Nonlinear pressure-velocity waveforms in large arteries, shock waves and wave separation

Oleg Ilyin

Dorodnicyn Computing Centre of Russian Academy of Sciences, Vavilova st. 40, 119333 Moscow, Russia

Abstract

The nonlinear inviscid 1D blood flow equations are studied analytically using the method of characteristics. The boundary value problem with a triangle-shaped boundary data at the aortic outlet is considered. The pressure-velocity profile, the shock conditions are derived as closed analytical expressions. Finally, the fully nonlinear wave separation expressions are obtained using the Riemann invariants. The results show good correspondence with the data from literature.

Keywords: Biological fluid dynamics, nonlinear waves

1. Introduction

The most popular 1D hemodynamic pressure-velocity propagation models are based on the quasilinear partial differential equations of the hyperbolic type [1] - [3]. This system of equations describe the blood motion in distensible vessels in terms of the unknown pressure and the blood velocity (or the vessel’s cross section area and the blood velocity). The solutions to 1D equations are usually obtained numerically for the branching arterial network [4] - [8] and they can be coupled with the solutions of the full 3D Navier-Stokes equations [9]. Since the difference ∆A between the systolic luminal area A₁ and the diastolic A₀ is small in many cases (∆A/A₀ is assumed to be negligible) then the equations can be linearized around A₀. The reduced equations are well known in sound propagation theory [10]. The linear theory allows to assess several properties of the flow which are important for the estimation of cardiovascular risk. Any change in the geometry of a vessel (bifurcation, lumen narrowing and other factors changing vascular impedance) causes an appearance of reflected waves. Simple and concise formulas for the wave separation into the forward and the backward waves can be obtained in the framework of the linear theory using the wave separation analysis [11] - [14] or the wave intensity analysis [13] - [15].

The perturbations travel along constant and parallel characteristics in the linear approximation therefore the waves do not change its initial form and...
shock-waves do not appear. Nevertheless, the nonlinear effects can play significant role in the behavior of pressure-velocity waves. For instance, the reported difference between the diameters of large vessels like a carotid artery in systole and diastole is 10% [16], then the relative difference between the lumen areas is about $\Delta A / A_0 \approx 0.2$. Hence, in general case the full nonlinear analysis should be performed. The existence of shock-waves has been proved numerically in [17], the formation of shock waves at distances between few centimeters and several meters from the aortic outlet depending on the elastic properties has been reported in [18]. The possibility of roll waves in collapsible tubes has been considered in [19]. The effects of friction coupled with nonlinearities has been studied in [21]-[22].

The goal of the paper is derive the closed analytical expression for the blood pressure, the luminal area and the velocity using the nonlinear equations for the mass and moment transfer. In contrast to the linear theory, where $\Delta A / A_0$ is assumed to be negligible, we will keep this terms but neglect the next terms, quadratic ones $(\Delta A / A_0)^2$. This assumption allows to keep main features of the nonlinear theory since the characteristics can converge or diverge. The analytical expression for the pressure-velocity waves is derived providing the quantitative information about the evolution of waves in time and space. It is shown that the wave significantly changes its shape while traveling along a vessel and shock waves can emerge. Moreover, with use of the Riemann invariants the closed nonlinear wave separation formulas are presented, the analytical expressions for the forward and backward waves are obtained.

2. Analytical solutions to 1D system for inviscid blood flow and shock wave formation

We consider the mass and the momentum conservation equations for inviscid blood motion in a distensible vessel with impermeable walls [3]

$$\frac{\partial A}{\partial t} + \frac{\partial Q}{\partial x} = 0,$$
$$\frac{\partial Q}{\partial t} + \frac{\partial}{\partial x} \left( \alpha Q^2 \frac{A}{A} \right) = -\frac{A}{\rho} \frac{\partial p}{\partial x},$$

(1)

where $Q,A,u$ are flow, cross-section area of a vessel and blood velocity respectively, $\alpha$ is profile shape factor and $\rho$ is blood density. We introduce the distensibility $D(A)$ which is by the definition [3]

$$D(A) = \frac{1}{A} \frac{\partial A}{\partial p}. $$

(2)

We suppose that the distensibility $D(A)$ is known function of $A$. In most applications this relationship is as follows

$$D(A) = \frac{D_0}{A^n}, \quad n > 0.$$

The velocity profile the blood is assumed to be flat $\alpha = 1$ and therefore $Q = uA$. From the definition of the distensibility [2] we have that

$$\frac{\partial p}{\partial x} = \frac{1}{D(A)A} \frac{\partial A}{\partial x}.$$
Then in terms of $u, A$ the equations (1) reduce to
\[ \frac{\partial A}{\partial t} + u \frac{\partial A}{\partial x} + A \frac{\partial u}{\partial x} = 0, \quad \frac{\partial u}{\partial t} + \frac{1}{\rho D(A)} A \frac{\partial A}{\partial x} + u \frac{\partial u}{\partial x} = 0. \quad (3) \]

For a sole forward traveling wave we have the following relation between the area of the vessel and the velocity (see the next paragraph)
\[ u = \int_{A_0}^A \frac{ds}{\sqrt{\rho D(s)}}. \quad (4) \]
where $A_0$ is the undisturbed vessel cross-section area (for the backward traveling wave we should use the substitute $u = -\int_{A_0}^A \frac{ds}{\sqrt{\rho D(s)}}$). Then the system of the equations (3) takes the form
\[ \frac{\partial A}{\partial t} + \left( \int_{A_0}^A \frac{ds}{\sqrt{\rho D(s)}} + \frac{1}{\sqrt{\rho D(A)}} \right) \frac{\partial A}{\partial x} = 0. \]

Now taking into account that $D(A) = D_0/A^n$ we finally obtain the equation
\[ \frac{\partial A}{\partial t} + \frac{1}{\sqrt{\rho D_0}} \left( \left( 1 + \frac{2}{n} \right) A^{n/2} - \frac{2}{n} A_0^{n/2} \right) \frac{\partial A}{\partial x} = 0. \quad (5) \]
which we solve jointly with the following triangle-shaped boundary condition (Fig. 2) at $x = 0$ for $t \in [0, T_0]$ where $T_0$ is one heartbeat
\[ A(t, x)|_{x=0} = A_0 + at, \quad t \in [0, t_0), \quad (6) \]
\[ A(t, x)|_{x=0} = A_0 + b(t_1 - t), \quad b = \frac{at_0}{t_1 - t_0}, \quad t \in [t_0, t_1], \quad (7) \]
\[ A(t, x)|_{x=0} = A_0, \quad t \in (t_1, T_0] \quad (8) \]
and
\[ A(t, x)|_{t=0} = A_0, \quad x > 0. \quad (9) \]
In practice $T_0 \approx 1s, t_1 \approx 0.3s$. The conditions (6)-(7) correspond to systole and (8) to diastole.

Previously the nonlinear study of the blood motion equations with use of the method of characteristics was presented in several papers [20]-[23]. The analytical expression for the pressure-velocity (or area-velocity) waves has not been presented before in the nonlinear case then our goal is to solve analytically the problem (3) and (6)-(9). We have for the equation (3) the formal solution along the characteristics
\[ A = \text{const}, \quad \frac{dt}{dx} = \frac{n\sqrt{\rho D_0}}{(n+2)A^{n/2} - 2A_0^{n/2}}. \]

Note that $A$ takes constant values on the characteristics $t(x)$. We can parameterize each of the characteristic curves by the value of $t(x)$ at $x = 0$ (integration
constant), thus we define \( t(0) \equiv s \). Now the parameter \( s \) marks the different characteristics. Then, after integration, we have

\[
A = f(s), \quad t = \frac{n\sqrt{\rho D_0}}{(n + 2)f(s)^{n/2} - 2A_0^{n/2}}x + s. \quad (10)
\]

Excluding the parameter \( s \) in the last expression we obtain

\[
A = f \left( t - \frac{n\sqrt{\rho D_0}x}{(n/2)A^{n/2} - 2A_0^{n/2}} \right).
\]

Now we are ready to obtain the explicit expressions for \( A \) using the boundary conditions (6)-(7). For \( t \in [0, t_0) \) we conclude that \( f(\cdot) \) takes the form

\[
f(z) = A_0 + az,
\]

then

\[
A = A_0 + a \left( t - \frac{n\sqrt{\rho D_0}x}{(n/2)A^{n/2} - 2A_0^{n/2}} \right).
\]

In general case, this transcendental equation can not be solved analytically. Then we adopt the following simplification. We assume that

\[
A^{n/2} = (A_0 + B)^{n/2} \approx A_0^{n/2} + (n/2)A_0^{n/2-1}B,
\]

then

\[
B = a \left( t - \frac{x}{c_0(1 + \frac{n+2}{2A_0}B)} \right),
\]

where \( c_0 = \sqrt{A_0^2/\rho D_0} \) is the pulse wave velocity in the linear theory. It is obvious that we have neglected the terms of the order \( O((\Delta A/A_0)^2) \). Solving the quadratic algebraic equation for \( B \) we finally deduce that

\[
A = A_0 + \frac{-(1 - akt) + \sqrt{(1 - akt)^2 - 4ak(c_0^{-1}x - t)}}{2k}, \quad (11)
\]

where \( k \equiv \frac{n+2}{2A_0} \). The expression (11) gives the solution of the considered problem for the boundary condition (6).

Similarly, the boundary condition (12) leads to the solution

\[
A = A_0 + \frac{-(1 - bk(t_1 - t)) + \sqrt{(1 - bk(t_1 - t))^2 + 4bk(c_0^{-1}x + (t_1 - t))}}{2k}, \quad (12)
\]

Now we need to obtain the conditions for the appearance of a shock wave. Since we have the solutions in explicit form then the shock wave appears for the
curve on the plane \((x,t)\) where \(f_x\) or \(f_t\) are infinite. For (11) we have the curve \(C_s\)
\[
C_s : 1 + akt = \sqrt{\frac{4a}{c_0}kx}.
\]
We conclude that the shock wave appears for the first time when the characteristic \(x = c_0t\) crosses \(C_s\). This happens at the point \(t_s\) defined by
\[
1 + akt_s = \sqrt{4akt_s},
\]
then
\[
ts_s = \frac{2A_0}{(n+2)a}, \quad x_s = c_0t_s.
\] (13)
If we assume that \(c_0\) lies in range from \(3 \text{ms}^{-1}\) to \(10 \text{ms}^{-1}\) and \(at_0 = 0.2A_0\) where \(t_0 = 0.15s\), then we have for the distance of shock \(x_s\) the estimate varying from \(1.8 \text{m}\) to \(6 \text{m}\) which similar to the one in [18].

The characteristics starting from \(x = 0, t \in [t_0, t_1]\) do not intersect each other and do not formate shock waves. Nevertheless, the shock waves can emerge in the domain 2 since the characteristics from the domain 1 travels to the domain 2 and meet the characteristics starting from \(x = 0, t \in [t_0, t_1]\). Obviously, this happens for \(x > x_s\) and \(t > t_s\). Therefore, if we consider the domain \(x < x_s\) then the absence of shock waves is guaranteed for the both domains (Fig. 1).

The typical evolution of waveform is presented in Fig. 2. Finally, compiling all the previous results we obtain the following proposition

The problem (3)-(7) has the following solution
\[
A(t,x) \approx A_0 + \frac{(1 - akt) + \sqrt{(1 - akt)^2 - 4ak(c_0^{-1}x - t)}}{2k},
\] (14)
which is valid for
\[
x < x_s \equiv c_0 \frac{2A_0}{(n+2)a}, \quad c_0t \geq x \geq c_0(1 + kat_0)(t - t_0)
\]
and
\[
A \approx A_0 + \frac{(1 - bk(t_1 - t)) + \sqrt{(1 - bk(t_1 - t))^2 + 4bk(c_0^{-1}x + (t_1 - t))}}{2k},
\] (15)
which is valid for
\[
x < x_s \equiv c_0 \frac{2A_0}{(n+2)a}, \quad c_0(1 + kat_0)(t - t_0) > x \geq c_0(t - t_1),
\]
where
\[
k \equiv \frac{n+2}{2A_0}, \quad c_0 \equiv \sqrt{\frac{A_0^n}{\rho D_0}}.
\]
Figure 1: The characteristics for the problem (5)-(9). The domain 1 corresponds to the characteristics starting from \( x = 0, t \in [0, t_0] \), the domain 2 corresponds to the characteristics starting from \( x = 0, t \in [t_0, t_1] \), where \( t_0 \) is the systole duration, \( x = 0 \) is the position of the aortic root, \( c_0 \) is the pulse wave velocity in the linear theory. The shock wave appears on the curve \( C_s \).
Figure 2: The initial symmetrical triangle shaped pulse-wave form at the aortic root (left picture) significantly changes its form after the wave travels for the time period of 0.35 s (right picture). The initial pulse wave has the physiological duration of 0.3s and \(c_0 = 4m/s\), the radius of the vessel equals \(1.5 \times 10^{-2}m\), \(A \approx 7 \times 10^{-4}m^2\). Note that the moment when the considered pulse-wave becomes a shock wave equals 0.4s. Non-symmetrical initial waves can also be considered using the formulas (11)-(12).

The blood velocity \(v_x \equiv u\) is calculated from the linearized version of the formula (3)

\[
u(t,x) = \frac{2}{n\sqrt{\rho D_0}}(A^{n/2} - A_0^{n/2}) \approx c_0 \frac{A - A_0}{A_0}
\]

and

\[
p(t,x) = p_0 + \frac{1}{nD_0}(A^n - A_0^n) \approx p_0 + \rho c_0^2 \frac{A - A_0}{A_0}.
\]

This solutions have the error of order \(O((\Delta A/A)^2)\) for \(n \neq 2\) and are exact for \(n = 2\). The backward wave can be obtained from the formulas above by replacing \(c_0^{-1}x\) with \(-c_0^{-1}x\).

Finally, let us mention one important fact. Since all the terms in the equations (1) (or (3)) depend on \(t, x\) only via \(p, A\) (or \(u, A\)) then the solutions (14)-(16) are invariant on shifts

\[(t, x) \rightarrow (t + t_0, x + x_0).
\]

The values of \(t_0, x_0\) are arbitrary constants. For the practical needs they can be used for fitting the experimental waveforms.

3. Exact Nonlinear Wave-Separation Formulas

We will derive the closed analytical expression for the separation of a pressure wave measured in an artery into a forward and backward (reflected wave)
components using the Riemann invariants. Our approach will be similar to the one presented in [22] but the final formulas presented in this paragraph have not encountered in literature.

The equations (3) can be recasted in the following form

\[ D_+ R_+ = 0, \quad D_- R_- = 0, \]

where

\[ D_+ = \frac{\partial}{\partial t} + \left( u + \frac{1}{\sqrt{\rho D(A)}} \right) \frac{\partial}{\partial x}, \quad D_- = \left( u - \frac{1}{\sqrt{\rho D(A)}} \right) \frac{\partial}{\partial x}, \]

\[ R_+ = u + \int_{A_0}^{A} \frac{ds}{\sqrt{\rho D(s)s}}, \quad R_- = u - \int_{A_0}^{A} \frac{ds}{\sqrt{\rho D(s)s}}. \]

After integration we obtain

\[ R_+ = u + \frac{2}{n\sqrt{\rho D_0}} \left( A^{n/2} - A_0^{n/2} \right), \quad R_- = u - \frac{2}{n\sqrt{\rho D_0}} \left( A^{n/2} - A_0^{n/2} \right). \] (19)

For the case of the sole forward wave we have that

\[ R_+ = 2u, \quad R_- = 0, \]

\[ u = \int_{A_0}^{A} \frac{ds}{\sqrt{\rho D(s)s}} \]

and for the backward wave

\[ R_+ = 0, \quad R_- = 2u, \]

\[ u = -\int_{A_0}^{A} \frac{ds}{\sqrt{\rho D(s)s}}. \]

Finally, integrating the right-hand sides of the expressions for the velocities we obtain for the forward and the backward waves respectively

\[ u = \pm \frac{2}{n\sqrt{\rho D_0}} \left( A^{n/2} - A_0^{n/2} \right). \] (20)

Let us remember that the pressure can be expressed via vessel cross-section area \( A \) using pressure-area relation \([2]\)

\[ p = p_0 + \frac{1}{nD_0} (A^n - A_0^n). \] (21)

Now we consider the wave at the point 2 composed of the forward and backward waves starting from the points 1 and 3 (Fig. 3). Let us denote the total blood pressure, the blood velocity and the vessel cross-section area of the composed wave at the point 2 as \( p, u, A \). Our goal is to separate this wave into the forward and the backward components, or to express the blood pressure of the forward
Figure 3: The forward and backward waves starting from the points 1 and 3 travel along the corresponding characteristic curves $C_+, C_-$ and meet at the point 2.
wave \( p_+ \) and the backward wave \( p_- \) in terms of \( p, u \). Using [21] we can express the Riemann invariants in [19] in terms of \( u, p \)

\[
R_+(u, p) = u + \frac{2}{n \sqrt{\rho D_0}} f(p), \quad R_-(u, p) = u - \frac{2}{n \sqrt{\rho D_0}} f(p),
\]

where \( f(p) = \sqrt{n D_0 (p - p_0) + A_0^n - A_0^{n/2}} \). Moreover we can inverse the relations and express \( u, p \) in terms of \( R_+, R_- \)

\[
p(R_+, R_-) = p_0 + \frac{1}{n D_0} \left\{ \left( \frac{n \sqrt{\rho D_0}}{4} (R_+ - R_-) + A_0^{n/2} \right)^2 - A_0^n \right\},
\]

\[
u(R_+, R_-) = \frac{R_+ + R_-}{2}.
\]

Now let us remember that the forward traveling wave is defined by the condition \( R_- = 0 \) while the backward wave is defined by \( R_+ = 0 \). Then we deduce that

\[
p_+ = p(R_+, 0) = p_0 + \frac{1}{n D_0} \left\{ \left( \frac{n \sqrt{\rho D_0}}{4} (R_+ + A_0^{n/2}) \right)^2 - A_0^n \right\}, \quad (25)
\]

\[
p_- = p(0, R_-) = p_0 + \frac{1}{n D_0} \left\{ \left( -\frac{n \sqrt{\rho D_0}}{4} R_- + A_0^{n/2} \right)^2 - A_0^n \right\}.
\]

We substitute the relations [22] for \( R_+ = R_+ (u, p), R_- = R_- (u, p) \) in [25]-[26] and after some algebra we obtain the full nonlinear wave-separation formulas

\[
\Delta p_+ (u, p) = \frac{\rho c_0^2}{2} \left\{ \frac{2}{n} g(\Delta p) + \frac{u}{c_0} \right\} \left\{ 1 + \frac{1}{4} g(\Delta p) + \frac{nu}{8c_0} \right\},
\]

\[
\Delta p_- (u, p) = \frac{\rho c_0^2}{2} \left\{ \frac{2}{n} g(\Delta p) - \frac{u}{c_0} \right\} \left\{ 1 + \frac{1}{4} g(\Delta p) - \frac{nu}{8c_0} \right\},
\]

where

\[
g(\Delta p) = \sqrt{\frac{n \Delta p}{\rho c_0^2}} + 1 - 1
\]

and \( \Delta p = p - p_0, \Delta p \pm = p_\pm - p_0 \). If we expand \( g(\Delta p) \) in Taylor series on \( \Delta p \) and keep only the first order term then \( g(\Delta p) \approx \frac{n \Delta p}{2c_0^2} \). Therefore the linear approximation of [27]-[28] gives the classical linear wave-separation formulas [11]

\[
\Delta p_+ = \frac{1}{2} (\Delta p + \rho c_0 u), \quad \Delta p_- = \frac{1}{2} (\Delta p - \rho c_0 u).
\]

The results of the analytical study can be validated on the data from the literature. We compare the waves calculated from the formulas [27]-[28] with the results presented in [23] where the nonlinear wave separation technique based on the numerical solution of the equations for the infinitesimal changes of the forward and the backward waves \( dp_\pm = \frac{c_\pm}{2} (du \pm \frac{dp}{p \mp p_\pm (p)}) \), where \( c_\pm (p) = u \pm c(p) \) (\( c(p) \) is known function) was adopted. The method [23] and the present
Figure 4: The pressure pulse wave (thick upper line, borrowed from [23]) and its separation in the forward and the backward components for $n = 4, c_0 = 4 \text{m/s}$. The lines with box markers correspond to the results obtained using (27)-(28), the other waves are obtained using the classic linear method and the non-linear (lines without markers and the lines with triangle boxes respectively) from [23]. The both nonlinear methods show very close results.
approach give very close results since the formulas (27)–(28) are the solutions of the equations in differential form [23], therefore the methods are equivalent. The comparison of the methods shows that the discrepancies are negligible except of some points (Fig. 4). Note that the linear separation formulas overestimate the forward and the backward waves if \( n \) grows. This is the consequence of the fact that the forward and the backward pressure waves in (27)–(28) behave as \( 1/\sqrt{n} \) if \( u/c_0 << g(\Delta p) \).

**Conclusions**

The nonlinear 1D equations for inviscid blood flow are investigated analytically. For the first time the closed expression for the nonlinear forward or backward waves were derived in the case of the assumption that the terms \( \Delta A/A_0 \) (the relative systolic change of the vessel’s cross-section area) are small. As a result, the shock formation conditions were obtained. Finally, the analytical expressions for the wave separation into the forward and the backward components were deduced.

The analytical nonlinear pulse wave expression can be used in the evaluation of the central pressure based on the pressure forms in other arteries. The main method for the assessment of the central (aortic) blood pressure from the application tonometry measurements in brachial (or radial) arteries is based on the application of the transfer functions [25]–[27]. This function matches the Fourier harmonics of the radial (brachial) and the aortic pressures and is derived using the regression analysis. Since the shape of the nonlinear pressure wave and its change during the propagation can be obtained analytically then the following approach can be tested. At the first step the separation of the blood pressure measured in a radial artery into the forward and the backward components is performed using the pressure profiles only [12]. Next, we estimate the free parameters \( t_0, x_0 \) (see [18]) for which the analytical profile has the best fit with the assessed forward component. Finally, the aortic pressure can be recovered by application a shift in variables \( (t, x) \rightarrow (t - \delta t, x - \delta x) \) where \( (\delta t, \delta x) \) are the approximate time and distance for the pulse wave to travel from the aortic root to the vessel under the consideration. This can be a problem for the future study.

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