DEDUCING THE MULTIDIMENSIONAL
SZEMERÉDI THEOREM FROM AN
INFINITARY REMOVAL LEMMA

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Abstract

We offer a new proof of the Furstenberg-Katznelson multiple recurrence theorem for several commuting probability-preserving transformations $T_1, T_2, \ldots, T_d : \mathbb{Z} \rightarrow (X, \Sigma, \mu)$ ([6]), and so, via the Furstenberg correspondence principle introduced in [5], a new proof of the multi-dimensional Szemerédi Theorem. We bypass the careful manipulation of certain towers of factors of a probability-preserving system that underlies the Furstenberg-Katznelson analysis, instead modifying an approach recently developed in [1] to pass to a large extension of our original system in which this analysis greatly simplifies. The proof is then completed using an adaptation of arguments developed by Tao in [13] for his study of an infinitary analog of the hypergraph removal lemma. In a sense, this addresses the difficulty, highlighted by Tao, of establishing a direct connection between his infinitary, probabilistic approach to the hypergraph removal lemma and the infinitary, ergodic-theoretic approach to Szemerédi’s Theorem set in motion by Furstenberg [5].

Contents

1 Introduction
2 Basic notation and preliminaries
3 The Furstenberg self-joining
4 Pleasant and isotropized extensions
1 Introduction

We give a new ergodic-theoretic proof of the multidimensional multiple recurrence theorem of Furstenberg and Katznelson \[6\], which their correspondence principle shows to be equivalent to the multidimensional Szemerédi Theorem.

**THEOREM 1.1** (Multidimensional multiple recurrence). Suppose that $T_1, T_2, \ldots, T_d : \mathbb{Z} \acts_X (X, \Sigma, \mu)$ are commuting probability-preserving actions and that $A \in \Sigma$ has $\mu(A) > 0$. Then

$$
\liminf_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} \mu(T_1^{-n}(A) \cap T_2^{-n}(A) \cap \cdots \cap T_d^{-n}(A)) > 0,
$$

and so, in particular, there is some $n \geq 1$ with

$$
\mu(T_1^{-n}(A) \cap T_2^{-n}(A) \cap \cdots \cap T_d^{-n}(A)) > 0.
$$

Our proof of Theorem 1.1 will call on some rather different ergodic-theoretic machinery from Furstenberg and Katznelson’s. Our main technical ingredients are the notions of ‘pleasant’ and ‘isotropized’ extensions of a system. Pleasant extensions were first used in \[1\] to give a new proof of the (rather easier) result that the ‘nonconventional ergodic averages’

$$
\frac{1}{N} \sum_{n=1}^{N} \prod_{i \leq d} f_i \circ T_i^n
$$

associated to $f_1, f_2, \ldots, f_d \in L^\infty(\mu)$ always converge in $L^2(\mu)$ as $N \to \infty$. (This was first shown by Tao in \[14\], although various special cases had previously been established by other methods \[2, 3, 15, 10, 11, 16, 4\].) Much of the present paper is motivated by the results used in \[1\] to give a new proof of this convergence. Isotropized extensions are a new tool developed for the present paper, but their analysis is closely analogous to that of pleasant extensions.
After passing to a pleasant and isotropized extension of our original system, the 
limit of (1) takes a special form, and in this paper it is by analyzing this expression 
that we shall prove positivity. It turns out that this special form enables us to 
make contact with the machinery developed by Tao in [13] for his infinitary proof 
of the hypergraph removal lemma. Since the hypergraph removal lemma offers 
a known route to proving the multidimensional Szemerédi Theorem (as shown, 
subject to some important technical differences, by Nagle, Rödl and Schacht [12] 
and by Gowers [8]), and this in turn is equivalent to multidimensional multiple 
recurrence, Tao’s work already offers a proof of multiple recurrence using his 
infinitary removal lemma. In a sense, our present contribution is to short-circuit 
the above chain of implications and give a near-direct proof of multiple recurrence 
using Tao’s ideas. Unfortunately, we have not been able to make a reduction to a 
simple black-box appeal to Tao’s result; rather, we formulate (Proposition 6.1) a 
closely-related result adapted to our ergodic theoretic setting, which then admits 
very similar proof. With this caveat, our work addresses the question of relating 
infinitary proofs of multiple recurrence and hypergraph removal explicitly raised 
by Tao at the beginning of Section 5 of [13]: it turns out that his ideas are not 
directly applicable to an arbitrary probability-preserving $\mathbb{Z}^d$-system, but becomes 
so only when we enlarge the system to lie in the special class of systems that are 
pleasant and isotropized.

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serious flaw was discovered in an earlier version.

2 Basic notation and preliminaries

Throughout this paper $(X, \Sigma)$ will denote a measurable space. Since our main re-
results pertain only to the joint distribution of countably many bounded real-valued 
functions on this space and their shifts under some measurable transformation, by 
passing to the image measure on a suitable shift space we may always assume that 
$(X, \Sigma)$ is standard Borel, and this will prove convenient for some of our later con-
structions. In addition, $\mu$ will always denote a probability measure on $\Sigma$. We shall 
write $(X^e, \Sigma^e)$ for the usual product measurable structure indexed by a set $e$, and 
$\mu^e$ for the product measure and $\mu^{^e}$ for the diagonal measure on this structure
respectively. We also write \( \pi_i : X^e \to X \) for the \( i \)-th coordinate projection whenever \( i \in e \). Given a measurable map \( \phi : (X, \Sigma) \to (Y, \Phi) \) to another measurable space, we shall write \( \phi \circ \mu \) for the resulting pushforward probability measure on \( (Y, \Phi) \).

If \( T : \Gamma \curvearrowright (X, \Sigma, \mu) \) is a probability-preserving action of a countable group \( \Gamma \), then by a \textbf{factor} of the quadruple \( (X, \Sigma, \mu, T) \) we understand a globally \( T \)-invariant sub-\( \sigma \)-algebra \( \Phi \leq \Sigma \). The \textbf{isotropy factor} is the sub-\( \sigma \)-algebra of those subsets \( A \in \Sigma \) such that \( \mu(A \Delta T^n(A)) = 0 \) for all \( \gamma \in \Gamma \), and we shall denote it by \( \Sigma^T \). If \( T_1, T_2 : \Gamma \curvearrowright (X, \mu) \) are two commuting actions of the same Abelian group, then we can define another action by \( (T_1^{-1}T_2)^\gamma := T_1^{-1}T_2^\gamma \), and then we write \( \Sigma^{T_1=T_2} \) for \( \Sigma^{T_1^{-1}T_2} \), and similarly if we are given a larger number of actions of the same group. The most important kind of morphism from one \( \Gamma \)-system \( T : \Gamma \curvearrowright (X, \Sigma, \mu) \) to another \( S : \Gamma \curvearrowright (Y, \Phi, \nu) \) is given by a measurable map \( \phi : X \to Y \) such that \( \nu = \phi \circ \mu \) and \( S \circ \phi = \phi \circ T \): we call such a \( \phi \) a \textbf{factor map}. In this case we shall write \( \phi : (X, \Sigma, \mu, T) \to (Y, \Phi, \nu, S) \). To a factor map \( \phi \) we can associate the factor \( \{ \phi^{-1}(A) : A \in \Phi \} \).

Our specific interest is in \( d \)-tuples of commuting \( \mathbb{Z} \)-actions \( T_i, i = 1, 2, \ldots, d \). Clearly these can be interpreted as the \( \mathbb{Z} \)-subactions of a single \( \mathbb{Z}^d \)-action corresponding to the \( d \) coordinate directions \( \mathbb{Z} \cdot e_i \leq \mathbb{Z}^d \).

Given these actions, we shall make repeated reference to certain factors assembled from the isotropy factors among the \( T_i \). These will be indexed by subsets of \( [d] := \{1, 2, \ldots, d\} \), or more generally by subfamilies of the collection \( \binom{[d]}{\geq 2} \) of all subsets of \( [d] \) of size at least 2. On the whole, these indexing subfamilies will be \textbf{up-sets} in \( \binom{[d]}{\geq 2} \): \( \mathcal{I} \subseteq \binom{[d]}{\geq 2} \) such that \( u \in \mathcal{I} \) and \( [d] \supseteq v \supseteq u \) imply \( v \in \mathcal{I} \). For example, given \( e \subseteq [d] \) we write \( \langle e \rangle := \{ u \in \binom{[d]}{\geq 2} : u \supseteq e \} \) (note the non-standard feature of our notation that \( e \in \langle e \rangle \) if and only if \( |e| \geq 2 \)):

up-sets of this form are \textbf{principal}. We will abbreviate \( \langle \{i\} \rangle \) to \( \langle i \rangle \). It will also be helpful to define the \textbf{depth} of a non-empty up-set \( \mathcal{I} \) to be \( \min\{|e| : e \in \mathcal{I} \} \).

The corresponding factors are obtained for \( e = \{i_1, i_2, \ldots, i_k\} \subseteq [d] \) with \( k \geq 2 \) by defining \( \Phi_e := \Sigma^{T_{i_1}=T_{i_2}=\ldots=T_{i_k}} \), and given an up-set \( \mathcal{I} \subseteq \binom{[d]}{\geq 2} \) by defining \( \Phi_\mathcal{I} := \bigvee_{e \in \mathcal{I}} \Phi_e \).

From the ordering among the factors \( \Phi_e \) it is clear that \( \Phi_\mathcal{I} = \Phi_\mathcal{A} \) whenever \( \mathcal{A} \subseteq \binom{[d]}{\geq 2} \) is a family that generates \( \mathcal{I} \) as an up-set, and in particular that \( \Phi_e = \Phi_{\langle e \rangle} \).

An \textbf{inverse system} is a family of probability-preserving systems \( T^{(m)} : \Gamma \curvearrowright \)}. 
Together with factor maps \( \psi_m : (X^{(m+1)}, \Sigma^{(m+1)}, \mu^{(m+1)}, T^{(m+1)}) \to (X^{(m)}, \Sigma^{(m)}, \mu^{(m)}, T^{(m)}) \), from this one can construct the inverse limit

\[
\lim_{m \to \infty} (X^{(m)}, \Sigma^{(m)}, \mu^{(m)}, T^{(m)})
\]

as described, for example, in Section 6.3 of Glasner [7].

Finally, the following distributional condition for families of factors will play a central rôle through this paper.

**DEFINITION 2.1** (Relative independence for factor-tuples). If \( \Sigma_i \geq \Xi_i \) are factors of \( (X, \Sigma, \mu) \) for each \( i \leq d \), then the tuple of factors \( (\Sigma_1, \Sigma_2, \ldots, \Sigma_d) \) is relatively independent over the tuple \( (\Xi_1, \Xi_2, \ldots, \Xi_d) \) if whenever \( f_i \in L^\infty(\mu|_{\Sigma_i}) \) for each \( i \leq d \) we have

\[
\int_X \prod_{i \leq d} f_i \, d\mu = \int_X \prod_{i \leq d} \mathbb{E}(f_i | \Xi_i) \, d\mu.
\]

### 3 The Furstenberg self-joining

It turns out that a particular \( d \)-fold self-joining of \( \mu \) both controls the convergence of the nonconventional averages (1) and then serves to express their limiting value. Given our commuting actions and any \( e = \{i_1 < i_2 < \ldots < i_k\} \subseteq [d] \), we define

\[
\mu^F_e(A_1 \times A_2 \times \cdots \times A_k) := \lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} \mu(T_{i_1}^{-n}(A_1) \cap T_{i_2}^{-n}(A_2) \cap \cdots \cap T_{i_k}^{-n}(A_k))
\]

for \( A_1, A_2, \ldots, A_k \in \Sigma \). That these limits always exist (and so this definition is possible) follows from the convergence of the nonconventional averages (1), although approaches to convergence that use this self-joining (as in [1], or for various special cases in [15] and [16]) actually handle both kinds of limits alternately in a combined proof of their existence by induction on \( k \).
Given the existence of the limits (1) and the assumption that \((X, \Sigma)\) is standard Borel, it is easy to check that \(\mu^F_e\) extends to a \(k\)-fold self-joining of \(\mu\) on \(\Sigma^\otimes e\). This is the Furstenberg self-joining of \(\mu\) associated to \(T_{i_1}, T_{i_2}, \ldots, T_{i_k}\). It is now clear from our definition that the assertion of Theorem 1.1 can be restated as being that if \(\mu(A) > 0\) then also \(\mu^F_{[d]}(A^d) > 0\). It is in this form that we shall prove it.

The following elementary properties of the Furstenberg self-joining will be important later.

**Lemma 3.1.** If \(e = \{i_1 < i_2 < \ldots < i_k\} \subseteq e' = \{j_1 < j_2 < \ldots < j_l\}\) then \(\pi_{i_1, i_2, \ldots, i_k} \circ \mu^F_e = \mu^F_{e'}\).

**Proof.** This is immediate from the definition: if \(A_{i_j} \in \Sigma\) for each \(j \leq k\) then

\[
(\pi_{i_1, i_2, \ldots, i_k} \circ \mu^F_e)(A_1 \times A_2 \times \cdots \times A_k) := \lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} \mu(T_{i_1}^{-n}(B_1) \cap T_{i_2}^{-n}(B_2) \cap \cdots \cap T_{i_k}^{-n}(B_1))
\]

where \(B_j := A_j\) if \(j \in e\) and \(B_j := X\) otherwise; but then this last average simplifies summand-by-summand directly to

\[
\lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} \mu(T_{j_1}^{-n}(A_1) \cap T_{j_2}^{-n}(A_2) \cap \cdots \cap T_{j_l}^{-n}(A_k)) =: \mu^F_e(A_1 \times A_2 \times \cdots \times A_k),
\]

as required. \(\square\)

**Lemma 3.2.** For any \(e \subseteq [d]\) the restriction \(\mu^F_e \mid_{\Phi^\otimes e}\) is just the diagonal measure \((\mu_{\Phi_e})^{\Delta_e}\).

**Proof.** If \(e = \{i_1 < i_2 < \ldots < i_k\}\) and \(A_j \in \Phi_e\) for each \(j \leq k\) then by definition we have

\[
\mu^F_e(A_1 \times A_2 \times \cdots \times A_k) = \lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} \mu(T_{i_1}^{-n}(A_1) \cap T_{i_2}^{-n}(A_2) \cap \cdots \cap T_{i_k}^{-n}(A_k))
\]

\[
= \lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} \mu(T_{i_1}^{-n}(A_1 \cap A_2 \cap \cdots \cap A_k))
\]

\[
= \mu(A_1 \cap A_2 \cap \cdots \cap A_k),
\]
as required.

It follows from the last lemma that whenever \( e \subseteq e' \) the factors \( \pi_i^{-1}(\Phi_e) \leq \Sigma^\otimes e' \) for \( i \in e \) are all equal up to \( \mu_{e'} \)-negligible sets. It will prove helpful later to have a dedicated notation for these factors.

**DEFINITION 3.3** (Oblique copies). For each \( e \subseteq [d] \) we refer to the common \( \mu_{[d]} \)-completion of the sub-\( \sigma \)-algebra \( \pi_i^{-1}(\Phi_e) \), \( i \in e \), as the **oblique copy** of \( \Phi_e \), and denote it by \( \Phi^F_e \). More generally we shall refer to factors formed by repeatedly applying \( \cap \) and \( \lor \) to such oblique copies as **oblique factors**.

It will be important to know that Furstenberg self-joinings behave well under inverse limits. The following is another immediate consequence of the definition, and we omit the proof.

**LEMMA 3.4.** If

\[
\ldots \rightarrow (X^{(m+1)}, \Sigma^{(m+1)}, \mu^{(m+1)}, T^{(m+1)}) \overset{\psi_m}{\rightarrow} (X^{(m)}, \Sigma^{(m)}, \mu^{(m)}, T^{(m)}) \rightarrow \ldots
\]

is an inverse system with inverse limit \((\tilde{X}, \tilde{\Sigma}, \tilde{\mu}, \tilde{T})\), then the Furstenberg self-joinings \(((X^{(m)})^d, (\Sigma^{(m)})^\otimes d, (\mu^{(m)})^F_{[d]}, T^\times d)\) with factor maps \( \phi^\times_{m} \) also form an inverse system with inverse limit \((\tilde{X}^d, \tilde{\Sigma}^\otimes d, \tilde{\mu}^F_{[d]}, \tilde{T}^\times d)\).

## 4 Pleasant and isotropized extensions

We now introduce the main technical definitions of this paper: that of ‘pleasant systems’, closely following [1], and alongside them the related notion of ‘isotropized systems’. Recall that to a commuting tuple of actions \( T_1, T_2, \ldots, T_d : \mathbb{Z} \curvearrowright (X, \Sigma, \mu) \) we associate the factors

\[
\Phi_e := \Sigma^{T_{i_1} = T_{i_2} = \ldots = T_{i_k}}
\]

indexed by subsets \( e = \{i_1, i_2, \ldots, i_k\} \subseteq [d] \).

**DEFINITION 4.1** (Pleasant system). A system \((X, \Sigma, \mu, T)\) is \((e, i)\)-**pleasant** for some \( i \in e \in \binom{[d]}{\geq 2} \) if the \( i \)-th coordinate projection \( \pi_i \) is relatively independent from
the other $\pi_j$, $j \in e$, over the factor $\pi_i^{-1}\left(\bigvee_{j \in e \setminus \{i\}} \Phi_{\{i,j\}}\right)$ under the Furstenberg self-joining $\mu_e^F$:

$$\int_{X^e} \prod_{j \in e} f_j \circ \pi_j \, d\mu_e^F = \int_{X^e} \left(\mathbb{E}_{\mu} \left(f_i \bigg| \bigvee_{j \in e \setminus \{i\}} \Phi_{\{i,j\}}\right) \circ \pi_i\right) \cdot \prod_{j \in e \setminus \{i\}} f_j \circ \pi_j \, d\mu_e^F$$

whenever $f_j \in L^\infty(\mu)$ for each $j \in e$.

It is **fully pleasant** if it is $(e, i)$-pleasant for every pair $i \in e$.

**DEFINITION 4.2** (Isotropized system). A commuting tuple of actions $T_1, T_2, \ldots, T_d : \mathbb{Z} \curvearrowright (X, \Sigma, \mu)$ is $(e, i)$-**isotropized** for some $i \in e \in ([d]_{\geq 2})$ if

$$\Phi_e \cap \left(\bigvee_{j \in [d] \setminus e} \Phi_{\{i,j\}}\right) = \bigvee_{j \in [d] \setminus e} \Phi_{e \cup \{j\}}$$

up to $\mu$-negligible sets.

It is **fully isotropized** if it is $(e, i)$-isotropized for every $(e, i)$.

Intuitively, both pleasantness and isotropizedness (say when $e = [d]$) assert that the factors $\Phi_{\{i\}}$ are ‘large enough’: in the first case, large enough to account for all of the possible correlations between the coordinate projections under the Furstenberg self-joining, and in the second to account for all of the possible intersection between $\Phi_e$ and the combination $\bigvee_{j \in [d] \setminus e} \Phi_{\{i,j\}}$ up to negligible sets. This notion of pleasantness is very similar to Definition 4.2 in [1], where ‘pleasant systems’ were first introduced as those in which the larger factors $\Sigma^{T_i} \vee \Phi_{\{i\}}$ were ‘characteristic’ for the asymptotic behaviour of the nonconventional averages (1) in $L^2(\mu)$. Here our emphasis is rather different, since we are concerned only with the integrals of these ergodic averages, rather than the functions themselves. For these integrals it turns out that we can discard the factors $\Sigma^{T_i}$ from consideration. This lightens some of the notation that follows, but otherwise makes very little difference to the work we must go through.

Notice that the subset $e \subseteq [d]$ is allowed to vary in both of the above definitions: this nuance is important, since the pleasantness property relating a proper subfamily of actions $T_i$, $i \in e$, is in general not a consequence of the pleasantness of the whole family, and similarly for isotropizedness.

The main goal of this section is the following proposition.
**PROPOSITION 4.3** (Simultaneously pleasant and isotropized extensions). Any commuting tuple of actions $T_1, T_2, \ldots, T_d : \mathbb{Z} \curvearrowright (X, \Sigma, \mu)$ admits an extension that is both fully pleasant and fully isotropized.

This will rely on a number of simpler steps, many closely following the arguments of [1]. We first show that any tuple of actions admits an $(e, i)$-pleasant extension and, separately, an $(e, i)$-isotropized extension.

The first of these results is proved exactly as was Proposition 4.6 in [1], and so we shall only sketch the proof here. The idea behind the proof is to construct of a tower of extensions, each accounting for the shortfall from pleasantness of its predecessor, and then the pass to the inverse limit.

**LEMMA 4.4** (Existence of an $(e, i)$-pleasant extension). Any commuting tuple of actions $T_1, T_2, \ldots, T_d : \mathbb{Z} \curvearrowright (X, \Sigma, \mu)$ admits an $(e, i)$-pleasant extension $(\tilde{X}, \tilde{\Sigma}, \tilde{\mu}, \tilde{T})$.

**Proof** We form $(\tilde{X}, \tilde{\Sigma}, \tilde{\mu}, \tilde{T})$ as the inverse limit of a tower of smaller extensions, each constructed from the Furstenberg self-joining of its predecessor. Let $(X^{(1)}, \Sigma^{(1)}, \mu^{(1)})$ be the Furstenberg self-joining $(X^e, \Sigma^e, \mu^e_F)$ and define on it the transformations

$$\tilde{T}_i := \prod_{j \in e} T_j$$

and

$$\tilde{T}_k := (T_k)^{X^e} \quad \text{for } k \neq i,$$

and interpret it as an extension of $(X, \Sigma, \mu, T)$ with the coordinate projection $\pi_i$ as factor map. We now see that if $f_j \in L^\infty(\mu)$ for each $j \in e$ then

$$\int_{X^e} \prod_{j \in e} f_j \circ \pi_j \ d\mu^e_F = \int_{X^e} E_\mu(f_i \circ \pi_i \mid (\pi_j)_{j \in e \setminus \{i\}}) \cdot \prod_{j \in e \setminus \{i\}} f_j \circ \pi_j \ d\mu^e_F,$$

and from the above definition that the factor of $X^e = X^{(1)}$ generated by $(\pi_j)_{j \in e \setminus \{i\}}$ is contained in $\bigvee_{j \in e \setminus \{i\}} \Phi^{(1)}_{(i,j)}$. If we now iterate this construction to form $(X^{(2)}, \Sigma^{(2)}, \mu^{(2)}, T^{(2)})$ from $(X^{(1)}, \Sigma^{(1)}, \mu^{(1)}, T^{(1)})$, and so on, then the approximation argument given for Proposition 4.6 of [1] shows that the inverse limit is $(e, i)$-pleasant. \hfill $\square$

**Remark** Since the appearance of [1], Bernard Host has given in [9] a method for constructing a pleasant extension of a system without recourse to an inverse limit.
However, we will make further use of inverse limits momentarily to construct an extension that is fully pleasant, rather than just \((e, i)\)-pleasant for some fixed \((e, i)\), and at present we do not know of any quicker construction guaranteeing this stronger condition.

A similar argument gives the existence of \((e, i)\)-isotropized extensions.

**Lemma 4.5** (Existence of \((e, i)\)-isotropized extension). Any commuting tuple of actions \(T_1, T_2, \ldots, T_d : \mathbb{Z} \rightrightarrows (X, \Sigma, \mu)\) admits an \((e, i)\)-isotropized extension \((\tilde{X}, \tilde{\Sigma}, \tilde{\mu}, \tilde{T})\).

**Proof** Once again we build this as an inverse limit. First form the relatively independent self-product \((X^{(1)}, \mu^{(1)}) := (X^{(2)} \otimes_{\Phi_e} \Sigma, \mu \otimes_{\Phi_e} \mu)\) with coordinate projections \(\pi_1, \pi_2\) back onto \((X, \Sigma, \mu)\), and interpret it as an extension of \((X, \Sigma, \mu)\) through the first of these. Choose arbitrarily some \(i \in e\), and now define the extended actions \(T_j^{(1)}\) on \(X^{(1)}\) by setting

\[
T_j^{(1)} := \begin{cases} T_j \times T_j & \text{if } j \not\in e, \\ T_j \times T_i & \text{if } j \in e; \end{cases}
\]

these all preserve \(\mu^{(1)}\), even in the latter case, because our product is relatively independent over the factor left invariant by each \(T_j^{-1}T_i\) for \(j \in e\).

We now extend \((X^{(1)}, \Sigma^{(1)}, \mu^{(1)}, T^{(1)})\) to \((X^{(2)}, \Sigma^{(2)}, \mu^{(2)}, T^{(2)})\) by repeating the same construction, and so on, to form an inverse series with inverse limit \((\tilde{X}, \tilde{\Sigma}, \tilde{\mu}, \tilde{T})\).

We will show that this has the desired property. Any \(f \in L^\infty(\mu |_{\Phi_e \cap (\bigvee_{j \in [d]\setminus e} \Phi_{e \cup j}))})\) may, in particular, be approximated in \(L^1(\mu)\) by finite sums of the form \(\sum_p \prod_{j \in [d]\setminus e} \phi_{p,j}\) with \(\phi_{j,p} \in L^\infty(\mu |_{\Phi_{e \cup j))})\). However, since \(\mu^{(1)}\) is joined relatively independently conditioned on \(\Phi_e\) and \(f\) is also \(\Phi_e\)-measurable, it follows that \(f \circ \pi_1 = f \circ \pi_2\) \(\mu^{(1)}\)-almost surely, and so in the extended system \((X^{(1)}, \Sigma^{(1)}, \mu^{(1)})\) we can alternatively approximate \(f \circ \pi_1\) by the functions \(\sum_p \prod_{j \in [d]\setminus e} \phi_{p,j} \circ \pi_2\); and now every \(\phi_{p,j} \circ \pi_2\) is both manifestly \(\Phi^{(1)}_{e \cup j}\)-measurable, since both \(T_j\) and \(T_i\) are simply lifted to \(T_j^{x_2}\) and \(T_i^{x_2}\), and manifestly \(\Phi^{(1)}_{e}\)-measurable, since all the transformations \(T_j^{(1)}\) defined above for \(j \in e\) agree on the second coordinate factor \(\pi_2^{-1}(\Sigma)\). Therefore \(f \circ \pi_1\) may be approximated arbitrarily well in \(L^1(\mu)\) by functions that are measurable with respect to \(\bigvee_{j \in [d]\setminus e} \Phi^{(1)}_{e \cup j}\). Now another simple approximation argument and the martingale convergence theorem show that the inverse limit system \((\tilde{X}, \tilde{\Sigma}, \tilde{\mu}, \tilde{T})\) is actually \((e, i)\)-isotropized, as required. \(\square\)
We will finish the proof of Proposition 4.3 using the following properties of stability under forming further inverse limits.

**Lemma 4.6 (Pleasantness of inverse limits).** If

\[ \ldots \rightarrow (X^{(m+1)}, \Sigma^{(m+1)}, \mu^{(m+1)}, T^{(m+1)}) \xrightarrow{\psi_{m+1}} (X^{(m)}, \Sigma^{(m)}, \mu^{(m)}, T^{(m)}) \rightarrow \ldots \]

is an inverse system with inverse limit \((\bar{X}, \bar{\Sigma}, \bar{\mu}, \bar{T})\) and \(i \in e \subseteq [d]\), then

- if \((X^{(m)}, \Sigma^{(m)}, \mu^{(m)}, T^{(m)})\) is \((e, i)\)-pleasant for infinitely many \(m\), then \((\bar{X}, \bar{\Sigma}, \bar{\mu}, \bar{T})\) is also \((e, i)\)-pleasant;
- if \((X^{(m)}, \Sigma^{(m)}, \mu^{(m)}, T^{(m)})\) is \((e, i)\)-isotropized for infinitely many \(m\), then \((\bar{X}, \bar{\Sigma}, \bar{\mu}, \bar{T})\) is also \((e, i)\)-isotropized.

**Proof** We give the proof for the retention of \((e, i)\)-pleasantness, the case of \((e, i)\)-isotropizedness being exactly analogous.

Since any \(1\)-bounded member of \(L^\infty(\bar{\mu})\) may be approximated arbitrarily well in \(L^1(\bar{\mu})\) by \(1\)-bounded members of \(L^\infty(\mu^{(m)})\), by a simple approximation argument it will suffice to prove that given \(m \geq 1\) and \(f_j \in L^\infty(\mu^{(m)})\) for each \(j \in e\) we have

\[ \int_X \prod_{j \in e} f_j \circ \pi_j \, d\bar{\mu}_e^F = \int_X E_{\bar{\mu}} \left( f_i \bigg| \bigvee_{j \in e \setminus \{i\}} \bar{\Phi}_{\{i,j\}} \right) \circ \pi_i \cdot \prod_{j \neq i} f_j \circ \pi_j \, d\bar{\mu}_e^F. \]

However, by definition and Lemma 3.4 we know that after choosing any \(m_1 \geq m\) for which \((X^{(m_1)}, \Sigma^{(m_1)}, \mu^{(m_1)}, T^{(m_1)})\) is \((e, i)\)-pleasant the above is obtained with \(\bigvee_{j \in e \setminus \{i\}} \Phi_{\{i,j\}}^{(m_1)}\) in place of \(\bigvee_{j \in e \setminus \{i\}} \bar{\Phi}_{\{i,j\}}\), and now letting \(m_1 \to \infty\) and appealing to the bounded martingale convergence theorem gives the result. \(\square\)

It now remains only to collect our different properties together using more inverse limits, whose organization is now rather arbitrary.

**Proof of Proposition 4.3** Pick a sequence of pairs \(((e_m, i_m))_{m \geq 1}\) from the finite set \(\{(e, i) : |e| \geq 2, i \in e \subseteq [d]\}\) in which each possible \((e, i)\) appears infinitely often. Now one last time form a tower of extensions

\[ \ldots \rightarrow (X^{(m+1)}, \Sigma^{(m+1)}, \mu^{(m+1)}, T^{(m+1)}) \rightarrow (X^{(m)}, \Sigma^{(m)}, \mu^{(m)}, T^{(m)}) \rightarrow \ldots \]

above \((X, \Sigma, \mu, T)\) in which \((X^{(m)}, \Sigma^{(m)}, \mu^{(m)}, T^{(m)})\) is \((e_{m+1}/2, i_{m+1}/2)\)-pleasant when \(m\) is odd and \((e_{m/2}, i_{m/2})\)-isotropized when \(m\) is even. The two parts of Lemma 4.6 now show that the resulting inverse limit extension has all the desired properties. \(\square\)
5 Furstenberg self-joinings of pleasant and isotropized systems

Having established that all systems have fully pleasant and isotropized extensions, it remains to explain the usefulness of such extensions for the proof of Theorem 1.1. This derives from the implications of these conditions for the structure of the Furstenberg self-joining.

**Lemma 5.1.** If the tuple \( T_1, T_2, \ldots, T_d : \mathbb{Z} \to (X, \Sigma, \mu) \) is fully pleasant and fully isotropized, \( I \subseteq \binom{[d]}{2} \) is an up-set and \( e \) is a member of \( \binom{[d]}{2} \setminus I \) of maximal size then the oblique copy \( \Phi_e^F \) and the oblique factor \( \Phi_I^F \) are relatively independent over \( \Phi_{I \cap \langle e \rangle}^F \) under \( \mu_{[d]}^F \).

**Proof** Suppose that \( F_1 \in L^\infty(\mu_{[d]}^F | \Phi_e^F) \) and \( F_2 \in L^\infty(\mu_{[d]}^F | \Phi_e^F) \). It will suffice to show that

\[
\int_{X^d} F_1 F_2 \, d\mu_{[d]}^F = \int_{X^d} E_\mu(F_1 | \Phi_{I \cap \langle e \rangle}^F) \cdot F_2 \, d\mu_{[d]}^F.
\]

Pick \( i \in e \). By Lemma 3.2 there is some \( f_1 \in L^\infty(\mu | \Phi_e) \) such that \( F_1 = f_1 \circ \pi_i \mu_{[d]}^F \)-almost surely.

Let \( \{a_1, a_2, \ldots, a_k\} \) be the antichain of minimal elements in \( I \); this clearly generates \( I \) as an up-set. Since \( e \notin I \) we must have \( a_j \setminus e \neq \emptyset \) for each \( j \leq k \). Pick \( i_j \in a_j \setminus e \) arbitrarily for each \( j \leq k \), so that, again by Lemma 3.2, \( \Phi_{a_j}^F = \pi_{i_j}^{-1}(\Phi_{a_j}) \) (up to \( \mu_{[d]}^F \)-negligible sets).

Now, since \( \Phi_I^F = \bigvee_{j \leq k} \Phi_{a_j}^F \), \( F_2 \) may be approximated arbitrarily well in \( L^1(\mu_{[d]}^F) \) by sums of products of the form \( \sum_p \prod_{j \leq k} \phi_{j,p} \circ \pi_{i_j} \) with \( \phi_{j,p} \in L^\infty(\mu | \Phi_{a_j}) \), and so by continuity and linearity it suffices to assume that \( F_2 \) is an individual such product term. This represents \( F_2 \) as a function of coordinates in \( X^d \) indexed only by members of \( [d] \setminus e \), and now we appeal to Lemma 3.1 and the pleasantness of \( \mu_{([d] \setminus e) \cup \{i\}}^F \) to deduce that

\[
\int_{X^d} F_1 \cdot \prod_{j \leq k} \phi_{j,p} \circ \pi_{i_j} \, d\mu_{[d]}^F = \int_{X([d] \setminus e) \cup \{i\}} E_\mu(f_1 | \bigvee_{j \in [d] \setminus e} \Phi_{\{i,j\}}) \cdot \prod_{j \leq k} \phi_{j,p} \circ \pi_{i_j} \, d\mu_{([d] \setminus e) \cup \{i\}}^F.
\]
However, now the property that \((X, \Sigma, \mu, T)\) is \((e, i)\)-isotropized and the fact that \(f_1\) is already \(\Phi_e\)-measurable imply that

\[
E_\mu \left( f_1 \mid \bigvee_{j \in [d] \setminus e} \Phi_{i,j} \right) = E_\mu \left( f_1 \mid \bigvee_{j \in [d] \setminus e} \Phi_{i \cup \{j\}} \right),
\]

and since each \(e \cup \{j\} \in \mathcal{I}\) (by the maximality of \(e\) in \(\mathcal{P}[d] \setminus \mathcal{I}\)), under \(\pi_i\) this conditional expectation must be identified with \(E_\mu(F_1 \mid \Phi_{\mathcal{I} \cap \langle e \rangle})\), as required. 

**Corollary 5.2.** If the tuple \(T_1, T_2, \ldots, T_d : \mathbb{Z} \rightarrow (X, \Sigma, \mu)\) is fully pleasant and fully isotropized and \(\mathcal{I}, \mathcal{I}' \subseteq (\mathbb{Z}^d)\) are up-sets then \(\Phi_{\mathcal{I}}^F\) and \(\Phi_{\mathcal{I}'}^F\) are relatively independent over \(\Phi_{\mathcal{I} \cap \mathcal{I}'}^F\) under \(\mu_{[d]}^F\).

**Proof.** This is proved for fixed \(\mathcal{I}\) by induction on \(\mathcal{I}'\). If \(\mathcal{I}' \subseteq \mathcal{I}\) then the result is clear, so now let \(e\) be a minimal member of \(\mathcal{I}' \setminus \mathcal{I}\) of maximal size, and let \(\mathcal{I}'' := \mathcal{I}' \setminus \{e\}\). It will suffice to prove that if \(F \in L^\infty(\mu_{[d]}^F | \Phi_{\mathcal{I}}^F)\) then

\[
E_{\mu_{[d]}^F}(F \mid \Phi_{\mathcal{I}}^F) = E_{\mu_{[d]}^F}(F \mid \Phi_{\mathcal{I}''}^F),
\]

and furthermore, by approximation, to do so only for \(F\) that are of the form \(F_1 \cdot F_2\) with \(F_1 \in L^\infty(\mu_{[d]}^F | \Phi_{\mathcal{I}}^F)\) and \(F_2 \in L^\infty(\mu_{[d]}^F | \Phi_{\mathcal{I}''}^F)\). However, for these we can write

\[
E_{\mu_{[d]}^F}(F \mid \Phi_{\mathcal{I}}^F) = E_{\mu_{[d]}^F}(E_{\mu_{[d]}^F}(F \mid \Phi_{\mathcal{I}''}^F) \mid \Phi_{\mathcal{I}}^F) = E_{\mu_{[d]}^F}(E_{\mu_{[d]}^F}(F_1 \mid \Phi_{\mathcal{I}''}^F) \cdot F_2 \mid \Phi_{\mathcal{I}}^F),
\]

and by Lemma 5.1

\[
E_{\mu_{[d]}^F}(F_1 \mid \Phi_{\mathcal{I}''}^F) = E_{\mu_{[d]}^F}(F_1 \mid \Phi_{\mathcal{I}'' \cap \langle e \rangle}^F).
\]

On the other hand \((\mathcal{I} \cup \mathcal{I}'') \cap \langle e \rangle \subseteq \mathcal{I}''\) (because \(\mathcal{I}''\) contains every subset of \([d]\) that strictly includes \(e\), since \(\mathcal{I}'\) is an up-set), and so Lemma 5.1 promises similarly that

\[
E_{\mu_{[d]}^F}(F_1 \mid \Phi_{\mathcal{I}'' \cap \langle e \rangle}^F) = E_{\mu_{[d]}^F}(F_1 \mid \Phi_{\mathcal{I}''}^F).
\]

Therefore the above expression for \(E_{\mu_{[d]}^F}(F \mid \Phi_{\mathcal{I}}^F)\) simplifies to

\[
E_{\mu_{[d]}^F}(E_{\mu_{[d]}^F}(F_1 \mid \Phi_{\mathcal{I}''}^F) \cdot F_2 \mid \Phi_{\mathcal{I}}^F) = E_{\mu_{[d]}^F}(E_{\mu_{[d]}^F}(F_1 \cdot F_2 \mid \Phi_{\mathcal{I}''}^F) \mid \Phi_{\mathcal{I}}^F)
\]

\[
= E_{\mu_{[d]}^F}(E_{\mu_{[d]}^F}(F \mid \Phi_{\mathcal{I}''}^F) \mid \Phi_{\mathcal{I}}^F) = E_{\mu_{[d]}^F}(F \mid \Phi_{\mathcal{I}''}^F) = E_{\mu_{[d]}^F}(F \mid \Phi_{\mathcal{I} \cap \mathcal{I}''}^F),
\]

by the inductive hypothesis applied to \(\mathcal{I}''\) and \(\mathcal{I}\), as required. 

\(\square\)
6 Completion of the proof

We have now set the stage for our analog of Tao’s infinitary hypergraph removal machinery. Observe first that the conclusion of Theorem 1.1 clearly holds for the commuting tuple \( T_1, T_2, \ldots, T_d : \mathbb{Z} \rightarrow (X, \Sigma, \mu) \) if it holds for any extension of that tuple. Therefore by Proposition 4.3 we may assume our commuting tuple is fully pleasant and fully isotropized, and so need only prove for such \( \mu \) that if \( \mu(A) > 0 \) then \( \mu_{\mu}(A^d) > 0 \). For these particular \( \mu \) Corollary 5.2 gives us a very precise picture of the joint law under \( \mu_{\mu}(d) \) of the poset of oblique factors \( \Phi_{\mu}^F \), and hence actually of the inverse image factors \( \pi_{\mu}^{-1}(\Phi_{\mu}) \) for \( I \subseteq \langle i \rangle \).

Note that we have not tamed all of the potentially wild structure of the joint distribution of the factors \( \Phi_{\mu} \) under \( \mu \), but only that of the associated oblique factors under the Furstenberg self-joining \( \mu_{\mu}(d) \). It seems quite likely that in some cases the factors \( \Phi_{\mu} \) of the original system can still exhibit a very complicated joint distribution, even after passing to a fully pleasant and isotropized extension. However, the understanding of the oblique copies is already enough to complete the proof of multiple recurrence using a relative of Tao’s ‘infinitary removal lemma’ in [13].

One of his chief innovations was an infinitary analog of the property of hypergraph removability for a collection of factors of a probability space (Theorem 4.2 of [13]). Here we shall actually make do with a more modest conclusion than his ‘removability’, but our argument will follow essentially the same steps. We shall derive Theorem 1.1 as the top case of the following inductive claim, tailored to our present needs.

**Proposition 6.1.** Suppose that \( I_{i,j} \) for \( i = 1, 2, \ldots, d \) and \( j = 1, 2, \ldots, k_i \) are collections of up-sets in \( \binom{d}{j} \) such that \( [d] \in I_{i,j} \subseteq \langle i \rangle \) for each \( i, j \), and suppose further that the sets \( A_{i,j} \in \Phi_{I_{i,j}} \) are such that

\[
\mu_{\mu}(d) \left( \prod_{i=1}^{d} \left( \bigcap_{j=1}^{k_i} A_{i,j} \right) \right) = 0.
\]

Then we must also have

\[
\mu \left( \bigcap_{i=1}^{d} \bigcap_{j=1}^{k_i} A_{i,j} \right) = 0.
\]

The following terminology will be convenient during the proof.
DEFINITION 6.2. A family \((\mathcal{I}_{i,j})_{i,j}\) has the property \(P\) if it satisfies the conclusion of the preceding proposition.

The conclusion of multiple recurrence follows from Proposition 6.1 at once:

**Proof of Theorem 1.1 from Proposition 6.1** Suppose that \(A \in \Sigma\) is such that \(\mu^F_{[d]}(A^d) = 0\). Then by the pleasantness of the whole system we have

\[
\mu^F_{[d]}(A^d) = \int_{X^d} \prod_{i=1}^d E_{\mu}(1_A | \Phi_{(i)}) \circ \pi_i \, d\mu^F_{[d]} = 0.
\]

Now the level set \(B_i := \{E_{\mu}(1_A | \Phi_{(i)}) > 0\}\) (of course, this is unique only up to \(\mu\)-negligible sets) lies in \(\Phi_{(i)}\), and the above equality certainly implies that also \(\mu^F_{[d]}(B_1 \times B_2 \times \cdots \times B_d) = 0\). Now, on the one hand, setting \(k_1 = 1, \mathcal{I}_{i,1} := \langle i \rangle\) and \(A_{i,1} := B_i\) for each \(i \leq d\), Proposition 6.1 tells us that \(\mu(B_1 \cap B_2 \cap \cdots \cap B_d) = 0\), while on the other we must have \(\mu(A \setminus B_i) = 0\) for each \(i\), and so overall \(\mu(A) \leq \mu(B_1 \cap B_2 \cap \cdots \cap B_d) + \sum_{i=1}^d \mu(A \setminus B_i) = 0\), as required. \(\square\)

It remains to prove Proposition 6.1. This will be done by induction on a suitable ordering of the possible collections of up-sets \((\mathcal{I}_{i,j})_{i,j}\), appealing to a handful of different possible cases at different steps of the induction. At the outermost level, this induction will be organized according to the depth of our up-sets (defined in Section 2).

Let us first illustrate how the above reduction to Proposition 6.1 and then the inductive proof of that proposition combine to give a proof of Theorem 1.1 in the simple case \(d = 3\).

**Example** Suppose that \(T_1, T_2, T_3 : \mathbb{Z} \to (X, \Sigma, \mu)\) is a fully pleasant and fully isotropized triple of actions and that \(A \in \Sigma\) has \(\mu^F_3(A^3) = 0\). We will show that \(\mu(A) = 0\). As in the above argument, we know that

\[
\mu^F_3(A^3) = \int_{X^3} (E_{\mu}(1_A | \Phi_{(1)}) \circ \pi_1) \cdot (E_{\mu}(1_A | \Phi_{(2)}) \circ \pi_2) \cdot (E_{\mu}(1_A | \Phi_{(3)}) \circ \pi_3) \, d\mu^F_3,
\]

and so we must actually have \(\mu^F_{[d]}(B_1 \times B_2 \times B_3) = 0\) where \(B_i := \{E_{\mu}(1_A | \Phi_{(i)}) > 0\}\). Clearly \(A\) is contained in \(B_1 \cap B_2 \cap B_3\) up to a \(\mu\)-negligible set, so it will suffice to show that this intersection is \(\mu\)-negligible.

Now, each of \(\Phi_{(1)} = \Phi_{\{1,2\}} \vee \Phi_{\{1,3\}}, \Phi_{(2)}\) and \(\Phi_{(3)}\) can be generated using intersections of members from countable generating sets in each \(\Phi_{\{i,j\}}\). Let

\[
B_{\{(i,j),1\}} \subseteq B_{\{(i,j),2\}} \subseteq \cdots
\]
be an increasing sequence of finite subalgebras of sets that generates \( \Phi_{\{i,j\}} \) up to \( \mu \)-negligible sets, and let

\[
\Xi_i^{(n)} := \Sigma_{T_1=T_2=T_3} \vee B_{\{i,j\},n} \vee B_{\{i,k\},n}
\]

when \( \{i, j, k\} = \{1, 2, 3\} \). By the martingale convergence theorem we have \( \mathbb{E}_\mu(1_{B_i} \mid \Xi_i^{(n)}) \to 1_{B_i} \) in \( L^2(\mu) \) as \( n \to \infty \). Now pick \( \delta \in (0, 1/3) \) and let \( C_i^{(n)} := \{ \mathbb{E}_\mu(1_{B_i} \mid \Xi_i^{(n)}) > 1 - \delta \} \), so for large \( n \) this set should be a \( \mu \)-approximation to \( B_i \), and observe in addition that \( \mathbb{E}_\mu(1_{C_i^{(n)} \setminus B_i} \mid \Xi_i^{(n)}) \leq \delta \) almost surely.

It easy to check from Corollary 5.2 that \( \Phi_i^{(n)} \) must be relatively independent from \( \pi_i^{-1}(\Xi_j^{(n)} \vee \Xi_k^{(n)}) \) over \( \pi_i^{-1}(\Xi_i^{(n)}) \) under \( \mu_{[3]} \) when \( \{i, j, k\} = \{1, 2, 3\} \), and from this we compute that

\[
\mu_{[3]}^{(n)}((C_1^{(n)} \times C_2^{(n)} \times C_3^{(n)}) \setminus \pi_i^{-1}(B_1)) = \int_{X^3} (\mathbb{E}_\mu(1_{C_i^{(n)} \setminus B_i} \mid \Xi_i^{(n)}) \circ \pi_1) \cdot 1_{C_2^{(n)} \setminus C_3^{(n)}} \cdot 1_{C_3^{(n)}} \mathbf{d}\mu_{[3]}^{(n)} \leq \delta \int_{X^3} 1_{C_1^{(n)}} \cdot 1_{C_2^{(n)}} \cdot 1_{C_3^{(n)}} \mathbf{d}\mu_{[3]}^{(n)} = \delta \mu_{[3]}^{(n)}(C_1^{(n)} \times C_2^{(n)} \times C_3^{(n)}).
\]

Therefore

\[
\mu_{[3]}^{(n)}(C_1^{(n)} \times C_2^{(n)} \times C_3^{(n)}) \leq \mu_{[3]}^{(n)}(B_1 \times B_2 \times B_3) + 3\delta \mu_{[3]}^{(n)}(C_1^{(n)} \times C_2^{(n)} \times C_3^{(n)}),
\]

and so since \( \delta < 1/3 \) we must have \( \mu_{[3]}^{(n)}(C_1^{(n)} \times C_2^{(n)} \times C_3^{(n)}) = 0 \) for all \( n \).

The importance of this is that for large \( n \) we have now approximated the sets \( B_i \) by sets \( C_i^{(n)} \) that lie in the simpler \( \sigma \)-algebras \( \Xi_i^{(n)} \) but nevertheless still enjoy the property that the measure \( \mu_{[3]}^{(n)}(C_1^{(n)} \times C_2^{(n)} \times C_3^{(n)}) \) is strictly zero. Since each \( B_{\{i,j\},n} \) is finite, for any given \( n \) we may write each \( C_i^{(n)} \) as a finite union of subsets of the form \( C_{i,p}^{(n)} = D_{i,p} \cap C_{i,j,p} \cap C_{i,k,p} \) with \( D_{i,p} \in \Sigma_{T_1=T_2=T_3} \) and \( C_{i,j,p} \in B_{\{i,j\},n} \) for every \( p \), and these must now also enjoy the property that

\[
\mu_{[3]}^{(n)}((D_{1,p_1} \cap C_{1,2,p_1} \cap C_{1,3,p_1}) \times (D_{1,p_2} \cap C_{2,1,p_2} \cap C_{2,3,p_2}) \times (D_{3,p_3} \cap C_{3,1,p_3} \cap C_{3,2,p_3})) = 0
\]

for all possible indices \( p_1, p_2, p_3 \).

Next the fact that \( \mu_{[3]}^{(n)}(\pi_i^{-1}(C) \triangle \pi_j^{-1}(C)) = 0 \) whenever \( C \in \Phi_{\{i,j\}} \) (Lemma 3.2) comes into play, allowing us for example to move the set \( C_{2,1,p_2} \) under the first
coordinate rather than the second in the above equation, and similarly. In this way we can re-arrange the above equation into the form

$$\mu^F_3(((D_{1,p_1} \cap D_{1,p_2} \cap D_{1,p_3}) \cap (C_{1,2,p_1} \cap C_{2,1,p_2} \cap C_{1,3,p_1} \cap C_{3,1,p_3})) \times (C_{2,3,p_2} \cap C_{3,2,p_3}) \times X) = 0.$$  

This equation involves the sets

$$D := D_{1,p_1} \cap D_{1,p_2} \cap D_{1,p_3} \in \Sigma_{T_1=T_2=T_3}, C_{1,2} := C_{1,2,p_1} \cap C_{2,1,p_2} \in \Phi_{(1,2), C_{1,3} := C_{1,3,p_1} \cap C_{3,1,p_3} \in \Phi_{(1,3)} \text{ and } C_{2,3} := C_{2,3,p_2} \cap C_{3,2,p_3} \in \Phi_{(2,3)}.$$  

Now, Corollary 5.2 tells us that the three oblique copies \( \Phi_{(1,2)} \) are relatively independent over \( \Phi^F_{(1,2,3)} \) under \( \mu^F_{[3]} \), and so we deduce from the above equation that

$$0 = \int_X (1_D \circ \pi_1) \cdot (\mu_1(1_{C_{1,2}} | \Sigma_{T_1=T_2=T_3} \circ \pi_1) \times (\mu_2(1_{C_{2,3}} | \Sigma_{T_1=T_2=T_3} \circ \pi_2) \cdot \mu^F_{[3]}) = \int_X 1_D \cdot \mu_1(1_{C_{1,2}} | \Sigma_{T_1=T_2=T_3}) \cdot \mu_2(1_{C_{2,3}} | \Sigma_{T_1=T_2=T_3}) d\mu$$

where the first and second line here are equal by Lemma 5.2 since all the functions involved are \( \Sigma_{T_1=T_2=T_3} \)-measurable.

However, this now implies that

$$\mu(D \cap \bigcap_{(i,j) \in (\mathbb{N}^2)} \{\mu_1(1_{C_{i,j}} | \Sigma_{T_1=T_2=T_3}) > 0\}) = 0$$

and hence that we must also have

$$\mu(D_{1,p_1} \cap D_{1,p_2} \cap D_{1,p_3} \cap C_{1,2,p_1} \cap C_{2,1,p_2} \cap C_{1,3,p_1} \cap C_{3,1,p_3} \cap C_{2,3,p_2} \cap C_{3,2,p_3}) = 0.$$  

Taking the union of these equations over triples of indices \( p_1, p_2, p_3 \) gives \( \mu(C_{i}^{(n)} \cap C_{j}^{(n)} = 0 \) for any \( n \), and so since the sets \( C_{i}^{(n)} \) approximate \( B_i \) as \( n \to \infty \), it follows that \( \mu(B_1 \cap B_2 \cap B_3) = 0 \), as required.

We now turn to full induction that generalizes the above argument, broken into a number of steps.

**Lemma 6.3 (Lifting using relative independence).** Suppose that all up-sets in the collection \( \mathcal{I}_{i,j} \) have depth at least \( k \), that all those with depth exactly \( k \) are principal, and that there are \( \ell \geq 1 \) of these. Then if property \( P \) holds for all similar collections having \( \ell - 1 \) up-sets of depth \( k \), then it holds also for this collection.
Proof Let \( I_{i_1,j_1} = \{ e_1 \}, I_{i_2,j_2} = \{ e_2 \}, \ldots, I_{i_k,j_k} = \{ e_k \} \) be an enumeration of all the (principal) up-sets of depth \( k \) in our collection. We will treat two separate cases.

First suppose that two of the generating sets agree; by re-ordering if necessary we may assume that \( e_1 = e_2 \). Clearly we can assume that there are no duplicates among the coordinate-collections \( (I_{i,j})_{j=1}^{k_i} \) for each \( i \) separately, so we must have \( i_1 \neq i_2 \). However, if we now suppose that \( A_{i,j} \in I_{i,j} \) for each \( i, j \) are such that

\[
\mu_{[d]}^F \left( \prod_{i=1}^{d} \left( \bigcap_{j=1}^{k_i} A_{i,j} \right) \right) = 0,
\]

then the same equality holds if we simply replace \( A_{i_1,j_1} \in \{ e_1 \} \) with \( A'_{i_1,j_1} := A_{i_1,j_1} \cap A_{i_2,j_2} \) and \( A_{i_2,j_2} \) with \( A'_{i_2,j_2} := X \). Now this last set can simply be ignored to leave an instance of a \( \mu_{[d]}^F \)-negligible product for the same collection of up-sets omitting \( I_{i_2,j_2} \), and so property \( P \) of this reduced collection completes the proof.

On the other hand, if all the \( e_i \) are distinct, we shall simplify the last of the principal up-sets \( I_{i_1,j_1} \) by exploiting the relative independence among the associated oblique copies of our factors. Assume for notational simplicity that \( (i_\ell, j_\ell) = (1,1) \); clearly this will not affect the proof. We will reduce to an instance of property \( P \) associated to the collection \( (I'_{i,j}) \) defined by

\[
I'_{i,j} := \begin{cases} \{ e_\ell \} \setminus \{ e_\ell \} & \text{if } (i,j) = (1,1); \\ I_{i,j} & \text{else,} \end{cases}
\]

which has one fewer up-set of depth \( k \) and so falls under the inductive assumption.

Indeed, we know from Corollary 5.2 that under \( \mu_{[d]}^F \) the set \( \pi^{-1}_i(A_{1,1}) \) is relatively independent from all the sets \( \pi^{-1}_i(A_{i,j}), (i,j) \neq (1,1), \) over the factor \( \pi^{-1}_i(\Phi_{\{e_i\}}) \), which is dense inside the relevant oblique copy \( \Phi_{\{e_i\}} \). Therefore

\[
0 = \mu_{[d]}^F \left( \prod_{i=1}^{d} \left( \bigcap_{j=1}^{k_i} A_{i,j} \right) \right) 
= \int_{X^d} \mathbb{E}_\mu(1_{A_{1,1}} | \Phi_{\{e_\ell\}}) \circ \pi_1 \cdot \prod_{j=2}^{d} 1_{\pi^{-1}_1(A_{1,j})} \cdot \prod_{i=2}^{d} \prod_{j=1}^{k_i} 1_{\pi^{-1}_i(A_{i,j})} \, d\mu_{[d]}^F.
\]

Setting \( A'_{1,1} := \{ \mathbb{E}_\mu(1_{A_{1,1}} | \Phi_{\{e_\ell\}}) > 0 \} \in \Phi_{\{e_\ell\}} \) and \( A'_{i,j} := A_{i,j} \) for \( (i,j) \neq (1,1) \), we have that \( \mu(A_{1,1} \setminus A'_{1,1}) = 0 \) and it follows from the above.
equality that also \( \mu^F_{[d]} \left( \prod_{i=1}^d \left( \bigcap_{j=1}^{k_i} A_{i,j} \right) \right) = 0 \), so an appeal to property P for the reduced collection of up-sets completes the proof. \( \square \)

**Remark**  The first very simple case treated by the above proof is the only step in the whole of the present section that is essentially absent from Tao’s arguments in Sections 6 and 7 of [13]. Nevertheless, it seems to be essential for the correct organization of the present argument, since we need to allow for which of our sets are lifted under which coordinate projections in the hypothesis that \( \mu^F_{[d]} \left( \prod_{i=1}^d \left( \bigcap_{j=1}^{k_i} A_{i,j} \right) \right) = 0 \).

**Lemma 6.4** (Lifting under finitary generation). Suppose that all up-sets in the collection \((\mathcal{I}_{i,j})_{i,j}\) have depth at least \(k\) and that among those of depth \(k\) there are \(\ell \geq 1\) that are non-principal. Then if property P holds for all similar collections having at most \(\ell - 1\) non-principal up-sets of depth \(k\), then it also holds for this collection.

**Proof**  Let \(\mathcal{I}_{i_1,j_1}, \mathcal{I}_{i_2,j_2}, \ldots, \mathcal{I}_{i_\ell,j_\ell}\) be the non-principal up-sets of depth \(k\), and now in addition let \(e_1, e_2, \ldots, e_r\) be all the members of \(\mathcal{I}_{i_\ell,j_\ell}\) of size \(k\) (so, of course, \(r \leq \binom{d}{k}\)). Once again we will assume for simplicity that \((i_\ell, j_\ell) = (1, 1)\).

We break our work into two further steps.

**Step 1**  First consider the case of a collection \((A_{i,j})_{i,j}\) such that for the set \(A_{1,1}\), we can actually find finite subalgebras of sets \(B_s \in \Phi_{\{e_s\}}\) for \(s = 1, 2, \ldots, r\) such that \(A_{i_\ell,j_\ell} \in B_1 \lor B_2 \lor \cdots \lor B_r \lor \Phi_{\mathcal{I}_{1,1} \cap (\geq k+1)}\) (so \(A_{1,1}\) lies in one of our non-principal up-sets of depth \(k\), but it fails to lie in an up-set of depth \(k + 1\) only ‘up to’ finitely many additional generating sets). Choose \(M \geq \max_{s \leq r} |B_s|\), so that we can certainly express

\[
A_{1,1} = \bigcup_{m=1}^M \left( B_{m,1} \cap B_{m,2} \cap \cdots \cap B_{m,r} \cap C_m \right)
\]

with \(B_{m,s} \in B_s\) for each \(s \leq r\) and \(C_m \in \Phi_{\mathcal{I}_{1,1} \cap (\geq k+1)}\). Inserting this expression into the equation

\[
\mu^F_{[d]} \left( \prod_{i=1}^d \left( \bigcap_{j=1}^{k_i} A_{i,j} \right) \right) = 0
\]

now gives that each of the \(M^r\) individual sets

\[
\left( B_{m,1} \cap B_{m,2} \cap \cdots \cap B_{m,r} \cap C_m \right) \cap \bigcap_{j=2}^{k_1} A_{1,j} \times \prod_{i=2}^d \left( \bigcap_{j=1}^{k_i} A_{i,j} \right)
\]

19
is $\mu_{\{d\}}$-negligible.

Now consider the family of up-sets comprising the original $I_{i,j}$ if $i = 2, 3, \ldots, d$ and the collection $\langle e_1 \rangle, \langle e_2 \rangle, \ldots, \langle e_r \rangle, I_{1,2}, I_{1,3}, \ldots, I_{1,k_1}$ corresponding to $i = 1$. We have broken the depth-$k$ non-principal up-set $I_{1,1} \cap \{ [d] \}_{k+1}$ into the higher-depth up-set $I_{1,1} \cap \{ [d] \}_{k+1}$ and the principal up-sets $\langle e_s \rangle$, and so there are only $\ell - 1$ minimal-depth non-principal up-sets in this new family. It is clear that for each $m \leq M^r$ the above product set is associated to this family of up-sets, and so an inductive appeal to property P for this family tells us that also

$$\mu\left( (B_{m,1} \cap B_{m,2} \cap \cdots \cap B_{m,r} \cap C_m) \cap \bigcap_{j=2}^{k_1} A_{1,j} \cap \bigcap_{i=2,j=1}^{d} A_{i,j} \right) = 0$$

for every $m \leq M^r$. Since the union of these sets is just $\bigcap_{i=1}^{d} \bigcap_{j=1}^{k_i} A_{i,j}$, this gives the desired negligibility in this case.

**Step 2** Now we return to the general case, which will follow by a suitable limiting argument applied to the conclusion of Step 1. Since any $\Phi_{\{e\}}$ is countably separated, for each $e$ with $|e| = k$ we can find an increasing sequence of finite subalgebras $B_{e,1} \subseteq B_{e,2} \subseteq \ldots$ that generates $\Phi_{\{e\}}$ up to $\mu$-negligible sets. In terms of these define approximating sub-$\sigma$-algebras

$$\Xi_{i,j}^{(n)} := \Phi_{I_{i,j} \cap \{ [d] \}_{k+1}} \vee \bigvee_{e \in I_{i,j} \cap \{ [d] \}_{k+1}} B_{e,n},$$

so for each $I_{i,j}$ these form an increasing family of $\sigma$-algebras that generates $\Phi_{I_{i,j}}$ up to $\mu$-negligible sets (indeed, if $I_{i,j}$ does not contain any sets of the minimal depth $k$ then we simply have $\Xi_{i,j}^{(n)} = \Phi_{I_{i,j}}$ for all $n$). Observe that by Corollary 5.2, for each $n$ we have that $\Phi_{I_{1,1}}^{F}$ and $\bigvee_{(i,j) \neq (1,1)} \pi_1^{-1}(\Xi_{i,j}^{(n)})$ are relatively independent over $\pi_1^{-1}(\Xi_{1,1}^{(n)})$.

Given now a family of sets $(A_{i,j})_{i,j}$ associated to $(I_{i,j})_{i,j}$, for each $(i, j)$ the conditional expectations $E_{\mu}(1_{A_{i,j}} \mid \Xi_{i,j}^{(n)})$ form an almost surely uniformly bounded martingale converging to $1_{A_{i,j}}$ in $L^2(\mu)$. Letting $B_{i,j}^{(n)} := \{ E_{\mu}(1_{A_{i,j}} \mid \Xi_{i,j}^{(n)}) > 1 - \delta \}$ for some small $\delta > 0$ (to be specified momentarily), it is clear that we also have $\mu(A_{i,j} \cap B_{i,j}^{(n)}) \to 0$ as $n \to \infty$. Let also

$$F := \prod_{i=1}^{d} \left( \bigcap_{j=1}^{k_i} B_{i,j}^{(n)} \right).$$
We now compute using the above-mentioned relative independence that
\[
\mu^F_{[d]}(F \setminus \pi_i^{-1}(A_{i,j})) = \int_{X^d} \left( \prod_{(i',j')} 1_{B_{i',j'}^{(n)}} \circ \pi_{i'} \right) - 1_{A_{i,j}} \circ \pi_i \cdot \left( \prod_{(i',j')} 1_{B_{i',j'}^{(n)}} \circ \pi_{i'} \right) \, d\mu^F_{[d]}
\]
\[
= \int_{X^d} (1_{B_{i,j}^{(n)}} \setminus A_{i,j}) \circ \pi_{i} \cdot \left( \prod_{(i',j') \neq (i,j)} 1_{B_{i',j'}^{(n)}} \circ \pi_{i'} \right) \, d\mu^F_{[d]}
\]
\[
= \int_{X^d} \left( E_{\mu}(1_{B_{i,j}^{(n)}} \setminus A_{i,j} \mid \Xi^{(n)}_{i,j} \circ \pi_{i}) \cdot \left( \prod_{(i',j') \neq (i,j)} 1_{B_{i',j'}^{(n)}} \circ \pi_{i'} \right) \right) \, d\mu^F_{[d]}
\]
for each pair \((i, j)\).

However, from the definition of \(B_{i,j}^{(n)}\) we must have
\[
E_{\mu}(1_{B_{i,j}^{(n)}} \setminus A_{i,j} \mid \Xi^{(n)}_{i,j} \circ \pi_{i}) \leq \delta 1_{B_{i,j}^{(n)}}
\]
almost surely, and therefore the above integral inequality implies that
\[
\mu^F_{[d]}(F \setminus \pi_i^{-1}(A_{i,j})) \leq \delta \int_{X^d} (1_{B_{i,j}^{(n)}} \circ \pi_{i} \cdot \left( \prod_{(i',j') \neq (i,j)} 1_{B_{i',j'}^{(n)}} \circ \pi_{i'} \right) \, d\mu^F_{[d]} = \delta \mu^F_{[d]}(F).
\]

From this we can estimate as follows:
\[
\mu^F_{[d]}(F) \leq \mu^F_{[d]} \left( \bigcap_{i=1}^{d} \left( \bigcap_{j=1}^{k_i} A_{i,j} \right) \right) + \sum_{(i,j)} \mu^F_{[d]}(F \setminus \pi_i^{-1}(A_{i,j})) \leq 0 + \left( \sum_{i=1}^{d} k_i \right) \delta \mu^F_{[d]}(F),
\]
and so provided we chose \(\delta < \left( \sum_{i=1}^{d} k_i \right)^{-1}\) we must in fact have \(\mu^F_{[d]}(F) = 0\).

We have now obtained sets \((B_{i,j}^{(n)})_{i,j}\) that are associated to the family \((\mathcal{I}_{i,j})_{i,j}\) and satisfy the property of lying in finitely-generated extensions of the relevant factors corresponding to the members of the \(\mathcal{I}_{i,j}\) of minimal size, and so we can apply the result of Step 1 to deduce that \(\mu \left( \bigcap_{i=1}^{d} \bigcap_{j=1}^{k_i} B_{i,j}^{(n)} \right) = 0\). It follows that
\[
\mu \left( \bigcap_{i=1}^{d} \bigcap_{j=1}^{k_i} A_{i,j} \right) \leq \sum_{i,j} \mu(A_{i,j} \setminus B_{i,j}^{(n)}) \to 0 \quad \text{as } n \to \infty,
\]
as required.
Proof of Proposition 6.1. We first take as our base case $k_i = 1$ and $I_{i,1} = \{[d]\}$ for each $i = 1, 2, \ldots, d$. In this case we know that for any $A \in \Phi[d]$ the pre-images $\pi_i^{-1}(A)$ are all equal up to negligible sets, and so given $A_1, A_2, \ldots, A_d \in \Phi[d]$ we have $0 = \mu_F(A_1 \times A_2 \times \cdots \times A_d) = \mu(A_1 \cap A_2 \cap \cdots \cap A_d)$.

The remainder of the proof now just requires putting the preceding lemmas into order to form an induction with three layers: if our collection has any non-principal up-sets of minimal depth, then Lemma 6.4 allows us to reduce their number at the expense only of introducing new principal up-sets of the same depth; and having removed all the non-principal minimal-depth up-sets, Lemma 6.3 enables us to remove also the principal ones until we are left only with up-sets of increased minimal depth. This completes the proof. \hfill \Box

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22
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