INTEGRATING CURVATURE: FROM UMLAUFSAZT TO $J^+$ INVARIANT.

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ABSTRACT. Hopf’s Umlaufsatz relates the total curvature of a closed immersed plane curve to its rotation number. While the curvature of a curve changes under local deformations, its integral over a closed curve is invariant under regular homotopies. A natural question is whether one can find some non-trivial densities on a curve, such that the corresponding integrals are (possibly after some corrections) also invariant under regular homotopies of the curve in the class of generic immersions. We construct a family of such densities using indices of points relative to the curve. This family depends on a formal parameter $q$ and may be considered as a quantization of the total curvature. The linear term in the Taylor expansion at $q = 1$ coincides, up to a normalization, with Arnold’s $J^+$ invariant. This leads to an integral expression for $J^+$.

Let $\Gamma$ be a closed oriented immersed plane curve $\Gamma : S^1 \to \mathbb{R}^2$. One of the fundamental notions related to $\Gamma$ is its curvature $\kappa$. Another important notion is that of a rotation number (or Whitney winding number) $\text{rot}(\Gamma)$, i.e. the number of turns made by the tangent vector as we follow $\Gamma$ along its orientation.

Hopf’s Umlaufsatz [2] is one of the simplest versions of the Gauss-Bonnet theorem and one of the fundamental theorems in the theory of plane curves. It relates two different types of data: local geometric characteristic of a plane curve – its curvature $\kappa$ – and a global topological characteristic – its rotation number $\text{rot}(\Gamma)$. Although the curvature of a plane curve changes under local deformations, the theorem states that its average (integral) over a closed curve is invariant under homotopies in the class of immersed curves:

**Theorem 1** (Hopf’s Umlaufsatz).

\begin{equation}
\frac{1}{2\pi} \int_{S^1} \kappa(t) \, dt = \text{rot}(\Gamma)
\end{equation}

A natural question is whether one can find some natural densities $\rho$ on $\Gamma$ such that the average $\int_{S^1} \kappa(t)\rho(t) \, dt$ is (possibly after some corrections) also

2010 Mathematics Subject Classification. 53A04, 57R42.
Key words and phrases. plane curves, curvature, rotation number, regular homotopy.
Both authors were partially supported by the ISF grant 1343/10.
invariant under local deformations of $\Gamma$. Since the rotation number is (up to normalization) the only invariant of $\Gamma$ in the class of immersed curves, we cannot expect such an expression to remain invariant under arbitrary homotopies. We can hope, however, that the result is invariant under regular homotopies in the class of \textit{generic} immersions, i.e. immersions with a finite set $X$ of transversal double points as the only singularities. Invariants of such a type were originally introduced by Arnold \cite{Arnold1} and include the celebrated $J^\pm$ and $St$ invariants (see \cite{Arnold1} for details).

We construct a family of such densities using the \textit{index} $\text{ind}_{\Gamma}(p)$ of a point $p$ relative to $\Gamma$. Given $p \in \mathbb{R}^2 \setminus \Gamma$, we define $\text{ind}_{\Gamma}(p)$ as the number of turns made by the vector pointing from $p$ to $\Gamma(t)$, as we follow $\Gamma$ along its orientation. This defines a locally-constant function on $\mathbb{R}^2 \setminus \Gamma$. See Figure 1a. Suppose that $\Gamma$ is generic. Then we can extend $\text{ind}_{\Gamma}$ to a $\frac{1}{2}\mathbb{Z}$-valued function on $\mathbb{R}^2$.

To define $\text{ind}_{\Gamma}(p)$ for $p \in \Gamma$, average its values on the regions adjacent to $p$ – two regions if $p$ is a regular point of $\Gamma$, and four regions if $p$ is a double point of $\Gamma$. See Figure 1b. For each double point $d = \Gamma(t_1) = \Gamma(t_2) \in X$, define

\[\theta_d \in (0, \pi)\] as the (non-oriented) angle between two tangent vectors $\Gamma'(t_1)$ and $-\Gamma'(t_2)$. For $q \in \mathbb{R} \setminus \{0\}$, define $I_q(\Gamma) \in \mathbb{R}[q^{1/2}, q^{-1/2}]$ by

\[I_q(\Gamma) = \frac{1}{2\pi} \left( \int_{\mathbb{S}^1} \kappa(t) \cdot q^{\text{ind}_{\Gamma}(\Gamma(t))} dt - \sum_{d \in X} \theta_d \cdot q^{\text{ind}_{\Gamma}(d)}(q^{1/2} - q^{-1/2}) \right)\]

\textbf{Theorem 2.} $I_q(\Gamma)$ is invariant under regular homotopies of $\Gamma$ in the class of generic immersions.

\textit{Proof.} Note that we can generalize all above notions and formulas to the case of a multi-component curve $\Gamma : \sqcup_n \mathbb{S}^1 \to \mathbb{R}^2$ (by a summation of indices relative to all components of $\Gamma$).

Let us smooth the original curve $\Gamma$ in each double point respecting the orientation to get a multi-component curve $\tilde{\Gamma} = \sqcup_n \tilde{\Gamma}_n$ without double points. Denote by $\text{ind}_{\tilde{\Gamma}}(p)$ the index of a point $p$ relative to $\tilde{\Gamma}$. Note that values of $I_q$ on $\Gamma$ and $\tilde{\Gamma}$ differ by an easily computable factor (which depends only on the regular homotopy class of $\Gamma$ in the class of generic immersions). Indeed,
consider a small neighborhood $U_d$ of a double point $d$ of index $i$, see Figure 1c. Under smoothing of $d$, the total curvature of $\tilde{\Gamma} \cap U_d$ differs from that of $\Gamma \cap U_d$ by $\pm(\pi - \theta_d)$ for the fragment with index $i \pm \frac{1}{2}$, see Figure 1c. Thus the integral part of $I_q$ changes by $\frac{1}{2\pi}(\pi - \theta_d)(q^i + \frac{1}{2} - q^i - \frac{1}{2})$. Also, the double point $d$ contributes $-\frac{1}{2\pi}\theta_d q^i(q^\frac{1}{2} - q^{-\frac{1}{2}})$ to $I_q(\Gamma)$. Smoothing removes $d$, so this summand disappears from $I_q(\tilde{\Gamma})$. Thus, the total change of $I_q$ under smoothing of $d$ equals $\frac{1}{2}q^i(q^\frac{1}{2} - q^{-\frac{1}{2}})$. Hence

$$I_q(\Gamma) = I_q(\tilde{\Gamma}) - \frac{1}{2} \sum_d q^{\text{indr}(d)}(q^\frac{1}{2} - q^{-\frac{1}{2}}).$$

Since $\sum_d q^{\text{indr}(d)}(q^\frac{1}{2} - q^{-\frac{1}{2}})$ is invariant under regular homotopies of $\Gamma$ in the class of generic immersions, it remains to prove the invariance of $I_q(\tilde{\Gamma}) = \sum_n I_q(\tilde{\Gamma}_n)$.

Note that $\text{ind}_\Gamma(\tilde{\Gamma}(t))$ is constant on each component $\tilde{\Gamma}_n$ of $\tilde{\Gamma}$, so

$$I_q(\tilde{\Gamma}_n) = \frac{1}{2\pi} \int_{S^1} \kappa_n(t) \cdot q^{\text{indr}(\tilde{\Gamma}_n(t))} dt = q^{\text{indr}(\tilde{\Gamma}_n(t))} \frac{1}{2\pi} \int_{S^1} \kappa_n(t) dt$$

and by Umlaufsatz (1) we get $I_q(\tilde{\Gamma}_n) = \pm q^{\text{indr}(\tilde{\Gamma}_n(t))}$, depending on $\text{rot}(\tilde{\Gamma}_n) = \pm 1$. Thus, $I_q(\tilde{\Gamma}_n)$ is invariant under regular homotopies of $\tilde{\Gamma}$. But a regular homotopy of $\Gamma$ in the class of generic immersions induces a regular homotopy of $\tilde{\Gamma}$ and the theorem follows.

□

Any two immersions with the same rotation number can be connected by regular homotopy in the class of generic immersions and a finite sequence of self-tangency and triple-point modifications, shown in Figure 2. Depending on orientations and indices of adjacent regions, one can distinguish several types of these modifications. Self-tangencies can be separated into direct (or dangerous) and opposite (or safe), shown in Figure 3a and 3b respectively. An index of a self-tangency modification is the index of two new-born double points (e.g., modifications in Figure 3 are of index $i$). Triple-point modifications can be separated into weak (or acyclic) and strong (or cyclic), shown

**Figure 2.** Self-tangency and triple-point modifications.
in Figure 3a and 3b respectively. An index of a triple-point modification is the minimum of indices of double points involved in this modification (e.g., modifications in Figure 4 are of index $i$). Invariants of regular homotopy classes of generic immersions are uniquely determined by their behavior under these modifications, together with normalizations on standard curves $K_i$ of $\text{rot}(K_i) = i$, $i = 0, \pm 1, \pm 2, \ldots$ shown in Figure 5. Basic invariants $J^\pm$ and $St$ of (regular homotopy classes of) generic plane curves were introduced axiomatically by Arnold [1]. In particular, $J^+$ is uniquely determined by the following axioms:

- $J^+$ does not change under an opposite self-tangency or triple-point modifications.
- Under a direct self-tangency modification which increases the number of double points, $J^+$ jumps by 2.

\footnote{Our indices of modifications differ from the ones of [3] by an $-1$ shift.}
• On the standard curves $K_i$ we have $J^+(K_0) = 0$ and $J^+(K_i) = -2(|i| - 1)$ for $i = \pm 1, \pm 2, \ldots$

In a similar way, $I_q(\Gamma)$ is uniquely determined by the following

**Theorem 3.** The invariant $I_q(\Gamma)$ satisfies the following properties:

- $I_q(\Gamma)$ does not change under opposite self-tangencies.
- Under direct self-tangencies of index $i$, the invariant $I_q(\Gamma)$ jumps by $-q^i(q^{1/2} - q^{-1/2})$.
- Under (both weak and strong) triple-point modifications of index $i$, $I_q(\Gamma)$ jumps by $-\frac{1}{2}q^{i+1}(q^{1/2} - q^{-1/2})^2$.
- We have $I_q(-\Gamma) = -I_q(\Gamma^{-1})$, where $-\Gamma$ denotes $\Gamma$ with the opposite orientation.
- On the standard curves $K_i$ we have $I_q(K_0) = \frac{1}{2}(q^{1/2} - q^{-1/2})$ and $I_q(K_i) = \frac{1}{2}(i - 1)q^{1/2} + \frac{1}{2}(i + 1)q^{-1/2}$ for $i = 1, 2, \ldots$

**Proof.** A straightforward computation verifies both the behavior of $I_q(\Gamma)$ under self-tangencies and triple-point modifications and its values on the curves $K_i$. To verify the behavior of $I_q(\Gamma)$ under an orientation reversal, note that $\text{ind}_{-\Gamma}(p) = -\text{ind}_{\Gamma}(p)$, which corresponds to the involution $q \rightarrow q^{-1}$ in terms of $q^{\text{ind}_{\Gamma}(\Gamma(t))}$ and $q^{\text{ind}_{\Gamma}(d)}$ of (2). Also, both terms in (2) change signs: the integral due to the change of parametrization, and the sum over double points due to the equality $q^{1/2} - q^{-1/2} = -\left((q^{-1})^{1/2} - (q^{-1})^{-1/2}\right)$.

Substituting $q = 1$ into (2), we readily obtain $I_1(\Gamma) = \frac{1}{2\pi} \int_S \kappa(t) \, dt = \text{rot}(\Gamma)$ and recover the classical Hopf Umlaufsatz, see Theorem 1. In this sense, invariant $I_q$ may be considered as a quantization of the total curvature (1). Let us study the next term $I'_1(\Gamma)$ of the Taylor expansion of $I_q(\Gamma)$ at $q = 1$.

**Proposition 4.** $I'_1(\Gamma)$ is related to Arnold’s $J^+$ invariant by

$$I'_1(\Gamma) = \frac{1}{2}(1 - J^+(\Gamma)).$$

**Proof.** Note that by Theorem 2 $I'_1(\Gamma)$ is invariant under regular homotopies of $\Gamma$ in the class of generic immersions. Differentiating at $q = 1$ expressions for jumps of $I_q(\Gamma)$ in Theorem 3 we immediately conclude that $I'_1(\Gamma)$ is invariant under opposite tangencies and triple-point modifications. Moreover, under direct tangencies, $I'_1(\Gamma)$ jumps by $-1$. Thus its behavior under all modifications is the same as that of $-\frac{1}{2}J^+(\Gamma)$ (up to an additive constant depending on $\text{rot}(\Gamma)$). A straightforward computation shows that $I'_1(\Gamma)$ takes values $I'_1(K_0) = \frac{1}{2}$ and $I'_1(K_i) = |i| - \frac{1}{2}$ for $i = \pm 1, \pm 2, \ldots$ on the standard curves $K_i$ and the proposition follows. \hfill \Box
Differentiating RHS of (2) at $q = 1$ and using Proposition 4 we get

**Corollary 5.** The following integral expression for $J^+$ holds:

$$J^+(\Gamma) = 1 - \frac{1}{\pi} \left( \int_{S^1} \kappa(t) \cdot \text{ind}_\Gamma(\Gamma(t)) \, dt - \sum_{d \in X} \theta_d \right).$$

**Remark 6.** An infinite family of invariants, called “momenta of index” $M_r$ together with their generating function $P_\Gamma(q) \in \mathbb{Z}[q, q^{-1}]$ were introduced by Viro in [3, Section 5]. A careful check of their behavior under self-tangencies and triple-point modifications, together with their values on the standard curves $K_i$, allow one to relate $P_\Gamma(q)$ to $I_q(\Gamma)$ as follows:

$$P_\Gamma(q) = (q^{\frac{1}{2}} - q^{-\frac{1}{2}})I_q(\Gamma) + \frac{1}{2} \sum_{d \in X} q^{\text{ind}_r(d)}(q^{\frac{1}{2}} - q^{-\frac{1}{2}})^2.$$

**Remark 7.** Our choice of the function $q^{\text{ind}_r}$ in the integral part of (2) was motivated by considerations of conciseness and convenience. In fact, one can use an arbitrary function $f : \frac{1}{2}\mathbb{Z} \to \mathbb{R}$ of $\text{ind}_\Gamma$ instead of $q^{\text{ind}_r}$ (with an appropriate change of the correction term) to produce an invariant. Namely, repeating the proof of Theorem 2 one can show that

$$Z(f, \Gamma) = \frac{1}{2\pi} \int_{S^1} \kappa(t) \cdot f(\text{ind}_\Gamma(\Gamma(t))) \, dt - \frac{1}{2\pi} \sum_{d \in X} \theta_d \cdot \left( f(\text{ind}_\Gamma(d) + 1/2) - f(\text{ind}_\Gamma(d) - 1/2) \right)$$

is an invariant of regular homotopy in the class of generic immersions, which does not change under opposite self-tangencies. Under direct self-tangencies of index $i$, $Z(f, \Gamma)$ jumps by $- \left( f(i + 1/2) - f(i - 1/2) \right)$. Under triple-point modifications of index $i$, it jumps by $-\frac{1}{2} \left( f(i+3/2) - 2f(i+1/2) + f(i-1/2) \right)$.

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