful remarks. Moreover we benefited from useful discussions with Johan Bijnens, Jens Bolte, Dmitri Diakonov, Stephen Fulling, and Ed Shuryak. TG acknowledges support from Det Svenska Vetenskapsrådet and SK is grateful for support from Deutsche Forschungsgemeinschaft under grant no. KE 888/1-1 and also from Crafoordska Stiftelsen under grant no. 20020681.
A Projected transport equations

In order to calculate the projected transport transport equations \((4.10)\) for the \(2 \times 2\) matrices \(b_{\pm}\) we have to evaluate the expressions,

\[
V_{\pm}^\dagger \gamma_{\mu} V_{\pm} (\partial_{\mu} b_{\pm}) , \quad V_{\pm}^\dagger V_{\pm} \frac{\partial b_{\pm}}{\partial t} , \quad V_{\pm}^\dagger \frac{\partial}{\partial t} V_{\pm} \quad \text{and} \quad V_{\pm}^\dagger \gamma_{\mu} \partial_{\mu} V_{\pm}^\dagger . \tag{A.1}
\]

We begin with the terms where the derivatives act on \(b_{\pm}\):

\[
V_{\pm}^\dagger \gamma_{\mu} a_{\mu} V_{\pm} = \frac{1}{2} \left( \mathbb{1}_2 , \left( \begin{array}{cc}
\pi_4 - i\sigma \pi & 0 \\
0 & a_4 + i\sigma a
\end{array} \right) \right) \left( \frac{1}{\Lambda} \right) \left( \begin{array}{cc}
\pi_4 + i\sigma \pi & 0 \\
0 & a_4 - i\sigma a
\end{array} \right) \left( \frac{1}{\Lambda} \right) = \frac{1}{2} \left[ \frac{\pi_4 - i\sigma \pi}{\Lambda} (a_0 + i\sigma a) + (a_0 - i\sigma a) \frac{\pi_4 + i\sigma \pi}{\Lambda} \right] \tag{A.2}
\]

\[
\Rightarrow \quad V_{\pm}^\dagger \gamma_{\mu} V_{\pm} (\partial_{\mu} b_{\pm}) = \frac{\partial \Lambda}{\partial p_{\mu}} (\partial_{\mu} b_{\pm}) .
\]

Due to normalisation, \(V_{\pm}^\dagger V_{\pm} = \mathbb{1}_2\), the second contribution is trivial, and together with the first one yields

\[
V_{\pm}^\dagger V_{\pm} \frac{\partial b_{\pm}}{\partial t} + V_{\pm}^\dagger \gamma_{\mu} V_{\pm} (\partial_{\mu} b_{\pm}) = \left( \frac{\partial}{\partial t} + \frac{\pi_{\mu}}{\Lambda} \partial_{\mu} \right) b_{\pm} : =\dot{b}_{\pm} , \tag{A.3}
\]

where the dot denotes a derivative along the flow with Hamiltonian \(\Lambda\). The remaining two terms yield

\[
V_{\pm}^\dagger \frac{\partial}{\partial t} V_{\pm} = V_{\pm}^\dagger \left( \frac{1}{\sqrt{2}} \left( \begin{array}{cc}
\frac{\partial \pi_4}{\partial t} + i\sigma \frac{\partial \pi}{\partial t} & 0 \\
0 & \frac{\partial \pi_4 - \sigma \pi}{\partial t}
\end{array} \right) - \frac{1}{\Lambda} \left( \pi_4 + i\sigma \pi \right) \frac{\partial \pi_4}{\partial t} \right) = \frac{1}{2} \left[ \frac{1}{\Lambda^2} (\pi_4 - i\sigma \pi) \left( \frac{\partial \pi_4}{\partial t} + i\sigma \frac{\partial \pi}{\partial t} \right) - \frac{1}{\Lambda^2} \pi_4 \frac{\partial \pi_4}{\partial t} \right] \tag{A.4}
\]

and

\[
V_{\pm}^\dagger \gamma_{\mu} \partial_{\mu} V_{\pm} = V_{\pm}^\dagger \left( \frac{1}{\sqrt{2}} \left( \frac{\partial}{\partial \xi} - i\sigma \nabla \right) \frac{\pi_4 + i\sigma \pi}{\Lambda} \right) = \frac{1}{2} \left[ \frac{1}{\Lambda^3} \pi_4 \left( \frac{\partial}{\partial \xi} \pi_4 - i\sigma \nabla \pi_4 \right) \right] \tag{A.5}
\]

\[= \frac{1}{2} \left[ \frac{1}{\Lambda^3} \pi_4 \left( \frac{\partial}{\partial \xi} \pi_4 - i\sigma \nabla \pi_4 \right) \right] . \]
Using the Hamilton-Jacobi equation (4.8) we can derive the useful relation
\[
\frac{\partial \pi_\mu}{\partial t} = \frac{\partial}{\partial t} \partial_\mu S = \partial_\mu \frac{\partial S}{\partial t} = -\frac{\pi_\mu}{\Lambda} \partial_\mu \pi_\nu, \tag{A.6}
\]
which we now use "backwards",
\[
-\frac{1}{\Lambda^3} \pi_\mu (\partial_0 \pi_\mu - i \sigma \nabla \pi_\mu) = \frac{1}{\Lambda^2} \left( \frac{\partial \pi_4}{\partial t} - i \sigma \frac{\partial \pi}{\partial t} \right). \tag{A.7}
\]
Hence,
\[
V^+ \gamma_\mu \partial_\mu V^+ = \frac{1}{2} \left[ \frac{\pi_\mu}{\Lambda^2} \frac{\partial \pi_\mu}{\partial t} + \frac{\partial_\mu \pi_\mu}{2 \Lambda} + \frac{i \sigma \pi_\mu}{2 \Lambda} \left( \partial_\mu \pi - \nabla \pi_4 + \nabla \pi + i \sigma (\nabla \times \pi) \right) \right]. \tag{A.8}
\]
and added to (A.4) a couple of terms drop out,
\[
\begin{align*}
V^+ \frac{\partial}{\partial t} V^+ + V^+ \gamma_\mu \partial_\mu V^+ = & \pi_\mu \frac{\partial \pi_\mu}{\partial t} + \frac{\partial_\mu \pi_\mu}{2 \Lambda} + \frac{i \sigma (\partial_\mu \pi - \nabla \pi_4 + \nabla \pi + i \sigma (\nabla \times \pi))}{2 \Lambda}
\end{align*}
\]
With
\[
\partial_0 \pi - \nabla \pi_4 = \partial_0 (\nabla S - gA) - \nabla (\partial_0 S - gA_0) = -g (\partial_0 A - \nabla A_0) = -gE \tag{A.10}
\]
and
\[
\nabla \times \pi = \nabla \times (\nabla S - gA) = -g \nabla \times A = -gB \tag{A.11}
\]
the non-scalar terms finally are given by
\[
-\frac{ig}{2 \Lambda} \sigma (B + E). \tag{A.12}
\]
The analogous calculation for \( b_- \) reads
\[
V_- \gamma_\mu a_\mu V_- = \frac{1}{2} \begin{pmatrix} \pi_\mu + i \sigma \pi_\mu & -a_\mu \end{pmatrix} \begin{pmatrix} 0 & a_0 - i \sigma a \\ a_0 + i \sigma a & 0 \end{pmatrix} \begin{pmatrix} \frac{\pi_4 - i \sigma \pi}{\Lambda} \\ -\frac{a}{2} \end{pmatrix}
\]
\[
= \frac{1}{2} \begin{pmatrix} -\frac{\pi_4 + i \sigma \pi}{\Lambda} (a_0 - i \sigma a) - (a_0 + i \sigma a) \frac{\pi_4 - i \sigma \pi}{\Lambda} \\ -\frac{\pi_4 + i \sigma \pi}{\Lambda} (a_0 - i \sigma a) - (a_0 + i \sigma a) \frac{\pi_4 - i \sigma \pi}{\Lambda} \end{pmatrix}
\]
\[
= -\frac{1}{2 \Lambda} (2a_0 \pi_4 + 2a \pi) = -\frac{\pi_\mu a_\mu}{\Lambda} \tag{A.13}
\]
\[
\implies V_- \gamma_\mu V_- (\partial_\mu b_-) = \frac{\partial \lambda_-}{\partial p_\mu} (\partial_\mu b_-)
\]
\[
\implies V_- \frac{\partial b_-}{\partial t} + V_- \gamma_\mu V_- (\partial_\mu b_-) = \left( \frac{\partial}{\partial t} + \frac{\pi_\mu}{\Lambda} \partial_\mu \right) b_- =: b_, \tag{A.14}
\]
where the dot now denotes a derivative along the flow with Hamiltonian \( \Lambda^- \). The other two terms yield

\[
V^\dagger \frac{\partial}{\partial t} V = V^\dagger \frac{1}{\sqrt{2}} \left( \frac{\partial \pi_4}{\partial t} - i\sigma \frac{\partial \pi}{\partial t} - \frac{\pi_4 - i\sigma \pi}{\Lambda} \frac{\partial \pi}{\partial t} \right)
\]

\[
= \frac{1}{2} \left[ \frac{1}{\Lambda^2} \left( \pi_4 + i\sigma \pi \right) \left( \frac{\partial \pi_4}{\partial t} - i\sigma \frac{\partial \pi}{\partial t} \right) - \frac{1}{\Lambda^2} \frac{\partial \pi}{\partial t} \right]
\]

\[
= \frac{1}{2\Lambda^2} \left[ -i\pi_4 \frac{\partial \pi}{\partial t} + i\sigma \pi \frac{\partial \pi}{\partial t} + i\sigma \left( \pi \times \frac{\partial \pi}{\partial t} \right) \right]
\]

(A.14)

and

\[
V^\dagger \gamma_\mu \partial_\mu V = V^\dagger \frac{1}{\sqrt{2}} \left( \left( \partial_0 + i\sigma \nabla \right) \frac{\pi_4 - i\sigma \pi}{\Lambda} \right)
\]

\[
= -\frac{1}{2} \left[ \left( \partial_0 + i\sigma \nabla \right) \frac{\pi_4 - i\sigma \pi}{\Lambda} \right]
\]

\[
= -\frac{1}{2} \left[ -\frac{1}{\Lambda^2} \pi_\mu \left( \partial_0 \pi_\mu + i\sigma \nabla \pi_\mu \right) \left( \pi_4 - i\sigma \pi \right) + \frac{1}{\Lambda} \left( \partial_0 + i\sigma \nabla \right) \right].
\]

(A.15)

Again we use the Hamilton-Jacobi equation (4.8),

\[
\frac{\partial \pi_\mu}{\partial t} = \frac{\partial}{\partial \mu} S = \partial_\mu \frac{\partial S}{\partial t} = -\partial_\mu \Lambda = \partial_\mu = \frac{\pi_\nu}{\Lambda} \partial_\mu \pi_\nu,
\]

(A.16)

concluding that

\[
V^\dagger \gamma_\mu \partial_\mu V = -\frac{1}{2} \left[ -\frac{1}{\Lambda^2} \left( \pi_4 \frac{\partial \pi_4}{\partial t} - i\sigma \pi \frac{\partial \pi_4}{\partial t} + i\pi_4 \color{red}{\partial_\mu} \frac{\partial \pi}{\partial t} + \color{red}{\pi} \frac{\partial \pi}{\partial t} + i\sigma \left( \color{red}{\partial \pi} \times \frac{\partial \pi}{\partial t} \right) \right)
\]

\[
+ \frac{1}{\Lambda} \left( \partial_0 \pi_4 - i\sigma \partial_0 \pi + i\sigma \nabla \pi_4 + \nabla \pi + i\sigma \left( \nabla \times \pi \right) \right)
\]

(A.17)

and together with (A.14) we obtain

\[
V^\dagger \frac{\partial}{\partial t} V + V^\dagger \gamma_\mu \partial_\mu V = \frac{\pi_\mu}{2\Lambda} \left( \frac{\partial \pi_4}{\partial t} - \frac{\partial \pi_4}{\partial t} \right) + i\sigma \left( \partial_0 \pi - \nabla \pi_4 - \nabla \times \pi \right).
\]

(A.18)

With (A.10) and (A.11) the non-scalar terms in this case read

\[
-\frac{i\sigma}{2\Lambda} \left( \sigma \left( B - E \right) \right) = -\frac{i\sigma}{2\Lambda} \left( \sigma \left( E - B \right) \right).
\]

(A.19)
Local Gaussian fluctuations vs. constant random fields

We show that averaging Weyl terms over stochastic fields – more precisely, independent locally Gaussian fields – is computationally equivalent to averaging over constant Gaussian fields.

When averaging a mean density \( \rho \), given by a Weyl term, over fields, say \( B(x) \), it is crucial that the Weyl term is given by an integral over the position variable \( x \), see (B.1). Therefore, let us now consider expressions of the form

\[
I := \int \int f(B(x), x) \, dx \, DB
\]  

with a Gaussian measure \( DB \). Think of the functional integral as defined by a suitably normalised continuum limit \( N \to \infty \) of its discretised analogue,

\[
I_N := \int \cdots \int \sum_{j=1}^{N} f(B_j, x_j) \prod_{k=1}^{N} \frac{e^{-B_k^2/(2\sigma^2)}}{\sqrt{2\pi\sigma^2}} \, dB_k,
\]

with lattice points \( x_j \) and \( B(x_j) =: B_j \). Then we obtain

\[
I_N = \sum_{j=1}^{N} \int f(B_j, x_j) \frac{e^{-B_j^2/(2\sigma^2)}}{\sqrt{2\pi\sigma^2}} \, dB_j = \int \sum_{j=1}^{N} f(B, x_j) \frac{e^{-B^2/(2\sigma^2)}}{\sqrt{2\pi\sigma^2}} \, dB.
\]

In the continuum limit we have thus derived the relation

\[
\int \int f(B(x), x) \, dx \, DB = \int \int f(B, x) \frac{e^{-B^2/(2\sigma^2)}}{\sqrt{2\pi\sigma^2}} \, dx \, dB,
\]

i.e. when interested in local Gaussian fluctuations in section 7.3 we may average over constant random fields instead.

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