Direction problems in affine spaces

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Abstract

This paper is a survey paper on old and recent results on direction problems in finite dimensional affine spaces over a finite field.

1 Introduction

Let $p$ be prime and $q = p^h$, $h \geq 1$. Then $\text{GF}(q)$ denotes the finite field of order $q$. A vector space of dimension $n$ over $\text{GF}(q)$ will be denoted as $V(n,q)$. The $n$-dimensional affine space over the field $\text{GF}(q)$, denoted as $\text{AG}(n, \text{GF}(q))$, shortly, $\text{AG}(n,q)$, is an incidence geometry of which the elements are the additive cosets of the vector subspaces of $V(n,q)$. The incidence is symmetrised set-theoretic containment. As such, a point of $\text{AG}(n,q)$ is represented by a unique vector of $V(n,q)$, a line is represented by a unique coset of a 1-dimensional vector subspace, etc. If $S$ is an element of the affine space, so represented by a coset of a vector subspace $\pi$, then the (affine) dimension of $S$ is defined as the vector dimension of $\pi$. Hence, the elements of $\text{AG}(n,q)$ have dimension $i$, $0 \leq i \leq n - 1$, and we call the elements of $\text{AG}(n,q)$ also affine subspaces.

The $n$-dimensional projective space over the field $\text{GF}(q)$, denoted as $\text{PG}(n, \text{GF}(q))$, shortly, $\text{PG}(n,q)$, is an incidence geometry of which the elements are the vector subspaces of the $(n+1)$-dimensional vector space $V(n+1,q)$. Incidence is symmetrised containment. As such, a point of $\text{PG}(n,q)$ is represented by a unique vector line, and hence up to a $\text{GF}(q)$ scalar, by a unique non-zero vector of $V(n+1,q)$. The zero vector represents the projective empty subspace. We call the elements of $\text{PG}(n,q)$ projective subspaces, and the projective dimension of a projective subspace $S$ is one less than its vector dimension.

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Consider now an affine space $\text{AG}(n, q)$. It is well known that in $\text{AG}(n, q)$ parallelism exists: given a point $P$ and an $i$-dimensional subspace $S$ not incident with $P$, then there exists a unique $i$-dimensional subspace $S'$ containing $P$ and not meeting $S$. We call $S$ and $S'$ parallel. It simply means that $S$ and $S'$ are cosets of the same $i$-dimensional vector subspace $\pi$ of $V(n, q)$. The set of directions of all affine spaces constitutes in fact a projective space $\text{PG}(n-1, q)$. The projective points of such a direction are simply called the points at infinity of the affine subspace $S$. Clearly, a line of $\text{AG}(n, q)$ has a point at infinity. We will often denote the projective space at infinity as $\pi_\infty$.

Let $U$ be a point set in $\text{AG}(n, q)$. We call a point $t \in \pi_\infty$ a direction determined by $U$ if there exist two different points $r, s \in U$ such that the affine line determined by $r$ and $s$ contains at infinity the point $t$. It is clear that a point set $U$ will determine a set of points at infinity, we denote the set of directions determined by $U$ as the set $U_D$. In the literature, when dealing with point sets in $\text{AG}(2, q)$, sometimes a determined direction is also called a determined slope, since a determined direction is the point at infinity of a line, and this point at infinity can indeed be represented by the slope of the line, and all points at infinity can be represented by all possible slopes (i.e. the elements of $\text{GF}(q) \cup \{\infty\}$). The following research questions have been addressed.

1. What are the possible sizes of $U_D$ given that $|U| = q^{n-1}$? What is the possible structure of $U_D$?

2. What are the possible sets $U$, $|U| = q^{n-1}$, given that $U_D$ (or its complement in $\pi_\infty$) or only $|U_D|$ is known?

3. Given that a set $N$ of directions is not determined by a set $U$, $|U| = q^{n-1} - \epsilon$, can $U$ be extended to a set $U'$, $|U'| = q^{n-1}$, such that $U'$ does not determine the given set $N$?

Directions problems have been studied in the past and also in affine spaces over arbitrary (non-finite) fields. A notorious example is [9]. A derived problem over finite fields has been addressed in [23] and [16]. We will focus on direction problems in affine spaces over finite fields. The earliest reference is probably the book of L. Rédei ([34]). This brings us seamlessly to the most used technique to study direction problems: the so-called polynomial method. However, this paper is mainly meant to be a survey paper on old and recent results on (and some applications of) direction problems in affine spaces. A recent and detailed survey paper on the polynomial method is e.g. [4].
2 Results in the affine plane \( \text{AG}(2, q) \)

Probably the first result to be mentioned is the following theorem. It addresses research question 1, and it appeared in [34], where it is stated in terms of the set of difference quotients of a function on a finite field, and it is obtained as a (non-trivial) consequence of the theory of so-called lacunary polynomials over finite fields.

**Theorem 2.1** Let \( U \) be a subset of the affine plane \( \text{AG}(2, p) \), \( p \) prime, such that \( |U| = p \) and \( U \) is not the set of points of a line. Then \( U \) determines at least \( \frac{p+3}{2} \) directions.

An alternative proof, including a characterization of the equality case, can be found in [31]. This characterization can be formulated as follows. Note that the following theorem is addressing research question 2.

**Theorem 2.2 ([31])** For every prime \( p > 2 \), up to affine transformation there is a unique set of \( p \) points in \( \text{AG}(2, p) \) determining \( \frac{p+3}{2} \) directions. Up to affine transformation, the point set is the graph of the function \( f(x) : \text{GF}(p) \to \text{GF}(p) : x \mapsto x^{p+1} \).

In some papers, Theorem 2.1 is referred to as a theorem of Rédei and Megyesi. However, László Megyesi and László Rédei have no joint paper. In the first version of the manuscript of [34], Rédei had only the bound \( \frac{p+1}{2} \), which was improved to \( \frac{p+3}{2} \) after a discussion with Megyesi. Since the arguments for the improvement were in fact already present in [34], Theorem 2.1 appears in the final version of the book, and no follow up paper by Rédei and Megyesi was written.\footnote{I am grateful to Tamás Szőnyi for this clarification}

The work of Rédei contains examples of point sets of which the number of determined direction can (easily) be computed. These examples are also attributed to Megyesi. For an integer \( d \mid q-1, 1 < d < q-1 \), let \( G = \text{GF}(q)^* \) be the multiplicative subgroup of \( \text{GF}(q) \) of order \( d \). Set

\[
U = \{(x, 0) : x \in G\} \cup \{(0, x) : x \notin G\},
\]

then \( U \) determines \( q + 1 - d \) directions. For \( q \) prime, this is the unique example characterized in Theorem 2.2.

The book of Rédei [34] is a treatise on so-called lacunary polynomials over finite fields. Its main objective is to characterize polynomials in one variable over a finite field under certain assumptions on its degree and on the absence
of terms of high degree. The direction result is obtained as a consequence of the developed theory.

It seems that the first improvement on Rédei’s original theorem only appears in 1995 in a paper by Blokhuis, Brouwer and Szőnyi, [15]. In 1999 these results are again improved by Blokhuis, Ball, Brouwer, Storme and Szőnyi, [14]. In 2003, an unresolved case in the main theorem of [14] is closed by Ball in [1]. We state here the main theorem of [1]. The interested reader can compare the different versions of this theorem throughout the papers [15, 14, 1]. Note that the following theorem is stated in terms of functions on GF(q). Clearly, a point set $U$ of points of AG(2, q), not determining all directions, corresponds, up to affine transformation, always to the graph of a function $f : \text{GF}(q) \to \text{GF}(q)$:

$$U_f = \{(x, f(x)) : x \in \text{GF}(q)\}.$$  

Also the converse is true: the graph of a function will not determine all directions.

**Theorem 2.3 ([1], Theorem 1.1)** Let $f$ be a function from $\text{GF}(q)$ to $\text{GF}(q)$, $q = p^h$ for some prime $p$, and let $N$ be the number of directions determined by $f$. Let $s = p^e$ be maximal such that any line with direction determined by $f$ that is incident with a point of the graph of $f$ is incident with a multiple of $s$ points of the graph of $f$. One of the following holds:

(i) $s = 1$ and $\frac{q^3 + 3}{2} \leq N \leq q + 1$;
(ii) $\text{GF}(s)$ is a subfield of $\text{GF}(q)$ and $\frac{q^2 + 1}{s} + 1 \leq N \leq \frac{q - 1}{s - 1}$;
(iii) $s = q$ and $N = 1$.

Moreover, if $s > 2$ then the graph of $f$ is GF(s)-linear, and all possibilities for $N$ can be determined explicitly (in principle).

The results in [15, 14, 1] are based on further elaboration of the techniques used in [34]. Essentially, the so-called Rédei-polynomial is associated to the affine point set. Its algebraic properties are derived form the geometric conditions and vice versa. These papers, together with other papers on blocking sets, which will be mentioned further in this survey, can be considered as the founding papers of the so-called polynomial method in finite geometry.

The original theorem of Rédei (Theorem 2.1) and its improvements in [31, 15, 14, 1] were further elaborated by Gács and Ball. The following theorem is due to Gács, and appeared in [28], which is the continuation of work started in [26, 27]. Note that [28] was submitted right after publication of [26, 27]. We give the original statement of the theorem.
Theorem 2.4 ([28], Theorem 1.3) For every prime $p$, besides lines and the example characterized by Lovász and Schrijver (see Theorem 2.3), any set of $p$ points in $\text{AG}(2, p)$ determines at least $\left\lceil \frac{2^{p-1}}{3} \right\rceil + 1$ directions.

Recall the examples of Megyesi. When $3 \mid q - 1$, the set $U$ determines exactly $2^{p-1}/3 + 2$ directions. Hence Theorem 2.4 is almost sharp.

In [40], the theorem of Rédei is reviewed, with the focus on applications. A classification of the example can also be found there. Based on this result, there is a slightly different proof of Theorem 2.2 in [29], and a generalization as follows.

Theorem 2.5 ([29], Theorem 1.3) Let $U$ be a subset of the affine plane $\text{AG}(2, q)$, $q = p^2$, $p$ prime, such that $|U_D| \geq \frac{q+3}{2}$ directions. Then either $U$ is, up to affine transformation, equivalent to the graph of the function $f(x) : \text{GF}(p) \to \text{GF}(p) : x \mapsto x^{q+1}/2$, or $U$ determines at least $q + p^2 + 1$ directions.

In [33], an example attaining the bound of Theorem 2.5 is reached, which makes this bound sharp. Note that the bound Theorem 2.5 is weaker but similar to the bound in Theorem 2.4.

Up to now, all results except for Theorem 2.2 and Theorem 2.3 (ii) give only information on the number of determined directions. A lot of attention has also been paid on characterization results of examples. There is however no sharp line between such characterization examples and examples only giving information on the number of determined directions. In many papers, both go together. We first mention some older results that provide characterizations.

Assume that $p$ is prime and that $f$ is any function from $\text{GF}(p) \to \text{GF}(p)$. Let $M(f)$ be the number of elements $c \in \text{GF}(p)$ such that $x \mapsto f(x) + cx$ is a permutation of $\text{GF}(p)$, which is equivalent with saying the $c$ is a non-determined direction of the graph of $f$.

It should be noted that permutation polynomials have been studied for their own interest. Let $f(x)$ be a permutation polynomial over $\text{GF}(q)$. The question for permutation polynomials $f(x)$ over $\text{GF}(q)$ the polynomial $f(x) + cx$ is a permutation polynomial for many values $c \in \text{GF}(q)$ is studied in [24].

A result of Szőnyi characterizes point sets contained in the union of two lines under certain assumptions on the determined directions. The following result of Szőnyi is a kind of generalization of Theorem 2.2.

Theorem 2.6 ([38]) If $M(f) \geq 2$, and the graph of $f$ is contained in the union of two lines, then after affine transformation, the graph of $f$ is equivalent to the example of Megyesi.
One of the most recent, if not the most recent paper on the direct ion problem in the plane is [5]. To state the results, we have to introduce the notation $I(f)$, which was also used in [28].

Consider again a function $f : \text{GF}(p) \to \text{GF}(p)$. By interpolation, any function determines a polynomial of degree at most $p - 1$ over $\text{GF}(p)$, and conversely, every such polynomial determines a function. A function $f$ corresponds with a polynomial $g(X) \in \text{GF}(p)[X]$ of a certain degree. Clearly, the function $f(X)^i$ for any $i$, is represented by $g(X)^i \in \text{GF}(p)[X]$. But since $x^p - x = 0$ for all $x \in \text{GF}(p)$, we may reduce $g(X)^i$ modulo $X^p - X$ to obtain a polynomial representation of $f^i(X)$. We call the degree of $g(X)^i$ modulo $X^p - X$ the degree of $f^i(X)$. Then $I(f)$ is defined as follows:

$$I(f) = \min \left\{ i + j : \sum_{x \in \text{GF}(p)} x^j f(X)^i \neq 0 \right\}.$$ 

In [5] it is explained why for all $n \leq I(f)$ implies that $f(X)^i$ has degree at most $p - 2 - n + i$. The following results are then found.

**Theorem 2.7 ([5], Theorem 2.4)** If $I(f) > \frac{p-1-2\epsilon}{t} + t - 2 + \epsilon$ for some integer $t$ then every line meets the graph of $f$ in at least $I(f) + 3 - t > \frac{p-1}{t} + 1$ points or at most $t - 1$ points.

The authors state the following conjecture, based on the proof of the previous theorem.

**Conjecture 2.8** If $I(f) > \frac{p-1-2\epsilon}{t} + t - 2 + \epsilon$ for some integer $t$, then the graph of $f$ is contained in an algebraic curve of degree $t - 1$.

The next theorem is actually the proof of this conjecture under extra assumptions.

**Theorem 2.9 ([5], Theorem 2.6)** If $I(f) > \frac{p-1-2\epsilon}{t} + t - 2 + \epsilon$ and there are $t - 1$ lines incident with at least $t$ points of the graph of $f$ then the graph of $f$ is contained in the union of these $t - 1$ lines.

Putting $t = 2$ in Theorem 2.7 yields the following corollary, which is a reformulation of Theorem 2.2

**Corollary 2.10 ([5], Theorem 3.1)** If $I(f) \geq \frac{p+1}{2}$ then $f$ is linear.

The next theorem is a generalization of a theorem in [28].
Theorem 2.11 ([5], Theorem 3.2) If \( I(f) \geq \frac{p+5}{3} \) then the graph of \( f \) is contained in an algebraic curve of degree 2.

In [11], information on \( M(f) \) in terms of the degree of \( f \) is obtained.

Up to now, all results related to research questions 1 and/or 2. The following result is a stability result dealing which addresses research question 3.

Theorem 2.12 ([39], Theorem 4) A set \( U \) of \( q-k > q-\frac{\sqrt{q}}{2} \) points of \( \text{AG}(2,q) \) for which \( |U_D| \leq \frac{q+1}{2} \), can be extended to a set \( U' \) of \( q \) points of \( \text{AG}(2,q) \) such that \( U_D = U'_D \).

The case \( q = p \) was handled separately in [40], using lacunary polynomials. The proof of Theorem 2.12 is much more dependent on algebraic geometric arguments. Several remarks with refinements and consequences under particular assumptions are found in [39].

(i) For \( q \) prime, the bound \( q-k > q-\frac{\sqrt{q}}{2} \) can be improved to \( q-k > q-\frac{p+45}{20} \).

(ii) The case \( k = 1 \) has a very short proof. Since \( n = 1 \), each \( y \not\in U_D \), there is a unique line \( L_y \) not meeting \( U \). From the short argument of the proof, it is deduced that all lines \( L_y, y \not\in U_D \), pass through a common point.

The following theorem is also found in [40].

Theorem 2.13 ([40]) Let \( U \) be a set of \( k \) points of \( \text{AG}(2,p), p \) prime, such that not all \( k \) points of \( U \) are collinear. Then \( |U_D| \geq \frac{k+3}{2} \).

Consider the affine plane \( \text{AG}(2,p) \), and consider a coset of a multiplicative subgroup \( H \leq \text{GF}(p)^* \). Set

\[
U = \{(x,0) : x \in H\} \cup \{(0,x) : x \in H\} \cup \{(0,0)\},
\]

then \( |U| = 2|H| + 1 \) and \( |U_D| = |H| + 2 = \frac{k+3}{2} \), hence, when \( k = 2d+1 \) with \( d \mid p - 1 \), the bound in the theorem is sharp. This “Megyesi-type” example is due to Aart Blokhuis.

The most recent result on planar direction problems is found in [25]. This paper actually addresses a variation on research question 1 in the plane. The authors consider a set \( U \) in \( \text{AG}(2,q) \) of less than \( q \) points, and derive a result similar to the results in [14] (which are part of Theorem 2.13).

Let \( q = p^h, p \) prime. Let \( U \) be a set of points of \( \text{AG}(2,q) \). Let \( d \) be a direction at infinity, then define \( s(d) \) as the greatest power of \( p \) such that each line \( l \) of direction \( d \) meets \( U \) in zero modulo \( s(d) \) points, and define

\[
s = \min\{s(y) : y \in U_D\}.
\]
Let \( U = \{(a_i, b_i) : 1 \leq i \leq |U|\} \). The Rédei-polynomial associated to the set \( U \) is defined as

\[
R(X, Y) = \prod_{i=1}^{|U|} (X - a_i Y + b_i) = X^n + \sum_{j=0}^{n-1} \sigma_{n-j}(Y)X^j
\]

The following proposition is in principle straightforward (as [25] is self contained, a proof can be found there). We describe it to introduce one more notion that is needed to formulate the main result from [25].

**Lemma 2.14** If \( y \in U_D \), then \( R(X, y) \in \text{GF}(q)[X] \setminus \text{GF}(q)[X^{p \cdot s(y)}] \). If \( y \not\in U_D \), then \( R(X, y) \mid X^q - X \).

The above observation leads to the definition of a polynomial \( H(X, Y) \) as follows. Consider \( R(X, Y) \) as a univariate polynomial over the ring \( \text{GF}(q)[Y] \). Since \( R(X,Y) \) is monic, division with remainder of \( X^q - X \) by \( R(X, Y) \) yields the quotient \( Q(X,Y) \) and remainder \( S(X,Y) \), define \( H(X,Y) := -S(X,Y) - X \). Properties of the polynomial \( H(X,Y) \) are then shown in [25], e.g. that \( H(X,Y) \) is a constant polynomial if \( |U_D| = 1 \). Suppose that \( |U_D| > 1 \), then define \( t(d) \) as the maximal power of \( p \) such that \( H(X, d) = f_d(X)^{t(d)} \) for some \( f_d(X) \not\in \text{GF}(q)[X] \), and define

\[
t = \min\{s(y) : y \in U_D\}.
\]

The main theorem can now be formulated.

**Theorem 2.15 ([25], Theorem 17)** Let \( U \) be an arbitrary set of points of \( \text{AG}(2, q) \). Assume that \( \infty \in U_D \), then one of the following holds.

(i) \( 1 \leq s \leq t < q \) and \( \frac{|U|-1}{t+1} + 2 \leq |U_D| \leq q + 1 \), or,

(ii) \( 1 < s \leq t < q \) and \( \frac{|U|-1}{t+1} + 2 \leq |U_D| \leq \frac{|U|-1}{s-1} \leq q + 1 \), or,

(iii) \( 1 \leq s \leq t = q \) and \( U_D = \{\infty\} \).

3 \hspace{1em} Planar direction problems, blocking sets and the polynomial method

A **blocking set** of a projective plane \( \Pi \) is set of points \( B \) such that any line of \( \Pi \) meets \( B \) in at least one point. We call a blocking set **trivial** if it contains a line, and minimal if no point of \( B \) can be deleted. The study of blocking sets of Desarguesian projective planes in particular is important in finite geometry, and results on blocking sets of the Desarguesian projective plane
PG(2, q) have many applications in the study of other substructures in finite projective spaces and finite classical polar spaces. We shortly describe in this section the connection between blocking sets, direction problems and the polynomial method.

Consider now a point set $U$ of size $q$ in the affine plane AG(2, q). The extension of AG(2, q) to the projective plane, by adding the slopes at infinity as points is well known. If $U$ is the set of points of a line, then $U$ determines one directions $d U \cup \{d\}$ is then a projective line, indeed meeting all lines of PG(2, q). So assume that $U$ is not contained in an affine line, then the set $B := U \cup U_D$ is a minimal blocking set of PG(2, q). Consider any line $l$ of PG(2, q), not the line at infinity. If $l$ has a slope $d \in U_D$, then $l$ meets $U$ in at least two points. Furthermore, since $|U| = q$, there exists at least one line $m$ on $d$ not meeting $U$. Hence $d$ cannot be removed from $B$. If $l$ has slope $d \notin U_D$, then every line on $d$ meets $U$ in exactly one point. Since $|U| = q$ and there are $q + 1$ lines on a point, on every point $P \in U$ there are at least two lines on $P$ meeting $U$ only in $P$. Finally, the line at infinity meets $U_D$. Hence, $B$ is a minimal blocking set of size $q + n$, where $n := |U_D|$, and there exists at least one line meeting $B$ in exactly $n$ points. We conclude that any point set $U$ of size $q$ gives rise to such a blocking set, which is called a blocking set of Rédei type.

The converse is not true: not every minimal blocking set is of Rédei type, and so is not constructed as $U \cup U_D$. However, the idea of Rédei blocking sets, the associated direction problem, and the results of Rédei have been inspiring to study small blocking sets. We first mention the following result of Bruen.

**Theorem 3.1 ([18])** Let $B$ be a blocking set of a finite projective plane $\Pi$ of order $n$. Then $|B| \geq n + \sqrt{n} + 1$, and $|B| = n + \sqrt{n} + 1$ if and only if $B$ is a Baer sub plane of $\Pi$.

An alternative proof, based on elementary counting techniques simplifies the proof Theorem 3.1. It can be found in [19].

Considering the examples of Megyesi, and Theorem 2.2 it is clear that for $p$ odd prime a minimal blocking set of size $p + \frac{p+3}{2} = \frac{3(p+1)}{2}$ exists. It was only shown by Blokhuis in 1994 that no blocking sets of size smaller than $\frac{3(p+1)}{2}$ exists.

**Theorem 3.2 ([12])** Assume that $q = p$ is an odd prime. If $B$ is a blocking set of PG(2, q) and $|B| \leq \frac{3p+1}{2}$, then $B$ contains all the points of a line.

The proof of this theorem in [12] is based on a generalization of a result of Rédei on lacunary polynomials. The result was generalized to planes of prime power order in the following theorem.
Theorem 3.3 ([13], Theorem 6) Assume that \( q = p^{2e+1}, p \) and odd prime, \( e \geq 1 \). Then a minimal blocking set of \( \text{PG}(2, q) \) has size at least \( q + \sqrt{pq} + 1 \) points.

The following examples of blocking sets are found in [34]. Let \( q = p^e, e > 1, \) and let \( \text{GF}(q) \) be a subfield of \( \text{GF}(q) \). Using \( f = T, \) the trace function from \( \text{GF}(q) \) \( \to \) \( \text{GF}(q_1) \), the graph of the function \( f \) determines \( \frac{q}{q_1} + 1 \) directions, so this construction yields a blocking set of size \( q + \frac{q}{q_1} + 1 \). Hence, for \( q = p^3, \) the bound of Theorem 3.3 is sharp.

More information on blocking sets of Desarguesian projective planes can be found in [17]. The situation for non-Desarguesian planes is more complicated, especially the construction of examples. Apart from [18] and [19], the papers [10, 11, 30] are interesting references for blocking sets of non-Desarguesian planes.

4 Direction problems in affine spaces

An early result on directions problems in affine spaces is found in [35]. It is motivated by the study of Rédei type blocking sets in projective spaces. The main theorem is the following. Its proof uses almost exclusively geometric and combinatorial arguments.

Theorem 4.1 ([35], Theorem 16) Let \( U \) be a set of points of \( \text{AG}(n, q) \), \( n \geq 3, q = p^h, |U| = q^k. \) Suppose that \( U \) is a \( \text{GF}(p) \)-linear set of points and that \( |U_D| < \frac{2p^3}{3} q^{k-1} + q^{k-2} + \cdots + q^2 + q. \) If \( (n - 1) \geq (n - k)h, \) then \( U \) is a cone with an \( (n - 1 - h(n - k)) \)-dimensional vertex at infinity and with a \( \text{GF}(p) \)-linear point set \( U_{(n-k)h} \) of size \( q^{(n-k)(h-1)}, \) contained in some affine \( (n - k)h \)-dimensional subspace of \( \text{AG}(n, q). \)

The following theorem is found in Sziklai, [36]. Note that Theorem 2.4 is used to prove it.

Theorem 4.2 ([36], Theorem 15) Let \( U \) be a set of \( p^2 \) points in \( \text{AG}(2, p), p \) prime, such that \( |U_D| < \frac{2p(p-1)}{3} + 2p. \) Then the set \( U \) is either a plane or a cylinder with the projective triangle as base if \( p > 11. \)

In Section 2 we have seen that many direction results in affine planes are obtained by studying the graph of a function in one variable over the finite field \( \text{GF}(q). \) The following result, which is probably (one of) the first results on direction problems in three dimensional affine spaces, generalizes this approach to functions in two variables, and can be found in [8]. It addresses research question 2.
Theorem 4.3 ([8], Theorem 2.4) Let \( q = p^h \), prime. If \( U \) is a set of \( q^2 \) points of \( AG(3, q) \) that does not determine at least \( p^e q \) directions for some \( e \in \mathbb{N} \cup \{0\} \), then every plane meets \( U \) in \( 0 \mod p^{e+1} \) points.

As in e.g. [15, 14, 1], a Rédei-polynomial is associated to the point set in \( AG(3, q) \) determined as the graph of a function \( f \) in two variables over \( GF(q) \). The Rédei-polynomial is now a polynomial in three variables, again lacunary, from which again strong algebraic properties can be derived. The use of Rédei-polynomials in more variables is further described in [7].

Theorem 4.3 is generalized and improved in [3].

Theorem 4.4 ([3], Theorem 1.3) Let \( q = p^h \) and \( 1 \leq p^e < q^{k-2} \), where \( e \) is a non-negative integer. If there are more than \( p^e (q - 1) \) directions not determined by a set \( U \) of \( q^{k-1} \) points in \( AG(k, q) \), then every hyperplane meets \( U \) in \( 0 \mod p^{e+1} \) points.

The generalization of Theorem 4.3 to general dimension is relatively straightforward from the arguments used to show the theorem. The improvement is based on using the representation \( GF(q) \times GF(q)^{k-1} \) for \( AG(k, q) \), and then associating a Rédei-polynomial to the set \( U \), which will, due to the used representation, be a polynomial in only two variables.

To construct a set of \( q^2 \) points in \( AG(3, q) \), that do not determine many directions, one could consider a set \( U \) of \( q \) points of \( AG(2, q) \), determining few directions, and then form a cylinder from \( U \). The easiest example is to take for \( U \) the set of \( q \) points on a line, then the corresponding set in \( AG(3, q) \) will be an affine plane. In [3], it is noted that this procedure is actually the only known way of constructing such sets in \( AG(3, q) \). Connecting this with Theorems 4.3 and 4.4, the following conjectures are explained in [3].

Conjecture 4.5 ([3], Conjecture 5.1) Let \( U \) be a set of \( p^2 \) points in \( AG(3, p) \), \( p \) prime, and let \( N \) be the set of non-determined directions. If \( |N| \geq p \), then \( U \) is the union of \( p \) parallel lines.

By Theorem 4.4 it follows that a set satisfying the conditions of the conjecture, meets every plane of \( AG(3, p) \) in \( 0 \mod p \) points. As such, the following conjecture is introduced in [3] as the strong cylinder conjecture.

Conjecture 4.6 ([3], Conjecture 5.2) Let \( U \) be a set of \( p^2 \) points in \( AG(3, p) \), \( p \) prime. If \( U \) has the property that every planes meets \( U \) in \( 0 \mod p \) points, then \( U \) is the union of \( p \) parallel lines.
As a final note, a generalization of this conjecture in terms of finite abelian groups is given in [3]. In [4], a weaker form of Conjecture 4.6 assuming on top that at least \( p \) directions are not determined by the set \( U \), is proposed as an open problem.

The theorems and corollary mentioned in Section 2 (Theorems 2.7, 2.9, 2.11 and Corollary 2.10) from [5] are purely planar results and continue on work started in [28]. The work in [6], contains some slight improvements of [28]. Its planar results were then improved further in [5] (and are therefore not explicitly mentioned in Section 2. In this section, we discuss that a generalization of the planar results to three dimensional affine spaces in [6].

Let \( U \) be a set of \( q \) points in \( \text{AG}(3, q) \). A line \( l \) at infinity is called *not determined by* \( U \) if every affine plane through \( l \) contains exactly one point of \( U \). A point set in \( \text{AG}(3, q) \) will now be studied as the graph of a pair of functions over the finite field \( \text{GF}(q) \). Then a line not determined by the graph \( \{(x, f(x), g(x)) : x \in \text{GF}(q)\} \), is corresponds with a pair \((c, d)\) such that \( f(x) + cg(x) + dx \) is a permutation polynomial.

Let \( p \) be prime and let \( f \) and \( g \) be two functions over \( \text{GF}(p) \). Define \( M(f, g) \) be the number of pairs \((a, b) \in \text{GF}(p)^2\) such that \( f(x) + ag(x) + bx \) is a permutation polynomial, and let

\[
I(f, g) = \min \left\{ k + l + m : \sum_{x \in \text{GF}(p)} x^k f(X)^l g(x)^m \neq 0 \right\}.
\]

The main result of the non-planar part of [6] is then the following theorem.

**Theorem 4.7 ([6], Theorem 3.2)** Let \( s = \lceil \frac{p - 1}{6} \rceil \). If \( M(f, g) > (2s + 1)(p + 2s)/2 \), then there are elements \( c, d, e \in \text{GF}(p) \) such that \( f(x) + cg(x) + dx + e = 0 \).

As a final remark, an example is provided in [6] that shows that the bound on \( M(f, g) \) is the right order of magnitude.

In [37], the notion of direction at infinity is generalized in a different way to arbitrary subspaces at infinity. Let \( U \) be a set of points of \( \text{AG}(n, q) \), and let \( k \leq n - 2 \) be a fixed integer. A projective subspace \( S \) of dimension \( k \) at infinity is determined by \( U \) if there is an affine subspace \( T \) of dimension \( k + 1 \) meeting the hyperplane at infinity in \( S \), is spanned by the point set \( U \cap T \). One observes easily that \( |U| \leq q^{n-1} \) if not all projective subspaces of dimension \( k \) at infinity are determined. The following theorem gives some information in the three-dimensional case.

**Theorem 4.8 ([37], Theorem 7)** Let \( U \) be a set of \( q^2 \) points of \( \text{AG}(3, q) \). Let \( U_L \) be the set of lines at infinity determined by \( U \), and let \( N \) be the set of non-determined lines at infinity. Then one of the following holds.
(i) The set $U$ determines all lines at infinity (so $|N| = 0$);
(ii) $|N| = 1$ and there is a parallel class of affine planes such that $U$ contains one (arbitrary) complete line in each of its planes;
(iii) $|N| = 2$ and the set $U$ together with two undetermined lines at infinity form a hyperbolic quadric or $U$ contains $q$ parallel lines;
(iv) $|N| \geq 3$ and then $U$ contains $q$ parallel lines.

Up to here, this section has been devoted only to results addressing research questions 1 and/or 2 (or variations on it). The following theorems deals with research question 3. It was first shown in [20] in three dimensions. The following formulation is taken from [4], where a proof for general $n$ is given using the representation $\text{GF}(q) \times \text{GF}(q)^{n-1}$ for $\text{AG}(n,q)$.

**Theorem 4.9** ([4], Theorem 6.8) Let $q = p^h$, $p$ and odd prime. A set $U$ of $q^{n-1} - 2$ points of $\text{AG}(n,q)$ that does not determine a set $D$ of at least $p + 2$ directions, can be extended to a set $U'$ of $q^{n-1}$ points, not determining the same set $D$ of directions.

The main motivation of the work in [21] was to study the problem of Theorem 4.9 in an alternative way.

**Theorem 4.10** ([21], Theorem 12) Let $q = p^h$, $p$ prime. Let $U$ be a set of $q^2 - \varepsilon$ points of $\text{AG}(3,q)$, where $\varepsilon < p$. Put $N = \pi_\infty \setminus U_D$ the set of non-determined directions. Then $N$ is contained in a plane curve of degree $\varepsilon^4 - 2\varepsilon^3 + \varepsilon$ or $U$ can be extended to a set $U'$, $|U'| = q^2$, $U_D = U'_D$.

**Theorem 4.11** ([21], Theorem 13) Let $n \geq 3$. Let $U \subset \text{AG}(n,q) \subset \text{PG}(n,q)$, $|U| = q^{n-1} - 2$. Put $N = \pi_\infty \setminus U_D$ the set of non-determined directions. Then $U$ can be extended to a set $U'$, $|U'| = q^{n-1}$, $U_D = U'_D$, or the points of $N$ are collinear and $|N| \leq \left\lfloor \frac{q+3}{2} \right\rfloor$, or the points of $N$ are on a (planar) conic curve.

## 5 Direction problems in affine spaces and special point sets in generalized quadrangles

A (finite) generalized quadrangle (GQ) is an incidence structure $\mathcal{S} = (\mathcal{P}, \mathcal{B}, \mathcal{I})$ in which $\mathcal{P}$ and $\mathcal{B}$ are disjoint non-empty sets of objects called points and lines (respectively), and for which $\mathcal{I} \subseteq (\mathcal{P} \times \mathcal{B}) \cup (\mathcal{B} \times \mathcal{P})$ is a symmetric point-line incidence relation satisfying the following axioms:

(i) each point is incident with $1 + t$ lines ($t \geq 1$) and two distinct points are incident with at most one line;
(ii) each line is incident with $1 + s$ points ($s \geq 1$) and two distinct lines are incident with at most one point;

(iii) if $x$ is a point and $L$ is a line not incident with $x$, then there is a unique pair $(y, M) \in \mathcal{P} \times \mathcal{B}$ for which $x \mathcal{I} M \mathcal{I} y \mathcal{I} L$.

The integers $s$ and $t$ are the parameters of the GQ and $\mathcal{S}$ is said to have order $(s, t)$. If $s = t$, then $\mathcal{S}$ is said to have order $s$. If $\mathcal{S}$ has order $(s, t)$, then $|\mathcal{P}| = (s + 1)(st + 1)$ and $|\mathcal{B}| = (t + 1)(st + 1)$ (see e.g. [32]). The dual $\mathcal{S}^D$ of a GQ $\mathcal{S} = (\mathcal{P}, \mathcal{B}, \mathcal{I})$ is the incidence structure $(\mathcal{B}, \mathcal{P}, \mathcal{I})$. It is again a GQ.

An ovoid of a GQ $\mathcal{S}$ is a set $\mathcal{O}$ of points of $\mathcal{S}$ such that every line is incident with exactly one point of the ovoid. An ovoid of a GQ of order $(s, t)$ has necessarily size $1 + st$. An partial ovoid of a GQ is a set $\mathcal{K}$ of points such that every line contains at most one point of $\mathcal{K}$. A partial ovoid $\mathcal{K}$ is called maximal if and only if $\mathcal{K} \cup \{p\}$ is not a partial ovoid for any point $p \in \mathcal{P} \setminus \mathcal{K}$, in other words, if $\mathcal{K}$ cannot be extended. It is clear that any partial ovoid of a GQ of order $(s, t)$ contains $1 + st - \rho$ points, $\rho \geq 0$, with $\rho = 0$ if and only if $\mathcal{K}$ is an ovoid.

Consider a non-singular quadratic form $f$ acting on $V(5, q)$, which is, up to coordinate transformation, unique. The points of $\text{PG}(4, q)$ that are totally singular with relation to $f$ constitute the parabolic quadric $Q(4, q)$. Since $f$ has Witt index two, there exist projective lines on $\text{PG}(4, q)$ that are completely contained in $Q(4, q)$. It is well known that $Q(4, q)$ is actually a GQ of order $q$. This GQ is one of the so-called finite classical generalized quadrangles. The motivation of [20] was to study the extendability of partial ovoids of $Q(4, q)$, $q$ odd. Using an alternative representation of $Q(4, q)$, this extendability problem translates directly to a stability question on sets of size $q^2 - \epsilon$ of $\text{AG}(3, q)$, not determining a given set of directions.

An oval of $\text{PG}(2, q)$ is a set of $q + 1$ points $\mathcal{C}$, such that no three points of $\mathcal{C}$ are collinear. When $q$ is odd, it is known that all ovals of $\text{PG}(2, q)$ are conics. When $q$ is even, several other examples and infinite families are known, see e.g. [22]. The GQ $T_2(\mathcal{C})$ is defined as follows. Let $\mathcal{C}$ be an oval of $\text{PG}(2, q)$, embed $\text{PG}(2, q)$ as a plane in $\text{PG}(3, q)$ and denote this plane by $\pi_\infty$. Points are defined as follows:

(i) the points of $\text{PG}(3, q) \setminus \text{PG}(2, q)$;
(ii) the planes $\pi$ of $\text{PG}(3, q)$ for which $|\pi \cap \mathcal{C}| = 1$;
(iii) one new symbol ($\infty$).

Lines are defined as follows:

(a) the lines of $\text{PG}(3, q)$ which are not contained in $\text{PG}(2, q)$ and meet $\mathcal{C}$ (necessarily in a unique point);
(b) the points of $C$.

Incidence between points of type (i) and (ii) and lines of type (a) and (b) is the inherited incidence of $\text{PG}(3,q)$. In addition, the point $\infty$ is incident with no line of type (a) and with all lines of type (b). It is straightforward to show that this incidence structure is a GQ of order $q$. The following theorem (see e.g. [32]) allows us to use this representation.

**Proposition 5.1** The GQs $T_2(C)$ and $Q(4,q)$ are isomorphic if and only if $C$ is a conic of the plane $\text{PG}(2,q)$.

Suppose now that $O$ is a (partial) ovoid of $Q(4,q)$. Using $Q(4,q) \cong T_2(C)$ if and only if $C$ is a conic, $O$ is equivalent with a set of points $U \cup \{\infty\}$, $U$ consisting exclusively of points of type (i), since any point of $Q(4,q)$ can play the role of the point $\infty$. Since no two points of $O$ are collinear in $Q(4,q)$, no two points of $U$ may determine a line of type (a) of $T_2(C)$, and since all lines of type (a) meet $\pi$ in a point of $C$, a (partial) ovoid of $Q(4,q)$ of size $q^2 + 1 - \epsilon$ is equivalent to a set $U$ of $q^2 - \epsilon$ points of $\text{AG}(3,q)$, not determining the points of a conic at infinity.

An immediate application of Theorem 4.4 is the following theorem.

**Theorem 5.2** ([3], Section 4) Let $U$ be a set of $q^2$ points in $\text{AG}(3,p)$, $p$ prime, whose non determined directions contain a conic. Then every plane of $\text{AG}(3,q)$ meets $U$ in $0 \mod p$ points.

And the previous theorem has the classification of all ovoids of $Q(4,p)$, $p$ prime, as a consequence.

**Corollary 5.3** ([3], Section 4) An ovoid of $Q(4,p)$, $p$ prime, is necessarily contained in a hyperplane of $\text{PG}(4,q)$, meeting $Q(4,q)$ in the points of an elliptic quadric.

Note that the previous corollary and theorem were first proved in [2], using a purely algebraic approach that is unrelated to direction problems.

The question addressed in [20] is whether a partial ovoid of $Q(4,q)$ of size $q^2 - 1$ can be maximal, i.e. whether the corresponding set $U$ of size $q^2 - 2$ can be non-extendable. As an immediate application of Theorem 4.9, the following theorem is described in [20].

**Theorem 5.4** ([20], Theorem 3) Let $q = p^h$, $p$ odd prime, $h > 1$. Then $Q(4,q)$ has no maximal partial ovoids of size $q^2 - 1$. 

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A (finite) partial geometry is a point-line geometry $S = (P, B, I)$ that is a generalization of a GQ. To define a partial geometry, the third GQ axiom is replaced by the following:

(iii) There exists a fixed integer $\alpha > 0$, such that if $x$ is a point and $L$ is a line not incident with $x$, then there are exactly $\alpha$ pairs $(y_i, M_i) \in P \times B$ for which $x \perp M_i \perp y_i \perp L$.

The integers $s, t$ and $\alpha$ are the parameters of $S$. The dual $S^D$ of a partial geometry $S = (P, B, I)$ is the incidence structure $(B, P, I)$. It is a partial geometry with parameters $s^D = t, t^D = s, \alpha^D = \alpha$. Clearly, a partial geometry with $\alpha = 1$ is a GQ.

We need some special pointsets in $PG(2, q)$ to describe our favorite partial geometries. An arc of degree $d$ of $PG(2, q)$ is a set $K$ of points such that every line of $PG(2, q)$ meets $K$ in at most $d$ points. If $K$ contains $k$ points, then it can also be called a $\{k, d\}$-arc. A typical example is a conic, which is a $\{q + 1, 2\}$-arc. The size of an arc of degree $d$ can not exceed $dq - q + d$. A $\{k, d\}$-arc $K$ for which $k = dq - q + d$, or equivalently, such that every line that meets $K$, meets $K$ in exactly $d$ points, is called maximal. With this definition, a conic is a non maximal $\{q + 1, 2\}$-arc, and it is well known that if $q$ is even, a conic, together with its nucleus, is a $\{q + 2, 2\}$-arc, which is complete. We mention that a $\{q + 1, 2\}$-arc is also called an oval, and a $\{q + 2, 2\}$-arc is also called a hyperoval. When $q$ is odd, all ovals are conics, and no $\{q + 2, 2\}$-arcs exist. Let $q = 2^h$, then every oval has a nucleus, and so can be extended to a hyperoval. Much more examples of hyperovals, different from a conic and its nucleus, are known. Maximal $\{k, d\}$-arcs exist for $d = 2^e, 1 \leq e \leq h$. Several infinite families and constructions are known. We refer to [22] for an overview, and detailed references to mentioned results here.

Let $q$ be even and let $K$ be a maximal $\{k, d\}$-arc of $PG(2, q)$. We define the incidence structure $T^*_2(K)$ as follows. Embed $PG(2, q)$ as a hyperplane $H_\infty$ in $PG(3, q)$. The points of $S$ are the points of $PG(3, q) \setminus H_\infty$. The lines of $S$ are the lines of $PG(3, q)$ not contained in $H_\infty$, and meeting $H_\infty$ in a point of $K$. The incidence is the natural incidence of $PG(3, q)$. One can check easily, using that $K$ is a maximal $\{k, d\}$-arc, that $T^*_2(K)$ is a partial geometry with parameters $s = q - 1, t = k - 1 = (d - 1)(q + 1)$, and $\alpha = d - 1$.

The definitions of (maximal) (partial) ovoid can be taken over for partial geometries. As in the GQ case, a (maximal) (partial) ovoid of $T^*_2(K)$ is equivalent with a (extendable) set of points $U$ of $AG(3, q)$ not determining the points of $K$ at infinity. A maximal $\{k, d\}$-arc can not be contained in a conic. Therefore Theorem [4,11] yields almost immediately the following theorem as a corollary.
Theorem 5.5 ([21], Corollary 18) Let $\mathcal{B}$ be a partial ovoid of size $q^2 - 2$ of the partial geometry $T_2^*(\mathcal{K})$, then $\mathcal{B}$ is always extendable to an ovoid.

Note that the following theorem is found in [8]. It is obtained indirectly as a corollary of Theorem 4.3 and some extra work.

Theorem 5.6 ([8], Corollary 3.2) Let $\mathcal{H}$ be a hyperoval of $\text{PG}(2,q)$, $q$ even. Let $\mathcal{O}$ be an ovoid of $T_2^*(\mathcal{H})$, then every plane of $\text{PG}(3,q)$ meets $\mathcal{O}$ in an even number of points. Moreover two planes meeting $\text{PG}(3,q) \setminus \text{AG}(3,q)$ in the same line intersect $\mathcal{O}$ either both in $0 \mod 4$ points or both in $2 \mod 4$ points.

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