Method of Relaxed Streamlined Upwinding for Hyperbolic Conservation Laws

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Abstract

In this work a new finite element based Method of Relaxed Streamlined Upwinding is proposed to solve hyperbolic conservation laws. Formulation of the proposed scheme is based on relaxation system which replaces hyperbolic conservation laws by semi-linear system with stiff source term also called as relaxation term. The advantage of the semi-linear system is that the nonlinearity in the convection term is pushed to the source term on right hand side which can be handled with ease. Six discrete velocity models are proposed for two dimensional relaxation system which symmetrically spread foot of the characteristics in all four quadrants. Proposed formulation gives exact diffusion vectors which are very simple. Moreover, the formulation is easily extendable from scalar to vector conservation laws. Various test cases are solved including Burgers equation (with convex and non-convex flux function), Euler equations and shallow water equations in one and two dimensions which demonstrate the robustness and accuracy of the proposed scheme. New test cases are proposed for Burgers equation, Euler and shallow water equations. Exact solutions are given for two dimensional Burgers test case which involve normal discontinuity and series of oblique discontinuities. Both four node quadrilateral element and three node triangular element are used to solve multidimensional problems. Error analysis of the proposed scheme shows optimal convergence rate. Moreover, spectral stability analysis is done which gives implicit expression for critical time step.

Keywords: Finite Element Method, Relaxation System, Burgers Equation, Euler Equations, Shallow Water Equations, Spectral Stability Analysis.

1. Introduction

Many natural processes are governed by hyperbolic conservation laws like high speed flows governed by compressible Euler equations, shallow water flows like flow in a canal, river flows etc are governed by shallow water equations, astrophysical flows or space weather governed by magnetohydrodynamic (MHD) equations etc. These equations describe the transport and propagation of waves (both linear and nonlinear) in space and time. Due to non-linear nature of convection term such equations admit discontinuous solution which precludes the possibility of finding closed form solution. Alternatively, numerical methods are used to solve these equations. In the literature of finite volume and finite difference framework upwind methods or upwind schemes are extensively used to solve hyperbolic conservation laws. Riemann solvers are one of the popular class of upwind schemes. Apart from Riemann solvers there are other upwind methods available in the literature like flux splitting methods, kinetic schemes and relaxation schemes. Upwind methods are also used in finite element framework where they are part of much larger group called as stabilized finite element methods which are used to solve hyperbolic conservation laws. There are various stabilized finite element methods available in the literature like Taylor Galerkin method, Streamlined-Upwind Petrov-Galerkin (SUPG) method, Discontinuous Galerkin method, Least-Square Galerkin method etc. For more details about these methods see [60,24,10,19,13]. Among all methods, SUPG method is one of the oldest stabilized scheme derived for convection dominated flows. Due to their desirable properties SUPG scheme is widely used stabilized finite element method for both compressible [53,54,27] as well as incompressible flows [12,26] (along with Pressure Stabilizing
Petrov Galerkin PSPG formulation \([52]\)). Different versions of SUPG scheme are derived like conservative variable and entropy variable based SUPG scheme \([28, 36]\). Kinetic theory based SUPG method called as KSUPG method is also developed for Burgers equation, Euler equations and ideal MHD equations \([29, 30]\).

Relaxation schemes introduced by Jin & Xin \([31]\) for hyperbolic conservation laws without source term are an attractive alternative to the other upwinds schemes. In the recent years, the simplicity of this scheme attracted many researchers around the world. Relaxation scheme is based on the relaxation system which replaces nonlinear convection term present in hyperbolic partial differential equation(s) by semi-linear system with stiff relaxation term on right hand side. This system is equivalent to original hyperbolic conservation law in the limit of vanishing relaxation parameter. Simple procedure present in Relaxation schemes to handle nonlinear convection term avoids more complex Riemann solvers, kinetic schemes and flux splitting methods. First and second order relaxation schemes are first introduced in \([31]\) while higher order relaxation schemes are discussed in \([46, 39, 47]\). The relaxation scheme for a hyperbolic conservation laws with source term are presented in \([17, 16, 21]\). Natalini \([43]\) interpreted Jin & Xin’s relaxation system as a discrete velocity Boltzmann equation with BGK model for the collision term. In \([3]\) Aregba-Driollet and Natalini introduced numerical schemes based on discrete velocity Boltzmann equation which are called as discrete velocity kinetic schemes. Relaxation schemes are also employed in the lattice Boltzmann framework \([6, 23]\). An alternative relaxation system for one dimensional conservation law proposed by Murthy \([42]\) retain the semi-linear structure of original relaxation system but at the same time satisfies the integral constraint which is more consistent than the original relaxation system. For more details about the relaxation schemes refer \([31, 44, 43, 3, 8, 15, 13, 25, 37]\) and the references there in.

Relaxation scheme is also used in finite element framework for one dimensional scalar and vector (elastodynamics) problems \([4]\). In this paper the relaxation based Streamlined-Upwind scheme, named as Method of Relaxed Streamlined-Upwinding (MRSU) is developed in finite element framework for hyperbolic conservation laws in one and two dimensions. The proposed scheme belongs to the class of stabilized finite element methods. Some of the salient features of proposed formulation are

1. The weak formulation of governing hyperbolic PDEs is started in conservation form which is obtained from relaxation system with an analogous procedure used in Relaxed scheme (instantaneous relaxation).
2. Only test space of convection part of the governing equation is enriched to obtain the required stabilization term.
3. In this analysis, group discretization or group formulation of flux function which are shown to be more accurate is used \([22, 41]\).
4. Proposed scheme can be easily extended from scalar to vector conservation laws.
5. Exact stabilization vectors can be obtained for both scalar as well as vector conservation laws. Importantly, no Jacobian matrices are involved in the stabilization terms.
6. Six symmetric discrete velocity models are proposed for two dimensional problems which includes four point along diagonal \((D4)\), nine point including rest particle \((AD9)\), eight point without rest particle \((AD8)\), four point along axis \((A4)\), four point along diagonal with one rest particle \((D5)\) and four point along axis with one rest particle \((A5)\) discrete velocity models.
7. Temporal discretization is done by using simple first order forward Euler method.
8. To show the efficacy of the proposed scheme various test cases of Burgers equation, Euler equations and shallow water equations are solved in one and two dimensions. Moreover, some new test cases for Burgers equation, Euler and shallow water equations are also introduced. In case of two dimensional Burgers equation a set of test cases involving normal and oblique discontinuity are proposed along with their exact solutions.
9. Error analysis of the proposed scheme shows optimal rate of convergence. Furthermore, spectral stability analysis gives implicit expression for critical time step.

This paper is arranged as follows. After introduction in section 1, section 2 describe governing hyperbolic conservation laws like Burgers equation, Euler and shallow water equations. Section 3 gives relaxation system for system of hyperbolic conservation laws. In section 4 relaxed formulation for hyperbolic conservation laws is explained which will be used to develop MRSU scheme. Section 5 explains two and three discrete velocity model for one dimensional equations whereas section 6 describe various symmetric discrete velocity models for two dimensional problems. In section 7 Chapman-Enskog type expansion of relaxation system is performed which gives stability condition for such
system. Section 8 gives weak formulation of MRSU scheme in detail. Section 9 describe temporal discritization of semi-discrete MRSU scheme followed by section 10 where simple gradient based shock capturing parameter is developed. In section 11 spectral stability analysis of the proposed scheme is carried out which gives expression for critical time step. In section 12 large number of numerical experiments are carried out for Burgers equation, Euler and shallow water equations which support author’s claim of robustness and accuracy in the proposed numerical scheme. Finally, this paper is concluded in section 13.

2. Governing Equations

The governing hyperbolic conservation laws are given as

\[
\frac{\partial U}{\partial t} + \frac{\partial G_i(U)}{\partial x_i} = 0 \quad \text{in} \quad \mathbb{R}^D \times \mathbb{R}_+ \tag{1}
\]

with appropriate initial and boundary conditions. Flux functions \( G_i(U) \) are functions of conserved variable \( U \). In case of Burgers equation both \( U \) and \( G_i(U) \)'s are scalar quantities whereas in case of Euler and Shallow water equations they are vector quantities.

For 3D Euler equations, \( U \) and \( G_i \)'s are given as

\[
U = \begin{cases} 
\rho \\
\rho u_1 \\
\rho u_2 \\
\rho u_3 \\
\rho E 
\end{cases}, \quad G_i = \begin{cases} 
\rho u_i \\
\delta_{i1} p + \rho u_1 u_i \\
\delta_{i2} p + \rho u_2 u_i \\
\delta_{i3} p + \rho u_3 u_i \\
p u_i + p u_i E 
\end{cases}, \quad i = 1, 2, 3
\]

where \( \rho, u_1, u_2, u_3, E, p \) are density, velocity components in \( x, y \) and \( z \) directions, total energy and pressure respectively and \( \delta_{ij} \) is a Kronecker delta. Total energy is given by

\[
E = \frac{p}{\rho (\gamma - 1)} + \frac{1}{2} \| u \|_{L_2}^2
\]

The eigenvalues of flux Jacobian matrices \( \frac{\partial G_i}{\partial U}, \quad i = 1, 2, 3 \) are

\[
u_i \pm \alpha, \quad \nu_i, \quad u_i, \quad u_i
\]

where \( \alpha = \sqrt{\gamma p}/\rho \) is acoustic speed.

For 2D shallow water equations, \( U \) and \( G_i \)'s are given as

\[
U = \begin{cases} 
h \\
h u_1 \\
h u_2 
\end{cases}, \quad G_i = \begin{cases} 
h u_i \\
h u_1 u_i + \frac{1}{2} \delta_{i1} g h^2 \\
h u_2 u_i + \frac{1}{2} \delta_{i2} g h^2 
\end{cases}, \quad i = 1, 2
\]

where \( h, u_1, u_2, g \) are water height, velocity components in \( x \) and \( y \) directions and acceleration due to gravity respectively. Here eigenvalues of flux Jacobian matrices \( \frac{\partial G_i}{\partial U}, \quad i = 1, 2 \) are

\[
u_i \pm \sqrt{g h}, \quad u_i
\]

3. Relaxation System for System of Hyperbolic Conservation Laws

Multidimensional system of hyperbolic conservation laws are given as

\[
\frac{\partial U}{\partial t} + \frac{\partial G_i(U)}{\partial x_i} = 0, \quad U(x, 0) = U_0(x) \tag{2}
\]
where $x \in \mathbb{R}^D$, $U \in \mathbb{R}^N$ and flux function $G_i(U) \in \mathbb{R}^N$ is nonlinear. As the above system of equations are hyperbolic so the Jacobian $\frac{\partial G}{\partial U}$ is diagonalizable.

Jin and Xin [31] proposed the following relaxation system for equation (2)

$$\frac{\partial U}{\partial t} + \frac{\partial W_i}{\partial x_i} = 0$$

$$\frac{\partial W_i}{\partial t} + \Omega^2_{ij} \frac{\partial U}{\partial x_j} = -\frac{1}{\epsilon} (W_i - G_i(U)), \quad i = 1, 2, \ldots, D$$

where $W_i \in \mathbb{R}^N$, $\Omega^2_{ij} \in \mathbb{R}^{N \times N}$ is a diagonal matrix with non-negative elements $\Omega^2_{ij}$, $i = 1, 2, \ldots, D$, $j = 1, 2, \ldots, N$ where $N$ is number of discrete velocities and $\epsilon$ is the relaxation time. The advantage of above relaxation system is convection term is linear and the nonlinearity is moved to the right hand side as source term. The solution of above relaxation system approaches solution of hyperbolic conservation law in the limit $\epsilon \to 0$ if following sub-characteristics condition is satisfied

$$\frac{\sigma^2_1}{\Omega^2_{11}} + \frac{\sigma^2_2}{\Omega^2_{22}} + \cdots + \frac{\sigma^2_N}{\Omega^2_{NN}} \leq 1 \quad (3)$$

where $\sigma_1, \sigma_2, \ldots, \sigma_N$ are the eigenvalues of Jacobian matrix $\frac{\partial G_i(U)}{\partial U}$.

4. Relaxed Formulation for Hyperbolic Conservation Laws

This section introduces relaxed formulation for one dimensional scalar hyperbolic conservation laws. Extension of this formulation for multidimensional scalar or vector conservation laws would be straightforward.

Consider the following scalar nonlinear hyperbolic conservation law in 1D

$$\frac{\partial U}{\partial t} + \frac{\partial G(U)}{\partial x} = 0 \quad \text{in} \; \mathbb{R} \times \mathbb{R}_+, \quad G(U) = U^2/2$$

with initial conditions $U(x, 0) = U_0(x)$. The relaxation system for above equation is

$$\frac{\partial U}{\partial t} + \frac{\partial W}{\partial x} = 0$$

$$\frac{\partial W}{\partial t} + \Lambda^2 \frac{\partial U}{\partial x} = -\frac{1}{\epsilon} (W - G(U)) \quad (4)$$

with initial conditions $U(x, 0) = U_0(x)$, $W(x, 0) = G(U_0(x))$, where $\lambda$ is a positive constant and $\epsilon$ is the relaxation time. Above set of equations can be written in matrix form as

$$\frac{\partial Q}{\partial t} + \Lambda \frac{\partial Q}{\partial x} = H$$

where

$$Q = \begin{bmatrix} U \\ W \end{bmatrix}, \quad A = \begin{bmatrix} 0 & 1 \\ \lambda^2 & 0 \end{bmatrix}, \quad H = \begin{bmatrix} 0 \\ \frac{1}{\epsilon} (G(U) - W) \end{bmatrix}$$

Matrix $A$ can be decomposed as

$$A = R \Lambda R^{-1}$$

where $R$ is a model matrix whose columns are the eigenvectors of $A$ and $\Lambda = \text{diag}(-\lambda, \lambda)$ is a spectral matrix with eigenvalues as diagonal entries. Introducing the Characteristics variable vector $f = R^{-1} Q$ above system can be written as

$$\frac{\partial f}{\partial t} + \Lambda \frac{\partial f}{\partial x} = R^{-1} H \quad (5)$$

where

$$f = R^{-1} Q = \begin{bmatrix} f_1 \\ f_2 \end{bmatrix} = \begin{bmatrix} \frac{U}{\epsilon} - \frac{W}{\epsilon^2} \\ \frac{U}{\epsilon} + \frac{W}{\epsilon^2} \end{bmatrix}$$
Let
\[
F = \{ F_1, F_2 \} = \{ \frac{U - G}{\epsilon}, \frac{U + G}{\epsilon} \}
\]
(6)

using this equation (5) can be written as
\[
\frac{\partial f}{\partial t} + \Lambda \frac{\partial f}{\partial x} = -\frac{(f - F)}{\epsilon}
\]
(7)

with initial conditions \( U(x, 0) = U_0(x), \ f(0) = F(U_0(x)) \).

For small value of \( \epsilon \) above equation looks similar to the classical Boltzmann equation with BGK model where \( \epsilon \) is performing a role of relaxation time. Here \( f \) represents the distribution function whereas \( F \) represents local Maxwellian distribution or equilibrium distribution. In the above equation discrete velocities \(-\lambda, \lambda\) are involved whereas in a classical Boltzmann equation velocities are continuous. Due to this Natalini \[3\] interpreted relaxation system (equation (7)) as a discrete velocity Boltzmann equation.

Splitting method \[31\] can be employed to solve equation (7) which can be written in two steps as
\[
\frac{\partial f}{\partial t} + \Lambda \frac{\partial f}{\partial x} = 0 : \text{Convection}
\]
\[
\frac{\partial f}{\partial t} = -\frac{(f - F)}{\epsilon} : \text{Relaxation}
\]

The solution of relaxation step is \( f = (f(0) - F)e^{-t/\epsilon} + F \). Assuming instantaneous relaxation to equilibrium i.e., \( \epsilon = 0 \) gives \( f = F \). Substituting this in convection step we get
\[
\frac{\partial F}{\partial t} + \Lambda \frac{\partial F}{\partial x} = 0
\]

which can be rewritten in conservation form as
\[
\frac{\partial P F}{\partial t} + \frac{\partial P(\Lambda F)}{\partial x} = 0
\]
(8)

Hyperbolic conservation law is recovered from above equation by taking moments
\[
\frac{\partial P F}{\partial t} + \frac{\partial P(\Lambda F)}{\partial x} = 0
\]
where
\[
P F = U, \ P(\Lambda F) = G
\]
(9)

and moment vector \( P = [1, 1, \cdots, 1] \). Equations (9) are called as Moment Relations. Schemes based on above procedure are called as instantaneous Relaxation schemes or Relaxed schemes. In the upcoming section, equation (8) will be used for deriving weak formulation of hyperbolic conservation laws.

5. Symmetric Discrete Velocity Models for One Dimensional Conservation Laws

Several choices of discrete velocity models are possible as long as the corresponding local Maxwellian distribution \( F \) satisfies the moment relations given by equation (9). In this section two models are considered for one dimensional hyperbolic conservation laws namely, two discrete velocity model with \( N = 2 \) and three discrete velocity model with \( N = 3 \).

Two discrete velocity model is already introduced in the previous section where two moving particles are present. In that case \( \Lambda \) is given as
\[
\Lambda = \text{diag}\{-\lambda, \lambda\}
\]
and the corresponding expression for \( F \) is given by equation (6) which satisfies the moment relations. Figure 1 (a) shows this model.
In case of three discrete velocity model there is a rest particle along with two moving particles (see figure 1(b)). In this case \( \Lambda \) is given as
\[
\Lambda = \text{diag}[-\lambda, 0, \lambda]
\]
Expression for \( F \) is given by
\[
F = \begin{cases} 
F_1 & \\
F_2 & \\
F_3 & 
\end{cases} = \begin{cases} 
\frac{U}{3} - \frac{G}{3} & \\
\frac{U}{3} & \\
\frac{U}{3} + \frac{G}{3} & 
\end{cases}
\]
Again, this satisfies the moment relations. Detailed derivation of \( F \) for three velocity model is given in Appendix.

6. Symmetric Discrete Velocity Models for Multidimensional Conservation Laws

Asymmetric discrete velocity model for multidimensional relaxation system is discussed in the literature [3]. This section introduces the symmetric discrete velocity models in two dimension which takes information symmetrically from all directions. Again, in two dimensions several choices of symmetric discrete velocity models are possible. In general, \( \Lambda \) matrices are constructed using \( N \) diagonal blocks corresponds to \( N \) discrete velocities as
\[
\Lambda_i = \text{diag}(\lambda_i^{(1)}, \lambda_i^{(2)}, \ldots, \lambda_i^{(N)})
\]
In order to admit kinetic entropy and satisfy the entropy inequality in the equilibrium limit \( \epsilon \to 0 \) by equation (7), Bochut [11] has characterized the space of Maxwellians. The Maxwellians are written as a linear combination of conserved variables and fluxes as
\[
F_i(U) = k_{i0}U + \sum_{j=1}^{D} k_{ij}G_j(U) \quad \text{for} \quad i = 1, 2, \ldots N
\]
where constants \( k_{ij} \) are chosen in such a way that consistency condition given by equations (9) is satisfied. The \( \Lambda \) matrices are constructed using orthogonal velocity method [31]. Here, diagonal entries \( \lambda_i^{(j)} \)’s are chosen such that they satisfy the following two conditions
\[
\sum_{j=1}^{N} \lambda_i^{(j)} = 0, \quad \forall \ i = 1, 2, \ldots D
\]
\[
\sum_{j=1}^{N} \lambda_i^{(j)}\lambda_j^{(l)} = 0, \quad \text{where} \quad i \neq j \quad \text{and} \quad \forall \ i, j = 1, 2, \ldots D
\]
with this the Maxwellian is obtained as
\[
F(U) = \frac{U}{N} + \sum_{j=1}^{2} \frac{G_j(U)}{\left(\sum_{i=1}^{N} |\lambda_i^{(j)}|\right)} \text{sgn}(\lambda_i^{(j)})
\]

Now, let's consider different symmetric discrete velocity models for two dimensional conservation laws. In this case six symmetric discrete velocity models are proposed viz., four point along diagonal (\(\mathcal{D}4\)) with \(N = 4\), nine point (\(\mathcal{D}9\)) with \(N = 9\), eight point (\(\mathcal{D}8\)) with \(N = 8\), four point along axis (\(\mathcal{D}4\)) with \(N = 4\), four point along diagonal with one rest particle (\(\mathcal{D}5\)) with \(N = 5\) and four point along axis with one rest particle (\(\mathcal{D}5\)) with \(N = 5\) as shown in figure 2(a). In all models magnitude of discrete velocities (characteristic speeds) in both \(x\) and \(y\) direction is same which is \(\lambda\) as shown in figure 2(a).

\(\mathcal{D}4\) symmetric relaxation system uses four characteristic speeds along diagonal from the four quadrant as shown in figure 2(a). In this case \(\Lambda\) matrices are given as

\[
\Lambda_1 = \text{diag}(-\lambda, \lambda, -\lambda, \lambda), \quad \Lambda_2 = \text{diag}(-\lambda, -\lambda, \lambda, \lambda)
\]

The characteristics variable and local Maxwellian distribution function are obtained as

\[
f = \begin{pmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \end{pmatrix} = \begin{pmatrix} \frac{U}{4} + \frac{W_1(U)}{4} + \frac{W_2(U)}{4} + \frac{W_3(U)}{4} \\ \frac{U}{4} + \frac{W_1(U)}{4} + \frac{W_2(U)}{4} + \frac{W_4(U)}{4} \\ \frac{U}{4} + \frac{W_1(U)}{4} + \frac{W_2(U)}{4} + \frac{W_3(U)}{4} \\ \frac{U}{4} + \frac{W_1(U)}{4} + \frac{W_2(U)}{4} + \frac{W_3(U)}{4} \end{pmatrix}, \quad F = \begin{pmatrix} F_1 \\ F_2 \\ F_3 \\ F_4 \end{pmatrix} = \begin{pmatrix} \frac{U}{4} - \frac{G_1(U)}{4} + \frac{G_2(U)}{4} \\ \frac{U}{4} + \frac{G_1(U)}{4} + \frac{G_2(U)}{4} \\ \frac{U}{4} - \frac{G_1(U)}{4} + \frac{G_2(U)}{4} \\ \frac{U}{4} + \frac{G_1(U)}{4} + \frac{G_2(U)}{4} \end{pmatrix}
\]

\(\mathcal{D}9\) symmetric relaxation system uses nine characteristic speeds (including rest particle) from the four quadrant as shown in figure 2(b). Here \(\Lambda\) matrices are given as

\[
\Lambda_1 = \text{diag}(-\lambda, 0, \lambda, \lambda, 0, -\lambda, -\lambda, 0),
\]

\[
\Lambda_2 = \text{diag}(-\lambda, -\lambda, -\lambda, 0, \lambda, \lambda, 0, 0)
\]

In this case the characteristics variable and local Maxwellian distribution function are obtained as

\[
f = \begin{pmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \\ f_5 \\ f_6 \\ f_7 \\ f_8 \\ f_9 \end{pmatrix} = \begin{pmatrix} \frac{U}{9} - \frac{W_1(U)}{9} - \frac{W_2(U)}{9} \\ \frac{U}{9} + \frac{W_1(U)}{9} + \frac{W_2(U)}{9} \\ \frac{U}{9} - \frac{W_1(U)}{9} - \frac{W_2(U)}{9} \\ \frac{U}{9} + \frac{W_1(U)}{9} + \frac{W_2(U)}{9} \\ \frac{U}{9} - \frac{W_1(U)}{9} - \frac{W_2(U)}{9} \\ \frac{U}{9} + \frac{W_1(U)}{9} + \frac{W_2(U)}{9} \\ \frac{U}{9} - \frac{W_1(U)}{9} - \frac{W_2(U)}{9} \\ \frac{U}{9} + \frac{W_1(U)}{9} + \frac{W_2(U)}{9} \end{pmatrix}, \quad F = \begin{pmatrix} F_1 \\ F_2 \\ F_3 \\ F_4 \\ F_5 \\ F_6 \\ F_7 \\ F_8 \\ F_9 \end{pmatrix} = \begin{pmatrix} \frac{U}{9} - \frac{G_1(U)}{9} - \frac{G_2(U)}{9} \\ \frac{U}{9} + \frac{G_1(U)}{9} + \frac{G_2(U)}{9} \\ \frac{U}{9} - \frac{G_1(U)}{9} - \frac{G_2(U)}{9} \\ \frac{U}{9} + \frac{G_1(U)}{9} + \frac{G_2(U)}{9} \\ \frac{U}{9} - \frac{G_1(U)}{9} - \frac{G_2(U)}{9} \\ \frac{U}{9} + \frac{G_1(U)}{9} + \frac{G_2(U)}{9} \\ \frac{U}{9} - \frac{G_1(U)}{9} - \frac{G_2(U)}{9} \\ \frac{U}{9} + \frac{G_1(U)}{9} + \frac{G_2(U)}{9} \end{pmatrix}
\]

\(\mathcal{D}8\) symmetric relaxation system uses eight characteristic speeds from the four quadrant as shown in figure 2(c) with

\[
\Lambda_1 = \text{diag}(-\lambda, 0, \lambda, \lambda, 0, -\lambda, -\lambda),
\]

\[
\Lambda_2 = \text{diag}(-\lambda, -\lambda, -\lambda, 0, \lambda, \lambda, 0)
\]
Figure 2: (a) Four point along diagonal (44), (b) nine point (99), (c) eight point (88), (d) four point along axis (44), (e) Four point along diagonal with one rest particle (45) and (f) Four point along axis with one rest particle (45) symmetric relaxation system
Here, the characteristics variable and local Maxwellian distribution function are obtained as

\[
f = \begin{pmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \\ f_5 \\ f_6 \\ f_7 \\ f_8 \end{pmatrix} = \begin{pmatrix} \frac{U}{2} - \frac{W_1(U)}{2} + \frac{W_2(U)}{2} \\ \frac{U}{2} + \frac{W_1(U)}{2} - \frac{W_2(U)}{2} \\ \frac{U}{2} + \frac{W_1(U)}{2} + \frac{W_2(U)}{2} \\ \frac{U}{2} - \frac{W_1(U)}{2} - \frac{W_2(U)}{2} \\ \frac{U}{2} - \frac{W_1(U)}{2} + \frac{W_2(U)}{2} \\ \frac{U}{2} + \frac{W_1(U)}{2} - \frac{W_2(U)}{2} \\ \frac{U}{2} - \frac{W_1(U)}{2} - \frac{W_2(U)}{2} \\ \frac{U}{2} + \frac{W_1(U)}{2} + \frac{W_2(U)}{2} \end{pmatrix}, \quad F = \begin{pmatrix} F_1 \\ F_2 \\ F_3 \\ F_4 \\ F_5 \\ F_6 \\ F_7 \\ F_8 \end{pmatrix} = \begin{pmatrix} \frac{U}{2} - \frac{G_1(U)}{2} - \frac{G_2(U)}{2} \\ \frac{U}{2} + \frac{G_1(U)}{2} - \frac{G_2(U)}{2} \\ \frac{U}{2} + \frac{G_1(U)}{2} + \frac{G_2(U)}{2} \\ \frac{U}{2} - \frac{G_1(U)}{2} + \frac{G_2(U)}{2} \\ \frac{U}{2} - \frac{G_1(U)}{2} - \frac{G_2(U)}{2} \\ \frac{U}{2} + \frac{G_1(U)}{2} - \frac{G_2(U)}{2} \\ \frac{U}{2} - \frac{G_1(U)}{2} + \frac{G_2(U)}{2} \\ \frac{U}{2} + \frac{G_1(U)}{2} + \frac{G_2(U)}{2} \end{pmatrix}
\]

\(4\) symmetric relaxation system uses four characteristic speeds along axis from the four quadrant as shown in figure (d) with

\[A_1 = \text{diag}(0, \lambda, 0, -\lambda), \quad A_2 = \text{diag}(-\lambda, 0, \lambda, 0)\]

In this case characteristic variable and local Maxwellian distribution function are obtained as

\[
f = \begin{pmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \end{pmatrix} = \begin{pmatrix} \frac{U}{2} - \frac{W_1(U)}{2} + \frac{W_2(U)}{2} \\ \frac{U}{2} + \frac{W_1(U)}{2} - \frac{W_2(U)}{2} \\ \frac{U}{2} + \frac{W_1(U)}{2} + \frac{W_2(U)}{2} \\ \frac{U}{2} - \frac{W_1(U)}{2} - \frac{W_2(U)}{2} \end{pmatrix}, \quad F = \begin{pmatrix} F_1 \\ F_2 \\ F_3 \\ F_4 \end{pmatrix} = \begin{pmatrix} \frac{U}{2} - \frac{G_1(U)}{2} - \frac{G_2(U)}{2} \\ \frac{U}{2} + \frac{G_1(U)}{2} - \frac{G_2(U)}{2} \\ \frac{U}{2} + \frac{G_1(U)}{2} + \frac{G_2(U)}{2} \\ \frac{U}{2} - \frac{G_1(U)}{2} + \frac{G_2(U)}{2} \end{pmatrix}
\]

\(5\) symmetric relaxation system uses four characteristic speeds along diagonal from the four quadrant and one rest particle as shown in figure (e) with

\[A_1 = \text{diag}(-\lambda, \lambda, -\lambda, 0), \quad A_2 = \text{diag}(-\lambda, -\lambda, \lambda, 0)\]

The characteristics variable and local Maxwellian distribution function are obtained as

\[
f = \begin{pmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \end{pmatrix} = \begin{pmatrix} \frac{U}{2} - \frac{W_1(U)}{2} + \frac{W_2(U)}{2} \\ \frac{U}{2} + \frac{W_1(U)}{2} - \frac{W_2(U)}{2} \\ \frac{U}{2} + \frac{W_1(U)}{2} + \frac{W_2(U)}{2} \\ \frac{U}{2} - \frac{W_1(U)}{2} - \frac{W_2(U)}{2} \end{pmatrix}, \quad F = \begin{pmatrix} F_1 \\ F_2 \\ F_3 \\ F_4 \end{pmatrix} = \begin{pmatrix} \frac{U}{2} - \frac{G_1(U)}{2} - \frac{G_2(U)}{2} \\ \frac{U}{2} + \frac{G_1(U)}{2} - \frac{G_2(U)}{2} \\ \frac{U}{2} + \frac{G_1(U)}{2} + \frac{G_2(U)}{2} \\ \frac{U}{2} - \frac{G_1(U)}{2} + \frac{G_2(U)}{2} \end{pmatrix}
\]

Finally, \(5\) symmetric relaxation system uses four characteristic speeds along axis from the four quadrant and one rest particle as shown in figure (f) with

\[A_1 = \text{diag}(0, \lambda, 0, -\lambda, 0), \quad A_2 = \text{diag}(-\lambda, 0, \lambda, 0, 0)\]

In this case characteristic variable and local Maxwellian distribution function are obtained as

\[
f = \begin{pmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \end{pmatrix} = \begin{pmatrix} \frac{U}{2} - \frac{W_1(U)}{2} + \frac{W_2(U)}{2} \\ \frac{U}{2} + \frac{W_1(U)}{2} - \frac{W_2(U)}{2} \\ \frac{U}{2} + \frac{W_1(U)}{2} + \frac{W_2(U)}{2} \\ \frac{U}{2} - \frac{W_1(U)}{2} - \frac{W_2(U)}{2} \end{pmatrix}, \quad F = \begin{pmatrix} F_1 \\ F_2 \\ F_3 \\ F_4 \end{pmatrix} = \begin{pmatrix} \frac{U}{2} - \frac{G_1(U)}{2} - \frac{G_2(U)}{2} \\ \frac{U}{2} + \frac{G_1(U)}{2} - \frac{G_2(U)}{2} \\ \frac{U}{2} + \frac{G_1(U)}{2} + \frac{G_2(U)}{2} \\ \frac{U}{2} - \frac{G_1(U)}{2} + \frac{G_2(U)}{2} \end{pmatrix}
\]

In all models the macroscopic variables are recovered by taking moments as

\[PF = U, \quad P(A, F) = G(U)\]

In the upcoming section these discrete velocity models in one and two dimensions will be compared based on expression for diffusion term and stability condition for coefficient of diffusion. Models with minimum numerical diffusion will be chosen for computation in both one and two dimensions.
7. Chapman-Enskog Type Expansion for the Relaxation System

A Chapman-Enskog type expansion for the relaxation system provides the condition under which the relaxation system is a dissipative approximation to the given hyperbolic conservation laws [17]. For one dimensional relaxation system with two discrete velocity model (equation (4)) this expansion is given as

\[
\frac{\partial U}{\partial t} + \frac{\partial G(U)}{\partial x} = \frac{\partial U}{\partial x} \left( \lambda^2 - (G'(U))^2 \right) + O(\epsilon^2)
\]

where \( G'(U) = \frac{\partial G(U)}{\partial x} \). First term on the right hand side represents the viscous dissipation term with coefficient of viscosity. Therefore, relaxation system provides a vanishing viscosity model for the original hyperbolic conservation laws. For stability, value of \( \lambda \) should be chosen such that

\[ \lambda^2 \geq (G'(U))^2 \]

In case of three discrete velocity model the stability condition become

\[ \frac{2}{3} \lambda^2 \geq (G'(U))^2 \]

Similarly, to obtain the stability conditions for two dimensional scalar conservation laws one can write the general Chapman-Enskog type expansion for two dimensional relaxation system as

\[
\frac{\partial U}{\partial t} + \frac{\partial G_1(U)}{\partial x} + \frac{\partial G_2(U)}{\partial y} =
\]

\[
\epsilon \frac{\partial}{\partial x} \left( \frac{\partial}{\partial x} \left[ P(\Lambda_1^2 \mathbf{F}') - (G'_1(U))^2 \right] + \frac{\partial}{\partial y} \left[ P(\Lambda_1 \Lambda_2 \mathbf{F}') - G'_1(U)G'_2(U) \right] \right)
\]

\[
+ \epsilon \frac{\partial}{\partial y} \left( \frac{\partial}{\partial x} \left[ P(\Lambda_1 \Lambda_2 \mathbf{F}') - G'_1(U)G'_2(U) \right] + \frac{\partial}{\partial y} \left[ P(\Lambda_2^2 \mathbf{F}') - (G'_2(U))^2 \right] \right)
\]

\[ + O(\epsilon^2) \]

Table 1 gives expressions for the terms \( P(\Lambda_1^2 \mathbf{F}') \), \( P(\Lambda_1 \Lambda_2 \mathbf{F}') \), \( P(\Lambda_2 \Lambda_1 \mathbf{F}') \) and \( P(\Lambda_2^2 \mathbf{F}') \) involved in the coefficient of diffusion for various symmetric relaxation models.

| Symmetric Model | \( P(\Lambda_1^2 \mathbf{F}') \) | \( P(\Lambda_1 \Lambda_2 \mathbf{F}') \) | \( P(\Lambda_2 \Lambda_1 \mathbf{F}') \) | \( P(\Lambda_2^2 \mathbf{F}') \) |
|-----------------|-----------------|-----------------|-----------------|-----------------|
| 9               | \( \lambda^2 \) | 0               | 0               | \( \lambda^2 \) |
| 9               | \( \frac{2}{3} \lambda^2 \) | 0               | 0               | \( \frac{2}{3} \lambda^2 \) |
| 9               | \( \frac{2}{3} \lambda^2 \) | 0               | 0               | \( \frac{2}{3} \lambda^2 \) |
| 4               | \( \frac{1}{2} \lambda^2 \) | 0               | 0               | \( \frac{1}{2} \lambda^2 \) |
| 5               | \( \frac{1}{2} \lambda^2 \) | 0               | 0               | \( \frac{1}{2} \lambda^2 \) |
| 5               | \( \frac{1}{2} \lambda^2 \) | 0               | 0               | \( \frac{1}{2} \lambda^2 \) |

Table 1: Coefficient of diffusion for symmetric relaxation models.

Therefore, to make dissipation positive following conditions must be satisfied

\[ \beta \lambda^2 \geq (|G'_1(U)|^2 + |G'_2(U)|^2) \]

where \( \beta \) is the coefficient of numerical diffusion (\( \lambda^2 \)). \( \beta = 1, 2/3, 3/4, 1/2, 4/5, 2/5 \) for \( 9, 4, 9, 8, 4, 5 \) models respectively.
For vector conservation laws the stability condition can be obtained following [35, 45, 58]. It can be shown that the solution of the relaxation system approaches exact solution of original hyperbolic partial differential equation as $\epsilon \to 0$ if sub-characteristics conditions given by equation (3) is satisfied. Following values of $\Omega$ based on the supremum eigenvalue of Jacobian matrices are chosen for Euler and shallow water equations.

For 1D Euler equations

$$\Omega_{ij} = \max(\sup|u + a|, \sup|u|, \sup|u - a|)$$

For 1D shallow water equations

$$\Omega_{ij} = \max(\sup|u + \sqrt{gh}|, \sup|u - \sqrt{gh}|)$$

For 2D Euler equations

$$\Omega_{ij} = \max(\sup|u_1 + a|, \sup|u_1|, \sup|u_1 - a|)$$
$$\Omega_{ij} = \max(\sup|u_2 + a|, \sup|u_2|, \sup|u_2 - a|)$$

For 2D shallow water equations

$$\Omega_{ij} = \max(\sup|u_1 + \sqrt{gh}|, \sup|u_1|, \sup|u_1 - \sqrt{gh}|)$$
$$\Omega_{ij} = \max(\sup|u_2 + \sqrt{gh}|, \sup|u_2|, \sup|u_2 - \sqrt{gh}|)$$

where $j = 1, 2, \ldots, N$ in all cases.

8. Weak MRSU formulation for Hyperbolic Conservation Laws

Weak MRSU formulation starts with equation (8) in conservation form. The standard Galerkin finite element approximation for equilibrium distribution function $F$ and the flux function $\Lambda_j F$ are

$$F \approx F^h = \sum_{\ell_i} N^h_{\ell_i} F, \quad \Lambda_j F \approx (\Lambda_j F)^h = \sum_{\ell_i} N^h_{\ell_i} (\Lambda_j F)$$

where group formation is used for flux function. Interpolation function $N^h \in C^0(\Omega)$ and the computational domain $\Omega$ is divided into $N_{el}$ number of elements as $\Omega = \bigcup_{i=1}^{N_{el}} \Omega_i$ such that $\Omega_i \cap \Omega_j = \emptyset$, $\forall i \neq j$.

Defining the suitable test and trial functions as

$$V^h = \{N^h \in H^1(\Omega) \text{ and } N^h = 0 \text{ on } \Gamma_D\}$$
$$S^h = \{f^h \in H^1(\Omega) \text{ and } F^h = F^h_D \text{ on } \Gamma_D\}$$

($\Gamma_D$ is the Dirichlet boundary) the weak formulation is written as, find $F^h \in S^h$ such that $\forall N^h \in V^h$

$$\int_{\Omega} N^h \left( \frac{\partial F^h}{\partial t} + \frac{\partial (\Lambda_j F)^h}{\partial x_j} \right) \, d\Omega + \sum_{e=1}^{N_{el}} \int_{\Omega_e} \sum_{j=1}^D \tau_i \Lambda_j \frac{\partial N^h}{\partial x_j} \cdot \left( \frac{\partial (\Lambda_j F)^h}{\partial x_j} \right) \, d\Omega$$
$$+ \sum_{e=1}^{N_{el}} \int_{\Omega_e} \delta^e \left( \frac{\partial N^h}{\partial x_j} \frac{\partial F^h}{\partial x_j} \right) \, d\Omega = 0$$

where $\tau_i = \frac{1}{2\max_{\Omega} \left| \frac{\partial x_i}{\partial r} \right|}$ with $\mathcal{D}$, $\mathbf{x}$ and $\mathbf{r}$ being dimension, physical coordinates and natural coordinates respectively. $\delta^e$ is the shock capturing parameter which will be defined later. First term is a standard Galerkin approximation. In the second expression, term inside square bracket is the enriched part of the test space which gives required diffusion. The last term is a shock capturing term which is used only for multidimensional vector problems and will be activated near high gradient region.

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By taking moments one can obtain weak MRSU formulation for hyperbolic conservation laws

\[ \int_{\Omega} N^h \left( \frac{\partial P^h}{\partial t} + \frac{\partial P(\Lambda, F)^h}{\partial x_i} \right) d\Omega + \sum_{e=1}^{N_e} \int_{\Omega_e} P \left[ \sum_{i} \tau_i \Lambda_i \frac{\partial N^h}{\partial x_i} \right] \left( \frac{\partial (\Lambda, F)^h}{\partial x_i} \right) d\Omega 
+ \sum_{e=1}^{N_e} \int_{\Omega_e} \delta^e \left( \frac{\partial N^h}{\partial x_i} \frac{\partial P^h}{\partial x_i} \right) d\Omega = 0 \]  

(11)

The moments are given as

\[ P^h = U^h, \quad P(\Lambda, F)^h = G_j^h \]

As discussed in the introduction part, only test space of convection term of the governing equation is enriched which gives required stabilization. Exact stabilization matrices are obtained for both scalar as well as vector conservation laws. In the above weak formulation, group discretization of flux vector is used. Group discretization was first introduced in [22, 41] which was later extended for FEM based flux corrected transport schemes for multidimensional conservation laws by [32, 33]. In SUPG scheme it is used by [7] which shows improvement in the accuracy of the solution.

Now lets evaluate coefficients of diffusion as well as cross-diffusion matrices given by second term in equation (11).

8.1. One Dimensional Equation(s)

The coefficient of diffusion for two discrete velocity model is obtained as

\[ P(\text{sgn}(\Lambda) \Lambda F^h) = \begin{cases} 
\lambda U^h & \text{for scalar conservation law} \\
\Omega_1 U^h, \forall j = 1, 2 & \text{for vector conservation laws}
\end{cases} \]

and for three discrete velocity model

\[ P(\text{sgn}(\Lambda) \Lambda F^h) = \begin{cases} 
\frac{2}{3} \lambda U^h & \text{for scalar conservation law} \\
\Omega_2 U^h, \forall j = 1, 2, 3 & \text{for vector conservation laws}
\end{cases} \]

Hence, for one dimensional relaxation system three discrete velocity model is less diffusive for both scalar as well as vector conservation laws which will be used for solving problems.

8.2. Two Dimensional Equation(s)

Table 2 shows the expressions for diffusion as well as cross-diffusion terms involved in MRSU scheme for scalar and vector conservation laws using various symmetric models. In case of vector conservation laws \( j = 1, 2, \cdots, N \).

It can be seen that the diffusion vectors are proportional to the supremum eigenvalues of the Jacobian matrices. One can use global supremum eigenvalue while solving problems but it produces excessive diffusion which in turn smears discontinuity. To reduce this diffusion a local supremum eigenvalue of the Jacobian matrix is used instead of global supremum. Second observation from the above diffusion expression is, among all two dimensional symmetric discrete velocity models, \( \Omega_5 \) model gives less diffusion for both scalar and vector conservation laws, hence, hereafter this model will be used for all computations.

Values of \( \lambda, \Omega_1, \) and \( \Omega_2 \) for one and two dimensional scalar as well as vector conservation laws are chosen according to the stability conditions discussed previously.

Theorem 8.1. In MRSU scheme using symmetric discrete velocity models, the constant factor involved in coefficient of numerical diffusion terms (excluding cross-diffusion terms) is given by \( \frac{N_c}{N} \) and it satisfies following inequality

\[ \frac{N_c}{N} \leq 1 \]

where \( N_c \) is number of non-zero diagonal elements of \( \Lambda \) matrix and \( N \) is the number of discrete velocities.
which can sense these high gradient regions and add additional diffusion near high gradient regions. Hence additional diffusion terms for symmetric models.

Proof: In general the diffusion term in MRSU scheme is written as

\[ P(\text{sgn}(\Lambda_i)\Lambda_i F^h) = P(\text{sgn}(\Lambda_i) F^h) \]

\[ = \left\{ \begin{array}{ll}
\frac{N}{2} U^h & \text{for scalar conservation law} \\
\frac{N}{2} \Omega_j U^h, j = 1, 2, \cdots, N & \text{for vector conservation laws}
\end{array} \right. \]

where in the last step we used expression of \( F^h \) for a discrete velocity model. Since \( N_z \leq N \) always, this implies \( \frac{N_z}{N} \leq 1 \).

Above mentioned inequality becomes equality only for two discrete velocity model in 1D and \( \mathcal{A}4 \) model in 2D where \( N_z = N \).

Theorem 8.2. Cross-diffusion vectors \( P(\text{sgn}(\Lambda_1)\Lambda_2 F^h) \) and \( P(\text{sgn}(\Lambda_2)\Lambda_1 F^h) \) vanishes in all symmetric models.

Proof: It is the direct consequence of condition given by equation (10).

| Symmetric Model | Scalar | Vector | Scalar | Vector | Scalar | Vector |
|----------------|--------|--------|--------|--------|--------|--------|
| \( \mathcal{A}4 \) | \( \frac{N}{2} U^h \) | \( \frac{N}{2} \Omega_j U^h \) | 0 | 0 | 0 | 0 |
| \( \mathcal{A}5 \) | \( \frac{N}{2} U^h \) | \( \frac{N}{2} \Omega_j U^h \) | 0 | 0 | 0 | 0 |

Table 2: Diffusion and cross-diffusion terms for symmetric models.

9. Temporal Discretization

For temporal discretization of semi-discrete scheme, forward Euler time discretization is used which can be obtained using Taylor series approximation in time about \( U^n \)

\[ U^{n+1} = U^n + \frac{\partial U^n}{\partial t} \Delta t + \frac{\partial^2 U^n}{\partial t^2} \frac{\Delta t^2}{2} + O(\Delta t^3) \]

where \( \Delta t = t^{n+1} - t^n \). From which forward Euler time discretization can be obtained as

\[ \frac{\partial U^n}{\partial t} = \frac{U^{n+1} - U^n}{\Delta t} + O(\Delta t) \]

The fully discretized system of equations are solved using Generalized Minimal Residual (GMRES) method.

The main purpose of this paper is to present new relaxation based stabilized finite element method for hyperbolic conservation laws. Here, first order scheme is presented, but, proposed scheme is extendable to any high order using higher order time integration like third/fourth order Runge-Kutta methods along with higher order interpolation function in space.

10. Shock Capturing Parameter

In case of multidimensional MRSU method, diffusion along streamline direction is not sufficient to suppress the oscillations near high gradient regions. Hence additional diffusion term with a shock capturing parameter is required which can sense these high gradient regions and add additional diffusion. There are many shock capturing parameters.
available in the literature [50, 51]. In this work a simple gradient based shock capturing parameter is presented as follows.

We define a simple element-wise gradient based shock capturing parameter \( \delta_e \) which introduces diffusion along high gradient direction. Figure 3 (a) shows a typical four node quadrilateral element. As shown in figure, the maximum change in \( \Phi \) (where \( \Phi \) could be density, temperature, pressure or even water height; in present work, density is used for Euler equations and water height is used for shallow water equations) occurs across node 1 and 3. The element based shock capturing parameter is then defined for node 1 and 3 as

\[
\delta_{1,3} = \frac{1}{D \sqrt{D}} \frac{1}{\| \nabla \Phi \|_\infty} \left( \left\| \nabla \Phi \right\|_\infty \right)
\]

where subscripts 1 and 3 represent node numbers. For nodes 2 and 4, it is defined as

\[
\delta_i = \frac{1}{D \sqrt{D}} \frac{1}{\| \nabla \Phi \|_\infty} \left( \Phi_{\text{max}} - \Phi_i \right)
\]

where \( i = 2, 4 \).

At element level matrix form, the shock capturing parameter is given by

\[
\delta = \begin{bmatrix}
\delta_{1} & 0 & 0 & 0 \\
0 & \delta_{2} & 0 & 0 \\
0 & 0 & \delta_{3} & 0 \\
0 & 0 & 0 & \delta_{4}
\end{bmatrix}
\]

The upper and lower bound on the value \( \| \nabla \Phi \|_\infty \) is given by

\[
0 \leq \| \nabla \Phi \|_\infty \leq \| \Phi \|_\infty
\]

It is important to note that, the addition of extra shock capturing term in the weak formulation makes the formulation inconsistent with the original equation. Thus, we define \( \delta^e \) such that as element size \( \Delta x \to 0 \), \( \delta^e \) should disappear. This condition is achieved by including \( \partial \Phi \) in the numerator, which vanishes as we refine the mesh. Similarly, one can define such delta parameter for triangular elements shown in figure 3(b).

10.1. Freezing of \( \delta \)

Calculation of \( \delta \) which is the global \( \delta^e \) at every time step can be computationally expensive. Moreover, it can be shown that it stagnates the convergence of the scheme [14]. The remedy to this problem is to freeze \( \delta \) if

\[
|\text{RES(current)} - \text{RES(previous)}| < \text{Tol}
\]

where RES and Tol are residual and desired tolerance value. This procedure results in the stagnation free convergence. Further, this procedure can be used for both steady state and transient problems.
11. Spectral Stability Analysis

Explicit numerical schemes are conditionally stable due to restriction given by CFL criteria. From computational point of view it is important to find the maximum stable time step also called as critical time step \( \Delta t_{cr} \) within which the scheme is stable. Various methods for stability analysis are available in the literature like von-Neumann and Spectral stability analysis. Unlike von-Neumann stability analysis, spectral stability analysis includes the boundary points and hence it is more accurate than that of von-Neumann stability analysis. In this section a spectral stability analysis of MRSU scheme is performed by stating following theorem.

**Theorem 11.1.** The critical time step \( \Delta t_{cr} \) of explicit MRSU scheme for one dimensional linear convection equation satisfy following inequality

\[
\Delta t_{cr} \leq \frac{1}{\varrho \left( \frac{1}{2} M^{-1} \left[ Cc + \frac{h}{2} D\lambda \right] \right)}
\]

where \( M, C \) and \( D \) are mass, convection and diffusion matrices given as

\[
M = \int_{\Omega} (N^h)^T N^h \, d\Omega, \quad C = \int_{\Omega} (N^h)^T \left( \frac{dN^h}{dx} \right) \, d\Omega, \quad D = \int_{\Omega} \left( \frac{dN^h}{dx} \right)^T \left( \frac{dN^h}{dx} \right) \, d\Omega
\]

whereas \( \varrho, c \) and \( \tilde{h} \) represent spectral radius of a matrix, constant wave speed and elemental length respectively.

**Proof:**

Fully discrete MRSU scheme for one dimensional linear convection equation is written as

\[
\frac{U^{n+1}}{U^n} = \left[ I - M^{-1} \Delta t \left( Cc + \frac{\tilde{h}}{2} D\lambda \right) \right]
\]

Let

\[
\mathcal{A} \triangleq \left[ I - M^{-1} \Delta t \left( Cc + \frac{\tilde{h}}{2} D\lambda \right) \right]
\]

be an amplification matrix. The error at \( n \)th time step \( E^n = U_{exact} - U^n \), \( \forall n \in \mathbb{R}^+ \) also satisfies equation (13), therefore

\[
\frac{E^{n+1}}{E^n} = \mathcal{A}
\]

which can be written as

\[
E^{n+1} = \mathcal{A} E^n = \mathcal{A}^2 E^{n-1} = \cdots = \mathcal{A}^{n+1} E^0
\]

where \( E^0 \) is the error at initial level. For numerical stability error should not grow in time which gives following stability condition

\[
\left| \frac{E^{n+1}}{E^0} \right| \leq 1 \Rightarrow ||\mathcal{A}^{n+1}|| \leq 1 \Rightarrow ||\mathcal{A}|| \leq 1 \Rightarrow \varrho(\mathcal{A}) \leq 1
\]

where \( \varrho(\mathcal{A}) \) is the spectral radius of an amplification matrix. After simplification the critical time step is obtained as

\[
\Delta t_{cr} \leq \frac{1}{\varrho \left( \frac{1}{2} M^{-1} \left[ Cc + \frac{h}{2} D\lambda \right] \right)}
\]

where we used the property of spectral radius of a matrix \( \varrho(\alpha \mathcal{A}) = \alpha \varrho(\mathcal{A}), \alpha \in \mathbb{R} \).

This result can be easily extended for higher dimensional linear convection equation.

As an example lets consider one dimensional linear convection equation with unity wave speed. The computational domain is \([0, 1]\). Initial condition is given as

\[
U(x, 0) = \begin{cases} \frac{1}{2} \left( 1 + \cos \left( \frac{\pi(x-0.2)}{0.12} \right) \right) & \text{for } |x - 0.2| \leq 0.12 \\ 0 & \text{Otherwise} \end{cases}
\]

The spectral radius of amplification matrix is computed numerically with 50 node points. Figure 4 shows the \( \Delta t \) vs. spectral radius. It can be observed that the critical value of time step is around \( 3.2 \times 10^{-3} \).
12. Numerical Experiments

In this section various 1D and 2D test cases are solved for Burgers equation, Euler and shallow water equations which shows the accuracy and robustness of the proposed scheme. These are the standard test cases which are chosen based on the complexity of the solution. Moreover, few new test cases for two dimensional problems are also introduced.

Remarks:

1. Residue plots are given for many steady state test cases where residue is calculated using relative $L_2$ error as

   \[
   \text{Residue} = \frac{\| U^{n+1} - U^n \|_{L_2}}{\| U^n \|_{L_2}}
   \]

2. Full Gauss-quadrature integration rule is used for linear elements in 1D as well as in both four node quadrilateral (Q4) and three node triangular (T3) linear elements in 2D.

12.1. Error Analysis using One Dimensional Convection Equation

Linear Lagrange interpolation function is used as a basis function in space. Experimental Order of Convergence (EOC) is calculated for a one dimensional convection equation with initial condition as a cosine wave (equation (14)) convecting with unity wave speed in $L_2$ and $H_1$ norm. Table 3 shows the EOC which is optimal for linear interpolation function $N^h \in C^0(\Omega)$.

| No. of Nodes | $L_2$ EOC | $H_1$ EOC |
|--------------|-----------|-----------|
| 40           | 0.1348    | 3.2934    |
| 80           | 0.0809    | 2.1803    |
| 160          | 0.0451    | 1.3226    |
| 320          | 0.0240    | 0.7521    |
| 640          | 0.0123    | 0.4134    |
| 1280         | 0.00615   | 0.2195    |
| 2560         | 0.00302   | 0.1154    |

Table 3: Convergence analysis of MRSU scheme.
12.2. 1-D Inviscid Burgers Test Cases with Smooth and Discontinuous Initial Data

1D inviscid Burgers equation is given by
\[ \frac{\partial U}{\partial t} + \frac{\partial U^2}{2 \partial x} = 0 \]

The domain is \([0, 1]\) for smooth initial condition and \([-1, 1]\) for discontinuous initial condition. Smooth initial condition is given by cosine pulse
\[ U(x, 0) = \begin{cases} \frac{1}{2} \left( 1 + \cos \left( \frac{\pi(x-0.5)}{0.3} \right) \right) & \text{for } |x-0.5| \leq 0.3 \\ 0 & \text{Otherwise} \end{cases} \]

The discontinuous initial condition is represented by a square wave as
\[ U(x, 0) = \begin{cases} 1 & \text{for } |x| < 1/3 \\ 0 & \text{for } 1/3 < |x| \leq 1 \end{cases} \]

In both cases number of node points are 200, final time is \(t = 0.3\) and CFL =0.5. Figure 5 (left) shows smooth initial profile (solid line) and MRSU solution after 0.3 seconds with circles. Figure 5 (right) shows discontinuous exact solution (solid line) and MRSU solution (circles).

12.3. Sod’s Shock Tube Problem

In this test case the domain is \([-10, 10]\). Sod’s shock tube problem consists of a left rarefaction, a right shock wave and a contact discontinuity which separates the rarefaction and shock wave. The initial conditions are given by
\[ (\rho, u, p)(x, 0) = \begin{cases} (3.857, 2.629, 10.3333) & \text{for } x < 1 \\ (1 + 0.2 \sin(5x), 0, 1) & \text{for } x \geq 1 \end{cases} \]

The number of node points are 200 and CFL number is 0.2. Final time is \(t = 0.01\). Figure 6 shows the density, velocity, pressure and Mach number plots. The solid line represent the exact solution while the circles are the numerical solution. Here, all the essential features like expansion wave, contact discontinuity and shock wave are captured reasonably well.

12.4. Shock-Entropy Wave Interaction

In this test case a moving shock wave with Mach number 3 interacts with sinusoidal density profile. The domain is \([0, 10]\). The initial conditions are given as
\[ (\rho, u, p)(x, 0) = \begin{cases} (3.857, 2.629, 10.3333) & \text{for } x < 1 \\ (1 + 0.2 \sin(5x), 0, 1) & \text{for } x \geq 1 \end{cases} \]
Final time is 1.8. Number of nodes used are 1000 and CFL number is 0.4. This test case involve both shock wave and smooth profile. MRSU solution (represented by star) is compared with reference solution (solid line).

Variant of this test case is also available which has a domain \([-1, 1]\) and the initial conditions are

\[
(\rho, u, p)(x, 0) = \begin{cases} 
(3.857, 2.629, 10.3333) & \text{for } x < -0.8 \\
(1 + 0.2 \sin(5\pi x), 0, 1) & \text{for } x \geq -0.8
\end{cases}
\]

Final time is \(t=0.47\). Number of nodes used and CFL number are same as before. The proposed scheme captures all the essential flow features of both test cases as shown in figure 7.
12.5. Woodward and Colella Blastwave Problem

It is one of the severe test case used to test the robustness and accuracy of the numerical scheme. The domain is [0, 1]. Initial conditions are given as $\rho = 1$ and $u = 0$ everywhere in the domain. Pressure is given as

\[
p(x, 0) = \begin{cases} 
1000 & \text{for } x \in [0, 0.1] \\
0.01 & \text{for } x \in [0.1, 0.9] \\
100 & \text{for } x \in [0.9, 1] 
\end{cases}
\]

The final time is 0.038. The solution consist of interaction of strong expansion with strong shock and contact waves. MRSU (star) resolves all the flow features with just 1000 nodes as shown in figure 8. The reference solution (shown in solid line) is generated with 10000 nodes Random choice method.

12.6. Dam Break Problem

Figure 9 shows the results of dam break problem. CFL = 0.25, number of nodes = 200 and final time = 50 sec. MRSU scheme can capture all the flow features like expansion region and hydraulic jump reasonably well with such a crude grid shown by circles. The exact solution is given by solid line.

12.7. Extreme Expansion Wave Problem

In this case CFL equals 0.3, number of nodes are 200 and final time is 20 second. Figure 10 shows the results of extreme expansion wave problem. The expansion waves are captured accurately.
12.8. Strong Shock Problem \[61\]

Figure 11 shows the results of strong shock problem. In this case CFL equals 0.4, number of nodes are 200 and final time is 40 second. MRSU scheme can capture discontinuous wavefronts reasonably well.

12.9. 2D Burgers Steady State Test Cases: Normal and Oblique Discontinuities

New set of test cases for two dimensional Burgers equation along with their exact solution are given. The proposed set of test cases can be used to test the accuracy and robustness of numerical algorithm. The domain is \([-0.2, 1] \times [0, 1]\). 2D Burgers equation is given by equation

\[
\frac{\partial U}{\partial t} + \left( \frac{U^2}{2} \right) \frac{\partial U}{\partial x} + \frac{\partial U}{\partial y} = 0
\]  
(15)

with following boundary conditions

\[
U(x, 0) = a \quad \text{for} \quad -0.2 < x < 0 \\
U(x, 0) = b \quad \text{for} \quad 0 < x < 1 \\
U(-0.2, y) = a \\
U(1, y) = b
\]
Now let's consider different values of \( a \) and \( b \) given in table 4 which corresponds to different solution shown in figure 12. From the given figures it is clear that cases (2)-(5) produces oblique discontinuity with decreasing discontinuity angle with respect to horizontal line whereas case (1) produce normal discontinuity. The exact solution is

\[
U(x, y) = \begin{cases} 
  a & \text{if } x < \frac{a+b}{2} y \\
  b & \text{if } x > \frac{a+b}{2} y
\end{cases}
\]  

(16)

72 × 60 and 144 × 120 quadrilateral meshes are used to solve case (1) and (5) using MRSU scheme. Figure 13 show contour plots.

12.10. 2D Burgers Test Case 1

Two dimensional Burgers equations is given by equation 15. The boundary conditions are:

\[
U(0, y) = 1 \text{ and } U(1, y) = -1, \quad 0 < y < 1
\]

and

\[
U(x, 0) = 1 - 2x, \quad 0 < x < 1
\]

Exact solution is given in [49]. The normal discontinuity is well captured using 64 × 64 and 128 × 128 grid as shown in figure 14.
Figure 13: 2D Burgers MRSU solution involving normal discontinuity (case (1), top row) and oblique discontinuity (case (5), bottom row).

Figure 14: 64 × 64 (left) and 128 × 128 (right) quadrilateral mesh.

12.11. 2D Burgers Test Case 2

In this test case domain is same as before but boundary conditions are given by,

$$U(0,y) = 1.5 \quad \text{and} \quad U(1,y) = -0.5, \quad 0 < y < 1$$

and

$$U(x,0) = 1.5 - 2x, \quad 0 < x < 1$$

Exact solution is given in [49]. The oblique discontinuity is captured quite accurately as shown in figure 15 with different grid size.
12.12. 2D Burgers Test Case with Non-convex Flux Function - KKP Rotating Wave [2]

The domain is $[-2, 2] \times [-2.5, 1.5]$. The 2D scalar conservation law with non-convex flux function is

$$\frac{\partial U}{\partial t} + \frac{\partial \sin U}{\partial x} + \frac{\partial \cos U}{\partial y} = 0$$

with initial conditions as

$$U(x, y, 0) = \begin{cases} 
3.5\pi & \text{if } x^2 + y^2 < 1 \\
\frac{\pi}{3} & \text{otherwise}
\end{cases}$$

Figure 16 shows the contour plots on mesh $\Delta x = \Delta y = 1/50$.

12.13. Shock Reflection Test Case [59]

In this test case the domain is rectangular $[0, 3] \times [0, 1]$. The boundary conditions are, inflow (left boundary): $\rho = 1, u_1 = 2.9, u_2 = 0, p = 1/1.4$. Post shock condition (top boundary): $\rho = 1.69997, u_1 = 2.61934, u_2 = -0.50633, p = 1.52819$. Bottom boundary is a solid wall where slip boundary condition is applied, i.e., $u \cdot n = 0$. At right boundary where flow is supersonic all primitive variables $\rho, u_1, u_2$ and $p$ are extrapolated. Pressure plots for $60 \times 20, 120 \times 40$ and $240 \times 80$ quadrilateral mesh along with the comparison of residue plots are given in figure 17.
Figure 17: Pressure contours (0.8:0.1:2.8) 60 × 20, 120 × 40 and 240 × 80 quadrilateral mesh using Q4 element. Residue plots are shown below.

Figure 18: Pressure contours using T3 element.
12.13.1. Oblique Shock Reflection over a Unstructured Triangular Mesh:

For triangular unstructured mesh (number of nodes: 2437 and number of triangles: 4680) the pressure contours are given in figure 18.

12.14. Half Cylinder Test Case

Four supersonic test cases with inflow Mach numbers 2, 3, 6 and 20 are tested on a half cylinder [56]. The domain is half circular. Left outer circle is inflow boundary, small circle inside the domain is a cylinder wall and the straight edges on right side are supersonic outflow boundaries (see figure 19). Pressure plots shows bow shock in front of the half-cylinder which is captured accurately at the right position in each case. These results are compared with existing results [9].

12.15. Double Mach Reflection Test Case

In the initial condition, Mach 10 shock wave makes an angle of 60° with the reflecting wall. The undisturbed air in front of shock has density 1.4 and pressure 1. Initial conditions and boundary conditions are given in [57]. Figure 20 shows density plot for two different meshes.

12.16. 15° Ramp Test Case [38]

In case of supersonic flow over a 15° ramp the inlet (left boundary) Mach number is 2, top and bottom boundaries are inviscid walls and the outlet (right boundary) is supersonic. Oblique shock wave over a wedge hits the top wall and is reflected back. Expansion waves interact with this reflected shock which results in weakening of shock strength. This shock reflects again from the bottom boundary. Figure 21 shows the pressure plots for structured quadrilateral (Q4) mesh.

12.17. Parallel Jet Flow [20]

The domain is [0, 1]² and the initial conditions are given as

\[ M = 4, \, \rho = 0.5 \, p = 0.25 \, \text{if} \, y > 0.5 \]
\[ M = 2.4, \, \rho = 1 \, p = 1 \, \text{if} \, y < 0.5 \]

where \( M \) is Mach number. Left boundary is the supersonic inflow and top, right and bottom boundaries are supersonic outflow where all the variables are extrapolated. Figure 22 shows the density and pressure plots for 51 × 51 and 101 × 101 node points. All the flow features like shock, expansion and contact are resolved quite well using MRSU scheme.

12.18. Modified Parallel Jet Flow

This is a slightly modified parallel jet flow test case which is used to demonstrate the capturing of contact wave when it is not aligned with the mesh. The domain is same but the initial conditions are

\[ M = 4, \, \rho = 0.5 \, p = 0.08 \, \text{if} \, y > 0.5 \]
\[ M = 2.8, \, \rho = 1 \, p = 1.3 \, \text{if} \, y < 0.5 \]

Boundary conditions are same as above. Figure 23 shows the density and pressure plots for 51 × 51 and 101 × 101 node points. The contact wave is captured well using MRSU scheme.

12.19. Circular Explosion Problem [55]

In this test case the domain is \([-1, 1]\)². The initial conditions represents two regions. First is inside the circle with radius 0.4 and second region is outside the circle. The initial conditions are

\[ (\rho, u_1, u_2, p)(x, y, 0) = \begin{cases} (1, 0, 0, 1) & \text{if} \, |r| \leq 0.4 \\ (0.125, 0, 0, 0.1) & \text{otherwise} \end{cases} \]

The final time is 0.2. The solution has circular shock, constantly moving away from the center. The circular expansion fan is moving towards the center. All these features are well captured by MRSU scheme. Figure 24 shows the density, pressure and velocity in x and y direction contours. One can see the shock, contact as well as expansion wave in density contours, whereas, only shock and expansion wave is present in the pressure contours as expected.
12.20. Flow Over a Bump

This is one of the difficult test case due to presence of stagnation point at the front and rear end of the bump. The bump height is 4% of the chord length. For the numerical simulation three different flow fields are considered, namely, Mach 0.5 (subsonic flow), Mach 0.85 (transonic flow) and Mach 1.4 (supersonic flow) over the bump.
12.20.1. Supersonic Flow with $M = 1.4$:

Figure 20 shows the pressure contours for Mach 1.4. In this test case shock appears from the leading edge of the bump which hits the top wall and then reflects back towards bottom wall. The reflected shock interacts with the shock generated from the trailing edge of the bump and it reflects again from the bottom boundary. MRSU captures all the essential flow features accurately.

12.20.2. Transonic Flow with $M = 0.85$:

Figure 21 shows the pressure contours for Mach 0.85 and the variation of Mach number along the bottom wall. The shock appears approximately at 86% of the bump from the front with upstream Mach number approximately 1.3.

12.20.3. Subsonic Flow with $M = 0.5$:

In this test case no shock wave appear. Figure 22 shows the pressure contours for Mach 0.5.

For both transonic as well as subsonic flow over the bump, Riemann invariant based boundary conditions are used.

12.21. Supersonic Flow Over a Reverse Bump

This is a new test case introduced here by reversing the bump. The inlet Mach number is 1.4. Expansion waves originates from the leading edge of the reverse bump. The curved surface of the reverse bump compresses the flow isentropically which generates Mach waves. The Mach waves coalesce to form an oblique shock at an angle of 50° approximately with the horizontal bottom wave which hits the inviscid top wall where no slip boundary condition is
applied. The incident shock wave reflects from the top wall. From the trailing edge of the reverse bump a second expansion wave originates which interacts with the reflected shock wave. The important features of this test case is to capture the shock wave and expansion waves correctly. Figure 28 shows the pressure contours on 240 × 80 quadrilateral mesh.
12.22. 2D Riemann Problems

Three Riemann problems involving all shock waves (case 1 and 2) and all expansion waves (case 3) are considered. Riemann problems are solved on domain $[0, 1]^2$. This square domain is divided into four quadrants where initial constant states are defined. These problems are proposed in such a way that the solution between these quadrant have only one wave like shock, contact etc. The initial conditions for Case 1 is

$$
(p, u_1, u_2, p)(x, 0) = \begin{cases}
(1.1, 0, 0, 1.1) & x \geq 0.5 \text{ and } y \geq 0.5 \\
(0.5065, 0.8939, 0, 0.35) & x \geq 0.5 \text{ and } y < 0.5 \\
(0.5065, 0, 0.8939, 0.35) & x < 0.5 \text{ and } y \geq 0.5 \\
(1.1, 0.8939, 0.8939, 1.1) & x \leq 0.5 \text{ and } y \leq 0.5 
\end{cases}
$$

and for Case 2

$$
(p, u_1, u_2, p)(x, 0) = \begin{cases}
(1.5, 0, 0, 1.5) & x \geq 0.5 \text{ and } y \geq 0.5 \\
(0.5323, 1.206, 0, 0.3) & x \geq 0.5 \text{ and } y < 0.5 \\
(0.5323, 0, 1.206, 0.3) & x < 0.5 \text{ and } y \geq 0.5 \\
(0.138, 1.206, 1.206, 0.029) & x \leq 0.5 \text{ and } y \leq 0.5 
\end{cases}
$$

Finally, for Case 3 the initial conditions are

$$
(p, u_1, u_2, p)(x, 0) = \begin{cases}
(1, 0.75, -0.5, 1) & x \geq 0.5 \text{ and } y \geq 0.5 \\
(2, 0.75, 0.5, 1) & x \geq 0.5 \text{ and } y < 0.5 \\
(3, -0.75, -0.5, 1) & x < 0.5 \text{ and } y \geq 0.5 \\
(1, -0.75, 0.5, 1) & x \leq 0.5 \text{ and } y \leq 0.5 
\end{cases}
$$

Figure 23 and 24 shows the density contours for case 1, 2 and case 3 respectively.

12.23. Circular Dam Break Test Case

In this test case the domain is $[-1, 1]^2$. The circular dam is placed at the center. The initial water height is

$$
h = \begin{cases}
2 & \text{If } x^2 + y^2 < 0.2 \\
1 & \text{Otherwise}
\end{cases}
$$

and $u_1 = u_2 = 0$. All boundary conditions are transmissive. Sudden breaking of circular dam creates radially outward propagating hydraulic jump and radially inward propagating depression wave. These waves are captured well as shown in figure 25.
12.24. Hydraulic Jump

This test case gives a hydraulic jump in a convergent wall section. In this case the flow is supercritical. The computational domain is \([0, 40] \times [0, 30]\) and it is discretized using \(80 \times 60\) node points. The initial conditions are, \(h = 1\ m\), \(u_1 = 8.57\ m/s\) and \(v_2 = 0\). The wall angle for the convergent section is \(8.95^\circ\). The boundary conditions are reflective at top and bottom walls whereas supercritical boundary conditions are imposed on inflow and outflow.
12.25. Hydraulic Jump Interaction in a Convergent Channel

This test case is a slight modification of previous test case where the walls are converging from both top and bottom with fixed angle 8.95° which form a convergent channel. Again, the flow is supercritical. The computational domain is [0, 50] × [0, 30] and it is discretized using 100 × 60 node points. The initial conditions are same as before. The boundary conditions are reflective at top and bottom walls whereas supercritical boundary conditions are applied on the inflow and outflow boundaries. Hydraulic jump generated from top and bottom wall intersects each other and then reflects from the top and bottom boundary. This interaction of hydraulic jump is captured well by MRSU scheme. Both contour plots for height and surface plots are given in figure 33.

12.26. Partial Dam Break Test Case [1]

This is a 2D test case for shallow water flows which simulates the partial dam break due to sudden opening of sluice gate in a rectangular channel. The domain is [0, 200 m]² and the length of sluice gate is 75m long as shown in
first figure of 34. The initial condition is

\[
h = \begin{cases} 
10 \text{ m} & \text{if } x < 100 \text{ m} \\
5 \text{ m} & \text{otherwise}
\end{cases}
\]

and \( u_1 = u_2 = 0 \). Boundary conditions are transmissive on left as well as right boundary, whereas reflective boundary conditions are used for remaining boundaries. Strong bore in the upstream and negative waves in the downstream directions are created due to sudden opening of sluice gate. The solution is plotted at \( t = 7.2 \text{ sec} \). The unstructured triangular mesh is used for the simulation. The number of nodes are 4238 and the number of elements are 8114. Second and third figure of 34 shows the contour with velocity vector plots and surface plot respectively.
Figure 30: Density contours for 2D Riemann problem case 3 using 400 × 400 mesh.

Figure 31: Dam break problem height contours (1.075:0.075:1.93) and surface plot using 100 × 100 quadrilateral mesh.

Figure 32: Hydraulic jump using 80 × 60 Q4 mesh and residue plot.
Figure 33: Contour and surface plots for hydraulic jump interaction problem using $100 \times 60$ mesh.

Figure 34: Unstructured mesh, contour plot along with velocity vector plot and surface plot for dam break problem.

13. Conclusions

A new stabilized finite element method namely Method of Relaxed Streamlined Upwinding is proposed for hyperbolic conservation laws. The proposed scheme is based on relaxation system which replaces hyperbolic conservation laws by semi-linear system with stiff source term. Six symmetric discrete velocity models are also proposed for two dimensional relaxation system which symmetrically spread foot of the characteristics in all four quadrants thereby
taking information symmetrically from all directions. Main aim of this paper is to introduce a new idea based on relaxation scheme which is not only potentially interesting but also easy to implement. There are several advantages of the proposed scheme.

1. MRSU gives exact diffusion vectors which is required for stabilization of the numerical scheme.
2. Diffusion vectors in MRSU scheme are simply solution vectors with supremum eigenvalue as a coefficient. This makes the scheme robust. Moreover, no complicated Jacobian matrices are involved in the diffusion terms.
3. Extension of MRSU scheme from scalar to vector conservation laws is direct.

The efficacy of the proposed scheme is shown by solving various one and two dimensional test cases for Burgers equation (which includes both convex and non-convex flux function), Euler equations and shallow water equations. Moreover, few new test cases are proposed for all the three equations. In case of two dimensional Burgers equation, set of test cases involving normal and oblique discontinuity are proposed along with the exact solutions. For Euler equations the variations in Mach number is from 0.5 to 20 which covers subsonic, transonic and supersonic regime. This shows the capability of proposed scheme in handling wide range of problems. Moreover, spectral stability analysis is carried out which gives expression for critical time step. Error analysis shows optimal convergence rate for the proposed scheme. The proposed scheme is easy to implement in the existing code of stabilized finite element methods without much modification. Also, extension to higher order MRSU scheme using higher order elements in space and higher order time integration would be straightforward.

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Appendix

Derivation of local Maxwellian Distribution (F) for one dimensional three discrete velocity model

Let $F = [F_1, F_2, F_3]^T$ be the local Maxwellian distribution function then, the moment relations (equation 9) are satisfied as

$$F_1 + F_2 + F_3 = U \quad (17)$$
$$\lambda F_1 + \lambda F_2 + \lambda F_3 = G(U)$$

Using three discrete velocity model the $\Lambda$ matrix is given as

$$\Lambda = \begin{bmatrix} -\lambda & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \lambda \end{bmatrix}$$

which makes the last moment relation as

$$F_3 - F_1 = \frac{G(U)}{\lambda} \quad (18)$$

Assuming $F$ as a linear combination of $U$ and $G(U)$ as

$$F = \begin{bmatrix} F_1 \\ F_2 \\ F_3 \end{bmatrix} = \begin{bmatrix} c_1 U + d_1 G(U) \\ c_2 U + d_2 G(U) \\ c_3 U + d_3 G(U) \end{bmatrix}$$
Using moment relations (17) and (18) following relations are obtained

\[
\sum_{i=1}^{3} c_i = 1 \\
\sum_{i=1}^{3} d_i = 0 \\
(c_3 - c_1) U + (d_3 - d_1) G(U) = \frac{G(U)}{\lambda}
\]

Choosing \( c_1 = c_2 = c_3 = \frac{1}{3} \) and \( d_1 = \frac{1}{2}, d_2 = 0 \), this gives

\[
F = \left\{ \begin{array}{l} 
\frac{U}{3} - \frac{G}{21} \\
\frac{U}{3} + \frac{G}{21}
\end{array} \right. 
\]

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