Frontiers and Exact Learning of $\mathcal{ELI}$ Queries under DL-Lite Ontologies

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Abstract

We study $\mathcal{ELI}$ queries (ELIQs) in the presence of ontologies formulated in the description logic $\mathcal{DL-Lite}$. For the dialect $\mathcal{DL-Lite}^H$, we show that ELIQs have a frontier (set of least general generalizations) that is of polynomial size and can be computed in polynomial time. In the dialect $\mathcal{DL-Lite}^E$, in contrast, frontiers may be infinite. We identify a natural syntactic restriction that enables the same positive results as for $\mathcal{DL-Lite}^H$. We use our results on frontiers to show that ELIQs are learnable in polynomial time in the presence of a $\mathcal{DL-Lite}^H$/restricted $\mathcal{DL-Lite}^E$ ontology in Angluin’s framework of exact learning with only membership queries.

1 Introduction

In the widely studied paradigm of ontology-mediated querying, a database query is enriched with an ontology that provides domain knowledge as well as additional vocabulary for query formulation [Bienvenu et al., 2013b; Calvanese et al., 2009]. We consider ontologies formulated in description logics (DLs) of the $\mathcal{DL-Lite}$ family and queries that are $\mathcal{ELI}$ queries (ELIQs) or, in other words, tree-shaped unary conjunctive queries (CQs). $\mathcal{DL-Lite}$ is a prominent choice for the ontology language as it underpins the OWL 2QL profile of the OWL ontology language [OWL Working Group, 2009]. Likewise, ELIQs are a prominent choice for the query language as they are computationally very well-behaved: without an ontology, they can be evaluated in polynomial time in combined complexity, in contrast to NP-completeness for unrestricted CQs. Moreover, in the form of $\mathcal{ELI}$ concepts they are a central building block of ontologies in several dialects of $\mathcal{DL-Lite}$ and beyond.

The aim of this paper is to study the related topics of computing least general generalizations (LGGs) of ELIQs under $\mathcal{DL-Lite}$ ontologies and learning ELIQs under $\mathcal{DL-Lite}$ ontologies in Angluin’s framework of exact learning [Angluin, 1987a; Angluin, 1987b]. Computing generalizations is a natural operation in query engineering that plays a crucial role in learning logical formulas [Plotkin, 1970; Muggleton, 1991], in particular in exact learning [ten Cate and Dalmau, 2021]. Exact learning, in turn, is concerned with constructing queries and ontologies. This can be challenging and costly, especially when logic expertise and domain knowledge are not in the same hands. Aiming at such cases, exact learning provides a systematic protocol for query engineering in which a learner interacts in a game-like fashion with an oracle, which may be a domain expert.

Our results on LGGs concern the notion of a frontier of an ELIQ $q$ w.r.t. an ontology $O$. Such a frontier is a set $F$ of ELIQs that generalize $q$, that is, $q \subseteq O q_T$ and $q_T \nsubseteq O q$ for all $q_T \in F$, where ‘$\subseteq O$’ denotes query containment w.r.t. $O$. Moreover, $F$ must be complete in that for all ELIQs $q'$ with $q \subseteq O q'$ and $q' \nsubseteq O q$, there is a $q_T \in F$ such that $q_T \subseteq O q'$. We are interested in computing a frontier that contains only polynomially many ELIQs of polynomial size, in polynomial time. This is possible in the case of ELIQs without ontologies as shown in [ten Cate and Dalmau, 2021]; for the simpler $\mathcal{EL}$ queries, the same had been observed earlier (also without ontologies) in [Baader et al., 2018; Kriegel, 2019]. In contrast, unrestricted CQs do not even admit finite frontiers [Nesetril and Tardif, 2000].

In exact learning, the learner and the oracle know and agree on the ontology $O$, and they also agree on the target query $q_T$ to use only concept and role names from $O$. The learner may ask membership queries where they produce an ABox $A$ and a candidate answer $a$ and ask whether $A, O \models q_T(a)$, that is, whether $a$ is an answer to $q_T$ w.r.t. $O$ on $A$. The oracle faithfully answers “yes” or “no”. Polynomial time learnability then means that the learner has an algorithm for constructing $q_T$, up to equivalence w.r.t. $O$, with running time bounded by a polynomial in the sizes of $q_T$ and $O$.

Learning with only membership queries, as described above and studied in this article, is a strong form of exact learning. In fact, there are not many cases where polynomial time learning with only membership queries is possible, ELIQs without ontologies being an important example [ten Cate and Dalmau, 2021]. Often, one would therefore also admit equivalence queries where the learner provides a hypothesis ELIQ $q_H$ and asks whether $q_H$ is equivalent to $q_T$ under $O$; the oracle answers “yes” or provides a counterexample, that is, an ABox $A$ and answer $a$ such that $A, O \models q_T(a)$ and $A, O \not\models q_H(a)$ or vice versa. This is done, for instance, in [Konev et al., 2018; Funk et al., 2021].

We consider as ontology languages the DLs $\mathcal{DL-Lite}^H$ and
DL-Lite\(^F\), equipped with role inclusions (also known as role hierarchies) and functional roles, respectively. Both dialects admit concept and role disjointness constraints and ELI concept on the right-hand side of concept inclusions [Calvanese et al., 2007; Kikot et al., 2011]. We show that DL-Lite\(^H\) admits polynomial frontiers that can be computed in polynomial time, and that DL-Lite\(^F\) does not even admit finite frontiers. We then introduce a fragment DL-Lite\(^{F-}\) of DL-Lite\(^F\) that restricts the use of inverse functional roles on the right-hand side of concept inclusions and show that it is as well-behaved as DL-Lite\(^H\). Both frontier constructions require a rather subtle analysis. We also note that adding conjunction results in frontiers of exponential size, even for very simple fragments of DL-Lite. One application of our results is to show that every ELIQ \(q\) can be characterized up to equivalence w.r.t. ontologies formulated in DL-Lite\(^H\) or DL-Lite\(^{F-}\) by only polynomially many data examples of the form \((A, a)\), labeled as positive if \(A, O \models q(a)\) and as negative otherwise.

We then consider in detail the application of our results in the context of exact learning and show that ELIQs can be learned in polynomial time w.r.t. ontologies formulated in DL-Lite\(^H\) or DL-Lite\(^{F-}\). The learning algorithm uses only membership queries provided that a seed query is available, that is, an ELIQ \(q_0\) such that \(q_0 \subseteq O \models q\). Such a seed query can be constructed using membership queries if \(O\) contains no concept disjointness constraints and obtained by a single initial equivalence query otherwise. We also show that ELIQs cannot be learned at all w.r.t. unrestricted DL-Lite\(^F\) ontologies using only membership queries, and that they cannot be learned with only polynomially many membership queries when conjunction is admitted.

Proof details are in the appendix.

Related Work.\footnote{This depends on admitting assertions \(\top(a)\) in ABoxes.} Exact learning of queries in the context of description logics has been studied in [Funk et al., 2021] while [Konev et al., 2018] considers learning entire ontologies, see also [Ozaki et al., 2020; Ozaki, 2020]. It is shown in [Funk et al., 2021] that a restricted form of CQs (that do not encompass all ELIQs) can be learned in polynomial time under EL\(^C\) ontologies using only membership queries, and that they cannot be learned with only polynomially many membership queries when conjunction is admitted.

2 Preliminaries

Ontologies and ABoxes. Let \(\mathbb{N}_C, \mathbb{N}_R,\) and \(\mathbb{N}_I\) be countably infinite sets of concept, role, and individual names. A role \(R\) is a role name \(r \in \mathbb{N}_R\) or the inverse \(r^-\) of a role name \(r\). An ELI concept is formed according to the syntax rule \(C, D ::= \top \mid A \mid C \cap D \mid \exists R.C\) where \(A\) ranges over concept names and \(R\) over roles. A basic concept \(B\) is an ELI concept of the form \(\top, A,\) or \(\exists R.\top\). When dealing with basic concepts, for brevity we may write \(\exists R\) in place of \(\exists R.\top\).

A DL-Lite\(^{H,F}\) ontology \(O\) is a finite set of concept inclusions (CIs) \(B \subseteq C\), role inclusions (RIs) \(R_1 \subseteq R_2\), concept disjointness constraints \(B_1 \cap B_2 \subseteq \bot\), role disjointness constraints \(R_1 \cap R_2 \subseteq \bot\), and functionality assertions \(\text{func}(R)\). Here, \(B, B_1,\) and \(R_2\) range over basic concepts, \(O\) over EL\(^C\) concepts, and \(R_1, R_2\) over roles. Superscript \(^H\) indicates the presence of role inclusions (also called role hierarchies) and superscript \(^F\) indicates functionality assertions, and thus it should be clear what we mean with a DL-Lite\(^H\) ontology and with a DL-Lite\(^{F}\) ontology. In fact, we are mainly interested in these two fragments of DL-Lite\(^{H,F}\).

A DL-Lite\(^{H,F}\) ontology is in normal form if all concept inclusions in it are of one of the forms \(A \subseteq B, B \subseteq A,\) and \(A \subseteq \exists R.A'\) with \(A, A'\) concept names or \(\top\) and \(B\) a basic concept. Note that CIs of the form \(\exists R \subseteq \exists S\) are not admitted and neither are CIs of the form \(A \subseteq \exists R.C\) with \(C\) a compound concept. An ABox \(A\) is a finite set of concept assertions \(A(a)\) and role assertions \(r(a, b)\) with \(A\) a concept name or \(\top, r\) a role name, and \(a, b\) individual names. We use \(\text{ind}(A)\) to denote the set of individual names used in \(A\).

As usual, the semantics is given in terms of interpretations \(I\), which we define to be a (possibly infinite and) non-empty set of concept and role assertions. We use \(\Delta^2\) to denote the set of individual names in \(I\), define \(A^I = \{a \mid A(a) \in I\}\) for all \(A \in \mathbb{N}_C,\) and \(r^I = \{(a, b) \mid r(a, b) \in I\}\) and \((r^-)^I = \{(b, a) \mid r(a, b) \in I\}\) for all \(r \in \mathbb{N}_R\). This definition of interpretation is slightly different from the usual one, but equivalent;\footnote{This depends on admitting assertions \(\top(a)\) in ABoxes.} its virtue is uniformity as every ABox is a finite interpretation. The interpretation function \(\cdot^I\) can be extended from concept names to EL\(^C\) concepts in the standard way [Baader et al., 2017]. An interpretation \(I\) satisfies a concept or role inclusion \(A_1 \subseteq A_2\) if \(A_1^I \subseteq A_2^I\), a concept or role disjointness constraint \(A_1 \cap A_2 \subseteq \bot\) if \(A_1^I \cap A_2^I = \emptyset\), and a functionality assertion \(\text{func}(R)\) if \(R^I\) is a partial function. It satisfies a concept or role assertion \(A\) if \(A^I \cap R^I\) is countable.

An interpretation is a model of an ontology or an ABox if it satisfies all inclusions, disjointness constraints, and assertions in it. We write \(O \models A_1 \subseteq A_2\) if every model of the ontology \(O\) satisfies the concept or role inclusion \(A_1 \subseteq A_2\) and \(O \models A_1 \equiv A_2\) if \(O \models A_1 \subseteq A_2\) and \(O \models A_2 \subseteq A_1\). If \(A_1\) and \(A_2\) are basic concepts or roles, then such consequences are decidable in \(\text{PTIME}\) both in DL-Lite\(^H\) and in DL-Lite\(^F\) [Artale et al., 2009]. An ABox \(A\) is satisfiable w.r.t. an ontology \(O\) if \(A\) and \(O\) have a common model. Deciding ABox satisfiability is also in \(\text{PTIME}\) in both DL-Lite\(^H\) and DL-Lite\(^F\).

A signature is a set of concept and role names, uniformly referred to as symbols. For any syntactic object \(O\) such as an ontology or an ABox, we use \(\text{sig}(O)\) to denote the symbols used in \(O\) and \(|O|\) to denote the size of \(O\), that is, the length
of a representation of $O$ as a word in a suitable alphabet.

**Queries.** An ELL concept $C$ can be viewed as an ELL query (ELIQ). An individual $a \in \text{ind}(A)$ is an answer to $C$ on an ABox $A$ w.r.t. an ontology $O$, written $A, O \models C(a)$, if $a \in C^2$ for all models $I$ of $A$ and $O$. We shall often view ELIQs as unary conjunctive queries (CQs) and also consider CQs that are not ELIQs. In this paper, CQs are always unary. A CQ thus takes the form $q(x_0) = \exists y \phi(x_0, y)$ with $\phi$ a conjunction of concept atoms $A(x)$ and role atoms $r(x, y)$ where $A \in A_C$ and $r \in R_N$. We use $\var{q}$ to denote the set of variables that occur in $q$. We may view $q$ as a set of atoms and may write $r^-(x, y)$ in place of $r(y, x)$. We call $x_0$ the answer variable and use the notation of an answer and the notation $A, O \models \var{q}$ also for CQs. The formal definition is in terms of homomorphisms as usual, details are in the appendix. ELIQs are in 1-to-1 correspondence with CQs whose Gaifman graph is a tree and that contain no self-loops and multi-edges. For example, the ELIQ $C = A \cap \exists r^-(\exists s.B \cap \exists r^2.A)$ is the CQ $q(x_0) = \{A(x), r(y, x), s(y, z), B(z), r(y, z'), A(z')\}$. We use $A_q$ to denote the ABox obtained from CQ $q$ by varying variables as individuals and atoms as assertions. A CQ $q$ is satisfying w.r.t. ontology $O$ if $A_q$ is.

For CQs $q_1$ and $q_2$ and an ontology $O$, we say that $q_1$ is contained in $q_2$ w.r.t. $O$, written $q_1 \subseteq_O q_2$ if for all ABoxes $A$ and $a \in \text{ind}(A)$, $A, O \models q_1(a)$ implies $A, O \models q_2(a)$. If $q_1, q_2$ are ELIQs, then this coincides with $q_1$ viewed as ELL concept being subsumed w.r.t. $O$ by $q_2$ viewed as an ELL concept [Baader et al., 2017]. We call $q_1$ and $q_2$ equivalent w.r.t. $O$, written $q_1 \equiv_O q_2$, if $q_1 \subseteq_O q_2$ and $q_2 \subseteq_O q_1$.

**O-saturatedness and O-minimality.** A CQ $q$ is $O$-saturated, with $O$ an ontology, if $A_q, O \models A(y)$ implies $A(y) \in q$ for all $y \in \var{q}$ and $A \in A_C$. It is $O$-minimal if there is no $x \in \var{q}$ such that $q \equiv_O q[\var{q}\backslash \{x\}]$ with $q[\var{q}\backslash \{x\}]$ the restriction of $q$ to the atoms that only contain variables in $S$. For a CQ $q$ and an ontology $O$ formulated in DL-Lite$^H$ or DL-Lite$^F$, one can easily find in polynomial time an $O$-saturated CQ $q'$ with $q \equiv_O q'$. To achieve $O$-minimality, we may repeatedly choose variables $x \in \var{q}$, check whether $A_q[q[\var{q}\backslash \{x\}], O \models q$, and if so replace $q$ with $q[\var{q}\backslash \{x\}]$. For ELIQs, the required checks can be carried out in PTIME in DL-Lite$^F$ [Bienvenu et al., 2013a], but are NP-complete in DL-Lite$^H$ [Kikot et al., 2011]. We conjecture that in DL-Lite$^H$, it is not possible to construct equivalent $O$-minimal ELIQs in polynomial time.

### 3 Frontiers in DL-Lite$^H$

We show that for every ELIQ $q$ and DL-Lite$^H$ ontology $O$ such that $q$ is satisfiable w.r.t. $O$, there is a frontier of polynomial size that can be computed in polynomial time. We also observe that this fails when DL-Lite$^H$ is extended with conjunction, even in very restricted cases.

**Definition 1.** A frontier of an ELIQ $q$ w.r.t. a DL-Lite$^{HF}$ ontology $O$ is a set of ELIQs $\mathcal{F}$ such that

1. $q \subseteq_O q_F$ for all $q_F \in \mathcal{F}$;
2. $q_F \not\subseteq_O q$ for all $q_F \in \mathcal{F}$;
3. for all ELIQs $q'$ with $q \subseteq_O q'$ and $q' \not\subseteq_O q$, there is a $q_F \in \mathcal{F}$ with $q_F \subseteq_O q'$.

It is not hard to see that finite frontiers that are minimal w.r.t. set inclusion are unique up to equivalence of the ELIQs in them, that is, if $\mathcal{F}_1$ and $\mathcal{F}_2$ are finite minimal frontiers of $q$ w.r.t. $O$, then for every $q_F \in \mathcal{F}_1$ there is a $q'_F \in \mathcal{F}_2$ such that $q_F \equiv_O q'_F$ and vice versa. The following is the main result of this section.

**Theorem 1.** Let $O$ be a DL-Lite$^H$ ontology and an ELIQ that is $O$-minimal and satisfiable w.r.t. $O$. Then a frontier of $q$ w.r.t. $O$ can be computed in polynomial time.

We note that Theorem 1 still holds when $O$-minimality is dropped as a precondition and Condition 2 of frontiers is dropped as well. For proving Theorem 1, we first observe that we can concentrate on ontologies that are in normal form.

**Lemma 1.** For every DL-Lite$^H$ ontology $O$, we can construct in polynomial time a DL-Lite$^H$ ontology $O'$ in normal form such that every $O$-minimal ELIQ $q$ is also $O'$-minimal and a frontier of $q$ w.r.t. $O$ can be constructed in polynomial time given a frontier of $q$ w.r.t. $O'$.

We now prove Theorem 1, adapting and generalizing a technique from [ten Cate and Dalmau, 2021]. Let $O$ and $q(x_0)$ be as in the formulation of the theorem, with $O$ in normal form. We may assume w.l.o.g. that $q$ is $O$-saturated. To construct a frontier of $q$ w.r.t. $O$, we consider all ways to generalize $q$ in a least general way where ‘generalizing’ means to construct from $q$ an ELIQ $q'$ such that $q \subseteq_O q'$ and $q' \not\subseteq_O q$ and ‘least general way’ that there is no ELIQ $q$ that generalizes $q$ and satisfies $q \subseteq_O q'$ and $q' \not\subseteq_O q$. We do this in two steps: the actual generalization plus a compensation step, the latter being needed to guarantee that we indeed arrive at a least general generalization.

For $x \in \var{q}$, we use $q_x$ to denote the ELIQ obtained from $q$ by taking the subtree of $q$ rooted at $x$ and making $x$ the answer variable. The construction that follows involves the introduction of fresh variables $x$, some of which are a ‘copy’ of a variable from $\var{q}$. We then use $x^+$ to denote that original variable.

**Step 1: Generalize.** For each variable $x \in \var{q}$, define a set $\mathcal{F}_0(x)$ that contains all ELIQs which can be obtained by starting with $q_x(x)$ and then doing one of the following:

(A) **Drop concept atom:**

1. choose an atom $A(x) \in q$ such that
   (a) there is no $B(x) \in q$ with $O \models B \subseteq A$ and $O \not\models A \subseteq B$ and
   (b) there is no $R(x, y) \in q$ with $O \models \exists R \subseteq A$;
2. remove all $B(x) \in q$ with $O \models A \equiv B$, including $A(x)$.

(B) **Generalize subquery:**

1. choose an atom $R(x, y) \in q$ directed away from $x_0$;
2. remove $R(x, y)$ and all atoms of $q_y$;
3. for each $q'(y) \in \mathcal{F}_0(y)$, add a disjoint copy $q''_F$ of $q'$ and the role atom $R(x, y')$ with $y''$ the copy of $y$ in $q''$. 


4. for every role $S$ with $\mathcal{O} \models R \sqsubseteq S$ and $\mathcal{O} \not\models S \sqsubseteq R$, add a disjoint copy $\tilde{q}_o$ of $q_o$ and the role atom $S(x, y')$ with $y'$ the copy of $y$ in $\tilde{q}_o$.

The definition of $x^+$ should be clear in all cases. In Point 3 of Case (B), for example, for every variable $z$ in $q$ that was renamed to $z'$ in $\tilde{q}$ set $z'^+ = z$. Note that $z^+$ is defined for all variables $z$ that occur in queries in $\mathcal{F}_0(x)$. Also note that, in Point 1b of (A), it is important to use $q$ rather than $q_x$ as $y$ could be a predecessor of $x$ in $q$.

**Step 2: Compensate.** We construct a frontier $\mathcal{F}$ of $q(x_0)$ by including, for each $p \in \mathcal{F}_0(x_0)$, the ELIQ obtained from $p$ by the following two steps. We write $x \sim_{\tilde{q}, \mathcal{O}}^R A$ if $\mathcal{A}_q, \mathcal{O} \models \exists R.A(x)$ and there is no $S(x, y) \in q$ with $\mathcal{O} \models S \sqsubseteq R$ and $\mathcal{A}_q, \mathcal{O} \models A(y)$.

**Step 2A.** Consider all $x \in \text{var}(p)$, roles $R$, and concept names $A$ such that $x^+ \sim_{\tilde{q}, \mathcal{O}}^R A$, $\mathcal{O} \models R \sqsubseteq S$, and $\mathcal{O} \models \exists S \sqsubseteq B$ implies $B(x) \in p$ for all concept names $B$. Add the atoms $S(x, z), A(z), R(x', z)$ where $z$ and $x'$ are fresh variables with $z^+ = z^+$, and add a disjoint copy $\tilde{q}$ of $q$, glue the copy of $x^+$ in $\tilde{q}$ to $x'$.

**Step 2B.** Consider every $S(x, y) \in p$ directed away from $x_0$ that was not added in Step 2A. Then $x^+$ and $y^+$ are defined. For every role $R$ with $A_q, \mathcal{O} \models R(x', y')$, add an atom $R(z, y)$, $z$ a fresh variable with $z^+ = z^+$, as well as a disjoint copy $\tilde{q}$ of $q$ and glue the copy of $x^+$ in $\tilde{q}$ to $z$.

This finishes the construction of the frontier $\mathcal{F}$ of $q$.

**Example 1.** Consider the DL-Lite$^H$ ontology $\mathcal{O} = \{A \sqsubseteq \exists r, \exists r \sqsubseteq A, r \sqsubseteq s\}$ and the ELIQ $q(x_0) = A(x_0) \land B(x_0)$. Then $\mathcal{F}$ contains the ELIQs $p_1$ and $p_2$ shown below:

$$
\begin{array}{ccc}
A, B & x_0 & B \\
\sim & & \sim \\
\ | & & |\\
\setminus & & \setminus \\
x & & x \\
\bigg\}\ |igg\} \\
\end{array}
$$

ELIQ $p_1$ is the result of dropping the concept atom $A(x_0)$ and $p_2$ is the result of dropping the concept atom $B(x_0)$. Step 2A adds an $r$-successor and an $s$-successor of $x_0$ in $p_2$ but only an $s$-successor in $p_1$ as $\mathcal{O} \models \exists r \sqsubseteq A$, and then attaches copies of $q$. Step 2B does nothing, as all role atoms have been added in Step 2A.$^2$

**Example 2.** Consider the DL-Lite$^H$ ontology $\mathcal{O} = \{r \sqsubseteq s\}$ and the ELIQ $q(x_0)$ shown on the left-hand side below:

$$
\begin{array}{ccc}
x_0 & x_0 & x_1 \\
| & & | \\
\sim & & \sim \\
\ | & & |\\
r & & x \\
\bigg\}\ |igg\} \\
\end{array}
$$

Then $\mathcal{F}$ contains only the ELIQ $p$ shown on the right-hand side. It is the result of dropping the concept atom $A(y)$ in $q_y$, then generalizing the subquery $r(x_0, y)$ in $q_{x_0} = q$, and then compensating. Step 2A of compensation adds nothing. Step 2B adds the two dashed role atoms and attaches copies of $q$ to $x_1$ and $x_2$.

$^2$Variables $x_2$ and $z_2$ can be dropped from $p_2$ resulting in an ELIQ that is equivalent w.r.t. $\mathcal{O}$. We did not include such optimizations in the compensation step to avoid making it more complicated.

**Lemma 2.** $\mathcal{F}$ is a frontier of $q(x_0)$ w.r.t. $\mathcal{O}$.

We next show that the constructed frontier is of polynomial size and that its computation takes only polynomial time.

**Lemma 3.** The construction of $\mathcal{F}$ runs in time polynomial in $||q|| + ||\mathcal{O}||$ (and thus $\sum_{p \in \mathcal{F}} ||p||$ is polynomial in $||q|| + ||\mathcal{O}||$).

We next observe that adding conjunction to DL-Lite destroys polynomial frontiers and thus Theorem 1 does not apply to DL-Lite$^H$ ontologies [Artale et al., 2009]. In fact, this already holds for very simple queries and ontologies, implying that also for other DLs that support conjunction such as $\mathcal{EL}$, polynomial frontiers are elusive. A conjunction of atomic queries (AQs) is a unary CQ of the form $q(x_0) = A_1(x_0) \land \cdots \land A_n(x_0)$ and a conjunction ontology is a set of CI's of the form $A_1 \sqcap \cdots \sqcap A_n \sqsubseteq A$ where $A_1, \ldots, A_n$ and $A$ are concept names.

**Theorem 2.** There are families of AQ's $q_1, q_2, \ldots$ and conjunctive ontologies $\mathcal{O}_1, \mathcal{O}_2, \ldots$ such that for all $n \geq 1$, any frontier of $q_n$ w.r.t. $\mathcal{O}_n$ has size at least $2^n$.

**4 Frontiers in DL-Lite$^F$**

We start by observing that frontiers of ELIQs w.r.t. DL-Lite$^F$ ontologies may be infinite. This leads us to identifying a syntactic restriction on DL-Lite$^F$ ontologies that regains finite frontiers. In fact, we show that they are of polynomial size and can be computed in polynomial time.

**Theorem 3.** There is an ELIQ $q$ and a DL-Lite$^F$ ontology $\mathcal{O}$ such that $q$ does not have a finite frontier w.r.t. $\mathcal{O}$.

In the proof of Theorem 3, we use the ELIQ $A(x)$ and $\mathcal{O} = \{A \sqsubseteq \exists r, \exists r \sqsubseteq A, r \sqsubseteq s, \text{func}(r^-)\}$.

The universal model $U_{\mathcal{O}, \mathcal{A}_q}$ of $\mathcal{A}_q$ and $\mathcal{O}$ is an infinite $r$-path on which every point has an $s$-successor. Now consider the following ELIQs $q_1, q_2, \ldots$ that satisfy $q_i \not\sqsubseteq \mathcal{O} \sqsubseteq q_i$: $q_i(x_1) = r(x_1, x_2), \ldots, r(x_{n-1}, x_n), s(x_n, y), s(x_n, y), r(x_{n-1}, x_n), A(x_1)^i$.

Any frontier $\mathcal{F}$ must contain a $p_i$ with $p_i \sqsubseteq q_i$ for all $i \geq 1$. We show that, consequently, there is no bound on the size of the queries in $\mathcal{F}$. We invite the reader to apply the frontier construction from Section 3 after dropping $\text{func}(r^-)$.

The proof actually shows that there is no finite frontier even if we admit the use of unrestricted CQ's in the frontier in place of ELIQs. To regain finite frontiers, we restrict our attention to DL-Lite$^F_-$ ontologies $\mathcal{O}$ such that if $B \sqsubseteq C$ is a CI in $\mathcal{O}$, then $C$ contains no subconcept of the form $\exists R.D$ with $\text{func}(R^-) \in \mathcal{O}$. We call such an ontology a DL-Lite$^F_-$ ontology. We again concentrate on ontologies in normal form.

**Lemma 4.** For every DL-Lite$^F_-$ ontology $\mathcal{O}$, we can construct in polynomial time a DL-Lite$^F_-$ ontology $\mathcal{O}'$ in normal form such that for every ELIQ $q$, a frontier of $q$ w.r.t. $\mathcal{O}$ can be constructed in polynomial time given a frontier of $q$ w.r.t. $\mathcal{O}'$. 

The main result of this section is as follows.

**Theorem 4.** Let \(\mathcal{O}\) be a DL-Lite\(^R\) ontology and \(q\) an ELIQ that is satisfiable w.r.t. \(\mathcal{O}\). Then a frontier of \(q\) w.r.t. \(\mathcal{O}\) can be computed in polynomial time.

To prove Theorem 4, let \(\mathcal{O}\) and \(q\) be as in the theorem, \(\mathcal{O}\) in normal form. We may assume w.l.o.g. that \(q\) is \(\mathcal{O}\)-minimal and \(\mathcal{O}\)-saturated. The construction of a frontier follows the same general approach as for DL-Lite\(^H\), but the presence of functional roles significantly complicates the compensation step. As before, we introduce fresh variables and rely on the mapping \(\chi\).

**Step 1: Generalize.** For each variable \(x \in \text{var}(q)\), define a set \(\mathcal{F}_0(x)\) that contains all ELIQs which can be obtained by starting with \(q_\chi(x)\) and then doing one of the following:

(A) **Drop concept atom:** exactly as for DL-Lite\(^H\).

(B) **Generalize subquery:**

1. choose an atom \(R(x, y) \in q\) directed away from \(x_0\);
2. remove \(R(x, y)\) and all atoms of \(\chi_q\);
3. if \(\text{func}(R) \notin \mathcal{O}\), then for each \(q'(y) \in \mathcal{F}_0(y)\) add a disjoint copy \(\hat{q}'\) of \(q'\) and the role atom \(R(x, y')\) with \(y'\) the copy of \(y\) in \(\hat{q}'\);
4. if \(\text{func}(R) \in \mathcal{O}\) and \(\mathcal{F}_0(y) \neq \emptyset\), then choose and add a \(q' \in \mathcal{F}_0(y)\) and the role atom \(R(x, y)\).

**Step 2: Compensate.** We construct a frontier \(\mathcal{F}\) of \(q(x_0)\) by including, for each \(p \in \mathcal{F}_0(x_0)\), the CQ obtained from \(p\) by the following two steps. For \(x \in \text{var}(q)\), \(R\) a role, and \(M\) a set of concept names from \(\mathcal{O}\), we write \(x \sim_{\mathcal{O}} M\) if \(M\) is maximal with \(\mathcal{A}_q, \mathcal{O} |\!|= \exists R. \bigcap M(x)\) and there is no \(R(x, y) \in q\) with \(\mathcal{A}_q, \mathcal{O} |\!|= \bigcap M(y)\).

Step 2A. Consider every \(x \in \text{var}(p)\), role \(R\), and set of concept names \(M = \{A_1, \ldots, A_k\}\) with \(\chi(x) \sim_{\mathcal{O}} M\) if \(\mathcal{O} |\!|= \exists R. \bigcap M(x)\) and there is no \(R(x, y) \in q\) with \(\mathcal{A}_q, \mathcal{O} |\!|= \bigcap M(y)\).

Step 2B. This step is iterative. For bookkeeping, we mark atoms \(R(x, y) \in p\) to be processed in the next round of the iteration. Marking is only applied to atoms \(R(x, y)\) directed away from \(x_0\) such that \(y^i\) is defined and if \(x^i\) is undefined then \(\text{func}(R^i) \notin \mathcal{O}\) or \(q\) contains no atom of the form \(R(y^i, z)\).

Start. Consider every \(R(x, y) \in p\) directed away from \(x_0\) with \(\text{func}(R^i) \notin \mathcal{O}\) or \(q\) contains no atom of the form \(R(y^i, z)\). Then \(x^i\) is defined. Extend \(p\) with atom \(R^{-}(y, x')\) and for all fresh variables \(z^i\) extend \(p\) with atom \(S(y, x, z)\) and glue the copy of \(y^i\) in \(\hat{q}\) to \(y\). Otherwise, do the following:

(i) For all atoms \(S(y^i, z) \in q\) with \(S(y^i, z) \neq R^{-}(y, x')\), extend \(p\) with atom \(S(y, z, z')\) and glue the copy of \(z^i\) to \(z\). Mark \(S(y, z')\). Let \(\mathcal{E}^+, \mathcal{E}^-\) be finite sets of data examples. We say that an ELIQ \(q\) fits \((\mathcal{E}^+, \mathcal{E}^-)\) w.r.t. a DL-Lite\(^H\) ontology \(\mathcal{O}\) if \((A, a) \in \mathcal{E}^+\) implies \(\mathcal{A}, \mathcal{O} |\!|= q(a)\) and \(\mathcal{A}, \mathcal{O} |\!|= q(a)\).
Algorithm 1 Learning ELIQs under DL-Lite ontologies

**Input** An ontology $\mathcal{O}$ in normal form and a CQ $q_0^\mathcal{O}$ satisfiable w.r.t. $\mathcal{O}$ such that $q_0^\mathcal{O} \subseteq \mathcal{O} \ q_\mathcal{O}$

**Output** An ELIQ $q_H$ such that $q_H \equiv_\mathcal{O} q_T$

$q_H := \text{treeify}(q_0^\mathcal{O})$

while there is a $q_F \in F_{q_H}$ with $q_F \subseteq \mathcal{O} q_\mathcal{O}$ do

$q_H := \text{minimize}(q_F)$

done

return $q_H$

and $(\mathcal{A}, a) \in E^+$ implies $\mathcal{A}, \mathcal{O} \not\models q(a)$. Then $(E^+, E^-)$ uniquely characterizes $q$ w.r.t. $\mathcal{O}$ if $q$ fits $(E^+, E^-)$ and every ELIQ $q'$ that also fits $(E^+, E^-)$ satisfies $q \equiv_\mathcal{O} q'$. The following is a consequence of Theorems 1 and 4, see also [ten Cate and Dalmau, 2021].

**Theorem 5.** Let $\mathcal{O}$ be an ontology formulated in DL-Lite$_H$ or DL-Lite$_{F-}$. Then for every ELIQ $q$ that is satisfiable w.r.t. $\mathcal{O}$, there are sets of data examples $(E^+, E^-)$ that uniquely characterize $q$ w.r.t. $\mathcal{O}$ and such that $||(E^+, E^-)||$ is polynomial in $|q| + |\mathcal{O}|$. If $\mathcal{O}$ is a DL-Lite$_F$ ontology, then $(E^+, E^-)$ is unique in polynomial time and the same holds for DL-Lite$_H$ if $q$ is $O$-minimal.

6 Learning ELIQs under Ontologies

We use our results on frontiers to show that ELIQs are polynomial time learnable under ontologies formulated in DL-Lite$_H$ and DL-Lite$_{F-}$, using only membership queries. We also present two results on non-learnability.

**Theorem 6.** ELIQs are polynomial time learnable under DL-Lite$_H$ ontologies and under DL-Lite$_{F-}$ ontologies using only membership queries.

If the ontology contains concept disjointness constraints, then this only holds true if the learner is provided with a seed CQ (definition given below).

For proving Theorem 6, let $\mathcal{O}$ be an ontology formulated in DL-Lite$_H$ or DL-Lite$_{F-}$ and $q_T(x_0)$ the target ELIQ known to the oracle. We may again assume $\mathcal{O}$ to be in normal form.

**Lemma 7.** In DL-Lite$_H$ and DL-Lite$_{F-}$, every polynomial time learning algorithm for ELIQs under ontologies in normal form that uses only membership queries can be transformed into a learning algorithm with the same properties for ELIQs under unrestricted ontologies.

The learning algorithm is displayed as Algorithm 1. It assumes a seed CQ $q_H^0$, that is, a CQ $q_H^0$ such that $q_H^0 \subseteq \mathcal{O} q_\mathcal{O}$ and $q_H^0$ is satisfiable w.r.t. $\mathcal{O}$. If $\mathcal{O}$ contains no disjointness constraints, then for $\Sigma = \text{sig}(\mathcal{O})$ we can use as the seed CQ

$q_H^0(x_0) = \{ (A(x_0) | A \in \Sigma \cap \mathcal{N}_C) \cup \{ r(x_0, x_0) | r \in \Sigma \cap \mathcal{N}_R \} \}.$

We can still construct a seed CQ $q_H^0$ in time polynomial in $|\mathcal{O}|$ if $\mathcal{O}$ contains no disjoint constraints on concepts (but potentially on roles); details are in the appendix. In the presence of concept disjointness constraints, a seed CQ can be obtained through an initial equivalence query.

The algorithm constructs and repeatedly updates a hypothesis ELIQ $q_H$ while maintaining the invariant $q_H \subseteq \mathcal{O} q_\mathcal{O}$. The initial call to subroutine treeify yields an ELIQ $q_H$ with $q_H \subseteq \mathcal{O} q_H \subseteq \mathcal{O} q_T$ to be used as the first hypothesis. The algorithm then iteratively generalizes $q_H$ by constructing the frontier $F_{q_H}$ of $q_H$ w.r.t. $\mathcal{O}$ in polynomial time and choosing from it a new ELIQ $q_H$ with $q_H \subseteq \mathcal{O} q_T$. In between, the algorithm applies the minimize subroutine to ensure that the new $q_H$ is $O$-minimal and to avoid an excessive blowup while iterating in the while loop.

We next detail the subroutines treeify and minimize. We define minimize on unrestricted CQs since it is applied to non-ELIQs as part of the treeify subroutine.

**The minimize subroutine.** The subroutine takes as input a unary CQ $q(x_0)$ that is satisfiable w.r.t. $\mathcal{O}$ and satisfies $q \subseteq \mathcal{O} q_\mathcal{O}$. It computes a unary CQ $q'$ with $q \subseteq \mathcal{O} q' \subseteq \mathcal{O} q_T$ using membership queries that is minimal in a strong sense.

Formally, minimize first makes sure that $q$ is $O$-saturated and then exhaustively applies the following operation:

*Remove atom.* Choose a role atom $r(x, y)$ in $q$ and let $q^-$ be the maximal connected component of $q \setminus \{ r(x, y) \}$ that contains $x_0$. Pose the membership query $\mathcal{A}_{\mathcal{O}} \models q(r(x_0))$. If the response is positive, continue with $q^-$ in place of $q$.

Clearly, the result of minimize is $O$-minimal.

**The treeify subroutine.** The subroutine takes as input a unary CQ $q(x_0)$ that is satisfiable w.r.t. $\mathcal{O}$ and satisfies $q \subseteq \mathcal{O} q_\mathcal{O}$. It computes an ELIQ $q'$ with $q \subseteq \mathcal{O} q' \subseteq \mathcal{O} q_\mathcal{O}$ by repeatedly increasing the length of cycles in $q$ and minimizing the obtained query; a similar construction is used in [ten Cate and Dalmau, 2021]. The resulting ELIQ is $O$-minimal.

Formally, treeify first makes sure that $q(x_0)$ is $O$-saturated and then constructs a sequence of CQs $p_1, p_2, \ldots$ starting with $p_1 = \text{minimize}(q)$ and then taking $p_{i+1} = \text{minimize}(p_i')$ where $p_i'$ is obtained from $p_i$ by doubling the length of some cycle. Here, a *cycle* in a CQ $q$ is a sequence $R_1(x_1, x_2), \ldots, R_n(x_{n-1}, x_1)$ of distinct role atoms in $q$ such that $x_1, \ldots, x_n$ are distinct. More precisely, $p_i'$ is the result of the following operation:

*Double cycle.* Choose a role atom $r(x, y) \in p_i$ that is part of a cycle in $p_i$ and let $p = p_i \setminus \{ r(x, y) \}$. The CQ $p_i'$ is then obtained by starting with $p$, adding a disjoint copy $p'$ of $p$ where $x'$ refers to the copy of $x \in \text{var}(p)$ in $p'$ and adding the role atom $r(x', y), r(x', y)$.

If $p_i$ contains no more cycles, treeify stops and returns $p_i$.

Returning to Algorithm 1, let $q_1, q_2, \ldots$ be the sequence of ELIQs that are assigned to $q_H$ during a run of the learning algorithm. We show in the appendix that for all $i \geq 1$, it holds that $q_i \subseteq \mathcal{O} q_T$, $q_i \subseteq q_{i+1}$ while $q_{i+1} \not\subseteq q_i$ and $|\text{var}(q_{i+1})| \geq |\text{var}(q_i)|$. This can be used to prove that the while loop in Algorithm 1 terminates after a polynomial number of iterations, arriving at a hypothesis $q_H$ with $q_H \equiv_\mathcal{O} q_T$.

We now turn to non-learnability results. Without a seed CQ, ontologies with concept disjointness constraints are not learnable using only polynomially many membership queries. A *disjointness ontology* is a DL-Lite$_{F-}$ ontology that only consists of concept disjointness constraints.
Theorem 7. AQ's are not learnable under disjointness ontologies using only polynomially many membership queries.

We next show that when we drop the syntactic restriction from DL-Lite$^{HF}$, then ELIQs are no longer learnable at all using only membership queries. Note that this is not a direct consequence of Theorem 3 as there could be an alternative approach that does not use frontiers.

Theorem 8. ELIQs are not learnable under DL-Lite$^{F}$ ontologies using only membership queries.

7 Outlook

A natural next step for future work is to generalize the results presented in this paper to DL-Lite$^{H,F}$, adopting the same syntactic restriction that we have adopted for DL-Lite$^{F}$, and additionally requiring that functional roles have no proper subroles. The latter serves to control the interaction between functional roles and role inclusions. Even with this restriction, however, that interaction is very subtle and the frontier construction becomes significantly more complex. Other interesting questions are whether ELIQs can be learned in polynomial time w.r.t. DL-Lite$_{core}$ ontologies and whether CQs can be learned w.r.t. DL-Lite$_{core}$ ontologies when both membership and equivalence queries are admitted.

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Additional Preliminaries

We introduce some additional preliminaries that are needed for the lemmas and proofs in the appendix.

We start with defining the semantics of conjunctive queries in full detail. A homomorphism \( h \) from interpretation \( I_1 \) to interpretation \( I_2 \) is a mapping from \( \Delta_{I_1} \) to \( \Delta_{I_2} \) such that \( d \in \Delta_{I_1} \) implies \( h(d) \in \Delta_{I_2} \) and \( (d,e) \in \tau_{I_1} \) implies \( (h(d),h(e)) \in \tau_{I_2} \). We use \( \text{img}(h) \) to denote the set \( \{ e \in \Delta_{I_2} \mid \exists d \in \Delta_{I_1} \colon h(d) = e \} \). For \( d_i \in \Delta_{I_1}, i \in \{1,2\} \), we write \( I_1, d_1 \rightarrow I_2, d_2 \) if there is a homomorphism \( h \) from \( I_1 \) to \( I_2 \) with \( h(d_1) = d_2 \). With a homomorphism from a CQ \( q \) to an interpretation \( I \), we mean a homomorphism from \( \mathcal{A}_q \) to \( I \). For a CQ \( q(x_0) \), we write \( q(x_0) \rightarrow (I,d) \) if there is a homomorphism \( h \) from \( q \) to \( I \) with \( h(x_0) = d \). Let \( q(x_0) \) be a CQ and \( I \) an interpretation. An element \( d \in \Delta_{I_2} \) is an answer to \( q \) in \( I \), written \( I \models q(d) \), if \( q(x_0) \rightarrow (I,d) \). Now let \( \mathcal{O} \) be an ontology and \( \mathcal{A} \) an ABox. An individual \( a \in \text{ind}(\mathcal{A}) \) is an answer to \( q \) on \( \mathcal{A} \) w.r.t. \( \mathcal{O} \), written \( \mathcal{A}, \mathcal{O} \models q(a) \), if \( a \) is an answer to \( q \) in every model of \( \mathcal{O} \) and \( \mathcal{A} \).

Let \( q(x_0) \) be an ELIQ. The codepth of a variable \( x \in \text{var}(q) \) is 0 if there is no \( R(x,x') \in q \) directed away from \( x_0 \) and is \( k + 1 \) if \( k \) is the maximum of the codepths of all \( x' \in \text{var}(q) \) with \( R(x,x') \in q \) directed away from \( x_0 \).

We next define universal models, first for DL-Lite\(^R\) and then for DL-Lite\(^F\). Let \( \mathcal{O} \) be a DL-Lite\(^R\) ontology and let \( \mathcal{A} \) be an ABox that is satisfiable w.r.t. \( \mathcal{O} \). For \( a \in \text{ind}(\mathcal{A}) \), concept names \( \mathcal{A} \), and \( R \) a role, we write \( a \sim_{\mathcal{A},\mathcal{O}}^R A \) if \( \mathcal{A}, \mathcal{O} \models \exists R.A(a) \) and there is no \( S(a,b) \in \mathcal{A} \) such that \( \mathcal{O} \models S \subseteq R \) and \( \mathcal{A}, \mathcal{O} \models A(b) \). Note that this is identical to the definition of `\( \sim \)` given in Section 3, but is formulated for ABoxes in place of ELIQs.

A trace for \( \mathcal{A} \) and \( \mathcal{O} \) is a sequence \( t = aR_1A_1R_2A_2\ldots R_nA_n \), \( n \geq 0 \) where \( a \in \text{ind}(\mathcal{A}) \), \( R_1,\ldots,R_n \) are roles that occur in \( \mathcal{O} \), and \( A_1,\ldots,A_n \) are sets of concept names that occur in \( \mathcal{O} \), such that \( a \sim_{\mathcal{A},\mathcal{O}}^R A_1 \) and \( \mathcal{O} \models A_i \subseteq \exists R_{i+1}.A_{i+1} \) for \( 1 \leq i < n \). Let \( T \) denote the set of all traces for \( \mathcal{A} \) and \( \mathcal{O} \). Then the universal model of \( \mathcal{A} \) and \( \mathcal{O} \) is

\[
\mathcal{U}_{\mathcal{A},\mathcal{O}} = \mathcal{A} \cup \{ A(a) \mid \mathcal{A}, \mathcal{O} \models A(a) \} \cup \\
\{ S(a,b) \mid R(a,b) \in \mathcal{A} \text{ and } \mathcal{O} \models R \subseteq S \} \cup \\
\{ B(tRA) \mid tRA \in T \text{ and } \mathcal{O} \models A \subseteq B \} \cup \\
\{ S(t,tRA) \mid tRA \in T \text{ and } \mathcal{O} \models R \subseteq S \}.
\]

For brevity, we write \( \mathcal{U}_{q,\mathcal{O}} \) instead of \( \mathcal{U}_{\mathcal{A}_q,\mathcal{O}} \) and \( x \sim_{\mathcal{A},\mathcal{O}}^R A \) instead of \( x \sim_{\mathcal{A}_q,\mathcal{O}}^R A \) for any conjunctive query \( q \).

Now let \( \mathcal{O} \) be a DL-Lite\(^F\) ontology and let \( \mathcal{A} \) be an ABox that is satisfiable w.r.t. \( \mathcal{O} \). For a set \( M \) of concept names, we write \( \bigwedge M \) as a shorthand for \( \bigwedge_{A \in M} A \). For \( a \in \text{ind}(\mathcal{A}) \), \( M, M' \) sets of concept names, and \( R \) a role, we write

- \( a \sim_{\mathcal{A},\mathcal{O}}^RM \) if \( \mathcal{A}, \mathcal{O} \models \exists R.\bigwedge M(a) \), \( M \) is maximal with this condition, and there is no \( R(a,b) \in \mathcal{A} \) such that \( \mathcal{A}, \mathcal{O} \models \bigwedge M(b) \);
- \( M \sim_{\mathcal{A},\mathcal{O}}^R M' \) if \( \mathcal{O} \models \bigwedge M \subseteq \exists R.\bigwedge M'(a) \) and \( M' \) is maximal with this condition.
The definition of ‘⇝’ in the first item is identical to the definition of ‘⇝’ given in Section 4, but is formulated for ABoxes in place of ELIQs. The maximality of $M$ is important when dealing with functional roles. If, for example, $O$ contains $A \ni \exists r. B_1$, $A \ni \exists r. B_2$, and $\text{func}(r)$ and $A = \{ A(a) \}$, then we have $a \sim_{A,O}^r \{ B_1, B_2 \}$, but not $a \sim_{A,O}^r \{ B_i \}$ for an $i \in \{ 1, 2 \}$. This helps to ensure that $a$ gets only a single r-successor in the universal model.

A trace for $A$ and $O$ is a sequence $t = a R_1 M_1 a R_2 M_2 \ldots a R_n M_n$, $n \geq 0$ where $a \in \text{ind}(A)$, $R_1, \ldots, R_n$ are roles that occur in $O$, and $M_1, \ldots, M_n$ are sets of concept names that occur in $O$, such that

(i) $a \sim_{A,O}^i M_i$

(ii) for $1 \leq i < n$, we have $M_i \sim_{O,i}^0 M_{i+1}$ and if $\text{func}(R_i) \in O$, then $R_{i+1} \neq R_i$.

Let $T$ denote the set of all traces for $A$ and $O$. Then the universal model of $A$ and $O$ is defined as $U_{A,O} = A \cup \{ A(a) \mid A,O \models A(a) \} \cup \{ A(\text{tRM}) \mid \text{tRM} \in T \text{ \text{ and } } A \in M \} \cup \{ R(t, tRM) \mid tRM \in T \}$.

In the following, we give three elementary lemmas that pertain to universal models and are used throughout the appendix. Their proof is entirely standard and omitted. The most important properties of universal models are as follows.

Lemma 8. Let $O$ be an ontology formulated in $DL-Lite_H^T$ or $DL-Lite^F$, and let $A$ an ABox that is satisfiable w.r.t. $O$. Then

1. $U_{A,O}$ is a model of $A$ and $O$;
2. $A, O \models q(\bar{a})$ iff $q(\bar{x}) \rightarrow (U_{A,O}, \bar{a})$ for all CQs $q(\bar{x})$ and all $\bar{a} \in \text{ind}(A)[\bar{x}]$.

We note that Lemma 8 ceases to hold when $O$ is formulated in $DL-Lite_H^T$ due to subtle interactions between role inclusions and functionality assertions.

The next lemma links query containment to the existence of homomorphisms into the universal model.

Lemma 9. Let $O$ be an ontology formulated in $DL-Lite_H^T$ or $DL-Lite^F$, and let $q_1(\bar{x}), q_2(\bar{y})$ be CQs that are satisfiable w.r.t. $O$. Then $q_1 \subseteq O q_2$ iff $q_2(\bar{y}) \rightarrow (U_{q_1,O}, \bar{x})$.

The following lemma states that any homomorphism from a CQ $q$ to some universal model $U_{A,O}$ can be extended to a homomorphism from $U_{q_1,O}$ to $U_{A,O}$.

Lemma 10. Let $O$ be an ontology formulated in $DL-Lite_H^T$ or $DL-Lite^F$, $A$ an ABox, and $q(x_0)$ a unary CQ, such that $A$ and $q$ are both satisfiable w.r.t. $O$. If $h$ is a homomorphism from $q$ to $U_{A,O}$ with $h(x_0) = a$ for some $a \in \text{ind}(A)$, then $h$ can be extended to a homomorphism $h'$ from $U_{q_1,O}$ to $U_{A,O}$ with $h'(x_0) = a$.

And finally, we show a property of $O$-minimal and $O$-saturated queries that we will use to show that the constructions indeed yield frontiers. Note that the property implies that every homomorphism $h$ from an ELIQ $q(x_0)$ to $U_{q_1,O}$ with $h(x_0) = x_0$ is injective.

Lemma 11. Let $O$ be an ontology in normal form formulated in $DL-Lite_H^T$ or $DL-Lite^F$, and $q(x_0)$ an ELIQ that is $O$-minimal, $O$-saturated and satisfiable w.r.t. $O$. Then $\text{var}(q) \subseteq \text{img}(h)$ for every homomorphism $h$ from $q$ to $U_{q_1,O}$ with $h(x_0) = x_0$.

Proof. Assume for contradiction that there is a variable $x \in \text{var}(q)$ with $x \not\in \text{img}(h)$. Let $q'$ be the restriction of $q$ to $\text{var}(q) \setminus \text{var}(q_z)$. We show that $h$ is also a homomorphism from $q$ to $U_{q',O}$, and thus that $q \equiv_O q'$. This contradicts the minimality of $q$.

First, observe that for all $y \in \text{var}(q)$, $h(y) \not\in \text{var}(q_z)$, since $q$ is connected and there is no $y' \in \text{var}(q)$ with $h(y') = x$.

Next let $y R_1 M_1 \ldots R_n M_n \in \Delta_{U_{q,O}}$ be a trace starting with some variable $y \in \text{var}(q')$. Then $y \sim_{q,O} R_1 M_1$ and therefore $A_q, O \models \exists R_1 \forall M_1(y)$ and there is no $R_1(y, y') \in q$ such that $A_q, O \models \forall M_1(y')$. Since $O$ is in normal form, there is a set of concept names $M$ such that $A_q, O \models \forall M \rightarrow \exists R_1 \forall M(y)$ and therefore $A_q, O \models \exists R_1 \forall M(y)$. Since $q'$ is a subset of $q$, $y \sim_{q,O} R_1 M_1$ and thus $y R_1 M_1 \ldots R_n M_n \in \Delta_{U_{q',O}}$.

Let $A(y) \in q$. If $h(y) \in \text{var}(q')$, then $A(h(y)) \in U_{q',O}$ by $O$-satisfaction of $q$. If $h(y)$ is a trace, then, by connectedness it is a trace below a variable of $q'$ and thus $A(h(y)) \in U_{q',O}$.

Let $r(y, y') \in q$. Again, by connectedness of $q$, $h(y)$ and $h(y')$ must be variables of $q'$ or traces starting with variables of $q'$. If both $h(y)$ and $h(y')$ are variables, then $r(h(y), h(y')) \in U_{q',O}$, since $h(y), h(y') \not\in \text{var}(q_z)$. If one or both of $h(y)$ and $h(y')$ are a trace, then since the trace-subtrees are identical, also $r(h(x_1), h(x_2)) \in U_{q',O}$.

Thus, $h$ is a homomorphism from $q$ to $U_{q',O}$ as required.

B Proofs for Section 2

We describe how to convert a $DL-Lite_H^T$-ontology $O$ into a $DL-Lite^F$-ontology $O'$ in normal form. We use $\mathcal{C}(O)$ to denote the set of all concepts that occur on the right-hand side of a concept inclusion in $O$. Note that $\mathcal{C}(O)$ is closed under sub-concepts. We introduce a fresh concept name $X_C$ for every complex concept $C \in \mathcal{C}(O)$, and set $X_A = A$ for concept names $A \in \mathcal{C}(O)$. The ontology $O'$ consists of the following concept and role inclusions:

- all role inclusions from $O$;
- $C \subseteq X_D$ for every $C \subseteq D \in O$;
- $X_{D_1 \sqcap D_2} \subseteq X_{D_1}$, for every $D_1 \sqcap D_2 \in \mathcal{C}(O)$ and $i \in \{ 1, 2 \}$;
- $X_{\exists R.C} \subseteq \exists R.X_C$, for every $\exists R.C \in \mathcal{C}(O)$;

Clearly, $O'$ can be computed in polynomial time. Moreover, it is easy to verify that for all $C \in \mathcal{C}(O)$, we have $O' \models X_C \subseteq C$. Regarding the relationship between $O$ and $O'$, we observe the following consequences of the definition of $O'$.

Lemma 12.

1. $O' \models O$, that is, every model of $O'$ is a model of $O$;
2. every model I of O can be extended into a model I' of O' by starting with I' = I and then setting for every complex concept C ∈ C(O), X_C = C^I.

Lemma 12 essentially says that O' is a conservative extension of O, but is slightly stronger in also making precise how exactly a model of O can be extended to a model of O'.

C  Proofs for Section 3

Lemma 1. For every DL-Lite^H ontology O, we can construct in polynomial time a DL-Lite^H ontology O' in normal form such that every O-minimal ELIQ q is also O'-minimal and a frontier of q w.r.t. O can be constructed in polynomial time given a frontier of q w.r.t. O'.

Proof. Let O be a DL-Lite^H ontology and let O' be the result of converting O into normal form as described before Lemma 12. Moreover, let q(x) be an ELIQ and F' a frontier of q w.r.t. O'. Let F be obtained from F' by including all ELIQs that can be obtained by taking an ELIQ p ∈ F' and then doing the following:

- for every atom X_C(x), C ∈ C(O), remove that atom and add a variable disjoint copy of C viewed as an ELIQ, gluing the root to x;

To prove that F is a frontier of q w.r.t. O, we show that the three conditions from the definition of frontiers are satisfied:

1. \( q \subseteq_O q_F \) for all \( q_F \in F \).

Assume to the contrary that there is a q_F(x) ∈ F with q \nsubseteq_O q_F. Then, there is an ABox \( \mathcal{A} \) and an individual \( a \in \text{ind}(\mathcal{A}) \) such that \( \mathcal{A}, O \models.q(a), \) but \( \mathcal{A}, O \nvdash q_F(a) \).

We may assume that concept names \( X_C \) do not occur in \( \mathcal{A} \) as they are not used in \( O, q, \) and \( q_F \). From O' \models O, we obtain \( \mathcal{A}, O' \models q(a) \). Since \( \mathcal{A}, O \nvdash q_F(a) \), there is a model I of A and O such that I \nvdash q_F(a). Let I' be the extension of I according to Point 2 of Lemma 12. Then I' is a model of O' and A. Now, let q^0_F(x) ∈ F' be the query from which q_F(x) was obtained in the construction of F. By construction of q_F and of I', I' \nvdash q^0_F(a), so A, O, O' \nvdash q^0_F(a). Thus, A witnesses that q \nsubseteq_O q^0_F, a contradiction to q_F being in F'.

2. q_F \nsubseteq_O q for all q_F ∈ F.

Let q_F(x) ∈ F and let q^0_F(x) ∈ F' be the ELIQ from which q_F(x) was obtained during the construction of F. Since q^0_F \nsubseteq_O q, there is an ABox \( \mathcal{A}' \) and an individual \( a \in \text{ind}(\mathcal{A}') \) such that \( \mathcal{A}', O' \models q^0_F(a), \) but \( \mathcal{A}', O' \nvdash q(a) \). Let the ABox \( \mathcal{A} \) be obtained by starting with \( \mathcal{A}' \) and adding \( C(b) \), for each concept assertion \( X_C(b) \in \mathcal{A}' \). Here, the addition of \( C(b) \), with C an ELI^H-concept, is defined as expected: view \( C(b) \) as a tree-shaped ABox \( A_C(b) \) that uses only fresh individual names, and then add this ABox gluing its root to b.

We aim to show A, O \models q_F(a) and A, O \nvdash q(a), witnessing q_F \nsubseteq_O q as required.

For the former, assume to the contrary that A, O \nvdash q_F(a). Then there is a model I of A and O such that I \nvdash q_F(a). Let I' be the extension of I according to Point 2 of Lemma 12. Then I' is a model of O' and, by construction of I' and of A from A', also a model of A'. Moreover, I \nvdash q_F(a) implies I' \nvdash q^0_F(a) by construction of q_F and of I'. This contradicts A', O' \models q^0_F(a).

It remains to show that A, O \nvdash q(a). Since A', O' \nvdash q(a), there is a model I of O and A' such that I \nvdash q(a). Since O' \models I, I is also a model of O. Since O' \models X_{3R,C} \subseteq 3R.C for all 3R.C \in \text{C}(O) and due to the construction of A from A', I is also a model of A. Thus, I witnesses A, O \nvdash q(a), as required.

3. For all ELIQs q' with q \subseteq_O q' \subseteq_O q, there is a q_F ∈ F with q_F \subseteq_O q'.

Let q' be an ELIQ with q \subseteq_O q' \subseteq_O q. From q \subseteq_O q', it follows that q' does not use the fresh concept names X_C(x) in O'. Consequently, we obtain from Lemma 12 that q \subseteq_O q' \subseteq_O q. There is thus a q^0_F(x) ∈ F' with q^0_F \subseteq_O q'. Assume that q_F ∈ F was obtained from q^0_F in the construction of F. It suffices to show that q_F \subseteq_O q'. Assume to the contrary that this is not the case. Then there is an ABox \( \mathcal{A}, A' \) and an individual \( a \in \text{ind}(\mathcal{A}) \) such that \( \mathcal{A}, O \models q_F(a), \) but \( \mathcal{A}, O \nvdash q(a) \). We can proceed as in Point 1 above to show that \( \mathcal{A}, O' \models q_F(a) \) and \( \mathcal{A}, O' \nvdash q(a) \), in contradiction to q^0_F \subseteq_O q'.

It remains to prove the second part of the Lemma: every O-minimal ELIQ q is also O'-minimal. Suppose that q is not O'-minimal, that is, there is a variable x such that q \subseteq_O q' where q' = q\{var(q) \setminus \{x\}\}. Clearly, we have q \subseteq_O q' as q' \subseteq_O q. Moreover, since q \subseteq_O q and Point 1 of Lemma 12, we have q' \subseteq_O q. Hence, q \equiv_O q' and q is not O-minimal.

Lemma 2. F is a frontier of q(x_0) w.r.t. O.

Proof. We show that F fulfills the three conditions of frontiers. For Condition 1, let p(x_0) be a query from F. Then, since q is satisfiable w.r.t. O, so is p. Hence it suffices to show p(x_0) \rightarrow (U_{q,C}, x_0) by Lemma 9.

We extend the mapping \( \downarrow \) to be defined on all variables of p by considering the yet unmapped variables added in Step 2A of the construction. Let z be such a fresh variable added for x \in \text{var}(p), roles R, S and concept name A. Then \( x^+ \leadsto_{R,C} A \) and by construction of \( U_{q,C} \), there is a trace \( x^+RA \in \Delta U_{q,C} \). Set \( z^+ = x^+RA \). Now \( \downarrow \) is defined on all variables of p and, by construction of p, it is a homomorphism from p to \( U_{q,C} \) with \( x_0^+ = x_0 \) as required.

We start the proof of the second condition of frontiers with the following claim:

Claim 1. p \subseteq_O q_x for all x \in \text{var}(q) and p(x) ∈ F_0(x).

Proof of Claim 1. We show the claim by induction on the codepth of x in q, matching the inductive construction of \( F_0 \). In the induction start, x has codepth 0. Then, by definition of codepth, there is no \( R(x, y) \in q \) that is directed away from x_0 and all p \in F_0(x) must be obtained by dropping a concept atom.

Let p(x) be a query from \( F_0(x) \) that is obtained by dropping the concept atom A(x) \in q. Then, by choice of A(x), there is no \( B(x) \in p \) with O \models B \subseteq A and no \( R(x, x') \in p \) with O \models 3R \subseteq A. Hence, A(x) \in q_x and A(x) \notin U_{p,C}, therefore p \nsubseteq_O q_x.
In the induction step, let \( x \) have codepth \( > 0 \), let \( p(x) \) be a query from \( F_0(x) \) and assume that the claim holds for all variables with smaller codepth. Let \( \cdot \cdot \cdot \) be the extension of the original \( \cdot \cdot \cdot \) for \( p \) to a homomorphism from \( U_{p,O} \) to \( U_{y',O} \) that exists by Lemma 10. If \( p \) is obtained by dropping a concept atom, then the same argument as in the induction start yields \( p \not\subseteq q_y \). If \( p \) is obtained by generalizing a subquery attached to a role atom \( R(x, y) \in \mathcal{Q}_x \), assume for contradiction that there is a homomorphism \( h \) from \( q_y \) to \( U_{p,O} \) with \( h(x) = x \).

From \( h \) we construct a homomorphism \( h' \) from \( q \) to \( U_{q',O} \) with \( h'(x_0) = x_0 \) by setting \( h'(z) = h(z)^+ \) for all \( z \in \text{var}(q_y) \) and \( h'(z) = z \) for all \( z \notin \text{var}(q_y) \). Since \( h \) must map \( y \) to an \( R \)-successor of \( x \) in \( U_{p,O} \), we may distinguish the following cases:

- \( h(y) \) is a \( z \in \text{var}(p) \) with \( z^+ \neq y \).

Then, by definition of \( \cdot \cdot \cdot \), \( z^+ = z \), and therefore, by construction of \( h' \), \( h'(y) = h'(z) = z \). Hence \( h' \) is a non-injective homomorphism from \( q \) to \( U_{q',O} \) with \( h'(x_0) = x_0 \), contradicting \( O \)-minimality or \( O \)-saturatedness of \( q \) by Lemma 11.

- \( h(y) \) is a trace \( h(x)SA \in \mathcal{D}U_{p,O} \) for some role \( S \) with \( O \models S \subseteq R \) and concept name \( A \).

Then, if \( h'(y) \) is also a trace, there must be a \( y' \in \text{var}(q) \) with \( y' \notin \text{img}(h') \), again contradicting \( O \)-minimality or \( O \)-saturatedness of \( q \) by Lemma 11.

If \( h'(y) \) is not a trace, but a successor \( y' \) of \( x \) with \( y' = y \), then \( h'(y') = h'(y) \), again contradicting \( O \)-minimality or \( O \)-saturatedness of \( q \) by Lemma 11.

If \( h'(y) = y \) and there is a \( y' \in \text{var}(q_y) \) with \( h'(y') = x \), then \( h'(y') = h'(x) \), again contradicting \( O \)-minimality or \( O \)-saturatedness of \( q \) by Lemma 11.

If \( h'(y) = y \) and there is no \( y' \in \text{var}(q_y) \) with \( h'(y') = x \), then we show a contradiction to \( O \)-minimality or \( O \)-saturatedness of \( q \) by constructing a homomorphism \( h'' \) from \( q \) to \( U_{q',O} \) with \( h''(x_0) = x_0 \) where \( q' \) is the restriction of \( q \) to \( \text{var}(q) \setminus \{y\} \). Note that by construction of \( h'' \), \( h''(z) = y \) implies \( z = y \).

Since \( h(x)SA = y \), there is no trace \( xSA \in \mathcal{D}U_{p,O} \). But, since \( h(x)SA \in \mathcal{D}U_{p,O} \) it must be that \( h(x) \not\rightarrow_{p,O} A \) and thus \( A_q \not| \rightarrow_{O} A \) implies \( A_q \not| \rightarrow_{O} A \).

Then, since \( q \) is connected, there must be a predecessor \( x' \) of \( x \) with \( h(x') = y \). Hence \( h'(x') = h'(x') = y' \), contradicting \( O \)-minimality of \( q \) by Lemma 11.

This completes the proof of Condition 2 of frontiers.

It remains to show that Condition 3 of frontiers is satisfied. Let \( q'(x_0) \) be an ELIQ that is satisfiable w.r.t. \( O \) such that \( q \subseteq q' \not\subseteq q \). We may assume w.l.o.g. that \( q' \) is \( O \)-saturated.

There is a homomorphism \( q' \) from \( q' \) to \( U_{q',O} \) with \( g(x_0) = x_0 \). We have to show that there is a \( p \) in \( F \) with \( p \subseteq q' \). To do this, we construct in three steps a homomorphism \( h \) from \( q' \) to \( U_{p,O} \) with \( h(x_0) = x_0 \) for some \( p \in F \). During all steps, we maintain the invariant

\[
\text{h(z)}^+ = g(z)
\]

for all variables \( z \in \text{var}(q') \) with \( h(z) \) defined and \( \cdot \cdot \cdot \cdot \) the extension of the original \( \cdot \cdot \cdot \) for \( p \) to a homomorphism from \( U_{p,O} \) to \( U_{h,O} \). In the first step of the construction, we define \( h \) for an initial segment of \( q' \).
Let $U \subseteq \text{var}(q')$ be the smallest set of variables (w.r.t. $\subseteq$) of $q'$ such that $x_0 \in U$ and $R(x, y) \in q'$ with $x \in U$, $g(y) \in \text{var}(q)$, and $S(g(x), g(y)) \in q$ directed away from $x_0$ implies $y \in U$. Let $q'_U$ be the restriction of $q'$ to the variables in $U$.

Claim 2. For all $x \in U$ with $q'_U \not\subseteq O \ q_0(x)$, there is a $p \in F_0(g(x))$ and a homomorphism $h'$ from $q'_U$ to $U_{p, O}$ that satisfies $(\ast)$.

Proof of Claim 2. Let $y = g(x)$. We show Claim 2 by induction on the codepth of $x$ in $q'_U$. In the induction start, $x$ has codepth 0. We distinguish the following cases:

- There is an $R(y, y') \in q_y$.
  Then let $p \in F_0(y)$ be constructed by generalizing the subquery attached to the role atom $R(y, y')$ and set $h'(x) = y$. Since $q$ is $O$-saturated, $A(y) \in U_{q, O}$ implies $A(y) \in U_{p, O}$. This and $y = g(x)$ implies that $h'$ is a homomorphism.

- There is no $R(y, y') \in q_y$.
  Then $q'_U \not\subseteq O \ q_0$ implies that there is an $A(y) \in q_y$ with $A(x) \notin U_{q'_U, O}$, and we must even find an $A$ with these properties such that there is no $B(y) \in q_y$ with $O \models B \subseteq A$ and $O \not\models A \subseteq B$. This implies that Property (a) of dropping concept atoms is satisfied. Property (b) is also satisfied, we have to argue that there is no $R(y, y') \in q_y$ with $O \models \exists R \subseteq A$. But if there is such an $R(y, y') \in q_y$, then $q'_y \rightarrow (U_{q'_U, O}, x')$ for some $S(x, x') \in q'_U$ with $O \models S \subseteq R$. This implies $A(x) \in U_{q'_U, O}$, a contradiction.

We may thus construct $p \in F_0(y)$ by dropping the concept atom $A(y)$. Set $h'(x') = g(x')$ for all $x' \in \text{var}(q'_U)$.

This completes the proof of Claim 2.

By Claim 2, there is a $p' \in F_0(x_0)$ such that $q'_U \rightarrow U_{p', O}$. Let $p \in F$ be the query that was obtained by applying the compensation step to $p'$. Then clearly also $q'_U \rightarrow U_{p, O}$. Define $h$ for all variables in $U$ according to the homomorphism that witnesses this. Let $h^-$ be the extension of the original $h^-$ for $p$ to a homomorphism from $U_{p, O}$ to $U_{q, O}$ which exists by Lemma 10.

We continue with the second step of the construction of $h$ which covers subtrees of $q'$ that are connected to the initial segment $q'U$ and whose root is mapped by $g$ to traces of $U_{q, O}$ (and as opposed to a variable from $\text{var}(q)$). Consider all atoms $R(x, x') \in q'$ with $h(x)$ defined, $h(x')$ undefined, and $g(x') \notin \text{var}(q)$. Before extending $h$ to $g(x')$, we first show that there is an atom $S(h(x)), z \in p$ with $O \models S \subseteq R$, added in Step 2A.

Since $g(x') \notin \text{var}(q)$, $g(x')$ must be a trace $g(x)SA$ in $\Delta H_{q, O}$ for some concept name $A$ and role $S$ with $O \models S \subseteq R$. Hence, $g(x) \rightarrow h^-(S, A)$.

We aim to show that Step 2A of compensation is applicable. To this end, take any concept name $B$ such that $O \models \exists R \subseteq B$. We have to show that $B(h(x)) \in p$. Assume to the contrary that $B(h(x)) \notin p$. Then, since $q$ is $O$-saturated, $B(g(x)) \in q$ and $p$ must be the result of dropping the concept atom $B(g(x))$. However, since $q'$ is $O$-saturated and $R(x, x') \in q'$, $B(x) \in q'$ and this contradicts Claim 2, since $x \in U$.

Hence, Step 2A adds the atoms $R(h(x)), z, B(z), S(z')$ with $z$ and $z'$ fresh variables and adds a disjoint copy $q^*$ of $q$, gluing the copy of $h(x)$ in $q^*$ to $z'$. Extend $h$ to the variables in $q''$ by setting $h'(z) = g(z)$ for all $z \in \text{var}(q'')$. If $g(x') \neq y'$, then extend $h'$ to the subtree below $x'$ by setting $h'(z) = g(z)$ for all $z \in \text{var}(q'')$. If $g(x') = y'$ and $O \models S \subseteq R$, then, by the induction hypothesis, there is a $p' \in F_0(y)$ and a homomorphism $h''$ from $q''$ to $U_{p', O}$ with $h''(x') = y'$. Extend $h''$ to the variables in $q''$ by mapping $q''_U$ according to $h''$ to the copy of $p'$ that was attached to $y$ in Point 3 of generalizing a subquery.

If $g(x') = y'$ and $O \not\models S \subseteq R$, then extend $h'$ to the variables in $q''_U$ by mapping $q''_U$ according to $g$ to the copy of $q''_U$ that was added in Point 4 of generalizing the subquery attached with the role $S$.

- For every $R(y, y') \in q_y$, $y' \rightarrow (U_{q''_U, O}, x')$ for some $S(x, x') \in q''_U$ with $O \models S \subseteq R$. Then there is an $A(y) \in q_y$ with $A(x) \notin U_{q''_U, O}$ and we must even find an $A$ with these properties such that there is no $B(y) \in q_y$ with $O \models B \subseteq A$ and $O \not\models A \subseteq B$. Thus, Property (a) of dropping concept atoms is satisfied. To show that Property (b) is also satisfied, we have to argue that there is no $R(y, y') \in q_y$ with $O \models \exists R \subseteq A$. But if there is such an $R(y, y') \in q_y$, then $q'_y \rightarrow (U_{q''_U, O}, x')$ for some $S(x, x') \in q''_U$ with $O \models S \subseteq R$. This implies $A(x) \in U_{q''_U, O}$, a contradiction.

In the third and final step of the construction of $h$, we consider the remaining subtrees of $q'$. Let $R(x, x') \in q'$ be directed away from $x_0$ with $h(x)$ defined and $h(x')$ undefined.

Then $h(x)$ was defined in the first step of the construction of $h$, and thus $x \in U$. As $h(x')$ was not defined in the second step $g(x') \in \text{var}(q)$. Therefore, since $x' \notin U$, $R(g(x), g(x')) \in U_{q, O}$ must be directed towards $x_0$. This implies that $x$ is not the root of $q$ and that there is an atom $T(x', x) \in q'$ directed away from $x_0$ with $g(x') = g(x')$. From $x \in U$ it follows that $x' \in U$ and therefore $h(x')$ and $h(x)$ were defined in the first step of the construction of $h$.

Hence, there is an atom $S(h(x'), h(x)) \in p$ directed away from $x_0$ with $O \models S \subseteq T$ that was not added in Step 2A.
Since by (s) $h(x'') = g(x'') = g(x')$ and $h(x) = g(x)$, $A_q, O \models R^-(h(x'')) = h(x')$. Therefore, Step 2B added an atom $R^-(z, h(x))$ to $p$, $z$ a fresh variable, and glued a copy $\hat{q}$ of $q$ to $z$. Set $h(x') = z$ and extend $h$ to the entire subtree $Q'$, by mapping all variables into the attached copy of $q$ according to $g$.

This completes the construction of $h$ and the proof that Condition 3 is satisfied. □

**Lemma 3.** The construction of $F$ runs in time polynomial in $\|q\| + \|O\|$ (and thus $\sum_{p \in F} \|p\|$ is polynomial in $\|q\| + \|O\|$).

**Proof.** In order to reduce notational clutter, we introduce some abbreviations used throughout the proof.

- $s = |\text{sig}(q)|$ denotes the number of concept and role names used in $q$;
- $o = ||O||$ denotes the size of $O$;
- for an ELIQ $p$, $n_p = |\text{var}(p)|$ denotes the number of variables in $p$;
- for a set $Q$ of queries, $n_Q$ denotes $\sum_{p \in Q} n_p$.

We assume without loss of generality that $s$ and $o$ are at least one.

We start with analyzing the size of the queries in $F_0(x)$ that are obtained as the result of the generalization step.

**Claim.** For every $x \in \text{var}(q)$,

$$n_{F_0}(x) \leq s \cdot o \cdot n_{q_x}^3.$$

**Proof of the claim.** The proof is by induction on the codeth of $x$ in $q$. For the base case, consider a variable $x$ of codeth 0 in $q$, that is, a leaf. In this case, only Step (A) is applicable, and it adds at most $s$ queries to $F_0(x)$, each with a single variable.

For the inductive step, consider a variable $x$ of codeth greater than 0. We partition $F_0(x)$ into $F_0^A(x)$ and $F_0^B(x)$, that is, the queries that are obtained by dropping a concept atom in Step (A) and the queries that are obtained by generalizing a subquery in Step (B), respectively, and analyze them separately, starting with $F_0^A(x)$. Clearly, every $p \in F_0^A(x)$ uses $n_{q_x}$ variables and there are at most $s$ queries in $F_0^A$. Thus, we have

$$n_{F_0^A(x)} \leq s \cdot n_{q_x}.$$ 

Next, we analyze $F_0^B(x)$. Each query in $F_0^B(x)$ is obtained by first picking, in Point 1, an atom $R(x, y)$ in $q_x$. Then, in Point 3, we add $\sum_{p \in F_0(y)} n_p$ variables and, in Point 4, we add at most $o$ copies of $q_y$. Thus, we obtain

$$n_{F_0^B(x)} \leq \sum_{R(x, y) \in q_x} (n_{q_x} + n_{F_0(y)} + o \cdot n_{q_y}).$$

Plugging in the induction hypothesis, we obtain

$$n_{F_0^B(x)} \leq \sum_{R(x, y) \in q_x} (n_{q_x} + s \cdot o \cdot n_{q_y}^3 + o \cdot n_{q_y}).$$

We simplify the right-hand side of (1) by making the following observations:

- $\sum_{R(x, y) \in q_x} n_{q_y} \leq n_{q_y} \cdot (n_{q_y} - 1)$,
- $\sum_{R(x, y) \in q_x} n_{q_y} = n_{q_y} - 1,$ and
- $\sum_{R(x, y) \in q_x} n_{q_y}^3 \leq \left(\sum_{R(x, y) \in q_x} n_{q_y}\right)^3 = (n_{q_y} - 1)^3$.

Here, the inequality is an application of the general inequality $\sum_{i} a_i^3 \leq \left(\sum_{i} a_i\right)^3$, for every sequence of non-negative numbers $a_1, \ldots, a_k$.

Using these observations, Inequality (1) can be simplified to:

$$n_{F_0^B(x)} \leq n_{q_x} \cdot (n_{q_x} - 1) + s \cdot o \cdot (n_{q_x} - 1)^3 + o \cdot (n_{q_x} - 1)$$

$$\leq s \cdot o \cdot (n_{q_x} - 1)^3 + (n_{q_x} - 1)^3 + (n_{q_x} - 1)$$

$$= s \cdot o \cdot n_{q_x}^3 + (n_{q_x} - 1)^3 + (n_{q_x} - 1).$$

Overall, we get

$$n_{F_0(x)} = n_{F_0^A(x)} + n_{F_0^B(x)}$$

$$\leq s \cdot o \cdot (n_{q_x} - 1)^3 + n_{F_0^A(x)} + (n_{q_x} - 1)^3 + (n_{q_x} - 1)$$

$$\leq s \cdot o \cdot n_{q_x}^3 + (n_{q_x} - 1)^3 + (n_{q_x} - 1).$$

In the last inequality, we used that $z^3 \geq z^2 + (z - 1)^3 - 1$, for all real numbers $z$. This finishes the proof of the claim.

We analyze now the compensation Step 2, in which the queries in $F_0(x_0)$ are further extended. We denote with $F_1$ the result of applying Step 2A to $F_0(x_0)$. In Step 2A, we add at most one variable and a copy of $q$ for every variable in $F_0(x_0)$ and every choice of a concept name $A$ and role names $S$ and $R$ that occurs in $O$. Therefore, we add at most $(1 + n_q) \cdot n_{F_0(x_0)} \cdot o^3$ variables in total. Using the claim, we get

$$n_{F_1} \leq n_{F_0(x_0)} + (1 + n_q) \cdot n_{F_0(x_0)} \cdot o^3$$

$$\leq s \cdot o \cdot n_{q_x}^3 \cdot (1 + (1 + n_q) \cdot o^3) \cdot (1 + n_q) \cdot o,$$

which is polynomial in $||q||$ and $||O||$. Moreover, the computation of $F$ can be carried out in polynomial time since all the involved queries are of polynomial size and consequences of $O$ can be decided in polynomial time. □

**Theorem 2.** There are families of AQ-s $q_1, q_2, \ldots$ and conjunctive ontologies $O_1, O_2, \ldots$ such that for all $n \geq 1$, any frontier of $q_n$ w.r.t. $O_n$ has size at least $2^n$.

**Proof.** For $n \geq 1$, let $q_n(x) = A_1(x) \land A_1^1(x) \land \cdots \land A_n(x) \land A_n^1(x)$

$$O_n = \{A_i \cap A_i^1 \subseteq A_1 \land A_1^1 \land \cdots \land A_n \land A_n^1 \mid 1 \leq i \leq n\}.$$

Suppose $F$ is a frontier of $q_n$ w.r.t. $O_n$. Let $p$ be any query that contains for each $i$ with $1 \leq i \leq n$ either $A_i(x)$ or $A_i^1(x)$. It suffices to show that $p \in F$.

Clearly, $q_n \subseteq O_n p \subseteq O_n q_n$ and thus Point 3 of the definition of frontiers implies that there is a $p' \in F$ with $p' \subseteq O p$.

We distinguish cases:
• $p'$ contains the atoms $A_i(x), A'_i(x)$ for some $i$. But then $p' \equiv_{\mathcal{O}} q_0$ and $p'$ cannot be in $\mathcal{F}$ by Point 2 of the definition of frontiers, a contradiction.

• $p'$ does not contain both atoms $A_i(x), A'_i(x)$ for any $i$. But then the ontology does not have an effect on the containment $p' \subseteq_{\mathcal{O}} p$ and hence every $A_i(x), A'_i(x)$ that occurs in $p$ must occur in $p'$. As $p'$ does not contain the atoms $A_i(x), A'_i(x)$ for any $i$, we actually have $p' = p$, which was to be shown.

\[ \square \]

\section{D Proofs for Section 4}

A CQ-frontier for an ELIQ $q$ w.r.t. $\mathcal{O}$ is a finite set of unary CQs that satisfies Properties 1–3 of Definition 1. Note that every frontier is a CQ-frontier, but not vice versa.

\textbf{Theorem 3.} There is an ELIQ $q$ and a DL-Lite\(F^\ast\) ontology $\mathcal{O}$ such that $q$ does not have a finite frontier w.r.t. $\mathcal{O}$.

\textbf{Proof.} Let $q(x) = A(x)$ and
\[
\mathcal{O} = \{ A \subseteq \exists r, \exists r' \subseteq \exists r, \exists r \subseteq \exists s, \text{func}(r^-) \}.
\]

The universal model $\mathcal{U}_{q,\mathcal{O}}$ of $\mathcal{A}$ and $\mathcal{O}$ is an infinite $r$-path in which every point has a single $s$-successor.

Suppose, for the sake of showing a contradiction, that $\mathcal{F}$ is a CQ-frontier of $q$ w.r.t. $\mathcal{O}$. We can assume w.l.o.g. that all queries in $\mathcal{F}$ are satisfiable w.r.t. $\mathcal{O}$, especially that they satisfy $\text{func}(r^-)$. Since $\mathcal{F}$ is finite, there is an $n \geq 1$ such that $|\text{var}(p)| < n$, for all $p \in \mathcal{F}$. Consider the following ELIQ $q'$:

\[
q'(x_1) = r(x_1, x_2), \ldots, r(x_{n-1}, x_n),
\]

\[
s(x, y), \quad s(x_n, y),
\]

\[
r(x'_1, x'_2), \ldots, r(x'_{n-1}, x'_n), A(x'_1).
\]

Note that $q' \not\subseteq q$ and that $q'$ satisfies $\text{func}(r^-)$.

By Property 3 of frontiers, there is a query $p(z) \in \mathcal{F}$ such that $p \subseteq_{\mathcal{O}} q'$. By Lemma 9, there is a homomorphism $h$ from $q'$ to $\mathcal{U}_{q,\mathcal{O}}$ with $h(x_1) = z$. We distinguish cases.

Suppose first that $h(x_i) \in \text{var}(p)$ for all $i$ with $1 < i < n$, then by the choice of $n$ there must be $1 < i < j \leq n$ such that $h(x_i) = h(x_j)$. Since $q'$ contains a directed $r$-path from $x_i$ to $x_j$ and $\mathcal{U}_{q,\mathcal{O}}$ does not contain edges between variables that are not part of $p$, this implies that $p$ must contain an $r$-cycle. Thus, $q \not\subseteq p$, violating Property 1 of frontiers.

Suppose now that $h(x_i) \notin \text{var}(p)$ for some $i$ with $1 \leq i \leq n$, that is, $h(x_i)$ is a trace starting with some $y \in \text{var}(p)$. Since $q'$ is an ELIQ, there is a $j < i$ such that $h(x_j) = y$ and $h(x_{j+1}), \ldots, h(x_i) \notin \text{var}(p)$. The structure of $q'$ and the structure of the anonymous part in universal models of $\mathcal{O}$ imply that $h(x'_j) = h(x_j)$.

We now show that $h(x_1) = h(x'_1)$. If $j = 1$, we are done. If $j > 1$, there are atoms $r(x_{j-1}, x_j)$ and $r(x'_{j-1}, x'_j)$ in $q'$. Since $h$ is a homomorphism, $h(x_j) = h(x'_j)$, and $p$ satisfies $\text{func}(r^-)$, we obtain $h(x_{j-1}) = h(x'_{j-1})$. Repeating this argument yields $h(x_1) = h(x'_1)$ as required. Since $h(x_1) = z$, we also have $h(x'_1) = z$. Since $h$ is a homomorphism and $A(x'_1) \in q'$, we have $A(z) \in p$ and thus $p \subseteq_{\mathcal{O}} q$, violating Property 2 of frontiers.

\textbf{Lemma 4.} For every DL-Lite\(F^\ast\) ontology $\mathcal{O}$, we can construct in polynomial time a DL-Lite\(F^\ast\) ontology $\mathcal{O}'$ in normal form such that for every ELIQ $q$, a frontier of $q$ w.r.t. $\mathcal{O}$ can be constructed in polynomial time given a frontier of $q$ w.r.t. $\mathcal{O}'$.

\textbf{Proof.} The proof of the Lemma is the same as the proof of Lemma 1, except for the verification that the constructed set $\mathcal{F}$ satisfies Property 2 of frontiers. So we detail this here.

\textbf{Claim.} $q_F \not\subseteq q$, for all $q_F \in \mathcal{F}$.

\textbf{Proof of the claim.} Let $q_F(x) \in \mathcal{F}$ and let $q'_F(x) \in \mathcal{F}'$ be the ELIQ from which $q_F(x)$ was obtained during the construction of $\mathcal{F}$. Since $q_F \not\subseteq q$, there is an ABox $\mathcal{A}'$ and an individual $a \in \text{ind}(\mathcal{A}')$ such that $\mathcal{A}', \mathcal{O}' \models q'_F(a)$, but $\mathcal{A}', \mathcal{O}' \not\models q(a)$. As in the proof of Lemma 1, the idea is to obtain an ABox $\mathcal{A}$ by starting with $\mathcal{A}'$ and adding $\mathcal{A}_C(b)$, for each concept assertion $X_C(b) \in \mathcal{A}'$. However, the addition of $\mathcal{A}_C(b)$, with $C$ an $\mathcal{ELI}$-concept has to respect the functionality assertions of $\mathcal{O}$. In order to achieve this, we define the addition of a tree-shaped ABox $\mathcal{B}$ with root $b_0$ to $\mathcal{A}$ at a inductively on the structure of $\mathcal{B}$ as follows:

1. for all $A(b_0) \in B$, add $A(a') \to A$;
2. for all $R(b_0, b') \in B$, let $b'$ be the sub-ABox of $B$ rooted at $b'\in\cdot$

(a) if $\text{func}(R) \in \mathcal{O}$ and there is an atom $R(a, a') \in A$, then add $b' \to A$ at $a'$;

(b) otherwise, add an atom $R(a, a')$ for a fresh individual $a'$ and add $b' \to A$ at $a'$.

Importantly, there can only be one atom $R(b, b')$ in the first case for the existential restriction, if $A$ satisfies the functionality assertions in $\mathcal{O}$. It should be clear that the resulting ABox also satisfies the functionality assertions if $A$ does.

We thus obtain the ABox $\mathcal{A}$ by starting with $\mathcal{A} = \mathcal{A}'$ and adding $\mathcal{A}_C(b)$ to $\mathcal{A}$ at $b$, for each concept assertion $X_C(b) \in \mathcal{A}'$. We aim to show $\mathcal{A}, \mathcal{O} \models q_F(a)$ and $\mathcal{A}, \mathcal{O} \not\models q(a)$, witnessing $q_F \not\subseteq q$ as required.

For the former, assume to the contrary that $\mathcal{A}, \mathcal{O} \not\models q_F(a)$. Then there is a model $\mathcal{I}$ of $\mathcal{A}$ and $\mathcal{O}$ with $\mathcal{I} \not\models q_F(a)$. Let $\mathcal{I}'$ be the extension of $\mathcal{I}$ according to Point 2 of Lemma 12. Then $\mathcal{I}'$ is a model of $\mathcal{O}'$ and, by construction of $\mathcal{I}'$ and of $\mathcal{A}$ from $\mathcal{A}'$, also a model of $\mathcal{A}'$. Moreover, $\mathcal{I} \not\models q_F(a)$ implies $\mathcal{I}' \not\models q'_F(a)$ by construction of $q_F$ and of $\mathcal{I}'$. This contradicts $\mathcal{A}', \mathcal{O}' \models q'_F(a)$.

It remains to show that $\mathcal{A}, \mathcal{O} \not\models q(a)$. Since $\mathcal{A}', \mathcal{O}' \not\models q(a)$, there is a model $\mathcal{I}$ of $\mathcal{O}'$ and $\mathcal{A}'$ such that $\mathcal{I} \not\models q(a)$. Since $\mathcal{O}' \models \mathcal{O}$, $\mathcal{I}$ is also a model of $\mathcal{O}$. Since $\mathcal{O}' \models \exists R.C \subseteq \exists R.C$ for all $\forall R.C \in E(\mathcal{O})$ and due to the construction of $\mathcal{A}$ from $\mathcal{A}'$, $\mathcal{I}$ is also a model of $\mathcal{A}$. Thus, $\mathcal{I}$ witnesses $\mathcal{A}, \mathcal{O} \not\models q(a)$, as required.

This finishes the proof of the claim and of the lemma.

\[ \square \]

\textbf{Lemma 5.} $\mathcal{F}$ is a frontier of $q(x_0)$ w.r.t. $\mathcal{O}$.

\textbf{Proof.} We show that $\mathcal{F}$ fulfills the three conditions of frontiers. For Condition 1, let $p(x_0)$ be a query from $\mathcal{F}$ and let $p_0(x_0) \in \mathcal{F}_0(x_0)$ be the query that was used to construct $p$
by applying the compensation step. First we observe that $p_0$ is satisfiable w.r.t. $O$, since $q$ is satisfiable w.r.t. $O$ and neither the dropping of a concept atom nor the generalizing of a sub-query introduces any violations of functionality assertions in $O$. Next, let $R(x, z) \in p$ be an atom that was added during Step 2A of the compensation step. Then either $\text{func}(R) \notin O$ or there is no atom $R(x', z') \in p$ and therefore Step 2A introduces no atoms that violate any functionality assertions. Similarly Step 2B ensures that all added atoms to not violate functionality assertions in $O$. Therefore $p$ is also satisfiable w.r.t. $O$. Hence it suffices to show $p(x_0) \rightarrow (U_q, O, x_0)$ by Lemma 9.

We extend the mapping $\cdot$ to be defined on all variables of $p$ by considering the yet unmapped variables added in Step 2A and Point (iii) of Step 2B. Let $u$ be a fresh variable with $u^+ \notin \text{var}(p)$ that was added because there is a $x \in \text{var}(p)$, a role $R$ and a set $M$ of concept names such that $x^+ \rightarrow_{\varnothing}^O M$. Then, by construction of $U_q$, there is a trace $x^+RM \in \Delta^M$. Set $u^+ = x^+RM$. Now $\cdot$ is defined on all variables of $p$ and, by construction of $p$, it is a homomorphism from $p$ to $U_q$ with $x_0^+ = x_0$ as required.

We start the proof of the second condition of frontiers with the following claim:

Claim 1. $p \not\subseteq q_x$ for all $x \in \text{var}(q)$ and $p(x) \in F_0(x)$.

Proof of Claim 1. We show the claim by induction on the codepth of $x$ in $q$, matching the inductive construction of $F_0$. In the induction step, $x$ has codepth 0. Then, by definition of codepth, there is no $R(x, y) \in q$ that is directed away from $x_0$ and all $p \in F_0(x)$ are obtained by dropping a concept atom.

Let $p(x)$ be a query from $F_0(x)$ that is obtained by dropping the concept atom $A(x) \in q$. Then, by choice of $A(x)$, there is no $B(x) \in p$ with $O \models B \subseteq A$ and no $R(x', x') \in p$ with $O \models R \subseteq A$. Hence $A(x) \in q_x$, but $A(x) \notin U_q$ and therefore $p \not\subseteq q_x$.

In the induction step, let $x$ have codepth $> 0$, let $p(x)$ be a query from $F_0(x)$ and assume that the claim holds for all variables with smaller codepth. Let $\cdot$ be the extension of the original $\cdot$ for $p$ to a homomorphism from $U_p$ to $U_q$, which exists by Lemma 10. If $p$ is obtained by dropping a concept atom, then the same argument as in the induction step yields $p \not\subseteq q_x$. If $p$ is obtained by generalizing the subquery attached to a role atom $R(x, y) \in q$, assume for contradiction that there is a homomorphism $h$ from $q$ to $U_q$ with $h(x) = x$. From $h$ we construct a homomorphism $h'$ from $q$ to $U_q$ with $h'(x_0) = x_0$ by setting $h'(z) = h(z)^+$ for all $z \in \text{var}(q)$ and $h'(z) = z$ for all $z \notin \text{var}(q)$. Since $h$ must map $y$ to a $R$-successor of $x$ in $U_p$, we may distinguish the following cases.

- $h(y)$ is a variable $z \in \text{var}(p)$ with $z^+ \neq y$. Then, by definition of $\cdot$, $z^+ = z$, and therefore, by construction of $h'$, $h'(y) = h(z) = z$. Hence, $h'$ is a non-injective homomorphism from $q$ to $U_q$ with $h'(x_0) = x_0$, contradicting $O$-minimality or $O$-saturatedness of $q$ by Lemma 11.

- $h(y)$ is a trace $h(x)RM \in U_p$ for some set $M$ of concept names.

If $h'(y)$ is also a trace, then there must be a $y' \in \text{var}(q)$ with $y' \notin \text{img}(h')$, again contradicting $O$-minimality or $O$-saturatedness of $q$ by Lemma 11.

If $h'(y) = y$ and there is no $y' \in \text{var}(q)$ with $h'(y') = x$, then we show a contradiction to $O$-minimality of $q$ by constructing a homomorphism $h''$ from $q$ to $U_q$ with $h''(x_0) = x_0$ where $q'$ is the restriction of $q$ to $\text{var}(q) \setminus \{\}$, Note that by construction of $h'$, $h''(z) = y$ implies $z = y$.

Since $h(x)RM = y$, there is no trace $xRM \in \Delta^M$. But, since $h(x)RM \in \Delta^M$, it must be that $h(x) \rightarrow_{\varnothing}^O M$ and thus $A_p, O \models \exists R \bigwedge M(h(x))$ and $A_p, O \models \exists R \bigwedge M(x)$. However, $x \not\rightarrow_{\varnothing}^O M$ because $R(x, y) \in q$ and $A_q, O \models \exists M(y)$.

Since $R(x, y) \notin q$ and by $O$-saturation of $q$ and normal form of $O$, it follows that $x \rightarrow_{\varnothing}^O M$ and therefore there is a trace $xRM \in \Delta^M$.

Construct $h''$ by setting $h''(z) = h'(z)$ for all $z \in \text{var}(q) \setminus \text{var}(q_y)$ and $h''(x) = xRM_1M_2 \ldots M_n$ for all $z \in \text{var}(q)\setminus\text{var}(q_y)$ if $h(z) = xRM_1M_2 \ldots M_n$. Since $R(x, y) \notin q$ and by $O$-saturation of $q$ and normal form of $O$, it follows that $x \rightarrow_{\varnothing}^O M$ and therefore there is a trace $xRM \in \Delta^M$.

Thus, $A(y'')$ was dropped during the construction of $p'$ from $q_{y''}$. This may only happen by dropping a concept atom. However, $R(x, y) \in q$ and $O \models \exists R \subseteq A$ contradicts Condition (b) of dropping a concept atom.

We have thus shown that $h$ cannot map $q_y$ entirely into the subtree below $y'$ in $U_p$. Consequently, there must be a $y'' \in \text{var}(q_y)$ with $h(y'') = x$. This yields $h'(y'') = h'(x) = x$ contradicting $O$-minimality of $q$ by Lemma 11.

This completes the proof of Claim 1.

We continue by using Claim 1 to show that $p \not\subseteq q$ for all $p \in F$. Let $p$ be a query from $F$ and assume for contradiction that $p \subseteq q$. Then, there is a homomorphism $h$ from $q$ to $U_p$ with $h(x_0) = x_0$. Let $\cdot$ be the extension of the original $\cdot$ for $p$ to a homomorphism from $U_p$ to $U_q$, which exists by Lemma 10. We compose $h$ and $\cdot$ to construct a homomorphism $h'$ from $q$ to $U_q$ with $h'(x_0) = x_0$. By Claim 1, there is no homomorphism that maps $q$ entirely into $U_{q'}$ for any $p' \in F_0(x_0)$. Hence, there must be an $x \in \text{var}(q)$ such that $h(x)$ is a fresh variable added in the compensation step.
This completes the proof of Condition 2 of frontiers.

It remains to show that Condition 3 of frontiers is satisfied. Let \( q'(x_0) \) be an ELIQ that is satisfiable w.r.t. \( O \) such that \( q \subseteq q' \not\subseteq q \). We may assume w.l.o.g. that \( q' \) is \( O \)-saturated and that it satisfies all functionality assertions in \( O \). If, in fact, \( q' \) contains atoms \( R(x, y_1), R(x, y_2) \) with \( y_1 \neq y_2 \) and \( \text{func}(R) \in O \), then we can identify \( y_1 \) and \( y_2 \), obtaining an ELIQ that is equivalent w.r.t. \( O \) to the original \( q' \).

There is a homomorphism \( q \) from \( q' \) to \( U_{q, O} \) with \( g(x_0) = x_0 \). We have to show that there is a \( p \in F \) with \( p \subseteq \sigma \) \( q' \). To do this, we construct in four steps a homomorphism \( h \) from \( q' \) to \( U_{p, O} \) with \( h(x_0) = x_0 \) for some \( p \in F \). During all steps, we maintain the invariant

\[
  h(z)^* = g(z) \tag{*}
\]

for all variables \( z \in \text{var}(q') \) with \( h(z) \) defined and \( ^* \) the extension of the original \( ^* \) for \( p \) to a homomorphism from \( U_{p, O} \) to \( U_{q, O} \). In the first step of the construction, we define \( h \) for an initial segment of \( q' \).

Let \( U \subseteq \text{var}(q') \) be the smallest set of variables (w.r.t. \( \subseteq \)) of \( q' \) such that \( x_0 \in U \) and, for all \( x \in U \) and \( R(x, y) \in q' \) directed away from \( x_0 \), we have: if \( R(g(x), g(y)) \) is an atom in \( q \) directed away from \( x_0 \), then \( y \in U \). Intuitively, \( U \) induces the maximal initial segment of \( q' \) that is mapped in a ‘direction-preserving’ way. Let \( q'' \) be the restriction of \( q' \) to the variables in \( U \).

**Claim 2.** For all \( x \in U \) with \( q'' \not\subseteq q_{g(x)} \), there is a \( p \in F_0(g(x)) \) and a homomorphism \( h' \) from \( q'' \) to \( U_{p, O} \) that satisfies \((*)\).

**Proof of Claim 2.** Let \( y = g(x) \). We show Claim 2 by induction on the codelpth of \( x \) in \( q'' \). In the induction start, \( x \) has codelpth 0. We distinguish the following cases:

- There is an \( R(y, y') \in q_y \).

Then let \( p \in F_0(g(x)) \) be constructed by generalizing the subquery attached to \( R(y, y') \) and set \( h'(x) = y \). Since \( q \) is \( O \)-saturated, \( A(y) \in U_{q, O} \) implies \( A(y) \in U_{p, O} \). This and \( y = g(x) \) implies that \( h' \) is a homomorphism.

- There is no \( R(y, y') \in q_y \).

Then \( q'' \not\subseteq O \cap q_y \) implies that there is an \( A(y) \in q_y \) with \( A(x) \notin U_{q''} \cap O \), and we must even find an \( A \) with these properties such that there is no \( B(y) \in q_y \) with \( O \models B \subseteq A \) and \( O \not\models A \subseteq B \). This implies that Property (a) of dropping concept atoms is satisfied. Property (b) is satisfied since there is no \( R(y, y') \in q_y \). and thus we may construct \( p \in F_0(g(x)) \) by dropping the concept atom \( A(y) \). Set \( h'(x) = y \).

In the induction step, let \( x \) have codelpth > 0 and assume that the claim holds for all variables of smaller codelpth. From \( q'' \not\subseteq O \), it follows that \( q_y \not\models (U_{q''} \cap O, x) \). We distinguish the following cases:

- There is an \( R(y, y') \in q_y \) such that \( q_y \not\models (U_{q''} \cap O, x') \) for all \( R(x, x') \in q_y \).

First assume \( \text{func}(R) \notin O \). Then let \( p \in F_0(g(x)) \) be constructed by generalizing the subquery attached to \( R(y, y') \). We construct the homomorphism \( h' \) from \( q'' \) to \( U_{p, O} \) by starting with \( h'(x) = y \) and continuing to map all successors of \( x \). Let \( S(x, x') \in q'' \). If \( g(x') \neq y' \), extend \( h' \) to the subtree below \( x' \) by setting \( h'(z) = g(z) \) for all \( z \in \text{var}(q_{x'}) \). If \( g(x') = y' \), then, by the induction hypothesis, there is a \( p' \in F_0(y') \) and a homomorphism \( h'' \) from \( q_{x'} \) to \( U_{p', O} \) with \( h''(x') = y' \). Extend \( h' \) to the variables in \( q_{x'} \) by mapping \( q_{x'} \) to \( h'' \) to the copy of \( p' \) that was attached to \( y \) in Point 3 of generalizing the subquery attached to \( R(y, y') \).

Now assume \( \text{func}(R) \in O \). Then there is at most one \( R(x, x') \in q'' \) with \( g(x') = y' \). If there is none, choose an arbitrary \( p \in F_0(y) \) constructed by generalizing the subquery attached to \( R(y, y') \) and extend \( h' \) as above. If there is a single such \( R(x, x') \), then, by the induction hypothesis, there is a \( p' \in F_0(y') \) and homomorphism \( h'' \) from \( q_{x'} \) to \( U_{p', O} \) with \( h''(x') = y' \). Let \( p \in F_0(g(x)) \) be constructed by generalizing the subquery attached to \( R(y, y') \) and attaching \( p' \) in Step 4, then extend \( h' \) as above.

- For every \( R(y, y') \in q_y \), \( q_y \not\models (U_{q''} \cap O, x') \) for some \( R(x, x') \in q'' \).

Then there is an \( A(y) \in q_y \) with \( A(x) \notin U_{q''} \cap O \) and we must even find an \( A \) with these properties and such that there is no \( B(y) \in q_y \) with \( O \models B \subseteq A \) and \( O \not\models A \subseteq B \). Thus, Property (a) of dropping concept atoms is satisfied. To show that Property (b) is also satisfied, we have to argue that there is no \( R(y, y') \in q_y \) with \( O \models \exists x \in A \). But by assumption for any such \( R(y, y') \in q_y \) we have \( q_{y'} \not\models (U_{q''} \cap O, x') \) for some \( R(x, x') \in q''_x \).

This implies \( A(x) \in U_{q''} \cap O \), a contradiction.

We may thus construct \( p \in F_0(g(x)) \) by dropping the concept atom \( A(y) \). Set \( h'(x) = g(x) \) for all \( x' \in \text{var}(q''_x) \).

This completes the proof of Claim 2.

By Claim 2, there is a \( p' \in F_0(x_0) \) such that \( q''_x \not\rightarrow U_{p'} \). Let \( p \in F \) be the query that was obtained by applying the
compensation step to \( p' \). Then clearly also \( q'' \rightarrow U_{p'.\varnothing} \). Define \( h \) for all variables in \( U \) according to the homomorphism that witnesses this and let \( h^1 \) be the extension of the original \( h \) for \( p \) to a homomorphism from \( U_{p.\varnothing} \) to \( U_{q.\varnothing} \).

We continue with the second step of the construction of \( h \) which covers parts of \( q' \) that are connected to the initial segment \( q^g \) and which are mapped to traces of \( U_{p.\varnothing} \) rather than to \( \var{q} \). Consider all atoms \( R(x, x') \in q' \) with \( h \) defined, \( h(x') \) undefined and \( h(x') \notin \var{q} \). Before extending \( h \) to \( x' \), we first show that there is an atom \( R(h(x), z) \in p \), added in Step 2A.

Since \( g(x') \notin \var{q}, g(x') \) must be a trace \( g(x')RM \in \Delta_{U_{q.\varnothing}} \) for some set \( M \), hence \( g(x) \sim_{q''} R_{q'\varnothing} M \). This implies that there is no \( R(g(x), y) \in q \) with \( A_q \models \bigwedge M(y) \). Furthermore, assume that there is a concept name \( B \) such that \( \exists \forall R \subseteq B \) but \( B(h(x)) \notin p \). Then, since \( q \) is \( O \)-saturated, \( B(g(x)) \in q \) and \( p \) must be the result of dropping the concept atom \( B(g(x)) \). However, since \( q \) is \( O \)-saturated, \( B(x') \in q' \), contradicting Claim 2, therefore there is no such concept name.

Hence, Step 2A adds a fresh variable \( z \in \var{p} \) and the atom \( R(h(x), z) \in p \) with \( z = g(x')RM \). We extend \( h \) to the initial segment \( q^g \) of \( g(x) \) that is mapped by \( g \) into the traces below \( g(x') \). Set \( h(\hat{x}) = zR_2M_2 \ldots R_nM_n \) for all \( \hat{x} \) in this initial segment with \( g(\hat{x}) = g(x)RM \). We finally add a \( R(x'', x''') \in q'' \) directed away from \( x_0 \) with \( g(x') \) a trace and \( g(x''') \in \var{q} \), then leave \( h(x''') \) undefined. The mapping \( h \) will be extended to the subtree \( q'' \) in the next steps. Note that \( h^1 \) is satisfied.

In the third step of the construction of \( h \), we consider all \( R(x, x') \in q' \) directed away from \( x_0 \) with \( h \) defined and \( h(x') \) undefined. Before defining \( h(x') \), we first show that

(a) there is an atom \( R(h(x), y) \in p \) directed towards \( x_0 \) such that \( \func(R^-) \notin \var{O} \) and

(b) \( g(x') \in \var{q} \).

Distinguish the following cases:

- \( g(x) \) is a trace.
  Then \( h(x) \) was defined in the second step. From the fact that \( h(x') \) was not defined in the second step, it follows that \( g(x') \in \var{q} \), as required for (b). Consequently, \( g(x) \) must be of the form \( g(x')R'\neg M \in U_{q.\varnothing} \) and \( h(x') \) was defined in the second step to be a fresh variable added to \( p \) in Step 2A of its construction and this variable is an \( R' \)-successor of some variable \( y \), that is, \( R'y, h(x') \in p \) directed away from \( x_0 \). We may thus use the inverse of this atom as the desired atom \( R(h(x), y) \) in (a). Since \( g(x')R'\neg M \) is a trace in \( U_{q.\varnothing} \) and due to the syntactic restriction adopted by the DL-Lite\( ^{-\neg} \)ontology \( \var{O} \), we further have \( \func(R') \notin \var{O} \).

- \( g(x) \) is not a trace, that is, \( g(x) \in \var{q} \).
  Since \( h(x') \) has neither been defined in the first nor in the second step, we must have \( x \in U \), \( g(x') \in \var{q} \) (as required for (b)) and \( R(g(x), g(x')) \in q \) is directed towards \( x_0 \). The latter implies that \( x \) is not the root of \( q' \), thus \( q' \) contains an atom \( S(x', x) \) directed away from \( x_0 \). From \( x_0 \), it follows by definition of \( \bigcup \) that \( x'' \in U \). Thus \( h(x'') \) and \( h(x') \) were both defined in the first step and, due to the formulation of that step, \( S(x'', x) \in q' \) directed away from \( x_0 \) implies that \( q \) contains the atom \( S(g(x''), g(x')) \) directed away from \( x_0 \). So \( S(g(x''), g(x')) \in q \) is directed away from \( x_0 \) and \( R(g(x), g(x')) \in q \) is directed towards \( x_0 \). Since \( q \) is a tree, this implies \( g(x'') = g(x') \) and \( S = R^- \). We have thus shown that \( R^- (x'', x) \) and \( R(x, x') \) are atoms in \( q' \) that are both directed away from \( x_0 \). Since \( q' \) satisfies all functionality assertions in \( \var{O} \), this implies \( \func(R) \notin \var{O} \). We use \( R(h(x), h(x'')) \in p \) as the desired atom \( R(h(x), y) \).

Consider the inverse \( R^- (y, h(x)) \) of the atom \( R(h(x), y) \in p \) that exists due to (a). In the start of Step 2B of the construction of \( p \), the inverse atom \( R^- (y, h(x)) \) is considered and leads to the introduction of an atom \( R(h(x), y') \), \( y' \) a fresh variable with \( y' \in q' \). Set \( h(x') = y' \). Note that \( R(h(x), h(x')) \in p \) was marked in Step 2B of the construction of \( p \).

In the final step of the construction of \( h \) we define \( h(x) \) for all remaining variables \( x \). We do this by repeatedly choosing atoms \( R(x, x') \in q' \) directed away from \( x_0 \) such that

1. \( h(x) \) and \( h(x') \) defined and
2. for all \( S(x', x'') \in q' \) directed away from \( x_0 \), \( h(x'') \) is undefined and there is at least one such \( S(x', x'') \).

If we choose such an \( R(x, x') \) directly after the third step of the construction of \( h \), then \( g(x') \in \var{q} \) due to (b) and \( R(h(x), h(x')) \in p \) was marked in Step 2B of the construction of \( p \). We implement our extension of \( h \) such that these conditions are always guaranteed when we choose an \( R(x, x') \in q' \) that satisfies Properties 1 and 2 above.

Let \( R(x, x') \in q' \) be an atom that satisfies Properties 1 and 2. First assume that \( \func(R^-) \notin \var{O} \). Then processing the marked atom \( R(h(x), h(x')) \in p \) in Step 2B of the construction of \( p \) results in a copy \( \hat{q} \) of \( q \) to be added to \( p \), with the copy of \( h(x') \) in \( \hat{q} \) glued to \( h(x') \). Define \( h \) for all variables \( x'' \in \var{q''} \) by setting \( h(x'') \) to be the copy of \( g(x'') \) in \( \hat{q} \) if \( g(x'') \) is a variable, or to be the trace \( h(x'')RM_1 \ldots R_nM_n \) if \( g(x'') \) is the trace \( g(x'')RM_1 \ldots R_nM_n \).

Now assume that \( \func(R^-) \notin \var{O} \). Consider each \( S(x', x'') \in q' \) directed away from \( x_0 \). We distinguish two cases:

- \( g(x'') \) is a trace.
  Since \( g(x'') \in \var{q} \), \( g(x'') \) must be of the form \( g(x')SM \). Thus, \( g(x') \sim_{q''} R_{q.\varnothing} M \) and by (d), \( h(x') \sim_{q''} R_{q.\varnothing} M \) and when the marked atom \( R(h(x), h(x')) \in p \) is processed in Step 2B of the construction of \( p \). Thus, Point (iii) of Step 2B adds to \( p \) atoms \( S(h(x'), u), S^-(u, y') \), and \( A(u) \) for every \( A \in M \). Additionally, processing the marked atom \( S^-(u, y') \) attaches a copy \( \hat{q} \) of \( q \) to \( y' \), since \( y' \in g(x') \sim_{q''} R_{q.\varnothing} M \) and therefore either \( \func(S) \notin \var{O} \) or there is no atom \( S(g(x'), z) \in q \).

Extend \( h \) by setting \( h(x'') = uR_2M_2 \ldots R_nM_n \) if \( g(x'') = g(x')SM \). For all \( x'' \in \var{q''} \) up until \( g(x'') = g(x') \). If there is a subtree
Proof. In order to reduce notational clutter, we introduce $g(x'')$ with $x'' \in \text{var}(g(x'))$ and $g(x'') = g(x')$, map it to $\tilde{y}$ and the attached $\tilde{q}$ by setting $h(x'') = \tilde{y}$ and all $h(z)$ for $z \in \text{var}(g(x''))$ to the copy of $g(z)$ in $\tilde{q}$, or the trace starting in $\tilde{q}$.

- $g(x'')$ is not a trace.

Since $g(x') \in \text{var}(q)$, this implies $S(g(x'), g(x'')) \in q$. It follows from (s) that $h(x') = g(x')$ and thus $S(h(x'), g(x'')) \in q$. When the marked atom $R(h(x), h(x'))$ is processed in Step 2B of the construction of $p$, then $S(h(x'), g(x'')) \in q$ is thus one of the atoms under consideration in Point (ii).

Since $R(x, x') \in q'$ and $S(x', x'') \in q'$ are both directed away from $x_0$ and $q'$ satisfies all functionality assertions in $O$, we have $S \neq R$. It follows from (s) that $h(x')^+ = g(x')$ As a consequence, $S(h(x'), g(x'')) \neq R^-(h(x'), h(x'))$. Thus, in Point (ii) of Step 2B an atom $S(h(x'), z')$ is added to $p$ with $z'$ fresh variable and $z'' = g(x'')$ and this atom is marked. Set $h(x'') = z''$ and leave the successors of $x''$ in $q'$ to be processed in subsequent iterations of the loop in step four of the construction of $h$.

This completes the construction of $h$ and the proof of Condition 3.



\textbf{Lemma 6.} The construction of $F$ runs in time polynomial in $|q| + |O|$ (and thus $\sum_{p \in F} |p|$ is polynomial in $|q| + |O|$).

\textbf{Proof.} In order to reduce notational clutter, we introduce some abbreviations used throughout the proof.

- $s = |\text{sig}(q)|$ denotes the number of concept and role names used in $q$;
- $o = |O|$ denotes the size of $O$;
- for an ELIQ $p$, $n_p = |\text{var}(p)|$ denotes the number of variables in $p$;
- for a set $Q$ of queries, $n_Q$ denotes $\sum_{p \in Q} n_p$.

We assume without loss of generality that $s$ and $o$ are at least one.

We start with analyzing the size of the queries in $F_0(x)$ that are obtained as the result of the ‘generalize’ step.

\textbf{Claim.} For every $x \in \text{var}(q)$, we have:

1. $|F_0(x)| \leq s \cdot n_{q_x}$;
2. $n_{F_0(x)} \leq s \cdot n_{q_x}^3$.

\textbf{Proof of the claim.} The proof of both points is by induction on the codepth of $x$ in $q$. We start with Point 1. For the base case, consider a variable $x$ of codepth 0 in $q$, that is, a leaf. In this case, only Step (A) is applicable, and it adds at most $s$ queries to $F_0(x)$.

For the inductive step, consider a variable $x$ of codepth greater than 0. We partition $F_0(x)$ into $F_0^A(x)$ and $F_0^B(x)$, that is, the queries that are obtained by dropping a concept atom in Step (A) and the queries that are obtained by generalizing a subquery in Step (B), respectively, and analyze them separately, starting with $F_0^A(x)$. Clearly, there are at most $s$ queries in $F_0^A(x)$, that is,

$$|F_0^A(x)| \leq s.$$ 

Next, we analyze $F_0^B(x)$. Each query in $F_0^B(x)$ is obtained by first picking, in Point 1, an atom $R(x, y)$ in $q_x$. If $\text{func}(R) \notin O$, we add 1 query to $F_0^B(x)$. Otherwise, we add $|F_0(y)|$ queries (in Point 4).

Thus, we obtain

$$|F_0^B(x)| \leq \sum_{R(x, y) \in q_x, \text{func}(R) \notin O} |F_0(y)| + \sum_{R(x, y) \in q_x, \text{func}(R) \in O} |F_0(y)|.$$ 

Using the fact that $n_{q_x} \geq 1$ and the induction hypothesis, we obtain

$$|F_0^B(x)| \leq s \cdot n_{q_x} = s \cdot \sum_{R(x, y) \in q_x} n_{q_y}.$$

The above sum can be simplified to $n_{q_x} - 1$. Hence, we obtain

$$|F_0(x)| = |F_0^A(x)| + |F_0^B(x)| \leq s + s \cdot (n_{q_x} - 1) = s \cdot n_{q_x}.$$

We now prove Point 2, again by induction on the codepth of $x$ in $q$. For the base case, consider a variable $x$ of codepth 0 in $q$, that is, a leaf. In this case, only Step (A) is applicable, and it adds at most $s$ queries of size 1 to $F_0(x)$.

For the inductive step, consider a variable $x$ of codepth greater than 0 and the same partition of $F_0(x)$ into $F_0^A(x)$ and $F_0^B(x)$ as before. Clearly, every $p \in F_0^A(x)$ uses $n_{q_x}$ variables and there are at most $s$ queries in $F_0^A(x)$. Thus, we have

$$n_{F_0^A(x)} \leq s \cdot n_{q_x}.$$ 

Next, we analyze $F_0^B(x)$. Each query in $F_0^B(x)$ is obtained by first picking, in Point 1, an atom $R(x, y)$ in $q_x$. If $\text{func}(R) \notin O$, we add $\sum_{R(x, y) \in q_x} n_{q_y}$ variables (in Point 3). Otherwise, we replace $q_y$ with some element of $F_0(y)$ (in Point 4). Thus, we obtain

$$n_{F_0^B(x)} \leq \sum_{R(x, y) \in q_x, \text{func}(R) \notin O} (n_{q_x} + n_{F_0(y)}) +$$

$$\sum_{R(x, y) \in q_x, \text{func}(R) \in O} (n_{q_x} \cdot |F_0(y)| + n_{F_0(y)}) \leq \sum_{R(x, y) \in q_x} (n_{q_x} \cdot |F_0(y)| + n_{F_0(y)}).$$

Plugging in the induction hypothesis from both Point 1 and 2, we obtain

$$n_{F_0(x)} \leq s \cdot n_{q_x} \sum_{R(x, y) \in q_x} n_{q_y} + s \cdot n_{q_x}^3.$$ 

We simplify the right-hand side of (2) by making the following observations:
Then note that the marking proviso “if variables. Using the claim, we get

\[ F(z) = \left( \sum_{(x,y) \in q} n_{q_y} \right)^3 = (n_x - 1)^3. \]

Here, the inequality is an application of the general inequality \( \sum a_i^3 \leq (\sum a_i)^3 \), for every sequence of non-negative numbers \( a_1, \ldots, a_k \).

Using these observations, Inequality (2) can be simplified to:

\[
\begin{align*}
F_0(x) &\leq s \cdot n_{q_x} \cdot (n_{q_x} - 1) + s \cdot (n_{q_x} - 1)^3 \\
&= s \cdot (n_{q_x}^3 - 2n_{q_x}^2 + 2n_{q_x} - 1).
\end{align*}
\]

Overall, we get

\[
F_0(x) = n_{F_0} + n_{F_0} \cdot o \leq s \cdot n_{q_x}^3 \cdot (1 + o).
\]

We now analyze Step 2B, applied to some query \( p \in F_1 \).

First of all note that the marking proviso “if \( R(x,y) \) is marked then \( y \) is defined and if \( x \) is undefined, then \( \text{func}(R^{-}) \notin O \) or \( q \) contains no atom of the form \( R(y^z, z) \)” is indeed satisfied.

Consider now an atom \( R(x,y) \in p \) that was marked in the Start phase. We distinguish two cases.

- If \( x \) is undefined, then the marking proviso implies that \( \text{func}(R^{-}) \notin O \) or \( q \) contains no atom of the form \( R(y^z, z) \). In the Step phase, we just unmark the atom and add a copy of \( q \), hence no iteration takes place, and the query size increases by \( n_q \).

- Otherwise, \( x \) is defined, and by definition of 4, we have \( R(x'^z, y'^z) \in q \). Now, the iterative process ensures that:

  - Whenever an atom \( S(y^z, z') \) is marked in (ii), then both \( y^z \) and \( z'^z \) are defined and \( S(y^z, z'^z) \in q \). Moreover, the condition ‘\( S(y^z, z) \notin R^{-}(y^z, x'^z) \)’ and the fact that \( q \) is an ELIQ ensure that every atom from \( q \) is ‘met’ at most once during the entire process.

  - Whenever an atom \( S^{-}(u, y^z) \) is marked in (iii), then \( u \) is undefined. Hence, the marking proviso implies that, in the Step phase, this atom is unmarked, a copy of \( q \) is added, and the iteration stops.

Overall, we obtain that, per role atom in \( p \), the marking process adds at most \( n_q \) role atoms in Step (ii), for each such atom and every \( \exists r.B \) in \( O \) one more role atom in Step (iii), and for each introduced variable at most one copy of \( q \). All this is polynomial in ||\( q || \) and ||\( O || \). Moreover, the computation of \( F \) can be carried out in polynomial time since all the involved queries are of polynomial size and consequences of \( O \) can be decided in polynomial time.

\[ \square \]

E Proofs for Section 5

Theorem 5. Let \( O \) be an ontology formulated in DL-Lite^H or DL-Lite^F+. Then for every ELIQ \( q \) that is satisfiable w.r.t. \( O \), there are sets of data examples \( (E^+, E^-) \) that uniquely characterize \( q \) w.r.t. \( O \) and such that \( ||(E^+, E^-)|| \) is polynomial in \( ||q|| + ||O|| \). If \( O \) is a DL-Lite^F+ ontology, then \( (E^+, E^-) \) can be computed in polynomial time and the same holds for DL-Lite^H if \( q \) is \( O \)-minimal.

Proof. Let \( O \) and \( q(x) \) be as in the theorem. By Theorems 1 and 4, we can compute in polynomial time a frontier \( F_q(x) \) for \( q \) w.r.t. \( O \). Let \( E^+ = \{(A_q, x)\} \) and \( E^- = \{(A_p, x) \mid p \in F_q(x)\} \). It is not hard to verify that \( q \) fits \( (E^+, E^-) \). We show that \( (E^+, E^-) \) in fact uniquely characterizes \( q \) w.r.t. \( O \).

Let \( q' \) be an ELIQ that fits \( (E^+, E^-) \). We have \( q \subseteq q' \) since \( (A_q, x) \) is a positive example. Moreover, since all data examples in \( E^- \) are negative examples for \( q' \), we know that \( q \not\subseteq q' \) for any \( p \in F_q(x) \). By Point 3 of the definition of frontiers, we can conclude that \( q' \not\subseteq q \). Thus \( q' \equiv O q \), as required.

F Proofs for Section 6

We start with showing how to construct a seed CQ in the case that the ontology \( O \) contains no concept disjointness constraints. This is in fact trivial if \( O \) contains no role disjointness constraints either, as then we can simply use

\[ q_{OP}^0(x_0) = \bigwedge_{A \in \Sigma \cap N_c} A(x_0) \land \bigwedge_{r \in \Sigma \cap N_k} r(x_0, x_0). \]

Here and in what follows, for brevity we use \( \Sigma \) to denote \( \text{sig}(O) \).

We consider now the case with role disjointness constraints (but still without concept disjointness). Let \( N = \{r_1, \ldots, r_m\} \) be the set of all role names \( r \in \Sigma \) such that \( \exists r \) is satisfiable w.r.t. \( O \). If, for example, \( O \) contains \( r \subseteq s \) and \( r \cap s \subseteq \bot \), then \( \exists r \) is not satisfiable w.r.t. \( O \).

Introduce variables \( x_0, \ldots, x_{2m} \) and let \( K_{2m+1} \) be the \( 2m+1 \)-clique that uses these variables as its vertices. It is known that for each \( n \geq 1 \), the \( n \)-clique \( K_n \) has at least \( n-1 \) Hamilton cycles that are pairwise edge-disjoint, see for instance the survey [Alspach et al., 1987]. We thus find in \( K_{2m+1} \) Hamilton cycles \( P_1, \ldots, P_m \), that are pairwise edge-disjoint. By directing the cycles, we may view each \( P_i \) as a set of directed edges \( (x_j, x_{j+1}) \). We then set

\[
q_{OP}^i(x_0) = \bigwedge_{(x_j, x_{j+1}) \in P_i} A(x_j) \land \bigwedge_{1 \leq j \leq 2m} \bigwedge_{(x_j, x_{j+1}) \in P_m} r_m(x_j, x_{j+1}).
\]
Clearly, \( q_{\delta}^0 \) has no multi-edges and thus satisfies all role disjointness constraints in \( \mathcal{O} \). Moreover, every variable has exactly one \( r \)-successor and exactly one \( r \)-predecessor for every role name \( r \in \mathcal{R} \). On the one hand, this implies that all functionality assertions in \( \mathcal{O} \) are satisfied. On the other hand, it means that there is a homomorphism from every target ELIQ \( q_{T} \) to \( q_{\delta}^0 \) because any \( q_{T} \) is required to be satisfiable w.r.t. \( \mathcal{O} \) and thus may only use role names from \( \mathcal{R} \).

If \( \mathcal{O} \) contains at least one concept disjointness constraint \( B_1 \cap B_2 \subseteq \bot \), then we cannot use the above \( q_{\delta}^0 \) as it is not satisfiable w.r.t. \( \mathcal{O} \), but we may obtain a seed query \( q_{\delta}^0 \) by viewing \( B_1 \cap B_2 \) as an ELIQ \( \delta \) in the obvious way and posing \( \delta \) as an equivalence query to the oracle. Since the target query is satisfiable w.r.t. \( \mathcal{O} \), the oracle is forced to return a positive counterexample \( (A, a) \), that is, a pair \( (A, a) \) such that \( A, \mathcal{O} \models q_{\delta}^0(a) \) and \( A, \mathcal{O} \not\models q_{\delta}^0(a) \). The desired query \( q_{\delta}^0 \) is \( (A, a) \) viewed as a CQ with answer variable \( a \). Note that when learning with equivalence queries, then in polynomial time learnability, the running time of the learning algorithm may also polynomially depend, at any given time, on the size of the largest counterexample returned by the oracle so far. This condition is satisfied by our algorithm.

**Lemma 7.** In DL-Lite\(^H\) and DL-Lite\(^F-\), every polynomial time learning algorithm for ELIQs under ontologies in normal form that uses only membership queries can be transformed into a learning algorithm with the same properties for ELIQs under unrestricted ontologies.

**Proof.** We show the lemma by converting a learning algorithm \( L' \) for ontologies in normal form into a learning algorithm \( L \) for unrestricted ontologies, relying on the normal form described in Lemma 12. Since \( L \) will ask a single query for every query asked by \( L' \), the lemma follows.

We start with DL-Lite\(^H\). Given a DL-Lite\(^H\) ontology \( \mathcal{O} \) and a signature \( \Sigma = \text{sig}(\mathcal{O}) \) with \( \text{sig}(q_{T}) \subseteq \Sigma \), algorithm \( L \) first computes the ontology \( \mathcal{O}' \) in normal form as per Lemma 12, choosing the fresh concept names so that they are not from \( \Sigma \). It then runs \( L' \) on \( \mathcal{O}' \) and \( \Sigma' = \Sigma \cup \text{sig}(\mathcal{O}') \). In contrast to \( L' \), the oracle still works with the original ontology \( \mathcal{O} \). To ensure that the answers to the queries posed to the oracle are correct, \( L \) modifies \( L' \) as follows.

Whenever \( L' \) asks a membership query \( A', \mathcal{O}' \models \text{ind}(A_{C(b)}) \), we may assume \( A' \) satisfies the functionality assertions from \( \mathcal{O} \), since otherwise the answer is trivially “yes”. Then, \( L \) instead asks the membership query \( A, \mathcal{O} \models \text{ind}(A_{C(b)}) \), and \( A \) is obtained from \( A' \) as follows. Start with \( A = A' \), and

\[ (*) \text{ add } C(b), \text{ for each concept assertion } X_C(b) \in A'. \]

Here, the addition of \( C(b) \) for an ELI-concept \( C \) to an ABox \( B \) is defined as expected in case of DL-Lite\(^H\) ontologies: View \( C(b) \) as a tree-shaped ABox \( A_{C(b)} \) with root \( b \) and assume without loss of generality that \( b \) is the only individual shared by \( B \) and \( A_{C(b)} \). Then take the union of \( A \) and \( A_{C(b)} \).

By the following claim, the answer to the modified membership query coincides with that to the original query.

**Claim 1.** \( A', \mathcal{O}' \models \text{ind}(A_{C(b)}) \) iff \( A, \mathcal{O} \models \text{ind}(A_{C(b)}) \) for all ELIQs \( q \) that only use symbols from \( \Sigma \).

**Proof of Claim 1.** For “if”, suppose that \( A, \mathcal{O} \models \text{ind}(A_{C(b)}) \) and \( \mathcal{I} \) is a model of \( A' \) and \( \mathcal{O}' \). We can assume that \( \Delta^2 \) does not mention any of the individuals that were introduced in the construction of \( A \). We will construct a model \( \mathcal{I}' \) of \( A \) and \( \mathcal{O} \) that has a homomorphism \( h \) from \( \mathcal{I}' \) to \( \mathcal{I} \) which is the identity on \( \Delta^2 \). This clearly suffices since \( \mathcal{I}' \models q(a) \).

The interpretation \( \mathcal{I}' \) has the following domain:

\[ \Delta^2 = \Delta^2 \cup \bigcup_{X_C(b) \in A'} \text{ind}(A_{C(b)}) \]

In order to define the interpretation of concept and role names, observe first that, for every \( X_C(b) \in A' \), there is a homomorphism \( h_{C(b)} : A_{C(b)}, b \rightarrow \mathcal{I}, b \) since \( \mathcal{I} \) is a model of \( A' \) and \( \mathcal{O}' \), and \( \mathcal{O}' \models X_C \subseteq C \). We combine all these homomorphisms into a mapping \( h : \Delta^2 \rightarrow \Delta^2 \) by taking

\[ h(c) = \begin{cases} c & \text{if } c \in \Delta^2, \\ h_{C(b)}(c) & \text{if } c \in \text{ind}(A_{C(b)}) \setminus \Delta^2. \end{cases} \]

Then, we set

\[ A^\mathcal{I'} = \{ d | h(d) \in A \} \]

and

\[ r^\mathcal{I'} = \{ (d, e) | (b(d), h(e)) \in r \}. \]

It is routine to verify that \( \mathcal{I}' \) is as required.

For “only if”, suppose that \( A', \mathcal{O}' \models \text{ind}(A_{C(b)}) \) and let \( \mathcal{I} \) be a model of \( A \) and \( \mathcal{O} \). Observe that the model \( \mathcal{I}' \) of \( \mathcal{O}' \) that can be obtained from \( \mathcal{I} \) as in Lemma 1 Point 2 coincides with \( \mathcal{I} \) on \( \Sigma \) and is additionally a model of \( A' \). It follows that \( \mathcal{I}' \models q(a) \) as required. This finishes the proof of Claim 1.

In the case of DL-Lite\(^F-\) ontologies, we follow the same strategy. However, the addition of \( A_{C(b)} \) in \( (*) \) has to respect the functionality assertions. In fact, not even \( A_{C(b)} \) necessarily satisfies the functionality assertions in \( \mathcal{O} \). We define the addition of a tree-shaped ABox \( B' \) with root \( b_0 \) to \( A \) at \( a \) inductively on the structure of \( B \) as follows:

1. for all \( A(b_0) \in B, add A(a) to A; \)
2. for all \( R(b_0, b') \in B, let B' be the sub-ABox of B rooted at b' and
   
   \[ (a) \text{ if } \text{func}(R) \in \mathcal{O} \text{ and there is an atom } R(a, a') \in A, \text{ then add } b' \text{ to } A \text{ at } a'; \]
   
   \[ (b) \text{ otherwise, add an atom } R(a, a') \text{ for a fresh individual } a' \text{ and add } b' \text{ to } A \text{ at } a'. \]

Note that there can only be one atom \( R(a, a') \) in Step 2(a) if \( A \) satisfies the functionality assertions in \( \mathcal{O} \). It should be also clear that the resulting ABox also satisfies the functionality assertions if \( A \) does.

Now, \( A \) is obtained from \( A' \) by starting with \( A = A' \) and

\[ (**) \text{ adding } A_{C(b)} \text{ to } A \text{ at } b, \text{ for each } X_C(b) \in A'. \]

By the following claim, the answer to the modified membership query coincides with that to the original query.

**Claim 2.** \( A', \mathcal{O}' \models \text{ind}(A_{C(b)}) \) iff \( A, \mathcal{O} \models \text{ind}(A_{C(b)}) \) for all ELIQs \( q \) that only use symbols from \( \Sigma \).

**Proof of Claim 2.** For “if”, suppose that \( A, \mathcal{O} \models \text{ind}(A_{C(b)}) \) and \( \mathcal{I} \) be a model of \( A' \) and \( \mathcal{O}' \). We can assume that \( \Delta^2 \) does
not mention any of the individuals that were introduced in the construction of $A$. We will construct a model $I'$ of $A$ and $O$ that has a homomorphism $h$ from $I'$ to $I$ which is the identity on $\Delta^2$. This clearly suffices since $I' \models \varphi(a)$.

Observe first that, for every $X_C(b) \in A'$, there is a homomorphism $h_{C(b)} : A_C(b), b \to I, b$ since $I$ is a model of $A'$ and $O'$, and $O' \models X_C \subseteq C$.

Let $F$ denote the set of fresh individuals introduced in the construction of $A$. Note that for every fresh element there is an $X_C(b) \in A'$ which `triggered' the addition of $d$ in some (possibly later) application of Step 2(b). We associate with every $d \in F$ an element $g(d) \in \Delta^2$ as follows:

- if $d$ was introduced in Step 2(b) triggered by $X_C(b) \in A'$, then set $g(d) = h_{C(b)}(b')$ where $b'$ is the element mentioned in Step 2.

We further associate with every $d \in F$ a tree-shaped interpretation $I_d$. Intuitively, $I_d$ is the unraveling of $I$ at $g(d)$, with the functionality assertions taken into account. Formally, the domain $\Delta^2_d$ consists of all sequences $a_0 R_1 a_1 \ldots R_n a_n$ such that

- $a_0 = g(d)$;
- $a_i \in \Delta^2$, for all $i$ with $0 \leq i \leq n$;
- $(a_i, a_{i+1}) \in R^2_{i+1}$, for all $i$ with $0 \leq i < n$;
- if $\text{func}(R^2_i) \in O$, then $R^2_i \neq R^2_i$, for all $i$ with $0 \leq i < n$;
- if $\text{func}(R^2_i) \in O$, there is no atom of shape $R_1(d, d')$ in $A_C(b)$, where $X_C(b) \in A'$ triggered the addition of $d$.

The interpretation of concept and role names is as follows:

$A^2_d = \{a_0 R_1 a_1 \ldots R_n a_n \in \Delta^2_d \mid a_n \in \Delta^2\}$ for all $A \in N_C$;

$r^2_d = \{(\pi, \pi r a) \mid \pi r a \in \Delta^2_d\} \cup \{(\pi r^2 a, \pi) \mid \pi r^2 a \in \Delta^2_d\}$ for all $r \in N_R$.

The interpretation $I'$ is then obtained by starting with $I' = I \cup A$, and then adding, for every $d \in F$, a copy $I_d$ of $I_d$ and gluing the copy of $g(d)$ in $I_d$ to $d$.

It is routine to verify that $I'$ is as required.

For “only if”, suppose that $A', O' \models \varphi(a)$ and let $I$ be a model of $A$ and $O$. Observe that the model $I'$ of $O'$ that can be obtained from $I$ as in Lemma 1 Point 2 coincides with $I$ on $\Sigma$ and is additionally a model of $A'$. It follows that $I' \models \varphi(a)$ as required. This finishes the proof of Claim 2.

We now work towards showing that the algorithm presented in Section 6 indeed learns ELIQs under ontologies formulated in DL-Lite$^H$ or DL-Lite$^{F-}$, in polynomial time. We start with analyzing the minimize subroutine.

**Lemma 13.** Let $q$ be a unary $CQ$ that is $O$-saturated and satisfiable w.r.t. $O$ such that $q \subseteq O$ for the target query $q_T(x_0)$, and let $q'(x_0) = \text{minimize}(q)$. Then

1. $q \subseteq O$ and $q' \subseteq O$ for $q_T$;
2. $|\var(q')| \leq |\var(q_T)|$;
3. $q'$ is $O$-minimal, connected, and $O$-saturated.

**Proof.** We start with Point 1. We have $q \subseteq q'$ since $q'$ is a subset of $q$. For $O' \subseteq O$, it suffices to observe that minimize ensures in each step that $A_{q'}, O' = q(x_0)$.

For Point 2, it suffices to show that $|\var(q')| \leq \var(h)$ for every homomorphism $h$ from $q_T$ to $U_{q', O}$ with $h(x_0) = x$.

We argue that for a contradiction that there is a homomorphism $h$ from $q_T$ to $U_{q', O}$ with $h(x_0) = x$ and $y \in \var(q')$ that is not in $\var(h)$. Choose some $r(x, y) \in q$ such that the distance from $x_0$ to $x$ is strictly smaller than that from $x_0$ to $y$, and let $y'$ be the restriction of $y \setminus \{r(x, y)\}$ to the atoms that contain only variables reachable from $x_0$ in $q \setminus \{r(x, y)\}$.

We argue that $h$ is a homomorphism from $q_T$ to $U_{q', O}$ which witnesses that $A_{q'}, O \models q(x_0)$, in contradiction to the construction of $q'$ and $y \in \var(q')$.

To see that $h$ is a homomorphism, first note that since $q_T$ is connected and $h(x_0) = x_0$, the range of $h$ contains only variables from $\var(q')$ and the subtrees below them that (consist of traces and) are added in the construction of the universal model. Next observe that for all $x_1, x_2 \in \var(q')$, the following holds by construction of universal models and since $q$ is $O$-saturated:

(a) $A(x_1) \in U_{q', O}$ iff $A(x_1) \in U_{q', -O}$;

(b) $r(x_1, x_2) \in U_{q', O}$ iff $r(x_1, x_2) \in U_{q', -O}$.

Since $O$ is in normal form and due to (a), it follows from the construction of universal models that the subtree in $U_{q', O}$ below each $x_1 \in \var(q') \setminus \{y\}$ is identical to the subtree in $U_{q', -O}$ below $x_1$. Moreover, the subtree in $U_{q', O}$ below $y$ can be obtained from the subtree in $U_{q', -O}$ below $y$ by dropping subtrees. It should thus be clear that, as required, $h$ is a homomorphism from $q_T$ to $U_{q', -O}$.

For Point 3, we start with $O$-minimality. Assume for a contradiction that $q'$ is not $O$-minimal, that is, there is a homomorphism $h$ from $q'$ to $U_{q''', O}$ with $h(x_0) = x_0$ where $q'' = \var(q') \setminus \{y\}$ for some variable $y \in \var(q)$. Choose some $r(x, y) \in q$ such that the distance from $x_0$ to $x$ is strictly smaller than that from $x_0$ to $y$ and let $y'$ be the restriction of $q \setminus \{r(x, y)\}$ to the atoms that contain only variables reachable from $x_0$ in $q \setminus \{r(x, y)\}$. We can show as above that $h$ is a homomorphism from $q_T$ to $U_{q', -O}$ and thus $A_{q'}, O \models q(x_0)$, in contradiction to the construction of $q'$ and $y \in \var(q')$.

Now for connectedness. Assume for a contradiction that $q'(x_0)$ is not connected and let $x$ be a variable that is in a different maximally connected component of $q'$ than $x_0$. Then $x$ is also in a different maximally connected component of $U_{q', -O}$ than $x_0$. By Point 1, there is a homomorphism $h$ from $q_T$ to $U_{q', -O}$ with $h(x_0) = x_0$. Since $q_T$ is connected, we must have $x \notin \text{img}(h)$, thus $\var(q') \notin \text{img}(h)$. But we have already seen in the proof of Point 2 that this is impossible.

Finally, $O$-saturatedness of $q'$ is clear given that the original $CQ q$ is $O$-saturated, $q'$ is a subquery of $q$, and during the construction of $q'$ we have not removed any concept atoms on any of the remaining variables.

We next turn to the treeify subroutine. We start with a preliminary.

**Definition 2.** An $ELI$-simulation from interpretation $I_1$ to interpretation $I_2$ is a relation $S \subseteq \Delta^2 \times \Delta^2$ such that for all $(d_1, d_2) \in S$, we have:
1. for all $A \in N_C$: if $A(d_1) \in I_1$, then $A(d_2) \in S$;
2. for all $r \in N_R$ and $R \in \{r, r^\sim\}$: if there is some $d'_1 \in \Delta^R$ with $R(d_1, d'_1) \in I_1$, then there is $d'_2 \in \Delta^R$ such that $(d'_1, d'_2) \in S$ and $R(d_2, d'_2) \in I_2$.

The following lemma gives an important property of simulations. The proof is standard and omitted.

**Lemma 14.** Let $O$ be a DL-Lite$^{HF}$ ontology, $A_1, A_2$ $A$-Boxes and $q(x)$ an ELIQ such that $A_1, A_2,$ and $q$ are satisfiable w.r.t. $O$. If there is an $\mathcal{ELI}^T$-simulation $S$ from $A_1$ to $A_2$ with $(a_1, a_2) \in S$, then $A_1, O \models q(a_1)$ implies $A_2, O \models q(a_2)$.

**Lemma 15.** Let $q(x_0)$ be a unary CQ that is $O$-saturated and satisfiable w.r.t. $O$ such that $q \subseteq O$ for the target query $q_T(x_0)$. Further let $p_1(x_0), p_2(x_0), \ldots$ be the sequence of CQs computed by treeify($q$). Then for all $i \geq 1$,
1. $p_i \subseteq q_T$;
2. $|\text{var}(p_{i+1})| > |\text{var}(p_i)|$;
3. $p_i$ is $O$-saturated and satisfiable w.r.t. $O$.

**Proof.** We show Point 1 by induction on $i$. The case $i = 1$ is immediate by Point 1 of Lemma 13 since $p_1 = \text{minimize}(q)$. Now let $i \geq 1$. By the induction hypothesis $p_i \subseteq q_T$ and thus $A_{p_i}, O \models q_T(x_0)$. By construction of $p_i$,

$$S = \{(x, x') \mid x \in \text{var}(p_i)\} \cup \{(x, x') \mid x \in \text{var}(p_i)\}$$

is an $\mathcal{ELI}^T$-simulation from $A_{p_i}$ to $A_{p'_i}$ with $(x_0, x_0) \in S$. Therefore, by Lemma 14, $A_{p'_i}, O \models q_T(x_0)$ and $p'_i \subseteq O$. Hence, by Point 1 of Lemma 13, $p_{i+1} \subseteq O$ for $p_{i+1} = \text{minimize}(p'_i)$.

For Point 2, define a homomorphism $q$ from $\text{var}(p_{i+1})$ to $\text{var}(p_i)$ by setting $g(x) = x$ for all $x \in \text{var}(p_i) \cap \text{var}(p_{i+1})$ and $g(x') = x$ for all $x' \in \text{var}(p_{i+1}) \setminus \text{var}(p_i)$. To establish that $|\text{var}(p_{i+1})| > |\text{var}(p_i)|$, we show the following three claims, proving that $g$ is surjective, but not injective. For an injective and surjective function, we use $g^{-1}$ to denote the inverse of $g$.

**Claim 1.** $g$ is surjective.

**Proof of Claim 1.** Suppose that $g$ is not surjective. Then some $y \in \text{var}(p_i)$ does not occur in the image of $g$. Choose some $r(x, y) \in p_i$ and define $p^-_i = p_i \setminus \{r(x, y)\}$. Clearly, $g$ is still a homomorphism from $p_{i+1}$ to $p^-_i$. By Lemma 10, we can extend $g$ to be a homomorphism from $U_{p_{i+1}, O}$ to $U_{p^-_i, O}$. By Point 1, there is a homomorphism $h$ from $q_T$ to $U_{p^-_i, O}$ with $h(x_0) = x_0$. Composing $h$ and $g$ yields a homomorphism $g' = g \circ h$ from $q_T$ to $U_{p^-_i, O}$ with $g'(x_0) = x_0$. Thus, $A_{p^-_i, O} \models q_T(x_0)$ which is in contradiction to the fact that $p_i$ is obtained by applying minimize, and that this operation would replace $p_i$ with $p^-_i$.

**Claim 2.** If $g$ is injective, then $r(x, y) \in p_i$ implies $r(g^-(x), g^- (y)) \in p_{i+1}$.

**Proof of Claim 2.** Suppose to the contrary that there is an $r(x, y) \in p_i$ with $r(g^-(x), g^- (y)) \notin p_{i+1}$. Then $g$ is also a homomorphism from $p_{i+1}$ to $p_i \setminus \{r(x, y)\}$ and using the same composition-of-homomorphisms argument as in the proof of Claim 1, we find a homomorphism $h$ from $q_T$ to $U_{p_i \setminus \{r(x, y)\}, O}$ with $h(x_0) = x_0$. Hence $p_i \setminus \{r(x, y)\} \subseteq O$. This contradicts the fact that $p_i = \text{minimize}(p'_i)$.

**Claim 3.** $g$ is not injective.

**Proof of Claim 3.** Let $R_1(x_1, x_2), \ldots, R_n(x_n, x_1)$ be a cycle in $p_i$ and $R_n(x_n, x_1)$ the atom that was chosen in the cycle doubling operation. Suppose for contradiction that $g$ is injective. The construction of $g$, together with $g$ being surjective and injective, implies that exactly one of $x_j, x'_j$ is in $\text{var}(p_{i+1})$ for all $j$ with $1 \leq j \leq n$. Assume that $x_n \in \text{var}(p_{i+1})$ (the case $x'_n \in \text{var}(p_{i+1})$ is analogous) and thus $g(x_n) = x_0$.

We prove by induction on $j$ that $x_j \notin \text{var}(p_{i+1})$ for $1 \leq j \leq n$, thus obtaining a contradiction to $x_n \in \text{var}(p_{i+1})$. For the induction start, assume to the contrary of what is to be shown that $x_1 \in \text{var}(p_{i+1})$. Then $g(x_1) = x_1$ and $R_n(x_n, x_1) \in p_i$ implies $R_n(x_n, x_1) \in p_{i+1}$ by Claim 2. This contradicts the construction of $p_{i+1}$ that removes $R_n(x_n, x_1)$.

For the induction step, let $j \geq 1$. By the induction hypothesis $x_j \notin \text{var}(p_{i+1})$ and thus $x'_j \in \text{var}(p_{i+1})$. Assume to the contrary of what is shown that $x_j \in \text{var}(p_{i+1})$. Then $g(x_j) = x_{j+1}$, and $R_j(x_j, x_{j+1}) \in p_i$ yield $R_j(x_j, x_{j+1}) \in p_{i+1}$ by Claim 2. This contradicts the construction of $p_{i+1}$.

This completes the proof of Claim 3 and thus Point 2 of the lemma.

We show Point 3 of the lemma by induction on $i$. In the induction start, $O$-saturatedness and satisfiability of $p_i$ w.r.t. $O$ follows from $O$-saturatedness and satisfiability of $q$ w.r.t. $O$ and the fact that minimize preserves those properties.

Now let $p_i$ be $O$-saturated and satisfiable w.r.t. $O$. Then $p'_i$ is also $O$ saturated, by construction. Moreover, we can construct a model $I$ of $A_{p'_i}$ and $O$ by starting with $I = A_{p'_i}$ and attaching the trace subtrees below every $x \in \text{ind}(A_{p_i})$ in $U_{A_{p'_i}, O}$ to $x$ and $x'$ in $I$, and then adding $(d, e)$ to $r^T$ whenever $(d, e) \in s^T$ and $O \models s \subseteq r$. The resulting interpretation $I$ is a model of $A_{p'_i}$ and $O$. In particular, $I$ satisfies all functionality assertions in $O$, because every element in $\Delta^T$ has the same number of $r$-successors and $r$-predecessors as its corresponding original element in $U_{A_{p'_i}, O}$.

Since minimize preserves $O$-saturatedness and satisfiability w.r.t. $O$, $p_{i+1}$ is therefore also $O$-saturated and satisfiable w.r.t. $O$. □

Point 2 of Lemma 15 and Point 2 of Lemma 13 imply that treeify terminates and thus eliminates all cycles in $q$ while maintaining $q \subseteq O$. The next lemma makes this precise.

**Lemma 16.** Let $q$ be a CQ that is $O$-saturated, satisfiable w.r.t. $O$, and satisfies $q \subseteq O$. Then $q' = \text{treeify}(q)$ is an ELIQ. Moreover, $\text{treeify}(q)$ runs in time polynomial in $|\text{var}(q_T)| + |q| + ||O||$.

**Proof.** Let $p_1, p_2, \ldots$ be the sequence of constructed queries. Recall that for all $i \geq 1$, $p_i$ is the result of applying minimize. Thus, by Point 2 of Lemma 13, $|\text{var}(p_i)| \leq |\text{var}(q_T)|$ for all $i \geq 1$. But by Point 2 of Lemma 15, $p_{i+1}$ has more variables than $p_i$, for every $i$. It follows that the length $n$ of the sequence of queries is at most $|\text{var}(q_T)|$ and thus treeify stops at $p_n = \text{treeify}(q)$. It also follows that $p_n$ does not contain
Lemma 15. For all variables reachable from \( q \), let \( \mathcal{H} \) be the restriction of \( \pi \) to \( \mathcal{H} \). We can minimize \( \pi' \) makes at most \( |\mathcal{H}| \) membership queries. It suffices to note that \( |\pi'| \leq 2 \cdot |\mathcal{H}| \) and that \( |\mathcal{H}| \leq |\mathcal{O}| \cdot ||\mathcal{H}|\) since \( \mathcal{H} \) is a homomorphism from \( q \) to \( \mathcal{O} \). 

We now analyze the main part of the algorithm. To this end, let \( q_0(x_0), q_2(x_0), \ldots \) be the sequences of hypotheses \( q_0 \) that the algorithm constructs. The following lemma summarizes their most important properties.

Lemma 17. For all \( i \geq 1 \),

1. \( q_i \subseteq \mathcal{O} \) \( q\)-free;
2. \( q_i \subseteq \mathcal{O} \) and \( q_{i+1} \not\subseteq \mathcal{O} \);
3. \( \operatorname{var}(q_i) \subseteq \operatorname{img}(h) \) for every homomorphism \( h \) from \( q_{i+1} \) to \( \mathcal{O} \) with \( h(x) = x \).

Proof. Point 1 is proved by induction on \( i \). For \( i = 1 \), \( q_1 = \operatorname{treeify}(q_0) \) and \( q_0 \subseteq \mathcal{O} \) imply \( q_i \subseteq \mathcal{O} \) by Point 3 of Lemma 15. For \( i > 1 \), recall that \( q_i = \operatorname{minimize}(q_{i-1}) \) for some \( q_{i-1} \subseteq \mathcal{O} \). Thus, \( q_i \subseteq \mathcal{O} \) follows by Point 1 of Lemma 13.

Point 2 follows from the fact that \( q_{i+1} \subseteq \mathcal{O} \) by \( q\)-free for \( q_{i-1} \) some element of a frontier of \( q_i \) w.r.t. \( \mathcal{O} \). By definition of frontiers and Point 1 of Lemma 13 it follows that \( q_i \subseteq \mathcal{O} \) and \( q_{i+1} \not\subseteq \mathcal{O} \), as required.

For Point 3, let \( h \) be a homomorphism from \( q_{i+1} \) to \( \mathcal{O} \) with \( h(x_0) = x_0 \). Assume to the contrary that \( y \in \var(x) \) is to be shown that some \( y \) is \( \var(x) \) in \( q_i \). Choose some \( y \in \var(x) \) such that the distance from \( x_0 \) to \( x \) is strictly smaller than that from \( x_0 \) to \( y \) and \( q_i \) be the restriction of \( q_i \setminus \{r(x,y)\} \) to \( \var(x) \). We can argue as in the proof of Lemma 13 that \( h \) is a homomorphism from \( q_i \) to \( \mathcal{O} \). By Point 10, we can extend \( h \) to be a homomorphism from \( q_i \) to \( \mathcal{O} \). By Point 1, there is a homomorphism \( g \) from \( q_i \) to \( \mathcal{O} \) with \( g(x_0) = x_0 \). Composing \( g \) and \( h \) yields a homomorphism \( g' \) from \( q_i \) to \( \mathcal{O} \) with \( g'(x_0) = x_0 \). Thus, \( \mathcal{A} \), \( \mathcal{O} \) satisfies \( ||\mathcal{O}|| \) times for each role atom in \( q_i \). Since \( q_i \) is a tree, \( \var(q_i) \subseteq \var(q_i) \), and \( q_i \) may only use symbols from \( \mathcal{O} \), the number of such atoms is bounded by \( ||\mathcal{O}|| \).

Theorem 7. \( A\mathcal{Q}^3 \)’s are not learnable under disjunctions on ontologies using only polynomially many membership queries.

Proof. To prove the theorem, we use a proof strategy that is inspired by basic lower bound proofs for abstract learning problems due to Angluin [Angluin, 1987b]. Essentially the same proof is given in [Funk et al., 2021] for a slightly different class of ontologies that allows only concept inclusions between arbitrary conjunctions of concept names.

Here, it is convenient to view the oracle as an adversary who maintains a set \( S \) of candidate target queries that the learner cannot distinguish based on the queries made so far. We have to choose \( S \) and the ontology carefully so that each membership query removes only few candidate targets from \( S \) and that after a polynomial number of queries there is still more than one candidate that the learner cannot distinguish.

For each \( n \geq 1 \), let \( \mathcal{O}_n = \{A_i \mid A_i \subseteq \mathcal{O} \} \) and \( S_n = \{q(x) \mid q(x) \subseteq \mathcal{O}_n \} \) for all \( i \) with \( 1 \leq i \leq n \). Note that \( S_n \) is a frontier of \( \mathcal{O}_n \), if only \( A\mathcal{Q}^3 \) queries using the concept names \( A_i \) and \( A_i' \) for all \( 1 \leq i \leq n \), are
considered for Condition 3. Clearly, $S_n$ contains $2^n$ queries.\footnote{In fact, it can be shown similar as in the proof of Theorem 2 that $S_n$ is contained in any frontier of $\bot$ w.r.t. $\mathcal{O}_n$. Hence, $\bot$ does not have polynomially sized frontiers w.r.t. disjointness ontologies.}

Assume to the contrary of what is to be shown that AQIs queries are learnable under disjointness ontologies using only polynomially many membership queries. Then there exists a learning algorithm and polynomial $p$ such that the number of membership queries needed to identify a target query $q_T$ is bounded by $p(n_1, n_2)$, where $n_1$ is the size of $q_T$ and $n_2$ is the size of the ontology. We choose $n$ such that $2^n > p(n_1, n_2)$, where $n_1$ is a polynomial such that every query $q \in S_n$ satisfies $|q| = r_2(m)$ and $n_2$ is a polynomial such that $r_2(m) > |\mathcal{O}_m|$ for every $m \geq 1$.

Now, consider a membership query posed by the learning algorithm with ABox and answer individual $(\mathcal{A}, a)$. The oracle responds as follows:

1. if $\mathcal{A}, \mathcal{O}_n \models q(a)$ for no $q \in S_n$, then answer no;
2. if $\mathcal{A}, \mathcal{O}_n \models q(a)$ for a single $q \in S_n$, then answer no and remove $q$ from $S_n$;
3. if $\mathcal{A}, \mathcal{O}_n \models q(a)$ for more than one $q \in S_n$, then answer yes.

Note that the third response is consistent since $\mathcal{A}$ must then contain $A_1(a)$ and $A'_i(a)$ for some $i$ and thus $\mathcal{A}$ is not satisfiable w.r.t. $\mathcal{O}_n$. Moreover, the answers are always correct with respect to the updated set $S_n$. Thus, the learner cannot distinguish the remaining candidate queries by answers to queries posed so far.

It follows that the learning algorithm removes at most $p(n_1(n), n_2(n))$ many queries from $S_n$. By the choice of $n$, at least two candidate concepts remain in $S_n$ after the algorithm is finished. Thus, the learner cannot distinguish between them and we have derived a contradiction.

**Theorem 8.** ELIQs are not learnable under DL-Lite$^X$ ontologies using only membership queries.

**Proof.** We use the same ontology $\mathcal{O}$ as in the proof of Theorem 3, that is,

$$\mathcal{O} = \{ A \sqsubseteq \exists r, \ }\exists r^- \sqsubseteq \exists r, \ }\exists r \sqsubseteq \exists s, \ \func(r^-) \}.$$ 

To show that ELIQs are not learnable under $\mathcal{O}$ using only membership queries we use an infinite set $\mathcal{H}$ of hypotheses (i.e., candidate target queries) that cannot be distinguished by a finite number of membership queries. Let

$$\mathcal{H} = \{ q^* \} \cup \{ q_n \ | \ n \text{ prime} \},$$

where $q^*(x_1) = \{ A(x_0), r(x_0, x_1), A(x_1) \}$ and each $q_n$ is defined as follows:

$$q_n(x_1) = \{ A(x_0), r(x_0, x_1), r(x_1, x_2), \ldots, r(x_{n-1}, x_n), s(x_n, y), s(x'_n, y), r(x'_1, x'_2), \ldots, r(x'_{n-1}, x'_n), A(x'_1) \}.$$ 

It is important to note that $q^* \subseteq_{\mathcal{O}} q_n$ for all $n \geq 1$. Intuitively, this makes it impossible for the learner to distinguish between $q^*$ being the target query and one of the $q_n$ being the target query. If, for example, the learner asks a membership query $\langle A^*, \mathcal{O} \models q_T(a) \rangle$, where $A^*$ is $q^*$ viewed as an ABox, then the oracle will answer ‘yes’ and the learner has not gained any additional information.

Now, the strategy of the oracle to answer a membership query $\langle A, \mathcal{O} \models q_T(a) \rangle$ is as follows:

1. if $\mathcal{A}, \mathcal{O} \models q^*(a)$, then reply “yes”;
2. otherwise, reply “no” and remove from $\mathcal{H}$ any $q$ that satisfies $\mathcal{A}, \mathcal{O} \models q(a)$.

An important aspect of this strategy is that, as proved below, only finitely many hypotheses $q$ are removed whenever Case 2 above applies. Consequently, after any number of membership queries, the set of remaining hypotheses $\mathcal{H}$ is infinite and contains $q^*$. The learner can then, however, not distinguish between $q^*$ and the remaining hypotheses and thus not identify the target query. In particular, the presence of $q^* \in \mathcal{H}$ prevents the learner from simply going through all $q_i \in \mathcal{H}$, asking membership queries with ABoxes that take the form of these queries, and identifying $q_i$ as the target query when the membership query for $q_i$ succeeds.

**Claim.** Let $\mathcal{A}$ an ABox and $a \in \text{ind}(\mathcal{A})$. If $\mathcal{A}, \mathcal{O} \not\models q^*(a)$, then $\mathcal{A}, \mathcal{O} \models q_n(a)$ for only finitely many primes $n$.

**Proof of the claim.** Let $\mathcal{A}$ an ABox and $a \in \text{ind}(\mathcal{A})$ such that $\mathcal{A}, \mathcal{O} \not\models q^*(a)$. Then $\mathcal{A}$ satisfies $\func(r^-)$. Suppose to the contrary of what we have to show that there are infinitely many primes $n$ such that $\mathcal{A}, \mathcal{O} \models q_n(a)$ and let $h_n$ be the witnessing homomorphisms from $q_n$ to $\mathcal{U}_{\mathcal{A}, \mathcal{O}}$ with $h(x_1) = a$.

- If $h_n(x_n) \notin \text{ind}(\mathcal{A})$ for some prime $n$ with $\mathcal{A}, \mathcal{O} \models q_n(a)$, then $h_n(x'_n) = h_n(x_n)$ due to the tree structure of the non-ABox part of $\mathcal{U}_{\mathcal{A}, \mathcal{O}}$. Since $r^-$ is functional, it follows that $h_n(x_j) = h_n(x'_j)$ for all $j$ with $1 \leq j \leq n$. Since $A(x_1) \in q_n$, also $A(h_n(x_1)) = A(h_n(x_1)) = A(a) \in \mathcal{A}$. Since also $A(x_0), r(x_0, x_1) \in q_n$, we have $A(h_n(x_0)), r(h_n(x_0), h_n(x_1)) \in \mathcal{U}_{\mathcal{A}, \mathcal{O}}$, and thus $h_n$ is a homomorphism from $q^*$ to $\mathcal{U}_{\mathcal{A}, \mathcal{O}}$ with $h(x_1) = a$. Hence, $\mathcal{A}, \mathcal{O} \models q^*(a)$, a contradiction.

- Otherwise, $h_n(x_n) \in \text{ind}(\mathcal{A})$ for all primes $n$ with $\mathcal{A}, \mathcal{O} \models q_n(a)$. Since $\mathcal{A}$ is finite, there is an element $b \in \text{ind}(\mathcal{A})$ such that $h_m(x_m) = b$ for infinitely many primes $m$. Thus, there is an $r$-path of length $m$ from $a$ to $b$ in $\mathcal{A}$ for infinitely many primes $m$. Since $\mathcal{A}$ is finite and satisfies $\func(r^-)$, this is only possible if $a = b, r(a, a) \in \mathcal{A}$, and $h_m(x_j) = a$ for all considered $m$ and all $j$ with $1 \leq j \leq m$. Since also $A(x_0), r(x_0, x_1) \in q_n$ and $\mathcal{A}$ satisfies $\func(r^-)$, we further have $h_n(x_0) = a$ for all primes $n$ with $\mathcal{A}, \mathcal{O} \models q_n(a)$ and $A(a) \in \mathcal{A}$. But then $\mathcal{A}, \mathcal{O} \models q^*(a)$, a contradiction.