Definable and Non-definable Notions of Structure

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Abstract

Definability is a key notion in the theory of Grothendieck fibrations that characterises when an external property of objects can be accessed from within the internal logic of the base of a fibration. In this paper we consider a generalisation of definability from properties of objects to structures on objects, introduced by Shulman under the name local representability.

We first develop some general theory and show how to recover existing notions due to Bénabou and Johnstone as special cases. We give several examples of definable and non-definable notions of structure, focusing on algebraic weak factorisation systems, which can be naturally viewed as notions of structure on codomain fibrations. Regarding definability, we give a sufficient criterion for cofibrantly generated awfs’s to be definable, generalising a construction of the universe for cubical sets, but also including some very different looking examples that do not satisfy finiteness in the internal sense, that exponential functors have a right adjoint. Our examples of non-definability include the identification of logical principles holding for the interval objects in simplicial sets and Bezem-Coquand-Huber cubical sets that suffice to show a certain definition of Kan fibration is not definable.

1 Introduction

1.1 Definability

In naïve category theory one often makes use of an external notion of set. For example, in locally small categories $\text{hom}(X,Y)$ is a set, a complete category is one with all small limits and the general adjoint functor theorem makes essential use of the solution set condition. This ties the definitions and results to an often unspecified theory of sets, typically understood to be Zermelo-Fraenkel set theory with choice. This use of set theory is often unnecessary and the link can be severed through the use of Grothendieck fibrations. We think of the base of the fibration as the foundation of mathematics where we are working. This could be “the” category of sets via a set indexed family fibration, but could also be a specific model of $\text{ZF}$, or more generally an elementary topos, or even more generally a category satisfying even weaker conditions. For example, in
this paper we will consider examples where the base is a locally cartesian closed
category, and examples where the base is the category of all small categories.

When working over a fibration, it is useful to know when an external property
of objects in the total category can be referred to from within the internal logic
of the base. This idea can be captured surprisingly well through an elegant
notion due to Bénabou referred to as definability \[Bèn85\].

Whereas Bénabou’s definition referred only to properties of objects, the same
idea can be applied to structures on objects. For example, on a fibration of
vertical maps \( V(\mathcal{E}) \to \mathcal{B} \) we could consider the class of maps with the property
of being a split epimorphism, and given a map, we can consider the collection of
all sections witnessing the map as a split epimorphism. To give a more extreme
e xample, when working in the internal logic of a topos we can talk about a given
object having the property of admitting a group structure, but it is more useful
to be able to talk about the collection of group structures on an object.

The concept of definability was generalised to certain structures by John-
stone [Joh02, Section B1.3] under the name comprehension schemes. A related
idea was also considered early on by Lawvere [Law70]. However, in this paper
we will consider an alternative definition due to Shulman [Shu19, Section 3].
Although our definition is based on and essentially equivalent to Shulman’s we
will give a reformulation that emphasises its role as a generalisation of the earlier
ideas by Bénabou and Johnstone. For this reason we will mainly use the termi-
nology definable, following Bénabou in place of Shulman’s locally representable.

1.2 Algebraic weak factorisation systems

The concept of weak factorisation system (wfs) is fundamental in homotopical
algebra, as a key ingredient in Quillen’s definition of model category [Qui67].
For any weak factorisation system on a category \( \mathcal{C} \), the class of right maps, gives
a class of objects in the codomain fibration \( \text{cod} : \mathcal{C} \to \mathcal{C} \) which is closed under
reindexing, including as a special case the fibrations in a model structure. As
observed by Shulman [Shu19], when working in the semantics of type theory it is
natural to ask when the fibrations in a model category are definable, or failing
that, when they can be replaced by a definable notion of structure. Given a
definable notion of structure, it is straightforward to construct universes that
can be used when modelling type theory. In the absence of a definable notion of
structure there is not a clear way to define universes for type theory in general.

Algebraic weak factorisation systems (awfs) are a structured version of wfs,
first introduced by Grandis and Tholen under the name natural weak factori-
sation system [GT06]. In an awfs the class of left maps is replaced by the
category of coalgebras for a comonad, and the class of right maps by a category
of algebras for a monad. In [BG16] Bourke and Garner gave a new alternative
definition of awfs, proving that is equivalent to the earlier definition. According
to this definition we can understand awfs’s as monadic notions of structure on a
codomain fibration, together with some extra structure in the form of a “com-
position functor.” Presented like this, we can naturally define awfs’s as being
definable simply when the underlying notion of structure is definable.
When studying the semantics of homotopy type theory constructively, e.g. as in \cite{Awo19,vdB22,BCH14,CCHM18,CH22,LOPS18,OP16}, it is usual to define the universe of small fibrations not as small maps that are Kan fibrations, but as small maps together with fibration structure. For this reason, it is more natural to consider Kan fibrations as part of an awfs, rather than as merely a class of maps in a wfs. Just as for wfs’s, when constructing the universe it is natural to ask that the awfs is definable. Although it is unclear whether definability is strictly necessary to model universes in type theory\footnote{We could also consider the weaker requirement that given a fibration structure on a map \( f \) we can witness \( f \) as a pullback of the universe map \( \tilde{U} \to U \) in a not necessarily unique way, and this may be sufficient for the semantics of type theory.}, it does appear to play an important role in the constructive models of type theory known to the author, including all those in the references above.

Aside from the semantics of type theory, the question of definability of awfs’s is an interesting topic for two reasons.

As explained above, definability is a rich topic in itself, and from this point of view awfs’s are a source of interesting examples both of definability and non-definability. In many of the other examples we will see in this paper definability is something that we can get “for free” from general arguments, often using local smallness of a fibration, or has no chance at all of holding. On the other hand, awfs’s provide examples of notions of structure where definability holds or does not hold for non trivial reasons. We will show that definability holds whenever an awfs is cofibrantly generated by a family of maps whose codomain is a “family of tiny objects,” recovering some known instances of definability as a corollary. We will also see some non trivial examples of awfs’s that are not definable.

Secondly, we can view the question of definability of a natural one within the field of awfs’s. Definable awfs’s have yet to be studied in detail, but we can already observe the following interesting property. One of the key properties of awfs’s that improves the situation with wfs’s is that left maps are closed under colimits, and right maps are closed under limits. More precisely, if we are given a diagram of right maps in a wfs, then the limit is not necessarily a right map. However, if we are given a diagram of maps that factors through the category of right maps in an awfs, then the limit is a right map, simply as corollary of the fact that the forgetful functor on right map structures is monadic, and so creates limits. In a definable awfs right maps are also stable under certain colimits, namely those for diagrams that factor through the category of right map structures and cartesian homomorphisms. This can be seen as a corollary of the fact that for a definable awfs the right maps are both monadic and comonadic in the following sense: the usual category of right maps and all homomorphisms is monadic, as for any awfs, whereas definability precisely tells us that the wide subcategory of cartesian homomorphisms is comonadic.
A note on set theoretic foundations

In some places of this paper we made use of categories of presheaves on large categories. This is unproblematic in the presence of sufficient large cardinals. However, the aforementioned presheaf categories are only used together with simple algebraic arguments that can easily be adapted into direct arguments that do not require the presheaf categories to exist. In this way the main results of this paper do not depend on large cardinals, or indeed on much set theory at all.

To avoid the use of the axiom of choice, we follow the convention that all categorical structure is “cloven.” That is, whenever we require the existence of a collection of such objects as limits, colimits and cartesian maps, we in fact require an operator assigning a choice of these objects.

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2 Some background and useful lemmas

We start with a few basic observations about discrete fibrations and adjunctions over a fibration including some key lemmas.

2.1 Discrete fibrations

Suppose we are given a discrete fibration \( P : \mathbb{C} \rightarrow \mathbb{D} \). For each \( D \in \mathbb{D} \) we define a “local” version, \( P_D \) as the following pullback.

\[
\begin{array}{ccc}
(P \downarrow D) & \longrightarrow & \mathbb{C} \\
\downarrow p_D & & \downarrow p \\
\mathbb{D}/D & \longrightarrow & \mathbb{D}
\end{array}
\]  

(1)

**Definition 2.1.** Given any functor \( F : \mathbb{C} \rightarrow \mathbb{D} \) and an object \( D \in \mathbb{D} \) we say the right adjoint to \( F \) is defined at \( D \) if we are given a terminal object of the comma category \((P \downarrow D)\).

**Lemma 2.2.** Let \( P : \mathbb{C} \rightarrow \mathbb{D} \) be a discrete fibration and \( D \) an object of \( \mathbb{D} \). The right adjoint to \( P \) is defined at \( D \) if and only if the discrete fibration \( P_D \) defined in (1) is representable as a presheaf.
Proof. We recall that a discrete fibration corresponds to a representable presheaf if and only if its domain has a terminal object. However, in (1) we explicitly described the domain of $P_D$ as the comma category $(P ↓ D)$.

One of the key ideas in our presentation of the general theory of definability will be the link between representability, existence of a right adjoint, and comonadicity, which will be a special case of the lemma below.

**Lemma 2.3.** The following are equivalent.

1. For every $D ∈ D$, the discrete fibration $P_D$ defined in (1) is representable as a presheaf.
2. $P$ has a right adjoint.
3. $P$ is comonadic.

Proof. For $1 ⇔ 2$ we apply Lemma 2.2, recalling that $P$ has a right adjoint if and only if the right adjoint is defined at $D$ for all objects $D$ of $D$.

To see $2 ⇒ 3$, note that discrete fibrations create connected limits and in particular create $P$-split equalizers and so we can apply Beck’s theorem to show $P$ is comonadic.

Although we are interested in discrete fibrations that have a right adjoint, we note in passing that discrete fibrations do not have left adjoints except in the trivial case.

**Proposition 2.4.** A discrete fibration $P : C → D$ has a left adjoint iff it is an isomorphism of categories.

Proof. Let $D ∈ D$. Note that we can explicitly describe the comma category $(D ↓ P)$ as follows. Each object $D → PC$ corresponds to an object $\hat{D}$ such that $P\hat{D} = D$ and a map $\hat{D} → C$ in $C$. A morphism is a map $C → C'$ making a commutative triangle as below

$$
\begin{array}{ccc}
D & \rightarrow & PC \\
& \searrow & \downarrow \\
& & PC'
\end{array}
$$

If such a morphism exists the two objects must have the same $\hat{D}$.

Hence if $(D ↓ P)$ has an initial object then $P^{-1}(D)$ contains exactly one object. It follows that $P$ is full, and $P$ is faithful in any case. Hence $P$ is an isomorphism of categories.

---

2 In general a Grothendieck fibration creates any limits that exist in its fibres and are preserved by reindexing.
2.2 Adjunctions over fibrations

Definition 2.5. An adjunction over $\mathcal{B}$ consists of fibrations $p : \mathcal{D} \to \mathcal{B}$ and $q : \mathcal{E} \to \mathcal{B}$ together with an adjunction $F \dashv G : \mathcal{D} \to \mathcal{B}$ such that $F$ and $G$ commute with $p$ and $q$, as illustrated below, and the unit and counit of the adjunction can be chosen to be vertical.

We say the adjunction is fibred if $F$ preserves cartesian maps.

Remark 2.6. We can view the above definition as an instance of adjunction for 2-categories, by considering the 2-category whose underlying 1-category is $\text{Cat}/\mathcal{B}$ and with 2-cells consisting of pointwise vertical natural transformations. See e.g. [KS74].

Often in this paper we will switch between fibred adjoints, adjoints over a fibration and ordinary adjoints in categories. The lemmas below make it clear when these turn out to be equivalent, allowing us to drop the distinction between the different definitions.

Lemma 2.7. Suppose we are given an adjunction $F \dashv G$ over $\mathcal{B}$. Then $G$ preserves cartesian maps.

Moreover, suppose $F$ has a partial right adjoint $G$ over $\mathcal{B}$ defined at an object $X$ of $\mathcal{D}$ over $I \in \mathcal{B}$. If $f : X' \to X$ is cartesian, then $G$ is also defined at $X'$ and the map $G(f) : GX' \to GX$ is cartesian.

Proof. By definition of partial right adjoint over $\mathcal{B}$, we have an object $FX$ in $\mathcal{E}$ and a vertical map $\epsilon_X : FGX \to X$ which is terminal in $(F \downarrow X)$. We define $\sigma : I \to J$ to be $q(f)$. We then have a diagram in $\mathcal{D}$ given by the solid lines below.

This gives us a unique vertical map $\epsilon_{X'}$, as in the dotted line above making a commutative square. We verify that this map is terminal in $(F \downarrow X')$.

Suppose we have an object $Y$ of $\mathcal{E}$ and map $h : FY \to X'$. By composing with $f$ we have an object of $(F \downarrow X)$ and so a unique map $t : Y \to GX$ making the commutative square below.

$$
\begin{array}{ccc}
FY & \xrightarrow{Ff} & FGX \\
\downarrow & & \downarrow \epsilon_X \\
X' & \xrightarrow{f} & X
\end{array}
$$
Now using the fact that $\bar{\sigma}$ is cartesian we get the dotted map $s$ in the commutative diagram below which is unique making the diagram commute and such that $q(s) = p(h)$.

\[
\begin{array}{ccc}
Y & \xrightarrow{s} & \sigma^*(GX) \\
\downarrow & & \downarrow \\
\sigma^*(GX) & \rightarrow & GX
\end{array}
\]

Finally the fact that $\epsilon_X \circ Fs = h$ follows from the fact that $f$ is cartesian. \hfill \Box

**Remark 2.8.** The first part of the above lemma is a folklore result, that appears as [Jac99, Exercise 1.8.5] for instance.

**Lemma 2.9.** Suppose we are given fibrations $p : D \rightarrow B$ and $q : E \rightarrow B$ and a functor $G : D \rightarrow E$ over $B$ such that $G$ has a left adjoint as a functor in $\mathbf{Cat}$. Then $G$ has a left adjoint over $B$ if and only if it preserves cartesian maps.

**Proof.** The implication ($\Rightarrow$) follows from Lemma 2.7

We show the implication ($\Leftarrow$). Let $E$ be an object of $E$. We have an initial object of $(E \downarrow G)$, say $\eta_{E} : E \rightarrow GFE$. Say that $\eta_{E}$ lies over a map $\sigma$ in $B$. We have a cartesian map $\bar{\sigma}(FE) : \sigma^*(FE) \rightarrow FE$. Since $G$ is fibred, $G(\bar{\sigma}(FE))$ is also cartesian and lies over $\sigma$. Hence there is a unique vertical map $\eta'_{E} : E \rightarrow G(\sigma^*(FE))$ making a commutative triangle, as illustrated below.

\[
\begin{array}{ccc}
E & \xrightarrow{\eta_{E}} & GFE \\
\downarrow \eta'_{E} & & \downarrow \sigma \\
G(\sigma^*(FE)) & \rightarrow & GFE
\end{array}
\]

By the initiality of $\eta_{E}$ in $(E \downarrow G)$, we have a unique map $t : FE \rightarrow \sigma^*(FE)$, as below.

\[
\begin{array}{ccc}
E & \xrightarrow{\eta_{E}} & GFE \\
\downarrow \eta'_{E} & & \downarrow t \\
G(\sigma^*(FE)) & \rightarrow & \sigma^*(FE)
\end{array}
\]

Write $\tau$ for the map $p(t)$ in $B$. Applying $q$ to the left hand diagram above, we see $\tau \circ \sigma = 1_{q(E)}$.

We can view both $t$ and $\bar{\sigma}(FE)$ as morphisms in $(E \downarrow G)$ and then compose them to get a morphism from the object $\eta_{E} : E \rightarrow GFE$ to itself, as illustrated below.

\[
\begin{array}{ccc}
E & \xrightarrow{\eta_{E}} & GFE \\
\downarrow \eta'_{E} & & \downarrow \sigma \\
G(\sigma^*(FE)) & \rightarrow & \sigma^*(FE)
\end{array}
\]
By initiality this morphism can only be the identity on $\eta_E$. We conclude that $\bar{\sigma}(FE) \circ t = 1_{FE}$, and so, applying $p$ we see $\sigma \circ \tau = 1_{p(FE)}$. We can now see that $\sigma$ is an isomorphism with inverse $\tau$. It follows that $\bar{\sigma}(FE)$ is also an isomorphism, and so $\bar{\sigma}_E : E \to G(\sigma^*(FE))$ is initial in $(E \downarrow G)$. By applying this for each object $E$, we can construct a left adjoint $F'$ that strictly commutes with $p$ and $q$ with a vertical unit $\eta'_E$.

We can immediately deduce the following proposition.

**Proposition 2.10.** Suppose we are given a fibred functor $U : D \to E$ between Grothendieck fibrations. Then $U$ is strictly monadic as a functor in $\textbf{Cat}$ if and only if it is strictly monadic for a monad over $\mathbb{B}$.

## 3 Notions of structure and definability

We now give the definition of notion of structure on a fibration, which is essentially equivalent to Shulman’s notion of fibred structure [Shu19, Section 3].

**Definition 3.1.** Given a fibred functor $\chi$, we write $\text{Cart}(\chi) : \text{Cart}(D) \to \text{Cart}(E)$ for the restriction to cartesian maps.

**Definition 3.2.** Suppose we are given a fibred functor between Grothendieck fibrations, as illustrated below.

\[
\begin{array}{ccc}
D & \xrightarrow{\chi} & E \\
p & \downarrow \quad & \downarrow q \\
\mathbb{B} & \xleftarrow{\chi^{-1}} & \mathbb{B}
\end{array}
\]

We say $\chi$ creates cartesian lifts if $\text{Cart}(\chi)$ is a discrete fibration.

We say a fibred notion of structure, or just notion of structure on a Grothendieck fibration $q : E \to \mathbb{B}$ is another fibration $p : D \to \mathbb{B}$ together with a functor $\chi$ from $D$ to $E$ that creates cartesian lifts.

We can understand the definition of notion of structure through the following proposition, whose proof is left as an exercise for the reader.

**Proposition 3.3.** A fibred functor $\chi : D \to E$ creates cartesian lifts if and only if for each $I \in \mathbb{B}$ the restriction of $\chi_I : D_I \to E_I$ to isomorphisms is a discrete fibration.

An object $X$ of $E_I$, is typically an $I$-indexed family $(X_i)_{i \in I}$ in some sense. We think of objects the fibre $\chi_I^{-1}(X)$ as a choice of structure on each object $X_i$ in the family. We think of morphisms in $D$ as families of structure preserving homomorphisms. The condition of creating cartesian lifts says that given an isomorphism $f : X_i \cong Y_i$ and a structure on $Y_i$ we can find a unique structure on $X_i$ making $f$ a structure preserving isomorphism.
Definition 3.4. We say a notion of structure, \( \chi \) is **definable** if \( \text{Cart}\chi \) has a right adjoint, as an ordinary functor between categories (not over \( \mathbb{B} \)).

Remark 3.5. It might look a little strange to only require the right adjoint to exist in categories, without even needing it to commute with the fibrations. We observe however, that this is also what happens with the notion of multi left adjoint \([\text{Die}79, \text{Section 3}]\). Namely, any functor between categories \( C \) and \( D \) corresponds to a unique fibred functor between set indexed family fibrations. A multi left adjoint to a functor \( F : C \to D \) is precisely a left adjoint to the corresponding functor \( \text{Fam}(C) \to \text{Fam}(D) \). A right adjoint to the functor \( \text{Fam}(C) \to \text{Fam}(D) \) in categories can be seen as a “multi right adjoint” following the same idea as multi left adjoints.

To better understand the definition of definability we give some alternative versions of the definition below.

Definition 3.6. Given \( X \in \mathbb{E} \) we write \( \tilde{\chi}_X \) for the presheaf on \( \mathbb{B}/q(X) \) defined as follows. Given an object \( \sigma : I \to q(X) \) in \( \mathbb{B}/q(X) \), we define \( \tilde{\chi}_X(\sigma) \) to be set of objects of \( \chi^{-1}(\sigma^*(X)) \).

Lemma 3.7. Let \( X \) be an object of \( \mathbb{E} \). Then the right adjoint to \( \text{Cart}(\chi) \) is defined at \( X \) if and only if the presheaf \( \tilde{\chi}_X \) is representable.

Proof. Note that we have an equivalence of categories \( \mathbb{B}/q(X) \cong \text{Cart}(\mathbb{D})/X \) and that \( \tilde{\chi}_X \) corresponds to the discrete fibration obtained by pulling back along the composition of the equivalence and \( \text{dom} : \text{Cart}(\mathbb{E})/X \to \text{Cart}(\mathbb{E}) \) as illustrated below.

```
\begin{array}{ccc}
\mathbb{B}/q(X) & \overset{\cong}{\longrightarrow} & \text{Cart}(\mathbb{D})/X \\
\downarrow & & \downarrow \\
\text{Cart}(\mathbb{E})/X & \longrightarrow & \text{Cart}(\mathbb{E}) \\
\end{array}
```

Hence we can apply Lemma 2.2 together with the equivalence in the left hand square above.

We note that when \( \mathbb{B} \) is a presheaf category we can describe the representing objects for \( \tilde{\chi}_X \) explicitly, as follows. We emphasise however that even though we can provide a concrete description, it can still happen that the maps constructed are not representing for \( \tilde{\chi}_X \) and in this case \( \tilde{\chi}_X \) is simply not representable at all.

Theorem 3.8. Suppose that the base category \( \mathbb{B} \) is a presheaf category and that \( \chi^{-1}(\{X\}) \) is a set for each object \( X \). Then for each object \( X \), we can construct an object \( J \) and map \( J \to p(X) \) such that if \( \tilde{\chi}_X \) is representable, then it can be represented by \( \sigma : J \to p(X) \).

Furthermore, we can construct a natural transformation from \( \tilde{\chi}_X \) to \( \text{y} \sigma \) which is an isomorphism precisely when \( \tilde{\chi}_X \) is representable.
Proof. Suppose that $B = \mathbf{Set}^{\text{op}}$ for some small category $C$. For each $c \in C$ the elements of $J(c)$ correspond to maps $y(c) \to J$. We can think of each such map as a pair consisting of a map $i : y(c) \to p(X)$ together with a map $y(c) \to J$ making a commutative triangle with the map $J \to p(X)$ that we have yet to define. However, we know that such commutative triangles must correspond precisely to objects in $\chi^{-1}(\{i^*(X)\})$. Hence we can just define $J(c)$ to consist of pairs $i, D$ where $i : y(c) \to p(X)$ and $D \in \chi^{-1}(\{i^*(X)\})$. \qed

**Proposition 3.9.** The following are equivalent.

1. $\tilde{\chi}_X$ is representable for every $X$.
2. $\text{Cart}(\chi)$ has a right adjoint.
3. $\text{Cart}(\chi)$ is comonadic.

Proof. We have shown (1 $\iff$ 2) in Lemma 3.7. The implication (2 $\implies$ 3) is directly from Lemma 2.3. \qed

The relation between local representability and colimits was pointed out by Shulman in [Shulman 2019, Proposition 3.18(iii)]. We observe that this can be seen as an instance of comonadic functors creating colimits:

**Lemma 3.10.** Suppose we are given a notion of structure $\chi : D \to E$. If $\chi$ is definable, then $\text{Cart}(\chi)$ (strictly) creates colimits.

Proof. If $\chi$ is definable, then $\text{Cart}(\chi)$ is comonadic by Proposition 3.9 and comonadic functors create colimits. \qed

We finish this section with a couple of useful lemmas that will be used later. The first says that in one sense the right adjoints witnessing definability are automatically stable under pullback. The second says that definable notions of structure are stable under pullback in the category of fibrations:

**Lemma 3.11.** Suppose that $B$ has pullbacks and $\chi$ creates cartesian lifts. If $\tilde{\chi}_X$ is representable for $X \in E_I$, then so is $\tilde{\chi}_{\sigma^*(X)}$ for $\sigma : J \to I$.

Moreover, if $\tilde{\chi}_X$ is representable by $\tau : K \to I$, then the representing object for $\chi_{\sigma^*(X)}$ can be explicitly described as the pullback $\sigma^*(\tau)$.

\[
\begin{array}{ccc}
\sigma^*(X) & \xrightarrow{\sigma(X)} & X \\
\downarrow & & \downarrow q \\
\sigma^*(\tau) & \xrightarrow{j} & K \\
\downarrow & & \downarrow \tau \\
J & \xrightarrow{\sigma} & I \\
\end{array}
\]
Proof. Suppose we are given an object of $\mathbb{B}/J$ of the form $\rho : L \to J$. Then maps from $\rho$ to $\sigma^*(\tau)$ in $\mathbb{B}/J$ correspond naturally to maps from $\sigma \circ \rho$ in $\mathbb{B}/I$. These correspond naturally to elements of $\bar{\chi}_X(\sigma \circ \rho)$. Note however that we have natural isomorphisms $\bar{\chi}_X(\sigma \circ \rho) \cong \bar{\chi}_{\sigma^*(X)}(\rho)$, since we can lift the isomorphism $(\sigma \circ \rho)^*(X) \cong \rho^*(\sigma^*(X))$ to a bijection between the objects of $\chi^{-1}((\sigma \circ \rho)^*(X))$ and those of $\chi^{-1}(\rho^*(\sigma^*(X)))$ using the assumption that $\chi$ creates cartesian lifts, and moreover the bijections are natural in $\rho$. We deduce that there is a natural correspondence between morphisms from $\rho$ to $\sigma^*(\tau)$ in $\mathbb{B}/J$ and elements of $\bar{\chi}_{\sigma^*(X)}(\rho)$, giving us the required isomorphism between $\bar{\chi}_{\sigma^*(X)}$ and the representable on $\sigma^*(\tau)$.

**Lemma 3.12.** Suppose we are given a strict pullback diagram in fibrations over $\mathbb{B}$, as below.

\[
\begin{array}{ccc}
\mathbb{C} \times_{\mathbb{E}} \mathbb{D} & \longrightarrow & \mathbb{D} \\
\rho^*(\chi) \downarrow & & \downarrow \chi \\
\mathbb{C} & \longrightarrow & \mathbb{E} \\
\rho \downarrow & & \downarrow \chi \\
\mathbb{B} & \longrightarrow & \mathbb{X}
\end{array}
\]

If $\chi$ creates cartesian lifts, then so does $\rho^*(\chi)$.

**Proof.** Note that a map $(f, g)$ in $\mathbb{C} \times_{\mathbb{E}} \mathbb{D}$ is cartesian if and only if $f$ is cartesian in $\mathbb{C}$ and $g$ is cartesian in $\mathbb{D}$. Hence we have a pullback diagram in $\text{Cat}$ as below.

\[
\begin{array}{ccc}
\text{Cart}(\mathbb{C} \times_{\mathbb{E}} \mathbb{D}) & \longrightarrow & \text{Cart}(\mathbb{D}) \\
\text{Cart}(\rho^*(\chi)) \downarrow & & \downarrow \text{Cart}(\chi) \\
\text{Cart}(\mathbb{C}) & \longrightarrow & \text{Cart}(\mathbb{E})
\end{array}
\]

However, discrete fibrations are stable under pullback, so if $\text{Cart}(\chi)$ is a discrete fibration, then so is $\text{Cart}(\rho^*(\chi))$. $\blacksquare$

4 Some examples of notions of structure

4.1 Full notions of structure

**Definition 4.1.** Let $q : \mathbb{E} \to \mathbb{B}$ be a fibration and $\mathcal{D} \subseteq \mathbb{E}$ a class of objects. We say $\mathcal{D}$ is closed under substitution if whenever $X \to Y \in \mathbb{E}$ is cartesian and $Y \in \mathcal{D}$ we also have $X \in \mathcal{D}$.

Note that a class of objects is closed under substitution if and only if the corresponding inclusion of a full subcategory $\mathcal{D} \hookrightarrow \mathbb{E}$ is a notion of structure. Following Shulman, we refer to notions of structure of this form as full. In this case definability recovers the definition of definability due to Bénabou [Bén85].

**Definition 4.2** (Bénabou). We say a class of objects closed under substitution is definable if the corresponding full notion of fibred structure is definable.
Proposition 4.3. Suppose \( D \subseteq \mathcal{E} \) is a definable class of objects. For each \( X \in \mathcal{E} \), the representing object for \( \bar{\chi}_X \) is a monomorphism as a map in \( \mathcal{B} \).

Proof. Note that for full notions of structure each presheaf \( \bar{\chi}_X \) is subterminal. The Yoneda embedding reflects subterminal objects, so the representing object is subterminal as an object of \( \mathcal{B}/q(X) \), which precisely says it is a monomorphism as a map in \( \mathcal{B} \). \( \Box \)

This tells us that the representing object for \( \bar{\chi}_X \) is a subobject of \( q(X) \). Unfolding the definitions, it is the largest subobject \( \sigma : I \hookrightarrow q(X) \) such that \( \sigma^*(X) \) belongs to \( D \).

We give some basic examples of full notions of structure and definability to illustrate the idea. See e.g. [Jac99, Section 9.6], [Str22 Section 12] or [Joh02, Section B1.3] for a more complete account.

Example 4.4. Given any category \( \mathcal{C} \) and any class of objects \( D \subseteq \mathcal{C} \), we can define a full notion of structure on the fibration of set or category indexed families on \( \mathcal{C} \). A family \( (X_i)_{i \in I} \) belongs to the class if \( X_i \) is an element of \( D \) for every \( i \in I \).

Classes of this form are always definable, assuming we have the axiom of full separation in the set theory where we are working. Given a set indexed family \( X := (X_i)_{i \in I} \), the representing object of \( \bar{\chi}_X \) is the subset of \( I \) defined as \( \{ i \in I \mid X_i \in D \} \).

For category indexed families, the representing object is a full subcategory with set of objects defined as for set indexed families.

Proposition 4.5. Let \( \mathcal{B} \) be a regular category. Then regular epimorphisms form a definable class for the codomain fibration \( \text{cod} : \mathcal{B} \to \mathcal{B} \).

Proof. First recall that in a regular category, regular epimorphisms are stable under pullback, which precisely says they are closed under substitution in the codomain fibration.

Now given \( f : X \to Y \), we have an image factorisation \( X \twoheadrightarrow \text{im}(f) \to Y \). We note that the square below is a pullback, e.g. by directly verifying the universal property.

\[
\begin{array}{ccc}
X & \to & X \\
\downarrow & & \downarrow f \\
\text{im}(f) & \to & Y
\end{array}
\]

Given any object \( Z \) and any map \( h : Z \to Y \) we have the commutative square below.

\[
\begin{array}{ccc}
h^*(X) & \to & X \\
\downarrow h^*(f) & & \downarrow \text{im}(f) \\
Z & \to & Y
\end{array}
\]
When $h^*(f)$ is a regular epimorphism, we get a unique diagonal filler $Z \to \text{im}(f)$, which witnesses $h^*(f)$ as a pullback of $X \to \text{im}(f)$, as below.

\[
\begin{array}{ccc}
h^*(X) & \to & X \\
\downarrow & & \downarrow \\
Z & \to & \text{im}(f) \\
\end{array}
\]

Conversely, given a pullback square as in the left hand square above, we can deduce that $h^*(f)$ is a regular epimorphism. □

More generally, given a pullback stable factorisation system on a category $\mathcal{B}$, the left class will always give a definable class with respect to $\text{cod} : \mathcal{B}^\to \to \mathcal{B}$.

**Example 4.6.** Let $q : \mathcal{E} \to \mathcal{B}$ be a fibration such that reindexing preserves any terminal objects that exist. E.g. set or category indexed families on a category, or any codomain fibration on a category with finite limits. Then the class objects $X$ of $\mathcal{E}$ that are terminal in their fibre category is closed under substitution.

**Example 4.7.** Let $q : \mathcal{E} \to \mathcal{B}$ be a fibration such that each fibre category has a terminal object and reindexing preserves monomorphisms and terminal objects. Again this includes set or category indexed families and any codomain fibration. An object $X$ of $\mathcal{E}_I$ is subterminal if the unique map $X \to 1_I$ is a monomorphism. Then the class of subterminal objects is closed under substitution.

**Proposition 4.8.** Suppose that $\mathcal{B}$ is a Heyting category. Then monomorphisms are a definable class in $\text{cod} : \mathcal{B}^\to \to \mathcal{B}$.

**Proof.** The intuitive idea is that subterminal objects can be described within the internal language of a Heyting category. An object is subterminal if any two elements of it are equal, namely if it satisfies the following sentence: $\forall x, y \in X \ x = y$. In the argument below we expand out the preceding sentence to an explicit categorical description, and check that it works.

Suppose we are given a map $f : X \to Y$. We have a diagonal map $\Delta_X : X \to X \times_Y X$. Write $p$ for the canonical map $X \times_Y X \to Y$. We then have a monomorphism $\forall_p \Delta_X : \forall_p X \to Y$. We check that this does give a representing object for $\bar{\chi}_f$.

First note that a map $h : Z \to Y$ factors through $\forall_p \Delta$ if and only if $\top \leq h^*(\forall_p \Delta_X)$ in the lattice of subobjects of $Z$. This is the case precisely when $\exists_h \top \leq \forall_p \Delta_X$ in subobjects of $Y$, which holds when $p^*(\exists_h \top) \leq \Delta_X$ in subobjects of $X \times_Y X$. Since image factorisation is stable under pullback, we have $p^*(\exists_h \top) \equiv \exists_{p^*(h)} \top$. Hence $p^*(\exists_h \top) \leq \Delta_X$ precisely when $\top \leq (p^*(h))^*(\Delta_X)$ in subobjects of $h^*(X) \times_Z h^*(X)$. However, one can calculate that $(p^*(h))^*(\Delta_X)$ is exactly the diagonal map $\Delta_{h^*(X)} : h^*(X) \to h^*(X) \times_Z h^*(X)$, which is equal to $\top$ precisely when $h^*(f)$ is a monomorphism. □
Example 4.9. Suppose we are given a fibration \( q : E \to B \). Then we can define a second fibration of vertical arrows \( V(\mathbb{E}) \to \mathbb{B} \). In any case the class of isomorphisms is a full notion of fibred structure on \( V(\mathbb{E}) \). If reindexing preserves monomorphisms, then they are also a full notion of structure.

If \( q \) is locally small, then both of these are definable classes.

**Theorem 4.10** (Bénabou). Let \( \mathbb{B} \) be a topos, together with a local operator \( j : \Omega \to \Omega \). The following classes of maps are definable classes of objects with respect to the codomain fibration on \( \mathbb{B} \).

1. Families of \( j \)-separated objects.
2. Families of \( j \)-sheaves.

**Example 4.11.** Let \((\mathcal{L}, \mathcal{R})\) be a weak factorisation system on a category \( \mathbb{B} \). Then the right class \( \mathcal{R} \) gives us a full notion of structure on the codomain fibration on \( \mathbb{B} \). We say a wfs is definable if the corresponding full notion of structure on \( \text{cod} : \mathbb{B} \to \mathbb{B} \) is definable.

As remarked by Shulman [Shu19, Example 3.17] a wfs is definable if it is cofibrantly generated by a set of maps with representable codomain, assuming the axiom of choice. We will see in Section 8.2 that the axiom of choice is strictly necessary.

### 4.2 Comprehension schemes

The idea of definability appears again in Johnstone’s notion of comprehension schemes. Although we make some adjustments to fit with the general theory of notions of structure, the idea essentially appears in [Joh02, Section B1.3]. In particular, in the definition below we include a requirement of isomorphism on objects to satisfy the definition of notion of structure. This is not required by Johnstone, who instead refers to it as a special case where the definition “works best.”

**Proposition 4.12.** Let \( q : E \to B \) be any fibration and \( F : C \to D \) an internal functor between internal categories in \( \mathbb{B} \). If \( F \) is an isomorphism on objects, then the corresponding \( F^* : \mathbb{E}^D \to \mathbb{E}^C \) between fibrations of diagrams defined by precomposing with \( F \), creates cartesian lifts.

**Proof.** Without loss of generality, \( C \) and \( D \) have the same object of objects and \( F_0 \) is the identity. We can write out the remaining data for the internal categories and internal functor as the following commutative diagram.

\[
\begin{array}{ccc}
C_1 & \xrightarrow{F_1} & D_1 \\
\downarrow{s} & & \downarrow{u} \\
C_0 & \xleftarrow{t} & D_0
\end{array}
\]

\[\text{Alternatively one can make Johnstone’s definition better behaved by working in univalent } \infty\text{-categories [Ste20].}\]
We can view a pair consisting of an object $Y$ of $E^D$ and a cartesian map into $F^*(Y)$ in $E^C$ as the solid lines in the upper diagram below, where the horizontal maps are cartesian over the maps in the lower square.

Cartesian lifts correspond precisely to vertical morphisms completing the diagram to a commutative cube, as in the dotted map above. However, such maps are uniquely determined by the universal property of the cartesian maps in the above diagram.

**Definition 4.13** (Johnstone). Given an isomorphism on objects internal functor $F$ in $B$, we say $q : E \to B$ satisfies the comprehension scheme for $F$ if the notion of structure $F^*$ is definable.

**Example 4.14.** Let $B$ be a category with finite limits and finite coproducts. Define internally in $B$ the inclusion functor from the discrete category on 2 objects, 2, to the category with two objects and a morphism from one to the other, denoted $\cdot \to \cdot$.

We can explicitly describe the diagram category $E^2$ as the pullback $E \times_B E$ and the diagram category $E^{\to}$ as the category of vertical maps $V(E)$. The functor $F^* : V(E) \to E \times_B E$ sends a vertical map to its domain and codomain.

We see that in this case a “structure” on a pair of objects $X, Y$ in the same fibre category $E_I$ is a vertical map from $X$ to $Y$.

This notion of structure is definable if and only if $q : E \to B$ is locally small.

**Example 4.15.** Let $B$ be category with finite limits and finite coproducts. We can construct internally in $B$ the category with two objects and a map between them $\cdot \to \cdot$ as well as the category with two objects and two maps between them $\cdot \rightrightarrows \cdot$. Furthermore, we can define the unique functor $F$ from $\cdot \rightrightarrows \cdot$ to $\cdot \to \cdot$ that is the identity on objects (and “collapses” the two morphisms).

As before, we can explicitly describe $E^{\to}$ as the category of vertical arrows. We explicitly describe $E^{\rightrightarrows}$ as the category of pairs of vertical arrows with the same domain and same codomain. We then have that $F^*$ is the inclusion of the
full subcategory of $E^{\to}$ of objects where the two arrows in the pair are equal. If $q$ satisfies the comprehension scheme for $F$ we say it has \textit{definable equality}.

**Theorem 4.16** (Johnstone). A fibration $q : E \to B$ is locally small if and only if it satisfies the comprehension scheme for all isomorphism on objects internal functors $F : C \to D$ in $B$.

\textit{Proof}. See the remark after [Joh02, Lemma B1.3.15]. \qed

\section*{4.3 (Co)Algebraic notions of structure}

We finally turn to the main source of motivating examples for this paper. These observations already appear in [Swa18b, Sections 4.3 and 5.1] and are minor variants of standard material, but we repeat them below for reference.

In the below, we assume we are given an arbitrary Grothendieck fibration $q : E \to B$.

**Definition 4.17.** An \textit{endofunctor over $B$} is a functor $T : E \to E$ such that $q \circ T = T$.

A \textit{pointed endofunctor over $B$} is an endofunctor $T$ over $B$ together with a natural transformation $\eta : 1_E \Rightarrow T$ such that $\eta_X : X \to TX$ is a vertical map for each $X \in E$.

A \textit{monad over $B$} is a pointed endofunctor $(T, \eta)$ over $B$ together with a natural transformation $\mu : T^2 \Rightarrow T$ such that $\mu_X : T(TX) \to TX$ is vertical for each $X \in E$, and such that $\eta$ and $\mu$ satisfy the usual monad laws, displayed below for reference.

\begin{equation*}
\begin{array}{c}
T 
\xrightarrow{\eta_T} T^2 
\xleftarrow{T\eta} T \\
\downarrow{\mu} \\
T \\
\end{array} 
\quad 
\begin{array}{c}
T^3 
\xrightarrow{T\mu} T^2 
\xleftarrow{T^2\mu} T \\
\downarrow{\mu_T} \\
T^2 \\
\end{array}
\end{equation*}

We say an endofunctor over $B$, $T : E \to E$ is \textit{fibred} if it preserves cartesian maps, and pointed endofunctors and monads are fibred if their underlying endofunctors are.

We dually define (fibred) copointed endofunctors and comonads.

We emphasise that in this definition monads over $B$ are not necessarily fibred (i.e. do not necessarily preserve cartesian maps) and in fact many natural examples of algebraic weak factorisation systems (to be covered in section 4.4) are not.

Monads over $B$ can be seen as a special case of the 2-categorical definition of monad [Str72] by working in the 2-category whose underlying 1-category is $\text{Cat}/B$ and whose 2-cells are pointwise vertical natural transformations. Fibred monads are monads in the usual 2-category of fibrations over $B$ (see e.g. [Jac99, Section 1.7]).
Definition 4.18. Let \( T \) be an endofunctor over \( \mathcal{B} \) and \( X \) an object of \( \mathcal{E} \). A \( T \)-algebra structure on \( X \) is a vertical map \( s : TX \to X \). A \( T \)-algebra is an object of \( \mathcal{E} \) together with \( T \)-algebra structure. This defines a category \( T \)-Alg, together with a forgetful functor \( \nu : T \text{-Alg} \to \mathcal{E} \).

Let \( (T, \eta) \) be a pointed endofunctor over \( \mathcal{B} \). An algebra structure on \( X \in \mathcal{E} \) is a (necessarily vertical) map \( s : TX \to X \) that satisfies the usual unit law, displayed below for reference.

\[
\begin{array}{ccc}
X & \xrightarrow{\eta_X} & TX \\
\downarrow & & \downarrow s \\
X & & \\
\end{array}
\]

We similarly define the category of \( (T, \eta) \)-algebras, again written as \( T \)-Alg when \( \eta \) is clear from the context.

Let \( (T, \eta, \mu) \) be a monad over \( \mathcal{B} \). An algebra structure on an object \( X \in \mathcal{E} \) is an algebra structure on the underlying pointed endofunctor \( s : TX \to X \) that additionally satisfies the usual multiplication law, displayed below for reference.

\[
\begin{array}{ccc}
T(TX) & \xrightarrow{T \mu_X} & TX \\
\downarrow & & \downarrow s \\
TX & \xrightarrow{s} & X \\
\end{array}
\]

We again write the category of algebras as \( T \)-Alg when \( \eta \) and \( \mu \) are clear from the context.

We dually define categories of coalgebras \( M \text{-Coalg} \) for endofunctors, co-pointed endofunctors and comonads \( M \) over \( \mathcal{B} \).

Lemma 4.19. Let \( T \) be an endofunctor, pointed endofunctor or monad over \( \mathcal{B} \). The forgetful functor \( \nu : T \text{-Alg} \to \mathcal{E} \) creates cartesian lifts. In particular the composition \( T \text{-Alg} \to \mathcal{B} \) is a Grothendieck fibration.

Proof. Suppose we are given a cartesian map \( f : X \to Y \) in \( \mathcal{E} \) together with an algebra structure on \( Y \). Together this gives us the solid lines in the diagram below, where \( f \) and \( T \) lie over the same map in \( \mathcal{B} \), and \( s \) is vertical.

\[
\begin{array}{ccc}
TX & \xrightarrow{Tf} & TY \\
\downarrow t & & \downarrow s \\
X & \xrightarrow{f} & Y \\
\end{array}
\]

However, since \( f \) is cartesian, there is a unique vertical map \( t : TX \to X \) making a commutative square as in the dotted arrow above. This is precisely an algebra structure on \( X \) making \( f \) a homomorphism of algebras. One can check that \( f \) remains cartesian as a map in \( T \text{-Alg} \).

Furthermore, one can check that if \( s \) satisfies the unit law for a pointed endofunctor or the multiplication law for a monad, then so does \( t \). \( \square \)
**Lemma 4.20.** Let $M$ be a fibred endofunctor, copointed endofunctor or comonad over $\mathcal{B}$. The forgetful functor $\nu : M\text{-Coalg} \rightarrow \mathcal{E}$ creates cartesian lifts. In particular the composition $M\text{-Coalg} \rightarrow \mathcal{B}$ is a Grothendieck fibration.

**Proof.** Suppose we are given a cartesian map $f : X \rightarrow Y$ in $\mathcal{E}$ together with a coalgebra structure on $Y$. Together this gives us the solid lines in the diagram below, where $f$ and $Mf$ lie over the same map in $\mathcal{B}$, and $s$ is vertical.

\[ \begin{array}{ccc}
X & \xrightarrow{t} & Y \\
\downarrow f & & \downarrow s \\
MX & \xrightarrow{Mf} & MY \\
\end{array} \]

Since $M$ is fibred, $Mf$ is cartesian, and so there is a unique vertical map $t : X \rightarrow MX$ making a commutative square as in the dotted arrow above. This is exactly a coalgebra structure on $X$ making $f$ a homomorphism of coalgebras. As before, one can check that $f$ remains cartesian as a map in $M\text{-Coalg}$ and that $t$ satisfies counit and comultiplication laws when $s$ does. \qed

**Remark 4.21.** We emphasise that Lemma 4.20 required the additional assumption that the comonad is fibred, so Lemmas 4.19 and 4.20 are not formally dual. The dual to Lemma 4.19 tells us that any forgetful functor $M\text{-Coalg} \rightarrow \mathcal{E}$ creates opcartesian lifts, whereas the dual to Lemma 4.20 tells us that if an endofunctor, pointed endofunctor or monad $T$ preserves opcartesian maps, then $T\text{-Alg} \rightarrow \mathcal{E}$ creates opcartesian lifts.

**Example 4.22.** Suppose we are given a choice of terminal object $1_I$ for each $I \in \mathcal{B}$ and reindexing preserves terminal objects.

This defines a fibred endofunctor over $\mathcal{B}$ by $T(X) := 1_{q(X)}$. An algebra structure on an object $X$ is simply a map $1_{q(X)} \rightarrow X$. We refer to algebras as pointed objects and write the category of algebras as $\mathcal{E}_*$. If the notion of structure $\mathcal{E}_* \rightarrow \mathcal{E}$ is definable, we say the fibration $\mathcal{E} \rightarrow \mathcal{B}$ admits comprehension [Law70].

**Example 4.23.** As a special case of Example 4.22 we can consider pointed objects in codomain fibrations. An object of $\mathcal{B} \rightarrow \rightarrow$ is a map $f$ of $\mathcal{B}$. A point of $f$ as an object of $\mathcal{B} \rightarrow$ is then precisely a section of $f$.

For a set indexed family fibration Fam($\mathcal{C}$) $\rightarrow \mathbf{Set}$, a pointed object is a family of objects $(C_i)_{i \in I}$ together with a choice of map $c_i : 1_C \rightarrow C_i$ in $\mathcal{C}$ for each $i \in I$.

A pointed object in a codomain fibration $\mathcal{B} \rightarrow \mathcal{B}$ is an object of $\mathcal{B} \rightarrow$, which is a map $f : X \rightarrow I$, together with a map from the identity on $I$ to $f$ in $\mathcal{B}/I$, which is just a section of $f$.

**Lemma 4.24.** For any category $\mathcal{B}$ with pullbacks, the forgetful functor from maps with sections to maps, $\nu : \mathcal{B} \rightarrow \mathcal{B} \rightarrow$ is definable.

Moreover, for a map $f : X \rightarrow I$, the representing object for $\nu f$ in $\mathcal{B}/I$ is simply $f$ itself.
Proof. For any \( f : X \to I \) in \( \mathcal{B} \), and any \( \sigma : J \to I \), sections of \( \sigma^*(f) : \sigma^*(X) \to J \) correspond precisely to maps \( J \to X \) making a commutative triangle by the universal property of the pullback.

\[
\begin{array}{ccc}
\sigma^*(X) & \longrightarrow & X \\
\downarrow & & \downarrow \\
J & \longrightarrow & J
\end{array}
\begin{array}{ccc}
J & \longrightarrow & X \\
\sigma & & \sigma \\
& \longleftarrow & \eta
\end{array}
\]

We also give a simple non-fibred example. Although it is not an awfs itself, it illustrates the essential idea why many natural examples of awfs’s are not fibred.

Example 4.25. Consider the codomain fibration \( \text{cod} : \mathcal{B} \to \mathcal{B} \). We define an endofunctor \( T \) as follows. Given \( f : X \to I \) we define \( T(f) \) to be the first projection \( \pi_0 : I^2 \to I \). We can visualise algebra structures on \( f \) as functions assigning for each pair \( i, j \) a fibre of \( f \) over \( i \). This picture can be made precise using the internal language of \( \mathcal{B} \).

Given a map \( \sigma : J \to I \), the pullback \( \sigma^*(\pi_0) \) is the projection \( J \times I \to J \), not \( J^2 \to J \). Hence the endofunctor does not preserve pullbacks.

The forgetful functor \( T\text{-Alg} \to \mathcal{B} \) still creates cartesian lifts, and in particular \( T\text{-Alg} \to \mathcal{B} \) is a fibration by Lemma 4.19.

Example 4.26. For an example of a coalgebraic notion of structure, we assume that the fibration \( q : \mathcal{E} \to \mathcal{B} \) has fibred coproducts and terminal objects and consider the endofunctor sending \( X \) to \( \mathbf{1} + \mathbf{1} \). A coalgebra structure on \( X \) is a 2-colouring, i.e. a partition of \( X \) into two pieces.

Theorem 4.27. Let \( q : \mathcal{E} \to \mathcal{B} \) be a locally small fibration, and suppose \( \mathcal{B} \) has all finite limits. The forgetful functors from categories of (co)algebras for fibred endofunctors, (co)pointed endofunctors and (co)monads are all definable.

Proof. We will show this for algebras, the proof for coalgebras being similar. [4]

Let \( P \) be a fibred endofunctor over \( \mathcal{B} \). Fix \( X \in \mathcal{E} \). We need to show that the presheaf on \( \mathcal{B}/q(X) \) sending \( \sigma : I \to p(X) \) to \( P \) algebra structures on \( \sigma^*(X) \) is representable. The set of algebra structures is by definition the hom set \( \mathcal{E}(P(\sigma^*(X)), \sigma^*(X)) \), which is naturally isomorphic to \( \mathcal{E}(\sigma^*(P(X)), \sigma^*(X)) \), since \( P \) is fibred. However, the latter presheaf is representable by the characterisations of local smallness in terms of representables.

Now suppose we are given a fibred pointed endofunctor \( \eta : \mathbf{1} \to P \) over \( \mathcal{B} \). We again need to show that the presheaf sending \( \sigma : I \to p(X) \) to \( (P, \eta) \) algebra structures on \( \sigma^*(X) \) is representable. Such an algebra structure is precisely a map \( f : P(\sigma^*(X)) \to \sigma^*(X) \) such that \( f \circ \eta_X = 1_X \). Observe that we can express the set of algebra structures as an equalizer in sets of the form

\[19\]

[4]With care it is also possible to deduce the result for coalgebras by duality.
$E(P(\sigma^*(X)), \sigma^*(X)) \Rightarrow E(\sigma^*(X), \sigma^*(X))$. Hence we can view the presheaf of algebra structures as an equalizer in presheaves defined pointwise as the preceding equalizer for each $\sigma$. Since $\mathbb{B}/p(X)$ has all finite limits and the Yoneda embedding preserves limits we can deduce that the presheaf of algebra structures is representable. Namely, the representing object is defined as an equalizer in $\mathbb{B}$ of the following form.

$$\hom(P(X), X) \Rightarrow \hom(X)$$

Finally, if $(P, \eta, \mu)$ is a monad, we can repeat the argument for pointed endofunctors, but now also need to specify the multiplication law as well as the unit law. Namely, the representing object is constructed as a limit in $\mathbb{B}/p(X)$ of the following form.

$$\hom(X) \quad \hom(P(X), X) \quad \hom(P^2(X), X)$$

\[ \square \]

### 4.4 Algebraic weak factorisation systems

Algebraic weak factorisation systems are an important tool for viewing classes of maps commonly considered in homotopical algebra as structure on a map, rather than a property of a map. In particular, they play an important role in providing a structured version of Kan fibration in cubical sets and simplicial sets [GS17, Swa10, Swa18, Awo19]. Although they are usually defined via functorial factorisations [GT06, Gar09], Bourke and Garner showed the definition is equivalent to one based on double categories [BG16]. We give a mild reformulation of their definition phrased in terms of notions of structure and some definitions from the theory of comprehension categories with relevance to the semantics of type theory.

We first note that notions of structure on codomain fibrations can be seen as comprehension categories, as used in the semantics of type theory [Jac99, Chapter 10].

**Definition 4.28** (Jacobs). A *comprehension category* is a Grothendieck fibration $p : E \to \mathbb{B}$ together with a fibred functor $\chi$ from $p$ to the codomain fibration on $\mathbb{B}$, as illustrated below.

![Comprehension Category Diagram](image-url)
Definition 4.29. We say a comprehension category is \textit{monadic} if $\chi$ is strictly monadic as a functor.

Remark 4.30. By Proposition \ref{prop:monadic} we do not need to distinguish between $\chi$ being monadic as a functor in $\textbf{Cat}$ or $\textbf{Cat}/\mathbb{B}$. We do not require the monad to preserve cartesian maps.

As a special case of Lemma \ref{lem:adjoint} we have:

Proposition 4.31. Any monadic comprehension category is a (necessarily monadic) notion of fibred structure on $\text{cod} : \mathbb{E} \rightarrow \mathbb{B}$.

Units are used in the theory of comprehension categories to model unit types in type theory. We recall the strict version of the definition.

Definition 4.32. A \textit{strict unit} is a functor $t : \mathbb{B} \rightarrow \mathbb{E}$ which is right adjoint to $p$ and such that $\chi(t(I)) = 1_I$ for all $I \in \mathbb{B}$.

Proposition 4.33. Every monadic comprehension category has a strict unit.

Proof. This follows from the fact that monadic functors create limits, noting that for each object $I$, $1_I$ is a terminal object in $\mathbb{B}/I$. \hfill $\square$

Suppose $\chi$ is a comprehension category with a strict unit $t$. Then we have the following commutative diagram in $\textbf{Cat}$.

\begin{equation}
\begin{array}{ccc}
\mathbb{E} & \xleftarrow{r} & \mathbb{B} \\
\downarrow{\chi} & & \downarrow{\chi} \\
\mathbb{B} & \xleftarrow{1} & \mathbb{B}
\end{array}
\end{equation}

Note that the bottom row is an internal category in $\textbf{Cat}$, with multiplication given by composition in $\mathbb{B}$. Viewed as a double category it is the double category of commutative squares in $\mathbb{B}$.

Definition 4.34. A \textit{composition functor} is a functor $\mathbb{E} \times_{\mathbb{B}} \mathbb{E} \rightarrow \mathbb{E}$ making the top row of (2) an internal category, and the whole square a functor.

We can think of composition functors as algebraic versions of $\Sigma$-types in the following sense. In the theory of comprehension categories we can implement $\Sigma$-types as an operation $\mathbb{E} \times_{\mathbb{B}} \mathbb{E} \rightarrow \mathbb{E}$ that commutes up to isomorphism with composition in $\mathbb{E}$, referred to as \textit{strong coproducts} by Jacobs \cite[Definition 10.5.2]{Jac99}. Jacobs’ definition of strong coproducts further requires that the operation is obtained from a dependent coproduct for the fibration $p$. However, we observe that we can satisfy this requirement by replacing the morphisms of $\mathbb{E}$ with those in $\mathbb{B}$ to make $\chi$ full and faithful. Since the morphisms of $\mathbb{E}$ are not
used in the interpretation of type theory this has no effect therein. We can also justify modifying Jacobs’ definition in this way by considering the construction of \( \Sigma \)-types in cubical sets \([\text{CCHM18}]\). A dependent coproduct in a fibration, \( \coprod_{\sigma} X \) is uniquely determined up to isomorphism by the map \( \sigma : I \to J \) and the object \( X \) in \( E_I \). However, \( \Sigma \)-types in cubical sets are implemented by defining a Kan fibration structure on the underlying \( \Sigma \)-type in the standard model of extensional type theory in presheaves. The Kan fibration structure depends on the fibration structures of both types given as input. Hence we should not expect it to be unique up to isomorphism of Kan fibration structures if we are only given the map \( \sigma : I \to J \) without a choice of fibration structure. It is however unique up to isomorphism of underlying presheaves.

Composition functors are stronger than necessary to obtain \( \Sigma \) types. In addition to Jacobs’ strongness condition, they also satisfy strict associativity as part of the definition of internal category, which is not needed for type theory. However, it is natural to consider \( \Sigma \)-types satisfying this additional requirement in the setting of cofibrantly generated awfs’s, where they occur automatically.

For the semantics of type theory it is useful to observe that any composition functor is automatically fibred, in the following sense.

**Proposition 4.35.** Any composition functor \(- \bullet - : E \times_B E \to E\) on a monadic comprehension category preserves cartesian maps in both arguments.

**Proof.** Suppose that \( f : X \to X' \) and \( g : Y \to Y' \) are composable and cartesian. Write \( \Gamma \) for \( p(Y') \) and \( \Delta \) for \( p(Y) \). Write \( \{-\} \) for the composition \( \text{dom} \circ \chi \). We then have the following commutative diagram in \( B \), where the upper commutative square is \( \chi(f) \), the lower commutative square is \( \chi(g) \), and the whole rectangle is the image under \( \chi \) of the composition \( g \bullet f \).

\[
\begin{array}{ccc}
\{X\} & \longrightarrow & \{X'\} \\
\downarrow & & \downarrow \\
\{Y\} & \longrightarrow & \{Y'\} \\
\downarrow & & \downarrow \\
\Delta & \longrightarrow & \Gamma
\end{array}
\]

Since \( \chi \) preserves cartesian maps, the upper and lower squares are both pullbacks. Hence the big rectangle is a pullback. However, any monadic fibred functor reflects cartesian maps (since it reflects vertical isomorphisms), and so \( g \bullet f \) is cartesian, as required.

**Theorem 4.36** (Bourke-Garner). The above definition of awfs corresponds precisely to the more usual definition (appearing e.g. in \([\text{Gar09}]\), which aside from a distributive law condition is the same as given by Grandis and Tholen under the name natural weak factorisation system \([\text{GT06}]\)).

**Proof.** This is a rephrasing of \([\text{BG16} \text{ Proposition 4}]\).
We also give a fibred version of the definition of awfs, as in [Swa18b]. Given a fibration \( p : \text{E} \to \text{B} \) note that we can also view \( V(\text{E}) \) as a double category, and similarly to before, we have the commutative diagram below.

\[
\begin{array}{ccc}
F & \xymatrix{ \text{dom} \circ \chi \ar[r] & \text{E} \ar[d]^\chi \ar[l]_{\text{cod} \circ \chi} } & E \\
V(\text{E}) & 1 \ar[l]_{\text{dom}} \ar[r]_{\text{cod}} & \text{E} \\
\end{array}
\]

Definition 4.37. We say a \textit{fibred composition functor} on a notion of structure \( \chi : F \to V(\text{E}) \) is a functor \( F \times_\text{E} F \to \text{E} \) over \( \text{B} \) making the top row of (3) an internal category and the whole diagram a double functor.

Definition 4.38. An \textit{algebraic weak factorisation system over a fibration} \( p : \text{E} \to \text{B} \) is a monadic notion of structure on \( V(\text{E}) \to \text{E} \) together with a fibred composition functor.

Definition 4.39. We say an algebraic weak factorisation system over \( p : \text{E} \to \text{B} \) is \textit{fibred} if it has a left adjoint that preserves the property of maps being cartesian over \( \text{B} \).

We say it is \textit{strongly fibred} if it has a left adjoint that preserves the property of maps being cartesian over \( \text{E} \).

4.5 Lifting structures

Let \( p : \text{E} \to \text{B} \) be a locally small bifibration and fix a vertical map \( m : A \to B \) in \( \text{E} \).

Definition 4.40. The \textit{lifting notion of structure generated by} \( m \) is the notion of structure on \( \text{cod} : V(\text{E}) \to \text{E} \) defined as follows. An object of \( m^\text{ith} \) is a pair consisting of \( f : X \to Y \in V(\text{E}) \) together with a section of the canonical map \( \text{hom}(B,X) \to \text{hom}(A,X) \times_{\text{hom}(A,Y)} \text{hom}(B,Y) \), with the map \( F \to V(\text{E}) \) given by projection.

Proposition 4.41. The lifting structure generated by \( m \) is a notion of structure on \( V(\text{E}) \to \text{E} \) and admits a fibred composition functor.

It follows that if a lifting notion of structure of a map is monadic, then it is automatically an awfs. We refer to awfs’s of this form as \textit{cofibrantly generated}.

Remark 4.42. By Proposition 2.11 to show that \( m^\text{ith} \to V(\text{E}) \) is monadic over \( \text{B} \) it suffices to show it is monadic as a functor in \textit{Cat}.

Theorem 4.43. Suppose \( p : \text{E} \to \text{B} \) is complete and cocomplete (as a fibration) and \( m^\text{ith} \to V(\text{E}) \) is an awfs. Then it is a fibred awfs (i.e. its left adjoint is fibred).
Proof. See \cite{Swa18b,Theorem 5.5.1].

Algebraic versions of the small object argument can be seen as proofs that certain lifting notions of structure are monadic.

**Theorem 4.44** (Garner). Suppose that $E \to B$ is a category indexed family fibration $\text{Fam}(C) \to \text{Cat}$ such that $C$ is cocomplete and one of the following conditions holds.

1. For every $X \in C$ there is a regular ordinal $\alpha$ for which $X$ is $\alpha$-presentable.
2. $C$ admits a proper well-copowered factorisation system $\mathcal{E}, \mathcal{M}$ such that for every $X \in C$ there is a regular ordinal $\alpha$ for which $X$ is $\alpha$-bounded with respect to $(\mathcal{E}, \mathcal{M})$.

Then for any family of maps $m$, the lifting notion of structure generated by $m$ is monadic.

Proof. See \cite{Gar09].

**Theorem 4.45** (Swan). Suppose that $E \to B$ is a codomain fibration $B \to B$ on a locally cartesian closed category $B$, that $m$ is a family of maps and one of the following conditions holds.

1. $B$ is locally cartesian closed, has exact quotients and $W$-types and satisfies WISC.
2. $B$ is an internal presheaf category in a locally cartesian closed category with finite colimits and disjoint coproducts and $m$ is a pointwise decidable monomorphism.

Then the lifting notion of structure generated by $m$ is monadic.

Proof. See \cite{Swa18c}. 

**Remark 4.46.** Regarding the connection between awfs’s and the semantics of type theory, we observe that writing $\chi$ for the forgetful functor $m^! \to V(E)$ the map $\text{dom} \circ \chi$ is right adjoint to the functor $1_{(-)} \to m^!$ sending each object of $E$ to the terminal object of its fibre. It follows that for each $I \in B$ the restriction of $m^! \to E_I$ admits comprehension in the sense of Example 4.22, and furthermore the resulting comprehension category with unit on $E_I$ as in \cite[Definition 10.4.7]{Jac99} is the same as that given in section 4.4.

5 Other characterisations of definability

5.1 Representable maps

**Theorem 5.1.** Suppose we are given a fibred functor $\chi : D \to E$ that creates cartesian lifts and a splitting on the fibration $E \to B$. Then we can define a splitting on $D$ that is strictly preserved by $\chi$. 24
Proof. Given an object $D$ of $\mathcal{D}$ and a map $\sigma : I \to p(D)$, we have a choice of object $\sigma^*(\chi(D))$ and cartesian map $\bar{\sigma} : \sigma^*(\chi(D)) \to \chi(D)$ in $\mathcal{E}$ over $\sigma$. We choose the splitting at $D$ to be the unique cartesian map over $\bar{\sigma}$ with codomain $D$. □

Definition 5.2. When $\mathcal{E}$ is split, we have a presheaf on $\mathbb{B}$, by mapping $I \in \mathbb{B}$ to the objects of $\mathcal{E}_I$. We denote this presheaf $\tilde{\mathcal{E}}$.

By Theorem 5.1 we similarly have another presheaf sending $I$ to the set of objects of $\mathcal{D}_I$, which we denote $\tilde{\mathcal{D}}$, and we have a natural transformation $\tilde{\chi} : \tilde{\mathcal{D}} \to \tilde{\mathcal{E}}$.

Proposition 5.3. When we are given a splitting of $\mathcal{E}$, the natural transformation $\tilde{\chi}$ in definition 5.2 is a representable map in presheaves over $\mathbb{B}$ if and only if $\chi$ is definable.

As an alternative to requiring splitness, one can consider a generalised definition of presheaf and representable map using 2-category theory that can be obtained from any Grothendieck fibration. This is one way of understanding the definition of local representability given by Shulman [Shu19, Definition 3.10].

5.2 Pullbacks of the notion of structure of sections

This characterisation is based on Awodey's universal fibrations [Awo19, Section 6.3]. We can understand this definition as follows. We saw in Lemma 4.24 that the notion of structure on $\text{cod} : \mathbb{B} \to \mathbb{B}$ of sections (Example 4.23) is always definable. We will see below that it is the universal example of definable notion of structure, in the sense that every other definable notion of structure is a pullback of this one. Hence we could alternatively define definable notions of structures as fibred functors $\zeta : \text{Cart}(\mathcal{E}) \to \text{Cart}(\mathbb{B} \to \mathbb{B})$ such that $\text{Cart}(\mathcal{D}) \to \text{Cart}(\mathcal{E})$ is the pullback of $\text{Cart}(\mathbb{B} \to \mathbb{B})$ along $\zeta$. This was already observed by Shulman [Shu19] Proposition 2.7, but for completeness we give a direct proof in our formulation here.

Theorem 5.4. Let $\mathbb{B}$ be a finitely complete category. A notion of fibred structure $\chi : \mathcal{D} \to \mathcal{E}$ over $\mathbb{B}$ is definable if and only if there are fibred functors $\text{Cart}(\mathcal{D}) \to \text{Cart}(\mathbb{B} \to \mathbb{B})$ and $\zeta : \text{Cart}(\mathcal{E}) \to \text{Cart}(\mathbb{B} \to \mathbb{B})$ making a (strict) pullback as illustrated below.

\[
\begin{array}{ccc}
\text{Cart}(\mathcal{D}) & \longrightarrow & \text{Cart}(\mathbb{B} \to \mathbb{B}) \\
\downarrow & & \downarrow \\
\text{Cart}(\mathcal{E}) & \longrightarrow & \text{Cart}(\mathbb{B} \to \mathbb{B})
\end{array}
\]

B

Proof. Suppose first that $\chi$ is definable. In this case we can assign for $X \in \mathcal{E}$ a representing object for $\tilde{\chi}_X$. We take $\zeta(X)$ to be the representing object in $\mathbb{B}/q(X)$. Now given a cartesian map $f : X \to Y$, we know by Lemma 4.11.
that \( q(f)^*(\zeta(Y)) \) is a representing object for \( \bar{\chi}_X \). This gives us a canonical isomorphism between \( \zeta(X) \) and \( q(f)^*(\zeta(Y)) \) over \( q(X) \). In turn this gives us a pullback square in \( \mathbb{B} \) with \( \zeta(X) \) on the left and \( \zeta(Y) \) on the right, i.e. a morphism in \( \text{Cart}(\mathbb{B}^{\to}) \). One can check that this construction preserves identities and composition giving a functor \( \zeta \).

Finally, we have for each \( X \) a bijection between sections of the map \( \zeta(X) \to q(X) \) and objects of \( \chi^{-1}(X) \). One can check this is natural, giving us a pullback square. For the converse, we recall that \( \mathbb{B}^{\to} \to \mathbb{B}^{\to} \) is always definable by Lemma 4.24. It follows that the same is true for \( \text{Cart}(\mathbb{B}^{\to}) \to \text{Cart}(\mathbb{B}^{\to}) \). But now using the pullback square we see that \( \text{Cart}(\chi) \) is definable by Lemma 3.12 and so \( \chi \) is too.

### 5.3 Small families of objects and universes

The main motivation for Shulman introducing local representability in [Shu19] was to study universes in models of type theory. In this section we recall, for reference, the relation between definability and universes in fibrations.

**Definition 5.5.** Let \( q : E \to B \) be a Grothendieck fibration, and an object \( V \) of \( E \), we say an object \( X \) of \( E \) is \( V \)-small if there exists a cartesian map \( X \to V \).

Note that \( V \) itself is \( V \)-small, since the identity map is cartesian. Also note that if an object \( U \) is \( V \)-small and another object \( X \) is \( U \)-small, then \( X \) is also \( V \)-small.

We now consider a collection of objects indexed by a class \( M \), say \((V_\alpha)_{\alpha \in M}\). Suppose further that every object \( X \) is \( V_\alpha \)-small for some \( \alpha \in M \). For this general definition it is technically possible to just take \( V_\alpha \) to be the class of all objects. However, in practice we usually assume extra conditions. For example, when working on a codomain fibration over a locally cartesian closed category, we might require class of \( V_\alpha \)-small maps to be closed under composition and dependent products. We can satisfy this over presheaf categories by assuming every set is contained in an inaccessible set, and taking \( V_\alpha \) to be the Hofmann-Streicher universe on an inaccessible ordinal \( \alpha \).

**Theorem 5.6.** A notion of structure \( \chi : D \to E \) is definable if and only if for all \( \alpha \in M \) we can find a cartesian map \( i : U_\alpha \to V_\alpha \) such that for every \( V_\alpha \)-small object \( X \) witnessed by a cartesian map \( f : X \to V_\alpha \) there is a natural correspondence between structures on \( X \) and maps \( q(X) \to q(U_\alpha) \) factoring \( q(f) \) through \( q(i) \).

**Proof.** By Lemma 2.7 and the assumption that every object is \( V_\alpha \)-small for some \( \alpha \), the right adjoint to \( \text{Cart}(\chi) \) is defined on all objects if and only if it is defined on each \( V_\alpha \). By Lemma 5.7 this is the same as each presheaf \( \bar{\chi}_{V_\alpha} \) being representable. However, expanding out the definition, this is precisely saying there is a natural correspondence between structures on \( X \) and maps \( q(X) \to q(U_\alpha) \) factoring \( q(f) \) through \( q(i) \).

Moreover, assume we can choose a canonical such \( \alpha(X) \) for each \( X \).
Note in particular that for presheaf categories we can obtain explicit descriptions of the $U_\alpha$ by Theorem 3.8.

By “truncating” the above theorem we get the following corollary.

**Corollary 5.7.** If $U_\alpha$ is as in the statement of Theorem 5.6 then an object $X$ is $U_\alpha$-small if and only if it is $V_\alpha$-small and admits at least one structure.

## 6 Fibrewise definability

There are different ways that a fibred awfs might be definable. We first note that the most direct definition of definability automatically holds in many situations:

**Theorem 6.1.** Let $q : E \to B$ be a locally small fibration. Every fibred awfs over $q$, regarded as a notion of structure on $V(E) \to B$, is definable.

**Proof.** This is a special case of Theorem 4.27. $\square$

Often this kind of definability is automatically true for cofibrantly generated awfs’s:

**Theorem 6.2.** Let $q : E \to B$ be a locally small fibration and suppose $B$ is locally cartesian closed. Every lifting notion of structure on $q$ is definable.

**Proof.** Write $h$ for the composition of maps

$$\text{hom}(A, X) \times_{\text{hom}(A, Y)} \text{hom}(B, Y) \longrightarrow I \times J \longrightarrow J$$

and $p$ for the canonical map $\text{hom}(B, Z) \to \text{hom}(m, f)$.

For any $\sigma : K \to J$, we have the following commutative diagram.

$$\begin{array}{ccc}
\text{hom}(B, \sigma^*(Y)) & \xrightarrow{\cong} & \sigma^*(\text{hom}(B, Y)) \\
\downarrow p' & & \downarrow \sigma(p) \\
\text{hom}(m, \sigma^*(f)) & \xrightarrow{\cong} & \sigma^*(\text{hom}(m, f))
\end{array}$$

Lifting structures on $\sigma^*(f)$ correspond precisely to sections of $p'$, which correspond precisely to maps $\sigma^*(\text{hom}(m, f)) \to \text{hom}(B, Y)$ making commutative triangles, as in the dotted diagonal arrow above. However, such maps correspond precisely to commutative triangles of the form below.

$$\begin{array}{ccc}
K & \xrightarrow{\sigma} & \prod_h (\text{hom}(B, Z)) \\
\sigma & & \Pi_h p
\end{array}$$

However, this verifies that $\prod_h p$ is indeed the representing object required to show that lifting structures are definable. $\square$
However, in practice for the semantics of type theory we are not so much interested in the entire fibred awfs, but only the ordinary awfs given by restriction to the terminal fibre $E_1$. This awfs is not necessarily definable, even if it is the restriction of a fibred awfs that is definable. Hence in this paper we mainly consider the following stronger notion, that we denote fibrewise definability.

**Definition 6.3.** Suppose we are given a fibration $q : E \to B$. A notion of structure on the fibration cod : $V(E) \to E$ is **fibrewise definable** if for each $I \in B$ the notion of structure on $E_I^* \to E_I$ given by restricting to the fibre category $E_I$ is definable.

This version of definability holds automatically for strongly fibred awfs’s, as shown below, but we will see some natural examples of fibred awfs’s where it does not.

**Theorem 6.4.** Let $q : E \to B$ be a locally small fibration. Suppose that each fibre category $E_I$ has dependent products. Then every strongly fibred awfs on $q$ is fibrewise definable.

**Proof.** We again use Theorem 4.27. 

### 7 Tiny Objects and Definable Awfs’s

Throughout this section we will assume that the Grothendieck fibration $q : E \to B$ is locally small and that $B$ has all finite limits. We write $\text{hom}(A, X)$ for the hom objects, following the convention that $A$ and $X$ can lie in different fibres. If $A$ is in the fibre over $I$ and $X$ in the fibre over $J$, then $\text{hom}(A, X)$ is isomorphic to $\text{hom}_{I \times J}(\pi_0^*(A), \pi_1^*(X))$, where $\pi_i$ are the projection maps out of $I \times J$. In particular, when $J$ is the terminal object, $\text{hom}(A, X)$ is the same as $\text{hom}_I(A, I^*(X))$.

We will give a sufficient criterion for a lifting notion of structure to be fibrewise definable, based on a definition of **family of tiny objects**. The argument is based on existing constructions in cubical sets [LOPS18, Awo19], but is more general in three respects:

1. By working in an arbitrary Grothendieck fibration we can not only use objects that are tiny in the internal sense, that exponentiation has a right adjoint, but by applying the the result to set indexed families, we can also use objects that are tiny in the external sense, that their hom set functor has a right adjoint.

2. Instead of focusing on the particular definition of Kan fibrations, we consider cofibrantly generated fibred awfs’s more generally. It turns out that for our sufficient criterion to apply, only the codomain of the generating family of left maps matters.

3. We consider not just individual tiny objects in a category, but families of tiny objects. Whereas the definition of Kan fibration only features one
tiny object, the interval, we will also see in Example 7.18 a definable awf s using the fact that the “family of all identity types” in a natural model can be seen as a family of tiny objects.

7.1 Tiny families of objects

Definition 7.1. Let $I$ and $J$ be elements of $B$ and $B$ an object of $E$. We say $B$ is tiny relative to $J$ if the functor $E_J \to B/(I \times J)$ defined as $\text{hom}(B, -)$ has a right adjoint. We say it is a tiny family of objects if it is tiny relative to $J$ for all $J$.

Example 7.2. Suppose that $E \to B$ is the fibration of set indexed families on a category $C$. Then a family $(B_i)_{i \in I}$ is tiny relative to $J$ (for any $J$) if and only if each object $B_i$ is externally tiny, i.e. the hom set functor $\text{hom}(B_i, -)$ has a right adjoint. If $C$ is a presheaf category then the tiny objects are precisely retracts of representables.

Example 7.3. Suppose that $E \to B$ is the fibration of category indexed families on a category $C$. Then a diagram $D : I \to C$ is tiny relative to $J$ when the functor $(D \downarrow -) : C^J \to \text{Cat}/(I \times J)$ has a right adjoint.

We can give the following sufficient criterion for diagrams of presheaves to be tiny over category indexed families.

Lemma 7.4. Suppose that $E \to B$ is the fibration of category indexed families on a presheaf category $\text{Set}^{op}$ and we are given a diagram $D : I \to C$. Then $y \circ D$ is tiny relative to $J$ for any small category $J$.

Proof. Given a representable $yc$ in $\text{Set}^{op}$, we see by the Yoneda lemma that $(y \circ D \downarrow yc) \cong (D \downarrow c)$. Hence if a right adjoint $G : \text{Cat}/I \to \text{Set}^{op}$ exists at all, we must have that $G(J \to I)(c)$ is the set of functors $(D \downarrow c) \to J$ over $I$, with the action on morphisms given by composition. One can check that this does indeed give a right adjoint.

Example 7.5. If $B \to B$ is a codomain fibration, and $B$ is tiny in the internal sense i.e. $(-)^B$ has a right adjoint, then the family $B \to 1$ is tiny relative to 1.

Example 7.5 is most useful when combined with the results below.

Lemma 7.6. Suppose that $B$ is locally cartesian closed. Suppose that $B \in E$ is tiny relative to $J$ and we are given a cartesian map $B' \to B$ over $\sigma : I \to K$, say. Then $B'$ is also tiny relative to $J$.

Proof. For each $X \in E_J$ we have the canonical pullback square below.

$$
\begin{array}{ccc}
\text{hom}(B', X) & \longrightarrow & \text{hom}(B, X) \\
\downarrow & & \downarrow \\
I & \underset{\sigma}{\longrightarrow} & J
\end{array}
$$
Hence we can factor the functor \( \text{hom}(B', -) \) as \( \text{hom}(B, -) : \mathbb{E}_1 \to \mathbb{E}/J \) followed by the functor \( \sigma^* : \mathbb{E}/J \to \mathbb{E}/I \) given by pullback. The former has a right adjoint by the assumption that \( B \) is tiny, and the latter by the assumption that \( \mathbb{E} \) is locally cartesian closed.

**Theorem 7.7** (Freyd–Yetter). Suppose that an object \( X \to 1 \) is tiny relative to 1 over a codomain fibration on a locally cartesian closed category with a classifier for regular monomorphisms.\(^6\) Suppose further that the map \( X \to 1 \) is a regular epimorphism. Then \( X \) is tiny relative to \( J \) for any object \( J \).

**Proof.** Essentially this is [Yet87, Theorem 1.4] aside from some rephrasing and the observations that the regular epimorphism condition is necessary\(^7\) and that a classifier for regular monomorphisms suffices for the proof in place of a subobject classifier.

It is well known that if the product functor \(- \times c\) is defined, then the representable \( y(c)\) is internally tiny. This was generalised to tiny families by Newstead [New18, Section 3.3].

**Theorem 7.8** (Newstead). We work over the codomain fibration on a presheaf category \( \text{Set}^{\text{op}} \). If \( f : X \to I \) is representable as a map of presheaves, then it is tiny relative to \( J \) for all \( J \).

**Proof.** Note that if \( f \) is representable, then we have a functor \( F : \int_c I \to \mathcal{C} \) such that for all \((c, i) \in \int_c I\) we have a pullback diagram of the form below:

\[
\begin{array}{ccc}
\text{y}(F(c, i)) & \longrightarrow & X \\
\downarrow & & \downarrow \\
\text{yc} & \longrightarrow & I
\end{array}
\]

For each \( J \), we can show that \( \text{hom}(X, -) \) is isomorphic to a functor obtained by reindexing along the functor \( F_J : \int_c I \times J \to \int_c J \) defined by \( F_J(c, i, j) := (F(c, i), j) \): for any \( c \in \mathcal{C} \) maps \( \text{yc} \to \text{hom}_{I \times J}(J^*(X), I^*(Y)) \) correspond, by the the adjunction between local exponentials and pullback, to maps \( \text{yc} \times_I X \to Y \), which correspond precisely to elements of \( Y(F(c, i)) \) in the fibre of \( j \).

\[
\begin{array}{ccc}
\text{Set}^{\text{op}}/J & \longrightarrow & \text{Set}^{\text{op}}/(I \times J) \\
\downarrow \cong & & \downarrow \cong \\
\int_c \text{Set}^{\text{op}}/J & \longrightarrow & \int_c \text{Set}^{\text{op}}/(I \times J)
\end{array}
\]

However, \( F_J \) has a right adjoint given by right Kan extension, so we are done. \( \square \)

---

\( ^6 \)Such categories are sometimes referred to as *quasitoposes*.

\( ^7 \)This was later noted by Yetter in an erratum to [Yet87].
Remark 7.9. Newstead also showed a converse statement when the small category $\mathcal{C}$ is Cauchy complete and has finite products. In this case every tiny family of objects is a representable map.

Corollary 7.10. Suppose that for an object $c$ of a small category $\mathcal{C}$, the product functor $- \times c$ exists. Then the representable object $yc$ is tiny as an object in presheaves.

Proof. If $- \times c$ exists then the unique map $yc \to 1$ is representable. \hfill $\square$

7.2 Definability from Tiny Codomain

We now use tininess to give examples of fibrewise definable lifting notions of structure.

We will use the following observation.

Lemma 7.11. Suppose we are given vertical maps $m : A \to B$, $f : X \to Y$ and $g : Z \to Y$ (where $f$ and $g$ necessarily lie in same the fibre). Write $g^*(f)$ for the pullback of $f$ along $g$. Then solutions to the universal lifting problem from $m$ to $g^*(f)$ correspond precisely to maps $\text{hom}(A,X) \times_{\text{hom}(A,Y)} \text{hom}(B,Z) \to \text{hom}(B,Y)$ making a commutative triangle as below.

$$
\begin{array}{c}
\text{hom}(A,X) \times_{\text{hom}(A,Y)} \text{hom}(B,Z) \\
\downarrow \text{hom}(A,X) \times_{\text{hom}(A,Y)} \text{hom}(B,Y)
\end{array}
$$

Proof. In the fibre category over $\text{hom}(A,X) \times_{\text{hom}(A,Y)} \text{hom}(B,Z)$ we can construct a commutative diagram of the following form.

$$
\begin{array}{c}
\sigma^*(A) \\
\downarrow \\
\sigma^*(B)
\end{array} \quad \begin{array}{c}
\tau^*(X) \\
\downarrow \\
\tau^*(Y)
\end{array}
$$

By the universal property of the pullback, this factors as two squares, below.

$$
\begin{array}{c}
\sigma^*(A) \\
\downarrow \\
\sigma^*(B)
\end{array} \quad \begin{array}{c}
\tau^*(g^*(X)) \\
\downarrow \\
\tau^*(Z)
\end{array} \quad \begin{array}{c}
\tau^*(X) \\
\downarrow \\
\tau^*(Y)
\end{array}
$$

One can check, for example by directly verifying the relevant universal property, that the left hand square is exactly the universal lifting problem from $m$ to $g^*(f)$.

Again applying the universal property of the pullback, diagonal fillers in the left hand square of (5) correspond precisely to diagonal fillers of (4). Maps
\(\sigma^*(B) \to \tau^*(X)\) correspond precisely to maps \(\hom(A, X) \times_{\hom(A, Y)} \hom(B, Z) \to \hom(B, X)\) by the universal property of \(\hom(B, X)\), and the upper and lower triangles commute for the diagonal filler if and only if the triangle in \((\Box)\) commutes.

**Theorem 7.12.** Suppose that \(\mathcal{B}\) is locally cartesian closed. Suppose we are given a vertical map \(m : A \to B\) in \(\mathcal{E}_I\) where \(B\) is tiny relative to \(J \in \mathcal{B}\). Then the restriction of the lifting notion of structure generated by \(m\) to \(J\) is definable.

**Proof.** Let \(G : \mathcal{B}/(I \times J) \to \mathcal{E}_J\) be the right adjoint to \(\hom(B, -)\).

Given \(f : X \to Y \in \mathcal{E}_J\), we will show that the presheaf \(\bar{\chi}_{\text{cod}(f)}\) from Definition 3.6 is representable.

Write \(p : \hom(m, f) \to \hom(B, Y)\) for the projection map, and the canonical map \(\hom(B, X) \to \hom(m, f)\) as \(t\). We construct the dependent product \(\prod_p t : \prod_p \hom(B, X) \to \hom(B, Y)\). Viewing this as a map in \(\mathcal{B}/(I \times J)\), we apply \(G\) to get a map \(G(\prod_p \hom(B, X)) \to G(\hom(B, Y))\). We pullback along the unit map \(\eta_Y : Y \to G(\hom(B, Y))\) to get a map \(Y \times_{G(\hom(B, Y))} G(\prod_p \hom(B, X)) \to Y\). We will show this is representing for the presheaf \(\bar{\chi}_{\text{q}(f)}\).

Fix a map \(g : Z \to Y\). Maps from \(g\) to \(f\) in \(\mathcal{E}_I/Y\) correspond naturally by the universal property of the pullback to maps \(h : Z \to G(\prod_p \hom(B, X))\) forming a commutative square as below.

\[
\begin{array}{ccc}
Z & \xrightarrow{h} & G(\prod_p \hom(B, X)) \\
\downarrow^g & & \downarrow \\
Y & \xrightarrow{\eta_Y} & G(\hom(B, Y))
\end{array}
\]

Passing across the adjunction, \(\hom(B, -) \dashv G\), we see that such maps correspond to the maps \(\hom(B, Z) \to \prod_p \hom(B, X)\) in the commutative square below.

\[
\begin{array}{ccc}
\hom(B, Z) & \xrightarrow{\hom(B, g)} & \prod_p \hom(B, X) \\
\downarrow & & \downarrow \\
\hom(B, Y) & \xrightarrow{} & \hom(B, Y)
\end{array}
\]

Rearranging, passing across the pullback-dependent product adjunction and simplifying allows us to apply Lemma 7.11 to show such diagrams correspond precisely to solutions of the universal lifting problem of \(m\) against \(g^*(f)\). □

**Corollary 7.13.** Let \(\mathcal{C}\) be a small category, and \((m_i : A_i \to yB_i)\) a family of maps in the presheaf category \(\text{Set}^{\mathcal{C}^{\text{op}}}\) with representable codomain. Then the awfs cofibrantly generated by the family is definable.

**Corollary 7.14.** For any small category \(\mathcal{C}\) with a wfs \((\mathcal{L}, \mathcal{R})\), there is a definable awfs on \(\text{Set}^{\mathcal{C}^{\text{op}}}\) such that the Yoneda embedding preserves and reflects left maps and right maps.
Proof. Take the generating left maps to be the image of the left maps in \( \mathcal{C} \) under the Yoneda embedding.

**Corollary 7.15.** Let \( \mathcal{B} \) be a locally cartesian closed category and let \( B \) be an internally tiny object in \( \mathcal{B} \). For any object \( I \) and any map \( m \) of the form \( A \to I^* (B) \) in the slice category \( \mathcal{B}/I \), the awfs cofibrantly generated by \( m \) is definable.

**Corollary 7.16.** Let \( \mathcal{E} \to \mathcal{B} \) be the category indexed families fibration for a category \( \mathcal{C} \) and let \( M : I \to \mathcal{C}^\to \) be a diagram of left maps that cofibrantly generates an awfs \( (L,R) \). If \( \text{cod} \circ M \) is tiny, then the awfs is definable.

Proof. Since \( \text{Cat} \) is not locally cartesian closed, we need to verify that the particular dependent product used in the proof of Theorem 7.12 exists, so that we can apply the same proof as before.

Write \( A \) for \( \text{cod} \circ M \) and \( B \) for \( \text{dom} \circ M \).

Explicitly, for each morphism \( f : X \to Y \) in \( \mathcal{C} \), we need to construct a dependent product along the canonical functor \( p : (A \downarrow X) \times_{(A \downarrow Y)} (B \downarrow Y) \to (B \downarrow Y) \). However, this functor is a discrete fibration, since it is a pullback of the discrete fibration \( (A \downarrow X) \to (A \downarrow Y) \). Hence the dependent product along \( p \) exists, and so we can continue following the same proof as in Theorem 7.12.

**Corollary 7.17.** For any small category \( \mathcal{C} \) with an awfs \( (L,R) \), there is a definable awfs on \( \text{Set}^{\mathcal{C}^\to} \) such that the Yoneda embedding lifts to functors from the categories of (co)algebra structures in \( \mathcal{C} \) to those in \( \text{Set}^{\mathcal{C}^\to} \).

Proof. We take the generating diagram of left maps to be the composition \( L\text{-Map} \to \mathcal{C}^\to \to \text{Set}^{\mathcal{C}^\to} \). It is clear by definition that composition with the codomain map factors through the Yoneda embedding.

**Example 7.18.** In Awodey’s natural models [Awo18, Section 2.4], intensional identity types are implemented as maps \( \text{id} : \tilde{U} \times_{\text{U}} \tilde{U} \) and \( i \) making a commutative square as below:

\[
\begin{array}{ccc}
\tilde{U} & \xrightarrow{i} & \tilde{U} \\
\downarrow{\delta} & & \downarrow{p} \\
\tilde{U} \times_{\text{U}} \tilde{U} & \xrightarrow{\text{id}} & \tilde{U}
\end{array}
\]

We can view the map \( \rho \), given by the universal property of the pullback below, as the “universal reflexivity map.”

\[
\begin{array}{ccc}
\tilde{U} & \xrightarrow{i} & \tilde{U} \\
\downarrow{\delta} & & \downarrow{p} \\
\tilde{U} \times_{\text{U}} \tilde{U} & \xrightarrow{\text{id}} & \tilde{U}
\end{array}
\]
8 Non definable examples
8.1 A review of Kan fibrations and Hurewicz Fibrations

Kan fibrations are one of the key ingredients to the standard model structure on simplicial sets \([\text{Qu}67]\). In this section we give a general definition of Kan fibration over a Grothendieck fibration, \(q : E \to B\). The definition is not the most general possible\(^8\), but is enough to cover most of the cases we will consider in this paper. For convenience we will assume that the base \(B\) has all finite limits and colimits, and that \(q\) has fibred products. We will write \(\hat{\times}\) for the pushout product on cartesian product. See e.g. \([\text{Rie}14, \text{Section 11.1}]\) for a standard reference on pushout product.

**Definition 8.1.** Let \(q : E \to B\) be a locally small fibration. Suppose we are given an interval object \(1 \to I \in E\) and a vertical monomorphism \(m : A \to B\) over \(I \in E\).

We say a vertical map \(f : X \to Y\) is a Kan fibration if it has the fibred right lifting property against the following family of maps: we first form the pushout products \(\delta_i \times_I m\) in \(E\) for \(i = 0, 1\), and then take their coproduct to obtain a vertical map over \(I + I\).

**Example 8.2.** We work over the set indexed family fibration on simplicial sets. We take the interval object to be \(\Delta_1\) and \(m\) to be the set indexed family of all boundary inclusions \(\partial \Delta_n \to \Delta_n\). In this way we obtain \([\text{ZG}67, \text{Chapter IV, Section 2, B}_2]\).

**Example 8.3.** We work over the codomain fibration on simplicial sets. We take the interval object again to be \(\Delta_1\). We take \(m\) to be the subobject classifier \(1 \to \Omega\) viewed as an object in \(\mathbf{Set}^{\Omega^{\text{op}}} / \Omega\). This gives us \([\text{GZ}67, \text{Chapter IV, Section 2, B}_3]\).

**Example 8.4.** As a generalisation of Example 8.3 we can work over an arbitrary topos with connected interval with disjoint endpoints, replace the subobject classifier with a classifier for a subclass of monomorphisms closed under composition and finite unions, and containing the endpoints of the interval. This gives the definition of Kan fibration in \([\text{OP}16]\).\(^9\)

\(^8\)Two possible generalisations are to replace an interval object with two global endpoints with a single generic point, as in \([\text{Awo}19]\), and to replace cartesian product with a general fibred monoidal product.

\(^9\)for Kan filling, rather than composition.
We also consider the following degenerate example:

**Definition 8.5.** A Hurewicz fibration is a Kan fibration where \( m \) is the unique map in \( E_1 \) from the initial object to the terminal object.

### 8.2 Full notions of structure and the axiom of choice

Our first examples of non-definable notions of structure will be full notions of structure arising from certain weak factorisation systems. The intuitive idea behind these results is that from any notion of structure we can obtain a full notion of structure by image factorisation, i.e. we can consider the class of objects “admitting at least one structure.” We can give a general rule of thumb that in the absence of the axiom of choice this is often unreasonable and can lead to non-definable full notions of structure, even when the original (non-full) notion of structure is definable. We first illustrate this idea with the very simple example of split epimorphisms. As we saw in Example 4.23 sections form a definable notion of structure on a codomain fibration.

**Theorem 8.6.** Suppose the class of split epimorphisms in \( \mathbb{B} \) is definable as a full notion of structure on \( \text{cod} : \mathbb{B}^{\rightarrow} \rightarrow \mathbb{B} \). Then every regular epimorphism splits.

**Proof.** Let \( f : X \rightarrow Y \) be a map in \( \mathbb{B} \) and let \( m : I \rightarrow Y \) be the representing object at \( f \). Note that if we pull \( f \) back along itself, then the projection map \( X \times_Y X \rightarrow X \) is a split epimorphism, with the diagonal map \( \Delta : X \rightarrow X \times_Y X \) as section. Hence \( f \) factors through \( m \). Now if \( f \) is a regular epimorphism, then it is left orthogonal to any monomorphism, giving us a section of \( m \). Hence \( f \) is a split epimorphism.

We now show the same basic idea applies when we consider the underlying wfs of a wide range of awfs’s in presheaf categories.

**Theorem 8.7.** Suppose we are given a weak factorisation system on a presheaf category \( \text{Set}^{\mathbb{C}_{op}} \) satisfying the following conditions:

1. The wfs is generated by locally decidable monomorphisms.
2. There is a left map \( m : A \rightarrow B \) such that there is \( c \in \mathbb{C} \) and \( x \in B(c) \) which does not lie in the image of \( m_c \).

If the notion of structure given by right maps (as in Example 4.11) is definable then the axiom of choice holds.

**Proof.** Suppose we are given a family of merely inhabited sets \((X_i)_{i \in I}\). We will construct a choice function for the family \((X_i)_{i \in I}\) from the assumption that the wfs is definable.

Let \( m : A \rightarrow B \) be a left map satisfying condition 2. Write \( \Omega_{\text{dec}} \) for the classifying object for locally decidable subobjects. For each \( i \in I \) we consider...
the factorisation $m$ given by the awfs cofibrantly by the following family of objects over the codomain fibration on $\textbf{Set}^{\textbf{C}^{\text{op}}}$.

\[
\begin{array}{ccc}
X_i & \xrightarrow{p_i} & \Omega_{\text{dec}} \times X_i \\
\downarrow & & \downarrow \\
\Omega_{\text{dec}} \times X_i & \xleftarrow{\Omega_{\text{dec}} \times p_i} & \\
\end{array}
\]

Intuitively the factorisation freely adds a filler for each lifting problem from a locally decidable monomorphism to $m$ and for each $x \in X_i$. Hence we can find a filler for each lifting problem, given a choice of $x \in X_i$.

We will write this factorisation as $(L_i, R_i)$, so we are considering the map $f_i := R_i m : K_i m \to B$.

Given a fixed $x \in X_i$ we can choose for each locally decidable monomorphism $n : C \to D$ a map $c : D \to \Omega_{\text{dec}}$ such that $n$ is the pullback of $p_i$ along $c$. Hence we can assign each left map an $L_i$-coalgebra structure and thereby a choice of diagonal filler for each lifting problem of a diagonal map against $f_i$. Since $X_i$ is merely inhabited it follows that there exists a function witnessing that $f_i$ has the right lifting property against each left map, and so is a right map. By the assumption of definability, it follows that the coproduct $\coprod_{i \in I} f_i : \coprod_{i \in I} K_i m \to I \times B$ is also a right map. Hence there is a function assigning a choice of filler for each lifting problem against $m$. In particular, for each $i \in I$ we have a choice of map $j_i$ for each of the following lifting problems.

\[
\begin{array}{ccc}
A & \xrightarrow{(i, L_i)} & \coprod_{i \in I} K_i m \\
\downarrow & & \downarrow \\
B & \xrightarrow{(i, 1_B)} & I \times B \\
\end{array}
\]

However, we can now read off from the explicit construction of $K_i m$ in presheaves [Swa18c] that each element of $K_i m(c)$ is either in the image of $L_i m$ or of the form $\text{sup}(z, \alpha)$ where $z$ belongs to $\Omega_{\text{dec}} \times X_i(c)$ and $\alpha$ is a dependent function to earlier constructed elements. We choose the object $c$ of $\mathcal{C}$ as in the condition on $m$ in the statement of the theorem to ensure the former case is not possible leaving only the latter case. In particular $\pi_1(z)$ belongs to $X_i$, giving us a choice function for the family $(X_i)_{i \in I}$.

We can use Theorem 3.4 to give a concrete example of a non definable weak factorisation system in simplicial presheaves. Given a small category $\mathcal{D}$, the category of simplicial presheaves is by definition the category of presheaves on $\mathcal{D} \times \Delta$. We can view this as the category of category of simplicial sets constructed internally in the presheaf topos $\textbf{Set}^{\mathcal{D}^{\text{op}}}$.

Following the work of Gambino, Henry, Sattler, Szumi\[GSS19, Hen19, GHSS21\] we can construct a model structure on simplicial presheaves using the internal logic of $\textbf{Set}^{\mathcal{D}^{\text{op}}}$. They define Kan fibrations as cofibrantly generated by the set of horn inclusions. We can read off an external description of the Kan fibrations as follows. We can equivalently view the category of simplicial presheaves as the category of functors
We can then read off the internal definition of Kan fibration as being the same as that given by the pointwise awfs [Rie11, Section 4.2] on Kan fibrations. Explicitly a natural transformation $f$ between functors $[\mathcal{D}, \text{Set}^\Delta^{op}]$ is a right map if we can assign Kan fibration structures $f_d$ for each object $d$ of $\mathcal{D}$ in such a way that for each morphism $\sigma : d \to d'$ in $\mathcal{D}$ is a morphism of Kan fibrations. That is, the diagonal fillers are chosen so that the triangles in the centre of each diagram below commute.

$$\begin{array}{ccc}
\Lambda_1^n & \longrightarrow & X_{d'} \\
\downarrow & \searrow & \downarrow X_d \\
\Delta_n & \longrightarrow & Y_{d'} \quad \Downarrow \quad \longrightarrow Y_d
\end{array}$$

We note that this definition gives a definable awfs. This follows from our general result, but already appears implicitly in the construction of the universe by Gambino and Henry [GH22]. However, it is commonly the case for the axiom of choice to fail in presheaf toposes, and since horn inclusions are locally decidable we can apply Theorem 8.7 to show that the full notion of structure from the wfs underlying the awfs is not definable. For instance, this applies for the very simple example of simplicial presheaves on the walking arrow $\cdot \to \cdot$, since $\text{Set}^{\rightarrow}$ is not boolean and so does not satisfy the internal axiom of choice. Finally we observe that in homotopical algebra it is common to consider two other definitions of Kan fibration in simplicial presheaves: projective and injective. The wfs of projective Kan fibrations is cofibrantly generated by a set of maps with representable codomain and so definable. For injective Kan fibrations the situation is in general unclear, but if $\mathcal{D}$ is an inverse category, as for simplicial presheaves on the walking arrow, then we can apply the construction of the universe in [Shu14, Section 12] to see that the wfs is definable, while the more sophisticated techniques of [Shu19] allow one to replace the wfs of injective Kan fibrations with a different, non-full, notion of structure, which is definable and has the same underlying class of maps for any small category $\mathcal{D}$.

### 8.3 Hurewicz fibrations in topological spaces and related examples

**Theorem 8.8.** The awfs of Hurewicz fibrations is not definable in any of the following categories:

1. Topological spaces
2. The function realizability topos
3. The Kleene–Vesley topos

**Proof.** We first consider topological spaces. We note that in the commutative cube below the top and bottom faces are pushouts and all side faces are pull-
backs.

We will show there are multiple Hurewicz fibration structures that all agree on the two pushout inclusions.

Suppose we are given a lifting problem as below:

\[
\begin{array}{c}
Z \times 1 \rightarrow Z \\
\downarrow \quad \downarrow \\
Z \times \delta_0 \rightarrow Z \times I
\end{array}
\]

For each \( c \in \mathbb{R} \), we define a diagonal filler \( j_c : Z \times I \rightarrow \mathbb{R} \times \mathbb{R} \) by the following formula:

\[
j_c(z, x) := (h(z) + c \min(k(z, 0) + 1, 0) \max(k(z, x) - 1, 0), k(z, x))
\]

Note that if the homotopy \( k \) factors through the inclusion \((-\infty, 1) \hookrightarrow \mathbb{R}\) then \( \max(k(z, x) - 1, 0) = 0 \) for all \( z, x \), and if it factors through the inclusion \((-1, \infty) \hookrightarrow \mathbb{R}\) then \( \min(k(z, 0) + 1, 0) = 0 \). In either case we have

\[
j_c(z, x) := (h(z), k(z, x)).
\]

However, it is easy to come up with examples of lifting problems where \( j_c \) is different for different values of \( c \). For example, this is the case whenever \( Z = 1 \) and \( k \) is defined by \( k(x) := 4x - 2 \).

For the function realizability and Kleene–Vesley topos, we simply use the embedding of countably based \( T_0\) spaces into the function realizability topos [Bau02], and observe that \( j_c \) is computable whenever \( h \) and \( k \) are.

8.4 Non definability of Kan fibrations from logical properties of the interval

We now give two classes of examples of non definable awfs's. In both cases we use the internal logic of a topos to construct similar examples to the one in Section 8.3 from certain logical principles. The first of these is that the interval admits a linear ordering, and the second a principle that we denote “detachable diagonal.” In both cases we will construct the Kan fibration structures in the internal logic of the topos, following Orton and Pitts [OP16].
8.4.1 Linear intervals and simplicial sets

It has already been shown by Sattler that the Kan fibrations of Example 8.3 are not definable in the category of simplicial sets, with a proof appearing in [vdBF22, Appendix D]. In this section we will see a new, more general proof of this fact.

Recall that the category of simplicial sets is the classifying topos for linear intervals with disjoint endpoints. In particular $\Delta^1_1$ is the generic such, with order relation given by degeneracy maps $\Delta^2 \to \Delta^1_1 \times \Delta^1_1$, and endpoints the face maps $\delta_0, \delta_1: \Delta^0 \Rightarrow \Delta^1_1$. We will show that in fact linearity of the intervals suffices to show non definability.

In the below, let $C$ be a topos and $(I, \leq, 0, 1)$ be a linear order with endpoints in $C$. Assume further that $(I, \leq, 0, 1)$ is non trivial in that the endpoint map $2 \to I$ is not a regular epimorphism. Equivalently, the following statement does not hold in the internal logic of $C$:

$$\forall x \in I \ x = 0 \lor x = 1$$

Note that any connected interval with disjoint endpoints is non trivial in this sense.

From linearity, we can show that $I \times I$ is the union of the two subobjects defined by $T_0 := \{(x, y) | x \geq y\}$ and $T_1 := \{(x, y) | x \leq y\}$. We define in the internal language a family of objects indexed over $I \times I$ by $Z_{x,y} := \{\varphi \in \Omega | x \geq y \rightarrow \varphi\}$. Note that the pullback of $Z$ along each of the inclusions $T_i \hookrightarrow I \times I$ is a trivial fibration, in the strongest sense, that we have a choice of lift against all monomorphisms.

**Lemma 8.9.** We construct two different fibration structures, in the sense of Example 8.3 on $Z \to I \times I$ that are equal when restricted to $T_0$ and when restricted to $T_1$.

**Proof.** We work in the internal logic of $C$. We will just define fillers for paths in the direction 0 to 1, the other direction being similar.

Suppose we are given $\psi \in \Omega$, a path $p: I \to I \times I$, an element $z_0$ of $Z_{p(0)}$ and a dependent function $f: \prod_{x:\psi} \psi \to Z_{p(x)}$ such that $\prod_{w:\psi} f(0, w) = z_0$.

For the first fibration structure, we define $q: \prod_{x:1} Z_{p(x)}$, as follows.

$$q(x) := \sum_{w:\psi} f(x, w) \lor (x = 0 \land z_0) \lor p(x) \in T_0 \lor$$

$$p(0) = (0, 0) \land \pi_1(p(x)) = 1 \land \prod_{w:\psi} f(x, w)$$

Note that the clause $p(x) \in T_0$ ensures that $q(x)$ belongs to $Z_{p(x)}$. We also need to check the boundary conditions. We clearly have $z_0 \to q(0)$. It remains to check $q(0) \to z_0$. To do this we show that each clause in the disjunction defining $q(0)$ implies $z_0$. For $\sum_{w:\psi} f(x, w)$ we apply the assumption that $\prod_{w:\psi} f(0, w) = z_0$. The second clause $x = 0 \land z_0$ is clear. For the third clause $p(0) \in T_0$, note
that this implies $z_0 = \top$. For the final clause we note that $p(0) = (0,0)$ and $\pi_1(p(0)) = 1$ gives a contradiction, making the final clause equal to $\bot$. We can similarly show the boundary condition for the partial elements.

For the second fibration structure, we define $r : \prod_{x : I} Z_{p(x)}$, as follows.

$$r(x) := \sum_{w : \psi} f(x, w) \lor (x = 0 \land z_0) \lor p(x) \in T_0 \lor \pi_1(p(0)) = 0 \land \pi_1(p(x)) = 1 \land \prod_{w : \psi} f(x, w)$$

A similar argument to before shows that $r$ satisfies the boundary conditions.

We check that $q$ and $r$ agree whenever $p$ lies entirely in $T_0$ and whenever it lies entirely in $T_1$. The former is trivial. For the latter, note that for any element of $T_1$ of the form $(x, y)$ we have by definition $x \leq y$ and so $y = 0$ if and only if $(x, y) = (0,0)$, so we can see that $q(x)$ and $r(x)$ are equivalent for all $x$.

We will show these give different values for a lifting problem against the pushout product of $2 \hookrightarrow I$ and $\delta_0 : 1 \rightarrow I$. In the internal logic we view this as a family of paths indexed by $I$, say a path $p_x : I \rightarrow I \times I$ for each $x \in I$, taken together with $\psi_x := (x = 0) \lor (x = 1)$ and partial elements that we need to define. We define $p_x(y) := (x, y)$ and take the partial elements $f$ to be constantly equal to $\top$.

We can then compute $q_x(1) = x = 0 \lor x = 1$ and $r_x(1) = \top$. However, composing with the inclusion $Z_{(x,1)} \hookrightarrow \Omega$ these give two different subobjects of $I$ by the non triviality condition on $\Omega$.

There are several possible ways to define Kan fibrations in simplicial sets that are known to give the same class of maps when working in a classical setting (see e.g. [GZ67, Chapter IV, Section 2]). By Corollary 7.13 we know that the awfs generated by the set indexed family of horn inclusions is definable. Also, as remarked by Shulman, the full notion of structure for the underlying wfs is definable, assuming the axiom of choice (which is strictly necessary by Theorem 8.7). However, none of the other commonly considered awfs’s on simplicial sets are definable.

**Corollary 8.10.** The awfs’s cofibrantly generated by the following classes of maps are not definable in simplicial sets:

1. Pushout product of mono and endpoint inclusion. (Example 8.1)
2. Pushout product of locally decidable mono and endpoint inclusion.
3. Pushout product of the set of subobjects of representables and endpoint inclusion.
4. Pushout product of boundary inclusions and endpoint inclusion. (Example 8.2)
5. *Fibred lifting problem against the coproduct of all horn inclusions with respect to the codomain fibration. This is same as enriched lifting problem for cartesian monoidal product.*

*Proof.* We directly considered $\mathbb{1}$ in Lemma 8.9. Note that in each awfs, each generating left map is a left map of $\mathbb{1}$ and so we do get two right map structures for each of the awfs’s. To show that the two right map structures are different in Lemma 8.9 we considered a lifting problem against the pushout product of $2 \hookrightarrow \mathbb{I}$ and $\delta_0 : 1 \rightarrow \mathbb{I}$. This is a left map in $2$, $3$ and $4$, so exactly the same proof applies for each of these. This only leaves $5$ for which we observe that a similar argument applies to $\mathbb{I} \times \delta_0$.

We also note that the same argument applies to Hurewicz fibrations:

**Corollary 8.11.** Let $\mathbb{C}$ be a topos and $(\mathbb{I}, \leq, 0, 1)$ be a non trivial linear order with endpoints in $\mathbb{C}$. Then Hurewicz fibrations are not definable.

### 8.4.2 Detachable diagonal and BCH cubical sets

We recall that BCH cubical sets [BCH14] possess a non-cartesian monoidal product, *separated product*. Bezem, Coquand and Huber defined Kan fibrations to be maps with the right lifting property against the category indexed family given by pushout product of a maps $0 \rightarrow \square_n$ a boundary inclusion $\partial \square_m \hookrightarrow \square_m$ and an endpoint inclusion (the last two can also be merged together to give an “open box inclusion”). A morphism is a pair of maps $\square_n \rightarrow \square_{n'}$ and $\square_m \rightarrow \square_{m'}$, which induces a morphism the corresponding maps in the pushout product. It is clear from the construction of the universe by Bezem, Coquand and Huber that this gives a definable awfs, although we can now also see definability as an instance of Corollary 7.16.

One might wonder what happens if instead of separated product we use cartesian product in the pushout product in the generating family of trivial cofibrations. We will give a concrete reason that this is a bad idea: the resulting awfs is not definable. Just as for simplicial sets, we will identify a logical property of the interval that suffices to carry out the argument.

We fist note that $\mathbb{I} \times \mathbb{I}$ can be constructed by “pasting a diagonal to $\mathbb{I} \otimes \mathbb{I}$.” More formally we have the following lemma.

**Lemma 8.12.** The following diagram is both a pushout and a pullback.

\[
\begin{array}{ccc}
1 + 1 & \xrightarrow{[(0,0),(1,1)]} & \mathbb{I} \otimes \mathbb{I} \\
\downarrow & & \downarrow \\
\mathbb{I} & \xrightarrow{\Delta} & \mathbb{I} \times \mathbb{I}
\end{array}
\]

*Proof.* E.g. this can be seen clearly using Pitts’ presentation of the category as 01-substitution sets [Pit15].
We can understand this pushout in the internal logic as follows. The lemma tells us directly that $I \times I$ can be written as the union of the subobjects $I \otimes I$ and $\Delta : I \hookrightarrow I \times I$ and that the intersection of these two subobjects is the inclusion of diagonal endpoints $2 \hookrightarrow I \times I$. In general, separated product $\otimes$ is not well behaved with respect to the internal logic of cubical sets, but in this case we can deduce from the above statement and purely formal reasoning in the Heyting algebra of subobjects that $I \otimes I$ can be defined from the diagonal inclusion via Heyting implication. Hence BCH cubical sets satisfy the following:

**Definition 8.13.** Let $\mathcal{B}$ be a topos with interval object $I$. We say $I$ has detachable diagonal if the following statement holds in the internal language of $\mathcal{B}$.

$$\forall i, j \in I, \quad i = j \vee (i = j \rightarrow i = 0 \vee i = 1)$$

**Remark 8.14.** Once again our main example of a topos with this property is in fact the classifying topos. To make this precise, note that given an interval $I$ with disjoint endpoints and detachable diagonal, we can define a binary relation $-\#-$ by taking $x \# y$ when $x = y \rightarrow x = 0 \vee x = 1$. This then defines a model for the following geometric theory:

$$\begin{align*}
x \# y & \vdash y \# x \\
x \# x & \vdash x = 0 \vee x = 1 \\
\vdash x \# 0 & \\
\vdash x \# 1 & \\
\vdash x = y & \\
\vdash x = 0 \vee x = 1 & \\
0 = 1 & \dashv \bot
\end{align*}$$

BCH cubical sets are the classifying topos for this theory with generic object $I$, where the binary relation $-\#-$ is the canonical map $I \otimes I \hookrightarrow I \times I$.

**Theorem 8.15.** Suppose that we are given a topos with connected interval object with disjoint endpoints and detachable diagonal.

Then the awfs cofibrantly generated by pushout product of monomorphisms and endpoint inclusions (with respect to cartesian product) is not definable.

**Proof.** We define for each $x, y \in I$ a set $P_{x,y}$ as follows. We first define

$$Q_{x,y} := \{0, 1 \in 2 \mid x = y\} + \{2, 3 \mid x = y \rightarrow x = 0 \vee x = 1\}$$

We define an equivalence relation $\sim$ on $Q$ by setting $0 \sim 2$ and $1 \sim 3$ when $x = y = 0$ and setting $0 \sim 3$ and $1 \sim 2$ when $x = y = 1$. We define $P_{x,y}$ to be the quotient $Q_{x,y}/\sim$.

Note that by assumption we can write $I \times I$ as a union of two subobjects: the diagonal $\{(x, y) \in I \times I \mid x = y\}$ and the subobject $C := \{(x, y) \in I \times I \mid x = y \rightarrow x = 0 \vee x = 1\}$. We will show that the restriction of $P$ to either subobject is isomorphic to the family constantly equal to 2, and so a Kan fibration, whereas $P$ itself is not.

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10 The only closed semi cartesian monoidal product fibred over a codomain fibration is cartesian product: any such monoidal product has a fibred right adjoint by closedness and so by Lemma [2.7] preserves opcartesian maps, and so we calculate $A \otimes_1 B \cong A \otimes_1 \sum_B 1 \cong \sum_B (B^*(A) \otimes_B 1) \cong A \times B$. 

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First consider the diagonal. In this case each $P_{x,x}$ contains equivalence classes $[0]$ and $[1]$. It can only contain the equivalence class $[2]$ when $x = 0 \lor x = 1$. However, in the former case $[2] = [0]$ and in the latter case $[2] = [1]$. Similarly, it can only contain an equivalence class $[3]$ when it is equal to either $[0]$ or to $[1]$. Hence $P_{x,x} \cong 2$.

Now consider the case where $(x,y) \in C$. In this case $P_{x,y}$ definitely contains the equivalence relations $[2]$ and $[3]$. However, it can only contain $[0]$ when it is identified with either $[2]$ or with $[3]$, and similarly for $[1]$. Hence we have $P_{x,y} \cong \{2, 3\} \cong 2$.

We now define a family of paths $p_x$ in $I \times I$ by setting $p_x(y) := (x, y)$. We define $z \in P_{x(0)}$ to be $[2]$. If $P$ is a Kan fibration, then we would have a family of fillers $j_x : \prod_{y \in I} P_{x,y}$. Note that by the explicit description of $P_{x,x}$ above, we have for all $x$ that $j_x(x) = [0]$ or $j_x(x) = [1]$. Since $j_0(0) = z_0 = [2] = [0]$, and using the connectedness of the interval, we have $j_1(1) = [0]$.

Now using the explicit description of $P_{x,y}$ for $(x,y) \in C$, we see that each $j_1(x)$ must be either equal to $[2]$ or to $[3]$. Again using $z_1$ and the connectedness of the interval, we have $j_1(1) = [2]$. However, $[0]$ and $[2]$ are not equal as elements of $P_{1,1}$, giving a contradiction.

**Remark 8.16.** Since we showed there is no fibration structure at all on the pushout, we can show that for presheaf categories where the interval has detachable diagonal the “canonical” universe, constructed in Theorem 3.3 is not fibrant. If it was, we would be able to construct a map from $I$ into the universe using the universal property of the pushout, and thereby pull back the fibration structure to the pushout.

9 Conclusion

Definability is a fundamental notion in the theory of Grothendieck fibrations that characterises when external properties and structure can be accessed from within the internal logic of the base of a fibration. It has appeared in many different guises over time. In this paper we gave a comprehensive overview unifying the theory of definability developed by Lawvere, Bénabou and Johnstone with the separate thread starting with Cisinski’s definition of local fibration [Cis14, Definition 3.7], further developed by Sattler [Sat17] and ending with Shulman’s local representability.

Algebraic weak factorisation systems can be viewed as monadic notions of structure equipped with a composition functor. As notions of structure they lie on the boarder between definability and non definability. On the side of definability we saw a general sufficient criterion that encompasses some very different looking examples of definable awfs’s. By applying our result to a codomain fibration, we recovered the definability of Kan fibrations in cubical sets [LOPS18, Awo19]. By applying to set indexed family fibrations, we obtained a different looking criterion, where the exponential functor used in the internal definition of tininess is replaced with a hom set functor. The theorem is phrased
as a general condition on awfs’s cofibrantly generated by a family of maps in a fibration, that includes Kan fibrations generated by a tiny interval, but also other examples. In particular in Example 7.18 we saw an example of awfs’s in natural models that made essential use of a tiny family of objects that is not simply generated by one tiny object. The general result includes most examples of cofibrantly generated awfs’s used in the semantics of homotopy type theory. However, we leave two interesting classes of examples as a direction for future work. The first is awfs’s cofibrantly generated by a double category, such as the definition of Kan fibration due to Van den Berg and Faber in [vdBF22], who gave a direct proof of definability. The second is examples where the role of exponential in Kan fibration is replaced by monoidal exponential, as in the definition of Kan fibration by Bezem, Coquand and Huber in [BCH14]. A promising approach is suggested by Nuyts and Devriese in [ND21], who showed that the relevant right adjoint to monoidal exponentiation is an instance of a general construction of transpension types in presheaf categories.

Our examples of non definable awfs’s included identifying logical principles satisfied by the interval that can be used to show the non definability of Kan fibrations. In both cases the main examples of simplicial sets, and BCH cubical sets respectively turned out to be classifying toposes for the structures that we considered. Simplicial sets have long been regarded as a very natural setting for studying the structure of topological spaces up to homotopy [GZ67], and they are used in the original model of homotopy type theory [KL21] so it is natural to ask if the same can be done when working constructively. We have seen that many reasonable definitions of Kan fibration in simplicial sets are non definable. However, we leave it as an open problem to either show that one of the non definable versions of Kan fibration can still be used to model univalent type theory in simplicial sets, or to show it causes an unavoidable obstruction, in which case it is necessary to use one of the definable versions of Kan fibration, as in [GSS19] or [vdBF22], or to avoid simplicial sets entirely in favour of other categories such as cubical sets. In BCH cubical sets we saw a more severe example of the kind of thing that can happen in the absence of definability. BCH cubical sets are a presheaf topos, and so very well behaved as a category, and possesses an obvious choice of interval object. As such, one might naively expect that as an alternative to the original monoidal definition of Kan fibration, it would be possible to construct a model of homotopy type theory using the cartesian definition of Kan fibration. However, the only apparent choice of universe classifying Kan fibrations fails to be a Kan fibration itself. We can see from this that definability is a key property to consider when constructing models of homotopy type theory.

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