Statistical Inference for Local Granger Causality

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Abstract

Granger causality has been employed to investigate causality relations between components of stationary multiple time series. We generalize this concept by developing statistical inference for local Granger causality for multivariate locally stationary processes. Our proposed local Granger causality approach captures time-evolving causality relationships in nonstationary processes. The proposed local Granger causality is well represented in the frequency domain and estimated based on the parametric time-varying spectral density matrix using the local Whittle likelihood. Under regularity conditions, we demonstrate that the estimators converge to multivariate normal in distribution. Additionally, the test statistic for the local Granger causality is shown

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to be asymptotically distributed as a quadratic form of a multivariate normal distribution. The finite sample performance is confirmed with several simulation studies for multivariate time-varying autoregressive models. For practical demonstration, the proposed local Granger causality method uncovered new functional connectivity relationships between channels in brain signals. Moreover, the method was able to identify structural changes in financial data.

*Keywords:* local Granger causality, multivariate locally stationary processes, time-varying spectral density matrix, local Whittle likelihood, brain signals.

1 Introduction

Statistical inference for cause and effect remains at the forefront of many studies including biology, medicine, physical systems, environmental science, public health, policy and finance. However, there remain challenges on inference because causality is notoriously difficult to establish. Granger causality, proposed in Granger (1963) and Granger (1969), is a milestone of causal inference in dynamic models. In broad terms, Granger causality from a time series \( \{Y_t\} \) to another series \( \{X_t\} \) measures the predictive ability from the series \( \{Y_t\} \) to \( \{X_t\} \). If the predictive ability of \( (X_s, Y_s)_{s<t} \) on \( X_t \) is not different from the predictive ability of \( (X_s)_{s<t} \) on \( X_t \), then there is “no Granger causal relationship” from the series \( \{Y_t\} \) to \( \{X_t\} \). Thus, Granger causality analysis is important for determining whether or not a set of variables contains useful information for improving the prediction of another set of variables.

Geweke (1982) and Geweke (1984) considered measures of linear dependence and feedback between components of a multivariate time series in both time and frequency domains. Hosoya (1991) proposed a refinement of the above measures, which also has a well-defined representation in the frequency domain. For nonstationary vector autoregressive (VAR) models, Sims et al. (1990) considered the Wald’s statistic for the hypothesis testing prob-
lem and elucidated its nonstandard asymptotic distribution. Granger causality for non-
stationary bivariate cointegrated processes has also been considered in Granger and Lin
(1993), among others. A thorough treatment of the Granger causality for multivariate
time series is discussed in Lütkepohl (2005).

In this paper, we propose a local Granger causality measure based on the locally stationary
process. A locally stationary process has a Cramér-like representation but its transfer
function is allowed to change over time. Formal models for the time-varying spectra of
nonstationary processes have been developed since this concept was introduced in Priestley
(1965). A more theoretically rigorous framework for multivariate locally stationary pro-
cesses have been formulated in Dahlhaus (2000). To estimate the time-varying spectral
density of a locally stationary process, Neumann and von Sachs (1997) developed a wavelet
estimator based on the pre-periodogram. The parameter estimation for an evolutionary
spectral is discussed in Dahlhaus and Giraitis (1998). Local inference for locally stationary
time series was investigated in Dahlhaus (2009). Moreover, Dahlhaus and Polonik (2009)
constructed the estimation theory for the weak convergence of the empirical spectral pro-
cesses. To the best of our knowledge, despite the recent progress on models that capture
nonstationary behavior, local Granger causality has not yet been developed. To address this
limitation, this paper undertakes the task of developing this local concept because many
time series phenomena display Granger causality behavior that changes over the course of
time (e.g., electroencephalograms and stock market indices). Thus, the contribution of this
paper is a rigorous framework for statistical inference for local Granger causality.

We focus on multivariate locally stationary processes to develop the statistical inference
for local Granger causality. Statistical inference for multivariate stationary processes has
been discussed in Hannan (1970), Taniguchi and Kakizawa (2000), Shumway and Stoffer,
(2004) and references therein. Taniguchi et al. (1996) developed a nonparametric method to test the cross-relationships between multiple time series. Sakiyama and Taniguchi (2004) discussed the discriminant analysis for multivariate locally stationary processes based on the likelihood ratio. Huang et al. (2004) proposed a SLEX model to develop a discriminant scheme that can extract local features of time series. Ombao et al. (2001) and Ombao et al. (2005) developed models for bivariate and multivariate nonstationary data using the SLEX basis which consists of well-localized Fourier-like waveforms.

As noted, the goal of this paper is to develop statistical inference for local Granger causality for multivariate locally stationary processes. In particular, local Granger causality is expressed in the frequency domain using the foundational ideas on Granger causality for stationary processes. We develop a procedure for parameter estimation based on the local Whittle likelihood and derive the asymptotic distribution of the estimators. Under regularity conditions, the estimates are shown to converge to multivariate normal in distribution. Parametric Granger causality, however, converges to normal or a quadratic form of normal random variables, which depends on the gradient of the causality measure. Several simulation studies were conducted to evaluate the finite sample performance for multivariate time-varying autoregressive models. To illustrate the potential impact of the proposed work, we analyzed the log-returns of the financial data and multichannel electroencephalogram (EEG) data. Using the proposed method, the local Granger causality analyses produced new insightful results in the data analyses.

The remainder of the paper is organized as follows. In Section 2, we propose the local Granger causality. The properties of the local Granger causality are detailed immediately behind the definition. In Section 3, we develop statistical inference for local Granger causality based on the local Whittle estimation for multivariate locally stationary processes.
Numerical results on finite sample performance of the estimator and the test statistic for local causality are reported in Section 4. In Section 5, we apply the proposed local Granger causality to EEG data and financial data. The proofs for the theoretical results in Section 3 are relegated to Section A in supplement.

1.1 Notations

$O_{m \times M}$ denotes a $m \times M$ zero matrix; $I_p$ denotes the $p \times p$ identity matrix; For any matrix $A$, let $\|A\|_\infty := \max_{1 \leq i \leq p} \sum_{j=1}^{p} |a_{ij}|$. For a square matrix $A$, $|A|$ denotes its determinant. $\overset{d}{\rightarrow}$ denotes the convergence in distribution. Additionally, let $l$ be a function such that

$$l(j) := \begin{cases} 1, & |j| \leq 1, \\ |j| \log^{1+\kappa} |j|, & |j| > 1, \end{cases}$$

for some constant $\kappa > 0$.

2 Local Granger Causality

In this section, we introduce the concept of local Granger causality (LGC) in the framework of locally stationary processes. Let $X_{t,T} = (X_{t,T}^{(1)}, \ldots, X_{t,T}^{(p)})^T$ be a sequence of $p$-dimensional multivariate stochastic processes

$$X_{t,T} = \sum_{j=-\infty}^{\infty} A_{t,T}(j) \epsilon_{t-j}, \quad (1)$$

where the sequences $\{A_{t,T}(j)\}_{j \in \mathbb{Z}}$ satisfy the following conditions: there exists a positive constant $C_A$ such that

$$\sup_{t,T} \|A_{t,T}(j)\|_\infty \leq \frac{C_A}{l(j)},$$

for
and there exists a sequence of functions $A(\cdot, j) : [0, 1] \to \mathbb{R}$ such that

(i) $\sup_u \|A(u, j)\|_{\infty} \leq \frac{C_A}{l(j)}$;

(ii) $\sup_j \|A_{t,T}(j) - A\left(\frac{t}{T}, j\right)\|_{\infty} \leq \frac{C_A}{l(j)} T^{-1}$;

(iii) $V\left(\|A(\cdot, j)\|_{\infty}\right) \leq \frac{C_A}{l(j)}$,

where $V(f)$ is the total variation of the function $f$ on the interval $[0, 1]$, i.e., $V$ is defined as

$$V(f) = \sup \left\{ \sum_{k=1}^{m} |f(x_k) - f(x_{k-1})|; 0 \leq x_0 < \cdots < x_m \leq 1, m \in \mathbb{N} \right\}.$$ 

The process (1) is usually referred to as the multivariate locally stationary process. We impose the following assumptions on the process (1) for the estimation theory later on.

**Assumption 2.1.** For the process in (1), let $\epsilon_t$ be independent and identically distributed with $E\epsilon_t = 0$ and $E\epsilon_t \epsilon_t^\top = K$, where the matrix $K$ exists and all elements are bounded by $C_K$. Furthermore, all elements in the $r$th moment of $\epsilon_t$ exist and bounded by $C_{\epsilon^r}$. There exists a finite constant $C > 0$ such that $C_{\epsilon^r} < C$.

Let $m$ and $M$ be two positive integers such that $p = m + M$. Suppose $X_{t,T} = (X_{t,T}^{(1)} \top, X_{t,T}^{(2)} \top \top, X_{t,T}^{(1)} \in \mathbb{R}^m, X_{t,T}^{(2)} \in \mathbb{R}^M$, has the time-varying spectral density matrix $f(u, \lambda)$ with the partition

$$f(u, \lambda) = \begin{pmatrix} f(u, \lambda)_{11} & f(u, \lambda)_{12} \\ f(u, \lambda)_{21} & f(u, \lambda)_{22} \end{pmatrix} := \frac{1}{2\pi} A(u, \lambda) K A(u, -\lambda)^\top, \quad u \in [0, 1],$$

where $A(u, \lambda) := \sum_{j=-\infty}^{\infty} A(u, j) \exp(i j \lambda)$. Let $\Sigma(u)$ be the one-step-ahead prediction error covariance matrix based on the time-varying spectral density matrix $f(u, \lambda)$ with the same
partition. By the Kolmogorov’s formula for multiple time series, we have
\[
\det \Sigma(u) = \exp \left( \frac{1}{2\pi} \int_{-\pi}^{\pi} \log \left( \det 2\pi f(u, \lambda) \right) \, d\lambda \right), \quad \text{for any } u \in [0, 1],
\]
(see Hannan (1970), p.162).

Let \( H(\tau_1, \tau_2) = \mathbb{P}(X_{t,T}^{(1)}, 1 \leq t \leq \tau_1; X_{t,T}^{(2)}, 1 \leq t \leq \tau_2) \) be the closed linear subspace generated by \( \{X_{t,T}^{(1)}, 1 \leq t \leq \tau_1; X_{t,T}^{(2)}, 1 \leq t \leq \tau_2\} \). Especially, we use \( H(\tau_1, 0) \) and \( H(0, \tau_2) \) to express the closed linear subspace generated by \( \{X_{t,T}^{(1)}, t \leq \tau_1\} \) and \( \{X_{t,T}^{(2)}, t \leq \tau_2\} \), respectively.

Introducing a companion process
\[
Y_{t,T}^{(2)} = X_{t,T}^{(2)} - E(X_{t,T}^{(2)} \mid H(t-1,t-1)),
\]
we propose the local Granger causality measure from \( \{X_{t,T}^{(2)}\} \) to \( \{X_{t,T}^{(1)}\} \) as
\[
\text{GC}^{(2\rightarrow1)}(u) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \text{FGC}(u, \lambda) \, d\lambda,
\]
where
\[
\text{FGC}(u, \lambda) = \log \left( \frac{|\hat{f}(u, \lambda)_{11}|}{|\hat{f}(u, \lambda)_{11} - 2\pi \hat{f}(u, \lambda)_{12} \hat{\Sigma}(u)_{22}^{-1} \hat{f}(u, \lambda)_{21}|} \right).
\]
Here, \( \hat{f}(u, \lambda) \) is the time-varying spectral density matrix of the process \( \{(X_{t,T}^{(1)} \top, Y_{t,T}^{(2)} \top) \top\} \) (see Propositions 2.1 and 2.2 below), and \( \hat{\Sigma}(u) \) is an \((M \times M)\)-matrix
\[
\hat{\Sigma}(u)_{22} = \Sigma(u)_{22} - \Sigma(u)_{21} \Sigma(u)_{11}^{-1} \Sigma(u)_{12}.
\]

The proposal of the local Granger causality (4) is motivated by Hosoya’s measure of causality (a term first coined in Granger and Lin (1995)) in combination with the non-stationary version of Kolmogorov’s formula by Dahlhaus (1996). The companion process \( \{Y_{t,T}^{(2)}\} \) in (3) is introduced in order to remove the possible effect brought by the nonorthogonality between residuals of predictions \( E(X_{t,T}^{(1)} \mid H(t-1,t-1)) \) and \( E(X_{t,T}^{(2)} \mid H(t-1,t-1)) \).
We now start to explain properties of the proposed local Granger causality. Denote by \( \Sigma_{t,T} \) the one-step-ahead prediction error covariance matrix, i.e.,

\[
\Sigma_{t,T} = \text{var}[X_{t,T} - E(X_{t,T} | H(t-1, t-1))].
\]

**Proposition 2.1.** The companion process \( \{Y_{t,T}^{(2)}\} \) is a locally stationary process with the time-varying spectral density

\[
f(u, \lambda) = \frac{1}{2\pi} \tilde{\Sigma}(u)\lambda^2.
\]

**Proof.** From the definition of \( \{X_{t,T}\} \) in (1), we have

\[
X_{t,T} - E(X_{t,T} | H(t-1, t-1)) = A_{t,T}(0) \epsilon_t.
\]

By the following formula (see Lemma 2.2 in Hosoya (1991))

\[
X^{(2)}_{t,T} - E(X^{(2)}_{t,T} | H(t, t-1)) = X^{(2)}_{t,T} - E(X^{(2)}_{t,T} | H(t-1, t-1)) - \Sigma_{t,T,21} \Sigma_{t,T,11}^{-1} \{X^{(1)}_{t,T} - E(X^{(1)}_{t,T} | H(t-1, t-1))\},
\]

we find that

\[
Y^{(2)}_{t,T} = \epsilon_t^{(2)} - \Sigma_{t,T,21} \Sigma_{t,T,11}^{-1} \epsilon_t^{(1)} = \left(-\Sigma_{t,T,21} \Sigma_{t,T,11}^{-1} I_M\right) \epsilon_t.
\]

In view of Example 2.3 (i) in Dahlhaus (2000), \( \{Y_{t,T}^{(2)}\} \) is locally stationary. A straightforward calculation gives the expression of \( f(u, \lambda) \) in (5).

Let \( H^{(2)}(\tau) \) be the closed linear subspace generated by \( \{Y_{t,T}^{(2)}, 1 \leq t \leq \tau\} \). The Hosoya measure is defined as

\[
HM_{t,T}^{(2 \rightarrow 1)} := \log \frac{\det \text{var}[X^{(1)}_{t,T} - E(X^{(1)}_{t,T} | H(t-1, 0))]}{\det \text{var}[X^{(1)}_{t,T} - E(X^{(1)}_{t,T} | \sigma\{H(t-1, 0) \cup H^{(2)}(t-1)\})]}. \tag{6}
\]
For any fixed $u \in [0, 1]$, the time-varying spectral matrix $f(u, \lambda)$ has a factorization

$$f(u, \lambda) = \frac{1}{2\pi} \Lambda(u, e^{-i\lambda})\Lambda(u, e^{i\lambda})^*, \quad z \in \mathcal{D},$$

(7)

(see Rozanov (1967)).

**Proposition 2.2.** Suppose all eigenvalues of $A(u, \lambda)A(u, -\lambda)^\top$ are bounded from below by some constant $C > 0$ uniformly in $u$ and $\lambda$, and all components of $A(u, \lambda)$ are differentiable in $u$ and $\lambda$ with bounded derivatives $(\partial/\partial u)(\partial/\partial \lambda)A(u, \lambda)_{ab}$ for $a, b \in \{1, 2, \ldots, p\}$. It holds that

$$|GC^{(2\rightarrow 1)}(t/T) - HM^{(2\rightarrow 1)}_{t,T}| = o_t(1) + O_T(1),$$

where the $o_t(1)$ term is uniform in $T$ and the $o_T(1)$ term is uniform in $t$.

**Proof.** From Proposition 2.1, we see that the process $\{(X^{(1)}_{t,T}, Y^{(2)}_{t,T})^\top\}$ is locally stationary. In view of Lemma 2.3 in Hosoya (1991), we see that the process has the time-varying spectral density matrix $f(u, \lambda)$ with $f(u, \lambda)_{11} = f(u, \lambda)_{11}$ and

$$f(u, \lambda)_{21} = f(u, -\lambda)^\top_{12} = \left(-\Sigma(u)_{21}\Sigma(u)^{-1}_{11} M\right) \Lambda(u, 0)\Lambda(u, e^{i\lambda})^{-1} \begin{pmatrix} f(u, \lambda)_{11} \\ f(u, \lambda)_{12} \end{pmatrix}.$$ (8)

A direct computation shows that the process $\{X^{(1)}_{t,T} - E(X^{(1)}_{t,T}) \mid \sigma(H(t-1, 0) \cup H^{(2)}(t-1))\}$ is still locally stationary and has the time-varying spectral density

$$f(u, \lambda)_{11} - f(u, \lambda)_{12}f(u, \lambda)^{-1}_{22}f(u, \lambda)_{21}.$$  

Inspection of Theorem 3.2 in Dahlhaus (1996) for the nonstationary version of Kolmogorov’s formula reveals that

$$\det \text{var}[X^{(1)}_{t,T} - E(X^{(1)}_{t,T}) \mid H(t-1, 0)] =$$
\[
\exp \left( \frac{1}{2\pi} \int_{-\pi}^{\pi} \log \left( \det 2\pi f(t/T, \lambda)_{11} \right) d\lambda \right) + o_t(1) + o_T(1), \quad (9)
\]

and

\[
\det \text{var} \left[ \mathbf{X}_{t,T}^{(1)} - E(\mathbf{X}_{t,T}^{(1)} \mid \sigma \{H(t-1,0) \cup \mathcal{H}^{(2)}(t-1)\}) \right]
= \exp \left( \frac{1}{2\pi} \int_{-\pi}^{\pi} \log \left( \det 2\pi \left( f(t/T, \lambda)_{11} - f(t/T, \lambda)_{12} \tilde{\Sigma}_{t,T}^{-1} f(t/T, \lambda)_{21} \right) \right) d\lambda \right)
+ o_t(1) + o_T(1). \quad (10)
\]

Combining (9) and (10) yields the desired result. \( \square \)

The local Granger causality measure can be regarded as the limit of that constructed by the Wigner-Ville spectrum. To be specific, let \( f_{t,T}(\lambda) \) be the Wigner-Ville spectrum of the process \( \{ \mathbf{X}_{t,T} \} \), i.e.,

\[
f_{t,T}(\lambda) = \frac{1}{2\pi} \sum_{s=-\infty}^{\infty} \text{Cov} \left( \mathbf{X}_{[t-s/2],T}, \mathbf{X}_{[t+s/2],T} \right) \exp(-i\lambda s),
\]

(see Martin and Flandrin (1985)). The measure of the Wigner-Ville spectrum now is

\[
\text{GC}_{t,T}^{(2\to1)} = \frac{1}{2\pi} \int_{-\pi}^{\pi} \text{FGC}_{t,T}(\lambda) d\lambda,
\]

where

\[
\text{FGC}_{t,T}(\lambda) = \log \left| \frac{|f_{t,T}(\lambda)_{11}|}{f_{t,T}(\lambda)_{11} - 2\pi f_{t,T}(\lambda)_{12} \tilde{\Sigma}_{t,T;22}^{-1} f_{t,T}(\lambda)_{21}} \right|
\]

and \( \tilde{\Sigma}(u) \) is an \((M \times M)\)-matrix

\[
\tilde{\Sigma}_{t,T;22} = \Sigma_{t,T;22} - \Sigma_{t,T;21} \Sigma_{t,T;11}^{-1} \Sigma_{t,T;12}.
\]
Proposition 2.3. Suppose $f(u, \lambda)$ is uniformly Lipschitz continuous with respect to $u$ and $\lambda$. For any sequence $t/T \to u$, We have
\[
\left| GC^{(2 \to 1)}(u) - GC^{(2 \to 1)}_{t,T} \right| = o(1).
\]

Proof. We only show that
\[
\frac{1}{2\pi} \int_{-\pi}^{\pi} \left( \log|f(u, \lambda)|_1 - \log|f_{t,T}(\lambda)|_1 \right) d\lambda = o(1),
\]
to see the difference in the numerator. The denominator can be proved similarly.

Let us consider the scalar process $\alpha^* X^{(1)}_{t,T}$ with the Wigner-Ville spectrum $\alpha^* f_{t,T}(\lambda) := \alpha^* f_{t,T}(\lambda) \alpha$ for any $\alpha \in \mathbb{C}^m$. Comparing it with $f^{\alpha}(u, \lambda) = \alpha^* f(u, \lambda) \alpha$, by Theorem 2.2 in Dahlhaus (1996), we see that
\[
\int_{-\pi}^{\pi} |f_{t,T}(\lambda) - f^{\alpha}(u, \lambda)|^2 d\lambda = o(1),
\]
which implies that
\[
\int_{-\pi}^{\pi} |f_{t,T}(\lambda) - f^{\alpha}(u, \lambda)| d\lambda = o(1),
\]
(11) since by the Cauchy-Schwarz inequality, we have
\[
\int_{-\pi}^{\pi} |f_{t,T}(\lambda) - f^{\alpha}(u, \lambda)| d\lambda \leq \sqrt{2\pi} \left( \int_{-\pi}^{\pi} |f_{t,T}(\lambda) - f^{\alpha}(u, \lambda)|^2 d\lambda \right)^{1/2}.
\]

By Taylor's expansion, we have
\[
\log|f_{t,T}(\lambda)|_1 = \log|f(u, \lambda)|_1 + \text{Tr} \left[ f(u, \lambda)^{-1}_1 (f_{t,T}(\lambda)_1 - f(u, \lambda)_1)_1 \right] + o \left( \text{Tr} \left[ f(u, \lambda)^{-1}_1 (f_{t,T}(\lambda)_1 - f(u, \lambda)_1)_1 \right] \right). \tag{12}
\]
Remembering that $f(u, \lambda) = \frac{1}{2\pi} \Lambda(u, e^{-i\lambda}) \Lambda(u, e^{i\lambda})^*$ from (17), we see that there exists an $m \times m$ Hermitian matrix $B$ such that $f(u, \lambda)^{-1}_1 = B^* B$, and thus
\[
\text{Tr} \left[ f(u, \lambda)^{-1}_1 (f_{t,T}(\lambda)_1 - f(u, \lambda)_1)_1 \right] = \text{Tr} \left[ B (f_{t,T}(\lambda)_1 - f(u, \lambda)_1) B^* \right], \tag{13}
\]
11
which is a sum of quadratic forms \( f_{t,T}^{\alpha}(\lambda) - f^{\alpha}(u, \lambda) \). Applying (11) to (13) yields
\[
\int_{-\pi}^{\pi} \text{Tr} \left[ f(u, \lambda)_{11}^{-1} (f_{t,T}(\lambda)_{11} - f(u, \lambda)_{11}) \right] d\lambda = o(1),
\]
and by observing (12), we conclude that
\[
\frac{1}{2\pi} \int_{-\pi}^{\pi} \log|f(u, \lambda)_{11}| - \log|f_{t,T}(\lambda)_{11}| d\lambda = o(1).
\]

Remark 2.4. The construction of a linear predictor in practice for the locally stationary process may be of interest to some readers. It can be shown that the predictor for the locally stationary process and that for the stationary approximation are asymptotically equivalent under adequate conditions. In contrast, we focus on the nonstationary version of Kolmogorov’s formula found in [Dahlhaus (1996)]. We elucidated that our local causality measure, as the limit of the measure constructed by the Wigner-Ville spectrum, is a unique measure for multivariate locally stationary processes.

3 Statistical Inference for Local Granger Causality

In this section, we develop the foundations for statistical inference for local Granger causality. The proofs of the theoretical results are relegated to Section A in supplement.

3.1 Local Whittle estimation

Let \( \{X_{t,T}\} \) be the multivariate locally stationary process defined by (11) with the time-varying spectral density \( f(u, \lambda) \) defined by (2). The starting point is local estimation by fitting a parametric spectral density model \( f_{\theta}(\lambda), \theta \in \Theta \subset \mathbb{R}^d \), to \( f(u, \lambda) \).
Consider the observation stretch \((X_{1:T}, \ldots, X_{T,T})\) and define \(I_T(u, \lambda)\) to be the pre-periodogram matrix
\[
I_T(u, \lambda) = \frac{1}{2\pi} \sum_{l:1 \leq [uT+1/2+l/2] \leq T} X_{[uT+1/2+l/2],T} X_{[uT+1/2-l/2],T}^T \exp(-i\lambda l) \tag{14}
\]
Note that the pre-periodogram \(I_T(u, \lambda)\) was first introduced in Neumann and von Sachs (1997).

We define the spectral divergence \(L(\theta, u)\) between the parametric spectral density and the time-varying spectral density as
\[
L(\theta, u) = \int_{-\pi}^\pi \log \det f_\theta(\lambda) + \text{Tr} \left( f(u, \lambda) f_\theta^{-1}(\lambda) \right) \, d\lambda. \tag{15}
\]
For any fixed \(u \in [0,1]\), define \(\theta_0(u)\) as
\[
\theta_0(u) := \arg \min_{\theta \in \Theta} L(\theta, u). \tag{16}
\]
Let \(u_k := k/T\). The sample analogue \(L_T\) of the spectral divergence is defined as
\[
L_T(\theta, u) = \frac{1}{T} \sum_{k=1}^T \frac{1}{b_T} K \left( \frac{u - u_k}{b_T} \right) \int_{-\pi}^\pi \log \det f_\theta(\lambda) + \text{Tr} \left( I_T(u_k, \lambda) f_\theta^{-1}(\lambda) \right) \, d\lambda, \tag{17}
\]
and the local Whittle estimator of \(\hat{\theta}_T(u)\) is defined as
\[
\hat{\theta}_T(u) := \arg \min_{\theta \in \Theta} L_T(\theta, u). \tag{18}
\]
We impose the following assumptions on the class of time-varying spectral densities and the kernel function \(K\) in (17) to investigate the asymptotic properties of the local Whittle estimator (18).

**Assumption 3.1.**
(i) The time-varying spectral density matrix \( f(u, \lambda) \) is continuously differentiable with respect to \( u \) for \( u \in (0, 1) \).

(ii) \( K : \mathbb{R} \to \mathbb{R} \) is a nonnegative, bounded symmetric continuous function of bounded variation with a compact support \([-1, 1]\) satisfying \( \int K(x) \, dx = 1 \). Let

\[
K_b(x) := \frac{1}{b} K\left(\frac{x}{b}\right),
\]

where \( b := b_T \to 0 \), as \( T \to \infty \).

We now specify the regularity conditions for the parametric model \( f_{\theta}(\lambda) \) and the local parameter \( \theta(u) \). For the brevity, let \( \theta := \theta(u) \) when \( u \) does not matter.

**Assumption 3.2.**

(i) For any fixed \( u \in [0, 1] \), \( \theta(u) \in \Theta \), where \( \Theta \) is a compact subset of \( \mathbb{R}^d \).

(ii) For any fixed \( u \in [0, 1] \), \( f_{\theta^{(1)}(u)} \neq f_{\theta^{(2)}(u)} \) on a set of positive Lebesgue measure, if \( \theta^{(1)}(u) \neq \theta^{(2)}(u) \).

(iii) The parametric spectral density matrix \( f_{\theta}(\lambda) \) is bounded away from 0 for each component, and is continuously differentiable with respect to \( \lambda \) for \( \lambda \in (-\pi, \pi) \).

(iv) For any \( \theta \in \Theta \), \( f_{\theta} \) is positive definite and it is twice continuously differentiable with respect to \( \theta \).

(v) For any fixed \( u \in [0, 1] \),

(v-a) \( \theta_0(u) \in \Theta \) is the unique minimizer of \( \mathcal{L}(\theta, u) \) and lies in the interior of \( \Theta \).
the matrices

\[
M_I^u = \int_{-\pi}^{\pi} \left[ \frac{\partial^2}{\partial \theta \partial \theta^\top} \text{Tr} \left\{ f_\theta^{-1}(\lambda) I_T(u, \lambda) \right\} + \frac{\partial^2}{\partial \theta \partial \theta^\top} \log \det f_\theta(\lambda) \right] d\lambda
\]

and

\[
M_f^u = \int_{-\pi}^{\pi} \left[ \frac{\partial^2}{\partial \theta \partial \theta^\top} \text{Tr} \left\{ f_\theta^{-1}(\lambda) f(u, \lambda) \right\} + \frac{\partial^2}{\partial \theta \partial \theta^\top} \log \det f_\theta(\lambda) \right] d\lambda
\]

are both positive definite.

First, let us consider the asymptotics for the sample analog of the spectral divergence \( L_T(\theta, u) \).

**Theorem 3.1.** Suppose Assumptions 2.1, 3.1 and 3.2 hold. For any \( u \in (0, 1) \), if \( b_T^{-1} = o(T(\log T)^{-6}) \) and \( b_T = o(T^{-1/5}) \), then we have

\[
\sqrt{Tb_T} (L_T(\theta, u) - L(\theta, u)) \overset{d}{\to} N(0, \mathbb{V}(u)).
\]

as \( T \to \infty \), where

\[
\mathbb{V}(u) = 4\pi \int_{-1}^{1} K(v)^2 dv \left( \int_{-\pi}^{\pi} \text{Tr} \left( f(u, \lambda) f_\theta^{-1}(\lambda) f(u, \lambda) f_\theta^{-1}(\lambda) \right) d\lambda 
+ \frac{1}{2} \sum_{r,t,v,w=1}^{p} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} \left( f_\theta^{rt}(\lambda_1) f_\theta^{vw}(\lambda_2) \tilde{\gamma}_{rtuw}(u; -\lambda_1, \lambda_2, -\lambda_2) \right) d\lambda_1 d\lambda_2 \right),
\]

(19)

where \( \tilde{\gamma} \) is the fourth-order spectral density of the process.

**Remark 3.2.** Inspection of the proof of Theorem 3.1 reveals that the main order of bias is \( O(T^{1/2}b_T^{-5/2}) \) and the asymptotic variance is of order \( O(T^{-1}b_T^{-1}) \). The optimal order of the bandwidth \( b_T \) can be determined by equating squared bias and variance. Thus, we obtain
the optimal order \( b_T = O(T^{-1/3}) \) and the mean square error is \( O(T^{-2/3}) \). This optimal order is similar to the one derived in Künsch (1989) in the context of statistical inference for stationary time series.

Let \( M_{f,0}^u \) be

\[
M_{f,0}^u = \int_{-\pi}^{\pi} \left[ \frac{\partial^2}{\partial \theta \partial \theta^\top} \text{Tr} \left\{ f_\theta^{-1}(\lambda) f(u, \lambda) \right\} + \frac{\partial^2}{\partial \theta \partial \theta^\top} \log \det f_\theta(\lambda) \right]_{\theta = \theta_0(u)} \, d\lambda.
\]

Now we establish the asymptotic normality of the local estimator \( \hat{\theta}_T(u) \).

**Theorem 3.3.** Suppose Assumptions 2.1, 3.1 and 3.2 hold. For any \( u \in (0, 1) \), if \( b_T^{-1} = o(T(\log T)^{-6}) \) and \( b_T = o(T^{-1/5}) \), then we have

\[
\sqrt{T} b_T (\hat{\theta}_T(u) - \theta_0(u)) \xrightarrow{d} N(0, \mathbb{V}(u)),
\]

as \( T \to \infty \), where \( \mathbb{V}(u) := (M_{f,0}^u)^{-1} \mathbb{V}^\theta(u)(M_{f,0}^u)^{-1} \) and

\[
\mathbb{V}^\theta(u)_{ab} = 4\pi \int_{-1}^{1} K(v)^2 \, dv \left( \int_{-\pi}^{\pi} \text{Tr} \left\{ f(u, \lambda) \left\{ \frac{\partial}{\partial \theta_a} f_\theta^{-1}(\lambda) \right\} f(u, \lambda) \left\{ \frac{\partial}{\partial \theta_b} f_\theta^{-1}(\lambda) \right\} \right\} \, d\lambda \right.

\left. + \frac{1}{2} \sum_{r,t,v,w=1}^{p} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} \left( \frac{\partial}{\partial \theta_a} f_\theta^r(\lambda_1) \cdot \frac{\partial}{\partial \theta_b} f_\theta^w(\lambda_2) \tilde{\gamma}_{rtvw}(u; -\lambda_1, \lambda_2, -\lambda_2) \right) \, d\lambda_1 \, d\lambda_2 \right),
\]

\((a, b = 1, \ldots, d)\), where \( \tilde{\gamma} \) is the fourth-order spectral density of the process.

**3.2 Inference for causality measures**

In this subsection, we develop the statistical inference for the local Granger causality measure (4) based on the parametric model \( \{ f_{\theta(u)} \mid \theta(u) \in \Theta \} \). Denote by \( \Sigma_{\theta(u)} \) the parametric
one-step-ahead prediction error matrix, and by $f_{\theta(u)}$ the parametric model for the companion process (3). Note that the matrix $f_{\theta(u)}$ is uniquely determined by the model $f_{\theta(u)}$ and the matrix $\Sigma_{\theta(u)}$ (see, e.g., (5) and (8)).

Suppose $f_{\theta(u)}$, $\Sigma_{\theta(u)}$ and $\hat{f}_{\theta(u)}$ have the same partition as (2). To make the statistical inference feasible, we impose the following assumption on the parametric models.

**Assumption 3.3.** For any $\theta \in \Theta$,

$$\int_{-\pi}^{\pi} \log |f_{\theta}(\lambda)| \, d\lambda > -\infty.$$  

Assumption 3.3 is usually referred to as the maximal rank condition. Under Assumption 3.3, $f_{\theta(u)}(\lambda)$ is non-degenerate a.e. and $\Sigma_{\theta(u)}$ is positive definite for any fixed $u \in [0, 1]$. Now the parametric local Granger causality for (4) is

$$GC^{(2\rightarrow 1)}(u; \theta) = \frac{1}{2\pi} \int_{-\pi}^{\pi} FGC(\lambda; \theta(u)) \, d\lambda, \quad (22)$$

where

$$FGC(\lambda; \theta) = \log \left| \frac{f_{\theta}(\lambda)_{11}}{f_{\theta}(\lambda)_{11} - 2\pi \hat{f}_{\theta}(\lambda)_{12} \hat{\Sigma}_{\theta,22}^{-1} \hat{f}_{\theta}(\lambda)_{21}} \right|.$$ 

The main results are described in the following.

**Theorem 3.4.** Suppose Assumptions [2.1], [3.1], [3.2] and [3.3] hold. If we have, for some $i = 1, \ldots, d$,

$$\frac{\partial}{\partial \theta_i} FGC^{(2\rightarrow 1)}(\lambda) \bigg|_{\theta = \theta_0(u)} \neq 0, \quad \text{for some } \lambda \in (-\pi, \pi), \quad (23)$$

uniformly in $u \in [0, 1]$, and if $b_T^{-1} = o(T(\log T)^{-6})$ and $b_T = o(T^{-1/5})$, then we have

$$\sqrt{Tb_T} \left( GC^{(2\rightarrow 1)}(u; \hat{\theta}_T) - GC^{(2\rightarrow 1)}(u; \theta_0) \right) \xrightarrow{d} N(0, \Psi^{GC}(u)),$$
where
\begin{align*}
V^{GC}(u) &= \left( \nabla^{2}GC(2\to1)(u; \theta_0) \right)^\top \nabla(u) \left( \nabla^{2}GC(2\to1)(u; \theta_0) \right), \\
\nabla^{2}GC(2\to1)(u; \theta_0) &= \left( \frac{\partial}{\partial \theta_1} GC(2\to1)(u; \theta_0), \ldots, \frac{\partial}{\partial \theta_d} GC(2\to1)(u; \theta_0) \right)^\top,
\end{align*}
and
\begin{align*}
\nabla^{2}GC(2\to1)(u; \theta_0) &= \left( \frac{\partial}{\partial \theta_i} FC(2\to1)(u; \theta_0) \mid_{\theta=\theta_0(u)} \right) \neq 0, \quad \text{a.e.} \, \lambda \in (-\pi, \pi].
\end{align*}

There are situations when condition (23) may not be satisfied. That is, for some \( u \in (0, 1) \),
\begin{align}
\frac{\partial}{\partial \theta} FC(2\to1)(\lambda) \mid_{\theta=\theta_0(u)} = 0, \quad \text{a.e.} \, \lambda \in (-\pi, \pi].
\end{align}

In this case, we centralize \( GC(2\to1)(u; \hat{\theta}_T) \) as \( CGC(u; \hat{\theta}_T) \), i.e.,
\begin{align}
CGC(u; \hat{\theta}_T) := GC(2\to1)(u; \hat{\theta}_T) - GC(2\to1)(u; \theta_0).
\end{align}

Then we have the following result.

**Theorem 3.5.** Suppose that Assumptions 2.1, 3.1, 3.2 and 3.3 hold. In addition, assume (24) with
\begin{align}
\mathcal{H}(u, \lambda) := \frac{\partial^2}{\partial \theta \partial \theta^\top} FC(2\to1)(\lambda) \mid_{\theta=\theta_0(u)} \neq O_{d \times d}, \quad \text{for some } \lambda \in (-\pi, \pi],
\end{align}
for \( u \in (0, 1) \). Let
\begin{align}
\mathcal{H}(u) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \mathcal{H}(u, \lambda) \, d\lambda.
\end{align}
Then if \( b_T^{-1} = o(T(\log T)^{-6}) \) and \( b_T = o(T^{-1/5}) \), the following result holds
\begin{align}
2Tb_T CGC(2\to1)(u; \hat{\theta}_T) \xrightarrow{d} \mathcal{N}(0, \mathcal{V}(u))^\top \mathcal{H}(u) \mathcal{N}(0, \mathcal{V}(u)),
\end{align}
for
where the normal distribution \( \mathcal{N}(0, \mathbb{V}(u)) \) is defined in Theorem 3.3. In particular, if \( \mathbb{V}(u)^{-1/2} \mathcal{H}(u) \mathbb{V}(u)^{-1/2} \) is an idempotent matrix, then the right hand side of (27) has a chi-squared distribution \( \chi^2_\nu \) with the degrees of freedom

\[
\nu = \text{Tr}(\mathbb{V}(u)^{-1/2} \mathcal{H}(u) \mathbb{V}(u)^{-1/2}).
\]

Example 1 (Time-varying vector autoregression model). Suppose the multivariate locally Gaussian stationary process (I) has the time-varying spectral density

\[
f(u, \lambda) = \frac{1}{2\pi} \left( I_2 + A(u) \exp(i\lambda) \right)^{-1} \begin{pmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{12} & \sigma_{22} \end{pmatrix} \left( I_2 + A(u)^\top \exp(-i\lambda) \right)^{-1},
\]

where \( A(u) = \begin{pmatrix} \alpha_{11}(u) & \alpha_{12}(u) \\ \alpha_{21}(u) & \alpha_{22}(u) \end{pmatrix} \) and \( \alpha_{12}(u) \equiv 0 \).

We adopt the following parametric spectral density \( f_\theta(\lambda) \) for model fitting.

\[
f_\theta(\lambda) = \frac{1}{2\pi} \left( I_2 + \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \exp(i\lambda) \right)^{-1} \begin{pmatrix} s_{11} & s_{12} \\ s_{12} & s_{22} \end{pmatrix} \left( I_2 + \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \exp(-i\lambda) \right)^{-1},
\]

where \( \theta = (a_{11}, a_{12}, a_{21}, a_{22}, s_{11}, s_{12}, s_{22})^\top \). From the definition (22), we have

\[
\text{GC}^{(2\rightarrow1)}(u; \theta) = \frac{1}{2\pi} \int_{-\pi}^{\pi} -\log \left| 1 - 2\pi f_\theta(\lambda)_{12} \left( \Sigma_{\theta,22} \right)^{-1} f_\theta(\lambda)_{21} f_\theta(\lambda)_{11}^{-1} \right| \, d\lambda.
\]

Thus, \( \text{FGC}^{(2\rightarrow1)}_\theta(\lambda) \) is

\[
\text{FGC}^{(2\rightarrow1)}_\theta(\lambda) = -\log \left| 1 - 2\pi f_\theta(\lambda)_{12} \left( \Sigma_{\theta,22} \right)^{-1} f_\theta(\lambda)_{21} f_\theta(\lambda)_{11}^{-1} \right|.
\]

A straightforward computation leads to

\[
\frac{\partial}{\partial \theta} \text{FGC}^{(2\rightarrow1)}_\theta(\lambda) \bigg|_{a_{12}=0} = 0, \quad \text{for any } \lambda \in (-\pi, \pi].
\]
In addition, it holds that
\[
\frac{\partial^2}{\partial a_{12}^2} \text{FGC}_{\theta}^{(2\rightarrow1)}(\lambda) \bigg|_{\theta=\theta_0(u)} = \frac{2(\sigma_{11} \sigma_{22} - \sigma_{12}^2)}{\sigma_{11}^4} \frac{|1 - \alpha_{11}(u) \exp(i\lambda)|^2}{|1 - \alpha_{22}(u) \exp(i\lambda)|^2} > 0;
\]
and for all \(\lambda \in (-\pi, \pi]\),
\[
\frac{\partial^2}{\partial \theta_i \partial \theta_j} \text{FGC}_{\theta}^{(2\rightarrow1)}(\lambda) \bigg|_{\theta=\theta_0(u)} = 0, \quad \text{for } \theta_i \neq a_{12} \text{ or } \theta_j \neq a_{12}.
\]
Let \(\hat{\theta}_T = (\hat{\alpha}_{11}(u), \hat{\alpha}_{12}(u), \hat{\alpha}_{21}(u), \hat{\alpha}_{22}(u), \hat{\sigma}_{11}, \hat{\sigma}_{12}, \hat{\sigma}_{22})^\top\) be the local Whittle estimator defined in (18). Applying Theorem 3.5 we obtain
\[
Tb_T \frac{\sigma_{11}^4 \left( \int_{-1}^1 K(v)^2 \, dv \right)^{-1}}{(1 + \alpha_{11}(u)^2 - 2\alpha_{11}(u)\alpha_{22}(u))(\sigma_{11} \sigma_{22} - \sigma_{12}^2)^2} \text{CGC}(u; \hat{\theta}_T) \xrightarrow{d} \chi_1^2,
\]
(28)
since
\[
\mathcal{H}(u)_{22} = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{\partial^2}{\partial a_{12}^2} \text{FGC}_{\theta}^{(2\rightarrow1)}(\lambda) \bigg|_{\theta=\theta_0(u)} \, d\lambda
\]
\[
= \frac{2 \left(1 + \alpha_{11}(u)^2 - 2\alpha_{11}(u)\alpha_{22}(u)\right)(\sigma_{11} \sigma_{22} - \sigma_{12}^2)^2}{(1 - \alpha_{22}(u)^2)^2 \sigma_{11}^4}
\]
and
\[
\mathcal{V}(u)_{22} = (1 - \alpha_{22}(u)^2) \int_{-1}^1 K(v)^2 \, dv.
\]

### 3.3 Hypothesis testing for causality measures

We now address the hypothesis testing problem for the local measures \(\text{GC}^{(2\rightarrow1)}\). Suppose that we want to test for local causality at a particular rescaled time \(u \in [0, 1]\). Define the local hypothesis \(H_0^{(2\rightarrow1)}\) to be
\[
H_0^{(2\rightarrow1)}: \text{GC}^{(2\rightarrow1)}(u) = c.
\]
(29)
We consider two cases of the null hypothesis (29): (i) $c = 0$, and (ii) $c > 0$.

Let us first consider the case (i) $c = 0$. For any fixed $u \in [0, 1]$, with the shorthand $	heta = \theta(u)$, we have

\[
GC^{(2 \to 1)}(u; \theta) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \log \left| f_{\theta}(\lambda)_{11} \right| |f_{\theta}(\lambda)_{11} - 2\pi f_{\theta}(\lambda)_{12} (\tilde{\Sigma}_{\theta,22})^{-1} f_{\theta}(\lambda)_{21}| \, d\lambda,
\]

and thus, $GC^{(2 \to 1)}(u; \theta) = 0$ if and only if

\[
|f_{\theta}(\lambda)_{11}| = |f_{\theta}(\lambda)_{11} - 2\pi f_{\theta}(\lambda)_{12} (\tilde{\Sigma}_{\theta,22})^{-1} f_{\theta}(\lambda)_{21}|.
\]  \(30\)

Here, $f_{\theta}(\lambda)_{11}$ and $2\pi f_{\theta}(\lambda)_{12} (\tilde{\Sigma}_{\theta,22})^{-1} f_{\theta}(\lambda)_{21}$ are both Hermitian. Thus, the equality (30) holds if

\[
2\pi f_{\theta}(\lambda)_{12} (\tilde{\Sigma}_{\theta,22})^{-1} f_{\theta}(\lambda)_{21} = O_{m \times m}.
\]

It follows that $f_{\theta}(\lambda)_{12} = O_{m \times M}$, since $(\tilde{\Sigma}_{\theta,22})^{-1}$ is positive definite. Since $f_{\theta}(\lambda)_{12} = O_{m \times M}$, it is straightforward to see that

\[
\frac{\partial}{\partial \theta} FG C^{(2 \to 1)}(\lambda) = 0,
\]

which is the case we considered in Theorem 3.5.

Accordingly, for the local hypothesis $H_0^{(2 \to 1)}: GC^{(2 \to 1)}(u) = 0$, we take

\[
S^\dagger(u) := 2T b_T^T GC^{(2 \to 1)}(u; \hat{\theta}_T)
\]

as the test statistic. Then we have the following result.

**Theorem 3.6.** Suppose Assumptions 2.1, 3.1, 3.2 and 3.3 hold. Under the null hypothesis $H_0^{(2 \to 1)}: GC^{(2 \to 1)}(u) = 0$, if $b_T^{-1} = o(T(\log T)^{-6})$ and $b_T = o(T^{-1/5})$, it holds that

\[
S^\dagger(u) \xrightarrow{d} \mathcal{N}(0, \mathbb{V}(u))^\top \mathcal{H}(u) \mathcal{N}(0, \mathbb{V}(u)).
\]

where $\mathcal{N}(0, \mathbb{V}(u))$ is defined in Theorem 3.3.
Remark 3.7. The matrix $\mathcal{H}(u)$ in (26) is unknown in general, but it is determinable from the parameterization of $f_{\theta}(\lambda)$. In practice, the matrix $\mathcal{H}(u)$ should be replaced with its plug-in version

$$\mathcal{H}(u; \hat{\theta}_T) := \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{\partial^2}{\partial \theta \partial \theta} \text{FGC}^{(2\rightarrow1)}(\lambda) \bigg|_{\theta=\hat{\theta}_T(u)} \, d\lambda$$

to construct an asymptotic $(1 - \alpha)$ confidence interval for $\text{GC}^{(2\rightarrow1)}(u)$. Instead, in some situation, for instance, as a continuation of Example 1, if we take

$$\tilde{S}^\dagger(u) := Tb_T \left( \text{CGC}^{(2\rightarrow1)}(u; \hat{\theta}_T) - c \right)$$

as a sample version of the left hand side in (28), then the confidence interval is $[0, \chi^2_{1,1-\alpha}]$, where $\chi^2_{1,1-\alpha}$ denotes the $(1 - \alpha)$ quantile of the chi-squared distribution with 1 degree of freedom. It is straightforward to see that the test by $\tilde{S}^\dagger(u)$ is consistent. If the null hypothesis is rejected, then the conclusion is that at level $\alpha$ there is sufficient evidence to suggest that there exists local Granger causality from one series to another at rescaled time $u \in [0, 1]$.

We now move on to the second case (ii) $c > 0$. The Wald type test statistic $S^\ast(u)$ is

$$S^\ast(u) = Tb_T \left( \text{GC}^{(2\rightarrow1)}(u; \hat{\theta}_T) - c \right) \left[ \left( \nabla \text{CGC}^{(2\rightarrow1)}(u; \hat{\theta}_T) \right) \right]^{-1} \left( \text{GC}^{(2\rightarrow1)}(u; \hat{\theta}_T) - c \right).$$

The following result is a direct consequence of Theorem 3.4.

Theorem 3.8. Suppose Assumptions 2.1, 3.1, 3.2 and 3.3 hold. Under the null hypothesis $H_0^{(2\rightarrow1)} : \text{GC}^{(2\rightarrow1)}(u) = c > 0$, if $b_T^{-1} = o(T (\log T)^{-6})$ and $b_T = o(T^{-1/5})$, we have

$$S^\ast(u) \overset{d}{\rightarrow} \chi^2_d.$$
where $\chi_d^2$ is a chi-squared distribution with the degrees of freedom $d$.

**Remark 3.9.** The covariance matrix $\mathbb{V}(u)$ in (31) is usually unknown, and thus, we have to construct a consistent estimator $\hat{\mathbb{V}}(u)$ instead. This can be done by following Keenan (1987) or Taniguchi (1982).

### 4 Numerical Simulations

In this section, we investigate the finite sample performance of our proposed local Whittle estimation and the hypothesis testing for non-causality of multiple time series.

#### 4.1 Finite sample performance of local Whittle estimation

Let $\{X_{t,T}\}$ be a multivariate locally stationary process defined by

$$
X_{t,T} = A\left(\frac{t}{T}\right)X_{t-1,T} + \epsilon_t, \quad \epsilon_t \sim \mathcal{N}(0, I_2).
$$

We consider two examples for the time-varying coefficient matrix $A(\cdot)$. Let $A^{(i)}(u)$ and $A^{(ii)}(u)$ be

$$
A^{(i)}(u) = \begin{pmatrix} 1/2 & a_{12}(u) \\ 0 & 1/2 \end{pmatrix}, \quad A^{(ii)}(u) = \begin{pmatrix} 7/10 & a_{12}(u) \\ 0 & 3/10 \end{pmatrix},
$$

where $a_{12}(u)$ is defined as

$$
a_{12}(u) =
\begin{cases}
0, & 0 \leq u \leq 1/\pi, \\
\pi(u - 1/\pi)/2, & 1/\pi \leq u \leq 2/\pi, \\
1/2, & 2/\pi \leq u \leq 1.
\end{cases}
$$

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Note that \( a_{12} \) is a piecewise continuous function. Additionally, \( a_{12} \) is not differentiable at \( 1/\pi \) and \( 2/\pi \), but \( a_{12}(t/T) \) does not take values on those points since they are irrational. Thus, \( a_{12} \) could be replaced by a differentiable function instead.

To study the finite sample performance of the local Whittle estimation, define \( f^{(i)}_\theta \) and \( f^{(ii)}_\theta \) to be

\[
f^{(*)}_\theta(\lambda) = \frac{1}{2\pi} A^{(*)}_\theta(\lambda)^{-1} \left( A^{(*)}_\theta(-\lambda)^T \right)^{-1}, \quad \bullet = (i) \text{ or } (ii),
\]

where \( A^{(i)}_\theta(u, \lambda) \) and \( A^{(ii)}_\theta(u, \lambda) \) are

\[
A^{(i)}_\theta(u, \lambda) = I_2 - \begin{pmatrix} 1/2 & \theta \\ 0 & 1/2 \end{pmatrix} \exp(-i\lambda), \quad A^{(ii)}_\theta(u, \lambda) = I_2 - \begin{pmatrix} 7/10 & \theta \\ 0 & 3/10 \end{pmatrix} \exp(-i\lambda).
\]

The local Whittle likelihood estimator \( \hat{\theta}_T(u) \) is defined as in (18), where the Epanechnikov kernel is used with the bandwidth \( b_T = 4T^{1/3} \). The procedure was examined over 100 simulations for \( T = 50 \) and \( T = 100 \), respectively.

Define \( \hat{\theta}^{(j)}_T(u) \) to be the estimate from the \( j \)-th dataset where \( j = 1, \ldots, 100 \). The numerical results of the mean, 5th and 95th percentile of the estimates \( \{ \hat{\theta}^{(j)}_T(u) \} \) are shown in Figure 1. Both results for Examples (i) and (ii) are similar. In this finite sample performance, we see that the parameter estimation loses some accuracy when \( u \) is close to edges of \([0,1]\). On the other hand, the estimate \( \hat{\theta}_T(u) \) works well around the center of the interval. The estimates capture the general form of the true function well. We can expect that the performance could be better if the change in the function \( a_{12}(u) \) is smaller.

Here, we remark that the estimation is asymptotically unbiased from the theoretical results. Compared with the case when \( T = 50 \), the confidence interval has narrower widths with the same coverage probability when \( T = 100 \). These observations justify our theoretical results.
Figure 1: Example (i) (left) and Example (ii) (right). The function $a_{12}$ is shown in red; The mean, 5th and 95th percentile of the estimates when $T = 50$ are shown in blue; The mean, 5th and 95th percentile of the estimates when $T = 100$ are shown in orange.

4.2 Testing for non-causality

Let us consider the model (32) again. We replace the function $a_{12}(u)$ in (33) by

$$a_{12}(u) = \begin{cases} 
0, & 0 \leq u \leq \frac{5}{2\pi}, \\
\frac{\pi}{2}(u - \frac{5}{2\pi}), & \frac{5}{2\pi} \leq u \leq 1.
\end{cases} \quad (34)$$

In this simulation study, we tested the local hypothesis

$$H_0^{(2\rightarrow1)} : GC^{(2\rightarrow1)}(u) = 0. \quad (35)$$

for the following values of rescaled time $u = 0.1, 0.3, 0.5, 0.7, 0.9$. The null hypothesis is rejected whenever

$$\tilde{S}^+(u) > \chi^2_{1,1-\alpha}, \quad (36)$$

which the details for have been considered in Remark 3.7.

The rejection probabilities of the test (36) based on 1000 simulations are reported in Table 1. The rejection probabilities when $u = 0.1$ is very close to the nominal significance level $\alpha$. However, the rejection probabilities when $u = 0.3, 0.5, 0.7$ seem conservative.
Table 1: Rejection probabilities of the test (36) for the model (32) with $a_{12}$ in (34).

| $\alpha$ | $u = 0.1$ | $u = 0.3$ | $u = 0.5$ | $u = 0.7$ | $u = 0.9$ |
|----------|------------|------------|------------|------------|------------|
| 1%       | 0.007      | 0.000      | 0.002      | 0.002      | 0.019      |
| 5%       | 0.043      | 0.020      | 0.015      | 0.030      | 0.065      |
| 10%      | 0.096      | 0.053      | 0.037      | 0.081      | 0.123      |
| 15%      | 0.152      | 0.085      | 0.071      | 0.111      | 0.186      |

compared with the nominal significance level. This stems from the different asymptotic distributions between $\alpha_{12} = 0$ and $\alpha_{12} \neq 0$ (see Theorems 3.4 and 3.5). Table 1 confirms that the local power for testing the null hypothesis (35) increase with $u$, because of the feature of the function $a_{12}$ in the model (32). In summary, the testing for non-local Granger causality can be seen as a methodology to find a predictive sign to detect the structural change of a dynamic model.

5 Data Analysis

In this section, we apply local Granger causality to two real datasets – EEG data and financial data.

5.1 EEG data

We provide a brief description of the data. The EEG signals are sampled at the rate of 100 Hertz. The recordings are taken from channels the central channels (C3, C4, Cz), parietal channels (P3, P4) and the temporal channels (T3, T4, T5) which correspond roughly to the
central, parietal and temporal brain cortical regions (See Figure 2). The original dataset has 32680 time points for each channel (i.e., the period of the observation is 326.8 seconds). This dataset was previously analyzed in Ombao et al. (2005) and Schröder and Ombao (2019). However, none of these two papers addressed the very important issue of causality. This is the first paper that examined local Granger causality features in this data.

![Figure 2: EEG channels.](image)

Local Granger causality was estimated and tested at every rescaled time point $u_k = 2.1k/326.8$ and $u_k = 4.2k/326.8$, due to the computational cost of the local Whittle estimation. This is equivalent to estimating and testing every 2.1 and 4.2 seconds respectively. We refer to these partial data as one at regular intervals of 2.1 seconds and 4.2 seconds. In Figure 2, we show the logarithm of local Granger causality between two specific channels P3 (left parietal) and T3 (left temporal). These two channels are of primary interest because the patient suffered from left temporal lobe epilepsy - though the precise location is quite close to the parietal lobe. Thus the seizure focus is the left temporal lobe and any abnormalities in the EEG are captured in the T3 and P3 channels. The 95% confidence intervals are shown below: the dashed one is computed from the data at regular intervals of
4.2 seconds; the dotted one is computed from the data at regular intervals of 2.1 seconds.

Figure 3: The logarithm of local Granger causality (LLGC) from the channel T3 to P3 (left) and that from the channel P3 to T3 (right) during the rescaled time $u \in [0, 1]$. The LLGC for data at regular intervals of 4.2 seconds is shown in black and that for those of 2.1 seconds are shown in orange. The dashed red line shows the 95% confidence levels computed from the data at regular intervals of 4.2 seconds, while the dotted red line shows the 95% confidence levels computed from the data at regular intervals of 2.1 seconds.

The partial data at regular intervals of 2.1 seconds and 4.2 seconds share a very similar move of the logarithm of local Granger causality and the 95% confidence intervals are also very similar. In general, the higher temporal resolution the sampling is, theoretically more accurate the estimates are. In our simulation results, the estimates from the data at regular intervals of 2.1 seconds are as good as that of 4.2 seconds. The analysis suggests that T3 does not cause P3 in the Granger sense, but P3 causes T3 in the Granger sense at latest after the rescaled time $u = 0.15$. This is a quite interesting finding because previous analyses have focused on the T3 channel because of the distinctly large amplitudes immediately post-seizure onset. However, the novel finding here is that the direction of Granger causality actually flows from P3 to T3. This suggests that, despite the relatively lower amplitude
changes in P3, it still explains the future large amplitude fluctuations in T3.

![Figure 4: Plots of the logarithms of local Granger causality from the channels C3, Cz, C4 in the column to the channels C3, Cz, C4 in the row. The 95% confidence intervals are below the dashed red lines; The dashed black lines show the logarithms of local Granger causalities.](image)

Next, we further investigate the local causalities between the central channels C3, Cz and C4 at regular intervals of 4.2 seconds. Figure 4 represents the numerical results of the causality from the column to the row. For example, the middle plot in the first row shows the causality from the channel C3 to Cz. Still, the 95% confidence intervals are below the dashed red lines and the logarithms of local Granger causalities are shown by the dashed black lines. From Figure 4, the conclusion is that channel the left central channel C3
does not cause central channel Cz. Moreover, the right central channel C4 does not cause channels C3 and Cz - in the local Granger sense. In other cases, local Granger causality changes across the evolution of the epileptic seizure which confirms the dynamic activity of the brain. This is a new finding since all previous analyses were limited to modeling dependence using only coherence which accounts for contemporaneous dependence; that is, there was no phase or lead-lag analysis. Moreover, this novel finding is quite interesting because a change in the causality structure was captured even before the onset of the epileptic seizure, which was approximately at $u = 0.5$.

![Figure 5: Plots of the logarithms of local Granger causality from the channels T3, T5, T4 in the column to the channels T3, T5, T4 in the row. The 95% confidence intervals are below the dashed red lines; The dashed black lines show the logarithms of local Granger causalities.](image-url)
Similarly, we studied the causalities between the temporal channels T3, T5, T4 at regular intervals of 4.2 seconds. The plots of the numerical results are shown in Figure [5]. Remember that the channels T3 and T4 are symmetrically located at both the left and right temporal cortical regions, respectively. The plots suggest that T3 and T4 do not cause each other in the local Granger sense. Furthermore, the channel T5, also on the left temporal cortical region, uniformly causes T3 in the data which is another interesting novel finding. It is already known to the neurologist that the patient has left temporal lobe epilepsy and that seizure events are generally initiated in the "left temporal"l region (which is the area covered by the T3 and T5 channels). Using the novel proposed concept of local Granger causality, the analysis produced a highly specific result of brain functional connectivity, that is, the direction goes from T5 to T3 and not the other way around. As an additional result, P3 and P4 do not cause each other uniformly in the local Granger sense at 95% confidence level.

5.2 Stock market data

The dataset is the weekly log-returns of the closing stock prices of two financial groups (Mitsubishi and Mizuho) in the Nikkei index. For brevity, two financial groups are denoted by A and B here. The weekly data are from 2006 January 1st to 2010 December 26th, so the length of the data is \( T = 260 \).

In our data analysis, we compute local Granger causality \( GC^{A\rightarrow B}(u) \) and \( GC^{B\rightarrow A}(u) \) for rescaled time \( u = 1/T, 2/T, \ldots, 1 \). In general, Granger causality is not symmetric (e.g., the analysis of EEG data) and we regard

\[
\left\{ \left( GC^{A\rightarrow B}(u), GC^{B\rightarrow A}(u) \right) \in \mathbb{R}^2 \right\}_{u=\frac{1}{T}, \frac{2}{T}, \ldots, 1}
\] (37)
as a point cloud in $\mathbb{R}^2$. We separate local Granger causality in (37) into two parts: (1) from 2006 January 1st to 2008 June 29th; (2) from 2008 July 6th to 2010 December 26th. The part (1) and part (2) have the same length, i.e., the length of each part is 130.

We apply computational topology tools, *persistence diagram, persistence barcode, persistence landscape*, to capture the feature of these data points of local Granger causality (See, e.g. [Fasy et al. (2014)](#), for the details of the persistence diagram and persistence barcode). The plots of the persistence diagram and the persistence barcode are shown in Figure 6. The permutation test is applied to the point clouds (1) and (2) to test for equal topologies of the point clouds (1) and (2). In other words, the null hypothesis is no statistical difference between the persistence landscapes of local Granger causality. The result is statistically significant at the significance level of 0.01. It is known that there is a financial crisis between 2007 and 2008. Through the analysis of local Granger causality, we detected the structural change of the causality between these two financial data. A theoretical justification of this approach will be left as future work.
Figure 6: (Above) the persistence diagram of local Granger causality (left); and the persistence barcode of local Granger causality (right) during the period (1). (Below) the persistence diagram of local Granger causality (left); and the persistence barcode of local Granger causality (right) during the period (2).

6 Conclusion

The primary contribution of this paper is statistical inference for local Granger causality for multivariate time series under the framework of multivariate locally stationary processes. Our proposed concept of local Granger causality is a generalization of Geweke’s measure
and Hosoya’s measure - both of which were developed only for stationary processes. Our proposed generalization is well characterized in the frequency domain and has the advantage of being able to capture time-evolving causality relationships. We developed a procedure for hypothesis testing for the existence of the local Granger causality from a parametric point of view. We demonstrate, through the analysis of real data, the efficiency and efficacy of this procedure to find the time-evolving aspects of the local Granger causality, which could be overlooked by the existing method for causality analysis.

In summary, we proposed a consistent method to detect the time change of the local Granger causality. While our proposed method is nonparametric, we note that a procedure for stationary processes was developed in Taniguchi et al. (1996). This could serve as an inspiration for constructing a nonparametric method for locally stationary processes to test for the local Granger causality, and compare the performance of both approaches. For multiple time series, to investigate the time change of the local Granger causality also suffers from the curse of dimensionality. There are many remaining challenges including dimension reduction in terms of causality between each component of multiple time series. In addition to the Lasso method in Tibshirani (1996), most penalized estimation procedures could be added to our parametric approach to shrink some minor causality between components. A frequency-specific local causality approach will also be elucidated in our future work.

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A. Proofs

In Section A, we provide proofs for results in Section 3. The fundamental properties of multivariate locally stationary processes are condensed in Section B. Some technical results to derive the asymptotic distributions are summarized in Section C.

A.1 Proof of Theorem 3.1

**Proof.** From equations (15) and (17), we have

\[ L_T(\theta, u) - L(\theta, u) = \]

\[ \left\{ \frac{1}{T} \sum_{k=1}^{T} \frac{1}{b_T} K \left( \frac{u - u_k}{b_T} \right) \right\} \int_{-\pi}^{\pi} \log \det f_\theta(\lambda) \, d\lambda + \text{Tr}\left( f(u, \lambda) f_\theta^{-1}(\lambda) \right) \, d\lambda \]

\[ + \frac{1}{T} \sum_{k=1}^{T} \frac{1}{b_T} K \left( \frac{u - u_k}{b_T} \right) \int_{-\pi}^{\pi} \text{Tr}\left\{ \left( f(u_k, \lambda) - f(u, \lambda) \right) f_\theta^{-1}(\lambda) \right\} \, d\lambda \]

\[ + \frac{1}{T} \sum_{k=1}^{T} \frac{1}{b_T} K \left( \frac{u - u_k}{b_T} \right) \int_{-\pi}^{\pi} \text{Tr}\left\{ \left( I_T(u_k, \lambda) - f(u_k, \lambda) \right) f_\theta^{-1}(\lambda) \right\} \, d\lambda \]

\[ = L_1 + L_2 + L_3, \quad \text{(say).} \]

Since \( K \) is a function of bounded variation, applying Lemma P5.1 in Brillinger (1981), it holds that

\[ \frac{1}{T} \sum_{k=1}^{T} \frac{1}{b_T} K \left( \frac{u - u_k}{b_T} \right) - 1 = \int_{0}^{1} \frac{1}{b_T} K \left( \frac{u - v}{b_T} \right) \, dv - 1 + O(T^{-1}) \]

\[ = \int_{\frac{u-1}{b_T}}^{\frac{u}{b_T}} K(x) \, dx - 1 + O(T^{-1}), \]

which implies that \( L_1 = O(T^{-1}) \), since the kernel \( K \) has a compact support.
Under Assumption 3.1 (i), $f$ is of bounded variation, and again, applying Lemma P5.1 in Brillinger (1981), we have

$$
\frac{1}{T} \sum_{k=1}^{T} \frac{1}{b_T} K \left( \frac{u - u_k}{b_T} \right) \left( f(u_k, \lambda) - f(u, \lambda) \right)
$$

$$
= \int_{0}^{1} \frac{1}{b_T} K \left( \frac{u - v}{b_T} \right) \left( f(v, \lambda) - f(u, \lambda) \right) dv + O(T^{-1})
$$

$$
= \int_{\mathbb{R}} K(x) \left( f(u - b_T x, \lambda) - f(u, \lambda) \right) dx + O(T^{-1}),
$$

$$
= \int_{\mathbb{R}} K(x) \left( -b_T x \frac{\partial}{\partial u} f(u, \lambda) + (b_T x)^2 \frac{\partial^2}{\partial u^2} f(u, \lambda) + O(b_T^3) \right) dx + O(T^{-1}).
$$

Since $K$ has a compact support and it is symmetric, we have

$$
\int_{-\infty}^{\infty} x K(x) dx = 0,
$$

which implies that

$$
L_2 = O(b_T^2) + O(T^{-1}),
$$

as $T \to \infty$. In summary, we have

$$
\sqrt{Tb_T} L_1 \to 0,
$$

$$
\sqrt{Tb_T} L_2 \to 0,
$$

since $b_T = o(T^{-1/5})$.

Finally, we apply Corollary C.7 to $L_3$ to show

$$
\sqrt{Tb_T} \left( \mathcal{L}_T(\theta, u) - \mathcal{L}(\theta, u) \right) \xrightarrow{d} \mathcal{N}(0, \mathbb{V}(u)).
$$

(38)

In fact, we only have to check Assumptions C.1 and C.2 for

$$
\phi(u, \lambda) = K(u) f_{\theta}^{-1}(\lambda),
$$

(39)
or equivalently, \(\psi(u, \lambda) = K(u)f_{\theta}^{ij}(\lambda)\) for \(i, j = 1, \ldots, p\), which is expressed in the Einstein notation. From the definition (2) of the time-varying spectral density matrix, \(f_{\theta}^{-1}(\lambda)\) is obviously Hermitian. Additionally, Assumption C.1 (ii) is satisfied if both \(K\) and \(f_{\theta}^{-1}(\lambda)\) are bounded functions of bounded variation, which follows Assumptions 3.1 (ii) and 3.2 (iii). Applying Corollary C.7 to (39), we obtain (38). \(\square\)

### A.2 Proof of Theorem 3.3

**Proof.** Note that we have

\[
\sqrt{T}b_T(\mathcal{L}_T(\theta, u) - \mathcal{L}(\theta, u)) \xrightarrow{d} \mathcal{N}(0, \mathcal{V}(u)).
\]

The consequence (20) follows, if the following conditions are guaranteed for the theorem, i.e.,

(i) both \(\mathcal{L}_T(\theta, u)\) and \(\mathcal{L}(\theta, u)\) are convex in \(\theta\) for each \(u\) and continuous in \(u\) for each \(\theta\);

(ii) \(\theta_0(u)\) is the unique minimizer of \(\mathcal{L}(\theta, u)\) for each \(u \in [0, 1]\)

According to (i), the convexity of \(\mathcal{L}_T(\theta, u)\) and \(\mathcal{L}(\theta, u)\) in \(\theta\) follows from Assumption 3.2 (v-b). Especially, note that \(\mathcal{L}_T(\theta, u)\) is a linear combination of \(\int_{-\pi}^{\pi} \log \det f_{\theta}(\lambda) + \text{Tr} \left( I_T(u_k, \lambda)f_{\theta}^{-1}(\lambda) \right) d\lambda\) with nonnegative coefficients, which implies that \(\mathcal{L}_T(\theta, u)\) is convex. The continuity of \(\mathcal{L}_T(\theta, u)\) and \(\mathcal{L}(\theta, u)\) in \(u\) follows from Assumption 3.1, i.e., the continuity of \(K\) and \(f(\cdot, \lambda)\). According to (ii), it is assumed in Assumption 3.2 (v-a). Since \(\theta_0(u)\) is the unique minimizer and \(f_{\theta}(\lambda)\) is twice continuously differentiable with respect to \(\theta\), again, (21) follows from Corollary C.7. \(\square\)
A.3 Proof of Theorem 3.4

Proof. For simplicity, denote

\[ f^{Z^\theta}_{(u)}(\lambda)_{11} := f_{\theta(u)}(\lambda)_{11} - 2\pi f_{\theta(u)}(\lambda)_{12} \left( \Sigma_{\theta(u),22} \right)^{-1} \tilde{f}_{\theta(u)}(\lambda)_{21}. \]

Accordingly, GC\(^{(2\rightarrow 1)}\)(\(u; \theta\)) in (22) is simply

\[ \text{GC}^{2\rightarrow 1}(u; \theta) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \log \left| \frac{f_{\theta(u)}(\lambda)_{11}}{f^{Z^\theta}_{\theta(u)}(\lambda)_{11}} \right| \, d\lambda. \]

Note that the domain of the integration is bounded, and under Assumption 3.3, \( \log |f_{\theta(u)}(\lambda)_{11}| \) is integrable in \( \lambda \) for \( u \in [0, 1] \), which implies that \( \log |f^{Z^\theta}_{\theta(u)}(\lambda)_{11}| \) is also integrable.

Now if we show \( f_{\theta(u)}(\lambda)_{11} \) and \( f^{Z^\theta}_{\theta(u)}(\lambda)_{11} \) are continuously differentiable with respect to \( \theta \), then applying the delta-method to (20) leads to the conclusion. We summarize the parametric expressions used in the causality measure. Suppose that \( \tilde{f}_{\theta(u)}(\lambda) \) admits the decomposition

\[ \tilde{f}_{\theta(u)}(\lambda) = \begin{bmatrix} f_{\theta(u)}(\lambda)_{11} & f_{\theta(u)}(\lambda)_{12} \\ f_{\theta(u)}(\lambda)_{21} & f_{\theta(u)}(\lambda)_{22} \end{bmatrix}. \]

With an abuse of notation, under Assumption 3.3, \( f_{\theta(u)} \), defined on the unit disk \( D \) in the complex plane, can be factorized as

\[ f_{\theta(u)}(z) = \frac{1}{2\pi} \Lambda_{\theta(u)}(z) \Lambda^{*}_{\theta(u)}(z), \quad z \in D. \] (40)

Especially, as shown in Rozanov (1967), it holds that

\[ \Sigma_{\theta(u)} = \Lambda_{\theta(u)}(0) \Lambda^{*}_{\theta(u)}(0). \] (41)

From Lemmas 2.2 and 2.3 in Hosoya (1991), we have

\[ f_{\theta(u)}(\lambda)_{11} = f_{\theta(u)}(\lambda)_{11}, \quad f_{\theta(u)}(\lambda)_{12} = f_{\theta(u)}(\lambda)_{21}^{*}, \] (42)
\[ \tilde{f}_{\theta(u)}(\lambda)_{21} = \begin{pmatrix} -\Sigma_{\theta(u),21} \Sigma_{\theta(u),11}^{-1} I_M \end{pmatrix} \Lambda_{\theta(u)}(0) \Lambda_{\theta(u)}(e^{i\lambda})^{-1} \begin{pmatrix} f_{\theta(u)}(\lambda)_{11} \\ f_{\theta(u)}(\lambda)_{21} \end{pmatrix}, \quad (43) \]

and

\[ \tilde{f}_{\theta(u)}(\lambda)_{22} = \frac{1}{2\pi} \tilde{\Sigma}_{\theta(u),22} := \frac{1}{2\pi} \left\{ \Sigma_{\theta(u),22} - \Sigma_{\theta(u),21} \Sigma_{\theta(u),11}^{-1} \Sigma_{\theta(u),12} \right\}. \quad (44) \]

The continuous differentiability of \( f_{\theta(u)}(\lambda)_{11} \) with respect to \( \theta \) directly follows from that of \( f_{\theta(u)}(\lambda) \) under Assumption 3.2 (iv). Note that \( \tilde{\Sigma}_{\theta(u),22} \) is a continuous function of \( \Sigma_{\theta(u)} \) from (41). Using the expression for \( \Sigma_{\theta(u)} \) in (41) and the relation in (40), the continuous differentiability of \( \Sigma_{\theta(u)} \) with respect to \( \theta \) follows from that of \( f_{\theta(u)}(\lambda) \). In addition, this implies the continuous differentiability of \( \tilde{f}_{\theta(u)}(\lambda)_{21} \) from (43), which in turn implies the continuous differentiability of \( f_{\theta(u)}(\lambda)_{11} \). This completes the proof of Theorem 3.4. \( \square \)

### A.4 Proof of Theorem 3.5

**Proof.** By Theorem 6.8 of Magnus and Neudecker (2007), we have

\[
\text{GC}^{(2 \rightarrow 1)}(u; \hat{\theta}_T) - \text{GC}^{(2 \rightarrow 1)}(u; \theta_0) = \nabla \text{GC}^{(2 \rightarrow 1)}(u; \theta_0) \hat{\theta}_T \text{GC}^{(2 \rightarrow 1)}(u; \theta_0)^	op (\hat{\theta}_T(u) - \theta_0(u)) \\
+ \frac{1}{2} (\hat{\theta}_T(u) - \theta_0(u)) \text{H}(u) (\hat{\theta}_T(u) - \theta_0(u)) + o_P \left( (\hat{\theta}_T(u) - \theta_0(u))^2 \right).
\]

Since \( \nabla \text{GC}^{(2 \rightarrow 1)}(u; \theta_0) = 0 \) from (21), we have

\[
Tb_T(\text{GC}^{(2 \rightarrow 1)}(u; \hat{\theta}_T) - \text{GC}^{(2 \rightarrow 1)}(u; \theta_0)) = \\
\frac{1}{2} \sqrt{Tb_T(\hat{\theta}_T(u) - \theta_0(u))} \text{H}(u) \sqrt{Tb_T(\hat{\theta}_T(u) - \theta_0(u))} + o_P(1).
\]

We arrived at the conclusion (27) by the continuous mapping theorem. \( \square \)
B Multivariate locally stationary processes

In Section B we review the basic properties of multivariate locally stationary processes. Especially, we evaluate the absolute differences of the covariances and higher order cumulants between a multivariate locally stationary process \( \{X_{t,T}\} \) and the approximate stationary process \( \{X^*(u)\} \) with a spectral density matrix \( f(u, \lambda) \).

Let \( l(j) \) be

\[
  l(j) := \begin{cases} 
    1, & |j| \leq 1, \\
    |j| \log^{1+\kappa}|j|, & |j| > 1,
  \end{cases}
\]

for some constant \( \kappa > 0 \). Let \( C \) be a generic constant in Appendix, and the following inequality is repetitively used in the proof.

\[
  \sum_{j=-\infty}^{\infty} \frac{1}{l(j)l(j+s)} \leq \frac{C}{l(s)}. \tag{45}
\]

The following assumption corresponds to Assumption 2.1 which is imposed for the multivariate locally stationary process \( \{X_{t,T}\} \) in the main text.

**Assumption B.1.** Suppose the multivariate locally stationary process \( X_{t,T} = (X_{t,T}^{(1)}, \ldots, X_{t,T}^{(d)}, \ldots, X_{t,T}^{(p)})^{\top} \) has a representation

\[
  X_{t,T} = \sum_{j=-\infty}^{\infty} A_{t,T}(j) \epsilon_{t-j}, \tag{46}
\]

where the sequences \( \{A_{t,T}(j)\}_{j \in \mathbb{Z}} \) and \( \{\epsilon_t\}_{t \in \mathbb{Z}} \) satisfy the following conditions: there exists a constant \( C_A \) such that

\[
  \sup_{t,T} \|A_{t,T}(j)\|_\infty \leq \frac{C_A}{l(j)}, \tag{47}
\]

and there exists a sequence of functions \( A(\cdot, j) : [0, 1] \to \mathbb{R} \) such that
(i) \( \sup_u \| A(u, j) \|_\infty \leq \frac{C_A}{l(j)} \);

(ii) \( \sup_j \sum_{t=1}^T \left\| A_t(T, j) - A\left( \frac{T}{t}, j \right) \right\|_\infty \leq CA \);

(iii) \( V\left( \| A(\cdot, j) \|_\infty \right) \leq \frac{C_A}{l(j)} \),

where \( V(f) \) is the total variation of the function \( f \) on the interval \([0, 1]\), i.e., \( V \) is defined as
\[
V(f) = \sup \left\{ \sum_{k=1}^m |f(x_k) - f(x_{k-1})|; \ 0 \leq x_0 < \cdots < x_m \leq 1, \ m \in \mathbb{N} \right\}.
\]

In addition, the \( \epsilon_t \) are assumed to be independent and identically distributed with \( E\epsilon_t = 0 \) and \( E\epsilon_t\epsilon_t^\top = K \), where the matrix \( K \) exists and all elements are bounded by \( C_K \). Furthermore, all the moments of \( \epsilon_t \) exist. All elements in \( r \)th moment of \( \epsilon_t \) are bounded by \( C_{\epsilon^{(r)}} < C \) for each \( r \geq 3 \) and some finite constant \( C > 0 \).

**Remark B.1.** Let us consider the time-varying spectral density matrix \( f(u, \lambda) = (f(u, \lambda)_{ij})_{i,j=1,\ldots,p} \) defined in (2), i.e.,
\[
f(u, \lambda) = \frac{1}{2\pi} A(u, \lambda)KA(u, -\lambda)^\top,
\]
where \( A(u, \lambda) = \sum_{j=-\infty}^{\infty} A(u, j) \exp(ij\lambda) \). The autovariance function \( \gamma(u, s) \) at \( u \) is
\[
\gamma(u, s) = \int_{-\pi}^{\pi} f(u, \lambda) \exp(i\lambda s) \, d\lambda = \sum_{j=-\infty}^{\infty} A(u, j)KA(u, j + s)^\top.
\]

Especially, the \((a, b)\)-element of the matrix \( \gamma \) is bounded by
\[
|\gamma(u, s)_{ab}| \leq \sum_{j=-\infty}^{\infty} \sup_u \| A(u, j) \|_\infty \| K \|_\infty \sup_u \| A(u, j + s)^\top \|_\infty
\leq C \sum_{j=-\infty}^{\infty} \frac{1}{l(j)l(j + s)}
\]

45
where the second inequality follows from Assumption B.1 (i) and the third inequality follows from (45).

**Remark B.2.** From (49), we can see that for any fixed \( u \in [0, 1] \) and any \((a, b)\)-element of the autovariance matrix, \(|\gamma(u, 0)_{ab}|\) is bounded, i.e.,

\[
|\gamma(u, 0)_{ab}| \leq C.
\]

Thus, the time-varying spectral density \( f(u, \lambda)_{jk} \) (\( 1 \leq j, k \leq p \)) are square-integrable for any fixed \( u \in [0, 1] \).

**Remark B.3.** Under Assumption B.1, the locally stationary process \( \{X_{t,T}\} \) has the following properties. Let \( X^{(d)}_{t,T} \) be the \( d \)th element of the vector \( X_{t,T} \). From (46), \( X^{(d)}_{t,T} \) has the expression

\[
X^{(d)}_{t,T} = \sum_{j=-\infty}^{\infty} \sum_{m=1}^{p} A_{t,T}(j) dm \epsilon_{t-j}^{(m)}.
\]

Thus, we obtain

\[
\begin{align*}
\text{Cov}(X^{(a)}_{t,T}, X^{(b)}_{t+s,T}) &= \sum_{j=-\infty}^{\infty} \sum_{l=-\infty}^{\infty} \sum_{m,n=1}^{p} A_{t,T}(j) am A_{t+s,T}(l) bn \text{Cov}\left(\epsilon_{t-j}^{(m)}, \epsilon_{t+s-l}^{(n)}\right) \\
&= \sum_{j=-\infty}^{\infty} \sum_{m,n=1}^{p} A_{t,T}(j) am K_{mn} A_{t+s,T}(j + s) bn \\
&= \sum_{j=-\infty}^{\infty} \left( A_{t,T}(j) K A_{t+s,T}(j) \right)_{ab}.
\end{align*}
\]

We first clarify the difference between (48) and (51) on discrete points \( u_k = k/T \) in the following.
Lemma B.4. Under Assumption [B.1] we have

\[
\sum_{k=1}^{T} \left| \text{Cov} \left( X_{[k+1/2-s/2],T}^{(a)}, X_{[k+1/2+s/2],T}^{(b)} \right) - \gamma(u_k, s)_{ab} \right| \leq C \left( 1 + \frac{1}{l(s)} \right),
\]

where \( C \) is a generic constant.

Proof. To evaluate (52), we use the expressions (50) and (48). Note that

\[
\sum_{k=1}^{T} \left| \text{Cov} \left( X_{[k+1/2-s/2],T}^{(a)}, X_{[k+1/2+s/2],T}^{(b)} \right) - \gamma(u_k, s)_{ab} \right|
\]

\[
\leq \sum_{k=1}^{T} \left| \sum_{j=-\infty}^{\infty} \sum_{m,n=1}^{\infty} \left( A_{[k+1/2-s/2],T}^{(j) am} K_{mn} A_{[k+1/2+s/2],T}^{(j) bn} - A(u_k, j)_{am} K_{mn} A_{[k+1/2+s/2],T}^{(j) bn} \right) \right|
\]

\[
+ \sum_{k=1}^{T} \left| \sum_{j=-\infty}^{\infty} \sum_{m,n=1}^{\infty} \left( A(u_k, j)_{am} K_{mn} A_{[k+1/2+s/2],T}^{(j) bn} \right. \right.
\]

\[
\left. - A(u_k, j)_{am} K_{mn} A(u_k, j + s)_{bn} \right| \right| .
\]

(53)

Considering the first term in the right hand side, we have

\[
\sum_{k=1}^{T} \left| \sum_{j=-\infty}^{\infty} \sum_{m,n=1}^{\infty} \left( A_{[k+1/2-s/2],T}^{(j) am} K_{mn} A_{[k+1/2+s/2],T}^{(j) bn} - A(u_k, j)_{am} K_{mn} A_{[k+1/2+s/2],T}^{(j) bn} \right) \right|
\]

\[
= \sum_{j=-\infty}^{\infty} \sum_{k=1}^{T} \left| \sum_{m,n=1}^{\infty} \left( A_{[k+1/2-s/2],T}^{(j)} - A(u_k, j) \right)_{am} K_{mn} A_{[k+1/2+s/2],T}^{(j) bn} \right|
\]

\[
\leq \sum_{j=-\infty}^{\infty} \sum_{k=1}^{T} \sum_{m,n=1}^{\infty} \left| A_{[k+1/2-s/2],T}^{(j)} - A(u_k, j) \right|_{am} \left| K_{mn} \right| \left| A_{[k+1/2+s/2],T}^{(j) bn} \right|
\]

\[
\leq \sum_{j=-\infty}^{\infty} \sum_{k=1}^{T} \sum_{m,n=1}^{\infty} \frac{C_K C_A^2}{l(j + s)} + \frac{C_K C_A^2}{l(j) l(j + s)}
\]

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\[ \leq CC_K C_A^2, \quad (54) \]

where the first inequality follows from \(|K_{mn}| \leq C_K, \ |A_{k+1/2+s/2,T}(j + s)b_n| \leq C_A/l(j + s)\) from (17), and

\[
\sum_{k=1}^{T} \sum_{m,n=1}^{p} \left| \left( A_{[k+1/2-s/2],T}(j) - A(u_k,j) \right)_{am} \right|
\leq \sum_{k=1}^{T} \sum_{m,n=1}^{p} \left| \left( A_{[k+1/2-s/2],T}(j) - A([k + 1/2 - s/2]/T,j) \right)_{am} \right|
+ \left| \left( A([k + 1/2 - s/2]/T,j) - A(u_k,j) \right)_{am} \right|
\leq C_A + \frac{C_A}{l(j)},
\]

where the second inequality follows from (ii) and (iii) in Assumption (B.1).

Also, it holds that

\[
\sum_{k=1}^{T} \sum_{j=-\infty}^{\infty} \sum_{m,n=1}^{p} \left( A(u_k,j)_{am} K_{mn} A_{[k+1/2+s/2],T}(j + s)b_n - A(u_k,j)_{am} K_{mn} A(u_k,j + s)b_n \right)
\]
\[
= \sum_{j=-\infty}^{\infty} \sum_{k=1}^{T} \sum_{m,n=1}^{p} A(u_k,j)_{am} K_{mn} \left( A_{[k+1/2+s/2],T}(j + s) - A(u_k,j + s) \right)_{bn}
\]
\[
\leq \sum_{j=-\infty}^{\infty} \sum_{k=1}^{T} \sum_{m,n=1}^{p} A(u_k,j)_{am} \left| K_{mn} \right| \left( A_{[k+1/2+s/2],T}(j + s) - A([k + 1/2 + s/2]/T,j + s) \right)_{bn}
\]
\[
+ \sum_{j=-\infty}^{\infty} \sum_{k=1}^{T} \sum_{m,n=1}^{p} A(u_k,j)_{am} \left| K_{mn} \right| \left( A([k + 1/2 + s/2]/T,j + s) - A(u_k,j + s) \right)_{bn}
\]
\[
\leq \sum_{j=-\infty}^{\infty} \frac{C_K C_A^2}{l(j)} + \sum_{j=-\infty}^{\infty} \frac{C_K C_A}{l(j)} \sum_{k=1}^{T} \left\| A([k + 1/2 + s/2]/T,j + s) - A(u_k,j + s) \right\|_{\infty}
\]
\[
\leq CC_K C_A^2 + \sum_{j=-\infty}^{\infty} \frac{C_K C_A^2}{l(j)l(j + s)}
\]
\[ \leq CC_KC_A^2 \left( 1 + \frac{1}{l(s)} \right), \tag{55} \]

where the last inequality follows from (45). Combining (53), (54) and (55), we obtain the desired result. \( \square \)

Generally, higher-order cumulants of the locally stationary process \( \{X_{t,T}\} \) can be approximated by those of the stationary process \( \{X^*(u)\} \) for \( u = t/T \) under Assumption B.1 in a similar manner as the autocovariance. To discuss higher-order cumulants, we introduce the notation \( X(u; s) \), which means the observation \( X(s), s \in \mathbb{Z} \), of the stationary process \( X^*(u) \).

Let \( \gamma_{a_1, \ldots, a_q}(u; t_1, \ldots, t_{q-1}) \) be the joint cumulant function of order \( q \), i.e.,

\[
\gamma_{a_1, \ldots, a_q}(u; t_1, \ldots, t_{q-1}) := \text{cum} \{ X^{(a_1)}(u; t + t_1), X^{(a_2)}(u; t + t_2), \ldots, X^{(a_{q-1})}(u; t + t_{q-2}), X^{(a_q)}(u; t) \}.
\]

To discriminate this notation from the autocovariance function, we do not use \( \gamma \) in boldface, although it is an extension of the autocovariance function to higher-orders.

**Lemma B.5.** Under Assumption B.1, we have

\[
\left| \text{cum} \left\{ X^{(a_1)}_{k+t_1,T}, X^{(a_2)}_{k+t_2,T}, \ldots, X^{(a_{q-1})}_{k+t_{q-1},T}, X^{(a_q)}_{k,T} \right\} - \gamma_{a_1, \ldots, a_q}(u_k; t_1, \ldots, t_{q-1}) \right| \\
\leq C \sum_{m=-\infty}^{\infty} \left( \sum_{i=1}^{q} \frac{l(m + t_i)}{\prod_{j=1}^{q} l(m + t_j)} + \frac{1}{\prod_{j=1}^{q} l(m + t_j)} \right),
\]

where \( C \) is a generic constant and \( t_q = 0 \).

**Remark B.6.** Lemma B.4 is a special case of Lemma B.5 when \( q = 2 \).
Proof. Under Assumption 13.1, there exists a constant $\tilde{C}_\varepsilon^{(r)}$ such that all cumulants of order $r$ are all bounded by $\tilde{C}_\varepsilon^{(r)}$, since all cumulants can be written in the form of polynomials of moments. Now, it holds that

$$
\left| \text{cum}(X_{k+1}^{(a_1)}, X_{k+2}^{(a_2)}, \ldots, X_{k+q-1}^{(a_q)}, X_{k}^{(a_q)}) - \sum_{i=1}^{q} \gamma_{a_1, \ldots, a_q}(u_k; t_1, \ldots, t_{q-1}) \right|
$$

$$
\leq \left| \sum_{j_1, \ldots, j_q = -\infty}^{\infty} \sum_{m_1, \ldots, m_q = 1}^{\infty} \left( A_{k+T_1}(j_1)_{a_1m_1} \cdots A_{k+T_2}(j_q)_{a_qm_q} - A(u_k; j_1)_{a_1m_1} \cdots A(u_k; j_q)_{a_qm_q} \right) \sum_{j_1, \ldots, j_q = -\infty}^{\infty} \sum_{m_1, \ldots, m_q = 1}^{\infty} A_{k+T_1}(j_1 + t_1)_{a_1m_1} \cdots A_{k+T_2}(j_q)_{a_qm_q}
$$

$$
\leq \tilde{C}_\varepsilon^{(q)} \sum_{j_q = -\infty}^{\infty} \sum_{m_1, \ldots, m_q = 1}^{\infty} \left| A_{k+T_1}(j_q + t_1)_{a_1m_1} \cdots A_{k+T_2}(j_q)_{a_qm_q} - A(u_k; j_q + t_1)_{a_1m_1} \cdots A(u_k; j_q)_{a_qm_q} \right|
$$

Note that, for $1 \leq i \leq q - 1$, we have

$$
\sum_{k=1}^{T} \left\| A_{k+T_1}(j_q + t_i) - A(u_k; j_q + t_i) \right\|_{\infty}
$$

$$
\leq \sum_{k=1}^{T} \left( \left\| A_{k+T_1}(j_q + t_i) - A(u_k; j_q + t_i) \right\|_{\infty} + \left\| A(u_k; j_q + t_i) - A(u_k; j_q + t_i) \right\|_{\infty} \right)
$$

$$
\leq C_A + \frac{C_A}{l(j_q + t_i)}.
$$
Thus, it holds
\[
\sum_{k=1}^{T} \left| \text{cum}(X_{k+t_1,T}^{(a_1)}, X_{k+t_2,T}^{(a_2)}, \ldots X_{k+t_{K-1},T}^{(a_{q-1})}, X_{k,T}^{(a_q)}) - \gamma_{a_1,\ldots,a_1}(u_k; t_1, \ldots, t_{q-1}) \right| \leq \tilde{C}_\epsilon^q C_A \sum_{j_q=-\infty}^{q} \left( \sum_{i=1}^{q-1} \frac{l(j_q + t_i)}{l(j_q) \prod_{j=1}^{q-1} l(j_q + t_j)} + \frac{1}{l(j_q) \prod_{j=1}^{q-1} l(j_q + t_j)} + \frac{1}{\prod_{j=1}^{q-1} l(j_q + t_j)} \right)
\]

We obtain the conclusion if we replace $j_q$ with $m$ and set $t_q = 0$.

\[\square\]

C  Empirical spectral process

In Section C, we consider the asymptotic distribution of the empirical spectral process for multivariate locally stationary processes.

We first impose the following assumptions on the matrix-valued functions $\phi$, which is to be considered later. Let $V_2(\cdot)$ be the total variation of bivariate functions, i.e.,
\[
V_2(f) = \sup \left\{ \sum_{k,l=1}^{m,n} |f(u_k, \lambda_l) - f(u_{k-1}, \lambda_l) - f(u_k, \lambda_{l-1}) + f(u_{k-1}, \lambda_{l-1})| : 0 \leq u_0 < \cdots < u_m \leq 1, 0 \leq \lambda_0 < \cdots < \lambda_n \leq \pi; \ m, n \in \mathbb{N} \right\}.
\]

Let $\Psi$ be a class of square-integrable functions, where the $L_2$-norm on $\psi \in \Psi$ is defined as
\[
\|\psi\|_{L_2}^2 = \int_0^1 \int_{-\pi}^{\pi} \psi(u, \lambda)^2 \, d\lambda \, du < \infty.
\]

The class $\Psi$ is considered for the elements of the matrix $\phi$.

For any class $\Phi := \{ \phi \in \mathbb{R}^{p \times p}; \phi_{ij} \in \Psi \text{ for } i, j = 1, \ldots, p \}$, let $\tau_{\infty, TV}$, $\tau_{TV, \infty}$, $\tau_{TV, TV}$ and $\tau_{\infty, \infty}$ be
\[
\tau_{\infty, TV} := \tau_{\infty, TV}(\Phi) = \sup_{\phi \in \Phi} \max_{1 \leq i, j \leq p} \sup_{u \in [0,1]} V(\phi_{ij}(u, \cdot)),
\]

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\[ \tau_{TV,\infty} := \tau_{TV,\infty}(\Phi) = \sup_{\phi \in \Phi} \max_{1 \leq i,j \leq p} \sup_{\lambda \in [0,\pi]} V(\phi_{ij}(\cdot, \lambda)), \]
\[ \tau_{TV,TV} := \tau_{TV,TV}(\Phi) = \sup_{\phi \in \Phi} \max_{1 \leq i,j \leq p} V_2(\phi_{ij}), \]
\[ \tau_{\infty,\infty} := \tau_{\infty,\infty}(\Phi) = \sup_{\phi \in \Phi} \max_{1 \leq i,j \leq p} \max_{u \in [0,1]} \sup_{\lambda \in [0,\pi]} |\phi_{ij}|. \]

**Assumption C.1.** Let \( \Phi \) be a class of \( p \times p \) matrix-valued continuous functions \( \phi(u, \cdot) \) on \([-\pi, \pi]\) such that for any \( \phi \in \Phi \), it holds that

(i) \( \phi(u, \cdot) = \phi^*(u, \cdot) \) for any fixed \( u \in [0, 1] \);

(ii) \( \tau_{\infty,TV}, \tau_{TV,\infty}, \tau_{TV,TV} \) and \( \tau_{\infty,\infty} \) are all finite.

For any function \( \psi \in \Psi \), let \( \psi_T \) be

\[ \psi_T(u, \lambda) = \frac{1}{b_T} \psi\left(\frac{u}{b_T}, \lambda\right), \]  

(56)

where \( b := b_T \to 0 \) as \( T \to \infty \). Let \( \Psi_T \) denotes the function class constituted by \( \psi_T \), i.e.,

\[ \Psi_T = \{ \psi_T(u, \lambda) = \frac{1}{b_T} \psi\left(\frac{u}{b_T}, \lambda\right); \psi \in \mathcal{L}_2 \}. \]  

(57)

**Assumption C.2.** For any \( \psi \in \Psi \), let \( \psi(\cdot, \lambda) \) be a positive, symmetric function of bounded variation such that \( \psi(\cdot, \lambda) \) has a compact support on \([-1, 1]\).

Let \( \mathcal{A}_T(u) \) and \( \mathcal{A}_T(u) \) be

\[ \mathcal{A}_T(u)_{ab} := \mathcal{A}_T(u; \psi)_{ab} = \frac{1}{T} \sum_{k=1}^{T} \int_{-\pi}^{\pi} \psi_T(u - u_k, \lambda) I_T(u_k, \lambda) d\lambda, \]  

(58)

\[ \mathcal{A}_T(u)_{ab} := \mathcal{A}_T(u; \psi)_{ab} = \frac{1}{T} \sum_{k=1}^{T} \int_{-\pi}^{\pi} \psi_T(u - u_k, \lambda) f(u_k, \lambda) d\lambda. \]  

(59)
The empirical spectral process $\xi_T(u)_{ab}$ is

$$\xi_T(u)_{ab} := \xi_T(u; \psi)_{ab} = \sqrt{Tb_T} \left( A_T(u; \psi) - \bar{A}_T(u; \psi) \right)_{ab}. \quad (60)$$

We use the first expression in (58), (59) and (60) when there is no confusion with $\psi$.

### C.1 Preliminary Computations

Let $\hat{\psi}$ be

$$\hat{\psi}(u, k) = \int_{-\pi}^{\pi} \psi(u, \lambda) \exp(-ik\lambda) \, d\lambda.$$ 

**Lemma C.1.** Let $\beta_T$ be a sequence of positive numbers such that $\beta_T \to 0$ as $T \to \infty$. Suppose

$$\limsup_{T \to \infty} \beta_T \sum_{s=-T}^{T} \sup_u |\hat{\psi}(u, -s)| < \infty. \quad (61)$$

Then, it holds that

$$|E A_T(u)_{ab} - \bar{A}_T(u)_{ab}| = O(T^{-1}b_T^{-1} \beta_T^{-1}).$$

**Remark C.2.** The condition

$$\sum_{s=-\infty}^{\infty} \sup_u |\hat{\psi}(u, -s)| < \infty \quad (62)$$

satisfies (61). However, if $\psi(u, \cdot)$ is only a function of bounded variation, then $\psi$ may not satisfy the condition (62). Under (61), we see that

$$\sum_{s=-T}^{T} \sup_u |\hat{\psi}(u, -s)| = O(\beta_T^{-1}),$$

which we use in the following evaluations.
Proof. From (14), we have

\[ I_T(u, \lambda)_{ab} = \frac{1}{2\pi} \sum_{l:1 \leq [uT+1/2+l/2] \leq T} X^{(a)}_{[uT+1/2+l/2],T} X^{(b)}_{[uT+1/2-l/2],T} \exp(-i\lambda l). \tag{63} \]

In expression (63), \( l \) depends on \( u \), but it can be naturally extended to

\[ I_T(u, \lambda)_{ab} = \frac{1}{2\pi} \sum_{l=1-T}^{T-1} X^{(a)}_{[uT+1/2+l/2],T} X^{(b)}_{[uT+1/2-l/2],T} \exp(-i\lambda l), \tag{64} \]

if we let \( X_{m,T} \equiv 0 \) for any \( m \leq 0 \) or \( m \geq T + 1 \). We shall use this expression (64) in the following proof. By Parseval’s identity, it holds that

\[
|E_A \mathcal{T}(u)_{ab} - \mathcal{A}_T(u)_{ab}| \\
\leq \frac{1}{T} \sum_{k=1}^{T} \int_{-\pi}^{\pi} \psi_T(u - u_k, \lambda) \left( EI_T(u_k, \lambda)_{ab} - f(u_k, \lambda)_{ab} \right) d\lambda \\
= \frac{1}{2\pi T} \sum_{k=1}^{T} \sum_{s=1-T}^{T-1} \hat{\psi}_T(u - u_k, \lambda) \left( E X^{(a)}_{[k+1/2+s/2],T} X^{(b)}_{[k+1/2-s/2],T} - \gamma(u_k, -s)_{ab} \right) \\
+ \left| \sum_{|s| \geq T} \hat{\psi}_T(u - u_k, \lambda) \gamma(u_k, -s)_{ab} \right| \\
\leq \frac{1}{2\pi T} \left| \sum_{k=1}^{T} \sum_{s=1-T}^{T-1} \hat{\psi}_T(u - u_k, \lambda) \left( \text{Cov}(X^{(a)}_{[k+1/2+s/2],T}, X^{(b)}_{[k+1/2-s/2],T}) - \gamma(u_k, -s)_{ab} \right) \right| \\
+ \frac{1}{2\pi T} \left| \sum_{|s| \geq T} \hat{\psi}_T(u - u_k, \lambda) \gamma(u_k, -s)_{ab} \right| \\
:= B_1 + B_2, \quad \text{(say).}
\]

By Lemma [3.4] it holds that

\[ B_1 \leq \frac{1}{2\pi b_T} \sum_{s=1-T}^{T-1} \sup_{u} |\hat{\psi}(u, -s)| \left| \frac{1}{T} \sum_{k=1}^{T} \text{Cov}(X^{(a)}_{[k+1/2+s/2],T}, X^{(b)}_{[k+1/2-s/2],T}) - \gamma(u_k, -s)_{ab} \right| \]

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\[
\leq \frac{C}{2\pi b_T T} \sum_{s=1-T}^{T-1} \sup_u |\hat{\psi}(u, -s)| \left(1 + \frac{1}{l(s)}\right)
= O(T^{-1}b_T^{-1} \beta_T^{-1}).
\]

Further, noting (49), we have
\[
B_2 \leq \frac{1}{2\pi b_T} \sup_u \sum_{|s| \geq T} |\hat{\psi}(u, -s)||\gamma(u, -s)_{ab}|
\leq \frac{C}{b_T \beta_T} \sum_{|s| \geq T} C l(s)
= O(T^{-1}b_T^{-1} \beta_T^{-1}),
\]
since \(\{l(j)^{-1}\}_{j \in \mathbb{N}}\) is a convergent series, and \(|l(s)| \geq T\) for \(|s| \geq T\). Therefore, we obtain the assertion.

Next, we evaluate the higher-order cumulants of \(\xi_T(u)\). We first clarify the bias between those of the time-varying process \(\{X_{t,T}\}\) and those of the approximate stationary process \(\{X(u, t)\}\), and then evaluate the higher order cumulants of the stationary process.

**Lemma C.3.** Let \(\beta_T\) be a sequence of positive numbers such that \(\beta_T \to 0\) as \(T \to \infty\). Suppose \(\hat{\psi}^{(1)}(\cdot, \lambda), \ldots, \hat{\psi}^{(q)}(\cdot, \lambda)\) are all functions of bounded variation and satisfy Assumption C.2 and

\[
\limsup_{T \to \infty} \beta_T \sum_{s=-T}^{T} \sup_u |\hat{\psi}^{(i)}(u, -s)| < \infty, \quad \text{for } i = 1, \ldots, q.
\]

If \(b_T \to 0\), \(Tb_T \to \infty\) and \(T^{-q/2} \beta_T^{-1} \to 0\) as \(T \to \infty\), then it holds that
\[
\text{cum}(\xi_T(u^{(1)}; \hat{\psi}^{(1)})_{a_1 b_1}, \ldots, \xi_T(u^{(q)}; \hat{\psi}^{(q)})_{a_q b_q}) = O(T^{1-q/2}b_T^{-1} \beta_T^{-q/2}).
\]

Especially, when \(q = 2\), we have
\[
\lim_{T \to \infty} \text{Cov}\left(\xi_T(u^{(1)}; \psi^{(1)})_{a_1 b_1}, \xi_T(u^{(2)}; \psi^{(2)})_{a_2 b_2}\right) = \\
2\pi \delta(u^{(1)}, u^{(2)}) \left( \int_{-\pi}^{\pi} \left( \int_{-\infty}^{\infty} \psi^{(1)}(v, \lambda) \psi^{(2)}(v, \lambda) \, dv \right) f(u^{(1)}, \lambda) f(u^{(2)}, \lambda)_{b_1 b_2} \, d\lambda \right. \\
+ \left. \int_{-\pi}^{\pi} \left( \int_{-\infty}^{\infty} \psi^{(1)}(v, \lambda) \psi^{(2)}(v, -\lambda) \, dv \right) f(u^{(1)}, \lambda) f(u^{(2)}, \lambda)_{b_1 a_2} \, d\lambda \right) \\
+ \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} \left( \int_{-\infty}^{\infty} \psi^{(1)}(v, \lambda_1) \psi^{(2)}(v, -\lambda_2) \, dv \right) \tilde{\gamma}_{a_1 a_2 b_1 b_2} (u^{(1)}; \lambda_1, \lambda_2, -\lambda_2) \, d\lambda_1 \, d\lambda_2 \right),
\]

where \(\tilde{\gamma}\) is the fourth-order spectral density of the process.

**Remark C.4.** The sequence \(\beta_T\) is used to alleviate the divergence of the harmonic series. There exists a sequence \(\beta_T\) such that \(\beta_T^{-1} = O(\log T)\) (See Remark C.5 below for details). For this sequence, the condition \(T^{-q/2} \beta_T^{-1} \to 0\) always holds true for \(q \geq 2\).

**Proof.** Using the expression (64), we have \(\mathcal{A}_T(u)_{ab}\) as

\[
\mathcal{A}_T(u)_{ab} = \frac{1}{2\pi T} \sum_{k=1}^{T} \sum_{s=-T}^{T-1} \hat{\psi}_T(u - u_k, -s) X^{(a)}_{[k+1/2+s/2],T} X^{(b)}_{[k+1/2-s/2],T},
\]

which is a linear combination of \(X^{(a)}_{[k+1/2+s/2],T} X^{(b)}_{[k+1/2-s/2],T}\). We apply Lemma [B.5] to compute the higher order cumulants. Actually, it holds that

\[
\text{cum}\left(\xi_T(u^{(1)}; \psi^{(1)})_{a_1 b_1}, \cdots, \xi_T(u^{(q)}; \psi^{(q)})_{a_q b_q}\right) \\
= \text{cum}\left(\frac{1}{T} \sum_{\kappa_1} X^{(a_1)}_{\kappa_1,T} X^{(b_1)}_{\kappa_1-s_1,T}, \frac{1}{T} \sum_{\kappa_2} X^{(a_2)}_{\kappa_2,T} X^{(b_2)}_{\kappa_2-s_2,T}, \cdots, \frac{1}{T} \sum_{\kappa_q} X^{(a_q)}_{\kappa_q,T} X^{(b_q)}_{\kappa_q-s_q,T}\right) \\
= \frac{1}{T^q} \sum_{\kappa_1, \cdots, \kappa_q} \text{cum}\left(X^{(a_1)}_{\kappa_1,T} X^{(b_1)}_{\kappa_1-s_1,T}, \cdots, X^{(a_q)}_{\kappa_q,T} X^{(b_q)}_{\kappa_q-s_q,T}\right),
\]

where for brevity, we let

\[
\kappa_1 := [k_1 + 1/2 + s_1/2], \quad \kappa_2 := [k_2 + 1/2 + s_2/2], \quad \ldots, \quad \kappa_q := [k_q + 1/2 + s_q/2].
\]

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To compute higher order cumulants, we have to consider all indecomposable partitions of the following table (See Brillinger (1981), Theorem 2.3.2):

\[
\begin{array}{c}
X^{(a_1)}_{\kappa_1,T} X^{(b_1)}_{\kappa_1-s_1,T} \\
X^{(a_2)}_{\kappa_2,T} X^{(b_2)}_{\kappa_2-s_2,T} \\
\vdots \\
X^{(a_q)}_{\kappa_q,T} X^{(b_q)}_{\kappa_q-s_q,T}
\end{array}
\]

In view of Lemma B.5 with some tedious computation, all indecomposable partitions can be approximated by those cumulants of the stationary process with a bias of lower order for a fixed \( q \geq 2 \).

We give a representative example of a partition below. The other partitions can be evaluated in the same manner. Without loss of generality, let \( q \) be odd. Suppose we evaluate the following cumulant:

\[
\frac{1}{T^q} \sum_{\kappa_1,\ldots,\kappa_q} \text{cum}(X^{(a_1)}_{\kappa_1,T}, X^{(a_2)}_{\kappa_2,T}) \text{cum}(X^{(b_2)}_{\kappa_2-s_2,T}, X^{(b_3)}_{\kappa_3-s_3,T}) \cdots \text{cum}(X^{(a_q)}_{\kappa_q,T}, X^{(b_1)}_{\kappa_1-s_1,T}).
\]

If we replace variables \( \kappa_2,\ldots,\kappa_q \) with \( \tau_2 := \kappa_2 - \kappa_1, \ldots, \tau_q := \kappa_q - \kappa_1 \), then we have

\[
\frac{1}{T^q} \sum_{\kappa_1,\tau_2,\ldots,\tau_q} \text{cum}(X^{(a_1)}_{\kappa_1,T}, X^{(a_2)}_{\kappa_1+\tau_2,T}) \text{cum}(X^{(b_2)}_{\kappa_1+\tau_2-s_2,T}, X^{(b_3)}_{\kappa_1+\tau_3-s_3,T}) \cdots \text{cum}(X^{(a_q)}_{\kappa_1+\tau_q,T}, X^{(b_1)}_{\kappa_1-s_1,T}). \tag{68}
\]

Applying Lemma B.5 (68) can be approximated by

\[
\frac{1}{T^q} \sum_{\kappa_1} T \sum_{\tau_2,\ldots,\tau_q} \gamma(u_{\kappa_1}, \tau_2) a_{1a_2} \gamma(u_{\kappa_1}, \tau_3-s_3-\tau_2+s_2) b_{2b_3} \cdots \gamma(u_{\kappa_1}, -s_1-\tau_q) a_q b_1. \tag{69}
\]

More precisely, the absolute bias between (68) and (69) is bounded by

\[
T^{-q} \sum_{i=1}^{q} C_i \left( 1 + \frac{1}{l(s_i)} \right).
\]
Returning back to the expression (67), we see that the full expression of the absolute bias is bounded by
\[
\frac{1}{2\pi b_T^2} \sum_{s_1, \ldots, s_q=1}^{T-1} \prod_{i=1}^q \sup_u |\hat{\psi}^{(i)}(u, -s_i)| \frac{1}{T^q} \sum_{\kappa_1, \tau_2, \ldots, \tau_q} \text{cum} \left( X_{\kappa_1, T}^{(a_1)}, X_{\kappa_1 + \tau_2, T}^{(a_2)} \right)
\]
\[
\text{cum} \left( X_{\kappa_1 + \tau_2 - s_3, T}^{(b_2)}, X_{\kappa_1 + \tau_3 - s_3, T}^{(b_1)} \right) \cdots \text{cum} \left( X_{\kappa_1 + \tau_q, T}^{(a_q)}, X_{\kappa_1 - s_1, T}^{(b_1)} \right)
\]
\[
- \frac{1}{T^q} \sum_{\kappa_1=1}^{T} \sum_{\tau_2, \ldots, \tau_q} \psi(u_{\kappa_1}, \tau_2)_{a_1} \psi(u_{\kappa_1}, \tau_3 - s_3 - \tau_2 + s_2)_{b_2} \cdots \psi(u_{\kappa_1}, -s_1 - \tau_q)_{a_q} b_1 \bigg| = O(T^{-q/b_T^q} \beta_{T}^{-q}).
\]

In summary, all cumulants of order \( q \) for \( \mathcal{A}_T \) can be approximated by those of the stationary process with a bias of order \( O(T^{-q/b_T^q} \beta_{T}^{-q}) \). Thus, the bias in those cumulants for \( \xi_T \) is \( O(T^{-q/2}b_T^{-q/2} \beta_{T}^{-q}) \). Furthermore, it holds that
\[
\text{cum} \left( \xi_T(u^{(1)}; \psi^{(1)}), \cdots, \xi_T(u^{(q)}; \psi^{(q)}) \right)_{a_q b_q} = O(T^{-q/2}b_T^{-q/2}),
\]
(70)
since \( \psi^{(1)}(\cdot, \lambda), \ldots, \psi^{(q)}(\cdot, \lambda) \) are all functions of bounded variation. A representative example of (70) is shown below.

Let us consider the case \( q = 2 \) for \( \xi_T \). Note that \( q \) is even now. We have three terms of the type (69), i.e.,

(i) the approximation for \( \text{cum}(X_{\kappa_1, T}^{(a_1)}, X_{\kappa_1 - s_1, T}^{(b_1)}) \text{cum}(X_{\kappa_2, T}^{(a_2)}, X_{\kappa_2 - s_2, T}^{(b_2)}) \):
\[
\frac{b_T}{T} \sum_{\kappa_1=1}^{T} \sum_{s_1, s_2, \tau_2} \hat{\psi}^{(1)}(u^{(1)})_{\kappa_1} \hat{\psi}^{(2)}(u^{(2)})_{\kappa_1} \gamma(u_{\kappa_1}, \tau_2)_{a_1 a_2} \gamma(u_{\kappa_1}, -s_1 + s_2)_{b_1 b_2};
\]
(71)

(ii) the approximation for \( \text{cum}(X_{\kappa_1, T}^{(a_1)}, X_{\kappa_2 - s_2, T}^{(b_2)}) \text{cum}(X_{\kappa_1 - s_1, T}^{(b_1)}, X_{\kappa_2, T}^{(a_2)}) \):

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\[ b_T \sum_{\kappa_1=1}^{T} \sum_{s_1,s_2,\tau_2} \hat{\psi}_T^{(1)}(u^{(1)} - u_{\kappa_1}, s_1) \hat{\psi}_T^{(2)}(u^{(2)} - u_{\kappa_1}, s_2) \]

\[ \sum_{s_1,s_2,\tau_2} \hat{\psi}_T^{(1)}(u^{(1)} - u_{\kappa_1}, s_1) \hat{\psi}_T^{(2)}(u^{(2)} - u_{\kappa_1}, s_2) \]

\[ \gamma(u_{\kappa_1}, \tau_2 - s_2) a_{1} a_{2} b_{1} b_{2} \gamma(u_{\kappa_1}, -s_1, \tau_2 - \tau_2 - s_2). \quad (73) \]

We first explain the term (71). By repeated application of the Parseval equality (see, e.g., the proof of Lemma 2.2 in Hosoya and Taniguchi (1982) for details) and by Lemma P5.1 in Brillinger (1981), the term (71) is equivalent to

\[ 2\pi b_T \int_{0}^{1} \int_{-\pi}^{\pi} \psi_T^{(1)}(u^{(1)} - u, \lambda) \psi_T^{(2)}(u^{(2)} - u, \lambda) f(u, \lambda)_{a_{1} a_{2}} f(u, \lambda)_{b_{1} b_{2}} d\lambda du + O(T^{-1}b_T^{-1}). \]

Under Assumption C.2, if \( u^{(1)} \neq u^{(2)} \), we have

\[ \int_{0}^{1} \int_{-\pi}^{\pi} \psi_T^{(1)}(u^{(1)} - u, \lambda) \psi_T^{(2)}(u^{(2)} - u, \lambda) f(u, \lambda)_{a_{1} a_{2}} f(u, \lambda)_{b_{1} b_{2}} d\lambda du = o(b_T^{-1}), \]

since the supports of \( \psi^{(1)} \) and \( \psi^{(2)} \) are compact. Thus, the term (71) converges to

\[ 2\pi \delta(u^{(1)}, u^{(2)}) \int_{-\pi}^{\pi} \left( \int_{-\infty}^{\infty} \psi^{(1)}(v, \lambda) \psi^{(2)}(v, \lambda) dv \right) f(u^{(1)}, \lambda)_{a_{1} a_{2}} f(u^{(1)}, \lambda)_{b_{1} b_{2}} d\lambda, \quad (74) \]

where \( \delta \) is a delta function such that \( \delta(a, b) = 1 \) if \( a = b \), and 0 otherwise. Similarly, the term (72) converges to

\[ 2\pi \delta(u^{(1)}, u^{(2)}) \int_{-\pi}^{\pi} \left( \int_{-\infty}^{\infty} \psi^{(1)}(v, \lambda) \psi^{(2)}(v, -\lambda) dv \right) f(u^{(1)}, \lambda)_{a_{1} a_{2}} f(u^{(1)}, \lambda)_{b_{1} a_{2}} d\lambda. \quad (75) \]
The term (73) converges to
\[ 2\pi \delta(u^{(1)}, u^{(2)}) \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} \left( \int_{-\infty}^{\infty} \psi^{(1)}(v, \lambda_1)\psi^{(2)}(v, -\lambda_2) \, dv \right) \]
\[ \tilde{\gamma}_{a_1a_2b_1b_2}(u^{(1)}; \lambda_1, \lambda_2, -\lambda_2) \, d\lambda_1 \, d\lambda_2, \quad (76) \]
by repeated application of the Parseval equality. Combining all terms (74), (75) and (76), we obtain the results of Lemma C.3.

\[ \square \]

C.2 Asymptotic Normality

Here, we show the asymptotic normality of the empirical spectral process \( \xi_T(\psi) \) in (60). To this goal, we adopt the idea in Dahlhaus and Polonik (2009) to use the Gaussian kernel as the mollifier with the property of being rapidly decreasing. Let \( G \) be the Gaussian kernel, that is,
\[ G(x) := \frac{1}{\sqrt{2\pi}} \exp \left( -\frac{1}{2} x^2 \right), \]
and \( G_\beta \) the mollifier
\[ G_\beta(x) = \frac{1}{\beta} G \left( \frac{x}{\beta} \right), \]
with \( \beta := \beta_T \to 0 \) as \( T \to \infty \). From the convolution theorem, the Fourier coefficients \( \hat{\psi}^{*T} \) of \( \psi^{*T} := \psi * G_\beta \) are
\[ \hat{\psi}^{*T}(u, k) = \hat{\psi}(u, k) \hat{G}_\beta(k), \quad k \in \mathbb{Z}. \quad (77) \]

**Remark C.5.** The remarkable feature of this manipulation is that
\[ \sum_{k \in \mathbb{Z}} \sup_{u \in [0, 1]} |\hat{\psi}^{*T}(u, k)| \leq \sum_{k \in \mathbb{Z}} \sup_{u \in [0, 1]} |\hat{\psi}(u, k)|, \]
since for any fixed \( k \in \mathbb{Z}, \)
\[ |\hat{G}_\beta(k)| = \left| \exp \left( -\frac{\beta^2 k^2}{2} \right) \right| \leq 1. \]
In addition, the following result holds.

\[ \sum_{k \in \mathbb{Z}} \sup_{u \in [0,1]} |\hat{\psi}^{*T}(u, k)| = O\left( \log(\beta^{-1}) \right). \tag{78} \]

If we take \( \beta_T \) as \( \beta_T = T^{-k} \) for any \( k \geq 1 \), then we have

\[ \sum_{k \in \mathbb{Z}} \sup_{u \in [0,1]} |\hat{\psi}^{*T}(u, k)| = O(\log T). \]

**Proof of Remark C.5.** For any \( 1 \leq i, j \leq p \), let \( \psi := \phi_{ij} \in \Psi \) as in Assumption C.1. Note that \( \psi(u, \cdot) \) is a continuous function of bounded variation.

(i) Let \( k \neq 0 \). From Jordan decomposition theorem, there exists a signed measure \( g_\psi \) such that

\[ \hat{\psi}(u, k) = \int_{-\pi}^{\pi} \frac{\exp(-ik\lambda) - 1}{-ik} g_\psi(u, d\lambda), \]

which leads to

\[ \sup_{u \in [0,1]} |\hat{\psi}(u, k)| \leq \frac{C}{|k|} \sup_{u \in [0,1]} V(\psi(u, \cdot)) \leq \frac{C\tau_{\infty, TV}}{|k|}. \tag{79} \]

(ii) Let \( k = 0 \).

\[ \sup_{u \in [0,1]} |\hat{\psi}(u, 0)| \leq 2\pi \sup_{u \in [0,1]} \sup_{\lambda \in [-\pi, \pi]} \psi(u, \lambda) \leq 2\pi \tau_{\infty, \infty}. \tag{80} \]

Combining (79) and (80) with the relation (77), we obtain

\[ \sup_{u \in [0,1]} |\hat{\psi}^{*T}(u, k)| \leq C \left( 1 + \sum_{k=1}^{\infty} \frac{1}{|k|} \exp\left( \frac{-\beta^2 k^2}{2} \right) \right) = O\left( \log(\beta^{-1}) \right). \]

Thus, the equation (78) is shown. \( \square \)

Next result shows that the \( \xi_T(u_k)_{ab} \) converges in finite dimensional distributions for \( k \geq 1 \).
Theorem C.6. Suppose Assumptions [B.1] and [C.1] hold. Let \( b_T \to 0 \) and \( T b_T \to \infty \), as \( T \to \infty \). For any \( q \), and \( u^{(1)}, \ldots, u^{(q)} \in [0, 1] \), it holds that

\[
(\xi_T(u^{(1)}; \psi^{(1)}), \ldots, \xi_T(u^{(q)}; \psi^{(q)}))^\top \overset{d}{\to} \mathcal{N}(0, (V_{jk})_{j,k=1,\ldots,q}), \quad \text{as } T \to \infty,
\]

where \( V_{jk} \) is

\[
V_{jk} = 2\pi \delta(u^{(j)}, u^{(k)})
\]

\[
\left( \int_{-\pi}^{\pi} \left( \int_{-\infty}^{\infty} \psi^{(j)}(v, \lambda) \overline{\psi^{(k)}(v, \lambda)} \, dv \right) f(u^{(j)}, \lambda)_{a_j a_k} \overline{f(u^{(j)}, \lambda)}_{b_j b_k} \, d\lambda + \int_{-\pi}^{\pi} \left( \int_{-\infty}^{\infty} \psi^{(j)}(v, \lambda) \overline{\psi^{(k)}(v, -\lambda)} \, dv \right) f(u^{(j)}, \lambda)_{a_j b_k} \overline{f(u^{(j)}, \lambda)}_{b_j a_k} \, d\lambda + \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} \left( \int_{-\infty}^{\infty} \psi^{(j)}(v, \lambda_1) \overline{\psi^{(k)}(v, -\lambda_2)} \, dv \right) \tilde{\gamma}_{a_j a_k b_j b_k} (u^{(j)}; \lambda_1, \lambda_2, -\lambda_2) \, d\lambda_1 \, d\lambda_2 \right),
\]

where \( \tilde{\gamma} \) is the fourth-order spectral density of the process.

Proof. First we show that

\[
\text{var}(\xi_T(u; \psi)_{ab} - \xi_T(u; \psi^*T)_{ab}) \to 0,
\]

which, in turn, shows that

\[
\xi_T(u; \psi)_{ab} - \xi_T(u; \psi^*T)_{ab} \to P 0. \quad (81)
\]

As in Remark [C.5], let \( \beta_T = T^{-k} \) for any \( k \geq 1 \). Following this choice, we have \( O(\beta_T/b_T) = o(1) \).

Note that

\[
\text{var}(\xi_T(u; \psi)_{ab} - \xi_T(u; \psi^*T)_{ab})
\]
\[ = T b_T \text{var} \left( \frac{1}{2\pi T} \right) \]
\[
\sum_{k=1}^{T} \sum_{s=1}^{T-1} \left\{ \hat{\psi}_T(u - u_k, -s) - \hat{\psi}_T^*(u - u_k, -s) \right\} \sum_{t \in T_s} \sum_{k+t,s,T} X_{k+t,T}^{(a)} X_{k+t+s,T}^{(b)} \]
\[
\leq b_T^{-1} \left( \sup_u \sum_{s=-\infty}^{\infty} |\hat{\psi}(u, -s) - \hat{\psi}^*(u, -s)| \right)^2 \]
\[
\leq C b_T^{-1} \sum_{s=-\infty}^{\infty} \frac{|\exp(-s^2 \beta_T^2/2) - 1|^2}{s^2}, \]

where the last inequality follows from (17). Since \( |\exp(-s^2 \beta_T^2/2) - 1| \leq \min(1, s^2 \beta_T^2/2) \), the order of the last term is \( O(\beta_T/b_T) = o(1) \). Thus, (81) is shown.

Now, we only have to consider the finite distributions of \( \xi_T(u; \psi^T) \). However, from Remark C.5, we find that the condition (65) is satisfied and thus the covariance matrix of \( \xi_T(u; \psi^T) \) can be expressed in the form of (66). Therefore, the proof is completed.

Finally, remembering the matrix \( \phi \) satisfies Assumption C.1, we define \( \mathcal{A}_T^0(u) \) and \( \mathcal{A}_T^2(u) \) as

\[ \mathcal{A}_T^0(u) := \frac{1}{T} \sum_{k=1}^{T} \int_{-\pi}^{\pi} \phi_T(u - u_k, \lambda) I_T(u_k, \lambda) \, d\lambda, \]
\[ \mathcal{A}_T^2(u) := \frac{1}{T} \sum_{k=1}^{T} \int_{-\pi}^{\pi} \phi_T(u - u_k, \lambda) f(u_k, \lambda) \, d\lambda, \]

and let \( \zeta_T(u) \) be

\[ \zeta_T(u) = \sqrt{T b_T} \text{Tr} \left( \mathcal{A}_T^0(u) - \mathcal{A}_T^2(u) \right). \]  

(82)

**Corollary C.7.** Suppose Assumptions [B.1], [C.1] and [C.2] hold. If \( b_T = o(1) \) and \( b_T^{-1} = o(T(\log T)^{-6}) \), then it holds that

\[ \left( \zeta_T(u^{(1)}), \cdots, \zeta_T(u^{(q)}) \right) ^\top \xrightarrow{d} N \left( \mathbf{0}, (\tilde{V}_{jk})_{j,k=1,\ldots,q} \right), \quad \text{as } T \to \infty, \]
where \( \tilde{V}_{jk} \) is given by

\[
\tilde{V}_{jk} = 4\pi \delta(u^{(j)}, u^{(k)}) \left( \int_{-\pi}^{\pi} \text{Tr} \left( \int_{-\infty}^{\infty} f(u^{(j)}, \lambda) \phi(v, \lambda) f(u^{(j)}, \lambda) \phi(v, \lambda) \, dv \right) \, d\lambda \right) + \frac{1}{2} \sum_{r,t;u,v=1}^{p} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} \int_{-\infty}^{\infty} \phi_{rt}(v, \lambda_1) \phi_{uv}(v, \lambda_2) \left( \tilde{\gamma}_{rtuv}(u^{(j)}; -\lambda_1, \lambda_2, -\lambda_2) \, dv \right) \, d\lambda_1 \, d\lambda_2,
\]

where \( \tilde{\gamma} \) is the fourth-order spectral density of the process.

**Proof.** From the definition of \( \zeta_T(u) \) in (82), we see that \( \zeta(u) \) is a linear combination of the processes \( \xi_T(u) \) in (60). With a similar computation to the latter part in Lemma A.3.3. in Hosoya and Taniguchi (1982), we obtain (83). \qed