Resonance identity, stability and multiplicity of closed characteristics on compact convex hypersurfaces

*Wei Wang†, Xijun Hu‡, Yiming Long§

1 Chern Institute of Mathematics
2 Key Lab of Pure Mathematics and Combinatorics of Ministry of Education
Nankai University
Tianjin 300071, The People’s Republic of China
3 Institute of Mathematics, Academy of Mathematics and Systems Science
Chinese Academy of Sciences, Beijing 100080, The People’s Republic of China

Abstract

There is a long standing conjecture in Hamiltonian analysis which claims that there exist at least \( n \) geometrically distinct closed characteristics on every compact convex hypersurface in \( \mathbb{R}^{2n} \) with \( n \geq 2 \). Besides many partial results, this conjecture has been only completely solved for \( n = 2 \). In this paper, we give a confirmed answer to this conjecture for \( n = 3 \). In order to prove this result, we establish first a new resonance identity for closed characteristics on every compact convex hypersurface \( \Sigma \) in \( \mathbb{R}^{2n} \) when the number of geometrically distinct closed characteristics on \( \Sigma \) is finite. Then using this identity and earlier techniques of the index iteration theory, we prove the mentioned multiplicity result for \( \mathbb{R}^6 \). If there are exactly two geometrically distinct closed characteristics on a compact convex hypersurface in \( \mathbb{R}^4 \), we prove that both of them must be irrationally elliptic.

Key words: Convex compact hypersurfaces, closed characteristics, Hamiltonian systems, resonance identity, multiplicity, stability.

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†Partially supported by NNSF and RFDP of MOE of China. E-mail: alexanderweiwang@yahoo.com.cn
‡Partially supported by NNSF of China (No. 10526038). E-mail: xjhu@amss.ac.cn
§Partially supported by the 973 Program of MOST, Yangzi River Professorship, NNSF, MCME, RFDP, LPMC of MOE of China, and Nankai University. E-mail: longym@nankai.edu.cn
1 Introduction and main results

In this paper, let $\Sigma$ be a fixed $C^3$ compact convex hypersurface in $\mathbb{R}^{2n}$, i.e., $\Sigma$ is the boundary of a compact and strictly convex region $U$ in $\mathbb{R}^{2n}$. We denote the set of all such hypersurfaces by $\mathcal{H}(2n)$. Without loss of generality, we suppose $U$ contains the origin. We consider closed characteristics $(\tau, y)$ on $\Sigma$, which are solutions of the following problem

$$\begin{cases}
\dot{y} = JN_{\Sigma}(y), \\
y(\tau) = y(0),
\end{cases}$$

(1.1)

where $J = \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix}$, $I_n$ is the identity matrix in $\mathbb{R}^{n}$, $\tau > 0$, $N_{\Sigma}(y)$ is the outward normal vector of $\Sigma$ at $y$ normalized by the condition $N_{\Sigma}(y) \cdot y = 1$. Here $a \cdot b$ denotes the standard inner product of $a, b \in \mathbb{R}^{2n}$. A closed characteristic $(\tau, y)$ is \textit{prime}, if $\tau$ is the minimal period of $y$. Two closed characteristics $(\tau, y)$ and $(\sigma, z)$ are \textit{geometrically distinct}, if $y(\mathbb{R}) \neq z(\mathbb{R})$. We denote by $\mathcal{T}(\Sigma)$ the set of all geometrically distinct closed characteristics on $\Sigma$. A closed characteristic $(\tau, y)$ is \textit{non-degenerate}, if 1 is a Floquet multiplier of $y$ of precisely algebraic multiplicity 2, and is \textit{elliptic}, if all the Floquet multipliers of $y$ are on $U = \{ z \in \mathbb{C} ||z| = 1 \}$, i.e., the unit circle in the complex plane.

There is a long standing conjecture on the number of closed characteristics on compact convex hypersurfaces in $\mathbb{R}^{2n}$:

$$\# \mathcal{T}(\Sigma) \geq n, \quad \forall \Sigma \in \mathcal{H}(2n).$$

(1.2)

Since the pioneering works [Rab1] of P. Rabinowitz and [Wei1] of A. Weinstein in 1978 on the existence of at least one closed characteristic on every hypersurface in $\mathcal{H}(2n)$, the existence of multiple closed characteristics on $\Sigma \in \mathcal{H}(2n)$ has been deeply studied by many mathematicians. When $n \geq 2$, besides many results under pinching conditions, in 1987-1988 I. Ekeland-L. Lassoued, I. Ekeland-H. Hofer, and A, Szulkin (cf. [EkL1], [EkH1], [Szul1]) proved

$$\# \mathcal{T}(\Sigma) \geq 2, \quad \forall \Sigma \in \mathcal{H}(2n).$$

In [LoZ1] of 2002, Y. Long and C. Zhu further proved

$$\# \mathcal{T}(\Sigma) \geq \left[ \frac{n}{2} \right] + 1, \quad \forall \Sigma \in \mathcal{H}(2n),$$

where we denote by $[a] \equiv \max\{k \in \mathbb{Z} | k \leq a \}$. Note that this estimate yields still only at least 2 closed characteristics when $n = 3$. We refer readers to the survey paper [Lon5] and the recent [Lon6] of Y. Long for earlier works and references on this conjecture. Our following main result in this paper gives a confirmed answer to the conjecture (1.2) for $n = 3$. 

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Theorem 1.1. There holds $\#T(\Sigma) \geq 3$ for every $\Sigma \in H(6)$.

One of the main ingredients of our proof of this theorem is a new resonance identity on closed characteristics. In [Eke1] of 1984, I. Ekeland discovered that there must exist a resonance condition relating the closed characteristics on $\Sigma \in H(2n)$ provided $\#T(\Sigma) < +\infty$. However, he did not state explicitly what the resonance condition is. Then in [Vit1] of 1989, C. Viterbo clarified such a resonance condition by establishing a mean index identity for closed characteristics on compact star-shaped hypersurfaces in $\mathbb{R}^{2n}$ provided all closed characteristics on $\Sigma$ together with their iterations are non-degenerate (cf. p.234 of [Eke3]). Note that in [Rad1] of 1989 and [Rad2] of 1992, a similar identity for closed geodesics on compact Finsler manifolds was established by H.-B. Rademacher. Motivated by these results, in the current paper we establish the following mean index identity for closed characteristics on every $\Sigma \in H(2n)$ when $\#T(\Sigma) < +\infty$. This yields hopefully an explicit version of what I. Ekeland discovered.

Theorem 1.2. Suppose $\Sigma \in H(2n)$ satisfies $\#T(\Sigma) < +\infty$. Denote all the geometrically distinct closed characteristics by $\{(\tau_j, y_j)\}_{1 \leq j \leq k}$. Then the following identity holds

$$\sum_{1 \leq j \leq k} \frac{\hat{x}(y_j)}{i(y_j)} = \frac{1}{2},$$

(1.3)

where $\hat{i}(y_j) \in \mathbb{R}$ is the mean index of $y_j$ given by Definition 3.14, $\hat{x}(y_j) \in \mathbb{Q}$ is the average Euler characteristic given by Definition 3.15 and Remark 3.16 below. Specially by (3.56) below we have

$$\hat{x}(y) = \frac{1}{K(y)} \sum_{0 \leq i \leq 2n-2} (-1)^{i(y^m)+i} k_i(y^m),$$

(1.4)

$K(y) \in \mathbb{N}$ is the minimal period of critical modules of iterations of $y$ defined in Proposition 3.13, $i(y^m)$ is the Morse index of a corresponding dual-action functional at the $m$-th iteration $y^m$ of $y$ (cf. Definition 3.3 and Proposition 3.5 below), $k_i(y^m)$ is the critical type numbers of $y^m$ given by Definition 3.11 below.

Remark 1.3. Note that $1/2$ in the right hand side of (1.3) comes from the average Euler characteristic for equivariant homology on the space of loops in $\mathbb{R}^{2n}$. In fact, we have

$$\frac{1}{2} = \lim_{N \to \infty} \frac{1}{N} \sum_{q \leq N} \dim H_q(\mathbb{C}P^\infty; \mathbb{Q}) = \lim_{N \to \infty} \frac{1}{N} \sum_{q \leq N} \dim H_q(S^\infty \times S^1 \{0\}; \mathbb{Q}).$$

Since the space of loops in $\mathbb{R}^{2n}$ is $S^1$-equivariantly homotopic to its origin $\{0\}$, the last term in the above expression represents the average Euler characteristic of the $S^1$-equivariant homology of the space of loops in $\mathbb{R}^{2n}$. For details, we refer to Section 5 below.
When all the closed characteristics on $\Sigma \in \mathcal{H}(2n)$ together with their iterations are nondegenerate, by Remark 3.16 below our identity (1.3) coincides with the identity (1.3) of Theorem 1.2 of [Vit1] as well as (153) in p.234 of [Eke3]. Thus for $\Sigma \in \mathcal{H}(2n)$ our Theorem 1.2 generalizes C. Viterbo’s result in [Vit1] to degenerate closed characteristics.

Note that in [HWZ1] of 1998, H. Hofer-K. Wysocki-E. Zehnder proved that $\# T(\Sigma) = 2$ or $\infty$ holds for every $\Sigma \in \mathcal{H}(4)$. In [Lon3] of 2000, Y. Long proved further that $\Sigma \in \mathcal{H}(4)$ and $\# T(\Sigma) = 2$ imply that both of the closed characteristics must be elliptic, i.e., each of them possesses four Floquet multipliers with two 1s and the other two locate on the unit circle too. Now as a by-product of our Theorem 1.2 we obtain a stronger result:

**Theorem 1.4.** Let $\Sigma \in \mathcal{H}(4)$ satisfy $\# T(\Sigma) = 2$. Then both of the closed characteristics must be irrationally elliptic, i.e., each of them possesses four Floquet multipliers with two 1s and the other two located on the unit circle with rotation angles being irrational multiples of $\pi$.

Because of above mentioned results and other indications, we suspect that the following conjectures hold:

**Conjecture 1.5.** For every integer $n \geq 2$, there holds

$$\{ \# T(\Sigma) \mid \Sigma \in \mathcal{H}(2n) \} = \{ n \} \cup \{ +\infty \}.$$  

It seems that for $n \geq 3$ there is no effective methods so far which can be used to prove that $\# T(\Sigma) > n$ implies $\# T(\Sigma) = \infty$.

Recall that a closed characteristic is irrationally elliptic, if it is elliptic and the linearized Poincaré map is suitably homotopic to the $\diamond$-product of one $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and $n - 1$ rotation $2 \times 2$ matrices with rotation angles being irrational multiples of $\pi$. Note that based upon our studies on the stabilities of closed characteristics on $\Sigma \in \mathcal{H}(2n)$, and closed geodesics on Finsler spheres, we tend to believe that the following may hold.

**Conjecture 1.6.** All the geometrically distinct closed characteristics on $\Sigma$ are irrationally elliptic for $\Sigma \in \mathcal{H}(2n)$ with $n \geq 2$ whenever $\# T(\Sigma) < \infty$.

The rest of this paper is arranged as follows.

(1) Motivated by the works [Kli1] and [Kli2] of W. Klingenberg, [GrM2] of D. Gromoll and W. Meyer, [Eke1] and [Eke3] of I. Ekeland and [Vit1] of C. Viterbo, for every $\Sigma \in \mathcal{H}(2n)$ with $\# T(\Sigma) < +\infty$, we shall construct a functional $\Psi_a$ for large $a > 0$ on the space of loops in $\mathbb{R}^{2n}$ and establish a Morse theory of this functional $\Psi_a$ to study closed characteristics on $\Sigma$.

As usual we use the Clarke-Ekeland dual action principle and a modification of the Ekeland index theory. Because in general such a dual action functional is not $C^2$, motivated by the studies
on closed geodesics and convex Hamiltonian systems, we follow [Eke1] to introduce a finite dimensional approximation to the space of loops in $\mathbb{R}^{2n}$ to get the enough smoothness. For the dual action principle, the origin becomes an accumulation point of its critical values. To estimate the contribution of the critical point at the origin to the Morse Series, we construct a special family of Hamiltonian functions which have better properties at the origin and infinity, and are homogenous in the middle. Such a construction allows us to give a precise understanding of the behavior of the dual action functional near the origin.

(2) In Section 2, fixing a hypersurface $\Sigma \in \mathcal{H}(2n)$ with $\# T(\Sigma) < +\infty$, we construct a family $\mathcal{A}$ of Hamiltonian functions by Proposition 2.4 using auxiliary functions satisfying conditions (i)-(iv) of Proposition 2.2. Using such Hamiltonian functions, we construct a functional $\Psi_a$ on the space of loops in $\mathbb{R}^{2n}$ for every $a > 0$ whose critical points are precisely all the closed characteristics on $\Sigma$ with periods less than $a$ and that the origin of the loop space is the only constant critical point of $\Psi_a$.

(3) In Section 3, we prove that for every fixed closed characteristic $(\tau, y)$ on $\Sigma$, the critical modules of all the functionals $\Psi_a$ produced by $H_a \in \mathcal{A}$ at its critical point corresponding to $(\tau, y)$ are isomorphic to each other whenever $a > \tau$. Therefore we can further require the Hamiltonian function in $\mathcal{A}$ to be homogeneous near such critical points so that the critical modules are periodic functions of the dimension. This homogeneity of the Hamiltonian function is realized by the condition (v) of Proposition 2.2.

(4) Using the properties of the Hamiltonian functions in $\mathcal{A}$, in Section 4, the property of the dual action functional near the origin is understood precisely and we show that the origin has in fact no homological contribution to the lower order terms in the Morse series.

(5) Using the homological information obtained in the Sections 2-4, in Section 5, we compute all the local critical modules of the dual action functional $\Psi_a$ and use such information to set up a Morse theory for all the closed characteristics on $\Sigma \in \mathcal{H}(2n)$. Together with the global homological information on the loop space we establish the claimed mean index identity (1.3) and prove Theorem 1.2.

(6) Using Theorem 1.2 together with the techniques developed in the index iteration theory we give proofs of Theorems 1.1 and 1.4 in Section 6.

Here we give a brief sketch for the proof of Theorem 1.1. Assuming that Theorem 1.1 does not hold, i.e., $\# T(\Sigma) \leq 2$, by [EkH1] or [LoZ1] we should have $\# T(\Sigma) = 2$. By our Theorem 1.2 the
two prime closed characteristics \((\tau_1, y_1)\) and \((\tau_2, y_2)\) must satisfy the following identity:

\[
\frac{\dot{\chi}(y_1)}{i(y_1)} + \frac{\dot{\chi}(y_2)}{i(y_2)} = \frac{1}{2}.
\]

(1.5)

Here it is well known that \(\dot{i}(y_j) > 2\) for \(j = 1\) and \(2\) always holds (cf. Theorem 1.7.7 of [Eke3] or Lemma 15.3.2 of [Lon4]). By [LoZ1] (cf. Theorem 15.5.2 of [Lon4]), at least one of \(\dot{i}(y_1)\) and \(\dot{i}(y_2)\) is irrational, say \(\dot{i}(y_1) \in \mathbb{R} \setminus \mathbb{Q}\) without loss of generality.

Then if \(\dot{\chi}(y_1) \neq 0\), we obtain \(i(y_2)\) is irrational too by (1.5). Thus by a careful derivation using the index iteration formulae of Y. Long in [Lon3] and estimates obtained by Y. Long and C. Zhu in [LoZ1], there should exist more than two closed characteristics which yields a contradiction.

If \(\dot{\chi}(y_1) = 0\), by the index iteration formulae of Y. Long in [Lon3], one can prove that the orbit \(y_2\) must satisfy

\[
\frac{\dot{\chi}(y_2)}{i(y_2)} \leq \frac{1}{4}.
\]

(1.6)

Then together with (1.5), it yields a contradiction too and proves the theorem.

In this paper, let \(\mathbb{N}, \mathbb{N}_0, \mathbb{Z}, \mathbb{Q}, \mathbb{R},\) and \(\mathbb{R}^+\) denote the sets of natural integers, non-negative integers, integers, rational numbers, real numbers, and positive real numbers respectively. Denote by \(a \cdot b\) and \(|a|\) the standard inner product and norm in \(\mathbb{R}^{2n}\). Denote by \(\langle \cdot, \cdot \rangle\) and \(\| \cdot \|\) the standard \(L^2\) inner product and \(L^2\) norm. For an \(S^1\)-space \(X\), we denote by \(X_{S^1}\) the homotopy quotient of \(X\) by \(S^1\), i.e., \(X_{S^1} = S^\infty \times_{S^1} X\), where \(S^\infty\) is the unit sphere in an infinite dimensional complex Hilbert space. In this paper we use only \(\mathbb{Q}\) coefficients for all homological modules. By \(t \rightarrow a^+\), we mean \(t > a\) and \(t \rightarrow a\).

2 A variational structure for closed characteristics

In the rest of this paper, we fix a \(\Sigma \in \mathcal{H}(2n)\) and assume the following condition on \(T(\Sigma)\):

(F) There exist only finitely many geometrically distinct closed characteristics \(\{(\tau_j, y_j)\}_{1 \leq j \leq k}\) on \(\Sigma\).

In this section, we transform the problem (1.1) into a fixed period problem of a Hamiltonian system and then study its variational structure. We introduce the following set:

**Definition 2.1.** Under the assumption (F), the set of periods on \(\Sigma\) is defined by

\[
\text{per}(\Sigma) = \{m\tau_j \mid m \in \mathbb{N}, 1 \leq j \leq k\}.
\]

Clearly \(\text{per}(\Sigma)\) is a discrete subset of \(\mathbb{R}^+\). Motivated by Definition 2.1 of [Eke1] and Lemma 2.2 of [Vit1], we construct the following auxiliary function to further define Hamiltonian functions.
Proposition 2.2. For any sufficiently small \( \theta \in (0, 1) \), there exists a function \( \varphi \equiv \varphi_0 \in C^\infty(\mathbf{R}, \mathbf{R}^+) \) depending on \( \theta \) which has 0 as its unique critical point in \([0, +\infty)\) such that the following hold

(i) \( \varphi(0) = 0 = \varphi'(0) \) and \( \varphi''(0) = 1 = \lim_{t \to 0^+} \frac{\varphi'(t)}{t} \).

(ii) \( \varphi(t) \) is a polynomial of degree 2 in a neighborhood of \(+\infty\).

(iii) \( \frac{d}{dt} \left( \frac{\varphi'(t)}{t} \right) < 0 \) for \( t > 0 \), and \( \lim_{t \to +\infty} \frac{\varphi'(t)}{t} < \theta \), i.e., \( \frac{\varphi'(t)}{t} \) is strictly decreasing for \( t > 0 \).

(iv) \( \min(\varphi'(t), \varphi''(t)) \geq \sigma \) for all \( t \in \mathbf{R}^+ \) and some \( \sigma > 0 \). Consequently, \( \varphi \) is strictly convex on \([0, +\infty)\).

(v) In particular, we can choose \( \alpha \in (1, 2) \) sufficiently close to 2 and \( c \in (0, 1) \) such that \( \varphi(t) = ct^\alpha \) whenever \( \varphi'(t) \in [\theta, 1-\theta] \) and \( t > 0 \).

Proof. We construct a \( \varphi \) satisfying (i)-(v).

Define \( \varphi_1(t) = \left( \frac{\alpha^2 - 7\alpha + 12}{12} \right) t^2 + (-\alpha^2 + 6\alpha - 8)t^3 + \left( \frac{\alpha^2 - 5\alpha + 6}{2} \right) t^4 \) for \( t \in (-\infty, 1] \) and \( \varphi_1(t) = t^\alpha \) for \( t \in [1, +\infty) \). Then \( \varphi_1 \in C^2 \) and \( \frac{\varphi_1''(t)}{t} \) is strictly decreasing for \( t > 0 \) and \( \alpha \in (1, 2) \). Note that \( \lim_{t \to 0^+} \varphi_1'(t) = \varphi_1''(0) = \alpha^2 - 7\alpha + 12 > 2 \) and \( \varphi_1(1) = \alpha < 2 \). Hence \( \varphi_2(t) \equiv \frac{\varphi_1'(t)}{\varphi_1(t)} \) satisfies (i) and (iii). Next we further modify \( \varphi_2 \) in a neighborhood of \(+\infty\).

Since \( \frac{\varphi_2'(t)}{t} \) tends to 0 when \( t \) goes to \(+\infty\), we obtain a \( T = T_0 > 1 \) sufficiently large such that

\[
\frac{\varphi_2'(t)}{t} = c\alpha t^{\alpha-2} \in \left( 0, \frac{\theta}{2\alpha - 1} \right), \quad \forall t \geq T,
\]

where \( c = c_\alpha = (\alpha^2 - 7\alpha + 12)^{-1} \). Now we define

\[
\varphi_3(t) = \begin{cases} 
\varphi_2(t), & t \in (-\infty, T]; \\
ct^\alpha + \alpha T^{\alpha-1}(t-T) + \frac{c(\alpha-1)T^{\alpha-2}}{2}(t-T)^2, & t \in [T, +\infty). 
\end{cases}
\]

Then \( \varphi_3 \in C^\infty(\mathbf{R} \setminus \{1, T\}, \mathbf{R}^+) \cap C^2(\mathbf{R}, \mathbf{R}^+) \). We can approximate \( \varphi_3 \) by a smooth function \( \varphi \) (cf. Theorem 2.5 of [Hir1]) such that \( \varphi = \varphi_3 \) holds outside a small neighborhood of \( \{1, T\} \), and \( \|\varphi - \varphi_3\|_{C^2} \) is small enough. Then it is easy to see that \( \varphi \) satisfies (i)-(iv) of the proposition.

Note that \( f(t) \equiv \frac{\varphi_1'(t)}{t} \) is a strictly decreasing function with \( f(1) = \frac{\alpha}{\alpha^2 - 7\alpha + 12} < 1 \). Since \( f(1) \) tends to 1 as \( \alpha \) goes to 2, if \( \alpha \) is chosen sufficiently close to 2, then \( f(t) > 1 - \theta \) holds for all \( t \in [0, 1] \). Together with (2.1) it is easy to verify that this \( \varphi \) satisfies (v).

Remark 2.3. 1° Note that in the above proof for (v), we can choose \( \alpha \in (1, 2) \) sufficiently close to 2, \( c \in (0, 1) \), and \( T \) sufficiently large such that \( \varphi(t) = ct^\alpha \) if and only if \( 1 + \delta \leq t \leq T - \delta \) for some \( \delta > 0 \). Note that the property (v) above is used only in the second part of the proof of Proposition 3.5 below to show that our index and nullity given by Definition 3.3 below coincide with those defined in [Eke1], [Eke3], and in our study in the Subsection 3.2 to obtain the periodic
property of critical modules at critical points. In the other parts of this paper we use functions \( \varphi \) which satisfy the properties (i)-(iv).

2° Note that in the following, we only need the definition of \( \varphi \) on \([0, +\infty)\). In Proposition 2.4 below, the parameter \( \vartheta \) given by Proposition 2.2 depends on the parameter \( a \), i.e., given an \( a > \hat{\tau} \) as in Proposition 2.4, we choose first the parameter \( \vartheta \in (0, \frac{1}{a} \min\{\hat{\tau}, \hat{\sigma}\}) \). Then we can choose the parameter \( \alpha \in (1, 2) \) depending on \( a \) and let \( \varphi \) to be homogeneous of degree \( \alpha \) and then modify it near 0 and \(+\infty\) such that (i)-(iv) in Proposition 2.2 hold. Here we do not require \( \varphi \) to satisfy (v) in Proposition 2.2. We denote such choices of \( \vartheta, \alpha \) and \( \varphi \) by \( \vartheta_a, \alpha_a \) and \( \varphi_a \) respectively to indicate their dependence on \( a \). In such a way, we can obtain a connected family of \( \varphi_a \) continuously depending on \( a \). Each \( \varphi_a \) in this family satisfies properties (i)-(iv) of Proposition 2.2. Moreover, the first and second derivatives of \( \varphi_a(t) \) with respect to \( t \) are also continuous in the parameter \( a \).

Note that under these choices, the coefficients of the polynomials in the proof of (ii) of Proposition 2.2 are continuous in \( a \). Here that \( \varphi_a \)s form a connected family in \( a \) is crucial in our study below, for example in the proofs of Proposition 3.2, Lemma 3.4 and Proposition 3.5.

Let \( j : \mathbb{R}^{2n} \to \mathbb{R} \) be the gauge function of \( \Sigma \), i.e., \( j(\lambda x) = \lambda \) for \( x \in \Sigma \) and \( \lambda \geq 0 \), then \( j \in C^3(\mathbb{R}^{2n} \setminus \{0\}, \mathbb{R}) \cap C^0(\mathbb{R}^{2n}, \mathbb{R}) \) and \( \Sigma = j^{-1}(1) \). Denote by \( \hat{\tau} = \inf\{s \mid s \in \text{per}(\Sigma)\} \) and \( \hat{\sigma} = \min\{|y|^2 \mid y \in \Sigma\} \).

**Proposition 2.4.** Let \( a > \hat{\tau}, \vartheta_a \in \left(0, \frac{1}{a} \min\{\hat{\tau}, \hat{\sigma}\}\right) \) and \( \varphi_a \) be a \( C^\infty \) function associated to \( \vartheta_a \) satisfying (i)-(iv) of Proposition 2.2. Define the Hamiltonian function \( H_a(x) = a\varphi_a(j(x)) \) and consider the fixed period problem

\[
\begin{align*}
\dot{x}(t) &= JH_a'(x(t)) \\
x(0) &= x(1)
\end{align*}
\] (2.3)

Then the following hold:

(i) \( H_a \in C^3(\mathbb{R}^{2n} \setminus \{0\}, \mathbb{R}) \cap C^1(\mathbb{R}^{2n}, \mathbb{R}) \) and there exist \( R, r > 0 \) such that

\[
 r|\xi|^2 \leq H_a''(x)\xi : \xi \leq R|\xi|^2, \quad \forall x \in \mathbb{R}^{2n} \setminus \{0\}, \xi \in \mathbb{R}^{2n}.
\]

(ii) There exist \( \epsilon_1, \epsilon_2 \in \left(0, \frac{1}{\tau}\right) \) and \( C \in \mathbb{R} \), such that

\[
\frac{\epsilon_1|x|^2}{2} - C \leq H_a(x) \leq \frac{\epsilon_2|x|^2}{2} + C, \quad \forall x \in \mathbb{R}^{2n}.
\]

(iii) Solutions of (2.3) are \( x \equiv 0 \) and \( x = py(\tau t) \) with \( \varphi_a(y) = \frac{r}{a} \), where \((\tau, y)\) is a solution of (1.1). In particular, nonzero solutions of (2.3) are in one to one correspondence with solutions of (1.1) with period \( \tau < a \).
(iv) There exists \( r_0 > 0 \) independent of \( a \) and there exists \( \mu_a > 0 \) depending on \( a \) such that

\[
H_a''(x) \xi \cdot \xi \geq 2a r_0 |\xi|^2, \quad \text{for} \quad 0 < |x| \leq \mu_a, \xi \in \mathbb{R}^{2n}.
\]

**Proof.** Since \( j(\lambda x) = \lambda j(x) \) for all \( x \in \mathbb{R}^{2n} \setminus \{0\} \) and \( \lambda \in \mathbb{R}^+ \), we have \( j'(\lambda x) = j'(x) \) and \( j''(\lambda x) = \lambda^{-1} j''(x) \). Hence for \( x = \lambda y \) with \( y \in \Sigma \) and \( \lambda \in \mathbb{R}^+ \) we have

\[
H_a''(x) \xi \cdot \xi = H_a''(\lambda y) \xi \cdot \xi = a \varphi_a''(\lambda y)(j'(\lambda y) \xi)^2 + a \varphi_a'(\lambda y) j''(\lambda y) \xi \cdot \xi = a \varphi_a''(\lambda)(j'(y) \xi)^2 + a \varphi_a'(\lambda) \lambda^{-1} j''(y) \xi \cdot \xi \geq a \sigma(j'(y) \xi)^2 + j''(y) \xi \cdot \xi, \quad (2.4)
\]

where the last inequality follows from (iv) of Proposition 2.2. Now fix \( y \in \Sigma \) and represent \( \mathbb{R}^{2n} = \mathbb{R} y \oplus T_y \Sigma \). We define a new norm in \( \mathbb{R}^{2n} \) by

\[
|z|_y^2 = \lambda^2 + |z_2|^2, \quad \forall z = \lambda y + z_2 \in \mathbb{R} y \oplus T_y \Sigma. \quad (2.6)
\]

Since any two norms on \( \mathbb{R}^{2n} \) are equivalent, we have

\[
C_1(y)^{-1} |z| \leq |z|_y \leq C_1(y) |z|, \quad (2.7)
\]

for some constant \( C_1(y) > 0 \) depending on \( y \). Note that \( j'(y) = N_{\Sigma}(y) \) by the fact that \( N_{\Sigma}(y) \cdot y = 1 \) and \( j'(y) \cdot y = j(y) = 1 \) for every \( y \in \Sigma \). Since \( j(\lambda y) = \lambda j(y) \), we have \( j'(y) \cdot y = j(y) \), hence \( j''(y)y = 0 \). For \( \xi = \lambda y + \xi_2 \), we have

\[
(j'(y) \xi)^2 + j''(y) \xi \cdot \xi = (j'(y) \cdot (\lambda y + \xi_2))^2 + j''(y)(\lambda y + \xi_2) \cdot (\lambda y + \xi_2) = (j'(y) \cdot \lambda y)^2 + j''(y) \xi_2 \cdot \xi_2 \geq \lambda^2 + C_2(y) |\xi_2|^2 \geq C_3(y) |\xi|^2,
\]

for some positive constants \( C_2(y) \) and \( C_3(y) \) depending on \( y \). Here the first inequality holds since \( \Sigma \) is strictly convex, hence \( j''(y) \mid_{T_y \Sigma} \) is positive definite. The last inequality follows from (2.6) and (2.7). By the compactness of \( \Sigma \) and (2.7) we have \( H_a''(x) \xi \cdot \xi \geq r |\xi|^2 \) for some \( r > 0 \). The compactness of \( \Sigma \) and (2.4) yield \( H_a''(x) \xi \cdot \xi \leq R |\xi|^2 \) for some \( R > 0 \). This proves (i).

For (ii), it suffices to consider \( |x| \) large. Hence suppose \( x = \lambda y \) for \( y \in \Sigma \) and \( \lambda > 0 \), then \( |x| = \lambda |y| \), so \( \lambda \) is large. By (ii) and (iii) of Proposition 2.2, we have \( H_a(x) = a \varphi_a(\lambda) = D_0 + D_1 \lambda + D_2 \lambda^2 \) for some \( 0 < 2D_2 < a \varphi_a(b) \). Hence \( H_a(x) = D_0 + \frac{D_1}{|y|^2} |x| + \frac{D_2}{|y|^4} |x|^2 \) with \( \frac{D_2}{|y|^4} < \frac{a \varphi_a(b)}{2|y|^2} < \frac{1}{2} \) by the definition of \( \sigma \). This proves (ii).
Clearly $x \equiv 0$ is the unique constant solution of (2.3). Suppose $x(t)$ is a nonconstant solution of (2.3), then $H_{a}(x(t)) = a\varphi_{a}(j(x(t))) = \text{const}$. Since $\varphi_{a}$ is strictly increasing, we have $j(x(t)) = \text{const}$. Let $\rho = j(x(t))$ and $y(t) = \rho^{-1}x\left(\frac{\rho}{a\varphi_{a}(\rho)}t\right)$. Then $j(y) = \rho^{-1}j(x) = \rho^{-1}\rho = 1$, hence $y(\mathbb{R}) \subset \Sigma$. Moreover, we have $\dot{y}(t) = JN_{\Sigma}(y(t))$ by (2.3). Hence $\frac{a\varphi_{a}(\rho)}{\rho}, y$ is a solution of (1.1). By (i) and (iii) of Proposition 2.2, we have $\varphi_{a}(\rho) < 1$. Hence $\tau = \frac{a\varphi_{a}(\rho)}{\rho} < a$. This together with $a\vartheta_{a} < \hat{\tau}$ proves one side of (iii). The other side of (iii) can be proved similarly and thus is omitted. Proposition 2.10 together with the proof of (i) and Proposition 2.2 (i) yield (iv).

In the following, we will use the Clarke-Ekeland dual action principle. As usual, the Fenchel transform of a function $F : \mathbb{R}^{2n} \rightarrow \mathbb{R}$ is defined by

$$F^{\ast}(y) = \sup\{x \cdot y - F(x) \mid x \in \mathbb{R}^{2n}\}. \quad (2.8)$$

Following Proposition 2.2.10 of [Eke3], Lemma 3.1 of [Eke1] and the fact that $F_{1} \leq F_{2} \iff F_{1}^{\ast} \geq F_{2}^{\ast}$, we have:

**Proposition 2.5.** Let $H_{a}$ be a function defined in Proposition 2.4 and $G_{a} = H_{a}^{\ast}$ the Fenchel transform of $H_{a}$. Then we have

(i) $G_{a} \in C^{2}(\mathbb{R}^{2n} \setminus \{0\}, \mathbb{R}) \cap C^{1}(\mathbb{R}^{2n}, \mathbb{R})$ and

$$G'_{a}(y) = x \iff y = H'_{a}(x) \Rightarrow H''_{a}(x)G''_{a}(y) = 1.$$ 

(ii) $G_{a}$ is strictly convex. Let $R$ and $r$ be the real numbers given by (i) of Proposition 2.4. Then we have

$$R^{-1}|\xi|^{2} \leq G''_{a}(y)\xi \cdot \xi \leq r^{-1}|\xi|^{2}, \quad \forall y \in \mathbb{R}^{2n} \setminus \{0\}, \xi \in \mathbb{R}^{2n}.$$ 

(iii) Let $\epsilon, \epsilon_{2}, C$ be the real numbers given by (ii) of Proposition 2.4. Then we have

$$\frac{|x|^{2}}{2\epsilon_{2}} - C \leq G_{a}(x) \leq \frac{|x|^{2}}{2\epsilon_{1}} + C, \quad \forall x \in \mathbb{R}^{2n}.$$ 

(iv) Let $r_{0} > 0$ be the constant given by (iv) of Proposition 2.4. Then there exists $\eta_{a} > 0$ depending on $a$ such that the following holds

$$G''_{a}(y)\xi \cdot \xi \leq \frac{1}{2ar_{0}}|\xi|^{2}, \quad \text{for} \quad 0 < |y| \leq \eta_{a}, \xi \in \mathbb{R}^{2n}.$$ 

(v) In particular, let $H_{a} = a\varphi_{a}(j(x))$ with $\varphi_{a}$ satisfying further (v) of Proposition 2.2. Then we have $G_{a}(\mu j(z)) = c_{1}\mu^{\beta}$ when $z \in \Sigma$ and $\mu j(z) \in \{H'_{a}(x) \mid H_{a}(x) = acj(x)^{\alpha}\}$, where $c$ is given by (v) of Proposition 2.2, $c_{1} > 0$ is some constant and $\alpha^{-1} + \beta^{-1} = 1$ holds with $\alpha = \alpha_{a}$ and $\beta = \beta_{a}$ depending on $a$. \hfill \blacksquare
Now we apply the dual action principle to problem (2.3). Let

\[ L_0^2(S^1, \mathbb{R}^{2n}) = \left\{ u \in L^2([0,1], \mathbb{R}^{2n}) \mid \int_0^1 u(t) dt = 0 \right\}. \tag{2.9} \]

Define a linear operator \( M : L_0^2(S^1, \mathbb{R}^{2n}) \to L_0^2(S^1, \mathbb{R}^{2n}) \) by

\[ \frac{d}{dt} Mu(t) = u(t), \quad \int_0^1 Mu(t) dt = 0. \tag{2.10} \]

The dual action functional on \( L_0^2(S^1, \mathbb{R}^{2n}) \) is defined by

\[ \Psi_a(u) = \int_0^1 \left( \frac{1}{2} Ju \cdot Mu + G_a(-Ju) \right) dt, \tag{2.11} \]

where \( G_a \) is given by Proposition 2.5.

By (ii) of Proposition 2.5 and the proof of Proposition 3.3 on p.33 of [Eke1], we have

**Proposition 2.6.** The functional \( \Psi_a \) is \( C^{1,1} \) on \( L_0^2(S^1, \mathbb{R}^{2n}) \). Suppose \( x \) is a solution of (2.3), then \( u = \dot{x} \) is a critical point of \( \Psi_a \). Conversely, suppose \( u \) is a critical point of \( \Psi_a \), then there exists a unique \( \xi \in \mathbb{R}^{2n} \) such that \( Mu - \xi \) is a solution of (2.3). In particular, solutions of (2.3) are in one to one correspondence with critical points of \( \Psi_a \).

**Proposition 2.7.** The functional \( \Psi_a \) is bounded from below on \( L_0^2(S^1, \mathbb{R}^{2n}) \).

**Proof.** For any \( u \in L_0^2(S^1, \mathbb{R}^{2n}) \), we represent \( u \) by its Fourier series

\[ u(t) = \sum_{k \neq 0} e^{k2\pi Jt} x_k, \quad x_k \in \mathbb{R}^{2n}. \tag{2.12} \]

Then we have

\[ Mu(t) = -J \sum_{k \neq 0} \frac{1}{2\pi k} e^{k2\pi Jt} x_k. \tag{2.13} \]

Hence

\[ \frac{1}{2} \langle Ju, Mu \rangle = -\frac{1}{2} \sum_{k \neq 0} \frac{1}{2\pi k} |x_k|^2 \geq -\frac{1}{4\pi} \|u\|^2. \tag{2.14} \]

By (2.11), we have

\[
\begin{align*}
\Psi_a(u) & = \int_0^1 \left( \frac{1}{2} Ju \cdot Mu + G_a(-Ju) \right) dt \\
& \geq \frac{1}{2} \langle Ju, Mu \rangle + \int_0^1 \left( \frac{|u|^2}{2\epsilon_2} - C \right) dt \\
& \geq \left( \frac{1}{2\epsilon_2} - \frac{1}{4\pi} \right) \|u\|^2 - C \\
& \geq C_4 \|u\|^2 - C \quad \tag{2.15}
\end{align*}
\]
for some constant $C_4 > 0$, where in the first inequality, we have used (iii) of Proposition 2.5. Hence the proposition holds.

By (2.15) and the proof of Lemma 5.2.8 of [Eke3], we have

**Proposition 2.8.** The functional $\Psi_a$ satisfies the Palais-Smale condition on $L^2_0(S^1, \mathbb{R}^{2n})$.

**Proposition 2.9.** $\Psi_a(u_a) < 0$ for every critical point $u_a \neq 0$ of $\Psi_a$.

**Proof.** By Proposition 2.4, we have $u_a = \dot{x}_a$ and $x_a = \rho_a y(\tau t)$ with

$$\frac{\varphi'_a(\rho_a)}{\rho_a} = \frac{\tau}{a}. \quad (2.16)$$

Hence we have

$$\Psi_a(u_a) = \int_0^1 \left( \frac{1}{2} J\dot{x}_a \cdot x_a + G_a(-J\dot{x}_a) \right) dt$$

$$= -\frac{1}{2}(H'_a(x_a), x_a) + \int_0^1 G_a(H'_a(x_a)) dt$$

$$= \frac{1}{2}a\varphi'_a(\rho_a)\rho_a - a\varphi_a(\rho_a). \quad (2.17)$$

Here the second equality follows from (2.3) and the third equality follows from (i) of Proposition 2.5 and (2.8).

Let $f(t) = \frac{1}{2}a\varphi'_a(t)t - a\varphi_a(t)$ for $t \geq 0$. Then we have $f(0) = 0$ and $f'(t) = \frac{a}{2}(\varphi''_a(t)t - \varphi'_a(t)) < 0$ since $\frac{d}{dt}(\varphi'_a(t)t) < 0$ by (iii) of Proposition 2.2. This together with (2.16) yield the proposition.

### 3 Critical modules for closed characteristics

In this section, we define the critical modules of closed characteristics and study some properties of them.

#### 3.1 Basic properties of critical modules

We have a natural $S^1$-orthogonal action on $L^2_0(S^1, \mathbb{R}^{2n})$ defined by

$$\theta \cdot u(t) = u(\theta + t), \quad \forall \theta \in S^1, t \in \mathbb{R}. \quad (3.1)$$

Clearly $\Psi_a$ is $S^1$-invariant. For any $\kappa \in \mathbb{R}$, we denote by

$$\Lambda^\kappa_a = \{ u \in L^2_0(S^1, \mathbb{R}^{2n}) \mid \Psi_a(u) \leq \kappa \}. \quad (3.2)$$

For a critical point $u$ of $\Psi_a$, we denote by

$$\Lambda_a(u) = \Lambda^\Psi_a(u) = \{ w \in L^2_0(S^1, \mathbb{R}^{2n}) \mid \Psi_a(w) \leq \Psi_a(u) \}. \quad (3.3)$$
Clearly, both sets are $S^1$-invariant. Since the $S^1$-action preserves $\Psi_a$, if $u$ is a critical point of $\Psi_a$, then the whole orbit $S^1 \cdot u$ is formed by critical points of $\Psi_a$. Denote by $\text{crit}(\Psi_a)$ the set of critical points of $\Psi_a$. Note that by the condition $(F)$, $(iii)$ of Proposition 2.4 and Proposition 2.6, the number of critical orbits of $\Psi_a$ is finite. Hence as usual we can make the following definition.

**Definition 3.1.** Suppose $u$ is a nonzero critical point of $\Psi_a$, and $\mathcal{N}$ is an $S^1$-invariant open neighborhood of $S^1 \cdot u$ such that $\text{crit}(\Psi_a) \cap (\Lambda_a(u) \cap \mathcal{N}) = S^1 \cdot u$. Then the $S^1$-critical modules of $S^1 \cdot u$ is defined by

$$C_{S^1, q}(\Psi_a, S^1 \cdot u) = H_{S^1, q}(\Lambda_a(u) \cap \mathcal{N}, (\Lambda_a(u) \setminus S^1 \cdot u) \cap \mathcal{N})$$

$$\equiv H_q((\Lambda_a(u) \cap \mathcal{N})_{S^1}, ((\Lambda_a(u) \setminus S^1 \cdot u) \cap \mathcal{N})_{S^1}),$$

(3.4)

where $H_{S^1, *}$ is the $S^1$-equivariant homology with rational coefficients in the sense of A. Borel (cf. Chapter IV of [Bor1]).

Note that this definition is independent of the choice of $\mathcal{N}$ by the excision property of the singular homology theory (cf. Definition 1.7.5 of [Cha1]). Recall that $X_{S^1}$ is defined at the end of Section 1.

We have the following proposition for critical modules.

**Proposition 3.2.** The critical module $C_{S^1, q}(\Psi_a, S^1 \cdot u)$ is independent of the choice of $H_a$ defined in Proposition 2.4 in the sense that if $x_i$ are solutions of (2.3) with Hamiltonian functions $H_{a_i}(x) \equiv a_i \varphi_{a_i}(\phi(x))$ for $i = 1$ and $2$ respectively such that both $x_1$ and $x_2$ correspond to the same closed characteristic $(\tau, y)$ on $\Sigma$. Then we have

$$C_{S^1, q}(\Psi_{a_1}, S^1 \cdot \dot{x}_1) \cong C_{S^1, q}(\Psi_{a_2}, S^1 \cdot \dot{x}_2), \quad \forall q \in \mathbb{Z}.$$  

(3.5)

In other words, the critical modules are invariant for all $a > \tau$ and $\varphi_a$ satisfying $(i)$-$(iv)$ of Proposition 2.2.

**Proof.** Fix a closed characteristic $(\tau, y)$ on $\Sigma$. We assume first $\tau < a_1 < a_2$. Let $\varphi_a$ be a family of functions satisfying $(i)$-$(iv)$ of Proposition 2.2 and $H_a(x) = a \varphi_a(\phi(x))$ satisfying Proposition 2.4 parametrized by $a \in [a_1, a_2]$. Without loss of generality we can assume $\varphi_a$ depends continuously on $a$ in the sense of Remark 2.3. For each $a \in [a_1, a_2]$, we denote by $x_a$ the corresponding solution of (2.3) with Hamiltonian $H_a$. Firstly we prove the following

**Claim.** For each $a \in [a_1, a_2]$ and $\varepsilon$ near 0, we have

$$|G_{a + \varepsilon}(y) - G_a(y)| = O(\varepsilon) + O(\varepsilon)|y|^2, \quad \forall y \in \mathbb{R}^{2n},$$

(3.6)

$$|G'_{a + \varepsilon}(y) - G'_a(y)| = O(\varepsilon) + O(\varepsilon)|y|, \quad \forall y \in \mathbb{R}^{2n},$$

(3.7)
where we denote by $B = O(\varepsilon)$ if $|B| < C|\varepsilon|$ for some constant $C > 0$.

In fact, fix an $a \in [a_1, a_2]$ and let $b \in (a - \varepsilon, a + \varepsilon)$. For any $y \in \mathbb{R}^{2n}$, we have $y = \lambda j'(\xi)$ for some $\lambda \geq 0$ and $\xi \in \Sigma$. Let $x = G'_b(y)$, then by (i) of Proposition 2.5, we have $\lambda j'(\xi) = y = H'_b(x) = b\varphi'_b(j(x))j'(x)$. Hence $x = \mu \xi$ for some $\mu > 0$. This yields

$$\lambda j'(\xi) = b\varphi'_b(j(x))j'(\xi).$$

(3.8)

Hence $\lambda = b\varphi'_b(j(x))$. Then $j(x) = (\varphi'_b)^{-1}(\lambda/b)$. Because $x = j(x)\xi$, we obtain

$$x = (\varphi'_b)^{-1}(\lambda/b)\xi, \quad G'_b(y) = x = (\varphi'_b)^{-1}(\lambda/b)\xi.$$  

(3.9)

Hence we have

$$|G'_{a+\varepsilon}(y) - G'_a(y)| = |(\varphi'_{a+\varepsilon})^{-1}(\lambda/(a+\varepsilon)) - (\varphi'_a)^{-1}(\lambda/a)||\xi|.$$  

It suffices to consider large $|y|$, where $(\varphi'_b)^{-1}$ is a polynomial of degree 1 and whose coefficients depend continuously on $b$ by (ii) of Proposition 2.2. Hence (3.7) holds.

For (3.6), we have

$$G_b(y) = x \cdot y - H_b(x) = \lambda (\varphi'_b)^{-1}(\lambda/b) - b\varphi_b((\varphi'_b)^{-1}(\lambda/b)).$$

As above for large $|y|$ by (ii) of Proposition 2.2 we may assume $\varphi_b$ is a polynomial of degree 2 and whose coefficients depend continuously on $b$, this proves (3.6) and the whole claim.

Now we have the following estimates:

$$|\Psi_{a+\varepsilon}(u) - \Psi_a(u)| \leq \int_0^1 |G_{a+\varepsilon}(-Ju) - G_a(-Ju)|dt = O(\varepsilon) + O(\varepsilon)||u||^2,$$

(3.10)

$$||\Psi'_{a+\varepsilon}(u) - \Psi'_a(u)||^2 = ||JG'_{a+\varepsilon}(u) - JG'_a(u)||^2 = O(\varepsilon) + O(\varepsilon)||u||^2.$$  

(3.11)

In particular, (3.10) and (3.11) imply that $b \mapsto \Psi_b$ is continuous in the $C^1$ topology. Note that the number of critical orbits of each $\Psi_b$ is finite. Hence by the continuity of critical modules (cf. Theorem 8.8 of [MaW1] or Theorem 1.5.6 on p.53 of [Cha1], which can be easily generalized to the equivariant sense), our proposition holds. Note that a similar argument as above shows that the critical modules are independent of the choice of $\varphi_a$ in $H_a(x) = a\varphi_a(j(x))$ whenever $a$ is fixed and $\varphi_a$ satisfies (i) to (iv) of Proposition 2.2.

We say that $\Psi_a$ with $a \in [a_1, a_2]$ form a continuous family of functionals in the sense of Proposition 3.2.

In order to compute the critical modules, as in p.35 of [Eke1] and p.219 of [Eke3] we introduce the following.
Definition 3.3. Suppose $u$ is a nonzero critical point of $\Psi_a$. Then the formal Hessian of $\Psi_a$ at $u$ is defined by

$$Q_a(v,v) = \int_0^1 (Jv \cdot Mv + G''_a(-Ju)Jv \cdot Jv)dt,$$

which defines an orthogonal splitting $L_0^2 = E_- \oplus E_0 \oplus E_+ \oplus L_0^2(S^1, \mathbb{R}^{2n})$ into negative, zero and positive subspaces. The index of $u$ is defined by $i(u) = \dim E_-$ and the nullity of $u$ is defined by $\nu(u) = \dim E_0$.

Next we show that the index and nullity defined as above are the Morse index and nullity of a corresponding functional on a finite dimensional subspace of $L_0^2(S^1, \mathbb{R}^{2n})$.

Lemma 3.4. Let $\Psi_a$ with $a \in [a_1, a_2]$ be a continuous family of functionals defined by (2.11). Then there exist a finite dimensional $S^1$-invariant subspace $X$ of $L_0^2(S^1, \mathbb{R}^{2n})$ and a family of $S^1$-equivariant maps $h_a : X \to X^\perp$ such that the following hold

(i) For $g \in X$, each function $h \mapsto \Psi_a(g + h)$ has $h_a(g)$ as the unique minimum in $X^\perp$.

(ii) Each $\psi_a$ is $C^1$ on $X$ and $S^1$-invariant. $g_a$ is a critical point of $\psi_a$ if and only if $g_a + h_a(g_a)$ is a critical point of $\Psi_a$.

(iii) If $g_a \in X$ and $H_a$ is $C^k$ with $k \geq 2$ in a neighborhood of the trajectory of $g_a + h_a(g_a)$, then $\psi_a$ is $C^{k-1}$ in a neighborhood of $g_a$. In particular, if $g_a$ is a nonzero critical point of $\psi_a$, then $\psi_a$ is $C^2$ in a neighborhood of the critical orbit $S^1 \cdot g_a$. The index and nullity of $\Psi_a$ at $g_a + h_a(g_a)$ defined in Definition 3.3 coincide with the Morse index and nullity of $\psi_a$ at $g_a$.

(iv) For any $\kappa \in \mathbb{R}$, we denote by

$$\tilde{\Lambda}_a^\kappa = \{g \in X \mid \psi_a(g) \leq \kappa\}.$$  

Then the natural embedding $\tilde{\Lambda}_a^\kappa \hookrightarrow \Lambda_a^\kappa$ given by $g \mapsto g + h_a(g)$ is an $S^1$-equivariant homotopy equivalence.

(v) The functionals $a \mapsto \psi_a$ is continuous in $a$ in the $C^1$ topology. Moreover $a \mapsto \psi_a''$ is continuous in a neighborhood of the critical orbit $S^1 \cdot g_a$.

Proof. By (ii) of Proposition 2.5, we have

$$(C'_a(u) - C'_a(v), u - v) \geq \omega |u - v|^2, \quad \forall a \in [a_1, a_2], \quad u, v \in \mathbb{R}^{2n},$$

for some $\omega > 0$. Hence we can use the proof of Proposition 3.9 of [Vit1] to obtain $X$ and $h_a$. In fact, $X$ is the subspace of $L_0^2(S^1, \mathbb{R}^{2n})$ generated by the eigenvectors of $-JM$ whose eigenvalues are less than $-\frac{\omega}{2}$ and $h_a(g)$ is defined by the equation

$$\frac{\partial}{\partial h}\Psi_a(g + h_a(g)) = 0.$$  

(3.15)
then (i)-(iii) follows from Proposition 3.9 of [Vit1]. (iv) follows from Lemma 5.1 of [Vit1].

We prove (v). As in [Vit1], (3.14) and the definition of $X$ yields

$$\langle \Psi_a'(u) - \Psi_a'(v), u-v \rangle \geq \frac{\omega}{2} ||u-v||^2, \quad \forall u-v \in X^\perp, \quad a \in [a_1, a_2]. \quad (3.16)$$

Hence we have

$$\frac{\omega}{2} ||h_{a+\epsilon}(g) - h_a(g)||^2 \leq \langle \Psi_a'(g + h_{a+\epsilon}(g)) - \Psi_a'(g + h_a(g)), h_{a+\epsilon}(g) - h_a(g) \rangle$$

$$= \langle \Psi_a'(g + h_a(g)) - \Psi_a'(g + h_a(g)), h_{a+\epsilon}(g) - h_a(g) \rangle$$

$$\leq (O(\epsilon) + O(\epsilon)||g + h_a(g)||^2)^\frac{1}{2} ||h_{a+\epsilon}(g) - h_a(g)||.$$ 

The second equality follows by (3.15) and the last inequality follows by (3.11). Hence $a \mapsto h_a(g)$ is continuous. We have $\psi_a(g) = \Psi_a(g + h_a(g))$ by definition, $\psi_a'(g) = \frac{\partial}{\partial g} \Psi_a(g + h_a(g))$ by (3.15). Hence the first statement of (v) follows from (3.10) and (3.11). The last statement of (v) follows from p.629 of [Vit1] and the implicit functional theorem with parameters.

Note that $\Psi_a$ is not $C^2$ in general, and then we can not apply Morse theory to $\Psi_a$ directly. After the finite dimensional approximation, the function $\psi_a$ has much better differentiability, which allows us to apply the Morse theory to study its property. Note that the above Lemma 3.4 is used only in Proposition 3.5 and Theorem 4.2 below.

**Proposition 3.5.** For all $b \geq a > \tau$, let $\Psi_b$ be a functional defined by (2.11), and $u_b = \dot{x}_b$ be the critical point of $\Psi_b$ so that $x_b$ corresponds to a fixed closed characteristic $(\tau, y)$ on $\Sigma$ for all $b \geq a$. Then the index $i(u_b)$ and the nullity $\nu(u_b)$ of the functional $\Psi_b$ at its critical point $u_b$ are constants for all $b \geq a$. In particular, when $H_b$ is $\alpha$-homogenous for some $\alpha \in (1, 2)$ near the image set of $x_b$, the index and nullity coincide with those defined by I. Ekeland in [Eke1] to [Eke3]. Specially $1 \leq \nu(u_b) \leq 2n - 1$ always holds.

**Proof.** We consider the nullity first. As in Proposition 3.6 of [Eke1], we have that $v \in L_0^2(S^1, \mathbb{R}^{2n})$ belongs to the null space of $Q_a$ if and only if $z = Mv - J_\xi$ is a solution of the linearized system

$$\dot{z}(t) = JH_a''(x_a(t))z(t), \quad (3.17)$$

for some unique $\xi \in \mathbb{R}^{2n}$. Denote by $R(t)$ the fundamental solution of (3.17). Then by Lemma 1.6.11 of [Eke3], we have

$$R(t)T_{y(0)}\Sigma \subset T_{y(\tau t)}\Sigma. \quad (3.18)$$

Clearly, we have

$$R(1)\dot{x}_a(0) = \dot{x}_a(0). \quad (3.19)$$
Let
\[ x_a(\rho, t) = \rho y\left(\frac{\tau t}{T_\rho}\right) \quad \text{with} \quad \tau = \frac{a\varphi'_a(\rho)}{\rho}. \] (3.20)
Then we have \( x_a(\rho, T_\rho) = x_a(\rho, 0) \). Differentiating it with respect to \( \rho \) and using (3.20), we get
\[ \tau \frac{d}{d\rho} \left( \frac{\rho}{\varphi'_a(\rho)} \right) \dot{x}_a(0) + R(1)\rho^{-1}x_a(0) = \rho^{-1}x_a(0). \]
Hence we have
\[ R(1)x_a(0) = x_a(0) - \rho\tau \frac{d}{d\rho} \left( \frac{\rho}{\varphi'_a(\rho)} \right) \dot{x}_a(0) = x_a(0) + \gamma \dot{x}_a(0), \] (3.21)
where \( \gamma < 0 \) since \( \frac{d}{d\rho} \left( \frac{\rho}{\varphi'_a(\rho)} \right) > 0 \) by (iii) of Proposition 2.2. For any \( w \in \mathbb{R}^{2n} \), we have
\[ H''_a(x_a)w = a\varphi''_a(j(x_a))(j'(x), w)j'(x) + a\varphi'_a(j(x_a))j''(x_a)w \]
\[ = a\varphi''_a(j(x_a))(j'(y), w)j'(y) + \tau j''(y)w. \] (3.22)
The last equality follows from (iii) of Proposition 2.4. Let \( z(t) = R(t)z(0) \) for \( z(0) \in T_{y(0)}\Sigma \). Then by \( (3.18) \), we have \( \dot{z}(t) = \tau j''(y(t))z(t) \). Therefore \( R(1)|_{T_{y(0)}\Sigma} \) is independent of the choice of \( H_a \) in Proposition 2.4. Summing up, we have proved that in an appropriate coordinates there holds
\[ R(1) = \begin{pmatrix} A & B \\ 0 & C \end{pmatrix} \quad \text{with} \quad A = \begin{pmatrix} 1 & \gamma \\ 0 & 1 \end{pmatrix}, \]
and \( C \) is independent of \( H_a \). This proves that \( \nu(\dot{x}_a) \) is constant for all \( H_a \) satisfying Proposition 2.4 with \( a > \tau \).

For any \( b > a > \tau \), by (v) of Proposition 2.2, we can construct a continuous family of \( \Psi_c \) with \( c \in [a, b] \) such that \( H_b \) is homogenous of degree \( \alpha = \alpha_b \) near the image set of \( x_b \). Now we can use Lemma 3.4 to obtain a continuous family of \( \psi_c \) such that \( \psi''_c(g_c) \) depends continuously on \( c \in [a, b] \), where \( g_c \) is the critical point of \( \psi_c \) corresponding to \( \dot{x}_c \). Because \( \dim \ker \psi''_c(g_c) = \nu(\dot{x}_c) = \text{constant} \), the index of \( \psi''_c(g_c) \) must be constant too. Because \( i(\dot{x}_b) \) and \( \nu(\dot{x}_b) \) coincide with the index and nullity defined by I. Ekeland (cf. (24) in p.219 of \[ Eke3 \]), our proposition holds.

In the following of this section, we fix a \( \Psi_a \). All the constructions below depend on this \( \Psi_a \). In order to simplify notations, we shall omit the subscript \( a \).

In order to relate the critical modules with the index and nullity of the critical point, we use the finite dimensional approximation introduced by I. Ekeland in \[ Eke1 \]. For \( \epsilon > 0 \), we define \( \Psi_{a,\epsilon}(v) \equiv \int_0^\epsilon (\frac{1}{2}Jv \cdot M_v + G_a(-Jv))dt \) for \( v \in L^2([0, \epsilon], \mathbb{R}^{2n}) \), where \( M_v(t) = \int_0^t v(s)ds \). Then we have
Proposition 3.6. (cf. Lemma 3.9 of [Eke1]) For $\xi \in \mathbb{R}^{2n}$ and $\epsilon > 0$, consider the problem
\[
\min \left\{ \Psi_{a,\epsilon}(v) \mid \int_0^\epsilon v(t)dt = \xi \right\}.
\] (3.23)
Then there exists $\epsilon_0 > 0$ such that for all $\epsilon < \epsilon_0$ and $\xi \in \mathbb{R}^{2n}$, the problem (3.23) has a unique solution $v(\epsilon, \xi)$ which is $C^2$ on $t$. We have $v(\epsilon, 0) = 0$ for all $\epsilon$ and if $\xi \neq 0$, then $v(\epsilon, \xi)(t) \neq 0$ for all $t$.

Now following I. Ekeland, we choose an $\iota \in \mathbb{N}$ such that $\iota^{-1} < \epsilon_0$ and let $\epsilon = \iota^{-1}$. Define
\[
(R^{2n})^\iota_0 = \left\{ (\xi_1, \ldots, \xi_\iota) \in (R^{2n})^\iota \mid \sum_{i=1}^\iota \xi_i = 0 \right\}.
\] (3.24)
Let $\Omega = \{ (\xi_1, \ldots, \xi_\iota) \in (R^{2n})^\iota_0 \mid \xi_i \neq 0, \forall i \}$. Let $p_\iota : L^2_0(S^1, \mathbb{R}^{2n}) \to (R^{2n})^\iota_0$ to be
\[
p_\iota v = \left( \int_0^{\iota \epsilon} vdt, \int_{\iota \epsilon}^{2\iota \epsilon} vdt, \ldots, \int_{(\iota - 1)\iota \epsilon}^{\iota \epsilon} vdt \right).
\] (3.25)
Let $r_\iota : (R^{2n})^\iota_0 \to L^2_0(S^1, \mathbb{R}^{2n})$ to be
\[
r_\iota(\xi_1, \ldots, \xi_\iota)(t) = v(\epsilon, \xi_k)(t - (k - 1)\epsilon), \quad (k - 1)\epsilon \leq t \leq k\epsilon,
\] (3.26)
where $v(\epsilon, \xi_k)$ is given by Proposition 3.6.

Lemma 3.7. (cf. Lemma 3.10 and 3.11 of [Eke1]) The functional $\Psi_a \circ r_\iota$ is $C^1$ on $(R^{2n})^\iota_0$ and $C^2$ on $\Omega$. If $(\xi_1, \ldots, \xi_\iota) \neq 0$ is a critical point of $\Psi_a \circ r_\iota$, then $r_\iota(\xi_1, \ldots, \xi_\iota) \neq 0$ is a critical point of $\Psi_a$. Conversely, if $u \neq 0$ is a critical point of $\Psi_a$, then $p_\iota u$ belongs to $\Omega$ and is a critical point of $\Psi_a \circ r_\iota$. Moreover, $u$ and $p_\iota u$ have the same index and nullity.

Now let $u \neq 0$ be a critical point of $\Psi_a$ with multiplicity $\text{mul}(u) = m$, i.e., $u$ corresponds to a closed characteristic $(m\tau, y) \subset \Sigma$ with $(\tau, y)$ being prime. Hence $u(t + \frac{1}{m}) = u(t)$ for all $t \in \mathbb{R}$ and the orbit of $u$, namely, $S^1 \cdot u \cong S^1/\mathbb{Z}_m \cong S^1$. Let $p : N(S^1 \cdot u) \to S^1 \cdot u$ be the normal bundle of $S^1 \cdot u$ in $L^2_0(S^1, \mathbb{R}^{2n})$ and let $p^{-1}(\theta \cdot u) = N(\theta \cdot u)$ be the fibre over $\theta \cdot u$, where $\theta \in S^1$. Let $DN(S^1 \cdot u)$ be the $\theta$ disk bundle of $N(S^1 \cdot u)$ for some $\theta > 0$ sufficiently small, i.e., $DN(S^1 \cdot u) = \{ \xi \in N(S^1 \cdot u) \mid \| \xi \| < g \}$ which is identified by the exponential map with a subset of $L^2_0(S^1, \mathbb{R}^{2n})$, and let $DN(\theta \cdot u) \cap DN(S^1 \cdot u)$ be the disk over $\theta \cdot u$. Clearly, $DN(\theta \cdot u)$ is $\mathbb{Z}_m$-invariant and we have $DN(S^1 \cdot u) = DN(u) \times_{Z_m} S^1$ where the $Z_m$ action is given by
\[
(\theta, v, t) \in Z_m \times DN(u) \times S^1 \mapsto (\theta \cdot v, \theta^{-1}t) \in DN(u) \times S^1.
\]
Hence for an $S^1$ invariant subset $\Gamma$ of $DN(S^1 \cdot u)$, we have $\Gamma/S^1 = (\Gamma_u \times_{Z_m} S^1)/S^1 = \Gamma_u/\mathbb{Z}_m$, where $\Gamma_u = \Gamma \cap DN(u)$. Let $\Gamma(\iota) = \text{im} r_\iota$ and for any $\kappa \in \mathbb{R}$, we denote by
\[
\Gamma(\iota)^\kappa = \{ u \in \Gamma(\iota) \mid \Psi_a(u) \leq \kappa \}.
\] (3.27)
Lemma 3.8. For any $\kappa \in \mathbb{R}$, we have a $\mathbb{Z}_s$-equivariant deformation retract from $\Lambda_0^\kappa$ to $\Gamma(\kappa)$.

**Proof.** For any $v \in L_0^2(S^1, \mathbb{R}^{2n})$, let $p_i(v) = (\xi_1, \ldots, \xi_i)$, we have

$$
\Psi_a(v) = \int_0^1 \left( \frac{1}{2} Jv \cdot Mv + G_a(-Jv) \right) dt
= \sum_{j=1}^i \int_{(j-1)\epsilon}^{j\epsilon} \left( \frac{1}{2} Jv \cdot \left( \int_0^{(j-1)\epsilon} v(s) ds + \int_{(j-1)\epsilon}^{j\epsilon} v(s) ds \right) + G_a(-Jv) \right) dt
= \sum_{j=1}^i \left( \Psi_{a,\epsilon}(v_j) + \frac{1}{2} J\xi_j \cdot \sum_{l=1}^{j-1} \xi_l \right),
$$

(3.28)

where $v_j(t) = v(t+(j-1)\epsilon)$. By Proposition 3.6, we have $\Psi_a(v) \geq \Psi_a(r, p_i v)$. Hence the deformation retract $F : \Lambda_0^\kappa \times [0, 1] \to \Lambda_0^\kappa$ is given by $(v, s) \mapsto sr, p_i v + (1-s)v$. This is well defined, since by Lemma 3.9 of [Eke1], we have $\Psi_{a,\epsilon}$ is strictly convex, hence

$$
\Psi_a(sr, p_i v + (1-s)v) \leq s \Psi_a(r, p_i v) + (1-s) \Psi_a(v) \leq \Psi_a(v).
$$

Clearly $F$ is $\mathbb{Z}_s$-equivariant and $F = \text{id}$ on $\Gamma(\kappa)$, hence the lemma holds.

As in p.51 of [Rad2], let

$$
D_i N(S^1 \cdot u) = DN(S^1 \cdot u) \cap \Gamma(\kappa), \quad D_i N(\theta \cdot u) = DN(\theta \cdot u) \cap \Gamma(\kappa).
$$

(3.29)

For a $\mathbb{Z}_m$-space pair $(A, B)$, let

$$
H_*(A, B)^{\pm \mathbb{Z}_m} = \{ \sigma \in H_*(A, B) \mid L_\sigma \sigma = \pm \sigma \},
$$

(3.30)

where $L$ is a generator of the $\mathbb{Z}_m$-action. Then by Lemma 3.8, as in Section 6 of [Rad2] or Section 3 of [BaL1], we have

**Lemma 3.9.** Suppose $u \neq 0$ is a critical point of $\Psi_a$ with $mul(u) = m$ and suppose $\Gamma(\kappa)$ is a finite dimensional approximation as above with $m|\iota$, i.e., $m$ is a factor of $\iota$. Then we have

$$
C_{S^1, s}(\Psi_a, S^1 \cdot u) \cong H_*(\Lambda_a(u) \cap DN(u))/\mathbb{Z}_m, \quad ((\Lambda_a(u) \setminus \{ u \}) \cap DN(u))/\mathbb{Z}_m)
\cong H_*(\Lambda_a(u) \cap D_i N(u))/\mathbb{Z}_m, \quad ((\Lambda_a(u) \setminus \{ u \}) \cap D_i N(u))/\mathbb{Z}_m)
\cong H_*(\Lambda_a(u) \cap D_i N(u), (\Lambda_a(u) \setminus \{ u \}) \cap D_i N(u))^{\mathbb{Z}_m}.
$$

(3.31)

**Proof.** For reader’s conveniences, we sketch a proof here and refer to Section 6 of [Rad2] or Section 3 of [BaL1] for related details.

By Definition 3.1, we have

$$
C_{S^1, s}(\Psi_a, S^1 \cdot u) \cong H_{S^1, s}(\Lambda_a(u) \cap DN(S^1 \cdot u), (\Lambda_a(u) \setminus \{ S^1 \cdot u \}) \cap DN(S^1 \cdot u)).
$$
Since all the isotropy groups $G_x = \{g \in S^1 \mid g \cdot x = x\}$ for $x \in DN(S^1 \cdot u)$ are finite, we can use Lemma 6.11 of [PaR1] to obtain

$$H^*_S((\Lambda_a(u) \cap DN(S^1 \cdot u))/S^1, ((\Lambda_a(u) \setminus \{S^1 \cdot u\}) \cap DN(S^1 \cdot u))/S^1)$$

$$\cong H^*((\Lambda_a(u) \cap DN(S^1 \cdot u))/S^1, ((\Lambda_a(u) \setminus \{S^1 \cdot u\}) \cap DN(S^1 \cdot u))/S^1)$$

By the condition (F) at the beginning of Section 2, a small perturbation on the energy functional can be applied to reduce each critical orbit to nearby non-degenerate ones. Thus similar to the proofs of Lemma 2 of [GrM1] and Lemma 4 of [GrM2], all the homological $\mathbb{Q}$-modules of each space pair in the above relations are all finitely generated. Therefore we can apply Theorem 5.5.3 and Corollary 5.5.4 on pages 243-244 of [Spa1] to obtain the same relation on homological $\mathbb{Q}$-modules:

$$H^*_S((\Lambda_a(u) \cap DN(S^1 \cdot u))/S^1, ((\Lambda_a(u) \setminus \{S^1 \cdot u\}) \cap DN(S^1 \cdot u))/S^1)$$

$$\cong H^*((\Lambda_a(u) \cap DN(S^1 \cdot u))/S^1, ((\Lambda_a(u) \setminus \{S^1 \cdot u\}) \cap DN(S^1 \cdot u))/S^1)$$

Now by Lemma 3.8, as in Section 6.2 of [Rad2] or Section 3 of [BaL1], we obtain

$$H^*_S((\Lambda_a(u) \cap DN(u))/\mathbb{Z}_m, ((\Lambda_a(u) \setminus \{u\}) \cap DN(u))/\mathbb{Z}_m)$$

$$\cong H^*_S((\Lambda_a(u) \cap DN(u))/\mathbb{Z}_m, ((\Lambda_a(u) \setminus \{u\}) \cap DN(u))/\mathbb{Z}_m).$$

Note that the same argument as in Section 6.3 of [Rad2], in particular Satz 6.6 of [Rad2], Lemma 3.6 of [BaL1] or Theorem 3.2.4 of [Bre1] yields

$$H^*_S((\Lambda_a(u) \cap DN(u))/\mathbb{Z}_m, ((\Lambda_a(u) \setminus \{u\}) \cap DN(u))/\mathbb{Z}_m)$$

$$\cong H^*_S((\Lambda_a(u) \cap DN(u))/\mathbb{Z}_m, ((\Lambda_a(u) \setminus \{u\}) \cap DN(u))/\mathbb{Z}_m).$$

The above relations together complete the proof of Lemma 3.9.

### 3.2 The periodic property of critical modules

By Proposition 3.2, we have that $C_{S^1 \cdot u}(\Psi_a, S^1 \cdot u)$ is independent of the choice of the Hamiltonian function $H_a$ whenever $H_a$ satisfies conditions in Proposition 2.4. Hence in order to compute the critical modules, we can choose $\Psi_a$ with $H_a$ being positively homogeneous of degree $\alpha = \alpha_a$ near the image set of every nonzero solution $x$ of $\{2.3\}$ which corresponding to some closed characteristic $(\tau, y)$ with period $\tau$ being strictly less than $a$. 

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In other words, for a given \( a > 0 \), we choose \( \vartheta \in (0,1) \) first such that \([a\vartheta, a(1-\vartheta)] \supset \text{per}(\Sigma) \cap (0,a)\) holds by the definition of the set \text{per}(\Sigma)\) and the assumption (F). Then we choose \( \alpha = \alpha_a \in (1,2) \) sufficiently close to 2 by (v) of Proposition 2.2 such that \( \varphi_a(t) = ct^\alpha \) for some constant \( c > 0 \) and \( \alpha \in (1,2) \) whenever \( \varphi_a(t) \in [\vartheta,1-\vartheta] \). In this subsection we suppose that \( \varphi_a \) possesses this property (v).

Now we consider iterations of critical points of \( \Psi_a \). Suppose \( u \neq 0 \) is a critical point of \( \Psi_a \) with \( \mu(u) = m \). By Propositions 2.4 and 2.6, we have \( u = \dot{x} \) with \( x \) being a solution of (2.3) and \( x = \rho y(\tau t) \) with \( \frac{\varphi_a(\rho)}{\rho} = \frac{\vartheta}{a} \). Moreover, \((\tau,y)\) is a closed characteristic on \( \Sigma \) with minimal period \( \frac{v}{m} \).

For any \( p \in \mathbb{N} \) satisfying \( pr < a \), the \( p \)th iteration \( u^p \) of \( u \) is given by \( \dot{x}^p \), where \( x^p \) is the unique solution of (2.3) corresponding to \((p\tau,y)\). Hence we have

\[
x(t) = \left( \frac{\tau}{(2\alpha a)^{1/2}} \right)^{\frac{1}{2}} y(\tau t), \quad x^p(t) = \left( \frac{p\tau}{(2\alpha a)^{1/2}} \right)^{\frac{1}{2}} y(p\tau t),
\]

\[
u(t) = \dot{x}(t) = \tau \frac{\alpha-1}{\alpha} (2\alpha a)^{1/2} \dot{y}(\tau t), \quad u^p(t) = \dot{x}^p(t) = (p\tau) \frac{\alpha-1}{\alpha} (2\alpha a)^{1/2} \dot{y}(p\tau t).
\]

These yield

\[
u^p(t) = p^{\frac{\alpha-1}{2}} u(pt).
\]

Choose two finite dimensional approximations \( \Gamma(u) \) and \( \Gamma(p\mu) \) as above and define the \( p \)th iteration \( \phi^p \) on \( D_{\text{per}}u \) by

\[
\phi^p : v(t) \mapsto p^{\frac{\alpha-1}{2}} v(pt).
\]

**Claim.** \( \phi^p \) maps \( D_{\text{per}}u \) into \( D_{\text{per}}u^p \) if the radii of the two normal disk bundles are suitably chosen.

In fact, clearly \( \phi^p(v) \in \text{DN}(u^p) \) and by Lemma 3.9 of [Eke1],

\[
v \left( t + \frac{k-1}{\ell} \right) = v \left( \frac{1}{\ell} \int_{k-1}^k v(s)ds \right)(t) \equiv v_k(t), \quad 1 \leq k \leq \ell, \quad 0 \leq t \leq \frac{1}{\ell}.
\]

Here \( v_k \) is the unique solution of

\[
v_k(t) = JH_a(M_{\frac{1}{lt}} v_k + \zeta_k), \quad \int_0^{\frac{1}{\ell}} v_k(t)dt = \int_{k-1}^k v(t)dt,
\]

where \( \zeta_k \) is uniquely determined by \( v_k \). Hence it suffices to show that

\[
\phi^p(v)(t) = JH_a(M_{\frac{1}{p\ell}} \phi^p(v)(t) + \zeta'), \quad t \in \left[ \frac{lt+j-1}{p\ell}, \frac{lt+j}{p\ell} \right],
\]

for some \( \zeta' \in \mathbb{R}^{2n} \) and \( 0 \leq l < p, \quad 1 \leq j < \ell \). An easy computation show that

\[
M_{\frac{1}{p\ell}} (\phi^p(v))(t) = p^{\frac{1}{2\alpha}} (M_{\frac{1}{\ell}} v)(pt).
\]
Then we have

\[
\phi^p(v)(t) = p^{\alpha - 1}v(pt) = p^{\alpha - 2}JH'_a((M_1^j v_j)(pt) + \xi_j) = JH'_a(p^{\alpha - 2}((M_1^j v_j)(pt) + \xi_j)) = JH'_a(M_{p^{\alpha - 1}}\phi^p(v)(t) + \xi').
\]

In the above computations, we have used that \(H_a\) and then \(H'_a\) are positively homogeneous of degrees \(\alpha\) and \(\alpha - 1\) respectively. This is true since by Proposition 3.6, all \(v \in D_i N(u^i)\) lies in an \(L^\infty\) neighborhood of \(u^i\) for \(1 \leq i \leq p\). This proves the claim.

We have

\[
\Psi_a(\phi^p(v)) = \int_0^1 \left( \frac{1}{2} J\phi^p(v)(t) \cdot M\phi^p(v)(t) + G_a(-J\phi^p(v)(t)) \right) dt = \int_0^1 \left( \frac{1}{2} Jp^{\alpha - 2}v(pt) \cdot p^{\alpha - 2}(Mv)(pt) + G_a(-Jp^{\alpha - 2}v(pt)) \right) dt = p^{\alpha - 2} \int_0^1 \left( \frac{1}{2} Jv(pt) \cdot (Mv)(pt) + G_a(-Jv(pt)) \right) dt = p^{\alpha - 2}\Psi_a(v). 
\]

(3.37)

Here the second equality follows from (3.36) and the third equality follows from (v) of Proposition 2.5. We define a new inner product \(\langle \cdot, \cdot \rangle_p\) on \(L^2_\delta(S^1, \mathbb{R}^{2n})\) by

\[
\langle v, w \rangle_p = p^{\alpha - 1} \langle v, w \rangle.
\]

(3.38)

Then \(\phi^p : D_i N(u) \to D_{pu} N(u^p)\) is an isometry from the standard inner product to the above one. Clearly \(\phi^p(D_i N(u))\) consists of points in \(D_{pu} N(u^p)\) which are fixed by the \(Z_p\)-action. Since the \(Z_p\)-action on \(D_{pu} N(u^p)\) are isometries and \(f = \Psi_a|_{D_{pu} N(u^p)}\) is \(Z_p\)-invariant, we have

\[
f''(v) = \begin{pmatrix} f'_{\phi^p(D_i N(u))} & 0 \\ 0 & * \end{pmatrix}, \quad \forall v \in \phi^p(D_i N(u)).
\]

(3.39)

Moreover, we have

\[
f'(v) = (f|_{\phi^p(D_i N(u))})', \quad \forall v \in \phi^p(D_i N(u)).
\]

(3.40)

Now we can apply the results by D. Gromoll and W. Meyer [GrM1] to the manifold \(D_{pu} N(u^p)\), with \(u^p\) as its unique critical point. Then \(mul(u^p) = pm\) is the multiplicity of \(u^p\) and the isotropy group \(Z_{pm} \subseteq S^1\) of \(u^p\) acts on \(D_{pu} N(u^p)\) by isometries. According to Lemma 1 of [GrM1], we have a \(Z_{pm}\)-invariant decomposition of \(T_{u^p}(D_{pu} N(u^p))\)

\[
T_{u^p}(D_{pu} N(u^p)) = V^+ \oplus V^- \oplus V^0 = \{(x_+, x_-, x_0)\}
\]

(3.41)
with \( \dim V^- = i(p) \), \( \dim V^0 = \nu(p) - 1 \) and a \( \mathbb{Z}_{pm} \)-invariant neighborhood \( B = B_+ \times B_- \times B_0 \) for \( 0 \) in \( T_u(p, N(u)) \) together with two \( \mathbb{Z}_{pm} \)-invariant diffeomorphisms

\[
\Phi : B = B_+ \times B_- \times B_0 \to \Phi(B_+ \times B_- \times B_0) \subset D_p N(u)
\]

and

\[
\eta : B_0 \to W(u) \equiv \eta(B_0) \subset D_p N(u)
\]

and \( \Phi(0) = \eta(0) = u \), such that

\[
\Psi_a \circ \Phi(x_+, x_-, x_0) = |x_+|^2 - |x_-|^2 + \Psi_a \circ \eta(x_0),
\]

(3.42)

with \( d(\Psi_a \circ \eta)(0) = d^2(\Psi_a \circ \eta)(0) = 0 \). As usual, we call \( W(u) \) a local characteristic manifold and \( U(u) = B_- \) a local negative disk at \( u \). By the proof of Lemma 1 of [GrM1], \( W(u) \) and \( U(u) \) are \( \mathbb{Z}_{pm} \)-invariant. It follows from (3.42) that \( u \) is an isolated critical point of \( \Psi_a|_{D_p N(u)} \). Then as in Lemma 6.4 of [Rad2], we have

\[
H_s(\Lambda_a(u) \cap D_p N(u), (\Lambda_a(u) \setminus \{u\}) \cap D_p N(u))
\]

\[
= H_s(U(u), U(u) \setminus \{u\}) \otimes H_s(W(u) \cap \Lambda_a(u), (W(u) \setminus \{u\}) \cap \Lambda_a(u)), \quad (3.43)
\]

where

\[
H_q(U(u), U(u) \setminus \{u\}) = \begin{cases} 
Q, & \text{if } q = i(u), \\
0, & \text{otherwise}.
\end{cases}
\]

(3.44)

Now we have the following proposition.

\textbf{Proposition 3.10.} Let \( u \neq 0 \) be a critical point of \( \Psi_a \) with \( \text{mul}(u) = 1 \). Then for all \( p \in \mathbb{N} \) and \( q \in \mathbb{Z} \), we have

\[
C_{S^1, q}(\Psi_a, S^1 \cdot u) \cong \left( H_{q-i(u)}(W(u) \cap \Lambda_a(u), (W(u) \setminus \{u\}) \cap \Lambda_a(u)) \right)^{\beta(u)p \mathbb{Z}_p},
\]

(3.45)

where \( \beta(u) = (-1)^{i(u) - i(u)} \). In particular, if \( u \) is non-degenerate, i.e., \( \nu(u) = 1 \), then

\[
C_{S^1, q}(\Psi_a, S^1 \cdot u) = \begin{cases} 
Q, & \text{if } q = i(u) \text{ and } \beta(u) = 1, \\
0, & \text{otherwise}.
\end{cases}
\]

(3.46)

\textbf{Proof.} Suppose \( \theta \) is a generator of the linearized \( \mathbb{Z}_p \)-action on \( U(u) \). Then \( \theta(\xi) = \xi \) if and only if \( \xi \in T_u(\phi^p(D_s N(u))) \). Hence it follows from (3.37) and (3.39) that \( \xi = (\phi^p)_*(\xi') \) for a unique \( \xi' \in T_u(D_s N(u))^- \). Hence the proof of Satz 6.11 in [Rad2] or Proposition 2.8 in [BaL1] yield this proposition. \( \blacksquare \)
Definition 3.11. Let \( u \neq 0 \) be a critical point of \( \Psi_a \) with \( \text{mul}(u) = 1 \). Then for all \( p \in \mathbb{N} \) and \( l \in \mathbb{Z} \), let

\[
\begin{align*}
  k_{l, \pm 1}(u^p) &= \dim \left( H_I(W(u^p) \cap \Lambda_a(u^p), (W(u^p) \setminus \{u^p\}) \cap \Lambda_a(u^p)) \right)^\pm Z_p, \\
  k_l(u^p) &= \dim \left( H_I(W(u^p) \cap \Lambda_a(u^p), (W(u^p) \setminus \{u^p\}) \cap \Lambda_a(u^p)) \right)^{\beta(u^p)}Z_p.
\end{align*}
\]

(3.47) (3.48)

\( k_l(u^p) \)'s are called critical type numbers of \( u^p \).

Note that by Proposition 3.5, we have \( k_{l, \pm 1}(u^p) = 0 \) if \( l \notin [0, 2n - 2] \).

Similar to Section 7.1 of [Rad2] or Theorem 2.11 of [BaLi1], we have

**Lemma 3.12.** Let \( u \neq 0 \) be a critical point of \( \Psi_a \) with \( \text{mul}(u) = 1 \). Suppose \( \nu(u^m) = \nu(u^{pm}) \) for some \( m, p \in \mathbb{N} \), then we have \( k_{l, \pm 1}(u^m) = k_{l, \pm 1}(u^{pm}) \) for all \( l \in \mathbb{Z} \).

**Proof.** We choose finite dimensional approximations \( \Gamma(\ell) \) and \( \Gamma(pt) \) as in Proposition 3.6 with \( m|\ell \) and let \( \phi^p : D_\ell N(u^m) \to D_{pt} N(u^{pm}) \) be the \( p \)th iteration map. By (3.38), \( \phi^p \) is an isometry under the modified metric. Hence by (3.37), we have

\[
\nu(u^m) - 1 = \dim \ker((\Psi_a|_{D_\ell N(u^m)})'' - I) = \dim \ker((\Psi_a|_{\phi^p(D_\ell N(u^m))})'' - I).
\]

Thus by (3.39) and the assumption \( \nu(u^m) = \nu(u^{pm}) \), we have that \( T_{u^{pm}}(\phi^p(D_\ell N(u^m))) \) contains the nullity space of the Hessian of \( \Psi_a|_{D_{pt} N(u^{pm})} \). Now by (3.40), we can use Lemma 7 of [GrM1] to obtain that \( \phi^p(W(u^m)) \equiv W(u^{pm}) \) is a characteristic manifold of \( \Psi_a|_{D_{pt} N(u^{pm})} \), where \( W(u^m) \) is a characteristic manifold of \( \Psi_a|_{D_\ell N(u^m)} \). By (3.37), we have

\[
\phi^p : \begin{cases} \ (W(u^m) \cap \Lambda_a(u^m), (W(u^m) \setminus \{u^m\}) \cap \Lambda_a(u^m)) \\ \to (W(u^{pm}) \cap \Lambda_a(u^{pm}), (W(u^{pm}) \setminus \{u^{pm}\}) \cap \Lambda_a(u^{pm})) \end{cases}
\]

is a homeomorphism. Suppose \( \theta \) and \( \theta_p \) generate the \( Z_m \) and \( Z_{pm} \) action on \( W(u^m) \) and \( W(u^{pm}) \) respectively. Then clearly \( \phi^p \circ \theta = \theta_p \circ \phi^p \) holds and it implies

\[
H_a(W(u^m) \cap \Lambda_a(u^m), (W(u^m) \setminus \{u^m\}) \cap \Lambda_a(u^m)) \equiv (W(u^{pm}) \cap \Lambda_a(u^{pm}), (W(u^{pm}) \setminus \{u^{pm}\}) \cap \Lambda_a(u^{pm})).
\]

Therefore our lemma holds.

**Proposition 3.13.** Let \( u \neq 0 \) be a critical point of \( \Psi_a \) with \( \text{mul}(u) = 1 \). Then there exists a minimal \( K(u) \in 2\mathbb{N} \) such that

\[
\begin{align*}
  \nu(u^{p+K(u)}) &= \nu(u^p), & i(u^{p+K(u)}) - i(u^p) &\in 2\mathbb{Z}, & \forall p \in \mathbb{N}, \\
  k_l(u^{p+K(u)}) &= k_l(u^p), & \forall p \in \mathbb{N}, & l \in \mathbb{Z}.
\end{align*}
\]

(3.50) (3.51)
We call $K(u)$ the minimal period of critical modules of iterations of the functional $\Psi_a$ at $u$.

**Proof.** As in the proof of Proposition 3.5, we denote by $R(t)$ the fundamental solution of (3.17). Then by Lemma 1.1 and 1.2 of [LoZ1], we have $i(u^p) = i(u, p) - n$ and $\nu(u^p) = \nu(u, p)$ for all $p \in \mathbb{N}$, where $(i(u, p), \nu(u, p))$ are index and nullity defined by C. Conley and E. Zehnder in [CoZ1], Y. Long and E. Zehnder in [LZe1] and Y. Long in [Lon1], cf. [Lon4]. Hence we have $\nu(u^p) = \dim \ker(R(1)^p - I_{2n})$. Denote by $\lambda_i = \exp(\pm 2\pi \frac{r_i}{s_i})$ the eigenvalues of $R(1)$ possessing rotation angles which are rational multiple of $\pi$ with $r_i, s_i \in \mathbb{N}$ and $(r_i, s_i) = 1$ for $1 \leq i \leq q$. Let $K(u)$ be twice of the least common multiple of $s_1, \ldots, s_q$. Then (3.50) holds. Note that the later conclusion in (3.50) follows from Theorem 9.3.4 of [Lon4].

In order to prove (3.51), it suffices to show

$$k_l(u^{m+qK(u)}) = k_l(u^m), \quad \forall q \in \mathbb{N}, \ l \in \mathbb{Z}, \ 1 \leq m \leq K(u).$$

(3.52)

In fact, assume that (3.52) is proved. Note that (3.51) follows from (3.52) with $q = 1$ directly when $p \leq K(u)$. When $p > K(u)$, we write $p = m + qK(u)$ for some $q \in \mathbb{N}$ and $1 \leq m \leq K(u)$. Then by (3.52) we obtain

$$k_l(u^{p+K(u)}) = k_l(u^{m+(q+1)K(u)}) = k_l(u^m) = k_l(u^{m+qK(u)}) = k_l(u^p),$$

i.e., (3.51) holds.

To prove (3.52), we fix an integer $m \in [1, K(u)]$. Let

$$A = \{s_i \in \{s_1, \ldots, s_q\} \mid s_i \text{ is a factor of } m\},$$

and let $m_1$ be the least common multiple of elements in $A$. Hence we have $m = m_1m_2$ for some $m_2 \in \mathbb{N}$ and $\nu(u^m) = \nu(u^{m_1})$. Thus by Lemma 3.12, we have $k_l(u^m) = k_{l, \beta}(u^{m_1})(u^{m_2})$. Since $m + pK(u) = m_1m_3$ for some $m_3 \in \mathbb{N}$, we have by Lemma 3.12 that $k_l(u^{m+pK(u)}) = k_{l, \beta}(u^{m+pK(u)})(u^{m_1})$. By (3.50), we obtain $\beta(u^{m+pK(u)}) = \beta(u^m)$, and then (3.52) is proved. This completes the proof.

Note that the above Proposition 3.13 could be established also without forcing the Hamiltonian to be homogeneous near its critical points. In fact, by Proposition 3.2, it holds for any Hamiltonian defined by Proposition 2.4.

### 3.3 Indices and Euler characteristics of closed characteristics

In the following, Let $\Psi_a$ by any function defined by (2.11) with $H_a$ satisfying Proposition 2.4, we do not require $H_a$ to be homogeneous anymore.
Definition 3.14. Suppose the condition (F) at the beginning of §2 holds. For every closed characteristic \((\tau, y)\) on \(\Sigma\), let \(a > \tau\) and choose \(\varphi_a\) to satisfy (i)-(iv) of Proposition 2.2. Determine \(\rho\) uniquely by \(\frac{\varphi'_a(\varphi)}{\rho} = \frac{\tau}{a}\). Let \(x = \rho y(\tau t)\) and \(u = \dot{x}\). Then we define the index \(i(\tau, y)\) and nullity \(\nu(\tau, y)\) of \((\tau, y)\) by

\[i(\tau, y) = i(u), \quad \nu(\tau, y) = \nu(u).\]

Then the mean index of \((\tau, y)\) is defined by

\[\hat{i}(\tau, y) = \lim_{m \to \infty} \frac{i(m\tau, y)}{m}.\]  \hfill (3.53)

Note that by Proposition 3.5, the index and nullity are well defined and is independent of the choice of \(a > \tau\) and \(\varphi_a\) satisfying (i)-(iv) of Proposition 2.2. Note that by Theorem 1.7.7 of [Eke3] (cf. Corollary 8.3.2 of [Lon4]), we have \(\hat{i}(\tau, y) > 2\).

For a prime closed characteristic \((\tau, y)\) on \(\Sigma\), we denote simply by \(y^m \equiv (m\tau, y)\) for \(m \in \mathbb{N}\). By Proposition 3.2, we can define the critical type numbers \(k_l(y^m)\) of \(y^m\) to be \(k_l(u^m)\), where \(u^m\) is the critical point of \(\Psi_a\) corresponding to \(y^m\). We also define \(K(y) = K(u)\), where \(K(u) \in \mathbb{N}\) is given by Proposition 3.13. Suppose \(\mathcal{N}\) is an \(S^1\)-invariant open neighborhood of \(S^1 \cdot u^m\) such that \(\text{crit}(\Psi_a) \cap (\Lambda_a(u^m) \cap \mathcal{N}) = S^1 \cdot u^m\). Then we make the following definition

**Definition 3.15.** The Euler characteristic \(\chi(y^m)\) of \(y^m\) is defined by

\[\chi(y^m) \equiv \chi((\Lambda_a(u^m) \cap \mathcal{N})_{S^1}, ((\Lambda_a(u^m) \setminus S^1 \cdot u^m) \cap \mathcal{N})_{S^1}) = \sum_{q=0}^{\infty} (-1)^q \dim C_{S^1, q}(\Psi_a, S^1 \cdot u^m).\]  \hfill (3.54)

Here \(\chi(A, B)\) denotes the usual Euler characteristic of the space pair \((A, B)\).

The average Euler characteristic \(\hat{\chi}(y)\) of \(y\) is defined by

\[\hat{\chi}(y) = \lim_{N \to \infty} \frac{1}{N} \sum_{1 \leq m \leq N} \chi(y^m).\]  \hfill (3.55)

The following remark shows that \(\hat{\chi}(y)\) is well-defined and is a rational number.

**Remark 3.16.** By (3.54), we have

\[\chi(y^m) = \sum_{q=0}^{\infty} (-1)^q \dim C_{S^1, q}(\Psi_a, S^1 \cdot u^m) = \sum_{l=0}^{2n-2} (-1)^l i(y^m) + l k_l(y^m).\]  \hfill (3.56)

Here the first equality follows from Definition 3.1. The second equality follows from Proposition 3.10 and Definition 3.11. Hence by (3.50) and Proposition 3.13 we have

\[\hat{\chi}(y) = \lim_{N \to \infty} \frac{1}{N} \sum_{1 \leq m \leq N \atop 0 \leq l \leq 2n-2} (-1)^l i(y^m) + l k_l(y^m)\]
Hence (3.57) implies
\[ \nu = \text{...} \]
Therefore \( \hat{\chi}(y) \) is well defined and is a rational number. In particular, if all \( y^m \)'s are non-degenerate, then \( \nu(y^m) = 1 \) for all \( m \in \mathbb{N} \). Hence the proof of Proposition 3.13 yields \( K(y) = 2 \). By (3.46) we have
\[ k_l(y^m) = \begin{cases} 1, & \text{if } i(y^m) - i(y) \in 2\mathbb{Z} \quad \text{and} \quad l = 0 \\ 0, & \text{otherwise.} \end{cases} \]
Hence (3.57) implies
\[ \hat{\chi}(y) = \begin{cases} (-1)^i(y), & \text{if } i(y^2) - i(y) \in 2\mathbb{Z}, \\ (-1)^{(i)(y)} \div 2, & \text{otherwise.} \end{cases} \] (3.58)

**Remark 3.17.** Note that \( k_l(y^m) = 0 \) for \( l \not\in [0, \nu(y^m) - 1] \) and it can take only values 0 or 1 when \( l = 0 \) or \( l = \nu(y^m) - 1 \). Moreover, the following facts are useful (cf. Lemma 3.10 of [BaL1], [Cha1] and [MaW1]):

(i) \( k_0(y^m) = 1 \) implies \( k_l(y^m) = 0 \) for \( 1 \leq l \leq \nu(y^m) - 1 \).
(ii) \( k_{\nu(y^m)-1}(y^m) = 1 \) implies \( k_l(y^m) = 0 \) for \( 0 \leq l \leq \nu(y^m) - 2 \).
(iii) \( k_l(y^m) \geq 1 \) for some \( 1 \leq l \leq \nu(y^m) - 2 \) implies \( k_0(y^m) = k_{\nu(y^m)-1}(y^m) = 0 \).
(iv) In particular, only one of the \( k_l(y^m) \)'s for \( 0 \leq l \leq \nu(y^m) - 1 \) can be non-zero when \( \nu(y^m) \leq 3 \).

### 4 Homological vanishing near the origin

In Section 3, we have studied nonzero critical points of \( \Psi_a \). This section is devoted to the study of the contribution of the origin to the Morse series of the functional \( \Psi_a \) on \( L^2(S^1, \mathbb{R}^{2n}) \). The main result in this section is motivated by Theorem 7.1 of [Vit1], but our proof is different from that in [Vit1].

We consider first the distribution of critical values of \( \Psi_a \). Note that a critical point \( u_a = u_a(\tau, y) \) of \( \Psi_a \) corresponds to a closed characteristic \((\tau, y)\) on \( \Sigma \) by Propositions 2.4 and 2.6. Therefore the critical value \( \Psi_a(u_a) = \Psi_a(u_a(\tau, y)) \) is a function of \((\tau, y), a\) and \( \varphi_a \) in Proposition 2.4.

**Proposition 4.1.** Let \( H_a(x) = a\varphi(j(x)) \) for \( a \in [a_1, a_2] \) be a family of functions given by Proposition 2.4 with the same \( \varphi \) satisfying (i)-(iv) of Proposition 2.2. For \( a \in [a_1, a_2] \), suppose \( u_a = u_a(\tau, y) \neq 0 \) is a critical point of \( \Psi_a \). Then \( \Psi_a(u_a) = \Psi_a(u_a(\tau, y)) \) is a function of the period \( \tau, a \) and \( \varphi \). Thus we denote it simply by \( \Psi_{a, \tau} \). Here we ignore the dependence of \( \Psi_{a, \tau} \) on \( \varphi \). Then we have the following properties of \( \Psi_{a, \tau} \).
\( (i) \) \( \Psi_{a, \tau} < 0 \) and \( \Psi_{a, \tau} \) is an increasing function of \( \tau \) when \( a \) is fixed.

\( (ii) \) \( \Psi_{a, \tau} \) is a strictly decreasing continuous function of \( a \) when \( \tau \) is fixed.

\( (iii) \) If \( a \in \text{per}(\Sigma) \), then we have \( \lim_{\lambda \to a^+} \Psi_{\lambda, a} = 0 \).

**Proof.** By Propositions 2.4 and 2.9, we have \( u_a = \dot{x}_a, \ x_a = \rho_a y(\tau t) \), and then (2.16) and (2.17) become

\[
\varphi'(\rho_a) = \frac{\tau}{a}, \tag{4.1}
\]

\[
\Psi_a(u_a) = \frac{1}{2}a\varphi'\left(\rho_a\right)^2 - a\varphi'\left(\rho_a\right). \tag{4.2}
\]

Note that \( \Psi_a(u_a) < 0 \) follows from Proposition 2.9. Note that by (4.1), \( \rho_a \) depends only on the period \( \tau \) of the closed characteristic, \( a \) and \( \varphi \), hence so does \( \Psi_a(u_a) \) by (4.2).

For \( (i) \), fix an \( a \in [a_1, a_2] \). Note that by (iii) of Proposition 2.2 and (4.1), \( \rho_a \) is a decreasing function of \( \tau \). Now let \( f(t) = \frac{1}{2}a\varphi'(t)t - a\varphi(t) \). As in the proof of Proposition 2.9, we obtain \( f(0) = 0 \) and \( f'(t) < 0 \) by (iii) of Proposition 2.2. Hence \( (i) \) holds.

For \( (ii) \), given \( \tau \in \text{per}(\Sigma) \) with \( \tau < a \), we may choose a fixed closed characteristic \( (\tau, y) \) on \( \Sigma \). Differentiating the equation (4.1) with respect to \( a \) yields

\[
(\varphi'(\rho_a) + a\varphi''(\rho_a)\rho'_a)\rho_a - a\varphi'(\rho_a)\rho'_a = 0.
\]

Hence we have

\[
\frac{d}{da}\Psi_a(u_a) = \frac{1}{2}\left(\varphi'(\rho_a)\rho_a - 2\varphi(\rho_a) + a\varphi''(\rho_a)\rho_a\rho'_a - a\varphi'(\rho_a)\rho'_a\right)
= -\varphi(\rho_a) < 0.
\]

Therefore \( (ii) \) holds. Note that in this proof we used the property that \( \varphi \) is independent of the choice of \( a \in [a_1, a_2] \).

For \( (iii) \), we have \( \rho_\lambda \to 0 \) as \( \lambda \to a^+ \) by (4.1), \( (i) \) and (iii) of Proposition 2.2. Hence (iii) holds by (4.2) and (i) of Proposition 2.2.

The main result in this section is

**Theorem 4.2.** Fix an \( a > 0 \) such that \( \text{per}(\Sigma) \cap (0, a) \neq \emptyset \). Then there exists an \( \varepsilon_0 > 0 \) small such that for any \( \varepsilon \in (0, \varepsilon_0] \) we have

\[
H_{S^1, q}(\Lambda^\varepsilon_0, \Lambda_a^{-\varepsilon}) = 0, \quad \forall q \leq I_0, \tag{4.3}
\]

if \( I_0 \) is the greatest integer in \( \mathbb{N}_0 \) such that \( I_0 < i(\tau, y) \) for all closed characteristics \( (\tau, y) \) on \( \Sigma \) with \( \tau \geq a \).
Proof. Let
\[ \tau(a) = \max\{\tau < a \mid \tau \in \text{per}(\Sigma)\}, \quad \varepsilon_0 = -\frac{1}{2}\Psi_{a, \tau(a)}. \]
Then by (i) of Proposition 4.1, there are no critical values of \( \Psi \) in the interval \([-\varepsilon_0, 0]\). Hence we have
\[ H_{S^1, q}(a, \varepsilon) \equiv H_{S^1, q}(0, \varepsilon), \quad \forall q \in \mathbb{Z}, \ v \in (0, \varepsilon_0]. \] (4.4)
In the following we assume \( \varepsilon \in (0, \varepsilon_0] \).

Note that by the same proof of Proposition 3.2, \( H_{S^1, q}(a, \varepsilon) \) is independent of the choice of \( \varphi_a \) in \( H_a(x) = a\varphi_a(j(x)) \) which satisfies (i)-(iv) of Proposition 2.2. Hence we can choose \( \varphi_a \equiv \varphi \) for any function \( \varphi \) satisfying (i)-(iv) of Proposition 2.2.

The rest part of the proof of this theorem is carried out in three steps.

Step 1. Claim: For every \( b > a \) there exists \( \hat{\varepsilon}_b \in (0, \varepsilon_0) \) such that
\[ H_{S^1, q}(a^\varepsilon, \Lambda_a^{-\varepsilon}) \equiv H_{S^1, q}(a^\varepsilon, \Lambda_b^{-\varepsilon}), \quad \forall q \leq I_0 \text{ and } \varepsilon \in (0, \hat{\varepsilon}_b]. \] (4.5)

In fact, by the above second paragraph, we may choose \( \varphi \) such that \( H_c(x) = c\varphi(j(x)) \) satisfies Proposition 2.4 for all \( c \in [a, b] \) with a fixed \( \varphi \).

By Lemma 3.4, we can choose a family of finite dimensional approximations \( h_c : X \to X^\perp \) and consider the functions \( \psi_c(g) = \Psi_c(g + h_c(g)) \) on the finite dimensional manifold \( X \). Moreover, we have \( H_{S^1, q}(a^\varepsilon, \Lambda_a^{-\varepsilon}) \equiv H_{S^1, q}(\tilde{\Lambda}_c^{-\varepsilon}) \) for any \( \varepsilon > 0 \) by (iv) of Lemma 3.4. Hence in order to prove (4.5), it suffices to prove
\[ H_{S^1, q}(\tilde{\Lambda}_a^{-\varepsilon}), \quad \forall q \leq I_0 \] (4.6)
for \( \varepsilon \in (0, \varepsilon_0) \) sufficiently small. Clearly, it suffices to prove that for any \( c \in [a, b] \) there exists \( \delta, \varepsilon' > 0 \) such that
\[ H_{S^1, q}(\tilde{\Lambda}_a^{-\varepsilon}), \quad \forall q \leq I_0. \] (4.7)
for any \( c_1, c_2 \in [c - \delta, c + \delta] \) and \( \varepsilon \in (0, \varepsilon') \). In the following, we fix a \( c \in [a, b] \).

We have two cases.

Case 1. \( c \notin \text{per}(\Sigma) \).

In this case, since \( \text{per}(\Sigma) \) is a discrete subset of \( \mathbb{R}^+ \) by Definition 2.1 and the assumption (F), we can find \( \delta > 0 \) such that \([c - \delta, c + \delta] \cap \text{per}(\Sigma) = \emptyset\). Hence nonzero critical points of \( \psi_\lambda \) are precisely those closed characteristics \((\tau, y)\) on \( \Sigma \) with period \( \tau < c \) for all \( \lambda \in [c - \delta, c + \delta] \). Let
\[ \tau_0 = \max\{\tau < c - \delta \mid \tau \in \text{per}(\Sigma)\}. \]
Then by Proposition 4.1 and the definition of $\tau_0$, we have
\[ \psi_{\lambda}(g_{\lambda}) \leq \Psi_{\lambda, \tau_0} \leq \Psi_{c-\delta, \tau_0} < 0 \]
for all nonzero critical points $g_{\lambda}$ of $\psi_{\lambda}$. Let $\varepsilon' = -\frac{1}{2}\Psi_{c-\delta, \tau_0}$. Then $\pm \varepsilon$ for $\varepsilon \in (0, \varepsilon']$ are regular values of $\psi_{\lambda}$ for all $\lambda \in [c - \delta, c + \delta]$. Moreover, by (2.15) and the definition of $\psi_{\lambda}$ in Lemma 3.4, we have $\psi_{\lambda}(g)$ goes to $+\infty$ as $\|g\|$ goes to $+\infty$. Hence we can choose $R > 0$ to be sufficiently large such that $\tilde{A}_{\lambda} \subset B_R(0)$ for all $\lambda \in [c - \delta, c + \delta]$, where $B_R(0)$ is the open ball in $X$ centered at the origin with the radius $R$. Then we have $\psi_{\lambda}(g)$ is continuous with respect to $\lambda$ uniformly for $g \in \bigcup_{\lambda \in [c - \delta, c + \delta]} \tilde{A}_{\lambda}$. Clearly, it follows from Lemma 3.4 that both $\psi_{\lambda}(g)$ and $\psi'_{\lambda}(g)$ are continuous for $(\lambda, g) \in [c - \delta, c + \delta] \times X$. Hence we can apply a slightly stronger version of Exercise 8.4 on p.203 of [MaW1] or Theorem 1.5.6 on p.53 of [Cha1] to obtain (4.7).

**Case 2.** $c \in \text{per}(\Sigma)$.

In this case, let
\[ \tau_0 = \max\{\tau < c \mid \tau \in \text{per}(\Sigma)\}, \quad \tau_1 = \min\{\tau > c \mid \tau \in \text{per}(\Sigma)\}. \]
Let $\delta < \frac{1}{2}\min\{c - \tau_0, \tau_1 - c\}$ to be determined later. Then nonzero critical points of $\psi_{\lambda}$ for $\lambda \in [c - \delta, c]$ consists of closed characteristics $(\tau, y)$ on $\Sigma$ with period $\tau < c$, and nonzero critical points of $\psi_{\lambda}$ for $\lambda \in (c, c + \delta]$ consists of closed characteristics $(\tau, y)$ on $\Sigma$ with period $\tau \leq c$. Then by Proposition 4.1, we have
\[ \psi_{\lambda}(g_{\lambda}) \leq \Psi_{\lambda, \tau_0} \leq \Psi_{c-\delta, \tau_0} < 0 \]
for all nonzero critical points $g_{\lambda}$ of $\psi_{\lambda}$ with period $\tau < c$ when $\lambda \in [c - \delta, c + \delta]$. Note that by (ii) and (iii) of Proposition 4.1, the critical value $\Psi_{\lambda, c}$ of $\psi_{\lambda}$ for $\lambda \in (c, c + \delta]$ with period $\tau = c$ is close to 0 when $\delta$ is small. Specially by Proposition 4.1, from the fact $\Psi_{c, \tau_0} < \frac{1}{2}\Psi_{c, \tau_0} < 0 = \lim_{\lambda \to c+} \Psi_{\lambda, c}$ we can choose $\delta > 0$ so small such that the following relation holds:
\[ \frac{1}{2}\Psi_{c, \tau_0} < \frac{1}{2}\Psi_{c-\delta, \tau_0} < \Psi_{c+\delta, c} < 0 = \lim_{\lambda \to c+} \Psi_{\lambda, c}. \]
Let $\varepsilon' = -\frac{1}{2}\Psi_{c-\delta, \tau_0}$. Therefore by our above discussion $\pm \varepsilon'$ are regular values of $\psi_{\lambda}$ for all $\lambda \in [c - \delta, c + \delta]$.

More precisely, we have Figure 4.1, where the $\mu$-axis denote the critical values of $\psi_{\lambda}$, $f_0(\lambda) = \Psi_{\lambda, \tau_0}$ for $\lambda \in [c - \delta, c + \delta]$ and $f_1(\lambda) = \Psi_{\lambda, c}$ for $\lambda \in (c, c + \delta]$. Both $f_0$ and $f_1$ are decreasing functions in $\lambda$ by (ii) of Proposition 4.1, and $\lim_{\lambda \to c+} f_1(\lambda) = 0$ by (iii) of Proposition 4.1. Moreover, by (i) of Proposition 4.1, the interior of the shaded part in the Figure 4.1 contains no critical values of
ψ_λ for λ ∈ [c − δ, c + δ]. Therefore −ε' is a common regular value of those ψ_λs. Since all critical values of ψ_λ are non-positive, ε' is a common regular value for all of ψ_λs too.

Hence by the same proof of Case 1, we have

\[ H_{S^1, q}(\tilde{\Lambda}_c^{\epsilon'}, \tilde{\Lambda}_c^{-\epsilon'}) \cong H_{S^1, q}(\tilde{\Lambda}_c^{\epsilon'}, \tilde{\Lambda}_c^{-\epsilon'}), \quad \forall q \in \mathbb{Z} \]  

(4.8)

for any \( c_1, c_2 \in [c - \delta, c + \delta] \).

Since 0 is the only critical value of ψ_λ in \((-\epsilon', \epsilon')\) for λ ∈ [c − δ, c], for any ε ≤ ε' we have

\[ H_{S^1, q}(\tilde{\Lambda}_c^{\epsilon}, \tilde{\Lambda}_c^{-\epsilon}) \cong H_{S^1, q}(\tilde{\Lambda}_c^{\epsilon}, \tilde{\Lambda}_c^{-\epsilon}), \quad \forall q \in \mathbb{Z}. \]  

(4.9)

But critical values of ψ_λ in \([-\epsilon', \epsilon']\) for λ ∈ (c, c + δ] are precisely 0 and Ψ_λ, c as indicated in the Figure 4.1.

If ε ∈ \((-\Psi_\lambda, c, \epsilon')\), then the interval \([-\epsilon', -\epsilon]\) contains no critical values of Ψ_λ. Hence (4.9) remains true for these ε and λ.

If ε ∈ (0, −Ψ_λ, c], we consider the exact sequence of the triple \((\tilde{\Lambda}_\lambda^\epsilon, \tilde{\Lambda}_\lambda^{-\epsilon}, \tilde{\Lambda}_\lambda^{-\epsilon'})\):

\[ H_{S^1, q}(\tilde{\Lambda}_\lambda^\epsilon, \tilde{\Lambda}_\lambda^{-\epsilon'}) \rightarrow H_{S^1, q}(\tilde{\Lambda}_\lambda^\epsilon, \tilde{\Lambda}_\lambda^{-\epsilon}) \rightarrow H_{S^1, q}(\tilde{\Lambda}_\lambda^\epsilon, \tilde{\Lambda}_\lambda^{-\epsilon}) \rightarrow H_{S^1, q-1}(\tilde{\Lambda}_\lambda^\epsilon, \tilde{\Lambda}_\lambda^{-\epsilon'}). \]  

(4.10)

Since \( \Psi_\lambda, c \) is the unique critical value of \( \psi_\lambda \) in \([-\epsilon', -\epsilon]\), as in Lemma 1.4.2 of [Cha1], we have

\[ H_{S^1, *}(\tilde{\Lambda}_\lambda^{-\epsilon}, \tilde{\Lambda}_\lambda^{-\epsilon'}) \cong \bigoplus_{i=1}^l C_{S^1, *}(\psi_\lambda, S^1 \cdot g_{\lambda, i}) \cong \bigoplus_{i=1}^l C_{S^1, *}(\Psi_\lambda, S^1 \cdot u_{\lambda, i}). \]  

(4.11)
Here $g_{\lambda, i}$ and $u_{\lambda, i}$ denote the critical points of $\psi_\lambda$ and $\Psi_\lambda$ with critical value $\Psi_{\lambda, c}$ respectively. The second isomorphism follows from (iv) of Lemma 3.4. By Proposition 3.10 and the definition of $I_0$, we have

$$C_{S^1, q}(\Psi_\lambda, S^1 \cdot u_{\lambda, i}) \cong 0, \quad \forall q \leq I_0, \ 1 \leq i \leq l.$$  

Hence (4.11) yields

$$H_{S^1, q}(\tilde{\Lambda}_\lambda^-, \tilde{\Lambda}_\lambda^\varepsilon) \cong 0, \quad \forall q \leq I_0. \quad (4.12)$$

Then by (4.10) we have

$$H_{S^1, q}(\tilde{\Lambda}_\lambda^-, \tilde{\Lambda}_\lambda^\varepsilon) \cong H_{S^1, q}(\tilde{\Lambda}_\lambda^\varepsilon, \tilde{\Lambda}_\lambda^-), \quad \forall q \leq I_0. \quad (4.13)$$

Since $\psi_\lambda$ has no critical value in $[\varepsilon, \varepsilon']$, we have

$$H_{S^1, q}(\tilde{\Lambda}_\lambda^\varepsilon, \tilde{\Lambda}_\lambda^-) \cong H_{S^1, q}(\tilde{\Lambda}_\lambda^\varepsilon, \tilde{\Lambda}_\lambda^-), \quad \forall q \in \mathbb{Z}. \quad (4.14)$$

Combining (4.8), (4.9), (4.13) and (4.14) we obtain (4.7). The proof of Step 1 is complete.

**Step 2.** Claim:

$$H_{S^1, q}(\Lambda_b^\varepsilon, \Lambda_b^-) \cong 0, \quad \forall q \leq I_0 \quad (4.15)$$

holds for some $b > a$ large enough and some $\varepsilon \in (0, \varepsilon_b]$ sufficiently small.

In fact, the proof is a modification of that of Theorem 3.8 in [Eke1] to the $S^1$-equivariant case. Considering $\psi_b$, we assume $b \notin \text{per}(\Sigma)$ and will determine $b$ later.

Firstly we approximate $\psi_b$ by an $S^1$-invariant $C^2$ function $\hat{\psi}$ satisfying the following conditions:

(i) $\hat{\psi}$ has the same critical points as $\psi_b$ outside a neighborhood $\Omega$ of 0. Hence $\hat{\psi}$ contains all nonzero critical points of $\psi_b$ as its critical points.

(ii) Each critical orbit $S^1 \cdot g$ of $\hat{\psi}$ contained in $\Omega$ is non-degenerate and $\hat{\psi}$ has Morse index $m^-(g) > I_0$ at the critical point $g$.

More precisely, we construct $\hat{\psi}$ as follows.

Following p.46 of [Eke1] and Proposition 2.5, we can approximate $G_b$ by a $C^2$ strictly convex function $\tilde{G}$ such that

$$\tilde{G}(x) = G_b(x), \quad \text{for } |x| \geq g_1, \quad (4.16)$$

$$(\tilde{G}''(x)\xi, \xi) \leq \frac{1}{br_0} |\xi|^2, \quad \text{for } |x| \leq g_2, \forall \xi \in \mathbb{R}^{2n}, \quad (4.17)$$

where $g_2 > g_1 > 0$ can be chosen as small as we want and $r_0$ is given by Proposition 2.5. Now we define a functional $\bar{\Psi}$ on $L^2_0(S^1, \mathbb{R}^{2n})$ as

$$\bar{\Psi}(u) = \int_0^1 \left( \frac{1}{2} Ju \cdot Mu + \tilde{G}(-Ju) \right) dt. \quad (4.18)$$
Denote by $\tilde{H}$ the Fenchel transform of $\tilde{G}$. Then critical points of $\tilde{\Psi}$ correspond to 1-periodic solutions of the equation $\dot{x} = J\tilde{H}'(x)$. Moreover, by choosing $\varrho_1$ small enough, nonzero critical points of $\Psi_b$ are also critical points of $\tilde{\Psi}$. Other critical points $u$ of $\tilde{\Psi}$ must satisfy $\|u\|_{C^0(S^1, \mathbb{R}^{2n})} < \varrho_2$. Hence such a critical point $u$ has index

$$i(u) \geq 2n \left[ \frac{br_0}{2\pi} \right] \equiv I_1(b),$$

(4.19)

by Definition 3.3, (4.17) and Proposition 1.4.14 on p.32 of [Eke3]. Now we can fix the $b$ in the Claim (4.15) to satisfy $I_1(b) > I_0$.

For any $\kappa \in \mathbb{R}$, we denote by

$$\Theta^\kappa = \{ u \in L^2_0(S^1, \mathbb{R}^{2n}) \mid \tilde{\Psi}(u) \leq \kappa \}.$$

(4.20)

By choosing $\|	ilde{\Psi} - \Psi_b\|_{C^1(L^2_0(S^1, \mathbb{R}^{2n}), \mathbb{R})}$ to be small enough, we can fix an $\varepsilon \in (0, \hat{\varepsilon}_b)$ such that $(-2\varepsilon, 0)$ contains no critical value of $\Psi_b$ and $\pm \varepsilon$ are regular values of $\tilde{\Psi}$. Then using a slightly stronger version of Exercise 8.4 of [MaW1] or Theorem 1.5.6 on p.53 of [Cha1] with continuous dependence on the parameter, we obtain

$$H_{S^1, q}(\Lambda_b^\varepsilon, \Lambda_b^{-\varepsilon}) \simeq H_{S^1, q}(\Theta^\varepsilon, \Theta^{-\varepsilon}), \quad \forall q \in \mathbb{Z}.$$

(4.21)

By Lemma 3.4, we can choose a finite dimensional approximation $\tilde{h} : X \to X^\perp$ and consider the function $\tilde{\psi}(g) = \tilde{\Psi}(g + \tilde{h}(g))$ on the finite dimensional manifold $X$. We have

$$H_{S^1, q}(\Theta^\varepsilon, \Theta^{-\varepsilon}) \simeq H_{S^1, q}(\tilde{\Theta}^\varepsilon, \tilde{\Theta}^{-\varepsilon}), \quad \forall q \in \mathbb{Z}.$$

(4.22)

by (iv) of Lemma 3.4, where $\tilde{\Theta}^\kappa = \{ g \in X \mid \tilde{\psi}(g) \leq \kappa \}$ for $\kappa \in \mathbb{R}$ and any critical point $g$ of $\tilde{\psi}$ with critical value in $[-\varepsilon, \varepsilon]$ has Morse index $m^-(g) \geq I_1(b)$ by (iii) of Lemma 3.4 and (4.19). By (iii) of Lemma 3.4, we have $\tilde{\psi} \in C^2(X, \mathbb{R})$ is $S^1$-invariant, and the $S^1$-action is $C^\infty$ on $X$. Hence by the Density Lemma of [Was1], $\tilde{\psi}$ can be $C^2$ approximated by a smooth $S^1$-invariant function $\psi$ whose critical orbits $S^1 \cdot g$ are non-degenerate when $\tilde{\psi}(g) \in [-\varepsilon, \varepsilon]$, i.e., $\tilde{\psi}$ is a Morse function there, and any critical point $g$ of $\tilde{\psi}$ with critical values in $[-\varepsilon, \varepsilon]$ has Morse index $m^-(g) \geq I_1(b)$. This finish the construction of $\tilde{\psi}$.

When $\tilde{\psi}$ is sufficiently close to $\tilde{\psi}$, we have by a slightly stronger version of Exercise 8.4 of [MaW1] or Theorem 1.5.6 on p.53 of [Cha1] again,

$$H_{S^1, q}(\tilde{\Theta}^\varepsilon, \tilde{\Theta}^{-\varepsilon}) \simeq H_{S^1, q}(\Delta^\varepsilon, \Delta^{-\varepsilon}), \quad \forall q \in \mathbb{Z},$$

(4.23)
where $\Delta^\kappa = \{ g \in X \mid \hat{\psi}(g) \leq \kappa \}$ for $\kappa \in \mathbb{R}$. Now by the Thom isomorphism (cf. p.77 of [Cha1]), we have

$$C_{S^1, q}(\hat{\psi}, S^1 \cdot g) \cong H_{q-m-}(BG_g, \theta),$$

(4.24)

where $G_g$ is the isotropy group of the critical orbit $S^1 \cdot g$ and $\theta$ is the orientation bundle of the negative bundle of $\hat{\psi}'(g)$. Hence we have

$$C_{S^1, q}(\hat{\psi}, S^1 \cdot g) \cong 0, \quad \forall q < I_1(b),$$

(4.25)

for any critical point $g$ of $\hat{\psi}$ with critical values in $[-\varepsilon, \varepsilon]$. Hence by the Morse inequality, we have

$$\sum_{i=1}^{l} \dim C_{S^1, q}(\hat{\psi}, S^1 \cdot g_i) \geq \dim H_{S^1, q}(\Delta^+, \Delta^-), \quad \forall q \in \mathbb{Z},$$

(4.26)

where we denote the critical orbits of $\hat{\psi}$ with critical values in $[-\varepsilon, \varepsilon]$ by $\{ S^1 \cdot g_1, \ldots, S^1 \cdot g_l \}$. Now combining (4.21)-(4.23), (4.25) and (4.26), we obtain the claim (4.15).

**Step 3.** Now (4.5) of Step 1 and (4.15) of Step 2 yield an $\varepsilon \in (0, \tilde{\varepsilon}_b]$ for some $b > a$ large enough such that (4.3) holds for this $\varepsilon$. Then by (4.4) we obtain (4.3) for all $\varepsilon \in (0, \epsilon_0]$ and then the proof of Theorem 4.2 is complete.

## 5 Proof of the Theorem 1.2

In this section, we give a proof for the Theorem 1.2 with $H_a(x) = a \varphi_a(j(x))$, where $\varphi_a$ satisfies (i)-(iv) of Proposition 2.2.

Let $\Psi_a$ be a functional defined by (2.11) for some $a \in \mathbb{R}$ large enough and let $\varepsilon > 0$ be small enough such that $[-\varepsilon, 0)$ contains no critical values of $\Psi_a$. We consider the exact sequence of the space pair $(\Lambda^a_\infty, \Lambda^{-\varepsilon}_a)$:

$$H_{S^1, q+1}(\Lambda^\infty_a, \Lambda^{-\varepsilon}_a) \to H_{S^1, q}(\Lambda^{-\varepsilon}_a) \to H_{S^1, q}(\Lambda^\infty_a) \to H_{S^1, q}(\Lambda^\infty_a, \Lambda^{-\varepsilon}_a)$$

(5.1)

for any $q \in \mathbb{Z}$. Let $I_0 \in \mathbb{N}_0$ be given by Theorem 4.2. Note that by Proposition 4.1, there are no critical values of $\Psi_a$ in $(0, +\infty)$. Hence by Theorem 4.2 we have

$$H_{S^1, q}(\Lambda^\infty_a, \Lambda^{-\varepsilon}_a) \cong H_{S^1, q}(\Lambda^\varepsilon_a, \Lambda^{-\varepsilon}_a) \cong 0, \quad \forall q \leq I_0.$$  

(5.2)

Therefore (5.1) implies

$$H_{S^1, q}(\Lambda^{-\varepsilon}_a) \cong H_{S^1, q}(\Lambda^\infty_a) \cong H_q(CP^\infty), \quad \forall q < I_0.$$  

(5.3)
The second isomorphism follows since \( \Lambda_a^\infty = L^2_0(S^1, \mathbb{R}^{2n}) \) is \( S^1 \)-equivariantly homotopic to a single point.

Let \( X \) be an \( S^1 \)-space such that the Betti numbers \( b_i(X) = \dim H_{S^1, i}(X; \mathbb{Q}) \) are finite for all \( i \in \mathbb{Z} \). As usual by Proposition 2.7, \( \Psi_a \) is bounded from below on \( L^2_0(S^1, \mathbb{R}^{2n}) \). Hence the \( S^1 \)-equivariant Morse series \( M(t) \) of the functional \( \Psi_a \) on the space \( \Lambda_a^{-\varepsilon} \) is defined as usual by

\[
M(t) = \sum_{q \geq 0, 1 \leq j \leq p} \dim C_{S^1, q}(\Psi_a, S^1 \cdot v_j)t^q, \quad (5.4)
\]

where we denote by \( \{S^1 \cdot v_1, \ldots, S^1 \cdot v_p\} \) the critical orbits of \( \Psi_a \) with critical values less than \( -\varepsilon \). Then the Morse inequality in the equivariant sense yields a formal power series \( Q(t) = \sum_{i=0}^{\infty} q_i t^i \) with nonnegative integer coefficients \( q_i \) such that

\[
M(t) = P(t) + (1 + t)Q(t), \quad (5.5)
\]

where \( P(t) \equiv P(\Lambda_a^{-\varepsilon})(t) \). For a formal power series \( R(t) = \sum_{i=0}^{\infty} r_i t^i \), we denote by \( R^L(t) = \sum_{i=0}^{L} r_i t^i \) for \( L \in \mathbb{N} \) the corresponding truncated polynomial. Using this notation, \( (5.5) \) becomes

\[
(-1)^L q_L = M^L(-1) - P^L(-1), \quad \forall L \in \mathbb{N}. \quad (5.6)
\]

Now we can give the following

**Proof of Theorem 1.2.** Firstly we choose \( \Psi_a \) as above and denote by \( \{u_1, \ldots, u_k\} \) the critical points of \( \Psi_a \) corresponding to \( \{y_1, \ldots, y_k\} \). Note that \( v_1, \ldots, v_p \) in \( (5.4) \) are iterations of \( u_1, \ldots, u_k \). Since \( C_{S^1, q}(\Psi_a, S^1 \cdot u_j^m) \) can be non-zero only for \( q = i(v_j^m) + l \) with \( 0 \leq l \leq 2n - 2 \) by Propositions 3.5 and 3.10, the formal Poincaré series \( (5.4) \) becomes

\[
M(t) = \sum_{1 \leq j \leq k, 0 \leq i \leq 2n-2, 1 \leq m_j < a/r_j} k_i(u_j^m)i(u_j^m)+l = \sum_{1 \leq j \leq k, 0 \leq i \leq 2n-2, 1 \leq m_j \leq K_j, aK_j+m_j < a/r_j} k_i(u_j^m)i(u_j^{aK_j+m_j})+l, \quad (5.7)
\]

where \( K_j = K(u_j) \) and \( s \in \mathbb{N}_0 \). The last equality follows from Proposition 3.13. Let \( I = I_0 - 2 \), where \( I_0 \) is given by \( (5.3) \) and consider the truncated polynomials \( M^I(t) \) and \( P^I(t) \).

Write \( M(t) = \sum_{h=0}^{\infty} w_h t^h \) and \( P^I(t) = \sum_{h=0}^{I} b_h t^h \). Then we have

\[
w_h = \sum_{1 \leq j \leq k, 0 \leq i \leq 2n-2, 1 \leq m \leq K_j} k_i(u_j^m) \# \{ s \in \mathbb{N}_0 | i(u_j^{aK_j+m}) + l = h \}, \quad \forall h \leq I + 1. \quad (5.8)
\]

Note that the right hand side of \( (5.7) \) contains only those terms satisfying \( sK_j + m_j < \frac{a}{r_j} \). Thus \( (5.8) \) holds only for \( h \leq I + 1 \) by \( (5.7) \).
Claim 1. $w_h \leq C$ for $h \leq I + 1$ with $C$ being independent of $a$.

In fact, we have

$$\# \{ s \in \mathbb{N}_0 \mid i(u_j^{sK_j+m}) + l = h \}$$

$$= \# \{ s \in \mathbb{N}_0 \mid i(u_j^{sK_j+m}) + l = h, \; |i(u_j^{sK_j+m}) - (sK_j + m)\hat{i}(u_j)| \leq 2n \}$$

$$\leq \# \{ s \in \mathbb{N}_0 \mid |h - l - (sK_j + m)\hat{i}(u_j)| \leq 2n \}$$

$$= \# \left\{ s \in \mathbb{N}_0 \bigg| \frac{h - l - 2n - m\hat{i}(u_j)}{K_j\hat{i}(u_j)} \leq s \leq \frac{h - l + 2n - m\hat{i}(u_j)}{K_j\hat{i}(u_j)} \right\}$$

$$\leq \frac{4n}{K_j\hat{i}(u_j)} + 2,$$  \hspace{1cm} (5.9)

where the first equality follows from the fact

$$|i(u_j^m) - m\hat{i}(u_j)| \leq 2n, \; \forall m \in \mathbb{N}, \; 1 \leq j \leq k,$$  \hspace{1cm} (5.10)

which follows from Theorems 10.1.2 and 15.1.1 of [Lon4]. Hence Claim 1 holds.

We estimate next $M^I(-1)$. By (5.8) we obtain

$$M^I(-1) = \sum_{h=0}^I w_h(-1)^h$$

$$= \sum_{1 \leq j \leq k, \; 0 \leq t \leq 2n-2} \sum_{1 \leq m \leq K_j} (-1)^{i(u_j^m) + t}k_l(u_j^m) \# \{ s \in \mathbb{N}_0 \mid i(u_j^{sK_j+m}) + l \leq I \}. \hspace{1cm} (5.11)$$

Here the second equality holds by (3.30).

Claim 2. There is a real constant $C' > 0$ independent of $a$ such that

$$\left| M^I(-1) - \sum_{1 \leq j \leq k, \; 0 \leq t \leq 2n-2} \sum_{1 \leq m \leq K_j} (-1)^{i(u_j^m) + t}k_l(u_j^m) \frac{I}{K_j\hat{i}(y_j)} \hat{i}(y_j) \right| \leq C', \hspace{1cm} (5.12)$$

where the sum in the left hand side of (5.12) equals to $I \sum_{1 \leq j \leq k} \frac{\hat{i}(y_j)}{i(y_j)}$ by (3.57).

In fact, we have the estimates

$$\# \{ s \in \mathbb{N}_0 \mid i(u_j^{sK_j+m}) + l \leq I \}$$

$$= \# \{ s \in \mathbb{N}_0 \mid i(u_j^{sK_j+m}) + l \leq I, \; |i(u_j^{sK_j+m}) - (sK_j + m)\hat{i}(u_j)| \leq 2n \}$$

$$\leq \# \{ s \in \mathbb{N}_0 \mid 0 \leq (sK_j + m)\hat{i}(u_j) \leq I - l + 2n \}$$

$$= \# \left\{ s \in \mathbb{N}_0 \bigg| 0 \leq s \leq \frac{I - l + 2n - m\hat{i}(u_j)}{K_j\hat{i}(u_j)} \right\}$$

$$\leq \frac{I - l + 2n}{K_j\hat{i}(u_j)} + 1.$$
On the other hand, we have
\[
\# \{ s \in \mathbb{N}_0 \mid i(u_j^{sK_j+m}) + l \leq I \} \\
\quad = \# \{ s \in \mathbb{N}_0 \mid i(u_j^{sK_j+m}) + l \leq I, \ |i(u_j^{sK_j+m}) - (sK_j + m)\hat{i}(u_j)| \leq 2n \} \\
\quad \geq \# \{ s \in \mathbb{N}_0 \mid i(u_j^{sK_j+m}) \leq (sK_j + m)\hat{i}(u_j) + 2n \leq I - l \} \\
\quad \geq \frac{I - l - 2n}{K_j\hat{i}(u_j)} - 2,
\]

where \( m \leq K_j \) is used. Combining these two estimates together with (5.11), we obtain (5.12).

Note that all coefficients in (5.5) are nonnegative, hence by Claim 1, we have \( q_I \leq w_I + 1 \leq C \).

By (5.3), we have
\[
P^I(t) = \sum_{0 \leq h \leq \frac{t}{2}} t^{2h}.
\]

By (5.6), we have
\[
(-1)^l q_I = M^I(-1) - P^I(-1) = M^I(-1) - \left( \left\lfloor \frac{I}{2} \right\rfloor + 1 \right).
\] (5.13)

By Theorem 1.7.7 of [Eke3] or Lemma 15.3.2 of [Lon4], we have \( \hat{i}(y_j) > 2 \) for \( 1 \leq j \leq k \). Hence \( i(m\tau_j, y_j) \equiv i(y_j^m) \to \infty \) as \( m \to \infty \) for \( 1 \leq j \leq k \). Now we let \( a \to +\infty \), then \( I = I_0 - 2 \to +\infty \) in Theorem 4.2. Note that by Claims 1 and 2, the constants \( C \) and \( C' \) are independent of \( a \). Hence dividing both sides of (5.13) by \( I \) and letting \( I \) tending to infinity yield
\[
\lim_{l \to \infty} \frac{1}{I} M^I(-1) = \frac{1}{2}.
\]

Hence (1.3) holds by (5.12).}

6 Proofs of the Theorems 1.1 and 1.4

In this section, we prove Theorems 1.1 and 1.4 based on Theorem 1.2 and the index iteration theory developed by Y. Long and his coworkers.

6.1 A brief review on an index theory for symplectic paths

In this subsection, we recall briefly an index theory for symplectic paths. All the details can be found in [Lon4].

As usual, the symplectic group \( \text{Sp}(2n) \) is defined by
\[
\text{Sp}(2n) = \{ M \in \text{GL}(2n, \mathbb{R}) \mid M^TJM = J \},
\]
whose topology is induced from that of \( \mathbb{R}^{4n^2} \). For \( \tau > 0 \) we are interested in paths in \( \text{Sp}(2n) \):

\[
\mathcal{P}_\tau(2n) = \{ \gamma \in C([0, \tau], \text{Sp}(2n)) \mid \gamma(0) = I_{2n} \},
\]

which is equipped with the topology induced from that of \( \text{Sp}(2n) \). The following real function was introduced in \([\text{Lon2}]\):

\[
D_\omega(M) = (-1)^{n-1} \omega^n \det(M - \omega I_{2n}), \quad \forall \omega \in U, \ M \in \text{Sp}(2n).
\]

Thus for any \( \omega \in U \) the following codimension 1 hypersurface in \( \text{Sp}(2n) \) is defined in \([\text{Lon2}]\):

\[
\text{Sp}(2n)_\omega^0 = \{ M \in \text{Sp}(2n) \mid D_\omega(M) = 0 \}.
\]

For any \( M \in \text{Sp}(2n)_\omega^0 \), we define a co-orientation of \( \text{Sp}(2n)_\omega^0 \) at \( M \) by the positive direction \( \frac{d}{dt}M e^{tJ} |_{t=0} \) of the path \( M e^{tJ} \) with \( 0 \leq t \leq 1 \) and \( \epsilon > 0 \) being sufficiently small. Let

\[
\begin{align*}
\text{Sp}(2n)_\omega^* & = \text{Sp}(2n) \setminus \text{Sp}(2n)_\omega^0, \\
\mathcal{P}_{\tau, \omega}^*(2n) & = \{ \gamma \in \mathcal{P}_\tau(2n) \mid \gamma(\tau) \in \text{Sp}(2n)_\omega^* \}, \\
\mathcal{P}_{\tau, \omega}^0(2n) & = \mathcal{P}_\tau(2n) \setminus \mathcal{P}_{\tau, \omega}^*(2n).
\end{align*}
\]

For any two continuous arcs \( \xi : [0, \tau] \rightarrow \text{Sp}(2n) \) with \( \xi(\tau) = \eta(0) \), it is defined as usual:

\[
\eta * \xi(t) = \begin{cases} 
\xi(2t), & \text{if } 0 \leq t \leq \tau/2, \\
\eta(2t - \tau), & \text{if } \tau/2 \leq t \leq \tau.
\end{cases}
\]

Given any two \( 2m_k \times 2m_k \) matrices of square block form \( M_k = \begin{pmatrix} A_k & B_k \\ C_k & D_k \end{pmatrix} \) with \( k = 1, 2 \), as in \([\text{Lon4}]\), the \( \bigcirc \)-product of \( M_1 \) and \( M_2 \) is defined by the following \( 2(m_1 + m_2) \times 2(m_1 + m_2) \) matrix \( M_1 \bigcirc M_2 \):

\[
M_1 \bigcirc M_2 = \begin{pmatrix}
A_1 & 0 & B_1 & 0 \\
0 & A_2 & 0 & B_2 \\
C_1 & 0 & D_1 & 0 \\
0 & C_2 & 0 & D_2
\end{pmatrix}.
\]

Denote by \( M^{\bigcirc k} \) the \( k \)-fold \( \bigcirc \)-product \( M \bigcirc \cdots \bigcirc M \). Note that the \( \bigcirc \)-product of any two symplectic matrices is symplectic. For any two paths \( \gamma_j \in \mathcal{P}_\tau(2n_j) \) with \( j = 0 \) and 1, let \( \gamma_0 \bigcirc \gamma_1(t) = \gamma_0(t) \bigcirc \gamma_1(t) \) for all \( t \in [0, \tau] \).

A special path \( \xi_n \in \mathcal{P}_\tau(2n) \) is defined by

\[
\xi_n(t) = \left( \begin{array}{cc}
2 - \frac{t}{\tau} & 0 \\
0 & (2 - \frac{t}{\tau})^{-1}
\end{array} \right)^{\otimes n} \quad \text{for } 0 \leq t \leq \tau.
\]
Definition 6.1. (cf. [Lon2], [Lon4]) For any $\omega \in U$ and $M \in \text{Sp}(2n)$, define
\[
\nu_\omega(M) = \dim \ker_{\mathbb{C}}(M - \omega I_{2n}).
\] (6.2)

For any $\tau > 0$ and $\gamma \in \mathcal{P}_\tau(2n)$, define
\[
\nu_\omega(\gamma) = \nu_\omega(\gamma(\tau)).
\] (6.3)

If $\gamma \in \mathcal{P}^*_{\tau,\omega}(2n)$, define
\[
i_\omega(\gamma) = \left[\text{Sp}(2n)^0_\omega : \gamma \ast \xi_n\right],
\] (6.4)

where the right hand side of (6.4) is the usual homotopy intersection number, and the orientation of $\gamma \ast \xi_n$ is its positive time direction under homotopy with fixed end points.

If $\gamma \in \mathcal{P}^0_{\tau,\omega}(2n)$, we let $\mathcal{F}(\gamma)$ be the set of all open neighborhoods of $\gamma$ in $\mathcal{P}_\tau(2n)$, and define
\[
i_\omega(\gamma) = \sup_{U \in \mathcal{F}(\gamma)} \inf \{i_\omega(\beta) | \beta \in U \cap \mathcal{P}^*_{\tau,\omega}(2n)\}.
\] (6.5)

Then
\[
(i_\omega(\gamma), \nu_\omega(\gamma)) \in \mathbb{Z} \times \{0, 1, \ldots, 2n\},
\]
is called the index function of $\gamma$ at $\omega$.

Note that when $\omega = 1$, this index theory was introduced by C. Conley-E. Zehnder in [CoZ1] for the non-degenerate case with $n \geq 2$, Y. Long-E. Zehnder in [LZe1] for the non-degenerate case with $n = 1$, and Y. Long in [Lon1] and C. Viterbo in [Vit2] independently for the degenerate case.

The case for general $\omega \in U$ was defined by Y. Long in [Lon2] in order to study the index iteration theory (cf. [Lon4] for more details and references).

For any symplectic path $\gamma \in \mathcal{P}_\tau(2n)$ and $m \in \mathbb{N}$, we define its $m$-th iteration $\gamma^m : [0, m\tau] \to \text{Sp}(2n)$ by
\[
\gamma^m(t) = \gamma(t - j\tau)\gamma(\tau)^j, \quad \forall j\tau \leq t \leq (j + 1)\tau, \quad j = 0, 1, \ldots, m - 1.
\] (6.6)

We still denote the extended path on $[0, +\infty)$ by $\gamma$.

Definition 6.2. (cf. [Lon2], [Lon4]) For any $\gamma \in \mathcal{P}_\tau(2n)$, we define
\[
(i(\gamma, m), \nu(\gamma, m)) = (i_1(\gamma^m), \nu_1(\gamma^m)), \quad \forall m \in \mathbb{N}.
\] (6.7)

The mean index $\hat{i}(\gamma, m)$ per $m\tau$ for $m \in \mathbb{N}$ is defined by
\[
\hat{i}(\gamma, m) = \lim_{k \to +\infty} \frac{i(\gamma, mk)}{k}.
\] (6.8)
For any $M \in \text{Sp}(2n)$ and $\omega \in U$, the splitting numbers $S_{M}^{\pm}(\omega)$ of $M$ at $\omega$ are defined by

$$S_{M}^{\pm}(\omega) = \lim_{\epsilon \to 0^+} i_{\omega} \exp(\pm \sqrt{-1} \epsilon)(\gamma) - i_{\omega}(\gamma), \quad (6.9)$$

for any path $\gamma \in \mathcal{P}_{\tau}(2n)$ satisfying $\gamma(\tau) = M$.

For a given path $\gamma \in \mathcal{P}_{\tau}(2n)$ we consider to deform it to a new path $\eta$ in $\mathcal{P}_{\tau}(2n)$ so that

$$i_{1}(\gamma^{m}) = i_{1}(\eta^{m}), \quad \nu_{1}(\gamma^{m}) = \nu_{1}(\eta^{m}), \quad \forall m \in \mathbb{N}, \quad (6.10)$$

and that $((i_{1}(\eta^{m}), \nu_{1}(\eta^{m})))$ is easy enough to compute. This leads to finding homotopies $\delta : [0,1] \times [0,\tau] \to \text{Sp}(2n)$ starting from $\gamma$ in $\mathcal{P}_{\tau}(2n)$ and keeping the end points of the homotopy always stay in a certain suitably chosen maximal subset of $\text{Sp}(2n)$ so that (6.10) always holds. In fact, this set was first introduced in [Lon2] as the path connected component $\Omega^{0}(M)$ containing $M = \gamma(\tau)$ of the set

$$\Omega(M) = \{ N \in \text{Sp}(2n) \mid \sigma(N) \cap U = \sigma(M) \cap U \text{ and } \nu_{\lambda}(N) = \nu_{\lambda}(M) \forall \lambda \in \sigma(M) \cap U \}. \quad (6.11)$$

Here $\Omega^{0}(M)$ is called the homotopy component of $M$ in $\text{Sp}(2n)$.

In [Lon2]-[Lon4], the following symplectic matrices were introduced as basic normal forms:

$$D(\lambda) = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}, \quad \lambda = \pm 2, \quad (6.12)$$

$$N_{1}(\lambda, b) = \begin{pmatrix} \lambda & b \\ 0 & \lambda \end{pmatrix}, \quad \lambda = \pm 1, b = \pm 1, 0, \quad (6.13)$$

$$R(\theta) = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix}, \quad \theta \in (0, \pi) \cup (\pi, 2\pi), \quad (6.14)$$

$$N_{2}(\omega, b) = \begin{pmatrix} R(\theta) & b \\ 0 & R(\theta) \end{pmatrix}, \quad \theta \in (0, \pi) \cup (\pi, 2\pi), \quad (6.15)$$

where $b = \left( \begin{array}{c} b_{1} \\ b_{2} \\ b_{3} \\ b_{4} \end{array} \right)$ with $b_{i} \in \mathbb{R}$ and $b_{2} \neq b_{3}$.

Splitting numbers possess the following properties:

**Lemma 6.3.** (cf. [Lon2] and Lemma 9.1.5 of [Lon4]) Splitting numbers $S_{M}^{\pm}(\omega)$ are well defined, i.e., they are independent of the choice of the path $\gamma \in \mathcal{P}_{\tau}(2n)$ satisfying $\gamma(\tau) = M$ appeared in (6.14). For $\omega \in U$ and $M \in \text{Sp}(2n)$, splitting numbers $S_{N}^{\pm}(\omega)$ are constant for all $N \in \Omega^{0}(M)$.

**Lemma 6.4.** (cf. [Lon2], Lemma 9.1.5 and List 9.1.12 of [Lon4]) For $M \in \text{Sp}(2n)$ and $\omega \in U$, there hold

$$S_{M}^{\pm}(\omega) = 0, \quad \text{if } \omega \notin \sigma(M), \quad (6.16)$$

$$S_{N_{1}(1,\omega)}^{\pm}(1) = \begin{cases} 1, & \text{if } a \geq 0, \\ 0, & \text{if } a < 0. \end{cases} \quad (6.17)$$
For any $M_i \in \text{Sp}(2n_i)$ with $i = 0$ and 1, there holds
\[ S^\pm_{M_0 \circ M_1}(\omega) = S^\pm_{M_0}(\omega) + S^\pm_{M_1}(\omega), \quad \forall \omega \in U. \] (6.18)

We have the following

**Theorem 6.5.** (cf. [Lon3] and Theorem 1.8.10 of [Lon4]) For any $M \in \text{Sp}(2n)$, there is a path $f : [0, 1] \to \Omega^0(M)$ such that $f(0) = M$ and
\[ f(1) = M_1(\omega_1) \diamond \cdots \diamond M_k(\omega_k), \] (6.19)
where each $M_i(\omega_i)$ is a basic normal form as in (6.12)-(6.15) for $1 \leq i \leq k$.

### 6.2 Multiplicity and stability of closed characteristics

Let $\Sigma \in \mathcal{H}(2n)$. Fix a constant $\alpha$ satisfying $1 < \alpha < 2$ and define the Hamiltonian function $H_\alpha : \mathbb{R}^{2n} \to [0, +\infty)$ by
\[ H_\alpha(x) = j(x)^\alpha, \quad \forall x \in \mathbb{R}^{2n}, \] (6.20)
where $j$ is the gauge function of $\Sigma$, i.e., $j(x) = \lambda$ if $x = \lambda y$ for some $\lambda > 0$ and $y \in \Sigma$ when $x \in \mathbb{R}^{2n} \setminus \{0\}$, and $j(0) = 0$.

Then $H_\alpha \in C^1(\mathbb{R}^{2n}, \mathbb{R}) \cap C^3(\mathbb{R}^{2n} \setminus \{0\}, \mathbb{R})$ is strictly convex and $\Sigma = H_\alpha^{-1}(1)$. It is well-known that the problem (1.1) is equivalent to the following given energy problem of the Hamiltonian system
\[
\begin{cases}
\dot{y}(t) = JH'_\alpha(y(t)), & H_\alpha(y(t)) = 1, \quad \forall t \in \mathbb{R}, \\
y(\tau) = y(0).
\end{cases}
\] (6.21)

Denote by $\mathcal{T}(\Sigma, \alpha)$ the set of all geometrically distinct solutions $(\tau, y)$ of the problem (6.21). Note that elements in $\mathcal{T}(\Sigma)$ defined in Section 1 and $\mathcal{T}(\Sigma, \alpha)$ are one to one correspondent to each other.

Let $(\tau, y) \in \mathcal{T}(\Sigma, \alpha)$. The fundamental solution $\gamma_y : [0, \tau] \to \text{Sp}(2n)$ with $\gamma_y(0) = I_{2n}$ of the linearized Hamiltonian system
\[ \dot{\xi}(t) = JH''_\alpha(y(t))\xi(t), \quad \forall t \in \mathbb{R}, \] (6.22)
is called the associated symplectic path of $(\tau, y)$. The eigenvalues of $\gamma_y(\tau)$ are called Floquet multipliers of $(\tau, y)$.

For any $(\tau, y) \in \mathcal{T}(\Sigma, \alpha)$ and $m \in \mathbb{N}$, we define its $m$-th iteration $y^m : \mathbb{R}/(m\tau\mathbb{Z}) \to \mathbb{R}^{2n}$ by
\[ y^m(t) = y(t - j\tau), \quad \forall j\tau \leq t \leq (j + 1)\tau, \quad j = 0, 1, 2, \ldots, m - 1. \] (6.23)
We still denote by $y$ its extension to $[0, +\infty)$.

We define via Definition 6.2 the following

$$S^+(y) = S^+_{\gamma_y}(1), \quad (6.24)$$
$$i(y, m), \nu(y, m) = (i(\gamma_y, m), \nu(\gamma_y, m)), \quad (6.25)$$
$$\hat{i}(y, m) = \hat{i}(\gamma_y, m), \quad (6.26)$$

for all $m \in \mathbb{N}$, where $\gamma_y$ is the associated symplectic path of $(\tau, y)$.

We have the following result:

**Theorem 6.6.** (cf. Theorem 15.1.1 of [Lon4] and references there in) Suppose $(\tau, y) \in T(\Sigma)$. Then we have

$$i(y^m) \equiv i(m\tau, y) = i(y, m) - n, \quad \nu(y^m) \equiv \nu(m\tau, y) = \nu(y, m), \quad \forall m \in \mathbb{N}, \quad (6.27)$$

where $i(m\tau, y)$ and $\nu(m\tau, y)$ are given by Definition 3.14. In particular, we have

$$\hat{i}(\tau, y) = \hat{i}(y, 1), \quad (6.28)$$

where $\hat{i}(\tau, y)$ is given by Definition 3.14. Hence we denote it simply by $\hat{i}(y)$.

Now we can prove Theorem 1.1 as follows:

For $n \geq 2$ and $\Sigma \in \mathcal{H}(2n)$ with $\# T(\Sigma) < +\infty$, using the index iteration theory developed by Y. Long and his coworkers, specially the common index jump theorem (Theorem 4.3 of [LoZ1], Theorem 11.2.1 of [Lon4]), we obtain the following estimate on the number $\# T(\Sigma)$ by Theorem 5.1 of [LoZ1] (Theorem 15.4.3 of [Lon4]):

$$\# T(\Sigma) \geq 2 S^+(x) - \nu(x, 1) \geq -1,$$

where $p_+$ counts the number of basic normal form $N_1(1, -1)$ appears in the basic normal form decomposition of $\gamma_x(\tau)$ in $\Omega^0(\gamma_x(\tau))$. This estimate indicates that the worst case for getting a
better estimate on \( \varrho_n(\Sigma) \) happens when \( p_+ = 2 \) holds. Here we have used Theorem 6.5 and Lemma 6.3.

Now if there are only two geometrically distinct closed characteristics on \( \Sigma \subset \mathbb{R}^6 \), together with (1.4) and (6.29), our above Theorem 1.2 can be used to either kill at least one of the possible \( N_1(1, -1) \)'s or to derive a contradiction when there are two \( N_1(1, -1) \)'s in the decomposition of \( \gamma_j(\tau_j) \) in \( \Omega^0(\gamma_j(\tau_j)) \). Thus the conjecture (1.2) holds for \( n = 3 \). More precisely we have

**Proof of Theorem 1.1.** Assume the contrary, i.e., by [EkH1] or [LoZ1] we assume \( \# T(\Sigma) = 2 \) for some \( \Sigma \in \mathcal{H}(6) \). We use the techniques in the index iteration theory developed by Y. Long and his coauthors (cf. [Lon4]), specially those techniques in the proof of Theorem 5.1 of [LoZ1] (cf. p.340 of [Lon4]) to reach a contradiction.

Denote the two prime closed characteristics on \( \Sigma \) by \((\tau_j, y_j)\) with the corresponding associated symplectic paths \( \gamma_j \equiv \gamma_{y_j} : [0, \tau_j] \to \text{Sp}(6) \) for \( j = 1, 2 \). Then by Lemma 1.3 of [LoZ1] or Lemma 15.2.4 of [Lon4], there exist \( P_j \in \text{Sp}(6) \) and \( M_j \in \text{Sp}(4) \) such that \( \gamma_j(\tau_j) = P_j^{-1}(N_1(1, 1) \circ M_j)P_j \).

By our Theorem 1.2, we obtain the following identity:

\[
\hat{\chi}(y_1) + \hat{\chi}(y_2) = \frac{1}{2},
\]

(6.31)

It is well known that \( \hat{i}(y_j) > 2 \) for \( j = 1 \) and 2 (cf. Theorem 1.7.7 of [Eke3] or Lemma 15.3.2 of [Lon4]).

By Theorems 1.1 and 1.3 of [LoZ1] (cf. Theorems 15.4.3 and 15.5.2 of [Lon4]) with \( n = 3 \), we may assume that \( y_1 \) has irrational mean index \( \hat{i}(y_1) \). Next we continue our study in two cases.

**Case 1.** The average Euler characteristic \( \hat{\chi}(y_1) \neq 0 \).

In this case, by (6.31) both \( y_1 \) and \( y_2 \) must possess irrational mean indices. Hence by Theorem 8.3.1 and Corollary 8.3.2 of [Lon4], each \( M_j \) can be connected to \( R(\theta_j) \circ N_j \) within \( \Omega^0(M_j) \) for some \( \theta_j \notin \mathbb{Q} \) and \( N_j \in \text{Sp}(2) \). Now by Lemma 6.4 we have

\[
S^+_{N_1(1, 1)}(1) = \nu_1(N_1(1, 1)) = 1, \quad S^+_{R(\theta_j)}(1) = \nu_1(R(\theta_j)) = 0,
\]

(6.32)

\[
2S^+_{N_j}(1) - \nu_1(N_j) \geq -1.
\]

(6.33)

Thus we obtain by (6.10), Lemma 6.3 and Lemma 6.4

\[
2S^+(y_j) - \nu(y_j, 1)
\]

\[
= 2S^+_{N_1(1, 1)}(1) - \nu_1(N_1(1, 1)) + 2S^+_{R(\theta_j)}(1) - \nu_1(R(\theta_j)) + 2S^+_{N_j}(1) - \nu_1(N_j)
\]

\[
= 1 + 2S^+_{N_j}(1) - \nu_1(N_j)
\]

\[
\geq 0,
\]

(6.34)
By Corollary 15.1.4 of [Lon4] we have \( i(y_j, 1) \geq 3 \) for \( j = 1, 2 \). Therefore for \( j = 1, 2 \), we obtain
\[
i(y_j, 1) + 2S^+(y_j) - \nu(y_j, 1) \geq 3. \tag{6.35}
\]
Now by the estimates (6.29) and (6.30), we get a contradiction
\[
2 = \#T(\Sigma) \geq \varrho_3(\Sigma) \geq 3, \tag{6.36}
\]
which completes the proof of Theorem 1.1 in Case 1.

**Case 2.** The average Euler characteristic \( \hat{\chi}(y_1) = 0 \).

In this case (6.31) becomes
\[
\frac{\hat{\chi}(y_2)}{i(y_2)} = \frac{1}{2}. \tag{6.37}
\]
Our above discussions in Case 1 can be applied to get (6.35) for \( j = 1 \). Thus by Corollary 1.1 of [LoZ1] (Theorem 15.4.4 of [Lon4]), we should get (6.36) whenever (6.35) holds for \( j = 2 \). This yields a contradiction.

Therefore now we assume that (6.35) does not hold for \( j = 2 \). Then as in Case 1, we denote the basic normal form decomposition of \( \gamma_{y_2}(\tau_2) \) in \( \Omega^0(\gamma_{y_2}(\tau_2)) \) by \( N_1(1, 1) \circ M_2 \). By theorem 6.5, the \( 4 \times 4 \) matrix \( M_2 \) is either the \( \circ \)-product of two matrices in (6.12)-(6.14) or one matrix in (6.15).

Therefore by Lemmas 6.3-6.4 and the first part of (6.32), we have
\[
2S^+(y_2) - \nu(y_2, 1) = 1 + S^+_{M_2}(1) - \nu_1(M_2) \geq 1 - p_+ \geq -1,
\]
where \( p_+ \) counts the number of basic normal form \( N_1(1, -1) \) appears in \( M_2 \), and both of the last two equalities hold simultaneously if and only if \( p_+ = 2 \), i.e.,
\[
M_2 = N_1(1, -1)^{\circ 2}. \tag{6.38}
\]
Because \( i(y_2, 1) \geq 3 \), we obtain that the only case for which (6.35) and consequently Theorem 1.1 does not hold is when (6.38) and
\[
i(y_2, 1) = 3 \tag{6.39}
\]
hold. Hence in the following, it suffices to derive a contradiction in this case.

Now by Theorem 8.3.1 of [Lon4], we obtain
\[
i(y_2, m) = m(i(y_2, 1) + 1) - 1 = 4m - 1, \quad \nu(y_2, m) = 3, \quad \forall m \in \mathbb{N}. \tag{6.40}
\]
By Theorem 6.6, we have
\[
i(y_2) = i(y_2, 1) - 3 = 0, \quad i(y_2^2) = i(y_2, 2) - 3 = 4, \quad \hat{i}(y_2) = 4. \tag{6.41}
\]
By Proposition 3.13, we obtain \( K(y_2) = 2 \). By Remark 3.17 and (1.4) with \( n = 3 \) we obtain

\[
\hat{\chi}(y_2) = \frac{1}{K(y_2)} \sum_{1 \leq m \leq 2} \sum_{0 \leq i \leq 2} (-1)^{i(y_m^2) + i} k_1(y_m^2)
\]

\[
= \frac{1}{2} (k_0(y_2) - k_1(y_2) + k_2(y_2) + k_0(y_2^2) - k_1(y_2^2) + k_2(y_2^2))
\]

\[
\leq 1.
\]

(6.42)

Now (6.37), (6.41) and (6.42) yield a contradiction:

\[
\frac{1}{2} = \frac{\hat{\chi}(y_2)}{i(y_2)} \leq \frac{1}{4},
\]

which proves Theorem 1.1 in Case 2.

The proof of Theorem 1.1 is complete.

Proof of Theorem 1.4. Using notations in the above proof of Theorem 1.1, we obtain (6.31) for the two prime closed characteristics \((\tau_1, y_1)\) and \((\tau_2, y_2)\) on \(\Sigma\) with \(M_j \in \text{Sp}(2)\) and \(M_1 = R(\theta_1)\) for some \(\theta_1 \in \mathbb{R} \setminus \pi \mathbb{Q}\). Then \(y_1\) is non-degenerate and then we obtain \(\hat{\chi}(y_1) \neq 0\) by (1.4) and (3.58). Therefore \(\tilde{i}(y_2)\) has to be irrational by (6.31).

Remark 6.7. Using notations in the proof of Theorem 1.1, by Theorem 1.4 for \(j = 1\) and 2 there exists \(P_j \in \text{Sp}(4)\) such that \(\gamma_j(\tau_j) = P_j^{-1}(N_1(1, 1) \circ R(\theta_j))P_j\) for some \(\theta_j \in (0, \pi) \cup (\pi, 2\pi)\) with \(\theta_j / \pi \notin \mathbb{Q}\). Then we obtain \(\nu(y_j^m) = 1\) and \(i(y_j^m) \in 2\mathbb{Z}\), specially \(y_j^m\) are all non-degenerate for all \(m \in \mathbb{N}\) and then \(K(y_j) = 2\) for \(j = 1, 2\). Thus we have \(\hat{\chi}(y_j) = 1\) for \(j = 1, 2\) by (1.4). Then together with (6.31) we obtain that \(\tilde{i}(y_1) / \tilde{i}(y_2)\) is irrational. Therefore in this sense, \(\Sigma\) behaves like a weakly non-resonant ellipsoid (cf. [Eke3]).

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