WELL-POSEDNESS AND OPTIMAL TIME-DECAY FOR COMPRESSIBLE MHD SYSTEM IN BESOV SPACE

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Abstract. In this paper, firstly, we prove the global well-posedness of three dimensional compressible magnetohydrodynamics equations for some classes of large initial data, which may have large oscillation for the density and large energy for the velocity and magnetic field. Secondly, we prove the optimal time decay for the compressible magnetohydrodynamics equations with low regularity assumptions about the initial data. Especially, we can obtain the optimal $L^2$ time decay rate when the initial data small in the critical Besov space (no small condition in space $H^{N/2+1}$). When we calculate the optimal time decay rate, we use differential type energy estimates in homogeneous Besov space, evolution in negative Besov space and the well-posedness results proved in the first part.

1. Introduction

Magnetic fields influence many fluids. Magnetohydrodynamics (MHD) is concerned with the interaction between fluid flow and magnetic field. The governing equations of isentropic compressible MHD system has the form

\[
\begin{cases}
\partial_t \rho + \text{div}(\rho u) = 0, \\
\partial_t (\rho u) + \text{div}(\rho u \otimes u) + \nabla P(\rho) = B \cdot \nabla B - \frac{1}{2} \nabla (|B|^2) + \mu \Delta u + (\lambda + \mu) \nabla \text{div} u, \\
\partial_t B + (\text{div} u) B + u \cdot \nabla B - B \cdot \nabla u = \nu \Delta B, \\
\text{div} B = 0, \\
(\rho, u, B)|_{t=0} = (\rho_0, u_0, B_0), \quad \text{div} B_0 = 0,
\end{cases}
\]

(1.1)

where $\rho = \rho(t, x)$ denotes the density, $x \in \mathbb{R}^N$, $t > 0$, $u \in \mathbb{R}^N$ is the velocity of the flow, and $B \in \mathbb{R}^N$ stands for the magnetic field. The constants $\mu$ and $\lambda$ are the shear and bulk viscosity coefficients of the flow, respectively, satisfying $\mu > 0$ and $\lambda + 2 \mu > 0$, the constant $\nu > 0$ is the magnetic diffusivity acting as a magnetic diffusion coefficient of the magnetic field. $P(\rho)$ is the scalar pressure function satisfying $P'(\bar{\rho}) > 0$.

Due to the physical importance and mathematical challenges, MHD system has been studied by many authors. There are a lot of work about weak solutions for compressible MHD equations. In a series of papers [7, 8, 9], X. Hu and D. Wang construct the global weak solutions with large initial data and they also studied the low mach number limit problem. A. Suen and D. Hoff [14], firstly, construct a global weak solutions when the initial data has small energy which is an important
progress. Then many authors in [17, 18] construct the global weak solutions when the initial data has small energy and they also allow the initial data contain vacuum. Besides the studies about weak solutions mentioned above, there are also a lot of investigations about classical and strong solutions. In 1984, S. Kawashima studied a very general symmetric hyperbolic-parabolic system which incorporate MHD system as a special one in his PhD thesis. He construct the global solution for MHD system when the initial data to be small in $H^4(\mathbb{R}^3)$ and he also get the time decay rate as follows

$$\| (\rho - \bar{\rho}, u, B - \bar{B}) \|_{L^2(\mathbb{R}^N)} \leq C (1 + t)^{N/(1/2p - 1/4)} \| (\rho_0 - \bar{\rho}, u_0, B_0 - \bar{B}) \|_{L^2 \cap L^p},$$

where $1 \leq p \leq 2N/(N + 1)$. After S. Kawashima’s investigation, recently, many authors [20, 21, 22, 23] studied the optimal time decay rate for the MHD system with the magnetic equilibrium state to be zero. Their studies get the optimal time decay rate for the derivatives of the solution which generalize the optimal time decay rate results for Navier-Stokes equations to the more complex MHD equations.

In 2000, R. Danchin did a seminal work in [25]. By scaling analysis R. Danchin give the definitions about critical space for compressible Navier-Stokes equations, then under this new framework he construct a global solution when the initial data to be small in critical Besov space. After this important work, there are a lot of further studies [4, 5, 6, 26]. In [27, 28], D. Bian, B. Yuan and B. Guo, firstly, construct the local solution for compressible MHD equations in critical Besov space framework. In [19], C. Hao analyze the linearized hyperbolic-parabolic system related to MHD equations carefully, then construct the global solution in the critical Besov space framework. Recently, there are some important work for incompressible MHD equations [10, 11, 12]. In these excellent papers, F. Lin, L. Xu and P. Zhang construct global solutions for incompressible MHD equations without magnetic diffusion by cutely using the dissipative structure of the system and the anisotropic Besov space technique. Then, X. Hu in paper [13] construct the global solutions for compressible MHD equations without magnetic diffusion. Due to the structure is very different for incompressible and compressible MHD equations, X. Hu’s work is very different to F. Lin, L. Xu and P. Zhang’s work and X. Hu used lots of structure information about the compressible MHD equations.

Through the above review, we find a problem that is all works in homogeneous Besov space framework just construct the global solution, however, there are few studies about the optimal time decay rate of the solution. Can we get optimal time decay rate under the low regularity assumptions about the initial data is an interesting question. For the Navier-Stokes equations, M. Okita [29] did some work about optimal time decay rate in Besov space framework, however, his results hold in the inhomogeneous Besov space framework and used complicated analysis about Green’s matrix. For the MHD system with non-trivial magnetic equilibrium state, we will meet the term $\sqrt{(\xi_1 + \xi_2 + \xi_3)^2 - C|\xi|^4}$ when we analyze the Green’s matrix, so we can hardly split the frequency to low part and high part. In this paper, we use differential type energy estimates in homogeneous Besov space and estimates in negative Besov space to get the optimal time decay rate about MHD system with non-trivial magnetic equilibrium state when the initial data to be small in space $B^{N/2-1,N/2} \times B^{N/2-1,N/2} \times B^{N/2-1,N/2}$. Our methods are completely different to M. Okita’s methods used in [29]. By a refined global well-posedness results and combing our decay argument, we obtain the optimal time decay rate.
when the initial data to be small in homogeneous Besov space which is more natural than inhomogeneous space.

The other parts of this paper are organized as follows. In section 2, we give a brief introduction about Besov space and the three main results of this paper. In Section 3, we prove the local well-posedness in critical Besov space framework by using Lagrange coordinate methods and prove that the local solution can propagate the smoothness of the initial data. In Section 4, we use Hoff’s energy methods to get a uniform estimates about the solution. In Section 5, we prove a blow up criterion and then combine the results from Section 3 to complete the proof of Theorem 2.6. In Section 6, we give the proof of a differential type inequality for a hyperbolic parabolic system. In Section 7, we get the estimates for the solution in the negative Besov space. In Section 8, combining the results from Section 6, we give the proof about Theorem 2.9 and Theorem 2.11. At last, for the reader’s convenience, we collect a lot of useful Lemmas in the Appendix.

2. Main Results and Some Preliminaries

In this section, we will introduce some notations and give the three main results of this paper. Firstly, let me give some basic knowledge about Besov space, which can be found in [1]. The homogeneous Littlewood-Paley decomposition relies upon a dyadic partition of unity. We can use for instance any \( \phi \in C^\infty(\mathbb{R}^N) \), supported in \( \mathcal{C} := \{ \xi \in \mathbb{R}^N, \, 3/4 \leq |\xi| \leq 8/3 \} \) such that

\[
\sum_{q \in \mathbb{Z}} \phi(2^{-q} \xi) = 1 \quad \text{if} \quad \xi \neq 0.
\]

Denote \( h = \mathcal{F}^{-1} \phi \), we then define the dyadic blocks as follows

\[
\Delta_q u := \phi(2^{-q}D)u = 2^{qN} \int_{\mathbb{R}^N} h(2^qy)u(x - y) \, dy, \quad \text{and} \quad S_q u = \sum_{k \leq q-1} \Delta_k u.
\]

The formal decomposition

\[
u = \sum_{q \in \mathbb{Z}} \Delta_q u
\]

is called homogeneous Littlewood-Paley decomposition. The above dyadic decomposition has nice properties of quasi-orthogonality: with our choice of \( \phi \), we have

\[
\Delta_k \Delta_q u = 0 \quad \text{if} \quad |k - q| \geq 2,
\]

\[
\Delta_k (S_{q-1} \Delta_q u) = 0 \quad \text{if} \quad |k - q| \geq 5.
\]

Let us now introduce the homogeneous Besov space.

**Definition 2.1.** We denote by \( S^s_h \) the space of tempered distributions \( u \) such that

\[
\lim_{j \to -\infty} S_j u = 0 \quad \text{in} \quad \mathcal{S}'.
\]

**Definition 2.2.** Let \( s \) be a real number and \( (p, r) \) be in \([1, \infty]^2\). The homogeneous Besov space \( B^s_{p,r} \) consists of distributions \( u \) in \( S^s_h \) such that

\[
\|u\|_{B^s_{p,r}} := \left( \sum_{j \in \mathbb{Z}} 2^{rjs} \|\Delta_j u\|_{L^p}^r \right)^{1/r} < +\infty.
\]
From now on, the notation $\dot{B}^s_p$ will stand for $\dot{B}^s_{p,1}$ and the notation $\dot{B}^s$ will stand for $\dot{B}^s_{2,1}$.

The study of non stationary PDE’s usually requires spaces of type $L^r_T(X) := L^r(0,T;X)$ for appropriate Banach spaces $X$. In our case, we expect $X$ to be a Besov spaces, so that it is natural to localize the equations through Littlewood-Paley decomposition. We then get estimates for each dyadic block and perform integration in time. However, in doing so, we obtain bounds in spaces which are not of type $L^r(T;B^s_p)$. This approach was initiated in [2] and naturally leads to the following definitions:

**Definition 2.3.** Let $(r, p) \in [1, +\infty]^2$, $T \in (0, +\infty]$ and $s \in \mathbb{R}$. We set

$$
\|u\|_{L^r_T(B^s_p)} := \sum_{q \in \mathbb{Z}} 2^{qs} \left( \int_0^T \|\Delta_q u(t)\|_{L^r_p}^r \, dt \right)^{1/r}
$$

and

$$
\dot{L}^r_T(\dot{B}^s_p) := \left\{ u \in L^r_T(\dot{B}^s_p), \|u\|_{L^r_T(\dot{B}^s_p)} < +\infty \right\}.
$$

Owing to Minkowski inequality, we have $\dot{L}^r_T(\dot{B}^s_p) \supseteq L^r_T(\dot{B}^s_p)$. That embedding is strict in general if $r > 1$. We will denote by $\dot{C}_T(\dot{B}^s_p)$ the set of function $u$ belonging to $\dot{L}^r_T(\dot{B}^s_p) \cap C([0,T];\dot{B}^s_p)$.

We will often use the following interpolation inequality:

$$
\|u\|_{L^r_T(\dot{B}^s_p)} \leq \|u\|_{\dot{L}^r_T(\dot{B}^s_p)}^{\theta} \|u\|_{L^r_T(\dot{B}^s_p)}^{1-\theta},
$$

with

$$
\frac{1}{r} = \frac{\theta}{r_1} + \frac{1-\theta}{r_2}
$$

and $s = \theta s_1 + (1-\theta)s_2$,

and the following embeddings

$$
\dot{L}^r_T(\dot{B}^{s/p}_p) \hookrightarrow L^r_T(C_0) \quad \text{and} \quad \dot{C}_T(\dot{B}^{s/p}_p) \hookrightarrow C([0,T] \times \mathbb{R}^N).
$$

Another important space is the hybrid Besov space, we give the definitions and collect some properties.

**Definition 2.4.** Let $s, t \in \mathbb{R}$. We set

$$
\|u\|_{B^{s,t}_{p,q}} := \sum_{k \leq R_0} 2^{ks} \|\Delta_k u\|_{L^r_p} + \sum_{k > R_0} 2^{kt} \|\Delta_k u\|_{L^r_p}
$$

and

$$
B^{s,t}_{p,q}(\mathbb{R}^N) := \left\{ u \in \mathcal{S}'(\mathbb{R}^N), \|u\|_{B^{s,t}_{p,q}} < +\infty \right\},
$$

where $R_0$ is a fixed constant.

**Lemma 2.5.** 1) We have $B^{s,s}_{2,2} = \dot{B}^s$.

2) If $s \leq t$ then $B^{s,t}_{p,p} = B^s_{p,p} \cap B^t_{p,p}$. Otherwise, $B^{s,t}_{p,p} = B^s_{p,p} + B^t_{p,p}$.

3) The space $B^{s,s}_{p,p}$ coincide with the usual inhomogeneous Besov space.

4) If $s_1 \leq s_2$ and $t_1 \geq t_2$ then $B^{s_1,t_1}_{p,p} \hookrightarrow B^{s_2,t_2}_{p,p}$.

5) Interpolation: For $s_1, s_2, \sigma_1, \sigma_2 \in \mathbb{R}$ and $\theta \in [0, 1]$, we have

$$
\|f\|_{B^{s_1+(1-\theta)s_2,\sigma_1+(1-\theta)\sigma_2}_{p,p}} \leq \|f\|_{B^{s_1}_{p,p}}^{\theta} \|f\|_{B^{s_2}_{p,p}}^{1-\theta}.
$$
From now on, the notation $B_{p,s}^{2,t}$ will stand for $B_{p,p}^{2,t}$ and the notation $B_{2,2}^{s,t}$ will stand for $B_{2,2}^{2,t}$.

Throughout the paper, we shall use some paradifferential calculus. It is a nice way to define a generalized product between distributions which is continuous in functional spaces where the usual product does not make sense. The paraproduct between $u$ and $v$ is defined by

$$T_{uv} := \sum_{q \in \mathbb{Z}} S_{q-1} u \Delta_q v.$$  

We thus have the following formal decomposition:

$$uv = T_{uv} + T_{vu} + R(u, v),$$  

with

$$R(u, v) := \sum_{q \in \mathbb{Z}} \Delta_q u \tilde{\Delta}_q v, \quad \tilde{\Delta}_q := \Delta_{q-1} + \Delta_q + \Delta_{q+1}.$$  

We will sometimes use the notation $T'_{uv} := T_{uv} + R(u, v)$.

For more information about Besov space and hybrid Besov space, we give reference [1, 2, 3, 4, 5]. Throughout this paper, $C$ stands for a “harmless” constant, and we sometimes use the notation $A \lesssim B$ as an equivalent of $A \leq CB$. The notation $A \approx B$ means that $A \lesssim B$ and $B \lesssim A$.

With these preparations, we can state our main results.

**Theorem 2.6.** Suppose the equilibrium state $\bar{\rho} > 0$, $\bar{B} = 0$ and dimension $N = 3$. Let $c_0$ be a small constant, $P_+ = \sup_{c_0/4 \leq \rho \leq 4c_0^{-1}} |P^{(k)}(\rho)|$ and $P_- = \inf_{c_0/4 \leq \rho \leq 4c_0^{-1}} |P'(\rho)|$. Assume that the initial data $(\rho_0, u_0)$ satisfies

$$\rho_0 - \bar{\rho} \in H^s \cap \dot{B}^{3/p}_{p,1}, \quad c_0 \leq \rho_0 \leq c_0^{-1},$$

$$u_0, B_0 \in H^{s-1} \cap \dot{B}^{3/p-1}_{p,1},$$

for some $p \in (3, 6)$ and $s \geq 3$. There exist five constants $c_1, c_2, c_3$ such that

$$\|\rho_0 - \bar{\rho}\|_{L^2} \leq c_1, \quad \|u_0\|_{\dot{B}^{3/p-1}_{p,1}} \leq \frac{c_2}{(1 + \|a_0\|_{\dot{B}^{3/p}_{p,1}})^{\delta}}, \quad \|B_0\|_{\dot{B}^{3/p-1}_{p,1}} \leq \frac{c_3}{(1 + \|a_0\|_{\dot{B}^{3/p}_{p,1}})^{\delta}},$$

$$\|u_0\|_{H^{-s}} \leq \frac{c_2}{(1 + \|a_0\|_{\dot{B}^{3/p}_{p,1}})^{\delta}(1 + \|\rho_0 - \bar{\rho}\|_{H^2}^2 + \|u_0\|_{L^2}^{4/3} + \|B_0\|_{L^2}^{4/3})},$$

$$\|B_0\|_{H^{-s}} \leq \frac{c_3}{(1 + \|a_0\|_{\dot{B}^{3/p}_{p,1}})^{\delta}(1 + \|\rho_0 - \bar{\rho}\|_{H^2}^2 + \|u_0\|_{L^2}^{4/3} + \|B_0\|_{L^2}^{4/3})},$$

for some $\delta \in \left(1 - \frac{2}{p}, \frac{4}{p}\right)$, then there exist a unique global solution $(\rho, u, B)$ satisfying

$$\rho \geq \frac{c_0}{4}, \quad \rho - \bar{\rho} \in C([0, \infty); H^s),$$

$$u, B \in C([0, \infty); H^{s-1}) \cap L^2(0, T; H^s),$$

for any $T > 0$.

The idea of the proof of the above theorem comes from the paper [6] which used the advantage of harmonic analysis and the structure variable called effective viscous flux. Our MHD case are more complex for the magnetic field and velocity field coupled with each other. We use the Lagrange coordinate technique to get
Remark 2.7. Isentropic Navier-Stokes system is well-posedness when the initial velocity field have large oscillation which is proved in [4, 5]. Until now, there are no such results for compressible MHD systems for the coupling between velocity and magnetic field make MHD system are more complex than Navier-Stokes equations. The above theorem allows the initial velocity field and magnetic field have large oscillation which is proved in [4, 5]. Until now, the there are no such results for compressible MHD systems for the coupling between velocity and magnetic field make MHD system are more complex than Navier-Stokes equations.

Remark 2.8. Theorem 2.6 is also important for us to get our next theorem for the optimal time decay rate. Using a differential type inequality for hyperbolic parabolic composite type linear system and estimating the equation in negative Besov space, we can prove Theorem 2.11. Combining the above well-posedness results and the techniques used in the proof of Theorem 2.11 we can get optimal time decay rate under homogeneous Besov space framework which is an important improvement for the study about optimal time decay rate.

**Theorem 2.9.** Suppose all the conditions in Theorem 2.6 are satisfied with \( p = 2 \). In addition, assume that
\[
\|\rho_0 - \overline{\rho}\|_{B^{1/2,3/2}} + \|u_0\|_{B^{1/2}} + \|B_0\|_{B^{1/2}} \leq \alpha_0,
\]
where \( \alpha_0 > 0 \) is a small positive number. If further, \((\rho_0 - \overline{\rho}, u_0, B_0) \in \dot{B}^{-s}_{2,\infty}\) for some \( s \in [0, 3/2] \), then for all \( t \geq 0 \), we have
\[
\|\rho(t) - \overline{\rho}\|_{\dot{B}^{-s}_{2,\infty}} + \|u(t)\|_{\dot{B}^{-s}_{2,\infty}} + \|B(t)\|_{\dot{B}^{-s}_{2,\infty}} \leq C,
\]
and
\[
\|\rho(t) - \overline{\rho}\|_{B^l} + \|u(t)\|_{B^l} + \|B(t)\|_{B^l} \leq C(1 + t)^{-\frac{1}{2+s}},
\]
for \(-s < l \leq 1/2\), where \( C > 0 \) is a positive constant depending on the initial data.

**Remark 2.10.** Taking \( l = 0 \) and \( s = 3/2 \), by embedding theorems, we will get
\[
\|\rho(t) - \overline{\rho}\|_{L^2} + \|u(t)\|_{L^2} + \|B(t)\|_{L^2} \leq C(1 + t)^{-\frac{3}{4}},
\]
Hence, Theorem 2.9 get the optimal time decay rate when the initial data to be small in almost critical Besov space and bounded in some homogeneous Sobolev space with negative regularity index. The results are highly reduce the requirements comparing to the previous works [20, 21, 22].

**Theorem 2.11.** Suppose the equilibrium state \( \overline{\rho} > 0 \) and \( \overline{B} = I \). Assume that \((\rho_0 - \overline{\rho}, u_0, B_0 - \overline{B}) \in B^{N/2-1,N/2+1} \times B^{N/2-1,N/2} \times B^{N/2-1,N/2} \) for an integer 
\( N \geq 3 \). Then there exists a constant \( \alpha_0 > 0 \) such that if
\[
\|\rho_0 - \overline{\rho}\|_{B^{N/2-1,N/2+1}} + \|u_0\|_{B^{N/2-1,N/2}} + \|B_0 - \overline{B}\|_{B^{N/2-1,N/2}} \leq \alpha_0,
\]
then the MHD system admits a unique global solution \((\rho, u, B)\) satisfying that for all \( t \geq 0 \)
\[
\|\rho - \overline{\rho}\|_{C(R^+;B^{N/2-1,N/2+1})} + \|(u, B)\|_{C(R^+;B^{N/2-1,N/2})}
+ \|\rho - \overline{\rho}\|_{L^1(R^+;B^{N/2+1,N/2+3})} + \|(u, B)\|_{L^1(R^+;B^{N/2+1,N/2+2})} \leq C\alpha_0.
\]
If further, \((\rho_0 - \bar{\rho}, u_0, B_0 - \bar{B}) \in \dot{B}^{-s}_{2,\infty}\) for some \(s \in [0, N/2]\) then for all \(t \geq 0\)
\[
\|\rho(t) - \bar{\rho}\|_{\dot{B}^{-s}_{2,\infty}} + \|u(t)\|_{\dot{B}^{-s}_{2,\infty}} + \|B(t)\|_{\dot{B}^{-s}_{2,\infty}} \leq C,
\]
and
\[
\|\rho(t) - \bar{\rho}\|_{\dot{B}^l} + \|u(t)\|_{\dot{B}^l} + \|B(t)\|_{\dot{B}^l} \leq C(1 + t)^{-\min\left(\frac{1}{2}, \frac{N/2 - 1}{s}\right)},
\]
for \(-s < l \leq N/2\), where \(C > 0\) is a positive constant depending on the initial data.

Remark 2.12. Theorem 2.11 give the optimal time decay rate when the equilibrium state of magnetic field is not zero. If we try to analyze the Green’s matrix for the linear system as Navier-Stokes system, we will encounter the term like \(\sqrt{(\xi_1 + \xi_2 + \xi_3)^2 - C|\xi|^4}\) in the eigenvalues of the Green’s matrix. This term make us hard to analyze the low and high frequency separately. In order to get Theorem 2.11, we use energy estimates under Besov space framework and combing the estimates in negative Besov space to avoid the analysis about the Green’s matrix. This pure energy method seems more flexible and simple than the method for analysis about Green’s matrix.

Remark 2.13. Recently, there is a paper [29] get the optimal time decay rate for Navier-Stokes system under inhomogeneous Besov space framework. We used a completely different method to obtain optimal time decay rate. Comparing to [29], our results need not the initial data to be small in \(B^{l}_1\) and we used homogeneous Besov space framework which seems more in accordance with R. Danchin’s seminal work [24]. Due to the reasons stated in Remark 2.12 the methods used in [29] seems hard to generalize to the MHD system with non-zero equilibrium state for magnetic field to get a result like Theorem 2.11.

Remark 2.14. The smallness conditions in Theorem 2.11 seems more restricted than Theorem 2.9 however, Theorem 2.9 can not instead of Theorem 2.11. The results in Theorem 2.11 tells us more things than Theorem 2.9. In Theorem 2.11 we also get a weak time decay rate when \(N = 3\) as follows
\[
\|\rho(t) - \bar{\rho}\|_{L^\infty} + \|u(t)\|_{L^\infty} + \|B(t)\|_{L^\infty} \leq C(1 + t)^{-1},
\]
which is slower than the optimal time decay rate \(-3/2\). How to get optimal time decay rate under low regularity assumptions is our future aim.

3. LOCAL EXISTENCE AND PROPAGATION OF REGULARITY

In this section, we will prove the local well-posedness and prove that the obtained solution can propagate the smoothness of the initial data. Without loss of generality, we can assume \(\bar{\rho} = 1\) from this section to the last section.

Let \(X_u\) be the flow associated to the vector field \(u\), that is the solution to
\[
X_u(t, y) = y + \int_0^t u(\tau, X_u(\tau, y)) d\tau.
\]
Denoting
\[
\hat{\rho}(t, y) = \rho(t, X_u(t, y)), \quad \hat{u}(t, y) = u(t, X_u(t, y)), \quad \hat{B}(t, y) = B(t, X_u(t, y)).
\]
Then we reformulate the MHD system in the Lagrangian coordinate to obtain

$$
\begin{aligned}
\partial_t (J_u \rho) &= 0, \\
J_u \rho_0 \partial_t \tilde{u} - \text{div} (\text{adj}(DX_u) (2 \mu D A_u \tilde{u} + \lambda \text{div} A_u \tilde{u} \text{Id}) + P(\rho) \text{Id})) \\
&= \text{div} \left( \text{adj}(DX_u) \tilde{B} \tilde{B}^T \right) - \frac{1}{\rho_0} \text{div} \left( \text{adj}(DX_u) |\tilde{B}|^2 \right), \\
J_u \partial_t \tilde{B} - \nu \text{div} \left( \text{adj}(DX_u) A_u^T \nabla \tilde{B} \right) \\
&= \text{div} \left( \text{adj}(DX_u) \tilde{B} \tilde{u}^T \right) - \text{div} \left( \text{adj}(DX_u) \tilde{u} \tilde{B} \right), \\
\text{div}(A_u \tilde{B}) &= 0,
\end{aligned}
$$

where $A_u = (D_u X_u)^{-1}$. During the transformation between Euler coordinates and Lagrangian coordinate, we used (3.1).

Next, we introduce $u_L$ the “free solution” to

$$
\begin{aligned}
\partial_t u_L - \mu \Delta u_L - (\lambda + \mu) \nabla \text{div} u_L &= 0, \\
u L |_{t=0} = u_0,
\end{aligned}
$$

and $B_L$ the “free solution” to

$$
\begin{aligned}
\partial_t B_L - \nu \Delta B_L &= 0, \\
\text{div} B_L &= 0, \\
B_L |_{t=0} &= B_0.
\end{aligned}
$$

Denoting

$$
L_{\rho_0}(\tilde{u}) = \partial_t \tilde{u} - \frac{1}{\rho_0} \text{div} (2 \mu D(\tilde{u}) + \lambda \text{div} \tilde{u} \text{Id}),
$$

and

$$
L_\nu(\tilde{B}) = \partial_t \tilde{B} - \nu \Delta \tilde{B}.
$$

We can rewrite the second and third equations in system (3.2) as follows

$$
\begin{aligned}
L_{\rho_0}(\tilde{u}) &= I_1(\tilde{u}, \tilde{u}) + \rho_0^{-1} \text{div}(I_2(\tilde{u}, \tilde{u}) + I_3(\tilde{u}, \tilde{u}) + I_4(\tilde{u})) + \rho_0^{-1} \text{div}(I_5(\tilde{u}, \tilde{B}) + I_6(\tilde{u}, \tilde{B})), \\
L_\nu(\tilde{B}) &= \nu L_\nu(\tilde{B}) + \text{div}(I_8(\tilde{u}, \tilde{B}) + I_9(\tilde{u}, \tilde{B}) + I_{10}(\tilde{u}, \tilde{B}, \tilde{B}) + I_{11}(\tilde{u}, \tilde{B}, \tilde{u}))
\end{aligned}
$$

where

$$
\begin{aligned}
I_1(v, w) &= (1 - J_v) \partial_t w, \\
I_2(v, w) &= (\text{adj}(DX_v) - \text{Id})(\mu Dw \cdot A_v + \mu A_v^T \nabla w + \lambda A_v^T : \nabla w), \\
I_3(v, w) &= \mu Dw \cdot (A_v - \text{Id}) + \mu (A_v^T - \text{Id}) \nabla w + \lambda (A_v^T - \text{Id}) : \nabla w, \\
I_4(v) &= \text{adj}(DX_v) P(J_v^{-1} \rho_0) \text{Id}, \\
I_5(v, w) &= \text{adj}(DX_v) ww^T, \\
I_6(v, w) &= \frac{1}{2} \text{adj}(DX_v) |w|^2, \\
I_7(v, w) &= (1 - J_v) \partial_t w, \\
I_8(v, w) &= \nu (\text{adj}(DX_v) - \text{Id}) A_v^T \nabla w, \\
I_9(v, w) &= \nu (A_v^T - \text{Id}) \nabla w, \\
I_{10}(v, w, k) &= \text{adj}(\nabla X_v) wk^T, \\
I_{11}(v, w, k) &= \text{adj}(DX_v) wk.
\end{aligned}
$$

In order to prove the local well-posedness, we define a map $\Phi : (v, b) \mapsto (\tilde{u}, \tilde{B})$ where $(\tilde{u}, \tilde{B})$ satisfies (3.7) with the $\tilde{u}$, $\tilde{B}$ on the right hand side changed to $v$, $b$. 


separately. Denoting \( \hat{u} = \tilde{u} - u_L \) and \( \hat{B} = \tilde{B} - B_L \), then we know that \( \hat{u} \) and \( \hat{B} \) satisfies

\[
L_{\rho_0}(\hat{u}) = (L_1 - L_{\rho_0})u_L + I_1(v, v) \\
+ \rho_0^{-1} \text{div}(I_2(v, v) + I_3(v, v) + I_4(v, b) + I_5(v, b) + I_6(v, b)), \\
L_{\nu}(\hat{B}) = I_7(v, b) + \text{div}(I_8(v, b) + I_9(v, b) + I_{10}(v, b, v) + I_{11}(v, b, v)), \\
\text{div}(\hat{B}) = \text{div}(Id - A_L)b, \\
(\hat{u}, \hat{B})|_{t=0} = (0, 0).
\]

(3.8)

In order to give a clear statement, we need to define

\[
E_p(T) = \left\{ v \in \tilde{L}(\{0, T\}; \tilde{B}^{N/p-1}) : \partial_t v, \nabla^2 v \in \tilde{L}^1(0, T; \tilde{B}^{N/p-1}) \right\}, \\
\tilde{B}_{E_p(T)}(u_L, B_L; R) = \left\{ v, b : \|v - u_L\|_{E_p(T)} + \|b - B_L\|_{E_p(T)} \leq R \right\}.
\]

Suppose that \( T, R \leq 1 \), and \( T, R \) small enough such that

\[
\int_0^T \|\nabla v\|_{B^{N/p}} dt \leq c < 1.
\]

(3.9)

**Step 1 : Stability of \( \tilde{B}_{E_p(T)}(u_L, B_L; R) \) for small enough \( R \) and \( T \).** Using Lemma [9.1] we could get

\[
\|\tilde{u}\|_{E_p(T)} \leq Ce^{C_{\rho_0,m} T}\left(\|I_1(v, v)\|_{L^1_p(\tilde{B}^{N/p-1})} + \|(L_1 - L_{\rho_0})u_L\|_{L^1_p(\tilde{B}^{N/p-1})} + \|I_2(v, v)\|_{L^1_p(\tilde{B}^{N/p})} + \|I_3(v, v)\|_{L^1_p(\tilde{B}^{N/p})} + \|I_4(v, b)\|_{L^1_p(\tilde{B}^{N/p})} + \|I_6(v, b)\|_{L^1_p(\tilde{B}^{N/p})}\right).
\]

Using Lemma [9.2] we could get

\[
\|\tilde{B}\|_{E_p(T)} \leq C\left(\|I_7(v, b)\|_{L^1_p(\tilde{B}^{N/p-1})} + \|I_8(v, b)\|_{L^1_p(\tilde{B}^{N/p})} + \|I_9(v, b)\|_{L^1_p(\tilde{B}^{N/p})} + \|I_{10}(v, b, v)\|_{L^1_p(\tilde{B}^{N/p})} + \|I_{11}(v, b, v)\|_{L^1_p(\tilde{B}^{N/p})}\right).
\]

There are many terms can be estimated similar to Navier-Stokes system as follows

\[
\|I_1(v, w)\|_{L^1_p(\tilde{B}^{N/p-1})} \leq C\|v\|_{L^1_p(\tilde{B}^{N/p})}\|\partial_t w\|_{L^1_p(\tilde{B}^{N/p})}, \\
\|(L_1 - L_{\rho_0})u_L\|_{L^1_p(\tilde{B}^{N/p-1})} \leq C\|u_L\|_{L^1_p(\tilde{B}^{N/p})}, \\
\|I_i(v, w)\|_{L^1_p(\tilde{B}^{N/p})} \leq C\|v\|_{L^1_p(\tilde{B}^{N/p})}\|w\|_{L^1_p(\tilde{B}^{N/p})}, \quad (i = 2, 3), \\
\|I_4(v)\|_{L^1_p(\tilde{B}^{N/p})} \leq CT\left(1 + \|\tilde{u}\|_{E_p(T)}(1 + \|v\|_{L^1_p(\tilde{B}^{N/p})})\right).
\]

Here, we only give the estimates for the terms different to Navier-Stokes systems.

\[
\|I_5(v, w)\|_{L^1_p(\tilde{B}^{N/p})} \leq C\left(\|\text{adj}(DX_c) - Id\|_{L^1_p(\tilde{B}^{N/p})} + 1\right)\|w\|_{L^1_p(\tilde{B}^{N/p})}^2 \\
\leq C\left(1 + \|v\|_{L^1_p(\tilde{B}^{N/p})}\right)\|w\|_{L^1_p(\tilde{B}^{N/p})}^2,
\]

\[
\|I_6(v, w)\|_{L^1_p(\tilde{B}^{N/p})} \leq C\left(1 + \|v\|_{L^1_p(\tilde{B}^{N/p})}\right)\|w\|_{L^1_p(\tilde{B}^{N/p})}^2,
\]

\[
\|I_7(v, w)\|_{L^1_p(\tilde{B}^{N/p})} \leq C\left(1 - \|v\|_{L^1_p(\tilde{B}^{N/p})}\right)\|\partial_t w\|_{L^1_p(\tilde{B}^{N/p})}^2,
\]

\[
\leq C\|v\|_{L^1_p(\tilde{B}^{N/p})}\|\partial_t w\|_{L^1_p(\tilde{B}^{N/p})}^2.
\]
We could reduce the above two inequalities further to get
\[
\|I_8(v, w)\|_{L^1_p(B_{p+1}^{N(p)})} \leq C\|\text{adj}(DX_v) - Id\|_{L^\infty_p(B_{p+1}^{N(p)})}^T \|\hat{A}_v^T - Id\|_{L^\infty_p(B_{p+1}^{N(p)})} \|w\|_{L^1_p(B_{p+1}^{N(p+1)})} \\
+ C\|\text{adj}(DX_v) - Id\|_{L^\infty_p(B_{p+1}^{N(p)})} \|w\|_{L^1_p(B_{p+1}^{N(p)})} \\
\leq C\|v\|_{L^1_p(B_{p+1}^{N(p+1)})} \|w\|_{L^1_p(B_{p+1}^{N(p+1)})} ,
\]

\[
\|I_9(v, w)\|_{L^1_p(B_{p+1}^{N(p)})} \leq C\|A_v^T - Id\|_{L^\infty_p(B_{p+1}^{N(p)})} \|w\|_{L^1_p(B_{p+1}^{N(p+1)})} \\
\leq C\|v\|_{L^1_p(B_{p+1}^{N(p+1)})} \|w\|_{L^1_p(B_{p+1}^{N(p+1)})} ,
\]

\[
\|I_{10}(v, w, k)\|_{L^1_p(B_{p+1}^{N(p)})} \leq C(1 + \|\text{adj}(\nabla X_v) - Id\|_{L^\infty_p(B_{p+1}^{N(p)})}) \|w\|_{L^1_p(B_{p+1}^{N(p)})} \|k\|_{L^1_p(B_{p+1}^{N(p)})} \\
\leq C(1 + \|v\|_{L^1_p(B_{p+1}^{N(p)})} \|k\|_{L^1_p(B_{p+1}^{N(p)})} \|w\|_{L^1_p(B_{p+1}^{N(p)})} ,
\]

\[
\|I_{11}(v, w, k)\|_{L^1_p(B_{p+1}^{N(p)})} \leq C(1 + \|v\|_{L^1_p(B_{p+1}^{N(p)})} \|k\|_{L^1_p(B_{p+1}^{N(p)})} \|w\|_{L^1_p(B_{p+1}^{N(p)})}.
\]

Combining the estimates about $I_1$ to $I_{11}$, we could finally get

\[
\|\hat{u}\|_{E_p(T)} \leq C e^{C_{\sigma_0} m T}(1 + \|a_0\|_{B_{p+1}^{N(p)}}^2) \left\{ T + \|a_0\|_{B_{p+1}^{N(p)}} \|uL\|_{L^1_p(B_{p+1}^{N(p+1)})} \\
+ \|\hat{v}\|_{L^1_p(B_{p+1}^{N(p+1)})} + \|uL\|_{L^1_p(B_{p+1}^{N(p+1)})} \|\hat{v}\|_{L^1_p(B_{p+1}^{N(p+1)})} + \|\hat{v}\|_{L^1_p(B_{p+1}^{N(p+1)})}^2 \\
+ \|\hat{v}\|_{L^1_p(B_{p+1}^{N(p+1)})} + \|uL\|_{L^1_p(B_{p+1}^{N(p+1)})}^2 \\
+ \|\hat{v}\|_{L^1_p(B_{p+1}^{N(p+1)})} + \|uL\|_{L^1_p(B_{p+1}^{N(p+1)})} \|\hat{v}\|_{L^1_p(B_{p+1}^{N(p+1)})} + \|B_{p+1}^{N(p+1)}\|_{L^1_p(B_{p+1}^{N(p+1)})} \right\},
\]

and

\[
\|\hat{B}\|_{E_p(T)} \leq C \left( \|uL\|_{L^1_p(B_{p+1}^{N(p+1)})} + \|\hat{v}\|_{L^1_p(B_{p+1}^{N(p+1)})} \|\partial_t B_{p+1}^{N(p+1)}\|_{L^1_p(B_{p+1}^{N(p+1)})} \\
+ \|\partial_t \hat{v}\|_{L^1_p(B_{p+1}^{N(p+1)})} + \|\hat{v}\|_{L^1_p(B_{p+1}^{N(p+1)})} \|\partial_t B_{p+1}^{N(p+1)}\|_{L^1_p(B_{p+1}^{N(p+1)})} \\
+ \|\hat{v}\|_{L^1_p(B_{p+1}^{N(p+1)})} + \|uL\|_{L^1_p(B_{p+1}^{N(p+1)})} \|\partial_t \hat{v}\|_{L^1_p(B_{p+1}^{N(p+1)})} + \|B_{p+1}^{N(p+1)}\|_{L^1_p(B_{p+1}^{N(p+1)})} \right) ,
\]

where $\hat{v} = v - u_L$, $\hat{b} = b - B_L$.

We could reduce the above two inequalities further to get

\[
\|\hat{u}\|_{E_p(T)} \leq C e^{C_{\sigma_0} m T}(1 + \|a_0\|_{B_{p+1}^{N(p)}}^2) \left\{ T + \|a_0\|_{B_{p+1}^{N(p)}} \|uL\|_{L^1_p(B_{p+1}^{N(p+1)})} \\
+ \|uL\|_{L^1_p(B_{p+1}^{N(p+1)})} + \|\hat{v}\|_{L^1_p(B_{p+1}^{N(p+1)})} \|\partial_t B_{p+1}^{N(p+1)}\|_{L^1_p(B_{p+1}^{N(p+1)})} \\
+ \|uL\|_{L^1_p(B_{p+1}^{N(p+1)})} + \|B_{p+1}^{N(p+1)}\|_{L^1_p(B_{p+1}^{N(p+1)})} \right\},
\]

and

\[
\|\hat{B}\|_{E_p(T)} \leq C \left( \|uL\|_{L^1_p(B_{p+1}^{N(p+1)})} + \|\hat{v}\|_{L^1_p(B_{p+1}^{N(p+1)})} \|\partial_t B_{p+1}^{N(p+1)}\|_{L^1_p(B_{p+1}^{N(p+1)})} \right) ,
\]

where $\hat{v} = v - u_L$, $\hat{b} = b - B_L$. 
Here, if we assume

\[ C_{\rho_0, m} T \leq \log 2, \quad T \leq R^2, \]

\[ \|a_0\|_{\dot{B}^{\beta}_{p,1}} \|u_L\|_{\dot{L}^1_{\beta}(\dot{B}^{\beta}_{p,1})} \leq R^2, \]

\[ \|\partial_t u_L\|_{L^1_{\beta}(\dot{B}^{\beta}_{p,1}^{-1})} + \|u_L\|_{L^1_{\beta}(\dot{B}^{\beta}_{p,1})} + \|u_L\|_{L^1_{\beta}(\dot{B}^{\beta}_{p,1})} \leq R, \]

\[ \|\partial_t B_L\|_{L^1_{\beta}(\dot{B}^{\beta}_{p,1}^{-1})} + \|B_L\|_{L^1_{\beta}(\dot{B}^{\beta}_{p,1})} \leq R, \]

\[ (1 + \|a_0\|_{\dot{B}^{\beta}_{p,1}}) R \leq \eta < \frac{1}{20C}, \]

then we know that

\[ \|\tilde{u}\|_{E_p(T)} \leq R, \quad \|\tilde{B}\|_{E_p(T)} \leq R. \]

Hence, \( \Phi \) is a self map on the ball \( \tilde{B}_{E_p(T)}(u_L, B_L; R) \).

**Step 2: Contraction estimates.** Taking \((v^1, b^1) \in \tilde{B}_{E_p(T)}(u_L, B_L; R)\) and \((v^2, b^2) \in \tilde{B}_{E_p(T)}(u_L, B_L; R)\), we define \((\tilde{u}^1, \tilde{B}^1) = \Phi(v^1, b^1)\) and \((\tilde{u}^2, \tilde{B}^2) = \Phi(v^1, b^1)\). Denote \(\delta \tilde{u} = \tilde{u}^2 - \tilde{u}^1, \delta \tilde{B} = \tilde{B}^2 - \tilde{B}^1, \delta v = v^2 - v^1\) and \(\delta b = b^2 - b^1\).

Through simple calculations, we have

\[ L_{\rho_0}(\delta \tilde{u}) = I_1(v^1, \delta v) + (J_{v^1} - J_{v^2}) \partial_t v^2 \]

\[ + \rho_0^{-1} \left\{ (I_2(v^2, v^2) - I_2(v^1, v^1)) + (I_3(v^2, v^2) - I_3(v^1, v^1)) \right. \]

\[ + (I_4(v^2) - I_4(v^1)) + (I_5(v^2, b^2) - I_5(v^1, b^1)) \]

\[ + (I_6(v^2, b^2) - I_6(v^1, b^1)) \right\}, \]

and

\[ L_{\rho}(\delta \tilde{B}) = (I_7(v^2, b^2) - I_7(v^1, b^1)) + \text{div} \left\{ (I_8(v^2, b^2) - I_8(v^1, b^1)) \right. \]

\[ + (I_9(v^2, b^2) - I_9(v^1, b^1)) + (I_{10}(v^2, b^2, v^2) - I_{10}(v^1, b^1, v^1)) \]

\[ + (I_{11}(v^2, b^2, v^2) - I_{11}(v^1, b^1, v^1)) \right\}. \]

Under the condition \(C_{\rho_0, m} T \leq \log 2\), using Lemma 9.1 and Lemma 9.2, we get

\[ \|\delta \tilde{u}\|_{E_p(T)} \leq C(1 + \|a_0\|_{\dot{B}^{\beta}_{p,1}}) \left( \|I_1(v^1, \delta v)\|_{L^1_{\beta}(\dot{B}^{\beta}_{p,1}^{-1})} \right. \]

\[ + \|I_2(v^2, v^2) - I_2(v^1, v^1)\|_{L^1_{\beta}(\dot{B}^{\beta}_{p,1}^{-1})} + \|I_3(v^2, v^2) - I_3(v^1, v^1)\|_{L^1_{\beta}(\dot{B}^{\beta}_{p,1})} \]

\[ + \|I_4(v^2) - I_4(v^1)\|_{L^1_{\beta}(\dot{B}^{\beta}_{p,1})} + \|I_5(v^2, b^2) - I_5(v^1, b^1)\|_{L^1_{\beta}(\dot{B}^{\beta}_{p,1})} \]

\[ + \|I_6(v^2, b^2) - I_6(v^1, b^1)\|_{L^1_{\beta}(\dot{B}^{\beta}_{p,1})} \right). \]
and
\[
\| \delta \hat{B} \|_{L^p(T)} \leq C \left( \| I_7(v^2, b^2) - I_7(v^1, b^1) \|_{L^p_v(B^{N/p-1}_{p,1})} \\
+ \| I_8(v^2, b^2) - I_8(v^1, b^1) \|_{L^p_v(B^{N/p}_{p,1})} \\
+ \| I_9(v^2, b^2) - I_9(v^1, b^1) \|_{L^p_v(B^{N/p}_{p,1})} \\
+ \| I_{10}(v^2, b^2, v^2) - I_{10}(v^1, b^1, v^1) \|_{L^p_v(B^{N/p}_{p,1})} \\
+ \| I_{11}(v^2, b^2, v^2) - I_{11}(v^1, b^1, v^1) \|_{L^p_v(B^{N/p}_{p,1})} \right).
\]

There are many terms appeared in the Navier-Stokes system, we only list the estimates
\[
\| I_1(v^1, \delta v) \|_{L^p_v(B^{N/p-1}_{p,1})} \leq C \| v^1 \|_{L^p_v(B^{N/p-1}_{p,1})} \| \delta \delta v \|_{L^p_v(B^{N/p-1}_{p,1})},
\]
\[
\| (J_{v^2} - J_{v^1}) \partial_t v^2 \|_{L^p_v(B^{N/p-1}_{p,1})} \leq C \| \delta v \|_{L^p_v(B^{N/p-1}_{p,1})} \| \partial_t v^2 \|_{L^p_v(B^{N/p-1}_{p,1})},
\]
\[
\| I_2(v^2, v^2) - I_2(v^1, v^1) \|_{L^p_v(B^{N/p-1}_{p,1})} \leq C \| (v^1, v^2) \|_{L^p_v(B^{N/p-1}_{p,1})} \| \delta v \|_{L^p_v(B^{N/p-1}_{p,1})},
\]
\[
\| I_3(v^2, v^2) - I_3(v^1, v^1) \|_{L^p_v(B^{N/p-1}_{p,1})} \leq C \| (v^1, v^2) \|_{L^p_v(B^{N/p-1}_{p,1})} \| \delta v \|_{L^p_v(B^{N/p-1}_{p,1})},
\]
\[
\| I_4(v^2) - I_4(v^1) \|_{L^p_v(B^{N/p-1}_{p,1})} \leq C(1 + \| a_0 \|_{B^{N/p-1}_{p,1}}) \| \delta v \|_{L^p_v(B^{N/p-1}_{p,1})}.
\]

Now, we analyze the different terms carefully. Since
\[
I_5(v^2, b^2) - I_5(v^1, b^1) = (\text{adj}(DX_{v^2}) - \text{adj}(DX_{v^1})) b^2 (b^2)^T
\]
\[
+ \text{adj}(DX_{v^2}) b^2 - b^1 (b^2)^T
\]
\[
+ \text{adj}(DX_{v^1}) b^1 (b^2 - b^1)^T,
\]
we have
\[
\| I_5(v^2, b^2) - I_5(v^1, b^1) \|_{L^p_v(B^{N/p-1}_{p,1})} \leq C \| \text{adj}(DX_{v^2}) - \text{adj}(DX_{v^1}) \|_{L^p_v(B^{N/p-1}_{p,1})} \| b^2 \|_{L^p_v(B^{N/p}_{p,1})}
\]
\[
+ C(1 + \| v^1 \|_{L^p_v(B^{N/p-1}_{p,1})}) \| \delta b \|_{L^p_v(B^{N/p}_{p,1})} \| (b^1, b^2) \|_{L^p_v(B^{N/p}_{p,1})}.
\]

\[I_6\] can be estimated same as \[I_5\]. Since
\[
I_7(v^2, b^2) - I_7(v^1, b^1) = (J_{v^2} - J_{v^1}) \partial_t b^2 + (1 - J_{v^1}) (\partial_t b^2 - \partial_t b^1),
\]
we know that
\[
\| I_7(v^2, b^2) - I_7(v^1, b^1) \|_{L^p_v(B^{N/p-1}_{p,1})} \leq C \| J_{v^2} - J_{v^1} \|_{L^p_v(B^{N/p}_{p,1})} \| \partial_t b^2 \|_{L^p_v(B^{N/p-1}_{p,1})}
\]
\[
+ C \| J_{v^2} - J_{v^1} \|_{L^p_v(B^{N/p}_{p,1})} \| \partial_t \delta b \|_{L^p_v(B^{N/p-1}_{p,1})}
\]
\[
\leq C \| \partial_t b^2 \|_{L^p_v(B^{N/p-1}_{p,1})} \| \delta v \|_{L^p_v(B^{N/p-1}_{p,1})}
\]
\[
+ C \| v^1 \|_{L^p_v(B^{N/p-1}_{p,1})} \| \partial_t \delta b \|_{L^p_v(B^{N/p-1}_{p,1})}.
\]

After some calculations, we have
\[
I_8(v^2, b^2) - I_8(v^1, b^1) = \nu(\text{adj}(DX_{v^2}) - \text{adj}(DX_{v^1})) A_{v^2}^T \nabla b^2
\]
\[
+ \nu(\text{adj}(DX_{v^2}) - \text{adj}(DX_{v^1})) (A_{v^2}^T - A_{v^1}^T) \nabla b^2
\]
\[
+ \nu(\text{adj}(DX_{v^1}) - \text{adj}(DX_{v^1})) A_{v^1}^T \nabla (b^2 - b^1),
\]
so we can get
\[
\|I_8(v^2, b^2) - I_8(v^1, b^1)\|_{L^p_t(B_{p,1}^{N/p+1})} \\
\leq C\|\delta v\|_{L^p_t(B_{p,1}^{N/p+1})}(1 + \|v^2\|^2_{L^p_t(B_{p,1}^{N/p+1})})\|\delta b\|_{L^p_t(B_{p,1}^{N/p+1})} \\
+ C\|v^1\|_{L^p_t(B_{p,1}^{N/p+1})}\|\delta v\|_{L^p_t(B_{p,1}^{N/p+1})}\|b^2\|_{L^p_t(B_{p,1}^{N/p+1})} \\
+ C\|v^1\|_{L^p_t(B_{p,1}^{N/p+1})}(1 + \|v^1\|^2_{L^p_t(B_{p,1}^{N/p+1})})\|\delta b\|_{L^p_t(B_{p,1}^{N/p+1})}.
\]

Due to
\[
I_9(v^2, b^2) - I_9(v^1, b^1) = \nu(A^T_{\alpha} - A^T_{\alpha'})\nabla b^2 + \nu(A^T_{\alpha} - Id)\nabla \delta b,
\]
we have
\[
\|I_9(v^2, b^2) - I_9(v^1, b^1)\|_{L^p_t(B_{p,1}^{N/p+1})} \leq C\|\delta v\|_{L^p_t(B_{p,1}^{N/p+1})}\|b^2\|_{L^p_t(B_{p,1}^{N/p+1})} \\
+ C\|v^1\|_{L^p_t(B_{p,1}^{N/p+1})}\|\delta b\|_{L^p_t(B_{p,1}^{N/p+1})}.
\]

Since
\[
I_{10}(v^2, b^2, v^2) - I_{10}(v^1, b^1, v^1) = (\text{adj}(\nabla X_{v^2}) - \text{adj}(\nabla X_{v^1}))b^2(v^2)^T \\
+ \text{adj}(\nabla X_{v^1})(b^2 - b^1)(v^2)^T \\
+ \text{adj}(\nabla X_{v^1})b^1(v^2 - v^1)^T,
\]
we easily have
\[
\|I_{10}(v^2, b^2, v^2) - I_{10}(v^1, b^1, v^1)\|_{L^p_t(B_{p,1}^{N/p+1})} \leq C(1 + \|v^2\|_{B_{p,1}^{N/p+1}})^2 \left\{ (T + \|v^1\|_{B_{p,1}^{N/p+1}})\|\delta v\|_{L^p_t(B_{p,1}^{N/p+1})} \\
+ \|\partial_t v^2\|_{L^2_t(B_{p,1}^{N/p+1})} + \|\partial_t v^1\|_{L^2_t(B_{p,1}^{N/p+1})}\|\delta b\|_{L^2_t(B_{p,1}^{N/p+1})} \\
+ \|v^1\|_{L^2_t(B_{p,1}^{N/p+1})}\|\partial_t \delta v\|_{L^2_t(B_{p,1}^{N/p+1})} + \|b^1\|_{L^2_t(B_{p,1}^{N/p+1})}\|\delta b\|_{L^2_t(B_{p,1}^{N/p+1})} \right\},
\]
and
\[
\|\delta \tilde{B}\|_{E_p(T)} \leq C \left\{ (\|\partial_t b^2\|_{L^2_t(B_{p,1}^{N/p+1})} + \|\partial_t b^1\|_{L^2_t(B_{p,1}^{N/p+1})})\|\delta v\|_{L^2_t(B_{p,1}^{N/p+1})} \\
+ \|v^1\|_{L^2_t(B_{p,1}^{N/p+1})}\|\partial_t \delta v\|_{L^2_t(B_{p,1}^{N/p+1})} + \|b^1\|_{L^2_t(B_{p,1}^{N/p+1})}\|\delta b\|_{L^2_t(B_{p,1}^{N/p+1})} \right\},
\]
Through some simple calculations, using conditions (3.12) with may be larger constant $C$, we have
\[\|\delta \tilde{u}\|_{E_p(T)} \leq C(1 + \|a_0\|_{\dot{B}^{N/p}_{p,1}}^2) R(\|\delta v\|_{E_p(T)} + \|\delta b\|_{E_p(T)})\]
and
\[\|\delta \tilde{B}\|_{E_p(T)} \leq CR(\|\delta v\|_{E_p(T)} + \|\delta b\|_{E_p(T)}).
\]
Hence, we get
\[\|\delta \tilde{u}, \delta \tilde{B}\|_{E_p(T)} \leq C(1 + \|a_0\|_{\dot{B}^{N/p}_{p,1}}^2) R(\|\delta v, \delta b\|_{E_p(T)})\]
Due to conditions (3.12), we finally get
\[\frac{1}{2} \|\delta v, \delta b\|_{E_p(T)} \leq \frac{1}{2} \|\delta v, \delta b\|_{E_p(T)}.
\]
Now, by contraction mapping theorem, we know that $\Phi$ admits a unique fixed point in $\dot{B}_{p,1}^{N/p}(u_L, B_L; R)$.

**Step 3:** Regularity of the density. Denoting $a = \rho - 1$, and we already know that $\rho = J_u^{-1} \rho_0$. So we have
\[a = (J_u^{-1} - 1)a_0 + a_0 + (J_u^{-1} - 1).
\]
Due to $u \in L^1(0, T; \dot{B}^{N/p+1}_{p,1})$, we know that $J_u^{-1}(t) - 1 \in C([0, T]; \dot{B}^{N/p}_{p,1})$. So we know that $a$ belongs to $C([0, T]; \dot{B}^{N/p}_{p,1})$. If
\[R \leq \frac{c_0}{4(1 + \|a_0\|_{\dot{B}^{N/p}_{p,1}})},
\]
we have
\[\rho(t) \geq (1 + a_0) - (1 + \|a_0\|_{L^\infty})(\|u_L\|_{L^1_t(\dot{B}^{N/p}_{p,1})}) + R
\geq (1 + a_0) - 2(1 + \|a_0\|_{\dot{B}^{N/p}_{p,1}})R
> 0,
\]
where $c_0$ defined as in Theorem 2.6 and $t \in [0, T]$.

**Step 4:** Uniqueness and continuity of the flow map Consider two couples of initial data $(\rho_0^1, u_0^1, B_0^1)$ and $(\rho_0^2, u_0^2, B_0^2)$. Define $\delta \rho = \rho^2 - \rho^1$, $\delta u = u^2 - u^1$ and $\delta B = B^2 - B^1$, we can easily find the equations for $\delta u$ and $\delta B$. Using similar methods as Navier-Stokes system and estimate the terms containing $B$ as in Step 3, we can finish this part. For there are no new gradients, we omit the details.

At this stage, by same reasons as in [30], we can get the following local well-posedness results for our MHD system as follows.

**Theorem 3.1.** Let $\bar{\rho} > 0$ and $c_0 > 0$. Assume that the initial data $(\rho_0, u_0, B_0)$ satisfies
\[\rho_0 - \bar{\rho} \in \dot{B}^{N/p}_{p,1}, \quad c_0 \leq \rho_0 \leq c_0^{-1},
\]
\[u_0, B_0 \in \dot{B}^{N/p-1}_{p,1}.
\]
Then there exists a positive time $T > 0$ such that if $p \in [2, 2n)$, the system has a unique solution $(\rho - \tilde{\rho}, u, B)$ satisfies

$$
\rho - \tilde{\rho} \in \tilde{C}([0, T]; B^N_{p,1}), \quad \frac{1}{2}\lambda_0 \leq \rho \leq 2\lambda_0^{-1},
$$

$$
u, B \in \tilde{C}([0, T]; B^{N/p}_{p,1} \cap L^1(0, T; B^{N/(p+1)}_{p,1})).
$$

**Remark 3.2.** In this remark, we expect to give a lower bound for the existence time in Theorem 3.1. Taking $m$ to be a large constant fixed in the proof of Theorem 3.1. Taking $C$ to be a large enough constant, let

$$\eta \leq \frac{1}{C}, \quad C_{\rho_0,m}T \leq \log 2, \quad (1 + \|a_0\|_{\dot{B}^{N/p}_{p,1}})R \leq \eta, \quad T \leq R^2,
$$

$$
\|a_0\|_{\dot{B}^{N/p}_{p,1}}\|u_0\|_{\dot{B}^{N/p-1}_{p,1}} \leq R^2, \quad \|u_0\|_{\dot{B}^{N/p-1}_{p,1}} \leq R, \quad \|B_0\|_{\dot{B}^{N/p-1}_{p,1}} \leq R,
$$

then the conditions (3.12) and (3.14) are satisfied. From the above conditions, we know that

$$\frac{1}{C^2(1 + \|a_0\|_{\dot{B}^{N/p}_{p,1}})^4}, \quad T \leq \frac{\log 2}{C_{\rho_0,m}}.
$$

Next, we calculate $C_{\rho_0,m}$ carefully as follows

$$
C_{\rho_0,m} \leq C \int_0^t \|S_m \nabla \left( \frac{1}{\rho_0} \right) \|_{\dot{B}^{N/p}_{p,1}}^2 d\tau
$$

$$
\leq C \int_0^t \|S_m \nabla \rho_0\|_{\dot{B}^{N/p}_{p,1}}^2 d\tau
$$

$$
\leq CT2^{2m}\|a_0\|_{\dot{B}^{N/p}_{p,1}}^2
$$

Combining (3.16) and (3.17), we know that we can take $T$ as

$$
T = \frac{\tilde{c}}{(1 + \|a_0\|_{\dot{B}^{N/p}_{p,1}})^4},
$$

where $\tilde{c}$ is a small enough positive constant.

Next, let us go to the second part of this section to prove the solution in Theorem 3.1 can propagate the smoothness of the initial data.

Before our proof, we need to introduce some notations which can also be found in [6, 31]. Define a weight function $\{w_k(t)\}_{k \in \mathbb{Z}}$ as follows

$$
\omega_k(t) = \sum_{t \geq k} 2^{k-t}(1 - 2^{-3t})^{1/2}, \quad k \in \mathbb{Z},
$$

where $c$ is a positive constant. We easily know that for any $k \in \mathbb{Z}$,

$$
\omega_k(t) \leq 2, \quad \omega_k(t) \sim \omega_k(t) \quad \text{if} \quad k \sim k',
$$

$$
\omega_k(t) \leq 2^{k-k'}\omega_{k'}(t) \quad \text{if} \quad k \geq k', \quad \omega_k(t) \leq 3\omega_{k'}(t) \quad \text{if} \quad k \leq k'.
$$

Now, we can introduce the following weighted Besov space.

**Definition 3.3.** Let $s \in \mathbb{R}$, $1 \leq p, r \leq +\infty$, $0 < T < +\infty$. The weighted Besov space $\dot{B}^{s}_{p,r}(\omega)$ is defined by

$$
\dot{B}^{s}_{p,r}(\omega) = \left\{ f \in \mathcal{S}'_m : \|f\|_{\dot{B}^{s}_{p,r}(\omega)} < +\infty \right\},
$$
where
\[ \|f\|_{\dot{B}_{p,r}^s(\omega)} := \\left\| \left( 2^{k\alpha} \omega_k(T) \|\Delta_k f\|_{L^p} \right) \right\|_k. \]

Obviously, \( \dot{B}_{p,r}^s \subset \dot{B}_{p,r}^s(\omega) \) and
\[ \|f\|_{\dot{B}_{p,r}^s(\omega)} \leq 2\|f\|_{\dot{B}_{p,r}^s}. \]

We also need to define the Chemin-Lerner type weighted Besov spaces as follows.

**Definition 3.4.** Let \( s \in \mathbb{R}, 1 \leq p, q, r \leq +\infty, 0 < T \leq +\infty. \) The weighted functional space \( \dot{L}_x^q(\dot{B}_{p,r}^s(\omega)) \) is defined as the set of all the distributions \( f \) satisfying
\[ \|f\|_{\dot{L}_x^q(\dot{B}_{p,r}^s(\omega))} := \|\left( 2^{k\alpha} \omega_k(T) \|\Delta_k f\|_{L^q(0,T;L^p)} \right) \|_r. \]

After this short introduction about weighted Besov space, we deduce some properties for the solution we obtained in the first part of this section. Obviously, we have
\[ \frac{c_0}{2} \leq \rho(t,x) \leq 2c_0^{-1}. \]

Since
\[ \|\tilde{u}\|_{\dot{L}_x^q(\dot{B}_{p,r}^{N/p-1})} + \|\tilde{u}\|_{\dot{L}_x^q(\dot{B}_{p,r}^{N/p+1})} \leq \frac{c_3}{(1 + \|a_0\|_{\dot{B}_{p,r}^{N/p}})^2}, \]
and
\[ \|u_L\|_{\dot{L}_x^q(\dot{B}_{p,r}^{N/p-1})} + \|u_L\|_{\dot{L}_x^q(\dot{B}_{p,r}^{N/p+1})} \leq \|a_0\|_{\dot{B}_{p,r}^{N/p-1}} \leq \frac{c_2}{(1 + \|a_0\|_{\dot{B}_{p,r}^{N/p}})^2}, \]
we know that
\[ \|u\|_{\dot{L}_x^q(\dot{B}_{p,r}^{N/p-1})} + \|u\|_{\dot{L}_x^q(\dot{B}_{p,r}^{N/p+1})} \leq \frac{c_2 + c_3}{(1 + \|a_0\|_{\dot{B}_{p,r}^{N/p}})}.
\]

For the magnetic field, we have
\[ \|B\|_{\dot{L}_x^q(\dot{B}_{p,r}^{N/p-1})} + \|B\|_{\dot{L}_x^q(\dot{B}_{p,r}^{N/p+1})} \leq R + \|B_0\|_{\dot{B}_{p,r}^{N/p}} \leq \frac{2c_3}{(1 + \|a_0\|_{\dot{B}_{p,r}^{N/p}})^2}.
\]

For the density, we know that
\[ \|a\|_{\dot{L}_x^q(\dot{B}_{p,r}^{3/p})} \leq \|Du\|_{\dot{L}_x^q(\dot{B}_{p,r}^{N/p})} \|a_0\|_{\dot{B}_{p,r}^{N/p}} + \|a_0\|_{\dot{B}_{p,r}^{3/p}} + \|Du\|_{\dot{L}_x^q(\dot{B}_{p,r}^{3/p})}.
\]

Hence, by some simple calculations, we obtain
\[ \|a\|_{\dot{L}_x^q(\dot{B}_{p,r}^{3/p})} \leq 2\|a_0\|_{\dot{B}_{p,r}^{N/p}} + \frac{c_2 + c_3}{(1 + \|a_0\|_{\dot{B}_{p,r}^{N/p}})^5}.
\]

So combining (3.19), (3.20), (3.21) and (3.22), for dimension \( N = 3 \), we know that the solution satisfies
\[ \frac{c_0}{2} \leq \rho(t,x) \leq 2c_0^{-1}, \]
\[ \|a\|_{\dot{L}_x^q(\dot{B}_{p,r}^{3/p})} \leq 2\|a_0\|_{\dot{B}_{p,r}^{N/p}} + \frac{c_2 + c_3}{(1 + \|a_0\|_{\dot{B}_{p,r}^{N/p}})^5}.
\]

(3.23)
At this stage, we estimate \( \|a\|_{L^\infty_T(\hat{B}^\beta_{p/(p)}(\omega))} \) carefully. As in [6], we have

\[
\sum_{j>M} 2^{j\frac{3}{p}} \|\Delta_j a_0\|_{L^p} \leq C 2^{-\frac{3}{p} M} \|a_0\|_{H^2},
\]

and

\[
\sum_{j\leq M} 2^{j\frac{3}{p}} \omega_j(T) \|\Delta_j a_0\|_{L^p} \leq C 2^{2M} T^\frac{3}{p} \|a_0\|_{H^2} + C 2^{-\frac{3}{p} M} \|a_0\|_{H^2}.
\]

So if we take \( M \) large enough and

\[(3.24) \quad T = \frac{c}{(1 + \|a_0\|_{\hat{B}^\beta_{p/(p)}})} (1 + \|a_0\|_{H^2})^{12}, \]

we have

\[(3.25) \quad \|a_0\|_{\hat{B}^\beta_{p/(p)}} \leq c', \]

where \( c, c' \) are small enough constants. Similar to \((3.22)\), we also have

\[(3.26) \quad \|a\|_{L^\infty_T(\hat{B}^\beta_{p/(p)}(\omega))} \leq \|a_0\|_{\hat{B}^\beta_{p/(p)}} + \frac{c_2 + c_3}{(1 + \|a_0\|_{\hat{B}^\beta_{p/(p)}})^2}.
\]

In the following proposition, we show that this solution allows us to propagate the regularity of the initial data in Sobolev space with low regularity.

**Proposition 3.5.** Let \( p \in (3,6) \) and \( 1 - \frac{2}{p} < \delta < \frac{3}{p} \). Assume that \( (\rho, u, B) \) is a solution for system \((1.1)\). If \( (a_0, u_0, B_0) \in H^{1-\delta} \times H^\delta \times H^\delta \), then

\[
\|a\|_{L^\infty_T(\hat{B}^\beta_{2,2})} \leq C \left( \|a_0\|_{H^{1-\delta}} + \|u_0\|_{H^{-\delta}} + \|B_0\|_{H^\delta} \right),
\]

\[(3.27) \quad \|(u,B)\|_{L^\infty_T(\hat{B}^\beta_{2,2})} + \|(u,B)\|_{L^1_T(\hat{B}^{\beta-\delta}_{2,2})} \leq C \left( 1 + \|a_0\|_{\hat{B}^\beta_{p/(p)}} \right) \left( \|\rho\|_{H^{1-\delta}} + TP + \|a_0\|_{H^{1-\delta}} \right).
\]

**Proof.** The proof can be divided into three steps. Set \( \beta = 1 - \delta \). Without loss of generality, we may assume that \( c, c_2, c_3, c', T \) (choose as in \((3.25)\) and \((3.24)\) ) small enough such that

\[(3.28) \quad \frac{c_2 + c_3}{(1 + \|a_0\|_{\hat{B}^\beta_{p/(p)}})^2} + 2C \left( c' + \frac{c_2 + c_3}{(1 + \|a_0\|_{\hat{B}^\beta_{p/(p)}})^2} \right) + 2C \frac{2c_2}{(1 + \|a_0\|_{\hat{B}^\beta_{p/(p)}})^2} + CT P \leq \frac{1}{2},
\]

where \( C \) is the constant appearing in the following estimates.

**Step 1 : Estimate for the Transport Equation.** This estimate is similar to the Navier-Stokes system, so we omit the details and only give the results as follows

\[(3.29) \quad \|a\|_{L^\infty_T(\hat{B}^\beta_{2,2})} \leq C \left( \|a_0\|_{\hat{B}^\beta_{2,2}} + \|u\|_{L^1_T(\hat{B}^{\beta+1}_{2,2})} \right).
\]

For details, we give the reference [6] on page 192.

**Step 2 : Estimate for the Momentum Equation.** Denote \( \bar{\mu} = \frac{4}{p} \) and \( \bar{\lambda} = \frac{3}{p} \),
apply the operator $\Delta_j$ to the momentum equation of (1.11), we can obtain
\[
\partial_t \Delta_j u - \nabla (\bar{\mu} \nabla \Delta_j u) - \nabla (\bar{\lambda} \Div \Delta_j u) = \Delta_j G + \Div ([\Delta_j, \bar{\mu}] \nabla u) + \nabla ([\Delta_j, \bar{\lambda}] \Div u)
\]
\[
+ \Delta_j \left( \frac{1}{\rho} B \cdot \nabla B \right) - \frac{1}{2} \Delta_j \left( \frac{1}{\rho} |B|^2 \right),
\]
where
\[
G = - u \cdot \nabla u - \frac{\bar{\rho} \tilde{P}'(\rho)}{\rho} \nabla u + \frac{\mu}{\rho^2} \nabla \rho \nabla u + \frac{\lambda}{\rho^2} \nabla \rho \Div u.
\]

Taking $L^2$ energy estimate for weighted Besov space, we have
\[
\|u\|_{L^2_t(B^s_{2,2})} + \|u\|_{L^2_t(B^s_{2,2})} \leq C(\|u_0\|_{B^{s-1}_{2,2}} + \|G\|_{L^2_t(B^s_{2,2})(\omega)})
\]
\[
+ C \left\| 2^{j_0} \bar{\omega}_j(T) \left( [\Delta_j, \bar{\mu}] \nabla u \right)_{L^2_t(B^s_{2,2})} + \|\Delta_j, \bar{\lambda} \nabla u \right\|_{L^2_t(B^s_{2,2})} \right\|_{L^2} \]
\[
+ C \left\| \frac{1}{\rho} B : \nabla B \right\|_{L^2_t(B^s_{2,2})} + C \left\| \frac{1}{\rho} \nabla |B|^2 \right\|_{L^2_t(B^s_{2,2})}.
\]

Similar to the Navier-Stokes system, we have
\[
\left\| 2^{j_0} \bar{\omega}_j(T) \left( [\Delta_j, \bar{\mu}] \nabla u \right)_{L^2_t(B^s_{2,2})} + \|\Delta_j, \bar{\lambda} \nabla u \right\|_{L^2_t(B^s_{2,2})} \right\|_{L^2} \]
\[
\leq C \|a\|_{L^\infty_t(B^{s_1}_{p,n}(\omega))} \|u\|_{L^2_t(B^{s_{2,2}}_{2,2})}
\]
\[
\text{and}
\]
\[
\|G\|_{L^2_t(B^s_{2,2})(\omega)} \leq C(\|u\|_{L^2_t(B^s_{2,2})} + |\nabla u|_{L^2_t(B^s_{2,2})} + \|\nabla u\|_{L^2_t(B^{s_{1}}_{2,2})(\omega)} + TP_+ \|a\|_{L^\infty_t(B^{s_{1}}_{2,2})}.
\]

Then, we estimate the term which is not appeared in the Navier-Stokes system as follows
\[
\left\| \frac{1}{\rho} B : \nabla B \right\|_{L^2_t(B^{s_{1}}_{2,2})} \leq C(1 + \|a\|_{L^\infty_t(B^{s_1}_{p,n})} \|B : \nabla B \|_{L^2_t(B^{s_{2,2}}_{2,2})} + \|a\|_{L^\infty_t(B^{s_1}_{p,n})} \|B \|_{L^2_t(B^{s_{2,2}}_{2,2})})
\]
\[
\leq C(1 + \|a\|_{L^\infty_t(B^{s_1}_{p,n})} \|B \|_{L^2_t(B^{s_{2,2}}_{2,2})} \|B \|_{L^2_t(B^{s_{2,2}}_{2,2})}.
\]

Plugging estimates (3.32), (3.33) and (3.34) into (3.31), we obtain
\[
\|u\|_{L^2_t(B^{s_{2,2}}_{2,2})} + \|u\|_{L^2_t(B^{s_1}_{2,2})} \leq C(\|u_0\|_{B^{s-1}_{2,2}} + C\|u\|_{L^2_t(B^{s_{2,2}}_{2,2})} \|u\|_{L^2_t(B^{s_{2,2}}_{2,2})})
\]
\[
+ C(1 + \|a\|_{L^\infty_t(B^{s_1}_{p,n})} \|u\|_{L^2_t(B^{s_{1}}_{2,2})} + CT P_+ \|a\|_{L^\infty_t(B^{s_1}_{2,2})} + \|u\|_{L^2_t(B^{s_{2,2}}_{2,2})})
\]
\[
+ C(1 + \|a\|_{L^\infty_t(B^{s_1}_{p,n})} \|B \|_{L^2_t(B^{s_{2,2}}_{2,2})} \|B \|_{L^2_t(B^{s_{2,2}}_{2,2})}.
\]

Taking $L^2$ energy estimates for equation (3.30), we obtain
\[
\|u\|_{L^2_t(B^{s_{2,2}}_{2,2})} \leq C(\|u_0\|_{B^{s-1}_{2,2}} + \|G\|_{L^2_t(B^{s_{2,2}}_{2,2})})
\]
\[
+ C \left\| 2^{j_0} \bar{\omega}_j(T) \left( [\Delta_j, \bar{\mu}] \nabla u \right)_{L^2_t(B^s_{2,2})} + \|\Delta_j, \bar{\lambda} \nabla u \right\|_{L^2_t(B^s_{2,2})} \right\|_{L^2} \]
\[
+ C \left\| \frac{1}{\rho} B : \nabla B \right\|_{L^2_t(B^s_{2,2})} + C \left\| \frac{1}{\rho} \nabla |B|^2 \right\|_{L^2_t(B^s_{2,2})}.
\]
The first two terms can be estimated as in the Navier-Stokes system, we only give the results as follows

\begin{equation}
\left\| 2^{\beta j} \left( \left\| \Delta_j, \nu \nabla u \right\|_{L^2_T(L^2)} + \left\| \Delta_j, \tilde{\lambda} \nabla u \right\|_{L^2_T(L^2)} \right) \right\|_{L^2}
\leq C \| a \|_{L^\infty_T(B^{3/2}_{p/r})} \| u \|_{L^1_T(B^{3/2+1}_{2,2})},
\end{equation}

\begin{equation}
\| G \|_{L^1_T(B^{3/2-1}_{2,2})} \leq C \left( \| u \|_{L^2_T(B^{3/2}_{p/r})} \| u \|_{L^1_T(B^{3}_{2,2})} + \| a \|_{L^\infty_T(B^{3/2}_{p/r})} \| u \|_{L^1_T(B^{3/2+1}_{2,2})} + TP_+ \| a \|_{L^\infty_T(B^{3/2}_{2,2})} \right).
\end{equation}

For the terms not appeared in the Navier-Stokes system, we can estimate as \(3.34\). Plugging \(3.37\), \(3.38\) and \(3.34\) into \(3.36\), we have

\begin{equation}
\| u \|_{L^2_T(B^{3/2-1}_{2,2})} \leq C \| u_0 \|_{B^{3/2-1}_{2,2}} + C \| a \|_{L^\infty_T(B^{3/2}_{p/r})} \| u \|_{L^1_T(B^{3/2+1}_{2,2})} + CP_T \| B \|_{L^2_T(B^{3/2}_{p/r})} \| B \|_{L^1_T(B^{3/2+1}_{2,2})} + C(1 + \| a \|_{L^\infty_T(B^{3/2}_{p/r})}) \| B \|_{L^2_T(B^{3/2}_{p/r})} \| B \|_{L^1_T(B^{3/2+1}_{2,2})}.
\end{equation}

**Step 3: Estimate for the Magnetic Field Equation.** Applying the operator \(\Delta_j\) to the third equation of system \(1.1\), we obtain

\[ \partial_t \Delta_j B - \nu \Delta_j B = -\Delta_j \text{div}(Bu^T - uB^T). \]

Performing \(L^2\) energy estimates, we could have

\begin{equation}
\| B \|_{L^2_T(B^{3/2-1}_{2,2})} + \| B \|_{L^1_T(B^{3/2+1}_{2,2})} \leq C \| B_0 \|_{B^{3/2-1}_{2,2}} + C \| \text{div}(Bu^T - uB^T) \|_{L^1_T(B^{3/2-1}_{2,2})}.
\end{equation}

Since

\begin{equation}
\| u \cdot \nabla B \|_{L^2_T(B^{3/2-1}_{2,2})} \leq C \| u \|_{L^2_T(B^{3}_{p/r})} \| \nabla B \|_{L^2_T(B^{3/2}_{2,2})}
\leq C \| u \|_{L^2_T(B^{3}_{p/r})} \| B \|_{L^2_T(B^{3}_{2,2})},
\end{equation}

and

\begin{equation}
\| B \cdot \nabla u \|_{L^1_T(B^{3/2-1}_{2,2})} \leq C \| B \|_{L^2_T(B^{3}_{p/r})} \| u \|_{L^2_T(B^{3}_{2,2})},
\end{equation}

we know that

\begin{equation}
\| B \|_{L^2_T(B^{3/2-1}_{2,2})} + \| B \|_{L^1_T(B^{3/2+1}_{2,2})} \leq C \| B_0 \|_{B^{3/2-1}_{2,2}} + C \| u \|_{L^2_T(B^{3}_{p/r})} \| B \|_{L^2_T(B^{3}_{2,2})} + C \| B \|_{L^2_T(B^{3}_{p/r})} \| u \|_{L^2_T(B^{3}_{2,2})}.
\end{equation}

Combining estimates \(3.35\) and \(3.33\), we will get

\begin{equation}
\| u \|_{L^1_T(B^{3/2+1}_{2,2})} + \| u \|_{L^2_T(B^{3/2}_{2,2})} + \| B \|_{L^1_T(B^{3/2+1}_{2,2})}
\leq C \left( \| u_0 \|_{B^{3/2-1}_{2,2}} + \| B_0 \|_{B^{3/2-1}_{2,2}} + TP_+ \| a \|_{L^\infty_T(B^{3/2+1}_{2,2})} \right).
\end{equation}

Combining estimates \(3.39\) and \(3.38\), we will have

\begin{equation}
\| u \|_{L^2_T(B^{3/2-1}_{2,2})} + \| B \|_{L^\infty_T(B^{3/2}_{2,2})}
\leq C \left( \| u_0 \|_{B^{3/2-1}_{2,2}} + \| B_0 \|_{B^{3/2-1}_{2,2}} + TP_+ \| a \|_{L^\infty_T(B^{3/2+1}_{2,2})} \right)
+ C \| a_0 \|_{B^{3/2}_{p/r}} \| u \|_{L^1_T(B^{3/2+1}_{2,2})}.
\end{equation}
At this stage, combining (3.44) and (3.45), we obtain
\[
\|u\|_{\dot{L}_{T}^{p}(\dot{B}_{2}^{s+1})} + \|B\|_{\dot{L}_{T}^{s}(\dot{B}_{2}^{s+1})} + \|u\|_{\dot{L}^{s+1}_{T}(\dot{B}_{2}^{s+1})} + \|B\|_{\dot{L}^{s+1}_{T}(\dot{B}_{2}^{s+1})} 
\leq C\left(1 + \|a_{0}\|_{\dot{B}^{s}_{p}/r} \right) \left(\|u_{0}\|_{\dot{B}^{s-1}_{2}} + \|B_{0}\|_{\dot{B}^{s-1}_{2}} + TP_{s} + \|a\|_{L_{T}^{s}(\dot{B}_{2}^{s+1})} \right).
\]  
(3.46)

At last, combining (3.29) and (3.46), we finally get the desired results (3.27). □

In the last part of this section, we prove the solution propagate the regularity of the initial data in Sobolev space with high regularity.

**Proposition 3.6.** Assume that \((\rho, u, B)\) is a solution of system (1.1) on \([0, T]\), which satisfies \(\rho \geq c_{0}\),  

\[
a \in \dot{L}_{T}^{s}(\dot{B}_{p}^{3/p}), \quad u, B \in \dot{L}_{T}^{s}(\dot{B}_{p}^{3/p}) \cap \dot{L}^{2}_{T}(\dot{B}_{p}^{3/p+1}),
\]  
(3.47)

where \(a = \rho - 1\). If \((a_{0}, u_{0}, B_{0}) \in H^{s} \times H^{s-1} \times H^{s-1}\) for \(s \geq 3\), then we have

\[
a \in \dot{L}_{T}^{s}(H^{s}), \quad u, B \in \dot{L}_{T}^{s}(H^{s-1}) \cap \dot{L}^{2}_{T}(H^{s+1}).
\]  
(3.48)

**Proof.** Considering (3.47), we can divide the time interval \([0, T]\) into finitely many small intervals \([T_{i}, T_{i+1}]\) with \(i = 0, 1, \ldots, N\) such that

\[
T_{i+1} - T_{i} \leq \epsilon, \quad \|a\|_{L^{\infty}(T_{i}, T_{i+1}; \dot{B}^{3/p}_{p}/r)} \leq \epsilon,
\]  
(3.49)

\[
\|(u, B)\|_{L^{1}(T_{i}, T_{i+1}; \dot{B}^{3/p+1}_{p})} \leq \epsilon,
\]  
for some \(\epsilon\) small enough. Here \(\omega^{i} = \{\omega^{i}_{k}\}\) stands for

\[
\omega^{i}_{k} = \sum_{\ell \geq k} 2^{k-i}(1 - e^{-c2^{2k}(T_{i+1} - T_{i})})^{\frac{1}{2}}.
\]  

Denote \(\dot{L}^{s}_{T_{i}}(\dot{B}^{s}_{p,r}) := \dot{L}^{s}(T_{i}, T_{i+1}; \dot{B}^{s}_{p,r})\). For the density, the estimate is similar to the Navier-Stokes equations, so we omit the details

\[
\|a\|_{\dot{L}_{T_{i}}^{s}(\dot{B}_{2}^{s})} \leq C\left(\|a(T_{i})\|_{\dot{B}_{2}^{s}} + \|u\|_{\dot{L}^{s}_{T_{i}}(\dot{B}_{2}^{s+1})}\right).
\]  
(3.50)

For the velocity field, as in the proof of Proposition 3.5, we have

\[
\|u\|_{\dot{L}^{s}_{T_{i}}(\dot{B}_{2}^{s+1})} + \|u\|_{\dot{L}^{s}_{T_{i}}(\dot{B}_{2}^{s})} \leq C\left(\|u(T_{i})\|_{\dot{B}_{2}^{s+1}} + \|G\|_{\dot{L}_{T_{i}}^{s}(\dot{B}_{2}^{s})} + \|a\|_{\dot{L}_{T_{i}}^{s}(\dot{B}_{2}^{s})}ight)
\]  
(3.51)

\[
\hat{C} \left[2^{s}\omega_{j}(t) \left(\|\Delta_{j}, \lambda \|_{L^{2}_{T_{i}}(L^{2})}, \|\Delta_{j}, \mu \|_{L^{2}_{T_{i}}(L^{2})}\right)\right] \leq C \left[\|u\|_{\dot{L}_{T_{i}}^{s}(\dot{B}_{2}^{s+1})} + \|G\|_{\dot{L}_{T_{i}}^{s}(\dot{B}_{2}^{s})}\right]
\]  
(3.52)

Similar to the Navier-Stokes system, we have

\[
\|G\|_{\dot{L}_{T_{i}}^{s}(\dot{B}_{2}^{s})} \leq C\left(\|u\|_{\dot{L}_{T_{i}}^{s}(\dot{B}_{2}^{s})} + \|u\|_{\dot{L}_{T_{i}}^{s}(\dot{B}_{2}^{s+1})}\right)
\]  
(3.53)
For the last two terms in (3.51) all can be estimated as follows

\[
\left\| \frac{1}{\rho} \cdot \nabla B \right\|_{L^2_t(\dot{B}^{s+1}_{2,2})} \leq C \| B \cdot \nabla B \|_{L^2_t(\dot{B}^{s+1}_{2,2})} + C \left\| \frac{a}{1 + a} B \cdot \nabla B \right\|_{L^2_t(\dot{B}^{s+1}_{2,2})}
\]

\[
\leq C \| B \|_{L^2_t(\dot{B}^{s}_{2,2})} \| B \|_{L^2_t(\dot{B}^{s}_{2,2})} + C \| a \|_{L^2_t(\dot{B}^{s}_{2,2})} \| B \cdot \nabla B \|_{L^2_t(\dot{B}^{s+1}_{2,2})}
\]

\[
+ C \| B \cdot \nabla B \|_{L^2_t(\dot{B}^{s}_{2,2})} \| a \|_{L^2_t(\dot{B}^{s}_{2,2})} \]

(3.54)

Plugging (3.52), (3.53) and (3.54) into (3.51), we will obtain

\[
\| u \|_{L^2_t(\dot{B}^{s+1}_{2,2})} + \| u \|_{L^2_t(\dot{B}^{s}_{2,2})} \leq C\left(\| u(T) \|_{\dot{B}^{s-1}_{2,2}} + \| a \|_{L^2_t(\dot{B}^{s}_{2,2})} \left(\| u \|_{L^2_t(\dot{B}^{s}_{2,2})} + (T_{i+1} - T_i)\right)\right)
\]

\[
+ C \| B \|_{L^2_t(\dot{B}^{s}_{2,2})} \| B \|_{L^2_t(\dot{B}^{s}_{2,2})} + C \| a \|_{\dot{B}^{s+1}_{2,2}} \| B \|_{L^2_t(\dot{B}^{s}_{2,2})}
\]

\[
+ C \| B \|_{L^2_t(\dot{B}^{s}_{2,2})} \| a \|_{L^2_t(\dot{B}^{s}_{2,2})} \]

(3.55)

For the equation of magnetic field, using similar methods as in Proposition 3.5, we obtain

\[
\| B \|_{L^2_t(\dot{B}^{s+1}_{2,2})} + \| B \|_{L^2_t(\dot{B}^{s}_{2,2})} \leq C\| B \|_{L^2_t(\dot{B}^{s}_{2,2})} + C \| u \cdot \nabla B \|_{L^2_t(\dot{B}^{s}_{2,2})}
\]

\[
+ C \| B \|_{L^2_t(\dot{B}^{s}_{2,2})} \]

(3.56)

Since

\[
\| u \cdot \nabla B \|_{L^2_t(\dot{B}^{s}_{2,2})} \leq C\left(\| u \|_{L^2_t(\dot{B}^{s}_{2,2})} \| \nabla B \|_{L^2_t(\dot{B}^{s}_{2,2})} + \| \nabla B \|_{L^2_t(\dot{B}^{s}_{2,2})} \right)\]

\[
\leq C\| u \|_{L^2_t(\dot{B}^{s}_{2,2})} \| B \|_{L^2_t(\dot{B}^{s}_{2,2})} + C \| B \|_{L^2_t(\dot{B}^{s}_{2,2})} \| u \|_{L^2_t(\dot{B}^{s}_{2,2})},
\]

and

\[
\| B \cdot \nabla u \|_{L^2_t(\dot{B}^{s}_{2,2})} \leq C\left(\| B \|_{L^2_t(\dot{B}^{s}_{2,2})} \| u \|_{L^2_t(\dot{B}^{s}_{2,2})} + \| u \|_{L^2_t(\dot{B}^{s}_{2,2})} \right)\]

\[
+ C \| u \|_{L^2_t(\dot{B}^{s}_{2,2})} \| B \|_{L^2_t(\dot{B}^{s}_{2,2})},
\]

we know that

\[
\| B \|_{L^2_t(\dot{B}^{s}_{2,2})} + \| B \|_{L^2_t(\dot{B}^{s}_{2,2})} \leq C\| B(T) \|_{\dot{B}^{s}_{2,2}} + C\| u \|_{L^2_t(\dot{B}^{s}_{2,2})} \| B \|_{L^2_t(\dot{B}^{s}_{2,2})} + C\| B \|_{L^2_t(\dot{B}^{s}_{2,2})} \| u \|_{L^2_t(\dot{B}^{s}_{2,2})},
\]

(3.57)

Combining (3.50), (3.55) and (3.57), we have

\[
\| a \|_{L^2_t(\dot{B}^{s}_{2,2})} + \| u \|_{L^2_t(\dot{B}^{s}_{2,2})} + \| B \|_{L^2_t(\dot{B}^{s}_{2,2})} \leq C\left(\| a \|_{L^2_t(\dot{B}^{s}_{2,2})} \left(\| a(T) \|_{\dot{B}^{s+1}_{2,2}} + \| u(T) \|_{\dot{B}^{s}_{2,2}} + \| B(T) \|_{\dot{B}^{s}_{2,2}}\right)\right)
\]

(3.58)

Estimate velocity as in (3.39) with regularity index to be \( s - 1 \) and combine the estimate about magnetic field \( B \), we finally get

\[
\| u \|_{L^2_t(\dot{B}^{s}_{2,2})} + \| B \|_{L^2_t(\dot{B}^{s}_{2,2})} \leq C\left(1 + \| a \|_{L^2_t(\dot{B}^{s}_{2,2})} \right)\| a(T) \|_{\dot{B}^{s}_{2,2}}
\]

\[
+ \| u(T) \|_{\dot{B}^{s}_{2,2}} + \| B(T) \|_{\dot{B}^{s}_{2,2}}.
\]

(3.59)
Combining (3.58) and (3.59), we arrive at
\[
\|a\|_{L^p_T(\dot{B}^{s-1}_{2,2})} + \|(u, B)\|_{L^p_T(\dot{B}^{s}_{2,2})} + \|(u, B)\|_{\dot{L}^p_T(\dot{B}^{s-1}_{2,2})} \\
\leq C(1 + \|a\|_{L^p_T(\dot{B}^{3/2}_{2,2})})(\|a(T_i)\|_{\dot{B}^{s-1}_{2,2}} + \|u(T_i)\|_{\dot{B}^{s-1}_{2,2}} + \|B(T_i)\|_{\dot{B}^{s-1}_{2,2}}).
\]
(3.60)

By induction, we know that
\[
\|a\|_{L^p_T(\dot{B}^{s}_{2,2})} + \|(u, B)\|_{L^p_T(\dot{B}^{s-1}_{2,2})} + \|(u, B)\|_{\dot{L}^p_T(\dot{B}^{s-1}_{2,2})} \\
\leq C(1 + \|a\|_{L^p_T(\dot{B}^{3/2}_{2,2})})^N + (\|a(T_i)\|_{\dot{B}^{s-1}_{2,2}} + \|u(T_i)\|_{\dot{B}^{s-1}_{2,2}} + \|B(T_i)\|_{\dot{B}^{s-1}_{2,2}}).
\]
(3.61)
Hence, we complete the proof. □

4. Hoff’s Energy Method

In two papers [15,16], D. Hoff construct a global weak solution for Navier-Stokes equations with discontinuous initial data with small energy. In the paper [14], A. Suen and D. Hoff generalize the results for Navier-Stokes equations to compressible MHD system. Here, we use the idea to our case. Comparing to [14], we remove the restriction on viscosity for we have stronger condition on the initial data.

We set \(\sigma(t) \coloneqq \min(1, t)\), define
\[
A_1(T) = \sup_{0 \leq t \leq T} \left( \|\nabla u(t)\|_{L^2}^2 + \|\nabla B(t)\|_{L^2}^2 \right) + \int_0^T \int_{\mathbb{R}^3} \rho|\dot{u}|^2 + |\dot{B}_i|^2 \, dx \, dt,
\]
\[
A_2(T) = \sup_{0 \leq t \leq T} \left( \int_{\mathbb{R}^3} \sigma(t)|\dot{u}|^2 \, dx + \sigma(t) \int_{\mathbb{R}^3} |B_i|^2 \, dx \right) \\
+ \int_0^T \int_{\mathbb{R}^3} \sigma(t)|\nabla \dot{u}|^2 \, dx \, dt + \int_0^T \int_{\mathbb{R}^3} \sigma(t)|\nabla B_i|^2 \, dx \, dt,
\]
and
\[
E(T) = \int_{\mathbb{R}^3} \sigma(t) \left( |\nabla B|^2 |B|^2 + |\nabla B|^2 |u|^2 + |\nabla u|^2 |B|^2 \right) \, dx,
\]
\[
H(T) = \int_0^T \int_{\mathbb{R}^3} |\nabla B| \, dx \, dt + \int_0^T \int_{\mathbb{R}^3} |\nabla u| \, dx \, dt \\
+ \int_0^T \int_{\mathbb{R}^3} \sigma(t)|\nabla B|^4 \, dx \, dt + \int_0^T \int_{\mathbb{R}^3} \sigma(t)|\nabla u|^4 \, dx \, dt.
\]

In the following sections, we denote
\[
\dot{f} = f_t + u \cdot \nabla f, \quad \mu' = \lambda + \mu,
\]
and
\[
C_0 = \|\rho_0 - \bar{\rho}\|_{L^2}^2 + \|u_0\|_{H^2}^2 + \|B_0\|_{H^1}^2.
\]
Throughout this section, we denote by \(C\) a constant depending only on \(\lambda, \mu, c_0, \bar{\rho}, \bar{P}_+, \bar{P}_-\)
and \(P_{-1}^-\).

Theorem 4.1. Let \((\rho, u, B)\) be a solution of system (1.1) satisfying
\[
\rho - \bar{\rho} \in C([0, T]; H^2), \quad u, B \in C([0, T]; H^2) \cap L^2(0, T; H^3).
\]
Then, there exists a constant $\epsilon_0$ depending only on $\nu, \lambda, \mu, c_0, \bar{\rho}, \bar{P}, P^+ - 1$ such that if the initial data $(\rho_0, u_0, B_0)$ satisfies

$$
c_0 \leq \rho_0(x) \leq c_0^{-1}, \quad x \in \mathbb{R}^3,
$$

$$
\|\rho - \bar{\rho}\|_{L^2}^2 + \|u_0\|_{H^1}^2 + \|B_0\|_{H^1}^2 \leq \epsilon_0,
$$

then we have

$$
\frac{c_0}{2} \leq \rho(t, x) \leq 2c_0^{-1}, \quad (t, x) \in [0, T] \times \mathbb{R}^3,
$$

$$
A(T) := A_1(T) + A_2(T) \leq \epsilon_0^2.
$$

Proof. Considering the assumption, there exists a $0 < T_0 \leq T$ such that the solution $(\rho, u, B)$ satisfies

$$
c_0 \leq \rho(t, x) \leq 2c_0^{-1}, \quad (t, x) \in [0, T_0] \times \mathbb{R}^3,
$$

$$
A_1(T_0) + A_2(T_0) \leq \epsilon_0^2.
$$

Without loss of generality, we assume that $T_0$ is a maximal time so that the above inequalities hold. In the following, we will give a refined estimates on $[0, T]$ for the solution. Due to the proof is too long, we divide the proof into several lemmas.

**Lemma 4.2.** ($L^2$ energy estimate)

$$
\int_{\mathbb{R}^3} |\rho - \bar{\rho}|^2 + |\rho|u|^2 + |B|^2 dx + \int_0^T \int_{\mathbb{R}^3} |\nabla u|^2 + |\nabla B|^2 dx dt \leq CC_0.
$$

The proof of this lemma is exactly the same as Lemma 2.2 in [19], so we omit it.

**Lemma 4.3.** ($H^1$ energy estimate)

$$
A_1(T_0) \leq CC_0 + C \int_0^{T_0} \int_{\mathbb{R}^3} |\nabla B|^2 |B|^2 + |\nabla B|^2 |u|^2 + |\nabla u|^2 |B|^2 dx dt
$$

$$
+ C \int_0^{T_0} \int_{\mathbb{R}^3} |\nabla u|^3 dx dt.
$$

Proof. Multiply the second equation in (1.1) by $\dot{u}$ and integrate over $\mathbb{R}^3$, we will obtain

$$
\int_{\mathbb{R}^3} \rho|\dot{u}|^2 \, dx = \int_{\mathbb{R}^3} (-\dot{u} \cdot \nabla P + \mu \Delta u \dot{u} + \mu' \nabla \text{div} u \dot{u} + \nabla (\frac{1}{2} |B|^2) \dot{u} + \text{div} (B B^T) \dot{u}) \, dx.
$$

(4.1)

By the continuity equation, we have

$$
\partial P + \text{div}(uP) = \text{div} (P - P') \dot{\rho}.
$$

(4.2)

There are some terms can be estimated as in the Navier-Stokes equations, so we omit the details and only give the results as follows

$$
\int_{\mathbb{R}^3} -\dot{u} \cdot \nabla P \, dx \leq \partial_t \int_{\mathbb{R}^3} \text{div} (P - P(\bar{\rho})) \, dx + C \|\nabla u\|_{L^2}^2.
$$

(4.3)

$$
\mu \int_{\mathbb{R}^3} \Delta u \dot{u} \, dx \leq -\frac{\mu}{2} \partial_t \int_{\mathbb{R}^3} |\nabla u|^2 \, dx + C \int_{\mathbb{R}^3} |\nabla u|^3 \, dx,
$$

(4.4)
\(\mu' \int_{\mathbb{R}^3} \nabla \text{div} \ u \, \dot{u} \, dx \leq -\frac{\lambda}{2} \partial_t \int_{\mathbb{R}^3} |\text{div} u|^2 \, dx + C \int_{\mathbb{R}^3} |\nabla u|^3 \, dx, \tag{4.5}\)

and

\[\int_{\mathbb{R}^3} \text{div} u (P - P(\bar{\rho})) \, dx \leq \|\text{div} u\|_{L^2} \|P(\rho) - P(\bar{\rho})\|_{L^2} \leq C \int_{\mathbb{R}^3} |\nabla u|^2 \, dx. \tag{4.6}\]

The following terms are not appeared in Navier-Stokes system

\[
\int_{\mathbb{R}^3} -\nabla \left( \frac{1}{2} |B|^2 \right) \dot{u} \, dx + \int_{\mathbb{R}^3} \text{div}(B B^T) \dot{u} \, dx \leq C \int_{\mathbb{R}^3} |\nabla B|^2 \|\dot{u}\| \, dx \\
\leq C \int_{\mathbb{R}^3} |\nabla B|^2 |B|^2 \, dx + \epsilon \int_{\mathbb{R}^3} |\dot{u}|^2 \, dx. \tag{4.7}\]

Multiply the third equation in (1.1) with \(B_t\) and integrate over \(\mathbb{R}^3\), we have

\[
\int_{\mathbb{R}^3} |B_t|^2 \, dx + \int_{\mathbb{R}^3} \text{div}(B u^T - u B^T) B_t \, dx = -\nu \partial_t \int_{\mathbb{R}^3} |\nabla B|^2 \, dx. 
\]

Integrate the above equality from 0 to \(T_0\), we will obtain

\[
\frac{\nu}{2} \int_{\mathbb{R}^3} |\nabla B|^2 \, dx + \int_0^{T_0} \int_{\mathbb{R}^3} |B_t|^2 \, dx \\
\leq CC_0 + C \int_0^{T_0} \int_{\mathbb{R}^3} |\nabla B|^2 |u|^2 + |\nabla u|^2 |B|^2 \, dxdt. \tag{4.8}\]

Combining (1.1), (4.3), (4.4), (4.5), (4.6), (4.7) and (4.8), we finally get the desired result.

\[\square\]

**Lemma 4.4.** \((H^2 \text{ energy estimate})\)

\[
A_2(T_0) \leq CC_0 + CA_1(T_0) + C \int_0^{T_0} \int_{\mathbb{R}^3} \sigma |\nabla u|^4 \, dxdt \\
+ C \int_0^T \int_{\mathbb{R}^3} \sigma (|u|^2 + |B|^2) (|B_t|^2 + |\dot{u}|^2 + |u|^2 (|\nabla B|^2 + |\nabla u|^2)) \, dxdt. 
\]

**Proof.** Take material derivative to the second equation of (1.1) to obtain

\[
\rho \dot{u}_t + \rho u \cdot \nabla \dot{u} + \nabla P_t + \text{div}(\nabla P \otimes u) + \nabla \left( \frac{1}{2} |B|^2 \right)_t + \text{div}(u \cdot \nabla (\frac{1}{2} |B|^2)) \\
- \text{div}(B B^T)_t = \mu \Delta u_t + \mu \text{div}(u \Delta u) \\
+ \mu' \text{div} \text{div} u + \mu' \text{div}(u \cdot \nabla u).
\]
Multiply \( \sigma(t) \dot{u} \) on both sides of the above equality, we will have

\[
\partial_t \left( \frac{\sigma}{2} \int_{\mathbb{R}^3} \rho |\dot{u}|^2 \, dx \right) - \frac{1}{2} \sigma' \int_{\mathbb{R}^3} \rho |\dot{u}|^2 \, dx = - \int_{\mathbb{R}^3} \sigma \dot{u} \left( \partial_j P_t + \text{div} (\partial_j P \, u) \right) \, dx = - \int_{\mathbb{R}^3} \sigma \dot{u} \left( \frac{1}{2} |B|^2_{x_j, t} + \text{div} \left( u \partial_j \left( \frac{1}{2} |B|^2 \right) \right) \right) \, dx
\]

\[
+ \int_{\mathbb{R}^3} \sigma \dot{u} \left( \text{div} (B^j B)_t + \text{div} (u \text{div} (B^j B)) \right) \, dx
\]

\[
+ \mu \sigma \int_{\mathbb{R}^3} \dot{u} \left( \Delta u_t + \text{div} (\Delta u \otimes u) \right) \, dx
\]

\[
+ \mu' \sigma \int_{\mathbb{R}^3} \dot{u} \left( \nabla \text{div} u_t + \text{div} (\nabla \text{div} u \otimes u) \right) \, dx.
\]

(4.9)

Similar to the \( H^1 \) estimate, there are some terms same as Navier-Stokes equations and we only give the results as follows

\[
- \int_{\mathbb{R}^3} \sigma \dot{u} \left( \partial_j P_t + \text{div} (\partial_j P \, u) \right) \, dx \leq C \sigma \| \nabla u \|_L^2 \| \nabla \dot{u} \|_L^2,
\]

(4.10)

\[
\mu \sigma \int_{\mathbb{R}^3} \dot{u} \left( \Delta u_t + \text{div} (\Delta u \otimes u) \right) \, dx \leq \sigma \int_{\mathbb{R}^3} \frac{\mu}{4} |\nabla \dot{u}|^2 \, dx + \sigma C \int_{\mathbb{R}^3} |\nabla u|^4 \, dx,
\]

(4.11)

\[
\mu' \sigma \int_{\mathbb{R}^3} \dot{u} \left( \nabla \text{div} u_t + \text{div} (\nabla \text{div} u \otimes u) \right) \, dx
\]

\[
\leq \sigma \int_{\mathbb{R}^3} \frac{\lambda}{2} |\text{div} \dot{u}|^2 + \frac{\mu}{4} |\nabla \dot{u}|^2 + C |\nabla u|^4 \, dx.
\]

(4.12)

For the terms not appeared in the Navier-Stokes equations, we have

\[
- \int_{\mathbb{R}^3} \sigma \dot{u} \left( \partial_j \left( \frac{1}{2} |B|^2 \right)_t + \text{div} \left( u \partial_j \left( \frac{1}{2} |B|^2 \right) \right) \right) \, dx
\]

\[
\leq \epsilon \int_{\mathbb{R}^3} \sigma |\nabla \dot{u}|^2 \, dx + C \int_{\mathbb{R}^3} \sigma |B|^2 (|B_t|^2 + |u|^2 |\nabla B|^2) \, dx.
\]

(4.13)

For the equation about magnetic field, multiplying \( \sigma B_t \) on both sides, we will have

\[
\frac{1}{2} \sigma \int_{\mathbb{R}^3} |B_t|^2 \, dx + \nu \int_{t_0}^{t_0} \int_{\mathbb{R}^3} \sigma |\nabla B_t|^2 \, dx dt
\]

\[
= - \int_{t_0}^{t_0} \int_{\mathbb{R}^3} \sigma B_t (\text{div} (B u^T - u B^T))_t \, dx dt + \frac{1}{2} \int_{t_0}^{t_0} \int_{\mathbb{R}^3} \sigma' |B_t|^2 \, dx dt.
\]

(4.14)
We can easily obtain the following estimates

\[
- \int_0^{T_0} \int_{\mathbb{R}^3} \sigma B_t(\text{div}(B u^T - u B^T)) dx dt = \int_0^{T_0} \int_{\mathbb{R}^3} \sigma \nabla B_t(B_t u^T + B u_t^T - u_t B^T) dx dt
\]

(4.15)

\[
\leq \epsilon \int_0^{T_0} \int_{\mathbb{R}^3} \sigma |\nabla B_t|^2 dx dt + C \int_0^{T_0} \int_{\mathbb{R}^3} \sigma (|B_t|^2 |u|^2 + |B|^2 |u_t|^2) dx dt
\]

\[
\leq \epsilon \int_0^{T_0} \int_{\mathbb{R}^3} \sigma |\nabla B_t|^2 dx dt + C \int_0^{T_0} \int_{\mathbb{R}^3} \sigma (|B_t|^2 |u|^2 + |B|^2 |u|^2 |\nabla u|^2) dx dt,
\]

where \( \epsilon \) is a small enough positive number. Combining estimates (4.9), (4.10), (4.11), (4.12), (4.13), (4.14) and (4.15), we will obtain

\[
\sigma \int_{\mathbb{R}^3} \rho |\dot{u}|^2 + |B_t|^2 dx + \int_0^{T_0} \int_{\mathbb{R}^3} \sigma (|\nabla u|^2 + |\nabla B_t|^2) dx dt
\]

\[
\leq C \int_0^{T_0} \int_{\mathbb{R}^3} \sigma |\nabla u||L^2|\nabla \dot{u}|L^2 dx dt + C \int_0^{T_0} \int_{\mathbb{R}^3} \sigma |\nabla u|^3 dx dt
\]

\[
+ C \int_0^{1 \wedge T_0} \int_{\mathbb{R}^3} \rho |\dot{u}|^2 dx dt + C \int_0^{1 \wedge T_0} \int_{\mathbb{R}^3} |B_t|^2 dx dt
\]

\[
+ C \int_0^{T_0} \int_{\mathbb{R}^3} \sigma (|u|^2 + |B|^2)(|B_t|^2 + |\dot{u}|^2 + |u|^2 (|\nabla B|^2 + |\nabla u|^2)) dx dt.
\]

We can easily obtain the following estimates

\[
\int_0^{T_0} \int_{\mathbb{R}^3} \sigma |\nabla u||L^2|\nabla \dot{u}|L^2 dx dt
\]

(4.17)

\[
\leq C \int_0^{T_0} \int_{\mathbb{R}^3} \sigma |\nabla u|^2 dx dt + C \int_0^{T_0} \int_{\mathbb{R}^3} \sigma |\nabla u|^2 dx dt
\]

\[
\leq CC_0 + CA_1(T_0),
\]

(4.18)

\[
\int_0^{1 \wedge T_0} \int_{\mathbb{R}^3} \rho |\dot{u}|^2 dx dt + \int_0^{1 \wedge T_0} \int_{\mathbb{R}^3} |B_t|^2 dx dt \leq CA_1(T_0).
\]

Substituting (4.17) and (4.18) into (4.16), we finally complete the proof. \( \square \)

**Lemma 4.5.**

\[
A_1(T_0) + A_2(T_0) \leq CC_0 + CC_0(C_0^{5/3} + C_0 A_2(T_0))A_1(T_0)^{3/2}
\]

\[
+ CC_0 A_1(T_0)^2 + CC_0^2 A_1(T_0)^{3/2}(\sigma |\nabla B|^2_{L^2})
\]

\[
+ C \int_0^{T_0} \int_{\mathbb{R}^3} |\nabla u|^3 + |\nabla B|^3 dx dt + C \int_0^{T_0} \int_{\mathbb{R}^3} \sigma (|\nabla u|^4 + |\nabla B|^4) dx dt.
\]
Proof. Combining Lemma 1.2, Lemma 1.3, and Lemma 1.4 we will have

\begin{align*}
A_1(T_0) + A_2(T_0) &\leq CC_0 + C \int_0^{T_0} \int_{\mathbb{R}^3} \sigma |\nabla u|^4 \, dx \, dt + C \int_0^{T_0} \int_{\mathbb{R}^3} |\nabla u|^3 \, dx \, dt \\
&+ C \int_0^{T_0} \int_{\mathbb{R}^3} \sigma(|u|^2 + |B|^2)(|B_t|^2 + |u_t|^2 + |\nabla B|^2 + |\nabla u|^2) \, dx \, dt \\
&+ C \int_0^{T_0} \int_{\mathbb{R}^3} (|B|^2 + |u|^2)(|\nabla B|^2 + |\nabla u|^2) \, dx \, dt.
\end{align*}

(4.19)

Next we need to estimate some typical terms on the right hand side.

**Term 1:** \( \int_0^{T_0} \int_{\mathbb{R}^3} |B|^2 |\nabla B|^2 \, dx \, dt \)

\[
\int_0^{T_0} \int_{\mathbb{R}^3} |B|^2 |\nabla B|^2 \, dx \, dt \leq \int_0^{T_0} \left( \int_{\mathbb{R}^3} |B|^6 \, dx \right)^{1/3} \left( \int_{\mathbb{R}^3} |\nabla B|^3 \, dx \right)^{2/3} \, dt \\
\leq C \int_0^{T_0} \int_{\mathbb{R}^3} |\nabla B|^3 \, dx \, dt + C \int_0^{T_0} \int_{\mathbb{R}^3} |B|^6 \, dx \, dt \\
\leq C \left( \int_{\mathbb{R}^2} |\nabla B|^2 \, dx \right)^2 \int_0^{T_0} \int_{\mathbb{R}^3} |\nabla B|^2 \, dx \, dt + C \int_0^{T_0} \int_{\mathbb{R}^3} |\nabla B|^3 \, dx \, dt \\
\leq CC_0 A_1(T_0)^2 + C \int_0^{T_0} \int_{\mathbb{R}^3} |\nabla B|^3 \, dx \, dt.
\]

**Term 2:** \( \int_0^{T_0} \int_{\mathbb{R}^3} \sigma |B|^2 |B_t|^2 \, dx \, dt \)

\[
\int_0^{T_0} \int_{\mathbb{R}^3} \sigma |B|^2 |B_t|^2 \, dx \, dt \leq \left( \int_0^{T_0} \int_{\mathbb{R}^3} |B|^6 \, dx \, dt \right)^{1/3} \left( \int_0^{T_0} \int_{\mathbb{R}^3} \sigma^{3/2} |B_t|^3 \, dx \, dt \right)^{2/3} \\
\leq \left( \int_0^{T_0} \left( \int_{\mathbb{R}^3} |\nabla B|^2 \, dx \right) \, dt \right)^{1/3} \left( \sigma \| B_t \|_{L^2}^2 \right)^{1/3} \left( \int_0^{T_0} \int_{\mathbb{R}^3} \sigma |B_t|^2 \, dx \, dt \right)^{2/3} \\
\leq CC_0^{1/3} A_1(T_0)^{4/3} A_2(T_0)^{1/3}.
\]

**Term 3:** \( \int_0^{T_0} \int_{\mathbb{R}^3} \sigma |B|^2 |u|^2 |\nabla B|^2 \, dx \, dt \)

\[
\int_0^{T_0} \int_{\mathbb{R}^3} \sigma |B|^2 |u|^2 |\nabla B|^2 \, dx \, dt \leq \int_0^{T_0} \int_{\mathbb{R}^3} \sigma (|B|^8 + |u|^8 + |\nabla B|^4) \, dx \, dt \\
\leq \int_0^{T_0} \int_{\mathbb{R}^3} \sigma |\nabla B|^4 \, dx \, dt + (\| u \|_{L^4}^4 + \| B \|_{L^4}^4) \int_0^{T_0} \sigma (\| u \|_{L^\infty}^4 + \| B \|_{L^\infty}^4) \, dt
\]

Since

\[
\| u \|_{L^4}^4 + \| B \|_{L^4}^4 \leq \| u \|_{L^2} \| \nabla u \|_{L^2}^3 + \| B \|_{L^2} \| \nabla B \|_{L^2}^3,
\]
and

\[
\int_0^{T_0} \sigma \|u\|_{L^\infty}^2 + \sigma \|B\|_{L^\infty}^4 \, dt \leq \int_0^{T_0} \sigma \|u\|_{L^5}^2 \|\nabla u\|_{L^8}^2 + \sigma \|B\|_{L^8}^2 \|\nabla B\|_{L^8}^2 \, dt \\
\leq \int_0^{T_0} \sigma \|\nabla u\|_{L^2}^2 (\|\hat{\rho} u\|_{L^2} + \|P(\rho) - P(\bar{\rho})\|_{L^6})^2 \, dt + \int_0^{T_0} \sigma \|\nabla B\|_{L^2}^2 \|\nabla^2 B\|_{L^2}^2 \, dt \\
\leq \sigma \|\hat{\rho} u\|_{L^2}^2 \int_0^{T_0} \|\nabla u\|_{L^2}^2 \, dt + CC_0^{5/3} + CC_0 \sigma \|\nabla^2 B\|_{L^2}^2,
\]

we can get the estimate

\[
\int_0^{T_0} \int_{\mathbb{R}^3} \sigma |B|^2 |u|^2 \|\nabla B\|^2 \, dx \, dt \leq \int_0^{T_0} \int_{\mathbb{R}^3} \sigma |\nabla B|^4 \, dx \, dt \\
+ CC_0 A_1(T_0)^2/3 + C C_0^2/3 + C C_0 \sigma \|\nabla^2 B\|_{L^2}^2 + C A_2(T_0).
\]

where we used Lemma 4.2 and the definition of $A_1, A_2$.

Using similar procedure as did for Term 1, we have

\[(4.20) \int_0^{T_0} \int_{\mathbb{R}^3} \sigma (|u|^2 + |B|^2) (|\nabla u|^2 + |\nabla B|^2) \, dx \, dt \leq CC_0^{1/3} A_1(T_0)^{1/3} A_2(T_0) \]

Using similar procedure as did for Term 2, we have

\[(4.21) \int_0^{T_0} \int_{\mathbb{R}^3} \sigma (|u|^2 + |B|^2) (|\nabla u|^2 + |\nabla B|^2) \, dx \, dt \leq CC_0^{1/3} A_1(T_0)^{1/3} A_2(T_0) \]

Using similar procedure as did for Term 3, we have

\[(4.22) \int_0^{T_0} \int_{\mathbb{R}^3} \sigma (|u|^2 + |B|^2) (|\nabla u|^2 + |\nabla B|^2) \, dx \, dt \leq CC_0 A_1(T_0)^{3/2} (C_0^{5/3} + C_0 \sigma \|\nabla^2 B\|_{L^2}^2 + C A_2(T_0)) \]

At last substitute (4.20), (4.21), (4.22) into (4.19) and do simple reduction, we will complete our proof.

Before going to the following part, we need to introduce some notations. Let

\[(4.23) F = (\lambda + \mu) \text{div} u - P(\rho) + P(\bar{\rho}),
\]

and

\[(4.24) \omega^{j,k} = u^j_{x_k} - u^k_{x_j}, \quad j, k = 1, 2, 3.
\]

Through simple calculations, we could find that

\[(4.25) \mu \Delta \omega^{j,k} = (\rho \hat{u}^j)_{x_k} - (\rho \hat{u}^k)_{x_j} - (B \cdot \nabla B^j)_{x_k} + (B \cdot \nabla B^k)_{x_j},
\]

and

\[(4.26) \Delta F = \text{div} g
\]

where $g^j = \rho \hat{u}^j + (\frac{1}{2}|B|^2)_{x_j} - \text{div}(B^j B)$ with $j = 1, 2, 3$. 

Lemma 4.6.

\[ E(T_0) \leq CC_0 + P_E(A(T_0)), \]

where \( A(T_0) = A_1(T_0) + A_2(T_0) \) and \( P_E \) is a polynomial function with 2 as the lowest order.

Proof. For the term \( \int_{\mathbb{R}^3} \sigma |\nabla B|^2 |u|^2 \, dx \), we have

\[ \int_{\mathbb{R}^3} \sigma |\nabla B|^2 |u|^2 \, dx \leq \sigma \|u\|_{L^4}^2 \int_{\mathbb{R}^3} |\nabla B|^2 \, dx \]

\[ \leq CA_1(T_0) \left( \sigma \|u\|_{L^4}^2 + \sigma \|\nabla u\|_{L^4}^2 \right). \tag{4.27} \]

For term \( \sigma \|u\|_{L^4}^2 \) which appeared in the above inequality, we have

\[ \sigma \|u\|_{L^4}^2 \leq \sigma \|u\|_{L^2}^{1/2} \|\nabla u\|_{L^2}^{3/2} \leq CC_0^{1/2} A_1(T_0)^{3/4}. \tag{4.28} \]

The analysis about \( \sigma \|\nabla u\|_{L^4}^2 \) is a little bit more complex, by \ref{4.23}, we know that

\[ \sigma \|\nabla u\|_{L^4}^2 \leq \sigma \|F\|_{L^4}^2 + \sigma \|\omega\|_{L^4}^2 + \sigma \|P(\rho) - P(\bar{\rho})\|_{L^4}^2. \tag{4.29} \]

For the first term on the right hand side of the above inequality, we have

\[ \sigma \|F\|_{L^4}^2 \leq \sigma \|F\|_{L^2}^{1/2} \|\nabla F\|_{L^2}^{3/2} \]

\[ \leq C \left( \|\nabla u\|_{L^2} + \|\rho - \bar{\rho}\|_{L^2} \right)^{1/2} \left( \int_{\mathbb{R}^3} |\nabla \dot{u}|^2 \, dx + \int_{\mathbb{R}^3} \sigma |\nabla B|^2 |B|^2 \, dx \right)^{3/4} \]

\[ \leq C \left( A_1(T_0)^{1/4} + C_0^{1/4} \right) \left( A_2(T_0)^{3/4} + E(T_0)^{3/4} \right). \tag{4.30} \]

For the second term on the right hand side of \ref{4.29}, we have

\[ \sigma \|\omega\|_{L^4}^2 \leq \sigma \|\omega\|_{L^2}^{1/2} \|\nabla \omega\|_{L^2}^{3/2} \]

\[ \leq \|\nabla u\|_{L^2}^{1/2} \left( \int_{\mathbb{R}^3} \sigma |\nabla \omega|^2 \, dx \right)^{3/4} \]

\[ \leq A_1(T_0)^{1/4} \left( \int_{\mathbb{R}^3} \sigma |\nabla \dot{u}|^2 \, dx + \int_{\mathbb{R}^3} \sigma |\nabla B|^2 |B|^2 \, dx \right)^{3/4} \]

\[ \leq A_1(T_0)^{1/4} \left( A_2(T_0)^{3/4} + E(T_0)^{3/4} \right). \tag{4.31} \]

For the third term on the right hand side of \ref{4.29}, we have

\[ \sigma \|P(\rho) - P(\bar{\rho})\|_{L^4}^2 \leq C \left( \int_{\mathbb{R}^3} |\rho - \bar{\rho}|^2 \, dx \right)^{1/2} \leq CC_0^{1/2}. \tag{4.32} \]

Substitute \ref{4.30}, \ref{4.31}, \ref{4.32} into \ref{4.29}, we will obtain

\[ \sigma \|\nabla u\|_{L^4}^2 \leq CC_0^{1/2} + C(A_1(T_0)^{1/4} + C_0^{1/4}) (A_2(T_0)^{3/4} + E(T_0)^{3/4}) \]

Combining \ref{4.28}, \ref{4.33} and \ref{4.27}, we finally have

\[ \int_{\mathbb{R}^3} \sigma |\nabla B|^2 |u|^2 \, dx \leq CA_1(T_0) \left( C_0^{1/2} A_1(T_0)^{3/4} + C_0^{1/2} \right) \]

\[ + \left( A_1(T_0)^{1/4} + C_0^{1/4} \right) \left( A_2(T_0)^{3/4} + E(T_0)^{3/4} \right). \tag{4.34} \]
Summing up (4.34) and (4.35), we obtain
\begin{equation}
(4.35)
\int_{\mathbb{R}^3} \sigma|\nabla u|^2 |B|^2 \, dx + \int_{\mathbb{R}^3} \sigma|\nabla B|^2 |B|^2 \, dx \\
\leq CA_1(T_0) \left( C_0^{1/2} A_1(T_0)^{3/4} + A_1(T_0)^{1/4} (A_2(T_0) + E(T_0))^{3/4} \right).
\end{equation}

Using similar arguments as for term (4.36), we could easily obtain our desired result. □

Through (4.36), we could easily obtain our desired result.

Using Young's inequality, we can get
\begin{equation}
(4.36)
E(T_0) \leq CC_0^4 A_1(T_0)^4 + CC_0^4 A_1(T_0) A_2(T_0) + CC_0 A_1(T_0)^4 \\
+ CA_1(T_0)^5 + CA_1(T_0)^4 A_2(T_0) + CC_0^4 A_1(T_0) A_2(T_0) + CC_0^4 A_1(T_0) A_2(T_0).
\end{equation}

Through (4.36), we could easily obtain our desired result.

At this stage, we could give the estimate about term $\sigma \|\nabla^2 B\|_{L^2}$ as follows
\begin{equation}
(4.37)
\int_{\mathbb{R}^3} \sigma|\nabla^2 B|^2 \, dx \leq C \int_{\mathbb{R}^3} \sigma|B|^2 \, dx + C \int_{\mathbb{R}^3} \sigma|\nabla B|^2 |u|^2 + \sigma|\nabla u|^2 |B|^2 \, dx \\
\leq CA_2(T_0) + CE(T_0) \\
\leq C(1 + C_0^{1/2}) A_2(T_0) + CA_1(T_0)^{5/4} A_2(T_0)^{3/4} \\
+ CC_0 A_1(T_0) A_2(T_0) + CA_1(T_0)^5 + CC_0 A_1(T_0)^4,
\end{equation}

where we used Lemma 4.6.

**Lemma 4.7.**

$$H(T_0) \leq CP_{HC}(C_0) + CP_{HA}(A(T_0)),$$

where $P_{HC}$ is a polynomial function with lowest order $\frac{3}{4}$ and $P_{HA}$ is a polynomial function with lowest order $\frac{3}{4}$.

**Proof.** We need to estimate every term appeared in the definition of $H(T_0)$. For
\begin{equation}
(4.38)
\int_0^{T_0} \int_{\mathbb{R}^3} \sigma|\nabla B|^4 \, dx \, dt, \quad \text{we have}
\end{equation}

\begin{equation}
(4.39)
\int_0^{T_0} \int_{\mathbb{R}^3} \sigma|\nabla^2 B|^2 \, dx \, dt \leq C \int_0^{T_0} \sigma \|\nabla^2 B\|_{L^2} \|\nabla^2 B\|_{L^2} \, dt \\
\leq CA_1(T_0)^{1/2} \sigma^{1/2} \|\nabla^2 B\|_{L^2} \int_0^{T_0} \sigma \|\nabla^2 B\|_{L^2} \, dt.
\end{equation}

For the last term of the above inequality, we have
\begin{equation}
(4.39)
\int_0^{T_0} \int_{\mathbb{R}^3} \sigma|\nabla^2 B|^2 \, dx \, dt \leq C \int_0^{T_0} \int_{\mathbb{R}^3} \sigma(|B|^2 + |\nabla B|^2 |u|^2 + |\nabla u|^2 |B|^2) \, dx \, dt \\
\leq A_1(T_0) + \int_0^{T_0} \int_{\mathbb{R}^3} \sigma(|\nabla B|^2 |u|^2 + |\nabla u|^2 |B|^2) \, dx \, dt.
\end{equation}
For the last two terms in the above inequality, we firstly have

\[
\int_0^{T_0} \int_{\mathbb{R}^3} \sigma |\nabla B|^2 |u|^2 \, dx \, dt \\
\leq \int_0^{T_0} \left( \int_{\mathbb{R}^3} \sigma |\nabla B|^4 \, dx \right)^{1/2} \left( \int_{\mathbb{R}^3} \sigma |u|^4 \, dx \right)^{1/2} \, dt \\
\leq \left( \int_0^{T_0} \int_{\mathbb{R}^3} \sigma |\nabla B|^4 \, dx \, dt \right)^{1/2} \left( \int_0^{T_0} \int_{\mathbb{R}^3} \sigma |u|^4 \, dx \, dt \right)^{1/2} \\
\leq H(T_0)^{1/2} \left( \int_0^{T_0} \left( \int_{\mathbb{R}^3} |u|^2 \, dx \right)^{1/2} \left( \int_{\mathbb{R}^3} |\nabla u|^2 \, dx \right)^{3/2} \, dt \right)^{1/2} \\
\leq H(T_0)^{1/2} C_0^{1/4} \left( \int_{\mathbb{R}^3} |\nabla u|^2 \, dx \right)^{1/4} \left( \int_0^{T_0} \int_{\mathbb{R}^3} |\nabla u|^2 \, dx \, dt \right)^{1/2} \\
\leq H(T_0)^{1/2} C_0^{3/4} A_1(T_0)^{1/4}.
\]

(4.40)

Similarly, we could get

\[
\int_0^{T_0} \int_{\mathbb{R}^3} \sigma |\nabla u|^2 |B|^2 \, dx \, dt \leq H(T_0)^{1/2} C_0^{3/4} A_1(T_0)^{1/4}.
\]

(4.41)

Substitute (4.40) and (4.41) into (4.39), we will have

\[
\int_0^{T_0} \int_{\mathbb{R}^3} \sigma |\nabla B|^2 |u|^2 \, dx \, dt \leq CA_1(T_0) + CH(T_0)^{1/2} C_0^{3/4} A_1(T_0)^{1/4}.
\]

(4.42)

Substitute estimates (4.37) and (4.42) into (4.38), we obtain

\[
\int_0^{T_0} \int_{\mathbb{R}^3} \sigma |\nabla B|^4 \, dx \, dt \leq CA_1(T_0)^{1/2}((1 + C_0)^{1/2} A_2(T_0) \\
+ A_1(T_0)^{5/4} A_2(T_0)^{3/4} + C_0^{1/4} A_1(T_0) A_2(T_0)^{3/4} \\
+ A_1(T_0)^5 + C_0 A_1(T_0)^4)^{1/2}(A_1(T_0) \\
+ C_0^{3/4} A_1(T_0)^{1/4} H(T_0)^{1/2})
\]

(4.43)

For the term \(\int_0^{T_0} \int_{\mathbb{R}^3} \sigma |\nabla u|^4 \, dx \, dt\), we have

\[
\int_0^{T_0} \int_{\mathbb{R}^3} \sigma |\nabla u|^4 \, dx \, dt \lesssim \int_0^{T_0} \int_{\mathbb{R}^3} \sigma (|\rho - \bar{\rho}|^4 + |F|^4 + |\omega|^4) \, dx \, dt.
\]

(4.44)

Concerning the first term on the right hand side of the above inequality, we have

\[
\int_0^{T_0} \int_{\mathbb{R}^3} \sigma |\rho - \bar{\rho}|^4 \, dx \, dt \lesssim C_0 + \int_0^{T_0} \int_{\mathbb{R}^3} \sigma |F|^4 \, dx \, dt.
\]

(4.45)

For the second term on the right hand side of (4.44), we have

\[
\int_0^{T_0} \int_{\mathbb{R}^3} \sigma |F|^4 \, dx \, dt \leq \int_0^{T_0} \sigma ||F||_{L^2} ||\nabla F||_{L^2}^2 \, dt \\
\leq \left( \int_{\mathbb{R}^3} |F|^2 \, dx \right)^{1/2} \left( \int_{\mathbb{R}^3} \sigma |\nabla F|^2 \, dx \right)^{1/2} \int_0^{T_0} \int_{\mathbb{R}^3} |\nabla F|^2 \, dx \, dt.
\]

(4.46)
Due to

\[(4.47) \quad \left( \int_{\mathbb{R}^3} |F|^2 \, dx \right)^{1/2} \leq \left( \int_{\mathbb{R}^3} |\nabla u|^2 \, dx \right)^{1/2} + \left( \int_{\mathbb{R}^3} |\rho - \bar{\rho}|^2 \, dx \right)^{1/2} \leq C(C_0 + A_1(T_0))^{1/2},\]

\[(4.48) \quad \left( \int_{\mathbb{R}^3} \sigma |\nabla F|^2 \, dx \right)^{1/2} \leq C \int_{\mathbb{R}^3} \sigma(|\dot{u}|^2 + |\nabla B|^2|B|^2) \, dx \leq CA_2(T_0) + CE(T_0),\]

and

\[(4.49) \quad \int_0^{T_0} \int_{\mathbb{R}^3} |\nabla F|^2 \, dx \, dt \leq C \int_0^{T_0} \int_{\mathbb{R}^3} |\dot{u}|^2 + |\nabla B|^2|B|^2 \, dx \, dt \leq CA_1(T_0) + C \left( \int_0^{T_0} \int_{\mathbb{R}^3} |\nabla B|^3 \, dx \, dt \right)^{2/3} \left( \int_0^{T_0} \int_{\mathbb{R}^3} |B|^6 \, dx \, dt \right)^{1/3} \leq CA_1(T_0) + CH(T_0)^{2/3}C_0^{1/2}A_1(T_0)^{1/6},\]

we can refine the estimate (4.47) to

\[(4.50) \quad \int_0^{T_0} \int_{\mathbb{R}^3} |\sigma F|^4 \, dx \, dt \leq C(C_0 + A_1(T_0))^{1/2}(A_2(T_0) + E(T_0))(A_1(T_0) + H(T_0)^{2/3}C_0^{1/2}A_1(T_0)^{1/6}).\]

Concerning the last term on the right hand side of (4.44), we have

\[(4.51) \quad \int_0^{T_0} \int_{\mathbb{R}^3} |\sigma|^4 \, dx \, dt \leq \int_0^{T_0} \sigma \|\omega\|_{L^2} \|\omega\|_{L^6}^3 \, dt \leq \|\omega\|_{L^2} \int_0^{T_0} \int_{\mathbb{R}^3} \sigma |\nabla \omega|^2 \, dx \, dt \int_0^{T_0} \int_{\mathbb{R}^3} |\nabla \omega|^2 \, dx \, dt \leq CA_1(T_0)^{1/2}(A_2(T_0) + E(T_0))(A_1(T_0) + H(T_0)^{2/3}C_0^{1/2}A_1(T_0)^{1/6}).\]

Substitute (4.45), (4.50) and (4.51) into (4.44), we obtain that

\[(4.52) \quad \int_0^{T_0} \int_{\mathbb{R}^3} |\nabla u|^4 \, dx \, dt \leq CC_0 + C(C_0 + A_1(T_0))^{1/2}(A_2(T_0) + E(T_0))(A_1(T_0) + H(T_0)^{2/3}C_0^{1/2}A_1(T_0)^{1/6} + CA_1(T_0)^{1/2}(A_2(T_0) + E(T_0))(A_1(T_0) + H(T_0)^{2/3}C_0^{1/3}A_1(T_0)^{1/3}).\]
Concerning the term $\int_0^T \int_{\mathbb{R}^3} |\nabla B|^3 \, dx \, dt$, we have
\begin{align*}
\int_0^T \int_{\mathbb{R}^3} |\nabla B|^3 \, dx \, dt & \leq C \int_0^T \|\nabla B\|_{L^2}^{3/2} \|\nabla B\|_{L^2}^{3/2} \, dt \\
& \leq \left( \int_0^T \|\nabla B\|_{L^2}^{6} \, dt \right)^{1/4} \left( \int_0^T \|\nabla B\|_{L^2}^{2} \, dt \right)^{3/4} \\
& \leq C \|\nabla B\|_{L^2} \left( \int_0^T \int_{\mathbb{R}^3} |\nabla B|^2 \, dx \, dt \right)^{1/4} \left( \int_0^T \int_{\mathbb{R}^3} |\nabla B|^2 \, dx \, dt \right)^{3/4} \\
& \leq C A_1(T_0)^{1/4} C_0^{1/4} \left( \int_0^T \int_{\mathbb{R}^3} (B_t^2 + |\nabla B|^2 |u|^2 + |\nabla u|^2 |B|^2) \, dx \, dt \right)^{3/4} \\
& \leq C A_1(T_0)^{1/4} C_0^{1/4} \left( A_1(T_0)^{3/4} + C_0^{2/3} A_1(T_0)^{4/3} H(T_0)^{1/3} \right).
\end{align*}
(4.53)

For the term $\int_0^T \int_{\mathbb{R}^3} |\nabla u|^3 \, dx \, dt$, we have
\begin{align*}
\int_0^T \int_{\mathbb{R}^3} |\nabla u|^3 \, dx \, dt & \leq \int_0^{\sigma(T_0)} \int_{\mathbb{R}^3} |\nabla u|^3 \, dx \, dt + \int_{\sigma(T_0)}^T \int_{\mathbb{R}^3} |\nabla u|^3 \, dx \, dt \\
& \leq \int_0^{\sigma(T_0)} \|\nabla u\|_{L^2}^{3/2} (\|\rho u\|_{L^2}^{3/2} + \|P(\rho) - P(\bar{\rho})\|_{L^2}^{3/2}) \, dt \\
& \quad + \left( \int_{\sigma(T_0)}^T \int_{\mathbb{R}^3} |\nabla u|^2 \, dx \, dt \right)^{1/2} \left( \int_{\sigma(T_0)}^T \int_{\mathbb{R}^3} |\nabla u|^4 \, dx \, dt \right)^{1/2} \\
& \leq C C_0^{1/4} A_1(T_0)^{3/4} + A_1(T_0)^{3/2} + C C_0^{1/2} H(T_0)^{1/2}.
\end{align*}
(4.54)

Combining estimates (4.43), (4.52), (4.53), (4.54) and (4.36), we could finish the proof by a long but tedious calculations. \qed

Now, combining estimate (4.51), Lemma 4.3, Lemma 4.7 and Lemma 4.8, we will obtain
\begin{align*}
A_1(T_0) + A_2(T_0) \leq C P_{AC}(C_0) + C P_{AA}(A_1(T_0) + A_2(T_0)),
\end{align*}
(4.55)
where $P_{AC}(\cdot)$ is a polynomial function with lowest order $\frac{3}{2}$ and $P_{AA}(\cdot)$ is a polynomial function with lowest order $\frac{3}{8}$. Since $A_1(T_0) + A_2(T_0) \leq \epsilon_0^{1/2}$ and $C_0 = \|\rho_0 - \bar{\rho}\|_{L^2}^{2} + \|u_0\|_{L^2}^{2} + \|B_0\|_{H^1}^{2} \leq \epsilon_0$, if $\epsilon_0 > 0$ is small enough, we have
\begin{align*}
A_1(T_0) + A_2(T_0) \leq C \epsilon_0^{1/4} \epsilon_0^{1/2} + C \epsilon_0^{1/6} \epsilon_0^{1/2} \leq \frac{1}{2} \epsilon_0^{1/2}.
\end{align*}
(4.56)

It remains to prove the lower and upper bound of the density. Set $\Gamma = \log(\rho)$ which satisfies
\begin{align*}
(\lambda + \mu) \dot{\Gamma} + (P(\rho) - P(\bar{\rho})) = -F.
\end{align*}
For $0 < t < \sigma(T_0)$, we will have

$$\int_0^{\sigma(T_0)} \|F\|_{L^\infty} dt \leq C \int_0^{\sigma(T_0)} \|F\|_{L^6}^{1/2} \|\nabla F\|_{L^6}^{1/2} dt$$

(4.57)

$$\leq C \int_0^{\sigma(T_0)} \left( \|\sqrt{\rho} \dot{u}\|_{L^2}^{1/2} + \|B \cdot \nabla B\|_{L^2}^{1/2} \right) \left( \|\rho \dot{u}\|_{L^6}^{1/2} + \|B \cdot \nabla B\|_{L^6}^{1/2} \right) dt$$

$$\leq C \int_0^{\sigma(T_0)} \left( \|\sqrt{\rho} \dot{u}\|_{L^2}^{1/2} + \|B \cdot \nabla B\|_{L^2}^{1/2} \right) \left( \|\nabla \dot{u}\|_{L^2}^{1/2} + \|B \cdot \nabla B\|_{L^6}^{1/2} \right) dt.$$}

By estimate (4.56), we have

$$\int_0^{\sigma(T_0)} \|\sqrt{\rho} \dot{u}\|_{L^2}^{1/2} \|\nabla \dot{u}\|_{L^2}^{1/2} dt \leq C \varepsilon_0^{1/4},$$

(4.58)

$$\int_0^{\sigma(T_0)} \|B \cdot \nabla B\|_{L^2}^{1/2} \|\nabla \dot{u}\|_{L^2}^{1/2} dt$$

$$= \int_0^{\sigma(T_0)} t^{-1/2} \left( \|\rho \dot{u}\|_{L^6}^{1/2} \right)^{1/4} \left( \|\nabla \dot{u}\|_{L^2}^{1/4} \right)^{1/4} dt$$

(4.59)

$$\leq C \varepsilon_0^{1/4} \int_0^{\sigma(T_0)} t^{-2/3} dt \left( \int_0^{\sigma(T_0)} \|\nabla \dot{u}\|_{L^2}^{1/4} dt \right)^{1/4}$$

$$\leq C \varepsilon_0^{1/4} \varepsilon_0^{1/8} \leq C \varepsilon_0^{3/8},$$
and
\[ \int_0^{\sigma(T_0)} \|B \cdot \nabla B\|_{L^2}^{1/2} \|B \cdot \nabla B\|_{L^6}^{1/2} \, dt \]
\[ \leq \int_0^{\sigma(T_0)} t^{-5/8} (t\|B \cdot \nabla B\|_{L^2}^2)^{1/4} \|B \cdot \nabla B\|_{L^6}^{1/4} (t\|\nabla^2 B\|_{L^6}^2)^3/8 \, dt \]
\[ + \int_0^{\sigma(T_0)} t^{-1/2} (t\|B \cdot \nabla B\|_{L^2}^2)^{1/4} \|B \cdot \nabla B\|_{L^6}^{1/4} \|\nabla B\|_{L^4} \, dt \]
\[ \leq C_0^{3/8} \int_0^{\sigma(T_0)} t^{-5/8} (t\|\nabla^2 B\|_{L^2}^2)^{3/8} \, dt + C_0^{1/4} \int_0^{\sigma(T_0)} t^{-1/2} t^{1/4} \|\nabla B\|_{L^4} \, dt \]
\[ \leq C_0^{3/8} \epsilon_0^{3/16} + C_0^{1/4} \epsilon_0^{9/16} \leq C_0^{3/16}, \]
where we used estimate (4.37) and (4.50). Substitute (4.58), (4.59), (4.60) and (4.61) into (4.57), we obtain
\[ \int_0^{T_0} \|F\|_{L^\infty} \, dt \leq C_0^{1/4}, \]
which implies that for \( t \leq \sigma(T_0) \),
\[ \inf(\log \rho_0(x)) - C_0^{1/4} - Ct \leq \log \rho(t, x) \leq \sup(\log(\rho_0(x))) + C_0^{1/4} + Ct. \]
So we can choose \( \epsilon_0, \tau \) small enough such that for \( t \leq \tau \leq \sigma(T_0) \),
\[ \frac{3}{4} \epsilon_0 < \rho(t, x) < \frac{3}{2} \epsilon_0^{-1}. \]
For \( \tau \leq t_1 \leq t_2 \leq T_0 \), we have
\[ \int_{t_1}^{t_2} \|F\|_{L^\infty} \, dt \leq C \int_{t_1}^{t_2} (\|\sqrt{\rho} u\|_{L^2}^{1/2} + \|B \cdot \nabla B\|_{L^2}^{1/2})(\|\nabla u\|_{L^2}^{1/2} + \|B \cdot \nabla B\|_{L^6}^{1/2}) \, dt \]
\[ \leq C (t_2 - t_1)^{1/2} \int_{t_1}^{t_2} \|\sqrt{\rho} u\|_{L^2}^{1/2} + \|B \cdot \nabla B\|_{L^2}^{2} + \|\nabla u\|_{L^2}^{2} + \|B \cdot \nabla B\|_{L^6}^{2} \, dt. \]
Now, we estimate the four integral terms on the right hand side of the above inequality. For the first two terms, we have
\[ \int_{t_1}^{t_2} \|\sqrt{\rho} u\|_{L^2}^{2} \, dt \leq A_2(T_0) \leq \epsilon_0^{1/2}, \quad \int_{t_1}^{t_2} \|\nabla u\|_{L^2}^{2} \, dt \leq A_2(T_0) \leq \epsilon_0^{1/2}, \]
The estimate about the third term appeared in (4.39), so we have
\[ \int_{t_1}^{t_2} \|B \cdot \nabla B\|_{L^2}^{2} \, dt = \int_{t_1}^{t_2} \int_{\mathbb{R}^3} |B|^2 |\nabla B|^2 \, dx \, dt \leq C_0^{1/2}. \]
For the fourth term, we have
\[ \int_{t_1}^{t_2} \|B \cdot \nabla B\|_{L^6}^{2} \, dt \leq \int_{t_1}^{t_2} \|\nabla B\|_{L^2}^{1/2} \|\nabla^2 B\|_{L^2}^{3/2} \, dt + \int_{t_1}^{t_2} \|\nabla B\|_{L^4} \, dt \]
\[ \leq \left( \int_{t_1}^{t_2} \|\nabla B\|_{L^2}^{2} \, dt \right)^{1/4} \left( \int_{t_1}^{t_2} \|\nabla^2 B\|_{L^2}^{2} \, dt \right)^{3/4} + C_0^{1/2} \]
\[ \leq C_0^{1/4} \epsilon_0^{3/8} + C_0^{1/2} \leq C_0^{1/2}. \]
Combining estimates (4.65), (4.66) and (4.67), we obtain
\[
\int_{t_1}^{t_2} \|F\|_{L^\infty} dt \leq C(t_2 - t_1)^{1/2} T_0^{1/2} \\
\leq \eta(t_2 - t_1) + C_\eta \epsilon_0,
\]
where \(\eta > 0\) is a small enough positive number. With estimate (4.68) at hand, we can mimic the Navier-Stokes case to obtain
\[
\frac{3}{4} c_0 < \rho(t, x) < \frac{3}{2} c_0^{-1},
\]
where \(0 \leq t \leq T_0\). With estimate (4.56) and (4.69), we can complete the proof by continuity argument. \(\square\)

5. Blow up Criterion and the Global Well-Posedness

In this part, we give a blow up criterion and then prove the local solution can be extended to a global one. Firstly, let me give the blow up criterion as follows.

**Theorem 5.1.** For dimension \(N = 3\), let \((\rho, u, B)\) be a solution of system (1.1) satisfying
\[
\rho(0) > 0, \quad \rho - \bar{\rho} \in C([0, T]; H^2), \\
u, B \in C([0, T]; H^2) \cap L^2(0, T; H^3).
\]
Let \(T^*\) be the maximal existence time of the solution. If \(T^* < +\infty\), then it is necessary that
\[
\limsup_{t \uparrow T^*} (\|\rho(t)\|_{L^\infty} + \|u(t)\|_{L^q} + \|B(t)\|_{L^q}) = +\infty,
\]
for any \(q \geq 6\).

**Proof.** We use the contradiction argument. Assume that \(T^* < +\infty\) and
\[
\sup_{t \in [0, T^*)} (\|\rho(t)\|_{L^\infty} + \|u(t)\|_{L^q} + \|B(t)\|_{L^q}) = M < +\infty.
\]
In what follows, we denote \(C\) to be a constant depending on \(T, M, \|u_0\|_{H^2}, \|B_0\|_{H^2}, \|\rho_0 - \bar{\rho}\|_{H^2}\). Firstly, from the energy estimates, we have
\[
\int_{\mathbb{R}^3} |\rho - \bar{\rho}|^2 + \rho|u|^2 + |B|^2 \, dx + \int_0^T \int_{\mathbb{R}^3} |\nabla u|^2 + |\nabla B|^2 \, dx dt \leq C. \tag{5.2}
\]
Considering both (5.1) and (5.2), for \(r \in [2, \infty]\), we have
\[
\|\rho - \bar{\rho}\|_{L^r_{x,t}((L^r)^*)} + \|\sqrt{\rho} u\|_{L^r_{x,t}(L^2)} + \|B\|_{L^r_{x,t}(L^2)} \leq C. \tag{5.3}
\]
Let \(v = L^{-1} \nabla P(\rho)\) to be a solution of the following elliptic system
\[
Lv := \mu \Delta v + \lambda \nabla \text{div} v = \nabla P(\rho). \tag{5.4}
\]
By elliptic estimate, for \(r \in [2, \infty]\), we can obtain
\[
\|\nabla v\|_{L^r} \leq C \|P(\rho) - P(\bar{\rho})\|_{L^r} \leq C, \quad \|\nabla^2 v\|_{L^r} \leq C \|\nabla \rho\|_{L^r}. \tag{5.5}
\]
Now, we introduce a new unknown \(w = u - v\). We can easily know that
\[
(\lambda + \mu) \Delta \text{div} w = (\lambda + \mu) \Delta u - \Delta (P(\rho) - P(\bar{\rho})),
\]
As in the Navier-Stokes equations \[32, 33\], we can easily obtain
\[
\rho \frac{\partial w}{\partial t} - \mu \Delta w + \lambda \nabla \text{div} w = \rho F + B \cdot \nabla B - \frac{1}{2} \nabla (|B|^2),
\]
where
\[
F = -u \cdot \nabla u + L^{-1} \text{div}(P(\rho)u) + L^{-1} \nabla ((\rho P'(\rho) - P(\rho))\text{div} u).
\]

Multiply (5.6) with \(\partial_t w\) and integrating by parts, we have
\[
\partial_t \int_{\mathbb{R}^3} \mu |\nabla w|^2 + \lambda |\text{div} w|^2 \, dx + \frac{1}{2} \int_{\mathbb{R}^3} \rho |\partial_t w|^2 \, dx \\
\leq C \|\sqrt{\rho F}\|^2_{L^2} + C \|B \cdot \nabla B\|^2_{L^2}.
\]
For the first term on the right hand side of the above inequality, we can estimate as in the Navier-Stokes equations \[32, 33\] to obtain
\[
\|\sqrt{\rho F}\|_{L^2} \leq C (1 + \|\nabla u\|_{L^2}^2) + \epsilon \|\sqrt{\rho \partial_t w}\|_{L^2},
\]
where \(\epsilon\) is a small enough positive number. Next, we need to estimate \(\|B \cdot \nabla B\|_{L^2},\)
\(\|u \cdot \nabla B\|_{L^2},\)
\(\|B \cdot \nabla u\|_{L^2}\) and \(\|u \cdot \nabla u\|_{L^2}\) to make our later estimate more clear. For the term \(\|B \cdot \nabla B\|_{L^2}\), we have
\[
\|B \cdot \nabla B\|_{L^2} \leq C \|B\|_{L^2} \|\nabla B\|_{L^2} + \epsilon \|\partial_t B\|_{L^2} + \epsilon \| \nabla \nabla B\|_{L^2} + \epsilon \|u \cdot \nabla B\|_{L^2} + \epsilon \|B \cdot \nabla u\|_{L^2}.
\]
Similarly, we have
\[
\|u \cdot \nabla B\|_{L^2} \leq C \|\nabla B\|_{L^2} + \epsilon \|\partial_t B\|_{L^2} + \epsilon \|u \cdot u\|_{L^2} + \epsilon \|B \cdot \nabla u\|_{L^2}.
\]
As in the Navier-Stokes equations \[32, 33\], we can easily obtain
\[
\|B \cdot \nabla u\|_{L^2} \leq C (1 + \|\nabla u\|_{L^2}^2) + \epsilon \|\sqrt{\rho \partial_t w}\|_{L^2} \\
\|u \cdot \nabla u\|_{L^2} \leq C (1 + \|\nabla u\|_{L^2}^2) + \epsilon \|\sqrt{\rho \partial_t w}\|_{L^2}.
\]
So summing up the above four estimates, we obtain
\[
\|B \cdot \nabla B\|_{L^2} + \|B \cdot \nabla u\|_{L^2} + \|u \cdot \nabla B\|_{L^2} + \|u \cdot \nabla u\|_{L^2} \\
\leq C (1 + \|\nabla u\|_{L^2}^2 + \|\nabla B\|_{L^2}^2) + \epsilon \|\sqrt{\rho \partial_t w}\|_{L^2} + \epsilon \|\partial_t B\|_{L^2}.
\]
Substitute (5.9) and (5.10) into (5.8), we will have
\[
\partial_t \int_{\mathbb{R}^3} \mu |\nabla w|^2 + \lambda |\text{div} w|^2 \, dx + \int_{\mathbb{R}^3} \rho |\partial_t w|^2 \, dx \\
\leq C (1 + \|\nabla u\|_{L^2}^2 + \|\nabla B\|_{L^2}^2) + \epsilon \|\sqrt{\rho \partial_t w}\|_{L^2} + \epsilon \|\partial_t B\|_{L^2}^2.
\]
Multiply \(\partial_t B\) to the third equation of system \[1.1\] and integrate by parts, we will obtain
\[
\partial_t \int_{\mathbb{R}^3} |\nabla B|^2 \, dx + \int_{\mathbb{R}^3} |\partial_t B|^2 \, dx \\
\leq C (1 + \|\nabla u\|_{L^2}^2 + \|\nabla B\|_{L^2}^2) + \epsilon \|\sqrt{\rho \partial_t w}\|_{L^2} + \epsilon \|\partial_t B\|_{L^2}^2,
\]
where we used estimate (5.10). If $\varepsilon$ is small enough, summing up (5.11) and (5.12), we will have
\[
\partial_t \int_{\mathbb{R}^3} \mu |\nabla w|^2 + \lambda |\text{div} w|^2 + |\nabla B|^2 \, dx + \int_{\mathbb{R}^3} \rho |\partial_t w|^2 \, dx
\]
\[
+ \int_{\mathbb{R}^3} |\partial_t B|^2 \, dx \leq C (1 + \|\nabla u\|_{L^2}^2 + \|\nabla B\|_{L^2}^2).
\]
Integrate the above inequality for time variable, we will obtain
\[
\|\nabla w\|_{L^\infty_T(L^2)} + \|\nabla B\|_{L^\infty_T(L^2)} + \|\sqrt{\tau} \partial_t w\|_{L^2_T(L^\infty)}
\]
\[
+ \|\partial_t B\|_{L^2_T(L^2)} + \|\nabla^2 w\|_{L^2_T(L^2)} \leq C.
\]
The above inequality combined with (5.5) implies that
\[
\|\nabla u\|_{L^\infty_T(L^2)} + \|\nabla u\|_{L^2_T(L^r)} \leq C \quad \text{for} \quad r \in [2, 6].
\]
For the term $\|\nabla^2 B\|_{L^2_T(L^2)}$, we will have
\[
\|\nabla^2 B\|_{L^2_T(L^2)} \leq C \|\partial_t B\|_{L^2_T(L^2)} + \|u \cdot \nabla B\|_{L^2_T(L^2)} + \|B \cdot \nabla u\|_{L^2_T(L^2)}
\]
\[
\leq C < +\infty,
\]
where we used (5.10) and (5.13). Hence, we have $\|\nabla B\|_{L^2_T(L^2)} \leq C < +\infty$ by interpolation. Now let us turn to the high order energy estimate. From the $H^2$ energy estimate (Lemma 4.4), we have
\[
\int_{\mathbb{R}^3} \sigma(t) (\rho |\dot{u}|^2 + |B_t|^2) \, dx + \int_0^T \int_{\mathbb{R}^3} \sigma(t) (|\nabla \dot{u}|^2 + |\nabla B_t|^2) \, dx dt
\]
\[
\leq CC_0 + C \int_0^T \int_{\mathbb{R}^3} \sigma(t) |\nabla u|^4 \, dx dt + C \int_0^T \int_{\mathbb{R}^3} |\nabla u|^3 \, dx dt
\]
\[
+ C \int_0^T \int_{\mathbb{R}^3} (|\nabla B|^2 |B|^2 + |\nabla B|^2 |u|^2 + |\nabla u|^2 |B|^2) \, dx dt
\]
\[
+ C \int_0^T \int_{\mathbb{R}^3} \sigma(t) (|u|^2 + |B|^2) (|B_t|^2 + |\dot{u}|^2 + |u|^2 (|\nabla B|^2 + |\nabla u|^2)) \, dx dt.
\]
Noting that
\[
\mu \Delta w + \lambda \nabla \text{div} w = \rho \dot{u} - B \cdot \nabla B + \frac{1}{2} \nabla (|B|^2),
\]
elliptic estimate yields that
\[
\|\nabla^2 w\|_{L^2} \leq C \|\rho \dot{u}\|_{L^2} + C \|B \cdot \nabla B\|_{L^2}.
\]
From (5.10) and (5.13), we obtain
\[
\int_0^T \int_{\mathbb{R}^3} |\nabla B|^2 |B|^2 + |\nabla B|^2 |u|^2 + |\nabla u|^2 |B|^2 \, dx dt \leq C < +\infty.
\]
Using same methods as for Navier-Stokes system \[6\], we have

\[
\int_0^T \int_{\mathbb{R}^3} \sigma(t)|B|^2|u|^2|\nabla B|^2 \, dx \, dt \\
\leq \int_0^T \left( \int_{\mathbb{R}^3} |B|^3|u|^3 \, dx \right)^{2/3} \left( \int_{\mathbb{R}^3} |\nabla B|^6 \, dx \right)^{1/3} \, dt \\
\leq \sup_{0 \leq t \leq T} \left( \int_{\mathbb{R}^3} |B|^3|u|^3 \, dx \right)^{2/3} \int_0^T |\nabla B|_{L^6}^2 \, dt \\
\leq C\|B\|_{L^2(\mathbb{R}^3)}^2 \|u\|_{L^6(\mathbb{R}^3)}^2 \|\nabla B\|_{L^6(\mathbb{R}^3)}^{1/2} \leq C < +\infty,
\]

we can use similar methods to obtain

\[
\int_0^T \int_{\mathbb{R}^3} \sigma(t)(|u|^2 + |B|^2)|u|^2(|\nabla B|^2 + |\nabla u|^2) \, dx \, dt \leq C < +\infty.
\]

Similar to the Navier-Stokes system \[6\], we can obtain

\[
\|\nabla u\|_{L^4} \leq C\|\nabla u\|_{L^6}(1 + \|\sqrt{\rho} \hat{u}\|_{L^3}), \quad \|\nabla u\|_{L^4} \leq C\|\nabla u\|_{L^6}^{3/2}.
\]

Plugging \((5.19), (5.21)\) and \((5.22)\) into \((5.16)\) and noting \(\|\nabla u(t)\|_{L^6} \in L^2(0, T)\) by \((5.14)\) and \((5.15)\), we deduce by Gronwall's inequality that

\[
\int_{\mathbb{R}^3} \sigma(t)(\rho|\hat{u}|^2 + |B_t|^2) \, dx + \int_0^T \int_{\mathbb{R}^3} \sigma(t)(|\nabla \hat{u}|^2 + |\nabla B_t|^2) \, dx \, dt \leq C.
\]

From the above inequality, elliptic estimate \((5.18)\) and Sobolev inequality, we have

\[
\|\nabla^2 u\|_{L^4(L^r)} \leq C, \quad \text{for} \quad r \in [2, 6],
\]

Using same methods as for Navier-Stokes system \[6\], we have

\[
\|\nabla^2 u\|_{L^4(L^r)} \leq C, \quad \|\nabla \rho\|_{L^r} \leq C, \quad \text{for} \quad r \in [2, 6].
\]

Since

\[
\|\nabla^2 B\|_{L^2} \leq C(\|B_t\|_{L^2} + \|u\|_{L^\infty} \|\nabla B\|_{L^2} + \|B\|_{L^\infty} \|\nabla u\|_{L^2}) \\
\leq C + \frac{1}{2}\|\nabla^2 u\|_{L^2} + \frac{1}{4}\|\nabla^2 B\|_{L^2},
\]

and

\[
\|\nabla^2 u\|_{L^2} \leq C\|\nabla^2 u\|_{L^2} + C\|\nabla^2 v\|_{L^2} \\
\leq C(\|\rho \hat{u}\|_{L^2} + \|B\|_{L^\infty} \|\nabla B\|_{L^2} + \|\nabla \rho\|_{L^2}) \\
\leq C + \frac{1}{4}\|\nabla^2 B\|_{L^2},
\]

we have

\[
\|\nabla^2 u\|_{L^2} + \|\nabla^2 B\|_{L^2} \leq C.
\]

By the above estimates, we easily know \(\|u\|_{L^\infty} + \|B\|_{L^\infty} \leq C\) by interpolation, so we will obtain

\[
\|\nabla^3 B\|_{L^4(L^2)} \leq C(\|\nabla B_t\|_{L^4(L^2)} + \|u\|_{L^\infty} \|\nabla^2 B\|_{L^4(L^2)} \\
+ \|B\|_{L^\infty} \|\nabla^2 u\|_{L^4(L^2)} + \|\nabla u \nabla B\|_{L^4(L^2)}) \leq C,
\]
where we also used
\[
\| \nabla u \nabla B \|_{L^2_t(L^2)} \leq \left( \int_0^T \left( \int_{\mathbb{R}^3} |\nabla u|^4 \, dx \right)^{1/2} \left( \int_{\mathbb{R}^3} |\nabla B|^4 \, dx \right)^{1/2} \, dt \right)^{1/2} \\
\leq C \| \nabla u \|_{L^\infty_t(L^4)} \| \nabla u \|_{L^2_t(L^6)} + C \| \nabla B \|_{L^\infty_t(L^4)} \| \nabla B \|_{L^2_t(L^4)} \\
\leq C < +\infty.
\]

Then, similar to Navier-Stokes system, we can get
\[
\int_0^T \| \nabla^3 w \|^2_{L^2} \, dt \leq C.
\]

From (5.29), (5.30), (5.27) and (5.28), we will obtain
\[
\| \nabla^2 u(t) \|^2_{L^2} + \| \nabla^2 B(t) \|^2_{L^2} + \int_0^T \| \nabla^3 u \|^2_{L^2} + \| \nabla^3 B \|^2_{L^2} \, dt \\
\leq C + C \int_0^T \| \nabla^2 \rho(t) \|^2_{L^2} \, dt.
\]

Form the continuity equation, we have
\[
dt \| \nabla^2 (\rho(t) - \tilde{\rho}) \|^2_{L^2} \leq C(1 + \| \nabla u(t) \|_{L^\infty}) \| \nabla^2 (\rho(t) - \tilde{\rho}) \|^2_{L^2} \\
+ C \| \nabla^3 u(t) \|^2_{L^2}.
\]

Summing up (5.29) and (5.30), we conclude by Gronwall’s inequality that for \(0 \leq t < T^*\),
\[
\| \rho(t) - \tilde{\rho} \|_{H^2} + \| u(t) \|_{H^2} + \| B(t) \|_{H^2} \leq C.
\]

This ensures that the solution can be continued after \(t = T^*\). \(\square\)

After we get the blow up criterion, we can give the proof about Theorem 2.6. From Theorem 3.1, Proposition 3.5 and Proposition 3.6, we obtain a solution \((\rho, u, B)\) of (1.1) satisfying
\[
\frac{c_0}{2} \leq \rho \leq 2\rho_0^{-1}, \quad \rho - \tilde{\rho} \in \dot{L}^\infty_T(\dot{B}^{3/p}_{p,1} \cap \dot{B}^{5}_{2,2}), \\
\rho, B \in \dot{L}^\infty_T(\dot{B}^{3/p}_{p,1} \cap \dot{B}^{5}_{2,2}) \cap \dot{L}^1_T(\dot{B}^{3/p+1}_{p,1} \cap \dot{B}^{5}_{2,2}).
\]

Moreover, it holds that
\[
\| \rho - \tilde{\rho} \|_{L^\infty_T(\dot{B}^{5}_{2,2})} \leq C(\| \rho_0 - \tilde{\rho} \|_{H^{1-\delta}} + \| u_0 \|_{H^{1-\delta}} + \| B_0 \|_{H^{3/2}}),
\]

\[
\|(u, B)\|_{L^\infty_T(\dot{B}^{3/p+1}_{p,1})} + \|(u, B)\|_{L^1_T(\dot{B}^{3/p+1}_{5/2,2})} \\
\leq C(1 + \| a_0 \|_{\dot{B}^{3/p}_{p,1}})(\| u_0 \|_{H^{1-\delta}} + \| B_0 \|_{H^{1-\delta}} + TP_+ \| \rho_0 - \tilde{\rho} \|_{H^{1-\delta}}),
\]

\[
\| a \|_{L^\infty_T(\dot{B}^{3/p}_{p,1})} \leq 2 \| a_0 \|_{\dot{B}^{3/p}_{p,1}} + \frac{c_2 + c_3}{(1 + \| a_0 \|_{\dot{B}^{3/p}_{p,1}})^{5/4}},
\]

\[
\| u \|_{L^\infty_T(\dot{B}^{3/p+1}_{p,1})} + \| u \|_{L^1_T(\dot{B}^{3/p+1}_{5/2,2})} \leq \frac{c_2 + c_3}{(1 + \| a_0 \|_{\dot{B}^{3/p}_{p,1}})^{5/4}},
\]

\[
\| B \|_{L^\infty_T(\dot{B}^{3/p+1}_{p,1})} + \| B \|_{L^1_T(\dot{B}^{3/p+1}_{5/2,2})} \leq \frac{2c_3}{(1 + \| a_0 \|_{\dot{B}^{3/p}_{p,1}})^{2}}.
\]
Then taking (5.41)

where (5.36)

Taking $c_2$ small enough, by Remark 5.2 and (3.24), the existence time $T$ has a lower bound

$T \geq \frac{c}{(1 + \|a_0\|_{\mathcal{B}^{3/p}})^4(1 + \|a_0\|_{H^2})^{1/2}}$

where $c$ is a small positive number.

Now, for any $T_1 \leq T$, we have

$\|(u, B)\|_{L_t^\infty (\mathcal{B}^{-\delta}_{2,1})} + \|(u, B)\|_{L_t^1 (\mathcal{B}^{2-\delta}_{2,1})}$

$\leq C(1 + \|a_0\|_{\mathcal{B}^{3/p}})(\bar{c}_2 + \bar{c}_3 + P_+ T_1 \|\rho_0 - \bar{\rho}\|_{H^{3-\delta}})$.

by the conditions on the solution just mentioned. For $r \in (1, 2 - \delta)$, we have

$\|\nabla u\|_{L_t^1(L^2)} \leq \sum_{j=-\infty}^{+\infty} 2^j \|\Delta_j u\|_{L_t^1(L^2)}$

$\leq \sum_{j \leq 0} 2^{j-1/2} \|\Delta_j u\|_{L_t^\infty(L^2)} + \sum_{j > 0} 2^j \|\Delta_j u\|_{L_t^1(L^2)}^{1/r} \|\nabla \Delta_j u\|_{L_t^1(L^2)}^{1/2r}$

$\leq C(\|u\|_{L_t^\infty(\mathcal{B}^{-\delta}_{2,2})} + \|u\|_{L_t^1(\mathcal{B}^{2-\delta}_{2,2})})$

$\leq C(1 + \|a_0\|_{\mathcal{B}^{3/p}})(\bar{c}_4 + \bar{c}_5 + P_+ T_1 \|\rho_0 - \bar{\rho}\|_{H^2})$.

Similarly, we have

$\|u\|_{L_t^1(L^2)} \leq C(1 + \|a_0\|_{\mathcal{B}^{3/p}})(\bar{c}_4 + \bar{c}_5 + P_+ T_1 \|\rho_0 - \bar{\rho}\|_{H^2})$,

$\|\nabla B\|_{L_t^1(L^2)} \leq C(1 + \|a_0\|_{\mathcal{B}^{3/p}})(\bar{c}_4 + \bar{c}_5 + P_+ T_1 \|\rho_0 - \bar{\rho}\|_{H^2})$,

and

$\|B\|_{L_t^1(L^2)} \leq C(1 + \|a_0\|_{\mathcal{B}^{3/p}})(\bar{c}_4 + \bar{c}_5 + P_+ T_1 \|\rho_0 - \bar{\rho}\|_{H^2})$.

Then taking $r = \frac{3}{2}$, we have

$\|(u, B)\|_{L_t^{3/2}(H^1)} \leq C(1 + \|a_0\|_{\mathcal{B}^{3/p}})(\bar{c}_4 + \bar{c}_5 + P_+ T_1 \|\rho_0 - \bar{\rho}\|_{H^2})$.

Hence, there exist $t_0 \in (0, T_1)$ such that

$\|(u, B)(t_0)\|_{H^1} \leq C(1 + \|a_0\|_{\mathcal{B}^{3/p}}) \left( \frac{\bar{c}_4 + \bar{c}_5}{T_1^{2/3}} + P_+ T_1^{1/3} \|\rho_0 - \bar{\rho}\|_{H^2} \right)$.

For density, using simply energy estimates, we will get

$\|\rho(t) - \bar{\rho}\|_{L^2} \leq C(1 + P_+)(c_1 + T_1^{1/2}(\|u_0\|_{L^2} + \|B_0\|_{L^2}))$. 
Take $T_1$ as

$$T_1 = \frac{c_0^3}{(5C)^3(1 + P_+)^3(1 + \|\rho_0 - \bar{\rho}\|_{H^2}^2 + \|(u_0, B_0)\|_{L^2}^2)(1 + \|a_0\|_{B^{3/2}_{p,1}}^p)^4},$$

such that $T_1 \leq T$ and

$$C(1 + \|a_0\|_{B^{3/2}_{p,1}}^p)_{T_1^{1/3}} \|\rho_0 - \bar{\rho}\|_{H^2} \leq \frac{\epsilon_0}{5},$$

$$C(1 + P_+) T_1^{1/2} (\|u_0\|_{L^2} + \|B_0\|_{L^2}) \leq \frac{\epsilon_0}{5}.$$

Then, we choose $c_1$, $c_2$ and $c_3$ small enough so that

$$C(1 + P_+) c_1 \leq \frac{\epsilon_0}{5}, \quad \frac{C(1 + \|a_0\|_{B^{3/2}_{p,1}}^p) c_2}{T_1^{3/3}} \leq \frac{\epsilon_0}{5}, \quad \frac{C(1 + \|a_0\|_{B^{3/2}_{p,1}}^p) c_3}{T_1^{3/3}} \leq \frac{\epsilon_0}{5}.$$

Through this choice of $c_1$, $c_2$ and $c_3$, by (5.33) and (5.44), we have

$$\|a(t_0)\|_{L^2} + \|u(t_0)\|_{H^1} + \|B(t_0)\|_{H^1} \leq \epsilon_0.$$

Theorem 5.1 implies that

$$\frac{\epsilon_0}{2} \leq \rho \leq 2\epsilon_0^{-1}, \quad \|u(t)\|_{L^\infty} + \|u(t)\|_{L^\infty} \leq C.$$

So the solution can be extended to a global one by Theorem 5.1.

6. Differential Type Estimates for Linear System

In this section, we want to get a differential type estimates for linear hyperbolic-parabolic system related to MHD system. Firstly, let us state how to get a related linear hyperbolic-parabolic system. Without loss of generality, we can assume $P'(1) = 1$ in the following three sections. If we suppose the equilibrium state of the magnetic field $B = I$, the equilibrium state of the density $\bar{\rho} = 1$ and denote $H = B - I$, $a = \rho - 1$, the MHD system (1.1) can be rewritten as

$$\begin{align*}
\partial_t a + u \cdot \nabla a + \text{div} u &= -a \text{div} u, \\
\partial_t u - \mu \Delta u - \mu' \nabla \text{div} u + \nabla a + \nabla (I \cdot H) - I \cdot \nabla H &= M_u, \\
\partial_t H - \nu \Delta H + (\text{div} u) I - I \cdot \nabla u &= -u \cdot \nabla H - (\text{div} u) H + H \cdot \nabla u, \\
\text{div} H &= 0.
\end{align*}$$

(6.1)

where $\mu' = \lambda + \mu$ and

$$M_u = -\frac{1}{2(1 + a)} \nabla (|H|^2) - \frac{a}{1 + a} I \cdot \nabla H + \frac{1}{1 + a} H \cdot \nabla H$$

$$+ \frac{a}{1 + a} \nabla (I \cdot H) - u \cdot \nabla u + \left( \frac{P'(1 + a)}{1 + a} - P'(1) \right) \nabla a$$

$$- \mu \frac{a}{1 + a} \Delta u - \mu' \frac{a}{1 + a} \nabla \text{div} u.$$

Denote $\omega = \Lambda^{-1} \text{div} u$, $\Omega = \Lambda^{-1} \text{curl} u$ and $E = \Lambda^{-1} \text{curl} H$ where $\Lambda f = \mathcal{F}[\xi^* \mathcal{F} f]$, $\text{curl} u = (\partial_j u^i - \partial_i u^j)_{ij}$ is a $N \times N$ matrix. With these notation, system (6.1) can
Proposition 6.1. For the above system, we can prove the following proposition.

\begin{equation}
\begin{aligned}
\partial_t a + u \cdot \nabla a + \Lambda \omega &= F, \\
\partial_t \omega + u \cdot \nabla \omega - (\mu + \mu') \Delta \omega - \Lambda a + I \cdot \nabla E &= G, \\
\partial_t \Omega - \mu \Delta \Omega - I \cdot \nabla E &= J, \\
\partial_t E - \nu \Delta E + \text{curl}(\omega I) - I \cdot \nabla \Omega &= K, \\
u &= -\Lambda^{-1} \nabla \omega - \Lambda^{-1} \text{div} \Omega, & B = -\Lambda^{-1} \text{div} E, & \text{div} B = 0,
\end{aligned}
\end{equation}

where \( \text{div} E = \sum_{i=1}^{N} \partial_i E_{ij} \) with entries \( E_{ij} \) of the matrix \( E \), and

\( F = -a \text{div} u, \)

\( G = \Lambda^{-1} \text{div} \left( -\frac{1}{2} \frac{1}{1+a} \nabla(|H|^2) + h(a) \nabla(I \cdot H) - h(a) I \cdot \nabla H \right) \)

\( + \frac{1}{1+a} H \cdot \nabla H + f(a) \nabla a - \mu h(a) \Delta u - \mu' h(a) \nabla \text{div} u - u \cdot \nabla u \),

\( J = \Lambda^{-1} \text{curl} \left( -\frac{1}{2} \frac{1}{1+a} \nabla(|H|^2) + h(a) \nabla(I \cdot H) - h(a) I \cdot \nabla H \right) \)

\( + \frac{1}{1+a} H \cdot \nabla H + f(a) \nabla a - \mu h(a) \Delta u - \mu' h(a) \nabla \text{div} u - u \cdot \nabla u \),

\( K = \Lambda^{-1} \text{curl}(-u \cdot \nabla H - (\text{div} u)H + H \cdot \nabla u). \)

At this stage, we can give the linearized system with convection terms as follows

\begin{equation}
\begin{aligned}
\partial_t a + v \cdot \nabla a + \Lambda \omega &= F, \\
\partial_t \omega + v \cdot \nabla \omega - (\mu + \mu') \Delta \omega - \Lambda a - I \cdot \nabla E &= G, \\
\partial_t \Omega - \mu \Delta \Omega - I \cdot \nabla E &= J, \\
\partial_t E - \nu \Delta E + \text{curl}(\omega I) - I \cdot \nabla \Omega &= K, \\
(a, \omega, \Omega, E)|_{t=0} &= (a_0, \omega_0, \Omega_0, E_0).
\end{aligned}
\end{equation}

For the above system, we can prove the following proposition.

**Proposition 6.1.** Let \((a, \omega, \Omega, E)\) be a solution of system (6.3) on \([0, T)\) for \( T > 0 \), \( 2 - \frac{N}{2} < s \leq 1 + N/2 \), \( V(t) = \int_0^t \|v(\tau)\|_{B^{N/2+1}} d\tau \). Then there exists a functional

\( \mathcal{E}_s(a, \omega, \Omega, E) \approx \|a(t)\|_{B^{s-2,s}} + \|\omega(t)\|_{B^{s-2,s-1}} + \|\Omega(t)\|_{B^{s-2,s-1}} + \|E(t)\|_{B^{s-2,s-1}}, \)

such that

\[
\frac{d}{dt} \left( e^{-CV(t)} \mathcal{E}_s(a, \omega, \Omega, E) \right) + ce^{-CV(t)} (\|a\|_{B^{s-2,s}} + \|\omega\|_{B^{s,s+1}} + \|\Omega\|_{B^{s,s+1}} + \|E\|_{B^{s,s+1}}) \leq C e^{-CV(t)} (\|F\|_{B^{s-2,s}} + \|G\|_{B^{s-2,s-1}} + \|J\|_{B^{s-2,s-1}} + \|K\|_{B^{s-2,s-1}}),
\]

where \( c \) is a small positive number and \( C \) depends on \( \mu, \lambda, s \).

**Proof.** Let \((a, \omega, \Omega, E)\) be a solution of the system (6.3) and \( \tilde{f} := e^{-\gamma V(t)} f \) for \( f = a, \omega, \Omega, E \). Applying the operator \( \Delta_k \) to both sides of system (6.3) and using
the new variable $(\tilde{a}, \tilde{\omega}, \tilde{\Omega}, \tilde{E})$, the system (6.3) can be transformed into

\[
\begin{aligned}
\partial_t \Delta_k \tilde{a} + \Delta_k (v \cdot \nabla \tilde{a}) + \Lambda \Delta_k \tilde{\omega} &= \Lambda_k \tilde{F} - \gamma V'(t) \Delta_k \tilde{a}, \\
\partial_t \Delta_k \tilde{\omega} + \Delta_k (v \cdot \nabla \tilde{\omega}) - (\mu + \mu') \Delta_k \tilde{\omega} &= -\Delta \Delta_k \tilde{a} - I \cdot \text{div} \Delta_k \tilde{E} = \Delta_k \tilde{G} - \gamma V'(t) \Delta_k \tilde{\omega}, \\
\partial_t \Delta_k \tilde{\Omega} - \mu \Delta \Delta_k \tilde{\Omega} - I \cdot \nabla \Delta_k \tilde{E} &= \Delta_k \tilde{J} - \gamma V'(t) \Delta_k \tilde{\Omega}, \\
\partial_t \Delta_k \tilde{E} - \nu \Delta \Delta_k \tilde{E} + \text{curl}(\Delta_k \tilde{\omega} I) - I \cdot \nabla \Delta_k \tilde{\Omega} &= \Delta_k \tilde{K} - \gamma V'(t) \Delta_k \tilde{E}.
\end{aligned}
\]

(6.4)

Take $0 < \beta < \min(\frac{2\mu + \lambda}{\lambda}, \nu)$, and denote

\[
\begin{aligned}
\alpha_k^2 &:= ||\Delta_k a||_{L^2}^2 + ||\Delta_k \omega||_{L^2}^2 + \frac{1}{2} ||\Delta_k \Omega||_{L^2}^2 + \frac{1}{2} ||\Delta_k E||_{L^2}^2 \\
&+ \beta (2 \mu + \lambda) ||\Delta \Delta_k a||_{L^2}^2 - 2 \beta |\Delta_k a| ||\Delta_k \omega||,
\end{aligned}
\]

then

\[
\alpha_k^2 \approx ||\Delta_k a||_{L^2}^2 + ||\Delta \Delta_k a||_{L^2}^2 + ||\Delta_k \omega||_{L^2}^2 + ||\Delta_k \Omega||_{L^2}^2 + ||\Delta_k E||_{L^2}^2.
\]

Using same methods in [19], for a small positive constant $\delta > 0$, we can obtain

\[
\begin{aligned}
\frac{d}{dt} \alpha_k(t) + (\delta 2^{k} + \gamma V'(t)) \alpha_k(t) &\lesssim ||\Delta_k \tilde{F}||_{L^2} + ||\Lambda \Delta_k \tilde{F}||_{L^2} + ||\Delta_k \tilde{G}||_{L^2} \\
&+ ||\Delta_k \tilde{J}||_{L^2} + ||\Delta_k \tilde{K}||_{L^2} + \kappa 2^{-k(s-1)} V'(t) ||(\tilde{a}, \tilde{\omega})||_{B^{-2,s-1} \times B^{s-2}}.
\end{aligned}
\]

(6.5)

Since $2 - \frac{N}{2} < s \leq 1 + \frac{N}{2}$, using Lemma 9.7, we have

\[
\begin{aligned}
(\Delta_k (v \cdot \nabla \tilde{a}) | \Delta_k \tilde{a}) &\leq C \kappa 2^{-k(s-2)} \alpha_k \|v\|_{B^{N/2+1}_{s} ||\tilde{a}||_{B^{s-2}},} \\
(\Delta_k (v \cdot \nabla \tilde{\omega}) | \Delta_k \tilde{\omega}) &\leq C \kappa 2^{-k(s-2)} \alpha_k \|v\|_{B^{N/2+1}_{s} ||\tilde{\omega}||_{B^{s-2}},} \\
(\Delta_k (v \cdot \nabla \tilde{\omega}) | \Lambda \Delta_k \tilde{k}) &\leq C \kappa 2^{-k(s-2)} \alpha_k \|v\|_{B^{N/2+1}_{s} ||\tilde{k}||_{B^{s-2},}} \\
(\Delta_k (v \cdot \nabla \tilde{a}) | \Lambda \Delta_k \tilde{\omega}) + (\Delta_k (v \cdot \nabla \omega) | \Lambda \Delta_k \tilde{a}) &\leq C \kappa 2^{-k(s-2)} ||\tilde{\omega}||_{B^{s-2}} + \kappa \|\tilde{a}\|_{B^{s-1}}. \quad (6.6)
\end{aligned}
\]

Combining the methods used to derive Proposition 3.1 in [19] and the above estimates (6.6), we obtain

\[
\begin{aligned}
\frac{d}{dt} \alpha_k(t) + (\delta 2^{k} + \gamma V'(t)) \alpha_k(t) &\lesssim ||\Delta_k \tilde{F}||_{L^2} + ||\Lambda \Delta_k \tilde{F}||_{L^2} + ||\Delta_k \tilde{G}||_{L^2} \\
&+ ||\Delta_k \tilde{J}||_{L^2} + ||\Delta_k \tilde{K}||_{L^2} + \kappa 2^{-k(s-2)} V'(t) ||(\tilde{a}, \tilde{\omega})||_{B^{-2,s-1} \times B^{s-2}}.
\end{aligned}
\]

(6.7)

Define

\[
E_s(\tilde{a}, \tilde{\omega}, \tilde{\Omega}, \tilde{E}) = \sum_{k \in \mathbb{Z}} 2^{k(s-1)} \alpha_k(t) + \sum_{k \in \mathbb{Z}} 2^{k(s-2)} \alpha_k(t),
\]

then we easily know that

\[
E_s(\tilde{a}, \tilde{\omega}, \tilde{\Omega}, \tilde{E}) \approx ||\tilde{a}||_{B^{-2,s}} + ||\tilde{\omega}||_{B^{-2,s-1}} + ||\tilde{\Omega}||_{B^{-2,s-1}} + ||\tilde{E}||_{B^{s-2,s-1}}.
\]

Perform the following calculation

\[
\sum_{k \in \mathbb{Z}} 2^{k(s-1)} \text{(6.5)} + \sum_{k \in \mathbb{Z}} 2^{k(s-2)} \text{(6.7)},
\]
then we will obtain
\[
\frac{d}{dt} \mathcal{E}_s(\tilde{a}, \tilde{\omega}, \tilde{\Omega}, \tilde{E}) + \delta(\|\tilde{a}\|_{B^{s,\infty}^s} + \|\tilde{\omega}\|_{B^{s,\infty}^s} + \|\tilde{\Omega}\|_{B^{s,\infty}^s} + \|\tilde{E}\|_{B^{s,\infty}^s}) \\
\lesssim \|\tilde{F}\|_{B^{s-2,s}} + \|\tilde{G}\|_{B^{s-2,s-1}} + \|\tilde{J}\|_{B^{s-2,s-1}} + \|\tilde{K}\|_{B^{s-2,s-1}} \\
+ V'(t)(\|\tilde{a}, \tilde{\omega}\|_{B^{s-2,s} \times B^{s-2,s-1}} - \gamma V'(t)(\|\tilde{a}, \tilde{\omega}\|_{B^{s-2,s} \times B^{s-2,s-1}}).
\]

If we take $\gamma$ large enough, we actually have
\[
\frac{d}{dt} \mathcal{E}_s(\tilde{a}, \tilde{\omega}, \tilde{\Omega}, \tilde{E}) + \delta(\|\tilde{a}\|_{B^{s,\infty}^s} + \|\tilde{\omega}\|_{B^{s,\infty}^s} + \|\tilde{\Omega}\|_{B^{s,\infty}^s} + \|\tilde{E}\|_{B^{s,\infty}^s}) \\
\lesssim \|\tilde{F}\|_{B^{s-2,s}} + \|\tilde{G}\|_{B^{s-2,s-1}} + \|\tilde{J}\|_{B^{s-2,s-1}} + \|\tilde{K}\|_{B^{s-2,s-1}}.
\]

Hence, we finally change the variables $(\tilde{a}, \tilde{\omega}, \tilde{\Omega}, \tilde{E})$ back to $(a, \omega, \Omega, E)$ then get our desired result.

If we take $\tilde{B} = 0$ instead of $\tilde{B} = 1$, denote $\omega = \Lambda^{-1} \div u$, $\Omega = \Lambda^{-1} \curl u$ and $E = \Lambda^{-1} \curl H$ again, the MHD system can be transformed to
\[
\begin{aligned}
\partial_t a + u \cdot \nabla a + \Lambda \omega &= L, \\
\partial_t \omega + u \cdot \nabla \omega - (2 \mu + \lambda) \Delta \omega - \Lambda a &= M, \\
\partial_t \Omega - \mu \Delta \Omega &= N, \\
\partial_t B - \nu \Delta B &= P, \\
u &= -\lambda^{-1} \nabla \omega - \Lambda^{-1} \div \Omega,
\end{aligned}
\tag{6.8}
\]
where
\[
L = -a \div u,
\]
\[
M = u \cdot \nabla \omega - \Lambda^{-1} \div \left( u \cdot \nabla u - h(a)(\mu \Delta u + \mu' \div \div u) - f(a) \nabla a \\
+ B \cdot \nabla B - h(a) B \cdot \nabla B - \frac{1}{2} \nabla(|B|^2) + \frac{1}{2} h(a) \nabla(|B|^2) \right),
\]
\[
N = u \cdot \nabla \omega - \Lambda^{-1} \curl \left( u \cdot \nabla u - h(a)(\mu \Delta u + \mu \div \div u) - f(a) \nabla a \\
+ B \cdot \nabla B - h(a) B \cdot \nabla B - \frac{1}{2} \nabla(|B|^2) + \frac{1}{2} h(a) \nabla(|B|^2) \right),
\]
\[
P = -\div (B u^T - u B^T).
\]
Hence, the linearized system has the following form
\[
\begin{aligned}
\partial_t a + v \cdot \nabla a + \Lambda \omega &= L, \\
\partial_t \omega + v \cdot \nabla \omega - (2 \mu + \lambda) \Delta \omega - \Lambda a &= M, \\
\partial_t \Omega - \mu \Delta \Omega &= N, \\
\partial_t B - \nu \Delta B &= P.
\end{aligned}
\tag{6.9}
\]
For the above linearized system, we have the following estimates.

**Proposition 6.2.** Let $(a, \omega, \Omega, B)$ be a solution of system (6.6) on $[0, T]$ for $T > 0$, $1 - \frac{N}{2} < s \leq 1 + N/2$, $V(t) = \int_0^t \|v(\tau)\|_{B^{N/2+1}} d\tau$ and $V(\infty)$ is bounded. Then there exists a functional
\[
\mathcal{E}_s(a, \omega, \Omega, B) \approx \|a(t)\|_{B^{s-1,s}} + \|\omega(t)\|_{B^{s-1,s}} + \|\Omega(t)\|_{B^{s-1,s}} + \|B(t)\|_{B^{s-1,s}},
\]
such that
\[
\frac{d}{dt} \left( e^{-CV(t)} a_k(a, \omega, \Omega, B) \right) + e^{-CV(t)} \left( \|a\|_{B^s} + \|\omega\|_{B^{s+1}} \right)
+ \|\Omega\|_{B^{s+1}} + \|B\|_{B^{s+1}} \leq C e^{-CV(t)} \left( \|L\|_{B^{-s+1}} + \|M\|_{B^{s+1}} \right)
+ \|N\|_{B^{-s+1}} + \|K\|_{B^{s+1}}),
\]
where \(c\) is a small positive number and \(C\) depends on \(\mu, s\).

**Proof.** From the linearized system (6.9), we see that it is similar to the Navier-Stokes system for the magnetic field has no couple with the velocity field. So this proposition can be proved using similar methods used in [24, 25]. Here, we just give a sketch of the proof. Let \((a, \omega, \Omega, B)\) be a solution of the system (6.9) and \(f := e^{-\gamma V(t)} f\) for \(f = a, \omega, \Omega, B\).

**Step 1 : low frequencies.** For \(k \leq k_0, k_0\) defined as in [25]. Define
\[
f_k^2 = \|\Delta_k \tilde{a}\|_{L^2}^2 + \|\Delta_k \tilde{\omega}\|_{L^2}^2 - \frac{1}{4}((\lambda + 2\mu)\Lambda \Delta_k \tilde{a} \Delta_k \tilde{\omega}),
\]
then we can prove
\[
\frac{1}{2} \frac{d}{dt} f_k + c(\lambda + 2\mu)2^{2k} f_k \leq C \|\Delta_k \tilde{L}\|_{L^2} + C \|\Delta_k \tilde{G}\|_{L^2} + C \kappa 2^{-k(s-1)}V'(t)(\|\tilde{a}\|_{B^{-s+1}} + \|\tilde{\omega}\|_{B^{s-1}}) - \gamma V'(t)f_k,
\]
where \(\sum_k \epsilon_k \leq 1\) and \(c\) is a small positive real number.

**Step 2 : high frequencies.** For \(k \geq k_0 + 1\), define
\[
f_k^2 = ((\lambda + 2\mu)\Lambda \Delta_k \tilde{a} \Delta_k \tilde{\omega})^2 + 2((\lambda + 2\mu)\Lambda \Delta_k \tilde{a} \Delta_k \tilde{\omega}),
\]
then we have
\[
\frac{1}{2} \frac{d}{dt} f_k + c(\lambda + 2\mu)^{-1} f_k \leq C \|\Delta_k \tilde{L}\|_{L^2} + C \|\Delta_k \tilde{M}\|_{L^2} + C \kappa 2^{-k(s-1)}V'(t)(\|\tilde{a}\|_{B^{-s+1}} + \|\tilde{\omega}\|_{B^{s-1}}) - \gamma V'(t)f_k,
\]
where \(\sum_k \epsilon_k \leq 1\) and \(c\) is a small positive real number.

**Step 3 : the damping effect.** If we take \(\gamma > 0\) large enough, we have
\[
\sum_{k \in \mathbb{Z}} \left( C \epsilon_k ((\tilde{a}, \tilde{\omega})\|_{B^{-s+1}} + \|\tilde{G}\|_{B^{s-1}}) - \gamma 2^{k(s-1)} f_k \right) \leq 0.
\]
Hence, from (6.10) and (6.11), we can obtain
\[
\frac{d}{dt} \left( \sum_{k \in \mathbb{Z}} 2^{k(s-1)} f_k \right) + c \sum_{k \in \mathbb{Z}} 2^{k(s-1)} \min(2^{2k}, (\lambda + 2\mu)^{-2}) \|\Delta_k \tilde{\omega}\|_{L^2} + c \|\tilde{a}(t)\|_{B^s} \leq C((\|\tilde{F}\|_{B^{-s+1}} + \|\tilde{G}\|_{B^{s-1}}).
\]

**Step 4 : the smoothing effect.** Denote \(g_k = \|\Delta_k \tilde{\omega}\|_{L^2}\), we then can obtain
\[
\frac{d}{dt} \left( \sum_{k \geq k_0 + 1} 2^{k(s-1)} g_k \right) + c \sum_{k \geq k_0 + 1} 2^{k(s-1)} \|\Delta_k \tilde{\omega}\|_{L^2}
\leq C \sum_{k \geq k_0 + 1} 2^{ks} \|\Delta_k \tilde{a}\|_{L^2} + C \|\tilde{G}\|_{B^{s-1}} + CV'(t)\|\tilde{\omega}\|_{B^{s-1}}.
For a small positive constant $\epsilon > 0$, performing the following calculations
\begin{equation}
(6.12) + \epsilon (6.13),
\end{equation}
we will obtain
\begin{equation}
\frac{d}{dt}e^{-\gamma V(t)} F_s(a, \omega) + c(\|\tilde{a}\|_{B^s} + \|\tilde{\omega}\|_{B^{s+1}}) 
\leq C\|\tilde{L}\|_{B^{s-1}} + C\|\tilde{M}\|_{B^{s-1}} + CV'(t)\|\tilde{\omega}\|_{B^{s-1}},
\end{equation}
where
\begin{equation}
F_s(a, \omega) = e^{\gamma V(t)} \left( \sum_{k \in \mathbb{Z}} 2^{k(s-1)} f_k + \epsilon \sum_{k \geq k_0 + 1} 2^{k(s-1)} g_k \right).
\end{equation}

Obviously, we know that
\begin{equation}
F_s(a, \omega) \approx \|a(t)\|_{B^{s-1}} + \|\omega(t)\|_{B^{s-1}}.
\end{equation}

Using the above identity, further more, we can obtain
\begin{equation}
\frac{d}{dt}e^{-\gamma V(t)} F_s(a, \omega) + c(\|\tilde{a}\|_{B^s} + \|\tilde{\omega}\|_{B^{s+1}}) 
\leq C\|\tilde{L}\|_{B^{s-1}} + C\|\tilde{M}\|_{B^{s-1}} + CV'(t)e^{-\gamma V(t)} F_s(a, \omega).
\end{equation}

Denote $C + K$ as $C$, we have
\begin{equation}
\frac{d}{dt}e^{-CV(t)} F_s(a, \omega) + ce^{-CV(t)}(\|a(t)\|_{B^s} + \|\omega(t)\|_{B^{s+1}}) 
\leq Ce^{-CV(t)}(\|L(t)\|_{B^{s-1}} + \|M(t)\|_{B^{s-1}}).
\end{equation}

**Step 5: the equation of $\Omega$ and $B$.** $\tilde{\Omega}$ satisfies
\begin{equation}
\partial_t \tilde{\Omega} - \mu \Delta \tilde{\Omega} = \tilde{N} - KV'(t)\tilde{\Omega}.
\end{equation}

Localizing the above equation, we find
\begin{equation}
\partial_t \Delta_q \tilde{\Omega} - \mu \Delta \Delta_q \tilde{\Omega} = \Delta_q \tilde{N} - KV'(t)\Delta_q \tilde{\Omega}.
\end{equation}

Hence, we can easily get
\begin{equation}
\frac{d}{dt}e^{-KV(t)}\|\Delta_q \tilde{\Omega}(t)\|_{L^2} + c2^{2q}\|\Delta_q \tilde{\Omega}(t)\|_{L^2} \leq C\|\Delta_q \tilde{N}(t)\|_{L^2}.
\end{equation}

Noting the definition of hybrid Besov space, we obtain
\begin{equation}
\frac{d}{dt}e^{-KV(t)}\|\tilde{\Omega}(t)\|_{B^{s-1}} + c\|\tilde{\Omega}(t)\|_{B^{s+1}} \leq C\|\tilde{N}(t)\|_{B^{s-1}}.
\end{equation}

As in Step 4, denote $2K$ as $C$, we have
\begin{equation}
\frac{d}{dt}e^{-CV(t)}\|\Omega(t)\|_{B^{s-1}} + ce^{-CV(t)}\|\Omega(t)\|_{B^{s+1}} \leq Ce^{-CV(t)}\|\tilde{N}(t)\|_{B^{s-1}}.
\end{equation}

Exactly the same as $\Omega$, for $B$, we have
\begin{equation}
\frac{d}{dt}e^{-CV(t)}\|B(t)\|_{B^{s-1}} + ce^{-CV(t)}\|B(t)\|_{B^{s+1}} \leq Ce^{-CV(t)}\|P(t)\|_{B^{s-1}}.
\end{equation}

Denote $E_s(a, \omega, \Omega, B) = F_s(a, \omega) + \|\Omega(t)\|_{B^{s-1}} + \|B(t)\|_{B^{s-1}}$. Obviously, we have
\begin{equation}
E_s(a, \omega, \Omega, B) \approx \|a(t)\|_{B^{s-1}} + \|\omega(t)\|_{B^{s-1}} + \|\Omega(t)\|_{B^{s-1}} + \|B(t)\|_{B^{s-1}}.
\end{equation}

Summing up $6.14$, $6.15$ and $6.16$, we complete the proof. \qed
7. Evolution in Negative Besov Space

In this section, we will derive the evolution of the negative Besov norms of the solution. The negative Besov space also used in [34], however, we derive a different form of estimates for our low regularity assumption.

**Proposition 7.1.** Let \((a, u, H)\) to be a solution of system (6.1), for \(s \in [0, N/2]\), we have

\[
\|\begin{pmatrix} a \cr u \cr H \end{pmatrix}(t)\|_{\dot{B}^{-s}_{2,\infty}}^2 \leq C \exp \left( C \int_0^t \|a\|_{\dot{B}^{N/2+1}} + \|u\|_{\dot{B}^{N/2+1, N/2+2}} + \|H\|_{\dot{B}^{N/2+1}} \, dt \right) \|\begin{pmatrix} a_0 \cr u_0 \cr H_0 \end{pmatrix}\|_{\dot{B}^{0}_{2,\infty}}^2.
\]

(7.1)

**Proof.** Applying the operator \(\Delta_q\) to system (6.1), we will have

\[
\begin{cases}
\partial_t \Delta_q a + \text{div}\Delta_q u = -\Delta_q (\text{div} u) - \Delta_q (u \cdot \nabla a), \\
\partial_t \Delta_q u - \mu \Delta_q u - \mu' \text{div}\Delta_q u + \nabla \Delta_q a + \nabla \Delta_q (I \cdot H) - \Delta_q (I \cdot \nabla H) = U_{\text{local}}, \\
\partial_t \Delta_q H - \nu \Delta_q \Delta_q H + \Delta_q (\text{div} u I) - \Delta_q (I \cdot \nabla u) = H_{\text{local}},
\end{cases}
\]

where

\[
U_{\text{local}} = -\frac{1}{2} \Delta_q \left( \frac{1}{1 + a} \nabla |H|^2 \right) - \Delta_q \left( \frac{a}{1 + a} I \cdot H \right) + \Delta_q \left( \frac{1 - a}{1 + a} \Delta_q H \right) + \Delta_q \left( \frac{a}{1 + a} \nabla (I \cdot H) \right) - \Delta_q (u \cdot \nabla u) - \mu \Delta_q \left( \frac{a}{1 + a} \Delta_q u \right) + \Delta_q \left( \frac{P'(1 + a) - P'(1)}{1 + a} \nabla a \right) - \mu' \Delta_q \left( \frac{a}{1 + a} \nabla \text{div} u \right),
\]

\[
H_{\text{local}} = -\Delta_q (u \cdot \nabla H) - \Delta_q (\text{div} u H) + \Delta_q (H \cdot \nabla u).
\]

Applying the operator \(\Lambda^{-s}\) to the above system and multiplying \(\Lambda^{-s}\Delta_q a, \Lambda^{-s}\Delta_q u\) and \(\Lambda^{-s}\Delta_q H\) to the first, second and third equation of the above system respectively, we will obtain

\[
(\partial_t \Lambda^{-s} \Delta_q a |\Lambda^{-s} \Delta_q a) + (\Lambda^{-s} \text{div} \Delta_q u |\Lambda^{-s} \Delta_q u) = W_1 + W_2,
\]

\[
(\partial_t \Lambda^{-s} \Delta_q u |\Lambda^{-s} \Delta_q u) - \mu (\Lambda^{-s} \Delta_q u |\Lambda^{-s} \Delta_q u) - \mu' (\Lambda^{-s} \text{div} \Delta_q u |\Lambda^{-s} \Delta_q u) + (\Lambda^{-s} \nabla \Delta_q a |\Lambda^{-s} \Delta_q a) + (\Lambda^{-s} \nabla \Delta_q (I \cdot H) |\Lambda^{-s} \Delta_q u) - (\Lambda^{-s} \Delta_q (I \cdot \nabla H) |\Lambda^{-s} \Delta_q u) = W_3 + W_4 + W_5 + W_6 + W_7,
\]

\[
(\partial_t \Lambda^{-s} \Delta_q H |\Lambda^{-s} \Delta_q H) - \nu (\Lambda^{-s} \Delta_q H |\Lambda^{-s} \Delta_q H) + (\Lambda^{-s} \Delta_q (\text{div} u I) |\Lambda^{-s} \Delta_q H) - (\Lambda^{-s} \Delta_q (I \cdot \nabla u) |\Lambda^{-s} \Delta_q H) = W_11 + W_12 + W_13.
\]

where

\[
W_1 = -(\Lambda^{-s} \Delta_q (\text{div} u) |\Lambda^{-s} \Delta_q a), \quad W_2 = -(\Lambda^{-s} \Delta_q (u \cdot \nabla a) |\Lambda^{-s} \Delta_q a),
\]

\[
W_3 = -\frac{1}{2} (\Lambda^{-s} \Delta_q \left( \frac{1}{1 + a} \nabla |H|^2 \right) |\Lambda^{-s} \Delta_q u),
\]

\[
W_4 = -(\Lambda^{-s} \Delta_q \left( \frac{a}{1 + a} I \cdot \nabla H \right) |\Lambda^{-s} \Delta_q u),
\]

\[
W_5 = -(\Lambda^{-s} \Delta_q (I \cdot \nabla u) |\Lambda^{-s} \Delta_q u).
\]
\[
W_5 = (\Lambda^{-s}\Delta_q(\frac{1}{1+a} H \cdot \nabla H)|\Lambda^{-s}\Delta_q u),
\]
\[
W_6 = (\Lambda^{-s}\Delta_q(\frac{a}{1+a} \nabla(I \cdot H))|\Lambda^{-s}\Delta_q u),
\]
\[
W_7 = -(\Lambda^{-s}\Delta_q(u \cdot \nabla u)|\Lambda^{-s}\Delta_q u),
\]
\[
W_8 = (\Lambda^{-s}\Delta_q((\frac{P'(1+a)}{1+a} - P'(1))\nabla a)|\Lambda^{-s}\Delta_q u),
\]
\[
W_9 = -\mu(\Lambda^{-s}\Delta_q(\frac{a}{1+a} \Delta u)|\Lambda^{-s}\Delta_q u),
\]
\[
W_{10} = -\mu'(\Lambda^{-s}\Delta_q(\frac{a}{1+a} \nabla \text{div} u)|\Lambda^{-s}\Delta_q u),
\]
\[
W_{11} = -(\Lambda^{-s}\Delta_q(u \cdot \nabla H)|\Lambda^{-s}\Delta_q B), \quad W_{12} = -(\Lambda^{-s}\Delta_q(\text{div} u H)|\Lambda^{-s}\Delta_q B),
\]
\[
W_{13} = (\Lambda^{-s}\Delta_q(H \cdot \nabla u)|\Lambda^{-s}\Delta_q B).
\]

Summing up the above three equalities and due to
\[
(\Lambda^{-s}\text{div}\Delta_q u|\Lambda^{-s}\Delta_q a) + (\Lambda^{-s}\nabla\Delta_q a|\Lambda^{-s}\Delta_q u) = 0,
\]
\[
(\Lambda^{-s}\nabla\Delta_q(I \cdot H)|\Lambda^{-s}\Delta_q u) + (\Lambda^{-s}\Delta_q(\text{div}u I)|\Lambda^{-s}\Delta_q H) = 0,
\]
\[
(\Lambda^{-s}\Delta_q(I \cdot \nabla H)|\Lambda^{-s}\Delta_q u) + (\Lambda^{-s}\Delta_q(I \cdot \nabla u)|\Lambda^{-s}\Delta_q H) = 0,
\]
we can easily obtain
\[
\frac{1}{2}\frac{d}{dt}\|\Lambda^{-s}\Delta_q a, \Lambda^{-s}\Delta_q u, \Lambda^{-s}\Delta_q H\|_{L^2}^2 + \mu\|\Lambda^{-s}\nabla\Delta_q u\|_{L^2}^2 + \mu'\|\Lambda^{-s}\text{div}\Delta_q u\|_{L^2}^2 + \nu\|\Lambda^{-s}\nabla\Delta_q H\|_{L^2}^2 = \sum_{i=1}^{13} W_i.
\]
(7.2)

Now, we need to estimate the right hand side of the above equality. For \(W_1\), we have
\[
W_1 \leq \|\Lambda^{-s}\Delta_q(\text{div}u)|_{L^2}\|\Lambda^{-s}\Delta_q a|_{L^2}
\]
\[
\leq C \left(\|T_a \text{div} u\|_{B^{-\frac{s}{2}}_{\infty}} + \|T \text{div} a\|_{B^{-\frac{s}{2}}_{\infty}} + \|R(a, \text{div} u)\|_{B^{-\frac{s}{2}}_{\infty}}\right) \|a\|_{B^{-\frac{s}{2}}_{\infty}}.
\]
(7.3)

The first term in the bracket can be estimated by using Lemma 9.4 as follows
\[
\|T_a \text{div} u\|_{B^{-\frac{s}{2}}_{\infty}} \leq C\|a\|_{B^{-\frac{s}{2}}_{\infty}} \|\text{div} u\|_{B^{N/2}}.
\]

The second term in the bracket can be estimated by using Remark 9.5 as follows
\[
\|T \text{div} a\|_{B^{-\frac{s}{2}}_{\infty}} \leq C\|\text{div}\|_{B^{N/2}} \|a\|_{B^{-\frac{s}{2}}_{\infty}}.
\]

The third term in the bracket can be estimated by using Lemma 9.6 as follows
\[
\|R(a, \text{div} u)\|_{B^{-\frac{s}{2}}_{\infty}} \leq C\|a\|_{B^{-\frac{s}{2}}_{\infty}} \|\text{div} u\|_{B^{N/2}}.
\]

Combining the above three estimates with (7.3), we find that
\[
W_1 \leq C\|a\|_{B^{-\frac{s}{2}}_{\infty}}^2 \|u\|_{B^{N/2+1}}.
\]
(7.4)
$W_2, W_7, W_8, W_9, W_{10}$ can be estimated similarly, so we only give the results as follows
\[ W_2 \leq C\|a\|_{B^{N/2+1}_2}\|u\|_{B^{N/2+1}_2}, \quad W_7 \leq C\|u\|_{B^{N/2+1}_2}^2, \quad W_8 \leq C\|a\|_{B^{N/2+1}_2}\|u\|_{B^{N/2+1}_2}, \quad W_9 + W_{10} \leq C\|u\|_{B^{N/2+1}_2}\|a\|_{B^{N/2+1}_2}\|u\|_{B^{N/2+1}_2}. \]

For $W_3$, we know that
\[ W_3 \leq C\|\Lambda^{-s}\Delta_q|H|^2\|L^2\|\Lambda^{-s}\Delta_q u\|L^2 \]
\[ \leq C\|H\|_{B^{N/2+1}_2}\|H\|_{B^{N/2+1}_2}\|u\|_{B^{N/2+1}_2} \]
\[ + C\|a\|_{B^{N/2+1}_2}\|H\|_{B^{N/2+1}_2}\|u\|_{B^{N/2+1}_2}\|u\|_{B^{N/2+1}_2}. \]

$W_5$ can be estimated similarly. For $W_4$, we have
\[ W_4 \leq \frac{a}{1+a} I : \nabla H \|B^{-s}_{2,\infty}\|u\|_{B^{-s}_{2,\infty}} \]
\[ \leq C\|H\|_{B^{N/2+1}_2}\|a\|_{B^{N/2+1}_2}\|u\|_{B^{N/2+1}_2}. \]

$W_6$ can be estimated similar to $W_4$. For $W_{11}$ to $W_{13}$, we have
\[ W_{11} + W_{12} + W_{13} \leq \|\Lambda^{-s}\Delta_q(H : \nabla u + \text{div} H)\|L^2\|\Lambda^{-s}\Delta_q H\|L^2 \]
\[ + \|\Lambda^{-s}\Delta_q(u : \nabla H)\|L^2\|\Lambda^{-s}\Delta_q H\|L^2 \]
\[ \leq C\|u\|_{B^{N/2+1}_2}\|H\|_{B^{N/2+1}_2}^2 + C\|H\|_{B^{N/2+1}_2}\|u\|_{B^{N/2+1}_2}\|H\|_{B^{N/2+1}_2}. \]

Summing all the estimates from $W_1$ to $W_{13}$ into (7.2), we will have
\[ \frac{d}{dt}\|\Lambda^{-s}\Delta_q a, \Lambda^{-s}\Delta_q u, \Lambda^{-s}\Delta_q H\|L^2 \]
\[ \leq C\|H\|_{B^{N/2+1}_2} + \|u\|_{B^{N/2+1,N/2+2}_2} + \|a\|_{B^{N/2+1}_2}\)(a, u, H)\|B^{-s}_{2,\infty}. \]

Integrate the above inequality, we will obtain
\[ \|\langle a, u, B\rangle\|_{B^{-s}_{2,\infty}} \leq \|(a_0, u_0, H_0)\|_{B^{-s}_{2,\infty}} \]
\[ + C\int_0^t \|\langle a\rangle_{B^{N/2+1}_2} + \|u\|_{B^{N/2+1,N/2+2}_2} + \|H\|_{B^{N/2+1}_2}\)(a, u, H)\|_{B^{-s}_{2,\infty}} d\tau. \]

Using Gronwall's inequality, we obtain (7.3). \[\square\]

Remark 7.2. If we consider the equilibrium state $\vec{B} = 0$ instead of $\vec{B} = I$ in Proposition 7.3, we can use same methods to get the following result (it is actually simpler).

Proposition 7.3. Let $(a, u, B)$ to be a solution of system (1.1), for $s \in [0, N/2]$, we have
\[ \|\langle a, u, B\rangle(t)\|_{B^{-s}_{2,\infty}} \leq C \exp \left( C\int_0^t \|\langle a\rangle_{B^{N/2+1}_2} + \|u\|_{B^{N/2+1,N/2+2}_2} \right. \]
\[ + \|B\|_{B^{N/2+1}_2} d\tau \left. \right) \|(a_0, u_0, B_0)\|_{B^{-s}_{2,\infty}}. \]

8. Optimal Time Decay Rate

In this section, we prove Theorem 2.11 and Theorem 2.9.
8.1. Proof about Theorem 2.11 Firstly, let us give the following global well-posedness results without proof for the proof is conventional and similar to [19].

**Theorem 8.1.** Let $N \geq 3$, $\rho > 0$, $\mu > 0$, $\lambda + 2\mu > 0$, $\nu > 0$, $P'(\cdot) > 0$ and $I$ to be an identity vector. Assume that $\rho_0 - \bar{\rho} \in B^{N/2-2,N/2}$ and $u_0, B_0 - I \in B^{N/2-2,N/2-1}$ with the condition $\text{div}B = 0$. Then, there exist a small number $\alpha_0$ and a constant $C$ such that if

$$
\|\rho_0 - \bar{\rho}\|_{B^{N/2-1,N/2+1}} + \|u_0\|_{B^{N/2-1,N/2}} + \|B_0 - I\|_{B^{N/2-1,N/2}} \leq \alpha_0,
$$

then the MHD system (6.7) has a unique global solution $(\rho, u, B)$ such that $(\rho - \bar{\rho}, u, B - I)$ belongs to

$$
E := \mathcal{C}(\mathbb{R}^+; B^{N/2-1,N/2+1} \times (B^{N/2-1,N/2})^{N+N}) \cap L^1(\mathbb{R}^+; B^{N/2-1,N/2+3} \times (B^{N/2+1,N/2+2})^{N+N}),
$$

and satisfies

$$
\|(\rho - \bar{\rho}, u, B - I)\|_E \leq C(\|\rho_0 - \bar{\rho}\|_{B^{N/2-1,N/2+1}} + \|u_0\|_{B^{N/2-1,N/2}} + \|B_0 - I\|_{B^{N/2-1,N/2}}).
$$

Here $C$ is independent of the initial data.

Applying Proposition 6.1 to the system (6.2) with $s = \frac{N}{2} + 1$, we will obtain

$$
\frac{d}{dt} \left( e^{-CV(t)} \mathcal{E}_{N/2+1}(a, \omega, \Omega, E) \right) + ce^{-CV(t)}(\|a\|_{B^{N/2+1,N/2+3}} + \|\omega\|_{B^{N/2+1,N/2+2}}
\|\Omega\|_{B^{N/2+1,N/2+2}} + \|E\|_{B^{N/2+1,N/2+2}} + \|F\|_{B^{N/2-1,N/2+1}}
+ \|G\|_{B^{N/2-1,N/2}} + \|J\|_{B^{N/2-1,N/2}} + \|K\|_{B^{N/2-1,N/2}})
\leq e^{-CV(t)}(\|F\|_{B^{N/2-1,N/2+1}})
$$

Now, we need to estimate the right hand side of the above inequality. Due to there are a lot of terms can be estimated similarly, we only give the estimates about some typical terms.

**Term 1 :** $a \text{div}u$.

$$
\|a \text{div}u\|_{B^{N/2-1,N/2+1}} \lesssim \|T_a \text{div}u\|_{B^{N/2-1,N/2+1}} + \|T_a \text{div}u\|_{B^{N/2-1,N/2+1}}
+ \|R(a, \text{div}u)\|_{B^{N/2-1,N/2+1}}
\lesssim \|a\|_{B^{N/2-1,N/2}} \|\text{div}u\|_{B^{N/2-1,N/2+1}} + \|\text{div}u\|_{B^{N/2-1,N/2}} \|a\|_{B^{N/2-1,N/2+1}}
+ \|a\|_{B^{N/2-1,N/2}} \|\text{div}u\|_{B^{N/2-1,N/2+1}}
\lesssim \|a\|_{B^{N/2-1,N/2}} \|u\|_{B^{N/2-1,N/2+1}}
\lesssim \alpha_0 \|u\|_{B^{N/2+1,N/2+2}}.
$$

In the proof of the above estimates, we used Lemma 9.4 and Lemma 9.6. In the following, we will use the two lemmas frequently without mentioned for it has been used in a similar way as the above situation.
Term 2: $u \cdot \nabla u$

$$
\|u \cdot \nabla u\|_{B^{N/2-1,N/2}} \lesssim \|T_u \nabla u\|_{B^{N/2-1,B/2}} + \|T \nabla u\|_{B^{N/2-1,N/2}} + \|R(u, \nabla u)\|_{B^{N/2-1,N/2}}
\lesssim \|u\|_{B^{N/2-1,N/2}} \|\nabla u\|_{\dot{B}^{N/2}} + \|\nabla u\|_{\dot{B}^{N/2}} \|u\|_{B^{N/2-1,N/2}} + \|u\|_{B^{N/2-1,N/2}} \|\nabla u\|_{\dot{B}^{N/2}}
\leq C \|u\|_{B^{N/2-1,N/2}} \|u\|_{B^{N/2+1,N/2}} + \|u\|_{B^{N/2+1,N/2}} \|u\|_{B^{N/2+1,N/2}} + \|u\|_{B^{N/2+1,N/2}} \|u\|_{B^{N/2+1,N/2}} + \|u\|_{B^{N/2+1,N/2}} \|u\|_{B^{N/2+1,N/2}}.
$$

Term 3: $h(a) \Delta u$

$$
\|h(a) \Delta u\|_{B^{N/2-1,N/2}} \leq C \|a\|_{\dot{B}^{N/2}} \|\Delta u\|_{B^{N/2-1,N/2}}
\leq C \|a\|_{B^{N/2-1,N/2+1}} \|u\|_{B^{N/2+1,N/2+2}}
\leq C \|a\|_{B^{N/2+1,N/2+2}}.
$$

Term 4: $f(a) \nabla a$

$$
\|f(a) \nabla a\|_{B^{N/2-1,N/2}} \leq C \|a\|_{B^{N/2-1,N/2}} \|\nabla a\|_{\dot{B}^{N/2}}
\leq C \|a\|_{B^{N/2-1,N/2+1}} \|a\|_{\dot{B}^{N/2}}
\leq C \|a\|_{\dot{B}^{N/2}}.
$$

Term 5: $\frac{1}{1+a} H \cdot \nabla H$

$$
\\|\frac{1}{1+a} H \cdot \nabla H\|_{B^{N/2-1,N/2}} \lesssim \|(1 - h(a)) H\|_{B^{N/2-1,N/2}} \|H\|_{B^{N/2+1,N/2}}
\lesssim (1 + \|a\|_{B^{N/2-1,N/2}}) \|H\|_{B^{N/2}} \|H\|_{B^{N/2+1,N/2}}
\lesssim \|a\|_{B^{N/2-1,N/2}} \|\nabla (I \cdot H)\|_{\dot{B}^{N/2}}
\lesssim \|a\|_{B^{N/2-1,N/2}} \|\nabla (I \cdot H)\|_{\dot{B}^{N/2}}
\lesssim a_0 \|H\|_{B^{N/2+1}}.
$$

Term 6: $h(a) \nabla (I \cdot H)$

$$
\|h(a) \nabla (I \cdot H)\|_{B^{N/2-1,N/2}} \lesssim \|h(a)\|_{B^{N/2-1,N/2}} \|\nabla (I \cdot H)\|_{\dot{B}^{N/2}}
\lesssim \|a\|_{B^{N/2-1,N/2}} \|\nabla (I \cdot H)\|_{\dot{B}^{N/2}}
\lesssim a_0 \|H\|_{B^{N/2+1}}.
$$

Term 7: $u \cdot \nabla H$

$$
\|u \cdot \nabla H\|_{B^{N/2-1,N/2}} \lesssim \|u\|_{B^{N/2-1,N/2}} \|H\|_{\dot{B}^{N/2}} \lesssim \|a\|_{B^{N/2-1,N/2}} \|H\|_{\dot{B}^{N/2}} \lesssim \|a\|_{B^{N/2-1,N/2}} \|H\|_{B^{N/2+1}}.
$$

Term 8: $H \cdot \text{div}u$

$$
\|H \cdot \text{div}u\|_{B^{N/2-1,N/2}} \lesssim \|H\|_{B^{N/2-1,N/2}} \|u\|_{\dot{B}^{N/2+1,N/2}} \lesssim \|a\|_{B^{N/2-1,N/2}} \|H\|_{\dot{B}^{N/2}} \lesssim \|a\|_{B^{N/2-1,N/2}} \|H\|_{B^{N/2+1}}.
$$

From the estimates of the above eight terms, we can get the estimates about $F, G, J, K$. Then using the relation $u = -\Lambda^{-1} \nabla \omega - \Lambda^{-1} \text{div} \Omega, H = -\Lambda^{-1} \text{div} E$ and $\text{div} H = 0$, we know that

$$
E_{N/2+1}(a, \omega, \Omega, E) \approx \|a(t)\|_{B^{N/2-1,N/2+1}} + \|u(t)\|_{B^{N/2-1,N/2}} + \|B(t)\|_{B^{N/2-1,N/2}},
$$
and

$$
\frac{d}{dt} E_{N/2+1}(a, \omega, \Omega, E) + c e^{-C_1 G(t)} \left( \|a\|_{B^{N/2+1,N/2+3}} + \|u\|_{B^{N/2+1,N/2+2}} + \|B\|_{B^{N/2+1,N/2+2}} \right)
\leq C e^{-C_2 G(t)} \alpha_0 \|a, u, B\|_{B^{N/2+1,N/2+3}} \chi(B^{N/2+1,N/2+3})^{N+1},
$$
where $C_1$, $C_2$ are two positive constants. Denoting $\mathcal{E}_{N/2+1}(a, \omega, \Omega, E)$ simply as $\mathcal{E}_{N/2+1}(t)$, and taking $\alpha_0$ small enough, we finally get

\[ \frac{d}{dt} e^{-CV(t)} \mathcal{E}_{N/2+1}(t) + C e^{-CV(t)} \|a, u, B\|_{B^N/2+1, N/2+3} \leq 0. \]

Finally, we need some interpolation estimates to close our proof. From the definition of hybrid Besov space, we know that

\[
\|a\|_{B^N/2-1, N/2+1} \approx \|a\|_{B^N/2-1} + \|a\|_{B^N/2+1},
\|u\|_{B^N/2-1, N/2+1} \approx \|u\|_{B^N/2-1} + \|u\|_{B^N/2+1},
\|u\|_{B^N/2+1, N/2+2} \approx \|u\|_{B^N/2+1} + \|u\|_{B^N/2+2},
\|H\|_{B^N/2-1, N/2+1} \approx \|H\|_{B^N/2-1} + \|H\|_{B^N/2+1},
\|H\|_{B^N/2+1, N/2+2} \approx \|H\|_{B^N/2+1} + \|H\|_{B^N/2+2}.
\]

From Lemma 9.3 we have

\[ \|a(t)\|_{B^N/2-1} \leq C \|a(t)\|_{B^N/2-1}^{\theta_1} \|a(t)\|_{B^N/2+1}^{1-\theta_1}, \]

where $\theta_1 = \frac{2}{N/2+1+\epsilon}$. Similarly, we also have

\[ \|u(t)\|_{B^N/2-1} \leq C \|u(t)\|_{B^N/2-1}^{\theta_1} \|u(t)\|_{B^N/2+1}^{1-\theta_1}, \]

\[ \|u(t)\|_{B^N/2} \leq C \|u(t)\|_{B^N/2}^{\theta_2} \|u(t)\|_{B^N/2+2}^{1-\theta_2}, \]

and

\[ \|H(t)\|_{B^N/2-1} \leq C \|H(t)\|_{B^N/2-1}^{\theta_1} \|H(t)\|_{B^N/2+1}^{1-\theta_1}, \]

\[ \|H(t)\|_{B^N/2} \leq C \|H(t)\|_{B^N/2}^{\theta_2} \|H(t)\|_{B^N/2+2}^{1-\theta_2}, \]

where $\theta_2 = \frac{2}{N/2+2+\epsilon}$. Combining the results in Proposition 7.3 and Theorem 8.1 we find

\[ \|a\|_{L^\infty(\mathbb{R}^+, B^N/2-1)} + \|u, H\|_{L^\infty(\mathbb{R}^+, B^N/2-1)} \leq C \left( \|a_0\|_{B^N/2-1} + \|u_0, H_0\|_{B^N/2-1} \right). \]

So we easily obtain

\[ \|a(t)\|_{B^N/2+1} \geq C \|a(t)\|_{B^N/2-1}^{1+\frac{2}{2+2+\epsilon}} = C \|a(t)\|_{B^N/2-1}^{1+\frac{2}{2+2+\epsilon}}; \]

\[ \|u(t)\|_{B^N/2+1} \geq C \|u(t)\|_{B^N/2-1}^{1+\frac{2}{2+2+\epsilon}} = C \|u(t)\|_{B^N/2-1}^{1+\frac{2}{2+2+\epsilon}}; \]

\[ \|H(t)\|_{B^N/2+1} \geq C \|H(t)\|_{B^N/2-1}^{1+\frac{2}{2+2+\epsilon}} = C \|H(t)\|_{B^N/2-1}^{1+\frac{2}{2+2+\epsilon}}; \]

\[ \|H(t)\|_{B^N/2+2} \geq C \|H(t)\|_{B^N/2-1}^{1+\frac{2}{2+2+\epsilon}} = C \|H(t)\|_{B^N/2-1}^{1+\frac{2}{2+2+\epsilon}}. \]

By taking $\alpha_0$ small enough in Theorem 8.1 we can assume

\[ \|a(t)\|_{B^N/2-1, N/2+1} \leq 1, \quad \|u(t), H\|_{B^N/2-1, N/2} \leq 1. \]
Due to the relationship between Sobolev space and Besov space, we obtain
\[
\|a(t)\|_{\dot{B}^{N/2-1}_{2,1}} \geq C\|a(t)\|_{\dot{B}^{N/2-1}_{2,1}}, \quad \|a(t)\|_{\dot{B}^{N/2+1}_{2,1}} \geq C\|a(t)\|_{\dot{B}^{N/2+1}_{2,1}},
\]

(8.3) \[\|u(t)\|_{\dot{B}^{N/2+1}_{2,1}} \geq C\|u(t)\|_{\dot{B}^{N/2+1}_{2,1}}, \quad \|u(t)\|_{\dot{B}^{N/2+2}_{2,1}} \geq C\|u(t)\|_{\dot{B}^{N/2+2}_{2,1}}, \]
\[\|H(t)\|_{\dot{B}^{N/2+1}_{2,1}} \geq C\|H(t)\|_{\dot{B}^{N/2+1}_{2,1}}, \quad \|H(t)\|_{\dot{B}^{N/2+2}_{2,1}} \geq C\|H(t)\|_{\dot{B}^{N/2+2}_{2,1}}.\]

Plugging (8.3) into (8.1), we obtain
\[
\frac{d}{dt}e^{-CV(t)}\mathcal{E}_{N/2+1}(t)
+ c \left( e^{-CV(t)} (\|a(t)\|_{\dot{B}^{N/2-1,1}_{2,1}} + \|u(t), H(t)\|_{\dot{B}^{N/2-1,1/2}}) \right)^{1 + \frac{2}{N/2+1}} \leq 0.
\]

Since \(\mathcal{E}_{N/2+1}(t) \approx \|a(t)\|_{\dot{B}^{N/2-1,1}_{2,1}} + \|u(t)\|_{\dot{B}^{N/2-1,1/2}} + \|H(t)\|_{\dot{B}^{N/2-1,1/2}},\) we finally get
\[
(8.4) \quad \frac{d}{dt}e^{-CV(t)}\mathcal{E}_{N/2+1}(t) + c \left( e^{-CV(t)}\mathcal{E}_{N/2+1}(t) \right)^{1 + \frac{2}{N/2+1}} \leq 0.
\]

Solving this differential inequality, we could obtain
\[
\mathcal{E}_{N/2+1}(t) \leq C e^{CV(t)} \left( \mathcal{E}_{N/2+1}(0) - \frac{2C}{N/2 + s - 1} + \frac{2C}{N/2 + s - 1} \right)^{\frac{N/2+1}{2+s}}.
\]

From Theorem 5.1 we know that \(V(t)\) is bounded by the initial data, there exists a constant \(C\) such that
\[
(8.5) \quad \|a(t)\|_{\dot{B}^{N/2-1,1}_{2,1}} + \|u(t), H(t)\|_{\dot{B}^{N/2-1,1/2}} \leq C(1 + t)^{-\frac{N/2-1+s}{2+s}}.
\]

Due to the relationship between Sobolev space and Besov space, we obtain
\[
\|\Lambda^{N/2-1} a(t)\|_{L^2} + \|\Lambda^{N/2-1} u(t), \Lambda^{N/2-1} H(t)\|_{L^2}
\leq C (\|a(t)\|_{\dot{B}^{N/2-1}_{2,1}} + \|u(t), H(t)\|_{\dot{B}^{N/2-1}_{2,1}})
\leq C (\|a(t)\|_{\dot{B}^{N/2-1,1}_{2,1}} + \|u(t), H(t)\|_{\dot{B}^{N/2-1,1/2}})
\leq C (1 + t)^{-\frac{N/2-1+s}{2+s}}.
\]

For \(\ell \in (-s, N/2 - 1),\) we have
\[
\|a(t)\|_{\dot{B}^\ell} \leq \|a(t)\|_{\dot{B}_{2,\infty}^{N/2-1}}^{\theta} \|a(t)\|_{\dot{B}_{2,\infty}^{N/2-1}}^{-\theta},
\]
where \(\theta = \frac{N/2-1-s}{N/2-1+s}.\) By (8.6), we then obtain
\[
\|a(t)\|_{\dot{B}^\ell} \leq \|a(t)\|_{\dot{B}_{2,\infty}^{N/2-1}}^{\frac{N/2-1-s}{N/2-1+s}} \|a(t)\|_{\dot{B}_{2,\infty}^{N/2-1}}^{-\frac{N/2-1-s}{N/2-1+s}}
\leq C (1 + t)^{-\frac{4s}{2+s}}.
\]

Similar to the above analysis for \(a(t),\) we also have the following estimates for \(u(t)\) and \(H(t)\)
\[
\|u(t), H(t)\|_{\dot{B}^\ell} \leq C (1 + t)^{-\frac{4s}{2+s}}.
\]

At this stage, we complete the proof of Theorem 2.11.
8.2. **Proof about Theorem 2.9.** Due to the proof about Theorem 2.9 is similar to the proof about Theorem 2.11, we only give the key different part the remaining part of the proof can be easily recovered.

**Theorem 8.2.** Let \( N \geq 3, \bar{\rho} > 0, \mu > 0, \lambda + 2\mu > 0, \nu > 0, P'(\cdot) > 0. \) Assume that \( \rho_0 - \bar{\rho} \in B^{N/2-1,N/2} \) and \( u_0, B_0 \in \dot{B}^{N/2-1} \) with the condition \( \text{div} B_0 = 0. \) Then, there exist a small number \( \alpha_0 \) and a constant \( C \) such that if

\[
\| \rho_0 - \bar{\rho} \|_{B^{N/2-1,N/2}} + \| u_0 \|_{\dot{B}^{N/2-1}} + \| B_0 \|_{\dot{B}^{N/2-1}} \leq \alpha_0,
\]

then the MHD system (1.1) has a unique global solution \((\rho, u, B)\) such that \((\rho - \bar{\rho}, u, B)\) belongs to

\[
E : = \dot{C}(\mathbb{R}^+; B^{N/2-1,N/2} \times (B^{N/2-1})^{N+N})
\]

\[
\cap L^1(\mathbb{R}^+; \dot{B}^{N/2} \times (\dot{B}^{N/2+1})^{N+N}),
\]

and satisfies

\[
\| (\rho - \bar{\rho}, u, B) \|_E \leq C(\| \rho_0 - \bar{\rho} \|_{B^{N/2-1,N/2}} + \| u_0 \|_{\dot{B}^{N/2-1}} + \| B_0 \|_{\dot{B}^{N/2-1}}).
\]

Here \( C \) is independent of the initial data.

Applying Proposition 6.2 to the system (6.3) with \( s = N/2 \), we will obtain

\[
\frac{d}{dt} (e^{-CV(t)} E_{N/2}(a, \omega, \Omega, B)) + ce^{-CV(t)} (\| a \|_{B^{N/2+1}} + \| \omega \|_{B^{N/2+1}} + \| \Omega \|_{B^{N/2+1}} + \| B \|_{B^{N/2+1}}) \leq C e^{-CV(t)} (\| L \|_{B^{N/2-1,N/2}} + \| M \|_{B^{N/2-1}} + \| N \|_{B^{N/2-1}} + \| P \|_{B^{N/2-1}}).
\]

Then estimate the right hand side of the above inequality exactly as in \( [25] \) and take \( \alpha_0 \) in Theorem 5.2 small enough, we can get

\[
\frac{d}{dt} e^{-CV(t)} E_{N/2}(a, \omega, \Omega, B) + ce^{-V(t)} \| a, u, B \|_{B^{N/2} \times (B^{N/2+1})^{N+N}} \leq 0.
\]

From Proposition 7.3 and Theorem 2.6 we know that

\[
\| a \|_{L^\infty(\mathbb{R}^+, B_{\infty}^{N/2})} + \| u, B \|_{L^\infty(\mathbb{R}^+, \dot{B}_{\infty}^{N})} \leq C \left( \| a_0 \|_{\dot{B}_{\infty}^{N/2}} + \| u_0, B_0 \|_{\dot{B}_{\infty}^{N}} \right).
\]

Now we can mimic the proof about Theorem 2.11 to prove Theorem 2.9. Hence, we complete the proof.

9. **Appendix**

In this section, for the reader’s convenience, we give some conventional results. Firstly, let us give the Lagrangian transform for terms appearing in the MHD system as follows

\[
\nabla_x(|B|^2) = J^{-1} \text{div}_y(\text{adj}(DX)|\tilde{B}|^2),
\]

\[
\tilde{B} \cdot \nabla_x B = \text{div}_x(B \otimes B) = J^{-1} \text{div}_y(\text{adj}(DX)^T \tilde{B} \otimes \tilde{B}),
\]

\[
\text{div}_u B = J^{-1} \text{div}_y(\text{adj}(DX) \tilde{u}) \tilde{B},
\]

\[
(\text{div}_x u) B + u \cdot \nabla_x B = \nabla_x (u B) = J^{-1} \text{div}_y(\text{adj}(DX) \tilde{u}) \tilde{B},
\]

\[
B \cdot \nabla_x u = \nabla_x (B \otimes u) = J^{-1} \text{div}_y(\text{adj}(DX)^\top \tilde{u} \otimes \tilde{B}).
\]
Then, let us look at the following Lamé system with nonconstant coefficients

\[ \partial_t v - 2a \text{div}(\mu D(v)) - b \nabla(\lambda \text{div} v) = f. \]

We assume that the following uniform ellipticity condition is satisfied:

\[ \alpha := \min \left( \inf_{(t,x) \in [0,T] \times \mathbb{R}^N} (2\alpha \mu + b\lambda)(t,x) \right) > 0. \]

Concerning this equation, we have the following lemma.

**Lemma 9.1.** \[30, 35\] Let \( a, b, \lambda, \mu \) be bounded functions satisfying (9.3). Assume that \( a\nabla\mu, b\nabla\lambda, \mu \nabla\mu \) and \( \lambda \nabla b \) are in \( L^\infty(0,T; \dot{B}_p^{n/p}) \) for some \( 1 < p < \infty \). There exist two constants \( \eta \) and \( \kappa \) such that if for some \( m \in \mathbb{Z} \) we have

\[ \min \left( \inf_{(t,x) \in [0,T] \times \mathbb{R}^N} S_m(2\alpha \mu + b\lambda)(t,x), \inf_{(t,x) \in [0,T] \times \mathbb{R}^N} S_m(\alpha \mu)(t,x) \right) \geq \frac{\alpha}{2}, \]

then the solutions to (9.2) satisfy for all \( t \in [0,T] \),

\[ \|v\|_{L^\infty_t(\dot{B}^{n/p}_p)} + \alpha\|v\|_{L^1_t(\dot{B}^{n/p+2}_p)} \leq C \left( \|v_0\|_{\dot{B}^1_p} + \|f\|_{L^1_t(\dot{B}^{n/p}_p)} \right) \exp \left( \frac{C}{\alpha} \int_0^t \|S_m(\mu \nabla a, a \nabla \mu, \lambda \nabla b, b \nabla \lambda)\|_{\dot{B}^{n/p+2}_p}^2 \, d\tau \right), \]

whenever \( -\min(n/p, n/p') < s \leq n/p - 1 \) is satisfied.

**Lemma 9.2.** \[35\] Let \( p \in [1, +\infty] \), \( 1 < \rho_2 < \rho_1 < +\infty \), and let \( v \) solve

\[ \begin{cases} \partial_t v - \nu \Delta v = f, \\ |v|_{t=0} = v_0. \end{cases} \]

Denote \( \rho_1' = (1 + 1/\rho_1 - 1/\rho_2)^{-1} \). Then there exist two positive constants \( c \) and \( C \) depending only on \( N \) and such that

\[ \|v\|_{L^{\rho_1/\rho_1'}_t(\dot{B}^{n+2/\rho_1}_{\rho_1'})} \leq \left( \sum_{q \in \mathbb{Z}} 2^{q^a} \|\Delta_q v_0\|_{L^p} \left( 1 - \frac{e^{-c\nu T 2^a \rho_1}}{c\nu \rho_1'} \right)^{1/\rho_1'} \right) \left( \sum_{q \in \mathbb{Z}} 2^{q(s-2+2/\rho_2)} \|\Delta_q f\|_{L^{\rho_2}_t(\dot{B}^0_{\rho_2})} \left( 1 - \frac{e^{-c\nu T 2^a \rho_1'}}{c\nu \rho_1''} \right)^{1/\rho_1''} \right). \]

In particular, we have

\[ \|v\|_{L^{\rho_1/\rho_1'}_t(\dot{B}^{n+2/\rho_1}_{\rho_1'})} \leq \frac{C}{\nu^{1/\rho_1}} \|v_0\|_{\dot{B}^1_p} + \frac{C}{\nu^{1/\rho_1'}} \|f\|_{L^{\rho_2}_t(\dot{B}^{n+2/\rho_2}_{\rho_1'})}. \]

Moreover \( v \) belongs to \( C([0,T]; \dot{B}^s_p) \).

At last, we will give some properties about Besov space.

**Lemma 9.3.** \[1\] A constant \( C \) exists which satisfies the following properties. If \( s_1 \) and \( s_2 \) are real numbers such that \( s_1 < s_2 \) and \( \theta \in (0,1) \), then we have, for any \( (p,r) \in [1, \infty]^2 \) and any \( u \in S'_h \),

\[ \|u\|_{\dot{B}^{s_1+\theta s_2}_{\rho_1}} \leq \|u\|_{\dot{B}^{s_1}_{\rho_1}}^{1-\theta} \|u\|_{\dot{B}^{s_2}_{\rho_2}}^\theta, \]

\[ \|u\|_{\dot{B}^{s_1+\theta s_2}_{\rho_1}} \leq \frac{C}{s_2 - s_1} \left( \frac{1}{\theta} + \frac{1}{1-\theta} \right) \|u\|_{\dot{B}^{s_2}_{\rho_2}}^{\theta} \|u\|_{\dot{B}^{s_1}_{\rho_1}}^{1-\theta}. \]
Lemma 9.4. There exists a constant $C$ such that for any real number $s$ and any $(p, r) \in [1, +\infty)^2$, we have, for any $(u, v) \in L^\infty \times \dot{B}^s_{p,r}$,

$$
\|T_u v\|_{\dot{B}^s_{p,r}} \leq C\|u\|_{L^\infty} \|v\|_{\dot{B}^s_{p,r}}.
$$

Moreover, for any $(s, t) \in \mathbb{R} \times (-\infty, 0)$ and any $(p, r_1, r_2) \in [1, +\infty]^3$, we have, for any $(u, v) \in \dot{B}^t_{\infty, r_1} \times \dot{B}_p^{s_1, r_2}$,

$$
\|T_u v\|_{\dot{B}^{s_1, r_2}_{p,r}} \leq \frac{C}{t} \|u\|_{\dot{B}^t_{\infty, r_1}} \|v\|_{\dot{B}_p^{s_1, r_2}},
$$

with $\frac{1}{r} = \min \left\{1, \frac{1}{r_1}, \frac{1}{r_2}\right\}$.

Remark 9.5. Using similar methods as in the proof of Lemma 9.4 we can get the following estimates

$$
\|T_u v\|_{\dot{B}^{-s}_{\infty, r}} \leq C\|u\|_{\dot{B}^s_{\infty, 1}} \|v\|_{\dot{B}^{-s}_{\infty, r}}.
$$

Lemma 9.6. There exists a constant $C$ which satisfies the following inequalities. Let $(s_1, s_2)$ be in $\mathbb{R}^2$ and $(p_1, p_2, r_1, r_2)$ be in $[1, +\infty]^4$. Assume that

$$
\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} \leq 1 \quad \text{and} \quad \frac{1}{r} = \frac{1}{r_1} + \frac{1}{r_2} \leq 1.
$$

If $s_1 + s_2$ is positive, then we have, for any $(u, v)$ in $\dot{B}^{s_1}_{p_1, r_1} \times \dot{B}^{s_2}_{p_2, r_2}$,

$$
\|R(u, v)\|_{\dot{B}^{s_1 + s_2}_{p, r}} \leq \frac{C}{s_1 + s_2} \|u\|_{\dot{B}^{s_1}_{p_1, r_1}} \|v\|_{\dot{B}^{s_2}_{p_2, r_2}}.
$$

When $r = 1$ and $s_1 + s_2 \geq 0$, we have, for any $(u, v)$ in $\dot{B}^{s_1}_{p_1, r_1} \times \dot{B}^{s_2}_{p_2, r_2}$,

$$
\|R(u, v)\|_{\dot{B}^{s_1 + s_2}_{p, 1}} \leq C\|u\|_{\dot{B}^{s_1}_{p_1, 1}} \|v\|_{\dot{B}^{s_2}_{p_2, 1}}.
$$

Lemma 9.7. Let $F$ be a homogeneous smooth function of degree $m$. Suppose that $-\frac{m}{2} < s_1, s_2, t_1, t_2 \leq 1 + \frac{m}{2}$. The following two estimates hold

$$
|(F(D)\Delta_k(v \cdot \nabla a)|F(D)\Delta_k a)| \leq C\epsilon_k 2^{-k\tilde{\alpha}_{s_1, t_2}(k)-m} \|v\|_{\dot{B}^{N/2+1}_{s_1, t_2}} \|a\|_{\dot{B}^{1, t_2}} \|F(D)\Delta_k a\|_{L^2},
$$

$$
|(F(D)\Delta_k(v \cdot \nabla b)|F(D)\Delta_k a)| \leq C\epsilon_k \|v\|_{\dot{B}^{N/2+1}_{s_1, t_2}} \|F(D)\Delta_k a\|_{L^2} \|b\|_{\dot{B}^{1, t_2}} \|\Delta_k b\|_{L^2}.
$$

where $(\cdot, \cdot)$ denotes the $L^2$-inner product, the operator $F(D)$ is defined by $F(D)f := \tilde{\Phi}^{-1} F(\xi) \tilde{\Phi} f$ and $\sum_{k \in \mathbb{Z}} \epsilon_k \leq 1$.

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