Hypergeometric functions related to Schur Q-polynomials and BKP equation

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Abstract

We introduce hypergeometric functions related to projective Schur functions $Q_\lambda$ and describe their properties. Linear equations, integral representations and Pfaffian representations are obtained. These hypergeometric functions are vacuum expectations of free fermion fields, and thus these functions are tau functions of the so-called BKP hierarchy of integrable equations.

1 Hypergeometric series

BKP hierarchy was invented in [1],[2] as a certain generalization of the KP hierarchy of integrable equations [3],[1]. In [4] (see also [5]) the role of projective Schur functions [6] as rational solutions of the BKP hierarchy was explained. Here we construct hypergeometric functions related to the projective Schur functions as certain tau functions of the one component BKP hierarchy. The consideration is going along the way of [7],[8],[9]. Let us notice that a different KP hierarchy of the root type B was constructed in the paper [10], we do not study this case in the present paper.

1.1 Neutral free fermions and BKP hierarchy [1],[2]

Let us consider neutral fermions \( \{ \phi_n, n \in \mathbb{Z} \} \), obeying the following canonical anticommutation relations

\[
[\phi_m, \phi_n]_+ = (-)^m \delta_{m,-n},
\]

where \([a, b]_+ = ab + ba\). In particular \(\phi_0^2 = 1/2\).

There are the right and the left vacuum vectors, \(|0\rangle\) and \(\langle 0|\) respectively, having the properties

\[
\phi_m|0\rangle = 0 \quad (m < 0), \quad \langle 0|\phi_m = 0 \quad (m > 0)
\]

We have a right and a left Fock spaces spanned, respectively, by the right and the left vacuum vectors and by right and left vectors

\[
\phi_{n_1} \cdots \phi_{n_k}|0\rangle, \quad \langle 0|\phi_{-n_k} \cdots \phi_{-n_1},
\]

where \(k = 1, 2, \ldots, \) and where we put

\[
n_1 > \cdots > n_k \geq 0
\]

According to [1], [2] we introduce the following operators (Hamiltonians) labeled by odd numbers \(n \in 2\mathbb{Z} + 1\):

\[
H_n = \frac{1}{2} \sum_{k=-\infty}^{\infty} (-1)^{k+1} \phi_k \phi_{-k-n}
\]
One can check that $H_n$ obey the following Heisenberg algebra relations
\[ [H_n, H_m] = \frac{n}{2} \delta_{m+n,0}, \tag{1.1.6} \]
where $[a, b] := ab - ba$ is a commutator.

By (1.1.1) we obtain
\[ [H_n, \phi_m] = \phi_{m-n} \tag{1.1.7} \]

We also notice that
\[ H_n |0\rangle = H_n \phi_0 |0\rangle = 0, \quad \langle 0 | H_{-n} = \langle 0 | \phi_0 H_{-n} = 0, \quad n > 0 \tag{1.1.8} \]

For a pair of collections of independent variables (so called BKP higher times, whose numbers we choose to be always odd numbers)
\[ t = (t_1, t_3, t_5 \ldots), \quad t^* = (t^*_1, t^*_3, t^*_5 \ldots) \tag{1.1.9} \]
one sets
\[ H(t) = \sum_{n=1,3,\ldots}^{+\infty} t_n H_n, \quad H^*(t^*) = \sum_{n=1,3,\ldots}^{+\infty} t^*_n H_{-n} \tag{1.1.10} \]

It is suitable to introduce the following fermionic field, which depends on a complex parameter $z$:
\[ \phi(z) = \sum_{-\infty}^{\infty} z^k \phi_k \tag{1.1.11} \]

For fermionic fields (1.1.11) the anticommutation relation (1.1.1) reads as
\[ [\phi(z), \phi(z')]_+ = \delta (-z, z') = \sum_{n \in \mathbb{Z}} \left( \frac{-z}{z'} \right)^n, \tag{1.1.12} \]

(the symbol $\delta$ denotes a Dirac $\delta$-function with the defying property $\frac{1}{2\pi} \oint f(z) \delta (-z/z') \frac{dz}{z} = f(-z')$).

The relation (1.1.7) yields
\[ [H_n, \phi(z)] = z^n \phi(z), \tag{1.1.13} \]
which in turn results in
\[ e^{H(t)} \phi(z) e^{-H(t)} = \phi(z) e^{\xi(t, z)}, \quad \xi(t, z) = \sum_{n=1,3,\ldots} z^n t_n \tag{1.1.14} \]

The formulae below are called bosonization formulae [2],[4]:
\[ \langle 0 | \phi(z) e^{H(t)} = \langle 0 | \phi_0 e^{H(t-\epsilon(z^{-1}))}, \tag{1.1.15} \]
\[ \sqrt{2} \langle 0 | \phi_0 \phi(z) e^{H(t)} = \langle 0 | e^{H(t-\epsilon(z^{-1}))}, \tag{1.1.16} \]

where
\[ \epsilon(z^{-1}) = \left( \frac{2}{z}, \frac{2}{3z^3}, \frac{2}{5z^5}, \ldots \right) \tag{1.1.17} \]

Furthermore, given $N > 1$, one obtains bosonization relations:
\[ \langle 0 | e^{H(t(x^N))} = 2^{\frac{N}{2}} \frac{\langle 0 | \phi \left( \frac{1}{x_N} \right) \cdots \phi \left( \frac{1}{x_1} \right) }{\Delta(x^N)}, \quad N \text{ even}, \tag{1.1.18} \]
\[ \langle 0 | e^{H(t|x^N)} \rangle = 2^{N+1} \frac{\langle 0 | \phi_0 \phi \left( \frac{1}{x_N} \right) \cdots \phi \left( \frac{1}{x_1} \right) \rangle}{\Delta(x^N)}, \quad N \text{ odd}, \]  

where

\[ \Delta(x^N) = \frac{\prod_{i<j}^N (1 - x_i x_j^{-1})}{\prod_{i<j}^N (1 + x_i x_j^{-1})} = \frac{\prod_{i<j}^N (x_i - x_j)}{\prod_{i<j}^N (x_i + x_j)} \]  

(1.1.19)

The following expression is a typical example of BKP tau-function

\[ \tau(t) = \langle 0 | e^{H(t)} g | 0 \rangle, \quad g = \prod_{k=1}^{2N} \Phi_k \exp \left( \sum_{n,m} b_{n,m} : \phi_n \phi_{-m} : \right), \]  

(1.1.21)

where each of \( \Phi_k \) is a linear combination of neutral fermions \( \phi_k \), and \( \exists N : b_{n,m} = 0, |n-m| > N \). Symbol \( : \) means normal ordering, i.e. \( : \phi_n \phi_{-m} : = \phi_n \phi_{-m} - \langle 0 | \phi_n \phi_{-m} | 0 \rangle \). The numbers \( b_n \) we consider satisfy \( b_{n,m} + b_{-m,-n} = 0 \).

We remind definitions [6] will be required in the sequel. A set of non increasing positive integers \( n_1 \geq n_2 \geq \cdots \geq n_k \geq 0 \) is called the partition of number \( n = |\lambda| = n_1 + \ldots + n_k \), and is denoted by \( \lambda = (n_1, n_2, \ldots, n_k) \); if \( n_k > 0 \), then \( k \) is called the length of the partition \( \lambda \) and denoted by \( l(\lambda) \). The number \( |\lambda| \) is called the weight of the partition \( \lambda \). Numbers \( n_1, n_2, \ldots \) are said to be the parts of the partition \( \lambda \). The partition of zero (i.e. \( n_1 = 0 \)) is denoted by \( \lambda = 0 \). The set of all partitions is usually denoted by \( P \).

The diagram of a partition \( \lambda \) (or the Young diagram \( \lambda \)) may be defined as the set of points (nodes) \((i,j) \in \mathbb{Z}^2 \) such that \( 1 \geq j \geq n_i \).

The conjugate of a partition \( \lambda \) is the partition \( \lambda' \) whose diagram is the transpose of the diagram \( \lambda \), i.e. the diagram obtained by reflection in the main diagonal.

There is different notation for partitions (Frobenius notation). Suppose that the main diagonal of the diagram \( \lambda \) consists of \( r \) nodes \((i,i) \) \((1 \leq i \leq r) \). Let \( \alpha_i = n_i - i \) be the number of nodes in the \( i \)th row of \( \lambda \) to the right of \((i,i)\), for \( 1 \leq i \leq r \), and let \( \beta_i = n_i' - i \) be the number of nodes in the \( i \)th column of \( \lambda \) below \((i,i)\), for \( 1 \leq i \leq r \). One has \( \alpha_1 > \alpha_2 > \cdots > \alpha_r \geq 0 \) and \( \beta_1 > \beta_2 > \cdots > \beta_r \geq 0 \). Then we denote the partition \( \lambda = (n_1, n_2, \ldots, n_k) \) by

\[ \lambda = (\alpha_1, \ldots, \alpha_r | \beta_1, \ldots, \beta_r) = (\alpha|\beta) \]  

(1.1.22)

Given set of variables \( t = (t_1, t_2, t_3, \ldots) \) and a given partition \( \lambda = (n_1, \ldots, n_k), n_k > 0 \), Schur function is defined as

\[ s_{\lambda}(t) = \det h_{n_i-i+j}(t)|_{i,j=1,\ldots,k}, \quad \sum_{k=0}^{\infty} z^k h_k(t) := e^{\sum_{m=1}^{\infty} z^m t_m} \]  

(1.1.23)

In what follows, we shall need only those sets of variables \( t \) where all variables with even numbers vanish: \( t = (t_1, 0, t_3, 0, t_5, \ldots) \). Keeping this in mind we shall write \( t = (t_1, t_3, t_5, \ldots) \) as in (1.1.9). (Below we shall use projective Schur functions which are defined as polynomials in variables \( t = (t_1, t_3, t_5, \ldots ) \).

A set of strictly vanishing positive integers \( \lambda = n_1 > n_2 > \cdots > n_k \geq 0 \) is called the partition with distinct parts or the strict partition. The set of all strict partitions is denoted by \( DP \).

We see that vectors (1.1.3) are labeled by strict partitions. Zero partition \( \lambda = 0 \) is related to vacuum vectors \( \langle 0 |, | 0 \rangle \).

If \( \lambda = (n_1, n_2, \ldots, n_k) \in DP \) and \( n_k \neq 0 \), one gets the so-called double of \( \lambda \), which is the partition \( \hat{\lambda} = (n_1, \ldots, n_k | n_1 - 1, \ldots, n_k - 1) \in P \) in the Frobenius notation.
Lemma 1 [4] Let \( n_1 > n_2 > \cdots > n_k \geq 0, k \) is even. If \( n_k = 0 \) we put \( \lambda = (n_1, n_2, \ldots, n_{k-1}) \), and if \( n_k > 0 \) we put \( \lambda = (n_1, n_2, \ldots, n_k) \). Then

\[
\langle 0 | e^{H(t)} \phi_{n_1} \cdots \phi_{n_k} | 0 \rangle = 2^{-\frac{k}{2}} Q_\lambda \left( \frac{t}{2} \right),
\]

(1.1.24)

where \( Q_\lambda \left( \frac{t}{2} \right) \) is the polynomial in variables \( t_1, t_3, t_5, \ldots \), which is called a projective Schur function and may be presented via the usual Schur function \( s_\lambda \) as

\[
2^{-\frac{k}{2}} Q_\lambda \left( \frac{t}{2} \right) = \sqrt{s_\lambda(t)},
\]

(1.1.25)

where \( s_\lambda(t) \) is the Schur function, and the partition \( \tilde{\lambda} \in P \) is the double of the strict partition \( \lambda \). Notice that \( Q_\lambda \left( \frac{t}{2} \right) = Q_\lambda \left( \frac{t_1}{2}, \frac{t_3}{2}, \frac{t_5}{2}, \ldots \right) \), while \( s_\lambda(t) = s_\lambda(t_1, 0, t_3, 0, t_5, 0, \ldots) \). □

We shall often use a special choice of times: \( t = t(x^{N_1}), t^* = t(y^{N_2}) \):

\[
mt_m = \sum_k N_1 \left( x_k^m - (-x_k)^m \right), \quad mt^*_m = \sum_k N_2 \left( y_k^m - (-y_k)^m \right)
\]

(1.1.26)

In case when it is not confusing we shall omit the superscripts \( N, N^* \), mainly we shall consider the case \( N_1 = N_2 = N \) with some \( N \).

Being rewritten as a function of \( x \) variables, \( Q_\lambda(x) \) becomes a well-known symmetric function, which was invented by Schur in the construction of the projective representations of the symmetric groups [11].

Also we shall need the so-called the product-of-the-hooks-length of the shifted diagram [6], which generalizes the notion of the factorial for the strict partition, and which we shall denote by \( H^*_\lambda \):

\[
H^*_\lambda = \left( \prod_{i=1}^k n_i! \right) \prod_{i<j} \frac{n_i + n_j}{n_i - n_j} = 2^{-\frac{k}{2}} \sqrt{H_\lambda}, \quad \frac{1}{\sqrt{H_\lambda}} = \sqrt{s_\lambda(t_\infty)} = 2^{-\frac{k}{2}} Q_\lambda \left( \frac{t_\infty}{2} \right),
\]

(1.1.27)

where \( \frac{t_\infty}{2} = (\frac{1}{2}, 0, 0, 0, \ldots) \), \( n_i \) are parts of \( \lambda \), and \( H_\lambda \) is the product-of-the-hooks-length of diagram [6] of the double of the partition \( \lambda \). The very last equality of (1.1.27) follows from (1.1.25), other equalities of (1.1.27) are extracted from [6]. The quantity \(|\lambda|! Q_\lambda(\frac{t_\infty}{2})\) has a combinatorial meaning: it counts the number of shifted standard tableaux of a given shape, see [6].

### 1.2 BKP tau-function of hypergeometric type and additional symmetries

Let us consider the following set of commuting fermionic operators

\[
B_k = \frac{1}{2} \sum_{n=-\infty}^{\infty} (-1)^n \phi_n \phi_{k-n} r(n) r(n-1) \cdots r(n-k+1), \quad k = 1, 3, \ldots,
\]

(1.2.1)

These operators are defined by a function \( r \) which satisfies the relation

\[
r(n) = r(1-n).
\]

(1.2.2)

Therefore it is enough to define \( r \) only for positive \( n \).

We check that \([B_m, B_k] = 0\) for each \( m, k \).
It is easy to check that
\[ \langle 0 | B_k = 0 \]  
(1.2.3)

Let us note that one can rewrite operators \( B_n \) in the form
\[ B_k = -\frac{1}{4\pi \sqrt{-1}} \oint \frac{dz}{z} \phi(-z) \left( \frac{1}{z} r(D) \right)^k \phi(z), \quad D := z \frac{d}{dz}, \]  
(1.2.4)

where operator \( r(D) \) acts on all functions of \( z \) from the right hand side according to the rule
\[ r(D) \cdot z^k = r(k) z^k. \]

For the collection of independent variables \( t^* = (t_1^*, t_3^*, \ldots) \) let
\[ B(t^*) = \sum_{n=1,3,\ldots} t_n^* B_n \]  
(1.2.5)

For a partition \( \lambda = (n_1, \ldots, n_k) \) and for a function of one variable \( r \), we introduce the notation
\[ r_{\lambda} = \prod_{i=1}^k r(1)r(2) \cdots r(n_i). \]  
(1.2.6)

We set \( r_0(M) = 1. \)

We have

**Lemma 2** Under conditions of the Lemma 1 the following formula holds
\[ \langle 0 | \phi_{-n_k} \cdots \phi_{-n_1} e^{-B(t^*)} | 0 \rangle = 2^{-\frac{k}{2}} r_{\lambda} Q_{\lambda} \left( \frac{t^*}{2} \right) \]  
(1.2.7)

Now let us consider the following tau-function (1.1.21) of the BKP hierarchy
\[ \tau_r(t, t^*) := \langle 0 | e^{H(t)} e^{-B(t^*)} | 0 \rangle \]  
(1.2.8)

Remark. Here \( B_k \) generate symmetry flows for BKP hierarchy (similar to the additional symmetries of KP [14]). The symmetries act on vacuum solution, parameters \( t_n^* \) play the role of group times.

Using Taylor expanding \( e^H = 1 + H + \cdots \) and **Lemma 2** we easily get

**Proposition 1** We have the expansion:
\[ \tau_r(t, t^*) = 1 + \sum_{\lambda \in DP} 2^{-l(\lambda)} r_{\lambda} Q_{\lambda} \left( \frac{t}{2} \right) Q_{\lambda} \left( \frac{t^*}{2} \right), \]  
(1.2.9)

where sum is going over all nonzero strict partitions (partitions with distinct parts \( DP \)). \( l(\lambda) \) is the length of the partition \( \lambda. \)

Obviously we get:
\[ \tau_r(t, t^*) = \tau_r(t^*, t) \]  
(1.2.10)

Also for \( a \in C \) tau function \( \tau_r(t, t^*) \) does not change if \( t_m \to a m t_m, t_n^* \to a^{-m} t_n^*, m = 1, 3, \ldots. \)

And it does not change if \( t_m \to a m t_m, m = 1, 3, \ldots, r(n) \to a^{-1} r(n). \)

For \( t = t(x^{N_1}), t^* = t^*(y^{N_2}) \), see (1.1.26), the sum (1.2.9) is restricted to the sum over partitions of the length \( l(\lambda) \leq N = \min(N_1, N_2). \)

Remark. Let us cite the recent paper [18] where a certain analog of Gessel theorem was proved, concerning actually the sum (1.2.9) with very specific choice of \( r: r(n) = 1, n < M \) and \( r(n) = 0, n \geq M. \) (This case describes certain rational solutions of BKP).
Remark. If there exists a function $\rho(n), n \in \mathbb{Z}$ such that

$$r(n) = \rho(-n)\rho(n-1),$$ (1.2.11)

the constraint (1.2.2) is satisfied. Then it is easy to check that

$$r_\lambda = \rho^{KP}_\lambda(0),$$ (1.2.12)

the $\rho^{KP}_\lambda(k)$ was used to construct hypergeometric functions related to Schur functions in the framework of KP theory [9],[8] and was defined as

$$\rho^{KP}_\lambda(k) := \prod_{i,j \in \lambda} \rho(k+j-i)$$ (1.2.13)

In (1.2.12) the product goes over all nodes of Young diagram $\tilde{\lambda}$ and the partition $\tilde{\lambda}$ is the double of the strict partition $\lambda \in PD$. In this case

$$\langle 0|\phi_{-n_k} \cdots \phi_{-n_1} e^{-B(t^*)}|0 \rangle = 2^{-\frac{4}{7}} r_\lambda Q_\lambda \left(\frac{t^*}{2}\right) = \rho^{KP}_\lambda(0) \sqrt{s_\lambda(t^*)}. $$ (1.2.14)

$$\tau_\rho(t, t^*) = 1 + \sum_{\lambda \in DP} 2^{-l(\lambda)} r_\lambda Q_\lambda \left(\frac{t}{2}\right) Q_\lambda \left(\frac{t^*}{2}\right) = 1 + \sum_{\lambda \in DP} \rho^{KP}_\lambda(0) \sqrt{s_\lambda(t)s_\lambda(t^*)},$$ (1.2.15)

where sum is going over all nonzero strict partitions (partitions with distinct parts $DP$), $r(n) = \rho(-n)\rho(n-1)$, while the partition $\tilde{\lambda} \in P$ is the double of the strict partition $\lambda \in PD$. The notation $\rho^{KP}_\lambda(0)$ is given by (1.2.13), and the notation $r_\lambda$ see in (1.2.6). $l(\lambda)$ is the length of the partition $\lambda$. □

**Proposition 2** Let us consider KP tau function of hypergeometric type $\tau_\rho(n, t, t^*)$ [8],[9],[7], which we denote $\tau^{KP}_r(n, t, t^*)$ in the present paper, while the notation $\tau_\rho(t, t^*)$ we keep for the BKP tau function (1.2.8). For $r(n) = r(1-n)$ we have

$$\tau^{KP}_r(0, t, t^*) = \tau^2_\rho(t, t^*),$$ (1.2.16)

or

$$\left(1 + \sum_{\lambda \in DP} 2^{-l(\lambda)} r_\lambda Q_\lambda \left(\frac{t}{2}\right) Q_\lambda \left(\frac{t^*}{2}\right)\right)^2 = 1 + \sum_{\lambda \in P} r^{KP}_\lambda(0) s_\lambda(t)s_\lambda(t^*),$$ (1.2.17)

where the notation $r_\lambda$ in the left hand side is due to (1.2.6), and the notation $r^{KP}_\lambda(0)$ in the right hand side is defined in (1.2.13). The sum in the left hand side is going over all nonzero partitions with distinct parts, while the sum in the right hand side is going over all nonzero partitions.

Let us consider additional set of commuting operators $\tilde{B}_k$:

$$\tilde{B}_k = \frac{1}{2} \sum_{n=-\infty}^{\infty} (-1)^{n+1} \phi_n \phi_{-k-n} r(n+1) \bar{r}(n+2) \cdots \bar{r}(n+k), \quad k = 1, 3, \ldots,$$ (1.2.18)

$$\tilde{B}(t^*) = \sum_{k=1,3,\ldots} \tilde{B}_k t^*_k.$$ (1.2.19)

One can also rewrite these operators in the form

$$\tilde{B}_n = -\frac{1}{4\pi \sqrt{-1}} \int \frac{dz}{z} \phi(-z) (\bar{r}(D) z)^n \phi(z), \quad D := z \frac{d}{dz},$$ (1.2.20)

where operator $r(D)$ acts on all functions of $z$ from the right hand side.
Proposition 3

\[ \langle 0 | e^{B(t)} e^{-B(t^*)} | 0 \rangle = \sum_{\lambda \in DP} (\bar{r} \lambda)^{2-\ell(\lambda)} Q_\lambda \left( \frac{t}{2} \right) Q_\lambda \left( \frac{t^*}{2} \right) \]  

(1.2.21)

Taking \( \bar{r} \lambda = 1 \) we get the useful relation:

\[ \exp \left( \sum_{n=1,3,5,\ldots} \frac{1}{2} n t_n t^*_n \right) = \sum_{\lambda \in DP} 2^{-\ell(\lambda)} Q_\lambda \left( \frac{t}{2} \right) Q_\lambda \left( \frac{t^*}{2} \right), \]  

(1.2.22)

where the l.h.s. one gets by evaluating the vacuum expectation value \( \langle 0 | e^{H(t)} e^{H(t^*)} | 0 \rangle \) with the help of (1.1.6) and (1.1.8).

Formula (1.2.22) yields the so-called vacuum tau-function.

The relation (1.2.22) will be of use throughout the text, whatever symbols \( t_k, t^*_k \) mean.

Formulas (1.2.1) (1.1.11) and (1.1.1) provide

\[ [B_n, \phi(z)] = - \left( \frac{1}{z} r(D) \right)^n \cdot \phi(z) \]  

(1.2.23)

Considering the exponents as its Taylor series we obtain

\[ e^{B(t^*)} \phi(z) e^{-B(t^*)} = \sum_{n \in \mathbb{Z}} \phi_n w(n, z) = e^{-\xi_r(t^*, z^{-1})} \cdot \phi(z), \]  

(1.2.24)

where operators

\[ \xi_r(t^*, z^{-1}) = \sum_{m=1,3,\ldots} t_m \left( \frac{1}{z} r(D) \right)^m, \quad D = z \frac{d}{dz} \]  

(1.2.25)

act on all functions of \( z \) on the right hand side, and

\[ w(n, z) = z^n \sum_{m=0}^{\infty} z^{-m} h_m (-t^*) r(n) \cdots r(n-m) = e^{-\xi_r(t^*, z^{-1})} \cdot z^n \]  

(1.2.26)

The formula (1.2.24) generalizes (1.1.14). In (1.2.24) \( \xi_r \) are operators which act on \( z \) variable of \( \phi(z) \).

1.3 \( H_0(T) \), fermions \( \phi(T, z) \) and bosonization rules

Let us introduce a set of variables \( T = \{ T_n, n \in \mathbb{Z}; T_n = -T_{-n} \} \). Put : \( \phi_m \phi_n := \phi_m \phi_n - \langle 0 | \phi_m \phi_n | 0 \rangle \) (it is said that the notation : \( A : \) serves for a normal ordering of an operator \( A \)).

Then we consider the operators

\[ H_0(T) = \frac{1}{2} \sum_{n \in \mathbb{Z}} (-)^{n+1} T_n : \phi_n \phi_{-n} := \sum_{n>0} (-)^{n+1} T_n \phi_n \phi_{-n} \]  

(1.3.1)

Then we see that \( e^{(-)^{n+1} T_n \phi_n \phi_{-n}} = 1 + (-)^n \phi_n \phi_{-n} \left( e^{-T_n} - 1 \right) \), and therefore we obtain

\[ e^{H_0(T)} \phi_n e^{-H_0(T)} = e^{-T_n} \phi_n \]  

(1.3.2)

For \( r \neq 0 \) we put

\[ r(n) = e^{T_{n-1} - T_n} \]  

(1.3.3)

We see that \( r(n) = r(1 - n) \). Then

\[ e^{-H_0(T)} B(t^*) e^{H_0(T)} = -H^*(t^*). \]  

(1.3.4)
Let $r \neq 0$. It is convenient to consider the fermionic operators:

$$\phi(T, z) := e^{H_0(T)} \phi(z) e^{-H_0(T)} = \sum_{n=-\infty}^{n=+\infty} e^{-T_n} z^n \phi_n$$ (1.3.5)

Given $N$ we use (1.1.26) to derive the bosonization rules below in a way similar to [1].

$$e^{-B(t^*(y^N))|0\rangle} = 2^N \frac{\phi(T, -y_1) \cdots \phi(T, -y_N)|0\rangle}{\Delta(y^N)}, \quad N \text{ even,} \quad \text{(1.3.6)}$$

$$e^{-B(t^*(y^N))|0\rangle} = 2^{N+1} \frac{\phi(T, -y_1) \cdots \phi(T, -y_N)\phi_0|0\rangle}{\Delta(y^N)}, \quad N \text{ odd} \quad \text{(1.3.7)}$$

Here $\Delta(x^N)$ see in (1.1.20), for $N = 1$ we put $\Delta(x^N) = 1$.

Therefore in variables $x, y$ (see (1.1.26)) one can write by (1.1.18), (1.1.19):

$$\tau_r(t(x^N), t^*(y^{N^*})) = \langle 0| e^{H(t(x^N))} e^{-B(t^*(y^{N^*}))}|0\rangle$$

$$= 2^{N+N^*} \frac{\langle 0| \phi(\frac{1}{x_N}) \cdots \phi(\frac{1}{x_1}) \phi(T, -y_1) \cdots \phi(T, -y_{N^*})|0\rangle}{\Delta(x^N)\Delta(y^{N^*})}, \quad N + N^* \text{ even} \quad \text{(1.3.8)}$$

$$= 2^{N+N^*+1} \frac{\langle 0| \phi_0 \phi(\frac{1}{x_N}) \cdots \phi(\frac{1}{x_1}) \phi(T, -y_1) \cdots \phi(T, -y_{N^*})|0\rangle}{\Delta(x^N)\Delta(y^{N^*})}, \quad N + N^* \text{ odd} \quad \text{(1.3.9)}$$

By (1.1.1), (1.1.2) we have

$$2^N \langle 0| \phi(\frac{1}{x_N}) \cdots \phi(\frac{1}{x_1})|0\rangle = \Delta(x^N), \quad 2^{N+1} \langle 0| \phi_0 \phi(\frac{1}{x_N}) \cdots \phi(\frac{1}{x_1})|0\rangle = \Delta(x^N), \quad \text{(1.3.10)}$$

where the first equality is for even $N$ while the second is for odd $N$, and where we put

$$\Delta(x^N) = \Delta = \prod_{i<j} \frac{1-x_jx_i^{-1}}{1+x_jx_i^{-1}}, \quad \frac{1-x_jx_i^{-1}}{1+x_jx_i^{-1}} = 1 + 2 \sum_{n=1}^{\infty} \left( -\frac{x_j}{x_i} \right)^n \quad \text{(1.3.11)}$$

At last we consider the different expression for the tau function of hypergeometric type, which we obtain due to (1.3.4). For non-vanishing $r$ we have

$$\tau_r(t, t^*) = \langle 0| e^{H(t)} \exp \left( \sum_{n=-\infty}^{\infty} (-)^n T_n : \phi_n \phi_{-n} : \right) e^{H^*(t^*)}|0\rangle$$

$$= 1 + \sum_{\lambda \in DP} 2^{-l(\lambda)} e^{-T_{n_1}-T_{n_2}-\cdots-T_{n_l}} Q_\lambda(t_2) Q_\lambda(t^*_{2}). \quad \text{(1.3.12)}$$

The sum is going over all different strict partitions $\lambda = (n_1, n_2, \ldots, n_l), \quad l = 1, 2, 3, \ldots$, excluding the partition of zero.

Tau-function (1.3.13) is linear in each $e^{T_n}$. With respect to the BKP dynamics the times $T_n$ have a meaning of integrals of motion.
1.4 Expressions and linear equations for the tau-functions \( \tau_r(\mathbf{t}(x^N), \mathbf{t}^*(y^{N^*})) \)

It is the well-known fact that tau functions solve Hirota bilinear equations [1],[2]. In this section we write down linear equations, which follow from the the bosonization formulae (1.3.8). These equations may be viewed as constraint on the totality of BKP tau functions, or, as someone prefer, may be viewed as a certain form of the so-called string equations.

First let us consider tau function (1.2.9) as a function of variables \( \mathbf{t}(\mathbf{x}) \) (see (1.1.26)) and of \( \mathbf{t}^* = (t_1, t_3, \ldots) \).

Let \( r'(n) := r(-n) \). By (1.2.25) we have

\[
\xi_{r'}(\mathbf{t}^*, x_i) = \sum_{m=1,3,\ldots} t_m (x_i r(D_i))^m, \quad D_i = x_i \frac{\partial}{\partial x_i} \tag{1.4.1}
\]

Let us note that \( [\xi_{r'}(\mathbf{t}^*, x_n), \xi_{r'}(\mathbf{t}^*, x_m)] = 0 \) for all \( n, m \). Below each \( e^{\xi_{r'}(\mathbf{t}^*, x_k)} \) means the Taylor series \( 1 + \xi_{r'}(\mathbf{t}^*, x_k) + \cdots \).

Then in cases \( \mathbf{t} = \mathbf{t}(x^N) \) (using (1.2.3), (1.3.6), (1.3.11) and (1.2.24)) we have the following representations

\[
\tau_r(\mathbf{t}(x^N), \mathbf{t}^*) = \Delta^{-1} e^{\xi_{r'}(\mathbf{t}^*, x_N^1)} \cdots e^{\xi_{r'}(\mathbf{t}^*, x_N^N)} \cdot \Delta, \tag{1.4.2}
\]

where each factor in the product of \( \Delta \) is the notation of the infinite series (1.3.12). Therefore the action of the pseudo-differential operators \( r(D_k), D_k = x_k \frac{\partial}{\partial x_k} \), are well defined on \( \Delta \) via the action on each monomial \( \prod_i x_i^{n_i} \), the action which is given by

\[
r(D_k) \cdot \prod_i x_i^{n_i} = r(n_k) \prod_i x_i^{n_i} \tag{1.4.3}
\]

For \( N = 1 \) the expression (1.4.2) takes a simple form:

\[
\tau_r(\mathbf{t}(x), \mathbf{t}^*) = 1 + r(1)x h_1(\mathbf{t}^*) + r(1)r(2)x^2 h_2(\mathbf{t}^*) + \cdots \tag{1.4.4}
\]

From (1.4.2) it follows that

\[
\left( \frac{\partial}{\partial t^*_m} - \sum_{i=1}^N (x_i r(-D_{x_i}))^m \right) \cdot \left( \Delta \tau_r(\mathbf{t}(x^N), \mathbf{t}^*) \right) = 0 \tag{1.4.5}
\]

Now let us consider the case \( \mathbf{t} = \mathbf{t}(x^N), \mathbf{t}^* = \mathbf{t}^*(y^{N^*}) \), see (1.1.26). We get

\[
\tau_r(\mathbf{t}(x^N), \mathbf{t}^*(y^{N^*})) = \sum_{\lambda \in \mathcal{P}} r_{\lambda} 2^{-(\ell(\lambda))} Q_{\lambda}(x^N) Q_{\lambda}(y^{N^*}) \tag{1.4.6}
\]

\[
= \frac{1}{\Delta(x^N)} \prod_{i=1,j=1}^N (1 + y_j x_i r(D_{x_i})) \cdot (1 - y_j x_i r(D_{x_i}))^{-1} \cdot \Delta(x_N) \tag{1.4.7}
\]

\[
= \frac{1}{\Delta(y^{N^*})} \prod_{i=1,j=1}^N (1 + x_i y_j r(D_{y_j})) \cdot (1 - x_i y_j r(D_{y_j}))^{-1} \cdot \Delta(y^{N^*}), \tag{1.4.8}
\]

where \( (1 - y_j x_i r(D_{x_i}))^{-1} \) and \( (1 - x_i y_j r(D_{y_j}))^{-1} \) are formal series \( 1 + y_j x_i r(D_{x_i}) + \cdots \) and \( 1 + x_i y_j r(D_{y_j}) + \cdots \) respectively.

Looking at (1.4.7), (1.4.8) one derives the following system of linear equations

\[
\left( D_{y_j} - 2 \sum_{i=1}^N \frac{y_j x_i r(D_{x_i})}{1 - (y_j x_i r(D_{x_i}))^2} \right) \cdot \left( \Delta(x_N) \tau_r(\mathbf{t}(x^N), \mathbf{t}^*(y^{N^*})) \right) = 0, \quad j = 1, \ldots, N^*, \tag{1.4.9}
\]
\[ \left( D_{x_i} - 2 \sum_{j=1}^{N^*} \frac{x_i y_j r(D_{y_j})}{1 - (x_i y_j r(D_{y_j}))^2} \right) \cdot \left( \Delta \left( y^{N^*} \right) \tau_r \left( t \left( x^N \right), t^* \left( y^{N^*} \right) \right) \right) = 0, \quad i = 1, \ldots, N, \]

(1.4.10)

where \( 1 - (y_j x_i r(D_{x_i}))^2 \) and \( 1 - (x_i y_j r(D_{y_j}))^2 \) are the formal series \( 1 + (y_j x_i r(D_{x_i}))^2 + \cdots \) and \( 1 + (x_i y_j r(D_{y_j}))^2 + \cdots \) respectively.

Also for the tau-function written in variables \( x^N, y^{N^*} \) (see (1.2.22)) we have

\[
\frac{1}{\Delta} \sum_{i=1}^{N} D_{x_i} \Delta \tau(t(x^N), T, t^*(y^{N^*})) = \frac{1}{\Delta} \sum_{i=1}^{N^*} \left( \frac{1}{y_i} D_{y_i} \right) \Delta^* \tau(t(x^N), T, t^*(y^{N^*})),
\]

(1.4.11)

where \( \Delta = \Delta(x^N) \) and \( \Delta^* = \Delta(y^{N^*}) \). This formula may be obtained by the insertion of the fermionic operator \( \text{res}_z : \phi(-z)z \frac{d}{dz} \phi(z) \) : inside the vacuum expectation value (1.3.13) (like it was done in [12]). These formulae can be also written in terms of higher BKP times, with the help of vertex operator action, see the subsection “Vertex operator action”. Then the relation (1.4.5) is the infinitesimal version of (1.9.5), while the relation (1.4.11) is the infinitesimal version of (1.9.6).

### 1.5 Vacuum expectation as a scalar product. Symmetric function theory consideration

It is well-known fact in the theory of symmetric functions, that there exists the scalar product, where the projective Schur functions are orthogonal

\[
< Q_\mu, Q_\lambda > = 2^{l(\lambda)} \delta_{\mu, \lambda},
\]

(1.5.1)

see [6], Chapter III. Let us note that projective Schur functions form a basis, see [6] for details.

One may obtain the following realization of this scalar product if functions \( Q_\lambda \) are written as functions of variables \( t = (t_1, t_2, \ldots) \) (sometimes below it is convenient to use the letter \( \gamma \) instead of the \( t \)). For functions \( Q_\lambda(\gamma) \) (moreover for any functions \( f = f(\gamma), g = g(\gamma) \)) we have the following realization of the scalar product:

\[
< Q_\mu, Q_\lambda >= \left( Q_\mu(\frac{\gamma}{2}) \cdot Q_\lambda(\frac{\gamma}{2}) \right) |_{\gamma = 0}, \quad < f, g >= \left( f(\frac{\gamma}{2}) \cdot g(\frac{\gamma}{2}) \right) |_{\gamma = 0},
\]

(1.5.2)

where

\[
\frac{\gamma}{2} = (\frac{\gamma_1}{2}, \frac{\gamma_2}{2}, \frac{\gamma_3}{2}, \ldots), \quad \frac{\bar{\gamma}}{2} = (\partial_{\gamma_1}, \frac{1}{3} \partial_{\gamma_3}, \ldots, \frac{1}{2n-1} \partial_{\gamma_{2n-1}}, \ldots)
\]

(1.5.3)

In particular we have \( < \gamma_1, \gamma_2 >= < 2\gamma_1, 2\gamma_2 >= \frac{2}{n} \delta_{\gamma_1, \gamma_2} \).

Let us deform the relation (1.5.1) as follows

\[
< Q_\mu, Q_\lambda >_r = 2^{l(\lambda)} r_{\lambda} \delta_{\mu, \lambda}
\]

(1.5.4)

By this formula and by (1.2.22) we obtain that \( \tau_r \) of (1.2.9) is the following scalar product of functions of variables \( \gamma \)

\[
\tau_r(t, t^*) = e^{\sum_{m=1,3,\ldots} \frac{1}{2} m t_m \gamma_m}, e^{\sum_{m=1,3,\ldots} \frac{1}{2} m t_m^* \gamma_m}_{r},
\]

(1.5.5)

notice that in (1.5.5) the variables \( t, t^* \) play the role of parameters.

Now let us show that scalar product (1.5.2) is equal to the following vacuum expectation value:
Proposition 4  For functions $f = f(\frac{\tau}{2}), g = g(\frac{\tau}{2})$

$$< f, g >= \langle 0 | f(\frac{H}{2})g(\frac{H^*}{2})|0 \rangle,$$  

(1.5.6)

where

$$\frac{H}{2} = (\frac{H_1}{1}, \frac{H_3}{3}, \ldots, \frac{H_{2n-1}}{2n-1}, \ldots), \quad \frac{H^*}{2} = (\frac{H_{-1}}{1}, \frac{H_{-3}}{3}, \ldots, \frac{H_{-2n+1}}{2n-1}, \ldots)$$  

(1.5.7)

The proof follows from the comparing of the formulae (1.1.6),(1.1.8) with (1.5.2). Also we have

Proposition 5  For functions $f = f(\frac{\tau}{2}), g = g(\frac{\tau}{2})$

$$< f, g >_r = \langle 0 | f(\frac{H}{2})g(\frac{B}{2})|0 \rangle,$$  

(1.5.8)

where

$$\frac{H}{2} = (\frac{H_1}{1}, \frac{H_3}{3}, \ldots, \frac{H_{2n-1}}{2n-1}, \ldots), \quad \frac{B}{2} = (B_1, B_3, \ldots, B_{2m-1}, \ldots)$$  

(1.5.9)

1.6  Hypergeometric functions

Let all parameters $b_k$ be not equal to negative integers. Denote $(1, 0, 0, \ldots)$ by $t_\infty$. Let

$$r(n) = \frac{\prod_{i=1}^p (a_i + n - 1)}{\prod_{i=1}^s (b_i + n - 1)}, n > 0, \quad t^* = t_\infty$$  

(1.6.1)

We have (see (1.2.9),(1.1.27)) :

$$\tau_r(t, t^*) = 1 + \sum_{\lambda \in DP} 2^{-l(\lambda)} Q_\lambda \left( \frac{t}{2} \right) \frac{1}{H_\lambda^*} \frac{\prod_{k=1}^p (a_k)_\lambda}{\prod_{k=1}^s (b_k)_\lambda},$$  

(1.6.2)

where the notation $(a)_\lambda$ is a version of Pochhammer Symbol attached to a partition:

$$(a)_\lambda := \prod_{i=1}^{l(\lambda)} (a_i)_n, \quad (a)_n := a(a+1) \cdots (a+n_i-1)$$  

(1.6.3)

If we take $t = (2x, \frac{2a_3}{3}, \frac{2a_5}{5}, \ldots)$, the sum is going over partitions of the length one, in this case $Q_{(n)} = 2^{x^n}, n > 0, Q_0 = 1, H_{(n)}^* = n!$, and we obtain the ordinary generalized hypergeometric function of one variable

$$\tau_r(t, t^*) = \sum_{n=0}^\infty \frac{\prod_{k=1}^p (a_k)_n}{\prod_{k=1}^s (b_k)_n} \frac{x^n}{n!} = _p F_s (a_1, \ldots, a_p; b_1, \ldots, b_s; x)$$  

(1.6.4)

Below we consider hypergeometric series related to Schur Q-functions with the following property: its square is a certain hypergeometric function related to the Schur functions $s_\lambda$ [7] via the Proposition 2, see (1.2.17). To achieve this property one should take a rational function $r$ solving $r(n) = r(1 - n)$. Let all parameters $\beta_k$ be non half integers. We take

$$r(n) = \frac{\prod_{k=1}^p \left( (n - \frac{1}{2})^2 - \alpha_k^2 \right)}{\prod_{k=1}^s \left( (n - \frac{1}{2})^2 - \beta_k^2 \right)}, \quad \rho(n) = \sqrt{-1}(\alpha - \frac{1}{2} - n)$$  

(1.6.5)

We obtain the following hypergeometric function

$$\tau_r(t, t^*) = 1 + \sum_{\lambda \in DP} \frac{2^{-l(\lambda)} \prod_{k=1}^p (\alpha_k + \frac{1}{2})_\lambda (-\alpha_k + \frac{1}{2})_\lambda}{H_\lambda^* \prod_{k=1}^s (\beta_k + \frac{1}{2})_\lambda (-\beta_k + \frac{1}{2})_\lambda} Q_\lambda \left( \frac{t}{2} \right).$$  

(1.6.6)
1.7 Integral representations of the scalar product $\langle , \rangle_r$

We need a function $\mu_r$ of one variable with the following properties:

$$\int \int \mu_r(zz^*)z^n dzd^* = \int \int \mu_r(zz^*)z^{*n} dzd^* = \delta_{n,0}$$

(1.7.1)

together with the relation

$$\int \int \mu_r(zz^*)z^n z^{*m} dzd^* = 2\delta_{n,m}r(1)r(2)\cdots r(n)$$

(1.7.2)

Remark. The way to find an appropriate $\mu_r$ is to solve the equation

$$\left(z^* - \frac{1}{z} r(-D)\right) \cdot \mu_r(zz^*) = 0, \quad D = z \frac{d}{dz}, \quad r(k) = r(1-k)$$

(1.7.3)

and to choose the integration domain $\Gamma$ should be chosen in such a way that the operator $\frac{1}{z} r(D), D = z \frac{d}{dz}$ is conjugated to the operator $\frac{1}{z} r(-D)$. $\square$

Using

$$\frac{1}{M!} \int \cdots \int \phi(z_M) \cdots \phi(z_1)|0\rangle \langle 0| \phi(z_M^*) \cdots \phi(z_1^*) \prod_{i=1}^M \mu_r(z_i, z_i^*) dz_i dz_i^* = \sum_{\lambda \in DP,l(\lambda) \leq M} 2^{l(\lambda)}|\lambda\rangle r_\lambda(\lambda)$$

(1.7.4)

we obtain for partitions $\lambda, \mu$ (both partitions have length $l(\lambda), l(\mu) \leq M$)

$$\frac{1}{M!} \int \cdots \int \Delta(z) \Delta(z^*) Q_\lambda(z) Q_\mu(z^*) \prod_{k=1}^M \mu_r(z_k, z_k^*) dz_k dz_k^* = 2^{l(\lambda)} r_\lambda \delta_{\lambda\mu} = \langle Q_\lambda, Q_\mu \rangle_r$$

(1.7.5)

where

$$\Delta(z) = \prod_{i < j} (z_i - z_j), \quad \Delta(z^*) = \prod_{i < j} (z_i^* - z_j^*)$$

(1.7.6)

With the help of equalities

$$e^{\sum_{n=1,3,\ldots}^{\infty} \sum_{k=1}^M z_k^n t_n} = \sum_{\lambda \in DP,l(\lambda) \leq M} 2^{-l(\lambda)} Q_\lambda(z^M) Q_\lambda(t),$$

(1.7.7)

$$e^{\sum_{n=1,3,\ldots}^{\infty} \sum_{k=1}^M z_k^{*n} t_n^*} = \sum_{\lambda \in DP,l(\lambda) \leq M} 2^{-l(\lambda)} Q_\lambda(z^*M) Q_\lambda(t^*)$$

(1.7.8)

we evaluate the integral

$$I_r(M, t, t^*) = \frac{1}{M!} \int \cdots \int \Delta(z) \Delta(z^*) \prod_{k=1}^M e^{\sum_{n=1,3,\ldots}^{\infty} (z_k^n t_n + z_k^{*n} t_n^*)} \mu_r(z_k, z_k^*) dz_k dz_k^*$$

(1.7.9)

We finely obtain

$$I_r(M, t, t^*) = 1 + \sum_{\lambda \in DP,l(\lambda) \leq M} 2^{-l(\lambda)} r_\lambda \left(\frac{t}{2}\right) Q_\lambda \left(\frac{t^*}{2}\right)$$

(1.7.10)

The restriction $l(\lambda) \leq M$ makes the difference between the r.h.s. of (1.7.10) and $\tau_r(t, t^*)$. However in case at least one of the sets $t, t^*$ has the form of (1.1.26) with $N$ or $N'$ no more then $M$, the integral $I_r(M, t, t^*)$ is the BKP tau function $\tau_r(t, t^*)$. In this case I would like to interpret the integral $I_r(M, t, t^*)$ as an analog of Borel sum for the divergent series $\tau_r$, similar we do in [13].

It may be interesting to apply this treatment to matrix models [15] and to statistical models where partition functions reduce to the similar type of integrals, see [16], [17] for examples of similar integrals.
1.8 Pfaffian formulae for the tau-functions $\tau_r(t(x), t^*)$, $\tau_r(t(x), t^*(y))$

First let us remind that a Pfaffian of a $2N$ by $2N$ skew symmetric matrix $S$ is defined as the square root of its determinant. We shall denote the Pfaffian as $\text{Pfaff}(S)$.

Now let us enumerate all neutral fermions in (1.3.8) from the left to the right as $\Phi_k$, $k = 1, 2, \ldots, 2N$ as it is written here

$$\langle 0|\phi(\frac{1}{x_N})\ldots\phi(\frac{1}{x_1})\phi(T, -y_1)\ldots\phi(T, -y_N)|0\rangle = \langle 0|\Phi_1\Phi_2\cdots\Phi_{2N-1}\Phi_{2N}|0\rangle$$

(1.8.1)

$\phi(\frac{1}{x_N}) = \Phi_1, \ldots, \phi(T, -y_N) = \Phi_{2N}$. With the help of Wick theorem (see [2] for instance) we obtain

$$\langle 0|\Phi_1\Phi_2\cdots\Phi_{2N-1}\Phi_{2N}|0\rangle = \text{Pfaff}(S), \quad S_{km} = \langle 0|\Phi_k\Phi_m|0\rangle$$

(1.8.2)

The matrix $S_{nm}$ consists of the following four blocks, $N$ by $N$ each:

$$S_{km} = \frac{1}{2}(x_m - x_k), \quad 1 \leq k, m \leq N; \quad S_{km} = \frac{1}{2}(y_m - y_k), \quad N + 1 \leq k, m \leq 2N,$$

(1.8.3)

and

$$S_{km} = -S_{mk} = \sum_{n=0}^{\infty} e^{-T_n}x_k^ny_n, \quad 1 \leq k \leq N, \quad N + 1 \leq m \leq 2N$$

(1.8.4)

where $e^{-T_n-1-T_n} = r(n)$. Thus in the case $t = t(x), t^* = t^*(y)$, we obtain the Pfaffian formula for (1.2.9)

**Proposition 6** For $r \neq 0$ and for $t(x^N), t^*(y^N)$ defined by (1.1.26) we have

$$\tau_r(t(x^N), t^*(y^N)) = \frac{\text{Pfaff}(S)}{\Delta(x^N)\Delta(y^N)}$$

(1.8.5)

where for the notation $\Delta(x^N)$ see (1.3.12). The matrix $S_{km}$ is defined by (1.8.3)-(1.8.4).

**Proposition 7** For $N$ even

$$\tau_r(t(x^N), t^*) = \frac{\text{Pfaff}(R)_{i,k=1}^{N}}{\Delta(x^N)}, \quad R_{ik} = \frac{1}{2x_i + x_k}r_r(t(x_i, x_k), t^*), \quad t_m(x_i, x_k) = \frac{2x_i^m}{m} + \frac{2x_k^m}{m}$$

(1.8.6)

where

$$\frac{1}{2x_i + x_k}r_r(t(x_i, x_k), t^*) = \langle 0|\phi(\frac{1}{x_i})\phi(\frac{1}{x_k})e^{-B(t^*)}|0\rangle$$

(1.8.7)

$$= \frac{1}{2} \sum_{n_i, n_k \geq 0} (x_k^{n_k}x_i^{n_i} - x_k^{n_i}x_i^{n_k})r_{n_k, n_i}Q_{n_k, n_i}(t^*)\frac{t^*}{2}$$

(1.8.8)

For $N$ odd the formula is almost the same, provided the additional $N + 1$th row and line are added to matrix $R$:

$$\tau_r(t(x^N), t^*) = \frac{\text{Pfaff}(R)_{i,k=1}^{N+1}}{\Delta(x^N)}, \quad R_{i,N+1} = -R_{N+1,i} = \tau_r(t(x_i), t^*), \quad t_m(x_i) = \frac{2x_i^m}{m}$$

(1.8.9)

(where for explicit expression of $\tau_r(t(x_i), t^*)$ see (1.4.4)).

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1.9 The vertex operator action. Linear equations II

Vertex operators $Z(z)$ act on the space $C[t_1, t_3, t_5, \ldots]$ of polynomials in infinitely many variables, and are defined by the formulae:

$$Z(z) = e^{\xi(t, z)} e^{-\frac{2}{\sqrt{-1}} \xi(t, \bar{z})}, \quad \xi(t, z) = z t_1 + z^3 t_3 + z^5 t_5 + \cdots, \quad \bar{\partial} := \left( \frac{\partial}{\partial t_1}, \frac{1}{3} \frac{\partial}{\partial t_3}, \frac{1}{5} \frac{\partial}{\partial t_5}, \ldots \right) \quad (1.9.1)$$

Let us introduce the operators (the generators of additional symmetries) which act on functions of $t$ variables:

$$\Omega_r(t^*) := -\frac{1}{2\pi \sqrt{-1}} \lim_{\epsilon \to 0} \oint Z(-z + \epsilon) \xi_r(t^*, z, D) Z(z) \frac{dz}{z} \quad (1.9.2)$$

(here parameters $t^*_k$ play the role of group times). For instance

$$\Omega_{r=1}(t^*) = 2 \sum_{n>0} (2n + 1) t_{2n+1} t^*_{2n+1} \quad (1.9.3)$$

We also consider

$$Z_{nn} = -\frac{1}{4\pi^2} \oint \frac{z^n}{(-z^*)^n} Z(-z^*) Z(z) \frac{dz dz^*}{zz^*} \quad (1.9.4)$$

Having in mind (1.3.3) it is convenient to have the second notation for our tau function:

$$\tau(t, T, t^*) := \tau_r(t, t^*)$$

Proposition 8 We have shift argument formulae

$$e^{\Omega_r(\gamma)} \cdot \tau_r(t, t^*) = \tau_r(t, t^* + \gamma) \quad (1.9.5)$$

We also have

$$e^{\sum_{n=-\infty}^{\infty} \gamma_n Z_{nn}} \tau(t, T, t^*) = \tau(t, T + \gamma, t^*), \quad (1.9.6)$$

In particular

$$e^{\Omega_r(t^*)} \cdot 1 = \tau_r(t, t^*), \quad (1.9.7)$$

$$e^{\sum_{n=-\infty}^{\infty} T_n Z_{nn}} \exp \left( 2 \sum_{n=1,3,\ldots} n t_n t^*_n \right) = \tau(t, T, t^*). \quad (1.9.8)$$

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