NysADMM: faster composite convex optimization via low-rank approximation

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Abstract
This paper develops a scalable new algorithm, called NysADMM, to minimize a smooth convex loss function with a convex regularizer. NysADMM accelerates the inexact Alternating Direction Method of Multipliers (ADMM) by constructing a preconditioner for the ADMM subproblem from a randomized low-rank Nyström approximation. NysADMM comes with strong theoretical guarantees: it solves the ADMM subproblem in a constant number of iterations when the rank of the Nyström approximation is the effective dimension of the subproblem regularized Gram matrix. In practice, ranks much smaller than the effective dimension can succeed, so NysADMM uses an adaptive strategy to choose the rank that enjoys analogous guarantees. Numerical experiments on real-world datasets demonstrate that NysADMM can solve important applications, such as the lasso, logistic regression, and support vector machines, in half the time (or less) required by standard solvers. The breadth of problems on which NysADMM beats standard solvers is a surprise: it suggests that ADMM is a dominant paradigm for numerical optimization across a wide range of statistical learning problems that are usually solved with bespoke methods.

1. Introduction
Consider the composite convex optimization problem
\[
\text{minimize}_{x \in \mathbb{R}^d} \ell(Ax; b) + r(x). \quad (1)
\]
We assume that \(\ell\) and \(r\) are convex and \(\ell\) is smooth. In machine learning, generally \(\ell\) is a loss function, \(r\) is a regularizer, \(A \in \mathbb{R}^{n \times d}\) is a feature matrix, and \(b \in \mathbb{R}^n\) is the label or response. Throughout the paper we assume that a solution to (1) exists. A canonical example of (1) is the lasso problem,
\[
\text{minimize} \; \frac{1}{2} \|Ax - b\|_2^2 + \gamma \|x\|_1, \quad (2)
\]
where \(\ell(Ax; b) = \frac{1}{2}\|Ax - b\|_2^2\) and \(r(x) = \gamma \|x\|_1\). We discuss more applications of (1) in Section 3.

The alternating directions method of multipliers (ADMM) is a popular algorithm to solve optimization problems of the form (1). However, when the matrix \(A\) is large, each iteration of ADMM requires solving a large subproblem. For example, consider the lasso where the loss \(\ell\) is quadratic. At each iteration, ADMM solves a regularized least-squares problem at a cost of \(O(nd^2)\) flops. On the other hand, it is not necessary to solve each subproblem exactly to ensure convergence: ADMM strategies that solve the subproblems inexactly are called inexact ADMM, and can be shown to converge when the sequence of errors is summable (Eckstein & Bertsekas, 1992). Unfortunately, it can be challenging even to satisfy this relaxed criterion. Consider again the lasso problem. At each iteration, inexact ADMM solves the regularized least-squares subproblem (4) approximately, for example, using the iterative method of conjugate gradients (CG). We call this method inexact ADMM with CG. The number of CG iterations required to achieve accuracy \(\epsilon\) increases with the square root of the condition number \(\kappa_2\) of the regularized Hessian, \(O(\sqrt{\kappa_2} \log(\frac{\kappa_2}{\epsilon}))\). Alas, the condition number of large-scale data matrices is generally high, and later iterations of inexact ADMM require high accuracy, so inexact ADMM with CG still converges too slowly to be practical.

In this work we show how to speed up inexact ADMM using preconditioned conjugate gradients (PCG) as a subproblem solver. We precondition with randomized Nyström preconditioning (Frangella et al., 2021), a technique inspired by recent developments in randomized numerical linear algebra (RandNLA). We call the resulting algorithm NysADMM (“nice ADMM”): inexact ADMM with PCG using randomized Nyström preconditioning. The Nyström preconditioner reduces the number of iterations required to solve the subproblem to \(\epsilon\)-accuracy to \(O(\log(\frac{1}{\epsilon}))\), independent of the condition number. For non-quadratic loss functions, NysADMM uses linearized inexact ADMM and accelerates the linear subproblem solve similarly.
1.1. Contributions

1. We provide a general algorithmic framework for solving large scale lasso, $\ell_1$-regularized logistic regression, and SVM problems.

2. Our theory shows that at each iteration only a constant number of matrix vector products (matvecs) are required to solve the ADMM subproblem, provided we have constructed the preconditioner appropriately. If the loss function is quadratic, only a constant number of matvecs are required to achieve convergence.

3. We develop a practical adaptive algorithm that increases the rank until the conditions of our theory are met, which ensures the theoretical benefits of the method can be realized in practice.

4. Even a preconditioner with lower rank often succeeds in speeding up inexact ADMM with PCG. Our analysis is also able to explain this phenomenon.

5. Our algorithm beats standard solvers such as glmnet, SAGA, and LIBSVM on large dense problems like lasso, logistic regression, and kernalized SVMs: it yields equally accurate solutions and often runs 2–4 times faster.

1.2. Related work

Our work relies on recent advancements in RandNLA for solving regularized least squares problems $(A^T A + \mu I)^x = A^T b$ for $x$, given a design matrix $A \in \mathbb{R}^{n \times d}$, righthand side $b \in \mathbb{R}^n$, and regularization $\mu \in \mathbb{R}$, using a sketch of the design matrix $A$ (Lacotte & Pilanci, 2020). NysADMM adapts the randomized Nyström preconditioner of (Frangella et al., 2021). These algorithms begin by forming a sketch $Y = A \Omega^T$ of $A$ (or $A^T$) with a random dimension reduction map $\Omega \in \mathbb{R}^{d \times s}$ (Martinsson & Tropp, 2020; Woodruff, 2014). For example, $\Omega$ may be chosen to have iid Gaussian entries. These algorithms obtain significant computational speedups by using a sketch size $s \ll \min\{n, d\}$ and working with the sketch in place of the original matrix to construct a preconditioner for the linear system. Frangella et al. (2021) and Lacotte & Pilanci (2020) show that these randomized preconditioners work well when the sketch size grows with the effective dimension (Equation (8)) of the Gram matrix (assuming, for Lacotte & Pilanci (2020) that we have access to a matrix square root). As the effective dimension is never larger than $d$ and often significantly smaller, these results substantially improve on prior work in randomized preconditioning (Meng et al., 2014; Rokhlin & Tygert, 2008) that requires a sketch size $s \gtrsim d$. Many applications require even smaller sketch sizes: for example, for NysADMM, a fixed sketch size $s = 50$ suffices even for extremely large problems.

We are not the first to use RandNLA to accelerate iterative optimization. Gower et al. (2019); Pilanci & Wainwright (2017) both use iterative sketching to accelerate Newton’s method, while Chowdhuri et al. (2020) use randomized preconditioning to accelerate interior point methods for linear programming. The approach taken here is closest in spirit to (Chowdhuri et al., 2020), as we also use randomized preconditioning. However, the preconditioner used in (Chowdhuri et al., 2020) requires the data matrix to have many more columns than rows, while ours can handle any (sufficiently large) dimensions.

NysADMM can solve many traditional machine learning problems, such as lasso, regularized logistic regression, and support vector machines (SVMs). In contrast, standard solvers for these problems use a wider variety of convex optimization techniques. For example, one popular lasso solver, glmnet (Friedman et al., 2010), relies on coordinate descent (CD), while solvers for SVMs, such as LIBSVM (Chang & Lin, 2011), more often use sequential minimal optimization (Platt, 1998), a kind of pairwise CD on the dual problem. For regularized logistic regression, especially for $\ell_1$ regularization, stochastic gradient algorithms are most commonly used (Defazio et al., 2014; Schmidt et al., 2017). Other authors propose to solve lasso with ADMM (Boyd et al., 2011; Yue et al., 2018). Our work, motivated by the ADMM quadratic programming framework of Stellato et al. (2020), is the first to accelerate ADMM with randomized preconditioning, thereby improving on the performance of standard CD or stochastic gradient solvers for each of these important classes of machine learning problems on large-scale dense data. Unlike Stellato et al. (2020), our work relies on inexact ADMM and can handle non-quadratic loss functions, which allows NysADMM to solve problems such as regularized logistic regression.

1.3. Organization of the paper

Section 2 introduces the NysADMM algorithm and necessary background from RandNLA. Section 3 lists a variety of applied problems that can be solved by NysADMM. Section 4 states the theoretical guarantees for NysADMM. Section 5 compares NysADMM and standard optimization solvers numerically on several applied problems. Section 6 summarizes the results of the paper and discusses directions for future work.

1.4. Notation and preliminaries

We call a matrix psd if it is positive semidefinite. The notation $a \geq b$ means that $a \geq Cb$ for some absolute constant $C$. Given a matrix $H$, we denote its spectral norm by $\|H\|$. We denote the Moore-Penrose pseudoinverse of a matrix $M$ by $M^\dagger$. For $\rho > 0$ and a symmetric psd matrix $H$, we define $H_\rho = H + \rho I$. We say a positive sequence
\( \{e^k\}_{k=1}^{\infty} \) is summable if \( \sum_{k=1}^{\infty} e^k < \infty \). We denote the Loewner ordering on the cone of symmetric psd matrices by \( \preceq \), that is \( A \preceq B \) if and only if \( B - A \) is psd.

2. Algorithm

2.1. Inexact linearized ADMM

To solve problem (1), we apply the ADMM framework. Algorithm 1 shows the standard ADMM updates, where the regularizer \( r = g + h \) is split into a smooth part \( g \) and a nonsmooth part \( h \).

Algorithm 1 ADMM

input feature matrix \( A \), response \( b \), loss function \( \ell \), regularization \( g \) and \( h \), stepsize \( \rho \)
repeat
\[
\hat{x}^{k+1} = \arg\min_x \{ \ell(Ax; b) + g(x) + \frac{\rho}{2} \| x - \hat{x}^k + u^k \|_2^2 \} \\
\tilde{x}^{k+1} = \arg\min_z \{ h(z) + \frac{\rho}{2} \| z - u^k \|_2^2 \} \\
u^{k+1} = u^k + \tilde{x}^{k+1} - \hat{x}^{k+1}
\]
until convergence
output solution \( x^* \) of problem (1)

In each iteration, two subproblems are solved sequentially to update variables \( x \) and \( z \). The \( z \)-subproblem often has a closed-form solution. For example, if \( h(x) = \| x \|_1 \), the \( z \)-subproblem is the soft thresholding, and if \( h \) is the indicator function of a convex set \( C \), the \( z \)-subproblem is projection onto the set \( C \).

There is usually no closed-form solution for the \( x \)-subproblem. Instead, it is usually solved inaccurately by an iterative scheme, especially for large-scale applications. To simplify the subproblem, inspired by linearized ADMM, we assume \( \ell \) and \( g \) are twice differentiable and notice that the \( x \) update is close to the minimum of a quadratic function given by the Taylor expansion of \( \ell \) and \( g \) at the current iterate:

\[
\hat{x}^{k+1} = \arg\min_x \{ \ell(A\hat{x}^k; b) + (x - \hat{x}^k)^T A^T \nabla \ell(A\hat{x}^k; b) \\
+ \frac{1}{2} (x - \hat{x}^k)^T A^T H^\ell(A\hat{x}^k; b) A (x - \hat{x}^k) + g(\hat{x}^k) \\
+ (x - \hat{x}^k)^T \nabla g(\hat{x}^k) + \frac{1}{2} (x - \hat{x}^k)^T H^g(\hat{x}^k)(x - \hat{x}^k) \\
+ \frac{\rho}{2} \| x - \hat{x}^k + u^k \|_2^2 \}.
\]  

Here \( H^\ell \) and \( H^g \) are the Hessian of \( \ell \) and \( g \) respectively. We assume throughout the paper that \( H^\ell \) and \( H^g \) are psd matrices, this is a very minor assumption, and is satisfied by all the applications we consider. The solution to this quadratic minimization may be obtained by solving the linear system

\[
(A^T H^\ell(A\hat{x}^k; b) A + H^g(\hat{x}^k) + \rho I)x = r^k
\]  

where

\[
r^k = \rho z^k - \rho \hat{x}^k + A^T H^\ell(A\hat{x}^k; b) A \hat{x}^k + H^g(\hat{x}^k) \hat{x}^k - A^T \nabla \ell(A\hat{x}^k; b) - \nabla g(\hat{x}^k).
\]

The inexact ADMM algorithm we propose solves (4) approximately at each iteration.

Algorithm 2 Inexact ADMM

input feature matrix \( A \), response \( b \), loss function \( \ell \), regularization \( g \) and \( h \), stepsize \( \rho \), positive summable sequence \( \{e^k\}_{k=0}^{\infty} \)
repeat
\[
\tilde{x}^{k+1} = \arg\min_x \{ \ell(A\tilde{x}^k; b) + \frac{\rho}{2} \| \tilde{x}^k - x + u^k \|_2^2 \} \\
u^{k+1} = u^k + \tilde{x}^{k+1} - \tilde{x}^{k+1}
\]
until convergence
output solution \( x^* \) of problem (1)

For a quadratic loss \( \ell \), when \( \sum_{k=0}^{\infty} e^k < \infty \) and under various other conditions, if optimization problem (1) has an optimal solution, the \( \{e^k\}_{k=0}^{\infty} \) sequence generated by Algorithm 2 converges to the optimal solution of (1) (Eckstein & Bertsekas, 1992; Eckstein & Yao, 2016).

From Boyd et al. (2011), quantity \( r_{d,k+1}^p = \rho (\tilde{z}^k - \tilde{z}^{k+1}) \) can be regarded as the dual residual and \( r_{d,k+1}^p = \tilde{z}^{k+1} - \tilde{z}^{k+1} \) can be viewed as the primal residual at iteration \( k + 1 \). This suggests that we can terminate the ADMM iterations when the primal and dual residuals become very small. The primal and dual tolerances can be chosen based on an absolute and relative criterion, such as

\[
\| r_{d,k}^p \|_2 \leq e^{abs} + e^{rel} \max \{ \| \tilde{z}^k \|_2, \| \tilde{z}^k \|_2 \} \\
\| r_{d,k}^p \|_2 \leq e^{abs} + e^{rel} \| \tilde{u}^k \|_2.
\]

The relative criteria \( e^{rel} \) might be \( 10^{-3} \) or \( 10^{-4} \) in practice. The choice of absolute criteria \( e^{abs} \) depends on the scale of the variable values. More details can be found in Boyd et al. (2011).

2.2. Randomized Nyström approximation and PCG

Nyström approximation constructs a low-rank approximation of a symmetric psd matrix \( H \). Let \( \Omega \in \mathbb{R}^{d \times s} \) be a test matrix (often, random Gaussian Frangella et al. (2021); Tropp et al. (2017)) with sketch size \( s \geq 1 \). The Nyström approximation with respect to \( \Omega \) is given by

\[
H(\Omega) = (H\Omega)(\Omega^T H\Omega)^{-1}(H\Omega)^T.
\]

The Nyström approximation \( H(\Omega) \) is symmetric, psd, and has rank at most \( s \) Lemma A.1. Naive implementation of the
NysADMM

Nyström approximation based on (6) is numerically unstable. Algorithm 4 in Appendix B states a stable procedure to compute a randomized Nyström approximation from Tropp et al. (2017).

Algorithm 4 returns the randomized Nyström approximation of matrix $H$ in the form of an eigendecomposition: $H_{\text{nys}} = U \Lambda U^T$. Let $\lambda_s$ be the $\text{th}$ eigenvalue. The randomized Nyström preconditioner and its inverse take the form

$$P = \frac{1}{\lambda_s + \rho} U (\Lambda + \rho I) U^T + (I - UU^T),$$

$$P^{-1} = (\lambda_s + \rho) U (\Lambda + \rho I)^{-1} U^T + (I - UU^T)$$  \hspace{1cm} (7)

(Frangella et al., 2021). In a slight abuse of terminology, we sometimes refer to the sketch size $s$ as the rank of the Nyström preconditioner. We will use the term sketch size and rank interchangeably throughout the paper. The Nyström preconditioner may be applied to vectors in $O(ds)$ time and only requires $O(ds)$ floating point numbers to store. The details of how to implement PCG with (7) are provided in Appendix B in Algorithm 5. We now provide some background on Nyström PCG and motivation for why we have paired it with ADMM.

Nyström PCG improves on standard CG both in theory and in practice for matrices with a small effective dimension (Frangella et al., 2021), which we now define. Given a symmetric psd matrix $H \in \mathbb{R}^{d \times d}$ and regularization $\rho > 0$, the effective dimension of $H$ is

$$d_{\text{eff}}(\rho) = \text{tr}(H (H + \rho I)^{-1}).$$  \hspace{1cm} (8)

The effective dimension may be viewed as smoothed count of the eigenvalues of $H$ greater than or equal to $\rho$. We always have $d_{\text{eff}}(\rho) \leq d$, and we expect $d_{\text{eff}}(\rho) \ll d$ whenever $H$ exhibits spectral decay.

In machine learning, most feature matrices naturally exhibit polynomial or exponential spectral decay (Derezinski et al., 2020), thus we expect that $d_{\text{eff}} \ll d$. The randomized Nyström preconditioner in Frangella et al. (2021) exploits the smallness of $d_{\text{eff}}(\rho)$ to build an highly effective preconditioner. Frangella et al. (2021) show that if (7) is constructed with a sketch size $s \geq d_{\text{eff}}(\rho)$, then the condition number of the preconditioned system is constant with high probability. An immediate consequence is that PCG solves the preconditioned system to $\epsilon$-accuracy in $O(\log( \frac{1}{\epsilon} ))$ iterations, independent of the condition number of $H$.

Observe the Hessian $A^T H^T A + H^g$ in the inexact ADMM subproblem (4) is formed from the feature matrix $A$. Based on the preceding discussion, we expect the Hessian to exhibit spectral decay and for the effective dimension to be small to moderate in size. Hence we should expect Nyström PCG to accelerate the solution of (4) significantly.

2.3. NysADMM

Integrating Nyström PCG with inexact ADMM, we obtain NysADMM, presented in Algorithm 3.

**Algorithm 3 NysADMM**

**Input** feature matrix $A$, response $b$, loss function $\ell$, regularization $g$ and $h$, stepsize $\rho$, positive summable sequence $\{\epsilon_k\}_{k=0}^\infty$

$[U, \Lambda] = \text{RandNyströmApprox}(A^T H^T A + H^g, s)$ \{use Algorithm 4 in Appendix B\}

**Repeat**

use Nyström PCG (Algorithm 5 in Appendix B) to find $\tilde{z}^{k+1}$ that solves (4) within tolerance $\epsilon_k$

$\tilde{z}^{k+1} = \text{argmin}_z \{ \ell (z) + \frac{\rho}{2} \| \tilde{z}^{k+1} - z + \tilde{u}^{k} \|^2 \}$

$\tilde{u}^{k+1} = \tilde{u}^{k} + \tilde{z}^{k+1} - \tilde{z}^{k}$

**Until** convergence

**Output** solution $x_*$ of problem (1)

Our theory for Algorithm 3, shows that if the sketch size $s \geq d_{\text{eff}}(\rho)$, then with high probability subproblem (4) will be solved to $\epsilon$-accuracy in $O(\log( \frac{1}{\epsilon} ))$ iterations (Corollary 4.2). When the loss $\ell$ is quadratic and the sequence of tolerances $\{\epsilon_k\}_{k=0}^\infty$ is decreasing with $\sum_{k=0}^\infty \epsilon_k < \infty$, NysADMM is guaranteed to converge as $k \to \infty$ with only a constant number of matvecs per iteration (Theorem 4.3). Table 1 compares the complexity of inexact ADMM with CG vs. NysADMM for $K$ iterations under the hypotheses of Theorem 4.3. NysADMM achieves a significant decrease in runtime over inexact ADMM with CG, as the iteration complexity no longer depends on the condition number $\kappa_2$.

| Method                | Complexity                                      |
|-----------------------|-------------------------------------------------|
| Inexact ADMM with CG  | $O \left( \sum_{k=1}^K T_{\text{mv}} \sqrt{\kappa_2 \log \left( \frac{\epsilon_k}{\epsilon_{K+1}} \right)} \right)$ |
| NysADMM               | $O \left( T_{\text{mv}} d_{\text{eff}}(\rho) + \sum_{k=1}^K T_{\text{mv}} \left( 4 + 2 \log \left( \frac{R}{\epsilon_{K+1}} \right) \right) \right)$ |

2.4. AdaNysADMM

Two practical problems remain in realizing the success predicted by the theoretical analysis of Table 1. These bounds are achieved by selecting the sketch size to be $d_{\text{eff}}(\rho)$, but the effective dimension is 1) seldom known in practice, and 2) often larger than required to achieve good convergence of NysADMM. Fortunately, a simple adaptive strategy for choosing the sketch size, inspired by Frangella et al. (2021), can achieve the same guarantees as in Table 1. This strategy
chooses a tolerance $\epsilon$ and doubles the sketch size $s$ until the empirical condition number $\frac{\hat{\lambda}_s + \rho}{\rho}$ satisfies

$$\frac{\hat{\lambda}_s + \rho}{\rho} \leq 1 + \epsilon.$$  

Theorem 4.4 guarantees that (9) holds when $s \geq d_{\text{eff}}(\rho)$ and that when (9) holds, the true condition number is on the order of $1 + \epsilon$ with high probability. We refer to (9) as the empirical condition number as it provides an estimate of the true condition number of the preconditioned system (Theorem 4.4).

Thus, to enjoy the guarantees of Theorem 4.4 in practice, we may employ the adaptive version of NysADMM, which we call AdaNysADMM. We provide the pseudocode for AdaNysADMM in Algorithm 7 in Appendix B. Furthermore, as we use a Gaussian test matrix, it is possible to construct a larger sketch from a smaller one. Hence the total computational work needed by the adaptive strategy is not much larger than if the effective dimension were known in advance. Indeed, AdaNysADMM differs from NysADMM only in the construction of the preconditioner. The dominant cost in forming the precondition is computing the sketch is $H\Omega$, which costs $O(T_{mv} d_{\text{eff}}(\rho))$. As AdaNysADMM reuses computation, the dominant complexity for constructing the Nyström preconditioner remains $O(T_{mv} d_{\text{eff}}(\rho))$. Consequently, the overall complexity of AdaNysADMM is the same as NysADMM in Table 1.

3. Applications

Here we discuss various applications that can be reformulated as instances of (1) and solved by Algorithm 3.

3.1. Elastic net

Elastic net generalizes lasso and ridge regression by adding both the $\ell_1$ and $\ell_2$ penalty to the least squares problem:

$$\text{minimize} \quad \frac{1}{2} ||Ax - b||^2_2 + \frac{1}{2}(1 - \gamma)||x||^2_2 + \gamma ||x||_1.$$

(10)

Parameter $\gamma > 0$ interpolates between the $\ell_1$ and $\ell_2$ penalties. NysADMM applies with $\ell(Ax; b) = \frac{1}{2} ||Ax - b||^2_2$, $g(x) = \frac{1}{2}(1 - \gamma)||x||^2_2$, and $h(x) = \gamma ||x||_1$. The Hessian matrices for $\ell$ and $g$ are $A^T A$ and $(1 - \gamma) I$ respectively.

3.2. Regularized logistic regression

Regularized logistic regression minimizes a logistic loss function together with an $\ell_1$ regularizer:

$$\text{minimize} \quad -\sum_i (b_i(Ax)_i - \log(1 + \exp((Ax)_i))) + \gamma ||x||_1.$$

NysADMM applies with $\ell(Ax; b) = -\sum_i (b_i(Ax)_i - \log(1 + \exp((Ax)_i)))$ and $h(x) = \gamma ||x||_1$. The inexact ADMM update chooses $x^{k+1}$ to minimize a quadratic approximation of the log-likelihood,

$$\text{minimize} \quad \frac{1}{2} \sum_i w_i^k (q_i^k - (Ax)_i)^2 + \frac{\rho}{2} ||x - z^k + u^k||^2_2,$$

where $w_i^k$ and $q_i^k$ depend on the current estimate $z^k$ as

$$w_i^k = \frac{1}{2 + \exp(-(Ax)^k)_i + \exp((Ax)^k)_i},$$

$$q_i^k = (Ax)^k_i + \frac{1}{2 + \exp(-(Ax)^k)_i} w_i^k.$$

Therefore, the solution of the $x$-subproblem can be approximated by solving the linear system

$$(A^T \text{diag}(w^k) A + \rho I) x = \rho \tilde{z}^k - \rho \tilde{u}^k + A^T \text{diag}(w^k) q^k.$$

Here $w^k$ and $q^k$ are the vectors for $w_i^k$ and $q_i^k$. The Hessian matrix of $\ell$ is given by $A^T \text{diag}(w^k) A$.

3.3. Support vector machine

To reformulate the SVM problem for solution with NysADMM, consider the dual SVM problem

$$\text{minimize} \quad \frac{1}{2} x^T \text{diag}(b) K \text{diag}(b) x - 1^T x$$

subject to $x^T b = 0$

$$0 \leq x \leq C.$$  

(12)

Variable $x$ is the dual variable, $b$ is the label or response, and $C$ is the penalty parameter for misclassification. For linear SVM, $K = A^T A$ where $A$ is a feature matrix; and for nonlinear SVM, $K$ is the corresponding kernel matrix. The SVM problem can be reformulated as (1) by setting $\ell(Ax; b) = \frac{1}{2} x^T \text{diag}(b) K \text{diag}(b) x$, $g(x) = -1^T x$, and $h$ is the indicator function for convex constraint set $x^T b = 0$, $0 \leq x \leq C$. The Hessian matrix for $\ell$ is $\text{diag}(b) K \text{diag}(b)$.

4. Convergence analysis

This section provides a convergence analysis for NysADMM. All proofs for the results in this section may be found in Appendix A. First we show Nyström PCG can solve any quadratic problem in a constant number of iterations.

**Theorem 4.1.** Let $H$ be a symmetric positive semidefinite matrix, $\rho > 0$ and set $H_\rho = H + \rho I$. Suppose we construct the randomized Nyström preconditioner with sketch size $s \geq 8 \left( \sqrt{d_{\text{eff}}(\rho)} + \sqrt{8 \log \left( \frac{16}{\delta} \right)} \right)^2$. Then

$$\kappa_2(P^{-1/2} H_\rho P^{-1/2}) \leq 8$$

with probability at least $1 - \delta$.
Theorem 4.1 strengthens results in Frangella et al. (2021), which provides sharp expectation bounds on the condition number of the preconditioned system, but gives loose high probability bounds based on Markov’s inequality. Our result tightens these bounds, showing that Nyström PCG enjoys an exponentially small failure probability.

As an immediate corollary, we can solve (4) with a few iterations of PCG using the Nyström preconditioner.

**Corollary 4.2.** Instate the hypotheses of Theorem 4.1 and let \( \tilde{x}_s \) denote the solution of (4). Then with probability at least \( 1 - \delta \), the iterates \( \{x_k\}_{k \geq 1} \) produced by Nyström PCG on problem (4) satisfy

\[
\frac{\|x_k - \tilde{x}_s\|_2}{\|\tilde{x}_s\|_2} \leq \left( \frac{1}{2} \right)^{t-1}.
\]

Thus, after

\[
t \geq \left\lceil \log \left( \frac{8\delta + \|g\|_2}{\log(2)} \right) \right\rceil
\]

iterations,

\[
\frac{\|x_k - \tilde{x}_s\|_2}{\|\tilde{x}_s\|_2} \leq \epsilon.
\]

Corollary 4.2 ensures that we can efficiently solve the subproblem to the necessary accuracy at each iteration. This result allows us to prove convergence of Nyström ADMM.

**Theorem 4.3.** Consider the problem in (1) with quadratic loss \( f(Ax; b) = \frac{1}{2} \|Ax - b\|_2^2 \) and the smooth part \( g \) of regularizer \( r \) has constant Hessian. Define initial iterates \( \tilde{x}^0, \tilde{z}^0, \tilde{u}^0 \in \mathbb{R}^d \), stepsize \( \rho > 0 \), and summable tolerance sequence \( \{\epsilon_k\}_{k=0}^\infty \subseteq \mathbb{R}_+ \). Assume at \( k \)th ADMM iteration, the norm of the righthand side of the linear system \( r_k \) is bounded by constant \( R \) for all \( k \). Construct the Nyström preconditioner with sketch size

\[
s \geq 8 \left( \sqrt{d_{\text{eff}}(\rho)} + \sqrt{8 \log \left( \frac{16}{\delta} \right)} \right)^2
\]

and solve problem (1) with Nyström ADMM, using \( T^k = 4 + \left\lceil 2 \log \left( \frac{R}{\sqrt{\rho}} \right) \right\rceil \) iterations for PCG at the \( k \)th ADMM iteration. Then with probability at least \( 1 - \delta \),

1. For all \( k \geq 0 \), each iterate \( \tilde{x}^{k+1} \) satisfies

\[
\frac{\|x^{k+1} - \tilde{x}^{k+1}\|_2}{\|\tilde{x}^{k+1}\|_2} \leq \epsilon^k,
\]

where \( x^{k+1} \) is the exact solution of (4).

2. As \( k \to \infty \), \( \{\tilde{x}^k\}_{k=0}^\infty \) converges to a solution of the primal (1) and \( \{\rho u^k\}_{k=0}^\infty \) converges to a solution of the dual problem of (1).

Theorem 4.3 establishes convergence of Nyström ADMM for a quadratic loss. The quadratic loss already covers many applications of interest including the lasso, elastic-net, and SVMs. We conjecture that a modification of our argument can show that Nyström ADMM converges linearly for any strongly convex loss, but we leave this extension to future work.

The next result makes rigorous the claims made in Section 2.4: it shows we can determine whether or not we have reached the effective dimension by monitoring the empirical condition number \( (\tilde{\lambda}_s + \rho)/\rho \).

**Theorem 4.4.** Suppose, for some user defined tolerance \( \epsilon > 0 \), the sketch size satisfies

\[
s \geq 8 \left( \sqrt{d_{\text{eff}}(\rho)} + \sqrt{8 \log \left( \frac{16}{\delta} \right)} \right)^2.
\]

Then the empirical condition number of the Nyström preconditioned system \( P^{-1/2}H_\rho P^{-1/2} \) satisfies

\[
\frac{\hat{\lambda}_s + \rho}{\rho} \leq 1 + \frac{\epsilon}{42}.
\]

Furthermore, with probability at least \( 1 - \delta \),

\[
\begin{align*}
&\frac{\|x^{\infty} - \tilde{x}_s\|_2}{\|\tilde{x}_s\|_2} \leq \epsilon, \\
&\frac{\kappa_2(P^{-1/2}H_\rho P^{-1/2} - \hat{\lambda}_s + \rho/\rho)}{\rho} \leq \epsilon.
\end{align*}
\]

Theorem 4.4 shows that once the empirical condition number is sufficiently close to 1, so too is the condition number of the preconditioned system. Hence it is possible to reach the effective dimension by doubling the sketch size of the Nyström approximation until the empirical condition number falls below the desired tolerance. Theorem 4.4 ensures the true condition number is close to this empirical estimate with high probability.

Theorem 4.4 also helps explain why sketch sizes much smaller than the effective dimension can succeed in practice. The point is best illustrated by instantiating an explicit parameter selection in Theorem 4.4, which yields the following corollary.

**Corollary 4.5.** Instate the hypotheses of Theorem 4.4 with \( \epsilon = 100 \). Then with a sketch size of \( s \geq d_{\text{eff}}(16\rho) \) the following holds

1. \( (\hat{\lambda}_s + \rho)/\rho \leq 1 + \frac{100}{42} \).

2. With probability at least \( 1 - \delta \),

\[
\begin{align*}
&\frac{\|x^{\infty} - \tilde{x}_s\|_2}{\|\tilde{x}_s\|_2} \leq \epsilon, \\
&\frac{\kappa_2(P^{-1/2}H_\rho P^{-1/2} - \hat{\lambda}_s + \rho/\rho)}{\rho} \leq \epsilon.
\end{align*}
\]

Corollary 4.5 shows that for a coarse tolerance of \( \epsilon = 100 \), a sketch size of \( s \geq d_{\text{eff}}(16\rho) \) suffices to ensure that the condition number of \( P^{-1/2}H_\rho P^{-1/2} \) is no more than around 100. Two practical observations cement the importance of this corollary. First, \( d_{\text{eff}}(16\rho) \) is often significantly smaller...
than $d_{\text{eff}}(\rho)$, possibly by an order of magnitude or more. Second, with a condition number around 100, PCG is likely to converge very quickly. In fact, for modest condition numbers, PCG is known to converge much faster in practice than the theory would suggest (Trefethen & Bau III, 1997). It is only when the condition number reaches around $10^3$, that convergence starts to slow. Thus, Corollary 4.5 helps explain why it is not necessary for the sketch size to equal the effective dimension in order for NysADMM to obtain significant accelerations.

5. Numerical experiments

In this subsection, we evaluate the performance of NysADMM to solve the standard lasso problem (2). Here we compare NysADMM with three standard lasso solvers, SSNAL (Li et al., 2018), mfIPM (Fountoulakis et al., 2014), and glmnet (Friedman et al., 2010). SSNAL is a Newton method based solver; mfIPM is an interior point method based solver and glmnet is a coordinate descent based solver. In practice, these three solvers and NysADMM rely on different stopping criteria. In order to make a fair comparison, in our experiments, the accuracy of a solution $x$ for (2) is measured by the following relative Karush–Kuhn–Tucker (KKT) residual (Li et al., 2018):

$$\eta = \frac{||x - \text{prox}_{\gamma A^T}(x - A^T(Ax - b))||}{1 + ||x|| + ||Ax - b||}.$$  

For a given tolerance $\epsilon$, we stop the tested algorithms when $\eta < \epsilon$. Note that stopping criterion (19) is rather strong: if $\eta \lesssim 10^{-4}$ for NysADMM, then the primal and dual gaps for ADMM are $\lesssim 10^{-4}$, which suffices for most applications. Indeed, for many machine learning problems, lower bounds on the statistical performance of the estimator (Loh, 2017) imply an unavoidable level of statistical error that is greater than this optimization error for most applications. Optimizing the objective beyond the level of statistical error (Agarwal et al., 2012; Loh & Wainwright, 2015) does not improve generalization. For standard lasso experiments, we fix the regularization parameter at $\gamma = 1$.

Table 3. Results for low precision lasso experiment.

| Task         | NysADMM | mfIPM | SSNAL | glmnet |
|--------------|---------|-------|-------|--------|
| STL-10       | 105     | 67    | 467   | 278    |
| CIFAR-10     | 251     | 655   | 692   | 391    |
| smallNorb-rf | 219     | 552   | 515   | 293    |
| E2006.train  | 313     | 875   | 903   | 554    |
| sector       | 235     | 678   | 608   | 396    |
| realsim-rf   | 193     | –     | 765   | 292    |
| rcv1-rf      | 226     | 563   | 595   | 273    |
| cod-rna-rf   | 208     | 976   | 865   | 324    |

Table 3 and Table 4 show results for lasso experiments. The average solution time for NysADMM, mfIPM, SSNAL, and
Table 4. Results for high precision lasso experiment.

| Task          | Time for $\epsilon = 10^{-2}$ (s) |
|---------------|------------------------------------|
|               | NysADMM | mfIPM | SSNAL | glmnet |
| STL-10        | 406     | 812   | 656   | 831    |
| CIFAR-10-rf   | 715     | 1317  | 1126  | 1169   |
| smallNorb-rf  | 596     | 896   | 768   | 732    |
| E2006.train   | 1657    | 1965  | 1446  | 2135   |
| sector        | 957     | 1066  | 875   | 1124   |
| realsim-rf    | 732     |       | 1035  | 922    |
| rcv1-rf       | 593     | 853   | 715   | 736    |
| cod-rna-rf    | 715     | 1409  | 1167  | 997    |

Table 5. Results for $\ell_1$-regularized logistic regression experiment.

| Task          | NysADMM (s) | SAGA (s) | APG (s) |
|---------------|-------------|----------|---------|
| STL-10        | 3012        | 6083     | 2635    |
| CIFAR-10-rf   | 7884        | 21256    | 17292   |
| p53-rf        | 528         | 2116     | 1880    |
| connect-4-rf  | 866         | 4781     | 7365    |
| smallnorb-rf  | 1808        | 6381     | 4408    |
| rcv1-rf       | 1237        | 3988     | 2759    |
| con-rna-rf    | 7528        | 21513    | 16361   |

Figure 1. Solution times for varying tolerance $\epsilon$ on STL-10.
compare NysADMM with the LIBSVM solver (Chang & Lin, 2011). LIBSVM uses sequential minimal optimization (SMO) to solve the dual SVM problem. We use the same stopping criteria as the LIBSVM solver, which stops the NysADMM method when the ADMM dual gap reaches $10^{-4}$ level. All SVM experiments use the RBF kernel. Table 6 shows the results of SVM experiments. On these problems, NysADMM is at least 3 times faster (and up to 58 times faster) than the LIBSVM solver. Consider problem formulation (12), with the RBF kernel. The Gram matrix $\text{diag}(b)K\text{diag}(b)$ is dense and approximately low rank: exactly the setting in which NysADMM should be expected to perform well. In contrast, the SMO-type decomposition in LIBSVM solver works better for sparse problems, as it updates only two variables at each iteration.

6. Conclusion

In this paper, we have developed a scalable new algorithm, NysADMM, that combines inexact ADMM and the randomized low-rank Nyström approximation to accelerate composite convex optimization. We show that NysADMM exhibits strong benefits both in theory and in practice. Our theory shows that when the Nyström preconditioner is constructed with an appropriate rank, NysADMM requires only a constant number of matvecs to solve the ADMM subproblem. We have also provided an adaptive strategy for selecting the rank that possesses a similar computational profile to the non-adaptive algorithm, and allows us to realize the theoretical benefits in practice. Further, numerical results demonstrate that NysADMM is as least twice as fast as standard methods on large dense lasso, regularized logistic regression, and kernalized SVM problems. More broadly, this paper shows the promise of recent advances in RandNLA to provide practical accelerations for important large-scale optimization algorithms.

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A. Proofs of main results

In this section we give the proofs for the main results of the paper: Theorem 4.1, Theorem 4.3, and Theorem 4.4.

A.1. Preliminaries

We start by recalling some useful background information and technical results that are useful for proving the main theorems. In order to obtain the exponentially small failure probabilities in Theorem 4.1 and Theorem 4.4 we take a different approach from the one in Frangella et al. (2021). The proofs are based on regularized Schur complements and approximate matrix multiplication. Our arguments are inspired by the techniques used to establish statistical guarantees for approximate kernel ridge regression via column sampling schemes (Alaoui & Mahoney, 2015; Bach, 2013).

A.1.1. Nyström approximation: properties

We start by recalling some important properties of the Nyström approximation (4). We shall also need the regularized Nyström approximation. Recall that \( \Omega \in \mathbb{R}^{d \times s} \) denotes the test matrix from which we construct the Nyström approximation. Given \( \sigma > 0 \), the regularized Nyström approximation with respect to \( \Omega \) is defined as

\[
H(\Omega)_\sigma = (H\Omega)(\Omega^TH\Omega + \sigma I)^{-1}(H\Omega)^T.
\]

Furthermore, let \( H = V\Lambda V^T \) be the eigendecomposition of \( H \) and define \( D_\sigma = H(H + \sigma I)^{-1} = \Lambda(\Lambda + \sigma I)^{-1} \). We shall see below that \( D_\sigma \) plays a crucial role in the analysis. The following lemmas are well known in the literature and summarize the properties of the Nyström and regularized Nyström approximation. Lemma A.1 may be found in Frangella et al. (2021) and Lemma A.2 in Alaoui & Mahoney (2015).

**Lemma A.1.** Let \( H(\Omega) \) be a Nyström approximation of a symmetric psd matrix \( H \). Then

1. The approximation \( H(\Omega) \) is psd and has rank at most \( s \).
2. The approximation \( H(\Omega) \) depends only on range(\( \Omega \)).
3. In the Loewner order, \( H(\Omega) \preceq H \).
4. In particular, the eigenvalues satisfy \( \lambda_j(H(\Omega)) \leq \lambda_j(H) \) for each \( 1 \leq j \leq d \).

**Lemma A.2.** Let \( H \) be a symmetric psd matrix, \( \sigma > 0 \). Define \( E = H - H(\Omega) \) and \( E_\sigma = H - H(\Omega)_\sigma \). Then the following hold.

1. \( H(\Omega)_\sigma \preceq H(\Omega) \preceq H \).
2. \( 0 \preceq E \preceq E_\sigma \).
3. If \( \|D_\sigma^{1/2}V^T(\frac{1}{s}\Omega\Omega^TV)VD_\sigma^{1/2} - D_\sigma\| \leq \eta < 1 \), then

\[
0 \preceq E_\sigma \preceq \frac{\sigma}{1-\eta} I.
\]

Lemma A.2 relates \( H(\Omega)_\sigma \) to \( H(\Omega) \) and \( H \). In particular, item 2 implies that \( \|E\| \leq \|E_\sigma\| \), so controlling \( E_\sigma \) controls \( E \). Item 3 shows that \( E_\sigma \) can be controlled by the spectral norm of the matrix

\[
D_\sigma^{1/2}V^T\frac{1}{s}\Omega\Omega^TVD_\sigma^{1/2} - D_\sigma.
\]

The spectral norm of (22) can be bounded by observing

\[
\mathbb{E} \left[ D_\sigma^{1/2}V^T \frac{1}{s}\Omega\Omega^TVD_\sigma^{1/2} \right] =
\]

\[
D_\sigma^{1/2}V^T \mathbb{E} \left[ \frac{1}{s}\Omega\Omega^TV \right] VD_\sigma^{1/2} =
\]

\[
D_\sigma^{1/2}V^TVD_\sigma^{1/2} = D_\sigma.
\]
Thus, $D_\sigma^{1/2}V^T\Omega^T V D_\sigma^{1/2}$ is an unbiased estimator of $D_\sigma$, and may be viewed as approximating the product of the matrices $D_\sigma^{1/2}V^T$ and $VD_\sigma^{1/2}$. Hence results from randomized linear algebra can bound the spectral norm of this difference.

In particular, it suffices to take a sketch size that scales with the effective dimension, using results on approximate matrix multiplication in terms of stable rank (Cohen et al., 2016).

### A.1.2. Approximate matrix multiplication in terms of the effective dimension

The condition in item 3 of Lemma A.2 follows immediately from theorem 1 of Cohen et al. (2016). Unfortunately, the analysis in that paper does not yield explicit constants. Instead we use a special case of their results due to Lacotte & Pilanci (2021) that provides explicit constants. Theorem A.3 simplifies theorem 5.2 in Lacotte & Pilanci (2021).

**Theorem A.3.** Let $\Psi \in \mathbb{R}^{s \times d}$ be a matrix with i.i.d. $N(0, \frac{1}{s})$ entries. Given $\delta > 0$, and $\tau \in (0, 1)$ it holds with probability at least $1 - \delta$ that

$$
\sup_{v \in \mathbb{S}^{d-1}} \langle v, (D_\sigma^{1/2}V^T \Psi V D_\sigma^{1/2} - D_\sigma)v \rangle \leq \tau + 2\sqrt{\tau},
$$

(26)

$$
\inf_{v \in \mathbb{S}^{d-1}} \langle v, (D_\sigma^{1/2}V^T \Psi V D_\sigma^{1/2} - D_\sigma)v \rangle \geq \tau - 2\sqrt{\tau},
$$

(27)

provided $s \geq \left( \sqrt{d_\text{eff}(\sigma)} + \sqrt{8 \log(16/\delta)} \right)^2$. Setting $\Psi = \frac{1}{\sqrt{s}} \Omega^T$, where $\Omega \in \mathbb{R}^{d \times s}$ has i.i.d. $N(0, 1)$ entries, Theorem A.3 yields the following corollary.

**Corollary A.4.** Let $\Omega \in \mathbb{R}^{d \times s}$ be a matrix with i.i.d. $N(0, 1)$ entries. Given $\delta > 0$, and $\tau \in (0, 1)$ it holds with probability at least $1 - \delta$ that

$$
\left\| D_\sigma^{1/2}V^T \frac{1}{s} \Omega^T V D_\sigma^{1/2} - D_\sigma \right\| \leq \tau + 2\sqrt{\tau}
$$

(28)

provided $s \geq \left( \sqrt{d_\text{eff}(\rho)} + \sqrt{8 \log(16/\delta)} \right)^2$. 

### A.1.3. Condition number of Nyström preconditioned linear system

The following result is a simpler version of proposition 5.2 in Frangella et al. (2021).

**Proposition A.5.** Let $\tilde{H} = U \hat{\Lambda} U^T$ be any rank-$s$ Nyström approximation, with $s$th largest eigenvalue $\hat{\lambda}_s$, and let $E = H - \tilde{H}$ be the approximation error. Construct the Nyström preconditioner $P$ as in (7). Then the condition number of the preconditioned matrix $P^{-1/2}H \rho P^{-1/2}$ satisfies

$$
\kappa_2(P^{-1/2}H \rho P^{-1/2}) \leq \frac{\hat{\lambda}_s + \rho + \|E\|}{\rho}.
$$

(29)

Proposition A.5 bounds the condition condition number of the Nyström preconditioned linear system in terms of $\hat{\lambda}_s, \rho$ and the approximation error $\|E\|$. We would like to emphasize that the bound in Proposition A.5 is deterministic.

### A.2. Proofs of Theorem 4.1 and Corollary 4.2

We start with two lemmas from which Theorem 4.1 follows easily. The first lemma and its proof appear in Frangella et al. (2021).

**Lemma A.6.** Let $H \in S_+^d(\mathbb{R})$ with eigenvalues $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_d$. Let $\rho > 0$ be regularization parameter, and define the effective dimension as in (8). Then the following statement holds.

Fix $\gamma > 0$. If $j \geq (1 + \gamma^{-1})d_\text{eff}(\rho)$, then $\lambda_j \leq \gamma \rho$.

**Lemma A.7.** Let $\epsilon > 0$ and $E = H - H(\Omega)$. Suppose we construct a randomized Nyström approximation from a standard Gaussian random matrix $\Omega$ with sketch size $s \geq 8 \left( \sqrt{d_\text{eff}(\epsilon)} + \sqrt{8 \log(16/\delta)} \right)^2$. Then the event

$$
\mathcal{E} = \{ \|E\| \leq 6\epsilon \},
$$

(30)

holds with probability at least $1 - \delta$. 

Theorem A.8. Consider a convex optimization problem in the primal form (P), minimize \( f(x) + h(Mx) \), where \( x \in \mathbb{R}^d \), \( M \in \mathbb{R}^{m \times d} \) has full column rank. Pick any \( y^0 \), \( z^0 \in \mathbb{R}^m \), and \( \rho > 0 \), and summable sequences

\[
\begin{align*}
\{\varepsilon^k\}_{k=0}^{\infty} & \subseteq [0, \infty), \quad \sum_{k=0}^{\infty} \varepsilon^k < \infty, \\
\{\nu^k\}_{k=0}^{\infty} & \subseteq [0, \infty), \quad \sum_{k=0}^{\infty} \nu^k < \infty, \\
\{\lambda^k\}_{k=0}^{\infty} & \subseteq (0, 2), \quad 0 < \inf \lambda^k \leq \sup \lambda^k < 2.
\end{align*}
\]

Proof. Let \( \Omega_s = \frac{1}{\sqrt{s}} \Omega \) and observe that \( H(\Omega_s) = H(\Omega) \). Now the conditions of Corollary A.4 are satisfied with \( \sigma = \epsilon \) and \( \tau = \delta \). Consequently with probability at least \( 1 - \delta \),

\[
\left\| D^{1/2}_e V^{T} \frac{1}{s} \Omega \Omega^{T} V D^{1/2}_e - D_e \right\| \leq \frac{1}{8} + \frac{\sqrt{2}}{2}.
\]

Hence applying Lemma A.2 with \( \sigma = \epsilon \) and \( \eta = \frac{1}{8} + \frac{\sqrt{2}}{2} \), we obtain

\[
\left\| H - H(\Omega_s) \right\| \leq 6\epsilon,
\]

with probability at least \( 1 - \delta \). Recalling our initial observation, we conclude the desired result. \( \square \)

A.2. Proof of Theorem 4.1

Proof. As \( s \geq 8 \left( \sqrt{d_{eff}(\rho)} + \sqrt{8 \log \left( \frac{4\rho}{\delta} \right)} \right)^2 \) we have that \( \|E\| \leq 6\rho \) with probability at least \( 1 - \delta \) by Lemma A.7. Furthermore, \( \lambda_s \leq \frac{\rho}{\delta} \) by item 3 of Lemma A.1 and Lemma A.6 with \( \gamma = 1/7 \). Combining this with Proposition A.5, we conclude with probability at least \( 1 - \delta \),

\[
\kappa_2(P^{-1/2}H_\rho P^{-1/2}) \leq \lambda_s + \rho \|E\| \rho \leq 1 + 6 + \frac{1}{t} \leq 8
\]

as desired. \( \square \)

A.2.2. Proof of Corollary 4.2

Proof. Let \( A = P^{-1/2}H_\rho P^{-1/2} \) and condition on the event that \( \kappa_2(A) \leq 8 \), which holds with probability at least \( 1 - \delta \). The standard theory for convergence of CG (Trefethen & Bau III, 1997) guarantees after \( t \) iterations that,

\[
\frac{\|x_t - \tilde{x}_s\|_A}{\|\tilde{x}_s\|_A} \leq 2 \left( \frac{\sqrt{\kappa_2(A)} - 1}{\sqrt{\kappa_2(A)} + 1} \right)^t
\]

where \( \|x\|_A = x^T Ax \). Theorem 4.1 guarantees that the Nyström preconditioned matrix satisfies \( \kappa_2(A) \leq 8 \), so the above display may be majorized as

\[
\frac{\|x_t - \tilde{x}_s\|_A}{\|\tilde{x}_s\|_A} \leq \left( \frac{1}{2} \right)^{t-1}.
\]

Now, from the elementary inequality

\[
\lambda_d(A) ||x||_2 \leq ||x||_A \leq \lambda_1(A) ||x||_2,
\]

we conclude

\[
\frac{\|x_t - \tilde{x}_s\|_2}{\|\tilde{x}_s\|_2} \leq \kappa_2(A) \left( \frac{1}{2} \right)^{t-1} \leq \left( \frac{1}{2} \right)^{t-4}.
\]

To obtain the claimed result, multiply both sides by \( \|\tilde{x}_s\|_2 \) and solve \( \|\tilde{x}_s\|_2 \left( \frac{1}{2} \right)^{t-4} = \epsilon \) for \( t \). \( \square \)

A.3. Proof of Theorem 4.3

This proof is a natural consequence of the following theorem from Eckstein & Bertsekas (1992).

Theorem A.8. Consider a convex optimization problem in the primal form (P), minimize \( f(x) + h(Mx) \), where \( x \in \mathbb{R}^d \), \( M \in \mathbb{R}^{m \times d} \) has full column rank. Pick any \( y^0 \), \( z^0 \in \mathbb{R}^m \), and \( \rho > 0 \), and summable sequences
The dual problem (D) of primal problem (P) is

\[ \text{maximize}_{y \in \mathbb{R}^m} - (f^*(-M^Ty) + g^*(y)). \]

Suppose the primal and dual ADMM iterates \( \{x^k\}_{k=0}^\infty, \{z^k\}_{k=0}^\infty, \) and \( \{y^k\}_{k=0}^\infty \) satisfy the update equations to within errors given by conform, for all \( k \) to

\[
\begin{align*}
\|x^{k+1} - \text{argmin}_x \{ f(x) + \langle y^k, Mx \rangle \} + \frac{1}{2}a_k \|Mx - z^k\|_2^2 \|_2 & \leq \varepsilon^k, \\n\|z^{k+1} - \text{argmin}_z \{ h(z) - \langle y^k, z \rangle \} + \frac{1}{2}b_k \|Mx^k - z + (1 - \lambda^k)z^k\|_2^2 \|_2 & \leq \nu^k, \\ny^{k+1} = y^k + \lambda^k Mx^{k+1} + (1 - \lambda^k)z^k - z^{k+1}.
\end{align*}
\]

Then if (P) has a Kuhn-Tucker pair, \( \{x^k\} \) converges to a solution of (P) and \( \{y^k\} \) converges to a solution of (D).

### A.3.1. Proof of Theorem 4.3

**Proof.** Consider optimization problem (1) and the associated NysADMM algorithm Algorithm 3. Suppose \( \{\tilde{x}^k\}_{k=0}^\infty, \{\tilde{z}^k\}_{k=0}^\infty, \) and \( \{\tilde{u}^k\}_{k=0}^\infty \) are generated by NysADMM iterations. Since \( \ell(Ax, b) \) is quadratic with respect to \( x \) and the smooth part \( g \) of regularizer \( r \) has constant Hessian, the \( x \)-subproblem of (1) is exactly the linear system (4).

Let \( x^{k+1} \) be the exact solution for the \( x \)-subproblem at iteration \( k \). For all \( k \geq 0 \), NysADMM iterate \( \tilde{x}^{k+1} \) satisfies \( \|x^{k+1} - x^{k+1}\|_2 \leq \varepsilon^k \). Let \( M = I, \nu^k = 0, \lambda^k = 1, y^k = \tilde{u}^k \) for all \( k \), and \( f(x) = \ell(Ax, b) + g(x) \). By Theorem A.8, \( \{\tilde{x}^k\}_{k=0}^\infty, \{\tilde{z}^k\}_{k=0}^\infty, \) and \( \{\tilde{u}^k\}_{k=0}^\infty \) satisfy condition (35). Therefore, if optimization problem (1) has a Kuhn-Tucker pair, \( \{\tilde{x}^k\} \) converges to a solution of (1) and \( \{\tilde{u}^k\} \) converges to a solution of the dual problem of (1).

Next, we derive the bound for the number of Nyström PCG iterations \( T^k \) required at NysADMM iteration \( k \). Note that in this case the Hessians of \( \ell \) and \( g \) are constant. We only need to sketch once for the constant linear system matrix \( A^T H^\ell (Ax; b) A + H^g (\tilde{x}^k) + \) and can reuse the sketch for all NysADMM iterations. Since the Nyström preconditioner is constructed with sketch size \( s \geq 8 \left( \sqrt{d_{\text{eff}}(\rho)} + \sqrt{8 \log \left( \frac{16}{\sqrt{\gamma}} \right)} \right) \), by Corollary 4.2, with probability at least \( 1 - \delta \), after

\[ T^k \geq \left\lceil \log \left( \frac{16\|x^{k+1}\|_2}{\varepsilon \rho} \right) \right\rceil \log(2) \]

Nyström PCG iterations, we have \( \|\tilde{x}^{k+1} - x^{k+1}\|_2 \leq \varepsilon^k \). Recall the righthand side of linear system (4) \( r^k \). The exact solution for the \( x \)-subproblem \( x^{k+1} \) at iteration \( k \) satisfies \( \|x^{k+1}\|_2 \leq \frac{\|r^k\|_2}{\rho} \). We have

\[ \left\lceil \log \left( \frac{16\|x^{k+1}\|_2}{\varepsilon \rho} \right) \right\rceil \leq \left\lceil \log \left( \frac{16\|r^k\|_2}{\varepsilon \rho} \right) \right\rceil \leq \left\lceil 2 \log \left( \frac{R}{\varepsilon \rho} \right) \right\rceil. \]

Further, by assumption, as \( \|r^k\|_2 \) is bounded by a constant \( R \) for all \( k \), we have

\[ \left\lceil \log \left( \frac{16\|r^k\|_2}{\varepsilon \rho} \right) \right\rceil \leq 4 + \left\lceil \log \left( \frac{R}{\varepsilon \rho} \right) \right\rceil \leq 4 + 2 \log \left( \frac{R}{\varepsilon \rho} \right). \]

This gives the bound for the number of Nyström PCG iterations \( T^k \) required at NysADMM iteration \( k \). \( \square \)

### A.4. Proof of Theorem 4.4

**Proof.** By hypothesis we have \( s > 8d_{\text{eff}}(\frac{\varepsilon \rho}{\lambda}) \), so Lemma A.6 with \( \gamma = 7 \) yields

\[ \hat{\lambda}_s \leq \lambda_s \leq \frac{1}{7} \frac{\varepsilon \rho}{6} = \frac{\varepsilon \rho}{42}. \]
Thus,
\[ \frac{\hat{\lambda}_s + \rho}{\rho} \leq 1 + \frac{\epsilon}{42}. \]

This gives the first statement. For the second statement we use our hypothesis on \( s \) to apply Lemma A.7 with tolerance \( \epsilon \rho / 6 \). From this we conclude \( \| E \| \leq \epsilon \rho \) with probability at least \( 1 - \delta \). Combining this with Proposition A.5 yields
\[
\kappa_2(P^{-1/2}H\rho^{-1/2}) - \frac{\hat{\lambda}_s + \rho}{\rho} \leq \epsilon,
\]
with probability at least \( 1 - \delta \). On the other hand, condition numbers always satisfy
\[
\kappa_2(P^{-1/2}H\rho^{-1/2}) \geq 1.
\]
Combining this with our upper bound on \( \hat{\lambda}_s \) gives
\[
\kappa_2(P^{-1/2}H\rho^{-1/2}) - \frac{\hat{\lambda}_s + \rho}{\rho} \geq 1 - (1 + \epsilon/42) = \epsilon/42.
\]
Hence with probability at least \( 1 - \delta \)
\[
\left| \kappa_2(P^{-1/2}H\rho^{-1/2}) - \frac{\hat{\lambda}_s + \rho}{\rho} \right| \leq \epsilon.
\]

\[ \square \]

**B. Randomized Nyström approximation and Nyström PCG**

In this section we give the algorithms from Frangella et al. (2021) for the randomized Nyström approximation and Nyström PCG.

**Algorithm 4 Randomized Nyström Approximation**

**input** psd matrix \( H \in \mathbb{S}_+^d(\mathbb{R}) \), sketch size \( s \)
\( \Omega = \text{randn}(d, s) \) \{Gaussian test matrix\}
\( \Omega = \text{qr}(\Omega, 0) \) \{thin QR decomposition\}
\( Y = H\Omega \) \{\( s \) matvecs with \( H \)\}
\( \nu = \text{eps} (\text{norm}(Y, 2)) \) \{compute shift\}
\( Y_\nu = Y + \nu \Omega \) \{add shift for stability\}
\( C = \text{chol}(\Omega^TY_\nu) \) \{Cholesky decomposition\}
\( B = Y_\nu / C \) \{triangular solve\}
\( [U, \Sigma, \sim] = \text{svd}(B, 0) \) \{thin SVD\}
\( \hat{\Lambda} = \max\{0, \Sigma^2 - \nu I\} \) \{remove shift, compute eigs\}
**output** Nyström approximation \( H_{\text{nys}} = U\hat{\Lambda}U^T \)

**C. AdaNysADMM**

In this section we give the adaptive algorithm for computing the randomized Nyström approximation adopted from Frangella et al. (2021). The adaptive algorithm has the benefit of reusing computation, in particular, we do not need to compute the sketch \( Y' \) from scratch. We simply add on the sketch that we have already computed. We also give the pseudo-code for AdaNysADMM that uses Algorithm 6 to compute the Nyström preconditioner.
Algorithm 5 Nyström PCG

**input** psd matrix $H$, righthand side $r$, initial guess $x_0$, regularization parameter $\rho$, sketch size $s$, tolerance $\varepsilon$

$[U, \Lambda] = \text{RandomizedNyströmApproximation}(H, s)$

$w_0 = r - (H + \rho I)x_0$

$y_0 = P^{-1}w_0$

$p_0 = y_0$

**while** $\|w\|_2 > \varepsilon$ **do**

$v = (H + \rho I)p_0$

$\alpha = (w_0^T y_0)/(p_0^T v)$

$x = x_0 + \alpha p_0$

$w = w_0 - \alpha v$

$y = P^{-1}w$

$\beta = (w^T y)/(w_0^T y_0)$

$x_0 \leftarrow x$, $w_0 \leftarrow w$, $p_0 \leftarrow y + \beta p_0$, $y_0 \leftarrow y$

**output** approximate solution $\hat{x}$

Algorithm 6 AdaptiveRandNysAppx

**input** symmetric psd matrix $H$, initial rank $s_0$, tolerance $\text{Tol}$

$Y = [ ]$, $\Omega = [ ]$, and $(\lambda_s + \rho)/\rho = \text{Inf}$

$m = s_0$

**while** $(\lambda_s + \rho)/\rho > \text{Tol}$ **do**

generate Gaussian test matrix $\Omega_0 \in \mathbb{R}^{n \times m}$

$[\Omega_0, \sim] = \text{qr}(\Omega_0, 0)$

$Y_0 = H\Omega_0$

$\Omega = [\Omega, \Omega_0]$ and $Y = [Y, Y_0]$

$\nu = \sqrt{n} \text{eps} \left( \text{norm}(Y, 2) \right)$

$Y_\nu = Y + \nu \Omega$

$C = \text{chol}(\Omega^T Y_\nu)$

$B = Y_\nu/C$

compute $[U, \Sigma, \sim] = \text{svd}(B, 0)$

$\Lambda = \max\{0, \Sigma^2 - \nu I\}$ {remove shift}

compute $(\lambda_s + \rho)/\rho$

$m \leftarrow s_0$, $s_0 \leftarrow 2s_0$ {double rank if tolerance is not met}

**if** $s_0 > s_{\text{max}}$ **then**

$s_0 = s_0 - m$ {when $s_0 > s_{\text{max}}$, reset to $s_0 = s_{\text{max}}$}

$m = s_{\text{max}} - s_0$

generate Gaussian test matrix $\Omega_0 \in \mathbb{R}^{n \times m}$

$[\Omega_0, \sim] = \text{qr}(\Omega_0, 0)$

$Y_0 = H\Omega_0$

$\Omega = [\Omega, \Omega_0]$ and $Y = [Y, Y_0]$

$\nu = \sqrt{n} \text{eps} \left( \text{norm}(Y, 2) \right)$ {compute final approximation and break}

$Y_\nu = Y + \nu \Omega$

$C = \text{chol}(\Omega^T Y_\nu)$

$B = Y_\nu/C$

compute $[U, \Sigma, \sim] = \text{svd}(B, 0)$

$\Lambda = \max\{0, \Sigma^2 - \nu I\}$

**break**

**end if**

**end while**

**output** Nyström approximation $(U, \hat{\Lambda})$
Algorithm 7 AdaNysADMM

**input** feature matrix $A$, response $b$, loss function $\ell$, regularization $g$ and $h$, stepsize $\rho$, positive summable sequence $\{\epsilon_k\}_{k=0}^{\infty}$

$[U, \hat{\Lambda}] = \text{AdaptiveRandNysAppx}(\ A^T H^s A + H^g, s) \ {\text{use Algorithm 6}}$

**repeat**

find $\hat{x}^{k+1}$ that solves (4) within tolerance $\epsilon^h$ by Nyström PCG

$\hat{x}^{k+1} = \arg\min_z \{ h(z) + \frac{\rho}{2} \| \hat{x}^{k+1} - z + \hat{u}^{k} \|_2^2 \}$

$\hat{u}^{k+1} = \hat{u}^{k} + x^{k+1} - z^{k+1}$

**until** convergence