Disordered $d$-wave superconductors with interactions

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Abstract

We study the localization properties of disordered $d$-wave superconductors by means of the fermionic replica trick method. We derive the effective non-linear $\sigma$-model describing the diffusive modes related to spin transport which we analyze by the Wilson-Polyakov renormalization group. A lot of different symmetry classes are considered within the same framework. According to the presence or the absence of certain symmetries, we provide a detailed classification for the behavior of some physical quantities, like the density of states, the spin and the quasiparticle charge conductivities. Following the original Finkel’stein approach, we finally extend the effective functional method to include residual quasiparticle interactions, at all orders in the scattering amplitudes. We consider both the superconducting and the normal phase, with and without chiral symmetry, which occurs in the so called two-sublattice models.

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I. INTRODUCTION

After the discovery of superconductivity at high temperatures in doped cuprate materials \[1\], a lot of efforts have been devoted to find a theoretical justification of this amazing property and to explain the complex phenomenological aspects of these “high $T_c$ superconductors”.

One of the novel features of these cuprates is the presence of gapless Landau-Bogoliubov quasiparticle excitations in the superconducting phase, due to the $d$-wave symmetry of the order parameter. This peculiarity makes these materials a suitable playground for studying the role of disorder in gapless superconductors.

Moreover, since cuprates are essentially two-dimensional, quantum interference and localization should strongly affect the quasiparticle transport and thermodynamic properties at low temperature \[2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15\]. Indeed, starting from a $d$-wave BCS Hamiltonian in the presence of disorder a multitude of different regimes and crossovers has been predicted from quantum interference, depending on the specific symmetry of the quasiparticles and of the disorder \[16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36, 37, 38, 39, 40, 41, 42\].

The present work, which revisits and extends the analysis in Ref. \[41\], would like to contribute to the widely studied but somehow still controversial topic of localization effects in these unconventional superconductors. There are several theoretical issues which are still open and motivate our investigation.

The first issue concerns the charge transport. Within a BCS description of the superconducting phase, charge is not a conserved quantity due to the gauge symmetry breaking. Therefore, although quasiparticles are gapless for a $d$-wave order parameter, nevertheless long-wavelength quasiparticle charge fluctuations are not diffusive in the presence of disorder, unlike in a normal metal. The diffusive modes just carry spin or energy, which remain conserved quantities. It is well known that, as disorder increases, quantum interference may lead to the Anderson localization in a normal metal \[43, 44, 45\]. While it is clear that such a phenomenon may suppress spin and thermal conductivities in a $d$-wave superconductor, the effects on the quasiparticle charge conductivity are not as settled yet. For this reason we extend existing quantum field theory approaches built to deal with the truly diffusive modes in a $d$-wave superconductor to include the charge modes, which acquire a mass term by the
onset of superconductivity. In this way we are able to calculate how the charge conductivity is modified by disorder in comparison with spin and thermal conductivities. We find the same corrections for all of them in all the cases under study but one, when time reversal symmetry is broken at half filling, although the conductivities are different at the Born level.

A second issue has to do with some controversial results about the quasiparticle density of states at the chemical potential in the presence of disorder. Within the self-consistent T-matrix approximation scheme\cite{18, 36}, it was found that the density of states vanishes linearly as the Fermi energy is approached in the pure system, and acquires instead a finite value in the presence of disorder. This was later used as the starting point to build up a standard field-theoretical approach based on the non-linear $\sigma$-model to cope with the quantum interference corrections not included within the self-consistent T-matrix approximation\cite{32}. This approach predicts localization and, eventually, a vanishing density of states.

This standard perturbative technique was nevertheless somewhat unsatisfactory. It was argued\cite{21, 22} that systems with nodes in the spectrum require a more careful analysis.

The quasiparticle spectrum in a $d$-wave superconductors can be described by 2-dimensional (2D) Dirac fermions, with conical spectrum. In the absence of interactions, the 2D quantum problem in the presence of disorder becomes effectively a 2D classical problem with the retarded-advanced frequency of the single particle Green’s function playing the role of an external field. This is the non-linear $\sigma$-model description\cite{46}. On the other hand, a classical zero frequency model in 2-dimensions with conical spectrum is analogous to a quantum problem of Dirac fermions in 1+1 dimension. Within this scheme the DOS is found to vanish (with a different behavior from the approach above) and no localization is predicted\cite{22}. In the language of Dirac fermions in 1+1 dimension, the disorder average within the replica trick method generates an effective interaction among the one-dimensional (1D) fermions, with all the complications that are known to occur. For instance, translated in the 1D language, the self-consistent T-matrix approach which generates a finite density of states at the Fermi energy is analogous to the Hartree-Fock approximation, which always leads to density-wave order parameters. However, it is known that Hartree-Fock fails completely in 1D, which poses serious doubt about the validity of the T-matrix approach even as a starting point of a perturbative treatment. From this point of view the controversy concerns, more deeply, the question of which model correctly describes the quantum interference effects. Indeed in the peculiar case in which at most pairs of opposite nodes are
coupled by disorder, the perturbation theory around the T-matrix saddle point solution does not contain any small parameter, like the inverse conductance in the conventional Anderson localization. However, in the most general case of disorder, when all four nodes are coupled, we find that a small parameter still exists being related to the anisotropy of the Dirac cones, suggesting a conventional field-theory treatment. Looking more carefully at the problem we find that the above controversial results are related within the non-linear $\sigma$-model to the presence (when only opposite nodes are coupled by disorder) or the absence (when all four nodes are coupled) of a Wess-Zumino-Witten term $[47, 48, 49]$.

A third interesting issue is the role of the nesting property in these kind of systems. Given a generic eigenfunction with energy $E$ and amplitude $\phi(i)_E$ at site $i = (n, m)$, the operator

$$O_\pi \phi(i)_E \equiv (-1)^{n+m} \phi(i)_E,$$

(1)

which shifts the momentum by $(\pi, \pi)$, generates the eigenfunction with energy $-E$ if nesting occurs. This implies an additional symmetry, which has the form of chiral symmetry, at $E = 0$, when the two wavefunctions $(1 \pm (-1)^{n+m})\phi_{E \rightarrow 0^+}$, defined on different sublattices, with $n + m$ even or odd, are both eigenvectors. The nesting property occurs when the operator $O_\pi$ anticommutes with the Hamiltonian, which is possible in models in which the Hamiltonian contains only terms which couple one sublattice with the other, so called two-sublattice models. In addition, the chiral symmetry further requires half-filling. Both conditions are quite strict and do not represent a common physical situation. Nevertheless chiral symmetry leads to quite different and somehow surprising scaling behaviors that are worth to be studied. It was seen, for instance, that this symmetry drastically changes the low energy density of states. Several models presenting a chiral symmetry were found to have isolated delocalized states at the band center at low dimensions. It was argued $[50, 51]$ that these models corresponds to a particular class of non-linear $\sigma$-models and it was shown that quantum corrections to the $\beta$ function which controls the scaling behavior of conductivity vanish at the band center at all orders in the disorder strength, leading to a metallic behavior at that value of chemical potential. Moreover, the $\beta$ function of the density of states was found to be finite, unlike in the standard Anderson localization. These scaling laws generate a divergent behavior at low energy of the density of states. The anomalous terms in the action, when chiral symmetry holds, were found to be connected with fluctuations of the staggered density of states $[52]$. The modes representing these fluctuations are massive
in standard non-linear $\sigma$-models, while they become diffusive in two-sublattice cases. For this reason in the conductance channels with both positive and negative frequencies acquire diffusive poles and contribute to quantum interferences corrections. This is what we find also in our two sublattices $d$-wave superconductive model that presents extended states at the band center which are associated with diffusive spin transport. The DOS diverges and shows a behavior similar to that of \cite{52}. Furthermore, we find an unexpected charge conductance behavior. As we have said before, although charge modes in $d$-wave superconductors are not diffusive, nevertheless quantum interference corrections affect the charge conductance. In particular, when chiral symmetry holds but time-reversal symmetry is broken, quasiparticle charge conductivity is suppressed, but spin and thermal conductivities stay finite, leading to a spin-metal but charge-insulator quasiparticle behavior. Moreover we saw that, even though magnetic fields or magnetic impurities introduce on-site terms in the Hamiltonian that spoil the full sublattice symmetry, staggered fluctuations are not totally suppressed introducing other symmetries in the model under study. We saw, for instance, that the problem of $d$-wave superconductors with chiral symmetry and magnetic impurities can be mapped to a $U(2n)$ non-linear $\sigma$-model and belongs accidentally to the same universality class as $d$-wave superconductors far from the nesting point embedded in a constant magnetic field.

From the point of view of the cuprate $d$-wave superconductors, it is not unlikely that chiral symmetry may play some role, especially in underdoped systems close to the half-filled Mott insulator. Indeed it is believed that the impurity potential is close to the unitary scattering limit \cite{30} which, by taking out one site, reduces to a random nearest-neighbor hopping. Furthermore, although the band structure does not have a perfect nesting, the superexchange interaction which stabilizes a Ne'el antiferromagnetic phase at half-filling may effectively reduce the energy scale at which deviations from perfect nesting get appreciable.

The last issue that we will discuss is the role of the residual quasiparticle interaction and its effects on the conductivity and on the density of states. Following the original Finkel’stein approach \cite{53}, which extended the effective functional method to the disordered electron-electron interacting systems, we introduce the effective quasiparticle scattering amplitudes in different channels, firstly considering systems without sublattice symmetry. We find that, consistently with the charge not being a conserved quantity, the singlet particle-hole channel does not contribute while the triplet channel does. On the other hand, the scattering
amplitude in the Cooper particle-particle channel acquires a factor 1/2 with respect to the normal metal state, which corresponds to the fact that only the phase of the order parameter is massless. We see that the effective interaction, which we assumed being repulsive, has a delocalizing effect enhancing the density of states. We extend the Finkel’stein model in order to include the nesting property, by introducing additional scattering amplitudes with \((\pi, \pi)\) momentum transferred, and we evaluate the new corrections to the density of states and to the conductivity. We also consider the metallic phase which can be of relevance for 2D metals or semimetals. Moreover, we notice an interesting fact which occurs at half-filling for a two-sublattice model in both the superconducting and normal phase. Depending on the sign of the interaction the staggered particle-hole fluctuations, being diffusive, can lead to a log-divergent staggered susceptibility, and a Stoner instability towards a spin or charge density wave.

II. THE MODEL

The characteristic feature of a \(d\)-wave superconductor is the existence of four nodal points where the order parameter vanishes. To study the low temperature transport properties of a \(d\)-wave superconductor, we consider the following model defined on a two-dimensional square lattice with lattice constant \(a\):

\[
\mathcal{H} = -\sum_{\langle ij \rangle} \sum_{\sigma} \left( t_{ij} e^{i\phi_{ij}} c_{i\sigma}^\dagger c_{j\sigma} + H.c. \right) + \sum_{\langle ij \rangle} \left[ \Delta_{ij} \left( c_{i\uparrow}^\dagger c_{j\downarrow}^\dagger + c_{j\uparrow}^\dagger c_{i\downarrow}^\dagger \right) + H.c. \right],
\]

(2)

where \(\langle ij \rangle\) means that the sum is restricted to nearest neighbor sites, \(c_{i\sigma}^\dagger\) creates an electron with spin \(\sigma = \uparrow, \downarrow\) at site \(i\), while \(c_{i\sigma}\) annihilates it. We take a gap function \(\Delta_{ij}\) of \(d\)-wave symmetry. The hopping matrix elements are independent random gaussian variables with average value \(t\) and variance \(ut\), and satisfy \(t_{ij} = t_{ji} \in \mathbb{R}\), and \(\phi_{ij} = -\phi_{ji}\), with \(\phi_{ij}\) zero or finite depending whether or not time reversal invariance holds. The spectrum of the Hamiltonian (2) possesses a nesting property. This implies an additional symmetry (chiral symmetry) at half filling. The localization properties are quite different whether chiral symmetry holds, which corresponds to the Fermi energy \(E_F = 0\) (half-filling), or broken, \(E_F \neq 0\). In the latter case, the localization properties of (2) are analogous to models in which on-site disorder is present or next-nearest neighbor hopping is included, which break chiral symmetry everywhere in the spectrum. For this reason while dealing with a bipartite lattice
that induces an higher degree of symmetry we can reduce the problem to the standard 
\textit{d}-wave case only by introducing an on site term in the Hamiltonian, thus spoiling chiral 
symmetry.

In the absence of randomness the quasiparticle spectrum has four nodes at \((\pm k_F, \pm k_F, 0)\).
In the vicinity of each gap node the Fourier transform of \(-t_{ji}\), namely \(\epsilon_k = -2t \cos(k_x a) - 2t \cos(k_y a)\), varies linearly perpendicularly to the Fermi surface while the Fourier transform 
of \(\Delta_{ij}\), that is \(\Delta_k = 2\Delta (\cos(k_x a) - \cos(k_y a))\), varies linearly parallel to the Fermi surface. 
Let us rotate the axes from \(k_x, k_y\) to \(k_1, k_2\) by \(\pi/4\) rotation, and define a Fermi velocity, \(v_1\) perpendicular to the Fermi surface, and a gap velocity \(v_2\) parallel to the same surface. Then, close to the nodes the quasiparticle spectrum is

\[
E_k \simeq \sqrt{v_1^2 k_1^2 + v_2^2 k_2^2},
\]

for nodes along \(k_1\) axis and

\[
E_k \simeq \sqrt{v_2^2 k_1^2 + v_1^2 k_2^2},
\]

for nodes along \(k_2\) axis. The spectrum, in the vicinity of each gap node takes the form 
of a Dirac cone whose anisotropy is measured by the ratio of the two velocities. As we 
will see afterward a strong anisotropy brings in a weak coupling regime the non-linear \(\sigma\)-
model representative of our disordered system, making the perturbation theory and the RG 
approach suitable tools of investigation.

We now analyze the disordered Hamiltonian (2) by using the replica trick method within 
the path integral formalism [54]. We introduce the vector Grassmann variables \(c_i\) and \(\bar{c}_i\) 
with components \(c_{i,\sigma,p,a}\) and \(\bar{c}_{i,\sigma,p,a}\), where \(i\) refers to a lattice site, \(\sigma\) to the spin, \(p = \pm\) is 
the index of positive (+\(\omega\)) and negative (−\(\omega\)) frequency components, and \(a = 1, \ldots, n\) is 
the replica index, as well as the Nambu spinors 

\[
\Psi_i = \frac{1}{\sqrt{2}} \begin{pmatrix} \bar{c}_i \\
\imath \sigma_y c_i \end{pmatrix} \quad \text{and} \quad \bar{\Psi}_i = [C\Psi_i]^t,
\]

with the charge conjugation matrix \(C = \imath \sigma_y \tau_1\). Here and in the following, the Pauli matrices \(\sigma_b\) 
\((b = x, y, z)\) act on the spin components, \(s_b\) \((b = 1, 2, 3)\) on the frequency (re-
tarded/advanced) components, and \(\tau_b\) \((b = 1, 2, 3)\) on the Nambu components \(\bar{c}\) and \(c\). The 
action corresponding to (2) is

\[
S = \sum_{ij} \bar{\Psi}_i \left( -t_{ij} e^{-\imath \phi_{ij} \tau_3} + i \Delta_{ij} \tau_2 s_1 - i \delta_{ij} \omega s_3 \right) \Psi_j
\]
where the source term $\omega \bar{\Psi}_i s_3 \Psi_i$ is introduced in order to reproduce positive and negative frequency propagators. As in the standard Abrikosov-Gorkov-Dzyalozinskii approach to superconductivity, the gap function couples fermions with opposite frequency $\omega$.

If magnetic impurities are present, we must add to Eq. (6) an additional spin-flip scattering term

$$- \sum_i \bar{\Psi}_i u_i \tau_3 \vec{\sigma} \cdot \vec{S}_i \Psi_i,$$

being $\vec{S}_i$ the impurity spin and $u_i$ the corresponding random potential. The same term, with $\vec{S}_i = \hat{B}$ and $u_i = B$, gives the Zeeman splitting in the presence of a constant magnetic field $\vec{B}$, which also breaks time-reversal invariance.

If an on site term is present or if we are far from half filling, we must add

$$- \sum_i \mu_i \bar{\Psi}_i \Psi_i,$$

which spoils chiral symmetry, $\mu_i$ being a random variable or $E_F$ respectively.

III. DISORDER AVERAGE

We derive the effective quantum field theory for the disordered $d$-wave superconducting model described in the previous section, following the work by Efetov, Larkin, and Khmel’nitsky. As usually, this field theory is a non-linear $\sigma$-model where the broken gauge symmetry enters as a reduction of the symmetry of the $Q$-matrix fields with respect to a normal metal.

The starting generating function is

$$Z = \int \mathcal{D} \bar{\Psi} \mathcal{D} \Psi \mathcal{D} t P(t) e^{-S},$$

where $P(t)$ is the gaussian probability distribution of the random bonds. Within the replica trick method, we can average $e^{-S}$ over disorder, obtaining

$$Z = \int \mathcal{D} \bar{\Psi} \mathcal{D} \Psi e^{-S_{\text{eff}}},$$

The effective action $S_{\text{eff}} = S_0 + S_{\text{imp}}$ is the sum of a regular part $S_0$ given by Eq.(6) with $t_{ij} = t$ (and $\phi_{ij} \equiv 0$) plus the impurity contribution

$$S_{\text{imp}} = -2 u^2 t^2 \sum_{\langle ij \rangle} (\bar{\Psi}_i \Psi_j)^2 = -2 u^2 t^2 \sum_{\langle ij \rangle} (\bar{\Psi}_i \Psi_j) (\bar{\Psi}_j \Psi_i)$$

(11)
since $\Psi_i \Psi_j = \overline{\Psi}_j \Psi_i$. By introducing the fields

$$X_i^{\alpha\beta} = \Psi_i^\alpha \overline{\Psi}_i^\beta,$$  
(12)

where $\alpha$ and $\beta$ is a multilabel for Nambu, advanced-retarded and replica components, we can write

$$S_{imp} = 2u^2 t^2 \sum_{(ij)} X_i^{\alpha\beta} X_j^{\beta\alpha} = 2u^2 t^2 \sum_{(ij)} Tr (X_i X_j).$$  
(13)

In Fourier components it becomes

$$S_{imp} = \frac{1}{V} \sum_{q \in BZ} \sum_{(ij)} w_q Tr (X_q X_{-q}),$$  
(14)

where $BZ$ means the Brillouin zone, $V$ the volume and

$$w_q = 2u^2 t^2 (\cos q_x a + \cos q_y a),$$  
(15)

$a$ being the lattice spacing. We can decouple Eq. (14) by an Hubbard-Stratonovich transformation, introducing an auxiliary field. However, since $w_q = -w_{q+(\pi,\pi)}$ and $w_q > 0$ if $q$ is restricted to the magnetic Brillouin zone ($MBZ$), we need to introduce two auxiliary fields defined within the $MBZ$, $Q_0_q = Q_{0-q}^\dagger$ and $Q_3_q = Q_{3-q}^\dagger$ [41, 52], through which

$$S_{imp} = \frac{1}{V} \sum_{q \in MBZ} \sum_{(ij)} \frac{1}{4w_q} Tr [Q_0_q Q_{0-q} + Q_3_q Q_{3-q}]$$

$$- \frac{i}{V} \sum_{q \in MBZ} Tr [Q_{0-q} X_{-q}^t + iQ_{3-q} X_{-q-(\pi,\pi)}^t].$$  
(16)

The above expression shows that $Q_0$ corresponds to smooth fluctuations of the auxiliary field, while $Q_3$ to staggered fluctuations. Namely, in the long-wavelength limit, the auxiliary field in real space is

$$Q_j = Q_{0j} + i(-1)^j Q_{3j},$$  
(17)

where $j$ is the site. However, to make sublattice symmetry more evident, it is convenient to use a unit cell which contains two sites, one for each sublattice. Indicating with $R$ a new unit cell vector and with $A$ and $B$ the labels for the two sublattices, we introduce a two component field

$$\Psi_R = \begin{pmatrix} \Psi_{AR} \\ \Psi_{BR} \end{pmatrix},$$  
(18)
through which we can rewrite Eq. (17) in the following way

\[ Q_R = Q_{0R} \gamma_0 + iQ_{3R} \gamma_3, \]  

(19)

where \( \gamma_b \ (b = 1, 2, 3) \) are Pauli matrices acting on the vector (18). Notice that, differently from the non-chiral case, \( Q \) here is not hermitian, in fact

\[ Q_R^\dagger = Q_{0R} \gamma_0 - iQ_{3R} \gamma_3 = \gamma_1 Q_R \gamma_1 = \gamma_2 Q_R \gamma_2, \]  

(20)

since \( Q_0 \) and \( Q_3 \) are both hermitian.

IV. SYMMETRIES

From Eq (16) we can deduce that a unitary transformation acting on the spinor \( \Psi_R \rightarrow T \Psi_R \) results into a rotation of the matrix field

\[ Q \rightarrow C T^d C^d Q T. \]  

(21)

Therefore, as originally done by Wegner [46], we analyze the symmetries underlying the theory in order to distinguish the soft or transverse modes from the massive or longitudinal modes, once we will define the vacuum states for the \( Q \) matrices by the saddle point approximation. Since the Hamiltonian parameters couple sites of the two different sublattices, we can consider generally two different global unitary transformations, one for sublattice A and another for sublattice B

\[ \Psi_A = T_A \Psi_A, \quad \Psi_B = T_B \Psi_B. \]

For frequency \( \omega = 0 \), the action is invariant if

\[ C T_A^d C^d H_{AB} T_B = H_{AB}, \]  

(22)

being \( C \) is the charge conjugation matrix \( i \sigma_y \tau_1 \), that implies

\[ C T_A^d C^d T_B = 1, \]

\[ C T_B^d C^d T_A = 1, \]  

(23)

valid even for non-superconducting states, as well as

\[ C T_A^d C^d \tau_2 s_1 T_B = \tau_2 s_1, \]

\[ C T_B^d C^d \tau_2 s_1 T_A = \tau_2 s_1, \]  

(24)
in the presence of a real superconducting order parameter. If time reversal symmetry is
broken, namely if the hopping parameter acquires a phase $e^{i\phi_{ij} \tau_3}$, we must further impose that
\[
CT_A^t \tau_3 T_B = \tau_3, \\
CT_B^t \tau_3 T_A = \tau_3.
\] (25)

In the presence of a constant magnetic field $B$, which introduces a Zeeman term $B_z \tau_3 \sigma_z$ in the Hamiltonian, we add the following constrains,
\[
CT_A^t \tau_3 T_B = \tau_3, \quad CT_A^t \tau_3 \sigma_z T_A = \tau_3 \sigma_z, \\
CT_B^t \tau_3 T_A = \tau_3, \quad CT_B^t \tau_3 \sigma_z T_B = \tau_3 \sigma_z,
\] (26)
and finally
\[
CT_A^t \tau_3 T_A = \tau_3, \quad CT_A^t \tau_3 \vec{\sigma} T_A = \tau_3 \vec{\sigma}, \\
CT_B^t \tau_3 T_B = \tau_3, \quad CT_B^t \tau_3 \vec{\sigma} T_B = \tau_3 \vec{\sigma},
\] (27)
in the presence of magnetic impurities, represented by the term $\vec{S} \cdot \vec{\sigma} \tau_3$ in the Hamiltonian with $\vec{S}$ a random vector variable.

If we are not at half filling or there are on-site impurities, namely if sublattice symmetry does not hold, $T_A$ and $T_B$ have to satisfy
\[
CT_A^t C^t T_A = 1, \quad CT_B^t C^t T_B = 1,
\] (28)
in order to be symmetry transformations. Finally, a finite frequency $\omega$, which acts as a symmetry breaking field, leads to the additional constrains
\[
CT_A^t C^t s_3 T_A = s_3, \quad CT_B^t C^t s_3 T_B = s_3.
\] (29)

All the above conditions imply that the unitary matrices $T$ belong to a group $G$ if $\omega = 0$ which is lowered to a group $H$ at $\omega \neq 0$.

The unitary transformations, $T_A$ and $T_B$, can be written as
\[
T_A = \exp \frac{W_0 + W_3}{2}, \quad T_B = \exp \frac{W_0 - W_3}{2},
\] (30)
with antihermitian $W$'s. Charge conjugacy invariance, through Eqs. (23), implies
\[
CW_0^t C^t = -W_0, \\
CW_3^t C^t = W_3,
\] (31) (32)
while the presence of superconducting term leads to the following constraints

\[ [\tau_2 s_1, W_{0(3)}] = 0. \]

(33)

If time reversal invariance is broken

\[ [\tau_3, W_{0(3)}] = 0. \]

(34)

If magnetic impurities are present, then

\[ [\tau_3 \vec{\sigma}, W_0] = \{\tau_3 \vec{\sigma}, W_3\} = 0, \]

(35)

while

\[ [\tau_3 \vec{\sigma} \cdot \vec{B}, W_0] = \{\tau_3 \vec{\sigma} \cdot \vec{B}, W_3\} = 0, \]

(36)

in the presence of a constant magnetic field. Finally, if chiral symmetry is broken by terms which couple the same sublattice, we must set \( W_3 = 0 \). In the presence of finite frequency \( \omega \) we have further to impose that

\[ [W_0, s_3] = \{W_3, s_3\} = 0. \]

(37)

V. SADDLE POINT

The full action, after the Hubbard-Stratonovich decoupling,

\[ S = -\sum_{k,q} \bar{\Psi}_k \left( i\omega s_3 \delta_{q0} - H^0_k \delta_{q0} + \frac{i}{V} Q_q \right) \Psi_{k+q} + \frac{1}{V} \sum_q \frac{1}{2w_q} Tr [Q_q Q_q^\dagger] , \]

(38)

by integrating over the Nambu spinors, transforms into

\[ S[Q] = \frac{1}{V} \sum_q \frac{1}{2w_q} Tr [Q_q Q_q^\dagger] - \frac{1}{2} Tr \ln \left[ i\omega s_3 - H^0 + iQ \right] , \]

(39)

where \( H^0 \) is the regular part of the Hamiltonian. In momentum space it reads

\[ H_k = \epsilon_k + i\Delta_k \tau_2 s_1 = E_k e^{2i\theta_k \tau_2 s_1}, \]

(40)

where

\[ E_k = \sqrt{\epsilon_k^2 + \Delta_k^2} \]

(41)

and

\[ \cos 2\theta_k = \frac{\epsilon_k}{E_k}, \quad \sin 2\theta_k = \frac{\Delta_k}{E_k} \]

(42)
Let us look for a saddle point $Q_{sp} \propto \sigma_0$ which has a $\tau_0s_3$ component $\Sigma$ as well as a $\tau_2s_1$ component $F$, both $k$ independent. Therefore

$$G_k^{-1} = i\omega s_3 - \epsilon_k - i\Delta_k\tau_2 s_1 + i\Sigma s_3 + iF\tau_2 s_1,$$

where we introduce explicitly the symmetry breaking term, namely $\omega s_3$. We notice that the new pairing order parameter is $\Delta_k - F$, so that, by defining

$$\tilde{E}_k = \sqrt{\epsilon_k^2 + (\Delta_k - F)^2},$$

as well as a modified $\tilde{\theta}_k$, we find the self-consistency equations

$$\Sigma = \frac{u^2t^2}{8} \sum_k Tr(G_k s_3),$$

$$F = \frac{u^2t^2}{8} \sum_k Tr(G_k \tau_2 s_1),$$

where

$$G_k = e^{-i\tilde{\theta}_k\tau_2 s_1} \frac{1}{-\tilde{E}_k + i(\omega + \Sigma)s_3} e^{-i\tilde{\theta}_k\tau_2 s_1}.$$

Therefore,

$$\Sigma = \frac{u^2t^2}{2} (\Sigma + \omega) \sum_k \frac{1}{\tilde{E}_k^2 + (\Sigma + \omega)^2},$$

$$F = -\frac{u^2t^2}{2} \sum_k \frac{\Delta_k - F}{\tilde{E}_k^2 + \Sigma^2} = \frac{u^2}{2} F \sum_k \frac{1}{\tilde{E}_k^2 + \Sigma^2},$$

where the last identity holds for $d$-wave order parameter. Notice that, for $s$-wave symmetry, these equations coincide with those found by Abrikosov, Gorkov and Dzyalozinskii [55]. The first equation implies that

$$\frac{\Sigma}{\Sigma + \omega} = \frac{u^2t^2}{2} \sum_k \frac{1}{\tilde{E}_k^2 + (\Sigma + \omega)^2},$$

which, inserted in the equation for $F$, leads to

$$\frac{F \omega}{\Sigma + \omega} = 0.$$

Being $\omega$ non zero, although infinitesimally small, this equation has the solution $F = 0$. Therefore, only $\Sigma \neq 0$ and such that

$$1 = \frac{u^2t^2}{2} \sum_k \frac{1}{\tilde{E}_k^2 + \Sigma^2}. $$
The above self-consistency equation leads to the Born approximation result
\[ \Sigma = \pi u^2 t^2 \nu = \frac{\pi}{4} w_0 \nu \propto e^{-\pi v_1 v_2 / u^2 t^2}, \] (50)
with \( \nu \) being the density of states at the chemical potential. Finally one gets the saddle point value
\[ Q_{sp} = \sigma_0 \tau_0 s_3 \Sigma. \] (51)

VI. TRANSVERSE MODES

Now we will consider the transformations that leave the total Hamiltonian unchanged while rotating the saddle point, that is to say the transformations that allows to move from a vacuum state to another one. The degrees of freedom of these transformations are the Goldstone modes which are massless and whose number is equal to
\[ \dim(G/H) = \dim(G) - \dim(H), \]
where G is the original symmetry group and H is the symmetry group that preserves the vacuum. The coset G/H tells us how many generators are broken. Since the saddle point has the same algebraic form of the frequency term in the action, the cosets related to Goldstone modes are obtained exactly by that transformations which are excluded in the lowering of symmetries due to finite frequency. In the following we will denote by \( T \) only those kind of transformations, represented in sublattice space by
\[ T = e^{W_0 \gamma_0 + W_3 \gamma_3}, \] (52)
where the \( \gamma \)'s are \( 2 \times 2 \) matrices acting on vectors. We adopt this exponential form, which is the adjoint representation of the group whose transverse modes belong to, because in this way small fluctuations from the saddle point can be easily written by Taylor expansion.

For each \( W_{0,3} \) we can separate the singlet term from the triplet one, writing
\[ W = W_S + i\mathbf{\sigma} \cdot \mathbf{W}_T, \] (53)
where the Pauli matrices \( \sigma_a, a = x, y, z \), act on spin space. In addition we rewrite \( W \) in \( \tau \)-components (Nambu space)
\[ W_S = W_{S0} \tau_0 + i \sum_{j=1}^{3} W_{Sj} \tau_j, \] (54)
\[ \vec{W}_T = \vec{W}_{T0}\tau_0 + i \sum_{j=1}^{3} \vec{W}_{Tj}\tau_j. \] (55)

Moreover, for each \( \tau \)-component, we write \((a = 0, 1, 2, 3)\),

\[ W_{S(T)a} = \sum_{\alpha=0}^{3} W_{S(T)aa} s_{\alpha}. \] (56)

Each component of \( W \) in Eq. (56) is a \( n \times n \) matrix in replica space. We denote by \( S \) and \( A \) the symmetric and the antisymmetric matrices in replica space and by \( R \) and \( I \) the real and imaginary ones. These matrices are collected in Appendix A, together with the corresponding cosets. The transverse components of the \( W \)-fields have to fulfill

\[ \{W_0, s_3\} = [W_3, s_3] = 0. \] (57)

VII. NON-LINEAR \( \sigma \)-MODEL

Here we derive the effective field theory describing the long wavelength transverse fluctuations of \( Q(R) \) around the saddle point. In general terms we may parametrize the \( Q \)-matrix as follows (cfr Eq (21))

\[ Q_{P}(R) = \tilde{T}(R)\dagger [Q_{sp} + P(R)] T(R) \equiv Q(R) + \tilde{T}(R)\dagger P(R) T(R), \] (58)

where \( T \) involves transverse massless fluctuations and \( P \) longitudinal massive ones, \( Q_{sp} = \Sigma s_3 \) being the saddle point. In Eq (58), we used the short notation

\[ \tilde{T}\dagger \equiv CT^tC = \gamma_1 T\dagger \gamma_1 = \gamma_2 T\dagger \gamma_2. \] (59)

Since only the \( T \)'s are diffusive, at the moment we concentrate just on them, neglecting the \( P \)'s and writing the action in terms of \( Q(R) = \tilde{T}(R)\dagger Q_{sp} T(R) \) alone, so that \( QQ\dagger = Q_{sp} \) is invariant, eventhough a term involving massless modes from integration over massive ones might appear. Afterwards we will reconsider this point. By integrating (38) over the Grassmann variables, we obtain the following action of \( Q \):

\[ -S[Q] = -\frac{1}{V} \sum_{q} \frac{1}{2w_q} Tr \left[ Q_{q} Q_{q}\dagger \right] + \frac{1}{2} Tr \ln \left[ i\omega s_3 - H^{(0)} + iQ \right]. \] (60)

Since global transformations leave invariant the action with \( \omega = 0 \), the effective theory in terms of \( Q \)-fields can be found by expanding to the second order in the gradients and to the
first order in $\omega$. To this end we rewrite the second term of $S[Q]$ as
\begin{equation}
\frac{1}{2} \operatorname{Tr} \ln \left( i\omega \tilde{T} s_3 T^\dagger - \tilde{T} H^{(0)} T^\dagger + iQ_{sp} \right). \tag{61}
\end{equation}

$H^{(0)}_{RR'}$ involves either $\gamma_1$ and $\gamma_2$, while $T$ involves $\gamma_0$ and $\gamma_3$. Then
\begin{align*}
\tilde{T}(R)H^{0}_{RR'} T(R')^\dagger &= H^{0}_{RR'} + \left( \tilde{T}(R')^\dagger - \tilde{T}(R)^\dagger \right) H^{0}_{RR'} \tag{62}
\end{align*}
\begin{align*}
&\simeq H^{0}_{RR'} - \tilde{T}(R) \nabla \tilde{T}(R)^\dagger \cdot (\vec{R} - \vec{R}') H^{0}_{RR'} \\
&+ \frac{1}{2} \sum_{ij} \tilde{T}(R) \partial_{ij} \tilde{T}(R)^\dagger (R_i - R_i')(R_j - R_j') H^{0}_{RR'} \equiv H^{0}_{RR'} + U_{RR'},
\end{align*}

with $\partial_{ij} = \frac{\partial^2}{\partial R_i \partial R_j}$, $R_i$ being the components of $\vec{R}$. Differently from the tight-binding Hamiltonian case \cite{52}, in which the term $(\vec{R} - \vec{R}') H^{0}_{RR'}$ is related to the charge current vertex, in BCS Hamiltonian, since the charge is not a conserved quantity, the spin current vertex is involved, since the spin is conserved. This can be seen writing the continuity equation
\begin{equation}
i\nabla \vec{J}(R) = [H^0, \rho_{\text{spin}}(R)], \tag{63}
\end{equation}

with $\rho_{\text{spin}}(R) = c_{R\uparrow}^\dagger c_{R\uparrow} - c_{R\downarrow}^\dagger c_{R\downarrow}$, from which we obtain the following expression for spin current on the basis \cite{55}
\begin{equation}
\vec{J}(R) = -i \sum_{R_1} (\vec{R} - \vec{R}_1) \bar{\psi}_RH^{0}_{RR_1}\sigma_z\psi_{R_1}. \tag{64}
\end{equation}

We have chosen $z$ as the spin quantization direction. Using Eq. \cite{52} the term \cite{61} in the action can be written as
\begin{align*}
\frac{1}{2} \operatorname{Tr} \ln \left( i\omega \tilde{T} s_3 T^\dagger - U - H^{(0)} + iQ_{sp} \right) \\
= -\frac{1}{2} \operatorname{Tr} \ln G + \frac{1}{2} \operatorname{Tr} \ln \left( 1 + G i\omega \tilde{T} s_3 T^\dagger - G U \right), \tag{65}
\end{align*}

where $G = (-H^{(0)} + iQ_{sp})^{-1}$ is the saddle point Green’s function. By expanding in $\omega$ the following term is found
\begin{equation}
\frac{i\omega}{4} \operatorname{Tr} \left( G \tilde{T} sT^\dagger \right) = \frac{\omega}{2w_0} \operatorname{Tr} \left( s_3 Q \right). \tag{66}
\end{equation}

The second order expansion in $U$ contains the terms:
\begin{align*}
-\frac{1}{2} \operatorname{Tr} (GU) \tag{67}
\end{align*}

and
\begin{align*}
-\frac{1}{4} \operatorname{Tr} (GU GU). \tag{68}
\end{align*}
Neglecting boundary terms coming from first derivatives and keeping in (67) the component of $U$ containing second derivatives, we get

$$-\frac{1}{4} Tr \left\{ \hat{T}(R) \partial_{ij} \hat{T}(R)^{-1} (R_i - R'_i) (R_j - R'_j) H^{(0)}_{RR'} G(R', R) \right\}.$$  \hfill (69)

Now let us consider the correlation function

$$\chi_{\mu,i}(R, R'; t, t_1, t_2) = \langle T \left[ c_{R_1}^\dagger(t) J_{R_1,R_2}(R) c_{R_2}^\dagger(t_1) J_{R_3,R_4}(R') c_{R_4}^\dagger(t_2) \right] \rangle,$$  \hfill (70)

where $\mu = 0, 1, 2$, and

$$J^0_{R_1,R_2}(R) = \delta_{RR_1} \delta_{RR_2} \sigma_z$$

the spin density matrix elements, while $J^i$, for $i = 1, 2$, are the matrix element components of the spin current and read

$$\tilde{J}_{R_1,R_2}(R) = -i (\tilde{R}_1 - \tilde{R}_2) H_{R_1,R_2}^0 \sigma_z \delta_{RR_1}.$$  \hfill (71)

By means of the continuity equation in the hydrodynamic limit we obtain the Ward identity

$$\sum_{RR'} \chi_{ji}(R, R'; E) = \sum_{RR'} (R_i - R'_i) (R_j - R'_j) Tr \left[ G(R, R'; E) H_{RR'}^0 \right].$$  \hfill (72)

Through Eq (72), Eq. (69) turns out to be

$$-\frac{\chi_{ij}^{++}}{16} Tr \left\{ \hat{T}(R) \partial_{ij} \hat{T}(R)^{-1} \right\},$$  \hfill (73)

which, integrating by part, is also equal to

$$-\frac{\chi_{ij}^{++}}{16} Tr \left\{ \hat{T}(R) \partial_i \hat{T}(R)^{-1} \hat{T}(R)^{-1} \partial_j \hat{T}(R) \right\}$$

$$= -\frac{1}{16} \chi_{ij}^{++} Tr (D_i D_j).$$  \hfill (74)

Here we have introduced the matrix $\tilde{D}(R)$ with the $i$-th component

$$D_i(R) = D_{0,i}(R) \gamma_0 + D_{3,i}(R) \gamma_3 \equiv \tilde{T}(R) \partial_i \tilde{T}(R)^{-1}.$$  \hfill (75)

The second term of the expansion in $U$, given in Eq. (68), is

$$-\frac{1}{4} Tr (G U G U) = \frac{1}{4} \sum_k \sum_R Tr \left\{ \tilde{D}(R) \cdot \tilde{J}_k \sigma_z G(k) \tilde{D}(R) \cdot \tilde{J}_k \sigma_z G(k) \right\}.$$  \hfill (76)
Since the Fourier components of the Green’s function and of the spin current operator in the long-wavelength limit, supposing \( t_{ij} = t_{ji} \in \mathbb{R} \) and \( \Delta_{ij} = \Delta_{ji} \in \mathbb{R} \), can be written in the following way

\[
G(k) = -\frac{\epsilon_k \gamma_1 + i \Delta_k \tau_2 s_1 \gamma_1 + i \Sigma s_3}{E_k^2 + \Sigma^2},
\]

(77)

\[
\vec{J}_k = \vec{\nabla}_k \epsilon_k \gamma_1 \sigma_z + i \vec{\nabla}_k \Delta_k \tau_2 s_1 \gamma_1 \sigma_z,
\]

(78)

and the positive and negative frequency Green’s functions are related by \( G^+ = s_1 G^- s_1 \), we have

\[
J^i_k \sigma_z G^+ D_j = \frac{1}{2} (D_j + \gamma_1 s_3 D_j s_3 \gamma_1) J^i_k \sigma_z G^+ + \frac{1}{2} (D_j - \gamma_1 s_3 D_j s_3 \gamma_1) J^i_k \sigma_z G^-,
\]

(79)

\( J^i_k \) being a component of \( \vec{J}_k \). Taking advantage of this result, Eq. (76) becomes

\[
\frac{1}{32} \chi^{ij}_+ Tr [D_i D_j + D_i s_3 \gamma_1 D_j s_3 \gamma_1] + \frac{1}{32} \chi^{ij}_- Tr [D_i D_j - D_i s_3 \gamma_1 D_j s_3 \gamma_1],
\]

(80)

which, summed to Eq. (74), gives

\[
\frac{1}{32} \left( \chi^{ij}_+ - \chi^{ij}_- \right) Tr [D_i D_j - D_i s_3 \gamma_1 D_j s_3 \gamma_1] = -\frac{\pi \sigma_{ij}}{32 \Sigma^2} Tr (\partial_\mu Q \partial_\mu Q^\dagger),
\]

(81)

where

\[
\sigma_{ij} = -\frac{1}{4\pi} \sum_k Tr \left[ J^i_k \left( G^+(k) - G^-(k) \right) J^j_k \left( G^+(k) - G^-(k) \right) \right]
\]

(82)

is the spin conductivity since \( \vec{J}_k \) is the spin current vertex. For \( i = j \), using Eqs. (77,78), we obtain the following quantity

\[
\sigma = \frac{\Sigma^2}{\pi V} \sum_k \left[ \vec{\nabla} \epsilon_k \cdot \vec{\nabla} \epsilon_k + \vec{\nabla} \Delta_k \cdot \vec{\nabla} \Delta_k \right] \left( E_k^2 + \Sigma^2 \right)^2 \approx \frac{1}{\pi^2} \frac{v_1^2 + v_2^2}{v_1 v_2},
\]

(83)

where, as previously defined, \( v_1 \) and \( v_2 \) are the velocities perpendicular and parallel to the Fermi surface. This is the quasiparticle conductivity in the Drude approximation [18, 36] which corresponds to the spin and the thermal conductivities since both energy and spin are conserved quantities, therefore fluctuations of energy and spin densities are diffusive. One can expect that in the limit in which pair order parameter goes to zero \( \sigma \) corresponds also to the charge quasiparticle conductivity in Born approximation since in that case the charge turns to be a conserved quantity as well. This is what we will find rigorously in Section IX.

Inserting Eq. (83) in Eq. (81) and adding Eq. (66), we arrive finally at the following non-linear \( \sigma \)-model action

\[
S[Q] = \frac{\pi}{32 \Sigma^2} \sigma \int dR Tr \left( \partial_\mu Q(R) \alpha_{\mu\nu} \partial_\nu Q^\dagger(R) \right) - \frac{\omega}{w_0} Tr \left( s_3 Q(R) \right).
\]

(84)
A particular metric $\alpha_{\mu\nu}$ appears, where $\mu, \nu = 1, 2$ denotes the directions $k_1$ and $k_2$, depending on which nodal contribution we include in Eq. (81) through Eq. (82). Specifically $\alpha_{\mu\nu} = \delta_{\mu\nu}$ for 4 nodes which is the model we have considered starting from Eq. 2. If we assume to suitably modify the disorder such that it couples quasiparticles belonging to only one node or two opposite nodes, namely if we consider only forward scattering in the presence of a random potential or including also backscattering processes, the metric to be considered is $\alpha_{\mu\nu} = \delta_{\mu\nu} \frac{v_1^2}{v_1^2 + v_2^2}$ for one node or $\alpha_{\mu\nu} = \delta_{\mu\nu} \frac{v_2^2}{v_1^2 + v_2^2}$ for two opposite nodes.

At this point it is important to discuss the differences which occur when disorder couples at most two opposite nodes, or the most generic case where all nodes are coupled together. We anticipate that the logarithmic terms, which appear upon integrating the gaussian propagator, derive from the expression

$$\frac{1}{\pi\sigma} \int \frac{d^2 k}{4\pi^2} \frac{1}{k_\mu \alpha_{\mu\nu} k_\nu} \equiv g \ln(\ldots),$$

where the effective coupling constant controlling the perturbative expansion is given by

$$g = \frac{1}{2\pi^2 \sigma}, \quad \text{for 4 nodes},$$

$$g = \frac{1}{\pi^2 \sigma} \frac{v_1^2 + v_2^2}{v_1 v_2}, \quad \text{for 1 node}.$$  

We readily see that, up to terms of order $u^4$, the disorder strength, for one node $g = 1$ or two opposite nodes $g = 0.5$, so that the system is never in a weak coupling regime and the saddle point physics with weak perturbative quantum interference effects is never realized not even as a crossover transient. This is the situation analyzed in Ref. 22 in which the authors found a vanishing density of states approaching the chemical potential with a disorder strength dependent exponent in the one node case while a universal exponent is found when two opposite nodes are coupled by the disorder. In Ref. 56, 57, 58, 59, 60 it is pointed out that this results depend on the presence of a Wess-Zumino-Witten term.

On the other hand, assuming for cuprates the generic 4-nodes case, since experimentally $v_2 \simeq v_1/15$, $g << 1$ and a perturbative expansion in $g$ is still meaningful. In the following we will consider the latter situation.

The full expression of the $Q$-matrix is expressed by Eq. 58 where the massive modes are

$$P = (P_{00}s_0 + P_{03}s_3) \gamma_0 + i (P_{31}s_1 + P_{32}s_2) \gamma_3,$$  

(88)
being all $P$’s a hermitian. Charge conjugation implies that $CP^t C^t = P$. Writing the free action of $Q_P(R)$ and expanding $w_q$ we obtain two contributions, a pure massive term and a term where massive and massless modes are mixed. Integrating over massive modes as in Ref. 52 we find another term which we have to add to the action for the transverse fluctuations

$$-rac{\pi}{2.32\Sigma^4{\Pi}} \int dR T r \left[ Q^\dagger(R) \bar{\nabla} Q(R) \gamma_3 \right] T r \left[ Q^\dagger(R) \bar{\nabla} Q(R) \gamma_3 \right]. \tag{89}$$

$\Pi$ is a parameter related to staggered density of states fluctuations [52]. This term exists only if chiral symmetry is present. Indeed a further term could appear, namely the Wess-Zumino-Witten term [56, 57, 58, 59, 60], which is calculated in detail in Ref. 60 within the same approach. It is found that this term accidentally cancels out thanks to the four-fold symmetry of the Dirac nodes. Notice that if such a symmetry of the spectrum is broken the WZW would appear again and we would take it into account obtaining very different scaling behaviors. However, normally, the system flows to the strong coupling regime [56, 57, 58, 59, 60], where all the nodes are locked and again the WZW term is canceled.

**VIII. RENORMALIZATION GROUP**

We now study the scaling behavior of our action by means of the Wilson-Polyakov renormalization group [61, 62]. We will also show how to evaluate the one loop correction to the conductivity, namely to the stiffness parameter of modes which acquire a mass, like the charge fluctuation inside the superconducting broken symmetry phase or the spin modes when spin isotropy is broken. According to the previous section the final expression of the action describing the transverse massless modes in the long-wavelength limit is [41]

$$S[Q] = \frac{\pi}{32} \sigma \int dR T r \left( \bar{\nabla} Q(R) \cdot \bar{\nabla} Q(R)^\dagger \right) - \frac{\omega}{\omega_0} \int dR T r \left( s_3 Q(R) \right) - \frac{\pi}{8 \cdot 32} \int dR T r \left[ Q^\dagger(R) \bar{\nabla} Q(R) \gamma_3 \right] T r \left[ Q^\dagger(R) \bar{\nabla} Q(R) \gamma_3 \right], \tag{90}$$

where we have rescaled the $Q$ matrices by a factor $\Sigma$ so that the new $Q$ is normalized to unity, i.e. $Q^t Q = I$ and $Q_{sp} = s_3$. Since $Q = T^\dagger s_3 T = s_3 T^2 = s_3 e^W$, at the gaussian level the first term in the action is simply

$$\frac{\pi \sigma}{32} \int dR T r \left( \bar{\nabla} Q^\dagger \bar{\nabla} Q \right) \simeq -\frac{\pi \sigma}{32} \int dR T r \left( \bar{\nabla} W \bar{\nabla} W \right). \tag{91}$$
while the last term is
\[
-\frac{\pi \Pi}{32 \cdot 8} \int dR Tr \left[ Q^I \tilde{\nabla} Q_3 \right] Tr \left[ Q^I \tilde{\nabla} Q_3 \right] \\
\approx -\frac{\pi \Pi}{64} \int dR Tr \left[ \tilde{\nabla} W_3 \right] Tr \left[ \tilde{\nabla} W_3 \right].
\] (92)

Depending on whether the \( W \) components defined by Eqs. (53-56) are real or imaginary matrices in replica space, which may be either symmetric or antisymmetric (see Appendix A), we find the following gaussian propagators for the diffusive modes
\[
\langle W_{qS}^{ab}(k)W_{qS}^{cd}(-k) \rangle = \pm D(k) (\delta_{ac} \delta_{bd} \pm \delta_{ad} \delta_{bc}) + D(k) \frac{\Pi}{\sigma + \Pi n} \delta_{q3} \delta_{SS} \delta_{i0} \delta_{a0} \delta_{ab} \delta_{cd},
\] (93)
where \( q = 0, 3 \) refers to the homogeneous or the staggered component, \( S = S, T \) for singlet or triplet term, \( i = 0, 1, 2, 3 \) refers to the \( \tau \)-components, \( \alpha = 0, 1, 2, 3 \) refers to the \( s \)-components, \( \alpha = 0, 1, 2, 3 \) refers to real (R) and imaginary (I) matrices, while the \( \pm \) sign inside the brackets refers to symmetric (S) or antisymmetric (A) matrices, \( n \) is the number of replicas and finally
\[
D(k) = \frac{1}{2\pi \sigma k^2}.
\] (94)

This propagator in two-dimensions will induce logarithmic singularities within the perturbative expansion. A standard way to handle those divergences is provided by the Renormalization Group (RG). In particular we here apply the Wilson-Polyakov RG procedure [61, 62, 63], which is particularly suitable to handle with the non-linear constraint \( QQ^I = I \). By this approach one assumes
\[
T(R) = T_f(R)T_s(R),
\]
where \( T_f \) involves fast modes with momentum \( q \in [\Lambda/s, \Lambda] \), while \( T_s \) involves slow modes with \( q \in [0, \Lambda/s] \), being \( \Lambda \) the high momentum cut-off, and \( s > 1 \) the rescaling factor (in terms of the elastic scattering time, \( \tau = \frac{1}{2\Sigma} \), \( \Lambda \) is given by \( \frac{\pi \sigma}{2\nu} \Lambda^2 \sim 1 \), \( \frac{\pi}{2\nu} \) being the diffusion constant at the Born level). The following equalities hold
\[
Tr \left[ \tilde{\nabla} Q^I \tilde{\nabla} Q \right] = Tr \left[ \tilde{\nabla} Q^I_f \cdot \tilde{\nabla} Q_f \right] \\
+ 2Tr \left[ \tilde{D}_s \gamma_1 Q_f \tilde{D}_s Q^I_f \gamma_1 \right] - 2Tr \left[ \tilde{D}_s \tilde{D}_s \right] \\
+ 4Tr \left[ \tilde{D}_s Q^I_f \tilde{\nabla} Q_f \right],
\] (95)
where $Q_f = \bar{T}^\dagger_f Q_{sp} T_f$ and $\bar{D}_s = T_s \bar{D}^\dagger T^\dagger_s$, as well as

$$\begin{align*}
&\text{Tr}\left[Q^\dagger \bar{D}Q_3\right] \cdot \text{Tr}\left[Q^\dagger \bar{D}Q_3\right] \\
&= \text{Tr}\left[(\bar{D}W_s + \bar{D}W_f) \gamma_3\right] \cdot \text{Tr}\left[(\bar{D}W_s + \bar{D}W_f) \gamma_3\right] .
\end{align*}$$

(96)

Since the fast and slow modes live in disconnected regions of momentum space, only the stiffness $\text{Tr}\left[\bar{D}Q^\dagger \bar{D}Q\right]$ generates corrections. By expanding in Eq. (95) the terms which couple slow and fast modes up to second order in $W_f$, after averaging over the fast modes within a one loop expansion, the stiffness generates the following contribution to the action for the slow modes

$$\frac{\pi\sigma}{32} \int dR \text{Tr}\left[\bar{D}Q^\dagger \bar{D}Q\right] + \langle S_1\rangle_f - \frac{1}{2} \langle S_2^2\rangle_f ,$$

(97)

where

$$S_1 = \frac{\pi\sigma}{32} \int dR 2\text{Tr}\left[\bar{D}Q_{sp} \bar{D}Q_{sp} W^2_f \gamma_1\right]$$

$$- 2\text{Tr}\left[\bar{D}Q_{sp} W_f \bar{D}W_f Q_{sp} \gamma_1\right]$$

(98)

and

$$S_2 = 4\frac{\pi\sigma}{32} \int dR \text{Tr}\left[\bar{D}W_f \bar{D}W_f\right] .$$

(99)

After the effective action for the slow modes have been obtained, the slow modes momenta are then rescaled back according to :

$$q \in \left[0, \frac{\Lambda}{s}\right] \rightarrow q' \in \left[0, \frac{\Lambda}{s}\right] .$$

where $q' \in [0, \Lambda]$ runs over the original momentum space. In this way the model is mapped onto another model defined onto the same range of momenta with renormalized parameters $\sigma(s)$ and $\Pi(s)$. The logarithmic singularities are controlled by the following dimensionless coupling constants

$$g = \frac{1}{2\pi^2 \sigma}, \quad c = \frac{1}{2\pi^2 \Pi} .$$

(100)

When chiral symmetry holds, namely when $W^3$ is massless, the new coupling $c$ has to be included. However, the combination $\sigma + n\Pi$ can be shown to represent the stiffness parameter of an abelian degree of freedom connected to the $\text{Tr}(W^3)$, which is finite and commutes with all other degrees of freedom. This implies that $\sigma + n\Pi$ is a constant of the RG flow, namely that

$$\beta_c = -\frac{c^2 \beta_g}{g^2 n} .$$

(101)
Since the theory is well behaved in the \( n \to 0 \) zero replica limit, this indirectly proves that

\[
\lim_{n \to 0} \beta_g = 0,
\]

namely that when chiral symmetry holds and when \( \text{Tr}(W^3) \) is massless the model stays metallic with a finite conductance.

The final results of the RG are collected in the Table I in which the \( \beta \) functions for \( g \) \( (\beta_g = d g / d \ln s) \) and for the density of states (DOS) \( (\beta_\rho = d \ln \rho / d \ln s) \) are listed for the different symmetry classes. The density of states scaling behavior is obtained through its expression in the \( Q \) matrix language

\[
\rho = \nu^8 Tr(s^3 Q),
\]

which allows a very simple loop expansion.

In Table I we also list the coset spaces \( G/H \) for the different classes (i) time reversal invariance is preserved with chiral symmetry [35, 41] or without [32, 64, 65]; (ii) time reversal symmetry is broken by introducing random phase with chiral symmetry [35, 41] or without [32]; (iii) a magnetic field is applied in the presence of chiral symmetry [41] or without it [32, 64, 65]; and finally (iv) in the presence of magnetic impurities with chiral symmetry [41] or in its absence [28].

**TABLE I: Coset spaces and \( \beta \) functions for the coupling \( g \) and for the DOS \( \rho \) in the different universality classes.**

| Coset space                                      | \( \beta_g \)         | \( \beta_\rho \)        |
|-------------------------------------------------|------------------------|-------------------------|
| Yes chiral, Yes \( \hat{T} \)                   | \( U(4n) \times U(4n) \times U(4n) \) | \( 8ng^2 \)            | \( (\Gamma/4 - 8n)g \) |
| Yes chiral, No \( \hat{T} \)                    | \( U(4n) / O(4n) \)    | \( 4ng^2 \)            | \( (-1 + \Gamma/4 - 4n)g \) |
| Yes chiral, magnetic field                       | \( O(4n) / O(2n) \times O(2n) \) | \( (2n - 1)g^2 \)      | \( -2ng \)                |
| Yes chiral, spin flip                            | \( U(2n) / U(n) \times U(n) \) | \( ng^2 \)             | \( -ng \)                 |
| No chiral, Yes \( \hat{T} \)                    | \( \text{Sp}(2n) \times \text{Sp}(2n) / \text{Sp}(2n) \) | \( 2(2n + 1)g^2 \)    | \( (-1 - 4n)g \)          |
| No chiral, No \( \hat{T} \)                     | \( \text{Sp}(2n) / U(2n) \) | \( (2n + 1)g^2 \)      | \( (-1 - 2n)g \)          |
| No chiral, magnetic field                        | \( U(2n) / U(n) \times U(n) \) | \( ng^2 \)             | \( -ng \)                 |
| No chiral, spin flip                             | \( O(2n) / U(n) \)     | \( (n - 1)g^2 / 2 \)   | \( (1 - n)g / 2 \)        |

According to Table I in the zero replica limit we obtain that, if chiral symmetry is absent and for non magnetic impurities, the conductance vanishes, and the DOS, which is finite
within the simplest Born approximation, is suppressed. As shown by Ref. [33], in the localized phase the DOS vanishes as $|E|$ or $E^2$ depending whether time reversal symmetry holds or not. Quite surprisingly, magnetic impurities give a delocalization correction to the conductance, as well as a DOS enhancement. On the contrary, if chiral symmetry is present, the conductance stays finite, or even increases in the presence of a magnetic field. Without magnetic field and in the absence of spin flip scattering, the DOS, according to the above $\beta$-function, diverges approximately like, $\rho(E) \sim \frac{1}{E} \exp \left[ -A \sqrt{-\ln E} \right]$, with $A > 0$ a model dependent constant [50, 52]. By a real space RG in the strong disorder regime [66, 67, 68] and through a supersymmetric [69] and replicated [70] random gauge field theory approach as well as within the standard non-linear $\sigma$-model [71], it has been recently argued that the correct asymptotic expression of the DOS is instead of the form $\rho(E) \sim \frac{1}{E} \exp \left[ -A (-\ln E)^{2/3} \right]$. In Ref. [71] the origin of the disagreement is identified into the existence of an infinite chain of relevant operators which are related to moments of the $Q$-field and which are coupled together in the RG equations.

IX. THE ACTION WITH VECTOR POTENTIALS

The quasiparticle charge modes, as well as the spin modes when magnetic impurities or a magnetic field are present, are not described by the non-linear $\sigma$-model (90), which only represents the truly massless diffusion modes. Nevertheless, charge and spin conductivities, $\sigma_c$ and $\sigma_s$, respectively, can be still evaluated through the stiffness of the corresponding modes, although they acquire a mass term. Alternatively, $\sigma_c$ and $\sigma_s$ can be determined by second derivatives of the action with respect to a source field which couples to the charge or to the spin current [72].

As explained in Appendix B the source field which couples to the charge quasiparticle current is the vector potential

$$A_c = \lambda^s (A^0 \tau_3 s_0 + A^1 \tau_3 s_1),$$

(103)

where $\lambda^s$ is a symmetric matrix in replica space, or, alternatively,

$$A_c = \lambda^a (A^0 \tau_3 \sigma_z s_0 + A^1 \tau_3 \sigma_z s_1),$$

(104)

with $\lambda^a$ an antisymmetric matrix. On the other hand the spin vector potential which couples
to the spin current is given by

$$A_s = \lambda^s (A^0 \tau_0 \sigma_z s_0 + A^1 \tau_0 \sigma_z s_1),$$

(105)
or alternatively

$$A_s = \lambda^a (A^0 \tau_0 s_0 + A^1 \tau_3 s_1).$$

(106)

In the hydrodynamic limit the action in the presence of a vector potential acquires a new term which, up to second order is $A$, is (see Appendix C)

$$S_A = \frac{\pi}{32} \sigma_c Tr \left[ \left( \nabla Q + i\frac{e}{c} [Q, A_c] \right) \left( \nabla Q^\dagger - i\frac{e}{c} [A_c, Q^\dagger] \right) - (\nabla Q \nabla Q^\dagger) \right],$$

(107)

for a charge vector potential, where $\sigma_c$ is the bare charge conductivity

$$\sigma_c = \frac{\Sigma^2 \pi V}{\sum_k \left[ \frac{(\nabla_k e)^2}{(E_k^2 + \Sigma^2)^2} \right]},$$

(108)

while for the spin case

$$S_A = \frac{\pi \sigma_s}{32} Tr \left[ \left( \nabla Q + i\frac{e}{2} [Q, A_s] \right) \left( \nabla Q^\dagger - i\frac{e}{2} [A_s, Q^\dagger] \right) - (\nabla Q \nabla Q^\dagger) \right],$$

(109)

where $\sigma_s$ is the bare spin conductivity, given by Eq.(83). In the presence of these terms in the action, the generating function $Z(A)$ depends now on $A$

$$Z(A) = \int DQ e^{-S-S_A}. \tag{110}$$

The Kubo formula both for charge and for spin conductivities is recovered by

$$\left. \left( \frac{\partial^2 \ln Z}{\partial A^0^2} - \frac{\partial^2 \ln Z}{\partial A^1^2} \right) \right|_{A=0}. \tag{111}$$

We can now calculate, through the one-loop expansion, the corrections to the charge and the spin conductivities as responses to the source fields. To this end we expand to the second order in $A$ the generating function

$$Z(A) = \int DQ e^{-S-S_A} \simeq \int DQ e^{-S} \left( 1 - S_{A1} - S_{A2} + \frac{1}{2} (S_{A1})^2 \right),$$

where $S_{A1}$ and $S_{A2}$ are

$$S_{A1} = \frac{i}{t} \int dR Tr \left( \nabla Q(R) \left[ Q^\dagger(R), A \right] \right), \tag{112}$$

$$S_{A2} = \frac{f^2}{t} \int dR Tr \left( \left[ A, Q(R) \right] \left[ Q^\dagger(R), A \right] \right). \tag{113}$$
and \( t = \frac{32}{\pi \sigma} \), \( f = \frac{1}{2} \) for the spin case and \( t = \frac{32}{\pi \sigma_c} \), \( f = \frac{e}{c} \) for the charge case.

Taking, for charge conductivity, the gauge \( (103) \), we can calculate the second derivatives of the generating function, \( \frac{\partial^2 Z(A)}{\partial A_0^2} \) and \( \frac{\partial^2 Z(A)}{\partial A_1^2} \), at one-loop level by expanding \( Q \) in terms of \( W \) and averaging with the quantum weight \( e^{-S} \). The gauge \( (104) \) gives the same results. For spin conductivity we take the expression \( (105) \) or alternatively \( (106) \) as the vector potential. Through Eq. \( (111) \) we find the one-loop quantum interference corrections for charge and for spin conductivity, which are summarized in the following Table II.

| Coset space                  | \( \delta \sigma_s/\sigma_s \) | \( \delta \sigma_c/\sigma_c \) |
|------------------------------|-------------------------------|-------------------------------|
| Yes chiral, Yes \( \hat{T} \) | \( \text{U}(4n) \times \text{U}(4n)/\text{U}(4n) \) | 0                             | 0                             |
| Yes chiral, No \( \hat{T} \)  | \( \text{U}(4n)/\text{O}(4n) \) | 0                             | \(-2g \ln s\)                 |
| Yes chiral, magnetic field   | \( \text{O}(4n)/\text{O}(2n) \times \text{O}(2n) \) | 0                             | 0                             |
| Yes chiral, spin flip        | \( \text{U}(2n)/\text{U}(2n) \times \text{U}(n) \) | 0                             | 0                             |
| No chiral, Yes \( \hat{T} \) | \( \text{Sp}(2n) \times \text{Sp}(2n)/\text{Sp}(2n) \) | \(-2g \ln s\)                 | \(-2g \ln s\)                 |
| No chiral, No \( \hat{T} \)  | \( \text{Sp}(2n)/\text{U}(2n) \) | \(-g \ln s\)                  | \(-g \ln s\)                  |
| No chiral, magnetic field    | \( \text{U}(2n)/\text{U}(2n) \times \text{U}(n) \) | 0                             | 0                             |
| No chiral, spin flip         | \( \text{O}(2n)/\text{U}(n) \) | \( g \ln s/2\)                | \( g \ln s/2\)                |

By this procedure we find that the one-loop corrections \( \delta \sigma_c/\sigma_c \) and \( \delta \sigma_s/\sigma_s \) coincide with \( \delta \sigma/\sigma \) in the absence of sublattice symmetry. When sublattice symmetry holds, quasiparticle charge conductivity may behave differently from spin conductivity, as it happens when time reversal symmetry is broken. Notice that, according to Table II quantum interference corrections in the diffusive modes influence also the stiffness of modes which are, on the contrary, not diffusive.

**X. THE RESIDUAL QUASIPARTICLE INTERACTION**

So far we have dealt with disorder in d-wave superconductors modeled by a BCS Hamiltonian for free Landau-Bogoliubov quasiparticles. However strong correlation is a crucial ingredient of the cuprates. Therefore it is important to understand the effects of the residual quasiparticle interactions even within the superconducting phase. Besides producing
dephasing scattering processes \[73, 74\], the interactions can renormalize both the density of states and the stiffness of the spin fluctuations. In this Section we extend our analysis to include the quasiparticle interactions, following the original work by Finkel’stein \[53, 75, 76\] for disordered metals. Moreover, we will extend the Finkel’stein model by including the nesting property, which requires to add interaction amplitudes with \((\pi, \pi)\) momentum transfer.

Let us first consider the following interaction contributions to the free energy

\[
T \sum_{|k| \ll k_F} \frac{\Gamma_1}{2} \bar{c}_n^\alpha(p_1) \bar{c}_m^\beta(p_2) c_{m-\omega}^\beta(p_2 - k) c_{n+\omega}^\alpha(p_1 + k),
\]

\[
T \sum_{|k| \ll k_F} \frac{\Gamma_2}{2} \bar{c}_n^\alpha(p_1) \bar{c}_m^\beta(p_2) c_{n+\omega}^\alpha(p_1 + k) c_{m-\omega}^\beta(p_2 - k),
\]

with \(\alpha\) and \(\beta\) spin indices and \(n, m\) and \(\omega\) Matsubara indices while \(p_1, p_2, k\) are the momenta involved. \(T\) is the temperature. The sum is performed both over momenta and frequencies. Indeed since interactions intermix the energies of the particles we should give up to a fixed energy description, used in the non-interacting case, introducing a discrete set of Matsubara frequencies.

\[\text{FIG. 1: Diagram of interaction in particle-hole channel}\]

These interactions can be rewritten distinguishing the singlet from the triplet channel in this way

\[
T \sum_{|k| \ll k_F} \frac{\Gamma_s}{2} \bar{c}_n(p_1) \sigma_0 c_{n+\omega}(p_1 + k) \bar{c}_m(p_2) \sigma_0 c_{m-\omega}(p_2 - k),
\]

\[
-T \sum_{|k| \ll k_F} \frac{\Gamma_t}{2} \bar{c}_n(p_1) \bar{\sigma} c_{n+\omega}(p_1 + k) \bar{c}_m(p_2) \bar{\sigma} c_{m-\omega}(p_2 - k),
\]

with \(\Gamma_t = \Gamma_2/2\) and \(\Gamma_s = \Gamma_1 - \Gamma_2/2\). By gaussian integration and using (5) we have

\[
e^T \sum \frac{\xi}{2} \bar{c}_n(p_1) \sigma c_{n+\omega}(p_1 + k) \bar{c}_m(p_2) \sigma c_{m-\omega}(p_2 - k) = \int dX e^{-\frac{1}{2} \sum_\omega (X(\omega)X(-\omega) - X_3(\omega)X_3(-\omega)) + 2i \sum \sqrt{-TT} (X_0(\omega)(\bar{\Psi}_n\gamma_0^\sigma\Psi_{n+\omega}) + X_3(\omega)(\bar{\Psi}_n\gamma_3^\sigma\Psi_{n+\omega}))},
\]

27
being \( X_0(-\omega) = X_0(\omega), \ X_3(-\omega) = -X_3(\omega) \) auxiliary Hubbard-Stratonovich fields and \( \Gamma = -\Gamma_s \) for the singlet particle-hole channel with \( \sigma = \sigma_0 \) or \( \Gamma = \Gamma_t \) with \( \sigma = \bar{\sigma} \) for the triplet particle-hole channel. In Eq. (118) the positive-negative energy index of the Nambu spinors have been extended to label the positive and negative Matsubara frequencies. By integrating the fermionic degrees of freedom, the full action including interaction is

\[
\frac{1}{2} Tr \ln \left( i\tilde{U} \hat{\epsilon} U^\dagger - \tilde{U} H^{(0)} U^\dagger + iQ_{sp} + 2i\sqrt{-T}\Gamma X_0 \tilde{U} \tau_0 \sigma^t U^\dagger \right)
\]

(119)

where \( \hat{\epsilon}_{nm} = \epsilon_n \delta_{nm} \equiv n\pi T \delta_{lk} \) with \( n, m \) odd integers. \( U \) is the unitary transformation, previously called \( T \) in the non-interacting case. Then, by expanding in terms of the saddle point Green’s functions, we find new terms in the action that represents the residual interactions in the p-h channels, namely

\[
-\frac{1}{2} \sum \left( X_0(\omega)^2 + X_3(\omega)^2 \right) - \sum \frac{\sqrt{-TT}}{2\pi} \nu \left( X_0(\omega) Tr(\tau_0 \sigma Q_{n,n+\omega}) + X_3(\omega) Tr(\tau_3 \sigma Q_{n,n+\omega}) \right).
\]

(120)

Finally, by integrating over the auxiliary fields \( X_0 \) and \( X_3 \) we get the following contributions to the free energy for the singlet channel

\[
-T \sum \frac{\pi^2 \nu^2}{8} \Gamma_s \sum_{l=0,3} \left( Tr(Q_{n,n+\omega}, l \sigma_0) Tr(Q_{m+\omega, m}, l \sigma_0) \right),
\]

(121)

and for the triplet channel

\[
T \sum \frac{\pi^2 \nu^2}{8} \Gamma_t \sum_{l=0,3} \left( Tr(Q_{n,n+\omega}, l \bar{\sigma}) Tr(Q_{m+\omega, m}, l \bar{\sigma}) \right).
\]

(122)

In the replica space the \( Q \) matrices contained in Eq. (121) and in Eq. (122) are diagonal and have the same indices since residual interactions is present at fixed disorder. For convenience we will put upper latin indices, like \( Q^{ab} \), to denote replicas. In \( d \)-wave superconductors, from \([T, \tau_2 s_1] = 0\) we have

\[
\tau_2 s_1 Q \tau_2 s_1 = -Q
\]

(123)

together with the condition

\[
C^t Q^l C = Q,
\]

(124)

where \( s_i, i = 0, 1, 2, 3, \) are Pauli matrices acting on the signs of Matsubara frequencies. For the singlet and \( \tau_0 \) and \( \tau_3 \) components, this means

\[
Q^{ab}_{s0,nm} = Q^{ba}_{s0,nm} = -Q^{ab}_{s0,-n-m},
\]

(125)

\[
Q^{ab}_{s3,nm} = -Q^{ba}_{s3,nm} = Q^{ab}_{s3,-n-m},
\]

(126)
having defined
\[ Q = Q_S \sigma_0 + i \vec{Q}_T \cdot \vec{\sigma} \]  
(127)
in spin space and
\[ Q_S = Q_{S0} \tau_0 + i \sum_{j=1,2,3} Q_{Sj} \tau_j, \quad Q_T = Q_{T0} \tau_0 + i \sum_{j=1,2,3} Q_{Tj} \tau_j \]  
(128)
in particle-hole space. Because of Eqs. (125, 126) the interaction in the p-h singlet channel is therefore
\[ \sum_{nm\omega} \sum_a Q^{aa}_{S0,n,n+\omega} Q^{aa}_{S0,m+m+\omega,m} + \sum_{m,n+\omega} Q^{aa}_{S3,n,n} \]  
(129)
By setting \(-n \rightarrow n + \omega\) in the last term, we recover the first with opposite sign, hence the sum is zero. More specifically, if we consider the transformation \(Q_{n,m} \rightarrow Q_{-m,-n}\) we find that the interaction terms only involve the symmetric components of the \(Q\)'s because of energy conservation. On the other hand, only the antisymmetric \(Q_S\) and the symmetric \(Q_T\) stay massless in the superconducting phase due to Eq. (123). This means that the singlet term, with \(\Gamma_s\), is suppress in \(d\)-wave superconductors by symmetry and only the triplet, with \(\Gamma_t\), survives. This is physically expected being charge fluctuations not diffusive.

We now take into account also the interaction in the Cooper channel. The diffusive cooperon represents fluctuation in the particle-particle channel with \(s\)-wave symmetry. Since the real part of the order parameter is already finite, fluctuations in the \(\tau_2 s_1\) channels are not diffusive, while only fluctuations in the \(\tau_1 s_1\) channel, corresponding to fluctuations of an \(is\) order parameter, stay massless. In the presence of residual interaction in the p-p channel, we must also consider the term
\[ T \sum_{|k| < k_F} \Gamma_c \left( \frac{\alpha}{2} \bar{c}_\alpha^\dagger(p_1) c^\dagger_\beta(k - p_1) c^\dagger_\beta(p_2) c^\alpha_{\omega-n}(k - p_2). \right) \]  
(130)
By introducing one more auxiliary field, \(Y^{\alpha\beta} = (Y^{\beta\alpha})^*\) with \(\alpha\) and \(\beta\) spin indices, the quantum weight due to p-p interaction can be rewritten as
\[ \int dY e^{-\frac{i}{2} \sum Y^\alpha_\beta(k) Y^\beta_\alpha(-k) + i \sum \sqrt{TT_\pi} (\alpha_\tau\alpha(p_1) Y^\alpha_\beta(p_1 + p_2) \bar{c}_{\omega+n}(p_2) + c^\dagger_\alpha(p_1) Y^\alpha_\beta(p_1 + p_2) \bar{c}_{\omega+n}(p_2))} \]  
(130)
From Eq. (5) the following equalities hold
\[ \bar{\Psi}(\tau_1 + i \tau_2) \Psi = -i c_\sigma y c, \]  
(131)
\[ \bar{\Psi}(\tau_1 - i \tau_2) \Psi = -i \bar{c}_\sigma y c, \]  
(132)
so that, calling $Y_R^{\alpha\beta} = \sum_\gamma Y^{\alpha\gamma} \sigma^{\gamma\beta}_y$ and $Y_L^{\alpha\beta} = \sum_\gamma \sigma^{\alpha\gamma}_y Y^{\gamma\beta}$, which implies $Y_L^{\beta\alpha} = (Y_R^{\alpha\beta})^*$, Eq. (130) becomes

$$
\int dY e^{-\frac{1}{2} \sum R \omega Y^{\alpha\beta}_R \omega + \frac{1}{2} \sum L \omega Y^{\alpha\beta}_L \omega + \sqrt{TT} \sigma^{\alpha\beta}_R \omega \psi_{-n, -n}^{\alpha\beta} + \sqrt{TT} \sigma^{\alpha\beta}_L \omega \psi_{n, -n}^{\alpha\beta} + \sum \sqrt{T} \Gamma_c (Q^{\alpha\beta}_{n, n, \tau} + Q^{\alpha\beta}_{m, m, \tau})},
$$

where $\tau^\pm = \tau_1 \pm i \tau_2$. Integrating over fermions we find

$$
\int dY e^{-\frac{1}{2} \sum R \omega Y^{\alpha\beta}_R \omega + i \sum \sqrt{TT} \sigma^{\alpha\beta}_R \omega \psi_{-n, -n}^{\alpha\beta} + \sqrt{TT} \sigma^{\alpha\beta}_L \omega \psi_{n, -n}^{\alpha\beta} + \sum \sqrt{T} \Gamma_c (Q^{\alpha\beta}_{n, n, \tau} + Q^{\alpha\beta}_{m, m, \tau})}
$$

and finally, after integration over $Y_R$, we obtain the following additional term to the free energy, representing the interaction in the Cooper channel

$$
T \sum \frac{\pi^2 \nu^2}{4} \Gamma_c \sigma_{\text{spin}} \{ Tr(Q_{n+\omega, -n\tau^+}) Tr(Q_{m+\omega, -m\tau^-}) \}. \quad (135)
$$

Also in this case the $Q$ matrices are diagonal in replica space and both of them have the same replica index. By the charge conjugacy relation $C^t Q^t C = Q$, the triplet terms don’t contribute since

$$
Q_{T1, nm}^{ab} = -Q_{T1, mn}^{ba}, \quad Q_{T2, nm}^{ab} = -Q_{T2, mn}^{ba}, \quad (136)
$$

so, if in Eq. (135) we transpose $Q_{n+\omega, -n}$ and put $-n \to n + \omega$ we will have triplet terms with opposite sign. This means that at the end only the following term remains

$$
T \sum \frac{\pi^2 \nu^2}{8} \Gamma_c \sum_{l=1,2} (Tr(Q_{n+\omega, -n\tau_l, \sigma_0}^{aa}) Tr(Q_{m+\omega, -m\tau_l, \sigma_0}^{aa})). \quad (137)
$$

This result is valid also in metal phase and reflects the local character of the $Q$ matrix.

### XI. Renormalization Group with Interactions

Now we calculate the corrections to the conductivity and to the density of states due to the interactions. Let us first consider the model without sublattice symmetry, for which the relevant interactions are $\Gamma_t$ and $\Gamma_c$ defined above.
A. Interactions with small momentum transfer

The properties of massless modes in the Matsubara frequency space are the following, having imposed antihermitianicity and conditions (31), (32).

\[ W_{S0,nm}^{ab} = W_{S0,mn}^{ab} = -W_{S1,nm}^{ba} = W_{S0,n-m-m}^{ba} = -W_{S0,n-m-n}^{ba}, \]

\[ W_{S1,nm}^{ab} = -W_{S1,mn}^{ab} = -W_{S1,n-m-m}^{ba} = W_{S1,n-m-n}^{ba}, \]

\[ W_{S2,nm}^{ab} = -W_{S2,mn}^{ab} = -W_{S2,n-m-m}^{ba} = W_{S2,n-m-n}^{ba}, \]

\[ W_{S3,nm}^{ab} = W_{S3,mn}^{ba} = W_{S3,n-m-m}^{ba} = -W_{S3,n-m-n}^{ba}, \]

\[ W_{T0,nm}^{ab} = W_{T0,mn}^{ba} = W_{T0,n-m-m}^{ba} = W_{T0,n-m-n}^{ba}, \]

\[ W_{T1,nm}^{ab} = -W_{T1,mn}^{ba} = -W_{T1,n-m-m}^{ba} = -W_{T1,n-m-n}^{ba}, \]

\[ W_{T2,nm}^{ab} = -W_{T2,mn}^{ba} = -W_{T2,n-m-m}^{ba} = -W_{T2,n-m-n}^{ba}, \]

\[ W_{T3,nm}^{ab} = -W_{T3,mn}^{ba} = -W_{T3,n-m-m}^{ba} = W_{T3,n-m-n}^{ba}, \]

where \( a \) and \( b \) are replica indices, while \( n \) and \( m \) are odd integer Matsubara indices with opposite sign. If we take \( n = -m \) we recover the symmetry properties derived in the previous chapters and reported in Appendix A. By these relations we can write the gaussian propagator

\[
\langle W_{S_i,nm}^{ab}(k)W_{S_i,rq}^{cd}(k) \rangle = (\pm) \frac{1}{2} \left( 1 - \lambda_n \lambda_m \right) D_{nm}(k)
\]

\[
\left( \delta_{nr}^{ac} \delta_{mq}^{bd} \pm \delta_{nr}^{ad} \delta_{mq}^{bc} (-)^i \delta_{n-r}^{ac} \delta_{m-q}^{bd} (-)^i \delta_{n-r}^{ad} \delta_{m-q}^{bc} \right), \tag{138}
\]

where \( S \) means \( S \) for singlet and \( T \) for triplet components, \( \delta_{nr}^{ac} = \delta_{ac} \delta_{nr} \), \( (\pm) \) are related to real or imaginary matrix elements of \( W \), \( \pm \) for symmetric or antisymmetric matrix, \( (-)^i \) the sign that \( W \) acquires changing the signs of Matsubara frequencies and this occurs only for modes proportional to \( \tau_1 \) and \( \tau_3 \), and finally

\[
D_{nm}(k) = \frac{1}{4\pi\nu} \frac{1}{Dk^2 + z|\epsilon_n - \epsilon_m|}, \tag{139}
\]

with \( \text{sign}(\epsilon_n) \equiv \lambda_n = -\lambda_m \) where \( \epsilon_n = n\pi T \) is a fermionic Matsubara frequency with odd integer \( n \). The factor \( z \) is the frequency renormalization and \( D = \sigma/(2\nu) \) the diffusion coefficient.

Let us introduce slow and fast modes in the spirit of Wilson Polyakov procedure, as we have done before,

\[
Q = \tilde{U}^\dagger_s Q_f U_s = \tilde{U}^\dagger_f \tilde{U}^\dagger_s Q_f U_f U_s, \tag{140}
\]
with $U = e^W$ (without chirality, $\tilde{U} = U$) and finally $Q_{spnm} = \lambda_n \delta_{nm}$.

$U_s$ contains only slow momentum fluctuations and

$$U_{snnm} = \delta_{nm}, \quad \text{if } (s_n \tau)^{-1} < |\epsilon_n| < \tau^{-1} \quad \text{or} \quad (s_m \tau)^{-1} < |\epsilon_m| < \tau^{-1},$$

where $\tau^{-1}$ acts as an energy cutoff and $s_e > 1$ is the rescaling factor. The massless fast modes satisfy by definition

$$W_{f nm}(k) = 0 \quad \text{if } \{Dk^2, |\epsilon_n|, |\epsilon_m|\} < (s_e \tau)^{-1}.$$

Now let us expand the interaction contributions to the full action, namely Eqs. (121, 122, 137) multiplied by $T^{-1}$, in terms of $W_f$, leaving slow $U_s$ unexpanded. In this way, besides the terms (98) and (99), also the following contributions should be evaluated in the one-loop expansion

$$S_{int}^1 = \frac{\pi^2 \nu}{8} \sum \nu \Gamma \text{Tr} \left( \tilde{U}_{n_1m_1}^{\dagger} \lambda_{n_1} W_{e_{m_1m_2}}^{eq} U_{m_2n_2}^{gd} \gamma_\sigma \right) \text{Tr} \left( \tilde{U}_{m_3n_3}^{\dagger} \lambda_{n_3} W_{e_{m_3m_4}}^{fh} U_{m_4n_4}^{hd} \gamma_\sigma \right) \delta(n_1 \mp n_2 \pm n_3 - n_4),$$

$$S_{int}^2 = \frac{\pi^2 \nu}{8} \sum \nu \Gamma \text{Tr} \left( \tilde{U}_{n_1m_1}^{\dagger} \lambda_{n_1} W_{e_{m_1m_2}}^{eq} W_{m_2m_3}^{gh} U_{m_3n_3}^{hd} \gamma_\sigma \right) \text{Tr} \left( Q_{n_3n_4}^{dd} \gamma_\sigma \right) \delta(n_1 \mp n_2 \pm n_3 - n_4),$$

where the upper indices are in the replica space, $\Gamma = \Gamma_t$, $\sigma = \sigma_\tau$, and we sum over $\tau = \tau_0, \tau_3$ and energies with constraint $\delta(n_1 - n_2 + n_3 - n_4)$ for p-h triplet channel and $\Gamma = \Gamma_s$, $\sigma = \sigma_0$, $\tau_1 = \tau_1, \tau_2$, with energy conservation law $\delta(n_1 + n_2 - n_3 - n_4)$ for p-p Cooper channel (in metal phase we would have also the singlet channel with $\Gamma = -\Gamma_s$, $\sigma = \sigma_0$, $\tau_1 = \tau_0, \tau_3$ and energy constraint $\delta(n_1 - n_2 + n_3 - n_4)$) and finally $Q = \tilde{U} Q_s U$ is the slow mode matrix. The sums in Eqs. (143, 144) are performed over all upper (replica) and lower (frequencies)
indices and over \( l \), according to the channel. Before starting the renormalization of all the parameters involved in the theory we can easily take into account all the ladders diagrams sketched in Fig. 4, substituting the bare particle-hole triplet scattering amplitude with

\[
\Gamma_t(q, \omega) = \Gamma_t \frac{Dq^2 + z|\omega|}{Dq^2 + (z + 2\nu \Gamma_t)|\omega|},
\]

which is the algebraic infinite ladder summation \cite{77}.

For Cooper amplitude we have instead, already after summing the first two diagrams, a logarithmic divergent term which does not depend on \( g \), the small parameter of the theory. To calculate one-loop corrections to conductivity due to the interactions, we have to consider the following averages over fast modes

\[
\langle S_{\text{int}}^1 \rangle - \langle S_1 S_{\text{int}}^1 \rangle - \langle S_2 S_{\text{int}}^1 \rangle + \frac{1}{2} \langle (S_2)^2 S_{\text{int}}^1 \rangle,
\]

where \( S^1 \) and \( S^2 \) are defined by Eq. (98) and Eq. (99). For instance, contracting the fast modes, the contribution coming from

\[
\langle S_{\text{int}}^1 \rangle = -\frac{\pi^2 \nu}{2} \sum \nu \Gamma(p, \epsilon_n \pm \epsilon_n) D_{m_1m_2}(p + p_1) \left( 1 - \lambda_{m_1} \lambda_{m_2} \right) \delta(n_1 \mp n_2 \pm n_3 - n_4)
\]

\[
\left\{ \text{Tr} \left( \lambda_{m_2} U_{m_2n_2}^{gd} (p_1 - p_2) \tau_l \sigma \tilde{U}_{n_1m_1}^{\text{de}} (p_2) \lambda_{m_1} U_{m_1n_4}^{\text{ed}} (-p_3) \tau_l \sigma \tilde{U}_{n_3m_2}^{\text{tdg}} (p_3 - p_1) \right) \right. 
\]

\[
\left. - (-1)^l \text{Tr} \left( \lambda_{m_2} U_{m_2n_2}^{gd} (p_1 - p_2) \tau_l \sigma \tilde{U}_{n_1m_1}^{\text{de}} (p_2) \lambda_{m_1} U_{m_1n_4}^{\text{ed}} (-p_3) \tau_l \sigma \tilde{U}_{n_3m_2}^{\text{tdg}} (p_3 - p_1) \right) \right\}
\]

which renormalizes the spin conductivity is obtained summing over fast momenta \( m_1 \) (the sum over slow momenta, instead, renormalizes the scattering amplitudes), giving simply

\[
-\pi^2 \nu n_\sigma \sum \nu \Gamma(p, \epsilon_n \pm \epsilon_n) D_{m_1m_2}(p + p_1) \text{Tr} \left( \tilde{U}_{n_2m_2}^{\text{tdg}} (-p_1) U_{m_2n_2}^{gd} (p_1) \right),
\]

with \( n_\sigma = 3 \) in the triplet p-h channel and \( n_\sigma = 1 \) in the Cooper one. Expanding the propagator \( D \) in terms of slow \( p_1 \), neglecting slow frequencies both in \( \Gamma \) and in \( D \), we have a factorized term

\[
-n_\sigma J \sum p_1^2 \text{Tr} \left( \tilde{U}_{n_2m_2}^{\text{tdg}} (-p_1) U_{m_2n_2}^{gd} (p_1) \right) = n_\sigma J \int d\tau \text{Tr} (AA),
\]
where \( A = \nabla \tilde{U} \tilde{U}^\dagger \) (notice that here, since chirality is not present, \( \tilde{U} = U \)) and
\[
J = \frac{1}{\pi} \int dp \, d\epsilon \, \Gamma(p, \epsilon) \left( \frac{D_p}{(D_p^2 + z|\epsilon|)^2} - \frac{2D_p^2 p^3}{(D_p^2 + z|\epsilon|)^3} \right).
\]
The other mean values in Eq. (146) give contributions of the same kind and also proportional to \( \int dr \, Tr(\Delta\Lambda\Lambda) \) (with chirality \( \int dr Tr(\Delta\gamma_1\Lambda\gamma_1) \)). Reminding that
\[
2Tr(\Delta\Lambda\Lambda - AA) = Tr(\nabla Q \nabla Q^\dagger),
\]
we obtain corrections to the stiffness coefficient.

To calculate corrections to the density of states, besides \( \langle S_\nu \rangle \), we will consider
\[
\langle S_{int}^1 S_\nu \rangle,
\]
where
\[
S_\nu = \frac{1}{2} \sum \, Tr(\epsilon_n \tilde{U}_{int}^{ab} \lambda_m \, W_{m1m2} \, W_{2m2m3} \, U_{m3n}).
\]

In order to calculate corrections to the interaction amplitudes at first order in \( g \), we need to consider the sum
\[
\langle S_{int} \rangle - \frac{1}{2} \langle (S_{int})^2 \rangle + \frac{1}{2} \langle (S_{int}^1)^2 S_{int}^2 \rangle + \frac{1}{2} \langle S_{int}^1 (S_{int}^2)^2 \rangle - \frac{1}{4} \langle (S_{int}^1 S_{int}^2)^2 \rangle,
\]
where \( S_{int} = S_{int}^1 + S_{int}^2 \). Because of the different structures of the two terms \( S_{int}^1 \) and \( S_{int}^2 \), we notice that \( \langle S_{int}^2 \rangle \) in the first term, \( \langle S_{int}^1 S_{int}^2 \rangle \) in the second term and the third term of Eq. (153) are vertex contributions (in Fig. 6, the corresponding diagrams are sketched using only the p-h triplet scattering amplitude) while the last two in Eq. (153) and \( \langle (S_{int}^2)^2 \rangle \) in the second term are bubble corrections, (see Fig. 7). In the normal phase the the first two diagrams of Fig. 7 and the first diagram of Fig. 6 vanish. There are other diagrams which give corrections to \( \Gamma_t \) in which \( \Gamma_c \) appears and viceversa, with the same topology but with different energy-momentum conservation laws.

Finally, the renormalization of \( z \) is determined by three contributions, the first two result from the corrections to the density of states \( \langle S_\nu \rangle \) and from Eq. (151) and the third from the expansion of \( \langle S_{int}^1 \rangle \), written explicitly in Eq. (148), in terms of the slow frequency \( n_2 \). This latter contribution cancels exactly all the terms of order higher than one in the interaction amplitudes coming from Eq. (151).
B. RG equations without chiral symmetry

The final renormalization group equations for $d$-wave superconductors, including residual quasiparticle interactions $\Gamma_t$ and $\Gamma_c$, in all the cases collected in Table II are the following:

a. No chiral, Yes $\hat{T}$. If time-reversal symmetry is preserved we obtain

$$
\frac{dg}{dl} = g^2 \left\{ -3\nu \Gamma_t f_2(z, z, z_2) - \nu \Gamma_c / z + 1 \right\}, \quad (154)
$$

$$
\frac{dz}{dl} = g \left\{ 3\nu \Gamma_t + \nu \Gamma_c - z / 2 \right\}, \quad (155)
$$

$$
\frac{d\nu \Gamma_t}{dl} = g \left\{ \nu (\Gamma_t + \Gamma_c) / 2 + 4\nu^2 (\Gamma_t \Gamma_t + \Gamma_t \Gamma_c) / z + 4\nu^3 (\Gamma_t \Gamma_c \Gamma_t) / z^2 - \nu \Gamma_t - \nu^2 \Gamma_t \Gamma_t \Gamma_t \right\}, \quad (156)
$$

$$
\frac{d\nu \Gamma_c}{dl} = g \left\{ \nu (3\Gamma_t - \Gamma_c) / 2 + 2\nu^2 (3\Gamma_t \Gamma_c z f_1(z, z_2) + \Gamma_c \Gamma_c) / z - \nu \Gamma_c \right\} - 2(\nu \Gamma_c)^2 / z, \quad (157)
$$
where, as usual, we have introduced the functions $f_1$ and $f_2$ which result from integrations of products of diffusion propagators, defined by

$$f_1(a,b) = \ln(a/b)/(a-b),$$  \hspace{1cm} (158)

$$f_2(a,b,c) = 2\left(b f_1(a,b) - c f_1(a,c)\right)/(b-c),$$  \hspace{1cm} (159)

and $z_2 = z + 2\nu\Gamma_t$. Notice that in Eq. (154) the last term is taken from Table II with $n = 0$ and divided by 2 since here we have adopted an energy scaling factor instead of a momentum scaling factor, being $l = \ln s_e$ with $s_e$ defined in Eq. (141). The contributions due to the non-vanishing density of states in the non-interacting case are singled out, they are the last two terms of Eq. (156) (corresponding to the first diagram of Fig. 6 and to the first two of Fig. 7) and the last term in the curly brackets of Eq. (157). In the normal phase, they disappear meanwhile the singlet particle-hole channel is turned on [53, 75]. The last term of Eq. (157) is due to the ladder summation. The scaling behavior of the density of states becomes

$$d\nu/dl = g\nu \left\{3\nu\Gamma_t f_1(z,z_2) + \nu\Gamma_c/z - 1/2\right\}. \hspace{1cm} (160)$$

Notice that the theory has only four parameters, $g, z, \nu\Gamma_t, \nu\Gamma_c$ since the density of states appears always multiplied with the scattering amplitudes. Summing and writing in terms of $u_t \equiv 2\nu\Gamma_t/z$ and $u_c \equiv 2\nu\Gamma_c/z$ we can separate from the set of equations the $z$-equation, obtaining

$$dg/dl = g^2 \left\{3(1-(1+u_t))\ln(1+u_t)/u_t - u_c/2 + 1\right\}, \hspace{1cm} (161)$$

$$dz/dl = gz \left\{3u_t + u_c - 1\right\}/2, \hspace{1cm} (162)$$

$$du_t/dl = g \left\{u_c/2 + 3u_t u_c/2 + u_t u_c u_t\right\}, \hspace{1cm} (163)$$

$$du_c/dl = g \left\{(3u_t - 2u_c)/2 + 3(u_c\ln(1+u_t) - u_t u_c)/2\right\} - (u_c)^2, \hspace{1cm} (164)$$

$$d\nu/dl = g\nu \left\{3\ln(1+u_t) + u_c - 1\right\}/2. \hspace{1cm} (165)$$

Supposing $T^{-1}$, the inverse of the temperature, being the coherence time for the quasiparticles, the integration of the equations above will run from $T$ to $\tau^{-1}$, the elastic scattering rate. Strictly speaking, half of the localization correction in the last term of Eq. (161), is cut off not by the temperature but by the inelastic scattering rate $\tau_{in}^{-1}$. However, $\tau_{in}^{-1}$, for some source of inelastic processes, it is found to be linear in $T$ [74]. At the first order in the
scattering amplitudes, supposing them so small, at the moment, that they do not flow fast, we have the following variation in the conductivity induced both by the disorder and by the interactions

$$\delta \sigma = \frac{1}{2\pi^2} \left( 1 - \frac{3}{2} u_t - \frac{1}{2} u_c \right) \ln(T\tau). \quad (166)$$

Solving now the whole set of equations (161-164) we find that, starting with a small and positive $u_{t0}$ (in Fig. 8 $u_{t0} = u_t(T \approx \tau^{-1}) = 0.15$, $u_{c0} = u_c(T \approx \tau^{-1}) = 0$ and $g_0 = g(T \approx \tau^{-1}) = 0.03$), $g$ increases for the presence of the quantum interferences but when the interactions become strong enough, after reaching a maximum, it decreases at low temperature, as shown in Fig. 8. The temperature of crossover depends on the interaction amplitudes. The smaller is the value of the positive $u_{t0}$, the higher is the peak of $g$, while $u_t$ diverges as in the normal phase [75, 76]. A negative starting value of $u_{t0}$, instead, reinforces the localization.

![Graph](image)

**FIG. 8:** Scaling behavior of $g$ of paragraph a, in the inserts $u_t$ and $u_c$.

b. **No chiral, No \( \hat{T} \).** If time reversal symmetry is broken, $\Gamma_c$ disappears from Eqs. (154-156, 160) and the last term of Eq. (154), which is the contribution due only to the disorder, is halved (see Table II). $u_t$ remains unrenormalized at first order in $g$ (Eq. (163) with $u_c = 0$), namely $du_t/dl = O(g^2)$, which is a result obtained also by Keldysh technique [78], so $u_t = u_{t0}$ at the one-loop level. The equation for $g$, exact in the interaction, becomes simply

$$dg/dl = g^2 \left\{ 3 (1 - (1 + u_{t0}) \ln(1 + u_{t0})/u_{t0}) + 1/2 \right\}, \quad (167)$$

implying that there is a positive critical value of $u_{t0}$ ($u_{t0}^* \approx 0.37$) for which $dg/dl = 0$ at lowest order in $g$. Above $u_{t0}^*$ the resistivity $g$ decreases, below $u_{t0}^*$ it increases.

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c. No chiral, magnetic field. If also the spin rotational invariance is broken, we have

\[ \frac{dg}{dl} = -g^2 \nu \Gamma_t f_2(z, z, z_2), \]  
\[ \frac{dz}{dl} = g \nu \Gamma_t, \]  
\[ \frac{d\Gamma_t}{dl} = -g \nu \Gamma_t / 2, \]  
\[ \frac{d\nu}{dl} = g \nu \Gamma_t f_1(z, z_2). \]  

Also in this case, as in the previous ones, we can eliminate \( z \) introducing \( u_t \), obtaining

\[ \frac{dg}{dl} = g^2 \left\{ 1 - (1 + u_t) \ln(1 + u_t)/u_t \right\}, \]  
\[ \frac{du_t}{dl} = -g u_t (1 + u_t)/2. \]

The solution is a transient since the scattering amplitude flows to zero and the conductivity changes value only in the meanwhile. For a small value of \( u_{t0} = u_t(T \simeq \tau^{-1}) \), we can retain only lowest order in \( u_t \) so that, given \( g_0 = g(T \simeq \tau^{-1}) \), the renormalized resistivity is given by

\[ g = g_0 e^{-u_{t0}}. \]  

d. No chiral, magnetic impurities. In the presence of magnetic impurities there are no corrections due to the interactions. Indeed both cooperons and triplet diffusons become massive.

C. Interactions with \((\pi, \pi)\) momentum transfer

At the nesting point where the staggered fluctuations become diffusive (i.e when the chiral symmetry holds), also quasiparticle interactions with \((\pi, \pi)\) momentum transfer must be included in the effective action. First notice that in the sublattice representation, the the contribution due to the interactions with small momentum transfer can be rewritten as

\[ \sum \frac{\pi^2 \nu}{32} \Gamma^0 \, Tr(Q_{\alpha\alpha}^{aa} \tau_1 \sigma \gamma_0) Tr(Q_{\alpha\alpha}^{aa} \tau_1 \sigma \gamma_0) \delta(n_1 \mp n_2 \pm n_3 - n_4), \]  

where \( \gamma_0 \) is the identity in the sublattice space and \( \Gamma^0 \) can be \(-\Gamma_s^0, \Gamma_t^0 \) and \( \Gamma_c^0 \).

Let us now consider interactions whose transferred momentum is \( q_\pi = (\pm \pi, \pm \pi) \), which also involve quasiparticles at the Fermi energy in the case of half filling,

\[ T \sum_{|k| < k_F} \frac{\Gamma_1^3}{2} \mathbf{r}_n^\alpha(p_1) \mathbf{r}_m^\beta(p_2) e_{m-\omega}^\beta(p_2 - k - q_\pi) e_{n+\omega}^\alpha(p_1 + k + q_\pi). \]
which led to Eq. (121), Eq. (122) and Eq. (137), we obtain the following additional
with $\Gamma^3$ only $\Gamma^3$ for the triplet and $\Gamma^3$ from the conditions (32), (33) and antihermitianicity, leading to
where now $\Gamma^3$ indicates the amplitude related to staggered modes. Repeating the steps
which led to Eq. (121), Eq. (122) and Eq. (137), we obtain the following additional
correction to the action

$$\sum \frac{\pi^2 \nu}{32} \Gamma^3 \text{Tr}(Q_{n1n2}^{aa} \tau_3) \text{Tr}(Q_{n3n4}^{aa} \tau_3) \delta(n_1 \pm n_2 \pm n_3 - n_4),$$

with $\Gamma^3$ taking the values $-\Gamma^3_s \equiv \Gamma^3_2/2 - \Gamma^3_1$ for the singlet p-h staggered channel, $\Gamma^3_t \equiv \Gamma^3_2/2$
for the triplet and $\Gamma^3_c$ for the p-p Cooper staggered channel. In the superconducting phase
only $\Gamma^3_t$ and $\Gamma^3_c$ are relevant. The properties of $W^3$ massless modes in energy space derive
from the conditions (32), (33) and antihermitianicity, leading to

$$W_{ab}^{S0,nm} = -W_{ab}^{S0,nn} = W_{ba}^{S0,mm} = W_{ab}^{S0,n-m} = W_{ba}^{S0,m-n},$$
$$W_{ab}^{S1,nm} = W_{ba}^{S1,mm} = W_{ba}^{S1,n-m} = -W_{ab}^{S1,m-n},$$
$$W_{ab}^{S2,nm} = W_{ab}^{S2,mm} = W_{ba}^{S2,n-m} = W_{ba}^{S2,m-n},$$
$$W_{ab}^{S3,nm} = -W_{ba}^{S3,mm} = -W_{ab}^{S3,n-m} = W_{ba}^{S3,m-n},$$
$$\tilde{W}_{ab}^{T0,nm} = -\tilde{W}_{ab}^{T0,mm} = \tilde{W}_{ba}^{T0,n-m} = \tilde{W}_{ba}^{T0,m-n},$$
$$\tilde{W}_{ab}^{T1,nm} = \tilde{W}_{ba}^{T1,mm} = -\tilde{W}_{ab}^{T1,n-m} = \tilde{W}_{ba}^{T1,m-n},$$
$$\tilde{W}_{ab}^{T2,nm} = \tilde{W}_{ab}^{T2,mm} = -\tilde{W}_{ba}^{T2,n-m} = -\tilde{W}_{ab}^{T2,m-n} = -\tilde{W}_{ab}^{T2,m-n},$$
$$\tilde{W}_{ab}^{T3,nm} = -\tilde{W}_{ab}^{T3,mm} = \tilde{W}_{ba}^{T3,n-m} = -\tilde{W}_{ab}^{T3,m-n} = \tilde{W}_{ab}^{T3,m-n} = \tilde{W}_{ab}^{T3,m-n},$$

where now $n$ and $m$ have same signs. By these properties we can write down the propagators,
reported in Appendix D, and evaluate Eq. (146), Eq. (151) and Eq. (153).

D. RG equations with chiral symmetry

The renormalization group equations in different symmetry classes in the presence of
chirality in the sublattice space are the following:
e. Yes chiral, Yes $\hat{T}$. If both time reversal symmetry and spin rotation invariance are preserved, we have, together with $\Gamma = g/c$ and $dc/dl = -4c^2$ (see Eqs. (100) (101)), the following equations

$$dg/dl = g^2 \left\{ -3\nu \Gamma_t^0 f_2(z, z, z_2) + 3\nu \Gamma_t^3 f_2(z, z, z_c) \right\}, \quad (180)$$

$$dz/dl = g \left\{ 3\nu \Gamma_t^0 - 3\nu \Gamma_t^3 + \nu \Gamma_d^0 + z\Gamma/8 \right\}, \quad (181)$$

$$d\nu \Gamma_t^0/dl = g \left\{ \nu \left( \Gamma_t^0 + \Gamma_t^3 + \Gamma_c^0 + \Gamma_c^3 \right)/2 + 4\nu^2 \left( \Gamma_t^0 \Gamma_t^3 + \Gamma_t^0 \Gamma_c^0 - \Gamma_t^3 \Gamma_c^3 \right)/z \right. \right.$$

$$+ 4\nu^3 \left( \Gamma_t^0 \Gamma_c^3 - \Gamma_t^3 \Gamma_c^0 \right)/z^2 \right.$$

$$+ \Gamma \nu \Gamma_t^0/4 + \Gamma \nu \Gamma_t^0 \Gamma_t^0/4z + \Gamma \nu \Gamma_t^0 \left( 1 + 2\nu \Gamma_t^0/z \right)^2/4 \right\}, \quad (182)$$

$$d\nu \Gamma_t^3/dl = g \left\{ \nu \left( \Gamma_t^0 + \Gamma_t^3 + \Gamma_c^0 + \Gamma_c^3 \right)/2 - 2\nu^2 \left( \Gamma_t^0 \Gamma_t^3 z f_1(z, z_2) + 3 \Gamma_t^3 \Gamma_t^3 \right) \right. \right.$$

$$+ \Gamma \nu (\Gamma_t^0 + \Gamma_t^0)/4 \right\} + 2(\nu \Gamma_t^0)^2/4, \quad (183)$$

$$d\nu \Gamma_c^0/dl = g \left\{ \nu \left( 3\Gamma_t^0 + 3\Gamma_t^3 - \Gamma_c^0 - \Gamma_c^3 \right)/2 + 2\nu^2 \left( 3\Gamma_t^0 \Gamma_c^3 - 3\Gamma_t^3 \Gamma_c^0 \right) \right. \right.$$

$$+ \Gamma \nu (\Gamma_t^0 + \Gamma_c^0)/4 \right\} - 2(\nu \Gamma_c^0)^2/4, \quad (184)$$

$$d\nu \Gamma_c^3/dl = g \left\{ \nu \left( 3\Gamma_t^0 + 3\Gamma_t^3 - \Gamma_c^0 - \Gamma_c^3 \right)/2 + 4\nu^2 \left( \Gamma_t^0 \Gamma_t^3 - \Gamma_t^3 \Gamma_t^3 \right)/z \right. \right.$$

$$+ 4\nu^3 \left( 3\Gamma_t^0 \Gamma_t^3 \Gamma_t^3 - \Gamma_t^3 \Gamma_t^0 \Gamma_t^0 \right)/z^2 \right.$$

$$+ \Gamma \nu \Gamma_c^0/4 - \Gamma \nu^2 \Gamma_t^0 \Gamma_c^0/4z + \Gamma \nu \Gamma_c^0 \left( 1 - 2\nu \Gamma_c^0/4 \right)^2/4 \right\}, \quad (185)$$

$$d\nu/dl = g \nu \left\{ 3\nu \Gamma_t^0 f_1(z, z_2) - 3\nu \Gamma_t^3/z + \nu \Gamma_c^0/z - \nu \Gamma_c^0 f_1(z, z_c) + z\Gamma/8 \right\}, \quad (186)$$

where $z_2 = z + 2\nu \Gamma_t^0$ and $z_c = z - 2\nu \Gamma_c^0$, which come from ladder summations in the triplet p-h channel, Eq. (125), and in the staggered p-p Cooper channel,

$$\Gamma_c^3(q, \omega) = \Gamma_c^3 \frac{Dq^2 + z|\omega|}{Dq^2 + (z - 2\nu \Gamma_c^0)|\omega|}, \quad (187)$$

The last term of Eq. (183) comes also from the ladder summation in the staggered p-h triplet channel that present the same log-divergence of the standard Cooper channel, Eq. (184), namely it does not depend on $g$. As a result, for repulsive interaction, the Stoner instability towards a spin-density wave is not destroyed by disorder. At the first order in the scattering amplitudes, introducing $u_t^0 = 2\Gamma_t^0/z, u_c^0 = 2\Gamma_c^0/z, u_t^1 = 2\Gamma_t^3/z$ and $u_c^1 = 2\Gamma_c^3/z$ the conductivity and the density of states are corrected in this way

$$\delta \sigma/\sigma = g \left[ 3(u_t^0 - u_t^1)/2 + (u_c^0 - u_c^1)/2 \right] \ln(T\tau)^{-1}, \quad (188)$$

$$\delta \nu/\nu = g \left[ 3(u_t^0 - u_t^1)/2 + (u_c^0 - u_c^1)/2 + \Gamma/8 \right] \ln(T\tau)^{-1}. \quad (189)$$
An effect of the SU(2) symmetry breaking is to reduce the triplet scattering amplitudes as magnetic field, namely the spin rotation invariance is broken, we have

- u

In the presence of magnetic impurities we have

If the d-wave superconductor is embedded in a constant magnetic field, namely the spin rotation invariance is broken, we have

An effect of the SU(2) symmetry breaking is to reduce the triplet scattering amplitudes as \( \Gamma^0_t \to \Gamma^0_t/3 \) and \( \Gamma^3_t \to 2\Gamma^3_t/3 \) in the conductance and in the DOS. Indeed for the standard modes the massless ones are those which commute with the Zeeman term, i.e. only one component in the triplet sector survives, while the staggered massless modes anticommute with the Zeeman term, i.e. there are two components in the triplet sector. Solving the equations above we obtain that for moderate values of the interactions \( g \) decreases, \( u_t^4 \) is finite and \( u_t^3 \) does not diverge if the starting value \( u_{t0}^3 \sim -g^{1/2} \) or if \( u_0^3 \) is positive and \( u_{t0}^3 \sim 0 \).

In the presence of magnetic impurities we have

\[
\frac{dg}{dl} = g^2 \nu \left\{ -3\Gamma^0_t f_2(z, z, z_2) + 3\Gamma^3_t/z \right\} /2, \tag{190}
\]
\[
\frac{dz}{dl} = g\nu \left\{ 3\Gamma^0_t - 3\Gamma^3_t + z(\Gamma/4 - 1)/2 \right\}, \tag{191}
\]
\[
\frac{dv\Gamma^0_t}{dl} = g \left\{ \nu \left( \Gamma^0_t + \Gamma^3_t \right)/2 + 4\nu^2 \left( \Gamma^0_t \Gamma^0_t - \Gamma^3_t \Gamma^3_t - \Gamma^0_t \Gamma^3_t + \Gamma^0_t \Gamma^3_t \right)/z \right\} - 4\nu^2 \Gamma^0_t \Gamma^3_t \Gamma^0_t /z^2
+ (\Gamma/4 - 1)\nu \Gamma^0_t + (\Gamma/4 - 1)\nu^2 \Gamma^0_t \Gamma^3_t /z + \Gamma \nu \Gamma^3_t \left( 1 + 2\nu \Gamma^0_t /z \right)^2 /4 \right\}, \tag{192}
\]
\[
\frac{dv\Gamma^3_t}{dl} = g \left\{ \nu \left( \Gamma^0_t + \Gamma^3_t \right)/2 - 2\nu^2 \left( \Gamma^0_t \Gamma^3_t z f_1(z, z_2) + 3\Gamma^3_t \Gamma^3_t \right)/z \right\}
+ (\Gamma/4 - 1)\nu \Gamma^1_t + \Gamma \nu \Gamma^3_t /4 \right\} + 2(\nu \Gamma^3_t)^2 /z, \tag{193}
\]
\[
\frac{dv}{dl} = g\nu \left\{ 3\nu \Gamma^0_t f_1(z, z_2) - 3\nu \Gamma^3_t /z + (\Gamma/4 - 1)/2 \right\}. \tag{194}
\]
\[ \frac{dv}{dl} = g\nu\Gamma_c^3 f_1(z, z_c). \] (203)

These equations are formally the same of Eq. (168-171), consistently to the fact that the soft modes live in the same manifold, \( U(2n)/U(n) \times U(n) \), where now \( n = \#\text{replicas} \times \#\text{positive frequencies} \). At low interaction regime we have the solution \( g = g_0 e^{\nu_0} \), similar to Eq. (174) except for the sign in front of the interaction amplitude.

E. In normal phase

The role of staggered fluctuations can also be considered in the metallic phase, extending the Finkel’stein model to include nesting effects. If superconductive order parameter is turned off and if staggered fluctuations are supposed to be massless, the singlet contributions stay relevant and the renormalization group equations become the following:

i. Yes chiral, Yes \( \hat{T} \). In the presence of time reversal symmetry we obtain

\[
\begin{align*}
dg/dl &= g^2 \left\{ \nu \Gamma_s^0 f_2(z, z, z_1) - \nu \Gamma_s^3 /z - 3\nu \Gamma_t^0 f_2(z, z, z_2) + 3\nu \Gamma_t^3 /z - 2\nu \Gamma_c^0 /z + 2\nu \Gamma_c^3 f_2(z, z, z_c) \right\}, \\
dz/dl &= g \left\{ \nu \Gamma_s^0 - \nu \Gamma_s^0 + 3\nu \Gamma_t^0 - 3\nu \Gamma_t^3 + 2\nu \Gamma_c^0 - 2\nu \Gamma_c^3 + z (1 + \Gamma/8) \right\}, \\
dnu\Gamma_t^0/dl &= g \left\{ \nu \left( \Gamma_s^0 + \Gamma_s^3 + \Gamma_t^0 + \Gamma_t^3 + 2\Gamma_c^0 + 2\Gamma_c^3 \right) /2 + 4\nu^2 \left( \Gamma_s^0 \Gamma_t^0 + \Gamma_s^0 \Gamma_t^0 + 2\Gamma_s^0 \Gamma_s^0 - \Gamma_s^0 \Gamma_t^3 \\
&\quad - \Gamma_s^0 \Gamma_t^3 \right) /z + 4\nu^3 \left( 2\Gamma_s^0 \Gamma_t^3 \Gamma_t^0 - \Gamma_s^0 \Gamma_t^3 \Gamma_t^0 + \Gamma_s^0 \Gamma_t^3 \Gamma_t^0 \right) /z^2 \\
&\quad + (2 + \Gamma/4) \left( \nu \Gamma_t^0 + \nu \Gamma_t^0 \right) + \Gamma \nu \Gamma_t^3 \left( 1 + 2\nu \Gamma_t^0 /z \right)^2 /4 \right\}, \\
dnu\Gamma_t^3/dl &= g \left\{ \nu \left( \Gamma_s^0 + \Gamma_s^3 + \Gamma_t^0 + \Gamma_t^3 + 2\Gamma_c^0 + 2\Gamma_c^3 \right) /2 - 2\nu^2 \left( \Gamma_s^0 \Gamma_t^0 \Gamma_t^0 + 3\Gamma_t^0 \Gamma_t^0 \\
&\quad - 2\Gamma_s^0 \Gamma_t^3 + 2\Gamma_s^3 \Gamma_t^3 \Gamma_t^0 \Gamma_t^0 + \Gamma_s^0 \Gamma_t^3 \Gamma_t^0 \Gamma_t^0 \right) /z \\
&\quad + (2 + \Gamma/4) \nu \Gamma_t^3 + \Gamma \nu \Gamma_t^3 /4 \right\} + 2(\nu \Gamma_t^0)^2 /z, \\
dnu\Gamma_s^0/dl &= g \left\{ \nu \left( -\Gamma_s^0 - \Gamma_s^3 + 3\Gamma_t^0 + 3\Gamma_t^3 + 2\Gamma_c^0 + 5\Gamma_c^3 \right) /2 + 4\nu^2 \left( \Gamma_s^0 \Gamma_s^0 - 3\Gamma_t^0 \Gamma_s^0 \\
&\quad + \Gamma_s^3 \Gamma_c^3 \Gamma_c^3 \Gamma_c^3 \right) /z + 4\nu^3 \left( -\Gamma_s^0 \Gamma_s^3 \Gamma_s^0 + 3\Gamma_s^0 \Gamma_s^3 \Gamma_s^0 \right) /z^2 \\
&\quad - 4\nu^2 \left( 2\Gamma_s^0 \Gamma_s^3 \Gamma_c^3 /z + \Gamma_s^0 \Gamma_s^3 \left( 1 - \Gamma_s^0 /z \right) \right) \\
&\quad + (2 + \Gamma/4) \left( \nu \Gamma_s^0 - \nu \Gamma_s^3 \Gamma_s^0 \Gamma_s^0 /z \right) + \nu \Gamma_s^3 \left( 1 - 2\nu \Gamma_s^0 /z \right)^2 /4 \right\}, \\
dnu\Gamma_s^3/dl &= g \left\{ \nu \left( -\Gamma_s^0 - \Gamma_s^3 + 3\Gamma_t^0 + 3\Gamma_t^3 + 2\Gamma_c^0 + 2\Gamma_c^3 \right) /2 + 2\nu^2 \left( -\Gamma_s^0 \Gamma_s^3 \Gamma_t^0 \Gamma_t^0 \Gamma_t^0 + \Gamma_s^0 \Gamma_s^3 \Gamma_t^0 \Gamma_t^0 \Gamma_t^0 \right) /z \\
&\quad + \Gamma_s^3 \Gamma_t^3 + 3\Gamma_t^3 \Gamma_s^3 \Gamma_t^0 \Gamma_t^0 \Gamma_t^0 + \Gamma_s^0 \Gamma_s^3 \Gamma_t^0 \Gamma_t^0 \Gamma_t^0 \right) /z + 2\nu \Gamma_s^3 \Gamma_s^3 \left( 1 - \Gamma_s^0 /z \right)^2 /4 \right\}.
\end{align*}
\]
applies in the diffusion propagators, modified by singlet p-h channel ladder summation
\[ \Gamma_0 / (2 + \Gamma^0) / \nu / \Gamma^3_s + \Gamma^0 / \Gamma^0_s / 4 - 2(\nu / \Gamma^3_s)^2 / z, \]
\[ d\nu / \Gamma^0_c / dl = g \left\{ \nu \left( \Gamma^0_s + \Gamma^0_s + 3\Gamma^0_t + 3\Gamma^3_t / 2 + 2\nu^2 \left( 3\Gamma^0_s \Gamma^0 \Gamma^0 f_1(z, z_2) - 3\Gamma^3_t \Gamma^0_s + \Gamma^3_s \Gamma^0 \Gamma^0 f_1(z, z_1) - 2\Gamma^0_t \Gamma^3_s \Gamma^0 f_1(z, z_1) + 2\Gamma^0_c \Gamma^0 f_1(z, z_1) + 1 / z \right) \right) \right\} + (2 + \Gamma^0 / \nu) \Gamma^0_c + \Gamma^0_c / 4 \} - 2(\nu / \Gamma^0_c)^2 / z, \]
\[ d\nu / \Gamma^3_c / dl = g \left\{ \nu \left( \Gamma^0_s + \Gamma^0_s + 3\Gamma^0_t + 3\Gamma^3_t / 2 + 4\nu^2 \left( 3\Gamma^3_s \Gamma^0 \Gamma^0 f_1(z, z_1) - f_1(z, z_1) \right) - 3\Gamma^3_t \Gamma^3_s \Gamma^3 + \Gamma^3_s \Gamma^3_s \Gamma^3 \right) / z^2 \right\} \]
\[ -4\nu^2 \Gamma^3_c \left( f_1(z_1, z_c) + f_1(z, z_1) - 2f_1(z, z_c) + \frac{1}{z} \right) \]
\[ + (2 + \Gamma^0 / 4) \left( \nu \Gamma^0_c - \nu \Gamma^0_c / \Gamma^3_c / z \right) + \Gamma^0_c / \Gamma^0_c (1 - 2\nu / \Gamma^3_c / z)^2 / 4 \right\} / z, \]
where \( z_1 = z_2 = \nu / \Gamma^0_s \) and \( z_2, z_c \) as before. \( z_1 \) is related to the compressibility \( \frac{\Delta q}{q} \), here it appears in the diffusion propagators, modified by singlet p-h channel ladder summation
\[ \Gamma^0_s(q, \omega) = \Gamma^0_s \frac{Dq^2 + z|\omega|}{Dq^2 + (z - 2\nu / \Gamma^0_s)|\omega|}. \]

If chiral symmetry is broken, the last contributions in the curly brackets proportional to \( (2 + \Gamma^0 / 4) \) and to \( \Gamma \) together with all the contributions proportions to the \( \Gamma^3 \)’s vanish and we obtain again the standard Finkel’stein equations. The equations above, although rather complex, hide a very amazing property, they in fact are symmetric with respect to the following transformation
\[ \Gamma^0_s = \Gamma^0_c \leftrightarrow -\Gamma^0_t, \]
\[ \Gamma^3_s = \Gamma^3_c \leftrightarrow -\Gamma^3_t. \]

This symmetric property represents the invariance with respect to the particle-hole symmetry transformation, \( c_{it} = d_{it}, c_{ti} = (-)^i d_{ti} \), that maps the charge to the spin and vice versa.

j. Yes chiral, No \( \hat{T} \). If time reversal symmetry is broken the renormalization group equations are found to be formally the same obtained in the superconductive case, when time reversal symmetry holds, Eqs.\((180, 186)\), with the substitutions
\[ \Gamma^0_s \rightarrow \Gamma^3_c, \]
\[ \Gamma^3_s \rightarrow \Gamma^0_c. \]
This accidentally simple equivalence of the equations is consistent with the fact that in both the cases the soft modes take values in the same coset $U(4n) \times U(4n)/U(4n)$.

**k. Yes chiral, magnetic field.** In the presence of a constant magnetic field we have

\[
\begin{align*}
dg/dl &= g^2 \left\{ \nu \Gamma^0_s f_2(z, z, z_1) - \nu \Gamma^0_x f_2(z, z, z_2) + 2 \nu \Gamma^3_t / z \right\}, \\
dz/dl &= g \left\{ -\nu \Gamma^0_s + \nu \Gamma^0_x - 2 \nu \Gamma^3_t \right\}, \\
d\nu \Gamma^0_t / dl &= g \left\{ \nu \left( \Gamma^0_0 / 2 - \Gamma^0_t / 2 + \Gamma^3_t \right) - 4 \nu^2 \Gamma^3_t \Gamma^3_t / z \right\}, \\
d\nu \Gamma^3_t / dl &= g \left\{ \nu \left( \Gamma^0_0 / 2 - 2 \nu^2 \left( 3 \Gamma^0_0 \Gamma^0_t z f_1(z, z_2) + 2 \Gamma^3_t \Gamma^3_t + \Gamma^0_0 \Gamma^3_t f_1(z, z_1) \right) / z \right) + 2 (\nu \Gamma^3_t)^2 \right\}, \\
d\nu \Gamma^0_s / dl &= g \left\{ \nu (-\Gamma^0_0 + \Gamma^0_t + 2 \Gamma^3_t) / 2 - 8 \nu^2 \Gamma^3_t \Gamma^0_s / z + 4 \nu^3 \Gamma^0_0 \Gamma^3_t \Gamma^0_s / z^2 \right\}, \\
dz/dl &= g \nu \left\{ -\nu \Gamma^0_s f_1(z, z_1) + \nu \Gamma^0_t f_1(z, z_2) - 2 \nu \Gamma^3_t \right\}.
\end{align*}
\]

**l. Yes chiral, magnetic impurities.** While with magnetic impurities we have

\[
\begin{align*}
dg/dl &= g^2 \left\{ \nu \Gamma^0_s f_2(z, z, z_1) + 2 \nu \Gamma^3_c f_2(z, z, z_c) + 1/2 \right\}, \\
dz/dl &= -g \left\{ \nu \Gamma^0_s + 2 \nu \Gamma^3_c + z / 2 \right\}, \\
d\nu \Gamma^0_s / dl &= g \left\{ \nu \Gamma^3_c - \frac{3}{2} \nu \Gamma^0_s + \nu^2 \Gamma^0_s \Gamma^0_s / z + 4 \nu^2 \Gamma^3_c \Gamma^3_c / z_c - 4 \nu^2 \left( 2 \Gamma^3_c \Gamma^3_c / z_c + \Gamma^0_s \Gamma^0_s \left( \frac{1}{z} - \frac{1}{z_c} \right) \right) \right\}, \\
d\nu \Gamma^3_c / dl &= g \left\{ \nu \Gamma^0_s / 2 + 4 \nu^2 \Gamma^0_s \Gamma^3_c (f_1(z_c, z_1) - f_1(z, z_1)) - \nu \Gamma^3_c + \nu^2 \Gamma^3_c \Gamma^3_c / z \right\} - 4 \nu^2 \Gamma^3_c \Gamma^3_c \left( f_1(z_1, z_c) + f_1(z, z_1) - 2 f_1(z, z_c) + \frac{1}{z} \right) \right\}, \\
d\nu / dl &= -g \nu \left\{ \nu \Gamma^0_s f_1(z, z_1) + 2 \nu \Gamma^3_c f_1(z, z_c) + 1/2 \right\}.
\end{align*}
\]

Notice that in this case by the transformation $\Gamma^0_s = \Gamma^3_c \rightarrow -\Gamma^0_t$ one obtains the same equations of the superconductive case with broken time reversal and chiral symmetries, analyzed in paragraph b, since also in the latter case the soft modes take values in the same manifold $Sp(2n) / U(2n)$.

Now we can also study the scaling behavior of the quantities $z_1 = z - 2 \nu \Gamma^0_s$ and $z_2 = z + 2 \nu \Gamma^0_t$ that are related by Ward identities to the compressibility and to the static spin susceptibility. As in [53, 76], because the ladder terms do not involve the small parameter $g$, we can consider the situation in which only the lowest order with respect to $\Gamma^0_s$ and $\Gamma^3_c$ (by choosing $\Gamma^0_c > g^{1/2}$ and $\Gamma^3_t < -g^{1/2}$) are retained in all the equations, consequently, we have for $z_1$ and $z_2$ the following equations:
(i) if time reversal symmetry holds
\[
d\left( z - 2\nu \Gamma_s^0 \right)/dl = 2g\nu \left( \Gamma_s^3 - 3\Gamma_t^3 - 2\Gamma_c^3 \frac{z}{z_c} + \frac{z}{2} \left( 1 + \frac{\Gamma}{8} \right) - \frac{\Gamma_t^3}{4} \right) \left( z - 2\nu \Gamma_s^0 \right)^2 / z^2,
\]
\[
d\left( z + 2\nu \Gamma_t^0 \right)/dl = 2g\nu \left( \Gamma_s^3 - \Gamma_t^3 + 2\Gamma_c^0 \frac{z}{z_c} + \frac{z}{2} \left( 1 + \frac{\Gamma}{8} \right) + \frac{\Gamma_t^3}{4} \right) \left( z + 2\nu \Gamma_t^0 \right)^2 / z^2,
\]

(j) if time reversal symmetry is broken
\[
d\left( z - 2\nu \Gamma_s^0 \right)/dl = 2g\nu \left( \Gamma_s^3 - 3\Gamma_t^3 + z\Gamma/16 - \Gamma\Gamma_t^3/4 \right) \left( z - 2\nu \Gamma_s^0 \right)^2 / z^2,
\]
\[
d\left( z + 2\nu \Gamma_t^0 \right)/dl = 2g\nu \left( \Gamma_s^3 - \Gamma_t^3 + 2\Gamma_c^0 \frac{z}{z_c} + z\Gamma/16 + \Gamma\Gamma_t^3/4 \right) \left( z + 2\nu \Gamma_t^0 \right)^2 / z^2,
\]

(k) with constant magnetic field
\[
d\left( z - 2\nu \Gamma_s^0 \right)/dl = -4g\nu \Gamma_t^3 \left( z - 2\nu \Gamma_s^0 \right)^2 / z^2,
\]
\[
d\left( z + 2\nu \Gamma_t^0 \right)/dl = 0,
\]

(l) with magnetic impurities
\[
d(z - 2\nu \Gamma_s^0)/dl = -2g\nu \left( 2\Gamma_c^3 \frac{z}{z_c} + \frac{z}{4} \right) \left( z - 2\nu \Gamma_s^0 \right)^2 / z^2.
\]

Notice that in all the cases, when chirality is spoiled, \( z_1 \) is not renormalized as already pointed out previously in Refs. 53, 76.

XII. CONCLUSIONS

In this work we have analyzed the role of disorder in \( d \)-wave superconductors, which have gapless Landau-Bogoliubov quasiparticle excitations. We have considered several universality classes, including the chiral symmetry which occurs at half-filling for a two-sublattice model. In addition, we have studied in details the effects of the residual quasiparticle interaction. The main results of this work are summarized in the following.

In the presence of non magnetic impurities the spin conductivity is suppressed by quantum interference corrections, in agreement with Ref. 32. The density of states vanishes in the insulating regime.

On the contrary, surprisingly, magnetic impurities gives a delocalization correction to the conductivity meanwhile enhancing the density of states 41.
If chiral symmetry is present, namely at half filling for a two-sublattice model, the spin stays delocalized in spite of disorder and the conductivity remains finite. The DOS diverges in the absence of magnetic field and magnetic impurities [41].

The quasiparticle charge conductivity, namely the optical conductivity at small frequency, has in general the same behavior as the spin conductivity. However, when chiral symmetry holds and time reversal symmetry is broken, the dirty d-wave superconductor behaves like a spin metal but charge insulator (at finite small frequency, excluding the Drude peak of the superflow), manifesting a sort of spin-charge separation.

Charge fluctuations as well as fluctuations of the real part of the order parameter, assuming the average value to be real, are not diffusive in a superconductor. Therefore the residual quasiparticle interaction written in terms of the diffusive modes only contains the spin-triplet particle-hole channel and the Cooper channel representing s-wave fluctuations of the imaginary part of the order parameter.

For repulsive interaction, particles and holes repel each other in the spin-triplet channel, hence opposing localization. In fact we find that a repulsive residual interaction gives a delocalizing correction to the conductivity and enhances the density of states. Namely the interaction can compete with quantum interferences due to the disorder.

According to the symmetry of the system, we have found that the interactions can be i) relevant, when both time reversal symmetry, $\hat{T}$, and spin rotational invariance, SU(2), are preserved, ii) marginal, when $\hat{T}$ is broken and SU(2) holds, iii) irrelevant, when both $\hat{T}$ and SU(2) are broken.

We have also studied $(\pi, \pi)$ momentum transfer interactions, since they are coupled to diffusive staggered spin fluctuations at half-filling. We find that the corrections to the conductivity due to the interaction at $(\pi, \pi)$ have opposite sign than the corrections coming from the interaction at small momentum. Moreover, we notice that an analog of the Anderson theorem for s-wave superconductors holds at half-filling for staggered density fluctuations. Namely, since these modes are diffusive, the staggered susceptibility remains log-divergent even in the presence of disorder. As a result, for repulsive interaction, the Stoner instability towards a spin-density wave is not destroyed by disorder.

In summary we have derived the complete sets of RG equations for many different symmetries at the lowest order in the small parameter $g$, which does not depend on the disorder, and at all orders in the interaction amplitudes, $\nu\Gamma_t, \nu\Gamma_c$, which are the real phenomenolog-
ical parameters, related instead to the impurity concentration through the density of states proportional to the scattering rate, \( \nu \propto \Sigma = \frac{1}{2\tau} \). We have derived the RG equations even in the normal phase when nesting effects occur. In the latter case we find that the compressibility is a renormalized quantity unlike the standard case \([53, 76]\) when the chiral symmetry is not present.

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**APPENDIX A: TRANSVERSE MODES**

In this Appendix we summarize the properties of the \( W \) matrices for soft modes needed to evaluate the gaussian averages. \( S \) (A) and \( R \) (I) mean symmetric (antisymmetric) in the replica space and real (imaginary) \( s_i \), \( i = 0, 1, 2, 3 \), components defined in Eqs.\((53, 56)\). Notice that the number of soft modes decreases by decreasing the symmetry since some of them become massive.

1. **Without sublattice symmetry**

   (a) **Without superconducting order parameter**

   i. **With time reversal symmetry**: \( \text{Sp}(4n)/\text{Sp}(2n) \times \text{Sp}(2n) \).

   \[
   \begin{array}{|c|c|c|}
   \hline
   W & s_1 & s_2 \\
   \hline
   W_0 & A, R & S, I \\
   W_{S0} & A, I & S, R \\
   W_{S1} & A, I & S, R \\
   W_{S2} & S, R & A, I \\
   W_{S3} & S, R & A, I \\
   W_{T0} & S, R & A, I \\
   W_{T1} & S, I & A, R \\
   W_{T2} & S, I & A, R \\
   W_{T3} & A, R & S, I \\
   \hline
   \end{array}
   \]
ii. Without time reversal symmetry: $U(4n)/U(2n) \times U(2n)$.

| $W_0$ | $s_1$ | $s_2$ |
|-------|-------|-------|
| $W_{s0}$ | A, R | S, I |
| $W_{s3}$ | S, R | A, I |
| $\tilde{W}_{T0}$ | S, R | A, I |
| $\tilde{W}_{T3}$ | A, R | S, I |

iii. With magnetic field: $U(2n) \times U(2n)/U(2n)$.

| $W_0$ | $s_1$ | $s_2$ |
|-------|-------|-------|
| $W_{s0}$ | A, R | S, I |
| $W_{s3}$ | S, R | A, I |
| $W_{T,0}$ | S, R | A, I |
| $W_{T,3}$ | A, R | S, I |

iv. With magnetic impurities: $U(2n)/U(n) \times U(n)$.

| $W_0$ | $s_1$ | $s_2$ |
|-------|-------|-------|
| $W_{s0}$ | A, R | S, I |
| $W_{s3}$ | S, R | A, I |

(b) With superconducting order parameter

i. With time reversal symmetry: $Sp(2n) \times Sp(2n)/Sp(2n)$.

| $W_0$ | $s_1$ | $s_2$ |
|-------|-------|-------|
| $W_{s0}$ | A, R |
| $W_{s1}$ | S, R |
| $W_{s2}$ | A, I |
| $W_{s3}$ | A, I |
| $\tilde{W}_{T0}$ | S, R |
| $\tilde{W}_{T1}$ | A, R |
| $\tilde{W}_{T2}$ | S, I |
| $\tilde{W}_{T3}$ | S, I |
ii. Without time reversal symmetry: $\text{Sp}(2n)/\text{U}(2n)$.

\[
\begin{array}{|c|c|c|}
\hline
W_0 & s_1 & s_2 \\
\hline
W_{S0} & A, R & \\
\hline
W_{S3} & A, I & \\
\hline
\bar{W}_{T0} & S, R & \\
\hline
\bar{W}_{T3} & S, I & \\
\hline
\end{array}
\]

iii. With magnetic field: $\text{U}(2n)/\text{U}(n) \times \text{U}(n)$.

\[
\begin{array}{|c|c|c|}
\hline
W_0 & s_1 & s_2 \\
\hline
W_{S0} & A, R & \\
\hline
W_{S3} & A, I & \\
\hline
W_{T,0} & S, R & \\
\hline
W_{T,3} & S, I & \\
\hline
\end{array}
\]

iv. With magnetic impurities: $\text{O}(2n)/\text{U}(n)$.

\[
\begin{array}{|c|c|c|}
\hline
W_0 & s_1 & s_2 \\
\hline
W_{S0} & A, R & \\
\hline
W_{S3} & A, I & \\
\hline
\hline
\end{array}
\]

2. With sublattice symmetry

(a) Without superconducting order parameter

i. With time reversal symmetry: $\text{U}(8n)/\text{Sp}(4n)$.

\[
\begin{array}{|c|c|c|}
\hline
W_0 & s_1 & s_2 \\
\hline
W_{S0} & A, R & S, I \\
\hline
W_{S1} & A, I & S, R \\
\hline
W_{S2} & A, I & S, R \\
\hline
W_{S3} & S, R & A, I \\
\hline
\bar{W}_{T0} & S, R & A, I \\
\hline
\bar{W}_{T1} & S, I & A, R \\
\hline
\bar{W}_{T2} & S, I & A, R \\
\hline
\bar{W}_{T3} & A, R & S, I \\
\hline
\end{array}
\]

\[
\begin{array}{|c|c|c|}
\hline
W_3 & s_0 & s_3 \\
\hline
W_{S0} & S, I & S, I \\
\hline
W_{S1} & S, R & S, R \\
\hline
W_{S2} & S, R & S, R \\
\hline
W_{S3} & A, I & A, I \\
\hline
\bar{W}_{T0} & A, I & A, I \\
\hline
\bar{W}_{T1} & A, R & A, R \\
\hline
\bar{W}_{T2} & A, R & A, R \\
\hline
\bar{W}_{T3} & S, I & S, I \\
\hline
\end{array}
\]
ii. Without time reversal symmetry: \( U(4n) \times U(4n)/U(4n) \).

| \( W_0 \) | \( s_1 \) | \( s_2 \) | \( W_3 \) | \( s_0 \) | \( s_3 \) |
|-----------|--------|--------|-----------|--------|--------|
| \( W_{S0} \) | A, R | S, I  | \( W_{S0} \) | S, I | S, I  |
| \( W_{S3} \) | S, R | A, I  | \( W_{S3} \) | A, I | A, I  |
| \( \tilde{W}_{T0} \) | S, R  | A, I  | \( \tilde{W}_{T0} \) | A, I | A, I  |
| \( \tilde{W}_{T3} \) | A, R | S, I  |

iii. With magnetic field: \( U(4n)/U(2n) \times U(2n) \).

| \( W_0 \) | \( s_1 \) | \( s_2 \) | \( W_3 \) | \( s_0 \) | \( s_3 \) |
|-----------|--------|--------|-----------|--------|--------|
| \( W_{S0} \) | A, R | S, I  | \( W_{T,0} \) | A, I | A, I  |
| \( W_{S3} \) | S, R | A, I  | \( W_{T,3} \) | S, I | S, I  |
| \( W_{T,0} \) | S, R  | A, I  | \( W_{T,0} \) | A, I | A, I  |
| \( W_{T,3} \) | A, R | S, I  |

iv. With magnetic impurities: \( Sp(2n)/U(2n) \).

| \( W_0 \) | \( s_1 \) | \( s_2 \) | \( W_3 \) | \( s_0 \) | \( s_3 \) |
|-----------|--------|--------|-----------|--------|--------|
| \( W_{S0} \) | A, R | S, I  | \( W_{S1} \) | S, R | S, R  |
| \( W_{S3} \) | S, R | A, I  | \( W_{S2} \) | S, R | S, R  |
| \( W_{S3} \) | A, I  |

(b) With superconducting order parameter

i. With time reversal symmetry: \( U(4n) \times U(4n)/U(4n) \).

| \( W_0 \) | \( s_1 \) | \( s_2 \) | \( W_3 \) | \( s_0 \) | \( s_3 \) |
|-----------|--------|--------|-----------|--------|--------|
| \( W_{S0} \) | A, R  |
| \( W_{S1} \) | S, R  |
| \( W_{S2} \) | A, I  |
| \( W_{S3} \) | A, I  |
| \( \tilde{W}_{T0} \) | S, R  |
| \( \tilde{W}_{T1} \) | A, R  |
| \( \tilde{W}_{T2} \) | S, I  |
| \( \tilde{W}_{T3} \) | S, I  |
ii. Without time reversal symmetry: $U(4n)/O(4n)$.

\[
\begin{array}{|c|c|c|}
\hline
W_0 & s_1 & s_2 \\
\hline
W_{S0} & A, R & \\
\hline
W_{S3} & A, I & \\
\hline
\tilde{W}_{T0} & S, R & \\
\hline
\tilde{W}_{T3} & S, I & \\
\hline
\end{array}
\quad
\begin{array}{|c|c|c|}
\hline
W_3 & s_0 & s_3 \\
\hline
W_{S0} & S, I & \\
\hline
W_{S3} & A, I & \\
\hline
\tilde{W}_{T0} & A, I & \\
\hline
\tilde{W}_{T3} & S, I & \\
\hline
\end{array}
\]

iii. With magnetic field: $O(4n)/O(2n) \times O(2n)$.

\[
\begin{array}{|c|c|c|}
\hline
W_0 & s_1 & s_2 \\
\hline
W_{S0} & A, R & \\
\hline
W_{S3} & A, I & \\
\hline
W_{T,0} & S, R & \\
\hline
W_{T,3} & S, I & \\
\hline
\end{array}
\quad
\begin{array}{|c|c|c|}
\hline
W_3 & s_0 & s_3 \\
\hline
W_{T,0} & A, I & \\
\hline
W_{T,3} & S, I & \\
\hline
W_{T,0} & A, I & \\
\hline
W_{T,3} & S, I & \\
\hline
\end{array}
\]

iv. With magnetic impurities: $U(2n)/U(n) \times U(n)$.

\[
\begin{array}{|c|c|c|}
\hline
W_0 & s_1 & s_2 \\
\hline
W_{S0} & A, R & \\
\hline
W_{S3} & A, I & \\
\hline
\hline
\end{array}
\quad
\begin{array}{|c|c|c|}
\hline
W_3 & s_0 & s_3 \\
\hline
W_{S1} & S, R & \\
\hline
W_{S2} & S, R & \\
\hline
\end{array}
\]

APPENDIX B: CHARGE AND SPIN VECTOR POTENTIALS

Let us consider an operator diagonal in the Nambu space. Namely

\[
\mathcal{A}_{ij} = \begin{pmatrix} A_{\uparrow,ij} & 0 \\ 0 & A_{\downarrow,ij} \end{pmatrix},
\]

where the matrix elements are matrices in the retarded/advanced and replica space. If we take $\mathcal{A}_{ii} = 0$, then such an operator corresponds to

\[
\sum_{ij} c_{i\uparrow}^\dagger A_{\uparrow,ij} c_{j\uparrow} - c_{i\downarrow}^\dagger A_{\downarrow,ij} c_{j\downarrow} = \sum_{ij} c_{i\uparrow}^\dagger \left[ \frac{1}{2} (A_{\uparrow,ij} - A_{\downarrow,ji}) + \frac{1}{2} \sigma_z (A_{\uparrow,ij} + A_{\downarrow,ji}) \right] c_{j\downarrow}.
\]

In the path integral formalism, a generic operator diagonal in the Nambu space,

\[
\sum_{ij} \Psi_i A_{ij} \Psi_j,
\]
with
\[ A_{ij} = \begin{pmatrix} A_{1,ij} & 0 \\ 0 & A_{2,ij} \end{pmatrix}, \]
corresponds instead to
\[ \frac{1}{2} \sum_{ij} \bar{e}_i \left[ A_{1,ji}^t + \sigma_y A_{2,ij} \sigma_y \right] c_j. \]

By comparison we have that
\[ \left[ A_{1,ji}^t + \sigma_y A_{2,ij} \sigma_y \right] = \left[ (A_{t,ij} - A_{\downarrow,ji}) + \sigma_z (A_{t,ij} + A_{\downarrow,ji}) \right]. \]

Suppose that the operators in question are currents. Then \( A_{ij} = -A_{ji} \), and the above relation reads
\[ \left[ -A_{1,ij}^t + \sigma_y A_{2,ij} \sigma_y \right] = \left[ (A_{t,ij} + A_{\downarrow,ij}) + \sigma_z (A_{t,ij} - A_{\downarrow,ij}) \right]. \]

In general we can consider either a charge current, implying \( A_{t,ij} = A_{\downarrow,ij} = A \), or a spin current, in which case \( A_{t} = -A_{\downarrow} = A \).

In the former case
\[ \left[ -A_{1,ij}^t + \sigma_y A_{2,ij} \sigma_y \right] = \left[ (A_{ij} + A_{\downarrow,ij}) + \sigma_z (A_{ij} - A_{\downarrow,ij}) \right], \]
while in the latter
\[ \left[ -A_{1,ij}^t + \sigma_y A_{2,ij} \sigma_y \right] = \left[ (A_{ij} - A_{\downarrow,ij}) + \sigma_z (A_{ij} + A_{\downarrow,ij}) \right]. \]

We therefore see that, if \( A \) (we assume the same property holds for \( A \)) is a symmetric matrix, the charge current operator is proportional to the identity in spin space and \( -A_{1,ij} + A_{2,ij} = 2A_{ij} \), namely \( A_{ij} = -A_{1,ij} = A_{ij} \), while the spin is proportional to \( \sigma_z \) and \( -A_{1,ij} + \sigma_y A_{2,ij} \sigma_y = 2\sigma_z A_{ij} \), implying \( A_{2,ij} = A_{1,ij} = -\sigma_z A_{ij} \).

In the opposite case of an antisymmetric \( A \), the charge current multiplies \( \sigma_z \) and \( [A_{1,ij} + \sigma_y A_{2,ij} \sigma_y] = 2\sigma_z A_{ij} \), leading to \( A_{1,ij} = -A_{2,ij} = \sigma_z A_{ij} \), while the spin is proportional to the identity and \( [A_{1,ij} + \sigma_y A_{2,ij} \sigma_y] = 2A_{ij} \), leading to \( A_{1,ij} = A_{2,ij} = A_{ij} \).

These relations imply that the charge current, if \( A \) is symmetric, is the identity in spin space, otherwise is proportional to \( \sigma_z \). For the spin current, the opposite occurs. Let us now see which are the vector potentials for charge and spin.
1. Current-current correlation function

Let us suppose to calculate the current-current correlation function \( \langle J(R)J(R') \rangle \). The current in Nambu spinor representation is

\[
\vec{J}(R) = -i \sum_{R_1} (\vec{R} - \vec{R}_1) \Psi_R \mathcal{H}_{RR_1} \Psi_{R_1} = \sum_{R_1R_2} \bar{\Psi}_{R_1} J_{R_1R_2}(R) \Psi_{R_2},
\]

(B1)

with

\[
J_{R_1R_2}(R) = -i(\vec{R}_1 - \vec{R}_2) \mathcal{H}_{R_1R_2} \delta_{R_1R}.
\]

(B2)

In the case of spin current we have \( \mathcal{H} = (t + i\Delta \tau_2 s_1) \sigma_z \), while for the charge current \( \mathcal{H} = t\tau_3 \).

The correlation function becomes

\[
\langle J(R)J(R') \rangle = \sum_{R_1R_2R_3R_4} \langle \bar{\Psi}_{R_1} J_{R_1R_2}(R) \Psi_{R_2} \bar{\Psi}_{R_3} J_{R_3R_4}(R') \Psi_{R_4} \rangle.
\]

(B3)

Let us introduce for simplicity multilabels indicating replica, hole-particle, spin, energy and position indices. The correlation function can be rewritten in this way

\[
\langle \bar{\Psi}^i J^i_{ij} \Psi^j \Psi^l J^l_{lm} \Psi^m \rangle = -J_{ij} \langle \Psi^j \bar{\Psi}^i \rangle J_{lm} \langle \Psi^m \bar{\Psi}^l \rangle + J_{ij} \langle \Psi^j \Psi^m \rangle J_{lm} \langle \bar{\Psi}^l \bar{\Psi}^i \rangle,
\]

(B4)

or in matrix language in the following way

\[
\langle \bar{\Psi} J \Psi \bar{\Psi} J \Psi \rangle = -Tr \left( J \langle \Psi \bar{\Psi} J \langle \Psi \bar{\Psi} \rangle \right) + Tr \left( J \langle \Psi \bar{\Psi} J \langle \Psi \bar{\Psi} \rangle \right).
\]

(B5)

Since

\[
\bar{\Psi} = (C\Psi)^t,
\]

(B6)

with \( C = i\tau_1 \sigma_y \), the correlation function becomes

\[
- Tr \left( J \langle \Psi \bar{\Psi} \rangle J \langle \Psi \bar{\Psi} \rangle + J \langle \Psi \bar{\Psi} \rangle C J^t C^t \langle \Psi \bar{\Psi} \rangle \right).
\]

(B7)

In terms of single particle Green’s functions

\[
- Tr \left( J G(J + C J^t C^t) G \right).
\]

(B8)

Let us define the following quantities, for spin

\[
J^S_{R_1R_2} = -i(R_1 - R_2)(t_{R_1R_2} + i\Delta_{R_1R_2} \tau_2 s_1),
\]

(B9)

and for charge

\[
J^C_{R_1R_2} = -i(R_1 - R_2)t_{R_1R_2},
\]

(B10)
such that $J = J^S \sigma_z$ for spin and $J = J^C \tau_3$ for charge. In both the cases the following relation holds

$$CJ^tC^t = J, \quad \text{(B11)}$$

therefore the correlation function can be reduced to

$$-2Tr (JGJ^G). \quad \text{(B12)}$$

Introducing a vector potential which is coupled to the current vertex, we need to evaluate

$$-Tr \left( J^K AG (J^K A + C(J^K A)^t C^t) G \right), \quad \text{(B13)}$$

where $K = C$ for the charge current vertex and $K = S$ for the spin current vertex. In the case of charge ($J^K = J^C$), with $A = (A^0 s_0 + A^1 s_1) \tau_3$ and in the case of spin ($J^K = J^S$), with $A = (A^0 s_0 + A^1 s_1) \sigma_z$, the following relation holds

$$C(J^K A)^t C^t = J^K A. \quad \text{(B14)}$$

Applying to (B13) the second derivative with respect to $A^0$ and to $A^1$, in the following combination

$$\left. \frac{\partial^2}{\partial A^0^2} \right|_{A=0} - \left. \frac{\partial^2}{\partial A^1^2} \right|_{A=0}, \quad \text{(B15)}$$

we obtain in both the cases

$$- (Tr(JG^+JG^+)-Tr(JG^-JG^+)) \propto \sigma, \quad \text{(B16)}$$

which is the charge (if $J = J^C \tau_3$) or the spin (if $J = J^S \sigma_z$) conductivity. Instead of using (B15), exploiting the $d$-wave symmetry, it is straightforward to show that Eq. (B16) comes easily from Eq. (B13) taking simply $A = (s_0 + i s_1) \tau_3$ for charge and $A = (s_0 + i s_1) \sigma_z$ for spin.

Let us consider in what follows only the spin case, with $J^K = J^S$, Eq. (B9), since in the $d$-wave superconductors only spin conductivity is conserved. If we now use the gauges $A = (A^0 s_0 + A^1 s_1) \tau_3$ and $A = A^0 s_3 \sigma_z + A^1 s_2$ we obtain

$$C(J_{RR'} A)^t C^t = -i(R - R')(t_{RR'} - i\Delta_{RR'} \tau_2 s_1) A, \quad \text{(B17)}$$

therefore the correlation function (B13) becomes ($\delta = R - R'$)

$$2Tr \left( \delta (t + i\Delta_2 s_1) A G \delta t AG \right). \quad \text{(B18)}$$
Taking advantage of $d$-wave symmetry and using the relations

$$\tau_0 s_2 G \tau_0 s_2 = \tau_3 s_1 G \tau_3 s_1, \quad \text{(B19)}$$

$$\tau_0 s_3 G \tau_0 s_3 = \tau_3 s_0 G \tau_3 s_0, \quad \text{(B20)}$$

which come from the following structure of the Green’s function at fixed disorder

$$G = [(\tau_0, \tau_3) \otimes s_0 + (\tau_2, \tau_1) \otimes s_1] \otimes (\gamma_1, \gamma_2) + i \Sigma[(\tau_2, \tau_1) \otimes s_2 \otimes \gamma_3 + (\tau_3, \tau_0) \otimes s_3 \otimes \gamma_0)],$$

(the terms $\tau_3 s_0, \tau_1 s_1, \tau_1 s_2, \tau_3 s_3$ disappears if time reversal symmetry is preserved), we can find that Eq. (B15) on Eq. (B18) gives

$$Tr (\delta t \tau_3 s_0 G \delta t \tau_3 s_0 G) - Tr (\delta t \tau_3 s_1 G \delta t \tau_3 s_1 G). \quad \text{(B21)}$$

Reminding that $J^C = -i \delta t$ and $G^- = s_1 G^+ s_1$, we can recognize Eq. (B21) as the Kubo formula for charge conductivity.

We have shown that the gauges $A = (A^0 s_0 + A^1 s_1) \tau_3$ and $A = (A^0 s_3 \sigma_z + A^1 s_2) \tau_0$ are equivalent to generate charge conductivity. The same calculation can be done to prove the equivalence of $A = (A^0 s_0 + A^1 s_1) \tau_0 \sigma_z$ and $A = (A^0 s_3 + A^1 s_2 \sigma_z) \tau_3$ to produce spin conductivity. At the end, the possible choices for the vector potentials, taking into account also the replica space through a symmetric tensor, $\lambda_a = \frac{1}{\sqrt{2}} (\delta a_1 \delta b_2 + \delta a_2 \delta b_1)$, or an antisymmetric one, $\lambda_a = \frac{i}{\sqrt{2}} (\delta a_1 \delta b_2 - \delta a_2 \delta b_1)$, are collected in the following table.

|     | Charge                  | Spin                    |
|-----|-------------------------|-------------------------|
| $A^0$ | $\lambda_0 \tau_0 s_0 \sigma_0$ | $\lambda_a \tau_3 s_0 \sigma_z$ | $\lambda_0 \tau_0 s_0 \sigma_0$ | $\lambda_a \tau_3 s_0 \sigma_z$ | $\lambda_a \tau_3 s_0 \sigma_0$ | $\lambda_a \tau_3 s_3 \sigma_0$ |
| $A^1$ | $\lambda_0 \tau_3 s_1 \sigma_0$ | $\lambda_a \tau_3 s_1 \sigma_z$ | $\lambda_0 \tau_0 s_3 \sigma_z$ | $\lambda_a \tau_0 s_3 \sigma_z$ | $\lambda_a \tau_3 s_0 \sigma_0$ | $\lambda_a \tau_3 s_3 \sigma_0$ |

**APPENDIX C: THE ACTION WITH VECTOR POTENTIALS**

1. **Charge vector potential**

Let us introduce a slow varying vector potential $\vec{A} \propto \tau_3$, taking $A_0$ and $A_1$ from the table above. The fields change in this way

$$c_R \rightarrow e^{iz \int_0^R \vec{A} \vec{R}'} d\vec{R} \approx e^{iz \bar{\vec{A}} \vec{R}} c_R, \quad \text{(C1)}$$

$$c_R^\dagger \rightarrow c_R^\dagger e^{-iz \int_0^{\vec{R}} \vec{A} \vec{R}'} d\vec{R} \approx c_R^\dagger e^{-iz \bar{\vec{A}} \vec{R}}, \quad \text{(C2)}$$
since \( A \) is a slow varying function, nearly constant on the lattice length. In the Hamiltonian this transformation is equivalent to changing the hopping term in this way

\[
t_{RR'} \rightarrow t_{RR'} e^{-i\frac{e}{c}\vec{\delta} \cdot \vec{A}(\vec{R}' - \vec{R})} \approx t_{RR'} \left( 1 - i\frac{e}{c} \vec{\delta} \cdot \vec{A} - \frac{e^2}{2c^2} (\vec{\delta} \cdot \vec{A})^2 \right), \tag{C3}
\]

where \( \vec{\delta} = \vec{R}' - \vec{R} \)

The interaction term, before the parametrization by \( \Delta \), is unaffected by this gauge transformation and so

\[
\Delta_{RR'} \rightarrow \Delta_{RR'}'. \tag{C4}
\]

The expressions in Eq. (C3) and Eq. (C4) represent the \( U(1) \) symmetry breaking. Now, returning to Eq. (61), we should consider in \( \tilde{T}_R H^0_{RR'} T_{R'} \) the transformed hopping term

\[
\tilde{T}_R t_{RR'} \left( 1 - i\frac{e}{c} \vec{\delta} \cdot \vec{A} - \frac{e^2}{2c^2} (\vec{\delta} \cdot \vec{A})^2 \right) T_{R'}^\dagger \approx \\
t_{RR'} \left( 1 + \tilde{T}_R \vec{\delta} \cdot \vec{\nabla} T_R^\dagger + \frac{1}{2} \tilde{T}_R (\vec{\delta} \cdot \vec{\nabla})^2 T_R^\dagger \right) - \tilde{T}_R t_{RR'} \left( i\frac{e}{c} \vec{\delta} \cdot \vec{A} + \frac{e^2}{2c^2} (\vec{\delta} \cdot \vec{A})^2 \right) T_{R'}^\dagger = \\
t_{RR'} + t_{RR'} \vec{\delta} \cdot \tilde{T}_R \vec{\nabla} T_R^\dagger + \frac{1}{2} \sum_{ij} \delta_i \delta_j T_R \delta_j T_R^\dagger + i\frac{e}{c} \tilde{T}_R t_{RR'} \vec{\delta} \cdot \vec{A} T_R^\dagger - \frac{e^2}{2c^2} \tilde{T}_R t_{RR'} (\vec{\delta} \cdot \vec{A})^2 T_R^\dagger.
\]

Besides the standard terms (the second and the third above) which bring to the non-linear \( \sigma \)-model as seen before, defining \( t = t_1 \gamma_1 + t_2 \gamma_2, \Delta = \Delta_1 \gamma_1 + \Delta_2 \gamma_2 \) and \( G = g + i \frac{\Sigma}{E^2 + \Sigma^2} s_3 \) with \( g(k) = -\frac{1}{E^2 + \Sigma^2} [(t_1 - i\Delta_1 \gamma_2 s_1) \gamma_1 + (t_2 - i\Delta_2 \gamma_2 s_1) \gamma_2] \), we have the following additional terms

1. \( i\frac{e}{c} \text{Tr} \left( G t \vec{\delta} \cdot \vec{A} \right) \),
2. \( i\frac{e}{c} \text{Tr} \left( G \tilde{T} \vec{\delta} \cdot \vec{A} \vec{\nabla} T^\dagger \right) \),
3. \( \frac{e^2}{2c^2} \text{Tr} \left( G \tilde{T} (\vec{\delta} \cdot \vec{A})^2 T^\dagger \right) \),
4. \( \frac{e^2}{2c^2} \text{Tr} \left( G \tilde{T} \vec{\delta} \cdot \vec{A} T^\dagger G \tilde{T} \vec{\delta} \cdot \vec{A} T^\dagger \right) \),
5. \( i\frac{e}{c} \text{Tr} \left( G t \vec{\delta} \cdot T \vec{\nabla} T^\dagger G \tilde{T} \vec{\delta} \cdot \vec{A} T^\dagger \right) \).

The first term is zero and using:

1. \( Q = \tilde{T}^\dagger \Sigma s_3 T \),
2. \( g = \frac{1}{2} (G^+ + G^-) \),
3. \( \frac{\Sigma}{E^2 + \Sigma^2} = \frac{1}{2t} (G^+ - G^-) \),
vi) Eq. (S3), v) the $d$-wave symmetry, implying that odd terms in $\Delta$ are zero under momentum integration and finally vi) the relation $(t_1\Delta_1 + \Delta_2 t_2)^2 = (\Delta_1^2 + \Delta_2^2)(t_1^2 + t_2^2)$, due to $t_1\Delta_2 = t_2\Delta_1$, we obtain, summing all terms and multiplying them for $-\frac{1}{2}$ (the coefficient in front of $Tr \ln(\varepsilon - H^0 + iQ)$), the following additional contribution, depending on the vector potential, in the action

$$S(A) = \frac{\pi}{32\Sigma^2} \sigma_c Tr \left[ \left( \nabla Q + i e c [Q, A] \right) \left( \nabla Q^\dagger - i e c [A, Q^\dagger] \right) - (\nabla Q \nabla Q^\dagger) \right],$$

with

$$\sigma_c = \sigma - \frac{\Sigma^2}{\pi V} \sum_k \left[ \frac{(\nabla_k \Delta_k)^2}{(E^2 + \Sigma^2)^2} \right], \quad (C5)$$

where $\sigma$ is given by Eq. (S3).

a. Bare charge conductivity

Let us suppose to have $A = A^0 s_0 + A^1 s_1$, to recover the Kubo formula we have to evaluate

$$\left. \left( \frac{\partial^2 \ln Z}{\partial A^\alpha} \right) \right|_{A=0} - \left. \left( \frac{\partial^2 \ln Z}{\partial A^\alpha} \right) \right|_{A=0}^2 - \left. \left( \frac{\partial^2 S(A)}{\partial A^\alpha} \right) \right|_{A=0} + \left. \left( \frac{\partial S(A)}{\partial A^\alpha} \right) \right|_{A=0}^2 \right|_0, \quad (C6)$$

with

$$Z(A) = \int DQ e^{-S_0 - S(A)}.$$

Since $S(A = 0) = 0$, the generating function at zero vector potential is again $Z(A = 0) = Z_0 = \int DQ e^{-S_0}$ and, introducing the notation $\langle ... \rangle_0$ to denote the quantum average with weight $e^{-S_0}$, we have

$$\left. \frac{\partial^2 \ln Z}{\partial A^\alpha} \right|_{A=0} = - \left. \left( \frac{\partial S(A)}{\partial A^\alpha} \right) \right|_{A=0}^2 - \left. \left( \frac{\partial^2 S(A)}{\partial A^\alpha} \right) \right|_{A=0} + \left. \left( \frac{\partial S(A)}{\partial A^\alpha} \right) \right|_{A=0}^2 \right|_0. \quad (C7)$$

The first term is zero, the second is the average of the operator

$$\left. \frac{\partial^2 S(A)}{\partial A^\alpha} \right|_{A=0} = \frac{e^2 \pi}{16 c^2 \Sigma^2} \sigma_c Tr \left( [Q(R), \tau_3 s_\alpha] [\tau_3 s_\alpha, Q(R)^\dagger] \right), \quad (C8)$$

while the third is the average of the square value of the following operator

$$\left. \frac{\partial S(A)}{\partial A^\alpha} \right|_{A=0} = i e c \left( \frac{\sigma_c \pi}{8 \Sigma^2} Tr (\nabla Q(R) Q(R)^\dagger \tau_3 s_\alpha) \right). \quad (C9)$$

At the saddle point,

$$Q(R) = Q_{sp} = \Sigma s_3,$$

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Eq. (C9) is zero and the action is simply
\[ S(A) = \frac{\pi e^2}{8c^2} \left( \sigma_c Tr(A^1) \right). \]
The bare conductivity is given by applying Eq. (C6) which yields
\[ -\frac{\partial^2 S(A)}{\partial A^0^2} \bigg|_{A=0} + \frac{\partial^2 S(A)}{\partial A^{12}^2} \bigg|_{A=0} = \frac{8\pi e^2}{c^2} \sigma_c, \]
where \( \sigma_c \) is given by Eq. (C5), more explicitly, using Eq. (83),
\[ \sigma_c = \frac{\Sigma^2}{\pi V} \sum_k \left[ \frac{(\nabla_k e_k)^2}{(E^2 + \Sigma^2)^2} \right] \sim \frac{1}{\pi^2} \frac{v_1}{v_2}. \]
At the Born level the charge conductivity that we found is in perfect agreement with the diagrammatic approach [36].

2. Spin vector potential

Let us now introduce a vector potential of this kind \( \vec{A} \propto \tau_0 \sigma_z \). The fields change as follows
\[ c_{R\uparrow} \rightarrow e^{i\frac{1}{2} \vec{A} \cdot \vec{R}} c_{R\uparrow}, \quad (C12) \]
\[ c_{R\downarrow} \rightarrow e^{-i\frac{1}{2} \vec{A} \cdot \vec{R}} c_{R\downarrow}, \quad (C13) \]
The parameters of the Hamiltonian becomes
\[ t_{RR'} \rightarrow t_{RR'} e^{-i\frac{1}{2} \vec{A} \cdot (\vec{R}' - \vec{R})} \simeq t_{RR'} \left( 1 - i\frac{1}{2} \vec{\delta} \cdot \vec{A} - \frac{1}{4} (\vec{\delta} \cdot \vec{A})^2 \right), \]
\[ \Delta_{RR'} \rightarrow \Delta_{RR'} \frac{1}{2} \left( 1 + e^{-i \vec{\delta} \cdot \vec{A} \cdot (\vec{R}' - \vec{R})} \right) \simeq \Delta_{RR'} \left( 1 - i\frac{1}{2} \vec{\delta} \cdot \vec{A} - \frac{1}{4} (\vec{\delta} \cdot \vec{A})^2 \right). \]
Although the parameters change differently, the two expansions to second order in \( A \) are equal and the additional term to the action due to the spin vector potential is
\[ S(A) = \frac{\pi \sigma_s}{32\Sigma^2} Tr \left[ \left( \nabla Q + i\frac{1}{2} [Q, A] \right) \left( \nabla Q^\dagger - i\frac{1}{2} [A, Q^\dagger] \right) - (\nabla Q \nabla Q^\dagger) \right], \]
where now the bare spin conductivity is \( \sigma_s = \sigma \), exactly the stiffness of spin fluctuations, given by Eq. (83).
In the presence of chiral symmetry the gaussian propagator becomes
\[
\langle W^{ab}_{p\delta_{in}}(k) W^{cd}_{p\delta_{ir}}(-k) \rangle = \frac{1}{2} \left( 1 - (-)^p \lambda_n \lambda_m \right)
\]
\[
\left[ (-)^p(\pm) D^{p}_{nm}(k) \left( \tilde{\delta}^{ac}_{m q} \delta^{bd}_{n r} (-)^p [\pm] \delta^{ad}_{p s} \delta^{bc}_{m q} (-)^i \tilde{\delta}^{ac}_{n r} \delta^{bd}_{m q} (-)^i \right) \right]_{\pm}
\]
\[
\Pi_{nr}(k) \delta_{p3} \delta_{nm} \delta_{nm} \delta_{nm} \delta_{nm},
\]
where \( p = 0, 3 \), depending on the \( \gamma \)-components, \( S = S, T \) for singlet or triplet component, \((\pm)\) are related to real or imaginary matrix elements of \( W_0 \) (the \( \gamma_0 \)-component), listed in Section XI A \([\pm]\) for symmetric or antisymmetric matrix elements (always referring to \( W_0 \), the sign \((-)^p\) takes care of the sign differences, when they occur, between the two components in sublattice space, \( W_0 \) and \( W_3 \), \((-)^i\) the sign that \( W \) acquires changing the signs of Matsubara frequencies (this occurs only for modes proportional to \( \tau_1 \) and \( \tau_3 \)), and finally

\[
D^0_{nm}(k) = \frac{1}{4\pi \nu \Delta k^2 + |\epsilon_n - \epsilon_m|}, \quad \text{with} \quad \lambda_n = -\lambda_m, \quad \text{(D2)}
\]
\[
D^3_{nm}(k) = \frac{1}{4\pi \nu \Delta k^2 + |\epsilon_n + \epsilon_m|}, \quad \text{with} \quad \lambda_n = \lambda_m, \quad \text{(D3)}
\]
\[
\Pi_{n,r}(k) = D^3_{nn}(k) \frac{\Pi k^2}{2\nu (\Delta k^2 + 2|\epsilon_r|)}, \quad \text{in the 0 replica limit.} \quad \text{(D4)}
\]

Using the charge conjugation of the unitary transformation \( U \)
\[
\tau_y \sigma_y U^t \tau_x \sigma_y = \tilde{U}^t, \quad \text{(D5)}
\]
we can derive the following relation
\[
Tr \left( U^{ab}_{m_1 m_2 n_1 n_2 \tau y \sigma y} \tilde{U}^{\tau z c d}_{m_3 m_4 n_3 n_4 \tau y \sigma y} \right) = \eta [\pm] Tr \left( U^{dc}_{m_4 m_3 \tau y \sigma y} \tilde{U}^{\tau z b a}_{m_2 m_1 \tau y \sigma y} \right), \quad \text{(D6)}
\]
with
\[
\eta = \begin{cases} 
-(-)^l & \text{in p-h singlet channel, } (l = 0, 3, \sigma = \sigma_0), \\
(-)^l & \text{in p-h triplet channel, } (l = 0, 3, \sigma = \tilde{\sigma}), \\
& \text{in p-p Cooper channel, } (l = 1, 2, \sigma = \sigma_0),
\end{cases}
\]
where \([\pm]\) refers to the symmetric or antisymmetric \( \tau_i-\sigma_j \)-component of \( W \), with \( i = 0, 1, 2, 3 \), \( j = 0, x, y, z \) (the same sign, \([\pm]\), which appear in the corresponding gaussian propagator).
For the following derivative operator $A = \nabla \tilde{U} \tilde{U}^\dagger$, whose charge conjugation condition is

$$\tau_1 \sigma_y A^\dagger \tau_1 \sigma_y = -\gamma_1 A \gamma_1,$$

where $\gamma_1$ is the first Pauli matrix on sublattice space, we can derive the following property, under the trace,

$$Tr (\tau_i \sigma_j \gamma_q \tau_i' \sigma_j' \gamma_p A_{ab}^{nm}) = -(-)^p(-)^q[\pm]_{ij}[\pm]_{i'j'} Tr (A_{ba}^{mn} \tau_i' \sigma_j' \gamma_p \tau_i \sigma_j \gamma_q),$$

useful to evaluate $\langle S_2^1 S_1^1 \rangle$ and $\frac{1}{2} \langle S_2 S_2 S_1^1 \rangle$, where $\gamma_q$ and $\gamma_p$ can be $\gamma_0$ and $\gamma_3$, matrices in sublattice space, while $\sigma_j$ and $\sigma_j'$ are identities or Pauli matrices in spin space.

Defining the quaternions $\bar{\tau}_i = \tau_0, i \tau_1, i \tau_2, i \tau_3$ and $\bar{\sigma}_j = \sigma_0, i \sigma_x, i \sigma_y, i \sigma_z$, we have also this sum rule

$$\sum_{i,j} (\pm)_{ij} [\pm]_{ij} Tr (M \bar{\tau}_i \bar{\sigma}_j) Tr (N \bar{\tau}_i \bar{\sigma}_j) = -4 Tr (MN).$$

Applying the relations seen above, we can obtain, for instance, the generic expression for the mean value of $S_1^1$ in the presence of chiral symmetry

$$\langle S_1^1 \rangle = -\frac{\pi^2 \nu}{8} \sum \nu \Gamma D_{m_1 m_2}^{k_1} (1 - (-)^{k_1} \lambda_{m_1} \lambda_{m_2}) \delta (n_1 \mp n_2 \pm n_3 - n_4)$$

$$\left\{Tr \left( \lambda_{m_2} tr \left( U_{m_2 n_2} \tau_1 \sigma U_{n_1 m_1}^{\dagger} \gamma_k \gamma_q \right) \lambda_{m_1} tr \left( U_{m_1 n_4} \tau_1 \sigma U_{n_3 m_2}^{\dagger} \gamma_k \gamma_q \right) \right)$$

$$- (-)^l Tr \left( \lambda_{m_2} tr \left( U_{m_2 n_2} \tau_1 \sigma U_{n_1 m_1}^{\dagger} \gamma_k \gamma_q \right) \lambda_{m_1} tr \left( U_{m_1 n_4} \tau_1 \sigma U_{n_3 m_2}^{\dagger} \gamma_k \gamma_p \right) \right) \right\},$$

where we have dropped, for simplicity, the momentum dependences and where $Tr$ means trace over all degrees of freedom except in sublattice space over which we shall trace by $tr$.

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