Continuous Constructive Fermionic Renormalization

M. Disertori and V. Rivasseau
Centre de Physique Théorique, CNRS UPR 14
Ecole Polytechnique
91128 Palaiseau Cedex, FRANCE

Abstract We build the two dimensional Gross-Neveu model by a new method which requires neither cluster expansion nor discretization of phase-space. It simply reorganizes the perturbative series in terms of trees. With this method we can for the first time define non perturbatively the renormalization group differential equations of the model and at the same time construct explicitly their solution.

I Introduction

The popular versions of renormalization and the renormalization group in field theory are based on differential equations (among which the most famous one is the Callan-Symanzik equation). However no non-perturbative version of these differential equations has been given until now.

On the other hand the renormalization group in statistical mechanics, for instance for spin systems after the works of Kadanoff and Wilson, relies on closely related but discretized equations. When block spining or other discretization of momentum space is used, the result is a discretized evolution of the effective action step by step. This point of view, in contrast with the first one, has led to rigorous non perturbative constructions for various models which have renormalizable power counting [R], but the methods always involved some discretization of phase space and the outcome is a discrete (not differential) flow equation. Furthermore, the rigorous discretization of phase space came with a price, namely the use of some technical tools such as cluster or Mayer expansions which are neither popular among theoretical physicists nor among mathematicians.
The proposal of Manfred Salmhofer to build a continuous version of the renormalization group for Fermionic theories [S] is therefore very interesting and welcome. Indeed Fermionic series with cutoffs are convergent (in contrast with Bosonic ones, which are Borel summable at best), and the continuous version of renormalization group which works so well at the perturbative level should therefore apply to them.

In this paper we realize the Salmhofer proposal on the particular example of the two-dimensional Gross-Neveu model. We rearrange Fermionic perturbation theory according to trees, an idea first developed in [AR2], perform subtractions only when necessary according to the relative scales of the subgraphs, and obtain (to our own surprise, quite easily) an explicit convergent representation of the model without any discretization or cluster or Mayer expansion. To prove the convergence requires only some well-known perturbative techniques of parametric representations ("Hepp’s sectors"), Gram’s bound on determinants and a crucial but rather natural concatenation of some intervals of integration for loop lines.

Therefore we can now consider that constructive theory for Fermions has been “reduced” to perturbation theory. Remark also that since the representation we use is an “effective” representation in the sense of [R], hence with subtractions performed only when necessary according to the relative scales of the subgraphs, we never meet the so called problem of “overlapping divergences” or classification of Zimmermann’s forests. In this sense constructive renormalization is easier than ordinary perturbative renormalization (which, from the constructive point of view, is flawed anyway because it generates renormalons).

Having an explicit convergent representation of the theory with a continuously moving cutoff, it is trivial both to define the continuous renormalization group equations which correspond to the variation of this cutoff and, at the same time, to check that our explicit representation is a solution of these equations.

Remark however we have not yet found the way to short-circuit our representation and to prove that the equations and their solutions exist by a purely inductive argument à la Polchinski [P] which would avoid an explicit formula for the solution. This is presumably possible but this question as well as the extension to other models, in particular to interacting Fermions models of condensed matter physics, is left for future investigation. It is also important to recall that we do not see at the moment how to extend this
method to Bosons, since there are no determinant and Gram’s bound for them.

II Model and Main Result

We consider the massive Gross-Neveu model \( GN_2 \), which describes \( N \) types of Fermions. These Fermions interact through a quartic term. Actually, the \( GN_2 \) action also requires a quadratic mass counterterm and a wave function counterterm in order for the ultraviolet limit to be finite. Therefore the bare action in a finite volume \( V \) is (using the notations of [FMRS]):

\[
S_V = \frac{\lambda}{N} \int_V d^2x \left[ \sum_a \bar{\psi}_a(x) \psi_a(x) \right]^2 + \delta m \int_V d^2x \left[ \sum_a \bar{\psi}_a(x) \psi_a(x) \right] + \delta \zeta \int_V d^2x \left[ \sum_a \bar{\psi}_a(x) i \not{\partial} \psi_a(x) \right]
\]

where \( \lambda \) is the bare coupling constant, \( \delta m \) and \( \delta \zeta \) are the bare mass and wave function counterterms, and \( a \) is the color index: \( a = 1, \ldots, N \). The action (II.1) and the power counting of the \( GN_2 \) model are like the ones for the Bosonic \( \phi^4 \) theory, except that, unlike the latter, the \( GN_2 \) theory is asymptotically free for \( N \geq 2 \), a condition which we assume from now on.

The free covariance in momentum space is

\[
C_{\gamma\delta}^{ab}(p) = \delta_{a,b} \left( \frac{1}{-\not{p} + m} \right)_{\gamma\delta} = \delta_{a,b} \left( \frac{\not{p} + m}{p^2 + m^2} \right)_{\gamma\delta}
\]

(II.2)

where \( \gamma, \delta \) are the spin indices, and \( a, b \) are the color indices. Most of the time we skip the inessential spin indices to simplify notation. The mass \( m \) is the renormalized mass. To avoid divergences, according to the notations of [KKS] we introduce an ultraviolet cut-off \( \Lambda_0 \) and (for later study of the renormalization group flow) a scale parameter \( \Lambda \) which plays the role of an infrared cutoff:

\[
C_{\Lambda_0}^{\Lambda}(p) = C(p) \left[ \eta \left( \frac{p^2 + m^2}{\Lambda_0^2} \right) - \eta \left( \frac{p^2 + m^2}{\Lambda^2} \right) \right].
\]

(II.3)

The cutoff function \( \eta \) might be any function which satisfies \( \eta(0) = 1 \), which is smooth, monotone and rapidly decreasing at infinity (this means faster
that any fixed power). For simplicity in this paper we restrict ourselves to the most standard case $\eta(x) = e^{-x}$. In this case both $C_{\Lambda_0}^\Lambda$ and its Fourier transform have explicit so called parametric representations:

$$
C_{\Lambda_0}^\Lambda(p) = \int_{\Lambda_0^{-2}}^{\Lambda^{-2}} (p + m) e^{-\alpha p^2 + m^2} d\alpha
$$

$$
C_{\Lambda_0}^\Lambda(x - y) = \pi \int_{\Lambda_0^{-2}}^{\Lambda^{-2}} \left( \frac{i(x - y)}{2\alpha^2} + \frac{m}{\alpha} \right) e^{-\alpha m^2 - |x-y|^2/\Lambda_0^2} d\alpha
$$

(II.4)

We define now the connected truncated Green functions, also called vertex functions, which are the coefficients of the effective action. The partition function with external fields $\xi, \bar{\xi}$ is

$$
Z_{V}^{\Lambda_0}(\xi, \bar{\xi}) = \int d\mu_{\quad C_{\Lambda_0}^\Lambda}(\psi, \bar{\psi}) e^{-S_{\Lambda_0}^\Lambda} <\psi, \xi> + <\psi, \bar{\xi}> <\psi, \bar{\psi}>
$$

(II.5)

The vertex function with $2p$ external points is:

$$
\Gamma_{2p}^{\Lambda_0}(\{y\}, \{z\}) := \Gamma_{2p}^{\Lambda_0}(y_1, ..., y_p, z_1, ..., z_p)
$$

(II.6)

$$
= \lim_{V \to \infty} \frac{\delta^{2p}}{\delta \xi(z_1) \cdot \delta \xi(z_p) \delta \xi(y_1) \cdot \delta \xi(y_p)} \left( \ln Z_{V}^{\Lambda_0} - F(C_{\Lambda_0}^\Lambda)^{-1}(\xi) \right)_{\xi=0}
$$

where $F(\xi) = <\xi, C_{\Lambda_0}^\Lambda \xi>$ is the bare propagator, and color indices are implicit. These functions (in fact distributions) form the coefficients of the effective action (expanded in powers of the external fields) at energy $\Lambda$ with UV cutoff $\Lambda_0$. Developing the exponential in $Z$ and attributing prime and double prime indices respectively to the mass and wave function counterterms we have:

$$
Z_{V}^{\Lambda_0}(\xi) = \sum_{p=0}^{\infty} \frac{1}{B^{2p}} \sum_{n, n', n''=0}^{\infty} \frac{(-1)^{n+n'+n''}}{n!n'!n''!} \sum_{a_i a' c_i d_i} \left( \frac{\Lambda}{N} \right)^{n} (\delta m)^{n'} (\delta \xi)^{n''}
$$

(II.7)

$$
\int_{V} d^2 y_1 ... d^2 y_p d^2 z_1 ... d^2 z_p d^2 x_1 ... d^2 x_n d^2 x'_1 ... d^2 x'_{n'} d^2 x''_1 ... d^2 x''_{n''} \prod_{i=1}^{p} \xi_{d_i}(z_i) \xi_{c_i}(y_i)
$$

\begin{align*}
& \left\{ y_{1,c_1} \ldots y_{p,c_p} x_{1,a_1} x_{1,b_1} \ldots x_{n,a_n} x_{n,b_n} x'_{1,a'_1} x'_{1,b'_1} \ldots x'_{n',a'_n} x'_{n',b'_n} \right. \\
& \left. \quad z_{1,d_1} \ldots z_{p,d_p} x_{1,a_1} x_{1,b_1} \ldots x_{n,a_n} x_{n,b_n} x''_{1,a''_1} x''_{1,b''_1} \ldots x''_{n'',a''_n} x''_{n'',b''_n} \right\}
\end{align*}

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where we used Cayley’s notation for the determinants:

\[
\{ \begin{array}{c}
u_{i,a} \\ v_{j,b}
\end{array} \} = \det(D_{ab}(u_i - v_j)) \quad (\text{II.8})
\]

and \(a_i, b_i, a_i', b_i', a_i'', b_i'', c_i, d_i\) are the color indices. By convention

\[
D_{ab}(u_i - v_j) := C_{ab}(u_i - v_j) \quad (\text{II.9})
\]

except when the second index is the one of a \(\psi\) field hooked to a \(\delta\zeta\) vertex. In this particular case the vertex has a so called derivative coupling, and therefore the propagator \(D\) bears a derivation, namely \(D_{ab}'(u_i - v_j) := i \not\partial_{v_j} C_{ab}'(u_i - v_j)\). This derived propagator is explicitly

\[
C_{ab}'(x - y) = \pi \int_{\Lambda_0^{-2}}^{\Lambda^{-2}} \left( -\frac{m(x-y)^2}{4\alpha^2} - \frac{1}{\alpha^2} \right) e^{-\alpha m^2-|x-y|^2/4\alpha} d\alpha 
\]

(II.10)

Expanding the determinant in (II.7) one obtains the usual perturbation theory in terms of Feynman graphs with the three types of vertices corresponding to the three terms of the action (II.1), and the logarithm is simply the sum over connected graphs. To see if a graph is connected, it is not necessary to know its whole structure but only a tree in it. Based on this remark the logarithm of (II.7) was computed in [AR2] using an expansion which is intermediate between the determinant form (II.7) and the fully expanded Feynman graphs. This expansion is based on a forest formula. Such formulas, discussed in [AR1], are Taylor expansions with integral remainders. They test the coupling or links (here the propagators) between \(n \geq 1\) points (here the vertices) and stop as soon as the final connected components are built. The result is therefore a sum over forests, which are simply defined as union of disjoint trees. A forest is therefore a (pedantic, but poetic) word for a Feynman graph without loops, and our point of view is that these are the natural objects to express Fermionic perturbation theory.

Here we use the most symmetric forest formula, the ordered Brydges-Kennedy Taylor formula, which states [AR1] that for any smooth function \(H\) of the \(n(n-1)/2\) variables \(u_l, \ l \in P_n = \{(i,j)|i,j \in \{1,\ldots,n\}, i \neq j\}\),

\[
H|_{u_1} = \sum_{o \rightarrow \mathcal{F}} \left( \int_{0 \leq w_1 \leq \ldots \leq w_k} \prod_{q=1}^k dw_q \right) \left( \prod_{q=1}^k \frac{\partial}{\partial u_{l_q}} H \right) (w_q^\mathcal{F}(w_q), l \in P_n) 
\]

(II.11)
where $o - \mathcal{F}$ is any ordered forest, made of $0 \leq k \leq n - 1$ links $l_1, ..., l_k$ over the $n$ points. To each link $l_q$ with $q = 1, ..., k$ of $\mathcal{F}$ is associated the parameter $w_q$, and to each pair $l = (i, j)$ is associated the weakening factor $w_{ij}^\mathcal{F}(w_q)$. These factors replace the variables $u_l$ as arguments of the derived function $\prod_{q=1}^{k} \frac{\partial}{\partial u_l} H$ in (II.11). These weakening factors $w_{ij}^\mathcal{F}(w)$ are themselves functions of the parameters $w_q, q = 1, ..., k$ through the formulas

$$
\begin{align*}
    w_{i,i}^\mathcal{F}(w) &= 1 \\
    w_{i,j}^\mathcal{F}(w) &= \inf_{l_q \in P_{i,j}^\mathcal{F}} w_q, \quad \text{if } i \text{ and } j \text{ are connected by } \mathcal{F} \\
    w_{i,j}^\mathcal{F}(w) &= 0 \quad \text{if } i \text{ and } j \text{ are not connected by } \mathcal{F}.
\end{align*}
$$

We apply this formula to the determinant in (II.7), inserting the interpolation parameter $u_l$ in the cut-off (but only between distinct vertices, so not for the “tadpole” lines):

$$
C_\Lambda^{\Lambda_0}(x, y, u) = \delta(x - y)C_\Lambda^{\Lambda_0}(x, x) + [1 - \delta(x - y)]C_\Lambda^{\Lambda_0}(x, y, u) \tag{II.13}
$$

$$
:= C_\Lambda^{\Lambda_0(u)}(x, y) \tag{II.14}
$$

$$
:= \pi \int_{\Lambda_0^{-2}(u)}^{\Lambda^{-2}} \left( \frac{\|x - y\|}{2\alpha^2} + \frac{m}{\alpha} \right) e^{-\alpha m^2 - |x-y|^2/4\alpha} d\alpha
$$

where

$$
\Lambda_0^{-2}(u) = \Lambda^{-2} + u(\Lambda_0^{-2} - \Lambda^{-2}). \tag{II.14}
$$

We use similar interpolation for the $C'$ propagators. When $u$ grows from 0 to 1, the ultraviolet cut-off of the interpolated propagator (between distinct vertices) grows therefore from $\Lambda$ to $\Lambda_0$. 
We define

\[ C_{\Lambda}^{\Lambda_0, u}(x, y, u) := \frac{\partial}{\partial u} C_{\Lambda}^{\Lambda_0}(x, y, u) = \pi \left( \frac{i(\dot{x} - \dot{y}) \Lambda_0^4(u)}{2} + m \Lambda_0^2(u) \right). \]

\[ C_{\Lambda}^{\Lambda_0, u}(x, y, u) := \frac{\partial}{\partial u} C_{\Lambda}^{\Lambda_0}(x, y, u) = \pi \left( \frac{|x - y|^2 \Lambda_0^6(u)}{4} + \frac{i m (\dot{x} - \dot{y}) \Lambda_0^4(u)}{2} - \Lambda_0^4(u) \right)(\Lambda^{-2} - \Lambda_0^{-2}) e^{-m^2 \Lambda_0^{-2}(u) - |x - y|^2 \Lambda_0^2(u)/4} \]

The derivative of \( \eta \) fixes \( C^u \) at an energy near \( \Lambda_0(u) \). We observe that for any fixed \( \epsilon' \) we have the scaled decay:

\[ |C^u(x, y)| \leq K \Lambda_0^3(u)(\Lambda^{-2} - \Lambda_0^{-2}) e^{-|x - y|/2(1 - \epsilon') \Lambda_0^m(u) - \epsilon' m^2 \Lambda_0^{-2}(u)/2} \]  

(II.16)

\[ |C'(x, y)| \leq K \Lambda_0^4(u)(\Lambda^{-2} - \Lambda_0^{-2}) e^{-|x - y|/2(1 - \epsilon') \Lambda_0^m(u) - \epsilon' m^2 \Lambda_0^{-2}(u)/2} \]

(II.17)

where \( K \) is a constant depending only on \( \epsilon' \) and \( \Lambda_0^m(u) := \sup[m, \Lambda_0(u)] \)

(II.18)

Applying this interpolation and the ordered forest formula (II.11) to the propagators in the determinant of (II.7) we obtain
\[ Z_\Lambda^{\Lambda_0}(\xi) = \sum_{p=0}^{\infty} \frac{1}{p!^2} \sum_{n,n',n''=0}^{\infty} \frac{1}{n!n'!n''!} \sum_{o-F} \sum_{\text{Col}} \sum_{\Omega} \left( \frac{\Lambda}{\epsilon} \right)^n (\delta m)^n' (\delta \xi)^n'' \epsilon(\mathcal{F}, \Omega) \]

\[ \int_V d^2 y_1 \ldots d^2 y_p d^2 z_1 \ldots d^2 z_p d^2 x_1 \ldots d^2 x_p \prod_{r=1}^{p} \xi_{dr}(z_r) \tilde{\xi}_{cr}(y_r) \]

\[ \int_{0 \leq w_1 \leq \ldots \leq w_k \leq 1} \left[ \prod_{q=1}^{k} D^{\Lambda_0,w_q}(\bar{x}_{l_q}, x_{l_q}) dw_q \right] [\det]_{\text{left}}(w^\mathcal{F}(w)) \]

(II.19)

where for simplicity the position of any vertex is simply denoted by the letter \( x \) and \( \bar{n} := n + n' + n'' \). \( \bar{x}_{l_q} \) and \( x_{l_q} \) are the ends of line \( l_q \). [\det]_{\text{left}} is the remaining determinant. Its entries correspond to the remaining fields necessary to complete each vertex of the forest into a quartic or quadratic vertex, according to its type (interaction or counterterm). For this model, remark that the sum \( \sum_{o-F} \) is performed only over the ordered forests that have, for each point \( x_i \) coordination number \( n(i) \leq 4 \) or \( n(i) \leq 2 \) depending of the type of the vertex (all other terms being zero). The additional sums over \( \text{Col} \) and \( \Omega \) correspond to coloring choices at each vertex and “fields versus antifields” choices at each line and vertex [AR2]. The sign \( \epsilon(\mathcal{F}, \Omega) \) comes in from the antisymmetry of Fermions and is computed in [AR2]: here we only need to know that it factorizes over the connected components of \( \mathcal{F} \). To find the expression for \( \ln Z \) we write \( Z \) as an exponential. In equation (II.19), the determinant factorizes over the ordered trees \( T_1 \ldots T_j \) forming the forest. Indeed one can resum all orderings of the ordered forest \( \mathcal{F} \) compatible with fixed orderings of its connected components, the trees \( T_1 \ldots T_j \). Furthermore the “weakening factor” \( w^\mathcal{F} \) vanishes between vertices belonging to different connected components. Hence:

\[ Z_\Lambda^{\Lambda_0}(\xi) = \sum_{p=0}^{\infty} \frac{1}{p!^2} \sum_{n,n',n''=0}^{\infty} \frac{1}{n!n'!n''!} \sum_{j=0}^{n} \sum_{n_1 + \ldots + n_j = n} \sum_{n_1' + \ldots + n_j' = n'} \sum_{n_1'' + \ldots + n_j'' = n''} \sum_{p_1 + \ldots + p_j + p' = p} \frac{n!n'!n''!}{n_1!n_j'!n_1''!n_j''!} \frac{p_1!2 \ldots p_j!2 p'!2}{p_1!2 \ldots p_j!2 p'!2} \]
\[(\xi, C^{A_\alpha}_\Lambda \xi)^{p'} \prod_{i=1}^{j} \left[ (\frac{\Lambda}{\lambda})^{n_i} (\delta m)^{n_i'} (\delta \zeta)^{n_i''} A(n_i, n_i', n_i'', p_i) \right] \]  

(II.20)

where

\[ A(n_i, n_i', n_i'', p_i) = \sum_{\mathcal{T}_i} \sum_{\text{Col}, \Omega} \epsilon(\mathcal{T}_i, \Omega_i) \int d^2 y_1 \ldots d^2 y_p d^2 z_1 \ldots d^2 z_p d^2 x_1 \ldots d^2 x_{\bar{n}_i} \]

\[ \prod_{r=1}^{p} \xi_{d_r}(z_r) \tilde{\xi}_{e_r}(y_r) \int_{0 \leq w_1 \leq \ldots \leq w_{\bar{n}_i-1} \leq 1} \left[ \prod_{q=1}^{\bar{n}_i-1} D^{A_\alpha_0, w_q}_\Lambda(\bar{x}_l_q, x_l_q) dw_q \right] \]

\[ [\text{det}]_{\text{left}, i}(w^T(w)) \]  

(II.21)

where \( \bar{n}_i \) is the number of vertices in the ordered tree \( \mathcal{T}_i \), which has therefore \( \bar{n}_i - 1 \) lines, \( p_i \) is the number of external fields of type \( y \) (and \( z \)) attached to the \( \mathcal{T}_i \), and \( p' \) is the number of free external propagators (not connected to any vertex) in the forest. This can be written as an exponential, hence

\[ \ln Z^{A_\alpha}_\Lambda(\xi) = (\xi, C^{A_\alpha}_\Lambda \xi) + \sum_{p=0}^{\infty} \frac{1}{p!} \sum_{n, n', n''=0}^{\infty} \frac{1}{n! n'! n''!} (\frac{\lambda}{\lambda})^{n} (\delta m)^{n'} (\delta \zeta)^{n''} \]

\[ \sum_{o=\mathcal{T}} \sum_{\text{Col}, \Omega} \epsilon(\mathcal{T}, \Omega) \int_{\mathcal{V}} d^2 y_1 \ldots d^2 y_p d^2 z_1 \ldots d^2 z_p d^2 x_1 \ldots d^2 x_{\bar{n}} \prod_{r=1}^{p} \xi_{d_r}(z_r) \tilde{\xi}_{e_r}(y_r) \]

\[ \int_{0 \leq w_1 \leq \ldots \leq w_{\bar{n}-1} \leq 1} \left[ \prod_{q=1}^{\bar{n}-1} D^{A_\alpha_0, w_q}_\Lambda(\bar{x}_l_q, x_l_q) dw_q \right] [\text{det}]_{\text{left}}(w^T(w)) \]  

(II.22)

where \( \mathcal{T} \) is an ordered tree over \( \bar{n} \) points, and the external points are all connected to the tree. Now, applying the definition (II.6), we obtain the vertex functions, for which the limit \( V \to \infty \) can be performed (because the external points hooked to the tree ensure convergence). The set

\[ E = \{(i_1, \ldots, i_p, j_1, \ldots, j_p) | i_1, \ldots, i_p, j_1, \ldots, j_p \in \{1, \ldots, \bar{n}\} \} \]  

(II.23)

fixes the internal points to which the \( 2p \) external lines hook.

We recall the well-known fact that the vertex functions in \( x \)-space are in fact distributions. For instance it is easy to see that when some of the external points \( i_k, j_k \) in the previous sum coincide, one has to factor out the product of the corresponding delta functions of the external arguments to obtain smooth functions. This little difficulty can be treated either by considering
the vertex functions in momentum space (they are then ordinary functions of external momenta, after factorization of global momentum conservation), or by smearing the vertex functions with test functions. Here we adopt this last point of view. The quantity under study is then $\Gamma_{\Lambda\Lambda_0}^{2p}$ smeared with smooth test functions $\phi_1(y_1), \ldots, \phi_p(y_p)$, $\phi_{p+1}(z_1), \ldots, \phi_{2p}(z_p)$:

$$\Gamma_{\Lambda\Lambda_0}^{2p}(\phi_1, \ldots, \phi_{2p}) \equiv \int d^2y_1 \ldots d^2y_p d^2z_1 \ldots d^2z_p$$

$$\Gamma_{\Lambda\Lambda_0}^{2p}(y_1, \ldots, y_p, z_1, \ldots, z_p)\phi_1(y_1)\ldots \phi_p(y_p)\phi_{p+1}(z_1)\ldots \phi_{2p}(z_p). \quad (II.24)$$

where we asked the test functions to have compact support: $\phi \in D(\mathbb{R}^2)$.

Remark that when some external antifield hooks to a $\delta\zeta$ vertex, the amputation by $C$ instead of $C'$ leaves a $\delta'$ distribution, which means a derivative acting on the corresponding test function.

We obtain the formula:

$$\Gamma_{\Lambda\Lambda_0}^{2p}(\phi_1, \ldots, \phi_{2p}) = \sum_{n,n',n''} \left( \lambda N \right)^n (\delta m)^{n'} (\delta \zeta)^{n''} \frac{1}{n!n'!n''!}$$

$$\sum_{o-T} \sum_{E} \sum_{Col, \Omega} \int d^2x_1 \ldots d^2x_n \phi_1(x_{i_1}) \ldots \phi_{2p}(x_{j_p})$$

$$\int_{0 \leq w_1 \leq \ldots \leq w_{n-1} \leq 1} \left[ \prod_{q=1}^{n-1} D_{\Lambda_0, w_q}(\bar{x}_{l_q}, x_{l_q}) dw_q \right] \left[ \det \left( \omega^T(w), E \right) \right]$$

where the propagator $D$ is now $C$ or $C'$ according to the discussion above.

When renormalization is introduced, it will be convenient to use the BPHZ subtraction prescription at 0 external momenta, which corresponds to integrate the vertex functions over all arguments except one. In this prescription one defines the renormalized coupling constant as the 4-vertex function of the full theory at zero external momenta:

$$\frac{\lambda_{ren}}{N} := \Gamma_{4,A}^{\Lambda\Lambda_0}(0,0,0,0) = \int d^2x_1 d^2x_2 d^2x_3 d^2x_4 \Gamma_{4}^{\Lambda\Lambda_0}(0,x_2,x_3,x_4) \quad (II.26)$$

Moreover we want the renormalized mass and wave function constant to be respectively $m$ and 1. This means that we impose the additional renormalization conditions:

$$\delta m_{ren} := \Gamma_{2}^{A\Lambda_0}(0,0) = \int d^2x_2 \Gamma_{2}^{\Lambda\Lambda_0}(0,x_2) = 0 \quad (II.27)$$
\[ \delta \zeta_{\text{ren}} := \hat{\phi} \Gamma^{\Lambda_0}_{2p}(0,0) = \int d^2x_2 i \not{x}_2 \Gamma^{\Lambda_0}_{2}(0,x_2) = 0 \quad (\text{II.28}) \]

With these conditions the whole theory (at fixed renormalized mass \( m \)) becomes parametrized only by \( \lambda_{\text{ren}} \), hence not only \( \lambda \) but also \( \delta m \) and \( \delta \zeta \) in (II.1) become functions of \( \lambda_{\text{ren}} \). This of course has a precise meaning only if we can construct the theory and solve the renormalization group flows, which is precisely what we are going to do. We can express the main result of this paper as a theorem on the existence of the ultraviolet limit of the vertex functions and of the renormalization group flows. Recall that the theory is not directly the sum but the Borel sum of the renormalized perturbation theory. In summary

**Theorem 1** The limit \( \Lambda_0 \to \infty \) of \( \Gamma^{\Lambda_0}_{2p}(\phi_1,\ldots,\phi_{2p}) \) exists and is Borel summable in the renormalized coupling constant \( \lambda_{\text{ren}} \), uniformly in \( N \) (where \( N \) is the number of colors). Since the parameter \( \Lambda \) varies continuously, the continuous renormalization group equations and in particular the \( \beta \) function are also well defined in the limit \( \Lambda_0 \to \infty \).

The first part of the theorem is similar to [FMRS], but the second part (the existence of the continuous renormalization group equations) is new. The rest of the paper is devoted to the proof of this theorem.

The precise bounds on the smeared vertex functions are given in theorem 3 below. They are uniform in \( N \) (and in fact proportional to \( N^{1-p} \)). Let us discuss also briefly the dependence in \( m \), the renormalized mass. For \( m \neq 0 \) fixed, we can define the physical scale of the system by putting \( m = 1 \). The theorem is then uniform in the infrared cutoff \( \Lambda \), including the point \( \Lambda = 0 \). In the case \( m = 0 \) our method requires a nonzero infrared cutoff \( \Lambda \neq 0 \). Since this cutoff is the only scale of the problem, we can then put it to 1: \( \Lambda = 1 \). In this last case, improperly called the “massless theory”, we know that there should be a non-perturbative mass generation [GN]. This mass generation has been proved rigorously for the model with fixed ultraviolet cutoff and large number \( N \) of components in [KMR], using the Matthews-Salam formalism of an intermediate Bosonic field and a cluster expansion with a small/large field expansion. Our result in the massless case \( m = 0 \) with a finite infrared cutoff \( \Lambda \) should therefore glue with the method and results of [KMR] to obtain at large \( N \) the mass generation of the full model without ultra-violet cutoff.

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III The Expansion

III.1 The continuous band structure

Remark that in (II.25)

\[ w_1 \leq w_2 \leq \ldots \leq w_{n-1} \implies \Lambda_0(w_1) \leq \Lambda_0(w_2) \ldots \leq \Lambda_0(w_{n-1}). \quad (III.1) \]

This naturally cuts the space of momenta into \( n \) bands \( B = \{1, \ldots, n\} \) (see Figure 2).

Looking at equation (II.4), we see that the covariance can be written as a sum of propagators restricted to single bands:

\[
C^\Lambda_0(p) = \sum_{k=1}^{n} C^\Lambda_0(w_k)(p) = \sum_{k=1}^{n} \int_{\Lambda_0^{-2}(w_{k-1})}^{\Lambda_0^{-2}(w_{k-1})} (p^2 + m)e^{-\alpha(p^2 + m^2)} d\alpha = C(p) \sum_{k=1}^{n} \eta^k \quad (III.2)
\]

where we defined

\[
\eta^k := e^{-\Lambda_0^{-2}(w_k)(p^2 + m^2)} - e^{-\Lambda_0^{-2}(w_{k-1})(p^2 + m^2)} \quad (III.3)
\]

and we adopted the convention

\[
w_0 = 0 \Rightarrow \Lambda_0^{-2}(w_0) = \Lambda^{-2} \quad w_n = 1 \Rightarrow \Lambda_0^{-2}(w_n) = \Lambda_0^{-2}. \quad (III.4)
\]
Similar formulas hold for \( C' \) but with an additional \( \not p \).

To cure the ultraviolet divergences we have to combine the divergent local parts of some subgraphs with counterterms and reexpress the series for \( G \) as an effective series in the sense of \( [R] \). For that purpose we use the band structure to distinguish the convergent and divergent subgraphs hence decide where renormalization is necessary.

### III.2 Notations

Now we fix some notations. It is convenient to give indices to the fields variables or the half-lines which correspond to these fields after Grassmann integration. We observe that there are several types of such variables, the half-lines which form the lines of the tree, the external variables (which correspond to amputated lines) and the entries (rows or columns) in the determinant \( \text{det}_{\text{left}} \). These entries will be called “loop fields” or “loop half-lines” since they form the usual loop lines of the Feynman graphs if one expands the determinant. We define \( E \) and \( L \) as the set of all external and loop half-lines. For each level \( i \) there is a tree-line \( l_i \) with two ends corresponding to two half-lines called \( f_i \) and \( g_i \). The loop fields \( \psi(x_g) \) and \( \bar{\psi}(x_f) \) are called \( h_f \) and \( h_g \), and (when expanded) the loop line \( \bar{\psi}(x_f)\psi(x_g) \) is called \( l_{fg} \) (it corresponds to a particular coefficient in the determinant \( \text{det}_{\text{left}} \)). Each tree half-line \( f_i \) or \( g_i \), each loop field \( h_f \) or \( h_g \) is hooked to a vertex called \( v_f \) or \( v_g \). We need also to care about the set \( S \) of special fields (or antifields) which are hooked to the \( \delta \zeta \) vertices and correspond to propagators \( C' \) which have different “power counting” than \( C \). Finally the index of the highest tree-line hooked to a vertex \( v \) is called \( i_v \).

Now \( \text{det}_{\text{left}} \) is the determinant of a matrix \( (n+1-p) \times (n+1-p) \). The corresponding loop fields can be labeled by an index \( a = 1, \ldots, 2n + 2 - 2p \). The matrix elements are \( D(x_f, x_g, w^{T}_{v_f,v_g}(w)) \). Therefore in terms of bands the line \( l_{fg} \) is restricted by the weakening factor \( w^{T}_{v_f,v_g}(w) \) to belong to the bands from 1 to the lowest index in the path \( P^{T}_{fg} \) (this path \( P^{T}_{fg} \) is defined in equation (II.13)). We call \( i^{T}_{fg} \) this index:

\[
\begin{align*}
i^{T}_{fg} &= \inf \{ q \mid l_q \in P^{T}_{fg} \} \quad \text{(III.5)} \\
D(x_f, x_g, w^{T}_{v_f,v_g}(w)) &= D(p) \sum_{k=1}^{i^{T}_{fg}} \eta^k(p) \quad \text{(III.6)}
\end{align*}
\]
By multilinearity one can expand the determinant in (II.25) according to the different bands in the sum (III.6) for each row and column.

\[ \det \left( w^T(w), E \right) = \sum_\mu \det \mathcal{M}(\mu) \quad (\text{III.7}) \]

where we define the attribution \( \mu \) as a collection of band indices for each loop field \( a \):

\[ \mu = \{ \mu(f_1), \ldots, \mu(f_{n+1-p}), \mu(g_1), \ldots, \mu(g_{n+1-p}) \} \quad \mu(a) \in B \quad \text{for} \quad a = 1 \ldots 2n + 2 - 2p. \quad (\text{III.8}) \]

Now, for each attribution \( \mu \) we need to exploit power counting. This requires notations for the various types of fields or half-lines which form the analogs of the quasi local subgraphs of \([R]\) in our formalism. We define:

\[ T_k = \{ l \in \mathcal{T} \mid i_{v_l} \geq k \} \]
\[ IT_k = \{ f_i, g_i \in \mathcal{T} \mid i \geq k \} \]
\[ IL_k = \{ a \in L \mid \mu(a) \geq k \} \]
\[ EE_k = \{ f, g \in E \mid i_{v_f}, i_{v_g} \geq k \} \]
\[ ET_k = \{ f_i | i_{v_f} \geq k, i < k \} \cup \{ g_i | i_{v_g} \geq k, i < k \} \]
\[ EL_k = \{ a \in L | i_{v_a} \geq k, \mu(a) < k \} \]
\[ \bar{N}_k = \{ v \text{ of type } \lambda | i_v \geq k \} \]
\[ N_k^\prime = \{ v \text{ of type } \delta m | i_v \geq k \} \]
\[ N_k'' = \{ v \text{ of type } \delta \zeta | i_v \geq k \} \]
\[ \bar{N}_k = \bar{N}_k \cup N_k^\prime \cup N_k'' , \quad G_k = IT_k \cup IL_k , \quad E_k = EE_k \cup ET_k \cup EL_k \]
\[ E_k'' = E_k \cap S , \quad T_k = \{ i_i \mid i \geq k \} \quad (\text{III.9}) \]

where we recall that \( S \) in the last definition is the set of those fields hooked to a vertex of type \( \delta \zeta \) which bear a derivation. We note \( |A| \) the number of elements in the set \( A \). For instance the reader can check that \( |IT_1| = 2n - 2 \) and that \( |T_1| = n - 1 \). Each \( G_k \) has \( c(k) \) connected components \( G^j_k, \ j = 1, \ldots, c(k) \). All the definitions in (III.9) can be restricted to each connected component. Applying power counting, the convergence degree for the subgraph \( G^j_k \) is

\[ \omega(G^j_k) = \frac{1}{2} (|E^j_k| + 2|N^j_k| - 4) \quad (\text{III.10}) \]
where we assumed that no external half-line hooked to a vertex of type $\delta \zeta$ bears a $i \not\partial$. To assure this for any $G_k^i$, we apply, for each vertex $v''$, the operator $i \not\partial$ (or $-i \not\partial$) to the highest tree half-line hooked to $v''$ (there is always at least one). In this way for all $k |E_k''| = 0$, and no loop line bears a gradient. Then $\mathcal{M}(\mu)$ is a matrix whose coefficients are

$$\mathcal{M}_{fg}(\mu)(x_f, x_g) = \delta_{\mu(f), \mu(g)} \int \frac{d^2 p}{(2\pi)^2} e^{-ip(x_f-x_g)} C(p) \eta^{\mu(f)}(p) W^{\mu(f)}_{v_f,v_g}$$  \hspace{1cm} (III.11)$$

where

$$W^{k}_{v,v'} = \begin{cases} 1 & \text{if } v \text{ and } v' \text{ are connected by } T_k \\ 0 & \text{otherwise} \end{cases}$$  \hspace{1cm} (III.12)$$

since we always have $D = C$ in the matrix $\mathcal{M}(\mu)$.

From (III.10) we see that there are three types of divergent subgraphs:
- for $|E_k^i| = 4$, $|N_k^i| = 0$ we have logarithmic divergence ($\omega(G_k^i) = 0$);
- for $|E_k^i| = 2$, $|N_k^i| = 0$ we have linear divergence ($\omega(G_k^i) = -1$);
- for $|E_k^i| = 2$, $|N_k^i| = 1$ we have logarithmic divergence ($\omega(G_k^i) = 0$).

In fact the divergent graphs are only those for which the algebraic structure of the external legs is of one of the three types in (II.1). For instance not all four-point subgraphs are divergent, but only those for which the flow of spin indices follows the flow of color indices [GK][FMRS]. Using the invariance of $\mathcal{L}$ under parity and charge conjugation one finds that all counterterms which are not of the three types in (II.1) are zero (this means that the corresponding subgraphs have 0 local part). Then renormalizing these subgraphs we improve power counting without generating new counterterms. In what follows, for simplicity, “divergent subgraph” always means subgraph with two or four external legs (this means we will renormalize some subgraph which does not need it but this does not affect the convergence of the series). Also for simplicity we change the definition of convergence degree (III.10) in

$$\omega'(G_k^i) = \frac{1}{2}(|E_k^i| - 4).$$  \hspace{1cm} (III.13)$$

To cure divergences, we apply to the amplitude of each divergent subgraph $g$ the operator $(1-\tau_g) + \tau_g$. In the momentum space $\tau_g$ is the Taylor expansion at order $-\omega(g)$ of the amplitude $\hat{g}(p)$ at $p = 0$. The operator $1-\tau_g$ makes the amplitude convergent when the UV cut-off is sent to infinity. The remaining
term $\tau_g \hat{g}$ gives a local counterterm for the coupling constant that depends on the energy of the external lines of $g$. At each vertex $v$, we can resum the series of all counterterms obtained applying $\tau_g$ to all divergent subgraphs (for different attributions $\mu$) that have the same set of external lines as $v$ itself. In this way we obtain an effective coupling constant which depends on the energy $\Lambda_0(\omega_{i_{v}})$ of the highest tree line hooked to the vertex $v$. This is true because after applying the $1-\tau_g$ operators, for each graph with nonzero amplitude the highest index at each vertex coincides with the highest tree index $i_{v}$ at each vertex! Indeed at vertices $v$ for which this is not true, there are loop fields with attribution $\mu$ higher than $i_{v}$. By (III.12) they must contract together forming tadpoles, which are set to zero by the $1-\tau_g$ operators. The corresponding graphs therefore disappear from the expansion.

For each attribution $\mu$ we define the set of divergent subgraphs as

$$D_{\mu} := \{ G_{i}^{k} | \omega'(G_{i}^{k}) \leq 0 \}. \quad (III.14)$$

The action of $\tau_g$ is

$$\tau_g \hat{g}(p_1, \ldots, p_k) = \sum_{j=0}^{\omega'(g)} \frac{1}{j!} \frac{d^j}{dt^j} \hat{g}(tp_1, \ldots, tp_k) |_{p=0} \quad k = 2, 4. \quad (III.15)$$

With this definition the effective constants $\lambda_{w}, \delta m_{w}, \delta \zeta_{w}$ turn out to be the vertex functions $\Gamma_4$, $\Gamma_2$ and $\delta \Gamma_2$ for an effective theory with infrared parameter $\Lambda = \Lambda_0(w)$:

**III.3 Effective constants**

In the space of positions, the operator $\tau_g$ is applied by partial integration to the product of external propagators $a(x_1, \ldots x_{v_e})$, as in [R]:

$$\tau_g^* (v_e) a(x_1, \ldots x_{v_e}) = \sum_{j=0}^{\omega'(g)} \frac{1}{j!} \frac{d^j}{dt^j} a(x_1(t), \ldots, x_{v_e}(t)) |_{t=0} \quad (III.16)$$

where $x_i(t) = x_{v_e} + t(x_i - x_{v_e})$, and $v_e$ is an external vertex of $g$ chosen as ‘reference vertex’. The choice of this reference vertex is given in section IV.3.1. As announced we find the three possible counterterms of (II.13). For $|E_i| = 4$ we have

$$\tau^* (x_1) \prod_{i=1}^{4} C(x_i; y_i) = \prod_{i=1}^{4} C(x_1, y_i), \quad (III.17)$$
so the counterterm is
\[
\int d^2x_1 \prod_{i=1}^{4} C^{a_i\alpha_i}(x_1, y_i) \int d^2x_2 d^2x_3 d^2x_4 \ g(0, x_2, x_3, x_4)_{a_1a_2a_3a_4} = \int d^2x_1 \prod_{i=1}^{4} C^{a_i\alpha_i}(x_1, y_i) \ \hat{g}(0, \ldots, 0) \quad (\text{III.18})
\]

This gives a coupling constant counterterm. For \(|E_i| = 2\) we have
\[
\tau^*(x_1)[C(x_1, y_1)C(x_2, y_2)] = C(x_1, y_1) \left[ C(x_1, y_2) + (x_2 - x_1)^\mu \frac{\partial}{\partial x_1} C(x_1, y_2) \right].
\]
\[
(\text{III.19})
\]

Integrating over internal points, we obtain a mass counterterm from the first term in the sum:
\[
\int d^2x_1 \prod_{i=1}^{2} C^{a_i\alpha_i}(x_1, y_i) \int d^2x_2 \ g^{a_1a_2}(0, x_2) = \int d^2x_1 \prod_{i=1}^{2} C^{a_i\alpha_i}(x_1, y_i) \ \hat{g}(0, \delta_{a_1a_2}) \quad (\text{III.20})
\]
where we applied the development
\[
\hat{g}(p) = f_1(p^2) + \gamma_5 f_2(p^2) + \gamma_5 \not{p} f_3(p^2) + \gamma_5 \not{p} f_4(p^2) \quad (\text{III.21})
\]
and we adopted for the gamma matrices the conventions in [FMRS]. By invariance under charge conjugation and parity \(f_2(0) = f_4(0) = 0\). For the second term in the sum we obtain a wave function counterterm:
\[
\int d^2x_1 C_{a_1a_1'}^{a_1}(x_1, y_1) \frac{\partial}{\partial x_1} C_{a_1a_2'}^{a_2}(x_1, y_2) \int d^2x_2 (x_2 - x_1)^\mu g^{a_1a_2}(x_1 - x_2) = \int d^2x_1 C_{a_1a_1'}^{a_1}(x_1, y_1) \frac{\partial}{\partial x_1} C_{a_1a_2'}^{a_2}(x_1, y_2) i \frac{\partial}{\partial p^\mu} \hat{g}(p)_{p=0} \delta_{a_1a_2} \quad (\text{III.22})
\]
Theorem 2 If we apply to each divergent subgraph \( g \in D_\mu \), for any attribution \( \mu \), the operator \((1 - \tau_g) + \tau_g = R_g + \tau_g\), the function (II.23) can be written as

\[
\Gamma_{2p}^{\Lambda_0}(\phi_1, \ldots, \phi_{2p}) = \sum_{n,n',n''=0}^{\infty} \frac{1}{n!n'!n''!} \sum_{o-T} \sum_{E,\mu} \sum_{Col,\Omega} \epsilon(T,\Omega) \int d^2x_1 \ldots d^2x_n
\]

\[
\int_{0 \leq w_1 \leq \ldots \leq w_{n-1} \leq 1} \prod_{q=1}^{n-1} dw_q \left[ \prod_{\nu} \left( \frac{\lambda_{w(\nu)}}{N} \right) \right] \left[ \prod_{\nu'} \delta m_{w(\nu')} \right] \left[ \prod_{\nu''} \delta \zeta_{w(\nu'')} \right]
\]

\[
\prod_{G_k^i \in D_\mu} R_{G_k^i} \left[ \prod_{q=1}^{n-1} D_{\Lambda}^{\Lambda_0,w_q}(\vec{x}_{t_q}, \vec{x}_{t_q}) \right] \det M(\mu) \phi_1(x_{i_1}) \ldots \phi_{2p}(x_{j_p})
\]

(III.23)

where the constants \( \lambda_w, \delta m_w, \delta \zeta_w \) are the ‘effective constants’, defined as:

\[
\frac{\lambda_w}{N} = \tilde{\Gamma}_4^{\Lambda_0(w),\Lambda_0}(0,0,0,0) = \int d^2x_2 d^2x_3 d^2x_4 \Gamma_{4}^{\Lambda_0(w),\Lambda_0}(0,x_2,x_3,x_4)
\]

\[
\delta m_w = \tilde{\Gamma}_2^{\Lambda_0(w),\Lambda_0}(0,0) = \int d^2x_2 \Gamma_{2}^{\Lambda_0(w),\Lambda_0}(0,x_2)
\]

\[
\delta \zeta_w = \partial \Gamma_{2}^{\Lambda_0(w),\Lambda_0}(p)|_{p=0} = \int d^2x_2 i \partial \Gamma_{2}^{\Lambda_0(w),\Lambda_0}(0,x_2)
\]

(III.24)

The effective constants are the vertex functions \( \Gamma_4, \Gamma_2 \) and \( \partial \Gamma_2 \) for an effective theory with infrared parameter \( \Lambda_0(w) \), and the renormalized constants correspond to the effective ones at the energy \( \Lambda \). (For the massive theory recall that we can use \( \Lambda = 0 \)).

\[
\lambda_{w=0} = \lambda_r = \lambda_r
\]

\[
\delta m_{w=0} = \delta m_r = 0
\]

\[
\delta \zeta_{w=0} = \delta \zeta_r = 0
\]

(III.25)

The reshuffling of perturbation theory performed by Theorem II can be proved by standard combinatorial arguments as in \([R]\) (the only difficulty was discussed above, when we remarked that the parameter \( w \) of the effective constants always corresponds to the highest tree line of the vertex. Otherwise the effective vertex generates a tadpole graph whose later renormalization gives 0).
IV Convergence of the series.

**Theorem 3** Let $\epsilon > 0$ be fixed. Suppose $\Lambda^m$ (defined below) belongs to some fixed compact $X$ of $]0, +\infty)$. The series (III.23) is absolutely convergent for $|\lambda_w|, |\delta m_w|, |\delta \zeta_w| \leq c$, $c$ small enough. This convergence is uniform in $\Lambda_0$ and $N$ (actually $\Gamma_{2p}$ is proportional to $N^{1-p}$). The ultraviolet limit $\Gamma_{2p}^\Lambda = \lim_{\Lambda_0 \to \infty} \Gamma_{2p}^{\Lambda \Lambda_0}$ exists and satisfies the bound:

$$|\Gamma_{2p}^\Lambda(\phi_1, \ldots \phi_{2p})| \leq (p!)^{5/2}[K(c, \epsilon, X)]^p (\Lambda^m)^{2-p} N^{1-p} \quad (IV.1)$$

where

$$\Lambda^m := \sup \{\Lambda, m\} , \quad (IV.2)$$

$$||\phi_i||_{\infty,2} := (||\phi_i||_{\infty} + ||\phi_i'|||_{\infty} + ||\phi_i''||_{\infty}) , \quad (IV.3)$$

$\Omega_i$ is the compact support of $\phi_i$, $K(c, \epsilon, X)$ is some function of $c, \epsilon$ and $X$, which tends to zero when $c$ tends to 0, $||\phi_i||_{\infty} = \sup_{x \in \Omega_i} |\phi_i(x)|$, $||\phi_1||_1 = \int d^2x |\phi_1(x)|$, and

$$d_T(\Omega_1, \ldots \Omega_{2p}) := \inf_{x_1 \in \Omega_1} d_T(x_1, \ldots x_{2p})$$

$$d_T(x_1, \ldots x_{2p}) := \inf_{u - T} \sum_{t \in T} |\bar{x}_t - x_t|. \quad (IV.4)$$

where in the definition of $d_T(x_1, \ldots x_{2p})$, called the “tree distance of $x_1, \ldots x_{2p}$”, the infimum over $u - T$ is taken over all unordered trees (with any number of internal vertices) connecting $x_1, \ldots x_{2p}$.

This bound means that one can construct in a non perturbative sense the ultraviolet limit of either the massive theory with any infrared cutoff $\Lambda$ including $\Lambda = 0$, or the weakly coupled massless theory with nonzero infrared cutoff $\Lambda$. To complete Theorem 1 from Theorem 3, one needs only to check Borel summability by expanding explicitly at finite order $n$ in $\lambda_{ren}$ and controlling the Taylor remainder. This additional expansion generates a finite number of Taylor operators $\tau_g$ for a finite number of non quasi-local subgraphs, which are responsible for the $n!$ of Borel summability [R]. Since this is rather standard we will not include this additional argument here. Finally the renormalization group equations are discussed in section V. The rest of the section is devoted to the proof of this theorem.
IV.1 Plan of the proof

To prove the theorem we show that the absolute value of the term \((n, n', n'')\) in the sum (excluding the effective constants) is bounded by \(K^n\). The strategy for the proof consists in moving the absolute value inside all sums and integrals, bounding the product of effective constants,

\[
\left[ \prod_v |\lambda_{w(v)}| \right] \left[ \prod_{v'} |\delta m_{w(v')}| \right] \left[ \prod_{v''} |\delta \zeta_{w(v'')}| \right] \leq c^n , \tag{IV.1}
\]

then taking \(c < K^{-1}\).

The loop determinant will be bounded by a Gram inequality, and we shall use the tree lines decay to bound the spatial integrals. Actually, we cannot move the absolute value directly inside the sum over attributions because \(#\{\mu\} \simeq \tilde{n}!\). In other words fixing the band index for each single half-line develops too much the determinant. The way to overcome this difficulty is to remark that the attributions contain much more information than necessary. We can in fact group the attributions into packets to reduce the number of determinants to bound. We observe that, if for the level \(i\) a connected component \(G^k_i\) has \(|EE^k_i| + |ET^k_i| \geq 5\), the subgraph is convergent and we do not need to know the band indices for the loop lines in that connected component. So for each convergent \(G^k_i:\)

- if \(|EE^k_i| + |ET^k_i| \geq 5\), we do not want to know anything on loop lines;
- if \(|EE^k_i| + |ET^k_i| < 5\), we just want to fix \(5 - |EE^k_i| - |ET^k_i|\) half-lines with energy lower than \(i\), but we are not interested on the energy of the other half-lines;

Instead of expanding the loop determinant over lines and columns as a sum over all attributions

\[
\det M = \sum_{\mu} \det M(\mu, E) \tag{IV.2}
\]

we write it as a sum over a smaller set \(\mathcal{P}\) (called the set of packets). These packets are defined by means of the function

\[
\phi : \{\mu\} \rightarrow \mathcal{P}
\]

\[
\mu \mapsto \mathcal{C} = \phi(\mu) \tag{IV.3}
\]

but this function must respect some constraints related to the future use of Gram’s inequality. This motivates the following definition:
Definition 1 The pair \((\mathcal{P}, \phi)\) is called a “Gram-compatible pair” if

\[
\forall C \in \mathcal{P}, \forall a, \exists J_a(C) \subset B \tag{IV.4}
\]

with the property \(\phi^{-1}(C) = \{\mu|\mu(a) \in J_a(C) \ \forall \ a\}\).

This definition ensures that there exists a matrix \(\mathcal{M}'\) such that

\[
\sum_{\mu \in \phi^{-1}(C)} \det \mathcal{M}(\mu) = \det \mathcal{M}'(C) \tag{IV.5}
\]

and that Gram’s inequality can be applied to \(\det \mathcal{M}'(C)\), as shown in Lemma 4.

IV.2 Construction of \(\mathcal{P}\)

We build first the partition \(\mathcal{P}\) of the set of attributions into packets. These packets should contain the informations we need over \(|E_{k}^j|\). In contrast with attributions there should be few of them; more precisely they should satisfy \(#\mathcal{P} \leq K^n\). Finally, together with the function \(\phi\), they should form a Gram-compatible pair. To define \(\mathcal{P}\) we introduce some preliminary definitions and notations.

To each ordered tree \(o - \mathcal{T}\) we can associate a rooted tree \(R_T\), which pictures the inclusion relation of the \(G_{k}^j[R]\). We can picture this tree with two types of vertices: crosses and dots. We recall that the leaves of a rooted tree are the vertices of the rooted tree with coordination number one. The leaves in our case are the dot-vertices and correspond exactly to the vertices \(v, v'\) or \(v''\) of the initial ordered tree \(T\). The other vertices of \(R_T\) are crosses. Each cross \(i\) corresponds to a line \(l_i\) of the initial ordered tree \(\mathcal{T}\), and has coordination number three, except the root which has coordination number two. To build \(R_T\) we take the lowest line in \(\mathcal{T}, l_1\), as root 1.
This line \( l_1 \), or root, separates \( T \) into two connected components \( T' \) and \( T'' \) possibly reduced to a single vertex. When \( T' \) or \( T'' \) is a single vertex, it gives a dot connected to 1. Otherwise it gives a cross, which is the lowest line of \( T \) in it. This procedure is repeated at each cross-vertex obtained, and generates \( R_T \).

Figure 4

Finally to complete the picture to each dot of \( R_T \) we hook all loop half-lines hooked to the corresponding vertex (there could be none). We define the ancestor of \( i \) \( A(i) \) as the cross-vertex just under \( i \) in \( R_T \) and we call \( v_a \), the dot-vertex to which the half-line \( a \) is hooked and \( i_a \) the cross-vertex connected to \( v_a \) (which represents a line of the initial tree!). For each cross-vertex \( i \) we define

\[
t_i := \{ l_j \in T | j \geq i, l_j \text{ connected to } l_i \text{ by } T_i \}
\]

(IV.6)

Figure 5

An example of a tree with its associated \( R_T \) is given in Figure 6:
For each tree line (cross-vertex) \( i \) and each connected component \( G_i^k \), no new line connects to \( t_i \) in the interval between \( i \) and \( A(i) \). Hence \( \omega(i)|G_i^{k'}| \geq \omega'(G_{A(i)}^{k''}) \forall i > i' > A(i) \) and \( T_{i'}^k \subset G_i^{k'} \subset G_i^k \). Therefore we can neglect what happens in this interval and generalize the definitions of (IV.9) for the internal lines of a subgraph \( G_i^k \). We define:

\[
\begin{align*}
g_i & := t_i \cup \{a \in L | i_a \geq i, v_a \in t_i, \mu(a) \geq A(i) + 1\} \\
eg e_i & := et_i \cup ee_i \cup el_i \\
et & := \{f_v | i_v \geq i, v_f \in t_i, i' < i\} \cup \{g_v | i_v \geq i, v_g \in t_i, i' < i\} \\
eee & := \{f, g \in E | i_v \geq i, i_v \geq i, v_f, v_g \in t_i\} \\
el & := \{a \in L | i_a \geq i, v_a \in t_i, \mu(a) \leq A(i)\}
\end{align*}
\]

This set of definitions (IV.7) concerns the connected component \( g_i \) above line \( i \). Remark that we defined as loop internal lines of \( g_i \), all loop lines higher than \( A(i) \). We also need some additional definitions concerning the other connected components:

\[
\begin{align*}
i(k) & := \inf \{j \geq i, v_j \in T_i^k\} \\
g_i^k & := g_i(k) \\
eg e_i^k & := et_i^k \cup ee_i^k \cup el_i^k \\
et_i^k & := et_i(k) ee_i^k := ee_i(k) el_i^k := el_i(k)
\end{align*}
\]

This second set of definitions is used only much later in the bounds when all connected components are considered at once.

**Definition 2** A *chain* \( C_{a,i} \) is the unique path in \( R_T \) from the half-line \( a \) to the cross-vertex \( i \) with \( i_a \geq_T i \):

\[
C_{a,i} := \{i'|i \leq_T i' \leq_T i_a\} \cup \{a\}
\]

(IV.9)
In the following, we write $i_a \geq^T i$ to specify that $v_a$ and $v_i$ are connected by $t_i$.

**Definition 3** a class $\mathcal{C}$ is a set of chains over $R_T$ with the properties:

\[
\forall C_{ai} \in \mathcal{C}, \forall C_{a'i'} \in \mathcal{C} \text{ one has } a \neq a' \\
\forall i \quad c_i \leq \max[0; 5 - |ee_i| - |et_i| - c'_i] \quad (IV.10)
\]

where we defined:

\[
c_i = \#\{C_{ai} \in \mathcal{C} | \ i \ fixed\} \\
c'_i = \#\{C_{a'i'} \in \mathcal{C} | i_a' \geq^T i, \ i' < i\} \quad (IV.11)
\]

So $c_i$ is the total number of chains arriving at $i$ and $c'_i$ is the total number of chains passing through $i$ and continuing further below. This definition ensures therefore that there are at most five chains passing through each cross $i$.

**Definition 4** The partition $\mathcal{P}$ is the set of all possible classes $\mathcal{C}$ over $R_T$.

To verify that this is a good definition, we have to prove three lemmas.

**Lemma 1** The cardinal of $\mathcal{P}$ is bounded by $K^n$.

**Proof:** we prove that $\mathcal{P} \subseteq \mathcal{P}'$ and $\#\mathcal{P}' \leq K^n$. We define $\mathcal{P}'$ as the set of all sets of chains $\mathcal{D}$, that are unions of five subsets (possibly empty) $Y_j$, where $Y_j$ is a set of completely disjoint chains (this means they have no cross and no dot in common).

\[
\mathcal{P}' := \{\mathcal{D}\} \quad \mathcal{D} := \bigcup_{j=1}^{5} Y_j. \quad (IV.12)
\]
To build a set of disjoint chains $Y_j$, we have at most three possible choices for each vertex: at each cross-vertex we can have no chain passing, a chain going right or left; at each dot-vertex touched by a chain, we have to choose among three (at most) loop half-lines. Putting all this together we have:

$$\#P' \leq (3^5)^n \cdot 2^{2p-n} \leq K^n$$  \hspace{1cm} (IV.13)

where the number 5 comes because each element of $C$ is made of five sets $Y_j$.

Figure 8 shows an example of disjoint sets built in this way.

Now we prove that $P \subseteq P'$ by induction on $i$. For each $C \in P$ we define $C(i)$ as the subset of $C$ that contains only chains ending in some point (cross-vertex) of the unique path connecting $i$ to the root.

$$C(i) := \{C_{a'i'} \in C \mid i' \leq_T i\}$$  \hspace{1cm} (IV.14)

This set satisfies the following induction law: if, for $A(i)$ there are five sets (eventually empty) of disjoint chains $Y_1(A(i))...Y_5(A(i))$ with

$$C(A(i)) = \cup_{j=1}^5 Y_j(A(i)),$$  \hspace{1cm} (IV.15)

then there are five sets $Y_1(i),...,Y_5(i)$ with $C(i) = \cup_j Y_j(i)$. This can be seen observing that $C(i)$ can be written as

$$C(i) = C(A(i)) \cup \{C_{a'i'} \in C \mid i' = i\}.$$  \hspace{1cm} (IV.16)

Among the five sets $Y_j$ forming $C(A(i))$ there are $c'_i$ ones containing chains passing through $i$: $Y_1(A(i)),...,Y_{c'_i}(A(i))$. If $c'_i + |ee_i| + |et_i| \geq 5$, there are

---

Figure 8

---
no chains ending at $i$ so $C(i) = C(A(i)) \subset P'$. If $c'_i + |ee_i| + |et_i| < 5$ there are $c_i$ chains ending at $i$ $C_{a_1,i}, ..., C_{a_{c_i},i}$, with $c_i \leq 5 - c'_i$, so we can define

$$Y_j(i) = Y_j(A(i)) \quad \text{for } j \leq c'_i,$$
$$Y_{c'_i+j}(i) = Y_{c'_i+j}(A(i)) \cup \{C_{a_j,i}\} \quad j = 1, ..., c_i,$$
$$Y_j(i) = Y_j(A(i)) \quad \text{for } j > c'_i + c_i. \quad \text{(IV.17)}$$

With these definitions we have

$$C(i) = \bigcup_{j=1}^{5} Y_j \subset P' \quad \text{(IV.18)}$$

Now, the hypothesis (IV.15) is true for the root $r$. In fact, by construction, we have at most five chains ending at $r$: $C_{a_1,r}, ..., C_{a_{5},r}$. If we define:

$$Y_1(r) = \{C_{a_1r}\}, ..., Y_5(r) = \{C_{a_5r}\} \quad \text{(IV.19)}$$

we have $C(r) = Y_1 \cup .. \cup Y_5 \subset P'$. Working the induction up to the leaves of $R_T$ completes the proof of the lemma. \hfill \Box

**Lemma 2** There exists a function $\phi : \{\mu\} \longrightarrow P$ which associates to each attribution $\mu = (\mu(1), \mu(2), ...)$ a class $C$ in $P$.

To define $\phi$ we fix an order over the half-lines and the lines of $R_T$. We do it turning around $R_T$ clockwise and we call $n(a)$ the index of $a$ in the ordering and $s_i$ the index of the line in $R_T$ connecting $i$ to $A(i)$.

![Figure 9](image)

We build the class $\phi(\mu)$ as a union of chains by induction, defining first the chains in $\phi(\mu)$ ending at the root, then the ones ending at the cross connected to the root by the line 1, and so on, following the ordering $s_i$. Therefore for each $i$ we consider the set $A_i = \{a \in cl_i | \nexists C_{ai} \in \phi(\mu) \text{ with } i' < i\}$ which is the set of loop half-lines that are external lines for $g_i$ and are not connected to a chain in $\phi(\mu)$ ending lower than $i$. 

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- If \([5 - |ee_i| - |et_i| - c'_i| > 0 \) and \(\# A_i < [5 - |ee_i| - |et_i| - c'_i] \) we have a divergent subgraph, and we add to the part already built of \(\phi(\mu)\) all the chains starting at an element of \(A_i\) and ending at \(i\), so

\[ c_i = \# A_i. \]  

(IV.20)

- If \(\# A_i \geq \text{max}[0, 5 - |ee_i| - |et_i| - c'_i] \), we have a convergent subgraph, so we put

\[ c_i = \text{max}[0, 5 - |ee_i| - |et_i| - c'_i] \]  

(IV.21)

and we add to the part already built of \(\phi(\mu)\) the \(c_i\) chains \(C_{a',i}\), with \(a' = a'_j, j = 1,...,c_i,\) which start at the \(c_i\) elements in \(A_i\) that have the lowest values of \(n(a)\), and end at \(i\).

In this way we obtain a set of chains with the two properties (IV.10). For each \(\mu, \phi(\mu)\) is an element of \(\mathcal{P}\) and \(\{\phi^{-1}(C)\}_{C \in P}\) is a partition of the set of attributions.

\[ \square \]

We call \(B_i\) the set of half-lines in \(A_i\) which are the starting points of chains in \(\phi(\mu)\) ending at \(i\) (see Figure 10). Therefore in the divergent case \(B_i = A_i\) and in the convergent case \(B_i = \{a'_j, j = 1,...,c_i\}.\) We also define

\[ eg_i(C) := et_i \cup ee_i \cup \{a|i_a \geq_T i \text{ and } a \in B_{i'} \text{ for some } i' \leq_T i\} \]  

(IV.22)

With this definition we have \(|eg_i(C)| = c_i + c'_i + |et_i| + |ee_i|\). Remark that in the divergent case \(|eg_i| \leq 4\), one has \(|eg_i| = |eg_i(C)|\), and in the convergent case one has \(|eg_i| \geq |eg_i(C)| \geq 5\). The next lemma describes the structure of the classes \(C\).

**Lemma 3** For each class \(C \in \mathcal{P}\) and each half-line \(a = 1,...,2n + 2 - 2p\) there exists a subset of band indices \(J_a(C) \subseteq B\) such that

\[ \phi^{-1}(C) = \{\mu|\mu(a) \in J_a(C) \forall a\}. \]  

(IV.23)
Proof  The existence of the $c_i$ chains $C_{ai}$ for $a \in B_i$ ending at $i$ implies a certain set of constraints on attributions. We distinguish two situations.

1) If $|eg_i(C)| \leq 4$ (divergent case)
   \begin{itemize}
   \item $\forall a \in B_i$, $\mu(a) \leq A(i)$;
   \item $\forall a \notin B_i$ with $i_a \geq T i$, $\mu(a) > A(i)$.
   \end{itemize}

2) If $|eg_i(C)| \geq 5$ (convergent case)
   \begin{itemize}
   \item $\forall a \in B_i$, $\mu(a) \leq A(i)$;
   \item $\forall a \notin B_i$ with $i_a \geq T i$, and $n(a) < \max_{a' \in B_{i'}} n(a')$, $\mu(a) > A(i)$;
   \end{itemize}

In any other case, there is no particular constraint. We observe that the underlined constraints for $\mu(a)$ are therefore determined by the chain structure and the ordering, but the crucial point is that they are independent from each other. Hence $J_a(C)$ is an interval in terms of band indices. Remark that if some chain in $C$ starts from $a$, it must end at some unique $i$, called $i'_a$. In this case we define $M(a, C) = A(i'_a)$. Otherwise we define $M(a, C) = i_a$.

Moreover for each $i'$ such that $a \notin B_{i'}$ we have two different lower bounds on $\mu(a)$, depending whether $g_{i'}$ is divergent or convergent. So the constraints in cases 1 and 2 simply mean $m(a, C) \leq \mu(a) \leq M(a, C)$, where

\begin{align*}
M(a, C) & = A(i'_a) \text{ if } a \in B_{i'_a}, \quad M(a, C) = i_a \text{ otherwise} \\
m(a, C) & = \sup_{i' \in I(a, C)} [A(i') + 1] \\
m(a, C) & = 1 \text{ if } I(a, C) = \emptyset
\end{align*}

(IV.24)

and

\begin{align*}
I(a, C) := \{i' | i_a \geq T i', a \notin B_{i'}, \text{ and, if } |eg_{i'}(C)| \geq 5, n(a) < \max_{a' \in B_{i'}} n(a')\}
\end{align*}

(IV.25)

In summary the constraints are expressed by

\begin{align*}
\phi^{-1}(C) & = \{\mu | \mu(a) \in J_a(C) \forall a\} \\
J_a(C) & = [m(a, C), M(a, C)]
\end{align*}

(IV.26)

$\square$
Example: As an example, in Figure 11 we have three chains $C_{a,3}$, $C_{a',2}$ and $C_{a'',1}$. The bands are: $i_1 + 1 \leq \mu(a) \leq i_2$, $i_0 + 1 \leq \mu(a') \leq i_1$, $\mu(a'') \leq i_0$.

![Figure 11]

We observe that, after packing the attributions into classes, the sets $T_i$, $t_i$, $ee_i$, $et_i$ are still well defined, but we can no longer define $g_i$ and $el_i$. We already defined $eg_i(C)$ in (IV.22). We add further definitions

$$g_i(C) := t_i \cup il_i(C)$$

$$il_i(C) := \{ a \in L \mid i_a \geq T, M(a, C) \geq A(i) + 1 \}$$

$$el_i(C) := \{ a \in L \mid i_a \geq T, M(a, C) \leq A(i) \}$$ (IV.27)

which generalize the notions of internal and external loop lines. Remark that $eg_i(C) = et_i \cup ee_i \cup el_i(C)$, and $|el_i(C)| = c_i + c'_i$. In the same way we extend these definitions to the other connected components

$$g^k_i(C) := g_{i(k)}(C) \ , \ il^k_i(C) := il_{i(k)}(C) \ , \ el^k_i(C) := el_{i(k)}(C)$$ (IV.28)

Furthermore the generalized definitions for the convergence degree and the set of divergent subgraphs after packing the attributions into classes become:

$$\omega(g_i(C)) := (|eg_i(C)| - 4)/2.$$  
$$D_C := \{ g_i(C) \mid \omega(g_i(C)) \leq 0 \}.$$ (IV.29)

We return now to the loop determinant in (III.23). Lemma 3 ensures that

$$\sum_{\mu \in \phi^{-1}(C)} det(M(\mu)) = det(M'(C))$$ (IV.30)
and that for each loop half-line \( a \) there exists a characteristic function
\[
\chi_a (C) : k \in B \rightarrow \{0, 1\} \quad \chi_a^k (C) = \begin{cases} 
0 & \text{if } k \not\in J_a (C) \\
1 & \text{if } k \in J_a (C) .
\end{cases}
\] (IV.31)

Therefore the matrix elements for \( M' (C) \) can be written
\[
M'_{fg} (x_f, x_g) = \int \frac{d^2 p}{(2\pi)^2} e^{-ip(x_f-x_g)} C(p) \sum_{k \in B} \chi^k_{a(f)} \chi^k_{a(g)} \eta^k (p) W^k_{v_f, v_g}
\]
\[
= \int \frac{d^2 p}{(2\pi)^2} F_f^{*} (p) G_g (p) \sum_{k} \chi^k_{a(f)} \chi^k_{a(g)} \eta^k (p) W^k_{v_f, v_g}
\]
where we omit for simplicity to write the dependence in \( C \), and we defined:
\[
F_f (p) = e^{ix fp} \frac{1}{(p^2 + m^2)^{\frac{7}{2}}} \quad G_g (p) = e^{ix gp} \frac{(-p^2 + m^2)}{(p^2 + m^2)^{\frac{7}{2}}} .
\] (IV.32)

\( v_a \) is the vertex to which the half-line \( a \) is hooked and \( \eta^k \) is the cutoff restricted to the band \( k \) (see equation (III.3)). Finally \( W^k \) is the \( \bar{n} \times \bar{n} \) matrix defined in equation (III.12). Our next lemma is crucial since it bounds the loop determinant without generating any factorial.

**Lemma 4** The matrix \( M' (C) \) satisfies the following Gram inequality:
\[
| \det M' (C) | \leq \prod_f \left[ \int \frac{d^2 p}{(2\pi)^2} \eta^f_{C} (p) |F_f (p)|^2 \right]^{\frac{1}{2}} \prod_g \left[ \int \frac{d^2 p}{(2\pi)^2} \eta^g_{C} (p) |G_g (p)|^2 \right]^{\frac{1}{2}},
\] (IV.33)

where the cutoff functions \( \eta^f_{C} (p) \) and \( \eta^g_{C} (p) \) corresponding to fields \( f \) and \( g \) are defined in equation (IV.44) below.

**Proof** The Gram inequality states:

If \( M \) is a \( n \times n \) matrix with elements \( M_{ij} = \langle f_i, g_j \rangle \) and \( f_i, g_j \) are vectors in a Hilbert space, we have \( | \det M | \leq \prod_{i=1}^{n} ||f_i|| \prod_{j=1}^{n} ||g_j|| \).

To apply Gram’s inequality, the matrix elements must be written as scalar products. We introduce the \( q \times q \) matrix \( 1_q \) which is not the identity, but the matrix with all coefficients equal to 1. It is obviously a positive symmetric matrix. We observe that the matrix \( W^k_{v,v'} \) is block diagonal with diagonal
blocks of type $1_{q_j}$, and $\sum q_j = \bar{n}$. Each block corresponds to all the vertices in a given connected component of $T_k$. Therefore $W$ itself is positive symmetric. We can define the symmetric matrix $(2n + 2 - 2p) \times (2n + 2 - 2p)$:

$$R_{ab}^k := \chi_a^k \chi_b^k$$

where $a$ and $b$ are the indices for the loop half-lines. By a permutation of field indices, we can list first the $q$ half-lines for which $\chi^k(C) = 1$. In this way the matrix $R$ becomes $2 \times 2$ block diagonal positive of the type

$$\begin{pmatrix} 1_q & 0 \\ 0 & 0 \end{pmatrix}.$$  

(IV.35)

Now we can group $W$ and $R$ in a unique matrix (tensor product)

$$\mathcal{W}_{v,a,v',b}^k := \chi_a^k \chi_b^k W_{v,v'}^k$$

(IV.36)

that is still positive as we can diagonalize separately $W$ and $R$. Hence the matrix

$$\sum_k \eta^k \mathcal{W}_{v,a,v',b}^k = T_{v,a,v',b}$$

(IV.37)

is symmetric positive, as is a linear combination (with positive coefficients $\eta^k$) of symmetric positive matrices; therefore we can take its square root (which is also positive symmetric):

$$T_{v,a,v',b} = \sum_{w,c} U_{v,a;w,c} U_{w,c,v',b}.$$  

(IV.38)

Now, we can write $\mathcal{M}'_{fg}$ as

$$\mathcal{M}'_{fg} = \int \frac{d^2 p}{(2\pi)^2} F_f^*(p) G_g(p) T_{v_f,a(f);v_g,a(g)}$$

$$= \int \frac{d^2 p}{(2\pi)^2} F_f^*(p) G_g(p) \sum_{v' s} U_{v_f,a(f);v',s} U_{v',s,v_g,a(g)}$$

(IV.39)

If we introduce the vectors

$$\mathcal{F}^f_{v',s}(p) = F_f(p) U_{v',s;v(f),a(f)} \quad \mathcal{G}^g_{v',s}(p) = G_g(p) U_{v',s;v(g),a(g)}$$

(IV.40)
we can write $\mathcal{M}_fg'$ as

$$
\mathcal{M}_fg' = \int \frac{d^2 p}{(2\pi)^2} \sum_{v',s} \mathcal{F}_{v's}^* \mathcal{G}_{v's}^g = \langle \vec{\mathcal{F}}', \vec{\mathcal{G}}^g \rangle .
$$

(IV.41)

Now we can apply Gram’s inequality:

$$
|\det \mathcal{M}_fg'| \leq \prod_{f=1}^{n+1-p} ||\vec{\mathcal{F}}'|| \prod_{g=1}^{n+1-p} ||\vec{\mathcal{G}}^g||
$$

(IV.42)

where

$$
||\vec{\mathcal{F}}'||^2 = \int \frac{d^2 p}{(2\pi)^2} \sum_{v',s} (\mathcal{F}_{v's}^f)^* (\mathcal{F}_{v's}^f) = \int \frac{d^2 p}{(2\pi)^2} \sum_{v',s} U_{v(f),a(f);v',s} U_{v',s;v(f),a(f)} |F_f|^2 = \int \frac{d^2 p}{(2\pi)^2} \sum_k \chi_{a(f)}^k W_{v(f),v(f)} \eta_{a(f)}^k |F_f|^2 = \int \frac{d^2 p}{(2\pi)^2} \sum_k \chi_{a(f)}^k |F_f|^2 = \int \frac{d^2 p}{(2\pi)^2} \sum_k \chi_{a(f)}^k |F_f|^2
$$

(IV.43)

as $(\chi_{a(f)}^k)^2 = \chi_{a(f)}^k$, and, as the bands in $\chi_{a}$ are adjacents, the cut-offs sum up (using equations (III.2-III.3) to give

$$
\eta_{\mathcal{C}}^a(p) = \left[ \eta \left( \frac{p^2 + m^2}{\Lambda_0(w_{M(a,\mathcal{C})})} \right) - \eta \left( \frac{p^2 + m^2}{\Lambda_0(w_{m(a,\mathcal{C})}-1)} \right) \right]
$$

(IV.44)

We can treat in the same way $\mathcal{G}$ and this achieves the proof of (IV.33). $\square$

**IV.3 Bound on the series**

We are now in the position to bound the series (III.23). After packing the attributions into packets we can put the absolute value inside the integrals and the sums and bound the product of effective constants by $c^\alpha$. Moreover, we observe that the two sums $\sum_{\text{Col},\Omega}$ in (III.23) are bounded by taking the supremum over $\text{Col}$ and $\Omega$ and multiplying by the number of elements. We have

$$
\#\{\Omega\} \leq 2^{2n+n'+n''-p} < 4^n 2^{-p}
$$

$$
\#\{\text{Col}\} \leq N^{n+1-p}
$$

(IV.45)
Indeed to estimate $\#\{\text{Col}\}$ remark that, once $T$ and $\Omega$ are known, the circulation of color indices is determined. If there are no external color indices fixed (vacuum graph), the attribution of color indices costs $N^2$ at the first four-point vertex (taken as root) and climbing inductively into the tree layer by layer a factor $N$ for each of the remaining four-point vertices of the tree (see [AR2]). The two-point vertices do not contribute as color is conserved at them. When we have fixed the $p$ independent external colors for the $2p$ external fields only $N^{n+1-p}$ choices remain.

We introduce some notations. Recalling the definitions (IV.27) and (IV.29) we say that a divergent subgraph $g_i(C) \in D_C$ is ‘D1PR’ (‘dangerous one particle reducible’) if, by cutting a single tree line, we can cut it into two subgraphs $g_j(C)$ and $g_j'(C)$, one of them, say $g_j(C)$, being a two point subgraph. The line to cut is then the tree line $l_{A(j)}$. In Figure 12 we show some examples of D1PR subgraphs, where tree lines are solid lines and loop half-lines are wavy.

![Figure 12](image)

All subgraphs that cannot be cut in this way are called D1PI (‘dangerous one particle irreducible’). We say that a four-point D1PI subgraph $g_i(C)$ is ‘open’ (as in [R]) if there exists a two-point subgraph $g_j(C) \in D_C$ (called its closure) with $j \leq_T i$ (then $g_i(C) \subset g_j(C)$) and they have two external lines in common (see Figure 13).

![Figure 13](image)
A four-point subgraph is called ‘closed’ if it is D1PI but not open. A two-point D1PI subgraph is always closed by definition. This classification of subgraphs is useful, as only closed subgraphs contribute in the product $\prod_{g \in D_C} (1 - \tau_g^*)$. Applying the definition of $\tau_g$ in the momentum space one can see that:

- if $g_i(C)$ is D1PR and $g_j(C)$ is the corresponding divergent subgraph, then
  \[
  \tau_{g_i(C)}(1 - \tau_{g_j(C)}) = 0 \tag{IV.46}
  \]
  so the renormalization of $g_i(C)$ is ensured by that of $g_j(C)$;

- if $g_i(C)$ is four-point and open, and $g_j(C)$ is the associated two-point subgraph containing it, then
  \[
  (1 - \tau_{g_j(C)})(\tau_{g_i(C)}) = 0. \tag{IV.47}
  \]

For any $g_i(C) \in D_C$ we know exactly which loop half-lines are external lines, therefore we can still apply the operator $1 - \tau_g^* = R_g^*$ to the external propagators, and distinguish closed subgraphs. Hence we define

\[
D_C^c := \{ g_i(C) \in D_C | g_i(C) \text{ closed} \} \tag{IV.48}
\]

and we apply $R_g^*$ only to $g \in D_C^c$. By the relation of partial order in $R_T$ we see that for each pair $g_i(C), g_{i'}(C) \in D_C^c$ we can only have that $g_i(C) \cap g_{i'}(C) = \emptyset$, or $g_i(C) \subseteq g_{i'}(C)$ (if $i' \leq i$). Hence $D_C^c$ has a forest structure. Following [R] we define the ‘ancestor’ of $g_i(C) \in D_C^c$, called $B(g_i(C))$, as the smallest subgraph in $D_C^c$ containing $g_i(C)$:

\[
B(g_i(C)) := g_{i'}(C), \quad i' = \max_{g_{i''}(C) \in D_C^c, g_i(C) \subseteq g_{i''}(C)} i''. \tag{IV.49}
\]

With all these bounds and definitions, the sum (III.23) becomes:

\[
|\Gamma^{\Lambda\Lambda_0}_{2p}(\phi_1, \ldots, \phi_{2p})| \leq \sum_{n,n',n''=0}^\infty N^{1-p} (cK)^\frac{\bar{n}}{n!n'!n''!} \sum_{E,C} \sum_{\phi_{i1}, \ldots, \phi_{ip}} \|
\int d^2x_1 \ldots d^2x_{\bar{n}} \int_{0 \leq w_1 \leq \ldots \leq w_{\bar{n}-1} \leq 1} \prod_{q=1}^{\bar{n}-1} dw_q \left[ \prod_{g \in D_C^c} R_g^* \left[ \prod_{q=1}^{\bar{n}-1} D_{\Lambda_0, w_q}(\bar{x}_{i_q}, x_{i_q}) \det M'(C) \phi_1(x_{i_1}) \ldots \phi_{2p}(x_{j_p}) \right] \right]
\]

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Before performing any bound we must study the action of $\prod_{g \in D_c^C} R_g^*$ on the tree propagators, the loop determinant and the external test functions. As the external half-lines for any subgraph cannot be of type $C'$ we will write $C$ instead of $D$ in the formulas. We distinguish two situations.

1) If $|eg(C)| = 4$ then $\omega(g_i(C)) = 0$ and the action of $R_g^*$ is:

$$
R_g^*(x_1) \prod_{i=1}^{4} C(x_i, y_i) := \sum_{i=2}^{4} R_{g_i}^0(x_1) \prod_{i=1}^{4} C(x_i, y_i) \quad (IV.51)
$$

$$
= C(x_1, y_1) \left[ \sum_{i=2}^{4} \prod_{2 \leq j < i} C(x_j, y_j) [C(x_i, y_i) - C(x_1, y_i)] \prod_{i<j \leq 4} C(x_1, y_j) \right]
$$

$$
= C(x_1, y_1) \left[ \sum_{i=2}^{4} \prod_{2 \leq j < i} C(x_j, y_j) [\delta_0 C(x_i, x_1, y_i)] \prod_{i<j \leq 4} C(x_1, y_j) \right]
$$

where we took as reference vertex $x_1$ and we defined $R_{g_i}^0$ as the operator that moves the external line with $i$ on the reference vertex $x_1$, and applies a difference $\delta_0 C(x_i, x_1, y_i)$ between two covariances on the line $i$.

2) If $|eg_i(C)| = 2$ then $\omega(g_i(C)) = -1$ and the action of $R_g^*$ is:

$$
R_g^*(x_1) C(x_1, y_1) C(x_2, y_2) := R_{g_1}^1(x_1) C(x_1, y_1) C(x_2, y_2)
$$

$$
= C(x_1, y_1) \left[ C(x_2, y_2) - C(x_1, y_2) - (x_2 - x_1)^{\mu} \frac{\partial}{\partial x_1^\mu} C(x_1, y_2) \right]
$$

$$
= C(x_1, y_1) [\delta_1 C(x_2, x_1, y_2)] \quad (IV.52)
$$

where we took as reference vertex $x_1$.

**IV.3.1 Choice of the reference vertex.**

Now, for each $g_i \in D_c^C$ we call the reference vertex $v_e(g_i(C))$. In this paper the choice of this vertex is adapted to the tree $T$, and is different from previous rules such as [R], chap.II. We adopt the following rule. We call the first external vertex of the graph, the one with position $x_{i_1}$ the root of the tree. We define $D_0^c$ and $D_1^c$ as the subsets of four-point and two-point subgraphs in $D_c^C$. 

35
For every subgraph $g \in D^{bc}_c$ and any vertex of $g$ there is a single path in $\mathcal{T}$ joining this vertex to the root. This path must contain a single well defined external line of $g$. The vertex to which this external line hooks is by definition our reference vertex for $g$ (see Figure 14).

![Figure 14](image)

This rule leaves us free of choosing in a different way the reference vertex for any two-point D1PI subgraph $D^{lc}_c$. The rule must ensure that open subgraphs and D1PR subgraphs are automatically renormalized by renormalization of their closure or proper parts. We decide to take as border vertex of any subgraph $g \in D^{lc}_c$ the one to which the highest of the two external half-lines of $g$ hooks. Remark that this external half-line is always a tree half-line in $\mathcal{T}$, so we know its scale. This rule, shown in Figure 15, fulfills the desired requirement, as will be shown below.

![Figure 15](image)

Finally we add a rule which is not strictly speaking necessary but simplifies the discussion: to compute the action of the renormalization operator we perform first all operations corresponding to two point subgraphs, then all operations corresponding to four point subgraphs, starting from the smallest
graphs towards the largest. This rule ensures that any external half-line of a subgraph \( g \) bearing one or two gradients because of the action of the Taylor operator for \( g \), cannot bear additional gradients from the later action of another Taylor operator for a different subgraph \( g' \).

### IV.3.2 Processes

Returning to equations (IV.51) and (IV.52), we start the renormalization for the two point subgraphs from the leaves of \( R_T \) (hence from the smallest subgraphs at highest energy) and go down. Then we perform the renormalization of four point subgraphs. Some of them may be already convergent due to renormalization of two point subgraphs. Also, even after fixing the reference vertex for each closed subgraph, there remains an arbitrary ordering of the other external lines for each four-point graph and a sum over three possible terms as shown in [IV.51]. Again some of these terms may themselves renormalize some lower four point subgraphs, so the outcome of the renormalization is difficult to capture in a single formula. We index the terms finally obtained by an index \( P \), called the process, which summarize all these choices made for the four point subgraphs. Hence

\[
\prod_{g \in D_C} (R_g) = \left( \sum_P \prod_{g \in D_C^0(P)} [R_g(P)] \right) \prod_{g \in D_C^1} R_g^1 \tag{IV.53}
\]

where \( D_C^0(P) \) is the subset of \( D_C^0 \) made of the subgraphs for which \( R_{g(C)} \neq 1 \), hence for which there is a non-trivial renormalization.

\[
R_g(P) = R_{g(i(P))}^0, \quad i(P) \in \{2, 3, 4\} \quad \text{if} \quad g \in D_C^0(P) \tag{IV.54}
\]

Hence in equation (IV.50), the absolute value inside the integrals can be bounded by:

\[
\left| \prod_{g \in D_C} R_g \left[ \prod_{q=1}^{n-1} D_{\Lambda,0}^{\alpha_{0,q}}(x_{l_q}, x_{l_q}) \det \mathcal{M}(\mathcal{C}) \phi_1(x_{i_1})...\phi_{2p}(x_{j_{2p}}) \right] \right| \leq \sum_P \prod_{q=1}^{n-1} \left| D_{\Lambda,0}^{\alpha_{0,q}}(x_{l_q}, x_{l_q}) \right| \det \mathcal{M}(\mathcal{C}) \left| \phi_1^\gamma(x_{i_1})...\phi_{2p}^\gamma(x_{j_{2p}}) \right| \tag{IV.55}
\]

where we defined \( D^\gamma, \mathcal{M}^\gamma, \phi^\gamma \) as the functions obtained after the application of \( \prod_{g \in D_C(P)} R_g(P) \). Again we bound the sum over processes \( P \) by the supremum times the number of possible processes. This number is bounded by
Indeed we recall that for any forest $\mathcal{F}$ of closed four-point subgraphs, we have $f(\mathcal{F}) \leq n-1$, where $f(\mathcal{F})$ is the number of four-point subgraphs in $\mathcal{F}$ [CR, Lemma C1]; this maximal number is not changed by adding $n' + n''$ two point vertices because of one particle irreducibility of the closed subgraphs.

From now on we work therefore with a fixed process $P$. We introduce some notations. We define $L^0_0(P)$ and $L^1_0$ as the set of loop half-lines which bear some single or double gradient respectively by some $R^0_0 g_i(P)$ or $R^1_0$ operator, $L^0 r_0(P)$ as the set of loop half-lines moved to the reference vertex by some $R^0_0 g_i(P)$ and $L^u_0(P)$ the loop half-lines left unchanged. In the same way we define the sets $T^0_0(P), T^1_0, T^r_0(P)$ and $T^u_0(P)$ for the tree half-lines, and $E^0_0(P), E^1_0, E^r_0(P)$ and $E^u_0(P)$ for the external half-lines. To avoid confusion, from now on we call $f_i$ and $\bar{f}_i$ the two half-lines forming the tree-line $l_i$.

**IV.3.3 Interpolations of the lines**

For a four-point subgraph the difference $\delta_0 C$ is expressed by

$$\delta_0 C(x, x_v, y) = \int_0^1 dt \frac{d}{dt} C(x(t), y)$$  \hspace{1cm} (IV.56)

For a two-point subgraph $\delta_1 C$ is expressed by

$$\delta_1 C(x, x_v, y) = \int_0^1 dt (1 - t) \frac{d^2}{dt^2} C(x(t), y)$$  \hspace{1cm} (IV.57)

This means that the external line hooked to $x$ has been hooked to the point $x(t)$ on any differentiable path joining $x$ to $x_v$ and has now a propagator (see Figure 16)

![Figure 16](image_url)

$$C^0(x(t), y) := \frac{d}{dt} C(x(t), y)$$  \hspace{1cm} (IV.58)
or (see Figure 17)

\[ C^1(x(t), y) := (1 - t) \frac{d^2}{dt^2} C(x(t), y). \]  

(IV.59)

Figure 17

In previous perturbative or constructive works, this path \( x(t) \) is always defined to be the linear segment connecting \( x \) to \( x_v \) hence is parametrized by

\[ x(t) := x_v + t(x - x_v) \quad x(0) = x_v \quad x(1) = x \]  

(IV.60)

But with the continuous band structure this obvious choice when applied to tree half-lines leads to difficulties. It is therefore more convenient to treat differently the loop, tree and external half-lines. Loop lines and external half-lines (except the root) do not affect spatial integration (recall that this spatial integration is performed using the decay of the tree lines). So for them we can choose the obvious linear interpolation that makes easier to factorize the matrix elements of \( M' \) as scalar products and to apply Gram’s inequality. For the tree lines it will be convenient to exploit the existence of \( T \) to choose a different path.

**IV.3.2.A: Loop lines** Now for each \( h_a \in L^0(P) \cup L^1 \) we define

\[ x_a(t) := x_{ve} + t(x_a - x_{ve}) \]  

(IV.61)

which is the obvious linear path.

The propagator for the line bearing the difference becomes respectively for a four-point and a two-point subgraph

\[ C^0(x_a(t), y) := \frac{d}{dt} C(x_a(t), y) = (x_a - x_v)^\mu \frac{\partial}{\partial x^\mu} C(x_a(t), y) \]  

(IV.62)
\[ C^1(x_a(t), y) := (1-t) \frac{d^2}{dt^2} C(x_a(t), y) = (x_a-x_v)^\mu (x_a-x_v)^\nu \frac{\partial}{\partial x^\mu} \frac{\partial}{\partial x^\nu} C(x_a(t), y). \] (IV.63)

If the second end of the line \( y \) is also moved we just apply the same formulas to \( C^0(x_a(t), y) \) and \( C^1(x_a(t), y) \), interpolating \( y \). Introducing these formulas into the matrix element \( \mathcal{M}^{\alpha \beta}_{fg} \), we can factorize it as the scalar product of \( F_f^r \) and \( G_g^r \) where

\[
\begin{align*}
F_f^r &= F_f \quad \text{for} \quad h_f \notin L^0(P) \cup L^1 \\
F_f^r &= \int dt (x_f-x_v)^\mu \frac{\partial}{\partial x^\mu} F_f(x_f(t)) \quad \text{for} \quad h_f \in L^0(P) \\
F_f^r &= \int dt (1-t)(x_f-x_v)^\mu (x_f-x_v)^\nu \frac{\partial}{\partial x^\nu} \frac{\partial}{\partial x^\mu} F_f(x_f(t)) \quad \text{for} \quad h_f \in L^1
\end{align*}
\]

where \( F_f, G_g \) are defined in (IV.32). The same definitions hold for \( G_g^r \). With these definitions the determinant is bounded by

\[ |\det \mathcal{M}^{\alpha \beta}_{fg}(C)| \leq \prod_f ||F_f^r|| \prod_g ||G_g^r|| \] (IV.65)

Now we can bound the norms using the following lemma.

**Lemma 5** The norms of \( F_f \) and \( G_g \) satisfy the bounds

\[
\begin{align*}
||F_f||_C^2 &\leq K [\Lambda_0(w_{M(f,C)}) - \Lambda_0(w_{M(f,C)-1})] \\
||G_g||_C^2 &\leq K [\Lambda_0(w_{M(g,C)}) - \Lambda_0(w_{M(g,C)-1})]
\end{align*}
\] (IV.66)

**Proof** Applying the definition (IV.44), \( ||F_f||_C^2 \) is written

\[
||F_f||_C^2 = \int \frac{d^2 p}{(2\pi)^2} \frac{1}{(p^2 + m^2)^{1/2}} \left[ \eta \left( \frac{p^2 + m^2}{\Lambda_0^2(w_{M(f,C)})} \right) - \eta \left( \frac{p^2 + m^2}{\Lambda_0^2(w_{M(f,C)-1})} \right) \right]
\]

\[
= \int \frac{d^2 p}{(2\pi)^2} (p^2 + m^2)^{1/2} \int_{\Lambda_0^{-2}(w_{M(f,C)-1})}^{\Lambda_0^{-2}(w_{M(f,C)})} d\alpha (-\eta'[p^2 + m^2 \alpha])
\]

\[
\leq \int_{\Lambda_0^{-2}(w_{M(f,C)})}^{\Lambda_0^{-2}(w_{M(f,C)-1})} d\alpha \alpha^{-\frac{3}{2}} \int_0^{\infty} dv \sqrt{v} [-\eta'(v)]
\]

\[
\leq K [\Lambda_0(w_{M(f,C)}) - \Lambda_0(w_{M(f,C)-1})]
\] (IV.67)

where, in the third line, we performed the change of variable \( v = \alpha (p^2 + m^2) \).

The same result holds for \( ||G_g||_C^2 \). \( \square \)
A similar argument can be performed for loop lines with some gradient applied. Each derivative adds a factor $\alpha^{-\frac{1}{2}}$ in the integral. With these definitions the determinant is bounded by

$$\prod_{a \in L^n(P) \cup L^e(P)} [\Lambda_0(w_M(a,C)) - \Lambda_0(w_m(a,C)-1)]^{\frac{1}{2}}$$

$$\prod_{a \in L^e(P)} |x_a - x_{v(a)}| [\Lambda_0^3(w_M(a,C)) - \Lambda_0^3(w_m(a,C)-1)]^{\frac{1}{2}}$$

$$\prod_{a \in L^1} |x_a - x_{v(a)}|^2 [\Lambda_0^5(w_M(a,C)) - \Lambda_0^5(w_m(a,C)-1)]^{\frac{1}{2}}$$

(IV.68)

**IV.3.2.B: External lines** The only external line essential in spatial integration is the root $y_1$, then we can choose this point as reference vertex for the whole graph so that it is never interpolated. For the other external lines we take again the easiest formula, the linear one. All gradients generated by moving the external lines in fact apply to the test functions. Therefore the product is bounded by

$$\prod_{h_{1e} \in E^e(P) \cup E^0(P), i_e \neq 1} ||\phi_{i_e}||_\infty \prod_{h_{1e} \in E^0(P)} |x_{i_e} - x_{v_i}| ||\phi_{i_e}'||_\infty$$

$$\prod_{h_{1e} \in E^1} |x_{i_e} - x_{v_i}|^2 ||\phi_{i_e}''||_\infty$$

(IV.69)

**IV.3.2.C: Tree lines** Now we consider tree lines.

We observe that the set of $f_i, \bar{f}_i \in T^{e_0}$ modifies the tree $T$ but does not disconnect it in the sense that it simply changes the hooking vertices of some line. On the other hand, interpolating each $f_i, \bar{f}_i \in T^{0} (P) \cup T^{1}$ with the linear rule (equation (IV.60)) in an intuitive sense “disconnects” the tree, since the point $x(t)$ in general no longer hooks to some point on a segment corresponding to a tree line. This defect would lead to difficulties when integrating over spatial positions. To avoid it we express the differences $\delta_0 C$ and $\delta_1 C$ using the connection between external vertices of any subgraph which is provided by the tree $T$ itself. But, as the tree $T$ is itself modified by renormalization, this process has to be inductive, starting from the smallest two point subgraphs of $D^{lc}$ and proceeding towards the biggest, then again from the smallest four point subgraphs of $D^{lc}(P)$ and proceeding towards the biggest.
Our induction creates progressively a new tree $T(P, J)$. To describe it, we number the subgraphs to renormalize in the order the operations are performed as $g_1, \ldots, g_r$. At the stage $1 \leq p \leq r$, before renormalization of $g_p$, the tree is called $T(P, J_{p-1})$ (we put $T(P, J_0) = T$). If the renormalization of $g_p$ as specified by the process $P$ does not act on a tree half-line external to $g_p$, we neither modify $J_{p-1}$ nor $T(P, J_{p-1})$. If the renormalization of $g_p$ results in some half-tree line $\tilde{f}_i$ or $f_i$ belonging to $T^0(P)$, as shown in Figure 18, case 1), we do not modify $J_{p-1}$, so we put $J_p = J_{p-1}$, but we modify $T(P, J_{p-1})$, that is we define $T(P, J_p)$ as $T(P, J_{p-1})$ but with the half line now hooked to the reference vertex of $g_p$.

Finally when the renormalization of $g_p$ interpolates a tree half-line $f_i$ or $\tilde{f}_i$, we modify both $J_{p-1}$ and $T(P, J_{p-1})$. There exists a unique path $\mathcal{P}_{x_f, x_v}$ joining the vertex $x_f$ where the half-line hooked to the fixed vertex of $g_p$. This path is made of $q$ lines and goes through $q + 1$ vertices with positions $x_0 = x_v, x_1, \ldots, x_q = x_f$. We interpolate the half line using this path instead of the linear path. This means that we write, if $g_p$ is a four point subgraph,

$$
\delta_0 C(x_f, x_v, y) = \sum_{j=1}^{q} \delta_0 C(x_j, x_{j-1}, y) \\
\quad = \sum_{j=1}^{q} (x_j - x_{j-1})^\mu \int_0^1 dt \frac{\partial}{\partial x_j(t)^\mu} C(x_j(t), y) \quad (IV.70)
$$

and if $g_p$ is a two point subgraph we write

$$
\delta_1 C(x_f, x_v, y) = \delta_0 C(x_f, x_v, y) - (x_f - x_v)^\mu \frac{\partial}{\partial x_v^\mu} C(x_v, y) \\
\quad = \sum_{j=1}^{q} \delta_0 C(x_j, x_{j-1}, y) - \sum_{j=1}^{q} (x_j - x_{j-1})^\mu \left[ \frac{\partial}{\partial x_j^\mu} C(x_{j-1}, y) - \sum_{k=1}^{j-1} \frac{\partial}{\partial x_k^\mu} C(x_{j-1}, y) \right] \\
\quad = \int_0^1 dt \sum_{j=1}^{q} (1 - t)(x_j - x_{j-1})^\mu(x_j - x_{j-1})^\nu \frac{\partial^2}{\partial x_j^\mu \partial x_j^\nu} C(x_j(t), y) \\
\quad + \int_0^1 dt \sum_{j=1}^{q} \sum_{k=1}^{j-1} (x_j - x_{j-1})^\mu(x_k - x_{k-1})^\nu \frac{\partial^2}{\partial x_j^\mu \partial x_k^\nu} C(x_j(t), y) \quad (IV.71)
$$
where for each $j$ (and $k \leq j$) we defined

$$x_j(t) := x_{j-1} + t(x_j - x_{j-1}) \ , \ x_k(t) := x_{k-1} + t(x_k - x_{k-1}) \ . \quad (IV.72)$$

Then we update $J$ and $T$. In the first case, where $g$ is a four point subgraph, we define $J_p = J_{p-1} \cup \{j\}$, where $j$ is the index of the line of $T(P,J_{p-1})$ chosen in (IV.71). In the second case, where $g$ is a two point subgraph, we define $J_p = J_{p-1} \cup \{j\} \cup \{k\}$, where $j$ and $k$ are the indices of the lines of $T(P,J_{p-1})$ chosen in (IV.71). Finally we update the tree according to Figure 18, case 2 and 3.

![Figure 18](image)

This means that, for $g_p$ a four point subgraph, in $\mathcal{T}(P,J_p)$ the external line hooked to $x_f$ is now hooked to the point $x_j(t)$ on the tree segment $[x_{j-1}, x_j]$ and has propagator

$$C^0(x_j(t), y) := (x_j - x_{j-1})^\mu \frac{\partial}{\partial \mu} C(x_j(t), y). \quad (IV.73)$$

For $g_p$ a two point subgraph, the external line previously hooked to $x_f$ is now hooked to the point $x_k(t)$ on the tree segment $[x_{k-1}, x_k]$ (with $k \leq j$) and has propagator

$$C^1(x_k(t), y) := (x_j - x_{j-1})^\mu (x_k - x_{k-1})^\nu \frac{\partial^2}{\partial \mu \partial \nu} C(x_k(t), y) \ k \neq j$$

$$C^1(x_j(t), y) := (1-t)(x_j - x_{j-1})^\mu (x_j - x_{j-1})^\nu \frac{\partial^2}{\partial \mu \partial \nu} C(x_j(t), y). \quad (IV.74)$$

Remark that the new tree $\mathcal{T}(P,J_p)$ has therefore one additional vertex and one line more than $\mathcal{T}(P,J_{p-1})$. The final tree built inductively in this way, $\mathcal{T}(P,J)$ is still a tree connecting all initial vertices, with at most $n - 1$
new vertices and new lines (as $n - 1$ is the maximal number of closed divergent
subgraphs).

The treatment of two or four point subgraphs and the rules for their fixed
vertex being different, we write $J = J^0 \cup J^1$, where $J^0$ is the set of indices $j$
for the interpolations associated to the renormalization of four point graphs,
and $J^1$ is the set of indices $j_g$ and $k_g$ for the interpolations associated to the
renormalization of two point graphs.

### IV.3.4 Bound on the sum over $J^1$ and on the associated distance
factors

Now, before going on, in order to reproduce the spatial decay between sup-
ports of Theorem 3, we take out a fraction $(1 - \epsilon)$ of the exponential decay
of each tree line in (II.16) and (II.17): this factor is bounded by

$$\prod_{i=1}^{\bar{n}-1} e^{-(1-\epsilon)|\bar{x}_i - x_i|\Lambda_0^m(w_i)} \leq e^{-\Lambda_m(1-\epsilon)dx_1...dx_p}$$

and we keep the remaining decay $e^{-(\epsilon/2)|\bar{x}_i - x_i|\Lambda_0^m(w_i)}$ (adjusting $\epsilon' = \epsilon/2$) for
two purposes: a fraction of this decay is used to perform spatial integration
and the other to bound the sum over $J^0$ and the distance factors generated by
equations (IV.62)-(IV.63) and (IV.73)-(IV.74). Therefore we have to bound,
for a fixed process $P$

$$\sum_{J^1} \sum_{J^0} \int dx \ A(x, J, P, \mathcal{T}) \ B(x, J, P, \mathcal{T})$$

where

$$A(x, J, P, \mathcal{T}) := \prod_{f_i, f_j \in T^1} |x_j - x_{j-1}| |x_k - x_{k-1}| \prod_{h_a \in L^1} |x_a - x_v|^2 \prod_{h_{i_e} \in E^1} |x_{i_e} - x_v|^2 \prod_{l \in T(P, J)} e^{-(\epsilon/4)|\bar{x}_l - x_l|\Lambda_0^m(w_l)}$$

$$B(x, J, P, \mathcal{T}) := \prod_{f_i, f_j \in T^0(P)} |x_j - x_{j-1}| \prod_{h_a \in L^0(P)} |x_a - x_v| \prod_{h_{i_e} \in E^0(P)} |x_{i_e} - x_v| \prod_{l \in T(P, J)} e^{-(\epsilon/4)|\bar{x}_l - x_l|\Lambda_0^m(w_l)}$$
where $J$ specifies in particular the distance factors $|x_j - x_{j-1}|$ and $|x_k - x_{k-1}|$, as explained above.

The strategy of the bound is to write

$$
\sum_{J_1} \sum_{j_0} \int dx \ A(x, J, P, T) \ B(x, J, P, T) \leq \sum_{J_1} \sum_{j_0} A(J, P, T) \int dx \ B(x, J, P, T)
$$

(IV.79)

where $A(J, P, T) := \sup_x A(x, J, P, T)$.

For each divergent subgraph $g_i \in D_c^c$ we define $t(i)$ as the index of the lowest tree line in the path of $T(P, J)$ joining $x_{v(g_i)}$ to the interpolated half-line which renormalize it. The next lemma proves that $A(J, P, T)$ is bounded by something independent of $J$:

**Lemma 6**

$$
A(J, P, T) \leq K^n \prod_{g_i \in D_c^{1c}} [\Lambda_0(w_{t(i)}) - \Lambda_0(w_{A(i)})]^{-2}
$$

(IV.80)

where $K$ is some $\epsilon$ dependent constant.

**Proof** For each loop or external line the difference $|x - x_v|$ can be bounded, applying several triangular inequalities, by the sum over the tree lines on the unique path in $T(P, J)$ connecting $x$ to $x_v$.

We observe that the same tree line $l_j$ can appear in several paths connecting different pairs of points $x_v, x$. Using the same fraction of its exponential decay many times might generate some unwanted factorials since $\sup_x x^n \exp(-x) = (n/e)^n$. To avoid this problem we define $D_j$ as the set of subgraphs $g_i \in D_c^{1c} \cup D_c^{0c}(P)$ which use the tree distance $|\bar{x}_{l_j} - x_{l_j}|$ to bound the norm of $|x - v_{x(g_i)}|$ or its square and we apply the relation

$$
e^{-\frac{2}{T}|\bar{x}_{l_j} - x_{l_j}| \Lambda_0^w(w_{l_j})} \leq e^{\frac{2}{T}|\bar{x}_{l_j} - x_{l_j}| \sum_{g_i \in D_j} [\Lambda_0(w_{t(i)}) - \Lambda_0(w_{A(i)})]}
$$

(IV.81)

With this expression a different decay factor is used for each subgraph. Applying this result and the inequalities $xe^{-x} \leq 1, x^2e^{-x} \leq 1$ completes the proof of the lemma. $\square$

It is proved in the next section that the sum and spatial integral

$$
\sum_{j_0} \int dx \ B(x, J, P, T)
$$

is in fact independent of $P$ and $J_1$ (and of the interpolation parameters $t$ that we omitted). Therefore the sum over $J_1$ will simply lead to the bound of the next section multiplied by the cardinal of the set $J_1$, that is by $|J_1|$. This is done thanks to the following lemma:
Lemma 7 We have $|J^1| \leq e^{2n}$.

Proof We consider the graphs $g_1, \ldots, g_r$ of $D^1_c$ in the order used for their renormalization in subsection IV.3.2.C. We define, for each such two point subgraph $g \in D^1_c$ the set $\mathcal{A}(g)$ of maximal subgraphs $g', g' \in D^1_c$, $g' \subset g$, and the reduced graph $g/D^1_c$ where each $g' \in \mathcal{A}(g)$ has been reduced to a single point.

We also count the number $L_g$ of lines on the unique path in the tree $T \cap g/D^1_c$ which joins the two external vertices of $g$. Remark that this number $L_g$ is independent of $J$ and that $\sum_{g \in D^1_c} L_g \leq \bar{n} - 1$. Finally we define the subset $\mathcal{A}'(g)$ of $\mathcal{A}(g)$ made of those $g'$ in $\mathcal{A}(g)$ which appear as reduced points on this unique path (see Figure 19).

By induction one can bound the number of choices in $J^1$ by

$$|J^1| \leq \prod_{g \in D^1_c} \left[ L_g + \sum_{g' \in \mathcal{A}'(g)} k_{g'} \sum_{j_{g'} = 1}^{j_g} 1 \right]$$

(IV.82)

where this product is again ordered from the smallest to the largest graphs.

Now for any positive increasing function $f$ we have $\sum_{j=1}^L f(j) \leq \int_1^{1+L} f(x)dx$ so that

$$|J^1| \leq \prod_{g \in D^1_c} \int_0^{1+L_g + \sum_{g' \in \mathcal{A}'(g)} y_{g'}} dx_g \int_0^{1+x_g} dy_g$$

(IV.83)

$$\leq \prod_{g \in D^1_c} \int_0^{1+L_g + \sum_{g' \in \mathcal{A}'(g)} y_{g'}} dx_g \int_0^{1+x_g} dy_g \leq e^{2r_1} \prod_{g \in D^1_c} e^{L_g} \leq e^{2n}$$

where in the last inequality, we bounded the last integral $\int_0^{1+x_{g_{r_1}}} dy_{g_{r_1}}$ by $e^{1+x_{g_{r_1}}}$ and then effectuate each integral exactly and bound each difference.
\[ e^z - 1 \text{ by } e^z. \] Finally we used the fact that every subgraph \( g' \) is in \( A'(g) \) for at most one \( g \), and the fact that \( 2r_1 \leq \bar{n} - 1 \) (again [CR, Lemma C1]). \( \square \)

**Remark:** This lemma does not apply to \( J^0 \). For a counter-example, the reader can look at the following graph and tree, for which \( J^0 \) can be of order \( K^n(n/5)! \).

![interpolated lines graph tree](image)

**Figure 20**

### IV.3.5 Bound on the sum over \( J^0 \), on the associated distance factors, and spatial integration of the vertices

It remains now to perform the sum over \( J^0 \) and the integral over the position of internal vertices, using the remaining tree decay in \( B \) which is \( \prod_{a \in T(P,J)} e^{-(\epsilon/4)|x_a - x_v|\Lambda_0^m(w_q)} \), and to check that the result, as announced, is independent of \( P, J \) and of the interpolation parameters \( t \). We recall that \( \Lambda_0^m(w) \) was defined in (II.18).

**Lemma 8**

\[
\sum_{J^0} \int dx B(x, J, P, T) \leq K^{\bar{n}} \prod_{g_i \in \mathcal{P}^0(P)} [\Lambda_0(w_{t(i)}) - \Lambda_0(w_{A(i)})]^{-1} \prod_{q=1}^{\bar{n}-1} (\Lambda_0^m(w_q))^{-2} 
\tag{IV.84}
\]

where \( K \) is some \( \epsilon \) dependent constant.

**Proof** First we divide one half of our remaining tree decay as in [IV.81]. This half will be used to bound each distance factor in \( \prod_{h_{a \in L^0(P)}} |x_a - x_v| \prod_{i_{e \in E^0(P)}} |x_{i_e} - x_v| \) as in Lemma 6, the sum over \( J^0 \) and the distance factors \( \prod_{f_{j} \in T^0(P)} |x_j - x_{j-1}| \).

As in Lemma 6, each distance factor in \( \prod_{h_{a \in L^0(P)}} |x_a - x_v| \prod_{h_{i_e \in E^0(P)}} |x_{i_e} - x_v| \) leads to a bound \( K[\Lambda_0(w_{t(i)}) - \Lambda_0(w_{A(i)})]^{-1} \).
Then we perform the spatial integrals from the leaves of the tree $\mathcal{T}(P, J)$ towards the root $x_1$, using the other half of the tree decay. In this inductive process when we meet an interpolated line hooked at some interpolated point $x_j(t)$ or $x_k(t)$ two different situations can occur, as pictured in Figure 21. The second situation (interpolated point not towards the root) can occur only for interpolations of two point subgraphs, hence associated to the $J^1$ indices. The first situation (interpolated point towards the root) must occur for all interpolations associated to four point subgraphs plus possibly some interpolation associated to two point subgraphs. This is the consequence of our rule for the preferred vertices of four point subgraphs (the interpolations associated to four point subgraphs always bring nearer to the root).

The sum over $J^0$ is performed in pieces; the sum over each index $j_g$ in $J^0$ is performed right after the spatial integration which has used the corresponding interpolated line. By the remark above, each sum over $j_g$ in $J^0$ occurs in the first situation of Figure 21.

![Figure 21](image)

The decay of any tree line $l_i = [x, y]$ with both ends $f_i, \bar{f}_i \in T^u(P) \cup T^{r0}(P)$ gives, by translation invariance, a factor:

$$
\int e^{-(\epsilon/8)|x-y|}\Lambda^m_{0}(w_i)\,d^2x = 128\pi\epsilon^{-2}[\Lambda^m_{0}(w_i)]^{-2}
$$

(IV.85)

Surprisingly, the same result holds when one or both ends of the line have been interpolated, as we explain now.

- In the first situation of Figure 21, $y$, the other end of the interpolated line, is connected to the root through the interpolated point (see Figure 21a). We can integrate over $y$ before integrating over $x_j$ and $x_{j-1}$. We use translation invariance to cancel the dependence from $x_j(t)$ in the interpolated covariance. The integral over the variable $t$ is bounded by 1. The spatial
integral over $y$ then gives the same factor as in (IV.85). Then we perform
the corresponding sum over $j$ in $J^0$ using
\[
\sum_{j=1}^{q} |x_j - x_{j-1}| e^{-\frac{\epsilon}{8} \sum_{j=1}^{q} |x_j - x_{j-1}|} \leq K [\Lambda_0(w_{t(i)}) - \Lambda_0(w_{A(i)})]^{-1}
\]
(IV.86)

WHen the point $x(t)$ is connected to the root through $y$ (see Figure 21b)
we have to compute the integral
\[
I = \int_0^1 dt \int d^2x_j d^2x_{j-1} e^{-(\epsilon/8)|x_j - x_{j-1}|\Lambda^m_0(w_j)} e^{-(\epsilon/8)|x_j(t) - y|\Lambda^m_0(w_k)}
\]
(IV.87)
where $x_j(t) = x_{j-1} + t(x_j - x_{j-1})$. Performing, for $t \neq 0$ the change of
variables
\[
v_1 = x_j(t) = tx_j + (1 - t)x_{j-1} \quad v_2 = \frac{1}{t}(x_j(t) - x_j(t)) = (x_{j-1} - x_j)
\]
(IV.88)
the integral becomes
\[
I = \int_0^1 dt \int d^2v_1 e^{-(\epsilon/8)|v_1|\Lambda^m_0(w_k)} \int d^2v_2 e^{-(\epsilon/8)|v_2|\Lambda^m_0(w_j)}
\]
\[
= [128\pi \epsilon^{-2}]^2 [\Lambda^m_0(w_j)]^{-2} [\Lambda^m_0(w_k)]^{-2}
\]
(IV.89)
This is again the same contribution as (IV.85), hence the same contribution
as if the line had not been interpolated!

Following the tree from the leaves to the root we can perform the integrals
over all positions in this way, except for $x_{i_1}$. This last point is integrated
using the test function $\phi_1$, which gives a factor $||\phi_1||_1$. Hence, the result of
all spatial integrations is $K^n \prod_{q=1}^{n-1} (\Lambda^m_0(w_q))^{-2}$ and the sum over $J^0$ has been
performed at a cost of $K^n$ (although $|J^0|$ can be very large). This completes
the proof of Lemma 8.

Since the result of Lemma 8 is independent of $J^1$, as announced, we can
apply Lemma 7. Let us collect the result of Lemmas 6-8, together with the
other factors remaining after spatial integration. The product over tree lines
propagators, is bounded by
\[
K^n (\Lambda^{-2} - \Lambda_0^{-2})^{n-1} e^{-(\epsilon/4)m^2\Lambda_0^{-2}(w_1)} \cdot \prod_{q=1}^{n-1} \Lambda^3_0(w_q) \prod_{f_q, j_q \in T^* \cup T^0(P)} \Lambda_0(w_{q}) \prod_{f_q, j_q \in T^1} \Lambda^2_0(w_{q})
\]
(IV.90)
where $\bar{K}$ is some $\epsilon$ dependent constant, and we included the scaling prefactors in (II.16)-(II.17). These factors are
- a factor $\Lambda_0(w)^3$ for each tree line,
- a factor $\Lambda_0(w)$ for each half-line in $T^0(P)$,
- a factor $\Lambda_0(w)^2$ for each half-line in $T^1$,
- a factor $\Lambda_0(w)$ for each half-line in $T'$, the set of half-lines hooked to a $\delta \zeta$ vertex which bears a derivative, hence has covariance $C'$.

Remark that we kept for only one line, the lowest one, the massive decay $e^{-(\epsilon/4)m^2\Lambda_0^2(w_1)}$ from bounds (II.16)-(II.17). It will be useful only to conclude the bound in the massive case when $\Lambda < m$. In this case the $n'$ vertices of type $\delta m$ may create some infrared difficulties if we were to replace directly for them the factor $(\Lambda_0^m(w_q))^{-2}$ by $(\Lambda_0(w_q))^{-2}$. We introduce the set $T'$ of the tree lines used for integration of the $\delta m$ vertices. There are $n'$ of them (or $n'-1$ if the root is of type $\delta m$, a case we will exclude for simplicity).

Recall that by (II.18) we have

\[ (\Lambda_0^m(w_q))^{-2} \leq (\Lambda_0(w_q))^{-2} \quad (IV.91) \]

\[ (\Lambda_0^m(w_q))^{-2} \leq m^{-1}(\Lambda_0(w_q))^{-1} \quad (IV.92) \]

We use the bound (IV.92) only for the lines of $T'$ when $\Lambda < m$. For all other cases we use the bound (IV.91).

**IV.3.6 Integration over the parameters $w_i$**

Now, putting everything together, we describe first the bound when $\Lambda > m$, hence $\Lambda_0^m = \Lambda$. Equation (IV.50) is bounded by

\[
|\Gamma_{2p}^\Lambda(\phi_1, \ldots, \phi_{2p})| \leq ||\phi_1|| \prod_{i=2}^{2p} ||\phi_i|| \leq N^{1-p} e^{-(1-\epsilon)\Lambda m d_P(\Omega_1, \ldots, \Omega_{2p})} (\Lambda^{-2} - \Lambda_0^{-2})^{\bar{n}-1} \sum_{n,n',n''=0}^\infty (eK)^n \frac{1}{n!n'n''!} \sum_{o=E,C} \int_{0 \leq w_1 \leq \ldots \leq w_{\bar{n}-1} \leq w_{\bar{n}}} \prod_{q=1}^{\bar{n}-1} dw_q \prod_{q=1}^{\bar{n}} \Lambda_0(w_q) \prod_{f_q, \bar{f}_q \in T' \cup T^0(P)} \Lambda_0(w_q) \prod_{f_q, \bar{f}_q \in T^1} \Lambda_0^2(w_q)
\]
The differences \[ 1 - \frac{\Lambda_0(w_{A(i)})}{\Lambda_0(w_{l(i)})} \] are dangerous as they appear with a negative exponent. They are the price to pay for implementing continuous renormalization group. Indeed, in this continuous formalism one has to perform renormalization even when the differences between internal and external energies of subgraphs are arbitrarily small. However, there is a natural solution to this problem: each subgraph to renormalize has necessarily loop lines and these loop lines, when evaluated in the continuous formalism by Gram’s inequality, give small factors precisely when the differences between internal and external energies of subgraphs become arbitrarily small.

In other words, we can cancel the dangerous differences with a negative exponent against the analogous differences with a positive exponent given by the loop lines. This is the purpose of the next lemma.

**Lemma 9** If \( g_i \in D^0_c(P) \) there is at least one loop line internal to \( g_i \) which satisfies \( \Lambda_0(w_{M(a,C)}) \leq \Lambda_0(w_{l(i)}) \) and \( \Lambda_0(w_{m(a,C)}) \geq \Lambda_0(w_{A(i)}) \). If \( g_i \in D^1_c \) there are at least two loop lines internal to \( g_i \) and which satisfy \( \Lambda_0(w_{M(a,C)}) \leq \Lambda_0(w_{l(i)}) \) and \( \Lambda_0(w_{m(a,C)}) \geq \Lambda_0(w_{A(i)}) \).

Assuming the lemma true, and using the relations \( \sqrt{\frac{1-x^2}{1-x}} \leq \sqrt{3} \) and \( \sqrt{\frac{1-x^2}{1-x}} \leq \sqrt{5} \), one obtains

\[
\prod_{a \in L^0(P) \cup L^0,0(P)} \Lambda_0^\frac{1}{2}(w_{M(a,C)}) \left[ 1 - \frac{\Lambda_0(w_{m(a,C)})}{\Lambda_0(w_{M(a,C)})} \right]^{\frac{1}{2}}
\]

\[
\prod_{a \in L^0(P)} \Lambda_0^\frac{3}{2}(w_{M(a,C)}) \left[ 1 - \frac{\Lambda_0^3(w_{m(a,C)})}{\Lambda_0^3(w_{M(a,C)})} \right]^{\frac{1}{2}}
\]

\[
\prod_{g \in D^0_c} \Lambda_0^{-1}(w_{l(i)}) \left[ 1 - \frac{\Lambda_0(w_{A(i)})}{\Lambda_0(w_{l(i)})} \right]^{-1}
\]

\[
\prod_{g \in D^1_c} \Lambda_0^{-2}(w_{l(i)}) \left[ 1 - \frac{\Lambda_0(w_{A(i)})}{\Lambda_0(w_{l(i)})} \right]^{-2}
\]

(IV.93)
\[
\prod_{a \in L^1} \left[ 1 - \frac{\Lambda_0^c(w_{m(a,c)} - 1)}{\Lambda_0^c(w_M(a,c))}\right]^{1/2} \prod_{g_i \in D_c^0} \left[ 1 - \frac{\Lambda_0(w_A(i))}{\Lambda_0(w_t(i))}\right]^{-1}
\] 
\[
\prod_{g_i \in D_c^1} \left[ 1 - \frac{\Lambda_0(w_{A(i)})}{\Lambda_0(w_t(i))}\right]^{-2} \leq 5^{n-1}
\]  \hspace{1cm} (IV.94)

where we bounded by 1 the loop lines differences that were not used to compensate some \(1 - \frac{\Lambda_0(w_{A(i)})}{\Lambda_0(w_t(i))}\) factor.

**Proof of Lemma 9** We observe that the lowest tree line \(l_{t(i)}\) in \(T_i(J,P)\) joining the interpolated line and the reference vertex is external line for the two subgraphs of \(g_i\), \(g_{t(i)1}\) and \(g_{t(i)2}\). One of these two subgraphs has for external line the reference external line of \(g_i\) and the other has for external line the interpolated line moved by the renormalization \(R^*\). But \(g_{t(i)1}\) and \(g_{t(i)2}\) must both have at least some additional external lines, otherwise \(g_i\) would be D1PR. By parity \(g_{t(i)1}\) and \(g_{t(i)2}\) must both have at least two such additional external lines.

We distinguish two cases:

- If \(g_i \in D_c^0(P)\), since there are at most two additional external lines of \(g_i\), we find that there must be at least two external half-lines of \(g_{t(i)1} \cup g_{t(i)2}\) different from \(l_{t(i)}\) which are internal in \(g_i\). If they are both loop half-lines we are done. If some of them is a tree half-line, the other half is external line for another subgraph of \(g_i\), \(g'\). Repeating the argument for \(g'\) (as \(|e| = 1\) is forbidden) we finally must find an associated loop half-line (see Figure 12).

- If \(g_i \in D_c^1\), since there was no additional external line of \(g_i\), then both \(g_{t(i)1}\) and \(g_{t(i)2}\) must have at least two external half-lines different from \(l_{t(i)}\) which are internal in \(g_i\). Either these four half-lines are loop half-lines and we are done, or some of them are tree lines, which we follow as above until we find the corresponding loop half-lines. \(\square\)

After applying the bound (IV.94) we can take the limit \(\Lambda_0 \rightarrow \infty\). Performing the change of variable \(u_i = 1 - w_i\) equation (IV.93) becomes:

\[
|\Gamma_{2p}(\phi_1, \ldots \phi_{2p})| \leq ||\phi_1||_1 \prod_{i=2}^{2p} ||\phi_i||_\infty 2^{e-(1-\epsilon)} \Lambda^m d_T(\Omega_1, ..., \Omega_{2p})
\]
\[
N^{1-p} \sum_{n, n', n''} \Lambda^{2-p-n'}(cK)^n \frac{1}{n! n'! n''!} \sum_{o-T E, C} \int_{0 \leq u_n \leq \cdots \leq u_1 \leq 1} \Pi_{q=1} u_q^\bar{n} d\bar{u}_q
\]

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hence is bounded away from 0. Then the integral (IV.95) becomes

$$\left[ \prod_{q=1}^{n-1} \frac{1}{\sqrt{u_q}} \right] \left[ \prod_{f_q, f_{\tilde{q}} \in T \cup T^0(P)} \frac{1}{\sqrt{u_q}} \right] \left[ \prod_{f_q, f_{\tilde{q}} \in T^1} \frac{1}{u_q} \right] \left[ \prod_{a \in L^u(P) \cup L^r(P)} (u_{M(a, c)})^{-\frac{1}{2}} \right] \left[ \prod_{a \in L^0(P)} (u_{M(a, c)})^{-\frac{1}{2}} \right] \left[ \prod_{a \in L^1} (u_{M(a, c)})^{-\frac{1}{2}} \right]$$

$$\left[ \prod_{g_i \in D_c^{\bar{c}}} \left( u_{t(i)} \right)^{\frac{1}{2}} \right] \left[ \prod_{g_i \in D_c^{\bar{c}}} \left( u_{t(i)} \right) \right]$$

(IV.95)

To factorize the integrals we perform the change of variable:

$$u_i = \beta_i u_{i-1} \quad \beta_i \in [0, 1]$$

(IV.96)

where by convention $u_0 = 1$. The Jacobian of this transformation is the determinant of a triangular matrix hence it is given by:

$$J = \beta_1 (\beta_1 \beta_2) ... (\beta_1 \beta_2 ... \beta_{n-2}) = \prod_{i=1}^{n-1} \beta_i^{n-1-i}.$$  (IV.97)

We absorb $\Lambda^{-n'}$ into the term $K^n$ since we recall that $\Lambda > m$ hence that $\Lambda^{-n'} = (\Lambda^m)^{-n'}$, and that in Theorem 3 $\Lambda^m$ remains in the compact $X$, hence is bounded away from 0. Then the integral (IV.95) becomes

$$|\Gamma_{2p}(\phi_1, ..., \phi_{2p})| \leq ||\phi_1||_1 \prod_{i=2}^{2p} ||\phi_i||_{\infty, 2} e^{(1-\epsilon)\Lambda^m} d \Omega_1 ... d \Omega_{2p}$$

(IV.98)

$$\left( \Lambda^m \right)^{2-p} N^{1-p} \sum_{n, n', n''=0}^{\infty} \frac{(cK)^n}{n! n'! n''!} \sum_{o, E, C} \int_0^1 ... \int_0^{n-1} \prod_{i=1}^{n-1} d \beta_i$$

$$\prod_{f_q, f_{\tilde{q}} \in T^0(P)} \left( \frac{1}{\sqrt{\beta_{q ... \beta_1}} \sqrt{\beta_{q ... \beta_1}} \sqrt{\beta_{q ... \beta_1}} \sqrt{\beta_{q ... \beta_1}}} \right)$$

$$\prod_{a \in L^u(P) \cup L^r(P)} (\beta_{M(a, c)} ... \beta_1)^{-\frac{1}{2}} \prod_{a \in L^0(P)} (\beta_{M(a, c)} ... \beta_1)^{-\frac{1}{2}} \prod_{a \in L^1} (\beta_{M(a, c)} ... \beta_1)^{-\frac{1}{2}}$$

$$\prod_{g_i \in D_c^{\bar{c}}} \left( \beta_{t(i)} ... \beta_1 \right)^{\frac{1}{2}} \prod_{g_i \in D_c^{\bar{c}}} \left( \beta_{t(i)} ... \beta_1 \right)^{\frac{1}{2}} \prod_{g_i \in D_c^{\bar{c}}} \left( \beta_{t(i)} ... \beta_1 \right)^{\frac{1}{2}}$$

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Each $\beta_i$ appears with the exponent $-1 + x_i$.

$$x_i = \frac{1}{2} [\bar{n} - i] - \frac{1}{2} |N_i^0| - \frac{1}{4} |IL_i(C)|$$

$$+ \frac{1}{2} [|S_i^0(C)| - |IT_i^0| - |IL_i^0(C)|] + [|S_i^1(C)| - |IT_i^1| - |IL_i^1(C)|]$$

(IV.99)

where we defined

$$IT_i^0 := \{ f_j, \bar{f}_j \in T^0(P) | j \geq i \}$$

$$IT_i^1 := \{ f_j, \bar{f}_j \in T^1 | j \geq i \}$$

$$IL_i(C) := \{ a \in L | M(a, C) \geq i \}$$

$$IL_i^0(C) := \{ a \in L^0(P) | M(a, C) \geq i \}$$

$$IL_i^1(C) := \{ a \in L^1 | M(a, C) \geq i \}$$

$$S_i^0(C) := \{ g_j \in D_{\mathcal{C}}^{bc}(P) | t(j) \geq i \}$$

$$S_i^1(C) := \{ g_j \in D_{\mathcal{C}}^{bc} | t(j) \geq i \}.$$ (IV.100)

c(i) is the number of connected components in $T_i(P)$. All these definitions can be restricted to the connected components: $IT_i^{0k}, IT_i^{1k}, IL_i^{1k}(C), IL_i^{0k}(C)$ $IL_i^{1k}(C)$, $S_i^{0k}(C)$ and $S_i^{1k}(C)$. We observe that $IL_i(C)$ corresponds to the set of half-lines that could have, in the class $C$, $\mu(a) \geq i$ and it is the equivalent of $IL_i$ defined in (III.9). $IT_i^0$ (respectively $IT_i^1$) and $IL_i^0(C)$ (respectively $IL_i^1(C)$) are the set of tree half-lines and loop half-lines at a level higher or equal to $i$, which are the interpolated external lines for some divergent subgraph in $D_{\mathcal{C}}^{bc}(P)$ (respectively in $D_{\mathcal{C}}^{bc}$), $S_i^0(C)$ (respectively $S_i^1(C)$) is the set of subgraphs in $D_{\mathcal{C}}^{bc}(P)$ (respectively in $D_{\mathcal{C}}^{bc}$) that have the internal tree line $l_{t(j)}$ of a level higher or equal to $i$. In the same way, we can define the equivalent of $EL_i$ and $E_i$ as

$$EL_i(C) := \cup_{k=1}^{c(i)} e_{l_i^k}(C)$$ (IV.101)

which is the set of loop half-lines that are forced to have $\mu(a) \leq i$, and

$$E_i(C) := \cup_{k=1}^{c(i)} e_{g_i^k}(C).$$ (IV.102)

The integral in the variable $d\beta_i$ can be performed only if the exponent of $\beta_i$ is bigger than $-1$. Using the relations

$$\bar{n} - i = \sum_{k=1}^{c(i)} |N_i^k| + |N_i'^k| + |N_i^{nk}| - 1$$
\[|E_i(C)| = |EL_i(C)| + |ET_i| + |EE_i| = 2|N_i| + 2 - |IL_i(C)|, \]

(IV.103)

the exponent of \(\beta_i\) can be written as \(-1 + \sum_{k=1}^{\sigma_i} x_i^k\), where

\[
x_i^k := \frac{1}{2} \left[ \frac{1}{2}(|E_i^k(C)| + 2|N_i^k| - 4) + [S_i^{0k}(C)| - |IT_i^{0k}(C)|] + 2[S_i^{1k}(C)| - |IT_i^{1k}| - |IL_i^{1k}(C)|] \right]
\]

(IV.104)

Remark that for any level \(i\) we have

\[
[S_i^0(C)| - |IT_i^0| - |IL_i^0(C)|] \geq 0 \quad [S_i^1(C)| - |IT_i^1| - |IL_i^1(C)|] \geq 0 \quad (IV.105)
\]

as each half-line in \(IT_i^0 (IT_i^1)\) or \(IL_i^0(C) (IL_i^1(C))\) is the external interpolated line for a subgraph \(g_j\). This subgraph \(g_j\) must have \(j > i\) hence have \(t(j) > i\).

Therefore for each half-line in one of these sets there is always at least one corresponding half-line in \(S_i^0(C) (S_i^1(C))\).

**Lemma 10** For any connected component in \(T_i^k\) we have \(x_i^k \geq 1/2\).

**Proof** We distinguish three situations.

- If \(|E_i^k(C)| \geq 5\), in fact, by parity of the number of external half-lines of any subgraph, \(|E_i^k(C)| \geq 6\) and then

\[
x_i^k \geq (1/4)(|E_i^k(C)| - 4) \geq 1/2.
\]

(IV.106)

- If \(|E_i^k(C)| = 4\), then there must be a subgraph \(g_j \in \mathcal{D}_c^{0k}(P)\) with \(j \geq i\) (\(j = i\) only if \(l_i\) belongs to the connected component \(T_i^k(J,P)\)) and \(A(j) < i\). Hence the interpolated line for \(g_j\) does not belong to \(IT_i^{0k}\) or \(IL_i^{0k}(C)\), but the corresponding internal line \(l_{(j)}\) belongs to \(S_i^{0k}\). Then

\[
[S_i^{0k}| - |IT_i^{0k}| - |IL_i^{0k}(C)| \geq 1 \quad (IV.107)
\]

and

\[
x_i^k = \frac{1}{2}[|N_i^k| + [S_i^{0k}| - |IT_i^{0k}| - |IL_i^{0k}(C)|] + 2[S_i^{1k}| - |IT_i^{1k}| - |IL_i^{1k}(C)|]\geq \frac{1}{2}.
\]

(IV.108)
Finally if $|E^k_i(C)| = 2$ one can see, by the same arguments, that
\[ |S^1_{ik}| - |IT^1_{ik}| - |IL^1_{ik}(C)| \geq 1 \quad (IV.109) \]
and
\[ x^k_i = \frac{1}{2}[-1 + |N^k_i| + |S^0_i| - |IT^0_i| - |IL^0_i(C)|] + 2[|S^1_i| - |IT^1_i| - |IL^1_i(C)|] \]

\[ \geq [-1 + 2|S^1_i| - |IT^1_i| - |IL^1_i(C)|] \geq \frac{1}{2}. \quad (IV.110) \]

Now we can perform the integrals in equation (IV.98) and we obtain
\[ |\Gamma^\Lambda_{2p}(\phi_1, \ldots, \phi_{2p})| \leq ||\phi_1||_1 \prod_{i=2}^{2p} ||\phi_i||_\infty \cdot 2e^{-(1-\epsilon)\Lambda m d_T(\Omega_1, \ldots, \Omega_{2p})} \quad (IV.111) \]

\[ (\Lambda^m)^{2-p} N^{1-p} \sum_{n,n',n''=0}^{\infty} (eK)^n \frac{1}{n!n'!n''!} \sum_{u-T} \sum_{\sigma} \sum_{E,C} \prod_{i=1}^{\hat{n}-1} \frac{1}{\sum_{k=1}^{c(i)} \beta_i^k} \]

where we wrote the sum over ordered trees as sum over unordered trees and sum over all possible orderings $\sigma$ of the tree. The sum $\sum_{\sigma}$ is over a set whose cardinal is bounded by $K^n$ so it’s sufficient to bound them with the supremum over this set, as we are interested in a theorem at weak coupling $\lambda$. However the sum over $E$ to attribute the $2p$ external lines to particular vertices runs over a set of at most $\hat{n}^{2p}$ (this is an overestimate!). This will lead to the factorial in Theorem 3. We remark however that a better bound on the behaviour of the vertex functions at large $p$ can presumably be obtained when the external points are sufficiently spread (not too closely packed), but we leave this improved estimate to a future study.

Moreover, we bound $\frac{1}{(\alpha)!} \sum_{u-T} f(T)$ by $\frac{\hat{n}^{\alpha-2}}{n!} \sup_{u-T} |f(T)|$ using Cayley’s theorem. Therefore, by Stirling’s formula, it’s enough to consider the unordered tree $T$ which gives the max$_{u-T} |f(T)|$. The sum that could still give some factorial is $\sum_{\sigma}$. To bound it we use the product of fractions obtained after integration on the $\beta_i$.

- if $|ET^k_i| \geq 5$ we have
\[ (|ET^k_i| + |EE^k_i| + |EL^k_i(C)| - 4) / 4 \geq (|ET^k_i| - 4) / 4 \geq \frac{|ET^k_i| + 1}{24} \quad (IV.112) \]
• if $|ET_i^k| < 5$ we have

$$x_i^k \geq 1/2 \geq \frac{|ET_i^k| + 1}{24} \quad (IV.113)$$

Now $|ET_i|$ depends on the (now unordered) tree $T$ and on its ordering $\sigma$. Therefore we call it from now on $|ET_i^\sigma|$. Recall that it is the total number of external tree half-lines of the subset $T_i^\sigma$ of $T$ made of the $\bar{n} - i$ highest lines in the permutation $\sigma$. Since $\sum_k (|ET_i^k| + 1) \geq |ET_i^\sigma| + 1$, equation (IV.111) becomes

$$|\Gamma_{2p}(\phi_1, \ldots, \phi_{2p})| \leq ||\phi_1||_1 \prod_{i=2}^{2p} ||\phi_i||_{\infty, 2} e^{-(1-\epsilon)\Lambda^m d_T(\Omega_1, \ldots, \Omega_{2p})}$$

$$(\Lambda^m)^{2-p} N^{1-p} \sum_{n,n',n''=0}^{\infty} \bar{n}^{2p} (cK)^{n+n'+n''} \sum_{\sigma} \prod_{i=1}^{\bar{n}-1} \frac{1}{|ET_i^\sigma| + 1} \quad (IV.114)$$

At this point we can apply a result of [CR] (Lemma A.1, B.3, B.4) which states that for any tree we have

$$\sum_{\sigma} \prod_{i=1}^{\bar{n}-1} \frac{1}{|ET_i^\sigma| + 1} \leq 4^\bar{n} \quad (IV.115)$$

For completeness let us recall the proof of this result. For each tree $T$ we can define a mapping $\xi$ of $T$ on a chain-tree with the same number of vertices:

$$\xi : T \rightarrow \xi_T \quad (IV.116)$$

To define $\xi$, we turn around $T$ starting from an arbitrary end line, and we number the lines in the order we meet them for the first time. The lines of $\xi_T$ are numbered in the same way and $\xi_T$ associates the lines with the same number.
Now we observe that the sum over the orders on $\mathcal{T}$ corresponds to the sum over all permutations of the indices in $\xi(\mathcal{T})$. Moreover Lemma B.3 in [CR] proves that for any connected or disconnected subgraph $R$ of $\mathcal{T}$, we have

$$ET(R) + 1 \geq c(\xi(\mathcal{T}))$$

(IV.117)

where $c(\xi(\mathcal{T}))$ is the number of connected components of the image of $R$ $\xi(\mathcal{T})$ and $ET(R)$ is the number of external half-lines of $R$ in $\mathcal{T}$. Finally we note that $\xi(T_i)$ is the set of lines with number $j \geq \bar{n} - i$ so we can write

$$\sum_{\sigma} \prod_{i=1}^{\bar{n}-1} \frac{1}{ET_{\sigma}^i} + 1 = \sum_{\sigma} \prod_{i=1}^{\bar{n}-1} \frac{1}{c(D_{\sigma}^i)} := \Delta_{\bar{n}} ,$$

(IV.118)

where $D_{\sigma}^i$ is the set of lines in the chain-tree $\xi(\mathcal{T})$, that have $\sigma(j) \geq \bar{n} - i$ (after the permutation $\sigma$). Now, applying Lemma B.4. in [CR], we obtain

$$\Delta_{\bar{n}} \leq 4^{\bar{n}}$$

(IV.119)

We recall that this can be proved by remarking that $\Delta_{\bar{n}}$ satisfies the inductive equation

$$\Delta_{\bar{n}} = \sum_{k=1}^{\bar{n}-1} \Delta_p \Delta_{\bar{n}-k} ,$$

(IV.120)

so that equation (IV.114) becomes

$$|\Gamma_{2p}^\Lambda(\phi_1, \ldots \phi_{2p})| \leq ||\phi_1||_1 \prod_{i=2}^{2p} ||\phi_i||_\infty 2e^{-(1-\epsilon)\Lambda_m d_T(\Omega_1, \ldots \Omega_{2p})}$$

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\[(\Lambda^m)^{2-p}N^{1-p} \sum_{n,n',n''=0}^{\infty} \bar{n}^{2p}(4cK)^{\bar{n}}\]  
(IV.121)

where \(K\) depends only on \(\epsilon\). Taking \(c\) small enough completes the proof of the theorem in the case \(\Lambda > m\), since \(\sum_{\bar{n}} \bar{n}^{2p}e^{-\bar{n}} \leq K^p(p!)^2\).

In the case \(\Lambda < m\), we have a few changes to perform. Replacing the lines of \(T'\) in (IV.90) by the bound (IV.92), keeping the massive decay factor \(e^{-(\epsilon/4)m^2\Lambda_0^2(w_1)}\) in (IV.90) and passing to the limit \(\Lambda_0 \to \infty\) we have the following changes: in (IV.95) we add the factors

\[(\Lambda/m)^{n'} \prod_{\ell_q \in T'} (u_q)^{-1/2} e^{-(\epsilon/4)m_1m^2\Lambda^{-2}}\]  
(IV.122)

The factor \((\Lambda/m)^{n'}\) exactly changes \(\Lambda^{2-p-n'}\) into \(\Lambda^{2-p}m^{-n'} = \Lambda^{2-p}(\Lambda^m)^{-n'}\). The factor \((\Lambda^m)^{-n'}\) is absorbed in \(K^{\bar{n}}\) since \(\Lambda^m\) in the hypothesis of Theorem 3 remains in the compact \(X\). Passing to the variables \(\beta_i\), the factor \([\prod_{\ell_q \in T'} (u_q)^{-1/2}]\) is bounded by the factor \(\prod_i |\beta_i|^{N'k_i}|\) in (IV.104), which was previously bounded by 1, hence not used at all. Finally the last integral over \(\beta_1\) becomes bounded, for \(p > 2\) by:

\[\Lambda^{2-p} \int_0^1 \beta_1^{(p-2)/2} \frac{d\beta_1}{\beta_1} e^{-(\epsilon/4)m^2\Lambda^{-2}}\]  
(IV.123)

Changing to the variable \(v = (\epsilon/4)m^2\beta_1\Lambda^{-2}\) we obtain for the final bound a factor

\[\left(4/\epsilon m^2\right)^{(p-2)/2} \int_0^{em^2\Lambda^{-2}/4} v^{(p-2)/2} \frac{dv}{v} e^{-v} \leq (\Lambda^m)^{2-p}K^p \sqrt{p!}\]  
(IV.124)

The case \(p = 2\) is easy and left to the reader. Hence Theorem 3 holds in every case, by combining the factor \(\sqrt{p!}\) with the factor \((p!)^2\) coming from the sum over \(E\). Remark that in the case \(m = 0\) we have never \(\Lambda < m\), hence the factor \((p!)^{5/2}\) can be replaced by \((p!)^2\).

V The Renormalization Group Equations

In this section we establish the renormalization group equations obtained when varying \(\Lambda\) and we check that for a fixed and small renormalized coupling constant, the effective constants remain bounded and small as predicted by
the well known perturbative analysis of the model, which is asymptotically free in the ultraviolet regime [MW].

The derivative \( \frac{\partial}{\partial \Lambda} \Gamma_{2p}^{\Lambda \Lambda_0}(\phi_1, ..., \phi_{2p}) \) can be written, using the expression (III.23), as:

\[
\frac{\partial}{\partial \Lambda} \Gamma_{2p}^{\Lambda \Lambda_0}(\phi_1, ..., \phi_{2p}) = T \Gamma_{2p}^{\Lambda \Lambda_0}(\phi_1, ..., \phi_{2p}) + L \Gamma_{2p}^{\Lambda \Lambda_0}(\phi_1, ..., \phi_{2p}).
\]

(V.125)

The first term \( T \Gamma_{2p}^{\Lambda \Lambda_0}(\phi_1, ..., \phi_{2p}) \) is the series obtained when the derivative falls on a tree line propagator (see Figure 23a):

\[
\begin{align*}
T \Gamma_{2p}^{\Lambda \Lambda_0}(\phi_1, ..., \phi_{2p}) &= \sum_{n,n',n''=0}^{\infty} \frac{1}{n!n'!n''!} \sum_{\mathcal{O}, \mu} \sum_{\mathcal{C}, \Omega} \epsilon(T, \Omega) \int d^2x_1...d^2x_n \\
&\quad \int_{0 \leq w_1 \leq \cdots \leq w_{n-1} \leq 1} \prod_{q=1}^{n-1} dw_q \left[ \prod_{v} \left( \frac{\lambda_{w(v)}}{N} \right) \right] \left[ \prod_{v' \neq v} \delta m_{w(v')} \right] \left[ \prod_{v''} \delta \zeta_{w(v'')} \right] \\
&\quad \prod_{G^i_t \in D_\mu} R_{G^i_t} \left[ \sum_{q=1}^{n-1} \frac{\partial}{\partial \Lambda} D_{\Lambda_0, w_q}(\bar{x}_{l_q}, x_{l_q}) \right] \\
&\quad \prod_{q \neq q'} D_{\Lambda_0, w_q}(\bar{x}_{l_q}, x_{l_q}) \det \mathcal{M}(\mu) \phi_1(x_{i_1})...\phi_{2p}(x_{j_p})
\end{align*}
\]  

(V.126)

\( \text{Figure 23} \)

The second term \( L \Gamma_{2p}^{\Lambda \Lambda_0}(\phi_1, ..., \phi_{2p}) \) is the series obtained when the derivative falls on a loop line in the determinant (see Figure 23b):

\[
\begin{align*}
L \Gamma_{2p}^{\Lambda \Lambda_0}(\phi_1, ..., \phi_{2p}) &= \sum_{n,n',n''=0}^{\infty} \frac{1}{n!n'!n''!} \sum_{\mathcal{O}, \mu} \sum_{\mathcal{C}, \Omega} \epsilon(T, \Omega) \int d^2x_1...d^2x_n \\
&\quad \int_{0 \leq w_1 \leq \cdots \leq w_{n-1} \leq 1} \prod_{q=1}^{n-1} dw_q \left[ \prod_{v} \left( \frac{\lambda_{w(v)}}{N} \right) \right] \left[ \prod_{v' \neq v} \delta m_{w(v')} \right] \left[ \prod_{v''} \delta \zeta_{w(v'')} \right] \\
&\quad \prod_{G^i_t \in D_\mu} R_{G^i_t} \left[ \prod_{q=1}^{n-1} D_{\Lambda_0, w_q}(\bar{x}_{l_q}, x_{l_q}) \phi_1(x_{i_1})...\phi_{2p}(x_{j_p}) \right]
\end{align*}
\]
where $\epsilon(f,g)$ is a sign coming from the development of the determinant. The convergence proofs of course extend to both terms of equation (V.125). Indeed, in the first one, the sum over the tree lines is bounded by a factor $\bar{n}$, and in the second one the sum is over the set of loop half-lines which is bounded by a factor $\bar{n}^2$. Therefore these sums cannot generate any factorial. Then we obtain the same bound as in (IV.114), with an additional factor $1/\Lambda$. This factor disappears when, as usual, the renormalization group equations are written as derivatives with respect to $\log \Lambda$ rather than $\Lambda$.

From these equations one can derive also equations for the flow of the effective constants defined in (III.24). For instance to obtain the flow of the effective coupling constant $\lambda$ which is the four-point vertex function at zero external momenta, we can use equations (V.125)-(V.127) in which we let $\phi_1 \to \delta(0), \phi_2, \phi_3, \phi_4 \to 1$. This is compatible with our $L_1-L_\infty$ bounds, so that everything remains bounded. We obtain in this way the famous continuous flow equation which gives the derivative of the coupling constant with respect to $\log \Lambda$:

$$\frac{\partial}{\partial \log \Lambda} N\hat{\Gamma}_4^{A}(0,0,0,0) = \frac{\partial}{\partial \log \Lambda} \lambda_{\Lambda} = \beta_2 \lambda_{\Lambda}^2 + O(c^3) + \lambda_{\Lambda}^2 O(\Lambda^{-\alpha}) \quad (V.128)$$

where

$$\beta_2 = -2(N-1)/\pi \quad (V.129)$$

is the first non trivial term corresponding to the four-point graph with one tree line and one loop line, and the last term $\lambda_{\Lambda}^2 O(\Lambda^{-\alpha})$ is an infrared correction to the asymptotic flow (see [FMRS]). The negative sign of $\beta_2$ is responsible for the asymptotic freedom of the model. Similar equations hold for the flow of $\delta m$ and $\delta \zeta$ (which remain bounded). For these equations up to one loop, see [MW] [GN] [GK] [FMRS]. For the computation up to two loops, we refer to [W].

From these renormalization group equations one can control the behavior of the effective constants and check that they remained bounded (until now this was assumed). The reader might be afraid that there is something circular in this argument. In fact this is not the case. Let us discuss for simplicity the massless case where the renormalized coupling $\hat{\Gamma}_4^{A\Lambda_0}(0,0,0,0)$
is only a function of $\Lambda_0/\Lambda$ and of the bare coupling $\lambda$. We know that it is analytic at the origin as function of the bare coupling $\lambda$ [AR2]. Therefore from (V.125)-(V.128) it is for small bare $\lambda$ and $\Lambda_0/\Lambda$ a monotone increasing function of the ratio $\Lambda_0/\Lambda$ (although this function might blow up in finite time).

Inverting the map from bare to renormalized couplings, one can prove that conversely for small renormalized coupling $\hat{\Gamma}^{\Lambda_0}(0,0,0,0)$ all the effective constants $\lambda_w$ remain bounded by the renormalized one. Therefore one can pass to the ultraviolet limit $\Lambda_0 \to \infty$, in analogy with the completeness of flows of vector fields on compact manifolds. Furthermore one can compute the asymptotic behavior of the bare coupling which tends to 0 as $1/(|\beta_2| \log(\Lambda_0/\Lambda))$. Similar arguments hold for the mass and wave function effective constants and achieve the proof of Theorem 1.

We recall for completeness that it is easy to build the Schwinger functions from the vertex functions and that the Osterwalder-Schrader axioms of continuous Euclidean Fermionic theories hold for the massive Gross-Neveu model at $\Lambda = 0$. The simplest proof is to remark that being the Borel sum of the renormalized expansion, the Schwinger functions we build are unique. Building them as limits of theories with different kinds of cutoffs prove the axioms since different sets of cutoffs violate different axioms [FMRS].

Acknowledgements

It is a pleasure to thank D. Brydges, C. de Calan, H. Knörrer, C. Kopper, G. Poirrot and M. Salmhofer for discussions related to this paper. In particular we thank C. Kopper for his careful reading of this preprint, which lead us to correct the initial version of subsections IV.3.4 and IV.3.5.

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