Laboratory application of sampling approaches to inverse scattering

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Abstract

This study presents an experimental investigation of the recently established generalized linear sampling method (GLSM) \cite{Pourahmadian2018} for non-destructive evaluation of damage in elastic materials. To this end, ultrasonic shear waves are generated in a prismatic slab of charcoal granite featuring a discontinuity interface induced by the three-point bending (3PB). The interaction of probing waves with the 3PB-induced damage gives rise to transient velocity responses measured on the sample's boundary by a 3D scanning laser Doppler vibrometer. Thus obtained waveform data are then carefully processed to retrieve the associated spectra of scattered displacement fields. On deploying multifrequency sensory data, the GLSM indicators are computed and their counterparts associated with the classical linear sampling method (LSM) \cite{Haltmeier2010} for comparative analysis. Verified with in-situ observations, the GLSM map successfully exposes the support of hidden scatterers in the specimen with a remarkable clarity and resolution compared to its predecessor LSM. It is further shown that the GLSM remains robust for sparse and partial-aperture data inversion, thanks to its rigorous formulation. For completeness, the one-sided reconstruction by both indicators is investigated.

\textit{Keywords:} waveform tomography, ultrasonic testing, non-destructive evaluation, (generalized) linear sampling method, material interfaces.

1. Introduction

Inverse scattering solutions are sought for uncovering geometrical and physical properties of hidden objects in a medium from remote (or boundary) observations of thereby scattered waveforms. In this context, waveform tomography of discontinuity surfaces bear direct relevance to (a) timely detection of degradation in safety-sensitive components, (b) in-situ monitoring of additive manufacturing processes, and (c) efficient energy mining from unconventional resources. Existing optimization-based approaches to waveform inversion typically incur high computational cost as a crucial obstacle to real-time sensing. Lately, non-iterative inverse scattering solutions \cite{Haltmeier2010, Pourahmadian2018, Pourahmadian2020} have been brought under the spotlight for their capabilities pertinent to fast imaging in highly scattering media \cite{Pourahmadian2020}. Spurred by the early study in \cite{Haltmeier2010}, such developments include: (i) the Factorization Method (FM) \cite{Kress2006, Kress2008}, (ii) the Linear Sampling Method (LSM) \cite{Haltmeier2010, Haltmeier2012}, (iii) MUSIC algorithms \cite{Healy1992}, (iv) the method of Topological Sensitivity (TS) \cite{Sokolowski1999, Sokolowski2000}, and (v) the Generalized Linear Sampling Method (GLSM) \cite{Pourahmadian2018, Pourahmadian2020}. Among these, the FM, LSM, and GLSM inherently carry a superior localization property that potentially leads to high-fidelity geometric reconstruction.

This study is focused on the GLSM indicator \cite{Pourahmadian2018, Pourahmadian2020} developed by building upon the factorization method and recent theories on design of imaging functionals \cite{Pourahmadian2018, Pourahmadian2020}. More specifically, the GLSM is a non-iterative, full-waveform approach to elastic-wave imaging of 3D discontinuity surfaces with non-trivial (generally heterogeneous and dissipative) interfacial condition. This indicator map – targeting geometric reconstruction of extended interfaces – is shown to be (a) agnostic with respect to the contact condition at the interface,

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(b) robust against measurement errors, and (c) flexible in terms of sensing parameters, e.g. the illumination frequency.

On the verification side, the effectiveness of sampling methods for elastic waveform tomography has been extensively examined by numerical simulations, see e.g., [3, 12]. A systematic experimental investigation of these imaging tools, however, is still lacking. To help bridge the gap, a few recent studies [14, 15] demonstrate successful performance of the classical linear sampling method in a laboratory setting. The present work augments these efforts by investigating the generalized linear sampling technique in an experimental campaign for the shape reconstruction of an extended damage zone from boundary data. In primary experiments, ultrasonic waves are induced in an intact slab of charcoal granite and the resulting velocity responses are captured by a 3D scanning laser Doppler vibrometer over the sample’s edges, furnishing the incident fields affiliated with every source location. The sample is then notched and fractured in the three-point-bending (3PB) configuration, then probed by ultrasonic waves in a similar fashion as in the primary experiments. The secondary measurements carry the scattering signature of 3PB-induced damage in the granite. The primary and secondary sensory data are then carefully processed and transformed into the frequency domain to compute the GLSM and LSM indicators and recover the support of damage zone. In this study, the data inversion is adapted to the testing configuration and the nature of measured waveforms. In particular, the reconstruction procedure is reformulated for multi-frequency inverse scattering, also the GLSM cost functional is carefully modified to accommodate for a highly asymmetric scattering operator resulting from the sparse sampling (of the incident surface). It is shown that the GLSM indicator successfully reconstructs the process zone’s geometry including the pre-manufactured notch and the (heterogeneous) mode I fracture induced by three-point bending. The performance of LSM and GLSM imaging functionals are compared. The influences of key testing parameters on the fidelity of reconstruction – including the source/measurement aperture and sensing resolution are also investigated for both indicators.

This paper is organized as follows. Section 2 formulates the direct scattering problem within the context of laboratory experiments, and provides an overview of the data inversion platform. Section 3 describes the experimental procedure and showcases the “raw” measurements. Section 4 includes a detailed account of signal processing in time and space in preparation for data inversion. Section 5 computes the (generalized) linear sampling functionals using multi-frequency data. Section 6 presents and discusses the results.

2. Theoretical foundation

This section briefly outlines two theories of inverse scattering considered in this study – namely, the classical linear sampling method [3, 16] and the recently developed generalized linear sampling technique [1, 17].

2.1. Problem statement

Let \( \mathcal{B} \subset \mathbb{R}^d \), \( d = 2, 3 \), denote a finite elastic body characterized by mass density \( \rho \), and Lamé parameters \( \mu \) and \( \lambda \), which henceforth is referred to as the baseline model. A set of unknown discontinuities \( \Gamma \) is embedded in \( \mathcal{B} \) whose support is possibly disjoint and of arbitrary shape. More specifically, \( \Gamma \) may be decomposed into \( N \) smooth open subsets \( \Gamma_n \), each of which may be arbitrarily extended to a closed Lipschitz surface \( \partial D_n \) enclosing a bounded simply connected domain \( D_n \subset \mathbb{R}^d \), so that \( \Gamma = \bigcup_{n=1}^{N} \Gamma_n \subset \bigcup_{n=1}^{N} \partial D_n \). The contact at the surface of \( \Gamma \) is characterized by a symmetric and heterogeneous interfacial stiffness matrix \( K(\xi), \xi \in \Gamma \), synthesizing the spatially varying nature of rough interfaces. Here, \( K \) is arbitrary and a priori unknown.

Assumption 2.1. In this study, the interfacial energy dissipation on \( \Gamma \) is assumed negligible during the course of ultrasonic measurements. This may be justified owing to the small amplitude of motion, and short period of observation in the experimental campaign.

The domain \( \mathcal{B} \) is excited by an ultrasonic source on its external boundary \( \partial \mathcal{B} \) so that the corresponding incident field \( u_i(\xi, t) \) in the baseline model is governed by
\[ \nabla \cdot [C : \nabla u^t](\xi, t) - \rho \ddot{u}(\xi, t) = 0, \quad (\xi \in \mathcal{B}, t \in (0, T]) \]
\[ n \cdot C : \nabla u^t(\xi, t) = g(\xi, t), \quad (\xi \in \partial \mathcal{B}, t \in (0, T]) \]
\[ u^t(\xi, t) = 0, \quad (\xi \in \partial \mathcal{B}_u, t \in (0, T]) \]
\[ u^t(\xi, 0) = 0, \quad (\xi \in \mathcal{F}, t = 0) \]

where the fourth-order elasticity tensor \( C = \lambda I_2 \otimes I_2 + 2\mu I_4 \) denotes the \( m \)th-order symmetric identity tensor; the single and double over-dots indicate first- and second- order time derivatives, respectively; \( T \) signifies the testing interval; \( n \) is the unit outward normal to the sample’s boundary \( \partial \mathcal{B} \); \( g(\xi, t) \) represents the external traction on the Neumann part of the boundary \( \partial \mathcal{B}_t \subset \partial \mathcal{B} \) which includes the source input; the displacement vanishes on the boundary’s Dirichlet part \( \partial \mathcal{B}_u \subset \partial \mathcal{B} \); and, \( \mathcal{F} \) denotes the observation surface \( S_{\text{obs}} \subset \partial \mathcal{B}_t \), and the corresponding scattered field may be computed as

\[ v(\xi, t) := |u - u^t|(\xi, t), \quad (\xi \in \mathcal{F}, t \in (0, T]) \]

satisfying

\[ \nabla \cdot [C : \nabla v](\xi, t) - \rho \ddot{v}(\xi, t) = 0, \quad (\xi \in \mathcal{B}\setminus \Gamma, t \in (0, T]) \]
\[ n_\Gamma \cdot C : \nabla v(\xi, t) = K(\xi)\{u\}(\xi, t) - t^f(\xi, \omega), \quad (\xi \in \Gamma, t \in (0, T]) \]
\[ n \cdot C : \nabla v(\xi, t) = 0, \quad (\xi \in \partial \mathcal{B}_t, t \in (0, T]) \]
\[ v(\xi, t) = 0, \quad (\xi \in \partial \mathcal{B}_u, t \in (0, T]) \]
\[ v(\xi, 0) = 0, \quad (\xi \in \mathcal{F}, t = 0) \]

where \( t^f = n_\Gamma \cdot C : \nabla u^t \) is the free-field traction on the surface of \( \Gamma \). The experiments are repeated for a set of ultrasonic excitations on the incident surface \( S_{\text{inc}} \subset \partial \mathcal{B}_t \).

To assist the inverse analysis, let us introduce the relevant function spaces as the following,

\[ H^{1/2}_t(\Gamma) := \{ f_\Gamma : f \in H^{1/2}(\partial \mathcal{D}) \}, \]
\[ \tilde{H}^{1/2}_t(\Gamma) := \{ f \in H^{1/2}(\partial \mathcal{D}) : \text{supp}(f) \subset \overline{\Gamma} \}, \quad (\xi \in \mathcal{F}, t \in (0, T]) \]

where \( \mathcal{D} = \bigcup_{n=1}^{N} D_n \) is a multiply connected Lipschitz domain of bounded support such that \( \Gamma \subset \partial \mathcal{D} \), and \( \Gamma := \Gamma \cup \partial \Gamma \) denotes the closure of \( \Gamma = \bigcup_{n=1}^{N} \Gamma_n \). Recall that every \( \Gamma_n \) is an open set (relative to \( \partial D_n \)) with a positive surface measure. Note that since \( v \in H^1(\mathcal{B} \setminus \Gamma)^3 \), then by trace theorems \( \{v\} \in \tilde{H}^{1/2}(\Gamma)^3 \).

2.2. Inverse solution

The (generalized) linear sampling indicators use the spectrum of scattered displacement field \( v \) on \( S_{\text{obs}} \) to non-iteratively reconstruct the support of hidden scatterers \( \Gamma \) via synthetic wavefront shaping. To this end, the scattering operator \( \Lambda : L^2(S_{\text{inc}})^3 \times L^2(\Omega)^3 \rightarrow L^2(S_{\text{obs}})^3 \times L^2(\Omega)^3 \) is constructed over a frequency bandwidth \( \Omega := [\omega_{\text{min}}, \omega_{\text{max}}] \subset \mathbb{R}^+ \) from test data as the following
\[ \Lambda(g)(\xi, \omega) = \int_{S^\text{inc}} V(\xi, y; \omega) \cdot g(y, \omega) \, dS_y, \quad g \in L^2(S^\text{inc})^3 \times L^2(\Omega)^3, \quad \xi \in S^\text{obs}, \ \omega \in \Omega. \] 

On denoting by \( F(\cdot) \) the Fourier transform operator, \( V_{ij}(\xi, y; \omega), i, j = 1, 2, 3, \) in \( (6) \) indicates the \( i^\text{th} \) component of the Fourier transformed displacement \( F(v)(\xi, \omega) \in L^2(S^\text{obs})^3 \times L^2(\Omega)^3 \) measured at \( \xi \in S^\text{obs} \) with frequency \( \omega \in \Omega \) due to excitation at \( y \in S^\text{inc} \) in the \( j^\text{th} \) direction.

In addition, let us consider the search volume \( \mathcal{S} \subset \mathcal{B} \subset \mathbb{R}^d \) in the (intact) baseline model, and define a set of trial dislocations \( L(x_o, R) \subset \mathcal{S} \) such that for every pair \( (x_o, R) \), \( L := x_o + RL \) specifies a smooth arbitrary-shaped fracture \( L \) at \( x_o \subset \mathcal{S} \) whose orientation is identified by a unitary rotation matrix \( R \in U(3) \). In this setting, the scattering pattern \( \Phi_L : \mathbb{R}^{1/2}(L)^3 \times L^2(\Omega)^3 \to L^2(S^\text{inc})^3 \times L^2(\Omega)^3 \) on \( S^\text{obs} \) – generated by \( L(x_o, R) \), as a sole scatterer in \( \mathcal{B} \), endowed with an admissible displacement density \( a(\xi, \omega) \in \mathbb{R}^{1/2}(L)^3 \times L^2(\Omega)^3 \) – is governed by

\[
\begin{align*}
\nabla \cdot [C : \nabla \Phi_L(\xi, \omega) + \rho \omega^2 \Phi_L(\xi, \omega) &= 0, \quad (\xi \in \mathcal{B} \setminus L, \omega \in \Omega) \\
\n\Phi_L(\xi, \omega) &= 0, \quad (\xi \in \partial \mathcal{B}_L, \omega \in \Omega) \\
\|\Phi_L(\xi, \omega)\| &= a(\xi, \omega), \quad (\xi \in L, \omega \in \Omega)
\end{align*}
\]

(7)

Given \( (7) \), one may generate a library of physically-consistent scattering patterns on \( S^\text{inc} \) for a grid of trial pairs \( (x_o, R) \) sampling \( \mathcal{S} \times U(3) \). The underpinning concept of wavefront shaping is that when the trial dislocation \( L \) is a subset of the true scatterers \( \Gamma \), its affiliated scattering pattern \( \Phi_L \in L^2(S^\text{inc})^3 \times L^2(\Omega)^3 \) may be recovered from experimental data by probing the range of operator \( \Lambda \) i.e., through solving

\[
\Lambda g \simeq \Phi_L, \quad g \in L^2(S^\text{inc})^3 \times L^2(\Omega)^3,
\]

(8)

for the wavefront densities \( g(\xi, \omega) \) on \( \xi \in S^\text{inc} \) at every frequency \( \omega \in \Omega \). In this setting, the principal theorem of linear sampling shines light on the unique behavior of \( g \) in terms of \( L \). This is accomplished by taking advantage of the factorization \( \Lambda = \mathcal{H}^* T \mathcal{H} \),

\[
\Lambda = \mathcal{H}^* T \mathcal{H},
\]

(9)

where \( (\cdot)^* \) indicates the adjoint operator, and

\[
\begin{align*}
\mathcal{H} : L^2(\partial \mathcal{B})^3 \times L^2(\Omega)^3 &\to H^{-\frac{1}{2}}(\Gamma)^3 \times L^2(\Omega)^3, \quad (\xi \in \Gamma, \omega \in \Omega) \\
\mathcal{H}^* : H^{\frac{1}{2}}(\Gamma)^3 \times L^2(\Omega)^3 &\to L^2(\partial \mathcal{B})^3 \times L^2(\Omega)^3, \\
T : H^{-\frac{1}{2}}(\Gamma)^3 \times L^2(\Omega)^3 &\to H^{\frac{1}{2}}(\Gamma)^3 \times L^2(\Omega)^3, \\
T(t^i) &:= \|v\|_{(\xi, \omega)}, \quad (\xi \in \Gamma, \omega \in \Omega)
\end{align*}
\]

(10)

This allows to rigorously characterize the solution \( g \) according to \( \Lambda \) Theorem 6.2] as the following.

**Theorem 2.1.** Given \( \Theta \), by assuming that the operator \( \mathcal{H}(\Gamma, \omega) \) in \( (10) \) is injective at frequency \( \omega \in \Omega \),

- If \( L \subset \Gamma \), then exists a density vector \( g_e \in L^2(S^\text{inc})^3 \times L^2(\Omega)^3 \) such that \( \|\Lambda g_e - \Phi_L\|_{L^2(S^\text{obs})} \leq \epsilon \) and \( \limsup_{\epsilon \to 0} \|\mathcal{H} g_e\|_{H^{-1/2}(\Gamma)} < \infty \).
- If \( L \notin \Gamma \), then \( \forall g_e \in L^2(S^\text{inc})^3 \times L^2(\Omega)^3 \) such that \( \|\Lambda g_e - \Phi_L\|_{L^2(S^\text{obs})} \leq \epsilon \), \( \lim_{\epsilon \to 0} \|\mathcal{H} g_e\|_{H^{-1/2}(\Gamma)} = \infty \).

**LSM indicator.** Theorem 2.1 of the linear sampling method poses two fundamental challenges in that: (i) the featured anomaly indicator \( \|\mathcal{H} g_e\|_{H^{-1/2}(\Gamma)} \) inherently depends on the support of unknown scatterers \( \Gamma \) since \( \mathcal{H} = \mathcal{H}(\Gamma) \), and (ii) construction of the wavefront density \( g_e \in L^2(S^\text{inc})^3 \times L^2(\Omega)^3 \) is implicit in the theorem 17 \( \mathcal{H} \). Conventionally, these issues are addressed by replacing \( \|\mathcal{H} g_e\|_{H^{-1/2}(\Gamma)} \) with \( \|g_e\|_{L^2(S^\text{inc})} \) which is, in turn, computed by way of Tikhonov regularization

\[
g_e = g_e := \min_{g \in L^2(S^\text{inc})} \left\{ \|\Lambda g - \Phi_L\|_{L^2(S^\text{obs})}^2 + \eta \|g\|_{L^2(S^\text{inc})}^2 \right\},
\]

(11)
where \( \eta = \eta(L) > 0 \) is a regularization parameter computable by the Morozov discrepancy principle \([19]\).

On the basis of \([11]\), the LSM indicator functional for every frequency \( \omega \in \Omega \) is constructed according to \([2]\) by

\[
L := \frac{1}{\|g_L\|_{L^2(S^{inc})}}.
\]

\( L = L(g_L, \omega) \) achieves its highest values at the loci of hidden scatterers \( \Gamma \). More specifically, the behavior of \( L \) within the search volume \( \mathcal{S} \subset \mathcal{B} \) may be characterized as the following,

\[
\begin{align*}
&\text{if } L \subset \Gamma \iff \liminf_{\eta \to 0} L(g_L, \omega) > 0, \\
&\text{if } L \subset \mathcal{S}\setminus\Gamma \iff \liminf_{\eta \to 0} L(g_L, \omega) = 0.
\end{align*}
\]

**GLSM indicator.** Approximations underlying the LSM imaging functional may lead to instability of the reconstruction, and sensitivity to measurement errors (see Section \([5]\)). To help meet the challenge, the GLSM \([1]\) takes advantage of the positive and self-adjoint operator \( \Lambda _t : \mathcal{L}^2(\partial \mathcal{B}_t)^3 \times \mathcal{L}^2(\partial \mathcal{B}_t)^3 \rightarrow \mathcal{L}^2(\partial \mathcal{B}_t)^3 \times \mathcal{L}^2(\Omega)^3 \), defined on the basis of the scattering operator \( \Lambda \) by

\[
\Lambda _t := \frac{1}{2} |\Lambda + \Lambda ^*| + \left| \frac{1}{2\gamma} (\Lambda - \Lambda ^*) \right|,
\]

with the affiliated factorization \([7]\)

\[
\Lambda _t = \mathcal{H}^* T_4 \mathcal{H},
\]

where the middle operator \( T_4 \) is coercive according to \([11]\, \text{Lemma 5.7} \) i.e., there exists a constant \( c > 0 \) independent of \( \mathcal{H} g \), such that

\[
\langle g_e, \Lambda _t g_e \rangle_{\mathcal{L}^2(\partial \mathcal{B}_t)^3} = \langle \mathcal{H}^* g_e, T_4 \mathcal{H} g_e \rangle_{\Gamma} \geq c \|\mathcal{H} g_e\|_{\mathcal{H}^{-1/2}(\Gamma)}^2, \quad \forall \mathcal{H} g_e \in \mathcal{H}^{-1/2}(\Gamma)^3.
\]

Here, \( \langle \cdot, \cdot \rangle_{\Gamma} \) denotes the duality product \( \langle \mathcal{H}^{-1/2}(\Gamma)^3, \mathcal{H}^{1/2}(\Gamma)^3 \rangle \). Thanks to \([16]\), the term \( \|\mathcal{H} g_e\|_{\mathcal{H}^{-1/2}(\Gamma)}^2 \) in Theorem \([2.1]\) may be safely replaced by \( \langle g_e, \Lambda _t g_e \rangle_{\mathcal{L}^2(\partial \mathcal{B}_t)^3} \) which is computable without prior knowledge of \( \Gamma \). Then, according to \([5]\, \text{Theorems 4.3} \) a robust solution to \([8]\) may be constructed by

\[
\gamma \in \arg \min \left\{ \|\Lambda \gamma - \Phi L\|_{L^2(S^{inc})}^2 + \gamma \|\Lambda _t g_e\|_{\mathcal{L}^2(\partial \mathcal{B}_t)^3}^2 + \delta \|\gamma\|_{L^2(\Omega)}^2 \right\},
\]

without the heuristics involved in the LSM approach. It should be mentioned that, in \([17]\), \( \delta > 0 \) is a measure of noise in data, and \( \gamma > 0 \) represents the regularization parameter defined in terms of \( \eta(L) \) of \([11]\) by

\[
\gamma(L) := \frac{\eta(L)}{\|\Lambda\|_{L^2} + \delta}.
\]

Note that the GLSM cost functional \([17]\) is convex \([5]\, \text{Theorem 4.1} \), and thus, its minimizer \( g_e \) can be computed without iterations. Similar to the LSM indicator \([12]\), the norm of penalty term in \([17]\) is used to identify the GLSM indicator as \( \langle g_e, \Lambda _t g_e \rangle_{\mathcal{L}^2(\partial \mathcal{B}_t)^3} + \delta \|g_e\|_{L^2(\Omega)}^2 \).\]

**Remark 2.1 (on the nature of \( \Lambda _t \)).** The operator \( \Lambda _t \) is symmetric, and thus, amenable to specific sensing configurations where \( S^{inc} = S^{obs} \subset \partial \mathcal{B}_t \). This implies that the loci of ultrasonic sources in experiments should coincide with the measurement points so that the discretized operator \( \Lambda \) is a square matrix. This may not be plausible or efficient in practice as evidenced in section \([3]\) where the observation grid is ten times more dense than the excitation grid.

This constraint may be relaxed by invoking Assumption \([2.1]\) where the system’s energy dissipation is presumed negligible during the testing period \((0, T)\) so that the operator \( \Lambda \) is normal \([7]\). In this setting, \([7]\, \text{Theorem 1.23} \) indicates that there exists a second factorization

\[
\Lambda = (\Lambda ^* \Lambda )^{\frac{1}{2}} T (\Lambda ^* \Lambda )^{\frac{1}{2}},
\]

such that the middle operator \( T \) is coercive, and the ranges of \( \mathcal{H}^* \) in \([5]\) and \( (\Lambda ^* \Lambda )^{1/4} \) coincide. As a result, the
term \( \| \mathcal{K} g_c \|_{L^2(\Gamma)}^2 \) in Theorem 2.1 may also be replaced by \( (g_c, (\Lambda^* \Lambda)^{1/2} g_c)_{L^2(\Sigma^{inc})}^2 \) which is computable from \( \Lambda \) notwithstanding of its symmetry condition. Following [5, Theorems 4.3], a solution to \( (8) \) is then generated by minimizing the modified GLSM cost functional, i.e.,

\[
g_c = \hat{g}_0 := \min_{g \in L^2(\Sigma^{inc})^3} \{ \| \mathcal{K} g - \Phi_L \|_{L^2(\Sigma^{obs})}^2 + \gamma (g, \mathcal{Y} g)_{L^2(\Sigma^{inc})} + \delta \gamma \| g \|_{L^2(\Sigma^{inc})}^2 \}, \quad \mathcal{Y} = (\Lambda^* \Lambda)^{1/2}. \tag{20}
\]

The new cost functional \( (20) \) is also convex and its minimizer \( g_0 = \hat{g}_0 (L, \omega) \) may be obtained non-iteratively as elucidated in section [3]. Following [3, Theorems 4.3], one may show that as \( \gamma \to 0 \), the solution \( g_0 \) remains bounded if and only if \( L \subset \Gamma \). More specifically, at every frequency \( \omega \in \Omega \),

\[
\text{if } L \subset \Gamma \iff \limsup_{\gamma \to 0} \limsup_{\delta \to 0} \left( (g_0, \mathcal{Y} g_0)_{L^2(\Sigma^{inc})} + \delta \| g_0 \|_{L^2(\Sigma^{inc})}^2 \right) < \infty,
\]

\[
\text{if } L \not\subset \Gamma \iff \liminf_{\gamma \to 0} \liminf_{\delta \to 0} \left( (g_0, \mathcal{Y} g_0)_{L^2(\Sigma^{inc})} + \delta \| g_0 \|_{L^2(\Sigma^{inc})}^2 \right) = \infty. \tag{21}
\]

Based on this, the (modified) GLSM indicator functional is defined by

\[
\mathcal{G}(g_0, \omega) := \frac{1}{\sqrt{(g_0, \mathcal{Y} g_0)_{L^2(\Sigma^{inc})} + \delta \| g_0 \|_{L^2(\Sigma^{inc})}^2}}, \tag{22}
\]

which reconstructs the support of hidden scatterers by achieving its highest values near \( \Gamma \).

3. Experimental campaign

Experiments are performed on a prismatic specimen of charcoal granite of dimensions 0.96m \( \times \) 0.3m \( \times \) 0.03m, mass density \( \rho = 2750\text{kg/m}^3 \), nominal Poisson’s ratio \( \nu = 0.23 \), and nominal Young’s modulus \( E = 62.6\text{GPa} \). These values are identified via a uniaxial compression test on a cylindrical sample of the same material.

The testing procedure involves three steps: (i) elastic-wave excitation and sensing in the baseline system, (ii) fracturing of the specimen, and (iii) elastic-wave testing of the damaged system.

Step 1. The ultrasonic experiments are first performed on the intact granite slab as shown in Fig. 1(a). Waveforms measured in this step furnish the “baseline” response of the system associated with the incident field \( u(\xi, t) \). This is required for computing the scattered field \( v(\xi, t) = u(\xi, t) - u(\xi, t) \), wherein \( u(\xi, t) \) represents the total field measurements in Step 3. Step 1 entails eight ultrasonic experiments where the sample is excited by an in-plane shear wave from one of the designated source locations \( s_1, s_2, \ldots, s_8 \) shown in Fig. 1(b). Shear waves are generated by a 0.5 MHz piezoelectric transducer (V151-RB by Olympus, Inc.) whose diameter of 32 mm is almost commensurate with the granite thickness. The transducer is aligned with the granite mid-plane along \( \xi_3 \) minimizing the out-of-plane excitation. The incident signal is a five-cycle burst of the form

\[
H(t_c) H(5-t_c) \sin (0.2\pi f_c t) \sin (2\pi f_c t), \tag{23}
\]

where \( f_c = 30\text{kHz} \) denotes the center frequency, and \( H \) is the Heaviside step function. The induced wave motion from each source location is measured by a 3D Scanning Laser Doppler Vibrometer (SLDV) as shown in Fig. 1(a). The PSV-400-3D SLDV system by Polytec, Inc. is capable of capturing the triaxial components of particle velocity on the surface of solids over a designated scanning grid. Its measurement (resp. spatial) resolution is better than 1\( \mu \text{m/s} \) (resp. 0.1mm) within the frequency range DC-1MHz, facilitating waveform sensing in the nanometer scale in terms of displacement [20].

Step 2. A notch of length 4cm and width 1.5mm is manufactured at the bottom center of specimen. The sample is then fractured in the three-point-bending (3PB) configuration by a closed-loop, servo-hydraulic, 1000KN MTS load frame such that the crack propagation is controlled by the crack mouth opening displacement (CMOD) measured by a clip gage. The loading process is monotonic with respect to the CMOD at a constant rate of 0.1\( \mu \text{m/s} \). The loading process is continued up to approximately 65% of the maximum force in the post-peak regime with the associated CMOD of 320\( \mu \text{m} \). Upon completion of the fracturing process, the specimen is unloaded and reconfigured according to Fig. 1(a).
Figure 1: Testing set-up: (a) a prismatic slab of charcoal granite is subject to ultrasonic testing prior to and after being fractured via three-point bending; (b) shear waves are generated by a piezoelectric source at \( s_i \) \( (i=1,2,\ldots,8) \), and the triaxial particle velocity field is captured by a 3D SLDV over the designated scanning grid \( \bigcup_{i=1}^{4} G_i \).

Step 3. The ultrasonic experiments are performed on the fractured specimen following the same procedure as in Step 1, i.e., the testing set-up involving the transducer locations, illuminating wavelet, and scanning area is as shown in Fig. 1.

Remark 3.1 (on the nature of wave motion). Measurements may be interpreted in the context of plane stress approximation – related to the elastic analysis of thin plates [21], whereby the particle motion is considered invariant through the thickness of specimen. In this setting, the effective Poisson’s ratio and Young’s modulus are respectively identified by \( \nu' = \nu/(1 + \nu) \) and \( E' = E(1 - \nu'^2) \) [21], resulting in the shear (S-) and compressional (P-) wave velocities

\[
c_s = \sqrt{\frac{E}{2(1 + \nu)\rho}} = 3041 \text{ m/s}, \quad c_p = \sqrt{\frac{E}{(1 - \nu^2)\rho}} = 4901 \text{ m/s}.
\]

Observe that the shear wavelength \( \lambda_s \) in the specimen may be approximated by 10 cm at 30 kHz, giving the shear-wavelength-to-plate-thickness ratio of \( \lambda_s/h \gtrsim 3.3 \). In this range, the phase error committed by the plane stress approximation is about 3% [22]. An in-depth experimental analysis of plane-stress wave propagation – in a specimen of similar dimensions and material properties, is provided in [23] where full-field waveform data are analyzed within the frequency range 10 – 40 kHz.

It should be mentioned that the sampling approaches to inverse scattering are full-waveform inversions [5], and thus, they do not rely on a specific mode of propagation, nor they require any such knowledge on the nature of wave motion. In this study, the plane-stress approximation implies that the data inversion may be conducted in a reduced-order space involving the in-plane components of the measured response as delineated in section 5.

As illustrated in Fig. 1(b), the scanning grid \( \bigcup_{i=1}^{4} G_i \) is in the immediate vicinity of the external boundary of specimen. More specifically, \( G_1 \) (resp. \( G_3 \)) is centered in the mid-right (resp. left) edge of the sample with 27 uniformly spaced measurement points over a span of 22 cm, while \( G_2 \) (resp. \( G_4 \)) is at the top (resp. bottom) center of the plate involving a uniform grid of 45 scan points over an interval of 38 cm. In light of Remark 3.1, this amounts to a spatial resolution of about 8 mm for ultrasonic measurements at 30 kHz in \( \xi_1 \) and \( \xi_2 \) directions.

At every scan point, the data acquisition is conducted for a time period of 1 ms at the sampling rate of 512 kHz. To minimize the impact of (optical and mechanical) random noise in the system, the measurements are averaged over an ensemble of 60 realizations at each scan point. Furthermore, signal enhancement and speckle tracking were enabled to avoid signal dropouts due to surface roughness.

Remark 3.2. Note that the observation grid is consistent with common configurations in practice where only a subset of the domain’s external boundary is accessible for (contact or non-contact) sensing. Recall that the (G)LSM indicators reconstruct the support of internal scatterers from boundary data. Thus, full-field ultrasonic measurements i.e., waveforms on the entire surface of specimen are not captured in this study. An image processing scheme for anomaly detection by way of full-field measurements is provided in [23].
Figure 2: SLDV measurements over the scanning grid $\bigcup_{i=1}^{4} G_i$: (a) particle velocity distribution $\dot{\mathbf{u}}_1(\ell, t = 0.25\text{ms})$ (resp. $\dot{\mathbf{u}}_2(\ell, t = 0.25\text{ms})$) in $\xi_1$ (resp. $\xi_2$) direction, where $\ell$ represents the counterclockwise arc length along the specimen’s edge as in Fig. 1(b), and (b) time history of the particle velocity response $[\dot{u}_1 \, \dot{u}_2](\ell = 0.6\text{m}, t)$ measured in the vicinity of transducer at $s_2$. Dots represent “raw” measurements and solid lines are the corresponding processed data according to section 4.

To demonstrate the acquired SLDV measurements, Fig. 2(a) displays a snapshot in time (at $t = 0.25\text{ms}$) of the particle velocity distributions $\dot{u}_1$ and $\dot{u}_2$ over the scanning grid $\bigcup_{i=1}^{4} G_i$ in $\xi_1$ and $\xi_2$ directions, respectively. These measurements are conducted on the intact specimen prior to fracturing. Note that the test data is plotted against the counterclockwise arc length $\ell$ around the specimen’s external boundary whose origin is at the bottom-right corner of the plate as shown in Fig. 1(b). Fig. 2(b) plots the time history of in-plane SLDV measurements at a fixed grid point with the affiliated arc length $\ell = 0.6\text{m}$ – in the immediate vicinity of the ultrasonic source $s_2$ indicated in Fig. 2(a). It should be mentioned that in Fig. 2 “raw” test data are shown with dots, while the processed data (according to section 4) are shown by the linearly interpolated solid lines.

Remark 3.3 (scattered field data). Recall that the (G)LSM indicators rely on the spectrum of scattered field $\mathbf{v}$ which may be directly computed from the free field $\mathbf{u}$ measured in Step 1, and total field $\mathbf{u}$ captured in Step 3. An effort was made to generate sufficiently similar incident waveforms (up to some simple post processing measures described in section 4) at each source location in both sensing steps. This is accomplished by exercising: (i) precise geometric alignment of the piezoelectric transducer, (ii) application of a thin and uniform layer of cyanoacrylate glue as couplant, and (iii) comparison of the incident waveforms captured in the vicinity of the transducer (before any reflections occur) prior to conducting the planned data acquisition.

4. Signal processing

This section aims to systemically extract the spectrum of scattered displacement response over the observation grid from the SLDV-measured particle velocity data. The results will be deployed in section 5 to reconstruct the support of 3PB-induced damage in the granite specimen. In this vein, “raw” measurement data are processed in three stages, involving: (1) spatiotemporal filtering and time integration, (2) synchronization of incidents and extraction of scattered fields, and (3) spectral analysis.

(1) spatiotemporal filtering and time integration. A band-pass filter of bandwidth 20kHz centered at 30kHz – consistent with the spectrum of excitation wavelet (23), is applied to the particle-velocity records at every scan point. Note that the filtered velocity signals are temporally smooth and differentiable as shown by solid lines in Fig. 2(b). At every snapshot in time, however, the spatial distribution of particle velocity over the scanning grid is contaminated with data points of exceptionally low signal-to-noise ratio – identified by sudden spikes in the observed waveforms e.g., see Fig. 2(a). To mitigate the spatial noise, first, a unified set of observation points are specified on $\bigcup_{i=1}^{4} G_i$ which remain invariant for both datasets obtained in Steps 1
Figure 3: Spatiotemporal scattered displacement field: (a) in-plane displacement distributions \(v_1(\ell, t = 0.25\text{ms})\) and \(v_2(\ell, t = 0.25\text{ms})\) in \(\xi_1\) and \(\xi_2\) directions, where \(\ell\) is the arc length, and (b) time history of the scattered displacement response \([v_1, v_2](\ell = 0.6\text{m}, t)\) computed in the vicinity of the ultrasonic source at \(s_2\). Dots are the response affiliated with the unified observation points, while the solid lines linearly interpolate the data points to clarify the waveforms.

and 3 of the testing procedure (prior to and after fracturing the specimen). Then, at every time sample, four linear interpolation functions are constructed independently on \(G_1, \ldots, G_4\) making use of (temporally filtered) velocity data points of admissible signal-to-noise ratio i.e., noisy points are excluded from the interpolation. In this setting, the velocity distribution at a given time may be computed over the unified observation points via the indicated interpolants. The resulting waveforms are spatially smooth as shown by solid lines in Fig. 2(a). A unified observation grid enables arithmetic operations between datasets of distinct sensing steps 1 and 3, which is required for the computation of scattered field. Thus-obtained velocity signals are then transformed into displacement data through numerical integration. The latter process, however, introduces a low-frequency drift i.e., integration constant in the results, which is eliminated by a high-pass filter of cut-off frequency 500Hz. In this way, one finds the spatiotemporally smooth “total” displacement fields corresponding to \(u(\xi, t)\) in (2) over \(S_{\text{obs}}\) which calls for further processing since the “scattered” field \(v(\xi, t)\) will be needed for the reconstructions of section 5.

(2) synchronization of incidents and extraction of scattered fields. To calculate the scattered field in light of remark 3.3, this stage aims to synchronize the time, and balance the magnitude of ultrasonic incidents between Steps 1 and 3 of experiments. Discrepancies in transducer’s physical input at different sensing steps – although curtailed by the measures indicated in the remark, are inevitable due to (a) perturbation of transducer-specimen coupling in reattachments, and (b) recalibration of the 3D SLDV system for ultrasonic tests of Step 3 (after fracturing the specimen). To address this problem, let us consider the (processed) incident displacement fields \(u'\) (related to the intact specimen) in the vicinity of every ultrasonic source \(s_1, \ldots, s_8\). The support of which is a subset of: (a) \(G_1\) near \(s_1, s_8, \text{ and } s_7\), (b) \(G_2\) in the immediate vicinity of \(s_2\) and \(s_3\), and (c) \(G_3\) in a neighborhood of \(s_4, s_5, \text{ and } s_6\). Then, the “reference” physical incidents (transducer inputs) are identified as the first 80-100 samples of displacement time histories in the indicated neighborhoods of \(s_1, \ldots, s_8\). Note that within this timeframe i.e., \([0 0.15]\text{ms}\) to \([0 0.2]\text{ms}\) depending on the source location, there is no fingerprint on the measured waveforms due to internal scatterers. In this setting, the displacement fields from every ultrasonic experiment in Step 3 are uniformly scaled (by a constant value) and shifted in time (by a fixed amount) so that the transducer inputs in Step 3 matches their counterparts in Step 1 for every source location. This leads to consistent ultrasonic data for both sensing steps, and one may now proceed to compute the scattered displacement fields by subtracting the total fields from their associated incidents fields. Fig. 3 illustrates the resulting scattered field distribution in time and space when the transducer is at \(s_2\).
(3) spectral analysis. This stage computes the spectrum of scattered displacement signals obtained in stage (2). Prior to the application of discrete Fourier transform, the problem of “spectral leakage” \[24\] due to the transient nature of measured waveforms should be addressed. In this vein, the displacement data are temporally windowed \[25\] using a tapered cosine i.e. Tukey window of the form \[26\], the transient nature of measured waveforms should be addressed. In this vein, the displacement data are temporally windowed \[25\] using a tapered cosine i.e. Tukey window of the form \[26\],

\[
\begin{equation}
\mathbf{w}(t, c) = \begin{cases}
\frac{1}{2} \left[ 1 + \cos \left( \frac{2\pi}{cT}(t - cT/2) \right) \right], & 0 \leq t < \frac{cT}{2} \\
1, & \frac{cT}{2} \leq t < T - \frac{cT}{2} \\
\frac{1}{2} \left[ 1 + \cos \left( \frac{2\pi}{cT}(t - T + cT/2) \right) \right], & T - \frac{cT}{2} \leq t \leq T
\end{cases}
\end{equation}
\]

where \( T \) signifies the observation interval \([0 \, 1] \text{ms}\); \( t \) is the sampled time vector of length 512, and \( 0 \leq c \leq 1 \) is the ratio of cosine-tapered length to the entire window length. Fig. \( 3(\text{b}) \) shows the scattered displacements at \( s_2 \) after the application of Tukey window \( \mathbf{w}(t, 0.2) \). Now that the support of windowed time signals is compact, one may safely proceed to compute the spectrum of scattered displacement fields via the fast Fourier transform. The resulting waveforms in the frequency domain will be used for the reconstruction in section \( 5 \).

5. Data Inversion

With the preceding data, one may generate the (G)LSM indicator maps in three steps, namely by: (i) constructing the discrete scattering operator \( \mathbf{\Lambda} \), (ii) computing the trial signature patterns affiliated with \( 7 \), and (iii) evaluating the (G)LSM imaging functionals \( 12 \) and \( 22 \) through non-iterative minimization of their corresponding cost functionals \( 11 \) and \( 20 \). These steps are elucidated in the following.

5.1. The discrete scattering operator

With reference to Fig. \( 1(\text{b}) \), the incident surface \( S^{\text{inc}} \) is sampled at \( N_s = 8 \) source locations \( \mathbf{y}_j \in \{s_1, s_2, \ldots, s_8\} \), while the observation grid \( S^{\text{obs}} = \bigcup_{i=1}^{N_p} \mathcal{G}_i \) is comprised of \( N_p = 144 \) measurement points \( \mathbf{\xi}_i \). In this setting, the spectrum of (in-plane) waveform data at \( N_x = 10 \) frequencies, specifically at \( \omega_\ell = 27, 28, \ldots, 36 \text{kHz} \), are deployed to generate the multi-frequency scattering operator \( \mathbf{\Lambda} \) as a \( 2N_xN_\omega \times N_xN_\omega \) matrix of components

\[
\mathbf{\Lambda}(2N_x\ell + 2i + 1; 2N_x\ell + 2i + 2, N_x\ell + j + 1) = \begin{bmatrix} F(v_1) \\ F(v_2) \end{bmatrix} (\mathbf{\xi}_i, \mathbf{y}_j; \omega_\ell),
\]

(25)

for

\[
\begin{align*}
\ell = 0, \ldots N_\omega - 1, \\
\end{align*}
\]

(26)

On recalling \( 6 \), here, \( F(v_i)(\mathbf{\xi}_i, \mathbf{y}_j; \omega_\ell), \ell = 1, 2, \) is the \( \ell^{\text{th}} \) component of the Fourier transformed displacement at the observation point \( \mathbf{\xi}_i \) and frequency \( \omega_\ell \) when the ultrasonic source is located at \( \mathbf{y}_j \).

5.2. A physics-based library of trial patterns

Let the search volume \( S \) be a \( 29 \text{cm} \times 29 \text{cm} \) square in the middle of specimen probed by a uniform \( 100 \times 100 \) grid of sampling points \( \mathbf{x}_o \) where the featured (G)LSM indicator functionals \( 12 \) and \( 22 \) are evaluated. In addition, the unit circle is sampled by 16 trial normal directions \( \mathbf{n} = \mathbf{R}\mathbf{n}_o \) wherein \( \mathbf{n}_o = (1, 0) \). Based on this, a total of \( M = 10000 \times 16 \) trial dislocations \( \mathbf{L} = \mathbf{x}_o + \mathbf{R}\mathbf{L} \) are generated for the specified pairs \( (\mathbf{x}_o, \mathbf{n}) \). Here, \( \mathbf{L} \) is a vertical crack of length 3mm. For each \( (\mathbf{x}_o, \mathbf{n}) \), the scattering signatures \( \mathbf{v}^{x_o\cdot\mathbf{n}}(\mathbf{\xi}, \omega) \) are computed separately for every \( \omega \in \Omega := \{27, 28, \ldots, 36\} \text{kHz} \) over the observation grid \( \mathbf{\xi} \in S^{\text{obs}} \) by solving

\[
\begin{align*}
\nabla \cdot \mathbf{C} : \nabla \mathbf{v}^{x_o\cdot\mathbf{n}}(\mathbf{\xi}, \omega) + \rho \omega^2 \mathbf{v}^{x_o\cdot\mathbf{n}}(\mathbf{\xi}, \omega) &= 0, & (\mathbf{\xi} \in \mathcal{B} \setminus L, \omega \in \Omega) \\
\mathbf{n} \cdot \mathbf{C} : \nabla \mathbf{v}^{x_o\cdot\mathbf{n}}(\mathbf{\xi}, \omega) &= 0, & (\mathbf{\xi} \in \partial \mathcal{B} \setminus S, \omega \in \Omega) \\
\mathbf{v}^{x_o\cdot\mathbf{n}}(\mathbf{\xi}, \omega) &= 0, & (\mathbf{\xi} \in \partial \mathcal{B} \setminus \mathcal{R}, \omega \in \Omega) \\
\mathbf{n} \cdot \mathbf{C} : \nabla \mathbf{v}^{x_o\cdot\mathbf{n}} &= |\mathbf{L}|^{-1} \delta(\mathbf{\xi} - \mathbf{x}_o) \mathbf{n}, & (\mathbf{\xi} \in \mathcal{L}, \omega \in \Omega)
\end{align*}
\]

(27)
Here, $\mathcal{B}$ represents the granite specimen, and $S$ represents the 2-cm long contact areas at the bottom of the plate where the wood supports meet the sample as shown in Fig. 1 (a).

These simulations are performed in three dimensions for the 0.96m $\times$ 0.3m $\times$ 0.03m granite plate via an elastodynamics code rooted in the boundary element method \[27, 10\]. For data inversion, however, only the in-plane components of the computed scattered fields are used in the following form

$$\Phi_{x_{s},n}(2N_p \ell + 2i + 1:2N_p \ell + 2i + 2) = \begin{bmatrix} v_{x_{s},n}^1 \\ v_{x_{s},n}^2 \end{bmatrix} (\xi_i^1; \omega \ell), \quad i = 0, \ldots N_p - 1, \quad \ell = 0, \ldots N_\omega - 1, \quad (28)$$

where $\Phi_{x_{s},n}$ is a $2N_pN_\omega \times 1$ vector. In this setting, the scattering equation (29) may be discretized as

$$\Lambda \mathbf{g}_{x_{s},n} = \Phi_{x_{s},n}. \quad (29)$$

**Remark 5.1.** It is worth noting that $\Phi_{x_{s},n}$ is invariant with respect to $\Lambda$. Hence, for computational efficiency, one may generate a $2N_pN_\omega \times M$ matrix $\Phi$,

$$\Phi(2N_p \ell + 2i + 1:2N_p \ell + 2i + 2, m) = \begin{bmatrix} v_{(x_{s},n)m}^1 \\ v_{(x_{s},n)m}^2 \end{bmatrix} (\xi_i^1; \omega \ell), \quad i = 0, \ldots N_p - 1, \quad \ell = 0, \ldots N_\omega - 1,$n

as the right hand side of scattering equation (29) – encompassing all choices of trial pairs $(x_{s,n})_m$, $m = 1, 2, \ldots M$.

5.3. The (generalized) linear sampling indicators

The scattering equation (29) is generally ill-posed due to (a) nonlinear nature of the inverse problem, (b) limited excitation and sensing apertures, (c) local (e.g., interfacial) modes of wave motion whose signature may not be found on $S_{\text{obs}}$ \[13\], and (d) noise in data. Accordingly, (29) is primarily solved by regularization e.g., through minimizing a designated (Tikhonov or GLSM) cost functional, or via sparse sampling.

5.3.1. The classical linear sampling indicator

Following \[1\], the Tikhonov-regularized solution $\mathbf{g}_{x_{s},n}^T$ to (29) is computed by non-iteratively minimizing the LSM cost functional,

$$\mathbf{g}_{x_{s},n}^T := \text{argmin}_{\mathbf{g}_{x_{s},n}} \left\{ \| \Lambda \mathbf{g}_{x_{s},n} - \Phi_{x_{s},n} \|_{L^2}^2 + \eta_{x_{s},n} \| \mathbf{g}_{x_{s},n} \|_{L^2}^2 \right\}, \quad (30)$$

where the regularization parameter $\eta_{x_{s},n}$ is obtained by way of Morozov discrepancy principle \[15\]. On the basis of (30), the LSM indicator functional is constructed as

$$\Sigma_{T}(x_{s}) = \frac{1}{\| \mathbf{g}_{x_{s}}^T \|_{L^2}}, \quad \mathbf{g}_{x_{s}}^T := \text{argmin}_{\mathbf{g}_{x_{s},n}} \| \mathbf{g}_{x_{s},n} \|_{L^2}. \quad (31)$$

The subscript $\Sigma$ indicates that the Tikhonov regularization is deployed to compute the LSM imaging functional.

5.3.2. The generalized linear sampling indicator

In light of (29), the GLSM-regularized solution $\mathbf{g}_{x_{s},n}^G$ to (29) is obtained through solving the linear system

$$\left( \Lambda^* \Lambda + \gamma_{x_{s},n} \left( \Lambda^* \Lambda \right)^{1/2} \left( \Lambda \Lambda^* \right)^{1/2} + \delta \gamma_{x_{s},n} I_{N_s N_\omega \times N_s N_\omega} \right) \mathbf{g}_{x_{s},n}^G = \Lambda^* \Phi_{x_{s},n}, \quad (32)$$

where $(\cdot)^*$ is the Hermitian operator, $\delta = 0.15 \| \Lambda \|_{L^2}$ indicates the estimated magnitude of noise in data, and the regularization parameter

$$\gamma_{x_{s},n} = \frac{\eta_{x_{s},n}}{\| \Lambda \|_{L^2} + \delta}; \quad (33)$$

wherein $\eta_{x_{s},n}$ is as in (30). As a result, $\mathbf{g}_{x_{s},n}^G$ is a $N_s N_\omega \times 1$ vector (or $N_s N_\omega \times M$ matrix for all the constructed right hand sides) identifying the distribution of wavefront densities over $S_{\text{inc}}$. In this setting, the GLSM imaging functional is computed according to (22) as the following,
\[ \mathcal{G}(x_o) = \frac{1}{\sqrt{(g_{x_o}^\mathcal{G} (\Lambda^* \Lambda)^{-1} g_{x_o}^\mathcal{G}) + \delta \| g_{x_o}^\mathcal{G} \|^2_{L^2}}}, \quad g_{x_o}^\mathcal{G} := \text{argmin}_{g_{x_o,n}} \| g_{x_o,n} \|_{L^2}. \] (34)

5.3.3. The linear sampling via direct inversion

We observed that for \( N_s = 8 \), the operator \( \Lambda \) in (29) is directly invertible owing to sparse sampling of \( S^{inc} \). In this setting, one may also construct the LSM indicator from the directly inverted solution,

\[ \mathcal{L}(x_o) = \frac{1}{\| g_{x_o, n} \|_{L^2}}, \quad g_{x_o, n} := \text{argmin}_{g_{x_o,n}} \| g_{x_o,n} \|_{L^2}, \quad g_{x_o,n} = \Lambda^{-1} \Phi_{x_o,n}. \] (35)

A comparative study of the linear sampling indicators \( \mathcal{L} \) and \( \mathcal{L}_T \) is included in Section 6.

The (generalized) linear sampling functionals canvas the support of 3PB-induced damage by achieving their highest values at sampling points that meet the support of newborn fractures \( \Gamma \) (or micro-cracked process zones), while remaining near zero everywhere else within the sampling region \( S \setminus \Gamma \).

5.3.4. The thresholded indicators

On introducing

\[ \mathcal{I}(x_o) := \begin{cases} 1 & \text{if } \mathcal{I}(x_o) > \tau_{tol} \times \max(\mathcal{I}), \\ 0 & \text{otherwise} \end{cases}, \quad \mathcal{I} \in \{ \mathcal{L}_T, \mathcal{L}, \mathcal{G} \}, \quad \tau_{tol} \in [0, 1[ \],

the thresholded imaging functionals may be expressed as

\[ \tilde{\mathcal{I}}(x_o) := \mathcal{I}(x_o) \mathcal{I}(x_o), \quad \mathcal{I} \in \{ \mathcal{L}_T, \mathcal{L}, \mathcal{G} \}. \] (36)

6. Results and discussion

Following [28], the 3PB-induced damage is exposed by spraying acetone on the back of specimen in a neighborhood of the pre-manufactured notch. While evaporating, the acetone reveals the “true” support of \( \Gamma \) as illustrated in Fig. 4. The latter is then compared with the reconstructed fractures \( \Gamma_L \) and \( \Gamma_G \) obtained by the LSM and GLSM indicators, respectively, according to Fig. 5.

**Figure 4:** Verification: (\( \Gamma \)) 3PB-induced fracture traced by acetone in a neighborhood of the pre-manufactured notch – weak traces are indicated by the dashed line, (\( \Gamma_L \)) recovered support of \( \Gamma \) by way of the classical linear sampling indicator \( \mathcal{L} \), and (\( \Gamma_G \)) reconstructed damage via the generalized linear sampling indicator \( \mathcal{G} \). \( \Gamma_L \) and \( \Gamma_G \) are extracted from Fig. 5.

6.1. Full aperture reconstruction

The spectrum of scattered displacement data \( F(v)(\xi_i, \omega) \) measured at 144 observation points \( \xi_i \in S^{obs} = \bigcup_{i=1}^4 \mathcal{G}_{i}, \ i = 0, \ldots, 143, \) for ten frequencies \( \omega_i = 27, 28, \ldots, 36 \)kHz, and eight source locations on \( S^{inc} = \{s_1, s_2, \ldots, s_8\} \) are deployed to compute the (G)LSM imaging functionals \( \mathcal{L}_T, \mathcal{L}, \) and \( \mathcal{G} \) according to (31), (35), and (34), respectively. Recall that the sampling region is a 29cm \( \times \) 29cm square in the middle of specimen. The resulting distributions are shown in Fig. 5. As mentioned earlier, the (G)LSM imaging functionals assume
their highest values in the vicinity of hidden scatterers $\Gamma$. It is worth mentioning that the caustics featured in the reconstructed maps of Fig. 5 are mostly governed by (i) illuminating wavelength, (ii) geometric symmetries of the domain, (iii) arrangement of sources and receivers, and (iv) mathematical properties of the associated cost functionals. Their intensity typically decreases when the source and measurement aperture along with the number of sources and receivers increase. An in-depth analysis of such focal regions for a related indicator known as the topological sensitivity is provided in [11].

Fig. 5 also includes the 60% thresholded maps $\tilde{L}_T$, $\tilde{L}_L$, and $\tilde{G}$ furnishing the support of sampling points $x_{o}$ that satisfy $I(x_{o}) > 0.6 \times \max(I)$, $I \in \{L_T, L, G\}$, according to (36). These results are used to approximate the support of damage $\Gamma_L$ and $\Gamma_G$ by the midline through the thresholded damage zone as shown in the figure. It is instructive to compare $\Gamma_L$ and $\Gamma_G$ with the “true” fracture boundary $\Gamma$ from Fig. 4 – also included as an inset in Fig. 5. Observe that both LSM and GLSM reconstructions indicate that the damage zone has advanced slightly further in the specimen compared to $\Gamma$. This may be justified by noting that acetone – used to recover $\Gamma$, detects only the sufficiently penetrable interfaces which may not include the tight contacts in the near tip region.

A comparative analysis of Fig. 5 indicates that the LSM functionals $L_T$ and $L$ result in quite similar reconstructions. In light of $L_T$ and $L$, however, observe that when the scattering operator $\mathbf{A}$ is invertible – here, thanks to the sparse sampling of $S^{inc}$, the direct-inversion-based operator $L$ leads to a “cleaner” reconstruction. In other words, the Tikhonov regularization, owing to its approximate nature, may intensify the caustics giving rise to a “noisy” $L_T$ reconstruction. Henceforth, we focus on the LSM maps constructed via direct inversion. The GLSM indicator $G$, on the other hand, successfully recovers the entire damage zone with a sharp localization in a neighborhood of $\Gamma$ and remarkably diminished reconstruction artifacts. This may be attributed to: (a) rigorous nature of the GLSM imaging functional which does not involve approximations underlying the LSM indicator, and (b) strong convexity of the GLSM cost functional (20), see e.g., [5, Theorem 4.3].

Figure 5: (Generalized) linear sampling indicators: (a) LSM indicator $L_T$ (31) computed via the Tikhonov regularization in the sampling region – a 29cm $\times$ 29cm square in the middle of specimen, and the associated thresholded indicator $\tilde{L}_T$ (36) with $\tau_{tol} = 0.6$, (b) LSM map $L$ (35) obtained via direct inversion and the corresponding $\tilde{L}$ thresholded at 60%, and (c) GLSM indicator map $G$ (34) and the affiliated $\tilde{G}$ similarly truncated at 60%. The inset shows the “ground-truth” support of $\Gamma$ from Fig 4. Here, full ultrasonic data is deployed for the reconstruction according to Fig. 4(b) where $S^{inc} = \{s_1, s_2, \ldots, s_8\}$ and $S^{obs} = \bigcup_{i=1}^{4} \mathcal{G}_i$ involving 144 measurement points.
Figure 6: LSM (top) versus GLSM (bottom) indicator maps computed from reduced data where $S_{\text{inc}} = \{s_1, s_2, \ldots, s_8\}$, while $S_{\text{obs}} = \bigcup_{i=1}^{3} \mathcal{G}_i$ is uniformly downsampled by a factor of: (a) three (corresponding to $N_p = 48$ measurement points), (b) five ($N_p = 28$), (c) seven ($N_p = 20$), and (d) nine ($N_p = 16$).

6.2. Reconstruction from reduced data

To examine the performance of (G)LSM indicators with sparse data, the measurement points on $S_{\text{obs}}$ are uniformly downsampled by a factor of $\beta \in \{3, 5, 7, 9\}$, so that a respective set of $N_p \in \{48, 28, 20, 16\}$ data points are used for the reconstruction – compared to $N_p = 144$ in Fig. 5. The resulting $\mathcal{L}$ and $\mathcal{G}$ distributions are shown in Fig. 6 for all $\beta$. Observe that while the GLSM indicator remains robust against downsampling, owing to its rigorous nature, the LSM indicator fails to retrieve the damage zone from sparse data, especially when $N_p \leq 20$. This is more evident in the 60% thresholded maps $\tilde{\mathcal{L}}$ and $\tilde{\mathcal{G}}$ shown in Fig. 7. Note that as the number of data points $N_p$ decreases, (a) caustics and reconstruction artifacts intensify in both maps which is rather expected in light of [11], and (b) image resolution decreases in the GLSM maps.

Partial source and “viewing” aperture

It is common in practice that a specimen is inaccessible from one side or, to the contrary, is only accessible from one side for ultrasonic testing. Imaging in such configurations are investigated in Fig. 8. In the top row, the specimen is assumed inaccessible from below for both excitation and measurement, and thus, the reconstruction is performed using data on three sides of the boundary $S_{\text{obs}} = \bigcup_{i=1}^{3} \mathcal{G}_i$ involving 99 measurement points for six source locations – i.e., $S_{\text{inc}} = \{s_1, s_2, s_3, s_4, s_5, s_8\}$. The LSM and GLSM indicators are able to recover most of the damage support. However, the GLSM functional appear to be more robust with less pronounced artifacts. In the bottom row, the specimen is presumed to be merely accessible from the top for ultrasonic illumination and sensing. In this setting, $\mathcal{L}$ and $\mathcal{G}$ are computed using limited data involving four ultrasonic sources on top $S_{\text{inc}} = \{s_1, s_2, s_3, s_4\}$, and 45 measurement points on $S_{\text{obs}} = \mathcal{G}_2$. In this case, the GLSM map successfully recovers the damage zone, while the LSM distribution canvases only a subset of the fracture support.

7. Conclusions

An experimental and data analysis framework is developed for in-situ waveform tomography of damage in elastic components. To this end, we take advantage of the recently established generalized linear sampling indicator for non-iterative, full-waveform reconstruction of a mode I fracture, induced via three-point bending, in a granite specimen using boundary observations of scattered ultrasonic waveforms. In this vein, transient
Figure 7: Thresholded indicator maps $\tilde{L}$ (top) and $\tilde{G}$ (bottom) associated with the LSM and GLSM distributions of Fig. 6. The number of (downsampled) measurement points $N_p$ is specified for every column (a)-(d). The insets in (a) are from Fig. 4, providing the “ground-truth” for the 3PB-induced fracture $\Gamma$. With reference to (36), the threshold in all cases is $\tau_{tol} = 0.6$.

Figure 8: Partial-aperture tomography: LSM $L$ (middle) and GLSM $G$ (right) indicator maps computed using limited data involving: (a) six ultrasonic sources on $S^{inc} = \{s_1, s_2, s_3, s_4, s_5, s_8\}$ and 99 measurement points on $S^{obs} = \bigcup_{i=1}^{3} G_i$ as shown in the top left panel, and (b) four sources on $S^{inc} = \{s_1, s_2, s_3, s_4\}$, and 45 points on $S^{obs} = G_2$ as depicted in the bottom left panel.

waves ranging from 20 to 40kHz are induced in the sample, and thus generated velocity responses are monitored by a 3D scanning laser Doppler vibrometer over the domain’s external boundary, which upon suitable signal processing furnish the spectra of scattered displacement fields over the designated scanning grid. Such sensory data are then deployed to compute the GLSM maps along with the classical LSM indicators for a comparative analysis. The results are verified against in-situ observations and shown to be successful in recovering the damage support. The GLSM, however, leads to a sharper localization and remarkably cleaner maps – with less-pronounced reconstruction artifacts. It is further demonstrated that the GLSM remains robust with reduced i.e., spatially downsampled data, as well as partial-aperture data e.g., when access to specimen for excitation and sensing is limited. In this study, the data inversion procedure is adapted for a multifrequency reconstruction. Given the transient nature of data, it would be interesting to extend the theory for a direct
implementation of this approach in the time domain. In this setting, a broadband dataset opens the door toward an in-depth analysis of multi-scale fracture networks in a damage zone.

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References

[1] F. Pourahmadian, B. B. Guzina, H. Haddar, Generalized linear sampling method for elastic-wave sensing of heterogeneous fractures, Inverse Problems 33 (5) (2017) 055007.

[2] F. Cakoni, D. Colton, The linear sampling method for cracks, Inverse Problems 19 (2003) 279–295.

[3] F. Cakoni, D. Colton, H. Haddar, Inverse Scattering Theory and Transmission Eigenvalues, SIAM, 2016.

[4] M. Bonnet, F. Cakoni, Analysis of topological derivative as a tool for qualitative identification, Inverse Problems 35 (10) (2019) 104007.

[5] F. Pourahmadian, H. Haddar, Differential tomography of micromechanical evolution in elastic materials of unknown micro/macrostructure, SIAM Journal on Imaging Sciences 13 (3) (2020) 1302–1330.

[6] R. Kress, Inverse scattering from an open arc, Math. Methods Appl. Sci. 18 (1995) 267–293.

[7] A. Kirsch, N. Grinberg, The Factorization Method for Inverse Problems, Oxford University Press, Oxford, 2008.

[8] Y. Boukari, H. Haddar, The factorization method applied to cracks with impedance boundary conditions, Inverse Probl Imag 7 (2013) 1123–1138.

[9] W. K. Park, Music-type imaging of small perfectly conducting cracks with an unknown frequency, J Phys Conf Ser 633 (1) (2015) 012005.

[10] F. Pourahmadian, B. B. Guzina, On the elastic-wave imaging and characterization of fractures with specific stiffness, Int. J Solids Struct. 71 (2015) 126–140.

[11] B. B. Guzina, F. Pourahmadian, Why the high-frequency inverse scattering by topological sensitivity may work, Proc. R. Soc. A 471 (2015) 20150187.

[12] L. Audibert, H. Haddar, The generalized linear sampling method for limited aperture measurements, SIAM Journal on Imaging Sciences 10 (2) (2017) 845–870.

[13] F. Pourahmadian, B. B. Guzina, H. Haddar, A synoptic approach to the seismic sensing of heterogeneous fractures: From geometric reconstruction to interfacial characterization, Computer Methods in Applied Mechanics and Engineering 324 (2017) 395 – 412.

[14] V. Baronian, L. Bourgeois, A. Recoquillay, Imaging an acoustic waveguide from surface data in the time domain, Wave Motion 66 (2016) 68–87.
[15] V. Baronian, L. Bourgeois, B. Chapuis, A. Recoquillay, Linear sampling method applied to non destructive testing of an elastic waveguide: theory, numerics and experiments, Inverse Problems 34 (7) (2018) 075006.

[16] F. B. Hassen, Y. Boukari, H. Haddar, Application of the linear sampling method to identify cracks with impedance boundary conditions, Inverse Probl. Sci. Eng. 21 (2013) 210 – 234.

[17] L. Audibert, H. Haddar, A generalized formulation of the linear sampling method with exact characterization of targets in terms of farfield measurements, Inverse Problems 30 (2014) 035011.

[18] T.-P. Nguyen, B. B. Guzina, Generalized linear sampling method for the inverse elastic scattering of fractures in finite bodies, Inverse Problems 35 (10) (2019) 104002.

[19] R. Kress, Linear integral equation, Springer, Berlin, 1999.

[20] Polytec, Inc., Basic Principles of Vibrometry, https://www.polytec.com/us/vibrometry/technology/ (2020, accessed 09/10/20).

[21] L. E. Malvern, Introduction to the Mechanics of a Continuous Medium, Prentice-Hall, Englewood Cliffs, 1969.

[22] H. Lamb, On waves in an elastic plate, Proc. R. Soc. A 93 (1917) 114 – 128.

[23] F. Pourahmadian, B. B. Guzina, On the elastic anatomy of heterogeneous fractures in rock, Int J Rock Mech Min 106 (2018) 259–268.

[24] D. J. Ewins, Modal testing: theory and practice, Research studies press, Letchworth, 1984.

[25] A. V. Oppenheim, R. W. Schafer, J. R. Buck, Discrete-time signal processing, Prentice Hall, Upper Saddle River, NJ, 1999.

[26] P. Bloomfield, Fourier analysis of time series: an introduction, John Wiley & Sons, 2004.

[27] M. Bonnet, Boundary Integral Equation Methods for Solids and Fluids, Wiley, 1999.

[28] F. Pourahmadian, Experimental validation of differential evolution indicators for ultrasonic waveform tomography, Mechanical Systems and Signal Processing, submitted: arXiv:2010.01813 (2020).