Unifying quantization for inhomogeneous integrable models

Anjan Kundu *
Saha Institute of Nuclear Physics, Theory Group
1/AF Bidhan Nagar, Calcutta 700 064, India.
Phone: +91-33-23375346(-ext. 2365)
Fax: +91-33-23374637

June 9, 2018

Abstract

Integrable inhomogeneous versions of the models like NLS, Toda chain, Ablowitz-Ladik model etc., though well known at the classical level, have never been investigated for their possible quantum extensions. We propose a unifying scheme for constructing and solving such quantum integrable inhomogeneous models including a novel inhomogeneous sine-Gordon model, which avoid the difficulty related to the customary non-isospectral flow by introducing the inhomogeneities through some central elements of the underlying algebra.

PACS numbers
02.30.lk, 03.65.Fd, 02.20.Sv, 02.20.Uw, 03.70.+k

Key words
inhomogeneous integrable equations, quantum integrability, Lax operator, R-matrix, unifying Yang-Baxter algebra, Bethe ansatz

1. Introduction

Over more than past two decades active interest has been focused in the study of inhomogeneous integrable models. Various types of space and time inhomogeneities have been introduced successfully in the well known classical integrable models like the nonlinear Schrödinger equation (NLS) through linear, quadratic, cylindrical, radial etc functions [1, 2, 3, 4], the Toda chain (TC) with arbitrary inhomogeneity [5] and the Ablowitz-Ladik (AL) model with linear in space and arbitrary in time functions [6, 7, 8] etc., preserving their integrability. The associated Lax operators, soliton solutions, Painlevé integrability criteria etc. have been extensively studied for such systems [9, 10]. Such inhomogeneities might induce intriguing effects like locally varying interactions, space-time dependence of soliton velocities, trapping of solitons in periodic movements etc. In the simplest example of linearly inhomogeneous NLS the solitons move with a uniform acceleration [1].

However, it is rather surprising that, investigations for the above inhomogeneous models have been carried out only at the classical level and no systematic effort has been made toward their quantization, except perhaps our own preliminary study [11]. A possible reason for this might be,
that the inhomogeneities in such models are believed to be generated through non-isospectral flows with space-time dependent spectral parameter $\lambda = \lambda(x,t)$. This in general would lead to a dynamical classical $r$-matrix and consequently to a quantum $R$-matrix with space-time dependence, taking us beyond the known formulation of quantum integrable systems.

We overcome this difficulty for some of the nonisospectral flows and propose a unifying quantization scheme for the integrable inhomogeneous NLS, TC and AL models together with a novel inhomogeneous sine-Gordon model. The idea of such quantization is based on our earlier scheme [12] as well as on the observation that, certain non-isospectral problems may be looked from a bit different angle by considering spectral parameter $\lambda$ still to be a constant, while relegating the inhomogeneities to a set of central elements of an underlying algebra, which ensures quantum integrability. These inhomogeneity elements, which may be interpreted as external classical fields, commute with all other basic operators and appear only in the quantum Lax operators, but not in the quantum $R$-matrix. Such fields might have arbitrary space-time dependence with suitable restrictions on their boundary and asymptotic conditions, imposed by the integrability. A positive outcome of this approach is that, the classical and the quantum $R$ matrices for such integrable inhomogeneous models remain the same as in their homogeneous counterparts.

Note that the algebraic Bethe ansatz solution of quantum integrable systems depends on the vacuum Lax operator as well as on the $R$-matrix elements. Therefore, since in our inhomogeneous extensions, the $R$-matrices are kept the same, the related result are affected only by the inhomogeneity in their Lax operators, which signifies the presence of impurities, defects, density fluctuations in the media, or the influence of variable external fields, depending on the physical situations [16].

We concentrate here on inhomogeneous quantum NLS, and TC models as well as AL and SG models, in their different forms. Note that homogeneous versions of all these models are quantizable [13, 14], where the first two sets of models are linked to the rational, while the last two to the trigonometric $R$-matrix.

The arrangement of this paper is as follows. We briefly review in sect. 2 various known forms of the classical inhomogeneous equations and in sect. 3 the basic structures of the quantum integrable systems. We present in sect. 4 the construction of quantum integrable extensions of inhomogeneous models in a unifying way and in sect. 5 show their exact and systematic solution through the Bethe ansatz method. Sect. 6 gives the concluding remarks.

2. **Classical integrable inhomogeneous models**

We briefly list different well known forms of the classically integrable inhomogeneous NLS, TC and AL models, where the details can be found in the cited references.

2.1 **Inhomogeneous NLS equations**

1. *NLS equation with linear inhomogeneity* (XNLS)

   An integrable NLS equation (NLSE) with linear $x$-dependent inhomogeneity was proposed in [1, 2] with non-isospectral flow $\dot{\Lambda} = \alpha$, $\Lambda(t)$ being the spectral parameter. The inhomogeneous equation allows uniformly accelerated soliton solution as well as time dependent wave-number and frequency of the enveloping wave [1].

2. *Cylindrical NLSE* (CNLS)

   An integrable cylindrically symmetric NLSE having an explicit time-dependent coefficient was proposed at the classical level, with non-isospectral dependence of spectral parameter:
\( \Lambda(x,t) = \frac{\partial \psi}{\partial t} + \frac{\psi}{t} \) [4].

3. **NLSE with \( t \)-dependent coupling (TNLS)**

It is shown through Painlevé analysis that the time-dependence of the nonlinear coupling in the integrable NLSE can be of the form \( F(t) = \frac{1}{at+b} \). This model also has the same non-isospectral flow as the previous case.

4. **Radially symmetric NLSE (RNLS)** and

5. **\( x \)-dependent nonlocal NLSE (NLNLS)** are known to be classically integrable [2, 4].

In both these cases the non-isospectral condition was taken to be \( \dot{\Lambda} = a\Lambda^2 \).

6. **NLSE with more general inhomogeneity (FNLS)**

Integrable inhomogeneous classical NLSE with more general \( x \)-dependent coefficient was proposed as \( i\psi_t + \psi_{xx} + 2(|\psi|^2 - F(x))\psi = 0 \), where \( F(x) \) can be linear, quadratic or in more general form, depending on the type of non-isospectrality given by the space-time dependent spectral parameter satisfying \( \Lambda_t = 2(\Lambda^2)_x F_x \) [3].

### 2.2 Inhomogeneous Ablowitz-Ladik model

The AL model was discovered first as a discrete NLSE. The inhomogeneous AL model was studied at the classical level in a series of papers [6, 7, 8], which describes the system to be in an external time-dependent and linear in space potential and induces an intriguing effect of trapping the soliton and forcing it to a periodic movement.

### 2.3 Inhomogeneous Toda chain

Classically integrable inhomogeneous Toda chain with varied space-time dependence, in of the form [5] \( u_{tt}(n) = g_1(n)e^{\delta(n-1)-u(n)} - g_1(n+1)e^{u(n)-u(n+1)} + \hat{g}_2(n) + \) boundary terms, allows different choices for inhomogeneity functions \( g_a(n,t), a = 1,2 \).

### 3. Quantum integrable systems

As is well known [13], the quantum integrability of a system is guaranteed by the quantum Yang-Baxter equation (YBE)

\[
R(\lambda - \mu)L_j(\lambda) \otimes L_j(\mu) = (I \otimes L_j(\mu))(L_j(\lambda) \otimes I)R(\lambda - \mu),
\]

with the quantum Lax operator \( L_j \) of the discretized model defined at all sites \( j = 1,2,\ldots,N \) and a site-independent \( R \)-matrix, together with the ultralocality condition: \( (I \otimes L_k(\mu))(L_j(\lambda) \otimes I) = L_j(\lambda) \otimes L_k(\mu), \ j \neq k \). The corresponding classical model with the related \( r \)-matrix: \( R(\lambda) \rightarrow I + hr(\lambda) + O(h^2) \) would satisfy the classical YBE \( \{ L_j(\lambda) \otimes L_k(\mu) \} = \delta_{jk}[r(\lambda - \mu), L_j(\lambda) \otimes L_k(\mu)] \) for the discretized Lax operator \( L_j(\lambda) \), which goes to the corresponding field Lax operator \( U(x,\lambda) \) at the continuum limit \( \Delta \rightarrow 0: L_j(\lambda) \rightarrow I + i\Delta U(x,\lambda) \). In such an approach therefore only the space-Lax operator \( L_j(\lambda) \) or its continuum version \( U(x,\lambda) \) together with the canonical relation play the central role, while the time-Lax operator \( V(x,\lambda) \) becomes insignificant. Moreover in quantum problems, unlike classical equations of motion, the main emphasis is on solving the eigenvalue problem of the Hamiltonian: \( H|n > = E_n|n > \), together with that of other conserved operators \( C_j, j = 1,2,\ldots \). These conserved operators are generated by the transfer matrix \( \tau(\lambda) = tr(\prod_j L_j(\lambda)) \) through expansions \( \tau(\lambda) = \sum C_{\pm j}\lambda^{\pm j} \) or through similar expansions of \( \log \tau(\lambda) \).

The algebraic Bethe ansatz, an exact method for quantum integrable systems solves therefore a general eigenvalue problem: \( \tau(\lambda)|n > = \Lambda_n|n > \). As an essential condition for the quantum
integrability, which follows from the ultralocality and the quantum YBE (1) the conserved operators including the Hamiltonian must commute mutually [13].

It is important to note that, the YBE (1) not only ensures the quantum integrability but also plays a central role in the exact Bethe ansatz (BA) solution and for this it is essential that, the $R$-matrix appearing in it must not depend on the space-time coordinates. Consequently, since the $R(\lambda - \mu)$-matrix is a function of $\lambda - \mu$, the spectral parameter can not dependent on space-time variables, even in inhomogeneous models. Therefore, for quantization of our inhomogeneous models we introduce inhomogeneity through Lax operator $L_j$, by keeping the spectral parameter constant and demanding that the changed Lax operator must satisfy the YBE (1) with the same $R$-matrix as in the original homogeneous model. A similar argument holds for the classical YBE at the classical limit. Therefore, it should be clear that we can consider inhomogeneous extensions for only those models, which are quantum integrable at their homogeneous limit. The inhomogeneous versions of integrable NLSE, TC and AL models discussed above are known only at the classical level as a non-isospectral problem, whereas inhomogeneous SG model seems to not have been studied even at the classical level. Our main concern now would be to look into the possible quantum extensions of these inhomogeneous models, which apparently have never been undertaken.

4. Unifying quantization for inhomogeneous models

We formulate a quantization scheme for inhomogeneous integrable models of both rational and trigonometric type based on an unifying quantum algebra [12]. The inhomogeneities are introduced in these models through central elements of the underlying algebra in a systematic way.

4.1 Rational class of inhomogeneous models:

Recall that both the homogeneous NLS and TC models are quantum integrable and associated with the well known rational $R(\lambda)$-matrix [14] given by its nontrivial elements as

$$a(\lambda) \equiv R_{11}^{11} = R_{22}^{22} = \lambda + \eta, \quad b(\lambda) \equiv R_{12}^{12} = R_{21}^{21} = \lambda, \quad c \equiv R_{12}^{21} = R_{12}^{21} = \eta,$$  \hspace{1cm} (2)

For constructing the intended inhomogeneous extensions of these models we start from a rational discrete general Lax operator [12]

$$L_{rn}(\lambda) = \left( \begin{array}{cc} c_1^0(\lambda + s_n^3) + c_1^1 & s_n^- \\ s_n^+ & c_2^0(\lambda - s_n^3) - c_2^1 \end{array} \right),$$  \hspace{1cm} (3)

which was shown to yield all integrable models with $2 \times 2$ Lax operators, belonging to the rational class. Note that $L$-operator (3) depends on operators $s$ with generalized spin algebra

$$[s_n^+; s_m^-] = \delta_{nm}(2m^+s_n^3 + m^-), \quad [s_n^3; s_m^\pm] = \pm\delta_{nm}(s_n^\pm)$$  \hspace{1cm} (4)

and a set of commuting operators $c_\alpha^\alpha, \alpha = 0, 1, a = 1, 2$, forming central elements $m^+ = c_1^0c_2^0, \ m^- = c_1^1c_2^0 + c_1^0c_2^1$, which commute with all other operators. We note that, algebra (4) dictated by YBE (1) guarantees the quantum integrability of all models realized through the Lax operator (3) and since the algebraic relations are valid locally at all sites $n = 1, 2, \ldots$, the central elements can be taken in general to be time as well as lattice-site dependent variables.
generalized HPR. We would see in the sequel that these inhomogeneous elements would in fact be crucial in introducing inhomogeneity in integrable quantum systems we intend to construct.

I. Quantum inhomogeneous NLS model

It is easy to see that when inhomogeneous elements are absent i.e. when $c_1^0 = c_2^0 = c_1^1 = -c_2^1 = 1$, giving $m^+ = 1, m^- = 0$, the generalized spin algebra (4) reduces simply to $su(2)$ spin algebra and through bosonic realization of its generators by the Holstein-Primakov representation (HPR) we can recover from (3) the Lax operator $L_{n}^{\text{lnls}}$ of the exact lattice version of the standard quantum NLS, introduced by Korepin and Izergin [15]. Therefore for constructing the corresponding inhomogeneous quantum model our strategy would be to follow a similar path, but with nontrivial elements $c_{a,n}^\alpha(t)$, for which we first find a realization of (4) in the form of generalized HPR

$$s_n^3 = s_n - N_n \Delta, \quad s_n^+ = f(N_n)\psi_n \sqrt{\Delta}, \quad s_n^- = \psi_n^\dagger f(N_n) \sqrt{\Delta},$$

$$\text{with} \quad f^2(N_n) = m^- + m_n^+ (2s_n - N_n \Delta), \quad N_n = \psi_n^\dagger \psi_n. \quad (5)$$

where $[\psi_n, \psi_m^\dagger] = \frac{1}{\Delta} \delta_{nm}$, $s_n$ is the spin parameter taken also to be site-dependent and $\Delta$ is the lattice spacing. (5) clearly recovers the standard HPR at the homogeneous limit. Therefore in analogy with [15] we show that, the generalized HPR yields an exact lattice version of the inhomogeneous quantum NLS model, for which we demonstrate first that, the associated $L$-operator (3) has a consistent continuum limit, which recovers the Lax operator of the corresponding quantum field model at $\Delta \to 0$. Making a particular reduction of the central elements as

$$c_1^0 = c_2^0 \equiv g_n(t), \quad c_1^1 = -c_2^1 = f_n(t), \quad \text{giving} \quad m^+ = g_n^2, m^- = 0, \quad (6)$$

where $f_n, g_n$ are space-time dependent arbitrary functions, we find that, at the high spin limit $s_n \to \frac{1}{\Delta} g_n^{-1}$ the generalized HPR (5) reduces (3) to

$$L_{n}^{\text{lnls}(inh)}(\lambda) = \Delta \sigma^3 L_n(\lambda) = I + \Delta \left( \begin{array}{cc} \Lambda_n - g_n \Delta N_n & \psi_n f^{(0)}(N_n) \sqrt{g_n} \\ -\psi_n^\dagger f^{(0)}(N_n) \sqrt{g_n} & (\Lambda_n + g_n \Delta N_n) \end{array} \right) \quad (7)$$

where $\Lambda_n = g_n \lambda + f_n$ and $f^{(0)}(N_n) = (2 - g_n \Delta^2 N_n)^{\frac{1}{2}}$. One can verify that the related discrete model is a quantum integrable system, since the Lax operator (7) associated with it together with the rational $R$-matrix (2) exactly satisfy the quantum YBE (1). We observe further that, at the field limit $\Delta \to 0$, when $\psi_n(t) \to \psi(\psi, t)$, $f_n \to f(x, t)$, $g_n \to g(x, t)$, the lattice Lax operator (7) reduces to: $L_{n}^{\text{lnls}(inh)} = I - i \Delta U^{\text{lnls}(inh)}(x, \lambda) + O(\Delta^2)$, recovering the corresponding field Lax operator

$$U^{\text{lnls}(inh)}(x, \lambda) = \left( \begin{array}{cc} i \Lambda & i \sqrt{2g} \psi \\ -i \sqrt{2g} \psi^\dagger & -i \Lambda \end{array} \right) \quad (8)$$

where $\Lambda = g(x, t) \lambda + f(x, t)$. We may check again that, (8) with quantum field operators $[\psi(x, t), \psi^\dagger(y, t)] = \delta(x - y)$ and $f(x, t), g(x, t)$ acting as classical background fields satisfies YBE (1) with the same $R$-matrix (2), up to the first order in $O(\Delta)$, which however is sufficient for the integrability of a quantum field model. Therefore the quantum Lax operator (8) clearly represents an inhomogeneous generalization of the quantum integrable NLS field model.
For the homogeneous lattice NLS model it has been established [15] that, at certain values of
the spectral parameter \( \lambda = \nu_i \), where the quantum determinant \( q_{\det}(L(\lambda)) \equiv \frac{1}{2} tr(L_n(\lambda)\sigma^2 L_0(\lambda + 1)\sigma^2) \) vanishes the Lax operator is realizable as a projector as well as an inverse projector. This important
property was shown to be useful for proving the local character of the discrete Hamiltonian, i.e. its interaction spreading over to only few nearest neighbors. For our inhomogeneous lattice NLS model we could show that \( q_{\det}(L(\lambda)) \) for the associated Lax operator (7) vanishes at \( \nu_1 = -\frac{1}{\Delta}, \nu_2^{(n)} = (g_n \Delta - \nu_1) \) and the \( L_n \)-operator at these points can be represented indeed as projectors and inverse projectors as

\[
L_{ij}^{\text{lnls}(inh)}(\nu_1) = \alpha_j \beta_i \quad \text{and} \quad L_{ij}^{\text{lnls}(inh)}(\nu_2^{(n)}) = \tilde{\alpha}_j \tilde{\beta}_i
\]

with \( \beta_1 = \tilde{\alpha}_2 = \Delta \psi_n \sqrt{g_n}, \quad \alpha_1 = \tilde{\beta}_2 = -\Delta \psi_n^* \sqrt{g_n}, \quad \alpha_2 = \beta_2 = f^{(0)}(N_n), \quad \tilde{\alpha}_1 = \tilde{\beta}_1 = f^{(0)}(N_n - 1). \)

However we note that the point \( \nu_2^{(n)} \) becomes site-dependent for this inhomogeneous model
and therefore the projector-representation can not be achieved for all \( L_n \) at a single value of \( \lambda \), which would lead inevitably to nonlocal interactions for its Hamiltonian in both classical as well as quantum cases. By choosing some particular forms for \( g_n \), e.g., taking them same for all odd \( n = 2k + 1 \), we can partially regulate the localization of the Hamiltonian, though such inhomogeneous models in general would lead to nonlocal interactions.

A further technique of pairwise grouping of the Lax operators at different sites for achieving
locality for the Hamiltonian has been shown for the homogeneous quantum lattice NLS. Whether the same technique is applicable also to our corresponding inhomogeneous case needs more detailed analysis, which we leave for future study.

At the classical limit, when field operators become classical functions one can show that,
(7) satisfies classical YBE with the classical rational \( r(\lambda) = \frac{1}{2\lambda} (I + \vec{\sigma} \cdot \vec{\sigma}) \)-matrix and represents therefore the Lax operator of a new integrable discretization of inhomogeneous classical NLSE. Similarly, at the classical limit (8) must also satisfy the continuum version of the classical
YBE: \( \{ U(x, \lambda) \otimes U(y, \mu) \} = \delta(x - y)[r(\lambda - \mu), U(x, \lambda) \otimes I + I \otimes U(x, \mu)] \), with the same \( r \)-matrix and would correspond to the classically integrable inhomogeneous NLS field models. Notice that, if in (8) one considers \( \Lambda \) to be the spectral parameter instead of \( \lambda \) entering \( r(\lambda) \)-matrix, the problem would look like a customary non-isospectral flow with nontrivial \( \Lambda_t, \Lambda_x \).

In fact if we rename \( \sqrt{2g} \psi = Q \), (8) would coincide at its classical limit with the Lax operator
of known inhomogeneous NLS models and hence would recover for different choices of the functions \( f(x, t), g(x, t) \), the inhomogeneous NLS equations proposed earlier. For example, i) \( g = 1, f = \alpha t \) would give linear time-dependence of \( \Lambda = \lambda + \alpha t \) with \( \dot{\Lambda} = \alpha \) reproducing the XNLS, while ii) \( g = \frac{1}{\tau}, f = \frac{\alpha}{\tau} \) would give the inhomogeneous CNLS. We can get also the inhomogeneous TNLS by multiplying the field variable in CNLS by a function of \( t \), which however is not a canonical step. We may consider now a more general situation iii) \( g = X(x), f = T(t)X(x) \), which gives \( \Lambda = X(x)(\lambda + T(t)) = g(x)h(t) \), coinciding with the separable solution of the FNLS [3]. Note that, the related non-isospectral picture satisfying the equation \( \dot{T} = \dot{X}(T - \lambda)^2 + F'(x)/X \), is compatible with our isospectral relations. Therefore a nontrivial solution [3]: \( X = ax + b, \dot{T} = a(T - \lambda)^2 + a_0 \), yielding the NLS with quadratic \( x \)-dependence: \( F(x) = a_0(\frac{1}{2}ax^2 + bx) + c \), should also be reachable from our construction.
In cases of radial and nonlocal NLS however it appears that, due to nonlinear nature of the non-isospectrality and more involved noncanonical fields, the known classical models RNLS and NLNLS would not coincide with the classical limits of our quantum inhomogeneous models. It is important to note that, in classical Lax pair approach non-isospectrality with space-time dependence appears in both the space ($U$) and time ($V$) Lax-operators and moreover the allowed transformations may not respect the canonicity of the fields. In the quantum case on the other hand the inhomogeneity can appear only in the space part $U$ or $L_j$, since the analog of $V$-operator is absent here. At the same time, the canonical structure defined in quantum models is fixed by the associated $R$-matrix and should remain unchanged under all transformations.

Nevertheless, as we have seen, apart from a simple noncanonical transformation of the field: $\psi \rightarrow Q = \sqrt{2g}\psi$, with $g$ a space-time dependent function, (8) in the classical limit maps into the known Lax operator for the inhomogeneous NLS equations.

**II. Quantum inhomogeneous Toda chain**

We intend to construct the quantum extension of the inhomogeneous Toda chain starting again from the same Lax operator (3), but choosing now the inhomogeneity fields $c_{1j}^1, c_{0j}^0$ arbitrary, while $c_{1j}^2 = c_{0j}^2 = 0$. This clearly gives both $m_j^\pm = 0$ reducing algebra (4) simply to

$$[s_n^+, s_m^-] = 0, \quad [s_n^3, s_m^\pm] = \pm \delta_{nm} s_n^\pm$$

which can be realized through canonical quantum fields $[u_n, p_m] = i\delta_{nm}$ of the Toda chain as

$$s_j^3 = -ip_j, \quad s_j^\pm = (G_j)^\pm e^{\mp u_j},$$

with arbitrary function $G_j$. This reduces (3) to the quantum Lax operator of the inhomogeneous Toda chain: $L_j^{TC(\text{inh})}(\lambda) = L_j(\lambda)/c_{0j}^0$ as

$$L_j^{TC(\text{inh})}(\lambda) = \begin{pmatrix} \lambda - i(p_j - g_{2(j)}) & e^{u_j} F_{1(j)} \\ e^{-u_j} F_{2(j)} & 0 \end{pmatrix},$$

with the inhomogeneous arbitrary functions

$$g_{2(j)} = i\frac{c_{1j}^1}{c_{0j}^0}, \quad F_{1(j)} = 1/(c_{1j}^0 G_j), \quad F_{2(j)} = G_j/c_{1j}^0.$$  

These space-time dependent functions obviously enter the Hamiltonian of the inhomogeneous quantum Toda chain as

$$H = \sum_n P_n^2 - g_{1(n+1)} e^{n(n)-u(n+1)}, \quad P_n = p_n - g_{2(n)}, \quad g_{1(n)} = F_{1(n-1)} F_{2(n)}$$

and the conserved total momentum operator as $P = \sum_n P_n$, generated through expansion of the transfer matrix $\tau(\lambda)$ using the Lax operator (12) as $P = C_{N-1}$, $H = C_{N-2}$ etc.

At the classical limit one can derive the evolution equation from this inhomogeneous Hamiltonian (14), which considering the relation obtained here: $-ip_j = \dot{u}_j - g_{2(j)}$ would recover the known inhomogeneous classical Toda chain equations for different choices of functions $g_{a(n)}, a = 1, 2$ [5].
4.2 Trigonometric class of inhomogeneous models:

since both AL and SG models belong to the trigonometric class, we focus now on the idea behind the construction of quantum integrable models associated with the well known trigonometric $R_{\text{trig}}$-matrix [14, 19], which is a $q$-deformation of (2) and given through its nontrivial elements as

$$a(\lambda) \equiv R_{11}^{11} = R_{22}^{22} = \sin(\lambda + \eta), \quad b(\lambda) \equiv R_{12}^{12} = R_{21}^{21} = \sin \lambda, \quad c \equiv R_{21}^{12} = R_{12}^{21} = \sin \eta,$$  

(15)

Following our earlier work [12] we start from a trigonometric generalization of (3):

$$L_t(\xi) = \left( \begin{array}{cc} \xi c_+^+ e^{i \alpha S^3} + \xi^{-1} c_1^- e^{-i \alpha S^3} & 2 \sin \alpha S^- \\ 2 \sin \alpha S^+ & \xi c^-^- e^{-i \alpha S^3} + \xi^{-1} c_2^+ e^{i \alpha S^3} \end{array} \right), \quad \xi = e^{i \alpha \lambda},$$  

(16)

with the $q$-spin operators $S$ satisfying a generalized $q = e^{i \alpha}$-deformed algebra

$$q^S S^\pm = q^{\pm 1} S^\pm q^{S^3}, \quad [S^+, S^-] = -\hbar \left( c_1^+ c_2^- q^{2 S^3} - c_1^- c_2^+ q^{-2 S^3} \right), \quad \hbar = q - q^{-1},$$  

(17)

with $c_a^+ a = 1, 2$. One can see that algebra (17) is a generalization of the well known quantum algebra $sl_q(2)$, which is recovered easily at the homogeneous limit $c_a^+ = c_a^- = 1$. It is interesting to show that the $L$-operator (16) together with the $R_{\text{trig}}$-matrix (15) satisfy the quantum YBE (1), which yields algebra (17) as the condition for the quantum integrability. It is also intriguing to note that, at the undeformed limit $q \to 1$, the trigonometric $L$-operator (16) goes to its rational limit (3), while the quantum algebra (17) reduces to the generalized spin algebra (4). The construction of inhomogeneous models for this trigonometric class follows also the idea adopted above, i.e. we consider all central elements $c_a^\pm, a = 1, 2$, appearing in (16), (17) to be space-time dependent functions, which may vary arbitrarily at different lattice points $j$ and this leads naturally to integrable lattice models with inhomogeneity. However since the underlying algebra (17) does not change even with such inhomogeneous extension, the system remains linked to the same trigonometric $R$-matrix and hence retains its exact solvability through the Bethe ansatz.

III. Quantum inhomogeneous sine-Gordon model

Note that an interesting realization of the quantum algebra (17) in the canonical variables: $[u, p] = i$, may be given as

$$S_3^3 = u, \quad S^+ = e^{-ip} g(u), \quad S^- = g(u) e^{ip},$$  

(18)

where

$$g^2(u) = \frac{1}{2 \sin^2 \alpha} \left( \kappa + \sin \alpha (s - u)(M^+ \sin \alpha (u + s + 1) + M^- \cos \alpha (u + s + 1)) \right)$$  

(19)

with arbitrary constant $\kappa$, spin constant $s$ and central elements $M^\pm = \pm \sqrt{1}(c_1^+ c_2^- \pm c_1^- c_2^+)$. The Lax operator (16) in realization (18) of its underlying algebra (17) would represent a generalized discrete sine-Gordon model, which for the trivial choice of $c_a^\pm = \frac{m}{\sqrt{2}}$, resulting $M^+ = m^2, M^- = 0$, would recover the known exact lattice SG proposed by Korepin at al [15]. We introduce now the inhomogeneities by considering the central elements to be different
at different lattice points: \( c_1^\pm = m_j e^{i \theta_j} \), \( c_2^\pm = m_j e^{-i \theta_j} \), which yields \( M_j^+ = m_j^2 \), \( M_j^- = 0 \), with variable mass \( m_j \) and constructs a novel inhomogeneous integrable lattice SG model linked to the same trigonometric \( R \) matrix [14]. It is noteworthy that, in this inhomogeneous case we find the expression for the quantum determinant as \( q \text{det} L(\xi) = tr(L_n(\xi)\sigma^2 L_n^*(\xi q^{-1})\sigma^2) = 1 + S_j(\xi q^{-1} + \xi^{-2} q) \), where \( S_j = \left( \frac{1}{4} m_j \Delta \right)^2 \) with explicit site-dependent inhomogeneity, which generalizes its homogeneous case. This leads to the nonlocal interactions of the Hamiltonian in both the classical as well as quantum cases. Recall that in the corresponding homogeneous limit one gets nonlocality only in the quantum case [15]. Nevertheless, due to the property of the Lax-operator (16): \( \sigma^2 L^*(\xi^*)\sigma^2 = L(\xi) \) the Hamiltonian of the inhomogeneous lattice SG model as in the homogeneous case is also hermitian.

For finding the corresponding inhomogeneous field model we may scale \( p \) and \( m_j \) by lattice constant \( \Delta \) and take the limit \( \Delta \to 0 \). This would derive from (16) through \( \sigma^1 L_t \to I + \Delta L \), the Lax operator \( L \) of the SG field model with variable mass parameter \( m = m(x, t) \). The corresponding Hamiltonian may be expressed as

\[
\mathcal{H} = \int dx \left[ m(x, t)(u_t)^2 + (1/m(x, t))(u_x)^2 + 8(m_0 - m(x, t) \cos(2\alpha u)) \right],
\]

(20)
describing an integrable inhomogeneous sine-Gordon field model with variable mass and placed in an external gauge field \( \theta(x) \). Such variable mass sine-Gordon equations may arise in physical situations [17] and therefore the related exact results become important.

IV. Quantum inhomogeneous Ablowitz-Ladik model

Considering in (16) a further \( p \)-deformation of the spin operators: \( S^\pm \to \tilde{S}^\pm = p^{-3} S^\pm \), which corresponds to a \( p, q \) deformed trigonometric \( R \)-matrix, and supposing \( c_1^+ = c_2^- = 0 \) and tuning \( p = q \), we get from the second commutator of (17) the relation

\[
(q^{-1} \tilde{S}^+ \tilde{S}^- - q \tilde{S}^- \tilde{S}^+) = -\hbar c_1^+ c_2^-.
\]

(21)
If we choose \( c_1^+ c_2^- = 1 \) and denote \( \tilde{S}^+ = b, \tilde{S}^- = -b^\dagger \), (21) reduces to a typical q-oscillator algebra [14]: \( bb^\dagger - q^2 b b^\dagger = q^2 - 1 \), which curiously yields the known commutator in the AL model: \( [b, b^\dagger] = \hbar (1 + b b^\dagger) \), by setting \( \hbar = q^2 - 1 \). Note that it is easy to introduce inhomogeneity into the quantum AL model by considering time dependence of central elements as \( c_1^+ = \frac{1}{c_2^-} = e^{i\tau(t)} \), while keeping the required condition \( c_1^+ c_2^- = 1 \) unchanged. This changes the Lax operator of the quantum AL model to its inhomogeneous version

\[
L_j^{\text{AL}(inh)}(\lambda) = \begin{pmatrix} \Lambda & -b_j \tilde{b}_j \cr b_j & \tilde{b}_j^\dagger \end{pmatrix}, \quad \Lambda = \xi e^{i\tau(t)},
\]

(22)
which at the classical limit coincides obviously with the Lax operator of [6, 7]. Mutually commuting conserved operators \( C_{\pm j}, j = N, N - 1, \ldots, 1 \), of this inhomogeneous quantum AL model can be obtained from the expansion of the transfer matrix \( \tau(\lambda) \), constructed using the Lax operator (22), while the Hamiltonian is constructed from the operators \( C_{N-2} + C_{2-N} \) together with a term related to the quantum determinant of (22) in the form

\[
H = \sum_n \tilde{b}_{n-1}^\dagger \tilde{b}_n + \tilde{b}_n^\dagger \tilde{b}_{n-1} + \frac{2\hbar}{\log(1 + \hbar)} \log(1 + \tilde{b}_n^\dagger \tilde{b}_n),
\]

(23)
Here we have performed a canonical transformation \( b_n \rightarrow \tilde{b}_n = be^{-i2n\hat{\Gamma}(t)} \) to remove explicit time dependence from the Hamiltonian. Though Hamiltonian (23) looks like the homogeneous one [14], due to \( \dot{b}_n \rightarrow \dot{b}_n + i2n\hat{\Gamma}(t)\tilde{b}_n \), an extra inhomogeneous term would appear in the evolution equation and therefore using the q-oscillator type commutation relation of AL model one can derive from Hamiltonian (23) the known inhomogeneous discrete NLS equation at the classical limit.

5. Exact Bethe ansatz solution

Exact solutions of the eigenvalue problem for quantum integrable systems: \( \tau(\lambda)|m> = \Lambda_m(\lambda)|m> \), given through the algebraic Bethe ansatz (ABA) [13], can be formulated almost in a model-independent way. The expression for the eigenvalues may be given in the form

\[
\Lambda_m(\lambda) = \alpha(\lambda) \prod_{j=1}^{m} f(\lambda_j - \lambda) + \beta(\lambda) \prod_{j=1}^{m} f(\lambda - \lambda_j), \quad \text{where} \quad f(\lambda) = \frac{a(\lambda)}{b(\lambda)}. \tag{24}
\]

The Bethe \( m \)-particle eigenstates are defined as \( |m> = \prod_{j=1}^{m} B(\lambda_j)|0> \), with the pseudovacuum state satisfying \( C(\lambda)|0> = 0 \), where \( B(\lambda) = T_{12}(\lambda), \quad C(\lambda) = T_{21}(\lambda) \) are the off-diagonal elements of the monodromy matrix \( T(\lambda) = \prod_n L_n(\lambda) \). For the discrete (and also for the exact discretized versions of) quantum integrable models the Bethe momenta \( \lambda_j \) are not arbitrary but should be determined from the Bethe equations

\[
\frac{\alpha(\lambda_j)}{\beta(\lambda_j)} = \prod_{k \neq j} \frac{a(\lambda_j - \lambda_k)}{a(\lambda_k - \lambda_j)}. \tag{25}
\]

Note that in (24, 25) the coefficients \( \alpha(\lambda) \) and \( \beta(\lambda) \) are the pseudovacuum eigenvalues of the diagonal elements of \( T(\lambda) \), i.e. \( T_{11}(\lambda) |0> = \alpha(\lambda) |0> \), \( T_{22}(\lambda) |0> = \beta(\lambda) |0> \) and therefore are the only model-dependent elements, since they are related to the Lax operator \( L_j(\lambda) \) of a concrete model. On the other hand the factors containing the function \( f(\lambda - \lambda_j) \), which are the major contributors in the above expressions, are given by the ratio of the \( R \)-matrix elements: \( a(\lambda) = R_{11}^{11}(\lambda) \) and \( b(\lambda) = R_{12}^{12}(\lambda) \) and therefore depend not on individual models, but on the class to which the models belong. Consequently, they remain the same for all models of the same class e.g., rational, trigonometric etc.

Therefore, we can solve the eigenvalue problem of the quantum inhomogeneous models by using the same formulas (24, 25) and even taking the factors \( f(\lambda) \) same as their corresponding homogeneous counterparts, since in our construction we could keep the quantum \( R \)-matrix same for both these cases. We have to remember however that, since the Lax operators are changed with the inclusion of inhomogeneity parameters, the expressions for \( \alpha(\lambda), \beta(\lambda) \) would be more complicated. Therefore for the Bethe ansatz solution of the exact lattice version of our inhomogeneous quantum NLS we may use (24, 25), taking the high spin limit of \( \alpha(\lambda) = \prod_n (g_n(\lambda + s_n) + f_n), \beta(\lambda) = \prod_n (g_n(\lambda - s_n) + f_n) \), which follows from the related Lax operator (7).

The model independent part for the NLS model as well as for the TC model, linked to the rational \( R \)-matrix, should naturally be given by \( a(\lambda) = \lambda + \alpha, \quad b(\lambda) = \lambda \). The model-dependent part for the quantum inhomogeneous TC model on the other hand, as evident from its Lax

---

```markdown
Here we have performed a canonical transformation \( b_n \rightarrow \tilde{b}_n = be^{-i2n\hat{\Gamma}(t)} \) to remove explicit time dependence from the Hamiltonian. Though Hamiltonian (23) looks like the homogeneous one [14], due to \( \dot{b}_n \rightarrow \dot{b}_n + i2n\hat{\Gamma}(t)\tilde{b}_n \), an extra inhomogeneous term would appear in the evolution equation and therefore using the q-oscillator type commutation relation of AL model one can derive from Hamiltonian (23) the known inhomogeneous discrete NLS equation at the classical limit.

5. Exact Bethe ansatz solution

Exact solutions of the eigenvalue problem for quantum integrable systems: \( \tau(\lambda)|m> = \Lambda_m(\lambda)|m> \), given through the algebraic Bethe ansatz (ABA) [13], can be formulated almost in a model-independent way. The expression for the eigenvalues may be given in the form

\[
\Lambda_m(\lambda) = \alpha(\lambda) \prod_{j=1}^{m} f(\lambda_j - \lambda) + \beta(\lambda) \prod_{j=1}^{m} f(\lambda - \lambda_j), \quad \text{where} \quad f(\lambda) = \frac{a(\lambda)}{b(\lambda)}. \tag{24}
\]

The Bethe \( m \)-particle eigenstates are defined as \( |m> = \prod_{j=1}^{m} B(\lambda_j)|0> \), with the pseudovacuum state satisfying \( C(\lambda)|0> = 0 \), where \( B(\lambda) = T_{12}(\lambda), \quad C(\lambda) = T_{21}(\lambda) \) are the off-diagonal elements of the monodromy matrix \( T(\lambda) = \prod_n L_n(\lambda) \). For the discrete (and also for the exact discretized versions of) quantum integrable models the Bethe momenta \( \lambda_j \) are not arbitrary but should be determined from the Bethe equations

\[
\frac{\alpha(\lambda_j)}{\beta(\lambda_j)} = \prod_{k \neq j} \frac{a(\lambda_j - \lambda_k)}{a(\lambda_k - \lambda_j)}. \tag{25}
\]

Note that in (24, 25) the coefficients \( \alpha(\lambda) \) and \( \beta(\lambda) \) are the pseudovacuum eigenvalues of the diagonal elements of \( T(\lambda) \), i.e. \( T_{11}(\lambda) |0> = \alpha(\lambda) |0> \), \( T_{22}(\lambda) |0> = \beta(\lambda) |0> \) and therefore are the only model-dependent elements, since they are related to the Lax operator \( L_j(\lambda) \) of a concrete model. On the other hand the factors containing the function \( f(\lambda - \lambda_j) \), which are the major contributors in the above expressions, are given by the ratio of the \( R \)-matrix elements: \( a(\lambda) = R_{11}^{11}(\lambda) \) and \( b(\lambda) = R_{12}^{12}(\lambda) \) and therefore depend not on individual models, but on the class to which the models belong. Consequently, they remain the same for all models of the same class e.g., rational, trigonometric etc.

Therefore, we can solve the eigenvalue problem of the quantum inhomogeneous models by using the same formulas (24, 25) and even taking the factors \( f(\lambda) \) same as their corresponding homogeneous counterparts, since in our construction we could keep the quantum \( R \)-matrix same for both these cases. We have to remember however that, since the Lax operators are changed with the inclusion of inhomogeneity parameters, the expressions for \( \alpha(\lambda), \beta(\lambda) \) would be more complicated. Therefore for the Bethe ansatz solution of the exact lattice version of our inhomogeneous quantum NLS we may use (24, 25), taking the high spin limit of \( \alpha(\lambda) = \prod_n (g_n(\lambda + s_n) + f_n), \beta(\lambda) = \prod_n (g_n(\lambda - s_n) + f_n) \), which follows from the related Lax operator (7).

The model independent part for the NLS model as well as for the TC model, linked to the rational \( R \)-matrix, should naturally be given by \( a(\lambda) = \lambda + \alpha, \quad b(\lambda) = \lambda \). The model-dependent part for the quantum inhomogeneous TC model on the other hand, as evident from its Lax
```
operator (12), is to be taken as $\alpha(\lambda) = \prod_n (\lambda + g_{2(n)})$, $\beta(\lambda) = 0$. However the basic problem associated with the TC model, namely the non-availability of the pseudovacuum, remains the same here as in the homogeneous case. Therefore for the solution of the eigenvalue problem in the inhomogeneous quantum TC model one has to adopt also the functional Bethe ansatz method [18], in place of the above algebraic approach.

For inhomogeneous quantum lattice SG and AL models also we can use the above Bethe ansatz result and since both of these models are associated with the trigonometric $R_{\text{trig}}$-matrix (15), we have $a(\lambda) = \sin(\lambda + \alpha)$, $b(\lambda) = \sin(\lambda)$. For inhomogeneous SG one has to use the pseudovacuum by taking the product of two adjacent sites, as in its homogeneous case [19], though the relevant details would be different here, since the mass $m_j$ becomes site-dependent.

For the inhomogeneous AL model due to an additional $p = q$-deformation one should use $b_{\pm}(\lambda) = q^{\pm}b(\lambda)$ in the above formulas and extra factors of $q^{-1}$ and $q$ should appear in the first and the second term of (24), respectively. Consequently another factor of $q^2$ also appears in the r.h.s. of the Bethe equations (25). The rest of the terms needed for the solution of this inhomogeneous AL model can be derived from its quantum Lax operator (22) giving $\alpha(\lambda) = (\xi e^{iT(t)})^N$, $\beta(\lambda) = (\xi e^{iT(t)})^{-N}$.

6. Concluding remarks

We have proposed quantum integrable extensions of the inhomogeneous NLS model, Toda chain and the Ablowitz-Ladik model, different forms of which are well known only at the classical level. We have also proposed an inhomogeneous SG model with variable mass, both at the exact lattice and the field limit, which are novel models at classical as well as at the quantum level. We have constructed such inhomogeneous quantum models exploiting the Yang-Baxter equation and avoiding the non-isospectrality problem by using central elements of the underlying algebra for introducing the inhomogeneity in a unified way. We have also indicated how to get the exact eigenvalue solutions of such quantum integrable inhomogeneous models systematically through the Bethe ansatz. It has been found that the classical limits of the quantum inhomogeneous models constructed here are close to known inhomogeneous equations, though their conventional forms can be reached only after some noncanonical transformations, permitted only at the classical level. The method presented here should be applicable for constructing inhomogeneous extension of some other quantum integrable models like the Liouville model, relativistic Toda chain etc.

Acknowledgment

The author likes to thank Prof Orlando Ragnisco of Univ. Rome-tre for many fruitful discussions.

References

[1] H. H. Chen and C. S. Liu, Phys. Rev. Lett. 37 (1976) 693
[2] F. Calogero and A. Degasperis, Nuov. Cim. 22 (1978) 270
[3] R. Balakrishnan, Phys. Rev. A 32 (1985) 1144
[4] A. V. Mikhailov and A. I. Yaremchuk, JETP Lett. 36 (1982) 78
   K. Porsezian and M. Lakshmanan, J. Math. Phys. 32 (1991) 2923 ; R. Radha and M.
   Lakshmanan, Chaos, Solitons & Fractals, 4 (1994) 181

[5] O. Ragnisco and R. Levi, Univ di Roma1 preprint no. 762 (1990)

[6] M. Bruschi, D. Levi and O. Ragnisco, Nuov. Cim. A 53 (1979) 21

[7] R. Scharf and A. R. Bishop Phys. Rev. A 43 (1991) 6535

[8] V. Konotop, O. Chubykalo and L. Vazquez, Phys. Rev. E 48 (1993) 563

[9] G. Quispel and D. Levi, CRM Proc. & Lect. Notes, 25 (2000) 363

[10] K. Porsezian, Pramana 48 (1997) 143

[11] Anjan Kundu, Theor. Math. Phys., 118 (1999) 333; J. Nonlin. Math. Phys. 8 (2001) 1

[12] Anjan Kundu, Phys. Rev. Lett. 82 (1999) 3936; Anjan Kundu and B. Basumallick Mod.
   Phys. Lett. A 7 (1992) 61

[13] L. D. Faddeev, Sov. Sc. Rev. C1 (1980) 107

[14] P. Kulish and E. K. Sklyanin, Lect. Notes in Phys. (ed. J. Hietarinta et al, Springer,Berlin,
    1982) vol. 151 p. 61.

[15] A. G. Izergin and V. E. Korepin, Nucl. Phys. B 205 [FS 5] (1982) 401

[16] P. Schmitteckert and U. Eckern, Europhys Lett. 30 (1995) 543; H. P. Eckle, A. Punnose
    and R. Römer, Europhys Lett. 39 (1997) 293; P. A. Bares, cond-mat/9412011; D. Sen and
    S. Lal, cond-mat/9811330

[17] D. Sen and S. Lal, e-print cond-mat/9811330

[18] E.K. Sklyanin Lect. Notes Phys.(Springer) 226 (1985) 196

[19] L.D. Faddeev, E.K. Sklyanin and L.A. Takhtajan, Teor. Mat. Fiz. 40 (1979) 194