Adaptive Estimation for Non-stationary Factor Models And A Test for Static Factor Loadings

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Abstract

This paper considers the estimation and testing of a class of high-dimensional non-stationary time series factor models with evolutionary temporal dynamics. In particular, the entries and the dimension of the factor loading matrix are allowed to vary with time while the factors and the idiosyncratic noise components are locally stationary. We propose an adaptive sieve estimator for the span of the varying loading matrix and the locally stationary factor processes. A uniformly consistent estimator of the effective number of factors is investigated via eigenanalysis of a non-negative definite time-varying matrix. A high-dimensional bootstrap-assisted test for the hypothesis of static factor loadings is proposed by comparing the kernels of the covariance matrices of the whole time series with their local counterparts. We examine our estimator and test via simulation studies and real data analysis.

Keywords: Time series factor model, local stationarity, high dimensional time series, test of static factor loadings, adaptive estimation.

1 Introduction

Technology advancement has made it easy to record simultaneously a large number of stochastic processes of interest over a relatively long period of time where the underlying data generating mechanisms of the processes are likely to evolve over the long observation time span. As a result both high dimensional time series analysis (Wei (2018)) and non-stationary time series analysis (Dahlhaus (2012)) have undergone unprecedented developments over the last two decades. This paper focuses on the following evolutionary linear factor model for a multivariate non-stationary time series:

\[ x_{i,n} = A(i/n)z_{i,n} + e_{i,n}, \]

(1.1)
where \( \{x_{i,n}\}_{i=1}^{n} \) is a \( p \)-dimensional observed time series, \( A(t) : [0, 1] \rightarrow \mathbb{R}^{p \times d(t)} \) is a matrix-valued function of possibly time-varying factor loadings and the number of factors \( d(t) \) is assumed to be a piecewise constant function of time, \( \{z_{i,n}\}_{i=1}^{n} \) is a \( d(i/n) \)-dimensional unobserved sequence of common factors and \( \{e_{i,n}\}_{i=1}^{n} \) are the idiosyncratic components. Here \( p = p_n \) may diverge to infinity with the time series length \( n \) and \( d(t) \) is typically much smaller than \( p \) uniformly over \( t \). Note that \( x_{i,n}, z_{i,n} \) and \( e_{i,n} \) are allowed to be locally-stationary processes for which the generating mechanism varies with time, see (4.4) for detailed formulation. Throughout the article we assume that \( \{e_{i,n}\} \) and \( \{z_{i,n}\} \) are centered.

The version of model (1.1) with constant loadings is among the most popular dimension reduction tools for the analysis of multivariate stationary time series (Stock and Watson (2011), Tsay (2013), Wei (2018)). According to the model assumptions adapted and estimation methods used, it seems that recent literature on linear time series factor models mainly falls into two types. The celebrated cross-sectional averaging method (summarized in Stock and Watson (2011)) which is popular in the econometric literature of linear factor models, exploits the assumption of weak dependence among the vector components of \( e_{i,n} \) and hence achieves de-noising via cross-sectional averaging. See for instance Stock and Watson (2002), Bai and Ng (2002), Bai (2003) and Forni et al. (2000) among many others. One advantage of the cross-sectional averaging method is that it allows for a very high dimensionality. In general the method requires that \( p \) diverges to achieve consistency and the estimation accuracy improves as \( p \) gets larger under the corresponding model assumptions. On the other hand, the linear factor model can also be fitted by exploring the relationship between the factor loading space and the auto-covariance or the spectral density matrices of the time series under appropriate assumptions. This method dates back at least to the works of Anderson (1963), Brillinger (2001) and Pena and Box (1987) among others for fixed dimensional multivariate time series and is extended to the high dimensional setting by the recent works of Lam et al. (2011), Lam and Yao (2012), Wang et al. (2019) and others. The latter method allows for stronger contemporary dependence among the vector components and is consistent when \( p \) is fixed under the requirement that the idiosyncratic components form a white noise.

To date, the literature on evolutionary linear factor models is scarce and the existing results are focused on extensions of the cross-sectional averaging method. Among others, Breitung and Eickmeier (2011) and Yamamoto and Tanaka (2015) considered testing for structural changes in the factor loadings and Motta et al. (2011) and Su and Wang (2017) considered evolutionary model (1.1) using the cross-sectional averaging method. Eichler et al. (2011) and Barigozzi et al. (2021) studied non-stationary dynamic factor models. In this paper, we shall extend the second estimation method mentioned in the last paragraph to the case of evolutionary factor loadings with locally stationary factor and idiosyncratic component time
series whose data generating mechanisms change smoothly over time. Using this framework, our approach to the factor model estimation contributes to the literature mainly in the following two aspects.

(a) To estimate the time-varying loading matrix, the prevailing approach in the literature is the local-constant kernel estimator, see for example Motta et al. (2011), Su and Wang (2017). It seems that it is difficult to extend the local-constant method to general local polynomial methods for factor models under the cross-sectional averaging set-up and therefore the estimation accuracy of the existing methods is not adaptive to the smoothness (with respect to time) of the factor loading matrix function. In this paper, we propose an alternative adaptive estimation method based on the method of sieves (Grenander (1981), Chen (2007)). The sieve method is computationally simple to implement and has the advantage of being adaptive to the unknown smoothness of the target function if certain linear sieves such as the Fourier basis (for periodic functions), the Legendre polynomials or the orthogonal wavelets are used (Chen (2007), Wang and Xiang (2012)). Specifically, we adapt the method of sieves to estimate the high-dimensional auto-covariance matrices of $x_{i,n}$ at each time point and subsequently estimate the span of the loadings $A(t)$ at each $t$ exploiting the relationship between $A(\cdot)$ and the kernel of the latter local auto-covariance matrices. We will show that the span of $A(\cdot)$ can be estimated at a rate independent of $p$ uniformly over time provided that all factors are strong with order $p^{1/2}$ Euclidean norms, extending the corresponding result for factor models with static loadings established in Lam et al. (2011)

(b) In most literature for time-varying factor models such as Motta et al. (2011), Su and Wang (2017), to estimate the time-varying loading matrix, it is assumed that the number of factors is constant over time. Typically further requirements on the factor process such as independence or time-invariance of its covariance matrix had to be added. In this paper, we model the factor process as a general locally stationary time series and allow the number of factors to be time-varying. Uniform consistency of the estimated span of the loading matrix as well as the number of factors will be established without assuming that the positive eigenvalues of the corresponding matrices are distinct which is commonly posited in the literature of factor models.

Testing whether $A(\cdot)$ is constant over time is important in the application of (1.1). In the literature, among others Breitung and Eickmeier (2011) proposed LR, LM and Wald statistics for testing static factor model against an alternative of piece-wise constant loadings, and Yamamoto and Tanaka (2015) improved the power of Breitung and Eickmeier (2011) by maximizing the test statistic over possible
numbers of the original factors. Assuming piece-wise stationarity, Barigozzi et al. (2018) estimated the change points of a factor model via wavelet transformations. Su and Wang (2017) considered an $L^2$ test of static factor loadings under the cross-sectional averaging method assuming that each component of $e_{i,n}$ is a martingale difference sequence. Here we shall propose a high-dimensional $L^\infty$ or maximum deviation test on the time-invariance of the span of $A(\cdot)$ which utilizes the observation that the kernel of the full-sample auto-covariance matrices coincides with all of its local counterparts under the null hypothesis of static span of loadings while the latter observation is likely to fail when the span of $A(\cdot)$ is time-varying. Using the uniform convergence rates of the estimated factor loadings established in this paper, the test statistic will be shown to be asymptotically equivalent to the maximum deviation of the sum of a high-dimensional locally stationary time series under some mild conditions. A multiplier bootstrap procedure with overlapping blocks is adapted to approximate the critical values of the test. The bootstrap will be shown to be asymptotically correct under the null and powerful under a large class of local alternatives. The assumptions of our testing procedure are different from those of the existing literature in the following two aspects.

(c) Under the null hypothesis of constant $A(\cdot)$, the common components of the time series, i.e. $A(i/n)z_{i,n}$, considered in the above-mentioned works are stationary or have time-invariant variance-covariance. With locally stationary $z_{i,n}$ assumed in this paper, under the null hypothesis, the common components are allowed to be locally stationary where their variance-covariance matrices can be smoothly time-varying.

(d) The validity of the tests of the above works is built on the divergence of both the length of time series $n$ and the dimension of the time series $p$. In contrast, our proposed test is proved to be asymptotically correct when $p$ is fixed or diverging slowly with $n$. Therefore, our test is suitable to investigate the loading matrix of fixed or moderately high-dimensional vector time series such as the COVID 19 death data investigated in Section 11.2 which records the daily COVID 19 deaths for $n = 591$ days in 57 places of the U.S. and three countries in the compacts of Free Association with the U.S (hence $p = 60$).

The paper is organized as follows. Section 2 introduces some notation. Section 3 explains the locally stationarity of the factor model (1.1). The assumptions of the model are investigated in Section 4. Sections 5 and 6 discuss the estimation of the evolutionary factor loading matrices and the test of static factor loadings, respectively, and the corresponding theoretical results are contained in Section 7. Section 8 investigates the power performance of the test of the static factor loading. Section 9 gives out methods
for tuning parameter selection. Numerical studies are displayed in Section 10. Section 11 illustrates a real world financial data application and a COVID 19 data analysis. Section 12 provides the proofs of Theorem 7.1, Theorem 7.2, Theorem 7.3, Proposition 7.3 and Theorem 7.4. The proofs of the remaining theorems, propositions, lemmas and corollaries are relegated to the online supplemental material.

2 Notation

For two series \(a_n\) and \(b_n\), write \(a_n \asymp b_n\) if the exists \(0 < m_1 < m_2 < \infty\) such that \(m_1 \leq \lim \inf \frac{|a_n|}{|b_n|} \leq \lim \sup \frac{|a_n|}{|b_n|} \leq m_2 < \infty\). Write \(a_n \preceq b_n\) (\(a_n \succeq b_n\)) if there exists a uniform constant \(M\) such that \(a_n \leq Mb_n\) (\(a_n \geq Mb_n\)). Let \(A := B\) represent ‘\(A\) is define as \(B\)’. For any \(p\) dimensional (random) vector \(v = (v_1, ..., v_p)\), write \(\|v\|_u = (\sum_{s=1}^{p} |v_s|^u)^{1/u}\), and the corresponding \(L^v\) norm \(\|v\|_{L^v} = (\mathbb{E}(|v|^v))^{1/v}\) for \(v \geq 1\). For any matrix \(F\) let \(\lambda_{\text{max}}(F)\) be its largest eigenvalue, and \(\lambda_{\text{min}}(F)\) be its smallest eigenvalue. Let \(\|F\|_F = (\text{trace}(F^T F))^{1/2}\) denote the Frobenius norm, and \(\|F\|_m\) be the smallest nonzero singular value of \(F\). Denote by \(\text{vec}(F)\) the vector obtained by stacking the columns of \(F\). Let \(\|F\|_2 = \sqrt{\lambda_{\text{max}}(FF^T)}\). In particular, if \(F\) is a vector, then \(\|F\|_2 = \|F\|_F\). For any vector or matrix \(A = (a_{ij})\) let \(|A|_\infty = \max_{i,j} |a_{ij}|\).

For any integer \(v\) denote by \(I_v\) the \(v \times v\) identity matrix. Write \(|\mathcal{I}|\) be the length of the interval \(\mathcal{I}\).

Let \(\mathcal{C}^K(\tilde{M})[0, 1]\) be the collection of functions \(f\) defined on \([0, 1]\) such that the \(K\)th order derivative of \(f\) is Lipschitz continuous with Lipschitz constant \(\tilde{M}\), \(\tilde{M} > 0\). Regarding the number of factors, let \(d = \sup_{0 \leq t \leq 1} d(t)\).

3 Locally stationarity of Model (1.1)

We allow the number of factors and the dimension of loading matrix of model (1.1) is time-varying. Since the number and the dimension are integers, it is sophisticated to define the ‘smoothly changing’ factor number or ‘smoothly changing’ dimensions directly, where the concept of ‘smoothly changing’ is the key assumption of locally stationary models and of nonparametric smoothing approaches. Moreover, in current literature many assumptions including stationarity and dependence strength, are not directly applicable to time series with possibly changing dimensions such as \(z_{i,n}\) in model (1.1). On the other hand, the dimension of \(x_{i,n}\) is time invariant, which is convenient to be assumed locally stationary. In the following we show that the evolutionary factor model (1.1) could be derived from a factor model with a fixed factor dimension \(d\) while the rank of the loading matrix varies with time. The latter representation is beneficial for the theoretical investigation of the model and provides insight on how smooth changes in the loading matrix cause the dimensionality of the factor space to change.
Consider a \( p \times d \) matrix \( \mathbf{A}^*(t) = (a_1^s(t), \ldots, a_d^s(t)) \) where \( a_s^x(t), 1 \leq s \leq d \) are \( p \) dimensional smooth functions, \( \sup_{t \in [0,1]} \| a_s^x(t) \|^2 \leq p^{1-\delta} \), and \( \inf_{t \in [0,1]} \| \mathbf{A}^*(t) \|_F \leq p^{1-\delta}, \sup_{t \in [0,1]} \| \mathbf{A}^*(t) \|_F \leq p^{1-\delta} \). The parameter \( \delta \) refers to the factor strength and will be discussed in condition (C1) later. Let \( \mathbf{z}_{i,n}^* \) and \( \mathbf{e}_{i,n} \) be \( d \) and \( p \) dimensional locally stationary time series. Here local stationarity refers to a slowly or smoothly time-varying data generating mechanism of a time series, which we refer to Dahlhaus (1997) and Zhou and Wu (2009) among others for rigorous treatments. Consider the following model with a fixed dimensional loading matrix

\[
\mathbf{x}_{i,n} = \mathbf{A}^*(i/n)\mathbf{z}_{i,n}^* + \mathbf{e}_{i,n}. \tag{3.1}
\]

Using singular value decomposition (SVD), model (3.1) can be written as

\[
\mathbf{x}_{i,n} = p^{\frac{1-\delta}{2}} \mathbf{U}(i/n) \Sigma(i/n) \mathbf{W}^\top(i/n) \mathbf{z}_{i,n}^* + \mathbf{e}_{i,n}, \tag{3.2}
\]

where \( \Sigma(i/n) \) is a \( p \times d \) rectangular diagonal matrix with diagonal \( \Sigma_{uu}(i/n) = \sigma_u(i/n) \) for \( 1 \leq u \leq d \), \( (\sigma_u(i/n))_1 \leq u \leq d \) are singular values of \( \mathbf{A}^*(i/n)/p^{\frac{1-\delta}{2}}, \mathbf{U}(i/n) \mathbf{U}^\top(i/n) = \mathbf{I}_p \) and \( \mathbf{W}^\top(i/n) \mathbf{W}(i/n) = \mathbf{I}_d \). Therefore \( \max_1 \leq u \leq d, \sup_{t \in [0,1]} \sigma_u(t) \) is bounded. Further, the \( (\sigma_u(i/n))_1 \leq u \leq d \) are ordered such that \( \sigma_1(i/n) \geq \ldots \geq \sigma_d(i/n) \geq 0 \). Let \( d(t) \) be the number of nonzero singular values \( \sigma_u(t) \) and equation (3.2) can be further written as

\[
\mathbf{x}_{i,n} = p^{\frac{1-\delta}{2}} \tilde{\mathbf{U}}(i/n) \tilde{\Sigma}(i/n) \tilde{\mathbf{W}}^\top(i/n) \mathbf{z}_{i,n}^* + \mathbf{e}_{i,n}, \tag{3.3}
\]

where \( \tilde{\Sigma}(i/n) \) is the matrix by deleting the rows and columns of \( \Sigma(i/n) \) where the diagonals are zero, and \( \tilde{\mathbf{U}}(i/n) \) and \( \tilde{\mathbf{W}}^\top(i/n) \) are the matrices resulted from the deletion of corresponding columns and rows of \( \mathbf{U}(i/n) \) and \( \mathbf{W}^\top(i/n) \), respectively. Let \( \mathbf{A}(i/n) = p^{\frac{1-\delta}{2}} \tilde{\mathbf{U}}(i/n) \tilde{\Sigma}(i/n) \) and \( \mathbf{z}_{i,n} = \tilde{\mathbf{W}}^\top(i/n) \mathbf{z}_{i,n}^* \), then \( \mathbf{A}(i/n) \) is a \( p \times d(i/n) \) matrix and \( \mathbf{z}_{i,n} \) is a length \( d(i/n) \) locally stationary vector. Notice that \( \mathbf{z}_{i,n} := \tilde{\mathbf{W}}^\top(i/n) \mathbf{z}_{i,n}^* \) in (3.2) is the collection of all factors in the sense that all the entries of \( \mathbf{z}_{i,n} \) are elements of \( \mathbf{z}_{i,n} \) for each \( i \). The fact that model (1.1) can be obtained from model (3.3) has the following implications. First, though we allow \( d(t) \) to jump from one integer to another over time, it is appropriate to assume \( \mathbf{x}_{i,n} \) of (1.1) is locally stationary since model (3.1) is locally stationary as long as \( \mathbf{z}_{i,n}^* \) and \( \mathbf{e}_{i,n} \) are locally stationary processes and each element of \( \mathbf{A}^*(t) \) is Lipschitz continuous. Notice that the dimensions of \( \mathbf{z}_{i,n}^* \) and \( \mathbf{A}^*(t) \) are time-invariant. Second, we observe from the SVD that \( \| \mathbf{A}^*(i/n) \|_m = \| \mathbf{A}(i/n) \|_m \) will be close to zero in a small neighborhood around the time when \( d(t) \) of model (1.1) changes in which
case it is difficult to estimate the rank of $A^\ast(\cdot)$ or the number of factors accurately. We will exclude such small neighborhoods in our asymptotic investigations. The varying $d(t)$ has been considered in the piecewise stationary factor models and factor models with piecewise constant loading matrix, where the jump points of $d(t)$ correspond to the change points of the factor models, see for instance Barigozzi et al. (2018) and Ma and Su (2018) among others.

4 Model Assumptions

Let $\tilde{z}_{i,n}$ be the vector consisting of all factors which appear during the considered period, and the components of $z_{i,n}$ are always elements of $\tilde{z}_{i,n}$. Adapting the formulation in Zhou and Wu (2009), we model the locally stationary time series $x_{i,n}$, $\tilde{z}_{i,n}$ and $e_{i,n}$, $1 \leq i \leq n$ as follows:

$$x_{i,n} = G(i/n, F_i), \quad \tilde{z}_{i,n} = Q(i/n, F_i), \quad e_{i,n} = H(i/n, F_i) \quad (4.4)$$

where the filtration $F_i = (e_{-\infty}, \ldots, e_{i-1}, e_i)$ with $\{e_i\}_{i \in \mathbb{Z}}$ i.i.d. random elements, and $G$, $Q$ and $H$ are $p$, $d$ and $p$ dimensional measurable nonlinear filters. The $j_{th}$ entry of the time series $x_{i,n}$, $\tilde{z}_{i,n}$ and $e_{i,n}$ can be written as $x_{i,j,n} = G_j(i/n, F_i)$, $\tilde{z}_{i,j,n} = Q_j(i/n, F_i)$ and $e_{i,j,n} = H_j(i/n, F_i)$. Let $\{e_i\}_{i \in \mathbb{Z}}$ be an independent copy of $\{e_i\}_{i \in \mathbb{Z}}$ and denote by $F_i^{(h)} = (e_{-\infty}, \ldots, e_h, e_{h+1}, \ldots, e_i)$ for $h \leq i$, and $F_i^{(h)} = F_i$ otherwise. The dependence measures for $x_{i,n}$, $\tilde{z}_{i,n}$ (or $z_{i,n}$ equivalently) and $e_{i,n}$ in $L^l$ norm are defined as (Zhou and Wu 2009)

$$\delta_G(l) := \sup_{t \in [0,1], i \in \mathbb{Z}, 1 \leq j \leq p} \delta_{G,l,j}(k) := \sup_{t \in [0,1], i \in \mathbb{Z}, 1 \leq j \leq p} \mathbb{E}^{1/l}(\|G_j(t, F_i) - G_j(t, F_i^{(i-k)})\|^l),$$

$$\delta_Q(l) := \sup_{t \in [0,1], i \in \mathbb{Z}, 1 \leq j \leq d} \delta_{Q,l,j}(k) := \sup_{t \in [0,1], i \in \mathbb{Z}, 1 \leq j \leq d} \mathbb{E}^{1/l}(\|Q_j(t, F_i) - Q_j(t, F_i^{(i-k)})\|^l),$$

$$\delta_H(l) := \sup_{t \in [0,1], i \in \mathbb{Z}, 1 \leq j \leq p} \delta_{H,l,j}(k) := \sup_{t \in [0,1], i \in \mathbb{Z}, 1 \leq j \leq p} \mathbb{E}^{1/l}(\|H_j(t, F_i) - H_j(t, F_i^{(i-k)})\|^l),$$

which quantify the magnitude of change of systems $G$, $Q$, $H$ in $L^l$ norm when the inputs of the systems $k$ steps ahead are replaced by their $i.i.d.$ copies. To formally state the requirements of model (1.1), we define further some notation. At time $i/n$, the $d(i/n)$ dimensional factors $z_{i,n} = (z_{i,1,n}, \ldots, z_{i,d(i/n),n})^\top$ where the index set $\{l_1, \ldots, l_{d(i/n)}\} \subset \{1, 2, \ldots, d\}$. Let $Q(t, F_i) = (Q_{i_1}(t, F_i), \ldots, Q_{i_d(t)}(t, F_i))^\top$, and $\Sigma_z(t, k) = \mathbb{E}(Q(t, F_{i+k})Q^\top(t, F_i))$, $\Sigma_e(t, k) = \mathbb{E}(H(t, F_{i+k})H^\top(t, F_i))$ denote the $k_{th}$ order auto-covariance of $z_{i,n}$ and $e_{i,n}$ at time $t$, respectively. Let $\Sigma_{xz}(t, k) = \mathbb{E}(Q(t, F_{i+k})H^\top(t, F_i))$, $\Sigma_{ez}(t, k) = \mathbb{E}(H(t, F_{i+k})Q^\top(t, F_i))$ and $\Sigma_z(t, k) = \mathbb{E}(G(t, F_{i+k})G^\top(t, F_i))$. By Section 3 we assume naturally $x_{i,n}$ is locally stationary which
implies that $\Sigma_x(t,k)$ is smooth with respect to $t$. We assume the following conditions for model (1.1).

(A1) Let $a_{ij}(t), 1 \leq i \leq p, 1 \leq j \leq d(t)$ be the $(i,j)_{th}$ element of $A(t)$. We assume there exists a sufficiently large constant $M$ such that

$$\sup_{t \in [0,1]} |a_{ij}(t)| \leq M. \quad (4.5)$$

(A2) Let $\sigma_{x,u,v}(t,k)$ be the $(u,v)_{th}$ element of $\Sigma_x(t,k)$. Assume $\sigma_{x,i,j}(t,k), 1 \leq i \leq p$ and $1 \leq j \leq p$, $1 \leq k \leq k_0$ belongs to a common functional space $\Omega$ which is equipped with an orthonormal basis $B_j(t)$, i.e. $\int_0^1 B_m(t)B_n(t)dt = 1(m = n)$, where $1(\cdot)$ is the indicator function. Assume $\Omega \in C^K(M)[0,1]$ for some $K \geq 2$. Moreover

$$\max_{1 \leq i \leq p, 1 \leq j \leq p} \sup_{t \in [0,1]} |\sigma_{x,i,j}(t,k) - \sum_{u=1}^{J_n} \tilde{\sigma}_{x,i,j,u}(k)B_u(t)| = O_p(g_{J_n,K,M}), \quad (4.6)$$

where $\tilde{\sigma}_{x,i,j,u}(k) = \int_0^1 \sigma_{x,i,j}(t,k)B_u(t)dt$, and $g_{J_n,K,M} \rightarrow 0$ as $J_n \rightarrow \infty$.

Condition (A1) concerns the boundedness of the loading matrix, while (A2) means $\Sigma_x(t,k)$ can be approximated by the basis expansion. The approximation error rate $g_{J_n,K,M}$ diminishes as $J_n$ increases. Often higher differentiability yields more accurate approximation rate.

We present condition (C1) on factor strength (c.f. Section 2.3 of Lam et al. (2011)) as follows.

(C1) Write $A(t) = (a_1(t), ..., a_{d(t)}(t))$ where $a_s(t), 1 \leq s \leq d(t)$ are $p$ dimensional vectors. Then

$$\sup_{t \in [0,1]} \|a_s(t)\|_2^2 \asymp p^{1-\delta} \text{ for } 1 \leq s \leq d(t) \text{ for some constant } \delta \in [0,1].$$

Besides, the matrix norm of $A(t)$ satisfies

$$\inf_{t \in [0,1]} \|A(t)\|_F \asymp p^{\frac{1-\delta}{2}}, \sup_{t \in [0,1]} \|A(t)\|_F \asymp p^{\frac{1-\delta}{2}}, \inf_{t \in \mathcal{T}_n} \|A(t)\|_m \geq \eta_n^{1/2} p^{\frac{1-\delta}{2}} \quad (4.7)$$

for a positive sequence $\eta_n = O(1)$ on a collection of intervals $\mathcal{T}_n \subset [0,1]$.

As in Lam et al. (2011) and Lam and Yao (2012), $\delta = 0$ and $\delta > 0$ in (C1) correspond to strong and weak factor strengths, respectively. Assumption (C1) means that the $d(t)$ factors in the model are of equal strength $\delta$ on $[0,1]$, and $\|A(t)\|_m$ is at least of the order $\eta_n^{1/2} p^{\frac{1-\delta}{2}}$ on $\mathcal{T}_n$ which enables us to correctly identify the number of factors on $\mathcal{T}_n$. By our discussions of model (3.1), $\mathcal{T}_n$ excludes small neighborhoods around the change points of the number of factors. See Remark 7.3 of Section 7 for the use of $\eta_n$ in practice. The assumptions for $z_{i,n}$ and $e_{i,n}$ are now in order.
(M1) The short-range dependence conditions hold for both $\mathbf{z}_{i,n}$ and $\mathbf{e}_{i,n}$ in $\mathcal{L}^l$ norm, i.e.

$$
\Delta_{Q,l,m} := \max_{1 \leq j \leq d} \sum_{k=m}^{\infty} \delta_{Q,l,j}(k) = o(1), \quad \Delta_{H,l,m} := \max_{1 \leq j \leq p} \sum_{k=m}^{\infty} \delta_{H,l,j}(k) = o(1)
$$

(4.8)

as $m \to \infty$ for some constant $l \geq 4$.

(M2) There exists a constant $M$ such that

$$
\sup_{t \in [0,1]} \max_{1 \leq u \leq d} \mathbb{E}|Q_u(t, F_0)|^4 \leq M, \quad \sup_{t \in [0,1]} \max_{1 \leq v \leq p} \mathbb{E}|H_v(t, F_0)|^4 \leq M.
$$

(M3) For $t, s \in [0,1]$, there exists a constant $M$ such that

$$
(\mathbb{E}|Q_u(t, F_0) - Q_u(s, F_0)|^2)^{1/2} \leq M|t - s|, 1 \leq u \leq d,
$$

(4.9)

$$
(\mathbb{E}|H_v(t, F_0) - H_v(s, F_0)|^2)^{1/2} \leq M|t - s|, 1 \leq v \leq p,
$$

(4.10)

$$
(\mathbb{E}|G_v(t, F_0) - G_v(s, F_0)|^2)^{1/2} \leq M|t - s|, 1 \leq v \leq p.
$$

(4.11)

Conditions (M1)-(M3) mean that each coordinate process of $\mathbf{z}_{i,n}$ and $\mathbf{e}_{i,n}$ are standard short memory locally stationary time series defined in the literature; see for instance Zhou and Wu (2009). Furthermore, the moment condition (M2) implies that for $0 \leq t \leq 1$, all elements of the matrices $\Sigma_{ze}(t,k)$ and $\Sigma_e(tk)$ are bounded in $\mathcal{L}^4$ norm.

We then postulate the following assumptions on the covariance matrices of the common factors $\mathbf{z}_{i,n}$ and the idiosyncratic components $\mathbf{e}_{i,n}$, which are needed for spectral decomposition:

(S1) For $t \in [0,1]$ and $k = 1, ..., k_0$, all components of $\Sigma_e(t,k)$ are 0.

(S2) For $k = 0, 1, ..., k_0$, $\Sigma_z(t,k)$ is full ranked on some sub-interval $\mathcal{I}_0$ of $[0,1]$.

(S3) For $t \in [0,1]$ and $k = 1, ..., k_0$, all components of $\Sigma_{ze}(t,k)$ are 0.

(S4) For $t \in [0,1]$ and $1 \leq k \leq k_0$, $\|\Sigma_{ze}(t,k)\|_F = o(\eta_n^{1/2} p^{1-\delta})$, where $\eta_n$ is the sequence defined in condition (C1).

Condition (S1) indicates that $(\mathbf{e}_{i,n})$ does not have auto-covariance up to order $k_0$ which is slightly weaker than the requirement that $(\mathbf{e}_{i,n})$ is a white noise process used in the literature. Condition (S2) implies that for $1 \leq i \leq n$, there exists a certain period that no linear combination of components of $\mathbf{z}_{i,n}$ is white noise that can be absorbed into $\mathbf{e}_{i,n}$. (S3) implies that $\mathbf{z}_{i,n}$ and $\mathbf{e}_{i+k,n}$ are uncorrelated for any $k \geq 0$. Condition
(S4) requires a weak correlation between \( z_{i+k,n} \) and \( e_{i,n} \). In fact it is the non-stationary extension of Condition (i) in Theorem 1 of Lam et al. (2011) and condition (C6) of Lam and Yao (2012). Though (C6) of Lam and Yao (2012) assumes a rate of \( o(p^{1-\delta}) \), it requires standardization of the factor loading matrix \( A \). If \( d(t) \) is piecewise constant with bounded number of change points, then \( |T_{\eta_n}| \to 1 \) as \( n \to \infty \) and \( \eta_n \to 0 \). If \( d(t) \equiv d \) we can assume that \( T_{\eta_n} = [0, 1] \) for some sufficiently small positive \( \eta_n > 0 \).

**Remark 4.1.** We now discuss the equivalent assumptions on the fix dimensional loading matrix model (3.1). Define \( \Sigma_{z^*}(t,k), \Sigma_{z^*e^*}(t,k), \Sigma_{e^*}(t,k) \) similarly to \( \Sigma(z)(t,k), \Sigma_{ze}(t,k), \Sigma_{e^*z^*}(t,k) \) and assume (S) for these quantities. By construction conditions (M) are satisfied. In particular, for model (3.1), (4.9) and (4.10) imply (4.11). Conditions (A) are satisfied if each element of \( A^*(t), \Sigma_{z^*}(t,k) \) and \( \Sigma_{z^*e^*}(t,k), k = 0, ..., k_0 \) has a bounded \( K \) order derivative. Condition (C1) is satisfied with \( T_{\eta_n} \) corresponding to the period when the smallest nonzero eigenvalue of \( \Sigma(z) \) exceeds \( \eta_n \).

### 5 Model Estimation

Observe from equation (1.1) and conditions (S1)-(S4) that

\[
\Sigma_x(t,k) = A(t)\Sigma_z(t,k)A^\top(t) + A(t)\Sigma_{ze}(t,k), \quad k \geq 1.
\]

Further define \( \Lambda(t) = \sum_{k=1}^{k_0} \Sigma_x(t,k)\Sigma_x^\top(t,k) \) for some pre-specified integer \( k_0 \) and we have

\[
\Lambda(t) = A(t)\left[ \sum_{k=1}^{k_0} (\Sigma_z(t,k)A^\top(t) + \Sigma_{ze}(t,k))(A(t)\Sigma_z^\top(t,k) + \Sigma_{ze}(t,k)) \right]A^\top(t).
\]

Therefore in principle \( A(t) \) can be identified by the null space of \( \Lambda(t) \). The use of \( \Lambda(t) \) was advocated in Lam et al. (2011), and in this paper we aim at estimating a set of time-varying orthonormal basis of this time-varying null space, which is identifiable up to rotation, to characterize \( A(t) \). Therefore we don’t impose the condition utilized by Lam et al. (2011) that the loading matrix is normalized for identification. As we discussed in the introduction, fitting factor models using relationships between the factor space and the null space of the auto-covariance matrices has a long history. In the following we shall propose a nonparametric sieve-based method for time-varying loading matrix estimation which is adaptive to the smoothness (with respect to \( t \)) of the covariance function \( \Sigma_x(t,k) \). For a pre-selected set of orthonormal basis functions \( \{B_j(t)\}_{j=1}^{\infty} \) satisfying the basis approximation condition (A2), we shall
approximate $\Sigma_x(t, k)$ by a finite but diverging order basis expansion

$$\Sigma_x(t, k) \approx \sum_{j=1}^{J_n} \int_0^t \Sigma_x(u, k) B_j(u) du B_j(t), \quad (5.3)$$

where the order $J_n$ diverges to infinity. The speed of divergence is determined by the smoothness of $\Sigma_x(t, k)$ with respect to $t$. Motivated by (5.3) we propose to estimate $\Lambda(t)$ by the following $\hat{\Lambda}(t)$:

$$\hat{\Lambda}(t) = \sum_{k=1}^{K_0} \hat{M}(J_n, t, k) \hat{M}^\top (J_n, t, k), \quad \text{where} \quad \hat{M}(J_n, t, k) = \sum_{j=1}^{J_n} \hat{\Sigma}_{x,j,k} B_j(t), \quad (5.4)$$

$$\hat{\Sigma}_{x,j,k} = \frac{1}{n} \sum_{i=1}^{n-k} \mathbf{x}_{i+k} \mathbf{x}_i^\top B_j(i/n). \quad (5.5)$$

Let $\lambda_1(\hat{\Lambda}(t)) \geq \lambda_2(\hat{\Lambda}(t)) \geq ... \geq \lambda_p(\hat{\Lambda}(t))$ denote the eigenvalues of $\hat{\Lambda}(t)$, and $\hat{\mathbf{v}}(t) = (\hat{\mathbf{v}}_1(t), ..., \hat{\mathbf{v}}_{d(t)}(t))$ where $\hat{\mathbf{v}}_i(t)$'s are the eigenvectors of $\hat{\Lambda}(t)$ corresponding to $\lambda_1(\hat{\Lambda}(t)), ..., \lambda_{d(t)}(\hat{\Lambda}(t))$. Then we estimate the column space of $\Lambda(t)$ by

$$\text{Span}(\hat{\mathbf{v}}_1(t), ..., \hat{\mathbf{v}}_{d(t)}(t)). \quad (5.6)$$

To implement this approach it is required to determine $d(t)$, which we estimate using a modified version of the eigen-ratio statistics advocated by Lam and Yao (2012); that is, we estimate

$$\hat{d}_n(t) = \arg\min_{1 \leq i \leq R(t)} \lambda_{i+1}(\hat{\Lambda}(t))/\lambda_i(\hat{\Lambda}(t)). \quad (5.7)$$

where $R(t)$ is the largest integer less or equal to $p/2$ such that

$$\frac{|\lambda_{R(t)}(\hat{\Lambda}(t))|}{\sqrt{\sum_{i=1}^p \lambda_i^2(\hat{\Lambda}(t))}} \geq C_0 \eta_n / \log n \quad (5.8)$$

for some positive constant $C_0$. Asymptotic theory established in Section 7 guarantees that the percentage of variance explained by each of the first $d(t)$ eigenvalues will be much larger than $\eta_n^2 / \log^2 n$ on $T_{\eta_n}$ with high probability which motivates us to restrict the upper-bound of the search range, $R(t)$, using (5.8).
6 Test for Static Factor Loadings

It is of practical interest to test $H_0 : \text{span}(A(t)) = \text{span}(A)$, where $A$ is a $p \times d$ matrix. In other words, one can find a time-invariant matrix $A$ to represent the factor loading matrices throughout time. Without loss of generality, we shall assume that $A(t) = A$ under the null hypothesis throughout the rest of the paper if no confusions will arise. Under the null hypothesis, $\Sigma_x(t, k) = A\Sigma_z(t, k)A^\top + A\Sigma_\varepsilon(t, k)$ for $k \neq 0$. Observe that testing $H_0$ is more subtle than testing covariance stationarity of $x_{i,n}$ as both $z_{i,n}$ and $e_{i,n}$ can be non-stationary under the null. By equation (1.1), we have, under $H_0$,

$$
\int_0^1 \Sigma_x(t, k) dt = A \int_0^1 (\Sigma_z(t, k)A^\top + \Sigma_\varepsilon(t, k)) dt, \quad k > 0.
$$

Consider the following quantity $\Gamma_k$ and its estimate $\hat{\Gamma}_k$:

$$
\Gamma_k = \int_0^1 \Sigma_x(t, k) dt \int_0^1 \Sigma_x^\top(t, k) dt, \quad \hat{\Gamma}_k = \left(\sum_{i=1}^{n-k} x_{i+k,n}x_{i,n}^\top/n\right)\left(\sum_{i=1}^{n-k} x_{i+k,n}x_{i,n}^\top/n\right)^\top.
$$

Let $\Gamma = \sum_{k=1}^{k_0} \Gamma_k$ and $\hat{\Gamma} = \sum_{k=1}^{k_0} \hat{\Gamma}_k$. Then the kernel space of $A$ can be estimated by the kernel of $\hat{\Gamma}$ under $H_0$. Let $\hat{f}_i$ and $f_i$, $i = 1, \ldots, p-d$ be the orthonormal eigenvectors of $\hat{\Gamma}$ and $\Gamma$ w.r.t. $(\lambda_{d+1}(\hat{\Gamma}), \ldots, \lambda_{p}(\hat{\Gamma}))$ and $(\lambda_{d+1}(\Gamma), \ldots, \lambda_{p}(\Gamma))$, respectively. Write $\mathbf{F} = (f_1, \ldots, f_{p-d})$, $\hat{\mathbf{F}} = (\hat{f}_1, \ldots, \hat{f}_{p-d})$.

The test is constructed by segmenting the time series into non-overlapping equal-sized blocks of size $m_n$. Without loss of generality, consider $n - k_0 = m_nN_n$ for integers $m_n$ and $N_n$. Define for $1 \leq j \leq N_n$ the index set $b_h = ((h-1)m_n + 1, \ldots, hm_n)$ and the statistic

$$
\hat{\Sigma}_{h,k} = \sum_{i \in b_h} x_{i+k,n}x_{i,n}^\top/m_n.
$$

Then we define the test statistic $\hat{T}_n$

$$
\hat{T}_n = \sqrt{m_n} \max_{1 \leq k \leq k_0} \max_{1 \leq i \leq n \leq N_n} \max_{1 \leq i \leq p-d} |\hat{f}_i^\top \hat{\Sigma}_{h,k}^x|_\infty,
$$

where $\hat{d}_n$ is an estimate of $d$. The test $\hat{T}_n$ utilizes the observation that the kernel space of the full sample statistic $\hat{\Gamma}$ coincides with that of $\Sigma_x(t, k)$ for each $t$ under the null while the coincidence is likely to fail under the alternative. Hence the multiplication of $\hat{f}_i^\top$, $1 \leq i \leq p-d$ and $\hat{\Sigma}_{h,k}^x$ should be small in $L^\infty$ norm uniformly in $h$ under the null while some of those multiplications are likely to be large under the alternative. We then propose a multiplier bootstrap procedure with overlapping blocks to determine the
critical values of $\hat{T}_n$. Define for $1 \leq i \leq m_n$, $1 \leq h \leq N_n$, the $(p - d) \times p$ vector

$$
\hat{l}_{i,h,k} = \text{vec}( (\hat{F}^\top \tilde{x}_{(h-1)m_n+i+k,n} x_{(h-1)m_n+i,n})^\top ),
$$

(6.3)
and the $k_0(p - d)p$-vector $\hat{l}_{i,h,\cdot} = (\hat{l}_{i,h,1}^\top, \ldots, \hat{l}_{i,h,k_0}^\top)$. Further define

$$
\hat{l}_i = (\hat{l}_{i,1,\cdot}^\top, \ldots, \hat{l}_{i,N_n,\cdot}^\top)
$$

(6.4)
for $1 \leq i \leq m_n$. Notice that $\hat{T}_n = \left| \sum_{i=1}^{m_n} \hat{l}_i / \sqrt{m_n} \right|_{\infty}$. For given $m_n$, let $\hat{s}_{j,w_n} = \sum_{r=j}^{j+w_n-1} \hat{l}_r$ and $\hat{s}_m = \sum_{r=1}^{m_n} \hat{l}_r$
for $1 \leq i \leq m_n$ where $w_n = o(m_n)$ and $w_n \to \infty$ is the window size. Define

$$
\kappa_n = \frac{1}{\sqrt{w_n(m_n - w_n + 1)}} \sum_{j=1}^{m_n-w_n+1} (\hat{s}_{j,w_n} - \frac{w_n}{m_n} \hat{s}_m) R_j
$$

(6.5)
where $\{R_i\}_{i \in \mathbb{Z}}$ are i.i.d. $N(0,1)$ independent of $\{x_{i,n}, 1 \leq i \leq n\}$. Then we have the following algorithm for testing static factor loadings:

**Algorithm for implementing the multiplier bootstrap:**

1. Select $m_n$ and $w_n$ by the Minimal Volatility (MV) method that will be described in Section 9.2.
2. Generate $B$ (say 1000) conditionally i.i.d. copies of $K_r = |\kappa_n^{(r)}|_{\infty}$, $r = 1, \ldots, B$, where $\kappa_n^{(r)}$ is obtained by (6.5) via the $r$th copy of i.i.d. standard normal random variables $\{R_i^{(r)}\}_{i \in \mathbb{Z}}$.
3. Let $K(r), 1 \leq r \leq B$ be the order statistics for $K_r, 1 \leq r \leq B$. Then we reject $H_0$ at level $\alpha$ if $\hat{T}_n \geq K_{\lfloor (1-\alpha)B \rfloor}$. Let $B^* = \min\{r : K(r) \geq \hat{T}_n\}$ and the corresponding $p$ value of the test can be approximated by $1 - B^*/B$.

To implement our test, $d$ will be estimate by

$$
\hat{d}_n = \arg\min_{1 \leq i \leq R} \lambda_{i+1}(\hat{\Gamma})/\lambda_i(\hat{\Gamma}).
$$

(6.6)
for some constant $R$ satisfying (5.8) with $\hat{\Lambda}(t)$ there replaced by $\hat{\Gamma}$.
7 Asymptotic Results

7.1 Theoretical Results for Model Estimation

Theorem 7.1 provides the estimation accuracy of $\hat{A}(t)$ by the sieve method.

Theorem 7.1. Assume conditions (A1), (A2), (C1), (M1), (M2), (M3) and (S1)–(S4) hold. Define $\nu_n = \sup_{1 \leq j \leq J_n} \text{Lip}_j + \sup_{1 \leq j \leq J_n} |B_j(t)|$, where Lip$_j$ is the Lipschitz constant of basis function $B_j(t)$. Write $\nu_n = \frac{J_n \sup_{1 \leq j \leq J_n} |B_j(t)|^2}{\sqrt{n}} + \frac{J_n \sup_{1 \leq j \leq J_n} |B_j(t)| \nu_n}{n} + g_{J_n,K,\tilde{M}}$, where the quantity $g_{J_n,K,\tilde{M}}$ is defined in condition (A2). Assume $\nu_n \delta = o(1)$. Then we have

$$\left\| \sup_{t \in [0,1]} (\hat{A}(t) - A(t)) \right\|_{L^1} = O(p^{2-\delta} \nu_n).$$

From the proof, we shall see that $\|A(t)\|_F$ is of the order $p^{2-2\delta}$ uniformly for $t \in [0,1]$. Hence if we further assume that $p^\delta \nu_n = o(1)$, then the approximation error of $\hat{A}(t)$ is negligible compared with the magnitude of $A(t)$. For orthonormal Legendre polynomials and trigonometric polynomials it is easy to derive that $\text{Lip}_j = O(j^2)$. Similar calculations can be performed for a large class of frequently-used basis functions. The first term of $\nu_n$ is due to the stochastic variation of $\hat{M}(J_n,t,k)$, while the second and last terms are due to the basis approximation.

Remark 7.1. Consider the case for $1 \leq k \leq k_0$, $1 \leq i, j \leq p$, $\sigma_{x,i,j}^{(1)}(t,k), ..., \sigma_{x,i,j}^{(v-1)}(t,k)$ are absolutely continuous in $t$ where $\sigma_{x,i,j}^{(m)}(t,k) = \frac{\partial^m \sigma_{x,i,j}(t)}{\partial t^m}$ for some positive integer $m$. We are able to specify the rate $g_{J_n,m+1,\tilde{M}}$ for various basis functions.

(a) Assume $\max_{1 \leq i,j \leq p, 1 \leq k \leq k_0} \frac{1}{\sqrt{1-t}} \sigma_{x,i,j}^{(v+1)}(t) dt \leq M < \infty$ for some constant $M$, $\sigma_{x,i,j}^{(v)}(t)$ is of bounded variation for all $i,j$, then the uniform approximation rate $g_{J_n,m+1,\tilde{M}}$ is $J_n^{-v+1/2}$ for normalized Legendre polynomial basis. If $\sigma_{x,i,j}(t,k)'s$ are analytic inside and on the Bernstein ellipse then the approximation rate decays geometrically in $J_n$. See for instance [Wang and Xiang (2012)] for more details.

(b) If $\sigma_{x,i,j}^{(v)}(t,k)$ is Lipschitz continuous, then the uniform approximation rate $g_{J_n,m+1,\tilde{M}}$ is $J_n^{-v}$ when trigonometric polynomials (if all $\sigma_{x,i,j}(t,k)$ can be extended to periodic functions) or orthogonal wavelets generated by sufficiently high order father and mother wavelets are used. See [Chen (2007)] for more details.

Write $\hat{B}(t) = (\hat{b}_d(t)+1(t), ..., \hat{b}_p(t))$ where $\hat{b}_s(t), d(t)+1 \leq s \leq p$ are eigenvectors of $\hat{A}(t)$ corresponding to $\lambda_{d(t)+1}(\hat{A}(t)), ..., \lambda_p(\hat{A}(t))$. Obviously $(\hat{v}_1(t), ..., \hat{v}_d(t), \hat{b}_d(t)+1(t), ..., \hat{b}_p(t))$ form a set of orthonormal
basis of \( \mathbb{R}^p \). Define \( V(t) = (v_1(t), \ldots, v_d(t)) \) where \( v_i(t) \)s are the eigenvectors of \( \Lambda(t) \) corresponding to the positive \( \lambda_i(\Lambda(t)) \)s and \( B(t) = (b_{d(t)+1}(t), \ldots, b_p(t)) \) with \( b_s(t) \), \( d(t) + 1 \leq s \leq p \) an orthonormal basis of null space of \( \Lambda(t) \). Therefore \( (v_1(t), \ldots, v_d(t), b_{d(t)+1}(t), \ldots, b_p(t)) \) form an orthonormal basis of \( \mathbb{R}^p \).

**Theorem 7.2.** Under conditions of Theorem 7.1, we have

(i) For each \( t \in T_{\eta_n} \) there exist orthogonal matrices \( \hat{O}_1(t) \in \mathbb{R}^{d(t) \times d(t)} \) and \( \hat{O}_2(t) \in \mathbb{R}^{(p-d(t)) \times (p-d(t))} \) such that

\[
\| \sup_{t \in T_{\eta_n}} \| V(t) \hat{O}_1(t) - V(t) \|_F^2 \|_{L^1} = O(\eta_n^{-1} p^\delta \nu_n),
\]

\[
\| \sup_{t \in T_{\eta_n}} \| \hat{B}(t) \hat{O}_2(t) - B(t) \|_F \|_{L^1} = O(\eta_n^{-1} p^\delta \nu_n).
\]

(ii) Furthermore, if \( \limsup_p \sup_{t \in (0,1)} \lambda_{\max}(\mathbb{E}(H(t,F_i)H^\top(t,F_i))) < \infty \) we have that

\[
p^{-1/2} \max_{1 \leq i \leq n} \| V(i/n) V^\top(i/n) x_{i,n} - \Lambda(i/n) z_{i,n} \|_2 = O_p(\eta_n^{-1} p^\delta/2 \nu_n + p^{-1/2}).
\]

Assertion (i) follows from Theorem 7.1 and a variant of Davis Kahan Theorem (Yu et al. (2015)) which does not require the separation of all non-zero eigenvalues. (i) involves orthogonal matrices \( \hat{O}_1(t) \) and \( \hat{O}_2(t) \) since it allows multiple eigenvalues at certain time points, which yields the non-uniqueness of the eigen-decomposition. Moreover, when all the factors are strong (\( \delta = 0 \)) the rate in (i) is independent of \( p \) and reduces to the uniform nonparametric sieve estimation rate for univariate smooth functions, which coincides with the well-known ‘blessing of dimension’ phenomenon for stationary factor models, see for example Lam et al. (2011). Further assuming \( p^\delta \nu_n = o(\eta_n) \), Theorem 7.2 guarantees the uniform consistency of the eigen-space estimator on \( T_{\eta_n} \). Through studying the estimated eigenvectors, Theorem 7.1 and Theorem 7.2 demonstrate the validity of the estimator (5.6). The next theorem discusses the property of estimated eigenvalues \( \lambda_i(\hat{\Lambda}(t)) \), \( i = 1, \ldots, p \).

**Theorem 7.3.** Assume \( \nu_n p^\delta = o(1) \) and that there exists a sequence \( g_n \to \infty \) such that \( \frac{\eta_n}{(g_n \nu_n p^\delta)^{1/2}} \to \infty \). Then under conditions of Theorem 7.2 we have that

(i) \( \| \sup_{t \in (0,1)} \max_{1 \leq j \leq d(t)} | \lambda_j(\hat{\Lambda}(t)) - \lambda_j(\Lambda(t)) | \|_{L^1} = O(p^{2-\delta} \nu_n) \).

(ii) \( \mathbb{P}(\sup_{t \in T_{\eta_n}} \max_{j=d(t)+1, \ldots, p} \lambda_j(\hat{\Lambda}(t)) \geq g_n^2 \eta_n^{-2} \nu_n^{-2} p^2) = O\left( \frac{1}{g_n} \right) \).

(iii) \( 1 - \mathbb{P}\left( \frac{\lambda_{j+1}(\hat{\Lambda}(t))}{\lambda_j(\hat{\Lambda}(t))} \geq \eta_n, j = 1, \ldots, d(t) - 1, \forall t \in T_{\eta_n} \right) = O\left( \frac{\nu_n p^\delta}{\eta_n} \right) \).
(iv) \( P(\sup_{t \in T_{\eta n}} \frac{\lambda_{d(t)+1}(\hat{\Lambda}(t))}{\lambda_{d(t)}(\Lambda(t))} \geq p^{2\delta} g_n^2 \nu^2_n / \eta_n^3) = O(\frac{1}{g_n} + \frac{\nu_n p^2}{\eta_n}). \)

Notice that term \( p^{2\delta} g_n^2 \nu^2_n / \eta_n^3 \) in (iv) is asymptotically negligible compared with the term \( \eta_n \) in (iii). If \( d(t) \) has a bounded number of changes, (iii) and (iv) of Theorem 7.3 indicate that the eigen-ratio estimator (5.7) is able to consistently identify the time-varying number of factors \( d(t) \) on intervals with total length approaching 1. We remark that most results on linear factor models require that the positive eigenvalues are distinct. Theorems 7.2 and 7.3 are sufficiently flexible to allow multiple eigenvalues of \( \Lambda(t) \). Theorem 7.3 yields the following consistency result for the eigen-ratio estimator (5.7).

Remark 7.2. If we assume that \( \sigma_{x,i,j}(t,k)'s \) are real analytic and normalized Legendre polynomials or trigonometric polynomials (when all \( \sigma_{x,i,j}(t,k) \) can be extended to periodic functions) are used as basis, we shall take \( J_n = M \log n \) for some large constant \( M \). Then Theorem 7.2 will yield an approximation rate of \( \frac{p^{2-\delta} \log n}{\sqrt{n}} \). Consider \( \eta_n = \log^c n \) for some fixed constant \( c > 0 \). For Theorem 7.2 the approximation rate of (i) is then \( \frac{p^{2-\delta} \log^{1+c} n}{\sqrt{n}} \) and of (ii) is \( p^{-1/2} + \frac{p^{2-\delta} \log^{1+c} n}{\sqrt{n}} \). Our rate achieves the rate of Lam et al. (2011) for stationary high dimensional time series except a factor of logarithm. For Theorem 7.3 the approximation rate is \( \frac{p^{2-\delta} \log n}{\sqrt{n}} \) for (i), and the lower-bounds are arbitrarily close to \( \frac{p^{2-\delta} \log^{2+2c} n}{n} \) for (ii) and \( \frac{p^{2-\delta} \log^{2+3c} n}{n} \) for (iv). These rates coincide that of Theorem 1 and Corollary 1 of Lam and Yao (2012) except a factor of logarithm. The above findings are consistent with the results in nonparametric sieve estimation for analytic functions where the uniform convergence rates for the sieve estimators (when certain adaptive sieve bases are used for the estimation) are only inferior than the \( n^{-1/2} \) parametric rate by a factor of logarithm. We shall also point out that the sieve approximation rates will be adaptive to the smoothness and will be slower when \( \sigma_{x,i,j}(t,k)'s \) are less smooth in which case \( g_{J_n,K,M} \) converges to zero at an adaptive but slower rate as \( J_n \) increases.

Proposition 7.1. Assume \( \nu_n p^\delta = o(1) \). Under the conditions of Theorem 7.2 we have

\[
P(\exists t \in T_{\eta n}, \hat{d}_n(t) \neq d(t)) = O(\frac{\nu_n p^\delta \log n}{\eta_n^2}).
\]

Hence \( \hat{d}_n(t) \) is uniformly consistent on \( T_{\eta n} \) if \( \frac{\nu_n p^\delta \log n}{\eta_n^2} = o(1) \).

Remark 7.3. In practice, if the estimated number of factors \( \hat{d}(t) \) does not change over time, then according to our discussions in Section 2, \( T_{\eta n} = [0, 1] \). Otherwise one can set

\[
T_{\eta n} = (0, 1) \cap \bigcap_{s=1}^{\hat{d}_n} \left( \hat{t}_s - \frac{1}{\log^2 n}, \hat{t}_s + \frac{1}{\log^2 n} \right)
\]

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where  \( \hat{t}_s, s = 1, \ldots r \) are the time points when  \( \hat{d}(t) \) changes. In fact, by condition (C1), equation (7.2) corresponds to choosing  \( \eta_n \propto \frac{1}{\log^4 n} \) when the eigenvalues of  \( A(t)/p(1-\delta)/2 \) are Lipschitz continuous.

### 7.2 Theoretical Results for Testing Static Factor Loadings

We discuss the limiting behaviour of  \( \hat{T}_n \) of (6.2) under  \( H_0 \) in this subsection. First, the following proposition indicates that with asymptotic probability one  \( \hat{d}_n \) equals  \( d \) under  \( H_0 \).

**Proposition 7.2.** Assume conditions (A1), (A2), (C1), (M1), (M2), (M3) and conditions (S1)-(S4) hold, and that  \( \frac{p^\delta \log n}{\sqrt{n}} = o(1) \). Then we have, under  \( H_0 \),

\[
P(\hat{d}_n \neq d) = O\left( \frac{p^\delta}{\sqrt{n}} \log n \right) = o(1) \quad \text{as} \quad n \to \infty.
\]

Next we define a quantity  \( \tilde{T}_n \) using the true quantities  \( F \) and  \( d \) to approximate  \( \hat{T}_n \),

\[
\tilde{T} = \sqrt{m_n} \max_{1 \leq k \leq k_0} \max_{1 \leq j \leq N_n} \max_{1 \leq i \leq p-d} \left| f_i^\top \hat{\Sigma}_{n,k} \right|_\infty.
\]

Recall the definition of  \( \hat{F} \) and  \( F \) in Section 6. We shall see from Proposition 7.3 below that  \( \hat{T} - \tilde{T} \) is negligible asymptotically under some mild conditions due to the small magnitude of  \( \| \hat{F} - F \|_F \).

**Proposition 7.3.** Let conditions (A1), (A2), (C1), (M1), (M3) and (S) be satisfied, and that  \( \frac{p^\delta \log n}{\sqrt{n}} = o(1) \). Assume the following condition (M'):

\[(M') \text{ There exists an integer } l > 2 \text{ and a universal constant } M \text{ such that } \max_{1 \leq j \leq d} \sum_{k=1}^\infty \delta_{Q,2l,j}(k) < \infty, \quad \max_{1 \leq j \leq p} \sum_{k=1}^\infty \delta_{H,2l,j}(k) < \infty \text{ and } \]

\[
\sup_{t \in [0,1]} \max_{1 \leq u \leq d} \mathbb{E}|Q_u(t,F_0)|^{2l} \leq M, \quad \sup_{t \in [0,1]} \max_{1 \leq v \leq p} \mathbb{E}|H_v(t,F_0)|^{2l} \leq M.
\]

Then there exists a set of orthonormal basis  \( F \) of the null space of  \( \Gamma \) such that for any sequence  \( g_n \to \infty \)

\[
P(|\tilde{T}_n - \tilde{T}| \geq g_n^2 \frac{p^\delta \sqrt{m_n}}{\sqrt{n}} \Omega_n) = O\left( \frac{1}{g_n} + \frac{p^\delta}{\sqrt{n}} \log n \right) \quad (7.4)
\]

where  \( \Omega_n = (np/m_n)^{1/l} \sqrt{p} \).

When  \( l \) is sufficiently large, the order of the rate in (7.4) is close to  \( p^\delta + 1/2 \sqrt{m_n} / \sqrt{p} \). Hence, in order for the error in (7.4) to vanish,  \( p \) can be as large as  \( O(n^a) \) for any  \( a < 1 \) when  \( \delta = 0 \). Furthermore, under the null  \( \tilde{T} \) is equivalent to the  \( L^\infty \) norm of the averages of a high-dimensional time series. Specifically, define  \( \tilde{I}_i \) by replacing  \( \hat{F} \) with  \( F \) in the definition of  \( I_i \) (c.f. (6.4)) with its  \( j \)th element denoted by  \( \tilde{i}_{i,j} \). Then straightforward calculations indicate that  \( \tilde{T} = \frac{\sum_{i=1}^{m_n} I_i}{\sqrt{m_n}} \) \( \infty \). Therefore we can approximate  \( \tilde{T} \) by the  \( L^\infty \)
norm of a certain mean zero Gaussian process via the recent development in high dimensional Gaussian approximation theory, see for instance Chernozhukov et al. (2013) and Zhang and Cheng (2018). Let \( y_i = (y_{i1}, \ldots, y_{ik_0} N_n (p-d)p) \) be a centered \( k_0 N_n (p-d)p \) dimensional Gaussian random vectors that preserved the auto-covariance structure of \( \tilde{l}_i \) for \( 1 \leq i \leq m \) and write \( y = \sum_{i=1}^{m_n} y_i / \sqrt{m_n} \).

**Theorem 7.4.** Assume conditions of Proposition 7.3 hold, \( \frac{\sqrt{m_n \log n}}{\sqrt{n}} \Omega_n = o(1) \) and assume the following conditions (a)-(f):

(a) \( \|A\|_\infty \lesssim 1 \).

(b) \( l \geq 4 \).

(c) \( (N_n p^2)^{1/l} \lesssim m_n^{\frac{3-25\zeta}{32}} \) and \( k_0 N_n (p-d)p \lesssim \exp(m_n^\zeta) \) for some \( 0 \leq \zeta < 1/11 \).

(d) Let \( \sigma_{j,j} = \sum_{i,l=1}^{m_n} \text{cov}(\tilde{l}_{ij}, \tilde{l}_{lj})/m_n \). There exists some constant \( \eta > 0 \) such that \( \min \sigma_{j,j} \geq \eta \), i.e., each component series of \( \tilde{l}_i \) is non-degenerated.

(e) The dependence measure for \( x_{i,n} \) satisfies

\[
\delta_{G,2l}(k) = O((((k+1) \log(k+1))^{-2}). \tag{7.5}
\]

(f) There exists a constant \( M_{2l} \) depending on \( l \) such that for \( 1 \leq i \leq n \), and for all \( p \) dimensional vector \( c \) such that \( \|c\|_2 = 1 \), the inequality \( \|c^T e_{i,n}\|_{L^2} \leq M_{2l} \|c^T e_{i,n}\|_{L^2} \) holds. Also \( \max_{1 \leq i \leq n} \lambda_{\max}(E(e_{i,n} e_{i,n}^T)) \) is uniformly bounded as \( n \) and \( p \) diverges.

Then under null hypothesis, we have

\[
\sup_{t \in \mathbb{R}} |\mathbb{P}(\tilde{T}_n \leq t) - \mathbb{P}(\|y\|_\infty \leq t)| \lesssim \iota(m_n, k_0 N_n p(p-d), 2l, (N_n p^2)^{1/l}), \tag{7.6}
\]

where \( \iota(\cdot) \) is defined as

\[
\iota(n, p, q, D_n) = \min(n^{-1/8} M^{1/2} \gamma^2 / 8 + \gamma + (n^{1/8} M^{-1/2} \gamma^{3/8})^{q/(1+q)} \left( \sum_{j=1}^{p} \Theta_{M,j,q} \right)^{1/(1+q)} + \Xi^{1/3} M (1 \vee \log(p/\Xi_M))^{2/3}), \tag{7.7}
\]

where \( \Theta_{M,j,q} = \sum_{k=M}^{\infty} \delta_{G,q,j}(k) \) and \( \Xi_M = \max_{1 \leq j \leq p} \sum_{k=M}^{\infty} k \delta_{G,j,q}(k) \), and the minimum is taken over
all possible values of $\gamma$ and $M$ subject to

$$n^{3/8}M^{-1/2}l_n^{-5/8} \geq \max\{D_n(n/\gamma)^{1/4}, l_n^{1/2}\},$$

with $l_n = \log(pn/\gamma) \vee 1$. Furthermore,

$$\sup_{t \in \mathbb{R}} |\mathbb{P}(\hat{T}_n \leq t) - \mathbb{P}(|y|_{\infty} \leq t)| = O \left( \left( \frac{p^6\sqrt{m_n \log n}}{\sqrt{n}} \Omega_n \right)^{1/3} + \iota(m_n, k_0N_np(p - d), 2l, (N_np^2)^{1/l}) \right). \quad (7.8)$$

Conditions (a)-(e) control the magnitude of the loading matrix, the dimension and dependence of the time series $x_{i,n}$ as well as non-degeneracy of the process $\hat{l}_i$. Those conditions are standard in the literature of high dimensional Gaussian approximation; see for instance Zhang and Cheng [2018]. Notice that if the dependence measures of $x_{i,n}$ in condition (e) satisfies $\delta_{G,2l}(k) = O((k + 1)^{-1+\alpha})$ for $\alpha > \frac{6}{u+8} \vee 1$ with $u = \frac{\log m_n/\log n}{1-\log m_n/\log n+2\log p/\log n}$, then $\iota(m_n, k_0N_np(p - d), 2l, (N_np^2)^{1/l}) = o(1)$. If further $\delta_{G,2l}(k)$ decays geometrically as $k$ increases, then $\iota(m_n, k_0N_np(p - d), 2l, (N_np^2)^{1/l})$ will be reduced to $m_n^{-1-11\zeta}/8$ for a constant $\zeta$ defined in condition (c). Condition (f) controls the magnitude of the $L^2$ norm of projections of $e_{i,n}$ by their $L^2$ norm which essentially requires that the dependence among the components of $e_{i,n}$ cannot be too strong. (f) is mild in general and is satisfied, for instance, if the components of $e_{i,n}$ are independent or $e_{i,n}$ is a sub-Gaussian random vector with a variance proxy that does not depend on $p$ for each $i$. Finally, we comment that Condition (d) is a mild non-degeneracy condition. For example, assuming (i) $e_{s,n}$ is independent of $e_{t,n}$, $z_{t,n}$ for $s \geq t$, $\min_{1 \leq i \leq n, 1 \leq j \leq p} \text{EX}_{i,j}^2 \geq \eta' > 0$, $\min_{1 \leq i \leq n} \lambda_{\text{min}}(\mathbb{E}(e_{i,n}e_{i,n}^\top)) \geq \eta'' > 0$ for some constants $\eta'$ and $\eta''$, and (ii) conditions (a)-(c), (e) and (f) hold, then condition (d) holds. To see this, note that under null hypothesis, for $1 \leq s \leq p - d$, $1 \leq k \leq k_0$, $1 \leq i, j \leq n$ and $1 \leq v \leq p$, we have for $i \neq j$,

$$\mathbb{E}(f_s^\top x_{i+k,n}x_{i,v,n}f_s^\top x_{i+k,n}x_{i,v,n}) = \mathbb{E}(f_s^\top e_{i+k,n}x_{i,v,n}f_s^\top e_{i+k,n}x_{i,v,n}) = 0, \quad (7.9)$$

$$\mathbb{E}((f_s^\top x_{i+k,n}x_{i,v,n})^2) = \mathbb{E}((f_s^\top e_{i+k,n}x_{i,v,n})^2) = f_s^\top (\mathbb{E}(e_{i+k,n}e_{i+k,n}^\top))f_s\mathbb{E}(x_{i,v,n}^2) \geq \eta'\eta''. \quad (7.10)$$

Consequently (7.9) and (7.10) imply condition (d).

### 7.3 Block Multiplier Bootstrap

The validity of the bootstrap procedure is supported by the following theorem:

**Theorem 7.5.** Denote by $W_{n,p} = (k_0N_p(p - d)p)^2$. Assume that the conditions of Theorem 7.4 hold,
$w_n \rightarrow \infty$, $w_n/m_n = o(1)$, and that there exist $q^* \geq l$ and $\epsilon > 0$ such that $\Theta_n := w_n^{-1} + \sqrt{w_n/m_n} W_{n,p}^{1/q^*} \lesssim W_{n,p}^{-\epsilon}$ and

(i) $\max_{1 \leq i \leq n} \max_{1 \leq j \leq p} \|x_{i,j,n}\|_{\mathcal{L}^{2q^*}} \leq M$ for some sufficiently large constant $M$.

(ii) Condition (e) of Theorem 7.4 holds with $l$ replaced by $q^*$.

Then we have that conditional on $x_{i,n}$ and under $H_0$,

$$\sup_{t \in \mathbb{R}} |\mathbb{P}(\hat{T}_n \leq t) - \mathbb{P}(\kappa_n|\infty \leq t|x_{i,n}, 1 \leq i \leq n)| = O_{p}\left(\left(p^3 \sqrt{m_n \log n} / \sqrt{n} \right)^{1/3} + \iota(m_n, k_0 N_n p - d, 2l, (N_n p^2)^{1/l}) + \Theta_n^{1/3} \log^{2/3}(W_{n,p})\right). \quad (7.11)$$

The condition $w_n^{-1} + \sqrt{w_n/m_n} W_{n,p}^{1/q^*} \lesssim W_{n,p}^{-\epsilon}$ holds if $w_n = o(n^\epsilon)$ for some $c > 0$ and $\sqrt{w_n/m_n} W_{n,p}^{1/2+\epsilon} = o(1)$. The convergence rate of (7.11) will go to zero and hence the bootstrap is asymptotically consistent under $H_0$ if the dependence between $x_{i,n}$ is weak such that $\iota(m_n, k_0 N_n p - d, 2l, (N_n p^2)^{1/l}) = o(1)$; see the discussion below (7.8) in Section 7.2.

8 Power

In this section we discuss the local power of our bootstrap-assisted testing algorithm in Section 7.3 for testing static factor loadings. For two matrices $A$ and $B$, let $A \odot B$ be the Hadamard product of $A$ and $B$. Let $J$ be a $p \times d$ matrix with all entries 1 where $d = \text{rank}(A)$. To simplify the proof, we further assume the following Condition (G):

(G1): $e_{i,n}$ is independent of $z_{s,n}$, $e_{s,n}$ for $i > s$.

(G2): the dependence measure of $z_{i,n}$ satisfies

$$\max_{1 \leq j \leq d} \delta_{Q_{A,j}}(k) = O(((k + 1) \log(k + 1))^{-2}). \quad (8.1)$$

(G3): the dependence measures of $e_i$ satisfy

$$\sum_{k=1}^{\infty} k \sup_{0 \leq t \leq 1} \|f^\top H(t, F_i) - f^\top H(t, F_i^{(i-k)})\|_{\mathcal{L}^4} = O(\log^2 p) \quad (8.2)$$

for all $p$ dimensional vector $f$ such that $\|f\|_F = 1$. 

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Consider the alternative hypothesis by \( \Gamma \). Let \( \Delta \) be an alternative hypothesis, some \( \delta > 0 \), \( \rho \) and \( \Omega \). The first is that the dependence measures satisfy \( \max_{1 \leq j \leq p} \delta_{HA,j}(k) = O(\chi^k) \) for some \( \chi \in (0,1) \); see Lemma 8.1 below. The second is that the components of process \( (e_{i,n}) \) are independent of each other, i.e., \( H_{ji}(\cdot, F_u) \) is independent of \( H_{jv}(\cdot, F_v) \) for all \( u, v \in \mathbb{N} \) as long as \( j_1 \neq j_2, 1 \leq j_1, j_2 \leq p \), and \( \sum_{k=1}^{\infty} \max_{1 \leq j \leq p} \delta_{HA,j}(k) = O(1) \).

**Lemma 8.1.** Under conditions (f), \( (M') \) and \( \max_{1 \leq j \leq p} \delta_{HA,j}(k) = O(\chi^k) \) for some \( \chi \in (0,1) \), \( (G3) \) holds.

We consider the following class of local alternatives:

\[
H_A : A(t) = A_n(t) := A \circ (J + \rho_n D(t)),
\]

(8.3)

where \( D(t) = (d_{ij}(t)) \) is a \( p \times d \) matrix, \( \max_{i,j} \sup_t |d_{ij}(t)| \ll 1 \), \( \sup_{t \in [0,1]} \| D(t) \circ A(t) \|_F = O(p^{1 - \delta_1}) \) for some \( \delta_1 \in [\delta, 1] \) and \( \rho_n \downarrow 0 \) controls the magnitude of deviation from the null. Notice that, under the alternative hypothesis, \( A_n(t) \) is a function of \( \rho_n \) and therefore we write

\[
\int_0^1 \Sigma_x(t, k) dt = \int_0^1 A_n(t) (\Sigma_z(t, k) A_n(t)^\top + \Sigma_{zz}(t, k)) dt := \gamma(\rho_n, k), \quad k > 0.
\]

Let \( \Gamma_k(\rho_n) = \gamma(\rho_n, k) \gamma(\rho_n, k)^\top \) and \( \Gamma(\rho_n) = \sum_{k=1}^{k_0} \Gamma_k(\rho_n) \). To simplify notation, write \( \Gamma(\rho_n), \rho_n > 0 \) as \( \Gamma_n \) with orthogonal bases \( (f_{1,n}, ..., f_{p-d,n}) := F_n \). Recall the centered \( k_0 N_n(p - d)p \) dimensional Gaussian random vectors \( y \) and \( y_i \), and the definition of \( \tilde{l}_i \) in Section 7.2. Define \( \tilde{l}_i \) in analogue of \( \tilde{l}_i \) with \( F \) replaced by \( F_n \).

**Theorem 8.1.** Consider the alternative hypothesis (8.3). Assume in addition that for \( t \in [0,1] \), \( \text{rank}(A(t)) = d, p^{\delta} n^{1/2} m_1^{1/2} \Omega_n = o(1) \), and \( \Delta_n = \left| \sum_{i=1}^{m_n} \mathbb{E} \tilde{l}_i/m_n \right| \propto p^{1 - \delta_1} \). Then assuming conditions of Theorem 7.5 and condition (G) hold, we have the following results:

(a) Under the considered alternative hypothesis, if \( \rho_n \sqrt{m_n \log n} \Omega_n = o(1) \) then

\[
\sup_{t \in \mathbb{R}} |\mathbb{P}(\tilde{T}_n \leq t) - \mathbb{P}(|y|_\infty \leq t)| = O\left( \left( \frac{p^{\delta} \rho_n^{1 - \delta_1 + \frac{p^{\delta}}{\sqrt{n}}} \sqrt{m_n \log n} \Omega_n}{\rho_n \sqrt{m_n \log n} \Omega_n} \right)^{1/3} + \left( \frac{p^{\delta} \rho_n^{1 - \delta_1 + \frac{p^{\delta}}{\sqrt{n}}} \sqrt{m_n \log n} \Omega_n}{\rho_n \sqrt{m_n \log n} \Omega_n} \right)^{1/2} \right)
\]

(8.4)
where the function $\iota$ is defined in (7.7). Furthermore, if $\sqrt{m_n \rho_n p^{1-d_1}} \to \infty$ then $\hat{T}_n$ diverges at the rate of $\sqrt{m_n \rho_n p^{-\frac{1-d_1}{2}}}$. 

(b) Assume $\Theta_n^{1/3} \log^{2/3}(\frac{W_n p}{\Theta_n}) = o(1)$, then

$$\sup_{t \in \mathbb{R}} |\mathbb{P}(|y|_\infty \leq t) - \mathbb{P}(|\kappa_n|_\infty \leq t | x_{i,n}, 1 \leq i \leq n)|$$

$$= O_p \left( \Theta_n^{1/3} \log^{2/3}(\frac{W_n p}{\Theta_n}) + \left( \frac{p^d}{\sqrt{n}} + \rho_n \sqrt{w_n \Omega_n} \right)^{l/(1+l)} \right). \quad (8.5)$$

Furthermore, $|\kappa_n|_\infty = O_p(\sqrt{w_n \rho_n p^{1-d_1}} \sqrt{\log n})$.

Therefore by (a) and (b) if $\sqrt{m_n \sqrt{w_n \log n}} \to \infty$, $\sqrt{m_n \rho_n p^{1-d_1}} \to \infty$, the power of the test approaches 1 as $n \to \infty$.

We introduce $\delta_1$ to allow that $D(t)$ has a different level of strength than that of $A$. For instance, if $D(t)$ has a bounded number of non-zero entries, then $\delta_1 = 1$ while $\delta$ can be any number in $[0, 1]$. The condition that $\Delta_n \asymp \rho_n p^{1-d_1}$ is mild. In fact, it is shown in equation (C.11) in the proof of Proposition C.2 in the supplemental material that $|\tilde{E}_i|_\infty = O(\rho_n p^{-\frac{1-d_1}{2}})$, which means $\Delta_n = O(\rho_n p^{-\frac{1-d_1}{2}})$. On the other hand, from the proof of Proposition C.2 the condition $\Delta_n \asymp \rho_n p^{1-d_1}$ will be satisfied if there exists at least one eigenvector of the null space of $\sum_{k=1}^{k_0} \Gamma_k$ with a number of order $p$ of its entries having the same order of magnitude.

9 Selection of Tuning Parameters

9.1 Selection of $J_n$ for the estimation of time-varying factor loading matrices.

We discuss the selection of $J_n$ for the estimation of time-varying factors. Since in practice $g_{J_n,K,M}$ is unknown, a data-driven method to select $J_n$ is desired. By Theorem 7.2 the residuals $\hat{e}_{i,n} = x_{i,n} - \hat{V}(\frac{i}{n})\hat{V}^\top(\frac{i}{n})x_{i,n}$, and $\hat{e}_{i,n} = (\hat{e}_{i,1,n}, ..., \hat{e}_{i,p,n})^\top$ is a $p$ dimensional vector. We select $J_n$ as the minimizer of the following cross validation standard $CV(J)$,

$$CV(J) = \sum_{i=1}^{n} \sum_{s=1}^{p} \frac{\hat{e}^2_{i,s,n}(J)}{(1 - v_{i,s}(J))^2} \quad (9.1)$$
where \( v_{i,s}(J) \) is the \( s \)th diagonal element of \( \hat{\mathbf{V}}(\hat{\mathbf{z}}) \mathbf{V}(\hat{\mathbf{z}}) \) obtained by setting \( J_n = J \), and \( \hat{e}_{i,s,n}(J) \), \( 1 \leq i \leq n \), \( 1 \leq s \leq p \) are also the residuals calculated when \( J_n = J \). The cross-validation has been widely used in the literature of sieve nonparametric estimation and has been advocated by for example Hansen (2014). We choose \( J_n = \arg\min_J(CV(J)) \) in our empirical studies.

### 9.2 Selection of tuning parameters \( m_n \) and \( w_n \) for testing static factor loadings

We select \( m_n \) by first select \( N_n \) and let \( m_n = [(n - k_0)/N_n] \). The \( N_n \) is chosen by the minimal volatility method as follows. For a given data set, let

\[
\hat{T}(N_n) = \sqrt{m_n} \max_{1 \leq k \leq k_0} \max_{1 \leq h \leq N_n} \max_{1 \leq i \leq p - \hat{d}_n} |\hat{f}_i^\top \hat{\Sigma}_{h,k}^x| \tag{9.2}
\]

be the test statistic obtained by using the integer parameter \( N_n \) with \( \hat{d}_n = \arg\min_{1 \leq i \leq R} \lambda_{i+1}(\hat{\Gamma})/\lambda_i(\hat{\Gamma}) \) where \( R \) is defined in (6.6). Consider a set of possible values for \( N_n \), which is denoted by \( \{J_1, ..., J_s\} \) where \( J_s \) are positive integers. For each \( J_v \), \( 1 \leq v \leq s \) we calculate \( \hat{T}(J_v) \) and hence the local standard error

\[
SE(\hat{T}(J_v), h) = \left( \frac{1}{2h} \sum_{u=-h}^{h} \left( \hat{T}(J_{u+l}) - \frac{1}{2h+1} \sum_{v=-h}^{h} \hat{T}(J_{u+l}) \right)^2 \right)^{1/2} \tag{9.3}
\]

where \( 1 + h \leq l \leq s - h \) and \( h \) is a positive integer, say \( h = 1 \). We then select \( N_n \) by

\[
\arg\min_{1+h \leq s-h} SE(\hat{T}(J_v), h) \tag{9.4}
\]

which stabilizes the test statistics. The idea behind the minimum volatility method is that the test statistic should behave stably as a function of \( N_n \) when the latter parameter is in an appropriate range. In our empirical studies we find that the proposed method performs reasonably well, and the results are not sensitive to the choice of \( N_n \) as long as \( N_n \) used is not very different from that chosen by (9.4).

After choosing \( N_n \) and hence \( m_n \), we then further choose \( w_n \) again by the minimal volatility method. In this case we first obtain the \( k_0 N_n(p - d)p \) dimensional vectors \( \{\hat{f}_i, 1 \leq i \leq m_n\} \) defined in Section 6. Then we select \( w_n \) by a multivariate extension of the minimal volatility method in Zhou (2013) as follows. We consider choosing \( w_n \) from a grid \( w_1 \leq ... \leq w_r \). For each \( w_n = w_i, 1 \leq i \leq r \) we calculate a \( k_0 N_n(p - d)p \) dimensional vector \( \mathbf{b}_{i,u} = \frac{1}{w_i(m_n - w_{i+1})} \sum_{j=1}^{u} (\hat{\mathbf{s}}_{j,w_i} - w_i \hat{\mathbf{s}}_{m,n})^\circ \) where \( \circ \) represents the Hadamard product and \( 1 \leq r \leq m_n - w_{r+1} \). Let \( \mathbf{B}_i = (\mathbf{b}_{i,1}^\top, ..., \mathbf{b}_{i,m_n-w_{r+1}}^\top) \) be a \( k_0 N_n(p - d)p(m_n - w_{r+1}) \times r \) matrix with its \( i \)th column \( \mathbf{B}_{i}^\circ \). Then for each row,
say \(i\)th row \(B_i\) of \(B\), we calculate the local standard error \(SE(B_{i\cdot}, h)\) for a given window size \(h\), see [9.3] for definition of \(SE\) and therefore obtain a \(r - 2h\) length row vector \((SE(B_{i\cdot,h+1}, h), ... SE(B_{i\cdot,r-h}, h))\). Stacking these row vectors we get a new \(k_0 N_n (p-d)p(m_n - w_r + 1) \times (r - 2h)\) matrix \(B^\dagger\). Let \(\text{colmax}(B^\dagger)\) be a \((r - 2h)\) length vector with its \(i\)th element being the maximum entry of the \(i\)th column of \(B^\dagger\). Then we choose \(w_n = w_{k+h}\) if the smallest entry of \(\text{colmax}(B^\dagger)\) is its \(k\)th element.

10 Simulation Studies

10.1 Estimating the time-varying factor models

In this subsection we shall examine the performance of our proposed estimator (5.6) for time-varying factor models, and compare it with that in Lam et al. (2011). The latter is equivalent to fixing \(J_n = 0\) in (5.3). We use normalized Legendre polynomials as our basis throughout our empirical studies. The method studied in Lam et al. (2011) is developed under the assumption of stationarity with static factor loadings and hence the purpose of our simulation is to illustrate that the methodology developed under stationarity does not directly carry over to the non-stationary setting. To demonstrate the advantage of the adaptive sieve method, our method is also compared with a simple local estimator of \(\Lambda(t)\), which was considered in the data analysis section in Lam et al. (2011) and we shall call it the local PCA method in our paper. Specifically, for each \(i\), \(\Lambda(\frac{i}{n})\) will be consistently estimated by

\[
\hat{\Lambda}(\frac{i}{n}) = \sum_{k=1}^{k_0} \hat{M}(\frac{i}{n}, k) \hat{M}^\top(\frac{i}{n}, k), \quad \hat{M}(\frac{i}{n}, k) = \frac{1}{2m+1} \sum_{j=i-m}^{j=i+m} x_{j+k} x_j^\top
\]

(10.5)

where \(m\) is the window size such that \(m \to \infty\) and \(m = o(n)\). The \(J_n\) of our method is selected by cross validation while \(m\) of the local PCA method is selected by the minimal volatility method. Define the following smooth functions:

\[
g_0(t) = 0.4(0.4 - 0.2t), \quad \alpha_1(t) = 1.3 \exp(t) - 1, \quad \alpha_2(t) = 0.6 \cos(\frac{\pi t}{3}), \quad \alpha_3(t) = -(0.5 + 2t^2), \quad \alpha_4(t) = 2 \cos(\frac{\pi t}{3}) + 0.6t.
\]

Let \(A = (a_1, ..., a_p)^\top\) be a \(p \times 1\) matrix with \(a_i = 1 + 0.2(i/p)^{0.5}\). Define the locally stationary process \(z_i = G_1(i/n, F_i)\) where \(G_1(t, F_i) = \sum_{j=0}^{\infty} g_0^j(t) \epsilon_{i-j}\) where filtration \(F_i = (\epsilon_{-\infty}, ..., \epsilon_i)\) and \((\epsilon_i)_{i \in \mathbb{Z}}\) is a
sequence of i.i.d. $N(0,1)$ random variables. We then define the time varying matrix

$$A(t) = \begin{pmatrix} A_1\alpha_1(t) \\ A_2\alpha_2(t) \\ A_3\alpha_3(t) \\ A_4\alpha_4(t) \end{pmatrix}.$$  

(10.6)

where $A_1, A_2, A_3$ and $A_4$ are the sub-matrices of $A$ which consist of the first round($p/5$)$_{th}$ rows, the (round($p/5$)+1)$_{th}$ to round($2p/5$)$_{th}$, (round($2p/5$)+1)$_{th}$ to (round($3p/5$))$_{th}$ and the (round($3p/5$)+1)$_{th}$ to $p_{th}$ rows of $A$, respectively. Let $e_{i,n} = (e_{i,1},...,e_{i,p})^\top$ be a $p \times 1$ vector with $(e_{i,j})_{i\in\mathbb{Z},1\leq j\leq p}$ i.i.d. $N(0,0.5^2)$, and $(e_{i,j})_{i\in\mathbb{Z},1\leq j\leq p}$ are independent of $(e_{i})_{i\in\mathbb{Z}}$.

We consider the cases that $p = 50, 100, 200, 500$ and $n = 1000, 1500$. The performances of the methods are measured in terms of the Root-Mean-Square Error (RMSE) and the average principal angle. The RMSE of the estimation is defined as

$$RMSE = \frac{1}{np} \sum_{i=1}^{n} \| \tilde{V}(i/n)\tilde{V}^\top(i/n)x_{i,n} - A(i/n)z_{i,n} \|_2^2.$$ 

The principle angle between $A(i/n)$ and its estimate $\hat{A}(i/n)$ is defined as $\gamma_{i,n} := \hat{A}_{i,n}^\top A_{i,n}$, where $A_{i,n}$ and $\hat{A}_{i,n}$ are the normalized $A(i/n)$ and $\hat{A}(i/n)$ respectively. The principle angle is a well-defined distance between spaces span($A(i/n)$) and span($\hat{A}(i/n)$). Finally, the average principle angle is defined as $\bar{\gamma} = \frac{1}{n} \sum_{i=1}^{n} \gamma_{i}$. We present the RMSE and the average principle angle of the three estimators using 100 simulation samples in Table 10.1 and 10.2 respectively. Our method achieves the minimal RMSE and average principle angle in all simulation scenarios among the three estimators. We choose $k_0 = 3$ in our simulation. Other choices $k_0 = 1,2,4$ yield the similar results are not reported here. As predicted by Theorem 7.2 when $\delta = 0$, RMSE in Table 10.1 decreases as $n$, $p$ increases and the average principle angle decreases with $n$ increases, and is independent of $p$.

Table 10.1: Mean and standard errors (in brackets) of simulated RMSE for our sieve method, the static loading method ($J_n = 0$) and Local PCA for model (10.6). The results are multiplied by 1000.

|       | $n = 1000$ | $n = 1500$ |
|-------|------------|------------|
|       | $J_n = 0$  | Local PCA  | $J_n = 0$  | Local PCA  |
| $p = 50$ | 228(4.5)   | 509(5.9)   | 251(3.1)   | 237(2.9)   |
| $p = 100$| 222(4.4)   | 494(3.9)   | 241(3.2)   | 239(3.6)   |
| $p = 200$| 221(4.0)   | 501(3.8)   | 242(2.7)   | 229(2.6)   |
| $p = 500$| 218(4.2)   | 490(3.3)   | 241(3.1)   | 225(2.8)   |

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Table 10.2: Mean and standard errors (in brackets) of simulated principle angles for our sieve method, the static loading method \((J_n = 0)\) and Local PCA for model (10.6). The results are multiplied by 1000.

| \(n = 1000\) | \(n = 1500\) |
|----------------|----------------|
| Sieve | \(J_n = 0\) | Local PCA | Sieve | \(J_n = 0\) | Local PCA |
|\(p = 50\) | 13.4(0.6) | 66.7(1.4) | 16.1(0.4) | 9.7(0.4) | 61.9(0.5) | 13.8(0.4)|
|\(p = 100\) | 13.5(0.6) | 64.8(0.8) | 15.6(0.4) | 10.3(0.4) | 63.0(0.8) | 13.8(0.3)|
|\(p = 200\) | 13.5(0.6) | 65.9(0.9) | 15.8(0.4) | 11.3(0.5) | 63.1(0.8) | 13.6(0.3)|
|\(p = 500\) | 13.7(0.6) | 65.3(0.9) | 15.9(0.4) | 10.6(0.4) | 61.6(0.5) | 13.8(0.3)|

10.2 Testing static loading matrix: type I error

We now examine our testing procedure in Section 6 to test the hypothesis of static factor loadings via \(B = 1000\) bootstrap samples. Define

\[
g_1(t) = 0.1 + 0.06t^2, \quad g_2(t) = 0.12 + 0.04t^2, \quad g_3(t) \equiv 0.15, \tag{10.7}
\]

\[
\alpha_1(t) \equiv 0.8, \alpha_2(t) \equiv 0.9, \alpha_3(t) \equiv 1. \tag{10.8}
\]

Let \(A\) be a \(p \times 3\) matrix with each element generated from \(2U(-1,1)\), and

\[
A(t) = \begin{pmatrix} A_1 \alpha_1(t) \\ A_2 \alpha_2(t) \\ A_3 \alpha_3(t) \end{pmatrix} \tag{10.9}
\]

where \(A_1, A_2\) and \(A_3\) are the sub-matrices of \(A\) which consist of the first \(\text{round}(p/3)\)th rows, the \((\text{round}(p/3) + 1)\)th to \(\text{round}(2p/3)\)th rows, and \((\text{round}(2p/3) + 1)\)th to \(p\)th rows, respectively. The factors \(z_{i,n} = (z_{i,1,n}, z_{i,2,n}, z_{i,3,n})^\top\) where \(z_{i,k,n} = 2 \sum_{j=0}^{\infty} d_k (i/n) \varepsilon_{i-j,k}\) for \(k = 1, 2, 3\), where \(\{\varepsilon_{i,k}\}\) are i.i.d. standard normal. We consider the following different sub-models for \(e_{i,n} = (e_{i,1,n}, \ldots, e_{i,s,n})^\top\):

(Model I) \((e_{i,s,n})_{i \in \mathbb{Z}}, 1 \leq s \leq p\) follow i.i.d. \(\frac{16}{25} \chi^2(5)/\sqrt{5/4}\)

(Model II) \(e_{i,s,n} = \frac{16}{25} \tilde{e}_{i-1,s,n} \tilde{e}_{i,s,n}\) where \((\tilde{e}_{i,s,n})_{i \in \mathbb{Z}}, 1 \leq s \leq p\) are i.i.d. \(N(0,1)\).

Both model I and model II are non-Gaussian. Furthermore, Model II is a stationary white noise process, see Zhou (2012). Table 10.3 reports the simulated Type I error based on 300 simulations for models I and II with different combinations of the dimension \(p\) and time series length \(n\). From Table 10.3 we see the simulated type I errors approximate its nominal level reasonably well.
Table 10.3: Simulated Type I errors for Model I and Model II

|       | Model I |       | Model II |
|-------|---------|-------|----------|
|       | n = 1000 | n = 1500 | n = 1000 | n = 1500 |
| level | 5% 10% 5% 10% | 5% 10% 5% 10% |
| p = 20 | 4.30% 10.00% 5.67% 10.67% | 5.33% 11.00% 5.67% 11.33% |
| p = 50 | 4.67% 11.33% 5.00% 10.00% | 4.00% 8.33% 5.33% 9.67% |
| p = 100 | 2.67% 5.33% 5.33% 10.67% | 3.00% 5.67% 5.00% 11.33% |

10.3 Testing static factor loadings: power

In this subsection, we examine the power performance of our testing procedure in Section 6 via $B = 1000$ bootstrap samples. Let $\tilde{A}$ be a $p \times 1$ matrix with each entry 1, and let $\tilde{A}_i$, $1 \leq i \leq 5$ be the sub-matrices of $\tilde{A}$ which consist of the first $(\text{round}((i-1)p)/5) + 1$ to $\text{round}(ip/5)$ rows of $\tilde{A}$, $1 \leq i \leq 5$. We then consider a time-varying normalized loading matrix $A(t)$ as follows. For a given $D \in \mathbb{R}$, let

$$
\alpha_1(t, D) = 1 - Dt, \alpha_2(t, D) = 1 - 2Dt, \alpha_3(t, D) = 1 - Dt^2, \alpha_4(t, D) = 1 - 2Dt^2, \alpha_5(t, D) \equiv 1,
$$

and we define the normalized loading matrix $A(t) = \tilde{A}_D(t)(\tilde{A}_D^T(t)\tilde{A}_D(t))^{-1/2}$ where

$$
\tilde{A}_D(t) = \begin{pmatrix}
\tilde{A}_1\alpha_1(t, D) \\
\tilde{A}_2\alpha_2(t, D) \\
\tilde{A}_3\alpha_3(t, D) \\
\tilde{A}_4\alpha_4(t, D) \\
\tilde{A}_5\alpha_5(t, D)
\end{pmatrix}.
$$

Let $(e_{i,s,n})_{i \in \mathbb{Z}, 1 \leq s \leq p}$ be i.i.d $N(0, 0.5^2)$, and $e_{i,n} = (e_{i,s,n}, 1 \leq s \leq p)^\top$. Let $(\epsilon_i)_{i \in \mathbb{Z}}$ be i.i.d. $N(0,1)$ that are independent of $(e_{i,s,n})_{i \in \mathbb{Z}, 1 \leq s \leq p}$. Then the locally stationary process $z_{i,n}$ we consider in this subsection is $z_{i,n} = 4 \sum_{j=0}^\infty g^j(i/n)\epsilon_{i-j}$, where the coefficient function $g(t) = 0.2 + 0.2t$. Observe that $D = 0$ corresponds to the situation when $A(t)$ is time invariant, and as $D$ increases we have larger deviations from the null. We examine the case of $p = 25, 50, 100, 150$, $n = 1500$ while increasing $D$ from 0 to 0.5. The results are displayed in Table 10.4 which supports that our test has a decent power. Each simulated rejection rate is calculated based on 100 simulated samples while for each sample the critical value is generated by block bootstrap algorithm using $B = 1000$ bootstrap samples in Section 6.
Table 10.4: Simulated rejection rate at different alternatives, with $B = 1000$

| $D$ | 0   | 0.1  | 0.2  | 0.3  | 0.4  | 0.5  |
|-----|-----|------|------|------|------|------|
|     | 5%  | 10%  | 5%  | 10%  | 5%  | 10%  | 5%  | 10%  | 5%  | 10%  | 5%  | 10%  |
| $p = 25$ | 6   | 13   | 18  | 35   | 96  | 97   | 100 | 100  | 100 | 100  |
| $p = 50$ | 7   | 12   | 14  | 27   | 91  | 95   | 100 | 100  | 100 | 100  |
| $p = 100$ | 4   | 11   | 17  | 33   | 79  | 87   | 100 | 100  | 100 | 100  |
| $p = 150$ | 7   | 12   | 9   | 25   | 55  | 65   | 100 | 100  | 100 | 100  |

11 Data Analysis

11.1 Analysis of option data

To illustrate the usefulness of our method we investigate the implied volatility surface of call option of Microsoft during the period June 1st, 2014-June 1st, 2019 with 1260 trading days in total. The data are obtained by OptionMetrics through the WRDS database. For each day we observe the volatility $W_t(u_i, v_j)$, where $u_i$ is the time to maturity and $v_j$ is the delta. A similar type of data has been studied by Lam et al. (2011).

We first examine whether the considered data set has a static loading matrix by performing our test procedure in Section 6. Specifically, we consider $u_i$ to take values at 30, 60, 91, 122, 152, 182, 273, 365, 547 and 730 for $i = 1, ..., 10$ and $v_j$ to take values at 0, 0.35, 0.40, 0.45, 0.50, 0.55 for $j = 1, ..., 5$. Then we write $\tilde{x}_t = vec(W_t(u_i, v_j)_{1 \leq i \leq 10, 1 \leq j \leq 5})$. Due to the well-documented unit root non-stationarity of $\tilde{x}_t$ (see e.g. Fengler et al. (2007)) we study the 50-dimensional time series

$$x_t := \tilde{x}_t - \tilde{x}_{t-1}.$$  

In our data analysis we choose $k_0 = 3$. Using the minimal volatility method we select $N_n = 4$ where $N_n = \frac{T - 3}{m_n}$ is the number of non-overlapping equal-sized blocks, and choose window size $w_n = 5$. Via $B = 10000$ bootstrap samples we obtain a $p-value$ of 6.08%. The small $p-value$ provides a moderate to strong evidence against the null hypothesis of static factor loadings.

We then apply our sieve estimator in Section 5 to estimate the time varying loading matrix. The cross validation method suggests the use of the Legendre polynomial basis up to 6th order. We find that during the considered period the number of factors, or the rank of the loading matrix, is either one or two at each time point. In Figure 11.1 we display the estimated number of factors at each time, and in Figure 11.2 we show the percentage of trace of $\Lambda(t)$ that is explained by the eigenvectors corresponding to the first and second largest eigenvalues, which reflects the time-varying structure of the loading matrix. The results
Figure 11.1: Number of factors for Microsoft call option

Figure 11.2: Proportion of trace of $\Lambda(t)$ that can be explained by the leading two eigenvectors for Microsoft call option

underpin that the loading matrix is time varying, and the leading two eigenvalues of $\Lambda(t)$ dominate the sum of all eigenvalues of $\Lambda(t)$.

11.2 Analysis of COVID-19 deaths in the U.S.

We consider the daily state-level new deaths of COVID 19 from the United States. The data can be obtained from https://data.cdc.gov/Case-Surveillance/United-States-COVID-19-Cases-and-Deaths-by-State. The dataset reports COVID 19 time series from 60 places, which include the 50 states, the District of Columbia, New York City, the U.S. territories of American Samoa, Guam, the Commonwealth of the Northern Mariana Islands, Puerto Rico, and the U.S Virgin Islands as well as three independent countries in compacts of free association with the United States, Federated States of Micronesia, Republic of the Marshall Islands, and Republic of Palau. It also records the New York City’s reported case and death counts separately from New York State’s counts. We consider the series of new death starting from Feb. 26, 2020 since Feb. 27, 2020 is the first day when a death case appears in this dataset. The series considered ends on Oct.8, 2021. We denote this 60-dimensional vector of length 591 by $\tilde{y}_t$, $t = 1, 2, \cdots, 591$. We study the differenced death series

$$\mathbf{y}_t = \tilde{y}_t - \tilde{y}_{t-1}$$

(11.11)
by our non-stationary factor model. We apply our test in Section 6 to examine whether the loading matrix of $y_t$ is static. We choose $k_0 = 3$. By selecting the number of non-overlapping equal-sized blocks $N_n = 7$ and window size $w_n = 24$ via the minimal volatility method, through $B = 10000$ bootstrap samples we obtain a $p$–value of 0.073% which provides a strong evidence against the null hypothesis of static factor loadings.

The sieve estimator in Section 5 is then applied to estimate the time varying loading matrix for $y_t$, with the cross validation method suggesting the use of the Legendre polynomial basis up to 7th order. During the considered period the number of factors varies from 1 to 5. For the state-level differenced new death of COVID 19 in USA, we depict the estimated number of factors in Figure 11.3 and the percentage of trace of $\Lambda(t)$ that is explained by the two leading eigenvalues in Figure 11.4. The results evidence that the loading matrix, as well as the number of factors are time varying, and the leading two eigenvalues of $\Lambda(t)$ could not dominate the sum of all eigenvalues of $\Lambda(t)$ in certain period.

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Online supplement

The online supplement contains the proofs of Auxiliary lemmas for Theorem 7.1, 7.2, 7.3, Proposition 7.1, Proposition 7.2, Theorem 7.5, Lemma 8.1 and Theorem 8.1.

12 Appendix

We now present some additional notation for the proofs. For any random variable $W = W(t, F_i)$, write $W(g) = W(t, F_{i}(g))$ where $F_{i}(g) = (\epsilon_{-\infty}, \ldots, \epsilon_{g-1}, \epsilon_{g}, \epsilon_{g+1}, \ldots, \epsilon_{i})$, and $(\epsilon_{i})_{i \in \mathbb{Z}}$ is an i.i.d copy of $(\epsilon_{i})_{i \in \mathbb{Z}}$.

Let $P_j = E(\cdot|F_j) - E(\cdot|F_{j-1})$ be the projection operator. Let $a_{uv}(t) = 0$ if $v \notin \{l_1, \ldots, l_{d(t)}\}$, where $l_i, 1 \leq i \leq d(t)$ is the index set of factors defined in Section 4.

Proof of Theorem 7.1.

Notice that for each $k \in 1, \ldots, k_0$, we have that

$$\sup_{t \in [0,1]} \|\hat{M}(J_n, t, k)\hat{M}^\top(J_n, t, k) - \Sigma_x(t, k)\Sigma_x^\top(t, k)\|_F \leq 2\|\Sigma_x(t, k)\|_F \|\hat{M}(J_n, t, k) - \Sigma_x(t, k)\|_F^2. \quad (S.1)$$

Notice that by Jansen’s inequality, Cauchy-Schwartz inequality and conditions (C1), (M2), (S1)–(S4), it follows that uniformly for $t \in [0,1]$,

$$\|\Sigma_x(t, k)\|_F^2 \leq 2\sum_{u=1}^{p} \sum_{v=1}^{p} \left( \sum_{u'=1}^{d} \sum_{v'=1}^{d} a_{uu'}(t)E(Q_{u'}(t, F_{i+k})Q_{v'}(t, F_{i}))a_{vv'}(t) \right)^2 + 2\|A(t)\Sigma_{ze}(t, k)\|_F^2 \leq C \sum_{u=1}^{p} \sum_{v=1}^{p} \left[ \sum_{u'=1}^{d} \sum_{v'=1}^{d} a_{uu'}(t)a_{vv'}(t) \right]^2 + o(p^2 - 2\delta) \leq C'd^2p^2 - 2\delta. \quad (S.2)$$

for some sufficiently large constant $C$ and $C'$ which depend on the constant $M$ in condition (M2). On the other hand, by Lemmas A.4, A.5 and A.6 in the online supplement we have that

$$\|\hat{M}(J_n, t, k) - \Sigma_x(t, k)\|_F^2 = O(p\nu_n). \quad (S.3)$$

Then the theorem follows from equations (S.1), (S.2) and (S.3). \qed

Proof of Theorem 7.2.
We first prove (i). It suffices to show that the $d(t)_th$ largest eigenvalue of $\Lambda(t)$ satisfies
\[
\inf_{\theta \in T_{\eta n}} \lambda_d(\Lambda(t)) \gtrsim \eta_n p^{2-2\delta}.
\] (S.4)

Then the theorem follows from Theorem 7.1 (S.4) and Theorem 1 of [Yu et al., 2015]. We now show (S.4).

Consider the QR decomposition of $A(t)$ such that $A(t) = V(t)R(t)$ where $V(t)^TV(t) = I_{d(t)}$ and $I_{d(t)}$ is an $d(t) \times d(t)$ identity matrix. Here $V(t)$ is a $p \times d(t)$ matrix and $R(t)$ is a $d(t) \times d(t)$ matrix. Then (5.2) of the main article can be written as
\[
A(t) = V(t)\tilde{A}(t)V^\top(t),
\] (S.5)

where
\[
\tilde{A}(t) = R(t)\left[\sum_{k=1}^{d(t)} (\Sigma_z(t, k)A^\top(t) + \Sigma_{ze}(t, k))(A(t)\Sigma_z^\top(t, k) + \Sigma_{ze}^\top(t, k))\right]R^\top(t).
\] (S.6)

We now discuss the $\lambda_d(t)(\tilde{A}(t))$. Write
\[
\Sigma_{Rz}(t, k) = R(t)\Sigma_z(t, k)R^\top(t), \Sigma_{Rze}(t, k) = R(t)\Sigma_{ze}(t, k),
\] (S.7)
\[
\Sigma_{Rz}(t, k_0) = (\Sigma_{Rz}(t, 1), \ldots, \Sigma_{Rz}(t, k_0)), \Sigma_{Rze}(t, k_0) = (\Sigma_{Rze}(t, 1), \ldots, \Sigma_{Rze}(t, k_0)).
\] (S.8)

Therefore we have
\[
\tilde{A}(t) = (\Sigma_{Rz}(t, k_0)(I_{k_0} \otimes V^\top(t))) + \Sigma_{Rze}(t, k_0))(\Sigma_{Rz}(t, k_0)(I_{k_0} \otimes V^\top(t))) + \Sigma_{Rze}(t, k_0))^\top
\] (S.9)

where $\otimes$ denotes the Kronecker product. On the other hand, by (C1) we have
\[
\sup_{t \in [0,1]} \|R(t)\|_2 \gtrsim p^{1-\delta}, \inf_{t \in [0,1]} \|R(t)\|_2 \gtrsim p^{1-\delta}, \sup_{t \in T_{\eta n}} \|R(t)\|_m \gtrsim \eta_n p^{1-\delta}, \inf_{t \in T_{\eta n}} \|R(t)\|_m \gtrsim \eta_n p^{1-\delta}.
\]

Then by conditions (M1), (M2), (M3), (S2) and (S4) we have
\[
\sup_{t \in [0,1]} \|\Sigma_{Rz}(t, k_0)\|_2 = o(p^{1-\delta}), \inf_{t \in [0,1]} \|\Sigma_{Rz}(t, k_0)\|_2 = o(p^{1-\delta}),
\]
\[
\sup_{t \in T_{\eta n}} \sigma_d(t)(\Sigma_{Rz}(t, k_0)) \gtrsim \eta_n p^{1-\delta}, \inf_{t \in T_{\eta n}} \sigma_d(t)(\Sigma_{Rz}(t, k_0)) \gtrsim \eta_n p^{1-\delta},
\]

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where $\sigma_{d(t)}(\Sigma)$ of a matrix $\Sigma$ denotes the $d(t)$th largest singular value of the matrix $\Sigma$. Therefore using (S.9) and the arguments in the proof of Theorem 1 of Lam et al. (2011) we have that

$$\inf_{t \in T_n} \lambda_{d(t)}(\hat{\Lambda}(t)) \gtrsim \eta_n p^{2-2\delta}.$$  
(S.10)

This shows (S.4). Therefore the assertion (i) of the Theorem follows. The assertion (ii) follows from result (i), condition (C1) and a similar argument of proof of Theorem 3 of Lam et al. (2011). Details are omitted for the sake of brevity. \(\square\)

**Proof of Theorem 7.3**

(i) follows immediately from Lemma A.1 in the online supplement and Theorem 7.1. For (ii), notice that by Theorem 7.2 we have that

$$\| \sup_{t \in T_n} \| \hat{\mathbf{B}}(t) - \mathbf{B}(t) \hat{\mathbf{O}}_2^\top(t) \|_F \|_{L^1} = O(\eta_n^{-1} p^\delta \nu_n),$$  
(S.11)

where $\mathbf{B}(t), \hat{\mathbf{B}}(t)$ and $\hat{\mathbf{O}}_2(t)$ are defined in Theorem 7.2. Since $\hat{\mathbf{O}}_2(t)$ is an $(p-d(t)) \times (p-d(t))$ orthonormal matrix, it is easy to check that the columns of $\mathbf{B}(t) \hat{\mathbf{O}}_2^\top(t)$ form an orthonormal basis of the null space of $\Lambda(t)$. Using this fact, we then follow the decomposition (A.5) of Lam and Yao (2012) to obtain that uniformly for $j = d(t) + 1, \ldots, p$

$$\lambda_j(\hat{\Lambda}) = K_{1,j}(t) + K_{2,j}(t) + K_{3,j}(t),$$  
(S.12)

with

$$\sup_{t \in T_n} \max_{d(t)+1 \leq j \leq p} |K_{1,j}(t)| \leq \sum_{k=1}^{k_0} \sup_{t \in T_n} \| \hat{\mathbf{M}}(J_n, t, k) - \Sigma_2(t, k) \|^2_F,$$  
(S.13)

$$\sup_{t \in T_n} \max_{d(t)+1 \leq j \leq p} |K_{2,j}(t)| \leq M_1 \sup_{t \in T_n} \| \hat{\Lambda}(t) - \Lambda(t) \|_F \| \hat{\mathbf{B}}(t) - \mathbf{B}(t) \hat{\mathbf{O}}_2^\top(t) \|_F,$$  
(S.14)

$$\sup_{t \in T_n} \max_{d(t)+1 \leq j \leq p} |K_{3,j}(t)| \leq M_2 \sup_{t \in T_n} \| \hat{\mathbf{B}}(t) - \mathbf{B}(t) \hat{\mathbf{O}}_2^\top(t) \|^2_F \| \Lambda(t) \|_F$$  
(S.15)

for some sufficiently large constants $M_1$ and $M_2$. For $K_{1,j}(t)$, by (S.3) and (S.13), we have that

$$\| \sup_{t \in T_n} \max_{d(t)+1 \leq j \leq p} |K_{1,j}(t)| \|_{L^1} = O(p^2 \nu_n^2).$$  
(S.16)

Notice that for any two random variables $X, Y$ with finite first moments, for any $c_1, c_2 > 0$, the following
inequality holds by Markov inequality

\[ P(|XY| \geq c_1 c_2 \mathbb{E}|X| \mathbb{E}|Y|) \leq P(|X| \geq c \mathbb{E}|X|) + P(|Y| \geq c \mathbb{E}|Y|) \leq 1/c_1 + 1/c_2. \]  

(S.17)

Meanwhile, \( \eta_n^{-1}(p \nu_n)^2 = p^{2-\delta} \nu_n (\eta_n^{-1} p^\delta \nu_n) \). Thereby by (S.17) and the results of Theorem 7.1, Theorem 7.2 and (S.14), we have that

\[ P\left( \sup_{t \in \mathcal{T}} \max_{d(t)+1 \leq j \leq p} |K_{2,j}(t)| \geq \eta_n^{-1}(p \nu_n)^2 g_n^2 \right) = O(1/g_n). \]  

(S.18)

Finally, by (S.2) we shall see that

\[ \sup_{t \in \mathcal{T}} \| \Lambda(t) \|_F \asymp p^{2-2\delta}, \]  

(S.19)

which together with Theorem 7.1 (S.15) and (S.17) provides the probabilistic bound for \( K_{3,j}(t) \)

\[ P\left( \sup_{t \in \mathcal{T}} \max_{d(t)+1 \leq j \leq p} |K_{3,j}(t)| \geq g_n^2 \eta_n^{-2}(p \nu_n)^2 \right) = O(1/g_n). \]  

(S.20)

As a result, (ii) follows from (S.16), (S.18) and (S.20).

For (iii), notice that by (S.10) and (S.19) it follows that

\[ \inf_{t \in \mathcal{T}} \| \Lambda(t) \|_m \gtrsim \eta_n p^{2-2\delta}, \sup_{t \in \mathcal{T}} \| \Lambda(t) \|_m \gtrsim \eta_n p^{2-2\delta}, \inf_{t \in \mathcal{T}} \| \Lambda(t) \|_2 \asymp p^{2-2\delta}, \sup_{t \in \mathcal{T}} \| \Lambda(t) \|_2 \asymp p^{2-2\delta}. \]  

(S.21)

Then by (i)

\[ P( \lambda_{j+1}(\hat{\Lambda}(t)) \geq \eta_n, j = 1, \ldots, d(t) - 1, \forall t \in \mathcal{T} ) \geq P( \sup_{t \in \mathcal{T}} \max_{1 \leq j \leq d(t)} |\lambda_j(\hat{\Lambda}(t)) - \lambda_j(\Lambda(t))| \leq \frac{\eta_n p^{2-2\delta}}{\tilde{M}_0} ), \]  

(S.22)

where \( \tilde{M}_0 \) is a sufficiently large constant such that \( \eta_n p^{2-2\delta} / \tilde{M}_0 < (1/2) \inf_{t \in \mathcal{T}} \lambda_{d(t)}(\Lambda(t)) \). Therefore by (i) and Markov inequality (iii) holds. Finally, (iv) follows from (i), (ii), (S.21) and (S.17).

Proof of Proposition 7.3

By Proposition 7.2, it suffices to prove everything on the event \( A_n := \{ \hat{d}_n = d \} \). Observe that \( \hat{T}_n = \| \sum_{i=1}^{m_n} \hat{I}_i \|_\infty \) and \( \hat{T}_n = \| \sum_{i=1}^{m_n} I_i \|_\infty \). Notice that for \( 1 \leq s_1 \leq N_n - 1, 1 \leq s_2 \leq p - d, 1 \leq s_3 \leq p, \)
1 \leq w \leq k_0$, the $(k_0(p-d)ps_1+(w-1)p(p-d)+(s_2-1)p+s_3)_{th}$ entry of $\hat{I}_i$ and $\hat{I}_i$ are $f^T_{s_2}x_{s_1m_n+i+w,n}x_{s_1m_n+i,s_3}$ and $\hat{f}^T_{s_2}x_{s_1m_n+i+w,n}x_{s_1m_n+i,s_3}$ respectively where $x_{i,s_2}$ is the $(s_2)_{th}$ entry of $x_{i,n}$. Then on event $A_n$,

$$
|\hat{T}_n - \hat{T}_n| = \max_{1 \leq s_1 \leq N_n-1, 1 \leq s_3 \leq p, 1 \leq w \leq k_0, 1 \leq s_2 \leq p-d} \left| (f^T_{s_2} - \hat{f}^T_{s_2}) \sum_{i=1}^{m_n} x_{s_1m_n+i+w,n}x_{s_1m_n+i,s_3} \right| / \sqrt{m_n}
$$

By the triangle inequality, condition $(M')$ and Cauchy inequality, it follows that

$$
\max_{1 \leq s_1 \leq N_n-1, 1 \leq s_3 \leq p, 1 \leq w \leq k_0, 1 \leq s_2 \leq p-d} \left| (f^T_{s_2} - \hat{f}^T_{s_2}) \sum_{i=1}^{m_n} x_{s_1m_n+i+w,n}x_{s_1m_n+i,s_3} \right| 
\leq \max_{1 \leq s_1 \leq N_n-1, 1 \leq s_3 \leq p, 1 \leq w \leq k_0, 1 \leq s_2 \leq p-d} \left| \sum_{u=1}^{p} (f_{s_2,u} - \hat{f}_{s_2,u}) \sum_{i=1}^{m_n} x_{s_1m_n+i+w,u}x_{s_1m_n+i,s_3} \right|
\leq \max_{1 \leq s_1 \leq N_n-1, 1 \leq s_3 \leq p, 1 \leq w \leq k_0, 1 \leq s_2 \leq p-d} \left( \sum_{u=1}^{p} (f_{s_2,u} - \hat{f}_{s_2,u})^2 \right)^{1/2} \left( \sum_{i=1}^{m_n} x_{s_1m_n+i+w,u}x_{s_1m_n+i,s_3} \right)^{1/2}
\leq \|\hat{F} - F\| \sum_{u=1}^{p} \left( \sum_{i=1}^{m_n} x_{s_1m_n+i+w,u}x_{s_1m_n+i,s_3} \right)^2 \left( \sum_{u=1}^{p} (f_{s_2,u} - \hat{f}_{s_2,u})^2 \right)^{1/2},
$$

where $f_{s_2,u}$ and $\hat{f}_{s_2,u}$ is the $u_{th}$ entry of $f_{s_2}$ and $\hat{f}_{s_2}$, respectively. Moreover, for $1 \leq s_1 \leq N_n-1, 1 \leq s_3 \leq p, 1 \leq w \leq k_0$,

$$
\| \sum_{u=1}^{p} \left( \sum_{i=1}^{m_n} x_{s_1m_n+i+w,u}x_{s_1m_n+i,s_3} \right)^2 \right|^{1/2} \right)^{1/2} \leq \sqrt{pm_n},
$$

Using the fact that

$$
\mathbb{E} \left( \max_{1 \leq s_1 \leq N_n-1, 1 \leq s_3 \leq p, 1 \leq w \leq k_0} \left( \sum_{u=1}^{p} \left( \sum_{i=1}^{m_n} x_{s_1m_n+i+w,u}x_{s_1m_n+i,s_3} \right)^2 \right)^{1/2} \right)^{1/2} \leq \sum_{1 \leq s_1 \leq N_n-1, 1 \leq s_3 \leq p, 1 \leq w \leq k_0} \left( \sum_{u=1}^{p} \left( \sum_{i=1}^{m_n} x_{s_1m_n+i+w,u}x_{s_1m_n+i,s_3} \right)^2 \right)^{1/2},
$$

(S.25)

(S.26)
together with \((S.25)\) we have
\[
\left\| \max_{1 \leq s_1 \leq N_n-1} \left\| \sum_{u=1}^{p} \sum_{i=1}^{m_n} x_{s_1 m_n+i+w,u} x_{s_1 m_n+i,s_3} \right\|^{1/2} \right\|_{L^1} = O((N_n p)^{1/4} \sqrt{p m_n}). \tag{S.27}
\]

Now the proposition follows from Proposition \(7.2\) \((S.23), (S.24)\) and \((S.27)\), Corollary \(B.3\) in the online supplement and an application of \((S.17)\).

**Proof of Theorem 7.4**

We first show \((7.6)\). To show \((7.6)\) we verify the conditions of Corollary 2.2 of \cite{ZhangCheng2018}. Recall the proof of Proposition 7.3 that for \(1 \leq s_1 \leq N_n-1, 1 \leq s_2 \leq p-d, 1 \leq s_3 \leq p, 1 \leq w \leq k_0\), the \((k_0(p-d)p s_1 + (w-1)p(p-d) + (s_2-1)p + s_3)\)th entry of \(\hat{T}_i\) is \(f_{s_2}^\top x_{s_1 m_n+i+w,u} x_{s_1 m_n+i,s_3}\) where \(x_{i,s_2}\) is the \((s_2)\)th entry of \(x_{i,n}\). Let \(\hat{I}_{i,s}\) be the \(s\)th entry of \(\hat{I}_i\). By the proof of Lemma A.3 in the online supplement, condition \((M')\), condition \((a)\) and the definition of \(x_{i,n}\), using the fact that \(\|f_i\|_F = 1\) for \(1 \leq i \leq p-d\), we get
\[
\delta_i(j,k,l) := \max_{1 \leq i \leq m_n} \max_{1 \leq s \leq N_n k_0(p-d)p} \mathbb{E}^{1/4}(|\hat{I}_{i,s} - \hat{I}_{(i-k),s}|) = O(((k+1) \log(k+1))^{-2}). \tag{S.28}
\]

Notice that \((S.28)\) satisfies \((10)\) of Assumption 2.3 of \cite{ZhangCheng2018}. Furthermore, by the triangle inequality, condition \((M')\) and Cauchy inequality, under null hypothesis, we have
\[
\|f_{s_2}^\top x_{s_1 m_n+i+w,u} x_{s_1 m_n+i,s_3}\|_{L^1} \leq \|f_{s_2}^\top e_{s_1 m_n+i+w}\|_{L^2} \|x_{s_1 m_n+i,s_3}\|_{L^2}.
\]

By triangle inequality and condition \((M')\), we have \(\|x_{s_1 m_n+i,s_3}\|_{L^2}\) is bounded for all \(s_1, i, s_3\). By conditions \((f)\) we have that
\[
\|f_{s_2}^\top e_{s_1 m_n+i+w}\|_{L^2} \leq M_2 \|f_{s_2}^\top e_{s_1 m_n+i+w}\|_{L^2}, \tag{S.29}
\]
\[
\|f_{s_2}^\top e_{s_1 m_n+i+w}\|_{L^2}^2 = f_{s_2}^\top \mathbb{E}(e_{s_1 m_n+i+w} e_{s_1 m_n+i+w}^\top) f_{s_2} \leq \max_{1 \leq i \leq n} \lambda_{\text{max}}(\text{cov}(e_{i,n} e_{i,n}^\top)) \lesssim 1, \tag{S.30}
\]
where the \(\lesssim\) in \((S.30)\) is uniformly for all \(i, n, p\), which is due to condition \((f)\). This leads to that
\[
\max_{1 \leq i \leq m_n} \mathbb{E}\left(\max_{1 \leq s \leq N_n k_0(p-d)p} |\hat{I}_{i,s}|^4 \right)
= \max_{1 \leq i \leq m_n} \mathbb{E}\left(\max_{1 \leq s_1 \leq N_n-1, 1 \leq s_2 \leq p, 1 \leq w \leq k_0, 1 \leq s_3 \leq p-d} |f_{s_2}^\top x_{s_1 m_n+i+w,u} x_{s_1 m_n+i,s_3}|^4 \right) \lesssim (N_n p^2)^{4/4} \tag{S.31}
\]
which with condition (c) shows that equation (12) of Corollary 2.2 of Zhang and Cheng (2018) holds. Finally, using \( \text{(S.28)} \), \( \text{(S.31)} \) and by similar arguments of Lemma A.3 in the online supplement and Lemma 5 of Zhou and Wu (2010), we shall see that 

\[
\max_{1 \leq j \leq k_0} \frac{N}{n} (p - d) \sigma_{j,j} \leq M < \infty
\]

for some constant \( M \). This verifies (9) of Zhang and Cheng (2018), which proves the first claim of the theorem.

To see the second claim of the theorem, writing 

\[
\delta_0 = \frac{g_n^2 p^\beta \sqrt{m_n}}{\sqrt{n}} \Omega_n,
\]

where \( g_n \) is a diverging sequence.

Notice that

\[
\mathbb{P}(\hat{T}_n \geq t) \leq \mathbb{P}(|\hat{T}_n - \tilde{T}_n| \geq \delta_0),
\]

\[
\mathbb{P}(\hat{T}_n \geq t) \geq \mathbb{P}(\tilde{T}_n \geq t + \delta_0, |\hat{T}_n - \tilde{T}_n| \leq \delta_0)
\]

\[
= \mathbb{P}(\tilde{T}_n \geq t + \delta_0) - \mathbb{P}(\hat{T}_n \geq t + \delta_0, |\hat{T}_n - \tilde{T}_n| \geq \delta_0)
\]

\[
\geq \mathbb{P}(\tilde{T}_n \geq t + \delta_0) - \mathbb{P}(|\hat{T}_n - \tilde{T}_n| \geq \delta_0).
\]

(S.33)

Therefore following the first assertion of Theorem 7.4 and Proposition 7.3, we have

\[
\sup_{t \in \mathbb{R}} |\mathbb{P}(\hat{T}_n \geq t) - \mathbb{P}(|y|_\infty \geq t)| \leq \sup_{t} |\mathbb{P}(|y|_\infty \geq t - \delta_0) - \mathbb{P}(|y|_\infty \geq t + \delta_0)|
\]

\[
+ O\left(\frac{1}{g_n} + \frac{p^\beta}{\sqrt{n}} \log n + \nu(m_n, k_0 N_n p(p - d), 2l, (N_n p^2)^{1/l})\right).
\]

(S.34)

Since \( |y|_\infty = \max(y, -y) \), by Corollary 1 of Chernozhukov et al. (2015), we have that

\[
\sup_t |\mathbb{P}(|y|_\infty \geq t - \delta_0) - \mathbb{P}(|y|_\infty \geq t + \delta_0)| = O(\delta_0 \sqrt{\log(n/\delta_0)}).
\]

(S.35)

Combining (S.34) and (S.35), and letting \( g_n = (\frac{p^\beta \sqrt{m_n} \log n}{\sqrt{n}} \Omega_n)^{-1/3} \), the second claim of Theorem 7.4 follows.

\[ \square \]

Supplemental Material for “Adaptive Estimation for Non-stationary Factor Models And A Test for Static Factor Loadings”

Abstract

Section 1 contains the proofs of Auxiliary lemmas for Theorem 7.1, 7.2, 7.3 for estimation of the null space of the evolutionary loading matrix, as well as Proposition 7.1. Section 2 includes the proof of Proposition 7.2 and Theorem 7.5 for testing the static factor loading. Finally, Section 3 proves Lemma 8.1 and Theorem 8.1.
1 Proofs of Proposition 7.1 and Auxiliary Lemmas for Theorem 7.1, 7.2, 7.3.

Proof of Proposition 7.1 Consider \( g_n = \frac{\eta^2}{\nu_n p^2 \log n} \). Then the conditions of Theorem 7.3 hold. Since \( \sum_{j=1}^{p} \lambda_j^2(\hat{A}(t)) = \|\hat{A}(t)\|_F^2 \), we shall discuss the magnitude of \( \|\hat{A}(t)\|_F^2 \). First, notice that by using similar and simpler argument of proofs of Theorem 7.2 it follows that \( \inf_{t \in T_n} \|\Lambda(t)\|_F \propto p^{2-2\delta} \). By Theorem 7.1 and (S.19) in the main article we shall see that

\[
\sup_{t \in T_n} \|\hat{A}(t)\|_F \propto p^{2-2\delta}, \quad \inf_{t \in T_n} \|\hat{A}(t)\|_F \propto p^{2-2\delta}. \tag{A.1}
\]

Define the events

\[
A_n = \left\{ \sup_{t \in T_n} \max_{1 \leq j \leq d(t)} |\lambda_j(\hat{A}(t)) - \lambda_j(\Lambda(t))| \leq \frac{\eta_n p^{2-2\delta}}{M_0} \right\}, \quad B_n = \left\{ \sup_{t \in T_n} \frac{\lambda_{d(t)+1}(\hat{A}(t))}{\lambda_{d(t)}(\Lambda(t))} \geq p^{2\delta} g_n^2 \sqrt{2/\eta_n^3} \right\},
\]

where as in the proof of Theorem 7.3 in the main article, \( M_0 \) is a sufficiently large constant such that \( \eta_n p^{2-2\delta}/M_0 < (1/2) \inf_{t \in T_n} \lambda_{d(t)}(\Lambda(t)) \). By the proof of Theorem 7.3 we shall see that

\[
\mathbb{P}(A_n) = 1 - O\left(\frac{\nu_n p^\delta}{\eta_n} \right), \quad \mathbb{P}(B_n) = 1 - O(1/g_n) - O\left(\frac{\nu_n p^\delta}{\eta_n} \right). \tag{A.3}
\]

Notice that on \( A_n \), by the choice of \( R(t) \),

\[
\inf_{t \in T_n} \min_{d(t)+1 \leq j \leq R(t)} \frac{\lambda_{j+1}(\hat{A}(t))}{\lambda_j(\hat{A}(t))} \geq \inf_{t \in T_n} \min_{d(t)+1 \leq j \leq R(t)} \frac{\lambda_{j+1}(\Lambda(t))}{\lambda_j(\Lambda(t))} \geq C_0 \frac{\eta_n p^{2-2\delta}}{\eta_n p^{2-2\delta} \log n} = C_0 / \log n \tag{A.4}
\]

for some sufficiently small positive constant \( C_0 \), where the last \( \geq \) is due to (A.1). Because on \( A_n \),

\[
\inf_{t \in T_n} \min_{1 \leq j \leq d(t)} \frac{\lambda_{j+1}(\hat{A}(t))}{\lambda_j(\hat{A}(t))} \geq \eta_n \quad \text{ (c.f. (S.22) in the main article)},
\]

and the fact that \( p^{2\delta} g_n^2 \sqrt{2/\eta_n^3} = o(\min(\eta_n, (\log n)^{-1}) \), we have that

\[
\mathbb{P}(\hat{d}_n(t) = d(t), \forall t \in T_n) \geq \mathbb{P}(A_n \cap B_n) = 1 - O\left(\frac{1}{g_n} \right) - O\left(\frac{\nu_n p^\delta}{\eta_n} \right). \tag{A.5}
\]

Therefore the proposition holds by plugging \( g_n = \frac{\eta^2}{\nu_n p^2 \log n} \) into the above equation. \( \square \)
1.1 Auxiliary Lemmas

The following lemma from [?] is useful for proving Theorem 7.3.

**Lemma A.1.** Let $M(n)$ be the space of all $n \times n$ (complex) matrices. A norm $\| \cdot \|$ on $M(n)$ is said to be unitary-invariant if $\| A \| = \| U AV \|$ for any two unitary matrices $U$ and $V$. We denote by $\text{Eig} \ A$ the unordered $n$-tuple consisting of the eigenvalues of $A$, each counted as many times as its multiplicity. Let $D(A)$ be a diagonal matrix whose diagonal entries are the elements of $\text{Eig} \ A$. For any norm on $M(n)$ define

$$\|(\text{Eig} \ A, \text{Eig} \ B)\| = \min_W \| D(A) - WD(B)W^{-1} \|$$

where the minimum is taken over all permutation matrices $W$. If $A, B$ are Hermitian matrices, we have for all unitary-invariant norms (including the Frobenius norm) the inequality

$$\|(\text{Eig} \ A, \text{Eig} \ B)\| \leq \| A - B \|.$$

In the following, recall that $a_{uv}(t) = 0$ if $v \notin \{l_1, ..., l_{d(t)}\}$, where $l_i, 1 \leq i \leq d(t)$ is the index set of factors defined in Section 4 of the main article.

**Lemma A.2.** Recall in the main article that

$$\delta_G,k(t) = \sup_{t \in [0,1], i \in \mathbb{Z}, 1 \leq j \leq p} \| G_j(t, F_i) - G_j(t, F_{i-k}) \|_{L^2}.$$  \hspace{1cm} (A.6)

Under conditions (A1), and (M1)–(M3), there exists a sufficiently large constant $M_0$, such that uniformly for $1 \leq u \leq p$ and $t \in [0,1]$,

$$\mathbb{E}|G_u(t, F_0)|^4 \leq M_0,$$  \hspace{1cm} (A.7)

$$\delta_G,k(t) = O(d \delta_Q,k(t) + \delta_H,k(t)).$$  \hspace{1cm} (A.8)

**Proof.** By definition we have that for $1 \leq u \leq p$,

$$G_u(t, F_i) = \sum_{v=1}^{d} a_{uv}(t) Q_v(t, F_i) + H_u(t, F_i).$$

Notice that here $d = \sup_{0 \leq t \leq 1} d(t)$ is fixed. Therefore, assumptions (A1), (M2) and triangle inequality lead to the first statement of boundedness of fourth moment of (A.7). Finally (A1) and (M3) lead to the assertion (A.8).  \hspace{1cm} \square
Lemma A.3. Consider the process $z_{i,u,n}z_{i+k,v,n}$ for some $k > 0$, and $1 \leq u, v \leq d$. Then under conditions (C1), (M1)–(M3) we have:

i) $\zeta_{i,u,v} = : \zeta_{u,v}(i_n, F_i)$ is a locally stationary process with associated dependence measures $\delta_{\zeta_{u,v},2}(h) \leq C(\delta_{Q,4}(h) + \delta_{Q,4}(h + k))$ (A.9)

for some universal constant $C > 0$ independent of $u, v$ and any integer $h$;

(ii) For any series of numbers $a_i, 1 \leq i \leq n$, we have

$$(\mathbb{E}\left[\left|\frac{1}{n} \sum_{i=1}^{n} a_i(\zeta_{i,u,v} - \mathbb{E}\zeta_{i,u,v})\right|^2\right])^{1/2} \leq \frac{2C}{n} (\sum_{i=1}^{n} a_i^2)^{1/2} \Delta_{Q,4,0}.$$ (A.10)

Proof. i) is a consequence of the Cauchy-Schwarz inequality, triangle inequality and conditions (M1), (M2). For ii), notice that

$$\sum_{i=1}^{n} a_i(\zeta_{i,u,v} - \mathbb{E}\zeta_{i,u,v}) = \sum_{i=1}^{n} a_i(\sum_{u=0}^{\infty} \mathbb{P}_{i+k-u} \zeta_{i,u,v}) = \sum_{u=0}^{\infty} \sum_{i=1}^{n} a_i \mathbb{P}_{i+k-u} \zeta_{i,u,v}. \quad (A.11)$$

By the property of martingale difference and i) of this lemma we have

$$\| \sum_{i=1}^{n} a_i \mathbb{P}_{i+k-u} \zeta_{i,u,v} \|_{L^2} = \sum_{i=1}^{n} a_i^2 \| \mathbb{P}_{i+k-u} \zeta_{i,u,v} \|_{L^2} \leq C^2 \sum_{i=1}^{n} a_i^2 (\delta_{Q,4}(u) + \delta_{Q,4}(u - k))^2. \quad (A.12)$$

By triangle inequality, inequalities (A.11) and (A.12), the lemma follows. \hfill \Box

Corollary 1.1. Under conditions (C1), (M1) and (M2) we have for each fixed $k > 0$, $1 \leq u, v \leq p$ and $1 \leq w \leq d$, $\psi_i = : \psi(\hat{i}, F_i) = e_{i+k,u,n}e_{i,v,n}$, $\phi_i = : \phi(\hat{i}, F_i) = z_{i+k,w,n}e_{i,v,n}$ and $\iota_i = : \iota(\hat{i}, F_i) = e_{i+k,u,n}z_{i,w,n}$ are locally stationary processes with associated dependence measures

$$\max(\delta_{\psi,2}(h), \delta_{\phi,2}(h), \delta_{\iota,2}(h)) \leq C(\delta_{Q,4}(h) + \delta_{H,4}(h) + \delta_{Q,4}(h + k) + \delta_{H,4}(h + k)). \quad (A.13)$$

for some universal constant $C > 0$ independent of $u, v$ and $w$.

Proof. The corollary follows from the same proof of Lemma A.3. \hfill \Box

To save notation in the following proofs, for given $J_n, k$ write $\hat{\mathbf{M}}(J_n, t, k)$ as $\hat{\mathbf{M}}$ if no confusion arises. Recall the definition of $\hat{\Sigma}_{x,j,k}$ and $\hat{\mathbf{M}}(J_n, t, k)$ in (5.5) and (5.4) in the main article. Observe the following
\[
\tilde{\Sigma}_{s,j,k} = V_{1,j,k} + V_{2,j,k} + V_{3,j,k} + V_{4,j,k}, \tag{A.14}
\]
\[
V_{1,j,k} = \frac{1}{n} \sum_{i=1}^{n-k} A(\frac{i+k}{n}) z_{i+k,n} A^\top (\frac{i}{n}) B_j (\frac{i}{n}), \tag{A.15}
\]
\[
V_{2,j,k} = \frac{1}{n} \sum_{i=1}^{n-k} A(\frac{i+k}{n}) z_{i+k,n} A^\top (\frac{i}{n}) B_j (\frac{i}{n}), \tag{A.16}
\]
\[
V_{3,j,k} = \frac{1}{n} \sum_{i=1}^{n-k} e_{i+k} z_{i,n} A(\frac{i}{n}) B_j (\frac{i}{n}), \quad V_{4,j,k} = \frac{1}{n} \sum_{i=1}^{n-k} e_{i+k} e_{i,n} A^\top (\frac{i}{n}). \tag{A.17}
\]

**Lemma A.4.** Under conditions (C1), (M1), (M2) and (M3) we have that
\[
\| \sup_{t \in [0,1]} \| \tilde{M}(J_n, t, k) - \mathbb{E} \tilde{M}(J_n, t, k) \|_F^2 \|_F^2 = O(\sqrt{\frac{J_n p \sup_{1 \leq j \leq J_n} |B_j(t)|^2}{\sqrt{n}}}).
\]

*Proof.* Using equations (A.14)-(A.17) we have that
\[
\tilde{M}(J_n, t, k) - \mathbb{E} \tilde{M}(J_n, t, k) = \sum_{j=1}^{J_n} \sum_{s=1}^{4} (V_{s,j,k} - \mathbb{E}(V_{s,j,k})) B_j(t) := \sum_{s=1}^{4} \tilde{V}_s(t), \tag{A.18}
\]
where \(\tilde{V}_s(t) = \sum_{j=1}^{J_n} (V_{s,j,k} - \mathbb{E}(V_{s,j,k})) B_j(t), \) for \(s = 1, 2, 3, 4.\) Consider the \(s = 1\) case and then
\[
\tilde{V}_1(t) := \sum_{j=1}^{J_n} (V_{1,j,k} - \mathbb{E}(V_{1,j,k})) B_j(t)
\]
\[
-\frac{1}{n} \sum_{j=1}^{J_n} B_j(t) \sum_{i=1}^{n-k} A(\frac{i+k}{n}) (z_{i+k,n} z_{i,n}^\top - \mathbb{E}(z_{i+k,n} z_{i,n}^\top)) A^\top (\frac{i}{n}) B_j (\frac{i}{n}). \tag{A.19}
\]
Consider \(\tilde{M}_j = \frac{1}{n} \sum_{i=1}^{n-k} A(\frac{i+k}{n}) (z_{i+k,n} z_{i,n}^\top - \mathbb{E}(z_{i+k,n} z_{i,n}^\top)) A^\top (\frac{i}{n}) B_j (\frac{i}{n}).\) Its \((u, v)_th, 1 \leq u \leq p, 1 \leq v \leq p\) element is
\[
\tilde{M}_{j,u,v} = \frac{1}{n} \sum_{i=1}^{n-k} \sum_{u'=1}^{d} \sum_{v'=1}^{d} a_{u'u'} (\frac{i+k}{n}) (z_{i+k,u',n} z_{i,v',n} - \mathbb{E}(z_{i+k,u',n} z_{i,v',n})) a_{v'v'} (\frac{i}{n}) B_j (\frac{i}{n}). \tag{A.20}
\]
Therefore it follows from the triangle inequality and Lemma A.3 that,
\[
\| \tilde{M}_{j,u,v} \|_F^2 \leq C \Delta Q_{4,0} \sup_{j,s} |B_j(t)| \sum_{u'=1}^{d} \sum_{v'=1}^{d} \sum_{i=1}^{n-k} a_{u'u'}^2 (\frac{i+k}{n}) a_{v'v'}^2 (\frac{i}{n}) \tag{A.21}
\]
for some sufficiently large constant $C$. Consequently by (C1) and Jansen’s inequality, we get

$$
\mathbb{E}\left(\|\tilde{M}_j\|_F^2\right) \leq \frac{C^2 \Delta^2 Q_{4,0} \sup_{j,t} |B_j(t)|^2}{n^2} \sum_{u=1}^{p} \sum_{v=1}^{p} \left( \sum_{J=1}^{J_n} B_j(t) \tilde{M}_{j,u,v} \right)^2 \\
\leq \frac{C^2 \Delta^2 Q_{4,0} d^2 \sup_{j,t} |B_j(t)|^2}{n^2} \sum_{i=1}^{n-k} \sum_{u=1}^{p} \sum_{v=1}^{p} \sum_{w=1}^{d} \sum_{u'=1}^{d} \sum_{v'=1}^{d} \sum_{w'=1}^{d} a_{u'u'}^2 \left( \frac{i+k}{n} a_{v'v}^2 \right) \\
\times \Delta_{Q,4,0} d^4 \sup_{j,t} |B_j(t)|^2 p^{2-2\delta} \frac{1}{n} \tag{A.22}
$$

for $1 \leq j \leq J_n$. On the other hand, since $(u,v)_{th}$ element of $\tilde{V}_1(t)$, which is denoted by $\tilde{V}_{1,u,v}(t)$, satisfies

$$
\tilde{V}_{1,u,v}(t) = \frac{1}{n} \sum_{j=1}^{J_n} B_j(t) \tilde{M}_{j,u,v}. \tag{A.23}
$$

Therefore by Jansen’s inequality it follows that

$$
\sup_{t \in [0,1]} \|\tilde{V}_1(t)\|_F^2 = \sup_{t, 1 \leq j \leq J_n} \sum_{u=1}^{p} \sum_{v=1}^{p} \left( \sum_{J=1}^{J_n} B_j(t) \tilde{M}_{j,u,v} \right)^2 \\
\leq \sup_{t, 1 \leq j \leq J_n} |B_j(t)|^2 \sum_{u=1}^{p} \sum_{v=1}^{p} \left( \sum_{J=1}^{J_n} \tilde{M}_{j,u,v} \right)^2 \\
\leq \sup_{t, 1 \leq j \leq J_n} |B_j(t)|^2 \sum_{u=1}^{p} \sum_{v=1}^{p} J_n \sum_{j=1}^{J_n} \tilde{M}_{j,u,v}^2 \\
\leq \sup_{t, 1 \leq j \leq J_n} |B_j(t)|^2 J_n \sum_{j=1}^{J_n} \|\tilde{M}_j\|_F^2. \tag{A.24}
$$

Therefore we have

$$
\mathbb{E}\left( \sup_{t \in [0,1]} \|\tilde{V}_1(t)\|_F^2 \right) \leq \sup_{t, 1 \leq j \leq J_n} |B_j(t)|^2 J_n \sum_{j=1}^{J_n} \mathbb{E}(\|\tilde{M}_j\|_F^2) \tag{A.25}
$$

Combining (A.22) we have that

$$
\mathbb{E}\left( \sup_{t \in [0,1]} \|\tilde{V}_1(t)\|_F^2 \right)^{1/2} = O\left( \frac{J_n \sup_{t, 1 \leq j \leq J_n} |B_j(t)|^2 p^{1-\delta}}{\sqrt{n}} \right). \tag{A.26}
$$
Similarly using Corollary \[\text{A.1}\] we have that

\[
\mathbb{E}\left(\sup_{t \in [0,1]} \|\hat{V}_s(t)\|_F^2\right)^{\frac{1}{2}} = O\left(\frac{J_n \sup_{1 \leq j \leq J_n} |B_j(t)|^2 p^{1-\delta/2}}{\sqrt{n}}\right), \quad s = 2, 3, \tag{A.27}
\]

and

\[
\mathbb{E}\left(\sup_{t \in [0,1]} \|\hat{V}_4(t)\|_F^2\right)^{\frac{1}{2}} = O\left(\frac{J_n p \sup_{1 \leq j \leq J_n} |B_j(t)|^2}{\sqrt{n}}\right). \tag{A.28}
\]

Then the lemma follows from \[\text{A.26}, \text{A.27}, \text{A.28}\] and triangle inequality. \(\square\)

**Lemma A.5.** Under conditions \((A1), (A2), (C1), (M1), (M2)\) and \((M3)\) we have that

\[
\|\sup_{t \in [0,1]} (\hat{\mathbf{M}}(J_n, t, k) - \Sigma^*_k(t))\|_F = O(J_n \sup_{t, 1 \leq j \leq J_n} |B_j(t)|^2 k p/n)
\]

where \(\Sigma^*_k(t) = \frac{1}{n} \sum_{j=1}^{J_n} n \sum_{i=1}^{n} \mathbb{E}(G(\frac{j + k}{n}, \mathcal{F}_{i+k})G(\frac{i}{n}, \mathcal{F}_i)^\top) B_j(\frac{i}{n}) B_j(t)\).

**Proof.** Consider the \((u, v)\)th element of \((\hat{\mathbf{M}}(J_n, t, k) - \Sigma^*_k(t))_{u,v}\). By definition, we have for \(1 \leq u, v \leq p\),

\[
(\hat{\mathbf{M}}(J_n, t, k) - \Sigma^*_k(t))_{u,v} = \frac{1}{n} \sum_{j=1}^{J_n} \sum_{i=1}^{n} \mathbb{E}\left((G_u(\frac{j + k}{n}, \mathcal{F}_{i+k}) - G_u(\frac{i}{n}, \mathcal{F}_i)) G_v(\frac{i}{n}, \mathcal{F}_i) B_j(\frac{i}{n}) B_j(t)\right)
\]

\[
+ \frac{1}{n} \sum_{j=1}^{J_n} \sum_{i=n-k+1}^{n} \mathbb{E}\left(G_u(\frac{i}{n}, \mathcal{F}_{i+k}) G_v(\frac{i}{n}, \mathcal{F}_i) B_j(\frac{i}{n}) B_j(t)\right). \tag{A.29}
\]

By condition \((M3), \text{Lemma A.2}\) and proof of Corollary 3.1 in \[\text{?}\] we have that uniformly for \(1 \leq u, v \leq p\),

\[
\sup_{t \in [0,1]} |(\hat{\mathbf{M}}(J_n, t, k) - \Sigma^*_k(t))_{u,v}| \leq M' J_n \sup_{t, 1 \leq j \leq J_n} |B_j(t)|^2 k/n \tag{A.30}
\]

for some sufficiently large constant \(M'\) independent of \(u\) and \(v\). Therefore by the definition of Frobenius norm, the lemma follows. \(\square\)

**Lemma A.6.** Let \(\nu_n = \sup_{1 \leq j \leq J_n} \text{Lip}_j + \sup_{t, 1 \leq j \leq J_n} |B_j(t)|\) where \(\text{Lip}_j\) is the Lipschitz constant of the basis function \(B_j(t)\). Then under conditions \((A1), (A2), (C1), (M1)\)–\((M3)\) we have that

\[
\sup_{t \in [0,1]} \|\Sigma^*_k(t) - \Sigma_x(t, k)\|_F = O\left(\frac{J_n \sup_{t, 1 \leq j \leq J_n} |B_j(t)| \nu_n}{n} + p g_{J_n, k, M}\right),
\]

where \(\Sigma^*_k\) is defined in Lemma \[\text{A.5}\].
In this section we define $\hat{\Sigma}_k(t) = \frac{1}{n} \sum_{j=1}^{J_n} \sum_{i=1}^{n} \Sigma_x(i,k) B_j(i) B_j(t)$. \hfill (A.31)

Define that $\hat{\Sigma}^*_k(t) = \sum_{j=1}^{J_n} \int_0^1 \Sigma_x(s,k) B_j(s) ds B_j(t)$. Notice that the $(u,v)_{th}$ element of $\Sigma^*_k(t) - \hat{\Sigma}^*_k(t)$ is

\[
(\Sigma^*_k(t) - \hat{\Sigma}^*_k(t))_{u,v} = \sum_{j=1}^{J_n} \left( \frac{1}{n} \sum_{i=1}^{n} \mathbb{E}(G_u(i, F_{i+k}) G_v(i, F_i)) B_j(i) \right) - \int_0^1 \mathbb{E}(G_u(s, F_{i+k}) G_v(s, F_i)) B_j(s) ds \right) B_j(t).
\] \hfill (A.32)

Notice that Lemma A.2 and Condition (A2) imply that there exists a sufficiently large constant $M'$ depending on $M_0$ of Lemma A.2 such that those Lipschitz constants of the functions

\[
\mathbb{E}(G_u(s, F_{i+k}) G_v(s, F_i)) B_j(s)
\]

are bounded by $M' \tau_n$ for all $1 \leq k \leq k_0$, $1 \leq u, v \leq p$. Then using similar argument to the proof of Lemma A.5 we obtain that

\[
\sup_{t \in [0,1]} \| \Sigma^*_k(t) - \hat{\Sigma}^*_k(t) \|_F = O(J_n \sup_{1 \leq j \leq J_n} |B_j(t)| p \tau_n) \quad (A.33)
\]

Similarly by using basis expansion (4.6) in condition (A2) of the main article we have that

\[
\sup_{t \in [0,1]} \| \Sigma^*_k(t) - \Sigma_x(t,k) \|_F = O(pg(J_n, K, \bar{G})) \quad (A.34)
\]

which completes the proof. \hfill \Box

## 2 Proof of Propositions 7.2, Theorem 7.5 and Auxiliary Results

In this section we define $\hat{w}_i$, $i = 1, \ldots, d$ as the orthonormal eigenvectors of $\sum_{k=1}^{k_0} \hat{\Gamma}_k$ w.r.t its largest $d$ eigenvalues: $(\lambda_1(\sum_{k=1}^{k_0} \hat{\Gamma}_k), \ldots, \lambda_d(\sum_{k=1}^{k_0} \hat{\Gamma}_k))$. Let $w_i$, $i = 1, \ldots, d$ be the orthonormal eigenvectors of $\sum_{k=1}^{k_0} \Gamma_k$ with respect to its $d$ positive eigenvalues: $(\lambda_1(\sum_{k=1}^{k_0} \Gamma_k), \ldots, \lambda_d(\sum_{k=1}^{k_0} \Gamma_k))$. Let $W = (w_1, \ldots, w_d)$, $F = (f_1, \ldots, f_{p-d})$, $\hat{W} = (\hat{w}_1, \ldots, \hat{w}_d)$ and $\hat{F} = (\hat{f}_1, \ldots, \hat{f}_{p-d})$, where $F$ and $\hat{F}$ are bases of null space of $\hat{\Gamma}$ and $\Gamma$, respectively. Therefore ($(w_i, 1 \leq i \leq d), (f_i, 1 \leq i \leq p-d)$) and ($(\hat{w}_i, 1 \leq i \leq d), (\hat{f}_i, 1 \leq i \leq p-d)$)
are both orthonormal bases for $\mathbb{R}^p$. Recall that $\hat{\Gamma} = \sum_{k=1}^{k_0} \hat{\Gamma}_k$ and $\Gamma = \sum_{k=1}^{k_0} \Gamma_k$ in Section 6 of the main article.

Since we are testing the null hypothesis of static factor loadings and are considering local alternatives where $A(\cdot)$ deviates slightly from the null in this section, we have assumed that $d(t) \equiv d$ for testing the static factor loadings in the main article, i.e., the number of factors is fixed, and therefore we assume that $\eta_n$ in conditions (C1), (S) satisfies $\eta_n \geq \eta_0 > 0$ for some positive constant $\eta_0$, and $\mathcal{T}_{\eta_n} = (0, 1)$ which coincides with discussions in Remark 7.3 of the main article.

**Corollary B.1.** Assume $(A1)$, $(A2)$, $(C1)$, $(M1)$–$(M3)$.

\[ \|\|\| \Gamma - \hat{\Gamma} \|\|_F \|_{L^1} = O_p\left( \frac{p^{2-\delta}}{\sqrt{n}} \right) \quad \text{(B.1)} \]

*Proof.* It suffices to show uniformly for $1 \leq k \leq k_0$,

\[ \|\|\| \Gamma_k - \hat{\Gamma}_k \|\|_F \|_{L^1} = O_p\left( \frac{p^{2-\delta}}{\sqrt{n}} \right). \quad \text{(B.2)} \]

By the proof of Lemma A.4, it follows that for $1 \leq k \leq k_0$,

\[ \left\| \frac{1}{n} \sum_{i=1}^{k-n} x_{i+k,n} x_{i,n}^\top - \frac{1}{n} \mathbb{E} \left( \sum_{i=1}^{k-n} x_{i+k,n} x_{i,n}^\top \right) \right\|_F \|_{L^2} = O\left( \frac{p}{\sqrt{n}} \right). \quad \text{(B.3)} \]

By the proof of Lemma A.5, it follows that

\[ \left\| \frac{1}{n} \sum_{i=1}^{k-n} \left( \mathbb{E} (x_{i+k,n} x_{i,n}^\top) - \Sigma_{x}(\frac{i}{n}, k) \right) \right\|_F = O\left( \frac{p}{n^2} \right), \quad \text{(B.4)} \]

\[ \left\| \frac{1}{n} \sum_{i=1}^{k-n} \Sigma_{x}(\frac{i}{n}, k) - \int_0^1 \Sigma_{x}(t,k) \, dt \right\|_F = O\left( \frac{p}{n} \right), \quad \text{(B.5)} \]

where to prove (B.5) we have used Lemma A.2. Then by (B.3) to (B.5) we have that

\[ \left\| \frac{1}{n} \sum_{i=1}^{k-n} x_{i+k,n} x_{i,n}^\top - \int_0^1 \Sigma_{x}(t,k) \, dt \right\|_F = O\left( \frac{p}{\sqrt{n}} \right), \quad \text{(B.6)} \]

which together with (S.2) in the main article and the definition of $\Gamma_k$ proves (B.2). Therefore the corollary holds.

**Corollary B.2.** Assume conditions $(A1)$, $(A2)$, $(C1)$, $(M1)$, $(M2)$, $(M3)$ and conditions $(S1)$–$(S4)$, then
there exist orthogonal matrices \( \hat{O}_3 \in \mathbb{R}^{d \times d} \), \( \hat{O}_4 \in \mathbb{R}^{(p-d) \times (p-d)} \) such that

\[
\| \| \hat{W} \hat{O}_3 - W \| \|_{F} = O\left(\frac{p^\delta}{\sqrt{n}}\right),
\]

\[
\| \| \hat{F} \hat{O}_4 - F \| \|_{F} = O\left(\frac{p^\delta}{\sqrt{n}}\right).
\]

**Proof.** By (S.2) in the main article and the triangle inequality we have that

\[
\sum_{k=1}^{k_0} \| (\int_0^1 \Sigma_x(t,k) dt)(\int_0^1 \Sigma_x(t,k) dt)^\top \|_F \propto p^{2-2\delta}. \tag{B.7}
\]

On the other hand, by Weyl’s inequality (\ref{weyl}) and (12) of Lam et al. (2011) we have that

\[
\sup_{t \in [0,1]} \| (\int_0^1 \Sigma_x(t,k) dt)(\int_0^1 \Sigma_x(t,k) dt)^\top \|_m \geq (\int_0^1 \inf_{t \in (0,1)} \| (\Sigma_x(t,1) dt)\|_m)^2. \tag{B.8}
\]

By (B.7), (B.8) and the proof of Theorem 7.2 we have that

\[
\lambda_d(\Gamma) \propto p^{2-2\delta}. \tag{B.9}
\]

As a result, the corollary follows from Corollary B.1, (B.9) and Theorem 1 of Yu et al. (2015). Details are omitted for the sake of brevity. \(\square\)

**Proof of Proposition 7.2**

**Proof.** It is follows from Corollaries B.1 and B.2, and the proof of Proposition 7.1. \(\square\)

**Corollary B.3.** Assume conditions of Proposition 7.2 hold, then under null hypothesis there exists an orthonormal basis \( \{f_i, 1 \leq i \leq p-d\} \) of null space of \( \Gamma \), such that

\[
\| \| \hat{F} - F \| \|_{F} = O\left(\frac{p^\delta}{\sqrt{n}}\right), \tag{B.10}
\]

where \( F = (f_1, \ldots, f_{p-d}) \).

**Proof.** Notice that by Corollary B.2 the exists a set of orthonormal basis \( G = (G_1, \ldots, G_{p-d}) \) of null space of \( \Gamma \) together with a \( (p-d) \times (p-d) \) orthonormal matrix \( \hat{O}_4 \) such that

\[
\| \| \hat{F} \hat{O}_4 - G \| \|_{F} = O\left(\frac{p^\delta}{\sqrt{n}}\right).
\]

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Take $F = G \tilde{O}_4^\top$ and the corollary is proved. \qed

Recall in Section 7.2 of the main article the $k_0N_n(p - d)p$ dimensional vectors $\tilde{l}_i$ which replaces $\hat{F}$ in $\hat{l}_i$ of (6.3) with $F$. In the following proofs, define

$$s_{j,w_n} = \sum_{r=j}^{j+w_n-1} \tilde{l}_i \text{ for } 1 \leq j \leq m_n - w_n + 1, \text{ and } s_{m_n} = \sum_{i=1}^{m_n} \tilde{l}_i. \quad (B.11)$$

Notice that by conditions (S1) and (S3), $\tilde{l}_i, 1 \leq i \leq m_n$ are mean-zero $N_n(p - d)p$ dimensional vectors.

**Proof of Theorem 7.5**

Define

$$\nu_n = \frac{1}{\sqrt{w_n(m_n - w_n + 1)}} \sum_{j=1}^{m_n - w_n + 1} (s_{j,w_n} - \frac{w_n}{m_n} s_{m_n}) R_j. \quad (B.12)$$

We shall show the following two assertions:

$$\sup_{t \in \mathbb{R}} |\mathbb{P}(|y|_{\infty} \leq t) - \mathbb{P}(|\nu_n|_{\infty} \leq t|x_{i,n}, 1 \leq i \leq n)| = O_p(\Theta_1^{1/3} \log^{2/3}(W_{n,p})) \quad (B.13)$$

$$\sup_{t \in \mathbb{R}} |\mathbb{P}(|\nu_n|_{\infty} \leq t|x_{i,n}, 1 \leq i \leq n) - \mathbb{P}(|\kappa_n|_{\infty} \leq t|x_{i,n}, 1 \leq i \leq n)| = O_p((p^{\delta + 1/2} \sqrt{w_n} \log n)^{(1/l)/l+1}). \quad (B.14)$$

The theorem then follows from (B.13), (B.14) and Theorem 7.4. In the remaining proofs, we write conditional on $\{x_{i,n}, 1 \leq i \leq n\}$ as conditional on $x_{i,n}$ for short if no confusion arises.

**Step (i): Proof of (B.13).** To show (B.13), we shall show that

$$\max_{1 \leq u,v \leq k_0N_n(p-d)} |\sigma_{u,v}^\nu - \sigma_{u,v}^Y|_{L^{q^*}} = O((w_n^{-1} + w_n/m_n W_{n,p}^{1/q^*})), \quad (B.15)$$

where $\sigma_{u,v}^\nu$ and $\sigma_{u,v}^Y$ are the $(u,v)$th entry of the covariance matrix of $\nu_n$ given $\tilde{l}_i$ and covariance matrix of $y$. Notice that (B.15) together with condition (d) implies that there exists a constant $\eta_0 > 0$ such that

$$\mathbb{P}(\max_{1 \leq u,v \leq k_0N_n(p-d)} \sigma_{u,v}^\nu \geq \eta_0) \geq 1 - O((w_n^{-1} + w_n/m_n W_{n,p}^{1/q^*})^q). \quad (B.16)$$

Since by assumption $w_n^{-1} + w_n/m_n W_{n,p}^{1/q^*} = o(1)$, it suffices to consider the conditional Gaussian approximation on the $\{x_{i,n}\}$ measurable event $\{\max_{1 \leq u,v \leq k_0N_n(p-d)} \sigma_{u,v}^\nu \geq \eta_0\}$. Then by the construction
of $y$ and Theorem 2 of Chernozhukov et al. (2015) (we consider the case $a_p = \sqrt{2 \log p}$ in there), (B.13) will follow.

Now we prove (B.15). Let $S_{j,w_n,s}$ and $S_{m,s}$ be the $s$th element of the vectors $s_{j,w_n}$ and $s_m$, respectively. By our construction, we have

$$
\sigma_{u,v} = \frac{1}{w_n(m_n - w_n + 1)} \left( \sum_{j=1}^{m_n-w_n+1} \left( S_{j,w_n,u} - \frac{w_n}{m_n} S_{m,u} \right) \left( S_{j,w_n,v} - \frac{w_n}{m_n} S_{m,v} \right) \right), \quad \sigma^Y_{u,v} = \mathbb{E} \frac{S_{m,u} S_{m,v}}{m_n}.
$$

(B.17)

On one hand, following the proof of Lemma 4 of Zhou (2013) using conditions (i), (ii) and Lemma 5 of Zhou and Wu (2010) we have that

$$
\max_{u,v} |E \sigma_{u,v} - \sigma^Y_{u,v}| = O \left( w_n^{-1} + \sqrt{\frac{w_n}{m_n}} \right).
$$

(B.18)

Now using (i), (ii) and a similar argument to the proof of Lemma 1 of Zhou (2013), Cauchy-Schwarz inequality and the fact that $\max_{1 \leq i \leq m_n} |X_i|^q \leq \sum_{i=1}^{m_n} |X_i|^q$ for any random variables $\{X_i\}_{1 \leq i \leq m_n}$, we obtain that

$$
\left\| \max_{u,v} \left| E \sigma_{u,v} - \sigma^Y_{u,v} \right| \right\|_{L^q} = O \left( \sqrt{\frac{w_n}{m_n}} W_{n,p}^{1/q^*} \right),
$$

(B.19)

$$
\left\| \max_{u,v} \left| \frac{w_n}{m_n(m_n-w_n+1)} \sum_{j=1}^{m_n-w_n+1} \left( S_{j,w_n,u} S_{j,w_n,v} - \mathbb{E}(S_{j,w_n,u} S_{j,w_n,v}) \right) \right| \right\|_{L^q} = O \left( \sqrt{\frac{w_n}{m_n}} W_{n,p}^{1/q^*} \right),
$$

(B.20)

$$
\left\| \max_{u,v} \left| \frac{1}{m_n(m_n-w_n+1)} \sum_{j=1}^{m_n-w_n+1} \left( S_{j,w_n,u} S_{m,v} - \mathbb{E}(S_{j,w_n,u} S_{m,v}) \right) \right| \right\|_{L^q} = O \left( \sqrt{\frac{w_n}{m_n}} W_{n,p}^{1/q^*} \right).
$$

(B.21)

Combining (B.19)-(B.21) we have

$$
\left\| \max_{u,v} \left| \sigma_{u,v} - \mathbb{E} \sigma_{u,v} \right| \right\|_{L^q} = O \left( \sqrt{\frac{w_n}{m_n}} W_{n,p}^{1/q^*} \right).
$$

(B.22)

Therefore (B.15) follows from (B.18) and (B.22).

Step(ii). We now show (B.14). It suffices to consider on the event $\{\hat{d}_n = d\}$. We first show that for $\epsilon \in (0, \infty)$

$$
\mathbb{P}(|v_n - \kappa_n|_\infty \geq \epsilon |x_{i,n}) = O_p((\epsilon^{-1})^{p^{3/2} + 1/2} \sqrt{\frac{w_n}{n}} (Nnp)^{1/2}).
$$

(B.23)
After that we then show for $\epsilon \in (0, \infty)$

$$\sup_{t \in \mathbb{R}} \mathbb{P}(|\mathbf{v}_n - t| \leq \epsilon | \mathbf{x}_{i,n}) = O_p(\epsilon \sqrt{\log(n/\epsilon)}).$$

(B.24)

Combining (B.23) and (B.24), and following the argument of (S.32) to (S.34) in the main article, we have

$$\sup_{t \in \mathbb{R}} \mathbb{P}(|\mathbf{v}_n| \leq t | \mathbf{x}_{i,n}, 1 \leq i \leq n) - \mathbb{P}(|\mathbf{v}_n| \leq t | \mathbf{x}_{i,n}, 1 \leq i \leq n) | = O_p((\epsilon^{-1} \frac{\hat{\nu}^{1/2} \sqrt{w_n}(N_n p)^{1/l}}{\sqrt{n}})^l + \epsilon \sqrt{\log(n/\epsilon)}).$$

(B.25)

Take $\epsilon = \left( \frac{\hat{\nu}^{1/2} \sqrt{w_n}}{\sqrt{n}} (N_n p)^{1/l} \right)^l / (l+1) \log^{-1/(2l+2)} n$, (B.14) follows.

To show (B.23) it suffices to prove that

$$\mathbb{E}\left( \left| \frac{1}{\sqrt{w_n(m_n - w_n + 1)}} \sum_{j=1}^{m_n-w_n+1} (\hat{s}_{j,w_n} - s_{j,w_n}) R_j \right|_\infty \right) = O_p\left( \frac{\hat{\nu}^{1/2} \sqrt{w_n}(N_n p)^{1/l}}{\sqrt{n}} \right),$$

(B.26)

and

$$\mathbb{E}\left( \left| \frac{1}{\sqrt{w_n(m_n - w_n + 1)}} \sum_{j=1}^{m_n-w_n+1} \frac{w_n}{m_n} (\hat{s}_{j,w_n} - s_{j,w_n}) R_j \right|_\infty \right) = O_p\left( \frac{\hat{\nu}^{1/2} \sqrt{w_n}(N_n p)^{1/l}}{\sqrt{n}} \right).$$

(B.27)

We now show (B.26), and (B.27) follows mutatis mutandis. Define $\hat{S}_{j,w_n}$ and $S_{j,w_n}$ as the $r$th component of the $k_0 N_n (p - d) p$ dimensional vectors $\hat{s}_{j,w_n}$ and $s_{j,w_n}$. Using the notation of proof of Theorem 7.4, it follows that

$$\max_{1 \leq s_1 \leq N_n - 1, 1 \leq s_2 \leq p - d} \max_{1 \leq s_3 \leq p_1, 1 \leq w \leq k_0} \left| \sum_{j=1}^{m_n-w_n+1} \sum_{i=j}^{m_n-w_n+1} (\hat{f}_{s_2}^\top - f_{s_2}^\top) x_{s_1 m_n+i+w,n} x_{s_1 m_n+i, s_3} R_j \right|.$$  

(B.28)

Notice that

$$\max_{1 \leq s_2 \leq p - d} \left| \sum_{j=1}^{m_n-w_n+1} \sum_{i=j}^{m_n-w_n+1} (\hat{f}_{s_2}^\top - f_{s_2}^\top) x_{s_1 m_n+i+w,n} x_{s_1 m_n+i, s_3} R_j \right|^2 \leq \sum_{1 \leq s_2 \leq p - d} \left\| \hat{f}_{s_2}^\top - f_{s_2}^\top \right\|^2 \sum_{j=1}^{m_n-w_n+1} \sum_{i=j}^{m_n-w_n+1} x_{s_1 m_n+i+w,n} x_{s_1 m_n+i, s_3} R_j \right|^2 \leq \left\| \hat{F} - F \right\|^2 \sum_{j=1}^{m_n-w_n+1} \sum_{i=j}^{m_n-w_n+1} x_{s_1 m_n+i+w,n} x_{s_1 m_n+i, s_3} R_j \right|^2.$$  

(B.29)
Therefore, by using Corollary [B.3] [B.28], [B.29] we have

\[
\left| \sum_{j=1}^{m_n-w_n+1} \left( \hat{s}_{j,w_n} - s_{j,w_n} \right) R_j \right|_\infty \leq \frac{p^\delta}{\sqrt{n}} \max_{1 \leq s_1 \leq N_n-1, \ 1 \leq s_3 \leq p, \ 1 \leq w \leq k_0} \left\| \sum_{j=1}^{m_n-w_n+1} \sum_{i=j}^{j+w_n-1} x_{s_1 m_n+i+w_n, s x_{s_1 m_n+i,s_3} R_j} \right\|_2. \tag{B.30}
\]

Notice that, conditional on \( x_{i,n} \), \( \sum_{j=1}^{m_n-w_n+1} \sum_{i=j}^{j+w_n-1} x_{s_1 m_n+i+w_n, x_{s_1 m_n+i,s_3} R_j} \) is a \( p \) dimensional Gaussian random vector for \( 1 \leq s_1 \leq N_n-1, 1 \leq s_3 \leq p, 1 \leq w \leq k_0 \) whose extreme probabilistic behavior depend on their covariance structure. Therefore we shall study

\[
\left\| \sum_{j=1}^{m_n-w_n+1} \sum_{i=j}^{j+w_n-1} x_{s_1 m_n+i+w_n, x_{s_1 m_n+i,s_3} R_j} \right\|_2.
\]

For this purpose, for any random variable \( Y \) write \( (\mathbb{E}(|Y|^l|x_{i,n}))^{1/l} = \|Y\|_{\mathcal{L}^l} \), then

\[
\mathbb{E} \left( \max_{1 \leq s_1 \leq N_n-1, \ 1 \leq s_3 \leq p, \ 1 \leq w \leq k_0} \left\| \sum_{j=1}^{m_n-w_n+1} \sum_{i=j}^{j+w_n-1} x_{s_1 m_n+i+w_n, x_{s_1 m_n+i,s_3} R_j} \right\|_2^{l/2} \right) \left| x_{i,n} \right|
\]

\[
\leq \sum_{1 \leq s_1 \leq N_n-1, \ 1 \leq s_3 \leq p, \ 1 \leq w \leq k_0} \mathbb{E} \left( \left( \sum_{s=1}^{p} \left\| \sum_{j=1}^{m_n-w_n+1} \sum_{i=j}^{j+w_n-1} x_{s_1 m_n+i+w_n, x_{s_1 m_n+i,s_3} R_j} \right\|_2^{2} \right)^{l/2} \right) \left| x_{i,n} \right|
\]

\[
= \sum_{1 \leq s_1 \leq N_n-1, \ 1 \leq s_3 \leq p, \ 1 \leq w \leq k_0} \left( \sum_{s=1}^{p} \left\| \sum_{j=1}^{m_n-w_n+1} \sum_{i=j}^{j+w_n-1} x_{s_1 m_n+i+w_n, x_{s_1 m_n+i,s_3} R_j} \right\|_2^{2} \right)^{l/2} \left| x_{i,n} \right|. \tag{B.31}
\]

By Burkholder inequality, conditions \((M')\), \((M1)\) and \((M3)\), we have that

\[
\left\| \left( \sum_{j=1}^{m_n-w_n+1} \sum_{i=j}^{j+w_n-1} x_{s_1 m_n+i+w_n, x_{s_1 m_n+i,s_3} R_j} \right) \right\|_{\mathcal{L}^l}^l \lesssim w_n \sqrt{m_n-w_n+1}. \tag{B.32}
\]

Therefore by straightforward calculations and equations \([B.30] - [B.32]\), we show \([B.26]\). As a consequence, \([B.14]\) follows. Finally \([B.24]\) follows from by Corollary 1 of \cite{Chernozhukov} and \([B.16]\), which
completes the proof. □

3 Proof of Lemma 8.1 and Theorem 8.1

Proof of Lemma 8.1

Let $f = (f_1, ..., f_p)^\top$ be a $p$ dimensional vector such that $\|f\|_F = 1$. Notice that by condition (f), there exists a constant $M_1$ such that

$$\|f^\top H(t, F_i) - f^\top H(t, F_i^{(i-k)})\|_{L^4} \leq 2\|f^\top H(t, F_i)\|_{L^4} \leq M_1. \tag{C.1}$$

On the other hand, by triangle inequality

$$\|f^\top H(t, F_i) - f^\top H(t, F_i^{(i-k)})\|_{L^4} \leq \sum_{j=1}^p f_j \|H_j(t, F_i) - H_j(t, F_i^{(i-k)})\|_{L^4} = O(\sqrt{p} \chi^k), \tag{C.2}$$

where the last equality is due to Cauchy-Schwartz inequality and Condition (M'). As a consequence, (C.1) and (C.2) lead to that

$$\sup_{0 \leq t \leq 1} \|f^\top H(t, F_i) - f^\top H(t, F_i^{(i-k)})\|_{L^4} = O(1 \wedge \sqrt{p} \chi^k). \tag{C.3}$$

Notice that when $k = \lfloor a \log p \rfloor$ for some sufficiently large constant $a$, $\sqrt{p} \chi^k = O(1)$. Therefore by (C.3),

$$\sum_{k=1}^{\lfloor a \log p \rfloor} k \sup_{0 \leq t \leq 1} \|f^\top H(t, F_i) - f^\top H(t, F_i^{(i-k)})\|_{L^4} + \sum_{k=\lfloor a \log p \rfloor + 1}^\infty k \sup_{0 \leq t \leq 1} \|f^\top H(t, F_i) - f^\top H(t, F_i^{(i-k)})\|_{L^4} = O(\log^2 p) + O(\sum_{k=\lfloor a \log p \rfloor + 1}^\infty k \chi^k) = O(\log^2 p),$$

which finishes the proof. □

Recall the quantities $\tilde{l}_i$ and $\tilde{l}_i$ defined in Section 8, and $\hat{l}_i$ in (6.4) in the main article. To prove the power result in Theorem 8.1 we need to evaluate the magnitude of $E\hat{l}_i$ under the local alternatives, which are determined by $E\tilde{l}_i$ and $E\tilde{l}_i$. Define $x_i^* = A(z_{i,n} + e_{i,n})$, and further define $\tilde{l}_i^*$ by replacing $x_{i,n}$ in $\tilde{l}_i$ with $x_{i,n}^*$, where under the local alternative (8.3) in the main article, $x_{i,n} = A \circ (J + \rho_n D(t))z_{i,n} + e_{i,n}$. Notice that, under the considered alternative, $y_i$ in fact preserves the auto-covariance structure of $\tilde{l}_i^*$. Then we prove Theorem 8.1 by the similar arguments to the proof of Theorems 7.4 and 7.5 but under...
local alternatives. We first present auxiliary propositions \(C.1\) and \(C.2\).

**Proposition C.1.** Write \(\Gamma(0)\) as \(\Gamma\). Under conditions of Theorem 8.1, we have that for each \(n\) there exists an orthogonal basis \(F = (f_1, \ldots, f_{p-d})\) such that

\[
\|F_n - F\|_F = O(\rho_n p^{(\delta-\delta_1)/2}),
\]  \hspace{1cm} (C.4)

where \(F\) is a basis of \(\Gamma\). Furthermore,

\[
\|\hat{F} - F_n\|_F = O\left(\frac{p^\delta}{\sqrt{n}}\right), \quad \|\hat{F} - F\|_F = O(\rho_n p^{(\delta-\delta_1)/2} + \frac{p^\delta}{\sqrt{n}}),
\]  \hspace{1cm} (C.5)

**Proof.** By the definition of \(\Gamma\) and \(\Gamma_n\), it follows that \(\|\Gamma_n - \Gamma\|_F = O(\rho_n p^{1-\delta_1} p^{3(1-\delta)/2})\). Therefore by the proof of Corollaries B.2 and B.3 for each \(n\) there exists an orthogonal basis \(F = (f_1, \ldots, f_{p-d})\) such that

\[
\|F_n - F\|_F = O\left(\frac{\|\Gamma_n - \Gamma\|_F}{p^{2-2\delta}}\right) = O(\rho_n p^{(\delta-\delta_1)/2}),
\]  \hspace{1cm} (C.6)

which shows \((C.4)\). By the similar argument to Corollary B.1 and triangle inequality, we have that

\[
\|\hat{F} - \Gamma_n\|_F = O\left(\frac{p^\delta}{\sqrt{n}}\right), \quad \|\hat{F} - \Gamma\|_F = O(\rho_n p^{(\delta-\delta_1)/2} + \frac{p^\delta}{\sqrt{n}}),
\]  \hspace{1cm} (C.7)

\[
\|\hat{F} - \Gamma\|_F \leq \|\hat{F} - \Gamma_n\|_F + \|\Gamma_n - \Gamma\|_F = O(\rho_n p^{(\delta-\delta_1)/2} + \frac{p^\delta}{\sqrt{n}}),
\]  \hspace{1cm} (C.8)

which further by the argument of \((C.6)\) yield that

\[
\|\hat{F} - F_n\|_F = O\left(\frac{\|\hat{F} - \Gamma_n\|_F}{p^{2-2\delta}}\right) = O\left(\frac{p^\delta}{\sqrt{n}}\right),
\]  \hspace{1cm} (C.9)

\[
\|\hat{F} - F\|_F = O\left(\frac{\|\hat{F} - \Gamma\|_F}{p^{2-2\delta}}\right) = O(\rho_n p^{(\delta-\delta_1)/2} + \frac{p^\delta}{\sqrt{n}}).
\]  \hspace{1cm} (C.10)

Therefore \((C.5)\) holds. \(\square\)

**Proposition C.2.** Under conditions of Theorem 8.1, we have that

\[
|\mathbb{E} \hat{l}_i|_\infty = O(\rho_n p^{1-\delta_1}), \quad |\mathbb{E} \hat{l}_i|_\infty = O(\rho_n p^{\frac{1-\delta_1}{2}}).
\]  \hspace{1cm} (C.11)

**Proof.** To show the first equation of \((C.11)\), notice that for \(1 \leq s_1 \leq N_n - 1, 1 \leq s_2 \leq p-d, 1 \leq s_3 \leq p, 1 \leq w \leq k_0\), the \((k_0(p-d)ps_1 + (w-1)p(p-d) + (s_2-1)p + s_3)_{th}\) entry of \(\mathbb{E} \hat{l}_i\) is \(\mathbb{E} (f^T_{s_2} x_{s_1 m_n + i + w, n} x_{s_1 m_n + i, s_3})\)
where \( x_{i,s_2} \) is the \((s_2)_{th}\) entry of \( x_{i,n} \). Then by Cauchy-Schwarz inequality we have that there exists a constant \( M \) such that uniformly for \( 1 \leq s_1 \leq N_n - 1, 1 \leq s_2 \leq p - d, 1 \leq s_3 \leq p, 1 \leq w \leq k_0, \)

\[
\left| \mathbb{E}(f_{s_2}^T x_{s_1 m_0 + i + w, n} x_{s_1 m_0 + i, s_3}) \right|
= \left| \mathbb{E}(f_{s_2}^T (\rho_n D (t_{i,w,n}) \circ A z_{s_1 m_0 + i + w, n} + e_{s_1 m_0 + i + w, n}) x_{s_1 m_0 + i, s_3}) \right|
\leq \rho_n M p^{\frac{1-d_1}{2}}, \tag{C.12}
\]

where \( t_{i,w,n} = (s_1 m_n + i + w)/n. \)

In the above argument we have used the fact that \( \|f_u\|_F = 1 \) for \( 1 \leq u \leq p - d \). Therefore the first equation of (C.11) follows from (C.12). Write \( h_{s_2} = f_{s_2} - f_{s_2,n} \). Similarly to (C.12), there exists a constant \( M \) such that uniformly for \( 1 \leq s_1 \leq N_n - 1, 1 \leq s_2 \leq p - d, 1 \leq s_3 \leq p, 1 \leq w \leq k_0, \)

\[
\left| \mathbb{E}(h_{s_2}^T x_{s_1 m_0 + i + w, n} x_{s_1 m_0 + i, s_3}) \right|
= \left| \mathbb{E}(h_{s_2}^T (A (t_{i,w,n}) z_{s_1 m_0 + i + w, n} + e_{s_1 m_0 + i + w, n}) x_{s_1 m_0 + i, s_3}) \right|
\leq \left| \mathbb{E}(h_{s_2}^T (A (t_{i,w,n}) z_{s_1 m_0 + i + w, n}) + e_{s_1 m_0 + i + w, n} x_{s_1 m_0 + i, s_3}) \right|
= O(\rho_n M p^{\frac{1-d_1}{2}}),
\]

where the last equality is due to (C.4); condition (C1), condition (f), Cauchy inequality and the submultiplicity of Frobenius norm. Therefore we have \( \| \mathbb{E} \hat{I}_i - \mathbb{E} \hat{I}_i \|_\infty = O(\rho_n p^{\frac{1-d_1}{2}}) \). Together with first equation of (C.11) we get

\[
\| \mathbb{E} \hat{I}_i \|_\infty = O(\rho_n p^{\frac{1-d_1}{2}}), \tag{C.13}
\]

and the proof is complete. \( \square \)

**Proof of Theorem 8.1**

We first show (a). Now a careful inspection of the proof of Proposition 7.3 shows that the proposition still holds under considered alternative hypothesis used in Theorem 7.4. Recall \( \hat{T} = \left| \frac{\sum_{i=1}^{m_n} \hat{I}_i}{\sqrt{m_n}} \right|_\infty. \) Define

\[
\tilde{T}_n = \left| \frac{\sum_{i=1}^{m_n} \hat{I}_i}{\sqrt{m_n}} \right|_\infty, \quad \tilde{T}_n^* = \left| \frac{\sum_{i=1}^{m_n} \hat{I}_i^*}{\sqrt{m_n}} \right|_\infty.
\]

Using Proposition C.1 and following the proof of Proposition 7.3 we have that for any sequence \( g_n \to \infty, \)

\[
\mathbb{P}(|\tilde{T}_n - \tilde{T}_n| \geq g_n^2 (\rho_n p^{(\delta - \delta_1)/2} + \frac{p^\delta}{\sqrt{n}} m_1^{1/2} \Omega_n) = O(\frac{1}{g_n} + \frac{p^\delta}{\sqrt{n}} \log n). \tag{C.14}
\]

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As a consequence, by the proof of Proposition 7.3, for any sequence $g_n$, the proof of Theorem 7.4 that under the alternative hypothesis,

$$\left| \tilde{T}_n - \tilde{T}_n^* \right| \leq \max_{1 \leq s_1 \leq N_n - 1, 1 \leq s_2 \leq p, 1 \leq w \leq k_0, 1 \leq s_3 \leq p - d} \sum_{i=1}^{m_n} \left| f_{s_2}^T \left( x_{s_1 m_n + i + w, n} x_{s_1 m_n + i, s_3} - x_{s_1 m_n + i + w, n}^* x_{s_1 m_n + i, s_3} \right) \right| / \sqrt{m_n}$$

$$\leq \max_{1 \leq s_1 \leq N_n - 1, 1 \leq s_2 \leq p, 1 \leq w \leq k_0} \sum_{u=1}^{p} \left( \sum_{i=1}^{m_n} (x_{s_1 m_n + i + w, u} x_{s_1 m_n + i, s_3} - x_{s_1 m_n + i + w, u}^* x_{s_1 m_n + i, s_3}) \right)^2 \|x\|^{1/2}, \quad (C.15)$$

where $x_{i,j}^*$ means the $j$th entry of $x_{i,n}^*$. Notice that

$$\left\| \sum_{u=1}^{p} \left( \sum_{i=1}^{m_n} (x_{s_1 m_n + i + w, u} x_{s_1 m_n + i, s_3} - x_{s_1 m_n + i + w, u}^* x_{s_1 m_n + i, s_3}) \right)^2 \right\|^{1/2} \|x\|^{1/2}$$

$$= \left\| \sum_{u=1}^{p} \left( \sum_{i=1}^{m_n} x_{s_1 m_n + i + w, u} x_{s_1 m_n + i, s_3} - x_{s_1 m_n + i + w, u}^* x_{s_1 m_n + i, s_3} \right)^2 \right\|^{1/2} \|x\|^{1/2}, \quad (C.16)$$

which can be further bounded by the summation of

$$\left\| 2 \sum_{u=1}^{p} \left( \sum_{i=1}^{m_n} x_{s_1 m_n + i + w, u} (x_{s_1 m_n + i, s_3} - x_{s_1 m_n + i, s_3}^*) \right)^2 \right\|^{1/2} \|x\|^{1/2}$$

and

$$\left\| 2 \sum_{u=1}^{p} \left( \sum_{i=1}^{m_n} x_{s_1 m_n + i + w, u} x_{s_1 m_n + i, s_3} - x_{s_1 m_n + i + w, u}^* x_{s_1 m_n + i, s_3} \right)^2 \right\|^{1/2} \|x\|^{1/2}. \quad (C.16)$$

Therefore, using the definition of $x_{i,n}^*$, straightforward calculations show that

$$\left\| \sum_{u=1}^{p} \left( \sum_{i=1}^{m_n} (x_{s_1 m_n + i + w, u} x_{s_1 m_n + i, s_3} - x_{s_1 m_n + i + w, u}^* x_{s_1 m_n + i, s_3}) \right)^2 \right\|^{1/2} \|x\|^{1/2} \approx \sqrt{m_n p_n}. \quad (C.17)$$

As a consequence, by the proof of Proposition 7.3 for any sequence $g_n \to \infty$,

$$\mathbb{P}(|\tilde{T}_n^* - \tilde{T}_n| \geq g_n p_n m_n^{1/2} \Omega_n) = O\left( \frac{1}{g_n} + \frac{p^d}{\sqrt{n}} \log n \right). \quad (C.18)$$

Notice that $y_i$ preserves the autocovariance structure of $\tilde{T}_n^*$. Then it follows from (C.14), (C.17) and the proof of Theorem 7.4 that under the alternative hypothesis,

$$\sup_{t \in \mathbb{R}} |\mathbb{P}(\tilde{T}_n \leq t) - \mathbb{P}(y_\infty \leq t)|$$

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\[\text{where by the definition of and the conditions on } \Delta_n, \text{ the indices } s_1, s_2, s_3, w \text{ can be chosen such that}\]
\[
\frac{1}{\sqrt{m_n}} \sum_{i=1}^{m_n} \mathbb{E}(f_{s_2,n}^T \mathbf{x}_{s_1,m_n+i+w,n} x_{s_1,m_n+i,s_3}) \asymp (\sqrt{m_n} \rho_n p^{-\frac{1}{2}}).
\] (C.22)

On the other hand, notice that
\[
\frac{1}{m_n} \sum_{i_1, i_2} \text{Cov}(f_{s_2,n}^T \mathbf{x}_{s_1,m_n+i_1+w,n} x_{s_1,m_n+i_1,s_3}, f_{s_2,n}^T \mathbf{x}_{s_1,m_n+i_2+w,n} x_{s_1,m_n+i_2,s_3}).
\] (C.23)

Consider the case when \( i_1 > i_2 \). Now using condition (G) and that
\[
\begin{align*}
x_{s_1,m_n+i_1+w,n} &= A(s_1 n + i_1 + w) z_{s_1,m_n+i_1+w,n} + e_{s_1,m_n+i_1+w,n}, \\
x_{s_1,m_n+i_2+w,n} &= A(s_1 n + i_2 + w) z_{s_1,m_n+i_2+w,n} + e_{s_1,m_n+i_2+w,n},
\end{align*}
\]
we have (let \( i'_1 = s_1 m_n + i_1, i'_2 = s_1 m_n + i_2 \) for short)
\[
\text{Cov}(f_{s_2,n}^T \mathbf{x}_{i'_1+w,x_{i'_1,s_3}, s_3}, f_{s_2,n}^T \mathbf{x}_{i'_2+w,x_{i'_2,s_3}, s_3}) = \text{Cov}(f_{s_2,n}^T A(\frac{i'_2 + w}{n}) z_{i'_2+w,x_{i'_2,s_3}}, f_{s_2,n}^T A(\frac{i'_2 + w}{n}) z_{i'_2+w,x_{i'_2,s_3}}) + \text{Cov}(f_{s_2,n}^T A(\frac{i'_2 + w}{n}) z_{i'_2+w,x_{i'_2,s_3}}, f_{s_2,n}^T e_{i'_2+w,x_{i'_2,s_3}}).
\] (C.24)
Notice that (C.4) and condition (C1) lead to

\[ \|f_{s_2,n}^\top A(\frac{k}{n})\|_F = O(\rho_n p^{\frac{1-\delta_1}{2}}) \tag{C.25} \]

uniformly for \(1 \leq s_2 \leq p - d\) and \(1 \leq k \leq n\). Therefore,

\[
\begin{align*}
&\text{Cov}(f_{s_2,n}^\top A(\frac{i_2'}{n} + \frac{w}{n})z_{i_1' + w, x_{i_1', s_3}}, f_{s_2,n}^\top A(\frac{i_2'}{n} + \frac{w}{n})z_{i_2' + w, x_{i_2', s_3}}) \\
= &\|f_{s_2,n}^\top A(\frac{i_2'}{n})\text{Cov}(z_{i_1' + w, x_{i_1', s_3}}, z_{i_2' + w, x_{i_2', s_3}})(f_{s_2,n}^\top A(\frac{i_2'}{n} + \frac{w}{n})\|_F \\
\leq &\rho_n^2 p^{1-\delta_1} \|\text{Cov}(z_{i_1' + w, x_{i_1', s_3}}, z_{i_2' + w, x_{i_2', s_3}})\|_F. \tag{C.26}
\end{align*}
\]

On the other hand using condition (G2) and condition (ii) of Theorem 7.5, by the proof of Lemma A.3 it follows for \(1 \leq u \leq d\), \(\leq v \leq p\), \(\Upsilon_{i,u,v} =: \Upsilon_{i,u,v}(\frac{i}{n}, F_i) = z_{i,u,n}x_{i+v,n}\) is a locally stationary process with associated dependence measures

\[ \delta_{\Upsilon_{u,v},2}(h) = O((((k + 1) \log(k + 1))^{-2}). \tag{C.27} \]

Therefore by Lemma 5 of Zhou and Wu (2010), uniformly for \(1 \leq i_1, i_2 \leq m_n\) and \(1 \leq s_3 \leq p\),

\[ \|\text{Cov}(z_{i_1' + w, x_{i_1', s_3}}, z_{i_2' + w, x_{i_2', s_3}})\|_F = O((|i_2' - i_1| \log(|i_2' - i_1|)^{-2}). \tag{C.28} \]

Together with (C.26) we have

\[ \frac{1}{m_n} \sum_{i_1', i_2'} \text{Cov}(f_{s_2,n}^\top A(\frac{i_2'}{n})z_{i_1' + w, x_{i_1'}}, f_{s_2,n}^\top A(\frac{i_2'}{n})z_{i_2' + w, x_{i_2'}}) = O(\rho_n^2 p^{1-\delta_1} \lor 1). \tag{C.29} \]

By similarly arguments using the proof of Lemma 5 of Zhou and Wu (2010) and condition (G3) we have

\[ \frac{1}{m_n} \sum_{i_1', i_2'} \text{Cov}(f_{s_2,n}^\top A(\frac{i_2'}{n} + \frac{w}{n})z_{i_1' + w, x_{i_1'}}, f_{s_2,n}^\top A(\frac{i_2'}{n} + \frac{w}{n})z_{i_2' + w, x_{i_2'}}) = O(\rho_n^2 p^{1-\delta_1} \log^2 p \lor 1) \tag{C.30} \]

By (C.21)–(C.23), (C.29) and (C.30) we show \(\hat{T}_n\) is diverging at the rate of \(\sqrt{m} \rho_n p^{1-\delta_1} \), which proves (a).

To show (b), we define

\[ s^*_j,w_n = \sum_{r=j}^{j+w_n-1} \tilde{l}_r, \quad s^*_m_n = \sum_{i=1}^{m_n} \tilde{l}_i \quad \text{for} \quad 1 \leq j \leq m_n - w_n + 1, \tag{C.31} \]
\[ \hat{s}_{j,w_n} = \sum_{i=j}^{j+w_n-1} \hat{l}_i, \quad \hat{s}_{m_n} = \sum_{i=1}^{m_n} \hat{l}_i \text{ for } 1 \leq j \leq m_n - w_n + 1. \] (C.32)

Then we define
\[ v_n^* = \frac{1}{\sqrt{w_n(m_n - w_n + 1)}} \sum_{j=1}^{m_n-w_n+1} (\hat{s}_{j,w_n} - \frac{w_n}{m_n} \hat{s}_{m_n}) R_j, \] (C.33)

\[ \hat{\kappa}_n = \frac{1}{\sqrt{w_n(m_n - w_n + 1)}} \sum_{j=1}^{m_n-w_n+1} (\hat{s}_{j,w_n} - \frac{w_n}{m_n} \hat{s}_{m_n}) R_j. \] (C.34)

Then by the proof of Theorem 7.5 we have that
\[ \sup_{t \in \mathbb{R}} |\mathbb{P}(|y_\infty \leq t) - \mathbb{P}(|v_n^*|_\infty \leq t|x_{i,n}, 1 \leq i \leq n)| = O_p(\Theta_n^{1/3} \log^{2/3}(\frac{W_n p}{\Theta_n})). \] (C.35)

Straightforward calculations using similar arguments to the proofs of step (ii) of Theorem 7.5 and equation (C.5) further show that under the considered local alternative hypothesis,
\[ \|\|\|v_n^* - \hat{\kappa}_n\|\|_{L^1,x_{i,n}}\|_L^1 = O(\rho_n \sqrt{w_n} \Omega_n), \] (C.36)

\[ \|\|\|\kappa_n - \hat{\kappa}_n\|\|_{L^1,x_{i,n}}\|_L^1 = O((\frac{\rho^\delta}{\sqrt{n}} + \rho_n \frac{\delta - \delta_1}{2}) \sqrt{w_n} \Omega_n). \] (C.37)

Therefore
\[ \|\|\|v_n^* - \kappa_n\|\|_{L^1,x_{i,n}}\|_L^1 = O((\frac{\rho^\delta}{\sqrt{n}} + \rho_n) \sqrt{w_n} \Omega_n). \] (C.38)

As a result, by the arguments of [B.25], [S.5] in the main article article follows.

To evaluate the magnitude of order of \( |\kappa_n|_\infty \), notice that \( \kappa_n \) is a Gaussian vector given \( \{x_{i,n}, 1 \leq i \leq n\} \).

Therefore almost surely
\[ \mathbb{E}(|\kappa_n|_\infty|x_{i,n}, 1 \leq i \leq n) \leq \sum_{j=1}^{m_n-w_n+1} (\hat{s}_{j,w_n} - \frac{w_n}{m_n} \hat{s}_{n})^{o_2} \psi_{\frac{1}{2}} |\| \sqrt{2 \log n}/\sqrt{w_n(m_n - w_n + 1)} \|. \] (C.39)

In the following we consider the event \( \{\hat{d}_n = d\} \). Observe that
\[ \|\sum_{j=1}^{m_n-w_n+1} (\hat{s}_{j,w_n} - \frac{w_n}{m_n} \hat{s}_{n})^{o_2} \psi_{\frac{1}{2}} \|_\infty \leq M_1 \|\| \sum_{j=1}^{m_n-w_n+1} (\hat{s}_{j,w_n} - \hat{s}_{j,w_n} - \frac{w_n}{m_n} (\hat{s}_{n} - \hat{s}_{n}))^{o_2} \psi_{\frac{1}{2}} \|_\infty \] (C.40)
\begin{align*}
+ M_1 \left( \sum_{j=1}^{m_n-w_n+1} (\hat{s}_{j,w_n} - \mathbb{E}\hat{s}_{j,w_n} - \frac{w_n}{m_n}(\hat{s}_n - \mathbb{E}\hat{s}_n))^2 \right)^{\frac{1}{2}} & \bigg|_{\infty} + M_1 \left( \sum_{j=1}^{m_n-w_n+1} (\mathbb{E}\hat{s}_{j,w_n} - \frac{w_n}{m_n}\mathbb{E}\hat{s}_n)^2 \right)^{\frac{1}{2}} \bigg|_{\infty} \\
\quad & := I + II + III
\end{align*}

for some large positive constant $M_1$, where $I$, $II$ and $III$ are defined in an obvious way.

For $I$, notice that similarly to the proof of Theorem 7.5 using Proposition C.1 we have

$$I/\sqrt{w_n(m_n-w_n+1)} = O_p\left( \frac{p^\delta}{\sqrt{n}} \sqrt{w_n\Omega_n} \right).$$  \hfill (C.41)

For $II$, by (C.29), (C.30) and a similar argument to the proof of Theorem 7.5 applied to $s_{j,w_n}$ and $s_n$, we shall see that

$$II/\sqrt{w_n(m_n-w_n+1)} = O_p(\rho_n p^{\frac{1-\delta_1}{2}} \log^2 p \vee \rho_n^2 p^{1-\delta_1} \vee 1).$$ \hfill (C.42)

For $III$, by Proposition C.2

$$III/\sqrt{w_n(m_n-w_n+1)} = O(\sqrt{\rho_n p^{\frac{1-\delta_1}{2}}}).$$ \hfill (C.43)

Therefore from (C.39) to (C.43), $|\kappa_n|_{\infty} = O_p(\sqrt{w_n\rho_n p^{1-\delta_1}} \sqrt{\log n})$, which completes the proof. \hfill \Box

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