WEIGHTED APPROXIMATION IN $\mathbb{C}$

JUJIE WU AND JOHN ERIK FORNÆSS

Abstract. We prove that if $\{\varphi_j\}$ is a sequence of subharmonic functions which are increasing to some subharmonic function $\varphi$ in $\mathbb{C}$, then the union of all the weighted Hilbert spaces $H(\varphi_j)$ is dense in the weighted Hilbert space $H(\varphi)$.

Mathematics Subject Classification (2010): 30D20, 30E10, 30H50, 31A05

Keywords: Holomorphic approximation; subharmonic function; Bergman kernel; $L^2$-estimate

CONTENTS

1. Introduction 1
2. Subharmonic functions in $\mathbb{C}$ 4
3. Comparison of weights 6
4. Proof of Theorem 1.3 8
5. Proof of Theorem 1.6 and Theorem 1.7 12
References 15

1. Introduction

Suppose $\varphi$ is a measurable function, locally bounded above on a domain $\Omega \subset \mathbb{C}^n$. Set

$$H(\Omega, \varphi) := \{f \in \mathcal{O}(\Omega) : \int_\Omega |f|^2 e^{-\varphi} d\lambda < +\infty\}$$

where $\mathcal{O}(\Omega)$ stands for the space of holomorphic functions on $\Omega$ and $d\lambda$ is the Lebesgue measure. Especially let $H(\varphi)$ denote the space of entire functions $f$ with $L^2(\mathbb{C}^n, \varphi)$ norm, i.e., $\|f\|^2_{L^2(\mathbb{C}^n, \varphi)} = \int_{\mathbb{C}^n} |f|^2 e^{-\varphi} d\lambda < +\infty$.

Since $f \in H(\Omega, \varphi)$, the function $|f|^2$ is plurisubharmonic (psh) and $\varphi$ is locally bounded above, we have

$$|f(w)| \leq \frac{C_n}{r^n} \|f\|_{L^2(B(w,r),0)} \leq \frac{C'_n}{r^n} \|f\|_{H(\Omega, \varphi)},$$

(1.1)
if the ball \( B(w, r) \subset \subset \Omega \) and for \( K \subset \subset \Omega \)
\[
\sup_K |f| \leq C \|f\|_{H(\Omega, \varphi)},
\]
where \( C \) depends only on \( K \) and \( \Omega \). So \( H(\Omega, \varphi) \) is a closed subspace of \( L^2(\Omega, \varphi) \) and thus a Hilbert space. Let \( K_{\Omega, \varphi}(z, w) \) denote the weighted Bergman kernel corresponding to the Hilbert space \( H(\Omega, \varphi) \). If \( \varphi = 0 \), then \( K_{\Omega}(z, w) := K_{\Omega,0}(z, w) \) is the classical kernel introduced by Stefan Bergman.

In 1971, B. A. Taylor \[5\] investigated weighted approximation results for entire functions on \( \mathbb{C}^n \). He proved:

**Theorem 1.1.** Let \( \varphi_1 \leq \varphi_2 \leq \varphi_3 \leq \cdots \) be psh functions on \( \mathbb{C}^n \), assume \( \varphi = \lim_{j \to \infty} \varphi_j \) is psh, and suppose that \( \int_K e^{-\varphi_j} d\lambda < \infty \) for every compact set \( K \). Then the closure of \( \bigcup_{j=1}^{\infty} H(\varphi_j + \log(1 + \|z\|^2)) \) in the Hilbert space \( L^2(\varphi + \log(1 + \|z\|^2)) \) contains \( H(\varphi) \).

In \[3\] we improved Taylor’s result as follows

**Theorem 1.2.** Let \( \varphi_1 \leq \varphi_2 \leq \varphi_3 \leq \cdots \) be psh functions on \( \mathbb{C}^n \) with psh limit. For any \( \epsilon > 0 \), let \( \bar{\varphi}_j = \varphi_j + \epsilon \log(1 + \|z\|^2) \) and \( \bar{\varphi} = \lim_{j \to +\infty} \bar{\varphi}_j \). Then \( \bigcup_{j=1}^{\infty} H(\bar{\varphi}_j) \) is dense in \( H(\bar{\varphi}) \).

It is an important question whether this theorem is true or false when \( \epsilon = 0 \). Here by using some potential theoretic properties of subharmonic functions we show that it holds in one dimension.

**Theorem 1.3.** Let \( \varphi_1 \leq \varphi_2 \leq \varphi_3 \leq \cdots \) be subharmonic functions on \( \mathbb{C} \). Suppose that \( \varphi = \lim_k \varphi_k \) and \( \varphi \) is locally bounded above. Then \( \bigcup_{k=1}^{\infty} H(\varphi_k) \) is dense in \( H(\varphi) \).

**Remark 1.4.** Let \( \varphi = \lim_k \varphi_k \). We define \( \phi = \varphi^* := \limsup_{\zeta \to z} \varphi(\zeta), z \in \mathbb{C}, \) which is the upper regularization of \( \varphi \). Then \( \phi \) is subharmonic, \( \varphi \leq \phi \), \( \varphi = \phi \) almost everywhere on \( \mathbb{C} \) and we have \( H(\varphi) = H(\phi) \) (See Theorem 3.4.2 in \[10\]).

We need the strong openness theorem as follows, see \[6\], \[7\], \[8\], \[9\].

**Theorem 1.5** (Strong openness theorem). Let \( V \subset \subset U \subset \mathbb{C}^n \) be an open set. Let \( \varphi_1 \leq \varphi_2 \leq \varphi_3 \leq \cdots \) be non-positive psh functions on \( U \) such that \( \varphi = \lim_k \varphi_k \) and \( \varphi \) is locally bounded above. If \( f \in \mathcal{O}(U) \) such that
\[
\int_U |f|^2 e^{-\varphi} d\lambda < \infty,
\]
then there exists \( j_0 \) so that when \( j \geq j_0 \)
\[
\int_V |f|^2 e^{-\varphi_j} d\lambda < \infty.
\]

For convenience, we will use \( K_{\varphi_j}(z, w) \) (resp. \( K_\varphi(z, w) \)) to denote the weighted Bergman kernel corresponding to the Hilbert space \( H(\varphi_j) := H(\mathbb{C}, \varphi_j) \) (resp. \( H(\varphi) := H(\mathbb{C}, \varphi) \)). As an application of the approximation theorem \([1,3]\) we will prove

**Theorem 1.6.** Let \( \varphi_1 \leq \varphi_2 \leq \varphi_3 \leq \cdots \) be subharmonic functions on \( \mathbb{C} \). Suppose that \( \varphi = \lim_k \varphi_k \) and \( \varphi \) is locally bounded above. Then

\[
\lim_j K_{\varphi_j}(z, z) = K_\varphi(z, z), \quad \forall z \in \mathbb{C}.
\]

Ligocka showed that the classical Bergman kernel of certain Hartogs domains can be expressed as the sum of a series of weighted Bergman kernels defined on another domain of lower dimension. We set

\[
\Omega_j = \{(z, w) \in \mathbb{C} \times \mathbb{C} : |w| < e^{-\varphi_j(z)}\}
\]

and

\[
\Omega = \{(z, w) \in \mathbb{C} \times \mathbb{C} : |w| < e^{-\varphi(z)}\}.
\]

We note that a domain \( \{(z, w) \in \mathbb{C} \times \mathbb{C} : |w| < e^{-\ell(z)}\} \) is open exactly when \( \ell(z) \) is upper semicontinuous. Hence we use \( \phi \) in the definition of \( \Omega \) and not \( \varphi \).

We denote by \( K_{\Omega_j}([z, t], (w, s)) \) (resp. \( K_\Omega([z, t], (w, s)) \)) the Classical Bergman kernel of the Hilbert space \( H(\Omega_j, 0) \) (resp. \( H(\Omega, 0) \)), where \( z, t, w, s \in \mathbb{C} \). Ligocka’s formula \([4]\) implies that

\[
K_{\Omega_j}([z, t], (w, s)) = \sum_{k=0}^{\infty} 2(k+1)K_{2(k+1)\varphi_j}(z, w)K_\varphi(t, s)^k, \quad z, t, w, s \in \mathbb{C}
\]

where \( K_{2(k+1)\varphi_j}(z, w) \) is the weighted Bergman kernel of the Hilbert space \( H(\mathbb{C}, 2(k+1)\varphi_j) \).

According to the result of Theorem \([1,6]\) we obtain that

**Theorem 1.7.** Let \( \{\varphi_j\} \) and \( \varphi \) be as in theorem \([1,6]\) then the sequence \( K_{\Omega_j}([z, t], (w, s)) \) converges to \( K_\Omega([z, t], (w, s)) \) locally uniformly in \( \Omega \times \Omega \).

The set-up of the paper is as follows: We prove in Section 2 an integral estimate for subharmonic weights. In Section 3 we discuss some potential theoretic properties of subharmonic functions which we will need. In Section 4, we prove the Main Theorem, Theorem \([1,3]\). In Section 5, we prove the convergence of the Bergman Kernel which can be seen as an application of the main Theorem \([1,3]\).
2. Subharmonic functions in $\mathbb{C}$

**Lemma 2.1.** Let $\alpha_i > 0$, $z_i \in \mathbb{C}$, $i = 1, \ldots, N$ and let $\alpha = \sum \alpha_i$. For any $z$ which is not one of the $z_i$ we have the inequality

$$\prod_{i=1}^{N} \left( \frac{1}{|z - z_i|} \right)^{\alpha_i} \leq \sum_{i=1}^{N} \frac{\alpha_i}{\alpha} \left( \frac{1}{|z - z_i|} \right)^{\alpha}.$$  

**Proof.**

$$\prod_{i=1}^{N} \left( \frac{1}{|z - z_i|} \right)^{\alpha_i} = e^{\sum_{i=1}^{N} \frac{\alpha_i}{\alpha} \log \left( \left( \frac{1}{|z - z_i|} \right)^{\alpha} \right)}$$

Since $\exp$ is convex,

$$\leq \sum_{i=1}^{N} \frac{\alpha_i}{\alpha} e^{\log \left( \left( \frac{1}{|z - z_i|} \right)^{\alpha} \right)}$$

$$= \sum_{i=1}^{N} \frac{\alpha_i}{\alpha} \left( \frac{1}{|z - z_i|} \right)^{\alpha}.$$  

\[\square\]

**Lemma 2.2.** Let $|z_0| < R$ and suppose $0 < \alpha < 2$. Then

$$\int_{|z| < R} \left( \frac{1}{|z - z_0|} \right)^{\alpha} d\lambda(z) \leq \frac{28R^2}{2 - \alpha}.$$  

**Proof.**

$$\int_{|z| < R} \left( \frac{1}{|z - z_0|} \right)^{\alpha} d\lambda(z) \leq \int_{|z| < 2R} \left( \frac{1}{|z|} \right)^{\alpha} d\lambda(z)$$

$$= 2\pi \frac{(2R)^{2-\alpha}}{2 - \alpha}$$

$$\leq 28R^2/(2 - \alpha).$$  

\[\square\]

As a direct consequence, we obtain

**Theorem 2.3.** Let $\alpha_i > 0$, $|z_i| < R$, $i = 1, \ldots, N$ and let $\alpha = \sum \alpha_i$. Suppose that $\alpha < 2$. Then

$$\int_{|z| < R} \prod_{i=1}^{N} \left( \frac{1}{|z - z_i|} \right)^{\alpha_i} d\lambda(z) \leq \frac{28R^2}{2 - \alpha}.$$  

**Lemma 2.4.** Let $K$ be a compact set in $\mathbb{R}^n$. Suppose $f(x, y) : K \times K \rightarrow \mathbb{R}$ is continuous. Let $\mu$ be a positive measure on $K$ with finite total mass $\alpha$. Let $\epsilon > 0$. Then there exists a finite positive measure $\sigma = \sum_{i=1}^{N} \rho_i \delta_{p_i}$ with $\sigma(K) = \alpha$ where the $p_i$ are points in $K$. Moreover if $\phi(x) = \int_{y \in K} f(x, y)d\mu(y)$ and $\psi(x) = \int_{y \in K} f(x, y)d\sigma(y)$ then $\psi(x) < \phi(x) + \epsilon$. 


Proof. Divide $K$ into finitely many small sets $K_i$ and pick $p_i \in K_i$. By uniform continuity of $f$ we can assume that for any $y \in K_i$ we have that $|f(x, y) - f(x, p_i)| < \varepsilon/C$ for any given $C$ (which will depend on $\alpha$.) Let $r_i = \mu(K_i)$. Define $\sigma = \sum_{i=1}^N r_i \delta_{p_i}$. Let $x \in K$. Define

$$\psi(x) = \int_{y \in K} f(x, y) d\sigma(y)$$

$$= \sum_i f(x, p_i) r_i$$

$$= \sum_i f(x, p_i) \int_{K_i} d\mu(y)$$

$$= \sum_i \int_{K_i} f(x, p_i) d\mu(y)$$

$$\leq \sum_i \int_{K_i} (f(x, y) + \varepsilon/C) d\mu$$

$$= \phi(x) + (\varepsilon/C \int_K d\mu)$$

$$= \phi(x) + \varepsilon \alpha/C.$$

We let $C = \alpha$. \hfill \Box

Theorem 2.5. Let $d\mu$ be a positive measure on the disc of radius $R$ with total mass $\alpha < 2$. Set $\phi(z) = \int_{|\zeta| < R} \log|z - \zeta| d\mu(\zeta)$. Then

$$\int_{|z| < R} e^{-\phi(z)} d\lambda(z) \leq \frac{28R^2}{2 - \alpha}.$$

Proof. Define $\psi_n(z, \zeta) = \max\{\log|z - \zeta|, -n\}$. Define

$$\phi_n(z) = \int_{|\zeta| < R} \psi_n(z, \zeta) d\mu(\zeta) \quad \text{for} \quad z \in \Delta(R).$$

Then $\phi_n : \Delta(R) \to \mathbb{R}$ is continuous and $\phi_n \searrow \phi$ pointwise for $z \in \Delta(R)$. Hence $e^{-\phi_n(z)} \nearrow e^{-\phi}$ on $\Delta(R)$. Therefore, in order to show that

$$\int_{|z| < R} e^{-\phi(z)} d\lambda(z) \leq \frac{28R^2}{2 - \alpha},$$

it suffices to show that

$$\int_{|z| < R} e^{-\phi_n(z)} d\lambda(z) \leq \frac{28R^2}{2 - \alpha} + \frac{1}{n} \quad \forall n.$$
We fix \( n \). Let \( \delta > 0 \). Since \( \psi_n \) is continuous, according to Lemma 2.4 we can find a finite positive measure \( \mu_n = \sum_{i=1}^{N} \alpha_i \delta z_i \) with total mass \( \alpha \) so that

\[
\tilde{\phi}_n := \int_{|\zeta|<R} \psi_n(z, \zeta) d\mu_n(\zeta) \leq \phi_n(z) + \delta.
\]

By Theorem 2.3 we know that

\[
\int_{|z|<R} \prod_{i=1}^{N} \left( \frac{1}{|z-z_i|} \right)^{\alpha_i} d\lambda(z) \leq \frac{28R^2}{2 - \alpha}.
\]

Hence

\[
\int_{|z|<R} e^{-\sum_{i} \alpha_i \log |z-z_i|} d\lambda \leq \frac{28R^2}{2 - \alpha}.
\]

So

\[
\int_{|z|<R} e^{-\tilde{\phi}_n} d\lambda \leq \frac{28R^2}{2 - \alpha}.
\]

Notice that \( \max\{\log |z-\zeta|, -n\} \geq \log |z-\zeta| \), then we have

\[
-\max\{\log |z-\zeta|, -n\} \leq -\log |z-\zeta|.
\]

Hence \( \int_{|z|<R} e^{-\tilde{\phi}_n} d\lambda \leq \frac{28R^2}{2 - \alpha} \). Choosing \( \delta \) small enough we get that

\[
\int_{|z|<R} e^{-\phi_n} d\lambda \leq \frac{28R^2}{2 - \alpha} + \frac{1}{n}.
\]

\[
\square
\]

3. Comparison of weights

**Lemma 3.1.** Suppose that \( |\zeta| < R \) and \( z \in \mathbb{C} \). Then

\[
\log |z-\zeta| \leq \frac{1}{2} \log(1 + |z|^2) + \log 2 + \frac{1}{2} \log(1 + R^2).
\]

**Proof.**

\[
\log |z-\zeta| \leq \log(|z| + |\zeta|) \\
\leq \log 2 + \max\{\log |z|, \log |\zeta|\} \\
\leq \log 2 + \frac{1}{2} \log(1 + |z|^2) + \frac{1}{2} \log(1 + |\zeta|^2).
\]

\[
\square
\]

**Proposition 3.2.** Let \( \mu \) be a nonnegative measure on the disc \( |\zeta| < R \) with

mass \( M \). Let \( \phi_1(z) = \int_{|\zeta|<R} \log |z-\zeta| d\mu(\zeta) \) and \( \phi_2(z) = M/2 \log(1 + |z|^2) \).

Suppose that \( \phi = \phi_1 + \sigma \) and \( \psi = \phi_2 + \sigma \). Then there exists constant \( C \) so that \( \|f\|_{\psi}^2 \leq C \|f\|_{\phi}^2 \). In particular, \( H(\phi) \subset H(\psi) \).
Proof. By the previous lemma we know
\[ \phi_1 \leq \phi_2 + M \left( \log 2 + \frac{1}{2} \log(1 + R^2) \right), \]
hence
\[ \phi \leq \psi + M \left( \log 2 + \frac{1}{2} \log(1 + R^2) \right). \]
It follows that \( e^{-\psi} \leq C e^{-\phi} \). Thus we have \( H(\phi) \subset H(\psi) \). \( \Box \)

For the other direction we need some extra hypothesis.

**Proposition 3.3.** Let \( \mu \) be a nonnegative measure on the disc \(|\zeta| < (R + \epsilon)\) \((\epsilon > 0 \text{ some constant})\) with mass \( \beta \in (0, 2) \). Let \( \alpha \) be the \( \mu \) mass of \( \Delta(R) \).

Suppose that \( \phi \) is subharmonic on \( \mathbb{C} \) and that \( \frac{1}{2\pi} \Delta \phi = \mu \) on \( \Delta(R + \epsilon) \). Let \( \phi_1(z) = \int_{|\zeta|<R} \log|z-\zeta|d\mu(\zeta) \) and \( \phi_2(z) = \alpha/2 \log(1+|z|^2) \). Write \( \phi = \phi_1 + \sigma \) and \( \psi = \phi_2 + \sigma \). Then \( H(\psi) \subset H(\phi) \).

We prove first a lemma:

**Lemma 3.4.** There exists a constant \( C \) so that if \( |z| \geq R + \frac{\epsilon}{2} \) and \( |\zeta| < R \), then
\[ \log|z - \zeta| \geq \frac{1}{2} \log(1 + |z|^2) - C. \]

**Proof.** We get
\[
\begin{align*}
\log|z - \zeta| &= \log|z| \left| 1 - \frac{\zeta}{z} \right| \\
&\geq \log|z| + \log \left( 1 - \frac{2R}{2R + \epsilon} \right) \\
&\geq \frac{1}{2} \log \left( |z|^2 \left( \frac{1}{R^2} + 1 \right) \right) - \frac{1}{2} \log \left( \frac{1}{R^2} + 1 \right) \\
&\quad + \log \left( 1 - \frac{2R}{2R + \epsilon} \right) \\
&\geq \frac{1}{2} \log(1 + |z|^2) - \frac{1}{2} \log \left( \frac{1}{R^2} + 1 \right) + \log \left( 1 - \frac{2R}{2R + \epsilon} \right).
\end{align*}
\]
The proof gives that we can choose \( C = \frac{1}{2} \log \left( \frac{1}{R^2} + 1 \right) - \log(1 - \frac{2R}{2R + \epsilon}) \). \( \Box \)

In order to prove the above Proposition 3.3 we also need the following well known Riesz Decomposition Theorem (see Ransford [10] Theorem 3.7.9).

**Theorem 3.5** (Riesz Decomposition Theorem). Let \( u \) be a subharmonic function on a domain \( D \) in \( \mathbb{C} \), with \( u \not\equiv -\infty \). Then, given a relatively compact open subset \( U \) of \( D \), we can decompose \( u \) as
\[ u = \int_{\zeta \in U} \log|z - \zeta|d\mu(\zeta) + h \]
on \( U \). Where \( \mu = \frac{1}{2\pi} \Delta u |_U \) and \( h \) is harmonic on \( U \).

We prove Proposition 3.3.

**Proof.** Let \( F \) be an entire function so that \( \int_C |F|^2 e^{-\psi} d\lambda < \infty \). If \( |z| > R + \frac{\epsilon}{2} \), by the previous Lemma,

\[
\phi_1 = \int_{|\zeta| < R} \log |z - \zeta| d\mu(\zeta) \\
\geq \frac{\alpha}{2} \log(1 + |z|^2) - \alpha C \\
= \phi_2 - \alpha C.
\]

Here \( C \) is the explicit constant from Lemma 3.4. It follows that

\[
\int_{|z| > R + \epsilon/2} |F|^2 e^{-\phi} d\lambda < e^{\alpha C} \int_{|z| > R + \epsilon/2} |F|^2 e^{-\psi} d\lambda < +\infty.
\]

On the disc of radius \( R + \epsilon \), according to Riesz decomposition theorem we can write \( \phi(z) = \int_{|\zeta| < R + \epsilon} \log |z - \zeta| d\mu(\zeta) + \tau := \Phi + \tau \) where \( \tau \) is a subharmonic function which is harmonic on the disc of radius \( R + \epsilon \). In particular we have that \( \tau \) is uniformly bounded on the disc of radius \( R + \frac{2}{3} \epsilon \). The result follows from Theorem 2.5 that

\[
\int_{|z| > R + \epsilon} e^{-\Phi} d\lambda < e^{\frac{2\pi}{2 - \beta} (2(R + \epsilon))^2} < +\infty
\]

and hence the same is true for the integral \( |F|^2 e^{-\Phi} \) on the disc of radius \( R + \frac{1}{2} \epsilon \). That means we have \( \int_C |F|^2 e^{-\phi} d\lambda < +\infty \). Thus

\[ H(\psi) \subset H(\phi). \]

\( \Box \)

### 4. Proof of Theorem 1.3

In this section we prove the Main Theorem, Theorem 1.3. We prove first the case when the upper regularization \( \phi \) is a harmonic function. We can suppose that \( \varphi_1 \) is not identically \(-\infty\).

**Lemma 4.1.** If \( \phi \) is harmonic, then there are constants \( c_1 \leq c_2 \leq \cdots, c_j \to 0 \) so that \( \varphi_j = \varphi + c_j = \phi + c_j \).

To prove the lemma, observe that \( \varphi_j - \phi \leq \varphi_j - \varphi \leq 0 \). Then \( \varphi_j - \phi \) must be constant. Thus we have \( \varphi = \phi \). The Lemma follows.

Then the theorem follows in the case when \( \phi \) is harmonic.

We can generalize this to the following case:

**Condition (A):** The upper regularization \( \phi \) of \( \varphi \) is a subharmonic function with the following property: \( \Delta \phi = \sum a_i \delta_{z_i} \) where \( z_i \) is a sequence in \( \mathbb{C} \) and \( a_i > 0 \).
Lemma 4.2. In the case of (A), there exist nonpositive constants $c_j$ so that $\varphi_j = \varphi + c_j = \phi + c_j$.

Proof. Fix $N$. We can write
\[ \phi = \sum_{i=1}^{N} a_i \log |z - z_i| + \psi_N \]
where $\psi_N$ is subharmonic and $\Delta \psi_N = \sum_{i>N} a_i \delta_{z_i}$. We get for any $j, N$
\[ \varphi_j - \sum_{i=1}^{N} a_i \log |z - z_i| \leq \psi_N \]
for $z \neq z_i$. Then $\varphi_j - \sum_{i=1}^{N} a_i \log |z - z_i|$ extends across $z_i$ as a subharmonic function $\psi^N_j$. That is $\varphi_j = \sum_{i=1}^{N} a_i \log |z - z_i| + \psi^N_j$ for any $N$ on $\mathbb{C}$. Thus $\varphi_j = -\infty$ at $z_i$ and $\Delta \varphi_j \geq \sum_{i=1}^{N} a_i \delta_{z_i}$. It follows that $\Delta \varphi_j \geq \Delta \phi$ on $\mathbb{C}$ for all $j$. So we can find some subharmonic function $\lambda_j$ such that $\varphi_j = \phi + \lambda_j$. But $\lambda_j \leq 0$ thus it must be constant. The Lemma follows and hence the theorem also follows in this case. \[ \square \]

Condition (B): Let $\phi$ be a subharmonic function on $\mathbb{C}$. Let $\mu$ denote the Laplacian of $\frac{1}{2\pi} \phi$. We say that $\phi$ satisfies condition (B) if there exist some constant $R > 0$ and $c > 0$ such that on the disc $|\zeta| < R + c$, the mass of $\mu$ is equal to $\beta$, with $0 < \beta < 2$ and the mass of $\mu$ on the disc $|\zeta| < R$ is $\alpha > 0$. According to Proposition 3.2 and Proposition 3.3 with the same notation as there we have the following Corollary:

Corollary 4.3. If $\phi$ satisfies the above condition (B), then the spaces $L^2(\mathbb{C}, \phi)$ and $L^2(\mathbb{C}, \psi)$ are the same. Moreover the norms are equivalent.

Lemma 4.4. Let $\varphi_1 \leq \varphi_2 \leq \varphi_3 \leq \cdots$ be subharmonic functions on $\mathbb{C}$ with $\varphi = \lim_{k} \varphi_k$. Suppose the upper regularization $\phi$ of $\phi$ satisfies the above Condition (B). Then $\bigcup_{k=1}^{\infty} H(\varphi_k)$ is dense in $H(\varphi)$.

To prove Lemma 4.4 we need the following $L^2$-estimate by Berndtsson (see [1]).

Lemma 4.5 ([1]). Let $\Omega \subset \mathbb{C}^n$ be a pseudoconvex domain and $\varphi \in psh(\Omega)$. Suppose $\psi$ is a $C^2$ real function satisfying
\[ ri\partial\bar{\partial}(\varphi + \psi) \geq i\partial\psi \wedge \bar{\partial}\psi \]
in the sense of distributions for some $0 < r < 1$. Then for each $\bar{\partial}$-closed $(0,1)$-form $v$, there is a solution $u$ of $\partial u = v$ which satisfies

\begin{equation}
(4.1) \quad \int_{\Omega} |u|^2 e^{\psi - \varphi} d\lambda \leq \frac{6}{(1 - r)^2} \int_{\Omega} |v|^2 e^{\bar{\partial}(\varphi + \psi)} e^{\psi - \varphi} d\lambda
\end{equation}

in the sense of distributions.

Proof of Lemma 4.4. Put $\frac{1}{2\pi} \Delta \phi = \mu$, $\frac{1}{2\pi} \Delta \varphi_j = \mu_j$ for each $j$. The mass of $\mu$ (resp. $\mu_j$) on the disc $\Delta(R)$ will be denoted by $\alpha$ (resp. $\alpha_j$). Since $\varphi_j$ is increasing to $\varphi$ and $\varphi = \phi$ a.e., we have that $\Delta \varphi_j$ converges to $\Delta \phi$ in the sense of distributions. That means we can find some $0 < c' < c$ with the mass of $\mu_j$ on the disc $\Delta(R + c')$ belongs to $(0,2)$ when $j$ is sufficiently large. Moreover the mass of $\mu_j$ on the disc $\Delta(R)$ is $\alpha_j > \alpha/2$ for all sufficiently large $j$. By using Riesz decomposition theorem we can write

$\phi = \bar{\varphi} + \int_{|\zeta| < R} \log |z - \zeta| d\mu(\zeta)$, \quad $\varphi_j = \bar{\varphi}_j + \int_{|\zeta| < R} \log |z - \zeta| d\mu_j(\zeta)$, \quad $\forall j$.

Here $\bar{\varphi}$ and $\bar{\varphi}_j$ are subharmonic functions on $\mathbb{C}$. Put

$\varphi_j' = \bar{\varphi}_j + \frac{\alpha_j}{2} \log(1 + |z|^2)$, \quad $\forall j$

and

$\phi' = \bar{\phi} + \frac{\alpha}{2} \log(1 + |z|^2)$.

By Corollary 4.3 we have that $H(\varphi_j') = H(\varphi_j)$ for each $j$ and $H(\phi') = H(\varphi)$. The following proof is very similar to [3], we make some changes. Here for convenience we replace $\log(1 + |z|^2)$ by $\log(e + |z|^2)$ without changing the spaces and the norms because of equivalence. Let $\chi : \mathbb{R} \rightarrow [0, 1]$ be a smooth function on $\mathbb{C}$ satisfying $\chi|_{(-\infty, \log \frac{1}{2})} = 1$, $\chi|_{(0, +\infty)} = 0$ and $|\chi'| \leq 3$. Set

$\psi = -\log \left(\log(e + |z|^2)\right)$.

Then we have

$i\partial \bar{\partial} \psi = -i \frac{\partial \bar{\partial} \log(e + |z|^2)}{\log(e + |z|^2)} + i \frac{\partial \log(e + |z|^2) \wedge \bar{\partial} \log(e + |z|^2)}{(\log(e + |z|^2))^2}$

and

$i\partial \psi \wedge \bar{\partial} \psi = i \frac{\partial \log(e + |z|^2) \wedge \bar{\partial} \log(e + |z|^2)}{(\log(e + |z|^2))^2}$.
For each $j$, put $\Phi_j := \varphi'_j + \frac{\alpha_j}{4} \psi = \tilde{\varphi}_j + \frac{\alpha_j}{4} \log(e + |z|^2) + \frac{\alpha_j}{4} \psi$ and $\Psi_j = \frac{\alpha_j}{4} \psi$. By calculation we have

\[
\begin{align*}
 i\partial\bar{\partial}(\Phi_j + \Psi_j) &\geq \frac{\alpha_j}{2} \partial\bar{\partial} \log(e + |z|^2) - \frac{\alpha_j}{2} \frac{\partial\bar{\partial} \log(e + |z|^2)}{\log(e + |z|^2)} \times \\
 &\geq \frac{\alpha_j}{2} \psi \\
 &= \frac{8}{\alpha_j} \partial\Psi_j \wedge \bar{\partial}\Psi_j
\end{align*}
\]

while $0 < \frac{\alpha_j}{8} < 1$. Let $f \in H(\varphi) = H(\varphi')$. We fix an $\epsilon > 0$. Put $v_\epsilon := f \cdot \bar{\partial} \chi(\log(-\psi) + \log \epsilon)$. Apply Lemma 4.5 especially for $\Omega = \mathbb{C}$ with $\varphi$ and $\psi$ replaced by $\Phi_j$ and $\Psi_j$ respectively, we then obtain a solution $u_{j, \epsilon}$ of $\partial u = v_\epsilon$ on $\mathbb{C}$ satisfying

\[
\int_{\mathbb{C}} |u_{j, \epsilon}|^2 e^{-\varphi'_j} d\lambda \leq \frac{6}{(1 - \frac{\alpha_j}{8})^2} \int_{\mathbb{C}} |f \partial\chi(\cdot)|^2 e^{\Psi_j - \Phi_j} d\lambda \\
\leq \frac{C}{\alpha_j} \epsilon^2 \int_{\frac{1}{8} \leq -\psi \leq \frac{1}{\epsilon}} |f|^2 e^{-\varphi'_j} d\lambda.
\]

Put $K := \{ z : z \in \mathbb{C}, -\psi \leq \frac{1}{\epsilon} \}$. Since $f \in H(\varphi)$, according to the strong openness theorem there exists $j_0$ so that when $j \geq j_0$ we have $\int_K |f|^2 e^{-\varphi} d\lambda < \infty$.

Next by using Lemma 3.1 and Proposition 3.2 we obtain that for some constant $C$, independent of $j$ and $K$

\[
\begin{align*}
\int_K |f|^2 e^{-\varphi'_j} d\lambda &= \int_K |f|^2 e^{-\frac{\alpha_j}{4} \log(e + |z|^2) - \tilde{\varphi}_j} d\lambda \\
&\leq C \int_K |f|^2 e^{-f_{|z|<K} \log(z - \zeta) d\mu_j(\zeta) - \tilde{\varphi}_j} d\lambda \\
&= C \int_K |f|^2 e^{-\varphi} d\lambda < \infty.
\end{align*}
\]

Set

\[
F_{j, \epsilon} = f \cdot \chi(\log(-\psi) + \log \epsilon) - u_{j, \epsilon}.
\]

Then $F_{j, \epsilon}$ is an entire function for each $j \geq j_0 \gg 1$ with

\[
\|F_{j, \epsilon}\|_{L^2(\mathbb{C}, \varphi'_j)} \leq (1 + \frac{C}{\sqrt{\alpha_j}}\epsilon) \|f\|_{L^2(K, \varphi'_j)} < +\infty.
\]
That is \( F_{j,\epsilon} \in \bigcup_{j=1}^{\infty} H(\varphi_j') = \bigcup_{j=1}^{\infty} H(\varphi_j') \). We also obtain
\[
\|F_{j,\epsilon} - f\|_{L^2(C,\varphi)}^2 \leq 2 \int_{-\psi \geq \frac{1}{\epsilon}} |f|^2 e^{-\varphi_j} d\lambda + C \int_C |u_{j,\epsilon}|^2 e^{-\varphi_j} d\lambda
\]
\[
\leq 2 \int_{-\psi \geq \frac{1}{\epsilon}} |f|^2 e^{-\varphi_j} d\lambda + C' \epsilon^2 \int_K |f|^2 e^{-\varphi_j} d\lambda.
\]

Still keeping \( \epsilon \) fixed, but letting \( j \to \infty \), we get
\[
\limsup_{j \to \infty} \|F_{j,\epsilon} - f\|_{L^2(C,\varphi)}^2 \leq 2 \int_{-\psi \geq \frac{1}{\epsilon}} |f|^2 e^{-\varphi_j} d\lambda + C' \epsilon^2 \int_K |f|^2 e^{-\varphi_j} d\lambda.
\]

Finally we let \( \epsilon \to 0 \). Then \( \bigcup_{j=1}^{\infty} H(\varphi_j) \) is dense in \( H(\varphi) \), which completes the proof. \( \square \)

For any subharmonic \( \phi \) on \( C \), we let \( \mu = \frac{1}{2\pi} \Delta \phi \) which is a locally finite positive measure on \( C \). Then \( \mu \) decomposes into a sum \( \mu = \mu_1 + \mu_2 \) where \( \mu_2 = \sum_i a_i \delta_{z_i} \) is an at most countable sum of Dirac masses and where \( \mu_1 \) has no point masses. Thus Theorem 1.3 follows as above, using Condition (B) because of the following lemma.

Lemma 4.6. Suppose \( \mu_1 \) is not identically zero. Then there exist a point \( z_0 \in C \) and \( 0 < r < s \) so that \( \mu(\Delta(z_0, r)) > 0, \mu(\Delta(z_0, s)) < 2 \).

Proof. The support of the measure \( \mu_1 \) is uncountable. Hence we can choose a point \( z_0 \) in the support of \( \mu_1 \) which is not one of the \( z_i \). Then \( \mu(\Delta(z_0, s)) \to 0 \) as \( s \to 0 \) while \( \mu(\Delta(z_0, r)) > 0 \) for all \( r > 0 \). \( \square \)

5. Proof of Theorem 1.6 and Theorem 1.7

Proof of theorem 1.6. We assume that if the upper regulation \( \phi \) of \( \varphi \) satisfies Condition (B). For other \( \phi \), we may use the same method as in the proof of Theorem 1.3 we skip the details. Let \( \chi, \psi, \Psi_j, \Phi_j \) and \( K \) as before in the proof of Lemma 4.4. The proof is similar to [2]. Set
\[
\lambda_\epsilon = \chi(\log(-\psi) + \log \epsilon), \quad \epsilon \ll 1.
\]

Let \( w \in B_R := \{|z| < R\} \). Applying Lemma 4.5 with \( \varphi \) and \( \psi \) replaced by \( \Phi_j \) and \( \Psi_j \) respectively, we get a solution \( u_{j,\epsilon} \) of
\[
\overline{\partial} u = K_{\varphi}(\cdot, w) \overline{\partial} \lambda_\epsilon
\]
such that
\[
\int_C |u_{j,\epsilon}|^2 e^{-\varphi_j} d\lambda \leq \frac{C}{\alpha_j} \epsilon^2 \int_{\frac{1}{\epsilon} \leq -\psi \leq \frac{1}{\epsilon}} |K_{\varphi}(\cdot, w)|^2 e^{-\varphi_j} d\lambda \leq \frac{C'}{\alpha_j} \epsilon^2 K_{\varphi}(w, w).
\]

The last inequality holds because of the following argument.
Since \(-\varphi_j\) monotonically decreases to \(-\varphi\), we can find appropriate control function of \(|K_\varphi(\cdot, w)|^2 e^{-\varphi_j}\) on \(K\). So we have for all large enough \(j\),

\[
\int_K |K_\varphi(\cdot, w)|^2 e^{-\varphi_j} d\lambda \rightarrow \int_K |K_\varphi(\cdot, w)|^2 e^{-\varphi} d\lambda
\]

in view of the Lebesgue dominated convergence theorem.

If \(\epsilon \ll 1\), then \(B_{R+1} \subset \{-\psi \leq \frac{1}{2\epsilon}\}\). Since \(u_{j,\epsilon}\) is holomorphic on \(\{-\psi \leq \frac{1}{2\epsilon}\}\), the mean value inequality yields

\[
|u_{j,\epsilon}(w)|^2 \leq C_n \int_{B_{R+1}} |u_{j,\epsilon}|^2 d\lambda
\]

\[
\leq C_{n,R} \int_{B_{R+1}} |u_{j,\epsilon}|^2 e^{-\varphi_j} d\lambda
\]

\[
\leq C_{n,R}' \int_{B_{R+1}} |u_{j,\epsilon}|^2 e^{-\varphi_j'} d\lambda
\]

\[
\leq \frac{C_{n,R}''}{\alpha_j} \epsilon^2 K_\varphi(w, w).
\]

It follows that

\[
f_{j,\epsilon} := \lambda \epsilon K_\varphi(\cdot, w) - u_{j,\epsilon}
\]

is an entire function satisfying

\[
|f_{j,\epsilon}(w)| \geq K_\varphi(w, w) - \frac{C_{n,R}}{\sqrt{\alpha_j}} \epsilon
\]

and

\[
\|f_{j,\epsilon}\|_{H(\varphi_j)} \leq \|K_\varphi(\cdot, w)\|_{L^2(K, \varphi_j)} + C \|u_{j,\epsilon}\|_{L^2(C, \varphi_j)}
\]

\[
\leq (1 + \frac{C}{\sqrt{\alpha_j}} \epsilon) \|K_\varphi(\cdot, w)\|_{L^2(K, \varphi_j)}
\]

\[
\leq (1 + \frac{C}{\sqrt{\alpha_j}} \epsilon) \sqrt{K_\varphi(w, w)}.
\]

Thus we have

\[
\frac{|f_{j,\epsilon}(w)|}{\|f_{j,\epsilon}\|_{H(\varphi_j)}} \geq \frac{K_\varphi(w, w) - C_{n,R,\alpha_j} \epsilon}{(1 + C_{\alpha_j} \epsilon) \sqrt{K_\varphi(w, w)}}
\]

that is

\[
\liminf_{j \to +\infty} K_{\varphi_j}(w, w) \geq K_\varphi(w, w).
\]

Since \(\varphi_j \leq \varphi\) we know that \(K_{\varphi_j}(w, w) \leq K_\varphi(w, w)\) for each \(j \geq 1\), thus we obtain

\[
\lim_{j \to +\infty} K_{\varphi_j}(w, w) = K_\varphi(w, w), \quad \forall w \in \mathbb{C}.
\]
This completes the proof. \[\square\]

**Proof of Theorem 1.7.** For each compact $F \subset \subset \mathbb{C}$, each fixed $w \in F$ and $z \in F$, according to the mean value inequality we know that

$$|K_{\varphi_j}(z, w) - K_{\varphi}(z, w)|^2$$

$$\leq C |K_{\varphi_j}(\cdot, w) - K_{\varphi}(\cdot, w)|^2_{H(U, 0)}$$

$$\leq C |K_{\varphi_j}(\cdot, w) - K_{\varphi}(\cdot, w)|^2_{H(U, \varphi)}$$

$$\leq C |K_{\varphi_j}(\cdot, w) - K_{\varphi}(\cdot, w)|^2_{H(\mathbb{C}, \varphi)}$$

$$= C \left( \int_{\mathbb{C}} |K_{\varphi_j}(\cdot, w)|^2 e^{-\varphi} d\lambda + \int_{\mathbb{C}} |K_{\varphi}(\cdot, w)|^2 e^{-\varphi} d\lambda - 2K_{\varphi_j}(w, w) \right)$$

$$\leq C \left( \int_{\mathbb{C}} |K_{\varphi}(\cdot, w)|^2 e^{-\varphi} d\lambda + K_{\varphi}(w, w) - 2K_{\varphi_j}(w, w) \right)$$

(5.1) $$= C(K_{\varphi_j}(w, w) - K_{\varphi_j}(w, w))$$

where $U$ is some neighborhood of the compact set $F$. By using the result of Theorem 1.4 we have $K_{\varphi_j}(z, w)$ pointwise converges to $K_{\varphi}(z, w)$ in $\mathbb{C} \times \mathbb{C}$. Similarly we have

$$\lim_{j \to \infty} K_{2(k+1)\varphi_j}(z, w) = K_{2(k+1)\varphi}(z, w) \quad \forall z, w \in \mathbb{C}. \tag{5.2}$$

On the other hand, from Ligocka’s formula

$$K_{\Omega_j}((z, t), (w, s)) = \sum_{k=0}^{\infty} 2(k + 1)K_{2(k+1)\varphi_j}(z, w) \langle t, s \rangle^k, \quad z, t, w, s \in \mathbb{C},$$

we can easily obtain that

$$2(k + 1)K_{2(k+1)\varphi_j}(z, w) = \frac{\partial^{2k}}{\partial t^{k} \partial s^{k}}K_{\Omega_j}([(z, t), (w, s)])|_{t=s=0}.$$ 

For each $(z_0, t_0) \in \Omega \subset \mathbb{C}^2$, there exist $r_1, r_2 > 0$ so that

$$(z_0, t_0) \in P := \Delta(z_0, r_1) \times \Delta(0, r_2) \subset \subset \Omega \subset \Omega_j, \quad j \geq 1.$$ 

Since $\varphi_j$ is increasing to $\varphi$ we know that for each $j \geq 1$

$$|K_{\Omega_j}([(z, t), (w, s)])| \leq K_{\Omega_j}([(z, t), (z, t)]^{\frac{1}{2}} K_{\Omega_j}([(w, s), (w, s)])^{\frac{1}{2}}$$

$$\leq K_{\Omega}([(z, t), (z, t)]^{\frac{1}{2}} K_{\Omega}([(w, s), (w, s)])^{\frac{1}{2}}.$$ 

Put $M := \sup_{j} \sup_{P} |K_{\Omega_j}([(z, t), (w, s)])|$, we have $M < +\infty$. According to the Cauchy integral formula we obtain

$$\left| \frac{\partial^{2k}}{\partial t^{k} \partial s^{k}}K_{\Omega_j}([(z, t), (w, s)]) \right|_{t=s=0} \leq C_k \frac{M}{r^{2k}}.$$
Let $0 < r_1' < r_1$, $0 < r_2' < r_2$, then for each $\epsilon > 0$, there exists $k_\epsilon \gg 1$, so that
\[
\sum_{k \geq k_\epsilon} 2(k + 1) \left| K_{2(k+1)\varphi_j}(z, w)(\overline{t})^k \right| < \epsilon, \quad \forall z, w \in \Delta(z_0, r_1'), \quad \forall t, s \in \Delta(0, r_2'),
\]
and
\[
\sum_{k \geq k_\epsilon} 2(k + 1) \left| K_{2(k+1)\varphi}(z, w)(\overline{t})^k \right| < \epsilon, \quad \forall z, w \in \Delta(z_0, r_1'), \quad \forall t, s \in \Delta(0, r_2').
\]

Thus we get
\[
\sum_{k=0}^{k_\epsilon} 2(k + 1) K_{2(k+1)\varphi_j}(z, w)(\overline{t})^k \rightarrow \sum_{k=0}^{k_\epsilon} 2(k + 1) K_{2(k+1)\varphi}(z, w)(\overline{t})^k
\]
by (5.2) we have $K_{\Omega_j}([z, t), (w, s)])$ pointwise converges to $K_{\Omega}([z, t), (w, s)])$ in $\Omega \times \Omega$. Because of (5.3) we know that the functions $K_{\Omega_j}([z, t), (w, s)])$ form a normal family in $\Omega \times \Omega$. From the normality and the pointwise convergence just proved, it is immediate that the convergence is uniform on compact subsets of $\Omega \times \Omega$. This completes the proof. □

**Acknowledgements** The authors were supported in part by the Norwegian Research Council grant number 240569, the first author was also supported by NSFC grant 11601120. The first author also thanks Professor Bo-Yong Chen for his valuable comments.

**References**

[1] Berndtsson, B.: Weighted estimates for the $\overline{\partial}$-equation. Complex Analysis and Geometry. Columbus, 1999. Ohio State University Mathematical Research Institute, 9, 43–57 (2001)
[2] Chen, B.Y.: Parameter dependence of the Bergman kernels. Adv. Math. 299, 108–138 (2016)
[3] Fornæss, J. E., and Wu, J. A Global Approximation Result by Bert Alan Taylor and the Strong Openness Conjecture in $\mathbb{C}^n$. J Geom Anal , 1–12 (2017)
[4] Ligocka E. On the Forelli-Rudin construction and weighted Bergman projections[J]. Studia Math, 94(3), 257–272(1989)
[5] Taylor, B.A.: On weighted polynomial approximation of entire functions. Pacific. J. Math. 36, 523–539 (1971)
[6] Guan, Q.A. and Zhou, X.Y., Strong openness conjecture and related problems for plurisubharmonic functions.[arXiv:1401.7158]
[7] Guan, Q.A. and Zhou, X.Y.: A proof of Demailly’s strong openness conjecture. Ann. Math. 182(2), 605–616 (2015)
[8] Guan, Q.A. and Zhou, X.Y.: Effectiveness of Demailly’s strong openness conjecture and related problems. Invent. math. 22, 635–676 (2015)
[9] Lempert, L.: Modules of square integrable holomorphic germs. Analysis meets geometry, 311-333, Trends Math., Birkhuser/Springer, Cham, 2017.
[10] Ransford, T.: Potential theory in the complex plane (Vol. 28). Cambridge university press (1995)
Corresponding author, Jujie Wu, E-mail address: jujie.wu@ntnu.no, School of Mathematics and Statistics, Henan University, Jinming Campus of Henan University, Jinming District, City of Kaifeng, Henan Province, P. R. China, 475001, Department of Mathematical Sciences, NTNU, Sentralbygg 2, Alfred Getz vei 1, 7034 Trondheim, Norway

John Erik Fornæss, E-mail address: john.fornass@ntnu.no, Department of Mathematical Sciences, NTNU, Sentralbygg 2, Alfred Getz vei 1, 7034 Trondheim, Norway