A SPECTRAL APPROACH TO YANG-MILLS THEORY

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Abstract. Yang–Mills theory in four dimensions is studied by using the Coulomb gauge. The Coulomb gauge Hamiltonian involves integration of matrix elements of an operator $\mathcal{P}$ built from the Laplacian and from a first-order differential operator. The operator $\mathcal{P}$ is studied from the point of view of spectral theory of pseudo-differential operators on compact Riemannian manifolds, both when self-adjointness holds and when it is not fulfilled. In both cases, well-defined matrix elements of $\mathcal{P}$ are evaluated as a first step towards the more difficult problems of quantized Yang–Mills theory.
1. Introduction

Although much progress has been made in the theoretical understanding of gauge theories of fundamental interactions,\textsuperscript{1−3} a number of outstanding unsolved problems remain. In particular, the mass gap problem has been stressed in the recent literature,\textsuperscript{4} which amounts to proving that, for any compact simple gauge group, quantum Yang–Mills theory on $\mathbb{R}^4$ has an Hamiltonian operator with no spectrum in the interval $(0, \delta)$ for some $\delta > 0$. One should then show that, starting from the classical action functional

$$I = -\frac{1}{4} \int \text{Tr}(F_{\mu\nu} F^{\mu\nu}) d^4x,$$

the corresponding Hamiltonian operator has energy spectrum bounded from below, with strictly positive lower bound. The solution of such a problem is very important not only to achieve internal consistency of the mathematical formalism, but also (if not mainly) for physical reasons: the existence of a mass gap would account for the nuclear force being strong but short-ranged. Moreover, the recent theory of glueballs\textsuperscript{5} relies entirely on action functionals like (1), without fermionic fields. In plain terms, it is possible for two or more gluons to combine into a strongly bound, neutral-coloured particle of pure glue. This (hypothetical) object is called a glueball.\textsuperscript{5} Moreover, a gluon can bind with a meson to form a hybrid. For example, a red quark and an anti-blue antiquark can bind with a blue/antired gluon. The lightest glueball allowed by quantum chromodynamics can be described by a circular tube of glue and has vanishing angular momentum.\textsuperscript{5} This has spherical symmetry while glueballs of other, elongated shapes have non-vanishing angular momenta and larger masses. Hybrids can announce their presence by yielding at least three $s$-wave mesons. They are in fact predicted to decay into one $s$-wave meson and another short-lived meson with internal angular momentum. The latter then breaks up into two $s$-wave mesons. The first (though uncertain) experimental evidence in favour of hybrids was obtained in 1994, when experimenters at Protvino found an object called $\pi(1800)$, emerging from collisions of pions with protons. This particle has the quantum properties and decay pattern expected for a hybrid. Recent research deals with special hybrids called exotics, which have combinations of internal angular momentum, parity and charge conjugation.
quantum numbers that are forbidden for mesons. For example, the simplest exotic has $J = 1$, $P = -1$ and $C = 1$. Inconclusive evidence also exists in favour of glueballs called $f_0(1500)$ and $f_2(1710)$, which would belong to the class of glueballs of mass in the range 1.500 to 1.800 MeV.\textsuperscript{5} The latest experiments tell us that glueball candidates and $q\bar{q}$ mesons have been found to be produced with different momentum and angular dependences in the central region of $pp$ collisions.\textsuperscript{6}

All this phenomenology can be described, in principle, with the help of a Yang–Mills Lagrangian. We have been therefore motivated, in our research, by the mass gap problem in four dimensions, although it will become clear that, for the time being, we only have a possibly new perspective in classical Yang–Mills theory. For this purpose, we have considered the Yang–Mills Lagrangian in the Coulomb gauge with the associated decomposition into electric and magnetic parts. Although the Coulomb gauge is non-covariant, it has the merit of leading to a quantum theory which is manifestly unitary. Moreover, well-established calculational recipes are available for writing down the Hamiltonian operator of the quantum theory,\textsuperscript{7} while recent work has shown that the Coulomb gauge can be viewed as the singular limit of the Landau–Coulomb interpolating gauge,\textsuperscript{8} adding therefore evidence in favour of such a gauge being very appropriate for the quantum theory.

Section 2 presents a review of the Coulomb gauge Hamiltonian, with emphasis on matrix elements of an operator $\mathcal{P}$ which plays a key role in Sec. 3. Here, attention is focused on compact Euclidean spacetime, for which discrete spectral resolutions of self-adjoint elliptic operators exist. The matrix elements of $\mathcal{P}$ among square-integrable functions are evaluated explicitly in such a case. Section 4 presents an assessment of the spectral approach to the mass gap.

2. Coulomb Gauge Hamiltonian

To help the general reader, we present here a brief review of the Coulomb gauge Hamiltonian in classical and quantum theory, relying on Ref. 7.
The Lagrangian $L$ of a classical Yang–Mills field can be expressed in terms of electric and magnetic fields, which are defined by

$$\vec{E}_i \equiv -\frac{d}{dt} \vec{A}_i - (\nabla_i + g \vec{A}_i \times) \vec{A}_0 = -\frac{d}{dt} \vec{A}_i + \frac{1}{g} D_i \vec{\omega},$$  \hspace{1cm} (2)$$

$$\varepsilon_{jki} \vec{B}_i \equiv \nabla_j \vec{A}_k - \nabla_k \vec{A}_j + g \vec{A}_j \times \vec{A}_k.$$  \hspace{1cm} (3)$$

In Eq. (2), $\vec{A}_0 = -\frac{\vec{\omega}}{g}$, and $D_i$ is the covariant derivative in the Coulomb gauge:

$$D_i \equiv \nabla_i + g \vec{A}_i \times,$$  \hspace{1cm} (4)$$

the cross denoting the isovector product in all our equations. Thus, the Lagrangian reads

$$L = \frac{1}{2} \int (\vec{E}_i \cdot \vec{E}_i - \vec{B}_i \cdot \vec{B}_i) d^3r,$$  \hspace{1cm} (5)$$

with corresponding Hamiltonian

$$H = \frac{1}{2} \int (\vec{E}_i \cdot \vec{E}_i + \vec{B}_i \cdot \vec{B}_i) d^3r.$$  \hspace{1cm} (6)$$

One can now exploit the constraint $\nabla_j \vec{A}_j = 0$ and decompose the electric field into a transverse part and a gradient term, i.e.

$$\vec{E}_i = \vec{E}_i^{tr} - \nabla_i \vec{\phi},$$  \hspace{1cm} (7)$$

where (hereafter we define the Laplacian as the operator $\Delta \equiv -\nabla^k \nabla_k$, with a minus sign in front of second derivatives to make it bounded from below)

$$\vec{E}_i^{tr} = -\frac{d}{dt} \vec{A}_i + \left( \delta_{ij} + \Delta^{-1} \nabla_i \nabla_j \right) (\vec{A}_j \times \omega),$$  \hspace{1cm} (8)$$

which satisfies $\nabla_i \vec{E}_i^{tr} = 0$, and

$$\vec{\phi} = \vec{A}_0 - g \Delta^{-1} (\vec{A}_j \times \nabla_j \vec{A}_0).$$  \hspace{1cm} (9)$$
Since the electric field has vanishing divergence with respect to the connection $\mathcal{D}$, i.e. $\mathcal{D}_i \vec{E}_i = 0$, one gets also an equation for $\vec{\phi}$, i.e.

$$-\vec{A}_i \times \vec{E}_i^{\text{tr}} + \frac{1}{g} \nabla_i \mathcal{D}_i \vec{\phi} = 0.$$  \hspace{1cm} (10)

Such an equation is solved upon inverting the operator $\nabla_i \mathcal{D}_i$, which yields (summation over repeated indices is understood)

$$\vec{\phi} = g(\nabla_k \mathcal{D}_k)^{-1} \vec{A}_i \times \vec{E}_i^{\text{tr}} = -g(\nabla_k \mathcal{D}_k)^{-1} \vec{A}_i \times \vec{\Pi}_i^{\text{tr}}.$$  \hspace{1cm} (11)

At this stage our analysis is still classical, but in the quantum theory the counterpart of Eq. (11) is more involved because zero-modes of the Faddeev–Popov operator play a role and their effect should be included.

Upon exploiting the transverse nature of $\vec{E}_i^{\text{tr}}$ and imposing fall-off conditions at infinity on $\vec{\phi}$, the classical Hamiltonian reads

$$H = \frac{1}{2} \int \left[ (\vec{E}_i^{\text{tr}})^2 + (\nabla_i \vec{\phi})^2 + (\vec{B}_i)^2 \right] d^3r.$$  \hspace{1cm} (12)

This is further simplified by using the identity

$$(\nabla_i \vec{\phi})(\nabla_i \vec{\phi}) = \nabla_i (\vec{\phi} \nabla_i \vec{\phi}) + \vec{\phi} \nabla \vec{\phi},$$  \hspace{1cm} (13)

jointly with Eq. (11). Thus, on defining

$$\vec{\Pi}_i^{\text{tr}},$$  \hspace{1cm} (14)

which represents the charge carried by $\vec{A}_i$, the classical Hamiltonian reads eventually

$$H = \frac{1}{2} \int \left[ (\vec{\Pi}_i^{\text{tr}})^2 + (\vec{B}_i)^2 \right] d^3r$$

$$+ \frac{g^2}{2} \int \sigma'_A(\vec{r}) |l, r\rangle \mathcal{P} |l', r'\rangle \sigma''_A(\vec{r}') d^3rd^3r'.$$  \hspace{1cm} (15)
In Eq. (15), $\mathcal{P}$ is the integro-differential operator defined by

$$\mathcal{P} \equiv (\nabla_i D_i)^{-1} \triangle (\nabla_j D_j)^{-1},$$

(16)

where the inverse of $\nabla_i D_i$ results from (11), and the Laplacian results from (13). The scheme obtained from (15) is not amenable to calculation unless one finds a convenient way of expressing the inverse of $\nabla_i D_i$. For this purpose, one defines

$$\Lambda \equiv \nabla_i (A_i \times) = A_i \times \nabla_i,$$

(17)

so that

$$\nabla_i D_i = -\triangle + g\Lambda,$$

(18)

and hence

$$(\nabla D_i)^{-1} = (-\triangle + g\Lambda)^{-1} = -\triangle^{-1} \left( I + g\Lambda\triangle^{-1} + (g\Lambda\triangle^{-1})^2 + O(g^3) \right)$$

$$= -\triangle^{-1} - g\triangle^{-1} \Lambda \Lambda^{-1} - g^2 \triangle^{-1} \Lambda \triangle^{-1} \Lambda \triangle^{-1} + O(g^3),$$

(19)

where we have applied the formula

$$(AB)^{-1} = B^{-1}A^{-1}$$

to the operators $A \equiv I - g\Lambda\triangle^{-1}$ and $B \equiv -\triangle$. The insertion of the (formal) expansion (19) into the definition (16) yields an algorithm for $\mathcal{P}$, i.e.

$$\mathcal{P} = \triangle^{-1} + \sum_{k=1}^{\infty} (k+1)g^k \triangle^{-1} (\Lambda\triangle^{-1})^k,$$

(20)

which is useful at small $g$ and for the analysis of spectral asymptotics.

In the quantum theory, one has instead to consider the un-renormalized coupling constant $g_0$ and the Faddeev–Popov determinant $^7$

$$\gamma \equiv \det(\nabla_i D_i).$$

(21)
The equal-time canonical commutation relations read (the smeared form is more rigorous but inessential for our purposes, which are not axiomatic)

\[
\left[ A_j(\vec{r}, t), \Pi^r_k(\vec{r}', t) \right] = i\delta^{lm} \left( \delta_{jk} + \Delta^{-1} \nabla_j \nabla_k \right) \delta^3(\vec{r} - \vec{r}'),
\]

and the Hamiltonian operator in the Coulomb gauge takes the form\(^7\)

\[
\hat{H} = \int \left[ \frac{1}{2} \gamma^{-1} \tilde{\Pi}^r_i \cdot \gamma \tilde{\Pi}^r_i + \frac{1}{2} B_i^2 \right] d^3r
\]

\[
+ \frac{g_0^2}{2} \int \gamma^{-1} \sigma^l(\vec{r}) \langle l, \vec{r} | \mathcal{P} | l', \vec{r}' \rangle \gamma \sigma^{l'}(\vec{r}') d^3rd^3r'.
\]

Note that, since quantum fields are operator-valued distributions,\(^9\) the Hamiltonian as in the form just written is not defined as an operator in a Hilbert space. Strictly, the product of local operators should be regularized, and all quantum formulae should be written with this understanding.

3. Structure of \(\mathcal{P}\)

Our ultimate goal is the investigation of the spectrum of \(\hat{H}\), with the associated role played by \(\gamma, \tilde{\Pi}^r_i, \tilde{B}_i, \sigma^l\). But this is still extremely difficult, and hence we resort to the analysis of the operator \(\mathcal{P}\) (see Eq. (16)) occurring in the classical theory, here written in the form

\[
\mathcal{P} = (\Delta - g\Lambda)^{-1} \Delta (\Delta - g\Lambda)^{-1},
\]

where one should bear in mind that \(\Lambda\) is the first-order differential operator defined in Eq. (17). Now two main cases can be distinguished in a framework where Minkowski space-time is replaced by a compact four-geometry \((M, g)\) without boundary. This is more relevant for Euclidean field theory, which is nevertheless an important branch of quantum field theory.\(^10,11\) By doing so, one may hopefully learn lessons about operators whose analysis is a mandatory step before being able to solve the original problem.
(i) Assume first that the operator $\mathcal{P}$ is self-adjoint. Since, under the above assumptions on $(M, g)$, the Laplacian $\triangle$ is self-adjoint, this means we are treating the isovector product $\vec{A}_i \times \nabla_i$ as a self-adjoint operator (see definition (17)), so that

$$
\mathcal{P}^\dagger = (\triangle^\dagger - g\Lambda^\dagger)^{-1} \triangle^\dagger (\triangle^\dagger - g\Lambda^\dagger)^{-1} = \mathcal{P}.
$$

(26)

One can then exploit theorems ensuring that the eigenfunctions $u_l$ of $\mathcal{P}$ form a complete orthonormal set. Any square-integrable function $\varphi \in L^2(M)$ can be then expanded according to ($c_l$ being the Fourier coefficients $c_l \equiv (u_l, \varphi)$)

$$
\varphi = \sum_{l=1}^{\infty} c_l u_l.
$$

(27)

The resulting mean value of $\mathcal{P}$ reads ($\lambda_l$ being its eigenvalues, for which $\mathcal{P} u_l = \lambda_l u_l$)

$$
(\varphi, \mathcal{P}\varphi) = \sum_{l, r=1}^{\infty} \lambda_l c_r^* c_l (u_r, u_l) = \sum_{l=1}^{\infty} \lambda_l |c_l|^2,
$$

(28)

while more general matrix elements read (with $b_l \equiv (u_l, \Phi)$)

$$
(\Phi, \mathcal{P}\varphi) = \sum_{l=1}^{\infty} \lambda_l b_l^* c_l,
$$

(29)

for all $\Phi$ and $\varphi \in L^2(M)$. The mean value of $\mathcal{P}$ is therefore positive if its spectrum is bounded from below, with positive lower bound.

(ii) Even when $(M, g)$ is compact, the operator $\triangle - g\Lambda$ may fail to be self-adjoint, since $\Lambda$ contains the effect of first-order covariant derivatives, which may be anti-self-adjoint. If this were the case, we can nevertheless exploit the expansion (20) to point out that $\mathcal{P}$ has the general structure

$$
\mathcal{P} = \triangle^{-1} + \mathcal{P}',
$$

(30)

where $\mathcal{P}'$ is a pseudo-differential operator (see Appendix) of order $-3$, since the lowest value of $k$ in the infinite sum in Eq. (20) involves the operator $\triangle^{-1} \Lambda \triangle^{-1}$. One can then
rely upon the work of Beals,\textsuperscript{12} who has studied spectral asymptotics of elliptic operators with a self-adjoint positive principal part, showing that the eigenvalues lie in a parabolic region around the positive axis. In our case the principal part of $\mathcal{P}$ is the inverse of the Laplacian, which is both self-adjoint and positive. Such a qualitative information should be supplemented by well known results about spectral resolutions of partial differential operators of positive order on compact manifolds.\textsuperscript{13} In other words, lack of self-adjointness of $\mathcal{P}$ makes it now impossible to expand $\varphi \in L^2(M)$ according to Eq. (27), while an expansion of $\varphi$ remains legitimate if a discrete spectral resolution of the Laplacian is used. For this purpose, let us denote by $v_l$ the eigenfunctions of the Laplacian with discrete eigenvalues $\mu_l$, for which

$$\triangle v_l = \mu_l v_l,$$

and ($C_l$ being the Fourier coefficient ($v_l, \varphi$))

$$\varphi = \sum_{l=1}^{\infty} C_l v_l. \tag{32}$$

With the help of the expansion (20), which is useful at small $g$, the mean value of the operator $\mathcal{P}$ is then found to take the form

$$\langle \varphi, \mathcal{P} \varphi \rangle = \sum_{l=1}^{\infty} \frac{1}{\mu_l} |C_l|^2$$

$$+ \sum_{l,r=1}^{\infty} C_r^* C_l \sum_{k=1}^{\infty} (k + 1) g^k \frac{1}{\mu_r} (v_r, (\Lambda \triangle^{-1})^k v_l). \tag{33}$$

Such a formula shows how the mean value of $\mathcal{P}$ changes on passing from the self-adjoint (see Eq. (28)) to the non-self-adjoint case. The formula (33) cannot be further simplified, because the operator $\Lambda$ does not commute with the inverse of the Laplacian. The first few powers of $\Lambda \triangle^{-1}$ in the sum over all positive values of the integer $k$ yield, for example,

$$\Lambda \triangle^{-1} v_l = \frac{1}{\mu_l} \Lambda v_l,$$
\[(\Lambda \triangle^{-1})^2 v_l = \frac{1}{\mu_l} \Lambda \triangle^{-1} \Lambda v_l,\]

and such formulae are useful if one needs to stop the analysis of \((\varphi, \mathcal{P}\varphi)\) at terms of order \(O(g^2)\). If the formula (33) is used, positivity of the mean value of \(\mathcal{P}\) amounts to proving that the following inequality holds:

\[
\sum_{l=1}^{\infty} \frac{1}{\mu_l} |C_l|^2 > - \sum_{l,r=1}^{\infty} C_r^* C_l \sum_{k=1}^{\infty} g^k \frac{1}{\mu_r} (v_r, (\Lambda \triangle^{-1})^k v_l). \tag{34}
\]

Similarly, for all \(\Phi\) and \(\varphi \in L^2(M)\), the formula (29) for the matrix element \((\Phi, \mathcal{P}\varphi)\) is now replaced by (with our notation, \(B_l \equiv (v_l, \Phi)\))

\[
(\Phi, \mathcal{P}\varphi) = \sum_{l=1}^{\infty} \frac{1}{\mu_l} B_l^* C_l
\]

\[
+ \sum_{l,r=1}^{\infty} B_r^* C_l \sum_{k=1}^{\infty} (k + 1) g^k \frac{1}{\mu_r} (v_r, (\Lambda \triangle^{-1})^k v_l). \tag{35}
\]

The effort of studying both self-adjoint and non-self-adjoint case for \(\mathcal{P}\) is especially valuable if one looks at sub-regions of \(M\) with smooth boundary, since then the particular boundary conditions chosen might or might not ensure self-adjointness of \(\mathcal{P}\).

4. Status of the Spectral Approach

In our paper we start from a physical motivation to move gradually towards the framework of pseudo-differential operators and their relevance for the Hamiltonian of classical Yang–Mills theory but on a background manifold with positive-definite metric. We have proposed a spectral-oriented perspective on the classical foundations of an outstanding open problem in modern quantum field theory, and we have evaluated the well-defined matrix elements of the fundamental operator \(\mathcal{P}\) in the self-adjoint and non-self-adjoint case. This step is important, since otherwise no hope exists of being able to evaluate the matrix elements occurring in the quantum theory. In particular, the use of discrete spectral resolutions of the Laplacian for studying matrix elements of \(\mathcal{P}\) in the non-self-adjoint case is a simple
but non-trivial technical point that we have advocated. At least three major problems are now in sight.

(i) What happens if the compact Riemannian four-manifold $M$ is replaced by a Lorentzian four-manifold, e.g. Minkowski space-time. Such a “decompactification limit” is crucial to recover the analysis of the original problem we are interested in.

(ii) What happens when the classical Hamiltonian (15) is replaced by the quantum Hamiltonian (24), with $|l, r\rangle$ being the eigenfunctionals of the position operator.

(iii) How to make manifest the role played by the gauge group in the quantum theory (in particular, the group structure of the operators $\Lambda$ and $P$).

We hope, however, that the spectral approach will at least show clearly the limits of what can be done to solve the mass gap problem, and it is our aim to devote our efforts to understand whether this is really the case.

Appendix

To help the reader who is not familiar with spectral theory, we here recall some basic concepts. A linear partial differential operator $P$ of order $d$ can be written in the form

$$P \equiv \sum_{|\alpha| \leq d} a_\alpha(x) D^\alpha_x,$$  \hspace{1cm} (A1)

where $|\alpha| \equiv \sum_{k=1}^{m} \alpha_k$, and

$$D^\alpha_x \equiv (-i)^{|\alpha|} \left( \frac{\partial}{\partial x_1} \right)^{\alpha_1} \cdots \left( \frac{\partial}{\partial x_m} \right)^{\alpha_m},$$  \hspace{1cm} (A2)

with $a_\alpha$ a $C^\infty$ function on $\mathbb{R}^m$ for all $\alpha$. The associated symbol is, by definition,

$$p(x, \xi) \equiv \sum_{|\alpha| \leq d} a_\alpha(x) \xi^\alpha,$$  \hspace{1cm} (A3)

i.e. it is obtained by replacing the differential operator $D^\alpha_x$ by the monomial $\xi^\alpha$. The pair $(x, \xi)$ may be viewed as defining a point of the cotangent bundle of $\mathbb{R}^m$, and the action of
$P$ on the elements of the Schwarz space $\mathcal{S}$ of smooth complex-valued functions on $\mathbb{R}^m$ of rapid decrease is given by\textsuperscript{13}

$$P f(x) \equiv \int e^{i(x-y) \cdot \xi} p(x, \xi) f(y) dy d\xi,$$

(A4)

where the $dy = dy_1...dy_m$ and $d\xi = d\xi_1...d\xi_m$ orders of integration cannot be interchanged, since the integral is not absolutely convergent.

Pseudo-differential operators are instead a more general class of operators whose symbol need not be a polynomial (and whose order is not necessarily positive) but has suitable regularity properties. More precisely, let $S^d$ be the set of all symbols $p(x, \xi)$ such that $p$ is smooth in the pair of variables $(x, \xi)$ with compact $x$ support, and for all $(\alpha, \beta)$ there exist constants $C_{\alpha,\beta}$ for which

$$|D_x^\alpha D_\xi^\beta p(x, \xi)| \leq C_{\alpha,\beta} (1 + |\xi|)^{d-|\beta|},$$

(A5)

for some real (not necessarily positive) value of $d$. The associated pseudo-differential operator, defined on the Schwarz space and taking values in the set of smooth functions on $\mathbb{R}^m$ with compact support,

$$P : \mathcal{S} \to C^\infty_c(\mathbb{R}^m)$$

is defined in a way formally analogous to the previous integral formula for $P f(x)$.

Ellipticity can be defined by means of a majorization obeyed by the inverse of the modulus of the symbol. In other words, let $U$ be an open subset with compact closure in $\mathbb{R}^m$, and consider an open subset $U_1$ whose closure $\overline{U}_1$ is properly included in $U : \overline{U}_1 \subset U$. If $p$ is a symbol of order $d$ on $U$, it is said to be elliptic on $U_1$ if there exists an open set $U_2$ which contains $\overline{U}_1$ and positive constants $F_i$ so that

$$|p(x, \xi)|^{-1} \leq F_1 (1 + |\xi|)^{-d}$$

(A6)

for $|\xi| \geq F_0$ and $x \in U_2$, where

$$|\xi| \equiv \sqrt{g^{ab}(x) \xi_a \xi_b} = \sqrt{\sum_{k=1}^m \xi_k^2}.$$

(A7)
The corresponding operator $P$ is then said to be elliptic.

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