On the mean square of the error term for the
two-dimensional divisor problem (I)

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Abstract. Suppose $a$ and $b$ are two fixed positive integers such that $(a,b) = 1$. In this paper we shall establish an asymptotic formula for the mean square of the error term $\Delta_{a,b}(x)$ of the general two-dimensional divisor problem.

1 Introduction

Suppose $1 \leq a \leq b$ are two fixed integers. Without loss of generality, we suppose $(a,b) = 1$. Define $d_{a,b}(n) := \sum_{n=h^a\!\!r^b} 1$. The general two-dimensional divisor problem is to study the error term

$$\Delta_{a,b}(x) := \sum_{n \leq x} d_{a,b}(n) - \zeta(b/a)x^{1/a} - \zeta(a/b)x^{1/b},$$

if $a \neq b$. If $a = b$, then the appropriate limit is to be taken in the above sum. This problem attracts the interests of many authors.

When $a = b = 1$, $\Delta_{1,1}(x)$ is the error term of the well-known Dirichlet divisor problem. Dirichlet first proved that $\Delta_{1,1}(x) = O(x^{1/2})$. The exponent $1/2$ was improved by many authors. The latest result reads (see Huxley\textsuperscript{[9]})

$$\Delta_{1,1}(x) \ll x^{131/416}(\log x)^{26947/8320}.$$  \hspace{1cm} (1.1)

For the lower bounds, the best results read

$$\Delta_{1,1}(x) = \Omega_+(x^{1/4}(\log x)^{1/4}(\log \log x)^{3+\log 4}/c \exp(-c\sqrt{\log \log \log x}))(c > 0)$$  \hspace{1cm} (1.2)

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and
\begin{equation}
\Delta_{1,1}(x) = \Omega_-(x^{1/4} \exp \left(c' (\log \log x)^{1/4} (\log \log \log x)^{-3/4} \right)) \quad (c' > 0),
\end{equation}
which are due to Hafner [5] and Corrádi and Kátai [2], respectively.

When \( a \neq b \), Richert [20] proved that
\begin{equation}
\Delta_{a,b}(x) \ll \begin{cases}
    x^{\frac{2}{a+b}}, & b \leq 2a, \\
    x^{\frac{2}{2a+2b}}, & b \geq 2a.
\end{cases}
\end{equation}

Better upper estimates can be found in [14, 15, 19, 21]. Hafner [6] proved that
\begin{equation}
\Delta_{a,b}(x) = \Omega_+ \left( x^{1/2(a+b)} (\log x)^{b/(a+b)} \log \log x \right)
\end{equation}
and
\begin{equation}
\Delta_{a,b}(x) = \Omega_- \left( x^{1/2(a+b)} e^{U(x)} \right),
\end{equation}
where
\[
U(x) = B (\log \log x)^{b/(a+b)(\log \log \log x)^{b/(a+b)} - 1}
\]
for some positive constant \( B > 0 \).

It is conjectured that the estimate
\begin{equation}
\Delta_{a,b}(x) = O \left( x^{1/2(a+b)+\varepsilon} \right)
\end{equation}
holds for any \( 1 \leq a \leq b, (a,b) = 1 \). When \( a = b = 1 \), the conjecture (1.7) is supported by the power moment results of \( \Delta_{1,1}(x) \). For the mean square of \( \Delta_{1,1}(x) \), Cramér [3] first proved the classical result
\begin{equation}
\int_1^T \Delta_{1,1}^2(x) \, dx = \frac{\zeta(3/2)^4}{6\pi^2 \zeta(3)} T^{3/2} + O(T^{5/4+\varepsilon}).
\end{equation}
The estimate \( O(T^{5/4+\varepsilon}) \) in (1.8) was improved to \( O(T(\log T) \log^5 T) \) in [22] and \( O(T \log^4 T) \) in [18]. The higher-power moments of \( \Delta_{1,1}(x) \) were studied in [7, 13, 23, 25, 26, 27]. When \( a \neq b \), by using the theory of the Riemann zeta-function, Ivić [12] proved that
\begin{equation}
\int_1^T \Delta_{a,b}^2(x) \, dx \begin{cases}
    \ll T^{1+1/(a+b)} \log^2 T, \\
    = \Omega(T^{1+1/(a+b)}).
\end{cases}
\end{equation}
He also conjectured that the asymptotic formula
\begin{equation}
\int_1^T \Delta_{a,b}^2(x) \, dx = c_{a,b} T^{1+1/(a+b)} (1 + o(1))
\end{equation}
holds for some positive constant \(c_{a,b}\). We note that the \(\Omega\)-result in (1.9) is also contained in a very general result of the second-named author (see Theorem 5 of [1]).

When \(a = b = 1\), the proofs of most power moment results of \(\Delta_{1,1}(x)\) mentioned above were started from the well-known truncated Voronoi’s formula (see, for example, [10])

\[
\Delta_{1,1}(x) = (\sqrt{2\pi})^{-1} x^{1/4} \sum_{n \leq N} \frac{d(n)}{n^{3/4}} \cos\left(4\pi \sqrt{xn} - \frac{\pi}{4}\right) + O(x^\varepsilon + x^{1/2+\varepsilon} N^{-1/2})
\]

for \(1 \ll N \ll x^A\), where \(A > 0\) is any fixed constant. Note that the infinite series \(\sum_{n=1}^\infty d(n)n^{-3/4} \cos(4\pi \sqrt{xn} - \pi/4)\) is conditionally convergent. When \(a \neq b\), Krätzel [14] provided a series representation of \(\Delta_{a,b}(x)\). However, Krätzel’s series converges only when \(b < 3a/2\), which is a significant restriction for many applications.

The aim of this paper is to prove Ivić’s conjecture (1.10). More precisely, we shall prove the following Theorem.

**Theorem.** Suppose \(1 \leq a < b\) are fixed integers for which \((a,b) = 1\). Then we have

\[
\int_1^T \Delta_{a,b}^2(x) \, dx = c_{a,b} T^{\frac{1+4b}{a+b}} + O(T^{\frac{1+4b}{a+b} - \frac{\varepsilon}{2(a+b)(a+b-1)}} \log^{7/2} T),
\]

where

\[
c_{a,b} := \frac{a^{b/(a+b)} b^{a/(a+b)}}{2(a+b+1)\pi^2} \sum_{n=1}^\infty g_{a,b}^2(n),
\]

\[
g_{a,b}(n) := \sum_{n=h^a r^b} h^{-\frac{a+2b}{2(a+b)+2b} r^{-\frac{a+2b}{2(a+b)+2b}}}.
\]

**Remark 1.** It is easy to see that the function \(g_{a,b}(n)\) is symmetric for \(a\) and \(b\), namely \(g_{a,b}(n) = g_{b,a}(n)\). The convergence of the infinite series \(\sum_{n=1}^\infty g_{a,b}^2(n)\) will be proved in Section 4.

**Remark 2.** Our theorem also holds for \(a = b = 1\). In this case we have

\[
g_{1,1}(n) = d(n)n^{-3/4}, \quad c_{1,1} = \frac{1}{6\pi^2} \sum_{n=1}^\infty d^2(n)n^{-3/2} = \frac{(\zeta(3/2))^4}{6\pi^2 \zeta(3)},
\]

where \(d(n) = d_{1,1}(n)\) is the Dirichlet divisor function. Hence our Theorem provides a new proof of Cramer’s classical result (1.8).

**Notations.** \(\mathbb{Z}\) denotes the set of all integers. For a real number \(u\), \([u]\) denotes the integer part of \(u\), \(\{u\}\) denotes the fractional part of \(u\), \(\psi(u) = \{u\} - 1/2\), \(\|u\|\) denotes the distance from \(u\) to the integer nearest to \(u\). \(\mu(n)\) is the Möbius function,
\((m, n)\) denotes the greatest common divisor of natural numbers \(m\) and \(n\). \(n \sim N\) means \(N < n \leq 2N\). \(\varepsilon\) always denotes a sufficiently small positive constant. \(SC(\Sigma)\) denotes the summation condition of the sum \(\Sigma\) when it is complicated. Finally, define

\[
\sum'_{\alpha \leq n \leq \beta} f(n) = \begin{cases} 
\sum_{\alpha < n < \beta} f(n), & \alpha \notin \mathbb{Z}, \beta \notin \mathbb{Z}, \\
\frac{f(\alpha)}{2} + \sum_{\alpha < n < \beta} f(n), & \alpha \in \mathbb{Z}, \beta \notin \mathbb{Z}, \\
\frac{f(\alpha)}{2} + \sum_{\alpha < n < \beta} f(n) + \frac{f(\beta)}{2}, & \alpha \in \mathbb{Z}, \beta \in \mathbb{Z}.
\end{cases}
\]

2 Two preliminary Lemmas

In order to prove our theorem, we need the following two Lemmas. Lemma 2.1 is well-known; see for example, Heath-Brown\[8\]. Lemma 2.2 is Theorem 2.2 of Min\[17\], see also Lemma 6 of Chapter 1 in \[24\]. A weaker version of Lemma 2.2 can be found in \[16\], which also suffices for our proof.

Lemma 2.1. Let \(H \geq 2\) be any real number. Then

\[
\psi(u) = -\sum_{1 \leq |h| \leq H} \frac{e(hu)}{2\pi ih} + O \left( \min(1, \frac{1}{H\|u\|}) \right).
\]

Lemma 2.2. Suppose \(A_1, \cdots, A_5\) are absolute positive constants, \(f(x)\) and \(g(x)\) are algebraic functions in \([a, b]\) and

\[
\frac{A_1}{R} \leq |f''(x)| \leq \frac{A_2}{R}, \quad |f'''(x)| \leq \frac{A_3}{RU}, \quad U \geq 1,
\]

\[
|g(x)| \leq A_4 G, \quad |g'(x)| \leq A_5 GU^{-1}, \quad U_1 \geq 1,
\]

\([\alpha, \beta]\) is the image of \([a, b]\) under the mapping \(y = f'(x)\), then

\[
\sum_{\alpha < n \leq \beta} g(n)e(f(n)) = e^{\pi i/4} \sum_{\alpha \leq n \leq \beta} b_u \frac{g(n_u)}{\sqrt{f''(n_u)}} e(f(n_u) - u n_u) + O \left( G \log(\beta - \alpha + 2) + G(b - a + R)(U^{-1} + U_1^{-1}) \right)
\]

\[
+ O \left( G \min \left[ \sqrt{R}, \max \left( \frac{1}{<\alpha>}, \frac{1}{<\beta>} \right) \right] \right),
\]

where \(n_u\) is the solution of \(f'(n) = u\),

\[
< t > = \begin{cases} 
\|t\|, & \text{if } t \text{ not an integer}, \\
\beta - \alpha, & \text{if } t \text{ an integer},
\end{cases}
\]

\[
b_u = \begin{cases} 
1, & \text{if } \alpha < u < \beta, \text{ or } \alpha, \beta \text{ not integers}, \\
1/2, & \text{if } \alpha \text{ or } \beta \text{ are integers},
\end{cases}
\]

\[
\sqrt{f''} = \begin{cases} 
\sqrt{f''}, & \text{if } f'' > 0, \\
i\sqrt{|f''|}, & \text{if } f'' < 0.
\end{cases}
\]
3 A Voronoi type formula of $\Delta(a, b; x)$

It suffices for us to evaluate the integral $\int_{T}^{2T} \Delta_{a,b}^{2}(x) dx$, where $T \geq 10$ is a large parameter.

It is well-known that (see for example, Ivić [3], eq.(14.46))

\[
\Delta_{a,b}(x) = f(a, b; x) + f(b, a; x) + O(1),
\]

where

\[
f(a, b; x) := - \sum_{m \leq x/(a+b)} \psi \left( \frac{x^{1/a}}{m^{b/a}} \right).
\]

Suppose $T \leq x \leq 2T$, $H$ is a parameter such that $T^c \ll H \ll T^{100(a+b)}$. By Lemma 2.1 we have

\[
f(a, b; x) = R_1(a, b; x) + R_2(a, b; x),
\]

\[
R_1(a, b; x) := \frac{1}{2 \pi i} \sum_{1 \leq |h| \leq H} \frac{1}{h} \sum_{m \leq x/(a+b)} e \left( \frac{hx^{1/a}}{m^{b/a}} \right),
\]

\[
R_2(a, b; x) := O \left( \sum_{m \leq x/(a+b)} \min \left( 1, \frac{1}{H \| x^{1/a} \| m^{b/a} } \right) \right).
\]

Define

\[
c := (2ab)^{ab}, \quad J := \left[ (L/(a+b) - \log L) \log^{-1} c \right], \quad L := \log T,
\]

\[
m_j := x^{1/(a+b)} c^{-j} \quad (j \geq 0).
\]

It is easy to see that

\[
c^J \sim T^{1/(a+b)} L^{-1}.
\]
We have

\begin{equation}
R_1(a, b; x) = \frac{1}{2\pi i} \sum_{1 \leq |h| \leq H} \frac{1}{h} \sum_{j=0}^{J} \sum_{m_j+1 < m \leq m_j} e\left(h^{1/a}x^{1/m^{b/a}}\right) + O(L^2)
\end{equation}

\begin{align*}
&= \frac{1}{2\pi i} \sum_{-H \leq h \leq -1} \frac{1}{h} \sum_{j=0}^{J} \sum_{m_j+1 < m \leq m_j} e\left(h^{1/a}x^{1/m^{b/a}}\right) \\
&\quad + \frac{1}{2\pi i} \sum_{1 \leq h \leq H} \frac{1}{h} \sum_{j=0}^{J} \sum_{m_j+1 < m \leq m_j} e\left(h^{1/a}x^{1/m^{b/a}}\right) + O(L^2)
\end{align*}

\begin{align*}
&= -\frac{1}{2\pi i} \sum_{1 \leq h \leq H} \frac{1}{h} \sum_{j=0}^{J} \sum_{m_j+1 < m \leq m_j} e\left(-h^{1/a}x^{1/m^{b/a}}\right) \\
&\quad + \frac{1}{2\pi i} \sum_{1 \leq h \leq H} \frac{1}{h} \sum_{j=0}^{J} \sum_{m_j+1 < m \leq m_j} e\left(h^{1/a}x^{1/m^{b/a}}\right) + O(L^2)
\end{align*}

\begin{align*}
&= -\frac{\Sigma_1}{2\pi i} + \frac{\Sigma_1}{2\pi i} + O(L^2),
\end{align*}

say, where

\[ \Sigma_1 = \sum_{1 \leq h \leq H} \frac{1}{h} \sum_{j=0}^{J} \sum_{m_j+1 < m \leq m_j} e\left(-h^{1/a}x^{1/m^{b/a}}\right). \]

Let

\[ S_{h,j}(x) := \sum_{m_j+1 < m \leq m_j} e\left(-h^{1/a}x^{1/m^{b/a}}\right). \]

Define

\[ c_1(a, b) := a^{b/2(a+b)b^{a/2(a+b)}}(a + b)^{-1/2}, \]
\[ c_2(a, b) := a^{b/(a+b)}b^{-b/(a+b)} + b^{a/(a+b)}a^{-a/(a+b)}. \]

It is easy to check that

\begin{equation}
(3.4) \quad c_1(a, b) = c_1(b, a), \quad c_2(a, b) = c_2(b, a).
\end{equation}

By Lemma 2.2 we get

\begin{equation}
(3.5) \quad S_{h,j}(x) = c_1(a, b)x^{\frac{1}{2(a+b)}} \sum_{n_j, h(a, b) \leq r \leq n_{j+1}, h(a, b)} h^{\frac{a}{2(a+b)}}r^{-\frac{2a+b}{2(a+b)}}
\end{equation}

\begin{align*}
&\times e\left(-c_2(a, b)x^{\frac{1}{2(a+b)}}(h^ar^b)^{\frac{1}{2(a+b)}} - \frac{1}{8}\right) + O(L),
\end{align*}
where
\[ n_{j,h}(a, b) := \frac{b}{a} h(2ab)^{(a+b)bj}. \]

Inserting (3.5) into \( \Sigma_1 \) we get

\begin{equation}
\Sigma_1 = c_1(a, b) x^{\frac{1}{2(a+b)}} \sum_{1\leq h \leq H} \sum_{\frac{bh}{a} \leq r \leq n_{J+1,h}(a, b)} h^{-\frac{a+2b}{2(a+b)}} r^{-\frac{2a+b}{2(a+b)}} \times e \left( -c_2(a, b) x^{\frac{1}{2(a+b)}} (h^a r^b) \frac{1}{\pi} - \frac{1}{8} \right) + O(L^2). \tag{3.6}
\end{equation}

From (3.3) and (3.6) we get

\begin{equation}
R_1(a, b; x) = R_1^*(a, b; x) + O(L^2), \tag{3.7}
\end{equation}

where

\[ R_1^*(a, b; x) = \frac{c_1(a, b)}{\pi} x^{\frac{1}{2(a+b)}} \sum_{1\leq h \leq H} \sum_{\frac{bh}{a} \leq r \leq n_{J+1,h}(a, b)} h^{-\frac{a+2b}{2(a+b)}} r^{-\frac{2a+b}{2(a+b)}} \times \cos \left( 2\pi c_2(a, b) x^{\frac{1}{2(a+b)}} (h^a r^b) \frac{1}{\pi} - \frac{\pi}{4} \right). \]

Define

\[ g(a, b; n, H, J) := \sum_{\frac{bh}{a} \leq r \leq n_{J+1,h}(a, b)} h^{-\frac{a+2b}{2(a+b)}} r^{-\frac{2a+b}{2(a+b)}}, \]

\[ g(a, b; n) := \sum_{\frac{bh}{a} \leq r \leq n_{J+1,h}(a, b)} h^{-\frac{a+2b}{2(a+b)}} r^{-\frac{2a+b}{2(a+b)}}. \]

It is easy to check that if \( h^a r^b \leq \min(H^{a+b}, T^{b/a})L^{-a-b^2/a-1}, bh/a \leq r \), then it follows that \( h \leq H, r \leq n_{J+1,h}(a, b) \). Thus

\begin{equation}
g(a, b; n, H, J) = g(a, b; n), \quad n \leq \min(H^{a+b}, T^{b/a})L^{-a-b^2/a-1}. \tag{3.8}
\end{equation}

So we have that

\[ R_1^*(a, b; x) = \frac{c_1(a, b)}{\pi} x^{\frac{1}{2(a+b)}} \sum_{1\leq n \leq H^{a+b} n_{J+1,h}(a, b)} g(a, b; n, H, J) \times \cos \left( 2\pi c_2(a, b) x^{\frac{1}{2(a+b)}} n^{\frac{1}{a+b}} - \frac{\pi}{4} \right) \]

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Similarly we have

\[ R_1(b, a; x) = R_1^*(b, a; x) + O(L^2), \]

where

\[ R_1^*(b, a; x) = \frac{c_1(b, a)}{\pi} x^{\frac{1}{2(a+b)}} \sum_{1 \leq n \leq H^{b_{n+1,d}(b, a)}} g^*(b, a; n, H, J) \times \cos \left( 2\pi c_2(b, a)x^{\frac{1}{a+b}n^{\frac{1}{a+b}}} - \frac{\pi}{4} \right), \]

\[ g(b, a; n, H, J) = \sum_{h=b^r_{n+1,d(b, a)}}^{H} \frac{h-b}{a+b} r - \frac{2b+a}{2(a+b)}, \]

\[ g(b, a; n) = \sum_{h=b^r_{n+1,d(b, a)}}^{H} \frac{h-b}{a+b} r - \frac{2b+a}{2(a+b)}. \]

It is easy to check that if that if \( h_{b^r_{n+1,d(b, a)}} \leq \min(H^{a+b}, T^{a/b})L^{-a-a^2/b-1}, ah/b \leq r, \) then \( h \leq H, r \leq n_{J+1,h(b, a)}. \) Thus

\[ g^*(b, a; n, H, J) = g(b, a; n), \quad n \leq \min(H^{a+b}, T^{a/b})L^{-a-a^2/b-1}. \]

Suppose \( z \) is a parameter such that \( T^z \ll z \leq \min(H^{a+b}, T^{a/b})L^{-b-b^2/a-1} \) and define

\[ R_{11}^*(a, b; x) = \frac{c_1(a, b)}{\pi} x^{\frac{1}{2(a+b)}} \sum_{1 \leq n \leq z} g(a, b; n) \times \cos \left( 2\pi c_2(a, b)x^{\frac{1}{a+b}n^{\frac{1}{a+b}}} - \frac{\pi}{4} \right), \]

\[ R_{12}^*(a, b; x) = R_1^*(a, b; x) - R_{11}^*(a, b; x), \]

\[ R_{11}^*(b, a; x) = \frac{c_1(b, a)}{\pi} x^{\frac{1}{2(a+b)}} \sum_{1 \leq n \leq z} g(b, a; n) \times \cos \left( 2\pi c_2(b, a)x^{\frac{1}{a+b}n^{\frac{1}{a+b}}} - \frac{\pi}{4} \right), \]

\[ R_{12}^*(b, a; x) = R_1^*(b, a; x) - R_{11}^*(b, a; x). \]

Recalling (3.4) we have

\[ R_{11}^*(a, b; x) + R_{11}^*(b, a; x) = \frac{c_1(a, b)}{\pi} x^{\frac{1}{2(a+b)}} \sum_{1 \leq n \leq z} \left( g(a, b; n) + g(b, a; n) \right) \times \cos \left( 2\pi c_2(a, b)x^{\frac{1}{a+b}n^{\frac{1}{a+b}}} - \frac{\pi}{4} \right). \]
From the definition of $g(a, b; n)$ and $g(b, a; n)$ we have

$$g(a, b; n) + g(b, a; n) = \sum_{n=h^{a,b}}' h^{-\frac{a+2b}{2(a+b)}} r^{-\frac{2a+b}{2(a+b)}} + \sum_{n=h^{r,b}}' h^{r^{-\frac{b+2a}{2(a+b)}}} r^{-\frac{2b+a}{2(a+b)}}$$

$$= \sum_{n=h^{a,b}}' h^{-\frac{a+2b}{2(a+b)}} r^{-\frac{2a+b}{2(a+b)}} + \sum_{n=r^{b,a}}' r^{-\frac{b+2a}{2(a+b)}} h^{r^{-\frac{2b+a}{2(a+b)}}}$$

$$= \sum_{n=h^{a,b}} h^{-\frac{a+2b}{2(a+b)}} r^{-\frac{2a+b}{2(a+b)}} = g_{a,b}(n).$$

Define

$$\Delta_{a,b}^*(x, z) := \frac{c_1(a, b)}{\pi} x^{-\frac{1}{2(a+b)}} \sum_{1 \leq n \leq z} g_{a,b}(n) \times \cos \left(2\pi c_2(a, b)x^{\frac{1}{a+b}} n^{\frac{1}{a+b}} - \frac{\pi}{4} \right).$$

Combining the above estimates we get

$$\begin{align*}
\Delta_{a,b}(x) &= \Delta_{a,b}^*(x, z) + E_{a,b}(x), \\
E_{a,b}(x) &:= R_{12}^*(a, b; x) + R_{12}^*(b, a; x) + R_2(a, b; x) + R_2(b, a; x) + O(L^2).
\end{align*}$$

The formula (3.11) can be viewed as a truncated Voronoi’s formula.

4 On the series $\sum_{n=1}^{\infty} g_{a,b}^2(n)$

In this section we shall prove that the infinite series $\sum_{n=1}^{\infty} g_{a,b}^2(n)$ is convergent. Without loss of generality, we suppose $a < b$. By the definition of $g_{a,b}(n)$ it is easy to see that

$$g_{a,b}(n) = n^{-\frac{2a+b}{(2a+2b)^{2}}} \sum_{n=h^{a,b}} h^{-\frac{b-a}{a}} = n^{-\frac{2a+b}{(2a+2b)^{2}}} g_{a,b}^*(n),$$

say, where

$$g_{a,b}^*(n) := \sum_{n=h^{a,b}} h^{-\frac{b-a}{a}}.$$
is a multiplicative function. So we have

\[
\sum_{n=1}^{\infty} \frac{g_{a,b}^2(n)}{n^s} = \prod_p \left( 1 + \sum_{\alpha=1}^{\infty} \frac{g_{a,b}^2(p^\alpha)}{p^{\alpha s}} \right) \quad (\Re s > 1).
\]

We shall show that

\[
\sum_{n=1}^{\infty} \frac{g_{a,b}^2(n)}{n^s} = \zeta(bs) \zeta(as + \frac{2(b-a)}{a}) G_{a,b}(s),
\]

where \(G_{a,b}(s)\) is analytic for \(\Re s > (3a - 2b)/a^2\).

We consider two cases.

Case 1. \(a = 1\)
If \(1 \leq \alpha \leq b - 1\), then \(p^\alpha = h^{a_r^b}\) implies \(h = p^\alpha, r = 1\). So
\[
g_{a,b}^*(p^\alpha) = p^{-\alpha(b-1)}.
\]

Suppose \(\alpha = mb\) with \(m \geq 1\). Then \(p^\alpha = h^{a_r^b}\) implies \(h = p^{ib}, r = p^{(m-j)b}, j = 0, 1, 2, \cdots, m\). So
\[
g_{a,b}^*(p^\alpha) = \sum_{j=0}^{m} p^{-j(b-1)}.
\]

Suppose \(\alpha = mb + u\) with \(m \geq 1\) and \(1 \leq u \leq b - 1\). Then \(p^\alpha = h^{a_r^b}\) implies \(h = p^{jb + u}, r = p^{(m-j)b}, j = 0, 1, 2, \cdots, m\). So
\[
g_{a,b}^*(p^\alpha) = \sum_{j=0}^{m} p^{-(jb+u)(b-1)} = p^{-u(b-1)} + p^{-u(b-1)} \sum_{j=1}^{m} p^{-j(b-1)}.
\]

From the above we get \((p \geq 2, \Re s = \sigma > 1)\)

\[
1 + \sum_{\alpha=1}^{\infty} \frac{g_{a,b}^2(p^\alpha)}{p^{\alpha s}} = 1 + \sum_{\alpha=1}^{b-1} \frac{p^{-2\alpha(b-1)}}{p^{\alpha s}} + \sum_{m=1}^{\infty} \frac{\left(\sum_{j=0}^{m} p^{-jb(b-1)}\right)^2}{p^{mb s}} + \sum_{m=1}^{\infty} \sum_{u=1}^{b-1} \frac{\left(\sum_{j=0}^{m} p^{-(jb+u)(b-1)}\right)^2}{p^{(mb+u)s}} = 1 + p^{-bs} + O(p^{-\sigma - 2(b-1)}).
\]

Inserting this formula into (4.2) we get that (4.3) is true for \(a = 1\).

Case 2. \(a \geq 2\)
Let \(k = \lceil b/a \rceil\). Then \((a, b) = 1\) implies that \(ka < b < ka + a\).
Consider the equation

\[(4.4)\]

\[ua + vb = \alpha,\quad u, v \in \mathbb{Z}.
\]

Let \(v(\alpha; a, b)\) denote the number of non-negative solutions of the equation (4.4). If \(v(\alpha; a, b) = 0\), then \(g_{a,b}^*(p^\alpha) = 0\). Especially, it is easy to see that if \(\alpha \leq b - 1, a \nmid \alpha\), then \(v(\alpha; a, b) = 0\). Hence

\[g_{a,b}^*(p^\alpha) = 0, \quad (\alpha \leq b - 1, a \nmid \alpha).
\]

If \(\alpha = ja, (j = 1, 2, \ldots, k)\), then the equation (4.4) has only one solution \((u, v) = (j, 0)\). So \(p^\alpha = h^a r^b\) implies that \(h = p^j, r = 1\), and hence

\[g_{a,b}^*(p^\alpha) = p^{(b-a)j}.\]

Later suppose \(\alpha \geq b\) and \(v(\alpha; a, b) > 0\). Let \((u_0, v_0)\) denote a special non-negative solution of (4.4) such that \(0 \leq u_0 < b\). Then all non-negative solutions of (4.4) are \((u, v) = (u_0 + bt, v_0 - at), 0 \leq t \leq v(\alpha; a, b)\). Hence we have

\[g_{a,b}^*(p^\alpha) = \sum_{j=0}^{v(\alpha; a, b)-1} p^{(u_0+jb)(b-a)j}.\]

Especially when \(b | \alpha, (0, \alpha/b)\) is a non-negative solution of (4.4). Thus

\[g_{a,b}^*(p^\alpha) = \sum_{j=0}^{v(\alpha; a, b)-1} p^{(u_0+jb)(b-a)j}, \quad \alpha \equiv 0(\text{mod } b).
\]

From the above we get

\[1 + \sum_{a=1}^{\infty} \frac{g_{a,b}^2(p^\alpha)}{p^{as}}
\]

\[= 1 + \sum_{j=1}^{k} \frac{p^{-2j(b-a)/a}}{p^{jas}} + \sum_{\alpha > b, b | \alpha} g_{a,b}^2(p^\alpha) + \sum_{\alpha > b, b \nmid \alpha} g_{a,b}^2(p^\alpha)
\]

\[= 1 + p^{-bs} + O(p^{-as - \frac{2(b-a)}{a}}).
\]

Inserting this formula into (4.2) we get that (4.3) is true in this case.

It is easy to check that \(\zeta((as + \frac{2b-2a}{a})\) has a simple pole at \(s = \frac{(3a - 2b)}{a^2}\). Since \(b > a \geq 1\), it is easy to see that \(\frac{(3a - 2b)}{a^2} < 1/b\). From (4.3) and Perron’s formula we get that for any \(X > 2\),

\[\sum_{X < n \leq 2X} g_{a,b}^2(n) \ll X^{1/b}.
\]
Hence we get
\[
\sum_{X < n \leq 2X} g_{a,b}^2(n) \ll X^{\frac{1}{2} - \frac{2a+b}{2(a+b)}} X^{-\frac{a}{(a+b)}},
\]
which implies the convergence of the infinite series \(\sum_{n=1}^{\infty} g_{a,b}^2(n)\). From (4.5) we also get
\[
\sum_{n \leq X} g_{a,b}^2(n) = \sum_{n=1}^{\infty} g_{a,b}^2(n) + O(X^{-\frac{a}{(a+b)}})
\]
and
\[
\sum_{n > X} g_{a,b}^2(n) = O(X^{-\frac{a}{(a+b)}}).
\]

5 On a special sum

Suppose \(a, b\) are fixed natural integers, \(T\) is a large parameter. Define
\[
S_{a,b}(T) : = \sum_{2} (h_1 h_2)^{-\frac{2b+a}{2a+2b}} (r_1 r_2)^{-\frac{2a+b}{2a+2b}}
\times \min\left(\frac{1}{T^{\frac{a}{a+b}}}, \frac{1}{|h_1^\frac{a}{a+b} r_1^{\frac{b}{a+b}} - h_2^\frac{a}{a+b} r_2^{\frac{b}{a+b}}|}\right),
\]
where
\[
SC(\Sigma_2) : h_1^{\frac{a}{a+b}} r_1^{\frac{b}{a+b}} \leq T^{-100(a+b)}, h_2^{\frac{a}{a+b}} r_2^{\frac{b}{a+b}} \leq T^{-100(a+b)},
0 < |h_1^{\frac{a}{a+b}} r_1^{\frac{b}{a+b}} - h_2^{\frac{a}{a+b}} r_2^{\frac{b}{a+b}}| < \frac{1}{10} h_1^{\frac{a}{2a+2b}} r_1^{\frac{b}{2a+2b}} h_2^{\frac{a}{2a+2b}} r_2^{\frac{b}{2a+2b}}.
\]

In this section we shall estimate the sum \(S_{a,b}(T)\), which is very important in our proof.

5.1 On a Diophantine inequality

Suppose \(\alpha\) and \(\beta\) are fixed non-zero real numbers, \(H_1 \geq 1, H_2 \geq 1, R_1 \geq 1, R_2 \geq 1\) are large real numbers, \(\delta > 0\). Let \(A(H_1, H_2, R_1, R_2; \delta)\) denote the number of the solutions of the inequality
\[
|h_1^\alpha r_1^\beta - h_2^\alpha r_2^\beta| \leq \delta, h_1 \sim H_1, h_2 \sim H_2, r_1 \sim R_1, r_2 \sim R_2.
\]
Lemma 5.1. We have
\[ A(H_1, H_2, R_1, R_2; \delta) \ll \delta(H_1 H_2)^{1-\alpha/2}(R_1 R_2)^{1-\beta/2} \]
\[ + (H_1 H_2 R_1 R_2)^{1/2}(\log 2H_1 H_2 R_1 R_2)^2, \]
where the implied constant depends only on \( \alpha, \beta \).

**Remark.** When \( H_1 = H_2, R_1 = R_2 \), Lemma 5.1 reduces to Lemma 1 of Fouvry and Iwaniec [4]. Here Lemma 5.1 is more general.

*Proof.* We follow the proof of Lemma 1 of Fouvry and Iwaniec [4]. Suppose \( u \) and \( v \) are two positive integers and let \( A_{u,v}(H_1, H_2, R_1, R_2; \delta) \) denote the number of solutions of the inequality (5.1) with \((r_1, r_2) = u, (h_1, h_2) = v\). Set \( r_j = m_j u, h_j = l_j v (j = 1, 2) \), then \((m_1, m_2, l_1, l_2)\) satisfies
\[
\frac{m_1^\alpha}{m_2^\alpha} - \frac{l_1^\alpha}{l_2^\alpha} \leq c(\alpha) c(\beta) \delta R_{2^{-\beta}} H_1^{-\alpha},
\]
\[
\frac{m_2^\beta}{m_1^\beta} - \frac{l_2^\beta}{l_1^\beta} \leq c(\alpha) c(\beta) \delta R_{1^{-\beta}} H_2^{-\alpha},
\]
where
\[
c(\gamma) = \begin{cases} 1, & \text{if } \gamma > 0, \\ 2^{-\gamma}, & \text{if } \gamma < 0. \end{cases}
\]

It is easy to see that \( \frac{m_1^\beta}{m_2^\beta} \) is \( c_1(\beta) u^2 (R_1/R_2)^{\beta-1} R_2^{-2} \)-spaced, where \( c_1(\beta) > 0 \) is some positive constant. So from (5.2) we get
\[ A_{u,v}(H_1, H_2, R_1, R_2; \delta) \ll \frac{H_1 H_2}{u^2} \left(1 + \frac{\delta R_{2^{-\beta}} H_1^{-\alpha}}{u^2 (R_1/R_2)^{\beta-1} R_2^{-2}}\right) \ll \frac{H_1 H_2}{u^2} + \frac{\delta H_1 H_2 R_1 R_2}{u^2 v^2 H_1^{-\alpha}}. \]

Similarly, \( \frac{m_2^\beta}{m_1^\beta} \) is \( c_2(\beta) u^2 (R_2/R_1)^{\beta-1} R_1^{-2} \)-spaced for some positive constant \( c_2(\beta) \), so from (5.3) we get
\[ A_{u,v}(H_1, H_2, R_1, R_2; \delta) \ll \frac{H_1 H_2}{u^2} \left(1 + \frac{\delta R_{1^{-\beta}} H_2^{-\alpha}}{u^2 (R_2/R_1)^{\beta-1} R_1^{-2}}\right) \ll \frac{H_1 H_2}{u^2} + \frac{\delta H_1 H_2 R_1 R_2}{u^2 v^2 H_2^{-\alpha}}. \]

From the above two estimates we get
\[
A_{u,v}(H_1, H_2, R_1, R_2; \delta) \ll \frac{H_1 H_2}{u^2} + \frac{\delta H_1 H_2 R_1 R_2}{u^2 v^2} \min(H_1^{-\alpha} R_1^{-\beta}, H_2^{-\alpha} R_2^{-\beta}) \ll \frac{H_1 H_2}{u^2} + \frac{\delta (H_1 H_2)^{1-\alpha/2} (R_1 R_2)^{1-\beta}}{u^2 v^2},
\]
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if we note that $\min(x, y) \leq x^{1/2}y^{1/2}$ for $x > 0, y > 0$.

Similarly we have

$$A_{u,v}(H_1, H_2; R_1, R_2; \delta) \ll \frac{R_1 R_2}{u^2} + \frac{\delta(H_1 H_2)^{1-\alpha/2}(R_1 R_2)^{1-\beta}}{u^2 v^2},$$

which combining (5.4) gives

$$A_{u,v}(D_1, D_2, N_1, N_2; \delta) \ll \frac{\delta(H_1 H_2)^{1-\alpha/2}(R_1 R_2)^{1-\beta}}{u^2 v^2} + \min(\frac{R_1 R_2}{u^2}, \frac{H_1 H_2}{v^2})$$

Summing over $u$ and $v$ completes the proof of Lemma 5.1. \qed

### 5.2 Estimate of the sum $S_{a,b}(T)$

In this subsection we estimate the sum $S_{a,b}(T)$. For simplicity, let

$$\eta := \frac{a}{\alpha+\beta} r_1^{\frac{a}{\alpha+\beta}} - \frac{b}{\alpha+\beta} r_2^{\frac{b}{\alpha+\beta}}.$$

By a splitting argument we get for some $(1,1,1,1) \ll (H_1, H_2, R_1, R_2) \ll (T^{\frac{100(a+b)}{a}}, T^{\frac{100(a+b)}{b}}, T^{\frac{100(a+b)}{c}}, T^{\frac{100(a+b)}{d}})$

that

$$S_{a,b}(T) \ll L^4 U_{a,b}(T; H_1, H_2, R_1, R_2),$$

where

$$U_{a,b}(T; H_1, H_2, R_1, R_2) := \sum_{3} \frac{1}{(h_1 h_2)^{\frac{a+b}{2(a+b)}} (r_1 r_2)^{\frac{a+b}{2(a+b)}}} \min(T^{\frac{1}{\alpha+\beta}}, |\eta|^{-1}),$$

$$SC(\Sigma_3) : h_j \sim H_j, r_j \sim R_j, (j = 1, 2), |\eta| < \frac{1}{10} h_1^{\frac{a}{\alpha+\beta}} r_1^{\frac{b}{\alpha+\beta}} h_2^{\frac{a}{\alpha+\beta}} r_2^{\frac{b}{\alpha+\beta}}.$$

By Lemma 5.1 with $(\alpha, \beta) = (a/(a+b), b/(a+b))$, the contribution of $T^{1/(a+b)}$ is

$$\ll \frac{T^{\frac{1}{a+b}}}{(H_1 H_2)^{\frac{a+b}{2(a+b)}} (R_1 R_2)^{\frac{a+b}{2(a+b)}}} \times A(H_1, H_2, R_1, R_2; T^{-\frac{1}{\alpha+\beta}})$$

$$\ll 1 + \frac{T^{\frac{1}{\alpha+\beta}}}{(H_1 H_2)^{\frac{a+b}{2(a+b)}} (R_1 R_2)^{\frac{a+b}{2(a+b)}}} L^2.$$
The condition \(SC(\Sigma_3)\) implies that \(h_1^a r_1^b \ll h_2^a r_2^b\). So by the mean value theorem we get for some \(x_0 \approx h_1^a r_1^b\) that

\[
|\eta| = \frac{1}{a + b} x_0^{\frac{1}{a+b}-1} |h_1^a r_1^b - h_2^a r_2^b| \gg (h_1^a r_1^b)^{\frac{1}{a+b}-1},
\]

which combining the inequality \(|\eta| \leq T^{-1/(a+b)}\) gives

\[H_1^a R_1^b \gg T^{1/(a+b-1)}.
\]

Similarly we have

\[H_2^a R_2^b \gg T^{1/(a+b-1)}.
\]

If \(a \leq b\), then combining the above estimates we get that the contribution of \(T^{1/(a+b)}\) is

\[
\ll 1 + \frac{T^{\frac{1}{a+b}}}{(H_1^a R_1^b)^{\frac{b}{2a+2b}} (H_2^a R_2^b)^{\frac{a}{2a+2b}} (H_1 H_2)^{\frac{b-a}{2a}}} L^2
\]

\[
\ll T^{\frac{1}{a+b} - \frac{a}{a+b} \frac{b}{a+b} - 1} L^2.
\]

If \(a > b\), then the contribution of \(T^{1/(a+b)}\) is

\[
\ll 1 + \frac{T^{\frac{1}{a+b}}}{(H_1^a R_1^b)^{\frac{b}{2a+2b}} (H_2^a R_2^b)^{\frac{a}{2a+2b}} (R_1 R_2)^{\frac{b-a}{2a}}} L^2
\]

\[
\ll T^{\frac{1}{a+b} - \frac{a}{a+b} \frac{b}{a+b} - 1} L^2.
\]

Namely, the the contribution of \(T^{1/(a+b)}\) is

\[
\ll T^{\frac{1}{a+b} - \frac{1}{a+b} \frac{1}{a+b} - 1} \frac{\min(a,b)}{\max(a,b)} L^2.
\]

(5.7)

Now we consider the contribution of \(1/|\eta|\). Suppose first that \(a \leq b\). By a splitting argument the contribution of \(1/|\eta|\) is

\[
\ll \frac{\mathcal{L}}{(H_1 H_2)^{\frac{a+b}{2a+2b}} (R_1 R_2)^{\frac{a+b}{2a+2b}} \delta} \times \mathcal{A}(H_1, H_2, R_1, R_2; \delta)
\]

\[
\ll \frac{\mathcal{L}}{\mathcal{L} + (H_1 H_2)^{\frac{a+b}{2a+2b}} (R_1 R_2)^{\frac{a+b}{2a+2b}}} L^3
\]

for some \(T^{-1/(a+b)} \ll \delta \ll H_1^{\frac{a}{2a+2b}} R_1^{\frac{b}{2a+2b}} H_2^{\frac{a}{2a+2b}} R_2^{\frac{b}{2a+2b}}\). From (5.6) we get

\[
\delta \gg (H_1^a R_1^b)^{\frac{1}{a+b}-1},
\]

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which combining $T^{-1/(a+b)} \ll \delta$ gives
\[
\delta^{-1} \ll \min \left( T^{\frac{1}{a+b}}, (H_1 R_1^{a+b-1}) \right).
\]

Thus the contribution of $1/|\eta|$ is
\[
\ll \mathcal{L} + \frac{1}{(H_1 R_1^{a+b}) (H_2 R_2^{a+b}) (H_1 H_2)^{b/a} \delta} \mathcal{L}^3
\]
\[
\ll \mathcal{L} + \frac{1}{(H_1 R_1^{a+b}) (H_1 H_2)^{b/a} \delta} \min \left( T^{\frac{1}{a+b}}, (H_1 R_1^{a+b-1}) \right) \mathcal{L}^3
\]
\[
\ll \mathcal{L} + \frac{(H_1 H_2)^{b/a}}{(H_1 R_1^{a+b})} \\times \min \left( T^{\frac{1}{a+b}}, (H_1 R_1^{a+b-1}) \right) \mathcal{L}^3
\]
\[
\ll T^{\frac{1}{a+b}} - \frac{b}{a(a+b)(a+b-1)} \mathcal{L}^3,
\]
where in the second step we used the fact $H_1 R_1^{b} \asymp H_2 R_2^{b}$.

Similarly if $a > b$, we can get that the contribution of $1/|\eta|$ is
\[
\ll T^{\frac{1}{a+b}} - \frac{a}{a(a+b)(a+b-1)} \mathcal{L}^3.
\]

Hence combining the above we see that the contribution of $1/|\eta|$ is
\[
(5.8) \quad \ll T^{\frac{1}{a+b}} - \frac{1}{(a+b)(a+b-1)} \min(a,b) \mathcal{L}^3.
\]

From (5.5), (5.7) and (5.8) we get the following

**Lemma 5.2.** Suppose $a$ and $b$ are fixed natural numbers, then we have
\[
S_{a,b}(T) \ll T^{\frac{1}{a+b}} - \frac{1}{(a+b)(a+b-1)} \min(a,b) \mathcal{L}^3.
\]

## 6 Proof of Theorem

In this section we shall prove our Theorem. It suffices to evaluate the integral $\int_T^{2T} \Delta_{a,b}^2(x) \, dx$ for $T \geq 10$. We always suppose $a < b$. 

6.1 Mean square of $\Delta^*_{a,b}(x, z)$

Suppose $H$ and $z$ are parameters such that

$$T^\varepsilon \ll H \ll T^{100(a+b)}, \quad T^\varepsilon \ll z \leq \min(H^{a+b}, T^{a/b})\mathcal{L}^{-b-b^2/a-1}.$$  

In this subsection we study the mean square of $\Delta^*_{a,b}(x, z)$.

By the elementary formula

$$(6.1) \quad \cos u \cos v = \frac{1}{2}(\cos (u - v) + \cos (u + v))$$

we may write

$$(6.2) \quad |\Delta^*_{a,b}(x, z)|^2 = \frac{c^2(a,b)}{\pi^2} x^{(a+b)\frac{1}{2}} \sum_{1 \leq n \leq z} g_{a,b}(n) g_{a,b}(m) \times \cos \left(2\pi c_2(a, b)x^{\frac{1}{a+b}}n^{\frac{1}{a+b}} - \frac{\pi}{4}\right) \cos \left(2\pi c_2(a, b)x^{\frac{1}{a+b}}m^{\frac{1}{a+b}} - \frac{\pi}{4}\right) = S_1(x) + S_2(x) + S_3(x),$$

where

$$S_1(x) = \frac{c^2(a,b)}{2\pi^2} x^{(a+b)\frac{1}{2}} \sum_{1 \leq n \leq z} g_{a,b}^2(n),$$

$$S_2(x) = \frac{c^2(a,b)}{2\pi^2} x^{(a+b)\frac{1}{2}} \sum_{n \leq z, m \leq z, n \neq m} g_{a,b}(n) g_{a,b}(m) \times \cos \left(2\pi c_2(a, b)x^{\frac{1}{a+b}}(n^{\frac{1}{a+b}} - m^{\frac{1}{a+b}})\right)$$

$$S_3(x) = \frac{c^2(a,b)}{2\pi^2} x^{(a+b)\frac{1}{2}} \sum_{n \leq z, m \leq z} g_{a,b}(n) g_{a,b}(m) \times \sin \left(2\pi c_2(a, b)x^{\frac{1}{a+b}}(n^{\frac{1}{a+b}} + m^{\frac{1}{a+b}})\right).$$

By (4.6) we get

$$(6.3) \quad \int_T^{2T} S_1(x) \, dx = \frac{c^2(a,b)}{2\pi^2} \sum_{n=1}^{\infty} g_{a,b}^2(n) \int_T^{2T} x^{\frac{1}{a+b}}dx + O(T^{\frac{1+a+b}{a+b}z^{-\frac{a}{a+b}}}).$$
By the first derivative test we get
\[ \int_T^{2T} S_3(x) \, dx \ll \sum_{n \leq z, m \leq z} g_{a,b}(n) g_{a,b}(m) \frac{T}{n^{a+b} + m^{a+b}} \]
\[ \ll T \sum_{n \leq z, m \leq z} g_{a,b}(n) g_{a,b}(m) \frac{1}{n^{2(a+b)} m^{2(a+b)}} \]
\[ \ll T \left( \sum_{n \leq z} g_{a,b}(n) \right)^2, \]
where in the second step we used the well-known inequality \( \alpha^2 + \beta^2 \geq 2\alpha\beta \). By Euler’s product we have for \( \Re s > 1 \) that
\[ \sum_{n=1}^{\infty} \frac{g_{a,b}(n)}{n^s} = \zeta(2s + a + 2b) \zeta(2s + 2a + 2b), \]
which can be continued meromorphically to the whole complex plane and has a double pole at \( s = 1/2(a + b) \). Thus Perron’s formula implies that
\[ \sum_{n \leq X} g_{a,b}(n) \ll X^{1/2(a+b)} \log X \]
for any \( X > 2 \), namely,
(6.4) \[ \sum_{n \leq X} g_{a,b}(n) n^{-1/2(a+b)} \ll \log^2 X. \]

From the above we get
(6.5) \[ \int_T^{2T} S_3(x) \, dx \ll TL^4. \]

Now we consider the contribution of \( S_2(x) \). Write
(6.6) \[ S_2(x) = S_{21}(x) + S_{22}(x), \]
where
\[ S_{2j}(x) = \frac{c_2^2(a, b)}{2\pi^2} x^{1/(a+b)} \sum_{2j} g_{a,b}(n) g_{a,b}(m) \times \cos \left( 2\pi c_2(a, b) x^{1/(a+b)} \left( n^{1/(a+b)} - m^{1/(a+b)} \right) \right) (j = 1, 2), \]
\[ SC(\Sigma_{21}) : n, m \leq z, |n^{1/(a+b)} - m^{1/(a+b)}| \geq \frac{(nm)^{1/(2a+2b)}}{10}, \]
\[ SC(\Sigma_{22}) : n, m \leq z, 0 < |n^{1/(a+b)} - m^{1/(a+b)}| < \frac{(nm)^{1/(2a+2b)}}{10}. \]
Similar to the case \( S_3(x) \), we have
\[
\int_T^{2T} S_{21}(x) dx \ll TL^4.
\]
By the first derivative test and Lemma 5.2 we get
\[
\int_T^{2T} S_{22}(x) dx \ll T \sum_{22} g_{a,b}(n) g_{a,b}(m) \min \left( \frac{1}{n^{\frac{1}{a+b}}}, \frac{1}{|n^{\frac{1}{a+b}} - n^{\frac{1}{a+b}}|} \right)
\ll TS_{a,b}(T) \ll T^{1+\frac{1}{a+b}} \frac{a}{b(a+b)(a+b-1)} L^7.
\]
From the above two estimates we get
\[
(6.7) \quad \int_T^{2T} S_{2}(x) dx \ll T^{1+\frac{1}{a+b}} \frac{a}{b(a+b)(a+b-1)} L^7.
\]
From (6.2), (6.3), (6.5) and (6.7) we have
\[
(6.8) \quad \int_T^{2T} |\Delta_{a,b}^*(x,z)|^2 dx = \frac{c_1^2(a,b)}{2\pi^2} \sum_{n=1}^{\infty} g_{a,b}(n) \int_T^{2T} x^{1/a+b} dx
+ O \left( T^{1+\frac{1}{a+b}} \frac{a}{b(a+b)(a+b-1)} L^7 \right).
\]

### 6.2 Mean squares of \( R_{12}^*(a, b; x) \) and \( R_{12}^*(b, a; x) \)

In this subsection we shall study the mean squares of \( R_{12}^*(a, b; x) \) and \( R_{12}^*(b, a; x) \). Recall that
\[
R_{12}^*(a, b; x) = \frac{c_1(a,b)}{\pi} x^{\frac{1}{2(a+b)}} \sum_{z \leq n \leq H^n, H(a,b)} g(a, b; n, H, J)
\times \cos \left( 2\pi c_2(a,b) x^{\frac{1}{a+b}} n^{\frac{1}{a+b}} - \frac{\pi}{4} \right).
\]
Thus
\[
|R_{12}^*(a, b; x)|^2 \ll x^{1/a+b} \sum_{z \leq n, m \leq H^n, H(a,b)} g(a, b; n, H, J) g(a, b; m, H, J)
\times e \left( c_2(a, b) x^{\frac{1}{a+b}} (n^{\frac{1}{a+b}} - m^{\frac{1}{a+b}}) \right)
= S_4(x) + S_5(x),
\]

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where
\[
S_4(x) = x^{\frac{1}{a+b}} \sum_{z \leq n \leq H^a n_{j+1}, H(a,b)} g^2(a, b; n, H, J),
\]
\[
S_5(x) = x^{\frac{1}{a+b}} \sum_{z \leq n, m \leq H^a n_{j+1}, H(a,b)} g(a, b; n, H, J)g(a, b; m, H, J)
\times e \left( c_2(a, b) x^{\frac{1}{a+b}} (n^{\frac{1}{a+b}} - m^{\frac{1}{a+b}}) \right)
\]

It is easy to see that
\[
(6.9) \quad g(a, b; n, H, J) \leq g(a, b; n) \leq g_{a,b}(n).
\]

By (4.7) we get
\[
(6.10) \quad \int_T^{2T} S_4(x) dx \ll T^{1 + \frac{1}{a+b}} z^{-\frac{a}{2(a+b)}}.
\]

By the first derivative test and (6.9) we get
\[
(6.11) \quad \int_T^{2T} S_5(x) dx \ll T \sum_{z \leq n, m \leq H^a T} g_{a,b}(n)g_{a,b}(m)
\times \min \left( T^{\frac{1}{a+b}}, \frac{1}{|n^{\frac{1}{a+b}} - m^{\frac{1}{a+b}}|} \right)
\ll T(\Sigma_4 + \Sigma_5),
\]

where
\[
\Sigma_4 := \sum_{\text{4}} g_{a,b}(n)g_{a,b}(m) \min \left( T^{\frac{1}{a+b}}, \frac{1}{|n^{\frac{1}{a+b}} - m^{\frac{1}{a+b}}|} \right),
\]
\[
\Sigma_5 := \sum_{\text{5}} g_{a,b}(n)g_{a,b}(m) \min \left( T^{\frac{1}{a+b}}, \frac{1}{|n^{\frac{1}{a+b}} - m^{\frac{1}{a+b}}|} \right),
\]
\[
SC(\Sigma_4) : z < n, m \leq H^a T, 0 < |n^{\frac{1}{a+b}} - m^{\frac{1}{a+b}}| \leq \frac{1}{10} n^{\frac{1}{2a+2b}} m^{\frac{1}{2a+2b}},
\]
\[
SC(\Sigma_5) : z < n, m \leq H^a T, |n^{\frac{1}{a+b}} - m^{\frac{1}{a+b}}| \geq \frac{1}{10} n^{\frac{1}{2a+2b}} m^{\frac{1}{2a+2b}}.
\]

By Lemma 5.2 we have
\[
(6.12) \quad \Sigma_4 \ll S_{a,b}(T) \ll T^{\frac{1}{a+b} - \frac{a}{6(a+b)(a+b-1)}} L^7.
\]
For $\Sigma_5$ by (6.4) we get

\[(6.13) \quad \Sigma_5 \ll \left( \sum_{n \leq H^aT} g_{a,b}(n)n^{-1/(2a+2b)} \right)^2 \ll \log^4 X.\]

From (6.10)-(6.13) we have

\[(6.14) \quad \int_T^{2T} |R_{12}^*(a,b;x)|^2 dx \ll T^{1+\frac{1}{a+b}} \frac{z^{-\frac{a}{b(a+b)}}}{\log(a+b)} + T^{\frac{1+a+b}{a+b}} \frac{z^{-\frac{a}{b(a+b)}}}{\log(a+b)(a+b-1)} L^7.\]

Similarly, we have

\[(6.15) \quad \int_T^{2T} |R_{12}^*(b,a;x)|^2 dx \ll T^{1+\frac{1}{a+b}} \frac{z^{-\frac{a}{b(a+b)}}}{\log(a+b)} + T^{\frac{1+a+b}{a+b}} \frac{z^{-\frac{a}{b(a+b)}}}{\log(a+b)(a+b-1)} L^7.\]

### 6.3 Mean squares of $R_2(a, b; x)$ and $R_2(b, a; x)$

In this subsection we shall study the mean squares of $R_2(a, b; x)$ and $R_2(b, a; x)$. Recall that

\[R_2(a, b; x) = O \left( \sum_{m \leq x^{1/(a+b)}} \min \left( 1, \frac{1}{H \| \frac{2^{1/a}}{m^{1/a}} \|} \right) \right).\]
So we have
\[
\int_T^{2T} R_2(a, b; x) dx \ll \sum_{m \leq 2T^{1/(a+b)}} \int_T^{2T} \min \left( 1, \frac{1}{H \| u \|} \right) m^b u^{a-1} du
\]
\[
\ll \sum_{m \leq 2T^{1/(a+b)}} m^b \left( \frac{T^{1/a}}{m^{b/a}} \right) \int_T^{2T} \min \left( 1, \frac{1}{H \| u \|} \right) m^{b/a} u^{a-1} du
\]
\[
\ll \sum_{m \leq 2T^{1/(a+b)}} m^b \left( \frac{T^{1/a}}{m^{b/a}} \right) \int_0^{T^{1/(a+b)}} \min \left( 1, \frac{1}{H \| u \|} \right) du
\]
\[
\ll T \sum_{m \leq 2T^{1/(a+b)}} \int_0^{1} \min \left( 1, \frac{1}{H \| u \|} \right) du
\]
\[
\ll T^{1+1/(a+b)} \int_0^{1/2} \min \left( 1, \frac{1}{H \| u \|} \right) du
\]
\[
\ll T^{1+1/(a+b)} \left( \int_0^{1/H} du + \int_{1/H}^{1/2} \frac{1}{Hu} du \right)
\]
\[
\ll T^{1+1/(a+b)} H^{-1} \mathcal{L},
\]
which combining the trivial bound \( R_2(a, b; x) \ll T^{1/(a+b)} \) gives
\[
(6.16) \quad \int_T^{2T} R_2^2(a, b; x) dx \ll T^{1+2/(a+b)} H^{-1} \mathcal{L}
\]

Similarly we have
\[
(6.17) \quad \int_T^{2T} R_2^2(b, a; x) dx \ll T^{1+2/(a+b)} H^{-1} \mathcal{L}.
\]

### 6.4 Completion of the proof of Theorem

In this subsection we complete the proof of Theorem. We take \( H = T^{10(a+b)} \) and \( z = T^{a/b} \mathcal{L}^{-b^{2}/a^{1}} \). From (3.11), (6.11)-(6.14) we get
\[
(6.18) \quad \int_T^{2T} E_{a,b}^2(x) dx \ll T^{1+10(b)} \frac{1}{(a+b)(a+b-1)} \mathcal{L}^7,
\]
which combining (6.8) and Cauchy’s inequality gives

\[ \int_T^{2T} E_{a,b}(x) \Delta_{a,b}^*(x,z)dx \ll T^{1+a+b} 
\]

(6.19) \quad \frac{a}{2b(a+b-1)} \mathcal{L}^{7/2}.

From (6.8), (6.18) and (6.19) we get

\[ \int_T^{2T} \Delta_{a,b}^*(x)dx = \frac{c^2(a,b)}{2\pi^2} \sum_{n=1}^\infty g_{a,b}(n) \int_T^{2T} x^{1+a+b}dx \]

+ \mathcal{O}(T^{1+a+b} 
\]

(6.20) \quad \frac{a}{2b(a+b-1)} \mathcal{L}^{7/2}).

From (6.20) and a splitting argument we get

\[ \int_T^1 \Delta_{a,b}^*(x)dx = \frac{c^2(a,b)}{2\pi^2} \sum_{n=1}^\infty g_{a,b}(n) \int_T^1 x^{1+a+b}dx \]

+ \mathcal{O}(T^{1+a+b} 
\]

(6.21) \quad \frac{a}{2b(a+b-1)} \mathcal{L}^{7/2})

= \frac{c^2(a,b)(a+b)}{2(1+a+b)\pi^2} \sum_{n=1}^\infty g_{a,b}(n)T^{1+a+b} 
\]

+ \mathcal{O}(T^{1+a+b} 
\]

(6.21) \quad \frac{a}{2b(a+b-1)} \mathcal{L}^{7/2})

= c_{a,b}T^{1+a+b} + \mathcal{O}(T^{1+a+b} 
\]

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