On super-rigid and uniformly super-rigid operators

Otmane Benchiheb · Fatimaezzahra Sadek · Mohamed Amouch

Abstract
An operator $T$ acting on a Banach space $X$ is said to be super-recurrent if for each open subset $U$ of $X$, there exist $\lambda \in \mathbb{K}$ and $n \in \mathbb{N}$ such that $\lambda T^n(U) \cap U \neq \emptyset$. In this paper, we introduce and study the notions of super-rigidity and uniform super-rigidity which are related to the notion of super-recurrence, we also investigate the basic properties of these two notions. In addition, we discuss the spectrum of these two operators. At the end, we study the super-recurrence, the super-rigidity and the uniform super-rigidity behaviors on finite-dimensional spaces.

Keywords Hypercyclicity · Supercyclicity · Transitivity · Recurrence · Super-recurrence

Mathematics Subject Classification Primary 47A16 · Secondary 37B20

1 Introduction and preliminaries

Throughout this paper, $X$ will denote a Banach space over the field $\mathbb{K}$ ($\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$). By an operator, we mean a linear and continuous map acting on $X$, and we denote by $B(X)$ the set of all operators acting on $X$.

A (discrete) linear dynamical system is a pair $(X, T)$ consisting of a Banach space $X$ and an operator $T$. The most important and studied notions in the linear dynamical system are those of hypercyclicity and supercyclicity.

We say that $T \in B(X)$ is hypercyclic if there exists a vector $x$ whose orbit under $T$;

$$\text{Orb}(x, T) := \{T^n x; n \in \mathbb{N}\},$$

is dense in $X$, in which case, the vector $x$ is called a hypercyclic vector for $T$. The set of all hypercyclic vectors for $T$ is denoted by $HC(T)$.

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In the context of separable Banach spaces, Birkhoff proved in [12] that the operator $T$ is hypercyclic if and only if it is topologically transitive; that is for each pair $(U, V)$ of nonempty open subsets of $X$ there exists $n \in \mathbb{N}$ such that

$$T^n(U) \cap V \neq \emptyset.$$ 

Hypercyclic operators have some noteworthy spectral properties. For instance, if $T$ is hypercyclic, then Kitai [31] proved that every component of the spectrum of $T$ must intersect the unit circle, while the adjoint Banach operator of a hypercyclic operator cannot have eigenvalue. This means that $\sigma_p(T^*) = \emptyset$, see [9, Proposition 1.7].

In 1974, Hilden and Wallen in [30] introduced the concept of supercyclicity. An operator $T$ acting on $X$ is said to be supercyclic if there exists $x \in X$ such that

$$\mathbb{K}\text{Orb}(x, T) := \{\lambda T^n x : \lambda \in \mathbb{K}, n \in \mathbb{N}\}$$

is dense in $X$. Such a vector $x$ is called a supercyclic vector for $T$. The set of all supercyclic vectors for $T$ is denoted by $\text{SC}(T)$.

As in the case of hypercyclicity, if $X$ is a separable Banach space, then $T$ is supercyclic if and only if for each pair $(U, V)$ of nonempty open subsets of $X$ there exist $\lambda \in \mathbb{K}$ and $n \in \mathbb{N}$ such that

$$\lambda T^n(U) \cap V \neq \emptyset,$$

see [9, Theorem 1.12].

For the spectral properties of a supercyclic operator, N. S. Feldman, V. G. Miller, and T. L. Miller proved that if $T$ is supercyclic, then there exists $R > 0$ such that the circle $\{z \in \mathbb{C} : |z| = R\}$, called a supercyclicity circle for $T$, intersects each component of the spectrum of $T$, see [9, Theorem 1.24] or [21]. Unlike the hypercyclicity case, the adjoint operator of a supercyclic operator $T$ can have an eigenvalue but not more then one. This means that either we have $\sigma_p(T^*) = \emptyset$ or there exists $\lambda$ such that $\sigma_p(T^*) = \{\lambda\}$.

For more information about hypercyclic and supercyclic operators and their properties, see the monographs [9] by Bayart and Matheron, [29] by Grosse-Erdmann and Peris and the references therein. Also, see the survey article [28] by Grosse-Erdmann and the recent book by Grivaux et al. [27].

Since 2004, many authors have been looking at several new concepts of hypercyclicity, which are stronger than hypercyclicity: frequent hypercyclicity [7, 8] by Bayart and Grivaux, $\mathcal{U}$-frequent hypercyclicity [33] by Shkarin, reiterative hypercyclicity [13] by Bonilla and Grosse-Erdman, and more generally $\mathcal{F}$-transitivity [10, 11] by Bes et al. In [1, 3–5] it was studied the dynamics of a set of operators instead of a single operator.

Another important notion in the linear dynamical system that has a long story is that of recurrence which introduced by Poincaré in [32]. Later, it have been studied by Gottschalk and Hedlund [26] and also by Furstenberg [22]. Recently, recurrent operators have been studied in [17, 18].

We say that $T \in B(X)$ is recurrent if for each open subset $U$ of $X$, there exists $n \in \mathbb{N}$ such that

$$T^n(U) \cap U \neq \emptyset.$$ 

A vector $x \in X$ is called a recurrent vector for $T$ if there exists an increasing sequence $(n_k) \subset \mathbb{N}$ such that

$$T^{n_k} x \longrightarrow x \quad \text{as} \quad k \longrightarrow \infty.$$
We denoted by \( \text{Rec}(T) \) the set of all recurrent vectors for \( T \).

In the last few years, recurrent operators take a lot of attentions by many authors.

Costakis et al. [17] have obtained that for any operator \( T \), if \( T \) is recurrent, then \( T^p \) and \( \lambda T \) are recurrent, for any \( p \geq 2 \) and any \( \lambda \in \mathbb{C} \) such that \( |\lambda| = 1 \); in fact

\[
\text{Rec}(T) = \text{Rec}(T^p) = \text{Rec}(\lambda T).
\]

Moreover, if the space \( X \) is complex, they showed that every component of the spectrum \( \sigma(T) \) meets the unit circle and the point spectrum \( \sigma_p(T^*) \) of its adjoint \( T^* \) is contained in the unit circle.

Galán et al. in [24] studied the product recurrence properties for weighted backward shifts on sequence spaces. In particular, they proved that if \( B \) is the backward shift on the Banach sequence spaces \( \ell^p(v) \) or \( c_0(v) \) such that

\[
\sup_{i \in \mathbb{N}} |w^{i+1}_i| \frac{v^{i+1}_i}{v^{i+1}_i} < \infty,
\]

then \( B \) admits a non-zero recurrent point \( x \), if and only if, \( B \) is transitive.

Grivaux et al. in [27] studied the uniform recurrence and his relation with topological weak mixing. In particular, they used the uniform recurrent notion to give a criterion for an operator \( T \) to be weakly topologically mixing.

Yin and Wei in [34] investigated the concept of recurrence of translation operators. In particular, they showed that the translation operator is hypercyclic, if and only if, it has a non-zero recurrent point.

Bonilla et al. in [14] studied the notion of Frequently recurrence operators.

Cardeccia and Muro, studied the notion of multiple recurrence and hypercyclicity in [16], see also the paper [15] for more interesting results specially in Weighted Shifts operators case.

A stronger notion of recurrence is that of rigidity. This notion has been introduced by Furstenberg and Weiss [23] in the ergodic theoretic setting. In the case of topological dynamical systems, the notions of rigidity and uniform rigidity have been introduced by Glasner and Maon [25]. These notions have also been studied in linear dynamics for example in [19, 20] by Eisner and Grivaux.

Recall that an operator \( T \in \mathcal{B}(X) \) is said to be rigid if there exists a strictly increasing sequence \( (n_k) \subset \mathbb{N} \) such that

\[
T^{n_k} x \longrightarrow x, \quad \text{as} \quad k \longrightarrow \infty, \quad \text{for all} \quad x \in X.
\]

This means that each vector of the space \( X \) is a recurrent vector for \( T \) with respect to the same sequence \( (n_k) \).

An operator \( T \) is said to be uniformly rigid if there exists a strictly increasing sequence \( (n_k) \subset \mathbb{N} \) such that

\[
\|T^{n_k} - I\| = \sup_{\|x\| \leq 1} \|T^{n_k} x - x\| \longrightarrow 0, \quad \text{as} \quad k \longrightarrow \infty.
\]

Clearly, we have that

\[
T \text{ is uniformly rigid } \Rightarrow T \text{ is rigid } \Rightarrow T \text{ is recurrent}.
\]

The converses of those implications does not hold in general, see [17].

Recently in [2], recurrent operators have been generalized to a large class of operators called super-recurrent operators. We say that \( T \in \mathcal{B}(X) \) is super-recurrent if for each open
subset $U$ of $X$, there exist $\lambda \in \mathbb{K}$ and $n \in \mathbb{N}$ such that

$$\lambda T^n(U) \cap U \neq \emptyset.$$  

A vector $x \in X$ is called a super-recurrent vector for $T$ if there exist an increasing sequence $(n_k) \subset \mathbb{N}$ and a sequence $(\lambda_k) \subset \mathbb{K}$ such that

$$\lambda_k T^{n_k}x \longrightarrow x \text{ as } k \longrightarrow \infty.$$ 

We denoted by $S\text{Rec}(T)$ the set of all recurrent vectors for $T$.

Taking into consideration the link between the recurrence and their deviations, we introduce in this paper the notions of super-rigidity and uniform super-rigidity which are related to the notion of super-recurrence and generalize the notions of rigidity and uniform rigidity.

In Sect. 2, we study the notion of super-rigidity. We give the relationship between super-rigid, rigid, super-recurrent, and recurrent operators. Also we prove that the super-rigidity is preserved under similarity and we give the relationship between the super-rigidity of an operator $T$ and its iterates by showing that $T$ is super-rigid if and only for each $p \geq 2$, the operator $T^p$ is super-rigid. At the end of the section, several spectral properties of super-rigid operators will be proven.

In Sect. 3, we introduce and study the notion of uniformly super-rigid operators. As in the super-rigidity’s case, we prove that the property of being uniform super-rigid is preserved under similarity and that $T$ is uniformly super-rigid if and only if $T^p$ is uniformly super rigid for all $p$. Moreover, we prove that the spectrum of a uniform-super-rigid operator has a specific property. This lead us to show that being invertible is a necessary condition of being uniformly super-rigid.

In Sect. 4, we characterize the super-recurrence in finite-dimensional spaces, and we show that in this case, the notions of super-recurrent, super-rigid and uniformly super-rigid are equivalent.

## 2 Super-rigid operators

In the following, we introduce the notion of super-rigid operators which generalizes the notion of rigidity and related to the notion of super-recurrence.

**Definition 2.1** An operator $T$ acting on $X$ is called super-rigid if there exist a strictly increasing sequence of positive integers $(n_k)_{k \in \mathbb{N}}$ and a sequence $(\lambda_k)_{k \in \mathbb{N}}$ of numbers such that

$$\lambda_k T^{n_k}x \longrightarrow x$$

for every $x \in X$.

**Example 2.2** For $1 < p < \infty$, let $X = \ell^p(\mathbb{N})$. Let $R$ be a strictly positive number and $(\lambda_k)$ a sequence of numbers such that $\lambda_k \in \{z \in \mathbb{C}: |z| = R\}$ for all $k$. Let $T$ be the operator defined on $\ell^p(\mathbb{N})$ by

$$T(x_1, x_2, \ldots) = (\lambda_1 x_1, \lambda_2 x_2, \ldots), \quad \text{for all } (x_k) \in \ell^p(\mathbb{N}).$$

Then $T$ is super-rigid. Indeed, this is since $R^{-1}T$ is recurrent, see [17, Theorem 5.4].

**Remark 2.3** Let $T$ be an operator acting on $X$. It is clear that if $T$ is rigid, then it is super-rigid with $\lambda_k = 1$ for all $k$, but the converse does not hold in general. Indeed, let $T$ be the operator defined in Example 2.2, then $T$ is super-rigid. However, $T$ is rigid if and only if $|\lambda_k| = 1$, for all $k$, see [17, Theorem 5.4].
Let $T$ be an operator acting on $X$. It is clear that $T$ is super-rigid $\Rightarrow T$ is super-recurrent.

However, the converse does not hold in general even if $T$ is supercyclic as we show in the next example.

**Example 2.4** Let $(e_n)_{n \in \mathbb{N}}$ be the canonical basis of $\ell^2(\mathbb{N})$ and $w = (\omega_n)_{n \in \mathbb{N}}$ be a bounded sequence of positive numbers. Let $B_w$ be the weighted backward shift operator, defined on $\ell^2(\mathbb{N})$ by

$$B_w(e_0) = 0 \quad \text{and} \quad B_w(e_n) = \omega_{n-1}e_n \quad \text{for all } n \geq 1.$$ 

$B_w$ is super-recurrent since it is supercyclic, see [9, Example 1.15]. However, $B_w$ cannot be super-rigid since $B_w(e_1) = 0$.

In the following proposition, we prove that the super-rigidity is preserved under similarity.

**Proposition 2.5** Let $T \in \mathcal{B}(X)$ and $S \in \mathcal{B}(Y)$. Assume that $T$ and $S$ are similar. Then $T$ is super-rigid on $X$ if and only if $S$ is super-rigid on $Y$.

**Proof** Since $T$ and $S$ are similar, then there exists a homeomorphism $\phi: X \rightarrow Y$ such that $S \circ \phi = \phi \circ T$. Let $y \in Y$, then there exists $x \in X$ such that $y = \phi x$. Since $T$ is super-rigid in $X$, there exist a strictly increasing sequence $(n_k)$ of positive integers and a sequence $(\lambda_k)$ of numbers such that $\lambda_k T^{n_k} x \rightarrow x$ as $k \rightarrow +\infty$. Since $\phi$ is continuous and $S \circ \phi = \phi \circ T$, it follows that $\lambda_k S^{n_k} y \rightarrow y$ as $k \rightarrow +\infty$, which means that $S$ is super-rigid. $\square$

Let $p$ be a nonzero fixed positive integer. In 1995, Ansari proved that $T$ is a hypercyclic (resp, supercyclic) operator on a separable Banach space if and only if $T^p$ is hypercyclic (resp, supercyclic), see [6]. Later, the same result was proven for recurrent operators, see [17, Proposition 2.3], and for super-recurrent operators, see [2, Theorem 3.11]. In the following theorem, we prove that this result remains true for super-rigid operators.

**Theorem 2.6** Let $T$ be an operator acting on $X$. Then $T$ is super-rigid if and only if $T^p$ is super-rigid for all $p \geq 2$.

**Proof** Assume that $T$ is super-rigid, then there exist a strictly increasing sequence $(n_k)$ of positive integers and a sequence $(\lambda_k)$ of numbers such that $\lambda_k T^{n_k} x \rightarrow x$ for every $x \in X$. Let $M := \sup_{k \in \mathbb{N}} \|\lambda_k T^{n_k}\|$. By the uniform boundedness principle we have $M < +\infty$. Let $x$ be a vector of $X$, then we have

$$\|\lambda_k^p T^{p n_k} x - x\| = \|\lambda_k^p T^{p n_k} x - \lambda_k^{p-1} T^{(p-1)n_k} x + \lambda_k^{p-1} T^{(p-1)n_k} x - \cdots + \lambda_k T^{n_k} x - x\|$$

$$\leq \|\lambda_k^p T^{p n_k} x - \lambda_k^{p-1} T^{(p-1)n_k} x\| + \cdots + \|\lambda_k T^{n_k} x - x\|$$

$$\leq \|\lambda_k^{p-1} T^{(p-1)n_k}\| \|\lambda_k T^{n_k} x - x\| + \cdots + \|\lambda_k T^{n_k} x - x\|$$

$$\leq \left( \sum_{i=0}^{p-1} M^i \right) \|\lambda_k^{n_k} x - x\|.$$ 

Since $T$ is super-rigid with respect to $(n_k)$ and $(\lambda_k)$, it follows that $\lambda_k^p T^{p n_k} x \rightarrow x$, which means that $T^p$ is super-rigid with respect to $(n_k)$ and $(\lambda_k^p)$.

$\square$
In the following, we will give a characterization of the spectrum of super-rigid operator. We begin with the following lemma.

**Lemma 2.7** Let $T$ be an operator acting on $X$ and $\lambda$ a nonzero number. Then, $T$ is super-rigid if and only if $\lambda T$ is super-rigid.

**Proof** Assume that $T$ is super-rigid, then there exists a strictly increasing sequence $(n_k) \subset \mathbb{N}$ and $(\lambda_k) \subset \mathbb{K}$ such that $\lambda_k T^{n_k} x \to x$ for every $x \in X$. Let $\mu_k = \lambda^{-n_k} \lambda_k$. Then

$$
\mu_k (\lambda T)^{n_k} x = \lambda^{-n_k} \lambda_k T^{n_k} x = \lambda_k T^{n_k} x \to x,
$$

for every $x \in X$. This implies that $\lambda T$ is super-rigid. $\Box$

The next proposition shows that the eigenvalues of a super-rigid operator has the same modulus.

**Proposition 2.8** Let $T$ be an operator acting on $X$. If $T$ is super-rigid, then

$$
\sigma_p(T) \subset \{ z \in \mathbb{C} : |z| = R \},
$$

for some strictly positive real number $R$.

**Proof** Assume that there exist $\lambda_1$ and $\lambda_2$ in the point spectrum of $T$ such that $|\lambda_1| < |\lambda_2|$. Let $m$ be a strictly positive real number such that $|\lambda_1| < m < |\lambda_2|$. Since $\lambda_1, \lambda_2 \in \sigma_p(T)$, it follows that there exist $x, y \in X \setminus \{0\}$ such that $Tx = \lambda_1 x$ and $Ty = \lambda_2 y$. By Lemma 2.7, the operator $\frac{\lambda}{m} T$ is super-rigid. Hence, there exist a sequence $(\mu_k) \subset \mathbb{K}$ and a sequence $(n_k) \subset \mathbb{N}$ such that

$$
\mu_k \left( \frac{\lambda_1}{m} \right)^{n_k} x \to x \quad (1)
$$

and

$$
\mu_k \left( \frac{\lambda_2}{m} \right)^{n_k} y \to y. \quad (2)
$$

By (1), we have $|\mu_k| \to +\infty$, and by (2), we have $|\mu_k| \to 0$, which is a contradiction. $\Box$

**Remark 2.9** Proposition 2.8 is not true in general for super-recurrent operators which are not super-rigid. Indeed, let $(e_n)_{n \in \mathbb{N}}$ be the canonical basis of $\ell^2(\mathbb{N})$ and $w = (\omega_n)_{n \in \mathbb{N}}$ be a bounded sequence of positive numbers. Suppose that $B_w$ is defined as in Example 2.4, then $B_w$ is super-recurrent. However, if $\omega_n = 1$, for all $n \geq 1$, and $B_w = B$ is the unweighted backward shift operator, then

$$
\sigma_p(B) = \mathbb{D},
$$

where, $\mathbb{D}$ is the unit open disk.

Let $T$ be a super-rigid operator acting on a Banach space $X$. Since any super-rigid operator is super-recurrent, it follows by [2, Theorem 4.2] that if $T$ is super-rigid, then there exists $R_1 > 0$ such that $\sigma_p(T^*) \subset \{ z \in \mathbb{C} : |z| = R_1 \}$. Moreover, by Proposition 2.8, there exists $R_2 > 0$ such that $\sigma_p(T) \subset \{ z \in \mathbb{C} : |z| = R_2 \}$. The next proposition shows that we have $R_1 = R_2$. This means that the eigenvalues of $T$ and the eigenvalues of its Banach adjoint are of the same modulus.

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Proposition 2.10 Let $T$ be an operator acting on $X$. If $T$ is super-rigid, then the eigenvalues of $T$ and the eigenvalues of its Banach adjoint are of the same modulus. This means that there exists some $R > 0$ such that

$$\sigma_p(T) \cup \sigma_p(T^*) \subset \{ z \in \mathbb{C} : |z| = R \}.$$  

Proof. Assume that there exist $\lambda_1 \in \sigma_p(T)$ and $\lambda_2 \in \sigma_p(T^*)$ such that $|\lambda_1| < m < |\lambda_2|$, where $m$ is a strictly positive real number. Since $\lambda_1 \in \sigma_p(T)$ and $\lambda_2 \in \sigma_p(T^*)$, it follows that there exist $x \in X \setminus \{0\}$ and $x^* \in X^* \setminus \{0\}$ such that $Tx = \lambda_1 x$ and $T^* x^* = \lambda_2 x^*$. By lemma 2.7, the operator $\frac{1}{m} T$ is super-rigid. Hence, there exist a sequence $(\mu_k) \subset \mathbb{K}$ and a sequence $(n_k) \subset \mathbb{N}$ such that $\mu_k \left( \frac{1}{m} T \right)^{n_k} y \to y$, for all $y \in X$. In particular, for $y = x$, we have

$$\mu_k \left( \frac{1}{m} T \right)^{n_k} x = \mu_k \left( \frac{\lambda_1}{m} \right)^{n_k} x \to x. \quad (3)$$

Since $x^*$ is an nonzero linear form on $X$, it follows that there exists $z \in X$ such that $x^*(z) \neq 0$. Since $\frac{1}{m} T$ is super-rigid and $x^*$ is continuous, we have

$$\mu_k \left( \frac{\lambda_2}{m} \right)^{n_k} x^*(z) \to x^*(z). \quad (4)$$

By (3), we have $|\mu_k| \to +\infty$ and by (4), we have $|\mu_k| \to 0$, which is a contradiction. \qed

3 Uniformly super-rigid operators

In the following, we introduce the notion of uniform super-rigidity which generalizes the notion of uniform rigidity.

Definition 3.1 An operator $T$ acting on $X$ is called uniformly super-rigid if there exist a strictly increasing sequence of positive integers $(n_k)_{k \in \mathbb{N}}$ and a sequence $(\lambda_k)_{k \in \mathbb{N}}$ of numbers such that

$$\|\lambda_k T^{n_k} - I\| = \sup_{\|x\| \leq 1} \|\lambda_k T^{n_k} x - x\| \to 0.$$  

Example 3.2 Let $T$ be an operator defined on $\mathbb{C}^n$ by: $T(x_1, \ldots, x_n) = (\lambda_1 x_1, \ldots, \lambda_n x_n)$, where $\lambda_i \in \mathbb{K}$ and $|\lambda_i| = R > 0$, for $1 \leq i \leq n$. Let $(x_1, \ldots, x_n) \neq (0, \ldots, 0)$ and $m \geq 0$, we have

$$R^{-m} T^m (x_1, \ldots, x_n) - (x_1, \ldots, x_n) = ((R^{-m} \lambda_1^m - 1) x_1, \ldots, (R^{-m} \lambda_n^m - 1) x_n).$$

Since $|\lambda_i| = R > 0$, for $1 \leq i \leq n$, it follows that there exists a strictly increasing sequence $(m_k)$ of positive integers such that

$$\|R^{m_k} T^{m_k} (x_1, \ldots, x_n) - (x_1, \ldots, x_n)\| \to 0, \quad \text{as } k \to \infty.$$  

This means that $T$ is a uniformly super-rigid operator.

Remark 3.3 Let $T$ be an operator acting on $X$ It is clear that if $T$ is uniform super-rigid, then it is uniform super-rigid with $\lambda_k = 1$ for all $k$. However, the converse does not hold in general. Indeed, let $T$ be the operator defined as in the Example 3.2, then $T$ is a uniformly super-rigid whenever $|\lambda_i| = R > 0$, for $1 \leq i \leq n$. But the operator $T$ is uniform rigid if and only if $|\lambda_i| = 1$, for $1 \leq i \leq n$, see [17, Section 4].
Lemma 3.4 Let $T$ be an operator acting on $X$. If $T$ is uniformly super-rigid, then $\lambda T$ is uniformly super-rigid for all nonzero number $\lambda$.

Proof If $T$ is uniformly super-rigid with respect to a sequence $(\mu_k)$ and a sequence $(n_k)$, then it is easy to show that $\lambda T$ is uniformly super-rigid with respect to $(\mu_k \lambda^{-n_k})$ and $(n_k)$. □

Remark 3.5 It is clear that each uniform super-rigid operator is super-rigid. However, the converse does not hold in general. Indeed, let $(\theta_k)_{k \in \mathbb{N}}$ be a sequence of real numbers such that
\[
\liminf_{n \to +\infty} \left( \sup_{k \in \mathbb{N}} |e^{2\pi i \theta_k} - 1| \right) \not\to 0.
\]

Let $R > 0$, and $\lambda_k = R e^{2\pi i \theta_k}$, for all $k \in \mathbb{N}$. Let $T$ be the operator defined as in Example 2.2. Then $T$ is super-rigid. On the other hand, assume that $T$ is uniformly super-rigid, then $R^{-1}T$ is uniformly rigid, which is not possible by [17, Theorem 5.4]. Hence, $T$ is super-rigid but not uniformly super-rigid.

In the following proposition, we prove that the uniform super-rigidity is preserved under similarity.

Proposition 3.6 Let $T \in \mathcal{B}(X)$ and $S \in \mathcal{B}(Y)$. Assume that $T$ and $S$ are similar. Then $T$ is uniformly super-rigid on $X$ if and only if $S$ is uniformly super-rigid on $Y$.

Proof Since $T$ and $S$ are similar, then there exists a homeomorphisms $\phi: X \to Y$ such that $S \circ \phi = \phi \circ T$.

Since $T$ is uniformly super-rigid, it follows that there exist a strictly increasing sequence of positive integers $(n_k)_{k \in \mathbb{N}}$ and a sequence $(\lambda_k)_{k \in \mathbb{N}}$ of numbers such that
\[
\|\lambda_k T^{n_k} - I\| = \sup_{\|x\| \leq 1} \|\lambda_k T^{n_k} x - x\| \to 0.
\]

Let $y$ be a nonzero vector of $Y$ and pick $x$ a nonzero vector of $X$ such that $y = \phi(x)$. Then
\[
\|\lambda_k S^{n_k} - I\| = \sup_{\|y\| \leq 1} \|\lambda_k S^{n_k} y - y\|
\]
\[
= \sup_{\|x\| \leq 1} \|\lambda_k S^{n_k} \circ \phi(x) - \phi(x)\|
\]
\[
= \sup_{\|x\| \leq 1} \|\phi(\lambda_k T^{n_k} x - x)\|
\]
\[
\leq \|\phi\| \sup_{\|x\| \leq 1} \|\lambda_k T^{n_k} x - x\| \to 0.
\]

Hence $S$ is uniformly super-rigid on $Y$. □

In the following theorem, we give the relationship between the uniform super-rigidity of an operator and its iterations.

Theorem 3.7 Let $T$ be an operator acting on $X$. Then $T$ is uniformly super-rigid if and only if $T^p$ is uniformly super-rigid, for all $p \geq 2$.

Proof Let $p$ be a strictly positive integer. Assume that $T$ is uniformly super-rigid, then there exist a strictly increasing sequence of positive integers $(n_k)_{k \in \mathbb{N}}$ and a sequence $(\lambda_k)_{k \in \mathbb{N}}$ of
numbers such that \( \| \lambda_k T^{n_k} - I \| \to 0 \). By Banach–Steinhaus theorem we have \( M := \sup_{n \in \mathbb{N}} \| \lambda_k T^{n_k} \| < +\infty \). It follows that
\[
\| \lambda_k^p T^{pn_k} - I \| \leq \| \lambda_k^{p-1} T^{(p-1)n_k} + \lambda_k^{p-2} T^{(p-2)n_k} + \cdots + 1 \| \| \lambda_k T^{n_k} - I \| \\
\leq \left( \sum_{i=0}^{p-1} M^i \right) \| \lambda_k T^{n_k} - I \| .
\]
This shows that \( T^p \) is uniformly super-rigid whenever \( T \) is uniformly super-rigid. \( \square \)

Let \( T \) be an operator acting on \( X \). If \( T \) is super-rigid, then by Proposition 2.10, there exists some \( R > 0 \) such that \( \sigma_p(T) \cup \sigma_p(T^*) \subset \{ z \in \mathbb{C} : |z| = R \} \). For uniformly super-rigid operators we have a significant strengthening of this result.

**Theorem 3.8** Let \( T \) be an operator acting on \( X \). If \( T \) is uniformly super-rigid, then there exists \( R > 0 \) such that
\[
\sigma(T) \subset \{ z \in \mathbb{C} : |z| = R \}.
\]

**Proof** Assume that \( T \) is uniformly super-rigid. Then there exist a sequence \( (\mu_k) \subset \mathbb{K} \) and a sequence \( (n_k) \subset \mathbb{N} \) such that \( \| \mu_k T^{n_k} - I \| \to 0 \) as \( k \to \infty \).

Note that we may suppose, using Lemma 3.4, that \( R = \| T \| \). Again, by Lemma 3.4, we may suppose that \( \| T \| = R = 1 \). By Proposition 2.10, we have \( \sigma_p(T) \cup \sigma_p(T^*) \subset \{ z \in \mathbb{C} : |z| = 1 \} \).

If \( \lambda \in \sigma(T) \cap \mathbb{D} \), then by the previous discussion, \( \lambda \) is necessarily in \( \sigma_T(T) \), the approximate point spectrum of \( T \).

By contradiction, assume that there exists \( \lambda \in \mathbb{K} \) such that \( \lambda \in \sigma(T) \cap \mathbb{D} \). Since \( \lambda \in \sigma(T) \), it follows by the spectral theorem that \( \lambda^p \in \sigma(T^p) \), for all \( p \geq 2 \). On the other hand, by Theorem 3.7, the operator \( T^p \) is uniformly super rigid for all \( p \geq 2 \). Hence \( \lambda^p \) is in the approximate point spectrum of \( T^p \). Thus, for all \( p \geq 2 \), there exists a sequence \( (x_k^{(p)})_{k \in \mathbb{N}} \subset \{ z \in \mathbb{C} : |z| = 1 \} \) such that \( \| T^p x_k^{(p)} - \lambda^p x_k^{(p)} \| \to 0 \) as \( k \to \infty \). Using this, one can find sequence \( (y_k)_{k \in \mathbb{N}} \) of \( X \) such that \( \| y_k \| \in \{ |z| = 1 \} \) and \( \| T^k y_k - \lambda y_k \| \leq \| a_k \| \), where \( (a_k) \) is such that \( (\mu_k |a_k + |\lambda|^k|) \to 0 \) as \( k \to \infty \). This implies that
\[
\| \mu_k T^k y_k \| \leq |\mu_k| |a_k + |\lambda|^k| \to 0 \text{ as } k \to \infty.
\]

On the other hand
\[
\| \mu_k T^{n_k} y_{n_k} - 1 \| \leq \| \mu_k T^{n_k} y_{n_k} - y_{n_k} \| \leq \| \mu_k T^{n_k} - I \| \to 0 \text{ as } k \to \infty,
\]
which is a contradiction. \( \square \)

**Remark 3.9** The inclusion in Theorem 3.8 could be a strict inclusion. Indeed, let \( \lambda \) be a non-zero number, then \( \lambda I \) is a uniformly super-rigid operator. On the other hand, let \( |\lambda| = R \). Then
\[
\sigma(T) = \{ \lambda \} \subset \{ z \in \mathbb{C} : |z| = R \}.
\]

**Corollary 3.10** Let \( T \) be an operator acting on \( X \). If \( T \) is uniformly super-rigid, then it is invertible.

**Proof** Assume that \( T \) is uniformly super-rigid. By Theorem 3.8, there exists \( R > 0 \) such that \( \sigma(T) \subset \{ z \in \mathbb{C} : |z| = R \} \). Hence, \( 0 \notin \sigma(T) \), which means that \( T \) is invertible. \( \square \)
4 Finite dimensional spaces

In this section, we will characterize the super-recurrence, super-rigidity, and uniform super-rigidity in finite-dimensional space.

In the following, if \( T \in B(\mathbb{K}^d) \), then we denote by \( A \) a matrix of \( T \). Moreover, if \( T \) is super-recurrent, let \( R > 0 \) be such that each component of the spectrum of \( T \) intersects the circle \( \{ z \in \mathbb{C} : |z| = R \} \), see [2, Theorem 4.1].

In the complex case, we have then the following theorem.

**Theorem 4.1** Let \( T \in B(\mathbb{C}^d) \). Then the following statements are equivalent:

1. \( T \) is super-recurrent;
2. \( T \) is super-rigid;
3. \( T \) is uniformly super-rigid;
4. \( A \) is similar to a diagonal matrix with entries of the same modulus.

**Proof** We need only to prove that (4) \( \Rightarrow \) (3) and (1) \( \Rightarrow \) (4).

(1) \( \Rightarrow \) (4): Assume that \( T \) is super-recurrent. Using the fact that \( \sigma_p(T) = \sigma(T) \) and [2, Theorem 4.1], we conclude that \( \sigma(T) = \{ \lambda_1, \ldots, \lambda_M \} \subset \{ z \in \mathbb{C} : |z| = R \} \), where each \( \lambda_i \) with multiplicities \( m_i \). By Jordan decomposition theorem, the matrix \( A \) is similar to a matrix of the form

\[
\begin{pmatrix}
A_1 & 0 & \cdots & 0 \\
0 & A_2 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & A_M
\end{pmatrix},
\]

where each \( A_j \) has the form \( A_j = \lambda_j I_{m_j} \) or the form

\[
A_j = \begin{pmatrix}
\lambda_j & 1 & 0 & \cdots & 0 \\
0 & \lambda_j & 1 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \cdots & \lambda_j & 1 \\
0 & 0 & \cdots & 0 & \lambda_j
\end{pmatrix}.
\]

Assume that there exists a block \( A_{j_0} \) has the form (5), then \( m_j \geq 2 \). Using [2, Proposition 3.7], we conclude that the operator presented by the matrix

\[
B = \begin{pmatrix}
\lambda_{j_0} & 1 \\
0 & \lambda_{j_0}
\end{pmatrix}
\]

is super-recurrent on \( \mathbb{C}^2 \). By straightforward induction, we have for all \( n \in \mathbb{N} \)

\[
B^n = \begin{pmatrix}
\lambda_{j_0}^n & n\lambda_{j_0}^{n-1} \\
0 & \lambda_{j_0}^n
\end{pmatrix}.
\]

Let \( (z_1, z_2) \in \mathbb{C}^2 \) be a super-recurrent vector for \( B \) with \( z_2 \neq 0 \). Then there exist \( (\mu_k) \subset \mathbb{C} \) and a strictly increasing sequence of positive integers \( (n_k) \) such that

\[
\mu_k \begin{pmatrix}
\lambda_{j_0}^{n_k} & n\lambda_{j_0}^{n_k-1} \\
0 & \lambda_{j_0}^{n_k}
\end{pmatrix} \begin{pmatrix}
z_1 \\
z_2
\end{pmatrix} \rightarrow (z_1, z_2).
\]
which implies that
\[ \mu_k \lambda_j^{nk} z_2 \rightarrow z_2 \quad \text{and} \quad \mu_k \lambda_j^{nk} z_1 + \mu_k n_k \lambda_j^{nk} z_2 \rightarrow z_1. \]

This is impossible since by hypothesis \( z_2 \neq 0 \).

(4) \( \Rightarrow \) (3): Assume that \( A \) is similar to a diagonal matrix with entries \( \lambda_1, \ldots, \lambda_d \in \mathbb{C} \) such that \( |z| = R \) for some \( R > 0 \). Then there exists an increasing sequence \( (n_k) \) such that \( (R^{-1} \lambda_i)^{n_k} \rightarrow 1 \), for \( 1 \leq i \leq d \). This implies that \( T \) is uniformly super-rigid.

\[ \square \]

In the real case, we have the following theorem.

Theorem 4.2 Let \( T \in \mathbb{R}^d \). The following assertions are equivalent:

(1) \( T \) is super-recurrent;
(2) \( T \) is super-rigid;
(3) \( T \) is uniformly super-rigid;
(4) \( A \) is similar to a matrix of the form
\[
\begin{pmatrix}
A_1 & 0 & \cdots & 0 \\
0 & A_2 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & A_M
\end{pmatrix},
\]
where each \( A_k \), \( 1 \leq k \leq M \), is a matrix with entry \( R \) or \( -R \), or a \( 2 \times 2 \) matrix of the form
\[
\begin{pmatrix}
a & b \\
-b & a
\end{pmatrix}, \quad a, b \in \mathbb{R}.
\]

Proof We need only to prove that (4) \( \Rightarrow \) (3) and (1) \( \Rightarrow \) (4).

(1) \( \Rightarrow \) (4): Assume that \( T \) is super-recurrent. By the Jordan decomposition, the matrix \( A \) is similar to a matrix which has the form of (6). We have then two cases.

(1): Each \( A_k \) has the form \( \lambda_k I_{m_k} \). In this case, it suffices to do the same method which was used in the complex case to conclude.

(2): Each \( A_k \) has the form
\[
\begin{pmatrix}
B & I_2 & 0 & \cdots & 0 \\
0 & B & I_2 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \cdots & B & I_2 \\
0 & 0 & \cdots & 0 & B
\end{pmatrix},
\]
where \( B \) has the form of (7). We claim that \( A_k = B \) for all \( k \), \( 1 \leq k \leq M \). Indeed, if there exists \( k_0 \) which different to \( B \), then by [2, Proposition 3.7], the operator presented by the matrix
\[
C = \begin{pmatrix} B & I_2 \\ 0 & B \end{pmatrix}
\]
is super-recurrent in \( \mathbb{R}^4 \). The same proof used in the complex case lead us to a contradiction.

Hence, if \( T \) is super-recurrent, then \( A \) is similar to a matrix which has the form of (6).

(4) \( \Rightarrow \) (3): It is not difficult to see that each matrix of that form is uniformly super-rigid. \( \square \)
Acknowledgements  The authors are sincerely grateful to the anonymous referees for their careful reading, critical comments and valuable suggestions that contribute significantly to improving the manuscript during the revision.

Data Availability Statement: Data availability statement is not applicable.

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