Jacobi’s Inversion Problem for Genus Two Hyperelliptic Integral II

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Abstract
In the previous paper, we reviewed the Rosenhain’s paper to the Jacobi’s inversion problem for the genus two hyperelliptic integral. In this paper, we review the Göpel’s paper to the Jacobi’s inversion problem for the genus two hyperelliptic integral.

1 Introduction
The doubly periodic function, the elliptic function, is obtained as the ratio of the one variable theta function. While, according to Abel, the elliptic function is obtained as the solution of the inversion problem of the elliptic integral. Similarly, the multiply periodic function such as the genus two hyperelliptic function, is obtained as the ratio of the two variables theta function. If we obtain such multiply periodic function as the inversion problem of the genus two hyperelliptic integral, Jacobi noticed that the naive solution of the inversion problem provides the non-single valued multiply periodic function \[1\]. Jacobi found that the symmetric combination of the hyperelliptic integral becomes the single valued function. Thus, Jacobi proposed the following Jacobi’s inversion problem

\[ u = \int^x \frac{dx}{\sqrt{f_5(x)}} + \int^{x'} \frac{dx'}{\sqrt{f_5(x')}} , \quad v = \int^x \frac{x\,dx}{\sqrt{f_5(x)}} + \int^{x'} \frac{x'\,dx'}{\sqrt{f_5(x')}} , \]

for the fifth-degree polynomial function \(f_5(x)\). For this inversion problem, Jacobi conjectured that the multiply periodic function, which is the solution of the Jacobi’s inversion problem, is given by the symmetric combination such as \(x + x', xx'\), which become the single valued functions of \(u\) and \(v\); furthermore, \(x + x', xx'\) are expressed as the ratio of the two variable theta functions \[2\].

The Jacobi’s inversion problem for the genus two case is solved by Göpel \[3, 4\] and independently by Rosenhain \[5, 6\].

In Rosenhain’s approach, the Riemann type addition formula of the hyperelliptic theta function is used. The key identities are the three quadratic theta identities, which parametrize five ratios of the square of hyperelliptic functions, which naturally gives the fifth-degree polynomial function \(f_5(x)\). While, in Göpel’s approach, the theta formula by the duplication method of the hyperelliptic theta function is used. The key identities are three quartic theta identities. One of these identities is the Kummer’s quartic identity, i.e., Kummer surface relation.

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In the previous paper, we reviewed the Rosenhain’s paper [7]. In this paper, we review Göpel’s paper.

2 The addition formulae of the genus two hyperelliptic theta functions -Duplication method-

The quadratic addition formula, which is called the duplication method, was first introduced by Jacobi [8] in 1828 for the the genus one elliptic theta functions. Göpel [3, 4] used this addition formula for the genus two hyperelliptic theta functions in 1847. Using this addition formula, we provided the full addition formula for the genus two hyperelliptic functions instead of the hyperelliptic theta functions [9].

The quartic addition formula, which is called the Riemann’s theta formula, was first introduced by Jacobi [10] in 1838 for the genus one elliptic theta functions. Rosenhain [5, 6] used this addition formula for the genus two hyperelliptic theta functions in 1850. Prym [11] named such addition formula as the Riemann’s theta formula because he writes down the proof according to the direct suggestion by Riemann [12] at Pisa in 1865. Using this addition formula, the full addition formula for the genus two hyperelliptic functions instead of the hyperelliptic theta functions are given by Kossak [13, 14].

The theta functions with two variables are defined by

\[ \vartheta \left[ \begin{array}{cc} a & c \\ b & d \end{array} \right] (x, y; \tau_{11}, \tau_{22}, \tau_{12}) = \sum_{m,n \in \mathbb{Z}} \exp \left\{ \pi i \left( \tau_{11} \left( m + \frac{a}{2} \right)^2 + \tau_{22} \left( n + \frac{c}{2} \right)^2 + 2 \tau_{12} \left( m + \frac{a}{2} \right) \left( n + \frac{c}{2} \right) \right) ight\} + 2 \pi i \left( (m + \frac{a}{2})(x + \frac{b}{2}) + (n + \frac{c}{2})(y + \frac{d}{2}) \right) \],

(2.1)

where we assume that \( \text{Im} \tau_{11} > 0, \text{Im} \tau_{22} > 0, (\text{Im} \tau_{11})(\text{Im} \tau_{22}) - (\text{Im} \tau_{12})^2 > 0 \) in order that the summation of \( m, n \in \mathbb{Z} \) becomes convergent. Renaming \( m \to m, n \to -n \), we can always choose \( \text{Im} \tau_{12} > 0 \), thus we assume \( \text{Im} \tau_{12} > 0 \).

We simply denote \( \vartheta \left[ \begin{array}{cc} a & c \\ b & d \end{array} \right] (x, y) = \vartheta \left[ \begin{array}{cc} a & c \\ b & d \end{array} \right] (x, y; \tau_{11}, \tau_{22}, \tau_{12}) \).

We also use the notation \( \Theta \left[ \begin{array}{cc} a & c \\ b & d \end{array} \right] (x, y) = \vartheta \left[ \begin{array}{cc} a & c \\ b & d \end{array} \right] (x, y; 2\tau_{11}, 2\tau_{22}, 2\tau_{12}) \). These have the following properties

\[ \vartheta \left[ \begin{array}{cc} a + 2 & c \\ b & d \end{array} \right] (x, y) = \vartheta \left[ \begin{array}{cc} a + 2 & c \\ b & d \end{array} \right] (x, y) = \vartheta \left[ \begin{array}{cc} a & c \\ b & d \end{array} \right] (x, y), \]

\[ \vartheta \left[ \begin{array}{cc} a & c \\ b + 2 & d \end{array} \right] (x, y) = (\sqrt{-1})^{2a} \vartheta \left[ \begin{array}{cc} a & c \\ b & d \end{array} \right] (x, y), \quad \vartheta \left[ \begin{array}{cc} a & c \\ b & d + 2 \end{array} \right] (x, y) = (\sqrt{-1})^{2c} \vartheta \left[ \begin{array}{cc} a & c \\ b & d \end{array} \right] (x, y). \]

In order to obtain the addition formula according to the duplication method, we consider
the product of the $\vartheta$ function in the form

$$
\vartheta \left[ \begin{array}{cc} a & c \\ b & d \end{array} \right] (x_1, y_1) \vartheta \left[ \begin{array}{cc} e & g \\ f & h \end{array} \right] (x_2, y_2) \\
= \sum_{m,n,p,q \in \mathbb{Z}} \exp \left\{ \pi i \left( \tau_{11}(m + \frac{a}{2})^2 + \tau_{22}(n + \frac{c}{2})^2 + 2\tau_{12}(m + \frac{a}{2})(n + \frac{c}{2}) \\
+ \tau_{11}(p + \frac{e}{2})^2 + \tau_{22}(q + \frac{g}{2})^2 + 2\tau_{12}(p + \frac{e}{2})(q + \frac{g}{2}) \right) \\
+ 2\pi i \left( (m + \frac{a}{2})(x_1 + \frac{b}{2}) + (n + \frac{c}{2})(y_1 + \frac{d}{2}) + (p + \frac{e}{2})(x_2 + \frac{f}{2}) + (q + \frac{g}{2})(y_2 + \frac{h}{2}) \right) \right\},
\right.
$$

where we use $m_1 = m + p, m_2 = m - p, n_1 = n + q, n_2 = n - q$. With $m_1 - m_2 = 2p = (\text{even number}), \{m_1, m_2\}$ are both even number or odd number. The pair $\{n_1, n_2\}$ are also both even number or odd number. Hence, we obtain 4 cases i) $m_1, m_2 : \text{even number}; n_1, n_2 : \text{even number},$ ii) $m_1, m_2 : \text{even number}; n_1, n_2 : \text{odd number},$ iii) $m_1, m_2 : \text{odd number}; n_1, n_2 : \text{even number},$ iv) $m_1, m_2 : \text{odd number}; n_1, n_2 : \text{odd number}$. Therefore, we obtain

$$
\vartheta \left[ \begin{array}{cc} a & c \\ b & d \end{array} \right] (x_1, y_1) \vartheta \left[ \begin{array}{cc} e & g \\ f & h \end{array} \right] (x_2, y_2) \\
= \Theta \left[ \begin{array}{cc} \frac{a+e}{2} & \frac{c+g}{2} \\ \frac{b+f}{2} & \frac{d+h}{2} \end{array} \right] (x_1 + x_2, y_1 + y_2) \Theta \left[ \begin{array}{cc} \frac{a-e}{2} & \frac{c-g}{2} \\ \frac{b-f}{2} & \frac{d-h}{2} \end{array} \right] (x_1 - x_2, y_1 - y_2) \\
+ \Theta \left[ \begin{array}{cc} \frac{a+e}{2} & \frac{c+g+2}{2} \\ \frac{b+f}{2} & \frac{d+h}{2} \end{array} \right] (x_1 + x_2, y_1 + y_2) \Theta \left[ \begin{array}{cc} \frac{a-e}{2} & \frac{c-g+2}{2} \\ \frac{b-f}{2} & \frac{d-h}{2} \end{array} \right] (x_1 - x_2, y_1 - y_2) \\
+ \Theta \left[ \begin{array}{cc} \frac{a+e+2}{2} & \frac{c+g}{2} \\ \frac{b+f}{2} & \frac{d+h}{2} \end{array} \right] (x_1 + x_2, y_1 + y_2) \Theta \left[ \begin{array}{cc} \frac{a-e+2}{2} & \frac{c-g}{2} \\ \frac{b-f}{2} & \frac{d-h}{2} \end{array} \right] (x_1 - x_2, y_1 - y_2) \\
+ \Theta \left[ \begin{array}{cc} \frac{a+e+2}{2} & \frac{c+g+2}{2} \\ \frac{b+f}{2} & \frac{d+h}{2} \end{array} \right] (x_1 + x_2, y_1 + y_2) \Theta \left[ \begin{array}{cc} \frac{a-e+2}{2} & \frac{c-g+2}{2} \\ \frac{b-f}{2} & \frac{d-h}{2} \end{array} \right] (x_1 - x_2, y_1 - y_2).
$$

Putting $x_1 = u_1 + u_2, x_2 = u_1 - u_2, y_1 = v_1 + v_2, y_2 = v_1 - v_2$, we express the above formula
in the form
\[
\vartheta \begin{bmatrix} a & c \\ b & d \end{bmatrix} (u_1 + u_2, v_1 + v_2) \vartheta \begin{bmatrix} e & g \\ f & h \end{bmatrix} (u_1 - u_2, v_1 - v_2) = \Theta \begin{bmatrix} \frac{a+e}{2} & \frac{c+g}{2} \\ b + f \end{bmatrix} (2u_1, 2v_1) \Theta \begin{bmatrix} \frac{a-e}{2} & \frac{c-g}{2} \\ b - f \end{bmatrix} (2u_2, 2v_2) + \Theta \begin{bmatrix} \frac{a+e+2}{2} & \frac{c+g+2}{2} \\ b + f \end{bmatrix} (2u_1, 2v_1) \Theta \begin{bmatrix} \frac{a-e}{2} & \frac{c-g}{2} \\ b - f \end{bmatrix} (2u_2, 2v_2) + \Theta \begin{bmatrix} \frac{a+e+2}{2} & \frac{c+g+2}{2} \\ b + f \end{bmatrix} (2u_1, 2v_1) \Theta \begin{bmatrix} \frac{a-e}{2} & \frac{c-g}{2} \\ b - f \end{bmatrix} (2u_2, 2v_2).
\]

(2.2)

In the special case of \( u_2 = 0, v_2 = 0 \), we obtain
\[
\vartheta \begin{bmatrix} a & c \\ b & d \end{bmatrix} (u_1, v_1) \vartheta \begin{bmatrix} e & g \\ f & h \end{bmatrix} (u_1, v_1) = \Theta \begin{bmatrix} \frac{a+e}{2} & \frac{c+g}{2} \\ b + f \end{bmatrix} (2u_1, 2v_1) \Theta \begin{bmatrix} \frac{a-e}{2} & \frac{c-g}{2} \\ b - f \end{bmatrix} (0, 0) + \Theta \begin{bmatrix} \frac{a+e+2}{2} & \frac{c+g+2}{2} \\ b + f \end{bmatrix} (2u_1, 2v_1) \Theta \begin{bmatrix} \frac{a-e}{2} & \frac{c-g}{2} \\ b - f \end{bmatrix} (0, 0) + \Theta \begin{bmatrix} \frac{a+e+2}{2} & \frac{c+g+2}{2} \\ b + f \end{bmatrix} (2u_1, 2v_1) \Theta \begin{bmatrix} \frac{a-e}{2} & \frac{c-g}{2} \\ b - f \end{bmatrix} (0, 0).
\]

(2.3)

We frequently use in the following form
\[
\begin{bmatrix} \vartheta[ a b ]^2(u, v) \\ \vartheta[ a b ]^2(u, v) \\ \vartheta[ a b ]^2(u, v) \\ \vartheta[ a b ]^2(u, v) \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix} \begin{bmatrix} \Theta[0 0 ](2u, 2v)\Theta[ a b ](0, 0) \\ \Theta[0 0 ](2u, 2v)\Theta[ a b ](0, 0) \\ \Theta[0 0 ](2u, 2v)\Theta[ a b ](0, 0) \\ \Theta[0 0 ](2u, 2v)\Theta[ a b ](0, 0) \end{bmatrix}.
\]

Here the Riemann matrix, which is the self-adjoint orthogonal matrix, emerges. We inversely solve the above in the following matrix form
\[
\begin{bmatrix} \Theta[0 0 ](2u, 2v)\Theta[ a b ](0, 0) \\ \Theta[0 0 ](2u, 2v)\Theta[ a b ](0, 0) \\ \Theta[0 0 ](2u, 2v)\Theta[ a b ](0, 0) \\ \Theta[0 0 ](2u, 2v)\Theta[ a b ](0, 0) \end{bmatrix} = \frac{1}{4} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \\ 1 & 1 & 1 & -1 \end{bmatrix} \begin{bmatrix} \vartheta[ a b ]^2(u, v) \\ \vartheta[ a b ]^2(u, v) \\ \vartheta[ a b ]^2(u, v) \\ \vartheta[ a b ]^2(u, v) \end{bmatrix}.
\]
We use the Göpel’s simplified notation of the form

\[
P_0 = \vartheta[\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}] (u, v), \quad P_1 = \vartheta[\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}] (u, v), \quad P_2 = \vartheta[\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}] (u, v), \quad P_3 = \vartheta[\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}] (u, v),
\]

\[
Q_0 = \vartheta[\begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}] (u, v), \quad Q_1 = \vartheta[\begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}] (u, v), \quad Q_2 = \vartheta[\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}] (u, v), \quad Q_3 = \vartheta[\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}] (u, v),
\]

\[
R_0 = \vartheta[\begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}] (u, v), \quad R_1 = \vartheta[\begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}] (u, v), \quad R_2 = \vartheta[\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}] (u, v), \quad R_3 = \vartheta[\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}] (u, v),
\]

\[
S_0 = \vartheta[\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}] (u, v), \quad S_1 = \vartheta[\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}] (u, v), \quad S_2 = \vartheta[\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}] (u, v), \quad S_3 = \vartheta[\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}] (u, v),
\]

\[
T = \Theta \left[ \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \right] (2u, 2v), \quad U = \Theta \left[ \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \right] (2u, 2v), \quad V = \Theta \left[ \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \right] (2u, 2v), \quad W = \Theta \left[ \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \right] (2u, 2v),
\]

\[
t = \Theta \left[ \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \right] (0, 0), \quad u = \Theta \left[ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \right] (0, 0), \quad v = \Theta \left[ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \right] (0, 0), \quad w = \Theta \left[ \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \right] (0, 0).
\]

Using the Göpel’s notation, we obtain

\[
\begin{pmatrix}
P_3^2 \\
P_2^2 \\
P_0^2 \\
Q_3^2 \\
Q_2^2 \\
Q_0^2 \\
R_3^2 \\
R_2^2 \\
R_0^2 \\
S_3^2 \\
S_2^2 \\
S_0^2
\end{pmatrix} = \frac{1}{4} \begin{pmatrix}
tT \\
vV \\
uU \\
wW \\
tT \\
vV \\
uU \\
wW \\
tT \\
vV \\
uU \\
wW \\
tT \\
vV \\
uU \\
wW \\
tT \\
vV \\
uU \\
wW \\
tT \\
vV \\
uU \\
wW
\end{pmatrix},
\]

\[
\begin{pmatrix}
P_3^2 \\
P_2^2 \\
P_0^2 \\
Q_3^2 \\
Q_2^2 \\
Q_0^2 \\
R_3^2 \\
R_2^2 \\
R_0^2 \\
S_3^2 \\
S_2^2 \\
S_0^2
\end{pmatrix} = \frac{1}{4} \begin{pmatrix}
vT \\
tV \\
wU \\
wW \\
vT \\
tV \\
wU \\
wW \\
vT \\
tV \\
wU \\
wW \\
vT \\
tV \\
wU \\
wW \\
vT \\
tV \\
wU \\
wW
\end{pmatrix},
\]

\[
\begin{pmatrix}
P_3^2 \\
P_2^2 \\
P_0^2 \\
Q_3^2 \\
Q_2^2 \\
Q_0^2 \\
R_3^2 \\
R_2^2 \\
R_0^2 \\
S_3^2 \\
S_2^2 \\
S_0^2
\end{pmatrix} = \frac{1}{4} \begin{pmatrix}
wT \\
uV \\
vU \\
tW \\
wT \\
uV \\
vU \\
tW \\
wT \\
uV \\
vU \\
tW \\
wT \\
uV \\
vU \\
tW \\
wT \\
uV \\
vU \\
tW
\end{pmatrix},
\]

\[
\begin{pmatrix}
P_3^2 \\
P_2^2 \\
P_0^2 \\
Q_3^2 \\
Q_2^2 \\
Q_0^2 \\
R_3^2 \\
R_2^2 \\
R_0^2 \\
S_3^2 \\
S_2^2 \\
S_0^2
\end{pmatrix} = \frac{1}{4} \begin{pmatrix}
1 & 1 & 1 & 1 \\
1 & -1 & 1 & -1 \\
1 & 1 & -1 & 1 \\
1 & -1 & 1 & -1 \\
1 & 1 & 1 & -1 \\
1 & -1 & 1 & -1 \\
1 & 1 & -1 & 1 \\
1 & -1 & 1 & -1 \\
1 & 1 & 1 & -1 \\
1 & -1 & 1 & -1 \\
1 & 1 & 1 & -1 \\
1 & -1 & 1 & -1 \\
1 & 1 & 1 & -1 \\
1 & -1 & 1 & -1 \\
1 & 1 & 1 & -1 \\
1 & -1 & 1 & -1 \\
1 & 1 & 1 & -1 \\
1 & -1 & 1 & -1
\end{pmatrix}.
\]

\[\text{3 The Kummer surface relation}\]

We use the Kummer’s quartic relation, i.e., Kummer surface relation, to solve the Jacobi’s inversion relation. Thus, we derive the Kummer’s quartic relation. By using the addition formula of \(P_1, S_1, P_2\) and \(S_2\), we obtain

\[
P_1^2 = tT + uU - vV - wW, \quad P_2^2 = tT + uU + vV - wW,
\]

\[
S_1^2 = wT + vU - uV - tW, \quad S_2^2 = wT - vU + uV - tW.
\]
We inversely solve in the form
\[
T = \frac{t(P^2_1 + P^2_2) - w(S^2_1 + S^2_2)}{2(t^2 - w^2)}, \quad U = \frac{u(P^2_1 - P^2_2) - v(S^2_1 - S^2_2)}{2(u^2 - v^2)}, \quad (3.3)
\]
\[
V = \frac{v(P^2_1 - P^2_2) - u(S^2_1 - S^2_2)}{2(u^2 - v^2)}, \quad W = \frac{w(P^2_1 + P^2_2) - t(S^2_1 + S^2_2)}{2(t^2 - w^2)}. \quad (3.4)
\]

Later, we use the following
\[
Q^2_1 = uT + tU - wV - vW, \quad R^2_1 = vT + wU - tV - uW. \quad (3.5)
\]

### 3.1 Proof of \(Q_1R_1 = aP_1S_1 + bP_2S_2\)

We will demonstrate that \(Q_1R_1 = aP_1S_1 + bP_2S_2\), \((a,b = \text{const.})\). Considering the square of this relation, we obtain \(Q^2_1R^2_1 = (aP_1S_1 + bP_2S_2)^2\). Using Eqs.\((3.3) - (3.5)\), the left-hand side term \(Q^2_1R^2_1\) is expressed as the polynomial of \(P_1, S_1, P_2, S_2\). While the right-hand side term \((aP_1S_1 + bP_2S_2)^2\) is of course the polynomial of \(P_1, S_1, P_2, S_2\). Hence, \(Q^2_1R^2_1 = (aP_1S_1 + bP_2S_2)^2\) gives the Kummer’s quartic relations of \(P_1, S_1, P_2, S_2\). If we can find one of the expressions, which is expressed as the linear combination of \(P_1S_1\) and \(P_2S_2\), it suffices for our purpose. For example, instead of using \(Q_1R_1\), we can use \(Q_2R_2\) because this also becomes the linear combination of \(P_1S_1\) and \(P_2S_2\). By using the addition formula, we obtain
\[
Q_1R_1 = \vartheta[\begin{array}{c} 1 \\ 0 \\ 0 \end{array}](u,v)\vartheta[\begin{array}{c} 0 \\ 1 \\ 0 \end{array}](u,v)
\]
\[
= \Theta[\begin{array}{c} \frac{1}{2} \frac{1}{2} \\ 0 \\ 2 \end{array}](2u,2v)\Theta[\begin{array}{c} \frac{1}{2} \\ \frac{1}{2} \\ 0 \end{array}](0,0) + \Theta[\begin{array}{c} \frac{1}{2} \frac{3}{2} \\ 0 \\ 2 \end{array}](2u,2v)\Theta[\begin{array}{c} \frac{1}{2} \frac{1}{2} \\ 0 \\ 0 \end{array}](0,0)
\]
\[
+ \Theta[\begin{array}{c} \frac{3}{2} \frac{1}{2} \\ 0 \\ 2 \end{array}](2u,2v)\Theta[\begin{array}{c} \frac{3}{2} \\ \frac{1}{2} \\ 0 \end{array}](0,0) + \Theta[\begin{array}{c} \frac{3}{2} \frac{3}{2} \\ 0 \\ 2 \end{array}](2u,2v)\Theta[\begin{array}{c} \frac{3}{2} \frac{1}{2} \\ 0 \\ 0 \end{array}](0,0),
\]
which provides
\[
Q_1R_1 = i\Theta[\begin{array}{c} \frac{1}{2} \frac{1}{2} \\ 0 \\ 0 \end{array}](2u,2v)\Theta[\begin{array}{c} \frac{1}{2} \frac{1}{2} \\ 0 \\ 0 \end{array}](0,0) - i\Theta[\begin{array}{c} \frac{1}{2} \frac{1}{2} \\ 0 \\ 0 \end{array}](2u,2v)\Theta[\begin{array}{c} \frac{1}{2} \frac{1}{2} \\ 0 \\ 0 \end{array}](0,0)
\]
\[
+ i\Theta[\begin{array}{c} \frac{1}{2} \frac{1}{2} \\ 0 \\ 0 \end{array}](2u,2v)\Theta[\begin{array}{c} \frac{1}{2} \frac{1}{2} \\ 0 \\ 0 \end{array}](0,0) - i\Theta[\begin{array}{c} \frac{1}{2} \frac{1}{2} \\ 0 \\ 0 \end{array}](2u,2v)\Theta[\begin{array}{c} \frac{1}{2} \frac{1}{2} \\ 0 \\ 0 \end{array}](0,0)
\]
\[
= i \left( \Theta[\begin{array}{c} \frac{1}{2} \frac{1}{2} \\ 0 \\ 0 \end{array}](2u,2v) - \Theta[\begin{array}{c} -\frac{1}{2} -\frac{1}{2} \\ 0 \\ 0 \end{array}](2u,2v) \right) \Theta[\begin{array}{c} \frac{1}{2} -\frac{1}{2} \\ 0 \\ 0 \end{array}](0,0)
\]
\[
- i \left( \Theta[\begin{array}{c} -\frac{1}{2} -\frac{1}{2} \\ 0 \\ 0 \end{array}](2u,2v) - \Theta[\begin{array}{c} \frac{1}{2} \frac{1}{2} \\ 0 \\ 0 \end{array}](2u,2v) \right) \Theta[\begin{array}{c} \frac{1}{2} \frac{1}{2} \\ 0 \\ 0 \end{array}](0,0),
\]
\[
\text{where we used}
\]
\[
\Theta[\begin{array}{c} \frac{1}{2} \frac{1}{2} \\ 0 \\ 0 \end{array}](0,0) = \Theta[\begin{array}{c} -\frac{1}{2} -\frac{1}{2} \\ 0 \\ 0 \end{array}](0,0), \quad \Theta[\begin{array}{c} \frac{1}{2} \frac{1}{2} \\ 0 \\ 0 \end{array}](0,0) = \Theta[\begin{array}{c} -\frac{1}{2} -\frac{1}{2} \\ 0 \\ 0 \end{array}](0,0). \quad (3.7)
\]
Similarly, we obtain

\[ P_1S_1 = \vartheta[ \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} ](u,v) \vartheta[ \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} ](u,v) \]
\[ = i \left( \Theta[ \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ 0 & 0 \end{bmatrix} ](2u,2v) - \Theta[ \begin{bmatrix} -\frac{1}{2} & -\frac{1}{2} \\ 0 & 0 \end{bmatrix} ](2u,2v) \right) \Theta[ \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ 0 & 0 \end{bmatrix} ](0,0) \]
\[ - i \left( \Theta[ \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} \\ 0 & 0 \end{bmatrix} ](2u,2v) - \Theta[ \begin{bmatrix} -\frac{1}{2} & \frac{1}{2} \\ 0 & 0 \end{bmatrix} ](2u,2v) \right) \Theta[ \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} \\ 0 & 0 \end{bmatrix} ](0,0), \quad (3.8) \]

\[ P_2S_2 = \vartheta[ \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} ](u,v) \vartheta[ \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} ](u,v) \]
\[ = i \left( \Theta[ \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ 0 & 0 \end{bmatrix} ](2u,2v) - \Theta[ \begin{bmatrix} -\frac{1}{2} & -\frac{1}{2} \\ 0 & 0 \end{bmatrix} ](2u,2v) \right) \Theta[ \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ 0 & 0 \end{bmatrix} ](0,0) \]
\[ + i \left( \Theta[ \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} \\ 0 & 0 \end{bmatrix} ](2u,2v) - \Theta[ \begin{bmatrix} -\frac{1}{2} & \frac{1}{2} \\ 0 & 0 \end{bmatrix} ](2u,2v) \right) \Theta[ \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} \\ 0 & 0 \end{bmatrix} ](0,0). \quad (3.9) \]

Combining these relations, we obtain three coupled equations

\[ Q_1R_1 = iBX - iAY, \quad (3.10) \]
\[ P_1S_1 = iAX - iBY, \quad (3.11) \]
\[ P_2S_2 = iAX + iBY, \quad (3.12) \]
\[ X = \left( \Theta[ \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ 0 & 0 \end{bmatrix} ](2u,2v) - \Theta[ \begin{bmatrix} -\frac{1}{2} & -\frac{1}{2} \\ 0 & 0 \end{bmatrix} ](2u,2v) \right), \quad (3.13) \]
\[ Y = \left( \Theta[ \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} \\ 0 & 0 \end{bmatrix} ](2u,2v) - \Theta[ \begin{bmatrix} -\frac{1}{2} & \frac{1}{2} \\ 0 & 0 \end{bmatrix} ](2u,2v) \right), \quad (3.14) \]
\[ A = \Theta[ \begin{bmatrix} \frac{1}{2} & 1 \\ 0 & 0 \end{bmatrix} ](0,0), \quad B = \Theta[ \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ 0 & 0 \end{bmatrix} ](0,0). \quad (3.15) \]

Expressing \( X \) and \( Y \) by \( P_1S_1 \) and \( P_2S_2 \), and substituting into the right-hand side of \( Q_1R_1 \), we complete the proof of the form

\[ Q_1R_1 = aP_1S_1 + bP_2S_2, \quad a = \frac{A^2 + B^2}{2AB}, \quad b = \frac{A^2 - B^2}{2AB}, \quad a^2 - b^2 = 1. \quad (3.16) \]

### 3.2 The expression of \( a \) and \( b \) with \( t, u, v, w \)

In order to express \( A \) and \( B \) as the function of \( t, u, v, w \), we consider the quantity

\[ P_3S_3 = \vartheta[ \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} ](u,v) \vartheta[ \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} ](u,v) \]
\[ = \left( \Theta[ \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ 0 & 0 \end{bmatrix} ](2u,2v) + \Theta[ \begin{bmatrix} -\frac{1}{2} & -\frac{1}{2} \\ 0 & 0 \end{bmatrix} ](2u,2v) \right) \Theta[ \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ 0 & 0 \end{bmatrix} ](0,0) \]
\[ + \left( \Theta[ \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} \\ 0 & 0 \end{bmatrix} ](2u,2v) + \Theta[ \begin{bmatrix} -\frac{1}{2} & \frac{1}{2} \\ 0 & 0 \end{bmatrix} ](2u,2v) \right) \Theta[ \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} \\ 0 & 0 \end{bmatrix} ](0,0). \quad (3.17) \]

By putting \( u = 0, v = 0 \), we obtain

\[ \vartheta[ \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} ](0,0) \vartheta[ \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} ](0,0) = 2\Theta[ \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ 0 & 0 \end{bmatrix} ](0,0)^2 + 2\Theta[ \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} \\ 0 & 0 \end{bmatrix} ](0,0)^2 = 2(A^2 + B^2). \quad (3.18) \]
Similarly, we consider the quantity

\[ Q_3 R_3 = \vartheta[\begin{array}{cc} 1 & 0 \\ 0 & 0 \end{array}] (u, v) \vartheta[\begin{array}{cc} 0 & 1 \\ 0 & 0 \end{array}](u, v) \]

\[ = (\Theta[\begin{array}{cc} \frac{1}{2} & \frac{1}{2} \\ 0 & 0 \end{array}](2u, 2v) + \Theta[\begin{array}{cc} -\frac{1}{2} & -\frac{1}{2} \\ 0 & 0 \end{array}](2u, 2v)) \Theta[\begin{array}{cc} \frac{1}{2} & \frac{1}{2} \\ 0 & 0 \end{array}](0, 0) \]

\[ + (\Theta[\begin{array}{cc} \frac{1}{2} & -\frac{1}{2} \\ 0 & 0 \end{array}](2u, 2v) + \Theta[\begin{array}{cc} -\frac{1}{2} & \frac{1}{2} \\ 0 & 0 \end{array}](2u, 2v)) \Theta[\begin{array}{cc} \frac{1}{2} & \frac{1}{2} \\ 0 & 0 \end{array}](0, 0). \]

(3.19)

By putting \( u = 0, v = 0 \), we obtain

\[ \vartheta[\begin{array}{cc} 1 & 0 \\ 0 & 0 \end{array}](0, 0) \vartheta[\begin{array}{cc} 0 & 1 \\ 0 & 0 \end{array}](0, 0) = 4 \Theta[\begin{array}{cc} \frac{1}{2} & \frac{1}{2} \\ 0 & 0 \end{array}](0, 0) \Theta[\begin{array}{cc} \frac{1}{2} & -\frac{1}{2} \\ 0 & 0 \end{array}](0, 0) = 4AB. \]

(3.20)

Therefore, the constant \( a \) is solved by the function of \( \vartheta[\begin{array}{cc} a & c \\ b & d \end{array}](0, 0) \) in the form

\[ a = \frac{A^2 + B^2}{2AB} = \frac{\vartheta[\begin{array}{cc} 1 & 1 \\ 0 & 0 \end{array}](0, 0) \vartheta[\begin{array}{cc} 0 & 0 \\ 0 & 0 \end{array}](0, 0)}{\vartheta[\begin{array}{cc} 1 & 0 \\ 0 & 0 \end{array}](0, 0) \vartheta[\begin{array}{cc} 0 & 1 \\ 0 & 0 \end{array}](0, 0)}. \]

(3.21)

Taking the square and use the addition formula, \( S_3(0, 0)^2 = \vartheta[\begin{array}{cc} 1 & 1 \\ 0 & 0 \end{array}](0, 0)^2 = 2(tw + uv) \),

\( P_3(0, 0)^2 = \vartheta[\begin{array}{cc} 0 & 0 \\ 0 & 0 \end{array}](0, 0)^2 = t^2 + u^2 + v^2 + w^2 \),

\( Q_3(0, 0)^2 = \vartheta[\begin{array}{cc} 1 & 0 \\ 0 & 0 \end{array}](0, 0)^2 = 2(tu + vw) \),

\( R_3(0, 0)^3 = \vartheta[\begin{array}{cc} 0 & 1 \\ 0 & 0 \end{array}](0, 0)^2 = 2(tv + uw) \), we obtain the expression of \( a^2 \) with \( t, u, v, w \)

\[ a^2 = \frac{\vartheta[\begin{array}{cc} 1 & 1 \\ 0 & 0 \end{array}](0, 0) \vartheta[\begin{array}{cc} 0 & 0 \\ 0 & 0 \end{array}](0, 0)^2}{\vartheta[\begin{array}{cc} 1 & 0 \\ 0 & 0 \end{array}](0, 0)^2 \vartheta[\begin{array}{cc} 0 & 1 \\ 0 & 0 \end{array}](0, 0)^2} = \frac{(tw + uv)(t^2 + u^2 + v^2 + w^2)}{2(tu + vw)(tv + uw)}. \]

(3.22)

By using \( a^2 - b^2 = 1 \), we obtain \( b^2 \) in the form

\[ b^2 = a^2 - 1 = \frac{(t^2 - u^2 - v^2 + w^2)(tw - uv)}{2(tu + vw)(tv + uw)}, \]

(3.23)

which provides

\[ a = \sqrt{\frac{(t^2 + u^2 + v^2 + w^2)(tw + uv)}{2(tu + vw)(tv + uw)}}, \quad b = \sqrt{\frac{(t^2 - u^2 - v^2 + w^2)(tw - uv)}{2(tu + vw)(tv + uw)}}. \]

(3.24)

3.3 The Kummer surface relation of \( P_1, S_1, P_2, S_2 \)

The Kummer’s quartic relation for \( P_1, S_1, P_2, S_2 \) is given by considering the square of \( Q_1 R_1 = aP_1 S_1 + bP_2 S_2 \) in the form \( Q_1^2 R_1^2 = a^2 P_1^2 S_1^2 + b^2 P_2^2 S_2^2 + 2abP_1 S_1 P_2 S_2 \) via Eqs. (3.3)-(3.5). After
the straightforward calculation, we obtain the following Kummer’s quartic relation

\[ P_1^4 + S_1^4 + P_2^4 + S_2^4 - 2F(P_1^2 P_2^2 + S_1^2 S_2^2) + 2C(P_1^2 S_2^2 + P_2^2 S_1^2) \]
\[ -2E(P_1^2 S_1^2 + P_2^2 S_2^2) - 4DP_1 P_2 S_1 S_2 = 0, \tag{3.25} \]
\[ E = \frac{t^2 u^2 + t^2 v^2 + u^2 w^2 + v^2 w^2 + 4twv}{2(tu + vw)(tv + uw)}, \tag{3.26} \]
\[ F = \frac{t^4 + u^4 + v^4 + w^4 - 2t^2 w^2 - 2u^2 v^2}{(t^2 + u^2 - v^2 - w^2)(t^2 - u^2 + v^2 - w^2)}, \tag{3.27} \]
\[ C = \frac{t^2 u^2 + t^2 v^2 + u^2 w^2 + v^2 w^2 - 4twv}{2(tu - vw)(tv - uw)}, \tag{3.28} \]
\[ D = \frac{(t^2 - w^2)^2(u^2 - v^2)^2(t^2 + u^2 + v^2 + w^2)(t^2 - u^2 - v^2 + w^2)(t^2 w^2 - u^2 v^2)}{(t^2 + u^2 - v^2 - w^2)(t^2 - u^2 + v^2 - w^2)(t^2 u^2 - v^2 w^2)(t^2 v^2 - u^2 w^2)}, \tag{3.29} \]

where \( C^2 - D^2 + E^2 + F^2 - 2CEF = 1 \) is satisfied.

If we use \( P_3(0,0) = a, S_3(0,0) = b, P_0(0,0) = c, S_0(0,0) = d \), we have the relations

\[ a^2 = t^2 + u^2 + v^2 + w^2, \quad b^2 = 2(tw + uv), \quad c^2 = t^2 - u^2 - v^2 + w^2, \]
\[ d^2 = 2(tw - uv), \]

by using the addition formula. Thus, we obtain the Kummer surface relation in the following Hudson’s standard form [15]

\[ P_1^4 + S_1^4 + P_2^4 + S_2^4 + A_1(P_1^2 S_2^2 + P_2^2 S_1^2) + B_1(P_1^2 P_2^2 + S_1^2 S_2^2) + C_1(P_1^2 S_1^2 + P_2^2 S_2^2) \]
\[ + 2D_1 P_1 P_2 S_1 S_2 = 0, \tag{3.30} \]
\[ A_1 = \frac{b^4 + c^4 - a^4 - d^4}{a^2 d^2 - b^2 c^2}, \quad B_1 = \frac{a^4 + c^4 - b^4 - d^4}{b^2 d^2 - a^2 c^2}, \quad C_1 = \frac{a^4 + b^4 - c^4 - d^4}{c^2 d^2 - a^2 b^2}, \]
\[ D_1 = \frac{abcd(a^2 + d^2 - b^2 - c^2)(b^2 + d^2 - a^2 - c^2)(c^2 + d^2 - a^2 - b^2)(a^2 + b^2 + c^2 + d^2)}{(a^2 d^2 - b^2 c^2)(b^2 d^2 - a^2 c^2)(c^2 d^2 - a^2 b^2)}, \tag{3.31} \]

where \( A_1^2 + B_1^2 + C_1^2 - D_1^2 - A_1 B_1 C_1 = 4 \) is satisfied.

### 4 The differential equation (Step I)

#### 4.1 The derivative formula

In this section, by using \( p = S_1/P_1, q = S_2/P_2 \) and \( s = P_1/P_2 \) of the form

\[ p = S_1/P_1 = \frac{\vartheta[1 \ 1 \ \vartheta[1 \ 1 \ \vartheta[0 \ 0 \ |(u, v) \]
\[ q = S_2/P_2 = \frac{\vartheta[1 \ 1 \ \vartheta[0 \ 0 \ |(u, v) \]
\[ s = P_1/P_2 = \frac{\vartheta[0 \ 0 \ \vartheta[0 \ 0 \ |(u, v) \]
\[ (4.1) \]

we first derive the differential equation of \( p = S_1/P_1 \) and \( q = S_2/P_2 \). By noticing the relation

\[ dp = \frac{1}{P_1^2}(dS_1 P_1 - dP_1 S_1), \quad dq = \frac{1}{P_2^2}(dS_2 P_2 - dP_2 S_2), \tag{4.2} \]
we consider the combination
\[
I = (dS_1 P_1 - dP_1 S_1) = d\vartheta[ 1 1 0 1 ](u, v)\vartheta[ 0 0 1 0 ](u, v) - d\vartheta[ 0 0 1 0 ](u, v)\vartheta[ 1 1 0 1 ](u, v), \tag{4.3}
\]
\[
J = (dS_2 P_2 - dP_2 S_2) = d\vartheta[ 1 1 1 0 ](u, v)\vartheta[ 0 0 1 0 ](u, v) - d\vartheta[ 0 0 1 0 ](u, v)\vartheta[ 1 1 1 0 ](u, v). \tag{4.4}
\]

We derive the derivative formula of the theta function by using the addition formula of the theta function. Thus, we consider
\[
I_1 = \vartheta[ 1 1 1 0 ](u_1 + u_2, v_1 + v_2)\vartheta[ 0 0 1 0 ](u_1 - u_2, v_1 - v_2)
- \vartheta[ 0 0 1 0 ](u_1 + u_2, v_1 + v_2)\vartheta[ 1 1 1 0 ](u_1 - u_2, v_1 - v_2)
= i \left( \Theta[ \frac{1}{2} \frac{1}{2} 0 0 ](2u_2, 2v_2) + \Theta[ -\frac{1}{2} -\frac{1}{2} 0 0 ](2u_1, 2v_1) \right) \left( \Theta[ \frac{1}{2} \frac{1}{2} 0 0 ](2u_2, 2v_2) - \Theta[ -\frac{1}{2} -\frac{1}{2} 0 0 ](2u_2, 2v_2) \right)
- i \left( \Theta[ \frac{1}{2} -\frac{1}{2} 0 0 ](2u_1, 2v_1 + \Theta[ -\frac{1}{2} \frac{1}{2} 0 0 ](2u_1, 2v_1) \right) \left( \Theta[ \frac{1}{2} -\frac{1}{2} 0 0 ](2u_2, 2v_2) - \Theta[ -\frac{1}{2} \frac{1}{2} 0 0 ](2u_2, 2v_2) \right). \tag{4.5}
\]

If we notice that
\[
\left( \Theta[ \frac{1}{2} \frac{1}{2} 0 0 ](2u_2, 2v_2) - \Theta[ -\frac{1}{2} -\frac{1}{2} 0 0 ](2u_2, 2v_2) \right),
\]
\[
\left( \Theta[ \frac{1}{2} -\frac{1}{2} 0 0 ](2u_2, 2v_2) - \Theta[ -\frac{1}{2} \frac{1}{2} 0 0 ](2u_2, 2v_2) \right),
\]
are odd functions, these odd functions become zero at \( u_2 = 0, v_2 = 0 \). Hence, by putting \( u_2 = du_1, v_2 = 0 \), and replacing \( u_1 \to u, v_1 \to v \), we obtain \( \partial \vartheta / \partial u \) in the form
\[
\left( \frac{\partial}{\partial u} \vartheta[ 1 1 1 0 ](u, v)\vartheta[ 0 0 1 0 ](u, v) - \frac{\partial}{\partial u} \vartheta[ 0 0 1 0 ](u, v)\vartheta[ 1 1 1 0 ](u, v) \right)
= i\alpha \left( \Theta[ \frac{1}{2} \frac{1}{2} 0 0 ](2u, 2v) + \Theta[ -\frac{1}{2} -\frac{1}{2} 0 0 ](2u, 2v) \right)
- i\beta \left( \Theta[ \frac{1}{2} -\frac{1}{2} 0 0 ](2u, 2v) + \Theta[ -\frac{1}{2} \frac{1}{2} 0 0 ](2u, 2v) \right)
= i\alpha f(2u, 2v) - i\beta g(2u, 2v), \tag{4.6}
\]
where
\[
f(2u, 2v) = \left( \Theta[ \frac{1}{2} \frac{1}{2} 0 0 ](2u, 2v) + \Theta[ -\frac{1}{2} -\frac{1}{2} 0 0 ](2u, 2v) \right), \tag{4.7}
\]
\[
g(2u, 2v) = \left( \Theta[ \frac{1}{2} -\frac{1}{2} 0 0 ](2u, 2v) + \Theta[ -\frac{1}{2} \frac{1}{2} 0 0 ](2u, 2v) \right), \tag{4.8}
\]
\[
\alpha = \left. \frac{d}{du} \left( \Theta[ \frac{1}{2} \frac{1}{2} 0 0 ](u, 0) - \Theta[ -\frac{1}{2} -\frac{1}{2} 0 0 ](u, 0) \right) \right|_{u=0}, \tag{4.9}
\]
\[
\beta = \left. \frac{d}{du} \left( \Theta[ \frac{1}{2} -\frac{1}{2} 0 0 ](u, 0) - \Theta[ -\frac{1}{2} \frac{1}{2} 0 0 ](u, 0) \right) \right|_{u=0}. \tag{4.10}
\]
Similarly, putting $u_2 = 0$, $v_2 = dv_1$, and replacing $u_1 \to u$, $v_1 \to v$, we obtain $\partial \theta / \partial v$ in the form

$$
\left( \frac{\partial}{\partial v} \theta \left[ \begin{array}{c} 1 \\ 1 \\ 0 \\ 0 \\ 1 \end{array} \right] (u, v) \theta \left[ \begin{array}{c} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{array} \right] (u, v) - \frac{\partial}{\partial v} \theta \left[ \begin{array}{c} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{array} \right] (u, v) \theta \left[ \begin{array}{c} 1 \\ 1 \\ 0 \\ 0 \\ 1 \end{array} \right] (u, v) \right)
$$

$$
= i \gamma \left( \theta \left[ \begin{array}{c} 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{array} \right] (2u, 2v) + \theta \left[ \begin{array}{c} -1 \\ -1 \\ 0 \\ 0 \\ 0 \end{array} \right] (2u, 2v) \right)
$$

$$
- i \delta \left( \theta \left[ \begin{array}{c} 1 \\ 1 \\ 0 \\ 0 \\ 1 \end{array} \right] (2u, 2v) + \theta \left[ \begin{array}{c} -1 \\ -1 \\ 0 \\ 0 \\ 1 \end{array} \right] (2u, 2v) \right)
$$

$$
= i \gamma f(2u, 2v) - i \delta g(2u, 2v), \tag{4.10}
$$

$$
\gamma = \frac{d}{dv} \left( \theta \left[ \begin{array}{c} 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{array} \right] (0, v) - \theta \left[ \begin{array}{c} -1 \\ -1 \\ 0 \\ 0 \\ 0 \end{array} \right] (0, v) \right) \bigg|_{v=0}, \tag{4.11}
$$

$$
\delta = \frac{d}{dv} \left( \theta \left[ \begin{array}{c} 1 \\ 1 \\ 0 \\ 0 \\ 1 \end{array} \right] (0, v) - \theta \left[ \begin{array}{c} -1 \\ -1 \\ 0 \\ 0 \\ 1 \end{array} \right] (0, v) \right) \bigg|_{v=0}. \tag{4.12}
$$

Thus, we obtain the derivative formula from the addition formula of the theta function in the form

$$
I = (dS_1 P_1 - dP_1 S_1) = du(\partial_u S_1 P_1 - \partial_u P_1 S_1) + dv(\partial_v S_1 P_1 - \partial_v P_1 S_1)
$$

$$
= if(2u, 2v)(\alpha du + \gamma dv) - ig(2u, 2v)(\beta du + \delta dv). \tag{4.13}
$$

We can obtain $J$ by the replacement of $u \to u+1$, $v \to v+1$ in Eq.(4.13). In the right-hand side of Eq.(4.13), we obtain

$$
I = (dS_1 P_1 - dP_1 S_1) = d\theta \left[ \begin{array}{c} 1 \\ 1 \\ 1 \\ 1 \\ 0 \end{array} \right] (u, v) \theta \left[ \begin{array}{c} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{array} \right] (u, v) - d\theta \left[ \begin{array}{c} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{array} \right] (u, v) \theta \left[ \begin{array}{c} 1 \\ 1 \\ 0 \end{array} \right] (u, v)
$$

$$
= \left( -1 \right) \left( d\theta \left[ \begin{array}{c} 1 \\ 1 \\ 1 \\ 1 \\ 0 \end{array} \right] (u, v) \theta \left[ \begin{array}{c} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{array} \right] (u, v) - d\theta \left[ \begin{array}{c} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{array} \right] (u, v) \theta \left[ \begin{array}{c} 1 \\ 1 \\ 0 \end{array} \right] (u, v) \right) = -J. \tag{4.14}
$$

While, in the right-hand side of Eq.(4.13), we obtain

$$
if(2u, 2v)(\alpha du + \gamma dv) - ig(2u, 2v)(\beta du + \delta dv)
$$

$$
= i \left( \theta \left[ \begin{array}{c} 1 \\ 1 \\ 1 \\ 2 \\ 0 \end{array} \right] (2u, 2v) \right) (\alpha du + \gamma dv)
$$

$$
- i \left( \theta \left[ \begin{array}{c} 1 \\ 1 \\ 1 \\ 2 \\ 1 \end{array} \right] (2u, 2v) \right) (\beta du + \delta dv)
$$

$$
= \left( -1 \right) \left( \theta \left[ \begin{array}{c} 1 \\ 1 \\ 1 \\ 1 \\ 0 \end{array} \right] (2u, 2v) \theta \left[ \begin{array}{c} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{array} \right] (2u, 2v) - \theta \left[ \begin{array}{c} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{array} \right] (2u, 2v) \theta \left[ \begin{array}{c} 1 \\ 1 \\ 0 \end{array} \right] (2u, 2v) \right) = -J. \tag{4.15}
$$
Thus, we obtain the derivative formula of $J$ in the form:

$$J = (dS_2P_2 - dP_2S_2) = du(\partial_u S_2P_2 - \partial_u P_2S_2) + dv(\partial_v S_2P_2 - \partial_v P_2S_2),$$

$$= i f(2u, 2v)(\alpha du + \gamma dv) + ig(2u, 2v)(\beta du + \delta dv). \tag{4.16}$$

Therefore, the necessary derivative formulae provide

$$(dS_1P_1 - dP_1S_1) = i f(2u, 2v)(\alpha du + \gamma dv) - ig(2u, 2v)(\beta du + \delta dv), \tag{4.17}$$

$$(dS_2P_2 - dP_2S_2) = i f(2u, 2v)(\alpha du + \gamma dv) + ig(2u, 2v)(\beta du + \delta dv). \tag{4.18}$$

Next, we express $f(2u, 2v)$, $g(2u, 2v)$ with the product of the original theta functions $P_0$, $P_1$, $\cdots$, $S_2$, $S_3$. Using Eq.$(3.17)$, we obtain

$$P_3S_3 = \theta[\begin{array}{cc} 1 & 1 \\ 0 & 0 \end{array}](u, v)\theta[\begin{array}{c} 0 \\ 0 \end{array}](u, v)$$

$$= \left(\theta[\begin{array}{c} \frac{1}{2} \\ 0 \end{array}][2u, 2v] + \theta[\begin{array}{c} -\frac{1}{2} \\ 0 \end{array}][2u, 2v]\right)\theta[\begin{array}{rr} \frac{1}{2} & \frac{1}{2} \\ 0 & 0 \end{array}](0, 0)$$

$$+ \left(\theta[\begin{array}{c} \frac{1}{2} \\ 0 \end{array}][2u, 2v] + \theta[\begin{array}{c} -\frac{1}{2} \\ 0 \end{array}][2u, 2v]\right)\theta[\begin{array}{rr} \frac{1}{2} & -\frac{1}{2} \\ 0 & 0 \end{array}](0, 0)$$

$$f(2u, 2v) = \theta[\begin{array}{rr} \frac{1}{2} & \frac{1}{2} \\ 0 & 0 \end{array}](0, 0) f(2u, 2v) + \theta[\begin{array}{rr} \frac{1}{2} & -\frac{1}{2} \\ 0 & 0 \end{array}](0, 0) g(2u, 2v). \tag{4.19}$$

By replacing $u \to u + 1, v \to v + 1$ in Eq.$(4.19)$, we obtain

$$P_0S_0 = \theta[\begin{array}{cc} 1 & 1 \\ 1 & 1 \end{array}](u, v)\theta[\begin{array}{c} 0 \\ 0 \end{array}](u, v)$$

$$= \left(\theta[\begin{array}{c} \frac{1}{2} \\ 2 \end{array}][2u, 2v] + \theta[\begin{array}{c} -\frac{1}{2} \\ 2 \end{array}][2u, 2v]\right)\theta[\begin{array}{rr} \frac{1}{2} & \frac{1}{2} \\ 0 & 0 \end{array}](0, 0)$$

$$+ \left(\theta[\begin{array}{c} \frac{1}{2} \\ 2 \end{array}][2u, 2v] + \theta[\begin{array}{c} -\frac{1}{2} \\ 2 \end{array}][2u, 2v]\right)\theta[\begin{array}{rr} \frac{1}{2} & -\frac{1}{2} \\ 0 & 0 \end{array}](0, 0)$$

$$= -\theta[\begin{array}{c} \frac{1}{2} \\ 2 \end{array}][0, 0] f(2u, 2v) + \theta[\begin{array}{c} \frac{1}{2} \\ 2 \end{array}][0, 0] g(2u, 2v). \tag{4.20}$$

Thus, we can express $f(2u, 2v)$ and $g(2u, 2v)$ with the product of the original theta functions in the form

$$f(2u, 2v) = \frac{P_3S_3 - P_0S_0}{2\theta[\begin{array}{rr} \frac{1}{2} & \frac{1}{2} \\ 0 & 0 \end{array}](0, 0)}, \quad g(2u, 2v) = \frac{P_3S_3 + P_0S_0}{2\theta[\begin{array}{rr} \frac{1}{2} & -\frac{1}{2} \\ 0 & 0 \end{array}](0, 0)}. \tag{4.21}$$

### 4.2 The differential equation (Step I)

By using Eqs.$(4.17)$, $(4.18)$, and $(4.21)$, we obtain the following differential equation

$$(dS_1P_1 - dP_1S_1) = i \frac{P_3S_3 - P_0S_0}{2\theta[\begin{array}{rr} \frac{1}{2} & \frac{1}{2} \\ 0 & 0 \end{array}](0, 0)}(\alpha du + \gamma dv) - i \frac{P_3S_3 + P_0S_0}{2\theta[\begin{array}{rr} \frac{1}{2} & -\frac{1}{2} \\ 0 & 0 \end{array}](0, 0)}(\beta du + \delta dv), \tag{4.22}$$

$$(dS_2P_2 - dP_2S_2) = i \frac{P_3S_3 - P_0S_0}{2\theta[\begin{array}{rr} \frac{1}{2} & \frac{1}{2} \\ 0 & 0 \end{array}](0, 0)}(\alpha du + \gamma dv) + i \frac{P_3S_3 + P_0S_0}{2\theta[\begin{array}{rr} \frac{1}{2} & -\frac{1}{2} \\ 0 & 0 \end{array}](0, 0)}(\beta du + \delta dv). \tag{4.23}$$

\[\text{In the Göpel’s expression, there is the overall minus sign.}\]
We can verify that Eq. (4.23) is obtained from Eq. (4.22) by replacing \( u \to u + 1, v \to v + 1 \). Under such replacement, \((dS_1 P_1 - dP_1 S_1) \to -(dS_2 P_2 - dP_2 S_2), (P_3 S_3 + P_0 S_0) \to (P_3 S_3 + P_0 S_0), (P_3 S_3 - P_0 S_0) \to -(P_3 S_3 - P_0 S_0)\), which provides the proof that Eq. (4.23) is obtained from Eq. (4.22). Thus, we obtain the following differential equation
\[
\frac{(dS_1 P_1 - dP_1 S_1) - (dS_2 P_2 - dP_2 S_2)}{P_3 S_3 + P_0 S_0} = -i(\beta du + \delta dv) = d\mu, \tag{4.24}
\]
\[
\frac{(dS_1 P_1 - dP_1 S_1) + (dS_2 P_2 - dP_2 S_2)}{P_3 S_3 - P_0 S_0} = i(\alpha du + \gamma dv) = d\nu. \tag{4.25}
\]
Which can be expressed in the form
\[
\frac{(P_1^2 dp - P_2^2 dq)}{P_3 P_2} = \frac{sdp - dq/s}{\phi} = d\mu, \tag{4.26}
\]
\[
\frac{(P_1^2 dp + P_2^2 dq)}{P_3 P_2} = \frac{sdp + dq/s}{\psi} = d\nu. \tag{4.27}
\]
\[
s = \frac{P_1}{P_2}, \quad \psi = \frac{P_3 S_3 - P_0 S_0}{P_1 P_2}, \quad \phi = \frac{P_3 S_3 + P_0 S_0}{P_1 P_2}. \tag{4.28}
\]
Therefore, we obtain the starting differential equation
\[
\frac{sdp - dq/s}{\phi} = d\mu, \quad \frac{sdp + dq/s}{\psi} = d\nu, \tag{4.29}
\]
\[
p = \frac{S_1}{P_1}, \quad q = \frac{S_2}{P_2}, \quad s = \frac{P_1}{P_2}, \quad \phi = \frac{P_3 S_3 + P_0 S_0}{P_1 P_2}, \quad \psi = \frac{P_3 S_3 - P_0 S_0}{P_1 P_2}. \tag{4.30}
\]

5 The differential equation with \( p \) and \( q \) (Step II)

5.1 The elimination of \( s \)-dependence in \( sdp \mp dq/s \)

By dividing \( P_1^2 P_2^2 \) in the Kummer surface relation Eq. (3.25), we obtain
\[
(1 - 2Ep^2 + p^4) s^4 + (1 - 2Eq^2 + q^4) - 2\left(F(1 + p^2 q^2) - C(p^2 + q^2) + 2Dpq\right)s^2 = 0, \tag{5.1}
\]
which provides
\[
(1 - 2Ep^2 + p^4)s^2 + \left(\frac{1 - 2Eq^2 + q^4}{s^2}\right) = 2\left(F(1 + p^2 q^2) - C(p^2 + q^2) + 2Dpq\right). \tag{5.2}
\]
This is given in the form
\[
\left(\sqrt{1 - 2Ep^2 + p^4} s \pm \sqrt{1 - 2Eq^2 + q^4}\right)^2
= 2\left(F(1 + p^2 q^2) - C(p^2 + q^2) + 2Dpq\right) \pm 2\sqrt{1 - 2Ep^2 + p^4} \sqrt{1 - 2Eq^2 + q^4}. \tag{5.3}
\]

\(^3\)In the Göpel’s expression, in the right-hand side of Eq. (4.23), there is the overall minus sign.

\(^4\)In Göpel’s expression, \((\pm 1)\) sign of \( sdp \mp dq/s \) is in the opposite
Introducing the function $\Delta$ of the form $\Delta(x) = \sqrt{1-2Ex^2+x^4}$, we can express the above in the form

$$\left(\frac{\Delta(p)}{s} + \frac{\Delta(q)}{s}\right)^2 = 2G(p,q) \pm 2\Delta(p)\Delta(q), \quad (5.4)$$

$$G(p,q) = \left(F(1+p^2q^2) - C(p^2 + q^2) + 2Dp\right). \quad (5.5)$$

Thus, we obtain the relation

$$\Delta(p) \frac{s}{s} + \Delta(q) \frac{s}{s} = \sqrt{2G(p,q) \pm 2\Delta(p)\Delta(q)}. \quad (5.6)$$

Therefore, in the combination of $(\Delta(p)s \pm \Delta(q)/s)$, $s$-dependence is eliminated. In order to eliminate $s$-dependence of the differential, we rearrange $(\Delta(p)s \pm \Delta(q)/s)$ in the form

$$s \frac{dp}{s} - \frac{dq}{s}$$

$$= \left(s\Delta(p) - \frac{\Delta(q)}{s}\right) \frac{1}{2} \left(\frac{dp}{\Delta(p)} + \frac{dq}{\Delta(q)}\right) + \left(s\Delta(p) + \frac{\Delta(q)}{s}\right) \frac{1}{2} \left(\frac{dp}{\Delta(p)} - \frac{dq}{\Delta(q)}\right)$$

$$= \sqrt{2G(p,q) - 2\Delta(p)\Delta(q)} \frac{1}{2} \left(\frac{dp}{\Delta(p)} + \frac{dq}{\Delta(q)}\right)$$

$$+ \sqrt{2G(p,q) + 2\Delta(p)\Delta(q)} \frac{1}{2} \left(\frac{dp}{\Delta(p)} - \frac{dq}{\Delta(q)}\right) \quad (5.7)$$

$$s \frac{dp}{s} + \frac{dq}{s}$$

$$= \left(s\Delta(p) + \frac{\Delta(q)}{s}\right) \frac{1}{2} \left(\frac{dp}{\Delta(p)} + \frac{dq}{\Delta(q)}\right) + \left(s\Delta(p) - \frac{\Delta(q)}{s}\right) \frac{1}{2} \left(\frac{dp}{\Delta(p)} - \frac{dq}{\Delta(q)}\right)$$

$$= \sqrt{2G(p,q) + 2\Delta(p)\Delta(q)} \frac{1}{2} \left(\frac{dp}{\Delta(p)} + \frac{dq}{\Delta(q)}\right)$$

$$+ \sqrt{2G(p,q) - 2\Delta(p)\Delta(q)} \frac{1}{2} \left(\frac{dp}{\Delta(p)} - \frac{dq}{\Delta(q)}\right) \quad (5.8)$$

Thus, we can express $sdp = \frac{dq}{s}$ only with $p$ and $q$.

### 5.2 The expression of $\psi$ and $\varphi$ with $p$ and $q$

**a) The expression of $(P_3^2S_3^2 + P_0^2S_0^2)/P_1^2P_2^2$ with $p$ and $q$**

Next, we express $(P_3^2S_3^2 + P_0^2S_0^2)/P_1^2P_2^2$ as the function of $p$ and $q$. By using the addition formula, we first express $P_3^2S_3^2, P_2^2S_2^2, P_0^2P_0^2, P_1^2P_2^2$ as the function of $T, U, V, W$ via Eqs.(2.5)–(2.8); in addition, by using Eq.(3.3) and Eq.(3.4), $P_3^2S_3^2, P_0^2S_0^2$ and $P_3^2S_2^2$ are expressed as the function of $P_1^2, S_1^2, P_2^2, S_2^2$. Thus, $(P_3^2S_3^2 + P_0^2S_0^2)/P_1^2P_2^2$ is expressed as the function of $p^2, q^2, s^2$ in the form

$$\psi^2 = \frac{P_3^2S_3^2 + P_0^2S_0^2}{P_1^2P_2^2} = \frac{(t^2w^2 - q^2w^2)(t^2v^2 - u^2w^2)}{(t^2 - w^2)^2(u^2 - v^2)^2}$$

$$\times \left(2C \left(1 - 2Ep^2 + p^4\right)s^2 + \frac{(1 - 2Eq^2 + q^4)}{s^2} - 4E(1 + p^2q^2) + 4(p^2 + q^2)\right). \quad (5.9)$$
The $s$-dependence is eliminated by using the Kummer’s quartic relation Eq. (5.2) of the form

\[(1 - 2Ep^2 + p^4)s^2 + \left(1 - \frac{2Eq^2 + q^4}{s^2}\right) = 2\left(F(1 + p^2q^2) - C(p^2 + q^2) + 2Dpq\right) = 2G(p, q)\]

Thus, we obtain the $s$-independent expression of \((P_3^2S_3^2 + P_0^2S_0^2)/P_1^2P_2^2\) in the form

\[\frac{1}{2}(\varphi^2 + \psi^2) = \frac{P_3^2S_3^2 + P_0^2S_0^2}{P_1^2P_2^2} = \frac{(tu^2 - v^2w^2)(tv^2 - u^2w^2)}{(t^2 - w^2)^2(u^2 - v^2)^2}\]

\[\times \left(4C(F(1 + p^2q^2) - C(p^2 + q^2) + 2Dpq) - 4E(1 + p^2q^2) + 4(p^2 + q^2)\right) \]

\[= b(1 + p^2q^2) - a(p^2 + q^2) + 2cpq, \quad (5.10)\]

where

\[a = \frac{C^2 - 1}{\sqrt{(C^2 - 1)(E^2 - 1)}}\quad , \quad b = \frac{CF - E}{\sqrt{(C^2 - 1)(E^2 - 1)}}\quad , \quad c = \frac{CD}{\sqrt{(C^2 - 1)(E^2 - 1)}}\]

and we used \(\sqrt{(C^2 - 1)(E^2 - 1)} = \frac{(t^2 - w^2)^2(u^2 - v^2)^2}{4(tu^2 - v^2w^2)(tv^2 - u^2w^2)}\). Thus, we obtain

\[\frac{1}{2}(\varphi^2 + \psi^2) = \frac{P_3^2S_3^2 + P_0^2S_0^2}{P_1^2P_2^2} = b(1 + p^2q^2) - a(p^2 + q^2) + 2cpq. \quad (5.11)\]

b) The expression of \(P_3^2S_3^2P_0^2S_0^2/P_1^4P_2^4\) with \(p\) and \(q\)

Similarly, we calculate

\[\varphi\psi = \frac{P_3^2S_3^2 - P_0^2S_0^2}{P_1^2P_2^2} = \frac{(tu + vw)(tv + uw)}{(t^2 - w^2)(u^2 - v^2)}\]

\[\times \left(\left(1 - 2Ep^2 + p^4\right)s^2 - \left(1 - \frac{2Eq^2 + q^4}{s^2}\right)\right)\]

\[= \frac{1}{2\sqrt{E^2 - 1}} \left(\left(1 - 2Ep^2 + p^4\right)s^2 - \left(1 - \frac{2Eq^2 + q^4}{s^2}\right)\right), \quad (5.12)\]

where \(2\sqrt{E^2 - 1} = \frac{(t^2 - w^2)(u^2 - v^2)}{(tu + vw)(tv + uw)}\) is used. Using Eq. (5.2) and Eq. (5.12), we obtain

\[\left(1 - 2Ep^2 + p^4\right)s^2 + \left(1 - \frac{2Eq^2 + q^4}{s^2}\right) = 2G(p, q), \quad (5.13)\]

\[\left(1 - 2Ep^2 + p^4\right)s^2 - \frac{(1 - 2Eq^2 + q^4)}{s^2} = 2\sqrt{E^2 - 1}\varphi\psi, \quad (5.14)\]

which provides

\[s^2 = \frac{G(p, q) + \sqrt{E^2 - 1}\varphi\psi}{1 - 2Ep^2 + p^4}, \quad \frac{1}{s^2} = \frac{G(p, q) - \sqrt{E^2 - 1}\varphi\psi}{1 - 2Eq^2 + q^4}, \quad (5.15)\]

Multiplying the first and the second term, we obtain

\[1 = \frac{G(p, q)^2 - (E^2 - 1)\varphi^2\psi^2}{(1 - 2Ep^2 + p^4)(1 - 2Eq^2 + q^4)}.\]
Thus, we obtain $\varphi^2\psi^2$ expressed with $p$ and $q$

$$(E^2 - 1)\varphi^2\psi^2 = G(p, q)^2 - (1 - 2Ep^2 + p^4)(1 - 2Eq^2 + q^4) = G(p, q)^2 - \Delta(p)^2\Delta(q)^2.$$ 

Hence, we obtain $\varphi\psi$ as the function of $p$ and $q$

$$\varphi\psi = \frac{P_3^2 S_3^2 - P_0^2 S_0^2}{P_1^2 P_2^2} = \sqrt{\frac{G(p, q)^2 - \Delta(p)^2\Delta(q)^2}{E^2 - 1}}. \quad (5.16)$$

Using Eq.(5.11) and Eq.(5.16), we obtain

$$K_1 = \frac{P_3^2 S_3^2 + P_0^2 S_0^2}{P_1^2 P_2^2} = \frac{1}{2}(\varphi^2 + \psi^2) = b(1 + p^2 q^2) - a(p^2 + q^2) + 2cpq, \quad (5.17)$$

$$K_2 = \frac{P_3^2 S_3^2 - P_0^2 S_0^2}{P_1^2 P_2^2} = \varphi\psi = \sqrt{\frac{G(p, q)^2 - \Delta(p)^2\Delta(q)^2}{E^2 - 1}}. \quad (5.18)$$

Thus, we obtain $P_3^2 S_3^2 P_0^2 S_0^2 / P_1^2 P_4^4$ in the form

$$\frac{P_3^2 S_3^2 P_0^2 S_0^2}{P_1^2 P_4^4} = \frac{K_1^2 - K_2^2}{4}$$

$$= \frac{1}{4} \left( b(1 + p^2 q^2) - a(p^2 + q^2) + 2cpq \right)^2 - \frac{G(p, q)^2 - \Delta(p)^2\Delta(q)^2}{E^2 - 1}$$

$$= \left( \frac{1}{2} b_1 (1 + p^2 q^2) - c_1 pq \right)^2, \quad (5.19)$$

$$b_1 = \frac{D}{\sqrt{(C^2 - 1)(E^2 - 1)}}, \quad c_1 = \frac{CE - F}{\sqrt{(C^2 - 1)(E^2 - 1)}},$$

where we used $C^2 - D^2 + E^2 + F^2 - 2CEF = 1$. Thus, we obtain

$$\frac{1}{4}(\varphi^2 - \psi^2) = \frac{P_3 S_3 P_0 S_0}{P_1 P_2^2} = \pm \left( \frac{1}{2} b_1 (1 + p^2 q^2) - c_1 pq \right). \quad (5.20)$$

For the sign ambiguity, we take $(+1)$ sign. Therefore, we obtain $\varphi^2$ and $\psi^2$ as the function of $p$ and $q$

$$\varphi^2 = \frac{P_3^2 S_3^2 + P_0^2 S_0^2 + 2P_3 S_3 P_0 S_0}{P_1^2 P_2^2} = (b + b_1)(1 + p^2 q^2) - a(p^2 + q^2) + 2(c - c_1)pq, \quad (5.21)$$

$$\psi^2 = \frac{P_3^2 S_3^2 + P_0^2 S_0^2 - 2P_3 S_3 P_0 S_0}{P_1^2 P_2^2} = (b - b_1)(1 + p^2 q^2) - a(p^2 + q^2) + 2(c + c_1)pq. \quad (5.22)$$

Thus, $\varphi$ and $\psi$ are expressed by the function of $p$ and $q$ in the form

$$\varphi = \sqrt{(b + b_1)(1 + p^2 q^2) - a(p^2 + q^2) + 2(c - c_1)pq}, \quad (5.23)$$

$$\psi = \sqrt{(b - b_1)(1 + p^2 q^2) - a(p^2 + q^2) + 2(c + c_1)pq}. \quad (5.24)$$
5.3 The differential equation with $p$ and $q$ (Step II)

Combining the previous results, we obtain the following differential equation, expressed with only $p$ and $q$

\[
\frac{1}{\varphi} \left\{ \sqrt{2G(p, q) - 2\triangle(p)\triangle(q)} \frac{1}{2} \left( \frac{dp}{\triangle(p)} + \frac{dq}{\triangle(q)} \right) + \sqrt{2G(p, q) + 2\triangle(p)\triangle(q)} \frac{1}{2} \left( \frac{dp}{\triangle(p)} - \frac{dq}{\triangle(q)} \right) \right\} = d\mu,
\]

\[
\frac{1}{\psi} \left\{ \sqrt{2G(p, q) + 2\triangle(p)\triangle(q)} \frac{1}{2} \left( \frac{dp}{\triangle(p)} + \frac{dq}{\triangle(q)} \right) + \sqrt{2G(p, q) - 2\triangle(p)\triangle(q)} \frac{1}{2} \left( \frac{dp}{\triangle(p)} - \frac{dq}{\triangle(q)} \right) \right\} = d\nu,
\]

(5.25)

\[
\varphi = \sqrt{(b + b_1)(1 + p^2q^2) - a(p^2 + q^2) + 2(c - c_1)pq},
\]

(5.27)

\[
\psi = \sqrt{(b - b_1)(1 + p^2q^2) - a(p^2 + q^2) + 2(c + c_1)pq},
\]

(5.28)

\[
G(p, q) = F(1 + p^2q^2) - C(p^2 + q^2) + 2Dpq, \quad \triangle(x) = \sqrt{1 - 2Ex^2 + x^4},
\]

(5.29)

\[
b = \frac{CF - E}{\sqrt{(C^2 - 1)(E^2 - 1)}}, \quad a = \frac{C^2 - 1}{\sqrt{(C^2 - 1)(E^2 - 1)}}, \quad c = \frac{CD}{\sqrt{(C^2 - 1)(E^2 - 1)}},
\]

\[
b_1 = \frac{D}{\sqrt{(C^2 - 1)(E^2 - 1)}}, \quad c_1 = \frac{CE - F}{\sqrt{(C^2 - 1)(E^2 - 1)}}.
\]

6 The differential equation with $y$ and $z$ (Step III)

6.1 The change of the functions from $p$, $q$ to $y$, $z$

Next, we change the function from $p$, $q$ into $y$, $z$ in such a way as the differential equation becomes separable. Because the combination of $\frac{dp}{\triangle(p)} \pm \frac{dq}{\triangle(q)}$ emerges, we change from $p$, $q$ to $y$, $z$ in such a way as $y$, $z$ satisfy the following differential equation

\[
\frac{dp}{\triangle(p)} = \frac{dy}{\triangle(y)} + \frac{dz}{\triangle(z)}, \quad \frac{dq}{\triangle(q)} = \frac{dy}{\triangle(y)} - \frac{dz}{\triangle(z)}.
\]

(6.1)

This is the differential equation, which provides the addition formula of the elliptic function for the Jacobi type the elliptic curve $y^2 = 1 - 2E x^2 + x^4 = \Delta(x)^2$. Thus, the addition formula gives

\[
p = \frac{y\Delta(z) + z\Delta(y)}{1 - y^2z^2}, \quad q = \frac{y\Delta(z) - z\Delta(y)}{1 - y^2z^2}.
\]

(6.2)

For more general Jacobi type elliptic curve $y^2 = 1 + \lambda_2 x^2 + \lambda_4 x^4$, we put $\Delta(x) = \sqrt{1 + \lambda_2 x^2 + \lambda_4 x^4}$ and the differential equation $\frac{dp}{\triangle(p)} = \frac{dy}{\triangle(y)} + \frac{dz}{\triangle(z)}$ provides the addition formula $p = \frac{y\Delta(z) + y\Delta(z)}{1 - \lambda_4 y^2 z^2}$. Using these functions, we obtain

\[
\frac{1}{2} \left( \frac{dp}{\triangle(p)} + \frac{dq}{\triangle(q)} \right) = \frac{dy}{\triangle(y)}, \quad \frac{1}{2} \left( \frac{dp}{\triangle(p)} - \frac{dq}{\triangle(q)} \right) = \frac{dz}{\triangle(z)}.
\]

(6.3)
6.2 $2G(p, q) \pm \Delta(p)\Delta(q)$ as the function of $y$ and $z$

Next, we calculate the necessary symmetric function of $p$ and $q$

$$pq = \frac{y^2\Delta(z)^2 - z^2\Delta(y)^2}{(1 - y^2z^2)^2} = \frac{y^2(1 - 2Ez^2 + z^4) - z^2(1 - 2Ey^2 + y^4)}{(1 - y^2z^2)^2}$$

$$= \frac{y^2 - z^2}{1 - y^2z^2}$$

$$p^2 + q^2 = \frac{2y^2\Delta(z)^2 + z^2\Delta(y)^2}{(1 - y^2z^2)^2} = \frac{2y^2(1 - 2Ez^2 + z^4) + z^2(1 - 2Ey^2 + y^4)}{(1 - y^2z^2)^2}$$

$$= \frac{2((1 + y^2z^2)(y^2 + z^2) - 4Ey^2z^2)}{1 - y^2z^2}$$

$$\Delta(p)\Delta(q) = \mp\frac{(1 + y^2z^2)^2 - 2E(1 + y^2z^2)(y^2 + z^2) + (y^2 + z^2)^2}{1 - y^2z^2}.$$  

(6.4)

(6.5)

(6.6)

For the ambiguity of sign in Eq. (6.6), we take $(-1)$ sign. Substituting the above expressions into

$$2G(p, q) \pm 2\Delta(p)\Delta(q) = 2\left(F(1 + p^2q^2) - C(p^2 + q^2) + 2Dpq\right) \pm 2\Delta(p)\Delta(q),$$

we obtain $2G(p, q) \pm 2\Delta(p)\Delta(q)$ as the function of $y$ and $z$

$$2G(p, q) \pm 2\Delta(p)\Delta(q)$$

$$= \frac{2(F + 1)}{(1 - y^2z^2)^2}\left(1 - 2\frac{(C + E - D)y^2}{(F + 1)} + y^4\right)\left(1 - 2\frac{(C + E + D)z^2}{(F + 1)} + z^4\right),$$

(6.7)

where we use the identity $C^2 - D^2 + E^2 + F^2 - 2CEF = 1$. It is quite surprising that the $y$ dependence and the $z$ dependence becomes separable in the second and the third term of Eq. (6.7). Thus, we obtain $2G(p, q) \pm 2\Delta(p)\Delta(q)$ as the function of $y$ and $z$

$$\sqrt{2G(p, q) - 2\Delta(p)\Delta(q)}$$

$$= \sqrt{\frac{2(F + 1)}{(1 - y^2z^2)^2}\left(1 - 2\frac{(C + E - D)y^2}{(F + 1)} + y^4\right)(1 - 2\frac{(C + E + D)z^2}{(F + 1)} + z^4)},$$

(6.8)

$$\sqrt{2G(p, q) + 2\Delta(p)\Delta(q)}$$

$$= \sqrt{\frac{2(F - 1)}{(1 - y^2z^2)^2}\left(1 - 2\frac{(C - E - D)y^2}{(F - 1)} + y^4\right)(1 - 2\frac{(C - E + D)z^2}{(F - 1)} + z^4)},$$

(6.9)

$$E_1 = \frac{(C - E - D)}{(F - 1)}, E_2 = \frac{(C + E - D)}{(F + 1)}, E_3 = \frac{(C + E + D)}{(F + 1)}, E_4 = \frac{(C - E + D)}{(F - 1)},$$

where we used the identity $C^2 - D^2 + E^2 + F^2 - 2CEF = 1$.

---

Göpel takes $(+1)$ sign. In such case, in connection with the sign of Eq. (6.10), we will see that the differential equation does not become of separable type.
6.3 The expression of $\varphi$ and $\psi$ with $y$ and $z$

Next, we calculate $\varphi^2$ and $\psi^2$ as the function of $y$ and $z$. First, we obtain

$$\varphi^2 = (b + b_1)(1 + p^2 q^2) - a(p^2 + q^2) + 2(c - c_1)pq$$

$$= (b + b_1) \left( 1 + p^2 q^2 - \frac{(C^2 - 1)}{(CF - E + D)}(p^2 + q^2) + \frac{2(CD - CE + F)}{(CF - E + D)}pq \right)$$

$$= \frac{b + b_1}{(1 - y^2 z^2)^2} \left( 1 - \frac{2(C - E - D)}{(F - 1)} y^2 + y^4 \right) \left( 1 - \frac{2(C + E + D)}{(F + 1)} z^2 + z^4 \right)$$

$$= \frac{b + b_1}{(1 - y^2 z^2)^2} (1 - 2E_1y^2 + y^4) (1 - 2E_3z^2 + z^4). \tag{6.10}$$

Similarly, we obtain

$$\psi^2 = (b - b_1)(1 + p^2 q^2) - a(p^2 + q^2) + 2(c + c_1)pq$$

$$= (b - b_1) \left( 1 + p^2 q^2 - \frac{(C^2 - 1)}{(CF - E - D)}(p^2 + q^2) + \frac{2(CD + CE - F)}{(CF - E - D)}pq \right)$$

$$= \frac{b + b_1}{(1 - y^2 z^2)^2} \left( 1 - \frac{2(C + E - D)}{(F - 1)} y^2 + y^4 \right) \left( 1 - \frac{2(C - E + D)}{(F + 1)} z^2 + z^4 \right)$$

$$= \frac{b + b_1}{(1 - y^2 z^2)^2} (1 - 2E_2y^2 + y^4) (1 - 2E_4z^2 + z^4), \tag{6.11}$$

where we used the identity $C^2 - D^2 + E^2 + F^2 - 2CEF = 1$. Thus, we obtain $\varphi$ and $\psi$ as the function of $y$ and $z$

$$\varphi = \frac{\sqrt{b + b_1}}{(1 - y^2 z^2)} \sqrt{(1 - 2E_1y^2 + y^4)(1 - 2E_3z^2 + z^4)}, \tag{6.12}$$

$$\psi = \frac{\sqrt{b - b_1}}{(1 - y^2 z^2)} \sqrt{(1 - 2E_2y^2 + y^4)(1 - 2E_4z^2 + z^4)}. \tag{6.13}$$

6.4 The differential equation with $y$ and $z$ (Step III)

Using Eqs. (6.3), (6.8), (6.9), (6.12) and (6.13), we can express the necessary quantities with $y$ and $z$

$$\frac{1}{2} \left( \frac{dp}{\triangle(p)} + \frac{dq}{\triangle(q)} \right) = \frac{dy}{\triangle(y)}, \quad \frac{1}{2} \left( \frac{dp}{\triangle(p)} - \frac{dq}{\triangle(q)} \right) = \frac{dz}{\triangle(z)}, \tag{6.14}$$

$$\sqrt{2G(p, q) - 2\Delta(p)\Delta(q)}$$

$$= \frac{\sqrt{2(F + 1)}}{(1 - y^2 z^2)} \sqrt{(1 - 2E_1y^2 + y^4)(1 - 2E_3z^2 + z^4)}, \tag{6.15}$$

$$\sqrt{2G(p, q) + 2\Delta(p)\Delta(q)}$$

$$= \frac{\sqrt{2(F - 1)}}{(1 - y^2 z^2)} \sqrt{(1 - 2E_2y^2 + y^4)(1 - 2E_4z^2 + z^4)}, \tag{6.16}$$

$$\varphi = \frac{\sqrt{b + b_1}}{(1 - y^2 z^2)} \sqrt{(1 - 2E_1y^2 + y^4)(1 - 2E_3z^2 + z^4)}, \tag{6.17}$$

$$\psi = \frac{\sqrt{b - b_1}}{(1 - y^2 z^2)} \sqrt{(1 - 2E_2y^2 + y^4)(1 - 2E_4z^2 + z^4)}. \tag{6.18}$$
Substituting these expressions into the following differential equations,

\[
\frac{1}{\varphi} \left\{ \sqrt{2G(p, q) - 2\triangle(p)\triangle(q)} \frac{1}{2} \left( \frac{dp}{\triangle(p)} + \frac{dq}{\triangle(q)} \right) + \sqrt{2G(p, q) + 2\triangle(p)\triangle(q)} \frac{1}{2} \left( \frac{dp}{\triangle(p)} - \frac{dq}{\triangle(q)} \right) \right\} = d\mu, \quad (6.19)
\]

\[
\frac{1}{\psi} \left\{ \sqrt{2G(p, q) + 2\triangle(p)\triangle(q)} \frac{1}{2} \left( \frac{dp}{\triangle(p)} + \frac{dq}{\triangle(q)} \right) + \sqrt{2G(p, q) - 2\triangle(p)\triangle(q)} \frac{1}{2} \left( \frac{dp}{\triangle(p)} - \frac{dq}{\triangle(q)} \right) \right\} = d\nu, \quad (6.20)
\]

we obtain the separable differential equation of \( y \) and \( z \) in the form

\[
d\mu = \frac{1 - y^2z^2}{\sqrt{b + b_1}} \frac{1}{\sqrt{(1 - 2E_1y^2 + y^4)(1 - 2E_3z^2 + z^4)}} \cdot \sqrt{\frac{2(F + 1)(1 - 2E_2y^2 + y^4)}{(1 - 2Ey^2 + y^4)(1 - 2E_1y^2 + y^4)}} dy + \sqrt{\frac{2(F - 1)(1 - 2E_4z^2 + z^4)}{(1 - 2E_2z^2 + z^4)(1 - 2E_4z^2 + z^4)}} dz.
\]

(6.21)

Similarly, we obtain

\[
d\nu = \frac{(1 - y^2z^2)}{\sqrt{b - b_1}} \frac{1}{\sqrt{(1 - 2E_2y^2 + y^4)(1 - 2E_4z^2 + z^4)}} \cdot \sqrt{\frac{2(F - 1)(1 - 2E_1y^2 + y^4)}{(1 - 2Ey^2 + y^4)(1 - 2E_2y^2 + y^4)}} dy + \sqrt{\frac{2(F + 1)(1 - 2E_3z^2 + z^4)}{(1 - 2E_2z^2 + z^4)(1 - 2E_3z^2 + z^4)}} dz.
\]

(6.22)

It is quite surprising that, if we use the functions \( y \) and \( z \), the differential equations become of the separable type.

## 7 The differential equation with \( y \) and \( y' \) (Step IV)

Next, we change the function in order that it provide the same type of Abelian differential.
7.1 The change of the function from $z$ to $y'$

The differential equation in the previous section provides

$$
\frac{d\mu}{\sqrt{b+b_1}} = \frac{\sqrt{2(F+1)(1-2E_2y^2+y^4)}}{\sqrt{(1-2Ey^2+y^4)(1-2E_1y^2+y^4)(1-2E_2y^2+y^4)}} dy
+ \frac{\sqrt{2(F-1)(1-2E_4z^2+z^4)}}{\sqrt{(1-2Ez^2+z^4)(1-2E_3z^2+z^4)(1-2E_4z^2+z^4)}} dz,
$$

(7.1)

$$
\frac{d\nu}{\sqrt{b-b_1}} = \frac{\sqrt{2(F-1)(1-2E_1y^2+y^4)}}{\sqrt{(1-2Ey^2+y^4)(1-2E_1y^2+y^4)(1-2E_2y^2+y^4)}} dy
+ \frac{\sqrt{2(F+1)(1-2E_3z^2+z^4)}}{\sqrt{(1-2Ez^2+z^4)(1-2E_3z^2+z^4)(1-2E_4z^2+z^4)}} dz.
$$

(7.2)

For the function $y$, we obtain the Abelian differential of the type

$$
\frac{dy}{\sqrt{(1-2Ey^2+y^4)(1-2E_1y^2+y^4)(1-2E_2y^2+y^4)}}.
$$

While, for the function $z$, we obtain the Abelian differential of the type

$$
\frac{dz}{\sqrt{(1-2Ez^2+z^4)(1-2E_3z^2+z^4)(1-2E_4z^2+z^4)}}.
$$

However, the type of the Abelian differential is different.

We keep $y$ to be the same function, yet we change the function $z$ into $y'$, which cause the Möbius transformation of $z^2$, and make the Abel function of the same type. We parametrize $2E$ with $\alpha$ in the form $2E = \alpha^2 + 1/\alpha^2$. For the constant $e$, we obtain $\Delta(e) = 1-2Ee^2+e^4 = (1-e^2/\alpha^2)(1-e^2/\beta^2)$, $\alpha\beta = 1$. We change the function from $y$, $z$ to $y'$, $y'$ ($y$ is not changed) by the following differential equation

$$
\frac{dz}{\Delta(z)} = \frac{dy'}{\Delta(y')} + \frac{de}{\Delta(e)} = \frac{dy'}{\Delta(y')},
$$

(7.3)

where we use $de = 0$ because $e$ is the constant. This provides

$$
z = \frac{y'\Delta(e) + e\Delta(y')}{1-e^2y'^2}.
$$

(7.4)

Next, we choose $e = \alpha$, thus we obtain $\Delta(e) = \Delta(\alpha) = 0$, hence $z$ is given as the function of $y'$

$$
z = \frac{\alpha\Delta(y')}{1-\alpha^2y'^2}.
$$

(7.5)

Considering the square of $z$, we obtain

$$
z^2 = \frac{\alpha^2 - y'^2}{1-\alpha^2y'^2}.
$$

(7.6)
Thus, \( y^2 \) is the Möbius transformation of \( z^2 \). Hence, we obtain

\[
\triangle(z)^2 = \left(1 - \frac{z^2}{\alpha^2}\right)\left(1 - \frac{z^2}{\alpha^2}\right) = \frac{(1 - \alpha^2)y'}{\alpha(1 - \alpha^2y^2)},
\]

which provides \( \Delta(z) \) as the function of \( y \) in the following form

\[
\Delta(z) = \frac{(1 - \alpha^2)y'}{\alpha(1 - \alpha^2y^2)},
\]

Using Eq. (7.5) and Eq. (7.8), we obtain the necessary quantities as the function of \( y \) and \( y' \)

\[
p = \frac{y\triangle(z) + z\triangle(y)}{1 - y^2z^2} = \frac{\beta(\alpha^4 - 1)yy' + \alpha\triangle(y)\triangle(y')}{1 - \alpha^2(y^2 + y'^2) + y^2y'^2},
\]

\[
q = \frac{y\triangle(z) - z\triangle(y)}{1 - y^2z^2} = \frac{\beta(\alpha^4 - 1)yy' - \alpha\triangle(y)\triangle(y')}{1 - \alpha^2(y^2 + y'^2) + y^2y'^2},
\]

\[
pq = \frac{y^2 - z^2}{1 - y^2z^2} = \frac{y^2 + y'^2 - \alpha^2(1 + y^2y'^2)}{1 - \alpha^2(y^2 + y'^2) + y^2y'^2}.
\]

We will use these relations in the next section.

**7.2 The differential equation with \( y \) and \( y' \) (Step IV)**

We calculate the following quantity as the function of \( y' \)

\[
\frac{1}{2}(z^2 + \frac{1}{z^2}) = \frac{1}{2}\left(\frac{\alpha^2 - y^2}{1 - \alpha^2y^2} + \frac{1 - \alpha^2y^2}{\alpha^2 - y^2}\right) = \frac{E - 2y^2 + Ey'^4}{1 - 2Ey'^2 + y'^4}.
\]

Thus, we obtain

\[
\frac{1 - 2E_3z^2 + z^4}{2z^2} = \frac{1}{2}(z^2 + \frac{1}{z^2}) - E_3 = \frac{(E - E_3) - 2(1 - EE_3)y^2 + (E - E_3)y'^4}{1 - 2Ey'^2 + y'^4}.
\]

Similarly, we obtain

\[
\frac{1 - 2E_4z^2 + z^4}{2z^2} = \frac{1}{2}(z^2 + \frac{1}{z^2}) - E_4 = \frac{(E - E_4) - 2(1 - EE_4)y^2 + (E - E_4)y'^4}{1 - 2Ey'^2 + y'^4}.
\]

By using the identity

\[
1 = E(E_1 + E_3) + E_1E_3 = 0, \quad 1 - E(E_2 + E_4) + E_2E_4 = 0, \quad \frac{E - E_4}{E - E_3} = \frac{F + 1}{F - 1},
\]

we obtain in the form

\[
\frac{1 - 2E_3z^2 + z^4}{2z^2} = \frac{(E - E_3) - 2(1 - EE_3)y^2 + (E - E_3)y'^4}{1 - 2Ey'^2 + y'^4}
\]

\[
= \frac{(E - E_3)(1 - 2E_1y^2 + y'^4)}{1 - 2Ey'^2 + y'^4}.
\]

\[
\frac{1 - 2E_4z^2 + z^4}{2z^2} = \frac{(E - E_4) - 2(1 - EE_4)y^2 + (E - E_4)y'^4}{1 - 2Ey'^2 + y'^4}
\]

\[
= \frac{(E - E_4)(1 - 2E_2y^2 + y'^4)}{1 - 2Ey'^2 + y'^4}.
\]
Taking the ratio of Eq. (7.14) and Eq. (7.15), we obtain the expression

\[
\frac{1 - 2E_3z^2 + z^4}{1 - 2E_4z^2 + z^4} = \left(\frac{F - 1}{F + 1}\right) \left(\frac{1 - 2E_1y^2 + y^4}{1 - 2E_2y^2 + y^4}\right), \quad (7.16)
\]

\[
\frac{1 - 2E_4z^2 + z^4}{1 - 2E_3z^2 + z^4} = \left(\frac{F + 1}{F - 1}\right) \left(\frac{1 - 2E_2y^2 + y^4}{1 - 2E_1y^2 + y^4}\right). \quad (7.17)
\]

Therefore, we obtain the differential equation of the same type Abelian differential

\[
d\mu = \frac{1}{\sqrt{b + b_1}} \left(\sqrt{\frac{2(F + 1)(1 - 2E_2y^2 + y^4)}{(1 - 2Ey^2 + y^4)(1 - 2E_1y^2 + y^4)}} \, dy + \sqrt{\frac{2(F - 1)(1 - 2E_4z^2 + z^4)}{(1 - 2Ez^2 + z^4)(1 - 2E_3z^2 + z^4)}} \, dz\right)
\]

\[
= \sqrt{\frac{2(F + 1)}{b + b_1}} \left(\frac{1 - 2E_2y^2 + y^4}{(1 - 2Ey^2 + y^4)(1 - 2E_1y^2 + y^4)} \, dy \right)
\]

\[
+ \sqrt{\frac{1 - 2E_2y^2 + y^4}{(1 - 2Ey^2 + y^4)(1 - 2E_1y^2 + y^4)} \, dy}\). \quad (7.18)
\]

Similarly, we obtain

\[
d\nu = \frac{1}{\sqrt{b - b_1}} \left(\sqrt{\frac{2(F - 1)(1 - 2E_1y^2 + y^4)}{(1 - 2Ey^2 + y^4)(1 - 2E_2y^2 + y^4)}} \, dy + \sqrt{\frac{2(F + 1)(1 - 2E_3z^2 + z^4)}{(1 - 2Ez^2 + z^4)(1 - 2E_4z^2 + z^4)}} \, dz\right)
\]

\[
= \sqrt{\frac{2(F - 1)}{b - b_1}} \left(\frac{1 - 2E_1y^2 + y^4}{(1 - 2Ey^2 + y^4)(1 - 2E_2y^2 + y^4)} \, dy \right)
\]

\[
+ \sqrt{\frac{1 - 2E_1y^2 + y^4}{(1 - 2Ey^2 + y^4)(1 - 2E_2y^2 + y^4)} \, dy}\). \quad (7.19)
\]

Thus, for \(dy\) and \(dy'\), we have the same type Abelian differential of the form

\[
dx \quad \sqrt{(1 - 2Ex^2 + x^4)(1 - 2E_1x^2 + x^4)(1 - 2E_3x^2 + x^4)}.
\]

8 The differential equation with \(x\) and \(x'\)(Step V)

8.1 The differential equation of the genus two Jacobi’s inversion problem

By changing the functions \(y, y'\) into \(x, x'\) in such a way as \(dx\) and \(dx'\) becomes the same Abelian differential of the fifth-degree hyperelliptic curve. For that purpose, we change the function in the form

\[
x = \left(\frac{1 - y^2}{1 + y^2}\right)^2, \quad x' = \left(\frac{1 - y'^2}{1 + y'^2}\right)^2. \quad (8.1)
\]

We denote in the following

\[
m = \frac{E + 1}{E - 1}, \quad m_1 = \frac{E_1 + 1}{E_1 - 1}, \quad m_2 = \frac{E_2 + 1}{E_2 - 1}. \quad (8.2)
\]

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Thus, we obtain the simple expression
\[
\frac{1 - m_2x}{1 - m_1x} = \left(\frac{E_1 - 1}{E_2 - 1}\right) \left(\frac{1 - 2E_2y^2 + y^4}{1 - 2E_1y^2 + y^4}\right), \quad \frac{1 - m_1x}{1 - m_2x} = \left(\frac{E_2 - 1}{E_1 - 1}\right) \left(\frac{1 - 2E_1y^2 + y^4}{1 - 2E_2y^2 + y^4}\right). \tag{8.3}
\]

Therefore, we obtain the simple expression
\[
\sqrt{1 - 2E_1y^2 + y^4} = \sqrt{\frac{E_1 - 1}{E_2 - 1}} \sqrt{1 - m_1x}, \quad \sqrt{1 - 2E_2y^2 + y^4} = \sqrt{\frac{E_2 - 1}{E_1 - 1}} \sqrt{1 - m_2x}, \quad \sqrt{1 - 2E_1y^2 + y^4} = \sqrt{\frac{E_1 - 1}{E_2 - 1}} \sqrt{1 - m_1x}, \quad \sqrt{1 - 2E_2y^2 + y^4} = \sqrt{\frac{E_2 - 1}{E_1 - 1}} \sqrt{1 - m_2x}. \tag{8.4}
\]

and
\[
\sqrt{1 - 2E_2y^2 + y^4} = \sqrt{\frac{E_2 - 1}{E_1 - 1}} \sqrt{1 - m_2x}, \quad \sqrt{1 - 2E_1y^2 + y^4} = \sqrt{\frac{E_1 - 1}{E_2 - 1}} \sqrt{1 - m_1x}. \tag{8.5}
\]

By differentiating \(y\), we obtain
\[
dx = -\frac{8((1 - y^2)y'dy}{(1 + y^2)^3}, \quad \sqrt{x} = \frac{1 - y^2}{1 + y^2}, \quad \sqrt{1 - x} = \frac{2y}{1 + y^2}, \quad \sqrt{1 - mx} = \sqrt{\frac{-2}{E - 1}} \sqrt{1 - 2Ey^2 + y^4}. \tag{8.6}
\]

Combining these relations, we obtain the connection of the differential of \(dx\) and \(dy\)
\[
\frac{dx}{\sqrt{x(1 - x)(1 - mx)}} = \frac{2\sqrt{2}\sqrt{-1}\sqrt{E - 1}}{\sqrt{1 - 2Ey^2 + y^4}} dy. \tag{8.7}
\]

Similarly, we obtain
\[
\frac{dx'}{\sqrt{x'(1 - x')(1 - mx')}} = \frac{2\sqrt{2}\sqrt{-1}\sqrt{E - 1}}{\sqrt{1 - 2Ey^2 + y^4}} dy'. \tag{8.8}
\]

Using Eqs. (7.18), (7.19), (8.7) and (8.8), we finally obtain the simplified differential equation
\[
\sqrt{\frac{2(F + 1)}{b + b_1}} \left(\sqrt{\frac{(1 - 2E_2y^2 + y^4)}{(1 - 2Ey^2 + y^4)(1 - 2E_1y^2 + y^4)}} dy + \sqrt{\frac{(1 - 2E_2y^2 + y^4)}{(1 - 2Ey^2 + y^4)(1 - 2E_1y^2 + y^4)}} dy'\right) = \frac{1}{2\sqrt{b + b_1}} \sqrt{\frac{(F + 1)(1 - E_2)}{(1 - E)(1 - E_1)}} \left(\sqrt{\frac{(1 - m_2x)}{x(1 - x)(1 - mx)}} dx + \sqrt{\frac{(1 - m_2x)}{x'(1 - x')(1 - mx')}} dx'\right) = d\mu, \tag{8.9}
\]
\[
\sqrt{\frac{2(F - 1)}{b - b_1}} \left(\sqrt{\frac{(1 - 2E_1y^2 + y^4)}{(1 - 2Ey^2 + y^4)(1 - 2E_2y^2 + y^4)}} dy + \sqrt{\frac{(1 - 2E_1y^2 + y^4)}{(1 - 2Ey^2 + y^4)(1 - 2E_2y^2 + y^4)}} dy'\right) = \frac{1}{2\sqrt{b + b_1}} \sqrt{\frac{(F + 1)(1 - E_2)}{(1 - E)(1 - E_1)}} \left(\sqrt{\frac{(1 - m_2x)}{x(1 - x)(1 - mx)}} dx + \sqrt{\frac{(1 - m_2x)}{x'(1 - x')(1 - mx')}} dx'\right) = d\nu. \tag{8.10}
\]

Therefore, we obtain the differential equation of the genus two Jacobi’s inversion problem
\[
\frac{(1 - m_2x)dx}{\sqrt{f_5(x)}} + \frac{(1 - m_2x')dx'}{\sqrt{f_5(x')}} = 2\sqrt{b + b_1} \sqrt{\frac{(1 - E)(1 - E_1)}{(F + 1)(1 - E_2)}} d\mu = d\hat{\mu}, \tag{8.11}
\]
\[
\frac{(1 - m_1x)dx}{\sqrt{f_5(x)}} + \frac{(1 - m_1x')dx'}{\sqrt{f_5(x')}} = 2\sqrt{b - b_1} \sqrt{\frac{(1 - E)(1 - E_2)}{(F - 1)(1 - E_1)}} d\nu = d\hat{\nu}, \tag{8.12}
\]
\[
f(x) = x(1 - x)(1 - mx)(1 - m_1x)(1 - m_2x). \tag{8.13}
\]
8.2 The expression of \( x \) and \( x' \) as the function of \( p \) and \( q \)

Next, we express \( x, x' \) with \( p, q \). From Eqs.\((7.9)-(7.11)\), we obtain the necessary quantities as the function of \( y \) and \( y' \)

\[
p + q = \frac{2\beta(\alpha^4 - 1)yy'}{1 - \alpha^2(y^2 + y'^2) + y^2y'^2}, \quad pq = \frac{y^2 + y'^2 - \alpha^2(1 + y'^2)}{1 - \alpha^2(y^2 + y'^2) + y^2y'^2},
\]

which provides

\[
1 + \alpha^2pq = \frac{(1 - \alpha^4)(1 + y'^2)}{1 + y^2y'^2 - \alpha^2(y^2 + y'^2)}, \quad 1 + \beta^2pq = \frac{(\beta^2 - \alpha^2)(y^2 + y'^2)}{1 + y^2y'^2 - \alpha^2(y^2 + y'^2)}.
\] \((8.14)\)

Thus, we obtain

\[
\frac{p + q}{1 + \alpha^2pq} = -\frac{2\beta yy'}{(1 + y^2y'^2)}, \quad \frac{p + q}{1 + \beta^2pq} = -\frac{2\alpha yy'}{(y^2 + y'^2)}.
\] \((8.15)\)

Hence, we can, in principle, express \( y^2 + y'^2 \) and \( yy' \) as the function of \( p \) and \( q \), which implies that \( x = (1 - y^2)^2/(1 + y^2)^2 \) and \( x' = (1 - y'^2)^2/(1 + y'^2)^2 \) can be expressed as the function of \( p \) and \( q \). For our purpose, we make the following combination

\[
f = \left( \frac{1 - yy'}{1 + yy'} \right)^2, \quad g = \left( \frac{y - y'}{y + y'} \right)^2,
\] \((8.16)\)

which provides the following nice factorization property

\[
f = \frac{(1 + \alpha p)(1 + \alpha q)}{(1 - \alpha p)(1 - \alpha q)} \quad g = \frac{(1 + \beta p)(1 + \beta q)}{(1 - \beta p)(1 - \beta q)}.
\] \((8.17)\)

Later, we use \( f + g, 1 + fg, \sqrt{fg} \). Thus, we calculate these quantities

\[
f + g = \frac{2(p^2q^2 + (\alpha - \beta)^2pq - (p^2 + q^2) + 1)}{(1 - \alpha p)(1 - \alpha q)(1 - \beta p)(1 - \beta q)}
\]

\[
eq \frac{2(p^2q^2 + (2E - 2)pq - (p^2 + q^2) + 1)}{(1 - \alpha p)(1 - \alpha q)(1 - \beta p)(1 - \beta q)},
\] \((8.18)\)

\[
1 + fg = \frac{2(p^2q^2 + (\alpha + \beta)^2pq + (p^2 + q^2) + 1)}{(1 - \alpha p)(1 - \alpha q)(1 - \beta p)(1 - \beta q)}
\]

\[
eq \frac{2(p^2q^2 + (2E + 2)pq + (p^2 + q^2) + 1)}{(1 - \alpha p)(1 - \alpha q)(1 - \beta p)(1 - \beta q)},
\] \((8.19)\)

\[
\sqrt{fg} = \pm \frac{\Delta(p)\Delta(q)}{(1 - \alpha p)(1 - \alpha q)(1 - \beta p)(1 - \beta q)},
\] \((8.20)\)

where we used \( \alpha^2 + \beta^2 = 2E, \alpha \beta = 1 \). We take \( +1 \) sign in Eq.\((8.20)\). Using Eq.\((8.16)\), we obtain

\[
\left( \frac{1 - yy'}{1 + yy'} \right) = \sqrt{f}, \quad \left( \frac{y - y'}{y + y'} \right) = \sqrt{g}.
\] \((8.21)\)

\(^6\text{Göpel take } (-1) \text{ sign in Eq.}(8.20).\)
thus, we obtain the desired form

\[
yy' = \frac{1 - \sqrt{f}}{1 + \sqrt{f}}, \quad \frac{y}{y'} = \frac{1 + \sqrt{g}}{1 - \sqrt{g}}.
\] (8.22)

Multiplying and dividing the first and the second term of Eq. (8.22), we obtain

\[
y^2 = \frac{(1 - \sqrt{f})(1 + \sqrt{g})}{(1 + \sqrt{f})(1 - \sqrt{g})}, \quad y'^2 = \frac{(1 - \sqrt{f})(1 - \sqrt{g})}{(1 + \sqrt{f})(1 + \sqrt{g})}.
\] (8.23)

Thus, we obtain the desired form

\[
\frac{1 - y^2}{1 + y^2} = \frac{\sqrt{f} - \sqrt{g}}{1 - \sqrt{f}g}, \quad \frac{1 - y'^2}{1 + y'^2} = \frac{\sqrt{f} + \sqrt{g}}{1 + \sqrt{f}g}.
\] (8.24)

Therefore, \(x\) and \(x'\) are expressed as the function of \(p\) and \(q\)

\[
x = \left(\frac{1 - y^2}{1 + y^2}\right)^2 = \frac{f + g - 2\sqrt{f}g}{1 + fg - 2\sqrt{f}g} = \frac{E(1 + p^2q^2) - (p^2 + q^2) - \Delta(p)\Delta(q)}{(E + 1)(pq + 1)^2}, \quad (8.25)
\]

\[
x' = \left(\frac{1 - y'^2}{1 + y'^2}\right)^2 = \frac{f + g + 2\sqrt{f}g}{1 + fg + 2\sqrt{f}g} = \frac{E(1 + p^2q^2) - (p^2 + q^2) + \Delta(p)\Delta(q)}{(E + 1)(pq + 1)^2}. \quad (8.26)
\]

### 8.3 The solution of the genus two Jacobi’s inversion problem

From the differential equation

\[
\frac{(1 - m_2 x)dx}{\sqrt{f_5(x)}} + \frac{(1 - m_2 x')dx'}{\sqrt{f_5(x')}} = d\tilde{\varphi}, \quad \frac{(1 - m_1 x)dx}{\sqrt{f_5(x)}} + \frac{(1 - m_1 x')dx'}{\sqrt{f_5(x')}} = d\tilde{\nu},
\]

we can rearrange in the standard Jacobi’s inversion problem of the form

\[
\frac{dx}{\sqrt{f_5(x)}} + \frac{dx'}{\sqrt{f_5(x')}} = dU_1, \quad \frac{xdx}{\sqrt{f_5(x)}} + \frac{x'dx'}{\sqrt{f_5(x')}} = dU_2,
\] (8.27)

where we obtain the expression \(U_1 = k_1 u_1 + k_2 u_2, \quad U_2 = k_3 u_1 + k_4 u_2\). We inversely express \(u_1 = \ell_1 U_1 + \ell_2 U_2, \quad u_2 = \ell_3 U_1 + \ell_4 U_2\). Thus, the solution of the Jacobi’s inversion problem is given by

\[
\varphi_{22}(U_1, U_2) = x + x' = \frac{2(E(1 + p^2q^2) - (p^2 + q^2))}{(E + 1)(pq + 1)^2}
\]

\[
= \frac{2E}{(E + 1)} - \frac{2(p^2 + q^2 - 2Epq)}{(E + 1)(1 + pq)^2}, \quad (8.28)
\]

\[
\varphi_{12}(U_1, U_2) = -xx' = \frac{-(E - 1)}{(E + 1)} + \frac{4(E - 1)pq}{(E + 1)(1 + pq)^2}, \quad (8.29)
\]

where

\[
p = \frac{S_1(u_1, u_2)}{P_1(u_1, u_2)} = \begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix} \begin{vmatrix} \ell_1 U_1 + \ell_2 U_2, \ell_3 U_1 + \ell_4 U_2 \end{vmatrix}, \quad (8.30)
\]

\[
q = \frac{S_2(u_1, u_2)}{P_2(u_1, u_2)} = \begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix} \begin{vmatrix} \ell_1 U_1 + \ell_2 U_2, \ell_3 U_1 + \ell_4 U_2 \end{vmatrix}. \quad (8.31)
\]
For the explicit functional form for $\varphi_{22}(U_1, U_2), \varphi_{12}(U_1, U_2)$, we must carefully determine $\ell_i, (i = 1, 2, 3, 4)$ by the $u_1 = 0, u_2 = 0$ value of various theta functions and derivative of the various theta functions.

9 Summary and Discussions

In the previous paper, we reviewed the Rosenhain’s paper to the Jacobi’s inversion problem for the genus two hyperelliptic integral. In this paper, we have reviewed the Göpel’s paper to the Jacobi’s inversion problem for the genus two hyperelliptic integral.

In the Rosenhain’s approach, the Riemann’s addition formula of the hyperelliptic theta function is used. The key identity of the Rosenhain’s approach is the three quadratic theta identities. In the Göpel’s approach, the addition formula by the duplication method is used. The key identity of the Göpel’s approach is the three quartic theta identities, Eq. (3.25), Eq. (5.11), Eq. (5.20). One of these identities, Eq. (3.25), is the quartic Kummer surface relation. Three quartic identities are given in the form

\[
\begin{align*}
P_1^4 + S_1^4 + P_2^4 + S_2^4 & - 2F(P_1^2 P_2^2 + S_1^2 S_2^2) + 2C(P_1^2 S_2^2 + P_2^2 S_1^2) \\
-2E(P_1^2 S_2^2 + P_2^2 S_1^2) - 4DP_1 S_1 P_2 S_2 &= 0, \\
P_3^2 S_3^2 + P_0^2 S_0^2 - b(P_1^2 P_2^2 + S_1^2 S_2^2) + a(P_1^2 S_2^2 + P_2^2 S_1^2) - 2c P_1 S_1 P_2 S_2 &= 0, \\
2P_3 S_3 P_0 S_0 - b_1(P_1^2 P_2^2 + S_1^2 S_2^2) + 2c_1 P_1 S_1 P_2 S_2 &= 0.
\end{align*}
\]

Starting from the genus two hyperelliptic theta functions, Göpel takes several steps to obtain the differential equation of the genus two Jacobi’s inversion equation. In step I, by using the addition formula of the genus two hyperelliptic theta functions, the derivative formula is obtained; thus the starting differential equation is obtained. In step II, by using three quartic theta identities, the differential equation of only $p$ and $q$ is obtained. In step III, by using the addition formula of the elliptic function of the form

\[
\frac{dp}{\Delta(p)} = \frac{dy}{\Delta(y)} + \frac{dz}{\Delta(z)} \quad \text{and} \quad \frac{dq}{\Delta(q)} = \frac{dy}{\Delta(y)} - \frac{dz}{\Delta(z)}
\]

, the separable type differential equation of $y$ and $z$ is obtained. In step IV, by using the addition formula of the elliptic function of the form

\[
\frac{dz}{\Delta(z)} = \frac{dy'}{\Delta(y')} + \frac{de}{\Delta(e)} = \frac{dy'}{\Delta(y')},
\]

with some special constant $e$, the differential equation of $y$ and $y'$ with the same type Abelian differential is obtained. In step V, via $x = \left(\frac{1 - y^2}{1 + y^2}\right)^2, x' = \left(\frac{1 - y'^2}{1 + y'^2}\right)^2$, the differential equation of $x$ and $x'$ with the Abelian differential of the fifth-degree hyperelliptic curve is obtained, i.e., the differential equation of the genus two Jacobi’s inversion problem is obtained.

In the paper on the comments of the Göpel’s paper, Jacobi pointed out the issue of the number of modules [16]. For the genus $g$ hyperelliptic curve $y^2 = \lambda_{2g+1} x^{2g+1} + \lambda_{2g} x^{2g} + \cdots + \lambda_1 x + \lambda_0$, we have the $2g + 2$ constant coefficients. However, we have the freedom to change, i) constant scale of $x$, ii) constant shift of $x$, iii) take the ratio of the coefficients, in order to provide the standard form. For example, in the $g = 1$ case, i) provides $y^2 = 4x^3 + \lambda_2 x^2 + \lambda_1 x + \lambda_0$. While, ii) provide $y^2 = 4x^3 - g_2 x - g_3$. In the step iii), $g_2^3/(g_2^3 - 27g_3^2)$ provides $\tau = \omega_3/\omega_1$ from $\omega_3$ and $\omega_1$ by taking the ratio of the constant coefficients. Thus, the number of the modules from the hyperelliptic curve is $2g + 2 - 3 = 2g - 1$. While, in the $g$-variable hyperelliptic theta function, in general, there are modules of the form $\tau_{ij}(= \tau_{ji}), (i, j = 1, 2, \cdots, g)$; thus, the number of the modules is $g(g + 1)/2$. Hence, for
$g \geq 3$, the number of the module for the hyperelliptic function $g(g + 1)/2$ becomes greater than that of the hyperelliptic curve $2g - 1$.

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