D=11 Supergravity Revisited

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I discuss two novel results in D=11 Supergravity. The first establishes, in two complementary ways, a no-go theorem that, in contrast to all D<11, a cosmological extension of the theory does not exist. The second deals with the structure of (on-shell) four-point invariants. These are important both for establishing existence of the lowest (2-loop) order candidate counter-terms in the theory proper, as well as for comparison with the form of eventual “zero-slope” QFT limit of M-theory.

I. Introduction

It is a particular pleasure for me to be present on this occasion to celebrate Dick Arnowitt’s ?-birthday. Although not (quite) Dick’s oldest collaborator in age, I have seniority in terms of years: our first publication was in 1953, 45 years ago, and our joint work began a couple of years before that, with an (as yet) unpublished manuscript. In the twelve year span between 1953 and 1965 we wrote some 30-odd papers, and (about) 85% of a book on general relativity, which I find useful in teaching to this day! I am also happy to see other (mutual) old collaborators here, including (in time ordered sequence) Charlie Misner, Mike Duff and Bruno Zumino, as well as other TAMU friends, to whose work I will in fact be referring.

There have been many changes in relativity since our old days; for one thing, the size of expert audiences has greatly increased as that subject moved towards center stage. For another,
“ADM”, originally regarded as a disreputable intrusion of quantum field theoretical ideas into classical gravity has ended up as an acronym whose meaning is barely remembered (in either camp), a sign of true acceptance! Since those days, I (unlike Dick) have strayed from the real world, and find myself currently in D=11, (supergravity, to boot), about which I will be speaking here. Supersymmetry itself has of course motivated much of Dick’s work from its earliest days; he just persists in believing we live in D=4.

II. D=11 Supergravity: Uniqueness

Supersymmetry, both as a global invariance but especially in its local, supergravity, context, is perhaps the most powerful and ubiquitous single invariance principle to have emerged in the past twenty years; it seems to underlie a wide variety of seemingly different phenomena, including, most recently, the dualities that have unified hitherto separate superstring models into a single M-theory. Sometimes, the very “threat” of supersymmetry is sufficient to reestablish deep results such as positivity of gravitational energy. While the mathematical tools that physicists use in supersymmetry are neither new nor complicated – basically Grassmann variables and Clifford algebras – yet there is clearly a lot left to understand in the unreasonable success of supersymmetry in physics. I believe that we still do not fully grasp at an intuitive level why the existence of a “Dirac square root” brings so many amazing “coincidences”, cancellations of everything from ghosts to infinities, uncanny dualities and even a preferred spacetime dimensionality. This is not the place to go through the vast literature of any one of the subsections of supersymmetry. Instead, I will confine myself here to some novel aspects of what is in some ways the quintessential super-system, D=11 supergravity (“Sugra”). I remind you of some basic history: Sugra, first discovered in D=4, followed by “degenerate” versions in D=2 (superstring), D=1 (superparticle), and D=3 (supermembrane), rapidly made its way up the dimensional ladder [1], ending with the “ultimate” rung of D=11 [2].
The reasons for this ceiling were in fact mathematically prosaic ones having to do with properties of Clifford algebras \[3, 1\] in signature \((D–1,1)\), but also reflected physical requirements that no massless fields with spin greater than 2, and no more than one graviton, be permitted. The former is due to the incompatibility of gravitational interactions of higher spin gauge fields \[4\] which has long been “understood”; more than one graviton is more intuitively seen to be a bad thing, but that can also be formalized.

Unlike its lower dimensional manifestations, \(D=11\) Sugra was also seen to be our “uniquely” unique QFT, in the sense that its hallmark requirement – equality of bose and fermi modes – necessarily adjoins to the graviton a single spin 3/2 fermion together with a (singlet) 3-form gauge potential \(A_{\mu \nu \alpha}\), with neither “N>1” extensions nor matter coupling permitted. Apart from the usual proliferation of 4-fermion terms (and one nonminimal coupling term), the action is simplicity itself, schematically

\[
I = \int d^{11}x [\kappa^{-2} R + \bar{\psi}_\mu \Gamma_{\mu \nu \alpha} D_\nu \psi_\alpha + F_{\mu \nu \alpha \rho}^2 + \kappa A \wedge F \wedge F] \tag{1}
\]

where \(F\) is the four-form curl of \(A\), \(D_\nu\) is the covariant derivative, and \(\kappa^2\) is the Einstein constant with obvious dimensions \(L^9\) in \(\hbar = c = 1\) units. Amusingly, the (metric-independent) Chern–Simons term in (1) seems to have been its first physics appearance, followed by similar ones on all lower (odd) \(D\); it is (uncharacteristically) \(P\) and \(T\) even. It was not for another few years that such terms would begin to emerge in the more familiar QED\(_3\) context.

I remind you that the degree of freedom count for Einstein gravity is \(D(D–3)/2=44\), the number of transverse-traceless spatial metric components, that of the form field counts the transverse spatial components \(A_{ijk}\) (invariance is under \(\delta A_{\mu \nu \alpha} = \partial_\mu \xi_{\nu \alpha}\)), so it is \((D–2)(D–3)(D–4)/3! = 84\), while the fermionic spinor-vector has \((D–3)\) transverse and \(\gamma–\)transverse for the spatial vector index times the usual Majorana spinor count \(2^{(D/2)–1} = 128\) (the \(\frac{1}{2}\) is for first-order). The corresponding
invariances are, symbolically

\[ \delta e_{\mu a} = \bar{\alpha}(x)\gamma_{a}\psi_{\mu}(x), \quad \delta \psi_{\mu} = [D_{\mu} + (\Gamma F)_{\mu}]\alpha(x) \]

\[ \delta A_{\mu\nu\alpha} = \bar{\alpha}(x)(\Gamma \psi(x))_{\mu\nu\alpha} \]

where \( \Gamma_{\mu_{1}...\mu_{n}} \) is a suitable \( n \)-index “gamma” matrix, \( e_{\mu a} \) the vielbein and \( \alpha(x) \) a Grassmanian parameter.

Uniqueness up to now has meant that, given the Einstein action as the geometrical term of the system, the rest of (1) necessarily follows. For example, even replacing the 4-form \( F \) by its equivalent dual 7-form does not lead to a consistent formulation despite its seemingly respecting the degree of freedom count. Indeed, there is not even any D=11 globally SUSY matter (highest spin <2) system, hence no sources of (1). As I said, this is in sharp contrast with all lower dimensional cases, including the “nearest neighbor” D=10. What I want to talk about here is recent work \[5\] on the remaining possible non-uniqueness, involving the replacement of the local Lorentz group underlying the Einstein gravity of (1) by the (anti) de Sitter one, through the introduction of a cosmological term \( \Lambda \int d^{11}x \sqrt{-g} \) for gravity – Einstein’s “biggest” but unavoidable, “mistake”. [As we know all too well, such a term arises always in ordinary QFT coupled to gravity, with a horribly wrong natural magnitude; originally, there was hope from the fact that this zero point energy is absent in supersymmetric QFT’s, but only if supersymmetry is unbroken.] In any case, the rapid construction of supergravities with cosmological terms (of anti-de Sitter sign) \[\square\] undercut this hope; indeed such models were possible for all dimensions in addition to the original D=4, including D=10. However, the apparent exception was D=11: on the one hand, arguments based on Clifford algebra seemed to forbid any simple such extension, but the more general “no-go” question remained open for a long time. Once D=11 supergravity reclaimed its rightful place, as the QFT limit of M-theory, and was no longer the enigma in a world of D=10 superstrings,
it became more important to settle it. I will now briefly sketch the two ways we used to do so, and refer to [5] for details. Our negative upshot means that D=11 Sugra is, beyond all its other amazing properties, the only QFT we know that forbids the presence of a cosmological term, and does so because of supersymmetry. Breaking the latter by simply including this term is forbidden by consistency considerations, unless of course one admits truly massive $\psi_\mu$ and $A$ fields.

Cosmological Sugra, when it does exist (for $D<11$), is based on a simultaneous extension of gravity and gravitino actions, the former with the usual $\int \Lambda \sqrt{-g}$ term, the latter with a “mass” term where $m \sim \sqrt{-\Lambda}$ is what requires adS sign for $\Lambda$. The reason this simultaneous deformation (at least) is needed is that since small gravitational excitations $h_{\mu\nu}$ about the vacuum (here adS) have the same excitation count as for $\Lambda=0$, so must the fermions. The quadratic graviton action is still gauge invariant under $\delta h_{\mu\nu} = \hat{D}_\mu \xi_\nu + \hat{D}_\nu \xi_\mu$ where $\hat{D}_\mu$ is the covariant derivative with respect to the background. This is precisely what is made possible by the mass term: gravitino excitations about adS also maintain their usual flat space gauge invariance, but with $\delta \psi_\mu = (\hat{D}_\mu + m \gamma_\mu) \epsilon(x) \equiv \mathcal{D}_\mu \epsilon(x)$, because (only) these extended covariant derivative commute, $[\mathcal{D}_\mu, \mathcal{D}_\nu] = 0$ if the mass is also “tuned” to adS as noted above.

1. The Noether way. One of the most useful, if seemingly pedestrian, tools we have had in building up nonabelian gauge theories from abelian ones – and also seeing when that is not possible – is the Noether procedure. Here one tries to gauge the simple linear theory by self-coupling its conserved current (if any!) in a possible infinite series of steps to reach a consistent nonlinear one. This is how one can get from Maxwell to Yang–Mills or from spin 2 fields to Einstein gravity or from spin 2 plus spin 3/2 to supergravity [5], but not, for example from spin 2 plus 5/2 to any consistent interacting model. To be sure, the starting point must be commensurate with the desired end: one cannot reach general relativity with a cosmological constant from the free theory in background...
flat space, but only if one starts with the free theory in a de Sitter (or adS) background. Here then, the challenge is to start with the assembly of free bose and fermi constituents in the AdS context, look for a Noether current associated with their global supersymmetry and attempt to bootstrap to the desired local invariance. That is, we try to mimic the way the known correct local Lorentz supergravity can be reached from its corresponding non-interacting components.

We already know how to start the linearized gravity and gravitino systems off as free gauge systems in the background that keep the correct bose-fermi equality; what about the form field? Because it is a form, it only depends on curls which of course do not change in nonflat geometries, so its count is already safe. Indeed, the only possible deformation here would also be a mass term $\sim m^2 A^2$, but (unlike its gravitino counterpart) that would break the invariance and therefore unacceptably raise the form field’s excitations from 84 to 120. So we have the desired starting point, three linearized systems so defined that their excitation content is correct also in the background. Can we define an initial “global SUSY” transformation for them? This is a priori possible, because there is a “constant”, Killing, spinor $\alpha(x)$ such that $D_\mu \alpha = 0$, consistent with $[D_\mu, D_\nu] = 0$. I emphasize that this “constant” $\alpha$ is not the same as the nonconstant $\epsilon(x)$ ($D_\mu \epsilon \neq 0$) under which the pure fermionic system is invariant by itself! So there is a candidate tranformation, but it is not an invariance because of the $F$-field. What happens is that the $m\bar{\psi}\psi$ term we had to introduce to maintain the “internal” gauge invariance of the gravitino action necessarily varies, though the global supersymmetry parameter $\alpha$ into a term that cannot, already on dimensional grounds, be cancelled by varying $F^2$, nor can we usefully alter its natural $\delta A \sim \bar{\alpha} \Gamma \psi$ variation. So the form field is the obstruction to so much as even an initial Noether current and there is no “zeroth” step.

2. Cohomology. This approach is complementary to the first; it is better suited to a different starting-point, the full $\Lambda = 0$ Sugra of (1). Suppose we immediately accept the full $\Lambda = 0$ action
as the starting-point of the desired extension, and look for a consistent deformation of this full nonlinear model with its “nonabelian” gauge invariance, that will include a cosmological term. More precisely, since this term is necessarily associated to a mass term $\sim m \bar{\psi} \Gamma \mu \nu \psi$, $m \sim \sqrt{-\Lambda}$ for the fermion as explained earlier, we begin the deformation process with terms linear in $m$, the cosmological one acting as a second order deformation. The beauty of the cohomological description is that we need not separately adjust the action and the transformation rules. If the deformation process is at all possible, it will reveal itself at one go. Here it is again the form field that blocks the process and forbids any extension of the desired type. For all lower $D$, consistent deformations exist. [This obstruction is also true with a dual 7-form description.] However, here if we adjoin to the original system $S_0$ (including ghost completions) a $\Delta S_1 \sim m \int \bar{\psi} \psi$, we find that we cannot even maintain the first order consistency $[S_0, \Delta S_1] = 0$ let alone use $\Delta S_2 \sim \Lambda \int \sqrt{-g}$ to cancel $[\Delta S_1, \Delta S_1]$ with $[S_0, \Delta S_2]$. Thus both approaches tell us independently that there is no extension of D=11 supergravity that contracts back to it.

It should be emphasized that, like all no-go theorems, ours is predicated on some assumptions that we believe to be reasonable; in particular, that a) the $m \to 0$ limit must be smooth (as for D<11), and b) no new dynamics beyond our three initial fields enters. When supergravity is broken or we compactify down to lower dimensions, $\Lambda$ can of course reappear!

III. D=11 Supergravity: On-shell Invariants

My second topic is the construction of on-shell invariants in D=11, and is still a work in progress [7]. The motivation is twofold: First, to determine the possible local counterterms that can be constructed, i.e., at what loop order does the theory begin to (or at least is able to) pay the price of depending on the (dimensional) Einstein coupling constant? Second, and potentially more important is to thereby discover what corrections to this limiting corner of M-theory should
be sought, much like determining corrections predicted by string theories to their zero-slope QFT limits. Unfortunately this is hard work because no formalism exists at D=11 to generate such invariants and a more arduous road, using physical arguments is needed. I shall only sketch our approach in the following.

There is one guaranteed way to generate an invariant in any theory: Consider the tree level amplitude (so no regularization worries appear) for some specific number of particle scattering, say the 4-point functions. All external legs are real, so we are “on-shell” for the invariants that express this amplitude. Thus at lowest order, at least, global supersymmetry is preserved by the effective action that expresses these amplitudes. This is of course a statement that the program must succeed, but not yet a concrete result. What enables us to proceed, apart from an awful lot of calculation, is the ability to cast the primitive scattering graphs, such as graviton-graviton or form-graviton, into expressions that are written entirely in terms of Riemann, or better, Weyl tensors for the gravity part. Here previous experience \[8\] in D=4 tells us that the Bel–Robinson (BR) tensor \[9\] will play an essential role, which helps. Another expected ingredient is the famous expression \[10\] of (D=10) string theory zero slope corrections to D=11 supergravity involving terms like \(t_8 R_1 \ldots R_4\), where \(t_8\) is an 8-index quantity made out of Kronecker deltas and the \(R\)'s represent Riemanns or Weyls whose indices they contract. The link between all those scalars can be obtained by means of another TAMU work \[11\], the exhaustive enumeration of quartic curvature invariants.

So the flow chart is more or less as follows: take all 3- and 4-point vertices in (1). Sticking to the bosonic sector, all we need are the \(\kappa h^3\) and \(\kappa^2 h^4\) gravity terms \((\kappa h_{\mu \nu} \equiv g_{\mu \nu} - \eta_{\mu \nu}\) and I omit showing derivatives), the 3-point \(\kappa h_{\mu \nu} T^{\mu \nu}(F, g = \eta)\) and 4-point \(\kappa^2 h h \frac{\delta T}{\delta g}\bigg|_{g=\eta}\) mixed vertices and finally the Chern–Simons 3-point \(\epsilon A F F\) interaction itself. Now draw all possible tree diagrams using
all these vertices, contracting the intermediate graviton or form propagator to a point. [Technically, this is all done in a systematic way in terms of the Mandelstam variables \(s, t, u\).] The contact vertices (\(hhFF\) and \(hhhh\)) are just there to keep gauge invariance (Ward identities) honest. So basically we have \((hh)(hh)\) factorization of the 4-graviton amplitude, say, into two graviton “stress tensors”. While the latter cannot be quite well-defined (you heard it first from ADM [12]) the contact terms save the day – as we know in the end they must\(^1\). So we will be able, using the \((s, t, u)\) derivatives that appear in the amplitude, to provide the correct \(R^4\) four-graviton effective action, presented as the sum of squares of BR-like currents. [But note that in this D>4 context, there is more than one of those!] At D=4 there is only one BR and the action reduces precisely to the famous maximally helicity-violating combination \((E_4^2 + P_4^2)\) where \((E_4, P_4)\) are the Euler and Pontryagin densities [14]. It also agrees with the \(t_8t_8 R^4\) term there, taking into account that to this (lowest) quartic order in \(h\), the D=8 Euler density \(E_8\) is a total divergence in any dimension, not just D=8. The matter (4-form) contributions can also be uniquely obtained for both the \(F^2R^2\) and \(F^4\) amplitudes (there is also a “bremsstrahlung” \(F^3R\) possible contribution representing graviton emission from some leg of the CS vertex). Strictly speaking, one should check global supersymmetry of the resulting expression, but that is of course guaranteed by our construction, and would only serve as a check on our arithmetic of the various coefficients. The reason it is hard to do explicitly is that it first requires knowledge of all amplitudes involving two gravitinos and two of our bosons, a possible but unattractive calculation. The internal checks on the pure \(R^4\) terms as well as the BR structures are really sufficient.

What is all this good for? There are two – equally important – applications:

First: is D=11 supergravity perhaps a miraculously finite theory? It can’t be just renormaliz-

\(^1\)This is a sort of realization of a notorious problem in MTW [3] relating the Bel-Robinson (BR) tensor to the graviton stress tensor’s double derivative.
able with its dimensional coupling constant $\kappa$, so it is either nonrenormalizable as lower dimensional Sugras are or all its candidate counterterms must vanish for some unknown reason. Now our construction shows that already at lowest possible (two-) loop order there is such an invariant. Whether its coefficient vanishes upon explicit calculation is of course a separate question, but apart from the expected absence of random cancellations (just as for 2-loop pure gravity in $D=4$ [15]) there is a remarkable new development. In a very recent paper, Bern, Dixon, and collaborators [16] have reduced supergravity loop calculations to super Yang–Mills ones in a very powerful way. Extrapolating their work beyond the $D=10$ barrier to super-matter systems then strongly suggests that this term does appear as a counterterm and hence dashes any hope that $D=11$ Sugra is different from the corresponding lower $D$ ones, all of which seem to go bad [16]. Let me explain parenthetically why 2 loops: In our expansion in powers of $\kappa^2$, tree level being $\sim \kappa^{-2}$, one-loop term would be $\sim \kappa^0$. But there is no possible local counterterm $\Delta I_1 = \int d^{11}x \Delta L_1$, since this would require an odd number of derivatives [17] (all this is of course in terms of dimensional regularization). The only one-loop candidate of dimension 11 is the Chern–Simons one, $\sim \epsilon \Gamma RRRRR$ which is parity-odd. At 2 loops we have $\kappa^2 d^{11}x$, so $\Delta L_2$ must have dimension 20, e.g., $\Delta L_2 \sim R^{10}$ or fewer $R$’s and more derivatives, like $R^4 \Box^6$ where $\Box^6$ is symbolic for derivatives acting on the curvatures. [In $D=4$, the 3-loop term $\sim \kappa^4 \int d^4x R^4$ was the lowest possible one [8], there being no 1- or 2-loop invariants available.]

The second application is in a way still more interesting, because it should find direct application in testing corrections of M-theory to its $D=11$ Sugra limit, somewhat like the zero slope corrections of string theory we mentioned earlier gave $R^4$ additions to $D=10$ Sugra (but not of course $D=11$ directly!) That is, whatever the right M-theory may be, it should not only reduce to this field theory, but produce additional effects necessarily starting with the above invariant this
Apart from the intrinsic value of these applications, I should add that learning to deal even with the purely gravitational sector has also taught us a number of hidden properties of (tree-level) general relativity, such as how its diffeomorphism invariance translates into the gauge-invariant structure of physical scattering amplitudes. This is just the sort of question that ADM were in fact aiming for (those curvatures are just glorified \( h^{TT}'s \)) long before supergravity came on the scene!

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