Point spread function estimation for blind image deblurring problems based on framelet transform

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Abstract

One of the most important issues in the image processing is the approximation of the image that has been lost due to the blurring process. These types of matters are divided into non-blind and blind problems. The second type of problem is more complex in terms of calculations than the first problems due to the unknown of original image and point spread function estimation. In the present paper, an algorithm based on coarse-to-fine iterative by $l_0 - \alpha l_1$ regularization and framelet transform is introduced to approximate the spread function estimation. Framelet transfer improves the restored kernel due to the decomposition of the kernel to different frequencies. Also in the proposed model fraction gradient operator is used instead of ordinary gradient operator. The proposed method is investigated on different kinds of images such as text, face, natural. The output of the proposed method reflects the effectiveness of the proposed algorithm in restoring the images from blind problems.

Keywords: Blind deblurring; Framelet; PSF estimation; Natural image; Fractional calculation.

1 Introduction

One of the most important data in human perception of the environment is the use of the image taken by the camera. With the growth of social networks such as Facebook and Instagram, images gained significant importance. With the spread of smartphones, it has become possible for everyone to take pictures, and this has led to the spread of image-based social networks such as Instagram and Pinterest. Due to the fact that shaking hands or other factors may cause a decrease in image quality during imaging, so improving the quality of the image is always one of the most

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important challenges for researchers in the field of computer science and mathematics. Among the most important issues to be studied in image enhancement is image blurring. These types of problems can be formulated as

\[ y = x \ast k + \epsilon, \quad (1.1) \]

where \( y \) and \( x \) in \( \mathbb{R}^{n \times m} \) denote blurred and original images and \( \epsilon \) shows noise. Also \( \ast \) stands for two dimensional convolution operator and \( k \in \mathbb{R}^{r \times s} \) represents point spread function (PSF), but why is this name chosen for this matrix? The image blurring without the process of adding noise can be seen in Fig. 1. From this figure, it can be seen that the structure of the PSF is effective in distributing the blurred image. Considering the concept of the two dimensional convolution operator, in fact according to the structure of the PSF and neighboring pixels, each of the pixels of the original image is changed.

![Image Blurring Without Adding Noise](image)

**Figure 1**: The image blurring without the process of adding noise.

Boundary pixel changes are affected by parts of the outside of the camera frame that do not appear at the output. This problem has been partially solved by considering different types of boundary conditions for the problems. For example, an image taken from the night sky is likely to have a black area outside the image. Boundary conditions that are considered according to the structure of the image are: zero, periodic, reflexive and anti-reflexive boundary conditions [1, 2]. Given that in most cases the problem is removed from the convolution form and written as a system of linear equations to solve the problems, each of the boundary conditions is effective on the matrix of coefficients of these linear equations. For example if boundary condition is considered as zero, the matrix of coefficients is a block toeplitz with toeplitz blocks (BTTB) matrix [3]. The noise factor is usually not considered in the calculations, however, in some types of blurring problems that have been affected by severe noise, in addition to the blurring process, the noise-removing process is also performed [4, 5]. According to the number of unknowns, the problem is divided into two types, the non-blind and blind problems. In the non-blind problems, blurred image and PSF are known. For example, an image taken from inside a train.
travelling at a constant speed has a motion PSF [6] and is considered the non-blind type of problems. In the second form, we have only the blurred image, and we have no information of the PSF. One of the most effective methods to solve this type of problem is to use the two-step method for image approximation and PSF approximation, so that using the iterative method starting from an initial approximation for the latent sharp image, which is normally blurred image, the PSF is approximated and then the approximation of latent sharp image is obtained using the approximated PSF; and this process continues until we find a suitable results of the PSF and the latent sharp image. The coarse-to-fine iterative method is one of the most widely used methods of this type [7]. Typically after this step to increase the quality of the output image, obtained PSF is used with a non-blind deblurring method. This step can be selected according to the type of image, as an example for saturation image, the saturation image deblurring algorithm [8] can have better performance. In this type of method, different choices are used for sentences including the PSF in the proposed model, for example the $l_0$ norm of the PSF or $l_0$ the gradient of PSF [9]. Selecting this norm according to the structure of the PSF is an efficient tool. With the development of new concepts in mathematics, a new method is always developed based on these concepts to solve real problems. The concept of frame is one of the practical concepts developed from the mathematical analysis and in recent years has been used in various topics of image processing. This concept has been extended to various articles on image deblurring problems, including non-blind and blind problems [10, 11]. In most of these articles, framelet transform is used on the image and then the algorithm is presented. In the present algorithm, which is studied in the next section, unlike the most articles, the penalized term $l_0 - \alpha l_1$ is used by framelet transform on the PSF. The reason for using this method is that images usually have sparse representations in the framelet transform domains. And this improves the PSF approximation because one point of the PSF is approximated at different frequencies. In order to observe the effect of framelet transform on the PSF, in Fig. 2, framelet transforms based on B-spline for different PSEs in Levin dataset [12] are shown. In addition, the method of subtracting two norms is an efficient method that has been considered in various articles [13, 14]. Perhaps the important problem with this type of the penalized term is that this method is non-convex. However, with specific methods, this problem can be turned into a few convex problems, so this problem can be easily solved. Another tool used in this paper is to use the fractional derivative. This concept is generalized to the ordinary derivative to fractional numbers [15]. This tool has been used in recent years in various articles on image processing as [16, 17]. More details of this concept are discussed in the following sections.

Notations: In this paper we use the following notations. $F(\cdot)$ and $F^{-1}(\cdot)$ are used for fast Fourier transform (FFT) and and inverse fast Fourier transform (IFFT), $\ast$ and $\langle \cdot, \cdot \rangle$ stand for two dimensional convolution operator and inner product. $H$ shows separable Hilbert space.

Outline: The organization of this paper is as follows: The concepts and tools used in this work are introduced in section 2. In Section 3, the proposed model based
on framelet transform and $l_0 - \alpha l_1$ regularization is introduced and numerical algorithms for the latent sharp image and PSF approximations are presented. In Section 4, the proposed algorithm is studied on different types of images and different tests are studied to evaluate the performance on the algorithm. A summary of the present method is given at the end of the paper in Section 5.

![Image of PSF and framelet transforms](image)

Figure 2: (a-e) Different types of PSFs from Levin dataset, (f-j) framelet transforms of PSFs.

## 2 Motivation and preliminaries

In the following, we explain the details of the proposed model to solve the blind image deblurring problems. Different metrics have been used in different papers to approximate the PSF, and despite having their own advantages, they have some disadvantages. In [18], $\|k\|_1$ is used for the PSF; this norm has a good sparse representation, but it causes the creation of a PSF with noise. $\|k\|_2^2$ is used in [19], despite the advantages of this metric such as convexity and fast speed in calculations and good noise suppression effect, this method creates a dense kernel. Another model that is being used to overcome the disadvantages of previous methods is the use of $\|k\|_0$ for example see [20]. In [21] this method is improved by adding a second phrase as $\|\nabla k\|_0$ and this change creates a PSF with good sparsity with less noise. Given the structure of the matrix of PSF and considering that most of its elements are zero and it can be considered as a sparse matrix. Using the $l_0$ norm is a suitable tool to approximate this matrix that is used in [9]. One way to approximate the $l_0$ is to use the $l_p$, $p \in (0, 1)$ because when $p \to 0$, $l_p \to l_0$. These non-convex metrics are used in [22, 23]. The ratio or difference of $l_1$ and $l_2$ is another method to approximate the $l_0$ that had been studied in various articles [24, 25]. Also $l_1 - l_2$ had been generalized by $l_1 - \alpha l_2$ in [14]. In the present algorithm, $l_0 - \alpha l_1$ metric for $\alpha \in [0, 1]$ is used instead of the $l_0$ metric. Obviously, when $\alpha \to 0$, $l_0 - \alpha l_1 \to l_0$. On the other hand, due to the domain of image pixels that are between zero and one, this metric will always be positive. The graph of this metric is plotted for $[-1, 1] \times [-1, 1]$ in Fig. 3. As can be seen from the figure, this is a non-convex metric. And that can cause
problems in finding a solution. To solve this problem, the proposed model is divided into several new sub-models using the methods that are described. The proposed model, which is introduced, has a suitable behavior in the output in sparsity and less noise. The results of the proposed algorithm are presented at each step, of coarse-to-fine in Fig. 4. The results show, as the size of PSF increases, the number of pixels with zero value increases. In the first iteration, approximately 45% of the pixels are equal to zero and in the last iteration, approximately 14% of the pixels are equal to zero. Then in the early stages of coarse-to-fine method for kernel approximation, it is necessary to reduce the effect of $l_0$-norm on the objective function and increase the effect by increasing the iteration. This proposed method is done by reducing the value of a control parameter between the two norms i.e., $\alpha$ in $l_0 - \alpha l_1$. On the other hand, the use of framelet transfer makes the edges of the PSF restored properly. Then the proposed method creates a proper approximations at the early stages of the PSF in coarse-to-fine algorithm. The details of the method are presented in the next section.

Figure 3: The plot of $l_0 - \alpha l_1$ metric for different values of $\alpha$.

Figure 4: Results of the proposed algorithm in coarse-to-fine steps.
2.1 Framelet transform

One of the fundamental concepts in the signal and image processing is the wavelet theory. In recent years, this discussion has been generalized and used in various articles. The concept of wavelet frames is briefly studied below and reader can find more information about it in [26, 27, 28].

Let \( \langle \cdot, \cdot \rangle \) stands for inner product in separable Hilbert space \( H \) then a countable set as \( X \subseteq H \) is called a tight frame if the following condition holds

\[
f = \sum_{\psi \in X} \langle \psi, f \rangle \psi, \quad \forall f \in L^2(\mathbb{R}).
\]

The wavelet system for a given set as \( \Psi = \{\psi_1, \ldots, \psi_n\} \) is defined as the following collection

\[
X(\Psi) := \{2^j \psi_i (2^j \cdot -k) | 1 \leq i \leq b; j, k \in \mathbb{Z}\}.
\]

Also if \( X(\Psi) \) be a tight frame in \( H \) then \( X(\Psi) \) is named tight wavelet frame and each element of \( \Psi \) is named a framelet. In the wavelet theory a compactly supported scaling function as \( \phi \) with refinement mask \( h_0 \) that \( F(\phi(2\omega)) = F(h_0)F(\phi(\omega)) \) is used to construct compactly supported wavelet tight frames. Here \( F \) stands the Fourier transform. Also a set of framelets for a given compactly supported scaling function as \( \phi \) to construct a wavelet tight frame are defined by

\[
F(\phi(2\omega)) = F(h_0)F(\phi(\omega)), \quad i = 1, \ldots, n,
\]

where \( F(h_0) \), \( i = 0, \ldots, n \) are considered as \( 2\pi \)-periodic. In the proposed algorithm the piecewise linear B-spline framelets [26] with the refinement mask \( F(h_0(\omega)) = \cos^2(\omega)2 \pi \) and two framelet masks \( F(h_1(\omega)) = \frac{-\sqrt{2}}{2} \sin(\omega) \) and \( F(h_2(\omega)) = \sin^2(\omega)2 \pi \) are used. The corresponding lowpass and highpass filters for these masks are considered as \( h_0 = \frac{1}{4}[1, 2, 1] \), \( h_1 = \frac{\sqrt{2}}{4}[1, 0, -1] \) and \( h_2 = \frac{1}{4}[-1, 2, -1] \), respectively. The details of wavelet frame transform can be found in [27, 28]. This transform is used in the next section on the PSE.

2.2 Fractional calculation

Fractional calculation is a concept that has been considered in recent years and has been studied in many different fields of science such as physics, mathematics and computer science. There are different definitions for fractional derivative such as Riemann–Liouville, Caputo, Caputo-Fabrizio and Grünwald–Letnikov (G-L) [15, 29]. As in the use of the ordinary derivative, the discrete form of the derivative is considered for image processing. In the use of the fractional derivative, the discrete form is also efficient. And considering that among the definitions of the fractional derivative, the discrete form of G-L definition is efficient, so this definition is used in image processing [30, 31].
Let $\Omega \subseteq \mathbb{R}^2$ be a bounded open set then for a real function $u : \Omega \rightarrow \mathbb{R}^2$ the fractional-order gradient is considered as

$$\nabla^\lambda u = (\nabla^\lambda_h u, \nabla^\lambda_v u)^T, \quad \lambda \in \mathbb{R}^+.$$  

By using G-L definition, we define

$$\nabla^\lambda_h u(i, j) = \sum_{l=0}^{L-1} (-1)^l C_{\lambda}^l u(i - l, j),$$

$$\nabla^\lambda_v u(i, j) = \sum_{l=0}^{L-1} (-1)^l C_{\lambda}^l u(i, j - l),$$

where

$$C_{\lambda}^l = \frac{\Gamma(\lambda + 1)}{\Gamma(l + 1)\Gamma(\lambda - l + 1)}.$$  

$\Gamma$ stands for gamma function. This definition is used in the next section to approximate the PSF.

3 Proposed model based on $l_0 - \alpha l_1$ regularization

Details of the proposed model are provided in this section and the numerical algorithms for solving the model are discussed.

3.1 Proposed model

The proposed objective function of the deblurring model is introduced as

$$\arg\min_{k, x} \|x \otimes k - y\|_2^2 + \gamma_1 P_x^\sigma + \gamma_2 P_k^\alpha,$$  

(3.1)

where

$$P_x^\sigma = \sigma \|x\|_0 + \|\nabla x\|_0,$$

$$P_k^\alpha = \|wk\|_0 - \alpha \|k\|_1,$$

$\gamma_1, \gamma_2$ and $\sigma$ are the regularization weights and $\alpha \in [0, 1]$; $w$ denotes the framelet transform matrix such that $w^T w = I$ and $\nabla$ stands for gradient operator where $\nabla x = (\nabla_h x, \nabla_v x)^T$. Also, horizontal and vertical derivatives are obtained using differential filters $\nabla_h = [1, -1]$ and $\nabla_v = [1, -1]^T$. As can be seen from the presented model, the part related to image restoration has not changed with the related papers [9], but the part of the PSF restoration has changed with terms described in the previous parts. This model is non-convex model, therefore, it is necessary to provide a suitable method for solving. The details of finding the answer is introduced in the following subsections.
3.2 Estimating image with blur kernel

To approximate the image from the PSF according to the proposed model, the following model needs to be solved.

\[
\argmin_x \| x \ast k - y \|_2^2 + \gamma_1 (\sigma \| x \|_0 + \| \nabla x \|_0).
\]  

(3.2)

The above model is solved in [9], but in the following, a summary of the method described in that paper is given. By using auxiliary variables \(a\) and \(b = (b_h, b_v)\) the problem (3.2) is rewritten as

\[
\argmin_{x, a, b} \| x \ast k - y \|_2^2 + \beta \| x - a \|_2^2 + \mu_1 \| \nabla x - b \|_2^2 + \gamma_1 (\sigma \| a \|_0 + \| \nabla b \|_0),
\]

where \(\beta\) and \(\mu_1\) are positive constant. Based on the new problem \(x\) is obtained by solving

\[
\argmin_x \| x \ast k - y \|_2^2 + \beta \| x - a \|_2^2 + \mu_1 \| \nabla x - b \|_2^2,
\]

which the closed-form solution for this subproblem is given by

\[
x = F^{-1}\left( \frac{F^*(k)F(y) + \beta F(a) + \mu_1 \sum_{i \in \{h,v\}} F^* (\nabla_i) F(b_i)}{F^*(k)F(k) + \beta + \mu_1 \sum_{i \in \{h,v\}} F^* (\nabla_i) F(\nabla_i)} \right),
\]

(3.3)

where * denotes the complex conjugacy. Also \(a\) and \(b\) are obtained by following subproblems

\[
\argmin_a \beta \| x - u \|_2^2 + \gamma_1 \sigma \| a \|_0,
\]

\[
\argmin_b \mu_1 \| \nabla x - b \|_2^2 + \gamma_1 \| b \|_0.
\]

These subproblems are pixel-wise minimization problem and the solutions are given by [32]

\[
a = \begin{cases} 
  x, & |x|^2 \geq \frac{\gamma_1 \sigma}{\beta}, \\
  0, & \text{otherwise},
\end{cases}
\]  

(3.4)

\[
b = \begin{cases} 
  \nabla x, & |\nabla x|^2 \geq \frac{\gamma_1}{\mu_1}, \\
  0, & \text{otherwise}.
\end{cases}
\]  

(3.5)

A summary of the described method is given in Algorithm 1.

Algorithm 1: Image restoration algorithm.

**Input:** Blurred image \(y\) and blur kernel \(k\).

\(x - y, \beta \leftarrow 2\gamma_1 \sigma\).

**repeat**

- obtain \(a\) by (3.4).
\[ \mu_1 \leftarrow 2 \gamma_1. \]

repeat
  
  obtain \( b \) by (3.5).
  
  obtain \( x \) by (3.3).

  \[ \mu_1 \leftarrow 2 \mu_1. \]

until \( \mu_1 > \mu_{\text{max}} \)

\[ \beta \leftarrow 2 \beta. \]

until \( \beta > \beta_{\text{max}} \)

\textbf{Output}: Intermediate latent image \( x \).

### 3.3 Estimating PSF with image

In order to approximate the PSF in model (3.1), it is necessary to solve the following problem

\[
\arg\min_k \| x \odot k - y \|_2^2 + \gamma_2 (\| wk_0 - \alpha \| k_1). \quad (3.6)
\]

Solving the above problem directly by using the intermediate latent image does not give a good output and in [7] recommends using the gradient operator to solve the above problem. Then based on the gradient operator, the following problem is proposed for (3.6).

\[
\arg\min_k \| \nabla x \odot k - \nabla y \|_2^2 + \gamma_2 (\| wk_0 - \alpha \| k_1). \]

But in the following, a general model based on the fraction gradient operator is presented to solve the problem (3.6) as

\[
\arg\min_k \| \nabla^\lambda x \odot k - \nabla^\lambda y \|_2^2 + \gamma_2 (\| wk_0 - \alpha \| k_1). \]

Specifically, when \( \lambda \) is equal to 1, the ordinary model by gradient operator is obtained. By introducing auxiliary variables \( c \) and \( d \) for \( wk \) and \( k \), respectively, we get

\[
\arg\min_{k,c} \| \nabla^\lambda x \odot k - \nabla^\lambda y \|_2^2 + \gamma_2 (\| c \|_0 - \alpha \| d \|_1),
\]

s.t. \( c = wk, \ d = k \)

The augmented form for the above problem can be get as

\[
\arg\min_{k,c} \| \nabla^\lambda x \odot k - \nabla^\lambda y \|_2^2 + \gamma_2 (\| wk - c \|_2 + \mu_2 \| wk - c \|_2) - \gamma_2 \alpha \| d \|_1 + \mu_3 \| d - k \|_2^2. \quad (3.7)
\]

To solve the above problem, in the first step, the solution of \( c \) is studied. By (3.7), the subproblem for \( c \) can be written as

\[
\arg\min_c \gamma_2 (\| wk - c \|_2 + \mu_2 \| wk - c \|_2).
\]
As discussed in the previous section, this problem is pixel-wise minimization problem and the solution is obtained as
\[
c = \begin{cases} \ w k, & |wk|^2 \geq \frac{\gamma_2}{\mu_2}, \\
0, & \text{otherwise.} \end{cases} \tag{3.8}
\]

In the next step, we approximate the value of \( k \). For this purpose by using (3.7), the following subproblem is obtained
\[
\arg\min_k \|\nabla^A x \circ k - \nabla^A y\|_2^2 + \mu_2 \|wk - c\|_2^2 + \mu_3 \|k - d\|_2^2. \tag{3.9}
\]

By using the optimal condition for (3.9) and fast Fourier transform, the closed-form of the final solution for \( k \) is written as
\[
k = F^{-1}\left\{ \frac{\sum_{i \in \{h,v\}} F^*(\nabla^A x) F(\nabla^A i) + \mu_3 F(d) + \mu_2 F(w^* c)}{\sum_{i \in \{h,v\}} F^*(\nabla^A i) F(\nabla^A x) + \mu_3 + \mu_2} \right\}. \tag{3.10}
\]

Now, the next subproblem that is needed to find \( d \) is studied. This subproblem is written as
\[
\arg\min_d -\gamma_2 \alpha \|d\|_1 + \mu_3 \|k - d\|_2^2.
\]

Similar to split Bregman iteration method [33, 34], the following iterative method is proposed to solve this problem
\[
a_{d}^{n+1} = \arg\min_d -\gamma_2 \alpha \|d\|_1 + \mu_3 \|d - k\|_2^2, \tag{3.11}
a_{b}^{n+1} = b^n + (k - a_{d}^{n+1}). \tag{3.12}
\]

The solution of problem (3.11) with condition \(|-\frac{\gamma_2 \alpha}{\mu_3}| < 1\) is obtained by using the proximal mapping for \( l_1 \)-norm as [35]
\[
a_{d}^{n+1} = k + b^n. \tag{3.13}
\]

The above processing can be expressed in the following algorithm. According to the structure of Algorithm 2, the value of \(|-\frac{\gamma_2 \alpha}{\mu_3}| \) is always lower than one, therefore, a local minimum value can be found for (3.11).

**Algorithm 2: PSF restoration algorithm.**

**Input:** Blurred image \( y \) and Intermediate latent image \( x \).

\( b = 0, k = 0, \mu_2 \leftarrow 2\gamma_2 \).

**repeat**

obtain \( c \) by (3.8).
\( \mu_3 \leftarrow 2\mu_2 \).

**repeat**

obtain \( d \) by (3.13).
\( b \leftarrow b + k - d \).
obtain \( k \) by (3.10).

\[
\mu_3 \leftarrow 2\mu_3 \quad \text{until} \quad \mu_3 > \mu_3^{\text{max}} \\
\mu_2 \leftarrow 2\mu_2. \\
\text{until} \quad \mu_2 > \mu_2^{\text{max}} \\
\text{Output: Intermediate kernel } k.
\]

### 3.4 Coarse-to-fine framework for PSF

In the previous subsections, the methods of image restoring by the PSF and PSF restoring by the image are studied. Using the above methods directly to restore image information does not always work. One of the most effective methods that is considered in various algorithms is the use of an image pyramid framework in a coarse-to-fine method \[7\]. Due to the fact that sharp edges is effective in approximating the PSF; so using the fraction gradient operator improves the kernel. This algorithm is given in Algorithm 3. According to the structure of the proposed algorithm, it is observed that by increasing the number of iterations and approaching the actual size of the image, the constant \( \alpha \) decreases to increase the effect of the \( l_0 \) norm. In the last step, to increase the quality of the restored image, depending on the type of image, a non-blind image deblurring algorithm based on the obtained PSF is used. Also, to reduce ringing artifacts, simple but efficient method based on Fourier domain restoration filter and extrapolated image that is introduced in \[36\] is used in the proposed algorithm. Also in this algorithm the proposed method for threshold of truncating gradients in \[7\] is used to estimate kernel with this difference that fractional gradient values are used instead of the ordinary gradient values. Numerical results related to the proposed algorithms are studied in the next section.

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**Algorithm 3: Coarse-to-fine framework.**

**Input:** Blurred image \( y \), the size of PSF, \( \lambda \).

Obtain \( k \) by the coarser level method.

**for** \( i = 1 \rightarrow 5 \) **do**

- Obtain \( x \) by Algorithm 1.
- Obtain \( k \) by Algorithm 2.
- \( \gamma_1 \leftarrow \max(\gamma_1/1.1, 10e^{-4}) \).
- \( \alpha \leftarrow \max(\alpha/1.1, 10e^{-4}) \).

**end for**

**Output:** \( k \) and intermediate image \( x \).
4 Experiment results

In order to evaluate the performance of the proposed algorithm, different types of images as text, face, natural, and low-light images are studied in this section. Also Windows 10-64bit, Intel(R) Core(TM) i3-5005U CPU @2.00GHz, by matlab 2014b have been used for the calculations. The results of the algorithm are evaluated using different tests such as Information content Weighted Structural Similarity Measure (IW-SSIM) [38], Multi-scale Structural Similarity (M-SSIM) [39], Feature Structural Similarity (F-SSIM) [40] and Peak Signal-to-Noise Ratio (PSNR).

Levin dataset: As a first example, we use Levin dataset. The results are given in Table 1. In this table the first number in parentheses stands for number of image and the second number denotes the number of kernel in dataset. The results in this table are compared with the results in [9, 37], and although in some cases the results of the proposed method have lower values compared to these methods, but in general the average of the proposed method has a better output compared to these methods.

| method in [9] | method in [37] | proposed method |
|---------------|----------------|-----------------|
| Levin dataset: | | |
| (1,1) 33.9744 0.9440 0.9649 0.8857 | 33.3215 0.9215 0.9509 0.8584 | 34.0847 0.9464 0.9672 0.8839 |
| (1,2) 31.8752 0.6811 0.6168 0.7585 | 31.6080 0.7404 0.8463 0.7733 | 31.9585 0.7452 0.8502 0.7699 |
| (1,3) 32.8359 0.8487 0.9114 0.8291 | 32.4375 0.8054 0.8681 0.8122 | 33.1591 0.8830 0.9307 0.8464 |
| (1,4) 31.1792 0.8943 0.9170 0.8265 | 31.5524 0.9119 0.9281 0.8460 | 30.8961 0.8587 0.8988 0.7998 |
| (1,5) 34.2503 0.9451 0.9657 0.8893 | 36.6390 0.9820 0.9983 0.9431 | 34.4087 0.9423 0.9633 0.8773 |
| (1,6) 31.8633 0.7944 0.8760 0.8031 | 32.3126 0.8331 0.9099 0.8198 | 32.3843 0.8733 0.9282 0.8370 |
| (1,7) 34.2585 0.9650 0.9690 0.9039 | 35.3590 0.9764 0.9806 0.9296 | 34.6982 0.9706 0.9756 0.9171 |
| (1,8) 31.6717 0.8902 0.9228 0.8379 | 32.0271 0.9335 0.9527 0.8755 | 31.2542 0.9268 0.9422 0.8708 |
| mean 32.6641 0.8697 0.9181 0.8417 | 33.1635 0.8882 0.9291 0.8573 | 32.8552 0.8833 0.9310 0.8586 |

Table 1: Results for Levin dataset.
Text image: Images containing text usually appear in images obtained by scanning a text. In recent years, with the expansion of the use of smart phones, scanning software has expanded [41]. Various factors can affect the quality of the output image, such as hand shake when using scanning software by smart phone. The results for two text images such that blurred by kernel 01 and 04 in Levin dataset are shown in Figs 5-7. The results of the proposed method are compared with [7, 9, 37, 42, 44]. In Fig. 5, the results for MS-SSIM, IW-SSIM and F-SSIM are given. Also Fig. 6 shows output deblurred images. The Fig. 6 shows the close approximation of the PSF by the proposed algorithm. Also realistic blurred text image includes car license plate is restored by the proposed algorithm and compared by methods in [37, 44, 45] in Fig 7. As can be seen from these figures, the proposed algorithm has a better output compared to the other methods.

Figure 5: Compare the results for IW-SSIM, M-SSIM and F-SSIM.
Face image: Another image that is considered in computer science is the face image. This type of image is used in face recognition in topics related to artificial intelligence, so increasing the image quality of the face before use can be considered. Numerical results related to the proposed method are given in Figs. 8-9 and are compared with some methods. The results show the efficiency of the proposed algorithm.
for face images. The results show that proposed algorithm compares favorably or even better against compared methods. It is also seen in Fig. 9 that increasing the kernel size in the algorithm, unlike the compared methods, has little effect on the quality of the restored image.

![Figure 8: Visual comparison for realistic blurred face image.](image)

![Figure 9: Restored face image with the used kernel sizes {25,35, 45, 55, 65}, from left to right, respectively. First row, method in [45]; second row, method in [37]; third row, proposed method.](image)

**Natural image:** Three real natural images are studied in this example. The results are given and compared with method in [37, 44, 45] in Fig.s 10 and 11. The results for this type of images show the efficiency of the proposed algorithm in restoring blurred nature images.

**Low-light image:** Restoring the low-light image is difficult due to their structure and
requires the design of a special algorithm for this type of image \cite{46, 48}. However, according to the results of the proposed method in Fig. 12, the results show that the proposed algorithm is efficient in recycling this type of images. In this comparison, we note that the algorithm in \cite{46} is designed specifically for this type of problems.

**Camera motion:** As a final example in this section we review a few examples of real camera motion dataset in \cite{47} \footnote{See: \url{https://webdav.tuebingen.mpg.de/pixel/benchmark4camerashake/}}. The results for this dataset are given in Figs. 13 and 14. The results of F-SSIM and PSNR values for the church image in this dataset are given in Fig. 13 and compared with methods in \cite{7, 8, 43, 44, 49, 50, 51}. In this diagram, the restored images of proposed and other methods are compared with around 200 ground truth images and the best result is reported as the main results. Also the restored images of the clock image with kernel 8 for the proposed and other algorithms are shown in Fig. 14.
Figure 12: Visual comparison for realistic blurred low-light image.

Figure 13: Compare the results for F-SSIM and PSNR with the kernels 1, 2, 3 and 4.
5 Conclusion

In this paper, the difference between two norms zero and one is used to approximate the point spread function in blind image deblurring problems. Also in the proposed algorithm, two concepts of framelet and fractional calculations have been used. The proposed algorithm is evaluated on different types of images with different PSF sizes. The results are compared with the other methods and the outputs show the efficiency of the proposed algorithm in deblurring performance.

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