Renormalizable supersymmetric gauge theory in six dimensions

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Abstract

We construct and discuss a $6D$ supersymmetric gauge theory involving four derivatives in the action. The theory involves a dimensionless coupling constant and is renormalizable. At the tree level, it enjoys $\mathcal{N}=(1,0)$ superconformal symmetry, but the latter is broken by quantum anomaly. Our study should be considered as preparatory for seeking an extended version of this theory which would hopefully preserve conformal symmetry at the full quantum level and be ultraviolet-finite.

1 Introduction

Higher-dimensional quantum field theories bear interest from different points of view and appear in numerous intertwining contexts, such as Kaluza-Klein approach, string theory, higher spin theory, etc.

Recently, one of the present authors suggested\cite{1} that some field theory in higher dimensions could play a role of fundamental microscopic theory. This hypothetical underlying higher-dimensional theory should, in particular, involve 3-brane classical solutions, which might be associated with our Universe in the spirit of\cite{2}. In contrast to other popular brane-Universe scenarios, like Randall-Sundrum scenario\cite{3}, the fundamental theory of the bulk in this case is not assumed to include gravity, the latter is expected to be generated as an effective theory living on the brane. Clearly, there should exist a mechanism of getting rid of the cosmological term which is known to be zero or very small. For ensuring this, the fundamental theory should be supersymmetric. Indeed, only supersymmetry can provide for the exact cancellation of quantum corrections to the energy density of the brane solution.

If we want the “ultimate” higher-dimensional theory to be renormalizable, the canonical dimension of the lagrangian should be greater than 4, i.e. it should involve higher derivatives. Higher derivative theories are known to have a problem of ghosts, which in many cases break unitarity and/or causality of the theory. However, a model study
performed in Refs. [1] indicated that in some cases, namely, when the theory enjoys *exact* conformal invariance, the ghosts are not so malignant and the theory might enjoy a unitary S-matrix to any order of perturbation theory.

We conclude that the conjectural fundamental QFT should preferably be a superconformal theory. This restricts the number of dimensions in the flat space-time where the theory is formulated by \( D \leq 6 \). Indeed, all standard superconformal algebras (involving the super-Poincaré algebra as a subalgebra) are classified (for instructive reviews see [4]). The highest possible dimension is six, which allows for the minimal \( \mathcal{N}=(1,0) \) conformal superalgebra and the extended chiral \( \mathcal{N}=(2,0) \) conformal superalgebra.

Thus a natural hypothesis is that the field theory in question lives in six dimensions and enjoys the highest possible superconformal (and super-Poincaré) symmetry with \( \mathcal{N}=(2,0) \). Unfortunately, no field theory with this symmetry group is known to date. A possible candidate is the superconformal theory of tensor (2,0) multiplet.\(^1\) However, the corresponding lagrangian (with a standard, linear realization of \( \mathcal{N}=(2,0) \) superconformal symmetry\(^2\)) is not constructed, and only indirect results concerning scaling behavior of certain operators have been obtained so far [7].

In this article, we derive the lagrangian for the 6D gauge theory with unextended \( \mathcal{N}=1 \) superconformal symmetry. This theory is conformal at the classical level and renormalizable. However, it is not finite: the \( \beta \) function does not vanish there and the conformal symmetry is broken at the quantum level by anomaly. In other words, the theory considered in this paper cannot be regarded as a viable candidate for the ultimate theory. However, its study represents a necessary preparatory step before tackling the problem of constructing and studying a possible extension of this theory, such that it would respect the superconformal symmetry (at least the \( \mathcal{N}=(1,0) \) one) at the full quantum level and so could be considered as the appropriate candidate.

The adequate technique for constructing the relevant 6D superfields and their interactions is the technique of harmonic superspace (HSS) [8] which was extended to six dimensions in [9–11]. In the next Section we briefly describe the HSS technique in six dimensions, derive the lagrangian of higher-derivative supersymmetric Yang-Mills theory and prove its conformal invariance. In Sect. 3, we derive the lagrangian in the component form. In Sect. 4, we calculate the \( \beta \) function of this theory at the 1-loop level. Its sign is positive so that the theory has the Landau pole.\(^3\) The last Section is devoted, as usual, to conclusions and speculations.

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\(^1\)This multiplet is closely related to the famous *M*-theory 5-brane [5, 6].

\(^2\)The nonlinear effective action of (2,0) tensor multiplet as the world-volume multiplet of *M*-theory 5-brane, with nonlinearly realized \( \mathcal{N}=(2,0) \) superconformal symmetry, was constructed in [6].

\(^3\)This fact might be understood as follows. The “usual” non-gauge and Abelian gauge theories in four dimensions have the Landau pole, while non-Abelian theories are asymptotically free. On the other hand, a non-gauge theory of the real scalar field with cubic (not bounded from below) potential \( \propto \phi^3 \) in six dimensions is known to be asymptotically free [12].
2 6D harmonic superspace.

The basic facts about the spinor representations of $SO(5,1)$ group and the $\mathcal{N}=1,6D$ superspace can be found in Appendix. What we actually need to know is that the standard $\mathcal{N}=1$ superspace (to be more precise, $\mathcal{N}=(1,0)$ superspace) involves the following coordinates

$$z = (x^M, \theta_i^a), \quad (M = 0, ..., 5, \quad a = 1, ..., 4, \quad i = 1, 2),$$

(2.1)

where the Grassmann coordinates $\theta_i^a$ obeys the reality condition

$$\overline{\theta_i^a} \equiv -C_b^a (\theta_i^b)^* = \theta^{ai}.$$  

(2.2)

Here the bar operation is the covariant conjugation defined in (A.5). The fact that $\bar{\theta}$ can be expressed via $\theta$ is a distinguishing feature of $6D$ superspace compared to $4D$ superspace.

The basic spinor derivatives of the $6D$, $\mathcal{N}=1$ superspace are

$$D^k_a = \partial^k_a - i \theta^k_{b} \partial_{ab}, \quad \{D^k_a, D^l_b\} = -2i \varepsilon^{kl} \partial_{ab},$$

(2.3)

where

$$\partial_{ab} = \frac{1}{2} (\gamma^M)_{ab} \partial_M, \quad \partial_M x^N = \delta^N_M, \quad \partial^k_a \theta^b_l = \delta^k_i \delta^b_a, \quad x^M = \frac{1}{2} (\gamma^M)_{ab} x^{ab}. (2.4)$$

The off-shell superfield constraints of the $6D$, $\mathcal{N}=1$ gauge theory have the following form [13, 14]:

$$\{\nabla^k_a, \nabla^l_b\} + \{\nabla^l_a, \nabla^k_b\} = 0,$$

(2.5)

where $\nabla^i_a = D^i_a + A^i_a (z)$ is the spinor covariant derivative; $A^i_a$ is the spinor superfield connection.

These constraints have been solved [10,11] in the framework of the HSS approach. To make the discussion self-contained and to establish the notation, we describe it briefly here.

Let us first observe that the symmetry group of the superspace $(x^M, \theta_i^a)$ involves besides Poincaré and supersymmetry transformations also $R$-symmetry $SU(2)$ transformations. The conventional superspace is a coset of the super-Poincaré transformations. It is natural to consider also the coset of the $R$-symmetry $SU(2)/U(1) = \mathbb{C} P^1 \equiv S^2$. It is parametrized by the harmonics $u^\pm i$ $(u^-_i = (u^+)^*, \ u^\pm_i u^-_i = 1)$. The harmonic $6D$, $\mathcal{N}=1$ superspace is parametrized by the coordinates

$$(z, u) = (x^M, \theta_i^a, u^\pm i).$$

The harmonic superspace in the analytic basis involves the harmonics and the so called analytic coordinates $Z_A = (x^M_A, \theta^\pm a)$

$$x^M_A = x^M + \frac{i}{2} \theta^a_i \gamma^M_{ab} \theta^b_k u_{+k} u_{-l}, \quad \theta^\pm a = u_{+}^a \theta^a k, (2.6)$$

3
It is convenient to define the following differential operators called spinor and harmonic derivatives (in the analytic basis):

\[ D^+_a = \partial_a - 2i\theta^a\partial_{ab} \]
\[ D^-_a = -\partial_a - 2i\theta^a\partial_{ab} \]
\[ D^0 = u^+_1 \frac{\partial}{\partial u^+_1} - u^-_1 \frac{\partial}{\partial u^-_1} + \theta^+_a \partial_a - \theta^-_a \partial_a \]
\[ D^{++} = \partial^{++} + i\theta^+_a \theta^+_b \partial_{ab} + \theta^+_a \partial_a - \theta^-_a \partial_a \]
\[ D^{--} = \partial^{--} + i\theta^-_a \theta^-_b \partial_{ab} + \theta^-_a \partial_a \]

where \( \partial_{\pm a} \theta^\pm_b = \delta^b_a \) and

\[ \partial^{++} = u^+_i \frac{\partial}{\partial u^-_i}, \quad \partial^{--} = u^-_i \frac{\partial}{\partial u^+_i} . \]

The following commutation relations hold

\[ \{ D^+_a , D^-_b \} = 2i \partial_{ab}, \quad [ D^{++} , D^{--} ] = D^0 \]
\[ [ D^{++} , D^+_a ] = [ D^{--} , D^-_a ] = 0, \quad [ D^{++} , D^-_a ] = D^+_a, \quad [ D^{--} , D^+_a ] = D^-_a . \]

We shall use the notation

\[ (D^\pm)^4 = -\frac{1}{24} \varepsilon^{abcd} D^\pm_a D^\pm_b D^\pm_c D^\pm_d \]

and the following conventions for the full and analytic superspace integration measures:

\[ d^{14} Z_A = d^6 x_A (D^-)^4 (D^+)^4, \quad d\zeta^{-4} = d^6 x_A (D^-)^4 . \]

The following simple identity,

\[ \frac{1}{2} (D^+)^4 (D^{--})^2 (D^+)^4 = \Box (D^+)^4, \quad \Box \equiv \partial^M \partial_M = \frac{1}{2} \varepsilon^{abcd} \partial_{ab} \partial_{cd} , \]

will be helpful for us. 4

## 2.1 Harmonic superfields and their interactions

A general 6D superfield depends on 8 odd coordinates \( \theta^a_i \) (or \( \theta^{\pm a} \)), which makes their component expansion rather complicated. There is, however, an important class of superfields, Grassmann-analytic superfields, which depend only on Grassmann-analytic (G-analytic) superfields, which depend only on

\[ (\zeta, u) = (x^M_A, \theta^{+a}, u^{\pm i}) \]

i.e. involves only half of the original Grassmann coordinates. The set (2.12) forms the closed superspace on which 6D, \( \mathcal{N}=(1,0) \) supersymmetry (and the full \( \mathcal{N}=(1,0) \) superconformal symmetry) can be realized and which is called “harmonic analytic superspace”. The structure of Grassmann-analytic (G-analytic) superfields is much simpler than that of a general superfield. The possibility to formulate the theory in terms of G-analytic superfields represents a crucial advantage of the HSS formalism. A certain disadvantage

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4What we will actually need is the equivalence of the differential operators \((1/2)(D^+)^4(D^{--})^2\) and \( \Box \) when acting on a Grassmann-analytic (see below) superfield \( \phi(\zeta, u) \).
is that the superfields depend now not only on the superspace coordinates, but also on harmonics \( u_i \). The experience shows, however, that the simplifications brought about by analyticity are more important than the complications coming from explicit harmonic dependence. A G-analytic superfield \( \phi(\zeta, u) \) satisfies the constraint \( D^a_+ \phi = 0 \).\(^5\) In the analytic basis \( D^a_+ \) is reduced to the partial derivative \( \partial/\partial \theta^a \) and this constraint simply means that \( \phi \) lives on the superspace (2.12).

The superfields can be classified according to their harmonic charge \( q \), the eigenvalue of \( D^0 \). By the full analogy with what is known for \( \mathcal{N} = 2 \) 4D theories [8], the 6D SYM theory is formulated in terms of the G-analytic anti-Hermitian superfield gauge potential which has charge +2 and is denoted \( V^{++} \). It defines the covariant harmonic derivative

$$
\nabla^{++} = D^{++} + V^{++}.
$$

(2.13)

The superfield gauge transformation uses the analytic anti-Hermitian matrix parameter \( \Lambda \)

$$
\delta_\Lambda V^{++} = D^{++} \Lambda + [V^{++}, \Lambda].
$$

(2.14)

It is convenient to introduce also non-analytic gauge connection \( V^{--} \) which can be obtained out of \( V^{++} \) as a solution of the harmonic zero-curvature equation

$$
D^{++} V^{--} - D^{--} V^{++} + [V^{++}, V^{--}] = 0.
$$

(2.15)

The connection \( V^{--} \) can be constructed as a series over products of \( V^{++} \) taken at different harmonic “points”,

$$
V^{--}(z, u) = \sum_{n=1}^\infty (-1)^n \int \prod_{i=1}^n du_i \frac{V^{++}(z, u_1) \cdots V^{++}(z, u_n)}{(u^+ u_1^+)(u_2^+ \cdots u_n^+)(u_1^+ u_2^+)),
$$

(2.16)

where the factors \((u^+ u_1^+)^{-1} \) etc are the harmonic distributions [8] and the central basis coordinates \( z \) are defined in (2.1). The connection \( V^{--} \) transforms as

$$
\delta_\Lambda V^{--} = D^{--} \Lambda + [V^{--}, \Lambda]
$$

(2.17)

under gauge transformations. It can be used to build up spinor and vector superfield connections

$$
A^a_-(V) = -D^+_a V^{--}, \quad A_{ab}(V) = -\frac{i}{2} D^+_a D^+_b V^{--}.
$$

(2.18)

In addition, one can define the covariant (1,0) spinor superfield strength,

$$
W^{+a} = -\frac{1}{6} \epsilon^{abcd} D^+_b D^+_c D^+_d V^{--}.
$$

(2.19)

The superfield action of the standard 6D, \( \mathcal{N}=1 \) gauge theory was constructed in [10],

$$
S = \frac{1}{f^2} \sum_{n=1}^\infty \frac{(-1)^{n+1}}{n} \int d^8 x d^8 \theta d u_1 \cdots d u_n \text{Tr} \left\{ V^{++}(z, u_1) \cdots V^{++}(z, u_n) \right\}.
$$

(2.20)

\(^5\)It is quite analogous to the habitual chirality constraint \( D_\alpha \phi = 0 \) in four dimensions.
where $f$ is the coupling constant of canonical dimension $-1$. The corresponding component Lagrangian in the Wess-Zumino gauge [see eq.(3.11) below] gives the standard equations of motion of the 2-nd order for the gauge fields and of the 1-st order for the fermions.

Let us derive the superfield equation of motion for the action (2.20). To this end, one should first represent the action as an integral over the analytic superspace by acting with the operator $(D^+)^4$ on the integrand [cf. (2.10)]. Taking the variation of the result over $\delta V^{++}$ and comparing it with (2.16), we obtain the equation

$$
F^{++} = \frac{1}{4} D^+ W^{+a} = (D^+)^4 V^{--} = 0 .
$$

(2.21)

The superfield $F^{++}$ is Grassmann-analytic. It is transformed as

$$
\delta_{\Lambda} F^{++} = [F^{++}, \Lambda]
$$

(2.22)

under gauge transformations.

It is very easy to write down now the superfield action with dimensionless coupling constant. It has the following form

$$
S = \frac{1}{2 g^2} \int d\zeta^{-4} d\bar{u} \text{Tr} \left( F^{++} ight)^2 .
$$

(2.23)

Indeed, the superfields $V^{++}, V^{--}$ are dimensionless. It follows that $F^{++}$ defined in (2.21) has canonical dimension 2. Hence $g$ is dimensionless. The action (2.23) is gauge invariant as follows immediately from (2.22).

The action (2.23) can be rewritten as an integral over the full 6$D$ harmonic superspace, in a few equivalent forms. The corresponding Lagrangians are the Chern-Simons type densities,

$$
S = \frac{1}{2g^2} \int d^{14} Z_A d\bar{u} \text{Tr} \left( V^{--} F^{++} \right) = \frac{1}{8g^2} \int d^{14} Z_A d\bar{u} \text{Tr} \left( A^{-}_a W^{+a} \right)
$$

$$
= \frac{1}{12g^2} \int d^{14} Z_A d\bar{u} \varepsilon^{abcd} \text{Tr} \left[ A_{ab}(V) A_{cd}(V) \right] .
$$

(2.24)

Note also that one can use in this model the alternative formalism with an auxiliary tensor superfield $H^{++}$

$$
S(F^{++}, H^{++}) = -\frac{1}{g^2} \int d\zeta^{-4} d\bar{u} \text{Tr} \left[ F^{++}(V^{++}) H^{++} + \frac{1}{2} \text{Tr} (H^{++})^2 \right] \quad (2.25)
$$

which is completely equivalent to the higher-order formalism.

To derive the equations of motion one should use the following tensor relation between arbitrary variations of harmonic connections:

$$
\delta V^{--} = \frac{1}{2} (\nabla^{--})^2 \delta V^{++} - \frac{1}{2} \nabla^{++} (\nabla^{--} \delta V^{--}),
$$

$$
\nabla^{--} \delta V^{++} = D^{--} \delta V^{++} + [V^{--}, \delta V^{++}] ,
$$

(2.26)

or equivalently

$$
\nabla^{--} \delta V^{++} = \nabla^{++} \delta V^{--}, \quad [\nabla^{++}, \nabla^{--}] = D^0 .
$$

(2.27)
The formula for $\delta V^{--}$ can be obtained by applying $\nabla^{--}$ to both sides of the first relation in (2.27) and using the second relation. The variation of $S$ is

$$
\delta S = \frac{1}{g^2} \int d\zeta^{-4} d\zeta^{-4} Tr (\delta F^{++} F^{++}) = \frac{1}{g^2} \int d^4 Z d\zeta^{-4} d\zeta^{-4} Tr (\delta V^{--} F^{++})
$$

$$
= \frac{1}{2g^2} \int d^4 Z d\zeta^{-4} Tr \left[ (\nabla^{--} \nabla^{--} \delta V^{--}) F^{++} \right]
$$

$$
= \frac{1}{4g^2} \int d\zeta^{-4} d\zeta^{-4} Tr \left[ \delta V^{++}(D^+)^4(\nabla^{--})^2(D^+)^4V^{--} \right] = 0 ,
$$

which leads to the equation

$$
(D^+)^4(\nabla^{--})^2(D^+)^4V^{--} = 0 .
$$

Note that in the process of deriving this equation, the second term in the formula for the variation $\delta V^{--}$ in (2.26) was omitted since it does not contribute by virtue of the important relation

$$
\nabla^{++} F^{++} = 0 ,
$$

which follows from the definition of $F^{++}$ in (2.21) and the harmonic zero-curvature condition (2.15).

### 2.2 Superconformal invariance

The action (2.23) is scale invariant which suggests its conformal invariance. In this subsection we prove it within the superfield HSS formalism.

Transformations of the $6D$, $N=(1,0)$ superconformal group $OSp(8^*|2)$ in the central basis have the form

$$
\delta x^{ab} = e^{ab} + \omega^a x^{ab} + \omega^b x^{ac} + a x^{ab} - \frac{i}{2} \epsilon^{ab}_{\theta \theta} - \frac{i}{2} \epsilon^{ab}_{\theta \pi} + \frac{1}{2} \eta^a \theta^b - \frac{1}{2} \eta^b \theta^a
$$

$$
\delta \theta^a_k = e^a + \omega^a_{\theta \theta} + \frac{1}{2} \theta^a_k \theta^b_k - L^a_k \theta^a_k + x^{ac} k_{cd} \theta^d_k + \frac{i}{2} \theta^a_k \theta^b_k k_{cd} - x^{ac} h_{ck} + i \eta^a_\pi \theta_k^c - \frac{i}{2} \eta^a_\pi \theta_k^c .
$$

The meaning of the group parameters is clear from their index structure and dimensions. In particular, $a$ is the dilatation parameter, $k_{ab}$ are the parameters of special conformal transformations, $\epsilon^{ab}_{\theta \theta}$ and $\eta^a_\pi$ are the parameters of $6D$ Poincaré and special conformal supersymmetries, $L^a_k$ are the parameters of $SU(2)$ rotations. The closeness of the transformations (2.31) can be directly checked.

The conformal transformation of the harmonics can be defined by the analogy with [8]

$$
\delta u^+_k = \Lambda^{++} u^-_k , \quad \delta u^-_k = 0 ,
$$

$$
\Lambda^{++} = i k_{ab} \theta^a \theta^b + 2 i \eta^+ \theta^+ + L^k u^+_k ,
$$

$$
D^- \Lambda^{++} = 2 i k_{ab} \theta^- \theta^b + 2 i \eta^- \theta^- + 2 i \eta^+ \theta^- + 2 L^k u^-_k u^+_k ,
$$

(2.32)
where \( \eta^+_a = \eta^-_a u^+_a \). They have the same closure as (2.31). The superconformal transformations of the harmonic derivatives are given by

\[
\delta D^{++} = -\Lambda^{++} D^0, \quad \delta D^{--} = -(D^{--} \Lambda^{++}) D^{--}, \quad (2.33)
\]

like in the 4D case [8].

Having the above transformations at hand, it is not difficult to find how the \( \mathcal{N}=(1,0) \) superconformal group is realized in the analytic basis

\[
\begin{align*}
\delta \theta^{a+} &= \epsilon^{a+} + \frac{1}{2} a \theta^{a+} + \omega^a_{\theta} \theta^{b+} + L^+ \theta^{a+} + x_A^{ac} k_{cd} \theta^{+d} - x^{ab}_A \eta^b_{-} + i \eta^a_- \theta^b \theta^{a+}, \\
\delta x^{ab}_A &= c^{ab} + \omega^a_{\theta} x^{ab}_A + \omega^b_{\theta} x^{ac}_A + ax^{ab}_A + i(\epsilon^a \theta^{b+} - \epsilon^b \theta^{a+}) - k_{cd} x^{ac}_A x^{bd}_A \\
\delta \theta^{a-} &= \epsilon^{a-} + \frac{1}{2} a \theta^{-a} + \omega^a_{\theta} \theta^{-b} + L^- \theta^{a-} - (x^{ac}_A - i \theta^{-a} \theta^{c}) \theta^{-d} k_{cd} \\
&= -x^{ac}_A \eta^c_+ + i \eta^a_+ \theta^{-c} \theta^{a+} - i \eta^a_- \theta^{-c} \theta^{a+} - i \eta^a_- \theta^{-c} \theta^{a+}.
\end{align*}
\]

Here \( L^{--} = L^{kl} u^+_k u^-_l \), etc. Using these transformation rules, it is easy to establish the transformation of the analytic superspace integration measure \( d\zeta^{-4} du = d^8 x_A d^4 \theta^{+} du \):

\[
\begin{align*}
\delta (d\zeta^{-4} du) &= (\partial_{ab} \delta x^{ab} + \partial^{a+} \Lambda^{++} - \partial_{a+} \delta \theta^{a+}) (d\zeta^{-4} du) \equiv 4 \Lambda d\zeta^{-4} du, \\
4 \Lambda &= 4a - 2x^{ab} k_{ab} - 4i \eta^a_+ \theta^{a+} - 2L^{+-}.
\end{align*}
\]

where we used

\[
\begin{align*}
\partial_{ab} \delta x^{ab} &= 6a - 3x^{ab} k_{ab} - 3i \eta^-_a \theta^{a+}, \quad \partial_{a+} \delta \theta^{a+} = 2a + 4L^{+-} - x^{ab} k_{ab} + 3i \eta^-_a \theta^{a+}, \\
\partial^{--} \Lambda^{++} &= 2i \eta^+_a \theta^{a+} + 2L^{++}, \quad D^{++} \Lambda = -\frac{1}{2} \Lambda^{++}. \quad (2.36)
\end{align*}
\]

Under the superconformal transformations given above the gauge potentials \( V^{\pm \pm} \) transform as

\[
\delta V^{++} = 0, \quad \delta V^{--} = -(D^{--} \Lambda^{++}) V^{--}, \quad (2.37)
\]

which mimics the transformation rules of the harmonic derivatives. The defining harmonic zero-curvature equation (2.15) is manifestly covariant with taking into account the relation

\[
D^{++} \Lambda^{++} = 0. \quad (2.38)
\]

Let us now verify the superconformal invariance of the action (2.23). The invariance of (2.23) under dilatations with the parameter \( a \) is evident. The invariance under the \( SU(2) \) transformations (with parameters \( L^{(ik)} \)) can be checked using the transformation properties in the analytic basis

\[
\begin{align*}
\delta V^{--} &= -2L^{+-} V^{--}, \quad \delta (D^+)^4 = 4L^{+-} (D^+)^4, \quad \delta F^{++} = 2L^{+-} F^{++}, \\
\delta (d\zeta^{-4} du) &= -2L^{+-} (d\zeta^{-4} du).
\end{align*}
\]

It is straightforward to obtain

\[
\delta_L S \sim \int d\zeta^{-4} du 2L^{+-} \text{Tr} \left( F^{++} \right)^2. \quad (2.40)
\]
Next we represent \( 2L^{+-} = D^{++}L^{-} \) and integrate by parts to rewrite \( \delta_L S \) as

\[
\delta_L S \sim -2 \int d\zeta^{-4} du \, L^{-} \text{Tr} \left( \nabla^{++} F^{++} F^{++} \right). \tag{2.41}
\]

This expression is vanishing as a consequence of the relation (2.30).

Let us now prove the invariance under the special conformal supersymmetry with the parameters \( \eta_i^a \). Since all other superconformal transformations are contained in the closure of this supersymmetry with itself and with the Poincaré supersymmetry and the action is manifestly invariant under the latter, the invariance under the conformal supersymmetry actually amounts to the invariance under the full superconformal group.

In the analytic basis, the covariant derivative \( D_+^a = \partial_{-a} \) transforms as

\[
\delta \eta D_+^a = -\frac{\partial \delta \theta^{-b}}{\partial \theta^{-a}} D_+^b = i(\eta^+ \cdot \theta^- + \eta^- \cdot \theta^+) D_+^a + i\eta^-_a \left( \theta^+ \cdot D^a \right) - i\eta^+_a \left( \theta^- \cdot D^a \right). \tag{2.42}
\]

Then it is straightforward to find

\[
\delta \eta (D^+)^4 = i(3\eta^+ \cdot \theta^- + 5\eta^- \cdot \theta^+) (D^+)^4 + \frac{i}{2} \epsilon^{abcd} \eta^+_a D^+_b D^+_c D^+_d. \tag{2.43}
\]

Taking into account that for the considered case

\[
D^{-} \Lambda^{++} = 2i(\eta^- \cdot \theta^+ + \eta^+ \cdot \theta^-) \tag{2.44}
\]

and using the transformation law (2.37) of \( V^{--} \), it is also straightforward to compute that

\[
\delta \eta F^{++} = i(\eta^+ \cdot \theta^- + 3\eta^- \cdot \theta^+) F^{++} + \frac{i}{6} \epsilon^{abcd} \eta^+_a D^+_b D^+_c D^+_d V^{--}. \tag{2.45}
\]

Despite the presence of two terms which, being taken separately, break analyticity, it is easy to check that this variation is still implicitly analytic: acting on it by \( D_+^a \) yields zero. Actually, it can be given the following manifestly analytic form

\[
\delta \eta F^{++} = 3i(\eta^+ \cdot \theta^-) F^{++} + i(D^+)^4 \left[ (\eta^+ \cdot \theta^-) V^{--} \right]. \tag{2.46}
\]

The analytic superspace integration measure is transformed as

\[
\delta \eta (d\zeta^{-4} du) = -4i(\eta^- \cdot \theta^+) (d\zeta^{-4} du), \tag{2.47}
\]

then the variation of the action (2.23), up to the overall renormalization factor, is as follows

\[
\delta \eta S \sim \int d\zeta^{-4} du (D^+)^4 \left[ 2i(\eta^+ \cdot \theta^- + \eta^- \cdot \theta^+) \text{Tr} \left( V^{--} F^{++} \right) \right] = 2i \int d^{14}Z du \left( \eta^+ \cdot \theta^- + \eta^- \cdot \theta^+ \right) \text{Tr} \left( V^{--} F^{++} \right). \tag{2.48}
\]
Then we represent

\[ \eta^+ \cdot \theta^- + \eta^- \cdot \theta^+ = D^{++}(\eta^- \cdot \theta^-), \]  

(2.49)

integrate by parts with respect to \( D^{++} \) and use the relations (2.30) and (2.15) in the form

\[ \nabla^{++} V^{--} = D^{--} V^{++}. \]

After these manipulations the variation acquires the form

\[ \delta \eta S \sim -2i \int d^{14} Z d\mu (\eta^- \cdot \theta^-) \text{Tr} (D^{--} V^{++} F^{++}). \]

(2.50)

Now one should again take off \((D^+)^4\) from the measure and apply it to the integrand. Clearly, when all four derivatives hit the expression under the trace, the result is zero. The only extra terms appear when one of the spinor derivatives hits \(\theta^-\) (and yields a Kronecker symbol) while three remaining ones hit \(D^{--} V^{++}\) under the trace. It is clear that the result is also vanishing. Thus we obtain the desired result

\[ \delta \eta S = 0. \]

(2.51)

Below we shall independently check that the component action is conformally invariant, which, together with the invariance under the Poincaré supersymmetry, also implies full superconformal invariance.

3 Component action

We now derive the component form of the action (2.23).

We start from the following component expansion of \(V^{++}\) written in the Wess-Zumino gauge

\[ V_{(WZ)}^{++} = \theta^+ a \theta^+ b A_{ab} + 2\sqrt{2}(\theta^+_a)^3 \psi^{-a} - 3(\theta^+_a)^4 D^{--}, \]

(3.1)

where

\[ (\theta^+_a)^3 = \frac{1}{6} \epsilon_{abcd} \theta^+ a \theta^+ b \theta^+ c \theta^+ d, \quad (\theta^+_a)^4 = -\frac{1}{24} \epsilon_{abcd} \theta^+ a \theta^+ b \theta^+ c \theta^+ d, \]

\[ \psi^{-a} = \psi^{ai} u_i^-, \quad D^{--} = D^{jk} u_i^- u_k^-. \]

(3.2)

The expansion (3.1) involves only the physical fields: gauge fields \(A_M\) (remind that \(A_{ab} = \frac{1}{2} A_M (\gamma_M)_{ab}\), gluino fields \(\psi^{ai}\) and a \(SU(2)\) triplet of the scalar fields \(D^{ik} = D^{ki}\). The particular numerical coefficients in (3.1) were introduced for further convenience.

If reducing the theory to 4 dimensions, we arrive at the \(\mathcal{N}=2\) vector multiplet, which can also be represented as the combination of the \(\mathcal{N}=1\) vector multiplet and the adjoint chiral multiplet. The 6D gauge field gives the 4D gauge field and a complex scalar, a 6D gluino field is split in two 4D gluinos and the field \(D^{ik}\) is decomposed as

\[ D^{ik} = \begin{pmatrix} \bar{F} & -D \\ -D & -F \end{pmatrix}, \]

(3.3)
where real $D$ is the auxiliary field of the $\mathcal{N}=1$ vector multiplet and complex $F$ is the auxiliary field of the adjoint chiral multiplet. $D$ and $F$ are auxiliary (i.e. non-dynamical) fields just in the theory based on the standard action (2.20) (and its 4-dimensional counterpart) because they enter it without derivatives. We will see that in the action (2.23) they become dynamical.

To find the component action, one must solve (2.15) with $V^{++}$ for $V^{--}$, act on that by $(D^+)^4$ to find $F^{++}$, substitute the latter into (2.23), and integrate the result over Grassmann and harmonic variables. The calculations are tedious (mainly because $V^{--}$ needed at the intermediate steps is not G-analytic), but feasible.

For solving (2.15) we decompose $V^{--}$ with respect to $\theta^{-a}$ with coefficients representing G-analytic superfields:

$$V^{--} = v^{--} + \theta^{-b}v^{-}_b + \theta^{-c}\theta^{-d}v_{cd} + (\theta^{-})^3_d v^{++} + (\theta^{-})^4 v^{++}$$  \hspace{1cm} (3.4)

and rewrite (2.15) as a set of rather cumbersome harmonic equations for the coefficients. This set is as follows

$$D^{++}v^{--} + \theta^{+a}v^{-}_a + \theta^{+a}\theta^{+b}[A_{ab}, v^{--}] + 2\sqrt{2}(\theta^+)^3_d [\psi^{-d}, v^{-}_b]$$
$$= \frac{3}{4} [D^{--}, v^{--}] = 0;$$
$$D^{++}v^{-}_a + 2\theta^{+a}v^{++} + \theta^{+a}\theta^{+c}[A_{ac}, v^{++}] + 2\sqrt{2}(\theta^+)^3_b [\psi^{-b}, v^{++}]$$
$$= 3\theta^+ [D^{--}, v^{++}] = 0;$$
$$D^{++}v^{++} + \theta^{+a}\theta^{+b}[A_{ab}, v^{++}] + 2\sqrt{2}(\theta^+)^3_a [\psi^{-a}, v^{++}]$$
$$= 3\theta^+ [D^{--}, v^{++}] = 0 (3.5)$$

with $\nabla_{ab} = \partial_{ab} - i[v_{ab}, \cdot]$. One solves these equations by decomposing the G-analytic coefficients with respect to $\theta^+$ and solving the resulting harmonic equations for the component fields. This procedure is rather boring, but straightforward. Actually, for constructing the action we need only the highest component $v^{++}$ in the expansion (3.4)

$$v^{++} = (D^+)^4 V^{--} = F^{++} = \lambda^{++} + \theta^{-a}\lambda^+_{a} + \theta^{+a}\theta^{+b}\lambda_{ab} + (\theta^+)^3\lambda^{-a} + (\theta^+)^4\lambda^{-}.$$  \hspace{1cm} (3.6)

For the component fields in (3.6) we obtain the following expressions

$$\lambda^{++} = -D^{++}, \quad \lambda^+_{a} = i\sqrt{2}(\gamma^M)^{ab}_{b} \nabla_M \psi^{++},$$
$$\lambda_{ab} = \frac{1}{2}(\gamma^M)^{ab}_{b} \left[ i\nabla_M D^{+-} + \nabla^N F_{NM} \right] + \epsilon_{abcd} \{ \psi^{-c}, \psi^{+d} \},$$
$$\lambda^{-a} = \sqrt{2}\nabla^a \psi^{+a} - \sqrt{2}F_{MN}(\sigma^M)^a_{b} \psi^{+b} u^{-}_i - \frac{4\sqrt{2}}{3} [\psi^{+a}, D^i] u^{-}_i$$
$$+ \sqrt{2} [\psi^{+a}, D^{kl}] u^{-}_i u^{+}_k u^{+}_l,$$
$$\lambda^{-} = -\nabla^2 D^{--} + 3[D^{--}, D^{+-}] - 2i \{ \psi^{-}, \gamma^M \nabla_M \psi^{-} \}.$$  \hspace{1cm} (3.7)
Here

\[ \nabla_M = \frac{1}{2}(\tilde{\gamma})^{ab}\nabla_{ab} = \partial_M - i[A_M, \cdot], \quad \nabla^2 = \nabla^M \nabla_M, \]

\[ F_{MN} = \partial_M A_N - \partial_N A_M - i[A_M, A_N] \]  

(3.8)

and \( \sigma^{MN} \) is defined in (A.10) (see Appendix).

As a warm-up, let us first reproduce the known component expression for the standard 6D SYM action (2.20). From (2.20), (2.16) and (2.10), the quadratic in fields part of the action can be represented as

\[ S_{\text{quadr}} = -\frac{1}{2} g^2 \int d^6 x \int d\zeta d^4 \theta^+ \text{Tr} \left\{ V^{+\dagger} F^{++} \right\} . \]  

(3.9)

Multiplying (3.1) by (3.6) with the linearized components (3.7), performing the Grassmann and harmonic integrations using the identities

\[ \int du u_i^+ u_j^- = \frac{1}{2} \epsilon_{ij}, \quad \int du u_i^+ u_k^+ u_j^- u_l^- = \frac{1}{6} (\epsilon_{ij} \epsilon_{kl} + \epsilon_{il} \epsilon_{kj}) \]  

(3.10)

and restoring the nonlinear terms by gauge invariance, we obtain

\[ S = \frac{1}{f^2} \int d^6 x \int d^4 \theta^+ \text{Tr} \left\{ -\frac{1}{2} F_{MN}^2 - \frac{1}{2} D^{ik} D_{ik} + i \psi^k \gamma^M \nabla_M \psi_k \right\} . \]

(3.11)

The component form of the higher derivative action (2.23) is also derived rather straightforwardly. After integrating over \( \theta^+ \), the action (2.23) is expressed in terms of the components of \( F^{++} \), eq. (3.6), as follows

\[ S = \frac{1}{2g^2} \int d^6 x d^4 \lambda \left\{ 2\lambda^+ \lambda^- - 2\lambda^+ \lambda^- - \epsilon^{abcd} \lambda_{ab} \lambda_{cd} \right\} . \]  

(3.12)

After substituting the expressions (3.7) and performing the integration over harmonics in (3.12) we obtain the sought component action:

\[ S = \frac{1}{g^2} \int d^6 x \text{Tr} \left\{ (D_M F_{ML})^2 + i \psi^j \gamma^M \nabla_M (\nabla)^2 \psi_j + \frac{1}{2} (\nabla_M D_{jk})^2 \right\} 
+ D_{ik} D^{kj} D^t + 2i D_{jk} (\psi^j \gamma^M \nabla_M \psi_k - \nabla_M \psi^j \gamma^M \psi_k) + (\psi^j \gamma^M \psi_j)^2 \n+ \frac{1}{2} \nabla_M \psi^j \gamma^M \sigma^{NS} [F_{NS}, \psi_j] - 2 \nabla^M F_{MN} \psi^j \gamma^N \psi_j \right\} . \]  

(3.13)

Let us discuss this result. Note first of all that the quadratic terms in the lagrangian are obtained from (3.11) by adding the extra box operator (it enters with negative sign, this makes the kinetic time-derivative terms positive definite in Minkowski space). It is immediately seen for the terms \( \propto D^2 \) and for the fermions. This is true also for the gauge part due to the identity

\[ \text{Tr} \left\{ (\nabla_M F_{MN})^2 \right\} = -\frac{1}{2} \text{Tr} \left\{ F_{MN} \nabla^2 F_{MN} \right\} - 2i \text{Tr} \left\{ F_{MN} F_{NS} F_{SM} \right\} . \]  

(3.14)
The appearance of the structure (3.14) was anticipated in [1] by lifting a 4D higher-derivative supersymmetric gauge lagrangian to six dimensions. Actually, only this higher-derivative gauge fields kinetic term is conformally invariant. 

As promised, the fields $D^{ik}$ become dynamical. They carry canonical dimension 2 and their kinetic term involves two derivatives. There is a cubic term $\propto D^3$. This sector of the theory reminds the renormalizable theory $(\phi^3)_6$. Gauge and fermion fields have the habitual canonical dimensions $[A_M] = 1$, $[\psi] = 3/2$. Their kinetic terms involve, correspondingly, 4 and 3 derivatives. The lagrangian involves also other interaction terms, all of them having the canonical dimension 6.

It is instructive to evaluate the number of on–shell degrees of freedom in this lagrangian. Consider first the gauge field. With the standard lagrangian $\propto \text{Tr}\{F_{MN}^2\}$, a six–dimensional gauge field $A_M$ has 4 on–shell d.o.f. for each color index. The simplest way to see this is to note that $A_0$ is not dynamical and we have to impose the Gauss law constraint on the remaining 5 spatial variables. For the higher-derivative theory, however, the presence of two extra derivatives doubles the number of d.o.f. and the correct counting is $2 \times 5 = 10$ before imposing the Gauss law constraint and $10 - 1 = 9$ after that. In addition, there are 3 d.o.f. of the fields $D_{ij}$ and we have all together 12 bosonic d.o.f. for each color index. The standard 6D Weyl fermion (with the lagrangian involving only one derivative) has 4 on–shell degrees of freedom. In our case, we have $4 \times 3 = 12$ fermionic d.o.f. due to the presence of three derivatives in the kinetic term. For sure, the numbers of bosonic and fermionic degrees of freedom on mass shell coincide.

Our theory is conformally invariant at the classical level. This was proven in the previous Section using superfield formalism. But conformal invariance of the component lagrangian (3.13) can be shown directly. As distinct from the consideration in the previous Section, we will use here the formalism where the coordinates do not change after the transformation and only the fields do (the “active” form of the transformations). A special conformal variation of a primary operator $\Phi$ is

$$\delta_C^* \Phi = \left[(2x^M \eta x^N - \eta x^2)\partial_N \Phi + 2x_N(\eta d + S^{MN})\Phi\right] \epsilon_M,$$  

(3.15)

where $d$ is the canonical dimension of $\Phi$ and $S^{MN}$ is the spin operator. The transformation law (3.15) has the same form in space of any dimension. It is easy to see that for a primary operator $O$ of canonical dimension $D$, $\int d^D x \delta_C O = 0$. Thus, conformal invariance of the action would be proven if the lagrangian density is transformed as in (3.15) with respect to conformal transformations (with $d = 6$). This is a nontrivial requirement and not any operator of canonical dimension $D$ satisfies it (for a good review see e.g. the lecture of R. Jackiw [16]). An example of a 4D action which is scale invariant, but not

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6 The four-derivative conformally invariant kinetic term of gauge fields was also considered in [15] where it appeared as an effective Lagrangian in a 6D non-conformal theory with standard two derivatives in the action. Its interpretation in our case is entirely different: it is present at the level of the microscopic Lagrangian and defines the free propagators of gauge fields.

7 In fact, the same check of the conformal invariance can be performed using the “passive” variations $\delta_C \Phi = 2x_N(\eta d + S^{MN})\Phi$, $\delta_C \nabla M = 2[(\epsilon \cdot \epsilon)\nabla M - x_M(\epsilon \cdot \nabla) + \epsilon_M(\epsilon \cdot \nabla)]$, $\delta_C (d^6 x) = -12(\epsilon \cdot \epsilon) d^6 x$, where “$\cdot$” denotes the scalar product.

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conformally invariant is
\[ O = \frac{[(\partial_\mu \phi)^2]^2}{\phi^4}. \]

The point is that the derivatives of primary operators do not enjoy the same transformation laws (3.15) as the primary operators. For example, if \( \Phi \) is a scalar of dimension \( d \), the variation of \( \partial_M \Phi \) under a special conformal transformation is
\[ \delta^*_C (\partial_M \Phi) = \delta^*_C (\partial_M \Phi) + 2d \Phi \epsilon_M. \]  
(3.16)

One can check, however, that most of the terms in (3.13) are transformed, up to a total derivative, according to the law (3.15). On the other hand, this is not true for the kinetic fermion terms and two last terms. Denoting the corresponding terms in the braces (with the attached signs and coefficients) in the consecutive order as \( (I) \), \( (II) \) and \( (III) \), we find that their variations contain, besides the standard pieces (3.15) and total derivative terms, also some extra pieces
\[ \hat{\delta}(I) = -4 \text{Tr} \left\{ F^{MN} \psi^i \gamma_N \psi_i \right\} \epsilon_M, \]
\[ \hat{\delta}(II) = -4 \text{Tr} \left\{ F^{MN} \psi^i \gamma_N \psi_i \right\} \epsilon_M, \]
\[ \hat{\delta}(III) = 8 \text{Tr} \left\{ F^{MN} \dot{\psi}^i \gamma_N \psi_i \right\} \epsilon_M. \]  
(3.17)

In deriving (3.17), we used the identity
\[ \gamma_M \sigma_{NS} + \sigma_{NS} \gamma_M = 2(\eta_{MS} \gamma_N - \eta_{MN} \gamma_S). \]  
(3.18)

We see that
\[ \hat{\delta} [(I) + (II) + (III)] = 0, \]  
(3.19)

which proves the conformal invariance of (3.13). As the action is supersymmetric by construction, this proves in a simple way the full superconformal invariance of the action (2.23), (3.13).

4 Charge renormalization

We proceed now to calculating (at the one–loop level) the \( \beta \) function of our theory. We will see that it does not vanish, which means that conformal invariance of the classical action is broken by quantum effects.

The simplest way to do this calculation is to evaluate 1–loop corrections to the structures \( \sim (\partial_M D)^2 \) and \( \sim D^3 \). The relevant Feynman graphs are depicted in Figs. 1, 2.

\( \text{8} \)It is interesting to note that there exists another independent conformal combination of these terms, \( \hat{\delta} [(III) + 2(II)] = 0. \)
For perturbative calculations, we absorb the factor $1/g$ in the definition of the fields. The relevant propagators are

\[
\begin{align*}
\langle A_M^A A_N^B \rangle &= -\frac{i\eta_{MN}\delta^{AB}}{p^4}, \\
\langle \psi^j A \psi^kB \rangle &= -\frac{i\epsilon^{jk}\delta^{AB}p_N\bar{\gamma}^N}{p^4}, \\
\langle D_{ik}^A D_{jl}^B \rangle &= -\frac{i\delta^{AB}p^2}{p^4}(\epsilon_{ij}\epsilon_{kl} + \epsilon_{il}\epsilon_{kj}),
\end{align*}
\]

where $A, B$ are color indices, $A_M = A_M^A t^A$, $\text{Tr}(t^A t^B) = \delta^{AB}/2$, etc. The vertices can be read out directly from the lagrangian.

Figure 1: Graphs contributing to the renormalization of the kinetic term. Thin solid lines stand for the particle $\mathcal{D}$, thick solid lines for fermions, and dashed lines for gauge bosons.

\[
\begin{align*}
\text{a)} & \quad \text{b)} \quad \text{c)}
\end{align*}
\]

Figure 2: The same for the $\mathcal{D}^3$ vertex.

Consider first the graphs in Fig. 1. They involve logarithmic and quadratic divergences. The latter do not appear if one uses dimensional regularization or any other regularization scheme that respects gauge invariance and supersymmetry. Indeed, the quadratically divergent pieces in Fig. 1 are associated with the contribution $\sim \text{Tr}\{D^2\}$ in the effective action. But this term is absent in the tree action (3.13) and cannot appear in the effective one.

The logarithmic divergences in the 2-point graphs are

\[
\Delta \mathcal{L}_{(2)}^{\text{eff}} = g^2c_V \left(-\frac{3}{2} - \frac{7}{6} + 4\right) \text{Tr}\{(\partial_M D_{jk})^2\} L = \frac{4g^2c_V}{3} \text{Tr}\{(\partial_M D_{jk})^2\} L,
\]

(4.2)
where\(^9\)

\[
L = \int_\mu^\Lambda \frac{d^6p_E}{(2\pi)^6p_E^6} = \frac{1}{64\pi^3} \ln \frac{\Lambda}{\mu}
\]

(4.3)

and three terms in the parentheses correspond to the contributions of the graphs in Fig. 1a,b,c.

The 3-point graphs in Fig. 2 involve only logarithmic divergence. We obtain

\[
\Delta L^{\text{eff}}_{(3)} = g^3 c_v \left( -\frac{9}{2} + \frac{3}{2} + \frac{32}{3} \right) \text{Tr} \{D_{ik} D^{kj} D^l_j\} L = \frac{23g^3 c_v}{3} \text{Tr} \{D_{ik} D^{kj} D^l_j\} L.
\]

(4.4)

The full 1-loop effective lagrangian in the \(D\) sector is

\[
L^{\text{eff}}_D = -\frac{1}{2} \text{Tr} \left\{ (\partial M D_{jk})^2 \right\} \left( 1 - \frac{8g^2 c_v}{3} L \right) - g \text{Tr} \{D_{ik} D^{kj} D^l_j\} \left( 1 - \frac{23g^2 c_v}{3} L \right).
\]

(4.5)

Absorbing the renormalization factor of the kinetic term in the field redefinition, we finally obtain\(^10\)

\[
g(\mu) = g_0 \left( 1 - \frac{11g_0^2 c_v}{3} L \right) = g_0 \left( 1 - \frac{11g_0^2 c_v}{192\pi^3} \ln \frac{\Lambda}{\mu} \right)
\]

(4.6)

for the effective charge renormalization.

The sign corresponds to the Landau zero situation, as in the ordinary 4-dimensional QED. It is amusing to observe that, if taking into account only the graphs in Fig. 1a and Fig. 2a, the coefficient would be zero. In other words, the purely bosonic 6\(D\) theory

\[
L = -\frac{1}{2} \text{Tr} \{ (\partial M D_{jk})^2 \} - g \text{Tr} \{D_{ik} D^{kj} D^l_j\}
\]

(4.7)

does not involve logarithmic divergences at the one–loop level.

5 Conclusions

In this paper we presented the first example of renormalizable higher-dimensional supersymmetric gauge theory. It is 6\(D\), \(N=(1,0)\) gauge theory with four derivatives in the action and dimensionless coupling constant. Though it is superconformally invariant at the classical level, the superconformal symmetry turns out to be broken in the quantum

---

\(^9\)We use here the simple momentum cutoff as an ultraviolet regularization. This procedure does not respect the symmetries of the theory and would not assure, e.g., the cancellation of the quadratic divergences, but, as long as the logarithmic one-loop divergences are concerned, it gives the same result as more sophisticated methods, and its use is legitimate. The situation is exactly the same as in the ordinary QED where the momentum cutoff gives a quadratically divergent photon mass, but assures a correct result for the one-loop logarithmic divergences in the effective charge.

\(^{10}\)This result agrees with a recent calculation of L. Casarin and A. Tseytlin performed by a different method [17].
case by conformal anomaly. As the result of this breaking, in accord with the arguments of [1], the quantum theory suffers from ghosts which cannot be entirely harmless. This raises the problem of searching for some extended theory, such that the (super)conformal symmetry is retained in it at the full quantum level. So this hypothetical theory would reveal the nice property of ultraviolet finiteness and could probably be considered as a candidate for the fundamental field theory. Exact conformal invariance may render the ghosts harmless [1].

What would the extended theory look like? We see only two options here:

- It may enjoy the maximal superconformal symmetry $\mathcal{N}=(2,0)$ in six dimensions. However, in this case it should depend on tensor rather than vector multiplets [5,7]. Unfortunately, to describe the tensor multiplet in the framework of HSS is not a trivial task and it is not solved yet. As a result, no microscopic Lagrangian for interacting $(2,0)$ tensor multiplet is known today...

- Another possibility to try is to add to the higher-derivative $6D, \mathcal{N}=\mathcal{N}=(1,0)$ supersymmetric gauge theory action (2.23) an off-shell action of $6D, \mathcal{N}=(1,0)$ hypermultiplet in some representation of the gauge group, with the same number of derivatives on fields. A similar extension of the non-conformal standard action of $\mathcal{N}=(1,0)$ gauge multiplet (having two derivatives) with the hypermultiplet in the adjoint representation is known to give rise to non-conformal $\mathcal{N}=(1,1)$ gauge theory [13]. In the HSS approach, the hypermultiplet is described by an analytic superfield $q^+ (\zeta, u)$ which is unconstrained and so contains off shell infinite towers of auxiliary fields with growing isospins (they come from the harmonic expansions). The higher-derivative action of $q^+$ could hopefully be constructed in such a way that it respects $\mathcal{N}=(1,0)$ superconformal symmetry (which is surely broken in the standard minimal $6D$ hypermultiplet action). However, the total higher-derivative $\mathcal{N}=(1,0)$ superconformal action of the gauge multiplet and hypermultiplet cannot be expected to possess neither $\mathcal{N}=(1,1)$ super Poincaré nor any extended superconformal supersymmetry. Indeed, $\mathcal{N}=(1,0)$ gauge multiplet and hypermultiplet can be combined only into a $\mathcal{N}=(1,1)$ gauge multiplet, while no superconformal extension is known for non-chiral $6D, \mathcal{N}=(1,1)$ Poincaré supersymmetry [4]. Nevertheless, such a coupled system could hopefully preserve its $\mathcal{N}=(1,0)$ superconformal symmetry at the full quantum level, thus being ultraviolet finite. This higher derivative $6D$ theory should reveal some rather unusual features, since the infinite towers of the “former” auxiliary fields collected in the harmonic expansion of $q^+$ should acquire kinetic terms and become propagating, like the “former” auxiliary field $D^{ik}$ of the vector multiplet.

A less ambitious but important task is to repeat the quantum calculations of Sect. 4 with making use of the manifestly supersymmetric techniques of the harmonic supergraphs [8]. This would allow one to obtain the charge renormalization for the whole superfield action (2.23), i.e. at once for all fields of the gauge multiplet. The supergraph techniques would be especially useful for exploring the quantum properties of the hypothetical higher-derivative gauge fields - hypermultiplet system.
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Appendix: 6D, N=1 superspace

The group $Spin(5,1)$ has two different irreducible spinor representations of complex dimension 4. An essential difference with the familiar $Spin(3,1)$ case (that also involves two different spinor representations) is that a complex conjugated $Spin(5,1)$ spinor belongs to the same representation of the group as the original one. To see it explicitly, one can express a $(1,0)$ Weyl spinor $\Psi^a, (a = 1, 2, 3, 4)$ in terms of two $SL(2, C)$ spinors

$$\Psi^a = \begin{pmatrix} \psi^\alpha \\ \bar{K}^\dot{\alpha} \end{pmatrix}.$$  \hfill (A.1)

The complex conjugation gives us

$$(\Psi^a)^* \equiv \bar{\Psi}^\dot{a} = \begin{pmatrix} \bar{\psi}^\dot{\alpha} \\ \bar{\kappa}^{\alpha} \end{pmatrix}.$$  \hfill (A.2)

This complex-conjugated representation of $Spin(5,1)$ is equivalent to some covariant $(1,0)$ spinor

$$\bar{\Psi}^\dot{a} = C^\dot{a}_a \bar{\Psi}^a, \quad \bar{\Psi}^a = -C^a_\dot{a} \bar{\Psi}^\dot{a},$$  \hfill (A.3)

where the charge-conjugation matrix is introduced and

$$C^a_\dot{a} C^\dot{a}_b = -\delta^a_b.$$  \hfill (A.4)

We shall use the covariant conjugation

$$\Psi^a \Rightarrow \bar{\Psi}^a = \begin{pmatrix} -K^\alpha \\ \bar{\psi}^{\dot{\alpha}} \end{pmatrix}$$  \hfill (A.5)

which has the unusual property $\bar{\Psi}^a = -\Psi^a$.

We choose the antisymmetric representation of the 6D Weyl matrices

$$(\gamma^M)_{ab} = -(\gamma^M)_{ba}, \quad \tilde{\gamma}^{ab} = \frac{1}{2} \varepsilon^{abcd} (\gamma_M)_{cd},$$  \hfill (A.6)

where $M = 0, 1, \ldots, 5$ and $\varepsilon^{abcd}$ is the totally antisymmetric symbol. All these matrices are real with respect to the covariant conjugation

$$(\gamma^M)_{ab} = C^a_\dot{a} C^\dot{a}_b (\gamma^M)_{cd},$$  \hfill (A.7)
The basic relations for these Weyl matrices are

\[(\gamma_M)_{ac}(\tilde{\gamma}_N)^{cb} + (\gamma_N)_{ac}((\tilde{\gamma}_M)^{cb} = -2\delta_a^b\eta_{MN}, \quad (A.8)\]
\[\varepsilon_{abcd} = \frac{1}{2}(\gamma_M)_{ab}(\gamma_N)_{cd}, \quad (A.9)\]

where \(\eta_{MN}\) is the metric of the 6D Minkowski space \((\eta_{00} = -\eta_{11} = \ldots = -\eta_{55} = 1)\) and \(\gamma_M = \eta_{MN}\gamma^N\).

The generators of the (1,0) spinor representation are \(S^{MN} = -\frac{1}{2}\sigma^{MN}\), where

\[(\sigma^{MN})^a_b = \frac{1}{2}(\tilde{\gamma}_M\gamma^N - \tilde{\gamma}_N\gamma^M)^a_b, \quad \sigma^{MN} = \sigma^{MN}. \quad (A.10)\]

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