Abstract. We introduce the notion of finite stature of a family \( \{H_\lambda\}_{\lambda \in \Lambda} \) of subgroups of a group \( G \). We investigate the separability of subgroups of a group \( G \) that splits as a graph of hyperbolic special groups with quasiconvex edge groups. We prove that when the vertex groups of \( G \) have finite stature, then quasiconvex subgroups of the vertex groups of \( G \) are separable in \( G \). We present some partial results in a relatively hyperbolic framework.

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1. Introduction

1.1. Stature and Statement of Main Result.

**Definition 1.1** (Stature). Let \( G \) be a group and let \( \{H_\lambda\}_{\lambda \in \Lambda} \) be a collection of subgroups of \( G \). Then \((G, \{H_\lambda\}_{\lambda \in \Lambda})\) has finite stature if for each \( \mu \in \Lambda \), there are...
finately many $H_\mu$-conjugacy classes of infinite subgroups of form $H_\lambda \cap C$, where $C$ is an intersection of (possibly infinitely many) $G$-conjugates of elements of $\{ H_\lambda \}_{\lambda \in \Lambda}$. We are especially interested in the finite stature of the vertex groups in a splitting of $G$ as a graph of groups. An attractive equivalent definition of finite stature in that case is described in Definition 3.7. Finite stature does not always hold for the vertex groups of a splitting, but it is easy to verify in the following simple case:

**Example 1.2.** If $G$ is a graph of groups such that each attaching map of each edge group is an isomorphism, then $G$ has finite stature with respect to its vertex groups. Note that this includes many nilpotent groups and solvable groups. In Proposition 3.28 we generalize this to provide a characterization of finite stature of the vertex groups when all edge groups in the splitting of $G$ are commensurable. This can be applied to tree $\times$ tree lattices to see that irreducibility corresponds to infinite stature of the vertex groups in the action on either factor.

The main result in this paper which is proven as Theorem 4.1 is the following:

**Theorem 1.3.** Let $G$ be the fundamental group of a graph of groups with finite underlying graph. Let $V$ be the collection of vertex groups of $G$. Suppose the following conditions hold:

1. each vertex group is word-hyperbolic and virtually compact special;
2. each edge group is quasiconvex in its vertex groups;
3. $(G, V)$ has finite stature.

Then each quasiconvex subgroup of a vertex group of $G$ is separable in $G$. In particular, $G$ is residually finite.

Though the vertex groups and edge groups of $G$ are hyperbolic, $G$ does not have to be (relatively) hyperbolic. For example, any free-by-cyclic group satisfies the assumption of Theorem 1.3, but not all such groups are relatively hyperbolic.

### 1.2. Height

We now recall the notion of height, which will sometimes play a facilitating role in determining finite stature.

**Definition 1.4 (Height).** A subgroup $H \leq G$ has finite height if there does not exist a sequence $\{ g_n \}_{n \in \mathbb{N}}$ with $g_i H \neq g_j H$ for $i \neq j$ and with $\cap_{n \in \mathbb{N}} H^{g_n}$ infinite. We use the notation $H^g = g H g^{-1}$.

A finite collection $\mathcal{H} = \{ H_1, \ldots, H_r \}$ of subgroups has finite height if each $H_k$ has finite height.

The above notion was introduced in [GMRS98], where the height $h$ is the number $0 \leq h \leq \infty$ that is the supremal length of a sequence with infinite intersection of conjugates. Particular attention was paid to word-hyperbolic groups, and it was shown in [GMRS98] that a quasiconvex subgroup of a word-hyperbolic group has finite height. Since then, this notion has been studied in other contexts, in the relatively hyperbolic setting [HW09], and most recently in a graded relatively hyperbolic setting [DM17].

As it examines infinite intersections of conjugates, finite stature is certainly related in various ways to finite height. And we will utilize finite height of various subgroups to prove finite stature for the edge groups in certain graphs of groups. The following crisply distinguishes the two notions:
Example 1.5 (Height/Stature when abelian). Let $G$ be abelian. Then $(G, \Lambda)$ has finite stature for any finite collection $\Lambda$. However, $(G, \Lambda)$ has infinite height precisely when $\Lambda$ has an element that is both infinite and infinite index in $G$.

Example 1.6. Let $M$ be a 3-manifold, and consider the splitting of $\pi_1 M$ arising from its JSJ splitting [JS79]. Then $\pi_1 M$ has finite stature relative to its vertex groups. Indeed, the intersection between adjacent vertex groups corresponds to the torus between them. And the intersection between adjacent edge groups is the normal cyclic subgroup associated to the Siefert fibration of their common vertex group. Finally, these normal cyclic subgroups intersect trivially for consecutive vertex groups. We conclude that the only multiple intersection in a vertex group arise from its tori, these normal cyclic subgroups, and the trivial group. Finally, we caution that this example is deceptive since its controlling feature is that length 3 lines in the Bass-Serre tree have trivial stabilizer.

1.3. Intuitive explanation of finite stature: As is clear from Theorem 1.3, we are particularly interested in the finite stature of the collection of vertex groups in a splitting of $G$ as a graph of groups. As the explanations in the text are more group theoretical, let us give a topological description of finite stature in this setting. Let $X$ be a graph of spaces associated to $G$ with underlying graph $\Gamma_X = \Gamma_G$, and where each vertex space $X_v$ has $\pi_1 X_v = G_v$ and each edge space $X_e$ has $\pi_1 X_e = G_e$, and the attaching maps are $\pi_1$-injections. We are interested in transsections which are defined in Definition 3.1, but are roughly “maximal product regions” in $X$ subject to having an underlying graph being isomorphic to a specific tree. More precisely, we are interested in immersions of graphs of spaces $Y \to X$ where the underlying graph $\Gamma_Y$ is a tree, and where all attaching maps are isomorphisms of groups. The reader should regard $Y$ as a product $Y_v \times \Gamma_Y$ where $Y_v$ is a vertex space of $Y$. We refer to $Y_v$ as the “cross-section” of $Y$. Maximality of the cross-section means that $Y \to X$ doesn’t extend to an immersion $Y' \to X$ with $\Gamma_{Y'} = \Gamma_Y$ such that the cross-section $Y'_v$ is bigger than $Y_v$ in the sense that $\pi_1 Y_v \to \pi_1 Y'_v$ is a proper inclusion. That $Y \to X$ is an “immersion” of graphs of spaces, means that the induced map $\tilde{Y} \to \tilde{X}$ induces an embedding between underlying Bass-Serre trees $\Gamma_{\tilde{Y}} \to \Gamma_{\tilde{X}}$. finite stature means that for each vertex space $X_v$ of $X$, there are finitely many distinct cross-sections mapping into $X_v$ up to homotopy. We refer to Example 1.7 and Figure 1. We ignore the cross-sections with $\pi_1 Y_v$ finite. Sometimes the underlying tree $\Gamma_Y$ is infinite, so $Y$ threatens to intersect $X_v$ in infinitely many cross-sections, yielding infinite stature. However, in many cases, the map $Y \to X$ factors as $Y \to \tilde{Y} \to X$ where $Y \to \tilde{Y}$ is a covering space and $\tilde{Y}$ has a finite underlying graph, in which case $Y$ yields finitely many cross-sections in each vertex space as desired.
Example 1.7. Consider the following simplistic example of a multiple HNN extension of the free group on $a, b, c, d, e$ with stable letters $\alpha, \beta, \gamma, \delta$.

$$\langle a, b, c, d, e, \alpha, \beta, \gamma, \delta, | a^\alpha = b, (b^2)^\beta = c, d^\gamma = d, (d^2)^\delta = e \rangle$$

The various transections are illustrated in Figure 1. Each transection can contribute several cross-sections within the vertex space.

In the case of the HNN extension $\langle a, t | a^t = aa \rangle$, there is a single immersed product region whose underlying graph is an infinite 3-valent tree. There are then infinitely many cross-sections in the vertex space, each corresponding to a circle $a^{2n}$. Hence the group has infinite stature with respect to its vertex subgroup. See Example 3.8.

Example 1.8 (Tubular Groups). A tubular group $G$ splits as a finite graph of groups with $\mathbb{Z}^2$ for each vertex group and $\mathbb{Z}$ for each edge group. We claim that $(G, \{G_v\})$ has infinite stature precisely when there is an embedding $BS(n, m) \hookrightarrow G$ with $n \neq \pm m$. Here $BS(n, m) = \langle a, t | t^{-1}a^nt = a^m \rangle$. To see this, consider an associated finite graph of groups $\Gamma'$ (that might be disconnected) formed as follows: Each edge $e$ of $\Gamma$ yields an edge of $\Gamma'$ with $G'_e = G_e$. There is a vertex $u$ of $\Gamma'$ for each maximal cyclic subgroup of $G_v$ that is commensurable with an edge group at $G_v$, and we let $G'_u$ be this maximal cyclic subgroup. The edges of $\Gamma'$ are attached to the vertices according to the inclusion of the edge groups. Note that a single vertex of $\Gamma$ can contribute multiple vertices of $\Gamma'$. We refer to Figure 2. There is a map $\Gamma' \to \Gamma$, that induces maps between graphs of spaces.

Every transection for $G$ induces a transection in $G'$ and vice-versa. As $G'$ is a graph of cyclic groups it is easy to see that it has finite stature if and only if it has no $BS(n, m)$ subgroup with $n \neq \pm m$ (see Lemma 3.28).

Finite stature of a tubular group does not allow us to prove residual finiteness, since there isn’t an adequate version of the Malnormal Special Quotient Theorem for abelian groups.

1.4. Connection to virtual specialness. In [HW18] we formulate a conjecture relating the notion of finite stature to virtual specialness of a compact nonpositively curved cube complex. Specifically, we show that $(\pi_1X, \{\pi_1U_i\})$ has finite stature when $X$ is a compact virtually special cube complex, and $\{U_i\}$ varies over the hyperplanes. We conjecture that the converse holds. We apply Theorem 1.3 there as a primary ingredient to prove the virtual specialness of certain nonpositively curved.
2.1.1. **Nonpositively curved cube complexes.** An $n$-dimensional cube is a copy of $[-\frac{1}{2}, +\frac{1}{2}]^n$. Its subcubes are the subspaces obtained by restricting some coordinates to $\pm \frac{1}{2}$. We regard a subcube as a copy of a cube in the obvious fashion. A cube complex $X$ is a cell complex obtained by gluing cubes together along subcubes, where all gluing maps are modeled on isometries. Recall that a flag complex is a simplicial complex with the property that a finite set of vertices spans a simplex if and only if they are pairwise adjacent. $X$ is nonpositively curved if the link of each 0-cube of $X$ is a flag complex. A CAT(0) cube complex $\tilde{X}$ is a simply-connected nonpositively curved cube complex.

2.1.2. **Hyperplanes.** A midcube is a subspace of an $n$-cube obtained by restricting one coordinate of $[-\frac{1}{2}, +\frac{1}{2}]^n$ to 0. A hyperplane $\tilde{U}$ is connected subspace of a CAT(0) cube complex $\tilde{X}$ such that for each cube $c$ of $\tilde{X}$, either $\tilde{U} \cap c = \emptyset$ or $\tilde{U} \cap c$ consists of a midcube of $c$. The carrier of a hyperplane $U$ is the subcomplex $N(\tilde{U})$ consisting of all closed cubes intersecting $U$. We note that every midcube of $\tilde{X}$ lies in a unique hyperplane, and $N(\tilde{U}) \cong \tilde{U} \times c^1$ where $c^1$ is a 1-cube. An immersed hyperplane $U \to X$ in a nonpositively curved cube complex is a map $\text{Stab}(\tilde{U}) \setminus \tilde{U} \to X$ where $\tilde{U}$ is a hyperplane of the universal cover $\tilde{X}$ of $X$. We similarly define $N(U) \to X$ via $N(U) = \text{Stab}(\tilde{U}) \setminus N(\tilde{U})$.

A map $\phi : Y \to X$ between nonpositively curved cube complexes is combinatorial if it maps open $n$-cubes homeomorphically to open $n$-cubes. A combinatorial map is a local-isometry if for each 0-cube $y$, the induced map $\text{link}(y) \to \text{link}(\phi(y))$ is an embedding of simplicial complexes, such that $\text{link}(y) \subset \text{link}(\phi(y))$ is full in the sense that if a collection of vertices of $\text{link}(y)$ span a simplex in $\text{link}(\phi(y))$ then they span a simplex in $\text{link}(y)$.

2.1.3. **Special Cube Complexes.** A nonpositively curved cube complex $X$ is special if each immersed hyperplane $U \to X$ is an embedding, and moreover $N(U) \cong U \times [-\frac{1}{2}, +\frac{1}{2}]$, each restriction $U \times \{\pm \frac{1}{2}\} \to X$ is an embedding, and if $U, V$ are hyperplanes of $X$ that intersect then 0-cube of $N(U) \cap N(V)$ lies in a 2-cube intersected by both $U$ and $V$.

Theorem 1.9. Let $G$ be hyperbolic relative to subgroups that are virtually f.g. free abelian by $\mathbb{Z}$. Suppose $G$ splits as a finite graph of groups whose edge groups are relatively quasiconvex and whose vertex groups are virtually sparse special. Then each relatively quasiconvex subgroup of each vertex group of $G$ is separable. In particular, $G$ is residually finite. Theorem 1.9 which is proven as Theorem 5.18 plays a role in the inductive proof of virtual specialness of certain relatively hyperbolic groups with quasiconvex hierarchies [Wis Sec 15].

2. Preliminaries

2.1. **Background on Cube Complexes.**

2.1.1. **Nonpositively curved cube complexes.** We emphasize that Theorem 1.3 applies to prove the residual finiteness of many groups that aren’t special. Moreover, the relatively hyperbolic variant given in the following application applies to groups that are not hyperbolic relative to virtually abelian subgroups, since the parabolics can be $\mathbb{Z}^n \rtimes Z$ where $\phi$ is an infinite order automorphism of $\mathbb{Z}^n$.
2.2. Cubical small cancellation. A cubical presentation \(\langle X \mid \{Y_i\}\rangle\) consists of a nonpositively curved cube complex \(X\), and a set of local isometries \(Y_i \to X\) of nonpositively curved cube complexes. We use the notation \(X^*\) for the cubical presentation above. As a topological space, \(X^*\) consists of \(X\) with a cone on each \(Y_i\) attached, so \(\pi_1X^* = \pi_1X/\langle\langle \{\pi_1Y_i\}\rangle\rangle\). We use the notation \(\tilde{X}^*\) for the universal cover of \(X^*\).

\[\|Y_i\|\] denotes the infimal length of an essential combinatorial closed path in \(Y_i\).

Let \(Y_i \to X\) and \(Y_j \to X\) be maps. A morphism \(Y_i \to Y_j\) is a map such that \(Y_i \to X\) factors as \(Y_i \to Y_j \to X\). It is an isomorphism if there is an inverse map \(Y_j \to Y_i\) that is also a morphism. Define an automorphism accordingly and let \(\text{Aut}(Y \to X)\) denote the group of automorphisms of \(Y \to X\).

A cone-piece of \(X^*\) in \(Y_i\) is a component of \(g\bar{Y}_j \cap \bar{Y}_i\) for some \(g \in \pi_1X\), where we exclude the case that \(i = j\) and \(g \in \text{Stab}(\bar{Y}_i)\) and there is a map \(\bar{g} : Y_i \to Y_i\) so that the following diagram commutes:

\[
\begin{array}{ccc}
\bar{Y}_i & \to & Y_i \\
g \downarrow & \Rightarrow & \bar{g} \downarrow \\
Y_i & \to & X \\
\end{array}
\]

For a hyperplane \(\bar{U}\) of \(\bar{X}\), let \(N(\bar{U})\) denote its carrier, which is the union of all closed cubes intersecting \(\bar{U}\). A wall-piece of \(X^*\) in \(Y_i\) is a component of \(\bar{Y}_i \cap N(\bar{U})\), where \(\bar{U}\) is a hyperplane that is disjoint from \(\bar{Y}_i\).

For instance, consider the presentation \(\langle a, b \mid (abb)^{20}, (baa)^{20}\rangle\), and regard it as a cubical presentation \(\langle X \mid Y_1, Y_2\rangle\) where \(X\) is a bouquet of circles and each \(Y_i\) is an immersed cycle. Then the path \(ab\) corresponds to a piece, since it appears as the intersection between distinct lines \(\bar{Y}_1, \bar{Y}_2\) in \(\bar{X}\). Likewise \(bb\) is a piece since it occurs as the intersection of two distinct translates of \(\bar{Y}_1\). However, \(bb\) is not a piece, since any two translates of \(\bar{Y}_1\) that contain \(bb\) are actually the same, and they differ by a translation that projects to an automorphism of \(Y_1\).

**Definition 2.1** (Small Cancellation). \(X^*\) satisfies the \(C'(\frac{1}{2})\) small cancellation condition if \(\text{diameter}(P) < \frac{1}{2}\|Y_i\|\) for every cone-piece or wall-piece \(P\) of \(Y_i\).

**Definition 2.2.** Let \(X^* = \langle X \mid \{Y_j\}\rangle\) and \(A^* = \langle A \mid \{B_j\}\rangle\) be cubical presentations. A map \(A^* \to X^*\) of cubical presentations is a local isometry \(A \to X\), so that for each \(j\) there exists \(i\) such that there is a map \(B_j \to Y_i\) so that the composition \(B_j \to A \to X\) equals \(B_j \to Y_i \to X\).

Given a cubical presentation \(X^*\) and a local isometry \(A \to X\), the induced presentation is the cubical presentation of the form \(A^* = \langle A \mid \{A \otimes_X Y_i\}\rangle\) where \(A \otimes_X Y_i\) is the fiber-product of \(A \to X\) and \(Y_i \to X\). We refer to Section 2.4.2 for the definition of fiber-product. It is immediate that there is a map of cubical presentations \(A^* \to X^*\).

The following is a slightly more restrictive version of the same notion treated in [Wis, Def 3.61 & Def 3.65]:

**Definition 2.3.** Let \(A^* = \langle A \mid \{B_j\}\rangle\) and \(X^* = \langle X \mid \{Y_i\}\rangle\). We say \(A^* \to X^*\) has liftable shells provided the following holds: Whenever \(QS \to Y_i\) is an essential closed path with \(|Q| > |S|_{Y_i}\) and \(Q \to Y_i\) factors through \(Q \to A \otimes_X Y_i \to Y_i\), there exists \(B_j\) and a lift \(QS \to B_j\), such that \(B_j \to A \to X\) factors as \(B_j \to Y_i \to X\).

The following is a restatement of a combination of [Wis, Thm 3.68 and Cor 3.72]:
Lemma 2.4. Let $X^*$ be $C'(\frac{1}{2\pi})$. Let $A^* \to X^*$ have liftable shells and suppose that $A^*$ is compact. Then $\pi_1 A^* \to \pi_1 X^*$ is injective, $A^* \to X^*$ lifts to an embedding $\tilde{A}^* \to \tilde{X}^*$, and moreover the map is a quasi-isometric embedding.

The following appears as [Wis] Lem 3.67.

Lemma 2.5. Let $\langle X \mid \{Y_i\}\rangle$ be a $C'(\frac{1}{2\pi})$ small-cancellation cubical presentation. Let $A \to X$ be a local isometry and let $A^*$ be the associated induced presentation. Suppose that for each $i$, each component of $A \otimes X Y_i$ is either a copy of $Y_i$ or is a contractible complex $K$ with diameter$(K) \leq \frac{1}{2} \|Y_i\|$. Then the natural map $A^* \to X^*$ has liftable shells.

2.3. Special quotients.

Definition 2.6. A collection of subgroups $\{H_1, \ldots, H_r\}$ of $G$ is malnormal provided that $H_i^g \cap H_j = \{1_G\}$ unless $i = j$ and $g \in H_i$. Similarly, the collection is almost malnormal if intersections of nontrivial conjugates are finite (instead of trivial). Note that this condition implies that $H_i \neq H_j$ (unless they are finite in the almost malnormal case).

The following appears as [Wis] Thm 12.2:

Theorem 2.7 (Malnormal Virtually Special Quotient). Let $G$ be a word-hyperbolic group with a finite index subgroup $J$ that is the fundamental group of a compact special cube complex. Let $\{H_1, \ldots, H_r\}$ be an almost malnormal collection of quasi-convex subgroups of $G$. Then there are finite index subgroups $\tilde{H}_1, \ldots, \tilde{H}_s$ such that: For any finite index subgroups $H'_1, \ldots, H'_s$ contained in the $H_1, \ldots, H_r$ the quotient: $G' = G/\langle\langle H'_1, \ldots, H'_s\rangle\rangle$ is a word-hyperbolic group with a finite index subgroup $J'$ that is the fundamental group of a compact special cube complex.

The following is a simplified restatement of [Wis] Lem 12.10:

Lemma 2.8. Let $\langle X \mid Y_1, \ldots, Y_k\rangle$ be a $C'(\frac{1}{2\pi})$ small-cancellation cubical presentation. Let $A_1 \to X$ and $A_2 \to X$ be based local isometries. Suppose $X^*$ has small pieces relative to $A_1, A_2$ in the following sense for each $1 \leq i \leq 2$ and $1 \leq i \leq k$: For each pair of lifts $A_j, Y_i$ to $\tilde{X}$, the piece $P$ between $A_j, Y_i$ satisfies: Either diameter$(P) < \frac{1}{8} \|Y_i\|$ or $\tilde{Y}_i \subset \tilde{A}_j$ and factor through a map $Y_i \to A_j$.

Let $\langle \pi_1 A_1 g_i \pi_1 A_2 \rangle$ be a collection of distinct double cosets in $\pi_1 X$. And suppose that for each chosen representative $g_i$ and each cone $Y_j$ we have $|g_i| < \frac{1}{8} \|Y_j\|$.

Let $G \to \tilde{G}$ denote the quotient $\pi_1 X \to \pi_1 X^*$. Then:

1) [Double Coset Separation] $\pi_1 A_1 g_i \pi_1 A_2 \neq \pi_1 A_1 \tilde{g}_j \pi_1 A_2$ for $i \neq j$.

Suppose moreover that $\langle \pi_1 A_1 g_i \pi_1 A_2 \rangle$ form a complete set of double cosets with the property that $\pi_1 A_1 \cap \pi_1 A_2$ is infinite. Then:

2) [Square Annular Diagram Replacement] If $\pi_1 A_1 \tilde{g} \cap \pi_1 A_2$ is infinite for some $\tilde{g} \in \tilde{G}$, then $\pi_1 A_1 \tilde{g} \pi_1 A_2 = \pi_1 A_1 \tilde{g}_i \pi_1 A_2$ for some (unique) $i$.

3) [Intersections of Images] $\pi_1 A_1 \tilde{g} \cap \pi_1 A_2 = \pi_1 A_1 \tilde{g}_i \cap \pi_1 A_2$ for each $i$.

Remark 2.9. The following hold under the assumption of Lemma 2.8:

1) Let $g \in G$ be arbitrary. If $\pi_1 A_1 \tilde{g} \cap \pi_1 A_2$ is infinite, then there exists $g'$ with $\tilde{g} = \tilde{g}'$ such that $\pi_1 A_1 \tilde{g}' \cap \pi_1 A_2 = \pi_1 A_1 \tilde{g}_i \cap \pi_1 A_2$.

2) Let $g \in G$ be arbitrary. If $(\pi_1 A_1)\tilde{g} \cap \pi_1 A_2$ is infinite then $(\pi_1 A_1)\tilde{g} \cap \pi_1 A_2 = \pi_1 A_1 \tilde{g} \cap \pi_1 A_2$. 


We only prove (1) as (2) is similar. Note that by Lemma 2.8 (2),
\[ \pi_1 A_1 \bar{g} \pi_1 A_2 = \pi_1 A_1 \bar{g}_i \pi_1 A_2. \]
Thus \( \bar{g} = \bar{a}_1 \bar{g}_i \bar{a}_2 \) for some \( a_1 \in A_1 \) and \( a_2 \in A_2 \). Let \( g' = a_1 g a_2. \) Then
\[ \pi_1 A_1 \bar{g} \cap \pi_1 A_2 = (\pi_1 A_1 \bar{g}_i \cap \pi_1 A_2)^{a_2} \]
where the third inequality follows from Lemma 2.8 (3).

2.4. **Superconvexity and fiber products.**

2.4.1. **Superconvexity.** The following are quoted from [Wis, Def 2.35 & Lem 2.36]:

**Definition 2.10.** Let \( X \) be a metric space. A subset \( Y \subset X \) is superconvex if it is convex and for any bi-infinite geodesic \( \gamma \), if \( \gamma \) is contained in the \( r \)-neighborhood \( N_r(Y) \) for some \( r > 0 \), then \( \gamma \subset Y \). A map \( Y \to X \) is superconvex if the map \( \bar{Y} \to \bar{X} \) is an embedding onto a superconvex subspace.

**Lemma 2.11.** Let \( H \) be a quasiconvex subgroup of a word-hyperbolic group \( G \). And suppose that \( G \) acts properly and cocompactly on a CAT(0) cube complex \( X \). For each compact subcomplex \( D \subset X \) there exists a superconvex \( H \)-cocompact subcomplex \( K \subset X \) such that \( D \subset K \).

The following is a consequence of [Wis, Lem 2.39]:

**Lemma 2.12.** Let \( Y \to X \) be compact and superconvex. Then there exists \( r \) bounding the diameter of every wall-piece in \( \langle X \mid Y \rangle \).

2.4.2. **Fiber Products.** We record the following from [Wis, Def 8.8 and Lem 8.9]:

**Definition 2.13** (fiber-product). Given a pair of combinatorial maps \( A \to X \) and \( B \to X \) between cube complexes, we define their fiber-product \( A \otimes_X B \) to be a cube complex, whose \( i \)-cubes are pairs of \( i \)-cubes in \( A, B \) that map to the same \( i \)-cube in \( X \). There is a commutative diagram:

\[
\begin{array}{ccc}
A \otimes_X B & \to & B \\
\downarrow & & \downarrow \\
A & \to & X
\end{array}
\]

Note that \( A \otimes_X B \) is the subspace of \( A \times B \) that is the preimage of the diagonal \( D \subset X \times X \) under the map \( A \times B \to X \times X \). For any cube \( Q \), the diagonal of \( Q \times Q \) is isomorphic to \( Q \) by either of the projections, and this makes \( D \) into a cube complex isomorphic to \( X \). We thus obtain an induced cube complex structure on \( A \otimes_X B \).

Our description of \( A \otimes_X B \) as a subspace of the cartesian product \( A \times B \) endows the fiber-product \( A \otimes_X B \) with the property of being a universal receiver in the following sense: Consider a commutative diagram as below. Then there is an induced map \( C \to A \otimes_X B \) such that the following diagram commutes:

\[
\begin{array}{ccc}
C & \to & B \\
\downarrow & & \downarrow \\
A \otimes_X B & \to & X \\
\end{array}
\]

where \( \bar{g} = \bar{a}_1 \bar{g}_i \bar{a}_2 \) for some \( a_1 \in A_1 \) and \( a_2 \in A_2 \). Let \( g' = a_1 g a_2. \) Then
\[ \pi_1 A_1 \bar{g} \cap \pi_1 A_2 = (\pi_1 A_1 \bar{g}_i \cap \pi_1 A_2)^{a_2} \]
where the third inequality follows from Lemma 2.8 (3).
Lemma 2.14. Let $A \to X$ and $B \to X$ be local isometries of connected nonpositively curved cube complexes. Suppose the induced lift of universal covers $\tilde{A} \subset \tilde{X}$ is a superconvex subcomplex. Then the noncontractible components of $A \otimes_X B$ correspond precisely to the nontrivial intersections of conjugates of $\pi_1(A,a)$ and $\pi_1(B,b)$ in $\pi_1X$.

Let $(X \setminus \{Y_i\})$ be a cubical presentation. Then any cone piece of $Y_i$ can be written as the universal cover of some component of $Y_j \otimes_Y Y_i$ in $\tilde{Y}_i$.

3. Stature, Depth and Big-trees

3.1. Big-trees and Stature. Let $G$ be the fundamental group of a finite graph of groups with underlying graph $\mathcal{G}$, and let $T$ be the associated Bass-Serre tree. A subtree of $T$ is nontrivial if it contains at least one edge. Note that for any nontrivial subtree $S \subset T$, the pointwise stabilizer of $S$, denoted by $\text{stab}(S)$, is the intersection of the pointwise stabilizers of edges in $S$. Consequently, $\text{stab}(S)$ equals the intersection of conjugates of edge groups of $G$ that correspond to the edges of $S$.

Definition 3.1. A big-tree is a nontrivial subtree $S \subset T$ such that

- $\text{stab}(S)$ is infinite;
- there does not exist a subtree $S' \subset T$ with $S \subset S'$ and $\text{stab}(S) = \text{stab}(S')$.

It follows from the definition that $G$ acts on the collection of big-trees of $T$. If two big-trees have the same pointwise stabilizer, then they are the same. Consequently, for a big-tree $S \subset T$, the fixed point set of $\text{stab}(S)$ is exactly $S$. Moreover, for two big-trees $S_1$ and $S_2$, there exists $g \in G$ such that $gS_1 = S_2$ if and only if $g\text{stab}(S_1)g^{-1} = \text{stab}(S_2)$.

Definition 3.2 (based big-trees, transsections and transfer isomorphisms). Choose a spanning tree in $\mathcal{G}$ and lift this tree to a subtree $T_\mathcal{G} \subset T$. This identifies vertex groups of $G$ with stabilizers of vertices in $T_\mathcal{G}$. For each vertex $v \in T$, choose $g_v \in G$ such that $g_vu \in T_\mathcal{G}$. Note that $g_au$ is unique but $g_a$ might not be unique.

A based big-tree $(S,v)$ consists of a big-tree $S \subset T$ and a vertex $v \in S$. An $(S,v)$-transsection is a subgroup $\text{stab}(g_vS) \leq \text{Stab}(g_vv)$, i.e. it is a subgroup of $\text{Stab}(g_vv)$ of the form $g_v, \text{stab}(S)g_v^{-1}$. Different choices of $g_v$ for $v$ yield different $(S,v)$-transsections, however, they are conjugate within $\text{Stab}(g_vv)$.

For two different vertices $v_1, v_2 \in S$, the inclusions $\text{Stab}(v_1) \leftrightarrow \text{stab}(S) \leftrightarrow \text{Stab}(v_2)$ induce an isomorphism between an $(S,v_1)$-transsection and an $(S,v_2)$-transsection. This is called a transfer isomorphism, which is well-defined up to conjugacy in the vertex groups.

Lemma 3.3. Let $(S_1,v_1)$ and $(S_2,v_2)$ be based big-trees. There exists $g \in G$ such that $g(S_1,v_1) = (S_2,v_2)$ if and only if there exist $g_1, g_2 \in G$ and $w \in T_\mathcal{G}$ such that $g_1v_1 = g_2v_2 = w$ and any $(S_1,v_1)$-transsection and $(S_2,v_2)$-transsection are conjugate in $\text{Stab}(w)$.

Proof. For $i \in \{1,2\}$, let $H_i \leq \text{Stab}(w)$ be an $(S_i,v_i)$-transsection.

Suppose $g(S_1,v_1) = (S_2,v_2)$. Pick $g_2 \in G$ such that $g_2v_2 = w \in T_\mathcal{G}$. We assume without loss of generality that $H_i = g_i\text{stab}(S_i)g_i^{-1}$ for $i = 1,2$, where $g_1 = g_2$. Thus $H_1 = g_2g(\text{stab}(S_1)g_1^{-1})g_2^{-1} = g_2\text{stab}(S_2)g_2^{-1} = H_2$.

Suppose $H_1$ and $H_2$ are conjugate in $\text{Stab}(w)$. We assume without loss of generality that $H_i = g_i\text{stab}(S_i)g_i^{-1}$. Hence $g_1\text{stab}(S_1)g_1^{-1} = hgh_2\text{stab}(S_2)g_2^{-1}h^{-1}$ for $h \in \text{Stab}(w)$. Let $k = g_1^{-1}hg_2$. Then $kv_2 = v_1$ and $kS_2 = S_1$. \qed
Definition 3.4 \((\mathcal{Y} \text{ and } \mathcal{Y}_v)\). Consider the action of \(G\) on the collection of based big-trees. For each \(G\)-orbit, we pick a representative \((S,u)\) and consider an \((S,u)\)-transection. Let \(\mathcal{Y}\) be the collection of all such transections. For a vertex group \(V\) of \(G\) (i.e. \(V = \text{Stab}(v)\) for some \(v \in T_G\) as above), let \(\mathcal{Y}_v \subset \mathcal{Y}\) be the sub-collection of \((S,v)\)-transections such that \(g_vv\) corresponds to the vertex group \(V\).

Definition 3.5. A big-tree \(S \subset T\) is lowest if it is not properly contained in another big-tree. A subtree \(S \subset T\) is high if \(|\text{Stab}(S)| = \infty\) and \(\text{Stab}(S)\) does not contain any pointwise stabilizer of a lowest big-tree as a finite index subgroup. Similarly, a transection in \(\mathcal{Y}\) is lowest or high, if the associated big-tree is lowest or high.

Lemma 3.6. Let \(V\) be a vertex group of \(G\). Pick two elements \(H_1, H_2\) in \(\mathcal{Y}_v\).

1. Suppose \(H_1, H_2\) are lowest. If \(H_1 \cap H_2^g\) is infinite for some \(g \in V\), then \(H_1 = H_2\) and \(H_1 = H_2^g\).
2. If \(H_1\) is lowest and \(H_2\) is arbitrary, then for any \(a_1, a_2 \in V\), either \(H_1^{a_1} \subset H_2^{a_2}\) or \(H_1^{a_1} \cap H_2^{a_2}\) is finite.

Proof. For \(i \in \{1,2\}\), suppose \(H_i\) is an \((S_i,v_i)\)-transection. Let \(v \in T_G\) be the vertex associated with \(V\). Suppose \(H_i = g_i \text{Stab}(S_i)g_i^{-1}\) for \(i \in \{1,2\}\), where \(g_i v_i = v\). Now we prove Part 1. Let \(S\) be the convex hull of \(g_1 S_1\) and \(g_2 S_2\). Then \(\text{Stab}(S) = H_1 \cap H_2^g\). Suppose \(\text{Stab}(S)\) is infinite. Since both \(S_1\) and \(S_2\) are lowest, we have \(S = g_1 S_1 = g_2 S_2\). Let \(k = g_1^{-1}g_2\). Then \(S_1 = k S_2\) and \(v_1 = kv_2\). Thus \(H_1 = H_2\). \((S_1,v_1) = (S_2,v_2)\) and \(g_1 = g_2\) by our choice of \(\mathcal{Y}_v\). Moreover, \(H_1^{a_1} \cap H_2^{a_2} = \text{Stab}(a_1 g_1 S_1)\) is finite.

We now prove Part 2. Let \(S\) be the convex hull of \(a_1 g_1 S_1\) and \(a_2 g_2 S_2\). Then \(H_1^{a_1} \cap H_2^{a_2} = \text{Stab}(S)\). If \(H_1^{a_1} \cap H_2^{a_2}\) is infinite, since \(a_1 g_1 S_1\) is lowest, \(S = a_1 g_1 S_1\). Hence \(a_2 g_2 S_2 \subset a_1 g_1 S_1\). Hence \(H_1^{a_1} = \text{Stab}(a_1 g_1 S_1) \subset \text{Stab}(a_2 g_2 S_2) = H_2^{a_2}\). \(\square\)

Definition 3.7. Let \(G\) act without inversions on a tree \(T\). Say \(G\) has finite stature (relative to the action \(G \acts T\)) if the action of \(G\) on the collection of based big-trees has finitely many orbits.

Note that \(G\) satisfies the above definition if and only if the collection \(\mathcal{Y}\) is finite.

Example 3.8. Let \(G = BS(1,2) = \langle a,t | a^t = a^2 \rangle\). The action of \(G\) on its Bass-Serre tree has finitely many orbits of big-trees, however, the action does not have finitely many orbits of based big-trees. Thus \(G\) does not have finite stature. On the other hand, if \(G = BS(1,1) = \langle a,t | a^t = a \rangle\), then \(G\) has finite stature with respect to its action on the Bass-Serre tree.

Lemma 3.9. Let \(G\) act without inversions on a tree \(T\). The following are equivalent:

1. \(G\) has finite stature with respect to the action \(G \acts T\).
2. For each vertex group \(V\) of \(G\), there are finitely many \(V\)-conjugacy classes of infinite subgroups of the form \(V \cap (\cap_{e \in E} \text{Stab}(e))\), where \(E\) is a collection of edges in \(T\).
3. \((G,V)\) has finite stature in the sense of Definition 1.1, where \(V\) is the collection of vertex groups of \(G\).

Recall that we have identified \(V\) with the stabilizer of a vertex \(v \in T_G\).

Proof. Note that an infinite subgroup \(H \leq V\) is of form \(V \cap (\cap_{e \in E} \text{Stab}(e))\) if and only if \(H\) is an \((S,v)\)-transection. Each element in \(\mathcal{Y}_v\) is of such form. Moreover, by
Lemma 3.10. Each $V$-conjugacy class of such subgroups contains exactly one element inside $\Upsilon_v$. Now the equivalence between (1) and (2) follows. The equivalence between (2) and (3) follows directly from definition.

In general, if $G$ splits as a graph of groups in two different ways, then it is possible that $G$ has finite stature under one splitting, but not the other splitting, see Example 3.31. However, when the splitting of $G$ is already clear, we will only write $G$ has finite stature for simplicity.

3.2. Depth and Stature. We now explore a notion measuring the maximal length of an increasing sequence of big trees. There are two variations according to whether the pointwise stabilizer of these big trees are commensurable.

Definition 3.10. Let $G$ be a group and let $\Lambda = \{H_1, \ldots, H_r\}$ be a collection of subgroups. The commensurable depth of $\Lambda$ in $G$, denoted $\delta_c(G, \Lambda)$, is the largest integer $d$, such that there is a strictly increasing chain $L_1 < \cdots < L_d$, where each $L_i$ is the intersection of finitely many conjugates of elements of $\Lambda$. If there are arbitrarily long such sequences, then we define $\delta_c(G, \Lambda) = \infty$. We say $\Lambda$ has finite commensurable depth in $G$ if $\delta_c(G, \Lambda) < \infty$.

Definition 3.11. Let $\Lambda$ and $G$ be as before. The depth of $\Lambda$ in $G$, denoted $\delta(G, \Lambda)$, is the largest integer $d$, such that there is a strictly increasing chain $L_1 < \cdots < L_d$ satisfying the conditions that $|L_1| = \infty$, each $L_i$ is an intersection of finitely many conjugates of elements of $\Lambda$, and $[L_i : L_{i+1}] = \infty$. If such $d$ does not exist, then we define $\delta(G, \Lambda) = \infty$.

Example 3.12. Note that $\delta_c(G, \Lambda) < \infty$ implies $\delta(G, \Lambda) < \infty$, but the converse may not be true. For instance if $G = \langle a, t | a^t = a^2 \rangle$, then the sequence $\langle a \rangle < \langle a^t^{-1} \rangle < \langle a^{t^{-2}} \rangle < \cdots$ shows that $H = \{ \langle a \rangle \}$ has infinite commensurable depth. However, $\delta(G, \{ \langle a \rangle \}) = 1$.

In the rest of this subsection, we return to the scenario where $G$ splits as a graph $G$ of groups. Recall that we have identified vertex groups of $G$ with vertex stabilizers of a subtree $T_G \subset T$. We assume in addition that each vertex group of $G$ is word-hyperbolic, and each edge group is quasiconvex in its associated vertex groups. Let $E$ be the collection of edge groups of $G$.

We recall the following fact which is proven in [GMRS98] (see also [HW09]):

Lemma 3.13. Let $\{H_1, \ldots, H_r\}$ be a collection of quasiconvex subgroups of the word-hyperbolic group $G$. Then $\{H_1, \ldots, H_r\}$ has finite height in $G$.

Note that each big-tree is uniformly locally finite by Lemma 3.13, thus $G$ has finite stature if and only if both of the following conditions hold:

1. There are finitely many $G$-orbits of big-trees in $T$;
2. $\text{Stab}(S)$ acts cocompactly on each big-tree $S$.

Lemma 3.14. Suppose each vertex group of $G$ is word-hyperbolic, and each edge group is quasiconvex in its vertex groups. Pick a finite subtree $S \subset T$ and a vertex $v \in S$. Then $\text{Stab}(S) \subset \text{Stab}(v)$ is quasiconvex.

Proof. It suffices to show that $\text{Stab}(S) \hookrightarrow \text{Stab}(e)$ is quasiconvex for a particular edge $e \subset S$, since quasiconvexity is transitive via the vertex groups. We induct on the number of edges in $S$. Let $e \subset S$ be containing a leaf. We remove $e$ from
to obtain another tree $S'$. Let $u = e \cap S'$. By induction, $\text{stab}(S')$ and $\text{stab}(e)$ are quasiconvex in $\text{Stab}(u)$, hence $\text{stab}(S) = \text{stab}(S') \cap \text{stab}(e)$ is quasiconvex in $\text{Stab}(u)$. So $\text{stab}(S)$ is quasiconvex in $\text{Stab}(e)$. □

We recall the following standard fact about hyperbolic groups.

Lemma 3.15. Let $G$ be a word-hyperbolic group. Then it contains only finitely many conjugacy classes of finite subgroups.

Lemma 3.16. Let $G$ be as in Lemma 3.14. If $G$ has finite stature, then $\delta_c(G, \mathcal{E}) < \infty$. Consequently, $\delta(G, \mathcal{E}) < \infty$.

Proof. Let $L_1 < \cdots < L_d$ be the chain in Definition 3.10. Thus there is a sequence of finite trees $T_1 \supset \cdots \supset T_d$ such that $L_i = \text{stab}(T_i)$. Let $V = \text{Stab}(v)$ be a vertex group with $v \in T_d$. Without loss of generality, we assume $v \in T_d$, thus each $L_i$ is an intersection of $V$ with an intersection of finitely many conjugates of elements of $\mathcal{E}$. Thus each $L_i$ is an $V$-conjugate of an element of $\Upsilon_V$. Moreover, if $i \neq j$, then $L_i$ and $L_j$ can not be conjugated to the same element of $\Upsilon_V$. This is because each $L_i$ is quasiconvex in a word-hyperbolic group $V$ (by Lemma 3.14) and a quasiconvex subgroup of a word-hyperbolic group is not conjugated to its proper subgroups (by Lemma 3.13). Since $\Upsilon_V$ is finite, we have a bound on the number of infinite order elements in the chain. By Lemma 3.15, there is an upper bound on the order of any finite subgroup. This bounds the length of any chain of finite subgroups. □

Remark 3.17.

(1) Even if $G$ satisfies the assumption of Lemma 3.14, the converse of Lemma 3.16 is not true. This can be seen by letting $G$ be the free by cyclic group with a non-standard splitting discussed in Example 3.31.

(2) Lemma 3.16 is not true for more general groups. For example, let $V = \langle a, s | a^s = a^2 \rangle$ and let $G = \langle V, t | a^t = a^2 \rangle$. Then $G$ has finite stature with respect to $V$, but $\delta_c(G, V) = \infty$.

Since $\delta_c(G, \mathcal{E}) < \infty$, the pointwise stabilizer of any subtree of $T$ can be expressed as an intersection of finitely many conjugates of edges groups. Thus the following two lemmas hold.

Lemma 3.18. Lemma 3.14 holds without the assumption that $S$ is finite. In particular, for any vertex group $V$, each element in $\Upsilon_V$ is quasiconvex.

Lemma 3.19. Suppose $G$ satisfies the assumption of Lemma 3.14. Then Lemma 3.9 still holds if we assume the collection $E$ there is finite.

Lemma 3.19 and Corollary 3.22 below imply the following.

Corollary 3.20. Suppose $G$ is hyperbolic and it splits as a graph of groups such that each vertex group is quasiconvex. Then $G$ has finite stature.

A proof of the following statement can be found in [GMRS98] or [HW09].

Lemma 3.21. Let $G$ be a hyperbolic group, and $A, B$ be quasiconvex subgroups. There are finitely many double cosets $B_gA$ such that $A \cap B^g$ is infinite.

In other words, if we consider the collection of all subgroups of form $A \cap B^g$ which are infinite, then there are finitely many $A$-conjugacy classes of such subgroups.

The next result follows from Lemma 3.13, Lemma 3.21 and Lemma 3.15.
Corollary 3.22. Let \( \{H_1, \ldots, H_r\} \) be a collection of quasiconvex subgroups of the word-hyperbolic group \( G \). Let \( K \) be a quasiconvex subgroup of \( G \). Then there are only finitely many \( K \)-conjugacy classes of subgroups of the form \( K \cap (\cap_{k=1}^n H_k) \).

3.3. Several observations for passing to finite index subgroups. We need the following lemmas later when we consider torsion-free finite index subgroups of the edge groups. For each \( E \in \mathcal{E} \), we choose a finite index normal subgroup \( E' \leq E \), and let \( E' = \{E'\}_{E \in \mathcal{E}} \). For each subtree \( S \subset T \), let \( E' \subseteq T \) denote the intersections of conjugates of elements of \( E' \) corresponding to edges of \( S \). Note that \( E'_S \) is well-defined since \( E' \leq E \) is normal for each \( E' \).

Lemma 3.23. Let \( G \) be as in Lemma 3.14. Then for any finite collection of edges \( \{e_i\}_{i=1}^n \) in \( T \), the index \( |\cap_{i=1}^n \text{stab}(e_i) : \cap_{i=1}^n E'_{e_i} | \) is uniformly bounded above. Consequently, \( \delta_\lambda(G, E') < \infty \) and \( \delta(G, E') = \delta(G, E) \).

Proof. By Lemma 3.16 \( \delta_\lambda(G, E) < \infty \). We claim \( |\text{stab}(S) : E'_S | \) is uniformly bounded above for any finite subtree \( S \). Note that there is a collection of edges \( \{e_i\}_{i=1}^m \subseteq S \) with \( m \leq \delta_\lambda(G, E) + 1 \) such that \( \text{stab}(S) = \cap_{i=1}^m \text{stab}(e_i) \). Thus for any edge \( e \in S \), \( \cap_{i=1}^m E'_{e_i} \subset \text{stab}(e) \), hence
\[
(3.24) \quad \cap_{i=1}^m E'_{e_i} : (\cap_{i=1}^m E'_{e_i}) \cap E'_e \leq |\text{stab}(e) : E'_e |
\]
Since there are only finitely many \( G \)-orbits of big-trees, there are finitely many isomorphism types of groups of form \( \text{stab}(S) \), each of which is f.g. by Lemma 3.18. Since each \( \cap_{i=1}^m E'_{e_i} \) is a subgroup of \( \text{stab}(S) \) with its index uniformly bounded above, there are finitely many isomorphism types of groups of form \( \cap_{i=1}^m E'_{e_i} \), and each of them is finitely generated. Since \( E'_S = \cap_{e \subset S} E'_{e} \), by Equation (3.24), we have \( |\text{stab}(S) : E'_S | \) is uniformly bounded above.

Finally, let \( S \) be the convex hull of \( \{e_i\}_{i=1}^n \). Then \( |\cap_{i=1}^n \text{stab}(e_i) : \cap_{i=1}^n E'_{e_i} | \leq |\text{stab}(S) : E'_S | \), which is uniformly bounded above by previous discussion.

Definition 3.25. The commensurator \( \mathbb{C}_G(H) \) of a subgroup \( H \) of \( G \), is the subgroup consisting of elements \( g \in G \) such that \( |H : H^g \cap H| < \infty \).

The following is proven in [KS96]:

Lemma 3.26. Let \( H \) be a quasiconvex subgroup of a word-hyperbolic group \( G \). Then \( H \) has finite index in the commensurator of \( H \) inside \( G \).

Lemma 3.27. Let \( G \) be as in Lemma 3.14. The following are equivalent:

1) \( G \) has finite stature;
2) for any vertex group \( V \) of \( G \) and its associated vertex \( v \in T \), there are finitely many \( V \)-conjugacy classes of infinite subgroups of \( V \) of the form \( E'_S \), where \( S \) is a finite subtree containing \( v \).

Proof. (1) \( \Rightarrow \) (2) is a consequence of Lemma 3.19 and Lemma 3.23. Now we assume (2). Let \( U' \) (resp. \( U \)) be the collection of infinite subgroups of \( V \) which are of form \( E'_S \) (resp. \( \text{stab}(S) \)) for a finite subtree \( v \in S \subset T \). Since \( |\text{stab}(S) : E'_S | \) is finite, each element of \( U' \) is quasiconvex in \( V \) by Lemma 3.14. By (2) and Lemma 3.26, each element of \( U' \) is finite index in \( \mathbb{C}_V(U') \) with index uniformly bounded above. Since \( \text{stab}(S) \) is contained in the commensurator \( \mathbb{C}_V(E'_S) \), \(|\mathbb{C}_V(E'_S) : \text{stab}(S)| \) is uniformly bounded above. Since there are finitely many \( V \)-conjugacy classes in \( \mathbb{C}(U') \), the same is true for \( U \), hence (1) follows by Lemma 3.19. \( \square \)
3.4. Stature, Depth and Cleanliness. In this subsection we examine the behavior of stature and commensurable depth for certain graphs of free groups. As in Examples 3.8 and 3.12, Baumslag-Solitar groups typically fail to have finite stature and fail to have finite commensurable depth. Irreducible lattices acting on products of trees \cite{Wis96, BM00} also have infinite stature. This failure is typical for general groups that split as a graph of groups where all edge groups are of finite index in the vertex groups:

**Proposition 3.28.** Let \( G \) split as a finite graph of groups where each edge group has finite index in its vertex groups. Then the following are equivalent:

1. \( G \) has finite stature with respect to its vertex groups.
2. \( G \) has finite commensurable depth with respect to its vertex groups.
3. there is a normal subgroup \( N \subset G \) such that \( N \) is of finite index in each vertex group, and the quotient \( G/N \) is a f.g. virtually free group.

It follows that \( G \) has a finite index subgroup that is isomorphic to \( N \rtimes F \) for some f.g. free group \( F \).

**Proof.** (3) \( \Rightarrow \) (1) and (2): Let \( N \subset G \) be the normal subgroup. For any subtree \( S \subset T \) of the Bass-Serre tree, we have \( N \subset \$\text{tab}(S) \subset V \). Hence \( \$\text{tab}(S) \subset V \) equals one of the finitely many subgroups of the vertex group \( V \) containing \( N \). Thus (1) and (2) follow.

(1) \( \Rightarrow \) (2): Let \( v \) be a vertex in the Bass-Serre tree \( T \). Let \( \mathcal{C} \) be the collection of subgroups of \( \text{Stab}(v) \) of form \( \$\text{tab}(S) \) where \( S \) is a finite subtree of \( T \) containing \( v \). Then each element of \( \mathcal{C} \) is a finite index subgroup of \( \text{Stab}(v) \). Note that finite index subgroups of \( \text{Stab}(v) \) of different index can not be conjugated (inside \( \text{Stab}(v) \)). Thus finite stature implies that the \( \left[ \text{Stab}(v) : H \right] \) is uniformly bounded from above for \( H \in \mathcal{C} \). Thus (2) follows.

(2) \( \Rightarrow \) (3): For any increasing sequence of finite subtrees \( S_1 \subset S_2 \subset \cdots \), the sequence \( \$\text{tab}(S_1) \geq \$\text{tab}(S_2) \geq \cdots \) stabilizes after finitely many proper inclusions. Let \( N' \) be the smallest subgroup arising in this way, and observe that \( N' \) is of finite index in \( V \) where \( v \) is a base vertex of \( S \) and \( V = \$\text{tab}(v) \). We now check that \( N' \) is a normal subgroup of \( G \). Observe that \( gN'g^{-1} = \$\text{tab}(gS) \). However \( N' = \$\text{tab}(S') \) where \( S' \) the smallest subtree containing \( S \cup gS \). Thus \( N' \) is normal. It follows that \( \$\text{tab}(T) = N' \). Finally, the quotient \( G/N' \) acts faithfully on the locally finite tree \( T \), and hence \( F = G/N' \) is virtually free. \( \square \)

Proposition 3.28 deceptively suggests that determining finite stature might be accessible and interpretable. However, we expect that:

**Conjecture 3.29.** Then there is no algorithm that takes as input a group \( G \) which splits as a finite graph of f.g. free groups, and outputs certification that \( G \) has infinite stature with respect to its vertex groups.

There is an algorithm that computes the stature when it is finite for the above \( G \), since one can repeatedly compute intersections of subgroups in a free group.

Let us turn now to some more restricted classes of graphs of free groups. Let \( G \) split as a finite graph of groups where all vertex and edge groups are f.g. free groups. The splitting is \textit{algebraically clean} if each edge group embeds as a free factor of each of its vertex groups. The splitting is \textit{geometrically clean} if it arises from a graph of spaces where each vertex space is a graph, and edge space is a
Example 3.31. Let $\phi$ be a pseudo-Anosov automorphism of $F_2$, the free group of two generators $a$ and $b$. Let $G = F \rtimes_{\phi} \mathbb{Z}$. $G$ has a splitting with the underlying graph being a circle. It is clear that $G$ has finite stature with respect to its standard splitting. However, we can change the graph of groups structure of $G$ by adding a new edge $e$ to its underlying graph along the vertex of the circle, such that the vertex group at the leaf of $e$ and the edge group of $e$ are the subgroup $\langle a \rangle$ in $F_2$. Since $\phi$ is pseudo-Anosov, $\phi^n(\langle a \rangle)$ and $\langle a \rangle$ are not conjugate inside $F_2$ for any $n \neq 0$. Thus there are infinitely many orbits of based big-trees under this new splitting. Hence $G$ does not have finite stature under such splitting.

Example 3.31 shows that finite stature may not hold in the algebraically clean case. When $G$ has an algebraically clean splitting, all its vertex groups and edge group are separable. This can be proven in various ways (e.g. doubling along a vertex group preserves algebraically clean). More fundamentally, residual finiteness
4. A separability result for graphs of hyperbolic special groups

In this section we prove the following theorem.

Theorem 4.1. Let $G$ split as a graph of groups with finite underlying graph. Suppose the following conditions hold:

1. each vertex group is word-hyperbolic and virtually compact special;
2. each edge group is quasiconvex in its vertex groups;
3. $(G, V)$ has finite stature, where $V$ is the collection of vertex groups of $G$.

Then each quasiconvex subgroup of a vertex group of $G$ is separable in $G$. In particular, $G$ is residually finite.

4.1. The depth reducing quotient. Let $G$ split as a finite graph of groups with underlying graph $G$. Let $\{V_i\}_{i=1}^n$ be the collection of vertex groups of $G$. Suppose there is an edge $E$ between $V_i$ and $V_j$ (it is possible that $i = j$). Then $E$ induces an isomorphism $\alpha_E : E_i \to E_j$ from a subgroup of $V_i$ to a subgroup of $V_j$. Note that $\alpha_E$ is a transfer isomorphism discussed in Definition 3.2.

Define a quotient of graphs of groups as follows. Let $\{q_i : V_i \to \tilde{V}_i\}_{i=1}^n$ be a collection of quotient maps. They are compatible if for any edge $E$ between $V_i$ and $V_j$, we have $\alpha_E(E_i \cap \ker q_i) = E_j \cap \ker q_j$. In this case, $\alpha_E$ descends to an isomorphism $\tilde{\alpha}_E : \tilde{E}_i \to \tilde{E}_j$, where $\tilde{E}_i = q_i(E_i)$. Define a new graph of groups with the same underlying graph $G$, vertex groups the $\tilde{V}_i$'s, and isomorphisms $\{\tilde{\alpha}_E\}$ between edge groups. Let $G$ be the fundamental group of this new graph of groups. There is an induced quotient homomorphism $G \to \tilde{G}$.

Proposition 4.2. Let $G$ be the fundamental group of a finite graph of groups and let $V$ and $\mathcal{E}$ be the collection of vertex groups and edge groups of $G$. Suppose

1. each vertex group is word-hyperbolic and virtually compact special;
2. each edge group is quasiconvex in its vertex groups;
3. $G$ has finite stature, and $\delta(G, \mathcal{E}) > 0$.

For each $V \in \mathcal{V}$, let $Q_V \leq V$ be a quasiconvex subgroup. Choose a finite index subgroup $V' \leq V$ for each $V \in \mathcal{V}$, and choose a finite index subgroup $E' \leq E$ for each $E \in \mathcal{E}$. Then there exists a collection of quotient homomorphisms $\{\phi_V : V \to \tilde{V}\}_{V \in \mathcal{V}}$ such that

1. $\tilde{V} = V/\langle\langle L_i^V \rangle\rangle$, where each $L_i^V$ is a finite index subgroup of a lowest transsection of $G$ in $V$, and the collection varies over representatives of all such lowest transections; moreover, each $L_i^V$ can be chosen such that it is contained in a given finite index subgroup of its associated lowest transsection;
2. for each edge group $E \to V$, the kernel $\ker(E \to \tilde{V})$ is generated by $V$-conjugates of $\{L_i^V\}$ that are contained in $E$.
(3) the collection \( \{ \phi_V : V \to \bar{V} \}_{V \in \mathcal{V}} \) is compatible, hence there is a quotient of graphs of groups \( \phi : G \to \bar{G} \) as above;

(4) \( \bar{V} \) is word-hyperbolic and virtually compact special for each \( V \);

(5) each edge group of \( \bar{G} \) is quasiconvex in the corresponding vertex groups;

(6) \( (G, \bar{V}) \) has finite stature;

(7) \( \delta(\bar{G}, \bar{E}) < \delta(G, E) \);

(8) \( \ker \phi|_V \leq V' \) for each \( V \in \mathcal{V} \) and \( \ker \phi|_E \leq E' \) for each \( E \in \mathcal{E} \);

(9) Each \( \bar{Q}_V \) is quasiconvex in \( \bar{V} \).

Moreover, let \( S \subseteq T \) be a finite subtree of the Bass-Serre tree of \( G \). Then we can assume that the \( G \)-equivariant map \( \phi_T : T \to \bar{T} \) to the Bass-Serre tree of \( \bar{G} \) has the property that \( \phi_T|_S \) is injective.

We now deduce Theorem 4.1 from Proposition 4.2.

Proof of Theorem 4.1. We induce on \( \delta(G, \mathcal{E}) \) and first look at the case \( \delta(G, \mathcal{E}) = 0 \). Then each element of \( \mathcal{E} \) is finite. By a covering space argument, \( G \) has a finite index subgroup which is a free product of a free group with a collection of groups which are fundamental groups of hyperbolic special cube complexes. Hence \( G \) is hyperbolic and virtually compact special. Any quasiconvex subgroup \( Q \) of a vertex group of \( G \) is quasiconvex in \( G \), hence \( Q \) is separable.

Now we assume \( \delta(G, \mathcal{E}) > 0 \). Let \( Q \) be a quasiconvex subgroup of a vertex group \( V \). By Proposition 4.2 there exists a quotient \( \phi : G \to \bar{G} \) which satisfies all the conditions there, in particular, \( \delta(\bar{G}, \bar{E}) < \delta(G, \mathcal{E}) \). Let \( \bar{Q} \) be the image of \( Q \) under \( V \to \bar{V} \). Pick \( g \in G - Q \), we claim that it is possible to choose \( \phi \) such that \( \bar{g} \notin \bar{Q} \). Assuming this claim, we can deduce the theorem as follows. By Proposition 4.2(6), \( Q \) is quasiconvex in \( \bar{V} \), hence \( \bar{Q} \) is separable in \( \bar{G} \) by induction, and Theorem 4.1 follows. Now we prove the claim. Let \( v \in T \) be the vertex associated with \( V \). Suppose \( g \) fixes \( v \). Since \( Q \) is separable in \( V \), there is a finite index normal subgroup \( \bar{V} \leq V \) such that \( \bar{g} \notin \bar{Q} \). By Proposition 4.2(5), we can choose \( \phi \) such that \( \ker(\bar{V}) \leq \bar{V} \), which implies \( \bar{g} \notin \bar{Q} \). Suppose \( g \) does not fix \( v \). Let \( S \) be the convex hull of \( v \) and \( g \bar{v} \). By the moreover statement of Proposition 4.2, we can assume \( \phi_T|_S \) is injective. Since \( \phi_T \) is \( G \)-equivariant, \( \bar{g} \) does not stabilize \( \phi_T(v) \), hence \( \bar{g} \notin \bar{V} \), in particular, \( \bar{g} \notin \bar{Q} \).

Remark 4.3. We can actually assume each vertex group of \( G \) acts geometrically on a \( CAT(0) \) cube complex. This is because of [Wis, Lem 7.14], which says that if \( A \) is word-hyperbolic and has a finite index subgroup that acts properly and cocompactly on a \( CAT(0) \) cube complex, then \( A \) acts properly and cocompactly on a \( CAT(0) \) cube complex.

Remark 4.4 (Intuition about compatibility). We describe how a characteristic subgroup of \( \text{stab}(S) \) for each lowest big-tree \( S \) yields a compatible collection of subgroups of the vertex groups of the graph of groups \( G \).

For each big tree \( S \), there is a short exact sequence \( 1 \to \text{stab}(S) \to \text{Stab}(S) \to K \to 1 \) where \( K \) acts faithfully on \( S \). A characteristic subgroup \( N \) of \( \text{stab}(S) \) (or more generally, a subgroup \( N \leq \text{stab}(S) \) that is invariant under conjugation by \( \text{Stab}(S) \)) yields a collection of conjugacy classes of subgroups in the vertex groups of \( G \) regarded as stabilizers of vertex groups of the Bass-Serre tree \( T \). More precisely, for a vertex \( v \) of \( T \), the translates \( qS \) that contain \( v \) yield a collection of subgroups of the vertex group \( G_v \). When \( v \in S \), the distinct conjugacy classes of such subgroups...
in $G_v$ correspond to the $\text{Stab}(S)$ orbits of $v$ in $S$. (More generally, for $v \in gS$, the analogous statement holds for $g^{-1}\text{Stab}(S)g$ orbits.) So the $\text{Stab}(S)$ orbits of $v$ in $S$ correspond to the collection of various subgroups we will quotient by in $\bar{v}$, and compatibility merely reflects that if $u, v$ are vertices that are joined by an edge $e$ in $gS$ for some $g \in G$, then the corresponding quotienting subgroups are isomorphic across that edge. Hence $G_{\bar{u}}, G_{\bar{v}}$ are compatible across $\bar{e}$.

Different constraints on the various vertex groups (to ensure some separability, or to lie in our chosen subgroups, or to ensure small-cancellation and hence preserve quasiconvexity and actual (instead of plausible) compatibility, are collected together from all the vertex groups of $S$, and we choose $N$ to respect these constraints.

From the viewpoint of the base space, we regard $G$ as the fundamental group of a graph of spaces. For simplicity, let us assume that $K$ is a free group, and let $N \leq \text{stab}(S)$ be a characteristic subgroup. Then $K \backslash S$ corresponds to a graph of spaces (which the reader should regard as an $X_N$ bundle over $\tilde{S} = K \backslash S$) that immerses into the graph of spaces for $G$. Plausible compatibility corresponds to the fact that the immersed graph of spaces is a bundle, and hence there is a direct isomorphism between the collection of subgroups of vertex groups on each side of an edge group.

In fact, we are describing “plausible compatibility” above: Namely that the subgroups we will kill are the same in a vertex group and an edge group. Compatibility says that killing them in the vertex group, induces the expected result on edge groups. It is a separate technical result that plausible compatibility yields compatibility under certain small cancellation hypotheses that we ensure.

A further important point is the use of Proposition 4.10 which shows that any finite tree with infinite point-wise stabilizer in $T$ is actually the image of a finite tree with infinite point-wise stabilizer in $T$. This ensures that the depth decreases, since we are quotienting by finite index subgroups of the point-wise stabilizers of lowest transections. This explains our interest in quotienting these particular subgroups.

4.2. Proof of Proposition 4.2. As in Definition 3.5, for each vertex group $V$, let $\{L_i^V\}_{i=1}^{\ell_v}$ be the collection of lowest elements in $\check{\Upsilon}_V$. Let $\{H_i^V\}_{i=1}^{h_v}$ be the collections of high elements in $\check{\Upsilon}_V$. The following is a consequence of Lemma 3.20 and Lemma 3.6.(1) (we recall commensurator and almost malnormality from Definition 3.25 and Definition 2.6).

**Lemma 4.5.** $\{C_V(L_i^V)\}_{i=1}^{\ell_v}$ is almost malnormal in $V$.

We may assume without loss of generality the $V'$ and the $E'$ of Proposition 4.2 are normal and torsion-free in their ambient groups. We may moreover assume $E' \leq V'$ whenever $E$ is an edge group of $V$.

By Remark 4.3, for each vertex group $V$, let $\tilde{X}_V$ be a $\text{CAT}(0)$ cube complex upon which $V$ acts properly and cocompactly. Let $X_{V'} = \tilde{X}_V / V'$. We do not assume $X_{V'}$ is virtually special (though it is true), since we only need $X_{V'}$ for cubical small cancellation theory.

The following holds by Lemma 3.6 and Lemma 3.15.

**Lemma 4.6.** Let $\tilde{X}_V^{L_i} \subset \tilde{X}_V$ be a $C_V(L_i^V)$-coconvex superconvex subcomplex for each lowest subgroup. Let $\tilde{X}_V^{H_i} \subset \tilde{X}_V$ be an $H_i^V$-coconvex superconvex subcomplex for each high subgroup. By possibly enlarging $\tilde{X}_V^{H_i}$, there exists $M > 0$ such that for any $g_1, g_2 \in V$ and any $i, i'$ we have:
(1) either \( \text{diam}(g_1\tilde{X}_L^V \cap g_2\tilde{X}_L^{H'}) < M \), or \( g_1\tilde{X}_L^V = g_2\tilde{X}_L^{H'} \);
(2) either \( \text{diam}(g_1\tilde{X}_L^V \cap g_2\tilde{X}_H^{H'}) < M \), or \( g_1\tilde{X}_L^V \subset g_2\tilde{X}_H^{H'} \).

We use \( \langle \cdots \rangle_H \) to denote the normal closure inside a subgroup \( H \) of a collection of elements. If \( H \) is clear, we will also write \( \langle \cdots \rangle \). We use \( \langle \cdots \rangle \) to denote the subgroup generated by a collection of elements.

**Lemma 4.7.** There exists a collection of finite index normal subgroups \( \{L'_V \subset \mathcal{C}_V(L'_V)\}_{V \in V} \) such that the following properties hold for each \( V \in V \), moreover, they hold for any deeper finite index normal subgroups of the \( \mathcal{C}_V(L'_V) \).

1. Each \( L'_V \leq V' \cap L'_V \). Moreover, for each edge group \( E \subset V \), any \( V' \)-conjugate of \( L'_V \) is either contained in \( E' \) or intersects \( E' \) trivially.
2. \( V = V/\langle \langle L'_1^V, \cdots, L'_d^V \rangle \rangle \) is word-hyperbolic and virtually compact special.

For a subgroup \( J \subset V \), we use \( \bar{J} \) to denote the image of \( J \) under \( V \rightarrow \bar{V} \).

3. Let \( E \) be any edge group of \( V \) and let \( E'_V = V' \cap E \). Then ker\( (E'_V \rightarrow \bar{V}) \) equals the normal closure in \( E'_V \) of all \( V' \)-conjugates of \( \{L'_V, \cdots, L'_d^V\}_{V \in V} \) that are contained in \( E'_V \). Moreover, \( E'_V \) is quasiconvex in \( \bar{V} \).

4. For each subtree \( S \) of the Bass-Serre tree \( T \), let \( E'_S \) be the subgroup defined right before Lemma 3.6.2 Then for each finite subtree \( S \) containing the vertex \( v \in T \) associated with \( V \) with \( E'_S \) infinite, any \( V' \)-conjugate of \( L'_V \) is either contained in \( E'_S \) or intersects \( E'_S \) trivially.

5. For any pair \( S_1 \) and \( S_2 \) of finite subtrees with infinite pointwise stabilizers with \( v \in S_1 \cap S_2 \), Remark 2.9 holds for \( E'_S \) and \( E'_S \) under the quotient homomorphism \( V' \rightarrow \bar{V}' \).

**Proof.** It suffices to prove that for each property, there exists a collection of subgroups satisfying that property, and the property still holds after passing to further finite index subgroups of elements in this collection.

We first ensure property \( \langle 1 \rangle \). Each infinite edge group in \( V \) is an element of \( \Upsilon \) up to conjugacy in \( V \). By Lemma 3.6.2, for any \( g \in V \), either \( L'_V \cap (E')^g \) is trivial, or \( L'_V \cap (E')^g \) is of finite index in \( L'_V \) (the index is uniformly bounded above independent of \( g \)). We choose \( L'_V \) to be contained in these finite index subgroups.

Property \( \langle 2 \rangle \) follows from Theorem 2.7.

Now we look at property \( \langle 3 \rangle \). We can assume \( E'_V \) is infinite. By property \( \langle 1 \rangle \), ker\( (V' \rightarrow \bar{V}) \leq V' \). For each \( L'_V \), the collection of all \( V' \)-conjugates of \( L'_V \) consists of finitely many \( V' \)-conjugacy classes. We pick a representative from each \( V' \)-conjugacy class and form the collection \( \{L'_V\}_{V \in V} \). Then ker\( (V' \rightarrow \bar{V}) = \langle L'_V : 1 \leq i \leq t_V, \lambda \in \Lambda_i \rangle \), where the subscript \( V' \) indicates normal closure inside \( V' \).

Each \( L'_V \) acts cocompactly on a translate of \( \tilde{X}_L^V \) (cf. Lemma 4.6.1, which we denote by \( \tilde{X}_V^{L_\lambda} \)). Let \( X_{L_\lambda} = \tilde{X}_V^{L_\lambda} / \tilde{L}_\lambda \). Each \( L'_V \) is residually finite since it is a subgroup of a virtually special group. Hence by Lemma 4.6.1 and Lemma 2.12, we can pass to finite index subgroup of \( L'_V \) such that the systole \( \|X_{L_\lambda}\| \) is large enough to ensure that the presentation \( \langle X_{V'}' \mid X_{L_\lambda} : 1 \leq i \leq t_V, \lambda \in \Lambda_i \rangle \) is \( C'(1/24) \) (note that \( \tilde{X}_V^{L_\lambda} \) is \( \mathcal{C}_V(\tilde{L}_\lambda) \)-invariant and we always choose \( \tilde{L}_\lambda \) to be normal in \( \mathcal{C}_V(\tilde{L}_\lambda) \), then all the cone-pieces correspond to the first situation of Lemma 1.6.1).

Note that \( E'_V \) acts cocompactly on a translate of \( \tilde{X}_V^{L_\lambda} \) or \( \tilde{X}_V^{H_\lambda} \) (cf. Lemma 4.6), which we denoted by \( \tilde{X}_V^{E'_V} \). Let \( X_{V'}^{E'_V} = \tilde{X}_V^{E'_V} / E'_{V'} \). Since \( E'_V \) is torsion-free, and \( E' \) is contained in \( E' \) as a finite index subgroup, property \( \langle 1 \rangle \) holds with \( E' \) replaced by...
$E'_i$. Thus by Lemma \ref{lem:2.14}, each component of the fiber product $X_{V'}^{E'_i} \otimes_{X_{V'}^{E'_i}} X_{V'}^{L_{1,\lambda}}$ is either a copy of $X_{V'}^{L_{1,\lambda}}$, or is a contractible complex. By passing to a further finite cover of each $X_{V'}^{L_{1,\lambda}}$, we can assume the diameter of contractible components in $X_{V'}^{E'_i} \otimes_{X_{V'}^{E'_i}} X_{V'}^{L_{1,\lambda}}$ is $\leq \frac{1}{2} \lVert X_{V'}^{L_{1,\lambda}} \rVert$ (note that contractible components in the fiber product do not change when we pass to a cover of $X_{V'}^{L_{1,\lambda}}$). By Lemma \ref{lem:3.56}, the map $(X_{V'}^{E'_i} \mid X_{V'}^{E'_i} \otimes_{X_{V'}^{E'_i}} X_{V'}^{L_{1,\lambda}} : 1 \leq i \leq \ell_V, \lambda \in \Lambda_i) \to (X_{V'} \mid X_{V'}^{L_{1,\lambda}} : 1 \leq i \leq \ell_V, \lambda \in \Lambda_i)$ has liftable shells. We have thus determined the appropriate $L_i'$ such that Lemma \ref{lem:2.4} ensures that property \(3\) holds for the edge group $E$. We repeat this argument for each infinite edge group of $V$ (there are only finitely many edge groups) to find the required collection satisfying property \(3\).

For property \(4\), we consider the collection of the various $\{E_S\}$ where $S$ ranges over all finite subtrees with infinite pointwise stabilizers based at $v$. By Lemma \ref{lem:3.27}, there are finitely many $V$-conjugacy classes of such subgroups. Thus property \(4\) can be arranged in the same way as property \(1\), using Lemma \ref{lem:3.6} and Lemma \ref{lem:3.23}.

It remains to arrange property \(5\). Let $\{E'_S\}$ be the collection in the previous paragraph. By our initial assumption that $E' \leq V'$ whenever $E$ is an edge group of $V$, we have $E'_S \leq V'$. Since $\{E'_S\}$ has only finitely many $V$-conjugacy classes, it has finitely many $V'$-conjugacy classes. Choose a representative from each $V'$-conjugacy class and form a collection $\{K_i\}_{i=1}^n$. Each $K_i$ is of finite index in a $V$-conjugate of an element in $\mathcal{T}_V$, thus each $K_i$ acts cocompactly on some translate of $X_{V'}^{L_{1,\lambda}}$ or $X_{V'}^{L'}$, which we shall denote by $\bar{X}_{V'}^{K_i}$. Let $X_{V'}^{K_i} = \bar{X}_{V'}^{K_i}/K_i$.

Since each $K_i$ is quasiconvex in $V'$ (Lemma \ref{lem:3.18}), there are finitely many double cosets $K_i g K_i$ such that $K_i^j \cap K_i$ is infinite (Lemma \ref{lem:3.21}). Let $\{K_i g_j^{(j)} K_i\}_{i=1}^\infty$ be the collection of such double cosets. We pass to further finite sheet cover of $X_{V'}^{L_{1,\lambda}}$ such that for any $i, j, \ell$, we have $M < \frac{1}{8} \lVert X_{V'}^{L_{1,\lambda}} \rVert$ and $|g_{ij}| < \frac{1}{8} \lVert X_{V'}^{L_{1,\lambda}} \rVert$, where $M$ is the constant in Lemma \ref{lem:3.27} \(2\). Then property \(5\) follows from Lemma \ref{lem:2.8} (the ‘factor through’ part of the assumption of Lemma \ref{lem:2.8} follows from property \(4\)).

For each vertex group $V$, we have determined in Lemma \ref{lem:4.7}, a collection $\{\hat{L}_i^{V'}\}_{i=1}^{\ell_V}$ which we call $L'$-subgroups. For each $V$ and $i$, we choose a finite index subgroup $L_i^{V'} \leq L_i^{V'}$ such that the collection $\{L_i^{V'}\}_{V \in \mathcal{V}, 1 \leq i \leq \ell_V}$ satisfies

\begin{enumerate}
  \item $L_i^{V'} \leq \hat{L}_i^{V'}$ for each $V$;
  \item $\{\hat{L}_i^{V'}\}$ is compatible with the transfer isomorphisms in Definition \ref{def:3.2} i.e. any transfer isomorphism from $L_i^{V'}$ to $L_i^{V''}$ maps $\hat{L}_i^{V'}$ to $\hat{L}_i^{V''}$.
\end{enumerate}

Such choice of finite index subgroups can be made in the following way. Define a relation over the set $\{L_i^{V'}\}_{V \in \mathcal{V}, 1 \leq i \leq \ell_V}$ such that $L_i^{V'} \sim L_i^{V''}$ if there is a transfer isomorphism between them. This is an equivalence relation by Definition \ref{def:3.2}. For each equivalence class, we pick a representative $L_i^{V'}$, and define $\hat{L}_i^{V'}$ to be a finite index characteristic subgroup of $L_i^{V'}$ such that $\hat{L}_i^{V'}$ is contained in the images of the $L$-subgroups under the transfer isomorphisms whose ranges are $L_i^{V'}$. Then we define the finite index subgroups of other elements in the equivalent class to be the image of $\hat{L}_i^{V'}$ under transfer isomorphisms.

Let $\hat{L}_i^{V'}$ be the normal closure of $\hat{L}_i^{V'}$ in $\mathcal{C}_V(L_i^{V'})$. Note that $\hat{L}_i^{V'} \leq \hat{L}_i^{V'}$, thus $\{\hat{L}_i^{V'}\}_{i=1}^{\ell_V}$ also satisfy Lemma \ref{lem:4.7}. Now we consider the quotient map $V \to \bar{V} = V/\langle \hat{L}_1^{V'}, \cdots, \hat{L}_{\ell_V}^{V'} \rangle$. Note that $\langle \hat{L}_1^{V'}, \cdots, \hat{L}_{\ell_V}^{V'} \rangle = \langle \hat{L}_1^{V'}, \cdots, \hat{L}_{\ell_V}^{V'} \rangle$ since $\langle \hat{L}_i^{V'} \rangle_{i \in \Lambda_i} \subset \langle \hat{L}_i^{V'} \rangle_{V'}$ for each $i$. We employ $\{\hat{L}_i^{V'}\}$ to facilitate the small-cancellation conditions.
Existence of the quotient map: Pick two vertex groups \( V_i \) and \( V_j \) such that there is an edge \( E \) between them. Let \( \alpha_E : E_i \to E_j \) be as in the beginning of Section 4.1. Let \( N_i = \ker(V_i \to \bar{V}_i) \). It suffices to show \( \alpha_E(E_i \cap N_i) = E_j \cap N_i \).

Let \( V'_i \leq V_i \) be our chosen finite index subgroup and let \( E'_i = V'_i \cap E_i \). Then 
\[
E_i \cap N_i = V'_i \cap N_i \cap E_i = N_i \cap E'_i \quad \text{since} \quad N_i \leq V'_i.
\]
By Lemma 4.7, \( N_i \cap E'_i \) is generated by all \( V_i \)-conjugates of \( \{ \bar{L}^V_k \}^k \) that are contained in \( E'_i \). Since \( E'_i \) is generated by all \( V_i \)-conjugates of \( \{ \bar{L}^V_k \}^k \), however, \( \alpha_E \) maps any \( V_i \)-conjugate of \( \bar{L}^V_k \) to an \( \bar{V}_j \)-conjugate of \( \{ \bar{L}^V_k \}^k \). Thus \( \alpha(E(N_i \cap E_i)) = \alpha(E(N_i \cap E'_i)) \subset N_i \cap E'_i \), and (3), conclusions (4), (3) and (5) of Proposition 4.2 hold. Proposition 4.2.

We have verified the compatibility of \( \{ \phi_{V_i} : V_i \to \bar{V}_i \} \). Hence by Lemma 4.7, conclusions 4, 7 and 9 of Proposition 4.2 hold. Proposition 4.2; holds by construction. Moreover, for \( V_i \), \( E_i \) and \( E'_i \) in Lemma 4.7 if an \( \bar{V}_i \)-conjugate of \( \{ \bar{L}^V_k \}^k \) is contained in \( E'_i \), then it has a finite index subgroup contained in \( E'_i \) hence it is contained in \( E'_i \) by (1). However, \( \ker(E \to E') = \ker(V \to \bar{V}) \cap E \) is generated by such conjugates by the discussion above, thus \( \ker(E \to E') \leq E' \) and conclusion of Proposition 4.2 holds. We deduce Proposition 4.2 in a similar way. Conclusion 9 of Proposition 4.2 can be arranged in a similar way as the quasiconvexity statement in Lemma 4.7.

\((G,V)\) has finite stature: Let \( E' \) be the collection of finite index subgroups of edge groups chosen at the beginning. By Lemma 4.23 \( \delta(G,E') < \infty \). Let \( \tilde{E} \) be the edge groups of \( \tilde{G} \) (they are quotients of \( E \)), and \( \tilde{E}' \) be the finite index subgroups of \( \tilde{E} \) that are images of elements of \( E' \). Let \( \tilde{T} \) be the Bass-Serre tree of \( \tilde{G} \). For each nontrivial subtree \( S \subset \tilde{T} \), we define \( \tilde{E}_S' \) in the same way as we defined \( E_S' \) for a subtree \( S \subset T \).

Let \( \phi : G \to \tilde{G} \) be the quotient map between graphs of groups induced by \( \{ \phi_{V_i} : V_i \to \bar{V}_i \} \). Recall that each \( \phi_{V_i} \) is formed by quotienting relators satisfying the properties of Lemma 4.7. Let \( \phi_T : T \to \tilde{T} \) be the \( G \)-equivariant map between the Bass-Serre trees of \( G \) and \( \tilde{G} \) induced by \( \phi \).

**Lemma 4.8.** Suppose there is a chain of finite nontrivial subtrees \( S_1 \subset \cdots \subset S_n \) with \( |S'_n| = \infty \). Choose a base vertex \( \bar{w} \) in \( S_1 \). Then for any \( w \in T \) with \( \phi_T(w) = \bar{w} \), there is a sequence of subtrees \( w \in S_1 \subset S_2 \subset \cdots \subset S_n \) such that \( \phi(E'_S) = \tilde{E}'_S \) and \( \phi \) maps \( S_i \) isomorphically to \( S_i \) for each \( i \).

**Proof.** We only prove the case \( n = 1 \) as the more general case is similar. We will induct on the number of edges in \( S_1 \). The case when \( S_1 \) has only one edge follows from the definition of quotients between graphs of groups. Now we consider more general \( S_1 \). Let \( e \subset S_1 \) be an edge containing a valence one vertex of \( S_1 \). We remove \( e \) from \( S_1 \) to obtain a smaller tree \( S_0 \). We can also assume the base point \( \bar{w} \) is inside \( S_0 \). By induction, we can lift \( S_0 \) to \( S_1 \subset T \) such that \( w \in S_0 \) and \( \phi(E'_S) = E'_S \).

Let \( \bar{v} = \bar{w} \cap \bar{S}_0 \), and let \( v \) be the lift of \( \bar{v} \) in \( S_0 \). Let \( e \subset T \) be a lift of \( e \) that contains \( v \). Up to conjugacy inside \( G \), we assume without loss of generality that \( \text{Stab}(v) \) is a vertex group \( V \) of \( G \) (recall that we have identified vertex groups of \( G \) with stabilizers of vertices in \( \tilde{T}_0 \subset T \)). We can then view \( E'_S \) and \( E'_S \) as subgroups of \( V' \).

Since \( E'_e \cap E'_S \) is infinite, by Lemma 4.7 and Remark 2.9 there exists \( g \in E' \) with \( g \in \ker \phi \) such that \( (E'_e)^g \cap E'_S \) is infinite and \( (E'_e)^g \cap E'_S = E'_e \cap E'_S \). Then
Lemma 4.9. now has the additional property that it is injective on their kernels are contained in these finite index subgroups. Then the resulting orbit. Up to conjugacy in $S$, a high subtree, there exists a subtree $\bar{S}$ containing $w$. It follows that $n+1 \leq \delta(G, E)$, hence $n \leq \delta(G, E) - 1$.

Injectivity of finite subtrees: We verify the moreover statement of Proposition 4.2. We assume without loss of generality that $S$ is connected. It suffices to show distinct edges of $S$ meeting a vertex $v$ is sent by $\phi_T$ to distinct edges. Let $\{e_i\}_{i=1}^n$ be the collection of edges of $S$ containing $v$ and in the same $\text{Stab}(v)$ orbit. Up to conjugacy in $G$, we assume that $\text{Stab}(v)$ is a vertex group $V$. Each $e_i$ corresponds to a coset $g_i E$ of an edge group in $V$. Since $E$ is separable in $V$, there exists a finite index normal subgroup $\hat{V} \leq V$ such that $g_i g_j^{-1} \notin EV$ for any $i \neq j$. Thus, $\ker(V \to \hat{V}) \leq \{g_i E\}$ project to distinct cosets in $\hat{V}$. We repeat this process of constructing $\hat{V}$ for other $\text{Stab}(v)$-orbits of edges in $S$ that contains $v$, as well as other vertices in $S$, to obtain a collection of finite index subgroups of certain vertex groups. By Proposition 4.2, it is possible to choose the $\phi_T^v$ such that their kernels are contained in these finite index subgroups. Then the resulting $\phi$ now has the additional property that it is injective on $S$. This concludes the proof of Proposition 4.2.

We record a consequence of the above construction.

Lemma 4.9. Let $\phi : G \to \bar{G}$ and $\phi_T : T \to \bar{T}$ be as above. Then

1. for any subtree $S \subset T$ with $\text{Stab}(S)$ infinite, $\phi(\text{Stab}(S))$ is commensurable to $S = \phi_T(S)$.
2. Pick finite subtree $S \subset T$ such that $\text{Stab}(S)$ is infinite and pick a vertex $\bar{w} \subset \bar{S}$. Then for any $w \in T$ with $\phi_T(w) = \bar{w}$, there exists a subtree $S \subset T$ containing $w$ such that $\phi(\text{Stab}(S))$ is commensurable to $\text{Stab}(\bar{S})$ and $S$ is a lift of $\bar{S}$.
3. Let $v \in T$ be a vertex and let $H_1$ and $H_2$ be two transsections in $\text{Stab}(v)$. If $|\phi(H_1) \cap \phi(H_2)| = \infty$, then there exists $g \in \ker(\text{Stab}(v) \to \text{Stab}(\bar{v}))$ such that $\phi(H_1^g \cap H_2) = \phi(H_1) \cap \phi(H_2)$ up to finite index subgroups.

Proof. For (1), by Lemma 3.16 and Lemma 3.19, we can assume all the subtrees involved are finite. It suffices to prove $\phi(E_S^v) = E_{\bar{S}}^v$. We induct on the number of edges in $S$. Suppose $S$ is a union of two smaller trees $S = S_1 \cup S_2$ such that $S_1$ and $S_2$ intersect in a vertex $v \in T$. Since $E_S = E_{S_1} \cap E_{S_2}$ is infinite, by Lemma 4.7, $\phi(E_{S_1}^v \cap E_{S_2}^v) = \phi(E_{S_1}^v) \cap \phi(E_{S_2}^v)$ (we view $E_{S_1}$ and $E_{S_2}$ as subgroups of $\text{Stab}(v)$, and up to conjugation, we can assume $v = v_A$ in Lemma 4.7). However, by induction, $\phi(E_{S_i}^v) = E_{\bar{S}_i}^v$ for $i = 1, 2$. Thus $\phi(E_S^v) = E_{\bar{S}}^v$. (2) is a consequence of
Lemma 4.8. (3) follows from Lemma 4.7 and that any transection of $G$ can be expressed as the pointwise stabilizer of a finite big-tree of $T$ (cf. Lemma 3.16).

We extract the following observation from the proof of Lemma 4.8.

**Proposition 4.10.** Let $\phi : G \to \bar{G}$ be the quotient map between graphs of groups induced by quotients $\{\phi_V : V \to \bar{V}\}_{V \in V}$ between their vertex groups. Let $\phi_T : T \to \bar{T}$ be the $G$-equivariant map between the Bass-Serre trees of $G$ and $\bar{G}$ induced by $\phi$.

Suppose the following holds whenever $H_1 \leq V$ is a $V$-conjugate of an edge group of $V$ and $H_2 \leq V$ is a transection in $V$: If $\phi_V(H_1) \cap \phi_V(H_2)$ is infinite then there exists $g \in V$ such that $\phi_V(H_1^g \cap H_2) = \phi_V(H_1) \cap \phi_V(H_2)$.

Then for any finite subtrees $S_1 \subset S_2 \subset \bar{T}$ with $|\text{stab}(S_2)| = \infty$, and for any lift $S_1 \subset T$ with $\phi(S_1) = \bar{S}_1$ and $\phi(\text{stab}(S_1)) = \text{stab}(\bar{S}_1)$, there is a lift $S_2$ with $S_1 \subset S_2$ and $\phi(S_2) = \bar{S}_2$ and $\phi(\text{stab}(S_2)) = \text{stab}(\bar{S}_2)$.

If we weaken the assumption so that there exists $g \in V$ and infinite intersection subgroups $H'_1 \subset H'_2 \subset H'_3$ such that $\phi_V(H'_1) \cap \phi_V(H'_2)$ is infinite then the second conclusion becomes $\phi(\text{stab}(S_i))$ is commensurable with $\text{stab}(\bar{S}_i)$ for each $i$.

It follows that a sequence of finite trees $S_1 \subset S_2 \subset \cdots$ with each $|\text{stab}(S_i)| = \infty$ lifts to a sequence $\bar{S}_1 \subset \bar{S}_2 \subset \cdots$ with $\phi$ mapping each $S_i$ isomorphically to $\bar{S}_i$ and each $\phi(\text{stab}(S_i)) = \text{stab}(\bar{S}_i)$.

## 5. Relatively Hyperbolic Application

### 5.1. Background on relative hyperbolicity.

We refer to [Hru10] for background on relative hyperbolic groups and the equivalence of various definitions of relative quasiconvexity.

**Theorem 5.1.** Let $G$ be hyperbolic relative to a collection of maximal parabolic subgroups $\mathcal{P}$ and let $H \leq G$ be relatively quasiconvex.

1. There are finitely many $H$-conjugacy classes of infinite subgroups of form $H \cap P^g$ with $g \in G$ and $P \in \mathcal{P}$, moreover, if $\mathcal{P}_H$ is a set of representatives of these conjugacy classes then $H$ is hyperbolic relative to $\mathcal{P}_H$.

2. Given $P \in \mathcal{P}$, there are finitely many $P$-conjugacy classes of infinite subgroups of form $P \cap H^g$ with $g \in G$.

The collection $\mathcal{P}_H$ is the induced peripheral structure on $H$.

**Proof.** The first assertion is [Hru10] Thm 9.1. Note that (1) implies that there are finitely many double cosets of form $HgP$ such that $H \cap P^g$ is infinite. Hence there are finitely many double cosets of form $PgH$ such that $P \cap H^g$ is infinite. Hence assertion (2) follows.

The following is a variant of [MP09] Prop 5.11:

**Theorem 5.2.** Let $G$ be hyperbolic relative to f.g. virtually abelian subgroups $\{P_i\}$, and let $Q$ be a relatively quasiconvex subgroup with its induced peripheral structure. Let $\{K_1, \ldots, K_m\}$ be a complete collection of representatives of maximal parabolic subgroups of $Q$. There exists a collection of finite index subgroups $\{\tilde{P}_i \leq P_i\}$ such that the following holds.

Let $\{\tilde{P}_i \leq \tilde{P}_i\}$ be a collection of subgroups such that $\tilde{P}_i \leq P_i$ is a finite index normal subgroup for each $i$. Let $Q^+ \leq G$ be the subgroup generated by the union of $Q$ and each conjugate of an element in $\{\tilde{P}_i\}$ having infinite intersection with $Q$. Then
(1) $Q^+$ is a full quasiconvex subgroup of $G$;
(2) $Q^+$ splits as a tree $T_m$ of groups whose vertex groups are \{$Q,A_1,\ldots,A_m$\}, and where $T_m$ is a wedge of $m$ edges.
(3) The central vertex of $T_m$ has vertex group $Q$, and for each $i$ there is an edge between $Q$ and $A_i$ whose edge group is $K_i$.
(4) Each $A_i$ has finite index in a maximal parabolic subgroup of $G$; and \{$A_1,\ldots,A_m$\} is the induced peripheral structure of $Q^+$ in $G$.

When $Q$ is parabolic, \{$K_i$\} = \{$Q$\} consists of a single element, $T_m$ is a single edge, and $A_i = (Q \cup g\bar{P}_ig^{-1})$ for some $g,i$. So it is a trivial splitting.

Proof. We assume $m \geq 1$ otherwise the theorem is trivial. First we claim there is a collection of finite index normal subgroups \{$\bar{P}_i$\} of \{$P_i$\} such that if $Q'$ is the subgroup of $G$ generated $Q$ and any conjugate of an element in \{$\bar{P}_i$\} having infinite intersection with $Q$, then $Q'$ satisfies all the requirements of the theorem. To see the claim, suppose first that \{$K_i$\} has a single representative of maximal parabolic subgroup, which is $K_1$. Suppose without loss of generality that $K_1 = Q \cap P_1$. Since f.g. virtually abelian groups are subgroup separable, $K_1$ is separable in $P_1$, so we can find finite index subgroup $P'_1 \leq P_1$ satisfying the assumption of [MP09 Thm 1.1]. Choose $\bar{P}_1 \leq P'_1$ which is finite index and normal in $P_1$. Then the subgroup $A_1$ of $P_1$ generated by $K_1$ and $\bar{P}_1$ also satisfies the assumption of [MP09 Thm 1.1]. Hence \{$Q \cup \bar{P}_1$\} is naturally isomorphic to $Q *_{Q \cap A_1} A_1 = Q *_{K_1} A_1$ as in [MP09 Thm 1.1]. And this subgroup is quasiconvex. The general case of $m \geq 1$ follows by the same argument together with the induction scheme in [MP09 Prop 5.11]. By Theorem [5.1] the induction terminates after finitely many steps.

We denote the vertex group of $Q'$ by \{$Q,A'_1,\ldots,A'_m$\}. Let \{$\bar{P}_1 \leq P_1$\} and $Q^+$ be as in the statement of the theorem. Then $Q^+ \leq Q'$. Suppose without loss of generality that $\bar{P}_1 \leq A'_i$ for $1 \leq i \leq m$ and suppose $A_i$ is generated by $\bar{P}_1$ and $K_i$. Then $Q^+ = (Q \cup (\bar{P}_1)^y)_{1 \leq i \leq m,g \in Q} = (Q \cup \{(A_i)^y\}_{1 \leq i \leq m, g \in Q}) = (Q \cup \{A_i\}_{i=1}^m)$. Thus Properties (2) and (3) hold. Now we verify (1). We only consider the case where $A_1 \subseteq A'_i$ and $A_i = A'_i$ for $2 \leq i \leq m$. Other cases are similar. Let $Q_1 = (Q \cup \{A_i\}_{i=2}^m)$. Then $Q^+=A_1 *_{K_1} Q_1$ and $Q' = A'_i *_{K_1} Q_1$. As $Q'$ is hyperbolic relative to f.g. virtually abelian subgroups, it suffices to show $Q^+$ is undistorted in $Q'$ [Hru10 Thm 1.4 and Thm 1.5]. Since each vertex group and edge group of $Q'$ is quasiconvex (hence undistorted) in $Q'$, we can view $Q'$ as a tree of spaces over its Bass-Serre tree with vertex spaces and edge spaces undistorted. Now $Q^+$ sits inside $Q'$ as a sub-tree of spaces, hence it is undistorted. Property (4) follows from (1) and Theorem [5.1].

Recall that a subgroup $H$ in a relatively hyperbolic group $G$ is loxodromic, if $H$ contains an infinite order element $h$ such that $h$ is not contained in a maximal parabolic subgroup of $G$. The following is a slightly more general form of [HW09 Cor 8.6], however, its proof is exactly the same as in [HW09].

**Theorem 5.3.** Suppose $G$ is relatively hyperbolic. Let $H_1, H_2$ be two relatively quasiconvex subgroups of $G$. Then there are only finitely many double cosets $H_1 g H_2$ such that $H_1 \cap g H_2 g^{-1}$ is loxodromic.

The following is a consequence of [HW09 Cor 8.6].

**Theorem 5.4.** Let \{$H_1,\ldots,H_r$\} be a collection of relatively quasiconvex subgroups of the word-hyperbolic group $G$. Then \{$H_1,\ldots,H_r$\} has finite height in $G$. 
Recall that the notion of finite height in the relatively hyperbolic setting is similar to Definition 1.4 except we replace “is infinite” there by “is loxodromic”.

The next result follows from Theorem 5.4 and Theorem 5.3.

**Corollary 5.5.** Let \( \{H_1, \ldots, H_r\} \) be a collection of relatively quasiconvex subgroups of a relatively hyperbolic group \( G \). Let \( K \) be a quasiconvex subgroup of \( G \). Then there are only finitely many \( K \)-conjugacy classes of loxodromic subgroups of form \( K \cap (\cap_{k=1}^r H_{k_i}^o) \).

5.2. **Virtually sparse specialness.** We recall the notion of a “sparse cube complex” which is a generalization of a compact cube complex that arises naturally for cubulations of groups that are hyperbolic relative to virtually abelian subgroups. We refer to [Wis, Sec 7.e] for more on this topic.

**Definition 5.6 (Quasiflat).** A quasiflat \( \tilde{F} \) is a locally finite CAT(0) cube complex with a proper action by a f.g. virtually abelian group \( P \) such that there are finitely many \( P \)-orbits of hyperplanes.

**Definition 5.7 (Sparse).** Let \( G \) be hyperbolic relative to f.g. virtually abelian groups \( \{P_i\} \). We say \( G \) acts cosparsely on a CAT(0) cube complex \( \tilde{X} \) if there is a compact subcomplex \( K \), and quasiflats \( \tilde{F}_i \) with \( P_i = \text{Stab}(\tilde{F}_i) \) for each \( i \), such that:

1. \( \tilde{X} = GK \cup \bigcup \tilde{G} \tilde{F}_i \)
2. for each \( i \) we have \( (\tilde{F}_i \cap GK) \subset P_i K' \) for some compact \( K' \).
3. for \( i, j \) and \( g \in G \), either \( \tilde{F}_i \cap g\tilde{F}_j \subset GK \) or else \( i = j \) and \( \tilde{F}_i = g\tilde{F}_j \).

It follows that translates of quasiflats are either equal or are coarsely isolated in the sense that diameter(\( g_i \tilde{F}_i \cap g_j \tilde{F}_j \)) is bounded by some uniform constant.

We will be especially interested in the case when \( G \) acts both properly and cosparsely. In particular, when \( G \) acts freely and cosparsely on \( \tilde{X} \), then the quotient \( X = G \backslash \tilde{X} \) is sparse. In this case, \( X \) is the finite union \( K \cup \bigcup \tilde{F}_i \) where \( K \) is compact and \( (\tilde{F}_i \cap \tilde{F}_j) \subset K \) for \( i \neq j \), and each \( \tilde{F}_i \) equals \( P_i \backslash \tilde{F}_i \) where \( P_i \) is a f.g. virtually abelian group acting freely on a quasiflat \( \tilde{F}_i \).

**Remark 5.8.** Suppose \( G = \pi_1 X \) for a sparse cube complex \( X \). Suppose \( G \) is hyperbolic relative to f.g. virtually abelian subgroups \( \{P_1, \ldots, P_n\} \) stabilizing quasiflats \( \{\tilde{F}_1, \ldots, \tilde{F}_n\} \) in \( \tilde{X} \), and \( X \) is sparse relative to \( \{F_1, \ldots, F_n\} \) with \( F_i = \tilde{F}_i / P_i \). Now we assume \( G \) is hyperbolic relative to another collection of virtually abelian subgroups \( \{P'_1, \ldots, P'_m\} \). If \( P'_i \) is conjugate to one of \( \{P_i\} \), then it stabilizes a translate of an element in \( \{\tilde{F}_i\} \), which we denote by \( \tilde{F}'_i \). If \( P'_i \) is not conjugate to one of \( \{P_i\} \), then \( P'_i \) is virtually \( \mathbb{Z} \), and \( P'_i \) acts cocompactly on a superconvex subcomplex \( \tilde{F}'_i \). Thus \( X \) is also sparse relative to \( \{F'_1, \ldots, F'_m\} \) with \( F'_i = \tilde{F}'_i / P'_i \). Thus we can always assume the sparse structure of the cube complex is compatible with a given collection of virtually abelian subgroups that \( G \) is relative hyperbolic to.

We refer to [SW15, Thm 7.2] for the following. A slightly weaker statement is expressed there, but the proof gives the following:

**Lemma 5.9.** Let \( G \) be hyperbolic relative to virtually abelian groups, and suppose \( G \) acts cosparsely on a CAT(0) cube complex \( \tilde{X} \). Let \( J \) be a full relatively quasiconvex subgroup of \( G \), and let \( \tilde{K}_o \) be a compact subcomplex of \( \tilde{X} \). Then there exists a convex subcomplex \( \tilde{Y} \subset \tilde{X} \) such that \( \tilde{K}_o \subset \tilde{Y} \) and such that \( J \) acts cosparsely on \( \tilde{Y} \).
Moreover, let \( \{P_i\} \) be the parabolic subgroups of \( G \) with \( |J \cap P_i| = \infty \), and let \( \{\tilde{F}_i\} \) be the corresponding quasiflats of \( G \), we may assume that
\[
(J\tilde{K}_0 \cup i J\tilde{F}_i) \subset \tilde{Y} \subset \mathcal{N}_r(J\tilde{K}_0 \cup i J\tilde{F}_i).
\]

Note that there might be large parts of \( \tilde{Y} \) that are in \( GK \) but not coarsely in \( J \).

It follows that for each \( m \geq 0 \) there is a uniform upper bound on \( \text{diameter}(\tilde{Y} \cap \mathcal{N}_m(\tilde{F}_k)) \) unless \( \tilde{F}_k \subset \tilde{Y} \). Indeed, if the latter statement does not hold, then \( g_i \tilde{F}_i \cap \mathcal{N}_m(\tilde{F}_k) \) lies in a finite neighborhood of \( J\tilde{K}_0 \). (Here \( K \) is the compact subcomplex such that \( GK \) contains the intersection of distinct \( G \)-translates of the various \( \tilde{F}_k \).) Thus if \( \tilde{F}_k \) has infinite coarse intersection with \( \tilde{Y} \) then it has infinite coarse intersection with \( J\tilde{x} \), and so \( |\text{Stab}_{J}(\tilde{F}_k)| = \infty \) and hence \( \tilde{F}_k \) is one of the quasiflats included in \( \tilde{Y} \).

The following is proven in [Wis, Thm 15.6]

**Lemma 5.10.** Let \( X \) be a virtually special nonpositively curved cube complex that is sparse. Suppose \( \pi_1 X \) is hyperbolic relative to subgroups \( P_1, \ldots, P_r \) stabilizing quasiflats \( \tilde{F}_1, \ldots, \tilde{F}_r \) of \( X \), where \( X \) is sparse relative to \( \tilde{F}_1, \ldots, \tilde{F}_r \), and each \( F_i = \hat{P}_i \setminus \tilde{F}_i \).

There exist finite index subgroups \( P'_1, \ldots, P'_n \) such that for any normal finite index or virtually-cyclic index subgroups \( P''_1 \subset P'_1 \), the quotient group \( G/\langle\langle P'_1, \ldots, P'_n \rangle\rangle \) is a word-hyperbolic group virtually having a quasiconvex hierarchy terminating in finite groups. Hence the quotient group is virtually compact special.

**Corollary 5.11.** Suppose \( G = \pi_1 X \) is hyperbolic relative to a collection of virtually abelian subgroups \( \{P_1, \ldots, P_n\} \), where \( X \) is a virtually special sparse cube complex. Suppose \( Q \leq G \) is quasiconvex. Then there exists \( \{P'_1, \ldots, P'_n\} \) with each \( P'_i \) being finite index in \( P_i \) such that for any \( \{\hat{P}_1, \ldots, \hat{P}_n\} \) with \( \hat{P}_i \leq P'_i \), \( [P'_1, P'_1] < \infty \) and \( \hat{P}_1 \leq P'_1 \), we have

1. \( \hat{G} = G/\langle\langle \hat{P}_1, \ldots, \hat{P}_n \rangle\rangle \) is word-hyperbolic and virtually compact special;
2. the image \( \hat{Q} \) of \( Q \) under \( G \to \hat{G} \) is quasiconvex;
3. \( \ker(Q \to \hat{G}) = \langle\langle Q \cap (P'_1) \rangle\rangle \).

**Proof.** By the above discussion, we assume the sparse structure of \( X \) is compatible with \( \{P_1, \ldots, P_n\} \). Let \( Q^+ \) be as in Theorem 5.2 with its representatives of maximal parabolic subgroups denoted by \( \{A_1, \ldots, A_m\} \). By Lemma 5.10 we choose \( \{P_i\} \) such that \( q : G \to \hat{G} = G/\langle\langle P_1, \ldots, P_n \rangle\rangle \) satisfies Conclusion (1). Moreover, we assume for each \( i, j \) and \( g \in G \), either \( A_j \cap (P_i)^g \) is trivial, or \( A_j \cap (P_i)^g = (P_i)^g \). Thus \( \{P_i\} \) induces a collection \( \{A_1, \ldots, A_m\} \) such that \( A_i \) is a finite index normal subgroup of \( A_i \). Suppose \( Q^+ \) stabilizes a superconvex subcomplex \( \tilde{Y} \subset \tilde{X} \) as in Lemma 5.5 and suppose each \( P_i \) stabilizes a quasiflat \( \tilde{F}_i \subset \tilde{X} \). Then there exists constant \( M \) such that for any \( g, g' \in G \) and \( i \), either \( g\tilde{F}_i \subset g'\tilde{Y} \), or \( g\tilde{F}_i \cap g'Y \) has diameter \( \leq M \). Since \( Q^+ \) is full, we can deduce from a liftable shell argument that it is contained. The claim \( \ker(Q^+ \to \hat{G}) = \langle\langle A_1, \ldots, A_m \rangle\rangle_{Q^+} \), the image \( Q^+ \) of \( Q \) under \( G \to \hat{G} \) is quasiconvex, and the proof of Conclusion (3) follows. Lemma 5.12 also implies that \( Q^+ \) is \( Q \) amalgamated with several f.g. virtually abelian groups along its maximal parabolic subgroups. Then each edge of \( Q^+ \) is contained in a maximal parabolic subgroup of \( Q^+ \). Thus \( Q \) is quasiconvex in \( Q^+ \) ([BWB Lem 4.9]) and conclusion (2) is true.
Lemma 5.12. Let $G = A \ast_C B$ and let $N \leq B$. Suppose the quotient $\bar{G} = G/\langle\langle N \rangle\rangle$ satisfies that $\ker(B \to \bar{G}) = N$. Then $\ker(A \to \bar{G}) = \langle\langle C \cap N \rangle\rangle_A$, and $\bar{G} = \bar{A} \ast_{\bar{C}} \bar{B}$, where $\bar{A} = A/\ker(A \to \bar{G})$, $\bar{B} = B/N$ and $\bar{C} = C/\ker(C \to \bar{G})$.

Proof. Note that $C \cap \ker(A \to \bar{G}) = \ker(C \to \bar{G}) = C \cap \ker(B \to \bar{G}) = C \cap N$. This together with $\langle\langle C \cap N \rangle\rangle_A \leq \ker(A \to \bar{G})$ implies that $C \cap \langle\langle C \cap N \rangle\rangle_A = C \cap N$. Let $\bar{A}' = A/\langle\langle C \cap N \rangle\rangle_A$. Then $A \to \bar{A}'$ and $B \to \bar{B}$ are compatible, hence induces a quotient of graphs of groups $q: A \ast_C B \to \bar{A}' \ast_{\bar{C}} \bar{B}$. Since $N \leq \ker(q)$, so $q$ is a composition $G \xrightarrow{q_1} \bar{G} \xrightarrow{q_2} \bar{A}' \ast_{\bar{C}} \bar{B}$. Since $q(A) = \bar{A}'$, we have $A \to A/\ker(A \to \bar{G}) \to \bar{A}'$ induced by $q$. Thus $A/\ker(A \to \bar{G}) = \bar{A}'$. This gives a homomorphism $h: \bar{A}' \ast_{\bar{C}} \bar{B} \to \bar{G}$. Then $h \circ q_2$ is identity since it is identity on $q_1(A)$ and $q_1(B)$. Thus $q_2$ is an isomorphism and the lemma follows. \hfill \square

We say a group $G$ is virtually sparse special if $G$ has a finite index torsion free subgroup $H$ such that $H$ is hyperbolic relative to f.g. virtually abelian and $H$ acts cocompactly on a $CAT(0)$ cube complex $X$ with the quotient $X/H$ being special.

The following is proved in [Wis, Thm 15.13].

Theorem 5.13. Suppose $G$ is virtually sparse special. Then any relatively quasi-convex subgroup of $G$ is separable.

We now discuss the virtually sparse specialness of certain amalgams.

Proposition 5.14. Let $G = E \ast_B A$ where $E$ is virtually sparse special, $A$ is a f.g. virtually abelian group and $B$ satisfies at least one of the following conditions

1. $B$ is a maximal parabolic subgroup of $E$;
2. $B$ is a maximal virtually cyclic group that is loxodromic in $E$.

Then $G$ is virtually sparse special.

The proof of Proposition 5.14 employs Theorem 5.16, which is a consequence of [Wis, Thm 18.15], as well as Lemma 5.17, which is a consequence of [Wis, Lem 7.56 and Rmk 7.57]. The original statement of [Wis, Thm 18.15] is under a more general condition called strongly sparse, however, this condition is satisfied for fundamental groups of virtually special compact cube complexes that are hyperbolic relative to abelian subgroups.

Definition 5.15. $G$ has an abelian hierarchy terminating in groups in a class $\mathcal{C}$ if $G$ belongs to the smallest class of groups $\mathcal{M}$ closed under the following conditions:

1. $\mathcal{C} \subset \mathcal{M}$;
2. if $H = A \ast_C B$ with $C$ being f.g. free-abelian and $A, B \subset \mathcal{M}$, then $H \in \mathcal{M}$;
3. if $H = A \ast_{C'} C''$ with $C$ being f.g. free-abelian and $A \in \mathcal{M}$, then $H \in \mathcal{M}$.

Theorem 5.16. Suppose that $G$ is hyperbolic relative to free-abelian subgroups, and that $G$ has an abelian hierarchy terminating in groups that are fundamental groups of virtually special compact cube complexes that are hyperbolic relative to abelian subgroups. Then $G$ is the fundamental group of a sparse cube complex $X$ that is virtually special.

Lemma 5.17. Let $X \to R$ be a local-isometry to a compact special cube complex. If $X$ is sparse then there is a local-isometry $X \to X'$ where $X'$ is compact and $\pi_1 X'$ is hyperbolic relative to abelian subgroups $\{P_i\}$ that contain the corresponding parabolic subgroups $\{P_i\}$ of the relatively hyperbolic structure of $\pi_1 X$. And there is a local-isometry $X' \to R$ such that $X \to R$ factors as $X \to X' \to R$. 

Moreover, we can assume that $\pi_1 X'$ splits over a tree $T_r$, whose central vertex group is $\pi_1 X$, whose edge groups are the $P_i$, and whose leaf vertex groups $P_i'$ are of the form $P_i \times \mathbb{Z}^{m_i}$ for some $m_i$.

Proof of Proposition 5.14. By assumption, $E$ has a finite index normal subgroup $E'$ which is the fundamental group of a sparse special cube complex. Moreover, by Theorem 5.13, we also assume $E'$ is hyperbolic relative to abelian subgroups. First we create a quotient of graph of groups $G = E \ast_B A = \tilde{E} \ast_B \tilde{A}$ such that

1. $\ker(E \to \tilde{E}) \subset E'$;
2. $\tilde{E}$ is virtually compact special;
3. both $\ker(A \to \tilde{A})$ and $\ker(B \to \tilde{B})$ are free abelian.

Let $\{P_1, \ldots, P_n\}$ be representatives of maximal parabolic subgroup of $G$ such that $P_i = B$. Let $\{P_1', \ldots, P_n'\}$ be as in Corollary 5.11. Let $\tilde{A} \leq A$ be a finite index abelian normal subgroup such that $\tilde{B} = \tilde{A} \cap B \leq P_i'$. Let $q_E$ be the quotient map $E \to \tilde{E} = E/\langle \tilde{B}, \tilde{P}_2, \ldots, \tilde{P}_n \rangle$, where $\tilde{P}_i \leq P_i'$ is a finite index abelian normal subgroup of $\tilde{P}_i$ with $\tilde{P}_i \leq \tilde{E}_0$ for $2 \leq i \leq n$. Let $q_A$ be the quotient map $A \to \tilde{A}$. Corollary 5.11(3) implies that $B \cap \ker(q_{E}) = B \cap \ker(q_{A}) = \tilde{B}$. Thus $q_A$ and $q_E$ induce the desired quotient $q$ with $\tilde{B} = B/\tilde{B}$.

As $A$ and $B$ are finite, we find finite index $\tilde{G}' \leq G'$ splitting as a graph of groups with underlying graph $\tilde{G}$ such that each edge group and vertex group of $\tilde{G}'$ is either trivial or isomorphic to $\tilde{E}_0$ which is a finite index normal torsion free subgroup of $\tilde{E}$ with $\tilde{E}_0 \leq q_{E}(E')$. Let $V_1$ (resp. $V_2$) be the collection of vertices of $\tilde{G}$ whose vertex groups are trivial (resp. isomorphic to $\tilde{E}_0$). Then $(\tilde{G}, V_1, V_2)$ is bipartite. Let $G' = q^{-1}(\tilde{G}')$. Then $G'$ splits as a graph of groups over $\tilde{G}$ such that

1. a vertex group of $G'$ is of type I (resp. II) if its associated vertex is in $V_1$ (resp. $V_2$), then each type I vertex group of $G'$ is isomorphic to $\tilde{A}_0 = \ker q_A$, each type II vertex group of $G'$ is isomorphic to $\tilde{E}_0 = (q_{E})^{-1}(\tilde{E}_0)$;
2. $\tilde{E}_0$ is the fundamental group of a sparse special cube complex and $\tilde{E}_0$ is hyperbolic relative to free abelian subgroups;
3. each edge group of $G'$ isomorphic to $\ker q_B$ and any edge group in a vertex group of type II is a maximal parabolic subgroup of this vertex group.

Now define a new graph of groups by enlarging each edge group and vertex group of $G'$ as follows. First we enlarge each type I vertex group of $G'$. Let $E_i$ be one such vertex group. Then we enlarge $E_i$ to $E_i^+$ for $1 \leq i \leq k_2$ such that

1. $E_i^+$ splits as a tree $T_r$ of groups where $T_r$ is an $r$-star with the central vertex group being $E_i$, each edge group being a peripheral subgroup of $E_i$ and each leaf vertex group being free abelian which contains its vertex group as a direct summand;
2. $E_i^+$ is the fundamental group of a compact special cube complex;
3. $E_i^+$ is hyperbolic relative to free abelian subgroups and each leaf vertex group of $E_i^+$ is a maximal parabolic subgroup of $E_i^+$.

Such enlargement is possible by Lemma 6.17. Second we enlarge each edge group. Let $B_i$ be an edge group. Then exactly one of its vertex groups is of type II, denoted by $E_j$. Since $E_j$ has a unique maximal parabolic subgroup $P$ containing $B_i$ such that $P = B_i \oplus \mathbb{Z}^{m_j}$, we enlarge $B_i$ to $P$. Last we enlarge each vertex group of type II. Let $A_i$ be one such vertex group and let $\{B_i\}_{i=1}^k$ be its edge groups. Since we
already enlarge $B_i$ to $P_i = B_i \oplus B'_i$, we enlarge $A_i$ to $A^+_i = A_i \oplus B'_i \oplus \cdots \oplus B'_k$. The boundary map $B_j \rightarrow A_i$ naturally extends to $P_j \rightarrow A^+_i$.

Let $G^+$ be the resulting new graph of groups from the previous paragraph. Note that $G^+$ is hyperbolic relative to the $A^+_i$ by [BW13, Thm A]. Thus $G^+$ is the fundamental group of a sparse cube complex $X$ that is virtually special by Theorem 5.16. Since there is a retraction $G^+ \rightarrow G'$ by our construction, $G'$ is quasi-isometrically embedded in $G^+$. Then it follows from [SW15, Thm 7.2] that $G'$ acts cosparsely on a convex subcomplex of $\tilde{X}$, which implies that $G'$ is sparse and virtually special, hence the proposition follows.

5.3. Quotienting in the relative hyperbolic setting. In this subsection we prove the following result.

**Theorem 5.18.** Let $G$ be hyperbolic relative to subgroups that are virtually f.g. free abelian by $\mathbb{Z}$. Suppose $G$ splits as a finite graph of groups whose edge groups are relative quasiconvex and whose vertex groups are virtually sparse special. Then each relatively quasiconvex subgroup of each vertex group of $G$ is separable. In particular, $G$ is residually finite.

The assumption of Theorem 5.18 implies that each intersection of a maximal parabolic subgroup of $G$ with a vertex group of $G$ is virtually f.g. abelian.

We need a preparatory fact for the proof of the above theorem, which may be useful for controlling stature in general situation.

**Lemma 5.19 (Full Splitting).** Suppose $G$ admits a splitting as in Theorem 5.18 with its collection of edge groups and vertex groups denoted by $E$ and $V$. We claim there is a new splitting of $G$ with the same underlying graph such that

1. each vertex/edge group of the old splitting is contained in the corresponding vertex/edge group of the new splitting;
2. each edge group of the new splitting is quasiconvex and full in its vertex group;
3. each vertex group of the new splitting is virtually sparse special.

**Proof.** Since each edge group is quasiconvex in $G$, so is each vertex group by [BW13, Lem 4.9]. First we describe a basic move. Let $E$ be an edge group and $V_1, V_2$ be its vertex groups (it is possible that $V_1 = V_2$). To simplify notation, we use the same letter for both the group and its Eilenberg–MacLane space. Suppose $E$ is not full in $V_1$. Let $P \subset V_1$ be a maximal parabolic subgroup such that $P' = P \cap E$ is infinite and is of infinite index in $P$. By [MP09, Thm 1.1], we can find a finite index subgroup $\hat{P} \leq P$ such that $P' \leq \hat{P}$ and the natural homomorphism $\hat{P} \ast P E \rightarrow V_1$ is injective with its image being relatively quasiconvex in $V_1$ (hence the image is also relatively quasiconvex in $G$). We also assume the moreover statement in [MP09, Thm 1.1] holds. Now consider the following commutative diagram of groups/Eilenberg–MacLane spaces:

$$
\begin{array}{ccc}
K & \rightarrow & \hat{P} \\
\downarrow & & \downarrow \\
E & \rightarrow & V_1
\end{array}
$$

where $K = \hat{P} \cap E$. We enlarge $E$ to $E' = (E \cup (K \times [0,1]) \cup \hat{P})/ \sim$ with left attaching map being $K \rightarrow E$ and the right attaching map being $K \rightarrow \hat{P}$. The old boundary map $E \rightarrow V_1$ naturally extends to a new boundary map $E' \rightarrow V_1$. 

realizing the injective homomorphism \( \hat{P} \ast_{\hat{P} \cap E} E \to V_1 \). We also enlarge \( V_2 \) to \( V_2' = (V_2 \cup (K \times [0,1]) \cup \hat{P}) / \sim \) with the left attaching map being \( K \to E \to V_2 \) and the right attaching map being \( K \to \hat{P} \). The boundary map \( E \to V_2 \) naturally extends to a new boundary map \( E' \to V_2' \), which represents the monomorphism \( \hat{P} \ast_{\hat{P} \cap E} E \to \hat{P} \ast_{\hat{P} \cap V_2} V_2 \).

Now we show \( V_2' \) is virtually sparse special. By Remark \( \ref{rem:5.8} \) and Proposition \( \ref{prop:5.14} \), it suffices to show \( \hat{P} \cap V_2 = \hat{P} \cap E \) is either a maximal parabolic subgroup of \( V_2 \), or a maximal loxodromic virtually cyclic subgroup. The subgroup \( P \subset V_1 \) in the previous paragraph can be written as \( P = P_G \cap V_1 \) where \( P_G \) is a maximal parabolic subgroup of \( G \). As \( \hat{P} \cap E = P_G \cap E \), it remains to show \( P_G \cap E = P_G \cap V_2 \).

Considering the subgroup \( H \) of \( P_G \) generated by \( V_1 \cap P_G \), \( E \cap P_G \) and \( V_2 \cap P_G \) (\( H \) has a graph of structure with these subgroups being its vertex/edge groups). Since \( P_G \) is virtually abelian by cyclic, so is \( H \). As \( P_G \cap E \) is of infinite index in \( P_G \cap V_1 \), \( P_G \cap E \not\subsetneq P_G \cap V_2 \) would indicate that \( H \) acts on a tree (which is its Bass-Serre tree) without any invariant vertices or lines, contradicting that \( H \) is virtually abelian by cyclic.

Now we have obtained a new graph of spaces and one readily verifies that all the boundary maps induces monomorphisms on the fundamental groups. The new graph of spaces deformation retracts onto the old one, so its fundamental group remains unchanged. The corresponding new graph of groups satisfies all the conditions in Theorem \( \ref{thm:5.15} \).

Now we show the conclusion of the lemma can be reached after finitely many basic moves. Let \( T \) be the Bass-Serre tree of \( G \) and let \( G = G/T \). Let \( P \) be a maximal parabolic subgroup of \( G \). Let \( S_P \) be the subtree of \( T \) spanned by vertices of \( T \) whose stabilizers intersect \( P \) in infinite subgroups. Note that \( S_P \) is \( P \)-invariant, and \( S_P \) also contains all edges of \( T \) whose stabilizers intersect \( P \) in infinite subgroups. We also know \( S_P/P \) is a finite graph by Theorem \( \ref{thm:5.11} \)(2). Since \( P \) is virtually free abelian by cyclic, one of the following hold:

(a) \( P \) stabilizes a vertex \( v \in T \);
(b) there is a \( P \)-invariant line \( \ell \subset T \).

Since \( G \) is relatively hyperbolic to virtually abelian subgroups, \( S_P/P \) is a tree with finitely many edges in case (a) and \( S_P/P \) is \( \ell/P \) together with finitely many finite trees attached to it in case (b).

Suppose case (a) holds. Let \( \bar{v} \in S_P/P \) be the image of the fixed vertex \( v \in T \) of \( P \). Note that \( S_P/P \) can be viewed as a tree of groups whose vertex and edge groups are decreasing as we move away from the base point \( \bar{v} \). Define the complexity for \( P \) to be the number of vertex groups and edge groups of \( S_P/P \) which are of infinite index in \( P \). If the complexity is 0, then there is no need to perform any move, otherwise there is an edge in \( e \subset S_P/P \) such that exactly one of its vertex groups is finite index in \( P \). Now apply the basic move to obtain a new splitting of \( G \) with the associated Bass-Serre tree denoted by \( T' \). We define \( S'_P \subset T' \) in the same way as \( S_P \). There is a natural \( G \)-equivariant graph morphism \( \phi : T \to T' \).

We claim \( \phi(S_P) = S'_{P} \). Indeed, it is immediate that \( \phi(S_P) \subset S'_{P} \) since a vertex with infinite \( P \)-stabilizer must map to a vertex with infinite \( P \)-stabilizer. Suppose \( \phi(S_P) \not\subsetneq S'_{P} \), then there is an edge \( e' \subset S'_{P} \) satisfying \( e' \not\subset \phi(S_P) \). Then there is an edge \( e \subset T \) such that either \( \text{Stab}(e') = \text{Stab}(e) \) or \( \text{Stab}(e') = \text{Stab}(e)_{B} \hat{P}_{0} \) where \( B \) is a maximal parabolic subgroup of \( \text{Stab}(e) \) and \( \hat{P}_{0} \) is of finite index in some maximal
parabolic subgroup of $G$. In the former case we have $P \cap \text{Stab}(e') = P \cap \text{Stab}(e)$, which implies that $e \in S_P$ and $\phi(e) = e'$. This leads to a contradiction. In the latter case by the moreover statement in [MP09, Thm 1.1], $P \cap \text{Stab}(e')$ is either conjugated (in $\text{Stab}(e')$) to a subgroup of $\text{Stab}(e)$, or conjugated to $P_0$. In either situation there exists $g \in \text{Stab}(e')$ such that $|(\text{Stab}(e'))^g \cap P| = \infty$. Thus $ge \subset S_P$ and $\phi(ge) = e'$, which yields a contradiction again. Thus the claim follows.

The claim implies that there is a surjective map $S_P/P \to S'_P/P$ and one readily sees that the complexity decreases. If $P$ stabilizes a line in $\ell \subset T$, then let $\ell/P$ be the core of $S_P/P$. We choose an edge $e \subset S_P/P$ such that exactly one of its vertex group is commensurable to a vertex group in the core, and run the same argument as before. Thus the $P$-complexity is 0 after finitely many steps. Then we deal with another maximal parabolic subgroup $P'$ in a different conjugacy class. The argument in the previous paragraph implies that $P$-complexity remains 0 when we decrease $P'$-complexity. Thus we are done after finitely many basic moves. \hfill \Box

The following is a main ingredient in the proof of Theorem 5.18.

**Proposition 5.20.** Let $G$ be as in Theorem 5.18. Let $E$ and $V$ be the collection of edge groups and vertex groups. Suppose in addition that $E$ is full in $V$ whenever $E$ is an edge group of $V$. For each $V \in V$ (resp. $E \in E$), we choose a finite index subgroup $V' \leq V$ (resp. $E' \leq E$). Let $S \subset T$ be a finite subtree of the Bass-Serre tree $T$ of $G$.

Then for each vertex group $V$, there is a quotient $\phi_V : V \to \bar{V}$ induced by quotienting finite index subgroups of its parabolic subgroups such that the following conditions hold for the collection $\{\phi_V : V \to \bar{V}\}_{V \in V}$:

1. these quotients are compatible, so there is an induced quotient $G \to \bar{G}$;
2. each $\bar{V}$ is hyperbolic and virtually compact special, moreover, each edge group of $\bar{V}$ is quasiconvex in $\bar{V}$;
3. $\bar{G}$ has finite stature;
4. for each edge group $E$ and each vertex group $V$, $\ker(E \to \bar{E}) \leq E'$ and $\ker(V \to \bar{V}) \leq V'$;
5. the induced map $\phi_T : T \to \bar{T}$ between the Bass-Serre trees is injective when restricted to $S$.

Now we deduce Theorem 5.18 from Proposition 5.20.

**Proof of Theorem 5.18.** It follows from Lemma 5.19 and Theorem 5.13 that it suffices to prove Theorem 5.18 in the special case that each edge group is full and quasiconvex. Given Theorem 5.13 this can be deduced from Proposition 5.20 and Theorem 4.1 in the same way as the proof of Theorem 4.1. \hfill \Box

The following will help control finite stature in Proposition 5.20 (3).

**Lemma 5.21.** Under the assumption of Proposition 5.20, let $S \subset T$ be a finite subtree of the Bass-Serre tree. Then $\text{Stab}(S)$ is a full quasiconvex subgroup of $\text{Stab}(v)$ for any vertex $v \in S$.

**Proof.** We induct on the number of edges in $S$. The case when $S$ is a vertex is clear. Let $S = S_1 \cup e$ where $S_1 \cap e = u$ is a vertex and $e$ is an edge and $S_1$ is a tree containing $u$. Let $H_1 \leq f_q H_2$ denote that $H_1$ is a full relatively quasiconvex subgroup of $H_2$. By induction, $\text{Stab}(S_1) \leq f_q \text{Stab}(u)$. Moreover, $\text{Stab}(e) \leq f_q \text{Stab}(u)$ as hypothesized in Proposition 5.20. Thus $\text{Stab}(S) = \text{Stab}(S_1) \cap f_q \text{Stab}(e) \leq f_q \text{Stab}(u)$.
any further finite index subgroups of $G$, hence $\$tab(S) \leq f,q \$tab(S_1)$. By induction, $\$tab(S_1) \leq f,q \$tab(v)$, hence $\$tab(S) \leq f,q \$tab(v)$.

**Proof of Proposition 5.20.** Let $\{P_{ij}^V\}$ be representatives of maximal parabolic subgroups of a vertex group $V \in V$ with the induced peripheral structure from $G$. We choose finite index subgroups $\tilde{P}_{ij}^V \leq P_{ij}^V$ such that the conclusions of Corollary 5.11 are satisfied for each edge group $E \leq V$ playing the role of $Q \leq G$ in Corollary 5.11.

We first describe the proof in the case where each $V$ is the fundamental group of a sparse cube complex $X_V$. We explain the general case at the end of the proof.

Let $\tilde{F}_{ij} \subset \tilde{X}_V$ be the quasiflat stabilized by $P_{ij}^V$. Then there is a constant $M$ such that for any $g_1, g_2 \in V$, either $g_1\tilde{F}_{ij} = g_2\tilde{F}_{ij}'$, or $g_1\tilde{F}_{ij} \cap g_2\tilde{F}_{ij}'$ has diameter $\leq M$.

By Corollary 5.11, $V$ contains finitely many $V$-conjugacy classes ofloxodromic transections. Let $\{J_i\}$ be representatives of these conjugacy classes. Each $J_i$ is the pointwise stabilizer of a finite subtree of $T$ and $\hat{J}_i$ is full and relatively quasiconvex in $V$ by Lemma 5.21. Choose a $J_i$-invariant convex subcomplex $\tilde{X}_i^V$ as in Lemma 5.9. Assume $M$ also satisfies that for any $g_1, g_2 \in V$, either $g_1\tilde{F}_{ij} \subset g_2\tilde{X}_i^V$, or $g_2\tilde{X}_i^V \cap g_1\tilde{F}_{ij}$ has diameter $\leq M$. By Theorem 5.3 there are finitely many double cosets $g^{-1}J_i g$ in $V$ such that $J_i \cap g^{-1}J_i g$ is loxodromic. Let $\{J_i g^{-1}J_i g\}$ be the collection of such double cosets. Choose finite index subgroups $\tilde{P}_{ij}^V \leq P_{ij}^V$ such that

1. for any $i, j, j'$, we have $M < \frac{1}{g}||\tilde{F}_{ij}/\tilde{P}_{ij}||$ and $|g_{ij}^{i,j'}| < \frac{1}{g}||\tilde{F}_{ij}/\tilde{P}_{ij}||$;
2. for any $i, j, j'$ and $g \in V$, either $(P_{ij})^g \leq J_{ij}$, or $(P_{ij}^V)^g \cap J_{ij} \neq 1$ (this is possible since each $J_i$ is full in $V$).

Then Remark 2.9 holds for intersections of conjugates of the $J_i$ if we quotient $V$ by any further finite index subgroups of $\{\tilde{P}_{ij}^V\}$.

Let $\{P_i\}_{i \in I}$ be representatives of maximal parabolic subgroups of $G$. Since each $P_i$ is residually finite, we choose a finite index normal subgroup $\tilde{P}_i \leq P_i$ such that for each $V \in V$ and $g \in G$, either $(\tilde{P}_i)^g \cap V$ is contained in a $V$-conjugate of an element in $\{\tilde{P}_{ij}^V\}$, or $(\tilde{P}_i)^g \cap V$ is the identity subgroup.

Let $\{\tilde{P}_{ij}^V\}$ be the collection of finite index subgroups of $\{P_{ij}^V\}$ induced by $\tilde{P}_i$. For each $V$, we consider the quotient $q_V : V \to \tilde{V}$ where:

$$\tilde{V} = V/\langle \langle \{\tilde{P}_{ij}^V\}\rangle \rangle_V = V/\langle \langle V \cap (\tilde{P}_i)^g\rangle \rangle_{g \in G, i \in I} \rangle_V.$$

Now Conclusion (2) holds by our choice of $\tilde{P}_i$. For any edge group $E \to V$ we have the following where the first equality follows from the choice of $\tilde{P}_i$ and Corollary 5.11(3) and the second equality is a notational restatement as above and the third holds since $E \leq V$.

$$\ker(E \to \tilde{V}) = \langle \langle E \cap (\tilde{P}_{ij}^V)^g \rangle \rangle_{g \in V} = \langle \langle E \cap (\tilde{P}_i)^g \rangle \rangle_{g \in G, i \in I} \rangle_E$$

The compatibility of the $q_V$ now follows, since $\ker(E \to \tilde{V})$ does not depend on $V$ as is clear in the last term above. Thus Conclusion (1) holds.

As Remark 2.9 holds for intersections of conjugates of the $J_i$, by Proposition 1.10 each transection of $\tilde{G}$ lifts to a transection of $G$, which is necessarily loxodromic in its vertex group since parabolic subgroups of vertex groups of $G$ have finite images. Moreover, being loxodromic in a vertex group implies being loxodromic in $G$ as
vertex groups have induced peripheral structure. Then Conclusion \[3.5\] follows from Corollary \[5.5\] and Lemma \[3.27\]. Given Theorem \[5.13\] the rest of the proof is similar to Proposition \[4.2\].

We now explain the modifications needed to handle the general case. Since virtually cosparse implies cosparse \[Wis\], we can assume \(V\) acts cosparsely on a \(CAT(0)\) cube complex \(\tilde{X}_V\). Let \(V \leq \tilde{V}\) be a finite index normal subgroup such that \(\tilde{V} = \pi_1(X_V)\) where \(X_V\) is a sparse cube complex. We do not use that \(X_V\) is virtually special; instead the cube complex \(X_V\) supports the cubical small cancellation theory computations. As in the proof of Proposition \[4.2\] we perform small cancellation computations inside \(\tilde{V}\), and the equalities between normal closures there induce equalities of normal closures in the whole group. \(\square\)

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