From the Zero–bias Anomaly to the Coulomb Blockade: 
an Exactly Solvable Model.

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Abstract

A microscopic theory of zero wavelength \((q = 0)\) interaction in finite–size systems is proposed. Its exact solution interpolates between the Coulomb blockade and the perturbative Altshuler–Aronov theory, in the strong and weak interaction limits respectively. The tunneling density of states and the quasiparticle life–time are calculated. The physical nature of the \((q = 0)\) component of the interaction is discussed.

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Much of the activity in the field of mesoscopic physics has focused on zero–dimensional ($d = 0$) weakly disordered systems, where the relevant energy scales turn out to be smaller than the Thouless energy, $E_c = \hbar D/L^2$, $D$ being the diffusivity and $L$ the system’s linear size (for a review see Ref. [1]). Studies of transport as well as of thermodynamic properties reveal the importance of electron–electron interactions [2]. The role of the latter and the interplay with disorder have been underlined by the perturbation theory developed by Altshuler and Aronov (AA) [3,4], who consequently explained the zero–bias anomaly in tunneling into a *macroscopic* conductor. Some ten years ago the observability of single electron effects in small capacitance devices has been proposed [5,6], leading to strong suppression of tunneling at low external bias. Nazarov [7] has pointed out to the common physics between the perturbative AA anomaly and the non–perturbative Coulomb blockade.

There are several interesting issues pertaining to $d = 0$ systems. One is the tunneling density of states (DOS). To study this quantity it is possible to employ perturbation theory in the spirit of AA; in doing so one should recall that the relevant wavenumbers are quantized with a special role played by the $q = 0$ component of the interaction. This component, pertaining to fluctuations of the global charge, is unscreened. When the interaction associated with this component is strong the phenomenological Coulomb blockade theory may be employed, but it is also desirable to have a microscopic theory at hand. Another interesting issue involves the electron inelastic life–time.

Here we present an *exactly solvable microscopic* model for interacting electrons in zero dimensional systems. Our model interpolates between the Coulomb blockade limit and the zero–bias anomaly. The results for the latter turn out to be in line with a simple generalization of the AA calculations [4]. We thus find the tunneling DOS as function of energy, temperature and coupling strength. We also evaluate the single particle life–time, associated with the width of the single particle levels. Finally we discuss the nature of the $q = 0$ interaction term, $V(q = 0)$. Relating it to the system’s capacitance we are able to establish a connection with previous studies of current fluctuations through an ultrasmall
We consider a system described by the Hamiltonian

\[ H = \sum_{\alpha} \epsilon_{\alpha} a_{\alpha}^{\dagger} a_{\alpha} + \frac{V}{2} \left( \sum_{\alpha} a_{\alpha}^{\dagger} a_{\alpha} - N_0 \right)^2, \]

where \( \{\epsilon_{\alpha}\} \) represent exact single particle eigenenergies; the parameter \( N_0 \) depends on the positive background and the external fields (e.g. gate voltages), and needs not to be an integer. The imaginary time single particle Green function is given by [10]

\[ G_{\alpha}(\tau_1, \tau_2, \mu) = -\frac{1}{Z(\mu)} \int \mathcal{D}[\psi_{\alpha}^{*}(\tau)\psi_{\alpha}(\tau)] e^{-\int_0^\beta d\tau [\sum_{\alpha} \psi_{\alpha}^{*}(\partial_{\tau}-\mu)\psi_{\alpha} + H[\psi_{\alpha}^{*}, \psi_{\alpha}]]} \psi_{\alpha}(\tau_1)\psi_{\alpha}^{*}(\tau_2), \]

where \( Z(\mu) \) is the partition function, given by the same integral without pre-exponential factors. Next we perform Hubbard–Stratonovich transformation by means of the imaginary Bose field \( \sigma(\tau) \), yielding

\[ G_{\alpha}(\tau_1, \tau_2, \mu) = -\frac{1}{Z(\mu)} \int \mathcal{D}[\sigma(\tau)] e^{\int_0^\beta d\tau \left[ \frac{\sigma^2(\tau)}{2m} + \sigma(\tau)N_0 \right]}
\int \mathcal{D}[\psi_{\alpha}^{*}\psi_{\alpha}] e^{-\int_0^\beta d\tau \sum_{\alpha} \psi_{\alpha}^{*}(\partial_{\tau}-\mu+\epsilon_{\alpha}+\sigma(\tau))\psi_{\alpha}(\tau_1)\psi_{\alpha}^{*}(\tau_2)}
= \frac{1}{Z(\mu)} \int \mathcal{D}[\sigma(\tau)] e^{\int_0^\beta d\tau \left[ \frac{\sigma^2(\tau)}{2m} + \sigma(\tau)N_0 \right]} Z[\sigma](\mu) G_{\alpha}^{[\sigma]}(\tau_1, \tau_2, \mu) \]

with the same transformations in \( Z(\mu) \). Here \( Z[\sigma](\mu) \) and \( G_{\alpha}^{[\sigma]}(\tau_1, \tau_2, \mu) \) are the partition function and the Green function, respectively, in the presence of a time dependent spatially uniform potential, \( \sigma(\tau) \), superimposed on the single particle problem [11]. Introducing the Matsubara representation (for the boson field \( \omega_m = 2\pi imT \), \( m = 0, \pm 1, \pm 2 \ldots \), \( \sigma_m = \beta^{-1} \int_0^\beta d\tau \sigma(\tau) \exp{\{\omega_m \tau\}} \) , we obtain

\[ Z[\sigma](\mu) = Z^{[0]}(\mu - \sigma_0) \]
\[ G_{\alpha}^{[\sigma]}(\tau_1, \tau_2, \mu) = G_{\alpha}^{[0]}(\tau_1, \tau_2, \mu - \sigma_0) e^{\int_{\tau_1}^{\tau_2} d\tau [\sigma(\tau) - \sigma_0]}, \]

where \( Z^{[0]}(\mu) \) and \( G_{\alpha}^{[0]}(\epsilon_n, \mu) \) are partition function and Green function of non-interacting electrons. Substituting in Eq.(3), we have

\[ G_{\alpha}(\tau_1, \tau_2, \mu) = \frac{1}{Z(\mu)} \int d\sigma_0 e^{\beta \frac{\sigma^2}{2m} + \sigma_0 N_0 - \Omega^{[0]}(\mu - \sigma_0)} G_{\alpha}^{[0]}(\tau_1, \tau_2, \mu - \sigma_0) \]
\int \prod_{m \neq 0} d\sigma_m \sum_{m \neq 0} \left[ \frac{\sigma_m \sigma_{-m} - \sigma_m}{2mT} - \frac{\sigma_m}{m} \exp(-\omega_m \tau_2) - \exp(-\omega_m \tau_1) \right]. \]
Here $\Omega^{[0]}$ is the (non–interacting) thermodynamic potential. The integral over the static component, $\sigma_0$, describes the smooth transition between the grand canonical ensemble with the chemical potential $\mu (V = 0)$ and the canonical ensemble with a given number of particles $[N_0] (V = \infty)$. For sufficiently large systems (or high temperature), $\frac{1}{\Delta} + \frac{1}{V} \gg \beta$ ($\Delta^{-1} \equiv -\partial^2 \Omega^{[0]} / \partial \mu^2$ – is the mean level spacing), differences between the two ensembles are negligible and the integral over $\sigma_0$ may be evaluated within a saddle point approximation (at $\sigma_0 = \sigma_0$). The $m \neq 0$ components are readily evaluated. The single particle Green function assumes the form

$$G_\alpha(\tau_1 - \tau_2, \mu) = G^{[0]}_\alpha(\tau_1 - \tau_2, \mu - \sigma_0) D(\tau_1 - \tau_2),$$

(6)

where the Green function of the auxiliary field $\sigma$, representing the Coulomb boson, is given by

$$D(\tau) = e^{-\frac{V}{2T} \left(|\tau| - \frac{\nu^2}{\tau}\right)}; \quad |\tau| \leq \beta$$

(7)

(it is periodic with a period $\beta$). In the Matsubara representation the Green function is superimposed with the Coulomb boson, see Fig. 1, to yield

$$G_\alpha(\epsilon_n, \mu) = T \sum_{\omega_m} G^{[0]}_\alpha(\epsilon_n - \omega_m, \mu) D(\omega_m),$$

(8)

where $\mu = \sigma_0$, $\epsilon_n = 2\pi i(n + 1/2)T$ and $D(\omega_m)$ is evaluated as the Fourier transform of $D(\tau)$ (in imaginary time), over the period $[0, \beta]$. The related spectral function is defined by

$$B(\omega) \equiv 2\Im D^A(\omega)$$

($D^A$ is the advanced Green function of the Coulomb boson) and is given by

$$B(\omega) = \sqrt{\frac{2\pi}{VT}} \left( e^{-\frac{(\omega + \frac{V}{2})^2}{2\nu T}} - e^{-\frac{(\omega - \frac{V}{2})^2}{2\nu T}} \right).$$

(9)

Employing the spectral representation of $G^{[0]}_\alpha, A^{[0]}_\alpha$, the total Green function assumes the following Lehmann representation

$$G_\alpha(\epsilon_n, \mu) = -\frac{1}{2} \int \int_{-\infty}^{\infty} \frac{d\epsilon'}{2\pi} \frac{d\omega'}{2\pi} B(\omega') A^{[0]}_\alpha(\epsilon', \mu) \frac{\coth\frac{\omega'}{2T} + \tanh\frac{\epsilon'}{2T}}{\epsilon_n - \omega' - \epsilon'}.$$ 

(10)
We are now in a position to evaluate the DOS, \( \nu(\epsilon) \equiv -\pi^{-1} \sum_\alpha \Im \mathcal{G}_\alpha^R(\epsilon) \). Analytically continuing \( \mathcal{G}_\alpha \), we finally find

\[
\nu(\epsilon) = \frac{1}{2} \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \mathcal{B}(\omega) \left( \tanh \frac{\omega - \epsilon}{2T} - \coth \frac{\omega}{2T} \right) \nu[0](\epsilon - \omega),
\]

where \( \nu[0] \) is the DOS related to \( \mathcal{G}_\alpha[0] \). For non–interacting electrons (with or without disorder), \( \nu[0](\epsilon) \approx \text{const} \), substituting Eq. (9) into Eq. (11), we obtain

\[
\frac{\nu(\epsilon)}{\nu[0]} = \sqrt{\frac{2\pi}{VT}} e^{-\frac{\nu}{2\pi}} \cosh \frac{\epsilon}{2T} \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} e^{-\frac{\omega^2}{2VT}}.
\]

Note that this result is independent of the disorder in the system. We have thus derived an expression for the DOS which interpolates between the perturbative zero bias anomaly and the semiclassical Coulomb blockade. In the limit of weak interaction \( (V \ll T) \) this reduces to

\[
\frac{\nu(\epsilon)}{\nu[0]} = 1 - \frac{V}{4T} \cosh^{-2} \frac{\epsilon}{2T}.
\]

This turns out to be the zero–bias anomaly of AA if one substitutes \( d = 0 \) in their analysis [4]. For strong interaction \( (V \gg T) \)

\[
\frac{\nu(\epsilon)}{\nu[0]} = \left\{ \begin{array}{ll}
\sqrt{\frac{2\pi T}{V}} e^{-\frac{\nu}{2\pi}} \cosh \frac{\epsilon}{2T} & \text{if } \epsilon \ll \sqrt{VT} \ll V, \\
1 - e^{-\frac{\nu}{2\pi}} & \text{if } \epsilon \gg V \gg T.
\end{array} \right.
\]

The exponential suppression of the tunneling DOS is a direct manifestation of the Coulomb blockade. In the extreme case of zero temperature one has

\[
\nu(\epsilon) \xrightarrow{T \to 0} \nu[0](\theta\left(|\epsilon| - \frac{V}{2}\right)).
\]

To discuss the quasiparticle lifetime due to global charge fluctuations, we need an explicit form of the electron Green function. To this end we substitute the spectral function of non–interacting electrons, \( \mathcal{A}_\alpha[0](\epsilon, \mu) = 2\pi \delta(\epsilon - \epsilon_\alpha + \mu) \), into Eq. (10), perform integrations and do analytical continuation \( (\epsilon_n \to \epsilon + i\delta) \). The result is

\[
\mathcal{G}_\alpha^R(\epsilon, \mu) = -i \sqrt{\frac{\pi}{2VT}} \left[ f(\epsilon_\alpha - \mu) w \left( \frac{x + V/2}{\sqrt{2VT}} \right) + [1 - f(\epsilon_\alpha - \mu)] w \left( \frac{x - V/2}{\sqrt{2VT}} \right) \right],
\]

(16)
where \( f(\epsilon) \) is the Fermi function, \( x = \epsilon - \epsilon_\alpha + \pi + i\delta \) and \( w(z) = \exp\{-z^2\} \text{erfc}(-iz) \) is the error function \([13]\). We note that the \( w(z) \) does not have poles at finite energies. We nevertheless may consider the width of the peak of the imaginary part of \( G^R \) as a measure of the quasiparticle life–time. As a result

\[
\frac{1}{\tau_{qp}} \propto \sqrt{VT}. \tag{17}
\]

Both the DOS and \( \tau_{qp} \) are measurable in tunneling experiment \([4]\).

Finally we discuss the nature of \( V \equiv V(q = 0) \). Consider the equivalent circuit shown in Fig. \([2]\). Its equilibrium noise spectrum \((T = 0)\) is given by the fluctuation–dissipation theorem

\[
\langle U_{tot}U_{tot} \rangle = i\omega \left( \frac{1}{i\omega C} + Z(\omega) \right). \tag{18}
\]

The voltage fluctuations on the system (capacitor) are

\[
U_C = U_{tot}/(1 + i\omega CZ),
\]

so the corresponding correlation function is

\[
K(\omega) \equiv e^2 \langle U_C U_C \rangle = \frac{e^2}{C} \frac{1}{1 + i\omega CZ(\omega)}.
\tag{19}
\]

We may now add to our original Hamiltonian, Eq. (1), a Gaussian noise term \( eU_C(t) \sum_\alpha a_\alpha^+ a_\alpha \) (the correlator of the noise is given by Eq. (19)). Integrating over the environment noise and following through the above derivation of \( G_\alpha \), we shall have to replace \( V \rightarrow V - K(\omega_m) \) in Eq. (5). It is natural to identify \( V = e^2/C \), where \( C \) is the total capacitance of the \( d = 0 \) system. In this case our noise–modified Coulomb boson (after Gaussian integration over \( \sigma_m \) in Eq. (5)) reproduces the result of Refs. \([8]\) and \([9]\) for fluctuations in ultrasmall capacitance junctions.

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\(^1\)Note that the AA result for the dephasing time \([4]\) scales as \( \frac{1}{\tau_\phi} \propto T^{2/(4-d)} D^{-d/(4-d)} \). For \( d = 0 \) it resembles our Eq. (17).
Note added after this work has been completed: in a recent preprint by Levitov and Shytov (SISSA N 9501130), these authors present an effective semiclassical action through which they study tunneling into a $d = 2$ system. Their $q = 0$ limit agrees with our exact result.
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FIGURES

FIG. 1. The single particle Green function in the presence of the Coulomb boson, $\mathcal{D}$.

FIG. 2. An equivalent circuit, consisting of the system’s capacitance $C$ and a noisy environment.