ALGEBRAS OF PARTIAL TRIANGULATIONS

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Abstract. We introduce two classes of algebras coming from partial triangulations of marked surfaces. The first one, called frozen algebra of a partial triangulation, is generally of infinite rank and contains frozen Jacobian algebras of triangulations of marked surfaces. The second one, called algebra of a partial triangulation, is always of (explicit) finite rank and contains classical Jacobian algebras of triangulations of marked surfaces and Brauer graph algebras. We classify the partial triangulations, depending on the complexity of their frozen algebras (some are free of finite rank, some are lattices over a formal power series ring and most of them are not finitely generated over their centre). For algebras of partial triangulations, we prove that they are symmetric when the surface has no boundary. From a more representation theoretical point of view, we prove that these algebras of partial triangulations are of tame representation type and we define a combinatorial operation on partial triangulation, generalizing Kauer moves of Brauer graphs and flips of triangulations, which give derived equivalences of the corresponding algebras.

1. Introduction

The aim of this paper is to introduce two new classes of algebras, called frozen and algebras of a partial triangulation, generalizing Brauer graph algebras on the one hand and Jacobian algebras coming from triangulations of surfaces on the other hand. Then we give some properties of these algebras which justify the interest of their study.

In the forties, Brauer introduced Brauer tree algebras which are of finite representation type. These algebras have been then generalized by various
authors to Brauer graph algebras. Brauer graph algebras are defined from the combinatorial datum of a ribbon graph and have many nice properties: they are finite dimensional and symmetric, they are of tame representation type with completely classified modules, and their tilting theory is well understood. For more details about Brauer graph algebras, see for example [AAC, Kau, Rog, WW]. Some generalizations, going in different direction than this paper, have already been proposed (see for example [GS]).

On the other hand, Jacobian algebras have been introduced more recently [DWZ], and in particular Jacobian algebras of triangulations of surfaces [LF, CILF]. These algebras are defined by using triangulations of oriented surfaces with marked points (with or without boundary). They share many nice properties with Brauer graph algebras. They are finite dimensional, they are symmetric when the surface have no boundary [Lad2], they are tame [GLFS]. Moreover, in certain cases, their derived equivalence classes are understood [Lad1].

We will now give an overview of this paper. All along, $k$ is a commutative ring with unit. We fix a compact connected oriented surface $\Sigma$ with or without boundary and a non-empty finite set $M$ of marked points. For each $M \in M$, we fix $m_M \in \mathbb{N}_{>0}$ and $\lambda_M \in k$ invertible. We have to exclude few degenerated cases for simplicity (see beginning of Section 2). A partial triangulation $\tau$ of $(\Sigma, M)$ is, roughly speaking, a subset of a triangulation of $(\Sigma, M)$. To any partial triangulation, in Section 2, we associate a quiver $Q_\tau$, the vertices of which are indexed by the edges of $\tau$ and the arrows of which winds counter-clockwisely around marked points. Then we get the frozen algebra $\Gamma_\tau = \Gamma_\lambda_\tau$ associated with the partial triangulation $\tau$ by factoring out some relations in the (complete) path algebra of $Q_\tau$.

In Section 4, we introduce the algebra $\Delta_\tau = \Delta_\lambda_\tau$ associated with $\tau$. It is the quotient of $\Gamma_\tau$ by the ideal generated by the idempotent corresponding to the boundary of $\Sigma$. An important structural result about $\Gamma_\tau$ and $\Delta_\tau$, which permits to do inductive arguments, is the following one:

**Theorem A** (Theorem 2.17 and Corollary 4.2). Let $\tau \subset \sigma$. Then we have

$$\Gamma_\tau \cong e_\tau \Gamma_\sigma e_\tau \quad \text{and} \quad \Delta_\tau \cong e_\tau \Delta_\sigma e_\tau$$

where $e_\tau$ is the idempotent of $\Gamma_\sigma$ or $\Delta_\sigma$ corresponding to arcs in $\tau$.

In Sections 5 and 6, we give the following results, which can be seen as the first motivation to introduce $\Gamma_\sigma$ and $\Delta_\sigma$:

**Theorem B** (Theorems 5.1 and 6.2). (1) If all $m_M$ are invertible in $k$ and $\sigma$ is a triangulation, then $\Gamma_\sigma$ (respectively $\Delta_\sigma$) is the frozen (respectively classical) Jacobian algebra of a quiver with potential.

(2) If $m_M = 1$ for every marked point $M$ and $\sigma$ is a triangulation, then $\Gamma_\sigma$ and $\Delta_\sigma$ correspond to the quiver with potential introduced in [LF].

(3) If $\sigma$ is sparse (that is “far” from a triangulation, see Definition 6.1), then $\Delta_\sigma$ is the Brauer graph algebra of the ribbon graph underlying to $\sigma$.

(4) Every Brauer graph algebra is obtained from a sparse partial triangulation of a surface without boundary.
In addition to Theorem B, notice that tiling algebras, introduced in [GM], are also special cases of algebras of partial triangulations. For the definition of a frozen Jacobian algebra of a quiver with potential, see for example [BIRS, DL1, DL2] (see also Section 5). We use the expression “classical Jacobian algebra” to refer to usual Jacobian algebras of a quiver with potential as defined in [DWZ].

In Section 3, we classify partial triangulations $\sigma$ in function of the complexity of $\Gamma^\sigma$. We prove that:

**Theorem C** (Theorem 3.1). If $\sigma$ is connected, we have

1. $\Gamma^\sigma$ is free of finite rank as a $k$-module if $\sigma$ is not connected to the boundary of $\Sigma$.
2. $\Gamma^\sigma$ is a lattice over $k[[x]]$ (i.e. free of finite rank) over $k[[x]]$ if
   - $\Sigma$ is a polygon with no puncture and at most one $m_M$ greater than 1;
   - or $\Sigma$ is a polygon with one puncture and $m_M = 1$ for all $M$ on the boundary;
   - or all $M$ on the boundary of $\Sigma$ satisfy $m_M = 1$ and all arcs in $\sigma$ are homotopic to a part of the boundary.
3. $\Gamma^\sigma$ is not finitely generated over its centre in any other case.

Notice that the second case generalizes slightly [DL1, DL2]. Moreover, if $\sigma$ is not connected, the result can be applied independently to each connected component.

The rest of the paper is dedicated to prove a certain number of properties of $\Delta^\sigma$, already known for Brauer graph algebras and for some Jacobian algebras coming from triangulations. In Theorem 4.5, we give a basis of $\Delta^\sigma$ and we deduce:

**Theorem D** (Corollary 4.6). The $k$-algebra $\Delta^\sigma$ is a free $k$-module of rank

$$\sum_{M \in \mathbb{M} \setminus \mathbb{P}} d_M (d_M - 1) + \sum_{M \in \mathbb{P}} m_M d_M^2 + f$$

where $\mathbb{P} \subset \mathbb{M}$ is the set of punctures (i.e. non-boundary marked points), for $M \in \mathbb{M}$, $d_M$ is the degree of $M$ in the graph $\sigma$ (without counting boundary components), and $f$ is the number of arcs in $\sigma$ with both endpoints on boundaries.

We also get the following property:

**Theorem E** (Theorem 4.9). If $\sigma$ has no arc incident to the boundary, then $\Delta^\sigma$ is a symmetric $k$-algebra (i.e. $\text{Hom}_k(\Delta^\sigma, k) \cong \Delta^\sigma$ as $\Delta^\sigma$-bimodules).

We then deal with two representation theoretical questions. In Section 7, we prove:

**Theorem F** (Theorem 7.1). If $k$ is an algebraically closed field, then $\Delta^\sigma$ is tame (or representation-finite).

We can even prove (Proposition 7.3) that, if $\Sigma$ has no boundary and $\sigma$ is a triangulation, then $\Delta^\sigma$ is of quasi-quaternion type in the sense of [Lad3] (see also [Erd, Sko]).

In Section 8, we define flips $\mu_u(\sigma)$ of a partial triangulation $\sigma$ with respect to most arcs $u$ (see Definition 8.1) which generalize flips for triangulations.
and Kauer moves for Brauer graph algebras. We also define coefficients $\mu_u(\lambda)_M$ for $M \in \mathcal{M}$. Then we obtain:

**Theorem G** (Theorem 8.4). *If $u$ is not close to the boundary (in the sense of Definition 8.1), then the algebras $\Delta^{\lambda}_\sigma$ and $\Delta^{\mu_u(\lambda)}_{\mu_u(\sigma)}$ are derived equivalent.*

Notice that most of arcs are not close to the boundary. Notice also that this theorem recovers the result for Brauer graph algebras, which is considered to be known ([Aih, Kau]), but not proven completely. The term Kauer move was introduced in [MS], where authors analyse the case of Brauer graph algebras coming from $m$-angulations of an oriented surface.

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2. Frozen algebras of partial triangulations

We consider a connected compact oriented bordered surface $\Sigma$ with non-empty finite set of marked points $\mathcal{M} \subset \Sigma$. For each marked point $M \in \mathcal{M}$, we fix an invertible scalar $\lambda_M \in k$ and a positive integer $m_M$. If a marked point $M \in \mathcal{M}$ is not on the boundary $\partial \Sigma$ of $\Sigma$, it is called a puncture. We define $\lambda_M := \prod_{M \in \mathcal{M}} \lambda_M$. If $\Sigma$ is a sphere (without boundary), $\# \mathcal{M} = 4$ and $m_M = 1$ for all $M \in \mathcal{M}$, we define $\nu_M = 1 - \lambda_M$. Otherwise, we set $\nu_M = 1$.

We assume that

- if $\Sigma$ is a sphere (without boundary), then $\# \mathcal{M} \geq 3$;
- if $\Sigma$ is a sphere (without boundary) and $\# \mathcal{M} = 3$, then $m_M > 1$ for all $M \in \mathcal{M}$;
- $\nu_M$ is invertible;
- if $\Sigma$ is a disc and $\# \mathcal{M} < 3$ then $\mathcal{M}$ contains at least one puncture $M$ satisfying $m_M > 1$ or at least two punctures.

An oriented edge of $(\Sigma, \mathcal{M})$ is a continuous map $\bar{u} : [0, 1] \to \Sigma$ such that

- $\bar{u}(\{0, 1\}) \subset \mathcal{M}$;
- $\bar{u}|_{(0, 1)}$ is injective from $(0, 1)$ to $\Sigma \setminus \mathcal{M}$;
- $\bar{u}$ is not homotopic to a constant path relatively to its endpoints.

The opposite of the oriented edge $\bar{u}$ is the oriented edge $-\bar{u}$ defined by $(-\bar{u})(t) = \bar{u}(1 - t)$. For an oriented edge $\bar{u}$ of $(\Sigma, \mathcal{M})$, we denote $s(\bar{u}) := \bar{u}(0)$, $t(\bar{u}) := \bar{u}(1)$ and we call $u := \{\bar{u}, -\bar{u}\}$ the corresponding (non-oriented) edge. We say that two oriented edges $\bar{u}$ and $\bar{v}$ are homotopic if they are homotopic relative to $\{s(\bar{v}) = s(\bar{u}), t(\bar{v}) = t(\bar{u})\}$ in $\Sigma \setminus \mathcal{M}$. Two (non-oriented) edges $u = \{\bar{u}, -\bar{u}\}$ and $v = \{\bar{v}, -\bar{v}\}$ are homotopic if $\bar{v}$ is homotopic to $\bar{u}$ or $-\bar{u}$.

We call boundary edge an edge homotopic to an edge entirely included in a boundary component and we call arc an edge which is not a boundary edge.

Two arcs of $(\Sigma, \mathcal{M})$ are compatible if they are not homotopic and they do not cross except maybe at their endpoints. A partial triangulation $\sigma$ of $(\Sigma, \mathcal{M})$ is a set of compatible edges containing boundary edges. An oriented
edge (respectively arc) of a partial triangulation $\sigma$ is a orientation of an edge (respectively arc) of $\sigma$.

In a partial triangulation, the set of oriented edges starting at a given $M \in M$ can be ordered counter-clockwisely up to cyclic permutation. Notice that if an edge $u$ has its two endpoints at $M$, its two orientations $\vec{u}$ and $-\vec{u}$ appear at different positions in this order. We call this order the cyclic order of oriented edges around $M$. Finally, we call as usual triangulation a maximal partial triangulation.

**Remark 2.1.** In some contexts, in particular concerning Brauer graph algebras, an oriented edge is called a half edge.

**Example 2.2.** In the following diagrams, we draw two partial triangulations. A partial triangulation of a disc and a partial triangulation of a torus.

![Diagram of partial triangulations](image)

In the first case, $\vec{u}$, $\vec{x}$ and $\vec{y}$ are boundary edges, while $\vec{v}$ and $\vec{w}$ are arcs. In the second case, $\vec{u}$ is an arc. In the first example, $A = s(\vec{u}) = s(\vec{v}) = t(\vec{w}) = t(\vec{y})$ and the cyclic order around $A$ is $\vec{u}, \vec{v}, -\vec{w}, -\vec{y}$.

We associate to the partial triangulation $\sigma$ a quiver:

**Definition 2.3.** We define $Q_\sigma$ to be a quiver with set of vertices the set of edges of $\sigma$. Then, each vertex $u$ has exactly two outgoing arrows, constructed in the following way: for both possible orientations $\vec{u}$ of the edge $u$, an arrow $[\vec{u}, \vec{v}]$ points to the next (oriented) edge $\vec{v}$ around $s(\vec{u})$.

If $M \in \mathcal{M}$ has at least one incident edge in $\sigma$ and the oriented edges starting at $M$ are ordered $\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_n$, we fix the following paths of $Q_\sigma$:

- $[\vec{u}_i, \vec{u}_j]$ := $[\vec{u}_i, \vec{u}_{i+1}][\vec{u}_{i+1}, \vec{u}_{i+2}] \cdots [\vec{u}_{j-1}, \vec{u}_j]$ composed of at least one arrow and at most $n$ arrows;
- $\lambda_M[\vec{u}_i, \vec{u}_j] := \lambda_M[\vec{u}_i, \vec{u}_{i+1}][\vec{u}_{i+1}, \vec{u}_{i+2}] \cdots [\vec{u}_{j-1}, \vec{u}_j]$ composed of at least $n(m_M - 1)$ arrows and at most $nm_M - 1$ arrows. It is $\lambda_M$ times the idempotent at $u_i$ if $\vec{u}_i = \vec{u}_j$ and $m_M = 1$.

Thus we get in particular $[\vec{u}_i, \vec{u}_j] \cdot [\vec{u}_i, \vec{u}_i] = [\vec{u}_i, \vec{u}_j] [\vec{u}_j, \vec{u}_i] = \lambda_M[\vec{u}_i, \vec{u}_i]^{m_M}$. 
Example 2.4. The quivers corresponding to partial triangulations of Example 2.2 are represented by dashed arrows in the following diagrams:

In the first diagram, the arrows of the quiver are
\[ [\vec{u}, \vec{v}] = a, \quad [\vec{v}, -\vec{u}] = b, \quad [-\vec{u}, -\vec{v}] = c, \quad [-\vec{v}, -\vec{x}] = d, \quad [\vec{x}, \vec{u}] = e, \quad [\vec{w}, -\vec{x}] = g, \quad [-\vec{y}, \vec{u}] = \alpha, \quad [-\vec{u}, \vec{x}] = \beta, \quad [-\vec{x}, \vec{y}] = \gamma, \]
and we also consider symbolic compositions like \( [\vec{u}, -\vec{y}] = abc \). Moreover, if we suppose that \( m_A = 1 \) and \( m_D = 3 \), we have for example \( [-\vec{y}, \vec{v}] = \lambda_A \alpha a \) and \( [-\vec{v}, \vec{u}] = \lambda_D d^2 \).

In the second diagram, we have \([\vec{u}, -\vec{u}] = a\) and \([-\vec{u}, \vec{u}] = b\) but also \([\vec{u}, \vec{u}] = ab\), \([\vec{u}, -\vec{u}] = ba\), \([\vec{u}, -\vec{u}] = \lambda_M (ab) \lambda_M^{-1} a\).

We construct a first algebra associated to \( \sigma \):

**Definition 2.5.** For each oriented edge \( \vec{u} \) of \( \sigma \), we denote
\[ C_\vec{u} := \lambda_{s(\vec{u})}[\vec{u}, \vec{v}]^{m_{s(\vec{u})}} - \lambda_{t(\vec{u})}[-\vec{u}, -\vec{v}]^{m_{t(\vec{u})}} \]
and we denote by \( \mathcal{I}_\sigma^\circ \) the ideal of \( kQ_\sigma \) generated by all possible \( C_\vec{u} \). We denote \( \Gamma_\sigma^\circ := kQ_\sigma / \mathcal{I}_\sigma^\circ \).

We say that a path \( \omega \) of \( Q_\sigma \) is \( C \)-irreducible if it cannot be written as \( \omega = \omega_1[\vec{u}, \vec{v}]^{m_{s(\vec{u})}} \omega_2 \) for some paths \( \omega_1 \) and \( \omega_2 \) and some oriented edge \( \vec{u} \).

**Example 2.6.** If we continue Example 2.4, we have, for the first example, if \( m_A = m_B = 1, m_C = 2 \) and \( m_D = 3 \),
\[ C_{\vec{u}} = \lambda_A abc \alpha - \lambda_B \beta e f, \quad C_{\vec{v}} = \lambda_A \alpha b c \alpha - \lambda_D d^3, \quad C_{\vec{w}} = \lambda_B f \beta e - \lambda_A \alpha c \alpha, \quad C_{\vec{x}} = \lambda_B \beta f e \beta - \lambda_C (\gamma g)^2, \quad C_{\vec{y}} = \lambda_C (\gamma g)^2 - \lambda_A \alpha c \alpha. \]
And we have, in the second example,
\[ C_{\vec{u}} = \lambda_M ((ab)^{m_M} - (ba)^{m_M}) \].

We give a convenient structural description of \( \Gamma_\sigma^\circ \).

**Proposition 2.7.** For each edge \( u \) of \( \sigma \), choose an orientation \( \vec{u} \) and denote \( C_\sigma := \sum_{u \in \sigma} \lambda_{s(\vec{u})}[\vec{u}, \vec{v}]^{m_{s(\vec{u})}} \in \Gamma_\sigma^\circ \). Then
(1) \( C_\sigma \) does not depend on the chosen orientations;
(2) there is an injection from \( k[x] \) to the centre of \( \Gamma_\sigma^\circ \) mapping \( x \) to \( C_\sigma \);
(3) \( \Gamma_\sigma^\circ \) is free over \( k[x] \) with basis consisting of the \( C \)-irreducible paths.

Proof of Proposition 2.7 is given in Subsection 10.1.
Definition 2.8. For $n \in \mathbb{Z}_{\geq 1}$ we fix a model $n$-gon without hole $\mathcal{P}(n)$ which is an (oriented) closed disc with $n$ marked points on its boundary, called its vertices. We number the vertices of $\mathcal{P}(n)$ from 1 to $n$ in the counter-clockwise order. We also fix a model $n$-gon with hole $\mathcal{P}^\circ(n)$ which is $\mathcal{P}(n) \setminus D$ where $D$ is a closed disc contained in the interior of $\mathcal{P}(n)$.

Let us consider a partial triangulation $\sigma$ of $(\Sigma, \mathcal{M})$. An $n$-gon without hole (respectively $n$-gon with hole) of $\sigma$ is an oriented continuous map $P : \mathcal{P}(n) \to \Sigma$ (respectively $P : \mathcal{P}^\circ(n) \to \Sigma$) satisfying:

- $P$ is injective on the interior of $\mathcal{P}(n)$ (respectively $\mathcal{P}^\circ(n)$);
- each side of $\mathcal{P}(n)$ (respectively $\mathcal{P}^\circ(n)$) is mapped injectively to an edge of $\sigma$;
- in the case of a polygon with hole, we cannot fill the hole: $P$ cannot be extended to a polygon without hole $P' : \mathcal{P}(n) \to \Sigma$ along the canonical inclusion $\mathcal{P}^\circ(n) \subset \mathcal{P}(n)$.

We call interior of $P$ the image by $P$ of the interior of $\mathcal{P}(n)$ (respectively $\mathcal{P}^\circ(n)$). If $P$ has no hole, we call set of punctures of $P$ the intersection $\mathcal{M}_P$ of $\mathcal{M}$ with the interior of $P$. If $P$ has a hole, we will use the symbolic notation $\#\mathcal{M}_P = \infty$.

We denote by $P_i$ the image by $P$ of the vertex number $i$ of $\mathcal{P}(n)$ (respectively $\mathcal{P}^\circ(n)$). We denote by $\overrightarrow{P_iP_{i+1}}$ the oriented edge of $\sigma$ on which is mapped the corresponding side of $\mathcal{P}(n)$ and $\overrightarrow{P_{i+1}P_i} = -\overrightarrow{P_iP_{i+1}}$. We call $\overrightarrow{P_iP_{i+1}}$ an oriented side of $P$. In the same way we define $\overrightarrow{P_nP_1}$ and $\overrightarrow{P_1P_n}$.

We call special monogon a polygon without hole $P$ with one side and one puncture $M$ satisfying $m_M = 1$, called special puncture.

Example 2.9. We continue with the figures of Example 2.4. The first one has five polygons:

- $P$ with sides $\overrightarrow{u}, \overrightarrow{w}$: we have $\mathcal{M}_P = \{D\}$;
- $Q$ with sides $-\overrightarrow{w}, \overrightarrow{x}, \overrightarrow{y}$: we have $\mathcal{M}_Q = \emptyset$;
- $R$ with sides $\overrightarrow{u}, \overrightarrow{x}, \overrightarrow{y}$: we have $\mathcal{M}_R = \{D\}$;
- $S$ with sides $\overrightarrow{u}, \overrightarrow{w}, \overrightarrow{v}, -\overrightarrow{v}$: we have $\mathcal{M}_S = \emptyset$;
- $T$ with sides $\overrightarrow{u}, \overrightarrow{x}, \overrightarrow{y}, \overrightarrow{v}, -\overrightarrow{v}$: we have $\mathcal{M}_T = \emptyset$.

The second has two polygons: a monogon $P$ with side $\overrightarrow{u}$ ($\mathcal{M}_P = \{N\}$) and a monogon $Q$ with side $-\overrightarrow{u}$ ($\#\mathcal{M}_Q = \infty$ as $Q$ has a hole).

The following proposition, proven in Subsection 10.2, gives a combinatorial description of polygons of $\sigma$.

Proposition 2.10. We consider a sequence of oriented edges $\overrightarrow{u_1}, \overrightarrow{u_2}, \ldots, \overrightarrow{u_n}$, with indices considered modulo $n$, such that $t(\overrightarrow{u_i}) = s(\overrightarrow{u_{i+1}})$ for $i = 1, \ldots, n$.

The following conditions are equivalent:

(i) There is an $n$-gon having oriented sides $\overrightarrow{u_1}, \overrightarrow{u_2}, \ldots, \overrightarrow{u_n}$ in this order.

(ii) The following conditions are satisfied:

- if $i \neq j$ then $\overrightarrow{u_i} \neq \overrightarrow{u_j}$;
- for each $i$, if $\overrightarrow{u_i}$ is an oriented boundary edge, then it is oriented clockwisely around the boundary;
- for any $i$ and $j$ such that $M := s(\overrightarrow{u_i}) = s(\overrightarrow{u_j})$, we have that $-\overrightarrow{u}_{i-1}, \overrightarrow{u}_i, -\overrightarrow{u}_{i+1}$ and $\overrightarrow{u}_j$ are ordered clockwisely around $M$;
- for any oriented boundary edge $\overrightarrow{v}$ and $i$ such that $M := s(\overrightarrow{u_i}) = s(\overrightarrow{v})$, we have that $-\overrightarrow{u}_{i-1}, \overrightarrow{u}_i$ and $\overrightarrow{v}$ are ordered clockwisely around $M$. 
Definition 2.11. Two $n$-gons without hole $P, P' : \mathcal{P}(n) \rightarrow \Sigma$ of $\sigma$ are said to be \textit{equivalent} if there exist an oriented automorphism $\psi$ of $\mathcal{P}(n)$ permuting the sides such that $P = P' \circ \psi$. Two $n$-gons with hole $P, P' : \mathcal{P}^o(n) \rightarrow \Sigma$ of $\sigma$ are said to be \textit{equivalent} if there exist two injection $\iota, \iota' : \mathcal{P}^o(n) \twoheadrightarrow \mathcal{P}^o(n)$ which map the sides of the $n$-gon to the sides of the $n$-gon and satisfy $P \circ \iota = P' \circ \iota'$.

Notice that this equivalence relation relates polygons which either have both hole, either have the same set of punctures. Moreover, an equivalence class of polygons is entirely determined by the sequence of its sides. From now on, we will consider polygons up to this equivalence relation. Using Proposition 2.10 permits to consider polygons as combinatorial objects.

We associate a second algebra to $\sigma$:

Definition 2.12. Consider a $n$-gon $P$. For $i \in \{1, 2, \ldots, n\}$, we call \textit{internal path winding around $P_i$} the path \[ \omega^P_i := [\overrightarrow{P_i P_{i+1}}, \overrightarrow{P_{i-1} P_i}] \]
and we call \textit{coefficiented external path winding around $P_i$} the (multiple of a) path \[ \xi^P_i := [\overrightarrow{P_i P_{i+1}}, \overrightarrow{P_{i-1} P_i}] \].

Notice that we have $\omega^P_i \xi^P_i = \xi^P_{i+1} \omega^P_{i+1}$ in $\Gamma^\sigma_\omega$. For $i \in \{1, 2, \ldots, n\}$, we denote
\[ \Gamma^\sigma_\omega \ni R_{P,i} := \omega^P_{i+1} \omega^P_i - \left\{ \begin{array}{ll} \xi^P_{i+2} \xi^P_{i+3} \cdots \xi^P_{i+2} \xi^P_{i-1}, & \text{if } M_P = \emptyset; \\ \lambda^M \omega^P_{i+1} \xi^P_{i+2} \cdots \xi^P_{i-1}, & \text{if } M_P = \{M\}; \\ 0, & \text{if } \#M_P \geq 2. \end{array} \right. \]
(Notice that, if $n = 1$ or $n = 2$, we have $M_P \neq \emptyset$).

Finally, we denote by $I_\sigma$ the ideal of $\Gamma^\sigma_\omega$ generated by all $R_{P,i}$ for all polygons of $\sigma$. We call \[ \Gamma_\sigma = \Gamma^\lambda_\omega := \Gamma^\sigma_\omega / I_\sigma \]
the \textit{frozen algebra associated to $\sigma$} where the completion is taken with respect to the ideal $J$ of $\Gamma^\sigma_\omega / I_\sigma$ generated by the arrows $q \in Q_1$ such that $q$ does not divide an idempotent in $\Gamma^\sigma_\omega / I_\sigma$ or equivalently $q$ is not of the form $[\overrightarrow{u, v}]$, $[\overrightarrow{u, v}]$ or $[\overrightarrow{v, u}]$ where $u$ is the oriented side of a special monogon and $v$ is the oriented edge pointing to the special puncture if it is in $\sigma$.

Example 2.13. We continue Examples 2.4, 2.6 and 2.9. For the first partial triangulation, in addition to relations $C_{\hat{u}}, C_{\hat{v}}, C_{\hat{w}}, C_{\hat{x}}$ and $C_{\hat{y}}$, we have the following relations:

- Coming from $P$: 
  \[ fab = \lambda_D f (\lambda_B \lambda_A \beta e c) a^2 \lambda_B \beta e \quad \text{and} \quad abf = \lambda_D a b (\lambda_A \lambda_B c a \beta e)^2 \lambda_A c a ; \]

- Coming from $Q$: 
  \[ ge = \lambda_A a a b \quad \text{and} \quad ec = \lambda_C \gamma g \gamma \quad \text{and} \quad eg = \lambda_B f \beta ; \]
The ideal \( J \) is generated by all arrows of \( Q_\sigma \). Notice that these relations are redundant. For instance, relations for \( \Gamma_\sigma \), the relations for \( Q \) and the relations from \( S \). For example, using relations of \( Q, S \) and relations of \( \Gamma_\sigma \), we find
\[
dbc = \lambda_A \beta \lambda_B \lambda_C = \lambda_A \beta \lambda_B \lambda_C \gamma g \gamma
\]
which is a relation coming from \( T \). This remark will be generalized in Theorem 2.16.

For the second partial triangulation, in addition to \( C_{\bar{u}} \), we get the relation \( a^2 = \lambda_N a (\lambda_M (ba)^{m_M} b)^{m_N} - 1 \) from \( P \) and \( b^2 = 0 \) from \( Q \). Using the second relation, the first relation can be simplified to
\[
a^2 = \begin{cases} 
\lambda_N a & \text{if } m_N = 1; \\
\lambda_N \lambda_M (ab)^{m_M} & \text{if } m_N = 2; \\
0 & \text{else.}
\end{cases}
\]
The ideal \( J \) is generated by \( a \) and \( b \) if \( m_N \geq 2 \) and by only \( b \) if \( m_N = 1 \). We find easily
\[
\Gamma_{\sigma} = \begin{cases} 
\left( b_{12} b_{22} / \left( b_{11} + b_{12}, b_{11} + b_{22}, b_{21} + b_{22}, b_{21} + b_{22}, b_{21} + b_{22} \right) \right) & \text{if } m_N = 1; \\
\left( a^2 - \lambda_N \lambda_M (ab)^{m_M}, a^2 - \lambda_N \lambda_M (ba)^{m_M}, b^2 \right) & \text{if } m_N = 2; \\
\left( (ab)^{m_M} - (ba)^{m_M}, a^2, b^2 \right) & \text{else.}
\end{cases}
\]
(in the first case, we separated the idempotents \( a/\lambda_N \) and \( 1-a/\lambda_N \). Notice that in all cases, the ideal contains all paths of length \( m_M + 1 \) so there is no need of completion here.

**Definition 2.14.** An arc \( u \) of a partial triangulation \( \sigma \) is said to be a reduction arc for a polygon \( P \) if it is connected to at least one vertex of \( P \) and if the interior of \( P \) intersects \( u \).

A \( n \)-gon \( P \) is said to be minimal if it does not have any reduction arc.

**Example 2.15.** In Example 2.9 minimal polygons are \( Q \) and \( S \).
We finish this Section by giving two important theorems to understand the algebra $\Gamma_\sigma$. These theorems are proven in Subsections 10.3 and 10.4.

**Theorem 2.16.** The ideal $I_\sigma$ is the ideal of $\Gamma_\sigma^\circ$ generated by all $R_{P,i}$ for $P$ minimal.

**Theorem 2.17.** Let $\tau \subset \sigma$ be an inclusion of partial triangulations. Let $e_\tau$ be the idempotent of $\Gamma_\sigma$ corresponding to the set of edges of $\tau$. There is an isomorphism of (non-unital) algebras

$$\varphi : \Gamma_\tau \cong e_\tau \Gamma_\sigma e_\tau$$

mapping each idempotent $e_u$ to $e_u$ and each arrow $[u, v]$ to the path $[u, v]$.

3. When is $\Gamma_\sigma$ module-finite over its centre?

In this section, we classify partial triangulations in three families depending on the complexity of $\Gamma_\sigma$. Recall the lattice over a commutative ring $R$ is an $R$-algebra which is free of finite rank as an $R$-module. The main theorem of this section is the following one:

**Theorem 3.1.** Let $E \subset \sigma$ be a connected component of $\sigma$. Let $e$ be the idempotent of $\Gamma_\sigma$ corresponding to $E$. Then:

1. If $E$ is disconnected from the boundary, then $e \Gamma_\sigma e$ is free of finite rank as a $k$-module.
2. If we are in one of the following three cases, then $e \Gamma_\sigma e$ is a lattice over $R := k[x]$:
   
   a. $(\Sigma, M)$ is a polygon without puncture and $m_M = 1$ for all $M \in M$ except at most one;
   
   b. $(\Sigma, M)$ is a polygon with one puncture and $m_M = 1$ for all $M \in M$ on the boundary;
   
   c. all marked points $M$ incident to $E$ satisfy $m_M = 1$ and any arc of $E$ is homotopic to a part of a boundary component.
3. In any other case, $e \Gamma_\sigma e$ is not finitely generated as a module over its centre.

Proof of Theorem 3.1 is given in Section 10.6. Notice that $\Gamma_\sigma$ is the direct product of the $e \Gamma_\sigma e$ corresponding to the connected components of $\sigma$ so Theorem 3.1 exhausts all cases. Notice also that Theorem 3.1 (2) generalizes slightly the observations of [DL1] and [DL2]. We give now a precise description of the lattices of Theorem 3.1 (b) when $\sigma$ contains only boundary edges. We fix $E$ and $e$ as in Theorem 3.1.

**Proposition 3.2.** Suppose that $E$ is a boundary component of $\Sigma$. Let $P$ be the $n$-gon corresponding to this boundary component. We have:

1. If $M_P = \emptyset$, $m_{P_i} = m$ and $m_{P_i} = 1$ for $2 \leq i \leq n$, then $e \Gamma_\sigma e \cong$

\[
\begin{bmatrix}
R' & R' & R' & \cdots & R' & t^{-m} R' \\
R' & \Delta & \Delta & \cdots & \Delta & R' \\
t^{m+1} R' & t^{m} R' & \Delta & \cdots & \Delta & R' \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
t^{m+1} R' & t^{m+1} R' & \Delta & \cdots & \Delta & R' \\
\end{bmatrix}_{n \times n}
\]
where $R' := k[[t]]$, $\Delta := k \oplus t^m R'$ and $R$ is identified to a subalgebra of $R'$ and $\Delta$ by mapping $x$ to $t^m$.

(2) If $\mathbb{M}_P = \{M_i\}$, $m_{P_i} = 1$ for $1 \leq i \leq n$ and $m_M = m$, then

$$e \Gamma_\sigma e \cong \begin{bmatrix}
R_m & R_m & R_m & \cdots & R_m & R_{m-1} \\
xR_{m-1} & R_m & R_m & \cdots & R_m & R_m \\
xR_m & xR_{m-1} & R_m & \cdots & R_m & R_m \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
xR_m & xR_m & xR_m & \cdots & R_m & R_m \\
xR_m & xR_m & xR_m & \cdots & xR_{m-1} & R_m \\
\end{bmatrix}_{n \times n}$$

where $R_j := \{(P, Q) \in R^2 \mid P - Q \in x^j R\}$ for $j \geq 0$. Notice that if $n = 1$, it degenerates to $e u \Gamma_\sigma e u \cong R_{m-1}$.

(3) If $\# \mathbb{M}_P > 1$ and $m_{P_i} = 1$ for $1 \leq i \leq n$, then

$$e \Gamma_\sigma e \cong \begin{bmatrix}
R' & R' & R' & \cdots & R' & R' & x^{-1} I \\
I & R' & R' & \cdots & R' & R' & R' \\
xR' & I & R' & \cdots & R' & R' & R' \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
xR' & xR' & xR' & \cdots & R' & R' & R' \\
xR' & xR' & xR' & \cdots & I & R' & R' \\
xR' & xR' & xR' & \cdots & xR' & I & R' \\
\end{bmatrix}_{n \times n}$$

where $R' := R[\varepsilon]/(\varepsilon^2)$ and $I := (\varepsilon, x) \subset R'$.

Proposition 3.2 is proven in Subsection 10.5.

4. ALGEBRAS OF A PARTIAL TRIANGULATIONS

In this section, we define algebras of partial triangulations and we give first results about these algebras. As usual, $\sigma$ is a partial triangulation of $(\Sigma, \mathbb{M})$. All results are proven in Subsections 10.7, 10.8, 10.9 and 10.10.

Definition 4.1. We call algebra associated to $\sigma$ the algebra

$$\Delta_\sigma = \Delta_\sigma^\lambda := \Gamma_\sigma^\lambda/(e_0)$$

where $e_0$ is the sum of the idempotent corresponding to boundary components.

First, notice that we can use the same techniques as for frozen algebras:

Corollary 4.2 (of Theorem 2.17). If $\tau \subset \sigma$ then $\Delta_\tau = e_\tau \Delta_\sigma e_\tau$.

By abuse of notation, we denote by $J$ the ideal of $\Delta_\sigma$ obtained by projection of $J$. We start by giving an alternative presentation of $\Delta_\sigma$, which does not involve completion. We consider the full subquiver $Q_\sigma$ of $Q_\sigma$ with vertices corresponding to arcs (i.e. non-boundary edges). For each oriented arc $\vec{v}$ of $\sigma$, we define a relation $R_{\vec{v}}$ in the following way. Let $P$ be the minimal polygon of $\sigma$ having $\vec{v}$ as an oriented side, and $\vec{u}$ (respectively $\vec{w}$) be the side of $P$ following (respectively preceding) $\vec{v}$. If one at least of $\vec{u}$ or $\vec{w}$ is a boundary edge, we put $R_{\vec{v}} = 0$. Otherwise, we put $R_{\vec{v}} = [\vec{u}, -\vec{w}] - f_{\vec{v}}$, where, denoting by $n$ the number of sides of $P$:

- if $n = 3$ and $\mathbb{M}_P = \emptyset$, $f_{\vec{v}} = |\vec{u}, \vec{w}|$;
For two elements $a$ and $b$, if $a 
eq b$, $n = 1$ and $M_P = \{M\}$ with $m_M = 1$, $f_{\bar{v}} = \lambda_M[\bar{u}, -\bar{v}]$;

- if $2 \notin k^\times$, $n = 1$ and $M_P = \{M\}$ with $m_M = 2$, $f_{\bar{v}} = \lambda_M \lambda_{\sigma(\bar{v})}[\bar{v}, \bar{v}]^{m_{\sigma(\bar{v})}}$;

- in any other case, $f_{\bar{v}} = 0$.

where any path of $Q_{\sigma}$ passing through a boundary component is $0$ in $kQ_{\sigma}$.

Then we have the following result:

**Theorem 4.3.** We have an isomorphism

$$\Delta_{\sigma} \cong \Delta_{\sigma}^\ast := \frac{kQ_{\sigma}}{JC_{\sigma} + (C_{\bar{v}}, R_{\bar{v}})_{\bar{v} \in \sigma}}$$

which maps $e_u$ to $e_{\bar{u}}$ for any $u \in \sigma$, such that $C_{\bar{u}}$ comes from the same element of $Q_{\sigma}$ on both side.

**Remark 4.4.** We will see that the isomorphism of Theorem 4.3 does not come from the identity of $kQ_{\sigma}$. In particular, the isomorphism of Corollary 4.2 cannot any more be realized by the naive identifications $[\bar{u}, \bar{v}]$ in $\Delta_{\sigma}^r$ and $\Delta_{\sigma}^s$.

We will give in Proposition 4.12 an other presentation of $\Delta_{\sigma}$ for which this naive identification still works.

We now give a convenient basis of $\Delta_{\sigma}$. Let $B$ the subset of $kQ_{\sigma}/(C_{\bar{v}})_{\bar{v} \in \sigma}$ consisting of $e_u$ for $u \in \sigma$ (a non-boundary edge), $e_u := e_u C_{\sigma}$ for $u$ an arc of $\sigma$ not connected to a boundary and $[\bar{u}, \bar{v}]$ where

- $\bar{u}$ and $\bar{v}$ are non-boundary oriented edges satisfying $s(\bar{u}) = s(\bar{v})$;

- $0 \leq \ell < m_{s(\bar{u})}$;

- If $\ell = m_{s(\bar{u})} - 1$ then $\bar{u} \neq \bar{v}$;

- If there is a boundary edge $\bar{b}$ with $s(\bar{b}) = s(\bar{u})$ then $\bar{u}, \bar{v}$ and $\bar{b}$ are strictly ordered counter-clockwisely around $s(\bar{u})$ and $\ell = 0$.

**Theorem 4.5.** (1) The set $B$ is mapped to $k$-bases of $\Delta_{\sigma}$ and $\Delta_{\sigma}^s$.

(2) For two elements $x$, $y$ of $B$, the product $xy$ is a scalar multiple of an element of $B$ in $\Delta_{\sigma}$ and in $\Delta_{\sigma}^s$.

(3) If $\sigma$ has no arc incident to the boundary, then for any $x \in \Delta_{\sigma}$, $Jx = xJ = 0$ if and only if $x \in (C_{\sigma})$.

We deduce the following easy corollary.

**Corollary 4.6.** The $k$-algebra $\Delta_{\sigma}$ is free of rank

$$\sum_{M \in \mathbb{M} \setminus \mathbb{P}} \frac{d_M(d_M - 1)}{2} + \sum_{M \in \mathbb{P}} m_M d_M^2 + f$$

where $\mathbb{P} \subset \mathbb{M}$ is the set of punctures, for $M \in \mathbb{M}$, $d_M$ is the degree of $M$ in the graph $\sigma$ (without counting boundary components), and $f$ is the number of arcs in $\sigma$ with both endpoints on boundaries.

**Example 4.7.** In Examples 2.4, 2.6, 2.13, the rank of the algebra of the left partial triangulation is $1 + 3 + 1 = 5$ and the one of the right partial triangulation is $4m_M$.

**Remark 4.8.** It is immediate that $\Delta_{\sigma}$ does not depend of the values of $m_M$ for $M \in \mathbb{M}$ on the boundary of $\Sigma$. This is reflected in the rank formula.

We also get the following theorem:
Theorem 4.9. If $\sigma$ has no arc incident to the boundary, then $\Delta_{\sigma}$ is a symmetric $k$-algebra (i.e., $\text{Hom}_k(\Delta_{\sigma}, k) \cong \Delta_{\sigma}$ as $\Delta_{\sigma}$-bimodules).

We construct a triangulation $\sigma'$ of a surface without boundary from the triangulation $\sigma$ having the property that $\Delta_{\sigma} = \Delta_{\sigma'}/(e_0)$ for an idempotent $e_0 \in \Delta_{\sigma'}$.

Definition 4.10. For each boundary of $\Sigma$ with $n$ marked points, we patch a $n$-gon with two punctures to form a marked surface $(\Sigma', M')$ without boundary that we call augmented surface of $(\Sigma, M)$. We call $\sigma'$ the partial triangulation of $(\Sigma', M')$ having the same edges and we call it its augmented partial triangulation.

Notice that it is immediate by definition that $\Delta_{\sigma'}^\lambda = \Gamma_{\sigma'}^\lambda$ does not depend on the choice of the $\lambda_M$'s for the added punctures $M$.

Finally, we give a variant of Theorem 4.3 which has the advantage to be compatible with Corollary 4.2, but gives a more complicated presentation of $\Delta_{\sigma}$. We keep notations introduced at the beginning of this section.

For each $\vec{v} \in \sigma$, we define $R_{\vec{v}} \in kQ_{\sigma'}/(C_{\vec{v}})_{\vec{v} \in \sigma}$. If one at least of $\vec{u}$, $\vec{w}$ is a boundary edge, we put $R_{\vec{v}} = 0$. Otherwise, we define $R_{\vec{v}} := [\vec{u}, -\vec{v}][\vec{v}, -\vec{w}]$.
where $f^*_{\otimes}$ depends on the situation as in Figure 4.11. On this figure, there are no hole or puncture other than the one depicted inside the drawn polygon (but there can be an arc in Case e from $P_3$ to $M$, and in Case f from $P_3$ to $P_3$). Moreover, for Case a, we can have that $P_1$, $P_2$, $P_3$ or $u$, $v$, $w$ are not distinct. In Cases b and c, $P_1$ and $P_2$ are not necessarily distinct. Let us define $f^*_{\otimes}$ as in Figure 4.11 (any path of $Q_\sigma$ passing through a boundary component is 0 in $kQ_\sigma$).

**Proposition 4.12.** We have the equality

$$\Delta_\sigma = \frac{kQ_\sigma}{JC_\sigma + (C_{\cdot\otimes}, R_{\otimes})_{\otimes\in\sigma}}.$$

Notice that in Proposition 4.12, in contrast to Theorem 4.3 where we need a non-trivial endomorphism of $kQ_\sigma$, we just give an alternative set of generators of the ideal of relations defining $\Delta_\sigma$.

5. **Case of triangulations**

Let us suppose in this section that $(\Sigma, M)$ admits at least one marked point on each boundary component and that $\sigma$ is a triangulation. We suppose also that for any $M \in M$, $m_M$ is invertible in $k$. We will prove that in this case the relations defining $\Gamma_\sigma$ and $\Delta_\sigma$ come from a potential. Up to Morita equivalence, $\Delta_\sigma$ is the algebra introduced by Labardini-Fragoso in [LF] and Cerulli Irelli, Labardini-Fragoso in [CILF] when $m_M = 1$ for all $M \in M$. For general definitions and results about Jacobian algebras and frozen Jacobian algebras, we refer to [DWZ] and [BIRS].

We call potential attached to $\sigma$ the following linear combination of cycles in $kQ_\sigma/[kQ_\sigma, kQ_\sigma]$:

$$W_\sigma = \sum_{P \text{ triangle of } \sigma} \omega_3^P \omega_2^P \omega_1^P - \sum_{M \in M} \lambda_M m_M \alpha_M$$

where the first sum runs over all (minimal) triangles of $\sigma$ and for each $M \in M$, $\alpha_M$ is the cycle running around $M$ with arbitrary starting vertex.

For any arrow $\alpha \in Q_{\sigma,1}$, and any cycle $u = u_1u_2\cdots u_n$ of $Q_\sigma$, we define the cyclic derivative

$$\partial_{\alpha}(u) := e_t(\alpha) \sum_{u_i=\alpha} u_{i+1}u_{i+2}\cdots u_{i-1}$$

and we extend this definition to $kQ_\sigma/[kQ_\sigma, kQ_\sigma]$.

Let us call an arrow $[\bar{u}, \bar{v}]$ frozen if $-\bar{v}$ and $\bar{u}$ are two consecutive oriented boundary edges winding counter-clockwisely around a hole and let us call $F'$ the set of frozen arrows. We call frozen Jacobian ideal of $W_\sigma$ the ideal $J(W_\sigma)$ of $kQ_\sigma$ generated by the elements $\partial_{\alpha}W_\sigma$ where $\alpha$ runs over non-frozen arrows of $Q_\sigma$. Adapting [BIRS], we call frozen Jacobian algebra of the frozen quiver with potential $(Q_\sigma, W_\sigma, F')$ the algebra

$$\mathcal{P}(Q_\sigma, W_\sigma, F) := \left( \frac{kQ_\sigma}{J(W_\sigma)} \right)^J.$$

The main theorem of this Section is:
Theorem 5.1. The identity map of $kQ_{\sigma}$ induces an isomorphism
$$\Gamma_{\sigma} \cong \mathcal{P}(Q_{\sigma}, W_{\sigma}, F).$$

Proof. Let $[\tilde{u}, \tilde{v}]$ be a non-frozen arrow of $Q_{\sigma}$. Then $\tilde{u}$ is not a boundary edge winding counter-clockwise around a hole. Thus, thanks to Proposition 10.6, there is a unique minimal polygon $P$ having $\tilde{u}$ as an oriented side. As $\sigma$ is a triangulation, it is immediate that $P$ is a triangle and $M_{P} = \emptyset$. Moreover, $P$ has also $-\tilde{v}$ as an oriented side and, up to numbering correctly the vertices of $P$, $[\tilde{u}, \tilde{v}] = \omega_{3}^{P}$. Notice also that $\omega_{3}^{P}$ is part of $\alpha_{M}$ only for $M = s(\tilde{u})$. Thus, an elementary computation gives $\partial_{\omega_{3}^{P}} W_{\sigma} = R_{P, 1}$. Moreover, all $R_{P, i}$ for minimal triangles $P$ are obtained in this way.

Let $u \in Q_{\sigma, 0}$. There is an orientation $\tilde{u}$ of $u$ such that $\tilde{u}$ is not a boundary edge winding counter-clockwise around a hole. Then taking the same notation as before for this $\tilde{u}$, we get
$$C_{\tilde{u}} = \lambda_{s(\tilde{u})}[\tilde{u}, \tilde{u}]^{m(\tilde{u})}_{\mathcal{Q}_{\sigma}} - \lambda_{t(\tilde{u})}[-\tilde{u}, -\tilde{u}]^{m(\tilde{u})}_{\mathcal{Q}_{\sigma}} = (\partial_{\omega_{1}^{P}} W_{\sigma})\omega_{3}^{P} - \omega_{3}^{P}(\partial_{\omega_{3}^{P}} W_{\sigma})$$
so, using Theorem 2.16, we get $\Gamma_{\sigma} \cong \mathcal{P}(Q_{\sigma}, W_{\sigma}, F)$. □

We deduce easily the following result:

Corollary 5.2. Using notation of Section 4 for $Q_{\sigma}$, and denoting by $W_{\sigma}$ the potential on $Q_{\sigma}$ induced by $W_{\sigma}$, we get
$$\Delta_{\sigma} \cong \mathcal{P}(Q_{\sigma}, W_{\sigma})$$
where $\mathcal{P}(Q_{\sigma}, W_{\sigma}) := \mathcal{P}(Q_{\sigma}, W_{\sigma}, \emptyset)$ is the usual Jacobian algebra.

Notice that in the case where there exist at least one special monogon in a triangulation, then $\Gamma_{\sigma}$ is not a basic algebra. We give a sketch of the method which permits to get a Morita equivalent basic Jacobian algebra. Suppose that there is in our triangulation a special monogon inducing a self-folded triangle. Let $\tilde{u}$ be the oriented arc enclosing the special monogon and $\tilde{v}$ the oriented arc pointing toward the special puncture. We can write the potential in the following way:
$$W_{\sigma} = [\tilde{v}, -\tilde{u}] [\tilde{u}, \tilde{v}] [-\tilde{v}, \tilde{v}] - \lambda_{t(\tilde{v})} [-\tilde{v}, \tilde{v}] - [\tilde{u}, \tilde{u}] \cdot [\tilde{u}, \tilde{v}] + W_{\sigma}$$
where $W_{\sigma}$ does not contain any occurrence of $[\tilde{u}, -\tilde{u}]$, $[\tilde{u}, \tilde{v}]$ or $[-\tilde{v}, -\tilde{v}]$. In particular $\partial_{[-\tilde{v}, -\tilde{v}]} W_{\sigma} = [\tilde{v}, -\tilde{u}] [\tilde{u}, \tilde{v}] - \lambda_{t(\tilde{v})} e_{v}$ and $\mathcal{P}(Q_{\sigma}, W_{\sigma}, F)$ is Morita equivalent to $(1 - e_{v}) \mathcal{P}(Q_{\sigma}, W_{\sigma}, F)(1 - e_{v})$. Let us denote
$$e_{u}^{*} := [\tilde{u}, \tilde{v}] [\tilde{u}, -\tilde{u}] / \lambda_{t(\tilde{v})} \quad \text{and} \quad e_{u}^{\approx} := e_{u} - e_{u}^{*}$$
which are orthogonal idempotents of $(1 - e_{v}) \mathcal{P}(Q_{\sigma}, W_{\sigma}, F)(1 - e_{v})$. We construct a quiver $Q'_{\sigma}$ in the following way: $Q'_{\sigma, 0} := Q_{\sigma, 0} \setminus \{u, v\} \cup \{u^{*}, u^{\approx}\}$ and $Q'_{\sigma, 1}$ consists of
• all arrows of $Q_{\sigma}$ which are not incident to $u$ or $v$;
• for each arrow $\alpha$ pointing toward $u$ except $[\tilde{v}, -\tilde{u}]$, two arrows $\alpha^{*}$ and $\alpha^{\approx}$ with $s(\alpha^{*}) = s(\alpha^{\approx}) = s(\alpha)$ and $t(\alpha^{*}) = u^{*}$ and $t(\alpha^{\approx}) = u^{\approx}$;
• for each arrow $\beta$ pointing from $u$ except $[\tilde{u}, \tilde{v}]$, two arrows $\beta^{*}$ and $\beta^{\approx}$ with $t(\beta^{*}) = t(\beta^{\approx}) = t(\beta)$ and $s(\beta^{*}) = u^{*}$ and $s(\beta^{\approx}) = u^{\approx}$.

Finally, we define a potential on $Q'_{\sigma}$ by $W'_{\sigma} := -\lambda_{t(\tilde{v})} [-\tilde{u}, \tilde{u}] + W_{\sigma}$ where
Any Brauer graph algebra is an immediate consequence of Theorem 4.3. Proof.

The following hold:

• $|−\vec{u}, \vec{u}|$ is obtained from $|−\vec{u}, \vec{u}|$ by substituting any arrow $\alpha$ pointing toward (respectively from) $u$ by $\alpha^\bullet$ (respectively $\alpha$) and replacing the product $[\vec{u}, \vec{v}][\vec{v}, −\vec{u}]$ as many times as it appears by a factor $\lambda_{s(\vec{v})}$;

• $W_\sigma$ is obtained from $W_\sigma$ by substituting any arrow $\alpha$ pointing toward (respectively from) $u$ by $\alpha^\bullet + \alpha^\infty$ (respectively $\alpha + \infty \alpha$).

Then, it is easy to see that there is an isomorphism of algebras

$$(1 − e_v)\mathcal{P}(Q_\sigma, W_\sigma, F)(1 − e_v) \cong \mathcal{P}(Q'_\sigma, W'_\sigma, F),$$

mapping

• common elements of $Q_\sigma$ and $Q'_\sigma$ to themselves;

• $e_u^\bullet$ to $e_v^\bullet$ and $e_u^\infty$ to $e_v^\infty$;

• any arrow $\alpha$ pointing toward (respectively from) $u$ to $\alpha^\bullet + \alpha^\infty$ (respectively $\alpha + \infty \alpha$);

• $[\vec{u}, \vec{v}].−\vec{v}, −\vec{v}[\vec{v}, −\vec{u}]$ to $\lambda_{s(\vec{v})}[\vec{u}, \vec{u}]e_u^\bullet$ for any $\ell \geq 0$.

Therefore, it gives a method to iteratively find a basic Jacobian algebra Morita equivalent to $\mathcal{P}(Q_\sigma, W_\sigma, F)$ as special mongons never share any arc under the hypotheses of this paper. We refer to [CILF] for a more direct construction of the basic quiver with potential in the case where $m_M = 1$ for any $M \in \mathbb{M}$.

As a corollary, we get:

**Corollary 5.3.** If $\sigma$ is a triangulation and $n_M = 1$ for any $M \in \mathbb{M}$, then the algebra $\Delta_\sigma$ is Morita equivalent to the classical Jacobian algebra corresponding to this triangulation in [CILF].

### 6. Case of Brauer graph algebras

**Definition 6.1.** We say that a partial triangulation $\sigma$ is sparse if

• no arc of $\sigma$ is incident to a marked point on the boundary of $\Sigma$;

• $\sigma$ does not contain any triangle without puncture;

• $\sigma$ does not contain any special mongon;

• if $2 \notin k^\times$, $\sigma$ does not contain any mongon enclosing a unique puncture $M$ with $m_M = 2$.

**Theorem 6.2.** The following hold:

(a) If $\sigma$ is sparse then $\Delta_\sigma \cong kQ_\sigma/I$ where $I$ is the ideal generated by the relations $C_{\vec{u}}$ and the relations $[\vec{u}, \vec{v}][−\vec{v}, −\vec{u}]$ for any triple of oriented arcs $\vec{u}, \vec{v}, \vec{w}$ satisfying $s(\vec{u}) = s(\vec{v})$ and $t(\vec{v}) = t(\vec{w})$. In other terms, $\Delta_\sigma$ is the Brauer graph algebra corresponding to the Brauer graph underlying to $\sigma$.

(b) Any Brauer graph algebra is $\Delta_\sigma$ for a partial triangulation of a marked surface.

**Proof.** (a) is an immediate consequence of Theorem 4.3.

(b) can be proven in the following classical way. Attach to each vertex $M$ of a Brauer graph a closed $2d_M$-gon $A_{\vec{v}_1}B_{\vec{v}_2}A_{\vec{v}_2}B_{\vec{v}_3} \cdots A_{\vec{v}_dM}B_{\vec{v}_dM}$ with $M$ at its centre, where $\vec{v}_1, \ldots, \vec{v}_dM$ are the oriented edges of the Brauer graph starting at $M$, cyclically ordered. Then, define

$$\Sigma := \bigcup_{M} A_{\vec{v}_1}B_{\vec{v}_2}A_{\vec{v}_3}B_{\vec{v}_4} \cdots A_{\vec{v}_dM}B_{\vec{v}_dM} \sim$$
where the equivalence relation ∼ identifies $A_\vec{v}B_\vec{v}$ with $B_{-\vec{v}}A_{-\vec{v}}$ for any oriented edge $\vec{v}$ of the Brauer graph. Then the Brauer graph can be embedded in $\Sigma$ by drawing each oriented edge $\vec{x}$ orthogonally to $A_\vec{x}B_\vec{x} = B_{-\vec{x}}A_{-\vec{x}}$. In this case, no marked point of $\Sigma$ is on the boundary and every polygon has a hole so $\sigma$ is sparse. □

Thanks to Theorem 6.2, we generalize (mildly) a result of Schroll [Sch]. For recall that the trivial extension of a $k$-algebra $A$ is the $k$-algebra $A \oplus \text{Hom}_k(A, k)$ with multiplication defined by $(a, f)(a', f') = (aa', af' + f'a)$.

Proposition 6.3. If $\sigma$ is a partial triangulation of $(\Sigma, \mathcal{M})$ such that every arc of $\sigma$ links two marked points on the boundary, without special monogon, then the trivial extension of $\Delta_{\sigma}$ is (canonically) isomorphic to the Brauer graph algebra with Brauer graph consisting of the arcs of $\sigma$ and multiplicity 1 at each vertex.

In Proposition 6.3 canonically means that $Q_{\sigma}$ is canoni
cally embedded in the quiver defining the Brauer graph algebra and that this embedding induces the isomorphism.

Proof. First of all, notice that all relations $R_{\vec{v}}$ of Proposition 4.12 are of the form $R_{\vec{v}} = [\vec{u}, -\vec{v}][\vec{v}, -\vec{w}]$ (indeed, $f_{\vec{v}}^*\vec{x}$ always goes through a boundary). Thus, up to adding some punctures (without incident arcs), we can suppose that every polygon $P$ has at least three punctures and we can suppose that $m_M = 1$ for any $M \in \mathcal{M}$. Thus, it is enough to check that $\Delta_{\sigma'}$ (see Definition 4.10) is the trivial extension of $\Delta_{\sigma}$ ($\sigma'$ is sparse so $\Delta_{\sigma'}$ is the Brauer graph algebra by Theorem 6.2). Using Definition 10.28 Lemma 10.33, there is an isomorphism $\text{Hom}_k(\Delta_{\sigma'}, k)$ to $\Delta_{\sigma'}$ given by
\[ e_u^* \mapsto e_u, \quad [\vec{u}, \vec{v}]^* \mapsto \lambda_{s(\vec{u})}[\vec{v}, \vec{u}], \quad c_u^* \mapsto e_u \]
(notice that in this case, there are no 2-special arcs). Moreover, $e_u \in \Delta_{\sigma}$ and $c_u \notin \Delta_{\sigma}$ for any arc $u$ of $\sigma$ and exactly one of $[\vec{u}, \vec{v}]$ and $[\vec{v}, \vec{u}]$ is in $\Delta_{\gamma}$ for any choice of two oriented arcs $\vec{u}, \vec{v} \in \sigma$ such that $s(\vec{u}) = s(\vec{v})$. It permits to conclude. □

7. Representation type of $\Delta_{\sigma}$

In this section, we prove the following result, generalizing [GLFS] in the case of triangulations of surfaces and [WW] in the case of Brauer graph algebras:

Theorem 7.1. If $k$ is an algebraically closed field, the algebra $\Delta_{\sigma}$ is representation tame for any partial triangulation $\sigma$ of $(\Sigma, \mathcal{M})$.

When $\Sigma$ has no boundary and $\sigma$ is a triangulation, we can be more precise. Recall the following definition from Ladhani [Lad3]:

Definition 7.2 ([Lad3]). An algebra $A$ is of quasi-quaternion type if it is of tame representation type, symmetric, indecomposable and for any $X \in \text{mod} A$, $\Omega^4 X \cong X$ where $\Omega$ is the syzygy functor in the stable category of $\text{mod} A$.

Proposition 7.3. If $k$ is an algebraically closed field, $\Sigma$ has no boundary and $\sigma$ is a triangulation of $(\Sigma, \mathcal{M})$, then $\Delta_{\sigma}$ is of quasi-quaternion type.
for any (minimal) triangle $\Delta$.

**Remark 7.4.** In [Lad3], Ladkani states a similar result. The final version of this paper, containing proofs, is not available yet.

Proposition 7.3 is a consequence of Theorems 4.9, 7.1 together with the following lemma, which will be proven at the end of Subsection 10.10.

**Proposition 7.5.** If $\Sigma$ has no boundary and $\sigma$ is a triangulation of $(\Sigma, M)$, then there is an exact sequence of $\Delta_\sigma$-bimodules of the form:

$$
\begin{array}{c}
0 \longrightarrow \Delta_\sigma \mathop{\longrightarrow}^\alpha \bigoplus_{u \in \sigma} \Delta_\sigma e_u \mathop{\longrightarrow}^\beta \bigoplus_{u \in \sigma} \Delta_\sigma e_{\bar{u}^+} \otimes e_{\bar{u}} \Delta_\sigma \\
0 \longrightarrow \Delta_\sigma \mathop{\longleftarrow}^\gamma \bigoplus_{u \in \sigma} \Delta_\sigma e_u \mathop{\longleftarrow}^\delta \bigoplus_{u \in \sigma} \Delta_\sigma e_{\bar{u}^+} \otimes e_{\bar{u}} \Delta_\sigma
\end{array}
$$

where $\bar{u}^+$ is the oriented arc following immediately $\bar{u}$ around $s(\bar{u})$ and $-^*$ is the duality of Theorem 4.3.

Notice that Proposition 7.3 does not need $k$ to be an algebraically closed field. From now on, we suppose that $k$ is an algebraically closed field. Following the same strategy than in [GLS], we use the known result for Brauer graph algebras and use the following result of Crawley-Boevey:

**Theorem 7.6 ([CB] Theorem B).** Let $A$ be a finite dimensional $k$-algebra, let $X$ be an irreducible algebraic variety over $k$ and let $g_1, \ldots, g_r : X \rightarrow A$ be morphisms of varieties. For $x \in X$, denote $A_x := A/(g_i(x))_{i=1,\ldots,r}$. If there exists a non-empty open subset $U$ of $X$ such that $A_x \cong A_{x'}$ for all $x, x' \in U$ and there exists $x_0 \in X$ such that $A_{x_0}$ is of tame representation type then $A_x$ is of tame representation type for $x \in U$.

To prove Theorem 7.6 let us take notations of Definition 4.10. The quotient $\Delta_{\sigma'} \rightarrow \Delta_\sigma$ induces a full and faithful functor $\text{mod} \Delta_{\sigma'} \rightarrow \text{mod} \Delta_\sigma$, so we can suppose that $\Sigma$ has no boundary. In the same way, for $\tau \subset \sigma$, the isomorphism $e_\tau \Delta_{\sigma'} e_{\tau'} \cong \Delta_{\tau'}$ of Corollary 4.2 gives a full and faithful functor $\Delta_\sigma \otimes \Delta_{\tau} : \text{mod} \Delta_{\tau'} \rightarrow \text{mod} \Delta_\sigma$ so we can suppose that $\sigma$ is a triangulation. In this case, minimal polygons of $\sigma$ are all triangles (Case a of Figure 4.11).

Define

$$A := kQ_{\sigma} / JC_{\sigma} + (C_{\bar{v}})_{\bar{v}}$$

which is obviously finite dimensional. We will define a family of functions $R_{\bar{v}} : k \rightarrow A$ indexed by oriented arcs $\bar{v}$ of $\sigma$ such that $A_{x} := A/(R_{\bar{v}(x)})_{\bar{v}}$ satisfies that $A_x \cong \Delta_\sigma$ if $x \neq 0$ and $A_0$ is a quotient of the Brauer graph algebra with Brauer graph coinciding with the partial triangulation. It will permit to conclude thanks to Theorem 7.6 because $A_0$ is representation tame.

We first state a key lemma:

**Lemma 7.8.** If $\sigma$ is not one of the two triangulations of Figure 7.7, there exists $p : Q_{\sigma,1} \rightarrow \mathbb{N}_{>0}$ satisfying that

1. for $M \in \mathbb{M}_r$, $\kappa := mM \sum_{i=1}^{d \sigma} p_{[\bar{u}_i, \bar{v}_{i+1}]}$ is a constant independent of $M$, where $\bar{u}_1, \ldots, \bar{u}_{d \sigma}$ are the oriented arcs starting at $M$ cyclically ordered;
2. For any (minimal) triangle $P$ of $\sigma$,
   - $\kappa_P := p_{\omega_P} + p_{\omega_P^0} + p_{\omega_P^1} > \kappa$ if $P$ is a special self-folded triangle;
\[ \kappa_P := p_{\omega^1_P} + p_{\omega^2_P} + p_{\omega^3_P} < \kappa \text{ else.} \]

We finish the construction before proving Lemma 7.8. For any non-zero \( x \). We consider the automorphism \( \psi \) of \( kQ_\sigma \) defined by \( \psi(q) := xq \) for any \( q \in Q_{\sigma,1} \). Then, for any oriented arc \( \vec{v} \) and the minimal triangle \( P \) with sides \( \vec{w}, \vec{u}, \vec{v} \), containing \( t \), we have \( \psi(C_{\vec{v}}) = x^t C_{\vec{v}}, \psi(JC_{\sigma}) = JC_{\sigma} \) and

\[
\psi(\mathcal{R}_{\vec{v}}) = \begin{cases} 
\kappa_{\vec{v}} - \kappa_{\vec{u}, \vec{v}}^\kappa \mathcal{R}_{\vec{v}}(x) & \text{if } P \text{ is special self-folded;} \\
\kappa_{\vec{v}} - \kappa_{\vec{u}, \vec{v}}^\kappa \mathcal{R}_{\vec{v}}(x) & \text{else.}
\end{cases}
\]

so \( \psi \) induces an isomorphism from \( \Delta_\sigma \) to \( A_x \).

Moreover, it is immediate that \( A_0 \) is the quotient of the Brauer graph algebra with Brauer graph induced by the partial triangulation modulo the idempotents corresponding to self-folded edges of special self-folded triangle. It concludes the proof of Theorem 7.1 except in the two cases of Figure 7.7. In this latter case, we use Theorem 8.4. Indeed, by [Ric1 Corollary 2.2], for self-injective algebras, derived equivalence implies stable equivalence, which obviously implies invariance of the representation type. As there are other triangulations of the sphere with four punctures, the result for cases of Figure 7.7 follows.

Proof of Lemma 7.8. Up to rescaling, we can choose the \( p_q \)'s in \( \mathbb{Q}_{\geq 0} \) and \( \kappa = 1 \) as these properties are invariant by scalar multiplication by positive integers. For \( \vec{u}, \vec{v} \) such that \( [\vec{u}, \vec{v}] \) is an arrow, denote \( p_{\vec{u}, \vec{v}} = 1/(d_s(\vec{u}, \vec{v})) \).

We start by proving that the \( p_{\vec{u}, \vec{v}} \)'s almost satisfy (1) and (2). Indeed (1) is satisfied. We know that \( 1/a + 1/b + 1/c < 1 \) except if \( (a, b, c) \) is in \( E = \mathfrak{G}_3 \{ (1, b, c), (2, 2, c), (2, 3, 3), (2, 3, 4), (2, 3, 5), (2, 3, 6), (2, 4, 4), (3, 3, 3) \} \).

So we should identify (minimal) triangles \( P \) of \( \sigma \) such that

\[
(d_{P,1} m_{P,1}, d_{P,2} m_{P,2}, d_{P,3} m_{P,3}) \in E.
\]

Let us take such a \( P \) and choose \( P_1 \) such that \( d_{P,1} m_{P,1} \leq d_{P,2} m_{P,2}, d_{P,3} m_{P,3} \).

If \( d_{P,1} m_{P,1} = 1 \), then it is immediate that \( P \) satisfies (2) (this is the case where \( P \) is a special self-folded triangle).
If \( d_{P_1} \in \{1, 2\} \), an easy case by case analysis using hypotheses on \((\Sigma, \mathcal{M})\) shows that each vertex \( N \) sharing an edge with \( P_1 \) satisfies \( d_{NM} \geq 4 \). Thus the only possibility which makes (2) fail is \((d_{P_1}m_{P_1}, d_{P_2}m_{P_2}, d_{P_3}m_{P_3}) = (2, 4, 4)\). In this case, we have \( p_{\omega_1} + p_{\omega_2} + p_{\omega_3} = \kappa \).

If \( m_{P_1} = 1 \) and \( d_{P_1} = 3 \), a quick analysis shows that at least one \( N \) connected to \( P_1 \) satisfies \( d_{NM} \geq 4 \). Thus, \( P \) satisfy (2) in this case. Then for any (minimal) triangle \( P \) of \( \sigma \),

- \( p_{\omega_1} + p_{\omega_2} + p_{\omega_3} > \kappa \) if \( P \) is a special self-folded triangle;
- \( p_{\omega_1} + p_{\omega_2} + p_{\omega_3} \leq \kappa \) else.

We get the \( \hat{P} \)'s by adding some small perturbations to the \( \hat{P} \)'s. It is immediate that it is possible if every vertex \( M \) with \( d_MM = 4 \) is in at least one triangle where the strict inequality is already satisfied (in this case, it is enough to add small numbers to the \( \hat{P} \)'s appearing in a triangle satisfying the equality and to compensate this addition by subtracting a small number to the \( \hat{P} \) appearing in a triangle satisfying the strict inequality).

Thus, we suppose that \( M \) with \( d_MM = 4 \) appears only in triangles \( P \) satisfying \( p_{\omega_1} + p_{\omega_2} + p_{\omega_3} = \kappa \). We have \( d_M = 4 \) and \( m_M = 1 \) (other cases have already been excluded). Denote by \( \bar{u}_1, \bar{u}_2, \bar{u}_3 \) and \( \bar{u}_4 \) the cyclically ordered oriented arcs starting at \( M \). According to our hypotheses, we can suppose that \( d_i(\bar{u}_1)m_i(\bar{u}_1) = d_i(\bar{u}_3)m_i(\bar{u}_3) = 2 \) and \( d_i(\bar{u}_2)m_i(\bar{u}_2) = d_i(\bar{u}_4)m_i(\bar{u}_4) = 4 \).

We consider two cases:

- If \( t(\bar{u}_2) = M \). In this case, \( \bar{u}_4 = -\bar{u}_2 \). As \( \sigma \) is a triangulation, using Proof of Lemma 10.3, there is a (minimal) triangle with sides \(-\bar{u}_3, \bar{u}_2, \bar{u}_3 \) in this order and therefore \( d_i(\bar{u}_3) = 1 \). In the same way, \( d_i(\bar{u}_1) = 1 \) and \((\Sigma, \mathcal{M})\) is a sphere with three punctures. It contradicts our hypothesis that, in this case, \( m_M \geq 2 \).

- If \( t(\bar{u}_2) \neq M \). As before, there is a (minimal) triangle with sides \(-\bar{u}_3, \bar{u}_2, \bar{v} \) in this order for a certain \( \bar{v} \). Notice that \( s(\bar{v}) = t(\bar{u}_2) \) and \( t(\bar{v}) = t(\bar{u}_3) \). So \( v \) is not a \( u_i \). So, using the hypotheses, \( d_i(\bar{u}_3) = 2 \) and \( m_i(\bar{u}_1) = 1 \). We get in the same way triangles with sides \(-\bar{u}_4, \bar{u}_3, -\bar{v} \) and \(-\bar{u}_1, \bar{u}_4, -\bar{v}' \) for a certain \( \bar{v}' \). An easy analysis proves that these four triangles cover \( \Sigma \) and therefore we are in the second case of Figure 7. It contradicts our hypothesis. \( \square \)

8. FLIPPING PARTIAL TRIANGULATIONS

We start by defining a combinatorial operation on partial triangulations of \( \sigma \).

**Definition 8.1.** Let \( \bar{u} \) be an oriented arc of \( \sigma \). If \( u \) is the only arc incident to \( s(\bar{u}) \), we define \( \bar{u}^+ := \emptyset \). If \( \bar{u} \) is followed by \( \bar{v} \) around \( s(\bar{u}) \) with \( \bar{v} \neq -\bar{u} \), we define \( \bar{u}^+ := \bar{v} \). If \( \bar{u} \) is followed by \(-\bar{u} \) and \( u \) is not the only arc incident to \( s(\bar{u}) \), we define \( \bar{u}^+ := (-\bar{u})^+ \). We says that \( u \) is close to the boundary if \( \bar{u}^+ \) or \((-\bar{u})^+ \) is a boundary edge.

Suppose now that \( \bar{u} \) is not close to the boundary. Let \( \tau \) be the partial triangulation obtained from \( \sigma \) by removing \( u \). The *mutation* or *flip* of \( \sigma \) with respect to \( u \) is the partial triangulation \( \mu_u(\sigma) \) obtained from \( \tau \) by adding the oriented arc \( \bar{u}^* \) constructed in the following way:
Figure 8.2. Flip of an oriented arc in a partial triangulation

(a) if $\vec{u}^+ \neq \emptyset$, it starts at $t(\vec{u}^+)$ just after $-\vec{u}^+$ in the cyclic ordering around $t(\vec{u})$, it follows $-\vec{u}^+$, winds around $s(\vec{u})$ counterclockwise;
if $\vec{u}^+ = \emptyset$, it starts at $s(\vec{u})$ at the same position as $\vec{u}$;

(b) if follows $\vec{u}$;
(c) if $(-\vec{u})^+ \neq \emptyset$ it winds around $t((-\vec{u})^+)$ just after $(-\vec{u})^+$;
if $(-\vec{u})^+ = \emptyset$, it ends at $t(\vec{u})$ at the same position as $\vec{u}$.

To summarize Definition 8.1, we depict in Figure 8.2 the three main possibilities (up to reorientation of $\vec{u}$).

We need also to define a mutation for coefficients:

**Definition 8.3.** We define coefficients $\mu_u(\lambda)_M = \lambda^*_M$ in the following way:

- If there exists a monogon $\vec{x}$ enclosing a unique puncture $M$ and either $\vec{u} = \pm \vec{x}$ and $\sigma$ does not contain an arc incident to $M$, or $\vec{u}$ is incident to $M$,
  \[ \lambda^*_M := -\lambda_M; \quad \lambda^*_{s(\vec{x})} := \nu_{s(\vec{x})}^{-1} \lambda_{s(\vec{x})}; \quad \lambda^*_N := \lambda_N \quad \text{for } N \in M \setminus \{s(\vec{x}), M\}. \]
- For $\vec{x} = \pm \vec{u}$, if $(-\vec{x})^+ = \emptyset$ and $\vec{x}$ is not a side of a self-folded triangle,
  \[ \lambda^*_{t(\vec{x})} := -\lambda_{t(\vec{x})}; \quad \lambda^*_{s(\vec{x})} := (-1)^{m_{s(\vec{x})}} \lambda_{s(\vec{x})}; \quad \lambda^*_N := \lambda_N \quad \text{for other } N \in M. \]
- In any other case, $\lambda^*_N := \lambda_N$ for any $N \in M$.

The operation which maps $\sigma$ to $\mu_u(\sigma)$ is sometimes called *Kauer move* as it was first introduced in [Kau] for Brauer graphs. The main theorem of this section generalizes [Kau] (see also [Aih]):

**Theorem 8.4.** If $u$ is an arc of a partial triangulation $\sigma$ of $(\Sigma, M)$ which is not close to the boundary, then $\Delta^\lambda_\sigma$ and $\Delta^\mu_u(\lambda)_\sigma$ are derived equivalent.

**Example 8.5.** We consider the two following partial triangulations of a disc with three punctures and no marked point on the boundary:
They are related by a flip so the following algebras, obtained for $\lambda_M = \lambda_N = \lambda_P = 1$ and $m_M = m_N = m_P = m$ are derived equivalent:

\[
\begin{pmatrix}
  y & x \\
  x & y
\end{pmatrix} \quad k \left( x^2 - (yx)^{m-1} y, y^2 \right)
\quad \quad \text{and} \quad \quad
\begin{pmatrix}
  \beta_2 \gamma \alpha - (\gamma_1 \gamma_2 \beta_2 \beta_1)^{m-1} \gamma_1 \gamma_2 \beta_2, \\
  \alpha \beta_1 - (\beta_1 \gamma_1 \gamma_2 \beta_2)^{m-1} \beta_1 \gamma_1 \gamma_2, \\
  \beta_1 \beta_2 - \alpha^{m-1} \gamma_1 \delta, \\
  \delta^m - (\gamma_2 \beta_2 \beta_1 \gamma_1)^{m}
\end{pmatrix}.
\]

**Remark 8.6.** We can of course reverse this construction to obtain an inverse of $\mu_\sigma$ which also gives a derived equivalence.

Before giving the proof, we deal with two particular cases coming from special monogons:

**Remark 8.7.**

- In case (F2), if $\vec{u}$ encloses a special monogon and the arc $v$ pointing to the special puncture is not in $\sigma$. By Proposition 9.1 and the discussion preceding it, $\Delta_\sigma^\lambda$ is Morita equivalent to $\Delta_{\sigma \cup \{v\}}^{\mu_\sigma(\lambda)}$. Then, we get easily that $\Delta_{\mu_\sigma(\sigma)}^{\mu_\sigma(\lambda)}$ is Morita equivalent to $\Delta_{\mu_\sigma(\sigma)}^{\mu_\sigma(\lambda)}$.

- If $\vec{u}$ is the double side of a special self-folded triangle, pointing toward the special puncture, $\mu_\sigma(\sigma) \cong \sigma$ and $\Delta_{\mu_\sigma(\sigma)}^{\mu_\sigma(\lambda)}$ is Morita equivalent to $\Delta_\sigma^\lambda$ according to Proposition 9.1 so the result is trivial in this case.

Notice that, in this case, $\Delta_{\mu_\sigma(\sigma)}^{\mu_\sigma(\lambda)}$ is not obtained by a non-trivial tilting. Let $\vec{v}$ be the oriented side enclosing the special monogon with special puncture $t(\vec{u})$ or $s(\vec{u})$. If $\vec{v}$ is not close to the boundary, a non-trivial tilting can be obtained by putting $\mu_\sigma(\sigma) = \mu_\sigma(\sigma)$ (notice that $\Delta_{\mu_\sigma(\sigma)}^{\mu_\sigma(\lambda)} \neq \Delta_{\mu_\sigma(\sigma)}^{\mu_\sigma(\lambda)}$). Indeed, using Proposition 9.1, $\Delta_\sigma^\lambda$ is Morita equivalent to $\Delta_{\mu_\sigma(\sigma)}^{\mu_\sigma(\lambda)}$, "exchanging" the idempotents $[\vec{v}, -\vec{v}]$, and $e_\sigma = [\vec{v}, -\vec{v}]$, so computing the algebra of $\mu_\sigma(\sigma)$ with respect to the coefficients $\mu_\sigma(\lambda)$ consists to a tilting at the idempotent $[\vec{v}, -\vec{v}]$ which is equivalent to $e_\sigma$.

In view of Remark 8.7, we suppose that, in Case (F2), $\vec{u}$ does not enclose a special monogon, and that, in Case (F1), $\vec{u}$ is not the double side of a special self-folded triangle.

To prove Theorem 8.4, we use the Okuyama-Rickard complex [1]. We prove that it is tilting under our assumptions. From now on, $\sigma, \vec{u}, \tau, \lambda^* := \mu_\sigma(\lambda)$ and $\lambda := \mu_\sigma(\sigma)$ are fixed as in Definitions 8.1 and 8.3. We denote

\[
\mathbf{e} := \begin{cases}
  e_\sigma - \lambda_M [\vec{u}, -\vec{u}] & \text{in case (F3), if $\vec{u}$ encloses a special monogon;} \\
  e_\sigma - \lambda_M [-\vec{u}, \vec{u}] & \text{in case (F3), if $-\vec{u}$ encloses a special monogon;} \\
  e_\sigma & \text{else,}
\end{cases}
\]

and $X_\mathbf{e}$ is the maximal indecomposable module supported by $\mathbf{e}$. 
We consider the following object of the homotopy category $K^b(\text{proj } \Delta_\sigma)$ of right $\Delta_\sigma$-modules: $T := e_\tau \Delta_\sigma \oplus P_u^*$ where $P_u^*$ is the following complex concentrated in degree 0 and 1:

$$e_{\vec{u}} + \Delta_\sigma \oplus e_{\vec{-u}} + \Delta_\sigma \xrightarrow{[\alpha_1 \alpha_2]} e_u \Delta_\sigma$$

where $\alpha_1 := [\vec{u}, \vec{u}]^+$ and $\alpha_2 := [-\vec{u}, (-\vec{u})^+]$ with the convention that $e_\emptyset = 0$, $[\vec{u}, \emptyset] = [-\vec{u}, \emptyset] = 0$. This complex can be understood as a projective presentation of $X_e$. In other terms, $[\alpha_1 \alpha_2]$ is a minimal right add$(e_\tau \Delta_\sigma)$-approximation of $e_u \Delta_\sigma$. Notice that $P_u^*$ contains a split direct summand if $u$ encloses a special monogon.

This is known that $T$ is a tilting complex when $\Delta_\sigma$ is symmetric. We prove that it is still the case here:

**Lemma 8.8.** The complex $T$ is tilting. In other terms,

(a) $\text{Hom}_{K^b(\text{proj } \Delta_\sigma)}(T, T[i]) = 0$ for any $i \neq 0$ where $[i]$ is the $i$-shift functor;

(b) $\Delta_\sigma$ is in the triangulated subcategory of $K^b(\Delta_\sigma)$ generated by $T$ (and therefore $K^b(\text{proj } \Delta_\sigma)$ is generated by $T$).

**Proof.** (a) The only possibly non-trivial terms of $\text{Hom}_{K^b(\text{proj } \Delta_\sigma)}(T, T[i])$ are

- $\text{Hom}_{K^b(\text{proj } \Delta_\sigma)}(e_\tau \Delta_\sigma, P_u^*[1])$: as there is no morphism from $e_\tau \Delta_\sigma$ to $X_e$, it is immediate that any element of $\text{Hom}_{K^b(\text{proj } \Delta_\sigma)}(e_\tau \Delta_\sigma, P_u^*[1])$ is homotopic to 0.
- $\text{Hom}_{K^b(\text{proj } \Delta_\sigma)}(P_u^*, e_\tau \Delta_\sigma[-1])$: such a non-zero morphism gives a non-zero map from $X_e$ to $e_\tau \Delta_\sigma$. But, using Theorem 4.5, the socle of $e_\tau \Delta_\sigma$ has to be concentrated on $\tau$ and arcs which are close to the boundary. As $u$ is not close to the boundary, we get a contradiction.
- $\text{Hom}_{K^b(\text{proj } \Delta_\sigma)}(P_u^*, P_u^*[1])$: it is similar to the previous case.
- $\text{Hom}_{K^b(\text{proj } \Delta_\sigma)}(e_\tau \Delta_\sigma, P_u^*[1])$: a morphism in this space is induced by a morphism from $e_{\vec{u}} + \Delta_\sigma \oplus e_{\vec{-u}} + \Delta_\sigma$ to $e_u \Delta_\sigma$. As $[\alpha_1 \alpha_2]$ is a right add$(e_\tau \Delta_\sigma)$-approximation, such a morphism is homotopic to 0.

(b) It is enough to prove that $e_u \Delta_\sigma$ is in the triangulated category generated by $T$. This is immediate: $e_u \Delta_\sigma$ is isomorphic to the cone of the canonical map $P_u^* \to e_{\vec{u}} + \Delta_\sigma \oplus e_{\vec{-u}} + \Delta_\sigma$.

Then, to prove Theorem 8.3 by the famous Theorem of Rickard [Ric2], it is enough to prove the following proposition:

**Proposition 8.9.** There is an isomorphism $\text{End}_{K^b(\text{proj } \Delta_\sigma)}(T) \cong \Delta^*_\sigma$.

Proposition 8.9 is proven in Subsection 10.11. We give a useful corollary which permits to get easily some isomorphisms:

**Corollary 8.10.** We suppose that $\Sigma$ has no boundary. Fix three partial triangulations $\sigma_1$, $\sigma_2$ and $\sigma^\circ$ of $(\Sigma, \mathcal{M})$ such that:

(a) $\sigma_1 \cup \sigma^\circ$ is a triangulation;

(b) $\sigma_1 \cap \sigma^\circ = \sigma_2 \cap \sigma^\circ = \emptyset$;

(c) any triangle of $\sigma_1 \cup \sigma^\circ$ that has a side in $\sigma^\circ$ has all its sides in $(\sigma_1 \cap \sigma_2) \cup \sigma^\circ$.

We also consider a second family of coefficient $(\mu_M)_{M \in \mathcal{M}}$.

Then, for any isomorphism $\psi_1 : \Delta^\mu_{\sigma_1} \to \Delta^\lambda_{\sigma_1}$ such that $\psi_1(e_u) = e_u$ for all $u \in \sigma_1$, there exists an isomorphism $\psi_2 : \Delta^\mu_{\sigma_2} \to \Delta^\lambda_{\sigma_2}$ such that $\psi_2(e_u) = e_u$. 

for all $u \in \sigma_2$ and $\psi_2|_{\Delta^u_{\sigma_1 \cap \sigma_2}} = \psi_1|_{\Delta^u_{\sigma_1 \cap \sigma_2}}$, where $\Delta^u_{\sigma_1 \cap \sigma_2}$ is identified to subalgebras of $\Delta^u_{\sigma_1}$ and $\Delta^u_{\sigma_2}$ via Corollary 5.2.

**Remark 8.11.** Proof of Corollary 8.10 only relies on case (F3). This is important as we will use it for the proof of case (F1).

**Proof.** Let us first suppose that $\sigma_2 \cup \sigma^o$ is a triangulation. Then, it is a classical fact that there exists a sequence of flips from $\sigma_1 \cup \sigma^o$ to $\sigma_2 \cup \sigma^o$ which do not involve arcs in their intersection $(\sigma_1 \cap \sigma_2) \cup \sigma^o$. Notice also that all flips involved are of type (F3) as other flips stabilize triangulations.

Notice that the first flip $u$ of the sequence can be applied to $\sigma_1$. Indeed, $u \notin (\sigma_1 \cap \sigma_2) \cup \sigma^o$, so by (c) the quadrilateral $u$ is a diagonal of has all its sides in $\sigma_1$. Moreover, the partial triangulation $\mu_u(\sigma_1)$ satisfies the same hypotheses (a), (b) and (c) as $\sigma$. Thus, we suppose that $\sigma_2 = \mu_u(\sigma_1)$, and the result is obtained by induction.

We consider the tilting object $T$ of $K^b(\proj \Delta^\lambda_{\sigma_1})$ defined for $u$. As $T$ is a projective presentation of $X_\sigma$, we get that $\psi_1^*(T)$ is the tilting object defined for $u$ in $K^b(\proj \Delta^\lambda_{\sigma_2})$. Then, using Proposition 8.9 we get an isomorphism

$$\psi_2 : \Delta^u_{\sigma_2} \xrightarrow{\varphi_2} \End_{K^b(\proj \Delta^\lambda_{\sigma_2})}(\psi_1^*(T)) \xrightarrow{(\psi_1^*)^{-1}} \End_{K^b(\proj \Delta^\lambda_{\sigma_1})}(T) \xrightarrow{\varphi_1} \Delta^\lambda_{\sigma_2}$$

(notice that we have $\mu_u(\lambda) = \lambda$ and $\mu_u(\mu) = \mu$ as we are in case (F3)). The fact that $\psi_2$ satisfies $\psi_2(e_u) = e_u$ for $u \in \sigma_2$ and $\psi_2|_{\Delta^u_{\sigma_1 \cap \sigma_2}} = \psi_1|_{\Delta^u_{\sigma_1 \cap \sigma_2}}$ is an immediate consequence of the construction of $\varphi_2$, $\varphi_1$ (see Lemma 10.38).

Finally, if $\sigma_2 \cup \sigma^o$ is not a triangulation, the result is obtained by completing $\sigma_2 \cup \sigma^o$ to a triangulation and using Corollary 4.2. \qed

**9. Dealing with special monogons**

Let us fix $\bar{u} \in \sigma$ enclosing a special monogon with special puncture $M$. We have that $[\bar{u}, -\bar{u}]/\lambda_M$ is an idempotent and therefore also $e_{s(\bar{u})} - [\bar{u}, -\bar{u}]/\lambda_M$. It leads to several observations. First of all, if $\sigma$ contains the arc $v$ joining $s(\bar{u})$ to $M$, $\Gamma_\sigma$ and $\Delta_\sigma$ are not basic. In fact $\Gamma_\sigma$ and $\Gamma_\sigma \setminus \{v\}$ are Morita equivalent in this case. This is also the case for $\Delta_\sigma$ and $\Delta_\sigma \setminus \{v\}$.

Another aspect of this remark is that, in view of Corollary 11.2, $e\Delta_\sigma e$ can be obtained as an algebra of partial triangulation for any idempotent $e$ except if $e \cdot e_{s(\bar{u})} = e_{s(\bar{u})} - [\bar{u}, -\bar{u}]/\lambda_M$ for a special monogon. The next proposition gives a change of basis which permit to get rid of this issue:

**Proposition 9.1.** We suppose that $\sigma$ does not contain the arc connecting $M$ to $s(\bar{u})$. For $N \in \mathbb{M}$, denote

$$\mu_N = \begin{cases} -\lambda_N & \text{if } N = M; \\
\nu_M^{-1} \lambda_N & \text{if } N = s(\bar{u}); \\
\frac{1}{\lambda_M} & \text{else.} \end{cases}$$

Then there is an isomorphism $\psi : \Delta^\mu_{\sigma} \to \Delta^\lambda_{\sigma}$ satisfying:

$$\psi([\bar{u}, -\bar{u}]) = [\bar{u}, -\bar{u}] - \lambda_M e_u;$$

$$\psi(e_v) = e_v$$

for any $v \in \sigma$;

$$\psi([\bar{x}, -\bar{x}]) = [\bar{x}, -\bar{x}]$$

for any $\bar{x} \neq \bar{u}$ special.

Before proving Proposition 9.1 we state the following corollary:
Corollary 9.2. For any idempotent $e$ of $\Delta_\sigma$, $e\Delta_\sigma e$ is Morita equivalent to an algebra of partial triangulation of $(\Sigma, M)$.

Proof. Using the preliminary remark, we can suppose that $\sigma$ does not contain any arc incident to a special puncture up to Morita equivalence. Then, Proposition 9.1 permits to exchange the role of idempotents $[\vec{u}, -\vec{u}]\lambda_M$ and $e_{s(\vec{u})} - [\vec{u}, -\vec{u}]\lambda_M$ for each special monogon to make sure that, in any case, $e \cdot e_{s(\vec{u})} \neq e_{s(\vec{u})} - [\vec{u}, -\vec{u}]\lambda_M$. Thus, Corollary 4.2 permits to conclude. □

Remark 9.3. Proposition 9.1 corresponds, using the vocabulary of tagged triangulation [CILF], to changing the tags at $M$.

The proof of Proposition 9.1 relies on this key lemma:

Lemma 9.4. Suppose that $\sigma$ contains a triangle without puncture or hole, with sides $-\vec{u}$, $\vec{v}$ and $-\vec{w}$ such that $s(\vec{v}) = t(\vec{v})$:

\[ \cdots \vec{v} \cdots \vec{w} \rightarrow M \rightarrow \cdots \vec{u} \cdots \]

Then Proposition 9.1 holds for $\sigma$.

Lemma 9.4 is proven in Subsection 10.12. We deduce Proposition 9.1.

Proof of Proposition 9.1. Thanks to Definition 4.10 and easy observations, we can suppose without loss of generality that $\Sigma$ has no boundary. The strategy is to use Corollary 8.10. Let $\sigma_2 = \sigma$ and take $\sigma^\circ = \{\vec{t}\}$ where $\vec{t}$ is the special arc pointing at $M$. Then we take a partial triangulation $\sigma_1$ which contains all special monogons of $\sigma$, which does not contain $\vec{t}$, which satisfies the hypothesis of Lemma 9.4 and which is maximal for these properties (it is an easy observation that taking all special monogons of $\sigma$ does not prevent to complete the partial triangulation as in Lemma 9.4). According to Lemma 9.4, there is an isomorphism $\psi_1 : \Delta_\mu^{\sigma_1} \rightarrow \Delta_\lambda^{\sigma_1}$ satisfying

\[
\psi_1([\vec{u}, -\vec{u}]) = [\vec{u}, -\vec{u}] - \lambda_M e_u;
\]

\[
\psi_1(e_v) = e_v \quad \text{for any } v \in \sigma;
\]

\[
\psi_1([\vec{x}, -\vec{x}]) = [\vec{x}, -\vec{x}] \quad \text{for any } \vec{x} \neq \vec{u} \text{ special}.
\]

Then we can apply Corollary 8.10 to get an isomorphism $\psi_2 : \Delta_\mu^{\sigma_2} \rightarrow \Delta_\lambda^{\sigma_2}$ satisfying the same conclusions. □

10. Proofs

10.1. Proof of Proposition 2.7: (1) The element $C_\sigma \in \Gamma_\sigma^\circ$ clearly does not depend on the choice of the orientations by definition of $I_\sigma^\circ$.

(2) Let two oriented edges $\vec{u}$ and $\vec{v}$ of $\sigma$ such that $s(\vec{u}) = s(\vec{v})$. We get

\[ [\vec{u}, \vec{v}]C_\sigma = [\vec{u}, \vec{v}] \cdot [\vec{v}, \vec{u}] = [\vec{u}, \vec{v}] = C_\sigma[\vec{u}, \vec{v}] \]
and, as $C_\sigma$ clearly commutes with idempotents of $Q_\sigma$, $C_\sigma$ is in the centre of $\Gamma_\sigma^0$. Moreover, we can grade $kQ_\sigma$ by letting the degree of an arrow $[\vec{u},\vec{v}]$ be $1/d_{s(\vec{u})}m_{s(\vec{u})}$ where $d_{s(\vec{u})}$ is the number of oriented edges of $\sigma$ starting at $s(\vec{u})$. Thus, it is clear that all the $C_\ell$ have degree 1 and $\Gamma_\sigma^0$ is graded. As the degree of $C_\ell^\sigma$ is $\ell$, the $C_\ell^\sigma$’s are linearly independent in $\Gamma_\sigma^0$ so $k[C_\sigma]$ is included in the centre of $\Gamma_\sigma^0$.

(3) Let us prove that for a path $\alpha$ and two scalar multiple of paths $\omega$ and $\omega'$, if $\alpha \omega = \alpha \omega'$ in $\Gamma_\sigma^0$ then $\omega = \omega'$ in $\Gamma_\sigma^0$. We denote by $\ell$ the minimal number of relations $C_\ell^\omega$ to apply to relate $\alpha \omega$ and $\alpha \omega'$ and by $n$ the length of $\alpha$. We will do an induction on $(\ell, n)$ and we will also prove that we need at most $\ell$ relations to go from $\omega$ to $\omega'$. If $\ell = 1$, the result is immediate. Suppose that $\ell > 1$.

Suppose that $n = 1$, i.e. $\alpha = [\vec{u}, \vec{v}]$ is an arrow of $Q_\sigma$. Fix a sequence of arrows $\alpha_0 = \alpha$, $\alpha_1, \ldots, \alpha_{\ell-1}, \alpha_\ell = \alpha$ and a sequence of scalar multiples of paths $\omega_0 = \omega_1, \ldots, \omega_{\ell-1}$, $\omega_{\ell} = \omega'$ such that $\alpha_i \omega_i$ is related with $\alpha_{i-1} \omega_{i-1}$ in one step (i.e. by one $C_{\alpha_i}^\omega$). If $\alpha_i = \alpha$ for some $i \neq 0, \ell$, applying the induction hypothesis is immediate. So we can suppose that $\alpha_1 = \alpha_2 = \cdots = \alpha_{\ell-1} = [\vec{u}, \vec{v}]$ where $[\vec{u}, \vec{v}]$ is the only other arrow starting at $u$. The first and last relations applied have to be $C_\vec{u}$, so one must have

$$\omega_1 = [\vec{v}], -\vec{u}[\omega_1'] \text{ and } \omega_{\ell-1} = [\vec{v}], -\vec{u}[\omega_{\ell-1}'].$$

These two scalar multiples of paths are equal in $\Gamma_\sigma^0$ in $\ell - 2$ steps so, by induction hypothesis, $\omega_1' = \omega_{\ell-1}'$ in $\Gamma_\sigma^0$. We have $\omega = [\vec{v}], \vec{u}[\omega_1']$ and $\omega' = [\vec{v}], \vec{u}[\omega_{\ell-1}']$ so the result is true in this case.

Suppose now that $\alpha = \alpha_0 \alpha'$ where $\alpha_0$ has length 1 and $\alpha'$ has length $n - 1$. By induction hypothesis, as $\alpha_0 \alpha' \omega$ and $\alpha_0 \alpha' \omega'$ are equal in $\Gamma_\sigma^0$, then $\alpha' \omega$ and $\alpha' \omega'$ are equal. Applying once again the induction hypothesis, we get $\omega' = \omega$ in $\Gamma_\sigma^0$.

According to (2) and by definition of $C$-irreducible paths, it is immediate that elements $C_\ell^\sigma \omega$ for $\ell \geq 0$ and $\omega$ a $C$-irreducible path generates $\Gamma_\sigma^0$ over $k$. Thus, it is enough to prove that they are linearly independent over $k$.

Let $E$ be the set of paths of $Q_\sigma$. Let $\sim$ be the smallest equivalence relation on $E$ such that $\omega_1 [\vec{u}, \vec{u}]^{m_{s(\vec{u})}} \omega_2 \sim \omega_1 [-\vec{u}, -\vec{u}]^{m_{s(\vec{u})}} \omega_2$ for any $\omega_1, \omega_2 \in E$ and any $\vec{u} \in \sigma$ such that $\omega_1 [\vec{u}, \vec{u}]^{m_{s(\vec{u})}} \omega_2 \neq 0$. Suppose that $C_\ell^\sigma \omega = \lambda C_\ell^\sigma \omega'$ for $\ell \leq \ell'$ and $\omega$, $\omega'$ two $C$-irreducible paths and $\lambda \in k$. Then, according to the previous discussion, we have $\omega = \lambda C_{\ell'}^\sigma - \ell \omega'$. As $\omega$ does not appear in any relation of $\Gamma_\sigma^0$, we get that $\ell' = \ell$, $\lambda = 1$ and $\omega = \omega'$. Thus, there is at most one multiple of a $C_\ell^\sigma \omega$ in each equivalence class of $E$. As relations relate only multiple of paths in the same equivalence class, it implies that the $C_\ell^\sigma \omega$’s are linearly independent over $k$. 

\[\square\]

10.2. Proof of Proposition \[2.10\] We need the following technical preliminaries to be able to construct easily polygons.

**Definition 10.1.** Let us define

$$\Delta^0 := \{(x, y) \in \mathbb{R}^2 \mid x, y \geq 0 \text{ and } x + y \leq 1\}$$

$$\subset \Delta := \{(x, y) \in \mathbb{R}^2 \mid x, y \geq 0 \text{ and } x + y \leq 1\}.$$

We call angle of $\sigma$ a continuous map $\hat{\phi} : \Delta \rightarrow \Sigma$ such that
\( \phi \) is injective and oriented on \( \Delta^0 \);
\( \phi \mathbb{I}_{[0,1] \times \{0\}} \) is an oriented edge called first side of \( \phi \);
\( \phi \mathbb{I}_{\{0\} \times [0,1]} \) is an oriented edge called second side of \( \phi \).

The interior of \( \phi \) is the image of \( \Delta^0 \).

We define now
\[
\Delta(1) := \{ (x, y) \in \mathbb{R}^2 | y \geq 0 \text{ and } x \geq 2y \text{ and } x + y \leq 1 \} \subset \Delta
\]
\[
\Delta(2) := \{ (x, y) \in \mathbb{R}^2 | x \geq 0 \text{ and } y \geq 2x \text{ and } x + y \leq 1 \} \subset \Delta.
\]

We say that two angles \( \hat{\alpha} \) and \( \hat{\beta} \) such that the second side of \( \hat{\alpha} \) is opposite to the first side of \( \hat{\beta} \) are compatible if
- \( \hat{\beta}^{-1}(\hat{\alpha}(\Delta)) \setminus \{(0,1)\} = \Delta(1) \);
- \( \hat{\alpha}^{-1}(\hat{\beta}(\Delta)) \setminus \{(0,1)\} = \Delta(2) \);
- \( \hat{\beta}|_{\Delta(1)} = \hat{\alpha}|_{\Delta(2)} \circ \psi \) where \( \psi : \Delta(1) \to \Delta(2) \) is the oriented homeomorphism defined by \( \varphi(x, y) = (y, 1 - x + y) \).

We will prove the following more precise version of Proposition 2.10:

**Lemma 10.2.** We consider a sequence of oriented edges \( \vec{u}_1, \vec{u}_2, \ldots, \vec{u}_n \), with indices considered modulo \( n \), such that \( t(\vec{u}_i) = s(\vec{u}_{i+1}) \) for \( i = 1 \ldots n \).

The following conditions are equivalent:

(i) There is a \( n \)-gon having oriented sides \( \vec{u}_1, \vec{u}_2, \ldots, \vec{u}_n \) in this order.

(ii) There exist a sequence \( \hat{\alpha}_1, \hat{\alpha}_2, \ldots, \hat{\alpha}_n \) of angles of \( \sigma \) such that
- the first side of \( \hat{\alpha}_i \) is \( \vec{u}_{i+1} \) and its second side is \( -\vec{u}_i \);
- if \( j \neq i - 1, i, i + 1 \) then the interiors of \( \hat{\alpha}_i \) and \( \hat{\alpha}_j \) do not intersect;
- \( \hat{\alpha}_{i+1} \) and \( \hat{\alpha}_i \) are compatible for any \( i \).

(iii) The following conditions are satisfied:
- if \( i \neq j \) then \( \vec{u}_i \neq \vec{u}_j \);
- for each \( i \), if \( \vec{u}_i \) is a boundary component, then it is oriented clockwise around the boundary;
- for any \( i \) and \( j \) such that \( M := s(\vec{u}_i) = s(\vec{u}_j) \), we have that \( -\vec{u}_{i-1}, \vec{u}_i, -\vec{u}_{i-1} \) and \( \vec{u}_j \) are ordered clockwise around \( M \);
- for any oriented boundary component \( \vec{v} \) and \( i \) such that \( M := s(\vec{v}) = s(\vec{u}_i) \), we have that \( -\vec{u}_{i-1}, \vec{u}_i \) and \( \vec{v} \) are ordered clockwise around \( M \).

**Proof.** \((i) \Rightarrow (ii)\) Suppose that \( \vec{u}_1, \ldots, \vec{u}_n \) forms a \( n \)-gon \( P : \mathcal{P}^i(n) \to \Sigma \).

It is immediate that we can take angles \( \hat{\alpha}_i^o \) in \( \mathcal{P}^o(n) \) satisfying the conditions expected for the \( \hat{\alpha}_i \)'s. Then the angles \( \hat{\alpha}_i := P \circ \hat{\alpha}_i^o \) satisfy the expected conditions.

\((ii) \Rightarrow (i)\) We denote by \( \overline{\mathcal{P}^o(n)} \) the closure of \( \mathcal{P}^o(n) \) in \( \mathcal{P}(n) \). We fix \( \hat{\alpha}_i \) as in \((ii)\). We also fix angles \( \hat{\alpha}_i^o \) of \( \overline{\mathcal{P}^o(n)} \) satisfying the same hypotheses and such that the maps \( \hat{\alpha}_i^o : \Delta \to \overline{\mathcal{P}^o(n)} \) are injective and their images cover entirely \( \overline{\mathcal{P}^o(n)} \) (this is easy). Let us prove that there is a unique map \( P : \overline{\mathcal{P}^o(n)} \to \Sigma \) such that \( \hat{\alpha}_i = P \circ \hat{\alpha}_i^o \) for every \( i \). First of all, as the images of the \( \hat{\alpha}_i^o \)'s cover \( \overline{\mathcal{P}^o(n)} \), such a map has to be unique if it exists. Suppose that \( \hat{\alpha}_i^o(x) = \hat{\alpha}_j^o(y) \) for some \( i, j \) and \( x, y \in \Delta \). If \( i = j \) then \( x = y \) by hypothesis and therefore \( \hat{\alpha}_i(x) = \hat{\alpha}_j(y) \). If \( i \neq j \), the hypotheses imply that \( j = i \pm 1 \), say \( j = i + 1 \) without loss of generality, and \( y = \psi(x) \) by definition of compatibility of angles (using also the fact that \( \hat{\alpha}_i^o \) and \( \hat{\alpha}_j^o \) are injective),
and then \( \hat{\alpha}_i(x) = \hat{\alpha}_j(y) \) also by definition of compatibility. We proved that \( P \) is well defined. By continuity of the \( \hat{\alpha}_i \)'s and the \( \hat{\alpha}_i^\circ \)'s, \( P \) is also continuous. The map \( P \) is also injective on the interior of \( \mathcal{P}^o(n) \) and maps injectively each side of \( \mathcal{P}^o(n) \) to an edge of \( \Sigma \) by definition of angles and compatibility. Finally, up to filling the hole of \( P \) if possible, \( P \) is an \( n \)-gon.

\( \text{(ii)} \Rightarrow \text{(iii)} \) We fix \( \hat{\alpha}_i \) as in \( \text{(ii)} \). Notice that \( \bar{u}_i = \hat{\alpha}_{i-1}|_{[0,1] \times \{0\}} \) so we immediately get that \( \bar{u}_i \neq \bar{u}_j \) by hypotheses on the angles and definition of compatible angles. In the same way, if \( \bar{u}_i \) is an oriented boundary edge, as it has a left neighbourhood which is in the image of \( \hat{\alpha}_{i-1} \), it has to winds around the boundary in the clockwise direction. Finally, suppose that \( M := s(\bar{u}_i) = s(\bar{u}_j) \). If \( i = j \), clearly \( \bar{u}_{i-1}, \bar{u}_i, -\bar{u}_{j-1} \) and \( \bar{u}_j \) are ordered clockwise around \( M \). If \( i \neq j \), we use the fact that \( -\bar{u}_{i-1} \) and \( \bar{u}_i \) are the second and first sides of \( \hat{\alpha}_{i-1} \) and the fact that the interior of \( -\bar{u}_{j-1} \) does not intersect the image of \( \hat{\alpha}_{i-1} \) to see that \( -\bar{u}_{i-1}, \bar{u}_i \) and \( -\bar{u}_{j-1} \) are ordered clockwise. The rest of the orderings comes by analogous arguments. The last point is proved in the same way.

\( \text{(iii)} \Rightarrow \text{(ii)} \) For each \( i \), let us fix an injective oriented map \( f_i : \Delta^{(1)} \rightarrow \Sigma \) such that

- \( f_i|_{[0,1] \times \{0\}} = \bar{u}_i \);
- the sets \( f_i(\Delta^{(1)} \setminus ([0,1] \times \{0\})) \) do not intersect any of the \( \bar{u}_i \)'s, the boundary of \( \Sigma \) and do not intersect each other.

It is clearly possible by the first two hypotheses of \( \text{(iii)} \). For each \( i \), we can choose an arc \( \gamma_i \) of \( \Sigma \) satisfying

- \( \gamma_i \) links \( f_i(2/3, 1/3) \) to \( f_{i+1}(2/3, 1/3) \);
- \( \gamma_i \) is homotopic to \( f_i([2/3, 1/3], (1,0)]) \cup f_{i+1}([0,0),(2/3, 1/3)]) \) relatively to its endpoints;
- the interiors of the \( \gamma_j \)'s do not intersect the \( \bar{u}_j \)'s and do not intersect each other.

This is because, by construction, the arcs

\[-\bar{u}_i, \ f_i([2/3, 1/3], (1,0))), \ f_{i+1}([0,0),(2/3, 1/3)]\], \( \bar{u}_{i+1} \)

are ordered clockwise around \( t(\bar{u}_i) = f_i(1,0) = f_{i+1}(0,0) = s(\bar{u}_{i+1}) \), and, according to the third and fourth conditions, there is neither another \( \bar{u}_j \) nor a boundary component which can be inserted between \( -\bar{u}_i \) and \( \bar{u}_{i+1} \). Then, we can construct angles \( \hat{\alpha}_i \) satisfying

- \( \hat{\alpha}_i|_{\Delta^{(1)}} = f_{i+1} \);
- \( \hat{\alpha}_i|_{\Delta^{(2)}} = f_i \circ \psi^{-1} \);
- the boundary of \( \hat{\alpha}_i(\Delta \setminus (\Delta^{(1)} \cup \Delta^{(2)})) \) is

\[ f_i([2/3, 1/3], (1,0)]) \cup f_{i+1}([0,0),(2/3, 1/3)]) \cup \gamma_i \]

where the last point is obtained using the fact that an homeomorphism of the circle can be extended to an homeomorphism of the disc. Then the \( \hat{\alpha}_i \)'s satisfy all the conditions. \( \square \)

10.3. **Proof of Theorem 2.16**. Before proving Theorem 2.16 we need to develop some tools:

The following lemma permits to understand better relations between \( J \) and external paths winding around polygons:
Lemma 10.3. (1) For two oriented edges \( \bar{u} \) and \( \bar{v} \) of \( \sigma \) starting at the same marked point, the following are equivalent:

(i) \( [\bar{u}, \bar{v}] \notin J \);

(ii) \( [\bar{u}, \bar{v}] \) is of the form \( [\bar{u}', -\bar{u}'], [\bar{v}', \bar{v}'] \) or \( [\bar{v}', -\bar{v}'] \) where \( \bar{u}' \) is the oriented side enclosing a special monogon and \( \bar{v}' \) is the oriented edge pointing to the special puncture if it is in \( \sigma \).

Let \( P \) be a polygon. We have

(2) \( \xi^P_\ell \notin J \) if and only if \( m_{P_\ell} = 1 \) and either \( P_\ell P_{\ell+1} \) or \( P \) is complementary to a special monogon.

(3) We have \( \xi^P_\ell \xi^P_{\ell+1} \notin J \) if and only if \( m_{P_\ell} = m_{P_{\ell+1}} = 1 \) and either \( P \) is a flat digon (i.e. a digon with oriented sides \( \bar{u} \) and \( -\bar{u} \)) or \( P \) is complementary to a special monogon.

(4) We have \( C_\sigma \in J \).

Proof. (1) (i) \( \Rightarrow \) (ii). Suppose that \( [\bar{u}, \bar{v}] \notin J \). The decomposition as product of arrows is

\[
[\bar{u}, \bar{v}] = [\tilde{u}_0, \tilde{u}_1][\tilde{u}_1, \tilde{u}_2] \cdots [\tilde{u}_{\ell-1}, \tilde{u}_\ell]
\]

where \( \tilde{u}_0 = \bar{u}, \tilde{u}_1, \ldots, \tilde{u}_\ell = \bar{v} \) are successive oriented edges of \( \sigma \) around the common starting point. None of the \( [\tilde{u}_{\ell-1}, \tilde{u}_\ell] \) is in \( J \) so for each of them, there exists a special monogon with oriented side \( \tilde{u}_i \) and possibly \( \tilde{v}_i \) pointing to the special puncture such that we are in one of the following three cases

- \( \tilde{u}_{\ell-1} = \tilde{v}_i \) and \( \tilde{u}_\ell = \tilde{v}_i \);
- \( \tilde{u}_{\ell-1} = \tilde{v}_i \) and \( \tilde{u}_\ell = -\tilde{v}_i \);
- \( \tilde{u}_{\ell-1} = \tilde{v}_i \) and \( \tilde{u}_\ell = -\tilde{v}_i \).

As we excluded the case where \( \Sigma \) is a sphere without boundary, \( \#M = 3 \) and \( m_M = 1 \) for some \( M \in M \), it is not possible that \( \tilde{v}_i \) and \( -\tilde{v}_i \) both enclose a special monogon. Thus, an easy analysis gives that \( \ell = 1 \) or \( \ell = 2 \) and the result follows.

(ii) \( \Rightarrow \) (i). Take \( \bar{u}' \) and \( \bar{v}' \) as in (ii) (\( \bar{v}' \) is not necessarily present). Let us prove that \( [\bar{u}', \bar{v}], [\bar{u}', -\bar{u}'] \) and \( [\bar{v}', -\bar{u}'] \) are not in \( J \). Let \( E \) be the set of paths of \( kQ_\sigma \) generated by \( e_{u'}, e_{v'}, [\bar{u}', \bar{v}'], [\bar{u}', -\bar{u}'] \) and \( [\bar{v}', -\bar{u}'] \). Consider the linear map

\[
\psi : kQ_\sigma \to k, \quad \omega \notin E \mapsto 0, \quad \omega \in E \mapsto \lambda_{\ell(\bar{v})}^a(\omega)
\]

where \( a(\omega) \) is the number of times \( \omega \) goes through \( u' \). By definition \( J \subset \ker \psi \) and \( [\bar{u}', \bar{v}'], [\bar{u}', -\bar{u}'], [\bar{v}', -\bar{u}'] \notin \ker \psi \) so it is enough to prove that \( \Gamma_\sigma + I_\sigma^0 \subset \ker \psi \). We clearly have \( C_\sigma \in \ker \psi \) for all \( \bar{u} \). Consider a relation \( R_{P,i} \in \Gamma_\sigma^0 \) coming from a \( n \)-gon \( P \).

If \( \omega_{\ell+1}^P \omega_{\ell}^P \in E \), by a similar argument as in the converse part of the proof, we get that \( P_\ell P_{\ell+1} = \bar{u}' \). The two only possibilities for \( P \) is then the special monogon itself or an induced self-folded triangle. In the first case, the relation is

\[
R_{P_i} = [\bar{u}', -\bar{u}']^2 - \lambda_{\ell(\bar{v})}[\bar{u}', -\bar{u}']
\]

and in the second case

\[
R_{P_i} = [\bar{v}', -\bar{u}'][\bar{u}', \bar{v}'] - \lambda_{\ell(\bar{v})}e_{v'}
\]

so in both case \( R_{P_i} \in \ker \psi \).
Suppose now that \( \alpha \) is a non-zero multiple of an element of \( E \). We know that \( \alpha \) is not a multiple of \( C_\sigma \) so, using the definition of \( R_{P,i} \), there are two cases:

(a) If \( \mathbb{M}_P = \{ M \} \), we need to have \( m_M = 1 \) and \( P \) is a monogon. So \( P \) is a special monogon and it is immediate that \( R_{P,i} \) is the one defined in (10.4).

(b) If \( \mathbb{M}_P = \emptyset \), we have \( n \geq 3 \) and \( \xi_{i+2}, \xi_{i+3}, \ldots, \xi_{i-1} \in E \). Let us discuss the possible values of \( \xi_{i+2}^P \):

- If \( \xi_{i+2}^P \) is a scalar multiple of \( e_{i,i'} \), it means that \( -\bar{u}', \bar{u}' \) are consecutive sides of \( P \). It is impossible thanks to Proposition 2.10.

- If \( \xi_{i+2}^P \) is a scalar multiple of \( [\bar{u}', \bar{v}] \), it means that \( -\bar{u}', \bar{v}' \) are consecutive sides of \( P \). Thus, \( -\bar{v}' \) should be the next side. It is impossible thanks to Proposition 2.10.

- In the same way it is impossible that \( \xi_{i+2}^P \) is a scalar multiple of \( [\bar{v}', -\bar{u}'] \).

- If \( \xi_{i+2}^P \) is a scalar multiple of \( [\bar{u}', -\bar{u}'] \), it means that \( -\bar{u}' \) is consecutive to \( -\bar{u}' \) which is only possible if \( P \) is a monogon. It contradicts \( n \geq 3 \).

- If \( \xi_{i+2}^P \) is a scalar multiple of \( e_{i,i'} \), it means that \( \bar{P}_i \bar{P}_{i+1} = \bar{v}' \) and \( \bar{P}_{i+1} \bar{P}_{i+2} = -\bar{v}' \). If \( n > 3 \), \( \xi_{i+3}^P \) should satisfy the same condition. It is impossible. So \( n = 3 \). As a consequence, \( \bar{P}_{i+1} \bar{P}_{i+2} \) is a loop starting at \( s(\bar{v}') \). As \( \mathbb{M}_P = \emptyset \), we get \( \bar{P}_{i+1} \bar{P}_{i+2} = \bar{u}' \). Finally, \( R_{P,i} \) is the relation defined in (10.5).

\[ \text{(2) First of all, if } \bar{P}_i \bar{P}_{i-1} = \bar{P}_i \bar{P}_{i+1} \text{ and } m_{P_i} = 1, \text{ then } \xi_{i}^P = |\bar{P}_i \bar{P}_{i-1}, \bar{P}_i \bar{P}_{i+1}| = \lambda_{P_i} e_{P_i \bar{P}_{i+1}} \text{ is multiple of an idempotent so is not in } J. \]

If \( P \) is a monogon with special complementary and \( m_{P_i} = 1 \) then \( \xi_{i}^P \notin J \) using (1). Suppose now that \( \xi_{i}^P \notin J \). If \( m_{P_i} > 1 \), we get that \( [\bar{P}_i \bar{P}_{i-1}, \bar{P}_i \bar{P}_{i+1}] \) divides \( \xi_{i}^P \) which is a contradiction thanks to (1). Suppose that \( m_{P_i} = 1 \) and \( \bar{P}_i \bar{P}_{i-1} \neq \bar{P}_i \bar{P}_{i+1} \). Then \( \lambda_{P_i} \xi_{i}^P = |\bar{P}_i \bar{P}_{i-1}, \bar{P}_i \bar{P}_{i+1}| \) is of the form \([\bar{u}, \bar{v}]\) or \([\bar{u}, -\bar{u}]\) or \([\bar{v}, -\bar{u}]\) where \( \bar{u} \) is enclosing a special monogon and \( \bar{v} \) is possibly pointing to the special puncture thanks to (1). By a similar reasoning as in (1) (ii) \( \Rightarrow \) (i), the only possibility is that \( \bar{P}_i \bar{P}_{i-1} = \bar{u} \) and \( \bar{P}_i \bar{P}_{i+1} = -\bar{u} \) and we are in the case of a monogon with special complementary.

(3) If \( \xi_{i}^P \neq J \) then \( \xi_{i}^P \neq J \) and \( \xi_{i+1}^P \neq J \) so \( m_{P_i} = m_{P_{i+1}} = 1 \) and \( P \) is a flat digon or a monogon with special complementary thanks to (2). The converse is immediate using the map \( \psi \) defined in (1) (ii) \( \Rightarrow \) (i).

(4) Each term of \( C_\sigma \) is multiple of a \([\bar{u}, \bar{u}]\) so this is immediate as, thanks to (1), \([\bar{u}, \bar{u}] \in J. \)

The following lemma tells us that there are always minimal polygons:

**Lemma 10.6.** For any oriented edge \( \bar{u} \) of \( \sigma \) such that \( \bar{u} \) is not a counter-clockwise oriented boundary edge, there exist a unique minimal polygon \( P \) having \( \bar{u} \) as an oriented edge, up to equivalence. Moreover, each internal path winding around a vertex of this polygon is an arrow of \( Q_\sigma \).

**Proof.** We prove by induction on \( n \) that there is a unique sequence \( \bar{u}_0, \bar{u}_1, \ldots, \bar{u}_n \) such that
• \( \bar{u}_0 = \bar{u} \);
• for each \( i \leq n \), if \( \bar{u}_i \) is a boundary edge then \( \bar{u}_i \) is oriented clockwise around the boundary;
• for each \( i < n \), \( s(\bar{u}_{i+1}) = t(\bar{u}_i) \) and \( [\bar{u}_{i+1}, -\bar{u}_i] \) is an arrow of \( Q_\sigma \).

For \( n = 0 \), it is obvious. Let us suppose the result proven for \( n \) and let us prove it for \( n + 1 \). By construction of \( Q_\sigma \), there is a unique \( \bar{u}_{n+1} \) satisfying that \( [\bar{u}_{n+1}, -\bar{u}_n] \) is an arrow of \( Q_\sigma \). Then, it is easy to see that \( \bar{u}_{n+1} \) can not be a counter-clockwise boundary component, as otherwise \( \bar{u}_n \) would be also one. It finishes the induction.

By definition, it is immediate that a polygon is minimal if and only if every internal path winding around a vertex in an arrow of \( Q_\sigma \). Thus, if a minimal polygon containing \( \bar{u} \) exists, using Proposition 2.10 it has to have sides \( \bar{u}_0, \ldots, \bar{u}_n \) where \( n \) is the smallest integer satisfying \( \bar{u}_n = \bar{u}_0 \). Conversely, let us fix the smallest possible \( n \) such that \( \bar{u}_n = \bar{u}_0 \). We have that the conditions of Proposition 2.10 are satisfied for \( (\bar{u}_i)_{0 \leq i \leq n-1} \). Notice in particular that for any \( i \) and for any oriented edge \( \vec{v} \) of \( \sigma \) such that \( s(\vec{v}) = s(\bar{u}_i) \), \( -\bar{u}_{i-1}, \bar{u}_i \) and \( \vec{v} \) are ordered clockwise around \( s(\bar{u}_i) \) as by definitions of arrows of \( Q_\sigma \), there is no oriented edge between \( -\bar{u}_{i-1} \) and \( \bar{u}_i \) in the clockwise order.

**Definition 10.7.** Suppose that \( (\Sigma, M) \) is a sphere without boundary and with four punctures and \( m_M = 1 \) for all \( M \in \mathbb{M} \). This is the case where \( v_M \neq 1 \). Let \( P \) be a n-gon. For \( 1 \leq \ell \leq n \), if \( \widehat{P}_{\ell+1}P_{\ell} \) encloses a special monogon, we call \( S^P_i \) the relation coming from this monogon. Otherwise we denote \( S^P_i = 0 \). We call special ideal of \( P \) the ideal \( S^P := (S^P_1, S^P_2, \ldots, S^P_n) \).

In any other case, we put \( S^P = 0 \).

**Lemma 10.8.** Suppose that an n-gon \( P \) satisfies \( \widehat{P}_iP_{i+1} = -\widehat{P}_jP_{j+1} \) for two sides \( P_iP_{i+1} \) and \( P_jP_{j+1} \) and one of the following conditions:
1. \( \#M_P > 0 \).
2. \( \#\exists M_P \leq 1 \) as the result is trivial for \( \#M_P \geq 2 \).

Then the following statements hold in \( \overline{\Gamma}_{\sigma}^J \):
1. \( (R_{P_i}) + S^P = (\omega_{i+1}^P \omega_i^P) + S^P \);
2. \( (R_{P_j}) + S^P = (\omega_{j+1}^P \omega_j^P) + S^P \);
3. for \( 1 \leq \ell \leq n \),
   - \( \omega_{\ell}^P (R_{P,\ell} - \omega_{\ell+1}^P \omega_\ell^P) \omega_{\ell+1}^P \in (\omega_{i+1}^P \omega_i^P, \omega_{j+1}^P \omega_j^P) \), if \( i + 1 = \ell = j + 1 = \ell = i + 1 \);
   - \( R_{P,\ell} - \omega_{\ell+1}^P \omega_\ell^P \in (\omega_{i+1}^P \omega_i^P, \omega_{j+1}^P \omega_j^P) \), else.

**Proof.** We can suppose that \( \#M_P \leq 1 \) as the result is trivial for \( \#M_P \geq 2 \). We also suppose that \( i = 1 \).

1. (a) We suppose first that \( n = 2 \): this is the case of a flat digon and we have \( M_P = \{ M \} \) for some \( M \in \mathbb{M} \). Hence \( (\Sigma, \mathbb{M}) \) is a sphere with three punctures. So \( m_M, m_{P_1}, m_{P_2} > 1 \). Thus:
   \[
   R_{P,1} = \omega_1^P \omega_1^P - \lambda_M \omega_2^P (\xi_2^P \xi_1^P)^{m_M - 1} \xi_2^P \\
   = \omega_1^P \omega_1^P - \lambda_M (\omega_2^P)^{m_{P_1} - 1}(\omega_1^P)^{m_{P_2} - 2} (\xi_2^P \xi_1^P)^{m_M - 2} \xi_2^P
   \]

which implies the result as \( \xi_2^P \in J \).
Figure 10.9. Special polygons in a sphere with four punctures

(b) Suppose now that \( n \geq 3 \). We suppose that \( j < n \). Indeed, we cannot have \( n = j = 2 \). Thus \( j = n \neq 2 \) is analogous to \( j = 2 \neq n \). We get easily

\[
R_{P,1} = \omega_2^P \omega_1^P - \alpha [P_j P_{j-1}]_1, P_3 P_4 [\omega_2^P \omega_1^P] P_5 P_6 P_j P_{j+1} P_{j+2}^P \beta
\]

where

\[
\alpha := \begin{cases} 
\xi_n^P \xi_{j-1}^P \ldots \xi_1^P & \text{if } M_P = \emptyset \text{ (and } j > 2), \\
\lambda_M (\xi_n^P \ldots \xi_2^P \xi_1^P \xi_{j-1}^P)^{m_M-1} & \text{if } M_P = \{M\} \text{ and } j = 2, \\
\lambda_M \omega_n^P (\xi_2^P \ldots \xi_n^P \xi_1^P \xi_{j-1}^P)^{m_M-1} \xi_n^P \xi_{j-2}^P \ldots \xi_{j-1}^P & \text{if } M_P = \{M\} \text{ and } j > 2,
\end{cases}
\]

\[
\beta := \xi_j^P \xi_{j+1}^P \ldots \xi_n^P.
\]

If at least one of \( \alpha, \beta \), \([P_j P_{j-1}]_1, P_3 P_4 [\omega_2^P \omega_1^P] P_5 P_6 P_j P_{j+1} P_{j+2}^P \beta \) is in \( J \), it is immediate that \((R_{P,1}) = (\omega_2^P \omega_1^P)\). Suppose that they are not in \( J \).

As \([P_j P_{j-1}]_1, P_3 P_4 [\omega_2^P \omega_1^P] P_5 P_6 P_j P_{j+1} P_{j+2}^P \beta \) encloses a special monogon.

More precisely, we are in one of these cases:

- \( j = 2 \) (and \( M_P = \{M\} \) as \( P_1 P_2 \) and \( P_j P_{j+1} \) are consecutive).
- \(-P_{j-1} P_j = -P_2 P_3 \), or \(-P_{j-1} P_j = -P_2 P_3 \) encloses a special monogon and \( j = 3 \) as \( P \) cannot have twice the same side.
- \(-P_{j-1} P_j = -P_2 P_3 \) and \( j > 3 \). In this case, as \( \alpha \notin J \), we have \( \xi_1^P \notin J \) so, as \( P \) is not a monogon, \( m_{P_1} = 1 \) and \( m_{P_3} = 1 \) as \( P_2 P_3 \). As two different (oriented) sides of \( P \) cannot be equal, we deduce that \( j = 4 \).

In the same way, as \([P_j P_{j+1}]_1, P_3 P_4 [\omega_2^P \omega_1^P] P_5 P_6 P_j P_{j+1} P_{j+2}^P \beta \) and \( j \neq n \), we have \( m_{P_1} = 1 \) and we are in one of these cases:

- \( n = j + 1 \) and \(-P_n P_{j+1}^P \) encloses a special monogon.
- \(-P_n P_{j+1}^P = P_{j+1} P_{j+2}^P, n = j + 2 \) and \( m_{P_n} = 1 \).

As \( P \) is not a monogon or a self-folded triangle without puncture, we get \( \omega_2^P \in J \) so, as \( \alpha \notin J \), we get either \( M_P = \emptyset \) or \( j = 2 \) and \( M_P = \{M\} \).

Moreover, in this last case, we get also \( m_M = 1 \) as \( \xi_1^P \in J \). We summarize all possible cases in Figure 10.6 (all on a sphere with four punctures and without boundary). In all cases, for all \( N \in M \), we have \( m_N = 1 \). We can simplify \( \alpha [P_j P_{j-1}]_1, P_3 P_4 [\omega_2^P \omega_1^P] P_5 P_6 P_j P_{j+1} P_{j+2}^P \beta \) in the following way:
If \( j = 2, \alpha \beta P_{j-1} P_{j+2} = \lambda_M \lambda_P \alpha \) where \( \alpha = e_{P\ell} P_{\ell} \),
- If \( j = 3, \alpha \beta P_{j-1} P_{j+2} = \lambda_P \lambda_M \alpha \) where \( \alpha = e_{P\ell} P_{\ell} \) is an idempotent modulo \( S^P \),
- If \( j = 4, \alpha \beta P_{j-1} P_{j+2} = \lambda_P \lambda_P \alpha \) where \( \alpha = e_{P\ell} P_{\ell} \),
- If \( n = j + 1, |P_{jn} P_{j+1} P_{j+2} | = \lambda_N \lambda_P \beta \) where \( \beta = e_{P\ell} P_{\ell} \) is idempotent modulo \( S^P \).

If \( n = j + 2, |P_{jn} P_{j+1} P_{j+2} | = \lambda_P \lambda_P \beta \) where \( \beta = e_{P\ell} P_{\ell} \).

So, in any case, \( R_{P\ell} = \omega_2 \omega_1 = -\lambda_M e_\alpha \omega_2 \omega_1 e_\beta \).

Thus we get

\[
((1 - e_\alpha) R_{P\ell}) + S^P = ((1 - e_\alpha) \omega_2 \omega_1) + S^P
\]

\[
(R_{P\ell} (1 - e_\beta)) + S^P = (\omega_2 \omega_1 (1 - e_\beta)) + S^P
\]

and \( e_\alpha R_{P\ell} e_\beta = \nu_M e_\alpha \omega_2 \omega_1 e_\beta \) so

\[
(e_\alpha R_{P\ell} e_\beta) + S^P = (e_\alpha \omega_2 \omega_1 e_\beta) + S^P
\]

as \( \nu_M \) is invertible. We conclude that \( R_{P\ell} + S^P = (\omega_2 \omega_1) + S^P \).

(2) This is the analogous to (1).

(3) If \( n = 2 \), this is an easy consequence of (1) and (2). So, up to swapping \( i = 1 \) and \( j \), we can suppose that \( j \neq n \).

(a) If \( j \neq 2 \), for any \( \ell \neq 1, 2, n, R_{P\ell} - \omega_{P\ell+1} \omega_{P\ell} \) is right divisible by

\[
\varepsilon_{n+2} \cdots \varepsilon_2 \varepsilon_1 \varepsilon_{P\ell} \cdots \varepsilon_{P\ell-1}
\]

so \( R_{P\ell} - \omega_{P\ell+1} \omega_{P\ell} \in (\omega_{P\ell+1} \omega_{P\ell}) \). In the same way, \( R_{P\ell} - \omega_{P\ell+1} \omega_{P\ell} \in (\omega_{P\ell+1} \omega_{P\ell}) \) if \( \ell \neq j - 1, j, j + 1 \). As \( j \neq 1, 2, n \), the only remaining possibilities are \( i = 1, j = 1, j + 1 = \ell = n = i - 1 \).

By symmetry of the situation, we suppose that \( \ell = 2 \) and \( j = 3 \) so that \( R_{P\ell} - \omega_{P\ell+1} \omega_{P\ell} \) is right divisible by

\[
\varepsilon_{n+2} \cdots \varepsilon_2 \varepsilon_1 \varepsilon_{P\ell} \cdots \varepsilon_{P\ell-1}
\]

and \( \omega_{P\ell} (R_{P\ell} - \omega_{P\ell+1} \omega_{P\ell}) \) is left divisible by

\[
\omega_{P\ell+2} \cdots \varepsilon_2 \varepsilon_1 \varepsilon_{P\ell} \cdots \varepsilon_{P\ell-1}
\]

so \( \omega_{P\ell} (R_{P\ell} - \omega_{P\ell+1} \omega_{P\ell}) = \omega_{P\ell} (R_{P\ell} - \omega_{P\ell+1} \omega_{P\ell}) \omega_{P\ell+1} \) \( = (\omega_{P\ell+1} \omega_{P\ell}) \). (b) if \( j = 2 \), for \( \ell \neq 1, n, \omega_{P\ell+1} \omega_{P\ell} - R_{P\ell} \) is right divisible by

\[
C_{\ell} \varepsilon_{n+2} \cdots \varepsilon_2 \varepsilon_1 \varepsilon_{P\ell} \cdots \varepsilon_{P\ell-1}
\]

so \( \omega_{P\ell+1} \omega_{P\ell} - R_{P\ell} \in (\omega_{P\ell+1} \omega_{P\ell}) \). For \( \ell \neq 2, 3 \), a similar computation gives the result. The last possibility is \( \ell = n = 3 \). In this case, \( j + 1 = \ell = i - 1 \) and \( \omega_{P\ell+1} \omega_{P\ell} - R_{P\ell} \omega_{P\ell+1} \) is right divisible by \( C_{\ell} \omega_{P\ell+1} = \xi_{P\ell+1} \omega_{P\ell} \omega_{P\ell+1} \) so

\[
(\omega_{P\ell+1} \omega_{P\ell} - R_{P\ell}) \omega_{P\ell+1} \in (\omega_{P\ell+1} \omega_{P\ell}) \].

In the same way, \( \omega_{P\ell} (\omega_{P\ell+1} \omega_{P\ell} - R_{P\ell}) \) \( (\omega_{P\ell+1} \omega_{P\ell} - R_{P\ell}) \) \( (\omega_{P\ell+1} \omega_{P\ell} - R_{P\ell}) \). \( \square \)

We will need the following easy observation:
Lemma 10.10. If \( P \) is a flat 4-gon (i.e. \( P_1^2 = -P_4^1 \) and \( P_2^3 = -P_3^4 \)) then we have the equalities
\[
(R_{P,1}, R_{P,2}) = (\omega_1^P \omega_2^P, \omega_1^P \omega_2^P) \quad \text{and} \quad (R_{P,3}, R_{P,4}) = (\omega_1^P \omega_2^P, \omega_1^P \omega_2^P).
\]

Proof. If we are not in the case of a sphere without boundary and with \( \#M = 3 \), then \( \#M_p \geq 1 \) and this is an easy consequence of Lemma 10.8.

So we suppose that \( \Sigma \) is a sphere without boundary and \( \#M = 3 \). We get easily
\[
R_{P,1} = \omega_1^P \omega_1^P - \lambda_1 \lambda_2 (\omega_1^P \omega_2^P)^{m_2} \omega_2^P \omega_2^P \quad \text{and} \quad R_{P,2} = \omega_1^P \omega_1^P - \lambda_1 \lambda_2 (\omega_1^P \omega_2^P)^{m_2} \omega_2^P \omega_2^P
\]
and, as we did the hypothesis that \( m_1, m_2, m_3 > 1 \), we get easily that \( \omega_1^P \omega_1^P \) is a strict factor of \( (\omega_1^P \omega_2^P)^{m_2} \omega_2^P \omega_2^P \) and \( \omega_2^P \omega_2^P \) is a strict factor of \( (\omega_1^P \omega_2^P)^{m_2} \omega_2^P \omega_2^P \). This gives the first equality by completion with respect to \( J \) (which contains all arrows in this case). The second equality is similar.

Let \( P \) be a polygon. We call reduction situation the datum of an oriented edge \( \hat{u} \) such that \( s(\hat{u}) = P_1 \) and two polygons \( P' \) and \( P'' \) such that one of the following holds:

(Ra) \( P' = P'' \) has \( n + 2 \) sides and \( P_{n+1} P_{n+2} = \hat{u} \) and \( P'_{n+2} P'_{n+1} = -\hat{u} \). Notice that either \( \#M_P = \#M_{P'} = \infty \), or \( \#M_P = \#M_{P'} \cup \{\hat{u}\} \).

(Rb) \( P' = P'' \) has \( n' > n + 2 \) sides and \( P_{n+1} P_{n+2} = \hat{u} \) and \( P'_{n+2} P'_{n+1} = -\hat{u} \).

(Rc) \( P' \) has \( n' \) sides and \( P'' \) has \( n'' \) sides with \( n' + n'' = n + 2 \) and \( P_{n+1} P_{n+2} = \hat{u} \) and \( P'_{n+2} P'_{n+1} = -\hat{u} \).

Reductions situations are illustrated on Figure 10.11. Notice that, for \( i = 1, \ldots, n' \) in case (Rc) and \( i = 1, \ldots, n + 1 \) in cases (Ra) and (Rb), we have \( \omega_i^P = \alpha_i^P \omega_i^P \beta_i^P \) and \( \xi_i^P = \beta_i^P \xi_i^P \alpha_i^P \) where
\[
\alpha_i = \begin{cases} 
\omega_i^P & \text{if } i = n + 1, \\
1 & \text{else,} 
\end{cases} \quad \beta_i = \begin{cases} 
\omega_i^P & \text{if } i = 1, \\
1 & \text{else,} 
\end{cases}
\]
in cases (Ra) and (Rb) and
\[
\alpha_i = \begin{cases} 
\omega_i^P & \text{if } i = n', n'' \neq 1, \\
\omega_i^P \omega_i^P & \text{if } i = n', n'' = 1, \\
1 & \text{else,} 
\end{cases} \quad \beta_i = \begin{cases} 
\omega_i^P & \text{if } i = 1, n'' \neq 1, \\
\omega_i^P \omega_i^P & \text{if } i = 1, n'' = 1, \\
1 & \text{else,} 
\end{cases}
\]
in case (Rc).

Lemma 10.12. In a reduction situation, for any \( i = 1, \ldots, \min(n', 1, n) \), there exists \( \kappa \) invertible in \( \mathcal{G}^j \) such that \( R_{P,1} \kappa - \alpha_i^P R_{P,1} \beta_i^P \in I_{P,u} + S^P \)

- \( I_{P,a} = (R_{P,n+1}, R_{P,n}) \) in cases (Ra) or (Rb);
- \( I_{P,a} = (R_{P,n}, R_{P,n'}) \) in case (Rc).

Proof. Let us consider the three cases separately:
(Ra) If \( \# \mathcal{M}_{P_r} \geq 1 \), we have \( R_{P,i} = \alpha_{i+1} \omega_i^{P_r} \omega_i^{P_r} \beta_i \) and thanks to Lemma 10.8, we get:

\[
\alpha_{i+1} \left( R_{P,i} - \omega_i^{P_r} \omega_i^{P_r} \right) \beta_i \in \left( \omega_1^{P_r} \omega_n^{P_r}, \omega_n^{P_r} \omega_{n'-1}^{P_r} \right) + S^{P_r} = I_{P,u} + S^{P_r}
\]

so \( R_{P,i} - \alpha_{i+1} R_{P,i} \beta_i \in I_{P,u} + S^{P_r} \).

So we suppose that \( \mathcal{M}_{P_r} = 0 \). Then, modulo \( R_{P,n+2} = R_{P',n'} \) we get

\[
\alpha_{n+1} \xi_{n+2}^{P_r} = \lambda_{P_{n+2}}^{P_r} \omega_1 \left( \omega_{n+2}^{P_r} \right)^{m_{n+2}}-1
\]

\[
= \lambda_{P_{n+2}}^{P_r} \xi_1^{P_r} \cdots \xi_{n+1}^{P_r} \left( \omega_{n+2}^{P_r} \right)^{m_{n+2}}-1
\]

\[
= \cdots \lambda_{P_{n+2}}^{P_r} \left( \xi_2^{P_r} \cdots \xi_{n+1}^{P_r} \right)^{m_{n+2}}-1 \omega_1^{P_r}
\]

Then, modulo \( R_{P',n'} \), for \( i = 1 \ldots n \), we have

\[
\alpha_{i+1} R_{P',i} \beta_i = \alpha_{i+1} \omega_i^{P_r} \omega_i^{P_r} \beta_i - \alpha_{i+1} \xi_{i+2}^{P_r} \cdots \xi_{n+1}^{P_r} \xi_{n+2}^{P_r} \xi_{i+1}^{P_r} \xi_{i+2}^{P_r} \cdots \xi_{n}^{P_r} \beta_i
\]

\[
= \omega_i^{P_r} \omega_{i+1}^{P_r} - \xi_{i+2}^{P_r} \cdots \xi_{n+1}^{P_r} \xi_{n+2}^{P_r} \xi_{i+1}^{P_r} \xi_{i+2}^{P_r} \cdots \xi_{n}^{P_r} \beta_i
\]

\[
= \omega_i^{P_r} \omega_{i+1}^{P_r} - \xi_{i+2}^{P_r} \cdots \xi_{n+1}^{P_r} \left( \xi_{i+2}^{P_r} \cdots \xi_{n+1}^{P_r} \right)^{m_{n+2}}-1 \omega_1^{P_r} \beta_i \xi_1^{P_r} \cdots \xi_{i-1}^{P_r}
\]

\[
= \omega_i^{P_r} \omega_{i+1}^{P_r} - \lambda_{P_{n+2}}^{P_r} \left( \xi_2^{P_r} \cdots \xi_{i+1}^{P_r} \right)^{m_{n+2}}-1 \xi_1^{P_r} \cdots \xi_{i-1}^{P_r}
\]

\[
= \omega_i^{P_r} \omega_{i+1}^{P_r} - \lambda_{P_{n+2}}^{P_r} C_0 \left( \xi_2^{P_r} \cdots \xi_{i+1}^{P_r} \right)^{m_{n+2}}-1 \xi_1^{P_r} \cdots \xi_{i-1}^{P_r} = R_{P,i}.
\]
(Rb) In this case, we get that \( \#M_{\nu} \geq 2 \) so we can use Lemma 10.8 as in the first case of (Ra).

(Rc) As \( 1 \leq i \leq n' - 1 \), if \( \#M_{\nu} > 1 \), we get that \( R_{P,i} = \alpha_i + 1 R_{P,i+1} \beta_1 \) so we can suppose that \( \#M_{\nu} \leq 1 \). If \( M_{\nu} = \{ M \} \), denote
\[
\theta = \lambda_{M} C_{\nu}(\xi_1^P \xi_2^\nu \cdots \xi_n^P, \alpha_\nu^p \beta_1)_{mM}^{-1}
\]
and if \( M_{\nu} = \emptyset \), denote \( \theta = \theta' = 1 \), so that, in both cases, \( \beta_1 \theta = \theta_1 \beta_1 \).

By an easy analysis, we have
\[
\alpha_i + 1 R_{P,i+1} \beta_1 = \alpha_i + 1 \omega_i^P \omega_{i+1}^P \xi_1^\nu \xi_2^\nu \cdots \xi_n^P, \alpha_\nu^p \beta_1
\]
\[
= \omega_i^P \omega_{i+1}^P \xi_1^\nu \xi_2^\nu \cdots \xi_n^P, \alpha_\nu^p \beta_1
\]
and
\[
= \omega_i^P \omega_{i+1}^P \xi_1^\nu \xi_2^\nu \cdots \xi_n^P, \alpha_\nu^p \beta_1 \theta_1 \xi_1^\nu \xi_2^\nu \cdots \xi_n^P.
\]

Write \( \gamma = \omega^P \) and \( \delta = \omega^P \) if \( n'' = 1 \) and \( \gamma = \delta = 1 \) else. If \( \#M_{\nu} > 1 \) then \( \gamma R_{P,i+1} \beta \delta = \gamma \omega_i^P \omega_n^P \delta = \alpha_n \beta_1 \) and therefore \( \alpha_i + 1 R_{P,i+1} \beta_1 = R_{P,i} \)
modulo \( R_{\nu, n'} \) and we conclude in this case.

If \( \#M_{\nu} \leq 1 \), let
\[
\eta' = \lambda_{N} C_{\nu}(\xi_1^P \xi_2^\nu \cdots \xi_n^P, \alpha_\nu^p \beta_1)_{mN}^{-1}
\]
and \( \eta = \lambda_{N} C_{\nu}(\xi_1^P \omega_1^P \omega_n^P \delta = \alpha_n \beta_1 \) and therefore \( \alpha_i + 1 R_{P,i} \beta_1 = R_{P,i} \)
modulo \( R_{\nu, n'} \) and we conclude in this case.

If \( \#M_{\nu} \leq 1 \), let
\[
\eta' = \lambda_{N} C_{\nu}(\xi_1^P \xi_2^\nu \cdots \xi_n^P, \alpha_\nu^p \beta_1)_{mN}^{-1}
\]
and \( \eta = \lambda_{N} C_{\nu}(\xi_1^P \omega_1^P \omega_n^P \delta = \alpha_n \beta_1 \) and therefore \( \alpha_i + 1 R_{P,i} \beta_1 = R_{P,i} \)
modulo \( R_{\nu, n'} \) and we conclude in this case.

If \( \#M_{\nu} > 1 \), then \( \gamma R_{P,i+1} \beta \delta = \gamma \omega_i^P \omega_n^P \delta = \alpha_n \beta_1 \) and therefore \( \alpha_i + 1 R_{P,i} \beta_1 = R_{P,i} \)
modulo \( R_{\nu, n'} \) and we conclude in this case.

If \( \#M_{\nu} \leq 1 \), let
\[
\eta' = \lambda_{N} C_{\nu}(\xi_1^P \xi_2^\nu \cdots \xi_n^P, \alpha_\nu^p \beta_1)_{mN}^{-1}
\]
and \( \eta = \lambda_{N} C_{\nu}(\xi_1^P \omega_1^P \omega_n^P \delta = \alpha_n \beta_1 \) and therefore \( \alpha_i + 1 R_{P,i} \beta_1 = R_{P,i} \)
modulo \( R_{\nu, n'} \) and we conclude in this case.

If \( \#M_{\nu} \leq 1 \), let
\[
\eta' = \lambda_{N} C_{\nu}(\xi_1^P \xi_2^\nu \cdots \xi_n^P, \alpha_\nu^p \beta_1)_{mN}^{-1}
\]
and \( \eta = \lambda_{N} C_{\nu}(\xi_1^P \omega_1^P \omega_n^P \delta = \alpha_n \beta_1 \) and therefore \( \alpha_i + 1 R_{P,i} \beta_1 = R_{P,i} \)
modulo \( R_{\nu, n'} \) and we conclude in this case.

If \( \#M_{\nu} \leq 1 \), let
\[
\eta' = \lambda_{N} C_{\nu}(\xi_1^P \xi_2^\nu \cdots \xi_n^P, \alpha_\nu^p \beta_1)_{mN}^{-1}
\]
and \( \eta = \lambda_{N} C_{\nu}(\xi_1^P \omega_1^P \omega_n^P \delta = \alpha_n \beta_1 \) and therefore \( \alpha_i + 1 R_{P,i} \beta_1 = R_{P,i} \)
modulo \( R_{\nu, n'} \) and we conclude in this case.

If \( \#M_{\nu} \leq 1 \), let
\[
\eta' = \lambda_{N} C_{\nu}(\xi_1^P \xi_2^\nu \cdots \xi_n^P, \alpha_\nu^p \beta_1)_{mN}^{-1}
\]
and \( \eta = \lambda_{N} C_{\nu}(\xi_1^P \omega_1^P \omega_n^P \delta = \alpha_n \beta_1 \) and therefore \( \alpha_i + 1 R_{P,i} \beta_1 = R_{P,i} \)
modulo \( R_{\nu, n'} \) and we conclude in this case.

If \( \#M_{\nu} \leq 1 \), let
\[
\eta' = \lambda_{N} C_{\nu}(\xi_1^P \xi_2^\nu \cdots \xi_n^P, \alpha_\nu^p \beta_1)_{mN}^{-1}
\]
and \( \eta = \lambda_{N} C_{\nu}(\xi_1^P \omega_1^P \omega_n^P \delta = \alpha_n \beta_1 \) and therefore \( \alpha_i + 1 R_{P,i} \beta_1 = R_{P,i} \)
modulo \( R_{\nu, n'} \) and we conclude in this case.

If \( \#M_{\nu} \leq 1 \), let
\[
\eta' = \lambda_{N} C_{\nu}(\xi_1^P \xi_2^\nu \cdots \xi_n^P, \alpha_\nu^p \beta_1)_{mN}^{-1}
\]
and \( \eta = \lambda_{N} C_{\nu}(\xi_1^P \omega_1^P \omega_n^P \delta = \alpha_n \beta_1 \) and therefore \( \alpha_i + 1 R_{P,i} \beta_1 = R_{P,i} \)
modulo \( R_{\nu, n'} \) and we conclude in this case.
Suppose that \( \nu \notin J \). A quick analysis, using Lemma 10.3 (2) proves that the only possibility is that \((\Sigma, M)\) is a sphere with four punctures with \(P'\) and \(P''\) as in Figure 10.13 and \(n_M = 1\) for \(M' \in M\). For the left diagram, we get \( \nu = \xi'_1 = \lambda_{P'_1} \xi_{P'_1} \xi_{P'_1} \) and \( e := [P'_2 P'_1 P'_2] / \lambda_Q \) is an idempotent modulo \( S^P \). For the right diagram, we get \( \nu = \lambda_{P_2} \lambda_{P_2} e \) where \( e := e_{P_2 P'_2} \). Finally, we find that, in both cases, modulo \( R_{P''} e'' \),

\[
\alpha_{i+1} R_{P', i} \beta_i = R_{P', i} (1 - \lambda_{M e})
\]

which induces the result, as \( \kappa := 1 - \lambda_{M e} \) satisfies \( \kappa^{-1} = 1 + \lambda_M e^{-1} \). \( \square \)

The following lemma permits to do inductions:

**Lemma 10.14.** (1) In a reduction situation, \( P' \) and \( P'' \) have less reduction arcs than \( P \).

(2) If \( P \) is a polygon of \( \sigma \) which is not minimal with reduction arc \( u \) then up to replacing \( P \) by an equivalent polygon \( \tilde{P} \), there exist a reduction situation involving \( P \) and an orientation \( \tilde{u} \) of \( u \).

**Proof.** First of all, (1) is immediate. Then (2) is an immediate consequence of Proposition 2.10. \( \square \)

Now, we can conclude the proof of Theorem 2.16.

**Proof of Theorem 2.16.** It is easy to prove, using Lemmas 10.14 and 10.12 by descending induction on \( N \gg 0 \) that relations coming from polygons having at most \( N \) reduction arcs and special monogons generate \( I_\sigma \). So \( I_\sigma \) is generated by relations coming from minimal polygons and special monogons.

Let \( P \) be a non-minimal special monogon. By hypothesis, a loop of \( \sigma \) can not cut \((\Sigma, M)\) into two special monogons, so \( S^P = 0 \) and thanks to Lemma 10.12, we conclude that the relation coming from \( P \) is in the ideal generated by relations coming from minimal polygons. \( \square \)

10.4. **Proof of Theorem 2.17.** Denote by \( \varphi^{\circ} : k Q_\tau \rightarrow e_\tau k Q_\sigma e_\tau \) the only injective morphism of algebras such that \( e_\tau \) for \( u \in \tau \) is mapped to \( e_u \) and \([\bar{u}, \bar{v}]\) is mapped to \([\bar{u}, \bar{v}]\) for all arrows \([u, v]\) of \( Q_\tau \).

First, \( \varphi^{\circ} \) induces an injective morphism \( \varphi : \Gamma_\sigma \rightarrow e_\tau \Gamma_\sigma e_\tau \). Indeed, the relation \( C_{\bar{u}} \) is mapped to \( C_{\bar{u}} \) for any oriented edge \( \bar{u} \). The injectivity is a clear consequence of Proposition 2.4.

Then, \( \varphi^{\circ} \) induces a morphism \( \varphi : \Gamma_\sigma \rightarrow e_\tau \Gamma_\sigma e_\tau \). Indeed, every relation defining \( \Gamma_\tau \) is also a relation defining \( \Gamma_\sigma \) (since every polygon of \( \tau \) is a polygon of \( \sigma \)). To prove that it is an isomorphism, it is enough to look at the case where the difference between \( \tau \) and \( \sigma \) is only one edge \( u \) (then the result comes by an immediate induction). Notice that \( C_\sigma = \varphi(C_\tau) + \lambda_{s(\bar{u})} [\bar{u}, \bar{u}]^{m(\bar{u})} \).

Let us prove that \( \varphi \) is an isomorphism. By Lemma 10.6 there are two minimal polygons \( P' \) and \( P'' \), unique up to equivalence, the first one having \( \bar{u} \) as a side and the second one having \( -\bar{u} \) as a side. We suppose that the number \( n' \) of sides of \( P' \) is greater or equal that the number \( n'' \) of sides of \( P'' \). If \( P' \) and \( P'' \) both involve only the edge \( u \), then we get easily that \( u \) forms a connected component of \( \sigma \) and the result is immediate. Otherwise, using Lemma 2.10 we construct a reduction situation involving a polygon \( P \) having the sides of \( P' \) and \( P'' \) except \( u \).
(a) We suppose first that we are in case (Ra) or in case (Rb) or in case (Rc) with $\#M_P, \#M_P' \in \{0, \infty\}$. We consider the following, clearly well defined, morphism of algebras:

$$\psi^\circ \circ : e_\tau kQ_\sigma e_\tau \to \Gamma_\tau$$

$$e_\nu \mapsto e_v$$

$$\nu \mapsto \nu$$

$$\nu, \bar{\nu} \in \tau, s(\nu) = s(\bar{\nu})$$

$$\nu_1^{P'} (\omega_1^{P'})^\ell \omega_{n-1}^{P'} \mapsto (\delta_{M_P, \emptyset} \delta_{\xi_3^P \xi_1^P} \cdots \xi_1^P)^\ell \omega_1^P$$

$$\omega_1^{P'}, \omega_{n+2}^{P'} \mapsto 0$$

$$\omega_1^{P'}, \omega_{n+2}^{P'} \mapsto 0$$

$$\omega_1^{P'}, \omega_{n+2}^{P'} \mapsto 0$$

$$\omega_1^{P'}, \omega_{n+2}^{P'} \mapsto 0$$

$$\omega_1^{P'}, \omega_{n+2}^{P'} \mapsto 0$$

We will prove that $\psi^\circ \circ$ induces an inverse of $\varphi$. Let us prove that it induces a morphism $\psi^\circ \circ : e_\tau \Gamma_\sigma e_\tau \to \Gamma_\tau$. If $\bar{\nu}$ is not an orientation of $\bar{\nu}$ then $\psi^\circ \circ (C_{\bar{\nu}}) = C_{\bar{\nu}} = 0$. We need to look at generators of $e_\tau (C_{\bar{\nu}}) e_\tau$ in each case:

(Ra) The generator of $e_\tau (C_{\bar{\nu}}) e_\tau$ are $\omega_1^{P'} (\omega_1^{P'})^\ell C_{\bar{\nu}} (\omega_1^{P'})^\ell \omega_{n-1}^{P'}$ for $\ell_1, \ell_2 \geq 0$

and we have

$$\psi^\circ \circ \left( \omega_1^{P'} (\omega_1^{P'})^\ell C_{\bar{\nu}} (\omega_1^{P'})^\ell \omega_{n-1}^{P'} \right)$$

$$= \psi^\circ \circ \left( \omega_1^{P'} (\omega_1^{P'})^\ell (\omega_{n-1}^{P'} \xi_1^P \omega_1^P - \xi_1^P \omega_{n-1}^{P'}) (\omega_1^{P'})^\ell \omega_{n-1}^{P'} \right)$$

$$= \left( \delta_{M_P, \emptyset} \delta_{\xi_3^P \xi_1^P} \cdots \xi_1^P \right)^\ell \omega_1^{P'} \left( \delta_{M_P, \emptyset} \delta_{\xi_3^P \xi_1^P} \cdots \xi_1^P \right)^\ell \omega_1^{P'}$$

We denote $\gamma := \left[ P_{\nu} P_{\nu-1}, P_{\nu+2} P_{\nu+3} \right]$. We have

$$\psi^\circ \circ \left( \omega_1^{P'} C_{\bar{\nu}} \omega_1^{P'} \right) = \psi^\circ \circ \left( \omega_1^{P'} (\omega_{n+1}^P \xi_1^P \omega_1^P - \omega_{n+1}^P \rho \omega_{n+2}^P) \omega_1^P \right)$$

$$= C_{\tau} \psi^\circ \circ \left( \omega_1^{P'} \omega_1^P \right) - \psi^\circ \circ \left( \omega_1^{P'} \omega_1^P \right) = 0;$$

$$\psi^\circ \circ \left( \omega_1^{P'} C_{\bar{\nu}} \omega_1^{P'} \right) = \psi^\circ \circ \left( \omega_1^{P'} (\omega_{n+1}^P \xi_1^P \omega_1^P - \omega_{n+1}^P \rho \omega_{n+2}^P) \omega_1^P \right)$$

$$= C_{\tau} \omega_1^P - \psi^\circ \circ \left( \omega_1^{P'} \omega_1^P \right) \rho \psi^\circ \circ \left( \omega_1^{P'} \omega_1^P \right) = 0;$$

$$\xi_2^P \omega_2^P = \omega_1^P$$

as $\#M_P \geq 2$. The two other cases are similar.

(Rc) With $n, n' > 1$. Recall that we supposed that $\#M_P, \#M_P' \in \{0, \infty\}$. The generators of $e_\tau (C_{\bar{\nu}}) e_\tau$ are $\omega_1^{P'} C_{\bar{\nu}} \omega_1^{P'}$, $\omega_1^{P'} C_{\bar{\nu}} \omega_1^{P'}$, $\omega_1^{P'} C_{\bar{\nu}} \omega_1^{P'}$ and $\omega_1^{P'} C_{\bar{\nu}} \omega_1^{P'}$. We have

$$\psi^\circ \circ \left( \omega_1^{P'} C_{\bar{\nu}} \omega_1^{P'} \right) = \psi^\circ \circ \left( \omega_1^{P'} (\omega_1^{P'} \xi_1^P \omega_1^P - \omega_1^{P'} \xi_1^P \omega_1^P) \omega_1^P \right)$$

$$= C_{\tau} \psi^\circ \circ \left( \omega_1^{P'} \omega_1^P \right) - \psi^\circ \circ \left( \omega_1^{P'} \omega_1^P \right) C_{\tau} = 0;$$
\[
\psi^\circ(\omega_1^P \mathcal{C}_u \omega_{n'}^P) = \psi^\circ(\omega_1^P (\omega_{n'}^P \xi_1 \omega_1^P - \omega_{n'}^P \xi_1 \omega_1^P) \omega_{n'}^P) \\
= C_r \omega_1^P - \psi^\circ(\omega_1^P \omega_{n'}^P) \epsilon_1 \psi^\circ(\omega_1^P \omega_{n'}^P) = \xi^P R_{P,1} = 0
\]

and the other two cases are similar.

(Rc) with \( n' = 1 \). The generators of \( e_\tau(C_g) e_\tau \) are \( \omega_1^P (\omega_{n'}^P) \ell_1 C_g(\omega_{n'}^P) \ell_2 \omega_{n'}^P \) for \( \ell_1, \ell_2 \in \mathbb{N} \). As \#\( \mathcal{M}_{P'} = \infty \), we have

\[
\psi^\circ(\omega_1^P (\omega_{n'}^P) \ell_1 C_g(\omega_{n'}^P) \ell_2 \omega_{n'}^P) \\
= \psi^\circ(\omega_1^P (\omega_{n'}^P) \ell_1 \left[ \omega_1^P \omega_{n'}^P \xi_1^P \omega_1^P - \omega_{n'}^P \xi_1^P \omega_1^P \right] (\omega_{n'}^P) \ell_2 \omega_{n'}^P) \\
= \delta_{t_1,0} \delta_{t_2,0} \delta_{M_{P'},0} \left( \omega_1^P \xi_1^P \xi_2^P \cdots \xi_1^P \omega_1^P - \xi_1^P \xi_2^P \cdots \xi_1^P \omega_1^P \right) + \delta_{t_1,1} \delta_{t_2,0} \delta_{M_{P'},0} \left( C_r \xi_2^P \cdots \xi_1^P \omega_1^P - \xi_2^P \cdots \xi_1^P C_r \right) \\
+ (\delta_{t_1,0} \delta_{t_2,1} - \delta_{t_1,1} \delta_{t_2,0}) \delta_{M_{P'},0} \left( \omega_1^P \xi_1^P \omega_1^P = 0. \right)
\]

We finished to prove that \( \psi^\circ : e_\tau \Gamma_{n'} e_\tau \rightarrow \Gamma_{\tau} \) is well defined. Let us prove that it induces a morphism \( \psi : e_\tau \Gamma_\sigma e_\tau \rightarrow \Gamma_{\tau} \). We need to prove that \( \psi^\circ(e_\tau e_\tau) = 0 \). Using Theorem \[2.16\] we know that \( I_\sigma \) is generated by relations coming for minimal polygons. Any minimal polygon of \( \sigma \) which does not contain \( u \) as a side also exists in \( t \) so we can focus on relations coming from \( P' \) and \( P'' \). According to Lemma \[10.12\] for \( i = 1, \ldots, \min(n' - 1, n) \), we have

\[
\psi^\circ(\alpha_{i+1} R_{P',i} \beta_i) \in \psi^\circ((R_{P,i}) + e_\tau (I_{P,u} + S^P) e_\tau) = \psi^\circ(e_\tau I_{P,u} e_\tau)
\]

and we have analogous observations for some relations coming from \( P' \) and \( P'' \). We look at relations coming from \( P' \) which cannot be simplified in that way:

(Ra) In this case, these relations are

- \( R_{P',1}(\omega_{n'}^P) \ell_1 \omega_{n' - 1}^P \) for \( \ell_1 \geq 0 \) if \( n > 1 \),
- \( \omega_{n'}^P (\omega_{n'}^P) \ell_2 \omega_{n' - 2}^P \) for \( \ell_2 \geq 0 \) if \( n > 1 \),
- \( \omega_{n'}^P (\omega_{n'}^P) \ell_1 R_{P',1}(\omega_{n'}^P) \ell_2 \omega_{n' + 1}^P \) for \( \ell_1, \ell_2 \geq 0 \) if \( n = 1 \),
- \( R_{P',1}(\omega_{n'}^P) \ell_1 \omega_{n' - 1}^P \) for \( \ell_1 \geq 0 \),
- \( \omega_{n'}^P (\omega_{n'}^P) \ell_2 R_{P',n'}(\omega_{n'}^P) \ell_1 \omega_{n'}^P \) for \( \ell_2 \geq 0 \).

Suppose first that \( \# \mathcal{M}_{P'} = 0 \). If \( n > 1 \), we get

\[
\psi^\circ((\omega_{n'}^P (\omega_{n'}^P) \ell_1 \omega_{n' - 1}^P) \\
= \psi^\circ((\omega_2^P \omega_1^P - \xi_3^P \xi_4^P \cdots \xi_{n'}^P (\omega_{n'}^P) \ell_1 \omega_{n' - 1}^P) \\
= \omega_2^P (\xi_3^P \cdots \xi_{n'}^P) \omega_1^P - \xi_3^P \cdots \xi_{n'}^P \omega_1^P \psi^\circ(\omega_1^P \lambda_1^P (\omega_{n'}^P) m_{P',n'} \ell - 1 \omega_{n' - 1}^P) \\
= (\xi_3^P \cdots \xi_{n'}^P \ell_1 \omega_2^P \omega_1^P - \lambda_1^P (\xi_3^P \cdots \xi_{n'}^P \ell - 1 \omega_{n' - 1}^P) \\
= (\xi_3^P \cdots \xi_{n'}^P \ell_1 \omega_2^P \omega_1^P = 0.
\]

In the same way, we prove that \( \psi^\circ(\omega_{n'}^P (\omega_{n'}^P) \ell_1 R_{P',n'}(\omega_{n'}^P)) = 0 \) and if \( n = 1 \),
\( \psi^\circ((\omega_{n'}^P (\omega_{n'}^P) \ell_1 (\omega_{n'}^P) \ell_2 \omega_{n'}) = 0 \). Similarly, it is easy to compute that
\( \psi^\circ(R_{P',n'}(\omega_{n'}^P) \ell_1 \omega_{n' - 1}^P) = \psi^\circ((\omega_{n'}^P \ell_1 R_{P',n'}(\omega_{n'}^P)) = 0. \)
In this case, thanks to Lemma 10.8, if $n$ is defined morphism of algebra.

It is easy to check that

$$\psi^0 (\omega_{i}^{P'} \omega_{i'}^{P'}) \in \omega^{P'} \omega^{P'}$$

$$= \omega^{P'} \omega^{P'}$$

where $P$ is the polygon with sides $P_{i+1}^{n+1}$, $P_{i+1}^{n+1}$, $P_{i+1}^{n+1}$ and $R_{i+1}^{n+1} = \omega_{i+1}^{P'}$ and $R_{i+1}^{n+1} = \omega_{i+1}^{P'}$ as #M > 1 and #M > 1 so the result is immediate in this case.

We get

$$\psi^0 (R_{P',n'}) = \psi^0 (\omega_{1}^{P'} \omega_{n'}^{P'} - \delta_{M,P'} \delta_{P'} \xi_{1}^{P} \xi_{2}^{P} \cdots \xi_{n'}^{P}) = 0,$$

and the same for $\omega_{1}^{P'} R_{P',n'-1}$.

We have to check relations $R_{P',1} (\omega_{1}^{P'} \omega_{n'}^{P'})$, $R_{P',n'}$ and $\omega_{1}^{P'} (\omega_{1}^{P'}) = R_{P',n'-1}$ for $\ell = 0$ or $\ell > 2$. It is clear, by definition of $\psi^0$, that $\psi^0 (R_{P',n'}) = 0$. As we made the hypothesis that #M > 1, we necessarily have #M > 1 and therefore #M > 1 so

$$\psi^0 (R_{P',1} (\omega_{1}^{P'} \omega_{n'}^{P'}) = 0$$

and if $\ell > 2$, $\psi^0 (R_{P',1} (\omega_{1}^{P'}) \omega_{n'}^{P'}) = 0$ by direct computation. The proof works analogously for $\omega_{1}^{P'} (\omega_{1}^{P'}) = R_{P',n'-1}$.

We have to check $\omega_{1}^{P'} (\omega_{1}^{P'}) = R_{P',1} (\omega_{1}^{P'})$ for $\ell_1, \ell_2 > 0$ and $R_{P',2}$. It is similar as before.

The only case which is not analogous for $P''$ is (Rc) when $n'' = 1$. In this case, $e_{\tau} (R_{P',1})$ is generated by relations $\omega_{1}^{P'} (\omega_{1}^{P'}) = R_{P',1} (\omega_{1}^{P'})$ for $\ell_1, \ell_2 > 0$. As in this case #M > 1, we get

$$\psi^0 (\omega_{1}^{P'} (\omega_{1}^{P'}) = 0$$

so we finished to prove that $\psi^0 (e_{\tau} I_{e_{\tau}} e_{\tau}) = 0$ and $\psi : e_{\tau} I_{e_{\tau}} e_{\tau} \to \Gamma$ is a well defined morphism of algebra.

It is immediate that $\psi \circ \psi = Id_{\Gamma}$. We also have $\varphi \circ \psi = Id_{\Gamma \circ e_{\tau}}$. It is enough to check the generators of $e_{\tau}$, $k Q_{e_{\tau}}$ one by one. It is easy in every case. Finally, $e_{\tau} \Gamma \circ e_{\tau} \cong \Gamma$. (b) We have to prove the isomorphism in case (Rc) when $0 < #M < \infty$ or $0 < #M < \infty$. We will do an induction on (#M, #M) with the product order. The cases ($0, 0$), ($0, \infty$), ($\infty, 0$) and ($\infty, \infty$) have been proved.
we can construct a partial triangulation \( \sigma \)

By minimality of \( P \) in (a). Suppose that \( 0 \) is a vertex of \( P \)

On the other hand, the minimal polygon \( P \)

\[ \text{Figure 10.15. Inductive argument: partial triangulation } \sigma' \]

in (a). Suppose that \( 0 < \# M_{P'} < \infty \). As \( P' \) is minimal, it is immediate that we can construct a partial triangulation \( \sigma' \) such that \( \sigma = \sigma' \setminus \{ v \} \) where \( s(\tilde{v}) \) is a vertex of \( P' \) and \( t(\tilde{v}) \in M_{P''} \) for an orientation \( \tilde{v} \) of \( v \). See Figure 10.15.

By minimality of \( P' \), it is clear that the reduction situation in \( \sigma' \) involving \( P' \) and \( v \) is of type (Ra) or (Rb). So we already know that \( e_{\sigma'} e_{\sigma} \cong \Gamma_{\sigma} \).

On the other hand, the minimal polygon \( P'' \) containing \( u \) in \( \sigma' \) has at least one puncture less than \( P' \) (indeed \( M_{P''} \subset M_{P'} \setminus \{ t(\tilde{v}) \} \)). Thus, by induction hypothesis, \( e_{\sigma'} e_{\sigma} e_{\tau'} \cong \Gamma_{\tau'} \) where \( \tau' = \sigma' \setminus \{ u \} \). As before, the reduction situation involving \( P \) and \( v \) in \( \tau' \) is of type (Ra) or (Rb) so we know that \( e_{\tau'} e_{\tau} \cong \Gamma_{\tau} \).

Finally, we have

\[ e_{\tau'} e_{\tau} = e_{\tau} e_{\tau'} = e_{\tau'} e_{\tau} = e_{\tau} e_{\tau} e_{\sigma} e_{\tau} = e_{\sigma} e_{\tau} e_{\sigma} e_{\tau} \cong e_{\tau'} e_{\tau} e_{\tau} \cong \Gamma_{\tau}. \]

A similar argument gives the induction step when \( 0 < \# M_{P''} < \infty \).

\[ \square \]

10.5. Proof of Proposition 3.2. Denote by \( E_{i,j} \) the matrix with entry 1 in cell \((i, j)\) and 0 everywhere else. In each case, we give the images of the generators:

(1) In this case, the isomorphism is generated by

\[ [\vec{P}_1 \vec{P}_2, \vec{P}_1 \vec{P}_n] \rightarrow \lambda_{P_1}^{n-2} t^{-m} E_{1,n} \]

\[ [\vec{P}_i \vec{P}_{i+1}, \vec{P}_i \vec{P}_{i-1}] \rightarrow t^{m} E_{i,i-1} \quad (1 < i \leq n) \]

\[ [\vec{P}_i \vec{P}_n, \vec{P}_i \vec{P}_1] \rightarrow \lambda_{P_1}^{2-n} t^{m+1} E_{n,1} \]

\[ [\vec{P}_i \vec{P}_{i-1}, \vec{P}_i \vec{P}_{i+1}] \rightarrow \lambda_{P_1} \lambda_{P_1}^{-1} E_{i-1,i} \quad (1 < i \leq n). \]

(2) In this case, the isomorphism is generated by

\[ [\vec{P}_1 \vec{P}_2, \vec{P}_1 \vec{P}_n] \rightarrow \lambda_M(0, x^{m-1}) E_{1,n} \]

\[ [\vec{P}_i \vec{P}_{i+1}, \vec{P}_i \vec{P}_{i-1}] \rightarrow \lambda_M(0, x^m) E_{i,i-1} \quad (1 < i \leq n) \]

\[ [\vec{P}_i \vec{P}_n, \vec{P}_i \vec{P}_2] \rightarrow \lambda_{P_1}^{-1} (x, x) E_{n,1} \]

\[ [\vec{P}_i \vec{P}_{i-1}, \vec{P}_i \vec{P}_{i+1}] \rightarrow \lambda_{P_1}^{-1} (1, 1) E_{i-1,i} \quad (1 < i \leq n). \]
(3) In this case, the isomorphism is generated by
\[ P_1 P_2, P_i P_n, P_1 P_{i-1}, P_i P_{i+1} \mapsto t, \epsilon E_{1,n}, \epsilon E_{i,i-1}, \lambda E_{i-1,i} \quad (1 < i \leq n). \]

The proof that these definitions give isomorphisms is each time elementary. In the first case the central element used is \( U := C_\sigma \). In the second and third case, it is \( U := \sum_{i=1}^n (\xi P_{i+1} \xi P_i \cdots \xi P_{i+1}^P) \). We have easily \( k[x] \cong k[U] \).

Then, the key point in each case is that it is easy to determine a \( k[U] \)-basis of \( e_\sigma \Gamma_\sigma e_\nu \) using relations for any pair of sides \( u \) and \( v \). \( \square \)

10.6. **Proofs of Theorem 3.1.** (1) This is a consequence of Corollary 4.6 proven later (indeed, in this case, \( e_\sigma \Gamma_\sigma e_\nu = e \Delta_\sigma e \)).

We will now prove (2). We need some preparation. Using the same strategy as for Proposition 5.24 we get the following lemma:

**Lemma 10.16.** (1) We consider the following triangulation \( \sigma \) of a disc with one puncture and one marked point on the boundary:

```
\begin{tikzpicture}
  \filldraw (0,0) circle (0.1);
  \filldraw (-1,1) circle (0.1);
  \filldraw (1,1) circle (0.1);
  \draw (-1,1) -- (1,1);
  \draw (-1,1) -- (-1,0);
  \draw (1,1) -- (1,0);
  \draw (-1,1) -- (0,0);
end{tikzpicture}
```

with \( m_{s(\bar{v})} = 1 \) and \( m_{t(\bar{v})} = m \) (and \( m > 1 \) by hypothesis). Then we get

\[ \Gamma_\sigma \cong \begin{bmatrix}
  R_{m-1} & 0 \times R \\
  0 \times t^{m-1}R & 0 \times R
\end{bmatrix} \]

using the notation of Proposition 5.24 (2).

(2) We consider the following triangulation \( \sigma \) of a disc with one puncture and two marked points on the boundary:

```
\begin{tikzpicture}
  \filldraw (0,0) circle (0.1);
  \filldraw (-1,1) circle (0.1);
  \filldraw (1,1) circle (0.1);
  \draw (-1,1) -- (1,1);
  \draw (-1,1) -- (-1,0);
  \draw (1,1) -- (1,0);
  \draw (-1,1) -- (0,0);
  \draw (-1,1) -- (-1,-1);
  \draw (1,1) -- (1,-1);
end{tikzpicture}
```

with \( m_{s(\bar{v}_1)} = m_{s(\bar{v}_2)} = 1 \) and \( m_{t(\bar{v}_1)} = m \). Then we get

\[ \Gamma_\sigma \cong \begin{bmatrix}
  R_m & R_{m-1} & 0 \times R & 0 \times R \\
  tR_{m-1} & R_m & 0 \times tR & 0 \times R \\
  0 \times t^mR & 0 \times t^{m-1}R & 0 \times R & 0 \times R \\
  0 \times t^mR & 0 \times t^{m-1}R & 0 \times tR & 0 \times R
\end{bmatrix}. \]

**Proof.** We just give the images of arrows of quivers:

(1) We put
We take the following notation:

Lemma 10.17. We need here a small part of the results of these articles, but in bigger
generality.

Proof. Let us number the vertices of the triangle \( P \) with sides \( \bar{u}, \bar{v}, \bar{w} \) in such a way that \( P_1 = M \). We consider the quiver \( Q'_\sigma \) obtained from \( Q_\sigma \) by removing \( [\bar{u}, \bar{v}] \). As \( R_{P_2} = \omega^3_2 \omega_1^2 - \lambda_M[\bar{u}, \bar{v}] \) and \( C_{\bar{u}}, C_{\bar{v}} \in (R_{P_1}, R_{P_2}, R_{P_3}) \), it is immediate that \( \Gamma_\sigma \) is \( kQ'_\sigma \) modulo all relations except \( R_{P_2}, C_{\bar{u}} \) and \( C_{\bar{v}} \).

As all relations defining \( \Gamma_\sigma' \) are relations in \( \Gamma_\sigma \), the inclusion \( \iota : kQ_{\sigma'} \subseteq kQ'_\sigma \) induces a morphism of algebras \( \varphi : \Gamma_\sigma' \to \iota \Gamma_\sigma \iota^{-1} \). Thanks to the relations coming from \( P \), any path of \( Q'_\sigma \) can be rewritten as a multiple of a path without factor \( \omega_2^3 \omega_1^2 \) or \( \omega_1^2 \omega_3^3 \). Thus, it is clear that \( \varphi \) is surjective.
Using the same argument, we have:
\[ e_{\sigma'}(R_{P_1}, R_{P_2})e_{\sigma'} = e_{\sigma'}(\omega_1^2 \omega_1^P - \xi_3^P, \omega_1^P \omega_2^P - \xi_2^P) e_{\sigma'} \]
\[ = e_{\sigma'}(\omega_1^2 \omega_1^P \omega_3^P - \xi_3^P \omega_3^P, \omega_1^P \omega_2^P \omega_3^P - \omega_2^P \xi_2^P) e_{\sigma'} \]
\[ = e_{\sigma'}(\omega_1^2 \xi_3^P - \xi_3^P \omega_3^P, \omega_1^P \omega_2^P \omega_3^P - \omega_2^P \xi_2^P) e_{\sigma'} \]
\[ = e_{\sigma'}(C_{\omega_1^P}, \omega_2^P \omega_1^P \omega_3^P - \omega_2^P \xi_2^P) e_{\sigma'} \]
so \( i^{-1}(e_{\sigma'}(I_{\sigma'}^o + I_{\sigma'}) e_{\sigma'}) = I_{\sigma'}^o + I_{\sigma'} \) and therefore \( \varphi \) is injective. We proved the first part of the statement.

With the same strategy, there is a surjective morphism of left \( \Gamma_{\sigma'} \)-modules \( \psi : \Gamma_{\sigma'}[-\bar{u}, \bar{u}] \to e_{\sigma'} \Gamma_{\sigma} e_{\sigma} \).

And, modulo \( (I_{\sigma'} + I_{\sigma'})[-\bar{u'}, \bar{u}] \), we have
\[ e_{\sigma'}(R_{P_1}, R_{P_3})e_{\sigma} = e_{\sigma'}(\omega_1^2 \omega_1^P - \xi_3^P) e_{\sigma} + e_{\sigma'}(\omega_1^P \omega_2^P \omega_3^P - \omega_2^P \xi_2^P)[-\bar{u'}, \bar{u}] \]
\[ = e_{\sigma'}(\omega_1^2 \omega_1^P - \xi_3^P) e_{\sigma} + e_{\sigma'}(\omega_1^P \omega_2^P \omega_3^P - \omega_2^P \xi_2^P)[-\bar{u'}, \bar{u}] \]
\[ = e_{\sigma'}(\omega_1^2 \omega_1^P - \xi_3^P) e_{\sigma} \]
so \( e_{\sigma'}(R_{P_1}, R_{P_2})e_{\sigma} \cap kQ_{\sigma'}[-\bar{u'}, \bar{u}] \subset (I_{\sigma'} + I_{\sigma'})[-\bar{u'}, \bar{u}] \) and therefore \( \psi \) is injective. The three other equalities are proved in the same way. \( \square \)

**Definition 10.18.** Suppose that we are in cases (b) or (c) of Theorem 3.1 (2). Let \( u_0 \) be in \( E \) which is not incident to a puncture (thus, by hypothesis, it is homotopic to a part of a boundary component). We consider a \( n \)-gon \( P \) of \( \sigma \) such that
- \( \bar{P}_n \bar{P}_1 \) is an orientation of \( u_0 \);
- \( M_P \) contains all punctures of \( (\Sigma, M) \);
- \( M_P \) contains all holes of \( \Sigma \) except the hole \( u_0 \) is incident to.

Then we say that \( \xi_1^P \xi_2^P \cdots \xi_n^P \) is a big cycle at \( u_0 \).

**Lemma 10.19.** Under the assumptions of Definition 10.18, we get

1. In cases (b) or (c) of Theorem 3.1 (2), all big cycles at \( u_0 \) are equal to the same element \( S_{u_0} \) of \( \Gamma_\sigma \).
2. Suppose that we are in the situation of Theorem 3.1 (2). For \( u \in E \), we denote:
\[ U_u := \begin{cases} 
    e_u C_\sigma & \text{in case (a) or if } u \text{ is incident to a puncture}; \\
    \lambda M S_u^M & \text{in case (b) if } u \text{ is not incident to the puncture } M; \\
    S_u & \text{in case (c)}.
\end{cases} \]

Then the element \( U_\sigma := \sum_{u \in E} U_u \) is in the centre of \( e\Gamma_\sigma e \).

**Proof.** (1) Let us take two polygons \( P_1 \) and \( P_2 \) as in Definition 10.18. First of all, the hypotheses imply immediately that \( P_1 \) and \( P_2 \) cannot pass through a puncture (otherwise \( P_1 \) or \( P_2 \) would not contain this puncture). Thus, thanks to Theorem 2.17, we can suppose that \( \sigma \) does not contain any arc incident to a puncture.

We will prove by induction on the the number of marked point on the boundary component incident to \( u_0 \) that the big cycles defined from \( P_1 \) and \( P_2 \) are equal in \( \Gamma_\sigma \). If this number is \( 1 \) or \( 2 \), we necessarily have \( P_1 = P_2 \) so the result is immediate. If this number is at least \( 3 \), it is then an easy combinatorial observation that there is a marked point \( M \) such that:
- \( M \) is on the boundary component \( u_0 \) is incident to;
• $M$ is not incident to $u_0$;
• $M$ is not incident to any (non-boundary) arc of $\sigma$.

Then, up to adding the arc $w$ if it is not in $\sigma$, the assumptions of Lemma 10.17 are satisfied. For $i = 1, 2$, we denote by $P'_i$ the polygon obtained from $P_i$ by replacing the sequence of sides $\vec{u}, \vec{v}$ by $-\vec{w}$ if it appears (it is possible using Proposition 2.10 and the assumptions about $P_i$). Then, it is an immediate consequence of the relations that the big cycle defined from $P_1$ is equal to the big cycle defined from $P'_1$ in $\Gamma_\sigma$. By induction hypothesis, the big cycles defined from $P'_1$ and $P'_2$ are equal in $\Gamma_\sigma'$, so they are equal in $\Gamma_\sigma$ thanks to Lemma 10.17.

(2) Case (a) is immediate so we focus on Cases (b) and (c). Let $[\vec{u}, \vec{v}]$ be an arrow of $Q_\sigma$ which links to arcs of $E$. Let us prove that $[\vec{u}, \vec{v}]U_\sigma[U_\sigma[\vec{u}, \vec{v}]]$

\[
\text{i.e. } [\vec{u}, \vec{v}]U_\sigma[U_\sigma[\vec{u}, \vec{v}]].
\]

We consider several cases:

(a) If $u$ and $v$ are both incident to punctures, it is an immediate consequence of the fact that $C_\sigma$ is central.

(b) If none of $u$ and $v$ is incident to a puncture. If there is a polygon $P$ having $-\vec{u}, \vec{v}$ as two consecutive sides and satisfying hypotheses of Definition 10.18 then, with $P_1 = s(\vec{u})$ we have

\[
[u, v]S_u = [\vec{u}, \vec{v}]x_1x_2 \cdots x_n \lambda_1 = [u, v]x_1x_2 \cdots x_n \lambda_1 = S_u[\vec{u}, \vec{v}]
\]

(we used that $m_{P_1} = 1$). If there is a polygon $P$ having $-\vec{u}, \vec{v}$ as two consecutive sides and satisfying hypotheses of Definition 10.18 then, with $P_1 = s(\vec{u})$ we have

\[
[u, v]S_u = [\vec{u}, \vec{v}]x_1x_2 \cdots x_n \lambda_1 = C_\sigma x_1x_2 \cdots x_n = x_1x_2 \cdots x_n \lambda_1 = S_u[\vec{u}, \vec{v}].
\]

If none of these two cases are satisfied, then it is an easy consequence of the hypotheses that $-\vec{u}$ and $\vec{u}$ are consecutive sides of a polygon $P'$ with $M_{P'} = \emptyset$ (because $u$ and $v$ are homotopic to parts of the boundary and $[\vec{u}, \vec{v}]$ is an arrow). Then there is a polygon $P$ satisfying the hypothesis of Definition 10.18 for $u$ (or for $v$) such that $-\vec{u}$ and $-\vec{v}$ are reduction arcs for $P$ of type (Ra) as in Figure 10.11. Moreover the polygon $P''$ defined by this reduction situation satisfy the hypotheses of Definition 10.18 for $u$ (or for $v$). Using relations in $P''$, we get $[\vec{u}, \vec{v}]S_u = [\vec{u}, \vec{v}]$ in any case.

(c) If $u$ is not incident to a puncture and $v$ is incident to a puncture. We are in Case (b) of Theorem 3.1 (2). Let $P$ be a polygon satisfying hypotheses of Definition 10.18 for $u$. As $[\vec{u}, \vec{v}]$ is an arrow, it is easy that $\vec{u}$ is an oriented side of $P$. Moreover $\vec{u}$ is a reduction arc for $P$ of type (Ra). Taking the notation of Figure 10.11 we get

\[
[u, v]U_v = \lambda_{P''} \omega_P P'' \left(\omega_{P''} \right)^{m_{P''}} \lambda_{P''} \left(\xi_2 \cdots \xi_1 \right)^{m_{P''}} [\vec{u}, \vec{v}] = U_u[\vec{u}, \vec{v}].
\]

(d) If $u$ is incident to a puncture and $v$ is not, this is similar as (c). \[\square\]

To finish the proof of Theorem 3.1 (2), we suppose that we are in one of the three cases (a), (b) or (c). Consider the central element $U_\sigma$ defined in Lemma 10.19. We will prove that $e_{\Gamma_\sigma}e$ is a $k[\pi]$-lattice by mapping $x$ to $U_\sigma$. Let $u$ and $v$ be two edges of $\sigma$. Let us prove that $e_{u_{\Gamma_\sigma} e_v}$ is free as a left $k[\pi]$-module. First of all, thanks to Theorem 2.17 we can suppose that
\[\sigma\] contains only \(u\) and \(v\) in addition to boundary edges. We do an induction on the number of boundary edges.

If \(u\) and \(v\) are both boundary edges, or if we are in one of the cases of Lemma 10.16, the result is immediate by Proposition 3.2 and Lemma 10.16. \(x = t^m\) in cases (a) or (b) and \(x = t\) in case (c). Otherwise, an easy combinatorial argument shows that the boundary contains a marked point \(M\) satisfying \(m_M = 1\) and neither \(u\) nor \(v\) is incident to \(M\). Then Lemma 10.17 permits to conclude the proof of Theorem 3.1 (2).

Finally, we prepare Theorem 3.1 (3).

**Lemma 10.20.** Let \(E_0 \subset E\) correspond to a boundary component and \(e_0\) be the corresponding idempotent. Let \(P\) be the \(n\)-gon of \(\sigma\) corresponding to this boundary component. If

\[\#\mathcal{M}_P > 0\text{ and } m_{P_i} > 1 \text{ for some } i,\]
or \(m_{P_i}, m_{P_j} > 1\) for \(i \neq j\),

then \(e_0 \Gamma_P e_0\) is not finitely generated over its centre.

**Proof.** To simplify, we suppose that \(m_{P_1} > 1\). For \(i = 1, \ldots, n\), we denote \(\xi_i := [P_i P_{i-1}, P_i P_{i+1}]\).

We suppose first that we are in the first case or in the second case with \(j \neq i \pm 1\). Let \(\alpha := \xi_1^{\ell_1} \xi_2^{\ell_2} \cdots \xi_n^{\ell_n}\). It is immediate that the \(\alpha^n\)'s form a linearly independent set of \(e_0 \Gamma_\sigma e_0\) as no path appearing in any relation defining \(\Gamma_\sigma\) is a factor of any \(\alpha^n\). As a consequence, if \(e_0 \Gamma_\sigma e_0\) was finitely generated over its centre, there would be an element \(U = \alpha^{\ell} + U'\) in the centre of \(e_0 \Gamma_\sigma e_0\) where \(\ell > 0\) and \(U'\) does not contain a multiple of \(\alpha^{\ell}\) as a summand. Again, no path appearing in any relation is a factor of \(\omega_1^{\ell}\). Therefore, any element of the complete path algebra equivalent to \(\omega_1^{\ell} U\) modulo the relations defining \(e_0 \Gamma_\sigma e_0\) has \(\omega_1^{\ell} \alpha^{\ell}\) as a term and cannot be divisible to the right by \(\omega_1^{\ell}\). It contradicts the fact that \(U\) is in the centre.

Let us now suppose that \(m_{P_1}, m_{P_2} > 1\) and \(\#\mathcal{M}_P = 0\) (so \(n \geq 3\) by hypothesis). We consider the following elements of the path algebra:

\[\beta_1 := \xi_1^{\ell_1} \xi_2^{\ell_2} \omega_1^{\ell_1} P_1\text{ and } \beta_2 := \xi_1^{\ell_1} \xi_2^{\ell_2} \xi_3^{\ell_3} \omega_1^{\ell_1} \cdots \xi_n^{\ell_n}\]

and we notice that \(\beta_1 = \beta_2\) in \(\Gamma_\sigma\) satisfy that any \(\beta_1^{\ell}\) can only be rewritten up to the relations as a linear combination of \(\beta_{i_1} \beta_{i_2} \cdots \beta_{i_\ell}\) for \(i_1, \ldots, i_\ell \in \{1, 2\}\).

In particular they are linearly independent. Thus, as before, we should have \(U = \beta_1^{\ell} + U'\) in the centre where \(\ell > 0\) and \(U'\) does not have any summand that is a scalar multiple of \(\beta_1^{\ell}\). If \(n > 3\) or \(m_{P_1} > 2\), we notice as before that \(\xi_1^{\ell} \omega_1^{\ell_1} \beta_1^{\ell}\) can only be rewritten as \(\xi_1^{\ell_1} \omega_1^{\ell_1} \beta_{i_1} \beta_{i_2} \cdots \beta_{i_\ell}\) for \(i_1, \ldots, i_\ell \in \{1, 2\}\).

In particular, it is never right divisible by \(\xi_1^{\ell_1} \omega_1^{\ell_1}\) so it contradicts the fact that \(U\) is in the centre. In the same way, if \(n = 3\) and \(m_{P_1} = 2\), \(\xi_1^{\ell_1} \omega_1^{\ell_1} \beta_1^{\ell}\) can only be rewritten as \(\xi_1^{\ell_1} \omega_1^{\ell_1} \beta_{i_1} \beta_{i_2} \cdots \beta_{i_\ell}\) or \(\lambda_1 \omega_1^{\ell_1} \omega_2^{\ell_2} \xi_2^{\ell_3} \omega_1^{\ell_1} \beta_{i_1} \beta_{i_2} \cdots \beta_{i_\ell}\) or \(\lambda_1 \omega_3^{\ell_1} \omega_2^{\ell_2} \xi_2^{\ell_3} \beta_{i_1} \beta_{i_2} \cdots \beta_{i_\ell}\) for \(i_1, \ldots, i_\ell \in \{1, 2\}\) so it is not right divisible by \(\xi_1^{\ell_1} \omega_1^{\ell_1}\) and \(U\) is not in the centre. \(\square\)

**Lemma 10.21.** Let \(E_0 \subset E\) correspond to a boundary component, \(P\) be the \(n\)-gon of \(\sigma\) corresponding to this boundary component. Suppose that \(P\) has at least two punctures or a hole and \(E\) contains at least a non-boundary arc
which is not homotopic to a part of a boundary. Then $e \Gamma_{\sigma} e$ is not finitely generated over its centre.

Proof. Using Lemma 10.20, we can suppose that $m_N = 1$ for any vertex $N$ of $P$. We suppose thanks to Theorem 2.17 that $E$ contains only one non-boundary arc $\vec{u}$ such that $s(\vec{u}) = P_1$ (as $E$ is connected). We will prove that $e_u \Gamma_{\sigma} e_u$ is not finitely generated over its centre. Let us distinguish two cases:

(a) If $t(\vec{u})$ is not a vertex of $P$. We have $\#M_P > 1$ so the elements

$$[\vec{u}, P_1 P_2] (\xi_1^P \xi_2^P \cdots \xi_n^P)^{\ell} \xi_1^P [P_1 P_2, \vec{u}]$$

do not have any factor appearing in a relation. The end of the reasoning is the same as in Proof of Lemma 10.20.

(b) If $t(\vec{u})$ is a vertex of $P$ and $\vec{u}$ is not homotopic to a part of $P$. The elements

$$[\vec{u}, P_1 P_2] (\xi_1^P \xi_2^P \cdots \xi_n^P)^{\ell} \xi_1^P [P_1 P_2, \vec{u}]$$

permit to conclude as before. Notice that in both case, $e_u \Gamma_{\sigma} e_u$ is infinitely generated as an algebra. □

Lemmas 10.20 and 10.24 conclude the proof of Theorem 5.1 (3).

10.7. Proof of Proposition 4.12 We start by giving two technical lemmas:

**Lemma 10.22.** In $\Delta_{\sigma}$, we have $C_{\sigma} J = 0$.

Proof. We can suppose that $\Sigma$ has no boundary as the result for $\sigma$ follows from the one for the partial triangulation $\sigma'$ of $\Sigma'$. We can also suppose that $\sigma$ is a triangulation as the result will be induced to any partial triangulation $\tau \subset \sigma$ thanks to Corollary 4.2.

As $C_{\sigma}$ is in the center of $\Delta_{\sigma}$, it is enough to prove that for an arrow $[\vec{u}, \vec{v}]$ of $Q_{\sigma}$ which is in $J$, we have $C_{\sigma} [\vec{u}, \vec{v}] = 0$. Using relations defining $\Delta_{\sigma}$, we get that $C_{\sigma} [\vec{u}, \vec{v}]$ is equal to

$$\lambda_{t(\vec{u})} [-\vec{u}, -\vec{u}]^{m_{t(\vec{u})}} [\vec{u}, \vec{v}] = \lambda_{s(\vec{u})} [\vec{u}, \vec{v}]^{m_{s(\vec{u})}} [\vec{u}, \vec{v}] = \lambda_{t(\vec{v})} [\vec{u}, \vec{v}] [-\vec{v}, -\vec{v}]^{m_{t(\vec{v})}}.$$

We will prove that $C_{\sigma} [\vec{u}, \vec{v}] \in C_{\sigma} J^2$ which is enough by completeness with respect to $J$. Under the hypothesis that $[\vec{u}, \vec{v}]$ is an arrow and $\sigma$ is a triangulation, the following cases are impossible:

- $\vec{v} = -\vec{u}$;
- $t(\vec{u}) = t(\vec{v}) \neq s(\vec{u})$ and $\vec{u} \neq \vec{v}$;
- $t(\vec{u}) = t(\vec{v}) = s(\vec{u})$

so we are in one of the following cases:

- If $s(\vec{u})$, $t(\vec{u})$ and $t(\vec{v})$ are distinct then, using Lemma 2.10, there is a polygon with sides $\vec{u}, -\vec{u}, \vec{v}, -\vec{v}$ and Lemma 10.10 gives $[-\vec{u}, -\vec{u}] [\vec{u}, \vec{v}] = 0$.
- If $\vec{u} = \vec{v}$. As $\vec{v}$ follows $\vec{u}$ around $s(\vec{u})$, we have $t(\vec{u}) \neq s(\vec{u})$ and Lemma 10.8 permits to conclude in the digon with sides $\vec{u}, -\vec{u}$.
- If $s(\vec{u}) = t(\vec{u}) \neq t(\vec{v})$. Let $P$ the triangle enclosed by $\vec{u}, \vec{v}, -\vec{v}$. Thanks to Lemma 10.8 if $\#M_P \geq 1$ then $[\vec{u}, \vec{v}] [\vec{v}, -\vec{u}] [\vec{u}, \vec{v}] = 0$, which permits to conclude. Suppose that $M_P = \emptyset$. Then $m_{s(\vec{v})} > 1$ as $[\vec{u}, \vec{v}] \in J$. So $C_{\sigma} [\vec{u}, \vec{v}] = [-\vec{u}, \vec{v}] [\vec{v}, -\vec{u}] [\vec{u}, \vec{v}] = \lambda_{t(\vec{v})} [-\vec{u}, \vec{v}] [-\vec{v}, -\vec{v}]^{m_{t(\vec{v})} - 1}$ is left
divisible by \([-\vec{u}, \vec{v}] \cdot [\vec{u}, \vec{v}][-\vec{u}, -\vec{v}] = [-\vec{u}, \vec{u}][-\vec{u}, \vec{v}] = [-\vec{u}, \vec{u}]^2 \cdot [\vec{u}, \vec{v}].\

If 

-\vec{u}

encloses more than one puncture then 

-\vec{u}, \vec{u}^2 = 0.

If it encloses one puncture then 

\(M \neq \emptyset\) and \(\Sigma\) is a sphere with three punctures. Thus, \(m_N > 1\) for all \(N \in \mathbb{M}\). Thus

\(\lambda^{2}L_{\sigma}(\vec{u})[-\vec{u}, -\vec{u}]^{\tau \sigma(\vec{u})^{-1}}[-\vec{u}, \vec{u}]^{\tau \sigma(\vec{u})^{-1}}[\vec{u}, \vec{v}]
\)

is right divisible by 

\(C_\sigma[\vec{u}, \vec{v}]^\tau[\vec{v}, \vec{v}]^{\tau \sigma(\vec{v})^{-1}} \in C_\sigma J^2\)

and the result follows.

\begin{itemize}
  \item If \(s(\vec{u}) = t(\vec{v}) \neq t(\vec{u})\), the reasoning is similar.
\end{itemize}

Let \(I\) be the kernel of the canonical projection \(kQ_\sigma \rightarrow \Delta_\sigma\). We denote 

\(I_0 := (C_\sigma) \vec{v} + C_\sigma J \subset I\).

**Lemma 10.23.** Let \(\vec{v}\) be an oriented arc of \(\sigma\), and \(P\) be the minimal \(n\)-gon containing \(\vec{v}\) with \(\vec{P}_1 \vec{P}_2 = \vec{v}\). Then we have

\((R_{P,1}) + S^P + I_0 + J I + IJ = (R^p_{\vec{v}}) + S^P + I_0 + J I + IJ.\)

**Proof.** Let \(\vec{u} := \vec{P}_2 \vec{P}_3\) and \(\vec{w} := \vec{P}_n \vec{P}_1\). Thus, \(\vec{u}\) follows immediately \(\vec{u}\) around \(P_2\) and \(\vec{w}\) follows immediately \(\vec{v}\) around \(P_1\).

Let us first suppose that \(\sigma\) contains only \(u, v\) and \(w\). If \(#\mathbb{M}_P \geq 2\), the result is immediate. So we suppose that \(#\mathbb{M}_P \leq 1\). If \(P\) has one puncture, we write \(\mathbb{M}_P = \{M\}\). Let us distinguish several cases:

\begin{itemize}
  \item[(a)] If \(P\) is a monogon (Case d). In this case \(\vec{u} = \vec{v} = \vec{w}\) and \(P\) has one puncture. We get easily that, modulo \(I_0\),

\([\vec{u}, -\vec{v}][\vec{v}, -\vec{w}] - R_{P,1} = \begin{cases} 
\lambda_M[\vec{v}, -\vec{v}] & \text{if } m_M = 1; \\
\lambda_M e_\sigma C_\sigma & \text{if } m_M = 2; \\
0 & \text{if } m_M > 2.
\end{cases}\)

(b) If \(P\) is a digon (Case b). In this case \(\vec{u} = \vec{w}\) and \(P\) has one puncture. Again, we easily find that, modulo \(I_0\),

\([\vec{u}, -\vec{v}][\vec{v}, -\vec{w}] - R_{P,1} = \begin{cases} 
\lambda_M e_\sigma C_\sigma & \text{if } m_M = 1; \\
0 & \text{if } m_M > 1.
\end{cases}\)

(c) If \(P\) is a triangle. The result is immediate if \(P\) has no puncture (Case a) so we consider the case where \(\mathbb{M}_P = \{M\}\). We have

\(R_{P,1} = \omega^P_2 \omega^P_1 - \lambda_M C_\sigma (\xi^P_2 \xi^P_1 \xi^P_3)^{\tau \sigma(\vec{v})^{-1}}\)

and, modulo \(I_0\), using Lemma [10.3], \(R_{P,1} = \omega^P_2 \omega^P_1\) except if \(m_M = m_P = 1\) and \(\vec{w} = -\vec{u}\). It induces the expected result (Case e or zero relation).

(d) If \(n \geq 4\) and \(P\) has a puncture or two non-consecutive sides coinciding. In this case, using Proposition [2.10] the assumptions of Lemma [10.8] are satisfied and the result easily follows (zero relation).

(e) If \(n = 4\), the sides of \(P\) are \(\vec{w}, \vec{v}, \vec{u}, -\vec{u}\) in this order and \(\mathbb{M}_P = \emptyset\). The case \(\vec{w} = -\vec{v}\) is an easy consequence of Proof of Lemma [10.10] so we suppose that \(\vec{w} \neq -\vec{v}\) (Case c). We have

\([\vec{u}, -\vec{v}][\vec{v}, -\vec{w}] - R_{P,1} = \lambda_P \lambda_P [\vec{u}, -\vec{v}][\vec{v}, -\vec{w}]^{\tau \sigma(\vec{w})^{-1}}[-\vec{v}, \vec{v}] [-\vec{v}, \vec{v}]^{\tau \sigma(\vec{v})^{-1}}[-\vec{v}, \vec{v}]^{\tau \sigma(\vec{v})^{-1}}[-\vec{v}, \vec{v}]^{\tau \sigma(\vec{v})^{-1}} = 0\)

Notice that, modulo \(I\), we have

\([\vec{u}, -\vec{v}][\vec{v}, -\vec{w}] = \lambda_P \lambda_P [\vec{u}, -\vec{v}][\vec{v}, -\vec{w}]^{\tau \sigma(\vec{w})^{-1}}[-\vec{v}, \vec{v}] [-\vec{v}, \vec{v}]^{\tau \sigma(\vec{v})^{-1}}[-\vec{v}, \vec{v}]^{\tau \sigma(\vec{v})^{-1}}[-\vec{v}, \vec{v}]^{\tau \sigma(\vec{v})^{-1}} = 0\)
using the fact that, by (b) in the digon with sides $-\vec{v}, -\vec{w}, [−\vec{u}, \vec{w}]$ is a multiple of $C_\sigma$ and $[\vec{u}, -\vec{v}] - R_{P,1} \in IJ$ (as $[-\vec{u}, \vec{w}] \in J$). The case $m_{P_3} = 1$ is direct.

(f) If $n = 4$, the sides of $P$ are $\vec{w}, \vec{v}, \vec{u}, -\vec{w}$ in this order and $M_P = \emptyset$, the reasoning is the same as (e) (Case c for $\vec{w}$).

(g) If $n = 5$ and the sides of $P$ are $\vec{w}, \vec{v}, \vec{u}, -\vec{u}, -\vec{w}$ in this order. The case $M_P \neq \emptyset$ is an easy consequence of Lemma 10.3. Suppose that $M_P = \emptyset$ (Case f). Using Proof of Lemma 10.3 we have that $t(\vec{u})$ and $s(\vec{w})$ are both only incident to one edge. Moreover, we have $s(\vec{v}) = t(\vec{v}) = s(\vec{u}) = t(\vec{v})$. So we get

$[\vec{u}, -\vec{v}] [\vec{v}, -\vec{w}] - R_{P,1} = \lambda P_3 \lambda P_3 [\vec{u}, -\vec{w}]$

and, thanks to (c) in the triangle $\vec{v}, \vec{u}, -\vec{u}$, modulo $I$, $[-\vec{u}, -\vec{u}] [\vec{v}, -\vec{u}] = 0$. So, if $m_{P_3} > 1$, we get that $[\vec{u}, -\vec{v}] [\vec{v}, -\vec{w}] - R_{P,1} \in IJ$ (as relations $R_{P,1}$ generating $I_{P,\vec{x}} x$ start or end at $x$ and neither $x$ and $u$, neither $x$ and $w$ can form a self-folded triangle). Thus, according to Lemma 10.12 we get $R_{P,1} - R_{P',1} \in IJ + IJ + S^P$. Therefore

$(R_{P,1}) + S^P + I_0 + JI + IJ = (R_{P',1}) + S^P + I_0 + JI + IJ$

where the last equality comes from the induction hypothesis.

We are now ready to prove Proposition 4.12.

Proof of Proposition 4.12 Summing the equalities of Lemma 10.23 for all $\vec{v}$, we get

$$(R_{P,\vec{v}})_{P,\ell} + I_0 + JI + IJ = (R_{P,\vec{v}})_{P,\ell} + (S^P)_{P,\ell} + I_0 + JI + IJ$$

where $P$ runs over all the minimal polygons, $\ell$ over all vertices of minimal polygons and $\vec{v}$ runs over all oriented arcs of $\sigma$. The left member is clearly included in $I$ by definition of $\Delta_\sigma$. It is immediate that $kQ_{\sigma}/I_0$ is a finitely generated $k$-module. Moreover, thanks to Theorem 2.16 up to completion with respect to $J$, $I$ is generated by $(C_{\vec{v}})_{\vec{v}}$ and $(R_{P,\vec{v}})_{P,\ell}$. Therefore, $I \subset I_0 + (R_{P,\vec{v}})_{P,\ell}$. Finally, the left member of (10.24) is $I$. As moreover, $(S^P)_{P,\ell} \subset (R_{P,\vec{v}})_{P,\ell}$, we rewrite (10.24) as follows:

$I = (R_{P,\vec{v}})_{\vec{v}} + C_\sigma J + (C_{\vec{v}})_{\vec{v}} + JI + IJ$
and we deduce by immediate induction that:

\[ I = (R^0_{\bar{v}}, C_{\bar{v}})_{\bar{v}} + C_{\sigma}J + \sum_{n'=0}^n J^{n'} I J^{n-n'} \]

for any \( n > 0 \). For \( n \) big enough, we have \( \sum_{n'=0}^n J^{n'} I J^{n-n'} \subset C_{\sigma}J \) so we finally get \( I = (R^0_{\bar{v}}, C_{\bar{v}})_{\bar{v}} + C_{\sigma}J \).

10.8. **Proof of Theorem 4.3**. Let \( \mathcal{P} \) be the set of minimal polygon of \( \sigma \). If \( \varepsilon \in \{0,1\}^P \), for any \( \bar{v} \in \sigma \), we define \( R^\varepsilon_{\bar{v}} = R_{\bar{v}} \) if \( \varepsilon(P) = 1 \) where \( P \) is the minimal polygon with oriented side \( \bar{v} \) and \( R^\varepsilon_{\bar{v}} = R^0_{\bar{v}} \) else. Finally, we put

\[ \Delta^\varepsilon_{\sigma} := \frac{k Q_{\sigma}}{JC_{\sigma} + (C_{\bar{v}}, R^\varepsilon_{\bar{v}})_{\bar{v} \in \sigma}}. \]

We will prove the following proposition:

**Proposition 10.25.** For any \( \varepsilon \in \{0,1\}^P \), there is an isomorphism \( \psi^\varepsilon : \Delta^\varepsilon_{\sigma} \rightarrow \Delta_{\sigma} \) satisfying \( \psi^\varepsilon(e_u) = e_u \) for any \( u \in \sigma \) and \( \psi^\varepsilon([\bar{u}]) = [\bar{u}]_{\sigma} \) for any \( \bar{u} \in \sigma \).

It implies Theorem 4.3 as \( \Delta^1_{\sigma} = \Delta^\varepsilon_{\sigma} \).

**Proof.** For \( \varepsilon \in \{0,1\}^P \), denote \( |\varepsilon| := \sum_{P \text{ minimal}} \varepsilon(P) \). We prove by induction on \((-\#\sigma, |\varepsilon|)\) the existence of \( \psi^\varepsilon \). As the case \( |\varepsilon| = 0 \) is trivial, we suppose that \( |\varepsilon| > 0 \). Let \( P \) be a minimal polygon of \( \sigma \) such that \( \varepsilon(P) = 1 \) and let \( \eta \in \{0,1\}^P \) be defined by \( \eta(P) = 0 \) and \( \eta(P') = \varepsilon(P') \) for any \( P' \neq P \). By induction hypothesis, the isomorphism \( \psi^\eta \) exists. If \( P \) does not correspond to Cases b with \( m_M = 1 \), c with \( m_{P_3} = 1 \), d with \( m_M = 2 \), e with \( m_{P_3} = m_M = 1 \) or f with \( m_{P_3} = m_{P_3} = 1 \) of Figure 4.11 the result is immediate as \( \Delta^\varepsilon_{\sigma} = \Delta^\eta_{\sigma} \).

If \( P \) is as in Case b with \( m_M = 1 \), we consider the partial triangulation \( \tau \) obtained from \( \sigma \) by adding an arc linking \( P_2 \) and \( M \). By induction hypothesis, there is an isomorphism \( \psi^\varepsilon : \Delta^\varepsilon_{\sigma} \rightarrow \Delta_{\tau} \) (where, for \( \varepsilon, P \) is replaced by the self-folded quadrilateral). Hence we have an isomorphism \( e_\sigma \psi^\varepsilon_{\varepsilon} e_\sigma : e_\sigma \Delta^\varepsilon_{\sigma} e_\sigma \rightarrow e_\sigma \Delta_{\sigma} e_\sigma = \Delta_{\sigma} \) by Corollary 1.2. Moreover, it is easy to observe that \( e_\sigma \Delta^\varepsilon_{\sigma} e_\sigma = \Delta^\sigma_{\sigma} \) so the result follows in this case. Case e with \( m_{P_3} = m_M = 1 \) is solved in the same way by adding an arc linking \( P_3 \) to \( M \). In Case f with \( m_{P_3} = m_{P_3} = 1 \) we can suppose by the same argument that there is an arc linking \( P_3 \) to \( P_5 \).

In each remaining case, we define \( \phi^\varepsilon : k Q_{\sigma} \rightarrow \Delta^\eta_{\sigma} \) and we prove that it induces an isomorphism \( \phi : \Delta^\varepsilon_{\sigma} \rightarrow \Delta^\eta_{\sigma} \) satisfying \( \phi^\varepsilon(e_u) = e_u \) for any \( u \in \sigma \) and \( \phi^\varepsilon([\bar{u}, \bar{u}]_{\sigma}) = [\bar{u}, \bar{u}]_{\sigma} \) for any \( \bar{u} \in \sigma \). It permits to conclude by putting \( \psi^\varepsilon = \psi^\eta \circ \phi \).

If \( P \) is as in Case d with \( m_M = 2 \) and \( 2 \) invertible in \( k \), for \( q \) an arrow of \( Q_{\sigma} \), we put

\[ \phi^\varepsilon(q) := \begin{cases} q - \lambda_M [-\bar{u}, -\bar{u}] / 2 & \text{if } q = [\bar{u}, -\bar{u}]; \\ q & \text{else.} \end{cases} \]

Indeed, \( \phi^\varepsilon([\bar{u}, -\bar{u}]^2) = 0 \). For any \( \bar{x} \in \sigma \) such that \( s(\bar{x}) = P_1 \) and \( x \neq u \), we get \(-\bar{u}, \bar{u}, [-\bar{u}, \bar{x}] = [\bar{x}, \bar{u}], [-\bar{u}, \bar{u}] = 0 \) (we use that in the case of a sphere with three punctures, \( m_N \geq 2 \) for any \( N \in \mathbb{M} \)). Hence we get \( \phi^\varepsilon([\bar{u}, \bar{u}^2]) = [\bar{u}, \bar{u}], \phi^\varepsilon([\bar{x}, -\bar{u}]^2) = [\bar{x}, -\bar{u}] \) and \( \phi^\varepsilon([\bar{u}, \bar{u}]_{\sigma}^2) = [\bar{u}, \bar{u}]_{\sigma} \) for
any $\vec{v} \in \sigma$. From these observations, $\varphi^o$ induces a well defined morphism from $\Delta_\sigma^x$ to $\Delta_\sigma^y$. It is clearly invertible.

If $P$ is as in Case $c$ with $m_{P_3} = 1$, denote $\tilde{\lambda} = \lambda_N$ and $\tilde{\nu} = \nu_M$ if the polygon with sides $-\vec{v}, -\vec{w}$ contains a unique puncture $N$, $m_N = 1$, and the minimal polygon $P'$ containing $-\vec{w}$ is a triangle or $\eta(P') = 1$. In any other case, write $\lambda = 0$ and $\tilde{\nu} = 1$. It permits to have the equalities $[-\vec{v}, \vec{w}][-\vec{w}, \vec{v}] = \lambda e_v C_\sigma$ and $[-\vec{w}, \vec{v}][-\vec{v}, \vec{w}] = \lambda e_w C_\sigma$ in $\Delta_\sigma^w$. Then write:

$$
\varphi^o(q) := \begin{cases} 
\tilde{\nu}^{-1}(q - \lambda P_3 [-\vec{v}, \vec{w}]) & \text{if } q = [\vec{v}, -\vec{w}]; \\
q & \text{else.}
\end{cases}
$$

We easily get that $\varphi^o(\mathcal{R}_{\vec{y}}) = \varphi^o(\mathcal{R}_{\vec{x}}) = \varphi^o(\mathcal{R}_{\vec{w}}) = \varphi^o(\mathcal{R}_{-\vec{w}}) = 0$. Then, an easy computation gives $\varphi^o([\vec{v}, \vec{v}]) = \tilde{\nu}^{-1}([\vec{v}, \vec{v}] - \delta_{m_{P_3}, 1} \lambda \lambda P_3 e_v C_\sigma)$ so using $C_\sigma J = 0$, $\varphi^o([\vec{v}, -\vec{w}]) = \tilde{\nu}^{-1}( [\vec{v}, \vec{w}] - \delta_{m_{P_3}, 1} \lambda \lambda P_3 [-\vec{v}, \vec{w}] )$. Then, we get $\varphi^o(\lambda P_3 [\vec{v}, \vec{v}]) = \varphi^o([\vec{v}, \vec{w}])$ and in the same way $\varphi^o(\lambda P_3 [-\vec{w}, -\vec{w}]) = e_w C_\sigma$. We now have to check relations $\mathcal{R}_{\vec{z}}^x$ when $\vec{z} \neq \pm \tilde{\nu}, \tilde{\nu}, \vec{w}$. The only case where we do not have trivially $\varphi^o(\mathcal{R}_{\vec{z}}^x) = 0$ is when $f_{\vec{z}}^o$ has $[\vec{v}, -\vec{w}]$ as a factor. In this case, an easy case by case analysis proves that there exist $f'$ multiple of a path which does not have $[\vec{v}, -\vec{w}]$ as a factor and $\vec{y} \in \sigma$ with $s(\vec{y}) = P_1$ such that $f_{\vec{z}} = [\vec{y}, -\vec{w}][f']$, and $t(\vec{y}) \neq P_1$ or $-\vec{w}, \vec{y}, -\vec{v}, \vec{v}$ are ordered around $P_1$, or $f_{\vec{z}}^o = f'[\vec{v}, \vec{y}]$, and $t(\vec{y}) \neq P_1$ or $-\vec{w}, -\vec{y}, \vec{y}, \vec{v}$ are ordered around $P_1$.

So it is enough to prove that $\varphi^o([\vec{y}, -\vec{w}]) = [\vec{y}, -\vec{w}]$ in the first case and $\varphi^o([\vec{v}, \vec{y}]) = [\vec{v}, \vec{y}]$ in the second case. By symmetry, we prove the first one. We use $\varphi^o([\vec{y}, -\vec{w}]) = [\vec{y}, \vec{v}]_{\varphi^o([\vec{v}, -\vec{w}])}$ and the easy observation that $[\vec{y}, \vec{y}][-\vec{v}, -\vec{w}] = \tilde{\lambda} [\vec{y}, -\vec{w}]$ so $[\vec{y}, \vec{v}][-\vec{v}, -\vec{w}] = [\vec{v}, -\vec{w}]$ in $\Delta_\sigma^y$. So $\varphi^o$ induces a morphism $\varphi : \Delta_\sigma^x \to \Delta_\sigma^y$, which is clearly invertible.

Finally, we consider Case $f$ where $m_{P_3} = m_{P_5} = 1$ and there is an arc $\vec{z}$ from $P_3$ to $P_5$. In this case, we put $\varphi^o(q) := \begin{cases} 
\tilde{\nu}_M^{-1}(q - \lambda P_3 [-\vec{v}, \vec{w}]) & \text{if } q = [\vec{v}, -\vec{w}]; \\
q & \text{else.}
\end{cases}$

It is immediate that $\varphi^o(\mathcal{R}_{\pm \vec{w}}) = \varphi^o(\mathcal{R}_{\vec{x}}) = \varphi^o(\mathcal{R}_{\vec{y}}) = \varphi^o(\mathcal{R}_{\vec{w}}) = 0$. We denote $\tilde{\lambda} := \lambda_N$ if $-\vec{v}$ encloses a special monogon with puncture $N$ and $\tilde{\lambda} = 0$ else. An easy computation gives $\varphi^o([\vec{v}, \vec{v}]) = \nu_M^{-1}([\vec{v}, \vec{v}] - \tilde{\lambda} \lambda P_3 e_v C_\sigma)$ so, using $C_\sigma J = 0$, $\varphi^o([\vec{v}, -\vec{w}]) = \nu_M^{-1}([\vec{v}, \vec{w}] - \delta_{m_{P_3}, 1} \lambda \lambda P_3 [-\vec{v}, \vec{w}])$. Using $[-\vec{v}, \vec{w}]^2 - \tilde{\lambda} [\vec{v}, \vec{w}] \in (C_\sigma)$ and $C_\sigma J = 0$, we deduce $\varphi^o(\lambda P_3 [\vec{x}, \vec{z}]) = e_x C_\sigma$ for $\vec{z} = \pm \vec{v}, -\vec{u}, -\vec{v}$. Moreover, $\varphi^o([\vec{x}, \vec{w}]) = [\vec{w}, \vec{w}]$ so $\varphi^o(\mathcal{R}_{\vec{x}}^o) = 0$ and $\varphi^o([\vec{v}, -\vec{w}]) = [\vec{v}, -\vec{w}]$ so all other $\mathcal{R}_{\vec{z}}^x$ are mapped to 0.

10.9. Proof of Theorem 4.3 for $\Delta_\sigma$. We start by proving the theorem for $\Delta_\sigma^x$.

Proof of Theorem 4.3 for $\Delta_\sigma$. We can check it for $\Delta_\sigma^y$ as it is immediate that it induces the result for $\Delta_\sigma$. So we suppose that $\Sigma$ has no boundary. We will use the presentation of Proposition 11.1.

For (3), Lemma 10.22 gives that if $x \in (C_\sigma)$ then $Jx = xJ = 0$. So it is enough to prove (3)’ If $Jx = xJ = 0$ then $x \in (C_\sigma)$.

We will prove at the same time (1) and (3)’. It is enough to check the result for $e_u \Delta_\sigma e_v$ for any pair of edges $u, v$. Thanks to Corollary 4.2, we can suppose that $\sigma$ contains only $u$ and $v$. If $u$ and $v$ are disconnected, it
is immediate that $e_u \Delta_\sigma e_v = 0$. Let us suppose that $u$ and $v$ are connected. We call $P$ the minimal polygon containing $\vec{u}$ and $P'$ the minimal polygon containing $-\vec{u}$.

Suppose first that $u = v$. If $u$ has two distinct endpoints, we immediately get, using $\mathcal{R}_{\vec{u}}^a$ and $\mathcal{R}_{-\vec{u}}$:

$$e_u \Delta_\sigma e_u = k[\omega_1^P, \omega_2^P]/(\omega_1^P \omega_2^P, \omega_2^P \omega_2^P, \lambda_{P_1}(\omega_1^P)^{m_{P_1}} - \lambda_{P_2}(\omega_2^P)^{m_{P_2}})$$

which clearly has a basis consisting of $e_u$, $e_u C_u$, $(\omega_1^P)^\ell$ for $1 \leq \ell \leq m_{P_1} - 1$ and $(\omega_2^P)^\ell$ for $1 \leq \ell \leq m_{P_2} - 1$ and it is immediate that if $a \in e_u \Delta_\sigma e_u$ satisfies $\omega_1^P = \omega_2^P = 0$ then $a \in (C_\sigma)$.

Let us now suppose that $u$ is a loop. Denote $x := [\vec{u}, -\vec{u}]$ and $y := [-\vec{u}, \vec{u}]$. We use the implicit notation $\mathbb{M}_P = \{M\}$ if $\#\mathbb{M}_P = 1$ and $\mathbb{M}_P = \{M', M''\}$ if $\#\mathbb{M}_P > 1$. Moreover, if $\#\mathbb{M}_P > 1$ we denote $m_M = \infty$ and if $\#\mathbb{M}_P > 1$ we denote $m_{M'} = \infty$. Denote $c := \lambda_{s(\vec{u})}(x_0)^{m_{s(\vec{u})}}$ and $c' := \lambda_{s(\vec{u})}(y_0)^{m_{s(\vec{u})}}$. We get easily:

- if $m_M > 1$ and $m_{M'} > 1$ then
  $$e_u \Delta_\sigma e_u = k(x, y)/(c - c', x^2 - \delta_{m_M, 2} \lambda_M c, y^2 - \delta_{m_{M'}, 2} \lambda_{M'} c, cx, cy);$$
- if $m_M = 1$ then $m_M = \infty$ and $e_u \Delta_\sigma e_u = k(x, y)/(c - c', x^2 - \lambda_M x, y^2);$
- if $m_{M'} = 1$ then $m_{M'} = \infty$ and $e_u \Delta_\sigma e_u = k(x, y)/(c - c', x^2, y^2 - \lambda_{M'} y)$.

The result follows in this case.

Suppose now that $u \neq v$. Let us choose orientations of $u$ and $v$ such that $s(\vec{u}) = s(\vec{v})$ and $\vec{v}$ follows immediately $\vec{u}$ around $s(\vec{u})$ (in the case where $u$ or $v$ is a loop). We distinguish several cases

(a) If $t(\vec{v})$, $s(\vec{u})$ and $t(\vec{u})$ are distinct. We get $[\vec{u}, -\vec{u}][\vec{u}, \vec{v}] = [\vec{u}, \vec{v}][-\vec{v}, -\vec{v}] = 0$ so, using the basis of $e_v \Delta_\sigma e_v$, any element of $e_u \Delta_\sigma e_v$ is a linear combination of elements of the form $[\vec{u}, \vec{v}][\vec{v}, \vec{v}]$ for $0 \leq \ell < m_{s(\vec{u})}$. Multiplying on the left by $[\vec{v}, \vec{u}]$ maps these elements to $[\vec{v}, \vec{v}]^{\ell + 1}$ which are linearly independent so the statement is true in this case. As $[\vec{v}, \vec{u}] \in J$, it also proves that no $x \in e_u \Delta_\sigma e_v \setminus \{0\}$ satisfies $Jx = 0$.

(b) If $t(\vec{v}) = t(\vec{u}) \neq s(\vec{u})$. In this case, $[\vec{u}, \vec{v}][-\vec{v}, -\vec{u}]$ and $[\vec{u}, -\vec{v}][\vec{v}, \vec{u}]$ are multiple of $C_\sigma$. Therefore, any element of $e_u \Delta_\sigma e_v$ is a linear combination of $[\vec{u}, \vec{v}][\vec{v}, \vec{v}]$ for $0 \leq \ell < m_{s(\vec{u})}$ and $[\vec{u}, -\vec{v}][-\vec{v}, -\vec{v}]$ for $0 \leq \ell < m_{t(\vec{u})}$. We denote $\lambda_M = \lambda_M$ if $\mathbb{M}_P = \{M\}$ and $m_M = 1$ and $\lambda_M = 0$ else. In the same way, $\lambda_{M'} = \lambda_{M'}$ if $\mathbb{M}_{P'} = \{M'\}$ and $m_{M'} = 1$ and $\lambda_{M'} = 0$ else.

Suppose that

$$\sum_{\ell=0}^{m_{s(\vec{u})}-1} \alpha_\ell [\vec{u}, \vec{v}][\vec{v}, \vec{v}]^\ell + \sum_{\ell=0}^{m_{t(\vec{u})}-1} \beta_\ell [-\vec{u}, \vec{v}][-\vec{v}, -\vec{v}]^\ell = 0.$$ 

Multiplying on the left by $[\vec{v}, \vec{u}]$ and using the structure of $e_v \Delta_\sigma e_v$, we get

$$\alpha_\ell = 0 \text{ for } \ell < m_{s(\vec{u})} - 1 \text{ and } \alpha_{m_{s(\vec{u})} - 1} + \beta_0 \tilde{\lambda}_M \lambda_{s(\vec{u})} = 0$$

and multiplying by $[-\vec{v}, -\vec{u}]$, we get

$$\beta_\ell = 0 \text{ for } \ell < m_{t(\vec{u})} - 1 \text{ and } \alpha_0 \tilde{\lambda}_{M'} \lambda_{t(\vec{u})} + \beta_{m_{t(\vec{u})} - 1} = 0.$$
If \( m_s(\vec{u}) > 1 \) or \( m_t(\vec{u}) > 1 \), we get that \( \alpha_\ell = 0 \) and \( \beta_\ell = 0 \) for any \( \ell \) so these elements are linearly independent. If \( m_s(\vec{u}) = m_t(\vec{u}) = 1 \), we deduce \( \nu_{\ell}^a \alpha_0 = \nu_{\ell}^a \beta_0 = 0 \) so the conclusion follows as \( \nu_{\ell}^a \) is invertible.

(c) If \( t(\vec{u}) = s(\vec{u}) \neq t(\vec{u}) \). Using \( R_{-\vec{u}}^\nu \) if \( \vec{u}, \vec{v} \) is a linear combination of \( \nu \vec{u}, \nu \vec{v} \) for \( \nu \in \{ e_u, \nu \vec{u}, \nu \vec{v} \} \) and every element of \( e_u \Delta_\nu e_v \) has the form \( \omega \vec{u}, \vec{v} \) for \( \omega \in e_u \Delta_\nu e_u \). We know that \( \omega \) is a linear combination of \( \lambda \vec{u}, \vec{v} \) for \( \lambda \in \{ e_u, \nu \vec{u}, \nu \vec{v} \} \) and \( 0 \leq \lambda < m_s(\vec{u}) \). Using \( R_{\vec{u}}^\nu \) and \( R_{-\vec{u}}^\nu \), we get

\[
[\vec{u}, -\vec{u}][\vec{u}, \vec{v}] = [\vec{u}, \vec{v}][\vec{u}, -\vec{u}][\vec{u}, \vec{v}]
\]

where we used the relations computed before in \( e_u \Delta_\nu e_v \). Using this identity, we deduce that \( \omega \vec{u}, \vec{v} \) is a linear combination of \( \lambda \vec{u}, \vec{v} \) for \( \lambda \in \{ e_u, -\vec{u}, \nu \vec{u}, \nu \vec{v} \} \) and every element of \( e_u \Delta_\nu e_v \) has the form \( \omega \vec{u}, \vec{v} \) for \( \omega \in e_u \Delta_\nu e_u \). We know that \( \omega \) is a linear combination of \( \lambda \vec{u}, \vec{v} \) for \( \lambda \in \{ e_u, -\vec{u}, \nu \vec{u}, \nu \vec{v} \} \) and \( 0 \leq \lambda < m_s(\vec{u}) \). Multiplying by \( \vec{u}, \vec{v} \) on the right, these elements are linearly independent. Notice that, using this argument, we see that the only possibility, up to rescaling, to have \( x \in e_u \Delta_\nu e_v \) satisfying \( x \vec{u} = 0 \) is to take \( x = [\vec{u}, -\vec{u}]_{m_t(\vec{u})-1} [-\vec{u}, \nu \vec{u}][\vec{u}, \vec{v}] \) if \( M_P = \emptyset \) and \( m_t(\vec{u}) = 1 \). In this case, according to the computation before, we get

\[
\lambda_{t(\vec{u})} x = [-\vec{u}, -\vec{u}]_{m_t(\vec{u})-1} [-\vec{u}, \nu \vec{u}][\vec{u}, -\vec{u}][\vec{u}, \vec{v}] = \lambda_{s(\vec{u})}^{-1} C_\nu \vec{u}, \vec{v} \in (C_\nu).
\]

(d) If \( t(\vec{v}) = s(\vec{u}) \neq t(\vec{u}) \). This is similar to the previous case.

(e) If \( t(\vec{v}) = s(\vec{u}) = t(\vec{u}) \). Using the structure of \( e_u \Delta_\nu e_v \), every element of \( e_u \Delta_\nu e_v \) can be written as a linear combination of \( \omega \vec{u}, \vec{v} \eta \) where \( \omega \in e_u \Delta_\nu e_u \) and \( \eta \in \{ e_v, [\vec{v}, \nu \vec{v}] \} \). Using the structure of \( e_u \Delta_\nu e_u \), \( \omega \) is a linear combination of \( \lambda \vec{u}, \vec{v} \) for \( \lambda \in \{ e_u, -\vec{u}, \nu \vec{u}, \nu \vec{v} \} \) and \( 0 \leq \lambda < m_s(\vec{u}) \). Using \( R_{\vec{u}}^\nu \), we have

\[
[\vec{u}, -\vec{u}][\vec{u}, \vec{v}] = \left\{ \begin{array}{ll}
\lambda_{M} e_v C_\nu & \text{if } M_P = \{ M \} \text{ and } M_M = 1; \\
0 & \text{else,}
\end{array} \right.
\]

and we deduce \( [\vec{u}, -\vec{u}][\vec{u}, \vec{v}] = [\vec{u}, -\vec{v}][\vec{u}, -\vec{u}][\vec{u}, \vec{v}] = 0 \). As a consequence, any element of \( e_u \Delta_\nu e_v \) is a linear combination of \( \epsilon \vec{u}, \vec{v} \eta \) for \( \epsilon \in \{ e_u, -\vec{u}, \nu \vec{u}, \nu \vec{v} \} \) and \( 0 \leq \epsilon < m_s(\vec{u}) \). It remains to prove that these elements are linearly independent. Suppose that \( \sum_{\epsilon, \eta} \mu_{\epsilon, \eta} \epsilon \vec{u}, \vec{v} \eta = 0 \). Let us denote \( x := [\vec{u}, -\vec{u}] \), \( y := [-\vec{u}, \vec{u}] \) and \( c := \lambda_{s(\vec{u})}(xy)^{m_s(\vec{u})} = \lambda_{s(\vec{u})}(yx)^{m_s(\vec{u})} \) in such a way that computations rules fit with the description of \( e_u \Delta_\nu e_v \) before.
Proposition 10.26. For any the ideal of relations of \( \Delta \) two paths of \( \phi \) we take the same notations as in Proof of Proposition 10.25 and we follow the same inductive argument. By Theorem 4.5 for \( 10.10 \) \( \sigma \) suppose now that \( \sigma \) follows the same inductive argument. By Theorem 4.5 for \( 10.10 \) we get \( \alpha = [\bar{u}, -\bar{v}] [\bar{v}, -\bar{u}] \) is equal to \( \lambda_{N'} x \) if \( \bar{v} \) encloses a special monogon with special puncture \( N' \) and \( \alpha = 0 \) else. Using the structure of \( e_u \Delta_\sigma e_u \), we get

\[
\mu_{\varepsilon, \ell, \varepsilon}(\bar{v}, -\bar{u}) = \begin{cases} 
-\lambda_{N'} \mu_{\varepsilon, \ell, [\bar{v}, -\bar{u}]} & \text{if } \bar{v} \text{ is special;} \\
0 & \text{else.}
\end{cases}
\]

Multiplying the equality by \( [\bar{v}, -\bar{u}] \), an analogous reasoning gives

- \( \mu_{\varepsilon, \ell, [\bar{v}, -\bar{u}]} = 0 \) if \( \ell < m_{s}(\bar{u}) - 1 \) or \( m_M > 1 \) or \( \varepsilon = e_u \);
- \( \mu_{y, m_s(\bar{u}) - 1, [\bar{v}, -\bar{u}]} = -\lambda_M \lambda_s(\bar{u}) (\mu_{e_v, 0, e_v} + \lambda_{N} \mu_{y, 0, e_v}) \) if \( m_M = 1 \) and \( -\bar{u} \) encloses a special monogon with puncture \( N \);
- \( \mu_{y, m_s(\bar{u}) - 1, [\bar{v}, -\bar{u}]} = -\lambda_M \lambda_s(\bar{u}) \mu_{e_v, 0, e_v} \) if \( m_M = 1 \) and \( -\bar{u} \) does not enclose a special monogon.

So we get \( \mu_{\varepsilon, \ell, y} = 0 \) if \( \varepsilon = e_u \) or \( m_M > 1 \) or \( \ell < m_{s}(\bar{u}) - 1 \) or \( -\bar{u} \) is not special or \( \bar{v} \) is not special. If \( m_M = 1 \), \( -\bar{u} \) is special and \( \bar{v} \) is special, we have \( \mu_{y, m_s(\bar{u}) - 1, e_v} = -\lambda_{N'} \mu_{y, m_s(\bar{u}) - 1, [\bar{v}, -\bar{u}]} = \lambda_{N'} \mu_{y, 0, e_v} \) which permits to conclude in any case as \( \mu_{y, 0, e_v} \) is invertible. Notice also that as \( [\bar{v}, -\bar{u}] \) and \( [\bar{v}, -\bar{u}] \) are in \( J \), we get that \( xJ = 0 \) is impossible for a non-zero \( x \in e_u \Delta_\sigma e_v \).

(2) is an easy consequence of (1) and Proposition 4.12 which states that the ideal of relations of \( \Delta_\sigma \) is generated by linear combinations of at most two paths.

Then we deduce this generalized version of Theorem 4.5 for \( \Delta_\sigma^* \):

**Proposition 10.26.** For any \( \varepsilon \in \{0, 1\}^P \), \( B \) is mapped to a basis of \( \Delta_\sigma^* \).

**Proof.** We take the same notations as in Proof of Proposition 10.25 and we follow the same inductive argument. By Theorem 4.5 for \( \Delta_\sigma \), \( B \) is mapped to a basis of \( \Delta_\sigma^0 \). Let \( \varepsilon \in \{0, 1\}^P \) such that \( \varepsilon \geq 0 \) and construct \( \eta \) as in Proof of Theorem 10.25. Then, it is immediate looking at the definition of \( \varphi^0 \) in each case that if \( B \) is mapped to a basis of \( \Delta_\sigma^0 \), then \( B \) is also mapped to a basis of \( \Delta_\sigma^* \).

(2) is then an easy consequence of (1) as before. \( \square \)

10.10. **Proof of Theorem 4.9** Finally, we will prove Theorem 4.9. We suppose now that \( \sigma \) has no arc incident to the boundary. We start with two lemmas.

**Lemma 10.27.** If \( \bar{u}, \bar{v} \in \sigma \) and \( s(\bar{u}) = s(\bar{v}) \), we have:

- if \( -\bar{v} \) encloses a special monogon with special puncture \( M \) and \( t(\bar{u}) \neq M \), \([\bar{u}, \bar{v}] [-\bar{v}, \bar{u}] = \lambda_M [\bar{u}, \bar{u}] \);  
- if \( \bar{u}, -\bar{u}, -\bar{v} \) form a self-folded triangle without puncture, then we have \([\bar{u}, \bar{v}] [-\bar{v}, \bar{u}] = [-\bar{u}, -\bar{v}] \);  
- if \( \bar{v}, -\bar{v} \) and \( \bar{u} \) are (strictly) ordered around \( s(\bar{u}) \), \( -\bar{v} \) encloses two punctures \( M \) and \( N \), and \( m_M = m_N = 1 \), we have \([\bar{u}, \bar{v}] [-\bar{v}, \bar{u}] = \lambda_M \lambda_N e_u C_\sigma \);
• in any other case, $[\vec{u}, \vec{v}][−\vec{v}, −\vec{u}] = 0$.

We also have:

• if $\vec{u}, −\vec{v}$ form a digon with one puncture $M$ and $m_M = 1$, then we have $[\vec{u}, \vec{v}][−\vec{v}, −\vec{u}] = \lambda_M e_a C_\sigma$;

• if $−\vec{u}, −\vec{v}$ form a digon with one puncture $M$ and $m_M = 1$ and $\vec{u}$ encloses a special monogon with special puncture $N$, then we have $[\vec{u}, \vec{v}][−\vec{v}, −\vec{u}] = \lambda_M \lambda_N e_a C_\sigma$;

• if $−\vec{v}$ encloses a special monogon with special puncture $M$ which is also enclosed by $−\vec{u}$ (not necessarily special), then we have $[\vec{u}, \vec{v}][−\vec{v}, −\vec{u}] = \lambda_M [\vec{u}, \vec{u}][\vec{u}, −\vec{u}]$;

• if $−\vec{v}$ encloses a special monogon with special puncture $M$ which is also enclosed by $\vec{u}$ (not necessarily special), then $[\vec{u}, \vec{v}][−\vec{v}, −\vec{u}] = \lambda_M [\vec{u}, −\vec{u}]$;

• if $−\vec{v} = \vec{u}$ encloses a unique puncture $M$ with $m_M = 2$ then we have $[\vec{u}, \vec{v}][−\vec{v}, −\vec{u}] = \lambda_M e_a C_\sigma$;

• in any other case, $[\vec{u}, \vec{v}][−\vec{v}, −\vec{u}] = 0$.

Proof. We can suppose that $\sigma$ contains only $u$ and $v$ thanks to Corollary 4.12. We start by $[\vec{u}, \vec{v}][−\vec{v}, \vec{u}]$. For this to be non-zero, we need $t(\vec{v}) = s(\vec{u}) = s(\vec{v})$ so $\vec{v}$ is a loop. Then studying the different possible orders of $\vec{v}$, $−\vec{v}$, $\vec{u}$ (and maybe $−\vec{u}$) around $s(\vec{v})$ gives easily the result using Proposition 4.12. The reasoning is analogous for $[\vec{u}, \vec{v}][−\vec{v}, −\vec{u}]$. The case $u = v$ is dealt in the same way.

\begin{definition}
We say that $\vec{u} \in \sigma$ is 2-special if it encloses a monogon containing two punctures $M_{\vec{u}}$ and $N_{\vec{u}}$ with $m_{M_{\vec{u}}} = m_{N_{\vec{u}}} = 1$. We define the subset $E$ of $\Delta_\sigma$ by

$$E := \{e_a C_\sigma \mid u \in \sigma \} \cup \{\lambda_{M_{\vec{u}}}^{-1} \lambda_{N_{\vec{u}}}^{-1} [\vec{u}, −\vec{u}] \mid \vec{u} \text{ is 2-special}\}.$$

We define the linear map $E^* : \Delta_\sigma \to k$ such that $E^*(x) = 1$ if $x \in E$ and $E^*(x) = 0$ if $x$ is an element of the basis without multiple in $E$ (it is possible as $E$ consists of multiples of elements of $B$).

\begin{lemma}
Suppose that $\sigma$ has no arc incident to a boundary. Let $u$ and $v$ be two elements of $B$ such that $a \in e_a \Delta_\sigma e_v$ and $b \in e_a \Delta_\sigma e_u$ for some $u$ and $v$. Let $\mu \in k$. Then $\mu ab \in E$ if and only if $\mu ba \in E$.

Proof. According to Corollary 4.12 we can suppose that $u$ and $v$ are the only arcs of $\Delta_\sigma$. Suppose that $u \neq v$.

According to Theorem 4.15 we can suppose that $s(\vec{u}) = s(\vec{v})$ and $a = [\vec{u}, \vec{v}][\vec{u}, \vec{v}]$. Suppose first that $b = [\vec{v}, \vec{v}][\vec{v}, \vec{w}]$ for $\vec{w} = ±\vec{u}$. Then we have $\mu ab = \mu [\vec{u}, \vec{u}]^{\vec{v} + \vec{v}}[\vec{u}, \vec{v}][\vec{v}, \vec{w}]$ and, according to Theorem 4.15 $\mu ab = e_a C_\sigma$ can happen in three situations:

(a) $\mu = \lambda_{s(\vec{u})}, \ell + \ell' = m_{s(\vec{u})} − 1$ and $\vec{w} = \vec{u}$;

(b) $\mu = \lambda_{s(\vec{u})}^{-1}, \ell + \ell' = m_{s(\vec{u})} − 1$, $\vec{w} = −\vec{u}$ and $\vec{u}$ encloses a special monogon with special puncture $M \neq t(\vec{v})$;

(c) $\mu = \lambda_{s(\vec{u})}^{-1}$, $\ell + \ell' = m_{s(\vec{u})}$, $\vec{w} = −\vec{u}$ and $\vec{u}$ encloses a special monogon with special puncture $M = t(\vec{v})$.

So we clearly have $\mu ba = e_a C_\sigma$.  

Moreover, \( \mu_{ab} = \lambda_{M_{ab}}^{-1} \lambda_{N_{ab}}^{-1} [\vec{u}, -\vec{w}] \) for a 2-special oriented arc \( \vec{u} \) can happen only if \( \vec{u} = \vec{u}, t(\vec{v}) \in \{ M_{ab}, N_{ab} \}, \vec{w} = -\vec{u}, \ell = \ell' = 0 \) and \( \mu = \lambda_{M_{ab}}^{-1} \lambda_{N_{ab}}^{-1} \). Then we have \( \mu_{ba} = e_v C_{\sigma} \).

Suppose now that \( b = [-\vec{v}, -\vec{w}]^{\ell'} [-\vec{v}, \vec{w}] \) with \( \vec{w} = \pm \vec{u} \). We have \( \mu_{ab} = \mu(\vec{u}, \vec{u}) f(\vec{u}, \vec{v}) [-\vec{v}, \vec{w}]^{\ell'} \) so using Lemma 10.27, the only cases where \( \mu_{ab} = e_v C_{\sigma} \) are the following:

(a) \( -\vec{v} \) encloses a special monogon with special puncture \( M \neq t(\vec{u}) \), \( \vec{w} = \vec{u} \), \( \ell + \ell' = m_{s(\vec{w})} - 1 \) and \( \mu = \lambda_{s(\vec{w})}^{-1} \lambda_{M}^{-1} \);
(b) \( -\vec{v} \) encloses a special monogon with special puncture \( M = t(\vec{u}) \), \( \vec{w} = \vec{u} \), \( \ell + \ell' = m_{s(\vec{w})} \) and \( \mu = \lambda_{s(\vec{u})}^{-1} \lambda_{M}^{-1} \);
(c) \( \vec{u}, -\vec{v}, \vec{u} \) are strictly ordered around \( s(\vec{u}) \), \( -\vec{v} \) is 2-special, \( \vec{w} = \vec{u} \), \( \ell = \ell' = 0 \) and \( \mu = \lambda_{M_{ab}}^{-1} \lambda_{N_{ab}}^{-1} \);
(d) \( \vec{u}, -\vec{v} \) form a digon with one puncture \( M \) such that \( m_M = 1 \), \( \vec{w} = -\vec{u} \), \( \ell + \ell' = 0 \) and \( \mu = \lambda_{M}^{-1} \);
(e) \( -\vec{v} \) is 2-special, \( \vec{u} \) is a special monogon with special puncture \( N_{-\vec{v}}, \vec{w} = -\vec{u} \), \( \ell = \ell' = 0 \) and \( \mu = \lambda_{M_{ab}}^{-1} \lambda_{N}^{-2} \); 
(f) \( -\vec{v} \) and \( \vec{u} \) are special monogons with special punctures \( M \) and \( N \), \( \vec{w} = -\vec{u} \), \( \ell + \ell' = m_{s(\vec{w})} - 1 \) and \( \mu = \lambda_{s(\vec{u})}^{-1} \lambda_{M}^{-1} \).

In cases (a), (b), (d) and (f), we have \( \mu_{ba} = e_v C_{\sigma} \). In cases (c) and (e), we have \( \mu_{ba} = \lambda_{M_{ab}}^{-1} \lambda_{N_{ab}}^{-1} [-\vec{v}, -\vec{w}] \) so \( \mu_{ab} \in E \).

Finally, the only case where \( \mu_{ab} = \lambda_{M_{ab}}^{-1} \lambda_{N_{ab}}^{-1} [\vec{w'}, -\vec{v}] \) for a 2-special oriented arc \( \vec{u} \) happens, again thanks to Lemma 10.27, when \( \vec{w} = \vec{w'}, -\vec{v} \) encloses \( N_{\vec{u}} \) and \( \mu = \lambda_{M_{ab}}^{-1} \lambda_{N_{ab}}^{-2} \). In this case, we have \( \mu_{ba} = e_v C_{\sigma} \).

The case \( u = v \) is analogous (and simpler).

**Definition 10.30.** For a \( k \)-algebra \( A \), we say that a \( k \)-linear map \( t : A \to k \) is a **trace** if it satisfies \( t(ab) = t(ba) \) for any \( a, b \in A \). We say that it is a **non-degenerate trace** if moreover the induced morphism of \( A \)-bimodule

\[
A \to \text{Hom}_k(A, k), \quad a \mapsto t(a-) 
\]

is an isomorphism.

We recall the following classical observation:

**Lemma 10.31.** Let \( M \) be a \( m \times \ell \) matrix with coefficient in \( k \). If there exists \( \mu \in k \setminus \{0\} \) such that \( \mu \delta = 0 \) for all maximal minors \( \delta \) of \( M \) then \( M \) has a non-zero kernel.

**Proof.** Up to adding rows of zeros, we can suppose that \( m \geq \ell \). If \( \ell = 1 \), the result is obvious. Let us suppose that \( \ell > 1 \). If \( \mu \delta = 0 \) for all \( (\ell - 1) \times (\ell - 1) \)-minors \( \delta' \) of \( M \) then the result is true by induction. So, without loss of generality, we can suppose that the upper left \( (\ell - 1) \times (\ell - 1) \) minor \( \delta' \) of \( M \) satisfy \( \mu \delta' \neq 0 \). For \( i = 1, \ldots, \ell \), let \( \delta_i := (-1)^i \mu \delta_i' \) where \( \delta_i' \) is the minor of \( M \) with rows \( 1, 2, \ldots, i - 1, i + 1, \ldots, \ell \) and columns \( 1, 2, \ldots, i - 1, i + 1, \ldots, \ell \). Then \( \delta \neq 0 \) is in the kernel of \( M \). \( \square \)

We deduce the following equivalent characterizations of non-degenerate traces:
**Lemma 10.32.** For a k-algebra $A$ which is free of finite rank over $k$ and a trace $t : A \to k$, the following are equivalent:

(i) $t$ is non-degenerate;

(ii) for any $\mu \in k$ and $a \in A$, $(\forall x \in A, t(ax) \in \mu k) \iff a \in \mu A$.

**Proof.** Let $\rho : A \to \text{Hom}_k(A, k)$ be defined by $\rho(a) = t(a-)$. If $\rho(a)$ is invertible, it is immediate that $\rho$ is injective. As $A$ and $\text{Hom}_k(A, k)$ have the same rank over $k$ and $\rho$ is injective, $\mu := \det(\rho) \in k$ is not a zero divisor by Lemma 10.33. Let $f \in \text{Hom}_k(A, k)$ and $b := \rho'f$ where $\rho'$ is the adjugate of $\rho$. Thus $\rho(b) = \mu f$. Using (ii), we then get $b = \mu a$ for some $a \in A$. Thus $\mu(f - \rho(a)) = 0$ so $f = \rho(a)$. \qed

**Lemma 10.33.** The linear map $\mathcal{E}^*$ is a non-degenerate trace.

**Proof.** As the product of any two elements of the basis is a multiple of an element of the basis thanks to Theorem 4.3 and using Lemma 10.29, we get immediately that $\mathcal{E}^*(xy) = \mathcal{E}^*(yx)$ for any $x, y \in \Delta_\sigma$ so $\mathcal{E}^*$ is a trace.

Let us prove that $\mathcal{E}^*$ is non-degenerate. We use the characterization of Lemma 10.32. Let $a \in \Delta_\sigma$ and $\mu \in k$ satisfying $\mathcal{E}^*(ax) \in \mu k$ for any $x \in \Delta_\sigma$. If $\mu$ is invertible, it is immediate that $a \in \mu \Delta_\sigma$.

Suppose that $\mu = 0$. If $a \neq 0$, then, as $J^n = 0$ for $n$ big enough, there exists $x, x' \in \Delta_\sigma$ such that $xx' \neq 0$ and $xx'J = Jxx' = 0$. Thanks to Theorem 10.9 (2), $xx' \in (C_\sigma)$. Then, up to multiplying by an idempotent, $xx'$ is a non-zero multiple of $e_uC_\sigma$ for $u \in \sigma$. So $\mathcal{E}^*(xx') \neq 0$. It is a contradiction.

If $\mu$ is not invertible, notice that $\Delta_{\sigma}' := \Delta_\sigma/(\mu \Delta_\sigma)$ is the algebra of $\sigma$ defined over the ring $k' := k/(\mu k)$ and through this identification, $\mathcal{E}^*$ is mapped to the corresponding trace over $\Delta_{\sigma}'$. Thus, applying the case $\mu = 0$ implies immediately that $a \in \mu \Delta_\sigma$. \qed

**Proof of Theorem 4.9.** It is an immediate consequence of Lemma 10.33. \qed

Finally, we prove Proposition 10.35.

**Proof of Proposition 10.35.** First of all, it is classical that the last four terms form an exact sequence, where $\alpha$ is the multiplication in $\Delta_\sigma$, $\beta(ae\underline{a} \otimes e\underline{a} b) = a \otimes [\underline{u}, \underline{u}^+]b - a[\underline{u}, \underline{u}^+] \otimes b$ and $\gamma$ is induced by the generating relations $R_\sigma$ (notice that the completion with respect to $J$ does not matter here). More precisely, if we define the morphism of $kQ_{\sigma}$-bimodules $\varphi : kQ_{\sigma} \to \bigoplus_{\underline{a} \in \sigma} \Delta_\sigma e_{\underline{a}} \otimes e_{\underline{a}}^+ \Delta_\sigma$ by $\varphi(q_1q_2 \cdots q_n) = \sum_{i=1}^n q_1 \cdots q_{i-1} \otimes q_{i+1} \cdots q_n$ for any path $q_1 \cdots q_n$, we let, for an oriented triangle $\vec{w}, \vec{v}, \vec{u}$ in $\sigma$, $\gamma(e_{\underline{u}} \otimes e_{\underline{v}}) = \varphi(R_\sigma)$. To conclude, it is enough to prove that $\gamma = \gamma^*$. Equivalently, we need to prove that

$$(\mathcal{E}^* \otimes \mathcal{E}^*)(\varphi(R_\sigma)b) = (\mathcal{E}^* \otimes \mathcal{E}^*)(\alpha \varphi(R_\sigma)b)$$

for two triangles $\vec{w}, \vec{v}, \vec{u}$ and $\vec{w'}, \vec{v'}, \vec{u'}$ in $\sigma$, $a \in e_{\underline{u}}Be_{\underline{v'}}$ and $b \in e_{\underline{w'}}Be_{\underline{u'}}$. Recall that, as $\sigma$ is a triangulation, we have $R_\sigma = [\underline{u}, -\underline{v}][\underline{v}, -\underline{w}] - [\underline{u}, \underline{w}].$
Let us consider first the term

\((\mathcal{E}^* \otimes \mathcal{E}^*)(b_\varphi(-\vec{u}, -\vec{v})a) = (\mathcal{E}^* \otimes \mathcal{E}^*)(b[\vec{u}, -\vec{v}] \otimes a + b \otimes [\vec{v}, -\vec{w}]a)\).

The first part \((\mathcal{E}^* \otimes \mathcal{E}^*)(b[\vec{u}, -\vec{v}] \otimes a)\) is non-zero if and only if \(a, b[\vec{u}, -\vec{v}] \in k\mathcal{E}\), which implies that \(\vec{w} = \pm \vec{w}'\) and \(\vec{w}' = \pm \vec{v}\). As there is at most one oriented triangle, that \(v = w'\) and \(w = u'\) are sides of, in the same order, we necessarily have \(\vec{w} = \vec{w}'\) and \(\vec{v} = \vec{u}\). So

\((\mathcal{E}^* \otimes \mathcal{E}^*)(b[\vec{u}, -\vec{v}] \otimes a) = (\mathcal{E}^* \otimes \mathcal{E}^*)(a \otimes [\vec{v}, -\vec{w}]b)\) (obviously this still holds when \((\mathcal{E}^* \otimes \mathcal{E}^*)(b[\vec{u}, -\vec{v}] \otimes a) = 0\) by symmetry of the reasoning). Moreover, in the same way,

\((\mathcal{E}^* \otimes \mathcal{E}^*)(b \otimes [\vec{v}, -\vec{w}]a) = (\mathcal{E}^* \otimes \mathcal{E}^*)(a[\vec{v}', -\vec{u}] \otimes b)\)

so \((\mathcal{E}^* \otimes \mathcal{E}^*)(b_\varphi([-\vec{u}, -\vec{w}]a) = (\mathcal{E}^* \otimes \mathcal{E}^*)(a\varphi([-\vec{v}', -\vec{u}][\vec{v}', -\vec{w}])b)\).

Let us now look at the second term \((\mathcal{E}^* \otimes \mathcal{E}^*)(b_\varphi(-\vec{u}, -\vec{w})a)\). It is immediate that, if it is non-zero, then there is an oriented arc \(\vec{x}\) such that \(s(\vec{x}) = s(\vec{u})\), \(\vec{x} = \pm \vec{w}'\) and \(\vec{x}^+ = \pm \vec{u}'\). As \(\sigma\) is a triangulation, the only possibility is that \(\vec{x} = \vec{w}'\) and \(\vec{x}^+ = -\vec{u}'\). Continuing this argument, we then find, in this case

\[
(\mathcal{E}^* \otimes \mathcal{E}^*)(b_\varphi([-\vec{u}, -\vec{w}]a) = \sum_{\omega_1[\vec{w}', -\vec{u}]\omega_2[-\vec{u}, \vec{w}]} \mathcal{E}^*(b_\varphi(\omega_1))\mathcal{E}^*(\omega_2a)
= \sum_{\omega_1[\vec{w}', -\vec{u}][\vec{w}]\omega_2[-\vec{u}, \vec{w}]} \mathcal{E}^*(a\varphi([-\vec{u}', -\vec{w}][\vec{u}', \vec{w}])b)
= (\mathcal{E}^* \otimes \mathcal{E}^*)(a\varphi([-\vec{u}', -\vec{w}][\vec{u}', \vec{w}])b).
\]

Combining both identities, we finally get, as necessary,

\((\mathcal{E}^* \otimes \mathcal{E}^*)(b_\varphi(\mathcal{R}_{\sigma})a) = (\mathcal{E}^* \otimes \mathcal{E}^*)(a\varphi(\mathcal{R}_{\sigma})b). \square\)

10.11. **Proof of Proposition 8.9.** Along this proof, we denote \(\lambda^* = \mu_u(\lambda)\). Moreover, every algebra of partial triangulation is computed with respect to \(\lambda\) if it is not specified.

We provide the following Lemma to simplify the situation:

**Lemma 10.34.** Suppose that \(\sigma \subset \sigma'\) for a partial triangulation \(\sigma'\) satisfying that \(\vec{u}^+\) and \((-\vec{u})^+\) have the same value computed with respect to \(\sigma\) or \(\sigma'\). Then \(\mu_u(\sigma) \subset \mu_u(\sigma')\) in an obvious way and, if \(T\) (respectively \(T'\)) denotes the tilting complex computed for \(\sigma\) (respectively \(\sigma'\) as before, we have

\[
\text{End}_{K^b(\mathcal{D}_\sigma)}(T) \cong \text{End}_{K^b(\mathcal{D}_{\sigma'})}(T')e_{\sigma^*}
\]

where \(e_{\sigma^*}\) is the idempotent of \(\text{End}_{K^b(\mathcal{D}_{\sigma'})}(T')\) corresponding to the set of arcs in \(\sigma^*\) \((e_{\sigma^*}\text{ corresponds to the summand }P^{\sigma^*}_u\text{ of }T')\).

**Proof.** First of all, it is clear under these assumptions that \(T \cong e_{\sigma^*}T'\text{ }e_{\sigma}\) as \(e_{\sigma^*}\text{ }\text{End}_{K^b(\mathcal{D}_{\sigma'})}(T')e_{\sigma^*}\times \Delta_{\sigma}\text{-modules. Moreover, }-e_{\sigma} : \text{add}(e_{\sigma^*}\Delta_{\sigma}) \rightarrow \text{proj}\Delta_{\sigma}\text{ is an equivalence of categories (as End}_{\Delta_{\sigma}}(e_{\sigma^*}\Delta_{\sigma}) = e_{\sigma^*}\Delta_{\sigma^*}e_{\sigma}) \cong \Delta_{\sigma}\text{ by Corollary 4.2. Hence, }-e_{\sigma} : K^b(\text{add}(e_{\sigma^*}\Delta_{\sigma})) \rightarrow K^b(\text{proj}\Delta_{\sigma})\text{ is also an equivalence of category. Moreover, it is immediate that }e_{\sigma^*}T' \text{ in }K^b(\text{add}(e_{\sigma^*}\Delta_{\sigma})), \text{ so }
\]

\[
e_{\sigma^*}\text{End}_{K^b(\mathcal{D}_{\sigma'})}(T')e_{\sigma^*} = \text{End}_{K^b(\text{add}(e_{\sigma^*}\Delta_{\sigma}))}(e_{\sigma^*}T')
\]

\[
\text{End}_{K^b(\mathcal{D}_{\sigma^*})}(e_{\sigma^*}T'e_{\sigma}) = \text{End}_{K^b(\mathcal{D}_{\sigma})}(T).
\] \square
In cases (F2) or (F3), the identity of $\alpha \psi$

**Proof.**

From now on, we identify $\Delta _{\tau }$ and $e_{\tau }\Delta _{\tau }e_{\tau }$ on the one hand and $\Delta _{\tau }^{\lambda ^{*}}$ and $e_{\tau }\Delta _{\tau }^{\lambda ^{*}}e_{\tau }$ on the other hand using Corollary [1.2] We also use notation of Section [1.3]. More precisely, we use $R_{\psi }$ and $f_{\psi }$ for $\sigma$ and $R_{\psi }^{*}$ and $f_{\psi }^{*}$ for $\sigma ^{*}$. We need the following lemmas:

**Lemma 10.36.** (1) In cases (F2) or (F3), in $\Delta _{\tau },$

- $[\bar{u},(-\bar{u})^{+}][(-\bar{u})^{+},(-\bar{u}^{*})^{+}] = [\bar{u},\bar{u}^{+}][(-\bar{u}^{*})^{+},(-\bar{u})^{+}]$.
- $|\bar{u}^{+},\bar{u}^{+}]: [-\bar{u},(-\bar{u})^{+}] = f_{\bar{u}^{*}}^{*}(\bar{u}^{*}),(-\bar{u})^{+}.$

(2) In case (F3), in $\Delta _{\tau },$

- $[\bar{u},\bar{u}^{+}][(-\bar{u}^{*})^{+},\bar{u}^{*}] = [\bar{u},(-\bar{u})^{+}][(-\bar{u}^{*})^{+},(-\bar{u})^{+}]$.
- $|(-\bar{u})^{+},\bar{u}^{+},[\bar{u},\bar{u}^{+}] = f_{\bar{u}^{*}}^{*}(\bar{u}^{*}),(-\bar{u})^{+}.$

(3) In case (F2), $[\bar{u},\bar{u}^{+}][(-\bar{u}^{*})^{+},\bar{u}^{*}] = 0$.

(4) In cases (F2) or (F3), the identity of $kQ_{\tau }$ induces an isomorphism $\psi : \Delta _{\tau }^{\lambda ^{*}} \rightarrow \Delta _{\tau }.$

**Proof.** (1), (2) and (3) are easy computations using Theorem [1.3] in Figure [10.35] (recall that in case (F2), $\bar{u}$ does not enclose a special monogon).

(4) The only case where it is not immediate is when $\lambda ^{*} \neq \lambda$. So we focus on case (F2) when $\bar{u}$ encloses a unique puncture $M$. In this case, we have $\lambda _{M}^{*} = -\lambda _{M}$. However, no relation of $\Delta _{\tau }^{\lambda ^{*}}$ does involve $\lambda _{M}^{*}$ (see Theorem [1.3] and recall that $\bar{u} \notin \tau$ and $m_{M} > 1$ by hypothesis). It permits to conclude.

**Lemma 10.37.** Suppose that we are in case (F1) or (F1'). Then there exists $\alpha \in e_{\bar{u}^{*}+}\Delta _{\tau }e_{\bar{u}^{*}+}$ and an isomorphism $\psi : \Delta _{\tau }^{\lambda ^{*}} \rightarrow \Delta _{\tau }$ such that

1. $[\bar{u},\bar{u}^{+}][(-\bar{u}^{*})^{+},\bar{u}^{*}] = [\bar{u},\bar{u}^{*}][\bar{u}^{*}]\alpha$;
2. $\psi([-\bar{u}^{+},\bar{u}^{*}]) = [-\bar{u}^{+},\bar{u}^{*}] - \alpha$;
3. $\psi(C_{\bar{u}^{*}}^{\bar{u}^{*}}) = C_{\bar{u}^{*}}^{\bar{u}^{*}}$;
4. $\alpha \bar{u}^{*}^{+},\bar{u}^{*} = (1 - \nu_{M})e_{\bar{u}^{*}+}C_{\bar{u}^{*}}$ or $\alpha \bar{u}^{*}^{+},\bar{u}^{*} = 0$;
5. for any element $\omega$ of the basis $B$ of the form $[\bar{u}^{*},x][\bar{x},\bar{x}]$, we have $\omega \alpha = (1 - \nu_{M})[-\bar{u}^{*},\bar{u}^{*}]\omega$,

or $\alpha \omega$ is multiple of an element of $B$ strictly longer than $[-\bar{u}^{*},\bar{u}^{*}]\omega$,

or $\alpha \omega = 0.$
Proof. We prove the result when \( \Sigma \) has no boundary as it implies the general result via Definition 4.10. Suppose first that we are in case (F 1) with \( m_{t(\vec{u})} > 2 \), or in case (F 1\')) with \( m_{t(\vec{u})} > 1 \). Then we have \([\vec{u}, \vec{u}^+][-\vec{u}^+, \vec{u}^+] = 0\). Thus, we can take \( \alpha = 0 \). No relation of \( \Delta_\tau \) (respectively \( \Delta_\tau^* \)) depends on \( \lambda_{t(\vec{u})} \) (respectively \( \lambda_{t(\vec{u})}^* \)). So, in case (F 1), we can take \( \psi(q) = q \) for any \( q \in Q_\tau \). Notice that in case (F 1\'), \([\vec{u}^+, \vec{u}^+] \) appears only in 0-relations and relations involving \( \lambda_{s(\vec{u})} \). So we can take \( \psi([-\vec{u}^+, \vec{u}^+]) = -[-\vec{u}^+, \vec{u}^+] \) and \( \psi(q) = q \) for any other \( q \in Q_\tau \) in case (F 1\').

We will now focus on cases (F 1) with \( m_{t(\vec{u})} = 2 \) and (F 1\') with \( m_{t(\vec{u})} = 1 \) (we have excluded (F 1) with \( m_{t(\vec{u})} = 1 \)).

(F 1) with \( m_{t(\vec{u})} = 2 \). We start by supposing that \( \sigma \) contains \( \vec{v} \) as follows:

\[
\vec{u}^+ = \vec{u}^+^t
\]

and we will check that we can choose:

- \( \alpha = \lambda_{t(\vec{u})} \vec{u}^+, -\vec{u}^+ \), \( \psi([-\vec{u}^+, \vec{u}^+]) = [-\vec{u}^+, \vec{u}^+] - \lambda_{t(\vec{u})} \vec{u}^+, -\vec{u}^+ \), \( \psi(q) = q \) for any other arrow \( q \) of \( Q_\tau \).

(1) is immediate. We will prove that \( \psi \) is an isomorphism, (2), (3) and (4) at the same time. Notice that \( \lambda_{t(\vec{u})}^* = -\lambda_{t(\vec{u})} \) and \( \lambda_M = \lambda_M \) for any other \( M \). For each puncture \( M \), we denote \( \lambda_M = \delta_{m_M} \lambda_M \). We denote \( \tilde{\lambda} = \lambda_N \) in the case of a sphere with four puncture where the only puncture \( N \) which is not on the diagram satisfies \( m_N = 1 \). In any other case, we denote \( \tilde{\lambda} = 0 \). Using Theorem 4.3, we get successively:

(a) \([\vec{u}^+, -\vec{u}^+, \vec{u}^+, \vec{v}] = 0 \). Indeed,

\[
[\vec{u}^+, -\vec{u}^+ | [\vec{u}^+, \vec{v}] = [\vec{u}^+, \vec{v}] | [\vec{v}, -\vec{u}^+] | [\vec{u}^+, \vec{v}].
\]

If \( (\Sigma, M) \) is not a sphere with three punctures,

\[
[\vec{v}, -\vec{u}^+ | [\vec{u}^+, \vec{v}] = \tilde{\lambda}_{t(\vec{u})} \vec{v} C_\tau \quad \text{so} \quad [\vec{u}^+, -\vec{u}^+ | [\vec{u}^+, \vec{v}] = 0
\]
as \( [\vec{u}^+, \vec{v}] \in J \). If \( (\Sigma, M) \) is a sphere with three punctures, we get

\[
[\vec{u}^+, -\vec{u}^+ | [\vec{u}^+, \vec{v}] = \tilde{\lambda}_{t(\vec{u})} \vec{v} C_\tau,
\]

but \( m_{t(\vec{v})} > 1 \) and \( 2m_{s(\vec{u})} = 2 \geq m_{s(\tilde{u})} \).

(b) \( \psi([-\vec{u}^+, \vec{v}] = [-\vec{u}^+, \vec{v}] \). Indeed, using (a), we get \( \psi([-\vec{u}^+, \vec{v}] = [-\vec{u}^+, \vec{v}] \)

and the result follows.

(c) \( \psi([-\vec{u}^+, \vec{v}] = [\vec{u}^+ \vec{v}]. \) Analogous as (b).

(d) \( \psi(\lambda_s(\vec{u}^+) | [\vec{u}^+, \vec{v}] = e_{\vec{u}^+} C_\tau \). If follows from (c).

(e) \( \psi(\lambda_t(\vec{u}^+) | [-\vec{u}^+, \vec{v}] = e_{\vec{u}^+} C_\tau \). Analogous as (d).
(f) \( \psi(\lambda_{s(\bar{v})}[\bar{u}, \bar{v}]^{s(\bar{v})}) = e_{\bar{u}}C_{\tau} \). Analogous as (d).

(g) \( \psi([-\bar{u}^+, \bar{u}^+]^2) = \lambda_{t(\bar{u})}^*e_{\bar{u}}+C_{\tau} \). Indeed, by (a),

\[
\psi([-\bar{u}^+, \bar{u}^+]^2) = ((-\bar{u}^+, \bar{u}^+) - \lambda_{t(\bar{u})}\bar{u}^+, -\bar{u}^+[2] \nonumber
= \lambda_{t(\bar{u})}e_{\bar{u}}+C_{\tau} - 2\lambda_{t(\bar{u})}e_{\bar{u}}+C_{\tau} + \lambda_{t(\bar{u})}^2[\bar{u}^+, -\bar{u}^+[2] 
= -\lambda_{t(\bar{u})}e_{\bar{u}}+C_{\tau} = \lambda_{t(\bar{u})}^*e_{\bar{u}}+C_{\tau} \nonumber.
\]

Then we get \( \psi(C_{\bar{u}^+}) = \psi(C_{\bar{u}}) = 0 \) and \( \psi(C_{\bar{u}}^+) = C_{\tau} \) by (d), (e) and (f).

We get \( \psi(R_{\bar{u}^+}) = 0 \) by (g). Finally, all other relations \( R_{\bar{u}^+} \) defining \( \Delta_{\bar{u}}^* \) involve \([-\bar{u}^+, \bar{v}], [\bar{v}, \bar{u}^+] \) and arrows other than \([-\bar{u}^+, \bar{u}^+] \) so we conclude that \( \psi(R_{\bar{u}^+}) = R_{\bar{u}} = 0 \) by (b) and (c). So \( \psi \) is a morphism. It is invertible with inverse satisfying \( \psi^{-1}([-\bar{u}^+, \bar{u}^+]) = [-\bar{u}^+, \bar{u}^+] + \lambda_{t(\bar{u})}\bar{u}^+, -\bar{u}^+[. \) The condition (4) comes from (a).

We proved the existence of \( \psi \) under the condition that \( \bar{v} \in \sigma \). The general case is obtained by applying Corollary 8.10 (see also Remark 8.11). Indeed, put \( \sigma_2 = \tau, \sigma^o = \{\bar{u}\} \). Let \( \sigma_1 \) be any partial triangulation which contains \( \bar{u}, \bar{u}^+ \) and \( \bar{v} \), which does not intersect \( \sigma^o \) and which is maximal for these properties. Then it is immediate that \( \sigma_1, \sigma_2 \) and \( \sigma^o \) satisfy the hypotheses of Corollary 8.10 (the isomorphism \( \Delta_{\bar{u}}^* \rightarrow \Delta_{\bar{u}} \) was just constructed). For (5), it is immediate via Corollary 4.12 that we can suppose that \( \omega \) is left divisible by \([\bar{u}^+, \bar{v}] \). Then, by (a), \( \omega = 0 \).

(F1') with \( m_{t(\bar{u})} = 1 \). We suppose first that \( \sigma \) contains \( \bar{v} \) as follows:

![Diagram](image)

and we will check that we can set:

- \( \alpha = \lambda_{t(\bar{u})}[\bar{u}^+, -\bar{u}^+[. \)
- \( \psi([-\bar{u}^+, \bar{u}^+]) = [-\bar{u}^+, \bar{u}^+] - \lambda_{t(\bar{u})}\bar{u}^+, -\bar{u}^+[. \)
- \( \psi([-\bar{u}^+, -\bar{u}^+[ = \nu_{\bar{u}}^{-1}[\bar{u}^+, -\bar{u}^+[. \)
- \( \psi([-\bar{u}^+, \bar{u}^+]) = [-\bar{u}^+, \bar{u}^+] - \lambda_{t(\bar{u})}\nu_{\bar{u}}^{-1}[\bar{u}^+, -\bar{u}^+[. \)
- \( \psi([\bar{u}, \bar{u}]) = \bar{u}^+, \bar{u}^+ - \lambda_{t(\bar{u})}\bar{u}^+, -\bar{u}^+[. \)
- \( \psi(q) = q \) for any other arrow \( q \) of \( Q_{\sigma} \).

Notice that \( \lambda_{t(\bar{u})}^* = -\lambda_{t(\bar{u})}, \lambda_{s(\bar{u})}^* = (1)^{m_{t(\bar{u})}}\lambda_{s(\bar{u})} \) and \( \lambda_{M} = \lambda_{\bar{M}} \) for any other \( M \). We use the same notation as in the latter case for \( \lambda \) and \( \bar{\lambda} \).

Identity (1) is immediate. We prove (2), (3) and (4). Using Theorem 4.3 we get successively:

(a) \( \psi([-\bar{u}^+, -\bar{u}^+[]) = \nu_{\bar{u}}^{-1}[\bar{u}^+, -\bar{u}^+[] - \lambda_{t(\bar{u})}\bar{u}^+, \bar{u}^+ + \lambda_{t(\bar{u})}\bar{v}e_{\bar{u}}+C_{\tau}). \) Indeed

\[
\psi([-\bar{u}^+, -\bar{u}^+[]) = ([\bar{u}^+, \bar{u}^+] - \lambda_{t(\bar{u})}\bar{u}^+, -\bar{u}^+[] \nu_{\bar{u}}^{-1}[\bar{u}^+, -\bar{u}^+[ 
= \nu_{\bar{u}}^{-1}[\bar{u}^+, -\bar{u}^+[ - \lambda_{t(\bar{u})}\bar{v}e_{\bar{u}}+C_{\tau})
\]
(b) \( \psi([-\vec{u}^+, \vec{u}^+]) = [-\vec{u}^+, \vec{u}^+] = \lambda_{t(\vec{u})} \tilde{\lambda}_{t(\vec{u}^+)} [\vec{u}^+, -\vec{u}^+] \). Indeed, using (a), \( C_J = 0 \) and \( \nu_M = 1 \) when \( m_{\vec{u}^+} > 1 \),
\[
\psi([-\vec{u}^+, \vec{u}^+]) = [-\vec{u}^+, -\vec{u}^+] = [-\vec{u}^+, \vec{u}^+] = \lambda_{t(\vec{u})} \tilde{\lambda}_{t(\vec{u}^+)} [\vec{u}^+, -\vec{u}^+]
\]
where we used at the end that \([-\vec{u}^+, -\vec{u}^+] [\vec{u}^+, -\vec{u}^+] \in JC_J = 0.

(c) \( \psi([\vec{u}^+, \vec{u}^+]) = [-\vec{u}^+, \vec{u}^+] \). Indeed
\[
[\vec{u}, \vec{u}^+] [-\vec{u}^+, -\vec{u}^+] \in J [-\vec{u}^+, \vec{u}^+] [-\vec{u}^+, \vec{u}^+] J \subseteq C_J = 0
\]
so \([\vec{u}, -\vec{u}^+] \cdot [\vec{u}, -\vec{u}^+] = [\vec{u}, -\vec{u}^+] [-\vec{u}^+, -\vec{u}^+] [-\vec{u}^+, -\vec{u}^+] = [-\vec{u}^+, -\vec{u}^+]
\]
so
\[
-\psi([\vec{u}^+, \vec{u}^+]) = \lambda_{t(\vec{u})} \tilde{\lambda}_{t(\vec{u}^+)} \lambda_{s(\vec{u})} \nu_M^{-1} \vec{u}^+] \vec{u}^+ \]
\[
= \lambda_{t(\vec{u})} \tilde{\lambda}_{t(\vec{u}^+)} \lambda_{s(\vec{u})} \nu_M^{-1} \vec{u}^+] \vec{u}^+
\]
\[
\psi([-\vec{u}^+, \vec{u}^+]) = \lambda_{t(\vec{u})} \tilde{\lambda}_{t(\vec{u}^+)} \lambda_{s(\vec{u})} \nu_M^{-1} \vec{u}^+] \vec{u}^+ \]
\[
\lambda_{t(\vec{u})} \tilde{\lambda}_{t(\vec{u}^+)} \lambda_{s(\vec{u})} \nu_M^{-1} \vec{u}^+] \vec{u}^+
\]
\[
\lambda_{t(\vec{u})} \tilde{\lambda}_{t(\vec{u}^+)} \lambda_{s(\vec{u})} \nu_M^{-1} \vec{u}^+] \vec{u}^+
\]
\[
\lambda_{t(\vec{u})} \tilde{\lambda}_{t(\vec{u}^+)} \lambda_{s(\vec{u})} \nu_M^{-1} \vec{u}^+] \vec{u}^+
\]
\[
\lambda_{t(\vec{u})} \tilde{\lambda}_{t(\vec{u}^+)} \lambda_{s(\vec{u})} \nu_M^{-1} \vec{u}^+] \vec{u}^+
\]
\[
\lambda_{t(\vec{u})} \tilde{\lambda}_{t(\vec{u}^+)} \lambda_{s(\vec{u})} \nu_M^{-1} \vec{u}^+] \vec{u}^+
\]
\[
\lambda_{t(\vec{u})} \tilde{\lambda}_{t(\vec{u}^+)} \lambda_{s(\vec{u})} \nu_M^{-1} \vec{u}^+] \vec{u}^+
\]
\[
\lambda_{t(\vec{u})} \tilde{\lambda}_{t(\vec{u}^+)} \lambda_{s(\vec{u})} \nu_M^{-1} \vec{u}^+] \vec{u}^+
\]
\[
\lambda_{t(\vec{u})} \tilde{\lambda}_{t(\vec{u}^+)} \lambda_{s(\vec{u})} \nu_M^{-1} \vec{u}^+] \vec{u}^+
\]
\[
\lambda_{t(\vec{u})} \tilde{\lambda}_{t(\vec{u}^+)} \lambda_{s(\vec{u})} \nu_M^{-1} \vec{u}^+] \vec{u}^+
\]
\[
\lambda_{t(\vec{u})} \tilde{\lambda}_{t(\vec{u}^+)} \lambda_{s(\vec{u})} \nu_M^{-1} \vec{u}^+] \vec{u}^+
\]
We conclude for (2), (3) and (4) as for (F1). We have to prove (5).

and we conclude with (b).

(l) \( \psi([\vec{u}^+, -\vec{u}^+][\vec{u}^+, \vec{v}]) = \psi([-\vec{u}^+, -\vec{v}]) \). Indeed,

\[
\psi([\vec{u}^+, \vec{v}][-\vec{u}, -\vec{u}^+]) = ([\vec{u}^+, \vec{v}] - \lambda_{t(\vec{a})}\tilde{\lambda}_{t(\vec{a}^+)})\vec{u}^+, -\vec{v})[-\vec{u}, -\vec{u}^+] \\
= [-\vec{u}^+, \vec{u}^+ + \lambda_{t(\vec{a})}\tilde{\lambda}_{t(\vec{a}^+)})\vec{u}^+, -\vec{v}] \\
= \nu_{\bar{\Sigma}}^{-1}(1 - \lambda_{t(\vec{a})}\tilde{\lambda}_{t(\vec{a}^+)})\vec{u}^+, -\vec{v}] \\
= -\vec{u}^+, -\vec{v}]
\]

and by (c) and \( \lambda_{st(\vec{a})} = (-1)^{m_s(t)}\lambda_{s(\vec{a})} \).

\[
\psi([-\vec{u}, -\vec{u}^+][\vec{a}^+, -\vec{a}^+]) = \psi([\vec{a}^+, \vec{a}^+]) \]

Analogous to (l).

We conclude for (2), (3) and (4) as for (F1). We have to prove (5).

- If \( t(\vec{x}) \neq s(\vec{x}) \), \( s(\vec{u}) \). We have

\[
\alpha([\vec{u}^+, \vec{a}] = \lambda_{t(\vec{a})}[\vec{u}^+, \vec{a}] + [\vec{u}^+, -\vec{u}^+][\vec{u}^+, \vec{x}].
\]

If \( (\Sigma, \mathcal{M}) \) is not a sphere with four punctures, it vanishes. Otherwise, we deduce:

\[
\alpha([\vec{u}^+, \vec{x}][\vec{x}, \vec{x}]^\ell = \lambda_{t(\vec{a})}\tilde{\lambda}_{t(\vec{a})}[\vec{u}^+, \vec{x}] + [\vec{u}^+, -\vec{u}^+][\vec{u}^+, \vec{x}]^\ell] \\
= \lambda_{t(\vec{a})}\tilde{\lambda}_{t(\vec{a})}\tilde{\lambda}_{t(\vec{a}^+)}\vec{u}^+, [\vec{x}, \vec{x}]^\ell
\]

as \( [\vec{u}^+, \vec{u}^+] = [-\vec{u}^+, -\vec{u}^+] \times \vec{x} = 0 \). If \( m_{t(\vec{a}^+)} > 1 \), this is strictly right divisible by \( [-\vec{u}^+, \vec{u}^+] \) \( \omega \) so it is longer. If \( m_{t(\vec{a}^+)} = 1 \) we have necessarily \( \ell = 0 \) and

\[
\alpha([\vec{u}^+, \vec{x}] = \lambda_{t(\vec{a})}\tilde{\lambda}_{t(\vec{a})}\tilde{\lambda}_{t(\vec{a}^+)}\tilde{\lambda}_{t(\vec{a}^+)}[-\vec{u}^+, \vec{x}] = (1 - \nu_{\bar{\Sigma}})[-\vec{u}^+, \vec{u}^+] \omega.
\]

- If \( t(\vec{x}) = s(\vec{x}) \) and \( \vec{x} \) is winding counter-clockwisely. If the triangle with sides \( \vec{u}^+, -\vec{u}^+ \) contains at least one puncture, then \( \alpha([\vec{u}^+, \vec{x}] = 0 \). Otherwise,

\[
\alpha([\vec{u}^+, \vec{x}][\vec{x}, \vec{x}]^\ell = \lambda_{t(\vec{a})}\tilde{\lambda}_{t(\vec{a})}[\vec{u}^+, \vec{x}] + [\vec{u}^+, -\vec{u}^+][\vec{u}^+, \vec{x}]^\ell] \\
= \lambda_{t(\vec{a})}\tilde{\lambda}_{t(\vec{a})}[\vec{u}^+, \vec{x}] + [\vec{u}^+, -\vec{u}^+][\vec{u}^+, \vec{x}]^\ell] \\
= \delta_{t,0}\lambda_{t(\vec{a})}\tilde{\lambda}_{t(\vec{a}^+)}[-\vec{u}^+, -\vec{x}]
\]

which is a strict multiple of \( [-\vec{u}^+, \vec{u}^+] \omega \) so it is longer (or 0).

- If \( t(\vec{x}) = s(\vec{x}) \) and \( \vec{x} \) is winding clockwise. According to the previous case for \( -\vec{x} \), the only possibility for \( \omega \) to be non-zero is when the
There exists a unique Lemma 10.38. \[ \psi \in \text{LAURENT DEMONET} \]

\[ \begin{aligned}
\alpha[\vec{u}^+, \vec{x}] &= \lambda_{t(\vec{u})} \lambda_s(\vec{u}^+) - \vec{u}^+, \vec{x}; [-\vec{x}, \vec{x}] = \lambda_{t(\vec{u})} \lambda_s(\vec{u}^+) \lambda_{t(\vec{u})} [-\vec{u}^+, \vec{x}]
\end{aligned} \]

which is strictly longer than \([-\vec{u}^+, \vec{u}^+]\) except if \(m_{t(\vec{u})} = 1\). In this case

\[ \alpha[\vec{u}^+, \vec{x}] = \lambda_{t(\vec{u})} \lambda_s(\vec{u}^+) \lambda_{t(\vec{u})} [-\vec{u}^+, \vec{x}] = (1 - \nu\lambda)[-\vec{u}^+, \vec{u}^+]\omega. \]

- If \(t(\vec{x}) = s(\vec{u})\). We have

\[ \alpha[\vec{u}^+, \vec{x}][\vec{x}, \vec{x}]^\ell = \lambda_{t(\vec{u})} \vec{u}^+, \vec{x}^+ \cdot [\vec{u}^+, -\vec{u}^+][\vec{u}^+, \vec{x}][\vec{x}, \vec{x}]^\ell \]

which admit a factor \([-\vec{x}, -\vec{u}^+]\) \([\vec{u}^+, \vec{x}] \in (C)\) so the only way that this does not vanish is \(\vec{x} = -\vec{u}^+\) and \(\ell = 0\). In this case,

\[ \alpha[\vec{u}^+, \vec{x}] = \lambda_{t(\vec{u})} \lambda_s(\vec{u}^+) \vec{x}^+ \epsilon_{\vec{u}, C} = \lambda_{t(\vec{u})} \lambda_s(\vec{u}^+) \epsilon_{\vec{u}, C}\]

which is strictly longer than \([-\vec{u}^+, \vec{u}^+]\) \(\omega\) except if \(m_{t(\vec{u})} = 1\). In this case,

\[ \alpha[\vec{u}^+, \vec{x}] = \lambda_{t(\vec{u})} \lambda_s(\vec{u}^+) \lambda_{t(\vec{u})} [-\vec{u}^+, \vec{x}] = (1 - \nu\lambda)[-\vec{u}^+, \vec{u}^+]\omega. \]

We generalize the notation of Lemma 10.37 by setting \(\alpha = 0\) and taking \(\psi\) as in Lemma 10.30 in cases (F2) or (F3). Thus, we always have

- \(\psi([-\vec{u}^+, \vec{u}^+]) = [-\vec{u}^+, \vec{u}^+] - \alpha\),
- \(\psi(C'_\tau) = C'_\tau\),
- \([-\vec{u}, (-\vec{u})^+][(-\vec{u})^+, (-\vec{u})^+] = [\vec{u}, \vec{u}^+] f_{\vec{u}, \vec{u}^+},
- \([\vec{u}^+, \vec{u}^+ [-\vec{u}, (-\vec{u})^+] = f_{\vec{u}^+, \vec{u}^+}(-\vec{u}^+)^+, (-\vec{u})^+,\)

and, in cases (F1), (F1'), (F3),

- \([\vec{u}, \vec{u}^+\vec{u}^+, \vec{u}^+] = [\vec{u}, \vec{u}^+] \alpha + [-\vec{u}, (-\vec{u})^+] f_{\vec{u}, \vec{u}^+},
- \([(-\vec{u})^+, \vec{u}^+ [-\vec{u}, \vec{u}^+] = f_{\vec{u}, \vec{u}^+} \vec{u}^+, \vec{u}^+\]

(in cases (F1) and (F1'), most of these equalities are trivial).

Then, we get the following maps in \(K^b(\text{proj } \Delta_\sigma)\):

- \(\pi_+ := \left( \begin{array}{cc}
\text{Id}_{\vec{u}^+, \Delta_\sigma} & 0 \\
0 & 0
\end{array} \right): P^\ast_\vec{u} \rightarrow e_{\vec{u}^+} \Delta_\sigma;
\)
- \(\pi_- := \left( \begin{array}{cc}
0 & -\text{Id}_{\vec{u}^+, \Delta_\sigma} \\
0 & 0
\end{array} \right): P^\ast_\vec{u} \rightarrow e_{(-\vec{u})^+} \Delta_\sigma;
\)
- \(\theta_+ := \left( \begin{array}{cc}
[-\vec{u}^+, \vec{u}^+] - \alpha & 0 \\
-f_{\vec{u}^+, \vec{u}^+} & 0
\end{array} \right): e_{\vec{u}^+, \Delta_\sigma} \rightarrow P^\ast_\vec{u}\) in cases (F1), (F1') or (F3);
- \(\theta_- := \left( \begin{array}{cc}
[-f_{\vec{u}^+, \vec{u}^+} & 0 \\
0 & 0
\end{array} \right): e_{(-\vec{u})^+} \Delta_\sigma \rightarrow P^\ast_\vec{u}\) in cases (F2) or (F3);
- \(\varepsilon := ([\vec{u}^+, -\vec{u}^+], [\vec{u}, -\vec{u}] + \vec{u}^+] \rightarrow P^\ast_\vec{u} \rightarrow P^\ast_\vec{u}\) in case (F1);
- \(\varepsilon := (0, [\vec{u}, -\vec{u}^+] \rightarrow P^\ast_\vec{u} \rightarrow P^\ast_\vec{u}\) in case (F1');
- \(\eta := -\left( \begin{array}{cc}
0 & 0 \\
0 & 0
\end{array} \right): P^\ast_\vec{u} \rightarrow P^\ast_\vec{u}\) in case (F2).

**Lemma 10.38.** There exists a unique \(\varphi^\circ : kQ^\ast \rightarrow \text{End}_{K^b(\text{proj } \Delta_\sigma)}(T)\) satisfying \(\varphi^\circ(x) = \psi(x)\) for \(x \in kQ_\tau\) and

- \(\varphi^\circ(e_{\vec{u}^+}) = \text{Id}_{P^\ast_\vec{u}}\);
- \(\varphi^\circ([-\vec{u}^+, \vec{u}^+]) = \pi_+\);
- \(\varphi^\circ([-(-\vec{u})^+, -\vec{u}^+]) = \pi_-\) in case (F3);
\[ \varphi^\circ ([\bar{u}^*, \bar{u}^*]) = \theta_+ \text{ in case (F1), (F1') or (F3)}; \]
\[ \varphi^\circ ([\bar{u}^*, -\bar{u}^*]) = \theta_- \text{ in cases (F2) or (F3)}; \]
\[ \varphi^\circ ([\bar{u}^*, -\bar{u}^*]) = \varepsilon \text{ in case (F1) or (F1')}; \]
\[ \varphi^\circ ([\bar{u}^*, -\bar{u}^*]) = \eta \text{ in case (F2)}. \]

Proof. Notice first that if \( \bar{u}^+ \) or \( -\bar{u}^* \) is a boundary component, some equality becomes trivial.

Arrows of \( Q_{\sigma^*} \) which are not in \( e_{\sigma}kQ_{\sigma^*}e_{\tau} \) are the one defined case by case. Thus, such a map, if it exists, is unique. For the well definition, it is enough to check each arrow of \( e_{\sigma}kQ_{\sigma^*}e_{\tau} \) which appear as a composition of arrows of \( Q_{\sigma^*} \). We consider all cases:

\[ ([\bar{u}^+, \bar{u}^+]) = ([\bar{u}^+, \bar{u}^+])(\bar{u}^*, \bar{u}^+) \text{ in cases (F1), (F1') or (F3)}; \]
\[ \varphi^\circ ([\bar{u}^+, \bar{u}^+])(\bar{u}^*, \bar{u}^+) = \pi_+, \theta_+ = \psi([\bar{u}^+, \bar{u}^+]) = \varphi^\circ ([\bar{u}^+, \bar{u}^+]); \]
\[ ([\bar{u}^+, -\bar{u}^+]) = ([\bar{u}^+, -\bar{u}^+])(\bar{u}^*, -\bar{u}^+) \text{ in case (F3)}: \]
\[ \varphi^\circ ([\bar{u}^+, -\bar{u}^+])(\bar{u}^*, -\bar{u}^+) = \pi_+, \theta_- = \psi([\bar{u}^+, -\bar{u}^+]) = \varphi^\circ ([\bar{u}^+, -\bar{u}^+]). \]

\[ \Box \]

Lemma 10.39. The following sequence is exact:

\[ e_{\bar{u}^+} \Delta_\sigma \oplus e_{(-\bar{u}^+)} \Delta_\sigma \xrightarrow{A} e_{\bar{u}^+} \Delta_\sigma \oplus e_{(-\bar{u}^+)} \Delta_\sigma \xrightarrow{\varphi^\circ} e_{\bar{u}} \Delta_\sigma \]

where

\[ A = \begin{pmatrix}
\begin{cases}
([\bar{u}^+, \bar{u}^+] - \alpha \ 0) & \text{in case (F1) or (F1')} \\
0 & 0
\end{cases}
& \begin{cases}
([\bar{u}^+, \bar{u}^+] - \alpha \ 0) & \text{in case (F2)} \\
0 & 0
\end{cases}
& \begin{cases}
([\bar{u}^+, \bar{u}^+] - \alpha \ 0) & \text{in case (F3)}
\end{cases}
\end{pmatrix}
\]

Proof. The composition vanishes by Lemmas 10.36 and 10.37. Let us prove the exactness:

(a) In case (F1) or (F1'). Let \( x \in e_{\bar{u}^+} \Delta_\sigma \) such that \([\bar{u}, \bar{u}^+]x = 0\) and write \( x = \sum_{b \in B} \mu b \). According to Lemma 10.37(5), for any \( \omega = [\bar{u}^+, \bar{x}] [\bar{x}, \bar{x}] \in B \), we have \( ([\bar{u}^+, \bar{u}^+] - \alpha) \omega = \kappa([-\bar{u}^+, \bar{u}^+] \omega + \omega') \) where \( \kappa \in \{1, \nu_B\} \) is invertible and \( \omega' \) is strictly longer than \([[-\bar{u}^+, \bar{u}^+] \omega] \) or 0. Thus, as the length of non-zero paths is bounded, an immediate induction permits to make the assumption, up to subtracting an element of \( \text{im} A \), that \( x = \sum_{b \in B'} \mu b \) where \( B' \) consists of the following elements of \( B \): \( e_{\bar{u}^+} \) and \( [\bar{u}^+, \bar{x}] [\bar{x}, \bar{x}] \) where \( s(\bar{x}) = s(\bar{u}^+) \), \( 0 \leq \ell < m_{\nu(\bar{u}^+)} \) and \( \ell \neq m_{\nu(\bar{u}^+)} - 1 \) if \( \bar{x} = \bar{u}^+ \). It is immediate that the left multiplication by \([\bar{u}, \bar{u}^+] \) is injective on \( B' \) so \( [\bar{u}, \bar{u}^+]x = 0 \) implies \( x = 0 \).

(b) In case (F2), notice that \( -\bar{u}^+ = \bar{u}^+ \). Let \((x, y) \in e_{\bar{u}^+} \Delta_\sigma \oplus e_{\bar{u}^+} \Delta_\sigma \). Modulo \( \text{im} A \), we can suppose that \( y \) is a linear combination of elements of \( B' \) where \( B' \) denotes the same subset of \( B \) as in (a). Then, reducing again modulo \( \text{im} A \), we can suppose that \( x \) is also a linear combination of elements of \( B' \). Then, if \((x, y) \) is in the kernel of \([\bar{u}, \bar{u}^+] \ [\bar{u}, -\bar{u}^+] \),
we get that \((x, y)\) is a linear combination of \(a := ([\bar{u}^+, -\bar{u}], 0)\) and \(b := ([\bar{u}^+, \bar{u}, -\bar{u}^+, -\bar{u}]).\) On the other hand, \(a = A([\bar{u}^+, \bar{u}], 0)\) and \(b = A(0, [-\bar{u}^+, -\bar{u}]).\)

(c) In case (F3). It is obtained by applying \(S_{\bar{u}} \otimes \Delta_\sigma\) to the standard projective bimodule resolution of \(\Delta_\sigma\) (see also Proposition 7.3). □

**Lemma 10.40.** The map \(\varphi^\circ : kQ_{\sigma^*} \to \text{End}_{K^b(\text{proj})}(\Delta_\sigma)(T)\) is surjective.

**Proof.** First of all, it is of course surjective onto \(e_\tau \text{End}_{K^b(\text{proj})}(\Delta_\sigma)(T)e_\tau \cong \Delta_\tau\) as \(\psi\) is surjective. If \(f \in \text{Hom}_{K^b(\text{proj})}(\Delta_\sigma)(P_u^*, e_\tau \Delta_\sigma),\) it is immediate that \(f\) factors through \(\pi_\pm,\) so \(f\) is in the image of \(\varphi^\circ (\pi_\pm = \pi_+ \eta\) in case (F2)).

If \(f \in \text{Hom}_{K^b(\text{proj})}(\Delta_\sigma)(e_\tau \Delta_\sigma, P_u^*),\) using the exact sequence of Lemma 10.39 and the fact that \(e_\tau \Delta_\sigma\) is projective, we get that \(f\) factors through \(\theta_\pm\) (we replace \(\theta_\pm\) by \(\eta \theta_\pm\) in case (F2)). So \(f\) is in the image of \(\varphi^\circ\) also in this case.

Finally, take \((f_2, f_1) \in \text{End}_{K^b(\text{proj})}(\Delta_\sigma)(P_u^*).\) This endomorphism induces via cokernel an endomorphism \(f_0\) of \(X_a.\) It is easy to see that \(f_0\) is a linear combination of the identity and morphisms induced by powers of \(\varepsilon\) (case (F1)) and \(\eta\) (case (F2)), but, up to an element of the image of \(\varphi^\circ\), we can suppose that \(f_0 = 0.\) Then, up to homotopy, we can suppose that \(f_1 = 0.\) So \(f_2\) factors through \(A\) of Lemma 10.39 and it permits to factor \((f_2, f_1)\) through \(\pi_\pm.\) Finally, \((f_2, f_1)\) is in the image of \(\varphi^\circ.\) □

**Lemma 10.41.** For any oriented edge \(\bar{v}\) of \(\sigma^*, \varphi^\circ(C_{\bar{v}}) = 0.\)

**Proof.** This is immediate if \(\bar{v} \neq \pm \bar{u}\) as in this case \(C_{\bar{v}} \in e_\tau Q_{\sigma^*} e_\tau\) and \(\varphi^\circ(C_{\bar{v}}) = \psi(C_{\bar{v}}) = 0.\) So we check \(C_{\bar{v}} = -C_{-\bar{u}}.\) In case (F3), we have

\[
\varphi^\circ(\lambda_{s(\bar{u})}([\bar{u}^+, \bar{u}^*]^{m_{s(\bar{u})}})) = \theta_\pm \psi([\bar{u}^+, -\bar{u}^+]) \pi_+
\]

\[
= \left(\begin{bmatrix} [\bar{u}^+, \bar{u}^+] & 0 \\ -f_{\bar{u}^*} & 0 \end{bmatrix}, 0 \right) \bar{u}^+, -\bar{u}^+[\begin{bmatrix} 1 & 0 \end{bmatrix}, 0]
\]

\[
= \left(\begin{bmatrix} \lambda_{s(\bar{u})}[-\bar{u}^+, -\bar{u}^+]^{m_{s(\bar{u})}} & 0 \\ -f_{\bar{u}^*} & 0 \end{bmatrix}, 0 \right) \bar{u}^+, -\bar{u}^+[\begin{bmatrix} 0 & 0 \end{bmatrix}, 0]
\]

and, in the same way,

\[
\varphi^\circ(\lambda_{s(-\bar{u})}([-\bar{u}^*, -\bar{u}^*]^{m_{s(-\bar{u})}})) = \left(\begin{bmatrix} [\bar{u}^+, \bar{u}^+] & 0 \\ -f_{\bar{u}^*} & 0 \end{bmatrix}, 0 \right) \bar{u}^+, -\bar{u}^+[\begin{bmatrix} 1 & 0 \end{bmatrix}, 0]
\]

Using the following homotopy:

\[
\begin{bmatrix} \bar{u}^+, \bar{u}^- \\ -[(-\bar{u})^+, -\bar{u}] \end{bmatrix} : e_u \Delta_\sigma \to e_{\bar{u}^+} \Delta_\sigma \oplus e_{(-\bar{u})^+} \Delta_\sigma,
\]

we get that the following endomorphism of \(P_u^*\) is homotopic to zero:

\[
\left(\begin{bmatrix} \lambda_{s(\bar{u})}([\bar{u}^+, \bar{u}^*]^{m_{s(\bar{u})}}) & [\bar{u}^+, \bar{u}^-] \\ -[(-\bar{u})^+, -\bar{u}] & \lambda_{s((-\bar{u})^+)}([-\bar{u}^+, (-\bar{u})^+]^{m_{s((-\bar{u})^+)}}] \end{bmatrix}, 0\right)
\]

which is equal to \(\varphi^\circ(C_{\bar{v}}),\) using the previous computation and Lemma 10.36.
In case (F1) or (F1'), we have

$$
\varphi^0(\lambda_{s(\bar{u}^*)}[^{\bar{u}^+},^{\bar{u}}]^{m_*(\bar{s}^*)}) = \theta_+ \psi([^{\bar{u}^+},^{-\bar{u}^+}]) \pi_+ \\
= (\left(-\bar{u}^+, \bar{u}^+\right) - \alpha) \psi([^{\bar{u}^+},^{-\bar{u}^+}]), 0) \\
= (\psi([-^{\bar{u}^+},^{\bar{u}^+}])(^{\bar{u}^+},^{-\bar{u}^+}), 0) \\
= (\psi(e_{\bar{u}^+}C_p^*), 0) = (\lambda_{s(\bar{u})}[^{\bar{u}^+},^{\bar{u}^+}]^{m_*(\bar{s})}, 0)
$$

(we use Lemma 10.37 at the last step). As we excluded the case where u is the special arc of a special monogon, we have

$$
\varphi^0(\lambda_{t(\bar{u}^*)}[-^{\bar{u}^*},^{-\bar{u}^*}]^{m_*(\bar{s}^*)}) = \lambda_{t(\bar{u}^*)}e^{m_*(\bar{s}^*)} = (0, -\lambda_{t(\bar{u})}[-^{\bar{u}},^{-\bar{u}}]^{m_*(\bar{s})}) \\
= (0, -\lambda_{S(\bar{u})}[^{\bar{u}},^{\bar{u}}]^{m_*(\bar{s})}).
$$

Using the homotopy $\bar{u}^+ \cdot \bar{u} : \epsilon_{-\bar{u}^+} \Delta_{\sigma} \rightarrow \epsilon_{\bar{u}^+} \Delta_{\sigma}$, we get that:

$$
\varphi^0(C_{\bar{u}^*}) = (\lambda_{S(\bar{u})}[^{\bar{u}^+},^{\bar{u}^+}]^{m_*(\bar{s})}, \lambda_{S(\bar{u})}[^{\bar{u}},^{\bar{u}}]^{m_*(\bar{s})})
$$

is homotopic to 0.

Finally, in case (F2), we have

$$
\varphi^0(\lambda_{s(\bar{u}^*)}[^{\bar{u}^+},^{\bar{u}}]^{m_*(\bar{s}^*)}) = \eta_\pi[^{\bar{u}^+},^{-\bar{u}^+}] \pi_+ \\
= \left(\begin{array}{cc}
\lambda_{S(\bar{u})}[^{\bar{u}^+},^{\bar{u}}]^{m_*(\bar{s})} & 0 \\
0 & 0
\end{array}\right)
$$

and

$$
\varphi^0(\lambda_{s(\bar{u}^*)}[-^{\bar{u}^*},^{-\bar{u}^*}]^{m_*(\bar{s}^*)}) = \theta_\pi^{-1}[^{\bar{u}^+},^{-\bar{u}^+}] \pi_+ \eta \\
= \left(\begin{array}{cc}
0 & -f_{s(\bar{u}^*)}[^{\bar{u}^+},^{-\bar{u}^+}] \\
0 & \lambda_{t(\bar{u}^*)}[^{\bar{u}^+},^{-\bar{u}^+}]^{m_*(\bar{s})}
\end{array}\right), 0
$$

thanks to Lemma 10.36. Using the homotopy

$$
\left[\begin{array}{c}
[^{\bar{u}^+},^{\bar{u}}] \\
[-^{\bar{u}^+},^{\bar{u}}]
\end{array}\right] : \epsilon_{\bar{u}^+} \Delta_{\sigma} \rightarrow \epsilon_{\bar{u}^+} \Delta_{\sigma} \oplus \epsilon_{[-\bar{u}^+]} \Delta_{\sigma},
$$

we get the following null-homotopic endomorphism of $P_n$:

$$
\left(\begin{array}{cc}
\lambda_{S(\bar{u})}[^{\bar{u}^+},^{\bar{u}}]^{m_*(\bar{s})} & [^{\bar{u}^+},^{\bar{u}}] \\
0 & -\lambda_{s(\bar{u})}[^{\bar{u}^+},^{\bar{u}}]^{m_*(\bar{s})}
\end{array}\right), 0
$$

which is $\varphi^0(C_{\bar{u}^*})$. \qed

**Lemma 10.42.** We have $\varphi^0(\mathcal{R}_{\bar{u}^*}^*) = \varphi^0(\mathcal{R}_{-\bar{u}^*}^*) = 0$.

**Proof.** We start by $\mathcal{R}_{-\bar{u}^*}^*$:

In cases (F2) or (F3), we have

$$
\varphi^0([-^{\bar{u}^+},^{\bar{u}}][^{\bar{u}^+},^{-\bar{u}^+}]) = \pi_+ \theta_- = f_{-\bar{u}^*} = \varphi^0(f_{-\bar{u}^*})
$$

and $\varphi^0(\mathcal{R}_{-\bar{u}^*}^*) = 0$.

Suppose that we are in case (F1). We have $\varphi^0([-^{\bar{u}^+},^{\bar{u}}][^{\bar{u}^+},^{-\bar{u}^+}]) = \pi_+ \varepsilon = ([^{\bar{u}^+},^{-\bar{u}^+}], 0)$ and $\varphi^0(f_{-\bar{u}^*}) = \varphi^0([-^{\bar{u}^+},^{\bar{u}}]) = [^{\bar{u}^+},^{-\bar{u}^+}]$ and therefore $\varphi^0(\mathcal{R}_{-\bar{u}^*}^*) = 0$ in this case. We used $\psi([-^{\bar{u}^+},^{\bar{u}^+}]) = [^{\bar{u}^+},^{-\bar{u}^+}]$, see in particular Case (i) of Proof of Lemma 10.37. In case (F1'), we have $\varphi^0([-^{\bar{u}^+},^{\bar{u}}][^{\bar{u}^+},^{-\bar{u}^+}]) = 0 = f_{-\bar{u}^*}$, so $\varphi^0(\mathcal{R}_{-\bar{u}^*}^*) = 0$. 

\[\square\]
Let us now consider \( R^*_{\bar{q}} \): in case (F3), this is similar as before. Suppose that we are in case (F1). We have
\[
\varphi^0([-\bar{u}^+, -\bar{u}^+][\bar{u}^+, \bar{u}^+]) = e \theta_+ = (\psi([-\bar{u}^+, -\bar{u}^+], 0) = (e_{\bar{u}} + C_\sigma, 0)
\]
and
\[
\varphi^0(f_{\bar{u}^+}) = \varphi^0([-\bar{u}^+, -\bar{u}^+]) = \theta_+[-\bar{u}^+, -\bar{u}^+] = (e_{\bar{u}} + C_\sigma, 0)
\]
so \( \varphi^0(R_{\bar{u}^+}) = 0 \). In case (F1'), \( \varphi^0([-\bar{u}^+, -\bar{u}^+][\bar{u}^+, \bar{u}^+]) = 0 = f_{\bar{u}^+}^* \) so the conclusion is immediate.

In case (F2), we have
\[
\varphi^0([\bar{u}^+, -\bar{u}^+][\bar{u}^+, \bar{u}^+]) = \eta^2 = (0, |\bar{u}, -\bar{u}|^2) = (0, f_{\bar{u}})
\]
and the only case where \( f_{\bar{u}} \neq 0 \) is the same that the case where \( f_{\bar{u}^+} \neq 0 \): Case d of figure [4.11] with \( m_M = 2 \) (as we excluded \( m_M = 1 \) in this case). In this case,
\[
\varphi^0(f_{\bar{u}^+}) = \varphi^0(\lambda Me_uC_\sigma) = -\lambda M\eta\theta_-\bar{u}^+[-\bar{u}^+, -\bar{u}^+\pi_+]
\]
which is homotopic to \((0, f_{\bar{u}}) = (0, \lambda Me_uC_\sigma)\) via the homotopy
\[
\left[\lambda M\bar{u}^+, \bar{u}[-\bar{\lambda}\bar{u}^+, -\bar{u}^+\right] : e\Delta_\sigma \to e_{\bar{u}}^*\Delta_\sigma + e_{\bar{u}^+}\Delta_\sigma.
\]
where \( \bar{\lambda} = \lambda_{\bar{u}} \) if the digon with sides \( \bar{u}^+ \) and \( \bar{u}^+ \) contains a unique puncture \( N \) and \( m_{\bar{s}(\bar{u})} = m_{\bar{s}(\bar{u}^+)} = m_N = 1 \), and \( \bar{\lambda} = 0 \) in any other case. \( \square \)

**Lemma 10.43.** Let \( x \in kQ_{\sigma^*}e_\tau \). If \( x \) vanishes in \( \Delta_{\sigma^*}^+ \), then \( \varphi^0(x) = 0 \).

**Proof.** First of all, if \( x \in kQ_\tau \), it is immediate as \( \varphi^0(x) = \psi(x) \).

Let \( x \in e_u kQ_\tau e_\tau \) such that \( x \) vanishes in \( \Delta_{\sigma^*}^+ \). We have \([-\bar{u}^+, \bar{u}^+]x = x^+ + y^+ \) and \([-(-\bar{u}^+), -\bar{u}^+]x = x^- + y^- \) with \( x^+, x^- \in kQ_\tau \) and \( y^+, y^- \in (R_{\bar{u}^+}^*, R_{\bar{u}^+}^*) \). It is immediate that the following map is injective:
\[
\varphi : \text{Hom}_{K^0(\text{proj})\Delta_{\sigma}}(\Delta_{\tau}, P_{\bar{u}^+}) \to \text{Hom}_{K^0(\text{proj})\Delta_{\sigma}}(\Delta_{\tau}, e_{\bar{u}}^*\Delta_\sigma + e_{(-\bar{u})^+}\Delta_\sigma)
\]
\[
f \mapsto (\pi_+f, \pi_-f)
\]
(notice that in case (F1), \( \pi_- = 0 \) and in case (F2), \( \pi_- = \pi_+\eta \)). Moreover,
\[
\varphi(\varphi^0(x)) = (\varphi^0([-\bar{u}^+, \bar{u}^+]x), \varphi^0([-(-\bar{u})^+, -\bar{u}^+]x))
\]
\[
= (\varphi^0(x^+) + \varphi^0(y^+), \varphi^0(x^-) + \varphi^0(y^-)) = (0, 0)
\]
where we used Lemma 10.42. So \( \varphi^0(x) = 0 \). \( \square \)

**Lemma 10.44.** The morphism \( \varphi^0 : kQ_{\sigma^*} \to \text{End}_{K^0(\text{proj})\Delta_{\sigma}}(T) \) induces a surjective morphism \( \varphi : \Delta_{\sigma^*}^+ \to \text{End}_{K^0(\text{proj})\Delta_{\sigma}}(T) \).

**Proof.** The surjectivity comes from Lemma 10.40. For the well-definition, we need to check that relations defining \( \Delta_{\sigma^*}^+ \) are mapped to 0. Using Lemmas 10.41, 10.42 and 10.43, we need to check the \( R^*_{\bar{u}} \)'s which are in \( \Delta_{\sigma^*}^+ e_{\bar{u}^+} \). In other terms, \( \bar{u} = -\bar{u}^+ \) always and \( \bar{u} = -(\bar{u})^+ \) in case (F3). By symmetry, it is enough to check \( R^*_{-\bar{u}^+} \).

We prove that \( \varphi^0(R^*_{-\bar{u}^+}) = 0 \) case by case:
(a) In case (F1),
\[ \varphi^o([\bar{u}^+, \bar{u}^+] [-\bar{u}^+, \bar{u}^+]) = 0, \]
which is homotopic, using \([\bar{u}^+, \bar{u}^+] : e_{\varphi} \Delta_{\sigma} \to e_{\bar{u}^+} \Delta_{\sigma},\) to \((-\alpha, -f_{-\bar{u}^+}).\)
Moreover,
\[ \varphi^o(f_{-\bar{u}^+}) = \varphi^o([-\bar{u}^+, -\bar{u}^+]) = -\lambda_{\iota([\bar{u}])} \epsilon_{\iota([\bar{u}])}^{-1} \]
\[ = -\lambda_{\iota([\bar{u}])} \delta_{\iota([\bar{u}]), 2} \bar{u}^+, -\bar{u}^+, f_{-\bar{u}^+} = (-\alpha, -f_{-\bar{u}^+}). \]

(b) In case (F1'),
\[ \varphi^o([-\bar{u}^+, \bar{u}^+] [-\bar{u}^+, \bar{u}^+]) = \psi([-\bar{u}^+, \bar{u}^+]) \pi_+ = (\psi([-\bar{u}^+, \bar{u}^+]), 0). \]

The homotopy \([\bar{u}^+, \bar{u}^+] : e_{\varphi} \Delta_{\sigma} \to e_{\bar{u}^+} \Delta_{\sigma}\) gives that \([-\bar{u}^+, \bar{u}^+], 0 = 0.\)
So, if \(m_{\iota([\bar{u}])} > 1, \varphi^o([-\bar{u}^+, \bar{u}^+] [-\bar{u}^+, \bar{u}^+]) \) is homotopic to 0. In this case, we also have \(\varphi^o(f_{-\bar{u}^+}) = 0.\) If \(m_{\iota([\bar{u}])} = 1, \varphi^o([-\bar{u}^+, \bar{u}^+] [-\bar{u}^+, \bar{u}^+])\)
is homotopic to \((\nu_{\iota([\bar{u}])} \lambda_{\iota([\bar{u}])}) \bar{u}^+, -\bar{u}^+, 0)\) (see Proof of Lemma 10.37). Moreover, we have
\[ \varphi^o(f_{-\bar{u}^+}) = \varphi^o(\lambda^o_{\iota([\bar{u}])} \bar{u}^+, \bar{u}^+]) = -\lambda_{\iota([\bar{u}])} \nu_{\iota([\bar{u}])} \nu_{\iota([\bar{u}])}^{-1} \bar{u}^+, -\bar{u}^+[\pi_+ \]
\[ = -\nu_{\iota([\bar{u}])} \lambda_{\iota([\bar{u}])} \bar{u}^+, -\bar{u}^+[0, 0). \]

(c) In cases (F2) or (F3),
\[ \varphi^o([-(-\bar{u}^+)] [-\bar{u}^+, \bar{u}^+]) = [-(-\bar{u}^+)] [-\bar{u}^+, \bar{u}^+]) \pi_+ \]
\[ = (\psi([-(-\bar{u}^+)] [-\bar{u}^+, \bar{u}^+]), 0). \]

The homotopy \([-(-\bar{u}^+)] : e_{\varphi} \Delta_{\sigma} \to e_{(-\bar{u}^+)} \Delta_{\sigma}\) gives
\[ ([\bar{u}^+, \bar{u}^+] [-\bar{u}^+, \bar{u}^+]) = 0, \quad \text{so,} \]
\[ \varphi^o([-(-\bar{u}^+)] [-\bar{u}^+, \bar{u}^+]) = ([0, -f_{-\bar{u}^+}], 0) = f_{-\bar{u}^+} \pi_+ \]
\[ = \varphi^o([-(-\bar{u}^+)], -(-\bar{u}^+), -\bar{u}^+[([0, -f_{-\bar{u}^+}], 0)]) = \varphi^o([-\bar{u}^+], -\bar{u}^+[\nu_{\iota([\bar{u}])} \lambda_{\iota([\bar{u}])}]). \square \]

The next lemma concludes Proof of Proposition 8.39.

**Lemma 10.45.** The morphism \(\varphi : \Delta^o_{\iota} \to \text{End}_{K^b(\text{proj} \Delta_{\sigma})}(T)\) is injective.

**Proof.** Let \(x \in \text{soc} \varphi.\) We have \(x = \lambda e_{\bar{u}^+} C_{\sigma^*}\) for some \(v \in \sigma^*\) and \(\lambda \in k,\)
thanks to Theorem 2.14. As \(\varphi\) coincide with \(\psi\) which is injective on \(\Delta^o_{\iota},\)
we get that \(\nu = u^+.\) Using Proof of Lemma 10.41, we get
\[ \varphi(e_{u^+} C_{\sigma^*}) = \begin{cases} (e_{\bar{u}^+} C_{\sigma}, 0) \quad & \text{in case (F1) or (F1')}; \\
(0, 0) \quad & \text{in case (F2)};
\end{cases} \]
and in every case, using the trace \(E^o\) of Definition 10.28 (see Lemma 10.33), we get that \(\varphi(\lambda e_{u^+} C_{\sigma^*}) = 0\) implies \(\lambda = 0\) (recall that a trace on \(\Delta_{\sigma}\) induces a trace on endomorphism rings of \(K^b(\text{proj} \Delta_{\sigma})\) by alternate sum of diagonal terms). \(\square\)
10.12. **Proof of Lemma 9.4.** We start by naming some elements of $\Delta_\sigma$. Denote

$$[\vec{u}, \vec{u}'] = [\vec{u}, -\vec{u}] - \lambda_M e_u$$

$$[-\vec{u}, \vec{v}] = \nu^{-1}_M([-\vec{u}, \vec{v}] - \lambda_M \vec{u}, \vec{v}]).$$

Then for any other arrow $[\vec{s}, \vec{t}]$ of $Q_\sigma$, denote $[\vec{s}, \vec{t}'] = [\vec{s}, \vec{t}]$ and extend this notation as before for any pair of oriented side starting at the same point. Finally, denote $\lambda_\sigma [\vec{s}, \vec{t}'] = \lambda_\sigma [\vec{s}, \vec{t}']$ for any $\vec{t} \neq \vec{s}$ in $\sigma$ such that $s(\vec{s}) = s(\vec{t})$.

Let us denote $\lambda_v = \lambda_P$ if $-\vec{v}$ encloses a special monogon with special puncture $P$ and $\lambda_v = 0$ in any other case. Denote also $\lambda_w = \lambda_Q$ if $\vec{w}$ encloses a special monogon with special puncture $Q$ and $\lambda_w = 0$ in any other case. Finally, denote $\lambda = \lambda_s(\vec{u})$ if $m_s(\vec{u}) = 1$ and $\lambda = 0$ else. Notice that $\lambda_M \lambda_v \lambda_w = 1 - \nu M$.

Then we prove the following identities in $\Delta_\sigma$:

(a) $[\vec{w}, \vec{v}] = \nu^{-1}_M ([\vec{w}, \vec{v}] - \lambda_M \lambda_w \lambda_v \vec{w}, \vec{v}]$. Indeed, using Proposition 4.12 and $C_\sigma J = 0$,

$$[\vec{w}, \vec{v}] = [\vec{w}, -\vec{w}][-\vec{w}, -\vec{v}] = \nu^{-1}_M [\vec{w}, -\vec{w}][\vec{w}, -\vec{v}] - \lambda_M \vec{w}, \vec{v}] $$

(b) $[\vec{v}, \vec{v}] \in (C_\sigma)$. We have

$$[\vec{v}, -\vec{v}] = e_v C_\sigma \in (C_\sigma)$$

and

$$[\vec{v}, -\vec{v}] = [\vec{v}, \vec{v}] - \vec{v}, \vec{w}] = -\vec{v}, \vec{w} \vec{v}, \vec{v} = -\vec{v}, [-\vec{v}, \vec{v}]$$

$$= \lambda_w - \vec{v}, [-\vec{v}, \vec{v}] = \lambda_w C_\sigma [-\vec{v}, \vec{v}] \in (C_\sigma)$$

So, using (a), we have, modulo $(C_\sigma)$,

$$[\vec{v}, \vec{v}] = [\vec{v}, \vec{v}][\vec{v}, -\vec{v}]$$

and, if $\nu_M \neq 1$, $[\vec{v}, \vec{v}] \in (C_\sigma)$ so the result follows.

(c) $[\vec{v}, \vec{v}'] = \nu^{-1}_M [\vec{v}, \vec{v}'] = \lambda_s(\vec{v}) \in (C_\sigma)$. This follows from (b) and $\lambda_s(\vec{v}) = \nu^{-1}_M \lambda_s(\vec{v})$.

(d) $[\vec{v}, \vec{v}'^2 - \lambda] \in (C_\sigma)$. It is an easy computation:

$$[\vec{v}, \vec{v}'^2] = \lambda^2_s(\vec{v}) e_v [\vec{v}, \vec{v}]^{2m_s(\vec{v}) - 2} = \begin{cases} \lambda^2_s(\vec{v}) e_v = \lambda_s(\vec{v}) [\vec{v}, \vec{v}] & \text{if } m_s(\vec{v}) = 1; \\ \lambda_s(\vec{v}) e_v C_\sigma & \text{if } m_s(\vec{v}) = 2; \\ 0 & \text{else.} \end{cases}$$
(e) \(|\vec{w}, \vec{v}'| = \nu_{M}^{-1}|\vec{w}, \vec{v}|\). Using (a), (c), (d) and \(C_{\sigma} J = 0\),

\(\begin{align*}
|\vec{u}, \vec{v}'| &= [\vec{w}, \vec{v}' \cdot |\vec{v}, \vec{v}'| = \nu_{M}^{2}(|\vec{w}, \vec{v}| - \lambda M \lambda w, \lambda v)]|\vec{w}, \vec{v}| = \nu_{M}^{2}|\vec{w}, \vec{v}|(\vec{w}, \vec{v} - \vec{u}, \vec{v})
= \nu_{M}^{2}(1 - \lambda M \lambda w, \bar{\lambda})(|\vec{w}, \vec{v}| \cdot |\vec{v}, \vec{v}'| = \nu_{M}^{-1}|\vec{w}, \vec{v}|, \vec{v}].
\end{align*}\)

(f) \([-\vec{u}, \vec{w}'][-\vec{w}, -\vec{v}] = |\vec{u}, \vec{v}'|\). Indeed, using (e),

\(\begin{align*}
[-\vec{u}, \vec{w}'][-\vec{u}, -\vec{v}] &= \nu_{M}^{2}([-\vec{u}, \vec{w}]((-\vec{w}, -\vec{v}) - \lambda M] |\vec{w}, \vec{v}|)
= \nu_{M}^{2}(|\vec{u}, \vec{v}| \cdot |\vec{w}, \vec{v}'| = |\vec{u}, \vec{v}'| = |\vec{u}, \vec{v}'|.
\end{align*}\)

(g) \([-\vec{w}, -\vec{v}'][-\vec{u}, \vec{v}'] = |\vec{w}, -\vec{v}'|\). This is similar as (f).

(h) \([-\vec{v}, -\vec{w}'] = [-\vec{v}, -\vec{w}] = |\vec{v}, \vec{v}'|[-\vec{u}, \vec{w}]'. By (c) and (d),

\(\begin{align*}
[-\vec{v}, -\vec{w}'] &= \nu_{M}^{-1}[[-\vec{v}, \vec{v}'] \cdot |\vec{v}, \vec{v}||\vec{v}, -\vec{w}']
= \nu_{M}^{-1}[\vec{v}, -\vec{w} - \lambda M] \vec{v}, -\vec{w}'] = \nu_{M}^{2}[\vec{v}, -\vec{w} - \lambda M] \vec{v}, -\vec{w}'
= \nu_{M}^{2}[\vec{v}, -\vec{w} - \lambda M \lambda w, \tilde{\lambda}] \vec{v}, -\vec{w}'] = \nu_{M}^{2}[\vec{v}, -\vec{w} - \lambda M \lambda w, \tilde{\lambda}] \vec{v}, -\vec{w}']
\end{align*}\)

and the second equality is trivial.

(i) \(|\vec{v}, \vec{v}'| = |\vec{x}, \vec{x}'|\) if \(s(\vec{z}) = s(\vec{v})\) and \(t(\vec{x})\) is enclosed by \(-\vec{v}\). By (c),

\(\begin{align*}
|\vec{v}, -\vec{v}'| &= \nu_{M}^{-1}[\vec{v}, \vec{v}'] \cdot |\vec{v}, \vec{v}'|\vec{v}, -\vec{v}'] = \nu_{M}^{-1}[\vec{v}, \vec{v}'] \cdot |\vec{v}, \vec{v}'|\vec{v}, -\vec{v}]
= \nu_{M}^{2}[\vec{v}, \vec{v}'] \cdot |\vec{v}, \vec{v}'|\vec{v}, -\vec{v}]
= \nu_{M}^{2}[\vec{v}, -\vec{w} - \lambda M \lambda w, \tilde{\lambda}] \vec{v}, -\vec{w}'] = \nu_{M}^{2}[\vec{v}, -\vec{w} - \lambda M \lambda w, \tilde{\lambda}] \vec{v}, -\vec{w}']
\end{align*}\)

and by (d), we have

\(\begin{align*}
|\vec{v}, -\vec{v}'] &= [\vec{v}, -\vec{v}'] \cdot |\vec{v}, \vec{v}'] = |\vec{v}, -\vec{v}'] \cdot |\vec{v}, \vec{v}'] \vec{v}, -\vec{v}'] = |\vec{v}, -\vec{v}'] \cdot |\vec{v}, \vec{v}'] = \tilde{\lambda} e_{v} C_{\sigma} [-\vec{v}, \vec{v}] = \tilde{\lambda} e_{v} C_{\sigma}
\end{align*}\)

so

\(\begin{align*}
|\vec{v}, -\vec{v}'| &= \nu_{M}^{2}[\vec{v}, -\vec{v}'] \cdot |\vec{v}, \vec{v}'] = |\vec{v}, -\vec{v}'] \cdot |\vec{v}, \vec{v}'] \vec{v}, -\vec{v}'] = \nu_{M}^{2}([\vec{v}, -\vec{v}] - \lambda M \lambda w, \lambda v)(2 - \lambda M \lambda w, \lambda v) e_{v} C_{\sigma})
= \nu_{M}^{2}([\vec{v}, -\vec{v}] - \lambda M \lambda w, \lambda v)(2 - \lambda M \lambda w, \lambda v) e_{v} C_{\sigma})
\end{align*}\)

Moreover, we have \(C_{\sigma} [-\vec{v}, \vec{x}] = [-\vec{v}, \vec{x}] \cdot [\vec{v}, \vec{v}][\vec{v}, \vec{x}] = \lambda w \vec{v}, \vec{x} \) so

\(\begin{align*}
|\vec{v}, \vec{x}'| &= \nu_{M}^{2}([\vec{v}, -\vec{v}] - \lambda M \lambda w, \lambda v)(2 - \lambda M \lambda w, \lambda v) e_{v} C_{\sigma})[-\vec{v}, \vec{x}]
= \nu_{M}^{2}(1 - \lambda M \lambda w, \lambda v)(2 - \lambda M \lambda w, \lambda v) e_{v} C_{\sigma})[-\vec{v}, \vec{x}]
= \nu_{M}^{2}(1 - \lambda M \lambda w, \lambda v)(2 - \lambda M \lambda w, \lambda v) e_{v} C_{\sigma})[-\vec{v}, \vec{x}]
= \nu_{M}^{2}(1 - \lambda M \lambda w, \lambda v)(2 - \lambda M \lambda w, \lambda v) e_{v} C_{\sigma})[-\vec{v}, \vec{x}]
\end{align*}\)

(j) \(|\vec{x}, -\vec{v}'| = |\vec{x}, -\vec{v} |\) if \(s(\vec{x}) = s(\vec{v})\) and \(t(\vec{x})\) is enclosed by \(-\vec{v}\). Same as (i).

(k) \([-\vec{v}, \vec{x}'| = [-\vec{v}, \vec{x}] |\vec{w}, \vec{x}] = \lambda w \vec{v}, \vec{x} \) if \(s(\vec{x}) = s(\vec{w})\) and \(t(\vec{x})\) is enclosed by \(\vec{w}\). Same as (i).

(l) \(|\vec{x}, \vec{w}'| = |\vec{x}, \vec{w}||\vec{x}, \vec{w} |\) if \(s(\vec{x}) = s(\vec{w})\) and \(t(\vec{x})\) is enclosed by \(\vec{w}\). Same as (i).
(m) $\lambda'_{s(\bar{x})}[\bar{x},\bar{x}]^{m_{s(\bar{x})}} = e_x C_\sigma$ for any $\bar{x} \in \sigma$ such that $s(\bar{x}) = s(\bar{u})$ and $\bar{x} \neq \pm \bar{u}$.

If $t(\bar{x})$ is enclosed by $-\bar{v}$ then using (i), we have

$$\lambda'_{s(\bar{x})}[\bar{x},\bar{x}]^{m_{s(\bar{x})}} = [\bar{x}, \bar{v}] \cdot [\bar{v}, \bar{x}'] = [\bar{x}, \bar{v}] \cdot [\bar{v}, \bar{x}] = e_x C_\sigma.$$

This is analogous if $t(\bar{x})$ is enclosed by $\bar{u}$. If $\bar{x} = \bar{v}$, take $\bar{y}$ such that $t(\bar{y})$ is enclosed by $-\bar{v}$. Thanks to (i), we have

$$\lambda'_{s(\bar{x})}[\bar{v},\bar{v}]^{m_{s(\bar{x})}} = [\bar{v}, \bar{y}]' \cdot [\bar{y}, \bar{v}] = [\bar{v}, \bar{y}] \cdot [\bar{v}, \bar{v}] = e_x C_\sigma.$$

This is analogous if $\bar{x} = -\bar{v}$, $\bar{x} = \bar{u}$ or $\bar{x} = -\bar{u}$.

(n) $[\bar{u},\bar{u}]^{m_{s(\bar{x})}} = [-\bar{u},-\bar{u}]^{m_{s(\bar{x})}}$. It follows from (e).

From these identities, we deduce that the following map is a morphism of algebras:

$$\varphi : \Delta^\mu \rightarrow \Delta^\lambda$$

$$e_x \mapsto e_x$$

for $x \in \sigma$;

$$[\bar{x}, \bar{y}] \mapsto [\bar{x}, \bar{y}]'$$

for $[\bar{x}, \bar{y}] \in Q_{\sigma,1}$.

Indeed, relations of the form $C_\bar{u}$ for $\Delta^\mu$ are mapped to 0 by $\varphi$ because of (m) and (n) if $s(\bar{x}) = s(\bar{u})$ or $t(\bar{x}) = s(\bar{v})$ and trivially otherwise. The relation coming from the special monogon enclosed by $\bar{u}$ is mapped to 0 easily. Relations coming from the triangle $-\bar{u}, \bar{v}, -\bar{w}$ are mapped to 0 thanks to (l), (g) and (h). Relations $R_{\bar{P},\bar{n}}$ coming from minimal polygons $\bar{P}$ completely enclosed by $-\bar{v}$ or by $\bar{w}$ are mapped to 0 thanks to (i), (j), (k) and (l) which permit to identify external paths winding around $s(\bar{v})$.

If we denote by $\psi : \Delta^\lambda \rightarrow \Delta^\mu$ the morphism obtained similarly, it is easy to prove that $\varphi$ and $\psi$ are inverse of each other by using (e), $\mu_M = -\lambda_M$ and $\nu'_M = -\nu^{-1}_M$ where $\nu'_M$ is computed for $\mu$.

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