On the geometry of classically integrable
two-dimensional non-linear sigma models

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Abstract

A master equation expressing the zero curvature representation of the equations of
motion of a two-dimensional non-linear sigma models is found. The geometrical prop-
erties of this equation are outlined. Special attention is paid to those representations
possessing a spectral parameter. Furthermore, a closer connection between integrabil-
ity and T-duality transformations is emphasised. Finally, new integrable non-linear
sigma models are found and all their corresponding Lax pairs depend on a spectral
parameter.

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1 Introduction

The problem of finding dynamical systems which are integrable is a fascinating subject in mathematics and theoretical physics. In classical mechanics integrability is understood as the possibility of finding as many conserved quantities as the number of degrees of freedom of the dynamical system. It happens, in some cases, that these conserved quantities lead to the exact solvability of the associated equations of motion. In field theory, however, an infinite number of conserved charges is required for integrability.

The Lax formulation of integrability provides a method for constructing conserved dynamical quantities. In this formulation, a two-dimensional field theory is considered to be classically integrable if a Lax pair \((\mathcal{A}_0, \mathcal{A}_1)\) can be found such that the linear system\(^1\)

\[
\begin{align*}
[\partial_0 + \mathcal{A}_0 (\lambda)] \Psi & = 0 \\
[\partial_1 + \mathcal{A}_1 (\lambda)] \Psi & = 0
\end{align*}
\]  

(1.1)
yields, as its consistency condition, the equations of motion of the two-dimensional theory under consideration. Here the matrices \(\mathcal{A}_0\) and \(\mathcal{A}_1\) depend on the fields of the theory and possibly on some free arbitrary parameter \(\lambda\), known as the spectral parameter. This parameter can be very useful in extracting conserved quantities. The fields \(\Psi\) can be either a column vector or a matrix of the same dimension as \(\mathcal{A}_0\) and \(\mathcal{A}_1\). The consistency condition (usually referred to as the zero curvature condition) of this linear system is clearly

\[\{\partial_0 \mathcal{A}_1 - \partial_1 \mathcal{A}_0 + [\mathcal{A}_0, \mathcal{A}_1]\} \Psi = 0.\]

The conserved quantities are then constructed using the so-called monodromy matrix

\[T (\lambda, \tau) = P \exp \left( - \int_0^{2\pi} \mathcal{A}_1 (\lambda, \sigma, \tau) \, d\sigma \right),\]

(1.2)

where \(P\) stands for the path-ordered exponential and we have chosen \(\sigma\) to be in the interval \([0, 2\pi]\). One can show that the traces of powers of the monodromy matrix, \(\text{Tr} [T^n (\lambda, \tau)]\), are independent of the time \(\tau\) and are in involution with respect to Poisson brackets:

\[\{\text{Tr} [T^n (\lambda_1, \tau)], \text{Tr} [T^n (\lambda_2, \tau)]\} = 0.\]

The proof of the first statement assumes the periodicity condition \(\mathcal{A}_0 (\lambda, 0, \tau) = \mathcal{A}_0 (\lambda, 2\pi, \tau)\). Expanding \(\text{Tr} [T^n (\lambda, \tau)]\) in powers of \(\lambda\) generates an infinite set of conserved charges (see [1, 2] for more details).

In this paper we would like to examine the question of integrability in two-dimensional non-linear sigma models. This is because there are only a handful cases of such theories which are known to be integrable (the principal chiral model, the Wess-Zumino-Witten model and their various modifications [3, 4, 5, 6]). It is therefore important to investigate whether other integrable models exist. Furthermore, the study of the properties of non-linear sigma models involves often the geometry of the target space on which these theories are defined. For instance, the renormalisation properties of these models constrains the geometry of the target space [7]. It will be shown in this paper that the requirement of integrability puts further constraints on the allowed target spaces. This could be of crucial importance to string theory as non-linear sigma models are supposed to describe the propagation of the massless modes of bosonic string theory [8]. In other words, the conditions for conformal

\[^1\text{Here, the two-dimensional coordinates are } (\tau, \sigma) \text{ with } \partial_0 = \frac{\partial}{\partial \tau} \text{ and } \partial_1 = \frac{\partial}{\partial \sigma}. \text{ In the rest of the paper, however, we will use the complex coordinates } (z = \tau + i\sigma, \bar{z} = \tau - i\sigma) \text{ together with } \partial = \frac{\partial}{\partial z} \text{ and } \bar{\partial} = \frac{\partial}{\partial \bar{z}}.\]
invariance at the quantum level (the vanishing of the beta functions) and the requirement of classical integrability of non-linear sigma models might reduce the number of possibilities for the spaces on which one can carry out the compactification of the extra dimensions of string theory.

We start this paper by giving the general framework for a zero curvature representation of the equations of motion of a two-dimensional non-linear sigma model. We derive a target space condition for this requirement and analyse its resulting geometry. In section 3, we provide some known solutions to this condition. Section 4 deals with the issue of introducing a spectral parameter in the Lax pair construction and further geometrical properties are analysed there. We then study, in section 5, the integrability of a non-linear sigma model which generalises the principal chiral sigma model. In section 6, the interplay between T-duality and integrability of non-linear sigma models is explored. This work is a continuation of an earlier investigation [9].

2 Zero curvature representation of non-linear sigma models

A two-dimensional non-linear sigma model is an interacting theory for some scalar fields $\varphi^i(z, \bar{z})$ as described by the action

$$ S = \int dz d\bar{z} Q_{ij}(\varphi) \partial \varphi^i \bar{\partial} \varphi^j . \quad (2.1) $$

The metric and the anti-symmetric tensor fields of this theory are defined as

$$ g_{ij} = \frac{1}{2} (Q_{ij} + Q_{ji}) , \quad b_{ij} = \frac{1}{2} (Q_{ij} - Q_{ji}) . \quad (2.2) $$

We will assume that the metric $g_{ij}$ is invertible and its inverse is denoted $g^{ij}$. Indices are raised and lowered using this metric. We will also define, respectively, the Christoffel symbols, the torsion and the generalised connection as follows

$$ \Gamma^k_{ij} = \frac{1}{2} g^{kl} (\partial_i g_{lj} + \partial_j g_{li} - \partial_l g_{ij}) , $$

$$ H^k_{ij} = \frac{1}{2} g^{kl} (\partial_i b_{lj} + \partial_j b_{li} + \partial_l b_{ij}) , $$

$$ \Omega^k_{ij} = \Gamma^k_{ij} - H^k_{ij} . \quad (2.3) $$

The equations of motion of this theory can be written as

$$ \mathcal{E}^l \equiv \bar{\partial} \partial \varphi^l + \Omega^l_{ij} \partial \varphi^i \bar{\partial} \varphi^j = 0 . \quad (2.4) $$

Let us now construct a linear system whose consistency conditions are equivalent to these equations of motion (a zero curvature representation). We take, as an ansatz, this linear system to have the following form

$$ \begin{bmatrix} \partial + \alpha_i (\varphi) \partial \varphi^i \end{bmatrix} \Psi = 0 , $$

$$ \begin{bmatrix} \bar{\partial} + \beta_j (\varphi) \bar{\partial} \varphi^j \end{bmatrix} \Psi = 0 , \quad (2.5) $$
where $\alpha_i$ and $\beta_i$ are two matrices depending on the fields $\varphi^i$. This form of the Lax pair is dictated by the fact that the equations of motions of the non-linear sigma model do not contain terms involving $\partial^2$ or $\bar{\partial}^2$.

The compatibility condition of the linear system takes then the form

$$\mathcal{F} \equiv (\beta_i - \alpha_i) \bar{\partial} \varphi^i + (\partial_i \beta_j - \partial_j \alpha_i + [\alpha_i, \beta_j]) \partial \varphi^i \bar{\partial} \varphi^j = 0 \ .$$

(2.6)

The non-linear sigma model enjoys a zero curvature representation of its equations of motion if this compatibility condition can be written as

$$\mathcal{F} = \mathcal{E}^i \mu_i = 0$$

(2.7)

for some matrices $\mu_i (\varphi)$. In order for this last relation to yield $\mathcal{E}^i = 0$ as the only non trivial possibility, the matrices $\mu_i$ have to be linearly independent and their number must be equal to the dimension of the target space of the non-linear sigma model.

The compatibility condition of the linear system yields the equations of motion of the two-dimensional non-linear sigma model, that is equation (2.7) holds, provided that the matrices $\alpha_i (\varphi), \beta_i (\varphi)$ and $\mu_i (\varphi)$ satisfy

$$\beta_i - \alpha_i = \mu_i \linebreak \partial_i \beta_j - \partial_j \alpha_i + [\alpha_i, \beta_j] = \Omega^l_{ij} \mu_l \ .$$

(2.8)

The first equation gives simply $\beta_i$ in terms of $\alpha_i$ and $\mu_i$

$$\beta_i = \alpha_i + \mu_i \ .$$

(2.9)

The second equation of the above set can then be written as

$$F_{ij} = - \left( \nabla_i \mu_j - \Omega^l_{ij} \mu_k \right) \ ,$$

(2.10)

where we have introduced, for later use, the field strength $F_{ij}$ and the gauge covariant derivative corresponding to the matrices $\alpha_i$

$$F_{ij} = \partial_i \alpha_j - \partial_j \alpha_i + [\alpha_i, \alpha_j] \linebreak \nabla_i X = \partial_i X + [\alpha_i, X] \ ,$$

(2.11)

where $X$ denotes any matrix valued quantity.

Equation (2.10) is at the centre of the ability to represent the equations of motion of a non-linear sigma model as a zero curvature condition of a linear system. The unknowns of the problem are the two sets of matrices $\alpha_i$ and $\mu_i$ and the generalised connection $\Omega^k_{ij}$.

Each triplet $(\alpha_i, \mu_i, \Omega^k_{ij})$ satisfying (2.10), yields a non-linear sigma model with a zero curvature representation (provided that one can extract $g_{ij}$ and $b_{ij}$ from the knowledge of $\Omega^k_{ij}$). However, equation (2.10) does not guarantee that the matrices $\alpha_i$ and $\mu_i$ will depend on a spectral parameter (which plays an important role in the construction of the conserved quantities of two-dimensional integrable theories). Let us now explore some properties of this central equation.

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2These equations are found by comparing the terms involving $\partial \bar{\partial} \varphi^i$ and $\partial \varphi^i \bar{\partial} \varphi^j$ on both sides of (2.7).
The geometry

The consistency relation \((\partial_i \partial_j \mu_k - \partial_j \partial_i \mu_k = 0)\) of equation (2.10) is given by
\[
\mathcal{R}^n_{jik} \mu_n = -D_j F_{ik} \equiv \nabla_j F_{ik} + [\mu_j, F_{ik}] - \Omega^m_{ij} F_{mk} - \Omega^m_{jm} F_{ik} ,
\]
where we have used the Bianchi identities \(\nabla_k F_{ij} + \nabla_j F_{ki} + \nabla_i F_{jk} = 0\). Here \(\mathcal{R}^n_{jik}\) is the generalised curvature tensor and is defined by
\[
\mathcal{R}^n_{jik} = \partial_i \Omega^n_{kj} - \partial_k \Omega^n_{ij} + \Omega^m_{im} \Omega^n_{mj} - \Omega^n_{km} \Omega^m_{ij} .
\]
We notice immediately that if \(F_{ij} = 0\) (that is, \(\alpha_i = M^{-1} \partial_i M\) for some invertible matrix \(M(\varphi)\)), then \(\mathcal{R}^n_{jik} \mu_n = 0\). Since the matrices \(\mu_i\) are assumed to be linearly independent, we have \(\mathcal{R}^n_{jik} = 0\) and the target space of the non-linear sigma model is, in this case, parallelisable.

In the context of string theory, non-linear sigma models describe the propagation of strings in non-trivial backgrounds. The consistency of this propagation is equivalent to the vanishing of the beta functions of the non-linear sigma model [10, 11, 12]. At the one loop level, these beta functions (in the absence of the dilaton field) are characterised by the generalised Ricci tensor \(\mathcal{R}^n_{i} = g^{jk} \mathcal{R}^n_{jik}\). If the non-linear sigma model admits a zero curvature representation then its generalised Ricci tensor satisfies
\[
\mathcal{R}^n_{i} \mu_n = -D^k F_{ik} .
\]
In particular, if the generalised Ricci tensor vanishes (namely, conformal invariance holds at the one loop level) then one has
\[
D^k F_{ik} = 0 .
\]
This last equation is a generalisation of the equations of motion of pure non-Abelian Yang-Mills gauge theory.

It is also interesting to split equation (2.10) into its symmetric and anti-symmetric parts. This yields
\[
0 = \nabla_i \mu_j + \nabla_j \mu_i - 2\Gamma^k_{ij} \mu_k ,
\]
\[
F_{ij} = -\frac{1}{2} (\nabla_i \mu_j - \nabla_j \mu_i) - H^k_{ij} \mu_k ,
\]
The first equation is a gauged version of a matrix valued Killing equation. Indeed, if \([\alpha_i, \mu_j] + [\alpha_j, \mu_i] = 0\) then this first equation is simply \(\partial_i \mu_j + \partial_j \mu_i - 2\Gamma^k_{ij} \mu_k = 0\). In this case the entries of \(\mu_i\) are Killing vectors (isometries) of the metric \(g_{ij}\).

The linear system (2.5) with \(\beta_i = \alpha_i + \mu_i\) could be made ‘more symmetric’ by writing \(\alpha_i = \gamma_i - \frac{1}{2} \mu_i\) and \(\beta_i = \gamma_i + \frac{1}{2} \mu_i\), for some matrices \(\gamma_i(\varphi)\). The linear system takes then the form
\[
\left[ \partial + \left( \gamma_i - \frac{1}{2} \mu_i \right) \partial \varphi^i \right] \Psi = 0
\]
\[
\left[ \partial + \left( \gamma_j + \frac{1}{2} \mu_j \right) \partial \varphi^j \right] \Psi = 0 .
\]
In terms of the matrices $\gamma_i$, the symmetric and anti-symmetric parts of equation (2.8) give
\[ \partial_i \mu_j + \partial_j \mu_i + [\gamma_i, \mu_j] + [\gamma_j, \mu_i] - 2 \Gamma^k_{ij} \mu_k = 0 \]
\[ \partial_i \gamma_j - \partial_j \gamma_i + [\gamma_i, \gamma_j] - \frac{1}{4} [\mu_i, \mu_j] + H^k_{ij} \mu_k = 0 \quad . \tag{2.18} \]

The advantage of working with the matrices $\gamma_i$ is that the derivatives of $\mu_i$ do not appear in the anti-symmetric part of the last set of equations.

Another geometric structure occurs when introducing the following change of variables
\[ \mu_i = 2g_{ij} \nu^j = (Q_{ij} + Q_{ji}) \nu^j \]
\[ \tilde{\alpha}_i = \alpha_i + Q_{il} \nu^l \quad . \tag{2.19} \]

In terms of the new variables $\nu^i$ and $\tilde{\alpha}_i$, equation (2.10) takes the form
\[ \nu^l \partial_i Q_{ij} + Q_{lj} \bar{\nabla}_j \nu^l + Q_{il} \bar{\nabla}_j \nu^l = Q_i k Q_{lj} [\nu^k, \nu^l] - \bar{F}_{ij} \quad , \tag{2.20} \]
where $\bar{F}_{ij} = \partial_i \tilde{\alpha}_j - \partial_j \tilde{\alpha}_i + [\tilde{\alpha}_i, \tilde{\alpha}_j]$ and $\bar{\nabla}_j \nu^l = \partial_j \nu^l + [\tilde{\alpha}_j, \nu^l]$. Notice that the left-hand side of this last equation is a gauged version of a matrix valued Lie derivative for the tensor $Q_{ij}$. With the new variables, $\tilde{\alpha}_i$ and $\nu^i$, the linear system is given by
\[ \left[ \partial + (-Q_{il} \nu^l + \tilde{\alpha}_i) \partial \varphi^i \right] \Psi = 0 \]
\[ \left[ \bar{\partial} + (Q_{kj} \nu^k + \tilde{\alpha}_j) \bar{\partial} \varphi^i \right] \Psi = 0 \quad . \tag{2.21} \]

Equation (2.20), when $\tilde{\alpha}_i = 0$ and when $\nu^i$ take values in a Lie algebra, is precisely the relation encountered in the context of Poisson-Lie duality and whose solution was given by Klimčík and Ševera in [13, 14].

Furthermore, equation (2.20) can be interpreted in the following way: Let us choose two currents such that
\[ I^a = (v^{-1})_i^a \partial \varphi^i \]
\[ \bar{I}^a = (w^{-1})_i^a \bar{\partial} \varphi^i \quad . \tag{2.22} \]
with $v^i_a$ and $w^i_a$ two field-dependent matrices and whose inverses are, respectively, $(v^{-1})^a_i$ and $(w^{-1})^a_i$. In terms of these currents, the equations of motion of the non-linear sigma model (2.1) are expressed as
\[ \mathcal{E}_i \equiv Q_{ij} w^j_a \partial \bar{I}^a + Q_{il} v^l_a \bar{\partial} I^a - \left[ v^i_a w^j_b \partial_i Q_{ij} - v^k_a \partial_k \left( Q_{ij} w^j_b \right) - w^k_b \partial_k \left( Q_{il} v^l_a \right) \right] I^a \bar{I}^b = 0 \quad . \tag{2.23} \]
In addition, these currents satisfy the Bianchi identities (stemming from the identity $\partial \bar{\partial} \varphi^i - \bar{\partial} \partial \varphi^i = 0$ in (2.22))
\[ \mathcal{B}^l \equiv w^i_a \partial \bar{I}^a - v^l_a \bar{\partial} I^a + \left( v^k_a \partial_k w^l_b - w^k_b \partial_k v^l_a \right) I^a \bar{I}^b = 0 \quad . \tag{2.24} \]

In terms of these currents the Lax pair (2.5) takes the form
\[ \left[ \partial + \alpha_i v^i_a I^a \right] \Psi = 0 \]
\[ \left[ \bar{\partial} + \beta_j w^j_b \bar{I}^b \right] \Psi = 0 \quad . \tag{2.25} \]
The consistency condition (the zero curvature condition) of this Lax pair is
\[ Z \equiv w^b_\beta \partial \beta^b - w^a_\alpha \partial \alpha^a + \left\{ v^k_\partial \partial_k \left( v^i_\beta \beta^i \right) - w^b_\alpha \partial_k \left( v^i_\alpha \partial_i \right) + v^i_\alpha \left[ \alpha^i_\beta \right] \right\} I^a \bar{I}^b = 0. \quad (2.26) \]

Equation (2.20) amounts then to demanding that
\[ Z = \nu^l E_l + \tilde{\alpha}^l B_l, \quad (2.27) \]
where
\[ \alpha^i = -Q_{il} \nu^l + \tilde{\alpha}_i \]
and
\[ \beta^i = Q^{il} \nu_l + \tilde{\alpha}_i. \]
This means that the Lax pair yields the equations of motion of the non-linear sigma model up to terms which identically vanish. The importance of working with currents will show up in the rest of the paper.

3 Known Solutions

As stated above, all the quantities entering equation (2.10) are unknowns. In order to find some solutions, we proceed by fixing some of these unknowns.

As a start, let us first check that this formalism reproduces the two well-known integrable non-linear sigma models, namely the principle chiral model and the Wess-Zumino-Witten model. These models are found by taking the following expressions for the matrices \( \alpha^i \) and \( \mu^i \)
\[ \alpha^i = x g^{-1} \partial_i g, \quad \mu^i = y g^{-1} \partial_i g, \quad (3.1) \]
where \( g(\varphi) \) is a Lie group element corresponding to some Lie algebra \( G \) defined by the commutation relations \([T_a, T_b] = f^{c}_{ab} T_c\). The indices of the Lie algebra \( a, b, c, \ldots \) have the same range as those of the target space of the sigma model \( i, j, k, \ldots \). We will use the fact that the gauge connection \( A_i = g^{-1} \partial_i g = e^a_i(\varphi) T_a \) satisfies the Bianchi identity \( \partial_i A_j - \partial_j A_i + [A_i, A_j] = 0 \). The inverses of the vielbiens \( e^a_i \) are denoted \( E^a_i \) and satisfy \( e^a_i E^b_i = \delta^a_b \) and \( e^a_i E^a_j = \delta^j_i \). Finally, the quantities \( x \) and \( y \) are two constant parameters which will provide the spectral parameter. We assume that \( x \) and \( y \) are different from zero.

Injecting the expressions of \( \alpha^i \) and \( \mu^i \) in (2.10) leads to
\[ \Gamma^k_{ij} = \frac{1}{2} E^k_a \left( \partial_i e^a_j + \partial_j e^a_i \right) \]
\[ H^k_{ij} = \kappa e^a_i e^b_j E^c_k f^e_{ab}, \quad (3.2) \]
where \( \kappa = -\frac{1}{y} \left( x^2 - x + xy - \frac{1}{2} y \right) \).

The above Christoffel symbols \( \Gamma^k_{ij} \) and torsion \( H^k_{ij} \) are those corresponding to the following metric \( g_{ij} \) and anti-symmetric tensor \( b_{ij} \)
\[ g_{ij} = \eta_{ab} e^a_i e^b_j \]
\[ H_{ijk} = \kappa \eta_{ac} f^d_{bc} e^a_i e^b_j e^c_k, \quad (3.3) \]
where \( \eta_{ab} \) is an invertible bilinear form of the Lie algebra \( G \) satisfying \( \eta_{ab} f^b_{cd} + \eta_{cb} f^b_{ad} = 0 \). Owing to the property that \( \partial_i e^a_j - \partial_j e^a_i + f^d_{bc} e^b_i e^c_j = 0 \), the torsion is a closed three form. Therefore \( b_{ij} \) exists locally.
To summarise, the Lax pair construction for the class of theories represented by the metric and the torsion in (3.3) is given by

\[
\left[ \partial + x \left( g^{-1} \partial_i g \right) \partial \varphi^i \right] \Psi = 0 \tag{3.4}
\]

\[
\left[ \bar{\partial} + \frac{x(2\kappa + 1)}{2x + 2\kappa - 1} \left( g^{-1} \partial_j g \right) \bar{\partial} \varphi^j \right] \Psi = 0 ,
\]

with \(x\) being the spectral parameter and \(\kappa\) a parameter defining the different models. The class of non-linear sigma models defined by (3.3) includes the principal chiral sigma model (\(\kappa = 0\)); the Wess-Zumino-Witten model (\(\kappa = \frac{1}{2}\)); and the non-conformally invariant Wess-Zumino-Witten model (\(\kappa \neq \frac{1}{2}\)).

Another interesting theory is found when the matrices \(\alpha_i\) and \(\mu_i\) are constant. In this case we take

\[
\alpha_i = x T_i \quad , \quad \mu_i = y T_i \quad ,
\]

where \([T_i, T_j] = f^k_{ij} T_k\). Replacing these in equation (2.10) yields

\[
\begin{align*}
\Gamma^k_{ij} &= 0 \\
H^k_{ij} &= \rho f^k_{ij} ,
\end{align*}
\]

where \(\rho = -\frac{1}{y} (x^2 + xy)\).

These relations yield a non-linear sigma model defined by

\[
\begin{align*}
g_{ij} &= \eta_{ij} \\
b_{ij} &= \frac{2}{3} \rho \eta_{kl} f^l_{ij} \varphi^k ,
\end{align*}
\]

where \(\eta_{ij}\) is the invertible bilinear form corresponding to the Lie algebra \([T_i, T_j] = f^k_{ij} T_k\) (\(\eta_{ij}\) must satisfy \(\eta_{kl} f^l_{ij} + \eta_{il} f^l_{kj} = 0\) in order for \(H_{ijk} = \rho \eta_{kl} f^l_{ij}\) to be totally anti-symmetric).

Therefore, the linear system for this non-linear sigma model (3.7) is given by

\[
\left[ \partial + x T_i \partial \varphi^i \right] \Psi = 0 \\
\left[ \bar{\partial} + \frac{\rho x}{x + \rho} T_j \bar{\partial} \varphi^j \right] \Psi = 0 ,
\]

where \(x\) plays the role of the spectral parameter while \(\rho\) is a free parameter\(^3\). The quantum properties of this model, thought for a while to be the dual of the principal chiral sigma model \([15, 16]\), have been studied in \([17]\).

In this paper, we will present other non-linear sigma models which admit a Lax pair representation. Some of these models are new integrable two-dimensional theories.

\(^3\)In fact the parameter \(\rho\) has no physical meaning as it can be simply absorbed by a rescaling of the field \(\varphi^i\). This operation leads to an overall factor in the Lagrangian.
4 Construction having a multiplicative spectral parameter

So far we have only given the conditions under which the equations of motion of a non-linear sigma model admit a zero curvature representation. However, as mentioned earlier these conditions (namely, equation (2.10)) do not guarantee the existence of a spectral parameter. For instance, the non-linear sigma model constructed in the context of Poisson-Lie T-duality \[13, 14\] admits a zero curvature representation but without a spectral parameter. Therefore, the class of non-linear sigma models enjoying a zero curvature representation is necessarily larger than the class of non-linear sigma models possessing the same property but with a spectral parameter.

As the presence of a spectral parameter in the Lax pair is of crucial importance in extracting conserved quantities \[1, 2\], we will now demand that our construction depends on such a spectral parameter.

We assume, in this section, that this spectral parameter enters in a multiplicative manner. Namely, we consider the following ansatz for the linear system

\[
\partial + x \hat{\alpha}_i (\varphi) \partial \varphi^i \Psi = 0 \\
\bar{\partial} + y(x) \hat{\beta}_j (\varphi) \bar{\partial} \varphi^j \Psi = 0 ,
\]

(4.1)

where \(x\) is our arbitrary spectral parameter and \(y(x)\) is a function of \(x\). The matrices \(\hat{\alpha}_i\) and \(\hat{\beta}_i\) are independent of \(x\).

The equivalence of the compatibility condition of the linear system and the equations of motion of the non-linear sigma model \((\mathcal{F} = \mu I \mathcal{E})\) results in the equality

\[
(y \hat{\beta}_i - x \hat{\alpha}_i) \bar{\partial} \partial \varphi^i + (y \partial_i \hat{\beta}_j - x \partial_j \hat{\alpha}_i + xy [\hat{\alpha}_i, \hat{\beta}_j]) \partial \varphi^i \bar{\partial} \varphi^j = \mu_i \left( \bar{\partial} \partial \varphi^i + \Omega^i_{ij} \partial \varphi^i \bar{\partial} \varphi^j \right) .
\]

(4.2)

Upon identification of the terms involving \(\bar{\partial} \partial \varphi^i\) and \(\partial \varphi^i \bar{\partial} \varphi^j\) on both sides of this equation we get

\[
\mu_i = y \hat{\beta}_i - x \hat{\alpha}_i \\
y \partial_i \hat{\beta}_j - x \partial_j \hat{\alpha}_i + xy [\hat{\alpha}_i, \hat{\beta}_j] = \Omega^i_{ij} \hat{\mu}_i .
\]

(4.3)

The first equation of this set determines the matrix \(\mu_i\) while the second leads to

\[
y \left( \partial_i \hat{\beta}_j - \Omega^i_{ij} \hat{\beta}_i \right) - x \left( \partial_j \hat{\alpha}_i - \Omega^j_{ij} \hat{\alpha}_i \right) + xy [\hat{\alpha}_i, \hat{\beta}_j] = 0 .
\]

(4.4)

The important point here is that this last equation should hold for any value of the spectral parameter \(x\) (we recall that \(\hat{\alpha}_i\), \(\hat{\beta}_i\) and \(\Omega^i_{ij}\) are independent of \(x\)). This requirement is fulfilled only if

\[
\left( \partial_i \hat{\beta}_j - \Omega^i_{ij} \hat{\beta}_i \right) = a [\hat{\alpha}_i, \hat{\beta}_j] \\
\left( \partial_j \hat{\alpha}_i - \Omega^j_{ij} \hat{\alpha}_i \right) = b [\hat{\alpha}_i, \hat{\beta}_j] .
\]

(4.5)

Equation (4.4) is of the form \(yA - xB + xyC = 0\) for three \(x\)-independent matrices \(A\), \(B\) and \(C\). There are, of course, various cases to be studied. We have analysed here only the case when the three matrices are proportional to each other \((A = aC, B = bC)\). The other cases lead, in general, to known integrable non-linear sigma models. In the rest of the paper, we will assume that both \(a\) and \(b\) are different from zero.
together with

\[ y(x) = \frac{bx}{x + a} \]  

(4.6)

for two parameters \(a\) and \(b\) (which will define the different non-linear sigma models). It is easy to see that by taking \(\tilde{\alpha}_i = \tilde{\beta}_i = g^{-1} \partial_i g, a = (2\kappa - 1)/2\) and \(b = (2\kappa + 1)/2\) one recovers the construction already given in (3.4). Similarly, by taking \(\tilde{\alpha}_i = \tilde{\beta}_i = T_i\) (where \([T_i, T_j] = f^k_{ij} T_k\)) and \(a = b = \rho\) we arrive at the linear system in (3.8).

**The geometry**

It is instructive to point out the special features of the theories which admit the Lax construction (4.1). We would like, in particular, to provide an interpretation for the two matrices \(\tilde{\alpha}_i\) and \(\tilde{\beta}_i\) of this Lax pair. In order to do this, we start by combining the two equations in (4.5) and taking the symmetric and anti-symmetric parts. This yields

\[
\begin{align*}
\partial_i K_j + \partial_j K_i - 2\Gamma^l_{ij} K_l &= 0 \\
\partial_i L_j - \partial_j L_i + 2H^l_{ij} K_l &= 0 \\
\partial_i L_j + \partial_j L_i - 2\Gamma^l_{ij} L_l &= [L_i, K_j] + [L_j, K_i] \\
\partial_i K_j - \partial_j K_i + 2H^l_{ij} L_l &= [L_i, L_j] - [K_i, K_j]
\end{align*}
\]  

(4.7)

where

\[
\begin{align*}
L_i &= b \tilde{\beta}_i + a \tilde{\alpha}_i \\
K_i &= b \tilde{\beta}_i - a \tilde{\alpha}_i
\end{align*}
\]  

(4.8)

The first two equations of the set (4.7) can be cast in the form

\[
K^i \partial_i g_{ij} + g_{ij} \partial_i K^l + g_{il} \partial_j K^l = 0 \\
K^i \partial_i b_{ij} + b_{ij} \partial_i K^l + b_{il} \partial_j K^l = - \left[ \partial_i \left( L_j + b_{ji} K^l \right) - \partial_j \left( L_i + b_{il} K^l \right) \right]
\]  

(4.9)

where \(K^i = g^{ij} K_j\). In terms of the tensor \(Q_{ij} = g_{ij} + b_{ij}\), these two equations combine to yield

\[
K^i \partial_i Q_{ij} + Q_{ij} \partial_i K^l + Q_{il} \partial_j K^l = - \left[ \partial_i \left( L_j + b_{ji} K^l \right) - \partial_j \left( L_i + b_{il} K^l \right) \right]
\]  

(4.10)

This last relation is precisely the condition needed for the non-linear sigma model (2.1) to enjoy the isometry symmetry \cite{18,19}

\[
\delta \varphi^i = \varepsilon^{AB} K^i_{AB}
\]  

(4.11)

where \(K^i_{AB}\) are the entries of the matrix \(K^i\) and \(\varepsilon^{AB}\) are constant infinitesimal parameters. We conclude that if a non-linear sigma model accepts a Lax pair representation of the form (4.1) then this model possesses an isometry symmetry (of course, for \(K^i \neq 0\)).
The conserved currents corresponding to the isometry transformation (4.11) are

\[
\begin{align*}
J &= (K_i - L_i) \partial \phi^i = -2a \hat{\alpha}_i \partial \phi^i \\
\bar{J} &= (K_i + L_i) \partial \phi^i = 2b \hat{\beta}_i \partial \phi^i .
\end{align*}
\]

(4.12)

In terms of the two currents \(J\) and \(\bar{J}\), the Lax pair (4.1) is given by

\[
\begin{align*}
\left[ \partial - \frac{x}{2a} J \right] \Psi &= 0 \\
\left[ \bar{\partial} + \frac{x}{2(x + a)} \bar{J} \right] \Psi &= 0 .
\end{align*}
\]

(4.13)

Moreover, by contacting both sides of equations (4.5) with \(\partial \phi^i \bar{\partial} \phi^j\), we obtain

\[
\begin{align*}
\partial \bar{J} + \frac{1}{2} [ J, \bar{J} ] &= 2b \hat{\beta}_i \mathcal{E}^i \\
\bar{\partial} J - \frac{1}{2} [ J, \bar{J} ] &= -2a \hat{\alpha}_i \mathcal{E}^i ,
\end{align*}
\]

(4.14)

where \(\mathcal{E}^i \equiv \bar{\partial} \partial \phi^i + \Omega^i_{\,j} \partial \phi^i \bar{\partial} \phi^j = 0\) are the equations of motion of the non-linear sigma model. These relations can be cast into the form

\[
\begin{align*}
\partial \bar{J} + \bar{\partial} J &= 2K_i \mathcal{E}^i \\
\bar{\partial} J - \partial \bar{J} + \left[ J, \bar{J} \right] &= 2L_i \mathcal{E}^i .
\end{align*}
\]

(4.15)

This last set of equations, suggests the study of three different cases:

1) \(K_i = 0\) and \(L_i \neq 0\):

In this case the two currents \(J\) and \(\bar{J}\) satisfy the equation \(\partial \bar{J} + \bar{\partial} J = 0\) independently of the equations of motion of the non-linear sigma model. Therefore, the equation \(\partial \bar{J} + \bar{\partial} J = 0\) is a Bianchi identity (a topological property) for the two currents. Furthermore, equations (4.7) reduce to

\[
\begin{align*}
\partial_i L_j - \partial_j L_i &= 0 \\
\partial_i L_j + \partial_j L_i - 2\Gamma^l_{ij} L_l &= 0 \\
2H^l_{ij} L_l &= [L_i, L_j] .
\end{align*}
\]

(4.16)

This set has a unique solution given by \(L_i = 2aT_i\), where \([T_i, T_j] = f^k_{ij} T_k\) and the integrable non-linear sigma model is the model defined in (3.7) with the identification \(a = \rho\).

2) \(L_i = 0\) and \(K_i \neq 0\):

Here the two currents \(J\) and \(\bar{J}\) are the conserved currents corresponding to the isometry symmetry \(\delta \phi^i = \varepsilon^{AB} K^i_{AB}\) of the non-linear sigma model and satisfy the Bianchi identity \(\partial \bar{J} - \bar{\partial} J + [ J, \bar{J} ] = 0\). In this case the set (4.17) gives

\[
\begin{align*}
\partial_i K_j + \partial_j K_i - 2\Gamma^l_{ij} K_l &= 0 \\
2H^l_{ij} K_l &= 0 \\
\partial_i K_j - \partial_j K_i + [K_i, K_j] &= 0 .
\end{align*}
\]

(4.17)
This has a unique solution given by \( K_i = g^{-1} \partial_i g \), for some Lie group element \( g(\varphi) \), and the integrable theory is the principal chiral non-linear sigma model (that is, equation (3.3) with \( \kappa = 0 \)). The parameter \( a \) in (4.13) is equal to \(-1/2\).

3) \( K_i \neq 0 \) and \( L_i \neq 0 \):
This case means that the integrable non-linear sigma model must possess the isometry symmetry (4.11) whose conserved currents are \( J \) and \( \bar{J} \) (namely, \( \partial J + \partial\bar{J} = 0 \) on shell). Moreover, the field strength \( \partial J - \partial\bar{J} + [J, \bar{J}] \) vanishes only when the equations of motion are obeyed and is no longer a Bianchi identity as in the previous case.

As seen earlier, the first two cases are unique and lead to known integrable non-linear sigma models. Therefore, any new integrable non-linear sigma model (having (4.13) as a Lax pair) must fit in this third class. Notice that the conformal and non-conformal WZW models are part of this third class. Indeed, by taking \( K_i = g^{-1} \partial_i g \) and \( L_i = 2\kappa g^{-1} \partial_i g \) (for \( \kappa \neq 0 \)) and injecting them into the set (4.7) one gets precisely equations (3.2) leading to the WZW models (3.3).

It is therefore natural to ask whether other non-linear sigma models, possessing (4.13) as a Lax pair, exist. We will provide below an example of a non-linear sigma model, different from (3.3) and (3.7), which have such a property.

**An Example**

The non-linear sigma model we consider here is based on the \( SU(2) \) Lie algebra. Its action is given by

\[
S = \int dz d\bar{z} \left\{ \partial r \bar{\partial} r + B(r) \delta_{ab} \partial n^a \bar{\partial} n^b + C(r) \epsilon_{abc} \partial n^a \bar{\partial} n^b \bar{\partial} n^c \right\} ,
\]

(4.18)

where \( r \) and \( n^a \) \((a = 1, 2, 3)\) are the fields of the non-linear sigma model subject to the constraint \( \delta_{ab} n^a n^b = 1 \). The \( SU(2) \) Lie algebra is \([T_a, T_b] = \epsilon_{abc} T_c\) with \( \epsilon_{123} = 1 \). We shall make no distinction between upper and lower \( SU(2) \) indices.

It is important to mention that this non-linear sigma model was first studied in [3] and later reexamined in [4]. The authors of [3] demanded that this model should be: firstly classically integrable and secondly it admits two commuting Kac-Moody algebras at the level of Poisson brackets (the Hamiltonian is quadratic in the Kac-Moody currents). Here we will require that the model is only classically integrable. Despite the fact that our requirement is less restrictive, the only solution we found is that of [3].

The equations of motion of this non-linear sigma model are

\[
\begin{align*}
0 &= -2\partial \bar{\partial} r + B' \delta_{ab} \partial n^a \bar{\partial} n^b + C' \epsilon_{abc} \partial n^a \bar{\partial} n^b \bar{\partial} n^c, \\
0 &= -2B \bar{\partial} n^b - B' \left( \partial \bar{\partial} n^b + \bar{\partial} r \bar{\partial} n^b \right) - 2Bn^b \delta_{ab} \partial n^c \bar{\partial} n^d \\
&\quad + \epsilon_{bac} \left[ 3C \partial n^a \bar{\partial} n^c + C' n^a \left( \partial \bar{\partial} n^c - \bar{\partial} r \bar{\partial} n^c \right) \right] - 3C \epsilon_{cde} n^b n^c \partial n^d \bar{\partial} n^e.
\end{align*}
\]

(4.19)

Here a prime denotes the derivative with respect to \( r \). The equations of motion for the field \( n^b \) are obtained by introducing a Lagrange multiplier for the constraint \( \delta_{ab} n^a n^b = 1 \).
The ansatz for the $SU(2)$ components of the two currents $J = J^a T_a$ and $\bar{J} = \bar{J}^a T_a$, appearing in the linear system (4.13), is taken to have the form

\[
J^a = n^a \partial r + \beta(r) \partial n^a + \gamma(r) \epsilon^{abc} n^b \partial n^c \\
\bar{J}^a = -n^a \bar{\partial} r - \beta(r) \bar{\partial} n^a + \gamma(r) \epsilon^{abc} n^b \bar{\partial} n^c
\] (4.20)

Our task now is the determination of the functions $B(r)$, $C(r)$, $\beta(r)$ and $\gamma(r)$ such that on shell (namely, when the equations of motion (4.19) are obeyed), these two currents satisfy

\[
\partial \bar{J}^a + \bar{\partial} J^a = 0 \\
\partial \bar{J}^a - \bar{\partial} J^a + \epsilon^{abc} J^b \bar{J}^c = 0
\] (4.21)

In practice, one extracts $\partial \bar{\partial} r$ and $\partial \bar{\partial} n^b$ from the equations of motion (4.19) and injects them into the above on-shell requirements (4.21).

We find that (4.21) holds provided that the following differential equations are obeyed:

\[
-1 + \beta' - \frac{\gamma}{B} C' = 0 \\
\gamma' - \frac{\gamma}{B} B' = 0 \\
- (1 - \gamma) - \beta' + \frac{\beta}{B} B' = 0 \\
\gamma' - \beta - \frac{\beta}{B} C' = 0 \\
2\gamma - \beta^2 - \gamma^2 - C' = 0 \\
2\beta (1 - \gamma) - B' = 0
\] (4.22)

The first two equations of this system are solved by

\[
B = b \gamma \\
C = -b (r - \beta) + c
\] (4.23)

where $b$ and $c$ are two arbitrary constants, with $b$ different from zero. The solutions to the whole system are then divided in three cases:

Case a)

\[
\gamma = 1 \quad , \quad \beta = \pm \sqrt{1 + b}
\] (4.24)

with $b$ arbitrary.

Case b)

\[
b = -1 \quad , \quad \gamma = \frac{(r - d)^2}{[(r - d)^2 - 1]} \quad , \quad \beta = -\frac{(r - d)}{[(r - d)^2 - 1]}
\] (4.25)

\[^5\text{We have used the fact that } \epsilon_{abc} \epsilon_{cde} = \delta_{ad} \delta_{be} - \delta_{ae} \delta_{bd} \text{ and other relations coming from the differentiation of the constraint } \delta_{ab} n^a n^b = 1. \text{ In particular, the relation } \epsilon^{bcd} n^a n^b \partial n^c \bar{\partial} n^d = \epsilon^{abc} n^a \bar{\partial} n^b \partial n^c.\]
where $d$ is a constant of integration.

Case c)

\[ b \neq -1 \quad , \]

\[ \gamma = \frac{\left[ 1 + be^{\eta a(r-d)} \right]^2}{\left[ 1 - be^{\eta a(r-d)} \right]^2 - 4e^{\eta a(r-d)}} \quad , \quad \beta = -\eta \sqrt{1 + b} \left[ 1 - b^2 e^{2\eta a(r-d)} \right] \left[ 1 - be^{\eta a(r-d)} \right]^2 - 4e^{\eta a(r-d)} \quad , \]

(4.26)

where $d$ is a constant of integration, $a = \frac{2}{\eta} \sqrt{(1 + b)}$ and $\eta^2 = 1$.

In summary, the $SU(2)$ integrable non-linear sigma model is given by

\[ S = \int dz d\bar{z} \left\{ \partial r \bar{\partial} r + b\gamma(r)\delta_{ab}\partial n^a \bar{\partial} n^b + \left[ b \beta(r) - (r - d) \right] + (c - bd) \right\} . \]

(4.27)

It is clear from the expressions of the two functions $\beta(r)$ and $\gamma(r)$ that the constant $d$ can be absorbed by the change of variable $r \rightarrow r + d$. Furthermore, the term involving the constant $(c - bd)$ in the action (4.27) is a total derivative and does not contribute to the equations of motion. Therefore, the integrable non-linear sigma model (4.27), up to some rescalings of the fields, is precisely the model of [3].

5 Generalisation of the principal chiral model

We have so far studied only the case when the spectral parameter enters the Lax pair in a multiplicative manner. What can one say now about the situations when this is not the case? Unfortunately, there is not much that one can say about a general non-linear sigma model. The integrability is described by the Lax pair (2.5) and the master equation (2.10) where one might find a spectral parameter hidden in the matrices $\alpha_i$ and $\beta_i$. However, a great deal can be learnt by exploring particular non-linear sigma models.

We start by studying a theory which generalises the principal chiral non-linear sigma model. The classical integrability of this model has already been investigated in the literature [20, 21]. Its study here is for two purposes: Firstly, we would like to highlight the group theory structure behind the integrability requirement. Secondly, the results of this example will be of use in the rest of the paper.

We consider the generalisation of the principal chiral non-linear sigma model as given by the action

\[ S(g) = \int dz d\bar{z} \Omega_{ab} \left( g^{-1} \partial g \right)^a \left( g^{-1} \bar{\partial} g \right)^b , \quad (5.1) \]

where $\Omega_{ab}$ is a constant matrix and $g^{-1} \partial g = (g^{-1} \partial g)^a T_a = A^a T_a$, $g^{-1} \bar{\partial} g = \left( g^{-1} \bar{\partial} g \right)^a T_a = \bar{A}^a T_a$ and $T_a$ are some Lie algebra generators, $[T_a, T_b] = f_{abc} T_c$, and $g$ an element in the corresponding Lie group. The equations of motion are conveniently expressed as

\[ \mathcal{E}_c = -\frac{1}{2} \left( \Omega_{cd} + \Omega_{dc} \right) \left( \partial \bar{A}^d + \bar{\partial} A^d \right) + \left[ \frac{1}{2} \left( \Omega_{cd} - \Omega_{dc} \right) f_{ab}^d + \left( \Omega_{ad} f_{bc}^d + \Omega_{db} f_{ac}^d \right) \right] A^a \bar{A}^b = 0 , \quad (5.2) \]

where use of the Bianchi identities

\[ \mathcal{B}^a = \partial A^a - \bar{\partial} A^a + f_{bc}^a A^b \bar{A}^c = 0 \quad , \quad (5.3) \]
The linear system, whose consistency condition is equivalent to the above equations of motion, could only be of the form
\[
\begin{align*}
(\partial + A^a P_a) \Psi &= 0 \\
(\bar{\partial} + \bar{A}^b Q_b) \Psi &= 0 ,
\end{align*}
\]
where \( P_a \) and \( Q_a \) are constant matrices. The spectral parameter, if it exists, is hidden in these matrices. The compatibility condition of this linear system is
\[
\mathcal{F} \equiv \frac{1}{2} (Q_a - P_a) \left( \partial \bar{A}^a + \bar{\partial} A^a \right) + \left( -\frac{1}{2} f^d_{ab} (Q_d + P_d) + [P_a, Q_b] \right) A^a \bar{A}^b = 0 .
\] (5.5)

We then demand that
\[
\mathcal{F} = R^a \mathcal{E}_a
\] (5.6)
for some linearly independent matrices \( R^a \). This leads firstly to the determination of the matrices \( P_a \) in terms of \( Q_a \) and \( R^a \) through
\[
P_a = Q_a + (\Omega_{ac} + \Omega_{ca}) R^c .
\] (5.7)
Secondly, it yields the condition
\[
[Q_a, Q_b] = f^d_{ab} Q_d + \Omega_{cd} f^d_{ab} R^c + \left( \Omega_{ad} f^d_{bc} + \Omega_{db} f^d_{ac} \right) R^c - (\Omega_{ac} + \Omega_{ca}) [R^c, Q_b] .
\] (5.8)

This last equation determines the non-linear sigma model (that is \( \Omega_{ab} \)) and its Lax pair (the matrices \( Q_a \) and \( R^a \)). It is clear that if a solution exists, then it must have a Lie algebra interpretation. We will show below that equation (5.8) does have solutions for particular examples.

**A three-dimensional example**

This particular solution to (5.8) is based on the \( SU(2) \) Lie algebra. We take the matrix \( \Omega_{ab} \) to be diagonal
\[
\Omega_{ab} = \text{diag} \left( L_1, L_2, L_3 \right) .
\] (5.9)
The linear system is taken to have the form
\[
\begin{align*}
\left( \partial + X^a_{\alpha} A^b T_a \right) \Psi &= 0 \\
\left( \bar{\partial} + Y^c_{\alpha} \bar{A}^d T_c \right) \Psi &= 0 ,
\end{align*}
\] (5.10)
where the \( SU(2) \) Lie algebra is \([T_a, T_b] = \epsilon_{abc} T_c\) with \( \epsilon_{123} = 1 \) and \( X^a_{\alpha} \) and \( Y^a_{\alpha} \) are constant \( 3 \times 3 \) matrices. In the notation of (5.4), we have \( P_a = X^b_{\alpha} T_b \), \( Q_a = Y^b_{\alpha} T_b \) and \( R^d = \frac{1}{2} (\Omega^{-1})^{da} (X^b_{\alpha} - Y^b_{\alpha}) T_b . \)
It is then found that equation (5.8) is satisfied for the following non-vanishing elements $X^a_b$ and $Y^a_b$:

$$
\begin{align*}
X^1_x &= \frac{1}{\sqrt{L_2} \sqrt{L_3}} \left[ \kappa_1 \sqrt{x + L_2} \sqrt{x + L_3} + \omega_1 \sqrt{x} \sqrt{x + L_1} \right], \\
X^2_x &= \frac{1}{\sqrt{L_2} \sqrt{L_3}} \left[ \kappa_2 \sqrt{x + L_1} \sqrt{x + L_3} + \omega_2 \sqrt{x} \sqrt{x + L_2} \right], \\
X^3_x &= \frac{1}{\sqrt{L_2} \sqrt{L_3}} \left[ \kappa_3 \sqrt{x + L_1} \sqrt{x + L_2} + \omega_3 \sqrt{x} \sqrt{x + L_3} \right], \\
Y^1_x &= \frac{1}{\sqrt{L_2} \sqrt{L_3}} \left[ \kappa_1 \sqrt{x + L_2} \sqrt{x + L_3} - \omega_1 \sqrt{x} \sqrt{x + L_1} \right], \\
Y^2_x &= \frac{1}{\sqrt{L_2} \sqrt{L_3}} \left[ \kappa_2 \sqrt{x + L_1} \sqrt{x + L_3} - \omega_2 \sqrt{x} \sqrt{x + L_2} \right], \\
Y^3_x &= \frac{1}{\sqrt{L_2} \sqrt{L_3}} \left[ \kappa_3 \sqrt{x + L_1} \sqrt{x + L_2} - \omega_3 \sqrt{x} \sqrt{x + L_3} \right],
\end{align*}
$$

(5.11)

where $\kappa^2 = \kappa^2_3 = 1$, $\kappa_3 = \kappa_1 \kappa_2$ and $\omega_1 = \kappa_2 \omega_3$, $\omega_2 = \kappa_1 \omega_3$, $\omega^2_3 = 1$. The spectral parameter is $x$. This solution was first found in [6] and later given in its present form in [21].

This solution might as well be extended to the case of the $SO(6)$ Lie algebra which is described by the commutations relations

$$
[J_a, J_b] = \epsilon_{abc} J_c, \quad [J_a, M_b] = \epsilon_{abc} M_c, \quad [M_a, M_b] = \epsilon_{abc} J_c.
$$

(5.12)

However, this Lie algebra can be cast in the form

$$
\begin{align*}
[J^+_a, J^+_b] &= \epsilon_{abc} J^+_c, \quad [J^+_a, J^-_b] = 0 \quad \text{with} \quad J^\pm_a = \frac{1}{2} (J_a \pm M_a).
\end{align*}
$$

(5.13)

We have therefore two copies of the $SU(2)$ Lie algebras and we conclude that the non-linear sigma model (5.11) defined by

$$
\Omega_{ab} = \text{diag} \left( L^+_1, L^+_2, L^+_3, L^-_1, L^-_2, L^-_3 \right)
$$

(5.14)

is integrable. The corresponding linear system is of the form (5.10) where now the indices $a, b, \ldots$ run from 1 to 6. The matrix elements $(X^1_1, X^2_2, X^3_3)$ and $(X^4_4, X^5_5, X^6_6)$ are obtained from (5.11) by replacing $(L_1, L_2, L_3)$ by $(L^+_1, L^+_2, L^+_3)$ and $(L^-_1, L^-_2, L^-_3)$, respectively. The determination of $Y^a_b$ is similar.

**A four-dimensional example**

So far, all the integrable non-linear sigma models listed in this paper are known. We will now explicitly construct a new integrable model. This is based on a four-dimensional non-semi-simple Lie algebra whose commutation relations are

$$
[J, P_i] = \epsilon_{ij} P_j, \quad [P_i, P_j] = \epsilon_{ij} T, \quad [T, J] = [T, P_i] = 0,
$$

(5.15)
with $\epsilon_{12} = 1$. We denote the generators of this Lie algebra as $T_a = \{P_1, P_2, J, T\}$ such that the Lie algebra is $[T_a, T_b] = f^c_{ab} T_c$. Although this algebra is non-semi-simple, it has nevertheless a non-degenerate bilinear form $\eta_{ab}$ obeying $\eta_{ab} f^c_{bd} + \eta_{cb} f^b_{ad} = 0$. It is given by

$$\eta_{ab} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & b & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix},$$

(5.16)

where $b$ is an arbitrary constant.

Using this invertible bilinear form one can, for instance, construct a principal chiral non-linear sigma model

$$S(g) = \int \text{d}z \text{d}\bar{z} \eta_{ab} (g^{-1} \partial g)^a (g^{-1} \bar{\partial} g)^b,$$

(5.17)

where as usual $(g^{-1} \partial g) = (g^{-1} \partial g)^a T_a = A^a T_a$ and $(g^{-1} \bar{\partial} g) = (g^{-1} \bar{\partial} g)^a T_a = \bar{A}^a T_a$. This is an integrable non-linear sigma model with the Lax pair

$$\left[ \partial + \frac{1}{1 + x} (g^{-1} \partial g) \right] \Psi = 0,$$

$$\left[ \bar{\partial} + \frac{1}{1 - x} (g^{-1} \bar{\partial} g) \right] \Psi = 0,$$

(5.18)

where $x$ is the spectral parameter.

Our aim is to generalise the above principal chiral non-linear sigma model. We consider the action

$$S(g) = \int \text{d}z \text{d}\bar{z} \Omega_{ab} (g^{-1} \partial g)^a (g^{-1} \bar{\partial} g)^b,$$

(5.19)

where $\Omega_{ab}$ is symmetric and of the form

$$\Omega_{ab} = \begin{pmatrix} L_1 & 0 & 0 & 0 \\ 0 & L_2 & 0 & 0 \\ 0 & 0 & b & L_3 \\ 0 & 0 & L_3 & 0 \end{pmatrix}.$$

(5.20)

This non-linear sigma model is integrable if a solution to (5.8) can be found. For this purpose, we seek a linear system of the form

$$\left( \partial + X^a_b A^b T_a \right) \Psi = 0,$$

$$\left( \bar{\partial} + Y^a_d \bar{A}^d T_c \right) \Psi = 0,$$

(5.21)

where we assume that the constant matrices $X^a_b$ and $Y^a_d$ are diagonal (that is, eight unknowns). It turns out that (5.8) yields six independent equations (for eight unknowns). These are

$$-2L_3 Y^2_2 X^1_1 + (L_1 - L_2) (X^4_4 - Y^4_4) + L_3 (X^4_4 + Y^4_4) = 0$$
Replacing these back into (5.22) allows the determination of $X_z$. This is a quadratic equation in $Y^2$. Equations of (5.22) yield then $P$ invariant under the rescaling in an iterative manner. The explicit solution to this set is too untidy to write down. We choose therefore to give it as our free parameters leaving us with six equations and six unknowns. One of these parameters reflects simply the fact that our Lie algebra is invariant under the rescaling $P_1 \to \lambda P_1$, $P_2 \to \lambda P_2$, $J \to J$, $T \to T^2$. Therefore, only one parameter is left and in order to simplify slightly the equations we take $X_1 = 1 + x$ and $Y_1 = 1 - x$. The parameter $x$ will play the role of the spectral parameter. The last two equations of (5.22) yield then

$$X_3^3 = \frac{1}{Y_3^2} \left[ 1 - \frac{x(L_2 - L_3)}{L_1} \right]$$

$$Y_3^3 = \frac{1}{X_3^2} \left[ 1 + \frac{x(L_2 - L_3)}{L_1} \right]$$

(5.23)

while the first and the second equations give

$$X_4^4 = \frac{(L_1 - L_2 - L_3)(1 - x)}{2(L_1 - L_2)} X_2^2 + \frac{(L_1 - L_2 + L_3)(1 + x)}{2(L_1 - L_2)} Y_2^2$$

$$Y_4^4 = \frac{(L_1 - L_2 + L_3)(1 - x)}{2(L_1 - L_2)} X_2^2 + \frac{(L_1 - L_2 - L_3)(1 + x)}{2(L_1 - L_2)} Y_2^2$$

(5.24)

Replacing these back into (5.22) allows the determination of $Y_2^2$

$$Y_2^2 = \frac{(L_3 - L_1 - L_2)}{(L_3 - L_1 + L_2)} X_2^2 + \frac{2(1 + x) L_2 (L_1 + x (L_2 - L_3))}{L_1 (L_3 - L_1 + L_2)} \frac{1}{X_2^2}$$

(5.25)

Finally, $X_2^2$ is obtained by solving the equation

$$L_1^2 (L_1 - L_3) (X_2^2)^4 - L_1 [(L_3 - L_1 - L_2) ((L_3 - L_2) x^2 - L_1) + 4 L_2 (L_1 - L_3) x] (X_2^2)^2$$

$$+ L_2 (1 + x)^2 (L_1 + x (L_2 - L_3))^2 = 0$$

(5.26)

This is a quadratic equation in $z = (X_2^2)^2$ whose solution is

$$\left( X_2^2 \right)^2 = \frac{1}{2L_1^2 (L_1 - L_3)} \left\{ L_1 \left[ (L_3 - L_1 - L_2) \left( (L_3 - L_2) x^2 - L_1 \right) + 4 L_2 (L_1 - L_3) x \right]\right.$$  
$$\pm \left\{ \left[ L_1 \left[ (L_3 - L_1 - L_2) \left( (L_3 - L_2) x^2 - L_1 \right) + 4 L_2 (L_1 - L_3) x \right] \right]^2$$

$$- 4L_1^2 L_2 (L_1 - L_3) (1 + x)^2 (L_1 + x (L_2 - L_3))^2 \right\}^{1/2} \right\} .$$

(5.27)
Once $X^2_a$ has been determined all the other elements $X^a_a$ and $Y^a_a$ are found.

We should mention that this solution is not valid for the two values of the spectral parameter $x = -1$ and $x = L_1/(L_3 - L_2)$ as they lead to $X^2_2 = 0$. Furthermore, there are special cases corresponding to particular values of $L_1$, $L_2$ and $L_3$ which are not discussed here. These particular cases are such that $L_1 - L_2 = 0$, $L_1 - L_3 = 0$, $L_3 - L_1 + L_2 = 0$ or $L_3 - L_1 - L_2 = 0$.

A five-dimensional example

We consider a five-dimensional non-semi-simple Lie algebra whose generators $T_a = \{P_1, P_2, P_3, J, T\}$ satisfy $[T_a, T_b] = f^{c}_{ab} T_c$ with the non-vanishing commutators being

$$[P_i, P_j] = \epsilon_{ijk} P_k + v^k \epsilon_{kij} J \quad , \quad [T_i, P_j] = v^k \epsilon_{kij} P_j \quad ,$$

where $\epsilon_{123} = 1$ and $v^k$ is an arbitrary constant vector.

This Lie algebra possesses an invertible bilinear form given by

$$\eta_{ab} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix} .$$

(5.29)

Of course, as in the above four-dimensional example, the principal chiral model constructed with this bilinear form is integrable.

The integrable non-linear sigma model we present here is described by the action

$$S(g) = \int d\z d\bar{\z} \Omega_{ab} \left( g^{-1} \partial g \right)^{a} \left( g^{-1} \bar{\partial} g \right)^{b} ,$$

(5.30)

where $\Omega_{ab}$ is a slight generalisation of the bilinear form (5.29) and is of the form

$$\Omega_{ab} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & L \\ 0 & 0 & 0 & L & 0 \end{pmatrix} .$$

(5.31)

This non-linear sigma model is integrable and its corresponding linear system is given by

$$\left( \partial + X^a_b A^b T_a \right) \Psi = 0$$

$$\left( \bar{\partial} + Y^c_d A^d T_c \right) \Psi = 0 \quad ,$$

(5.32)

The condition of integrability (5.8) leads to a long list of equations which we will not write down. Here we will only give the result of a computer based investigation.
where the constant matrices $X^a_b$ and $Y^a_b$ are

$$X^a_b = \begin{pmatrix} x & 0 & 0 & 0 & 0 \\ 0 & x & 0 & 0 & 0 \\ 0 & 0 & x & 0 & 0 \\ 0 & 0 & 0 & y & 0 \\ 0 & 0 & 0 & 0 & 1 + L (x - 1) \end{pmatrix}$$

$$Y^a_b = \frac{1}{2x - 1} \begin{pmatrix} x & 0 & 0 & 0 & 0 \\ 0 & x & 0 & 0 & 0 \\ 0 & 0 & x & 0 & 0 \\ 0 & 0 & 0 & 2x^2 - y (2x - 1) & 0 \\ 0 & 0 & 0 & 0 & (2x - 1) - L (x - 1) \end{pmatrix} . \tag{5.33}$$

The parameters $x$ and $y$ are arbitrary.

**Klimčík’s solution**

The most important ingredient in the construction of [24] is a linear operator $R$ acting on the generators of a simple Lie algebra $\mathcal{G}$. This operator is required to satisfy two relations. Firstly

$$[RX, RY] = R ([X, Y]_R) + [X, Y] \quad \text{for} \quad X, Y \in \mathcal{G} , \tag{5.34}$$

where

$$[X, Y]_R = [RX, Y] + [X, RY] . \tag{5.35}$$

Secondly, $R$ verifies the skew-symmetry condition

$$< RX, Y >_{\mathcal{G}} + < X, RY >_{\mathcal{G}} = 0 , \tag{5.36}$$

where $<, >_{\mathcal{G}}$ is the Killing-Cartan form on the Lie algebra $\mathcal{G}$.

An $R$ operator satisfying (5.34) and (5.36) was given in [24]. This is constructed as follows: Let the Lie algebra $\mathcal{G}$ be generated by $(H^\mu, B^\alpha, C^\alpha)$ where $\alpha$ labels the positive roots and $H^\mu$ are the generators of the Cartan subalgebra. The step generators $E^\alpha$ and $E^{-\alpha}$ are defined, up to a normalisation factor, as

$$E^\alpha = B^\alpha + iC^\alpha , \quad E^{-\alpha} = B^\alpha - iC^\alpha . \tag{5.37}$$

The linear operator $R$ acts on the generators of the Lie algebra $\mathcal{G}$ as follows:

$$RH^\mu = 0 , \quad RB^\alpha = C^\alpha , \quad RC^\alpha = -B^\alpha . \tag{5.38}$$

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7This solution was published during the reviewing process of the present paper.
For instance, in the case of the Lie algebra $\text{SU}(2)$, with generators $\vec{T} = \{T_1, T_2, T_3\}$, we have

$$R\vec{T} = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} T_1 \\ T_2 \\ T_3 \end{pmatrix} = \begin{pmatrix} T_2 \\ -T_1 \\ 0 \end{pmatrix} ,$$

(5.39)

Similarly, for $G = \text{SU}(3)$ the operator $R$ acts as

$$RT_1 = T_2 , \quad RT_2 = -T_1 , \quad RT_3 = 0 , \quad RT_4 = T_5 , \quad RT_5 = -T_4 , \quad RT_6 = T_7 , \quad RT_7 = -T_6 , \quad RT_8 = 0 .$$

(5.40)

We will show below that (at least for the $\text{SU}(2)$ case) the above action of the operator $R$ is not the most general one.

The integrable non-linear sigma model constructed in [24] is described by the action

$$S(g) = \int d\tau d\bar{\tau} < g^{-1} \partial g , (I - \varepsilon R)^{-1} g^{-1} \bar{\partial} g >_G ,$$

(5.41)

where $g$ is the group element corresponding to the Lie algebra $G$, $I$ is the identity operation on the generators of $G$ and $\varepsilon$ is a free parameter. The equations of motion of this theory can be cast into the form

$$E \equiv \partial \bar{\partial} J - \bar{\partial} \partial J + \varepsilon \left[ J, \bar{J} \right]_{R} = 0 ,$$

(5.42)

where the two currents $J$ and $\bar{J}$ are defined as

$$J = (I + \varepsilon R)^{-1} g^{-1} \partial g , \quad \bar{J} = -(I - \varepsilon R)^{-1} g^{-1} \bar{\partial} g .$$

(5.43)

These two currents satisfy the Bianchi identity

$$B \equiv -(\partial \bar{\partial} J + \bar{\partial} \partial J) + \varepsilon \left[ J, \bar{J} \right]_{R} + (\varepsilon^2 - 1) \left[ J, \bar{J} \right]_{R} + \varepsilon \left[ RJ, \bar{J} \right] + \varepsilon \left[ J, RJ \right] .$$

(5.44)

These are found by writing $g^{-1} \partial g = (I + \varepsilon R) J$ and $g^{-1} \bar{\partial} g = -(I - \varepsilon R) \bar{J}$ and demanding that the identity $\partial \left( g^{-1} \bar{\partial} g \right) - \bar{\partial} \left( g^{-1} \partial g \right) + \left[ g^{-1} \partial g , g^{-1} \bar{\partial} g \right] = 0$ holds.

Finally, the Lax pair corresponding to this non-linear sigma is given by

$$\left[ \partial - \left( \varepsilon^2 - \varepsilon R - \frac{1 + \varepsilon^2}{1 + x} \right) J \right] \Psi = 0$$

$$\left[ \bar{\partial} + \left( \varepsilon^2 + \varepsilon R - \frac{1 + \varepsilon^2}{1 - x} \right) \bar{J} \right] \Psi = 0$$

(5.45)

with $x$ being the spectral parameter. The zero curvature condition of this linear system is

$$\mathcal{F} = -\frac{x(1 + \varepsilon^2)}{1 - x^2} (1 + x \varepsilon R) \mathcal{E} + \frac{1 + x^2 \varepsilon^2}{1 - x^2} \mathcal{B} .$$

(5.46)

We see that the second term vanishes identically while the first is equivalent to the equations of motion.
Of course the action (5.41) is a particular case of the generalised chiral principal model (5.1). For example, in the case of the Lie algebra $SU(2)$, the model (5.41) leads to

$$\Omega_{ab} = \frac{1}{1 + \varepsilon^2} \begin{pmatrix} 1 & -\varepsilon & 0 \\ \varepsilon & 1 & 0 \\ 0 & 0 & 1 + \varepsilon^2 \end{pmatrix}. \quad (5.47)$$

The anti-symmetric part of $\Omega_{ab}$ yields a total derivative in the action and one is left with the integrable model of Cherednik [9]. For completeness, we also give the matrices $P_a = X_b^b T_b$ and $Q_a = Y_a^b T_b$ appearing in the linear system (5.4) by comparison with (5.45)

$$X_a^b = \frac{1}{1 + x} \begin{pmatrix} 1 & x\varepsilon & 0 \\ -x\varepsilon & 1 & 0 \\ 0 & 0 & 1 - x\varepsilon^2 \end{pmatrix}, \quad Y_a^b = \frac{1}{1 - x} \begin{pmatrix} 1 & x\varepsilon & 0 \\ -x\varepsilon & 1 & 0 \\ 0 & 0 & 1 + x\varepsilon^2 \end{pmatrix}, \quad (5.48)$$
where the lower index labels the rows.

Similarly, the tensor $\Omega_{ab}$ corresponding to the $SU(3)$ case is given by

$$\Omega_{ab} = \frac{1}{1 + \varepsilon^2} \begin{pmatrix} 1 & -\varepsilon & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \varepsilon & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 + \varepsilon^2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 - \varepsilon & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \varepsilon & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 - \varepsilon & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \varepsilon & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 + \varepsilon^2 & 0 \end{pmatrix}. \quad (5.49)$$

Here the anti-symmetric part does not result in a total derivative in the action. The explicit form of the $SU(3)$ matrices $P_a = X_b^b T_b$ and $Q_a = Y_a^b T_b$ of the linear system (5.4) are

$$X_a^b = \frac{1}{1 + x} \begin{pmatrix} 1 & x\varepsilon & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -x\varepsilon & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 - x\varepsilon^2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & x\varepsilon & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -x\varepsilon & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & x\varepsilon & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -x\varepsilon & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 - x\varepsilon^2 \end{pmatrix}. \quad (5.50)$$

The matrix $Y_a^b$ is obtained from the expression of $X_a^b$ by replacing $x$ with $-x$ and $\varepsilon$ with $-\varepsilon$.

As mentioned above, it seems that the action of the operator $R$ as given in (5.38) is not the most general one. Indeed, for the case of the Lie algebra $SU(2)$ we have found that the most general linear operator $R$, satisfying the two conditions (5.34) and (5.36), is given by

$$R\vec{T} = \begin{pmatrix} 0 & a & c \\ -a & 0 & b \\ -c & -b & 0 \end{pmatrix} \begin{pmatrix} T_1 \\ T_2 \\ T_3 \end{pmatrix}, \quad (5.51)$$
where \( a \) and \( b \) are two arbitrary parameters and \( c = \sqrt{1 - a^2 - b^2} \). In this case, the action \( (5.41) \) yields an integrable non-linear sigma model of the form \( (5.1) \) with

\[
\Omega_{ab} = \frac{1}{1 + \varepsilon^2} \begin{pmatrix}
1 + \varepsilon^2 & -\varepsilon (a + \varepsilon b c) & \varepsilon (\varepsilon ab - c) \\
\varepsilon (a - \varepsilon b c) & 1 + \varepsilon^2 c^2 & -\varepsilon (b + \varepsilon ac) \\
\varepsilon (\varepsilon ab + c) & \varepsilon (b - \varepsilon ac) & (1 + \varepsilon^2 a^2)
\end{pmatrix}.
\]  
(5.52)

The corresponding Lax pair as read from \( (5.45) \) results in a linear system of the form \( (5.4) \) with

\[
X_a^b = \frac{1}{1 + x} \begin{pmatrix}
1 - x \varepsilon^2 b^2 & x \varepsilon (a + \varepsilon b c) & -x \varepsilon (\varepsilon ab - c) \\
-x \varepsilon (a - \varepsilon b c) & 1 - x \varepsilon^2 c^2 & x \varepsilon (b + \varepsilon ac) \\
-x \varepsilon (\varepsilon ab + c) & -x \varepsilon (b - \varepsilon ac) & (1 - x \varepsilon^2 a^2)
\end{pmatrix}.
\]  
(5.53)

The matrix \( Y_a^b \) is obtained from the expression of \( X_a^b \) by replacing \( x \) with \( -x \) and \( \varepsilon \) with \( -\varepsilon \).

Finally, we should also mention that the integrability of the model \( (5.41) \) holds even when the Lie algebra \( G \) is non-semi-simple provided that the bilinear form \( <, >_G \) is invertible (the invertibility of the bilinear form is used in deriving the equations of motion). This is so because the proof of the integrability of the non-linear sigma model \( (5.41) \) relies on the two relations \( (5.34) \) and \( (5.36) \), satisfied by the \( R \) operator, and on the existence of an invertible bilinear form on the Lie algebra \( G \). As an example, the non-semi-simple Lie algebra \( (5.15) \) with the invertible bilinear form \( (5.16) \) possesses an \( R \) operator of the form \( R^T \) given by

\[
R^T = \begin{pmatrix}
0 & 1 & 0 & -a \\
-1 & 0 & -d & 0 \\
a & d & c & -bc \\
0 & 0 & 0 & -c
\end{pmatrix}.
\]  
(5.54)

where \( a, c \) and \( d \) are arbitrary parameters (\( b \) is the parameter already appearing in the invariant bilinear form \( (5.16) \)). The resulting integrable non-linear sigma model, as written in \( (5.1) \), has a matrix \( \Omega_{ab} \) given by

\[
\Omega_{ab} = \frac{1}{1 + \varepsilon^2} \begin{pmatrix}
1 & -\varepsilon & \varepsilon(a-cd) & 0 \\
\varepsilon & 1 & \varepsilon(d+ca) & 0 \\
-\varepsilon(a-cd) & \varepsilon(d+ca) & \frac{b+\varepsilon^2 c^2 (b-d^2-a^2)}{1+\varepsilon^2} & 0 \\
0 & 0 & \frac{b+\varepsilon^2 c^2 (b-d^2-a^2)}{1+\varepsilon^2} & 0
\end{pmatrix}.
\]  
(5.55)

The Lax pair of this integrable non-linear sigma model is of the form \( (5.4) \) with \( P_a = X_a^b T_b \) and \( Q_a = Y_a^b T_b \) and where

\[
X_a^b = \frac{1}{1 + x} \begin{pmatrix}
1 & x \varepsilon & 0 & \frac{xc(\varepsilon d-a)}{1+\varepsilon c} \\
x \varepsilon & 1 & 0 & \frac{-xc(d+ca)}{1+\varepsilon c} \\
\frac{xc(a+\varepsilon a)}{1+\varepsilon c} & \frac{xc(d-\varepsilon a)}{1+\varepsilon c} & \frac{1+\varepsilon c x(c+\varepsilon)}{1+\varepsilon c} & \frac{-xc(1+\varepsilon^2 a^2)}{1+\varepsilon c-2xc} \\
0 & 0 & 0 & \frac{1+\varepsilon c x(c+\varepsilon)}{1+\varepsilon c}
\end{pmatrix}.
\]  
(5.56)

with the lower index counting the rows. The matrix \( Y_a^b \) is obtained by substituting \( x \) with \( -x \) and \( \varepsilon \) with \( -\varepsilon \) in the expression of \( X_a^b \).

\[\text{\textsuperscript{8}} \text{There are other } \mathcal{R} \text{ operators for the algebra } (5.15). \text{ We have chosen to write down the simplest of them.}\]
6 Duality and integrability

In this section we will put forward a method for constructing new integrable non-linear sigma models starting from already integrable ones. This is based on the concept of T-duality [25] that certain two-dimensional non-linear sigma have.

In order to illustrate the important role of T-duality in the construction of integrable non-linear sigma models, we will start by considering a simple example. The discussion of the general case will be treated somewhere else [26] (Abelian T-duality in the context of integrability has been used in [5] in a very particular non-linear sigma model). We will consider the non-linear sigma model of the previous section whose action is

\[ S(g) = \int \! dz \! \bar{z} \Omega_{ab} \left( g^{-1} \partial g \right)^a \left( g^{-1} \bar{\partial} g \right)^b , \]  

(6.1)

This theory has the symmetry transformation \( g \rightarrow Lg \), where \( L \) is a constant element of the Lie group to which \( g \) belongs. The duality transformations are found by first gauging this symmetry and adding a Lagrange multiplier term which constrains the gauge field strength to vanish [27, 28, 29]. This results in the first order action

\[ S_1(g, B, \bar{B}, \chi) = \int \! dz \! \bar{z} \left[ \Omega_{ab} \left( g^{-1} \partial g + g^{-1} B g \right)^a \left( g^{-1} \bar{\partial} g + g^{-1} \bar{B} g \right)^b + \chi_a \left( \partial B^a - \bar{\partial} B^a + f^a_{bc} B^b \bar{B}^c \right) \right] . \]  

(6.2)

Here \( B = B^a T^a \) and \( \bar{B} = \bar{B}^a T^a \) are the gauge fields with the gauge transformation \( B \rightarrow LBL^{-1} - \partial LL^{-1}, \bar{B} \rightarrow LBL^{-1} - \bar{\partial} LL^{-1} \). The Lagrange multiplier \( \chi_a \), transforming in the adjoint representation, imposes the pure gauge condition \( \partial B^a - \bar{\partial} B^a + f^a_{bc} B^b \bar{B}^c = 0 \) whose solution is \( B = h^{-1} \partial h \) and \( \bar{B} = h^{-1} \bar{\partial} h \). Upon replacing this back into the action \( S_1 \) one gets the relation \( S_1 = S(hg) \) and by choosing \( h = 1 \) (thanks to the local gauge symmetry \( g \rightarrow Lg, h \rightarrow hL^{-1} \)) one concludes that the action \( S_1 \) is equivalent to the action \( S \).

The dual action is obtained by keeping the Lagrange multiplier and eliminating, instead, the gauge fields (through their equations of motion). This procedure yields, after the gauge choice \( g = 1 \), the dual theory

\[ \tilde{S}(\chi) = \int \! dz \! \bar{z} \left( M^{-1} \right)^{ab} \partial \chi_a \bar{\partial} \chi_b \]  

\[ M_{ab} \equiv \Omega_{ab} + \chi_c f^c_{ab} . \]  

(6.3)

We will show now that if the original theory (6.1) is integrable then its dual (6.3) is also integrable. It is convenient, for this purpose, to introduce the two currents

\[ J^a = \left( M^{-1} \right)^{ba} \partial \chi_b , \quad \bar{J}^a = - \left( M^{-1} \right)^{ab} \bar{\partial} \chi_b . \]  

(6.4)

In terms of these, the equations of motion of the dual theory (6.3) are

\[ \tilde{\mathcal{E}}^a \equiv \partial \bar{J}^a - \bar{\partial} J^a + f^a_{bc} J^b \bar{J}^c = 0 . \]  

(6.5)

We notice that these are the Bianchi identities (6.3) of the original theory with \( (J^a, \bar{J}^a) \) interchanged with \( (A^a, \bar{A}^a) \). Furthermore, these currents satisfy the Bianchi identity (stemming
from \( \partial \bar{\partial} \chi_a - \bar{\partial} \partial \chi_a = 0 \)
\[
\tilde{B}_c \equiv -\frac{1}{2} (\Omega_{cd} + \Omega_{dc}) \left( \partial J^d + \bar{\partial} \bar{J}^d \right) - \frac{1}{2} (\Omega_{cd} - \Omega_{dc}) f^d_{ab} \left( \Omega_{ad} f^d_{bc} + \Omega_{db} f^d_{ac} \right) J^a \bar{J}^b \]

Again these are a linear combination of the equations of motion (5.2) and the Bianchi identities (5.3) of the original sigma model with \((J^a, \bar{J}^a)\) interchanged with \((A^a, \bar{A}^a)\).

Therefore, for the dual non-linear sigma model, the linear combination
\[
\tilde{B}_c + \left[ \frac{1}{2} (\Omega_{cd} - \Omega_{dc}) + \chi_a f^a_{cd} \right] \tilde{E}^d = -\frac{1}{2} (\Omega_{cd} + \Omega_{dc}) \left( \partial J^d + \bar{\partial} \bar{J}^d \right) - \frac{1}{2} (\Omega_{cd} - \Omega_{dc}) f^d_{ab} \left( \Omega_{ad} f^d_{bc} + \Omega_{db} f^d_{ac} \right) J^a \bar{J}^b \tag{6.7}
\]
takes exactly the form of the equations of motion (5.2) of the original theory with the exchange \((J^a, \bar{J}^a) \leftrightarrow (A^a, \bar{A}^a)\).

As already shown in equation (2.27), the zero curvature condition stemming from a Lax pair is a linear combination of the equations of motion of the sigma model and some corresponding Bianchi identities. Moreover, the equations of motion and the Bianchi identities of the dual theory are simply linear combinations of those of the original theory with the exchange \((J^a, \bar{J}^a) \leftrightarrow (A^a, \bar{A}^a)\). Consequently, if the original non-linear sigma model (6.1) is integrable (namely, if equation (5.8) is satisfied) then its dual theory (6.3) is also integrable. The Lax pair of the dual theory is
\[
(\partial + J^a P_a) \Psi = 0 \quad (\bar{\partial} + \bar{J}^b Q_b) \Psi = 0 \tag{6.8}
\]
This is simply the Lax pair of the original theory with \((J^a, \bar{J}^a)\) and \((A^a, \bar{A}^a)\) interchanged. Indeed, if the original non-linear sigma model (6.1) is integrable (that is when (5.8) is satisfied) then the zero curvature condition of this last linear system is
\[
\tilde{Z} = R^a \tilde{B}_a + \left[ Q_a + (\Omega_{ba} + \chi_c f^c_{ba}) R^b \right] \tilde{E}^a \tag{6.9}
\]
where we have used the relation \(P_a = Q_a + (\Omega_{ac} + \Omega_{ca}) R^c\) and the integrability condition (5.8). We notice that the first term vanishes identically and the second term yields the equations of motion of the dual non-linear sigma model.

**Examples**

1) The first example which enters into the class of non-linear sigma models (6.1) is of course the principal chiral non-linear sigma model for which the matrix \(\Omega_{ab} = \eta_{ab}\), where \(\eta_{ab}\) is the invariant bilinear form of the underlying Lie algebra (that is, \(\eta_{ab} f^b_{cd} + \eta_{bc} f^b_{ad} = 0\)). Its corresponding dual non-linear sigma model is written in (6.3) with
\[
M_{ab} = \eta_{ab} + \chi_c f^c_{ab} \tag{6.10}
\]
The Lax pair of the dual of the principal chiral non-linear sigma mode is read from (3.4), for \( \kappa = 0 \), according to the above prescription. This is therefore given by

\[
\begin{align*}
\{ \partial + x \left[ (M^{-1})^{ab} \partial \chi_a \right] T_b \} \Psi &= 0 \\
\{ \bar{\partial} + \frac{x}{2x-1} \left[ - (M^{-1})^{cd} \bar{\partial} \chi_d \right] T_c \} \Psi &= 0 ,
\end{align*}
\]

(6.11)

where \( x \) is the spectral parameter.

2) The second example we consider is the \( SU(2) \)-based non-linear sigma model whose corresponding matrix \( \Omega_{ab} \) is given in (5.9) and for which

\[
M_{ab} = \delta_{ab} L_b + \chi_c \epsilon_{cab} .
\]

(6.12)

Its dual partner is given by the action (6.3) with the matrix \( M^{-1} \) explicitly given by

\[
M^{-1} = \frac{1}{D} \begin{pmatrix}
\chi_1^2 + L_2 L_3 & \chi_1 \chi_2 - \chi_3 L_3 & \chi_1 \chi_3 + \chi_2 L_2 \\
\chi_1 \chi_2 + \chi_3 L_3 & \chi_2^2 + L_1 L_3 & \chi_2 \chi_3 - \chi_1 L_1 \\
\chi_1 \chi_3 - \chi_2 L_2 & \chi_2 \chi_3 + \chi_1 L_1 & \chi_3^2 + L_1 L_2 \\
\end{pmatrix},
\]

(6.13)

where \( D = L_1 L_2 L_3 + L_1 \chi_1^2 + L_2 \chi_2^2 + L_3 \chi_3^2 \). The Lax pair of this dual theory is then given by

\[
\begin{align*}
\{ \partial + X_b^a \left[ (M^{-1})^{eb} \partial \chi_e \right] T_a \} \Psi &= 0 \\
\{ \bar{\partial} + Y_d^c \left[ - (M^{-1})^{df} \bar{\partial} \chi_f \right] T_c \} \Psi &= 0 ,
\end{align*}
\]

(6.14)

where \( T_a \) are the generators of the \( SU(2) \) Lie algebra and the matrices \( X_b^a \) and \( Y_b^a \) are as given in (6.11).

7 Conclusion

The question of classical integrability of two-dimensional non-linear sigma models has been addressed in this paper. We have first focused on the issue of representing the equations of motion of the non-linear sigma model as a zero curvature condition of a linear system. This is regardless of whether the Lax pair depended or not on a spectral parameter. This requirement resulted in a master equation with some interesting geometrical properties. In particular, it is shown that in the case when the matrices involved in this equation are Lie algebra valued matrices, this master equation is a generalisation of an equation encountered in the context of Poisson-Lie T-duality. It is therefore hopeful that a general solution along the lines of [13, 14] might be found to this master equation.

We have then put special emphasis on those constructions admitting a spectral parameter. Two situations emerged from this analyses. The first consists of those constructions where the spectral parameter enters in a multiplicative way. The geometry of the integrable non-linear sigma models is in this case tractable. The isometry symmetry of these sigma model plays an essential role and is responsible for their integrability. Even in this simplified version, the general solution to the master equation remains a challenging problem. The second situation
concerns those constructions for which the spectral parameter does not manifest itself in a multiplicative manner. Here, we have studied only the integrability of a generalisation of the principal chiral non-linear sigma model. The master equation for these models, (5.8), has a certain Lie algebra structure and one hopes that this might be helpful in finding solutions. We have carried out a computer assisted study and found two new integrable non-linear sigma models. These are given in (5.20) and (5.31) and are based on two non-semi-simple Lie algebras of dimension four and five, respectively.

Always in the context of the generalised principal chiral non-linear sigma model, we have given a brief summary of a new integrable non-linear sigma model that has recently been found by Klimčík [24]. This solution to the master equation holds for any simple Lie algebra and relies on an $R$ operator which acts on the generators of the Lie algebra. We have pointed out that the action of the $R$ operator as given in [24], equation (5.38), is not the most general one. We have worked out the most general $R$ operator for the case of the $SU(2)$ Lie algebra. This has led to a more general integrable non-linear sigma model (5.52) in comparison with the one in (5.47). However, the problem of constructing the most general $R$ operator for other Lie algebras is still an open issue. Furthermore, it is shown that the $R$ operator can also be extended to non-semi-simple Lie algebras. This was carried out for a particular example resulting in another integrable non-linear sigma model (5.55).

We have also shown that there is a connection between T-duality and integrability of non-linear sigma models. More precisely, if a non-linear sigma model is integrable and admits a T-duality transformation then its dual is also integrable. This might not sound surprising as two non-linear sigma models related by a T-duality transformation are by definition equivalent. However, it is not obvious how to find the Lax pair associated to the dual theory starting from the Lax pair of the original non-linear sigma model. We have given here the recipe for this passage. T-duality is therefore a mean for constructing new integrable non-linear sigma models.

Among the open problems that could be addressed in the light of this work would be the extension of the standard procedure of the dressing transformations [30, 1, 2] encountered in the principal chiral sigma model [31, 32, 33] to the new integrable models found in this paper. The construction of the conserved charges is also another challenging issue.

Finally, we should mention that the study of the integrability of non-linear sigma models carried out here could be of interest to string theory in its quest for integrable string backgrounds [34, 35, 36].

Acknowledgments: I would like to thank Péter Forgács, Max Niedermaier and Paul Sorba for very useful discussions and Anastasia Doikou, Andreas Fring and Ctirad Klimčík for correspondence. The pertinent remarks of an anonymous referee are also here acknowledged.

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