DIRICHLET DUALITY AND
THE NONLINEAR DIRICHLET PROBLEM
ON RIEMANNIAN MANIFOLDS

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ABSTRACT

In this paper we study the Dirichlet problem for fully nonlinear second-order equations on a riemannian manifold. As in our previous paper [HL4] we define equations via closed subsets of the 2-jet bundle where each equation has a natural dual equation. Basic existence and uniqueness theorems are established in a wide variety of settings. However, the emphasis is on starting with a constant coefficient equation as a model, which then universally determines an equation on every riemannian manifold which is equipped with a topological reduction of the structure group to the invariance group of the model. For example, this covers all branches of the homogeneous complex Monge-Ampère equation on an almost complex hermitian manifold $X$.

In general, for an equation $F$ on a manifold $X$ and a smooth domain $\Omega \subset X$, we give geometric conditions which imply that the Dirichlet problem on $\Omega$ is uniquely solvable for all continuous boundary functions. We begin by introducing a weakened form of comparison which has the advantage that local implies global. We then associate to $F$ two natural “conical subequations”: a monotonicity subequation $M$ for $F$ and the asymptotic interior of $F$. If $X$ carries a global $M$-subharmonic function, then weak comparison implies full comparison. The asymptotic interior of $F$ is used to formulate boundary convexity and provide barriers. In combination the Dirichlet problem becomes uniquely solvable as claimed.

A considerable portion of the paper is concerned with specific examples. They include a wide variety of equations which make sense on any riemannian manifold, and many which hold universally on almost complex or quaternionic hermitian manifolds, or topologically calibrated manifolds.

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# TABLE OF CONTENTS

1. Introduction.
2. $F$-Subharmonic Functions.
3. Dirichlet Duality and the Notion of a Subequation.
4. Subequations Locally Equivalent to Constant Coefficient Subequations.
5. The Riemannian Hessian – A Canonical Splitting of $J^2(X)$.
6. Universal Subequations on Manifolds with Topological $G$-Structure.
7. Strictly $F$-Subharmonic Functions.
8. Comparison Theory – Local to Global.
9. Strict Approximation and Monotonicity Subequations.
10. A Comparison Theorem for $G$-Universal Subequations.
11. Strictly $F$-Convex Boundaries and Barriers.
12. The Dirichlet Problem – Existence.
13. The Dirichlet Problem – Summary Results.
14. Universal Riemannian Subequations.
15. The Complex and Quaternionic Hessians.
16. Geometrically Defined Subequations – Examples.

**Appendices:**

A. Equivalent Definitions of $F$-Subharmonic.
B. Elementary Properties of $F$-Subharmonic Functions.
C. The Theorem on Sums.
D. Some Important Counterexamples.
1. Introduction

In a recent article [HL4] the authors studied the Dirichlet problem for fully nonlinear equations of the form $F(Hess\ u) = 0$ on smoothly bounded domains in $\mathbb{R}^n$. Our approach employed a duality which enabled us to geometrically characterize domains for which one has existence and uniqueness of solutions for all continuous boundary data. These results covered, for example, all branches of the real, complex and quaternionic Monge-Ampère equations, and all branches of the special Lagrangian potential equation.

Here we shall extend these results in several ways. First, all results in [HL4] are shown to carry over to riemannian manifolds with an appropriate topological reduction of the structure group. For example, we treat the complex Monge-Ampère equation on almost complex manifolds with hermitian metric. Second, the results are extended to equations involving the full 2-jet of functions. There still remains a basic notion of duality, and a geometric form of $F$-boundary convexity. Existence and uniqueness theorems are established, and a large number of examples are examined in detail.

In [HL4] our approach was to replace the function $F(Hess\ u)$ by a closed subset $F \subset \text{Sym}^2(\mathbb{R}^n)$ of the symmetric $n \times n$ matrices subject only to the positivity condition

$$F + \mathcal{P} \subset F$$

(1.1)

where $\mathcal{P} \equiv \{ A \in \text{Sym}^2(\mathbb{R}^n) : A \geq 0 \}$ (cf. [Kr]). Such an $F$ was called a Dirichlet set in [HL4] but will be called a subequation here. A $C^2$-function $u$ on a domain $\Omega$ is $F$-subharmonic if $Hess_x u \in F$ for all $x \in \Omega$, and it is $F$-harmonic if $Hess_x u \in \partial F$ for all $x$. (The reader might note the usefulness of this approach in treating other branches of the equation $\det(Hess\ u) = 0$.) The important point is to extend these notions to upper semi-continuous $[-\infty, \infty)$-valued functions. This was done [HL4] by using the dual subequation

$$\tilde{F} \equiv -\sim(F) = \sim(-F).$$

(1.2)

The class of subaffine functions, or $\tilde{\mathcal{P}}$-subharmonic functions, plays a key role since (1.1) is equivalent to $F + \tilde{F} \subset \tilde{\mathcal{P}}$.

Each subequation $F$ has an associated asymptotic interior $\overline{F}$, and using this we introduced a notion of strictly $\overline{F}$-convex hypersurfaces. For the $q$th branch of the complex Monge-Ampère equation for example, this is just classical $q$-pseudo-convexity. It is then shown in [HL4] that for any domain $\Omega$ whose boundary is strictly $\overline{F}$- and $\overline{\tilde{F}}$-convex, solutions to the Dirichlet Problem exist for all continuous boundary data. Furthermore, uniqueness holds on arbitrary domains $\Omega$.

In this paper the results from [HL4] will be generalized in several ways. To begin we shall work on a general manifold $X$ and consider subequations given by closed subsets

$$F \subset J^2(X)$$

of the 2-jet bundle which satisfy three conditions. The first is the positivity condition

$$F + \mathcal{P} \subset \mathcal{P}$$

(P)

where $\mathcal{P}_x$ is the set of 2-jets of non-negative functions with critical value zero at $x$. The second is the negativity condition

$$F + \mathcal{N} \subset F,$$

(N)
where $\mathcal{N}$ is the jets of non-positive constant functions. The third is a mild topological condition, which is satisfied in all interesting cases. (See §2.) We always assume our subequations $F$ satisfy these conditions. For the pure second-order constant coefficient subequations considered in [HL$_4$] the positivity conditions (P) and (1.1) are equivalent and automatically imply the other two conditions (N) and (T).

Here the dual subequation $\tilde{F}$ is defined by (1.2) exactly as in [HL$_4$]. While duality continues to be important, it is not used to define $F$-subharmonic and $F$-harmonic functions in the upper semi-continuous setting. This is because subaffine (i.e., co-convex) functions do not satisfy the maximum principle on general riemannian manifolds. So we use instead the equivalent [HL$_4$, Remark 4.9] viscosity definition (cf. [CIL]). The $F$-subharmonic functions have most of the important properties of classical subharmonic functions, such as closure under taking maxima, decreasing limits, uniform limits, and upper envelopes. This is discussed in Section 2 and Appendices A and B.

The notion of strict $\tilde{F}$-convexity of an oriented hypersurface generalizes to this context. This is discussed below in the introduction and treated in Section 11.

An important class of subequations treated in this paper are those constructed on riemannian manifolds, with an appropriate reduction of structure group, from constant coefficient equations on $\mathbb{R}^n$. It is a basic fact that on riemannian manifolds $X$, there is a canonical bundle splitting

$$J^2(X) = \mathbb{R} \oplus T^*X \oplus \text{Sym}^2(T^*X)$$

given by the riemannian hessian. This enables us for example to carry over any $O_n$-invariant constant coefficient subequation $F \subset \text{Sym}^2(\mathbb{R}^n)$ (the so-called Hessian equations in [CNS$_2$]) to all riemannian manifolds. That is, any purely second order subequation on $\mathbb{R}^n$ which depends only on the eigenvalues of the hessian carries over to general riemannian manifolds.

Much more generally however, consider a constant coefficient subequation, i.e., a subset

$$F \subset \mathbb{R} \oplus \mathbb{R}^n \oplus \text{Sym}^2(\mathbb{R}^n)$$

satisfying (P), (N) and (T), and its compact invariance group

$$G \equiv \{ g \in O_n : g(F) = F \}.$$ 

Suppose $X$ is a riemannian manifold with a topological $G$-structure. This means that $X$ is provided with an open covering $\{U_\alpha\}_\alpha$ and an orthogonal tangent frame field $e^\alpha = (e_1^\alpha, ..., e_n^\alpha)$ on each $U_\alpha$ so that each change of framing $G_{\alpha\beta} : U_\alpha \cap U_\beta \to G \subset O_n$ is valued in the subgroup $G$. Then from the splitting (1.3), the subequation $F$ can be transplanted to a globally defined subequation $F$ on $X$. Such subequations are called $G$-universal and make sense on any riemannian manifold with a topological $G$-structure.

An important new ingredient is the concept of a convex monotonicity cone for $F$ (Definition 9.4). This is a subset $M \subset J^2(X)$ which is a convex cone (with vertex 0) at each point and satisfies

$$F + M \subset F.$$ 

The subequation $P$ defined by requiring that the riemannian hessian be $\geq 0$ is a monotonicity cone for all purely second-order subequations. The corresponding $P$-subharmonic functions are just the riemannian convex functions. (Note that these exist globally on $\mathbb{R}^n$ and, in fact, on any complete simply-connected manifold of non-negative sectional curvature.) Similarly, on an almost complex hermitian manifold $X$ the subequation $P^\mathbb{C}$, defined by requiring the hermitian symmetric part of Hess $u$ to be $\geq 0$, is a monotonicity cone for any $U_n$-universal subequation depending only
on the hermitian symmetric part. The corresponding $P^C$-subharmonic functions are hermitian plurisubharmonic functions on $X$.

Our first main result is the following.

**Theorem 13.1.** Suppose $F$ is a $G$-universal subequation on a riemannian manifold $X$ provided with a topological $G$-structure as above. Suppose there exists a $C^2$ strictly $M$-subharmonic function on $X$ where $M$ is a monotonicity cone for $F$.

Then for every domain $\Omega \subset \subset X$ whose boundary is strictly $\overline{\Omega}$- and $\overline{\omega}$-convex, both existence and uniqueness hold for the Dirichlet problem. That is, for every $\varphi \in C(\partial \Omega)$ there exists a unique $F$-harmonic function $u \in C(\overline{\Omega})$ with $u|_{\partial \Omega} = \varphi$.

The simplest case of this, where $F$ is a constant coefficient equation in euclidean $\mathbb{R}^n$ (and $G = \{1\}$), already generalizes the main result in [HL]. Here comparison holds for any subequation which is gradient independent since the squared distance to a point can be used to construct the required $M$-subharmonic function. (See Theorem 13.4.) A similar comment holds for any complete manifold $X$ of non-positive sectional curvature.

In general, requiring the existence of a strictly $M$-subharmonic function (or something similar) is an intuitively necessary global hypothesis; for example in the case where $G = \text{SO}_n$, one is free to arbitrarily change the riemannian geometry inside a domain $\Omega$ while preserving the $F$-convexity of $\partial \Omega$. However, even for quite regular metrics – domains in $S^3 \times S^3$ – examples in Appendix D show that uniqueness fails without this hypothesis.

Theorem 13.1 applies in a quite broad context. We point out some examples here. They will be discussed in detail in the latter sections of the paper.

**Example A. (Parallelizable manifolds).** Even when $F$ admits absolutely no symmetries, i.e. $G = \{1\}$, the theorem has broad applicability. Suppose $X$ is a riemannian manifold which is parallelizable, that is, on which there exist global vector fields $V_1, ..., V_n$ which form a basis of $T_xX$ at every point. Applying the Schwartz process we can find global fields $e_1, ..., e_n$ which are pointwise orthonormal everywhere. The single open set $U = X$ with $e = (e_1, ..., e_n)$ is a $G = \{1\}$-structure, and every constant coefficient subequation $F$ in $\mathbb{R}^n$ can be carried over to a subequation $F$ on $X$.

Note that many manifolds are indeed parallelizable. For example, all orientable 3-manifolds have this property.

**Example B. (General riemannian manifolds).** Any constant coefficient subequation $F \subset \mathbb{R} \oplus \mathbb{R}^n \oplus \text{Sym}^2(\mathbb{R}^n)$ which is invariant under the action of $O_n$ carries over to all riemannian manifolds. There are, for example, many invariant functions of the riemannian hessian which yield universal “purely second-order” equations. Associated to each $A \in \text{Sym}^2(\mathbb{R}^n)$ is its set of ordered eigenvalues $\lambda_1 \leq \cdots \leq \lambda_n$. These eigenvalues have the property that $\lambda_q(A + P) \geq \lambda_q(A)$ for any $P \in \mathcal{P}$. Thus for example, the set $\mathbf{P}_q = \{A : \lambda_q(A) \geq 0\}$ gives a subequation on any riemannian manifold. This subequation $\mathbf{P}_q$ is the $q$th branch of the Monge-Ampère equation.

There are many other equations of this type. In fact, let $\Lambda \subset \mathbb{R}^n$ be any subset which is invariant under permutations of the coordinates and satisfies the positivity condition: $\Lambda + \mathbb{R}_+^n \subset \Lambda$ where $\mathbb{R}_+ = \{t \geq 0\}$. Then this set, considered as a relation on the eigenvalues of the riemannian hessian, determines a universal subequation on every riemannian manifold.

A classical case is $F(\sigma_k) = \{\lambda : \sigma_1(\lambda) \geq 0, ..., \sigma_k(\lambda) \geq 0\}$ (the principal branch of $\sigma_k(\lambda) = 0$ where $\sigma_k$ denotes the $k$th elementary symmetric function). One can compute that the convexity of $\partial \Omega$, for this equation and its dual, corresponds to the $F(\sigma_{k-1})$-convexity of its second fundamental form. For domains in $\mathbb{R}^n$ this result was proved in [CNS2]. It is generalized here in two ways. We establish the result for all branches of the equation and we carry it over to general riemannian manifolds. (See Theorem 14.4.)
We are also able to treat inhomogeneous subequations such as \( \text{Hess} \, u \geq 0 \) and \( \det \text{Hess} \, u \geq f(x) \) for a positive function \( f \) on \( X \) (and the analogous subequations for the other \( \sigma_k \) as above). This follows from Theorem 10.1 on local affine equivalence.

**Example C. (Almost complex hermitian manifolds).** Consider \( C^n = (R^{2n}, J) \) where \( J : R^{2n} \to R^{2n} \) represents multiplication by \( \sqrt{-1} \). To any \( A \in \text{Sym}^2(R^{2n}) \) one can associate the hermitian symmetric part \( A^G = \frac{1}{2}(A - JAJ) \) which has \( n \) real eigenvalues \( \lambda_1 \leq \cdots \leq \lambda_n \) occurring with multiplicity 2 on \( n \) complex lines. The entire discussion in Example B now applies. Any permutation-invariant subset \( \Lambda \subset R^n \) satisfying the \( R^o_\Lambda \)-positivity condition \( \Lambda + R^o_\Lambda \subset \Lambda \) gives a natural subequation on any manifold with \( U_n \)-structure, i.e., any almost complex manifold with a compatible riemannian metric. This includes for example all branches of the complex Monge-Ampère equation. We note that almost complex hermitian manifolds play an important role in modern symplectic geometry.

One can also treat the Dirichlet problem for Calabi-Yau-type equations in this context of almost complex manifolds. See Example 4.10 for example.

One can also consider \( U_n \)-invariant functions of the skew-hermitian part \( A^{sk} = \frac{1}{2}(A + JAJ) \). An important case of this is the Lagrangian equation discussed below.

**Example D. (Almost quaternionic hermitian manifolds).** A discussion parallel to that in Example C holds with the complex numbers \( C \) replaced by the quaternions \( H = (R^{4n}, I, J, K) \). In particular Theorem 13.1 applies to the Dirichlet problem for all branches of the quaternionic Monge-Ampère equation on almost quaternionic hermitian manifolds.

**Example E. (Grassmann structures).** Fix any closed subset \( G \subset G(p, R^n) \) of the Grassmannian of \( p \)-planes in \( R^n \), and consider the subequation \( F(G) \) defined for \( (r,p,A) \in R \oplus R^n \oplus \text{Sym}^2(R^n) \) by the condition

\[
\text{tr}_\xi A \equiv \text{tr}(A|_\xi) \geq 0 \quad \text{for all } \xi \in G.
\]

Here the \( F(G) \)-subharmonic functions are more appropriately called \( G \)-plurisubharmonic. Now \( F(G) \) carries over to a subequation on any riemannian manifold with \( G \)-structure where

\[
G \equiv \{ g \in O_n : g(G) = G \}.
\]

For example, if \( G = G(p, R^n) \), then \( G = O_n \) and the corresponding subequation, which makes sense on any riemannian manifold, states that the sum of any \( p \) eigenvalues of \( \text{Hess} \, u \) must be \( \geq 0 \). This is called geometric \( p \)-convexity or \( p \)-plurisubharmonicity.

A particularly interesting example is given by the \( U_n \)-invariant set \( \text{LAG} \subset G(n, C^n) \) of Lagrangian subspaces of \( C^n = (R^{2n}, J) \). On any almost complex hermitian manifold, this gives rise to a notion of Lagrangian subharmonic and Lagrangian harmonic functions with an associated Dirichlet problem which is solvable on Lagrangian convex domains. This is discussed in more detail below.

Another rich set of examples comes from almost calibrated manifolds such as manifolds with topological \( G_2 \) and \( \text{Spin}_7 \) structures. These are also discussed at the end of the introduction.

For all such structures Theorem 13.1 gives the following general result. Call a domain \( \Omega \subset X \) strictly \( G \)-convex if it has a strictly \( G \)-psh defining function. This holds if \( \partial \Omega \) is strictly \( G \)-convex (cf. (1.4) below) and there exists \( f \in C^2(\Omega) \) which is strictly \( G \)-psh.

**THEOREM 16.1.** Let \( X \) be a riemannian manifold with topological \( G \)-structure so that the \( G \)-universal subequation \( F(G) \) is defined on \( X \). Then on any strictly \( G \)-convex domain \( \Omega \subset X \)
the Dirichlet problem for $F(G)$-harmonic functions is uniquely solvable for all continuous boundary data.

Equivalence of Subequations

Theorem 13.1 actually holds for a much broader class of subequations, namely those which are locally affinely equivalent to constant coefficient subequations. The notion of (ordinary) equivalence is expressed in terms of automorphisms of the 2-jet bundle. An (linear) automorphism of $J^2(X)$ is a smooth bundle isomorphism $\Phi : J^2(X) \to J^2(X)$ which has certain natural properties with respect to the short exact sequence $0 \to \text{Sym}^2(T^*X) \to J^2(X) \to J^1(X) \to 0$. (See Definition 4.1 for details). The automorphisms form a group, and two subequations $F,F' \subset J^2(X)$ are called equivalent if there exists an automorphism $\Phi$ with $\Phi(F) = F'$.

A subequation on a general manifold $X$ is said to be locally equivalent to a constant coefficient subequation if each point $x$ has a coordinate neighborhood $U$ so that $F|_U$ is equivalent to a constant coefficient subequation $U \times F$ in those coordinates.

Any $G$-universal subequation on a riemannian manifold with topological $G$-structure is locally equivalent to a constant coefficient subequation. Theorem 13.1 remains true if one replaces the “$G$-universal assumption” with the assumption that $F$ is locally equivalent to a constant coefficient subequation. This is a strictly broader class of equations. For example, a generic subequation defined by a smoothly varying linear inequality in the fibres of $J^2(X)$ is locally equivalent to constant coefficients.

The notion of equivalence can be further substantially broadened by using the affine automorphism group. This is the fibrewise extension of the linear automorphisms by the full group of translations in the fibres $J^2_x(X)$. Theorem 13.1 holds for subequations which are only locally affinely equivalent to $G$-universal subequations. This allows one to treat inhomogeneous equations with variable right hand side. (See Example 4.10.)

Note. The notion of equivalence is a quite weak relation. An equivalence of subequations $\Phi : F \to F'$ does not induce a correspondence between $F$-subharmonic functions and $F'$-subharmonic functions. In fact, for a $C^2$-function $u$, $\Phi(J^2u)$ is almost never the 2-jet of a function.

Comparison Results

This paper contains a number of other possibly quite useful existence and uniqueness results which lead to Theorem 13.1 above.

A central concept in this subject is that of comparison for upper semi-continuous functions, which is treated in Section 8.

Definition. We say that comparison holds for $F$ on $X$ if for all compact subsets $K \subset X$ and functions

$$u \in F(K) \quad \text{and} \quad v \in \tilde{F}(K),$$

the Zero Maximum Principle holds for $u + v$ on $K$, that is,

$$u + v \leq 0 \quad \text{on} \quad \partial K \quad \Rightarrow \quad u + v \leq 0 \quad \text{on} \quad K. \quad (ZMP)$$

If comparison holds for $F$ on $X$, then one easily deduces the uniqueness of solutions to the Dirichlet problem for $F$-harmonic functions on every compact subdomain.
Obviously comparison for small compact sets $K$ does not imply that comparison holds for arbitrary compact sets. However, this is true for a weakened form of comparison involving a notion of strictly $F$-subharmonic functions.

Consider a subequation $F$ on a riemannian manifold $X$. For each constant $c > 0$ we define $F^c \subset F$ to be the subequation whose fibre at $x$ is

$$F^c_x \equiv \{ J \in F_x : \text{dist}(J, \sim F) \geq c \}$$

where “dist” denotes distance in the fibre $J_x(X)$. We define an upper semi-continuous function $u$ on $X$ is to be strictly $F$-subharmonic if for each point $x \in X$ there is a neighborhood $B$ of $x$ and a $c > 0$ such that $u$ is $F^c$-subharmonic on $B$.

**Definition.** We say that weak comparison holds for $F$ on $X$ if for all compact subsets $K \subset X$ and functions

$$u \in F^c(K) \text{ (some } c > 0) \quad \text{and} \quad v \in \tilde{F}(K),$$

the Zero Maximum Principle holds for $u + v$ on $K$. We say that local weak comparison holds for $F$ on $X$ if every $x \in X$ has some neighborhood on which weak comparison holds.

It is proved (see Theorem 8.3) that:

*Local weak comparison implies weak comparison.*

Then in Section 10 we prove that:

*Local weak comparison holds for any subequation which is locally affinely equivalent to a constant coefficient subequation.*

Taken together we have that global weak comparison holds for all such subequations. In particular, we have that

Weak comparison holds for $G$-universal subequations on a riemannian manifold.

We then establish, under certain global conditions, that weak comparison implies comparison.

**THEOREM 9.7.** Suppose $F$ is a subequation for which local weak comparison holds. Suppose there exists a $C^2$ strictly $M_F$-subharmonic function on $X$ where $M_F$ is a monotonicity cone for $F$. Then comparison holds for $F$ on $X$.

Combined with our notions of boundary convexity, we prove the following.

**THEOREM 13.3.** Suppose comparison holds for a subequation $F$ on $X$.

Then for every domain $\Omega \subset X$ whose boundary is strictly $\overline{F}$- and $\overline{\tilde{F}}$-convex, both existence and uniqueness hold for the Dirichlet problem.

In Section 9 we introduce the concept of strict approximation for $F$ on $X$ and show that if $X$ admits a $C^2$ strictly $M_F$-subharmonic function (where $M_F$ is a monotonicity cone for $F$ as above), then strict approximation holds for $F$ on $X$. Furthermore, we show that weak comparison plus strict approximation implies comparison.

**Boundary Convexity**

An important part of this paper is the formulation and study of the notion of boundary convexity for a general fully non-linear second-order equation. This concept is presented in Section 11. It strictly generalizes the boundary convexity defined in [CNS2] and in [HL4].

8
In the Grassmann examples this boundary convexity condition is particularly transparent and geometric. Suppose $F(G)$ is a purely second-order equation, defined as in Example E by a subset $G \subset G(p, TX)$ of the Grassmann bundle. Then boundary convexity for a domain $\Omega$ becomes the requirement that the second fundamental form $II_{\partial\Omega}$ of $\partial\Omega$ satisfy
\[
\text{tr}_\xi II_{\partial\Omega} \geq 0
\]
for all $\xi \in G$ such that $\xi \subset T(\partial\Omega)$. (When there are no $G$-planes in $T_x\partial\Omega$, $G$-convexity automatically holds at $x$.) This convexity automatically implies convexity for the dual subequation.

Domains with convex boundaries in the riemannian sense are $F$-convex for any purely second-order subequation.

Some subequations $F$ have the property that every boundary is $F$-convex. These include the $p$-Laplace-Beltrami subequation,
\[
\|\nabla u\|^2 \Delta u + (p - 2)(\nabla u)^t (\text{Hess } u)(\nabla u) \geq 0
\]
for $1 \leq p < \infty$, and the infinite Laplace-Beltrami subequation
\[
(\nabla u)^t (\text{Hess } u)(\nabla u) \geq 0.
\]

For the general minimal surface subequation $(1 + \|\nabla u\|^2)\Delta u - (\nabla u)^t (\text{Hess } u)(\nabla u) \geq 0$ strict boundary convexity is equivalent to strictly positive mean curvature with respect to the interior normal.

Existence without Comparison

Certain methods which go back to Bremermann and Walsh [B], [W] enable us to prove existence theorems in the absence of a monotonicity cone which includes cases without uniqueness. In particular, we prove in section 12 that if $X = K/G$ is a riemannian homogeneous space and $F \subset J^2(X)$ is a $K$-invariant subequation, then existence holds for all continuous boundary data on any domain which is strictly $\overline{F}$ and $\overline{\nabla F}$ convex. This applies in particular to the $p$-Laplace-Beltrami and infinite Laplace-Beltrami subequations mentioned above, where all domains are strictly $\overline{F}$ and $\overline{\nabla F}$ convex.

The euclidean version of this result is stated as Theorem 12.7. It establishes existence for any constant coefficient subequation $\mathbf{F}$ on all strictly $\overline{\mathbf{F}}$ and $\overline{\nabla \mathbf{F}}$-convex domains (when they exist). We follow this with an Example 12.8 of a second order equation where uniqueness does in fact fail.

Further Examples.

There are many geometrically interesting subequations which are covered by the results above. We examine a few more examples here. We start with a general observation.

**Inhomogeneous Equations.** The methods above apply to any subequation which is locally affinely equivalent to a constant coefficient equation. This greatly extends the equations that one can treat. For example, suppose that $F$ is a $G$-universal subequation with a monotonicity cone $M$ on a manifold $X$, and let $J$ be any smooth section of the 2-jet bundle $J^2(X)$. Then $F_J \equiv F + J$ (fiber-wise translation) is also a subequation having the same asymptotic interior $\overline{F_J} = \overline{F}$ and also having $M$ as a monotonicity cone.
As a simple but interesting example, suppose that $F$ is one of the branches of the homogeneous Monge-Ampère equation (in the real, complex or quaternionic case), and choose $J_x = f(x)I$ where $f$ is an arbitrary smooth function on $X$ (and $I$ is the identity section of $\text{Sym}^2(T^*X)$). Then one can treat the inhomogeneous equation

$$\lambda_k(\text{Hess } u) = f(x)$$

under the same conditions that one can treat the homogeneous equation $\lambda_k(\text{Hess } u) = 0$. For the principal branch $\mathcal{P}$, $\mathcal{P}^\mathbb{C}$ etc., and $f \equiv -1$, this yields functions which are quasi-convex, quasi-psh, etc. The higher branches with variable $f$ are more interesting.

Similar remarks hold for any purely second-order subequation, such as those below. However, many other interesting subequations arise from local affine equivalence (cf. Example 4.10).

**Example F. (Almost calibrated manifolds).** An important class of Grassmann structures (discussed in Example E) are given by calibrations. A calibration is a $p$-form $\phi$ with the property that $\phi(\xi) \leq 1$ for all $\xi \in G(p, \mathbb{R}^n)$. The associated set is

$$G(\phi) \equiv \{ \xi \in G(p, \mathbb{R}^n) : \phi(\xi) = 1 \}$$

and the associated group is

$$G(\phi) = \{ g \in O_n : g^*(\phi) = \phi \}.$$  

Any riemannian manifold with a topological $G(\phi)$-structure will carry a global $p$-form $\tilde{\phi}$, called an *almost calibration*, which is of type $\phi$ at every point but is not necessarily closed. Some of the subequations already referred to can be defined in this way. For example, almost complex hermitian geometry arises from $\phi = \omega$, the standard Kähler form. There are however many others. Several interesting examples are given next.

**Example G. (Almost Hyperkahler Manifolds).** Here we consider a $4n$-dimensional manifold $X$ equipped with a subbundle $Q \subset \text{Hom}(TX, TX)$ generated by *global* sections $I, J, K$ satisfying the standard quaternion relations: $I^2 = J^2 = K^2 = -1$, and $IJ = K$, etc. and equipped with a compatible riemannian metric $g$. The topological structure group is $\text{Sp}_n$. This is a special case of Example D above so the quaternionic plurisubharmonic functions are defined, and when the hyperKähler structure is integrable, they coincide with the ones used by Alesker and Verbitsky to study the Quaternionic Monge-Ampere equation in the principal-branch case. (See [A1,2], [AV])

However a manifold with a topological $\text{Sp}_n$-structure carries other almost calibrations such as the generalized Cayley calibration: $\Omega = \frac{1}{2} \{ \omega_I^2 + \omega_J^2 - \omega_K^2 \}$ introduced in [BH].

**Example H. (Almost Calabi-Yau Manifolds).** This is an almost complex hermitian manifold with a global section of $\Lambda^{n,0}(X)$ whose real part $\Phi$ has comass $\equiv 1$. This is equivalent to having topological structure group $\text{SU}_n$. In addition to the structures discussed in Case 2, these manifolds carry functions associated with the *Special Lagrangian calibration* $\Phi$. The $\Phi$-submanifolds are called *Special Lagrangian submanifolds* and the $\Phi$-subharmonic functions are said to be *Special Lagrangian subharmonic*.

**Example I. (Almost $G_2$ manifolds).** Let $\text{Im}O = \mathbb{R}^7$ denote the imaginary Cayley numbers. This space is acted on by the group $G_2$ of automorphisms of $O$ which preserves the 3-form

$$\varphi(x, y, z) = \langle x \cdot y, z \rangle$$

called the *associative calibration*. Any 7-dimensional manifold $X$ with a topological $G_2$-structure carries an associated riemannian metric and a global (non-closed) calibration $\varphi$. There also exists a
degree \(4\) calibration \(\psi = *\varphi\) on \(X\). One then has \(\varphi\) and \(\psi\) subharmonic and harmonic functions on \(X\) and one can consider the Dirichlet Problem on bounded domains. We note that these structures exists in abundance. In [LM, page 348] it is shown that for any 7-manifold \(X\)

\[ X \text{ has a topological } G_2 \text{ structure } \iff X \text{ is spin.} \]

**Example J. (Almost Spin}_7\text{ manifolds).** On the Cayley numbers \(O = \mathbb{R}^8\) there is a 4-form of comass one defined by

\[ \Phi(x, y, z, w) = \langle (x \cdot y) \cdot z - x \cdot (y \cdot z), w \rangle \]

and preserved by the subgroup \(\text{Spin}_7 \subset \text{SO}_8\) (cf. [HL1], [H], [LM]). This determines a non-closed calibration \(\Phi\) on any 8-manifold \(X\) with a topological \(\text{Spin}_7\)-structure. In [LM, page 349] it is shown that for any 8-manifold \(X\)

\[ X \text{ has a topological } \text{Spin}_7 \text{ structure } \iff X \text{ is spin and } p_1(X)^2 - 4p_2(X) + 8\chi(X) = 0 \]

for an appropriate choice of orientation on \(X\). Here \(p_k(X)\) is the \(k\)th Pontrjagin class and \(\chi(X)\) denotes the Euler class of \(X\).

**Example K. (Lagrangian subhamonicity).** Suppose \((X, J)\) is an almost complex hermitian manifold of real dimension \(2n\). Then, as mentioned in Example E, there is a natural Grassmann structure \(\text{LAG} \subset G(n, TX)\) consisting of the Lagrangian \(n\)-planes. This gives rise to the Lagrangian subequation defined in terms of the Riemannian hessian by the condition that

\[ \text{tr} \left\{ \text{Hess } u_\xi \right\} \geq 0 \text{ for all } \xi \in \text{LAG} \]

Interestingly, there is a beautiful polynomial operator which vanishes on the Lagrangian harmonic functions, and so the Dirichlet problem here can be considered to be for this operator. Furthermore, just as in the Monge-Ampère case, this operator has many branches, each of which is another \(U_n\)-invariant subequation on the manifold. This follows from Gårding’s theory of hyperbolic polynomials [G], [HL4].

**Example L. (Equations involving the dependent variable but independent of the gradient).** Many purely second-order equations can be enhanced to ones which involve the dependent variable \(u\) and our theory continues to apply. For example, consider the \(O_n\)-universal subequation \(F\) defined by requiring

\[ f(u, \lambda_q(A)) \geq 0 \]

where \(f(x, y)\) is non-increasing in \(x\) and non-decreasing in \(y\). As a special case, consider

\[ \lambda_q(A) - \varphi(u) \geq 0 \]

where \(\varphi\) is monotone non-decreasing. The dual subequation \(\tilde{F}\) is given by

\[ \lambda_{n-q+1} + \varphi(-u) \geq 0. \]

If \(\varphi(0) = 0\), then the required convexity of the boundary is that

\[ \min\{\lambda_q(II_{\partial\Omega}), \lambda_{n-q}(II_{\partial\Omega})\} > 0. \]
Another interesting $O_n$-universal subequation is

$$F \equiv \{(r,p,A) : A \geq 0 \text{ and } \det A - e^r \geq 0\}$$

which is discussed in Remark 12.9. Its dual equation is

$$\tilde{F} = \{(r,p,A) : -A \not< 0 \text{ or } -|\det A| + e^{-r} \geq 0\}.$$  

Gårding hyperbolic polynomials

Gårding’s beautiful theory of hyperbolic polynomials [G], when applied to homogeneous polynomials $M$ on $\text{Sym}^2(\mathbb{R}^n)$, fits perfectly into this paper. It unifies and generalizes many of our constructions. We give here a brief sketch of how this works, and refer to [HL7] for full details.

By definition a homogeneous polynomial $M$ of degree $m$ on $\text{Sym}^2(\mathbb{R}^n)$ is hyperbolic with respect to the identity $I$ if for all $A \in \text{Sym}^2(\mathbb{R}^n)$ the polynomial $s \mapsto M(sI + A)$ has exactly $m$ real roots. The negatives of these roots are called the $M$-eigenvalues of $A$. It is useful to order these eigenvalues

$$\lambda_1^M(A) \leq \cdots \leq \lambda_m^M(A)$$

and normalize so that $M(I) = 1$. Then the polynomial factors as $M(sI + A) = \prod_{k=1}^m (s + \lambda_k^M(A))$.

Using the ordered eigenvalues we can define branches

$$F_k^M = \{A \in \text{Sym}^2(\mathbb{R}^n) : \lambda_k^M(A) \geq 0\}$$

which satisfy $F_1^M \subset F_2^M \subset \cdots$. The principal branch $F^M \equiv F_1^M$ is the connected component of $\{M \neq 0\}$ containing $I$. Gårding proves that:

1. The principal branch $F^M$ is a convex cone.
2. $F_k^M + F^M \subset F_k^M$ for all $k$.

Under the positivity assumption: $F^M + \mathcal{P} \subset F^M$ we then have that

Each $F_k^M$ is a constant coefficient subequation for which $F^M$ is a monotonicity cone.

When $M = \det$ we get the branches of the Monge-Ampère equation (Example B). However, this applies to many other interesting cases (such as Examples C, D and K). Furthermore, for each subset $\Lambda \subset \mathbb{R}^n$ as in 14.1 below, one can construct a subequation $F_{M,\Lambda}$ using the $\lambda_k^M$ (see [HL7]).

If the polynomial $M$ is invariant under a subgroup $G \subset O_n$, then these subequations carry over to any manifold with a topological $G$-structure.

Parabolic Subequations

The methods and results of this paper carry over effectively to parabolic equations. Suppose $X$ is a riemannian manifold equipped with a $G$-universal subequation $F$ for $G \subset O_n$. We assume $F$ is induced from a universal model

$$F = \{J \in \mathfrak{J}^2 : f(J) \geq 0\}$$

where $f : \mathfrak{J}^2(X) \to \mathbb{R}$ is $G$-invariant, $\mathcal{P}$- and $\mathcal{N}$-monotone, and Lipschitz in the reduced variables $(p,A)$. Then on the riemannian product $X \times \mathbb{R}$ we have the associated $G$-universal parabolic subequation $H$ defined by

$$f(J) - p_0 \geq 0$$

12
where $p_0$ denotes the $u_t$ component of the 2-jet of $u$. The $H$-harmonic functions are solutions of the equation $u_t = f(u, Du, D^2 u)$. Interesting examples which can be treated include:

(i) $f = \text{tr}A$, the standard heat equation $u_t = \Delta u$ for the Laplace-Beltrami operator on $X$.

(ii) $f = \lambda_q(A)$, the $q$th ordered eigenvalue of $A$. This is the natural parabolic equation associated to the $q$th branch of the Monge-Ampère equation.

(iii) $f = \text{tr}A + \frac{1}{|p|^2 + \epsilon} p^tAp$ for $k \geq -1$ and $\epsilon > 0$. When $X = \mathbb{R}^n$ and $k = -1$, the solutions $u(x,t)$ of the associated parabolic equation, in the limit as $\epsilon \to 0$, have the property that the associated level sets $\Sigma_t \equiv \{x \in \mathbb{R}^n : u(x,t) = 0\}$ are evolving by mean curvature flow. (See the classical papers of Evans-Spruck [ES] and the very nice account in [E].)

(iv) $f = \text{tr}\{\arctan A\}$. When $X = \mathbb{R}^n$, solutions $u(x,t)$ have the property that the graphs of the gradients: $\Gamma_t \equiv \{(x,y) \in \mathbb{R}^n \times \mathbb{R}^n = C^n : y = D_x u(x,t)\}$ are Lagrangian submanifolds which evolve the initial data by mean curvature flow. (See [CCH] and references therein.)

Straightforward application of the techniques in this paper shows that:

Comparison holds for all $G$-universal subequations $H$ on $X \times \mathbb{R}$.

By standard techniques one can prove more. Consider a compact subset $K \subset \{t \leq T\} \subset X \times \mathbb{R}$ and let $K_T \equiv K \cap \{t = T\}$ denote the terminal time slice of $K$. Let $\partial_0 K \equiv \partial K - \text{Int}K_T$ denote the parabolic boundary of $K$. Here $\text{Int}K$ denotes the relative interior in $\{t = T\} \subset X \times \mathbb{R}$. We say that parabolic comparison holds for $H$ if for all such $K$ (and $T$)

$$u + v \leq p \quad \text{on} \quad \partial_0 K \quad \Rightarrow \quad u + v \leq p \quad \text{on} \quad \text{Int}K$$

for all $u \in H(K)$ and $v \in \bar{H}(K)$. Then one has that:

Parabolic comparison holds for all $G$-universal subequations $H$ on $X \times \mathbb{R}$.

Under further mild assumptions on $f$ which are satisfied in the examples above, one also has existence results. Consider a domain $\Omega \subset X$ whose boundary is strictly $\overline{F}$ and $\overline{F}$-convex. Set $K = \overline{\Omega} \times [0,T]$. Then

For each $\varphi \in C(\partial_0 K)$ there exists a unique function $u \in C(K)$ such that $u|_{\text{Int}K}$ is $H$-harmonic and $u|_{\partial_0 K} = \varphi$.

One also obtains corresponding long-time existence results. Details will appear elsewhere.

A brief outline of the paper. Section 2 along with Appendices A and B provide a self-contained treatment of general $F$-subharmonic functions and their properties. Section 3 introduces the concept of a subequation and discusses the natural duality among subequations. Section 4 discusses automorphisms of the 2-jet bundle and a weak notion of equivalence for subequations. Riemannian manifolds are considered in Section 5 where it shown that there is a natural splitting of the 2-jet bundle induced by the riemannian hessian. This splitting gives many geometric examples of subequations which are purely second-order. In Section 6 “universal” subequations are constructed. Suppose $F$ is a constant coefficient subequation with compact invariance group $G \subset O_n$, and $X$ is a riemannian manifold with a topological $G$-structure. Then the model $F$ induces a $G$-universal subequation $F$ on $X$ with the property that $F$ is locally equivalent to $F$. In Section 7 a notion of strictly $F$-subharmonic functions is introduced for upper semi-continuous functions. In Section 8 a weak form of comparison is defined using this notion of strictness. It is shown that if weak comparison holds locally, then it is true globally. Section 9 addresses the question of when weak comparison implies comparison. We first note that this holds whenever approximation by strictly $F$-subharmonic functions is possible. The main discussion concerns how this “strict
approximation” can be deduced from a form of monotonicity. This yields many geometric examples. In Section 10 we prove that weak comparison holds for any subequation which is locally (weakly) equivalent to a constant coefficient subequation, and in particular for the \( G \)-universal subequations constructed in Section 6. The proof relies on the Theorem on Sums stated in Appendix C. In Section 11 we introduce the notion of the asymptotic interior \( \overline{F} \) for a general subequation \( F \). This provides a notion of strict \( F \)-boundary-convexity which implies the existence of barrier families essential to existence proofs. For \( G \)-universal subequation the existence of these barriers is actually equivalent to strict \( F \)-convexity. Section 12 addresses the existence question for the Dirichlet problem. Assuming strict boundary convexity for both the subequation \( F \) and its dual \( \overline{F} \), several existence theorems are proved. Section 13 compiles and summarizes the results established for the Dirichlet Problem.

The remaining sections are devoted to examples. Section 14 examines \( O_n \)-universal subequations. These are subequations that makes sense on any riemannian manifold. Particular attention is paid to subequations which are purely second-order. Analogous results in the complex and quaternionic case are examined in Section 15. Section 16 discusses the subequations \( F(\mathcal{G}) \) geometrically defined by a closed subset \( \mathcal{G} \) of the grassmannian.

We note that in [AFS] and [PZ] standard viscosity theory has been retrofitted to riemannian manifolds by using the distance function, parallel translation, Jacobi fields, etc.. In our approach this machinery in not necessary. We get by with the standard viscosity techniques (cf. [CIL], [C]).

2. F-Subharmonic Functions.

Let \( X \) be a smooth \( n \)-dimensional manifold. Denote by \( J^2(X) \to X \) the bundle of 2-jets whose fibre at a point \( x \) is the quotient

\[
J^2_x(X) = C^\infty_x / C^\infty_{x,3}
\]

where \( C^\infty_x \) denotes the germs of smooth functions at \( x \) and \( C^\infty_{x,3} \) the subspace of germs which vanish to order three at \( x \). Given a smooth function \( u \) on \( X \), let \( J^2_xu \in J^2_x(X) \) denote its 2-jet at \( x \). and note that \( J^2_xu \) is a smooth section of the bundle \( J^2(X) \).

The bundle on 1-jets \( J^1(X) \) is defined similarly and has a natural splitting \( J^1(X) = \mathbb{R} \oplus T^*X \) with \( J^1_xu = (u(x), (du)_x) \). There is a short exact sequence of bundles

\[
0 \to \text{Sym}^2(T^*X) \to J^2(X) \to J^1(X) \to 0 \quad (2.1)
\]

Here \( \text{Sym}^2(T^*_xX) \) is embedded as the space of 2-jets of functions with critical value zero at the point \( x \), i.e.,

\[
\text{Sym}^2(T^*_xX) \cong \{ J^2_xu : u(x) = 0, (du)_x = 0 \}.
\]

Note that if \( u \) is such a function, and \( V, W \) are vector fields near \( x \), then \( (\text{Hess}_xu)(V, W) = V \cdot W \cdot u = W \cdot V \cdot u + [V, W] \cdot u = W \cdot V \cdot u = (\text{Hess}_xu)(W, V) \) is a well-defined symmetric form on \( T_xX \). However, for functions \( u \) with \((du)_x \neq 0\), there is no natural definition of \( \text{Hess}_xu \), i.e., the sequence (2.1) has no natural splitting. Choices of splittings correspond to definitions of a hessian, and there is a canonical one for each riemannian metric as we shall see in §5.

At a minimum point \( x \) for a smooth function \( u \), we have \((du)_x = 0\) (so that \( \text{Hess}_xu \in \text{Sym}^2(T^*X) \) is well defined), and \( \text{Hess}_xu \geq 0 \). The isomorphism

\[
\{ H \in \text{Sym}^2(T^*X) : H \geq 0 \} \cong \{ J^2_xu : u \geq 0 \ \text{near} \ x \ \text{and} \ u(x) = 0 \} \quad (2.2)
\]
defines a cone bundle
\[ \mathcal{P} \subset \text{Sym}^2(T^*X) \subset J^2(X) \]
with \( \mathcal{P}_x \) defined by (2.2).

Given an arbitrary subset \( F \subset J^2(X) \) a function \( u \in C^2(X) \) will be called \( F \)-\textbf{subharmonic} if its 2-jet satisfies
\[ J^2_x u \in F_x \quad \text{for all } x \in X, \]
and \textbf{strictly} \( F \)-\textbf{subharmonic} if its 2-jet satisfies
\[ J^2_x u \in (\text{Int} F)_x \quad \text{for all } x \in X, \]

These notions are of limited interest for general sets \( F \)

**Definition 2.1.** A subset \( F \subset J^2(X) \) satisfies the \textbf{Positivity Condition} if
\[ F + \mathcal{P} \subseteq F \quad (P) \]
A subset \( F \subset J^2(X) \) which satisfies (P) will be called \( \mathcal{P} \)-\textbf{monotone}.

Note that condition (P) implies that
\[ \text{Int} F + \mathcal{P} \subset \text{Int} F. \]

Monotonicity is a key concept in this paper. For arbitrary subsets \( M, F \subset J^2(X) \) we say that \( F \) is \( M \)-\textbf{monotone} or that \( M \) is a \textbf{monotonicity set for} \( F \) if
\[ F + M \subset F \]

It is necessary and quite useful to extend the definition of \( F \)-subharmonic to non-differentiable functions \( u \). Let \( \text{USC}(X) \) denote the set of \([-\infty, \infty]\)-valued, upper semicontinuous functions on \( X \).

**Definition 2.2.** A function \( u \in \text{USC}(X) \) is said to be \( F \)-\textbf{subharmonic} if for each \( x \in X \) and each function \( \varphi \) which is \( C^2 \) near \( x \), one has that
\[ \begin{cases} u - \varphi \leq 0 & \text{near } x_0 \\ = 0 & \text{at } x_0 \end{cases} \quad \Rightarrow \quad J^2_x \varphi \subset F_x. \quad (2.3) \]

Note that if \( u \in C^2(X) \), then
\[ u \in F(X) \quad \Rightarrow \quad J^2_x u \in F_x \quad \forall \ x \in X \quad (2.4) \]

since the test function \( \varphi \) may be chosen equal to \( u \) in (2.3). The converse is not true for general subsets \( F \). However, we have the following.

**Proposition 2.3.** Suppose \( F \) satisfies the Positivity Condition (P) and \( u \in C^2(X) \). Then
\[ J^2_x u \in F_x \quad \text{for all } x \in X \quad \Rightarrow \quad u \in F(X). \quad (2.5) \]

**Proof.** Assume \( J^2_x u \in F_{x_0} \) and and \( \varphi \) is a \( C^2 \)-function such that
\[ \begin{cases} u - \varphi \leq 0 & \text{near } x_0 \\ = 0 & \text{at } x_0 \end{cases} \]
Since \((\varphi - u)(x_0) = 0\), \(d(\varphi - u)_{x_0} = 0\), and \(\varphi - u \geq 0\) near \(x_0\), we have \(J_{x_0}^2(\varphi - u) \in P_{x_0}\) by definition. Now the Positivity Condition implies that \(J_{x_0}^2 u \in J_{x_0}^2 u + P_{x_0} \subset F_{x_0}\). This proves that \(u \in F(X)\).

Because of Proposition 2.3 we must assume that \(F\) satisfies the Positivity Condition. Otherwise the definition of \(F\)-subharmonicity would not extend the natural one for smooth functions.

There is an equivalent definition of \(F\)-subharmonic functions which is quite useful. We record it here and prove it in Appendix A.

**Lemma 2.4.** Fix \(u \in \text{USC}(X)\). Then \(u \notin F(X)\) if and only if there exists a point \(x_0 \in X\), local coordinates \(x\) at \(x_0\), \(\alpha > 0\) and a quadratic function \(q(x) = r + \langle p, x - x_0 \rangle + \frac{1}{2}\langle A(x - x_0), x - x_0 \rangle\) with \(J_{x_0}^2(q) \notin F_{x_0}\) so that

\[
u(x) - q(x) \leq -\alpha|x - x_0|^2 \quad \text{near } x_0 \quad \text{and} \quad \nu = 0 \quad \text{at } x_0\]

**Remark 2.5.** The positivity condition is rarely used in proofs. This is because without it \(F(X)\) is empty and the results are trivial. For example, the positivity condition is not required in the following theorem. (\(F\) need only be closed.)

It is remarkable, at this level of generality, that \(F\)-subharmonic functions share many of the important properties of classical subharmonic functions.

**Theorem 2.6. Elementary Properties of \(F\)-Subharmonic Functions.** Let \(F\) be an arbitrary closed subset of \(J^2(X)\).

(A) (Maximum Property) If \(u, v \in F(X)\), then \(w = \max\{u, v\} \in F(X)\).

(B) (Coherence Property) If \(u \in F(X)\) is twice differentiable at \(x \in X\), then \(J_{x}^2 u \in F_{x}\).

(C) (Decreasing Sequence Property) If \(\{u_j\}\) is a decreasing \((u_j \geq u_{j+1})\) sequence of functions with all \(u_j \in F(X)\), then the limit \(u = \lim_{j \to \infty} u_j \in F(X)\).

(D) (Uniform Limit Property) Suppose \(\{u_j\} \subset F(X)\) is a sequence which converges to \(u\) uniformly on compact subsets to \(X\), then \(u \in F(X)\).

(E) (Families Locally Bounded Above) Suppose \(\mathcal{F} \subset F(X)\) is a family of functions which are locally uniformly bounded above. Then the upper semicontinuous regularization \(v^*\) of the upper envelope

\[v(x) = \sup_{f \in \mathcal{F}} f(x)\]

belongs to \(F(X)\).

**Proof.** See Appendix B.

**Cautionary Note 2.7.** Despite the elementary proofs of the properties in Theorem 2.6, illustrating how well adapted Definition 2.2 is to nonlinear theory, there are difficulties with the linear theory. If \(F_1\) and \(F_2\) are \(P\)-monotone subsets, then the (fibrewise) sum \(F_1 + F_2\) is also obviously a \(P\)-monotone subset. However, the property

\[u \in F_1(X), v \in F_2(X) \implies u + v \in (F_1 + F_2)(X)\]
is difficult to deduce from Definition 2.2 even in the basic case where $F_1 = F_2 = F_1 + F_2$ is the linear subequation on $\mathbb{R}^n$ defined by $\Delta u \geq 0$.

3. Dirichlet Duality and the Notion of a Subequation.

The following concept is the lynchpin for the Dirichlet Problem.

**Definition 3.1.** Given a subset $F \subset J^2(X)$ the **Dirichlet dual** $\tilde{F}$ of $F$ is defined by

$$\tilde{F} = \sim (- \text{Int} F) = -(\sim \text{Int} F) \quad (3.1)$$

Note the obvious properties:

1. $F_1 \subset F_2 \Rightarrow \tilde{F}_2 \subset \tilde{F}_1$.
2. $\tilde{F}_1 \cap F_2 = \tilde{F}_1 \cup \tilde{F}_2$.
3. $\tilde{\tilde{F}} = F$ provided that $F = \text{Int} F$.

Thus to have a true duality with $\tilde{\tilde{F}} = F$ we must assume that $F = \text{Int} F$. For simplicity we also want to compute the dual fibrewise in the jet bundle. Consequently we will assume the following three topological conditions on $F$, combined as condition (T).

(i) $F = \text{Int} F$, (ii) $F_x = \text{Int} F_x$, (iii) $\text{Int} F_x = (\text{Int} F)_x \quad (T)$

It is then easy to see that the fibre of $\tilde{F}$ at $x$ is given by $-(\sim \text{Int} F_x) = -(\sim \text{Int} F_x) = (\tilde{F}_x)$, so there is no ambiguity in the notation $\tilde{F}_x$.

**Definition 3.2.** A subset $F \subset J^2(X)$ satisfying (T) will be called a **T-subset**.

**Lemma 3.3.** Suppose $F \subset J^2(X)$ has the property that $F = \text{Int} F$. Then

(a) $F$ satisfies Condition (P) $\iff \tilde{F}$ satisfies Condition (P).
(b) $F$ satisfies Condition (T) $\iff \tilde{F}$ satisfies Condition (T).

**Proof.** Assertion (b) is straightforward. To prove (a) we use another property.

**Lemma 3.4.** Suppose that $F$ is a T-subset. Then

$$\tilde{F}_x + J = \tilde{F}_x - J \quad \text{for all } J \in J^2_x(X) \quad (4)$$

**Proof.** Fix $J \in J^2_x(X)$. Then $J' \in \tilde{F}_x + J \iff -J' \notin \text{Int}(F_x + J) = \text{Int} F_x + J \iff -(J' + J) \notin \text{Int} F_x \iff J' + J \in \tilde{F}_x \iff J' \in \tilde{F}_x - J$. ■

**Corollary 3.5.** Suppose $F$ is a T-subset of $J^2(X)$ and $M$ is an arbitrary subset of $J^2(X)$. Then $F$ is $M$-monotone $\iff \tilde{F}$ is $M$-monotone.

**Proof.** Fix $J \in M_x$ and assume $F_x + J \subset F_x$, or equivalently $F_x \subset F_x - J$. By (1) this implies that $F_x - J \subset \tilde{F}_x$. By (4) we have $\tilde{F}_x - J = \tilde{F}_x + J$ so that $\tilde{F}_x + J \subset \tilde{F}_x$. The converse follows from (3). ■

**Proof of (a).** Take $M = \mathcal{P}$ in Corollary 3.5. ■
Given a closed subset $F \subset J^2(X)$, note that
\[ \partial F = F \cap (\sim \text{Int} F) = F \cap (\sim \widetilde{F}) \quad (3.2) \]

**Definition 3.6.** A function $u \in \text{USC}(X)$ is said to be $F$-harmonic if
\[ u \in F(X) \quad \text{and} \quad -u \in \widetilde{F}(X). \]

Thus, a function $u \in C^2(X)$ is $F$-harmonic if and only if
\[ J_x^2 u \in \partial F_x \quad \text{for all} \quad x \in X. \quad (3.3) \]

Note also that an $F$-harmonic function is automatically continuous. The topological condition (T) implies that $\widetilde{\widetilde{F}} = F$ and hence that
\[ u \text{ is } F\text{-harmonic} \quad \iff \quad -u \text{ is } \widetilde{F}\text{-harmonic}. \]

The focal point of this paper is the **Dirichlet Problem**, abbreviated (DP). Given a compact subset $K \subset X$ and a function $\varphi \in C(\partial K)$, a function $u \in C(K)$ is a solution to (DP) if
\[ (1) \quad u \text{ is } F\text{-harmonic on } \text{Int} K \quad \text{and} \quad (2) \quad u|_{\partial K} = \varphi. \]

**Example 3.7.** Consider the one variable first order subset $F$, defined by $|p| \leq 2|x|$, and its dual $\widetilde{F}$ defined by $|p| \geq 2|x|$. Of the conditions (P) and (T), the subset $F$ satisfies all but condition (ii) of (T) and its dual $\widetilde{F}$ satisfies all but (iii) of (T). Both functions $x^2 - 1$ and $-x^2 + 1$ are $F$-harmonic and have the same boundary values on $[-1,1]$. Thus without the full condition (T), solutions of the Dirichlet Problem may not be unique.

**Subequations**

Uniqueness for the Dirichlet problem requires a third hypothesis. Consider the elementary canonical splitting
\[ J^2(X) = \mathbb{R} \oplus J_{\text{reduced}}^2(X) \]
where $\mathbb{R}$ denotes the 2-jets of (locally) constant functions and $J_{\text{reduced}}^2(X)_x = \{ J_x^2 u : u(x) = 0 \}$ is the space of reduced 2-jets at $x$, with the short exact sequence of bundles
\[ 0 \rightarrow \text{Sym}^2(T^*X) \rightarrow J_{\text{reduced}}^2(X) \rightarrow T^*X \rightarrow 0 \]
which is the same as (2.1) except for the trivial factor $\mathbb{R}$.

We define
\[ \mathcal{N} \subset \mathbb{R} \subset J^2(X) \]
to have fibres $\mathcal{N}_x = \mathbb{R}^- = \{ c \in \mathbb{R} : c \leq 0 \}$.

**Definition 3.8.** A subset $F \subset J^2(X)$ satisfies the **Negativity Condition** if
\[ F + \mathcal{N} \subseteq F. \quad (N) \]

A subset $F \subset J^2(X)$ satisfying (N) will be called $\mathcal{N}$-**monotone**.

Now we can add to Lemma 3.2 a third conclusion
(c) $F$ satisfies Condition (N) $\iff \bar{F}$ satisfies Condition (N).

by taking $M = \mathcal{N}$ in Corollary 3.5.

In this paper the main results concerning existence and uniqueness for the Dirichlet problem assume that $F$ satisfies (P), (T) and (N). This is formalized as follows.

**Definition 3.9.** By a subequation $F$ on a manifold $X$ we mean a subset

$$F \subset J^2(X)$$

which satisfies the three conditions (P), (T) and (N).

**Proposition 3.10.**

$$F \text{ is a subequation } \iff \bar{F} \text{ is a subequation}$$

**Proof.** See (a), (b) and (c) above.

Our investigation of the (DP) for a subequation involves two additions subequations, which are constructed from the original and have two additional properties. One is the cone property:

$$J \in F \Rightarrow tJ \in F \text{ for all } t > 0,$$

i.e., each fibre $F_x$ is a cone with vertex at the origin. Stronger yet is the convex cone property:

each fibre $F_x$ is a convex cone with vertex at the origin.

**Definition 3.11.** A closed subset $F \subset J^2(X)$ having properties (P), (T), (N) and the cone property will be called a cone subequation. If is also has the convex cone property it will be called a convex cone subequation.

**Constant Coefficient Subequations**

The 2-jet bundle on $X = \mathbb{R}^n$ is canonically trivialized by

$$J^2_x = (u(x), D_x u, D^2_x u) \quad (3.4)$$

where

$$D_x u = \left( \frac{\partial u}{\partial x_1}(x), \ldots, \frac{\partial u}{\partial x_n}(x) \right) \quad \text{and} \quad D^2_x u = \left( \left( \frac{\partial^2 u}{\partial x_i \partial x_j}(x) \right) \right)$$

are the first and second derivatives of $u$ at $x$. That is, for any open subset $X \subset \mathbb{R}^n$ there is a canonical trivialization

$$J^2(X) \cong X \times \mathbb{R} \times \mathbb{R}^n \times \text{Sym}^2(\mathbb{R}^n) \quad (3.4)'$$

with fibre

$$J^2 = \mathbb{R} \times \mathbb{R}^n \times \text{Sym}^2(\mathbb{R}^n)$$

The notation $J = (r, p, A) \in J^2$ will be used for the coordinates on $J^2$.

**Definition 3.12.** Consider a subset $F \subset J^2$ which satisfies (P), (N) and $F = \text{Int}F$. By a constant coefficient subequation with model fibre $\mathcal{F}$ we mean a subequation on an open subset $X \subset \mathbb{R}^n$ of the form $F = X \times \mathcal{F} \subset J^2(X)$. For simplicity we shall often denote this subequation simply by $F$ (with the euclidean coordinates implied), and refer to the $F$-subharmonic functions as $F$-subharmonic functions.

Note that our assumptions on $F$ imply conditions (P), (N), and (T) for $F$.

This paper will be principally concerned with second-order partial differential subequations on (riemannian) manifolds which generalize the constant coefficient subequations on euclidean space.
4. Subequations Locally Equivalent to Constant Coefficient Subequations.

To effectively generalize the notion of constant coefficient equations to general manifolds we need the notion of equivalence of subequations, determined by the automorphisms of the 2-jet bundle.

Definition 4.1. (a) An automorphism of $J^2_{\text{red}}(X)$ is a bundle isomorphism $\Phi : J^2_{\text{red}}(X) \to J^2_{\text{red}}(X)$ such that with respect to the short exact sequence

$$0 \longrightarrow \text{Sym}^2(T^*X) \longrightarrow J^2_{\text{red}}(X) \longrightarrow T^*X \longrightarrow 0 \quad (4.1)$$

we have

$$\Phi(\text{Sym}^2(T^*X)) = \text{Sym}^2(T^*X) \quad (4.2)$$

so there is an induced bundle automorphism $h = h_\Phi : T^*X \longrightarrow T^*X \quad (4.3)$

and we further require that on $\text{Sym}^2(T^*X)$ we have

$$\Phi(A)(v, w) = A(h^t v, h^t w) \equiv A^h(v, w) \quad \text{for } v, w \in TX. \quad (4.4)$$

(b) An automorphism of $J^2(X) = \mathbb{R} \oplus J^2_{\text{red}}(X)$ is a bundle isomorphism $\Phi : J^2(X) \to J^2(X)$ which the direct sum of the identity on $\mathbb{R}$ and an automorphism of $J^2_{\text{red}}(X)$.

Lemma 4.2. The automorphisms of $J^2(X)$ form a group. They are the sections of the bundle of groups whose fibre at $x \in X$ is the group of automorphisms of $J^2_x(X)$ defined by (4.2), (4.3) and (4.4) above.

Proof. We first show that the composition of two automorphisms of $J^2(X)$ is again an automorphism. Suppose $\Psi$ and $\Phi$ are bundle automorphisms. Then $\Psi \circ \Phi$ clearly satisfies condition (1) and one sees easily that $h_{\Psi \circ \Phi} = h_\Psi \circ h_\Phi$. Finally, $(\Psi \circ \Phi)(A)(v, w) = \Psi(\Phi(A))(v, w) = \Phi(A)(h_\Psi^t v, h_\Psi^t w) = A(h_\Psi^t \circ h_\Phi^t v, h_\Psi^t \circ h_\Phi^t w) = A((h_\Psi \circ h_\Phi)^t v, (h_\Psi \circ h_\Phi)^t w) = A(h_{\Psi \circ \Phi}^t v, h_{\Psi \circ \Phi}^t w)$. The proof that the inverse of an automorphism is an automorphism is similar.

Definition 4.3. Two subequations $F, F' \subset J^2(X)$ are equivalent if there exists an automorphism $\Phi : J^2(X) \to J^2(X)$ with $\Phi(F) = F'$.

Proposition 4.4. With respect to any splitting

$$J^2(X) = \mathbb{R} \oplus T^*X \oplus \text{Sym}^2(T^*X)$$

of the short exact sequence (4.1), a bundle automorphism has the form

$$\Phi(r, p, A) = (r, hp, A^h + L(p)) \quad (4.5)$$

where $h$ is a smooth section of the bundle $\text{End}(T^*X)$ and $L$ is a smooth section of the bundle $\text{Hom}(T^*X, \text{Sym}^2(T^*X))$.

Proof. Obvious.

Example 1. The trivial 2-jet bundle on $\mathbb{R}^n$ has fibre

$$J^2 = \mathbb{R} \times \mathbb{R}^n \times \text{Sym}^2(\mathbb{R}^n).$$
with automorphism group

\[ \text{Aut}(J^2) \equiv \text{GL}_n \times \text{Hom} (\mathbb{R}^n, \text{Sym}^2(\mathbb{R}^n)) \]

where the action is given by

\[ \Phi(h, L)(r, p, A) = (r, hp, hAh^t + L(p)) \]

and the group law is

\[ (\tilde{h}, \tilde{L}) \cdot (h, L) = (\tilde{h}h, \tilde{L}Lh^t + L \circ h) \]

**Example 2.** Given a local coordinate system \((x_1, \ldots, x_n)\) on an open set \(U \subset X\), the canonical trivialization

\[ J^2(U) = U \times \mathbb{R} \times \mathbb{R}^n \times \text{Sym}^2(\mathbb{R}^n) \] (4.6)

is determined by \(J_2^2 u = (u, Du, D^2 u)\) evaluated at \(x\), where \(Du = (u_1, \ldots, u_n)\) and \(D^2 u = ((u_{ij}))\) (cf. Remark 2.8). With respect to this splitting, every automorphism is of the form

\[ \Phi(u, Du, D^2 u) = (u, hDu, h \cdot D^2 u \cdot h^t + L(Du)) \] (4.7)

where \(h \in \text{GL}_n\) and \(L_x : \mathbb{R}^n \to \text{Sym}^2(\mathbb{R}^n)\) is linear for each point \(x \in X\).

**Definition 4.5.** A subequation \(F \subset J^2(X)\) is **locally equivalent to a constant coefficient subequation** if each point \(x\) has a coordinate neighborhood \(U\) so that \(F|_U\) is equivalent to a constant coefficient subequation \(U \times F\) in those coordinates.

**Lemma 4.6.** Suppose \(X\) is connected and \(F \subset J^2(X)\) is locally equivalent to a constant coefficient subequation. Then there is a subequation \(F \subset J^2\), unique up to equivalence, such that \(F\) is locally equivalent to \(U \times F\) on every distinguished coordinate chart.

**Proof.** In the overlap of any two distinguished charts \(U_1 \cap U_2\) choose a point \(x\). Then the local equivalences \(\Phi_1\) and \(\Phi_2\), restricted to \(F_x\), determine an equivalence from \(F_1\) to \(F_2\). Thus the local constant coefficient equations on these charts are all equivalent, and they can be made equal by applying the appropriate constant equivalence on each chart. \(\blacksquare\)

**Remark 4.7.** One reason that the notion of equivalence employed in Definition 4.3 is natural is that this notion is induced by diffeomorphisms. Namely, if \(\varphi\) is a diffeomorphism fixing a point \(x_0\), then in local coordinates (as in Example 2 above) the right action on \(J^2_{x_0}\), induced by the pull-back \(\varphi^*\) on 2-jets, is given by (4.7) where \(h_{x_0}\) is the transpose on the Jacobian matrix \(((\frac{\partial x'}{\partial x}))\)

and \(L_{x_0}(Du) = \sum_{k=1}^n u_k \frac{\partial^2 \varphi^b}{\partial x_i \partial x_j}(x_0)\).

**Cautionary Note.** A local equivalence \(\Phi : F \to F'\) does not take \(F\)-subharmonic functions to \(F'\)-subharmonic functions. In fact, for \(u \in C^2\), \(\Phi(J^2 u)\) is almost never the 2-jet of a function. It happens if and only if \(\Phi(J^2 u) = J^2 u\).

**Remark 4.8. (Affine Automorphisms).** The automorphism group \(\text{Aut}(J^2)\) can be naturally extended by the translations of \(J^2\). This enhanced group is called the **affine automorphism group** and is defined to be the inverse image \(\text{Aut}_{\text{aff}}(J^2) = \pi^{-1}(\text{Aut}(J^2))\) of \(\text{Aut}(J^2) \subset \text{GL}(J^2)\) under the surjective group homomorphism \(\pi : \text{Aff}(J^2) \to \text{GL}(J^2)\). Using this group we greatly enhance the family of equations that are locally equivalent to constant coefficient equations. These equations will be called **locally affinely equivalent to a constant coefficient equation.**

21
Note that any affine automorphism $\tilde{\Phi}$ can be written in the form

$$\tilde{\Phi} = \Phi + J$$

(4.8)

where $\Phi$ is a (linear) automorphism and $J$ is a section of the bundle $J^2(X)$.

**Lemma 4.9.** Suppose $F$ is a subequation on a coordinate chart $U$ which is affinely equivalent to a constant coefficient equation $U \times F$. Write the affine equivalence $\tilde{\Phi}$ as in (4.8) above so that

$$J \in F_x \iff \Phi_x(J) + J_x \in F$$

for $x \in U$. Then

$$J \in \tilde{F}_x \iff \Phi_x(J) - J_x \in \tilde{F}$$

**Proof.** $J \in \tilde{F}_x \iff -J \notin \text{Int}F_x \iff \Phi_x(-J) + J_x \notin F \iff -\{\Phi_x(J) - J_x\} \notin F \iff \Phi_x(J) - J_x \in \tilde{F}$

**Example 4.10.** Let $X$ be an almost complex hermitian manifold and consider the pure second-order $U_n$-universal subequation $F$ given by

$$A_C + I \geq 0 \quad \text{and} \quad \det_C\{A_C + I\} \geq 1.$$ 

Let $f > 0$ be a smooth positive function on $X$ and write $f = h^{-2n}$. Consider the global affine equivalence of $J^2(X)$ given by

$$\tilde{\Phi}(r,p,A) = (r, hp, (hI)A(hI))^t + (h^2 - 1)I = (r, hp, h^2A + (h^2 - 1)I)$$

and set $F_f = \tilde{\Phi}(F)$. Then

$$(r, p, A) \in F_f \iff h^2(A_C + I) \geq 0 \quad \text{and} \quad \det_C\{h^2(A_C + I)\} \geq 1$$

$$\iff (A_C + I) \geq 0 \quad \text{and} \quad \det_C\{(A_C + I)\} \geq f$$

so the $F_f$-harmonic functions are functions $u$ with $\text{Hess}_C u + I \geq 0$ (quasi-plurisubharmonic) and $\det_C\{\text{Hess}_C u + I\} = f$. If $X$ is actually a complex manifold of dimension $n$ with Kähler form $\omega$, this last equation can be written in the more familiar form

$$\left(\frac{1}{i} \partial \bar{\partial} u + \omega\right)^n = f \omega^n.$$
5. The Riemannian Hessian – A Canonical Splitting of $J^2(X)$.

5.1. The riemannian hessian. Assume now that $X$ is equipped with a riemannian metric $g$. Then, using the riemannian connection, any $C^2$-function $u$ on $X$ has a canonically defined hessian at every point given as follows. Fix $x \in X$ and vector fields $V$ and $W$ defined near $x$. Define

$$(\text{Hess}_x u)(V, W) \equiv V \cdot W \cdot u - (\nabla_V W) \cdot u$$

where the RHS is evaluated at $x$. Since $(\nabla_V W) \cdot u - (\nabla_W V) \cdot u = \left[ V, W \right] \cdot u$, we see that $(\text{Hess}_x u)(V, W)$ is symmetric and depends only on the values of $V$ and $W$ at the point $x$, i.e., Hess $u$ is a section of $\text{Sym}^2(T^*X)$. Of course at a critical point the riemannian hessian always agrees with the hessian defined in Section 2.

This riemannian hessian has a simple expression in terms of local coordinates $x = (x_1, ..., x_n)$ on $X$, namely

$$(\text{Hess}_u) \left( \frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j} \right) = \frac{\partial^2 u}{\partial x_i \partial x_j} - \sum_{k=1}^n \Gamma^k_{ij}(x) \frac{\partial u}{\partial x_k}$$

where $\Gamma^k_{ij}$ denote the Christoffel Symbols of the metric connection defined by the relation $\nabla_{\frac{\partial}{\partial x_i}} \frac{\partial}{\partial x_j} = \sum_{k=1}^n \Gamma^k_{ij} \frac{\partial}{\partial x_k}$. The equation (5.2) can be written more succinctly as

$$\text{Hess} u = D^2 u - \Gamma^x(Du)$$

where $Du$ and $D^2u$ are the first and second derivatives of $u$ in the coordinates $x$ and $\Gamma^x : \mathbb{R}^n \to \text{Sym}^2(\mathbb{R}^n)$ denotes the linear Christoffel map defined above.

5.2. The canonical splitting. The riemannian hessian, combined with the exterior derivative, determines a bundle isomorphism

$$\Psi : J^2(X) \longrightarrow \mathbb{R} \oplus T^*X \oplus \text{Sym}^2(T^*X)$$

defined on a $C^2$-function $u$ at a point $x$ by

$$\Psi(u) \equiv (u(x), (du)_x, \text{Hess}_x u).$$

Note that the right hand side depends only on the 2-jet of $u$ at $x$, and hence $\Psi$ is a well-defined bundle map.

5.3. Local trivializations associated to framings. Let $e_1, ..., e_n$ be a local framing of the tangent bundle $TX$ on some neighborhood $U$. Then the canonical splitting (5.3) determines a local trivialization of $J^2(X)$ by composing $\Psi$ with the isomorphism

$$\Phi : \mathbb{R} \oplus T^*U \oplus \text{Sym}^2(T^*U) \longrightarrow \mathbb{R} \oplus \mathbb{R}^n \oplus \text{Sym}^2(\mathbb{R}^n)$$

given at $x$ by

$$\Phi(u, du, \text{Hess} u) \equiv (r, p, H) \quad \text{where} \quad r \equiv u, \quad p \equiv (e_1(u), ..., e_n(u)), \quad \text{and} \quad H_{ij} \equiv (\text{Hess} u)(e_i, e_j).$$
Suppose in particular that we choose the framing $\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n}$ for local coordinates defined on $U$. Then from (5.2)' we have that the local trivialization is

$$(\Phi \circ \Psi)(u) = (u, Du, D^2u - \Gamma_x(Du)).$$

(5.5)

### 5.4. Transformations associated to a change of framing.

Suppose we are given a change of framing $e = h\xi$ over the neighborhood $U$, that is,

$$e_i(x) = \sum_{j=1}^{n} h_{ij}(x)\xi_j(x)$$

(5.6)

for a smooth function $h : U \to \text{GL}_n(\mathbb{R})$. Then the associated trivializations are related by

$$(r, p, H) = (r, h(x) \cdot \xi, h(x) \cdot \bar{H} \cdot h^t(x))$$

(5.7)

as one can see by noting for example that

$$(\text{Hess } u)(e_i, e_j) = (\text{Hess } u)\left(\sum_k h_{ik}\xi_k, \sum_\ell h_{j\ell}\xi_\ell\right) = \sum_{k,\ell} h_{ik}(\text{Hess } u)(\xi_k, \xi_\ell) h_{j\ell}. $$

### 5.5. Pure Second-Order Subequations.

On a riemannian manifold $X$ these are the subequations which involve only the riemannian hessian and not the value of the function or its gradient. That is, with respect to the riemannian splitting (5.3) a subequation of the form

$$F = \mathbb{R} \oplus T^*X \oplus F',$$

for a closed subset $F' \subset \text{Sym}^2(T^*X)$ will be called **pure second-order**. Said differently, a subequation is pure second order if for $C^2$ functions $u$ it is defined by the requirement that $\text{Hess } u \in F'$.

Note that condition (P) for $F$ is equivalent to

$$F' + \mathcal{P} \subset F', \text{ which implies that } F' + \text{Int}\mathcal{P} \subset \text{Int}F'$$

Since $A + \epsilon I$ approximates any $A \in F'$, it follows that Condition (P) implies

$$(i) \quad F = \overline{\text{Int}F} \quad \text{and} \quad (ii) \quad F_x = \overline{\text{Int}F_x}.$$

In summary: A closed subset $F' \subset \text{Sym}^2(T^*X)$ is a pure second-order subequation if and only if

$$F' + \mathcal{P} \subset F' \quad \text{and} \quad \text{Int}F'_x = (\text{Int}F')_x \quad \text{for all } x.$$

(5.8)

### 5.6. The complex and quaternionic hessians.

Let $X$ be a riemannian manifold equipped with a pointwise orthogonal almost complex structure $J : TX \to TX$. On this hermitian almost complex manifold one can define the **complex hessian** of a $C^2$-function $u$ by

$$\text{Hess}^C u \equiv \frac{1}{2} \{ \text{Hess } u - J(\text{Hess } u)J \}.$$

(5.9)
This is a hermitian symmetric quadratic form on the complex tangent spaces of \(X\). In particular its eigenvalues are real with even multiplicity and its eigenspaces are \(J\)-invariant. (See §15 for more details.)

Analogously suppose \(X\) is equipped with a hermitian almost quaternionic structure, i.e., orthogonal bundle maps \(I, J, K : TX \rightarrow TX\) satisfying the standard quaternionic identities: \(I^2 = J^2 = K^2 = -1, \, IJ = -JI = K\), etc. Then one can define the **quaternionic hessian**

\[
\text{Hess}^H u \equiv \frac{1}{4} \{\text{Hess} u - I(\text{Hess} u)I - J(\text{Hess} u)J - K(\text{Hess} u)K\}.
\] (5.10)

A number of basic subequations are defined in terms of these Hessians. Primary among them are the following.

### 5.7. The Monge-Ampère subequations

A classical subequation, which is defined on any riemannian manifold \(X\) and will play an important role in this paper, is the (real) **Monge-Ampère subequation**

\[
P \equiv P^R \equiv \{J^2(u) : \text{Hess}^R u \geq 0\}
\] (5.11)

When \(X\) carries an almost complex or quaternionic structure as above, we also have the associated **complex Monge-Ampère subequation**

\[
P^C \equiv \{J^2(u) : \text{Hess}^C u \geq 0\}
\] (5.12)

and **quaternionic Monge-Ampère subequation**

\[
P^H \equiv \{J^2(u) : \text{Hess}^H u \geq 0\}
\] (5.13)

One easily checks that these sets are in fact subequations and have the convex cone property. They provide important examples of monotonicity cones for many other subequations and play an important role in the study of those subequations (See Section 8). In addition they typify an important general construction which we now present.

### 5.8. Geometrically defined subequations – Grassmann structures

Consider the Grassmannian \(G(p, T_x X)\) of unoriented \(p\)-planes \(\xi\) through the origin in \(T_x X\). If \(X\) is a riemannian manifold, then we can identify \(\xi \in G(p, T_x X)\) with orthogonal projection \(P_\xi \in \text{Sym}^2(T^*_x X)\) onto \(\xi\).

Given \(A \in \text{Sym}^2(T^*_x X)\) and \(\xi \in G(p, T_x X)\), the **\(\xi\)-trace** of \(A\) is defined by

\[
\text{tr}_\xi A = \langle A, P_\xi \rangle
\] (5.14)

using the natural inner product on \(\text{Sym}^2(T^*_x X)\). Equivalently,

\[
\text{tr}_\xi A = \text{trace} \left( A \big|_{\xi} \right)
\] (5.15)

where \(A \big|_{\xi} \in \text{Sym}^2(\xi)\) is the restriction of \(A\) to \(\xi\).

Note that for any closed subset \(G_x \subset G(p, T_x X)\) the set

\[
F(G_x) = \{A \in \text{Sym}^2(T^*_x X) : \text{tr}_\xi A \geq 0 \quad \forall \xi \in G_x\}
\]

is a closed convex cone with vertex at the origin in \(\text{Sym}^2(T^*_x X)\). Moreover, \(F_x\) automatically satisfies positivity since if \(P \geq 0\), \(\text{tr}_\xi P \geq 0\) for all \(\xi \in G(p, T_x X)\).
Definition 5.1. Given a closed subset \( \mathcal{G} \) of the Grassmann bundle \( G(p, TX) \), the convex cone subequation \( F(\mathcal{G}) \) defined by:

\[
\text{tr}_x \text{Hess}_x u \geq 0 \quad \text{for all } \xi \in \mathcal{G}_x \subset G(p, T_x X)
\]

for \( C^2 \) functions \( u \), is said to be geometrically defined by \( \mathcal{G} \). Moreover, \( F(\mathcal{G}) \)-subharmonic functions will be referred to as \( \mathcal{G} \)-plurisubharmonic functions.

5.9. Gradient independent subequations. On a riemannian manifold \( X \) there are the subequations which only involve the riemannian hessian and the value of the function. That is, with respect to the riemannian splitting (5.3) a subset of the form

\[
F = T^*X \oplus F',
\]

for a closed subset \( F' \subset \mathbb{R} \oplus \text{Sym}^2(T^*X) \) is said to be gradient-independent. Note that

\[
F' \equiv \mathcal{N} + \mathcal{P} \subset \mathbb{R} \oplus \text{Sym}^2(T^*X)
\]

provides an example of a gradient-independent subequation. It is also a convex cone subequation.

In general a closed subset \( F = T^*X \oplus F' \) satisfies (P) and (N) if and only if

\[
F' + \mathcal{N} + \mathcal{P} \subset F'.
\]

The topological conditions (T)(i) and (T)(ii) then follow from (P) and (N) by using \( (r, A) = \lim_{\epsilon \to 0} (r - \epsilon, A + \epsilon I) \). This proves that \( F \subset J^2(X) \) is a gradient-independent subequation if and only if \( F \) is of the form \( F = T^*X \oplus F' \) and \( F' \) satisfies:

\[
F' \text{ is closed, } F' + \mathcal{N} + \mathcal{P} \subset F', \quad \text{and } \text{Int} F'_x = (\text{Int} F')_x \text{ for all } x.
\]

Said differently, a closed subset \( F' \subset \mathbb{R} \oplus \text{Sym}^2(T^*X) \) with \( \text{Int} F'_x = (\text{Int} F')_x \) for all \( x \), is a (gradient-independent) subequation if and only if \( F' \) is \( (\mathcal{N} + \mathcal{P}) \)-monotone.

6. Universal Subequations on Manifolds with Topological \( G \)-Structure.

In this section we construct the subequations of principal interest in the paper, namely, subequations on riemannian manifolds which are “universally determined”. The construction starts with a “universal model”, which is a constant coefficient subequation

\[
F \subset J^2.
\]

The idea is to find subequations \( F \subset J^2(X) \) which are locally equivalent to the constant coefficient subequations with model fibre \( F \).

To accomplish this we consider the compact invariance group

\[
G = G(F) \equiv \{ g \in O_n : g(F) = F \}
\]

where \( O_n \) acts on \( J^2 \) by

\[
g(r, p, A) \equiv (r, gp, gAg^t).
\]

The main point is that whenever a riemannian manifold \( X \) is given a topological \( G \)-structure, we can construct the desired subequation by using this structure and the canonical splitting of \( J^2(X) \) in the pervious section. Subequations \( F \) constructed from \( F \) in this way will be called \( G \)-universal.
Recall that for a fixed subgroup $G \subseteq \text{O}_n$, a topological $G$-structure on $X$ is a family of $C^\infty$ local trivializations of $TX$ over open sets in a covering $\{U_\alpha\}$ of $X$ whose transition functions have values in $G$, i.e., the transition function from the $\alpha$-trivialization to the $\beta$-trivialization is just a map $G^{\beta,\alpha} : U_\alpha \cap U_\beta \rightarrow G$. A local trivialization on $U_\alpha$ is simply a choice of framing $e_1^\alpha, \ldots, e_n^\alpha$ for $TX$ over $U_\alpha$, and one can think of these trivializations as a family of admissible framings for $TX$.

By using the riemannian hessian, each local framing $e_1^\alpha, \ldots, e_n^\alpha$ for $TX$ also induces a trivialization of the 2-jet bundle $\Phi_\alpha : J^2(U_\alpha) \rightarrow U_\alpha \times \mathbb{R} \times \mathbb{R}^n \times \text{Sym}^2(\mathbb{R}^n)$ defined by

$$\Phi_\alpha(J^2 u) = (x, u(x), (e_1^\alpha(u), \ldots, e_n^\alpha(u)), (\text{Hess} u)(e_i^\alpha, e_j^\alpha)).$$

for $C^2$-functions $u$ on $U_\alpha$. (Here $\Phi_\alpha$ is the composition $\Phi_\alpha = \Phi'_\alpha \circ \Psi$ where $\Psi$ is defined in Section 5.2 and $\Phi'_\alpha$ is the map associated to the framing $\{e_k^\alpha\}$ as in Section 5.3).

**Lemma 6.1.** Suppose $F \subset J^2$ is a constant coefficient subequation with invariance group $G$, and that $X$ is furnished with a topological $G$-structure. Then the condition on a 2-jet $J \in J^2_2(X)$ that

$$\Phi_\alpha(J) \in F$$

is independent of the choice of admissible framing $e_1^\alpha, \ldots, e_n^\alpha$.

**Proof.** This follows from the frame change formula (5.7). 

**Corollary 6.2.** There is a well-defined second order subequation

$$F \subset J^2(X)$$

induced by the universal model $F \subset J^2$, such that for $u \in C^2(X)$:

$$J^2 u \in F \iff (u, (e_1(u), \ldots, e_n(u)), (\text{Hess} u)(e_i, e_j)) \in F$$

for each admissible framing $e_1, \ldots, e_n$. Moreover, $\bar{F}$ is induced by $\bar{F}$.

**Proof.** It is easy to show that

(a) $F$ satisfies (P) $\iff$ $F$ satisfies (P).

(b) $F$ satisfies (N) $\iff$ $F$ satisfies (N).

(c) $F$ satisfies (T) $\iff$ $F$ satisfies (T).

(d) $\bar{F}$ is induced by $\bar{F}$.

Note that $F$ is $G$-invariant if and only if $\bar{F}$ is $G$-invariant.

**Definition 6.3.** Suppose $F \subset J^2$ is a universal model invariant under a subgroup $G \subset \text{O}_n$, and let $X$ be a riemannian manifold with a topological $G$-structure. Then the subequation $F$ induced by $F$ on $X$ via the ansatz (6.5) will be referred to as a $G$-universal subequation constructed from $F$.

For simplicity the $F$-subharmonic functions on $X$ will be called $F$-subharmonic.
THEOREM 6.4. A \( G \)-universal subequation \( F \) constructed from \( F \) is locally equivalent to the constant coefficient subequation \( F \).

Proof. It suffices to prove the following.

Lemma 6.5. Suppose \( x = (x_1, ..., x_n) \) is a local coordinate system on \( U \) and that \( e_1, ..., e_n \) is an admissible frame on \( U \). Let \( h \) denote the \( \text{GL}_n \)-valued function on \( U \) defined by

\[
e_i = \sum_{j=1}^{n} h_{ij} \frac{\partial}{\partial x_j}
\]

and recall the Christoffel operator \( \Gamma \) from Subsection 5.1. Then a \( C^2 \)-function \( u \) is \( F \)-subharmonic on \( U \) if and only if

\[
(u, hD\!u, h(D^2u - \Gamma(Du))h^t) \in F \text{ on } U.
\]

(6.6)

Remark 6.6. Fix \( J \in J^2_x \) with \( x \in U \). Then by using the standard isomorphism \( J^2_x \cong \mathbb{R} \times \mathbb{R}^n \times \text{Sym}^2(\mathbb{R}^n) \) induced by the coordinate system \( x = (x_1, ..., x_n) \) to represent \( J \) as \( J = (r, p, A) \), Lemma 6.5 can be restated as saying that

\[
J \in F_x \iff (r, h^t p, h(A - \Gamma(p))h^t) \in F
\]

(6.7)

Proof. First note that

\[
e_i(u) = \sum_{j=1}^{n} h_{ij} \frac{\partial u}{\partial x_j}
\]

or simply \( e(u) = hD\!u \). Then note that

\[
(Hess u)(e_i, e_j) = (Hess u) \left( \sum_{k=1}^{n} h_{ik} \frac{\partial}{\partial x_k}, \sum_{\ell=1}^{n} h_{j\ell} \frac{\partial}{\partial x_\ell} \right)
= \sum_{k, \ell} h_{ik} (Hess u) \left( \frac{\partial}{\partial x_k}, \frac{\partial}{\partial x_\ell} \right) h_{j\ell}.
\]

By (5.2) the matrix

\[
(Hess u) \left( \frac{\partial}{\partial x_k}, \frac{\partial}{\partial x_\ell} \right) = D^2u - \Gamma(Du)
\]

so that the matrix \( (Hess u)(e_i, e_j) \) equals \( h(D^2u - \Gamma(Du))h^t \). Finally, apply (6.5).

Example 1. Note that the construction above with \( G = \{1\} \) will apply to any riemannian manifold \( X \) on which there exist \( n \) vector fields which are orthonormal at every point. For example, this is true on any orientable riemannian 3-manifold. It is also true of any Lie group with a metric invariant under left translations by elements of the group.

Further interesting examples are given by less drastic reductions of the structure group.

Example 2. If \( X \) is an almost complex manifold with a compatible (hermitian) metric, then one can consider \( F = \mathbb{R} \times \mathbb{R}^{2n} \times \mathbb{P}_C \) where \( \mathbb{P}_C \subset \text{Sym}^2(\mathbb{R}^n) \) are the matrices whose hermitian symmetric part is \( \geq 0 \) (cf. (5.9) and Section 15). This leads to solving the the Dirichlet problem for the homogeneous complex Monge-Ampere equation on domains in \( X \).
7. Strictly F-Subharmonic Functions.

Consider a second order subequation \( F \subset J^2(X) \) on a manifold \( X \). For the more general case of subsets which are just \( \mathcal{P} \)-monotone see Remark 7.8. Strict subharmonicity for \( C^2 \)-functions is unambiguous.

**Definition 7.1.** A function \( u \in C^2(X) \) is said to be strictly \( F \)-subharmonic on \( X \) if

\[
J_x^2 u \in \text{Int} F \quad \text{for all } x \in X.
\]

(7.1)

There is more than one way to extend this notion to functions \( u \in F(X) \) which are not \( C^2 \). The choice made in Definition 7.4 will be shown to be useful in the next section 8 and in discussing the Dirichlet Problem.

Fix a metric on the vector bundle \( J^2(X) \). (If \( X \) is given a riemannian metric, this metric together with the canonical splitting (5.4) determines a metric on \( J^2(X) \).

**Definition 7.2.** For each \( c > 0 \) we define the \( c \)-strict subset \( F^c \subset F \) by

\[
F^c_x = \{ J \in F_x : \text{dist}(J, \sim F_x) \geq c \}
\]

where \( \text{dist} \) denotes distance in the fibre \( J^2_x(X) \).

**Lemma 7.3.** Suppose that \( F \) is a subequation. Then the set \( F^c \) is both \( \mathcal{P} \)- and \( \mathcal{N} \)-monotone. However, \( F^c \) does not necessarily satisfy (T).

**Proof.** Note that for \( J \in J^2_x(X) \),

\[
J \in F^c_x \iff B_x(J, c) \subseteq F_x
\]

(7.3)

where \( B_x(J, c) \subset J^2_x(X) \) is the closed metric ball of radius \( c \) about the point \( J \) in the fibre. Suppose now that \( J_i \in F^c_x \) is a sequence converging to \( J \) at \( x \). Then \( B_x(J_i, c) \subseteq F_x \) for all \( i \), and since \( F \) is closed we conclude that \( B_x(J, c) \subseteq F_x \). Hence \( F^c \) is closed.

Suppose now that \( P \in \mathcal{P}_x \). Then by Condition (P) for \( F \) we see that \( B_x(J, c) \subseteq F_x \implies B_x(J + P, c) = B_x(J, c) + P \subseteq F_x \), and so Condition (P) holds for \( F^c \). The proof that \( F^c \) satisfies Condition (N) is the same.

**Example.** Let \( K \) denote the union of the unit disk \( \{|z| \leq 1\} \) with the interval \( [1, 2] \) on the \( x \)-axis in \( \mathbb{R}^2 \). Let \( K_c = \{ z \in \mathbb{R}^2 : \text{dist}(x, K) \leq c \} \). Define \( F \) by requiring \( p \in K_c \). Then \( F^c \) is easily seen to be equal to \( K \). Now \( F \) is a subset which satisfies (P), (N) and (T), whereas \( F^c \) is a subset which satisfies (P) and (N) but not (T).

**Definition 7.4.** A function \( u \) on \( X \) is strictly \( F \)-subharmonic if for each point \( x \in X \) there is a neighborhood \( B \) of \( x \) and \( c > 0 \) such that \( u \) is \( F^c \)-subharmonic on \( B \). Let \( F_{\text{strict}}(X) \) denote the space of such functions.

For a \( C^2 \) function it is easy to see that the two definitions of strictly \( F \)-subharmonicity, given in Definitions 7.1 and 7.4 agree (see Remark 7.8). Moreover, it is easy to see that Definition 7.4 is independent of the choice of metric on the bundle \( J^2(X) \).

Strictly \( F \)-subharmonic functions are stable under smooth perturbations.

**Lemma 7.5.** (Stability). Suppose \( u \in F^c(X) \) and \( \psi \in C^2(X) \). For each precompact open subset \( Y \subset X \)

\[
u + \delta \psi \in F^c(Y) \quad \text{if } \delta \text{ is sufficiently small.}
\]
\textbf{Remark 7.8.} Choose \( \delta > 0 \) so that
\begin{equation}
\delta \| J^2_x \psi \| < \frac{\epsilon}{2} \quad \text{for all } x \in Y.
\end{equation}

Fix \( J \in F^c_x \) with \( x \in Y \). By (7.3) we have \( B_x(J, c) \subset F_x \). Note then that \( B_x(J + \delta J^2_x \psi, \frac{\epsilon}{2}) = B_x(J, \frac{\epsilon}{2}) + \delta J^2_x \psi \) is contained in \( B_x(J, c) \subset F_x \) by (7.5). Again using (7.3) this shows that \( J + \delta J^2_x \psi \in F^\delta_x \).

\textbf{Corollary 7.6.} Suppose \( u \in F_{\text{strict}}(X) \) and \( \psi \in C^\infty_{\text{opt}}(X) \). Then
\begin{equation*}
u + \delta \psi \in F_{\text{strict}}(X) \quad \text{if } \delta \text{ is sufficiently small.}
\end{equation*}

We shall need the following two properties.

\textbf{Lemma 7.7.}
\begin{enumerate}[(i)]
\item \( u, v \in F_{\text{strict}}(X) \implies \max\{u, v\} \in F_{\text{strict}}(X) \)
\item If \( F \) satisfies the negativity condition (N), then \( u \in F_{\text{strict}}(X) \) and \( c > 0 \implies u - c \in F_{\text{strict}}(X) \)
\end{enumerate}

The proof is straightforward and omitted.

\textbf{Remark 7.8. (\( \mathcal{P} \)-monotone subsets).} The results of this section remain true for \( \mathcal{P} \)-monotone subsets and for \((\mathcal{P} + \mathcal{N})\)-monotone subsets which are not necessarily subequations (i.e., condition (T) may not be satisfied). This is important in Section 11, on boundary convexity, where the results are applied to a \( \mathcal{P} \)-monotone subset \( \overline{F} \) which is open. Everything is straightforward except the proof of the assertion
\begin{equation}
\psi \in C^2(X) \cap F_{\text{strict}}(X) \implies J^2_x \psi \in \text{Int} F \quad \text{for all } x \in X.
\end{equation}

In general (7.6) cannot be proved by establishing that \( F^c \subset \text{Int} F \). For example, take \( X = \mathbb{R} \) and define \( F \) by requiring \( p \geq 0 \) if \( x \leq 0 \) and \( p \geq 1 \) if \( x > 0 \). Then the point \( x = 0, p = \frac{1}{2} \) belongs to \( F^\frac{1}{2} \) but not \( \text{Int} F \).

\textbf{Proof of (7.6).} Fix \( x_0 \in X \) and a trivialize \( J^2(X) = U \times J^2 \) on a neighborhood \( U \) of \( x_0 \) via a choice of orthonormal frame field. For such a choice the fibre metric on \( J \) constant. Now \( \psi \) is \( c \)-strict for some \( c > 0 \) in a smaller neighborhood \( U \) of \( x_0 \), i.e., \( B(J^2_x \psi, c) \subset F_x \) for \( x \in U \). By continuity we have \( |J^2_x \psi - J^2_{x_0} \psi| \leq \frac{c}{2} \) on a neighborhood \( V \) of \( x_0 \), and therefore \( B(J^2_{x_0} \psi, \frac{c}{2}) \subset F_x \) for \( x \in W \equiv U \cap V \). Thus \( W \times B(J^2_{x_0} \psi, \frac{c}{2}) \subset F \) and in particular \( J^2_x \psi \in \text{Int} F \).

The following elementary example shows that \( J^2_x \psi \in \text{Int}(F_x) \) for \( x \) near \( x_0 \) (rather than \( J^2_{x_0} \psi \in (\text{Int} F)_{x_0} \) as in (7.1)) is not sufficient to guarantee that \( \psi \) is \( c \)-strict near \( x_0 \). Let \( F \) be the one variable subequation defined by \( \{|p| \leq |x|\} \cup \{(0) \times [-1, 1]\} \). Take \( \psi \equiv 0 \). Then:
\begin{enumerate}
\item \( \psi \) is not strictly \( F \)-subharmonic since \( J^2 \psi \notin \text{Int} F \),
\item \( \psi \) is not \( c \)-strict near \( x = 0 \) since \( F^c_x = \emptyset \) for \( 0 < |x| < c \), but
\item \( J^2_x \psi \in \text{Int}(F_x) \) for all \( x \).
\end{enumerate}
8. Comparison Theory – Local to Global.

In this section we begin our analysis of the uniqueness/comparison question for $F$-harmonic functions with given boundary values on a compact subset $K$ of $X$. Let us set the notation

$$F(K) \equiv \{ u \in USC(K) : u|_{\text{Int}K} \in F(\text{Int}K) \}$$

Then the comparison principle can be stated as follows:

For all $u \in F(K)$ and $-w \in \tilde{F}(K)$,

$$u \leq w \text{ on } \partial K \Rightarrow u \leq w \text{ on } K$$

(8.1)

However, we prefer to state it in the following form which invokes duality.

**Definition 8.1.** We say that **comparison holds for** $F$ on $X$ if for all compact sets $K \subset X$, whenever

$$u \in F(K) \quad \text{and} \quad v \in \tilde{F}(K),$$

the **Zero Maximum Principle** holds for $u + v$ on $K$, that is,

$$u + v \leq 0 \text{ on } \partial K \Rightarrow u + v \leq 0 \text{ on } K \quad (ZMP)$$

If comparison holds, it is immediate that

**Uniqueness for the Dirichlet Problem holds**, that is:

If two $F$-harmonic functions agree on $\partial K$, they must agree on $K$.

Local comparison does not imply global comparison. However, for a weakened form of comparison, local does imply global. This is the main result of this section.

**Definition 8.2.** We say that **weak comparison holds for** $F$ on $X$ if for all compact subsets $K \subset X$, and functions

$$u \in F^c(K), \quad v \in \tilde{F}(K), \quad c > 0$$

the Zero Maximum Principle (ZMP) holds for $u + v$ on $K$. We say that **local weak comparison holds for** $F$ on $X$ if for all $x \in X$, there exists a neighborhood $U$ of $x$ such that weak comparison holds for $F$ on $U$.

**THEOREM 8.3.** Suppose that $F$ is a subequation on a manifold $X$. If local weak comparison holds for $F$ on $X$, then weak comparison holds for $F$ on $X$.

**Proof.** Suppose weak comparison fails for $F$ on $X$. Then there exist $c > 0$, $u \in F^c(X)$, $v \in \tilde{F}(X)$ and compact $K \subset X$ with

$$u + v \leq 0 \text{ on } \partial K \text{ but } \sup_{K}(u + v) > 0.$$  

Choose a maximum point $x_0 \in \text{Int}K$ and let $M = u(x_0) + v(x_0) > 0$ denote the maximum value. Fix local coordinates $x$ on a neighborhood of $x_0$ containing $U = \{ x : |x - x_0| < \rho \}$. By choosing $\rho$ sufficiently small we can assume that weak comparison holds for $F$ on $U$.

By the Stability Lemma 7.5, we have $u' = u - \delta |x - x_0|^2 \in F^\sharp(U)$ if $\delta$ is chosen small enough. Now $x_0$ is the unique maximum point for $u' + v$ on $U$, with maximum value $M$. Choose
Thus, weak comparison fails for $F$ on $U$, contrary to assumption.

Weak comparison can be strengthened as follows. Define

$$F_{\text{strict}}(K) = \{ u \in \text{USC}(K) : u|_{\text{Int}K} \in F_{\text{strict}}(\text{Int}K) \}$$ (8.2)

**Lemma 8.4.** Suppose weak comparison holds for $F$ on $X$. For all compact sets $K \subset X$, if $u \in F_{\text{strict}}(K)$ and $v \in F(K)$, then $u + v$ satisfies the (ZMP).

**Proof.** We assume $u + v \leq 0$ on $\partial K$. Since $u \in \text{USC}(K)$, for each $\delta > 0$ the set

$$U_\delta = \{ x \in K : u(x) + v(x) < \delta \}$$

is an open neighborhood of $\partial K$ in $K$. Exhaust $\text{Int}K$ by compact sets $K_\epsilon$ with $\text{Int}K = \bigcup \epsilon K_\epsilon$. Then $\partial K_\epsilon \subset U_\delta$ for $\epsilon > 0$ small. Now $u - \delta + v \leq 0$ on $\partial K_\epsilon$. However, $u - \delta$ is $F^c$-subharmonic on $\text{Int}K_\epsilon$ for some $c > 0$. Hence, (WC) for $K_\epsilon$ states that $u - \delta + v \leq 0$ on $K_\epsilon$. Thus $u - \delta + v \leq 0$ on $\text{Int}K$. Hence $u + v \leq 0$ on $K$.

**9. Strict Approximation and Monotonicity Subequations.**

In this section we discuss certain global approximation techniques which can be used to deduce comparison from weak comparison. Consider a general second order subequation $F$ on a manifold $X$.

**Definition 9.1.** We say that **strict approximation** holds for $F$ on $X$ if for each compact set $K \subset X$, each function $u \in F(X)$ can be uniformly approximated by functions in $F_{\text{strict}}(K)$.

When strict approximation is available, it is easy to show that weak comparison implies comparison.

**THEOREM 9.2. (Global Comparison).** Suppose $F$ is a subequation on a manifold $X$. Assume that both local weak comparison and the strict approximation property hold for $F$ on $X$. Then comparison holds for $F$ on $X$.

**Proof.** Suppose $u \in F(K)$, $v \in F(K)$, and $u + v \leq 0$ on $\partial K$. By Theorem 8.3 we can assume that weak comparison holds. By strict approximation, for each $\epsilon > 0$, there exists $\overline{\sigma} \in F_{\text{strict}}(K)$ with $u - \epsilon \leq \overline{\sigma} \leq u + \epsilon$ on $K$. Now Property (N) implies that $\overline{\sigma} - \epsilon \in F_{\text{strict}}(K)$. Note that $\overline{\sigma} - \epsilon + v \leq u + v \leq 0$ on $\partial K$. Lemma 8.4 states that $\overline{\sigma} - \epsilon + v \leq 0$ on $K$. This proves that $u + v$ satisfies the (ZMP).

**Example 9.3. (The Eikonal Equation).** This subequation $F$ on $\mathbb{R}^n$ is defined by $|\nabla u| \leq 1$. Given $u \in F(K)$, set $u_\epsilon = (1 - \epsilon)u$. Then $u_\epsilon \in F^\epsilon(K)$ because if $\varphi$ is a test function for $(1 - \epsilon)u$ at $x_0$, then $\frac{1}{1-\epsilon}\varphi$ is a test function for $u$ at $x_0$. Thus $|\nabla \varphi(x_0)| \leq 1 - \epsilon$. 

32
In contrast to this example, for the geometric subequations $F$ that are of primary interest in this paper, the approximations $u_\epsilon$ will be of the form

$$u_\epsilon = u + \epsilon \psi \quad 0 < \epsilon \leq \epsilon_0 \quad (9.1)$$

where $\psi$ is a $C^2$-function independent of $u$. The function $\psi$ will be referred to as an **approximator** for $F$.

Suppose $M \subset J^2(X)$ is a subset such that the fibre-wise sum

$$F + M \subset F \quad \text{and} \quad \epsilon M \subset M \quad \text{for} \ 0 < \epsilon \leq \epsilon_0. \quad (9.2)$$

If $\psi$ is a $C^2$-function which is strictly $M$-subharmonic, then $\psi$ is an approximator for $F$ (see the proof of Theorem 9.5). However, condition (9.2) implies that at each point $x$

$$F_x + \alpha_1 J_1 + \alpha_2 J_2 \subset F_x \quad \text{for all} \ \alpha_1 > 0, \alpha_2 > 0 \quad \text{and} \ J_1, J_2 \in M_x. \quad (9.3)$$

Hence we might as well assume that $M$ is a convex cone.

**Definition 9.4.** A subset $M \subset J^2(X)$ will be called a **convex monotonicity cone for** $F$ if

1. $M$ is a convex cone with vertex at the origin, and
2. $F + M \subset F$.

**Theorem 9.5.** Suppose $M$ is a convex monotonicity cone for $F$ as above. If there exists $\psi \in C^2(X)$ which is strictly $M$-subharmonic, then strict approximation holds for $F$ on $X$.

**Proof.** It will suffice to establish the following.

**Assertion 9.6.** For each compact subset $K \subset X$, there exists $\delta > 0$ such that

$$u + \epsilon \psi \in F^\delta(K) \quad \text{for all} \ u \in F(K) \quad \text{and for all} \ \epsilon > 0. \quad \text{To begin note that} \ u - \varphi \text{ has local maximum 0 at a point } x \text{ if and only if } u + \epsilon \psi - (\varphi + \epsilon \psi) \text{ has local maximum 0 at } x. \quad \text{Hence we must show that under the hypothesis } J^2_x \varphi \in F_x \text{ we have that } J^2_x (\varphi + \epsilon \psi) = J^2_x \varphi + \epsilon J^2_x \psi \in F^\delta_x, \text{ in other words that } F_x + \epsilon J^2_x \psi \subset F^\delta_x.$$

Since $\psi \in C^2(X)$, $J^2_x \psi$ is a continuous function of $x$. Hence $\{J^2_x \psi : x \in K\}$ is compact. That $\psi$ is strictly $M$-subharmonic implies that $\{J^2_x \psi : x \in K\}$ is a compact subset of $\text{Int} M$. Take $\delta = \text{the distance from } \{J^2_x \psi : x \in K\} \text{ to } \sim \text{Int} M$. Then $B(J^2_x \psi, \delta) \subset M_x \text{ for all } x \in K \text{ where } B \text{ denotes the ball in the fibre.}$

Suppose now that $J \in F_x$ and $x \in K$. Then

$$B(J + \epsilon J^2_x \psi, \epsilon \delta) = J + \epsilon B(J^2_x \psi, \delta) \subset F_x + M_x \subset F_x$$

as desired. $\square$

Combining Theorem 9.2 with Theorem 9.5 yields the version of Global Comparison that will be used in this paper.

**Theorem 9.7.** Suppose $F$ is a subequation on a manifold $X$. Assume that $X$ supports a $C^2$ function which is strictly $M$-subharmonic, where $M$ is a monotonicity cone for $F$. Then local weak comparison for $F$ implies global comparison for $F$ on $X$.

**Remark 9.8.** (Circular Monotonicity Cones). In a situation where the $C^2$-function $\psi$ is given, the simplest monotonicity cone to consider is one whose fibre at each point $x$ is a circular
cone $C(J)$ about $J \equiv J^2_2\psi$. If $\delta = \text{dist}(J^2_2\psi, \sim \text{Int} M)$, as in the above proof, then the circular cone can be taken to be the cone $C^\delta(J)$ on the ball $B(J, \delta)$. The cross-section of this cone $C^\delta(J)$ by the hyperplane (through $J$) perpendicular to $J$, is a ball of radius $R$ in this hyperplane, where, setting $\gamma = 1/R$, one calculates that

$$\delta = \frac{|J|}{\sqrt{1 + \gamma^2|J|^2}}$$

This same cone will be denoted by $C_\gamma(J)$ when $\gamma$ is to be emphasized.

**Lemma 9.9.** Suppose that $F$ is a subequation with $F_x \neq \emptyset$ and $F_x \neq J^2_2(X)$, and fix $J \in J^2_2(X)$. The following are equivalent.

(1) $F_x$ is $C_\gamma(J)$-monotone.

(2) The boundary $\partial F_x$ can be graphed over the hyperplane $J^\perp$ with graphing function $f$ which is $\gamma$-Lipschitz, i.e., for $J_0, J \in J^\perp$

$$J_0 + tJ \in F_x \iff t \geq f(J_0),$$

and for all $J_0, J_0 \in J^\perp$

$$-\gamma|J_0| \leq f(J_0 + J_0) \leq \gamma|J_0|$$

The elementary proof is left to the reader.

**Remark 9.10.** Suppose $F$ is a universal model (i.e., a constant coefficient subequation) which is $G$-invariant. If $F$ has a convex monotonicity cone $M$ which is $G$-invariant, then on a manifold $X$ with topological $G$-structure, the induced subequation $F$ and the induced convex monotonicity cone $M$ satisfy $F + M \subset F$, i.e., $M$ is a convex monotonicity cone for $F$ on $X$.

**Remark 9.11.** The global comparison Theorems 9.2 and 9.7 are useful even locally. In the next Section 10 we will prove that weak comparison holds for any constant coefficient subequation $F$ on $\mathbb{R}^n$. Consequently, if $F$ has a monotonicity cone $M$ with non-empty interior, then local comparison holds for $F$. To prove this, fix a point $x_0$ and pick a point $(r, p, A) \in \text{Int} M$ above $x_0$. Let $\psi$ denote the quadratic function whose 2-jet at $x_0$ equals $(r, p, A)$. Then $\psi$ is strictly $M$-subharmonic in a neighborhood of $x_0$.

Given a constant coefficient subequation $F$, we ask the question:

When does $F$ have a monotonicity cone $M$ with interior? (9.4)

We might as well assume that $M = \overline{\text{Int} M}$. Now $M$ need not satisfy conditions (P) or (N), i.e. $M$ need not be a subequation. (For example let $M = C(J)$ in Remark 9.8.) However, $M' = M + (\mathbb{R}_- \times \{0\} \times \mathcal{P})$ satisfies:

(1) $M' = \overline{\text{Int} M'}$ and $M'$ is both $\mathcal{N}$- and $\mathcal{P}$-monotone, i.e., $M'$ is a subequation.

(2) $F + M \subset F$.

In summary $M'$ is a convex conical monotonicity subequation for $F$. Thus we now ask the question:

When does $F$ have a convex conical monotonicity subequation $M$? (9.5)

Each such $M$ is a convex cone with interior, and $M$ contains $\mathbb{R}_- \times \{0\} \times \mathcal{P}$. Hence $M$ can be thought of as a fattening of $\mathbb{R}_- \times \{0\} \times \mathcal{P}$ to a set with interior.
In the next subsection we present a few of the basic examples of monotonicity subequations $M$ on a manifold. They are all based on a universal model $M$ satisfying Definition 9.4. We leave it to the reader to explicitly describe the model $M$ in the examples.

Examples of Strict Approximation Using Monotonicity.

Most of the following examples are purely second order. We use the canonical splitting from Section 5.2:

$$J^2(X) \cong \mathbb{R} \oplus T^*X \oplus \text{Sym}^2(T^*X). \quad (9.6)$$

on the riemannian manifold $X$. They are the subequations $F \subset J^2(X)$ which are the pull-backs of subsets $F' \subset \text{Sym}^2(T^*X)$ under the projection $J^2(X) \to \text{Sym}^2(T^*X)$ induced by (9.6). (See Subsection 5.5.)

Example 9.12. (The Real Monge-Ampère Monotonicity Subequation). For all pure second order subsets $F$ the Positivity Condition (P) is equivalent to $F$ being $P$-monotone where

$$P = \mathbb{R} \oplus T^*X \oplus \mathcal{P}$$

is the Monge-Ampère subequation discussed in Subsection 5.7. For a general riemannian manifold $X$, strict $P$-subharmonicity is the same as strict convexity.

THEOREM 9.13. Suppose $X$ is a riemannian manifold which supports a strictly convex $C^2$ function. Then strict approximation holds for every pure second order subequation $F$ on $X$.

For example in $\mathbb{R}^n$ the function $|x|^2$ is strictly $P$-subharmonic. More generally if $X$ has sectional curvature $\leq 0$, the function $\delta(x) = \text{dist}(x,x_0)^2$ is strictly $P$-convex up to the first cut point of $x_0$. In particular, if $X$ is complete and simply connected, then $\delta$ is globally strictly $P$-convex. Of course on any riemannian manifold $X$ the function $\delta$ is strictly $P$-subharmonic in a neighborhood of $x_0$ since $\text{Hess}_{x_0}\delta = 2I$. Thus strict approximation holds for all pure second order subequations in these cases.

Example 9.14. (The Complex Monge-Ampère Monotonicity Subequation). Suppose now that $(X,J)$ is a hermitian almost complex manifold and consider the projection

$$J^2(X) \to \text{Sym}^2_{\mathbb{C}}(T^*X)$$

given by projecting onto $\text{Sym}^2(T^*X)$ and then taking the Hermitian symmetric part (cf. 5.6). A subequation defined by pulling back a subset of $\text{Sym}^2_{\mathbb{C}}(T^*X)$ will be called a complex hessian subequation. Each such $F$ is $P^\mathbb{C}$-monotone where $P^\mathbb{C}$ is the complex Monge-Ampère subequation defined in 5.7.

The $C^2$ functions $\psi$ on $X$ which are strictly $P^\mathbb{C}$-subharmonic are just the classical $\mathbb{C}$-plurisubharmonic functions if $X$ is a complex manifold. We will use this terminology even if $J$ is not integrable.

THEOREM 9.15. Suppose $F$ is a complex hessian subequation on a hermitian almost complex manifold $X$. If $X$ supports a strictly $\mathbb{C}$-plurisubharmonic function of class $C^2$, then strict approximation holds for $F$ on $X$.

Example 9.16. (The Quaternionic Monge-Ampère Monotonicity Subequation). We leave it to the reader to formulate the analogous result on an almost quaternionic hermitian manifold which supports a strictly $\mathbb{H}$-plurisubharmonic function.

Note that $P \subset P^\mathbb{C} \subset P^\mathbb{H}$ so the corresponding subharmonic functions are progressively easier to find.
Example 9.17. (Geometrically Defined Monotonicity Subequations). Suppose $M^G$ is geometrically defined by a subset $G$ of the Grassmann bundle $G(p, TX)$ as in section 5.8. For example the three Monge-Ampère cases over $K = \mathbb{R}, \mathbb{C}$ and $\mathbb{H}$ are geometrically defined by taking $G$ to be the Grassmannian of $K$-lines $G(1, TX), G\mathbb{C}(1, TX) \subset G(2, TX)$ and $G\mathbb{H}(1, TX) \subset G(4, TX)$ respectively. Other important examples are given by taking $G$ to be all of $G(p, TX), G\mathbb{C}(p, TX)$ and $G\mathbb{H}(p, TX)$ for general $p$.

Any calibration $\phi$ of degree $p$ on a riemannian manifold $X$ determines the subset

$$G(\phi) = \{ \xi \in G(p, TX) : \phi(\xi) = 1 \}$$

of calibrated $p$-planes, and hence a convex cone subequation geometrically defined by $G(\phi)$. That is, a $C^2$ function $u$ is $G(\phi)$-plurisubharmonic if

$$\text{tr}_\xi \text{Hess} u \geq 0 \text{ for all } \xi \in G(\phi).$$

THEOREM 9.18. Suppose $G$ is a closed subset of the Grassmann bundle $G(p, TX)$. If $X$ supports a $C^2$ strictly $G$-plurisubharmonic function, then strict approximation holds for any subequation on $X$ which is $M^G$ monotone.

Of course, one such equation is $M^G$ itself.

Example 9.19. (Gradient Independent Subequations). These are subsets $F \subset J^2(X)$ which are the pull-backs of subsets $F' \subset \mathbb{R} \oplus \text{Sym}^2(T^*X)$ under the projection $J^2(X) \to \mathbb{R} \oplus \text{Sym}^2(T^*X)$ induced by (9.4). See Section 5.9. All such sets are $M^-$-monotone where

$$M^- \equiv \mathbb{R}^- \oplus T^*X \oplus P.$$  \hspace{1cm} (9.7)

if and only if they satisfy (P) and (N). Hence, gradient independent subequations are automatically $M^-$-monotone.

More generally, if $M'$ is any one of the pure second order monotonicity subequations discussed above, then

$$M = \mathbb{R}^- \oplus T^*X \oplus M'$$

is a gradient independent monotonicity subequation. Each subequation $F$ which is $M$ monotone must be gradient independent. The same $\psi$’s used in the previous examples will work for $F$. This is because on a compact subset $K$, $\psi - c$ is strictly negative for $c >> 0$. More precisely, Theorem 9.8 (and its complex and quaternionic versions) continues to hold for the more general gradient independent subequations.

10. A Comparison Theorem for $G$-Universal Subequations.

The main result of this section can be stated as follows.

THEOREM 10.1. Suppose $F$ is a subequation on a manifold $X$ which is locally affinely equivalent to a constant coefficient subequation $F$. Then weak comparison holds for $F$ on $X$.

A case of particular interest in this paper is the following. Suppose $X$ is a riemannian manifold equipped with a topological $G$-structure, for a closed subgroup $G \subset O_n$. Suppose $F \subset J^2$ is a constant coefficient, $G$-invariant subequation on $\mathbb{R}^n$. Let $F \subset J^2(X)$ denote the associated $G$ universal subequation on $X$ given by Corollary 6.2.
Corollary 10.2. Suppose \( F \) is a \( G \)-universal subequation on \( X \). Then weak comparison holds for \( F \)-subharmonic functions on \( X \).

**Proof.** On any coordinate chart \( U \) with an admissible local framing \( F \) is equivalent to \( F \) (by Lemma 6.5).

**Remark.** The following proof of local weak comparison uses the Theorem on Sums, which is discussed in Appendix C. The argument for equations which are honestly constant coefficient is particularly easy — see Corollary C.3.

**Proof of Theorem 10.1.** For clarity we first present the proof in the case where \( F \) is locally (linearly) equivalent to a constant coefficient equation as in Definition 4.5. By Theorem 8.3 we need only prove weak comparison on a chart \( U \). The argument for equations which are honestly constant coefficient is

By (10.4) we have

\[
\left( \begin{array}{c} A \varepsilon \\ B \varepsilon \end{array} \right) = \frac{3}{\epsilon} \begin{pmatrix} I & -I \\ -I & I \end{pmatrix},
\]

The equivalence of \( F \) and \( F \) says that

\[
\alpha' \equiv (r'_{\varepsilon}, p'_{\varepsilon}, A'_{\varepsilon}) \in F_{x_{\varepsilon}}^c \quad \text{and} \quad \beta' \equiv (s'_{\varepsilon}, q'_{\varepsilon}, B'_{\varepsilon}) \in \tilde{F}_{y_{\varepsilon}}
\]

where

\[
r'_{\varepsilon} = r_{\varepsilon}, \quad p'_{\varepsilon} = h(x_{\varepsilon})p_{\varepsilon}, \quad A'_{\varepsilon} = h(x_{\varepsilon})A_{\varepsilon}h(x_{\varepsilon})^t + L_{x_{\varepsilon}}(p_{\varepsilon}) \quad (10.5)
\]

\[
s'_{\varepsilon} = s_{\varepsilon}, \quad q'_{\varepsilon} = h(y_{\varepsilon})q_{\varepsilon}, \quad B'_{\varepsilon} = h(y_{\varepsilon})B_{\varepsilon}h(y_{\varepsilon})^t + L_{y_{\varepsilon}}(q_{\varepsilon}). \quad (10.6)
\]

Since \( \alpha' \in F_{\varepsilon}^c \), we also have for any \( P_{\varepsilon} \geq 0 \) that \( \alpha'' \equiv (r_{\varepsilon} - M_{\varepsilon}, p'_{\varepsilon}, A'_{\varepsilon} + P_{\varepsilon}) \in F_{\varepsilon}^c \). By (10.4) we have \( -\beta' \notin \text{Int} F \). Hence

\[
0 < c \leq \text{dist}(\alpha'', -\beta') = |\alpha'' + \beta'|. \quad (10.7)
\]

To complete the proof we show that \( \alpha'' + \beta' \) converges to zero. The first component of \( \alpha'' + \beta' \) is \( r - M_{\varepsilon} + s_{\varepsilon} \) which is zero by (10.1).

The second component of \( \alpha'' + \beta' \) is

\[
p'_{\varepsilon} + q'_{\varepsilon} = h(x_{\varepsilon})\frac{(x_{\varepsilon} - y_{\varepsilon})}{\epsilon} - h(y_{\varepsilon})\frac{(x_{\varepsilon} - y_{\varepsilon})}{\epsilon} = \left(h(x_{\varepsilon}) - h(y_{\varepsilon})\right)\frac{(x_{\varepsilon} - y_{\varepsilon})}{\epsilon}
\]

which converges to zero as \( \epsilon \to 0 \) by (10.2).
It remains to find \( P_\epsilon \geq 0 \) so that the third component of \( \alpha'_\epsilon + \beta'_\epsilon \), namely \( A'_\epsilon + P_\epsilon + B'_\epsilon \), converges to zero. This will contradict (10.7).

Multiplying both sides in (10.3) by
\[
\begin{pmatrix}
h(x_\epsilon) & 0 \\
0 & h(y_\epsilon)
\end{pmatrix}
\] on the left and
\[
\begin{pmatrix}
h(x_\epsilon) & 0 \\
0 & h(y_\epsilon)
\end{pmatrix}^t
\] on the right

gives
\[
\begin{pmatrix}
h(x_\epsilon)A_\epsilon h(x_\epsilon)^t \\
h(y_\epsilon)B_\epsilon h(y_\epsilon)^t
\end{pmatrix}
\leq \frac{3}{\epsilon} \left( \begin{pmatrix}
h(x_\epsilon)h(x_\epsilon)^t & -h(x_\epsilon)h(y_\epsilon)^t \\
-h(y_\epsilon)h(x_\epsilon)^t & h(y_\epsilon)h(y_\epsilon)^t
\end{pmatrix} \right).
\]

Restricting these two forms to the diagonal now yields
\[
\frac{1}{3} h(x_\epsilon)A_\epsilon h(x_\epsilon)^t + h(y_\epsilon)B_\epsilon h(y_\epsilon)^t + P_\epsilon \leq \frac{3}{\epsilon} \left( \begin{pmatrix}
h(x_\epsilon) - h(y_\epsilon) \\
h(x_\epsilon) - h(y_\epsilon)
\end{pmatrix} \right)
\]
\[
\leq \lambda \frac{|x_\epsilon - y_\epsilon|^2}{\epsilon} \cdot I
\]
for some \( \lambda > 0 \).

Thus there exists \( P_\epsilon \in \mathbf{P} \) so that
\[
h(x_\epsilon)A_\epsilon h(x_\epsilon)^t + h(y_\epsilon)B_\epsilon h(y_\epsilon)^t + P_\epsilon = \frac{\lambda}{\epsilon} |x_\epsilon - y_\epsilon|^2 \cdot I.
\]

It now follows from the definitions in (10.5) and (10.6) that
\[
A'_\epsilon + B'_\epsilon + P_\epsilon = \frac{\lambda}{\epsilon} |x_\epsilon - y_\epsilon|^2 \cdot I + L_{x_\epsilon}(p_\epsilon) + L_{y_\epsilon}(q_\epsilon).
\]

However,
\[
|L_{x_\epsilon}(p_\epsilon) + L_{y_\epsilon}(q_\epsilon)| = \left| (L_{x_\epsilon} - L_{y_\epsilon}) \left( \frac{x_\epsilon - y_\epsilon}{\epsilon} \right) \right|
\]
\[
\leq \| L_{x_\epsilon} - L_{y_\epsilon} \| \frac{|x_\epsilon - y_\epsilon|}{\epsilon}
\]
\[
= O \left( \frac{|x_\epsilon - y_\epsilon|^2}{\epsilon} \right)
\]

Using (10.2) this shows that
\[
A'_\epsilon + B'_\epsilon + P_\epsilon \equiv \frac{|x_\epsilon - y_\epsilon|^2}{\epsilon} \to 0 \quad \text{as } \epsilon \searrow 0.
\]

This completes the proof in the case of linear equivalence.

Suppose now that our local equivalence is affine and can be written in the form \( \tilde{\Phi}_x = \Phi_x + J_0(x) \) where \( \Phi \) is a linear equivalence as above (cf. (4.8)). Then the proof above goes through essentially unchanged except that, in light of Lemma 4.9, we must replace (10.4) with
\[
\alpha'_\epsilon + J_0(x_\epsilon) \equiv (r'_\epsilon, p'_\epsilon, A'_\epsilon) + J_0(x_\epsilon) \in \mathbf{F}^c
\]
and
\[
\beta'_\epsilon - J_0(y_\epsilon) \equiv (s'_\epsilon, q'_\epsilon, B'_\epsilon) - J_0(y_\epsilon) \in \overline{\mathbf{F}}
\]
(10.4')

We now observe that \( J_0(x_\epsilon) - J_0(y_\epsilon) \to 0 \), and the rest of the proof is exactly as written above. 

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38
Combining Corollary 0.2 with Theorem 9.7 yields comparison for a wide class of $G$-universal subequations.

**THEOREM 10.3.** Suppose $F$ is a $G$-universal subequation on a manifold $X$. If $X$ supports a $C^2$ strictly $M$-subharmonic function, where $M$ is a monotonicity cone for $F$, then comparison holds for $F$ on $X$.

11. Strictly F-Convex Boundaries and Barriers.

In this section we introduce the notion of a strictly $F$-convex boundary for a general subequation $F$. This convexity implies, and in the $G$-universal setting is equivalent to, the existence of barriers. Strictly $F$-convex boundaries can be characterized geometrically using a riemannian metric and the associated second fundamental form of $\partial \Omega$.

Our first step is to replace $F$ by an (asymptotically) smaller open set which is a cone. Suppose $F \subset J^2(X)$ is a subequation.

**Definition 11.1.** The asymptotic interior $\overline{F}$ of $F$ is the set of all $J \in J^2(X)$ for which there exists a neighborhood $\mathcal{N}(J)$ in (the full manifold) $J^2(X)$ and a number $t_0 > 0$ such that

$$t \cdot \mathcal{N}(J) \subset F$$  \hspace{1cm} (11.1)

Note that:

- If $F$ is a cone, then $\overline{F} = \text{Int} F$,  \hspace{1cm} (11.2)
- but otherwise $\overline{F}$ is smaller than $F$ asymptotically and may be empty.

**Proposition 11.2.** The asymptotic interior $\overline{F}$ is an open cone in $J^2(X)$ satisfying Conditions (P) and (N).

**Proof.** Obviously $\overline{F}$ is open and is a cone (i.e. $t \overline{F} = \overline{F}$ for all $t > 0$). To prove (P) suppose $J \in \overline{F}$ and $P_{x_0} \in P_{x_0} \subset J^2_{x_0}(X)$. Extend $P_{x_0}$ to a smooth section $P$ of $\mathcal{P}$ near $x_0$. Note that the fibre-wise sum $\mathcal{N}(J) + P$ is a neighborhood $\mathcal{N}(J + P_{x_0})$ of $J + P_{x_0}$. (The fibre at $x$ of the sum is defined to be empty if $\mathcal{N}(J)_x$ is empty.) Finally note that $t \mathcal{N}(J + P_{x_0})_x = t \mathcal{N}(J)_x + tP_x \subset F$ if $t \geq t_0$, since $t \mathcal{N}(J)_x \subset F$ if $t \geq t_0$. The proof of (N) is similar. 

**Remark.** A $C^2$-function $u$ with $J^2_x u \in \overline{F}$ for all $x$ will be called strictly $\overline{F}$ subharmonic. This is the notion required for boundary convexity and barriers for the Dirichlet problem. Thus the question of when the closure of $\overline{F}$ is a subequation with interior $= \overline{F}$ can be avoided.

11.1 Boundary Convexity. Suppose $M \subset \subset X$ is a smooth oriented hypersurface in $X$. By a defining function for $M$ we mean a smooth function $\rho$ defined on a neighborhood of $M$ such that

$$M = \{x : \rho(x) = 0\} \quad \text{and} \quad \nabla \rho \neq 0$$

defines the orientation on $M$. (If $\Omega$ is a domain with smooth boundary $\partial \Omega$, then $\rho < 0$ on $\Omega$.)

**Definition 11.3.** The oriented hypersurface $M$ is strictly $\overline{F}$-convex (or, for simplicity, strictly $F$-convex) at $x \in M$ if any one of the following conditions holds.

1. There exists a local defining function $\rho$ for $M$ near $x$ such that

$$J^2_x \rho \in \overline{F}$$  \hspace{1cm} (11.3)
(so that \( \rho \) is strictly \( \overline{F} \)-subharmonic near \( x \)).

(2) There exists a local defining function \( \rho \) for \( M \) near \( x \) and \( t_0 > 0 \) such that

\[
J^2_x \rho + t(d\rho)_x \circ (d\rho)_x \in \overline{F}
\tag{11.4}
\]

for all \( t \geq t_0 \).

(3) Given any defining function \( \rho \) for \( M \) near \( x \), there exists \( t_0 > 0 \) such that

\[
J^2_x \rho + t(d\rho)_x \circ (d\rho)_x \in \overline{F}
\tag{11.4}
\]

for all \( t \geq t_0 \).

Note that if (11.4) holds for \( t = t_0 \), then it holds for all \( t \geq t_0 \) by positivity for \( \overline{F} \).

**Proposition 11.4.** Conditions (1), (2), and (3) above are equivalent.

**Proof.** (1) \( \Rightarrow \) (3): Assume that (1) is true for the defining function \( \rho \). Any other local defining function for \( M \) is of the form \( \tilde{\rho} = u\rho \) for some smooth function \( u > 0 \). At \( x \in M \) we have \( d\tilde{\rho} = ud\rho \) and \( J^2\tilde{\rho} = uJ^2\rho + du \circ d\rho \). Hence with \( \epsilon > 0 \) we have

\[
J^2\tilde{\rho} + t\tilde{\rho} \circ d\tilde{\rho} = uJ^2\rho + du \circ d\rho + tu^2d\rho \circ d\rho
\]

\[
= u(J^2\rho - \epsilon \cdot I) + (tu^2d\rho \circ d\rho) + du \circ d\rho + u\epsilon \cdot I
\]

at \( x \). Now for \( \epsilon > 0 \) sufficiently small we have \( u(J^2\rho - \epsilon \cdot I) \in \overline{F} \) by the assumption on \( \rho \), the openness of \( \overline{F} \) and the cone property for \( \overline{F} \). For all \( t \) sufficiently large, the remaining term in the second line above lies in \( \text{IntP}_x \subset \text{Sym}^2(T^*_x X) \). Now apply the positivity condition (P) for \( \overline{F} \).

(2) \( \Rightarrow \) (1): Assume that (2) is true for the defining function \( \rho \) for \( M \) and consider \( \rho_t \equiv \rho + \frac{1}{2}t\rho^2 \) for \( t \geq 0 \). Then \( \rho_t \) is also a defining function for \( M \) and it has 2-jet

\[
J^2\rho_t = J^2\rho + td\rho \circ d\rho \quad \text{on} \quad M.
\]

Hence \( \rho_t \) satisfies (1) if \( t \geq t_0 \).

In particular, this proves that the existence of \( t_0 \) satisfying (11.4) is a condition on \( M \) independent of the defining function \( \rho \).

**Corollary 11.5.** If \( M \) is compact with a global defining function \( \rho \) (for example, if \( M \) is the boundary of a precompact domain \( \Omega \)), then for all \( t \) sufficiently large, the defining function \( \rho_t \equiv \rho + \frac{1}{2}t\rho^2 \) is strictly \( F \)-subharmonic in a neighborhood of \( M \).

**Proof.** Since \( \rho + \frac{1}{2}t\rho^2 \) is strictly \( \overline{F} \)-subharmonic in a neighborhood \( U_x \) of \( x \) in \( M \) for \( t \geq \text{some} \ t_x \), we can pass to a finite covering of \( M \) by such sets \( U_x \) and take \( t \) to be the largest \( t_x \) in the family. Then \( \rho_t \) is strictly \( F \)-subharmonic at all points of \( M \) and hence in a neighborhood of \( M \).

The question of a global strictly \( F \)-subharmonic defining function on a neighborhood of \( \overline{\Omega} \) will be discussed later.

**Example 11.6.** The strictness condition: \( J^2_x \rho \in \overline{F} \) in (1) is not independent of the defining function \( \rho \). Suppose \( F = \mathcal{P} \), the constant coefficient subequation defined by requiring at least one eigenvalue of \( D^2u \) be \( \geq 0 \). The defining function \( \rho(x) = 1 - |x|^2 \) for the unit sphere \( S^{n-1} \) (with inwardly pointing gradient) is not \( \mathcal{P} \)-subharmonic, whereas the defining function \( \rho + \rho^2 \) is strictly \( \mathcal{P} \)-subharmonic near \( S^{n-1} \).
11.2. Barriers. The existence of a family of barriers at a boundary point $x_0 \in \partial \Omega$ of a domain $\Omega$ is needed for establishing boundary regularity for the Dirichlet problem.

**Definition 11.7.** Suppose that $\rho$ is a defining function for a hypersurface $M$ near $x_0$. We say that $\rho$ defines a family of $F$-barriers for $M$ at $x_0$ if there exist $t_0 > 0$, $\epsilon_0 > 0$, $\delta_0 > 0$ and $r_0 > 0$ such that the function

\[ \beta(x) = \lambda + C \left( \rho(x) - \epsilon \frac{|x - x_0|^2}{2} \right) \]  

is strictly $F$-subharmonic on $B(x_0, r_0)$ for all $\lambda \in \mathbb{R}$, $C \geq C_0 \equiv \max\{t_0, \lambda/\delta_0\}$ and $0 < \epsilon \leq \epsilon_0$.

**Lemma 11.8. (Existence of Barriers).** Suppose $\Omega \subset X$ is a domain with smooth boundary which is strictly $F$-convex at $x_0 \in \partial \Omega$. Then there exists a local defining function $\rho$ for $\partial \Omega$ near $x_0$ which defines a family of $F$-barriers for $\partial \Omega$ at $x_0$.

**Proof.** Choose a local defining function $\rho$ for $\partial \Omega$ near $x_0$ with

\[ J = J_{x_0}^2 \rho \in \mathcal{F}. \]

By the definition of $\mathcal{F}$ there exists a neighborhood $\mathcal{N}(J)$ and $t_0 > 0$ such that $t \cdot \mathcal{N}(J) \subset \text{Int} \mathcal{F}$ for all $t \geq t_0$. By shrinking we may assume that $\mathcal{N}(J) = B(x_0, r_0) \times N$ is a product neighborhood in some local coordinate system. Pick $\delta_0 > 0$ small enough so that there exists a vertically smaller neighborhood $\mathcal{N}'(J)$ with

\[ \mathcal{N}'(J) + \delta \subset \mathcal{N}(J) \quad \text{for all } 0 \leq \delta \leq \delta_0. \]

Pick $\epsilon_0 > 0$ and shrink $r_0$ if necessary so that

\[ J_{x}^2 - \epsilon \frac{|x - x_0|^2}{2} \in \mathcal{N}'(J) \quad \forall x \in B(x_0, r_0) \quad \text{and} \quad \forall 0 \leq \epsilon \leq \epsilon_0 \]

and hence

\[ \delta + J_{x}^2 - \epsilon \frac{|x - x_0|^2}{2} \in \mathcal{N}(J) \quad \forall x \in B(x_0, r_0), \quad \forall 0 \leq \epsilon \leq \epsilon_0 \quad \text{and} \quad \forall 0 \leq \delta \leq \delta_0. \]

Suppose $C \geq C_0 = \max\{t_0, \lambda/\delta_0\}$. Choose $\delta = \lambda/C$ (which is $\leq \delta_0$). Then for $|x - x_0| \leq r_0$, $0 \leq \epsilon \leq \epsilon_0$, the function $\beta$ defined by (11.5) has its 2-jet at $x$ in $CN(J) \subset \text{Int} \mathcal{F}$. \hfill \blacksquare

**Remark 11.9. (Existence of Barriers – Refinements).** Actually, less than is stated in Definition 11.7 is required for the application to the Dirichlet problem. Namely, one only needs a continuous defining function $\rho$ such that for each $\lambda \in \mathbb{R}$, there exist $C_0 \geq 0$, $\epsilon_0 > 0$ and $r_0 > 0$ so that

\[ \beta(x) = \lambda + C \left( \rho(x) - \epsilon \frac{|x - x_0|^2}{2} \right) \in F_{\text{strict}}(\overline{B(x_0, r_0) \cap \Omega}) \]

is strictly $F$-subharmonic on $B(x_0, r_0)$ for all $C \geq C_0$ and $0 < \epsilon \leq \epsilon_0$. (See Section 7 for the definition of $F_{\text{strict}}(K)$.)

This applies to the intersection $\Omega = \Omega_1 \cap \Omega_2$ where $\Omega_1, \Omega_2 \subset X$ are two domains with smooth, strictly $\mathcal{F}$-convex boundaries. Fix $x_0 \in \partial \Omega_1 \cap \partial \Omega_2$ and $\lambda \in \mathbb{R}$. Assume that there exist local defining functions $\rho_k$ for each $\partial \Omega_k$ near $x_0$ and constants $r_0 > 0$, $\epsilon_0 > 0$ and $C_0 \geq 0$ such that $\beta_k(x) = \lambda + C_k \left( \rho_k(x) - \epsilon |x - x_0|^2 \right) \in F_{\text{strict}}(\overline{B(x_0, r_0) \cap \Omega})$ for all $C \geq C_0$ and $0 < \epsilon \leq \epsilon_0$. Then

\[ \max\{\beta_1, \beta_2\} \in F_{\text{strict}}(\overline{B(x_0, r_0) \cap \Omega_1 \cap \Omega_2}) \]
for all \( C \geq C_0 \) and \( 0 < \epsilon \leq \epsilon_0 \) by Lemma 7.7(i).

This remark is particularly useful in establishing existence for parabolic equations.

We now point out that the asymptotic interior \( \overrightarrow{F} \) could have been defined using another section \( J_1 \) of \( J^2(X) \) as the vertex rather than the zero section. Define the asymptotic interior of \( F \) based at \( J_1 \), denoted \( \overrightarrow{F}_{J_1} \), to be the set of \( J \) for which there exists a neighborhood \( N(J) \) of \( J \) in \( J^2(X) \) and \( t_0 > 0 \) such that

\[
J_1 + tN(J) \subset F \quad \text{for all} \quad t \geq t_0.
\]

(11.8)

**Lemma 11.10.** \( \overrightarrow{F}_{J_1} = \overrightarrow{F}_{J_2} \) for any two section \( J_1 \) and \( J_2 \).

The proof has been given above using (11.6) in the important case where \( J_1 \equiv 0 \) and \( J_2 \equiv \lambda \). The proof in general is left to the reader.

### 11.3. A Geometric Characterization of strict \( \overrightarrow{F} \)-Convexity.

Given a riemannian manifold \( X \), Proposition 11.4 enables us to characterize an \( \overrightarrow{F} \)-convex hypersurface \( M \) in terms of its second fundamental form \( II_M \) with respect to \( -n \) where \( n \) is the orienting unit normal. (If \( M = \partial \Omega \), \( -n \) is inward pointing.) Recall the canonical decomposition of the 2-jet bundle given in Section 5.2:

\[
J^2(X) = \mathbb{R} \oplus T^*X \oplus \text{Sym}^2(T^*X).
\]

(11.9)

**Proposition 11.11.** The hypersurface \( M \) is strictly \( \overrightarrow{F} \)-convex at a point \( x \in M \) if and only if

\[
(0, n, tP_n \oplus II_M) \in \overrightarrow{F} \quad \text{for all} \quad t \geq \text{some} \quad t_0.
\]

(11.10)

where \( P_n \) denotes orthogonal projection onto the normal line \( \mathbb{R} \cdot n \) at \( x \).

**Note.** Blocking with respect to the decomposition \( T_xX = \mathbb{R} \cdot n \oplus T_x(M) \), (11.10) can be rewritten

\[
\left( 0, (1, 0), \begin{pmatrix} t & 0 \\ 0 & II_M \end{pmatrix} \right) \in \overrightarrow{F} \quad \text{for all} \quad t \geq \text{some} \quad t_0.
\]

(11.10)'

**Proof.** Choose \( \rho \) to be the signed distance function on a neighborhood \( U \) of \( M \). That is,

\[
\rho(y) = \begin{cases} -\text{dist}(y, \partial \Omega) & \text{if} \quad y \in U^- \\ +\text{dist}(y, \partial \Omega) & \text{if} \quad y \in U^+ \end{cases}
\]

where \( U - M = U^+ \cup U^- \) and signs are chosen so that \( \nabla \rho = n \) on the hypersurface \( M \). Then it is a standard calculation (cf. \([HL_2, (5.7)]\)) that

\[
\text{Hess}_x \rho = \begin{pmatrix} 0 & 0 \\ 0 & II_M \end{pmatrix}.
\]

with respect to the splitting \( T_xX = (\mathbb{R} \cdot \nabla \rho) \oplus T_xM \). Since \( \nabla \rho = n \) at \( x \) the assertion follows directly from Definition 11.3 and Proposition 11.4.

### 11.4. G-Universal Subequations.

We note, to begin, that the construction of the asymptotic interior is compatible with the construction of a \( G \)-universal subequation from a constant coefficient subequation.
Lemma 11.12. Suppose \( F \subset J^2(X) \) is a subequation defined by a \( G \)-invariant universal model \( F \) as in Corollary 6.2. Then the asymptotic interior \( \overline{F} \) of \( F \) is the bundle of open cones induced by the universal model \( \overline{F} \) via (6.3), where \( \overline{F} \) is the asymptotic interior of \( F \).

Proof. Exercise.

Our notion of strict boundary convexity for a subequation \( F \) is sufficient to insure the existence of a family of \( F \)-barriers (Lemma 11.8). Might there be other different conditions which also insure the existence of barriers? For \( G \)-universal subequations which are independent of the \( r \)-variable the answer is no. The next Proposition states that in this case the converse holds, that is, strict boundary convexity is equivalent to the existence of a family of barriers. This result provides justification for the definition of boundary convexity that we have chosen for a general subequation \( F \).

Proposition 11.13. Suppose \( F \) is a \( G \)-universal subequation on a manifold \( X \) depending only on the reduced 2-jets, (i.e., of the form \( F = \mathbb{R} \times F_0 \) for \( F_0 \subset T^*X \oplus \text{Sym}^2(T^*X) \)), and consider a domain \( \Omega \subset X \). Suppose \( \rho \) is a defining function for \( \partial \Omega \) at \( x_0 \in \partial \Omega \). If \( \rho \) defines a family of \( F \)-barriers for \( \partial \Omega \) at \( x_0 \), then \( J^2_{x_0} \rho \in \overline{F}_{x_0} \), i.e., \( \partial \Omega \) is strictly \( F \)-convex at \( x_0 \).

Proof. For clarity we give the proof when \( F \) is a constant coefficient subequation on an open set \( X \subset \mathbb{R}^n \). The hypothesis is that : there exist \( C_0 \geq 0 \), \( \epsilon_0 > 0 \), and \( r_0 > 0 \) such that for \( C \geq C_0 \), \( 0 \leq \epsilon \leq \epsilon_0 \), and \( |x-x_0| \leq r_0 \), the reduced 2-jet of the function \( \beta(x) = C(\rho(x) - \frac{\epsilon}{2}|x-x_0|^2) \) belongs to \( \text{Int}F \). Set \( J^2_{x_0} \rho = (p(x), A(x)) \) and \( J_0 = J^2_{x_0} \rho = (p_0, A_0) \).

Then by hypothesis

\[
C(p(x) - \epsilon(x-x_0), A(x) - \epsilon \text{Id}) \in \text{Int}F \quad \text{for all} \quad C \geq C_0, 0 < \epsilon \leq \epsilon_0, \quad \text{and} \quad |x-x_0| \leq r_0. \quad (11.11)
\]

Define a neighborhood \( \mathcal{N}(J_0) \) by requiring that

\[
|p - p_0| \leq \bar{r} \quad \text{and} \quad A - A_0 \geq \epsilon \text{Id}
\]

where \( J = (p, A) \). We show that there are sufficient 2-jets given by (11.11) to fill out \( C\mathcal{N}(J_0) \), thus proving that \( C\mathcal{N}(J_0) \subset \text{Int}F \) if \( C \geq C_0 \).

Consider the map \( \Phi(x) = p(x) - \epsilon(x-x_0) \), and note that \( \Phi(x_0) = p_0 \). The derivative \( \Phi'(x_0) = A_0 - \epsilon \text{Id} \) is non-singular for a dense set of \( \epsilon \), so we may choose \( 0 < \epsilon < \epsilon_0 \) so that \( \Phi \) is a diffeomorphism in a neighborhood of \( x_0 \). Pick \( 0 < \epsilon < \epsilon \) and then pick \( 0 < r < r_0 \) so that

1. \( \Phi \) restricted to \( B(x_0, r_0) \) is a diffeomorphism, and
2. \( A_0 - A(x) \geq - (\epsilon - \epsilon) \text{Id} \) if \( |x-x_0| < r \).

Now pick \( \bar{r} > 0 \) so that \( B(p_0, \bar{r}) \subset \Phi(B(x_0, r_0)) \). Then given \( p \in B(p_0, \bar{r}) \), there exists a unique \( x \in B(x_0, r_0) \) satisfying \( p = p(x) - \epsilon(x-x_0) \). By (11.11) this proves that

\[
C(p, A(x) - \epsilon \text{Id}) \in \text{Int}F \quad \text{for all} \quad C \geq C_0.
\]

Since \( (p, A) \in \mathcal{N}(J_0) \) defined above, we have \( A \geq A_0 - \epsilon \text{Id} \). Also by (2) we have \( A_0 - \epsilon \text{Id} \geq A(x) - \epsilon \text{Id} \). The positivity condition for \( \text{Int}F \) now implies that \( C(p, A) \in \text{Int}F \) for all \( C \geq C_0 \).
12. The Dirichlet Problem – Existence.

Throughout this section we assume that $F$ is a subequation on a manifold $X$ and that $\Omega \subset X$ is a domain with smooth boundary $\partial \Omega$. Furthermore, we assume that both $F_{\text{strict}}(\Omega)$ and $\tilde{F}_{\text{strict}}(\Omega)$ contain at least one function bounded below. (See (8.2) for this notation.) This assumption is “minor”, for example it is obvious locally for all subequations with $F_x \neq \emptyset$ and $\tilde{F}_x \neq \emptyset$ for all $x$.

Our key assumption in the existence theorems is that $\partial \Omega$ is both $\rightarrow F$ and $\rightarrow \tilde{F}$ strictly convex.

**Definition 12.1.** Given a boundary function $\varphi \in C(\partial \Omega)$, consider the Perron family

$$F(\varphi) \equiv \{ u \in \text{USC}(\Omega) : u|_\Omega \in F(\Omega) \text{ and } u|_{\partial \Omega} \leq \varphi \}$$

and define the Perron function

$$U(x) \equiv \sup\{u(x) : u \in F(\varphi)\}.$$ 

to be the upper envelope of the Perron family.

We shall begin by isolating all the conclusions that hold only under the assumption of weak comparison. This is done in the next theorem and its corollary. In the two subsequent theorems the remaining gap in existence is filled in two different ways. In the first we see that, assuming comparison, the gap is easy to fill. In the second we assume constant coefficients and apply an argument of Walsh [W] to fill the gap.

The method of proof for the next theorem is the classical barrier argument. In the case where $F = P_C$ on $C^n$ these arguments can be found as far back as Bremermann [B] except for the “bump argument” for part (3) due to Bedford and Taylor [BT1]. This argument was rediscovered by Ishii [I].

**Theorem 12.2.** Suppose that $\partial \Omega$ is both $\rightarrow F$ and $\rightarrow \tilde{F}$ strictly convex, and that weak comparison holds for $F$ on $X$. Given $\varphi \in C(\partial \Omega)$, the Perron function $U$ satisfies:

1. $U^* = U = U^* = \varphi$ on $\partial \Omega$,
2. $U = U^*$ is $F$-subharmonic on $\Omega$,
3. $-U^*$ is $\tilde{F}$-subharmonic on $\Omega$.

**Corollary 12.3.** If $U$ is lower semicontinuous on $\Omega$, i.e., if $U^* = U$, then:

(a) $U \in C(\overline{\Omega})$,
(b) $U$ is $F$ harmonic on $\Omega$,
(c) $U = \varphi$ on $\partial \Omega$.

i.e., $U$ solves the Dirichlet problem on $\overline{\Omega}$ for boundary values $\varphi$.

**Theorem 12.4.** Assume that comparison holds for $F$, and suppose that $\partial \Omega$ is both $\rightarrow F$ and $\rightarrow \tilde{F}$ strictly convex. Then for each $\varphi \in C(\partial \Omega)$ the Perron function $U$ solves the Dirichlet problem on $\Omega$ for boundary values $\varphi$.

**Theorem 12.5.** Suppose that $F$ is a constant coefficient subequation on $X = \mathbb{R}^n$, or more generally suppose that $X = K/G$ is a riemannian homogeneous space and that $F$ is a subequation which is invariant under the natural action of the Lie group $K$ on $J^2(X)$.

If $\partial \Omega$ is both $\rightarrow F$ and $\rightarrow \tilde{F}$ strictly convex, then existence holds for the Dirichlet problem on $\Omega$ as above.
Remark 12.6. There is a hidden hypothesis in these existence results; namely, both $\overrightarrow{F}$ and $\overleftarrow{F}$ must be non-empty in order for there to exist any domain $\Omega$ whose boundary is both $\overrightarrow{F}$ and $\overleftarrow{F}$ strictly convex. For example, the Eikonal subequation $|p| \leq 1$ has $\overrightarrow{F} = \emptyset$, and, as is well known, existence fails for general $\varphi \in C(\partial B)$ where $B$ is a ball in $\mathbb{R}^n$. However, Theorem 12.5 does apply to the infinite Laplacian, defined by taking $F$ to be the closure of the set $\{(r, p, A) \in J^2(\mathbb{R}^n) : \langle Ap, p \rangle > 0\}$. This example is self-dual, i.e., $\overleftarrow{F} = F$, and it is also a cone with $\overrightarrow{F} = F$. It is easy to check that $(r, p, A) \in \text{Int} F$ if and only if either $\langle Ap, p \rangle > 0$ or $A > 0$ and $p = 0$. Hence, strictly convex functions are strictly $F$-subharmonic and define domains for which existence holds.

Now for the proofs.

Lemma $F$. 
$$U^*|_{\Omega} \in F(\Omega)$$

Lemma $\overleftarrow{F}$. 
$$-U_*|_{\Omega} \in \overleftarrow{F}(\Omega)$$

Proposition $F$. Suppose $\partial \Omega$ is strictly $\overrightarrow{F}$-convex at $x_0 \in \partial \Omega$. For each $\delta > 0$ small, there exists $u \in F(\varphi)$ with the additional properties:
- $u$ is continuous at $x_0$,
- $u(x_0) = \varphi(x_0) - \delta$,
- $u \in F_{\text{strict}}(\overline{\Omega})$.

Proposition $\overleftarrow{F}$. Suppose $\partial \Omega$ is strictly $\overleftarrow{F}$-convex at $x_0 \in \partial \Omega$. For each $\delta > 0$ small, there exists $\overline{u} \in F(-\varphi)$ with the additional properties:
- $\overline{u}$ is continuous at $x_0$,
- $\overline{u}(x_0) = -\varphi(x_0) - \delta$,
- $\overline{u} \in F_{\text{strict}}(\overline{\Omega})$.

Corollary $F$. 
$$\varphi(x_0) \leq U_*(x_0)$$

Corollary $\overleftarrow{F}$. 
$$U^*(x_0) \leq \varphi(x_0)$$

Conclusion 1. (Boundary Continuity) We have $U_* = U = U^* = \varphi$ on $\partial \Omega$. In particular, $U$ is continuous at each point of $\partial \Omega$.

Proof. By Corollaries $F$ and $\overleftarrow{F}$ we have $\varphi(x_0) \leq U_*(x_0) \leq U(x_0) \leq U^*(x_0) \leq \varphi(x_0) \forall x_0$. ■

Conclusion 2. 
$$U^* \in F(\varphi).$$

Proof. Corollary $\overleftarrow{F}$ together with Lemma $F$. ■

Conclusion 3. 
$$U = U^* \text{ on } \overline{\Omega}.$$ 

Proof. We have $U^* \leq U$ on $\overline{\Omega}$ since $U^* \in F(\varphi)$. ■

Proof of Lemma $F$. Because of the families locally bounded above property, it suffices to show that $F(\varphi)$ is uniformly bounded above. Pick $\psi \in F_{\text{strict}}(\overline{\Omega})$ bounded below. Pick $c >> 0$ so
that $\psi - c \leq -\varphi$ on $\partial \Omega$. By Lemma 7.2' we have $\psi - c \in \tilde{F}_{\text{strict}}(\Omega)$. Given $u \in F(\varphi)$ we have $u + (\psi - c) \leq 0$ on $\partial \Omega$. Weak comparison for $\tilde{F}$ implies that $u + \psi - c \leq 0$ on $\Omega$. Hence, $u \leq \inf_{\Omega} \psi + c$ for all $u \in F(\varphi)$. 

**Proof of Proposition F.** Assume $\psi \in F_{\text{strict}}(\Omega)$ is bounded below on $\Omega$. Since $\partial \Omega$ is strictly $\tilde{F}$-convex at $x_0$, Lemma 11.8 states that there exist a local defining function $\rho$ for $\partial \Omega$ near $x_0$, and $r > 0, \epsilon_0 > 0$ and $C_0 > 0$ such that in some local coordinates

\[
\beta(x) \equiv \varphi(x_0) - \delta + C(\rho(x) - \epsilon|x - x_0|^2) \in F_{\text{strict}}(B(x_0, r) \cap \Omega) \\
\forall C \geq C_0 \quad \text{and} \quad \forall \epsilon \leq \epsilon_0.
\]

(12.1)

Shrink $r > 0$ so that

\[
\varphi(x_0) - \delta < \varphi(x) \quad \text{on} \quad \partial \Omega \cap B(x_0, r). \quad (12.2)
\]

Pick $N > \sup_{\partial \Omega} |\varphi| + \sup_{\Omega} \psi$ so that

\[
\psi - N < \varphi - \delta \quad \text{on} \quad \partial \Omega. \quad (12.3)
\]

Choose $C$ so large that on $A \equiv (B(x_0, r) \sim B(x_0, r/2)) \cap \Omega$ we have

\[
\beta < \psi - N \quad (12.4)
\]

This is possible since $\rho(x) - \epsilon|x - x_0|^2$ is strictly negative on $A$ and $\psi - N$ is bounded below on $A$.

By (12.4) we have that

\[
\underline{u}(x) = \max\{\beta, \psi - N\}
\]

is a well defined function on $\Omega$ which is equal to $\psi - N$ outside $B(x_0, r/2)$.

Now we have $\psi - N \in F_{\text{strict}}(\Omega)$, because Condition (N) is satisfied. Thus Lemma 7.7(i) and condition (12.1) imply that $\underline{u} \in F_{\text{strict}}(\Omega)$.

Outside the set $B(x_0, r/2) \cap \Omega$ we have $\underline{u} = \psi - N$ which is $\leq \varphi$ on $\partial \Omega$ by (12.3), while on $B(x_0, r/2) \cap \partial \Omega$ we still have $\psi - N \leq \varphi$, but also

\[
\beta = \varphi(x_0) - \delta + C(\rho(x) - \epsilon|x - x_0|^2) = \varphi(x_0) - \delta - C\epsilon|x - x_0|^2 \leq \varphi(x_0) - \delta \leq \varphi(x)
\]

by (12.1). Thus $\underline{u}|_{\partial \Omega} \leq \varphi$. This proves that $\underline{u} \in F(\varphi)$. Finally note that $\beta(x_0) = \varphi(x_0) - \delta$ which is $> \psi(x_0) - N$ by (12.3). Continuity of $\beta$ and the upper semicontinuity of $\psi$ implies that $\beta > \psi - N$ in a neighborhood of $x_0$. That is, $\underline{u} = \beta$ in a neighborhood of $x_0$. Thus $\underline{u}$ is continuous at $x_0$, and $\underline{u}(x_0) = \varphi(x_0) - \delta$.

**Proof of Proposition $\tilde{F}$.** This is merely Proposition F with an exchange of roles.

**Proof of Corollary F.** Since $\underline{u} \in F(\varphi)$, we have $u \leq U$. Hence, $u_* \leq U_*$. By the continuity of $u$ at $x_0$ and the fact that $\underline{u}(x_0) = \varphi(x_0) - \delta$, we have $\varphi(x_0) - \delta \leq U_*(x_0)$ for all $\delta > 0$ small.

**Proof of Corollary $\tilde{F}$.** Here we use weak comparison again. Choose $u \in F(\varphi)$. Since $\tilde{\pi} \leq -\varphi$ on $\partial \Omega$, we have $u + \tilde{\pi} \leq 0$ on $\partial \Omega$. Since $\tilde{\pi} \in \tilde{F}_{\text{strict}}(\Omega)$, weak comparison implies that $u + \tilde{\pi} \leq 0$ on $\Omega$. Therefore, $U + \tilde{\pi} \leq 0$ on $\Omega$, i.e. $U \leq -\tilde{\pi}$ on $\Omega$. Since $\tilde{\pi}$ is continuous at $x_0$ and $\tilde{\pi}(x_0) = -\varphi(x_0) - \delta$, this implies that $U^*(x_0) \leq \varphi(x_0) + \delta$ for all $\delta > 0$ small.

**Proof of Lemma $\tilde{F}$.** Note that the Conclusions 1, 2 and 3 are now established. Suppose $-U_*^* \notin \tilde{F}(\Omega)$. Then by Lemma 2.4 there exist $x_0 \in \Omega$, $\epsilon > 0$, and $\psi \in C^2$ near $x_0$ so that in local coordinates
(1) \(-U_* - \psi \leq -\epsilon |x-x_0|^2\) near \(x_0\)
(2) \(= 0\) at \(x_0\)

but
\[ J^2\psi \notin \tilde{F}_{x_0}, \quad \text{i.e.,} \quad -J^2\psi \in \text{Int}F_{x_0}. \]

Then there exist \(r > 0, \delta > 0\) so small that
\[ u \equiv -\psi + \delta \text{ is } F\text{-subharmonic on } B(x_0, r) \]

Moreover, (1) implies that for \(\delta > 0\) sufficiently small,
\[ (1)' \quad u < U_* \text{ on a neighborhood of } \partial B(x_0, r) \]

Since \(U_* \leq U\), statement \((1)'\) implies that the function
\[ u' \equiv \begin{cases} U & \text{on } \overline{\Omega} - B(x_0, r) \\ \max\{U, u\} & \text{on } B(x_0, r) \end{cases} \]

is \(F\)-subharmonic on \(\Omega\). Conclusion 1 says that \(U = \varphi\) on \(\partial\Omega\), which implies \(u' \in F(\varphi)\). Therefore, \(u' \leq U\) on \(\overline{\Omega}\), which implies in turn that \(u \leq U\) on \(B(x_0, r)\).

Now statement (2) above says \(U_*(x_0) = u(x_0) - \delta\). Pick a sequence \(x_k \to x_0\) with \(\lim_{k \to \infty} U(x_k) = U_*(x_0)\). Then
\[ (i) \lim_{k \to \infty} U(x_k) = u(x_0) - \delta \]
\[ (ii) \lim_{k \to \infty} u(x_k) = u(x_0). \]

This implies that \(u(x_k) > U(x_k)\) for all \(k\) large, contradicting the fact that \(u \leq U\) on \(B(x_0, r)\).

This completes the proof of Theorem 12.2 and its Corollary 12.3. Interior continuity is all that remains in showing that \(U|_{\overline{\Omega}}\) is \(F\)-harmonic.

**Proof of Theorem 12.4. (Assuming Comparison):** By Corollary F and Lemma \(\tilde{F}\)

\[-U_* \in \tilde{F}(-\varphi).\]

In particular
\[ U - U_* \leq 0 \quad \text{on } \partial\Omega. \]

Since \(U|_{\Omega} \in F(\Omega)\) and \(-U_*|_{\Omega} \in \tilde{F}(\Omega)\), comparison implies
\[ U - U_* \leq 0 \quad \text{on } \overline{\Omega}, \]

that is, \(U \leq U_* \leq U\). Since \(U = U^*\) we are done.

**Proof of Theorem 12.5. (Assuming Constant Coefficients):** We suppose that \(X = \mathbb{R}^n\) and that \(F\) has constant coefficients. Let \(\Omega_\delta \equiv \{x \in \Omega : \text{dist}(x, \partial\Omega) > \delta\}\) and \(C_\delta \equiv \{x \in \overline{\Omega} : \text{dist}(x, \partial\Omega) < \delta\}\). Suppose \(\epsilon > 0\) is given. By the continuity of \(U\) at points of \(\partial\Omega\) and the compactness of \(\partial\Omega\) it follows easily that there exists a \(\delta > 0\) such that
\[ \text{if } |y| \leq \delta, \text{ then } U_y \leq U + \epsilon \text{ on } C_{2\delta} \quad (12.5) \]
where \( U_y(x) \equiv U(x + y) \) is the \( y \)-translate of \( U \) and where we define \( U \) to be \(-\infty\) on \( \mathbb{R}^n - \overline{\Omega} \). We claim that

\[
\begin{align*}
\text{if } |y| \leq \delta, \text{ then } U_y \leq U + \epsilon \text{ on } \overline{\Omega} \quad (12.6)
\end{align*}
\]

Setting \( z = x + y \), this implies that

\[
\text{if } z \in \overline{\Omega}, x \in \overline{\Omega}, \text{ and } |z - x| \leq \delta, \text{ then } U(z) \leq U(x) + \epsilon
\]

and therefore by symmetry that \(|U(z) - U(x)| \leq \epsilon\). Thus the proof is complete once (12.6) is established.

To prove (12.6), note first that \( U_y - \epsilon \in F(\Omega_\delta) \) for each \(|y| < \delta\) by the translation invariance of \( F \) and property (N). Since \( U_y \leq U + \epsilon \) on the collar \( C_{2\delta} \), one has

\[
g_y \equiv \max\{U_y - \epsilon, U\} \in F(\Omega).
\]

Now (12.5) implies that \( g_y = U \) on \( C_{2\delta} \). Therefore,

\[
g_y \in \mathcal{F}(\varphi),
\]

and hence \( g_y \leq U \) on \( \overline{\Omega} \). This proves that

\[
U_y - \epsilon \leq g_y \leq U \quad \text{on } \Omega_\delta.
\]

Combined with (12.5) this proves (12.6).

This Walsh argument evidently applies to domains \( \Omega \) in any riemannian homogeneous space \( X = G/H \) provided that the equation \( F \) is invariant under the action of \( G \) on \( J^2(X) \).

Our existence result for general constant coefficient subequations can be restated as follows.

**THEOREM 12.7**. Suppose that \( F \subset J^2(\mathbb{R}^n) \) is a constant coefficient subequation, and that \( \Omega \) is a domain with smooth boundary \( \partial\Omega \) which is both \( \overline{F} \) and \( \overline{\mathcal{F}} \) strictly convex. Given \( \varphi \in C(\partial\Omega) \), let \( U \) denote the \( F \)-Perron function on \( \Omega \) with boundary values \( \varphi \), and let \( V \) denote the \( \overline{\mathcal{F}} \)-Perron function on \( \overline{\Omega} \) with boundary values \(-\varphi\). Then both \(-V\) and \( U \) are solutions to the Dirichlet problem for \( F \) with boundary values \( \varphi \). Furthermore, any other such solution \( u \) satisfies

\[
-V \leq u \leq U \quad \text{on } \overline{\Omega}.
\]

**Proof**. This follows easily from Theorem 12.5.

**Example 12.8. (Existence without Uniqueness)**. Consider the dual subequations

\[
\begin{align*}
F &\equiv \{ (r, p, A) : A - \frac{1}{2}|p|^\frac{3}{2} (I + P_{[p]}) \geq 0 \} \\
\overline{F} &\equiv \{ (r, p, A) : A + \frac{1}{2}|p|^\frac{1}{2} (I + P_{[p]}) \geq 0 \}
\end{align*}
\]

(12.7)

(where \( P_{[p]} \) us orthogonal projection onto the \( p \)-line). First note that the asymptotic interior of both \( F \), and of \( \overline{F} \), is \( \text{Int}\mathcal{P} \). Consequently, the boundaries \( \partial\Omega \) which are strictly \( \overline{\mathcal{F}} \)- and \( \overline{\mathcal{F}} \)-convex are precisely the boundaries which are classically strictly convex. Consider now the Dirichlet problem
for the $R$-ball $\Omega = \{ x : |x| < R \}$ with boundary-value function $\varphi = 0$ (hence $\tilde{\varphi} = -\varphi = 0$ also). Then Theorem 12.7 applies. Moreover,

- The $F$-Perron function is $U \equiv 0$.
- The $\tilde{F}$-Perron function is $V(x) \equiv \frac{1}{12} (R^3 - |x|^3)$.

Thus, in particular, while existence and weak comparison hold (cf. Theorem 10.1 or Corollary C.3), local comparison fails here.

**Proof.** Obviously $U \equiv 0$ is $F$-harmonic. Moreover,

$$-U(x) + \frac{\epsilon}{2} |x|^2 = \frac{\epsilon}{2} |x|^2 \text{ is strictly } \tilde{F} \text{ subharmonic.} \tag{12.8}$$

Assuming (12.8), weak comparison implies that on $\overline{\Omega}$:

$$u(x) + (-U(x) + \frac{\epsilon}{2} |x|^2) \leq \frac{\epsilon}{2} R^2 \text{ for all } u \in \mathcal{F}(\varphi).$$

Thus, $u \leq U$ for all $u \in \mathcal{F}(\varphi)$, i.e., $U$ is the $F$-Perron function for $\varphi = 0$.

Similarly, calculation shows that $V(x) = \frac{1}{12} (R^3 - |x|^3)$ is $\tilde{F}$-harmonic. Moreover,

$$-V(x) + \frac{\epsilon}{2} |x|^2 \text{ is strictly } F \text{ subharmonic.} \tag{12.9}$$

Assuming (12.9), weak comparison implies that on $\overline{\Omega}$:

$$(-V(x) + \frac{\epsilon}{2} |x|^2) + v(x) \leq \frac{\epsilon}{2} R^2 \text{ for all } v \in \tilde{\mathcal{F}}(-\varphi).$$

Thus, $v \leq V$ for all $v \in \tilde{\mathcal{F}}(-\varphi)$, i.e., $V$ is the $\tilde{F}$-Perron function for $-\varphi = 0$.

To prove (12.8) simply note that the Hessian of $-U(x) + \frac{\epsilon}{2} |x|^2$ is $A = \epsilon \cdot I$ and apply the definition. To prove (12.9) note that for the function $-V(x) + \frac{\epsilon}{2} |x|^2$

$$p = \left( \frac{|x|}{\epsilon} + \epsilon \right) x \quad \text{and} \quad A = \left( \frac{|x|}{\epsilon} + \epsilon \right) \cdot I + \frac{|x|}{\epsilon} P[x].$$

The eigenspaces for $A - \frac{1}{2} |p| \frac{1}{2} (I + P[p])$ are:

- $[x]$ with eigenvalue $\frac{|x|}{\epsilon} + \epsilon - \left( \frac{|x|^2}{4} + \epsilon |x| \right) \frac{1}{2}$ and $[x]^\perp$ with eigenvalues $\frac{|x|}{\epsilon} + \epsilon - \frac{1}{2} \left( \frac{|x|^2}{4} + \epsilon |x| \right) \frac{1}{2}$.

It is easy to see that these eigenvalues are $> 0$ on $\overline{\Omega}$.

Despite the above examples of strict approximation, it must fail in general for this equation since otherwise uniqueness would hold. More precisely, $U$ has no $F$-strict approximation and $V$ has no $\tilde{F}$-strict approximation.

**Remark 12.9. (Refinements).**

1) Note that in the proof of Proposition F we only needed the barrier $\beta$ defined in (12.1) to be strict on the $\Omega$-portion of the ball, that is we only really needed

$$\beta \in F_{\text{strict}}(\Omega \cap B(x_0, r)).$$

This can sometimes be established even though $\partial \Omega$ is not strictly $\overline{F}$-convex. For example, suppose $F$ is defined by

$$A \geq 0 \quad \text{and} \quad \det A \geq e^r \tag{12.10}$$
and note that $F$ is defined by $A > 0$ and $r < 0$.

Consider $\Omega = \{ x : |x| < R \}$ and set $\rho(x) = \frac{1}{2}(|x|^2 - R^2)$. Note that with $\lambda = \varphi(x_0) - \delta$ the barrier satisfies $\beta(x) \leq \lambda$ on the $\Omega$-portion of $B(x_0, r)$. Then it is easy to see that for $C > 0$ sufficiently large, we have $\beta \in F_{\text{strict}}(\Omega \cap B(x_0, r))$. This is because $J^2 \beta = (\beta(x), C(x-2\epsilon(x-x_0)), C(1-2\epsilon)I)$, so that $\det A - e^\epsilon \geq C^n(1-2\epsilon)^n - e^\lambda > 0$ for all $x \in \Omega \cap B(x_0, r)$. Thus Proposition F and Corollary F are valid for this $F$ in spite of the fact that $\partial \Omega$ is not strictly $F$-convex. In fact, Proposition F and Corollary F are false for the subequation $H \equiv \text{closure}\{F\}$ unless $\varphi(x_0) \leq 0$. It is now easy to show that existence and uniqueness hold for the subequation $F$ defined by (12.10), since $\psi(x) = \frac{1}{2}(|x|^2 - R^2)$ is a good approximator for $F$.

2) Notice that the weakened form of Proposition F with $u \in F(\Omega)$ but not necessarily strict, is all that is needed to prove Corollary F. However, to prove Corollary $\tilde{F}$ from Proposition $\tilde{F}$ weak comparison was used so that the strictness of $\pi \in F_{\text{strict}}(\Omega)$ cannot be dropped.

13. The Dirichlet Problem – Summary Results.

We present here several summary results which follow from the work above. We make the following standing hypotheses in this section.

(i) $F$ is a subequation on a riemannian manifold $X$.
(ii) $\Omega \subset\subset X$ is a domain with smooth boundary $\partial \Omega$.
(iii) Both $F_{\text{strict}}(\Omega)$ and $\tilde{F}_{\text{strict}}(\Omega)$ have at least one function bounded below.

Consider the following

**DIRICHLET PROBLEM for $F$:**

Given $\varphi \in C(\partial \Omega)$, consider the Perron function

$$ U \equiv \sup_{u \in F(\varphi)} u \quad \text{where} \quad F(\varphi) = \{ u \in F(\Omega) : u|_{\partial \Omega} \leq \varphi \}. $$

**EXISTENCE.** For each $\varphi \in C(\partial \Omega)$ the Perron function satisfies

- $U$ is $F$-harmonic
- $U = \varphi$ on $\partial \Omega$
- $U \in C(\Omega)$

**UNIQUENESS.** $U$ is the only function with these three properties.

In what follows, $M_F$ denotes a monotonicity cone for $F$.

**THEOREM 13.1.** Suppose $F$ is a $G$-universal subequation where $X$ is provided with a topological $G$-structure. Suppose there exists a $C^2$ strictly $M_F$-subharmonic function on $X$.

Then for every domain $\Omega \subset\subset X$ whose boundary is strictly $\overline{F}$- and $\overline{\tilde{F}}$-convex, both existence and uniqueness hold for the Dirichlet problem.

**THEOREM 13.1’**. Theorem 13.1 also holds for any subequation $F$ which is locally affinely equivalent to a $G$-universal subequation on $X$. 

50
THEOREM 13.2. Suppose $F$ is a subequation for which weak comparison holds. Suppose there exists a $C^2$ strictly $M_F$-subharmonic function on $X$.

Then for every domain $\Omega \subset \subset X$ whose boundary is strictly $\overline{F}$- and $\overline{\tilde{F}}$-convex, both existence and uniqueness hold for the Dirichlet problem.

THEOREM 13.3. Suppose comparison holds for a subequation $F$ on $X$.

Then for every domain $\Omega \subset \subset X$ whose boundary is strictly $\overline{F}$- and $\overline{\tilde{F}}$-convex, both existence and uniqueness hold for the Dirichlet problem.

THEOREM 13.4. Let $F$ be a constant coefficient subequation on $\mathbb{R}^n$.

(Existence.) Then for every domain $\Omega \subset \subset X$ whose boundary is strictly $\overline{F}$- and $\overline{\tilde{F}}$-convex, existence holds for the Dirichlet problem.

(Comparison.) If $F$ is pure second-order, or more generally, independent of the gradient, then comparison holds for $F$ on $\mathbb{R}^n$.

This provides a new proof of existence and uniqueness for pure second-order subequations, established in [HL4] using a result of Slodkowski [S1].

Proof. The existence is just a restatement of Theorem 12.7. Since weak comparison holds for any constant coefficient subequation, comparison will follow from strict approximation. A closed subset $F \subset J^2$ is a pure second-order subequation if and only if $M = \mathbb{R} \times \mathbb{R}^n \times \mathcal{P}$ is a monotonicity set for $F$. Since $|x|^2$ is strictly $M$-subharmonic, strict approximation holds for $F$. More generally, recall that a closed subset $F \subset J^2$ is independent of the gradient if and only if $M \equiv \mathbb{R}_- \times \mathbb{R}^n \times \mathcal{P}$ is a monotonicity set for $F$. In this case the function $|x|^2 - R^2$ is strictly $M$-subharmonic on the ball of radius $R$ about the origin. Since each compact subset $K \subset \mathbb{R}^n$ is contained in such a ball, strict approximation holds for $F$.

More generally we have the following (see Theorem 9.13).

THEOREM 13.5. Let $X = K/G$ be a riemannian homogeneous space and suppose $F$ is a subequation which is invariant under the natural action of the Lie group $K$ on $J^2(X)$.

(Existence.) Then for every domain $\Omega \subset \subset X$ whose boundary is strictly $\overline{F}$- and $\overline{\tilde{F}}$-convex, existence holds for the Dirichlet problem.

(Comparison.) If $X$ supports a strictly convex $C^2$-function, then comparison holds for $F$ on $X$. 

51
14. Universal Riemannian Subequations.

In this section we consider subequations defined on any riemannian manifold by the requirement that $\text{Hess}_x u \in \mathbf{F}$ (for $u \in C^2$), where $\mathbf{F}$ is a closed subset of $\text{Sym}^2(R^n)$ which is $O_n$-invariant. Recall from Section 5.5, that for such purely second-order closed subsets of $J^2(X)$, condition (N) is automatic and condition (P) implies condition (T).

Each $O_n$-invariant closed subset $\mathbf{F} \subset \text{Sym}^2(R^n)$ determines a closed subset $\Lambda$, invariant under the permutation group $\pi_n$ and consists of all $n$-tuples of eigenvalues of $A$ where $A \in \mathbf{F}$. Conversely, each closed subset $\Lambda \subset R^n$ invariant under $\pi_n$ determines a closed $O_n$-invariant subset $\mathbf{F} \subset \text{Sym}^2(R^n)$ namely $\mathbf{F} \equiv \{ A : \lambda(A) \in \Lambda \}$. Moreover,

$$\mathbf{F} \text{ satisfies (P) } \iff \Lambda + R^n_+ \subset \Lambda$$

(14.1)

where $R^n_+ = [0, \infty)$. The implication from left to right in (14.1) is obvious since $\Lambda$ can be taken to be the subset of diagonal elements in $\mathbf{F}$ and $R^n_+$ the set of diagonal elements in $\mathcal{P}$. To prove the reverse implication consider the ordered eigenvalues $\lambda_1(A) \leq \cdots \leq \lambda_n(A)$

(14.2)

of $A \in \text{Sym}^2(R^n)$. Here $\lambda_k(A)$, the $k$th smallest eigenvalue, is a well defined continuous function on $\text{Sym}^2(R^n)$. The standard fact needed is the monotonicity of the ordered eigenvalues:

$$A \leq B \Rightarrow \lambda_k(A) \leq \lambda_k(B) \text{ for all } k$$

(14.3)

which follows from the minimax definition of $\lambda_k(A)$.

Set

$$\lambda(A) = (\lambda_1(A), \ldots, \lambda_n(A)).$$

Now assume $A \in \mathbf{F}$ and $P \geq 0$ so that $B = A + P \geq A$. Monotonicity (14.3) says that $\lambda(B)$ equals $\lambda(A)$ plus a vector in $R^n_+$. Since $\lambda(A) \in \Lambda$, the assumption $\Lambda + R^n_+ \subset \Lambda$ implies that $\lambda(B) \in \Lambda$. Hence $B \in \mathbf{F}$, that is, $\mathbf{F}$ satisfies (P), and (14.1) is proved.

**Definition 14.1.** Suppose $\Lambda$ is a closed subset of $R^n$ invariant under the permutation group $\pi_n$. If $\Lambda$ satisfies

$$\Lambda + R^n_+ \subset \Lambda$$

(14.4)

then $\Lambda$ will be referred to as a **positive** (or $R^n_+$-**monotone**) set in $R^n$.

In the final Remark 14.11 we give a canonical description of all possible $R^n_+$-monotone sets. Otherwise, the remainder of this section is devoted to examples. Our discussion is confined to subsets $\mathbf{F} \subset \text{Sym}^2(R^n)$, but each $\mathbf{F}$ corresponds to a universal subequation in riemannian geometry, and our language reflects that fact.

**Example 14.2. (Monge-Ampère – The Principal Branch).** The Monge-Ampère equation $\det(A) = 0$ gives rise to several subequations or branches. The principal branch is just the subequation $\mathcal{P}$. In terms of the ordered eigenvalues it is given by:

$$\lambda_1(A) \geq 0.$$
The dual subequation $\widetilde{P}$ is defined by
\[ \lambda_n(A) \geq 0. \]
In [HL4] we proved that on open sets $X \subset \mathbb{R}^n$, a function $u$ is $\widetilde{P}$-subharmonic if and only if $u$ is subaffine, that is,
\[ u \leq a \text{ on } \partial K \implies u \leq a \text{ on } K \]
for each affine function $a$ and each compact subset $K \subset X$. Thus $u$ is $P$-harmonic if and only if $u$ is convex and $-u$ is subaffine. Since $\partial \widetilde{P} \subset \{ \det = 0 \}$, the subequation $\widetilde{P}$ is another branch of the Monge-Ampère equation.

**Example 14.3. (The Other Branches of the Monge-Ampère Equation).** Define the $q$th branch $P_q$ by the condition that
\[ \lambda_q(A) \geq 0 \quad \text{(at least } n - q + 1 \text{ eigenvalues } \geq 0). \]
This gives $n$ branches or subequations
\[ P = P_1 \subset \cdots \subset P_q \subset \cdots \subset P_n = \widetilde{P} \]
for the Monge-Ampère equation since, for each $q$,
\[ \partial P_q \subset \{ \det = 0 \} \]
and the positivity condition
\[ P_q + P \subset P_q \]
is satisfied by (14.3). Duality becomes
\[ \widetilde{P}_q = P_{n-q+1} \]
since $\lambda_q(-A) = -\lambda_{n-q+1}(A)$. The principle branch $P$ is a monotonicity subequation for each branch $P_q$. Since each $P_q$ is a cone, we have
\[ \overline{P}_q = P_q. \]
Thus the boundary of a domain $\Omega \subset X$ is strictly $P_q$-convex at a point $x$ iff its second fundamental form $II_{\partial \Omega}$ (with respect to the interior-pointing normal) has at least $n - q$ principle curvatures $> 0$. Thus, $\partial \Omega$ is strictly $\overline{P}_q = P_{n-q+1}$-convex if $II_{\partial \Omega}$ has at least $q - 1$ principle curvatures $> 0$.

**Theorem 13.1** gives us the following result.

**Theorem 14.4.** Let $\Omega \subset \subset X$ be a domain with smooth boundary in a riemannian manifold $X$. Suppose $\Omega$ admits a smooth strictly convex global defining function. Then the Dirichlet problem for every branch of the real Monge-Ampère equation is uniquely solvable for all continuous boundary functions.

**Example 14.5. (Geometric $p$-Plurisubharmonicity).** Let $G(p, \mathbb{R}^n)$ denote the grassmannian of all $p$-dimensional subspaces of $\mathbb{R}^n$. Define $F(G(p, \mathbb{R}^n))$ by requiring that
\[ \text{tr}_W A \geq 0 \quad \text{for all } W \in G(p, \mathbb{R}^n) \quad (14.5) \]
As with all geometrically defined subequations, this is a convex cone subequation. Function with are $F(G(p, \mathbb{R}^n))$-subharmonic are more appropriately called **geometrically $p$-plurisubharmonic** since one can prove that a function $u$ is $F(G(p, \mathbb{R}^n))$-subharmonic if and only if its restriction to every minimal $p$-dimensional submanifold of $X$ is subharmonic with respect to the induced riemannian metric (or $\equiv -\infty$).

The subequation $F(G(p, \mathbb{R}^n))$ can be written in terms of the ordered eigenvalues as
\[
\lambda_1(A) + \cdots + \lambda_p(A) \geq 0,
\]
i.e., all $p$-fold sums of the eigenvalues are $\geq 0$.

The dual subequation $\tilde{F}(G(p, \mathbb{R}^n))$ can be described by either of the two equivalent conditions:
\[
\text{tr}_W A \geq 0 \quad \text{for some } W \in G(p, \mathbb{R}^n)
\]
\[
\lambda_{n-p+1}(A) + \cdots + \lambda_n(A) \geq 0.
\]

Both the subequation $F(G(p, \mathbb{R}^n))$ and its dual $\tilde{F}(G(p, \mathbb{R}^n))$ are branches of a polynomial equation. Define a polynomial $M_p$ on $\text{Sym}^2(\mathbb{R}^n)$ by
\[
M_p(A) = \prod_{|I|=p} \left( \lambda_{i_1}(A) + \cdots + \lambda_{i_p}(A) \right)
\]  
(14.6)

where the prime indicates that the product is over all multi-indices $I = (i_1, \ldots, i_p)$ with $i_1 < i_2 < \cdots < i_p$. Then
\[
A \in \partial F(G(p, \mathbb{R}^n)) \quad \text{or} \quad A \in \partial \tilde{F}(G(p, \mathbb{R}^n)) \quad \Rightarrow \quad M_p(A) = 0.
\]

The equation $M_p(A) = 0$ has many branches, or subequations. Let
\[
\lambda_I(A) \equiv \lambda_{i_1} + \cdots + \lambda_{i_p}
\]
denote the $I$th $p$-fold sum of ordered eigenvalues. These $N = \binom{n}{p}$ real numbers can be ordered as
\[
\lambda_1(p, A) \leq \cdots \leq \lambda_N(p, A).
\]

Define $F_k, k = 1, \ldots, N$ by the condition that
\[
\lambda_k(p, A) \geq 0.
\]

That is, $A \in F_k$ has at least $N - k + 1$ $p$-fold sums of its eigenvalues $\geq 0$. Note that if $B \in F_1 = F(G(p, \mathbb{R}^n))$ (the principal branch) and $A \in F_k$, then $A + B \in F_k$. That is
\[
F(G(p, \mathbb{R}^n)) \text{ is a monotonicity subequation for each branch } F_k.
\]

Since $F_k$ is already a cone, we have that $\overline{F}_k = F_k$. The dual is given by
\[
\overline{F}_k = F_{N-k+1}.
\]

Note that by (11.6)' strict boundary convexity of a domain $\Omega$ means that
\[
\left( \begin{array}{cc}
t & 0 \\
0 & I_{\partial \Omega} \end{array} \right) \in F_k \quad \text{for all } t > 0 \text{ sufficiently large.}
\]
Note that the top \( \binom{n}{p-1} \) \( p \)-fold sums of eigenvalues of this matrix are automatically positive for large \( t \). Hence, if \( \binom{n}{p} - k + 1 \leq \binom{n}{p-1} \) every boundary is automatically strictly \( F_k \)-convex. Otherwise, \( F_k \)-boundary convexity means that \( \Omega \) has at least \( \binom{n}{p} - k + 1 - \binom{n-1}{p-1} \) \( p \)-fold sums of its principal curvatures \( \geq 0 \).

We leave it to the reader to formulate corollaries of Theorem 13.1 for these equations (as we did for the Monge-Ampère equation above).

**Remark 14.6.** Each \( A \in \text{Sym}^2(\mathbb{R}^n) \) acts as a derivation \( D_A \) on \( \Lambda^p \mathbb{R}^n \) and \( D_A \in \text{Sym}^2(\Lambda^p \mathbb{R}^n) \).

The polynomial \( M_p \) is the restriction of the determinant on \( \text{Sym}^2(\Lambda^p \mathbb{R}^n) \) to \( \text{Image}(D_A) \)

\[
M_p(A) = \det(D_A).
\]

**Example 14.7. (Elementary Symmetric Functions).** Recall that the cone \( \mathcal{P} \) can be defined by requiring that

\[
\sigma_1(A) \geq 0, \sigma_2(A) \geq 0, \ldots, \sigma_n(A) \geq 0
\]

where \( \sigma_k(A) \) is the \( k \)th elementary symmetric function of the eigenvalues of \( A \). For each \( k, 1 \leq k \leq n \) the condition

\[
\sigma_1(A) \geq 0, \sigma_2(A) \geq 0, \ldots, \sigma_k(A) \geq 0
\]

defines a convex cone subequation \( F(\sigma_k) \). One can show that this set \( F(\sigma_k) \) is exactly the closure of the connected component of the complement of \( \{ \sigma_k(A) = 0 \} \) which contains the identity \( I \).

The equation \( \{ \sigma_k(A) = 0 \} \) has \( k - 1 \) other branches

\[
F(\sigma_k) = F_1(\sigma_k) \subset F_2(\sigma_k) \subset \cdots \subset F_k(\sigma_k),
\]

(14.7) each of which is \( F(\sigma_k) \)-monotone, and for which \( \widetilde{F}_j(\sigma_k) = F_{k-j+1}(\sigma_k) \). (See the general discussion in [HL\textsuperscript{7}].) The last branch \( F_k(\sigma_k) \), which is the dual to \( F_1(\sigma_k) \), is given by the condition

\[
\sigma_1(A) \geq 0 \text{ or } -\sigma_2(A) \geq 0 \text{ or } -\sigma_3(A) \geq 0 \text{ or } \ldots \text{ or } (-1)^{k-1}\sigma_k(A) \geq 0.
\]

**THEOREM 14.8.** Let \( \Omega \subset X \) be a domain with smooth boundary in a riemannian manifold \( X \). Suppose \( \Omega \) is globally strictly \( F(\sigma_k) \)-convex, that is, suppose there is a strictly \( F(\sigma_k) \)-subharmonic defining function for \( \Omega \). Then for every branch of the equation \( \sigma_k(\text{Hess}u) = 0 \), the Dirichlet problem is uniquely solvable for all continuous boundary data.

**Proof.** The existence of a strictly \( F(\sigma_k) \)-subharmonic defining function \( \rho \) implies that the boundary is strictly \( F(\sigma_k) \)-convex, and from the inclusions (14.7) it is also \( F_j(\sigma_k) \)-convex for each \( j \). Since \( \widetilde{F}_j(\sigma_k) = F_{k-j+1}(\sigma_k) \), the boundary convexity hypothesis of Theorem 13.1 is satisfied. Since \( F(\sigma_k) \) is a monotonicity subequation of \( F_j(\sigma_k) \) and \( \rho \) is strictly \( F(\sigma_k) \)-subharmonic, the other hypothesis is satisfied and Theorem 13.1 applies.

**Example 14.9. (The Special Lagrangian Potential Equation).** This is the subequation \( F_c \) defined by the condition

\[
\text{tr arctan} A = \arctan(\lambda_1(A)) + \cdots + \arctan(\lambda_n(A)) \geq \frac{c\pi}{2},
\]

for \( c \in (-n, n) \). Since arctan is an odd function, the dual equation is again of this form. That is,

\[
\bar{F}_c = F_{-c}
\]
For values of $c$ where $F_c$ is convex, the Dirichlet problem for this equation was studied in detail by Caffarelli, Nirenberg and Spruck [CNS$_2$] who established existence, uniqueness and regularity. Existence, uniqueness and continuity for all other branches $F_c$ were established in [HL$_4$].

One computes that the asymptotic interior of $F_c$ is

$$\overline{F_c^\gamma} = \text{Int} P_q = \{ A : \lambda_q(A) > 0 \}$$

($\lambda_q(A)$ is the $q$th ordered eigenvalue) where $q$ is the unique integer such that

$$\frac{n-c}{2} \leq q < \frac{n-c}{2} + 1$$

(14.9)

THEOREM 14.10. Let $X$ be a riemannian $n$-manifold $X$ on which there exists some global strictly convex function. Let $\Omega \subset X$ be a domain with smooth boundary and suppose $\partial \Omega$ is strictly $P_q$-convex for an integer $q$ satisfying

$$1 \leq q < \frac{n}{2} + 1$$

(14.10)

Then the Dirichlet problem for $F_c$-harmonic functions is uniquely solvable on $\Omega$ for all continuous $\varphi \in \partial \Omega$ and for all $c$ with

$$|c| < n - 2q + 2$$

(14.11)

Proof. Recall that strict $P_q$-convexity implies strict $P_{q'}$-convexity for $q \leq q' \leq n$. Suppose $c \geq 0$ and (14.9) holds. Let $\bar{q}$ be the integer satisfying (14.9) with $c$ replaced by $-c \leq 0$. Then $\bar{q} \geq q$. Hence, by the first remark and (14.8) the hypothesis on the boundary is satisfied and Theorem 13.1 applies to $F_c$. As $c$ descends to zero, the $q$ in (14.9) increases, so by the initial remark, the boundary hypothesis continues to be satisfied.

The subequations $F_c$ are related to the equation

$$\text{Im} \left\{ e^{-i\theta} \det C(I + i\text{Hess } u) \right\} = 0.$$  

(14.12)

(for fixed $\theta$) which arises in Special Lagrangian geometry [HL$_1$]. If $u$ satisfies (14.12), then the graph $\{ y = \nabla u \}$ in $\mathbb{R}^n \times \mathbb{R}^n$ is absolutely volume-minimizing. Note that $\text{Im} \left\{ e^{-i\theta} \det C(I + iA) \right\} = \text{Im} \left\{ e^{-i\theta} \prod_k (I + i\lambda_k(A)) \right\} = 0$ yields the congruence

$$\sum_k \arctan(\lambda_k(A)) \equiv \theta \pmod{\pi}$$

Thus the equation (14.12) has many disjoint connected sheets corresponding to the subequations $F_{2\left(\frac{\theta}{\pi} + k\right)}$ for either $n$ or $n-1$ integer values of $k$. The maximal (and minimal) values were treated in [CNS$_2$] and the other values in the euclidean case in [HL$_4$].

An interesting case where all of the above applies is the cotangent bundle $T^*K$ of a Lie group $K$. Fixing an orthonormal framing $e_1, ..., e_n$ with respect to a left-invariant metric gives a splitting

$$T^*K = K \times \mathbb{R}^n$$

which determines in an obvious way a hermitian almost complex structure and an invariant $(n,0)$-form, i.e., an almost Calabi-Yau structure. As above we can solve the Dirichlet problem for the
special Lagrangian potential equation on $\mathcal{P}_q$-convex domains $\Omega \subset K$ for $q$ as above. For each solution $u$ the graph of $\nabla u$ in $T^*K$ will be a special Lagrangian submanifold.

**Remark 14.11. (A canonical form for general $R_{+}^{n}$-monotone sets $\Lambda$).** Consider the hyperplane $H$ in $\mathbb{R}^n$ perpendicular to $e = (1, \ldots, 1)$. The boundary of the positive orphan $R_{+}^{n}$ is a graph over $H$. More precisely for $\mu = (\mu_1, \ldots, \mu_n) \in H$, set $\|\mu\|^+ = -\mu_{\min}$ where $\mu_{\min} = \min\{\mu_1, \ldots, \mu_n\}$. Then

$$\partial R_{+}^{n} = \{\mu + \|\mu\|^+ e : \mu \in H\}.$$  

Similarly

$$\partial R_{-}^{n} = \{\mu - \|\mu\|^- e : \mu \in H\},$$  

where $\|\mu\|^- = \mu_{\max} = \max\{\mu_1, \ldots, \mu_n\}$.

One can characterize the positive sets $\Lambda$ as follows. There exists a function $f : H \to \mathbb{R}$, invariant under the action of $\pi_n$ on $H$, which is $\| \cdot \|^\pm$-Lipschitz, i.e.,

$$-\|\mu\|^\pm \leq f(\lambda + \mu) - f(\lambda) \leq \|\mu\|^\pm \text{ for all } \lambda, \mu \in H$$  

(14.13)

such that

$$\Lambda = \{\mu + te : \mu \in H \text{ and } t \geq f(\mu)\}.$$  

(14.14)

Finally note that $\|\lambda + \mu\|^\pm \leq \|\lambda\|^\pm + \|\mu\|^\pm$ and $\|\mu\|^- = \| - \mu\|^+.$

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**15. The Complex and Quaternionic Hessians.**

Virtually all the results of the previous section carry over directly to the complex and quaternionic hessians, that is, to the case of $U_n$ and $Sp_1 \cdot Sp_n$ invariant equations.

**The Complex Case.** Consider $C^n = (\mathbb{R}^{2n}, J)$ where $J : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ denotes the standard complex structure. Then we have the set of hermitian symmetric matrices

$$\text{Sym}_C^2(C^n) = \{A \in \text{Sym}^2(\mathbb{R}^{2n}) : AJ = JA\}$$

and the natural projection

$$\pi : \text{Sym}^2(\mathbb{R}^{2n}) \longrightarrow \text{Sym}_C^2(C^n) \quad \text{given by } \pi(A) = \frac{1}{2}(A - JAJ).$$

If $A \in \text{Sym}_C^2(C^n)$ and $A(e) = \lambda e$, then $A(Je) = \lambda Je$, and so $C^n$ decomposes into a direct sum of $n$ complex eigenlines with eigenvalues $\lambda_1(A) \leq \cdots \leq \lambda_n(A)$. Let $\mathcal{P}^C = \pi(\mathcal{P}) = \{\lambda_1(A) \geq 0\}$ be the cone of positive hermitian symmetric matrices.

Note that closed subsets $F \subset \text{Sym}_C^2(C^n)$ invariant under $U_n$ are uniquely determined by their eigenvalue set $\Lambda \subset \mathbb{R}^n$ (invariant under $\pi_n$). Moreover, one again has monotonicity of the ordered eigenvalues, so that positiveness for invariant sets can be expressed as

$$\pi^{-1}(F) + \mathcal{P} \subset \pi^{-1}(F) \quad \iff \quad F + \mathcal{P}^C \subset F \quad \iff \quad \Lambda + R_{+}^{n} \subset \Lambda.$$  

Thus each positive $\pi_n$ invariant set $\Lambda \subset \mathbb{R}^n$ determines a pure second order $U_n$-invariant subequation $\pi^{-1}(F)$. All these equations carry over to any complex (or almost complex) manifold $X$ with a hermitian metric.

The complex Monge-Ampère equation

$$\det_C A = \lambda_1(A) \cdots \lambda_n(A)$$
behaves exactly as in the real case with principle branch $\mathcal{P}^C$ defined by $\{\lambda_1(A) \geq 0\}$ and the remaining branches defined by $\lambda_k(A) \geq 0$. The cone $\mathcal{P}^C$ is a monotonicity subequation for each of the branches. The $\mathcal{P}^C$-subharmonic functions are exactly the classical plurisubharmonic functions on $X$. The boundary of a domain is strictly $\mathcal{P}^C$-convex if and only if it is classically strictly pseudoconvex. Theorem 13.1 now gives the following.

**THEOREM 15.4.** Let $\Omega \subset \subset X$ be a domain with smooth boundary in an almost complex hermitian manifold $X$. Suppose $\Omega$ admits a smooth strictly plurisubharmonic global defining function. Then the Dirichlet problem for every branch of the complex Monge-Ampère equation is uniquely solvable for all continuous boundary functions.

Furthermore, if $X$ carries some strictly plurisubharmonic function on a neighborhood of $\overline{\Omega}$, then one can uniquely solve the Dirichlet problem for the branch $\mathcal{P}_q$ if and only if the second fundamental form of $\partial \Omega$ at least $\max\{q, n - q + 1\}$ principal curvatures $> 0$ at each point.

This generalizes a result of Hunt and Murray [HM] and Slodkowski [S1] to the case of almost complex manifolds.

The discussion of geometric $p$-plurisubharmonicity, elementary symmetric functions and the special Lagrangian potential equation all carry over, and analogues of Theorems 14.8 and 14.10 hold.

**The Quaternionic Case.** Consider $\mathbb{H}^n = (\mathbb{R}^{4n}, I, J, K)$ where $I, J, K : \mathbb{R}^{4n} \to \mathbb{R}^{4n}$ denote right scalar multiplication by the unit quaternions $i, j, k$. Then we have the set of quaternionic hermitian symmetric matrices

$$\text{Sym}^2_{\mathbb{H}}(\mathbb{H}^n) = \{ A \in \text{Sym}^2(\mathbb{R}^{4n}) : AI = IA, AJ = JA, AK = KA \}$$

and the natural projection

$$\pi : \text{Sym}^2(\mathbb{R}^{2n}) \longrightarrow \text{Sym}^2(\mathbb{C}^n) \quad \text{given by} \quad \pi(A) = \frac{1}{4}(A - IAI - JAJ - IKI).$$

If $A \in \text{Sym}^2_{\mathbb{H}}(\mathbb{H}^n)$ and $A(e) = \lambda e$, then $A(e) = \lambda Ie$, $A(Je) = \lambda Je$ and $A(Ke) = \lambda Ke$, and so $\mathbb{H}^n$ decomposes into a direct sum of $n$ quaternionic eigenlines with eigenvalues $\lambda_1(A) \leq \cdots \leq \lambda_n(A)$. Let $\mathcal{P}^H = \pi(\mathcal{P}) = \{\lambda_1(A) \geq 0\}$ be the cone of positive quaternionic hermitian symmetric matrices. The discussion now completely parallels the one given for the complex case above. The discussion carries over with no change.
16. Geometrically Defined Subequations – Examples.

The subequations $F(\mathcal{G})$ which are geometrically defined by a subset $\mathcal{G}$ of the Grassmann bundle $G(p, TX)$ (as in Definition 5.1) are always convex cone subequations. As we have seen the universal subequations $F(\mathcal{G})$ where $\mathcal{G}$ is one of the grassmannians $G(p, \mathbb{R}^n)$, $G_C(p, \mathbb{C}^n)$ or $G_H(p, \mathbb{H}^n)$ for $p = 1, \ldots, n$ are particularly interesting. These are principal branches of a polynomial equation $M_p = 0$. However, there are many additional interesting examples. For some of them there is no known polynomial operator.

We begin with the general result. Fix a closed subset $\mathcal{G} \subset G(p, \mathbb{R}^n)$. Set $G = \{g \in O_n : g(\mathcal{G}) = \mathcal{G}\}$ and let $X$ be a riemannian manifold with topological $G$-structure so that the $G$-universal subequation $F(\mathcal{G})$ is defined on $X$. A domain $\Omega \subset X$ is strictly $\mathcal{G}$-convex if it admits a strictly $\mathcal{G}$-plurisubharmonic defining function. The existence and topological structure of such domains has been studied in [HL_{2,3}]. It is shown there that if $\partial \Omega$ is strictly $\mathcal{G}$-convex and if $\Omega$ supports a strictly $\mathcal{G}$-plurisubharmonic function, then $\Omega$ is itself strictly $\mathcal{G}$-convex. From Theorem 13.1 we have the following.

**THEOREM 16.1.** Let $X$ be a riemannian manifold with topological $G$-structure. Then for any strictly $\mathcal{G}$-convex domain $\Omega \subset X$ the Dirichlet problem for $F(\mathcal{G})$-harmonic functions is uniquely solvable for all continuous boundary data.

**Example 16.2. (Calibrations).** Fix a form $\phi \in \Lambda^p \mathbb{R}^n$ with comass 1, i.e.,
\[
\|\phi\|_{\text{comass}} = \sup \left\{ \frac{\|\phi\|_W}{\text{vol}_W} : W \in G(p, \mathbb{R}^n) \right\} \leq 1.
\]
Given such a form, we define
\[
\mathcal{G}(\phi) = \left\{ W \in G(p, \mathbb{R}^n) : \frac{\|\phi\|_W}{\text{vol}_W} = 1 \right\}
\]
Let $G_\phi = \{g \in SO_n : g^*\phi = \phi\}$ and note that $G_\phi$ preserves $\mathcal{G}(\phi)$. Therefore, when $X$ has a topological $G_\phi$-structure, Theorem 16.1 applies. In this case the $\mathcal{G}(\phi)$-plurisubharmonic and $F(\mathcal{G}(\phi))$-harmonic functions are simply called $\phi$-plurisubharmonic and $\phi$-harmonic functions. Specific examples are given at the end of the introduction.

Note that if $X$ has a topological $G_\phi$-structure, the form $\phi$ determines a global smooth $p$-form $\phi$ on all of $X$ with $\|\phi\|_{\text{comass}} \equiv 1$. If $d\phi = 0$, then $\phi$ is a standard calibration [HL_{1}], and all $\phi$-submanifolds are automatically minimal. For this case an analysis of $\phi$-plurisubharmonic functions, $\phi$-convexity, $\phi$-positive currents, etc. is carried out in detail in [HL_{2,3}].

**Example 16.3. (Lagrangian Plurisubharmonicity).** Take $\mathcal{G} \equiv \text{LAG} \subset G_R(n \mathbb{C}^n)$ to be the set of Lagrangian $n$-planes in $\mathbb{C}^n$. These planes have many equivalent descriptions. They occur, for example, as tangent planes to graphs of gradients over $\mathbb{R}^n \subset \mathbb{C}^n$ and also as rotations of $\mathbb{R}^n \subset \mathbb{C}^n$ by an element of the unitary group $U_n$. In particular, the invariance group which fixes LAG is $U_n$. There exists a polynomial $M$ on $\text{Sym}^2(\mathbb{R}^{2n})$ which is $I$-hyperbolic and of degree $2^n$ with the top branch of $\{M = 0\}$ equal to $F(\text{LAG})$. This is discussed in a separate paper [HL_{6}].

**Example 16.4. $F(\mathcal{G}(\phi))$ as a Monotonicity Subequation.** To be specific consider the associative calibration $\phi$ on $\mathbb{R}^7 = \text{Im}O$ and the set $\mathcal{G}(\phi)$ of associative 3-planes. The convex cone subequation $F(\mathcal{G}(\phi))$ is not the principal branch of a polynomial equation $\{M = 0\}$. Nevertheless, one can construct subequations $\mathcal{F}$ which are $\mathcal{G}(\phi)$-monotone. For example, define $\mathcal{F}$ by the condition that $A \in \mathcal{F}$ if
\[
\exists \xi, \eta \in \mathcal{G}(\phi) \text{ with } \xi \perp \eta \text{ and } \text{tr}_\xi A \geq 0, \text{ tr}_\eta A \geq 0.
\]
Appendix A. Equivalent Definitions F-Subharmonic.

In this appendix we assume the following for each fibre $F_x \subset J^2 = \mathbb{R} \times \mathbb{R}^n \times \text{Sym}^2(\mathbb{R}^n)$:

If $(r, p, A + \alpha \cdot I) \in F_x$ for all $\alpha > 0$ small, then $(r, p, A) \in F_x$. \hfill (A.1)

The Positivity Condition is not required, nor is $F$ required to be closed.

**Remark.** Assuming the Positivity Condition (P), Condition (A.1) is equivalent to requiring that the $\text{Sym}^2$-fibres of $F$ are closed. That is

Each $F_{x, r, p} = \{ A \in \text{Sym}^2(\mathbb{R}^n) : (x, r, p, A) \in F \}$ is closed. \hfill (A.1)'

This is a well defined notion on a manifold $X$. Note that if $A \in F_{x, r, p}$, then for each $\varepsilon > 0$ there exists $A_\varepsilon \in F_{x, r, p}$ with $A_\varepsilon - A \leq \varepsilon \cdot I$. By (P) this implies $A + \varepsilon \cdot I \in F_{x, r, p}$. Assuming (A.1) this proves that $A \in F_{x, r, p}$, i.e., this proves (A.1)'.

**Proposition A.1.** Suppose that $u \in \text{USC}(X)$ and $x_0 \in X$. Let $x = (x_1, ..., x_n)$ be local coordinates on a neighborhood of $x_0$. Then the following Conditions I, II, III, and IV are equivalent.

I. For all $\varphi \in C^2$ near $x_0$,

\[
(1) \quad \left\{ \begin{array}{l}
    u - \varphi \leq 0 \quad \text{near } x_0 \\
    u - \varphi = 0 \quad \text{at } x_0 
\end{array} \right. \quad \Rightarrow \quad J^2_{x_0} \varphi \in F_{x_0}
\]

II. For all $(r, p, A) \in J^2$,

\[
(2) \quad \left\{ \begin{array}{l}
    u(x) - \left[ r + \langle p, x - x_0 \rangle + \frac{1}{2} \langle A(x - x_0), x - x_0 \rangle \right] \leq 0 \quad \text{near } x_0 \\
    u(x) - \left[ r + \langle p, x - x_0 \rangle + \frac{1}{2} \langle A(x - x_0), x - x_0 \rangle \right] = 0 \quad \text{at } x_0 
\end{array} \right. \quad \Rightarrow \quad (r, p, A) \in F_{x_0}
\]

III. For all $(r, p, A) \in J^2$,

\[
(3) \quad \left\{ \begin{array}{l}
    u(x) - \left[ r + \langle p, x - x_0 \rangle + \frac{1}{2} \langle A(x - x_0), x - x_0 \rangle + o(|x - x_0|^2) \right] \leq 0 \quad \text{near } x_0 \\
    u(x) - \left[ r + \langle p, x - x_0 \rangle + \frac{1}{2} \langle A(x - x_0), x - x_0 \rangle + o(|x - x_0|^2) \right] = 0 \quad \text{at } x_0 
\end{array} \right. \quad \Rightarrow \quad (r, p, A) \in F_{x_0}
\]

IV. For all $(r, p, A) \in J^2$ and $\alpha > 0$,

\[
(4) \quad \left\{ \begin{array}{l}
    u(x) - \left[ r + \langle p, x - x_0 \rangle + \frac{1}{2} \langle A(x - x_0), x - x_0 \rangle \right] \leq -\alpha |x - x_0|^2 \quad \text{near } x_0 \\
    u(x) - \left[ r + \langle p, x - x_0 \rangle + \frac{1}{2} \langle A(x - x_0), x - x_0 \rangle \right] = -\alpha |x - x_0|^2 \quad \text{at } x_0 
\end{array} \right. \quad \Rightarrow \quad (r, p, A) \in F_{x_0}
\]

**Proof.** (I $\Rightarrow$ II): Given $(r, p, A) \in J^2$ satisfying (2), set

\[\varphi = r + \langle p, x - x_0 \rangle + \frac{1}{2} \langle A(x - x_0), x - x_0 \rangle.\]

Since the quadratic function $\varphi$ satisfies (1), the condition I implies $(r, p, A) = J^2_{x_0} \varphi \in F_{x_0}$. 

60
(II ⇒ III): Given \((r, p, A) \in J^2\) satisfying (3), it follows that \(\forall \alpha > 0\)

\[ u(x) - \left[ r + \langle p, x - x_0 \rangle + \frac{1}{2} \langle A(x - x_0), x - x_0 \rangle \right] \leq \alpha |x - x_0|^2 \quad \text{near } x_0 \]

or equivalently

\[ u(x) - \left[ r + \langle p, x - x_0 \rangle + \frac{1}{2} \langle (A + 2\alpha I)(x - x_0), x - x_0 \rangle \right] \leq 0 \quad \text{near } x_0 \quad \text{at } x_0. \]

However, this is just (2) for \((r, p, A + 2\alpha I)\). Hence by II we have \((r, p, A + 2\alpha I) \in F_{x_0}\) for all \(\alpha > 0\), proving that \((r, p, A) \in F_{x_0}\) because of the assumption (A.1).

(III ⇒ I): Given \(\varphi\) satisfying (1), the Taylor series for \(\varphi\) satisfies (3).

(II ⇒ IV): This holds since (4) ⇒ (2).

(IV ⇒ II): Suppose that \((r, p, A) \in J^2\) satisfies (2). Equivalently,

\[ u(x) - \left[ r + \langle p, x - x_0 \rangle + \frac{1}{2} \langle (A + 2\alpha I)(x - x_0), x - x_0 \rangle \right] \leq -\alpha |x - x_0|^2 \quad \text{near } x_0 \]

That is, \((r, p, A + 2\alpha I)\) satisfies (4). Therefore, by IV, \((r, p, A + 2\alpha I) \in F_{x_0}\) for all \(\alpha > 0\). Finally (A.1) implies that \((r, p, A) \in F_{x_0}\).
Appendix B. Elementary Properties of F-Subharmonic Functions.

The proof of Theorem 2.6, which lists the elementary properties of F-subharmonic functions, is given in this appendix. As explained in Remark 2.6, the positivity condition is not needed in these proofs. For Properties (A) and (B) absolutely no assumptions on F are required. For Property (C) we only require that the Sym\(^2\)-fibres of F be closed (cf. (A.1)'). Not surprisingly, for (D), (E) and (F) the full hypothesis that F be closed is used.

Proofs.

(A) The condition that max\{u, v\} - \varphi \leq 0 near \(x_0\) with equality at \(x_0\) implies that for one of the functions u, v, say u, we have \(u(x_0) = \varphi(x_0)\). In this case, \(u - \varphi \leq 0\) near \(x_0\) with equality at \(x_0\).

Hence, \(J_{x_0}^2\varphi \in F_{x_0}\).

(B) This follows from Condition III in Proposition A.1.

The remaining properties are proved by a common method which uses Lemma 2.4.

(C) Recall the basic fact that if \(\{v_j\} \subset USC(X)\) is a decreasing sequence with limit v, then

\[
\lim_{j \to \infty} \left\{ \sup_K v_j \right\} = \sup_K v
\]

(B.1)

This is proven as follows. Given \(\delta > 0\), the upper semi-continuity of \(v_j\) implies that the set \(K_j = \{x \in K : v_j(x) \geq \sup_K v + \delta\}\) is compact. The sets \(K_j\) are decreasing since \(\{v_j\}\) is decreasing. The pointwise convergence of \(\{v_j\}\) to \(v\) implies that \(\bigcap_j K_j = \emptyset\). Hence, there exists \(j_0\) with \(K_j = \emptyset\) for all \(j \geq j_0\). That is, \(v_j(x) < \sup_K v + \delta\) for all \(j \geq j_0\) and \(x \in K\).

Suppose now that \(u \notin F(X)\). Then by Lemma 2.4 there exists \(x_0 \in X\), local coordinates \(x\) about \(x_0\), \(\alpha > 0\), and a quadratic function \(\varphi(x) = r + \langle p, x - x_0 \rangle + \frac{1}{2}\langle A(x - x_0), x - x_0 \rangle\) such that

\[
\begin{align*}
\text{near } x_0 & \quad \text{and} \\
\lim_{j \to \infty} x_j &= x_0. \\
\lim_{j \to \infty} x_j &= x_0.
\end{align*}
\]

(B.3)

In particular, \(x_j \in IntB\) is an interior maximum point for \(u_j - \varphi\). Set

\[
r_j = u_j(x_j), \quad p_j = D_{x_j} \varphi = p + A(x_j - x_0), \quad A_j = D_{x_j}^2 \varphi = A.
\]

This proves that

\[
(r_j, p_j, A_j) \in F_{x_j}
\]

since \(u_j \in F(X)\).

Applying (B.1) to \(K = B\) yields that \(r_j = \sup_B u_j \setminus \sup_B u = r\). Hence, \(\lim_{j \to \infty} r_j = r\).

This proves that

\[
\lim_{j \to \infty} (x_j, r_j, p_j, A_j) = (x_0, r, p, A).
\]

(B.4)
Since $F$ is closed, this proves that $(x_0, r, p, A) \in F_{x_0}$, contrary to hypothesis.

(D) Since (B.1) is valid if $\{v_j\}$ converges uniformly to $v$, the proof of (D) is essentially the same as the proof of (C), except easier.

(E) Suppose $u \notin F(X)$. Then exactly as in the previous proofs we have (B.2) for $v^*$. Since

$$ r = v^*(x_0) = \lim_{k \to \infty} \sup_{|y-x_0| \leq \frac{1}{k}} \left\{ \sup_{f \in \mathcal{F}} f(y) \right\}, $$

it follows easily that there exists a sequence $y_k \to x_0$ in $\mathbb{R}^n$ and a sequence $f_k \in \mathcal{F}$ such that

$$ \lim_{k \to \infty} f_k(y_k) = r. $$

Choose a maximum point $x_k$ for $f_k - \varphi$ over $B$, a small closed ball about $x_0$ on which the condition (B.2) holds. By taking a subsequence we may assume that $x_k \to \bar{x} \in B$. Now

$$ f_k(y_k) - \varphi(y_k) \leq f_k(x_k) - \varphi(x_k). \quad (B.5) $$

The left hand side has limit zero. Hence,

$$ 0 \leq \liminf_{k \to \infty} f_k(x_k) - \varphi(\bar{x}) \\ \leq \limsup_{k \to \infty} v(x_k) - \varphi(\bar{x}) \leq v^*(\bar{x}) - \varphi(\bar{x}). \quad (B.6) $$

since by definition $f_k \leq v$. This proves that $\bar{x} = x_0$ by (B.2). In particular, each $x_k$ is interior to $B$ for $k$ large. Therefore, since each $f_k$ is $F$-subharmonic, we find that

$$ (f_k(x_k), D_{x_k} \varphi, D_{x_k}^2 \varphi) \in F_{x_k}. $$

Note that (B.6) implies $\lim_{k \to \infty} f_k(x_k) = v^*(x_0) = \varphi(x_0)$. Since $F$ is closed we conclude that

$$ J_{x_0}^2 \varphi = \lim_{k \to \infty} (f_k(x_k), D_{x_k} \varphi, D_{x_k}^2 \varphi) \in F_{x_0}, $$

which is a contradiction.  

\[ \blacksquare \]
Appendix C. The Theorem on Sums.

In this appendix we recall the fundamental Theorem on Sums, which plays a key role in viscosity theory. We restate the result in a form which is particularly suited to our use.

Fix an open subset $X \subset \mathbb{R}^n$, and let $F, G \subset J^2(X)$ be two second order partial differential subequations. Recall that a function $w \in \text{USC}(X)$ satisfies the Zero Maximum Principle on a compact subset $K \subset X$ if

\[ w \leq 0 \text{ on } \partial K \implies w \leq 0 \text{ on } K. \]

(ZMP)

**THEOREM C.1.** Suppose $u \in F(X)$ and $v \in G(X)$, but $u + v$ does not satisfy the Zero Maximum Principle (ZMP) on a compact set $K \subset X$. Then there exist a point $x_0 \in \text{Int}K$ and a sequence of numbers $\epsilon \downarrow 0$ with associated points $z_\epsilon = (x_\epsilon, y_\epsilon) \to (x_0, x_0)$ in $X \times X$, and there exist

\[ J_{x_\epsilon} \equiv (r_\epsilon, p_\epsilon, A_\epsilon) \in F_{x_\epsilon} \quad \text{and} \quad J_{y_\epsilon} \equiv (s_\epsilon, q_\epsilon, B_\epsilon) \in G_{y_\epsilon}, \]

such that

(1) \quad $r_\epsilon = u(x_\epsilon)$, $s_\epsilon = v(y_\epsilon)$, and $r_\epsilon + s_\epsilon = M_\epsilon \searrow M_0 > 0$,

(2) \quad $p_\epsilon = \frac{x_\epsilon - y_\epsilon}{\epsilon}$, $q_\epsilon = \frac{y_\epsilon - x_\epsilon}{\epsilon} = -p_\epsilon$, and $\frac{|x_\epsilon - y_\epsilon|^2}{\epsilon} \to 0$

(3) \quad $-\frac{3}{\epsilon}I \leq \begin{pmatrix} A_\epsilon & 0 \\ 0 & B_\epsilon \end{pmatrix} \leq \frac{3}{\epsilon} \begin{pmatrix} I & -I \\ -I & I \end{pmatrix}$

**Remark C.2.** In fact, we have

\[ J_{x_\epsilon} \in \overline{J^{2,+}_x}u \quad \text{and} \quad J_{y_\epsilon} \in \overline{J^{2,+}_y}v, \]

where $J^{2,+}_x u$ denotes the upper 2-jet of $u$ at $x$.

Restricting the right hand inequality in (3) to the diagonal yields

(3)$'$ \quad $A_\epsilon + B_\epsilon = -P_\epsilon$ \quad where \quad $P_\epsilon \geq 0$.

This is enough to prove a result which lies between weak comparison and comparison for a constant coefficient subequation $F$ on $\mathbb{R}^n$.

**Corollary C.3.** Suppose $H$ and $F$ are constant coefficient subequations with $H \subset \text{Int}F$. Suppose $K$ is a compact subset of $\mathbb{R}^n$. If $u \in H(K)$ and $v \in \tilde{F}(K)$, then

\[ u + v \leq 0 \text{ on } \partial K \quad \Rightarrow \quad u + v \leq 0 \text{ on } K \]

**Proof.** By (1), (2), and (3)$'$ we have

\[ (-s_\epsilon, -q_\epsilon, -B_\epsilon) = (r_\epsilon - M_\epsilon, p_\epsilon, A_\epsilon + P_\epsilon). \]

Now $(r_\epsilon, p_\epsilon, A_\epsilon) \in H$ implies $(r_\epsilon - M_\epsilon, p_\epsilon, A_\epsilon + P_\epsilon) \in \text{Int}F$.

However, $(s_\epsilon, q_\epsilon, B_\epsilon) \in \tilde{F}$ says that $(-s_\epsilon, -q_\epsilon, -B_\epsilon) \notin \text{Int}F$. 

64
Appendix D. Some Important Counterexamples.

One might wonder whether comparison, or at least uniqueness for the Dirichlet problem, can be established if one weakens the assumption that there exist a strictly $M$-subharmonic function where $M + F \subset F$. Consider for example the case of domains which are strictly $F$-convex (i.e., for which there exists a globally strictly $F$-subharmonic defining function) but the condition $F + F \subset F$ is not satisfied. We give here an example of a strongly $P_3$-convex domain in a non-negatively curved space where comparison and, in fact, uniqueness for the Dirichlet problem fail.

Consider the standard riemannian 3-sphere $S^3 = \{ x \in \mathbb{R}^4 : \| x \| = 1 \}$ and the great circle $\gamma = \{ (x_1, x_2, 0, 0) : x_1^2 + x_2^2 = 1 \} \subset S^3$. Define

$$\delta : S^3 \to \mathbb{R} \quad \text{by} \quad \delta(x) \equiv \text{dist}(x, \gamma)$$

where distance is taken in the 3-sphere metric. The level sets $T_s \equiv \delta^{-1}(s)$ for $0 < s < \pi/2$ are flat tori, which are orbits of the obvious $T^2$ torus action on $\mathbb{R}^4 = \mathbb{R}^2 \times \mathbb{R}^2$. The eigenvalues $\lambda_1(s), \lambda_2(s)$ of the second fundamental form $II_s$ of $T_s$ are constant on $T_s$ and, by the Gauss curvature equation, satisfy

$$\lambda_1(s) = -\frac{1}{\lambda_2(s)}.$$

Straightforward calculation (cf. [HL2, (5.7)]) shows that the riemannian Hessian

$$\text{Hess} \delta = \begin{pmatrix} 0 & 0 \\ 0 & II_s \end{pmatrix} \quad \text{where} \quad \delta = s \in (0, \pi/2).$$

from which it follows easily that

$$\text{Hess} \left( \frac{1}{2} \delta^2 \right) = \begin{pmatrix} 1 & 0 \\ 0 & \delta \cdot II_s \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & s\lambda(s) & 0 \\ 0 & 0 & -\frac{s}{\lambda(s)} \end{pmatrix}$$

where $\delta = s \in (0, \pi/2)$. As $s \searrow 0$, $\lambda(s) = \frac{1}{s} + O(1)$, and so

$$\text{Hess} \left( \frac{1}{2} \delta^2 \right) \bigg|_{s=0} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

In particular we see the following. Let

$$U \equiv S^3 - \tilde{\gamma}$$

where $\tilde{\gamma} = \{ (0, 0, x_3, x_4) : x_3^2 + x_4^2 = 1 \}$ is the “opposite” or “focal” geodesic to $\gamma$. Then

$$\text{Hess} \left( \frac{1}{2} \delta^2 \right) \quad \text{has 2 strictly positive eigenvalues everywhere on} \ U, \quad (D.1)$$

$$\text{Hess} \left( -\frac{1}{2} \delta^2 \right) \quad \text{has 1 eigenvalue} \geq 0 \quad \text{everywhere on} \ U. \quad (D.2)$$

**Conclusion D.1. (Co-convex $\nRightarrow$ Maximum Principle).** Example (D.2) shows that on spherical domains the Maximum Principle fails for co-convex, i.e. $\mathcal{P}$-subharmonic functions (where $\mathcal{P}$ means at least one eigenvalue $\geq 0$). In euclidean space In euclidean space co-convex functions do satisfy the maximum principle. They are called *subaffine functions* and play an important role in [HL4].

65
Consider now the product
\[ U \times U \subset S^3 \times S^3 \]
and define \( \delta_k = \delta \circ \pi_k \) where \( \pi_k : U \times U \to U \) denotes projection onto the \( k \)th factor. Set
\[ \rho = \frac{1}{2} \delta_1^2 + \frac{1}{2} \delta_2^2 \]
and note that
\[ \text{Hess } \rho = \text{Hess } \left( \frac{1}{2} \delta_1^2 \right) \oplus \text{Hess } \left( \frac{1}{2} \delta_2^2 \right). \]
In particular, \( \rho \) has four strictly positive eigenvalues everywhere on \( U \times U \). In the terminology of §7 this means that \( \rho \) is strictly \( P_2 \)-subharmonic on \( U \times U \). (Recall that \( P_q \)-subharmonic means that \( \text{Hess } u \) has at least \( n - q = 6 - q \) eigenvalues \( \geq 0 \).) Therefore the domain
\[ \Omega_c \equiv \{(x, y) \in U \times U : \rho(x, y) \leq c\} \text{ is strictly } P_2 \text{ convex } \quad (D.3) \]
for \( 0 < c < \frac{\pi^2}{8} \).

Consider now the functions
\[ u_1 = -\frac{1}{2} \delta_1^2 \quad \text{and} \quad u_2 = -\frac{1}{2} \delta_2^2. \]
By (D.2) have the following. Recall that \( u \) is \( P_q \)-harmonic if \( \lambda_{q+1} \equiv 0 \) where \( \lambda_1 \leq \cdots \leq \lambda_6 \) are the eigenvalues of \( \text{Hess } u \).

**Lemma D.2.** Each \( \text{Hess } u_k, k = 1, 2 \) has two negative eigenvalues, three zero eigenvalues and one non-negative eigenvalue at every point of \( U \times U \). In particular,
\[ u_1, u_2 \in P_2(U \times U) \quad \text{and} \quad -u_1, -u_2 \in P_1(U \times U). \]
Furthermore, each \( u_k \) is \( P_2, P_3 \) and \( P_4 \)-harmonic, whereas each \( -u_k \) is \( P_1, P_2 \) and \( P_3 \)-harmonic on \( U \times U \).

**Conclusion D.3. (Comparison Fails).** Each domain \( \Omega_c \) is strictly \( P_2 \)-convex. Furthermore there are smooth functions
\[ u_1 \in P_2(\Omega_c) \quad \text{and} \quad u_2 \in \bar{P}_2(\Omega_c) \]
(since \( P_2(\Omega_c) \subset P_3(\Omega_c) = \bar{P}_2(\Omega_c) \)) such that
\[ u_1 + u_2 \equiv -c < 0 \text{ on } \partial \Omega_c \quad \text{and} \quad \sup_{\Omega_c}(u_1 + u_2) = 0. \]
Hence, the Comparison Principle fails for \( P_2 \) on each \( \Omega_c \).

**Conclusion D.4. (Uniqueness fails for the Dirichlet problem).** Each domain \( \Omega_c \) is strictly \( P_3 \)-convex (since \( P_2 \Rightarrow P_3 \)). By the lemma we have that both functions \( u_1 \) and \( c - u_2 \) are \( P_3 \)-harmonic on \( \Omega_c \), and
\[ u_1 \equiv c - u_2 \text{ on } \partial \Omega_c. \]

**Remark D.5. (Existence without uniqueness (again)).** Note that the domains \( \Omega_c \) are strictly \( \bar{P}_2 \) and \( \bar{P}_2 \)-convex (since \( P_k = \bar{P}_k \) and \( P_2 \Rightarrow P_3 \)). Since the isometry group of \( S^3 \times S^3 \) is transitive and preserves these subequations, Theorem 12.5 guarantees the existence of solutions to the Dirichlet Problem for all continuous boundary data on each \( \Omega_c \). However, as seen above, uniqueness fails in general.
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