OPTIMALITY RESULTS FOR A SPECIFIC FRACTIONAL PROBLEM

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Abstract. In this paper, one minimizes a fractional function over a compact set. Using an exact separation theorem, one gives necessary optimality conditions for strict optimal solutions in terms of Fréchet subdifferentials. All data are assumed locally Lipschitz.

1. Introduction. The separation theorems for convex sets play a key role in functional analysis and optimization theory. In fact, many crucial results with their proofs are based on separation arguments which are applied to convex sets (see [6]). In [8], Zheng, Yang and Zou proposed a related approach (an exact separation theorem) which can be considered as a generalization of the convex separation theorem to nonconvex sets and used as a powerful tool for deducing optimality conditions in nonconvex optimization. In order to use it, one should have an empty intersection between the sets and each set is considered near one of its elements; which is not the case in the extremal principle in [2].

In this paper, we are concerned with the following nonconvex fractional programming problem

\[(P) : \begin{align*}
\min_x \varphi(x) &= \frac{f(x)}{g(x)} \\
\text{Subject to : } x &\in \Omega
\end{align*}\]

where \(X\) is an Asplund space, \(\Omega\) is a compact subset of \(X\), \(f : X \to \mathbb{R}\) and \(g : X \to \mathbb{R}\) are Lipschitz continuous functions such that \(g(x) \neq 0\), for all \(x \in \Omega\).

The point \(\bar{x} \in \Omega\) is said to be a strict local optimal solution of \((P)\) iff there exists a neighborhood \(U\) of \(\bar{x}\) such that

\[\varphi(x) > \varphi(\bar{x}), \forall x \in (U \cap \Omega) \setminus \{\bar{x}\}.
\]

With the help of the mentioned exact separation theorem [8] (see Theorem 2.3 below), one gives necessary optimality conditions for \((P)\) in terms of Fréchet subdifferentials. Throughout this work, we use standard notations. We denote by \(X^*\)

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the topological dual of $X$ with the canonical dual pairing $\langle \cdot, \cdot \rangle$; $\| (x, y) \| := \| x \| + \| y \|$ is the $l_1$-norm of $(x, y) \in X \times X$; $B_X$ and $B_{X^*}$ stand for the closed unit balls in the space and dual space; and $w^*$ denotes the weak* topology on the dual space. For a multifunction $F : X \rightrightarrows X^*$, the expressions
\[
\limsup_{x \to x^*} F (x) := \left\{ x^* \in X^* \setminus \exists x_k \to x^*, \exists x^*_k \overset{w^*}{\to} x^*_k \in F (x_k) \ \forall k \in \mathbb{N} \right\}
\]
and
\[
\liminf_{x \to x^*} F (x) := \left\{ x^* \in X^* \setminus \forall x_k \to x^* \exists x^*_k \overset{w^*}{\to} x^*_k \in F (x_k) \ \forall k \in \mathbb{N} \right\}
\]
signify, respectively, the sequential Painlevé-Kuratowski upper/outer and lower/inner limits in the norm topology in $X$ and the weak* topology in $X^*$; $\mathbb{N} := \{ 1, 2, \ldots \}$.

The rest of the paper is organized in this way: Section 2 contains basic definitions and preliminary material from nonsmooth variational analysis. Section 3 addresses main results (optimality conditions). A conclusion is given Section 4.

2. Preliminaries. In this section, we give some definitions, notations and results, which will be used in the sequel. For a subset $D \subseteq X$, $\text{cl} \ D$, $\text{co} \ D$, $\overline{\text{co}} \ D$ and cone $D$ stand for the closure, the convex hull, the closure of the convex hull and the convex cone generated by $D$, respectively. The following definitions are crucial for our investigation.

Definition 2.1. [2] Let $\Omega \subset X$ be locally closed around $\bar{x} \in \Omega$. Then the Fréchet normal cone $\hat{N}(\bar{x}; \Omega)$ and the Mordukhovich normal cone $N(\bar{x}; \Omega)$ to $\Omega$ at $\bar{x}$ are defined by
\[
\hat{N}(\bar{x}; \Omega) := \left\{ x^* \in X^* : \limsup_{x \to \bar{x}} \frac{\langle x^*, x - \bar{x} \rangle}{\| x - \bar{x} \|} \leq 0 \right\},
\]
\[
N (\bar{x}; \Omega) := \limsup_{x \to \bar{x}} \hat{N} (x; \Omega) = \left\{ x^* \in X^* \setminus \exists x_k \overset{1}{\to} \bar{x}, \exists x^*_k \overset{w^*}{\to} x^* \text{ with } x^*_k \in \hat{N} (x_k; \Omega) \ \forall k \in \mathbb{N} \right\} .
\]
where $x \to \bar{x}$ stands for $x \to \bar{x}$ with $x \in \Omega$.

Directly from (1), it follows that for any locally closed set $\Omega$ and any $\bar{x} \in \Omega$
\[
\hat{N}(\bar{x}; \Omega) \subset N(\bar{x}; \Omega).
\]
(2)
The set $\Omega$ is called regular at $\bar{x}$ if (2) holds as equality. Remark that the set of regular sets includes all convex sets.

Definition 2.2. [2, 4] Let $\varphi : X \to \overline{\mathbb{R}}$ be a lower semicontinuous function around $\bar{x}$.
1. The Fréchet subdifferential of $\varphi$ at $\bar{x}$ is
\[
\hat{\partial} \varphi (\bar{x}) := \left\{ x^* \in X^* : \liminf_{x \to \bar{x}} \frac{\varphi (x) - \varphi (\bar{x}) - \langle x^*, x - \bar{x} \rangle}{\| x - \bar{x} \|} \geq 0 \right\} .
\]
2. The Mordukhovich subdifferential of $\varphi$ at $\bar{x}$ is defined by
\[
\partial \varphi (\bar{x}) := \limsup_{x \to \bar{x}} \hat{\partial} \varphi (x),
\]
3. The singular subdifferential of $\varphi$ at $\bar{x}$ is defined by
\[
\partial^\infty \varphi (\bar{x}) := \left\{ x^* \in X^* : (x^*, 0) \in N ((\bar{x}, \varphi (\bar{x})); epi (\varphi)) \right\} .
\]
where $x \overset{x^*}{\to} \bar{x}$ means that $x \to \bar{x}$ with $\varphi (x) \to \varphi (\bar{x})$. Here, $epi (\varphi)$ denotes the epigraph of $\varphi$ defined by
\[
epi (\varphi) = \{ (x, \lambda) \in X \times \mathbb{R} : \varphi (x) \leq \lambda \} .
\]
One clearly has

\[ \tilde{N}(\bar{x}; \Omega) = \partial \delta(\bar{x}; \Omega), \quad N(\bar{x}; \Omega) = \partial \delta(\bar{x}; \Omega), \]

where \( \delta(\cdot; \Omega) \) is the indicator function of \( \Omega \).

A function \( \varphi : X \to \mathbb{R} \) is called lower regular [4] at \( \bar{x} \) with \( \|\varphi(\bar{x})\| < +\infty \) if \( \partial \varphi(\bar{x}) = \partial \varphi(\bar{x}) \). Besides the classical cases of convex functions and those strictly differentiable at \( \bar{x} \) (in particular, smooth functions), the latter class includes substantially broader collections of functions encountered in variational analysis and optimization; see [2, 4, 7]. Since \( X \) is Asplund, if \( \varphi \) is lower regular at \( \bar{x} \) and locally Lipschitzian around this point, one gets \( \partial \varphi(\bar{x}) \neq \emptyset \); see [2, Corollary 2.25]. Remark that both the lower regularity and the locally Lipschitzity around \( \bar{x} \) are automatic when \( \varphi \) is convex and continuous around \( \bar{x} \).

Remark 1. [3] 1. For any closed set \( \Omega \subset X \) and \( \bar{x} \in \Omega \), one has

\[ N_c(\bar{x}; \Omega) = \text{co} \ N(\bar{x}; \Omega) \]

and for any Lipschitz continuous function \( \varphi : X \to \mathbb{R} \) around \( \bar{x} \), one has

\[ \partial_c \varphi(\bar{x}) = \text{co} \ \partial \varphi(\bar{x}) \]

where \( N_c(\bar{x}; \Omega) \) and \( \partial_c \varphi(\bar{x}) \) denote respectively the Clarke’s normal cone and the Clarke’s subdifferential.

2. The Fréchet normal cone \( \tilde{N}(\bar{x}; \Omega) \) is always convex while the Mordukhovich normal cone \( N(\bar{x}; \Omega) \) is nonconvex in general.

The following result is a separation theorem, which can be considered as a generalization of the convex separation theorem to nonconvex sets. We will use it in our investigation of optimality conditions.

Theorem 2.3. [8] Let \( X \) be an Asplund space and \( A, A_1, \ldots, A_n \) be nonempty closed (not necessarily convex) subsets of \( X \) such that \( A \) is compact and \( A \cap \bigcap_{i=1}^{n} A_i \) = \( \emptyset \). Let \( 1 \leq p, q \leq +\infty \) be such that

\[ \frac{1}{p} + \frac{1}{q} = 1. \]

Then, for any \( \varepsilon \in ]0, +\infty[ \) and \( \rho \in ]0, 1[ \) there exist \( a \in A, a_i \in A_i \) and \( a_i^* \in X^* \), \( i = 1, \ldots, n \), such that the following statements hold:

1. \[ \left( \sum_{i=1}^{n} \|a_i - a\|^p \right)^{\frac{1}{p}} \leq \gamma_p(A_1, \ldots, A_n, A) + \varepsilon. \]

2. \[ \begin{cases} a_i^* \in \tilde{N}(A_i, a_i), & i = 1, \ldots, n, \\ -\sum_{i=1}^{n} a_i^* \in \tilde{N}(A, a) \text{ and } \left( \sum_{i=1}^{n} \|a_i^*\|^q \right)^{\frac{1}{q}} = 1. \end{cases} \]

3. \[ \rho \left( \sum_{i=1}^{n} \|a_i - a\|^p \right)^{\frac{1}{p}} \leq \sum_{i=1}^{n} \langle a_i^*, a - a_i \rangle. \]
Here, $\gamma_p(A_1, \ldots, A_n, A)$ denotes the \((p\text{-weighted})\) non-intersect index of $A_1, \ldots, A_n, A$ defined by

$$
\gamma_p(A_1, \ldots, A_n, A) = \inf \left\{ \left( \sum_{i=1}^{n} \|a_i - a\|^p \right)^{\frac{1}{p}} : a \in A, a_i \in A_i, i = 1, \ldots, n \right\}.
$$

3. Main result. Theorem 3.1 provides necessary optimality conditions for the optimization problem $(P)$.

**Theorem 3.1.** Let $\bar{x} \in \Omega$ be a strict local optimal solution of $(P)$. Then, there exist sequences $\{v_k\} \subset \Omega$ and $\{\omega_k\} \subset \Omega$ such that

$$
\lim_{k \to +\infty} \varphi(v_k) = \varphi(\bar{x}), \quad \lim_{k \to +\infty} v_k = \bar{x}, \quad \lim_{k \to +\infty} w_k = \bar{x} \quad \text{and} \quad 0 \in \partial \varphi(v_k) + \tilde{N}(\Omega, w_k).
$$

- Letting $k \to \infty$, we get

$$
0 \in \partial \varphi(\bar{x}) + N(\Omega, \bar{x}).
$$

- If in addition $\tilde{\partial}(f(v_k)g)(v_k) \neq \emptyset$, we deduce that

$$
\tilde{\partial}(f(v_k)g)(v_k) \subseteq \tilde{\partial}(g(v_k)f)(v_k) + \tilde{N}(\Omega, w_k).
$$

**Proof.** For each $k \in \mathbb{N}$, let

$$
A = \Omega \times \left\{ \varphi(\bar{x}) - \frac{1}{k+1} \right\} \quad \text{and} \quad A_1 = epi(\varphi).
$$

Since $\varphi$ is lower semicontinuous and since $\Omega$ is compact, one deduces that $A$ and $A_1$ are closed subsets of $X \times \mathbb{R}$, $A$ is a compact subset of $X \times \mathbb{R}$ and that $\varphi(\bar{x}) > -\infty$. Moreover, since $(\bar{x}, \varphi(\bar{x})) \in epi(\varphi)$, one has

$$
\gamma_1(A_1, A) = \inf \{ \|a_1 - a\| : a \in A, a_1 \in A_1 \} \leq \| (\bar{x}, \varphi(\bar{x})) - (\bar{x}, \varphi(\bar{x}) - \frac{1}{k+1}) \| \leq \frac{1}{k+1}.
$$

- Let us prove that

$$
A \cap A_1 = \emptyset.
$$

By contrary, suppose that there exists $x_k \in \Omega$ such that

$$
\left( x_k, \varphi(\bar{x}) - \frac{1}{k+1} \right) \in epi(\varphi).
$$

Then,

$$
\varphi(x_k) \leq \varphi(\bar{x}) - \frac{1}{k+1} < \varphi(\bar{x}).
$$

A contradiction.

- Applying Theorem 2.3, for each fixed $k \in \mathbb{N}$, there exist $w_k \in \Omega$, $v_k \in \Omega$, $(v_k, \alpha_k) \in epi(\varphi)$ and $(v_k^*, \beta_k) \in X \times \mathbb{R}$ such that

$$
\|(v_k^*, -\beta_k)\|_{\infty} = 1, \quad \left( \left( w_k, \varphi(\bar{x}) - \frac{1}{k+1} \right) - (v_k, \alpha_k) \right) < \gamma_1(A_1, A) + \frac{1}{k+1} < \frac{2}{k+1} \quad \text{and} \quad (v_k^*, -\beta_k) \in \tilde{N}(epi(\varphi), (v_k, \alpha_k)) \cap -\tilde{N} \left( A, \left( w_k, \varphi(\bar{x}) - \frac{1}{k+1} \right) \right).
$$
It follows from (3) that \((v_k, \alpha_k)\) is not an interior point of \(\text{epi} \, (\varphi)\); consequently,
\[
\alpha_k = \varphi (v_k). \tag{6}
\]
Then, using (4), one obtains the following inequalities:
\[
\|w_k - v_k\| \leq \frac{2}{k+1} \quad \text{and} \quad |\varphi (v_k) - \varphi (x)| \leq \frac{3}{k+1}.
\]
This implies that
\[
\varphi (v_k) > -\infty, \quad \lim_{k \to +\infty} \varphi (v_k) = \varphi (x) \quad \text{and} \quad \lim_{k \to +\infty} \|v_k - w_k\| = 0.
\]
- Since \(w_k \in \Omega\) and since \(\Omega\) is compact, taking a subsequence if necessary, we can assume that \(\{w_k\}\) converges to a point \(\bar{w}\) in \(\Omega\). Consequently,
\[
\lim_{k \to +\infty} v_k = \lim_{k \to +\infty} w_k = \bar{w}.
\]
The continuity of the function \(\varphi\) allows us to deduce that \(\varphi (\bar{w}) = \varphi (x)\).
Since \((\bar{w}, \alpha_k) \in \text{epi} \, (\varphi)\) and \((v_k, -\beta_k) \in \widehat{N} \, (\text{epi} \, (\varphi), (v_k, \alpha_k))\), by Proposition 1.31 (b) of [1], one has
\[
\beta_k \geq 0.
\]
- Let us prove that \(\beta_k \neq 0\). By contrary, suppose that \(\beta_k = 0\). From (3) – (6), we get
\[
\|(v_k^*, 0)\|_\infty = 1 \quad \text{and} \quad (v_k^*, 0) \in \widehat{N} \, (\text{epi} \, (\varphi), (v_k, \varphi (v_k))). \tag{7}
\]
By [1, Corollary 1.31.2], we have
\[
\widehat{N} \, (\text{epi} \, (\varphi), (v_k, \varphi (v_k))) = \left( \bigcup_{\lambda \geq 0} \lambda \left( \partial \varphi (v_k), -1 \right) \right) \cup (\partial^\infty \varphi (v_k), 0). \tag{8}
\]
Combining (7) and (8), we deduce that \(v_k^* \in \partial^\infty \varphi (v_k)\) and \(v_k^* \neq 0\). Since \(\varphi\) is locally Lipschitzian around \(x\), we have by [2, Corollary 1.81] that \(\partial^\infty \varphi (v_k) = \{0\}\), which is a contradiction.
- Setting \(\pi_k^* = \frac{v_k^*}{\beta_k}\), from (5), one has
\[
(\pi_k^*, -1) \in \widehat{N} \, (\text{epi} \, (\varphi), (v_k, \alpha_k)) \cap -\widehat{N} \left( A, \left( w_k, \varphi (x) - \frac{1}{k+1} \right) \right).
\]
Thus,
\[
\pi_k^* \in \partial \varphi (v_k) \quad \text{and} \quad (\pi_k^*, -1) \in -\widehat{N} \left( A, \left( w_k, \varphi (x) - \frac{1}{k+1} \right) \right) = -\widehat{N} (\Omega, w_k) \times \mathbb{R}.
\]
That is,
\[
\pi_k^* \in \partial \varphi (v_k) \cap -\widehat{N} (\Omega, w_k).
\]
Consequently,
\[
0 \in \partial \varphi (v_k) + \widehat{N} (\Omega, w_k). \tag{9}
\]
- Let us prove that
\[
0 \in \partial \varphi (x) + N (\Omega, x).
\]
Using the Lipschitz property of \(\varphi\), we can find \(s_k^* \in \partial \varphi (v_k)\) and \(\alpha_k^* \in \widehat{N} (\Omega, w_k)\) such that
\[
0 = s_k^* + \alpha_k^*, \quad \|s_k^*\| \leq L \quad \text{and} \quad (s_k^*, -1) \in \widehat{N} \, (\text{gr} \, (\varphi), (v_k, \varphi (v_k)));
\]
where \( L \) is the Lipschitz constant of \( \varphi \). Since \( X \) is Asplund and the sequence \( \{s_k^*\} \) is bounded, it is weak* sequentially compact. Letting \( k \to \infty \), taking a subsequence if necessary, we may assume that \( s_k^* \to * \in X \) such that
\[
s^* \in -N(\Omega, x), \quad \|s^*\| \leq L \text{ and } (s^*, -1) \in N(gr(\varphi), (x, \varphi(x))).
\]
Thus,
\[
s^* \in \partial \varphi(x) \quad \text{and} \quad s^* \in -N(\Omega, x).
\]
Then,
\[
0 \in \partial \varphi(x) + N(\Omega, x).
\]
- Suppose that \( \hat{\partial}(f(v_k)g)(v_k) \neq \emptyset \). Let us prove that
\[
\hat{\partial}(f(v_k)g)(v_k) \subseteq \hat{\partial}(g(v_k)f)(v_k) + \hat{N}(\Omega, w_k).
\]
By [4, Theorem 3.11], from (9), we deduce that
\[
0 \in \bigcap_{x^* \in \hat{\partial}(f(v_k)g)(v_k)} \left[ \frac{\partial (g(v_k)f)(v_k) - x^*}{g(v_k)} \right] + \hat{N}(\Omega, w_k).
\]
Since \( \hat{N}(\Omega, w_k) \) is a cone, one deduces that for every \( x^* \in \hat{\partial}(f(v_k)g)(v_k) \), one has
\[
0 \in \left[ \hat{\partial}(g(v_k)f)(v_k) - x^* \right] + \hat{N}(\Omega, w_k).
\]
Finally, one gets
\[
\hat{\partial}(f(v_k)g)(v_k) \subseteq \hat{\partial}(g(v_k)f)(v_k) + \hat{N}(\Omega, w_k).
\]

Next we present a consequence of Theorem 3.1 in case \( g \) is convex continuous around \( x \).

**Corollary 1.** Let \( \overline{x} \in \Omega \) be a strict optimal solution of \( (P) \). Suppose that \( f(\overline{x}) > 0 \) and that \( g \) is convex continuous around \( \overline{x} \). Then, there exist sequences \( \{v_k\} \subset \Omega \) and \( \{w_k\} \subset \Omega \) such that
\[
\lim_{k \to +\infty} \varphi(v_k) = \varphi(\overline{x}), \quad \lim_{k \to +\infty} v_k = \overline{x}, \quad \lim_{k \to +\infty} w_k = \overline{x},
\]
and
\[
\hat{\partial}(f(v_k)g)(v_k) \subseteq \hat{\partial}(g(v_k)f)(v_k) + \hat{N}(\Omega, w_k).
\]

**Proof.** It is well known from convex analysis that the subdifferential of every convex function is nonempty at a point of continuity [5, Proposition 1.11].

4. **Conclusion.** In this paper, we are concerned with a fractional optimization problem \( (P) \). Assuming data locally Lipschitz, one investigates necessary optimality conditions. With the help of an exact separation theorem [8], one gives necessary optimality conditions for \( (P) \) in terms of Fréchet subdifferentials.

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