RUBIO DE FRANCIA EXTRAPOLATION THEOREM AND RELATED TOPICS IN THE THEORY OF ELLIPTIC AND PARABOLIC EQUATIONS. A SURVEY

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Abstract. We give a brief overview of the history of the Sobolev mixed norm theory of linear elliptic and parabolic equations and the recent development in this theory based on the Rubio de Francia extrapolation theorem. A self contained proof of this theorem along with other relevant tools of Real Analysis are also presented as well as an application to mixed norm estimates for fully nonlinear equations.

1. Introduction

The goal of this paper is to show how the extrapolation theorem of Rubio de Francia combined with more or less standard techniques from the theory of partial differential equations allows one to get estimates for solutions in Sobolev spaces with mixed norms “for free”. We give a complete proof of this theorem in spite of the fact that it can be found in a few articles and books. The reason is that, if you try to learn how this extrapolation theorem is proved by reading books or articles on real or harmonic analysis, you will be buried under mountains of very beautiful and fascinating results and it is not an easy task to sort out which only very few of them are actually needed to prove the extrapolation theorem. We collected these few with complete proofs including the extrapolation theorem on twelve pages.

The author’s interest in equations in spaces with mixed norms arose in connection with stochastic partial differential equations where these norms and embedding theorems allow one to obtain useful information on solutions (see [35]). See also [25] and [47] for applications to the Navier-Stokes equations and [50] and [51] for applications to other problems. The spaces used in [35] were of type $L_q(\mathbb{R}, H^p_{\mu,\theta}(\mathbb{R}^d_+)), q \geq p$, where $H^p_{\mu,\theta}(\mathbb{R}^d_+)$ is an $H^p_\mu$ type space with weights allowing certain blow up near the boundary of the half space $\mathbb{R}^d_+$. In [35] there is a restriction $q \geq p \geq 2$ dictated by the presence of stochastic terms. In [36] stochastic terms from [35] are eliminated and $p, q$ are arbitrary in $(1, \infty)$. The range of weights in [36], dictated by the applications to stochastic partial differential equations, was later extended to the optimal one in [34].

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If there are no stochastic terms and the coefficients are independent of time, then we are dealing with the heat equation. As far as we know, the first explicitly stated result on a priori estimates for the heat equation in spaces \( L_q([0, T], W^2_p(\mathbb{R}^d)) \) with \( p, q \in (1, \infty) \) appeared in [47] with a somewhat sketchy proof based on the Calderón-Zygmund theorem for operator-valued kernels which at that time was only published for locally summable kernels and formally was not applicable. Complete proofs for more general case of the coefficients which depend only on \( t \) appeared in [37] (one of the first papers with mixed norms and the coefficients measurable in \( t \)).

Probably the earliest mentioning of the results of that kind appeared in [49]. The proof in [47] uses the classical result for \( q = p \) using which the above mentioned a priori estimate from [47] can be easily obtained from a general result of [5] proved (and stated earlier in [49]) for abstract analytic semigroups. This a priori estimate is also contained as a particular case in Theorem 2.1 of [25] which is proved on the basis of a general theorem on invertibility of the sum of two resolvent-commuting operators (see also [23], [24]).

Before that the case \( p \in (1, \infty), q = 1 \) was considered in [45]. Several proofs and generalizations of the result from [47] were given in [36] where equations with measurable in time coefficients in \( \mathbb{R}^d \) and \( \mathbb{R}^d_+ \) are treated and in the latter case again weights are introduced.

In [37] a systematic approach to parabolic equations in \( \mathbb{R}^{d+1} \) in spaces \( L_q(\mathbb{R}, W^2_p(\mathbb{R}^d)) \) is given on the basis of the Calderón-Zygmund theorem. The coefficients are supposed to be independent of the space variable and measurable with respect to the time variable. If \( p = \infty \), it is natural to replace the space \( L_q(\mathbb{R}, W^2_p(\mathbb{R}^d)) \) with \( L_q(\mathbb{R}, C^{2+\alpha}(\mathbb{R}^d)) \) and ask if the results still hold. In [38] we prove that the answer to this question is indeed positive. The approach in [38] is quite different from [5] and is based on simple estimates of the heat potentials and well-known properties of the Hardy-Littlewood maximal function.

It is also worth mentioning the articles [7] and [30] containing a quite extensive references where similar issues are treated in a very general setting of homogeneous spaces and analytic semigroups. The semigroup approach or the approach based on resolvent-commuting operators seem to produce no results for parabolic equations with coefficients measurable in time. Good sources of references and discussions of methods and obtained results are also found in [9], [28], [50], and [51].

Up until recently, excluding [38], [40], [50], and [51] and a few references therein, in most other papers concerning \( L_q(L_p) \)-spaces the methods heavily depend on the properties of the elliptic part of the operators, which are supposed to be independent of \( t \) and have well behaving resolvent or generate a “good” semigroup. However, in [1] (also see references therein) there is a general theorem allowing one to treat the case when the coefficients are continuous in \( t \). These restrictions exclude parabolic equations with coefficients measurable or even VMO in \( t \) (even if they are independent of \( x \),...
the case considered in [38]). In particular, in [28] the authors only consider equations with VMO coefficients independent of time, although combining their results with [1] would include equations with coefficients continuous in $t$. By the way, in the particular case that $q = p$ this also does not allow one to cover the results of [2], where the coefficients are in $VMO(\mathbb{R}^{d+1})$.

Speaking about the case $q = p$, which is not our main interest here, it is worth saying that there is a quite extensive literature about linear equations and systems with VMO coefficients. The interested reader can consult [3], [4], [9], [26], [28], [46], [48], [11, 12, 13, 14], and the references therein.

In what concerns the mixed norm results with time dependent coefficients, for quite a long time the power of summability with respect to the time variable was assumed to be greater than that with respect to the space variables. Of course, in divergence form equations such a restriction was not necessary due to duality arguments. A remarkable step forward in this annoying problem occurred in 2017 when the authors of [20] noticed that one can use the Muckenhoupt $A_p$-weights and the Rubio de Francia extrapolation theorem to get rid of the restriction on the powers of summability even in the case of equations with coefficients measurable in time variable.

Actually, the Muckenhoupt weights were already used before for similar purposes. It seems that [27] was one of the first if not the first article using the Muckenhoupt weights and the Rubio de Francia extrapolation theorem in the theory of evolution or elliptic equations in spaces with mixed norms. The authors deal with higher-order parabolic equations with time independent continuous coefficients and use the approach based on the so-called $R$-boundedness. In [29] the solvability is established of the same type of equations, with time independent coefficients belonging to VMO, in spaces with $A_p$-weight as the underlying measures. Again the authors use the Muckenhoupt weights and the Rubio de Francia extrapolation theorem to check the $R$-boundedness of certain families of operators.

In [20] the authors consider parabolic higher-order equations with the coefficients measurable in time and uniformly continuous in the space variables. They derive $L_p(L_q)$-estimates for arbitrary $p, q \in (1, \infty)$. The $A_p$-weights and Rubio de Francia extrapolation theorem are used again to check the $R$-boundedness of certain operators but also to use Mikhlin’s multiplier theorem and extrapolate with respect to $t$, which was a natural (for people who knew the extrapolation theorem) but absolutely new (for those, like the author, unfamiliar with weights and extrapolation) idea. The starting point in [20] is the case $q = p$ where Mikhlin’s theorem is used for equations with constant coefficients. The results of [20] are generalized in [21] for parabolic systems satisfying the Legendre-Hadamard condition.

In the author’s opinion even more important step forward was done in [15], where, as in many cases, the general ideology based on the Fefferman-Stein theorem ([18]) is more or less standard albeit its implementations vary. The starting point consists of pointwise estimates of the sharp functions of the derivatives of solutions, which together with an appropriate version of
the Fefferman-Stein theorem (proved in [15]) allows one to avoid using singular integrals, Mikhlin’s theorem, \( R \)-boundedness, \( H^\infty \)-calculus, and some other notions and tools from functional analysis. In [20], after a semigroup corresponding to the operators with constant coefficient is constructed in \( L_q \)-spaces with weights, the authors use their special technique based on the fact that \((-\Delta)^{m/2}\) has a bounded \( H^\infty \)-calculus of angle \(< \pi/2\) and \( L_q(\mathbb{R}^d) \) has finite cotype and also the \( R \)-boundedness of special kind of operators, proved earlier in [19], to obtain their main result. In the proof of the \( R \)-boundedness they used the extrapolation theorem.

The equations in [15] have the coefficients measurable in \( t \) and almost VMO in \( x \), much more general than in [20]. Furthermore, the authors of [15] considered divergence and non-divergence equations whose coefficients are measurable in one spatial variable.

The approach developed in [15] turned out to be applicable even to fully nonlinear equations. See [17], where the first to date weighted and mixed-norm Sobolev estimates are presented for fully nonlinear elliptic and parabolic equations in the whole space under a relaxed convexity condition with almost VMO dependence on space-time variables. The corresponding interior and boundary estimates are also obtained.

The rest of the paper is organized as follows. In the next section we present with almost all details the proof of mixed-norm estimates for the Laplacian. This proof is based on the results in Section 3 about partitions and stopping times, Section 4 about Muckenhoupt weights, Section 5 containing the proof of Hardy-Littlewood maximal function theorem in \( L_p \)-spaces with weights, Section 6 devoted to the proof of the Rubio de Francia extrapolation theorem, and Section 7 about a generalized Fefferman-Stein theorem. In the concluding Section 8 we state without proof a result from recent paper [17] about mixed norm estimates for fully nonlinear equations.

2. Illustration

Let \( \mathbb{R}^d \) be a \( d \)-dimensional Euclidean space with \( d \geq 2 \) of points \( x = (x_1, \ldots, x_d) = (x_1, x') \), where \( x' \in \mathbb{R}^{d-1} \). We consider Laplace’s equation

\[
\Delta u = f. \tag{2.1}
\]

Squaring both sides of (2.1) and integrating over \( B_r := \{ x : |x| < r \} \) one easily shows (see, for instance, Exercise 1.1.5 in [41]) that, if \( u \) is smooth in \( B_r \) and \( u = 0 \) on \( \partial B_r \), then

\[
\sum_{i,j=1}^d \| D_{ij} u \|_{L_2(B_r)}^2 \leq \| f \|_{L_2(B_r)}^2, \tag{2.2}
\]

where \( D_{ij} = D_i D_j, D_i = \partial / \partial x_i \). Then one proves (see, for instance, Exercise 1.3.23 in [41]) that for any \( f \in L_2(B_r) \) there is a unique \( u \in W_2^2(B_r) \)
vanishing on $\partial B_r$, such that $\Delta u = f$ in $B_r$. Moreover, it holds that
\[
r^{-4}\|u\|_{L^2(B_r)}^2 + r^{-2}\sum_i \|D_i u\|_{L^2(B_r)}^2 + \sum_{i,j} \|D_{ij} u\|_{L^2(B_r)}^2 \leq 5\|f\|_{L^2(B_r)}, \tag{2.3}
\]
where 5 is certainly not the best constant.

By using this fact and considering $u\zeta$, where $\zeta$ is a cut-off function one proves (see Exercise 2.4.6 of [41] in which take $u$ independent of $t$) that, if $u \in W^2_2(B_R)$ is harmonic in $B_R$, then for any $r \in (0,R)$
\[
\|D^2 u\|_{L^2(B_r)} \leq N(d)(R-r)^{-2}\|u\|_{L^2(B_R)}, \tag{2.4}
\]
where and below by $D^m u$ we mean the collection of all $m$-th order partial derivatives of $u$ and by it $L_p$-norm we mean the $L_p$-norm of $|D^m u|$, the latter being the Euclidean norm of $D^m u$ as a vector in an appropriate Euclidean space. One can iterate this estimate by applying it to the derivatives of $u$ , which are harmonic in $B_R$ along with $u$. Then one sees that, for any $m \geq 1$,
\[
\|D^m u\|_{L^2(B_r)} \leq N(d,r,R,m)\|D^2 u\|_{L^2(B_R)}.
\]
For $m$ large enough, by Sobolev embedding theorems it follows that
\[
\max_{B_1} |D^3 u| \leq N(d)\|D^2 u\|_{L^2(B_2)} \tag{2.5}
\]
if $u \in W^2_2(B_2)$ is harmonic in $B_2$.

By using the notation
\[
\int_A f \, dx = \frac{1}{|A|} \int_A f \, dx,
\]
where $|A|$ is the volume of $A$, we infer that for the functions $u$ as in (2.5) and $r \in (0,1)$ we have
\[
\int_{B_r} \int_{B_r} |D^2 u(x) - D^2 u(y)|^2 \, dx \, dy \leq r^2 N(d) \int_{B_2} |D^2 u|^2 \, dx.
\]
By using scaling we obtain the following important estimate (we have just repeated in the simplest situation the proof of Theorem 4.2.6 of [41]).

**Lemma 2.1.** Let $\nu \geq 2$ and $r \in (0,\infty)$ be some constants and let $u \in W^2_2(B_{\nu r})$ be harmonic in $B_{\nu r}$. Then with a constant $N = N(d)$ we have
\[
\int_{B_r} \int_{B_r} |D^2 u(x) - D^2 u(y)|^2 \, dx \, dy \leq N\nu^{-2} \int_{B_{\nu r}} |D^2 u|^2 \, dx. \tag{2.6}
\]

Now we have an analog of Theorem 4.3.1 of [41].

**Theorem 2.2.** Let $\nu \geq 2$ and $r \in (0,\infty)$ be some constants and let $u \in W^2_2(B_{\nu r})$. Then with a constant $N = N(d)$ we have
\[
\int_{B_r} \int_{B_r} |D^2 u(x) - D^2 u(y)|^2 \, dx \, dy \leq N\nu^d \int_{B_{\nu r}} |\Delta u|^2 \, dx + N\nu^{-2} \int_{B_{\nu r}} |D^2 u|^2 \, dx. \tag{2.7}
\]
Proof. Let \( v \in W^2_2(B_{\nu r}) \) be the solution of \( \Delta v = \Delta u \), that equals zero on \( \partial B_{\nu r} \). By taking into account (2.3), we get

\[
\int_{B_r} \int_{B_r} |D^2 v(x) - D^2 v(y)|^2 \, dx \, dy \leq 4 \int_{B_r} |D^2 v|^2 \, dx
\]

\[
\leq 4 \nu^d \int_{B_{\nu r}} |D^2 v|^2 \, dx \leq N \nu^d \int_{B_{\nu r}} |\Delta u|^2 \, dx.
\]

(2.8)

On the other hand, \( w := u - v \) is harmonic and by Lemma 2.1

\[
\int_{B_r} \int_{B_r} |D^2 w(x) - D^2 w(y)|^2 \, dx \, dy \leq N \nu^{-2} \int_{B_{\nu r}} |D^2 w|^2 \, dx
\]

\[
\leq N \nu^{-2} \int_{B_{\nu r}} |D^2 w|^2 \, dx + N \nu^{-2} \int_{B_{\nu r}} |D^2 v|^2 \, dx,
\]

which combined with (2.8) leads to (2.7) and proves the theorem.

Then let \( B \) be the collection of open balls in \( \mathbb{R}^d \) and introduce the notation

\[
f^#(x) = \sup_{B \in B, B \ni x} \int_B \int_B |f(x') - f(x'')| \, dx' \, dx'',
\]

\[
Mf(x) = \sup_{B \in B, B \ni x} \int_B |f(y)| \, dy.
\]

The function \( f^# \) is called the sharp function of \( f \) and \( Mf \) is its maximal function.

In light of Hölder’s inequality and the possibility of changing the origin Theorem 2.2 implies the following estimate, the direct analogs of which play a crucial role in the theory of linear and fully nonlinear equations with VMO main coefficients.

**Theorem 2.3.** There is a constant \( N = N(d) \) such that for any \( \nu \geq 2 \) and \( u \in W^2_{2,\text{loc}}(\mathbb{R}^d) \) we have

\[
(D^2 u)^# \leq N \nu^{d/2}(M(|\Delta u|^2))^{1/2} + N \nu^{-1}(M(|D^2 u|^2))^{1/2}.
\]

(2.9)

The approach based on estimates involving sharp and maximal functions was first suggested in [39].

We can raise both sides of (2.9) to the power \( p > 2 \) and then integrate through by using any measure \( w(dx) \), but, if this measure has a density \( w \in A_{p/2} \) (see the definition of \( A_p \) in Section 4), then, on the one hand, the Hardy-Littlewood theorem turns out to hold for such a measure (see Theorem 5.3) and one can estimate the integrals of \( (M(|\Delta u|^2))^{p/2} \) and \( (M(|D^2 u|^2))^{p/2} \) just with the integrals of \( |\Delta u|^p \) and \( |D^2 u|^p \) against \( w(dx) \), respectively. On the other hand the Fefferman-Stein theorem turns out to be true (see Theorem 7.4), which allows one to estimate the integral of \( |D^2 u|^p \) through the integral of the \( p \)th power of its sharp function. Finally, having \( \nu^{-1} \) on the right allows us to absorb what is coming from the last term in (2.9) into the integral of \( |D^2 u|^p \) (provided that it is finite) and leads to the following in which \([u]_{A_{p/2}}\)
is the $A_{p/2}$-constant of the $A_{p/2}$-weight $w$, which affects both constants in the Hardy-Littlewood and the Fefferman-Stein theorems.

**Theorem 2.4.** Let $p > 2$ and $w$ be an $A_{p/2}$-weight on $\mathbb{R}^d$. Then there is a constant $N = N(d, p, [w]_{A_{p/2}})$ such that for any $u \in W^2_p(\mathbb{R}^d)$ we have

$$
\int_{\mathbb{R}^d} |D^2 u|^p w \, dx \leq N \int_{\mathbb{R}^d} |\Delta u|^p w \, dx. \quad (2.10)
$$

Of course, raising (2.9) to any power $p > 2$ and applying the usual versions of the Hardy-Littlewood and the Fefferman-Stein theorems rather than their weighted counterparts leads to the classical estimate

$$
\|D^2 u\|_{L_p} \leq N(p, d) \|\Delta u\|_{L_p}, \quad \forall u \in W^2_p(\mathbb{R}^d), \quad (2.11)
$$

first obtained by using the Calderón-Zygmund theorem. This is what was usually done in parts of the theory related to VMO conditions and using the Fefferman-Stein theorem.

What is amazing is that (2.10) implies not only (2.11) for $p > 2$, but also the following result by just a mere reference to a simple corollary (see Theorem 6.2) of the Rubio de Francia extrapolation theorem (Theorem 6.1). Note that Theorem 2.5, is a very particular case of the results in [15].

To state this result, for $i \in \{1, \ldots, d\}$ and $p_i \in (1, \infty)$, introduce

$$
\|f\|_{L_{p_1, \ldots, p_d}}^{p_d} := \int_{\mathbb{R}} \left( \cdots \left( \int_{\mathbb{R}} |f|^{p_1} \, dx_1 \right)^{p_2/p_1} \, dx_2 \right)^{p_3/p_2} \cdots \right)^{p_d/p_{d-1}} \, dx_d. \quad (2.12)
$$

**Theorem 2.5.** Let $p_1, \ldots, p_d \in (2, \infty)$. Then there is a constant $N = N(d, p_1, \ldots, p_d)$ such that for any $u \in W^2_{2, \text{loc}}(\mathbb{R}^d)$ we have

$$
\|D^2 u\|_{L_{p_1, \ldots, p_d}} \leq N \|\Delta u\|_{L_{p_1, \ldots, p_d}}, \quad (2.13)
$$

provided that $\|D^2 u\|_{L_{p_1, \ldots, p_d}} + \|Du\|_{L_{p_1, \ldots, p_d}} + \|u\|_{L_{p_1, \ldots, p_d}}$ is finite.

**Remark 2.6.** This “provided that ... is finite” appears because we obtained (2.10) for $u \in W^2_2(\mathbb{R}^d)$ rather than $u \in W^2_{2, \text{loc}}(\mathbb{R}^d)$ and the derivation of (2.13) first proceeds for $u \zeta$, where $\zeta$ is a cut-off function and then sending $\zeta \to 1$. Similarly to the case of (2.11), (2.13) is false if we drop the above provision. The example $u = x_1^2 - x_2^2$ shows this.

To derive some direct consequences of Theorem 2.5 consider equation (2.1) in $\mathbb{R}^d_+ = \{x = (x_1, x') : x_1 \in [0, \infty), x' \in \mathbb{R}^{d-1}\}$ with either zero Dirichlet or zero Neumann condition. By using either odd or even continuation across the plane $\{x_1 = 0\}$ one obtains the following estimates for $p_1, p_2 > 2$ in both cases:

$$
\int_{\mathbb{R}^{d-1}} \left( \int_0^\infty |D^2 u|^{p_1} \, dx_1 \right)^{p_2/p_1} \, dx' \leq N \int_{\mathbb{R}^{d-1}} \left( \int_0^\infty |\Delta u|^{p_1} \, dx_1 \right)^{p_2/p_1} \, dx', \quad (2.14)
$$
\[
\int_0^\infty \left( \int_{\mathbb{R}^{d-1}} |D^2 u|^{p_2} \, dx' \right)^{p_1/p_2} \, dx_1 \leq N \int_0^\infty \left( \int_{\mathbb{R}^{d-1}} |\Delta u|^{p_2} \, dx' \right)^{p_1/p_2} \, dx_1.
\] (2.15)

Of course, these estimates are only true provided that conditions similar to the one in Theorem 2.5 are satisfied.

Above one can also replace \(dx_1\) with \(w(x_1) \, dx_1\), where \(w\) is any \(A_{p_1/2}\)-weight (see Theorem 6.2) and then the constants \(N\) depend also on \([w]_{A_{p_1/2}}\).

By the way, we know that in \((0, \infty)\) the functions \(x^q\) are \(A_p\)-weights iff \(q \in (-1, p-1)\).

The outlined way of proving (2.11) is similar to what is done in the part of the Sobolev-space theory of linear and fully nonlinear elliptic and parabolic equations which rely on the Fefferman-Stein theorem. There is another part where they use an approach based on the theory of viscosity solutions and which is out of the scope of the present review. Still it would be interesting to know if one can obtain, say Theorem 8.2, by using the methods of the theory of viscosity solutions.

One may be not satisfied with the restriction \(p_i > 2\) in (2.13), (2.14), and (2.15). There are two known ways around it:

(a) Take \(p_0 \in (1, 2)\) and first prove (or use) the fact that for any \(f \in L_p(B_r)\) equation (2.1) in \(B_r\) with zero boundary condition on \(\partial B_r\) has a unique solution \(u \in W^2_p(B_r)\) and a natural analog of (2.3) holds with \(L_p(B_r)\) in place of \(L^2(B_r)\). Then by just repeating what is below (2.3) we come to the conclusion that (2.13), (2.14), and (2.15) hold for any \(p_i \in (p_0, \infty)\), and since \(p_0 > 1\) is arbitrary, they hold for any \(p_1, p_2 \in (1, \infty)\).

(b) Take \(p_0 \in (1, 2)\) and use (2.11) (which, as we pointed out, follows from Theorem 2.3 for \(p > 2\) and then for \(p \in (1, 2)\) by duality) to derive (2.4) for \(u \in W^2_{p_0}(B_R)\) with \(L_p\) in place of \(L^2\) as in Exercise 2.4.6 of [41] in which take \(u\) independent of \(t\). This will lead to Lemma 2.1 for the same range of \(\nu\) with power of summability 2 replaced by \(p_0\) and then lead to Theorem 2.2 with the same replacement of 2 by \(p\) but for \(\nu \geq 4\). This new restriction comes from the fact that this time we do not want to use the solvability of (2.1) in balls and instead take a \(\zeta \in C_0^\infty(B_{\nu r})\) such that \(\zeta = 1\) on \(B_{\nu r/2}\), solve the equation \(\Delta v = \zeta \Delta u\) in \(\mathbb{R}^d\) and define \(w = u - v\) which turns out to be harmonic in \(B_{\nu r/2}\). Then an obvious modification of the short proof of Theorem 2.2 proves it with \(p_0\) in place of 2. After that one proceeds as in (a). This way is used even in the first step for equations with VMO coefficients when the solvability of equations in question in balls or cylinders is either unknown or hard to obtain, in particular, because of necessity to deal with boundary conditions (higher-order case).

3. Partitions and stopping times

Fix some integers \(k_1, \ldots, k_d \geq 1\) and call any

\[ C_{l,x} = x + [0, l^{k_1}) \times \ldots \times [0, l^{k_d}), \quad x \in \mathbb{R}^d, l > 0 \]
a “cube” or half-closed “cube” of size \( l \). Observe that if \( k_1 = ... = k_d = 1 \), then we deal just with usual cubes whose edges are parallel to the coordinate axes and which are common in the theory of elliptic equations. If \( k_1 = 2 \) and \( k_2 = ... = k_d = 1 \), we are dealing with parabolic “cubes” often used in the theory of second-order parabolic equations.

Next, we introduce a subset \( \Omega \) of \( \mathbb{R}^d \) which is, actually a product of some \( \mathbb{R} \) and some shifted \([0, \infty)\). Namely, we assume that we are given \( \Omega \subset \mathbb{R}^d \) such that, for each \( l > 0 \), it is the disjoint union of some “cubes” of size \( l \) which belong to \( \Omega \). The simplest example is \( \Omega = \mathbb{R}^d \). The reader is advised to always have in mind this basic example.

For \( n \in \mathbb{Z} = \{0, \pm 1, ...\} \) introduce the following families of “dyadic cubes” by

\[
C_n = \{ C_n(i_1, ..., i_d) : i_1, ..., i_d \in \mathbb{Z}, C_n(i_1, ..., i_d) \subset \Omega \},
\]

where

\[
C_n(i_1, ..., i_d) = [i_1 2^{-k_1 n}, (i_1 + 1)2^{-k_1 n}) \times ... \times [i_d 2^{-k_d n}, (i_d + 1)2^{-k_d n}].
\]

Also set \( \mathcal{C} = \bigcup_{n \in \mathbb{Z}} C_n \).

Define \( \mathcal{F}_n \) as the collection of subsets of \( \Omega \) consisting of an empty set and of the unions of some elements of \( \mathcal{C}_n \). Obviously \( \mathcal{F}_n \subset \mathcal{F}_m \) for \( n \leq m \).

If \( \tau = \tau(x) \) is a function on \( \Omega \) with values in \( \{\infty, 0, \pm 1, \pm 2, ...\} \), we call \( \tau \) a stopping time (relative to \( \{\mathcal{F}_n\} \)) if, for each \( n = 0, \pm 1, \pm 2, ... \),

\[
\{x : \tau(x) = n\} \in \mathcal{F}_n
\]

(that is \( \{x : \tau(x) = n\} \) is either empty or else is the union of some elements of \( \mathcal{C}_n \)). The simplest example of a stopping time is given by \( \tau(x) \equiv 0 \).

If \( \tau \) is a stopping time we denote by \( \mathcal{F}_\tau \) the collection of Borel \( A \) such that, for any \( n \in \mathbb{Z} \) we have

\[
A \cap \{\tau = n\} \in \mathcal{F}_n.
\]

Observe that if we are given two stopping times \( \tau \) and \( \sigma \) and \( \sigma \leq \tau \), then \( \mathcal{F}_\sigma \subset \mathcal{F}_\tau \) since

\[
A \cap \{\tau = n\} = \bigcup_{k=\infty}^{n} A \cap \{\sigma = k\} \cap \{\tau = n\}
\]

and \( A \cap \{\sigma = k\} \in \mathcal{F}_k \subset \mathcal{F}_\tau \). Obviously the intersection of two sets in \( \mathcal{F}_\tau \) belongs to \( \mathcal{F}_\tau \). An easy and useful fact is that \( \Omega, \{\tau < \infty\} \in \mathcal{F}_\tau \). Also a useful fact to remember is that if \( A \in \mathcal{F}_\tau \), then \( A \cap \{\tau < \infty\} \) is a disjoint union of \( A \cap \{\tau = n\} \in \mathcal{F}_n \) and each \( A \cap \{\tau = n\} \) is either empty or is a disjoint union of some \( C \in \mathcal{C}_n \) such that \( C = C \cap \{\tau = n\} \), the latter showing that \( C \in \mathcal{F}_\tau \).

We assume that we are given a measure \( \mu \) on Borel subsets of \( \Omega \) such that, for any \( x \in \Omega \),

\[
\lim_{l \to \infty} \mu(C_{l,x}) = \infty.
\]
Whenever it makes sense we use the notation
\[ f_A = \int_A f \mu(dx) := \frac{1}{\mu(A)} \int_A f(x) \mu(dx) \quad \left( \begin{array}{c} 0 \\ 0 := 0 \end{array} \right) \]
for the average value of \( f \) over \( A \).

Next, for each \( x \in \Omega \) and \( n \in \mathbb{Z} \), there exists (a unique) \( C \in \mathbb{C}_n \) such that \( x \in C \). We denote this \( C \) by \( C_n(x) \). The reader can check himself (or consult, for instance, [41] for this and a few more simple properties of introduced objects) that for any \( \lambda > 0 \) and \( f \geq 0 \), such that \( f_{C_n(x)} \to 0 \) as \( n \to -\infty \) for any \( x \),
\[ \tau_\lambda(x) = \inf \{ n : f_{C_n(x)} > \lambda \} \quad (\inf \emptyset := \infty) \tag{3.2} \]
is a stopping time.

We also use the notation
\[ f_{|n}(x) = f_{C_n(x)} = \int_{C_n(x)} f(y) \mu(dy). \]
If we are also given a stopping time \( \tau \), we let
\[ f_{|\tau}(x) = f_{|\tau(x)}(x) \]
for those \( x \) for which \( \tau(x) < \infty \) and \( f_{|\tau}(x) = f(x) \) otherwise.

We suppose that \( \mu \) satisfies the “doubling condition”: for any \( n, C \in \mathbb{C}_n \), and \( C' \in \mathbb{C}_{n-1} \) such that \( C \subset C' \) we have
\[ \mu(C') \leq N_0 \mu(C), \tag{3.3} \]
where \( N_0 \) is a constant independent of \( n, C, C' \). One of consequences of this condition is that for \( f \geq 0 \) on the set where \( \tau_\lambda(x) = n \) we have \( f_{|n-1}(x) \leq \lambda \)
and
\[ f_\tau(x) = \frac{1}{\mu(C_n(x))} \int_{C_n(x)} f \mu(dy) \leq \frac{N_0}{\mu(C_{n-1}(x))} \int_{C_{n-1}(x)} f \mu(dy) \leq N_0 \lambda. \tag{3.4} \]

Another consequence of (3.1) and (3.3) is that \( \mu(C_{l,x}) > 0 \) for any \( x \in \Omega \) and \( l > 0 \).

In the following lemma by \( I_{A,\tau<\infty} \) we mean the indicator function of the set \( \{ x \in A : \tau(x) < \infty \} \).

**Lemma 3.1.** (i) Let \( f \) be Borel on \( \Omega \), \( f \geq 0 \), let \( \tau \) be a stopping time, and let \( A \in \mathcal{F}_\tau \). Then
\[ \int_{\Omega} f_{|\tau}(x) I_{A,\tau<\infty} \mu(dx) = \int_{\Omega} f(x) I_{A,\tau<\infty} \mu(dx). \tag{3.5} \]

(ii) Let \( f \) be Borel on \( \Omega \), \( f \geq 0 \), and let \( \lambda > 0 \) be a constant. Assume that \( f_{|n}(x) \to 0 \) as \( n \to -\infty \) at any \( x \). Then for \( \tau = \tau_\lambda \) defined in (3.2) we have
\[ \lambda I_{\tau<\infty} < f_{|\tau}(x) I_{\tau<\infty} \leq N_0 \lambda, \tag{3.6} \]
and for any \( A \in \mathcal{F}_\tau \)
\[ N_0^{-1} \lambda^{-1} \int_{\Omega} f(x) I_{A,\tau<\infty} \mu(dx) \leq \mu(\{ x \in A : \tau(x) < \infty \}) \]
\[ \leq \lambda^{-1} \int_{\Omega} f(x) I_{A, \tau < \infty} \mu(dx). \] (3.7)

Proof. (i) Owing to the additivity of the integral, it suffices to prove (3.5) with \( \tau = n \) in place of \( \tau < \infty \) and, since the set \( \{ x \in A : \tau(x) = n \} \) is the disjoint union of some \( C \in \mathbb{C}_n \), it only remains to observe that for such \( C \) we have \( C \cap \{ \tau = n \} = C \) and

\[ \int_{\Omega} f_{\tau} (x) I_{C, \tau = n} \mu(dx) = \int_{\Omega} f_{C} I_{C} \mu(dx) = \int_{C} f \mu(dx) = \int_{\Omega} f_{I_{C, \tau = n}} \mu(dx). \]

(ii) Relations (3.6) follow from the definition of \( \tau \) and (3.4). The first inequality in (3.7) follows from (3.5) and (3.6). The second one follows from Chebyshev's inequality and (3.5) because

\[ \mu(\{ x \in A : \tau(x) < \infty \}) = \mu(\{ x \in \Omega : f_{\tau} I_{A, \tau < \infty} > \lambda \}). \]

The lemma is proved.

Define the maximal “dyadic” function of \( f \) by

\[ Mf(x) = \sup_{n<\infty} |f|_{n}(x), \] (3.8)

so that \( Mf = M|f| \). Observe that, if \( f \geq 0 \), \( \{ x : Mf(x) > \lambda \} = \{ x : \tau_{\lambda} < \infty \} \), where \( \tau_{\lambda} \) is taken from (3.2). Therefore, Lemma 3.1 (ii) implies the following.

**Corollary 3.2.** Under the conditions of Lemma 3.1 (ii)

\[ N^{-1}_{0} \lambda^{-1} \int_{\Omega} f(x) I_{Mf > \lambda} \mu(dx) \leq \mu(\{ x \in A : Mf(x) > \lambda \}) \]

\[ \leq \lambda^{-1} \int_{\Omega} f(x) I_{Mf > \lambda} \mu(dx). \] (3.9)

The following standard consequence of the second inequality in (3.9) is left to the reader as an exercise (see Exercise 3.2.7 and the hint to it in [41]).

**Corollary 3.3.** Let \( f_{C} \in L_{1}(\mu) \) for any \( C \in \mathbb{C} \). Then (Lebesgue differentiating theorem) \( f_{n} \to f \) (\( \mu \)-a.e.) and, in particular, \( |f| \leq \mathcal{M}f \) (\( \mu \)-a.e.).

It is worth noting that both inequalities in (3.7) are crucial in the proof of the reverse Hölder’s inequality for Muckenhoupt’s weights, and the right inequality is used in a crucial way in proving the Fefferman-Stein theorem in Sect 7.

The following remark will not be used in the future. It was hard not to make it.

**Remark 3.4.** The first inequality in (3.7) is instrumental not only in proving the reverse Hölder inequality for \( A_p \)-weights but its version (3.9) also is crucial in the proof of the first part of a remarkable Zygmund-Stein result that, for any Borel \( f \geq 0 \) and \( \lambda_0 > 0 \),

\[ \int_{\Omega} \mathcal{M}f I_{Mf > \lambda_0} \mu(dx) < \infty \implies \int_{\Omega} f I_{f > \lambda_0} \log(f/\lambda_0) \mu(dx) < \infty, \] (3.10)
\[
\int_{\Omega} fI_{f > \lambda} \log(f/\lambda_0) \, \mu(dx) < \infty \quad \implies \quad \int_{\Omega} Mf_{Mf > 2\lambda_0} \mu(dx) < \infty. \tag{3.11}
\]

Here (3.10) is obtained just by integrating with respect to \( \lambda \in (\lambda_0, \infty) \) the first inequality in (3.9), where on the left \( Mf \) is replace with a smaller quantity \( f \). To prove (3.11) use the second inequality in (3.9), which after observing that \( M(fI_{f \leq \lambda}) \leq \lambda \) implies that

\[
\mu(\{x : Mf(x) > 2\lambda\}) \leq \mu(\{x : M(fI_{f > \lambda})(x) > \lambda\})
\]

\[
\leq \lambda^{-1} \int_{\Omega} f(x)I_{f > \lambda} \, \mu(dx).
\]

Then again integrate with respect to \( \lambda \in (\lambda_0, \infty) \) the inequality between the extreme terms. After that one will only need to estimate \( 2\lambda_0\mu(\{x : MfI_{f > \lambda}(x) > 3\lambda_0/2\}) \leq (4/3) \int_{\Omega} fI_{f > 3\lambda_0/2} \, \mu(dx) \),

where the last term is finite due to the condition in (3.11).

4. **Muckenhoupt’s weights**

We use the setting of Section 3, but the doubling condition here is stronger than (3.3). We need (3.3) to hold for a collection of subsets of \( \Omega \) which contains all translates of “contracted” interiors of \( C \) which keep them in \( \Omega \). For \( x \in \Omega \) and \( l > 0 \) denote

\[
D_{l,x} = x + (0, l^{k_1}) \times ... \times (0, l^{k_d}).
\]

Assume that we have a family \( Q \) of open subsets of \( \Omega \) which contains all \( D_{l,x} \) and is such that for any \( Q \in Q \) there exists a \( D_{l,x} \) satisfying

\[
Q \subset D_{l,x}, \quad \mu(D_{2l,x}) \leq N_0 \mu(Q). \tag{4.1}
\]

We suppose that

\[
\mu(\partial \Omega) = 0 \tag{4.2}
\]

and then, as is easy to see, the new doubling condition (4.1) implies (3.3) and, moreover, \( \mu(C_{2l,x}) \leq N_0 \mu(C_{l,x}) \) for any \( x \in \Omega \) and \( l > 0 \) (just in case, observe that the sets \( D_{l,x} \) are open).

**Definition 4.1** (Muckenhoupt’s weights). Let \( w(x) \) be a function on \( \Omega \) such that \( 0 < w \leq \infty \) (\( \mu \)-a.e.) and \( wQ < \infty \) for any \( Q \in Q \). We call it an \( A_1 \)-weight (relative to \( Q \) or relative to \( (Q, \mu) \)) if there is a constant \( N \) such that

\[
wQ \leq Nw(x) \quad \forall x \in Q, \forall Q \in Q. \tag{4.3}
\]

on \( \Omega \). The least constant \( N \) satisfying (4.3) is called the \( A_1 \)-constant of \( w \) denoted by \([w]_{A_1}\). For \( p \in (1, \infty) \) we call \( w \) an \( A_p \)-weight if there is a constant \( N \) such that

\[
wQ\left(\left(w^{-1/(p-1)}\right)_Q\right)^{p-1} \leq N \quad \forall Q \in Q. \tag{4.4}
\]
The least constant $N$ satisfying (4.4) is called the $A_p$-constant of $w$ denoted by $[w]_{A_p}$.

Set

$$w(A) = \int_A w \, d\mu,$$

and denote $L_p(w) = L_p(\Omega, w(dx))$.

**Remark 4.2.** Hölder’s inequality implies that, if $p \in (1, \infty)$, $A$ is Borel, and

$$\infty > w > 0 \text{ (}\mu\text{-a.e.}\text{)} \text{ on } A,$$

then

$$1 = \int_A w^{1/p} w^{-1/p} \, d\mu \leq (w_A)^{1/p} \left( (w^{-1/(p-1)})_A \right)^{(p-1)/p}.$$  \hfill (4.5)

Therefore, $[w]_{A_p} \geq 1$ if $w \in A_p$. This obviously holds for $p = 1$ as well.

Also note that one can replace what is raised to the power $p - 1$ in (4.4) with $(w^{-1/(p-1)})_A \, \mu(A)/\mu(Q)$ for any Borel $A \subset Q$. Then (4.5) implies that

$$[w]_{A_p} \frac{w(A)}{w(Q)} \geq \left( \frac{\mu(A)}{\mu(Q)} \right)^p.$$  \hfill (4.6)

We obtained this result for $p > 1$. It is also true for $p = 1$ since, for $w \in A_1$ we have $[w]_{A_1} \, w \geq w_Q$ on $Q$ and hence

$$\mu(A) = \int_A \frac{1}{w} w \, d\mu \leq [w]_{A_1} \frac{1}{w_Q} w(A),$$

which is (4.6) for $p = 1$.

An important consequence of (4.6) is that for $p \in [1, \infty)$, $w \in A_p$, and any $\alpha \in (0, 1)$ there exists $\beta \in (0, 1)$ depending only on $\alpha$, $p$, and $[w]_{A_p}$, such that, for any Borel $S \subset Q \in \mathcal{Q}$

$$\mu(S) \leq \alpha \mu(Q) \implies w(S) \leq \beta w(Q).$$  \hfill (4.7)

On gets (4.7) by inspecting the following version of (4.6)

$$[w]_{A_p} \left( 1 - \frac{w(S)}{w(Q)} \right) \geq \left( 1 - \frac{\mu(S)}{\mu(Q)} \right)^p \geq (1 - \alpha)^p.$$

**Remark 4.3.** In the future we several times say that a constant entering an estimate depends only on ..., and $[w]_{A_p}$. It is important to emphasize that, given a $K_0$, in these situations the constant can be chosen to depend only on ..., and $K_0$, provided that $[w]_{A_p} \leq K_0$.

As a simple exercise based on (4.3) or Hölder’s inequality one proves that $A_p \subset A_q$ if $1 \leq p \leq q$. Here is an extension of this for $A_1$-weights (when $v \equiv 1$).

**Lemma 4.4.** If $w, v \in A_1$ and $p \in (1, \infty)$, then $w v^{1-p} \in A_p$ and $[w v^{1-p}]_{A_p} \leq [w]_{A_1} [v]_{A_1}^{p-1}$.
Thus, on the other hand, by the second part of (3.7)
\[
A
\]
Note that
\[
\tau
\]
use the results of Section 3 in the new setting to prove (4.8) for Theorem 4.5.

Q
us that we may also assume that
\[
Q
\]
there exist constants
\[
N
\]
\[
\epsilon
\]
\[
\beta
\]
concentrate on the case that
\[
Q
\]
observe that, since
\[
Q
\]
\[
Q
\]
pointwise limit as
\[
\epsilon
\]
\[
D
\]
\[
A
\]
Proof. Since
\[
A
\]
\[
A
\]
\[
A
\]
\[
A
\]
and by (4.6) we have
\[
\bar{w}_{D_{1,x}} \leq [w]_{A_p} N_0^{-1} w_Q.
\]
This convinces us that we may concentrate on the case that
\[
Q = D_{1,x}.
\]
Changing scales, perhaps different along different axes (and using the rules of changing the variables in the Lebesgue integrals), if necessary, convinces us that we may also assume that
\[
Q = D_{1,x}.
\]
Then we observe that
\[
I_Q
\]
the pointwise limit as
\[
\epsilon \downarrow 0
\]
of the indicators of
\[
x + [\epsilon, 1)^d
\]
so that we only need to prove (4.8) for
\[
Q = x + [\epsilon, 1)^d
\]
and changing the scales again reduces our task to proving (4.8) only for
\[
Q = C_{1,x} = x + [0, 1)^d
\]
In that case, as is easy to see, we can take
\[
x
\]
as the new origin, replace
\[
\Omega
\]
with
\[
\Omega' = [0, \infty)^d
\]
and use the results of Section 3 in the new setting to prove (4.8) for
\[
Q = [0, 1)^d
\]

\[I := \int_{\Omega'} \bar{w} I_{C, \tau_k + \epsilon < \infty} \mu(dx) \leq \int_{\Omega'} \bar{w} I_{C, \tau_k < \infty} \mu(dx) \leq N_0 \lambda^k w_Q \mu(C).\]

On the other hand, by the second part of (3.7)
\[I \geq \lambda^{k+1} w_Q \mu(C \cap \{\tau_{k+1} < \infty\}).\]
Thus,
\[\frac{\mu(C \cap \{\tau_{k+1} < \infty\})}{\mu(C)} \leq \frac{1}{2}.
\]
By (4.7)
\[w(C \cap \{\tau_{k+1} < \infty\}) \leq \beta w(C)\]
and, since \( \{\tau_k < \infty\} \in \mathcal{F}_{\tau_k} \), the set \( \{\tau_k < \infty\} \) is represented as a disjoint union of \( C \in \mathcal{F}_{\tau_k} \cap \mathcal{C} \), so that

\[
w(\{\tau_k+1 < \infty\}) \leq \beta w(\{\tau_k < \infty\}).
\]

It follows that

\[
w(\{\tau_k < \infty\}) \leq \beta^k w(\{\tau_0 < \infty\}) = \mu(Q) \beta^k w_Q.
\]

Now, observe that \( (\mu \text{-a.e.}) \) on the set \( \{\tau_k < \infty, \tau_{k+1} = \infty\} \) we have

\[
w \leq \lambda_{k+1} w_Q.
\]

Also \( (\mu \text{-a.e.}) \) \( \bigcup_{k=0}^{\infty} \{\tau_k < \infty, \tau_{k+1} = \infty\} = \{\tau_0 < \infty\} \) since \( w < \infty \) \( (\mu \text{-a.e.}) \). Obviously, the terms in the above union are disjoint.

Therefore,

\[
\int_Q w^{1+\varepsilon} \mu(dx) = \int_Q w^{1+\varepsilon} I_{\tau_k=\infty} \mu(dx) + \sum_{k=0}^{\infty} \int_Q w^{1+\varepsilon} I_{\tau_k<\infty, \tau_{k+1}=\infty} \mu(dx)
\]

\[
\leq \mu(Q)(w_Q)^{1+\varepsilon} + \sum_{k=0}^{\infty} \lambda^{(k+1)\varepsilon}(w_Q) \int_Q w I_{\tau_k<\infty, \tau_{k+1}=\infty} \mu(dx)
\]

\[
\leq \mu(Q)(w_Q)^{1+\varepsilon} \left[ 1 + \sum_{k=0}^{\infty} \lambda^{(k+1)\varepsilon} \beta^k \right].
\]

We see that it only remains to choose \( \varepsilon > 0 \) so small that the last series converges. The theorem is proved.

The following somewhat sharper statement than (4.7) will be needed only in the applications of the theory of \( A_p \)-weights rather than in the proof of the Rubio de Francia theorem.

**Corollary 4.6.** If \( p \in (1, \infty) \) and \( w \in A_p \), then there exists \( \beta = \beta(p, N_0, [w]_p) \in (0, 1) \) and \( N = N(p, N_0, [w]_p) \) such that for any Borel \( S \subset Q \subset \mathbb{Q} \) we have

\[
\frac{w(S)}{w(Q)} \leq N \left( \frac{\mu(S)}{\mu(Q)} \right)^\beta.
\]

(4.9)

Indeed, by Hölder’s inequality

\[
w(S) = \int_S w \mu(dx) \leq \left( \int_Q w^{1+\varepsilon} \mu(dx) \right)^{1/(1+\varepsilon)} \mu^{\varepsilon/(1+\varepsilon)}(S)
\]

\[
= \mu^{1/(1+\varepsilon)}(Q) \left[ (w^{1+\varepsilon})_Q \right]^{1/(1+\varepsilon)} \mu^{\varepsilon/(1+\varepsilon)}(S)
\]

\[
\leq N \mu^{1/(1+\varepsilon)}(Q) w_Q \mu^{\varepsilon/(1+\varepsilon)}(S) = N w(Q) \mu^{-\varepsilon/(1+\varepsilon)}(Q) \mu^{\varepsilon/(1+\varepsilon)}(S),
\]

which is what is claimed with \( \beta = \varepsilon/(1+\varepsilon) \).

For the reader’s orientation, we point out that the weights satisfying (4.9) are called \( A_{\infty} \)-weights.

The following result is crucial in deriving the Hardy-Littlewood theorem with \( A_p \)-weights.
Theorem 4.7 (Self improving property). If \( p \in (1, \infty) \) and \( w \in A_p \), then there exists 
\[
q = q(p, N_0, [w]_{A_p}) \in (1, p)
\]
such that \( w \in A_q \). Furthermore, \([w]_{A_q}\) is estimated by a constant \( N = N(p, N_0, [w]_{A_p})\).

Proof. Note that for \( p \in (1, \infty) \) the condition \( w \in A_p \) is equivalent to 
\[
w - \frac{1}{p-1} \in A_p/p-1
\]
and
\[
[w - \frac{1}{p-1}]_{A_p/p-1} = [w]_{A_p}.
\]
By Theorem 4.5 there exist \( \varepsilon > 0 \) and \( N_1 \) depending only on \( p, N_0, [w]_{A_p} \) such that 
\[
\left( w - \frac{1+(1+\varepsilon)/(p-1)}{p-1} \right)_Q \leq N_1 \left( [w]_{A_p/p-1} \right)_{Q}^{1+\varepsilon}.
\]
Obviously, \( (1+\varepsilon)/(p-1) = 1/(q-1) \) for some \( q \in (1, p) \) and the above inequality means that 
\[
\left( [w]_{A_p} \right)^{q-1} \leq N_1^{q-1} \left( [w]_{A_p/p-1} \right)^{p-1}.
\]
By multiplying both parts by \( w_Q \), we get that \( w \in A_q \) and \([w]_{A_q} \leq N_1^{q-1}[w]_{A_p}\).

The lemma is proved.

Define the Hardy-Littlewood maximal operator by
\[
Mf(x) = \sup_{Q \in \mathcal{Q}, Q \ni x} (|f|_Q) = \sup_{Q \in \mathcal{Q}} I_Q(x)(|f|)_Q.
\]
Since the \( Q \)'s are open, \( I_Q(x) \) are lower semicontinuous, so is \( Mf \) and, hence, it is Borel measurable. Also for an \( A_p \)-weight \( w \) set
\[
Mwf(x) = \sup_{Q \in \mathcal{Q}, Q \ni x} \frac{1}{w(Q)} \int_Q |f| w(dy).
\]

Lemma 4.8. Let \( p \in [1, \infty) \) and \( w \in A_p \). Then for any \( Q \in \mathcal{Q} \) and measurable \( f \geq 0 \) we have
\[
(f_Q)^p w_Q \leq [w]_{A_p} (f^p w)_Q.
\]

Proof. If \( p = 1 \), (4.11) follows from (4.6) and the linearity of the integral. If \( p > 1 \), by Hölder’s inequality
\[
(f_Q)^p w_Q = ((f w^{1/p})^{p-1})^p w_Q \leq (f^p w)_Q \left( (w^{-1/(p-1)})_Q \right)^{p-1} w_Q
\]
and (4.11) follows in light of our definitions. The lemma is proved.

Observe that (4.11) implies the following.

Corollary 4.9. For \( p \in [1, \infty) \), \( w \in A_p \), and Borel \( f \geq 0 \) we have
\[
(Mf)^p \leq [w]_{A_p} Mwf^p.
\]
5. The case of \( \mu \) with a “real” doubling property on \( \mathbb{Q} \)

In this section we assume that \( \mu \) satisfies the conditions (3.1) and (4.2) from Sections 3 and 4 and satisfies an even stronger doubling condition than (4.1) because we are going to use a covering lemma similar to the Vitali one. Let \( \Omega^0 \) be the interior of \( \Omega \).

Introduce \( \mathbb{Q} \) as the collection of
\[
C_l(x) = x + (-l^{k_1}/2, l^{k_1}/2) \times \cdots \times (-l^{k_d}/2, l^{k_d}/2), \quad x \in \Omega, l > 0,
\]
that lie entirely in \( \Omega \). Actually, this is the same collection as the collection of \( D_{l,x} \), \( x \in \Omega, l > 0 \), from Section 4. We use a different notation which allows us to express the doubling condition more easily. Suppose that for any \( C_l(x) \in \mathbb{Q} \) we have
\[
\mu(C_{4l}(x) \cap \Omega) \leq N_0 \mu(C_l(x)). \tag{5.1}
\]
All \( A_p \)-weights in this section are \( A_p \)-weights relative to these \( \mathbb{Q} \), \( \mu \).

Observe that, if \( C_l(x) \in \mathbb{Q} \), then \( C_l(x) = D_{l,y} \) for a \( y \in \Omega \) and \( D_{2l,y} \subset C_{4l}(x) \cap \Omega^o \). Therefore, condition (5.1) implies that condition (4.1) is satisfied by the above \( \mathbb{Q} \) and we can use the results of Section 4. In particular, (4.6) implies the following.

**Corollary 5.1.** For any \( p \in [1, \infty) \) and \( w \in A_p \), the measure \( w(A) \) satisfies the doubling condition (5.1) with constant \( N_0^p[w]_{A_p} \).

**Lemma 5.2** (Maximal inequality). If \( p \in [1, \infty) \) and \( w \in A_p \), then there exists a constant \( N \), depending only on \( p, N_0 \), and \( [w]_{A_p} \), such that for any \( \lambda > 0 \) and Borel \( f \)
\[
w(\{ x : \mathbb{M}_w f(x) > \lambda \}) \leq N \lambda^{-1} \int_{\Omega} |f| w(dx). \tag{5.2}
\]

Proof. We may assume that \( f \geq 0 \) and \( f \in L_1(w) \). Define \( A(\lambda) = \{ x \in \Omega : \mathbb{M}_w f(x) > \lambda \} \). This set is open since \( \mathbb{M}_w f \) is lower semicontinuous. Take a compact set \( K \subset A(\lambda) \). Then by the definition of \( A(\lambda) \) for any \( x \in K \) there exists a \( Q \in \mathbb{Q} \) such that \( x \in Q \) and \( \int_{Q} f \, w(dx) > \lambda w(Q) \). Then, of course, \( Q \subset A \).

By the compactness of \( K \), there is a finite collection \( Q_1, \ldots, Q_n \in \mathbb{Q} \) covering \( K \) and such that for each \( Q = Q_i \) we have \( \int_{Q} f \, w(dx) > \lambda w(Q) \).

Now we use a Vitali’s covering argument. If \( Q = C_l(x) \in \mathbb{Q} \), then define its size to be \( l \) and set \( Q' = C_{4l}(x) \cap \Omega^o \). Then denote by \( \hat{Q}_1 \) any of \( Q_i \) which has the largest size and set it aside. Next, introduce \( \hat{Q}_2 \) as one of the remaining \( Q_i \) which has the largest size between those \( Q_i \) that have no intersection with \( \hat{Q}_1 \). It may happen that no such \( Q_i \) exist. Then it is almost obvious that \( Q_i \subset \hat{Q}_1' \) for any \( i \). If \( \hat{Q}_2 \) exists, we proceed further.

If we have already defined \( \hat{Q}_1, \ldots, \hat{Q}_k \), then we define \( \hat{Q}_{k+1} \) as one of the “cubes” in the family of
\[
\{ Q_1, \ldots, Q_n \} \setminus \{ \hat{Q}_1, \ldots, \hat{Q}_k \}, \tag{5.3}
\]
which are disjoint from $\hat{Q}_1, \ldots, \hat{Q}_k$, which has the largest size (between those in (5.3), that are disjoint from $\hat{Q}_1, \ldots, \hat{Q}_k$). In finitely many steps we will come to a $k$ for which any “cube” in family (5.3) intersects one of $\hat{Q}_1, \ldots, \hat{Q}_k$ or else the family is empty. In the second case, obviously, for any $i$

$$Q_i \subset \bigcup_{j=1}^k \hat{Q}_j'. \tag{5.4}$$

It turns out that (5.4) also holds for any $i$ in the first case. Indeed, if, for a fixed $i$, $Q_i$ has a nonempty intersection with $\hat{Q}_j$, then define $r = r(i)$ as the smallest such $j$ and observe that, if $r = 1$, then as has been pointed out above, $Q_i \subset \hat{Q}_1'$ and (5.4) holds. If $r > 1$, then the size of $Q_i$ is no greater than that of $\hat{Q}_r$ by the choice of $\hat{Q}_r$ and because $Q_i$ has no intersection with $\hat{Q}_1, \ldots, \hat{Q}_{r-1}$ by the definition of $r$. Now as above, this combined with $\hat{Q}_r \cap Q_i \neq \emptyset$ implies that $Q_i \subset \hat{Q}_r'$. This proves (5.4).

It follows that

$$K \subset \bigcup_{j=1}^k \hat{Q}_j'.$$

Finally, use the fact that, owing to the doubling property in Corollary 5.1, we have $w(Q') \leq N w(Q)$, where the constant $N$ depends only on $p, N_0$, and $[w]_{A_p}$, so that

$$w(K) \leq \sum_{j=1}^k w(\hat{Q}_j') \leq N \sum_{j=1}^k w(\hat{Q}_j) \leq N \lambda^{-1} \sum_{j=1}^k \int_{\hat{Q}_j} f \, w(dx) \leq N \lambda^{-1} \int_{\Omega} f \, w(dx).$$

By taking a sequence of compact sets $K_m \uparrow A(\lambda)$ and passing to the limit, we get (5.2) in our particular case. The lemma is proved.

We can now prove one of the fundamental results of the theory.

**Theorem 5.3.** (i) If $p \in [1, \infty)$ and $w \in A_p$, then the operator $\mathcal{M} f$ is of weak $(p, p)$-type with respect to measure $w$. More precisely, there exists a constant $N$ depending only on $p, N_0$, and $[w]_{A_p}$ such that for any $\lambda > 0$ and Borel $f \geq 0$ we have

$$w(\{x : \mathcal{M} f > \lambda\} \leq N \lambda^{-p} \int_{\Omega} f^p w(dx).$$

(ii) If $p \in (1, \infty)$ and $w \in A_p$, then $\mathcal{M} f$ is a bounded operator in $L_p(w)$, that is there exists a constant $N$ depending only on $p, N_0$, and $[w]_{A_p}$ such that for any $f \in L_p(w)$ we have

$$\|\mathcal{M} f\|_{L_p(w)} \leq N \|f\|_{L_p(w)}.$$
Proof. (i) By Lemma 5.2 and Corollary 4.9
\[ w(\{ x : Mf > \lambda^{1/p} \} \leq w(\{ x : M_w|f|^{p} > [w]_{A_p}^{-1}\lambda \} \leq N[w]_{A_p}^{-1}\int \Omega f^p w(dx) \]
and it only remains to make an obvious substitution.

(ii) By Theorem 4.7 there exists \( q = q(p, N_0, [w]_{A_p}) \in (1, p) \) such that \( w \in A_q \). Then assertion (ii) implies that \( Mf \) is a weak \((q, q)\)-operator with respect to measure \( w \). The fact that it also maps bounded functions into bounded ones with norm one is almost obvious. Now our assertion follows from the Marcinkiewicz interpolation theorem. The theorem is proved.

6. Rubio de Francia extrapolation theorem

We work in the framework of Section 5 and suppose that \( \mu \) satisfies (3.1), (4.2), and (5.1).

**Theorem 6.1.** Let \( p, q \in (1, \infty) \), \( K_q \) be a constant from \((1, \infty)\) and \( f, g \) be Borel nonnegative functions of \( \Omega \). Then there exists \( K_p = K(p, q, K_q, N_0) \geq 1 \) such that if
\[ \int_{\Omega} f^p w_p \mu(dx) \leq \int_{\Omega} g^p w_p \mu(dx) \tag{6.1} \]
for any \( w_p \in A_p \) with \([w]_{A_p} \leq K_p\), then
\[ \int_{\Omega} f^q w_q \mu(dx) \leq 4^q \int_{\Omega} g^q w_q \mu(dx) \tag{6.2} \]
for any \( w_q \in A_q \) with \([w]_{A_q} \leq K_q\).

Proof. We follow the proof of Theorem 2.5 of [15] or the proof of Theorem 1.4 of [8], which are streamlined versions of the proof of Theorem IV.5.19 of [22]. The proof is striking.

Denote by \( N_1 \) the constant \( N \) which suits Theorem 5.3 (ii) with \( q \) in place of \( p \) for any \( w \in A_q \) with \([w]_{A_q} \leq K_q \). Also set \( q' = q/(q - 1) \) and denote by \( N_2 \) the constant \( N \) which suits Theorem 5.3 (ii) with \( q' \) in place of \( p \) for any \( w \in A_{q'} \) with \([w]_{A_{q'}} \leq K^{1/(q-1)}_q \).

Set
\[ K_p = 2^p N_1^{p-1} N_2, \]
and assuming that (6.1) holds for all \( w \in A_p \) with \([w]_{A_p} \leq K_p\) we prove (6.2) for any \( w_q \in A_q \) such that \([w_q]_{A_q} \leq K_q\).

We fix such a \( w_q \) and for nonnegative \( h \in L_q(w_q) \) define
\[ \Re h(x) = \sum_{k=0}^{\infty} \frac{M_k h(x)}{2^k N_1^k}, \]
where \( M_k \) is the \( k \)-th iteration of \( M \): \( M^0 h = h, M^{k+1} = MM^k \).

Observe that, obviously, \( h \leq \Re h, \| \Re h \|_{L_q(w_q)} \leq 2\| h \|_{L_q(w_q)}, \| M \Re h \| \leq 2N_1 \Re h \), so that, if \( \| h \|_{L_q(w_q)} > 0 \), then \( \Re h \in A_1 \) and
\[ [\Re h]_{A_1} \leq 2N_1. \]
Next, we know that \( w_q^{1-q'} \in A_{q'} \) and \( M \) is bounded on \( L_{q'}(w_q^{1-q'}) \). Since
\[
[w_q^{1-q'}]_{A_{q'}} = ([w_q]_{A_q})^{1/(q-1)} \leq K_q^{1/(q-1)},
\]
we have that
\[
\|Mh\|_{L_{q'}(w_q^{1-q'})} \leq N_2 \|h\|_{L_{q'}(w_q^{1-q'})}.
\]
This means that the operator \( M'h = w_q^{-1}M(hw_q) \) is bounded in \( L_{q'}(w_q) \) with norm less than \( N_2 \). Indeed, for \( h \in L_{q'}(w_q) \) we have
\[
\|M'h\|_{L_{q'}(w_q)} = \|M(hw_q)\|_{L_{q'}(w_q^{1-q'})} \leq N_2 \|hw_q\|_{L_{q'}(w_q^{1-q'})} = N_2 \|h\|_{L_{q'}(w_q)}.
\]
This allows us to introduce the operator
\[
R'h(x) = \sum_{k=0}^{\infty} \frac{(M')^k h(x)}{2^k N_2^k}, \quad h \in L_{q'}(w_q), \quad h \geq 0,
\]
and claim that \( h \leq R'h, \|R'h\|_{L_{q'}(w_q)} \leq 2\|R'h\|_{L_{q'}(w_q)}, M'R'h \leq 2N_2 R'h, \) so that, if \( \|h\|_{L_{q'}(w_q)} > 0 \), then \( w_q R'h \in A_1 \) and
\[
[w_q R'h]_{A_1} \leq 2N_2.
\]
Then observe that while proving (6.2) we may certainly assume that \( 0 < \|g\|_{L_q(w_q)} < \infty \). Take a nonnegative test function \( h \in L_{q'}(w_q) \) such that \( \|h\|_{L_{q'}(w_q)} > 0 \). From the properties of \( R \) and \( R' \) and Lemma 4.4 we get that the function
\[
w_p := (Rg)^{1-p} w_q R'h
\]
is an \( A_p \)-weight and
\[
[w_p]_{A_p} \leq [Rg]^{p-1}_{A_1} [w_q R'h]_{A_1} \leq 2^p N_2^{p-1} N_2 = K_p.
\]
Furthermore, since \( \|Rg\|_{L_q(w_q)} \leq 2 \|g\|_{L_q(w_q)} \) and \( \|R'h\|_{L_{q'}(w_q)} \leq 2 \|h\|_{L_{q'}(w_q)} \), by Hölder’s inequality
\[
\int_\Omega (Rg) w_q R'h \mu(dx) \leq 4 \|g\|_{L_q(w_q)} \|h\|_{L_{q'}(w_q)}.
\]
(6.3)

Now, by Hölder’s inequality and by assumption
\[
\int_\Omega f h w_q \mu(dx) = \int_\Omega f (h w_q / w_p) w_p \mu(dx) \leq \|f\|_{L_p(w_p)} \|h w_q / w_p\|_{L_{p/(p-1)}(w_p)} \leq \|g\|_{L_p(w_p)} \|h w_q / w_p\|_{L_{p/(p-1)}(w_p)},
\]
where, in light of \( g \leq Rg \) and (6.3),
\[
\|g\|_{L_p(w_p)}^p = \int_\Omega g^p (Rg)^{1-p} R'h w_q (dx)
\]
\[
\leq \int_\Omega (Rg) R'h w_q (dx) \leq 4 \|g\|_{L_q(w_q)} \|h\|_{L_{q'}(w_q)},
\]
and, in light of \( h \leq R'h \) and (6.3),
\[
\|h w_q / w_p\|_{L_{p/(p-1)}(w_p)}^p = \int_\Omega h^p/(p-1) (Rg)^p (R'h)^{p/(1-p)} (Rg)^{1-p} R'h w_q (dx)
\]
\[
\leq \int_\Omega (Rg) R'h w_q (dx) \leq 4 \|g\|_{L_q(w_q)} \|h\|_{L_{q'}(w_q)}.
\]
\[ \int_{\Omega} (\mathbb{R}g) \mathbb{R}' h \, w_q (dx) \leq 4 \| g \|_{L_q(w_q)} \| h \|_{L_q'(w_q)}. \]

Hence,

\[ \int_{\Omega} f h \, w_q (dx) \leq 4 \| g \|_{L_q(w_q)} \| h \|_{L_q'(w_q)} \]

and the arbitrariness of \( h \in L_q'(w) \) proves (6.2). The theorem is proved.

To extract the most important for us consequences of this theorem we split the coordinates of \( x \) into several groups.

We take integers \( m \geq 2, d_1, \ldots, d_m \geq 1 \) such that \( d_1 + \ldots + d_m = d \), define \( l_0 = 0, l_{i+1} = l_i + d_{i+1}, i = 0, \ldots, m-1 \), and express the points in \( \mathbb{R}^d \) as

\[ x = (x_1, \ldots, x_d) = (\bar{x}_1, \ldots, \bar{x}_m), \]

where \( \bar{x}_i = (x_{l_{i-1}+1}, \ldots, x_{l_i}) \). Then \( \mathbb{R}^d = \mathbb{R}^{d_1} \times \ldots \times \mathbb{R}^{d_m} \) and, accordingly, \( \Omega = \Omega_1 \times \ldots \times \Omega_m \), where \( \Omega_i \) is the projection of \( \Omega \) on \( \mathbb{R}^{d_i} \). We use the notation \( \bar{x}_i = (x_{l_{i-1}+1}, \ldots, x_{l_i}) \) for generic points in \( \mathbb{R}^{d_i} \) and set

\[ \bar{\Omega}_i = \Omega_1 \times \ldots \times \Omega_m = \{ \bar{x}_i = (\bar{x}_1, \ldots, \bar{x}_m) \}. \]

Let \( Q_i \) be the family of projections on \( \mathbb{R}^{d_i} \) of elements of \( Q \) from Section 5, so that \( Q = Q_1 \times \ldots \times Q_m \) and if \( C_l(x) \in Q \), then \( C_l(x) = C_l(\bar{x}_1) \times \ldots \times C_l(\bar{x}_m) \), where \( C_l(\bar{x}_i) \in Q_i \). We set

\[ Q^i = Q_i \times \ldots \times Q_m. \]

We assume that, for each \( i \), we are given a measure \( \mu_i \), which satisfies conditions (3.1) and (4.2) relative to \( \Omega_i \) and satisfy the following doubling condition: if \( C_l(\bar{x}_i) \in Q_i \), then

\[ \mu_i (C_{d_l}(\bar{x}_i) \cap \Omega_i) \leq N_0^{1/m} \mu_i (C_l(\bar{x}_i)). \quad (6.4) \]

Observe that \( N_0 \) in (4.1) or \( N_0^{1/m} \) in (6.4) are not necessarily the best constants for which these conditions are valid and we use \( N_0^{1/m} \) in (6.4) just in order to make our last assumption that \( \mu = \mu_1 \times \ldots \times \mu_m \) somewhat consistent. We set

\[ \mu^i = \mu_i \times \ldots \times \mu_m. \]

Next, we introduce \( A_p(Q_i, \mu_i) \) as \( A_p \)-weights relative to \( Q_i, \mu_i \). One easily checks that if \( w_i \) are \( A_p \)-weights relative to \( Q_i, \mu_i \), then \( w_i w_j \) is an \( A_p \)-weights relative to \( Q_i \times Q_j, \mu_i \times \mu_j \) and

\[ [w_i w_j]_{A_p} \leq [w_i]_{A_p} [w_j]_{A_p}. \]

Finally, for \( i \in \{1, \ldots, m\} \), \( p_i \in (1, \infty) \), and weights \( w_i \) given on \( \Omega_i \) introduce

\[ \| f \|_{L[p_1, \ldots, p_m]}^{p_1, \ldots, p_m} (w_1, \ldots, w_m) := \int_{\Omega_m} \left( \cdots \left( \int_{\Omega_2} \left( \int_{\Omega_1} |f|^{p_1} w_1 \, d\bar{x}_1 \right)^{p_2/p_1} w_2 \, d\bar{x}_2 \right)^{p_3/p_2} \cdots \right)^{p_m/p_{m-1}} w_m \, d\bar{x}_m. \quad (6.5) \]
Theorem 6.2. Let $K^*, p_k ∈ (1, ∞)$, $w_k ∈ A_{p_k}(Ω_k, μ_k)$, $|w_k|_{p_k} ≤ K^*$, $k = 1, \ldots, m$, and let $u, g$ be measurable functions on $Ω$. Then there exists a constant $K_0 = K_0(d, m, p_1, \ldots, p_m, N_0, K^*) ≥ 1$ such that if

$$\|u\|_{L_{p_1}(w)} ≤ \|g\|_{L_{p_1}(w)}$$

for every $w ∈ A_{p_1}(Q, μ)$ with $[w]_{p_1} ≤ K_0$, then we have

$$\|u\|_{L_{p_1(\cdots, p_m(w_1, \ldots, w_m)} ≤ 4^m \|g\|_{L_{p_1(\cdots, p_m(w_1, \ldots, w_m)}}}.$$

Proof. We follow the proof of Corollary 2.7 in [15] or Theorem 8.1 in [17].

If for fixed $0 ≤ j ≤ m − 1$, $K_{j+1} ≥ 1$, and two nonnegative functions $U_j$ and $G_j$ on $\hat{Ω}^{j+1}$ it holds that (a)

$$\int_{Ω^{j+1}+1} U_j^{p_j} w(\hat{x}^{j+1}) μ^{j+1}(d\hat{x}^{j+1}) ≤ 4^{(j-1)p_j} \int_{Ω^{j+1}+1} G_j^{p_j} w(\hat{x}^{j+1}) μ^{j+1}(d\hat{x}^{j+1})$$

for every $w ∈ A_{p_j}(Q^{j+1}, μ^{j+1})$ as long as

$$[w]_{p_j} ≤ K_j = K(p_j, p_{j+1}, K^*K_{j+1}, N_0), \quad (6.7)$$

then we have (b)

$$\int_{Ω^{j+1}+1} U_j^{p_j+1} w(\hat{x}^{j+1}) μ^{j+1}(d\hat{x}^{j+1}) ≤ 4^{p_j+1} \int_{Ω^{j+1}+1} G_j^{p_j+1} w(\hat{x}^{j+1}) μ^{j+1}(d\hat{x}^{j+1}) \quad (6.8)$$

for every $w ∈ A_{p_{j+1}}(Q^{j+1}, μ^{j+1})$ with $[w]_{p_{j+1}} ≤ K^*K_j+1$.

We define $K_m = 1$ and define $K_j$ for $j = 0, 1, \ldots, m − 1$ recurrently by the equation in (6.7).

Also set $U_0(x) = u(x)$,

$$U_j(\hat{x}^{j+1}) = \left(\int_{Ω_j} U_{j-1}^{p_j}(\hat{x}^{j}) w_j(\hat{x}_j) μ_j(d\hat{x}_j)\right)^{1/p_j}, \quad 1 ≤ j ≤ m − 1,$$

and similarly we introduce $G_j$’s by taking $g$ in place of $u$. To prove the theorem, it suffices to prove that (b) holds for $j = m − 1$ because $w_m ∈ A_{p_m}(Ω_m, μ_m)$ and $[w_m]_{A_{p_m}} ≤ K^* = K^*K_m$. We are going to use the induction on $j = 0, 1, \ldots, m − 1$.

Observe that (b) holds for $j = 0$ by assumption. Suppose that it holds for a $j ∈ \{0, 1, \ldots, m − 2\}$. Then (6.8) also holds for

$$w(\hat{x}^{j+2}) ∈ A_{p_{j+1}}(Q^{j+2}, μ^{j+2})$$

if $w(\hat{x}^{j+2}) ∈ A_{p_{j+1}}(Q^{j+2}, μ^{j+2})$ and $[w(\hat{x}^{j+2})]_{A_{p_{j+1}}} ≤ K_{j+1}$, because then $[w(\hat{x}^{j+1})]_{A_{p_{j+1}}} ≤ K^*K_{j+1}$. Remarkably, this implies that (a) holds with $j + 1$ in place of $j$. Then (b) also holds with $j + 1$ in place of $j$. This justifies the induction and proves the theorem.

Remark 6.3. By relabeling the coordinates one sees that Theorem 6.2 holds true if the repeated integrals in (6.5) are taken in any other order.
7. Generalized Fefferman-Stein theorem

We keep working in the setting of Section 5 with $\mu$ satisfying (3.1), (4.2), and (5.1). All $A_p$-weights are also taken from there. Since, obviously, $Mf \leq Mf$ (cf. (3.8) and (4.10)) the following is a corollary of Theorem 5.3 (ii).

**Theorem 7.1** (Hardy-Littlewood). Let $p \in (1, \infty)$ and $w \in A_p$. Then for any $f \in L^p(w)$ we have

$$\|Mf\|_{L^p(w)} \leq N\|f\|_{L^p(w)},$$

(7.1)

where $N$ depends only on $p$, $N_0$, and $[w]_{A_p}$.

Next result is Lemma 2.8 of [17]. Its proof given for the sake of completeness is also taken from there. Recall that $C$ is introduced in Section 3.

**Lemma 7.2.** Let $\gamma \in (0, 1]$, $vI_C \in L_1(\mu)$ for any $C \in C$, and let $v|_n \to 0$ as $n \to -\infty$ on $\Omega$. Assume that $|u| \leq v$ and for any $C \in C$ there exists a measurable function $u^C$ given on $C$ such that $|u| \leq u^C \leq v$ on $C$ and, for any $x \in C$

$$\left(\int_C \int_C |u^C(z) - u^C(y)|^\gamma \mu(dz)\mu(dy) \right)^{1/\gamma} \leq g(x).$$

(7.2)

Let $p \in [1, \infty)$ and $w$ be an $A_p$-weight. Then for any $\lambda > 0$ we have

$$w(\{x : |u(x)| \geq \lambda\}) \leq N\nu^{-\beta}\lambda^{-\gamma\beta} \int_\Omega g^\beta(x)I_{Mw(x) > \alpha \lambda} w(dx),$$

(7.3)

where $\alpha = (2N_0)^{-1}$ and $\nu = 1 - 2^{-\gamma}$ and the constants $\beta \in (0, 1)$ and $N$ depend only on $p, N_0$, and $[w]_{A_p}$.

Proof. Obviously we may assume that $u \geq 0$. Fix a $\lambda > 0$ and define

$$\tau(x) = \inf \{n \in \mathbb{Z} : v|_n(x) > \alpha \lambda\}.$$

We know that $\tau$ is a stopping time and if $\tau(x) < \infty$, then

$$v|_n(x) \leq \lambda/2, \quad \forall n \leq \tau(x).$$

We also know that $v|_n \to v \geq u$ (a.e.) as $n \to \infty$. It follows that (a.e.)

$$\{x : u(x) \geq \lambda\} = \{x : u(x) \geq \lambda, \tau(x) < \infty\} = \{x : u(x) \geq \lambda, v|_\tau(x) \leq \lambda/2\} = \bigcup_{n \in \mathbb{Z}} \bigcup_{C \in \mathcal{F}_n^\tau} A_n(C),$$

where

$$A_n(C) := \{x \in C : u(x) \geq \lambda, v|_n(x) \leq \lambda/2\},$$

and $\mathcal{F}_n^\tau$ is the family of disjoint elements of $C_n$ such that

$$\{x : \tau(x) = n\} = \bigcup_{C \in \mathcal{F}_n^\tau} C.$$
Next, for each \( n \in \mathbb{Z} \) and \( C \in \mathcal{C}_n \) on the set \( A_n(C) \), if it is not empty, we have \( v_{|n} = v_C \) and on \( A_n(C) \)
\[
 u^\gamma - (v_C)^\gamma \geq \lambda^\gamma (1 - 2^{-\gamma}) = \nu \lambda^\gamma.
\]
We use this and the inequality \( |a - b|^\gamma \geq |a| - |b| >= \gamma \) and conclude that for \( x \in A_n(C) \),
\[
 \int_C |u^C(x) - u^C(y)|^\gamma \mu(dy) \geq (u^C(x))^\gamma - \int_C (u^C(y))^\gamma \mu(dy) \\
 \geq u^\gamma(x) - \int_C v^\gamma(y) \mu(dy) \geq u^\gamma(x) - (v_C(x))^\gamma \geq \nu \lambda^\gamma,
\]
so that by Chebyshev’s inequality
\[
 \mu(A_n(C)) \leq \nu^{-1} \lambda^{-\gamma} \int_C \int_C |u^C(z) - u^C(y)|^\gamma \mu(dz) \mu(dy).
\]
It follows by assumption (7.2) that
\[
 \frac{\mu(A_n(C))}{\mu(C)} \leq \nu^{-1} \lambda^{-\gamma} g^\gamma(x)
\]
for any \( x \in \Omega \). Corollary 4.6 implies that
\[
 w(A_n(C)) \leq N_{w,\beta} \nu^{-\beta} \lambda^{-\gamma \beta} g^{\gamma \beta}(x) w(C).
\]
Since this holds for any \( x \in C \),
\[
 w(A_n(C)) \leq N_{w,\beta} \nu^{-\beta} \lambda^{-\gamma \beta} \int_C g^{\gamma \beta}(x) w(dx).
\]
Hence,
\[
 w\{ x : u(x) \geq \lambda \} \leq N_{w,\beta} \nu^{-\beta} \lambda^{-\gamma \beta} \sum_{n \in \mathbb{Z}} \sum_{C \in \mathcal{F}_n} \int_C g^{\gamma \beta} w(dx) \\
 = N_{w,\beta} \nu^{-\beta} \lambda^{-\gamma \beta} \int_\Omega g^{\gamma \beta} I_{\tau < \infty} w(dx).
\]
It only remains to observe that \( \{ \tau < \infty \} = \{ Mv > \alpha \lambda \} \). The lemma is proved.

Remark 7.3. It is worth saying a few words about the history of Lemma 7.2, which is the core of the approach based on the Fefferman-Stein theorem. In case \( \gamma = 1 \), \( u = u^C = v \), \( w(x) \equiv 1 \), and slightly more general right-hand side of (7.2), estimate (7.3) with \( \beta = 1 \) becomes Lemma 3.2.9 of [41] and serves there as one of the mains tools of obtaining a priori estimates for Sobolev solutions of linear elliptic and parabolic equations with continuous or VMO or else almost VMO coefficients. The point is that (7.2) for \( D^2u \) in place of \( u^C \) amounts to pointwise estimating the so-called sharp function of \( D^2u \) (cf. Theorems 2.2, 2.3) and (7.3) easily leads to estimates of the \( L_p \)-norms of \( D^2u \) through the norm of its sharp function (cf. the
proof of Theorem 7.4). In this form Lemma 7.2 was designed to use the Fefferman-Stein theorem instead of explicit singular integral representation of the derivatives of solutions and subsequent application of the Coifman-Rochberg-Weiss commutator theorem for singular integrals in order to treat equations with main VMO coefficients, which was first done in [6] and later continued in an avalanche of papers. Using the Fefferman-Stein theorem does not require any explicit representation formulas and turned out to be applicable to many linear and fully nonlinear equations.

In its initial form Lemma 7.2 turned out to provide a crucial information even in the case when the coefficients are only measurable with respect to one variable and almost VMO with respect to the others. This line of research started in [32] and [33] and continued in many papers, see, for instance, [11, 12, 13, 14], and the references therein. In particular, in [31] it is allowed for all main coefficients to depend in a measurable way on time and one space coordinate apart from one which is supposed to be measurable in time and VMO in spatial variables. In the elliptic case in [16] the coefficients are measurable with respect to two spatial variables.

The need to generalize Lemma 7.2 and add $\mu_C$ still with $\gamma = \beta = 1, w \equiv 1$ comes when one wants to allow the direction in which the coefficients are only measurable to depend on $x$. This was first noted in [42] for nondivergence type equations and first used in [10] for divergence equations.

In a remarkable paper [15] the authors proved the version of Lemma 7.2 with $\gamma = 1$ but with $A_p$-weights. This allowed the authors to get mixed norms estimates just by referring to Theorem 6.2.

The necessity to have $\gamma \in (0,1]$ (actually, very small one) came when we started applying the same methodology to fully nonlinear equations. Then Lemma 7.2 with $\beta = 1, w \equiv 1$, appeared in [43].

In its final form Lemma 7.2 was needed in [17] when we investigated fully nonlinear equations in the spaces with mixed norms.

Whenever it makes sense define the sharp “dyadic” function of $f$ by

$$f_\gamma^\#(x) = \sup_{C \in \mathcal{C}, C \ni x} \left( \int_C \int_C |f(z) - f(y)|^\gamma \mu(dz) \mu(dy) \right)^{1/\gamma}.$$ 

**Theorem 7.4** (Fefferman-Stein). Let $p \in (1, \infty)$ and $w \in A_p$. Then for any $f \in L_p(w)$ we have

$$\|f\|_{L_p(w)} \leq N \|f_\gamma^\#\|_{L_p(w)}, \quad (7.4)$$

where the constant $N$ depends only on $p, N_0, \gamma, \text{ and } [w]_{A_p}$.

**Proof.** In Lemma 7.2 we take $u = f, v = u^C = |f|, \text{ and } g = f_\gamma^\#$. Then we plug $\lambda^{1/p}$ in place of $\lambda$ in (7.3) and integrate with respect to $\lambda \in (0, \infty)$. This yields

$$\|f\|_{L_p(w)}^p \leq N \int_{\Omega} (f_\gamma^\#)^{\gamma \beta} (\mathcal{M}f)^{p-\gamma \beta} w(dx).$$
By using Hölder’s inequality, we obtain
\[ \|f\|_{L^p(w)}^p \leq N\|f\|_{\gamma_{L^p(w)}}\|\mathcal{M}f\|_{L^p(w)}^{p-\gamma}. \]
After that it only remains to use (7.1). The theorem is proved.

Remark 7.5. As is easy to see the condition \( w \in A_p \) can be replaced by \( w \in A_\infty \). This is, actually, the form in which Theorem 7.4 is proved in [17].

8. AN APPLICATION TO FULLY NONLINEAR PARABOLIC EQUATIONS

Here we consider functions on \( \mathbb{R}^{d+1}_{+,+} = \{(t, x) = (t, x_1, x') : t \geq 0, x_1 \geq 0, x' \in \mathbb{R}^{d-1}\} \).

We concentrate on parabolic equations in \( \mathbb{R}^{d+1}_{+,+} \) with zero Dirichlet boundary condition.

Denote by \( S \) the set of symmetric \( d \times d \) matrix, fix \( \delta \in (0, 1) \), and by \( S_\delta \) denote the subset of \( S \) consisting of matrices whose eigenvalues are between \( \delta \) and \( \delta^{-1} \).

Let \( A \) be a countable set and suppose that on \( \mathbb{R}^{d+1}_{+,+} \) for each \( \alpha \in A \) we are given an \( S_\delta \)-valued measurable function \( a^\alpha(t, x) = (a^\alpha_{ij}(t, x)) \). For \( u'' = (u''_{ij}) \in S \) and \((t, x) \in \mathbb{R}^{d+1}_{+,+}\) introduce
\[ F(u'', t, x) = \sup_{\alpha \in A} \sum_{i,j=1}^d a^\alpha_{ij}(t, x)u''_{ij}. \]

For functions \( u = u(t, x) \) having two derivatives in \( x \) set \( F[u] = F[u](t, x) = F(D^2u(t, x), t, x) \). Also denote \( \partial_t = \partial/\partial t \).

For \((t, x) \in \mathbb{R}^{d+1}_{+,+}\) and \( r > 0 \) denote \( B_r = \{x \in \mathbb{R}^d : |x| < r\}, B_r(x) = x + B_r, B^+_r(x) = B_r(x) \cap \{x^1 > 0\}, \)
\[ C^+_r(t, x) = [t, t+r^2) \times B^+_r(x). \]

Next assumption contains a parameter \( \theta \in (0, 1) \) which will be specified later.

Assumption 8.1. There is an \( R_0 \in (0, \infty) \) such that for any \( \alpha \in A \), \( z \in \mathbb{R}^{d+1}_{+,+} \), and \( r \in (0, R_0) \), one can find \( \bar{a}^\alpha \in S_\delta \) (independent of \((t, x)\)) such that
\[ \int_{C^+_r(z)} \sup_{\alpha \in A} |a^\alpha(t, x) - \bar{a}^\alpha| \, dx \, dt \leq \theta. \]

Here is one of the results from [17] obtained by combining the above results and results from [44].

Theorem 8.2. Let \( p_1, p_2, p_3 > d + 1 \), and \( u \in W^{1,2}_{1,loc} (\mathbb{R}^{d+1}_{+,+}). \) Suppose that \( u \) vanishes on \( \{x_1 = 0\} \). Finally, take \( q \in (-1, p_1/(d + 1) - 1) \). Then there
exists \( \theta = \theta(d, \delta, p_1, p_2, p_3, q) \in (0, 1) \) such that, if Assumption 8.1 is satisfied with this \( \theta \), then

\[
\int_0^\infty \left( \int_{\mathbb{R}^{d-1}} \left( \int_0^\infty x_1^q \left( |D^2 u| + |Du| + |u| \right)^{p_1} \, dx_1 \right)^{p_2/p_1} \, dx' \right)^{p_3/p_2} \, dt
\leq N \int_0^\infty \left( \int_{\mathbb{R}^{d-1}} \left( \int_0^\infty x_1^q |\partial_t u + F[u] - u|^{p_1} \, dx_1 \right)^{p_2/p_1} \, dx' \right)^{p_3/p_2} \, dt,
\]  

(8.1)

provided that the left-hand side is finite, where \( N \) depends only on \( d, \delta, p_1, p_2, p_3, q, \) and \( R_0 \). The one-dimensional example of \( F[u] = D^2 u \) and \( u(t, x) = \sinh x \) shows that (8.1) is wrong without the additional assumption on its left-hand side.

Remark 8.3. The reader understands that one has similar estimates for the integrals with respect to \( x_1, x', \) and \( t \) mixed in any other order (cf. Remark 6.3). For some readers \( \partial_t u + F[u] - u \) may look unusual in comparison with \( F[u] - \partial_t u - u \). One gets the corresponding result for the latter operator by changing \( t \to -t \).

Remark 8.4. In [34] the authors consider linear \( F \) with coefficients depending only on time in a measurable way and prove a priori estimates similar to the one in Theorem 8.2, however, for any \( p_1 = p_2, p_3 > 1 \) and \( q \in (-1, 2p_1 - 1) \). The latter range is much wider than ours \( (-1, p_1/(d + 1) - 1) \), but our operators are much more general and we have three integrals.

Also note that the range \( (p_1 - 1, 2p_1 - 1) \) was used in [36] to build the solvability theory of parabolic equations in Sobolev spaces with weights with the highest order of derivatives being an arbitrary given number: positive, negative, integral or fractional.

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