Poisson’s Equation in Nonlinear Filtering

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Abstract—The goal of this paper is to gain insight into the equations arising in nonlinear filtering, as well as into the feedback particle filter introduced in recent research. The analysis is inspired by the optimal transportation literature and by prior work on variational formulation of nonlinear filtering. The construction involves a discrete-time recursion based on the successive solution of minimization problems involving the so-called forward variational representation of the elementary Bayes’ formula. The construction shows that the dynamics of the nonlinear filter may be regarded as a gradient flow, or a steepest descent, for a certain energy functional with respect to the Kullback-Leibler divergence pseudo-metric. The feedback particle filter algorithm is obtained using similar analysis. This filter is a controlled system, where the control is obtained via consideration of the first order optimality conditions for the variational problem. The filter is shown to be exact, i.e., the posterior distribution of the particle matches exactly the true posterior, provided the filter is initialized with the true prior.

I. INTRODUCTION

The aim of this paper is to describe an application of optimal transportation theory to nonlinear filtering. The contributions of this paper are two-fold: One, to show that the dynamics of the K-S equation are a gradient flow for a certain variational problem, with respect to the Kullback-Leibler divergence. Two, the variational problem is used to derive the feedback particle filter, first introduced in [8] (see also [7], [6], [5]).

To expose the main ideas, it is useful to restrict our attention to the following special case in which the state is constant:

\[
\begin{align*}
\text{d}X_t &= 0, \\
\text{d}Z_t &= h(X_t) \text{d}t + \text{d}W_t,
\end{align*}
\]

(1a)

(1b)

where \(X_t \in \mathbb{R}^d\) is the state at time \(t\), \(Z_t \in \mathbb{R}^1\) is the observation process, \(h(\cdot)\) is a \(C^1\) function, and \(\{W_t\}\) is a standard Wiener process. The state is constant, and has initial condition distributed as \(X_0 \sim p_0\).

The objective of the filtering problem is to estimate the posterior distribution of \(X_t\) given the history \(\mathcal{Z}_t := \sigma(Z_s : s \leq t)\). The posterior is denoted by \(p^*\), so that for any measurable set \(A \subseteq \mathbb{R}^d\),

\[
\int_A p^*(x, t) \text{d}x = P\{X_t \in A \mid \mathcal{Z}_t\}.
\]

The evolution of \(p^*(x, t)\) is described by the Kushner–Stratonovich (K-S) partial differential equation

\[
dp^* = (h - \hat{h})^T (dZ_t - \text{d}t) p^*,
\]

(2)

with initial condition \(p_0\), where \(\hat{h}_t = \int h(x)p^*(x, t) \text{d}x\). The theory of nonlinear filtering is described in the classic monograph [4].

Although our analysis is restricted to a particular model with a static state process, it can be extended to broader classes of filtering problems, subject to technical conditions discussed in Remark 1. The main technical condition concerns the existence of a solution and certain a priori bounds for a Poisson’s equation. For the model considered in this paper, bounds are obtained based on a Poincaré, or spectral gap, inequality (see the bound \(p_1(\lambda_0)\) in Assumption A2).

The first part of the paper concerns the construction of the gradient flow. A time-stepping procedure is introduced, consisting of successive minimization problems in the space of probability densities. The weak form of the nonlinear filter is derived via analysis of the first-order optimality conditions for these problems. The derivation shows the nonlinear filter dynamics may be regarded as a gradient flow, or a steepest descent, for a certain energy functional with respect to the Kullback-Leibler divergence.

The second part of the paper is concerned with derivation of the feedback particle filter algorithm, based again on the analysis of the first variation. The algorithm is shown to be exact. That is, the posterior distribution of the particle matches exactly the true posterior, provided the filter is initialized with the true prior.

The remainder of this paper is organized as follows. The time-stepping procedure is introduced in Sec. II and properties of its solution established; the gradient flow result – convergence is the solution of the time-stepping procedure to weak solution of the K-S equation (2) – appears in Sec. III; and the feedback particle filter algorithm in Sec. IV.

Notation: \(C^k\) is used to denote the space of \(k\)-times continuously differentiable functions; \(C^k_c\) denotes the subspace of functions with compact support. \(L^1\) is used to denote the space of functions that are bounded a.e. (Lebesgue); \(L^2(\mathbb{R}^d; \rho)\) the Hilbert space of functions on \(\mathbb{R}^d\) that are square-integrable with respect to density \(\rho\); \(H^k(\mathbb{R}^d; \rho)\) denotes the Hilbert space of functions whose first \(k\) derivatives.
\( \text{(defined in the weak or distributional sense) are in } L^2(\mathbb{R}^d;\rho), \text{ and } H^1_0(\mathbb{R}^d;\rho) = \{ \phi \in H^1(\mathbb{R}^d;\rho) \mid \int \phi(x)\rho(x)dx = 0, \} \)

For a function \( f, \nabla f = \frac{\partial f}{\partial x} \) is used to denote the gradient and \( D^2 f = \frac{\partial^2 f}{\partial x^2} \) is used to denote the Hessian. The derivatives are interpreted in the weak sense.

\[ \int \phi(x)\rho(x)dx = -\int (g(x) - \hat{g})\rho(x)dx, \]

where \( \rho > 0 \) is a given density, \( g \) is a given function, and \( \hat{g} = \int g(x)\rho(x)dx. \)

II. TIME-STEPPING PROCEDURE

The time-stepping procedure involves a sequence of minimization problems in the space of probability densities with finite second moment. This space is denoted as \( \mathcal{P} \).

We consider a finite time interval \([0, T] \) with an associated discrete-time sequence \( \{0, t_1, t_2, \ldots, t_N\} \) of sampling instants, with \( 0 \leq t_1 \leq \cdots \leq t_N = T \). The corresponding increments are given by \( \Delta t_n = t_n - t_{n-1}, n = 1, \ldots, N \).

A realization of the stochastic process \( Z_t \), the solution of SDE (1b), sampled at discrete times, is written as \( \{Z_0, Z_1, Z_2, \ldots, Z_N\} \). We use \( \Delta Z_n = Z_n - Z_{n-1} \) to define the discrete-time observation process, and let

\[ Y_n = \frac{\Delta Z_n}{\Delta t_n}. \]

In discrete time, \( Y_n \) is viewed as the observation made at time \( t_n \). We eventually let \( N \to \infty \) and simultaneously let \( \Delta t_n \to 0 \), where

\[ \Delta t_n = \max \{ \Delta t_n : n \leq N \}. \]

The elementary Bayes theorem is used to obtain the posterior distribution, expressed recursively as

\[ \rho_0(x) = \rho_0^*(x), \]

\[ \rho_n(x) = \rho_{n-1}(x) \exp\left( -\phi_n(x) \right) \int \rho_{n-1}(y) \exp\left( -\phi_n(y) \right) dy, \]

where \( \phi_n(x) = \Delta t_n (Y_n - h(x))^2 \). Note that the \( \{\rho_n\} \) are random probability measures since they depend on the discrete-time process \( \{Z_n\} \). In particular, \( \rho_0 \) is measurable w.r.t. \( \sigma(Z_i : i = 0, \ldots, n) \). This observation should be kept in mind when dealing with various parameters associated with the \( \rho_n \), e.g., norm bounds for functions in \( L^p(\mathbb{R}^d;\rho_n) \).

The variational formulation of the Bayes recursion is the following time-stepping procedure: Set \( \rho_0 = \rho_0^* \in \mathcal{P} \) and inductively define \( \{\rho_n\}_{n=1}^N \in \mathcal{P} \) by taking \( \rho_n \in \mathcal{P} \) to minimize the functional

\[ L_n(\rho) \equiv D(\rho \mid \rho_{n-1}) + \frac{\Delta t_n}{2} \int \rho(x)(Y_n - h(x))^2 dx, \]

where \( D \) denotes the relative entropy or Kullback–Leibler divergence,

\[ D(\rho \mid \rho_{n-1}) = \int \rho(x)\ln\left( \frac{\rho(x)}{\rho_{n-1}(x)} \right) dx. \]

The proof that \( \rho_n \), as defined in (5), is in fact the minimizer is straightforward: By Jensen’s formula, \( L_n(\rho) \geq -\int (\rho_{n-1}(y) \exp(-\phi_n(y)) dy \) with equality if and only if \( \rho = \rho_n \). The optimizer \( \rho_n \) is in fact the “twisted distribution” that arises in the theory of large deviations for empirical means [2]. Although the optimizer is known, a careful look at the first order optimality equations associated with \( \rho_n \) leads to i) the nonlinear filter (2) for evolution of the posterior (in Sec. III), and ii) a particle filter algorithm for approximation of the posterior (in Sec. IV).

Throughout the paper, the following assumptions are made for the prior distribution \( \rho_0^* \) and for function \( h \):

**Assumption A1** The probability density \( \rho_0^* \in \mathcal{P} \) is of the form \( \rho_0^*(x) = e^{-\theta(x)} \), where \( \theta(x) \in C^2 \), \( \nabla \theta(x) = O(|x|) \) and \( |\nabla^2 \theta(x)| \to \infty \) as \( |x| \to \infty \), and \( D^2 \theta_0 \in L^\infty \).

**Assumption A2** The function \( h \in C^2 \) with \( h, \nabla h, D^2 h \in L^\infty \).

Under assumption A1, the density \( \rho_0 = \rho_0^* \) is known to admit a spectral gap (or Poincaré inequality) [1]; That is, for some \( \lambda_0 > 0 \), and for all functions \( f \in H^1(\mathbb{R}^d;\rho_0) \) with \( \int f \rho_0 \) \( dx = 0 \),

\[ \int |f(x)|^2 \rho_0(x) dx \leq \frac{1}{\lambda_0} \int |\nabla f(x)|^2 \rho_0(x) dx \quad [\text{PI}(\lambda_0)] \]

The following proposition shows that the minimizers all admit a uniform spectral gap. The proof is omitted on account of space.

**Proposition 1:** Under Assumption (A1)-(A2),

(i) The minimizer \( \rho_n \) is of the form \( \rho_n = e^{-\gamma_n(x)} \), where \( \gamma_n(x) \in C^2 \), \( \nabla \gamma_n(x) = O(|x|) \) and \( |\nabla^2 \gamma_n(x)| \to \infty \) as \( |x| \to \infty \), and \( D^2 \gamma_n \in L^\infty \).

(ii) Suppose \( f \in L^2(\mathbb{R}^d;\rho_{n-1}) \). Then \( f \in L^2(\mathbb{R}^d;\rho_n) \) with

\[ \int \rho_n(x)|f(x)|^2 dx \leq C \exp(\alpha|\Delta t_n|) \int \rho_{n-1}(x)|f(x)|^2 dx, \]

where the constants \( C, \alpha \) are uniformly bounded in \( n \).

(iii) The ratio \( \frac{\rho_n}{\rho_{n-1}} \in H^1(\mathbb{R}^d;\rho_{n-1}) \).

(iv) There exists \( \lambda > 0 \), such that \( \rho_n \) satisfies PI(\lambda) for each \( n \).

The sequence of minimizers \( \{\rho_n\} \) is used to construct, via a piecewise-constant interpolation, a density function \( \rho^{(N)}(x, t) \) for \( t \in [0, T] \): Define \( \rho^{(N)}(x, t) \) by setting \( \rho^{(N)}(x, t_n) = \rho_n(x) \), and taking \( \rho^{(N)}(x, t) \) to be constant on each time interval \([t_{n-1}, t_n]\) for \( n = 1, 2, \ldots, N \).

The following section is concerned with convergence analysis for the limit, as \( \Delta t_n \to 0 \). Before describing the analysis, we present a few preliminaries concerning a certain Poisson’s equation. This equation is fundamental to both the nonlinear filter (in Sec. III) and the particle filter algorithm (in Sec. IV).

**A. Poisson’s Equation**

We are interested in obtaining a solution \( \phi \) of Poisson’s equation,

\[ \nabla \cdot (\rho(x)\nabla \phi(x)) = -(g(x) - \hat{g})\rho(x), \]


\[ \int \phi(x)\rho(x)dx = 0, \]

where \( \rho > 0 \) is a given density, \( g \) is a given function, and \( \hat{g} = \int g(x)\rho(x)dx. \)
Let $H^1_0(\mathbb{R}^d; \rho) = \{ \phi \in H^1(\mathbb{R}^d; \rho) \mid \int \phi(x) \rho(x) \, dx = 0 \}$. A function $\phi \in H^1_0(\mathbb{R}^d; \rho)$ is said to be a weak solution of Poisson’s equation (8) if
\[
\int \nabla \phi(x) \cdot \nabla \psi(x) \rho(x) \, dx = \int (g(x) - g_0) \psi(x) \rho(x) \, dx,
\]
for all $\psi \in H^1(\mathbb{R}^d, \rho)$.

The existence-uniqueness result for the weak solution of Poisson’s equation is described next; its proof appears in [5].

**Theorem 1.** Suppose $\rho(x) = e^{-\|x\|^2}$ satisfies $\Pi(\lambda)$.

(i) If $g \in L^2(\mathbb{R}^d; \rho)$, then there exists a unique weak solution $\phi \in H^1_0(\mathbb{R}^d; \rho)$ satisfying (9).

(ii) If $g \in H^1(\mathbb{R}^d; \rho)$ and $D^2 g \in L^\infty$, then the weak solution has higher regularity: $\phi \in H^2(\mathbb{R}^d, \rho)$ with
\[
\int |(D^2 \phi)(\nabla \phi)| \rho(x) \, dx \leq C(\lambda; \rho) \int |\nabla g|^2 \rho(x) \, dx,
\]
where $C(\lambda; \rho) = \lambda^{-2} \sqrt{\lambda} + \|D^2(g)\|_{L^2}$.

**III. NONLINEAR FILTER**

The analysis proceeds by first obtaining the first variation as described in the following Lemma. The proof is omitted on account of space.

**Lemma 1 (First-order optimality condition):** Consider the minimization problem (6) under Assumptions (A1)-(A2). The minimizer $\rho_n$ satisfies the Euler-Lagrange equation
\[
\int \rho_n \left( - \nabla \nabla \cdot \zeta + \nabla \cdot \zeta - (\Delta \rho - h \rho) \nabla \cdot \zeta \right) \, dx = 0
\]
for each vector field $\zeta \in L^2(\mathbb{R}^d; \rho_{n-1})$.

We are now prepared to state the main theorem concerning the limit of the sequence of densities $\{\rho^{(N)}(x,t)\}$. For the purpose of the proof, an alternate form of the E-L equation is more useful. For a given function $g \in L^2(\mathbb{R}^d; \rho_{n-1})$, let $\zeta \in L^2(\mathbb{R}^d; \rho_{n-1})$ denote the weak solution of
\[
\nabla \cdot (\rho_{n-1}(x) \zeta(x)) = - (g(x) - \int \rho_{n-1}(x) g(x) \, dx) \rho_{n-1}(x).
\]
Such a solution exists by Theorem 1 (i). The E-L equation (12) can then be expressed as
\[
\int \rho_n(g(x) \, dx = \int \rho_{n-1}(x) g(x) \, dx + \int \rho_n(x) [\Delta \rho - h \rho] \nabla h(x) \cdot \zeta(x) \, dx.
\]
Let us suppose now $\Delta \rho \to 0$ uniformly, so that $\Delta \rho \to 0$ as $N \to \infty$, where the maximum step size $\Delta \rho$ was introduced in (3). It then follows that $\rho^{(N)}(x,t) \to \rho(x,t)$ a.s. for all $t \in [0,T]$. Such a (sub-sequential) limit exists because of Prop. 1. In fact for the special case of the signal process (1a) considered in this paper, the limiting density is given by the following explicit formula:
\[
\rho(x,t) = (\text{const.}) \exp \left( \frac{1}{2} (h(x))^2 \right) \rho_0(x).
\]

The proof of the following theorem appears in Appendix V-A. Notationally, $\langle f, \rho_t \rangle = \int f(x) \rho(x,t) \, dx$ and $\hat{h}_t = \int h(x) \rho(x,t) \, dx$.

**Theorem 2:** The density $\rho$ is a weak solution of the nonlinear filter with prior $\rho_0 = \rho_0^0$. That is, for any test function $f \in C_c(\mathbb{R}^d)$,
\[
\langle f, \rho_t \rangle = \langle f, \rho_0 \rangle + \int_0^t \langle (h - \hat{h}_s)(dZ_s - \hat{h}_s \, ds), f \rangle.
\]

**Remark 1:** The considerations of this section highlight the variational underpinnings of the nonlinear filter for the special case, $dX_t = 0$.

For a general class of diffusions, the time-stepping procedure is modified as follows: Set $\rho_0 = \rho_0^0$ and inductively define $\{\rho_n\}_{n=1}^N \subset \mathcal{P}$ by taking $\rho_n \in \mathcal{P}$ to minimize the functional (6),
\[
I_n(\rho) = D(\rho \mid \mathcal{P}[\rho_{n-1}]) + \frac{\Delta t}{2} \int \rho(x)(h(x) - Y(x))^2 \, dx,
\]
where $\mathcal{P}[\rho_{n-1}]$ is the “push-forward” from time $t_{n-1}$ to $t_n$, i.e., $\mathcal{P}[\rho_{n-1}]$ is the probability density of $X_n$, given $\rho_n$ as the (initial) density of $X_{n-1}$. For the special case considered in this section, $\mathcal{P}[\rho_{n-1}] = \rho_{n-1}$.

The proof procedure is easily modified to derive the counterpart of the E-L equation (12) and the nonlinear filter (16) for a general class of diffusions. The hard part is to establish, in an a priori manner, the spectral bound $\Pi(\lambda)$ in Prop. 1. Derivation of the spectral bound for the general case will be a subject of future work. Note that the bound is needed to obtain a unique solution of the Poisson equation.

The following section shows that both the variational analysis and the Poisson equation are also central to construction of a particle filter algorithm in continuous time.

**IV. FEEDBACK PARTICLE FILTER**

The objective of this section is to employ the time-stepping procedure to construct a particle filter algorithm.

A particle filter is comprised of $N$ stochastic processes $\{X^i_t \mid i \leq N\}$: The value $X^i_t \in \mathbb{R}^d$ is the state for the $i$th particle at time $t$. For each time $t$, the empirical distribution formed by the “particle population” is used to approximate the posterior distribution. This is defined for any measurable set $A \subset \mathbb{R}^d$ by
\[
\rho^{(N)}(A,t) = \frac{1}{N} \sum_{i=1}^N 1\{ X^i_t \in A \}.
\]

The model for the particle filter is assumed here to be a controlled system,
\[
dX^i_t = a(X^i_t, t) \, dt + K(X^i_t, t) \, dZ_t,
\]
where $a(X^i_t, t)$ and $K(X^i_t, t)$ represent the drift and diffusion coefficients, respectively.

The objective is to develop an algorithm that can efficiently generate samples from $\rho^{(N)}(A,t)$.

The algorithm proceeds as follows:

1. **Initialization:** At time $t=0$, each particle is initialized according to the prior distribution $\rho_0(X_0)$.
2. **Prediction:** At each time step $t$, the state of each particle is updated according to the system dynamics $X^i_{t+\Delta t} = a(X^i_t, t) \Delta t + K(X^i_t, t) \Delta Z_t$.
3. **Importance Sampling:** The weight of each particle is updated according to the likelihood $w^i_{t+\Delta t} = \frac{1}{N} \int \rho^{(N)}(X^i_{t+\Delta t}, t+\Delta t) \, dx$
4. **Resampling:** If the variance of the particle weights exceeds a predetermined threshold, the particles are resampled according to their weights.

The algorithm is iterated for each time step $t \\in [0,T]$. The final state of each particle approximates the posterior distribution $\rho^{(N)}(X,T)$.

**V. CONCLUSION**

The results presented in this paper highlight the importance of variational techniques in the construction of particle filters. The nonlinear filter considered is a special case of the general class of diffusions, and the spectral bound $\Pi(\lambda)$ is a key component in establishing the uniqueness of the solution.

The algorithm developed can be used to approximate the posterior distribution of a stochastic system, which is a critical component in many applications, such as sensor fusion and robotics.

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where the functions $K(x,t), u(x,t)$ are $\mathbb{R}^d$-valued. It is assumed that the initial conditions $\{X_i^j\}_{i=1}^N$ are i.i.d., independent of $\{X_t, Z_t\}$, and drawn from the initial distribution $\rho^*(x_0) \equiv \rho_0(x)$ of $X_0$.

We impose the following admissibility requirements on the control input $U^i_t$ in (17):

**Definition 1 (Admissible Input):** The control input $U^i_t$ is admissible if the following conditions are met: (i) The random variables $u(x,t)$ and $K(x,t)$ are $\mathcal{F}_t = \sigma(Z_s : s \leq t)$ measurable for each $t$. (ii) For each $t$, $E[|u|^2] \leq E[\sum_i |u_i(X^i_t, t)|^2] < \infty$ and $E[|K|^2] \leq E[\sum_i |K_i(X^i, t)|^2] < \infty$.

There are two types of conditional distributions of interest in our analysis:

(i) $p(x,t)$ is the conditional distribution of $X^i_t$ given $\mathcal{F}_t$.

(ii) $p^*(x,t)$ is the conditional distribution of $X^i$ given $\mathcal{F}_t$.

The functions $\{u(x,t), K(x,t)\}$ are said to be optimal if $p \equiv p^*$. That is, given $p^*(\cdot,0) = p(\cdot,0)$, our goal is to choose $\{u,K\}$ in the feedback particle filter so that the evolution equations of these conditional distributions coincide.

The optimal functions are obtained from the time-stepping procedure introduced in Sec. II. Recall that at step $n$ of the procedure, the distribution $p_n$ is obtained upon minimizing the functional (6), repeated below:

$$I_n(p) = D(p|p_{n-1}) + \frac{\Delta n}{2} \int p(x)(Y_n - h(x))^2 \, dx.$$  

The optimizer has an explicit representation given in (5). The key is to construct a diffeomorphism $\phi = s_n(x)$ such that $p = s_n^\#(p_{n-1})$, where $s_n^\#$ denotes the push-forward operator. The push-forward of a probability density $p$ by a smooth map $s$ is defined through the change-of-variables formula

$$\int g(s(x)) \rho(x) \, dx = \int g(\phi(x)) \rho(x) \, dx,$$

for all continuous and bounded test functions $g$.

The particle filter equations are obtained from the first-order optimality conditions for $s_n$. For this purpose, we look at the cumulative objective function, defined for $N \geq 1$ by

$$J(N)(\xi) = \frac{1}{N} \sum_{n=1}^N \left( I_{n+1}(s_n^\#(p_{n-1})) - \frac{\Delta n}{2} y_n^2 \right),$$  

where $\xi = (s_1, s_2, \ldots, s_N)$ denotes a sequence of diffeomorphisms. The objective is to construct a minimizer, denoted as $\hat{\xi} = (\hat{s}_1, \hat{s}_2, \ldots, \hat{s}_N)$, and consider the limit as $N \to \infty$, $\Delta N \to 0$. Note the sequence $\{p_{n-1}(\cdot)\}_{n=1}^N$ is assumed given (see $s^\#$). Its limit, which we denote as $\rho(x,t)$, see (15), is equal to $p^*(x,t)$, the posterior distribution of $X^i_t$ given $\mathcal{F}_t$, by Theorem 2.

The calculations in Appendix V-B provide the following characterization of the optimal functions $\{u,K\}$:

(i) The function $K$ is a solution to

$$\nabla \cdot (pK) = - (h - \hat{h}) p,$$  

(ii) The function $u$ is obtained as

$$u(x,t) = - \frac{1}{2} \int K(x,t)(h(x) + \hat{h}) + \Omega(x,t),$$  

where $\hat{h} = \int h(x)p(x,t) \, dx$ and $\Omega = (\Omega_1, \Omega_2, \ldots, \Omega_d)$ is a $\mathbb{R}^d$-valued function with

$$\Omega_i(x,t) := \frac{1}{2} \sum_{k=1}^d K_i(x,t) \frac{\partial K_j}{\partial x_k}(x,t).$$

This in particular yields the following feedback particle filter algorithm – obtained upon substituting $\rho$ by $p$, the posterior distribution of $X^i_t$ given $\mathcal{F}_t$:

**Feedback particle filter** (in Stratonovich form) is given by

$$dX^i_t = K(X^i_t, t) \circ dU^i_t,$$  

where

$$dU^i_t = dZ_i - \frac{1}{2}(h(X^i_t) + \hat{h}) \, dt, \quad \hat{h} := E[h(X^i_t)|\mathcal{F}_t].$$

The gain function is expressed as

$$K(x,t) = \nabla \phi(x,t),$$

and it is obtained at each time $t$ as a solution of Poisson’s equation:

$$\nabla \cdot (p(x,t) \nabla \phi(x,t)) = -(h(x) - \hat{h})p(x,t), \quad \int \phi(x,t)p(x,t) \, dx = 0,$$

where $p$ denotes the conditional distribution of $X^i_t$ given $\mathcal{F}_t$.

This algorithm requires approximations in numerical implementation since both the gain $K$ and the conditional mean $\hat{h}$ depend upon the density $p$ to be estimated. This is resolved by replacing $p$ by the empirical distribution (IV) to obtain

$$\hat{h} \approx \frac{1}{N} \sum_{i=1}^N h(X^i_t) =: \hat{h}(\cdot).$$

Likewise, a Galerkin algorithm is used to obtain a finite-dimensional approximation of the gain function $K$; cf., [5].

The following theorem shows that, in absence of these approximations, the feedback particle filter is exact. Its proof appears in [5].

**Theorem 3:** Under Assumptions (A1)-(A2), the feedback particle filter (21) is exact. That is, provided $p(\cdot,0) = p^*(\cdot,0)$, we have for all $t \geq 0$,

$$p(\cdot,t) = p^*(\cdot,t).$$

**Remark 2:** The extension of the feedback particle filter to the general nonlinear filtering problem is straightforward. In particular, consider the filtering problem

$$dX_t = a(X_t) \, dt + dB_t,$$

$$dZ_t = h(X_t) \, dt + dW_t,$$

where $X_t \in \mathbb{R}^d$ is the state at time $t$, $Z_t \in \mathbb{R}$ is the observation, $a(\cdot)$, $h(\cdot)$ are $C^1$ functions, and $\{B_t\}$, $\{W_t\}$ are mutually independent standard Wiener processes.
For the solution to this problem, the feedback particle filter is given by
\[ dX_i^t = a(X_i^t)\, dt + dB_i^t + K(X_i^t, t) \circ dB_i^t, \]
where the formulae for \( K \) and \( \tilde{f} \) are as before. The extension of the Theorem 3 to this more general case requires a well-posedness analysis of the solution of Poisson’s equation. The key is to obtain a priori spectral bounds (see also Remark 1) which will be a subject of future publication. 

V. APPENDIX

The convergence proofs here require bounds in the almost-sure and \( L^2 \) senses.

Recall that we consider a finite time interval \([0, T]\), and for each \( N \) we consider a discrete-time sequence \( \{0, t_1, t_2, \ldots, t_N\} \) with \( 0 \leq t_1 \leq \ldots \leq t_N = T \), and denote \( \Delta_n = t_n - t_{n-1} \). We let \( \Delta N = \max_n \Delta_n \), which is assumed to vanish as \( N \to \infty \).

We use \( C > 0 \) to denote a constant that may depend on \( N \) and on the process path \( \{Z_t\} \), but is uniformly bounded in \( L^2 \). Recall that the densities \( \rho_n, \ldots, \rho_N \) are random objects that depend on the samples \( Z_0, \ldots, Z_N \). In particular, the observation process has continuous sample paths, so there exists such a \( C \) for which \( |Z_t| \leq C \) for all \( t \in [0, T] \).

\[ \text{A. Proof of Theorem } 2 \]

We are given a test function \( f \in C_c \). So, \( f \in L^2(\mathbb{R}^d; \rho_n) \) for all \( n \in \{1, 2, \ldots, N\} \). Furthermore, there exists a uniform bound,
\[ \|f\|_{L^2(\mathbb{R}^d; \rho_n)} < \|f\|_{L^2} < C \quad \forall n. \quad (22) \]
Denote \( \hat{f}_n = f \rho_n(x) f(x) \, dx \).

Let \( \xi_n \in L^2(\mathbb{R}^d; \rho_{n-1}) \) be the weak solution of
\[ \nabla \cdot (\rho_{n-1}(x) \xi_n(x)) = -(f(x) - \hat{f}_{n-1}) \rho_{n-1}(x). \quad (23) \]
Such a solution exists by Theorem 1, and moreover,
\[ \int \rho_{n-1} |\xi_n|^2 \, dx < (\text{const}) \int \rho_{n-1} |f - \hat{f}_{n-1}|^2 \, dx < C, \quad (24) \]
where the (const.) is independent of \( n \) (by Prop. 1 (iv)), and using Prop. 1 (iii),
\[ \int \rho_n(x) |\xi_n(x)|^2 \, dx \leq C \exp(\alpha |\Delta N|) \int \rho_{n-1}(x) |\xi_n(x)|^2 \, dx. \quad (25) \]
Using the E-L equation (14) with \( g = f \) and \( \zeta_n = \xi_n \), for \( n = 1, 2, \ldots, N \):
\[ \hat{f}_n - \hat{f}_{n-1} = \int \rho_n(x) [\Delta Z_n - h(x)\Delta n] \nabla h(x) \cdot \xi_n(x) \, dx, \]
and, upon summing,
\[ \hat{f}_N = \hat{f}_0 + \sum_{n=1}^N \int \rho_n(x) [\Delta Z_n - h(x)\Delta n] \nabla h(x) \cdot \xi_n(x) \, dx. \quad (26) \]
The proof is completed by showing that, as \( \Delta_n \to 0 \), the summation converges to the Ito integral in (16), where the convergence is in \( L_2 \). The details of this technical but straightforward calculation are omitted on account of space.

B. Derivation of the feedback particle filter

We consider the cumulative objective function (18), repeated below:
\[ J^{(N)}(\xi) = N \sum_{n=1}^N \left( I_n(s_n^R(\rho_n^{-1})) - \frac{\Delta n}{2} \frac{dY_n}{\nu} \right), \quad (27) \]
where \( \xi = (s_1, s_2, \ldots, s_N) \) denotes a sequence of diffeomorphisms. The sequence \( \{\rho_n^{-1}(x)\}_{n=1}^N \) is assumed given here (see (5)). The objective is to construct a minimizer, denoted as \( \hat{\chi} = (\hat{\chi}_1, \hat{\chi}_2, \ldots, \hat{\chi}_N) \), and consider the limit as \( N \to \infty, \Delta_n \to 0 \).

The calculations in this section are strictly formal. Generally, the technicalities are downplayed in the interest of succinctly describing the main calculations. The Einstein’s tensor notation is employed for some of the more laborious calculations.

This optimization problem (27) can be considered term-by-term since \( \{\rho_n^{-1}\} \) is fixed for fixed \( N \) and \( \Delta_n \). With these parameters fixed, and attention focused to the \( n \)th summand, we recast the optimization problem as one over \( s_n \) as follows:
\[ I_n(s_n) = -\int \rho_{n-1}(x) \ln(\det(Ds_n(x))) \, dx - \int \rho_{n-1}(s_n(x)) \, dx + \frac{\Delta_n}{2} \int \rho_{n-1}(x)(Y_n - h(s_n(x)))^2 \, dx, \quad (28) \]
where we have used the identity \( \rho_n(s_n(x)) \det(Ds_n(x)) = \rho_{n-1}(x) \). As in the initial problem formulation, the minimizer is denoted as \( \hat{\chi}_n \). The first-order conditions for optimality appears in the following Lemma. Given \( \nu \in C_c(\mathbb{R}^d, \mathbb{R}^d) \), the directional derivative is denoted
\[ \delta I_n(\chi) \cdot \nu = \frac{d}{d\epsilon} I_n(\chi + \epsilon \nu) \bigg|_{\epsilon = 0}. \]

Lemma 2 (First-Order Optimality Conditions): Consider the minimization problem (28) under Assumptions (A1)-(A2). The first-order optimality condition for the minimizer \( \hat{\chi}_n(x) \) is given by
\[ 0 = \delta I_n(\chi) \cdot \nu = \int \rho_{n-1}(x) tr(D\hat{\chi}_n^{-1}(x)D\nu(x)) \, dx + \int \rho_{n-1}(x) \frac{1}{\rho_{n-1}(\hat{\chi}_n(x))} \nabla \rho_{n-1}(\hat{\chi}_n(x)) \cdot \nu(x) \, dx + \int \rho_{n-1}(x) (\Delta Z_n - h(\hat{\chi}_n(x)) \Delta n) \nabla h(\hat{\chi}_n(x)) \cdot \nu(x) \, dx. \quad (29) \]

Proof: The three terms in (29) are obtained by explicitly evaluating the derivatives \( \frac{d}{d\epsilon} I_n(\chi + \epsilon \nu) \), at \( \epsilon = 0 \), for the three terms in (28):
(i) The first term is given by
\[ -\int \rho_{n-1} \left[ \ln(\det(D\hat{\chi}_n)) + \ln(\det(I + \epsilon D\hat{\chi}_n^{-1} D\nu)) \right] \, dx. \]
Therefore, for the first term,
\[
\frac{d}{de} \left[ \cdots \right]_{e=0} = -\int \rho_{n-1} \frac{d}{de} \ln(\det(I + e D\chi_{n}^{-1} Dv)) \left|_{e=0} \right. dx
\]
\[
= -\int \rho_{n-1}(x) \nabla \rho_{n-1}(\chi_{n}(x)) \cdot v(x) dx.
\]

(ii) The second term is obtained by a direct calculation
\[
\frac{d}{de} \left[ \cdots \right]_{e=0} = -\int \rho_{n-1}(x) \nabla \rho_{n-1}(\chi_{n}(x)) \cdot v(x) dx.
\]

(iii) Similarly for the third term,
\[
\frac{d}{de} \left[ \cdots \right]_{e=0} = -\int \rho_{n-1}(x) \nabla \rho_{n-1}(\chi_{n}(x)) \cdot v(x) dx.
\]

Since our interest is in the limit as \( \Delta t \to 0 \) and \( N \to \infty \), we now restrict to diffeomorphisms of the form \( \chi_{n}(x) = x + K(x,n)\Delta t + u(x,n)\Delta t, \) where the appropriate function spaces are: \( K \in H^1(\mathbb{R}^d; \rho_{n-1}) \) and \( u \in H^1(\mathbb{R}^d; \rho_{n-1}). \) Starting from (29),
\[
\delta I_n(\chi_n) \cdot v = E_{\varepsilon}(n) \Delta z_t + E_{\Delta}(n) \Delta t_n + O(\Delta n^2),
\]
where, denoting \( K(x,n) = (K_1(x,n), \ldots, K_d(x,n)), u(x,n) = (u_1(x,n), \ldots, u_d(x,n)) \) and expressing \( v(x) = v(x,n) = (v_1(x), \ldots, v_d(x)) \), the following equations give expressions for \( E_{\varepsilon} \) and \( E_{\Delta} \) (expressed using Einstein’s tensor notation):
\[
E_{\varepsilon} = -\int \rho_{n-1} \frac{\partial}{\partial x_i} \left( \frac{1}{\rho_{n-1}} \frac{\partial}{\partial x_j} (\rho_{n-1} K_j) \right) v_i dx
\]
\[
= -\int \rho_{n-1} \frac{\partial}{\partial x_i} v_i dx,
\]
(31)
\[
E_{\Delta} = \frac{1}{2} \int \rho_{n-1} \frac{\partial^2 \ln \rho_{n-1}}{\partial x_i \partial x_j} K_j K_i v_i dx + \int \frac{\partial}{\partial x_j} \left( \rho_{n-1} \frac{\partial K_j}{\partial x_k} \frac{\partial K_k}{\partial x_i} \right) v_i dx.
\]
(32)

We now return to the objective function \( J^{(N)}(s) \) defined in (27). For any fixed \( N \), the first order optimality condition for the minimizer \( \chi = (\chi_1, \chi_2, \ldots, \chi_N) \) is now immediate:
\[
0 = \delta J^{(N)}(\chi) \cdot v = \sum_{n=1}^{N} E_{\varepsilon}(n) \Delta z_t + E_{\Delta}(n) \Delta t_n
\]
\[
+ \sum_{n=1}^{N} \left[ O(\Delta t^2), O(\Delta z^2), O(\Delta n^2) \right],
\]
(33)
where \( v(x) = (v(x,1), \ldots, v(x,N)) \) and \( v(:,n) \in C^1(\mathbb{R}, \mathbb{R}^d) \) is an arbitrary perturbation. Recall now, \( \chi_n(x) = x + K(x,n)\Delta t_n + u(x,n)\Delta t_n. \) The sequence \( \rho_{n}, \{ K(x,n) \}, \{ u(x,n) \} \) and \( \{ v(x,n) \} \) are used to construct, via interpolation, \( \rho^{(N)}(x,t), K^{(N)}(x,t), u^{(N)}(x,t) \) and \( v^{(N)}(x,t) \), respectively. Recall \( \rho^{(N)} \to \rho(x,t) \), given in (15). Likewise we formally denote the limit of \( K^{(N)}(x,t), u^{(N)}(x,t) \) and \( v^{(N)}(x,t) \) as \( K(x,t), u(x,t) \) and \( v(x,t) \), respectively.

With this notation the right-hand-side of (33), as \( N \to \infty \), is expressed as an Itô integral,
\[
\int_0^T \int \rho(x,s) \left( \frac{\partial}{\partial x_i} \left( \frac{1}{\rho} \frac{\partial}{\partial x_j} (\rho K_j) \right) + \frac{\partial h}{\partial x_i} \right) v(x,s) dx ds
\]
\[
- \int_0^T \int \rho(x,s) \left( \frac{1}{\rho} \frac{\partial}{\partial x_j} (\rho u_j) - h \frac{\partial h}{\partial x_j} \right) + \frac{\partial^2 h}{\partial x_i \partial x_j} K_j K_i
\]
\[
+ \frac{1}{2} \frac{\partial^3 \ln \rho}{\partial x_i \partial x_j \partial x_k} K_j K_k - \frac{1}{\rho} \frac{\partial}{\partial x_j} \left( \rho K_j \frac{\partial K_k}{\partial x_i} \right) v(x,s) dx ds.
\]

Since \( \delta J^{(N)}(\chi) \cdot v = 0 \) by optimality, and \( v \) is arbitrary, we obtain weak-sense differential equations for \( K \) and \( u \). The following two equations follow, also defined in the weak sense:
\[
\frac{\partial}{\partial x_i} \left( \frac{1}{\rho} \frac{\partial}{\partial x_j} (\rho K_j) \right) = -\frac{\partial h}{\partial x_i},
\]
(34)
\[
\frac{\partial}{\partial x_i} \left( \frac{1}{\rho} \frac{\partial}{\partial x_j} (\rho u_j) \right) = h \frac{\partial h}{\partial x_i} - \frac{\partial^2 h}{\partial x_i \partial x_j} K_j - \frac{1}{2} \frac{\partial^3 \ln \rho}{\partial x_i \partial x_j \partial x_k} K_j K_k
\]
\[
+ \frac{1}{\rho} \frac{\partial}{\partial x_j} \left( \rho K_j \frac{\partial K_k}{\partial x_i} \right) v(x,s) dx ds.
\]
(35)

The BVP (19) is obtained by integrating (34) once:
\[
\frac{\partial}{\partial x_j} (\rho K_j) = (h - \hat{h}) \rho,
\]
where \( \hat{h} = \int h(x) \rho(x) dx. \) Using this the right-hand-side of (35) is simplified, and the resulting equation is given by
\[
\frac{\partial}{\partial x_j} \left( \frac{1}{\rho} \frac{\partial}{\partial x_i} (\rho u_j) \right) = \frac{\partial h}{\partial x_i} + \frac{1}{2} \frac{\partial}{\partial x_j} \left( \rho K_j \frac{\partial K_k}{\partial x_i} \right) v(x,s) dx ds.
\]
(36)

It is readily verified, by direct substitution, that (36) admits a closed-form solution:
\[
u_j = -K_j \frac{(h + \hat{h})}{2} + \frac{1}{2} \frac{\partial K_j}{\partial x_k} v(x,s) dx ds.
\]
This gives (20).

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