Majorana Modes of Giant Vortices

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We study Majorana zero modes bound to giant vortices in topological superconductors or topological insulator/normal superconductor heterostructures. By expanding in inverse powers of a large winding number $n$, we find an analytic solution for asymptotically all $n$ zero modes required by the index theorem. Contrary to the existing estimates, the solution is not pinned to the vortex boundary and is composed of the warped lowest Landau level states. While the dynamics which shapes the zero modes is a subtle interference of the magnetic effects and Andreev reflection, the solution is very robust and is determined by a single parameter, the vortex radius. The resulting local density of states has a number of features which give remarkable signatures for an experimental observation of the Majorana fermions in two dimensions.

Majorana quasiparticle excitations in various condensed matter systems are in a spotlight of theoretical and experimental studies for over a decade. A renown example of the Majorana quasiparticles are the zero-energy states bound to the vortices in a topological superconductor or on the interface between a topological insulator and a normal superconductor. The giant vortices of large winding number $n$ are of particular interest since they host multiple zero modes and can be studied to use highly nontrivial systems of interacting Majorana states such as the Sachdev-Ye-Kitaev model. The vortices with $n > 1$ have already been observed in mesoscopic semiconductors and can be engineered in the specially designed heterostructures. Unambiguous identification of the zero modes in the vortex core poses a great challenge for the modern experimental techniques inferred from the Jackiw-Rossi theory of charged massless two-component Dirac fermion in 2 + 1 dimensions described by the Lagrange density

$$L_{JR} = i\bar{\psi}\gamma^\mu D_\mu \psi + \frac{1}{2}\left(\bar{\psi}\gamma^\mu \phi + \bar{\psi}\psi^\star\right),$$

where $D_\mu = \partial_\mu + iA_\mu$ is the gauge covariant derivative, the Dirac matrices reduce to the Pauli matrices $\gamma^\mu = (\sigma_3, i\sigma_2, -i\sigma_1)$, $\psi^\star = -i\sigma_1\psi^\star$ is the charge conjugate spinor, and $\phi$ is a scalar field of charge 2 representing the pair potential. For the static zero-energy states the field equations for the spinor components read

$$D_\pm\psi^\pm + \phi\psi^\pm = 0,$$

where the chiral derivatives are $D_\pm = D_1 \pm iD_2$. We are interested in the solution of Eq. (2) in the background of the axially symmetric Abrikosov vortex of the winding number $n$, which implies the following field configuration in polar coordinates $\phi(r, \theta) = f(r)e^{in\theta}$, $A_\theta = -na(r)/2$, $A_r = 0$, with $f(0) = a(0) = 0$ and $f(\infty) = f_\infty$, $a(\infty) = 1$. Then the negative chirality equation does not have a normalizable solution and the $n$ zero modes of positive chirality can be written as follows

$$\xi^+_l = \frac{1}{\sqrt{2}} \left( e^{il\theta} \psi^+_1 + e^{i(n-1-l)\theta} \psi^+_n \right),$$

$$\eta^+_l = \frac{i}{\sqrt{2}} \left( e^{i\theta} \psi^+_1 - e^{i(n-1-l)\theta} \psi^+_n \right),$$

where $0 \leq l \leq n/2 - 1$ for even $n$ and $0 \leq l \leq (n-1)/2$, $\eta^+_{(n-1)/2} = 0$ for odd $n$. The partial wave amplitudes satisfy the following equations

$$\left( \frac{d}{dr} - \frac{l + n/2}{r} \right) \psi^+_l + f\psi^+_n = 0,$$

$$\left( \frac{d}{dr} - \frac{n - l - 1}{r} + \frac{n/2}{r^2} \right) \psi^+_{n-1-l} + f\psi^+_l = 0.$$

After identification of $\psi^+_l$ and $\psi^+_{n-1-l}$ with the components of the Nambu spinor, and of $f$ with the pair potential the above system reproduces the Bogoliubov-de-Gennes equations for the Majorana vortex zero modes of
the effective Dirac fermion at zero chemical potential in the condensed matter systems (see e.g. [18]).

Let us outline the main idea of our approach. The solution of Eq. (4) requires an explicit form of the vortex fields. In general the functions \(a(r)\) and \(f(r)\) can be systematically obtained only through the numerical calculation within the self-consistent Bogoliubov-de-Gennes formalism [19]. However, the vortex structure drastically simplifies for \(n \gg 1\). In this limit the vortices evolve into the thin-wall flux tubes [20] with the nonlinear dynamics confined to a narrow boundary layer outside the vortex core [15]. The boundary layer depth is given by the maximal of the magnetic penetration length \(\delta\) and the correlation length \(\xi\) of the superconductor, while the core radius grows with \(n\) as \(r_n = 2^{3/4}\sqrt{n}\xi\), where \(\zeta = \sqrt{\delta/\xi}\) is the geometric average of the scales. Inside and outside the core the dynamics of the gauge and scalar fields is the geometric average of the scales. Inside and outside the core the dynamics of the gauge and scalar fields linearizes up to the corrections exponentially suppressed for large \(n\). Inside the boundary layer the asymptotic vortex solution does not depend on the winding number and gets the corrections in powers of \(1/n\), based on the effective field theory idea of scale separation has been developed in Refs. [15, 16]. In the present work it is applied to the analysis of Eq. (4) in the giant vortex background. Throughout the paper we consistently use the universal aspects of the leading order result and neglect the model-dependent corrections.

It is convenient to decouple the gauge field by a field redefinition

\[
\psi_i^+(r) = u_i(r) \exp \left( -\frac{n}{2} \int_0^r \frac{a(r')}{r'} dr' \right)
\]

(5)

and to transform the system Eq. (4) into the second order equation

\[
\left[ \frac{d^2}{dr^2} - \left( \frac{n-1}{r} + \frac{f'}{f} \right) \frac{d}{dr} + \frac{l}{r} \left( \frac{n-l}{r} + \frac{f'}{f} \right) - f^2 \right] u_i = 0,
\]

(6)

where \(f' = df/dr\). At \(r > r_n\) the scalar and gauge fields exponentially approach their vacuum values and the normalizable solution of Eq. (4) reads

\[
\psi_i^+(r) \propto K_\mu(r/\sqrt{2}\delta),
\]

(7)

where \(K_\mu(z)\) is the \(\mu\)th modified Bessel function with \(\mu = \sqrt{n^2/4-l(l-1)}\), and the relation \(f_\infty = 1/\sqrt{2}\delta\) is used. Thus, the solution exponentially decays outside the core indicating that the zero modes are localized in the vortex core or on its boundary.

The field dynamics inside the vortex core is determined solely by the gauge interaction giving the universal solution [10]

\[
f(r) = f_0 \exp \left[ \frac{n}{2} \left( \ln \left( \frac{r^2}{r_n^2} \right) - \frac{r^2}{r_n^2} + 1 \right) \right],
\]

\[a(r) = \frac{r_n^2}{r^2},\]

(8)

where \(r < r_n\), \(f_0\) is an inessential integration constant, and \(a(r)\) corresponds to a homogeneous magnetic field. Though the pair potential in Eq. (8) is exponentially suppressed, it is a singular perturbation since the order of the Bogoliubov-de-Gennes equations for vanishing \(f\) is reduced. Indeed, for \(r < r_n\) the logarithmic derivative term \(f'/f = n/r(1 - r^2/r_n^2)\) in Eq. (6) is not suppressed and must be kept to get two regular solutions at \(r = 0\). These solutions are

\[
u^+_i(r) = r^l, \quad \nu_i^{+2}(r) = r^{2n-l}E_\nu \left( \frac{nr^2}{2r_n^2} \right),
\]

(9)

where \(E_\nu(z)\) is the \(\nu\)th exponential integral with \(\nu = 1 + l - n\). The behavior of the second solution at large \(n\) is quite peculiar. For \(l < n/2\) it reduces to \(\nu_i^{+2}(r) \sim r^l\), i.e. the two solutions are degenerate up to the exponentially suppressed terms. For \(l > n/2\), however, it transforms into \(\nu_i^{+2}(r) \sim r^{2n-l}e^{-nr^2/2r_n^2}\) and is the only solution which gives an unsuppressed contribution to \(\psi_i^+(r)\). The gauge field factor in Eq. (5) inside the core equals to \(e^{-nr^2/4r_n^2}\) and for the partial waves in the large-\(n\) limit we finally get

\[
u_i^+(r) \sim N_l \begin{cases} 
  r^l e^{-nr^2/4r_n^2}, & l < n/2, \\
  r^{2n-1} e^{-3nr^2/4r_n^2}, & l > n/2,
\end{cases}
\]

(10)

where \(N_l\) is the normalization factor. Eq. (10) describes two groups of approximately Gaussian peaks. For \(l < n/2\) the peaks of the width \(\sigma = r_n/\sqrt{n} = 2^{3/4}\zeta\) are centered at \(\tilde{r}_l = \sqrt{2l/n}r_n\), while for \(l > n/2\) the peaks have the width \(\sigma' = \sigma/\sqrt{3}\) and are centered at \(\tilde{r}_l' = \sqrt{(2/3)(2-l)/n}r_n\). The solutions with \(l \approx n/2\) are localized inside the boundary layer where the nonlinear effects are essential and an explicit analytical solution is not available. At the same time, for known functions \(f(r)\) and \(a(r)\) the solution is given by

\[
\psi_i^+(r) \propto r_2 \exp \left[ -\int_{r_0}^r \left( \frac{na(r')}{2r'} + f(r') \right) dr' \right],
\]

(11)

where \(n\) is assumed to be even. It describes a non-Gaussian peak of an \(O(\delta)\) width. Note that the functions \(f(r)\) and \(a(r)\) do not depend on \(n\) inside the boundary layer and are known explicitly for Ginzburg-Landau theory in the integrable limits of large, critical, or small values of the Ginzburg-Landau parameter \(\kappa = \delta/\xi\) [16].

Let us now discuss the physical nature of the solution Eq. (10). For \(l < n/2\) it corresponds to the lowest Landau level states formed by an approximately homogeneous magnetic field inside the vortex core. Each of
these states encircles an even number of the flux quanta. Hence, only about $n/2$ of the Landau states fit into the vortex core and are not affected by the pair potential. For larger $l$ the effect of Andreev reflection on the localization of the states increases and for $l \approx n$ it exceeds the effect of the magnetic field. This follows e.g. from a comparison of the exponential factor in Eq. (3) due to the magnetic field to the one of $u_l^{(2)}$ due to the pair potential. As it has been pointed out for $l > n/2$ the Andreev reflection squeezes the Gaussian peaks by the factor $\sqrt{3}$ and displaces them towards the center of the vortex with the innermost position $\sqrt{2/3} r_n$ of the maximal angular momentum partial wave. Remarkably such a significant effect is achieved in the region where the pair potential is exponentially small i.e. the Andreev reflection in this case is a long-range phenomenon. It can be attributed to the singular character of the vanishing pair potential limit for the Bogoliubov-de-Gennes equations discussed above.

We are now able to compute the experimentally observable radial density of states $\rho(r) = 2\pi \sum_{l=0}^{n-1} (\psi_l^+(r))^2$ for the Majorana zero modes. At large $n$ the effect of the poorly approximated $l \approx n/2$ states is negligible and the sum converges to the function

$$\rho(r) \sim \rho_0 \left\{ \begin{array}{c} 1/2, \quad r/r_n < \sqrt{2/3}, \\ 2, \quad \sqrt{2/3} < r/r_n < 1, \end{array} \right. \quad (12)$$

where $\rho_0 = 2n/r_n^2 = 1/\sqrt{2} \zeta^2$ does not depend on $n$. The function $\rho(r)$ for a few finite values of the winding number is plotted in Fig. 1. There the $l = n/2$ states are approximated by the Gaussian peaks of the width $\sigma$ centered at $r_n$, which does not significantly affect the distribution even for the moderate values of $n$. As we can observe the convergence to the asymptotic result is very fast for $r < \sqrt{2/3} r_n$ and slow for $\sqrt{2/3} r_n < r < r_n$ but the characteristic shape of the spatial distribution becomes evident already at $n = 4$. Thus, it is more important to estimate the accuracy of our prediction for the local density of states at a given moderately large value of $n$, i.e. the accuracy of each line in Fig. 1. Inside the core where most of the states are localized the accuracy of the method is exponential and for $n = 4$ the estimated error is about a few percent. At the core boundary the accuracy deteriorates due to the dependence of the $l \approx n/2$ states on the exact form of the pair potential. A conservative estimate of the uncertainty for an individual state can be done by evaluating the factor $e^{-f(r')}d\sigma'$ in Eq. (11) where the integral runs over the boundary layer of the depth $\delta$. Approximating $f(r)$ with $f_\infty = 2$ we get the correction factor $e^{-1/2}\sqrt{\pi}$ corresponding to a 30% error. This, however, affects only the tail of the distribution at $r > r_n$ where the states with $l \approx n/2$ give the dominant contribution. Thus, our analysis is reasonably accurate already for $n = 4$. The vortices with such winding number have already been observed experimentally [9].

So far we have considered the case $\kappa = \mathcal{O}(1)$. For $\kappa \gg 1$ there appears another condition on the allowed values of $n$. The method [15, 16] relies on the scale hierarchy $\delta/r_n \ll 1$. This scale ratio is proportional to $\sqrt{\kappa}/n$. For $\kappa > n$ the magnetic field is expelled from the vortex core and the vortex cannot be considered as a thin-wall flux tube. At the same time, the superconductors with the large value of the Ginzburg-Landau parameter may not be ideal for the experimental realization of the giant vortices. Indeed, the free energy in this case grows with $n$ as $n^2 \ln \kappa$ while for $\kappa = \mathcal{O}(1)$ it scales as $n/\kappa$ [16]. This makes the giant vortices for large $\kappa$ much less stable against the decay into the elementary vortices and, hence, more difficult to create in an experiment.

In any case, an experimental realization of the giant Abrikosov vortices with $n = \mathcal{O}(10)$ may not be an easy task. At the same time the hard wall giant vortices of arguably very large $n$ can be created by a magnetic flux flowing through a hole in the superconducting film on the surface of the topological insulator. Such a design has been originally suggested in Ref. [8] for a physical realization of the Sachdev-Ye-Kitaev model on the hole boundary, but it can also be an ideal place for the study of two-dimensional Majorana zero modes in the hole interior. The result Eq. (12) in this case should be adjusted. The term $f'/f$ in Eq. (6) now gets a very large positive constant contribution proportional to the ratio of the Cooper pair chemical potential to the superconductor energy gap. This effectively makes the Andreev reflection short-range so that all the states with $l \gtrsim n/2$ get localized on the hole edge. The radial density of states now takes the form

$$\rho(r) \sim \frac{\rho_0}{2} \left( 1 + \frac{R}{2} \delta(r - R) \right), \quad (13)$$

where $\rho_0 = 2n/R^2$ and $R$ is the hole radius. The delta-
function in the above equation is in fact an approximation of
the non-Gaussian peak with the width of order $\delta \ll R$,
and $R$ should be taken larger than $r_n$ for a given $n$ and $\zeta$
to have a stable vortex configuration.

Though the spatial distributions in Eq. (12) and Eq. (13) are quite similar, the physical properties of the
two systems are qualitatively different. For Abrikosov
vortices the parameter $\rho_0$ which defines the average den-
sity of states is $n$-independent and completely determined
by the intrinsic properties of the superconductor through
the geometric average $\zeta$ of the magnetic penetration and
the correlation length. By contrast, for the hard-wall vor-
tices the parameter $\rho_0$ is quantized in the units of $2/R^2$
and is proportional to the number of the magnetic flux
quanta. Thus, it can be discretely changed by the vari-
ation of the applied magnetic field $B$. The corresponding
average rate of the density variation inside the core evalu-
ates to

$$\frac{d\rho}{dB} = \pi K_J,$$

(14)

where $K_J$ is the Josephson constant (the inverse of the
magnetic flux quantum).

Our solution Eq. (10) is qualitatively different from the
existing analysis where the role of the magnetic field on
the formation of the vortex states has been neglected.
This is indeed justified for an elementary vortex and
for the large values of the Ginzburg-Landau parameter
$\kappa \gg 1$ when the vortex states are predominantly formed
through the Andreev reflection and are localized near
the core boundary [21–23]. However, with the increas-
ing winding number the magnetic flux through the vor-
tex core grows and for the giant vortices with $n \gg \kappa$
the zero modes are formed through a fine interplay be-
tween the magnetic effects and the long-range Andreev
reflection resulting in a set of the warped lowest Landau
level states. Note that the effect of the magnetic field on
the delocalization of the zero modes for the elementary
vortex lattice has been recently discussed in Ref. [24].

To conclude, we have applied an advanced asymptotic
method based on the scale separation and the expansion
in inverse powers of the winding number to find the ana-
lysical solution for the Majorana zero modes of the giant
vortices. In the case of the Abrikosov vortices the solu-
tion reveals a nontrivial dynamical origin and a simple
universal structure, provided the vortex winding number
exceeds the value of the Ginzburg-Landau parameter $\kappa$
of the superconductor. It is not sensitive to the form of
the pair potential and is completely determined by a sin-
gle parameter, the vortex radius, which can be directly
measured in the experiment. The resulting local density
of states is confined to the vortex core, where the non-
zero modes are magnetically gapped. The density has a
characteristic profile which can be used as a signature for
the identification of the Majorana zero modes by scan-
ning electron microscopy [12]. For the hard-wall giant
vortices in the specially designed heterostructures [8] we
have found that a half of the zero modes are pinned to
the vortex edge with the other half filling the vortex core.
The dependence of the energy on the applied magnetic
field, characteristic of the non-zero modes, can therefore
be used for a clear identification of the Majorana core
states. Moreover, the density of the zero modes is quan-
tized and changes discreetly under the variation of the
magnetic field with the universal average rate given by
the Josephson constant. These features establish the gi-
ant vortices as an ideal laboratory for the search of com-
pelling experimental evidence of the Majorana fermions
in two dimensions.

Acknowledgments. We would like to thank Joseph
Maciejko for many useful discussions. The work of L.G.
was supported through by NSERC. The work of A.P. was
supported in part by NSERC and the Perimeter Institute
for Theoretical Physics.

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