ON THE ZERO-IN-THE-SPECTRUM CONJECTURE

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Abstract. We prove that the answer to the "zero-in-the-spectrum" conjecture, in its form, suggested by J. Lott, is negative. Namely, we show that for any \( n \geq 6 \) there exists a closed \( n \)-dimensional smooth manifold \( M^n \), so that zero does not belong to the spectrum of the Laplace-Beltrami operator acting on the \( L^2 \) forms of all degrees on the universal covering \( \tilde{M} \).

1. The Main result

M. Gromov formulated the following conjecture (cf. [G1], p. 120; [G2], p. 21, Problem 0.5. F1, and also [G2], p. 238):

Conjecture A. Let \( M \) be a closed aspherical manifold; is it true that zero is always in the spectrum of the Laplace-Beltrami operator \( \Delta_p \), acting on the square integrable \( p \)-forms on the universal covering \( \tilde{M} \), for some \( p \)?

If the Strong Novikov Conjecture holds for the fundamental group \( \pi_1(M) \), then \( 0 \in \text{Spec}(\Delta_p) \) for some \( p \), cf. [L], p. 371. Hence a counterexample to Conjecture A would be also a counterexample to the Strong Novikov Conjecture.

G. Yu obtained in [Yu1], [Yu2] results, confirming Conjecture A under some additional assumptions.

In 1991 J. Lott raised a more general "zero-in-the-spectrum" question:

Conjecture B. Is it true, that for any complete Riemannian manifold \( M \) zero is always in the spectrum of the Laplace-Beltrami operator \( \Delta_p \), acting on the square integrable \( p \)-forms on \( M \), for some \( p \)?

We refer to the survey articles [L] and [Lu].

J. Lott showed in [L] that Conjecture B is true for manifolds of low dimension and also for some classes of higher dimensional manifolds.

In this article we give negative answers to Conjecture B and also to a version of Conjecture A where one removes the assumption of asphericity of \( M \). We prove the following Theorem.

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Theorem 1. For any \( n \geq 6 \) there exists a closed \( n \)-dimensional smooth manifold \( M \), so that for any \( p = 0, 1, \ldots, n \) the zero does not belong to the spectrum of the Laplace-Beltrami operator

\[
\Delta_p : \Lambda^p(\tilde{M}) \to \Lambda^p(\tilde{M}),
\]
acting on the space of \( L^2 \)-forms \( \Lambda^p(\tilde{M}) \) on the universal covering \( \tilde{M} \) of \( M \).

Our proof of Theorem 1 will be based on the fact that it can be restated in an equivalent form using the notion of extended \( L^2 \)-homology, introduced in [F1]:

Theorem 2. For any \( n \geq 6 \) there exists a closed orientable \( n \)-dimensional smooth manifold \( M \), so that extended \( L^2 \)-homology \( H^p(M; \ell^2(\pi)) = 0 \) vanishes for all \( p \). Here \( \pi \) denotes the fundamental group \( \pi = \pi_1(M) \), and \( \ell^2(\pi) \) denotes the \( L^2 \)-completion of the group ring \( \mathbb{C}[\pi] \).

Equivalence between Theorem 1 and Theorem 2 can be established as follows. Zero not in the spectrum of the Laplacian \( \Delta_p : \Lambda^p(\tilde{M}) \to \Lambda^p(\tilde{M}) \) for all \( p \) is equivalent to vanishing of the extended \( L^2 \)-cohomology \( H^*(M; \ell^2(\pi)) \), cf. [F1], according to the De Rham Theorem for extended cohomology, cf. section 7 of [F2] and also [S]. Vanishing of the extended \( L^2 \)-cohomology is equivalent to vanishing of the extended \( L^2 \)-homology \( H^*(M; \ell^2(\pi)) \), because of the Poincaré duality, cf. [F1], Theorem 6.7.

The proof of Theorem 2 is based on the following Theorem:

Theorem 3. There exists a finite 3-dimensional polyhedron \( Y \) with fundamental group \( \pi_1(Y) = \pi = F \times F \times F \), where \( F \) denotes a free group with two generators, such that the extended \( L^2 \)-homology \( H^p(Y; \ell^2(\pi)) = 0 \) vanishes for all \( p = 0, 1, \ldots \).

The strategy of our proof of Theorems 2 and 3 is similar to the method used by M.A. Kervaire [K], who constructed smooth homology spheres with prescribed fundamental groups. Our proof uses \( L^2 \)-analogue of the Hopf exact sequence.

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Our purpose is to show that there exists a 3-dimensional cell complex $Y$, obtained from $X$ by first taking a bouquet with finitely many copies of $S^2$ and then adding a finite number of 3-dimensional cells, so that
\[ H_i(Y; \ell^2(\pi)) = 0 \quad \text{for any} \quad i = 0, 1, \ldots. \] (1)

**B. $L^2$-Hopf exact sequence.** First we will calculate the extended $L^2$ homology of $X$ using the spectral sequence constructed in Theorem 9.7 of \[F1\]. We will work in the von Neumann category $C_{\pi}$ of Hilbert representations of $\pi$, cf. \[F2\], §2, example 5. We will denote by $E_i$ the universal covering of $X$.

By Theorem 9.7 of \[F1\], there exists a spectral sequence in the abelian category $\pi$ of Hilbert representations of $\pi$, cf. \[F2\], §1. Let $\tilde{X}$ be the universal covering of $X$. We will use the functors
\[ Tor^p_q(\ell^2(\pi), H_q(\tilde{X}; C)) \]
with values in the extended abelian category $\mathcal{E}(\pi)$, which are defined in \[F2\], page 660 under the assumption that the homology modules $H_q(\tilde{X}; C)$ of the universal covering admit finite free resolutions. In our case only two of these homology modules can be nonzero (for $q = 0$ and $q = 2$), and (since $H_2(\tilde{X}; C) = C \otimes \pi_2(X)$) our assumption (b) guarantees this finiteness condition for $q = 2$. The functor $Tor^p_q(\ell^2(\pi), H_q(\tilde{X}; C))$ can be denoted by
\[ \ell^2(\pi) \otimes_{\pi} H_q(\tilde{X}; C). \] (2)

It is an analog of the tensor product, cf. \[F2\], §6. Note that in general it takes values in the extended category $\mathcal{E}(\pi)$, i.e. it may have a nontrivial torsion part.

By Theorem 9.7 of \[F1\], there exists a spectral sequence in the abelian category $\mathcal{E}(\pi)$ with the following properties:

- the initial term $E^2_{p,q}$ equals $Tor^p_q(\ell^2(\pi), H_q(\tilde{X}; C))$.
- The spectral sequence converges to the extended homology $H_{p+q}(X; \ell^2(\pi))$.

For $q = 0$ we have $H_0(\tilde{X}; C) = C$, and $Tor^p_q(\ell^2(\pi), C)$ can be also understood as the extended $L^2$ homology of the Eilenberg - MacLane space $K(\pi, 1)$. We will use notation
\[ Tor^p_q(\ell^2(\pi), C) = H_q(\pi; \ell^2(\pi)). \] (3)

It is an analog of the group homology.

Since $X$ is two-dimensional, the spectral sequence contains only two rows ($q = 0$ and $q = 2$) and may have only one nontrivial differential. Hence we obtain the following isomorphisms:
\[ H_0(X; \ell^2(\pi)) \simeq H_0(\pi; \ell^2(\pi)) \quad \text{and} \quad H_1(X; \ell^2(\pi)) \simeq H_1(\pi; \ell^2(\pi)). \] (4)

These are Hurewicz type isomorphisms. The first nontrivial differential of the $E^2$-term is $d_2 : E^2_{3,0} \to E^2_{0,2}$. Here $E^2_{3,0} = H_3(\pi; \ell^2(\pi))$ and $E^2_{0,2} = \ell^2(\pi) \otimes_{\pi} H_2(\tilde{X}; C)$. Using the Hurewicz isomorphism $H_2(\tilde{X}) \simeq \pi_2(\tilde{X}) \simeq \pi_2(X)$, we may write
\[ E^2_{0,2} = \ell^2(\pi) \otimes_{\pi} \pi_2(X). \]
and the above differential is
\[ d_2 : \mathcal{H}_3(\pi; \ell^2(\pi)) \to \ell^2(\pi) \otimes_{\pi} \mathcal{H}_2(X). \] (5)

Note also that this differential must be a monomorphism (viewed as a morphism of the abelian category \( \mathcal{E}(\mathcal{C}_\pi) \)), since \( \mathcal{H}_3(X; \ell^2(\pi)) = 0 \) (recall that \( X \) is two-dimensional).

The spectral sequence above yields the following exact sequence
\[ 0 \to \mathcal{H}_3(\pi; \ell^2(\pi)) \xrightarrow{d_2} \ell^2(\pi) \otimes_{\pi} \mathcal{H}_2(X) \xrightarrow{h} \mathcal{H}_2(X; \ell^2(\pi)) \to \mathcal{H}_2(\pi, \ell^2(\pi)) \to 0. \] (6)

It is an \( L^2 \) analog of the Hopf’s exact sequence.

We conclude (using \( \mathcal{H}_2 \) and our assumptions (a)) that
\[ \mathcal{H}_0(X; \ell^2(\pi)) = \mathcal{H}_1(X; \ell^2(\pi)) = 0 \]
and \( \mathcal{H}_2(X; \ell^2(\pi)) \) can be found from the exact sequence
\[ 0 \to \mathcal{H}_3(\pi; \ell^2(\pi)) \to \ell^2(\pi) \otimes_{\pi} \mathcal{H}_2(X) \xrightarrow{h} \mathcal{H}_2(X; \ell^2(\pi)) \to 0. \] (7)

C. We will now specialize our discussion to the following group
\[ \pi = F \times F \times F, \]
where \( F \) is a free group with two generators. We will denote the free generators of the factor number \( r \) (where \( r = 1, 2, 3 \)) by \( a_1^r, a_2^r \). We will fix the presentation of \( \pi \) given by 6 generators \( a_1^1, a_2^1, a_1^2, a_2^2, a_1^3, a_2^3 \) and the following 12 relations
\[ (a_i^k, a_j^l) = 1, \quad \text{for} \quad k \neq l, \quad k, l \in \{1, 2, 3\}, \quad i, j \in \{1, 2\}, \]
where \((v, w) = vwv^{-1}w^{-1}\) denotes the commutator.

\( \pi \) satisfies condition (a) above, as follows from the Kunneth theorem for the extended \( L^2 \)-cohomology, cf. Appendix, Theorem \( \mathbb{H} \). Here we use that \( \mathcal{H}_j(F; \ell^2(F)) \) is nonzero only for \( j = 1 \) and has no torsion; hence the terms containing the periodic product in formula \( \mathbb{H} \), vanish; cf. Proposition \( \mathbb{H} \), statement (b).

Let us show that this group \( \pi \), together with its specified presentation, satisfies condition (b). The two-dimensional complex \( X \) constructed out of this presentation will have one zero-dimensional cell \( e^0 \), six 1-dimensional cells \( e_1^1, e_1^2, e_1^3 \) and 12 two-dimensional cells \( e_{ij}^{12}, e_{ij}^{13}, e_{ij}^{23} \). Here \( e_i^k \) denotes the 1-cell corresponding to the generator \( a_i^k \) and \( e_{ij}^{kl} \) denotes the 2-cell corresponding to the relation \( (a_i^k, a_j^l) = 1 \).

Let \( 0 \to C_2 \to C_1 \to C_0 \to 0 \) denote the chain complex of the universal covering \( \tilde{X} \). The boundary homomorphism acts as follows
\[ \partial e_i^k = (a_i^k - 1)e^0, \quad \partial e_{ij}^{kl} = (a_i^k - 1)e_j^l - (a_j^l - 1)e_i^k. \]

Using the Hurewicz isomorphisms \( \pi_2(X) = \pi_2(\tilde{X}) = H_2(\tilde{X}) \), we may compute the group \( \pi_2(X) \), viewed as a \( \mathbb{Z}[\pi] \)-module, as the kernel of \( \partial : C_2 \to C_1 \). Let
\[ x \in C_2, \quad x = \sum_{ij} \lambda_{ij}^{12} e_{ij}^{12} + \sum_{ij} \lambda_{ij}^{13} e_{ij}^{13} + \sum_{ij} \lambda_{ij}^{23} e_{ij}^{23}, \quad \lambda_{ij}^{kl} \in \mathbb{Z}[\pi], \]
be an element with $\partial x = 0$. Then the following equations hold
\[
\sum_i \lambda_{ij}^{12}(a_i^1 - 1) = \sum_i \lambda_{ij}^{23}(a_i^2 - 1),
\]
\[
\sum_j \lambda_{ij}^{12}(a_j^2 - 1) + \sum_j \lambda_{ij}^{13}(a_j^3 - 1) = 0,
\]
\[
\sum_i \lambda_{ij}^{13}(a_i^1 - 1) + \sum_i \lambda_{ij}^{23}(a_i^2 - 1) = 0.
\]
Hence we may write
\[
\lambda_{ij}^{12} = \sum_k \mu_{ijk}^{12}(a_k^1 - 1), \quad \mu_{ijk}^{12} \in \mathbb{Z}[\pi],
\]
\[
\lambda_{ij}^{23} = \sum_k \mu_{ijk}^{23}(a_k^1 - 1), \quad \mu_{ijk}^{23} \in \mathbb{Z}[\pi],
\]
\[
\lambda_{ij}^{13} = \sum_k \mu_{ijk}^{13}(a_k^2 - 1), \quad \mu_{ijk}^{13} \in \mathbb{Z}[\pi].
\]
Therefore we obtain
\[
\mu_{ijk}^{12} = \mu_{jki}^{23} = -\mu_{ikj}^{13}. \quad (8)
\]
Conversely, any system $\mu_{ijk}^{rs} \in \mathbb{Z}[\pi]$ satisfying (8) determines a cycle $x \in C_2, \partial x = 0$. This proves that $\pi_2(X)$ is a free $\mathbb{Z}[\pi]$-module of rank 8 with the basis
\[
x_{ijk} = (a_i^1 - 1)e_{jk}^{23} - (a_j^2 - 1)e_{ik}^{13} + (a_k^3 - 1)e_{ij}^{12}, \quad i, j, k \in \{1, 2\}. \quad (9)
\]
Note that the Eilenberg-MacLane complex $K = K(\pi, 1)$ is $B \times B \times B$, where $B$ is the bouquet of two circles; $\tilde{K}$ is obtained from $X$ by adding 8 three-dimensional cells $e_{ijk}$, where $i, j, k \in \{1, 2\}$, which correspond to different triple products of 1-dimensional cells of $B$. It is easy to see that the boundary of $e_{ijk}$ is given by
\[
\partial e_{ijk} = x_{ijk} \in \pi_2(X).
\]
The chain complex of the universal covering $\tilde{K}$ is $0 \to C_3 \to C_2 \to C_1 \to C_0 \to 0$, where $C_3$ is the free $\mathbb{Z}[\pi]$-module generated by the cells $e_{ijk}$ and the rest is the same as the chain complex of $X$.

For a discrete group $\pi$ we will denote by $C^*_R(\pi) \subset C^*_c(\pi)$ the real part of the reduced $C^*$-algebra, i.e. the norm closure of the real group ring $\mathbb{R}[\pi] \subset C[\pi]$.

D. Proposition. Let $F$ be the free group with generators $a_1, a_2$. Then there exist $u_1, u_2 \in C^*_R(F) \subset C^*_c(F)$ such that
(i) $u_1(a_1 - 1) + u_2(a_2 - 1) = 0$,
(ii) for any pair $v_1, v_2 \in \ell^2(\pi)$ with
\[
v_1(a_1 - 1) + v_2(a_2 - 1) = 0 \quad (10)
\]
there exists a unique $w \in \ell^2(\pi)$ such that
\[
v_1 = wu_1, \quad v_2 = wu_2.
\]
Here we consider $F$ as a subgroup of $\pi = F \times F \times F$ identifying it with one of the factors. The reduced $C^*$-algebra $C^*_r(F) \subset C^*_r(\pi)$ acts in the usual way on $\ell^2(\pi)$.

**Proof.** For convenience, we will assume in the proof that $F$ is the third factor in $\pi$. Consider the standard complex

$$
\ell^2(F) \oplus \ell^2(F) \xrightarrow{d} \ell^2(F), \quad (v_1, v_2) \mapsto v_1(a_1 - 1) + v_2(a_2 - 1).
$$

(11)
calculating the extended $L^2$ homology of the bouquet $S^1 \vee S^1$ of two circles with coefficients in $\ell^2(F)$. Since $F$ is not amenable, we know from Brooks’ theorem that $d$ is an epimorphism, i.e. $\mathcal{H}_0(S^1 \vee S^1; \ell^2(F)) = 0$. The Euler characteristic calculation shows that $\ker d = \mathcal{H}_1(S^1 \vee S^1; \ell^2(F))$ is one dimensional, i.e. it is isomorphic to $\ell^2(F)$.

Here we use the fact that the von Neumann algebra $\mathcal{N}(F)$ is a factor.

Let $P : \ell^2(F) \oplus \ell^2(F) \to \ell^2(F) \oplus \ell^2(F)$ be the orthogonal projection onto $\ker d$. We claim that the element $P(1, 0)$ belongs to

$$
C^*_R(F) \oplus C^*_R(F) \subset C^*_r(F) \oplus C^*_r(F) \subset \ell^2(F) \oplus \ell^2(F).
$$

(12)
Let $d^*$ be the adjoint of $d$. Then $\ker d = \ker(d^*d)$. Moreover, the image of $d^*d$ is closed and thus zero is an isolated point in the spectrum of $d^*d$. Hence we may use the holomorphic functional calculus (Cauchy’s formula) in order to express the projector $P$ as

$$
P = \frac{1}{2\pi i} \int_\Gamma (z - d^*d)^{-1}dz,
$$

where $\Gamma$ is a small circle around the origin. This explains that $P(v_1, v_2)$ belongs to $C^*_r(F) \oplus C^*_r(F)$ (cf. (12)), assuming that $v_1, v_2$ lie in the reduced $C^*$-algebra $C^*_r(F)$. Moreover, since the operator $d^*d$ is real, we obtain that $P(v_1, v_2) \in C^*_R(F) \oplus C^*_R(F)$, for $v_1, v_2 \in C^*_R(F)$.

We will set now

$$
(u_1, u_2) = P(1, 0).
$$

Then (i) is clearly satisfied.

We want to show that the restriction of $P$ on the first summand $\ell^2(F)$ in (11) gives an isomorphism $P : \ell^2(F) \to \ker d$. Since both ker $d$ and $\ell^2(F)$ have von Neumann dimension one, and the spectrum of $P$ contains only 0 and 1, we conclude that it is enough to show that $P(v, 0) = 0$ for $v \in \ell^2(F)$ implies $v = 0$. If $P(v, 0) = 0$ i.e. $(v, 0) \in (\ker d)^\perp$ then $(v, \ker d) = 0$, i.e. $v$ is orthogonal to the projection of $\ker d$ on the first summand $\ell^2(F)$. From this we will obtain that necessarily $v = 0$ if we will show that the projection of $\ker d$ on the first summand is dense.

Let $f_i : \ell^2(F) \to \ell^2(F)$ be operator $x \mapsto x(a_i - 1)$, where $i = 1, 2$. It is clear that $f_1$ and $f_2$ are injective and hence their images are dense. We claim that $f_1^{-1}(\text{im} \ f_2)$ is dense. If not, let $H$ denote the orthogonal complement to the closure of $f_1^{-1}(\text{im} \ f_2)$. Then we may apply Proposition 2.4 from [2]; it implies that $H$ must intersect im $f_2$, which is impossible. Hence it follows that the projection of $\ker d$ on the first summand $\ell^2(F)$ (which coincides with $f_1^{-1}(\text{im} \ f_2)$) is dense.
As a result we obtain from the above arguments that for any pair \((v_1, v_2) \in \ker d\) (i.e. which is a solution of (10)) there exists \(w \in \ell^2(F)\), so that \(P(w, 0) = (v_1, v_2)\), i.e. \(v_1 = wu_1\) and \(v_2 = wu_2\). This is in fact a part of our statement (ii).

In order to prove (ii) in full generality, observe that 
\[
\ell^2(\pi) = \ell^2(F) \hat{\otimes} \ell^2(F) \hat{\otimes} \ell^2(F),
\]
(cf. appendix) and thus (using the Kunneth theorem for extended \(L^2\) homology, cf. Theorem 5) we find that the kernel of the operator
\[
d : \ell^2(\pi) \oplus \ell^2(\pi) \rightarrow \ell^2(\pi), \quad (v_1, v_2) \mapsto v_1(a_1 - 1) + v_2(a_2 - 1),
\]
equals \(\ell^2(F) \hat{\otimes} \ell^2(F) \hat{\otimes} \mathcal{H}_1(S^1 \vee S^1; \ell^2(F))\). (ii) now follows. \(\square\)

E. Now we describe the kernel of the Hurewicz homomorphism
\[
h : \ell^2(\pi) \hat{\otimes}_\pi \pi_2(X) \rightarrow \mathcal{H}_2(X; \ell^2(\pi)).
\]
Let \(u_i^s \in C^*_r(\pi)\), where \(s = 1, 2, 3\) and \(i = 1, 2\), denote the element given by Proposition D applied to the factor \(F \subset \pi\) number \(s = 1, 2, 3\). Here we consider \(C^*_r(F)\) as being canonically embedded into the von Neumann algebra \(\mathcal{N}(\pi)\).

We claim that the kernel of the Hurewicz homomorphism \(h\) is generated by the element
\[
y = \sum_{ijk} u_i^1 u_j^2 u_k^3 x_{ijk} \in C^*_r(\pi) \hat{\otimes}_\pi \pi_2(X).
\]

More precisely, our statement is that any element \(x \in \ell^2(\pi) \hat{\otimes}_\pi \pi_2(X)\) with \(h(x) = 0\) has the form \(x = \mu y\) for some \(\mu \in \ell^2(\pi)\).

Note that the product \(\mu y\) has sense because the coefficients of \(y\) in the basis \(x_{ijk}\) belong to \(C^*_r(\pi) \subset C^*_r(\pi)\).

First, it is easy to check (using (11)) that \(h(y) = 0\).

Let
\[
x = \sum_{ijk} \mu_{ijk} x_{ijk} \in \ell^2(\pi) \hat{\otimes}_\pi \pi_2(X), \quad h(x) = 0,
\]
be an arbitrary element of \(\ker h\), where \(\mu_{ijk} \in \ell^2(\pi)\). Using (12), we obtain (equating to zero the coefficients of the cells \(e_{jk}^{23}\)) that for any pair of indices \(j, k\) holds
\[
\sum_{i=1}^2 \mu_{ijk}(a_i^1 - 1) = 0.
\]
Hence, applying Proposition D, we conclude that there exist \(\mu_{ijk} \in \ell^2(\pi)\) such that
\[
\mu_{ijk} = \mu_{jk} u_i^1.
\]
We write again \( h(x) = 0 \), equating to zero the coefficients of the cells \( e_{ik}^{13} \) and using (13). We obtain that for any pair of indices \( i, k \) holds
\[
\sum_j \mu_{jk} u_i^1 (a_j^2 - 1) = \left[ \sum_j \mu_{jk} (a_j^2 - 1) \right] u_i^1 = 0. \tag{16}
\]
Note that \( w u_i^1 = 0 \) for \( w \in \ell^2(\pi) \) implies \( w u_i^2 = 0 \) (using (10)) and from the uniqueness statement in Proposition D, (ii), we obtain that \( w = 0 \). Therefore (16) implies
\[
\sum_j \mu_{jk} (a_j^2 - 1) = 0
\]
and hence using Proposition D,
\[
\mu_{jk} = \mu_k u_j^2, \quad \text{where} \quad \mu_k \in \ell^2(\pi).
\]
Substitute again \( \mu_{ijk} = \mu_k u_i^1 u_j^2 \) into \( h(x) = 0 \) and equating to zero the coefficients of the cells \( e_{ik}^{13} \) we obtain
\[
\left[ \sum_k \mu_k (a_k^3 - 1) \right] u_i^1 u_j^2 = 0, \quad \text{and hence} \quad \sum_k \mu_k (a_k^3 - 1) = 0. \tag{17}
\]
Using Proposition D as above we finally obtain
\[
\mu_k = \mu u_k^3, \quad \text{where} \quad \mu \in \ell^2(\pi).
\]
Therefore, we find that \( \mu_{ijk} = \mu u_i^1 u_j^2 u_k^3 \) and \( x = \mu y \).

F. Our goal is to show that one may add 8 cells of dimension 3 to the bouquet \( X \vee S^2 \) such that the obtained 3-dimensional complex \( Y \) will have all trivial extended \( L^2 \) homology
\[
H_j(Y; \ell^2(\pi)) = 0, \quad j = 0, 1, \ldots.
\]

For the proof, let’s examine again the exact sequence (7):
\[
0 \to H_3(\pi; \ell^2(\pi)) \xrightarrow{\phi^*} \ell^2(\pi) \otimes_{\pi} \pi_2(X) \xrightarrow{h} H_2(X; \ell^2(\pi)) \to 0. \tag{18}
\]
As we know, \( \phi \) maps the generator \( y \) of \( H_3(\pi; \ell^2(\pi)) \) according to formula (14), i.e. \( \phi \) is given by a matrix with entries in \( C^*_R(\pi) \subset C^*_r(\pi) \). Let
\[
Q : \ell^2(\pi) \otimes_{\pi} \pi_2(X) \to \ell^2(\pi) \otimes_{\pi} \pi_2(X)
\]
denote the orthogonal projection onto \( (\text{im} \phi)^\perp \), the orthogonal complement of the image of \( \phi \). Since \( X \) is two-dimensional, \( H_2(X; \ell^2(\pi)) \) in has no torsion and therefore \( \text{im} \phi \) is closed. Note that \( (\text{im} \phi)^\perp \) coincides with \( \ker(\phi^*) \). Since the image of \( \phi \phi^* \) is closed we conclude that zero is an isolated point in the spectrum of \( \phi \phi^* \) and hence we may write
\[
Q = \frac{1}{2\pi i} \int_{\Gamma} (z - \phi \phi^*)^{-1} dz,
\]
where $\Gamma$ is a small circle round zero. Therefore, in the basis $x_{ijk}$ the projector $Q$ is given a $(8 \times 8)$-matrix with entries in $C^*_R(\pi)$.

The projective $C^*_R(\pi)$-module determined by $Q$ is stably free; we know that adding a free one-dimensional module (generated by $y$) makes it free. Therefore we may consider the bouquet $X_1 = X \cup S^2$ so that $\mathcal{H}_2(X_1; \ell^2(\pi)) = \mathcal{H}_2(X; \ell^2(\pi)) \oplus \ell^2(\pi)$ and $\pi_2(X_1) = \pi_2(X) \oplus \mathbb{Z}[\pi]$. Thus, the exact sequence (18) for $X_1$

$$0 \to \mathcal{H}_3(\pi; \ell^2(\pi)) \xrightarrow{\gamma} \ell^2(\pi) \otimes_\pi \pi_2(X_1) \xrightarrow{h} \mathcal{H}_2(X_1; \ell^2(\pi)) \to 0.$$  

will have the following property: the orthogonal projection

$Q_1 : \ell^2(\pi) \otimes_\pi \pi_2(X_1) \to \ell^2(\pi) \otimes_\pi \pi_2(X_1)$

onto $(\text{im } \psi)^{\perp}$ is given by a $(9 \times 9)$-matrix with entries in $C^*_R(\pi)$ which determines a free $C^*_R(\pi)$-module of rank 8.

We may reformulate the last statement as follows: there exists a $\mathbb{Z}[\pi]$-homomorphism

$$\gamma : (\mathbb{Z}[\pi])^8 \to C^*_R(\pi) \otimes_\pi \pi_2(X_1)$$  

such that the following composite

$$\ell^2(\pi) \otimes_\pi (\mathbb{Z}[\pi])^8 \xrightarrow{1 \otimes \gamma} \ell^2(\pi) \otimes_\pi \pi_2(X_1) \xrightarrow{h} \mathcal{H}_2(X_1; \ell^2(\pi))$$  

is an isomorphism. Now we will use the fact that the rational group ring $\mathbb{Q}[\pi]$ is dense in $C^*_R(\pi)$ with respect to the operator norm topology. Hence we may approximate $\gamma$ by a $\mathbb{Z}[\pi]$-homomorphism

$$\gamma_1 : (\mathbb{Z}[\pi])^8 \to \mathbb{Q}[\pi] \otimes_\pi \pi_2(X_1)$$

so that the similar composition (21) is an isomorphism. Finally, we may multiply $\gamma_1$ by a large integer $N$ to obtain a $\mathbb{Z}[\pi]$-homomorphism

$$\gamma_2 : (\mathbb{Z}[\pi])^8 \to \mathbb{Z}[\pi] \otimes_\pi \pi_2(X_1) = \pi_2(X_1)$$

such that the composition

$$(\ell^2(\pi))^8 = \ell^2(\pi) \otimes_\pi (\mathbb{Z}[\pi])^8 \xrightarrow{1 \otimes \gamma_2} \ell^2(\pi) \otimes_\pi \pi_2(X_1) \xrightarrow{h} \mathcal{H}_2(X_1; \ell^2(\pi))$$

is an isomorphism.

Let $z_1, \ldots, z_8 \in \pi_2(X_1)$ be images of a free basis of $(\mathbb{Z}[\pi])^8$ under $\gamma_2$. Realize each $z_j$ by a continuous map $f_j : S^2 \to X_1$, where $j = 1, \ldots, 8$, and let

$$Y = X_1 \cup e_1^3 \cup \cdots \cup e_8^3$$

be obtained from $X_1$ by glueing 8 three-dimensional cells to $X_1$ along $f_1, \ldots, f_8$. We claim that

$$\mathcal{H}_j(Y; \ell^2(\pi)) = 0 \quad \text{for all} \quad j = 0, 1, \ldots .$$  

In order to show this, we note that $\mathcal{H}_j(Y, X; \ell^2(\pi))$ vanishes for all $j \neq 3$ and the 3-dimensional extended $L^2$ homology $\mathcal{H}_3(Y, X; \ell^2(\pi))$ equals $(\ell^2(\pi))^8$. The boundary homomorphism $\partial : \mathcal{H}_3(Y, X; \ell^2(\pi)) \to \mathcal{H}_2(X; \ell^2(\pi))$ is an isomorphism since it coincides
with (22). Hence (23) follows from the homological exact sequence of the pair \((Y, X)\). This completes the proof of Theorem 3.

**G.** Now we may complete the proof of Theorem 2. We have constructed above a finite 3-dimensional polyhedron \(Y\). For any \(n \geq 6\) we may embed \(Y\) into \(\mathbb{R}^{n+1}\) as a subpolyhedron. Let \(N \subset \mathbb{R}^{n+1}\) be the regular neighborhood of \(Y \subset \mathbb{R}^{n+1}\). We will define \(M\) as the boundary of \(N\), i.e. \(M = \partial N\).

First note that the inclusion \(M \to N\) induces an isomorphism of the fundamental groups and thus \(\pi_1(M) = \pi = F \times F \times F\), where \(F\) is a free group in two generators. We want to show that

\[
\mathcal{H}_j(M; \ell^2(\pi)) = 0, \quad \text{for all} \quad j = 0, 1, \ldots
\]

In the exact homological sequence

\[
\ldots \to \mathcal{H}_j(M; \ell^2(\pi)) \to \mathcal{H}_j(N; \ell^2(\pi)) \to \mathcal{H}_j(N, M; \ell^2(\pi)) \to \ldots
\]

we have \(\mathcal{H}_j(N; \ell^2(\pi)) = 0\). Also, \(\mathcal{H}_j(N, M; \ell^2(\pi)) \simeq \mathcal{H}^{n+1-j}(N; \ell^2(\pi))\) by the Poincaré duality (cf. [F1]) and \(\mathcal{H}^{n+1-j}(N; \ell^2(\pi)) = 0\) because of (23) using the Universal Coefficients Theorem (cf. [F1]). Hence, (24) follows.

**Appendix: Kunneth theorem for extended \(L^2\) cohomology**

1. A *Hilbert category* \(\mathcal{C}\) is defined as an additive subcategory of the category of Hilbert spaces and bounded linear maps, such for any morphism \(f : H \to H'\) of \(\mathcal{C}\) the inclusion \(\ker(f) \subset H\) belongs to \(\mathcal{C}\) and also the adjoint map \(f^* : H' \to H\) belongs to \(\mathcal{C}\), cf. [F1]. It is shown in [F1] that any Hilbert category can be canonical embedding into an abelian category \(\mathcal{E}(\mathcal{C})\), called the *extended abelian category*.

Let \(\mathcal{C}, \mathcal{C}'\) and \(\mathcal{C}''\) be three Hilbert categories and let

\[
\hat{\otimes} : \mathcal{C} \times \mathcal{C}' \to \mathcal{C}''
\]

be a covariant functor of two variables (the "tensor product") such that

(a) for \(H \in \text{Ob}(\mathcal{C})\) and \(H' \in \text{Ob}(\mathcal{C}')\) the image \(H \otimes H'\) has as the underlying Hilbert space the tensor product of Hilbert spaces \(H\) and \(H'\);

(b) if \(f : H \to H_1\) is a morphism of \(\mathcal{C}\) and \(f' : H' \to H'_1\) is a morphism of \(\mathcal{C}'\) then \(f \otimes f' : H \otimes H' \to H_1 \otimes H'_1\) is the tensor product of bounded linear maps \(f\) and \(f'\).

Recall that the tensor product if Hilbert spaces \(H \otimes H'\) is defined as the Hilbert space completion of the algebraic tensor product \(H \otimes H'\) with respect to the following scalar product \(\langle v \otimes w, v' \otimes w' \rangle = \langle v, v' \rangle \cdot \langle w, w' \rangle\).

Suppose that \((C, d)\) and \((C', d')\) are chain complexes in \(\mathcal{C}\) and \(\mathcal{C}'\) correspondingly. We assume that all chain complexes are graded by non-negative integers and have a finite length. Their tensor product \((C, d) \otimes (C', d')\) (defined in the usual way) is a chain complex in \(\mathcal{C}''\). \((C, d) \otimes (C', d')\) is a projective chain complex in the abelian category \(\mathcal{E}(\mathcal{C}'')\) and its extended homology \(\mathcal{H}_* (C \otimes C')\) is an object of the extended category.
Our purpose is to express the extended homology of \((C, d) \hat\otimes (C', d)\) in terms of the extended homology \(H_*(C)\) of \((C, d)\) and \(H_*(C')\) of \((C', d)\).

2. Example. Suppose that \(G\) and \(H\) are discrete groups. Let \(C_G\) denote the category of Hilbert representations of \(G\). Recall, that an object of \(C\) is a Hilbert space with a unitary \(G\)-action which can be continuously and \(G\)-equivariantly imbedded into a finite direct sum \(\ell^2(G) \oplus \cdots \oplus \ell^2(G)\); morphisms of \(C\) are bounded linear maps commuting with the \(G\)-action. Then we have the tensor product functor
\[
\hat\otimes : C_G \times C_H \to C_{G \times H}
\] which is of a primary interest for us.

3. Tensor and periodic products. Given a tensor product \((25)\), it defines two bifunctors \(E(C) \times E(C') \to E(C')\), which we now describe. Let \(\mathcal{X} = (\alpha : A' \to A) \in \text{Ob}(E(C))\) and \(\mathcal{Y} = (\beta : B' \to B) \in \text{Ob}(E(C'))\) be two objects with \(\alpha\) and \(\beta\) injective. Consider the following chain complex in \(C''\)
\[
0 \to A' \hat\otimes B' \xrightarrow{(-1 \hat\otimes \beta)} (A' \hat\otimes B) \oplus (A \hat\otimes B') \xrightarrow{(\alpha \hat\otimes 1, 1 \hat\otimes \beta)} A \otimes B \to 0.
\] (27)
In other words, we view the objects \(\mathcal{X}\) and \(\mathcal{Y}\) as chain complexes of length 1 and then \((27)\) is the tensor product of these chain complexes. The extended homology of \((27)\) in dimension 0 will be called the tensor product of \(\mathcal{X}\) and \(\mathcal{Y}\):
\[
\mathcal{X} \hat\otimes \mathcal{Y} = ((\alpha \hat\otimes 1, 1 \hat\otimes \beta) : (A' \hat\otimes B) \oplus (A \hat\otimes B') \to A \hat\otimes B).
\] (28)
The extended homology of \((27)\) in dimension 1 will be called the periodic product of \(\mathcal{X}\) and \(\mathcal{Y}\):
\[
\mathcal{X} \ast \mathcal{Y} = \left(\begin{array}{c}
-1 \hat\otimes \beta \\
\alpha \hat\otimes 1
\end{array}\right) : A' \hat\otimes B' \to Z,
\] (29)
where
\[
Z = \ker \left(\begin{array}{c}
-1 \hat\otimes \beta \\
\alpha \hat\otimes 1
\end{array}\right) : (A' \hat\otimes B) \oplus (A \hat\otimes B') \to A \hat\otimes B).
\] (30)
It is easy to see that \(\mathcal{X} \hat\otimes \mathcal{Y}\) and \(\mathcal{X} \ast \mathcal{Y}\) are covariant functors of two variables.

Proposition 4. Let \(\hat\otimes : C \times C' \to C''\) be a tensor product functor \((25)\). Let \(\mathcal{X} \in \text{Ob}(E(C))\) and \(\mathcal{Y} \in \text{Ob}(E(C'))\). Then
(a) \(\mathcal{X} \hat\otimes \mathcal{Y}\) is projective if both \(\mathcal{X}\) and \(\mathcal{Y}\) are projective;
(b) \(\mathcal{X} \ast \mathcal{Y} = 0\) if \(\mathcal{X}\) or \(\mathcal{Y}\) is projective;
(c) \(\mathcal{X} \hat\otimes \mathcal{Y}\) is torsion if \(\mathcal{X}\) or \(\mathcal{Y}\) is torsion;
(d) If \(C''\) is a finite von Neumann category then \(\mathcal{X} \ast \mathcal{Y}\) is torsion for any \(\mathcal{X}\) and \(\mathcal{Y}\).
Proof. Statements (a) and (b) follow directly from the definitions.

Let’s prove (c) assuming that $\mathcal{X} = (\alpha : A' \to A)$ is torsion, i.e. $\text{im} \alpha \subset A$ is dense. From the definition of the tensor product $\otimes$ it follows that then the image of $\alpha \otimes 1 : A' \otimes B \to A \otimes B$ is dense and hence from (28) we see that $\mathcal{X} \otimes \mathcal{Y}$ is torsion.

It is enough to prove (d), assuming that both $\mathcal{X}$ and $\mathcal{Y}$ are torsion. Let $\mathcal{X} = (\alpha : A' \to A)$ and $\mathcal{Y} = (\beta : B' \to B)$ with $\alpha$ and $\beta$ injective and having dense images. Then $A'$ is isomorphic to $A$ and $B'$ is isomorphic to $B$ (cf. [F2], §2). Therefore (d) will follow if we can show that $Z$ (given by (31)) is isomorphic to $A \otimes B$. The projection of $Z$ on the first coordinate gives a morphism $Z \to A' \otimes B$ which is injective (obviously) and has a dense image (this follows from Proposition in §2 of [F2]). Hence we obtain (using Lemma in §2 of [F2]) that $Z$ is isomorphic to $A' \otimes B \simeq A \otimes B$.  

Theorem 5 (Kunneth formula). Extended homology $\mathcal{H}_n(C \otimes C')$ of a tensor product, where $(C, d)$ is a chain complex in $\mathcal{C}$ and $(C', d)$ is a chain complex in $\mathcal{C}'$, equals

$$\mathcal{H}_n(C \otimes C') = \bigoplus_{i+j=n} \mathcal{H}_i(C) \otimes \mathcal{H}_j(C') \oplus \bigoplus_{i+j=n-1} \mathcal{H}_i(C) \circ \mathcal{H}_j(C').$$  

(31)

Proof. Let $Z_i \subset C_i$ and $Z'_i \subset C'_i$ denote the subspaces of cycles.

We have the decomposition $C_i = Z_i \oplus Z_i^\perp$; the boundary homomorphism vanishes on $Z_i$ and maps $Z_i^\perp$ into $Z_{i-1}$. Let’s denote by $D_i$ the short chain complex $D_i = (d : Z_{i+1} \to Z_i)$, where $Z_i$ stands in degree $i$ and $Z_{i+1}$ stands in degree $i+1$. Then $C \simeq \bigoplus_{i=0}^\infty D_i$, i.e. $C$ is isomorphic to the direct sum of the chain complexes $D_i$.

Similarly, we define chain complexes $D'_j = (d : Z'_{j+1} \to Z'_j)$ and $C' \simeq \bigoplus_{j=0}^\infty D'_j$. Hence we obtain

$$C \otimes C' \simeq \bigoplus_{i,j} (D_i \otimes D'_j), \quad \mathcal{H}_n(C \otimes C') = \bigoplus_{i,j} \mathcal{H}_n(D_i \otimes D'_j).$$  

(32)

Now we observe that $D_i$ has nontrivial homology only in dimension $i$ and $\mathcal{H}_i(D_i) = \mathcal{H}_i(C)$; similarly, $D'_j$ has nontrivial homology only in dimension $j$ and $\mathcal{H}_j(D'_j) = \mathcal{H}_j(C')$. Therefore $D_i \otimes D'_j$ has nontrivial homology only in dimensions $i + j$ and $i + j + 1$, and

$$\mathcal{H}_{i+j}(D_i \otimes D'_j) = \mathcal{H}_i(C) \otimes \mathcal{H}_j(C'), \quad \mathcal{H}_{i+j+1}(D_i \otimes D'_j) = \mathcal{H}_i(C) \circ \mathcal{H}_j(C').$$  

(33)

according to our definition of the tensor and periodic products. Formula (31) now follows by combining (32) and (33).
Theorem 6 (Kunneth formula for extended $L^2$ homology). Let $X$, $X'$ be finite cell complexes with $\pi = \pi_1(X)$, $\pi' = \pi_1(X')$. Then

$$\mathcal{H}_n(X \times X'; \ell^2(\pi \times \pi')) \simeq \bigoplus_{i+j=n} \mathcal{H}_i(X; \ell^2(\pi)) \otimes \mathcal{H}_j(X'; \ell^2(\pi')) \oplus \bigoplus_{i+j=n-1} \mathcal{H}_i(X; \ell^2(\pi)) \ast \mathcal{H}_j(X'; \ell^2(\pi'))$$

(34)

where the tensor and periodic products are understood with respect to functor (26).

Proof. Let $C_\ast(\tilde{X})$ and $C_\ast(\tilde{X}')$ be the cell chain complexes of the universal coverings $\tilde{X}$ and $\tilde{X}'$. We apply the previous Theorem to chain complexes $C = \ell^2(\pi) \tilde{\otimes}_\pi C_\ast(\tilde{X})$ and $C' = \ell^2(\pi') \tilde{\otimes}_{\pi'} C_\ast(\tilde{X}')$. Note that $C$ is a chain complex in category $C_\pi$ (cf. Example above) and $\mathcal{H}_n(C) = \mathcal{H}_n(X; \ell^2(\pi))$. Similarly $C'$ is a chain complex in $C_{\pi'}$ and $\mathcal{H}_n(C') = \mathcal{H}_n(X'; \ell^2(\pi'))$. Formula (34) follows from (31) using the isomorphism $\ell^2(\pi) \tilde{\otimes} \ell^2(\pi') = \ell^2(\pi \times \pi')$ and the fact that the chain complex $C_\ast(\tilde{X}) \otimes_{\mathbb{Z}} C_\ast(\tilde{X}')$ over $\mathbb{Z}[\pi \times \pi']$ is isomorphic to $C_\ast(\tilde{X} \times \tilde{X}')$, where we consider the obvious product cell structure on $X \times X'$. \hfill \Box

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