QCD Corrections in $\Gamma_{\text{sl}}(B)$

Nikolai Uraltsev

INFN, Sezione di Milano, Milan, Italy

Abstract

Short-distance expansion of the total semileptonic $B$ widths is reviewed for the OPE-conformable scheme employing low-scale running quark masses. The third- and fourth-order BLM corrections are given and the complete resummation of the BLM series presented. The effect of higher perturbative orders with running quark masses is found very small. Numerical consequences for $|V_{cb}|$ are addressed.

*On leave of absence from Department of Physics, University of Notre Dame, Notre Dame, IN 46556, U.S.A. and St. Petersburg Nuclear Physics Institute, Gatchina, St. Petersburg 188300, Russia
Total decay rates of heavy flavors are dominated by short-distance physics; nonperturbative effects originating from a typical hadronic momentum scale are described by the OPE. The semileptonic $B$ widths $\Gamma_{sl}(b \to c)$ and $\Gamma_{sl}(b \to u)$ offer an unmatched accuracy in determination of the corresponding CKM elements. To make a full use of the infrared-free nature of perturbative corrections to the widths they should be computed in the context of the Wilsonian OPE; in particular this suggests using the running quark masses normalized at a low scale $\mu \ll m_b$ [1]. The perturbative corrections (i.e., the $\alpha_s$-expansion of the Wilson coefficients of the unit operator $\bar{Q}Q$) explicitly depend in this approach on the separation (normalization) scale $\mu$.

The $\mu$-dependence of the Wilson coefficients is uniquely determined [2, 3] by normalization point independence of the inclusive width and by known $\mu$-dependence of quark masses and local heavy quark operators in the OPE series. Below the expressions are collected for the scheme based on running ‘kinetic’ heavy quark masses and other operators defined via the small velocity (SV) sum rules [4, 5]. These definitions are complete even beyond perturbation theory; likewise they are valid to any perturbative order.

1 Perturbative corrections

Considering pure short-distance (perturbative) effects, we denote

$$\Gamma_{sl}^{\text{pert}}(b \to c) = \frac{G_F^2 m_b^5(\mu)}{192 \pi^3} |V_{cb}|^2 (1 + A_{\text{ew}}) \left\{ z_0(r) + \bar{a}_1(r; \mu) \frac{\alpha_s(m_b)}{\pi} + \bar{a}_2(r; \mu) \left( \frac{\alpha_s(m_b)}{\pi} \right)^2 + \bar{a}_3(r; \mu) \left( \frac{\alpha_s(m_b)}{\pi} \right)^3 + \ldots \right\},$$

where $z_0(r)$ is the tree-level phase space factor and $r = m_c^2(\mu)/m_b^2(\mu)$:

$$z_0(r) = 1 - 8r + 8r^3 - r^4 - 12r^2 \ln r.$$  

In what follows $\alpha_s$ is the $\overline{\text{MS}}$ strong coupling normalized at $m_b$, unless indicated explicitly.

The electroweak corrections $A_{\text{ew}}$ describe the ultraviolet renormalization of the semileptonic Fermi interaction and, being the QED-counterpart of the familiar factors $c_{\pm}$ in nonleptonic Lagrangian, has a well-known logarithm of $M_Z/m_b$ due to virtual photon exchanges. It depends only on the fermion charges and is the same as in $\beta$-decays of neutron or hyperons [6]:

$$1 + A_{\text{ew}} \simeq \left( 1 + \frac{\alpha}{\pi} \ln \frac{M_Z}{m_b} \right)^2.$$  

The $\mu$-dependence of the perturbative factors $\bar{a}_1(r; \mu)$ and $\bar{a}_2(r; \mu)$ through the second order in $\alpha_s$ is given by

$$\bar{a}_1(r; \mu) = a_1^{(0)}(r) + [5 \Lambda_1 + 3 p_1] z_0(r) + \left[ 2(\sqrt{r} - r) \Lambda_1 + (1 - r)p_1 \right] \frac{d z_0(r)}{d r}$$

$$\bar{a}_2(r; \mu) = a_2^{(0)}(r) + [5 \Lambda_1 + 3 p_1] a_1^{(0)}(r) + \left[ 2(\sqrt{r} - r) \Lambda_1 + (1 - r)p_1 \right] \frac{d a_1^{(0)}(r)}{d r}$$
\[
\begin{align*}
+ \left[ 5\Lambda_2 + 3p_2 + 10\Lambda_1^2 + 4p_1^2 + \frac{25}{2} \Lambda_1 p_1 \right] z_0(r) \\
+ \left[ 2(\sqrt{r} - r)\Lambda_2 + (1 - r)p_2 + (1 + 6\sqrt{r} - 7r)\Lambda_1^2 + \left( \frac{1}{4r} + 2 - \frac{9}{4} r \right) p_1^2 + \\
\left( \frac{1}{\sqrt{r}} + 3 + 4\sqrt{r} - 8r \right) \Lambda_1 p_1 \right] \frac{d^2 z_0(r)}{dr^2} + \frac{1}{2} \left[ 2(\sqrt{r} - r)\Lambda_1 + (1 - r)p_1 \right]^2 d^2 z_0(r) \ , \tag{5}
\end{align*}
\]

where \( \bar{a}_k^{(0)}(r) \equiv \bar{a}_k(r; 0) \) are the coefficients perturbatively computed at \( \mu = 0 \), i.e. for the pole quark masses and the pole-type definition of the kinetic and higher-order operators. The \( b \to u \) case corresponding to \( r = 0 \) amounts to using \( z_0 = 1 \), \( a_1 = -\frac{2}{3} \left( \pi^2 - \frac{25}{4} \right) \) while discarding all terms with derivatives.

Four parameters \( \Lambda_{1,2} \) and \( p_{1,2} \) entering above denote the coefficients in the perturbative expansion of \( \overline{\Lambda}(\mu)/m_b \) and \( \mu_2^2(\mu)/m_b^2 \) to first and second order in \( \alpha_s(M)/\pi \), respectively. In the the kinetic mass scheme they are [7]

\[
\Lambda_1 = \frac{4}{3} C_F \frac{\mu}{m_b}, \quad \Lambda_2 = \Lambda_1 \left[ \frac{\beta_0}{2} \left( \ln \frac{M}{2\mu} + \frac{8}{3} \right) - C_A \left( \frac{\pi^2}{6} - \frac{13}{12} \right) \right], \tag{6}
\]
\[
p_1 = C_F \frac{\mu_2^2}{m_b^2}, \quad p_2 = p_1 \left[ \frac{\beta_0}{2} \left( \ln \frac{M}{2\mu} + \frac{13}{6} \right) - C_A \left( \frac{\pi^2}{6} - \frac{13}{12} \right) \right]. \tag{7}
\]

Here

\[
C_F = \frac{4}{3}, \quad C_A = N_c = 3, \quad \beta_0 = \frac{11}{3} C_A - \frac{2}{3} n_f = 9.
\]

The general structure of Eqs. (4) and (5) holds in higher orders in \( \alpha_s \), but the explicit coefficients change and the terms proliferate. This does not happen for the BLM corrections\(^1\) to any order, for which the translation between different \( \mu \) takes basically the same form as for the first-order coefficient \( a_1(r; \mu) \). The only dependence on the order \( k \) in the BLM expansion is the constant accompanying the power of the renorm-group logs \( \left( \ln \frac{m}{2\mu} + \frac{8}{3} \right)^{k-1}, \left( \ln \frac{m}{2\mu} + \frac{13}{6} \right)^{k-1} \) in Eqs. (6), (7) (details can be found in Ref. [5]). For the third order BLM, in particular, they were given in [7]:

\[
\Lambda_3^{\text{BLM}} = \Lambda_1 \left( \frac{\beta_0}{2} \right)^2 \left[ \left( \ln \frac{M}{2\mu} + \frac{8}{3} \right)^2 + \frac{67}{36} - \frac{\pi^2}{6} \right] , \quad p_3^{\text{BLM}} = p_1 \left( \frac{\beta_0}{2} \right)^2 \left[ \left( \ln \frac{M}{2\mu} + \frac{13}{6} \right)^2 + \frac{10}{9} - \frac{\pi^2}{6} \right]. \tag{8}
\]

Although the expressions for even higher orders in BLM is straightforward, in practice it is simpler to evaluate the BLM corrections in the widths directly in the running mass scheme. The general formalism is described in [5], and using an additional technical trick given in Appendix simplifies necessary integrations.

\(^1\)Perturbative subseries including the coefficients of the form \( a_n = \left( \frac{\beta_0}{2} \right)^{n-1} \hat{a}_n \), where the coefficients are viewed as polynomials in \( n_f \).
2 Numerical estimates

For the second-order non-BLM coefficient we have

\[
\bar{a}_2(r; \mu) - \bar{a}_2^{(0)}(r; \mu) \big|_{\text{non-BLM}} \equiv \delta_2^{(0)}(r; \mu), \quad \delta_2^{(0)}(0.25^2; 1/4.6) \simeq -1.35
\]  

(9)

The simple interpolating expression accounting for the mass dependence of \( \bar{a}_1^{(0)} \),

\[
\delta_2^{(0)}(r; \mu) \simeq -\frac{9.9}{m_b} z_0(r) \left[ 1 - 4.5 (\sqrt{r} - 0.25) + 13 (\sqrt{r} - 0.25)^2 + 0.7 \left( \frac{\mu}{m_b} - \frac{1}{4.6} \right) - 12 (\sqrt{r} - 0.25) \left( \frac{\mu}{m_b} - \frac{1}{4.6} \right) + 71 (\sqrt{r} - 0.25)^2 \left( \frac{\mu}{m_b} - \frac{1}{4.6} \right) \right]
\]

(10)

is accurate enough in the relevant domain of quark masses and factorization scale \( \mu^2 \).

Using the evaluation \( \bar{a}_2^{(0)} \simeq 0.9 \) of the second-order non-BLM coefficients by Czarnecki and Melnikov \( [9] \) we obtain

\[
\bar{a}_2^{\text{non-BLM}}(0.25^2; 1 \text{ GeV}) \simeq -0.65 z_0(0.25^2)
\]

(11)

The second-order BLM coefficient has been addressed in the literature more than once; here the shift between \( \mu = 0 \) and \( \mu = 1 \text{ GeV} \) constitutes about \( 0.78 \beta_0 \) for \( r \approx 0.25^2 \), and

\[
\bar{a}_2^{\text{BLM}}(r; \mu) \simeq -0.69 \beta_0 z_0(r) \quad \text{at } r = 0.25^2 \quad \text{and} \quad \frac{\mu}{m_b} = \frac{1}{4.6}
\]

(12)

All BLM coefficients can be readily computed numerically along the lines of Ref. \([5]\) using Eq. (A.8) given below in Appendix. The required expression for the first-order perturbative correction with non-zero gluon mass is given in Ref. \([10]\). This yields

\[
\bar{a}_3^{\text{BLM}}(0.25^2; \mu) \simeq -0.52 \beta_0^3 z_0(r), \quad \bar{a}_4^{\text{BLM}}(0.25^2; \mu) \simeq -0.27 \beta_0^3 z_0(r) \quad \text{at } \frac{\mu}{m_b} = \frac{1}{4.6}
\]

(13)

BLM-wise, it is not difficult to perform the all-order resummation of the perturbative corrections in terms of the running masses. The result reads

\[
A^{\text{BLM}}(r; \mu) \equiv \frac{1}{z_0(r)} \left\{ z_0(r) + \bar{a}_1(r; \mu) \frac{\alpha_s(m_b)}{\pi} + \bar{a}_2^{\text{BLM}}(r; \mu) \left( \frac{\alpha_s(m_b)}{\pi} \right)^2 + \bar{a}_3^{\text{BLM}}(r; \mu) \left( \frac{\alpha_s(m_b)}{\pi} \right)^3 + \ldots \right\} \simeq 0.880;
\]

(14)

it is independent of the scale for \( \alpha_s \) one starts with.

The distribution of the gluon virtualities \( Q \) in \( \Gamma_{s\ell}(b \to c) \) is illustrated in Fig. 1 in the case of pole masses \( (\mu = 0) \) and for \( \mu = 1 \text{ GeV}, \mu = 1.5 \text{ GeV} \) and \( \mu = 2 \text{ GeV} \). The effect of removing the infrared domain is self-manifest. Moreover, it is evident that a too significant part of the perturbative corrections with pole masses comes from gluon momenta below 500 MeV, which would then cast legitimate doubts on the numerical results.

\[\text{Recent analysis by DELPHI } [8] \text{ yielded } m_b(1 \text{ GeV}) \simeq 4.6 \text{ GeV and } r \approx 0.06.\]
Figure 1: Gluon momentum scale distribution in $\Gamma_{sl}(b \to c)$. Solid, dashed and dot-dashed lines correspond to $\mu = 1$ GeV, 1.5 GeV and 2 GeV, respectively; lighter short-dashed line illustrates the case of $\mu = 0$ (pole masses). The area under each curve gives the first-order perturbative coefficient.

The growing component of the BLM series becomes sign-alternating starting the third or fourth order, this is a result of the infrared-free form of the width in the OPE-motivated approach. Consequently, the second-order approximation already turns out quite accurate, and the final number is practically reproduced by including the third-order term. It is also worth noting that using instead of $\alpha_s^{\overline{MS}}(m_b)$ a better physically justified expansion parameter [11, 3], for example the $V$-scheme $\alpha_s^{(V)}(m_b)$ or the ‘dipole radiation’ $\alpha_s^{(\omega)}(m_b)$ yields a nearly precise perturbative factor already to the first order:

$$A^{(V)}(r; \mu) \simeq (1 - 0.105 - 0.018 + 0.017 - ...) .$$  \hspace{1cm} (15)

The case of $\Gamma_{sl}^{\text{pert}}(b \to u)$ is technically simpler. Here the full second-order corrections are known analytically [12], $a_2^{(0)}(0) \simeq 5.54$, we have $\delta_2^{(0)} \simeq -6.9$ and, therefore

$$a_1(0; 1 \text{ GeV}) \simeq -0.29, \quad a_2^{\text{non-BLM}}(0; 1 \text{ GeV}) \simeq -1.35, \quad a_2^{\text{BLM}}(0; 1 \text{ GeV}) \simeq 0.44 \beta_0 ;$$  \hspace{1cm} (16)

third- and fourth-order BLM corrections amount to

$$a_3^{\text{BLM}}(0; 1 \text{ GeV}) \simeq 1.23 \beta_0^2, \quad a_4^{\text{BLM}}(0; 1 \text{ GeV}) \simeq 2.2 \beta_0^3 .$$  \hspace{1cm} (17)

A meaningful BLM summation for $b \to u$ would require, however incorporating the four-fermion operator [13] $\bar{b} \gamma_\mu (1-\gamma_5) u \bar{u} \gamma_\nu (1-\gamma_5) b$ ($\delta_{\mu\nu} - \delta_{\mu0} \delta_{\nu0}$) as well – it is strongly enhanced and has significant negative anomalous dimension. Accounting for this operator affects already the tree phase space coefficient at order $\mu^3/m_b^3$. 

4
\section{\(\Gamma_{sl}(B)\)}

In the complete OPE prediction for the semileptonic width, there remains a minor arbitrariness in incorporating the power corrections from the chromomagnetic and Darwin terms until their coefficients are computed with the \(\mathcal{O}(\alpha_s)\) accuracy.\footnote{The coefficient for the kinetic operator equals to the one in the parton term, while for the \(LS\) operator it is a combination of the latter and the coefficient for the chromomagnetic operator, see Ref. [2].} There are reasons to think that the factorized form has some advantages:

\[ \Gamma_{sl}(B) = \Gamma_0 A^{pert} \left[ z_0(r) \left( 1 - \frac{\mu^2 - \mu_G^2 + \tilde{\rho}_{1}^{3} + \rho_{LS}^{3}}{2m_b^2} \right) - 2(1 - r)^4 \frac{\mu^2 - \tilde{\rho}_{1}^{3} + \rho_{LS}^{3}}{m_b^2} + D(r) \frac{\tilde{\rho}_{D}^{3}}{m_b^3} + \ldots \right], \]

where \(D(r)\) can be obtained from Ref. [14]:

\[ D(r) = 8 \ln r + \frac{34}{3} - \frac{32}{3} r - 8r^2 + \frac{32}{3} r^3 - \frac{10}{3} r^4 \approx -18.3 z_0 \]

The kinetic and chromomagnetic expectation values here are finite-\(m_Q\) matrix elements in actual \(B\) mesons. (As argued in Ref. [15], their shift compared to the limit \(m_Q \to \infty\) is strongly suppressed.)

To give the numerical prediction, we evaluate this at the canonical values \(m_b(1\, \text{GeV}) = 4.6\, \text{GeV},\ \mu^2(1\, \text{GeV}) = 0.4\, \text{GeV}^2,\ \mu_G^2(1\, \text{GeV}) = 0.35\, \text{GeV}^2,\ \tilde{\rho}_D^{3}(1\, \text{GeV}) = 0.12\, \text{GeV}^3\) and \(\rho_{LS}^{3}(1\, \text{GeV}) = -0.15\, \text{GeV}^3\) using \(r = 0.25^2\) and \(\alpha_s(m_b) = 0.22:\)

\[ |V_{cb}| \simeq V_{cb}^{(0)} \left( \frac{\text{Br}_{sl}(B)}{0.105} \right)^{\frac{3}{2}} \left( \frac{1.55\, \text{ps}}{\tau_B} \right)^{\frac{3}{2}} (1 - 4.8 [\text{Br}(B \to X_u \ell \nu) - 0.0018]) , \quad V_{cb}^{(0)} \simeq 0.0421 . \]

The above evaluation uses the full second-order perturbative corrections with BLM resummation Eq. (14), where the added non-BLM term is computed at \(\alpha_s = 0.25\). The resulting value of \(V_{cb}^{(0)}\) acquires extra factors 0.993, 1.002 or 1.004 if one discards \(\mathcal{O}(\alpha_s^3)\) and higher terms, including only the third-order BLM correction, or both third and fourth orders, respectively. The dependence on the heavy quark parameters is as follows [16]:

\[ |V_{cb}| = V_{cb}^{(0)} \times [1 + 0.4 (\alpha_s(m_b) - 0.22)] \times \]

\[ [1 - 0.65 (m_b(1\, \text{GeV}) - 4.6\, \text{GeV}) + 0.40 (m_c(1\, \text{GeV}) - 1.15\, \text{GeV}) + 0.01 (\mu^2 - 0.4\, \text{GeV}^2) + 0.10 (\tilde{\rho}_D^{3} - 0.12\, \text{GeV}^3) + 0.05 (\mu_G^2 - 0.35\, \text{GeV}^2) - 0.01 (\rho_{LS}^{3} + 0.15\, \text{GeV}^3) ] . \]

It should be noted that the Darwin expectation value \(\tilde{\rho}_{D}^{3}\) appearing above is not a true matrix element normalized at 1 GeV, but is rather understood as the one extrapolated to \(\mu = 0: \) to two loops \(\tilde{\rho}_{D}^{3} \approx \rho_{D}^{3}(1\, \text{GeV}) - 0.1\, \text{GeV}^3;\) the perturbative coefficients have been determined correspondingly. Modifying equations for using \(\rho_D^{3}(1\, \text{GeV})\) proper is
straightforward. This does not change the value of $|V_{cb}|$ by any appreciable amount. The only drawback of the adopted simplified option is that the apparent extracted value of $\hat{\rho}_D^3$ is strongly correlated with the approximation applied.

To summarize, the perturbative corrections to the total semileptonic widths of beauty particles are modest and allow precision theoretical control when viewed as the Wilson coefficient of the leading heavy quark operator in the OPE. This assumes using well-defined low-scale running masses; these masses can be accurately extracted from experiment. As was pointed out long ago [3], the actual short-distance perturbative corrections to semileptonic widths are only about 10 percent, and can be accurately evaluated already at the one-loop level adopting a physically motivated perturbative coupling.

Acknowledgments: I am grateful to M. Battaglia, D. Benson, I. Bigi, P. Gambino, T. Mannel and P. Roudeau for useful discussions. This work was supported in part by the NSF under grant number PHY-0087419.

Appendix

Here a brief description of technicalities involved is given following Ref. [5]. Computing the BLM corrections is most straightforward in the generalized dispersive approach [17] employing the dispersion relation for the dressed gluon propagator:

$$\frac{\alpha_s(k^2)}{k^2} \left( \delta_{\mu\nu} - c \frac{k_\mu k_\nu}{k^2} \right) = -\frac{1}{\pi} \text{Im} \int \frac{d\lambda^2}{\lambda^2} \frac{\alpha_s(-\lambda^2)}{k^2 + \lambda^2} \left( \delta_{\mu\nu} - c \frac{k_\mu k_\nu}{k^2} \right).$$  \hspace{1cm} (A.1)

The form of the integrand suggests considering for a generic observable $A$ the one-loop diagram with a non-zero gluon mass $\lambda$

$$A(\lambda^2) = 1 + \frac{\alpha_s}{\pi} A_1(\lambda^2)$$ \hspace{1cm} (A.2)

and integrating it over $\lambda^2$ with the weight $\rho(\lambda^2) \frac{d\lambda^2}{\lambda^2} = -\frac{1}{\pi^2} \text{Im} \alpha_s(-\lambda^2) \frac{d\lambda^2}{\lambda^2}$:

$$A_{\text{BLM}} = 1 + \int \frac{d\lambda^2}{\lambda^2} A_1(\lambda^2) \rho(\lambda^2).$$ \hspace{1cm} (A.3)

A simple trick appears useful. Expressing

$$A_1(\lambda^2) = A_1(0) \frac{M^2}{M^2 + \lambda^2} - \left( A_1(0) \frac{M^2}{M^2 + \lambda^2} - A_1(\lambda^2) \right)$$

with an arbitrary $M$, the integral of the first term yields identically $A_1(0) \frac{\alpha_s(M^2)}{\pi}$ and, therefore

$$A_{\text{BLM}} = 1 + A_1(0) \frac{\alpha_s(M^2)}{\pi} - \int \frac{d\lambda^2}{\lambda^2} \rho(\lambda^2) \left( A_1(0) \frac{M^2}{M^2 + \lambda^2} - A_1(\lambda^2) \right).$$ \hspace{1cm} (A.4)
The standard BLM expansion amounts to using literal one-loop \( \alpha_s(k^2) \):
\[
\alpha_s(k^2) = \frac{\alpha_s(Q^2)}{1 + \frac{\beta_0 \alpha_s(Q^2)}{4\pi} \ln \frac{k^2}{Q^2}}.
\] (A.5)

As had been shown in [18], in this setting the dispersive approach reproduces the literal BLM series.

Taking \( M^2 = e^{\frac{5}{3}} m_Q^2 \) we obtain expansion directly in terms of \( \alpha_s^{\overline{\text{MS}}} (m_Q) \):
\[
A_{BLM}^{\overline{\text{MS}}} = 1 + A_1(0) \frac{\alpha_s(m_Q)}{\pi}
+ \int_{-\infty}^{\infty} \frac{dt}{1 + \beta_0 \alpha_s(t - \frac{5}{3})^2} \left( A_1(0) \frac{1}{1 + e^{-\frac{\lambda^2}{3}}} - A_1(e^t m_Q^2) \right)
- \frac{4}{\beta_0} \left[ \frac{m_Q^2}{m_Q^2 - \Lambda_V^2} A_1(0) - A_1(-\Lambda_V^2) \right],
\] (A.6)

where \( \alpha_s \equiv \alpha_s^{\overline{\text{MS}}}(m_Q) \), and
\[
\Lambda_V^2 = m_Q^2 e^{-\frac{4\pi}{\beta_0 \alpha_s(m_Q)} + \frac{5}{3}}.
\] (A.7)

This form is convenient for numerical integration. It also allows the straightforward expansion in \( \alpha_s(m_Q) \):
\[
A_{BLM}^{\overline{\text{MS}}} = 1 + A_1(0) \frac{\alpha_s(m_Q)}{\pi}
+ \sum_{n=0}^{\infty} \frac{4}{\beta_0} \left( \frac{\beta_0 \alpha_s(m_Q)}{4\pi} \right)^{n+2} \times
\sum_{k=0}^{n} (-\pi^2)^k C_{n+1}^{2k+1} \cdot \int \frac{d\lambda^2}{\lambda^2} \left[ \ln \frac{m_Q^2}{\lambda^2} + \frac{5}{3} \right]^{n-2k} \left( A_1(0) \frac{m_Q^2}{m_Q^2 + e^{-\beta_0 \alpha_s(m_Q)} - \Lambda_V^2} - A_1(\lambda^2) \right).
\] (A.8)

Since the infrared part is removed from the one-loop diagram for the total width, the corresponding \( A_1(\lambda^2) \) is a real analytic function in the vicinity of zero, and no ambiguity in Eq. (A.6) is encountered. Nevertheless, as follows from the generating integral (A.6) the series (A.8) is only asymptotic.

Computing BLM corrections with the running masses \( m_Q(\mu) \) requires expressions for the order-\( \alpha_s \) terms in \( \overline{\Lambda}_{\text{pert}}(\mu) \) and \( \mu^2_{\text{pert}}(\mu) \). They are given by integrating the SV spectral density (Eqs. (16), (19) of Ref. [5]) with the proper power of energy:
\[
\overline{\Lambda}_{\text{pert}}(\mu; \lambda) = \frac{16 \alpha_s}{9\pi} \vartheta (\mu^2 - \lambda^2) \left[ (1 - \frac{\lambda^2}{4\mu}) \sqrt{\mu^2 - \lambda^2} - \frac{3\pi}{8} \lambda + \frac{3}{4} \lambda \arctan \frac{\lambda}{\sqrt{\mu^2 - \lambda^2}} \right],
\]
\[
\mu^2_{\text{pert}}(\mu; \lambda) = \frac{4 \alpha_s}{3\pi} \vartheta (\mu^2 - \lambda^2) \left( \frac{\mu^2 - \lambda^2}{\mu} \right)^{3/2}.
\] (A.9)

As discussed in Refs. [18, 17], the quantity \( -\lambda^2 \frac{dA_1(\lambda^2)}{\lambda^2} \) can be qualitatively viewed as describing the contribution of gluons with virtuality \( \lambda^2 \) in the one-loop process. This convention has been adopted in Fig. 1.
References

[1] I. Bigi, M. Shifman, N. Uraltsev and A. Vainshtein, *Phys. Rev.* **D50** (1994) 2234.

[2] I.I. Bigi, M. Shifman, N.G. Uraltsev and A. Vainshtein, *Phys. Rev.* **D52** (1995) 196.

[3] N.G. Uraltsev, *Int. J. Mod. Phys.* **A11** (1996) 515.

[4] I. Bigi, M. Shifman, N. Uraltsev and A. Vainshtein, *Phys. Rev.* **D56** (1997) 4017.

[5] N.G. Uraltsev, *Nucl. Phys.* **B491** (1997) 303.

[6] A. Sirlin, *Nucl. Phys.* **B71** (1974) 29, and *Rev. Mod. Phys.* **50** (1978) 573.

[7] A. Czarnecki, K. Melnikov and N. Uraltsev, *Phys. Rev. Lett.* **80** (1998) 3189 and hep-ph/9708372.

[8] M. Battaglia et al., hep-ph/0210319.

[9] A. Czarnecki and K. Melnikov, *Phys. Rev. Lett.* **78** (1997) 3630; *Phys. Rev.* **D59** (1999) 014036.

[10] P. Ball, M. Beneke and V. Braun, *Phys. Rev.* **D52** (1995) 3929.

[11] M. Shifman and N.G. Uraltsev, *Int. Journ. Mod. Phys.* **A10** (1995) 4705.

[12] T. van Ritbergen, *Phys. Lett.* **B454** (1999) 353.

[13] I. Bigi and N. Uraltsev, *Nucl. Phys.* **B423** (1994) 33.

[14] M. Gremm and A. Kapustin, *Phys. Rev.* **D55** (1997) 6924.

[15] N. Uraltsev, *Phys. Lett.* **B545** (2002) 337.

[16] N. Uraltsev, hep-ph/0210044, Proc. ICHEP-2002, Amsterdam, the Netherlands.

[17] Yu.L. Dokshitzer, G. Marchesini and B.R. Webber, *Nucl. Phys.* **B469** (1996) 93.

[18] P. Ball, M. Beneke and V.M. Braun, *Nucl. Phys.* **B452** (1995) 563.