Free Knots and Parity

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Abstract

We consider knot theories possessing a parity: each crossing is decreed odd or even according to some universal rule. If this rule satisfies some simple axioms concerning the behaviour under Reidemeister moves, this leads to a possibility of constructing new invariants and proving minimality and non-triviality theorems for knots from these classes, and constructing maps from knots to knots.

Our main example is virtual knot theory and its simplification, free knot theory. By using Gauss diagrams, we show the existence of non-trivial free knots (counterexample to Turaev’s conjecture), and construct simple and deep invariants made out of parity. Some invariants are valued in graph-like objects and some other are valued in groups.

We discuss applications of parity to virtual knots and ways of extending well-known invariants.

The existence of a non-trivial parity for classical knots remains an open problem.

Keywords: knot, link, free knot, parity,

1 Introduction

The present paper describes the state-of-art of the theory of free knots and links, a drastic simplification of virtual knot theory. The main tool in our investigation is the concept of parity. If there is a possibility to distinguish between two types of crossings, even ones and odd ones in some knot theory, this will allow us to construct new and powerful invariants and improve some well-known invariants. In particular, parity was first used to prove that free knots are in general non-trivial.

The first example of parity, called Gaussian, comes from the chord diagram of a link: a chord (and, thus, the crossing corresponding to it) is even whenever the number of chords linked with it, is even. Summing up the properties of the Gaussian parity, one can axiomatize the notion of parity for crossings. Most of theorems we present here, work for different parities in different situations.

Here we would like to emphasize the point of view that if there is a possibility to distinguish between different types (not necessarily two types) of crossings when looking at a diagram, this information can be used to improve many known invariants, \cite{12,13,14,15}. For rigorous proofs of most of the theorems given here, see \cite{12,13,14}.

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The concepts of the present paper have some far-reaching generalizations in other situations. In particular, parity is very helpful in the theory of graph-links, see review \[7\] in the present volume.

The present paper is organized as follows. In the next section we describe the notions of virtual knots and free knots.

Section 3 is devoted to the first examples and applications of parity; we construct “forgetting mappings” and reprove the theorem on “closedness” of the class of even diagrams.

Then in section 4 we redefine the concept of parity in terms of homology of graphs.

Section 5 deals with invariants of free knots and virtual knots and minimality results.

We conclude the section by a very simply constructed invariant of free knots valued in a certain group, see \[18\]; this invariant gives an obstruction for a free knot to be slice.

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### 2 Virtual knots. Free Knots

Virtual knots were first introduced by Kauffman \[10\] as an extension of classical knots into knots in thickened surfaces considered up to isotopy and stabilizations/destabilizations. Virtual knots have a simple diagrammatic descriptions by planar diagrams and Reidemeister moves. Another description of virtual knots comes from Gauss diagrams \[6\]: one admits all possible Gauss diagrams (not necessarily realizing planar Gauss codes) and factorizes them by formal Reidemeister moves.

A virtual diagram is a generic immersion of a framed 4-graph in \( \mathbb{R}^2 \) with each vertex endowed with a classical crossing structure, i.e. one pair of opposite half edges is defined to form an overcrossing; the other two half-edges form an undercrossing; when drawing a diagram on the plane, intersections of different edges of the framed 4-graphs in interior points are called virtual crossings, and they are encircled.

By abusing notation, we shall allow a 4-graph (four-valent graph) to have connected free components homeomorphic to a circle; this agrees with the fact that an unknot may have a crossingless diagram. We shall denote the one-component crossingless diagram by \( \Gamma_0 \).

A virtual link is an equivalence class of virtual diagrams by usual Reidemeister moves applied to classical crossings and the detour move. The detour move changes the immersion of an edge of the graph: one takes an edge fragment drawn on the plane (which has only virtual crossings and self-crossings inside) and redraws it arbitrarily in a generic way with all new crossings specified as virtual.

So, every virtual knot or link has a “frame” being a framed 4-graph, whence virtual crossings “are not there”. As shown in \[6\], classical links embed into virtual links, i.e., every two classical knots which are equivalent as virtual knots are equivalent as classical knots (isotopic).

In fact, embedding theorems of such sort saying that the equivalence relation on a subset coincides with the induced equivalence relation coming from a set, can be proved by constructing projection “maps” from the whole set to a subset. The concept of parity can be used in this direction in many cases, see Theorem \[4\] ahead.

We are now going to turn to a simplification of virtual knots obtained by “forgetting” all structures except framing at classical crossings.
Free knots and links (also known as homotopy classes of Gauss words, resp., Gauss phrases, following Turaev [22]) are defined as follows.

By a framed 4-graph we mean a 4-valent graph where for each vertex we fix a way of splitting of the four emanating half-edges into two pairs of edges called (formally) opposite.

Such graphs naturally arise from knot (or virtual knot) projections or from generically immersed curves in 2-surfaces.

A free link is an equivalence class of framed 4-graphs modulo the following Reidemeister moves:

1) addition/removal of a loop formed by two half-edges of the same edge approaching a vertex from non-opposite sides;
2) addition/removal of a bigon with two edges being non-opposite at both vertices;
3) the triangular move pulling a piece of unicursal curve through a crossing formed by two other pieces of unicursal curves, see Fig. 1.

The relation of half-edges to be opposite allows one to define the notion of component and to count the number of unicursal components of a framed graph, which turns out to be invariant under Reidemeister moves. A free knot is a single-component free link.

There are obvious “forgetting maps” from virtual links to free links and from homotopy classes of curves on 2-surfaces to free links; classical knots (or planar curves) project to free unknots.

Thus, the study of invariants of free links may enlarge our knowledge about virtual knots or knots in surfaces.

For two-component free links there is an obvious invariant: the parity of number of the intersection points; for an n-component free links there is a collection of mutual parities which can be encoded by a graph on n vertices.

Several years ago, V.G. Turaev [20] conjectured that all free knots are trivial.

This conjecture was recently disproved by myself and (independently, just some days later) A.Gibson [4]. To do that, I used the concept of parity.

The existence of a non-trivial parity for classical knots remains an open problem. For different examples of parity, see [14].

3 The concept of parity

We shall encode knots and virtual knots [10] by Gauss diagrams [6]. We say that two chords of a chord diagram are linked if both ends of one of them lie in different connected components of the two-component set obtained from the circle by deleting the ends of the second components. We say that a chord c of a Gauss diagram is even if the number of chords it is linked with, is even, and odd, otherwise. We shall denote the set of chords linked with c by $E_c$; any chord is assumed not to be linked with itself; by $E_a + E_b$ we shall mean $E_a \cup E_b \setminus (E_a \cap E_b)$. For two chords $a, b$, we write $\langle a, b \rangle = 1$ if $a$ and $b$ are linked, and $\langle a, b \rangle = 0$, otherwise. Here + stays for the Boolean sum of sets.

We say that a vertex of the four-valent framed graph corresponding to the Gauss diagram is odd whenever the chord corresponding to this vertex is odd.

Let us summarize the properties of even and odd chords with respect to their behaviour under Reidemeister moves. It is easy to see that

1. The parity of a vertex participating in the first Reidemeister move, is even.
The first Reidemeister move corresponds to an addition/removal of an even vertex, and the pairwise incidence relation of the remaining vertices does not change.

2. The second Reidemeister move adds (removes) two vertices \( a, b \) of the same parity. Indeed, for such vertices \( a, b \) herewith \( E_a + E_b \) is either \( \emptyset \) or coincides with \( a + b \); after applying a Reidemeister move the parity of the remaining vertices does not change. Neither does the pairwise incidence of the remaining vertices.

3. The sum of parities (modulo 2) of the three vertices participating in a third Reidemeister move is even. The parity of a vertex participating in a third Reidemeister move does not change after this move is applied.

   Indeed, when performing the third Reidemeister move, we have three vertices (chords) \( a, b, c \), for which \( E_a + E_b + E_c \subset \{ a, b, c \} \mid E_a + E_b + E_c \mid = 0 \text{ or } 2 \). After performing the move, we get instead of \( a, b, c \) three vertices \( a', b', c' \) with pairwise switched incidences (w.r.t. \( a, b, c \)) the remaining incidences are unchanged: for \( f, g \notin \{ a, b, c \} : \langle f, g \rangle \) remains unchanged:

   a) \( \langle f, a \rangle = \langle f, a' \rangle ; \quad \langle f, b \rangle = \langle f, b' \rangle ; \quad \langle f, c \rangle = \langle f, c' \rangle \) and

   b) the number of odd vertices amongst \( a, b, c \) is even (is equal to zero or two).

We call the parity of vertices of the Gauss diagram (and corresponding chords) the Gaussian parity.

**Definition 1.** By a parity we shall mean a function on the set of all four-valent framed graphs which satisfies the conditions above (assuming all vertices not taking part in a Reidemeister move preserve their parity).

**Remark 1.** Here, the vertices of a 4-valent graph are not enumerated, so the parity function on should be symmetric with respect to the action of the group of symmetries of the graph respecting the framing.

In the case when we deal not merely with free knots or links, but with knots having some decorations, by parity we shall mean the property of crossings satisfying the same axioms with respect to the Reidemeister moves. Thus, for instance, the second parity axiom for the second Reidemeister move requires that the parity is the same for both vertices of any “bigon” in the case of free knots, whence for virtual knots, we shall require this only for those pairs of crossings which can participate in a second Reidemeister move, that is, for two crossings of opposite signs \( (\bigotimes, \bigotimes) \) forming a bigon.

Certainly, each parity defined for free links induces a parity for virtual knots, but not vice versa.

In the sequel, we shall state some theorems using parity. Unless specified otherwise, such theorems will hold for any parity.

The existence of non-trivial parity for classical knots remains an open problem.

### 3.1 The functorial map \( f \)

It turns out that the parity axioms listed above lead to a simple and powerful map on the set of free knots and, more generally, on the set of virtual knots.

Let \( K \) be a virtual knot diagram. Let \( f \) be a diagram obtained from \( K \) by making all odd crossings virtual. In other words, we remove all odd chords.

The following theorem follows from definitions.
**Theorem 1.** The map $f$ is a well-defined map on the set of all virtual knots. For a virtual knot diagram $K$, $f(K) = K$ iff all crossings of $K$ are even. Otherwise, the number of classical crossings of $f(K)$ is strictly less than the number of classical crossings of $K$.

### 3.2 Other examples of parities. The parity hierarchy

The Gaussian parity (defined via intersection as above) is not the only parity for knots. We shall give two more examples which can be applied in different situations.

The first example deals with two-component free links. A crossing of a 2-component free link is called *even* if it is formed by two branches of the same component; otherwise it is called *odd*.

Another example deals with the set of virtual knots with all crossings having *even* Gaussian parity. It follows from Theorem 1 that this set is closed with respect to the Reidemeister moves, i.e., whenever two diagrams having even crossings are equivalent, they are equivalent by a sequence of Reidemeister moves where all intermediate diagrams have all even crossings. Denote this class of knots by $\mathcal{V}^1$; evidently, it includes all classical knots. We shall show that there exists a natural *filtration* on the set of all virtual knots:

$$
\mathcal{V}^0 \supset \mathcal{V}^1 \supset \mathcal{V}^2 \supset \cdots \supset \mathcal{V}^n \supset \cdots,
$$

which starts with the set of $\mathcal{V}^0$ of all virtual knots and has as a limit some set $\mathcal{V}^\infty$ of "knots of index zero" including all classical knots.

So, let $D$ be a virtual diagram, and let $G(D)$ be the corresponding Gauss diagram. We shall endow the diagram $G$ with signs and arrows in the usual way (the sign plus corresponds to $\overarc{\cdot}$ and the sign minus corresponds to $\underarc{\cdot}$; an arrow is directed from the pre-image of the arc forming an overpass to the pre-image of the arc forming an underpass).

With each classical crossing we associate its *index* which will be a non-negative integer. Let $X$ be a classical crossing of $D$, and let $c(X)$ be the corresponding (oriented) chord of $G(D)$. Consider all chords of the diagram $G(D)$, which are linked with $c(X)$. Let us calculate the sum of signs of all those chords intersecting $c(X)$ from the left to the right and subtract the sum of signs of those chords intersecting $c(X)$ from the right to the left.

The *index of $c$* is the absolute value of the above quantity; we shall denote it by $\text{ind}(c(X))$.

Clearly, a chord is even whenever its index is even.

Let us collect some facts concerning the index; the proof can be obtained by a simple check:

**Statement 1.** 1. Any chord taking part in a Reidemeister-1 move has index 0.

2. In the Reidemeister-2 move indices of the two chords are the same.

3. Like parity, the index is invariant under the third Reidemeister move $\text{ind}(a) = \text{ind}(a')$, $\text{ind}(b) = \text{ind}(b')$, $\text{ind}(c')$; moreover, if some three chords $a, b, c$ take part in a third Reidemeister move then $\text{ind}(a) + \text{ind}(b) + \text{ind}(c) = 0$, see Fig. 1.

4. The index remains unchanged under any Reidemeister move for those chords not participating in this move.

5. All crossings of a classical diagram have index zero.

The properties described above show that the index can be used for defining parities.
Indeed, from Statement 1 we see that for diagrams from $\mathfrak{V}^1$ we may introduce the following parity: let $K$ be a knot diagram from $\mathfrak{V}^1$; we decree those crossings of $K$ having index divisible by four, to be even, and those having index congruent to two modulo four, to be odd.

From Statement 1 it follows that the parity defined in this way, satisfies all the axioms.

Applying $f$ to knots from $\mathfrak{V}^1$ with respect to the parity described above (Gaussian parity) we get the set $\mathfrak{V}^2$ of diagrams with all indices divisible by four.

Arguing as above, we define the sets $\mathfrak{V}^k$ of diagrams with indices divisible by $2^k$, $k \in \mathbb{N}$. Let $\mathfrak{V}^\infty$ be the set of diagrams with all crossings having index zero.

The following theorem holds

**Theorem 2.** Let $K, K'$ be two diagrams of virtual knots from $\mathfrak{V}^k$ (where $k$ is a positive integer or $\infty$), representing equivalent virtual knots. Then there is a chain of diagrams $K = K_0 \to K_1 \cdots \to K_n = K'$ from $\mathfrak{V}^k$ where every two adjacent diagrams differ by a Reidemeister move (or a detour move).

The proof of Theorem 2 is obtained as follows. Starting from an arbitrary chain connecting $K$ to $K'$, we consequently apply the map $f$ for the Gaussian parity until the chain belongs to $\mathfrak{V}^2$, then we consequently apply the other $f$ to get a sequence from $\mathfrak{V}^2$, and so on, until we get a chain from $\mathfrak{V}^k$. Since all of our maps $f$ (applied to different $\mathfrak{V}^k$) are well defined, each two consecutive elements in each of our chains will be connected by a Reidemeister move or a detour move.

In the case $k = \infty$ it is sufficient to apply the trick from Theorem 7 finitely many times, also. Indeed, consider an arbitrary chain connecting $K$ to $K'$. Assume the maximal number of chords of diagrams in this chain does not exceed $m < 2^m$. If an index of some chord is divisible by $2^m$ then this index is equal to zero. So, it will be sufficient to apply the map $f$ $m$ times to get a sequence of diagrams from $\mathfrak{V}^\infty$.

Note that the class $\mathfrak{V}^\infty$ is quite interesting: it is an “approximation” of classical knots in the set of virtual knots. All invariants defined on the set $\mathfrak{V}^\infty$, can be translated to virtual knots by using $f$ (in virtue of Theorem 2). Obviously, all classical diagrams are diagrams belong to $\mathfrak{V}^\infty$. A non-classical knot diagram $D$ from $\mathfrak{V}^\infty$ is given in Fig. 2.

### 4 Parity as homology

Consider a framed 4-graph $\Gamma$ with one unicursal component. The homology group $H_1(\Gamma, \mathbb{Z}_2)$ is generated by “halves” corresponding to vertices: for every vertex $v$ we have two halves of the graph $\Gamma_{v,1}$ and $\Gamma_{v,2}$, obtained by smoothing at this vertex, see Fig. 3. If the set of framed 4-graphs (possibly, with some further decorations at vertices) is endowed with a parity, we may assume that we are given the following cohomology class $h$: for each of the halves $\Gamma_{v,1}, \Gamma_{v,2}$ we
set $h(\Gamma_{v,1}) = h(\Gamma_{v,2}) = p(v)$, where $p(v)$ is the parity of the vertex $v$. Taking into account that every two halves sum up to give the cycle generated by the whole graph, we have defined a “characteristic” cohomology class $h$ from $H_1(\Gamma, \mathbb{Z}_2)$.

Collecting the properties of this cohomology class we see that

1. For every framed 4-graph $\Gamma$ we have $h(\Gamma) = 0$.
2. If $\Gamma'$ is obtained from $\Gamma$ by a first Reidemeister move adding a loop then for every basis $\{\alpha_i\}$ of $H_1(\Gamma, \mathbb{Z}_2)$ there exists a basis of the group $H_1(\Gamma, \mathbb{Z}_2)$ consisting of one element $\beta$ corresponding to the loop and a set of elements $\alpha'_i$ naturally corresponding to $\alpha_i$.
   Then we have $h(\beta) = 0$ and $h(\alpha_i) = h(\alpha'_i)$.
3. Let $\Gamma'$ be obtained from $\Gamma$ by a third increasing Reidemeister move. Then for every basis $\{\alpha_i\}$ of $H_1(\Gamma, \mathbb{Z}_2)$ there exists a basis in $H_1(\Gamma', \mathbb{Z}_2)$ consisting of one “bigon” $\gamma$, the elements $\alpha'_i$ naturally corresponding to $\alpha_i$, and one additional element $\delta$, see Fig. [4] left.
   Then the following holds: $h(\alpha_i) = h(\alpha'_i), h(\gamma) = 0$. 

Figure 2: A non-classical diagram from $\mathcal{F}^\infty$

Figure 3: The graphs $\Gamma_{v,1}$ and $\Gamma_{v,2}$
4. Let $\Gamma'$ be obtained from $\Gamma$ by a third Reidemeister move. Then there exists a graph $\Gamma''$ with one vertex of valency 6 and the other vertices of valency 4 which is obtained from either of $\Gamma$ or $\Gamma'$ by contracting the “small” triangle to the point. This generates the mappings $i : H_1(\Gamma, \mathbb{Z}_2) \to H_1(\Gamma'', \mathbb{Z}_2)$ and $i' : H_1(\Gamma', \mathbb{Z}_2) \to H_1(\Gamma'', \mathbb{Z}_2)$, see Fig. 4 right.

Then the following holds: the cocycle $h$ is equal to zero for small triangles, besides that if for $a \in H_1(\Gamma, \mathbb{Z}_2), a' \in H_1(\Gamma', \mathbb{Z}_2)$ we have $i(a) = i'(a')$, then $h(a) = h(a')$.

Thus, every parity for free knots generates some $\mathbb{Z}_2$-cohomology class for all framed 4-graphs with one unicursal component, and this class behaves nicely under Reidemeister moves.

The converse is true as well. Assume we are given a certain “universal” $\mathbb{Z}_2$-cohomology class for all four-valent framed graphs satisfying the conditions 1)-4) described above. Then it originates from some parity. Indeed, it is sufficient to define the parity of every vertex to be the parity of the corresponding half. The choice of a particular half does not matter, since the value of the cohomology class on the whole graph is zero. One can easily check that parity axioms follow.

This point of view allows to find parities for those knots lying in $\mathbb{Z}_2$-homologically nontrivial manifolds. For more details, see [14].

5 Invariants and minimality examples

As an example showing the power of the notion of parity we present the following theorem.

**Theorem 3.** Let $K$ be a four-valent framed graph with one unicursal component such that all chords of $K$ are odd and no second decreasing Reidemeister move can be applied to $K$. Then $K$ is a minimal diagram of the corresponding free link in the following strong sense: for any diagram $K'$ equivalent to $K$ there is a smoothing of $K'$ isomorphic to the graph $K$.

We shall call four-valent framed graphs from the formulation of Theorem 3 irreducibly odd.

In fact, we shall see that irreducibly odd diagrams are minimal in a stronger sense, see Corollary 1 ahead.

To prove this theorem, we shall introduce an invariant $\left[\right]$ valued in linear combinations of some graphs (more precisely, equivalence classes of graphs), which allows to reduce all the Reidemeister moves to the second Reidemeister move only.
Let $\mathcal{G}$ be the set of all equivalence classes of framed graphs with one unicursal component modulo second Reidemeister moves. Consider the linear space $\mathbb{Z}_2\mathcal{G}$.

Let $\Gamma$ be a framed graph, let $v$ be a vertex of $\Gamma$ with four incident half-edges $a, b, c, d$, s.t. $a$ is opposite to $c$ and $b$ is opposite to $d$ at $v$.

By smoothing of $\Gamma$ at $v$ we mean any of the two framed 4-graphs obtained by removing $v$ and repasting the edges as $a - b$, $c - d$ or as $a - d$, $b - c$, see Fig. 5.

Herewith, the rest of the graph (together with all framings at vertices except $v$) remains unchanged.

We may then consider further smoothings of $\Gamma$ at several vertices.

Consider the following sum

$$[\Gamma] = \sum_{s \text{ even}, 1 \text{ comp}} \Gamma_s,$$

which is taken over all smoothings in all even vertices, and only those summands are taken into account where $\Gamma_s$ has one unicursal component.

Thus, if $\Gamma$ has $k$ even vertices, then $[\Gamma]$ will contain at most $2^k$ summands, and if all vertices of $\Gamma$ are odd, then we shall have exactly one summand, the graph $\Gamma$ itself.

Consider $[\Gamma]$ as an element of $\mathbb{Z}_2\mathcal{G}$. In this case it is evident, for instance, that if all vertices of $\Gamma$ are even then $[\Gamma] = [\Gamma_0]$; by construction, all summands in the definition of $[\Gamma]$ are equal to $[\Gamma_0]$, it can be easily checked that the number of such summands is odd.

Now, we are ready to formulate the main result of the present section:

**Theorem 4.** If $\Gamma$ and $\Gamma'$ represent the same free knot then in $\mathbb{Z}_2\mathcal{G}$ the following equality holds: $[\Gamma] = [\Gamma']$.

**Theorem 4** yields the following

**Corollary 1.** Let $\Gamma$ be an irreducibly odd framed 4-graph with one unicursal component. Then any representative $\Gamma'$ of the free knot $K_\Gamma$, generated by $\Gamma$, has a smoothing $\tilde{\Gamma}$ having the same
number of vertices as $\Gamma$. In particular, $\Gamma$ is a minimal representative of the free knot $K_\Gamma$ with respect to the number of vertices.

Proof of the Corollary. By definition of $[\Gamma]$ we have $[\Gamma] = \Gamma$. Thus if $\Gamma'$ generates the same free knot as $\Gamma$ we have $[\Gamma'] = \Gamma$ in $\mathbb{Z}_2 \mathfrak{S}$.

Consequently, the sum representing $[\Gamma']$ in $\mathfrak{S}$ contains at least one summand which is equivalent to $\Gamma$ in $\mathbb{Z}_2 \mathfrak{S}$. Thus $\Gamma'$ has at least as many vertices as $\Gamma$ does.

Moreover, the corresponding smoothing of $\Gamma'$ is a diagram, which is equivalent to $\Gamma$ by second Reidemeister moves. The irreducibility condition yields that one of smoothings of $\Gamma'$ coincides with $\Gamma$.

The invariant $[\cdot]$ is constructive. To see that, one should just look at the structure of the set $\mathbb{Z}_2 \mathfrak{S}$.

Example 1. The simplest example of an irreducibly odd graph (which is minimal according to Theorem 3) is depicted in Fig. 6.

Example 2. A free link whose minimality can be established analogously by using $\{\cdot\}$ is shown in Fig. 7.

Remark 2. In the case of free links with a parity, the bracket $[\cdot]$ does not work verbatim: by definition, for any free link $L$ with all odd crossings we have $[L] = 0$ since the only even smoothing of $L$ gives more than one component. Nevertheless, there is a simple modification of $[\cdot]$, denoted by $\{\cdot\}$, which is the sum over all even smoothings, which yield links with no split components. $\{\cdot\}$ allows to prove minimality theorems for links with arbitrarily many components, for more details, see [14].

5.1 The set $\mathbb{Z}_2 \mathfrak{S}$

Having a framed 4-graph, one can consider it as an element of $\mathbb{Z}_2 \mathfrak{S}$. It is natural to try simplifying it: we call a graph in $\mathbb{Z}_2 \mathfrak{S}$ irreducible if no decreasing second Reidemeister move can be applied to it. The following theorem is trivial.
Theorem 5. Every 4-valent framed graph $G$ with one unicursal component considered as an element of $\mathbb{Z}_2 \mathcal{G}$ has a unique irreducible representative, which can be obtained from $G$ by consecutive application of second decreasing Reidemeister moves.

This allows to recognize elements $\mathbb{Z}_2 \mathcal{G}$ easily, which makes the invariants constructed in the previous subsection digestable.

In particular, the minimality of a framed 4-graph in $\mathbb{Z}_2 \mathcal{G}$ is easily detectable: one should just check all pairs of vertices and see whether any of them can be cancelled by a second Reidemeister move (or in $\mathbb{Z}_2 \mathcal{G}$ one should also look for free loops).

5.2 The “even” Kauffman bracket

We have shown that the parity consideration allows to construct new invariants. In fact, parity considerations allow to improve some existing invariants, as well. Below we show how to strengthen the Kauffman bracket by using parity, for details see [14]. A construction of a generalized Alexander polynomial and a generalized quandle are given in D.M.Afanasiev’s paper [1].

We shall explicitly write down all formulae for virtual knot theory. The generalization of the Kauffman bracket given below works also in some other cases, when for every crossing, one can distinguish between the two ways of smoothing, $A$ and $B$, so that these smoothings “respect the Reidemeister moves as in the classical case”; also note that these generalizations work for the case of graph-links [7, 8, 9].

We recall the definition of the Kauffman bracket. For a virtual diagram $K$, we set

$$\langle K \rangle = \sum_s a^{\alpha(s)} \beta(s) (-a^2 - a^{-2})^{\gamma(s) - 1},$$  \hspace{1cm} (2)

where $\alpha(s)$ and $\beta(s)$ are the numbers of smoothings of type $A: \overline{\times} \rightarrow \underbar{\times}$ and of type $B: \overline{\times} \rightarrow \overline{\times}$, respectively, and $\gamma(s)$ denotes the number of circles (unicursal components) in the state $s$. 

Figure 7: Minimal free two-component link
(after smoothing). Note that in the definition of the Kauffman bracket we *smooth all crossings*. Thus, the value of the Kauffman bracket is just a Laurent polynomial in $a$.

Consider the free module $F$ over $\mathbb{Z}[a, a^{-1}]$ generated by all framed 4-graphs.

Let $\tilde{F}$ be the quotient of $F$ modulo the following relation

1. The second Reidemeister move,
2. The relation $L \sqcup \mathbb{O} = (-a^2 - a^{-2})L$, where $L$ stays for any framed 4-graph and $L \sqcup \mathbb{O}$ denote the split sum of $L$ with a circle.

The algorithmic recognizability of elements from $\tilde{F}$ is similar to that of $\mathbb{Z}_2 \mathcal{G}$.

Let us use the Gaussian parity for virtual knots, and let us construct the even Kauffman bracket taking values in the module $\tilde{F}$, as follows.

Now, by *states* we shall mean those states considered by smoothings at *even crossings*.

We set

$$\langle K \rangle_{\text{even}} = \sum_{s_{\text{even}}} a^{\alpha(s) - \beta(s)} K_s,$$

where $K_s$ is the free link obtained from the diagram $K$ by smoothing with respect to the state $s$. Here $K_s$ is considered as an element from $\tilde{F}$.

It is easy to see that the invariant $\langle \cdot \rangle$ (as well as $\{ \cdot \}$) is a specification of the bracket defined above.

Then the following theorem takes place

**Theorem 6.** The bracket $\langle \cdot \rangle_{\text{even}}$ is invariant under $\Omega_2, \Omega_3$ (and the detour move). When applying $\Omega_1$ the bracket $\langle \cdot \rangle_{\text{even}}$ gets multiplied by $(-a)^{\pm 3}$. The following normalization for $\langle \cdot \rangle_{\text{even}}$ is invariant under all Reidemeister moves: $X_{\text{even}}(K) = (-a)^{-3w(K)} \langle K \rangle_{\text{even}}$, where $w(K)$ stays for the writhe number of the oriented diagram $K$.

We call $X_{\text{even}}(K)$ the even Jones polynomial of the virtual knot $K$.

**Remark 3.** The even Jones polynomial is a generalization of the invariant $\{ \cdot \}$ for any knot theory possessing a parity and a distinguished rule for $A$- and $B$-type smoothing respecting the Reidemeister moves.

Moreover, for the Gaussian parity, the polynomial $X_{\text{even}}(K)$ is a generalization of the usual Jones polynomial for classical knots and for knots with all crossings being even. In this case, in the definition of $\langle \cdot \rangle_{\text{even}}$, all elements $K_s$ are free links, which in the module $\tilde{F}$ are multiples of the unknot with coefficients being powers of the polynomial $(-a^2 - a^{-2})$. Considering the generator of the module $\tilde{F}$, corresponding to the unknot, as 1, we get the standard Jones polynomial.

The invariance proof is analogous to the invariance proof of the usual Kauffman bracket; for details, see [14].

### 5.3 Atoms and Parity

Theorem 3 and Corollary 1 deal with only those free knots having odd chords. However, the class of knots having with all even chords is very important. These are knots corresponding to *orientable atoms*. We shall describe the notion of atom and its connection to the Gaussian parity, and also construct an example of a trivial link with all even chords.

In many situations, it is easier to find *links* rather than *knots* with desired non-triviality properties. So, we shall first define a map from free 1-component links to $\mathbb{Z}_2$-linear combinations.
of 2-component links, and then we shall study the latter by an invariant similar to that constructed in [14].

Ideologically, the first map is a simplified version of Turaev’s cobracket [19] which establishes a structure of Lie coalgera on the set of curves immersed in 2-surfaces (up to some equivalence, the Lie algebra structure was introduced by Goldman in a similar way). We shall use a simplification of Turaev’s construction adopted to the case of free knots and links.

The second map takes a certain state sum for a 2-component free link, where we distinguish between two types of crossings, and smooth only crossings of the first type. What should these “two types” mean, will be discussed later.

In some sense, the invariant [·] of free knots constructed in [14] is a diagrammatic extension of a terrifically simplified Alexander polynomial (we forget about the variable and signs taking \( \mathbb{Z}_2 \)-coefficients). The invariant \{·\} suggested in the present paper is in the same sense an extension of the terrifically simplified Kauffman bracket, but again we use diagrams as coefficients.

Altogether, these two constructions (the bracket and Turaev’s \( \Delta \)) provide an example of non-trivial and minimal diagrams of free knots with orientable atoms.

**Definition 2.** An atom (originally introduced by Fomenko, [3]) is a pair \((M, \Gamma)\) consisting of a 2-manifold \(M\) and a graph \(\Gamma\) embedded in \(M\) together with a colouring of \(M \setminus \Gamma\) in a checkerboard manner. An atom is called orientable if the surface \(M\) is orientable. Here \(\Gamma\) is called the frame of the atom, whence by genus (atoms and their genera were also studied by Turaev [21], and atom genus is also called the Turaev genus [21]) (Euler characteristic, orientation) of the atom we mean that of the surface \(M\).

Having an atom \(V\), one can construct a virtual link diagram out of it as follows. Take a generic immersion of atom’s frame into \(\mathbb{R}^2\), for which the formally opposite structure of edges coincides with the opposite structure induced from the plane.

Put virtual crossings at the intersection points of images of different edges and restore classical crossings at images of vertices ‘as above’. Obviously, since we disregard virtual crossings, the most we can expect is the well-definiteness up to detours. However, this allows us to get different virtual link types from the same atom, since for every vertex \(V\) of the atom with four emanating half-edges \(a, b, c, d\) (ordered cyclically on the atom) we may get two different clockwise-orderings on the plane of embedding, \((a, b, c, d)\) and \((a, d, c, b)\). This leads to a move called virtualisation.

**Definition 3.** By a virtualisation of a classical crossing of a virtual diagram we mean a local transformation shown in Fig. 8.

The above statements summarise as
**Proposition 1.** (see, e.g., [13]). Let $L_1$ and $L_2$ be two virtual links obtained from the same atom by using different immersions of its frame. Then $L_1$ differs from $L_2$ by a sequence of detours and virtualisations.

At the level of Gauss diagrams, virtualisation is the move that does not change the writhe numbers of crossings, but inverts the arrow directions. So, atoms just keep the information about signs of Gauss diagrams, but not of their arrows.

A further simplification comes when we want to forget about the signs and pass to flat virtual links (see also [24]): in this case we don’t want to know which branch forms an overpass at a classical crossing, and which one forms an underpass. So, the only thing we should remember is its frame with opposite edge structure of vertices (the $A$-structure). Having that, we take any atom with this frame and restore a virtual knot up to virtualisation and crossing change.

The $A$-structure of an atom’s frame is exactly a 4-valent framed graph.

This perfectly agrees with the fact that free links are virtual links modulo virtualization and crossing changes.

Having a framed 4-graph, one can consider all atoms which can be obtained from it by attaching black and white cells to it. In fact, it turns out that for a given framed 4-graph either all such surfaces are orientable or they are all non-orientable.

To see this, one should introduce the source-sink orientation. By a source-sink orientation of a 4-valent framed graph we mean an orientation of all edges of this graph in such a way that for each vertex some two opposite edges are outgoing, whence the remaining two edges are incoming.

The following statement is left to the reader as an exercise

**Example 3.** Let $G$ be a framed 4-graph. Then the following conditions are equivalent:

1. $G$ admits a source-sink orientation
2. At least one atom obtained from $G$ by attaching black and white cells is orientable.
3. All atoms obtained from $G$ by attaching black and white cells are orientable.

Moreover, if $G$ has one unicursal component, then each of the above conditions is equivalent to the following: every chord of the corresponding Gauss diagram $C(D)$ is even.

Theorem 1 leads to a new proof of a partial case the following

**Theorem 7** (First proved by O.Ya. Viro and V.O.Manturov, 2005, first published in [7]). The set of virtual links with orientable atoms is closed. In other words, if two virtual diagrams $K$ and $K'$ have orientable atoms and they are equivalent, then there is a sequence of diagrams $K = K_0 \rightarrow K_1 \cdots \rightarrow K_n = K'$ all having orientable atoms where $K_i$ is obtained from $K_i$ by a Reidemeister move.

Indeed, for the case of knots, this is a direct corollary from Theorem 1 in view of the exercise above; the general case of this theorem can be proved by a modification of parity, called relative parity, investigated by D.Yu.Krylov and the author, [12].

We give two examples: for a planar 4-valent framed graph we present a source-sink orientation (left picture, Fig. 9), and for a non-orientable 4-valent framed graph (right picture, Fig 9, the artefact of immersion is depicted by a virtual crossing) we see that the source-sink orientation taken from the left crossing leads to a contradiction for the right crossing.

5.4 **Turaev’s Cobracket and Its Even Analogue**

There is a simple and fertile idea due to Goldman [5] and Turaev [19] of transforming two-component curves into one-component curves and vice versa.
Here we simplify Turaev’s idea for our purposes and we shall call the map we are going to construct “Turaev’s $\Delta$”. Let $\Theta_l$ be the set of all equivalence classes of framed graphs with $l$ unicursal components by the second Reidemeister move and the relation that takes every framed 4-graph with a split component to 0. We shall construct a map from $\mathbb{Z}_2\Theta$ to $\tilde{\mathbb{Z}}_2\Theta$ as follows.

In fact, to define the map $\Delta$, one may require for a free knot to be oriented. However, we can do without.

Given a framed 4-graph $G$. We shall construct an element $\Delta(G)$ from $\mathbb{Z}_2\Theta$ as follows. For each crossing $c$ of $G$, there are two ways of smoothing it. One way gives a knot, and the other smoothing gives a 2-component link $G_c$. We take the one giving a 2-component link and write

$$\Delta(G) = \sum_c G_c \in \mathbb{Z}_2\Theta$$

(4)

**Theorem 8.** $\Delta(G)$ is a well defined mapping from $\mathbb{Z}_2\Theta$ to $\tilde{\mathbb{Z}}_2\Theta$.

The proof is standard and follows Turaev’s original idea. One should consider the three Reidemeister move. The first move adds a new summand which has a free loop (the latter assumed to be trivial in $\tilde{\mathbb{Z}}_2\Theta$); for the second Reidemeister move we get two new identical summands, which cancel each other because we are dealing with $\mathbb{Z}_2$ coefficients. For the third Reidemeister moves the LHS and the RHS will lead to the summands identical up to second Reidemeister moves.

**Statement 2.** The free knot $K_1$ shown in Fig. 7 is minimal.

To see the minimality, one takes $\Delta(K_1)$ which, as an element of $\mathbb{Z}_2\Theta$, coincides with the link shown in Fig. 7 there is exactly one summand obtained by smoothing at $x$; all other summands cancel by symmetry over $\mathbb{Z}_2$. Now, considering $\{\Delta(K_1)\}$, we get the desired claim.

Analogously one defines the maps $\Delta_{\text{even}}$ and $\Delta_{\text{odd}}$ corresponding to smoothing at even (resp., odd) crossings.

6 Parity and groups

It turns out that the parity argument can be used to construct some extremely simple invariants of free knots; moreover, these invariants turn out to be obstructions for a free knot to be slice.

The idea goes as follows. Consider the group $G = \langle a, b, b'|a^2 = b^2 = b'^2 = 1, ab = b'a \rangle$. For a Gauss diagram $D$, we shall construct a word $\gamma(D)$ in the alphabet $a, b, b'$ such that the conjugacy class of this word in $G$ is an invariant of the corresponding free knot. Let $D$ be an oriented chord diagram on a circle $C$, and let $X$ be a fixed point on $C$ distinct from chord ends.
We say that an odd chord $c$ of $D$ is of the first type if it is linked with an odd number of odd chords, and of the second type if it is linked with an odd number of even chords.

With the pointed diagram $D$ we associate a word in the alphabet $a, b, b'$ in the following way. We start walking along the orientation of the circle $C$ from the point $X$. Every time when we meet a chord end, we write a letter $a, b$, or $b'$ depending on whether the corresponding chord is even, first type odd or second type odd.

Denote the obtained word by $\gamma(D)$; this word generates an element $[\gamma(D)]$ in the group $G$; sometimes we shall abbreviate notation and denote $[\gamma(D)]$ just by $\gamma(D)$.

**Theorem 9.** If pointed chord diagrams $(D, X)$ and $(D', X')$ generate equivalent (oriented) free knots then $[\gamma(D)] = [\gamma(D')]$ in $G$.

The proof of this theorem is a simple check of the invariance under Reidemeister moves. Theorem 9 has an obvious

**Corollary 2.** The conjugacy class of the element $[\gamma(D, X)]$ in $G$ is an invariant of free knots.

Indeed, moving the fixed point through a chord end, we change the word cyclically, which means conjugation in $G$.

The group $G$ admits a simple combinatorial description. Its Cayley graph looks like a vertical strip on a squared paper between $x = 0$ and $x = 1$: we choose the point $(0, 0)$ to be the unit in the group; the multiplication by $a$ on the right is chosen to one step in a horizontal direction (to the right if the first coordinate of the point is equal to zero, and to the left if this first coordinate is equal to one), the multiplication by $b$ is one step upwards if the sum of coordinates is even and one step downwards if this sum is odd, and the multiplication by $b'$ is one step downwards if the sum of coordinates is even and one step upwards if the sum of coordinates is even, see Fig. 11.

With each pointed chord diagram one associates an element from $G$ having coordinates $(0, 4l)$. Moreover, the conjugacy class of the element $(0, 4l)$ for $l \neq 0$ consists of the two elements: $(0, 4l)$ and $(0, -4l)$. Thus, for each long free knot one gets an integer-valued invariant, equal to $l$; we
shall denote this invariant by \( l(K) \); each compact free knot has, in turn, the invariant equal to \(|l|\); we shall denote the latter by \( L(K) \).

In Fig. 12 we depict a free knot \( K_1 \) with \( l(K_1) = 4 \).

So, it is possible to prove non-triviality of some free knots by using parity arguments in a much simpler ways than in the initial sections of the present paper.

In some consequent paper we shall prove that \( L(K) \) is a \textit{cobordism invariant of free knots}, i.e., if \( L(K) \) is non-zero then the free knot \( K \) is not null-cobordant.

This invariant was strengthened in [18].

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