Definite Integral of $\alpha$-Fractal Functions

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**Abstract.** In this article the integration of the $\alpha$-fractal interpolation function $f^\alpha$ corresponding to any continuous function $f$ on a compact interval $I$ of $\mathbb{R}$ is estimated although there is no explicit form of $\alpha$-fractal interpolation function till now. Some results related to the definite integral of $f^\alpha$ are established. Also the flipped $\alpha$-fractal function $f^\alpha_f$ corresponding to the continuous function $f$ is constructed and a result is proved that relates the definite integrals of the fractal functions $f^\alpha_f$ and $f^\alpha$.

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1. Introduction

There are objects in nature whose geometric structure is very irregular and complicated. Such as surface of broken stone, the boundary when a drop of ink falls on a paper, trees, system of blood vessels, mountain ranges, coastlines, smoke etc. Benoit B. Mandelbrot [5] named these objects as fractals.

Barnsley [1] developed the fractal theory through introducing the concept of fractal interpolation function (FIF) and constructed the FIF using Hutchinson’s operator [4] on an iterated function system (IFS), whose attractor is the graph of a continuous function interpolating a certain data set. FIFs whose graphs are fractals have been broadly used in approximation theory, interpolation theory, financial series, computer graphics, signal processing etc.

Later, Navascués [7, 8], introduced the concept of $\alpha$-fractal interpolation function $f^\alpha$ which is a fractal perturbation corresponding to a continuous function $f \in C(I)$ defined on a compact interval $I$ of $\mathbb{R}$. It is discussed in details the theory and applications of $\alpha$-fractal interpolation function in the literature [8, 11, 12]. The function $f^\alpha$ is continuous but in general nowhere differentiable.


Barnsley et. al. [3] introduced the calculus of FIF and showed that integral of a FIF is also a FIF. In [14], the fractional order integral of FIF is discussed and it is shown that FIF can be integrated on any closed interval \([a, b] \subset [0, \infty)\). In the same paper, it is also proved that the fractional order integral of FIF is still a FIF on the interval \([0, b](b > 0)\).

In [13], a method for finding the numerical integration of affine fractal function is mentioned and an upper bound of error in computation of the integration of an affine fractal function with the integration of classical one is placed.

In the present paper, in Section 3 we estimate the definite integral of the \(\alpha\)-fractal function \(f^\alpha\) corresponding to a continuous function \(f\) on a compact interval \(I\). We obtain some results related to the definite integral of the function \(f^\alpha\). In Section 4 we constructed the flipped \(\alpha\)-fractal function \(f_{F}^{\alpha}\) corresponding to the continuous function \(f\) on the compact interval \(I_F\) which is the flipped version of the interval \(I\) about the \(y\)-axis. We proved that the definite integral of \(f_{F}^{\alpha}\) on \(I_F\) is equal to the definite integral of \(f^\alpha\) on \(I\).

2. Definitions and Notations

2.1. Fractal Interpolation Functions

Let \(N \geq 2\) be an integer. Consider a set of interpolation points \(D = \{(x_i, y_i) \in I \times \mathbb{R} : i = 0, 1, ..., N\}\), where \(\Delta : x_0 < x_1 < ... < x_N\) is a partition of the closed interval \(I = [x_0, x_N]\). Set \(I_i = [x_{i-1}, x_i]\) for \(i = 1, 2, ..., N\). Let \(L_i : I \to I_i, i = 1, 2, ..., N\), be contraction homeomorphisms with

\[
L_i(x_0) = x_{i-1}, L_i(x_N) = x_i,
\]

\[
|L_i(x) - L_i(y)| \leq a|x - y|,
\]

for all \(x, y \in I\) and for some \(0 \leq a < 1\). Let \(F_i : I \times \mathbb{R} \to \mathbb{R}, i = 1, 2, ..., N\), be continuous functions satisfying the join-up conditions

\[
F_i(x_0, y_0) = y_{i-1}, F_i(x_N, y_N) = y_i,
\]

\[
|F_i(x, y) - F_i(x, y')| \leq |\alpha_i||y - y'|,
\]

for all \(x \in I\) and for all \(y, y' \in \mathbb{R}\) and for some \(0 \leq |\alpha_i| < 1, i = 1, 2, ..., N\).

Define the mappings \(W_i : I \times \mathbb{R} \to I_i \times \mathbb{R}; i = 1, 2, ..., N\) by, for all \((x, y) \in I \times \mathbb{R}\),

\[
W_i(x, y) = (L_i(x), F_i(x, y)).
\]

**Theorem 2.1.** [2] The iterated function system \(\{I \times \mathbb{R}; W_i(x, y) : i = 1, ..., N\}\) defined in (2.1) admits a unique attractor \(G\), where \(G\) is the graph of a continuous function \(g : I \to R\) which obeys \(g(x_i) = y_i\) for \(i = 1, 2, ..., N\).

Let \(C^*(I) = \{f \in C(I) : f(x_0) = y_0, f(x_N) = y_N\}\) and \(C^{**}(I) = \{f \in C(I) : f(x_i) = y_i; i = 0, 1, 2, ..., N\}\).

The Read-Bajraktarvic (RB) operator \(T : C^*(I) \to C^{**}(I)\) defined by (see [6])

\[(Tf)(x) = F_i(L_i^{-1}(x), f(L_i^{-1}(x))); x \in I_i, i = 1, 2, ..., N,\]
is a contraction with contractivity factor \( |\alpha|_\infty = \max\{|\alpha_i| : i = 1, \ldots, N\} < 1 \). Due to Banach fixed point theorem, \( T \) has a unique fixed point \( g \) (say) which also interpolates the points of \( D \). This function \( g \) is called a fractal interpolation function or simply a fractal function corresponding to the IFS \( (2.1) \) and this unique function \( g \) satisfies the fixed point equation

\[
g(x) = F_i(L_i^{-1}(x), g(L_i^{-1}(x))); x \in I_i, i = 1, 2, \ldots, N. \tag{2.2}
\]

The free parameters \( \alpha_i, i = 1, 2, \ldots, N \) are the vertical scaling factors of the transformation \( W_i \) and the corresponding vector \( \alpha = (\alpha_1, \alpha_2, \ldots, \alpha_N) \) is called the scale vector of the IFS \( (2.1) \).

### 2.2. \( \alpha \)-Fractal Interpolation Functions

Let \( f \in C(I) \). Consider the IFS defined by the iterated mappings

\[
L_i(x) = a_i x + e_i \tag{2.3}
\]

and

\[
F_i(x, y) = \alpha_i y + f(L_i(x)) - \alpha_i b(x), \tag{2.4}
\]

where \( a_i = \frac{x_i - x_{i-1}}{x_N - x_0}, e_i = \frac{x_N x_{i-1} - x_0 x_i}{x_N - x_0}, 0 \leq |\alpha_i| < 1, i = 1, 2, \ldots, N \) and \( b \in C(I) \), known as base function which satisfy \( b(x_0) = f(x_0), b(x_N) = f(x_N) \).

Let \( f^\alpha (= f^\alpha_{\alpha,b}) \) be the continuous function whose graph is the attractor of the IFS represented by (2.3) and (2.4). This \( f^\alpha \) is called the \( \alpha \)-fractal interpolation function or simply \( \alpha \)-fractal function of \( f \) with respect to the base function \( b \) and the partition \( \Delta \). From (2.2), \( f^\alpha \) satisfies the fixed point equation

\[
f^\alpha(x) = f(x) + \alpha_i(f^\alpha - b)(L_i^{-1}(x)), \tag{2.5}
\]

for all \( x \in I_i, i = 1, 2, \ldots, N \). From (2.5), it is easy to deduce that

\[
|f^\alpha - f|_\infty \leq \frac{|\alpha|_\infty}{1 - |\alpha|_\infty} \|f - b\|_\infty.
\]

For \( \alpha = 0 \), the fractal function \( f^\alpha \) agrees with \( f \). More discussion of \( \alpha \)-fractal function for different choices of \( \alpha \) can be found in [7, 9, 10].

### 3. Definite Integral of \( \alpha \)-Fractal Functions

In this section, the definite integral \( \int_{x_0}^{x_N} f^\alpha(x)dx \) of the fractal function \( f^\alpha \) corresponding to any continuous function \( f \) defined on a compact interval \( I = [x_0, x_N] \) is estimated and some results related to the definite integrals of fractal functions are established.

**Theorem 3.1.** Let \( f^\alpha \) be the fractal function of \( f \in C(I) \) with base \( b \in C(I) \), then

\[
\int_{x_0}^{x_N} f^\alpha(x)dx = \frac{1}{1 - \lambda} \int_{x_0}^{x_N} f(x)dx - \frac{\lambda}{1 - \lambda} \int_{x_0}^{x_N} b(x)dx, \tag{3.1}
\]

where \( \lambda = \sum_{i=1}^{N} \alpha_i \alpha_i \).
Proof. For \( x \in I_i, i = 1, ..., N \), the self-referential equation for \( f^\alpha \):

\[
f^\alpha(x) = f(x) + \alpha_i(f^\alpha - b)(L_i^{-1}(x)).
\]

Then

\[
\int_{x_0}^{x_N} f^\alpha(x)dx = \int_{x_0}^{x_N} f(x)dx + \sum_{i=1}^{N} \alpha_i \int_{x_{i-1}}^{x_i} (f^\alpha - b)(L_i^{-1}(x))dx.
\]

Letting \( L_i^{-1}(x) = z \),

\[
\int_{x_0}^{x_N} f^\alpha(x)dx = \int_{x_0}^{x_N} f(x)dx + \left[ \sum_{i=1}^{N} a_i \alpha_i \right] \int_{x_0}^{x_N} (f^\alpha - b)(z)dz
\]

\[
= \int_{x_0}^{x_N} f(x)dx + \lambda \int_{x_0}^{x_N} (f^\alpha - b)(x)dx,
\]

where \( \lambda = \sum_{i=1}^{N} a_i \alpha_i \). Therefore

\[
\int_{x_0}^{x_N} f^\alpha(x)dx = \frac{1}{(1-\lambda)} \int_{x_0}^{x_N} f(x)dx - \frac{\lambda}{(1-\lambda)} \int_{x_0}^{x_N} b(x)dx.
\]

\[\square\]

In the following example, we calculate the definite integral of the fractal function \( f^\alpha(x) = (x^3 + x)^\alpha \), where \( \alpha = (0.2, -0.3, 0.5, 0.3, 0.4) \) with base function \( b(x) = 2x \), over the interval \([0, 1]\). The function \( f(x) = x^3 + x \) and its fractal function on \([0, 1]\) are shown in Figure 1.

Example. Let \( \Delta : 0 < 0.2 < 0.4 < 0.6 < 0.8 < 1 \) be a partion of \( I = [0, 1] \), then \( a_i = 0.2, i = 1, 2, ..., 5 \), and \( \lambda = \sum_{i=1}^{5} a_i \alpha_i = 0.22 \). Applying Theorem 3.1

\[
\int_{0}^{1} (x^3 + x)^\alpha dx = \frac{1}{(1-\lambda)} \int_{0}^{1} (x^3 + x)dx - \frac{\lambda}{(1-\lambda)} \int_{0}^{1} 2xdx
\]

\[
= \frac{1}{(1-0.22)} \left[ \frac{x^4}{4} + \frac{x^2}{2} \right]_0^1 - \frac{0.22}{(1-0.22)} \left[ x^2 \right]_0^1
\]

\[
= \frac{53}{78}.
\]

Corollary 3.2. Let the scale vector \( \alpha = (\alpha_1, \alpha_2, ..., \alpha_N) \) be such that \( \sum_{i=1}^{N} \alpha_i = 0 \). Then, for any uniform partition \( \Delta : x_0 < x_0 + h < x_0 + 2h < ... < x_0 + Nh = x_N \) of \( I = [x_0, x_N] \) and for any base function \( b \),

\[
\int_{x_0}^{x_N} f^\alpha(x)dx = \int_{x_0}^{x_N} f(x)dx.
\]

Proof. For uniform partition, \( a_i = \frac{x_i - x_{i-1}}{x_N - x_0} = \frac{h}{x_N - x_0} = \frac{1}{N}, i = 1, 2, ..., N \). The desired result is obtained by setting \( \lambda = \sum_{i=1}^{N} a_i \alpha_i = \frac{1}{N} \sum_{i=1}^{N} \alpha_i = 0 \) in Theorem 3.1. \[\square\]
Example. Consider $\Delta : 0 < 0.2 < 0.4 < 0.6 < 0.8 < 1$ as a partition of $I = [0,1]$. Let $f(x) = \frac{1}{x+1}$ and $b(x) = 1 - \frac{x}{2}$ be defined on $I = [0,1]$. Let $f^\alpha$ be the fractal function of $f$ with base function $b$. Suppose the scale vector $\alpha = (-0.2, 0.4, 0.3, -0.6, 0.1)$. Here $a_i = 0.2, i = 1, 2, ..., 5$, and $\lambda = \sum_{i=1}^{5} a_i \alpha_i = 0$. Therefore, applying Corollary 3.2

$$\int_0^1 f^\alpha(x) dx = \int_0^1 f(x) dx$$

$$= \int_0^1 \frac{dx}{x+1}$$

$$= \log 2.$$

Theorem 3.3. Let $\alpha = (\alpha_1, \alpha_2, ..., \alpha_N)$ and $\beta = (\beta_1, \beta_2, ..., \beta_N)$ be two scale vectors with $\sum_{i=1}^{N} \alpha_i = \sum_{i=1}^{N} \beta_i$. Suppose that $f^\alpha$ and $f^\beta$ are respectively the $\alpha$-fractal function and $\beta$-fractal function of $f$ with the base function $b$. Then, for any uniform partition $\Delta : x_0 < x_0 + h < x_0 + 2h < ... < x_0 + Nh = x_N$ of $I = [x_0, x_N]$,

$$\int_{x_0}^{x_N} f^\alpha(x) dx = \int_{x_0}^{x_N} f^\beta(x) dx.$$

Proof. For uniform partition, $a_i = \frac{x_i-x_{i-1}}{x_N-x_0} = \frac{h}{x_N-x_0} = \frac{1}{N}, i = 1, 2, ..., N$ and for the scale vectors $\alpha$ and $\beta$, $\lambda_\alpha = \frac{1}{N} \sum_{i=1}^{N} a_i \alpha_i = \frac{1}{N} \sum_{i=1}^{N} \alpha_i = \frac{1}{N} \sum_{i=1}^{N} \beta_i = \sum_{i=1}^{N} a_i \beta_i = \lambda_\beta$.

Therefore, applying Theorem 3.1

$$\int_{x_0}^{x_N} f^\alpha(x) dx = \frac{1}{(1-\lambda_\alpha)} \int_{x_0}^{x_N} f(x) dx - \frac{\lambda_\alpha}{(1-\lambda_\alpha)} \int_{x_0}^{x_N} b(x) dx$$

$$= \frac{1}{(1-\lambda_\beta)} \int_{x_0}^{x_N} f(x) dx - \frac{\lambda_\beta}{(1-\lambda_\beta)} \int_{x_0}^{x_N} b(x) dx$$

$$= \int_{x_0}^{x_N} f^\beta(x) dx.$$
Theorem 3.4. If $|\alpha|_\infty \to 0$, then
\[ \int_{x_0}^{x_N} f^\alpha(x) dx \to \int_{x_0}^{x_N} f(x) dx \]
and if $\alpha = 0$, then
\[ \int_{x_0}^{x_N} f^\alpha(x) dx = \int_{x_0}^{x_N} f(x) dx. \]

Proof. For first one, use $\lambda = \sum_{i=1}^{N} a_i \alpha_i \leq \sum_{i=1}^{N} a_i |\alpha|_\infty \to 0$ as $|\alpha|_\infty \to 0$ in Theorem 3.1. The second part is straight forward. □

Theorem 3.5. Let $f_b^\alpha, g_b^\alpha$ be the fractal functions of $f, g \in \mathcal{C}(I)$ with the base functions $b, \tilde{b} \in \mathcal{C}(I)$ respectively, then
\[ \int_{x_0}^{x_N} (\gamma f + \delta g)^\alpha_{\gamma b + \delta \tilde{b}}(x) dx = \gamma \int_{x_0}^{x_N} f_b^\alpha(x) dx + \delta \int_{x_0}^{x_N} g_b^\alpha(x) dx, \]
for any $\gamma, \delta \in \mathbb{R}$.

Proof. For $x \in I_i, i = 1, ..., N$, the self-referential equations for $f_b^\alpha$ and $g_b^\alpha$ are respectively
\[ f_b^\alpha(x) = f(x) + \alpha_i (f_b^\alpha - b)(L_i^{-1}(x)) \]
and
\[ g_b^\alpha(x) = g(x) + \alpha_i (g_b^\alpha - \tilde{b})(L_i^{-1}(x)). \]
The equality
\[ (\gamma f + \delta g)^\alpha_{\gamma b + \delta \tilde{b}} = \gamma f_b^\alpha + \delta g_b^\alpha \] (3.2)
is obtained from the uniqueness of the self-referential equation below
\[ (\gamma f_b^\alpha + \delta g_b^\alpha)(x) = (\gamma f + \delta g)(x) + \alpha_i \left[ (\gamma f_b^\alpha + \delta g_b^\alpha) - (\gamma b + \delta \tilde{b}) \right] (L_i^{-1}(x)). \]
The result is follows from the equation (3.2). □

Setting $\gamma = -1$ and $\delta = 0$ in Theorem 3.5, we get the following result.

Corollary 3.6. Let $f_b^\alpha$ be the fractal function of $f$ with base function $b$, then
\[ \int_{x_0}^{x_N} (-f)_{-b}^\alpha(x) dx = - \int_{x_0}^{x_N} f_b^\alpha(x) dx. \]

A supporting example to the Corollary 3.6 is given below.

Example. Let $I = [0, 1]$ and $\Delta : 0 < 0.2 < 0.4 < 0.6 < 0.8 < 1$ be a partition of $I$. Let $f(x) = x^3$ and $b(x) = x^2$ be defined on $I = [0, 1]$. Let $f_b^\alpha$ and $(-f)_{-b}^\alpha$ be the fractal functions of $f$ and $-f$ with base functions $b$ and $-b$ respectively. Suppose the scale vector $\alpha = (-0.1, 0, 0.1, 0.2, 0.3)$. Here
\( a_i = 0.2, i = 1, 2, \ldots, 5 \), and \( \lambda = \sum_{i=1}^{5} a_i \alpha_i = 0.1 \). Therefore, using Theorem 3.1

\[
\int_{0}^{1} (x^3)_{x^2}^{\alpha}(x) dx = \int_{0}^{1} f_{b}^{\alpha}(x) dx
\]

\[
= \frac{1}{(1 - \lambda)} \int_{0}^{1} f(x) dx - \frac{\lambda}{(1 - \lambda)} \int_{0}^{1} b(x) dx
\]

\[
= \frac{1}{(1 - 0.1)} \int_{0}^{1} x^3 dx - \frac{0.1}{(1 - 0.1)} \int_{0}^{1} x^2 dx
\]

\[
= \frac{13}{54}.
\]

and

\[
\int_{0}^{1} (-x^3)_{-x^2}^{\alpha}(x) dx = \int_{0}^{1} (-f)_{-b}^{\alpha}(x) dx
\]

\[
= \frac{1}{(1 - \lambda)} \int_{0}^{1} (-f(x)) dx - \frac{\lambda}{(1 - \lambda)} \int_{0}^{1} (-b(x)) dx
\]

\[
= \frac{1}{(1 - 0.1)} \int_{0}^{1} (-x^3) dx - \frac{0.1}{(1 - 0.1)} \int_{0}^{1} (-x^2) dx
\]

\[
= -\frac{13}{54}.
\]

**Lemma 3.1.** Let \( f_{b}^{\alpha} \) be the \( \alpha \)-fractal function of \( f \in C(I) \) with the base function \( b \in C(I) \). If \( g \in C(I) \) is linear, then \( g \circ f_{b}^{\alpha} \) is also \( \alpha \) -fractal function of \( g \circ f \) with the base function \( g \circ b \). In other words,

\[
(g \circ f)_{g \circ b} = g \circ f_{b}^{\alpha}.
\]

**Proof.** For \( x \in I_i, i = 1, \ldots, N \), we have

\[
f_{b}^{\alpha}(x) = f(x) + \alpha_i (f_{b}^{\alpha} - b)(L_i^{-1}(x)).
\]

Then, for \( x \in I_i, i = 1, \ldots, N \),

\[
(g \circ f_{b}^{\alpha})(x) = g(f_{b}^{\alpha}(x)) = (g \circ f)(x) + \alpha_i (g \circ f_{b}^{\alpha} - g \circ b)(L_i^{-1}(x)).
\]

Since \((g \circ f)(x_0) = (g \circ b)(x_0)\) and \((g \circ f)(x_N) = (g \circ b)(x_N)\), the uniqueness of the self-referential equation (3.3) gives the desired result.

**Theorem 3.7.** Let \( f_{b}^{\alpha} \) be the \( \alpha \)-fractal function of \( f \in C(I) \) with the base function \( b \in C(I) \). If \( g \in C(I) \) is linear, then

\[
\int_{x_0}^{x_N} (g \circ f_{b}^{\alpha})(x) dx = \frac{1}{(1 - \lambda)} \int_{x_0}^{x_N} (g \circ f)(x) dx - \frac{\lambda}{(1 - \lambda)} \int_{x_0}^{x_N} (g \circ b)(x) dx,
\]

where \( \lambda = \sum_{i=1}^{N} a_i \alpha_i \).
Proof. Since \( g \circ f_b^\alpha \) is the \( \alpha \)-fractal function of \( g \circ f \) with the base function \( g \circ b \), using Theorem 3.1
\[
\int_{x_0}^{x_N} (g \circ f_b^\alpha)(x)dx = \int_{x_0}^{x_N} (g \circ f)^\alpha_{gb}(x)dx
\]
\[
= \frac{1}{(1-\lambda)} \int_{x_0}^{x_N} (g \circ f)(x)dx - \frac{\lambda}{(1-\lambda)} \int_{x_0}^{x_N} (g \circ b)(x)dx.
\]

\[\square\]

4. Construction of Flipped \( \alpha \)-Fractal Function

Definition 4.1. Let \( I = [x_0, x_N] \) and \( I_F = [-x_N, -x_0] \). We can define the flipped function of \( f \in \mathcal{C}(I) \) about \( y \)-axis as
\[
f_F(-x) = f(x), \text{ for all } x \in I.
\]
or equivalently
\[
f_F(x) = f(-x), \text{ for all } x \in I_F.
\]

Let \( f^\alpha(= f^\alpha_{\Delta,b}) \) be the \( \alpha \)-fractal function associated to \( f \) with the base function \( b \) and the partition \( \Delta \). Recall the fixed point equation for \( f^\alpha \):
\[
f^\alpha(x) = f(x) + \alpha_i(f^\alpha - b)(L_i^{-1}(x)),
\]
for all \( x \in I_i, i = 1, 2, ..., N \).
Let \( \Delta_F : -x_N < -x_N-1 < ... < -x_0 \) is a partition of the closed interval \( I_F = [-x_N, -x_0] \). Set \( I_{Fi} = [-x_{N+1-i}, -x_{N-i}] \) for \( i = 1, 2, ..., N \). Let \( f_F \in \mathcal{C}(I_F) \) such that \( f_F(-x) = f(x) \), for all \( x \in I \) and let \( \alpha_F = (\alpha_N, \alpha_{N-1}, ..., \alpha_1) \). Define the RB operator \( T_{\Delta_F,b_F} : \mathcal{C}^*(I_F) \to \mathcal{C}^{**}(I_F) \) by
\[
(T_{\Delta_F,b_F}g)(x) = f_F(x) + \alpha_N+1-i(g - b_F)((L_F)^{-1})(x), x \in I_{Fi},
\]
where \( L_{Fi}(x) = a_{Fi}x + e_{Fi}, a_{Fi} = a_{N+1-i}, e_{Fi} = -e_{N+1-i}, i = 1, 2, ..., N \) and \( b_F \in \mathcal{C}(I_F) \) satisfying \( b_F(-x) = b(x) \), for all \( x \in I \). We can easily show that \( T_{\Delta_F,b_F} \) is a contraction map.

Let \( f^\alpha_F(= f^\alpha_{\Delta_F,b_F}) \) be the \( \alpha_F \)-fractal function of \( f_F \) with respect to \( I_F \).

From (4.1), \( f_F^\alpha \) satisfies the fixed point equation
\[
f_F^\alpha(x) = f_F(x) + \alpha_{N+1-i}(f_F^\alpha - b_F)((L_F)^{-1}(x)), x \in I_{Fi}, i = 1, 2, ..., N.
\]

From (4.2), for \( i = 1, 2, ..., N \)
\[
f_F^\alpha(-x) = f_F(-x) + \alpha_{N+1-i}(f_F^\alpha - b_F)((L_F)^{-1}(-x)), x \in I_{N+1-i}.
\]
Since \( (L_F)^{-1}(-x) = -L_{N+1-i}(x), i = 1, 2, ..., N \), then (4.3) becomes, for \( i = 1, 2, ..., N \)
\[
f_F^\alpha(-x) = f_F(-x) + \alpha_{N+1-i}(f_F^\alpha - b_F)(-L_{N+1-i}^{-1}(x)), x \in I_{N+1-i}.
\]
This above equation is the same as
\[
f_F^\alpha(-x) = f_F(-x) + \alpha_i(f_F^\alpha - b_F)(-L_i^{-1}(x)), x \in I_i, i = 1, 2, ..., N.
\]
Therefore, for $i = 1, 2, ..., N$

$$f^\alpha_F(-x) = f_F(-x) + \alpha_i f^\alpha_F(-L_i^{-1}(x)) - \alpha_i b_F(-L_i^{-1}(x)), x \in I_i$$
$$= f(x) + \alpha_i f^\alpha_F(-L_i^{-1}(x)) - \alpha_i b(L_i^{-1}(x)), x \in I_i.$$

Let $g$ be the flipped function of $f^\alpha_F$, then $f^\alpha_F(-x) = g(x), x \in I$. Now, for all $x \in I_i, i = 1, 2, ..., N$

$$g(x) = f^\alpha_F(-x)$$
$$= f(x) + \alpha_i f^\alpha_F(-L_i^{-1}(x)) - \alpha_i b(L_i^{-1}(x))$$
$$= f(x) + \alpha_i g(L_i^{-1}(x)) - \alpha_i b(L_i^{-1}(x))$$
$$= f(x) + \alpha_i (g - b)(L_i^{-1}(x)).$$

The uniqueness of the fixed point equation of $f^\alpha$ implies that

$$g(x) = f^\alpha(x) = f^\alpha_F(-x), x \in I_i, i = 1, 2, ..., N.$$

Thus we see that $f^\alpha_F$ is the flipped function of $f^\alpha$ about $y$-axis. We say that $f^\alpha_F$ is the flipped $\alpha$-fractal function of $f$.

**Example.** Figure 2 corresponds to the function $f(x) = \sqrt{x}$ on the interval $[0, 1]$ and its flipped function $(\sqrt{x})_F$ on $[-1, 0]$. The fractal function $f^\alpha$ associated with $f$ corresponding to the partition $\Delta : 0 < 0.2 < 0.4 < 0.6 < 0.8 < 1$, the scaling vector $\alpha = (0.3, 0.5, 0.2, 0.15, 0.02)$, and the base function $b(x) = x$ and the flipped $\alpha$-fractal function of $f$ are shown in Figure 3.

![Figure 2. The function \(\sqrt{x}\) and its flipped function \((\sqrt{x})_F\)](image)

**Theorem 4.1.** Suppose that $I = [x_0, x_N]$ be any compact interval. Let $f^\alpha$ be the fractal function of $f \in C(I)$ with base $b \in C(I)$, then

$$\int_{-x_N}^{-x_0} f^\alpha_F(x) dx = \int_{x_0}^{x_N} f^\alpha(x) dx.$$
Proof. For \( x \in I_{F_i}, i = 1, \ldots, N \), the self-referential equation for \( f_{F}^{\alpha} \) is

\[
f_{F}^{\alpha}(x) = f_{F}(x) + \alpha_{N+1-i}(f_{F}^{\alpha} - b_{F})((L_{F})_{i}^{-1}(x)).
\]

Then

\[
\int_{-x_{N}}^{-x_{0}} f_{F}^{\alpha}(x)dx = \int_{-x_{N}}^{-x_{0}} f_{F}(x)dx + \sum_{i=1}^{N} \alpha_{N+1-i} \int_{-x_{N+1-i}}^{-x_{N-i}} (f_{F}^{\alpha} - b_{F})((L_{F})_{i}^{-1}(x))dx.
\]

Letting \((L_{F})_{i}^{-1}(x) = z\),

\[
\int_{-x_{N}}^{-x_{0}} f_{F}^{\alpha}(x)dx = \int_{-x_{N}}^{-x_{0}} f_{F}(x)dx + \left[ \sum_{i=1}^{N} a_{N+1-i} \alpha_{N+1-i} \right] \int_{-x_{N}}^{-x_{0}} (f_{F}^{\alpha} - b_{F})(z)dz
\]

\[
= \int_{-x_{N}}^{-x_{0}} f_{F}(x)dx + \lambda_{F} \int_{-x_{N}}^{-x_{0}} (f_{F}^{\alpha} - b_{F})(x)dx,
\]

where \( \lambda_{F} = \sum_{i=1}^{N} a_{N+1-i} \alpha_{N+1-i} = \sum_{i=1}^{N} a_{i} \alpha_{i} = \lambda \).
Therefore
\[
\int_{-x_N}^{-x_0} f^\alpha_F(x) dx = \frac{1}{(1 - \lambda)} \int_{-x_N}^{-x_0} f_F(x) dx - \frac{\lambda}{(1 - \lambda)} \int_{-x_N}^{-x_0} b_F(x) dx
\]
\[
= \frac{1}{(1 - \lambda)} \int_{-x_N}^{-x_0} f(-x) dx - \frac{\lambda}{(1 - \lambda)} \int_{-x_N}^{-x_0} b(-x) dx
\]
\[
= \frac{1}{(1 - \lambda)} \int_{x_0}^{x_N} f(x) dx - \frac{\lambda}{(1 - \lambda)} \int_{x_0}^{x_N} b(x) dx
\]
\[
= \int_{x_0}^{x_N} f^\alpha(x) dx.
\]

\[\blacksquare\]

The following example is an application of the Theorem 4.1

**Example.** Consider the integral \( \int_{-1}^{0} (x^2)^\alpha_{-x}(x) dx \), where \( \alpha = (0.2, -0.1, 0, 0.3, 0.4) \).

To apply the Theorem 4.1, first we need to find out \( \int_{0}^{1} (x^2)^\alpha_{x}(x) dx \). For this, let \( \Delta : 0 < 0.2 < 0.4 < 0.6 < 0.8 < 1 \) be a partition of \( I = [0, 1] \), then \( a_i = 0.2, i = 1, 2, ..., 5 \), and \( \lambda = \sum_{i=1}^{5} a_i \alpha_i = 0.16 \). Using Theorem 3.1 and applying the Theorem 4.1,

\[
\int_{-1}^{0} (x^2)^\alpha_{-x}(x) dx = \int_{0}^{1} (x^2)^\alpha_{x}(x) dx
\]
\[
= \int_{0}^{1} x^2 dx - \frac{\lambda}{(1 - \lambda)} \int_{0}^{1} x dx
\]
\[
= \frac{1}{3(1 - 0.16)} [x^3]_0^1 - \frac{0.16}{2(1 - 0.16)} [x^2]_0^1
\]
\[
= \frac{19}{63}.
\]

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