A GENERALIZATION OF THE METHODS OF BRASS, HARBORTH, AND NIENBORG

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Abstract. In 1995, Brass, Harborth and Nienborg disproved a conjecture of Erdős when they showed that a $C_4$-free subgraph of the hypercube, $Q_n$, can have at least $(\frac{1}{2} + \omega(1))e(Q_n)$ edges. In this paper, we generalize the idea of Brass, Harborth and Nienborg to provide good constructions of $Q_3$-free subgraphs of $Q_n$ for some small values of $n$.

1. Main Idea

The idea is to generalize the idea of Brass, Harborth and Nienborg [1] and apply it to other graphs. We start with a base graph $G_k \subset Q_k$ which is $H$-free as well as a colored $Q_m$ graph. We then split $G_k$ into $G_k^{(1)}$ and $G_k^{(2)}$ with a (not necessarily complete) parallel set of edges in between them. We then consider our colored graph $Q_m$. Since it is bipartite, we replace each vertex in one partite set with $G_k^{(1)}$ and the other with $G_k^{(2)}$. We then use the colors of the edges in $Q_m$ to somehow determine which edges to include in our new $G_{k+m-1} \subset Q_{k+m-1}$ which will be $H$-free, assuming our algorithm works correctly.

2. The BHN construction

We start with an $aoe$-coloring of $Q_m$. All $a$-colored edges are replaced with a copy of $P$. For the $e$- and $o$-colored edges, we note that $G_k^{(i)}$ is again bipartite. We divide each of these into bipartite sets and add all edges in one bipartite set wherever we see an $e$-colored edge and all edges in the other set when we see an $0$-colored edge. The new graph will be $C_4$-free.

It is interesting to note that assuming $G_k$ is maximal (but not necessarily maximum), the resulting construction will also be maximal (I think).

2.1. Why it works. We wish to try and find a $C_4$ in our newly constructed graph. Clearly, such a $C_4$ cannot be a subgraph of $G_k^{(i)}$ since $G_k^{(i)}$ is $C_4$-free.

Next, suppose our $C_4$ contains one edge in some $G_k^{(1)}$ and another in some $G_k^{(2)}$ with two edges going in between them. If the two edges came from an $a$-labeled edge in $G_m$, then notice that $G_k^{(1)} \cup G_k^{(2)} \cup P$ forms a copy of $G_k$ and so there is no $C_4$. If it comes from an $e$- or o-colored edge, then we see that since the two edges are adjacent in $G_k^{(i)}$, they must come from different
partite sets and so one of the edges in our e- or o-edge is not present in our new graph.

Finally, the only case left is that all 4 of our vertices come from different copies of $G_k^{(i)}$. Then our each of our vertices are in the same partite set and this graph forms a 4-cycle in $Q_m$ so it contains at least one e-edge and one o-edge. But since these contain edges from different partite sets, one of them is absent.

3. First Idea

We wish to copy the idea of Brass, Harborth and Nienborg exactly. The only difference is that we now require two things from our aeo-coloring of $G_m$:

1. Each $Q_3$ contains at least one e-edge and one o-edge.
2. Each $Q_2$ contains at least one e-edge or one o-edge.

To see that this creates a $Q_3$-free subgraph isn’t much harder than before.

1. No $Q_3$ can be contained entirely inside a $G_k^{(i)}$.

2. If our $Q_3$ is contained within a single edge of $Q_m$ then if that edge was an a-edge, we once again have a copy of $G_k$. If it’s an e- or o-edge, it is actually missing not one by two of the edges in between.

3. If our $Q_3$ is contained within a $C_4$ of $Q_m$, then it has at least one e- or o-edge. Since there are two possible edges running between those in our new graph and they come from different partite sets, one of them is not in our new graph.

4. Finally, the only case is that our $Q_3$ is part of a $Q_3$ in $G_m$, but as in the last case, this contains one e- and one o-edge and so one of these is not an edge.

3.1. Why this doesn’t work. Since $\text{ex}(Q_3, Q_2)=9$, each cube in our graph must contain at least 3 e- or o-edges. Thus, our $G_m$ must be at least 1/4 non-a edges. Thus, at every step we are omitting at least 1/8 of the possible edges within these edges. This means that we would need to find a parallel class with at least 5/6 of the possible edges to stay above 3/4 of the total edges in the graph.
3.2. **Salvaging Something?** Although this doesn’t work asymptotically, we may be able to use it for small values of \( n \). Furthermore, since the construction is not likely to be maximal (see case 2 from before) we may be able to go back and add edges by hand. This should be a much simpler job for my java program.

We can *aeo*-color a \( Q_3 \) by removing the edges \([00*], [1*1] \) and \([1*0]\) and assigning two of them color \( e \) and one of them color \( o \). Let \( G_k \) be a \( Q_3 \)-free subgraph of \( Q_k \). Let \( e_k \) be the number of edges in this graph and \( p_k \) be the number of edges in a parallel class. (We wish to choose the parallel class with the largest number of edges or we get nothing).

Using our technique, we can create a \( Q_3 \)-free \( G_{k+2} \). When creating this, we will have exactly 4 \( G_k^{(1)} \)s and 4 \( G_k^{(2)} \)s. The total number of edges in these is \( 4(e_k - p_k) \). We will be adding \( 9p_k \) edges corresponding to our \( a \)-colored edges and \( 3 \cdot 2^{k-2} \) edges corresponding to our \( e \)- and \( o \)-colored edges. This gives the recurrence

\[
e_{k+2} = 4e_k + 5p_k + 3 \cdot 2^{k-2}.
\]

For \( k = 5 \), we may use the unique construction given by Offner\[2\] which has 72 out of 80 possible edges and a parallel class with all 16 edges. Plugging this into the equation above gives \( e_7 = 392 \). Since there are 448 total edges in \( Q_7 \), using that notation this gives \( c(Q_3, 7) \leq 56 \), which improves at least on the 1993 bound by Graham, Harary, Livingston and Stout\[3\] of 62.

Applying this to \( Q_4 \) gives us \( c(Q_3, 6) \leq 24 \), which is worse than the exact result of 22. In a \( Q_3 \)-free \( Q_6 \) omitting 22 edges, we must have one parallel class with \( \lceil 22/6 \rceil = 3 \) omitted edges. So if we take this construction we have \( e_6 = 192 - 22 = 70 \) and \( p_6 = 32 - 3 = 29 \) which gives \( e_8 = 873 \) and so \( c(Q_3, 8) \leq 1024 - 873 = 151 \).

I have no idea how this compares to known bounds.

3.2.1. **A simpler construction.** Actually, an easier way to do this is to just take a \( C_4 \) with one edge labeled \( e \) and the other 3 labeled \( a \). This gives the recurrence

\[
e_{k+1} = 2(e_k - p_k) + 3p_k + 2^{k-2} = 2e_k + p_k + 2^{k-2}.
\]

Plugging in our result for \( e_6 \) above with \( p_6 = 29 \), this gives \( e_7 = 395 \) and so \( c(Q_3, 7) \leq 53 \). Then \( p_7 \geq 57 \) and so \( c(Q_3, 8) \leq 145 \).

Edit: This actually gives \( e_7 = 385 \), so the result is worse.

3.2.2. **A more elaborate construction.** Another *aeo*-coloring of \( Q_4 \) is possible using the following construction:

| \( e \) edges | \(*00*) | \(*11*) | \[1*0]\] | \[0*1]\] | \( o \) edges | \[00*\] | \[11*\] | \[101*\] | \[010*\] |
|---|---|---|---|---|---|---|---|---|---|
| | | | | | | | | | |
and the rest of the edges are colored with $a$. This yields the recursion:

$$e_{k+3} = 8e_k + 16p_k + 2^{k+1}.$$ 

Also note at this point that we may replace $e_k$ with the number of non-edges in $G_k$ and $p_k$ with the number of non-edges in our parallel class and the recursive formula remains the same. Recall that $e_4 = 3$ and $p_4 = 0$ (using the number of omitted edges) so this gives $e_7 = 8 \cdot 3 + 32 = 56$ gives us the same upper bound for $c(Q_3, 7)$ as before. However, for $k = 5$ we have $e_k = 8$ and $p_k = 0$ so we get $c(Q_3, 8) \leq 128$ which is a substantial improvement on the bound above. This also gives $c(Q_3, 9) \leq 352$.

I ran the $G_7$ construction through my program which verified that it was indeed $Q_3$-free with omitted edges:

$$[*000000], [000*010], [0000*11], [*000011], [000*101], [*000101], [000110], [0001*00], [*001001], [*001010], [*001100], [*001111], [001000*], [0*10001], [0*10010], [00101*0], [0*10100], [0*10111], [0*11000], [00110*1], [0*11011], [0*11101], [001111*], [0*11110], [010000*], [01001*0], [01010*1], [010111*], [0110*01], [0111*10], [0111*00], [10000*0], [10001*0], [10010*1], [100111*], [101*010], [1010*11], [1011*01], [110*001], [110*010], [11*0001], [11*0010], [11*0100], [11*0101], [11*0111], [1101*00], [11*1000], [11*1011], [11*1101], [11*1110], [11100*0], [11101*0], [11110*1], [111111*]$$

I tried to run a perturbation algorithm on this graph to see if we could perhaps do slightly better by first removing 2 edges and then adding edges to the resulting graph, but after 20 hours, my computer spat out the same graph. To try first deleting 3 edges, I estimate that it would take my machine a bit over 2 years.

We may also give the more general recurrence. Since there are $k$ parallel classes and $e_k$ total edges in our $G_k$, we may find $p_k \leq \lfloor e_k/k \rfloor$ and so

$$e_{k+3} \leq 8e_k + 16\lfloor e_k/k \rfloor + 2^{k+1}.$$ 

Here is a table of bounds up to the point where we start getting a proportion of edges larger than 1/4, taking the lower bound from [2]:

| $k$ | Bound |
|-----|-------|
| 2   | 4     |
| 3   | 13    |
| 4   | 56    |
| 5   | 128   |
| 6   | 352   |
| \( k \) | LB  | UB  | \( \frac{\text{LB}}{e(Q_k)} \approx 0.11607 \) | \( \frac{\text{UB}}{e(Q_k)} \approx 0.115995 \) |
|-----|-----|-----|---------------------------------|---------------------------------|
|    7 | 52  | 56  | \( \frac{52}{438} \approx 0.11607 \) | \( \frac{56}{438} = 0.125 \) |
|    8 | 119 | 128 | \( \frac{119}{1,024} \approx 0.11621 \) | \( \frac{128}{1,024} = 0.125 \) |
|    9 | 268 | 352 | \( \frac{268}{2,304} \approx 0.11632 \) | \( \frac{352}{2,304} \approx 0.15278 \) |
|   10 | 596 | 832 | \( \frac{596}{5,120} \approx 0.11641 \) | \( \frac{832}{5,120} \approx 0.16250 \) |
|   11 | 1,312 | 1,792 | \( \frac{1,312}{11,264} \approx 0.11648 \) | \( \frac{1,792}{11,264} \approx 0.15909 \) |
|   12 | 2,863 | 4,464 | \( \frac{2,863}{24,576} \approx 0.11650 \) | \( \frac{4,464}{24,576} \approx 0.18164 \) |
|   13 | 6,204 | 10,032 | \( \frac{6,204}{53,248} \approx 0.11651 \) | \( \frac{10,032}{53,248} \approx 0.18840 \) |
|   14 | 13,363 | 21,024 | \( \frac{13,363}{114,688} \approx 0.11652 \) | \( \frac{21,024}{114,688} \approx 0.18331 \) |
|   15 | 28,635 | 49,856 | \( \frac{28,635}{245,760} \approx 0.11652 \) | \( \frac{49,856}{245,760} \approx 0.20286 \) |
|   16 | 61,088 | 108,976 | \( \frac{61,088}{524,288} \approx 0.11652 \) | \( \frac{108,976}{524,288} \approx 0.20786 \) |
|   17 | 129,812 | 224,976 | \( \frac{129,812}{1,114,112} \approx 0.11652 \) | \( \frac{224,976}{1,114,112} \approx 0.20193 \) |
|   18 | 274,896 | 517,552 | \( \frac{274,896}{2,739,296} \approx 0.11652 \) | \( \frac{517,552}{2,739,296} \approx 0.21937 \) |
|   19 | 580,336 | 1,111,856 | \( \frac{580,336}{4,980,736} \approx 0.11652 \) | \( \frac{1,111,856}{4,980,736} \approx 0.22323 \) |
|   20 | 1,221,760 | 2,273,680 | \( \frac{1,221,760}{10,485,760} \approx 0.11652 \) | \( \frac{2,273,680}{10,485,760} \approx 0.21684 \) |
|   21 | 2,565,696 | 5,124,736 | \( \frac{2,565,696}{22,020,996} \approx 0.11652 \) | \( \frac{5,124,736}{22,020,996} \approx 0.23273 \) |
|   22 | 5,375,744 | 10,879,712 | \( \frac{5,375,744}{46,137,344} \approx 0.11652 \) | \( \frac{10,879,712}{46,137,344} \approx 0.23581 \) |
|   23 | 11,240,192 | 22,105,536 | \( \frac{11,240,192}{96,368,992} \approx 0.11652 \) | \( \frac{22,105,536}{96,368,992} \approx 0.22915 \) |
|   24 | 23,457,722 | 49,096,752 | \( \frac{23,457,722}{201,926,092} \approx 0.11652 \) | \( \frac{49,096,752}{201,926,092} \approx 0.24387 \) |
|   25 | 48,870,400 | 103,338,816 | \( \frac{48,870,400}{419,430,400} \approx 0.11652 \) | \( \frac{103,338,816}{419,430,400} \approx 0.24638 \) |
|   26 | 101,650,432 | 208,999,264 | \( \frac{101,650,432}{872,415,232} \approx 0.11652 \) | \( \frac{208,999,264}{872,415,232} \approx 0.23956 \) |
|   27 | 211,120,128 | 459,059,616 | \( \frac{211,120,128}{1,811,939,628} \approx 0.11652 \) | \( \frac{459,059,616}{1,811,939,628} \approx 0.25335 \) |
We can also take the construction of a $Q_2$-free subgraph of $Q_5$ and divide the non-edges into $e$ and $o$ edges so that each $Q_3$ contains at least one $e$-edge and one $o$-edge. This can be done as follows:

\[
\begin{array}{|c|c|c|}
\hline
\text{e edges} & \text{o edges} \\
\hline
[*0001] & [*1000] & [*0100] & [*0010] \\
[*0111] & [*1110] & [*1101] & [*0111] \\
[1010*] & [1101*] & [0000*] & [0111*] \\
[0*101] & [0*010] & [1*000] & [1*111] \\
[10*10] & [11*01] & [00*11] & [01*00] \\
[001*1] & [010*1] & [111*0] & [100*1] \\
\hline
\end{array}
\]

Since there are $80 - 24 = 56$ $a$ edges and 24 non-$a$ edges. This yields the recursion:

\[ e_{k+4} = 16e_k + 40p_k + 24 \cdot 2^{k-2}. \]

This will in general give worse bounds than the $aeo$-colored $Q_4$, but it gives a better result for $k = 5$ since $p_5 = 0$. This gives $c(Q_3, 9) \leq 320$ which will in turn give smaller values for $c(Q_3, 9 + 3k)$.

We may also be able to improve some other small bounds by taking a $Q_2$-free $Q_m$ and then dividing up the non-edges into $e$- and $o$-edges. For instance, it would give $c(Q_3, 10) \leq 736$ which seems to be the last number where improvements would be possible in this manner.

4. A General Construction for $Q_3$-free subgraphs of the hypercube

We denote an edge as before in the form $[x_1x_2 \cdots x_{i-1} \ast x_{i+1} \cdots x_n]$. Where $x_j \in \{0, 1\}$. We denote two function for an edge:

- $p(e)$ is the number of ones before the $\ast$ minus the number of ones afterward.
- Then let

\[ A = \{ e : p(e) \equiv 0 \pmod{4} \}. \]

For the second part, we could consider instead edges where $p(e) \equiv 1, 2, \text{ or } 3 \pmod{4}$ and the argument would still apply. Hence $A$ contains at most $1/4$ of the edges of $Q_n$.

It remains to show that $A$ contains at least one edge from each $Q_3$ in $Q_n$. Denote a $Q_3$ by $a \ast b \ast c \ast d$ where $a, b, c, d$ are strings of zeros and ones. Let $|s|$ denote the number of ones in a string $s$.

**Case 1:** $|a| + |b| + |c| + |d| \equiv 0 \pmod{2}$. Then consider the edges:

- $a \ast b0c0d$
- $a \ast b1c1d$.

Both of them are part of our $Q_3$ and one of them has $p(e) \equiv 0$.

**Case 2:** $|a| + |b| + |c| + |d| \equiv 1 \pmod{2}$. Then consider the edges:

- $a1b \ast c0d$
- $a0b \ast c1d$.

Again, both of them are part of our $Q_3$ and one of them has $p(e) \equiv 0$. Hence $Q_n - A$ contains no $Q_3$. 
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