1. Introduction

In this paper we study the geometry of the sections for solutions to the Monge-Ampere equation
\[ \det D^2 u = f, \quad u : \overline{\Omega} \to \mathbb{R} \] convex,
which are centered at a boundary point \( x_0 \in \partial \Omega \). We show that under natural local assumptions on the boundary data and the domain, the sections
\[ S_h(x_0) = \{ x \in \overline{\Omega} | \ u(x) < u(x_0) + \nabla u(x_0) \cdot (x - x_0) + h \} \]
are “equivalent” to ellipsoids centered at \( x_0 \), that is, for each \( h > 0 \) there exists an ellipsoid \( E_h \) such that
\[ cE_h \cap \overline{\Omega} \subset S_h(x_0) - x_0 \subset CE_h \cap \overline{\Omega}, \]
with \( c, C \) constants independent of \( h \).

The situation in the interior is well understood. Caffarelli showed in [C1] that if
\[ 0 < \lambda \leq f \leq \Lambda \] in \( \Omega \),
and for some \( x \in \Omega \),
\[ S_h(x) \subset \subset \Omega, \]
then \( S_h(x) \) is equivalent to an ellipsoid centered at \( x \) i.e.
\[ kE \subset S_h(x) - x \subset k^{-1}E \]
for some ellipsoid \( E \) of volume \( h^{n/2} \) and for a constant \( k > 0 \) which depends only on \( \lambda, \Lambda, n \).

This property provides compactness of sections modulo affine transformations. This is particularly useful when dealing with interior \( C^{2,\alpha} \) and \( W^{2,p} \) estimates of strictly convex solutions of
\[ \det D^2 u = f \]
when \( f > 0 \) is continuous (see [C2]).

Sections at the boundary were also considered by Trudinger and Wang in [TW] for solutions of
\[ \det D^2 u = f \]
but under stronger assumptions on the boundary behavior of \( u \) and \( \partial \Omega \), and with \( f \in C^\alpha(\overline{\Omega}) \). They proved \( C^{2,\alpha} \) estimates up to the boundary by bounding the mixed derivatives and obtained that the sections are equivalent to balls.

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2. Statement of the main Theorem.

Let \( \Omega \) be a bounded convex set in \( \mathbb{R}^n \). We assume throughout this note that
\[
B_\rho(\rho e_n) \subset \Omega \subset \{ x_n \geq 0 \} \cap B_\rho,
\]
for some small \( \rho > 0 \), that is \( \Omega \subset (\mathbb{R}^n)^+ \) and \( \Omega \) contains an interior ball tangent to \( \partial \Omega \) at 0.

Let \( u : \Omega \to \mathbb{R} \) be convex, continuous, satisfying
\[
\det D^2 u = f, \quad \lambda \leq f \leq \Lambda \quad \text{in} \quad \Omega.
\]
We extend \( u \) to be \( \infty \) outside \( \Omega \).

By subtracting a linear function we may assume that
\[
x_{n+1} = 0 \quad \text{is the tangent plane to} \quad u \quad \text{at} \quad 0,
\]
in the sense that
\[
u \geq 0, \quad u(0) = 0,
\]
and any hyperplane \( x_{n+1} = \epsilon x_n, \quad \epsilon > 0 \) is not a supporting hyperplane for \( u \).

In this paper we investigate the geometry of the sections of \( u \) at 0 that we denote for simplicity of notation
\[
S_h := \{ x \in \Omega : \quad u(x) < h \}.
\]

We show that if the boundary data has quadratic growth near \( \{ x_n = 0 \} \) then, as \( h \to 0 \), \( S_h \) is equivalent to a half-ellipsoid centered at 0.

Precisely, our main theorem reads as follows.

**Theorem 2.1.** Assume that \( \Omega, u \) satisfy (2.1)-(2.3) above and for some \( \mu > 0 \),
\[
\mu|x|^2 \leq u(x) \leq \mu^{-1}|x|^2 \quad \text{on} \quad \partial \Omega \cap \{ x_n \leq \rho \}.
\]
Then, for each \( h < c(\rho) \) there exists an ellipsoid \( E_h \) of volume \( h^{n/2} \) such that
\[
k E_h \cap \overline{\Omega} \subset S_h \subset k^{-1} E_h.
\]
Moreover, the ellipsoid \( E_h \) is obtained from the ball of radius \( h^{1/2} \) by a linear transformation \( A_h^{-1} \) (sliding along the \( x_n = 0 \) plane)
\[
A_h E_h = h^{1/2} B_1
\]
\[
A_h(x) = x - \nu x_n, \quad \nu = (\nu_1, \nu_2, \ldots, \nu_{n-1}, 0),
\]
with
\[
|\nu| \leq k^{-1}|\log h|.
\]
The constant \( k \) above depends on \( \mu, \lambda, \Lambda, n \) and \( c(\rho) \) depends also on \( \rho \).

Theorem 2.1 is new even in the case when \( f = 1 \). The ellipsoid \( E_h \), or equivalently the linear map \( A_h \), provides information about the behavior of the second derivatives near the origin. Heuristically, the theorem states that in \( S_h \) the tangential second derivatives are bounded from above and below and the mixed second derivatives are bounded by \( |\log h| \). This is interesting given that \( f \) is only bounded and the boundary data and \( \partial \Omega \) are only \( C^{1,1} \) at the origin.

**Remark.** Given only the boundary data \( \varphi \) of \( u \) on \( \partial \Omega \), it is not always easy to check condition (2.4). Here we provide some examples when (2.4) is satisfied:

1) If \( \varphi \) is constant and the domain \( \Omega \) is included in a ball included in \( \{ x_n \geq 0 \} \).
2) If the domain \( \partial \Omega \) is tangent of order 2 to \( \{ x_n = 0 \} \) and the boundary data \( \varphi \) has quadratic behavior in a neighborhood of 0.

3) \( \varphi, \partial \Omega \in C^3 \) at the origin, and \( \Omega \) is uniformly convex at the origin.

We obtain compactness of sections modulo affine transformations.

**Corollary 2.2.** Under the assumptions of Theorem 2.1, assume that

\[
\lim_{x \to 0} f(x) = f(0)
\]

and

\[ u(x) = P(x) + o(|x|^2) \quad \text{on} \ \partial \Omega \]

with \( P \) a quadratic polynomial. Then we can find a sequence of rescalings

\[
\tilde{u}_h(x) := \frac{1}{h} \tilde{u}(h^{1/2}A_h^{-1}x)
\]

which converges to a limiting continuous solution \( \tilde{u}_0 : \tilde{\Omega}_0 \to \mathbb{R} \) with \( kB_1^+ \subset \Omega_0 \subset k^{-1}B_1^+ \)

such that

\[
\det D^2 \tilde{u}_0 = f(0)
\]

and

\[
\tilde{u}_0 = P \quad \text{on} \ \tilde{\Omega}_0 \cap \{ x_n = 0 \},
\]

\[
\tilde{u}_0 = 1 \quad \text{on} \ \partial \tilde{\Omega}_0 \cap \{ x_n > 0 \}.
\]

In a future work we intend to use the results above and obtain \( C^{2,\alpha} \) and \( W^{2,p} \) boundary estimates under appropriate conditions on the domain and boundary data.

### 3. Preliminaries

Next proposition was proved by Trudinger and Wang in [TW]. Since our setting is slightly different we provide its proof.

**Proposition 3.1.** Under the assumptions of Theorem 2.1, for all \( h \leq c(\rho) \), there exists a linear transformation (sliding along \( x_n = 0 \))

\[
A_h(x) = x - \nu x_n,
\]

with

\[
\nu_n = 0, \quad |\nu| \leq C(\rho)h^{-\frac{n}{n+1}},
\]

such that the rescaled function

\[
\tilde{u}(A_h x) = u(x),
\]

satisfies in

\[
\tilde{S}_h := A_h S_h = \{ \tilde{u} < h \}
\]

the following:

(i) the center of mass of \( \tilde{S}_h \) lies on the \( x_n \)-axis;

(ii) \( k_0 h^{n/2} \leq |\tilde{S}_h| = |S_h| \leq k_0^{-1}h^{n/2} \).
(iii) the part of \( \partial \tilde{S}_h \) where \( \{ \tilde{u} < h \} \) is a graph, denoted by

\[
\tilde{G}_h = \partial \tilde{S}_h \cap \{ \tilde{u} < h \} = \{(x', g_h(x'))\}
\]

that satisfies

\[
g_h \leq C(\rho)|x'|^2
\]

and

\[
\frac{h}{2}|x'|^2 \leq \tilde{u} \leq 2\mu^{-1}|x'|^2 \quad \text{on } \tilde{G}_h.
\]

The constant \( k_0 \) above depends on \( \mu, \lambda, \Lambda \) and the constants \( C(\rho), c(\rho) \) depend also on \( \rho \).

In this section we denote by \( c, C \) positive constants that depend on \( n, \mu, \lambda, \Lambda \). For simplicity of notation, their values may change from line to line whenever there is no possibility of confusion. Constants that depend also on \( \rho \) are denote by \( c(\rho), C(\rho) \).

**Proof.** The function

\[
v := \frac{\Lambda}{\mu^{n+1}} x_n^2 - C(\rho)x_n
\]

is a lower barrier for \( u \) in \( \Omega \cap \{x_n \leq \rho\} \) if \( C(\rho) \) is chosen large.

Indeed, then

\[
v \leq u \quad \text{on } \partial \Omega \cap \{x_n \leq \rho\},
\]

\[
v \leq 0 \leq u \quad \text{on } \Omega \cap \{x_n = \rho\},
\]

and

\[
\det D^2 v > \Lambda.
\]

In conclusion,

\[
v \leq u \quad \text{in } \Omega \cap \{x_n \leq \rho\},
\]

hence

\[
(3.1) \quad S_h \cap \{x_n \leq \rho\} \subset \{v < h\} \subset \{x_n > c(\rho)(\mu|x'|^2 - h)\}.
\]

Let \( x_h^* \) be the center of mass of \( S_h \). We claim that

\[
(3.2) \quad x_h^* \cdot e_n \geq c_0(\rho)\bar{h}^\alpha, \quad \alpha = \frac{n}{n+1},
\]

for some small \( c_0(\rho) > 0 \).

Otherwise, from [3.1] and John’s lemma we obtain

\[
S_h \subset \{x_n \leq C(n)c_0 \bar{h}^\alpha \leq h^\alpha\} \cap \{|x'| \leq C_1 h^{\alpha/2}\},
\]

for some large \( C_1 = C_1(\rho) \). Then the function

\[
w = \varepsilon x_n + \frac{h}{2} \left( \frac{|x'|}{C_1 h^{\alpha/2}} \right)^2 + \Lambda C_1^{2(n-1)} \bar{h} \left( \frac{x_n}{h^\alpha} \right)^2
\]

is a lower barrier for \( u \) in \( S_h \) if \( c_0 \) is sufficiently small.

Indeed,

\[
w \leq \frac{h}{4} + \frac{h}{2} + \Lambda C_1^{2(n-1)} (C(n)c_0)^2 h \quad \text{in } S_h,
\]

and for all small \( h \),

\[
w \leq \varepsilon x_n + \frac{h^{1-\alpha}}{C_1^2} |x'|^2 + C(\rho)c_0 \frac{x_n}{h^\alpha} \leq \mu|x'|^2 \leq u \quad \text{on } \partial \Omega,
\]
and
\[ \det D^2 w = 2\Lambda. \]

Hence
\[ w \leq u \quad \text{in} \quad S_h, \]
and we contradict that 0 is the tangent plane at 0. Thus claim (3.2) is proved.

Now, define
\[ A_h x = x - \nu x_n, \quad \nu = \frac{x_h' \cdot e_n}{x_h' \cdot e_n}, \]
and
\[ \tilde{u}(A_h x) = u(x). \]

The center of mass of \( \tilde{S}_h = A_h S_h \) is
\[ \tilde{x}_h^* = A_h x_h^* \]
and lies on the \( x_n \)-axis from the definition of \( A_h \). Moreover, since \( x_h^* \in S_h \), we see from (3.1)-(3.2) that
\[ |\nu| \leq C(\rho) \left( \frac{x_h' \cdot e_n}{x_h' \cdot e_n} \right)^{1/2} \leq C(\rho) h^{-\alpha/2}, \]
and this proves (i).

If we restrict the map \( A_h \) on the set on \( \partial \Omega \) where \( \{u < h\} \), i.e. on
\[ \partial S_h \cap \partial \Omega \subset \{x_n \leq |x'|^2/\rho\} \cap \{|x'| < Ch^{1/2}\}, \]
we have
\[ |A_h x - x| = |\nu| x_n \leq C(\rho) h^{-\alpha/2} |x'| \leq C(\rho) h^{1-\alpha} |x'|, \]
and part (iii) easily follows.

Next we prove (ii). From John’s lemma, we know that after relabeling the \( x' \) coordinates if necessary, (3.3)
\[ D_h B_1 \subset \tilde{S}_h - \tilde{x}_h^* \subset C(n) D_h B_1 \]
where
\[ D_h = \begin{pmatrix} d_1 & 0 & \cdots & 0 \\ 0 & d_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & d_n \end{pmatrix}. \]

Since
\[ \tilde{u} \leq 2\mu^{-1} |x'|^2 \quad \text{on} \quad \tilde{G}_h = \{(x', g_h(x'))\}, \]
we see that the domain of definition of \( g_h \) contains a ball of radius \((\mu h/2)^{1/2}\). This implies that
\[ d_i \geq c_1 h^{1/2}, \quad i = 1, \cdots, n-1, \]
for some \( c_1 \) depending only on \( n \) and \( \mu \). Also from (3.2) we see that
\[ \tilde{x}_h^* \cdot e_n = x_h^* \cdot e_n \geq c_0(\rho) h^\alpha \]
which gives
\[ d_n \geq c(n) \tilde{x}_h^* \cdot e_n \geq c(\rho) h^\alpha. \]
We claim that for all small $h$,

$$\prod_{i=1}^{n} d_i \geq k_0 h^{n/2},$$

with $k_0$ small depending only on $\mu, n, \Lambda$, which gives the left inequality in (ii).

To this aim we consider the barrier,

$$w = \epsilon x_n + \sum_{i=1}^{n} \frac{c h}{d_i} \left( \frac{x_i}{d_i} \right)^2.$$

We choose $c$ sufficiently small depending on $\mu, n, \Lambda$ so that for all $h < c(\rho)$,

$$w \leq h \quad \text{on } \partial \tilde{S}_h,$$

and on the part of the boundary $\tilde{G}_h$, we have $w \leq \tilde{u}$ since

$$w \leq \epsilon x_n + \frac{c}{c^2} |x'|^2 + c h \left( \frac{x_n}{d_n} \right)^2 \leq \frac{H}{4} |x'|^2 + chC(n) \frac{x_n}{d_n} \leq \frac{H}{4} |x'|^2 + ch^{1-n} C(\rho) |x'|^2 \leq \frac{H}{2} |x'|^2.$$

Moreover, if our claim does not hold, then

$$\det D^2 w = (2ch)^n (\prod d_i)^{-2n} > \Lambda,$$

thus $w \leq \tilde{u}$ in $\tilde{S}_h$. By definition, $\tilde{u}$ is obtained from $u$ by a sliding along $x_n = 0$, hence 0 is still the tangent plane of $\tilde{u}$ at 0. We reach again a contradiction since $\tilde{u} \geq w \geq \epsilon x_n$ and the claim is proved.

Finally we show that

$$|\tilde{S}_h| \leq C h^{n/2}$$

for some $C$ depending only on $\lambda, n$. Indeed, if

$$v = h \quad \text{on } \partial \tilde{S}_h,$$

and

$$\det D^2 v = \lambda$$

then

$$v \geq u \geq 0 \quad \text{in } \tilde{S}_h.$$

Since

$$h \geq h - \min_{\tilde{S}_h} v \geq c(n, \lambda) |\tilde{S}_h|^{2/n}$$

we obtain the desired conclusion.

In the proof above we showed that for all $h \leq c(\rho)$, the entries of the diagonal matrix $D_h$ from (3.3) satisfy

$$d_i \geq ch^{1/2}, \quad i = 1, \ldots, n - 1$$
\[ d_n \geq c(\rho)h^\alpha, \quad \alpha = \frac{n}{n+1} \]

\[ ch^{n/2} \leq \prod d_i \leq Ch^{n/2}. \]

The main step in the proof of Theorem 2.1 is the following lemma that will be proved in the remaining sections.

**Lemma 3.2.** There exist constants \( c, c(\rho) \) such that

\[ d_n \geq ch^{1/2}, \]

for all \( h \leq c(\rho) \).

Using Lemma 3.2 we can easily finish the proof of our theorem.

**Proof of Theorem 2.1.** Since all \( d_i \) are bounded below by \( ch^{1/2} \) and their product is bounded above by \( Ch^{n/2} \) we see that

\[ Ch^{1/2} \geq d_i \geq ch^{1/2} \quad i = 1, \ldots, n \]

for all \( h \leq c(\rho) \). Using (3.3) we obtain

\[ \tilde{S}_h \subset Ch^{1/2}B_1. \]

Moreover, since

\[ \tilde{x}_h^* \cdot e_n \geq d_n \geq ch^{1/2}, \quad (\tilde{x}_h^*)' = 0, \]

and the part \( \tilde{G}_h \) of the boundary \( \partial S_h \) contains the graph of \( \tilde{g}_h \) above \( |x'| \leq ch^{1/2} \), we find that

\[ ch^{1/2}B_1 \cap \tilde{\Omega} \subset \tilde{S}_h, \]

with \( \tilde{\Omega} = A_h\Omega \), \( \tilde{S}_h = A_hS_h \). In conclusion

\[ ch^{1/2}B_1 \cap \tilde{\Omega} \subset A_hS_h \subset Ch^{1/2}B_1. \]

We define the ellipsoid \( E_h \) as

\[ E_h := A_h^{-1}(h^{1/2}B_1), \]

hence

\[ cE_h \cap \tilde{\Omega} \subset \tilde{S}_h \subset CE_h. \]

Comparing the sections at levels \( h \) and \( h/2 \) we find

\[ cE_{h/2} \cap \tilde{\Omega} \subset CE_h \]

and we easily obtain the inclusion

\[ A_hA_{h/2}^{-1}B_1 \subset CB_1. \]

If we denote

\[ A_h x = x - \nu_h x_n \]

then the inclusion above implies

\[ |\nu_h - \nu_{h/2}| \leq C, \]

which gives the desired bound

\[ |\nu_h| \leq C |\log h| \]

for all small \( h \).
We introduce a new quantity \( b(h) \) which is proportional to \( d_n h^{-1/2} \) and which is appropriate when dealing with affine transformations.

**Notation.** Given a convex function \( u \) we define
\[
b_u(h) = h^{-1/2} \sup_{S_h} x_n.
\]
Whenever there is no possibility of confusion we drop the subindex \( u \) and use the notation \( b(h) \).

Below we list some basic properties of \( b(h) \).

1) If \( h_1 \leq h_2 \) then
\[
\left( \frac{h_1}{h_2} \right)^{1/2} \leq \frac{b(h_1)}{b(h_2)} \leq \left( \frac{h_2}{h_1} \right)^{1/2}.
\]

2) A rescaling
\[
\tilde{u}(Ax) = u(X)
\]
given by a linear transformation \( A \) which leaves the \( x_n \) coordinate invariant does not change the value of \( b \), i.e.
\[
b_{\tilde{u}}(h) = b_u(h).
\]

3) If \( A \) is a linear transformation which leaves the plane \( \{x_n = 0\} \) invariant the values of \( b \) get multiplied by a constant. However the quotients \( b(h_1)/b(h_2) \) do not change values i.e.

\[
\frac{b_{\tilde{u}}(h_1)}{b_{\tilde{u}}(h_2)} = \frac{b_u(h_1)}{b_u(h_2)}.
\]

4) If we multiply \( u \) by a constant, i.e.
\[
\tilde{u}(x) = \beta u(x)
\]
then
\[
b_{\tilde{u}}(\beta h) = \beta^{-1/2} b_u(h),
\]
and
\[
\frac{b_{\tilde{u}}(\beta h_1)}{b_{\tilde{u}}(\beta h_2)} = \frac{b_u(h_1)}{b_u(h_2)}.
\]

From (3.3) and property 2 above,
\[
c(n)d_n \leq b(h)h^{1/2} \leq C(n)d_n,
\]
hence Lemma 3.2 will follow if we show that \( b(h) \) is bounded below. We achieve this by proving the following lemma.

**Lemma 3.3.** There exist \( c_0, c(\rho) \) such that if \( h \leq c(\rho) \) and \( b(h) \leq c_0 \) then
\[
(3.5) \quad \frac{b(th)}{b(h)} > 2,
\]
for some \( t \in [c_0, 1] \).
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This lemma states that if the value of \(b(h)\) on a certain section is less than a critical value \(c_0\), then we can find a lower section at height still comparable to \(h\) where the value of \(b\) doubled. Clearly Lemma 3.3 and property 1 above imply that \(b(h)\) remains bounded for all \(h\) small enough.

The quotient in \(\text{3.5}\) is the same for \(\tilde{u}\) which is defined in Proposition 3.1. We normalize the domain \(\tilde{S}_h\) and \(\tilde{u}\) by considering the rescaling \(v(x) = \frac{1}{h} \tilde{u}(h^{1/2}Ax)\) where \(A\) is a multiple of \(D_h\) (see \(\text{3.3}\)), \(A = \gamma D_h\) such that \(\det A = 1\).

Then
\[c h^{-1/2} \leq \gamma \leq C h^{-1/2},\]
and the diagonal entries of \(A\) satisfy
\[a_i \geq c, \quad i = 1, 2, \ldots, n - 1,\]
\[c b_n(h) \leq a_n \leq C b_n(h).\]

The function \(v\) satisfies
\[\lambda \leq \det D^2 v \leq \Lambda,\]
\[v \geq 0, \quad v(0) = 0,\]
is continuous and it is defined in \(\tilde{\Omega}_v\) with
\[\tilde{\Omega}_v := \{v < 1\} = h^{-1/2} A^{-1} \tilde{S}_h.\]
Then
\[x^* + cB_1 \subset \tilde{\Omega}_v \subset cB_1^+,\]
for some \(x^*,\) and
\[ct^{n/2} \leq |S_t(v)| \leq Ct^{n/2}, \quad \forall t \leq 1,\]
where \(S_t(v)\) denotes the section of \(v\). Since \(\tilde{u} = h \quad \text{in} \quad \partial \tilde{S}_h \cap \{x_n \geq C(\rho)h\},\)
then
\[v = 1 \quad \text{on} \quad \partial \tilde{\Omega}_v \cap \{x_n \geq \sigma\}, \quad \sigma := C(\rho)h^{1-\alpha}.\]
Also, from Proposition 3.1 on the part \(G\) of the boundary of \(\partial \Omega_v\) where \(\{v < 1\}\) we have
\[\frac{1}{\mu} \sum_{i=1}^{n} a_i^2 x_i^2 \leq v \leq 2 \mu^{-1} \sum_{i=1}^{n} a_i^2 x_i^2.\]

In order to prove Lemma 3.3 we need to show that if \(\sigma, a_n\) are sufficiently small depending on \(n, \mu, \lambda, \Lambda\) then the function \(v\) above satisfies
\[b_v(t) \geq 2 b_v(1)\]
for some \(1 > t \geq c_0.\)

Since \(\alpha < 1,\) the smallness condition on \(\sigma\) is satisfied by taking \(h < c(\rho)\) sufficiently small. Also \(a_n\) being small is equivalent to one of the \(a_i, 1 \leq i \leq n - 1\) being large since their product is 1 and \(a_i\) are bounded below.
In the next sections we prove property (3.7) above by compactness, by letting $\sigma \to 0$, $a_i \to \infty$ for some $i$. First we consider the 2D case and in the last section the general case.

4. The 2 dimensional case.

In order to fix ideas, we consider first the 2 dimensional case.

We study the following class of solutions to the Monge-Ampere equation. Fix $\mu > 0$ small, $\lambda$, $\Lambda$. We denote by $D_\sigma$ the set of convex, continuous functions $u : \overline{\Omega} \to \mathbb{R}$ such that

\begin{equation}
\lambda \leq \det D^2 u \leq \Lambda; \tag{4.1}
\end{equation}

\begin{equation}
0 \in \partial \Omega, \quad B_\mu(x_0) \subset \Omega \subset B_{1/\mu}^+ \quad \text{for some } x_0; \tag{4.2}
\end{equation}

\begin{equation}
\mu h^{n/2} \leq |S_h| \leq \mu^{-1} h^{n/2}; \tag{4.3}
\end{equation}

\begin{equation}
u(h) := (\sup_{S_h} x) h^{-1/2} > M \tag{4.4}
\end{equation}

Property (3.7) easily follows from the proposition above. Indeed, by choosing $M = 2\mu^{-1} > 2\nu(1)$ we prove the existence of a section $h \geq c_0$ such that

\begin{equation}b(h) = (\sup_{S_h} x) h^{-1/2} > M \tag{4.4}
\end{equation}

for some $h \geq c_0$.

Proposition 4.1. Assume $n = 2$. For any $M > 0$ there exists $c_0$ small depending on $M, \mu, \lambda, \Lambda$, such that if $u \in D_\sigma$ and $\sigma \leq c_0$, then

\begin{equation}b(h) := (\sup_{S_h} x) h^{-1/2} > M \tag{4.4}
\end{equation}

We prove Proposition 4.1 by compactness. First we discuss briefly the compactness of bounded solutions to Monge-Ampere equation. For this we need to introduce solutions with possibly discontinuous boundary data.

Let $u : \Omega \to \mathbb{R}$ be a convex function with $\Omega \subset \mathbb{R}^n$ bounded and convex. We denote by

$$\Gamma_u := \{(x, x_{n+1}) \in \Omega \times \mathbb{R} \mid x_{n+1} \geq u(x)\}$$

the upper graph of $u$.

Definition 4.2. We define the values of $u$ on $\partial \Omega$ to be equal to $\varphi$ i.e

$$u|_{\partial \Omega} = \varphi,$$

if the upper graph of $\varphi : \partial \Omega \to \mathbb{R} \cup \{\infty\}$

$$\Phi := \{(x, x_{n+1}) \in \partial \Omega \times \mathbb{R} \mid x_{n+1} \geq \varphi(x)\}.$$
is given by the closure of $\Gamma_u$ restricted to $\partial\Omega \times \mathbb{R}$,
\[
\Phi := \overline{\Gamma_u} \cap (\partial\Omega \times \mathbb{R}).
\]

From the definition we see that $\varphi$ is always lower semicontinuous. The following comparison principle holds: if $w : \overline{\Omega} \rightarrow \mathbb{R}$ is continuous and
\[
\det D^2 w \geq \Lambda \geq \det D^2 u, \quad w|_{\partial\Omega} \leq u|_{\partial\Omega},
\]
then
\[
w \leq u \quad \text{in } \Omega.
\]
Indeed, from the continuity of $w$ we see that for any $\varepsilon > 0$, there exists a small neighborhood of $\partial\Omega$ where $w - \varepsilon < u$. This inequality holds in the interior from the standard comparison principle, hence $w \leq u$ in $\Omega$.

Since the convex functions are defined on different domains we use the following notion of convergence.

**Definition 4.3.** We say that the convex functions $u_m : \Omega_m \rightarrow \mathbb{R}$ converge to $u : \Omega \rightarrow \mathbb{R}$ if the upper graphs converge
\[
\overline{\Gamma_{u_m}} \rightarrow \overline{\Gamma_u} \quad \text{in the Hausdorff distance.}
\]

Similarly, we say that the lower semicontinuous functions $\varphi_m : \partial\Omega_m \rightarrow \mathbb{R}$ converge to $\varphi : \partial\Omega \rightarrow \mathbb{R}$ if the upper graphs converge
\[
\Phi_m \rightarrow \Phi \quad \text{in the Hausdorff distance.}
\]

Clearly if $u_m$ converges to $u$, then $u_m$ converges uniformly to $u$ in any compact set of $\Omega$, and $\Omega_m \rightarrow \Omega$ in the Hausdorff distance.

**Remark:** When we restrict the Hausdorff distance to the nonempty closed sets of a compact set we obtain a compact metric space. Thus, if $\Omega_m$, $u_m$ are uniformly bounded then we can always extract a subsequence $m_k$ such that $u_{m_k} \rightarrow u$ and $u_{m_k}|_{\partial\Omega_{m_k}} \rightarrow \varphi$.

Next lemma gives the relation between the boundary data of the limit $u$ and $\varphi$.

**Lemma 4.4.** Let $u_m : \Omega_m \rightarrow \mathbb{R}$ be convex functions, uniformly bounded, such that
\[
\lambda \leq \det D^2 u_m \leq \Lambda
\]
and
\[
u = \overline{u_m} - l \leq 0 \quad \text{in } \Omega_m.
\]

Then
\[
\lambda \leq \det D^2 u \leq \Lambda,
\]
and the boundary data of $u$ is given by $\varphi^*$ the convex envelope of $\varphi$ on $\partial\Omega$.

**Proof.** Clearly $\Phi \subset \overline{\Gamma_u}$, hence $\Phi^* \subset \overline{\Gamma_u}$. It remains to show that the convex set $K$ generated by $\Phi$ contains $\overline{\Gamma_u} \cap (\partial\Omega \times \mathbb{R})$.

Indeed consider a hyperplane
\[
x_{n+1} = l(x)
\]
which lies strictly below $K$. Then, for all large $m$
\[
\{u_m - l \leq 0\} \subset \Omega_m,
\]
and by Alexandrov estimate we have that
\[
u_m - l \geq -Cd_{m}^{1/n}
\]
where $d_m(x)$ represents the distance from $x$ to $\partial \Omega_m$. By taking $m \to \infty$ we see that

$$u - l \geq -Cd^{1/n}$$

thus no point on $\partial \Omega$ below $l$ belongs to $\overline{\Gamma}_u$. 

\[\square\]

In view of the lemma above we introduce the following notation.

**Definition 4.5.** Let $\varphi : \partial \Omega \to \mathbb{R}$ be a lower semicontinuous function. When we write that a convex function $u$ satisfies

$$u = \varphi \text{ on } \partial \Omega$$

we understand

$$u|_{\partial \Omega} = \varphi^*$$

where $\varphi^*$ is the convex envelope of $\varphi$ on $\partial \Omega$.

Whenever $\varphi^*$ and $\varphi$ do not coincide we can think of the graph of $u$ as having a vertical part on $\partial \Omega$ between $\varphi^*$ and $\varphi$.

It follows easily from the definition above that the boundary values of $u$ when we restrict to the domain

$$\Omega_h := \{ u < h \}$$

are given by

$$\varphi_h = \varphi \text{ on } \partial \Omega \cap \{ \varphi \leq h \} \subset \partial \Omega_h$$

and $\varphi_h = h$ on the remaining part of $\partial \Omega_h$.

The comparison principle still holds. Precisely, if $w : \overline{\Omega} \to \mathbb{R}$ is continuous and

$$\det D^2w \geq \Lambda \geq \det D^2u, \quad w|_{\partial \Omega} \leq \varphi,$$

then

$$w \leq u \text{ in } \Omega.$$ 

The advantage of introducing the notation of Definition 4.5 is that the boundary data is preserved under limits.

**Proposition 4.6 (Compactness).** Assume

$$\lambda \leq \det D^2u_m \leq \Lambda, \quad u_m = \varphi_m \text{ on } \partial \Omega_m,$$

and $\Omega_m$, $\varphi_m$ uniformly bounded.

Then there exists a subsequence $m_k$ such that

$$u_{m_k} \to u, \quad \varphi_{m_k} \to \varphi$$

with

$$\lambda \leq \det D^2u \leq \Lambda, \quad u = \varphi \text{ on } \partial \Omega.$$ 

Indeed, we see that we can also choose $m_k$ such that $\varphi^*_{m_k} \to \psi$. Since $\varphi_{m_k} \to \varphi$ we obtain

$$\varphi \geq \psi \geq \varphi^*,$$

and the conclusion follows from Lemma 4.3.

Now we are ready to prove Proposition 4.1.

**Proof of Proposition 4.1.** If $c_0$ does not exist we can find a sequence of functions $u_m \in D_{1/m}$ such that

$$b_{u_m}(h) \leq M, \quad \forall h \geq \frac{1}{m},$$
By Proposition 4.6 there is a subsequence which converges to a limiting function $u$ satisfying (4.1)-(4.2)-(4.3) and (see Definition 4.5) $u = \varphi$ on $\partial \Omega$ with
\begin{equation}
\varphi = 1 \quad \text{on} \quad \partial \Omega \setminus \{0\}, \quad \varphi(0) = 0,
\end{equation}
and moreover $u$ has an obstacle by below in $\Omega$
\begin{equation}
u \geq \frac{1}{M^2}x_2^2.
\end{equation}

We consider the barrier
$$w := \delta(|x_1| + \frac{1}{2}x_2^2) + \frac{\Lambda}{\delta}x_2^2 - Nx_2$$
with $\delta$ small depending on $\mu$, and $N$ large so that
$$\frac{\Lambda}{\delta}x_2^2 - Nx_2 \leq 0 \quad \text{in} \quad B_1^+.$$
Then
$$w \leq \varphi \quad \text{on} \quad \partial \Omega,$$
and
$$\det D^2 w > \Lambda.$$
Hence
$$w \leq u \quad \text{in} \quad \Omega$$
which gives
$$u \geq \delta|x_1| - Nx_2.$$ 

Next we construct another explicit subsolution $v$ such that whenever $v$ is above the two obstacles
$$\delta|x_1| - Nx_2, \quad \frac{1}{M^2}x_2^2,$$
we have
$$\det D^2 v > \Lambda \quad \text{and} \quad v \leq 1.$$ 
Then we can conclude that
$$u \geq v,$$
and we show that this contradicts the lower bound on $|S_h|$. 

We look for a function of the form
$$v := rf(\theta) + \frac{1}{2M^2}x_2^2,$$
where $r, \theta$ represent the polar coordinates in the $x_1, x_2$ plane.

The domain of definition of $v$ is the angle
$$K := \{\theta_0 \leq \theta \leq \pi - \theta_0\}$$
with $\theta_0$ small so that
$$\frac{1}{2M^2}x_2^2 \leq \frac{1}{2}(\delta|x_1| - Nx_2) \quad \text{on} \quad \partial K \cap B_\mu.$$

In the set
$$\{v \geq \frac{1}{M^2}x_2^2\}$$
i.e. where
$$\frac{1}{r} \geq \frac{\sin^2 \theta}{2M^2f}$$
we have

\[ \det D^2 v = \frac{1}{r} (f'' + f') \frac{\sin^2 \theta}{M^2} \geq \frac{1}{r} (f'' + f') \frac{\sin^2 \theta_0}{2M^2}. \]  

We let

\[ f(\theta) = \sigma e^{C_0 |\frac{\pi}{2} - \theta|}, \]

where \( C_0 \) is large depending on \( \theta_0, M, \Lambda \) so that (see (4.7))

\[ \det D^2 v > \Lambda \]
in the set where

\( \{ v \geq \frac{1}{M^2 x_2^2} \} \).

On the other hand we can choose \( \sigma \) small so that

\[ v \leq \delta |x_1| - N x_2 \text{ on } \partial K \cap B_{\mu} \]
and

\[ v \leq 1 \text{ on the set } \{ v \geq \frac{1}{M^2 x_2^2} \}. \]

In conclusion

\[ u \geq v \geq \varepsilon x_2, \]
hence

\[ u \geq \max\{ \varepsilon x_2, \delta |x_1| - N x_2 \}. \]

This implies

\[ |S_h| \leq C h^2 \]
for all small \( h \) and we contradict that

\[ |S_h| \geq \mu h, \quad \forall h \in [0, 1]. \]

\[ \square \]

5. THE HIGHER DIMENSIONAL CASE

In higher dimensions it is more difficult to construct an explicit barrier as in Proposition 4.1 in the case when in (3.6) only one \( a_i \) is large and the others are bounded. We prove our result by induction depending on the number of large eigenvalues \( a_i \).

Fix \( \mu \) small and \( \lambda, \Lambda \). For each increasing sequence

\[ \alpha_1 \leq \alpha_2 \leq \ldots \leq \alpha_{n-1} \]
with

\[ \alpha_1 \geq \mu, \]
we consider the family of solutions

\[ \mathcal{D}_\sigma^\mu(\alpha_1, \alpha_2, \ldots, \alpha_{n-1}) \]
of convex, continuous functions \( u : \Omega \to \mathbb{R} \) that satisfy

\[ \lambda \leq \det D^2 u \leq \Lambda \text{ in } \Omega, \quad u \geq 0 \text{ in } \Omega; \]

\[ 0 \in \partial \Omega, \quad B_{\mu}(x_0) \subset \Omega \subset B_{1/\mu}^+ \text{ for some } x_0; \]

\[ \mu h^{n/2} \leq |S_h| \leq \mu^{-1} h^{n/2}; \]

\[ u = 1 \text{ on } \partial \Omega \setminus G; \]
and
\[ \mu \sum_{i=1}^{n-1} \alpha_i^2 x_i^2 \leq u \leq \mu \sum_{i=1}^{n-1} \alpha_i^2 x_i^2 \quad \text{on } G, \]

where \( G \) is a closed subset of \( \partial \Omega \) which is a graph in the \( e_n \) direction and is included in boundary in \( \{ x_n \leq \sigma \} \).

For convenience we would like to add the limiting solutions when \( \alpha_{k+1} \to \infty \) and \( \sigma \to 0 \). We denote by
\[ D_{\mu}^0(\alpha_1, \ldots, \alpha_k, \infty, \infty, \ldots, \infty) \]
the class of functions \( u : \Omega \to \mathbb{R} \) that satisfy properties (5.1)-(5.2)-(5.3) and (see Definition 4.5) \( u = \varphi \) on \( \partial \Omega \) with
\[ \varphi = 1 \quad \text{on } \partial \Omega \setminus G; \]

(5.6) \[ \mu \sum_{i=1}^{k} \alpha_i^2 x_i^2 \leq \varphi \leq \min\{1, \mu \sum_{i=1}^{k} \alpha_i^2 x_i^2\} \quad \text{on } G, \]

where \( G \) is a closed set
\[ G \subset \partial \Omega \cap \{ x_i = 0, \ i > k \}, \]
and if we restrict to the space generated by the first \( k \) coordinates then
\[ \{ \mu \sum_{i=1}^{k} \alpha_i^2 x_i^2 \leq 1 \} \subset G \subset \{ \mu \sum_{i=1}^{k} \alpha_i^2 x_i^2 \leq 1 \}. \]

We extend the definition of \( D_{\mu}^0(\alpha_1, \alpha_2, \ldots, \alpha_{n-1}) \) to include also the pairs with
\[ \mu \leq \alpha_1 \leq \ldots \leq \alpha_k < \infty, \quad \alpha_{k+1} = \ldots = \alpha_{n-1} = \infty \]
for which \( \sigma = 0 \) i.e. \( D_{\mu}^0(\alpha_1, \alpha_2, \ldots, \alpha_k, \infty, \ldots, \infty) \).

Proposition 4.6 implies that if
\[ u_m \in D_{\sigma_m}^\mu(a_m^1, \ldots, a_m^{n-1}) \]
is a sequence with \( \sigma_m \to 0 \) and \( a_{k+1}^m \to \infty \)
for some fixed \( 0 \leq k \leq n-2 \), then we can extract a convergent subsequence to a function \( u \) with
\[ u \in D_{\sigma}^\mu(a_1, \ldots, a_l, \infty, \ldots, \infty), \]
for some \( l \leq k \) and \( a_1 \leq \ldots \leq a_l \).

**Proposition 5.1.** For any \( M > 0 \) and \( 1 \leq k \leq n-1 \) there exists \( C_k \) depending on \( M, \mu, \lambda, \Lambda, n, k \) such that if \( u \in D_{\sigma}^\mu(\alpha_1, \alpha_2, \ldots, \alpha_{n-1}) \) with
\[ \alpha_k \geq C_k, \quad \sigma \leq C_k^{-1} \]
then
\[ b(h) = (\sup_{S_h} x_n)h^{-1/2} \geq M \]
for some \( h \) with \( C_k^{-1} \leq h \leq 1 \).

As we remarked in the previous section, property (5.7) and therefore Lemma 3.3 follow from Proposition 5.1 by taking \( k = n-1 \) and \( M = 2\mu^{-1} \).

We prove the proposition by induction on \( k \).
Lemma 5.2. Proposition 5.1 holds for \( k = 1 \).

Proof. By compactness we need to show that there does not exist \( u \in D_0^\mu(\infty, \ldots, \infty) \) with \( b(h) \leq M \) for all \( h \).

The proof is almost identical to the 2 dimensional case. One can see as before that

\[
u \geq \max\{\delta|x'| - N x_n, \frac{1}{M^2}x^2_n\}
\]

and then construct a barrier of the form

\[
v = rf(\theta) + \frac{1}{2M^2}x^2_n, \quad \theta_0 \leq \theta \leq \frac{\pi}{2}
\]

where \( r = |x| \) and \( \theta \) represents the angle in \([0, \pi/2]\) between the ray passing through \( x \) and the \([x_n = 0]\) plane.

Now,

\[
det D^2 v = \frac{f'' + f}{r} \left( \frac{f \cos \theta - f' \sin \theta}{r \cos \theta} \right)^{n-2} \sin^2 \theta \frac{M^2}{M^2}.
\]

We have

\[
\frac{f}{r} > \frac{\sin^2 \theta}{2M^2} \quad \text{on the set } \{v > \frac{1}{M^2}x^2_n\}
\]

and we choose a function of the form

\[
f(\theta) := \nu e^{C_0(\frac{\pi}{2} - \theta)}
\]

which is decreasing in \( \theta \).

Then

\[
det D^2 v > \frac{f'' + f}{f} \left( \frac{\sin^2 \theta_0}{2M^2} \right)^{n-1} > \Lambda
\]

if \( C_0 \) is chosen large.

We obtain as before that

\[
u \geq \max\{\delta|x'| - N x_n, \epsilon x_n\}
\]

which gives

\[
|S_h| \leq Ch^n
\]

and we reach a contradiction.

\[\square\]

Now we prove Proposition 5.1 by induction on \( k \).

Proof of Proposition 5.1. In this proof we denote by \( c, C \) positive constants that depend on \( M, \mu, \lambda, \Lambda, n \) and \( k \).

We assume that the statement holds for \( k \) and we prove it for \( k + 1 \).

It suffices to show the existence of \( C_{k+1} \) only in the case when \( \alpha_k < C_k \), otherwise we use the induction hypothesis.

If no \( C_{k+1} \) exists then we can find a limiting solution

\[
u \in D_0^{\tilde{\mu}}(1,1, \ldots, 1, \infty, \ldots, \infty)
\]

with

\[(5.8) \quad b(h) < Mh^{1/2}, \quad \forall h > 0\]

where \( \tilde{\mu} \) depends on \( \mu \) and \( C_k \).

We show that such a function \( u \) does not exist.
Denote 
\[ x = (y, z, x_n), \quad y = (x_1, \ldots, x_k) \in \mathbb{R}^k, \quad z = (x_{k+1}, \ldots, x_{n-1}) \in \mathbb{R}^{n-1-k}. \]

On the \( \partial \Omega \) plane we have
\[ \varphi \geq w := \delta|x'|^2 + \delta|z| + \frac{\Lambda}{\delta^{n-1}}x_n^2 - N x_n \]
for some small \( \delta \) depending on \( \tilde{\mu} \), and \( N \) large so that
\[ \frac{\Lambda}{\delta^{n-1}}x_n^2 - N x_n \leq 0 \text{ on } B^+_{1/\tilde{\mu}}. \]

Since \( \det D^2 w > \Lambda \), we obtain \( u \geq w \) on \( \Omega \) hence (5.9)
\[ u(x) \geq \delta|z| - N x_n. \]

We look at the section \( S_h \) of \( u \). From (5.8)-(5.9) we see that (5.10)
\[ S_h \subset \{ x_n > 1 \mathcal{N}(\delta|z| - h) \} \cap \{ x_n \leq M h^{1/2} \}. \]

We notice that an affine transformation \( x \rightarrow Tx \),
\[ Tx := x + \nu_1 z_1 + \nu_2 z_2 + \ldots + \nu_{n-k-1} z_{n-k-1} + \nu_{n-k} x_n \]
with \( \nu_1, \nu_2, \ldots, \nu_{n-k} \in \text{span}\{e_1, \ldots, e_k\} \)
i.e a sliding along the \( y \) direction, leaves the \( z, x_n \) coordinate invariant together with the subspace \( (y, 0, 0) \).

The section \( \tilde{S}_h := TS_h \) of the rescaling
\[ \tilde{u}(Tx) = u(x) \]
satisfies (5.10) and \( \tilde{u} = \tilde{\varphi} \) on \( \partial \tilde{S}_h \) with
\[ \tilde{\varphi} = \varphi \quad \text{on } \tilde{G} := \{ \varphi \leq h \} \subset G, \]
\[ \tilde{\varphi} = h \quad \text{on } \partial \tilde{S}_h \setminus \tilde{G}. \]

From John’s lemma we know that \( S_h \) is equivalent to an ellipsoid \( E_h \). We choose \( T \) an appropriate sliding along the \( y \) direction, so that \( TE_h \) becomes symmetric with respect to the \( y \) and \( (z, x_n) \) subspaces, thus
\[ \tilde{x}_h^n + c(n)|\tilde{x}_h|^{|n AB_1 \subset \tilde{x}_h \subset C(n)|\tilde{x}_h|^{|1/n AB_1, \quad \text{det } A = 1 \]
and the matrix \( A \) leaves the \( y \) and the \( (z, x_n) \) subspaces invariant.

By choosing an appropriate system of coordinates in the \( y \) and \( z \) variables we may assume
\[ A(y, z, x_n) = (A_1 y, A_2(z, x_n)) \]
with
\[ A_1 = \begin{pmatrix} \beta_1 & 0 & \cdots & 0 \\ 0 & \beta_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \beta_k \end{pmatrix} \]
with \( 0 < \beta_1 \leq \cdots \leq \beta_k \), and
\[
A_2 = \begin{pmatrix}
\gamma_{k+1} & 0 & \cdots & 0 & \theta_{k+1} \\
0 & \gamma_{k+2} & \cdots & 0 & \theta_{k+2} \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \cdots & \gamma_{n-1} & \theta_{n-1} \\
0 & 0 & \cdots & 0 & \theta_n
\end{pmatrix}
\]

with \(\gamma_j, \theta_n > 0\).

Next we use the induction hypothesis and show that \(\tilde{S}_h\) is equivalent to a ball.

**Lemma 5.3.** There exists \(C_0\) such that
\[
\tilde{S}_h \subset C_0 h^{n/2} B_1^+.
\]

**Proof.** Using that
\[
|\tilde{S}_h| \sim h^{n/2}
\]
we obtain
\[
\tilde{x}_h^* + ch^{1/2} AB_1 \subset \tilde{S}_h \subset Ch^{1/2} AB_1.
\]

We need to show that \(\|A\| \leq C\).

Since \(\tilde{S}_h\) satisfies (5.10) we see that
\[
\tilde{S}_h \subset \{(z, x_n) | \leq Ch^{1/2}\},
\]
which together with the inclusion above gives \(\|A_2\| \leq C\) hence
\[
\gamma_j, \theta_n \leq C, \quad |\theta_j| \leq C.
\]

Also \(\tilde{S}_h\) contains the set
\[
\{(y, 0, 0) | |y| \leq \tilde{\mu}^{1/2} h^{1/2}\} \subset \tilde{G},
\]
which implies
\[
\beta_i \geq c > 0, \quad i = 1, \ldots, k.
\]

We define the rescaling
\[
w(x) = \frac{1}{h} \tilde{u}(h^{1/2} Ax)
\]
which is defined in a domain \(\Omega_w := h^{-1/2} A^{-1} \tilde{S}_h\) such that
\[
B_c(x_0) \subset \Omega_w \subset B_c^+, \quad 0 \in \partial \Omega_w,
\]
and \(w = \varphi_w\) on \(\partial \Omega_w\) with
\[
\varphi_w = 1 \quad \text{on} \quad \partial \Omega_w \setminus G_w,
\]
\[
\tilde{\mu} \sum \beta_i^2 x_i^2 \leq \varphi_w \leq \min\{1, \tilde{\mu}^{-1} \sum \beta_i^2 x_i^2\} \quad \text{on} \quad G_w,
\]
where \(G_w := h^{-1/2} A^{-1} \tilde{G}\).

This implies that
\[
w \in \mathcal{D}_0^\mu(\beta_1, \beta_2, \ldots, \beta_k, \infty, \ldots, \infty)
\]
for some value \(\tilde{\mu}\) depending on \(\mu, M, \lambda, \Lambda, n, k\).

We claim that
\[
b_u(h) \geq c_*.
\]

First we notice that
\[
b_u(h) = b_0(h) \sim \theta_n.
\]
Since
\[ \theta_n \prod \beta_i \prod \gamma_j = \det A = 1 \]
and
\[ \gamma_j \leq C, \]
we see that if \( b_u(h) \) (and therefore \( \theta_n \)) becomes smaller than a critical value \( c \), then
\[ \beta_k \geq C_k(\bar{\mu}, \bar{M}, \lambda, \Lambda, n), \]
with \( \bar{M} := 2\bar{\mu}^{-1} \), and by the induction hypothesis
\[ b_w(\hat{h}) \geq \bar{M} \geq 2b_w(1) \]
for some \( \hat{h} > C_k^{-1} \). This gives
\[ \frac{b_u(h\hat{h})}{b_u(h)} = \frac{b_w(\hat{h})}{b_w(1)} \geq 2, \]
which implies \( b_u(h\hat{h}) \geq 2b_u(h) \) and our claim follows.

Next we claim that \( \gamma_j \) are bounded below by the same argument. Indeed, from the claim above \( \theta_n \) is bounded below and if some \( \gamma_j \) is smaller than a small value \( \tilde{c}_* \) then
\[ \beta_k \geq C_k(\bar{\mu}, \bar{M}_1, \lambda, \Lambda, n) \]
with
\[ \bar{M}_1 := \frac{2M}{\bar{\mu}c_*}. \]
By the induction hypothesis
\[ b_w(\hat{h}) \geq \bar{M}_1 \geq \frac{2M}{c_*}b_w(1), \]
and with \( \eta > 0 \) depending on \( q_0 \) and \( \mu, M, \lambda, \Lambda, n, k \).

Proof. From Lemma \[5.3\] we see that after performing a linear transformation \( T \) (siding along the \( y \) direction) we may assume that
\[ S_h \subset C_0 h^{1/2} B_1. \]
Let
\[ w(x) := \frac{1}{h} u(h^{1/2}x) \]
for some small $h \ll p$.

Then

\[ S_1(w) := \Omega_w = h^{-1/2}S_h \subset B_{C_0}^+ \]

and our hypothesis becomes

\[ w \geq \frac{p}{h^{1/2}}(|z| - qx_n), \]

Moreover the boundary values $\varphi_w$ of $w$ on $\partial \Omega_w$ satisfy

\[ \varphi_w = 1 \quad \text{on} \quad \partial \Omega_w \setminus G_w \]

\[ \mu |y|^2 \leq \varphi_w \leq \min\{1, \hat{\mu}^{-1}|y|^2\} \quad \text{on} \quad G_w, \]

where $G_w := h^{-1/2}\{\varphi \leq h\}$.

Next we show that $\varphi_w \geq v$ on $\partial \Omega_w$ where $v$ is defined as

\[ v := \delta |x|^2 + \frac{\Lambda}{\delta^{n-1}}(z_1 - qx_n)^2 + N(z_1 - qx_n) + \delta x_n, \]

and $\delta$ is small depending on $\hat{\mu}$ and $C_0$, and $N$ is chosen large such that

\[ \frac{\Lambda}{\delta^{n-1}}t^2 + Nt \]

is increasing in the interval $|t| \leq (1 + q_0)C_0$.

From the definition of $v$ we see that

\[ \det D^2 v > \Lambda. \]

On the part of the boundary $\partial \Omega_w$ where $z_1 \leq qx_n$ we use that $\Omega_w \subset B_{C_0}$ and obtain

\[ v \leq \delta(|x|^2 + x_n) \leq \varphi_w. \]

On the part of the boundary $\partial \Omega_w$ where $z_1 > qx_n$ we use (5.11) and obtain

\[ 1 = \varphi_w \geq C(|z| - qx_n) \geq C(z_1 - qx_n) \]

with $C$ arbitrarily large provided that $h$ is small enough. We choose $C$ such that the inequality above implies

\[ \frac{\Lambda}{\delta^{n-1}}(z_1 - qx_n)^2 + N(z_1 - qx_n) < \frac{1}{2}. \]

Then

\[ \varphi_w = 1 > \frac{1}{2} + \delta(|x|^2 + x_n) \geq v. \]

In conclusion $\varphi_w \geq v$ on $\partial \Omega_w$ hence the function $v$ is a lower barrier for $w$ in $\Omega_w$. Then

\[ w \geq N(z_1 - qx_n) + \delta x_n \]

and, since this inequality holds for all directions in the $z$-plane, we obtain

\[ w \geq N(|z| - (q - \eta)x_n), \quad \eta := \frac{\delta}{N}. \]

Scaling back we get

\[ u \geq p'(|z| - (q - \eta)x_n) \quad \text{in} \quad S_h. \]

Since $u$ is convex and $u(0) = 0$, this inequality holds globally, and the lemma is proved.
We remark that Lemma 5.4 can be used directly to prove Proposition 4.1 and Lemma 5.2.

End of the proof of Proposition 5.1. From (5.9) we obtain an initial pair \((p, q_0)\) which satisfies the hypothesis of Lemma 5.4. We apply this lemma a finite number of times and obtain that

\[ u \geq \epsilon (|z| + x_n), \]

and we contradict that \(\tilde{S}_h\) is equivalent to a ball of radius \(h^{1/2}\).

\[ \square \]

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Department of Mathematics, Columbia University, New York, NY 10027
E-mail address: savin@math.columbia.edu