SOLID VON NEUMANN ALGEBRAS

NARUTAKA OZAWA

Dedicated to Professor Masamichi Takesaki on the occasion of his 70th birthday.

Abstract. We prove that the relative commutant of a diffuse von Neumann subalgebra in a hyperbolic group von Neumann algebra is always injective. It follows that any non-injective subfactor in a hyperbolic group von Neumann algebra is non-$\Gamma$ and prime. The proof is based on $C^*$-algebra theory.

1. Introduction

Recall that a von Neumann algebra is said to be diffuse if it does not contain a minimal projection. We say a von Neumann algebra $\mathcal{M}$ is solid if for any diffuse von Neumann subalgebra $\mathcal{A}$ in $\mathcal{M}$, the relative commutant $\mathcal{A}' \cap \mathcal{M}$ is injective. A solid von Neumann algebra is necessarily finite. We prove the following theorem which answers a question of Ge [Ge] that whether the free group factors are solid.

Theorem 1. The group von Neumann algebra $L\Gamma$ of a hyperbolic group $\Gamma$ is solid.

Recall that a factor $\mathcal{M}$ is said to be prime if $\mathcal{M} \cong \mathcal{M}_1 \bar{\otimes} \mathcal{M}_2$ implies either $\mathcal{M}_1$ or $\mathcal{M}_2$ is finite dimensional. The existence of such factors was proved by Popa [Po2] who showed the group factors of uncountable free groups are prime. The case for countable free groups had remained open for some time, but was settled by Ge [Ge]. This was generalized by Ştefan [St] to their subfactors of finite index. Our theorem gives a further generalization. Indeed, combined with a result of Popa [Po3] (Proposition 7 in this paper), we obtain the following proposition.

Proposition 2. A subfactor of a solid factor is again solid and a solid factor is non-$\Gamma$ and prime unless it is injective.

Notably, this provides infinitely many prime $\text{II}_1$-factors (with the property (T)). Indeed thanks to a theorem of Cowling and Haagerup [CH], for lattices $\Gamma_n$ in $\text{Sp}(1, n)$, we have $L\Gamma_m \not\cong L\Gamma_n$ whenever $m \neq n$. However, we are unaware of any non-injective solid factor with the Haagerup property other than the free group factor(s). This proposition also distinguishes the factor $(L\mathcal{F}_\infty \bar{\otimes} L_\infty [0, 1]) \ast L\mathcal{F}_\infty$ from the free group factor $L\mathcal{F}_\infty$, which answers a question of Shlyakhtenko.

Date: February 7, 2003.

2000 Mathematics Subject Classification. Primary 46L10; Secondary 20F67.

Key words and phrases. Group von Neumann algebras, hyperbolic groups, solid, prime.

The author was supported by the JSPS Postdoctoral Fellowships for Research Abroad.
2. Preliminary Results on Reduced Group $C^*$-algebras

For a discrete group $\Gamma$, we denote by $\lambda$ (resp. $\rho$) the left (resp. right) regular representation on $\ell_2 \Gamma$ and let $C^*_\lambda \Gamma$ (resp. $C^*_\rho \Gamma$) be the $C^*$-algebra generated by $\lambda(\Gamma)$ (resp. $\rho(\Gamma)$) in $B(\ell_2 \Gamma)$ and $L\Gamma = (C^*_\lambda \Gamma)''$ be its weak closure. The $C^*$-algebra $C^*_\lambda \Gamma$ (resp. the von Neumann algebra $L\Gamma$) is called the reduced group $C^*$-algebra (resp. the group von Neumann algebra).

The study of $C^*$-norms on tensor products was initiated in the 50’s by Turumaru, but the first substantial result was obtained by Takesaki [Ta] who showed the minimal tensor norm is the smallest among the possible $C^*$-norms on a tensor product of $C^*$-algebras. He also introduced the notion of nuclearity and found that the reduced group $C^*$-algebra $C^*_\lambda \mathbb{F}_2$ of the free group $\mathbb{F}_2$ on two generators is not nuclear. Namely, the $*$-homomorphism

$$C^*_\lambda \mathbb{F}_2 \otimes C^*_\rho \mathbb{F}_2 \ni \sum_{i=1}^n a_i \otimes x_i \mapsto \sum_{i=1}^n a_i x_i \in \mathbb{B}(\ell_2 \mathbb{F}_2)$$

is not continuous w.r.t. the minimal tensor norm. Yet, Akemann and Ostrand [AO] proved a remarkable theorem that it is continuous if one composes it with the quotient map $\pi$ from $\mathbb{B}(\ell_2 \mathbb{F}_2)$ onto the Calkin algebra $\mathbb{B}(\ell_2 \mathbb{F}_2)/\mathbb{K}(\ell_2 \mathbb{F}_2)$. By a completely different argument, Skandalis (Théorème 4.4 in [Sk]) proved the same for all discrete subgroups in connected simple Lie groups of rank one, and Higson and Guentner (Lemma 6.2.8 in [HG]) for all hyperbolic groups. In summary,

**Theorem 3.** Let $\Gamma$ be a hyperbolic group or a discrete subgroup in a connected simple Lie group of rank one. Then, the $*$-homomorphism

$$\nu_\Gamma: C^*_\lambda \Gamma \otimes C^*_\rho \Gamma \ni \sum_{i=1}^n a_i \otimes x_i \mapsto \pi(\sum_{i=1}^n a_i x_i) \in \mathbb{B}(\ell_2 \Gamma)/\mathbb{K}(\ell_2 \Gamma)$$

is continuous w.r.t. the minimal tensor norm on $C^*_\lambda \Gamma \otimes C^*_\rho \Gamma$.

The crucial ingredient in the proof was the amenability of the action of $\Gamma$ on a suitable boundary which is ‘small at infinity’. For the information on amenable actions, we refer the reader to the book [AR] of Anantharaman-Delaroche and Renault. Since $\Gamma$ acts amenably on a compact set, $C^*_\lambda \Gamma$ is embeddable into a nuclear $C^*$-algebra and thus has the property (C) of Archbold and Batty (Theorem 3.6 in [AB]). Although we do not need this fact, we mention that the property (C) is equivalent to exactness by a deep theorem of Kirchberg [Ki]. By Effros and Haagerup’s theorem (Theorem 5.1 in [EH]), the property (C) implies the local reflexivity. In summary,

**Lemma 4.** Let $\Gamma$ be as above. Then, $C^*_\lambda \Gamma$ is locally reflexive, i.e., for any finite dimensional operator system $E \subset (C^*_\lambda \Gamma)^{**}$, there is a net of unital completely positive maps $\theta_i: E \to C^*_\lambda \Gamma$ which converges to $\text{id}_E$ in the point-weak$^*$ topology.
3. Proof of Theorem

We recall the following principle \[\text{Ch}\]; if \(\Psi: A \to B\) is a unital completely positive map and its restriction to a \(C^*\)-subalgebra \(A_0 \subset A\) is multiplicative, then we have \(\Psi(ab) = \Psi(a)\Psi(x)\Psi(b)\) for any \(a, b \in A_0\) and \(x \in A\).

Let \(\mathcal{N} \subset \mathcal{M}\) be finite von Neumann algebras with a faithful trace \(\tau\) on \(\mathcal{M}\). Then, there is a normal conditional expectation \(\varepsilon_{\mathcal{N}}\) from \(\mathcal{M}\) onto \(\mathcal{N}\), which is defined by the relation \(\tau(\varepsilon_{\mathcal{N}}(a)x) = \tau(ax)\) for \(a \in \mathcal{M}\) and \(x \in \mathcal{N}\). This implies that a von Neumann subalgebra of a finite injective von Neumann algebra is again injective. Moreover, \(\varepsilon_{\mathcal{N}}\) is unique in the sense that any trace preserving conditional expectation from \(\mathcal{M}\) onto \(\mathcal{N}\) coincides with \(\varepsilon_{\mathcal{N}}\). Indeed, for any \(a \in \mathcal{M}\) and \(x \in \mathcal{N}\), we have

\[
\tau(\varepsilon'(a)x) = \tau(\varepsilon'(ax)) = \tau(ax) = \tau(\varepsilon_{\mathcal{N}}(a)x).
\]

We say a von Neumann subalgebra \(\mathcal{M}\) in \(\mathcal{B}(\mathcal{H})\) satisfies the condition (AO) if there are unital ultraweakly dense \(C^*\)-subalgebras \(B \subset \mathcal{M}\) and \(C \subset \mathcal{M}'\) such that \(B\) is locally reflexive and the \(*\)-homomorphism

\[
\nu: B \otimes C \ni \sum_{i=1}^{n} a_i \otimes x_i \mapsto \pi\left(\sum_{i=1}^{n} a_i x_i\right) \in \mathcal{B}(\mathcal{H})/\mathcal{K}(\mathcal{H})
\]

is continuous w.r.t. the minimal tensor norm on \(B \otimes C\). We have seen in Section 2 that the group von Neumann algebra \(\mathcal{L}\Gamma\) satisfies the condition (AO) whenever \(\Gamma\) is a hyperbolic group or a discrete subgroup in a connected simple Lie group of rank one.

Lemma 5. Let \(B \subset \mathcal{M}\) and \(C \subset \mathcal{M}'\) be unital ultraweakly dense \(C^*\)-subalgebras with \(B\) locally reflexive and let \(\mathcal{N} \subset \mathcal{M}\) be a von Neumann subalgebra with a normal conditional expectation \(\varepsilon_{\mathcal{N}}\) onto \(\mathcal{N}\). Assume that the unital completely positive map

\[
\Phi_{\mathcal{N}}: B \otimes C \ni \sum_{i=1}^{n} a_i \otimes x_i \mapsto \sum_{i=1}^{n} \varepsilon_{\mathcal{N}}(a_i)x_i \in \mathcal{B}(\mathcal{H})
\]

is continuous w.r.t. the minimal tensor norm on \(B \otimes C\). Then \(\mathcal{N}\) is injective.

Proof. Since \(B \otimes_{\min} C \subset \mathcal{B}(\mathcal{H}) \otimes_{\min} C\) and \(\mathcal{B}(\mathcal{H})\) is injective, \(\Phi_{\mathcal{N}}\) extends to a unital completely positive map \(\tilde{\Psi}: \mathcal{B}(\mathcal{H}) \otimes_{\min} C \to \mathcal{B}(\mathcal{H})\). Then, \(\Psi\) is automatically a \(C\)-bimodule map. Put \(\psi(a) = \Psi(a \otimes 1)\) for \(a \in \mathcal{B}(\mathcal{H})\). Then, for every \(a \in \mathcal{B}(\mathcal{H})\) and \(x \in C\), we have

\[
x\psi(a) = \Psi(1 \otimes x)\Psi(a \otimes 1) = \Psi(a \otimes x) = \Psi(a \otimes 1)\Psi(1 \otimes x) = \psi(a)x.
\]

Hence, \(\psi\) maps \(\mathcal{B}(\mathcal{H})\) into \(C' = \mathcal{M}\). It follows that \(\tilde{\psi} = \varepsilon_{\mathcal{N}}\psi: \mathcal{B}(\mathcal{H}) \to \mathcal{N}\) is a unital completely positive map such that \(\tilde{\psi}|_B = \varepsilon_{\mathcal{N}}|_B\). Let \(I\) be the set of all triples \((\mathcal{E}, \mathcal{F}, \varepsilon)\), where \(\mathcal{E} \subset \mathcal{N}\) and \(\mathcal{F} \subset \mathcal{N}_*\) are finite subsets and \(\varepsilon > 0\) is arbitrary. The set \(I\) is directed by the order relation \((\mathcal{E}_1, \mathcal{F}_1, \varepsilon_1) \leq (\mathcal{E}_2, \mathcal{F}_2, \varepsilon_2)\) if and only if...
$E_1 \subset E_2$, $F_1 \subset F_2$ and $\varepsilon_1 \geq \varepsilon_2$. Let $i = (E, F, \varepsilon) \in I$ and let $E \subset N$ be the finite dimensional operator system generated by $E$. We note that $E \subset M = pB^*$ and $\varepsilon_N^*(F) \subset M_* = pB^*$, where $p \in B^{**}$ is the central projection supporting the identity representation of $B$ on $H$. Since $B$ is locally reflexive (cf. Lemma 4), there is a unital completely positive map $\theta_i : E \to B$ such that for $a \in E$ and $f \in F$, we have

$$|\langle \varepsilon_N \theta_i(a), f \rangle - \langle a, f \rangle| = |\langle \theta_i(a), \varepsilon_N^*(f) \rangle - \langle a, \varepsilon_N^*(f) \rangle| < \varepsilon.$$  

Take a unital completely positive extension $\bar{\theta}_i : B(H) \to B(H)$ of $\theta_i$ and let $\sigma_i = \bar{\psi} \theta_i : B(H) \to N$. It follows that for $a \in E$ and $f \in F$, we have

$$|\langle \sigma_i(a), f \rangle - \langle a, f \rangle| = |\langle \varepsilon_N \theta_i(a), f \rangle - \langle a, f \rangle| < \varepsilon.$$  

Therefore, any cluster point, in the point-ultraweak topology, of the net $\{\sigma_i\}_{i \in I}$ is a conditional expectation from $B(H)$ onto $N$. □

We now prove Theorem 6 or more precisely,

**Theorem 6.** A finite von Neumann algebra $M$ satisfying the condition (AO) is solid.

**Proof.** Let $A$ be a diffuse von Neumann subalgebra in $M$. Passing to a subalgebra if necessary, we may assume $A$ is abelian and prove the injectivity of $N = A' \cap M$. It suffices to show $\Phi_N$ in Lemma 3 is continuous on $B \otimes_{\min} C$. Since $A$ is diffuse, it is generated by a unitary $u \in A$ such that $\lim_{k \to \infty} u^k = 0$ ultraweakly. Let $\Psi_n(a) = n^{-1} \sum_{k=1}^n u^k au^{-k}$ for $a \in B(H)$ and let $\Psi : B(H) \to B(H)$ be its cluster point in the point-ultraweak topology. It is not hard to see that $\Psi|_M$ is a trace preserving conditional expectation onto $N$ and hence $\Psi|_M = \varepsilon_N$. It follows that for any $\sum_{i=1}^n a_i \otimes x_i \in M \otimes M'$, we have

$$\Psi(\sum_{i=1}^n a_i \otimes x_i) = \sum_{i=1}^n \varepsilon_N(a_i)x_i = \Phi_N(\sum_{i=1}^n a_i \otimes x_i).$$  

Since $\lim_{k \to \infty} u^k = 0$ ultraweakly, we have $K(H) \subset \ker \Psi$. This implies $\Psi = \bar{\Psi} \nu$ for some unital completely positive map $\bar{\Psi} : B(H)/K(H) \to B(H)$. Since $\nu$ in the condition (AO) is continuous on $B \otimes_{\min} C$, so is $\Phi_N = \bar{\Psi} \nu$. □

The following proposition was communicated to us by Popa [Po3]. We are grateful to him for allowing us to present it here.

**Proposition 7.** Assume the type II$_1$ factor $M$ (with separable predual) contains a non-injective von Neumann subalgebra $N_0'$ such that $N_0' \cap M^\omega$ is a diffuse von Neumann algebra, where $M^\omega$ is an ultrapower algebra of $M$. Then there exists a non-injective von Neumann subalgebra $N_1' \subset M$ such that $N_1' \cap M$ is diffuse.
**Proof.** Replacing it with a subalgebra if necessary, we may assume the non-injective von Neumann subalgebra \( \mathcal{N}_0 \) is generated by a finite set \( \{x_1, x_2, \ldots, x_m\} \). By Connes’ characterizations of injectivity (Theorem 5.1 in [Co]), it follows that there exists \( \varepsilon > 0 \) such that if \( \{x'_1, \ldots, x'_m\} \subset \mathcal{M} \) are so that \( \|x'_i - x_i\|_2 \leq \varepsilon \) then \( \{x'_i\}_i \) generates a non-injective von Neumann subalgebra in \( \mathcal{M} \). Indeed, if there existed injective von Neumann algebras \( B_k \subset \mathcal{M} \) such that \( \lim_k \|x_i - \varepsilon B_k(x_i)\|_2 = 0, \forall i \), then for any \( \sum_{j=1}^n a_j \otimes y_j \in \mathcal{N}_0 \otimes \mathcal{M}' \), we would have

\[
\| \sum_{j=1}^n a_j y_j \|_{\mathcal{B}(\mathcal{H})} \leq \liminf_{k \to \infty} \| \sum_{j=1}^n \varepsilon B_k(a_j) y_j \|_{\mathcal{B}(\mathcal{H})} = \liminf_{k \to \infty} \| \sum_{j=1}^n \varepsilon B_k(a_j) \otimes y_j \|_{B_k \otimes_{\text{min}} \mathcal{M}'} \leq \| \sum_{j=1}^n a_j \otimes y_j \|_{\mathcal{N}_0 \otimes_{\text{min}} \mathcal{M}'},
\]

which would imply \( C^*(\mathcal{N}_0, \mathcal{M}') \cong \mathcal{N}_0 \otimes_{\text{min}} \mathcal{M}' \) and the injectivity of \( \mathcal{N}_0 \) (cf. the proof of Lemma 5).

Since \( \mathcal{N}_0 \cap \mathcal{M}^\omega \) is diffuse, it follows by induction that there exists a sequence of mutually commuting, \( \tau \)-independent two dimensional abelian \( * \)-subalgebras \( \mathcal{A}_n \subset \mathcal{M} \), with minimal projections of trace 1/2, such that

\[
\|\varepsilon_{\mathcal{A}_{n+1} \cap \mathcal{M}}(x_i) - x_i\|_2 < \varepsilon/2^{n+1}, \forall i.
\]

But if we denote by \( B_n = \mathcal{A}_1 \vee \mathcal{A}_2 \vee \ldots \vee \mathcal{A}_n \) then we also have

\[
\|\varepsilon_{\mathcal{B}_n \cap \mathcal{M}}(x_i) - \varepsilon_{\mathcal{B}_n \cap \mathcal{M}}(\varepsilon_{\mathcal{A}_{n+1} \cap \mathcal{M}}(x_i))\|_2 < \varepsilon/2^{n+1}, \forall i.
\]

Since \( \varepsilon_{\mathcal{B}_n \cap \mathcal{M}} \circ \varepsilon_{\mathcal{A}_{n+1} \cap \mathcal{M}} = \varepsilon_{\mathcal{B}_n \cap \mathcal{M}} \), if we denote by \( \mathcal{A} = \vee_n \mathcal{B}_n \) and take into account that \( \varepsilon_{\mathcal{A} \cap \mathcal{M}} = \lim_n \varepsilon_{\mathcal{B}_n \cap \mathcal{M}} \) (see e.g., [Po1]), then by triangle inequalities we get

\[
\|x_i - \varepsilon_{\mathcal{A} \cap \mathcal{M}}(x_i)\|_2 \leq \varepsilon, \forall i.
\]

Thus, if we take \( \mathcal{N}_1 \) to be the von Neumann algebra generated by

\[
x'_i = \varepsilon_{\mathcal{A} \cap \mathcal{M}}(x_i), \ 1 \leq i \leq m,
\]

then \( \mathcal{N}_1 \) satisfies the required conditions. \( \square \)

4. **Remark**

We remark the possibility that, for a hyperbolic group \( \Gamma \), the \( * \)-homomorphism

\[
\mathcal{L}\Gamma \otimes C_p^*\Gamma \ni \sum_{i=1}^n a_i \otimes x_i \mapsto \pi(\sum_{i=1}^n a_i x_i) \in \mathcal{B}(\ell_2\Gamma)/\mathbb{K}(\ell_2\Gamma)
\]

may be continuous w.r.t. the minimal tensor norm. If this is the case, then it would follow that a von Neumann subalgebra \( \mathcal{N} \subset \mathcal{L}\Gamma \) is injective if and only if

\[
C^*(\mathcal{N}, C_p^*\Gamma) \cap \mathbb{K}(\ell_2\Gamma) = \{0\},
\]

which would reprove our results (modulo Theorem 2.1 in [Co]).
Acknowledgment. The author would like to thank Professor Sorin Popa for showing him Proposition 7 and Professor Nigel Higson for providing him the references [HG] and [Sk]. This research was carried out while the author was visiting the University of California at Los Angeles under the support of the Japanese Society for the Promotion of Science Postdoctoral Fellowships for Research Abroad. He gratefully acknowledges the kind hospitality of UCLA.

References

[AO] C. A. Akemann and P. A. Ostrand. On a tensor product $C^*$-algebra associated with the free group on two generators. J. Math. Soc. Japan 27 (1975), no. 4, 589–599.

[AR] C. Anantharaman-Delaroche and J. Renault. Amenable groupoids. With a foreword by Georges Skandalis and Appendix B by E. Germain. Monographies de L’Enseignement Mathématique 36. Geneva, 2000.

[AB] R. J. Archbold and C. J. K. Batty. $C^*$-tensor norms and slice maps. J. London Math. Soc. (2) 22 (1980), no. 1, 127–138.

[Ch] M.-D. Choi. A Schwarz inequality for positive linear maps on $C^*$-algebras. Illinois J. Math. 18 (1974), 565–574.

[Co] A. Connes. Classification of injective factors. Cases $II_1$, $II_\infty$, $III_\lambda$, $\lambda \neq 1$. Ann. of Math. (2) 104 (1976), no. 1, 73–115.

[CH] M. Cowling and U. Haagerup. Completely bounded multipliers of the Fourier algebra of a simple Lie group of real rank one. Invent. Math. 96 (1989), no. 3, 507–549.

[EH] E. G. Effros and U. Haagerup. Lifting problems and local reflexivity for $C^*$-algebras. Duke Math. J. 52 (1985), no. 1, 103–128.

[Ge] L. Ge. Applications of free entropy to finite von Neumann algebras. II. Ann. of Math. (2) 147 (1998), no. 1, 143–157.

[HG] N. Higson and E. Guentner. Group $C^*$-algebras and $K$-theory. CIME Lecture Notes. Preprint 2002.

[Ki] E. Kirchberg. On subalgebras of the CAR-algebra. J. Funct. Anal. 129 (1995), no. 1, 35–63.

[Po1] S. Popa. On a problem of R. V. Kadison on maximal abelian $*$-subalgebras in factors. Invent. Math. 65 (1981/82), no. 2, 269–281.

[Po2] S. Popa. Orthogonal pairs of $*$-subalgebras in finite von Neumann algebras. J. Operator Theory 9 (1983), no. 2, 253–268.

[Po3] S. Popa. Private Communication. January 2003.

[Sk] G. Skandalis. Une notion de nucléarité en $K$-théorie (d’après J. Cuntz). K-Theory 1 (1988), no. 6, 549–573.

[Șt] M. B. Ștefan. The primality of subfactors of finite index in the interpolated free group factors. Proc. Amer. Math. Soc. 126 (1998), no. 8, 2299–2307.

[Ta] M. Takesaki. On the cross-norm of the direct product of $C^*$-algebras. Tôhoku Math. J. (2) 16 (1964), 111–122.

Department of Mathematical Sciences, University of Tokyo, Komaba, 153-8914
E-mail address: narutaka@ms.u-tokyo.ac.jp