Extracting Nucleon Magnetic Moments and Electric Polarizabilities from Lattice QCD in Background Electric Fields

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Abstract

Nucleon properties are investigated in background electric fields. As the magnetic moments of baryons affect their relativistic propagation in constant electric fields, electric polarizabilities cannot be determined without knowledge of magnetic moments. We devise combinations of baryon two-point functions in external electric fields to isolate both observables. Using an ensemble of anisotropic gauge configurations with dynamical clover fermions, we demonstrate how magnetic moments and electric polarizabilities can be determined from lattice QCD simulations in background electric fields. We obtain results for both the neutron and proton. Our study is currently limited to electrically neutral sea quarks.

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I. INTRODUCTION

Understanding low-energy properties of hadrons directly from QCD remains a challenging endeavor. In this low-energy regime, quark and gluon interactions must be treated non-perturbatively, which ultimately results in their confinement into hadrons. After three decades of dedicated work, lattice QCD has evolved into a tool to address the non-perturbative dynamics underlying hadrons and their interactions, see [1] for an overview. Electromagnetic moments and multipole polarizabilities are low-energy properties of hadrons with transparent physical meaning. These properties characterize the distribution of charge and magnetism within a hadron, and the response of the charge and magnetism distributions to external fields, respectively. Low-energy properties of hadrons can be elegantly described using an effective theory of QCD, based upon treating pseudoscalar mesons as the Goldstone modes arising from spontaneous chiral symmetry breaking. A picture of hadrons emerges from chiral dynamics: that of a hadronic core surrounded by a pseudoscalar meson cloud. In part, the electromagnetic properties of hadrons encode the distribution of charged mesons, and the stiffness of the charged meson cloud. Chiral dynamics consequently makes predictions for the form of electromagnetic observables. Confirming these predictions both experimentally and from the lattice will be a milestone in our understanding of non-perturbative QCD dynamics.

Computation of hadronic electromagnetic properties using lattice QCD can be accomplished in two different ways. The current insertion method can be used to determine hadronic matrix elements of the electromagnetic current. This method is ideal for the computation of electromagnetic form factors, but is limited in the extraction of multipole moments due to the available lattice momentum, which, for periodic boundary conditions, is quantized in units of $2\pi/L$, where $L$ is the size of the lattice. Determination of multipole moments relies on a long extrapolation to vanishing momentum transfer. For multipole polarizabilities, the temporal extent of current lattices makes direct computation of the Compton scattering tensor infeasible. Were lattices long enough to allow the computation of matrix elements with two current insertions, the extraction of polarizabilities would still require the long extrapolation to zero momentum. Alternately, hadronic properties can be determined using the background field method [4–6]. With this method, one determines simple lattice two-point functions in the presence of classical external fields. Observables are then determined from the behavior of hadronic parameters with the strength of the external field.

We focus our attention on simulations of spin-half baryons in external electric fields. Neutral hadrons in electric fields have been investigated with lattice QCD using the quenched approximation at pion masses greater than 500 MeV [7, 8]. There has also been a fully dynamical calculation for the neutron in an electric field using a pion mass of 760 MeV [9]. These calculations do not employ constant electric fields but attempt to mitigate effects from field gradients by imposing Dirichlet boundary conditions on the quark fields in the time and/or space directions. Such an approach leads to uncertainties that are difficult to quantify, and even the possibility to restore chiral symmetry due to boundary critical phenomena. These calculations, moreover, do not treat effects resulting from the baryon spin. In this work, we

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1 For isovector form factors, the restriction to quantized momentum transfer can be lifted by imposing isospin twisted boundary conditions on the quark fields [2, 3]. There is, however, no known method to handle the isoscalar contribution.
correctly treat the baryon spin, and thereby demonstrate how to determine nucleon magnetic moments and electric polarizabilities using lattice QCD in background electric fields. We consider both the neutron and proton. For the latter, we use the relativistic generalization of the method proposed in [10], which relies on the determination of the behavior of correlators using hadronic effective actions. We have also employed this method recently for pseudoscalar meson electric polarizabilities [11]. A salient feature of our computation is that it utilizes a periodic lattice action with everywhere constant electric fields. Our calculations of nucleon magnetic moments and electric polarizabilities include effects from dynamical quarks, however, they are restricted to electrically neutral sea quarks.

We organize our presentation in the following manner. First in Sect. II, we analytically determine the form of baryon correlation functions in external electric fields. We specialize to the case of a uniform electric field, and derive results for both neutral and charged baryons. A key observation of this section is that baryon electric polarizabilities cannot be determined without knowledge of their magnetic moments. In Sect. III, we provide the pertinent details of our lattice computations, and implementation of the background field. In Sect. IV, we present our analysis of nucleon correlation functions calculated in background electric fields using lattice QCD. For both the neutron and proton, we demonstrate that the measured correlation functions agree with the analytic expectations from the hadronic theory, and that magnetic moments and electric polarizabilities can be extracted from the data. The Appendix is devoted to the analysis of unpolarized neutron correlators, from which consistent results are obtained. These results, however, do not permit the determination of the electric polarizability—only a combination of the polarizability and square of the magnetic moment. A brief conclusion in Sect. V ends our work.

II. BARYON CORRELATION FUNCTIONS

To extract properties of nucleons in background electric fields, we must first understand the expected behavior of their two-point correlation functions. In this section, we determine baryon two-point functions using single-hadron effective actions. The functional forms deduced for these two-point functions can then be utilized to fit baryon correlators computed with lattice QCD. From these fits, one can deduce hadronic parameters, such as the magnetic moments and electric polarizabilities.

To arrive at a uniform electric field of the form \( \tilde{E} = E \hat{z} \), we use the Euclidean space vector potential

\[
A_\mu = (0, 0, -E x_4, 0).
\]  

While there are other gauge equivalent choices, we find Eq. (1) particularly useful.\(^2\) The analytic continuation, \( \mathcal{E} \rightarrow -iE_M \), is needed to recover Minkowski space results. As our interests lie only with quantities perturbative in the strength of the field, this analytic continuation can be performed trivially, see [15]. A Euclidean formulation is natural from the point of view of lattice gauge theory simulations; moreover, the Euclidean formulation removes instabilities due to non-perturbative effects, i.e. the Schwinger mechanism [16]. With

\(^2\) On a torus, many of the gauge equivalent choices in infinite volume are no longer equivalent, but differ by their holonomy. External fields with non-vanishing holonomy lead to new interactions that are finite volume artifacts [12–14]. As the current study is restricted to one lattice volume, we postpone the investigation of finite volume effects to future work.
the vector potential specified, we can determine the baryon two-point functions. As neutral and charged baryons propagate differently in electric fields, we handle each separately.

A. Neutral Baryons

We consider first the case of a neutral spin-half particle of mass $M$ described by the field $\psi(x)$. The Euclidean space correlation function in the hadronic theory we denote by $G_{\alpha\beta}(x_4, \mathcal{E})$, which is given by

$$G_{\alpha\beta}(x_4, \mathcal{E}) = \int dx \langle 0 | \psi_\alpha(x) \bar{\psi}_\beta(0) | 0 \rangle \mathcal{E},$$

where the subscript denotes that the correlation function is calculated in the background electric field. Integrating over all space projects the correlator onto vanishing three-momentum, which is a good quantum number. For lattice QCD with spatially periodic boundary conditions, a sum over lattice sites accomplishes the same thing.

The energy of the neutral particle, $E(\mathcal{E})$, depends on the strength of the electric field. For weak fields, the energy has the expansion

$$E(\mathcal{E}) = M + \frac{1}{2} 4\pi \alpha_E \mathcal{E}^2 + \ldots,$$

where $\alpha_E$ is the electric polarizability, and the ellipsis denotes higher-order terms in even powers of the field. The quadratic Stark shift is positive due to our Euclidean space treatment. The magnetic moment, $\mu$, is also important, and its coupling to $\psi(x)$ is entirely anomalous. The single-hadron effective action for a neutral spin-half particle takes the form

$$S_E = \int d^4x \psi(x) \left[ \partial^\mu + E(\mathcal{E}) - \frac{\mu(\mathcal{E})}{4M} \sigma_{\mu\nu} F_{\mu\nu} \right] \psi(x).$$

The electromagnetic field strength tensor is $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. For a background electric field, $\sigma_{\mu\nu} F_{\mu\nu} = \vec{K} \cdot \vec{E}$, where $\vec{K} = i\vec{\gamma}_4$ is the generator of boosts in the spin-half representation of the Lorentz group. The magnetic moment coupling has been written as $\mathcal{E}$-dependent. In small fields, $\mu(\mathcal{E})$ has a perturbative expansion in even powers of the field, and satisfies the zero-field relation, $\mu(0) = \mu$. Using the effective action in Eq. (4) to determine the unpolarized two-point function, we arrive at

$$\text{Tr}[G(x_4, \mathcal{E})] = Z(\mathcal{E}) \exp \left[ -x_4 E_{\text{eff}}(\mathcal{E}) \right],$$

where the effective energy, $E_{\text{eff}}(\mathcal{E})$, depends on the magnetic moment, and is given by

$$E_{\text{eff}}(\mathcal{E}) = M + \frac{1}{2} \mathcal{E}^2 \left( 4\pi \alpha_E - \frac{\mu^2}{4M^2} \right).$$

We have retained terms in the effective energy only up to second order in the electric field. While the magnetic moment contribution arises from a relativistic effect, the resulting shift in baryon energy is comparable to that due to the polarizability.\(^3\) The Appendix is devoted

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\(^3\) In the chiral limit, the contribution from the magnetic moment term is suppressed relative to the electric polarizability by a factor of $m_\pi/M$. This suppression, however, owes to the singular behavior of the polarizability in that limit, namely $\alpha_E \sim 1/m_\pi$. Because our lattice pion mass is larger than physical, we will make no assumption about the dominance of the polarizability near the chiral limit.
to the extraction of $E_{\text{eff}}(\mathcal{E})$, using unpolarized neutron correlation functions. The correlator in Eq. (5), however, does not allow access to the electric polarizability without knowledge of the magnetic moment.

To extract both the magnetic moment and electric polarizability, we use the boost projection operators

$$\mathcal{P}_\pm = \frac{1}{2}(1 \pm K_3),$$

where $K_3$ is the boost operator in the $\hat{z}$-direction. The boost-projected correlation functions are given by

$$G_\pm(x_4, \mathcal{E}) \equiv \text{Tr}[\mathcal{P}_\pm G(x_4, \mathcal{E})]$$

$$= Z(\mathcal{E}) \left(1 \pm \frac{\mathcal{E}\mu}{2M^2}\right) \exp \left[-x_4 E_{\text{eff}}(\mathcal{E})\right].$$

With the additional electric field dependence present in the amplitude, one can separate the electric polarizability from the magnetic moment by simultaneously analyzing both boost-projected correlators. The method we employ to accomplish this will be detailed below.

### B. Charged Baryons

Consider now a spin-half baryon with charge $Q$. The magnetic moment, $\mu$, is a sum of two terms, $\mu = Q + \tilde{\mu}$. The piece proportional to the charge is the Dirac magnetic moment, while that denoted by $\tilde{\mu}$ is the anomalous magnetic moment. Including terms relevant for a uniform external field, the relativistic single-particle action for a charged baryon has the form

$$S_E = \int d^4x \bar{\psi}(x) \left[ D + E(\mathcal{E}) - \frac{\tilde{\mu}(\mathcal{E})}{4M^2} \sigma_{\mu\nu}F_{\mu\nu} \right] \psi(x),$$

where the electromagnetically gauge covariant derivative is $D_\mu = \partial_\mu + iQA_\mu$, and higher-order terms in the field strength appear parametrically in the $\mathcal{E}$-dependent couplings in Eq. (10). The parameter $E(\mathcal{E})$ is the charged particle’s rest energy, which has the weak field expansion in Eq. (3). The anomalous magnetic moment coupling, $\tilde{\mu}(\mathcal{E})$, has a weak field expansion in even powers of the field, and satisfies the zero-field relation, $\tilde{\mu}(0) = \tilde{\mu}$.

As with the neutral baryons, it is beneficial to consider boost-projected correlation functions. For charged baryons, these have the form

$$G_\pm(x_4, \mathcal{E}) = Z(\mathcal{E}) \left(1 \pm \frac{\tilde{\mu}\mathcal{E}}{2M^2}\right) D \left(x_4, E_{\text{eff}}(\mathcal{E})^2 = Q\mathcal{E}, \mathcal{E}\right),$$

where the function $D(x_4, E^2, \mathcal{E})$ is the relativistic propagator function

$$D(x_4, E^2, \mathcal{E}) = \int_0^\infty ds \sqrt{\frac{Q\mathcal{E}}{2\pi \sinh(Q\mathcal{E}s)}} \exp \left[-\frac{1}{2}Q\mathcal{E}x_4^2 \coth(Q\mathcal{E}s) - \frac{1}{2}E^2s\right].$$

When the charge is set to zero, we recover the neutral baryon correlation functions in Eq. (9). As one can see, to determine charged baryon electric polarizabilities, we must also deduce their anomalous magnetic moments. This can be done utilizing both boost-projected correlation functions. Obviously the fits are delicate, however, the function in Eq. (12) also describes the propagation of a charged scalar in an electric field. Our previous study demonstrated that such fits can be carried out [11].
III. LATTICE DETAILS

To demonstrate our method for extracting nucleon magnetic moments and electric polarizabilities from lattice two-point functions, we have employed an ensemble of anisotropic gauge configurations with $(2+1)$-flavors of dynamical clover fermions [17, 18]. Our ensemble consists of 200 lattices of size $L^3 \times \beta = 20^3 \times 128$. After an initial 1000 thermalization trajectories, the lattices were chosen from an ensemble of 7000 spaced either by 20 or 40 to minimize autocorrelations. The lattice spacing in the spatial directions is $a_s = 0.123\, \text{fm}$ [17, 18], with a non-perturbatively tuned anisotropy parameter of $\xi \equiv a_s/a_t = 3.5$, where $a_t$ is the temporal lattice spacing. The finer temporal spacing is a crucial feature for this study, as it allows us to fit more complicated functional forms for the time-dependence of correlation functions. On the ensemble, the renormalized strange quark mass is near the physical value, while the renormalized light quark mass leads to a pion mass of $m_\pi \approx 390\, \text{MeV}$.

On each configuration, we compute at least 10 propagators for each of the up, down, and strange quarks with random spatial source locations. Multiple inversions were made efficient using the EigCG inverter [19]. Interpolating fields at the source are generated from gauge-covariantly Gaussian-smeared quark fields [20, 21] on a stout-smeared [22] gauge field in order to optimize the overlap onto the ground state. Interpolating fields at the sink are constructed from local quark fields. Each propagator is located with source time at $\tau_{\text{src}} = 0$. Randomization of the source time location, while improving the statistical sampling, would complicate the determination of charged baryon correlation functions, as their two-point functions are no longer time-translationally invariant. The correlator given in Eq. (11) is explicitly a function of the sink time-slice and not simply a function of the source-sink separation. The full dependence on source time is given in [15].

To implement the background field on the lattice, we modify the $SU(3)$ color gauge links, $U_\mu(x)$, for each quark flavor by multiplying by the color-singlet Abelian links, $U_\mu^{(E)}(x)$, for the external field, namely

$$U_\mu(x) \rightarrow U_\mu(x)U_\mu^{(E)}(x)U_\mu^{(E)\perp}(x),$$

where $U_\mu^{(E)}(x) = \exp[iqA_\mu(x)]$, with $q$ as the quark electric charge, and $A_\mu(x)$ given in Eq. (1). The additional transverse links are given by

$$U_\mu^{(E)\perp}(x) = \exp[iqE_\beta x_3 \delta_\mu 4 \delta x_4, \beta - 1].$$

These additional links together with the quantization condition for a torus [23]

$$\mathcal{E} = \frac{2\pi n}{q_d \beta L},$$

ensure that the flux through every elementary plaquette is $\mathcal{E}$ [24, 25]. In the quantization condition, $q_d = -1/3$ is the electric charge of the down quark, and $n$ must take on integer values. As the multiplication in Eq. (13) is carried out on pre-existing gauge configurations, the sea quarks remain electrically neutral. This approximation is imposed because of computational restrictions which will not be rectified in the near future without a significant increase in resources.

Using Eq. (13), we computed propagators for nine values of the field strength, $n$, corresponding to the integer appearing in the quantization condition, Eq. (15). We use $n = 0$, which corresponds to a vanishing external field, as well as $n = \pm 1, \ldots, \pm 4$. On
our lattices, the expansion parameter governing the deformation of a hadron’s pion cloud is given by [15] \((e \mathcal{E}/m^2) = 0.18 n^2\). From the size of this parameter, we anticipate the need to include terms beyond quadratic order in the electric field expansion of hadron energies. In our analysis, we include terms up to quartic order. Larger lattices will be required for better control over systematics relating to the electric field expansion of observables.

IV. LATTICE RESULTS

Nucleon two-point functions were obtained for each source location on a given configuration. Results for multiple source locations on each configuration were then source averaged, yielding lattice correlation functions we denote by: \(g_{\pm}(x_4, \mathcal{E})_i\). Here \(i\) labels the configuration, and \(\pm\) refers to the boost projection. This procedure was repeated for each value of the external field. We have performed multiple differing procedures to analyze the data, of which we detail only one method thoroughly in the text. Consistent results were obtained from the other procedures.

To enforce invariance under parity transformations, under which \(\mathcal{E} \to -\mathcal{E}\), we took the geometric mean of correlators calculated at \(n\) and \(-n\) on each configuration. Specifically from the set of \(g_{\pm}(x_4, \mathcal{E})_i\), we form

\[
g_{\pm}(x_4, \mathcal{E})_i = \sqrt{g_{\pm}(x_4, \mathcal{E})_i g_{\pm}(x_4, -\mathcal{E})_i},
\]

for \(\mathcal{E} \geq 0\). This reduces the nine field values to five, corresponding to the integers \(n = 0, \ldots, 4\). This ensemble of correlation functions was then used to generate 200 bootstrap ensembles. For the ensemble averaged correlation functions, we use the same notation but without a configuration label, namely \(g_{\pm}(x_4, \mathcal{E})\). Fits to the bootstrapped ensemble are performed as described below.

Fits to correlation functions in vanishing electric field are often guided by effective mass plots. Ordinarily one looks for a plateau in the effective mass, \(M_{\text{eff}}(t)\), to ascertain when the excited state contributions have dropped out of the correlator. For the boost projected correlation functions, we define two different effective masses

\[
M_{\text{eff}}^\pm(t) = -\log \frac{g_{\pm}(t + 1, \mathcal{E})}{g_{\pm}(t, \mathcal{E})}.
\]
The situation is quite simple for the case of vanishing electric field; thus we handle this first. When the external field vanishes, the two effective masses are identical, \( M^+_{\text{eff}}(t) = M^-_{\text{eff}}(t) \equiv M_{\text{eff}}(t) \), and the standard analysis applies. Due to our choice of anisotropic lattices, the physical length of the time-direction is too short for the correlator to exhibit a plateau. In Fig. 1, we show the effective mass plot for the nucleon in zero electric field. Statistical noise dominates the correlator beyond the window of time depicted. To extract the mass of the ground state, we perform a two-state fit. The fit function, \( \mathcal{G}(t, \mathcal{E} = 0) \), has the form

\[
\mathcal{G}(t, 0) = Z(0) \exp(-tM) + Z'(0) \exp(-tM'),
\]

where the parameters \( Z(0) \), and \( M \) arise from the ground state, while the primed parameters account for excited state contamination. We use a correlated chi-squared analysis to fit the time dependence of the bootstrap ensemble of correlation functions. As the amplitude parameters \( Z(0) \), and \( Z'(0) \) enter the fit function linearly, we utilize variable projection (see [26] for references) to reduce the number of fit parameters from four down to two. The fit to the zero field nucleon correlation function has also been shown in the figure. The fit window has been determined by comparing single and double effective masses, see [26–28] for details on the latter.

**Neutron**

For non-vanishing electric fields, fit functions for the neutron and proton differ considerably. For the neutron, the fit function is similar in form to the zero-field case, however, there are two distinct fit functions corresponding to the boost projectors \( \mathcal{P}_\pm \), namely

\[
\mathcal{G}_\pm(t, \mathcal{E}) = Z(\mathcal{E}) \left[ 1 \pm \frac{\mu(\mathcal{E})\mathcal{E}}{2M^2} \right] \exp[-tE_{\text{eff}}(\mathcal{E})] + Z'(\mathcal{E}) \left[ 1 \pm \frac{\mu'(\mathcal{E})\mathcal{E}}{2M'^2} \right] \exp[-tE'_{\text{eff}}(\mathcal{E})].
\]

The unprimed parameters are those of the ground state, while the primed parameters account for excited state contributions. Notice both fit functions are identical to Eq. (18) for vanishing electric field. For a fixed non-zero value of the electric field strength, there are six fit parameters, three for the ground state: \( Z(\mathcal{E}) \), \( E(\mathcal{E}) \), \( \frac{\mu(\mathcal{E})\mathcal{E}}{2M^2} \), and similarly three for the excited state contribution. To perform the fits, variable projection is again utilized to remove the overall amplitudes, \( Z(\mathcal{E}) \) and \( Z'(\mathcal{E}) \). This reduces the number of fit parameters from six down to four. To determine the remaining four parameters, we perform simultaneous fits to both boost projected correlators for each value of the external field. In principle, such fits should take into account correlations between the boost projected correlators. We find, however, that the off-diagonal correlations between boost projected correlators are an order of magnitude smaller than the diagonal ones. Thus we treat the boost-projected correlators as uncorrelated, and fit them to the function in Eq. (19) taking into account correlations in time. In Fig. 2, we show the effective mass plots for the boost projected neutron correlation functions. Along with these plots, we show the effective masses resulting from the simultaneous fit to both boost projected correlators. Details of the fits to correlation functions, and the extracted parameters are collected in Table I.

The correlation function fits are carried out on the bootstrap ensemble. In particular, we arrive at an ensemble of ground state energies and ground state magnetic couplings for each magnitude of the electric field \( \mathcal{E} \). These ensembles we generically denote by \( \{ \mathcal{O}_i(\mathcal{E}) \} \), where \( i \) indexes the bootstrap ensemble, \( i = 1, \ldots, N \), and \( \mathcal{O} \) represents either the ground
FIG. 2: Effective mass plots for the boost-projected neutron correlation functions. For each value of the electric field strength, the curved band shows the result of the simultaneous fit to both boost-projected correlation functions using Eq. (19). The band accounts for the uncertainty in the extracted ground state energy, $E(E)$. The flat band shows the extracted value of $E(E)$ with the uncertainty.

state energy $E$, or magnetic coupling $\mu$. As the ensembles of configurations for different field strengths are generated from the same underlying lattice configurations, correlations between the energies for different field strengths will be significant and we account for these. On the bootstrap ensemble of energies and magnetic couplings, we perform electric-field correlated fits to the function $O(E)$, where for the case of the ground state energy, $O(E) = E(E)$, with

$$E(E) = M + \frac{1}{2}4\pi\alpha E E^2 - \frac{1}{4!}(4\pi)^2\alpha E E E^4,$$

(20)
TABLE I: Summary of fit results for neutron two-point functions using the time window: $5 \leq t/a \leq 28$. All quoted values are averages over the bootstrap ensemble, and are given in dimensionless lattice units. For the electric polarizability, $\alpha_{E}^{\text{latt}} = \alpha_{E}(2\alpha_{f.s.} a_{t} a^{2})^{-1}$, with the fine-structure constant $\alpha_{f.s.} = e^{2}/4\pi$, while for the magnetic moment, we have defined $\mu_{\text{latt}} = \mu(2e(a_{t}M)^{3}/(a_{t}M_{N}))^{-1}$, with the physical magnetic moment, $\mu$, given in units of nuclear magnetons, $\mu_{N} = \frac{e}{2M_{N}}$. For the fits, $\chi^{2}/d$ is the chi-squared per degree of freedom, and $1 - P$ is the integrated chi-squared. The first half of the table summarizes the time-correlated fits to the energies and magnetic couplings in each field, while the second half summarizes the field-correlated fits. The two differing fits to the latter are denoted by I and II, and are described in the text. The second uncertainty on polarizabilities and magnetic moments is an estimate of the systematic due to the choice of fit window as explained in the text.

| $N$ | $a_{t}E$ | $\mu_{\text{latt}}$ | $\chi^{2}/d$ | $1 - P$ |
|-----|---------|----------------|-------------|---------|
| 0   | 0.2056(7)| -              | 0.55        | 0.96    |
| 1   | 0.2074(6)| -51(4)        | 0.70        | 0.93    |
| 2   | 0.2142(6)| -52(2)        | 0.91        | 0.65    |
| 3   | 0.2240(6)| -50(1)        | 1.1         | 0.24    |
| 4   | 0.2375(6)| -47(1)        | 1.5         | 0.02    |

| $N$ | $a_{t}M$ | $\alpha_{E}^{\text{latt}}$ | $\chi^{2}/d$ | $1 - P$ | $\mu_{\text{latt}}$ | $\chi^{2}/d$ | $1 - P$ |
|-----|---------|----------------------------|-------------|---------|----------------|-------------|---------|
| I   | 0.2055(2)| 40(1)(2)                  | 0.3         | 0.9     | -52(1)(1)    | 0.6         | 0.7     |
| II  | 0.2054(3)| 42(3)(2)                  | 0.3         | 0.9     | -52(2)(1)    | 0.7         | 0.6     |

and for the case of the ground state magnetic coupling, $O(\mathcal{E}) = \mu(\mathcal{E})$, with

$$
\mu(\mathcal{E}) = \mu + \overline{\mu}_{E} \mathcal{E}^{2} + \overline{\mu}_{EEE} \mathcal{E}^{4}.
$$

(21)

With the ensemble average quantities denoted by $\overline{O}(\mathcal{E}) = \frac{1}{N} \sum_{i} O_{i}(\mathcal{E})$, we minimize the correlated chi-squared, namely

$$
\chi^{2} = \sum_{\mathcal{E},\mathcal{E}'} \left[ \overline{O}(\mathcal{E}) - O(\mathcal{E}) \right] C_{\mathcal{E},\mathcal{E}'}^{-1} \left[ \overline{O}(\mathcal{E}') - O(\mathcal{E}') \right],
$$

(22)

with the field-strength correlation matrix, $C_{\mathcal{E},\mathcal{E}'}$, given by

$$
C_{\mathcal{E},\mathcal{E}'} = \frac{1}{N-1} \sum_{i=1}^{N} \left[ \overline{O}(\mathcal{E}) - O_{i}(\mathcal{E}) \right] \left[ \overline{O}(\mathcal{E}') - O_{i}(\mathcal{E}') \right].
$$

(23)

Because all fit parameters, enter the fit functions $O(\mathcal{E})$ linearly, the chi-squared minimization can be done analytically. Fits to the energy function are carried out on the bootstrap ensemble, resulting fit parameters are averaged, and the uncertainties from fitting and bootstrapping are added in quadrature. The same is done for the magnetic moment function. We find that the best fits result from taking $\mu_{E} = 0$, and results quoted for the neutron use this constraint. Furthermore we perform two different field-correlated fits as follows: (I) a
FIG. 3: Electric field strength dependence of extracted neutron parameters. The two different fits (I and II) are described in the text. The bands on the energy plots show the uncertainty in the extracted value for the polarizability, while the bands on the magnetic coupling plots show the uncertainty in the extracted magnetic moment.

fit to all five field strengths using Eqs. (20) and (21), (II) the same fit function but with the largest field strength excluded. Finally, to estimate the systematics due to the choice of fit window, we performed uncorrelated fits to the electric field dependence of meson energies determined on adjacent fit windows. We chose the nine fit windows obtained by varying the start and end times by one unit in either direction. On each time window, we determined the electric polarizability and magnetic moment. The systematic uncertainty on these observables due to the fit window is estimated as the standard deviation of the extracted observables over the various adjacent windows. Details of the correlated electric field fits and extracted parameters are tabulated in Table I.

From the extracted parameters, we can investigate the electric field dependence of the energies and magnetic moment couplings. This is done in Fig. 3, where we plot the field strength dependence of these quantities. The plots, moreover, show the results of the two fits (I and II) to the electric field dependence. The values of the extracted parameters are consistent with naïve expectations. For the magnetic moment of the neutron in units of nuclear magnetons, we find $\mu_{n}^{\text{conn}}(m_{\pi} = 390\text{MeV}) = -1.69(6)(2)(5)\mu_{N}$. We have appended a superscript to reflect that our computation includes only connected contributions. The three uncertainties quoted are from: (i) statistics and fitting, (ii) the systematic due to the fit window, and (iii) conversion to units of physical nuclear magnetons. For (i), we take the largest value of the uncertainty from the two fits to the field-strength dependence. There are additional sources of systematic uncertainty that are unaccounted for, namely the effects of finite lattice spacing and finite lattice volume. For the neutron electric polarizability, we find $\alpha_{E}^{n,\text{conn}}(m_{\pi} = 390\text{MeV}) = 3.1(2)(2)(3)\times 10^{-4}\text{fm}^{3}$. The first two uncertainties quoted are as for the magnetic moment, while the third arises from scale setting.
FIG. 4: Effective mass plots for the boost-projected proton correlation functions. For each value of the electric field strength, the curved upper band shows the result of the simultaneous fit to both boost-projected correlation functions using Eq. (24). The band accounts for the uncertainty in the extracted ground state rest energy, $E(E)$. The curved lower band shows the contribution to the fit from only the ground state, while the flat band shows just the extracted value of $E(E)$ with its uncertainty.

Proton

For the proton, we perform a similar analysis. Fits to proton correlation functions are carried out using a two-state fit function. This allows us to remove excited state contamination. We perform simultaneous time-correlated fits to both boost projected proton
TABLE II: Summary of fit results for proton two-point functions using the time window: $6 \leq t/a \leq 28$. Tabulated entries are as in Table I, with the exception that we have denoted the magnetic couplings as anomalous using tildes.

| $P$ | $n$ | $a_t E(\mathcal{E})$ | $\tilde{\mu}^{\text{latt}}(\mathcal{E})$ | $\chi^2/d$ | $1-P$ |
|-----|-----|----------------------|----------------------|-----------|-------|
| 0   |     | 0.2052(8)            | -                    | 0.53      | 0.96  |
| 1   |     | 0.2072(5)            | 53(4)                | 0.62      | 0.98  |
| 2   |     | 0.2118(6)            | 48(2)                | 0.81      | 0.80  |
| 3   |     | 0.2198(7)            | 46(1)                | 0.84      | 0.76  |
| 4   |     | 0.2293(9)            | 41(1)                | 1.5       | 0.02  |

| $P$ | $a_t M$ | $\alpha_E^{\text{latt}}$ | $\chi^2/d$ | $1-P$ | $\tilde{\mu}^{\text{latt}}$ | $\chi^2/d$ | $1-P$ |
|-----|---------|------------------------|-------------|-------|------------------|-------------|-------|
| I   | 0.2053(2) | 32(1)(1)             | 0.13        | 0.98  | 52(1)(1)         | 1.3         | 0.3   |
| II  | 0.2054(3) | 31(3)(4)             | 0.16        | 0.96  | 52(3)(1)         | 1.7         | 0.2   |

correlation functions using the fit function

$$G_x(t, \mathcal{E}) = Z(\mathcal{E}) \left[ 1 \pm \frac{\tilde{\mu}(\mathcal{E}) E}{2 M^2} \right] D \left( t, E_{\text{eff}}(\mathcal{E})^2 \mp \mathcal{E}, \mathcal{E} \right)$$

$$+ Z'(\mathcal{E}) \left[ 1 \pm \frac{\tilde{\mu}'(\mathcal{E}) E}{2 M^2} \right] D \left( t, E_{\text{eff}}'(\mathcal{E})^2 \mp \mathcal{E}, \mathcal{E} \right), \quad (24)$$

with $D(x, E^2, \mathcal{E})$ as the relativistic propagator function given in Eq. (12). As the overall amplitudes $Z(\mathcal{E})$ and $Z'(\mathcal{E})$ enter the fit function linearly, we utilize variable projection to eliminate them from the simultaneous fits. For each value of the electric field $\mathcal{E}$, there are then four parameters in the fit: the ground state rest energy $E(\mathcal{E})$, the ground state anomalous magnetic coupling $\tilde{\mu}(\mathcal{E})$, as well the rest energy and anomalous magnetic coupling for the excited state. In Fig. 4, we show the effective mass plots for the boost projected proton correlation functions. Along with these plots, we show the effective masses resulting from the simultaneous fit to both boost projected correlators using Eq. (24). Details of the fits to proton correlation functions, and the extracted parameters from the fits are collected in Table II.

We perform fits to proton correlations functions on the entire bootstrap ensemble. This enables us to form an ensemble of extracted parameters for each value of the field strength. In particular, we consider the ensemble of extracted ground state rest energies, $\{E_i(\mathcal{E})\}$, and ground state anomalous magnetic moments, $\{\tilde{\mu}_i(\mathcal{E})\}$. Collectively we denote these ensembles by $\mathcal{O}_i(\mathcal{E})$, with the ensemble average denoted by $\overline{\mathcal{O}}(\mathcal{E})$. Electric field correlated fits are performed using the fit functions in Eqs. (20) and (21). For the latter it is the electric field dependence of the anomalous couplings that is being fit. Furthermore, we extract the anomalous magnetic moment using the constraint $\overline{\mu}_{EEE} = 0$, which results in better fits. For both observables, we perform the fit using all the data (fit I), and excluding results for the largest field strength (fit II). Results of the fits are collected in Table II. The electric field strength dependence of the extracted rest energies and anomalous magnetic couplings are shown in Fig. 5. Also depicted are the field-correlated fits to these
quantities. The values of the extracted parameters are again consistent with naïve expectations. For the magnetic moment of the proton in units of nuclear magnetons, we find
\[ \mu_p^{\text{conn}}(m_\pi = 390 \text{MeV}) = 2.69(15)(1)(5) [\mu_N] \]. We have appended a superscript to reflect that our computation includes only connected contributions; we have also added in the Dirac contribution. The three uncertainties quoted are as before: (i) statistics and fitting, (ii) the systematic due to the fit window, and (iii) conversion to physical units. For (i), we take double the larger value of the uncertainty from the two fits to the field-strength dependence because of the comparatively poor quality of fit. For the proton electric polarizability, we find
\[ \alpha_p^E(m_\pi = 390 \text{MeV}) = 2.4(2)(3)(2) \times 10^{-4} \text{fm}^3 \], where the last uncertainty arises from scale setting.

Finally let us compare results for the neutron and proton. Within the uncertainty, the extracted anomalous magnetic moments are exactly isovector. The isovector combination of moments, moreover, is independent of the electric charges of the sea quarks due to strong isospin symmetry. For the nucleon isovector magnetic moment, we find
\[ \mu_V(m_\pi = 390 \text{MeV}) \sim 4.4 [\mu_N] \]. While this is smaller than the physical moment, chiral corrections drive the magnetic moment downward. Studies at additional pion masses are necessary to extrapolate to the physical point. For the electric polarizabilities, our results show both isovector and isoscalar components, however, the latter is the dominant one. This is also seen experimentally and from chiral perturbation theory. The smaller isovector component, \[ \alpha_V^E(m_\pi = 390 \text{MeV}) \sim 0.7 \times 10^{-4} \text{fm}^3 \], receives smaller chiral corrections, and is less sensitive to the electric charges of the sea (but not independent). While values for the electric polarizabilities of the neutron and proton are smaller than experiment, chiral perturbation theory suggests large corrections as one nears the chiral limit. Additionally including contributions from sea quark electric charges will drive both polarizabilities upwards. It will be interesting to carry out simulations at additional quark masses and with electrically charged sea quarks to observe this behavior.
V. CONCLUSION

Above we investigate the relativistic propagator for spin-half particles in classical electric fields. The presence of magnetic moments affects the behavior of two-point correlation functions, and we use this observation to devise a method to determine magnetic moments and electric polarizabilities from lattice QCD simulated in background electric fields. Using anisotropic gauge configurations with dynamical clover fermions, we perform such computations. In the Appendix, we obtain results for the neutron using unpolarized lattice correlation functions. Such results, however, do not allow one to determine the electric polarizability—only a combination of the electric polarizability and the square of the magnetic moment. The separation of these terms requires treatment of baryon spin, which is afforded by studying boost-projected correlators. Our analysis of boost-projected lattice correlation functions demonstrates that nucleon magnetic moments and electric polarizabilities can be extracted from simulations in background electric fields. This applies to both the neutron and proton.

There are a number of refinements possible to our computation that would reduce the systematic uncertainties. Currently our calculations are limited to electrically neutral sea quarks. Various methods could be employed to remedy this situation. Furthermore studies on larger volumes will not only reduce finite volume effects, but allow the implementation of smaller values of the quantized external field strength. Simulations at various values of the quark mass will allow for chiral extrapolations to make contact with the physical QCD point. We intend to carry out this work in the future. Finally our approach can be used to study the magnetic moments and electric polarizabilities of the remaining members of the baryon octet.

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Appendix A: Analysis of Unpolarized Neutron Correlation Functions

Here we present the analysis of unpolarized neutron correlation functions. These correlation functions are obtained simply by taking the average of boost-projected correlators. On each configuration, for each value of the electric field strength, we form the unpolarized, source-averaged lattice correlation function, \( \overline{g}(x_4, E)_i \), given by \( \overline{g}(x_4, E)_i = \frac{1}{2} [g_+(x_4, E)_i + g_-(x_4, E)_i] \). Parity invariance is enforced by taking the geometric mean of correlators obtained for a given field value and its negative. Specifically we form

\[
\overline{g}(x_4, E)_i = \sqrt{g(x_4, E)_i \overline{g}(x_4, -E)_i},
\]

(A1)
FIG. 6: Effective mass plots for unpolarized neutron correlation functions. For each value of the electric field strength, the curved band shows the result of the fit to unpolarized correlation functions using Eq. (A3). The band accounts for the uncertainty in the extracted ground-state effective energy, $E_{\text{eff}}(E)$. The flat band shows the extracted value of $E_{\text{eff}}(E)$ with the uncertainty.

for $E \geq 0$. This ensemble of unpolarized correlators was then used to generate 200 bootstrap ensembles for $n = 0, \ldots, 4$. The average unpolarized correlator is similarly denoted but without the subscript referring to configuration number, namely by $\overline{g}(x_4, E)$. The standard effective mass is then formed

$$M_{\text{eff}}(t) = -\log \frac{\overline{g}(t + 1, E)}{\overline{g}(t, E)}, \quad (A2)$$

and is used to guide spectroscopic analysis of the unpolarized correlators.

For a given value of the electric field, $E$, we extract the effective energy, $E_{\text{eff}}(E)$ given in Eq. (6), using a two-state fit function of the form

$$\overline{g}(t, E) = Z(E) \exp [-tE_{\text{eff}}(E)] + Z'(E) \exp [-tE'_{\text{eff}}(E)]. \quad (A3)$$

Notice the fit function contains the same parameters as in Eq. (19); however, without the boost projection, we cannot disentangle the magnetic moment contribution. While there are four parameters to fit in Eq. (A3), we utilize variable projection to eliminate the amplitudes $Z(E)$ and $Z'(E)$, leaving just two parameters: the effective energy of the ground and excited states. Time-correlated fits are performed, with results shown in Fig. 6. Fit details and extracted parameters are collected in Table III.

Fits are carried out on the entire bootstrap ensemble enabling us to form an ensemble of extracted effective energies of the ground state, $\{E_{\text{eff}, i}(E)\}$. The average of this ensemble we denote $E_{\text{eff}}(E)$. Using the fit function

$$E_{\text{eff}}(E) = M + A_E E^2 + B_E E^4, \quad (A4)$$

electric field correlated fits are performed using Eq. (22). We perform two fits: a fit using all the data (fit I), and a fit that excludes results for the largest field strength (fit II). Results of
TABLE III: Summary of fit results for unpolarized neutron two-point functions using the time window: $5 \leq t/a \leq 28$. All quoted values are averages over the bootstrap ensemble, and are given in dimensionless lattice units. The quantity $A_{E}^{\text{latt}}$ is the pseudo-polarizability defined in Eq. (A5). For the fits, $\chi^2/d$ is the chi-squared per degree of freedom, and $1 - P$ is the integrated chi-squared. The first half of the table summarizes the time-correlated fits to the effective energies in each field, while the second half summarizes the field-correlated fits. The two differing fits to the latter are denoted by I and II, and are described in the text. The second uncertainty on the pseudo-polarizability is an estimate of the systematic due to the choice of fit window as explained in the text.

| $N$ | $n$ | $a_t E_{\text{eff}}(E)$ | $\chi^2/d$ | $1 - P$ |
|-----|-----|------------------|-----------|--------|
| 0   | 2041(7) | 0.50             | 0.97      |
| 1   | 2058(7) | 0.67             | 0.88      |
| 2   | 2082(8) | 0.97             | 0.50      |
| 3   | 2130(7) | 0.70             | 0.85      |
| 4   | 2204(8) | 0.70             | 0.85      |

| $N$ | $a_t M$ | $A_{E}^{\text{latt}}$ | $\chi^2/d$ | $1 - P$ |
|-----|---------|---------------------|-----------|--------|
| I   | 0.2045(3) | 17(1)(1)           | 0.53      | 0.75   |
| II  | 0.2044(3) | 20(4)(3)           | 0.52      | 0.72   |

FIG. 7: Electric field strength dependence of the neutron effective energy. The two different fits (I and II) are described in the text. The bands show the uncertainty in the extracted value for the pseudo-polarizability.

the fits are collected in Table III. The coefficient of the term quadratic in the field strength is not the electric polarizability because it includes contributions from the magnetic moment. For this reason, we call this coefficient the pseudo-polarizability; and, in our choice of lattice units, the pseudo-polarizability is given by

$$A_{E}^{\text{latt}} = \alpha_{E}^{\text{latt}} - \frac{a_t M}{2\xi^2}(\mu_{latt})^2,$$

with $\xi$ as the anisotropy factor. The systematic due to the fit window is estimated by performing uncorrelated fits on the adjacent fit windows obtained by varying the start and end times by one unit.
The extracted value for the pseudo-polarizability is roughly half the size of the electric polarizability, see Table I. Because of the sign of the magnetic moment contribution, we expect the pseudo-polarizability to be less than the electric polarizability. The values of the magnetic moment and electric polarizability extracted from boost-projected correlators can be used to find a value for the pseudo-polarizability. Using the results of Table I in Eq. (A5), we find

\[ I : A_{\text{E}}^{\text{latt}} = 17(1)(2), \quad \text{and} \quad II : A_{\text{E}}^{\text{latt}} = 19(3)(2), \]  

(A6)

for the two field-correlated fits. These values are concordant with those found in Table III from analyzing the unpolarized neutron correlators.

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