ON GROMOV’S DIHEDRAL RIDIGIDITY CONJECTURE AND STOKER’S CONJECTURE

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Abstract. In this paper, we prove Gromov’s strong dihedral rigidity conjecture for convex Euclidean polyhedra in all dimensions. As a consequence, we answer positively the Stoker conjecture for convex Euclidean polyhedra in all dimensions.

1. Introduction

The main purpose of this paper is to solve (a strong version of) Gromov’s dihedral rigidity conjecture (Conjecture 1.4) for convex Euclidean polyhedra in all dimensions (Theorem 1.5). As a consequence, we answer positively the Stoker conjecture for convex Euclidean polyhedra in all dimensions (Theorem 1.6).

The Gromov dihedral rigidity conjecture is a conjecture concerning comparisons of scalar curvature, mean curvature and dihedral angles for Riemannian metrics on polyhedra. It can be viewed as scalar curvature analogue of the Alexandrov’s triangle comparisons for spaces whose sectional curvature is bounded below [1, 2]. The dihedral rigidity conjecture is one of the fundamental conjectures among the extensive list of conjectures and open questions on scalar curvature formulated by Gromov [6, 8, 9]. It has profound implications in geometry and mathematical physics. For example, it implies the positive mass theorem, a foundational result in general relativity and differential geometry [18, 19] [24] (cf. [13, Section 5] and [22, Discussion after Theorem 1.7]).

A cousin of the dihedral rigidity conjecture is Gromov’s dihedral extremality conjecture (Conjecture 1.2). In our previous joint paper with Yu [22], the authors completely settled Gromov’s dihedral extremality conjecture for convex polyhedra in all dimensions [22, Theorem 1.9]. In fact, Yu and the authors proved a more general theorem on comparisons of scalar curvatures, mean curvatures and dihedral angles between two compact manifolds with corners (possibly of different dimensions) [22, Theorem 1.7 & 1.8], which implies Gromov’s dihedral extremality conjecture for any n-dimensional convex polyhedron with n ≥ 2. Moreover, the results in [22] also imply a weaker version of Gromov’s dihedral rigidity conjecture for convex polyhedra in all dimensions. More precisely, under the conditions of Gromov’s dihedral rigidity conjecture (Conjecture 1.3), Yu and the authors showed that the metric g is Ricci flat. Note that for 3-dimensional manifolds, Ricci flatness coincides with flatness. As a consequence, Yu and the
authors solved Gromov’s dihedral rigidity conjecture for any 3-dimensional convex polyhedron [22, Theorem 1.9]. However the higher dimensional case of Gromov’s dihedral rigidity conjecture remained open.

In this paper, we solve Gromov’s strong dihedral rigidity conjecture (Conjecture 1.4) for convex Euclidean polyhedra in all dimensions. As a consequence, we answer positively the Stoker conjecture for convex Euclidean polyhedra in all dimensions.

Before we state our main results, let us recall Gromov’s dihedral extremality and dihedral rigidity conjectures for convex Euclidean polyhedra. Given a Riemannian metric $g$ on an oriented manifold $M$ with corners or more generally with polytope boundary (Definition 2.1), we shall denote the scalar curvature of $g$ by $\text{Sc}(g)$, the mean curvature of each face $F_i$ of $M$ by $H_g(F_i)$, and the dihedral angle function of two adjacent faces $F_i$ and $F_j$ by $\theta_{ij}(g)$. Here the dihedral angle $\theta_{ij}(g)_x$ at a point $x \in F_i \cap F_j$ is defined as follows.

**Definition 1.1.** Write $F_{ij} = F_i \cap F_j$. Let $u$ and $v$ be the unit inner normal vector of $F_{ij}$ with respect to $F_i$ and $F_j$ at $x \in F_{ij}$, respectively. Let $\theta_{ij}(g)_x$ be either the angle of $u$ and $v$, or $\pi$ plus this angle, depending on the vector $(u + v)/2$ points inward or outward, respectively. See Figure 1.

Here the angle $\theta_{ij}(g)_x$ takes value in $(0, \pi) \cup (\pi, 2\pi)$. Roughly speaking, if $M$ is convex at $x$, then $\theta_{ij}(g)_x < \pi$; and if $M$ is concave at $x$, then $\theta_{ij}(g)_x > \pi$.

We remark that if one allows the corner structure of the manifold $M$ to have degeneracy, then $\theta_{ij}(g)_x = \pi$ if $u = -v$, and $\theta_{ij}(g)_x = 0$ if $u = v$.

**Conjecture 1.2** (Gromov’s dihedral extremal conjecture for convex polyhedra, [7, Section 7]). Let $P$ be a convex polyhedron in $\mathbb{R}^n$ and $g$ the Euclidean metric on $P$. If $\bar{g}$ is a smooth Riemannian metric on $P$ such that

1. $\text{Sc}(\bar{g}) \geq \text{Sc}(g) = 0$,
2. $H_{\bar{g}}(F_i) \geq H_g(F_i) = 0$ for each face $F_i$ of $P$, and
3. $\theta_{ij}(\bar{g}) \leq \theta_{ij}(g)$ on each $F_{ij} = F_i \cap F_j$,

then we have

$\text{Sc}(\bar{g}) = 0, H_{\bar{g}}(F_i) = 0$ and $\theta_{ij}(\bar{g}) = \theta_{ij}(g)$

for all $i$ and for all $i \neq j$.

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Our sign convention for the mean curvature is that the mean curvature of the standard round sphere viewed as the boundary of a Euclidean ball is positive.
Conjecture 1.3 (Gromov’s dihedral rigidity conjecture for convex polyhedra, [6, Section 2.2]). Let $P$ be a convex polyhedron in $\mathbb{R}^n$ and $g$ the Euclidean metric on $P$. If $\overline{g}$ is a smooth Riemannian metric on $P$ such that

1. $\text{Sc}(\overline{g}) \geq \text{Sc}(g) = 0$,
2. $H_{\overline{g}}(F_i) \geq H_g(F_i) = 0$ for each face $F_i$ of $P$, and
3. $\theta_{ij}(\overline{g}) \leq \theta_{ij}(g)$ on each $F_{ij} = F_i \cap F_j$,

then $\overline{g}$ is also a flat metric.

In fact, Gromov proposed a stronger version of the dihedral rigidity conjecture for convex Euclidean polyhedra [9, Section 3.18].

Conjecture 1.4 (Gromov’s strong dihedral rigidity conjecture). Let $P$ be a convex polyhedron in $\mathbb{R}^n$ and $g$ the Euclidean metric on $P$. If $\overline{g}$ is a smooth Riemannian metric on $P$ such that

1. $\text{Sc}(\overline{g}) \geq \text{Sc}(g) = 0$,
2. $H_{\overline{g}}(F_i) \geq H_g(F_i) = 0$ for each face $F_i$ of $P$, and
3. $\theta_{ij}(\overline{g}) \leq \theta_{ij}(g)$ on each $F_{ij} = F_i \cap F_j$,

then $\overline{g}$ is flat and all codimension one faces of $(P, \overline{g})$ are flat; moreover, at every point $x \in P$, the manifold $(P, \overline{g})$ is locally isometric to $(P, g)$.

The first main result of this paper is the following theorem, which solves Gromov’s strong dihedral rigidity conjecture for convex Euclidean polyhedra (Conjecture 1.4).

Theorem 1.5. Let $P$ be a convex polyhedron in $\mathbb{R}^n$ and $g$ the Euclidean metric on $P$. If $\overline{g}$ is a smooth Riemannian metric on $P$ such that

1. $\text{Sc}(\overline{g}) \geq \text{Sc}(g) = 0$,
2. $H_{\overline{g}}(F_i) \geq H_g(F_i) = 0$ for each face $F_i$ of $P$, and
3. $\theta_{ij}(\overline{g}) \leq \theta_{ij}(g)$ on each $F_{ij} = F_i \cap F_j$,

then $\overline{g}$ is flat and all codimension one faces of $(P, \overline{g})$ are flat; moreover, at every point $x \in P$, the manifold $(P, \overline{g})$ is locally isometric to $(P, g)$.

We shall prove Theorem 1.5 by using twisted Dirac operators. In fact, we will prove a more general dihedral rigidity theorem (Theorem 2.3) that allows comparisons of possibly different manifolds, which includes Theorem 1.5 as a special case.

As a consequence of Theorem 1.5, we answer positively the Stoker conjecture for convex Euclidean polyhedra in all dimensions. The Stoker conjecture states that the dihedral angles of a convex Euclidean polyhedron determine the angles of each face [20]. More precisely, we have the following theorem.

Theorem 1.6. If $P_1$ and $P_2$ are two convex Euclidean polyhedra of the same combinatorial type such that all corresponding dihedral angles are equal, then all corresponding face angles\(^2\) are equal.

\(^2\)Here the face angles refer to the dihedral angles of each codimension one face (thought of as a polyhedron itself).
Let us mention briefly some of the previous work on the Stoker conjecture. There have been many attempts to solve the Stoker conjecture in the past fifty years. The conjecture was verified in some special cases. For example, Karcher verified the Stoker conjecture for a special class of 3-dimensional convex polyhedra [12]. There is also an analogous conjecture for convex hyperbolic polyhedra, which has also been known in some special cases. For example, Andreev [3] proved the Stoker conjecture for convex hyperbolic polyhedra when all dihedral angles are less than \( \pi/2 \). Furthermore, Mazzeo and Montcouquiol proved an infinitesimal version of the Stoker conjecture for both the Euclidean and hyperbolic cases [16, Theorem 1]. We should also mention that the analogue of Stoker’s conjecture for convex spherical polyhedra is false, due to counterexamples of Schlenker [17].

So far, we have been mainly concerned with convex polyhedra in Euclidean spaces. In fact, we can apply the same method of proof for Theorem 1.5 to prove analogous rigidity results for manifolds with smooth boundary. In particular, we have the following rigidity theorem for Euclidean balls.

**Theorem 1.7.** Suppose \((B, g)\) is a ball of radius \( r > 0 \) in the Euclidean space \( \mathbb{R}^n \). Let \((N, \overline{g})\) be a compact oriented \( n \)-dimensional spin manifold with smooth boundary and \( f: (N, \overline{g}) \to (B, g) \) a smooth map such that

1. \( \text{Sc}(\overline{g})_x \geq \text{Sc}(g)_{f(x)} = 0 \) for all \( x \in N \),
2. \( H_{\overline{g}}(\partial N)_y \geq H_g(\partial B)_{f(y)} = \frac{n-1}{r} \) for all \( y \in \partial N \),
3. \( f \) is distance-non-increasing on \( N \),
4. the degree of \( f \) is nonzero,

then \( f \) is an isometry.

Here \( f: N \to B \) is said to be distance-non-increasing at \( x \in N \) if \( \|df\|_x \leq 1 \), where \( df: TN \to TB \) is the tangent map.

The paper is organized as follows. In Section 2, we introduce a notion of manifolds with polytope boundary, which is a class of manifolds that includes for example all polyhedra. We prove a general dihedral rigidity theorem for flat manifolds with polytope boundary. Consequently, we prove Gromov’s strong dihedral rigidity conjecture for convex Euclidean polyhedra (Theorem 1.5) and the Stoker conjecture for convex Euclidean polyhedra (Theorem 1.6). In Section 3, we prove a scalar-mean rigidity theorem for Euclidean balls (Theorem 1.7).

**Acknowledgments.** We would like to thank Tian Yang for helpful comments.

2. Dihedral Rigidity of flat manifolds

In this section, we prove Gromov’s strong dihedral rigidity conjecture for convex Euclidean polyhedra (Theorem 1.5) and the Stoker conjecture for convex Euclidean polyhedra (Theorem 1.6).

2.1. Manifolds with polytope boundary. In this subsection, we introduce a notion of manifolds with polytope boundary.
Recall that $n$-dimensional smooth manifolds with corners are locally modeled on $[0, \infty)^k \times \mathbb{R}^{n-k}$ with $0 \leq k \leq n$. More precisely, let $M$ be a Hausdorff space. A chart $(U, \varphi)$ (possibly with corners) for $M$ is a homeomorphism $\varphi$ from an open subset $U$ of $M$ to an open subset of $[0, \infty)^k \times \mathbb{R}^{n-k}$ for some $0 \leq k \leq n$. Two charts $(U_1, \varphi_1)$ and $(U_2, \varphi_2)$ are $C^\infty$-related if either $U_1 \cap U_2$ is empty or the map

$$\varphi_2 \circ \varphi_1^{-1}: \varphi_1(U_1 \cap U_2) \to \varphi_2(U_1 \cap U_2)$$

is a diffeomorphism (of open subsets in $[0, \infty)^k \times \mathbb{R}^{n-k}$ and $[0, \infty)^k_1 \times \mathbb{R}^{n-k_1}$). A system of pairwise $C^\infty$-related charts of $M$ that covers $M$ is called an atlas of $M$. A smooth manifold with corners is a Hausdorff space equipped with a maximal atlas of charts.

Similarly, we introduce the following notion of manifolds with polytope boundary, which are locally modeled on $n$-dimensional polyhedra in $\mathbb{R}^n$. For a given Hausdorff space $X$, a polytope chart $(U, \varphi)$ for $X$ is a homeomorphism $\varphi$ from an open subset $U$ of $M$ to an open subset of an $n$-dimensional polyhedron in $\mathbb{R}^n$. Two polytope charts $(U_1, \varphi_1)$ and $(U_2, \varphi_2)$ are $C^\infty$-related if either $U_1 \cap U_2$ is empty or the map

$$\varphi_2 \circ \varphi_1^{-1}: \varphi_1(U_1 \cap U_2) \to \varphi_2(U_1 \cap U_2)$$

is a diffeomorphism (of open subsets of $n$-dimensional polyhedra). Again, a system of pairwise $C^\infty$-related charts of $X$ that covers $X$ is called an atlas of $X$.

**Definition 2.1.** A smooth manifold with polytope boundary is a Hausdorff space equipped with a maximal atlas of polytope charts.

We also introduce the following notion of corner maps between manifolds with polytope boundary.

**Definition 2.2.** A map $f: N \to M$ between manifolds with polytope boundary is called a corner map if

1. $f$ is smooth map\(^3\) between manifolds with polytope boundary;
2. $f$ maps faces to faces, that is, for each codimension one face $F_i$ of $N$, we have $f(F_i) \subseteq F_i$ for some codimension one face $F_i$ of $M$.
3. For any collection of codimension one faces, say, $\{\overline{F_1}, \ldots, \overline{F_k}\}$ of $N$, if $x \in \cap_{j=1}^k \overline{F_j}$, then the tangent map

$$f_*: T_xN \to T_{f(x)}M$$

restricted to the linear subspace $\mathfrak{N}_x \subset T_xN$ is injective and

$$\cap_{j=1}^k T F_j \bigcap f_* (\mathfrak{N}_x) = 0,$$

where $\mathfrak{N}_x$ is the linear subspace of $T_xN$ spanned by the normal vectors of $\{\overline{F_j}\}_{j=1}^k$ at $x$, and $F_j$ is the corresponding codimension one face in $M$ such that $f(F_j) \subseteq F_j$.

\(^3\) A smooth map between manifolds with polytope boundary if and only if $f$ is the restriction of a smooth map $\varphi: N' \to M'$, where $N'$ and $M'$ are two open manifolds which respectively contain $N$ and $M$ as their submanifolds.
2.2. Dihedral Rigidity of flat manifolds with polytope boundary. In this section, we prove a general rigidity theorem that allows comparison of possibly different manifolds. This answers positively Gromov’s flat corner domination conjecture [9, Section 3.18].

**Theorem 2.3.** Suppose $M$ is a compact oriented $n$-dimensional flat submanifold with polytope boundary in $\mathbb{R}^n$ with the flat metric $g$ such that

(a) $M$ has nonzero Euler characteristic,
(b) each of its codimension one face is convex, that is, its second fundamental form is non-negative,
(c) all of its dihedral angles are $\leq \pi$,

Let $(N, \tilde{g})$ be a compact oriented $n$-dimensional spin manifold with polytope boundary. If $f : (N, \tilde{g}) \to (M, g)$ is a corner map such that

1. $f$ is distance-non-increasing on $\partial N$,
2. the degree of $f$ is nonzero,
3. $\text{Sc}(\tilde{g})_x \geq \text{Sc}(g)_{f(x)} = 0$ for all $x \in N$,
4. $H_{\tilde{g}}(\tilde{F}_i)_y \geq H_g(F_i)_{f(y)}$ for all $y$ in each codimension one face of $N$, $i$, $j$ of $N$ and all $z \in \tilde{F}_i \cap \tilde{F}_j$,
5. $\theta_{ij}(\tilde{g})_z \leq \theta_{ij}(g)_{f(z)}$ for all $\tilde{F}_i, \tilde{F}_j$ and all $z \in \tilde{F}_i \cap \tilde{F}_j$,

then $(N, \tilde{g})$ is also flat. Furthermore, the following hold.

(i) If a codimension one face $F_i$ of $M$ is flat, then the corresponding face $\tilde{F}_i$ of $N$ is also flat.

(ii) Suppose $\bar{x}$ is a point in the intersection of $m$ codimension one faces of $N$ whose unit inner normal vectors at $x$ are denoted by $\bar{\nu}_1, \ldots, \bar{\nu}_m$. Let $\nu_1, \ldots, \nu_m$ be the unit inner normal vectors at $f(\bar{x})$ of the corresponding faces of $M$. Then we have

$$\langle \nu_i, \nu_j \rangle_g = \langle \bar{\nu}_i, \bar{\nu}_j \rangle_{\tilde{g}}, \quad \forall i, j = 1, \ldots, m. \quad (2.1)$$

**Remark 2.4.** We point out one subtlety of part (ii). For example, in a manifold $N$ with polytope boundary, it is possible for a point $x$ to lie in the intersection of more than $n$ codimension one faces of $N$, where $n = \dim N$. In this case, to deduce the equalities in line (2.1) requires solving an a priori over-determined linear system. Such a phenomenon does not occur for manifolds with corners. Since every vertex of an $n$-dimensional manifold with corners is the intersection of precisely $n$ codimension one faces. These faces pairwise intersect, hence the equalities in line (2.1) simply become the equalities of corresponding dihedral angles in the case of manifolds with corners.

**Remark 2.5.** It will be clear from the proof that the conclusion in part (ii) of Theorem 2.3 holds under much weaker assumptions. More precisely, suppose $(M, g)$ is a compact $n$-dimensional manifold with polytope boundary satisfying conditions (a), (b), (c) and the curvature operator of $(M, g)$ is non-negative. If

\[\text{The notation } \tilde{F}_i \text{ and } F_i \text{ means that the map } f \text{ takes the face } \tilde{F}_i \text{ of } N \text{ to the face } F_i \text{ of } M.\]
\(f: (N, \mathcal{F}) \to (M, g)\) is a spin\(^5\) area-non-increasing\(^6\) corner map satisfying conditions (1) – (5), then part (\textit{ii}) holds.

**Remark 2.6.** We point out that, if \((M, g)\) is flat and all of its codimension one faces are flat, then the condition (1) for requiring \(f\) to be distance-non-increasing on \(\partial N\) is \textit{not} needed. Indeed, in this case, any rescaling on \(g\) does not change the scalar curvature, mean curvature and dihedral angles of \((M, g)\). Therefore, by a rescaling of \(g\) if necessary, we can make \(f: (N, \mathcal{F}) \to (M, g)\) distance-non-increasing (hence also area-non-increasing) on the entire \(N\). Alternatively, we can also directly apply the estimates from [22, Theorem 1.8].

**Proof of Theorem 2.3.** The odd dimensional case can easily be reduced to the even dimensional case by considering \(N \times [0, 1]\) and \(M \times [0, 1]\) with the obvious product metrics. Therefore, we can assume without loss of generality that \(N\) and \(M\) are even-dimensional.

Let \(S_N\) and \(S_M\) be the corresponding spinor bundle over \(N\) and \(M\). Consider the vector bundle \(S_N \otimes f^*S_M\) over \(N\), which carries a natural Hermitian metric and a unitary connection \(\nabla\) compatible with Clifford multiplication by elements of \(\mathbb{C}\ell(TN) \otimes f^*\mathbb{C}\ell(TM)\). We will denote the Clifford multiplication of a vector \(\mathfrak{v} \in TN\) by \(\mathfrak{c}(\mathfrak{v})\) and the Clifford multiplication of a vector \(v \in f^*TM\) by \(c(v)\).

Consider the associated Dirac operator \(D\) on \(S_N \otimes f^*S_M\):

\[
D = \sum_{i=1}^{n} \mathcal{C}(e_i)(\overline{\nabla}_{e_i} \otimes 1 + 1 \otimes \nabla_{e_i})
\]

where \(\overline{\nabla}\) and \(\nabla\) are the spinor connections on \(S_N\) and \(f^*S_M\).

Let \(B\) be the local boundary condition on each codimension one face \(\overline{F}_i\) of \(N\) given by

\[
(\overline{\mathfrak{c}} \otimes \mathfrak{c})(\overline{e}_n(x)) \otimes c(e_n(x))\varphi(x) = -\varphi(x), \forall x \in F_i,
\]

(2.2)

where \(\overline{\mathfrak{c}}\) (resp. \(\mathfrak{c}\)) is the \(\mathbb{Z}_2\)-grading operator on \(S_N\) (resp. \(f^*S_M\)), and \(e_n(x)\) (resp. \(\overline{e}_n(x)\)) is the unit inner normal vector of \(\overline{F}_i\) at \(x\) (resp. of \(F_i\) at \(f(x)\)). By [22], under the given assumptions of the current theorem, we have the following.\(^7\)

- \(D\) with respect to the boundary condition \(B\) is essentially self-adjoint and Fredholm ([22, Theorem 3.9]).

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\(^5\)A smooth map \(f: N \to M\) is called a spin map if the second Stiefel–Whitney classes of \(TM\) and \(TN\) are related by

\[w_2(TN) = f^*(w_2(TM)).\]

Equivalently, \(f: N \to M\) is a spin map if \(TN \oplus f^*TM\) admits a spin structure.

\(^6\)Here \(f\) is area-non-increasing if \(\|
\wedge^2 df\|_x \leq 1\) for every \(x \in N\), where \(\wedge^2 df: \wedge^2 TN \to \wedge^2 TM\) is the second exterior product of \(df\).

\(^7\)Some of the main results of [22] were only stated for manifolds with corners. However, a key step of the proofs in [22] is to analyze Dirac type operators that arise from asymptotically conical metrics. Since a Riemannian metric on a manifold with polytope boundary is also asymptotically conical near its singular points, the same analysis from [22] also applies to Dirac type operators on manifolds with polytope boundary. In particular, all the main results in [22] also hold for manifolds with polytope boundary.
The Fredholm index of $D$ is equal to $\deg(f) \cdot \chi(M)$, where $\deg(f)$ is the degree of the map $f$ and $\chi(M)$ is the Euler characteristic of $M$ ([22, Theorem 3.19]), which is non-zero by assumption.

Therefore there exists a non-zero section $\varphi$ of $S_N \otimes f^*S_M$ satisfying the boundary condition $B$ such that $D\varphi = 0$. Further estimates ([22, Proposition 2.6]) show that $\varphi$ is parallel with respect to the connection $(\nabla \otimes 1 + 1 \otimes \nabla)$ (see the proof of [22, Theorem 1.8]). By the Sobolev embedding theorem, $\varphi$ is a smooth section of $S_N \otimes f^*S_M$ over $N$.

Since $(M, g)$ is a codimension zero flat submanifold of $\mathbb{R}^n$, there exist $n$ parallel sections of $TM$, denoted by $\{v_1, v_2, \ldots, v_n\}$, such that they form an orthonormal basis of $T_xM$ at every point $x \in M$. We also use the same notation to denote the standard basis in $\mathbb{R}^n$. Let $\Lambda$ be the collection of all subsets of $\{1, 2, \ldots, n\}$. For $\lambda \in \Lambda$, we define

$$w_\lambda = \wedge_{i \in \lambda} v_i \in \Lambda^*TM$$

Note that $\{w_\lambda\}_{\lambda \in \Lambda}$ are parallel sections of $\Lambda^*TM$ such that they form an orthonormal basis of $\Lambda^*T_xM$ at every point $x \in M$.

With the section $\varphi$ of $S_N \otimes f^*S_M$ from above, we define

$$\varphi_\lambda = (1 \otimes c(w_\lambda)) \varphi.$$ 

Since $w_\lambda$ is parallel (with respect to the connection $\nabla$), we see that $\varphi_\lambda$ is parallel (with respect to the connection $\overline{\nabla} \otimes 1 + 1 \otimes \nabla$). Note that $\overline{\nabla} \otimes 1 + 1 \otimes \nabla$ is a Hermitian connection that preserves the inner product on $S_N \otimes f^*S_M$. Therefore, for any pair of elements $\lambda, \mu \in \Lambda$, the function $\langle \varphi_\lambda(x), \varphi_\mu(x) \rangle$ (as $x$ varies over $N$) is a constant function. In particular, we may assume without loss of generality that $|\varphi_\lambda(x)| = 1$ for all $x \in N$ and all $\lambda \in \Lambda$.

**Claim.** The parallel sections $\{\varphi_\lambda\}_{\lambda \in \Lambda}$ are mutually orthogonal.

Note that $\dim(S_N \otimes f^*S_M) = 2^n = |\Lambda|$, where $|\Lambda|$ is the cardinality of the set $\Lambda$. Thus if the claim holds, then the curvature form of $S_N \otimes f^*S_M$ vanishes. Since $M$ is flat, the curvature form of $S_N \otimes f^*S_M$ is equal to the curvature form $R^{S_N}$ of $S_N$. By [4, Theorem 2.7], we have

$$R^{S_N}_{X,Y} \sigma = \frac{1}{2} R^{\overline{g}}_{X,Y} \cdot \sigma, \quad \text{for all } \sigma \in \Gamma(S_N) \text{ and } X, Y \in \Gamma(TN),$$

where $R^{\overline{g}}$ is the curvature form of the Levi-Civita connection on $TN$ with respect to $\overline{g}$. It follows that $R^{\overline{g}} = 0$, that is, $\overline{g}$ is flat.

Now we prove the claim. For each $\lambda \in \Lambda$ and $x \in M$, we denote by $V_\lambda$ the subspace in $T_xM \cong \mathbb{R}^n$ spanned by $\{v_i\}_{i \in \lambda}$.

Let $\lambda$ and $\mu$ be two distinct members of $\Lambda$. Without loss of generality, we assume there exists $1 \leq k \leq n$ such that $k \in \mu$ and $k \notin \lambda$. Equivalently, we have $v_k \in V_\lambda^\perp \cap V_\mu$. Consider the linear function $L(y) = \langle y, v_k \rangle$ on $M$ (viewed as a subspace of $\mathbb{R}^n$), which attains its minimum at some point $x \in M$, since $M$ is compact. Note that $f : \partial N \to \partial M$ is surjective, since $\deg(f) \neq 0$. In particular, there is a point $\overline{x} \in \partial N$ such that $f(\overline{x}) = x$. Now there are two cases to consider.
Case I. If $x$ is in the interior of a codimension one face $F_i$, then the unit inner normal vector $u$ of $F_i$ at $x$ is equal to $v_k$. In this case, let $\pi$ be the unit inner normal vector at $\pi$ of the corresponding face $\overline{F}_i$ of $N$. Recall that the section $\varphi$ satisfies the following boundary condition at $\pi$:

$$(\overline{\epsilon} \otimes \epsilon)(\overline{\epsilon}(\overline{u}) \otimes c(u))\varphi(\pi) = -\varphi(\pi).$$

Note that for a vector $v \in \mathbb{R}^n$, we have

$$c(w_\lambda)c(v) = \begin{cases} (-1)^{|\lambda|}c(v)c(w_\lambda), & \text{if } v \in V_\lambda^+, \\ (-1)^{|\lambda|-1}c(v)c(w_\lambda), & \text{if } v \in V_\lambda, \end{cases}$$

where $|\lambda|$ is the cardinality of the set $\lambda$. Therefore $\varphi_\lambda$ and $\varphi_\mu$ satisfy the following equations at $\pi$:

$$(\overline{\epsilon} \otimes \epsilon)(\overline{\epsilon}(\overline{u}) \otimes c(u))\varphi_\lambda(\pi) = -\varphi_\lambda(\pi),$$

$$(\overline{\epsilon} \otimes \epsilon)(\overline{\epsilon}(\overline{u}) \otimes c(u))\varphi_\mu(\pi) = \varphi_\mu(\pi).$$

It follows that $\langle \varphi_\lambda, \varphi_\mu \rangle$ vanishes at $\pi$, hence everywhere on $N$.

Case II. Suppose $x$ lies in the interior of the intersection of $m$ codimension one faces. Let $u_1, \ldots, u_m$ be the set of unit inner normal vectors of these $m$ codimension one faces. Then in this case, the vector $v_k$ from above lies in the linear span of $u_1, \ldots, u_m$ in $T_x M$. Since $\deg(f) \neq 0$, it follows from the definition of corner maps that there exists $\pi \in f^{-1}(x)$ such that $\pi$ lies in the intersection of $m$ codimension one faces of $N$. Indeed, $\deg(f) \neq 0$ implies that the map $f|_\partial: \partial N \to \partial M$ has nonzero degree. Since $f$ maps codimension $k$ faces of $N$ to codimension $k$ faces of $M$ for any $1 \leq k \leq n$, it follows by induction that for each codimension $m$ face $F_\theta$ of $M$, there exists a codimension $m$ face $\overline{F}_\theta$ of $N$ such that $f$ maps $\overline{F}_\theta$ to $F_\theta$ with nonzero degree. This in particular implies that $f$ maps $\overline{F}_\theta$ surjectively onto $F_\theta$. To summarize, we see that there exists $\pi \in f^{-1}(x)$ such that $\pi$ lies in the intersection of $m$ codimension one faces of $N$. We denote the corresponding unit inner normal vectors of these faces at $\pi$ by $\overline{u}_1, \ldots, \overline{u}_m$. Since $v_k$ lies in the linear span of $u_1, \ldots, u_m$, we have

$$v_k = \sum_{i=1}^{m} a_i u_i$$

for some numbers $a_1, \ldots, a_m \in \mathbb{R}$. Accordingly, we define

$$\overline{\pi} := \sum_{i=1}^{m} a_i \overline{u}_i.$$ 

At the point $\pi$, the section $\varphi$ satisfies multiple boundary conditions, that is,

$$(\overline{\epsilon} \otimes \epsilon)(\overline{\epsilon}(\overline{u}_i) \otimes c(u_i))\varphi(\pi) = -\varphi(\pi), \ \forall i = 1, \ldots, m.$$

Equivalently, we have

$$(\overline{\epsilon}(\overline{u}_i) \otimes 1)\varphi(\pi) = -(1 \otimes cc(u_i))\varphi(\pi), \ \forall i = 1, \ldots, m.$$
Since \((\overline{\mathcal{C}}(\vec{\pi}) \otimes 1)\) commutes with \(1 \otimes c(w_{\lambda})\) and \(1 \otimes c(w_{\mu})\), we have

\[
(\overline{\mathcal{C}}(\vec{\pi}) \otimes 1)\varphi_{\lambda}(x) = -(1 \otimes ec(v_{k}))\varphi_{\lambda}(\vec{\pi}),
\]
and

\[
(\overline{\mathcal{C}}(\vec{\pi}) \otimes 1)\varphi_{\mu}(x) = (1 \otimes ec(v_{k}))\varphi_{\mu}(\vec{\pi}).
\]

Note that \(\mathcal{C}(\vec{\pi})(-\mathcal{C}(\vec{\pi})^*) = \mathcal{C}(\vec{\pi})^2 = |\vec{\pi}|^2\) and similarly \(c(v_{k})(-c(v_{k})^*) = |v_{k}|^2 = 1\).

It follows that

\[
|\vec{\pi}|^2_\mathcal{G}(\varphi_{\lambda}(\vec{\pi}), \varphi_{\mu}(\vec{\pi})) = -\langle \varphi_{\lambda}(\vec{\pi}), \varphi_{\mu}(\vec{\pi}) \rangle.
\]

Since \(|\vec{\pi}|^2_\mathcal{G} \geq 0\), this implies that \(\langle \varphi_{\lambda}(\vec{\pi}), \varphi_{\mu}(\vec{\pi}) \rangle = 0\), hence \(\langle \varphi_{\lambda}, \varphi_{\mu} \rangle = 0\) everywhere on \(N\). This proves the claim, hence shows that \((N, \vec{g})\) is flat.

Now we shall prove part \((i)\), that is, the following claim.

**Claim.** If a face \(F_i\) of \(M\) is flat, then the corresponding face \(\overline{F}_i\) in \(N\) is also flat.

Let \(\{s_{\alpha}\}_{1 \leq \alpha \leq 2^{n/2}}\) be a set of parallel sections of \(f^*S_M\) such that they form an orthonormal basis of \((f^*S_M)_x\) for any point \(x \in N\). Hence we can write

\[
\varphi = \sum_{\alpha} s_{\alpha} \otimes s_{\alpha},
\]

where \(s_{\alpha}\) are sections of \(S_N\). Since \(\varphi\) is parallel, each \(s_{\alpha}\) is parallel with respect to \(\nabla\).

From the above, we see that

\[
\{\varphi_{\lambda} = \sum_{\alpha} s_{\alpha} \otimes w_{\lambda}s_{\alpha}\}_{\lambda \in \Lambda}
\]
forms a basis of \((S_N \otimes f^*S_M)_{\vec{\pi}}\) at every point \(\vec{\pi} \in N\). It follows that

\[
\{s_{\alpha}\}_{1 \leq \alpha \leq 2^{n/2}}
\]
forms a basis of \((S_N)_{\vec{\pi}}\) at every \(\vec{\pi} \in N\). That is, \(\{s_{\alpha}\}\) is linearly independent at any point in \(N\).

At the face \(\overline{F}_i\) of \(N\), the section \(\varphi\) satisfies the boundary condition

\[
(\overline{\mathcal{C}}(e_{n}) \otimes 1)\varphi = -(1 \otimes ec(e_{n}))\varphi,
\]

where \(e_{n}\) (resp. \(e_{n}\)) is the inner unit normal vector field of \(\overline{F}_i\) (resp. the corresponding codimension one face \(F_i\) in \(M\)). Therefore we have

\[
\sum_{\alpha} \overline{\mathcal{C}}(e_{n})s_{\alpha} \otimes s_{\alpha} = \sum_{\alpha} s_{\alpha} \otimes ec(e_{n})s_{\alpha}.
\]

Note that \(e_{n}\) is parallel. Let \(X\) be an arbitrary tangent vector field along \(\overline{F}_i\). By applying \((\nabla_X \otimes 1 + 1 \otimes \nabla_X)\) to both sides of the above equality, we obtain

\[
\sum_{\alpha} \overline{\mathcal{C}}(\nabla_X e_{n})s_{\alpha} \otimes s_{\alpha} = 0.
\]

Therefore \(c(\nabla_X e_{n})s_{\alpha} = 0\) for all \(\alpha \in \{1, 2, \ldots, 2^{n/2}\}\). It follows that \(\nabla_X e_{n} = 0\) for all tangent vector fields \(X\) along \(\overline{F}_i\), that is, the second fundamental form of \(\overline{F}_i\) vanishes. As we have shown that \(N\) is flat, this implies that \(\overline{F}_i\) is also flat.
Now let us prove part (ii). By assumption, \( \mathbf{x} \) is a point in the intersection of \( m \) codimension one faces of \( N \) whose unit inner normal vectors at \( \mathbf{x} \) are denoted by \( \nu_1, \ldots, \nu_m \). Let \( \nu_1, \ldots, \nu_m \) be the unit inner normal vectors at \( f(\mathbf{x}) \) of the corresponding faces of \( M \). Again, let \( \varphi \) be the parallel section of \( S_N \otimes f^* S_M \) from above. By the above discussion, we have
\[
(\mathbf{c}(g) \otimes 1) \varphi(\mathbf{x}) = -(1 \otimes \mathbf{c}(\nu_j)) \varphi(\mathbf{x}), \quad \forall j = 1, \ldots, m.
\]
For \( a = (a_1, a_2, \ldots, a_m) \in \mathbb{R}^m \), we define
\[
\nu_a := \sum_{j=1}^{m} a_j \nu_j \quad \text{and} \quad \nu_a := \sum_{j=1}^{m} a_j \nu_j.
\]
Clearly, we have
\[
(\mathbf{c}(g) \otimes 1) \varphi(\mathbf{x}) = -(1 \otimes \mathbf{c}(\nu_a)) \varphi(\mathbf{x}).
\]
By taking vector norms of both sides, we obtain
\[
|\nu_a|^2 g \cdot |\varphi(\mathbf{x})|^2 = |\nu_a|^2 g \cdot |\varphi(\mathbf{x})|^2
\]
since \( \mathbf{c}(\nu_a)(\mathbf{c}(\nu_a)^*) = \mathbf{c}(\nu_a)^2 = |\nu_a|^2 g \) and \( \mathbf{c}(\nu_a)^* = \mathbf{c}(\nu_a)^2 = |\nu_a|^2 g \). It follows that \( |\nu_a|_g = |\nu_a|_g \), since \( |\varphi(\mathbf{x})| \neq 0 \).

Consider the two symmetric quadratic forms \( Q, \overline{Q} : \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R} \) defined by
\[
\overline{Q}(a, b) := \langle \nu_a, \nu_b \rangle_g \quad \text{and} \quad Q(a, b) := \langle \nu_a, \nu_b \rangle_g.
\]
The above discussion shows that
\[
Q(a, a) = \overline{Q}(a, a), \quad \forall a \in \mathbb{R}^m.
\]
By the polarization identity, we have
\[
Q(a, b) = \frac{1}{4}(Q(a+b, a+b) - Q(a-b, a-b)),
\]
\[
\overline{Q}(a, b) = \frac{1}{4}(\overline{Q}(a+b, a+b) - \overline{Q}(a-b, a-b)).
\]
Hence \( Q \) and \( \overline{Q} \) are identical. In particular, we have
\[
\langle \nu_i, \nu_j \rangle_g = \langle \nu_i, \nu_j \rangle_g, \quad \forall i, j = 1, \ldots, m.
\]
This finishes the proof.

Observe that in the proof of Theorem 2.3 above, if there is a point \( x \) of \( M \) such that the inner normal vectors (of codimension one faces) at \( x \) span the whole tangent space \( T_x M \), then we can deduce the flatness of \((N, g)\) only using this point. More precisely, we have the following theorem (Theorem 2.8) that generalizes Theorem 2.3 to a class of flat manifolds that admit vertices.

**Definition 2.7.** Let \( M \) be an \( n \)-dimensional manifold with polytope boundary. We say a point \( x \) of \( M \) is a *vertex* if \( x \) lies in the intersection of at least \( n \) codimension one faces of \( M \).
Theorem 2.8. Suppose $M$ is a compact oriented $n$-dimensional flat manifold such that

(a) $M$ has nonzero Euler characteristic,
(b) each of its codimension one face is convex, that is, its second fundamental form is non-negative,
(c) all of its dihedral angles are $\leq \pi$,
(d) $M$ admits a vertex.

Let $(N, \overline{g})$ be a compact oriented $n$-dimensional spin manifold with polytope boundary. If $f: (N, \overline{g}) \to (M, g)$ is a corner map such that

(1) $f$ is distance-non-increasing on $\partial N$,
(2) the degree of $f$ is nonzero,
(3) $\text{Sc}(\overline{g}) \geq \text{Sc}(g)$ for all $x \in N$,
(4) $H_g(F_i) \geq H_g(f(F_i))$ for all $y$ in each codimension one face $F_i$ of $N$,
(5) $\theta_{ij}(\overline{g}) \leq \theta_{ij}(g)$ for all $F_i, F_j$ and all $z \in F_i \cap F_j$,

then $(N, \overline{g})$ is also flat. Furthermore, if a codimension face $F_i$ of $M$ is flat, then the corresponding face $\overline{F}_i$ of $N$ is also flat.

Remark 2.9. Either Theorem 2.3 or Theorem 2.8 has its own advantages in terms of applications. The only difference between Theorem 2.8 and Theorem 2.3 is the slightly different assumptions on the manifold $M$. On one hand, in Theorem 2.8, $M$ is not necessarily a codimension zero submanifold of $\mathbb{R}^n$; furthermore, $M$ is allowed to be non-simply connected. On the other hand, in Theorem 2.3, $M$ is assumed to a codimension zero submanifold of $\mathbb{R}^n$, but $M$ is allowed to have smooth boundary (in particular not necessarily admits a vertex).

Proof. Let $\widetilde{M}$ be the universal cover of $M$. Let $\widetilde{N}$ be the pullback covering space of $N$ via the map $f: N \to M$. Let us denote the corresponding map $\widetilde{N} \to \widetilde{M}$ by $\widetilde{f}$.

Let $\varphi$ be the parallel section of $S_N \otimes f^* S_M$ as in the proof of Theorem 2.3. Clearly, $\varphi$ lifts to a parallel section of $S_{\widetilde{N}} \otimes \widetilde{f}^* S_{\widetilde{M}}$, which we still denote by $\varphi$. Since $\widetilde{M}$ is flat and simply connected, there exist $n$ parallel sections of $T\widetilde{M}$, denoted by $\{v_1, v_2, \ldots, v_n\}$, such that they form an orthonormal basis of $T_x \widetilde{M}$ at every point $x \in \widetilde{M}$. With the same notation as in the proof of Theorem 2.3, let $\Lambda$ be the collection of all subsets of $\{1, 2, \ldots, n\}$. For $\lambda \in \Lambda$, we define

$$w_\lambda = \wedge_{i \in \lambda} v_i \in \Lambda^* T\widetilde{M}$$

Note that $\{w_\lambda\}_{\lambda \in \Lambda}$ are parallel sections of $\Lambda^* T\widetilde{M}$ such that they form an orthonormal basis of $\Lambda^* T_x \widetilde{M}$ at every point $x \in \widetilde{M}$. We define a collection of non-zero parallel sections

$$\varphi_\lambda = (1 \otimes w_\lambda) \varphi, \ \forall \lambda \in \Lambda.$$
Since \( \varphi_\lambda \) is parallel, we see that \( \langle \varphi_\lambda(x), \varphi_\mu(x) \rangle \) (as \( x \) varies over \( \tilde{N} \)) is a constant function \( \tilde{N} \), for any \( \lambda, \mu \in \Lambda \).

By assumption, \( \tilde{M} \) admits a vertex, say, \( x \in \tilde{M} \). By the same argument for Case II in the proof of Theorem 2.3, there exists \( \tilde{x} \in f^{-1}(x) \) such that \( \tilde{x} \) is also a vertex. Let \( \{u_1, u_2, \ldots, u_m\} \) be the inner normal vectors of the codimension one faces intersecting at \( x \). Since \( x \) is a vertex, \( u_1, u_2, \ldots, u_m \) span \( T_x \tilde{M} \). In particular, \( v_k(x) \) is a linear combination of \( u_1, u_2, \ldots, u_m \) for any \( k = 1, \ldots, n \).

From the same argument in the proof of Theorem 2.3, we obtain that

\[
\langle \varphi_\lambda(\tilde{x}), \varphi_\mu(\tilde{x}) \rangle = 0, \quad \forall \lambda, \mu \in \Lambda, \; \lambda \neq \mu.
\]

Therefore \( \{\varphi_\lambda\}_{\lambda \in \Lambda} \) forms a parallel basis of \( S_{\tilde{N}} \otimes \tilde{f}^* S_{\tilde{M}} \). It follows that \( \tilde{N} \) is flat, hence \( N \) is flat.

Now suppose a codimension face \( F_i \) of \( M \) is flat. The same computation from the proof of part (i) of Theorem 2.3 shows the corresponding face \( \tilde{F}_i \) of \( N \) is also flat. This finishes the proof. \( \square \)

Now let us deduce Theorem 1.5 and Theorem 1.6 from Theorem 2.3.

\textbf{Proof of Theorem 1.5.} It is an immediate consequence of Theorem 2.3 that \( (P, \mathfrak{g}) \) is flat and all codimension one faces of \( (P, \mathfrak{g}) \) are flat. In particular, the local geometry of \( (P, \mathfrak{g}) \) near each codimension \( k \) face \( F^k \) is completely determined by the unit inner normal vectors of codimension one faces that contain \( F^k \). By part (ii) of Theorem 2.3, the relative positions of these unit inner normal vectors coincide with the relative positions of the unit inner normal vectors of the corresponding faces of \( (P, g) \). This proves that \( (P, \mathfrak{g}) \) and \( (P, g) \) are locally isometric, hence finishes the proof of Theorem 1.5. \( \square \)

\textbf{Proof of Theorem 1.6.} Let \( P_1 \) and \( P_2 \) be two convex polyhedra in \( \mathbb{R}^n \). By taking direct product with the unit interval \([0, 1]\) if necessary, we assume without loss of generality that \( n \) is even. Since the combinatorial types of \( P_1 \) and \( P_2 \) are the same, there is a homeomorphism \( f : P_1 \to P_2 \) that preserves the combinatorial structures and matches the dihedral angles. The map \( f \) may not be smooth (as a map between two manifolds with polytope boundary), but it can be chosen to be a diffeomorphism (in fact locally a linear map asymptotically) away from the codimension two faces. We identify the spinor bundle \( S_{P_1} \otimes f^* S_{P_2} \) with the bundle of differential forms \( \Lambda^* T \mathbb{R}^n \) over \( P_1 \). By [22, Theorem 3.9], the twisted Dirac operator associated to the bundle \( S_{P_1} \otimes f^* S_{P_2} \) is essential self-adjoint. To more precise, the proof of [22, Theorem 3.9] is based on the analysis on manifolds with conical metrics. In particular, the analysis is carried out near (but not including) conical singularities. Hence all main results in [22] apply to the current situation. In particular, the de Rham operator on \( \Lambda^* T \mathbb{R}^n \) subject to the boundary condition

\[
(\tilde{c} \otimes c)(\tilde{c}(c_n) \otimes c(e_n)) \sigma = -\sigma
\]

for differential forms \( \sigma \) at each codimension one face is essentially self-adjoint and Fredholm (cf. [22, Theorem 3.9]). By [22, Theorem 3.19], the Fredholm index of the de Rham operator subject to the above boundary condition is non-zero.
Hence there is a non-zero parallel section of $S_{P_1} \otimes f^* S_{P_2} \cong \wedge^* \mathbb{R}^n$ on $P_1$ satisfying the above boundary condition.

By part (ii) of Theorem 2.3 (or rather its proof), at any given vertex $x \in P_1$, the relative positions of the unit inner normal vectors of codimension one faces at $x$ coincide with the relative positions of the unit inner normal vectors of the corresponding faces of $P_2$. It follows that the corresponding face angles of $P_1$ and $P_2$ coincide. This finishes the proof. □

3. RIGIDITY OF MANIFOLDS WITH SMOOTH BOUNDARY

In this section, we investigate rigidity results for manifolds with smooth boundary. In particular, we prove a scalar-mean rigidity theorem for Euclidean balls (Theorem 1.7).

**Proposition 3.1.** Let $(M, g)$ and $(N, \mathcal{F})$ be two compact $n$ dimensional manifolds with smooth boundary, and $f : N \rightarrow M$ a spin map. Suppose

1. $M$ has nonzero Euler characteristic,
2. the curvature operator of $(M, g)$ is non-negative,
3. the second fundamental form of $\partial M$ is strictly positive,
4. $f$ is area-non-increasing on $N$ and $f$ is distance-non-increasing on $\partial N$,
5. the degree of $f$ is nonzero,
6. $\text{Sc}(\mathcal{F})_x \geq \text{Sc}(g)_{f(x)} = 0$ for all $x \in N$,
7. $H_{\mathcal{F}}(F_i)_{y} \geq H_{g}(F_i)_{f(y)}$ for all $y \in \partial N$.

Then $f : \partial N \rightarrow \partial M$ is a local isometry.

**Proof.** The odd dimensional case can easily be reduced to the even dimensional case by considering $N \times [0, 1]$ and $M \times [0, 1]$ with the obvious product metrics. Therefore, we can assume without loss of generality that $N$ and $M$ are even-dimensional.

Suppose $f(y) = x \in \partial M$ for a given point $y \in \partial N$. We diagonalize the metric at $y$ and $x$, and choose an orthonormal basis $\{e_j\}_{1 \leq j \leq n-1}$ of $T_y(\partial N)$ and an orthonormal basis $\{e_j\}_{1 \leq j \leq n-1}$ of $T_x(\partial M)$ such that $f_* e_j = \alpha_j e_j$ for some $\alpha_j \in [0, 1]$, since $f$ is distance-non-increasing on $\partial N$. We denote by $e_n$ (resp. $e_n$) the unit inner normal vector of $\partial N$ at $y$ (resp. of $\partial M$ at $x$).

Let $A$ be the second fundamental form of $\partial M$. Let $\varphi$ and $c$ be the boundary Clifford actions, that is, $\varphi(\varphi_j) = \varphi(\varphi_n)\varphi(\varphi_j)$ and $c(e_j) = c(e_n)c(e_j)$.

Since $A$ is strictly positive, there is an invertible endomorphism $L \in \text{End}(T_x(\partial M))$ such that $A = L^2$, that is,

$$A(e_i, e_j) = \langle Le_i, Le_j \rangle.$$

Let us write

$$Le_i = \sum_{1 \leq j \leq n-1} L_{ij} e_j.$$
and

\[ L e_i = \sum_{1 \leq j \leq n-1} \langle L e_i, f_\ast \bar{e}_j \rangle \bar{e}_j = \sum_{1 \leq j \leq n-1} L_{ij} \alpha_j e_j. \]

We have the following inequality (cf. \cite[Lemma 2.3]{22})

\[
\sum_{i,j} A(f_\ast \bar{e}_i, e_j) \sigma(\bar{e}_i) \otimes c_\partial(e_j) \\
= \sum_{i,j,k} \langle L(f_\ast \bar{e}_i), e_k \rangle_M \langle L(e_j), e_k \rangle_M \sigma(\bar{e}_i) \otimes c_\partial(e_j) \\
= \sum_k \sigma(Le_k) \otimes c_\partial(Le_k) \\
= -\frac{1}{2} \sum_k \left( \sigma(Le_k)^2 \otimes 1 + 1 \otimes c_\partial(Le_k)^2 - \left( \sigma(Le_k) \otimes 1 + 1 \otimes c_\partial(Le_k) \right)^2 \right) \\
\leq -\frac{1}{2} \sum_k \sigma(Le_k)^2 \otimes 1 - \frac{1}{2} \sum_k 1 \otimes c_\partial(Le_k)^2,
\]

where the last inequality follows from the fact the element

\[
(\sigma(Le_k) \otimes 1 + 1 \otimes c_\partial(Le_k))
\]

is skew-symmetric, hence its square is non-positive. Note that

\[
\sum_k c_\partial(Le_k)^2 = \sum_k L_{kj} L_{kj} c_\partial(e_j)^2 + \sum_k \sum_{i \neq j} L_{ki} L_{kj} c_\partial(e_i)c_\partial(e_j) \\
= -\sum_j A_{jj} + \sum_{i \neq j} A_{ij} c_\partial(e_i)c_\partial(e_j) = -H_g,
\]

and

\[
\sum_k \sigma(Le_k)^2 = \sum_k \alpha_j^2 L_{kj} L_{kj} \sigma(e_j)^2 + \sum_k \sum_{i \neq j} \alpha_i \alpha_j L_{ki} L_{kj} c_\partial(e_i)c_\partial(e_j) \\
= -\sum_j A_{jj} \alpha_j^2 + \sum_{i \neq j} \alpha_i \alpha_j A_{ij} c_\partial(e_i)c_\partial(e_j) \\
= -\sum_j A_{jj} \alpha_j^2 \geq -\sum_j A_{jj} = -H_g.
\]

To summarize, we have

\[
\sum_{i,j} A(f_\ast \bar{e}_i, e_j) \sigma(\bar{e}_i) \otimes c_\partial(e_j) \leq f^\ast(H_g)
\]

at \( y \in \partial N \).

Let \( \varphi \) be a parallel section of \( S_N \otimes f^\ast S_M \) as in the proof of Theorem 2.3. The fact that \( \varphi \) satisfies \( D\varphi = 0 \) together with the assumptions on comparisons of scalar curvature and mean curvature show that the above inequality

\[
\sum_{i,j} A(f_\ast \bar{e}_i, e_j) \sigma(\bar{e}_i) \otimes c_\partial(e_j) \leq f^\ast(H_g)
\]
becomes equality at \( \varphi \) (cf. [22, Lemma 2.3 & Proposition 2.6]), that is,
\[
\left( \sum_{i,j} A(f_\ast \vec{e}_i, e_j) \vec{e}_i \otimes e_j(e_j) \right) \varphi = f_\ast (H(g)) \cdot \varphi. \tag{3.1}
\]
It follows that
\[
\sum_j A_{jj} \alpha_j^2 = \sum_j A_{jj}.
\]
Since \( A \) is strictly positive, \( A_{jj} > 0 \) for all \( 1 \leq j \leq n - 1 \). Therefore \( \alpha_j = 1 \) for all \( 1 \leq j \leq n - 1 \), which shows that \( f_\ast : T_y(\partial N) \to T_x(\partial M) \) is an isometry. It follows that \( f : \partial N \to \partial M \) is a local isometry. This finishes the proof. \( \square \)

Now we prove Theorem 1.7.

**Proof of Theorem 1.7.** First, it is an immediate consequence of [22, Theorem 1.7] that the mean curvature of \( \partial N \) is
\[
H_\bar{g}(\partial N)_y = H_\bar{g}(\partial M)_{f(y)} = \frac{n - 1}{r}
\]
for all \( y \in \partial N \). Furthermore, by Theorem 2.3, \( (N, \bar{g}) \) is flat.

For any point \( \bar{x} \) in \( N \), let \( \alpha : [0, \ell_1] \to N \) be the shortest path from \( \bar{x} \) to \( \partial N \), which is geodesic in \( N \). Let \( \dot{\alpha} \) be the unit tangent vector along \( \alpha \). Since the length of \( \alpha \) attains the minimum among all paths from \( \bar{x} \) to \( \partial N \), \( \dot{\alpha}(\ell_1) \) coincides with the unit outer normal vector of \( \partial N \) at \( \alpha(\ell_1) \). For any vector field \( U \) along \( \alpha \), we have the second variation of the length of \( \alpha \) is non-negative, that is,
\[
0 \leq \delta^2 |\alpha|(U, U) = \langle \nabla_U U, \dot{\alpha} \rangle_0^\ell_1 + \int_0^{\ell_1} \left( ||\nabla_a U||^2 - \langle \nabla_a U, \dot{\alpha} \rangle \right). \tag{3.2}
\]

Let \( V \) be a unit tangent vector in \( T_{\alpha(\ell_1)}(\partial N) \). We denote still by \( V \) the parallel translate of \( V \) along \( \alpha \). Set \( U(\alpha(s)) = sV(\alpha(s))/\ell_1 \) for \( s \in [0, \ell_1] \). By line (3.2), we obtain
\[
0 \leq \delta^2 |\alpha|(U, U) = \int_0^{\ell_1} \frac{1}{\ell_1^2} ds - \frac{1}{r}.
\]
It follows that \( \ell_1 \leq r \), that is,
\[
\sup_{\bar{x} \in N} \text{dist}(\bar{x}, \partial N) \leq r. \tag{3.3}
\]
where \( \text{dist}(\bar{x}, \partial N) \) is the distance between \( \bar{x} \) and \( \partial N \).

Let \( p \) be a preimage in \( N \) of the origin \( o_B \) of \( B \). Since \( f \) is distance-non-increasing, we have
\[
\text{dist}(p, \partial N) \geq \text{dist}(f(p), \partial M) = r.
\]
By [15, Theorem 1.1], it follows that \( N \) is isometric to \( B \). By Proposition 3.1, the map \( f : \partial N \to \partial B \) is a local isometry. Now it is not difficult to use the condition that \( f \) is distance-non-increasing to deduce that \( f \) is an isometry from \( N \) to \( B \).

Alternatively, we can also show \( f : N \to B \) is an isometry by a direct computation as follows. As \( N \) is flat, every geodesic in \( N \) ends on \( \partial N \). Let \( p \) be the preimage of the origin \( o_B \) of \( B \) chosen before. Let \( \beta : [0, \ell_2] \to N \) be the longest geodesic from \( p \) to \( \partial N \). For any vector \( V \in T_{\beta(\ell_2)}(\partial N) \), there is a unique
Jacobi field $J_V$ along $\beta$ such that $J_V(p) = 0$ and $J_V(\beta(\ell_2)) = V$. Let $\dot{\beta}$ be the unit tangent vector along $\beta$. Since the length of $\beta$ attains the maximum among all geodesics from $p$ to $\partial N$, $\dot{\beta}(\ell_2)$ coincides with the unit outer normal vector of $\partial N$ at $\beta(\ell_2)$. Since $f$ is distance-non-increasing and locally isometric on the boundary, $f$ maps $\dot{\beta}(\ell_2)$ to the normal vector of $\partial M$. This, together with the fact that $f$ preserves the mean curvature, implies $f$ preserves the second fundamental form. Therefore the second variation of arc length becomes

$$0 \geq \delta^2 |\beta| (J_V, J_V) = \langle \nabla_V V, \dot{\beta}(\ell_2) \rangle + \int_0^{\ell_2} \left( |\nabla_{\dot{\beta}} J_V|^2 - \langle \nabla_{\dot{\beta}} J_V, \dot{\beta} \rangle^2 \right) dt = -\frac{1}{r} |V|^2 + \int_0^{\ell_2} \left( |\nabla_{\dot{\beta}} J_V|^2 - \langle \nabla_{\dot{\beta}} J_V, \dot{\beta} \rangle^2 \right).$$

Since $N$ is flat, the differential equation for the Jacobi field becomes

$$\nabla_{\dot{\beta}} \nabla_{\dot{\beta}} J_V = 0.$$ 

As $\nabla_{\dot{\beta}} \dot{\beta} = 0$, we see that $|\nabla_{\dot{\beta}} J_V|^2 - \langle \nabla_{\dot{\beta}} J_V, \dot{\beta} \rangle^2$ is constant along $\beta$. In particular, at $\beta(\ell_2)$, we have that

$$|\nabla_{\dot{\beta}(\ell_2)} V|^2 - \langle \nabla_{\dot{\beta}(\ell_2)} V, \dot{\beta}(\ell_2) \rangle^2 = |\nabla_V \dot{\beta}(\ell_2)|^2 - \langle \nabla_V \dot{\beta}(\ell_2), \dot{\beta}(\ell_2) \rangle^2 = \frac{1}{r^2} |V|^2, \quad (3.4)$$

since the connection $\nabla$ is torsion free. Therefore we have

$$0 \geq -\frac{1}{r} |V|^2 + \frac{\ell_2}{r^2} |V|^2,$$

that is, $\ell_2 \leq r$. Now since $\text{dist}(p, \partial N) = r$, the shortest geodesic from $p$ to $\partial N$ also has length $r$. Thus every geodesic from $p$ to $\partial N$ has length $r$. In particular, $f$ maps every geodesic from $p$ to $\partial N$ isometrically to a radial line segment of $B$.

Let $\gamma(t)$ be an arbitrary geodesic in $N$ such that $\gamma(0) = p$ and $\gamma(r) \in \partial N$. Let $f \gamma$ be the image of $\gamma$ in $B$, which is a radial line segment. For any $V \in T_{\gamma(r)}(\partial N)$ and $f_* V \in T_{f \gamma(r)}(\partial N)$, there is a unique Jacobi field $J_V$ along $\gamma$ such that

$$J_V(\gamma(0)) = 0 \text{ and } J_V(\gamma(r)) = V,$$

and a unique Jacobi field $J_{f_* V}$ along $f \gamma$ such that

$$J_{f_* V}(f \gamma(0)) = 0 \text{ and } J_{f_* V}(f \gamma(r)) = f_* V.$$ 

Since we already show that $f$ maps geodesics near $\gamma$ to geodesics near $f \gamma$, $f$ maps the Jacobi field $J_V$ to $J_{f_* V}$.

Let $\dot{\gamma}$ be the unit tangent vector of $\gamma$, then $f_* \dot{\gamma}$ is the unit tangent vector of $f \gamma$. By the previous argument, $\dot{\gamma}(r)$ is the unit outer normal vector at $\gamma(r)$. We first notice that

$$\frac{d^2}{dt^2} \langle J_V, \dot{\gamma} \rangle = \frac{d}{dt} \langle \nabla_{\dot{\gamma}} J_V, \dot{\gamma} \rangle = 0.$$ 

Thus $\langle J_V, \dot{\gamma} \rangle$ is a linear function in $t$. As $J_V(\gamma(0)) = 0$ and $J_V(\gamma(r)) \perp \dot{\gamma}(r)$, we have that $\langle J_V, \dot{\gamma} \rangle$ is identically zero. Therefore

$$\frac{d^2}{dt^2} |J_V|^2 = 2 |\nabla_{\dot{\gamma}} J_V|^2 = 2 (|\nabla_{\dot{\gamma}} J_V|^2 - \langle \nabla_{\dot{\gamma}} J_V, \dot{\gamma} \rangle) = \frac{1}{r} |V|^2.$$
As $|J_V(\gamma(0))| = 0$, we have $|J_V(\gamma(t))| = t|V|/r$.

Recall that $f_*(J_V(\gamma(t))) = J_{f_*V}(f\gamma(t))$. The same computation above shows that $|J_{f_*V}(f\gamma(t))| = t|f_*V|/r = t|V|/r$ and $(J_{f_*V}, f_*) = 0$. It follows that $f: T_xN \to T_{f(x)}B$ is an isometry for any $x \in \gamma$ and any geodesic $\gamma$ from $p$ to $\partial N$.

It remains to show that any point $\bar{x} \in N$ lies on a geodesic from $p$ to $\partial N$. Let $p$ be the preimage of $o_B$ from before. There exists a piecewise smooth path $\gamma$ minimizing the distance from $p$ to $\bar{x}$. The path $\gamma$ is of the following form $\gamma_1 \cup \gamma_2 \cup \cdots \cup \gamma_k$, where each $\gamma_i$ is either a geodesic in $N$ or a geodesic in $\partial N$.

Assume on the contrary that $k > 1$, that is, $\gamma_1$ is a geodesic from $p$ to $\partial N$ and $\gamma_2$ is a geodesic on $\partial N$. By the previous construction, there is a family of geodesics $\gamma_1, s$ from $p$ to $\partial N$ such that

$$\frac{d}{ds}(\gamma_1, s) = J_{\gamma_2},$$

where $J_{\gamma_2}$ is the unique Jacobi field along $\gamma_1, s$ generated by the tangent vector $\dot{\gamma}_2$ of $\gamma_2$ as constructed previously. In particular, the length of $\gamma_1, s$ is $r$ for any $s$. This contradicts the minimality of the length of $\gamma$. Therefore $k = 1$, and $p$ and $q$ are connected by a geodesic $\gamma$ inside $N$. Now the geodesic $\gamma$ extends to a geodesic that ends on $\partial N$.

To summarize, we have shown that $f: N \to B$ is a local isometry at any point in $N$, hence a Riemannian covering map. It follows that $f: N \to B$ is an isometry, since $B$ is simply connected. This finishes the proof. $\square$

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