On gravitomagnetic precession around black holes

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ABSTRACT
We compute exactly the frequency of Lense–Thirring precession for point masses in the Kerr metric, for arbitrary black hole mass and specific angular momentum. We show that this frequency, for point masses at or close to the innermost stable orbit, and for holes with moderate to extreme rotation, is less than, but comparable to the rotation frequency. Thus, if the quasi-periodic oscillations observed in the modulation of the X-ray flux from some black holes candidates, BHCs, are due to Lense–Thirring precession of orbiting material, we predict that a separate, distinct QPO ought to be observed in each object.

Key words:

1 INTRODUCTION
The large effective area, very high time-resolution and excellent telemetry of the Rossi X-ray Timing Explorer (RXTE) have made possible the discovery of Quasi Periodic Oscillations (QPOs) in the range $\sim 100–1200$ Hz from a variety of accreting collapsed objects, weakly magnetic neutron stars (see van der Kis 1998 for a review) and, more surprisingly, in black hole candidates (BHCs, Morgan et al. 1997; Remillard et al. 1997). It has been recently suggested (Cui et al. 1998) that these QPOs in BHCs arise through Lense–Thirring (LT, 1918) precession of matter from the accretion disks.

Since motion of a point mass in a Kerr metric allows an exact treatment, and a detailed comparison of Keplerian and Lense–Thirring frequencies has not been explicitly carried out in the literature up to now, it seems worthwhile to derive these quantities for an arbitrary black hole mass and specific angular momentum. This allows us to clarify the meaning of the precession frequency, about which some confusion seems to be present in the literature. This is the aim of this Letter. In the last section we shall also discuss the problems which are raised by this computation, regarding the interpretation of Cui et al. (1998).

2 BOUND ORBITS IN THE KERR METRIC
In what follows we will consider a test particle of unit mass in motion inside a Kerr spacetime. The metric, in Boyer-Lindquist coordinates (Boyer & Lindquist 1967) and in units

\begin{equation}
G = c = 1 \text{ is}
\end{equation}

\begin{equation}
\begin{aligned}
ds^2 &= -(1 - \frac{2Mr}{\rho^2})dt^2 - \frac{4aMr\sin^2\theta}{\rho^2}dtd\phi + \frac{\rho^2}{\Delta}dr^2 \\
&\quad + \rho^2d\theta^2 + \frac{\Lambda \sin^2\theta}{\rho^2}d\phi^2,
\end{aligned}
\end{equation}

where

\begin{equation}
\rho^2 = r^2 + a^2 \cos^2\theta
\end{equation}

\begin{equation}
\Delta = r^2 + a^2 - 2Mr
\end{equation}

\begin{equation}
\Lambda = -\Delta a^2 \sin^2\theta + (r^2 + a^2)^2.
\end{equation}

$M$ and $a$ are, respectively, the mass and the specific angular momentum of the black hole.

As Carter (1968) first demonstrated, the equation of motion can be separated, and the resulting equations become:

\begin{equation}
\rho^2 \ddot{r} = \pm \sqrt{R(r)}
\end{equation}

\begin{equation}
\rho^2 \ddot{\theta} = \pm \sqrt{\Theta(\theta)}
\end{equation}

\begin{equation}
\rho^2 \ddot{\phi} = (L\sin^{-2}\theta - aE) + a\Delta^{-1}P
\end{equation}

\begin{equation}
\rho^2 \dot{t} = a(L - aE\sin^2\theta + (r^2 + a^2)\Delta^{-1}P,
\end{equation}

with

\begin{equation}
\Theta = Q - \cos^2\theta[a^2(1 - E^2) + L^2 \sin^{-2}\theta],
\end{equation}

\begin{equation}
P = E(r^2 + a^2) - La,
\end{equation}

\begin{equation}
R = P^2 - \Delta[r^2 + Q + (L - aE)^2].
\end{equation}

The dot denotes differentiation with respect to the proper time $\tau$; signs in (4) and (5) can be chosen independently. $E$, $L$ and $Q$ are the three constants of the particle motion: $E$ and $L$ are, respectively, the energy and the angular momentum in the azimuthal direction as seen by an observer at
rest at infinity; \( Q \) is related to Carter’s constant of motion (see e.g. Chandrasekhar [1983] and de Felice [1980]) and characterizes the \( \theta \) motion.

As Wilkins [1972] showed, bound motion is possible only if \( E^2 < 1 \) and \( Q \geq 0 \); moreover, for given \( Q \) and \( L \) and \( |E| < 1 \), there may be at most one region of binding. Analysis of the \( \theta \) effective potential shows that every orbit either remains in the equatorial plane (\( Q = 0 \)), or crosses it repeteadly (\( Q > 0 \)). For every bound motion, introducing the angle-action variables, we can define the three fundamental proper frequencies

\[
1/\tau_{\phi,p} = \nu_{\phi,p}, \quad 1/\tau_{\theta,p} = \nu_{\theta,p}, \quad 1/\tau_{r,p} = \nu_{r,p},
\]

where \( \tau_{\phi,p}, \tau_{\theta,p} \) and \( \tau_{r,p} \) are the proper time periods for \( \phi, \theta \) and \( r \) motions respectively. Unlike the Newtonian case of particle motion around a spherically symmetric central object, where all orbits close and the three fundamental frequencies are equal, in the Kerr field (\( a \neq 0 \)) there is no degeneracy, i.e.

\[
\nu_{\phi,p} \neq \nu_{\theta,p} \neq \nu_{r,p}.
\]

The same is also true for coordinate frequencies \( \nu_{\phi}, \nu_{\theta} \) and \( \nu_{r} \).

Let us first consider a circular geodesic in the equatorial plane (\( \theta = \pi/2 \)). We have, for the coordinate angular velocities measured by an observer static at infinity [Bardeen et al. 1972]

\[
\Omega_r = \Omega_\theta = 0
\]

the angular velocity \( \Omega_\phi \) deviates from its Keplerian value at small radii. The upper sign refers to prograde orbits and the lower to retrograde ones. If we slightly perturb a circular orbit introducing velocity components in the \( r \) and \( \theta \) directions, we can compute the coordinate frequencies of the small amplitude oscillations within the plane (the epicyclic frequency \( \Omega_r \)) and in the perpendicular direction (the vertical frequency \( \Omega_\theta \)) (Okazaki, Kato & Fukue [1987], Kato [1990], de Felice & Usseglio-Tomasset [1990], Perez et al. [1997]):

\[
\Omega_r^\phi = \Omega_\theta^\phi \left[ 1 + 4 \frac{aM^{1/2}}{r^{3/2}} + 3\frac{a^2}{r} \right]
\]

\[
\Omega_\theta^\phi = \Omega_r^\phi \left[ 1 - 6M + 8aM^{1/2}r^{-1/2} - 3a^2 \right] r^2(5r^{3/2} \pm aM^{1/2})^2.
\]

In the case of the Schwarzschild metric (\( a = 0 \)), there is a partial degeneracy, as the vertical frequency coincides with the azimuthal one. The epicyclic frequency, instead, is always lower than the other two, reaching a maximum for \( r = 8M \) and going to zero at \( r = 6M \) (Okazaki et al. 1987). These qualitative behaviour of the epicyclic frequency is preserved in the Kerr field (\( a \neq 0 \)), and is a key feature for the existence of trapped diskoisismic g-modes (Perez et al [1997]).

### 3 Spherical Orbits and Frame Draging

We now confine ourselves to the study of those orbits with constant \( r \) which are arbitrarily (not infinitesimally) lifted over the equatorial plane, i.e. with a finite value of \( Q \).

The conditions for the stability of a spherical orbit with radius \( r = r_0 \) are (see eq. (9))

\[
R(r_0) = 0
\]

\[
\frac{\partial R}{\partial r}(r = r_0) = 0
\]

\[
\frac{\partial^2 R}{\partial r^2}(r = r_0) < 0.
\]

Conditions (9) and (10) introduce two relations between \( r \) and the constants of motion \( E, L \) and \( Q \), reducing the free parameters that characterize the orbit to two; thus, given a specific Kerr black hole (i.e. given the values of \( M \) and \( a \)), a spherical orbit is completely determinated, for example, by specifying its radius and the value of \( Q \), which fixes the amplitude of motion in the \( \theta \) direction (Wilkins [1972]).

The motion is open, since the two fundamental frequencies \( \nu_\phi \) and \( \nu_\theta \) (proper or coordinate) are incommensurable; then the Fourier spectra of every function of the position of the test particle will contain a superposition of the two fundamental frequencies and of all their harmonics and will be of the kind

\[
\sum_{l=\infty}^{\infty} \sum_{m=-\infty}^{\infty} C_{lm} e^{i(l\nu_\phi + m\nu_\theta)t + \beta},
\]

where \( \beta \) is an arbitrary phase.

Therefore, the most natural signals to look for in such a system are the two fundamental coordinate frequencies themselves and the difference between them, that, as we will show, coincide with the unique correct definition of precession frequency of the nodes of a spherical orbit.

In fact we can compute exactly the coordinate period of the \( \theta \) motion: if we call \( \vartheta_\pm \) (with \( \vartheta_- < \vartheta_+ \)) the two roots of the equation \( \theta(\vartheta_0) = 0 \), we see from (3) that the particle oscillates on the coordinate sphere between the angles \( \vartheta_- \) and \( \pi/2 + \vartheta_+ \). Dividing (8) by (3) and integrating, we obtain

\[
\tau_\vartheta = 4 \left\{ \left[ K(k) - E(k) \right] \left[ \frac{z_+}{\beta} \right]^{1/2} E + \right.
\]

\[
+ \frac{K(k)}{a\sqrt{2z_+}} \left[aL + \frac{P(r^2 + a^2)}{\Delta} - Ea^2 \right] \right\}
\]

where \( \beta = 1 - E^2, k^2 = z_-/z_+ \) (with \( z_\pm = \cos^2 \theta_\pm \)) and \( K(k) \) and \( E(k) \) are the elliptic integrals of the first and second kind, respectively.

The change of azimuth during one quarter oscillation of latitude is given by

\[
\Delta \varphi = \frac{1}{a\sqrt{2z_+}} \left\{ \Pi(-z_-) + \left[ a \left( 2MrE - aL \right) K(k) \right] \right\}
\]

where \( \Pi(k) \) is the elliptic integral of the third kind.

An orbit is called co-revolving (or prograde) if \( \Delta \varphi > 0 \), counter-revolving (or retrograde) if \( \Delta \varphi < 0 \). If the \( \theta \) and \( \phi \) frequencies were the same, \( \Delta \varphi \) would equal \( \pi/2 \); it means that we can define

\[
\frac{\nu_\phi}{\nu_\theta} = |\Delta \varphi|/\pi/2.
\]
The angle by which the nodes of a spherical orbit are dragged during each nodal period is therefore

$$\Delta \Omega = 2\pi \frac{\nu_\phi}{\nu_\theta} - 1$$

(15)

and, consequently, the coordinate precession frequency of the nodes (or Frame-Dragging frequency) is

$$\nu_{FD} = \frac{\Delta \Omega}{\theta_0} = |\nu_\phi - \nu_\theta|.$$  

(16)

We stress here that this definition is different from the one given in eq. (2) of Cui, Zhang and Chen (1998) ($\nu_{FD} = \nu_\phi \Delta \Omega / 2\pi$), with which it coincides only far from the source, where $\nu_\theta \approx \nu_\phi$. But in this case it would be sufficient to consider the weak-field limit, the well known Lense-Thirring equation [Lense & Thirring 1913], and this point clearly frustrates the aim of their work. In fact, approaching the horizon (to which the innermost stable orbit tends in the limit $a \rightarrow 1$) their definition leads to a divergence of the frame dragging frequency, while it can be shown that

$$\lim_{r \rightarrow r_{hor}} 2\pi \nu_{FD} = \Omega_{BH}$$

where $\Omega_{BH}$ is the angular velocity of the Black Hole (Christodoulou & Ruffini 1971; Misner, Thorne & Wheeler 1973), i.e. the angular velocity of the Zero Angular Momentum Observers on the horizon. This is a relevant difference between this work and that of Cui et al. (1998).

We chose to set, in our calculations, $M = 7 M_\odot$, in order to compare our results with the 300Hz QPO observed from GRO J1655-40 (Remillard et al. 1997), the BH for which the mass is most accurately measured (Orosz & Bailyn 1997). We considered only direct (i.e. prograde) orbits. In Figure 1 we plot the three frequencies calculated at selected radii as functions of $a$ for $Q = 1$. The radii are $r_i$, the radius of the innermost stable circular orbit, which has been calculated solving the quartic equation

$$\frac{\partial^2 R(r)}{\partial r^2} |_{r=r_i} = 0,$$

where $r_{peak} = r_i/\eta$ (with $\eta$ slowly varying from 0.62 to 0.76 as $a$ goes from $-1$ to 1), the radius of maximum surface emissivity of the disk (Page & Thorne 1974), $2r_i$ and $2r_{peak}$.

The value of $Q = 1$ (which corresponds to an ‘opening angle’ of the orbit over the equatorial plane which varies from about $3^\circ$ for $a = 0.5$ to about $5^\circ$ for $a = 0.99$) was chosen for simplicity, because, exploring the whole range $Q = 0.01 \rightarrow 10$, we found the relative changes in the frequencies raising from $\sim 2\%$ ($a = 0.5$) to a maximum of only $\sim 5\%$ ($a = 0.99$). It is immediately seen that for decreasing radii and increasing values of $a$ the splitting of $\nu_\phi$ and $\nu_\theta$ increases dramatically. Correspondingly the frame dragging frequency increases, reaching values that are comparable to $\nu_\theta$ for $r = 2r_{peak}$ and $2r_i$, or even larger than $\nu_\theta$ for $r = r_{peak}$ and $r_i$.

Figure 2 shows the three frequencies $\nu_\phi$, $\nu_\theta$ and $\nu_{FD}$ for selected values of the angular momentum of the black hole ($a = 0.5$, 0.9, 0.95, 0.998). These graphs represent the frequency changes that would take place if, for a given black hole, the orbital radius of the precessing matter changed (see Section 4).

These results, obtained in a fully general–relativistic framework, admit a simple interpretation in terms of a Newtonian analogy. In the classical gravitational potential due to a spherical star, $\propto 1/r$, the frequencies of motion for a bound orbit in the azimuthal ($\phi$), radial and latitudinal ($\theta$) directions are all equal; this well–known property assures that all orbits close in this potential. Whenever a small perturbation is introduced, such as that due to the star’s oblateness, this property is lost and the $\phi$–frequency $\nu_\phi$ becomes different from the $\theta$–frequency $\nu_\theta$. Then the spectrum emitted by a source on this orbit will contain all harmonics of the type $n\nu_\phi + m\nu_\theta$, with $n,m$ integers; of these, the line with, most likely, the largest amplitude is that at frequency $\nu_\phi - \nu_\theta$. This, in particular, is the classical precession frequency due to a Newtonian star not being perfectly spherical. Since in classical mechanics departures from spherical symmetry are always modest, we always find $\nu_\phi - \nu_\theta \ll \nu_\theta, \nu_\phi$. But in the gravitational field around a fastly rotating black hole such departures are much more significant, implying that this inequality is no longer satisfied. In other words, as departures from a Newtonian potential increase, whether because we are moving to a strong–field limit, or because the black hole is rotating faster, we expect to move toward a situation where $\nu_\phi - \nu_\theta \approx \nu_\theta \approx \nu_\phi$. This is exactly what we see happening in Figures 1 and 2.

4 APPLICATION TO BLACK HOLE CANDIDATES

By using the black hole mass and angular momentum which have been measured (or indirectly inferred) for several BHCs, Cui et al. (1998) find a reasonably good agreement of the predicted point–mass precession frequencies at the radius where the disk emissivity is highest, with the observed QPO frequencies. If disk precession is not confined to such a radius and differential precession takes place at a frequency close to the local frame dragging frequency, it remains to be demonstrated that a sufficiently narrow QPO peak matching the observations can be generated as a result of the different precession frequencies that might take place at different radii. More crucially, it is well–known that in viscous accretion disks Lense–Thirring precession of the whole disk is strongly damped (Bardeen and Petterson 1975, Pringle 1992 and references therein). However there appear to be precession modes that are strongly confined to the innermost disk regions and only weakly damped (Markovic & Lamb 1998); these are currently being investigated in greater detail. The mechanism responsible for the excitation of these modes remains an open question. The perspectives for some kind of resonant excitation driven from an azimuthal asymmetry do not appear promising in consideration of the black hole “no hair theorem”.

An alternative possibility is that in the innermost disk region there are individual blobs moving like test–particles in the BHC field, executing also LT–precession, and modulating the observed X–ray flux either through occultation or because they are self–luminous. The existence of discrete blobs is of course not an embarrassment for this argument, since their existence is required in all scenarios trying to explain QPOs, in particular those involving weakly magnetic neutron stars in Low Mass X–ray Binaries (LMXRBS). This tendency of the disk to form discrete blobs seems to be independent of the nature of the accreting source; for instance, Krolik (1998) suggests that it be due to a need to circum-
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Figure 1. The three coordinate frequencies ($\nu_\phi$, dashed line, $\nu_\theta$, dot-dashed line and $\nu_{FD}$, solid line) of spherical orbits at four different radii as functions of the dimensionless angular moment of an $M = 7 M_\odot$ black hole. $r_i$ is the radius of the innermost stable orbit and $r_{\text{peak}}$ is the radius of maximum surface emissivity of the disk. $Q$ is always set equal to 1.

Vent the local instabilities of the disk, irrespective of the properties of the accreting source. So this tendency might be present in accretion disks surrounding both neutron stars and black holes.

By pushing further the analogy with neutron star LMXRBs, where high frequency QPOs are very often observed and successfully interpreted in terms of the Keplerian frequency (note that in LMXRBs $\nu_\phi \simeq \nu_\theta$, see below), one would conclude that if blobs’ precession is responsible for the QPO observed in BHCs, there is no obvious reason why QPOs reflecting the $\phi$ and $\theta$-components of the orbital motion should not be there. Indeed, if, according to the model of Cui et al. (1998), the ~300 Hz QPOs of GRS J1655-40 originate from the frame dragging frequency of blobs off the equatorial plane in the innermost disk regions, then the $\phi$ and $\theta$ frequencies of the orbital motion are $\nu_\phi \simeq 970$ and 950 Hz and $\nu_\theta \simeq 670$ and 650 Hz, while $a \simeq 0.88$ and 0.96, respectively in the case in which frame dragging QPOs are produced at the innermost stable orbit or the radius of highest disk emissivity. The difference between $\nu_\phi$ and $\nu_\theta$ is large and two well-separated QPO peaks might be expected. These signals, however, have not been detected yet. Similar considerations would apply to the case of the ~67 Hz QPOs from GRS 1915+105† (Morgan et al. 1997).

The application of beat frequency models, BFMs, to those neutron stars systems that show twin kHz QPO peaks, allows us to identify the higher frequency kHz QPO ($\sim 800 - 1200$ Hz) as arising directly from the Keplerian motion of blobs at the inner edge of the disk (moreover the neutron star spin frequency is inferred from the difference frequency of the twin kHz QPOs). Stella & Vietri (1998) noticed that the precession frequency of these blobs, as derived from the neutron star parameters inferred from BFMs, agrees well with a broad peak around 20 – 35 Hz that is apparent in the power spectra of three sources. In this model, therefore, $\nu_\phi - \nu_\theta \sim 20 - 35$ Hz, a separation that is comparable to or smaller than the width of the higher frequency kHz QPO peak. Therefore it is not surprising that the signals at $\nu_\phi$ and $\nu_\theta$ are difficult to disentangle in the case of neutron

† The harmonic content of the three fundamental frequencies in the problem at hand will depend on the mechanism responsible for the generation of the signal(s) (e.g. self-luminous vs. occulting blobs) and on geometry, and is beyond the scope of this Letter.
star LMXRBs. It should also be noticed that, around neutron stars with weak magnetic fields, a natural mechanism exists to lift matter off the equatorial plane, through the interaction with a spinning, tilted magnetic dipole moment (Vietri and Stella 1998).

In the frame dragging interpretation of BHC QPOs, very specific predictions are also made in relation to the changes in $\nu_\phi$ and $\nu_\theta$ that results from changes in the frame dragging frequency $\nu_\phi - \nu_\theta$ (cf. Fig. 2). This would provide indeed a sensitive diagnostic to confirm the interpretation and study with unprecedented detail the motion of matter close to event horizon of a Kerr black hole.

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