An Augmented Lagrangian Based Parallelizable Nonconvex Solver for Bilinear Model Predictive Control  
(draft manuscript)  
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Abstract— Nonlinear model predictive control is widely adopted to manipulate bilinear systems, and bilinear models are ubiquitous in chemical process, mechanical system and quantum physics, to name a few. Running an MPC controller in real-time requires solving a non-convex optimization problem at step. In this work, we propose a novel parallel augmented Lagrangian based bilinear MPC solver via a novel horizon splitting scheme. The resulting algorithm converts the non-convex MPC control problem into a set parallelizable multi-parametric quadratic programming (mpQP) and an equality constrained QP problem. The mpQP solution can be pre-computed offline to enable efficient online computation. The proposed algorithm is validated on a building simulation and is deployed on a TI C2000 LaunchPad to emulate the bilinear DC motor speed control.

I. INTRODUCTION

Bilinear systems were originally introduced in [1], [2] to model the interaction between the inputs and the states. These dynamics may result from the linearization of a nonlinear input affine system, and are mainly used to model the connection, spinning in chemical processes, mechanical systems and physics [3], [4]. Additionally, by means of the concept of Carleman linearization [5], bilinear systems are proved to be capable of modelling general nonlinear system [6]. Meanwhile, with resource to sophisticated tools such as Lie algebra [7, Chapter 2] and Volterra series [8], the bilinear control theory has been explored in depth and found various successful applications [9]–[13].

Nonlinear model predictive control (NMPC) is one of the most successful controllers used in bilinear systems [14]–[17], which optimizes its input by predicting the system evolution [18], and that can therefore enforce inputs/states constraint satisfaction. An efficient solver is a must to run NMPC in real-time, whose main enablers are typically the parallelization via distributed algorithms, the explicit MPC and the warm-start strategies.

The distributed algorithm decouples the NMPC problem into a set of small scale problems that can be solved in parallel. Such decoupling procedure mainly leverages the linear equality constraints appeared in the NMPC problems, which can reflect the topology of a network system or naturally emerges in the time direction by introducing dummy variables. The latter approach is the horizon splitting method [19], [20], or sometimes termed Schwarz decomposition [21]. It splits the predictive trajectories into short sequential sequences, where the linear couplings naturally enforce the equality between the initial states and the terminal state of two adjacent short sequences, whence the name come. A classical approach used in distributed optimization is based on dual decomposition, where gradient-based method [22], [23] and semi-smooth Newton method [24] are used to solve the concave dual problem. Another famous approach is the Alternating Direction Method of Multiplier, which parallelize the computation by introducing auxiliary variables [25], [26]. These two aforementioned methods lacks convergence guarantees in nonconvex case, and hence only applicable to linear systems. In [27], an augmented Lagrangian based distributed optimization algorithm is proposed, and is applied to parallelize the computation of MPC problems in [28], [29]. However, even though being parallelized, these algorithms requires a solution to multiple non-convex optimization problems in each iteration, which are still numerically intense. Via the scope of horizon splitting, tools beyond distributed algorithm are leveraged to further improve the efficiency. The banded structure of the KKT system is the most investigated object in this setup. In [30], a general parallel solver is summarized by a binary-tree-structured algorithm. In [20], an approximation scheme is introduced to develop a parallel Ricatti solver. However, these algorithms still handles the nonconvex problem directly, hence are still numerically challenging.

Another category of method leverages the super-linear convergence property to improves the real-time efficiency of the solver. These methods not strictly define the “warm-start” strategy, and usually offer an good initialization based on the solution information gathered from the preceeding time setp. The elemental method directly shifts solution from last iteration [31], and then a Newton iteration ensure efficient local convergence. Under the umbrella of sequential quadratic programming (SQP), the sensitivity information of the local solution is further used to initialize the KKT system, where the initial guess of active constraint is the most challenging object. In [32], the piece-wise affine property of linear MPC is used to estimate the change of the active constraint. This idea is generalized in [33] under the name of real-time iteration,
where a sensitivity analysis of the local solution is used to give a piece-wise affine update of the control law.

Instead of solving the NMPC directly online, the explicit MPC approaches treat the MPC control law as a nonlinear mapping from initial state to the control input. In a linear MPC set up, the optimal control law is locally affine [34], [35], and this piece-wise affine parametric solution is first used to pre-compute the MPC control law offline in [35]. However, this algebraic property only holds for linear system, and its application to nonlinear MPC is limited without approximation [36].

In this work, we proposed a new augmented Lagrangian based algorithm, which typically combines the idea the idea horizon splitting and explicit MPC. Different from the standard horizon splitting approach, a novel interlacing horizon splitting is introduced. The advantages of the proposed controller are summarized as follows:

1) Each iteration of the proposed algorithm only need call parametric QP solution in parallel and a sparse linear equation system. And the size of the parametric QP solution is only depends on the system constraints, and fully independent from the system dynamics and prediction horizon.

2) An novel interlacing horizon splitting scheme is introduced. The resulting problem size is the same as the original NMPC problem without introducing auxiliary variables.

3) The optimization will not fail even with an infeasible initial state, and output an input satisfying input constraint.

After introducing the notations and background knowledge in the rest of this Section II the bilinear MPC control problem is introduced in Section III after which parallel non-convex solver is introduced in Section III-A. In particular, Section III-A introduces a novel interlacing horizon splitting scheme, based on which the solver is elaborated in Section III-B. The convergence property of the proposed solver is studied in Section III-C. The efficacy of the proposed algorithm is investigated in Section IV where an efficient real-time MPC solver is used as a benchmark.

### A. Notations

We use the symbols $S^n_+$ and $S^n_{2+}$ to denote the set of symmetric, positive semi-definite and symmetric, positive definite matrices in $\mathbb{R}^{n \times n}$. For a given matrix $\Sigma \in S^n_+$, the notation $\|x\|_\Sigma = \sqrt{x^T \Sigma x}$ is used, although this notation should be used with caution, since $\|\cdot\|_\Sigma$ is only a norm if $\Sigma \in S^n_{2+}$. Moreover, we call a function $c : \mathbb{R}^n \to \mathbb{R} \cup \{ \infty \}$ strongly convex with matrix parameter $\Sigma \in S^n_+$, if the inequality

$$c(tx + (1-t)y) \leq tc(x) + (1-t)c(y) - \frac{1}{2}t(1-t)\|x - y\|^2_\Sigma$$

is satisfied for all $x, y \in \mathbb{R}^n$ and all $t \in [0, 1]$. Notice that this definition contains the standard definition of convexity as a special case for $\Sigma = 0$. Similarly, $c$ is called strictly convex, if there exists a continuous and strictly monotonously increasing function $\alpha : \mathbb{R} \to \mathbb{R}$ with $\alpha(0) = 0$ such that

$$c(tx + (1-t)y) \leq tc(x) + (1-t)c(y) - \frac{1}{2}t(1-t)\|x - y\|^2_\Sigma$$

is satisfied for all $x, y \in \mathbb{R}^n$ and all $t \in [0, 1]$. Notice that all convex functions in this paper are assumed to be closed and proper [37]. For a vector $x \in \mathbb{R}^n$, we denote by $[x]_i$ its $i$-th element. Set $\mathbb{Z}^n_+$ denotes the range of integers from $i$ to $j$ with $i \leq j$. The Kronecker product of two matrices $A \in \mathbb{R}^{k \times l}$ and $B \in \mathbb{R}^{m \times n}$ is denoted by $A \otimes B$ and $\text{vec}(A)$ denotes the vector that is obtained by stacking all columns of $A$ into one long vector. The reverse operation is denoted by mat, such that $\text{mat}(\text{vec}(A)) = A$. The unit matrix in $\mathbb{R}^{n \times n}$ is denoted by $I_n$, $\text{blkdiag}(H_1, \ldots, H_n)$ construct a block-diagonal matrix whose diagonal blocks are accordingly $H_i$.

### B. Preliminaries

We first recap some existing results from the field of multi-parametric quadratic programming (mpQP) used later in this paper. A generic convex mpQP can be written in the form of

$$\min_x \frac{1}{2}x^T Q x + \theta^T S x$$

s.t. $A x \leq b + C \theta$, (1a)

with decision variables $x \in \mathbb{R}^{n_x}$ and parameters $\theta \in \mathbb{R}^{n_p}$. Here, matrices $Q \in S^{n_x}_+$, $S \in \mathbb{R}^{n_p \times n_x}$, $A \in \mathbb{R}^{m \times n_x}$, $C \in \mathbb{R}^{m \times n_p}$ and vector $b \in \mathbb{R}^m$ are given data. Moreover, we denote by $\Omega$ the set of all parameter $\theta$ for which (1) is feasible. For a strictly convex mpQP (1), it has been shown (see, e.g., [38]) that $\Omega$ is a polytope while the solution map $x^*(\theta) : \mathbb{R}^{n_p} \to \mathbb{R}^{n_x}$ is a continuous piecewise affine (PWA) function of the parameters. This indicates the Lipschitz-continuity of $x^*(\cdot)$, i.e., there exists a positive constant $\eta > 0$ such that for any $\theta_1, \theta_2 \in \Omega$, we have

$$\|x^*(\theta_1) - x^*(\theta_2)\| \leq \eta \|\theta_1 - \theta_2\|.$$  (2)

We then recall some definitions from the field of nonlinear programming (NLP). Let us consider NLPs in a generic form

$$\min_x f(x) \quad \text{s.t.} \quad \begin{cases} g(x) = 0 \quad | \lambda \\ h(x) \leq 0 \quad | \kappa. \end{cases}$$

Throughout the rest of this paper, we write down the Lagrangian multipliers right after the constraints such that $\lambda \in \mathbb{R}^{n_g}$ and $n_g \geq \kappa \geq 0$ denote respectively the Lagrangian multipliers of the equality constraints and inequality constraints. Functions $f : \mathbb{R}^{n_x} \to \mathbb{R}$, $g : \mathbb{R}^{n_g} \to \mathbb{R}^{n_g}$ and $h : \mathbb{R}^{n_h} \to \mathbb{R}^{n_h}$ are assumed twice continuously differentiable. Then, we introduce the definition of regular KKT point for NLP (3)

**Definition 1** For a given KKT point $(x^*, \lambda^*, \kappa^*)$, it is called a regular KKT point if the linear independence constraint qualification (LICQ), the second order sufficient condition (SOSC) and the strict complementarity condition (SCC) hold.

For a given feasible $x$, we denote by $A(x)$ the active set at $x$, i.e., the index set that includes the equality constraints and the
inequality constraints that holds equality at $x$. If a given KKT point $(x^*, \lambda^*, \kappa^*)$ is regular, there exist an open neighbourhood $B(x^*)$ around $x^*$ such that the active set is fixed for any $x \in B(x^*)$, i.e., $A(x) = A(x^*)$ [39]. The regularity at the KKT points guarantees the local convergence property when a Newton-type method is applied to solve (3) [40].

II. PROBLEM FORMULATION

This paper concerns the discrete-time time-invariant bilinear dynamics:

$$x_{k+1} = Ax_k + Bu_k + \sum_{i=1}^{n_u} C_i x_k [u_k]_i + B_w w_k$$  \hspace{1cm} (4)

with state $x_k \in \mathbb{R}^{n_x}$, control inputs $u_k \in \mathbb{R}^{n_u}$ and disturbance $w_k$ at time instant $k$. For the sake of simplicity, we group the bilinear coefficient matrices $C = \begin{bmatrix} C_1^\top, ..., C_{n_u}^\top \end{bmatrix}^\top \in \mathbb{R}^{n_x \times n_u \times n_x}$ and assume that the states and control inputs are subject to the polyhedral constraints

$$x_k \in \mathcal{X} := \{ x \in \mathbb{R}^{n_x} | P_x x \leq p_x \},$$

and $u_k \in \mathcal{U} := \{ u \in \mathbb{R}^{n_u} | P_u u \leq p_u \}.$

An MPC controller can be thus designed by recursively solving the following optimal control problem in a receding horizon fashion,

$$\min_{x_0, x, u} \sum_{k=0}^{N-1} \ell_k(x_k, u_k) + \ell_N(x_N)$$  \hspace{1cm} (5a)

subject to:

$$x_0 = x(t),$$

$$ \forall k \in \{0, 1, ..., N - 1\}$$

$$x_{k+1} = Ax_k + Bu_k + \sum_{i=1}^{n_u} C_i x_k [u_k]_i + B_w w_k \quad | \lambda_k$$

$$x_{k+1} \in \mathcal{X}, \quad u_k \in \mathcal{U}$$  \hspace{1cm} (5d)

with $x = \begin{bmatrix} x_1^\top, ..., x_N^\top \end{bmatrix}^\top$, $u = \begin{bmatrix} u_1^\top, ..., u_{N-1}^\top \end{bmatrix}^\top$ and prediction horizon $N \in \mathbb{Z}_{>0}$. Here, the stage cost $\ell_k(\cdot, \cdot)$, $k \in \mathbb{Z}_{0}^{N-1}$ and terminal cost $\ell_N(\cdot)$ are quadratic and strongly convex, i.e.,

$$\ell_k(x, u) = \frac{1}{2} x^\top Q_k x + q_k^\top x + \frac{1}{2} u^\top R_k u + r_k^\top u,$$

$$\ell_N(x) = \frac{1}{2} x^\top Q_N x + q_N^\top x,$$

with $Q, P \in \mathbb{S}^+_{n_x}$ and $R \in \mathbb{S}^+_{n_u}$. Notice that although its objective is strongly convex, solving nonconvex Problem (5) is challenging due to the bilinear dynamics (5c).

III. ALGORITHM DEVELOPMENT

In this section, we will first introduce the novel interleaving horizon splitting scheme, after which the parametric nonconvex solver is elaborated. The convergence property of the proposed solver is studied in Section III-C, and this section is summarized with an intuitive interpretation of the proposed algorithm.

A. Interlacing horizon splitting Based Reformulation

This section presents the interleaving horizon splitting scheme used later to develop a parallelizable parametric solver to deal with (5). As depicted in Figure 1 its main idea is to bind the $k$-th input $u_k$ with state $x_{k+1}$. To this end, we introduce shorthands $\xi_0 = x_0$ and $\xi_k = [u_1^\top, x_1^\top]^\top$ for all $k \in \mathbb{Z}_1^N$ with associated constraint sets $\Xi_0 = \{ \xi \in \mathbb{R}^{n_x} : \xi_0 = x(t) \}$ and $\Xi_k = \{ \xi \in \mathbb{R}^{n_x+n_u} : \xi_0 = x(t) \}$ with $P_{\xi} = \text{diag}(P_u, P_x)$ and $p_{\xi} = [p_u^\top, P_x^\top]^\top$. Moreover, we denote the decoupled objective by

$$F_0(\xi_0) = \frac{1}{2} \xi_0^\top Q_0 \xi_0 + \xi_0^\top \xi_0,$$

$$F_k(\xi_k) = \frac{1}{2} \| \xi_k \|_{\text{diag}(R_{k-1}, Q_k)}^2 + [v_{k-1}^\top, q_k^\top]^\top \xi_k, \quad k \in \mathbb{Z}_1^N$$

and summarize the bilinear dynamics (4) by

$$D_k \xi_k + E_k \xi_{k+1} + (S_{k+1} \xi_{k+1} \otimes I_{n_u})^\top G_k \xi_k = d_k$$

with coefficient matrices

$$d_k = -B_w w_k, \quad k \in \mathbb{Z}_0^{N-1}$$

$$D_0 = A, \quad D_k = [0_{n_x \times n_u}, A], \quad k \in \mathbb{Z}_1^{N-1}$$

$$E_k = [B, -I_{n_u}], \quad S_k = [I_{n_u}, 0_{n_u \times n_x}], \quad k \in \mathbb{Z}_0^{N-1}$$

and

$$G_0 = C, \quad G_k = [0_{n_x \times n_u}, C_1^\top, ..., 0_{n_x \times n_u}, C_n^\top]^\top$$

for all $k \in \mathbb{Z}_1^N$. Accordingly, Problem (5) can be rewritten as

$$\min_{\xi} \sum_{k=0}^{N} F_k(\xi_k)$$  \hspace{1cm} (6a)

s.t. $(S_{k+1} \xi_{k+1} \otimes I_{n_u})^\top G_k \xi_k + D_k \xi_k + E_k \xi_{k+1} = d_k \quad | \lambda_k, \quad k \in \mathbb{Z}_0^{N-1}$$

$$\xi_k \in \Xi_k, \quad k \in \mathbb{Z}_0^N.$$  \hspace{1cm} (6b)

In comparison with the existing splitting schemes for nonlinear MPC [19]–[21], [29], which reformulate (5) as a series coupled small scale bilinear MPC problems with shorter horizons, the reformulation (6) based on the proposed horizon splitting does not introduce auxiliary variables. More importantly, the proposed horizon splitting scheme results in $N$ decoupled convex mpQP subproblems by dualizing the bilinear dynamics (6b). Due to the independence of the prediction horizon, the mpQP problem is much more scalable than the mpQP solution used in explicit linear MPC, and these solution maps...
of the subproblems can be pre-computed offline with a fixed memory requirement.

**B. Augmented Lagrangian Based Parallelizable Solver**

As Problem 6 is nonconvex, we design our primal-dual parallelizable solver for dealing with 5 by using the augmented Lagrangian to avoid the issue of duality gap [41, Chapter 11.K]. For a given primal dual trajectory $\xi$, we denote the augmented Lagrangian function with proximal weight $\rho \succ 0$ of the subproblems can be pre-computed offline with a fixed memory requirement.

$$L^\rho(\xi, \lambda, \bar{\xi}) = L_0^\rho(\xi_0, \bar{\xi}_0, \bar{\xi}_1) + \sum_{k=1}^{N-1} L_k^\rho(\xi_k, \bar{\xi}_k, \bar{\xi}_k, \bar{\xi}_k, \bar{\xi}_k, \bar{\xi}_k)$$

with

$$L_0^\rho(\xi_0, \bar{\xi}_0, \bar{\xi}_1) := F_0(\xi_0) + \lambda_0^T \left[D_0 + (S_1 \bar{\xi}_1 \otimes I_{n_y})^T G_0 \right] \xi_0 + \frac{\rho}{2} \| \xi_0 - \bar{\xi}_0 \|^2$$

$$L_k^\rho(\xi_k, \bar{\xi}_k, \bar{\xi}_k) := F_k(\xi_k) + \lambda_{k-1}^T \left[E_{k-1} \cdot \left( \lambda_{k-1}^T \left( G_{k-1} \bar{\xi}_k \right) \right) \right] \xi_k + \lambda_k^T \left[D_k + (S_{k+1} \bar{\xi}_{k+1} \otimes I_{n_y})^T G_k \right] \xi_k + \frac{\rho}{2} \| \xi_k - \bar{\xi}_k \|^2$$

$$L_N^\rho(\xi_N, \bar{\xi}_N, \bar{\xi}_N) := F_N(\xi_N) + \frac{\rho}{2} \| \xi_N - \bar{\xi}_N \|^2 + \lambda_{N-1}^T \left[ (E_{N-1} + \lambda_{N-1}^T G_{N-1}) \cdot S_N \right] \xi_N$$

with $\rho \succ 0$ and $\lambda := [\lambda_0^T, \lambda_1^T, \ldots, \lambda_N^T]^T$. If the primal-dual solution $(\xi^*, \lambda^*)$ of (6) is a regular KKT point, we have solving (6) equivalent to solving

$$\max_{\lambda} - \sum_{k=0}^{N-1} \lambda_k d_k + \min_{\bar{\xi}} L^\rho(\xi, \lambda, \bar{\xi})$$

s.t. $\xi \in \Xi = \Xi_0 \times \ldots \times \Xi_N$.

As $L^\rho$ is decoupled, our main idea to develop a parallelizable algorithm solving (6) is to design a primal-dual algorithm to solve the dual problem (7) with iteratively updating $\bar{\xi}$. Algorithm 1 outlines the main steps of the proposed algorithm for solving (6). Step 1) deals with decoupled problem (8) in parallel, which have explicit solutions as convex mpQPs. Particularly, their solution maps are piece-wise affine functions and can be pre-computed offline (See Section [IV-A]). Based on the local solutions $y$, Step 2) evaluates the sensitivities including the Hessian approximation of the Lagrangian $L^\rho$, the gradients of the decoupled objective $F_j$ and the bilinear dynamic residual $c_k$. The active Jacobian $P^{\rho}_{jk}$ is constructed based on active set at local solutions $\xi_k$. The terminal condition is given at Step 3). It is clear that if the conditions hold, we have the iterate $(y, \lambda)$ satisfies the first order optimality condition

$$\| \nabla_\xi L^\rho(y, \lambda, y) \| = O(\epsilon)$$

and the primal feasibility condition

$$\| D_k y_k + E_k y_{k+1} + (S_{k+1} y_{k+1} \otimes I_{n_y})^T G_k y_k \| = O(\epsilon)$$

for all $k \in \Xi_0^{N-1}$ up to a small error of order $O(\epsilon)$. Step 4) deals with a structured equality-constrained QP (9). In order to overcome the potential infeasibility caused by the linearization of nonlinear dynamic in constraint (10c), we introduce a decouple slackness $s_k$ for each active constraint (10b). This makes QP (10) always feasible no matter whether the original problem (6) is feasible or not. Accordingly, if one applies Algorithm 1 as an online solver for MPC, the resulting MPC controller is always feasible, i.e., the iteration of Algorithm 1 can be never stuck and it is independent of the initial condition $x(t)$. The implementation details of Algorithm 1 will be elaborated in Section [IV].

**Algorithm 1 Augmented Lagrangian Based Online Solver for Bilinear MPC**

**Input:** an initial guess of $(z, \lambda)$, a stop tolerance $\epsilon > 0$.

**Repeat:**

1) Solve decoupled problems in parallel

$$\min_{y_k \in \Xi_k} L_0^\rho(y_k, \lambda_0, x(t), z_1), \quad \min_{y_k \in \Xi_k} L_k^\rho(y_k, \lambda_k, z_k, z_{k+1}, \bar{\xi}_k), \quad k \in \Xi_0^{N-1}$$

(8a) (8b)

2) Evaluate sensitivities

$$H \approx \nabla_\xi L^\rho(y, \lambda, y), \quad g_k = \nabla F_k(y_k), \quad k \in \Xi_0^N$$

(9a)

$$c_k = D_k y_k + E_k y_{k+1} + (S_{k+1} y_{k+1} \otimes I_{n_y})^T G_k y_k - d_k$$

(9b)

and the active Jacobian $P^{\rho}_{jk}$ at local solution $\xi_k$.

3) Terminate if $\max_k \| c_k \| \leq \epsilon$ and $\max_k \| y_k - z_k \| \leq \epsilon$ hold.

4) Solve equality-constrained QP

$$\min_{y_k, \lambda_k} \frac{1}{2} \Delta y^T H \Delta y + \sum_{k=1}^{N} \{ s_k^T \Delta y_k + \mu \| s_k \|^2 \} \quad (10a)$$

s.t. $\Delta y_0 = 0$ (10b)

$$c_k + D_k \Delta y_k + E_k y_{k+1} + (S_{k+1} y_{k+1} \otimes I_{n_y})^T G_k \Delta y_k + \nabla F_k(\xi_k) \cdot S_{k+1} \Delta y_{k+1} = 0 \quad (10c)$$

$$P_{jk}^\rho \Delta y_k = s_k, \quad k \in \Xi_0^{N-1} \quad (10d)$$

5) Update $z^+ = y + \Delta y$ and $\lambda^+ = \lambda^Q$.

**Remark 1** ADMM can be used to solve a bi-convex optimization problem [26, Chapter 9.2]. However, in order to guarantee convergence, it cannot consider input/state constraint and the stage cost $J(x, u)$ has to be separable in $x$ and $u$.

**Remark 2** As discussed above, the proposed solver is always feasible even when the initial state makes the NMPC problem infeasible. This phenomenon was also observed in the compositions of the projection operator [42], [43], which converges a point that is closest to all the sets. However, in the nonconvex setup, the property of convergent results are unclear and remain for future research.
C. Local Convergence Property

This section discusses the local convergence property of Algorithm 1. The main idea of establishing the local convergence guarantee of Algorithm 1 follows two facts: local mpQPs have Lipschitz-continuous solution map and the coupled QP (10) is equivalent to Newton-type method. To this end, we introduce the following lemma to first establish the quadratic contraction of the solution of (10).

Lemma 1 Let the KKT point \((\xi^*, \lambda^*)\) of Problem (6) be regular such that the solution \(\xi^*\) is a local minimizer. Moreover, let the Hessian evaluation \(H\) satisfy

\[
H = \nabla_{\xi \xi} \mathcal{L}^0(y, \lambda, y) + O(||y - z||).
\]

and let \(\mu\) in (10) satisfy \(\frac{1}{\mu} \leq O(||y - z||)\). The solution of (10) locally satisfies,

\[
\|z^+ - \xi^*\| \leq \alpha \|y - \xi^*\|^2, \quad \|\lambda^+ - \lambda^*\| \leq \alpha \|y - \xi^*\|^2.
\]

Here, the term “local” means that the initial guess of primal-dual iterates are located in a small neighbourhood around the local solution \((\xi^*, \lambda^*)\). The proof of Lemma 1 follows the fact that the active sets are not changed locally based on the assumptions [44]. Then, the standard analysis of Newton’s method [40, Chapter 3.5] gives

\[
\left| \begin{array}{c}
\xi^+ - \xi^*

\lambda^+ - \lambda^*
\end{array} \right| \leq \frac{1}{\mu} \left| \begin{array}{c}
H - \nabla_{\xi \xi} \mathcal{L}^0(y, \lambda, y) \cdot O(||y - \xi^*||)

O(||y - \xi^*||^2)
\end{array} \right|
\]

Based on condition (11), we have that there exists a constant \(\alpha > 0\) such that the local quadratic contraction (12) holds. Then, we can summarize the convergence result as follows.

Theorem 1 Let all assumptions in Lemma 1 be satisfied. The iterates \(y\) of Algorithm 1 converges locally with quadratic rate.

Proof: As discussed in Section 3B, we have the local solution map \(y^*(z, \lambda)\) of convex mpQPs (8) are Lipschitz continuous such that we have

\[
\|y^+ - \xi^*\| \leq \eta \left| \begin{array}{c}
z^+ - \xi^*

\lambda^+ - \lambda^*
\end{array} \right|
\]

with \(y^+\) the iterate starting Algorithm 1 at \((z^+, \lambda^+)\). Substituting (12) into the inequality above yields

\[
\|y^+ - \xi^*\| \leq \alpha \cdot \eta \|y - \xi^*\|^2,
\]

which is sufficient to establish the local quadratic convergence of iterates \(y\).

D. Discussions

We would like to wrap up this section by explaining the key idea behind the proposed algorithm in a more intuitive way. In nonlinear programming, the main challenge comes from the detection of the active set. The proposed algorithm shifts this computational burden to the small local convex mpQP problems (8), whose problem size is independent of the prediction horizon and the explicit solution can be precomputed offline (See Section IV-A). With the estimated active sets, the coupled QP problem (10) reproduces a penalized SQP step and updates the estimate of dual variables in the system dynamics. Regarding the fact the Lagrange multiplier is a local parameter, the proximal term \(\frac{1}{2} ||\xi_k - \xi_k^*||^2\) avoids an aggressive primal update regarding the dual estimate \(\lambda_k\). In summary, the benefits of can be summarized as follows:

- It brings the efficiency of explicit MPC into a NMPC setup. The integration of the explicit mpQP solution can, on one side, return an accurate primal solution when good estimates of dual variables \(\lambda_k\) are given. On the other side, the active set estimation is pre-computed in the mpQP solution, which significantly improves the real-time efficiency.
- It retains the SQP structure, which make it compatible with any existing acceleration strategy developed for real-time SQP, such as warm-start.
- It is highly parallelizable. Not only the mpQP solutions in (8) can be executed in parallel, the equality constrained coupled QP can be solved in parallel [20], [24], [45] regarding the preceding benefit.

IV. IMPLEMENTATION DETAILS

This section elaborates the implementation details for Algorithm 1 with a particular emphasis on run-time aspects and limited memory requirement. Here, the implementation of Steps 3) and 5) turns out to be straightforward such that we focus on the implementation of Step 1), 2) and 4).

A. Local mpQP

We summarize the local mpQPs (8) into a uniform form

\[
\mathcal{P}(\theta_k) : \min_{y_k \in \mathbb{Z}_k} \frac{1}{2} y_k^T Q_k y_k + \theta_k^T y_k
\]

with parametric inputs \(\theta_k \in \mathbb{R}^{n_x+n_u}\) and coefficient matrices \(Q_k = \text{diag}(R_{k-1}, Q_k) + \rho I_{n_x+n_u}\) for all \(k \in \mathbb{Z}_k^+\). Here, the first problem is omitted as constraint \(\Xi_0\) enforces the initial condition \(\xi_0 = x(t)\). Based on the formulation of \(\mathcal{L}_k^0\), we can work out the explicit form of \(\theta_k\) as follows,

\[
\theta_k = [r_{k-1}^T; q_k^T]^T + (E_{k-1} + \text{mat}(G_{k-1} z_{k-1}) \cdot S_k)^T \lambda_{k-1} + [D_k + (S_{k+1} z_{k+1} \otimes I_{n_u})^T G_k]^T \lambda_k - \rho z_k
\]

\[
\theta_N = [r_{N-1}^T; q_N^T]^T - \rho z_N + (E_{N-1} + \text{mat}(G_{N-1} z_{N-1}) \cdot S_N)^T \lambda_{N-1}.
\]

Evaluating these parameters only require matrix-vector multiplications such that the complexity is \(\mathcal{O}(N \cdot (n_x+n_u)^2)\). In this paper, we use the enumeration-based multi-parametric QP algorithm from [46] for generating solution maps \(\xi_k^* : \mathbb{R}^{n_x+n_u} \rightarrow \mathbb{R}^{n_x+n_u}\) of (13). The complexity of pre-processing the small-scale QPs (13) depends on the number \(N_{R,k}\) of critical regions over which the PWA optimizers \(\xi_k^*(\cdot)\) are defined [38]. Here, we assume that each parametric QP is post-processed, off-line, to obtain binary search trees [47] in \(\mathcal{O}(N_{R,k}^2)\) time. Once the trees are constructed, they provide a fast evaluation of the solution maps in (13) in time that is logarithmic in the number of regions, thus establishing the \(\mathcal{O}(\sum_k \log_2(N_{R,k}))\) on-line computational bound. The
memory requirements are directly proportional to the number of critical regions with each region represented by a finite number of affine half-spaces.

B. Sensitivities Evaluation

Step 2) of Algorithm 1 evaluates the sensitivities \( g_k, c_k, \tilde{P}_k \) and \( H \). As we consider the quadratic cost, the gradients \( g_k \) can be easily evaluated with analytical form. Moreover, the primal feasibility residual \( c_k \) and active Jacobian \( \tilde{P}_k \) are also straightforward. Therefore, we focus on the computation of the Hessian matrix \( H \) in this subsection.

As we used an interleaving horizon splitting scheme, the exact Hessian \( \nabla_{\xi} L(y, \lambda, y) \) is not block diagonal with respect to each \( y_k \) but banded block diagonal. However, due to the cross over block caused by the bilinear dynamic, we can work out each block analytically as follows

\[
\nabla_{\xi} L(y, \lambda, y) = \begin{bmatrix}
Q_0 & S_{0,1} & 0 & \cdots & 0 \\
S_{1,0} & Q_1 & S_{1,2} & \cdots & 0 \\
0 & \ddots & \ddots & \ddots & \ddots \\
0 & 0 & \ddots & \ddots & \ddots \\
0 & 0 & 0 & \ddots & S_{N-1,N} & Q_N \\
\end{bmatrix}
\]

with blocks

\[
S_{0,1} = S_{1,0} = [C_1^{T} \lambda_0, \ldots, C_n^{T} \lambda_0, 0_{n_x \times n_x}] \in \mathbb{R}^{n_x \times (n_u+n_x)} \\
S_{k,k+1} = S_{k,k+1}^{T} = [C_{k}^{T} \lambda_k, \ldots, C_{n_u}^{T} \lambda_k, 0_{n_u \times n_u}] \in \mathbb{R}^{(n_u+n_x) \times (n_u+n_x)}
\]

for all \( k \in \mathbb{Z}^{N-1}. \) It is clear that evaluating the exact Hessian is equivalent to evaluate \( C_k^{T} \lambda_k \) for all \( i \in \mathbb{Z}_i^{n_u} \) and \( k \in \mathbb{Z}_k^{N-1}. \) Therefore, its computational complexity is only \( O(N n_u n_x^2). \)

In practice, some heuristics can be adopted to achieve a better numerical robustness on the convergence performance of Algorithm 1 such as enforcing \( H \approx \nabla_{\xi} L(y, \lambda, y) \geq 0 \) with adding a regularization term, i.e., \( H = \nabla_{\xi} L(y, \lambda, y) + \sigma I \) with \( \sigma \geq 0. \)

C. Coupled QP

The coupled QP (10) has no inequality constraints such that solving (10) is equivalent to deal with the resulting linear KKT system

\[
\begin{bmatrix}
H + \rho \tilde{P}_k \tilde{P}_k^T & J^T \\
J & 0
\end{bmatrix}
\begin{bmatrix}
\Delta y \\
\lambda^{QP}
\end{bmatrix} =
\begin{bmatrix}
-g \\
-c
\end{bmatrix}
\]

with

\[
\begin{bmatrix}
\tilde{D}_0 & \tilde{E}_0 \\
\tilde{D}_1 & \tilde{E}_1 \\
\vdots & \ddots \\
\tilde{D}_N & \tilde{E}_N
\end{bmatrix}
\]

and for all \( k \in \mathbb{Z}_k^{N}, \)

\[
\tilde{D}_k = D_k + (S_{k+1} z_{k+1} \otimes I_{n_a})^T G_k, \\
\tilde{E}_k = E_k + \text{mat}(G_k z_k) \cdot S_{k+1}.
\]

If we change the layout of the KKT matrix \( \mathcal{H} \) by resorting \( w \) as

\[
(\Delta y_0, \lambda_0^{QP}, \Delta y_1, \lambda_1^{QP}, \ldots, \Delta y_{N-1}, \lambda_{N-1}^{QP}, \Delta y_N),
\]

the structure of the KKT matrix \( \mathcal{H} \) can be further exploit, which is essentially banded-diagonal such that the Schur-complement based back-forward sweeps can be used to deal with it efficiently. In order to illustrate this idea, we consider \( N = 2 \) such that the resulting KKT system is

\[
\begin{bmatrix}
Q_0 & S_{0,1} & 0 & \cdots & 0 \\
S_{1,0} & Q_1 & S_{1,2} & \cdots & 0 \\
0 & \ddots & \ddots & \ddots & \ddots \\
0 & 0 & \ddots & \ddots & \ddots \\
0 & 0 & 0 & \ddots & S_{N-1,N} & Q_N \\
\end{bmatrix}
\begin{bmatrix}
\Delta y_0 \\
\lambda_0^{QP} \\
\Delta y_1 \\
\lambda_1^{QP} \\
\Delta y_2 \\
\end{bmatrix} =
\begin{bmatrix}
g_0 \\
-c_0 \\
g_1 \\
c_1 \\
g_2
\end{bmatrix}
\]

with \( Q_k = \text{diag}(R_{k-1}, Q_k) + \mu(\tilde{P}_k)^T \tilde{P}_k. \) We start the backward sweep by considering the whole KKT matrix as a 2x2 block matrix. Then, applying the Schur complement with respect to the lower left block \( \mathcal{Q}_2 \) yields a reduced KKT matrix

\[
\begin{bmatrix}
Q_0 & S_{0,1} & 0 & \cdots & 0 \\
S_{1,0} & Q_1 & S_{1,2} & \cdots & 0 \\
0 & \ddots & \ddots & \ddots & \ddots \\
0 & 0 & \ddots & \ddots & \ddots \\
0 & 0 & 0 & \ddots & S_{N-1,N} & Q_N \\
\end{bmatrix}
\begin{bmatrix}
\Delta y_0 \\
\lambda_0^{QP} \\
\Delta y_1 \\
\lambda_1^{QP} \\
\Delta y_2 \\
\end{bmatrix} =
\begin{bmatrix}
g_0 \\
-c_0 \\
g_1 \\
c_1 \\
g_2
\end{bmatrix}
\]

Applying the Schur complement once more results in a reduced KKT system with respect to only \( (\Delta y_0, \lambda_0) \) such that we can apply the initial condition \( \Delta y_0 = 0 \) and start the forward sweeps to get primal-dual solution \( (\Delta y, \lambda). \) This method has been shown that it is equivalent to the Riccati recursion in dealing with LQR problems [48]. As the update of right-hand side of the KKT system only requires matrix vector multiplication, we can have the computational complexity is dominated by the matrix update, which is \( O(N(n_x + n_u)^3). \)

V. NUMERICAL RESULTS

The parametric QP solution of the proposed algorithm is generated by the multi-parametric toolbox (MPT) [49].

A. Bilinear Building Control

In this part, the proposed algorithm is compared with an efficient optimal control solver ACADOS [50]. The code generation in ACADOS is based on the SQP method with exact Hessian and full condensing. Both the proposed method and the ACADOS code uses the mirror method to regularize the indefinite QP problem [51]. To ensure a fair comparison, both solvers use the same initialization in the first iteration and are warm started with the solution from the last iteration. The computational time is the sum of the CPU time returned by the solver. And the results in this section are generated by a laptop with Intel i7-11800H and 32 GB memory.

This comparison studies the building control problem, whose model refers to a room model reported in [52]. The
The states are indoor temperature, indoor wall temperature, external wall temperature and the supply air temperature. The control input is the valve position in the air handling unit, where the heat transfer between the air and the hot water flowing in the heating coil results in the bilinear term in the system dynamics. The disturbances \( w \) are respectively the solar radiation and the outdoor temperature. 

In building control, a common practice is to apply certainty equivalence control, which uses weather forecast in the MPC formulation. Meanwhile, the building evolves under the actual weather condition that is similar but not identical to the weather forecast. The following result is generated while respecting this fact, we ran 100 Monte-Carlo test with recorded weather from tomorrow.io in winter (Figure 2(a) and (b) plot one sampled weather condition). The weather forecast used in the MPC problem is the recorded weather perturbed by zero-mean random noise, while the simulation uses the recorded weather. The prediction horizon is set to 10 with an objective of minimizing energy consumption, whose loss function is

\[
\ell_k(x_k, u_k) = u_k^2.
\]

The indoor temperature is bounded within \([22, 24]°C\) to ensure the occupant comfort. The control input (i.e. fractional valve position) is bounded within \([0, 1]\).

**Remark 3** It is possible to define an objective as \(\ell_k(x_k, u_k) = |u_k|\). The resulting local problem can be reformulated as a linear program, and thus also a parametric QP. We use a quadratic loss function here to avoid unnecessary confusion.

To show the efficiency of the proposed algorithm, the proposed algorithm is not parallelized. The comparison of maximal and mean solution time on different convergence tolerance are summarized in Table 1. Different tolerance uses the same set of sampled weather to ensure a consistent comparison. The maximal solution time mainly occur at the first step of each Monte-Carlo run, where good initialization are not available yet (i.e. cold-start), the proposed algorithm uses on average 20\% of maximal solution time in ACADOS, and thus shows significantly better performance. The mean solution reflects the averaged performance with good initial solution guess (i.e. warm-start), where the mean solution time of the proposed algorithm only costs on average 65\% of the solution time given by ACADOS under different convergence tolerance.

| Tolerance | 10^{-4} | 10^{-5} | 10^{-6} |
|-----------|---------|---------|---------|
| sol time (ms) | max | mean | max | mean | max | mean |
| **ACADOS** | 4.603 | 0.179 | 4.985 | 0.196 | 5.140 | 0.203 |
| proposed | 0.931 | 0.103 | 0.979 | 0.130 | 0.982 | 0.148 |

**Table 1**

**Statistics of the solution time at different tolerance**

Besides the comparison of the numerical efficiency, the property that the proposed algorithm always output a solution even when the problem is feasible can be useful in practice, which typically the case in building control. Due to the uncertain occupant behaviour, such as open the window, the indoor climate can be significantly perturbed resulting in an infeasible state for the MPC problem. Consider that the occupant opens the window at 10:00 A.M to bring in fresh air, which leads to a sudden drop of indoor temperature (Figure 2(c)). Such sudden temperature drops causes infeasibility, which leads to the failures of the ACADOS solver. However, the proposed algorithm can still give reasonable control input and quickly recovers the indoor temperature to a comfort level.

### B. Bilinear DC Motor Control with C2000

In this part, the proposed algorithm is deployed on an embedded system, Texas Instrument C2000 LaunchPad F28379D. The dynamics of the bilinear DC motor is

\[
\begin{align*}
\frac{dx_1}{dt} &= -\frac{R_a}{L_a} x_1 - \frac{K_m}{L_a} x_2 u + \frac{V_s}{L_a} \\
\frac{dx_2}{dt} &= -\frac{B}{J} x_2 + \frac{K_m}{J} x_1 u - \frac{T_e}{J},
\end{align*}
\]

where states \(x_1, x_2\) are respectively rotor current and angular velocity, the control input \(u\) is stator current. The armature resistance \(R_a\) is 10 ohm with an inductance \(L_a\) of 60 mH. The motor constant \(K_m\) is 0.2297 \(V/(A\text{rad/s})\), the damping ratio \(B\) is 0.0024, the inertia \(J\) is 0.008949. \(V_s\) is the stator voltage and \(T_e\) is the external torque, which are respectively 60 V and 1.46 Nm. We design a speed controller such that the MPC problem has an objective of \(\ell_k(x_k, u_k) = (x_{k,2} - v_{ref})\), where \(v_{ref}\) denotes the reference speed. The angular velocity is bounded within \([16, 20]\text{rad/s}\). The continuous dynamics is discretized by Euler method with a sampling time at 10ms. The prediction horizon is set to 3. The emulated control results are shown in Figure 3 where the motor follows the reference trajectory while respecting the speed constraint. The proposed algorithm can execute in real-time with a maximal and an averaged execution time at 2.088 ms and 1.764 ms respectively.

### VI. Conclusion

This paper proposes a novel augmented Lagrangian based nonconvex solver for bilinear model predictive control. The proposed algorithm brings the benefit of explicit MPC to
nonlinear MPC via a novel horizon splitting scheme. The proposed algorithm is validated by building simulation and is deployed on C2000 embedded system.

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