A Note on the Common Spectral Properties for Bounded Linear Operators

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Abstract. Let $X$ and $Y$ be Banach spaces, $A : X \to Y$ and $B, C : Y \to X$ be bounded linear operators. We prove that if $A(BA)^2 = ABACA = ACABA = (AC)^2A$, then

$$\sigma_*(AC) \setminus \{0\} = \sigma_*(BA) \setminus \{0\}$$

where $\sigma_*$ runs over a large of spectra originated by regularities.

1. Introduction

Throughout this paper $\mathcal{L}(X,Y)$ denotes the set of all bounded linear operators acting from a complex Banach space $X$ into another one, $Y$, and $\mathcal{L}(X)$ is a short for $\mathcal{L}(X,X)$. Given two operators $A \in \mathcal{L}(X,Y)$ and $B \in \mathcal{L}(Y,X)$, Jacobson’s Lemma asserts that

$$\sigma(AB) \setminus \{0\} = \sigma(BA) \setminus \{0\} \quad (1)$$

where $\sigma(\cdot)$ denotes the ordinary spectrum.

Several works have been devoted to equality (1) by showing that $AB - I$ and $BA - I$ share many spectral properties. See [2, 3, 5, 6, 13, 15, 16, 18, 19] and the references therein. Barnes in [2] extended (1) to other part of the spectrum and showed that $AB - I$ and $BA - I$ share some spectral properties. In [3], Benhida and Zerouali investigated equation (1) for various Taylor joint spectra. For $A$ and $B$ satisfying $ABA = A^2$ and $BAB = B^2$, Schmoeger [15, 16] and Duggal [7] showed that $A, B, AB$ and $BA$ share spectral properties. Corach et al. [6] investigated common properties for $ac - 1$ and $ba - 1$ where $a, b$ and $c$ are elements in associative ring such that $aba = aca$. For bounded linear operators $A$, $B$ and $C$, Zeng and Zhong [19] studied spectral properties for $AC$ and $BA$ under the condition $ABA = ACA$. If $C = I$ in the last condition, one can retrieve Schmoeger’s result. For operators $A, B, C$ and $D$ satisfying $ACD = DBD$ and $BDA = ACA$, Yan and Fang [17] investigated spectral properties for $AC$ and $BD$. Recently, [5] studied common properties for $ac$ and $ba$ for elements in a ring satisfying $a(ba)^2 = abaca = acaba = (ac)^2a$. 

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Theorem 3.7] that for every $n$, the empty set is taken to be infinite.

It is easy to see that (see [1]). One says that $T$ if $T$ and $T$ descent to the main results of the paper. In Theorem 3.1 we prove that if $A \in \mathcal{L}(X, Y)$ and $B, C \in \mathcal{L}(Y, X)$ satisfy

$$A(BA)^2 = ABACA = ACABA = (AC)^2A,$$

In section two we give basic definitions and notation which we need in the sequel. Section 3 is devoted to the main results of the paper. In Theorem 3.1 we prove that if $A \in \mathcal{L}(X, Y)$ and $B, C \in \mathcal{L}(Y, X)$ satisfy

$$A(BA)^2 = ABACA = ACABA = (AC)^2A,$$

where $\sigma$, runs over a large of spectra originated by regularities.

2. Basic definitions and notations

For an operator $T \in \mathcal{L}(X)$, let $\mathcal{N}(T)$ and $\mathcal{R}(T)$ stand for the kernel, respectively the range of $T$. An operator $T \in \mathcal{L}(X)$ is said to be an upper semi-Fredholm operator if $\mathcal{R}(T)$ is closed and $\dim \mathcal{N}(T) < \infty$, and $T$ is said to be a lower semi-Fredholm operator if $\text{codim} \mathcal{N}(T) < \infty$. One says that $T$ is a Fredholm operator if $\dim \mathcal{N}(T) < \infty$ and $\text{codim} \mathcal{N}(T) < \infty$. If $T$ is either upper or lower semi-Fredholm then $T$ is said semi-Fredholm operator. In this case the index of $T$ is defined by $\text{ind}(T) = \dim \mathcal{N}(T) - \dim \mathcal{R}(T)$.

The ascent of $T$, asc$(T)$, is the smallest nonnegative integer $n$ for which $\mathcal{N}(T^n) = \mathcal{N}(T^{n+1})$, i.e.; $\text{asc}(T) = \inf\{n \in \mathbb{Z}_+ : \mathcal{N}(T^n) = \mathcal{N}(T^{n+1})\}$. If no such integer exists, we shall say that $T$ has infinite ascent. In a similar way, the descent of $T$, dsc$(T)$, is defined by $\text{dsc}(T) = \inf\{n \in \mathbb{Z}_+ : \mathcal{R}(T^n) = \mathcal{R}(T^{n+1})\}$ and if no such integer exists, we shall say that $T$ has infinite descent. We say that $T$ is left Drazin invertible if $\text{asc}(T) < \infty$ and $\mathcal{R}(T^{\text{asc}(T)+1})$ is closed and $T$ is right Drazin invertible if $\text{dsc}(T) < \infty$ and $\mathcal{R}(T^{\text{dsc}(T)})$ is closed. If $T$ is both left and right Drazin invertible, then $T$ is said to be Drazin invertible; which is equivalent to $\text{asc}(T) = \text{dsc}(T) < \infty$ (see [1]). One says that $T$ is upper semi-Browder if $T$ is upper semi-Fredholm with finite ascent, and $T$ is lower semi-Browder if $T$ is lower semi-Fredholm with finite descent. If $T$ is both upper and lower semi-Browder then $T$ is said to be Browder operator (see [14]).

For each $n \in \mathbb{Z}_+$, let $c_n(T) = \dim \mathcal{R}(T^n)/\mathcal{R}(T^{n+1})$ and $c'_n(T) = \dim \mathcal{N}(T^{n+1})/\mathcal{N}(T^n)$. It was proved in [8, Lemma 3.2] that for every $n$, we have

$$c_n(T) = \dim X/(\mathcal{R}(T) + \mathcal{N}(T^n))$$

and $c'_n(T) = \dim \mathcal{N}(T^n) \cap \mathcal{R}(T)$. It is easy to see that $\{c_n(T)\}$ and $\{c'_n(T)\}$ are decreasing sequences and $\text{dsc}(T) = \inf\{n \in \mathbb{Z}_+ : c_n(T) = 0\}$, $\text{asc}(T) = \inf\{n \in \mathbb{Z}_+ : c'_n(T) = 0\}$.

Following [12], the essential descent $\text{dsc}_{e}(T)$ of $T$ is defined by $\text{dsc}_{e}(T) = \inf\{n \in \mathbb{Z}_+ : c_n(T) < \infty\}$, and the essential ascent $\text{asc}_{e}(T)$ of $T$ is defined by $\text{asc}_{e}(T) = \inf\{n \in \mathbb{Z}_+ : c'_n(T) < \infty\}$, where the infimum over the empty set is taken to be infinite.

Let $\mathcal{N}^{\infty}(T)$ and $\mathcal{R}^{\infty}(T)$ denote the hyper-kernel and the hyper-range of $T$ defined by

$$\mathcal{N}^{\infty}(T) = \bigcup_{n=1}^{\infty} \mathcal{N}(T^n) \quad \text{and} \quad \mathcal{R}^{\infty}(T) = \bigcap_{n=1}^{\infty} \mathcal{R}(T^n).$$

One says that $T$ is semi-regular if $\mathcal{R}(T)$ is closed and $\mathcal{N}^{\infty}(T) \subseteq \mathcal{R}(T)$.

For each $n \in \mathbb{Z}_+$, $T \in \mathcal{L}(X)$ induces a linear maps $\Gamma_n$ from the space $\mathcal{R}(T^n)/\mathcal{R}(T^{n+1})$ into $\mathcal{R}(T^{n+1})/\mathcal{R}(T^{n+2})$. The dimension of the null space of $\Gamma_n$ will be denoted by $k_n(T)$, i.e., $k_n(T) = \dim \mathcal{N}(\Gamma_n)$. It follows from [8, Theorem 3.7] that for every $n$,

$$k_n(T) = \dim((\mathcal{R}(T^n) \cap \mathcal{N}(T))/(\mathcal{R}(T^{n+1}) \cap \mathcal{N}(T)))$$

$$= \dim(\mathcal{R}(T) + \mathcal{R}(T^{n+1}))/\mathcal{R}(T) + \mathcal{N}(T^n)).$$
Let
\[ k(T) = \sum_{n=0}^{\infty} k_n(T). \]
Then it follows from [8, Theorem 3.7] that \( k(T) = \dim N(T)/(N(T) \cap R^c(T)) = \dim (R(T) + N^e(T))/R(T). \) The stable nullity \( c(T) \) and the stable defect \( c'(T) \) of \( T \) are defined by
\[ c(T) = \sum_{n=0}^{\infty} c_n(T) \text{ and } c'(T) = \sum_{n=0}^{\infty} c'_n(T). \]
Then we have \( c(T) = \dim X/R^c(T) \) and \( c'(T) = \dim R^c(T). \)

According to [11], the degree of stable iteration of \( T \in \mathcal{L}(X) \) is defined by
\[ \text{dis}(T) = \inf \{ n \in \mathbb{Z}_+ : k_n(T) = 0 \text{ for all } m \geq n \}, \]
and the degree of essential stable iteration of \( T \) ([18]) is defined as
\[ \text{dis}_e(T) = \inf \{ n \in \mathbb{Z}_+ : k_n(T) < \infty \text{ for all } m \geq n \}. \]

**Definition 2.1.** Let \( R \) be a non-empty subset of \( \mathcal{L}(X) \). \( R \) is called a regularity if it satisfies the following two conditions:

- i) if \( n \in \mathbb{N} \), then \( A \in R \) if and only if \( A^n \in R \);
- ii) if \( A, B, C \) and \( D \) are mutually commuting operators in \( \mathcal{L}(X) \) such that \( AC + BD = I \), then \( AB \in R \) if and only if \( A \in R \) and \( B \in R \).

A regularity \( R \subset \mathcal{L}(X) \) assigns to each \( T \in \mathcal{L}(X) \) a subset of \( \mathbb{C} \) defined by
\[ \sigma_{R}(T) = \{ \lambda \in \mathbb{C} : T - \lambda I \notin R \} \]

and called the spectrum of \( T \) corresponding to the regularity \( R \). We note that every regularity \( R \) contains all invertible operators, so that \( \sigma_{R}(T) \subseteq \sigma(T) \). In general, \( \sigma_{R}(T) \) is neither compact nor non-empty (see [10, 12, 14]).

The regularities \( R_i \), where \( 1 \leq i \leq 15 \), were introduced and studied in [10, 12, 14] but are in a different form. Regularity \( R_{18} \) was introduced by [4], while \( R_{16}, R_{17} \) and \( R_{19} \) were introduced by [18].

**Definition 2.2.**
\[
\begin{align*}
R_1 &= \{ T \in \mathcal{L}(X) : c(T) = 0 \}, \\
R_2 &= \{ T \in \mathcal{L}(X) : c(T) < \infty \}, \\
R_3 &= \{ T \in \mathcal{L}(X) : \exists d \in \mathbb{Z}_+ \text{ such that } c_d(T) = 0 \text{ and } R(T^{d+1}) \text{ is closed} \}, \\
R_4 &= \{ T \in \mathcal{L}(X) : c_d(T) < \infty, \forall n \in \mathbb{Z}_+ \}, \\
R_5 &= \{ T \in \mathcal{L}(X) : \exists d \in \mathbb{Z}_+ \text{ such that } c_d(T) < \infty \text{ and } R(T^{d+1}) \text{ is closed} \}, \\
R_6 &= \{ T \in \mathcal{L}(X) : c(T) = 0 \text{ and } \sigma_{R}(T) \text{ is closed} \}, \\
R_7 &= \{ T \in \mathcal{L}(X) : c'(T) < \infty \text{ and } R(T) \text{ is closed} \}, \\
R_8 &= \{ T \in \mathcal{L}(X) : \exists d \in \mathbb{Z}_+ \text{ such that } c_d'(T) = 0 \text{ and } R(T^{d+1}) \text{ is closed} \}, \\
R_9 &= \{ T \in \mathcal{L}(X) : c_d'(T) < \infty \text{ for every } n \in \mathbb{Z}_+ \text{ and } R(T) \text{ is closed} \}, \\
R_{10} &= \{ T \in \mathcal{L}(X) : \exists d \in \mathbb{Z}_+ \text{ such that } c_d'(T) < \infty \text{ and } R(T^{d+1}) \text{ is closed} \}, \\
R_{11} &= \{ T \in \mathcal{L}(X) : k(T) = 0 \text{ and } \sigma_{R}(T) \text{ is closed} \}, \\
R_{12} &= \{ T \in \mathcal{L}(X) : k(T) < \infty \text{ and } \sigma_{R}(T) \text{ is closed} \}, \\
R_{13} &= \{ T \in \mathcal{L}(X) : \exists d \in \mathbb{Z}_+ \text{ such that } k_a(T) = 0 \text{ for every } n \geq d \text{ and } R(T^{d+1}) \text{ is closed} \}, \\
R_{14} &= \{ T \in \mathcal{L}(X) : k_a(T) < \infty \text{ for every } n \in \mathbb{Z}_+ \text{ and } R(T) \text{ is closed} \}, \\
R_{15} &= \{ T \in \mathcal{L}(X) : \exists d \in \mathbb{Z}_+ \text{ such that } k_a(T) < \infty \text{ for every } n \geq d \text{ and } R(T^{d+1}) \text{ is closed} \}, \\
R_{16} &= \{ T \in \mathcal{L}(X) : \exists d \in \mathbb{Z}_+ \text{ such that } c_d(T) = 0 \text{ and } R(T) + N(T^d) \text{ is closed} \}, \\
R_{17} &= \{ T \in \mathcal{L}(X) : \exists d \in \mathbb{Z}_+ \text{ such that } c_d(T) < \infty \text{ and } R(T) + N(T^d) \text{ is closed} \}, \\
R_{18} &= \{ T \in \mathcal{L}(X) : \exists d \in \mathbb{Z}_+ \text{ such that } k_a(T) = 0 \text{ for every } n \geq d \text{ and } R(T) + N(T^d) \text{ is closed} \}, \\
R_{19} &= \{ T \in \mathcal{L}(X) : \exists d \in \mathbb{Z}_+ \text{ such that } k_a(T) < \infty \text{ for every } n \geq d \text{ and } R(T) + N(T^d) \text{ is closed} \}.
\]
Thus follows from (2) that

\[ R_3 = \{ T \in \mathcal{L}(X) : \text{dsc}(T) < \infty \text{ and } \mathcal{R}(T^{\text{dsc}(T)+1}) \text{ is closed} \}, \]

\[ R_5 = \{ T \in \mathcal{L}(X) : \text{asc}(T) < \infty \text{ and } \mathcal{R}(T^{\text{asc}(T)+1}) \text{ is closed} \}, \]

\[ R_8 = \{ T \in \mathcal{L}(X) : \text{asc}(T) < \infty \text{ and } \mathcal{R}(T^{\text{asc}(T)+1}) \text{ is closed} \}, \]

\[ R_{10} = \{ T \in \mathcal{L}(X) : \text{dis}(T) < \infty \text{ and } \mathcal{R}(T^{\text{dis}(T)+1}) \text{ is closed} \}, \]

\[ R_{13} = \{ T \in \mathcal{L}(X) : \text{dis}(T) < \infty \text{ and } \mathcal{R}(T^{\text{dis}(T)+1}) \text{ is closed} \}, \]

\[ R_{15} = \{ T \in \mathcal{L}(X) : \text{dis}(T) < \infty \text{ and } \mathcal{R}(T^{\text{dis}(T)+1}) \text{ is closed} \}. \]

The operators of \( R_1, R_2, R_3, R_4 \) and \( R_5 \) are surjective, lower semi-Browder, right Drazin invertible, lower semi-Fredholm and right essentially Drazin invertible operators, respectively. The operators of \( R_6, R_7, R_8, R_9 \) and \( R_{10} \) are bounded below, upper semi-Browder, left Drazin invertible, upper semi-Fredholm and left essentially Drazin invertible operators, respectively. The operators of \( R_{11}, R_{12} \) and \( R_{13} \) are semi-regular, essentially semi-regular and quasi-Fredholm operators. The operators of \( R_{15} \) are the operators with eventual topological uniform descent.

3. Main results

The following is our main result.

**Theorem 3.1.** Let \( A \in \mathcal{L}(X, Y) \) and \( B, C \in \mathcal{L}(Y, X) \) such that \( A(BA)^2 = ABACA = ACABA = (AC)^2A \). Then

\[ \sigma_{R_i}(AC) \setminus \{ 0 \} = \sigma_{R_i}(BA) \setminus \{ 0 \} \text{ for } 1 \leq i \leq 19. \]

The proof of our main result uses several auxiliary lemmas.

**Lemma 3.2.** Let \( A \in \mathcal{L}(X, Y) \) and \( B, C \in \mathcal{L}(Y, X) \) such that \( A(BA)^2 = ABACA = ACABA = (AC)^2A \). Let \( Q \) be a polynomial. Then we have

1) \( ABAR(Q(CA - I)) \subseteq \mathcal{R}(Q(AB - I)) \);
2) \( ABAN(Q(CA - I)) \subseteq N(Q(AB - I)) \);
3) \( ACAR(Q(BA - I)) \subseteq \mathcal{R}(Q(AC - I)) \);
4) \( ACAN(Q(BA - I)) \subseteq N(Q(AC - I)) \).

**Proof.** It is easy to see that for each \( k \in \mathbb{Z_+}, \)

\[ ABA(CA - I)^k = (AB - I)^kABA \text{ and } ACA(BA - I)^k = (AC - I)^kACA. \]  \hspace{1cm} (2)

Then

\[ ABAQ(CA - I) = Q(AB - I)ABA \text{ and } ACAQ(BA - I) = Q(AC - I)ACA. \]  \hspace{1cm} (3)

1) Let \( x \) belongs to \( \mathcal{R}(Q(CA - I)). \) Then there exists some \( y \in X \) such that \( Q(CA - I)y = x. \) Hence it follows from (2) that \( ABAx = ABAQ(CA - I)x = Q(AB - I)ABAx \) which belongs to \( \mathcal{R}(Q(AB - I)). \) Thus \( ABAQ(CA - I) \subseteq \mathcal{R}(Q(AB - I)). \)

2) Let \( x \in N(Q(CA - I)). \) Then \( Q(CA - I)x = 0. \) It follows from (2) that \( Q(AB - I)ABAx = ABAQ(CA - I)x = 0. \) Thus \( ABAx \in N(Q(AB - I)). \)

Using (3), (3) and 4) go similarly. \( \Box \)
Lemma 3.3. Let $A \in \mathcal{L}(X, Y)$ and $B, C \in \mathcal{L}(Y, X)$ such that $A(BA)^2 = ABACA = ACABA = (AC)^2 A$. Then
\[ c_n(AC - I) = c_n(BA - I) \text{ for all } n \in \mathbb{Z}^+. \]
In particular, $c(AC - I) = c(BA - I)$.

Proof. Let
\[ \Gamma_{AC} : \mathcal{R}((BA - I)^n)/\mathcal{R}((BA - I)^{n+1}) \rightarrow \mathcal{R}((AC - I)^n)/\mathcal{R}((AC - I)^{n+1}) \]
be the linear application defined by
\[ \Gamma_{AC}(x + \mathcal{R}((BA - I)^{n+1})) = ACAx + \mathcal{R}((AC - I)^{n+1}). \]

Since $ACAR((BA - I)^n) \subseteq \mathcal{R}((AC - I)^n)$ by Lemma 3.2, part 3), then $\Gamma_{AC}$ is well defined. We shall show that $\Gamma_{AC}$ is injective.

Let $x \in \mathcal{R}((BA - I)^n)$ such that $\Gamma_{AC}(x) = 0$. Then $ACAx \in \mathcal{R}((AC - I)^{n+1})$. Hence $CACAx \in \mathcal{R}((CA - I)^{n+1})$.

From Lemma 3.2, part 1), we have $ABACACAx \in \mathcal{R}((AB - I)^{n+1})$. Then
\[ (BA)^4 x = BABACACAx \in \mathcal{R}((BA - I)^{n+1}). \]

Since $x \in \mathcal{R}((BA - I)^n)$ then $x = (BA - I)^{n} z$ for some $z \in X$. Hence
\[
\begin{align*}
x &= (BA)^4 x - ((BA)^4 - I)x \\
&= (BA)^4 x - ((BA)^3 + (BA)^2 + (BA) + I)(BA - I)x \\
&= (BA)^4 x - ((BA)^3 + (BA)^2 + (BA) + I)(BA - I)^{n+1} z \\
&= (BA)^4 x - (BA - I)^{n+1} z, \quad (BA)^3 + (BA)^2 + (BA) + I \in \mathcal{R}((BA - I)^{n+1}).
\end{align*}
\]

Thus $\Gamma_{AC}$ is injective and consequently
\[ c_n(BA - I) \leq c_n(AC - I). \tag{4} \]

In similar way, we show that
\[ c_n(CA - I) \leq c_n(AB - I). \tag{5} \]

Finally,
\[
\begin{align*}
c_n(BA - I) &\leq c_n(AC - I) \\
&= c_n(CA - I) \quad \text{([18, Lemma 3.9]} \\
&\leq c_n(AB - I) \text{ by (5)} \\
&= c_n(BA - I) \quad \text{([18, Lemma 3.9].}
\end{align*}
\]

Therefore $c_n(BA - I) = c_n(AC - I)$ for all $n \in \mathbb{Z}^+$. In particular, $c(AC - I) = c(BA - I)$. \hfill \Box

For $T \in \mathcal{L}(X)$, let $\sigma_{des}(T)$ and $\sigma_{des}'(T)$ be, respectively, the descent spectrum and the essential descent spectrum of $T$ defined by
\[ \sigma_{des}(T) = \{ \lambda \in \mathbb{C} : dsc(T) = \infty \} \text{ and } \sigma_{des}'(T) = \{ \lambda \in \mathbb{C} : dsc(T) = \infty \}. \]

The following is an immediate consequence of Lemma 3.3.

Corollary 3.4. Let $A \in \mathcal{L}(X, Y)$ and $B, C \in \mathcal{L}(Y, X)$ such that $A(BA)^2 = ABACA = ACABA = (AC)^2 A$. Then
\[ \sigma_{AC} \setminus \{ 0 \} = \sigma_{BA} \setminus \{ 0 \}, \text{ for } \sigma_e \in \{ \sigma_{des}, \sigma_{des}' \}. \]

Lemma 3.5. Let $A \in \mathcal{L}(X, Y)$ and $B, C \in \mathcal{L}(Y, X)$ such that $A(BA)^2 = ABACA = ACABA = (AC)^2 A$. Then
\[ c_n'(AC - I) = c'_n(BA - I) \text{ for all } n \in \mathbb{Z}^+. \]

In particular, $c'(AC - I) = c'(BA - I)$.  


Proof. Let

\[ \Psi_{ACA} : \mathcal{N}((BA - I)^{n+1})/\mathcal{N}((BA - I)^n) \to \mathcal{N}((AC - I)^{n+1})/\mathcal{N}((AC - I)^n) \]

be the linear application defined by

\[ \Psi_{ACA}(x + \mathcal{N}((BA - I)^n)) = ACAx + \mathcal{N}((AC - I)^n). \]

Since ACAN((BA - I)^{n+1}) \subseteq \mathcal{N}((AC - I)^{n+1}) by Lemma 3.2, part 4), then \( \Psi_{ACA} \) is well defined.

Now we show that \( \Psi_{ACA} \) is injective. Let \( x \in \mathcal{N}((BA - I)^{n+1}) \) such that \( \Psi_{ACA}(x) = 0 \), which means that \( ACAx \in \mathcal{N}((AC - I)^n) \). Hence \( CACAx \in \mathcal{N}((CA - I)^n) \). It follows from Lemma 3.2, part ii), that \( ABACACAx \in \mathcal{N}((AB - I)^n) \). Then

\[ (BA)^4x = BABACACAx \in \mathcal{N}((BA - I)^n). \]

Hence

\[ x = (BA)^4x - ((BA)^4 - I)x = (BA)^4x - [(BA)^3 + (BA)^2 + (BA) + I](BA - I)x \in \mathcal{N}((BA - I)^n). \]

Which implies that \( \Psi_{ACA} \) is injective and then

\[ c_n'(BA - I) \leq c_n'(AC - I). \]

Similarly, we prove that

\[ c_n'(CA - I) \leq c_n'(AB - I). \]

Finally,

\[ c_n'(BA - I) \leq c_n'(AC - I) = c_n'(CA - I) \quad \text{[18, Lemma 3.10]} \leq c_n'(AB - I) \quad \text{by (7)} = c_n'(BA - I) \quad \text{[18, Lemma 3.10];} \]

Therefore \( c_n'(BA - I) = c_n'(AC - I) \) for all \( n \in \mathbb{Z}_+ \). In particular, \( c'(AC - I) = c'(BA - I). \)

For \( T \in \mathcal{L}(X) \) let \( \sigma_{asc}(T) \) and \( \sigma_{asc}'(T) \) be respectively the ascent spectrum and the essential ascent spectrum of \( T \) defined by

\[ \sigma_{asc}(T) = \{ \lambda \in \mathbb{C} : \text{asc}(T) = \infty \} \quad \text{and} \quad \sigma_{asc}'(T) = \{ \lambda \in \mathbb{C} : \text{asc}(T) = \infty \}. \]

Then the following is an immediate consequence of Lemma 3.5

**Corollary 3.6.** Let \( A \in \mathcal{L}(X, Y) \) and \( B, C \in \mathcal{L}(Y, X) \) such that \( A(BA)^2 = ABACA = ACABA = (AC)^2A \). Then

\[ \sigma_{asc}(AC \setminus \{0\}) = \sigma_{asc}(BA \setminus \{0\}), \quad \text{for} \quad \sigma_{asc} \in \{ \sigma_{asc}, \sigma_{asc}' \}. \]

**Lemma 3.7.** Let \( A \in \mathcal{L}(X, Y) \) and \( B, C \in \mathcal{L}(Y, X) \) such that \( A(BA)^2 = ABACA = ACABA = (AC)^2A \). Then

\[ k_n(AC - I) = k_n(BA - I) \quad \text{for all} \quad n \in \mathbb{Z}_+. \]

**Proof.** Let \( \Phi_{ACA} \) be the linear application from \( \mathcal{R}(BA - I) + \mathcal{N}((BA - I)^{n+1})/\mathcal{R}(BA - I) + \mathcal{N}((BA - I)^n) \) to \( \mathcal{R}(AC - I) + \mathcal{N}((AC - I)^{n+1})/\mathcal{R}(AC - I) + \mathcal{N}((AC - I)^n) \) defined by

\[ \Phi_{ACA}(x + \mathcal{R}(BA - I) + \mathcal{N}((BA - I)^n)) = ACAx + \mathcal{R}(BA - I) + \mathcal{N}((AC - I)^n). \]
Since, by Lemma 3.2, parts 3) and 4),

$$\text{ACA}(R(BA - I)) + N((BA - I)^{n+1}) \subseteq R(BA - I)) + N((BA - I)^n),$$

then $\Phi_{ACA}$ is well defined.

We prove that $\Phi_{ACA}$ is injective. Let $x \in R(BA - I) + N((BA - I)^{n+1})$ such that $\Phi_{ACA}(x) = 0$. Then $ACA x \in R(AC - I) + N((AC - I)^n)$. So, there exist some $y \in R(BA - I)$ and $z \in N((AC - I)^n)$ such that $ACA x = y + z$. Then $ACA x = Cy + Cz \in R(ACA - I) + N((CA - I)^n)$. Thus by Lemma 3.2, parts 1) and 2), we get that $ABACACA x \in R(AB - I) + N((AB - I)^n)$ and consequently $(BA)^4 x = BABAACA x \in R(AB - I) + N((BA - I)^n)$. Then $x = (BA)^4 x - ((BA)^4 - I)x$

$$(8)$$

Hence $\Phi_{ACA}$ is injective. Thus

$$k_n(BA - I) \leq k_n(AC - I).$$

(8)

In similar way, we show that

$$k_n(CA - I) \leq k_n(AB - I).$$

(9)

Therefore,

$$k_n(BA - I) \leq k_n(AC - I)$$

$$= k_n(CA - I) \quad \text{(18, Lemma 3.8)}$$

$$\leq k_n(AB - I) \quad \text{(by (9))}$$

$$= k_n(BA - I) \quad \text{(18, Lemma 3.8).}$$

\[ \square \]

**Lemma 3.8.** Let $A \in \mathcal{L}(X, Y)$ and $B, C \in \mathcal{L}(Y, X)$ such that $A(BA) = ABACA = ACABA = (AC)^2 A$. Then for all $n \in \mathbb{Z}_+$, $R((AC - I) + N((AC - I)^n)$ is closed if and only if $R((BA - I) + N((BA - I)^n)$ is closed.

In particular $R(AC - I)$ is closed if and only if $R(BA - I)$ is closed.

**Proof.** Assume that $R((AC - I) + N((AC - I)^n)$ is closed. Let $\{x_p\}$ be a sequence in $R((BA - I) + N((BA - I)^n)$ which converges to $x \in X$. Then $ACA x_p$ converges to $ACA x$. Since $ACA x \in R(AC - I) + N((AC - I)^n) \subseteq R((AC - I) + N((AC - I)^n)$ by Lemma 3.2, part 3) and 4), then $ACA x_p$ belongs to $R((AC - I) + N((AC - I)^n)$. Since $R(AC - I) + N((AC - I)^n)$ is closed and $ACA x_p$ converges to $ACA x$.

Thus

$$ACA x \in R((AC - I) + N((AC - I)^n)$$

$$CA x \in R((AC - I) + N((AC - I)^n)$$

$$ABCA(ACA) \in R((BA - I) + N((BA - I)^n) \quad \text{(by Lemma 3.2)}$$

$$(BA)^4 x = ABA(ACA) x \in R((BA - I) + N((BA - I)^n).$$

Therefore $R((BA - I) + N((BA - I)^n)$ is closed.

The opposite implication goes similarly. \[ \square \]

**Lemma 3.9.** Let $A \in \mathcal{L}(X, Y)$ and $B, C \in \mathcal{L}(Y, X)$ such that $A(BA) = ABACA = ACABA = (AC)^2 A$. Then for all $n \in \mathbb{N}$, $R((AC - I)^n)$ is closed if and only if $R((BA - I)^n)$ is closed.
Proof. As in the presentation before [2, Proposition], for each \( n \in \mathbb{N} \) there exists \( B_n \) and \( C_n \in \mathcal{L}(Y, X) \) such that
\[
(I - AC)^n = I - AC_n \quad \text{and} \quad (I - BA)^n = I - B_n A.
\]
Indeed, we have \( B_n = \sum_{k=1}^{n} (-1)^{k-1} \binom{n}{k} B(AB)^{k-1} \) and \( C_n = \sum_{k=1}^{n} (-1)^{k-1} \binom{n}{k} (CA)^{k-1} C \). It is easy to check that
\[
A(B_n A)^2 = AB_n AC_n A = AC_n AB_n A = (AC_n)^2 A.
\]
Then it follows from Lemma 3.8 that \( \mathcal{R}((AC - I)^n) \) is closed if and only if \( \mathcal{R}((BA - I)^n) \) is closed.

**Proof of Theorem 3.1**: The proof follows at once from Lemmas 3.2-3.9.

4. Applications and concluding remarks

A bounded operator \( T \in \mathcal{L}(X) \) is said to be **upper semi-Weyl** operator if \( T \) is upper semi-Fredholm with \( \text{ind}(T) \leq 0 \), and \( T \) is said to be **lower semi-Weyl** operator if \( T \) is lower semi-Fredholm with \( \text{ind}(T) \geq 0 \). If \( T \) is both upper and lower semi-Fredholm then \( T \) is said to be **Weyl** operator. Then \( T \) is weyl operator precisely when \( T \) is a Fredholm operator with index zero. The **upper semi-Weyl spectrum** \( \sigma_{uw}(T) \), the **lower semi-Weyl spectrum** \( \sigma_{lw}(T) \) and the **Weyl spectrum** \( \sigma(T) \) of \( T \) are defined by
\[
\sigma_{uw}(T) = \{ \lambda \in \mathbb{C} : T - \lambda I \text{ is not upper semi-Weyl} \},
\]
\[
\sigma_{lw}(T) = \{ \lambda \in \mathbb{C} : T - \lambda I \text{ is not lower semi-Weyl} \},
\]
\[
\sigma(T) = \sigma_{uw}(T) \cup \sigma_{lw}(T).
\]
From Lemma 3.3 and Lemma 3.5 we deduce the following result

**Proposition 4.1.** Let \( A \in \mathcal{L}(X, Y) \) and \( B, C \in \mathcal{L}(Y, X) \) such that \( A(BA)^2 = ABACA = ACABA = (AC)^2 A \). Then
\[
\sigma_{uw}(AC) \setminus \{0\} = \sigma_{uw}(BA) \setminus \{0\} \quad \text{for} \quad \sigma_{uw}, \sigma_{lw}, \sigma_{uw}.
\]

An operator \( T \in \mathcal{L}(X) \) is said to be **Riesz** operator if \( T - \lambda I \) is a Fredholm operator for all \( 0 \neq \lambda \in \mathbb{C} \). Then the following proposition is an immediate consequence of Theorem 3.1

**Proposition 4.2.** Let \( A \in \mathcal{L}(X, Y) \) and \( B, C \in \mathcal{L}(Y, X) \) such that \( A(BA)^2 = ABACA = ACABA = (AC)^2 A \). Then \( AC \) is a Riesz operator if and only if \( BA \) is a Riesz operator.

Following [21], an operator \( T \in \mathcal{L}(X) \) is said to be **generalized Drazin-Riesz** operator if there exists \( S \in \mathcal{L}(X) \) such that
\[
TS = ST, \quad STS = S \quad \text{and} \quad T^2 S - T \quad \text{is a Riesz operator}.
\]
The operator \( S \) is called a **generalized Drazin-Riesz inverse of \( T \)**.

**Theorem 4.3.** Let \( A \in \mathcal{L}(X, Y) \) and \( B, C \in \mathcal{L}(Y, X) \) such that \( A(BA)^2 = ABACA = ACABA = (AC)^2 A \). Then \( AC \) is generalized Drazin-Riesz invertible if and only if \( BA \) is generalized Drazin-Riesz invertible. In this case, if \( S \) is a generalized Drazin-Riesz inverse of \( AC \) then \( BS^2 A \) is a generalized Drazin-inverse of \( BA \).

**Proof.** Assume that \( AC \) is generalized Drazin-Riesz invertible. then there exists \( S \in \mathcal{L}(X) \) such that \( S(AC) = (AC)S, S(AC)S = S \) and \( (AC)^2 S - AC \) is Riesz. Set \( T = BS^2 A \) and we shall show that
\[
T(BA) = (BA)T, \quad T(BA)T = T \quad \text{and} \quad (BA)^2 T - BA \quad \text{is Riesz operator.}
For the first equality, we have
\[
T(BA) = BS^2 A(BA) \\
= BS^2(AC)S^2(AC)A(BA) \\
= BS^4(AC)^2 A(CA) \\
= B(AC)^3 S^4 A \\
= B(AB)S^2 A \\
= BAT.
\]

For the second,
\[
T^2(BA) = BS^2 ABS^2 ABA \\
= BS^2 ABS^2 (AC)S^2(AC)ABA \\
= BS^2 ABS^2 (AC)S^2(AC)ACA \\
= BS^2 AB(AC)(AC)S^4 ACA \\
= BS^2 AC(AC)(AC)S^4 ACA \\
= BS^2 ACS^2 ACA \\
= BS^2 A \\
= T.
\]

Set \( P = ACS - I = SAC - I \). Then
\[
T(BA)^2 - BA = BS^2 A(BA)^2 - BA \\
= BS^2 (AC)^2 A - BA \\
= BS^4 CA - BA \\
= B(SAC - I)A \\
= BPA.
\]

Hence it remains to show that \( BPA \) is a Riesz operator. We have
\[
(PA)(PA)B(PA) = (SCA - A)B(SACA - A)(SACSA - A) \\
= (SCA - A)B(SACABA - ABA)(CSA - A) \\
= (SCA - A)B(SACACA - ABA)(CSA - A) \\
= [(SACA - A)B(SACACA) - (SCA - A)BABAB](CSA - A) \\
= [(SCA - A)B(SACACA) - (SCA - A)BA(ACA)](CSA - A) \\
= (SCA - A)B(SACACA - ACA)(CSA - A) \\
= (SCA - A)B(SACA - A)(ACSA - A) \\
= (PA)(BP)C(PA).
\]

In the same way, one can prove that
\[
(PA)(BP)C(PA) = (PA)(BP)C(PA) = (PA)(BP)(PA) = (PA)(BP)(PA) = (PA)(BP)(PA).
\]

Since \((PA)(BP)C = (AC)^2 S - AC\) is a Riesz operator by assumption, then it follows from Proposition 4.2 that \( B(PA) \) is a Riesz operator. Therefore \( BA \) is generalized Drazin-Riesz invertible and \( BS^2 A \) is a generalized Drazin-inverse of \( BA \).

In similar way, we prove the opposite implication. \(\square\)

**Remark 4.4.** If \( A \) and \( B \) ∈ \( L(X) \) such that \( ABA = A^2 = BAB = B^2 \), then
\[
A(BA)^2 = ABAIA = AIABA = (AI)^2 A \quad (10)
\]
and
\[
B(AB)^2 = BABIB = BIBAB = (BI)^2 B. \quad (11)
\]

Then it follows from (10) and (11) that \( A, B, BA \) and \( AB \) share above spectral properties. So we retrieve the results of [7].
In the following two examples, the common spectral properties for $AC$ and $BA$ can only followed directly from the above results, but not from the corresponding ones in [7, 9, 15, 16, 19].

**Example 4.5.** Let $P$ be a non trivial idempotent on $X$. Let $A$, $B$ and $C$ defined on $X \oplus X \oplus X$ by

$$A = \begin{pmatrix} 0 & I & 0 \\ 0 & P & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} I & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad C = \begin{pmatrix} I & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$ 

Then $A(BA)^2 = ABACA = ACABA = (AC)^2A$, while $ABA \neq ACA$ and $BAB \neq B^2$.

**Example 4.6.** Let $A$ and $B$ be as in Example 4.5 and let $C$ be defined on $X \oplus X \oplus X$ by

$$C = \begin{pmatrix} 0 & 0 & 0 \\ P & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$ 

Then $A(BA)^2 = ABACA = ACABA = (AC)^2A$, while $ABA \neq ACA$ and $BAB \neq B^2$.

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