1. Introduction

We work over an algebraically closed field $k$ of characteristic zero. All logarithmic structures are assumed fine and saturated, and $\text{Log}$ denotes the algebraic stack parameterizing fine and saturated logarithmic structures as in [Ols03].

1.1. Statement of result. We are given a proper, fine and saturated logarithmic scheme $X=(X,M)$ with projective underlying scheme $\overline{X}$. In [GS13, Che14, AC] a stack $\overline{\mathcal{M}}_{\Gamma}(X)$ of stable logarithmic maps of numerical type $\Gamma$ is described. The purpose of this paper is to complete a proof of the following theorem:

**Theorem 1.1.1.** The stack $\overline{\mathcal{M}}_{\Gamma}(X)$ is a proper algebraic stack with projective coarse moduli space. The map $\overline{\mathcal{M}}_{\Gamma}(X) \to \overline{\mathcal{M}}_{\Gamma}(\overline{X})$ to the stack of stable maps of $\overline{X}$ with the underlying numerical data $\Gamma=(g,n,\beta)$ is representable.
This result was proven in [GS13, Theorems 0.1 and 0.2] under the assumption that the sheaf of groups \( \mathcal{M}^{gp} \) associated to the characteristic monoid \( \mathcal{M} = M/\mathcal{O}_X^* \) is globally generated. It was also proven in [AC, Theorem 3.15] under the stronger assumption that \( \mathcal{M} \) itself is globally generated. Many cases of interest are covered by the results in [GS13, AC], but significantly the case of a toroidal embedding with self intersections is not. In the cited papers it was hoped that the result would hold in general, which Theorem 1.1.1 provides.

The following key properties, which are part of Theorem 1.1.1, were shown in [GS13, Theorem 0.1] for Zariski logarithmic structures \( X \) and [AC, Theorem 3.15] when the characteristic monoid \( \mathcal{M} \) is globally generated; the general case is proved in [Wis14].

**Theorem 1.1.2 ([Wis14]).**

1. \( \overline{\mathcal{M}}_\Gamma(X) \) is algebraic and locally of finite type over \( \mathbb{C} \);
2. the map \( \overline{\mathcal{M}}_\Gamma(X) \to \overline{\mathcal{M}}_\Gamma(L(X)) \) is representable by algebraic spaces.

To complete a proof of Theorem 1.1.1, it remains to show that for general \( X \),

1. the stack \( \overline{\mathcal{M}}_\Gamma(X) \) is of finite type, see Proposition 1.5.5;
2. \( \overline{\mathcal{M}}_\Gamma(X) \) is separated and satisfies the weak valuative criterion for properness, see Proposition 1.4.3.

The two statements above are proven in this paper by reducing to the case where the characteristic monoid \( \mathcal{M} \) is globally generated. This case was shown in [AC, Corollary 3.11], by further reducing to the rank one case treated in [Che14].

**Remark 1.1.3.** In [AC, GS13] it is shown that the map \( \overline{\mathcal{M}}_\Gamma(X) \to \overline{\mathcal{M}}_\Gamma(L(X)) \) is finite under the assumptions made in those papers, and we believe this should hold in general.

### 1.2. Method.

The main problem is boundedness, namely statement (1) listed above. The problem eluded standard approaches of étale descent. Instead we use a form of non-flat logarithmic étale descent.

Our strategy is to use the “virtual birational invariance” of the moduli spaces, proven in [AW13] when \( X \) is logarithmically smooth. Specifically, we construct a proper and logarithmically étale morphism \( Y \to X \) such that the characteristic sheaf \( \overline{\mathcal{M}}_Y \) is globally generated (Proposition 1.3.1). We then show that the map of moduli spaces \( \overline{\mathcal{M}}(Y) \to \overline{\mathcal{M}}(X) \) is surjective (Proposition 1.4.2). We further show that for each numerical datum \( \Gamma \) on \( X \) there is a finite collection of numerical data \( \Gamma_i \) on \( Y \) such that \( \bigsqcup \overline{\mathcal{M}}_{\Gamma_i}(Y) \to \overline{\mathcal{M}}_{\Gamma}(X) \) is surjective (Proposition 1.5.5). Since \( \overline{\mathcal{M}}_{\Gamma_i}(Y) \) is proper it follows that \( \overline{\mathcal{M}}_{\Gamma}(X) \) is bounded, as required.
We now proceed to describe the steps in more detail.

1.3. **The Artin fan of** $X$. Olsson [Ols03] associates to the logarithmic structure $X$ a canonical morphism $X \to \text{Log}$ to the stack of logarithmic structures. Under mild assumptions on $X$ there is an initial factorization of this map through a strict, representable, étale map $\mathcal{X} \to \text{Log}$. Following [AW13] we call $\mathcal{X}$ the *Artin fan of* $X$.

The construction of $\mathcal{X}$ has its origin in unpublished notes on gluing Gromov–Witten invariants by Q. Chen and by M. Gross. It is closely related to what is known as the *Kato fan* $F(X)$ of $X$ [Kat94, Sections 9 and 10], and to the associated *generalized polyhedral cone complex* $\Sigma(X)$ defined in [Thu07, ACP12]. A more complete picture of the relationship between these objects, as well as with Berkovich spaces, is given in [Uli13]. The simplest cases of Artin fans were used previously in [ACFW11, ACW10, CMW12].

Artin fans are used in the following statement, which is our key reduction step.

**Proposition 1.3.1.** There exists a representable, projective, birational, and logarithmically étale morphism $\mathcal{Y} \to \mathcal{X}$ such that the sheaf of characteristic monoids $\mathcal{M}_Y$ is globally generated. Writing $Y = X \times_\mathcal{X} \mathcal{Y}$, we have a projective and logarithmically étale morphism $Y \to X$ such that the characteristic sheaf $\mathcal{M}_Y$ is globally generated.

See Corollary 4.4.3.

1.4. **Moduli of prestable maps.** Following [AW13, Sections 3.1,3.2], we define a moduli stack $\mathcal{M}(\mathcal{X})$ of prestable maps with target $\mathcal{X}$, a stack $\mathcal{M}(\mathcal{Y})$ of prestable maps with target $\mathcal{Y}$, and a stack $\mathcal{M}'(\mathcal{Y} \to \mathcal{X})$ of prestable maps which are relatively stable for $\mathcal{Y} \to \mathcal{X}$. All three are shown in [AW13, Corollary 3.1.3 and Proposition 1.5.2] to be logarithmically étale over the stack $\mathcal{M}$ of prestable curves. There is a tautological diagram of stacks

$$
\begin{array}{ccc}
\mathcal{M}(\mathcal{Y}) & \longrightarrow & \mathcal{M}(\mathcal{X}) \\
\downarrow & & \downarrow \\
\mathcal{M}'(\mathcal{Y} \to \mathcal{X}) & \longrightarrow & \mathcal{M}(\mathcal{X}) \\
\downarrow & & \downarrow \\
\mathcal{M}(\mathcal{Y}) & & 
\end{array}
$$

with strict vertical arrows and cartesian square. The morphism $\mathcal{M}'(\mathcal{Y} \to \mathcal{X}) \to \mathcal{M}(\mathcal{X})$ is birational [AW13, Proposition 5.3.1]. We prove in
Corollary 4.5.3 that it satisfies the valuative criterion for properness and is surjective. This gives in particular the following.

**Proposition 1.4.2.** The morphism $\overline{M}(Y) \to \overline{M}(X)$ is proper and surjective.

A simple argument then shows the following, see Section 4.6:

**Proposition 1.4.3.** $\overline{M}(X)$ is separated and satisfies the weak valuative criterion for properness.

1.5. **Numerical data.** If $\overline{M}(Y)$ were of finite type we would now be done. As it is anyway a disjoint union of connected components of finite type, it will be sufficient to show that finitely many of those components map to each connected component of $\overline{M}(X)$. To do this, we identify numerical data on $\overline{M}(X)$ that admit finitely many lifts to $\overline{M}(Y)$ with each lift corresponding to a component of $\overline{M}(Y)$. These numerical data include the genus, the number of marked points, the homology class of the curve, and contact information associated to each marked point. These are encoded in terms of logarithmic points:

1.5.1. **Moduli of logarithmic points.** The logarithmic numerical data of $X$ are, by definition, the connected components of the logarithmic evaluation stack $\wedge_{\mathbb{N}}X$ parameterizing standard logarithmic points in $X$. The evaluation stack $\wedge_{P}X$ for an arbitrary sharp monoid $P$ is constructed and explicitly described in [ACGM10]. Our use of this stack is limited to the case $\wedge_{\mathbb{N}}X$, which we denote simply by $\wedge X$.

The formation of $\wedge X$ is covariantly functorial in $X$, so the morphism $Y \to X$ induces a morphism $\wedge Y \to \wedge X$.

**Proposition 1.5.2.** The morphism $\wedge Y \to \wedge X$ induces a bijection of connected components.

See Corollary 5.2.6.

1.5.3. **Degrees.** To bound $\beta_{Y}$ we have

**Proposition 1.5.4.** Let $f : C \to Y$ be a stable logarithmic map whose stabilization $f' : C' \to X$ has discrete data $\Gamma$. Let $L$ be a relatively ample line bundle for $Y/X$, and denote by $L_{Y}$ its pullback to $Y$. Then $\deg_{C} f^{*}L_{Y}$ is constant on $\overline{M}_{\Gamma}(X)$, and determined combinatorially by $\Gamma$.

See Proposition 5.3.1. By a standard argument, given $\Gamma$ there are only finitely many possibilities for $\beta_{Y}$ with image class $\beta$ and fixed $\beta_{Y} \cdot c_{1}(L_{Y})$: see Proposition 5.3.2. Together with Propositions 1.4.2 and 1.5.2 this implies:
Proposition 1.5.5. For each numerical datum $\Gamma$ on $X$ there is a finite collection of numerical data $\Gamma_i$ on $Y$ such that $\bigsqcup \overline{\mathcal{M}}_{\Gamma_i}(Y) \to \overline{\mathcal{M}}_{\Gamma}(X)$ is surjective.

These propositions together provide our main theorem.

Proof of Theorem 1.1.1. By Proposition 1.3.1 we have $\overline{\mathcal{M}}_Y$ is globally generated, so each $\overline{\mathcal{M}}_{\Gamma_i}(Y)$ is proper by either [AC, Proposition 5.8] or [GS13, Theorem 0.2]. We rely on the properties enumerated in Section 1.1. The stack $\overline{\mathcal{M}}_F(X)$ is algebraic, locally of finite type over $\mathbb{C}$, and separated. Since the image of a proper algebraic stack in a separated algebraic stack is proper, Proposition 1.5.5 implies that $\overline{\mathcal{M}}_F(X)$ is proper. The map $\overline{\mathcal{M}}_F(X) \to \overline{\mathcal{M}}_L(X)$ is representable by Theorem 1.1.2.

2. The stack $\overline{\mathcal{M}}_F(X)$

Let $X$ be a logarithmic scheme that is projective over $k$ and let $S$ be another logarithmic scheme over $k$. A prestable logarithmic map over $S$ with target $X$ consists of a logarithmic curve $C \to S$ in the sense of [Kat00, Ols07], along with a logarithmic morphism $C \to X$. It is customary to indicate such a map by $C \to X$, suppressing the remaining data from the notation. A prestable logarithmic map $C \to X$ is stable if the map $C \to X$ of underlying schemes is Kontsevich stable.

There are at least three distinct ways to create a groupoid of stable or prestable logarithmic maps. First, note that given a prestable logarithmic map $C \to X$ over $S$, its pullback along a logarithmic morphism $S' \to S$ is a logarithmic map $C' \to X$ over $S'$. This defines a groupoid over the category of logarithmic schemes, which we denote temporarily by $\mathcal{L}(X)$. Second, one can consider only strict arrows $S' \to S$, namely arrows obtained by pullback along $S' \to S$. This forms a groupoid over the category of schemes, by sending a prestable logarithmic map $C \to X$ over $S$ to the scheme $S$. We denote this groupoid temporarily by $\mathcal{L}^{str}(X)$. A key result is the following:

Theorem 2.1 (See [GS13, Theorem 2.4], [Che14, Theorem 2.1.10], [Wis14, Corollary 1.1.2]). The groupoid $\mathcal{L}^{str}(X)$ is an algebraic stack, locally of finite type over $k$.

The stack $\mathcal{L}^{str}(X)$ has a canonical logarithmic structure, since an object $C \to X$ over $S = (\mathcal{S}, M_S)$ defines a logarithmic structure on $\mathcal{S}$. The stack $\mathcal{L}^{str}(X)$ is rather large, because it includes all possible choices of logarithmic structures $S$ on $\mathcal{S}$, and fails to be proper. A better behaved substack of minimal prestable logarithmic maps is defined
in [Che14, AC] when the characteristic sheaf \( \mathcal{M}_X \) is globally generated, in [GS13] when \( X \) is a Zariski logarithmic scheme, and in [Wis14] in general. It is denoted \( \mathcal{M}(X) \). It obtains a canonical logarithmic structure by restriction from \( \mathcal{L}^{\text{str}}(X) \), and we denote by \( \mathcal{M}(X) \) this stack with its logarithmic structure; in particular \( \mathcal{M}(X) \) can be viewed as a groupoid over the category of logarithmic schemes. These have the following key properties:

**Theorem 2.2** ([Wis14, Corollary 1.1.2]).

1. The stack \( \mathcal{M}(X) \) is an open substack of \( \mathcal{L}^{\text{str}}(X) \). In particular it is algebraic and locally of finite type over \( k \).
2. We have an isomorphism \( \mathcal{M}(X) \simeq \mathcal{L}(X) \) of groupoids over the category of logarithmic schemes.
3. The morphism \( \mathcal{M}(X) \to \mathcal{M}(X) \) is representable by algebraic spaces.

This immediately implies Theorem 1.1.2.

The second statement justifies naming \( \mathcal{M}(X) \) the logarithmic stack of prestable logarithmic maps. Concretely it says that every prestable logarithmic map \( C \to X \) over a logarithmic scheme \( S \) is canonically the pullback along a logarithmic morphism \( S \to S^{\text{min}} \) of a minimal prestable logarithmic map \( C^{\text{min}} \to X^{\text{min}} \) over \( S^{\text{min}} \), and the underlying map of schemes \( S \to S^{\text{min}} \) is the identity. The first statement then tells us that the groupoid of prestable logarithmic maps with target \( X \) is a logarithmic algebraic stack.

For a prestable map \( f : C \to X \) over \( S \), we denote by \( g \) the arithmetic genus of the fibers of \( C \to S \), by \( \beta \) the curve class \( \int C \), and by \( c_i, i = 1, \ldots, n \) the logarithmic numerical data (or contact orders) of \( C \to X \) at the \( n \) marked points, introduced in Section 1.5.1. These data are locally constant, so \( \mathcal{M}(X) \) breaks into open and closed substacks: \( \mathcal{M}(X) = \bigsqcup_{\Gamma} \mathcal{M}_{\Gamma}(X) \), with \( \Gamma = (g, \{c_i\}, \beta) \).

To each prestable logarithmic map \( C \to X \) over \( S \) we have an associated map \( C \to X \) of underlying schemes, giving a morphism \( \mathcal{M}(X) \to \mathcal{M}(X) \). This restricts to morphisms \( \mathcal{M}_{\Gamma}(X) \to \mathcal{M}_{\Gamma}(X) \), where \( \Gamma = (g, n, \beta) \).

Finally we denote by \( \mathcal{M}(X) \) the open substack of stable logarithmic maps, which again decomposes as \( \mathcal{M}(X) = \bigsqcup_{\Gamma} \mathcal{M}_{\Gamma}(X) \). By definition, the morphism \( \mathcal{M}(X) \to \mathcal{M}(X) \) restricts to \( \mathcal{M}(X) \to \mathcal{M}(X) \), and this again decomposes into morphisms \( \mathcal{M}_{\Gamma}(X) \to \mathcal{M}_{\Gamma}(X) \).
3. Artin fans

We extend the construction of [AW13] to logarithmic schemes which are not logarithmically smooth.

3.1. The Artin fan of a logarithmic scheme. An Artin fan is a logarithmic Artin stack that is logarithmically étale over a point. An Artin fan whose tautological morphism to Log (which is necessarily étale) is representable will be said to have faithful monodromy. Logarithmic morphisms between Artin fans are always logarithmically étale (see [AMW12, Lemma A.7]).

It was shown in [AW13, Section 2.2] that a logarithmically smooth scheme $X$ admits an initial factorization of the map $X \to \text{Log}$ through a representable, étale morphism $\mathcal{X} \to \text{Log}$. This stack $\mathcal{X}$ is called the Artin fan of $X$. In fact, every locally connected logarithmic scheme admits an Artin fan, as Proposition 3.1.1 will show below.

If $\sigma$ is a fine, saturated, sharp monoid then define $A_\sigma$ to be the Artin fan representing the functor

$$X \mapsto \text{Hom}(\sigma^\vee, \Gamma(X, \overline{M}_X))$$

on logarithmic schemes. It is straightforward to see that $A_\sigma$ is representable by the stack quotient of the toric variety associated to $\sigma$ by its dense torus. Note that $\Gamma(A_\sigma, \overline{M}_{A_\sigma}) = \sigma^\vee$, so that

$$\text{Hom}(A_\sigma, A_\tau) = \text{Hom}(\sigma, \tau).$$

We write $A = A_{\mathfrak{m}}$.

**Proposition 3.1.1.** Let $X$ be a logarithmic algebraic stack that is locally connected in the smooth topology. Then there is an initial factorization of the map $X \to \text{Log}$ through an étale morphism $\mathcal{X} \to \text{Log}$ that is representable by algebraic spaces.

**Proof.** The collection of all smooth $U \to X$ (with smooth morphisms between them) for which the factorization exists is closed under colimits. It is therefore sufficient to show that any geometric point $x$ of $X$ has a smooth neighborhood for which the desired factorization exists. Passing to a smooth neighborhood of $x$ we may assume that $X$ is connected and that

$$\Gamma(X, \overline{M}_X) \to \Gamma(x, \overline{M}_X)$$

is bijective. We write $\sigma = \Gamma(X, \overline{M}_X)^\vee$. We will show that in this situation $A_\sigma$, with the map $f : X \to A_\sigma$ associated to the bijection $\sigma^\vee \to \Gamma(X, \overline{M}_X)$, satisfies the required universal property.
Consider a map $g : X \to \mathcal{X}$ where $\mathcal{X} \to \text{Log}$ is étale and representable by algebraic spaces. We wish to construct a unique map $s : \mathcal{A}_\sigma \to \mathcal{X}$ making the diagram

\[
\begin{array}{ccc}
X & \xrightarrow{g} & \mathcal{X} \\
\downarrow{f} & & \downarrow{s'} \\
\mathcal{A}_\sigma & \xrightarrow{s} & \text{Log}
\end{array}
\] (3.1.3)

commute. Set $\mathcal{X}' = \mathcal{X} \times_{\text{Log}} \mathcal{A}_\sigma$. Notice that (3.1.3) has a unique lift when $X$ is replaced by $x$ since $f$ sends $x$ to the unique closed point of $\mathcal{A}_\sigma$. Given $\xi : Z \to \mathcal{A}_\sigma$ we write $\Gamma(Z, \mathcal{X}')$ for the lifts of $\xi$ to $Z \to \mathcal{X}'$. By [AW13, Corollary 2.4.3], the map

$$\Gamma(\mathcal{A}_\sigma, \mathcal{X}') \to \Gamma(x, \mathcal{X}')$$

is a bijection. This provides the map $s$ and proves that it is unique; all that is left is to verify that diagram (3.1.3) commutes, i.e., that $sf = g$. The locus in $X$ where this equality holds is open—it is the locus where two sections of an étale sheaf coincide—and contains $x$. Moreover, because a strict map from a logarithmic scheme to an Artin fan must be constant on logarithmic strata, the locus of $X$ on which $sf = g$ is a union of strata. The connectedness of $X$ and the bijectivity of (3.1.2) guarantee that the only open union of strata containing $x$ is $X$ itself.

**Proposition 3.1.4.** Let $\mathcal{X}$ be an Artin fan and let $f : \mathcal{A}_\sigma \to \mathcal{X}$ be a morphism of Artin fans. Then there is a factorization of $f$ through a strict morphism $\mathcal{A}_\tau \to \mathcal{X}$, which is minimal with respect to open embeddings. This factorization is unique, up to an $\mathcal{X}$-isomorphism which is not necessarily unique. That is, if there is another such factorization through $\mathcal{A}_\tau$, then there is a commutative diagram of the following form:

\[
\begin{array}{ccc}
\mathcal{A}_\sigma & \xrightarrow{\sim} & \mathcal{A}_{\tau'} \\
\downarrow{r} & & \downarrow{r'} \xrightarrow{\sim} \\
\mathcal{A}_\tau & \xrightarrow{f} & \mathcal{X}
\end{array}
\]

**Proof.** First we show the existence of $\mathcal{A}_\tau$. Since strict maps $\mathcal{A}_\tau \to \mathcal{X}$ cover $\mathcal{X}$ we choose one whose image contains the image of the closed point of $\mathcal{A}_\sigma$. Replacing $\tau$ by a face, we can assume that the closed point of $\mathcal{A}_\tau$ and the closed point of $\mathcal{A}_\sigma$ map to the same point of $\mathcal{X}$. By definition, the projection $\mathcal{A}_\tau \times_{\mathcal{X}} \mathcal{A}_\sigma \to \mathcal{A}_\sigma$ therefore has a section over the closed point of $\mathcal{A}_\sigma$. By [AW13, Corollary 2.4.3] this extends to a section over all of $\mathcal{A}_\sigma$. This proves the existence.
Given two such factorizations, say through $A_\tau$ and $A_\tau'$, applying the existence statement proved above to the map $A_\tau \to X$ gives an $X$-morphism $A_\tau \to A_\tau'$. Since both $A_\tau$ and $A_\tau'$ are strict over $X$ this map must also be strict, hence an open embedding. Since the image contains the closed point it is actually an isomorphism.

In the construction above, we select the map $A_\tau \to A_\tau'$ so that the composition $A_\sigma \to A_\tau \to A_\tau'$ agrees with the already specified map $A_\sigma \to A_{\tau'}$ at the closed point of $A_\sigma$. Since a section of $A_{\tau'} \times_X A_\sigma$ over $A_\sigma$ is determined by what it does to the closed point, this implies that the two maps $A_\sigma \to A_{\tau'}$ agree and completes the proof.

3.2. Étale covers of Artin fans. It was shown in [AW13, Section 2.4], following [Ols07, Remark 5.26], that an Artin fan $X$ is covered by strict maps $A_\sigma \to X$. Moreover, $A_\sigma \times_X A_{\sigma'}$ is an Artin fan, so it also has a cover of this form. Therefore every Artin fan has a presentation as a colimit of strict maps between Artin fans $A_\sigma$.

An inclusion of faces $\sigma \subset \tau$ induces a strict open embedding $A_\sigma \subset A_\tau$, and the assignment $\sigma \mapsto A_\sigma$ respects intersections of faces. Therefore, given a fan in the sense of [Ful98, Section 1.4] or [CLS11, Definition 3.1.2], we may define an Artin fan $A_\Sigma$ by gluing together the $A_\sigma$ for $\sigma \in \Sigma$ according to the way they intersect inside of $\Sigma$. This permits us to give

**Definition 3.2.1.** A **subdivision** of an Artin fan $X$ is a morphism of Artin fans $Y \to X$ whose base change via any map $A_\sigma \to X$ is isomorphic to $A_\Sigma$ for some subdivision $\Sigma$ of $\sigma$.

Since the morphisms $\varphi : A_\sigma \to X$ cover $X$, we may construct a map $Y \to X$ by constructing compatible maps $Y_\varphi \to A_\sigma$. The meaning of compatibility here is that the $Y_\varphi$ should be stable under pullback via face maps $A_\tau \to A_\sigma$. We use this idea to construct several refinements of $X$.

3.3. A substitute for functoriality of Artin fans. While Artin fans are functorial with respect to strict morphism of logarithmic schemes, they appear not to be functorial in any natural way with respect to general logarithmic morphisms. In this section we adapt the construction of Section 3.1 to achieve a weak form of functoriality that will be suitable for our application in Proposition 4.5.2.

Let $X$ be a scheme equipped with a **morphism** of logarithmic structures $M'_X \to M_X$. Let $\text{Log}_{\Delta^1}$ be the universal example of an algebraic stack with these data [Ols05, Theorem 2.4], so that there is a tautological map $X \to \text{Log}_{\Delta^1}$. We show that there is an initial factorization of this map through an étale map $X \to \text{Log}_{\Delta^1}$.
Proposition 3.3.1. Let $X$ be a locally connected scheme equipped with a morphism of logarithmic structures. The corresponding map $X \to \text{Log}_{\Delta^1}$ admits an initial factorization through a representable étale map $X \to \text{Log}_{\Delta^1}$.

Proof. We begin by noting that the collection of all étale $Y \to X$ such that $Y \to \text{Log}_{\Delta^1}$ has an initial factorization through some $Y \to \text{Log}_{\Delta^1}$ is closed under colimits. As the universal property characterizing this factorization respects colimits, it will be sufficient to work étale-locally in $X$. We may therefore assume that there is a geometric point $x$ of $X$ for which the maps

$$
\begin{align*}
\Gamma(X, M_X) &\to \Gamma(x, M_X) \\
\Gamma(X, M'_X) &\to \Gamma(x, M'_X)
\end{align*}
$$

are bijections. Set $\sigma = \Gamma(X, M_X)^\vee$ and $\tau = \Gamma(X, M'_X)^\vee$. The map $\sigma \to \tau$ induces a map $\varphi : A_\sigma \to A_\tau$ and moreover gives a map $A_\sigma \to \text{Log}_{\Delta^1}$. In order to emphasize the map to $\text{Log}_{\Delta^1}$, we write $A_\sigma \to \tau \simeq A_\sigma$ here.

Lemma 3.3.2. The map $A_\sigma \to \text{Log}_{\Delta^1}$ is étale and the collection of all such maps is an étale cover of $\text{Log}_{\Delta^1}$.

Proof. As both stacks are locally of finite presentation, we only need to verify the lifting property. Consider a diagram

$$(3.3.3)$$

$$(3.3.4)$$

in which $S'$ is an infinitesimal extension of $S$. The map $S' \to \text{Log}_{\Delta^1}$ gives a morphism of logarithmic structures $M_{S'} \to M'_{S'}$ on $S'$. The commutativity of the square induces a commutative square

$$(3.3.4)$$

where the vertical arrow on the right is the restriction of the map of characteristic monoids $\overline{M}_{S'} \to \overline{M}'_{S'}$ to $S$ and the horizontal arrows are charts. But $\text{ét}(S') = \text{ét}(S)$ and under this identification $\overline{M}_{S} = \overline{M}_{S'}$ and $\overline{M}'_{S} = \overline{M}'_{S'}$. With these substitutions, $(3.3.4)$ gives the diagonal arrow lifting $(3.3.3)$ and shows it is unique.
The assertion that the $A_{\sigma \to \tau}$ cover $\text{Log}_{\Delta^1}$ translates into the following familiar facts:

1) the characteristic monoid of a fine, saturated logarithmic structure possesses a chart locally, and
2) a morphism of fine, saturated logarithmic structures with charts by $\sigma^\vee$ and $\tau^\vee$ may be induced locally from a morphism $\sigma \to \tau$.

Returning to the proof of Proposition 3.3.1, our reduction guarantees we have a morphism $X \to A_{\sigma \to \tau}$ over $\text{Log}_{\Delta^1}$. To see that $A_{\sigma \to \tau}$ has the right universal property, we repeat the argument of Proposition 3.1.1. We consider a commutative diagram

$$
\begin{array}{ccc}
X & \longrightarrow & \mathcal{X} \\
\downarrow & & \downarrow \\
A_{\sigma \to \tau} & \longrightarrow & \text{Log}_{\Delta^1}
\end{array}
$$

in which $\mathcal{X}$ is étale over $\text{Log}_{\Delta^1}$. Replacing $\mathcal{X}$ with $\mathcal{X} \times_{\text{Log}_{\Delta^1}} A_{\sigma \to \tau}$, we immediately reduce to the case where there is a map $\mathcal{X} \to A_{\sigma \to \tau}$ that is compatible with the rest of the diagram, and the problem is to show there is exactly one section of this map making the rest of the diagram commute. By assumption, a unique section exists at the geometric point $x$ of $A_{\sigma \to \tau}$.

$\text{Corollary 3.3.5. Let } X \to Y \text{ be a morphism of logarithmic schemes. Suppose that } \mathcal{Y} \text{ is the Artin fan of } Y \text{ and } \mathcal{X} \text{ is the Artin fan of } X \text{ relative to } \text{Log}_{\Delta^1}. \text{ Then there is a canonical morphism } \mathcal{X} \to \mathcal{Y} \text{ making the diagram below commute:}

$$
\begin{array}{ccc}
X & \longrightarrow & \mathcal{X} \\
\downarrow & & \downarrow \\
Y & \longrightarrow & \mathcal{Y}
\end{array}
$$

4. Subdivisions

The goal of this section is to show that essentially any logarithmic scheme has a logarithmic modification with globally generated characteristic monoid. We begin by defining our terms.

$\text{Definition 4.1. A logarithmic modification of logarithmic Artin stacks is a proper, surjective, logarithmically étale morphism.}$
A \textit{representable} logarithmic modification of a logarithmically smooth scheme is a modification, but in general logarithmic modifications need not be representable (they include certain root stack constructions) nor need they be birational (when the target is not logarithmically smooth). Examples of logarithmic modifications appear in Sections 4.2 and 4.3 below.

4.2. \textbf{Star subdivision}. Let \( \sigma \) be a fine, saturated, sharp monoid and \( x \in \sigma \) an element. For each face \( \tau \) of \( \sigma \) not containing \( x \), let \( \tau' \) be the saturated submonoid of \( \sigma \) generated by \( \tau \) and \( x \). The \( \tau' \) and all of their faces constitute a fan, called the star subdivision of \( \sigma \), and denoted \( x \cdot \sigma \).

This construction is functorial with respect to inclusion of faces containing \( x \). That is, if \( \sigma \subset \tau \) is the inclusion of a face containing \( x \), then \( x \cdot \sigma \) is canonically a subfan of \( x \cdot \tau \).

To translate this to Artin fans, we must subdivide at vectors \( x \) stable under monodromy, and further replace such vectors \( x \) with their Artin fan analogues. As in Section 3.1 we write \( A = A_N \). The following definition is adapted from \cite[Section 5.3]{Wlo03}.

\textbf{Definition 4.2.1.} Let \( \mathcal{X} \) be an Artin fan. We will call a morphism of Artin fans \( A \to \mathcal{X} \) a vector of \( \mathcal{X} \). We call a vector \( x \) of \( \mathcal{X} \) stable if, whenever \( x \) factors through some strict map \( A_\sigma \to \mathcal{X} \), that factorization is unique.

Assuming \( x \) is stable, we construct the star subdivision \( \mathcal{X}' \) as follows: For any map \( \varphi : A_\sigma \to \mathcal{X} \), let

\[
\mathcal{X}'_{\varphi} = \begin{cases} 
A_\sigma & \text{if } x \text{ does not lift to } A_\sigma \\
A_{x,\sigma} & \text{if } x \text{ lifts to } A_\sigma
\end{cases}
\]

where \( A_{x,\sigma} \) denotes the star subdivision of \( A_\sigma \) with respect to the unique lift of \( x \) to \( A_\sigma \). Since \( x \) is stable, this construction is compatible with strict \( \mathcal{X} \)-morphisms \( A_\varphi \to A_\sigma \), hence glues to give a global construction.

\textbf{Proposition 4.2.2.} Star subdivision is projective.

\textit{Proof.} Let \( \phi_x : \mathcal{X}' \to \mathcal{X} \) be a star subdivision given by a stable vector \( x : A \to \mathcal{X} \). Note that \( \phi_x \) is representable and birational. It suffices to produce a \( \phi_x \)-ample line bundle over \( \mathcal{X}' \). Let \( E \subset \mathcal{X}' \) be the exceptional divisor. Since \( x \) is stable, such \( E \) is a well-defined prime divisor over \( \mathcal{X}' \). We first notice that \( E \) is \( \mathbb{Q} \)-cartier. This could be checked locally via the toric geometry over each chart, see \cite[11.1.6(b)]{CLS11}. Let \( L \) be the line bundle associated to \( -k \cdot E \) for some sufficiently divisible...
positive integer $k$. By [Gro61, 4.6.4] to see that $L$ is $\phi_x$-ample, it suffices to check the statement locally over $\mathcal{X}$. By taking base change to a covering of $\mathcal{X}$, we may assume that $\mathcal{X} = \mathcal{A}_\sigma$. Note that $\mathcal{A}_\sigma$ is given by the global quotient of the affine toric varieties $\mathbb{A}_\sigma$ by its maximal torus. The ampleness follows from the fact that star subdivisions induce equivariant projective modifications of toric varieties in which the Cartier divisor $-kE$ is ample.

4.3. Barycentric subdivision. For a fine, saturated, sharp monoid $\sigma$, let $B(\sigma)$ be the barycentric subdivision of $\sigma$ (see, e.g., [CLS11, Exercise 11.1.10]). The fan $B(\sigma)$ is automatically simplicial. We obtain a map $\mathcal{A}_{B(\sigma)} \to \mathcal{A}_\sigma$ that is stable under base change via face maps, by definition. By descent we obtain a map $B(\mathcal{X}) \to \mathcal{X}$ that we call the barycentric subdivision of $\mathcal{X}$.

**Proposition 4.3.1.** A barycentric subdivision of Artin fans is projective.

**Proof.** The barycenters of the maps $\mathcal{A}_{\sigma} \to \mathcal{X}$ are stable, so that barycentric subdivision may be achieved as iterated star subdivision (along the barycenters of the cones, in order of decreasing dimension).

4.4. Resolution. The following lemma is essentially a restatement of [AT13, Lemma 2.4.6 (1)]:

**Lemma 4.4.1.** Let $\mathcal{X}$ be an Artin fan and $B\mathcal{X}$ its barycentric subdivision. Every map $x : \mathcal{A} \to B\mathcal{X}$ is stable.
Proof. Suppose we have a strict map $\mathcal{A}_\sigma \to B\mathcal{X}$ and two maps $x, y : \mathcal{A} \to \mathcal{A}_\sigma$ that have isomorphic images in $B\mathcal{X}$. If $\tau$ is the minimal face of $\sigma$ containing $x$ then we must have a pair of maps $\mathcal{A}_\tau \supseteq \mathcal{A}_\sigma$ that compose to the same map $B\mathcal{X}$. But recall that $\sigma$ is simplicial, say of dimension $d$, and corresponds to a flag of $d$ faces of a monoid $\omega$, for some strict $\mathcal{A}_\omega \to \mathcal{X}$. A face $\tau \subset \sigma$ is characterized uniquely by the dimensions of the faces in the corresponding subflag of $\omega$. A fortiori, the two inclusions $\tau \subset \sigma$ must coincide.

Theorem 4.4.2. Any Artin fan $\mathcal{X}$ has a projective subdivision $\mathcal{Y} \to \mathcal{X}$ admitting a strict map $\mathcal{Y} \to \mathcal{A}^S$, for some finite set $S$.

Proof. Step 1. Let $\mathcal{X}$ be an Artin fan. Consider the barycentric subdivision, $B\mathcal{X} \to \mathcal{X}$, a projective morphism by Proposition 4.3.1. By Lemma 4.4.1, every vector of $B\mathcal{X}$ is stable.

Step 2. Since every vector in $B\mathcal{X}$ is stable we can resolve singularities by the same procedure used to resolve singularities in toric and toroidal geometry, see e.g. [KKMSD73, Theorem 11*, page 94]. Indeed, if $\mathcal{A}_\sigma \to \mathcal{X}$ is any strict map, then any element of $\sigma$ corresponds to a stable vector of $\mathcal{X}$, hence can be used as the center of a star subdivision. We can apply the familiar procedure to resolve toric singularities individually to each map $\mathcal{A}_\sigma \to \mathcal{X}$. With further star subdivision we maintain the property that every vector is stable. After a finite number of subdivisions, we obtain a subdivision $\mathcal{Y} \to \mathcal{X}$ where $\mathcal{Y}$ is smooth (and in particular simplicial) with the property that every vector $y : \mathcal{A} \to \mathcal{Y}$ is stable. Since the procedure involves only star subdivisions, it is projective.

Step 3. Consider the set $S$ of strict maps $\mathcal{A} \to \mathcal{Y}$. These correspond to the codimension-1 strata of $\mathcal{Y}$. We construct a map $\mathcal{Y} \to \mathcal{A}^S$, by working with charts $\mathcal{A}^r \to \mathcal{Y}$. If $\mathcal{A}^r \to \mathcal{Y}$ is any strict map, we note that $\mathcal{A}^r$ is the coproduct of $r$ copies of $\mathcal{A}$ as a monoidal stack, and we have $r$ open embeddings $\mathcal{A} \to \mathcal{A}^S$ associated to the $r$ compositions $\mathcal{A} \to \mathcal{A}^r \to \mathcal{Y}$. As $\mathcal{A}^r$ is the coproduct (as a monoidal stack) of its $r$ open embeddings of $\mathcal{A}$, this map extends to a map $\mathcal{A}^r \to \mathcal{A}^S$. This construction respects $\mathcal{Y}$-morphisms $\mathcal{A}^s \to \mathcal{A}^r$, hence descends to a map $\mathcal{Y} \to \mathcal{A}^S$.

All that is left now is to verify that the map $\mathcal{Y} \to \mathcal{A}^S$ constructed above is strict. It is sufficient to show that each of the maps $\mathcal{A}^r \to \mathcal{A}^S$ is strict. Recall that every vertex of $\mathcal{Y}$ is stable, so that each component of $\mathcal{A}^r$ maps to a distinct component of $\mathcal{A}^S$. From this it is obvious that the map $\mathcal{A}^r \to \mathcal{A}^S$ is the open embedding of $r$ components. ⊤
With notation as in the statement and proof of the theorem, consider the set $S' \subset S$ consisting of strict maps $A \to \mathcal{Y}$ such that the composite $A \to X$ is not strict. These correspond to the exceptional divisors of $\mathcal{Y} \to \mathcal{X}$: each of the generators of $\mathcal{A}^{S'}$ pulls back to a line bundle and section $(\mathcal{L}_i, s_i)$ on $\mathcal{Y}$ vanishing along an exceptional divisor of the projective morphism $\mathcal{Y} \to \mathcal{X}$.

**Corollary 4.4.3.** Let $X$ be a logarithmic scheme. Then there is a logarithmic modification $\Psi : Y \to X$ with relatively ample line bundle $L$, as well as line bundles and sections $(L_i, s_i)$ on $Y$ vanishing along subschemes $E_i \subset Y$, having the following properties:

(i) The morphism $\Psi$ is projective, logarithmically étale, and surjective.

(ii) $\Psi$ is an isomorphism away from the locus $\cup_i E_i$.

(iii) We have $L = \bigotimes L_i^{\otimes m_i}$ with $m_i$ negative.

(iv) $Y$ has Deligne-Faltings logarithmic structure.

(v) If $X$ is logarithmically smooth, then the underlying structure $Y$ is smooth in the usual sense.

**Proof.** Let $\mathcal{X}$ be the Artin fan of $X$, let $\mathcal{Y}$ be given by Theorem 4.4.2 and take $Y = \mathcal{Y} \times_\mathcal{X} X$. By the theorem, this gives (i), (iv), and (v) immediately. For the $L_i$, $s_i$, and $E_i$ we simply pull back $\mathcal{L}_i$, $s_i$, and $\mathcal{E}_i$ from $\mathcal{Y}$. This gives (ii). Recall that the exceptional divisor of any star subdivision is anti-ample. Since the composition of projective morphisms is projective, there is a linear combination, with positive coefficients, of the pullbacks of these divisors which is anti-ample for $\mathcal{Y} \to \mathcal{X}$. Since every divisor $(s_i)$ corresponding to an element of $S'$ appears in such an exceptional divisor, there exist negative integers $m_i$ such that $L = \bigotimes L_i^{\otimes m_i}$ is relatively ample for $\mathcal{Y} \to \mathcal{X}$. Then (iii) is obtained by taking $L$ to be the pull-back of $\mathcal{L}$. $\diamondsuit$

**4.5. Stable maps into subdivisions.** Let $\mathcal{X}$ be an Artin fan. Recall from Section 1.4 that $\mathcal{M}(\mathcal{X})$ is the moduli stack parameterizing prestable logarithmic maps to $\mathcal{X}$ and $\mathcal{M}'(\mathcal{Y} \to \mathcal{X})$ parameterizes prestable maps which are relatively stable for $\mathcal{Y} \to \mathcal{X}$. An object of $\mathcal{M}'(\mathcal{Y} \to \mathcal{X})(S)$ is a diagram

$$
\begin{array}{ccc}
C & \longrightarrow & \mathcal{Y} \\
\downarrow & & \downarrow \\
\widehat{C} & \longrightarrow & \mathcal{X}
\end{array}
$$

(4.5.1)
of pre-stable logarithmic maps over $S$ where $C \to \overline{C}$ is a logarithmic modification and the automorphism group of this diagram relative to the bottom arrow $\overline{C} \to \mathcal{X}$ is finite. In other words, the map $C \to \mathcal{Y} \times_{\mathcal{X}} \overline{C}$ is stable and $C \to \overline{C}$ is a contraction of rational components. The morphism $\mathcal{M}'(\mathcal{Y} \to \mathcal{X}) \to \mathcal{M}(\mathcal{X})$ under consideration sends a diagram (4.5.1) to $\overline{C} \to \mathcal{X}$. See [AW13, Section 3.1], where this morphism is shown to be birational, for a more thorough discussion.

**Proposition 4.5.2.** Let $\mathcal{Y} \to \mathcal{X}$ be a modification of Artin fans. Any diagram

\[
\begin{array}{ccc}
S' & \to & \mathcal{M}'(\mathcal{Y} \to \mathcal{X}) \\
\downarrow & & \downarrow \\
S & \to & \mathcal{M}(\mathcal{X})
\end{array}
\]

admits a unique lift after passing to a (not necessarily representable) logarithmic modification $S' \to S$.

**Proof.** The map $S \to \mathcal{M}(\mathcal{X})$ corresponds to a logarithmic curve $\overline{C}$ over $S$ and a map $\overline{C} \to \mathcal{X}$. Applying Corollary 3.3.5 to the map $\overline{C} \to S \times \mathcal{X}$ we obtain a diagram of Artin fans:

\[
\begin{array}{ccc}
\mathcal{Y} & \to & \\
\downarrow & & \\
\mathcal{C} & \to & \mathcal{X} \\
\downarrow & & \\
S & \to & 
\end{array}
\]

Take $\mathcal{C} = \mathcal{C} \times_{\mathcal{X}} \mathcal{Y}$, with the fiber product formed in the category of fine, saturated, logarithmic algebraic stacks; this is the pullback of a subdivision of $\mathcal{X}$, hence is a subdivision of $\overline{C}$. After a logarithmic modification of $S$, we can assume that $S$ is smooth (Theorem 4.4.2), $\mathcal{C} \to S$ is equidimensional [AK00, Lemmas 4.1 and 4.3], and therefore that $\mathcal{C}$ is flat over $S$ [AK00, Remark 4.6]. By [AK00, Proposition 5.1] we can ensure as well that the fibers of $\mathcal{C} \to S$ are reduced by replacing the integral lattice of $S$ with a finite index sublattice.\(^1\)

Now let $C = \mathcal{C} \times_S \overline{C}$. We show that $C$ is a logarithmic curve [ACG+10, Definition 4.5] over $S$. We must verify the following properties:

\(^1\)Note that this corresponds to a root stack construction, so that $S' \to S$ is not necessarily representable.
(1) $C$ is logarithmically smooth over $S$: It is the composition of a logarithmically étale map $C \to \overline{C}$ (the base change of the log. étale map $Y \to X$) and a logarithmically smooth map $\overline{C} \to S$.

(2) $C \to S$ has connected fibers: Since $\overline{C}$ has connected fibers over $S$, it is sufficient to show that $C \to \overline{C}$ has connected fibers. This follows by base change from the connectedness of the fibers of $Y \to X$.

(3) $C \to S$ is integral in the logarithmic sense: Since $C \to \overline{C}$ and $S \to S$ are strict, this is immediate from the flatness of the map $C \to S$.

(4) $C \to S$ has reduced, 1-dimensional fibers: The map $C \to C \times_S S$ is smooth of relative dimension 1 and $C \to S$ has reduced 0-dimensional fibers.

(5) $C$ is proper over $S$: The map $C \to \overline{C}$ is proper (it is a subdivision), so $C \to \overline{C}$ is proper, and $\overline{C} \to S$ is proper by hypothesis. Therefore $C$ lifts $\overline{C} \to X$ to a diagram (4.5.1). It is the base change of subdivision, so it is a logarithmic modification. Furthermore, any component of $C$ contracted in $\overline{C}$ is stabilized by the map to $Y$. Therefore this diagram lifts $\overline{C} \to X$ to a point of $\mathcal{M}′(Y \to X)$.

We verify that this lift is unique. Suppose that $C'$ is another lift. By the universal property of fiber product, we obtain a map $f : C' \to C = \overline{C} \times_X Y$. By the definition of $\mathcal{M}(Y \to X)$, this map is stable. On the other hand, the map $C' \to \overline{C}$ is a logarithmic modification of logarithmic curves, hence is a contraction of semistable components. Thus, $C \to \overline{C}$ is stable and a contraction of semistable components, hence it is an isomorphism.

\textbf{Corollary 4.5.3.} Assume that $Y$ is a subdivision of an Artin fan $X$. Then the morphism

$$\mathcal{M}′(Y \to X) \to \mathcal{M}(X)$$

is birational and satisfies the valuative criterion for properness.

\textbf{Proof.} Birationality was proved in [AW13, Proposition 5.3.1]. The valuative criterion is immediate from Proposition 4.5.2.

\hfill \blacklozenge

\textbf{4.6. The valuative criterion.}

\textbf{Proof of Proposition 1.4.3.} Let $R$ be a discrete valuation ring and $K$ be the fraction field of $R$. Consider an object $f : \text{Spec } K \to \overline{M}$, which we would like to extend, possibly after base change, to a unique object $\text{Spec } R \to \overline{M}(X)$. The object $f$ corresponds to a logarithmic morphism $f : S := (\text{Spec } K, M_S) \to \overline{M}(X)$. 

Consider the composition $f : S \to \mathcal{M}(\mathcal{X})$ of $f$ with $\mathcal{M}(\mathcal{X}) \to \mathcal{M}(\mathcal{X})$, and the cartesian diagram 1.4.1. By Proposition 4.5.2 there is a logarithmic modification $S' \to S$ and a unique lift of $f$ to $f' : S' \to \mathcal{M}(Y \to \mathcal{X})$, giving rise to a unique lift $f' : S' \to \overline{\mathcal{M}}(Y)$. Choosing a point on $S'$ we may assume $S' = \text{Spec} \ K$. By the valuative criterion of $\mathcal{M}(Y)$, see [AC, Corollary 3.11], after taking a further finite extension of $R$, we have a logarithmic scheme $S'' = (\text{Spec} \ R, M_{S''})$ extending $S'$, and a unique lift $\tilde{f}' : S'' \to \overline{\mathcal{M}}(Y)$ of $f' : S' \to \overline{\mathcal{M}}(Y)$. Composing with $\overline{\mathcal{M}}(Y) \to \overline{\mathcal{M}}(X)$ we obtain an arrow $\tilde{f} : S'' \to \overline{\mathcal{M}}(X)$:

\[
\begin{array}{ccc}
S' & \xrightarrow{f'} & \overline{\mathcal{M}}(Y) \\
\downarrow & & \downarrow \\
S'' & \xrightarrow{\tilde{f}'} & \overline{\mathcal{M}}(X)
\end{array}
\]

Applying Proposition 4.5.2 to $S' \to \mathcal{M}(\mathcal{X})$ we obtain that, after replacing $S'$ by a logarithmic modification $S''$, any extension $\tilde{f}$ lifts to $\overline{\mathcal{M}}(Y)$, and since $\tilde{f}'$ is unique we obtain that $\tilde{f}$ is unique, as needed.

\section{5. Boundedness of Numerical Data}

In this section we will identify locally constant numerical data $\Gamma$ on $\overline{\mathcal{M}}(X)$ such that each $\overline{\mathcal{M}}_\Gamma(X)$ is of finite type. In addition to the genus $g$ of the source curve, the number $n$ of marked points, and the homology class $\beta$ of the curve’s image in $X$, we also have evaluation maps

$\overline{\mathcal{M}}(X) \to \wedge X \to \wedge \mathcal{X}$

associated to each marking. The choice of a connected component of $\wedge \mathcal{X}$ for each marked point gives one more locally constant datum. Let $\Gamma = (g, n, \beta, \varphi)$ where $\varphi \in \pi_0(\wedge \mathcal{X})^n$. We write $\overline{\mathcal{M}}_\Gamma(X)$ for the open and closed substack of $\overline{\mathcal{M}}(X)$ with these numerical data.

Select a logarithmic modification $Y \to X$, obtained by base change from a subdivision of Artin fans $\mathcal{Y} \to \mathcal{X}$, as in Corollary 4.4.3. The irreducible components $E_i$ of the exceptional locus of $\mathcal{Y} \to \mathcal{X}$ are non-singular divisors which are unions of logarithmic strata. We denote their pre-images on $Y$ by $E_i$ and the corresponding line bundles by $L_i$.

Write $\overline{\mathcal{M}}_\Gamma(Y)$ for the open and closed substack of $\overline{\mathcal{M}}(Y)$ lying above $\overline{\mathcal{M}}_\Gamma(X)$. The following proposition, whose proof occupies the rest of this section, will complete the proof of our main theorem:
Proposition 5.1. The algebraic stack $\mathcal{M}_\Gamma(Y)$ is of finite type.

Recall that if the genus $g$, number of markings $n$, degree with respect to some ample line bundle on $Y$, and a component of $\wedge Y$ for each marked point are fixed in $\Xi$, then $\mathcal{M}_\Xi(Y)$ is of finite type [GS13, Theorem 3.12] or [AC, Corollary 3.13]. We will show that $\mathcal{M}_\Gamma(Y)$ is a union of only finitely many $\mathcal{M}_\Xi(Y)$. Obviously, once $\Gamma$ is fixed, $g$ and $n$ are fixed. The first step of our argument will be to show that the components of $\wedge Y$ map bijectively to the components of $\wedge X$, so that once a component of $\wedge X$ is fixed in $\Gamma$ there is a unique component of $\wedge Y$ lying above it. Finally, we will show that the degree in $Y$ is bounded by the choice of $\Gamma$.

5.2. Boundedness of contact orders. Recall that a family of logarithmic points parameterized by a logarithmic scheme $(X, M_X)$ is simply a line bundle $L$ on $X$. Equivalently, a family of logarithmic points parameterized by $X$ may be viewed as an augmentation $M_X \to M'_X$ of the logarithmic structure of $X$ with $M'_X = M_X \times \mathbb{N}$. A logarithmic point of $(Y, M_Y)$ parameterized by $(X, M_X)$ is a logarithmic morphism $(X, M'_X) \to (Y, M_Y)$, where $(X, M'_X)$ is a family of logarithmic points parameterized by $(X, M_X)$.

Proposition 5.2.1. Suppose that the following diagram of logarithmic algebraic stacks is cartesian:

$$
\begin{array}{ccc}
Y' & \longrightarrow & Y \\
\downarrow & & \downarrow \\
X' & \longrightarrow & X 
\end{array}
$$

Then the diagram below is cartesian as well:

$$
\begin{array}{ccc}
\wedge Y' & \longrightarrow & \wedge Y \\
\downarrow & & \downarrow \\
\wedge X' & \longrightarrow & \wedge X 
\end{array}
$$

Proof. This is immediate from the modular description of the stack of logarithmic points. ♠

We evaluate $\wedge A_\sigma$ for a fine, saturated, sharp monoid $\sigma$. Consider a map $f : (X, M'_X) \to A_\sigma$. This corresponds to a homomorphism of monoids,

$$
\sigma^\vee \to \Gamma(X, M'_X) \times \text{Hom}(X, \mathbb{N}).
$$
The map $\sigma^\vee \to \text{Hom}(X, \mathbb{N})$ may be viewed as a locally constant function $\varphi : X \to \sigma$. For each $\varphi \in \sigma$ we therefore obtain an open and closed substack $\wedge_\varphi A_\sigma$. The element $\varphi$ is called the contact order.

Recall that each element $\xi$ of $\sigma^\vee$ corresponds to a line bundle and section $(L_\xi, s_\xi)$ on $A_\sigma$. If the pairing $(\varphi, \xi)$ is non-zero at a point $x$ of $X$ then $f^*s_\xi$ vanishes identically on $X$. Let $A_\varphi \subset A_\sigma$ be the closed substack defined by the vanishing of $s_\xi$ for all $\xi$ such that $(\varphi, \xi) > 0$. Then the map $(X, M_X) \to A_\sigma$ factors uniquely through $A_\varphi$. This gives a map $\wedge_\varphi A_\sigma \to A_\varphi$.

This map is not an isomorphism. In order to lift a map $X \to A_\varphi$ to $\wedge_\varphi A_\sigma$ we must find a line bundle $L$ on $X$, together with isomorphisms $f^*L_\xi \simeq L^{(\varphi, \xi)} \otimes L_\alpha$ (where $\alpha$ is the image of $\xi$ in $\Gamma(X, \mathbb{M}_X)$) for all $\xi \in \sigma^\vee$. The choices of such $L$ form a gerbe banded by $\mu_d$ where $d$ is the greatest common divisor of the $(\varphi, \xi)$. If $\varphi = 0$ then it is a gerbe banded by $\mathbb{G}_m$. This proves the following proposition:

**Proposition 5.2.2.** For any $\varphi \in \sigma$, the map $\wedge_\varphi A_\sigma \to A_\varphi$ is a gerbe banded by $\mathbb{G}_m$ if $\varphi = 0$ and otherwise by $\mu_d$, where $d$ is the greatest common divisor of the $(\varphi, \xi)$, with $\xi \in \sigma^\vee$.

**Corollary 5.2.3.** The connected components of $\wedge A_\sigma$ are in bijection with the elements of $\sigma$.

**Corollary 5.2.4.** If $\mathcal{X}$ is an Artin fan then the connected components of $\wedge \mathcal{X}$ are in bijection with the isomorphism classes of maps $A \to \mathcal{X}$.

**Proof.** We may present $\mathcal{X}$ as a colimit of a diagram of strict maps among the $A_\sigma$. Since $\pi_0(\mathcal{X}, -), \wedge( -)$, and $\text{Hom}(A, -)$ all respect strict colimits of Artin fans, the problem reduces to the case $\mathcal{X} = A_\sigma$. In that case we only need to recall that $\text{Hom}(A, A_\sigma) = \sigma$, functorially in $\sigma$. \hfill ♠

**Corollary 5.2.5.** If $\sigma \to \tau$ is a morphism of fine, saturated, sharp monoids and $\varphi \in \sigma$ has image $\psi \in \tau$ then the induced map $\wedge_\varphi A_\sigma \to \wedge_\psi A_\tau$ is of finite type.

**Proof.** Since the map is certainly locally of finite type, it is sufficient to show it is quasi-compact. But $\wedge_\varphi A_\sigma$ and $\wedge_\psi A_\tau$ are, respectively, gerbes over $A_\varphi$ and $A_\psi$. The map $A_\varphi \to A_\psi$ is certainly quasi-compact. \hfill ♠

**Corollary 5.2.6.** Let $\mathcal{Y} \to \mathcal{X}$ be a subdivision. Then the induced map $\wedge \mathcal{Y} \to \wedge \mathcal{X}$ is a bijection on connected components and is of finite type.

\footnote{$A_\varphi$ depends only on the face of $\sigma$ that contains $\varphi$.}
Proof. A subdivision induces a bijection $$\text{Hom}(A, Y) \rightarrow \text{Hom}(A, X)$$ on the sets of connected components. To show the map is of finite type, it is sufficient to work étale-locally in $$X$$. We may therefore assume $$X = A \tau$$. The subdivision $$Y$$ of $$A \tau$$ has an open cover by finitely many $$A \sigma$$. This reduces us to showing that the maps $$\wedge \varphi A \sigma \rightarrow \wedge \varphi A \tau$$ are of finite type, as we did in the previous corollary. ♠

5.3. Boundedness of the curve classes. Let $$f : C \rightarrow Y$$ be an object of $$\overline{M}(Y)$$. Denote by $$c_j(E_i)$$ the contact order of the $$j$$-th marking with the exceptional divisor $$E_i$$ as in Corollary 4.4.3. These numbers are uniquely determined by the induced maps to $$\wedge Y$$, hence by $$\Gamma$$. The following is a restatement of Proposition 1.5.4. Recall that $$L$$ is a relatively ample line bundle for $$Y$$ over $$X$$ and that $$L \cong \bigotimes L \otimes m_i$$ for negative integers $$m_i$$. 

**Proposition 5.3.1.** Let $$f : C \rightarrow Y$$ be a point of $$\overline{M}(Y)$$. The values $$c_j(E_i)$$ determine $$\deg_C(L)$$.

**Proof.** We have 
$$\deg_C(L) = \sum_i m_i \deg_C(E_i) = \sum_i m_i \deg_C(E_i).$$

This quantity is locally constant on $$\mathfrak{M}(Y)$$, which is logarithmically smooth. We can therefore replace $$C$$ with a deformation that is smooth and intersects the $$E_i$$ properly. In this case $$\deg_C(E_i) = \sum_j c_j(E_i)$$, so $$\deg_C(L) = \sum_{i,j} m_i c_j(E_i)$$ is determined by the $$c_j(E_i)$$, as required. ♠

**Proposition 5.3.2.** Fix an ample line bundle $$M$$ on $$X$$ and $$f : C \rightarrow Y$$ a point of $$\overline{M}(Y)$$. Denote the projection from $$Y$$ to $$X$$ by $$\pi$$. Then $$L \otimes \pi^*M$$ is ample on $$Y$$ and 
$$\deg(f^*(L \otimes \pi^*M)) = \deg(f^*\pi^*M) + \sum_{i,j} m_i c_j(E_i).$$

In particular, the degree of $$f$$ with respect to $$L \otimes \pi^*M$$ is determined combinatorially by the image of $$(C, f)$$ in $$\overline{M}(X)$$.

**Proof.** We have 
$$\deg(f^*L \otimes f^*\pi^*M) = \deg(f^*\pi^*M) + \deg(f^*L)$$
and $$\deg(f^*L)$$ was computed in the last proposition. ♠

We conclude that $$\Gamma$$ bounds the degree of $$f : C \rightarrow Y$$ as well as its contact orders along the logarithmic divisors. Therefore $$\overline{M}_\Gamma(Y)$$ is of finite type. This completes the proof of Proposition 5.1.
As the map \( \overline{\mathcal{M}}_\Gamma(Y) \to \overline{\mathcal{M}}_\Gamma(X) \) is proper, we deduce that \( \overline{\mathcal{M}}_\Gamma(X) \) is of finite type as well.

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(Abramovich) Department of Mathematics, Brown University, Box 1917, Providence, RI 02912, U.S.A.

E-mail address: abramovic@math.brown.edu

(Chen) Department of Mathematics, Columbia University, Rm 628, MC 4421, 2990 Broadway, New York, NY 10027, U.S.A.

E-mail address: q.chen@math.columbia.edu

(Marcus) Mathematics and Statistics, The College of New Jersey, Ewing, NJ 08628, U.S.A.

(Wise) University of Colorado, Boulder, Boulder, Colorado 80309-0395, USA

E-mail address: jonathan.wise@colorado.edu