Lemaître-Tolman-Bondi model: fractality, bang time, and Hubble law

I. Initial conditions and compatibility of density and velocity laws

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Received, 1999 / accepted, 1999

Abstract. We start a systematic study of the Lemaître-Tolman-Bondi (LTB) model as applied to the large scale structure and its evolution. Here we study three possible initial conditions of the LTB models which are asymptotically FRW at large scales: bang time, fractal density (with fractal dimension $D = 2$), and velocity law. Any two of these determine the third one. Fractal density and simultaneous bang time provide a quantitative estimate for the scale beyond which the deflection from the linear Hubble law is small. This border may be identified with the zero-velocity surface. For fractal density and linear Hubble law it is shown that the bang time is necessarily non-simultaneous.

Key words: Cosmology:theory – Large Scale Structure of the Universe

1. Introduction

Lemaître-Tolman-Bondi (LTB) models are exact solutions of Einstein’s equations for 1) spherical symmetry, 2) pressureless matter (dust) and 3) motion with no particle layers intersecting. Originally studied by Lemaître (1933), Tolman (1934) and Bondi (1947), these models are the simplest generalization of the Friedmann-Robertson-Walker (FRW) models with a non-zero density gradient.

At least three cosmological applications of LTB models have been discussed. The first relates to the evolution of inhomogeneities and peculiar velocities in an expanding universe. Important results have been derived under the assumption of "unique bang time" (Olson & Silk 1979), when every mass shell has been simultaneously created. So, Silk & Wilson (1979\textsuperscript{a}, 1979\textsuperscript{b}), Olson & Silk (1979), and Olson & Stricland (1990) studied the formation of galactic clusters from small density and velocity perturbations (implicitly, non-simultaneous bang is used in Silk & Wilson (1979b)). It was shown that after a long time the initial conditions are forgotten and a universal density profile is formed. The LTB model has been applied to the determination of the mass density parameter $\Omega_o$ from the local peculiar velocity field (Silk 1974), and of the mass of the Virgo cluster (Hoffman et al. 1980; Tully & Shaya 1984; Teerikorpi et al. 1992; Ekholm et al. 1999).

The second area of cosmological application was developed by Ellis et al. (1985), Maartens (1996) and Mustapha et al. (1998). They use the FRW model for small (observable) scales and assume the LTB model for large scales.

The third area is the modeling of fractal matter distributions within general relativity (Bonnor 1972, Ribeiro 1992, 1992\textsuperscript{a}, 1992\textsuperscript{b}, Humphreys et al. 1998\textsuperscript{a}, Matravers, 1998\textsuperscript{a}). This has gained impact from redshift surveys revealing fractality with the fractal dimension $D \approx 2$ in the space distribution of galaxies up to distances of $100h^{-1}$ Mpc ($h = H_0/100\text{km s}^{-1}\text{Mpc}^{-1}$) (see Baryshev et al. 1994, Sylos Labini et al. 1998\textsuperscript{a}), and confirming the scale invariant de Vaucouleurs (1970) law. This leads to a new application of the LTB models, pointed out by Bonnor (1972), where the fractal structure is treated as spherically symmetrical inhomogeneities centered in every galaxy.

Baryshev et al. (1998\textsuperscript{a}) showed that the linear perturbation theory for the growth of density fluctuations leads to a non-linear Hubble law if all matter is in the fractals. Then the observed linear Hubble law (at scales $< 100h^{-1}$ Mpc) requires a very low background (FRW) density, $\Omega_o < 10^{-3}$. The exact LTB calculations by Humphreys et al. (1998\textsuperscript{a}) and Matravers (1998\textsuperscript{a}) gave the same conclusions. Another explanation for the linear Hubble law within the fractal structure (the “Hubble-de Vaucouleurs paradox”) was proposed by Baryshev et al. (1998): uni-
Recently, Teerikorpi et al. (1998) showed from Tully-Fisher as predicted by a fractal with dimension and the energy-momentum tensor has the form:

$$8 \pi G c^4 T^1_0 = \frac{1}{R^2} \left(-e^{-\lambda} R'^2 + 2 R \ddot{R} + \dot{R}^2 + 1\right) = 0,$$

$$8 \pi G c^4 T^2_0 = -\frac{e^{-\lambda}}{R} \left(\dot{R}' - \frac{\dot{R} \lambda'}{2}\right) + \frac{\ddot{R} \lambda'}{2} + \frac{\dot{R}^2}{4} + \frac{\dot{R}}{R} = 0,$$

$$T^1_0 = \frac{e^{-\lambda}}{R} \left(2 \ddot{R}' - \dot{\lambda} R'\right) = 0,$$

$$T^2_0 = T^3_0,$$

where $' = \frac{\partial}{\partial r}$ and $' = \frac{\partial}{\partial t}$. Einstein’s equation Eq. (3) defines the function $\lambda(r, t)$:

$$e^{\lambda(r, t)} = \frac{R'^2(r, t)}{f^2(r)}.$$

$f(r)$ is one of the undetermined functions of the models. Here $' = \frac{\partial}{\partial r}$ and $' = \frac{\partial}{\partial t}$. Via Eq. (6), the $T^1_0$-component reduces to the equation of motion

$$2 \ddot{R}(r, t) R(r, t) + \dot{R}^2(r, t) + 1 - f^2(r) = 0.$$  

The first and second integrals of Eq. (8) are:

$$R^2(r, t) = f^2(r) - 1 + \frac{F(r)}{2 R(r, t)},$$

$$\pm t + t_R(r) = \int^{R(r, t)}_{R(t_0)} \frac{d\tilde{R}}{\sqrt{c^2 (f^2(r) - 1) + \frac{F(r)}{2R}}},$$

where $F(r)$ and $t_R(r)$ are the second and third undetermined functions of the models. $t_R(r) t(r)$ is the bang time $t_R(r)$. Equation (9) gives for the density:

$$8 \pi G c^4 T^0_0(r, t) = \frac{dF(r)}{2 R^2(r, t)} - \frac{1}{2} \frac{\partial(R(r, t))}{\partial r}.$$

Eq. (11) shows that the models become singular by two different causes, defined by $R(r, t) = 0$ (which corresponds to the bang time $t_R(r)$) and $R'(r, t) = 0$ (which corresponds to the layer intersection time $t_{R'}(r)$). The bang time (Silk & Wilson 1979a) and layer intersection time function (for flat LTB model see Gromov 1997), respectively, correspond to these singularities. The gravitating mass $M_{grav}$ and the invariant mass $M_{inv}$ of the dust are defined by the energy-momentum tensor:

$$M_{grav}(R(r, t)) = \frac{4 \pi}{c^2} \int^{R(r, t)}_{R(t_0)} T^0_0(y) y^2 dy = M_{grav}(r),$$

and

$$M_{inv}(r) = \frac{4 \pi}{c^2} \int^{r}_{0} T^0_0(x, t) \sqrt{-g(x, t)} dx,$$
where \( \sqrt{-g} = \frac{R^c R^e}{2} \). Substituting \( \rho \) from the definition of \( T^t_t \) and Eq. (11) into Eq. (12) and Eq. (13) we obtain

\[
M_{\text{grav}}(r) = \left( \frac{c^2}{4G} \right) F(r), \quad \text{and} \quad F(0)=0. \quad \text{So,}
\]

\[
M_{\text{inv}}(r) = \int_0^r \frac{dM_{\text{grav}}(x)}{f(x)}, \quad f(r) = \frac{M'_{\text{grav}}}{M_{\text{inv}}}. \quad \text{(14)}
\]

One of the interpretations of the function \( f(r) \), given by Bondi (1947), relates it to the curvature and components of the Einstein tensor:

\[
K^1_1 = 2 \frac{f}{R} \frac{df}{dR}, \quad K^2_2 = K^3_3 = \frac{f^2 - 1}{R^2} + \frac{f}{R} \frac{df}{dR}, \quad K = \frac{2}{R^2} \frac{d}{dR} \left( R \left( f^2 - 1 \right) \right).
\]

The space curvature is equal to zero if and only if \( f = 1 \).

The LTB models are defined up to some transformation of the co-moving coordinate \( \psi \): \( \psi \rightarrow \tilde{\psi} \), which decreases the number of undetermined functions from 3 to 2 (Just 1960, Just & Kraus 1962). See also Hellaby & Lake (1985) and Hellaby (1987) for interesting examples. These two functions should be chosen from the set:

\[
t_R, \quad t, \quad \rho_0, \quad f, \quad R_0, \quad \tilde{R}_0. \quad \text{(15)}
\]

The transformation is not unique and may be chosen according to the character of the problem (see Gromov 1996).

The transformation \( \psi \) is time independent, so it can be used to fix one of the functions from the set Eq. (15). In this case one speaks about a parametrization of the LTB model. We give two examples. The first is often used:

\[
R(r, 0) = r. \quad \text{(16)}
\]

This implies that \( F(r) \) is defined by the initial density profile \( \rho(r, 0) \). It was fully studied by Liu (1990a, 1990b, 1991), and used e.g. by Ribeiro (1992a, 1992b, 1993) and Gonçalves & Moss (1997). An alternative transformation is:

\[
M_{\text{inv}}(r) G/c^2 = r. \quad \text{(17)}
\]

It was first used by Eardley (1974) and studied by Gromov (1996, 1997).

The bang time \( t_R \) is used as one of the initial conditions by Silk & Wilson (1979a), Olson & Silk (1979), Olson & Stricland (1990).

3. Zero-velocity surface in the LTB model

We use two different sets of initial conditions:

A) bang time \( t_R(r) \) and initial density profile \( \rho_0(r) \), and
B) initial density \( \rho_0(r) \) and velocity \( \tilde{R}_0(r) \) profile.

They are interconnected as may be seen from a simple analogy of an apple dropping from an apple tree. If the initial position and velocity, and an equation of motion are given, we can calculate the time it will reach the ground. For the LTB models it is the same. If we start with \( t_R(r) \) and \( \rho_0(R) \) as initial conditions, we can calculate the velocity profile \( \tilde{R}(r) \) for \( t = 0 \): since a particle must reach the centre by the time \( t_R(r) \), it must have a predefined velocity at \( t = 0 \). Since the density profile is also given, the gravitational potential becomes fixed by the same initial conditions. As from each two functions the third may be derived, all three cannot be selected freely.

3.1. Dimensionless equations

Now we restate the models in terms of dimensionless quantities. We use the following characteristic values:

\[
l_0 = c, \quad t_0 = \frac{1}{H_0}, \quad \Omega_0 = \frac{\rho(\infty)}{\rho_{cr}}, \quad M_0 = \frac{4 \pi}{3} \frac{\rho(\infty)}{l_0^3} l_0^3.
\]

and dimensionless variables:

\[
\xi = \frac{R}{l_0}, \quad \eta = \frac{r}{l_0}, \quad \tau = \frac{t}{t_0}, \quad \delta(\xi) = \frac{\rho(R)}{\rho(\infty)} \mu(\eta) = \frac{M_{\text{grav}}}{M_0} = 3 \int_0^\eta \delta(x) x^2 dx,
\]

where \( l_0 \) and \( t_0 \) are the characteristic length and time, \( \Omega_0 \) is the density parameter of the FRW background, \( H_0 \) is the value of the Hubble parameter at the moment of the initial conditions, \( \rho_0(r) \) is the dust density, \( \rho_{cr} \) is the critical density, \( M_0 \) is the characteristic mass, \( \xi \) and \( \eta \) are dimensionless Euler and Lagrangian coordinates, \( \delta(\xi) \) is the dimensionless density, and \( \mu(\eta) \) is the dimensionless mass of the dust. In terms of these quantities, the bang time is written as \( \tau(\eta) \). The index \( \xi \) reminds us that the bang time is the time required for the particle to come from its initial position \( \xi_0 \) to \( \xi = 0 \).

In this section we show how one can use the first and second integrals, Eq. (11) and Eq. (12), to calculate the zero-velocity surface for LTB models if we relax the often used assumption of unique bang time.

We use an effective dimensionless gravitating mass \( \mu^* \):

\[
\mu^* = \Omega_0 \mu = 2 M_{\text{grav}} G/c^2 l_0. \quad \text{(18)}
\]

In terms of dimensionless variables the first integral of the equation of motion Eq. (11) becomes

\[
\dot{\xi}^2(\eta, \tau) = f^2(\eta) - 1 + \frac{\mu^*}{\delta(\eta)}. \quad \text{(19)}
\]

The form of the second integral Eq. (12) depends on the sign of \( f^2(\eta) - 1 \). For \( f^2(\eta) - 1 < 0 \) (closed models):

\[
\pm \tau + \tau(\eta) = \frac{\mu^*(\eta)^{-1/2}}{A^{3/2}} \left( \arcsin \sqrt{A} - \sqrt{A} \sqrt{1-A} \right); \quad \text{(20)}
\]
for \( f^2(\eta) - 1 = 0 \) (flat models):
\[
\xi^{3/2} = \xi_0^{3/2} \pm \frac{3}{2} \tau \sqrt{\mu};
\]
(21)
and for \( f^2(\eta) - 1 > 0 \) (open models):
\[
\pm \tau + \tau_\xi(\eta) = \frac{\mu^*(\eta)^{-1/2}}{(-A)^{3/2}} \left( -\text{arcsinh}\sqrt{-A} + \sqrt{-A\sqrt{-A}} \right)
\]
(22)
where \( \tau_\xi(\eta) \) is defined by the solution of Eq. (30):
\[
A = \frac{1 - f^2(\xi)}{\mu^*(\eta)} \xi(\eta, \tau).
\]
(23)

3.2. The closed and open models with arbitrary bang time

In this and the remaining sections we concentrate on initial conditions. The co-moving coordinate is chosen as \( \xi_0 \). By solving Eq. (19) for \( f^2 \) and substituting this into Eq. (20), the expression for bang time \( \tau_\xi \) of the closed and open models may be rewritten in the form (taking into account that for a fixed moment, \( \xi \) may be used as a co-moving coordinate):
\[
\tau_\xi = \sqrt{\frac{\xi_0^3}{\mu^*}} \Psi, \quad B = \frac{\xi_0\xi_0^2}{\mu^*} \geq 0,
\]
(24)
and for closed models \( 0 \leq B < 1 \)
\[
\Psi(B) \equiv \Psi^c(B) = \frac{\text{arcsinh}\sqrt{1-B} - \sqrt{1-B} \sqrt{B}}{(1-B)^{3/2}},
\]
(25)
while for open models \( B > 1 \)
\[
\Psi(B) \equiv \Psi^{op}(B) = -\frac{\text{arcsinh}\sqrt{B-1} + \sqrt{B-1} \sqrt{B-1}}{(B-1)^{3/2}}.
\]
(26)

Eq. (24) allows us to rewrite equation Eq. (19) as
\[
B = (f^2 - 1) \frac{\xi_0}{\mu^*} + 1.
\]
(27)

It follows from Eq. (24) that \( B = 0 \) corresponds to the following set of initial conditions: if \( \xi_0 = 0 \), \( \text{grad}_\xi(\xi_0 = 0) = 0 \), when
\[
\lim_{\xi_0 \to 0} B = \lim_{\xi_0 \to 0} \left( \frac{\xi_0}{\xi_0^2} \right)^2 \geq 0;
\]
(28)

whereas in the case of \( \xi_0 \neq 0 \), \( B = 0 \) implies \( \xi_0 = 0 \). The velocity of the particle is equal to zero at the boundary \( \xi_{ZV} \), see Figs. 1-3. For both cases \( f^2 = 1 - \mu^*/\xi_0 \geq 0 \) for \( B = 0 \), which implies the inequality \( \xi_0 \geq \mu^* \) for \( B = 0 \). Note that this restricts the kind of particular TB model in which the nonequality may be satisfied: because \( f^2 \geq 0 \), it follows that \( f^2 - 1 < 0 \). So, \( B = 0 \) may be satisfied only in the closed model.

The limit \( B \to 1 \) corresponds to \( f \to 1 \), so that open and closed models both have a common limit which coincides with the flat model: \( \Psi^f(B) = \frac{1}{2} \).

Olson & Silk (1979) defined the boundary between open and closed TB models with a unique bang time as a place where \( f = 1 \). We also postulate a set of initial conditions (e.g. fractal density and Hubble law), which produce the following sequence of particular models: a closed domain of the model which has a position around a centre ("core") and open model farther out from the centre ("shell"). The two domains are separated by the flat one located on the surface where \( f^2 = 1 \). For the closed "core" \( 0 \leq B < 1 \), so, from Eq. (25) and Eq. (27) it follows that
\[
\frac{2}{3} < \Psi^c(B) \leq \text{arcsin}(1) \approx 1.57,
\]
(29)
where \( \frac{2}{3} \) corresponds to the boundary of the closed model, the flat model, (this boundary we denote by \( \xi_{f1} \)) and \( \text{arcsin}(1) \) corresponds to the zero-velocity surface \( \xi_{ZV} \).

For arbitrary bang time, the zero-velocity surface is defined by the solution of Eq. (30):
\[
\tau_\xi = \left( \frac{\xi_0^3}{\mu^*} \right)^{1/2} \text{arcsin}(1).
\]
(30)

The solution of Eq. (30) may be real or complex depending on initial conditions, i.e. bang time and density profile. A complex solution means that the zero-velocity surface lies in the centre of symmetry. If the solution is real (and positive) then the zero-velocity surface is found at a finite distance from the centre and separates the collapsing region of the closed part of the model from the expanding region. In the domain \( 0 \leq \xi_0 < \xi_{ZV} \) we can introduce some other LTB model corresponding to the initial density profile, also closed. Humphreys et al. (1998a) first demonstrated how to construct the LTB model for that domain.

The above approach utilizes the coordinate criterium for the existence of a central collapsing domain. Using Eq. (24), we can also define a second form for this criterium, the mass criterium. From Eq. (24) it follows that the two limits of function \( \Psi(B) \) correspond to two characteristic masses. The mass \( \mu_{ZV} \),
\[
\mu_{ZV}(\xi_0) = (1.57/\tau_\xi)^2 \xi_0^3,
\]
(31)
corresponds to the low limit of radial Euler coordinate \( \xi_{ZV} \). This designates the beginning of the domain \( \xi_0 > \xi_{ZV} \) from which all particles can collapse at time \( \tau_\xi \). Similarly, the characteristic mass corresponding to the flat model (or to the upper boundary of the closed model, which is the same thing) has the form
\[
\mu_{f1}(\xi_0) = (2/3 \tau_\xi)^2 \xi_0^3.
\]
(32)
This criterium can be stated as follows: if the graph of the mass, corresponding to a given initial density profile, intersects the graph of \( \mu_{ZV} \), then \( \xi_{ZV} > 0 \).
3.3. The flat LTB model

We now turn to the simplest initial condition, \( f = 1 \). If \( \tau_\xi = \text{const} \), the flat LTB model reduces to the flat FRW model. As shown by Gromov (1997), for the flat LTB model the bang time may be given in the form:

\[
\tau_\xi(\mu) = \left( \frac{\lambda}{\mu^3} \right) \mu \int_0^\mu d\mu \rho_0(y),
\]

which immediately implies that \( \rho_0(\xi_0) = \text{const} \) for simultaneous bang time. In any other case, \( \rho_0(\xi_0) \neq \text{const} \) and the bang time is not constant. As was shown above, \( \xi_{ZV} \) may not be equal to zero (and the bang is not simultaneous) if and only if the LTB model is closed, so the domain of definition of the flat model is the whole region \( \xi_0 \geq 0 \).

4. LTB models for a fractal density distribution

In this section we study the LTB models with initial conditions given by the fractal density profile and Hubble law.

4.1. Fractal density distribution and the Hubble law: Hubble-de Vaucouleurs paradox

Two fundamental empirical laws have been established from extragalactic data. First, there is the power law density-distance relation (de Vaucouleurs law) which corresponds to fractal structure (Mandelbrot [1982] with fractal dimension \( D \approx 2 \) up to the depth of available catalogues \( \approx 100 \, h^{-1} \, \text{Mpc} \) (Sylos Labini et al. 1998). Second, Cepheids, TF-distance indicator and Type Ia supernovae confirm the linearity of Hubble’s redshift-distance law within the same distances where the fractality exists.

Baryshev et al. (1998) emphasized that the linear redshift-distance relation inside the fractal (inhomogeneous) matter distribution creates the so-called Hubble-de Vaucouleurs (HdeV) paradox. It means that the interpretation of the Hubble law within FRW cosmological models as a consequence of a homogeneous galaxy distribution disagrees with new data on the galaxy distribution for a scale interval from 1 to 100 Mpc. We emphasize that the paradox exist for small distance scales (up to 100 Mpc), i.e., for redshifts less than 0.03. Hence the arguments of Abdalla et al. (1999) on essential relativistic corrections do not explain the paradox.

Two solutions of the HdeV paradox are previously known. The first one (Baryshev et al. 1998) is based on uniform dark matter starting just from the halos of galaxies, in which case the standard FRW model works. But then the fractal distribution of luminous galaxies can appear only from special initial perturbations of FRW. \(^2\)

The second solution is a very low value for the global average density (Baryshev et al. 1998; Humphreys et al. 1998b). However, if the upper cut-off scale of the fractal structure is large, the low density contradicts the estimated density of the baryonic luminous and dark matter.

4.2. On the applicability of the LTB model to fractals

The LTB model has proved useful for understanding the kinematics of galaxies around individual mass concentrations. For example, Teerikorpi et al. (1992) could put in evidence the expected behaviour in the Virgo supercluster: 1) Hubble law at large distances, 2) retardation at smaller distances, 3) zero-velocity surface, and 4) collapsing galaxies at still smaller distances.

Bonnor (1972) was the first to apply the LTB model to the hierarchical cosmology. He used de Vaucouleurs’s density law \( \rho \sim d^{-\gamma} \) with \( \gamma = 1.7 \). Ribeiro (1992a, 1992b, 1993) has developed a numerical approach to solving LTB equation for fractal galaxy distribution. Humphreys et al. (1998b) gave a relation between number counts and redshifts for LTB models with large scale FRW behaviour.

However, the application of LTB models to a fractal distribution leads to a conceptual problem, because the original LTB formulation contained a central point of the universe, around which the density distribution is isotropic. In a fractal distribution (Mandelbrot [1982] there is no unique centre, but every object of the structure may be treated as a local centre which accommodates the LTB centre. Every structure point is surrounded by a spherically symmetric (in average) matter distribution.

In this sense, the application of the LTB model to fractals means that there is an infinity of LTB exemplars with centres on every structure point. Their initial conditions are slightly different, because for any fixed scale the average density is approximately constant. For different scales the density is a power law. This excludes geocentrism and makes possible the use of LTB models as an exact general relativistic cosmological model where expansion of space becomes scale dependent. \(^3\)

4.3. Simultaneous bang time

We showed in Sect. 3 that a unique bang time and constant density imply open FRW models. Here, with non-linear LTB models, we study density perturbations with arbitrary amplitude on the FRW background and show how the initial fractal density changes the models.

Humphreys et al. (1998b) used a density profile which needed junction conditions for densities corresponding to...
The solution of Eq. (30) with unique bang time and initial conditions. \( \xi_{ZV} \), the solution of Eq. (34), depends on the initial conditions \( (34) \) and \( (35) \). The upper curve is the function \( \log(\tau_{FRW}(\xi_{\Omega}/\mu^{*}))^{1/2} \) for \( \Omega = 0.001 \) and \( \mu = 0.02 \) (see Table 3), while the lower curve corresponds to \( \Omega = 0.001 \) and \( \mu = 0.02 \) (Table 4). The upper line is max \( \log(\Psi^{z}) = \log(\arcsin(1)) = 0.196 \). The lower line is min \( \log(\Psi^{z}) = \log(\frac{1}{2}) = -0.176 \). The galaxy scale \( \epsilon = 2 \cdot 10^{-6} \). The curve intersects the upper line at \( \xi_{TB} \). It is seen that \( \Omega = 0.001 \) and \( \mu = 0.02 \) produce the intersection at a scale > galaxy scale, which corresponds to real (and positive) solution of Eq. (30), but \( \Omega = 0.001 \) and \( \mu = 0.002 \) do not produce it. In the last case the solution of Eq. (30) is imaginary.

The coordinate criterium of existence of \( \xi_{ZV} > 0 \) for simultaneous bang time \( \tau_{FRW} = 0.98 \), \( \Omega = 0.01 \) and \( \mu = 0.02 \) (see Table 3). The model is defined for \( \xi_{0} \). Here \( \xi_{0} \geq \xi_{ZV} \). At the boundary \( \xi_{ZV} \) velocity \( \dot{\xi}(\xi_{ZV}) = 0 \), see Fig. 4.

to different scales. We consider the analytical case of a smooth density profile. The fractal density on the FRW background and simultaneous bang are the initial conditions of our LTB models:

\[
\delta(\xi, 0) = \frac{A}{\epsilon + \xi_{0}} + 1, \quad (34)
\]

\[
\frac{1}{(1 - \Omega_{0})^{3/2}} \sqrt{\frac{1 - \Omega_{0}}{\Omega_{0}}} \left[ \frac{1}{\Omega_{0}} - \arcsin \sqrt{\frac{1 - \Omega_{0}}{\Omega_{0}}} \right] \quad (35)
\]

Here \( \epsilon = R_{galaxy}/l_{b} \approx 10 Kpc/5 \cdot 10^{6} Kpc = 2 \cdot 10^{-6} \). Above the galactic scale, Eq. (24) describes the fractal density law with \( D = 2 \). The density contrast \( (\rho_{galaxy}/\rho_{FRW}) \) is \( \delta(\xi = 0) \approx 10^{-24} \text{g/cm}^{3}/10^{-29} \text{g/cm}^{3} = 10^{5} \). So, \( A \approx 0.2 \). For our calculations we use \( A = 0.002, 0.02, 0.2, 2 \), which imply the amplitude of the density \( \delta(0) = 10^{3}, 10^{4}, 10^{5}, 10^{6} \), and we use \( \Omega_{0} = 0.001, 0.01, 0.1, 0.99 \), which imply \( \tau_{FRW} = 0.997, 0.98, 0.898, 0.688 \).

The properties of the LTB models with initial conditions Eq. (24), Eq. (34) were studied in Sect. 3. Now we apply the results of Sect. 3.2. to the initial conditions with given parameters. For the chosen values of \( A \) and \( \Omega_{0} \) LTB models have a closed "core" and open "shell". But only for \( \Omega = 0.001 \) and \( A = 0.002 \) does Eq. (30) have a complex solution (Fig.1). This means that only these parameters produce the TB model with fractal density and simultaneous bang time with \( \xi_{ZV} \). All other cases of the adopted parameter values, Eq. (30) has a real solution and \( \xi_{ZV} > 0 \) (see Tables 1 - 4).

Figs. 2 and 3 illustrate the coordinate and mass criteria for the existence of \( \xi_{ZV} > 0 \) for \( A = 0.02 \) and \( \Omega = 0.01 \).

Our solution depends on \( \Omega_{0} \): \( \mu^{*} \sim \Omega_{0} \) and \( \tau_{FRW} = \tau_{FRW}(\Omega_{0}) \). Tables 1 - 4 show characteristic values of \( \xi_{ZV} \) and \( \xi_{fl} \) for different \( A \) and \( \Omega_{0} \). Here \( l_{b} = c/H_{0} = 5000 \text{Mpc} \), \( H_{0} = 60 \text{km/s/Mpc} \).

Finally, we calculate the velocity \( \dot{\xi}(\xi) \), produced by the initial conditions Eq. (24) and Eq. (37). For both domains of the model, closed and open, \( \dot{\xi}_{0} = B(\mu^{*}/\Omega_{0}) \),

\[
\dot{\xi}(\xi) = \frac{\dot{A}}{\dot{\epsilon} + \dot{\xi}_{0}} + 1, \quad \dot{\xi} = \xi. \quad (37)
\]

The bang time in this situation is calculated by two formulas, corresponding to the closed and open domains:

\[
\tau_{\xi}^{cl}(\xi_{0}) = \frac{1}{(1 - B)^{3/2}} \left[ -\sqrt{B} \sqrt{1 - B} + \arcsin \sqrt{1 - B} \right], \quad (38)
\]

\[
\tau_{\xi}^{op}(\xi_{0}) = \frac{1}{(B - 1)^{3/2}} \left[ \sqrt{B} \sqrt{B - 1} - \arcsinh \sqrt{B - 1} \right], \quad (39)
\]

where from Eq. (22) and Eq. (37), \( B = \xi_{0}^{2}/\mu^{*}(\xi_{0}) \). For \( A = 0.02 \) and \( \Omega_{0} = 0.01 \) the bang time \( \tau_{\xi}(\xi_{0}) \) is shown in Fig. 5.
Fig. 4. Velocity produced by initial conditions Eq. (34) and Eq. (37) (simultaneous (FRW) bang time and given density profile) for $\Omega_0 = 0.001, 0.01, 0.1$ (from left to right) and $A = 0.02$. Different $\Omega_0$ produce different $\xi_{ZV}$, in which $\xi(\xi_{ZV}) = 0$. Recall that log $\xi = -1$ corresponds to 500 Mpc and the size of the Local Group (1 Mpc) is encountered at log $\xi_0 = -3.7$.

Fig. 5. The non-simultaneous bang time $\tau_3(\xi_0)$ produced by initial conditions $A$ with $\Omega_0 = 0.01, A = 0.02$. At infinity the time of collapse $\tau_3(\xi_0) \rightarrow \tau_{FRW} = 0.98$, indicated by the horizontal line.

5. Discussion and conclusions

We have studied the LTB model solution for two pairs of initial conditions related to observations: (bang time function, density profile) and (density profile, velocity function) which both permit one to treat the LTB problem as the Cauchy problem. The second pair may be obtained from observations at the present epoch, while from the first pair, with bang time function assumed, one may predict the present-day velocity function. A discussion of this mathematical side is given by Gromov [1999]. We show that for most parameter values of the model, there is the zero-velocity surface.

Our study is much guided by Bondi’s (1947) idea when he said “The assumption of spherical symmetry supplies us with a model which lies between the completely homogeneous models of cosmology and the actual universe with its irregularities.”

As the “actual universe with its irregularities” has turned out to be fractal, at least in its luminous matter and in scales up to 100 Mpc or more, we follow some earlier works in representing fractality with spherically symmetrical systems of different scales. The interesting conceptual difficulties will be treated in Paper III. Here we have assumed that the fractal representation is adequate and complement our previous work (Baryshev et al. [1998]) on the Hubble law within fractals. The qualitative conclusions of that paper, based on the linear regime, are confirmed by our exact LTB solutions. In particular, a fractal distribution of matter with $D = 2$, smoothly going over to the FRW background, generally results in a large deviation from the linear Hubble law, when there is a unique bang time. Only if $\Omega_0 \approx 0.001$, does a reasonable density contrast $A = 0.2$ produce an acceptable Hubble law.

As the above mentioned two pairs of initial conditions are interconnected, one may start from the linear Hubble law as the velocity function, and derive the required bang time function. This we have done, and conclude that in this manner in the frame of the LTB models it is thus possible to have a linear velocity - distance relation when matter distribution is fractal. Physically, this would mean that different spherical shells are created at different moments given by the bang time function.

Thus the list of possible solutions of the Hubble-de Vaucouleurs paradox (Baryshev et al. [1998]) in the frame of the LTB description of fractality now includes 1) a very low cosmic density, or 2) a dominating smooth dark matter, or 3) non-simultaneous bang time.

Acknowledgements. We thank the referee for valuable comments, C. Hellaby for kindly sending us his paper, and M. Hanski for a useful discussion. The work was supported by the Center for Cosmoparticle Physics “Cosmion” (project “Cosmoparticle physics”), the Russian program “Integration” (project N.578), and the Academy of Finland (project “Cosmology in the Local Galaxy Universe”).

References

Abdalla, E., Mohayace, R., Ribeiro, M.B., astro-ph/9910003
Baryshev, Yu., Sylos Labini F., Montuori, M., Pietronero, L., 1994, Vistas in Astronomy 38, 419
Baryshev, Yu., Sylos Labini F., Montuori, M., Pietronero, L., Teerikorpi, P., 1998, Fractals 6 No.3, 231
\[
\log(\tau_{F,RW} \sqrt{\frac{\xi_0}{\mu^*}})
\]
