The classification of ERP $G_2$-structures on Lie groups

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Abstract

A complete classification of left-invariant closed $G_2$-structures on Lie groups which are extremally Ricci pinched (i.e., $\frac{d}{dt} = \frac{1}{6}|\tau|^2 \varphi + \frac{1}{6} * (\tau \wedge \tau)$), up to equivalence and scaling, is obtained. There are five of them, they are defined on five different completely solvable Lie groups and the $G_2$-structure is exact in all cases except one, given by the only example in which the Lie group is unimodular.

Keywords $G_2$-structures · Extremally Ricci pinched · Laplacian solitons

Mathematics Subject Classification 53C30 · 53C25 · 53C29

1 Introduction

A $G_2$-structure on a seven-dimensional differentiable manifold $M$ is a differential 3-form $\varphi$ on $M$ which is positive (or definite), in the sense that $\varphi$ (uniquely) determines a Riemannian metric $g$ on $M$ together with an orientation. Playing a role analogous in a way to that of almost Kähler structures in almost Hermitian geometry, closed $G_2$-structures (i.e., $d\varphi = 0$) have been studied by several authors (see, e.g., [6–8, 10, 17, 18]). They appear as natural candidates to be deformed via the Laplacian flow $\frac{d}{dt}\varphi(t) = \Delta\varphi(t)$ toward a torsion-free $G_2$-structure (i.e., $d\varphi = 0$ and $d * \varphi = 0$) producing a Ricci-flat Riemannian metric with holonomy contained in $G_2$ (see [16] for an account of recent advances). The torsion of a closed $G_2$-structure $\varphi$ is completely determined by the 2-form $\tau = -* d * \varphi$, which in addition satisfies that $d * \varphi = \tau \wedge \varphi$.

The following remarkable curvature estimate for closed $G_2$-structures on a compact manifold $M$ was discovered by Bryant (see [6, Corollary 3]):

$$\int_M \text{scal}^2 \ast 1 \leq 3 \int_M |\text{Ric}|^2 \ast 1,$$

(1)
where \( \text{scal} \) and \( \text{Ric} \), respectively, denote scalar curvature and Ricci tensor of \((M, g)\). Bryant called extremally Ricci pinched (ERP for short) the structures at which equality holds in (1) (see [6, Remark 13]) and proved that they are characterized by the following neat equation:

\[
\dd t = \frac{1}{6} |\tau|^2 \varphi + \frac{1}{6} \ast (\tau \wedge \tau).
\]  

(2)

In the compact case, this is actually the only way in which \( \dd t \) can quadratically depend on \( \tau \) (see [6, (4.66)]). A non-necessarily compact \((M, \varphi)\) satisfying (2) is also called ERP.

The first examples of ERP \(G_2\)-structures in the literature are homogeneous (i.e., the automorphism group \( \text{Aut}(M, \varphi) := \{ f \in \text{Diff}(M) : f^* \varphi = \varphi \} \) acts transitively on \( M \)). In [6], Bryant gave the first example on the homogeneous space \( SL_2(\mathbb{C}) \ltimes \mathbb{C}^2 / SU(2) \), which indeed admits a compact locally homogeneous quotient (the only compact ERP structure known so far), a second one was found on a unimodular solvable Lie group in [14] and a curve was given in [11]. (Each structure in the curve is actually equivalent to Bryant’s example, although the Lie groups involved are pairwise non-isomorphic.) Three more examples were recently given on non-unimodular solvable Lie groups in [15]. We refer to [4] for examples of ERP \(G_2\)-structures which are not homogeneous.

It is worth pointing out that homogeneous \(G_2\)-geometry includes the following particular features:

- Torsion-free \(G_2\)-structures are necessarily flat by [2].
- Closed \(G_2\)-structures are only allowed on non-compact manifolds by [18].
- Einstein closed \(G_2\)-structures do not exist by [3, 9].
- The only other possibility for a quadratic dependence is to have \( \dd t = \frac{1}{7} |\tau|^2 \varphi \), whose existence is an open problem (see [14]).
- Estimate (1) does not hold in general; examples with \( \text{scal}^2 > 3 |\text{Ric}|^2 \) were found in [15].

In this paper, we continue the study of left-invariant ERP \(G_2\)-structures on Lie groups initiated in [15], where it was shown that the ERP condition (2) requires very strong structure constraints on the Lie algebra. By using such a structure theorem (see Theorem 2.1) as a starting point, we have obtained a complete classification.

As usual, two manifolds endowed with \(G_2\)-structures \((M, \varphi)\) and \((M', \varphi')\) are called equivalent if there exists a diffeomorphism \( f : M \rightarrow M' \) such that \( \varphi = f^* \varphi' \). Two Lie groups endowed with left-invariant \(G_2\)-structures \((G, \varphi)\) and \((G', \varphi')\) are called equivariantly equivalent if there exists an equivalence \( f : G \rightarrow G' \) which is in addition a Lie group isomorphism. (This holds if and only if \( \varphi = df|_e^* \varphi' \), where \( df|_e : g \rightarrow g' \) is the corresponding Lie algebra isomorphism.)

We fix a seven-dimensional vector space \( \mathfrak{g} \), and for each Lie bracket \( \mu \) on \( \mathfrak{g} \), we consider \((G_\mu, \varphi)\), the simply connected Lie group \( G_\mu \) with Lie algebra \((\mathfrak{g}, \mu)\) endowed with the left-invariant \(G_2\)-structure defined by the positive 3-form on \( \mathfrak{g} \) given by

\[
\varphi := e^{127} + e^{347} + e^{567} + e^{135} - e^{146} - e^{236} - e^{245} = \omega \wedge e^7 + \rho^+,
\]  

(3)

where \( \{e_1, \ldots, e_7\} \) is a basis of \( \mathfrak{g} \) (orthonormal with respect to the inner product \( \langle \cdot, \cdot \rangle_\varphi \) induced by \( \varphi \)).
Table 1 Structure coefficients

| \( \mu_B \) | \( de^2 = 0, \) \( de^3 = \frac{1}{3}e^{37}, \) \( de^4 = \frac{1}{3}e^{47}, \) \( de^1 = -\frac{1}{6}e^{17}, \) \( de^2 = -\frac{1}{6}e^{27}, \) |
|---|---|
| \( \mu_{M1} \) | \( de^5 = \frac{2}{3}e^{14} + \frac{1}{3}e^{23} + \frac{1}{6}e^{57}, \) \( de^6 = \frac{1}{3}e^{13} - \frac{1}{3}e^{24} + \frac{1}{6}e^{67}. \) |
| \( \mu_{M2} \) | \( de^5 = \frac{2}{3}e^{13} + \frac{1}{3}e^{24} + \frac{1}{6}e^{45} - \frac{\sqrt{3}}{6}e^{65}, \) \( de^6 = \frac{1}{3}e^{13} - \frac{1}{3}e^{24} + \frac{1}{6}e^{45} + \frac{1}{3}e^{67}. \) |
| \( \mu_{M3} \) | \( de^5 = \frac{2}{3}e^{13} + \frac{1}{3}e^{24} + \frac{1}{6}e^{45} - \frac{\sqrt{3}}{6}e^{65}, \) \( de^6 = \frac{1}{3}e^{13} - \frac{1}{3}e^{24} + \frac{1}{6}e^{45} + \frac{1}{3}e^{67}. \) |
| \( \mu_J \) | \( de^1 = \frac{1}{3}e^{14} + \frac{1}{3}e^{23} - \frac{1}{3}e^{35}, \) \( de^2 = \frac{1}{3}e^{14} + \frac{1}{3}e^{23} - \frac{1}{3}e^{35}, \) \( de^3 = \frac{1}{3}e^{14} + \frac{1}{3}e^{23} - \frac{1}{3}e^{35}, \) \( de^4 = \frac{1}{3}e^{14} + \frac{1}{3}e^{23} - \frac{1}{3}e^{35}, \) \( de^5 = \frac{1}{3}e^{14} + \frac{1}{3}e^{23} - \frac{1}{3}e^{35}, \) |
| \( \mu_\tau \) | \( de^1 = -\frac{1}{6}e^{17} - \frac{1}{2}e^{27}, \) \( de^2 = -\frac{1}{6}e^{27} + \frac{1}{2}e^{17}, \) \( de^3 = \frac{1}{3}e^{14} + \frac{1}{3}e^{23} + \frac{1}{6}e^{57} + \frac{1}{2}(r + t)e^{67}, \) \( de^6 = \frac{1}{3}e^{13} - \frac{1}{3}e^{24} + \frac{1}{6}e^{67} - \frac{1}{2}(r + t)e^{57}. \) |

**Theorem 1.1** Any Lie group endowed with a left-invariant ERP \( G_2 \)-structure is equivalent up to scaling to \((G_\mu, \varphi)\), where \( \mu \) is exactly one of the following Lie brackets given in Table 1:

\[
\mu_B, \mu_{M1}, \mu_{M2}, \mu_{M3}, \mu_J.
\] (4)

Moreover, in order to obtain a classification up to equivariant equivalence and scaling, exactly the structures \((G_\mu, \varphi)\), \( r, t \in \mathbb{R} \), \( (r, t) \neq (0, 0) \), must be added to the list (4) (see also Table 1). The structures \((G_\mu, \varphi)\) are all equivalent to \((G_\mu, \varphi)\) and the family of Lie algebras \( \mu_\tau, r, t \in \mathbb{R} \) is pairwise non-isomorphic. (Note that \( \mu_0 = \mu_B \).)

More friendly descriptions of these Lie algebras are given in Examples 5.1, 5.2, 5.3, 4.1, 4.2 and (15) (see (7) first). The proof of this theorem follows from Propositions 3.1, 4.3 and 5.4 and is developed in Sects. 3, 4 and 5, after some preliminary material given in Sect. 2.

We now list some interesting properties of the ERP \( G_2 \)-structures obtained in the classification:
• \( \mu_B \) was originally found in [6], \( \mu_j \) in [14] and the \( \mu_{M_i} \)'s in [15], and the curve from [11] belongs to the 2-parameter family \( \mu_r \).

• They are all steady Laplacian solitons and expanding Ricci solitons (see [15, Corollary 4.8]).

• They all have torsion 2-form equal to \( \tau = e^{12} - e^{56} \).

• With the only exception of \( \mu_j \), they are all exact; indeed,

\[
\varphi = d_\mu \left( 3\tau - (\text{tr} A_1)^{-1} e^{34} \right), \quad A_1 := \text{ad}_\mu e_7 |_{\text{sp}(e_7, e_4)}.
\]

Note that \((\text{tr} A_1)^{-1} = \frac{3}{2}, \frac{\sqrt{30}}{3}, 3, \sqrt{6}\) for \( \mu \) given by \( \mu_B \) (or \( \mu_r \)), \( \mu_{M1}, \mu_{M2}, \mu_{M3} \), respectively. On the contrary, one obtains that \((G_{\mu_j}, \varphi)\) is not exact by using that the ERP condition implies that \( e^{347} \) would be exact, which is impossible since \( de^i \not\perp e^{347} \) for any \( i, j \).

• \( \frak{h} := \text{sp}\{e_1, \ldots, e_6\} \) is always a unimodular ideal, and the corresponding SU(3)-structures \((\frak{h}, \omega, \rho^+)\) are all half-flat (i.e., \( d\omega^2 = 0 \) and \( d\rho^+ = 0 \)). It is in addition coupled for \( \mu_B \) (i.e., \( d\omega = \frac{1}{2} \rho^+ \)). For \( \mu_j \), it is straightforward to see that the corresponding SU(3)-structure on the ideal \( \frak{h}_j := \{ e_1 - e_2 + e_7 \}^\perp \) is symplectic half-flat (i.e., \( d\omega_1 = 0 \) and \( d\rho_1^+ = 0 \)).

• All Lie algebras in (4) are completely solvable, and the only unimodular one is \( \mu_j \). It was proved in [11, Theorem 6.7] that \( G_{\mu_j} \) is the only unimodular Lie group admitting an ERP \( G_{2} \)-structure. The question of whether \( G_{\mu_j} \) admits a lattice is still open.

• For each \( \mu \) in (4), the simply connected Lie group \( G_\mu \) is the only Lie group with Lie algebra \( \mu \); indeed, the center of \( G_\mu \) is trivial since the center of the Lie algebra is so and the exponential function is a diffeomorphism.

• The Betti numbers of each Lie algebra are given in Table 2, together with some information on the respective nilradical \( n \).

• Concerning symmetries, we have included in Table 2 for each of the examples both the subgroup of automorphisms of \( \varphi \) and the subgroup of isometries of \( \langle \cdot, \cdot \rangle_\varphi \) which are also Lie group automorphisms (see Sect. 6 for a more detailed study of symmetries).

### 2 Preliminaries

A left-invariant \( G_2 \)-structure on a Lie group is determined by a positive 3-form on the Lie algebra \( \frak{g} \), which will always be given by

\[
\varphi := e^{127} + e^{347} + e^{567} + e^{135} - e^{146} - e^{236} - e^{245}
= \omega_7 \wedge e^7 + \omega_3 \wedge e^3 + \omega_4 \wedge e^4 + e^{347},
\]

\[
(5)\]

| \( b_1 \) | \( b_2 \) | \( b_3 \) | \( b_4 \) | \( b_5 \) | \( b_6 \) | dim \( n \) | nilp. deg. | \( \text{Aut}(\mu) \cap G_2 \) | \( \text{Aut}(\mu) \cap O(7) \) |
|-------|-------|-------|-------|-------|-------|--------|----------|----------------|----------------|
| \( \mu_B \) | 1 | 2 | 2 | 2 | 2 | 0 | 6 | 2-Step | \( S^1 \times S^1 \) |
| \( \mu_{M1} \) | 1 | 0 | 0 | 1 | 1 | 0 | 6 | 4-Step | \( Z_2 \times Z_2 \) |
| \( \mu_{M2} \) | 2 | 1 | 0 | 1 | 2 | 1 | 5 | 3-Step | \( Z_2 \times Z_2 \times Z_2 \) |
| \( \mu_{M3} \) | 2 | 2 | 2 | 2 | 1 | 5 | 2-Step | \( Z_4 \) |
| \( \mu_j \) | 3 | 3 | 1 | 1 | 3 | 3 | 4 | Abelian | \( SL_2(Z_3) \) |
| \( \mu_r \) | 1 | 2 | 2 | 2 | 2 | 0 | 6 | 2-Step | \( S^1 \times S^1 \) |
where \( \{e_1, \ldots, e_7\} \) is an orthonormal basis of \( \mathfrak{g} \) and
\[
\omega_7 := e^{12} + e^{56}, \quad \omega_3 := e^{26} - e^{15}, \quad \omega_4 := e^{16} + e^{25}.
\]

The following is the main structure result in [15]. Let \( \theta \) denote the usual representation of \( \mathfrak{g} \mathfrak{l}_4(\mathbb{R}) \) on \( \Lambda^2(\mathbb{R}^4)^* \). That is, \( \theta(E)\alpha = -\alpha(E\cdot \cdot) - \alpha(\cdot, E\cdot) \) for all \( E \in \mathfrak{g} \mathfrak{l}_4(\mathbb{R}) \) and \( \alpha \in \Lambda^2(\mathbb{R}^4)^* \).

**Theorem 2.1** [15, Theorem 4.7 and Proposition 4.9] Every Lie group endowed with a left-invariant ERP \( G_z \)-structure is equivariantly equivalent (up to scaling) to some \((G, \varphi)\) with torsion \( \tau = e^{12} - e^{56} \), such that \( \varphi \) is as in (5) and the following conditions hold for the Lie algebra \( \mathfrak{g} \) of \( G \):

\[
\begin{align*}
(i) & \quad \mathfrak{h} := \text{sp}\{e_1, \ldots, e_6\} \text{ is a unimodular ideal, } \mathfrak{g}_0 := \text{sp}\{e_7, e_3, e_4\} \text{ is a Lie subalgebra,} \\
(ii) & \quad \theta(\text{ade}_7|_{\mathfrak{g}_0}) \tau = \frac{1}{3}\omega_7, \theta(\text{ade}_3|_{\mathfrak{g}_0}) \tau = \frac{1}{3}\omega_3 \text{ and } \theta(\text{ade}_4|_{\mathfrak{g}_0}) \tau = \frac{1}{3}\omega_4, \\
(iii) & \quad \theta(\text{ade}_7|_{\mathfrak{g}_0})\omega_7 + \theta(\text{ade}_3|_{\mathfrak{g}_0})\omega_3 + \theta(\text{ade}_4|_{\mathfrak{g}_0})\omega_4 = \tau + \text{(trade}_3|_{\mathfrak{g}_0})\omega_7.
\end{align*}
\]

Conversely, if \( \mathfrak{g} \) satisfies conditions (i)–(iii), then \((G, \varphi)\) is an ERP \( G_z \)-structure with torsion \( \tau = e^{12} - e^{56} \).

Structurally, it follows from the above theorem that, up to equivariant equivalence, the Lie bracket \( \mu \) of the Lie algebra \( \mathfrak{g} \) of any ERP \((G, \varphi)\) with \( \tau = e^{12} - e^{56} \) is given by:
\[
\mu = (A_1, A, B, C),
\]
in the sense that \( \mu \) is determined by the 2 \( \times \) 2 matrix \( A_1 := \text{ad}_\mu e_7|_{\mathfrak{g}_0} \) and the three 4 \( \times \) 4 traceless matrices \( A := \text{ad}_\mu e_7|_{\mathfrak{g}_0}, B = \text{ad}_\mu e_3|_{\mathfrak{g}_0}, C := \text{ad}_\mu e_4|_{\mathfrak{g}_0} \), which must satisfy conditions (ii) and (iii). The Jacobi condition, on the other hand, is equivalent to
\[
[A, B] = AB + cC, \quad [A, C] = bB + dC, \quad [B, C] = 0, \quad A_1 = \begin{bmatrix} a & b \\ c & d \end{bmatrix}.
\]

It was obtained in [15, Section 5] that there are only three possibilities for the underlying vector space \( \mathfrak{n} \) of the nilradical of \( \mu \) and that the following additional conditions must hold in each case (up to equivariant equivalence):

- **\( \mathfrak{n} = \mathfrak{g}_1 \) (dim \( \mathfrak{n} = 4 \))**: This is equivalent to \( \mu \) unimodular and one has that \( A_1 = 0 \), the matrices \( A, B, C \) are all symmetric, they pairwise commute and \( \{ \sqrt{3}A, \sqrt{3}B, \sqrt{3}C \} \) is orthonormal relative to the usual inner product \( \text{tr}XY^\dagger \). In particular, only one Lie algebra (up to isomorphism) shows up in this case.

- **\( \mathfrak{n} = \mathbb{R}e_4 \oplus \mathfrak{g}_1 \) (dim \( \mathfrak{n} = 5 \))**: \( A, B \) are symmetric, \( C \) is nilpotent, \( a = b = c = 0 \) and \( d > 0 \).

- **\( \mathfrak{n} = \mathfrak{h} \) (dim \( \mathfrak{n} = 6 \))**: \( A_1 \) and \( A \) are normal, \( B \) and \( C \) are nilpotent and
  - either \( A_1 = \begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix} \) with \( a \leq d, a + d > 0, \)
  - or \( A_1 = \begin{bmatrix} a & b \\ -b & a \end{bmatrix} \), with \( a > 0, b \neq 0 \).
Up to equivalence, only five examples of left-invariant ERP $G_2$-structures on a Lie group were known, one with dim $n = 4$ and two in each of the other two cases (see [15, Section 5]).

We recall from [15, Proposition 4.4] that in the non-unimodular case (i.e., dim $n = 5, 6$), the equivariant equivalence among the set of closed $G_2$-structures $(G_\mu, \varphi)$, where $\mu = (A_1, A, B, C)$ is as in (7), is determined by the action of the subgroup $U_{h, r} = U_0 \cup U_0 g \subset G_2$ (see [15, Lemma 2.3]), where $G_2 := \{ h \in \text{GL}_7(\mathbb{R}) : h^* \varphi = \varphi \} \subset \text{SO}(7)$,

$$U_0 := \left\{ \begin{bmatrix} 1 & h_1 & h_2 \\ h_3 & 1 & h_4 \\ h_5 & h_6 & 1 \end{bmatrix} : h_i \in \text{SO}(2), h_1 h_2 h_3 = I \right\},$$

and

$$ge_7 = -e_7, \quad g_1 := g|_{h_1} = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}, \quad g_2 := g|_{h_2} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ -1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}.$$

The action is given as follows (see [15, (30)]): If $h \in U_0$, say with $h_1 = \begin{bmatrix} x & y \\ -y & x \end{bmatrix}$, $x^2 + y^2 = 1$ and $h_4 := \begin{bmatrix} h_2 & 0 \\ 0 & h_3 \end{bmatrix}, h_2, h_3 \in \text{SO}(2)$, then

$$h \cdot \mu = (h_1 A_1 h_1^{-1}, h_4 A h_4^{-1}, h_4 (xB - yC) h_4^{-1}, h_4 (yB + xC) h_4^{-1}),$$

and

$$g \cdot \mu = (-g_1 A_1 g_1^{-1}, -g_2 A g_2^{-1}, g_2 B g_2^{-1}, -g_2 C g_2^{-1}).$$

### 3 Case dim $n = 4$

In this section, we obtain a classification of left-invariant ERP $G_2$-structures on Lie groups with nilradical of dimension 4, up to equivariant equivalence and scaling. Recall from Sect. 2 that there is a unique Lie group $G$ involved in this case. Since a $G_2$-structure $\varphi$ is ERP if and only if $-\varphi$ is so, it is enough to consider $G_2$-structures with a given orientation.

As explained in [16, Section 6], each $\mu$ in the algebraic subset $L \subset \Lambda^2 \mathfrak{g}^* \otimes \mathfrak{g}$ of all Lie brackets on $\mathfrak{g}$ is identified with $(G_\mu, \varphi)$, where $G_\mu$ denotes the simply connected Lie group with Lie algebra $(\mathfrak{g}, \mu)$ and $\varphi$ is defined by (5). The orbit $\text{GL}_+^+(\mathbb{R}) \cdot \mu$ therefore parametrizes the set of all left-invariant $G_2$-structures on $G_\mu$ with the same orientation as $\varphi$, due to the equivariant equivalence:

$$(G_{h \cdot \mu}, \varphi) \simeq (G_\mu, \varphi(h \cdot h^* h)), \quad \forall h \in \text{GL}_7(\mathbb{R}).$$

Note that two elements in $L$ are equivariantly equivalent if and only if they belong to the same $G_2$-orbit, and that they are in the same $O(7)$-orbit if and only if they are equivariantly isometric as Riemannian metrics. It is known that both assertions hold, without the word
‘equivariantly’, for completely real solvable Lie brackets (see [1]). In light of Theorem 2.1, any \((G_{\mu}, \varphi)\) will be assumed from now on to have the structure \(\mu = (A_1, A, B, C)\) as in (7).

Assume that \((G_{\mu}, \varphi)\) and \((G_{h\mu}, \varphi)\) are both ERP with \(\tau_{\mu} = \tau_{h\mu} = e^{12} - e^{56}\) for some \(h \in \text{GL}_7(\mathbb{R})\), det \(h > 0\). Since both \((G_{\mu}, \langle \cdot, \cdot \rangle)\) and \((G_{h\mu}, \langle \cdot, \cdot \rangle)\) are solvsolitons by [15, Corollary 4.8], it follows from the uniqueness of solvsolitons (up to equivariant isometry and scaling) on a given Lie group (see [13] or [5]) that we can assume \(h \in \text{SO}(7)\). Thus, \(\text{Ric}_{h\mu} = h\text{Ric}_{\mu} h^{-1}\) and so by [15, Proposition 3.9, (iv)],

\[
h = \begin{bmatrix} h_1 \\ h_2 \end{bmatrix}, \quad h_1 \in \text{O}(3), \quad h_2 \in \text{O}(4), \quad \det h_1 = \det h_2.
\]

with respect to the decomposition \(\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1\).

Relative to the matrix \(J\) with respect to the basis \(\{e_1, e_2, e_5, e_6\}\), given by:

\[
J := \begin{bmatrix} 0 & -1 \\ 1 & 0 \\ 0 & 1 \\ -1 & 0 \end{bmatrix},
\]

one obtains the following classic orthogonal decompositions:

\[
\mathfrak{sl}_4(\mathbb{R}) = \mathfrak{so}(4) \oplus \text{sym}_0(4), \quad \text{sym}_0(4) = 1 \oplus 2, \quad \mathfrak{sp}(2, \mathbb{R}) = \mathfrak{u}(2) \oplus 1, \quad \mathfrak{gl}_2(\mathbb{C}) = \mathfrak{u}(2) \oplus 2,
\]

where

\[
1 := \left\{ \begin{bmatrix} a & b & e & f \\ b & -a & -f & e \\ e & -f & c & d \\ f & e & d & -c \end{bmatrix} : a, \ldots, f \in \mathbb{R} \right\}, \quad (12)
\]

and 2 has the following orthogonal basis:

\[
T_7 := \begin{bmatrix} -1 \\ -1 \\ 1 \\ 1 \end{bmatrix}, \quad T_3 := \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \quad T_4 := \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{bmatrix}. \quad (13)
\]

Note that \(|T_i|^2 = \frac{1}{9}\) for all \(i\) and

\[
\theta(T_7) \tau = \frac{1}{3} \omega_7, \quad \theta(T_3) \tau = \frac{1}{3} \omega_3, \quad \theta(T_4) \tau = \frac{1}{3} \omega_4. \quad (14)
\]

According to the structural results given in Sect. 2, each ERP \(G_2\)-structure with \(\dim \mathfrak{n} = 4\) is therefore determined by an abelian subalgebra \(\mathfrak{a} \subseteq \text{sym}_0(4)\) endowed with an orthogonal basis \(\{A, B, C\}\) such that \(|A|^2 = |B|^2 = |C|^2 = \frac{1}{3}\) and

\[
A = E + T_7, \quad B = F + T_3, \quad C = G + T_4,
\]

for some uniquely determined \(E, F, G \in \mathfrak{g}\). Thus, \(\{E, F, G\}\) is orthogonal and \(|E|^2 = |F|^2 = |G|^2 = \frac{2}{9}\). The only known example in this case is \(\mu_j := (0, A, B, C)\) (see [15, Example 5.4] and [14, Example 4.7]), where
In particular, $G_{\mu_j}$ is the only unimodular Lie group admitting an ERP $G_2$-structure by [11, Theorem 6.7] (see also [15, Proposition 5.2]). In the following proposition, we show in addition that $G_{\mu_j}$ admits exactly one ERP $G_2$-structure up to equivariant equivalence and scaling (cf. [5, Remark 5.3]).

**Proposition 3.1** $(G_{\mu_j}, \varphi)$ is the unique ERP $G_2$-structure up to equivariant equivalence and scaling among the class of unimodular Lie groups (or equivalently, on Lie groups with nilradical of dimension 4) endowed with a $G_2$-structure.

**Proof** Let $a$ denote the abelian subalgebra associated with $\mu_j$ and assume that $\bar{a} \subset \text{sym}_0(4)$ is another ERP abelian subalgebra with corresponding basis

$$\{ \bar{A} = \bar{E} + T_7, \bar{B} = \bar{F} + T_3, \bar{C} = \bar{G} + T_4 \}.$$

It is well known that there exists $h_2 \in \text{SO}(4)$ such that $h_2 A h_2^{-1} = \bar{a}$. If $h_1 \in \text{SO}(3)$ is defined in terms of the above basis by $h_2 A h_2^{-1} = h_1 \bar{A}$ and so on, then $\bar{\mu} = h \cdot \mu_j$, where $h$ is as in (11). Thus, $h_j = I$ can be assumed up to equivariance equivalence, since there is an $u \in U_{g_1, r}$ (see [15, (10)]) such that $u|_{g_0} = h_1^{-1}$, hence

$$h_2 (E + T_7) h_2^{-1} = \bar{E} + T_7, \quad h_2 (F + T_3) h_2^{-1} = \bar{F} + T_3, \quad h_2 (G + T_4) h_2^{-1} = \bar{G} + T_4.$$

It follows from (12) and (13) that

$$\langle \bar{A} e_1, e_1 \rangle = a - \frac{1}{6}, \quad \langle \bar{A} e_2, e_2 \rangle = -a - \frac{1}{6}, \quad \text{for some } a \in \mathbb{R},$$

but since $\text{Spec}(\bar{A}) = \{-\frac{1}{6}, \frac{1}{2}\}$, we obtain that $a = 0$ and $e_1, e_2$ are both eigenvectors of $\bar{A}$ with eigenvalue $-\frac{1}{6}$. Thus, $\bar{E}$ has the form

$$\bar{E} = \frac{1}{6} \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ c & d \\ d & c \end{bmatrix}, \quad c, d \in \mathbb{R}, \quad c^2 + d^2 = \frac{1}{9},$$

and so there exists
such that \( u_2 E u_2^{-1} = E \) (see also (9)). Note that \( u_2 T_i u_2^{-1} = T_i \) for all \( i \). This allows us to assume that \( E = E \), up to equivariant equivalence. It now follows from (16) that \( h_2 \) commutes with \( A \) and so \( h_2 e_5 = \pm e_5 \), which implies that

\[
\overline{F} e_5 = -T_3 e_5 = -\frac{1}{6} e_2, \quad \overline{G} e_5 = -T_4 e_5 = -\frac{1}{6} e_1.
\]

By (12), the matrices \( \overline{F} \) and \( \overline{G} \) considerably simplify as follows:

\[
\overline{F} = \frac{1}{6} \begin{bmatrix}
a & b & 0 & 1 \\
b - a & -1 & 0 \\
0 & -1 & 0 & 0 \\
1 & 0 & 0 & 0
\end{bmatrix}, \quad \overline{G} = \frac{1}{6} \begin{bmatrix}
a' & b' & -1 & 0 \\
b' - a' & 0 & -1 \\
-1 & 0 & 0 & 0 \\
0 & -1 & 0 & 0
\end{bmatrix},
\]

and condition \( [\overline{B}, \overline{C}] = 0 \) gives that

\[
\overline{B} = \frac{1}{6} \begin{bmatrix}
a & b & 0 & 2 \\
b - a & 0 & 0 \\
0 & 0 & 0 & 0 \\
2 & 0 & 0 & 0
\end{bmatrix}, \quad \overline{C} = \frac{1}{6} \begin{bmatrix}
-b & a & 0 & 0 \\
0 & a & b & -2 \\
0 & 0 & 0 & 0 \\
0 & -2 & 0 & 0
\end{bmatrix}, \quad a^2 + b^2 = 2.
\]

Notice that \( \mu_j \) corresponds to \( a = 0, b = -\sqrt{2} \). Finally, recall from (16) that \( \overline{B} \) and \( \overline{C} \) are, respectively, conjugate to \( B \) and \( C \), so \( \text{tr} \overline{B} = \frac{1}{18} a \) cannot depend on \( a \), so \( a = 0 \), and since if \( a = 0 \), then \( \text{tr} \overline{C} = \pm \frac{\sqrt{2}}{18} \) depending on whether \( b = \pm \sqrt{2} \), we obtain that \( \mu = \mu_j \), concluding the proof. \( \square \)

### 4 Case \( \dim n = 5 \)

We classify in this section, up to equivariant equivalence and scaling, all left-invariant ERP \( G_2 \)-structures on Lie groups with nilradical of dimension 5. There are only two known examples in this case, which we next describe.

From now on, all matrices in \( \mathfrak{gl}_6(\mathbb{R}) \) will be written in terms of the orthogonal basis of \( \Lambda^2 \mathfrak{g}_1^* \) defined by

\[
\mathcal{B} := \{ \tau, \overline{\omega}_3, \overline{\omega}_4, \omega_7, \omega_3, \omega_4 \}, \quad \text{where} \quad \overline{\omega}_3 := e^{26} + e^{15}, \quad \overline{\omega}_4 := e^{16} - e^{25}.
\]

see (6) for the definition of the \( \omega_i \)'s. Note that each element in \( \mathcal{B} \) has norm equal to \( \sqrt{2} \).

**Example 4.1** [15, Example 5.5] Let \( (G_{\mu_{M2}}, \varphi) \) be the ERP \( G_2 \)-structure with Lie bracket \( \mu_{M2} \) given by

\[
(A_{1})_{M2} = \frac{1}{2} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad A_{M2} = \frac{1}{2} \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}, \quad B_{M2} = \frac{1}{6} \begin{bmatrix} -1 & 2 & 1 \\ 2 & 1 & 0 \end{bmatrix}, \quad C_{M2} = \frac{1}{2} \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & -1 & 0 \end{bmatrix}.
\]

It follows by an easy computation that
The above two examples are the only ERP

Example 4.2 [15, Example 5.8] The ERP $G_2$-structure $(G_{\mu M}, \varphi)$ has Lie bracket $\mu M$ given by

$$(\mu M)_{A_1} = \frac{1}{6} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \mu M_{A_2} = \frac{1}{12} \begin{bmatrix} -2 & 0 & -\sqrt{2} \\ 0 & -2 & 0 \\ -\sqrt{2} & 0 & 2 \end{bmatrix},$$

$$\mu M_{B_1} = \frac{1}{6} \begin{bmatrix} 0 & \sqrt{2} & 0 \\ \sqrt{2} & 0 & 1 \\ 0 & 1 & -\sqrt{2} \end{bmatrix}, \quad \mu M_{C_1} = \frac{1}{12} \begin{bmatrix} -\sqrt{2} & 0 & 2-\sqrt{6} \\ 0 & \sqrt{2} & 0 \\ 2+\sqrt{6} & 0 & -2+\sqrt{6} \end{bmatrix}.$$
Proposition 4.3 Every Lie group endowed with a left-invariant ERP $G_2$-structure and having five-dimensional nilradical is equivariantly equivalent (up to scaling) to either $(G_{\mu_{32}}, \varphi)$ or $(G_{\mu_{33}}, \varphi)$.

Proof We can assume that the Lie bracket $\mu$ of an ERP $G_2$-structure $(G_{\mu}, \varphi)$ is given by 

$$\mu = (A_1, A, B, C),$$

where $A$, $B$ are symmetric, $C$ is nilpotent and $A_1 = \begin{bmatrix} 0 & \delta \\ \delta & 0 \end{bmatrix}$, $\delta > 0$ (see Sect. 2). Since $\text{tr}A = \text{tr}B = \text{tr}C = 0$, the matrices $\theta(A), \theta(B), \theta(C)$ have the form as in (18) with respect to the basis $B$ given in (17).

It follows from Theorem 2.1 (ii) and (iii) that $3\theta(A), 3\theta(B)$ and $3\theta(C)$ are, respectively, given by

$$\begin{bmatrix} 1 & 0 & 0 \\ a_{24} & a_{23} & a_{26} \\ a_{34} & a_{33} & a_{36} \end{bmatrix}, 
\begin{bmatrix} 0 & 1 & 0 \\ b_{24} & b_{23} & b_{26} \\ b_{34} & b_{33} & b_{36} \end{bmatrix}, 
\begin{bmatrix} 0 & 0 & 0 \\ 0 & c_{23} & c_{25} \\ 0 & 0 & c_{34} \end{bmatrix}.$$

for some $a_{ij}, b_{ij}, c_{ij} \in \mathbb{R}$. Now, since $A_1, A, B, C$ satisfy Jacobi and $\theta$ is a representation, we know that if

$$R := [\theta(A), \theta(C)] - \delta \theta(C), 
S := [\theta(A), \theta(B)], 
T := [\theta(B), \theta(C)],$$

then $R = S = T = 0$. From the first columns of $R, S$ and $T$, we easily obtain that

$$c_{25} = b_{26}, c_{35} = b_{36}, b_{24} = a_{25}, b_{34} = a_{35}, c_{24} = a_{26}, c_{34} = a_{36}, c_{45} = 0.$$

We list below the other null expressions provided by $R = S = T = 0$ that will be needed next:

$$T(4..6, 2..3) = \frac{1}{9} \begin{bmatrix} -3b_{26} & -a_{35}c_{23} \\ b_{35} & a_{35}c_{23} \end{bmatrix} = 0$$

(19)

$$R(4..6, 2..3) = \frac{1}{9} \begin{bmatrix} -6a_{26} & -a_{34}c_{23} \\ b_{35} & a_{34}c_{23} \end{bmatrix} = 0$$

(20)

$$T(6, 5) = \frac{1}{9} (b_{26}^2 + b_{36}^2 + b_{25}^2 + b_{35}^2 + b_{25}a_{24} + b_{35}a_{34} - 1) = 0$$

(21)

$$R(6, 4) = \frac{1}{9} (a_{26}^2 + a_{36}^2 + a_{24}^2 + a_{34}^2 + a_{24}b_{24} + a_{34}b_{35} + 9\delta^2 - 1) = 0.$$  

(22)

The rest of the proof will be divided into two cases, and each case will lead us to one of the known examples.

We first assume that $c_{23} = 0$. It follows from (19), (20) and $\delta > 0$ that

$$b_{26} = b_{36} = a_{25} = a_{35} = a_{26} = a_{36} = 0, b_{25} = -2a_{24}, b_{35} = -2a_{34}.$$
From (21) and (22), we have that \( a_{24}^2 + a_{34}^2 = \frac{1}{2} \) and \( \delta = \frac{\sqrt{6}}{6} \); hence, \( \theta(A), \theta(B) \) and \( \theta(C) \) are, respectively, equal to

\[
\begin{bmatrix}
1 & 0 & 0 \\
0 & a_{24} & 0 \\
0 & a_{34} & 0
\end{bmatrix}
\begin{bmatrix}
1/3 \\
1/3 \\
0
\end{bmatrix}
\begin{bmatrix}
0 & 1 & 0 \\
0 & -2a_{24} & 0 \\
0 & -2a_{34} & 0
\end{bmatrix}
\begin{bmatrix}
0 & 0 & 2 \\
0 & 0 & 2a_{24} \\
0 & 0 & 2a_{34}
\end{bmatrix}
\]

where \( a_{24}^2 + a_{34}^2 = \frac{1}{2} \). Note that when \( a_{24} = 0 \) and \( a_{34} = \frac{\sqrt{3}}{2} \), we obtain \( \theta(A_{M3}), \theta(B_{M3}) \) and \( \theta(C_{M3}) \) from Example 4.2, and by acting with \( h := \begin{bmatrix} I \\ u \\ u^{-1} \end{bmatrix} \in U_6 \) as in (9), we have that \( \theta(h_1A_{M3}h_1^{-1}), \theta(h_1B_{M3}h_1^{-1}) \) and \( \theta(h_1C_{M3}h_1^{-1}) \) are given by

\[
\begin{bmatrix}
2 & 0 & 0 \\
\sqrt{2s} & 0 & 0 \\
\sqrt{2c} & 0 & 0
\end{bmatrix}
\begin{bmatrix}
0 & 1 & 0 \\
0 & -\sqrt{2s} & 0 \\
0 & -\sqrt{2c} & 0
\end{bmatrix}
\begin{bmatrix}
0 & 0 & 2 \\
0 & 0 & \sqrt{2s} \\
0 & 0 & \sqrt{2c}
\end{bmatrix}
\]

where \( h_1 := \begin{bmatrix} u \\ u^{-1} \end{bmatrix} \) and \( \theta(h_1) = \begin{bmatrix} c & s \\ -s & c \\ 1 \\ 1 \end{bmatrix} \), \( c^2 + s^2 = 1 \). This implies that we have covered all the examples with \( c_{23} = 0 \). In other words, if \( c_{23} = 0 \), then the ERP \( G_2 \)-structure \( (A_1, A, B, C) \) is equivariantly equivalent to \( \mu_{M3} \).

Suppose now that \( c_{23} \neq 0 \). Equation (19) implies that \( b_{25} = b_{35} = 0 \) and

\[
a_{25} = \frac{3b_{36\delta}}{c_{23}}, \quad a_{35} = -\frac{3b_{36\delta}}{c_{23}}, \quad b_{26}(c_{23}^2 - 9\delta^2) = 0, \quad b_{36}(c_{23}^2 - 9\delta^2) = 0.
\]

By (21), \( b_{26}^2 + b_{36}^2 = 1 \); thus, \( c_{23}^2 = 9\delta^2 \). On the other hand, from (20) we have that

\[
a_{26} = -\frac{6a_{36\delta}}{c_{23}}, \quad a_{36} = \frac{6a_{36\delta}}{c_{23}}, \quad a_{24}(c_{23}^2 - 36\delta^2) = 0, \quad a_{34}(c_{23}^2 - 36\delta^2) = 0.
\]

Hence, \( a_{24} = a_{34} = 0 \) and so (22) gives that \( 9\delta^2 = 1 \). The matrices \( \theta(A), \theta(B) \) and \( \theta(C) \) therefore read as follows:

\[
\begin{bmatrix}
1 & 0 & 0 \\
0 & c_{23}b_{36} & 0 \\
0 & -c_{23}b_{26} & 0
\end{bmatrix}
\begin{bmatrix}
1/3 \\
1/3 \\
0
\end{bmatrix}
\begin{bmatrix}
0 & 1 & 0 \\
0 & c_{23}b_{36} & 0 \\
0 & -c_{23}b_{26} & 0
\end{bmatrix}
\begin{bmatrix}
0 & 0 & 1 \\
0 & b_{26} & b_{36} \\
1 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
0 & 0 & 1 \\
0 & b_{26} & b_{36} \\
1 & 0 & 0
\end{bmatrix}
\]

\[
\begin{bmatrix}
1 & 0 & 0 \\
0 & c_{23}b_{36} & 0 \\
0 & -c_{23}b_{26} & 0
\end{bmatrix}
\begin{bmatrix}
1/3 \\
1/3 \\
0
\end{bmatrix}
\begin{bmatrix}
0 & 1 & 0 \\
0 & -c_{23}b_{36} & 0 \\
0 & b_{26} & b_{36}
\end{bmatrix}
\begin{bmatrix}
0 & 0 & 1 \\
0 & b_{26} & b_{36} \\
1 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
0 & 0 & 1 \\
0 & b_{26} & b_{36} \\
1 & 0 & 0
\end{bmatrix}
\]

\[
\begin{bmatrix}
1 & 0 & 0 \\
0 & c_{23}b_{36} & 0 \\
0 & -c_{23}b_{26} & 0
\end{bmatrix}
\begin{bmatrix}
1/3 \\
1/3 \\
0
\end{bmatrix}
\begin{bmatrix}
0 & 1 & 0 \\
0 & c_{23}b_{36} & 0 \\
0 & -c_{23}b_{26} & 0
\end{bmatrix}
\begin{bmatrix}
0 & 0 & 1 \\
0 & b_{26} & b_{36} \\
1 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
0 & 0 & 1 \\
0 & b_{26} & b_{36} \\
1 & 0 & 0
\end{bmatrix}
\]

\[
\begin{bmatrix}
1 & 0 & 0 \\
0 & c_{23}b_{36} & 0 \\
0 & -c_{23}b_{26} & 0
\end{bmatrix}
\begin{bmatrix}
1/3 \\
1/3 \\
0
\end{bmatrix}
\begin{bmatrix}
0 & 1 & 0 \\
0 & -c_{23}b_{36} & 0 \\
0 & b_{26} & b_{36}
\end{bmatrix}
\begin{bmatrix}
0 & 0 & 1 \\
0 & b_{26} & b_{36} \\
1 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
0 & 0 & 1 \\
0 & b_{26} & b_{36} \\
1 & 0 & 0
\end{bmatrix}
\]
where \( a_{25}^2 + a_{35}^2 = 1, \delta = \frac{1}{2}, \) and \( c_{23} = \pm 1. \) Note that when \( a_{25} = -1, a_{35} = 0 \) and \( c_{23} = -1, \) we obtain \( \theta(A_{M_2}), \theta(B_{M_2}), \theta(C_{M_2}) \) from Example 4.1, and if we act with \( h \in U_0 \) as in the case when \( c_{23} = 0, \) then \( \theta(h_1A_{M_2}h_1^{-1}), \theta(h_1B_{M_2}h_1^{-1}) \) and \( \theta(h_1C_{M_2}h_1^{-1}) \) are, respectively, as follows:

\[
\begin{bmatrix}
1 & 0 & 0 \\
0 & -c & 0 \\
0 & s & 0
\end{bmatrix}
\]

\[
\begin{bmatrix}
0 & 1 & 0 \\
-c & 0 & s \\
s & 0 & c
\end{bmatrix}
\]

\[
\begin{bmatrix}
0 & 0 & 1 \\
1 & 0 & c \\
0 & 0 & -1
\end{bmatrix}
\]

The above matrices are therefore covered only when \( c_{23} = -1. \) To reach the cases with \( c_{23} = 1, \) we have to act with

\[
\tilde{h} := \begin{bmatrix}
1 \\
-l \\
-u^{-1}
\end{bmatrix} \in U_0, \quad \tilde{h}_1 := \begin{bmatrix}
u \\
-u^{-1}
\end{bmatrix}, \quad \theta(\tilde{h}_1) = \begin{bmatrix}
1 & c & s \\
-s & c & 0 \\
0 & 0 & 1
\end{bmatrix}
\]

\[
c^2 + s^2 = 1,
\]

to obtain that \( \theta(\tilde{h}_1A_{M_2}\tilde{h}_1^{-1}), \theta(\tilde{h}_1B_{M_2}\tilde{h}_1^{-1}) \) and \( \theta(\tilde{h}_1C_{M_2}\tilde{h}_1^{-1}) \) are, respectively, equal to

\[
\begin{bmatrix}
1 & 0 & 0 \\
0 & c & 0 \\
0 & -s & 0
\end{bmatrix}
\]

\[
\begin{bmatrix}
0 & 1 & 0 \\
-c & 0 & s \\
s & 0 & c
\end{bmatrix}
\]

\[
\begin{bmatrix}
0 & 0 & 1 \\
1 & 0 & s \\
0 & s & 0
\end{bmatrix}
\]

Hence if \( c_{23} \neq 0, \) then \( (A_1, A, B, C) \) is equivariant equivalent to \( \mu_{M_2}, \) which completes the proof of the proposition. \( \square \)

### 5 Case dim \( n = 6 \)

Just as in Sect. 4, we prove here that the two known examples of ERP \( G_2 \)-structures on Lie groups are actually the only ones with a six-dimensional nilradical, up to equivalence and scaling. However, up to equivariant equivalence, a 2-parameter family around one of these examples must be added to complete the classification.

**Example 5.1** (see [15, Example 5.7] and [6, Example 1]) Let \( \mu_B \) be the ERP \( G_2 \)-structure given by

\[
\begin{bmatrix}
1 & 0 & 0 \\
0 & c & 0 \\
0 & -s & 0
\end{bmatrix}
\]

\[
\begin{bmatrix}
0 & 1 & 0 \\
-c & 0 & s \\
s & 0 & c
\end{bmatrix}
\]

\[
\begin{bmatrix}
0 & 0 & 1 \\
1 & 0 & s \\
0 & s & 0
\end{bmatrix}
\]
\[(A_1)_B = \frac{1}{3} \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad A_B = \frac{1}{6} \begin{bmatrix} -1 & -1 \\ -1 & 1 \end{bmatrix}, \quad B_B = \frac{1}{3} \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \quad C_B = \frac{1}{3} \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 \end{bmatrix}.\]

from which follows that
\[\theta(A_B) = \frac{1}{3} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \theta(B_B) = \frac{1}{3} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \theta(C_B) = \frac{1}{3} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}.\]

Example 5.2 [11, Example 6.4] For each pair \(r, t \in \mathbb{R}\), let \(\mu_{rt}\) denote the \(G_2\)-structure given by
\[(A_1)_{rt} = \frac{1}{3} \begin{bmatrix} 1 & -r \\ r & 1 \end{bmatrix}, \quad A_{rt} = \frac{1}{6} \begin{bmatrix} -1 & -2t \\ 2t & 1 \\ 2(r + t) & -1 \end{bmatrix}, \quad B_{rt} = \frac{1}{3} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \quad C_{rt} = \frac{1}{3} \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \end{bmatrix}.\]

Note that when \(r = t = 0\), one recovers the example \(\mu_B\) above. It follows from [15, Proposition 3.5] that \((G_{\mu_r}, \varphi)\) is equivalent to \((G_{\mu_B}, \varphi)\) and consequently ERP for all \(r, t \in \mathbb{R}\). Furthermore, since
\[\mathbb{R}^* \text{Spec(ad } \mu_{rt} e_7|_B) = \mathbb{R}^* \{\frac{1}{3} + ir, \frac{1}{3} + ir, -\frac{1}{6} + 2ir, -\frac{1}{6} + ir, \frac{1}{6} - i(r + t), \frac{1}{6} - i(r + t)\}\]
is an isomorphism invariant, one obtains that the family of Lie algebras \(\{\mu_{rt} : r, t \in \mathbb{R}\}\) is pairwise non-isomorphic. In particular, the family of \(G_2\)-structures \(\{(G_{\mu_r}, \varphi) : r, t \in \mathbb{R}\}\) is pairwise non-equivariantly equivalent.

It is straightforward to check that \(\theta(A_{\mu_r}), \theta(B_{\mu_r})\) and \(\theta(C_{\mu_r})\) are, respectively, given by
\[
\begin{bmatrix}
\frac{1}{3} & 0 & 0 & r + 2t & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
-\frac{1}{3} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -r & 0 & 0 \\
0 & 0 & 0 & r & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
\end{bmatrix},
\begin{bmatrix}
\frac{1}{3} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
\end{bmatrix},
\begin{bmatrix}
\frac{1}{3} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
\end{bmatrix}.
\]

Example 5.3 [15, Example 5.8]) Consider the ERP \(G_2\)-structure \(\mu_{M1}\), where
\[ (A_1)_{M_1} = \frac{1}{\sqrt{30}} \begin{bmatrix} \sqrt{30} & 0 \\ 0 & 2\sqrt{30} \end{bmatrix}, \quad A_{M_1} = \frac{1}{60} \begin{bmatrix} -10 - \sqrt{30} & 0 & -2\sqrt{5} & 0 \\ 0 & -10 + \sqrt{30} & 0 & -2\sqrt{5} \\ -2\sqrt{5} & 0 & 10 - \sqrt{30} & 0 \\ 0 & -2\sqrt{5} & 0 & 10 + \sqrt{30} \end{bmatrix} \]

\[ B_{M_1} = \frac{1}{\sqrt{30}} \begin{bmatrix} 0 & -\sqrt{5} & 0 & 5 - \sqrt{30} \\ 5\sqrt{5} & 0 & 5 & 0 \\ 0 & 5 + \sqrt{30} & 0 & \sqrt{5} \\ 5 & 0 & -5\sqrt{5} & 0 \end{bmatrix}, \quad C_{M_1} = \frac{1}{50} \begin{bmatrix} -\sqrt{5} & 0 & 5 - \sqrt{30} & 0 \\ 0 & \sqrt{5} & 0 & -5 + \sqrt{30} \\ 5 + \sqrt{30} & 0 & \sqrt{5} & 0 \\ 0 & -5 - \sqrt{30} & 0 & -\sqrt{5} \end{bmatrix} \]

It easily follows that \( \theta(A_{M_1}), \theta(B_{M_1}) \) and \( \theta(C_{M_1}) \) are, respectively, equal to

\[
\begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & -\sqrt{30} & 0 & 50 \\
2\sqrt{5} & 0 & 0 & 0 \\
0 & -\sqrt{30} & 0 & 0
\end{bmatrix}
\begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & -\sqrt{30} & 0 & 0 \\
6\sqrt{5} & 0 & 0 & 0 \\
0 & -\sqrt{30} & 0 & 0
\end{bmatrix}
\begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & -\sqrt{30} & 0 & 0 \\
0 & -4\sqrt{5} & 0 & 0 \\
0 & -\sqrt{30} & 0 & 0
\end{bmatrix}
\]

We are now ready to prove the main result in this section.

**Proposition 5.4** Any Lie group endowed with a left-invariant ERP \( G_2 \)-structure and having a six-dimensional nilradical is equivariantly equivalent (up to scaling) to either \( (G_{\mu_{M_1}}, \varphi) \) or some \( (G_{\mu_{t}}, \varphi), r, t \in \mathbb{R} \).

**Proof** It follows from Sect. 2 that we can assume that the Lie bracket \( \mu \) of an ERP \( G_2 \)-structure \((G, \varphi)\) is given by \( \mu = (A_1, A, B, C) \), where \( A_1 \) and \( A \) are normal and \( B \) and \( C \) nilpotent.

We first consider the case when \( A_1 \) and \( A \) are symmetric, so it can be assumed that \( A_1 = \begin{bmatrix} \alpha & 0 \\ 0 & \delta \end{bmatrix} \), where \( \alpha + \delta > 0 \) and \( \delta \geq \alpha \) (see Sect. 2). In much the same way as in the proof of Proposition 4.3, we first compute the form of \( \theta(A), \theta(B) \) and \( \theta(C) \) by applying the conditions provided by Theorem 2.1, (i) and (ii). From the nullity of the first column of each of the Jacobi condition matrices

\[ R := [\theta(A), \theta(B)] - \alpha \theta(B) = 0, \quad S := [\theta(A), \theta(C)] - \delta \theta(C) = 0, \quad T := [\theta(B), \theta(C)] = 0, \]

we obtain that

\[
\begin{align*}
c_{25} &= b_{26}, & c_{35} &= b_{36}, & c_{24} &= a_{26}, & c_{34} &= a_{36}, \\
b_{24} &= a_{25}, & b_{34} &= a_{35}, & b_{45} &= 3\alpha, & c_{45} &= b_{46} = 0,
\end{align*}
\]

and so the matrices \( 3\theta(A), 3\theta(B) \) and \( 3\theta(C) \) are, respectively, given by

\[
\begin{bmatrix}
1 & 0 & 0 \\
0 & 0 & 0 \\
0 & a_{24} & a_{25} \\
0 & a_{25} & a_{35} \\
0 & a_{26} & a_{36}
\end{bmatrix}
\begin{bmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & a_{24} & a_{25} \\
0 & a_{25} & a_{35} \\
1 & a_{26} & a_{36}
\end{bmatrix}
\begin{bmatrix}
0 & 0 & 1 \\
0 & 0 & 0 \\
-3a & 0 & 0 \\
0 & 0 & b_{46} \\
1 & -3a & 0
\end{bmatrix}
\begin{bmatrix}
0 & 0 & 1 \\
0 & 0 & 0 \\
-3a & 0 & 0 \\
0 & 0 & b_{46} \\
1 & -(a_{24} + b_{23}) & -3\delta
\end{bmatrix}
\]

\[ \Box \]
Note that \( b_{23} \) and \( \alpha \) cannot simultaneously vanish, since in that case \( \theta(B) \) and so \( B \) would be symmetric, which is a contradiction as \( B \) is nilpotent.

We write below the remaining expressions given by the nullity of \( R, S, T \) that are needed in the proof:

\[
S[4..5, 2..3] = \frac{1}{9} \begin{bmatrix}
-6\delta a_{26} - a_{34}c_{23} & -6\delta a_{36} + a_{24}c_{23} \\
-3\delta b_{26} - a_{35}c_{23} & -3\delta b_{36} + a_{25}c_{23}
\end{bmatrix} = 0,
\tag{24}
\]

\[
R[4..5, 2..3] = \frac{1}{9} \begin{bmatrix}
-6\alpha a_{25} - a_{34}b_{23} & -6\alpha a_{35} + a_{24}b_{23} \\
3\alpha(a_{24} - b_{25}) - a_{35}b_{23} & 3\alpha(a_{34} - b_{35}) + a_{23}b_{23}
\end{bmatrix} = 0,
\tag{25}
\]

\[
R[6, 4] = \frac{1}{9}(a_{25}a_{26} + a_{35}a_{36} - a_{24}b_{26} - a_{36}b_{36}) = 0,
\tag{26}
\]

\[
R[3, 2] = \frac{1}{9}(a_{34}a_{25} + a_{35}b_{25} + a_{36}b_{26} - a_{24}a_{35} - a_{25}b_{35} - a_{36}b_{36} + 3ab_{23}) = 0.
\tag{27}
\]

\[
R[5, 4] = \frac{1}{9}(a_{25}^2 + a_{35}^2 - b_{25}a_{24} - b_{35}a_{34} - 1 + 9)\alpha^2 = 0,
\tag{28}
\]

\[
T[5, 2..3] = \frac{1}{9} \begin{bmatrix}
b_{23}b_{36} - b_{35}c_{23} - 3a_{26}\alpha & -b_{23}b_{26} + b_{25}c_{23} & -3a_{36}\alpha
\end{bmatrix} = 0,
\tag{29}
\]

\[
S[6, 2..3] = \frac{1}{9} \begin{bmatrix}
3\delta(2a_{24} + b_{25}) - a_{36}c_{23} & 3\delta(2a_{34} + b_{35}) + a_{26}c_{23}
\end{bmatrix} = 0,
\tag{30}
\]

\[
T[6, 2..3] = \frac{1}{9} \begin{bmatrix}
-2b_{23}(a_{34} + b_{35}) + 3a_{25}\delta - b_{36}c_{23} & 2b_{23}(a_{24} + b_{25}) + 3a_{35}\delta + b_{26}c_{23}
\end{bmatrix} = 0.
\tag{31}
\]

The proof is divided into two steps, depending on the value of \( c_{23} \). The case when \( c_{23} \neq 0 \) leads to a contradiction, and when \( c_{23} = 0 \), the value of \( b_{23} \) determines if the ERP \( G_2 \)-structure \( \mu \) is either equivariant equivalent to \( \mu_B \) or \( \mu_M \).

We first assume \( c_{23} \neq 0 \), so from (24),

\[
a_{34} = -\frac{6\delta}{c_{23}} a_{26}, \quad a_{24} = \frac{6\delta}{c_{23}} a_{36}, \quad a_{35} = -\frac{3\delta}{c_{23}} b_{26}, \quad a_{25} = \frac{3\delta}{c_{23}} b_{26}.
\]

Replacing these values in (25) and (26), and keeping in mind that \( b_{23} \) and \( \alpha \) cannot simultaneously vanish, we obtain that \( a_{26} = a_{36} = b_{26} = b_{36} = 0 \). Consequently, we have that \( 0 = S[3, 2] = \frac{1}{9}\delta c_{23} \), which is a contradiction.

Assume now that \( c_{23} = 0 \); thus, \( a_{26} = a_{36} = b_{26} = b_{36} = 0 \) and by (24) and from (30), we therefore obtain that \( b_{25} = -2a_{24} \) and \( b_{35} = -2a_{34} \). If \( b_{23} = 0 \), then \( \alpha \neq 0 \), and it follows from (25) that \( a_{24} = a_{34} = a_{25} = a_{35} = 0 \) and

\[
0 = S[6, 4] = -\frac{1}{9} + \delta^2, \quad 0 = T[6, 5] = -\frac{1}{9} + \alpha\delta.
\]

Hence, \( \delta = \alpha = \frac{1}{3} \) and so \( \theta(A) = \theta(A_B), \theta(B) = \theta(B_B), \theta(C) = \theta(C_B) \), that is, we obtain that \( \mu = \mu_B \). On the contrary, if \( b_{23} \neq 0 \), then by (25),
On the other hand, if we act on but at least one of them is not symmetric, so the condition:

\[ \frac{9a}{b_2} a_{34}, \quad \frac{9a}{b_2} a_{24}; \]

therefore, from (25) and (27), it follows that

\[ a_{24}^2 + a_{34} = 6a^2, \quad b_{23}^2 = 54\alpha^2, \]

but (28) and the fact that \( \alpha + \delta > 0 \) imply that \( \alpha^2 = \frac{1}{30} \) and by (31),

\[ \alpha = \frac{\sqrt{30}}{30}, \quad \delta = 2\frac{\sqrt{30}}{30}, \quad b_{23} = \frac{3}{5} e\sqrt{5}, \quad e := \pm 1, \]

from which follows that

\[
\theta(A) = \begin{bmatrix}
\frac{1}{30} & & & \\
10a_{24} & 5\sqrt{6}e a_{34} & 0 \\
0 & 5\sqrt{6}e a_{24} & 0 \\
0 & 0 & 0
\end{bmatrix},
\]

\[
\theta(B) = \begin{bmatrix}
\frac{1}{30} & & & \\
\frac{6\sqrt{5}e}{5} & -\frac{5\sqrt{6}e a_{34}}{5} & 0 \\
-\frac{20a_{24}}{\sqrt{30}} & -20a_{34} & -\frac{\sqrt{30}}{5} \\
0 & 0 & 0
\end{bmatrix},
\]

\[
\theta(C) = \begin{bmatrix}
\frac{1}{30} & & & \\
0 & 0 & 5 \\
0 & 0 & 5a_{14} \\
5 & 5a_{24} & 5a_{34} & -\frac{\sqrt{30}}{5}
\end{bmatrix},
\]

Let us denote by \( \theta(A)(e, a_{24}, a_{34}), \theta(B)(e, a_{24}, a_{34}) \) and \( \theta(C)(e, a_{24}, a_{34}) \) the above matrices. Note that

\[ \theta(A)\left(1, \frac{1}{\sqrt{5}}\right) = \theta(A_{M1}), \quad \theta(B)\left(1, \frac{1}{\sqrt{5}}\right) = \theta(B_{M1}), \quad \theta(C)\left(1, \frac{1}{\sqrt{5}}\right) = \theta(C_{M1}). \]

If we act on \( \mu_{M1} \) with \( h \in U_0 \) as in (9), then we obtain the equivariantly equivalent \( G_2 \)-structure for which the matrices \( \theta(h_1 A_{M1} h_1^{-1}), \theta(h_1 B_{M1} h_1^{-1}) \) and \( \theta(h_1 C_{M1} h_1^{-1}) \) are, respectively, given by

\[ \theta(A)\left(1, \frac{s}{\sqrt{5}}, \frac{c}{\sqrt{5}}\right), \quad \theta(B)\left(1, \frac{s}{\sqrt{5}}, \frac{c}{\sqrt{5}}\right), \quad \theta(C)\left(1, \frac{s}{\sqrt{5}}, \frac{c}{\sqrt{5}}\right). \]

On the other hand, if we act on \( \mu_{M1} \) with \( \tilde{h} \in U_0 \) as in (9), then we obtain

\[ \theta(A)\left(-1, \frac{s}{\sqrt{5}}, \frac{c}{\sqrt{5}}\right), \quad \theta(B)\left(-1, \frac{s}{\sqrt{5}}, \frac{c}{\sqrt{5}}\right), \quad \theta(C)\left(-1, \frac{s}{\sqrt{5}}, \frac{c}{\sqrt{5}}\right). \]

Hence, we reach all the above matrices, from which follows that if \( c_{23} = 0 \) and \( b_{23} \neq 0 \), then \( \mu \) is equivariantly equivalent to \( \mu_{M1} \).

Secondly, we consider the case when \( \mu = (A_1, A, B, C) \), where \( A_1 \) and \( A \) are both normal, but at least one of them is not symmetric, so \( A_1 = \begin{bmatrix} \gamma & \beta \\ -\beta & \delta \end{bmatrix} \), where either \( \beta = 0 \) or \( \alpha = \delta \) (see Sect. 2). In this case, by using Theorem 2.1, (i) and (ii) and the equations provided by only the first columns of the following matrix equations determined by the Jacobi condition:
[θ(A), θ(B)] = αθ(B) − βθ(C),  [θ(A), θ(C)] = βθ(B) + δθ(C),  [θ(B), θ(C)] = 0,

one obtains that all the conditions in (23) hold plus the following extra ones:

\[ a_{45} = 0, \quad a_{46} = 0, \quad a_{56} = 3β, \]

and so the matrices \(3θ(A), 3θ(B)\) and \(3θ(C)\) are, respectively, given by

\[
\begin{bmatrix}
1 & 0 & 0 \\
- a_{23} & a_{24} & a_{25} & a_{26} \\
0 & a_{25} & a_{35} & 3β \\
0 & a_{26} & a_{36} & -3β
\end{bmatrix}
\begin{bmatrix}
0 & 1 & 0 \\
- b_{23} & a_{25} & b_{25} & b_{26} \\
0 & a_{25} & -3a & b_{35} & b_{36} \\
0 & b_{26} & b_{36} & 0
\end{bmatrix}
\begin{bmatrix}
0 & 0 & 1 \\
- c_{23} & a_{26} & b_{26} & -(a_{24} + b_{25}) \\
0 & a_{26} & a_{36} & 3β \\
0 & b_{26} & b_{36} & 0
\end{bmatrix}
\]

From the following fact:

\[
θ(E) = \begin{bmatrix}
0 & 0 & a - b \\
- b - a & 0 & 0 \\
0 & a + b & -(a + b)
\end{bmatrix}
\]

we obtain that \(adeγ|_{\mathfrak{h}} = \tilde{A} + D\), where \(\tilde{A}\) and \(D\) are, respectively, the symmetric and skew-symmetric parts, given by

\[
\tilde{A} := \begin{bmatrix}
α & 0 \\
0 & δ \\
S(A)
\end{bmatrix}, \quad D := \frac{1}{6}
\begin{bmatrix}
0 & 6β \\
-6β & 0 \\
- a_{23} - 3β & a_{23} + 3β & 0
\end{bmatrix}
\]

and \(S(A)\) denotes the symmetric part of \(A\) (see (18)). Since \(D ∈ \text{Der}(\mathfrak{h}) \cap \mathfrak{su}(3)\) and commutes with \(adeγ|_{\mathfrak{h}}\) as \(A\) is normal, it follows from [15, Proposition 3.5] that \((A_1, A, B, C)\) is equivalent as a \(G_2\)-structure to \((S(A_1), S(A), B, C)\), which is therefore ERP. By the case worked out above, \((S(A_1), S(A), B, C)\) must be precisely \(μ_B\), since the other possibility, \(μ_{M_1}\) (up to equivariance equivalence), does not admit any derivation in \(\mathfrak{su}(3)\) (see Sect. 6). This implies that \(μ = μ_r\), where \(r = -3β\), \(t = \frac{1}{2}a_{23} + \frac{3}{2}β\), completing the proof of the proposition.

□

Remark 5.5 It can be directly shown that, up to equivalence, a Lie group with an ERP \(G_2\)-structure has to be completely solvable by only using the structure results obtained in [15] and the fact that they are solvsolitons (see [15, Corollary 1.2]). Indeed, it follows from Theorem 2.1, (ii) that \(θ(A), θ(B)\) and \(θ(C)\) have, respectively, the form
and according to [13, Theorem 4.8], $\theta(A)$ is always normal; in addition, $\theta(B)$ is normal if $\dim n = 4, 5$ and $\theta(C)$ must also be normal if $\dim n = 4$. A very easy computation therefore gives that $a_{45} = a_{46} = 0$, and in addition, $b_{45} = b_{56} = 0$ and $c_{46} = c_{56} = 0$, respectively. It is now straightforward to show that they must be symmetric in order to commute (which is mandatory by [13, Theorem 4.8]) and satisfy the Jacobi condition, and so $A$ and $B$ are symmetric if $\dim n = 5$ and they are all symmetric if $\dim n = 4$. In particular, the corresponding solvable Lie groups are completely solvable. In the case when $\dim n = 6$, one obtains that the ERP structure is equivalent to one on the completely solvable Lie group obtained by replacing $A$ with the symmetric part $S(A)$ of $A$, as shown at the end of the proof of Proposition 5.4.

6 Symmetries

We aim in this section to provide some insight on the symmetries of each of the six ERP $G_2$-structures obtained in the classification Theorem 1.1.

The isometry group of a left-invariant Riemannian metric on an $n$-dimensional Lie group can be very tricky to compute, even in the completely solvable case. It is not hard to see, however, that $\text{Iso}(G_\mu, \langle \cdot, \cdot \rangle) = KG_\mu$, where $K := \text{Iso}(G_\mu, \langle \cdot, \cdot \rangle)_e$ is the isotropy subgroup at the identity and $G_\mu$ also denotes the subgroup of left translations. The following conditions are also easily seen to be equivalent:

(i) $G_\mu$ is normal in $\text{Iso}(G_\mu, \langle \cdot, \cdot \rangle)$.

(ii) $\text{Iso}(G_\mu, \langle \cdot, \cdot \rangle) = K \ltimes G_\mu$.

(iii) $K = \text{Aut}(G_\mu) \cap \text{Iso}(G_\mu, \langle \cdot, \cdot \rangle)$, which is identified with the group

$$\text{Aut}(\mu) \cap \text{O}(n), \quad \text{O}(n) := \text{O}(g, \langle \cdot, \cdot \rangle),$$

of orthogonal automorphisms of the Lie algebra.

This is known to hold if $\mu$ is unimodular and completely solvable (see [12]). In any case, the subgroup of isometries of $(G_\mu, \langle \cdot, \cdot \rangle)$ given by

$$\left( \text{Aut}(\mu) \cap \text{O}(n) \right) \ltimes G_\mu$$

is always present and less difficult to compute.

On the other hand, given a Lie group $(G_\mu, \varphi)$ endowed with a left-invariant $G_2$-structure, we can also consider the subgroup of automorphisms of $(G_\mu, \varphi)$ given by:

$$\left( \text{Aut}(\mu) \cap G_2 \right) \ltimes G_\mu \subset \text{Aut}(G_\mu, \varphi) \subset \text{Iso}(G_\mu, \langle \cdot, \cdot \rangle),$$

and since $G_2 \subset \text{SO}(7)$ ($\langle \cdot, \cdot \rangle := \langle \cdot, \cdot \rangle_\varphi$), we obtain that
Note that \((\text{Aut}(\mu) \cap G_2) \ltimes G_\mu = \text{Aut}(G_\mu, \varphi)\) also holds in the unimodular completely solvable case.

In this light, we compute in what follows the two groups given in (32) for each of the ERP structures appearing in Theorem 1.1. It is straightforward to show that the Lie algebra \(\text{Der}(\mu) \cap \mathfrak{g}(7)\) of \(\text{Aut}(\mu) \cap O(7)\) is always zero except for \(\mu_B\) and \(\mu_r\), where it coincides with \(u_0\), the two-dimensional abelian Lie algebra of the Lie group \(U_0 \simeq S^1 \times S^1\) given in (8). This implies that the groups in (32) are all finite in the other four cases.

The results of our computations of \(G_2\)-automorphisms can be summarized as follows. All the matrices below are written in terms of the basis \(\{e_7, e_3, e_4, e_1, e_2, e_5, e_6\}\):

- \(\text{Aut}(\mu_B) \cap G_2 = \text{Aut}(\mu_r) \cap G_2 = U_0 \simeq S^1 \times S^1\).
- \(\text{Aut}(\mu_{M1}) \cap G_2 = \langle f_0 \rangle \simeq \mathbb{Z}_2\), where \(f_0 := \text{Dg}(1, 1, 1, -1, -1, -1, -1)\).
- \(\text{Aut}(\mu_{M2}) \cap G_2 = \langle f_0 \rangle \simeq \mathbb{Z}_2\).
- \(\text{Aut}(\mu_{M3}) \cap G_2 = \langle f_1 \rangle \simeq \mathbb{Z}_4\), where \(f_1|_{\mathbb{Z}_8} := \text{Dg}(1, -1, -1)\) and

\[
\begin{bmatrix}
0 & -1 \\
1 & 0 \\
0 & -1 \\
1 & 0
\end{bmatrix},
\]

- \(\text{Aut}(\mu_j) \cap G_2 \simeq \text{SL}_2(\mathbb{Z}_3)\), the binary tetrahedral group of order 24. Indeed, it is easy to check that this group has order 24, only one element of order 2 and none of order 12, a condition that characterizes \(\text{SL}_2(\mathbb{Z}_3)\) among the groups of order 24.

**Remark 6.1** From the original presentation of \((G_{\mu}, \varphi)\) as a homogeneous space \((G/K, \psi)\) endowed with a \(G\)-invariant \(G_2\)-structure \(\psi\) given in [6, Example 1], where

\[G/K = (\text{SL}_2(\mathbb{C}) \ltimes \mathbb{C}^2)/\text{SU}(2),\]

we obtain that \(\text{Aut}(G_{\mu}, \varphi)\) actually contains a six-dimensional subgroup isomorphic to \(\text{SL}_2(\mathbb{C})\) and that \(\text{SU}(2) \subset \text{Aut}(G_{\mu}, \varphi)\). Since \(\text{Aut}(\mu_B) \cap G_2 = S^1 \times S^1\), this shows that there are indeed automorphisms of \((G_{\mu}, \varphi)\) which are not compositions of Lie group automorphisms and left translations. We do not know if this is also the case for the other examples, except for the unimodular case \(\mu_j\) (cf. Corollary 6.2).

On the other hand, concerning isometries we have obtained the following:

- \(\text{Aut}(\mu_B) \cap O(7) = \langle f_2 \rangle \ltimes U_0 \simeq \mathbb{Z}_2 \ltimes (S^1 \times S^1)\), where \(f_2 := \text{Dg}(1, 1, -1, 1, -1, -1)\).
- \(\text{Aut}(\mu_r) \cap O(7) = U_0 \simeq S^1 \times S^1\).
- \(\text{Aut}(\mu_{M1}) \cap O(7) = \langle f_0, f_3 \rangle \simeq \mathbb{Z}_2 \times \mathbb{Z}_2\), where \(f_3 := \text{Dg}(1, -1, -1, -1, -1)\).
- \(\text{Aut}(\mu_{M2}) \cap O(7) = \langle f_0, f_2, f_4 \rangle \simeq \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2\), where \(f_4|_{\mathbb{Z}_8} := I\) and

\[
\begin{bmatrix}
1 & 0 & -1 \\
-1 & 0 & 1
\end{bmatrix},
\]
The classification of $\mathbf{G}_2$-structures on Lie groups

- Aut($\mu_{M3}) \cap O(7) = \langle f_3, f_5, f_6 \rangle \leq D_4 \times \mathbb{Z}^2$, where $D_4$ is the dihedral group of degree four (and order 8), $f_5|_{\mathfrak{g}_1} := Dg(1, -1, 1), f_6|_{\mathfrak{g}_1} := Dg(1, 1, -1)$ and

$$f_5|_{\mathfrak{g}_1} := \frac{1}{3} \begin{bmatrix}
0 & \sqrt{6} & 0 \\
-\sqrt{6} & 0 & -\sqrt{3} \\
-\sqrt{3} & 0 & \sqrt{6}
\end{bmatrix}, \quad f_6|_{\mathfrak{g}_1} := \frac{1}{3} \begin{bmatrix}
-\sqrt{6} & 0 & -\sqrt{3} \\
0 & -\sqrt{6} & 0 \\
-\sqrt{3} & 0 & \sqrt{6}
\end{bmatrix}. $$

Indeed, the generators satisfy the following relations:

$$f_3^2 = f_5^4 = (f_3 f_5)^2 = e, \quad f_3 f_6 = f_6 f_3, \quad f_5 f_6 = f_6 f_5.$$

- Aut($\mu_{J}) \cap O(7) \approx S_4 \ltimes \mathbb{Z}^4$, where $S_4$ is the symmetric group of degree four (and order 24), since it is isomorphic to the centralizer in $O(4)$ of the maximal torus $\langle A, B, C \rangle$ of $\mathfrak{sl}_4(\mathbb{R})$ (see (15)).

**Corollary 6.2** The groups of automorphisms and isometries of the $\mathbf{G}_2$-structure $(G_{\mu}, \varphi)$ are, respectively, given by

$$\text{Aut}(G_{\mu}, \varphi) = \text{SL}_2(\mathbb{Z}_3) \ltimes G_{\mu}, \quad \text{Iso}(G_{\mu}, \langle \cdot, \cdot \rangle) = \left( S_4 \ltimes \mathbb{Z}_2^4 \right) \ltimes G_{\mu}.$$

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