Cross product in N Dimensions - the doublewedge product

Carlo Andrea Gonano and Riccardo Enrico Zich
Politecnico di Milano, Energy Department, via La Masa 34, 20156 Milan, MI, Italy
(Dated: August 26, 2014)

The cross product \( \times \) frequently occurs in Physics and Engineering, since it has large applications in many contexts, e.g. for calculating angular momenta, torques, rotations, volumes etc. Though this mathematical operator is widely used, it is commonly expressed in a 3-D notation which gives rise to many paradoxes and difficulties. In fact, instead of other vector operators like scalar product, the cross product is defined just in 3-D space, it does not respect reflection rules and invokes the concept of “handedness”. In this paper we are going to present an extension of cross product in an arbitrary number N of spatial Dimensions, different from the one adopted in the Exterior Algebra and explicitly designed for an easy calculus of moments.

PACS numbers: 45.20.-d, 45.10.Na, 02.40.Yy, 45.20.da
Keywords: cross product, pseudovector, N Dimensions, dimensional, moment, N-D, wedge product, doublewedge

INTRODUCTION
In this report we present a summary of a Master Thesis, published in Italian, concerning the extension of cross product \( \times \) in N Dimensions [1]. To indicate that new operator we use the doublewedge \( \wedge \wedge \) symbol, which resemble the Grassmann’s wedge product \( \wedge \) and a notation for cross product commonly adopted in Italy (see also [2]). Here our task is to show the main drawbacks and difficulties of 3-D cross product and to introduce a user-friendly N-D notation, suitable also for students.

Very brief historical notes
The history of cross product is strictly related to that of vector calculus [3,4]. In 1773, Lagrange calculated the volume of a tetrahedra finding cross product via analysis, but “vectors” haven’t been invented yet. In 1799, C. F. Gauss and C. Wessel represented complex numbers like arrows on a plane and in 1840 H.G. Grassmann introduced the exterior product and a wedge \( \wedge \) as its symbol. That could be considered as the first cross product definition, but for Grassmann the operation’s result is not a vector; though, it’s an area or volume with an oriented boundary (Fig.1). In his External Algebra Grassmann also performs the first N-D extension of his operator \( \wedge \), making it to act on many vectors at the same time, e.g:

\[
(a \wedge b \wedge d) \wedge c = a \wedge (b \wedge c \wedge d)
\]  

The result of this operation is generally interpreted as the signed (hyper-)volume of a N-D parallelogram whose edges are N vectors. In 1843, W. R. Hamilton invented the quaternions to describe rotations in 3-D and in 1846 he adopted the terms scalar and vector referring to real and imaginary parts of a quaternion. The vector part of a product between quaternions with null real parts is equal to cross product. In 1881-84, J.W. Gibbs wrote for his students the Elements of Vector Analysis [5], where modern vector calculus is explained and in 1901 his disciple E.B. Wilson published Vector Analysis [6], which had a large diffusion. In Gibbs’s notation the cross product is indicated with a \( \times \) and it’s considered a vector. Shortly, from the end of the XIX century there were many different contributions to the development of vector calculus, though interpretations and notations were not uniform. We can mention W.K. Clifford, O. Heaviside, G. Peano, G. Ricci-Curbastro and T. Levi-Civita just to cite some who worked on that topic. Nowadays cross \( \times \) and wedge \( \wedge \) products are well distinct operators and employed in different fields, though they share similar algebraic properties.

3-D CROSS PRODUCT DEFINITION AND USES
The cross product is an operation between two vectors \( \vec{a} \) and \( \vec{b} \) and in 3-D it is defined as \( \vec{p} = \vec{a} \times \vec{b} \) with:

\[
\vec{p}^T = [a_2b_3 - a_3b_2, a_3b_1 - a_1b_3, a_1b_2 - a_2b_1]
\]  

FIG. 1. Different interpretations of cross and wedge product
The cross product frequently appears in Physics and Engineering, since it’s used for the calculus of moments, rotation axes, volumes, etc.

\[ \vec{M} = \vec{r} \times \vec{F} \]  

\text{torque or moment of a force (3)}

\[ \vec{\phi}_{A\rightarrow B} = \varphi \frac{\vec{a} \times \vec{b}}{||\vec{a} \times \vec{b}||} \]  

A-towards-B rotation-vector (4)

\[ V = (\vec{a} \times \vec{b})^T \cdot \vec{c} \]  

volume of parallelepiped \( \vec{a}, \vec{b}, \vec{c} \) (5)

Actually it’s one of the most widespread mathematical operator in Mechanics and it’s suitable for many applications.

**LIMITS AND DIFFICULTIES FOR 3-D CROSS PRODUCT**

Though it is commonly used, the cross product presents some “oddities”, e.g., you need the concepts of clock-wise sense and right-hand. Furthermore, this operator is not always so easy to use: the most frequent mistake is to confuse the signs (+ or −?) and in practice you have to memorize long identities like:

\[ \vec{a} \times (\vec{b} \times \vec{c}) = (\vec{c}^T, \vec{a}) \vec{b} - (\vec{b}^T, \vec{a}) \vec{c} \]  

(6)

\[ (\vec{a} \times \vec{b})^T \cdot (\vec{c} \times \vec{d}) = (\vec{a}^T, \vec{c}) (\vec{b}^T, \vec{d}) - (\vec{a}^T, \vec{d}) (\vec{b}^T, \vec{c}) \]  

(7)

Re-demonstrate them every time is a long work, since it requires to explicit coordinates for each vector, permutation of indices etc., and you risk confusion with letters and signs: mistake is in ambush.

Moreover, we are going to show some more serious paradoxes concerning cross product.

**3-D Rotation-vectors**

While sum and scalar product between vectors are operations easy to be extended in N-D, the cross product is defined just in 3-D and it’s often used to express rotation-vectors. Those kind of vectors can not be summed with the tip-tail rule, unlike common (polar) vector; in fact rotations don’t sum because they don’t commute:

\[ \vec{\phi}_{A\rightarrow B} + \vec{\phi}_{B\rightarrow C} \neq \vec{\phi}_{A\rightarrow C} \]  

(8)

Usually, those originated by cross product are called axial vectors or pseudovectors.

**Alice through the looking-glass**

If we place a set of “true” vectors, like radii, velocities, forces etc., in front of the mirror they are simply reflected, instead of moments and pseudovector in general. In fact cross product doesn’t respect reflection rules and the specular image of a right hand is a left one and counterclock-wise looks clock-wise.

**Flatland - a 2-D world**

In Flatland\[^{[7]}\] E. A. Abbott describes life and customs of people in a 2-D world: in this universe vectors can be summed together and projected, areas are calculated, rotations are clock-wise or counterclock-wise, reflection is possible... but cross product does not exist; otherwise, 2-D inhabitants should have great fantasy to imagine a 3\(^{rd}\) dimension to contain a vector orthogonal to their plane.

By the way, in 2-D a single scalar number is sufficient to describe a force’s moment:

\[ M = M(r, \vec{F}) = r_1 F_2 - r_2 F_1 \]  

(9)

With such a definition, this operation respects all algebraic properties of cross product, but the result is a scalar.

**4-D space**

In a 4-D space each vector has 4 components and in order to construct a cross product \( \vec{p}' = \vec{a} \times \vec{b} \) we have to impose that \( \vec{p}' = [p_1, p_2, p_3, p_4] \) is perpendicular to vectors \( \vec{a} \) and \( \vec{b} \) and that its magnitude is equal to the
area between them:
\[ \vec{p}^T \cdot \vec{a} = 0; \quad \vec{p}^T \cdot \vec{b} = 0; \quad \| \vec{p} \| = \| \vec{a} \times \vec{b} \| \quad (10) \]
But these are just 3 equations, and we have 4 unknowns: the problem has 1 degree of indetermination. In fact, in 4-D there is an infinity of vectors \( \vec{p} \) that satisfy these requirements: rotation axes are not unique!

So, cross product maybe exists just in 3-D, or it’s not a vector.

**N-D CROSS PRODUCT**

As we have seen, in 3-D cross product can give some troubles. Now we desire to extend it in N spatial Dimensions and want it to satisfy some conditions:

- **Moment calculus**: the new operation should involve just 2 vectors at time.
  In fact, differently from the exterior product \( \land \), it must be of practical utility in Physics for calculating moments rather than volumes or determinants.

- **Analogy**: the algebraic properties of the new operator should be analogous to those of the classic 3-D cross product.

- **N-D validity**: the new operation must be valid in every positive integer number N of spatial Dimensions.

- **User-friendly**: the N-D notation should be general and of easy use, allowing simpler counts.

Moreover, we would like to solve some of the paradoxes previously mentioned, re-interpreting the concept of cross product itself.

**Definition of N-D cross product**

We notice that in Mechanics the angular velocity is sometime written like a pseudo-vector \( \vec{\omega} \), other times like a matrix \( \vec{\Omega} \), and the latter can be constructed also in N-D. For example, for two points \( P \) and \( Q \) on a rigid body we can write the velocities \( \vec{v} \) as:

\[
\vec{v}_P - \vec{v}_Q = \vec{\omega} \times (\vec{x}_P - \vec{x}_Q) \quad \text{3-D notation} \quad (11)
\]
\[
\vec{v}_P - \vec{v}_Q = \vec{\Omega} \cdot (\vec{x}_P - \vec{x}_Q) \quad \text{N-D notation} \quad (12)
\]
In 3-D it’s possible to pass from one notation to the other using the Levi-Civita \( \varepsilon_{ijk} \) anti-symmetric 3-tensor:

\[
\omega_i = -\frac{1}{2} \sum_{j=1}^{3} \sum_{k=1}^{3} (\varepsilon_{ijk} \Omega_{jk}) \quad (13)
\]
\[
\Omega_{ij} = -\sum_{k=1}^{3} (\varepsilon_{ijk} \omega_k) \quad (14)
\]
However, using a tensor of rank 3 could be heavy for somebody, so we can write more simply:

\[
\vec{\Omega} = [\vec{\omega} \times] = \begin{bmatrix}
0 & -\omega_3 & \omega_2 \\
\omega_3 & 0 & -\omega_1 \\
-\omega_2 & \omega_1 & 0
\end{bmatrix} \quad (15)
\]
Is it possible a similar reasoning with moments?

Let’s observe the z-component of a moment \( M_z = r_x F_y - r_y F_x \): we notice that subscript \( z \) doesn’t appear neither in the force nor in the radius, so \( \vec{M} \), rather than “around z axis”, looks to be “from \( x \) to \( y \)”. If we assemble the moment in a matrix form, we get:

\[
\vec{M} = [\vec{M} \times] = \begin{bmatrix}
0 & -M_3 & M_2 \\
M_3 & 0 & -M_1 \\
-M_2 & M_1 & 0
\end{bmatrix} \quad (16)
\]
\[
\vec{M} = \begin{bmatrix}
0 & r_2 F_1 - r_1 F_2 & r_3 F_1 - r_1 F_3 \\
r_1 F_2 - r_2 F_1 & 0 & r_3 F_2 - r_2 F_3 \\
r_1 F_3 - r_3 F_1 & r_3 F_2 - r_2 F_3 & 0
\end{bmatrix} \quad (17)
\]
It’s straightforward to demonstrate that:

\[
M_{ij} = F_i r_j - F_j r_i \quad (18)
\]
Since vectors \( \vec{F} \) and \( \vec{r} \) can have any dimension N, we define the **N-D cross product** as the difference of dyads:

\[
\vec{M} = \vec{r} \hat{\times} \vec{F} = [\vec{F} \hat{T}] - [\vec{r} F^T] \quad (19)
\]
It can be easily verified that the new operator respects all the required algebraic properties; just the result is no more a vector but an anti-symmetric matrix or 2-tensor. For full theory, see [1].

**\LaTeX** command for the doublewedge symbol**

In order to distinguish the N-D cross product from the 3-D \( \times \) and the wedge \( \land \) ones, we introduced the new symbol \( \hat{\times} \), called “**doublewedge**”. In order to write the doublewedge in \LaTeX, you can create (or copy-paste) a macro in the document preamble:

\[
\text{\newcommand{\doublewedge}{\overset{\wedge}{\land}} \text{\scriptsize{$\wedge$}};}}\}
\]
Then, to display the symbol, just write \text{\doublewedge}.

**Algebraic properties**

The N-D cross product or doublewedge product has many algebraic properties in common with the 3-D one, as previously required.

- **anti-commutativity**:

\[
\vec{a} \hat{\times} \vec{b} = -\vec{b} \hat{\times} \vec{a} \quad (20)
\]
• distributivity over addition:
\[ \vec{a} \wedge (\vec{b} + \vec{c}) = \vec{a} \wedge \vec{b} + \vec{a} \wedge \vec{c} \] (21)

• compatibility with scalar multiplication:
\[ (\alpha \vec{a}) \wedge (\beta \vec{b}) = \alpha \beta [\vec{a} \wedge \vec{b}] \quad \forall \alpha, \beta \in \mathbb{C} \] (22)

Differently from the cross and wedge products, the \( \wedge \) operation cannot be repeated over itself, since its inputs are vectors and the output is a matrix.

**Main algebraic identities**

In table I we report the main mathematical identities involving cross product with both 3-D and N-D notations.

**APPLICATIONS AND CONSEQUENCES**

In this section we bring some sparse examples regarding the application of \( \wedge \) product in different contexts. For details see [1].

**Perpendicular component of a vector**

The perpendicular component \( \vec{F}_\perp \) of a vector \( \vec{F} \) on any other \( \vec{r} \) can be calculated as:
\[ \vec{F}_\perp = \vec{F} - \frac{1}{r^2} (\vec{r} \cdot \vec{F}) \vec{r} \quad \Rightarrow \quad \vec{r} \cdot \vec{F}_\perp = 0 \] (23)

The same equation can be re-written as:
\[ \vec{F}_\perp = \frac{1}{r^2} \left( [\vec{F} r^2] - [\vec{r} \vec{F} r^2] \right) \cdot \vec{r} = \frac{1}{r^2} [\vec{r} \wedge \vec{F}] \cdot \vec{r} \] (24)

This result is a particular case of the identity:
\[ [\vec{r} \wedge \vec{F}] \cdot \vec{c} = ([\vec{F} r^2] - [\vec{r} \vec{F} r^2]) \cdot \vec{c} \] (25)
\[ [\vec{r} \wedge \vec{F}] \cdot \vec{c} = \vec{F} (\vec{r} \cdot \vec{c}) - \vec{r} (\vec{F} \cdot \vec{c}) \] (26)

Let’s notice that we derived it in 2 rows. The 3-D equivalent identity is:
\[ (\vec{r} \times \vec{F}) \times \vec{c} = \vec{F} (\vec{r} \cdot \vec{c}) - \vec{r} (\vec{F} \cdot \vec{c}) \] (27)

but to demonstrate it with 3-D formalism it’s a longer task (try to believe).

**Angular momenta and inertia matrices**

Given a body defined on a lagrangian domain \( \Omega_x \), its angular momentum \( \vec{L}_0 \) with respect to a pole \( \vec{x}_0 \) is:
\[ \vec{L}_0 = \int_{\Omega_x} (\vec{x} - \vec{x}_0) \wedge (\rho \vec{v}) \, d\Omega \] (28)

where \( \rho \) and \( \vec{v} \) are the mass density and velocity respectively. In 3-D, for a rigid body holds:
\[ \vec{L}_0 = m (\vec{x}_G - \vec{x}_0) \times \vec{v}_0 + \vec{T}_0 \vec{\omega} \] (29)

where \( \vec{x}_G \) is the center of gravity and \( \vec{T}_0 \) is the 3-D inertia matrix, defined as:
\[ \vec{T}_0 = \int_{\Omega_x} \rho \begin{pmatrix} y^2 + z^2 & -xy & -xz \\ -xy & z^2 + x^2 & -yz \\ -xz & -yz & x^2 + y^2 \end{pmatrix} \, d\Omega \] (30)

where \([x; y; z] = \vec{x} - \vec{x}_0 \). Let’s notice that in (30) indices are misleading, in fact:
\[ \mathcal{I}_{xx} = \int_{\Omega_x} \rho (y^2 + z^2) \, d\Omega_x \neq \int_{\Omega_x} \rho x^2 \, d\Omega_x \] (31)

With N-D notation, instead, the inertia matrix \( \vec{T}_0 \) is compactly defined as:
\[ \vec{T}_0 \triangleq \int_{\Omega_x} \rho [\Delta x_0 \Delta x_0^T] \, d\Omega \quad \text{with:} \quad \Delta \vec{x}_0 = \vec{x} - \vec{x}_0 \] (32)

Let’s notice that the N-D inertia matrix \( \vec{T}_0 \) is conceptually similar to the matrix of covariances \( \sigma_{ij} \) used in Statistics.

With N-D notation the Eq.(29) will look:
\[ \vec{L}_0 = m (\vec{x}_G - \vec{x}_0) \wedge \vec{v}_0 + \vec{T}_0 \vec{\Omega} - \left( \vec{T}_0 \vec{\Omega} \right)^T \] (33)

**Volume calculus: the 3-indices product**

In 3-D, the signed volume \( V \) of a parallelepiped whose edges are vectors \( \vec{a}, \vec{b}, \vec{c} \) can be calculated as:
\[ V = (\vec{a} \times \vec{b}) \cdot \vec{c} = (\vec{b} \times \vec{c}) \cdot \vec{a} = (\vec{c} \times \vec{a}) \cdot \vec{b} \] (34)
\[ V = \begin{vmatrix} b_3 a_2 - a_3 b_2 c_1 + (b_1 a_3 - a_1 b_3) c_2 \\ + (b_2 a_1 - a_2 b_1) c_3 \end{vmatrix} \] (35)

In N-D for a hyper-parallelepiped with edges \( \vec{v}_1, \vec{v}_2, \ldots, \vec{v}_N \) the signed hyper-volume is:
\[ V = \det |\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_N| \] (36)

However, if we want to determine a 3-D volume in an N-D space it’s convenient to define the 3-indices product:
\[ \overline{A}_{ijk} \cdot \vec{c} = A_{ij} c_k + A_{jk} c_i + A_{ki} c_j \] (37)
\[ \overline{A}_{ijk} \cdot \vec{c} = \overline{A}_{jki} \cdot \vec{c} = \overline{A}_{kij} \cdot \vec{c} \] (38)
TABLE I. Main mathematical identities for cross product

| 3-D notation | N-D notation |
|--------------|--------------|
| \( \vec{M} = \vec{r} \times \vec{F} \) | \( \vec{M} = \vec{r} \wedge \vec{F} = [\vec{r} \vec{F}^T] - [\vec{r} \vec{F}^T] \) |
| \( M_{iz} = F_i r_y - r_x F_y \) | \( M_{ij} = F_i r_j - r_i F_j \) |
| \( (\vec{r} \times \vec{F}) \times \vec{c} = \vec{F} (\vec{r}^T \cdot \vec{c}) - \vec{r} (\vec{F}^T \cdot \vec{c}) \) | \( [\vec{r} \wedge \vec{F}] \cdot \vec{c} = \vec{F} (\vec{r}^T \cdot \vec{c}) - \vec{r} (\vec{F}^T \cdot \vec{c}) \) |
| \( \vec{M} \times \vec{c} = [\vec{M} \times] \vec{c} \) | \( \vec{M} \cdot \vec{c} \) |
| \( V = (\vec{a} \times \vec{b}) \cdot \vec{c} \) | \( V = [\hat{a} \wedge \hat{b}] \cdot \vec{c} \) |
| \( (\vec{a} \times \vec{c}) \cdot (\vec{b} \times \vec{c}) \cdot \vec{a} = (\vec{a} \times \vec{b}) \cdot \vec{b} \) | \( [\hat{a} \wedge \hat{b}] \cdot [\hat{b} \wedge \hat{c}] \cdot [\hat{c} \wedge \hat{a}] \cdot \vec{d} = (\vec{a} \wedge \vec{b}) \wedge (\vec{b} \wedge \vec{c}) \cdot \vec{d} \) |
| \( (\vec{L} \vec{a} \times (\vec{L} \vec{b})) = \det(\vec{L}) \left( L^{-T} \cdot (\vec{a} \times \vec{b}) \right) \) | \( (\vec{L} \vec{a} \wedge (\vec{L} \vec{b}) = \vec{L} \left[ \hat{a} \wedge \hat{b} \right] \vec{L}^T \) |
| \( (\vec{a} \times \vec{b}) \cdot (\vec{c} \times \vec{d}) = (\vec{a} \cdot \vec{c})(\vec{b} \cdot \vec{d}) - (\vec{a} \cdot \vec{d})(\vec{b} \cdot \vec{c}) \) | \( \frac{1}{2} [\hat{a} \wedge \hat{b}] : [\hat{c} \wedge \hat{d}] = (\vec{a} \cdot \vec{c})(\vec{b} \cdot \vec{d}) - (\vec{a} \cdot \vec{d})(\vec{b} \cdot \vec{c}) \) |

where \( i,j,k \in \{1,2,\ldots,N\} \) are arbitrary indices. It’s quite straightforward to verify that in 3-D holds:

\[
V = (\vec{a} \times \vec{b}) \cdot \vec{c} = [\hat{a} \wedge \hat{b}] \cdot \vec{c} \quad (39)
\]

Anyway we remember that the \( \wedge \) operator was conceived for the calculus of moments rather than volumes.

**Power calculus: the matrix contraction**

In Mechanics the power \( P \) transferred to a rotating body is the scalar product of its angular velocity \( \vec{\omega} \) and the applied torque \( \vec{M} \)

\[
P = \vec{M} \cdot \vec{\omega} \quad (40)
\]

Since both \( \vec{\omega} \) and \( \vec{M} \) are pseudovectors, with the N-D formalism the power will be calculated by the contraction of matrices \( \vec{M} \) and \( \vec{\omega} \)

\[
P = \frac{1}{2} \vec{M} : \vec{\omega} = \frac{1}{2} \sum_{i=1}^{N} \sum_{j=1}^{N} M_{ij} \Omega_{ij} \quad (41)
\]

The basic idea is quite similar to the tensor contraction adopted in Relativity.

**3-D and N-D curl**

Curl is the differential operator analogous to cross product, and in 3-D it suffers for the same problems, since it generates pseudovectors.

\[
\vec{\nabla} \times \vec{v} = \left[ \frac{\partial v_3}{\partial x_2} - \frac{\partial v_2}{\partial x_3} ; \frac{\partial v_1}{\partial x_3} - \frac{\partial v_3}{\partial x_1} ; \frac{\partial v_2}{\partial x_1} - \frac{\partial v_1}{\partial x_2} \right] \quad (42)
\]

The extension in N-D is instantaneous:

\[
\vec{\nabla} \wedge \vec{v} = \left[ \frac{\partial \vec{v}}{\partial \vec{x}} \right]^T - \left[ \frac{\partial \vec{v}}{\partial \vec{x}} \right] \quad (43)
\]

\[
[\vec{\nabla} \wedge \vec{v}]_{ij} = \frac{\partial v_i}{\partial x_j} - \frac{\partial v_j}{\partial x_i} = v_{i/j} - v_{j/i} \quad (44)
\]

Even in this case it can be verified that the new operator satisfies all the required differential properties.

**The magnetic field \( \vec{B} \) is not a vector**

The magnetic field \( \vec{B} \) is often involved with cross product and curl: is it a “true” vector? Look at Faraday’s law and Lorentz force equations in 3-D:

\[
\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \quad \vec{F}_B = -Q_e \vec{B} \times \vec{v} \quad (45)
\]

We know, from definition, that \( \vec{E}, \vec{F}_B \) and \( \vec{v} \) are true vectors and, using N-D notation, \( (45) \) will look:

\[
\vec{\nabla} \wedge \vec{E} = -\frac{\partial \vec{B}}{\partial t} \quad \vec{F}_B = -Q_e \vec{B} \wedge \vec{v} \quad (46)
\]

Thus, the magnetic field \( \vec{B} \) is a not a vector, but a pseudovector, and, in a wider N-D view, it is a matrix or 2-tensor. The use of \( \vec{B} \)-tensor is not new, but it seems not to be always understood: for further details see \[\text{ }\]

**CONCLUSION**

With the usual 3-D notation the cross product exhibits many limits and difficulties, since it produces pseudovectors. In order to simplify calculations we defined the N-D cross product and introduced the \( \wedge \) symbol, solving some paradoxes and showing that moments are actually better described by matrices rather than by vectors. In this paper we reported just a summary of a more complete work \[\text{ }\] which also includes the N-D curl extension. We underline that the use of 2-tensors instead of pseudovectors \[\text{ }\] is not a completely new idea, but it seems not to be so widespread or understood, even in Relativity and Quantum Mechanics.

The N-D notation for cross product was explicitly conceived to help students with counts and we are confident
that it will be a practical tool also in classic Mechanics and Geometry.

ACKNOWLEDGMENTS

We thanks Prof. Antonella Abbá, Prof. Sonia Leva, Riccardo Albi, Giorgio Fumagalli, Andrea Gatti, Pietro Giuri and Alessandro Nicolai for their careful reviews and Prof. Marco Mussetta for his precious help and support. We also would like to signal some authors who have independently come to conclusions analogous to ours in different ways: [10] [11].

[1] C. A. Gonano, *Estensione in N-D di prodotto vettore e rotore e loro applicazioni*, Master’s thesis, Politecnico di Milano (2011).
[2] T. Levi-Civita and U. Amaldi, *Lezioni di meccanica razionale*, Vol. I (Zanichelli editore Bologna, 1949).
[3] M. J. Crowe, *A History of Vector Analysis: The Evolution of the Idea of a Vectorial System* (University of Notre Dame press, 1967).
[4] M. J. Crowe, “A History of Vector Analysis,” (2002), talk at University of Louisville.
[5] J. W. Gibbs, “Elements of Vector Analysis - Arranged for the Use of Students of Physics,” (1881-1884), note for students, privately printed.
[6] E. B. Wilson, *Vector analysis - A text-book for the use of students of mathematics and physics* (Yale Bicentennial publication, 1901).
[7] E. A. Abbott, *Flatland: A Romance of Many Dimensions* (Seely & Co., 1884).
[8] C. A. Gonano and R. E. Zich, “Magnetic monopoles and Maxwell’s equations in N Dimensions,” in *Electromagnetics in Advanced Applications (ICEAA), 2013 International Conference on* (2013) pp. 1544-1547.
[9] E.g. for magnetic field $\mathbf{B}$ and angular momentum $\mathbf{L}$.
[10] A. McDavid and C. McMullen, “Generalizing Cross Products and Maxwell’s Equations to Universal Extra Dimensions,” [arXiv:hep-ph/0609260 [hep-ph]].
[11] P. Guio, “Levi-Civita symbol and cross product vector/tensor,” (2011), original note developed for a course on Physics of Astrophysics.
[12] A. Gray, “Vector Cross Products on Manifolds,” Transactions of the American Mathematical Society 141, pp. 465-504 (1969).
[13] M. Hage-Hassan, “Inertia tensor and cross product In n-dimensions space,” (2006), [arXiv:math-ph/0604051 [math-ph]].
[14] M. Manarini, “Estensione della formula del doppio prodotto vettoriale agli spazi a più di tre dimensioni. Una formula di calcolo integrale ed un teorema della divergenza per i bivettori,” Rend. Semin. Mat. Univ. Padova **10**, 1-20 (1939).
[15] A. P. Morando and S. Leva, *Note di teoria dei Campi Vettoriali* (Esulapeco, Bologna, 1998).
[16] A. Palatini, “Concetto di vettore generalizzato prodotto interno, prodotto esterno, divergenza e rotore. Teoremi generali della divergenza, del rotore e di Stokes,” Rend. Semin. Mat. Univ. Padova 4, 122-139 (1933).
[17] Z. K. Silagadze, “Feynman’s derivation of Maxwell equations and extra dimensions,” Ann. Fond. Louis de Broglie **27**, 241-255 (2002), Special issue on contemporary electrodynamics.
[18] Z. K. Silagadze, “Multi-dimensional vector product,” J. Phys. A: Math. Gen. **35**, 4949-4953 (2002).