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Long-time behavior and Darwinian optimality for an asymmetric size-structured branching process

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Abstract
We study the long time behavior of an asymmetric size-structured measure-valued growth-fragmentation branching process that models the dynamics of a population of cells taking into account physiological and morphological asymmetry at division. We show that the process exhibits a Malthusian behavior; that is that the global population size grows exponentially fast and that the trait distribution of individuals converges to some stable distribution. The proof is based on a generalization of Lyapunov function techniques for non-conservative semi-groups. We then investigate the fluctuations of the growth rate with respect to the parameters guiding asymmetry. In particular, we exhibit that, under some special assumptions, symmetric division is sub-optimal in a Darwinian sense.

Keywords Branching process · Cell division · Growth-fragmentation · Long-time behavior · Measure-valued process · Population dynamics

Mathematics Subject Classification 60J80 · 92D25

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1 Introduction

The aim of this paper is to study the long-time behavior of an asymmetric size-structured growth-fragmentation branching process for population dynamics. This work is motivated by recent biological experiments (Proenca et al. 2018; Stewart et al. 2005; Wang et al. 2010) that detected asymmetry in cell division for the species *Escherichia coli*. *E. coli* is a rod shaped bacterium that grows exponentially with some elongation rate then divides roughly in the middle into two genetically identical daughter cells. Each daughter cell therefore creates a new pole at division and inherits the other pole from its mother. After two divisions, it is possible to distinguish sister cells: one has inherited the old pole of its mother while the other one has inherited the new pole of its mother. The former is called the old pole cell, and the latter the new pole cell. It is possible to track experimentally the status (old pole or new pole) of each cell together with their sizes along time and lineages, see (Proenca et al. 2018; Stewart et al. 2005; Wang et al. 2010). These experiments showed that there is a statistically significant difference between the elongation rates of the old pole and new pole cells (Delyon et al. 2018). This is called physiological asymmetry throughout this paper. There is also a statistically significant difference between the sizes at birth of the old pole and new pole cells. This phenomenon is called morphological asymmetry. To date, the biological mechanisms leading to these behavioral differences are not yet understood. The aim of this paper is to propose a model for the dynamics of a population of cells taking into account both physiological and morphological asymmetry, and to compare its theoretical properties to that of the symmetric model. In particular, we study how these asymmetric properties influence the growth speed of the population.

Let us introduce informally our model. We consider a cell population where every individual is represented by two traits \((x, p)\), where \(x\) is its size and \(p \in \{0, 1\}\) is its status, typically 0 for the old pole cell and 1 for the new pole cell. These traits and the number of individuals in the population evolve randomly in continuous time as follows:

- Each individual divides (i.e. dies and gives birth to two new individuals) independently from the others (conditionally to the past) following an exponential clock with size-dependent intensity \(B\);
- Between divisions, the size of an individual of trait \((x, p)\) grows exponentially with status-dependent elongation rate \(\alpha_p\) and its status remains constant;
- At division, an individual of trait \((x, p)\) dies and gives birth to two individuals of trait \((\theta_0 x, 0)\) and \((\theta_1 x, 1)\), with \(\theta_0 + \theta_1 = 1\).

The parameters \(\alpha_0\) and \(\alpha_1\) represent the possibly different elongation rates of old pole and new pole cells and models physiological asymmetry. The parameters \(\theta_0\) and \(\theta_1\) represent the proportion of the size of the mother inherited by each sister cell, thus taking into account morphological asymmetry. These parameters will not be assumed to be random even if our results can be easily generalized to this hypothesis. To retrieve a fully symmetric model, simply take \(\alpha_0 = \alpha_1\) (physiological symmetry) and \(\theta_0 = \theta_1 = 1/2\) (morphological symmetry).

This model belongs to the class of growth-fragmentation dynamics that has attracted a lot of attention in the literature, see for instance (Michel 2006a, b; Olivier 2017;
Bertoin and Watson 2018; Campillo et al. 2017; Doumic et al. 2015; Guillemin et al. 2004; Cloez 2017; Bansaye et al. 2019) and references therein, among many others. However, it is more general than classical size-structured growth-fragmentation models as, to our best knowledge, it is the first model to take physiological and morphological asymmetry into account.

As may be expected in the classical study of growth-fragmentation models, we begin by demonstrating that our model exhibits Malthusian behavior. This result states that the population size grows exponentially fast and that the trait distribution converges to some stable distribution. The exponential growth rate $\lambda$ is the eigenvalue of some non-local and non-diffusive operator. Although existence and uniqueness of these eigenvalues are expected for such branching models, there is no simple and systematic method to prove it. Proving such results for related growth-fragmentation models is tricky and has attracted a large amount of research in recent years (Bertoin and Watson 2018; Bertoin 2019; Doumic Jauffret and Gabriel 2010; Balagué et al. 2013; Cáceres et al. 2011; Mischler and Scher 2016; Marguet 2019; Bansaye et al. 2019). Most of the techniques used in these works cannot be applied to our problem. For instance they require regularity conditions on the operators imposing in particular that the distribution of $(\theta_0, \theta_1)$ cannot be deterministic. Instead, we use the approach of Bansaye et al. (2019); Cloez and Gabriel (2019) based on irreducibility and Lyapunov functions. It is not straightforward that the Malthusian behavior holds for deterministic values of $(\theta_0, \theta_1)$. Indeed, in this setting, the physiological symmetric model $(\alpha, \alpha, \theta, \theta)$ for equal elongation rates $\alpha_0 = \alpha_1$ does not exhibit a Malthusian behavior. The population grows exponentially fast but the size distribution does not stabilize. It is shown in Bernard et al. (2019); Gabriel and Martin (2019) that the latter oscillates at frequencies that depend on the initial configuration. It is because of this atypical property that our demonstration of Malthusian behaviour is delicate. Asymmetry of the elongation rate therefore guarantees the vanishing of the initial condition as well as the absence of oscillation. That is an interesting first conclusion of our results from a biological point of view. As we aim to study the influence of asymmetry, we also study the variability of the growth rate of the population with respect to the variation of $\alpha_1 - \alpha_0$ and $\theta_1 - \theta_0$. From an evolutionary point of view (Metz et al. 1995; Metz 2006; Dieckmann and Law 1996; Geritz et al. 1998), the Malthusian rate $\lambda$ is called the fitness and determines if a mutant population can invade a resident one: a mutant with a larger fitness should invade the resident population. We compute the partial derivatives of $(\alpha_1, \alpha_0, \theta_1, \theta_0) \mapsto \lambda(\alpha_1, \alpha_0, \theta_1, \theta_0)$ similarly as in Campillo et al. (2017); Gaubert and Lepoutre (2015); Michel (2006b); Olivier (2017) for related models. As in these works, these formulas involve unknown quantities such as eigenvectors. To overcome this problem, numerical simulations are often used. Instead, we focus on the particular case $B(x) = x$ which includes the idea that large cells divide faster than small cells. For this special division rate, we establish new formulas for the asymptotic distribution, even in the symmetric case. We then extend some results of Zaidi et al. (2015); Hall and Wake (1989, 1990) which establish some explicit formulas for the asymptotic distribution. To derive them, we show that a clever transformation of the quantities involved satisfies a functional equation with known solutions. As a consequence, we show that in the presence of morphological asymmetry, physiological
symmetry is suboptimal in a Darwinian sense. That is an interesting second conclusion of our results from a biological point of view.

This paper is organized as follows. In Sect. 2, we define the measure-valued branching process modeling the physiologically and morphologically asymmetric cell division, and we make the connection with semigroup theory and partial differential equations. In Sect. 3, we prove our first main result concerning the long-time behavior of the measure-valued branching process. In Sect. 4, we study the sensitivity of the Malthusian parameter as a function of the parameters of the model, we prove our second main result and we establish several explicit formulas in the special case $B(x) = x$. Finally we conclude the paper with a discussion in Sect. 5 that ties our results back to the biological literature and shows how the biological understanding has been advanced.

2 Definition of the model and main results

In this section, we precisely define our asymmetric size-structured branching process and state our main results regarding its asymptotic behavior: existence of eigenelements, which can be interpreted biologically as the Malthusian behavior and sensibility analysis of these eigenelements with respect to the asymmetry parameters. In the special case where the division rate is the identity function, we state in addition the Darwinian sub-optimality of physiological symmetry in the presence of morphological asymmetry.

2.1 Asymmetric branching process

In this section, we define the measure-valued branching process we use to model physiologically and morphologically asymmetric cell division. It can be seen either as a branching process (Harris 1964), a piecewise deterministic Markov process (Davis 1993) or a stochastic differential equation with jumps (Ikeda and Watanabe 2014). Therefore we detail the model in these three frameworks. We also introduce here all our notation and explain the link of our model with the partial differential equations theory.

Branching process path-wise construction Throughout the paper, we use the classical Ulam-Harris-Neveu notation (Dawson 1993) to identify each individual in a genealogical tree. Let

$$\mathcal{U} = \bigcup_{n \in \mathbb{N}} \{0, 1\}^n,$$

denote the set of all the descendants of the original (unique) ancestor. The original ancestor is labeled by $\emptyset$ and is identified to $\{0, 1\}^0$. When an individual $u \in \mathcal{U}$ dies (divides), it gives birth to two descendants labelled $u0$, $u1$. We denote by $b_u$ and $d_u$ the birth and the death dates of individual $u$. Let $\mathcal{V}_t$ be the set of alive individuals at
time $t \geq 0$; that is

$$\mathcal{V}_t = \{u \in \mathcal{U} \mid b_u \leq t < d_u\}.$$  

We denote its cardinal by $N_t$; this represents the number of alive individuals at time $t$. Every individual $u \in \mathcal{U}$ at time $t \in [b_u, d_u)$ possesses a trait $Y_t^u = (X_t^u, P_t^u)$, where $X_t^u \in \mathbb{R}^+$ is the size of individual $u$ at time $t$ and $P_t^u \in \{0, 1\}$ is its status and encodes that individuals may have two different dynamics. The global population is described through the punctual measure

$$Z_t = \sum_{u \in \mathcal{V}_t} \delta_{Y_t^u} \in \mathcal{M}_f^P(\mathbb{R}^+ \times \{0, 1\}),$$

where $\mathcal{M}_f^P(\mathbb{R}^+ \times \{0, 1\})$ denotes the set of positive and finite punctual measures on $\mathbb{R}^+ \times \{0, 1\}$.

Let us now describe the population random dynamics. For all $t \geq 0$, we set $\mathcal{F}_t = \sigma\{\mathcal{V}_s, (Y_s^u)_{u \in \mathcal{V}_s}, \forall s \leq t\}$, the $\sigma$-field generated by the traits of all individuals born before time $t$ and up to time $t$ (or their death, whichever comes first). Conditionally on $\mathcal{F}_t$, we have that

- For all $u \in \mathcal{V}_t$ and $t \leq r < d_u$, the size of individual $u$ up to its death grows exponentially with a status-dependent growth rate: we have

  $$X_r^u = X_t^u \exp(\alpha_{P_t^u} (r - t)), \quad P_r^u = P_t^u,$$

  where $\alpha_0 > 0$ and $\alpha_1 > 0$ are the (possibly) different elongation rates modeling physiological asymmetry. The status is constant until death.

- For all $u \in \mathcal{V}_t$, the death dates $d_u$ are independent random variables with distribution given by

  $$\mathbb{P}(d_u > t + s \mid \mathcal{F}_t) = \exp\left(- \int_0^s B(X_t^u \exp(\alpha_{P_t^u} r))dr\right),$$

  where $B$ is a measurable function from $\mathbb{R}^+$ onto $\mathbb{R}^+$ representing the size-dependent division rate. Indeed, as stated above $X_t^u \exp(\alpha_{P_t^u} r)$ is the size at time $t + r$ of individual $u$ given that it did not die between dates $t$ and $r$.

- For all $u \in \mathcal{V}_t$, we have $b_{u0} = b_{u1} = d_u$, meaning that individual $u$ dies (divides) and at the same time gives birth to two individuals $u0$ and $u1$, and for $i \in \{0, 1\}$,

  $$Y_{b_{ui}}^u = (\theta_i X_{d_u -}, i) = (\theta_i X_{b_u} e^{\alpha_{P_t^u} (d_u - b_u)}, i).$$

This equation means that the two new individuals get a different status, individual $u0$ with status 0 inherits a proportion $\theta_0$ of the size at death of individual $u$, and individual $u1$ with status 1 inherits a proportion $\theta_1$ of the size at death of individual $u$. One has $\theta_0 > 0$, $\theta_1 > 0$ and $\theta_0 + \theta_1 = 1$. Allowing $\theta_0$ and $\theta_1$ to differ from $1/2$ models morphological asymmetry.
This model is well defined until the explosion time $T$ such that $N_T = +\infty$. We will show in Lemma 1 below, that $T = \infty$ when the division rate $B$ is locally bounded.

The process $(Z_t)_{t \geq 0}$ belongs to the class of measure-valued piecewise deterministic Markov processes introduced in Cloez et al. (2020) and satisfies a stochastic differential equation with jumps, as detailed below. In the sequel, we denote $\mathbb{E}_{(x,p)}$ and $\mathbb{P}_{(x,p)}$ respectively the expectation and probability conditionally to $Z_0 = \delta(x,p)$. For any measurable function $f : \mathbb{R}_+ \times \mathbb{R}_+ \times \{0, 1\} \to \mathbb{R}$, set

$$Z_t(f) = Z_t(f_t) = \sum_{u \in \mathcal{V}_t} f(t, Y_t^u) = \sum_{u \in \mathcal{V}_t} f_t(Y_t^u).$$

**Piecewise deterministic Markov process framework** The only source of randomness of the process comes from the division clocks $d_u$. The special form of the distribution of the division clocks yields that the measure-valued process $(Z_t)_{t \geq 0}$ is a measure-valued piecewise deterministic Markov process. Its local characteristics, as defined in Cloez et al. (2020), are as follows. For any punctual measure $\zeta = \sum_{j=1}^n \delta(x_j, p_j) \in \mathcal{M}_P^+(\mathbb{R}_+ \times \{0, 1\})$, and $t \geq 0$,

- The flow is defined by
  $$\Phi(\zeta, t) = \sum_{j=1}^n \delta_{(x_j e^{\alpha p_j t}, p_j)},$$

- The jump intensity is $\lambda(\zeta) = \sum_{j=1}^n B(x_j)$,

- The Markov jump kernel is given by
  $$Q(\zeta, A) = \sum_{k=1}^n \frac{B(x_k)}{\sum_{j=1}^n B(x_j)} \mathbf{1}_A(\zeta - \delta(x_k, p_k) + \delta(\theta_0 x_k, 0) + \delta(\theta_1 x_k, 1)),$$

  for all Borel subset $A$ of $\mathcal{M}_P^+(\mathbb{R}_+ \times \{0, 1\})$.

**Stochastic differential equation framework** The dynamics of the measure-valued process $(Z_t)$ can also be described in terms of stochastic differential equation with jumps. Let $\mathcal{N}(ds, du, dl)$ be a Poisson point measure on $\mathbb{R}_+ \times \mathcal{U} \times \mathbb{R}_+$ of intensity $ds n(du) dl$ where $ds$, $dl$ are Lebesgue measures and $n(du)$ the counting measure on $\mathcal{U}$. If $f : \mathbb{R}_+ \times \mathbb{R}_+ \times \{0, 1\} \to \mathbb{R}$ is a bounded measurable function with bounded measurable derivatives then one has

$$Z_t(f_t)$$

$$= \sum_{u \in \mathcal{V}_t} f_t(Y_t^u) = \sum_{u \in \mathcal{V}_t} f_t(X_t^u, P_t^u)$$

$$= \sum_{u \in \mathcal{V}_0} f_0(Y_0^u) + \int_0^t \sum_{u \in \mathcal{V}_s} (\partial_s f_s(Y_s^u) + \alpha p_s X_s^u \partial_x f_s(Y_s^u)) ds$$

$$+ \int_{\mathcal{V}_t \times \mathcal{U} \times \mathbb{R}_+} f_s(\theta_0 X_s^u, 0) + f_s(\theta_1 X_s^u, 1) - f_s(Y_s^u) \mathcal{N}(ds, du, dl).$$
See (Tran 2006; Fournier and Méléard 2004; Bansaye and Méléard 2015; Marguet 2019) for details.

**Transitions semi-group** We can naturally associate to \((Z_t)_{t \geq 0}\) the semigroup \((M_t)_{t \geq 0}\) defined for any non-negative measurable function \(f : \mathbb{R}_+ \times \{0, 1\} \rightarrow \mathbb{R}\) by

\[
M_t f(x, p) = \mathbb{E} \left[ \sum_{u \in \mathcal{V}_t} f(Y^u_t) \mid Z_0 = \delta(x, p) \right] = \mathbb{E}_{(x, p)} \left[ \sum_{u \in \mathcal{V}_t} f(Y^u_t) \right],
\]

which describes the mean behavior of \(Z_t(f)\). We will see in Lemma 1, that \(M_t\) also acts on bounded functions \(f\) through Eq. (3). Let us define \(\mathcal{C}^1_+((0, +\infty) \times \{0, 1\})\) the space of non negative and continuous functions with continuous derivative with respect to the variable \(x \in \mathbb{R}_+\). In Lemma 2, we show that the extended generator \(A\) of \((M_t)_{t \geq 0}\) reads

\[
A f(x, p) = \alpha_p x \partial_x f(x, p) + B(x) \left( f(\theta_0 x, 0) + f(\theta_1 x, 1) - f(x, p) \right),
\]

for all \(f \in \mathcal{C}^1_+((0, +\infty) \times \{0, 1\})\). The dual semigroup \(M^*_t \mu = \mu M_t\) describes the mean behavior of the process \(Z\), that is \(\mu M_t = \mathbb{E}[Z_t \mid Z_0 = \mu]\), for \(\mu \in \mathcal{M}_+(\mathbb{R}_+ \times \{0, 1\})\). Let us now end this subsection by a link with partial differential equation theory. If we define \((\mu_t^0), (\mu_t^1)\) by the equality

\[
\mu M_t = \mu_t^0(dx, dp) = \delta_0(dp)\mu_t^0(dx) + \delta_1(dp)\mu_t^1(dx),
\]

we obtain the following system of growth-fragmentation equations satisfied by \((\mu_t^0, \mu_t^1)\): for all \(p \in \{0, 1\}\), one has

\[
\partial_t \mu_t^0(x) + \partial_x \left( \alpha_p x \mu_t^0(x) \right) + B(x) \mu_t^0(x) = \frac{1}{\theta_p} B \left( \frac{x}{\theta_p} \right) \mu_t^0 \left( \frac{x}{\theta_p} \right) + \frac{1}{\theta_p} B \left( \frac{x}{\theta_p} \right) \mu_t^1 \left( \frac{x}{\theta_p} \right).
\]

Equation (4) is a system of growth-fragmentation equations with growth rates variability. To our knowledge, this equation (or more precisely, this system of equations) has never been introduced before. Our main result (see Theorem 1) states that the solution of Eq. (4) converges at exponential speed to some stable distribution in some weighted \(L^1\) norm.

### 2.2 Existence of eigenelements and spectral gap inequality

In this section we state our main results concerning the eigenelements of the general asymmetric model. We first make two assumptions to avoid atypical behaviors.

**Assumption 1**

(i) Function \(B\) is a positive and continuous function on \((0, +\infty)\) such that

\[
\lim_{x \to 0} B(x) = 0, \quad \lim_{x \to \infty} B(x) = +\infty.
\]
(ii) Elongation rates verify $\alpha_0 \neq \alpha_1$.

The first assumption states that small cells do not divide and very large cells divide at once. To see informally the necessity of this type of assumptions, consider the simpler process $X_t \geq 0$ modeling a single cell lineage (without branching) with a constant division rate ($i.e.$ jump rate) $B$. This process increases exponentially between jumps and jumps from $X_T - \Theta X_T$ at Poissonian times $T$; where $\Theta$ is a random variable taking value in $\{\theta_0, \theta_1\}$. This process is then the exponential of a Levy process. Thus, it has 3 possible asymptotic behaviors: convergence to infinity, convergence to 0 or oscillation without convergence (Bertoin 1996, Corollary 2 p.190). To avoid such trivial behavior, we assume that small cells do not divide and large cells divide faster.

The second assumption is the physiological asymmetry assumption which is necessary to avoid oscillation of the size distribution depending on the initial state as shown in Bernard et al. (2019); Gabriel and Martin (2019).

Our first significant result concerns the existence of eigenelements and the convergence of the semigroup at exponential speed.

**Theorem 1** Suppose Assumptions 1 hold. Then there exist a probability measure $\gamma$, a measurable function $h : (0, +\infty) \times \{0, 1\} \to (0, +\infty)$ and $\lambda > 0$ such that $\gamma(h) = 1$ and

$$M_t h = e^{\lambda t} h, \quad \gamma M_t = e^{\lambda t} \gamma. \quad (5)$$

Moreover, $h \leq V$, $\gamma(V)$ is finite and there exist $C, \omega > 0$ such that for all $t \geq 0$ and measure $\mu$, one has

$$\sup_{\|f/V\|_\infty \leq 1} \left| e^{-\lambda t} \mu M_t f - \mu(h) \int f d\gamma \right| \leq C e^{-\omega t} \mu(1 + V), \quad (6)$$

where $V : x \mapsto x^q + \frac{1}{x^q}$ for some $q > 0$, and the supremum in (6) is taken over all measurable functions $f$ such that $f/V$ is bounded by 1.

In other words, this theorem reads

$$\mathbb{E}_{(x, i)} \left[ \sum_{\mu \in V_i} f(X_i^\mu, P_i^\mu) \right] = h(x, i) e^{\lambda t} \gamma(f) + O(e^{(\lambda - \omega)t}),$$

where $O$ is the Landau notation; this mean that $e^{-(\lambda - \omega)t} O(e^{(\lambda - \omega)t})$ is uniformly bounded over measurable functions $f$ such that $f/V$ is bounded.

Setting $f = 1$, we see that the mean number of individuals grows exponentially at rate $\lambda$, which is called Malthusian behavior in population dynamics. Parameter $\lambda$ is called the Malthusian parameter. In addition we can prove that $\lambda$ is between $\alpha_0$ and $\alpha_1$; see Remark 1.

Inequality (6) ensures the uniqueness (up to multiplicative constants) of the eigenelements. More precisely, if there exists a measurable function $\tilde{h}$, bounded by $V$, and a number $\tilde{\lambda}$ such that for all $t \geq 0$ (or at least one), $M_t \tilde{h} = e^{\tilde{\lambda} t} \tilde{h}$ then by choosing
\( \mu = \delta \) and \( f = \tilde{h} \) in Eq. (6) we see that \( \tilde{\lambda} = \lambda \) and \( \tilde{h} = \gamma(\tilde{h})h \). Similarly, probability measure \( \gamma \) is unique.

When \( \alpha_0 = \alpha_1 \), the existence and uniqueness of a unique triplet \((\lambda, h, \gamma)\) of eigenelements that satisfies (5) was proven in Doumic Jauffret and Gabriel (2010) for the symmetric equation (with one cell population) but the convergence (6) is false (Bernard et al. 2019). Using this result, we prove in Lemma 10 and Lemma 11 that existence and uniqueness also hold true for our system of two cell equations.

We can go further than exhibiting the mean behavior of the process. Indeed, we can study the variation of the Malthusian parameter as a function of the parameters of the model. To do so, we introduce the following change of variable. Let \( \alpha_1 = \alpha + \epsilon \), \( \alpha_0 = \alpha - \epsilon \) in such a way that \( \alpha = (\alpha_0 + \alpha_1)/2 \) and \( \epsilon = (\alpha_1 - \alpha_0)/2 \), \( \theta = \theta_0 \) (recall that \( \theta_1 = 1 - \theta_0 \)) and \( \mathbf{u} = (\alpha, \epsilon, \theta) \in (0, +\infty) \times (0, 1) \).

We now study the eigenelements \((\lambda, \gamma, h)\) as functions of \( \mathbf{u} \). However, for the sake of simplicity we do not highlight this dependence on the parameter \( \mathbf{u} \) in the notation (it will be done in Sect. 4).

**Theorem 2** Under Assumptions 1 (i) and if \( B \) is \( C^1 \) then \( h \) is \( C^2 \) and

1. \[
\frac{\partial \lambda}{\partial \alpha} = \int_{(0,1]} \int_0^{+\infty} x \partial_x h(x, p) \gamma(dx, dp),
\]

2. \[
\frac{\partial \lambda}{\partial \epsilon} = \int_{(0,1]} \int_0^{+\infty} (2p - 1)x \partial_x h(x, p) \gamma(dx, dp),
\]

3. \[
\frac{\partial \lambda}{\partial \theta} = \int_{(0,1]} \int_0^{+\infty} B(x) \left[ \partial_x h(\theta x, 0) - \partial_x h((1 - \theta)x, 1) \right] \gamma(dx, dp).
\]

On the one hand, this result establishes the regularity of the Malthusian parameter. On the other hand, it extends some results of Michel (2006a, b) to our asymmetric framework. In the symmetric case \( \alpha_0 = \alpha_1 = \alpha \), the eigenvalue is \( \lambda = \alpha \) and the eigenfunction is \( h : (x, p) \mapsto x \). From Theorem 2 (ii) the influence of physiological asymmetry is related to the asymptotic mean size of the cells. Unfortunately, the asymptotic measure \( \gamma \) is generally unknown. From Theorem 2 (i), the malthusian parameter \( \lambda \) is increasing with \( \alpha \). There are no similar arguments in the non-symmetric case.

### 2.3 The particular case \( B(x) = x \)

In the special case where the division rate \( B \) equals the identity function, we obtain more explicit results such as the shape of \( \gamma \), the moments of \( \gamma \), etc which generalize
results of Hall and Wake (1989, 1990). These results are in Sect. 4. Together with Theorem 2, these additional properties yield the Darwinian sub-optimality of symmetry which reads as follows.

**Theorem 3** Let \( \theta \in (0, 1) \) be such that \( 1 - \theta < \theta \) and \( \alpha \in (0, +\infty) \). At \( u = (\alpha, 0, \theta) \), we have

\[
\frac{\partial \lambda}{\partial \epsilon} < 0.
\]

Theorem 3 implies that if a cell divides into two daughter cells with morphological asymmetry (\( \theta \neq 1/2 \)), then it is better, in the Darwinian sense that the Malthusian parameter is increased, that the two daughter cells have different elongation rates and thus also exhibit physiological asymmetry. More precisely, it is advantageous for the largest cell at division to elongate faster.

The rest of this paper is dedicated to the proofs of our main results theorems 1, 2, 3 and some additional results.

### 3 Malthusian behavior: eigenelements of the semi-group

This section is dedicated to the proof of Theorem 1 exhibiting the eigenelements of the semi-group of our asymmetric branching process. Biologically speaking, it establishes the Malthusian behavior of the asymmetric model. We start with preliminary results concerning the non-explosion of the process and its infinitesimal generator in Sect. 3.1, and then proceed to the proof in Sect. 3.2 by using the approach developed in Bansaye et al. (2019).

#### 3.1 Non explosion and martingale properties

We first establish that under a mild condition on the division rate \( B \), the process does not explode in finite time. To that end, we introduce the notation:

\[
\bar{\alpha} = \max(\alpha_0, \alpha_1), \quad \underline{\alpha} = \min(\alpha_0, \alpha_1).
\]

**Lemma 1** If \( B \) is locally bounded on all intervals of the type \([0, M]\), for all \( M \geq 0 \), then the population does not explode and for all \((x, p) \in \mathbb{R}_+ \times \{0, 1\} \) and \( T > 0 \), one has

\[
\mathbb{E}_{(x, p)}[N_T] \leq \exp\left(T \sup_{y \leq xe^{\bar{\alpha}T}} B(y)\right).
\]

In particular, \((M_t)_{t \geq 0}\) acts on bounded and measurable functions.

**Proof** Starting with one cell with size \( x \) and status \( p \), all its descendants have size lower than \( xe^{\bar{\alpha}T} \) up to time \( T \). Using for instance the Gillespie algorithm, one can
couple our model on \([0, T]\) with a simple Yule process \((\Upsilon_t)_{0 \leq t \leq T}\) with branching rate

\[
B = \sup_{y \leq e^{\alpha T}} B(y),
\]

in such a way that the number \(N_t\) of individuals in the original process at time \(t \leq T\) is bounded by \(\Upsilon_t\). See for instance (Harris 1964, Section 8 p.105) for the definition and properties of Yule process. Finally as \(\mathbb{E}_1[\Upsilon_T] \leq e^{TB}\), one obtains

\[
\mathbb{E}_{(x, p)}[N_T] \leq \mathbb{E}_1[\Upsilon_T] \leq e^{TB}.
\]

As this quantity is finite for all \(T\), the process does not explode in finite time. \(\square\)

An alternative proof could be to use the SDE (2) with a stopping time argument as in (Fournier and Méléard 2004, Theorem 4.1).

Let us define the operator \(A\) acting on the space of \(C^1\) functions \(f\) by

\[
Af(x, p) = \alpha p x \partial_x f(x, p) + B(x) \left( f(\theta_0 x, 0) + f(\theta_1 x, 1) - f(x, p) \right),
\]

for every \((x, p) \in \mathbb{R}_+ \times \{0, 1\}\).

In the following lemma, we derive a Duhamel type formula (variation of constants formula for semigroups) describing the evolution of \((M_t)_{t \geq 0}\). Consequently, we show that operator \(A\) is, in the sense stated in this lemma, the generator of \((M_t)_{t \geq 0}\).

**Lemma 2** Assume that \(B\) is locally bounded over all intervals of the type \([0, M]\), for all \(M \geq 0\).

(i) For all \((x, p) \in \mathbb{R}_+ \times \{0, 1\}\), \(t \geq 0\) and measurable functions \(f\) such that \(M_t f\) is well defined we have

\[
M_t f(x, p) = f(x e^{\alpha p t}, p) e^{-\int_0^t B(x e^{\alpha p s}) \, ds} + \int_0^t e^{-\int_s^t B(x e^{\alpha p s'}) \, ds'} B(x e^{\alpha p s}) M_{t-s} f(\theta_0 x e^{\alpha p s}, 0) \, ds + \int_0^t e^{-\int_s^t B(x e^{\alpha p s'}) \, ds'} B(x e^{\alpha p s}) M_{t-s} f(\theta_1 x e^{\alpha p s}, 1) \, ds.
\]

(ii) For all bounded \(C^1\) functions \(f\) such that \(Af\) is bounded, we have that

\[
\left( Z_t(f) - Z_0(f) - \int_0^t Z_s(Af) \, ds \right)_{t \geq 0}
\]

is a martingale.

**Proof** (i) We split the expression of \(M_t f(x, p)\) depending on the ancestor individual being still alive at time \(t\) or not. Using the branching property, we obtain

\[
M_t f(x, p) = \mathbb{E}_{(x, p)} \left[ \sum_{u \in V_t} f(Y^u_t) 1_{d_0 > t} \right] + \mathbb{E}_{(x, p)} \left[ \sum_{u \in V_t} f(Y^u_t) 1_{d_0 \leq t} \right]
\]
\begin{equation*}
= f(xe^{\alpha p t}, p)\mathbb{P}(d\theta > t)
+ \mathbb{E}_{(x,p)} \left[ \mathbb{E}_{(\theta_0 x e^{\alpha p d\theta}, 0)} \left[ \sum_{u \in \mathcal{V}_t-d\theta} f(Y^u_{t-d\theta}) \mathbf{1}_{d\theta \leq t} \right] \right]
+ \mathbb{E}_{(x,p)} \left[ \mathbb{E}_{(\theta_1 x e^{\alpha p d\theta}, 1)} \left[ \sum_{u \in \mathcal{V}_t-d\theta} f(Y^u_{t-d\theta}) \mathbf{1}_{d\theta \leq t} \right] \right]
= f(xe^{\alpha p t}, p)e^{-\int_0^t B(xe^{\alpha p s})ds} \\
+ \int_0^t e^{-\int_0^s B(xe^{\alpha p s})ds} B(xe^{\alpha p s}) M_{t-s} f(\theta_0 x e^{\alpha p s}, 0)ds \\
+ \int_0^t e^{-\int_0^s B(xe^{\alpha p s})ds} B(xe^{\alpha p s}) M_{t-s} f(\theta_1 x e^{\alpha p s}, 1)ds.
\end{equation*}

(ii) Fix a bounded $C^1$ function $f$ and $(x, p) \in \mathbb{R}_+ \times \{0, 1\}$. From (i), we have that $t \mapsto M_t f(x, p)$ is derivable at time $t = 0$ and
\[
\frac{\partial M_t f}{\partial t} \bigg|_{t=0} (x, p) = A f(x, p).
\]

If $Af$ is further bounded, by the semigroup (or Markov) property and the dominated convergence theorem, we have that $t \mapsto M_t f(x, p)$ is derivable at every time $t$ and
\[
\frac{\partial M_t f}{\partial t} (x, p) = M_t Af(x, p).
\]

Finally (ii) is a consequence of the previous equation and Markov property (as in (Ethier and Kurtz 2009, Proposition 1.7 p. 162)).

Note again that Lemma 2 (ii) could be proved through the SDE (2).

### 3.2 Proof of Theorem 1

To prove Theorem 1, we use the approach developed in Bansaye et al. (2019). To do so, we have to verify that the semigroup $(M_t)_{t \geq 0}$ satisfies (Bansaye et al. 2019, Assumptions A) which are the existence of Lyapunov functions, a mass ratio inequality and a Doeblin minoration condition. These three steps are described in the next three subsections.

#### 3.2.1 Lyapunov functions

Let $V, \phi : (0, +\infty) \times \{0, 1\} \to (0, +\infty)$ defined by
\[
\phi(x, p) = x + \frac{1}{x}, \quad V(x, p) = x^q + \frac{1}{x^q},
\]
for some \(q \geq (\bar{\alpha} + 2)/\alpha > 1\). Note that we have \(V \geq \phi \geq 0\), and that both \(V\) and \(\phi\) belong to \(C^1_c((0, +\infty) \times \{0, 1\})\).

The aim of this subsection is to show that \(V\) and a well-suited function \(\psi\) introduced in Eq. (13) below verify (Bansaye et al. 2019, Assumption (A0) (A1) (A2)), that roughly speaking states that \(\psi \leq V\), functions \(M_t V, M_t \psi\) are locally bounded and

\[
M_t V \leq \alpha V + \theta 1K \psi, \quad M_t \psi \geq \beta \psi,
\]

for some \(\beta > \alpha\) and \(\tau > 0\), where \(K \subset (0, +\infty)\) is a compact set. These assumptions guaranty some compactness (or tightness) property for the dynamics of the semigroup.

We start with establishing drift properties for \(\phi\) and \(V\), based on straightforward analytic calculations.

**Lemma 3** *Under Assumption 1, there exist \(a, b, \zeta \in \mathbb{R}\) such that \(b > a\) and*

\[
A \phi \geq b \phi \quad \text{and} \quad A V \leq a V + \zeta \phi.
\]

**Proof** Let \(\phi_0 : (x, p) \mapsto x\) and \(\phi_1 : (x, p) \mapsto 1/x\). As the division rate \(B\) is non negative, we have

\[
A \phi_0 \geq \alpha \phi_0, \quad A \phi_1 \geq -\bar{\alpha} \phi_1.
\]

Thus, we have the first inequality with \(b = -\bar{\alpha}\). Now, we write again \(V = V_0 + V_1\) with \(V_0 : (x, p) \mapsto x^q\) and \(V_1 : (x, p) \mapsto 1/x^q\) and separate the calculations. On the one hand, we have

\[
AV_0(x, p) = x^q \left(q \alpha p - B(x)(1 - \theta_0^q - \theta_1^q)\right),
\]

and as \(B(x) \to \infty\) as \(x\) tends to infinity, there exists \(N\) such that for all \(x \geq N\),

\[
q \bar{\alpha} - B(x)(1 - \theta_0^2 - \theta_1^2) \leq b - 1.
\]

Recall that \(b = -\bar{\alpha}\) so that \(-(b - 1) > 1\) and \(q > 1\). Thus for all \(x \geq \max\{1, N\}\), one has

\[
AV_0(x, p) - (b - 1) V(x, p) \leq -(b - 1)x^{-q} \leq -(b - 1)x \leq -(b - 1)\phi(x, p).
\]

Therefore \(AV_0 - (b - 1) V\) is bounded by \(\zeta_0 \phi\) for \(\zeta_0\) large enough. On the other hand, we have

\[
AV_1(x, p) \leq V_1(x, p) \left(-q \alpha + B(x)(\theta_0^{-q} + \theta_1^{-q} - 1)\right) \\
\leq V_1(x, p)(b - 2 + B(x)(\theta_0^{-q} + \theta_1^{-q} - 1)),
\]

as \(q \geq (\bar{\alpha} + 2)/\alpha\) and \(b = -\bar{\alpha}\). Similarly as above, as \(\lim_{x \to 0} B(x) = 0\) we obtain that \(AV_1 - (b - 1) V\) is bounded by \(\zeta_1 \phi\) for some large enough \(\zeta_1 > 0\). We then obtain the desired result by setting \(a = b - 1\) and \(\zeta = \zeta_0 + \zeta_1\). \(\square\)
Lemma 3 above almost gives the sufficient drift conditions of (Bansaye et al. 2019, Proposition 2.2) to verify (Bansaye et al. 2019, Assumption (A0) (A1) (A2)). However, a lower bound is missing. The rest of this subsection is dedicated to adapting arguments of the type (Cloez and Gabriel 2019) to prove (Bansaye et al. 2019, Assumption (A0) (A1) (A2)) in our setting.

Lemma 4 Under Assumption 1, for every $t \geq 0$, $M_t V$ and $M_t \phi$ are finite. Moreover,

$$M_t V \leq e^{(a+\zeta)t} V, \quad M_t \phi \geq e^{bt} \phi,$$

and

$$M_t V \leq e^{at} V + \frac{\zeta}{a-b} M_t \phi,$$

where $a$, $b$, $\zeta$ are the constants from Lemma 3.

Proof We begin the proof by a standard localization argument (as in Meyn and Tweedie 1993) to prove that martingale properties of Lemma 2 extend to non-bounded functions, then we use Gronwall lemma. Let $m > 0$ and set

$$\tau_m = \inf \{ t \geq 0 | \exists u \in V_t, \ X_t^u \notin [1/m, m] \}.$$

As $V$ and $AV$ are bounded over $[1/m, m]$, it follows from Lemma 2 that

$$\left( Z_{t \wedge \tau_m} (V) - Z_0 (V) - \int_0^{t \wedge \tau_m} Z_s (AV) ds \right)_{t \geq 0}$$

is a martingale. Now from Lemma 3, one has $AV \leq CV$ for $C = a + \zeta$, thus $(e^{-C(t \wedge \tau_m)} Z_{t \wedge \tau_m} (V))_{t \geq 0}$ is a supermartingale (see (Ethier and Kurtz 2009, Corollary 3.3 p. 66) for instance) one obtains

$$\mathbb{E} (x, p) \left[ e^{-Ct} Z_t (V) \right] \leq \liminf_{m \to \infty} \mathbb{E} (x, p) \left[ e^{-C(t \wedge \tau_m)} Z_{t \wedge \tau_m} (V) \right] \leq V(x, p).$$

We deduce from this inequality and from $\phi \leq V$, that $Z_t (V)$ and $Z_t (\phi)$ are integrable and $M_t V \leq e^{(a+\zeta)t} V$. As a consequence, using Lemmas 2, 3, the preceding localization argument, and now dominated convergence, we obtain

$$M_t V \leq V + \int_0^t (a M_s V + \zeta M_s \phi) ds,$$

and

$$M_t \phi \geq \phi + b \int_0^t M_s \phi ds.$$
On the first hand, Eq. (12) and Gronwall Lemma entail the second equation in (9). On
the other hand, Eq. (11) and Gronwall Lemma entail

$$M_t V \leq e^{at} V + \zeta \int_0^t e^{a(t-s)} M_s \phi ds.$$  

Applying operator $M_t$ to the second functional inequality in (9) taken at $t - s$ yields
$M_s \phi \leq e^{-b(t-s)} M_t \phi$ and ends the proof of Eq. (10).

Fix now $\tau > 0$ and set

$$\psi = e^{-(a+\zeta)\tau} M_\tau \phi.$$  

We have the straightforward inequality

$$\psi \leq e^{-(a+\zeta)\tau} M_\tau V \leq V,$$

and we can now prove that Eq. (8) hold.

**Lemma 5** There exist $\beta > \alpha$, $\theta > 0$ and a compact set $K$ such that

$$M_\tau V \leq \alpha V + \theta 1_K \psi, \quad (14)$$

and

$$M_\tau \psi \geq \beta \psi. \quad (15)$$

**Proof** Let $K = \{ V \leq R \psi \}$, where constant $R$ will be fixed below. Lemma 4 gives
(14) and (15) with

$$\alpha = e^{a\tau} + \frac{1}{R}, \quad \beta = e^{b\tau}, \quad \theta = \frac{\xi}{a-b}.$$  

By choosing a sufficiently large $R$, one gets the desired inequality $\beta > \alpha$ (recall that
$b > a$ from Lemma 3). It remains to prove that $K$ is a compact set. To that end, let us
show that

$$\lim_{x \to 0} \frac{\psi(x, p)}{V(x, p)} = \lim_{x \to \infty} \frac{\psi(x, p)}{V(x, p)} = 0. \quad (16)$$

From Lemma 2 (i) and Lemma 4, we have

$$e^{(a+\zeta)\tau} \psi(x, p) = M_\tau \phi(x, p)$$

$$= \phi(x e^{a \tau}, p) e^{-\int_0^\tau B(x e^{a \tau}) ds}$$

$$+ \int_0^\tau e^{-\int_0^s B(x e^{a \xi}) ds} B(x e^{a \xi}) M_{s-x} \phi(\theta_0 x e^{a \xi}, 0) ds$$
\[ + \int_0^\tau e^{-\int_0^{s'} B(x e^{ap's})ds'} B(x e^{ap's}) M_{t-s} \phi(\theta_1 x e^{ap's}, 1) ds \]
\[ \leq \phi(x e^{ap't}, p) e^{-\int_0^t B(x e^{ap's})ds} \]
\[ + \int_0^\tau e^{-\int_0^{s'} B(x e^{ap's})ds'} B(x e^{ap's}) e^{(\alpha+\zeta)(\tau-s)} V(\theta_0 x e^{ap's}, 0) ds \]
\[ + \int_0^\tau e^{-\int_0^{s'} B(x e^{ap's})ds'} B(x e^{ap's}) e^{(\alpha+\zeta)(\tau-s)} V(\theta_1 x e^{ap's}, 1) ds. \]

Then using Assumption 1 and the definitions of \( \phi \) and \( V \), we obtain the limits (16). \( \square \)

**Remark 1** When the existence of eigenelements is known, the preceding inequalities allow to give some bounds on eigenvalues \( \lambda \). More precisely as \( \phi_0 : x \mapsto x \) is bounded by \( V \), \( Z_t(\phi_0) \) is integrable and from Lemma 2 and

\[ \alpha \phi_0 \leq A \phi_0 \leq \alpha \phi, \]

we find that \( (e^{\alpha t} Z_t(\phi_0))_{t \geq 0} \) and \( (e^{\alpha t} Z_t(\phi_0))_{t \geq 0} \) are sub and super-martingales. Integrating over \( \mathbb{E}_Y \), we find

\[ \alpha \leq \lambda \leq \alpha. \]

### 3.2.2 Mass ratio inequality

We now show the mass-ratio inequality that states that almost all cells will grow with almost the same speed. This corresponds to (Bansaye et al. 2019, Assumption (A4)) or Eq. (17).

**Lemma 6** For any compact set \( K \subset (0, +\infty) \), there exist \( d > 0 \) such that for all \( (x, m) \) and \( (y, p) \) in \( K \times \{0, 1\} \) and \( t \geq 0 \), one has

\[ M_t \psi(x, m) \geq d M_t \psi(y, p). \] (17)

**Proof** The proof is based on the approach developed in Cloez and Gabriel (2019). Let us fix \( (x, m) \) and \( (y, p) \) in \( K \times \{0, 1\} \) and some time

\[ T > \frac{1}{\alpha} \log \left( \frac{\max(K) - \min(K)}{\min(\theta_0, \theta_1)} \right). \]

There exists \( k \in \mathbb{N} \) and \( s \leq T \) such that

\[ y = \theta_0^k \theta_p e^{\alpha_0(T-s)ap's} x. \]
Let use the notation of the path-wise construction introduced in Sect. 2.1. Let \( v = 01 \ldots 1 \) with \( k \) times the digit 0. We have

\[
M_T f(x, m) \geq \mathbb{E}_{(x,m)} \left[ 1_{b_v \leq T, d_v \geq T + s} \sum_{u \in \mathcal{V}_T^v} f(X_T^u, P_T^u) \right],
\]

where \( \mathcal{V}_T^v \) is the set of individuals that are issued from individual \( v \) and alive at time \( T \). As \( Y_{b_v+s} = (y, p) \), the strong Markov property gives

\[
M_T f(x, m) \geq \mathbb{E}_{(x,m)} \left[ 1_{b_v \leq T, d_v \geq T + s} e^{b(T-b_v-s)} \right].
\]

Then for \( f = M_r \psi \) and \( r \geq 0 \) we have

\[
M_{r+T} \psi(x, m) \geq M_{r+T} \psi(y, p) \mathbb{E}_{(x,m)} \left[ 1_{b_v \leq T, d_v \geq T + s} e^{b(T-b_v-s)} \right].
\]

Now \( (x, y) \mapsto \mathbb{E}_{(x,m)} \left[ 1_{b_v \leq T, d_v \geq T + s} e^{b(T-b_v-s)} \right] \) (note that \( s \) and \( k \) hence \( v \) depend on \( y \)) is a continuous function on a compact set and then has a lower bound \( d_0 \). Hence the result holds for any \( t = r + T \geq T \). Now set \( t \leq T \). One has

\[
M_t \psi(y, p) \leq M_t V(y, p) \leq e^{(a+\zeta) t} V(y, p) \leq e^{(a+\zeta) t} \frac{\sup_{K} V}{\inf_{K} \psi} M_t \psi(x, m),
\]

hence the result also holds true for \( t \leq T \). \( \square \)

### 3.2.3 Doeblin minoration

In this subsection, we show the Doeblin minoration condition (Bansaye et al. 2019, Assumption (A3)). This assumption is an irreducibility and aperiodicity type assumption.

**Lemma 7** For any compact set \( K \subset (0, +\infty) \), there exist a probability measure \( \nu \) and \( c > 0 \) such that

\[
\forall (x, p) \in K \times \{0, 1\}, \quad \delta_{(x, p)} M_\tau \geq c \nu,
\]

where \( \tau \) is defined in Eq. (13).

**Proof** Let \( \delta < \min(K) \). We show that starting from one cell of size in \( K \), there is a (possibly very small but) positive probability that at time \( \tau \), there is at least one cell which size is uniformly distributed on \( I = [\delta', \delta] \), for any \( \delta' < \delta \) fixed.
Using the notation of the path-wise construction of Sect. 2.1, we have, for any non-negative measurable function $f$,

$$
\delta_{(x,p)} M_{\tau} f \geq \sum_{k \geq 0} \mathbb{E}_{(x,p)} \left[ f \left( e^{\alpha_0 b_0} e^{\alpha_0 (d_0 - b_0)} e^{\alpha_1 (\tau - d_0)} \theta_0 \theta_1^k x, 1 \right) \mathbf{1}_{b_{01\cdots 1} \leq \tau < d_{01\cdots 1}} \right] = \sum_{k \geq 0} \mathbb{E}_{(x,p)} \left[ f \left( e^{b_0 (\alpha_p - \alpha_0)} e^{d_0 (\alpha_0 - \alpha_1)} e^{\alpha_1 \tau} \theta_0 \theta_1^k x, 1 \right) \mathbf{1}_{b_{01\cdots 1} \leq \tau < d_{01\cdots 1}} \right],
$$

where $01 \cdots 1$ contains one 0 succeeded by $k$ times 1. The only source of randomness in the last expectation are $b_0, d_0, b_{01\cdots 1}$ and $d_{01\cdots 1}$. As $B$ is non-negative, the couple $(b_0, d_0)$ admits a density $(b, d) \mapsto \varphi_x (b, d)$ with respect to the Lebesgue measure. Moreover, this density is positive over the set

$$(b, d) \in (0, \tau)^2 \mid d > b,$$

and $(b, d, x) \mapsto \varphi_x (b, d)$ is continuous. One also has

$$\mathbb{P}_{(x,p)} (b_{01\cdots 1} \leq \tau < d_{01\cdots 1}) > 0,$$

and thus the latter probability is uniformly lower bounded over $K$ by some positive constant (depending on $k$). Then, by a change of variable (on $d_0$), for some constants $c_k, c > 0$, one obtains

$$\sum_{k \geq 0} \mathbb{E}_{(x,p)} \left[ f \left( e^{b_0 (\alpha_p - \alpha_0)} e^{d_0 (\alpha_0 - \alpha_1)} e^{\alpha_1 \tau} \theta_0 \theta_1^k x, 1 \right) \mathbf{1}_{b_{01\cdots 1} \leq \tau < d_{01\cdots 1}} \right] \geq \sum_{k \geq 0} c_k \int e^{\tau (\alpha_0 - \alpha_1)} e^{\alpha_1 \tau} \theta_0 \theta_1^k x \ f (u, 1) du \geq c \int \frac{f (u, 1)}{\delta - \delta'} du.$$

The result is proved by setting $v (f) = \int_1^\infty \frac{f (u, 1)}{\delta - \delta'} du$. \qed

**Remark 2** Lemma 7 is the only step where we used Assumption 1 (ii). When $\alpha_0 = \alpha_1$, Lemma 7 is not satisfied. We can see in the proof that the change of variable is no longer possible because $d \mapsto e^{d \times 0}$ is constant. As shown in Gabriel and Martin (2019); Bernard et al. (2019), in such a model, the distribution of the process is concentrated on a comb that depends on the initial conditions and then does not verify the Doeblin assumption. This is why eigenvalues exist (see Doumic Jauffret and Gabriel 2010) but the convergence does not hold (see Gabriel and Martin 2019).

We can now turn to the proof of Theorem 1.

**Proof of Theorem 1** Theorem 1 is now a consequence of (Bansaye et al. 2019, Theorem 2.1) and Lemmas 5, 6 and 7 that establish that (Bansaye et al. 2019, Assumptions A) hold in our context. \qed
4 Variations of the principal eigenvalue

This section is dedicated to the study of the variations of the principal eigenvalue $\lambda$ from Theorem 1 with explicit formulas in the special case $B(x) = x$. In particular, it contains the proof of Theorems 2 and 3. More specifically, recall the decompositions $\alpha_1 = \alpha + \epsilon, \alpha_0 = \alpha - \epsilon$ in a such way that $\alpha = (\alpha_0 + \alpha_1)/2$ and $\epsilon = (\alpha_1 - \alpha_0)/2, \theta = \theta_0$. Theorems 2 and 3 describe the variations of the map $u = (\alpha, \epsilon, \theta) \in (0, +\infty)^3 \times (0, 1) \mapsto \lambda$ and show that under some suitable assumptions, physiological symmetry is sub-optimal.

This section is organized as follows. We begin by proving the regularity of eigenelements and then prove Theorem 2 in Sect. 4.1. Then we give a general formula for the eigenmeasure in Sect. 4.2. Finally, we conclude this part by studying the special case where $B$ is the function $x \mapsto x$ and by proving Theorem 3 in Sect. 4.3.

4.1 Regularity of the eigenelements and proof of Theorem 2

We begin by proving that the eigenfunction $h$ defined in Theorem 1 is smooth enough.

Lemma 8 Under Assumption 1, $h$ is in $C^1$ and

$$Ah = \lambda h.$$  

Moreover, if $B$ is in $C^p$ then $h$ is in $C^{p+1}$.

Proof We begin by showing that $x \mapsto h(x, p)$ is continuous. From Lemma 2 (i), we have

$$e^{\lambda t} h(x, p) = h(xe^{\alpha_p t}, p) e^{\int_0^t B(xe^{\alpha_p s}) ds} + \int_0^t B(xe^{\alpha_p s}) e^{\int_0^s B(xe^{\alpha_p \tau}) d\tau} e^{\lambda(t-s)} (h(\theta_0 xe^{\alpha_p s}, 0) + h(\theta_1 xe^{\alpha_p s}, 1)) ds.$$  

Thus the function $\Xi$ defined by

$$\Xi(t, x) := \int_0^t B(xe^{\alpha_p s}) e^{\int_0^s B(xe^{\alpha_p \tau}) d\tau} e^{\lambda(t-s)} (h(\theta_0 xe^{\alpha_p s}, 0) + h(\theta_1 xe^{\alpha_p s}, 1)) ds$$

verifies

$$\Xi(t, x) \leq \int_0^t B(xe^{\alpha_p s}) e^{\int_0^s B(xe^{\alpha_p \tau}) d\tau} e^{\lambda(t-s)} (V(\theta_0 xe^{\alpha_p s}) + V(\theta_1 xe^{\alpha_p s})) ds,$$

as $h \leq V$. This function then tends to 0 when $t \to 0$. Then fixing $y > 0$ and choosing $t$ such that $y = xe^{\alpha_p t}$, we get

$$h(x, p) = h(y, p) a(x/y, x) + b(x/y, x)$$
where \( a, b \) are two functions satisfying
\[
\lim_{y \to x} a(x/y, x) = 1, \quad \lim_{y \to x} b(x/y, x) = 0.
\]

Thus \( x \mapsto h(x, p) \) is continuous as \( h(y, p) \) tends to \( h(x, p) \) as \( y \) tends to \( x \). The proof of the differentiation is similar. Indeed, we have
\[
\begin{align*}
|h(xe^{\alpha p t})| &= e^{\lambda t} h(x, p) - \int_0^t B(xe^{\alpha p s})e^{\int_0^s B(xe^{\alpha p r})dr} e^{\lambda (t-s)} (h(\theta_0 xe^{\alpha p s}) + h(\theta_1 xe^{\alpha p s}))ds.
\end{align*}
\]

As \( h \) is continuous, we can differentiate the right member of the right hand side. Then we obtain that \( h \) can be differentiated and one has (punctually) \( \mathcal{A}h = \lambda h \). This yields that \( \partial_x h \) is continuous and then \( h \) is \( C^1 \). The functional equation \( \mathcal{A}h = \lambda h \) and the inequality \( h \leq V \) permit to bound \( \partial_x h \) and then dominated convergence implies the last regularity property.

From now on, we study the eigenelements \((\lambda, \gamma, h)\) from Theorem 1 as functions of \( u \). We highlight this dependence on the parameter \( u \) by denoting \( \mathcal{A}_u \) instead of \( \mathcal{A} \) the extended generator of the semigroup \((M_t)_{t \geq 0}\) defined in (7) (the term extended generator is used from Lemma 2 and definitions in (Meyn and Tweedie 1993, Section 1.3), Davis (1993) which are closely related). It is defined for \( C^1 \) functions \( f \) by
\[
\mathcal{A}_u f(x, p) = (\alpha - (1 - 2p)\epsilon) x \partial_x f(x, p) + B(x)(f(\theta x, 0)
+ f((1 - \theta)x, 1) - f(x, p)).
\]

Similarly, we denote by \((\lambda_u, h_u, \gamma_u)\) the eigenelements of Theorem 1 and also use the notation \( \lambda(u) = \lambda_u \). Using Lemma 8, we can now prove the continuity of the eigenvectors with respect to the parameters \( u \).

**Lemma 9** If \( B \) is in \( C^1 \) then the maps \( u \mapsto \lambda_u, u \mapsto h_u \) and \( u \mapsto \gamma_h(h_u) \) are continuous on \((0, +\infty)^2 \times (0, 1)\) for every \( v \in (0, +\infty)^2 \times (0, 1) \).

**Proof** Let \((u_n)\) be any sequence converging to some fixed \( u \in (0, +\infty)^2 \times (0, 1) \). We will show that \((\lambda_{u_n}, h_{u_n})\) tends to \((\lambda_u, h_u)\) as \( n \) tends to infinity.

By Remark 1, \((\lambda_u)\) is bounded and by Lemma 8, \( h_{u_n} \) is in \( C^2 \). Using \( \mathcal{A}_{u_n} h_{u_n} = \lambda_{u_n} h_{u_n} \) and \( h_{u_n} \leq V \), we can bound \( h_{u_n}(x, p), \partial_x h_{u_n}(x, p) \) and \( \partial_x^2 h_{u_n}(x, p) \) locally in \( x \) uniformly in \( n \). Then Arzelà-Ascoli theorem ensures that \((\lambda_{u_n}, h_{u_n}, \partial_x h_{u_n})\) is relatively compact (for the compact convergence); see for instance (Royden and Fitzpatrick 1988, Problem 9 p. 210)). Let us consider any limiting value \((\lambda, h, g)\) of \((\lambda_{u_n}, h_{u_n}, \partial_x h_{u_n})\). Since the convergence is uniform (over compact sets from Arzelà-Ascoli theorem), we have \( g = \partial_x h \). Now, taking the (punctual) limit in \( h_{u_n} \leq V \) and \( \mathcal{A}_{u_n} h_{u_n} = \lambda_{u_n} h_{u_n} \) shows that \( h \leq V \) and \( \mathcal{A}_u h = \lambda h \). Then uniqueness of eigenelements shows that \( \lambda = \lambda_u \) and \( h = h_u \). Indeed, from Lemma 2, for any \( n \geq 1 \),
the process
\[
\left( Z_{t \wedge \tau_n}(h) - Z_0(h) - \int_0^{t \wedge \tau_n} Z_s(\lambda h) ds \right)_{t \geq 0} = \left( Z_{t \wedge \tau_n}(h) - Z_0(h) - \int_0^{t \wedge \tau_n} \lambda Z_s(h) ds \right)_{t \geq 0}
\]
is a martingale, where
\[
\tau_n = \inf\{ t \geq 0 | \exists u \in V_t, X_t^u \geq n \}.
\]
This sequence of stopping times allows to bypass the unbounded character of \( h \). Let us consider \( Z_0 = \delta x_0 \), for some \( x_0 > 0 \). We have \( \lim_{n \to \infty} \tau_n = +\infty \) because of Lemma 1 and that the support of \( Z_t \) is included into \([0, x_0 e^{m(x_0, x_1)t}]\). Then, using optional stopping time theorem and using dominated convergence (with \( Z_{t \wedge \tau_n}(h) \leq Z_t(V) \) and Lemma 4), we obtain
\[
M_t h - h - \int_0^t \lambda M_s h ds = 0,
\]
where \((M_t)_{t \geq 0}\) is the semigroup associated to parameter \( u \). This last equality means \( M_t h = e^{\lambda t} h \) and then, as \( h \leq V \), by uniqueness \( h = h_u \) and \( \lambda = \lambda_u \). We then conclude that
\[
\lim_{n \to \infty} (\lambda_{u_n}, h_{u_n}) = (\lambda_u, h_u).
\]
Finally \( \lim_{n \to \infty} \gamma_v(h_{u_n}) = \gamma_v(h_u) \) because of \( \lim_{n \to \infty} h_{u_n} = h_u, h_{u_n} \leq V \) and dominated convergence.

We can now differentiate the eigenvalue; namely we are now able to prove Theorem 2.

**Proof of Theorem 2** The proof is based on the equality
\[
\lambda_u - \lambda_v = \frac{\gamma_v((A_u - A_v) h_u)}{\gamma_v(h_u)}.
\]
(i) We note \( u = (\alpha + \delta, \epsilon, \theta) \) and \( v = (\alpha, \epsilon, \theta) \). We have
\[
\gamma_v((A_u - A_v) h_u) = \int_0^{+\infty} \int_{[0,1]} \delta x \partial_x h_u(x, p) \gamma_v(dx, dp),
\]
that allows us to conclude by dividing by \( \delta \), taking the limit \( \delta \to 0 \) and using Lemma 9. (ii) We note \( u = (\alpha, \epsilon + \delta, \theta) \) and \( v = (\alpha, \epsilon, \theta) \). We have
The eigenmeasure Lemma 10 and one last time.

4.2 Explicit eigenmeasure: a general formula

Before focusing on the special case \( B(x) = x \), let us establish here a link between the classical symmetric model with one population and \( \epsilon = 0 \), \( \theta = 1/2 \) (as in Michel 2006a; Hall and Wake 1989 for instance) and our asymmetric model. From now on and for the rest of the paper, we will assume that \( \epsilon = 0 \), that means that \( \alpha_0 = \alpha_1 \).

**Lemma 10** The eigenmeasure \( \gamma_u \) is given by

\[
\gamma_u(dx, dp) = \gamma_u^0(x)\delta_0(dp)dx + \gamma_u^1(x)\delta_1(dp)dx,
\]

with

\[
\gamma_u^0(x) = e^{-\int_1^x B(r) + 2a \frac{dr}{ar}} dr \int_0^{x/\theta} e^{\int_1^r B(r) + 2a \frac{dr}{ar}} \frac{1}{\alpha \theta y} B(y)U_u(y)dy,
\]

\[
\gamma_u^1(x) = e^{-\int_1^x B(r) + 2a \frac{dr}{ar}} dr \int_0^{x/(1-\theta)} e^{\int_1^r (1-\theta) B(r) + 2a \frac{dr}{ar}} \frac{1}{\alpha (1-\theta)y} B(y)U_u(y)dy,
\]

where \( U_u \) satisfies:

\[
\alpha xU_u(x) + (2\alpha + B(x))U_u(x) = \frac{1}{1-\theta} B \left( \frac{x}{1-\theta} \right) U_u \left( \frac{x}{1-\theta} \right) + \frac{1}{\theta} B \left( \frac{x}{\theta} \right) U_u \left( \frac{x}{\theta} \right),
\]

and \( \int_0^{+\infty} U_u(x)dx = 1. \)
Lemma 10 is a cornerstone in the proof of Theorem 3. In addition it is interesting by itself. Indeed, it can trivially be generalized for random divisions (random $\theta$) and using results of Hall and Wake (1989, 1990), we can exhibit some explicit formulas for $\gamma$ for explicit distributions of $\theta$ (for instance $\theta$ uniformly distributed in $(0, 1)$).

**Proof** In this proof we denote the eigenmeasure (by abuse of notation):

$$\gamma_u(dx, dp) = \gamma_u^0(x) dx \delta_0(dp) + \gamma_u^1(x) dx \delta_1(dp).$$

Since $u = (\alpha, 0, \theta)$ we have $\lambda_u = \alpha$. Using Eq. (18) with $f(x, 0) = 0$ and $f(x, 1) = f(x)$ we obtain:

$$\alpha \int_0^{+\infty} f(x) \gamma_u^1(x) dx = \int_0^{+\infty} A_u f(x, 1) \gamma_u^1(x) dx + \int_0^{+\infty} A_u f(x, 0) \gamma_u^0(x) dx$$

$$= \int_0^{+\infty} (\alpha x \partial_x f(x) + B(x)(f((1 - \theta)x) - f(x))) \gamma_u^1(x)$$

$$+ \int_0^{+\infty} B(x) f((1 - \theta)x) \gamma_u^0(x) dx$$

$$= \int_0^{+\infty} f(x) \left( -\alpha (\gamma_u^1(x) + x \partial_x \gamma_u^1(x)) - B(x) \gamma_u^1(x) \right)$$

$$+ \frac{1}{1 - \theta} B \left( \frac{x}{1 - \theta} \right) \gamma_u^1 \left( \frac{x}{1 - \theta} \right)$$

$$+ \frac{1}{1 - \theta} B \left( \frac{x}{1 - \theta} \right) \gamma_u^0 \left( \frac{x}{1 - \theta} \right)$$

We deduce that

$$\alpha x \partial_x \gamma_u^1(x) + (2\alpha + B(x)) \gamma_u^1(x) = \frac{1}{1 - \theta} B \left( \frac{x}{1 - \theta} \right) \gamma_u^1 \left( \frac{x}{1 - \theta} \right)$$

$$+ \frac{1}{1 - \theta} B \left( \frac{x}{1 - \theta} \right) \gamma_u^0 \left( \frac{x}{1 - \theta} \right).$$

We write $\mathcal{U}_u(x) = \gamma_u^1(x) + \gamma_u^0(x)$ and we conclude by solving the previous equation. In particular the measures $(\gamma_u^0, \gamma_u^1)$ are absolutely continuous with respect to the Lebesgue measure. □

**Lemma 11** We have, for every $x > 0$,

$$h(x, 0) = h(x, 1) = x.$$

**Proof** By linearity $h = h(\cdot, 0) + h(\cdot, 1)$ is a positive eigenvector of the classical fully symmetric equation (one population, $\epsilon = 0$, $\theta = 1/2$) and then, by uniqueness, $h : x \mapsto x$; see (Doumic Jauffret and Gabriel 2010) for details. Now, for $i \in \{0, 1\}$ and $x > 0$, we set

$$g_i(x) = g(x, i) = -g(x, 1 - i) = h(x, i) - h(x) = -h(x, 1 - i).$$
We will show \( g_i = 0 \). On the one hand, as \( h(\cdot, i) \geq 0 \), \( g(\cdot, i) = -g(\cdot, 1 - i) \), and \( h(0) = 0 \), we necessarily have \( g(0, i) = 0 \). Using the eigenvector equations, we have

\[
\alpha g_0(x) = \alpha x g_0'(x) + B(x)(g_0(\theta_0 x, 0) - g_0(\theta_1 x, 1) - g_0(x)),
\]

and by integration,

\[
g_0(x) = \int_0^x \frac{(\alpha + B(u))g_0(u) + B(u)g_0(\theta_1 u) - B(u)g_0(\theta_0 u)}{\alpha u} \, du.
\]

Then, there exists \( C > 0 \), such that for any \( \delta \in [0, 1] \),

\[
\zeta(\delta) := \sup_{x \in [0, \delta]} \frac{|g_0(x)|}{\sqrt{x}} \leq C \sqrt{\delta} \zeta(\delta),
\]

where \( C \) does not depend on \( \delta \) nor \( g_0 \) but only on \( \sup_{[0,1/\min(\theta_0, \theta_1)]} B \) and \( \alpha \). Consequently if \( C \sqrt{\delta} < 1 \) then \( \zeta(\delta) = 0 \). Thus, \( g(x) = 0 \) on \([0, \delta]\). Iterating this argument, we find \( g_0(x) = 0 \) for all \( x > 0 \).

\[
\boxdot
\]

4.3 The particular case \( B(x) = x \)

In all this section, we assume that \( B \) is the identity function \( x \mapsto x \), which verifies all our assumptions. To prove our main result Theorem 3, we give some explicit formulas in this special case. More precisely Theorem 4 gives an explicit expression of \( \gamma \), Lemma 12 its moments and Lemma 13 its logarithmic moments. The section finishes by the proof of Theorem 3. All these explicit results are not necessary for proving Theorem 3 but have an interest by themselves, to derive statistical estimators by the method of moments for example. A reader who is only interested in the proof of Theorem 3 can therefore directly jump to the end of this section.

We recall that, in what follows, we already only consider the physiologically symmetric case \( \epsilon = 0 \), that is \( \alpha_0 = \alpha_1 \). We are interested in the local behavior of \( \lambda \) at \( u = (\alpha, 0, \theta) \). By Lemma 10, to explicit \( \gamma \) at this point, it is enough to study the eigen-measure \( U_u \) of the physiologically symmetric model (namely the eigen-measure of the model developed in Michel (2006a), Hall and Wake (1989) which is indeed a solution of (20)).

Theorem 4 The solution \( U_u \) of Eq. (20) is given by:

\[
U_u(x) = \frac{K}{\alpha x^2} \sum_{n=0}^{+\infty} \mathbb{E} \left[ \prod_{k=1}^{n} \left( \frac{1}{1 - \frac{1}{Q_k} 1/Q_n} \exp \left( -\frac{1}{Q_n} \frac{x}{\alpha} \right) \right) \right],
\]

where \( K \) is a normalizing constant, \( Q_k = \prod_{j=1}^{k} \Theta_j \) where \((\Theta_j)_{j \geq 1}\) is an i.i.d. sequence of random variables with distribution \( \mathbb{P}(\Theta = \theta) = \theta \) and \( \mathbb{P}(\Theta = 1 - \theta) = 1 - \theta \).
This result generalizes part of the results in Hall and Wake (1989, 1990). As we will see in its proof, using (Guillemin et al. 2004), we can also simplify this expression in some special cases. However, we will not use this explicit expression to calculate the moments of $\gamma$.

**Proof** We have

$$\left(\alpha x U_u(x)\right)’ + (x + \alpha)U_u(x) = \frac{1}{\theta} B\left(\frac{x}{\theta}\right) U_u\left(\frac{x}{\theta}\right) + \frac{1}{1-\theta} B\left(\frac{x}{1-\theta}\right) U_u\left(\frac{x}{1-\theta}\right).$$

By multiplying by $x$ we obtain

$$\alpha x^2 U_u'(x) + 2\alpha x U_u(x) + x^2 U_u(x) = \frac{x}{\theta} B\left(\frac{x}{\theta}\right) U_u\left(\frac{x}{\theta}\right) + \frac{x}{1-\theta} B\left(\frac{x}{1-\theta}\right) U_u\left(\frac{x}{1-\theta}\right).$$

We define $Z(x) = Ax^2 U_u(x)$ where $A$ is a normalising constant; we obtain:

$$\alpha Z'(x) + Z(x) = Z\left(\frac{x}{\theta}\right) + Z\left(\frac{x}{1-\theta}\right).$$

We now take the Laplace transform $\mathcal{L}(Z(z)) = \int_{0}^{+\infty} e^{-zx} Z(x) dx$ to obtain

$$\mathcal{L}(Z(z)) = \frac{1}{\alpha z + 1} (\theta \mathcal{L}(\theta z) + (1-\theta) \mathcal{L}((1-\theta)z)). \quad (22)$$

Let $I$ be a random variable with distribution $Z$. Equation (22) is equivalent to the following equality in distribution:

$$I \overset{d}{=} E + \Theta I,$$

where $P(\Theta = \theta) = \theta$, $P(\Theta = 1-\theta) = 1-\theta$ and $E$ is exponentially distributed with parameter $1/\alpha$. This equation was studied in particular in (Guillemin et al. 2004, Section 3). By using (Guillemin et al. 2004, Proposition 5) with the following notation, coming from their setting: $\beta \in (0, 1)$, $X = \ln(\Theta) / \ln(\beta)$ and let $E_0 \sim \mathcal{E}(1)$, we obtain

$$\frac{I}{\alpha} \overset{d}{=} E_0 + \beta^x \frac{I}{\alpha},$$

which yields Eq. (21).

Let us continue now with two lemmas on the calculation of moments of the eigenmeasure.

**Lemma 12** Set $m_p = \int_{0}^{+\infty} x^p U_u(x) dx$, for $p \in \mathbb{N}$. We have

$$m_0 = 1, \quad m_1 = \alpha, \quad m_2 = \frac{-\alpha^2}{\theta \log(\theta) + (1-\theta) \log(1-\theta)}.$$
and for all $p > 2$:

$$m_p = \frac{-\alpha^2}{\theta \log(\theta) + (1 - \theta) \log(1 - \theta)} \prod_{q=2}^{p-1} \left( \frac{\alpha(q - 1)}{1 - \theta q - (1 - \theta)q} \right).$$

**Proof** Since $U_u$ is the density of a probability measure, we have $m_0 = 1$. Now, let us define the generator

$$B f(x) = \alpha x f'(x) + B(x)(f(x\theta) + f((1 - \theta)x) - f(x)). \quad (23)$$

Then, for every $C^1$ function we have

$$\int_0^\infty B f(x) U_u(x) dx = \alpha \int_0^\infty U_u(x) f(x) dx.$$

Using now, $f : x \mapsto 1$ gives

$$\alpha = \alpha \int_0^{+\infty} U_u(x) dx = \int_0^{+\infty} B(x) U_u(x) dx = \int_0^{+\infty} x U_u(x) dx$$

and so $m_1 = \alpha$. Now for $p \geq 1$ and $f : x \mapsto x^p$, we have

$$\alpha m_p = \alpha \int_0^{+\infty} x^p U_u(x) dx$$

$$= \int_0^{+\infty} U_u(x) \left( \alpha px^{p-1} + x(\theta^p x^p + (1 - \theta)^p x^p - x^p) \right) dx$$

$$= \alpha(p - 1) m_p + m_{p+1}(\theta^p + (1 - \theta)^p - 1),$$

which yields for $p > 1$,

$$m_{p+1} = \frac{\alpha (p - 1)}{1 - \theta^p - (1 - \theta)^p} m_p. \quad (24)$$

To complete the demonstration, we need to calculate $m_2$. However, Eq. (24) holds for real numbers $p$, and then taking the limit $p \to 1$ ends the proof. $\square$

**Lemma 13** For all $p \in \mathbb{N}$, let us define $l_p = \int_0^{+\infty} x^p \log(x) U_u(x) dx$. Then we have

$$l_0 = 1 + 2(\log(\theta) + \log(1 - \theta)), \quad l_1 = \log(\theta)\alpha + \log(1 - \theta)\alpha,$$

and for all $p \in \mathbb{N}^*$,

$$l_{p+1} = \frac{1}{\theta^p + (1 - \theta)^p} \left( \alpha(1 - p) l_p - m_{p+1}(\theta^p \log(\theta) + (1 - \theta)^p \log(1 - \theta)) - \alpha m_p \right).$$
**Proof** The proof is similar to that of Lemma 12 using \( f : x \mapsto x^p \log(x) \). \( \square \)

We now end the paper with the proof of Theorem 3.

**Proof of Theorem 3** Let \( u = (\alpha, 0, \theta) \), then we have \( \lambda(u) = \alpha \) and \( h_u(x, m) = x \). By Theorem 2 (ii), we obtain

\[
\frac{\partial \lambda(u)}{\partial \epsilon} = \frac{\partial \lambda(\alpha, 0, \theta)}{\partial \epsilon} = \int_0^{+\infty} x (\gamma^1_u(x) - \gamma^0_u(x)) dx.
\]

By Lemma 10, we have

\[
\int_0^{+\infty} x \gamma^1_u(x) dx = \int_0^{+\infty} \int_0^{+\infty} e^{-\frac{x-1}{\alpha}} x \int_0^{x/(1-\theta)} (1-\theta) z e^{\frac{(1-\theta)z-1}{\alpha}} B(z) U_u(z) dz dx
\]

\[
= \int_0^{+\infty} \int_0^{+\infty} e^{-\frac{x-1}{\alpha}} x (1-\theta) z e^{\frac{(1-\theta)z-1}{\alpha}} B(z) U_u(z) dx dz
\]

\[
= \int_0^{+\infty} G(z(1-\theta)) B(z) U_u(z) dz,
\]

where

\[
G(t) = te^\frac{t-1}{\alpha} \int_t^{+\infty} e^{-\frac{x-1}{\alpha}} x dx.
\]

So we obtain

\[
\int_0^{+\infty} x (\gamma^1_u(x) - \gamma^0_u(x)) dx = \int_0^{+\infty} (G(z(1-\theta)) - G(z\theta)) B(z) U_u(z) dz. \tag{25}
\]

We conclude by studying the monotonicity of \( G \). We rewrite

\[
G(t) = te^\frac{t-1}{\alpha} \int_t^{+\infty} e^{-\frac{x-1}{\alpha}} x dx,
\]

and we obtain

\[
G'(t) = \left(1 + \frac{t}{\alpha}\right) e^\frac{t-1}{\alpha} \int_t^{+\infty} e^{-\frac{x-1}{\alpha}} x dx - 1.
\]

We apply Jensen inequality to the strictly convex function \( x \mapsto 1/x \) and the measure \((\alpha e^{-t/\alpha})^{-1} 1_{(t, +\infty)}(x) e^{-\frac{x}{\alpha}} dx\), that allows to obtain

\[
(\alpha e^{-t/\alpha})^{-1} \int_t^{+\infty} e^{-\frac{x}{\alpha}} x dx > \frac{1}{(\alpha e^{-\frac{1}{\alpha}})^{-1} \int_t^{+\infty} x e^{-\frac{x}{\alpha}}}
\]

\[
= \frac{1}{(\alpha e^{-\frac{1}{\alpha}})(\alpha e^{-\frac{1}{\alpha}} + \alpha^2 e^{-\frac{1}{\alpha}})} = \frac{1}{t + \alpha}.
\]
It comes that
\[
\int_t^{+\infty} e^{-\frac{x}{\alpha}} \frac{e^{-\frac{x}{\alpha}}}{x} \, dx > \alpha e^{-\frac{t}{\alpha}} + \alpha,
\]
and that \(G'(t) > 0\). Equation (25) allows us to conclude the proof. \(\Box\)

**Remark 3** Mimicking the proof of Theorem 3, we can show that, for general division rate \(B\), if
\[
G : u \mapsto ue^{\int_1^u \frac{B(r)}{ar} \, dr} \int_u^{+\infty} e^{-\int_1^x \frac{B(r)}{ar} \, dx} \, dx
\]
is increasing (which seem at least right for \(x^p\) for \(p \leq 1\)) then the conclusion of Theorem 3 also holds true.

## 5 Discussion

The aim of this paper was to study the long-time behavior of an asymmetric size-structured growth-fragmentation branching process for cell population dynamics. We showed in Theorem 1 that the process exhibits a Malthusian behavior, in the sense that the global population size grows exponentially fast and that the trait distribution of individuals converges to some stable distribution. We then investigated the fluctuations of the growth rate with respect to the parameters guiding morphological and physiological asymmetry in Theorem 2. Finally, in the very special case where the division rate is the identity function with respect to the length of the cell, we exhibited in Theorem 3 that in the presence of morphological asymmetry, physiological symmetry is sub-optimal in the sense that a small deviation from physiological asymmetry in the appropriate direction yields an increase of the growth rate.

This work was motivated by biological experiments (Proenca et al. 2018; Stewart et al. 2005; Wang et al. 2010) that detected and quantified a significant physiological asymmetry in cell division for *E. coli*. More specifically, they highlighted that the old-pole cells have smaller elongation rates than the new-pole cells, which corresponds to \(\alpha_0 < \alpha_1\) or equivalently \(\epsilon > 0\) in our model. A natural question is then to understand how this asymmetry influences the dynamics and try to understand why *E. coli* dynamics parameters deviate from symmetry.

Our study is a first step to shed some light on these problems. On the one hand, we show, through Theorem 1, that taking into account the asymmetry discovered by Proenca et al. (2018); Stewart et al. (2005); Wang et al. (2010), allows to avoid abnormal oscillations in the asymptotic behavior of classical division models (Bernard et al. 2019; Gabriel and Martin 2019). Asymmetry is therefore more realistic from an empirical and theoretical point of view. On the other hand, we were successful in proving that physiological symmetry is sub-optimal in the presence of morphological asymmetry, for a special class of parameters. However, we obtained that if \(\theta = \theta_0 > 1/2\) then the growth rate of the population locally increases when \(\alpha_0 > \alpha_1\), and locally
decreases when $\alpha_0 < \alpha_1$ which may not be biologically realistic. To better understand the Darwinian optimality of asymmetry, it should be successful to generalize Theorem 3 in a more general framework. For instance, assuming a linear division rate $B$ may be too restrictive to be realistic; see for instance (Doumic et al. 2015). Also, perhaps we should consider even more general models such as size-age models, incremental models (Taheri-Araghi et al. 2015; Hall et al. 1991; Sinko and Streifer 1967).... Unfortunately such generalizations seem very challenging at this stage from the mathematical point of view. Finally, to biologically justify the asymmetry from our approach, one must consider that, from an evolutionary point of view, cells naturally optimize their growth rates. This assumption may not be valid because, for example, there may be physical barriers that do not allow cells to have all the range of parameters studied here. These aspects are also beyond the scope of this paper but deserve attention.

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