Abstract. We study the classification of smooth toroidal compactifications of nonuniform ball quotients in the sense of Kodaira and Enriques. Moreover, several results concerning the Riemannian and complex algebraic geometry of these spaces are given. In particular we show that there are compact complex surfaces which admit Riemannian metrics of nonpositive curvature, but which do not admit Kähler metrics of nonpositive curvature. An infinite class of such examples arise as smooth toroidal compactifications of ball quotients.

1. Introduction

Let $\tilde{M}$ be a symmetric space of noncompact type, and let $\text{Iso}_0(\tilde{M})$ denote the connected component of the isometry group of $\tilde{M}$ containing the identity. Recall that $\text{Iso}_0(\tilde{M})$ is a semi-simple Lie group. A discrete subgroup $\Gamma \subset \text{Iso}_0(\tilde{M})$ is a lattice in $\tilde{M}$ if $\tilde{M}/\Gamma$ is of finite volume. When $\Gamma$ is torsion free, then $\tilde{M}/\Gamma$ is a finite volume manifold or a locally symmetric space. A lattice $\Gamma$ is uniform (nonuniform) if $\tilde{M}/\Gamma$ is compact (noncompact).

The theory of compactifications of locally symmetric spaces or varieties has been extensively studied, see for example [4]. In fact, locally symmetric varieties of noncompact type often occur as moduli spaces in algebraic geometry and number theory, see [1]. For technical reasons this beautiful theory is mainly developed for quotients of symmetric spaces or varieties by arithmetic subgroups. For arithmetic subgroups of semi-simple Lie groups a nice reduction theory is available [4]. Among many other things, the aforementioned theory can be used to deduce their finite generation, the existence of finitely many conjugacy classes of maximal parabolic subgroups, and the existence of neat subgroups of finite index.

The celebrated work of Margulis [19] implies that lattices in any semi-simple Lie group of real rank bigger or equal than two are arithmetic subgroups. This important theorem does not cover many interesting cases such as lattices in the complex hyperbolic space $\mathbb{C}H^n$, where non-arithmetic lattices are known to exist by the work of Mostow and Mostow-Deligne; see [6] and the bibliography therein.

It is thus desirable to develop a theory of compactifications of locally symmetric varieties modeled on $\mathbb{C}H^n$ regardless of the arithmeticity of the defining torsion free lattices. A compactification of finite-volume complex-hyperbolic manifolds as a complex spaces with isolated normal singularities was obtained by Siu and Yau in [24]. This compactification may be regarded as a generalization of the Baily-Borel compactification defined for arithmetic lattices in $\mathbb{C}H^n$. A toroidal compactification for finite-volume complex-hyperbolic manifolds was described by Hummel and...
2. Toroidal Compactifications and the Kodaira-Enriques Classification

Let $PU(1, 2)$ denote the connected component of $\text{Iso}(CH^2)$ containing the identity. Let $\Gamma$ be a nonuniform torsion-free lattice of holomorphic isometries of the complex hyperbolic plane $CH^2$, i.e., $\Gamma \leq PU(1, 2)$. Recall that the locally symmetric space $CH^2/\Gamma$ has finitely many cusp ends $A_1, ..., A_n$ which are in one to one correspondence with conjugacy classes of the maximal parabolic subgroups of $\Gamma$ [8]. The set of all parabolic elements of $\Gamma$ can be written as a disjoint union of subsets $\Gamma_x$, where $\Gamma_x$ is the set of all parabolic elements in $\Gamma$ having $x$ as unique fixed point. Here $x$ is a point in the natural point set compactification of $CH^2$. 

The constructions of both Siu-Yau and Hummel-Schroeder rely on the theory of nonpositively curved Riemannian manifolds. The key point here is that the structure theorems for finite-volume manifolds of negatively pinched curvature, or more generally for visibility manifolds [7], can be used as a substitute of the reduction theory for arithmetic subgroups.

In this paper we study torsion-free nonuniform lattices in the complex hyperbolic plane $CH^2$ and their toroidal compactifications. Let $\Gamma$ be a lattice as above and let $CH^2/\Gamma$ denote its toroidal compactification. When $CH^2/\Gamma$ is smooth, it is a compact Kähler surface [15]. It is then of interest to place these smooth Kähler surfaces in the framework of the Kodaira-Enriques classification of complex surfaces [3]. The main purpose of this paper is to prove the following:

**Theorem A.** Let $\Gamma$ be a nonuniform torsion-free lattice in $CH^2$. There exists a finite subset $F \subset \Gamma$ of parabolic isometries for which the following holds: for any normal subgroup $\Gamma' \triangleleft \Gamma$ with the property that $F \cap \Gamma'$ is empty, then $CH^2/\Gamma'$ is a surface of general type with ample canonical line bundle. Moreover, $CH^2/\Gamma'$ admits Riemannian metrics of nonpositive sectional curvature but it cannot support Kähler metrics of nonpositive sectional curvature.

An outline of the paper follows. Section II starts with a summary of the results of Hummel and Schroeder [15]. Such results are then combined with the Kodaira-Enriques classification to prove that when the lattice $\Gamma$ is sufficiently small then $CH^2/\Gamma$ is a surface of general type with ample canonical bundle. In section III we present some examples of a surfaces of general type which do not admit any nonpositively curved Kähler metric, but whose underlying smooth manifolds admit Riemannian metrics of nonpositive curvature. Finally the proof of Theorem A is given.

In section IV we show how Theorem A, combined with the theory of semi-stable curves on algebraic surfaces [23], can be used to address the problem of the projective-algebraicity of minimal compactifications (Siu-Yau) of finite-volume complex-hyperbolic surfaces. The results of section IV are then summarized in Theorem B. The result obtained is effective.

The projective-algebraicity of minimal compactifications is proved, through $L^2$-estimates for the $\overline{\partial}$-operator, by Mok in [20]. This analytical approach works in any dimension.
obtained by adjoining points at infinity corresponding to asymptotic geodesic rays. Thus, given a cusp $A_i$, let us consider the associated maximal parabolic subgroup $\Gamma_{x_i} \leq \Gamma$ and the horoball $HB_{x_i}$ stabilized by $\Gamma_{x_i}$. We then have that $HB_{x_i}/\Gamma_{x_i}$ is naturally identified with $A_i$.

Recall that after choosing an Iwasawa decomposition \cite{7} for $PU(1, 2)$, we get a identification of $\partial HB$ with the three dimensional Heisenberg Lie group $N$. Moreover, $N$ comes equipped with a left invariant metric and then we may view $\Gamma_{x_i}$ as a lattice in $Iso(N)$. The cusps $A_1, ..., A_n$ are then identified with $N/\Gamma_{x_i} \times [0, \infty)$, for $i = 1, ..., n$.

The isometry group of $N$ is isomorphic to the semi-direct product $Iso(N) = N \rtimes U(1)$. We say that a lattice in $Iso(N)$ is rotation free if it is a lattice in $N$, i.e., if it is a lattice of left translations. A parabolic isometry $\phi \in \Gamma$ is called unipotent if it acts as a translation on its invariant horospheres.

We now briefly summarize some of the results of Hummel \cite{15} and Hummel-Schroeder \cite{14}.

**Theorem 1** (Hummel-Schroeder). Let $\Gamma$ be a nonuniform torsion-free lattice in $CH^2$. Then, there exists a finite subset $F \subset \Gamma$ of parabolic isometries such that for any normal subgroup $\Gamma' \triangleleft \Gamma$ with the property that $F \cap \Gamma'$ is empty, then $CH^2/\Gamma'$ is smooth and Kähler.

Furthermore, using a cusp closing technique arising from Riemannian Geometry they were able to prove:

**Theorem 2** (Hummel-Schroeder). Let $\Gamma$ be a nonuniform torsion-free lattice in $CH^2$. Then, there exists a finite subset $F' \subset \Gamma$ of parabolic isometries such that $F' \supset F$ for which the following holds. For any normal subgroup $\Gamma' \triangleleft \Gamma$ with the property that $F' \cap \Gamma'$ is empty, then $CH^2/\Gamma'$ admits a Riemannian metric of nonpositive sectional curvature.

A few remarks about these results. A nonuniform torsion-free lattice in $CH^2$ admits a smooth toroidal compactification if its parabolic isometries are all unipotent. In the arithmetic case this is achieved by choosing a neat subgroup of finite index \cite{1}. It is also interesting to observe that we have plenty of normal subgroups satisfying the requirements of Theorem \cite{1} and \cite{2} in fact $PU(1, 2)$ is linear and then residually finite by a fundamental result of Mal’cev \cite{18}. Finally, it is interesting to notice that in general one expects the strict inclusion $F' \supset F$ to hold. Explicit examples can be derived from the construction of Hirzebruch \cite{12}.

For simplicity, a compactification as in Theorem \cite{2} will be referred as toroidal Hummel-Schroeder compactification.

**Proposition 2.1.** Let $M$ be a finite-volume complex-hyperbolic surface which admits a toroidal Hummel-Schroeder compactification. Then the Euler number of $M$ is strictly positive.

**Proof.** The idea for the proof goes back to an unpublished result of J. Milnor about the Euler number of closed four dimensional Riemannian manifolds having sectional curvatures along perpendicular planes of the same sign; see the paper by S. S. Chern \cite{5}. Let $(\overline{M}, \overline{g})$ be the Riemannian manifold obtained by closing the cusps of $M$ under the condition of nonpositive curvature \cite{14}. Let $\Omega$ be its curvature matrix. We can always choose \cite{5} an orthonormal frame $\{e_i\}_{i=1}^4$ such that:

$$R_{1231} = R_{1241} = R_{1232} = R_{1242} = R_{1332} = R_{1341} = 0.$$
It follows that
\[ Pf(\Omega) = \Omega_1^1 \wedge \Omega_3^3 - \Omega_1^3 \wedge \Omega_3^2 + \Omega_2^1 \wedge \Omega_3^2 \]
\[ = \{R_{1221}R_{3443} + R_{1243}^2 + R_{1331}R_{2442} + R_{1342}^2 + R_{1441}R_{2332} + R_{1234}^2 \} d\mu, \]
where \( Pf(\Omega) \) is the Pfaffian of the skew symmetric matrix \( \Omega \). The statement is now a consequence of Chern-Weil theory.

We can now use the Kodaira-Enriques classification of closed smooth surfaces \[3\] to derive the following theorem. The proof is in the spirit of the theory of nonpositively curved spaces.

**Theorem 2.2.** Let \( M \) be a finite-volume complex-hyperbolic surface which admits a toroidal Hummel-Schroeder compactification. Then \( \overline{M} \) is a surface of general type without rational curves.

**Proof.** Since \( \overline{M} \) admits a Riemannian metric of nonpositive sectional curvature, the Cartan-Hadamard theorem \[22\] implies that the universal cover of \( \overline{M} \) is diffeomorphic to the four dimensional euclidean space. Consequently, \( \overline{M} \) is aspherical and then it cannot contain rational curves. Moreover, the second Betti number of \( \overline{M} \) is even since by construction it admits a Kähler metric. By the Kodaira-Enriques classification \[3\] we conclude that the Kodaira dimension of \( \overline{M} \) cannot be negative.

From Proposition \[21\] we know that the Euler number of \( \overline{M} \) is strictly positive. The minimal complex surfaces with Kodaira dimension equal to zero and positive Euler number are simply connected or with finite fundamental group. Since \( \pi_1(\overline{M}) \) is infinite, the Kodaira dimension of \( \overline{M} \) is bigger or equal than one.

The fundamental group of an elliptic surface with positive Euler number is completely understood in terms of the orbifold fundamental group of the base of the elliptic fibration. More precisely, denoting by \( \pi : S \to C \) the elliptic fibration, if \( S \) has no multiple fibers then \( \pi \) induces an isomorphism \( \pi_1(S) \simeq \pi_1(C) \). In the case where we allow multiple fibers we have the isomorphism \( \pi_1(S) \simeq \pi_1^{\text{ orb}}(C) \). For these results we refer to \[9\]. We are now ready to show that \( \overline{M} \) cannot be an elliptic surface. When \( S \) has multiple fibers, the group \( \pi_1(S) \) has always torsion and then it cannot be the fundamental group of a nonpositively curved manifold. If we assume \( \pi_1(\overline{M}) \simeq \pi_1(C) \), the fact that \( \pi_1(\overline{M}) \) grows exponentially \[24\] forces the genus of the Riemann surface \( C \) to be bigger or equal than two. Since all closed geodesics in a manifold of nonpositive curvature are essential in \( \pi_1 \), we have that the fundamental group of the flats introduced in the compactification injects in \( \pi_1(\overline{M}) \) and then by assumption in \( \pi_1(C) \). By elementary hyperbolic geometry this would imply that \( \mathbb{Z} \oplus \mathbb{Z} \) acts as a discrete subgroup of \( \mathbb{R} \), which is clearly impossible.

**Corollary 2.3.** A toroidal Hummel-Schroeder compactification has ample canonical line bundle.

**Proof.** By Theorem \[22\] we know that \( \overline{M} \) is a minimal surface of general type without rational curves. The corollary follows from Nakai’s criterion for ampleness of divisors on surfaces \[3\]. More precisely, since for a minimal surface of general type the self-intersection of the canonical divisor is strictly positive \[3\], it suffices to show that \( K_{\overline{M}} \cdot E > 0 \) for any effective divisor \( E \). Thus, let \( E \) be an irreducible divisor
and assume $K_{\mathbb{C}} \cdot E = 0$. By the Hodge index theorem we must have $E \cdot E < 0$. By the adjunction formula $E$ must be isomorphic to a smooth rational curve with self-intersection $-2$.

In the arithmetic case, part of the results contained in Theorem 2.2 can be derived from a theorem of Tai, see [1]. Furthermore, similar results for the so-called Picard modular surfaces are obtained by Holzapfel in [13].

3. Examples

In this section we present examples of surfaces of general type which do not admit nonpositively curved Kähler metrics, but such that their underlying smooth manifolds do admit Riemannian metrics with nonpositive Riemannian curvature. In order to do this one needs to understand the restrictions imposed by the nonpositive curvature assumption on the holomorphic curvature tensor.

Thus, define

$$p = 2\text{Re}(\xi), \quad q = 2\text{Re}(\eta)$$

where

$$\xi = \xi^\alpha \partial_\alpha, \quad \eta = \eta^\alpha \partial_\alpha.$$ 

In real coordinates we have

$$R(p, q, q, p) = R(h_{ijk}p^h q^i q^j p^k)$$

while in complex terms

$$R(\xi + \bar{\xi}, \eta + \bar{\eta}, \eta + \bar{\eta}, \xi) = R(\xi, \bar{\eta}, \eta, \bar{\eta}) + R(\xi, \bar{\eta}, \bar{\eta}, \xi) + R(\bar{\xi}, \eta, \eta, \bar{\xi}) + R(\bar{\xi}, \bar{\eta}, \eta, \bar{\xi}).$$

We then have

$$R(h_{ijk}p^h q^i q^j p^k) = R_{\alpha \beta \gamma \delta} \xi^\alpha \eta^\beta \xi^\gamma \xi^\delta + R_{\alpha \beta \gamma \delta} \xi^\alpha \eta^\beta \xi^\gamma \xi^\delta + R_{\alpha \beta \gamma \delta} \xi^\alpha \eta^\beta \xi^\gamma \xi^\delta + R_{\alpha \beta \gamma \delta} \xi^\alpha \eta^\beta \xi^\gamma \xi^\delta \leq 0.$$ 

If we assume the Riemannian sectional curvature to be nonpositive we have

$$R(h_{ijk}p^h q^i q^j p^k) = R_{\alpha \beta \gamma \delta} \xi^\alpha \eta^\beta \xi^\gamma \xi^\delta + R_{\alpha \beta \gamma \delta} \xi^\alpha \eta^\beta \xi^\gamma \xi^\delta \leq 0.$$
In complex dimension two, the right hand side of the above equality reduces (after some manipulations) to

\[
R_{\alpha \beta \gamma} (\xi^\alpha \eta^\beta - \eta^\alpha \xi^\beta) (\xi^\delta \eta^\gamma - \eta^\delta \xi^\gamma)
= R_{\alpha \gamma \beta} \eta^\beta (\xi^\alpha \eta^\gamma - \eta^\alpha \xi^\gamma) + 4 \Re \{ R_{\alpha \gamma \beta} (\xi^\alpha \eta^\gamma - \eta^\alpha \xi^\gamma) (\xi^\delta \eta^\gamma - \eta^\delta \xi^\gamma) \}
+ 2 R_{\alpha \gamma \beta} (\xi^\alpha \eta^\gamma - \eta^\alpha \xi^\gamma) (\xi^\delta \eta^\gamma - \eta^\delta \xi^\gamma) + 4 \Re \{ R_{\alpha \gamma \beta} (\xi^\alpha \eta^\gamma - \eta^\alpha \xi^\gamma) (\xi^\delta \eta^\gamma - \eta^\delta \xi^\gamma) \}
+ 2 R_{\alpha \gamma \beta} (\xi^\alpha \eta^\gamma - \eta^\alpha \xi^\gamma) (\xi^\delta \eta^\gamma - \eta^\delta \xi^\gamma) + 4 \Re \{ R_{\alpha \gamma \beta} (\xi^\alpha \eta^\gamma - \eta^\alpha \xi^\gamma) (\xi^\delta \eta^\gamma - \eta^\delta \xi^\gamma) \}
+ 2 R_{\alpha \gamma \beta} (\xi^\alpha \eta^\gamma - \eta^\alpha \xi^\gamma) (\xi^\delta \eta^\gamma - \eta^\delta \xi^\gamma) + 4 \Re \{ R_{\alpha \gamma \beta} (\xi^\alpha \eta^\gamma - \eta^\alpha \xi^\gamma) (\xi^\delta \eta^\gamma - \eta^\delta \xi^\gamma) \}
+ \Re \{ R_{\alpha \gamma \beta} (\xi^\alpha \eta^\gamma - \eta^\alpha \xi^\gamma) (\xi^\delta \eta^\gamma - \eta^\delta \xi^\gamma) \}.
\]

Following Siu-Mostow [21], we choose the ansatz

\[
\xi^1 = ia, \quad \xi^2 = -i, \quad \eta^1 = a, \quad \eta^2 = 1
\]

where \(a\) is a real number. We get the inequality

\[
R_{\alpha \beta \gamma} 4 a^4 - 2 R_{\beta \alpha \gamma} 4 a^2 + R_{\alpha \beta \gamma} 4 \leq 0.
\]

Since nonpositive Riemannian sectional curvature implies nonpositive holomorphic sectional curvature, we conclude that

\[
(R_{\alpha \beta \gamma})^2 \leq R_{\alpha \beta \gamma} R_{\alpha \beta \gamma} R_{\alpha \beta \gamma} R_{\alpha \beta \gamma}.
\]

**Theorem 3.1.** A toroidal Hummel-Schroeder compactification does not admit any Kähler metric with nonpositive Riemannian sectional curvature.

**Proof.** Let us proceed by contradiction. Consider one of the elliptic divisors added in the compactification. By the properties of submanifolds of a Kähler manifold [17], we have that the holomorphic sectional curvature tangent to the elliptic divisor has to be zero. Let us denote such a holomorphic sectional curvature by \(R_{\alpha \beta \gamma}\). By the inequality (1), we conclude that \(R_{\alpha \beta \gamma} = 0\). As a result, the Ricci curvature tangent to the elliptic divisor has to be zero. We conclude that

\[
K_M \cdot \Sigma = \int_\Sigma c_1(K_M) = 0,
\]

which contradicts the ampleness of \(K_M\), see Corollary 2.3.

\(\square\)

Combining Theorems 2.2 and 3.1 with Corollary 2.3 we have thus proved Theorem A.

4. **PROJECTIVE–ALGEBRAICITY OF MINIMAL COMPACTIFICATIONS**

Let \(\overline{M}\) be a smooth toroidal compactification of a finite-volume complex-hyperbolic surface \(M\) and let \(\Sigma\) denote the compactifying divisor. The set \(\Sigma\) is exceptional and it can be blow down. The resulting complex surface, with isolated normal singularities, it is usually referred as the minimal compactification of \(M\) [24]. In this section we address the problem of the projective-algebraicity of minimal compactifications of finite-volume complex-hyperbolic surfaces. This is motivated by a beautiful example of Hironaka, see [11] page 417, which shows that by contracting a smooth elliptic divisor on an algebraic surface one can obtain a nonprojective
complex space. In the arithmetic case, the projective-algebraicity of minimal compactifications of finite-volume complex-hyperbolic surfaces it is known by the work of Baily and Borel, see [4].

For completeness, we recall the theory of semi-stable curves on algebraic surfaces and logarithmic pluricanonical maps as developed by Sakai in [23].

Let $\overline{M}$ be a smooth projective surface. Let $\Sigma$ be a reduced divisor having simple normal crossings on $\overline{M}$.

**Definition 1.** The pair $(\overline{M}, \Sigma)$ is called minimal if $\overline{M}$ does not contain an exceptional curve $E$ of the first kind such that $E \cdot \Sigma \leq 1$.

We consider the logarithmic canonical line bundle $L = K_{\overline{M}} + \Sigma$ associated to $\Sigma$. Given any integer $k$, define $P_m = \dim \mathcal{H}^0(\overline{M}, \mathcal{O}(mL))$. If $P_m > 0$, we define the $m$-th logarithmic canonical map $\Phi_{mL}$ of the pair $(\overline{M}, \Sigma)$ by

$$\Phi_{mL}(x) = [s_1(x), ... , s_N(x)],$$

for any $x \in \overline{M}$ and where $s_1, ..., s_N$ is a basis for the vector space $\mathcal{H}^0(\overline{M}, \mathcal{O}(mL))$.

At this point one introduces the notion of logarithmic Kodaira dimension exactly as in the closed smooth case. We denote this numerical invariant by $\overline{k}(M)$ where $M = \overline{M} \setminus \Sigma$. We refer to [16] for further details.

**Definition 2.** A curve $\Sigma$ is semi-stable if has only normal crossings and each smooth rational component of $\Sigma$ intersects the other components of $\Sigma$ in more than one point.

The following proposition gives a numerical criterion for a minimal semi-stable pair $(\overline{M}, \Sigma)$ to be of log-general type. For the proof we refer to [23].

**Proposition 4.1.** Given a minimal semi-stable pair $(\overline{M}, \Sigma)$ we have that $\overline{k}(M) = 2$ if and only if $L$ is numerically effective and $L^2 > 0$.

We can now state one of the main results contained in [23]. In what follows, we denote by $\mathcal{E}$ the set of irreducible curves $E$ in $\overline{M}$ such that $L \cdot E = 0$.

**Theorem 3** (Sakai). Let $(\overline{M}, \Sigma)$ be a minimal semi-stable pair of log-general type. The map $\Phi_{mL}$ is then an embedding modulo $\mathcal{E}$ for any $m \geq 5$.

It is then necessary to characterize the irreducible divisors in $\mathcal{E}$. In particular, we need the following proposition.

**Proposition 4.2.** Let $(\overline{M}, \Sigma)$ be a minimal semi-stable pair with $\overline{k}(M) = 2$. Let $E$ be an irreducible curve such that $L \cdot E = 0$. If $E$ is not contained in $\Sigma$ then $E \cong \mathbb{C}P^1$ and $E \cdot E = -2$.

**Proof.** Under these assumptions we know that $L^2 > 0$. By the Hodge index theorem

$$L^2 > 0, \quad L \cdot E = 0 \quad \implies \quad E^2 < 0.$$ 

But now $L \cdot E = 0$ which implies

$$K_{\overline{M}} \cdot E = -\Sigma \cdot E \leq 0.$$ 

We then have $K_{\overline{M}} \cdot E = 0$ if and only if $E$ does not intersect $\Sigma$. In this case $p_a(E) = 0$ and then $E \cong \mathbb{C}P^1$ and $E^2 = -2$. Assume now that $K_{\overline{M}} \cdot E < 0$, then $K_{\overline{M}} \cdot E = E^2 = -1$ and therefore $E$ is an exceptional curve of the first kind such that $E \cdot \Sigma = 1$. This contradicts the minimality of the pair $(\overline{M}, \Sigma)$. \qed
We are now ready to prove the main results of this section. Let \( \mathbb{C}H^2/\Gamma \) be a finite-volume complex-hyperbolic surface that admits a smooth toroidal compactification as in Theorem 2.2. We then have that \( \mathbb{C}H^2/\Gamma \) is a surface of general type with compactification divisor consisting of smooth disjoint elliptic curves.

**Proposition 4.3.** Let \( \mathbb{M} \) be a minimal surface of general type. Let \( \Sigma \) be a reduced divisor whose irreducible components consist of disjoint smooth elliptic curves. Then, \((\mathbb{M}, \Sigma)\) is a minimal semi-stable pair with \( k(M) = 2 \).

**Proof.** Recall that the canonical divisor of any minimal complex surface of non-negative Kodaira dimension is numerically effective [3]. It follows that the adjoint divisor \( L \) is numerically effective. An elliptic curve on a minimal surface of general type has negative self intersection. Moreover, for a minimal surface of general type it is known that the self-intersection of the canonical divisor is strictly positive [3]. By the adjunction formula
\[
L^2 = K^2_{\mathbb{M}} - \Sigma^2 > 0.
\]
By Proposition 4.1 we conclude that \( k(M) = 2 \). \( \square \)

Let \( \mathbb{C}H^2 \backslash \Gamma_1 \) be a finite-volume complex-hyperbolic surface which admits a smooth toroidal compactification \( \mathbb{M}_1 \). Let \((\mathbb{M}_1, \Sigma_1)\) be the associated minimal semi-stable pair. By Theorem A, we can find a normal subgroup of finite index \( \Gamma_2 \triangleleft \Gamma_1 \) such that the toroidal compactification \( \mathbb{M}_2 \) of \( \mathbb{C}H^2/\Gamma_2 \) is a minimal surface of general type with compactification divisor \( \Sigma_2 \). Since \( \pi: \mathbb{C}H^2/\Gamma_2 \rightarrow \mathbb{C}H^2/\Gamma_1 \) is an unramified covering we conclude that \( k(M_1) = k(M_2) [10] \). But by proposition 4.3 we know that \( k(M_2) = 2 \), it follows that \((\mathbb{M}_1, \Sigma_1)\) is a minimal semi-stable pair of log-general type. Let us summarize this argument into a proposition.

**Proposition 4.4.** Let \((\mathbb{M}, \Sigma)\) be a smooth pair arising as the toroidal compactification of a finite-volume complex-hyperbolic surface. The pair \((\mathbb{M}, \Sigma)\) is minimal and log-general.

The following theorem is the main result of the present section.

**Theorem B.** Let \((\mathbb{M}, \Sigma)\) be a smooth pair arising as the toroidal compactification of a finite-volume complex-hyperbolic surface. Then, the associated minimal compactification is projective algebraic.

**Proof.** By Proposition 4.3 the minimal pair \((\mathbb{M}, \Sigma)\) is log-general. By Theorem [3] we know that \( \Phi_{mE} \) is an embedding modulo \( \mathcal{E} \) for any \( m \geq 5 \). We clearly have that \( \Sigma \) is contained in \( \mathcal{E} \). We claim that there are no other divisors in \( \mathcal{E} \). Assume the contrary. By Proposition 4.2 any other curve in \( \mathcal{E} \) must be a smooth rational divisor \( E \) with self-intersection minus two. The adjunction formula gives \( K_{\mathbb{M}}E = 0 \) which implies \( \Sigma \cdot E = 0 \). This is clearly impossible. By Theorem 3 for \( m \geq 5 \), the map
\[
\Phi_{mE}: \mathbb{M} \rightarrow \mathbb{C}P^{N-1}
\]
gives a realization of the minimal compactification as a projective-algebraic variety. \( \square \)
For an approach to the projective-algebraicity problem through $L^2$-estimates for
the $\bar{\partial}$-operator we refer to the aforementioned paper of Mok [20].

Acknowledgements. I would like to thank Professor Claude LeBrun for his con-
tant support and for constructive comments on the paper. I also would like to
thank Professor Klaus Hulek for some useful bibliographical suggestions and the
referee for pertinent comments on the manuscript.

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