On Linear Part of Filled-Section in Splicing

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1 Introduction

In [1] and [2], using splicing Hofer, Wysocki and Zehnder have introduced a smooth model for GW theory near a nodal map $f: S = C_- \cup C_+ \to M$ in the setting of Sc-Banach manifolds in polyfold theory. The basic idea is to introduce, for each gluing parameter $R = R_\theta \in [R_0, \infty)$ with $\theta \in S^1$, the anti-gluing $S^R_- = C_- \ominus R C_+$ as the counter part of the usual gluing $S^R_+ = C_- \oplus R C_+$; then to define a family of isomorphisms $T^R/\hat{T}^R$ between the spaces of maps/sections over $S$ to the corresponding ones over $S^R = S^R_- \cup S^R_+$. The usual $\bar{\partial}_J$-operator, $\Phi^R_+$, acting on the maps on $S^R_+$ then is extended into an operator, denoted by $\Phi^R = (\Phi^-_+, \Phi^R_+)$ acting on the maps on $S^R_-$ in such a way that the index of the linearizations of $\Phi^-_+$ is equal to zero so that the zero set of $\Phi^R$, at least virtually, can be identified with the usual moduli space of $J$-holomorphic maps in the GW-theory with the domain $S^R_+$. The filled-section $\Psi = \{\Psi^R\}$ is obtained from $\Phi^R$ by the conjugation: $\Psi = \hat{T}^{-1} \circ \Phi \circ T = \{\hat{T}^{-1} \circ \Phi^R \circ T\}$. Unlike the $\bar{\partial}_J$-operators $\{\Phi^R_+\}$, the domains and targets of $\{\Psi^R\}$ are independent of $R$, this make it possible to take derivatives for these filled-sections along $R$-direction even at $R = \infty$. However since $T/\hat{T}$ is only an Sc-isomorphism, it is expected that there is a loss of differentiability for $\Psi$ in general. In actuality, the derivatives of filled-section defined in [1, 2] have a loss of differentiability in the nonlinear part of $\Psi$, and do not converge fast enough as $R \to \infty$ in the "linear part". This together with
other difficulties prevents from proving that $\Psi$ is smooth in the usual setting of Banach analysis. Instead it was proved in [2] that the filled-section there is $C^\infty$-smooth with respect to the gluing profile $R = \phi(r) = e^{1/r} - e^{1/r_0}$.

Since in the splicing above the operator $\Phi^R$ is free to be chosen as long as some basic requirements are satisfied, it is natural to ask what the most natural choices are and if it is possible to define the filled-section using such choices in the framework of the usual Banach analysis. This sequence of papers is a report of our work with an affirmative to this question.

While the operator $\Phi^R_+$ acting on the maps on finite cylinder $S^R_+$ is required to be the standard $\bar{\partial}_J$-operator, the desired operator $\Phi^R_-$ on $S^R_- \simeq \mathbb{R}^1 \times S^1$ cannot act on $u_- \ominus u_+$ alone and has to be nonlinear (comparing with [2]). In cylindrical coordinated $(t, s)$ on $S^R_-, \Phi^R_-$ will have the form $\Phi^R_-(u_- \ominus_R u_+) = \partial_t(u_- \ominus_R u_+) + \omega^R(t)(u_- \ominus_R u_+, u_- \ominus_R u_+) + J(u_- \ominus_R u_+ \circ \Gamma)\partial_s(u_- \ominus_R u_+)$. Here $\Gamma^R: S^R_- \rightarrow \hat{S}^R_-$ is the transfer map that transforms the almost complex structure along $u_- \ominus_R u_+$ to the one along $u_- \ominus_R u_+$, and $u_- \ominus_R u_+$ is the extended gluing of $u_-$ and $u_+$ with domain $\mathbb{R}^1 \times S^1$; and $\omega^R(t)$ is part of the connection matrix $\omega^R(t)$ with $\omega^R(t) = 0$ so that the linear part $\partial_t$ of $\Phi^R$ from the trivial connection is replaced by the covariant derivative from the $(R, t)$-dependent connection given by $\omega^R(t)$ such that on $u_- \ominus_R u_+$, it is still same as $\partial_t$ (see the definition in next section).

The ideas to deal with linear and nonlinear part of $\Phi^R$ are quite different. In this paper we only define the linear part of $\Phi^R$, denoted by $\Phi^R_\parallel = ((\Phi^R_L)_-, (\Phi^R_L)_+)$. Since $(\Phi^R_L)_+ = \partial_t$, its study has its own interests in analysis beyond the need in the Gromov-Witten type theories.

Let $\Psi^R_L = (T^R)^{-1} \circ \Phi^R_\parallel \circ T^R$ be the corresponding operator. Then the main theorem of this paper is the following theorem.

**Theorem 1.1** Using the gluing profile $R = e^{1/r} - e^{1/r_0}$, the filled-section $\Psi^R_L = \{\Psi^R_L^\theta\}$ above with $r \in [0, r_0)$ (hence $R = R_\theta \in (R_0, \infty]$ ) and $\theta \in S^1$ is of class $C^1$.

We now briefly explain the main idea of the definition of $\Psi^R_L$. The main issue here is to make the proper choice of $\Phi^R_{L, -}$ so that the derivatives of $\Psi^R_L$ converge or decay fast enough.

To this end, we need change the matrix of splicing $T_\beta$ used to define the total gluing map $T^R$: instead of using cut-off functions $\beta$ and $1 - \beta$ as entries of $T_\beta$ to define $T^R$ based on partition of unit (at least for $u_- \oplus u_+$ in [1, 2] with the support of $\beta'$ lying on the sub-cylinder of length 1 with the distance
to the boundary circles of \( C_\pm \), a pair of \( R \)-dependent cut-off functions \( \beta^R = (\beta_-^R, \beta_+^R) \) is chosen without the condition that \( \beta_-^R + \beta_+^R = 1 \) in such a way that (i) \( |(\beta_\pm^R)'| \sim o(1/(R^{1/2} \cdot \ln^2 R)) \); (ii) \( (\beta_\pm^R)' \) has the support with length \( \sim 2R^{1/2} \cdot \ln^2 \) with the distance \( \sim (R - 3R^{1/2} \cdot \ln^2) \) to the nearest boundary circle of \( S^R \) so that the splicing matrix \( T_\beta \) has two “splicing regions”, where no-trivial splicing takes place (see the definition in next section).

The purpose of this construction of the splicing matrix is to create a situation such that (i) and (ii) above together has the following effect on the ”unwanted terms” in the derivatives of \( \Psi^R_{-L} \): they become favorable terms in one splicing region but have worse convergent rate in the other. The key step then is to introduce the connection matrix \( \omega^R \) to get rid of these new unwanted terms.

We remark that using the splicing matrix \( T_\beta \) and the connection matrices \( \omega^R(t) \) above, the linear splicing here permits the interpretation as a pair of traveling rank 2 (or complex line) bundles with non-trivial curvatures (see Sec. 2.5). The more details of this and its relation with the corresponding linear splicing in the setting of [2] will be given somewhere else.

Theorem 1.1 will be proved in Sec. 3 and Sec. 4 after the basic definitions of splicing are given in Sec.2.

The nonlinear part of the filled-section in the setting of Banach analysis will be given in [3].

Only elementary Banach analysis is used in this paper which can be found in Lang’s book [4]. In last section, we have collected some of basic analytic facts used in this paper and its companion [3].

## 2 Basic definitions of the splicing

The basic analytic set-up used in this paper and its companion is the space \( L^p_{k,\delta}(C_\pm; C^n) \) of \( L^p_k \)-functions with \( \delta \)-exponential decay. We use such a space as a local model for \( L^p_k \)-maps near a nodal stable map with fixed values at its ends. In order to have the “right” analytic set-up to accommodate the geometric operator \( \Phi \), in particular to have the right dimension of the zero set of \( \Phi^R_{+} \), it is necessary to allow the ends moving; in the case with fixed ends the constrains should be imposed to the image of the total gluing accordingly. However, in this paper and its companion, we will suppress this and related aspects in order to concentrated on the main issues here: defining the filled-section, for which the total gluing, the main construction of this section only
serves as an intermediate step.

The constructions in subsections 2.1 and 2.2 are essentially the same as those in [2].

2.1 Total gluing of the nodal surface \( S \)

We start with the definitions of the total gluing of the given nodal surface \( S \).

For the purpose of this paper, we only need to consider the germ of the given nodal surface, still denoted by \( S = C_− \cup_{d=0} C_+ \) with the double point \( d = d_± \). Thus each component \((C_±, d_±)\) is identified with the standard disk \((D, 0)\) canonically up to a rotation. Identify \((C_−, d_−)\) with \((-∞, 0) \times S^1, -∞ \times S^1\) (\(L_− \times S^1, -∞ \times S^1\)) canonically up to a rotation by considering the double point \( d_− \) as the \( S^1 \) at \(-∞\) of the half cylinder \( L_− \times S^1 \). Here we have denoted the negative half line \((-∞, 0)\) by \( L_− \). Similarly \((C_+, d_+) \simeq ((0, ∞), ) \times S^1, ∞ \times S^1) = (L_+ \times S^1, ∞ \times S^1)\).

- **Cylindrical coordinates on \( C_± \):**

  By the identification \( C_± \simeq L_± \times S^1 \), each \( C_± \) has the cylindrical coordinates \((t±, s±) \in L_± \times S^1\).

  Let \( a = (R, θ) \in [0, ∞) \times S^1 \) be the gluing parameter. To define the total gluing/deformation \( S^a = S^{(R, θ)} \) with gluing parameter \( R \neq ∞ \), we introduce the \( a\)-dependent cylindrical coordinates \((t^{a±, s^{a±}}) \) on \( C_± \) by the formula \( t± = t^{a±} ± R \) and \( s± = s^{a±} ± θ \). In the following if there is no confusion, we will denote \( t^{a±} \) by \( t \) and \( s^{a±} \) by \( s \) for both of these \( a\)-dependent cylindrical coordinates.

  Thus the \( t\)-range for \( L_− \) is \((-∞, R)\) and the \( t\)-range for \( L_+ \) is \((-R, ∞)\) with the intersection \( L_− \cap L_+ = (-R, R)\).

- **Total gluing \( S^a = (S^a_−, S^a_+) \):**

  In term of the \( a\)-dependent cylindrical coordinates \((t, s), C_− = (-∞, R) \times S^1 \) and \( C_+ = (-R, ∞) \times S^1 \).

  Then \( S^a_+ \) is defined to be the finite cylinder of length \( 2R \) obtained by gluing \((-1, R) \times S^1 \subset C_− \) with \((-R, 1) \times S^1 \subset C_+ \) along the ”common” region \((-1, 1) \times S^1 \) by the identity map in term of the \( a\)-dependent coordinates \((t, s)\). Similarly, \( S^a_− \) is the infinite cylinder defined by gluing \((-∞, 1) \times S^1 \subset C_− \) with \((-1, ∞) \times S^1 \subset C_+ \) along \((-1, 1) \times S^1 \) by the identity map.

  Geometrically, both \( S^a_± \) are obtained by first cutting each \( C_± \) along the circle at \( t = 0 \) into two sub-cylinders, then gluing back the sub-cylinders in
with the corresponding ones in $C_+$ along the same circle with a relative rotation of angle $2\theta$. Set $S_\infty = S$.

The total gluing can also be described as follows. We may consider $S = S_\infty = C_- \cup C_+ = L_- \times S^1 \cup L_+ \times S^1$ to be the surface by gluing $C_-$ and $C_+$ along the the circles $\{-\infty\} \times S^1$ and $\{\infty\} \times S^1$ by identity map. Then $S^{\infty, \theta}$ is defined to be the surface still by gluing $C_-$ and $C_+$ along these circles but by a relative rotation of angle $2\theta$. Then $S^a$ is defined to be $S^a = (S^{\infty, \theta})^R$.

Now the cylindrical coordinates $(t_\pm, s_\pm)$ on $C_\pm$ as well as the $a$-dependent cylindrical coordinates $(t, s)$ become the corresponding ones on each $S^a_\pm$ with the relation: $t_\pm = t \pm R$ and $s_\pm = s \pm \theta$.

### 2.2 Splicing matrix by HWZ

The splicing matrix used in [1, 2] will be denoted by $T_\alpha$ which is defined as follows.

Fix a smooth cut-off function $\alpha : \mathbb{R}^1 \to [0, 1]$ with the property that $\alpha(t) = 1$ for $t < -1$, $\alpha(t) = 0$ if $t > 1$ and $\alpha' \leq 0$.

Then

$$T_\alpha = \begin{bmatrix} \alpha & -(1 - \alpha) \\ (1 - \alpha) & \alpha \end{bmatrix}.$$  

Though it is not necessary, $\alpha$ can be chosen such that $\alpha(t) + \alpha(-t) = 1$.

### 2.3 Splicing matrix $T_\beta$

To defined $T_\beta$, we need to choose a length function depending $R$. For the purpose of this paper, the length function $L(R) = L_1(R) = R^{1/2} \cdot \ln^2 R$. In general, for any positive integer $k$, $L_k(R) = R^{k/(k+1)} \cdot \ln^2 R$.

The splicing matrix $T_\beta$ used in this paper is defined by using a new pair of cut-off function $\beta = (\beta_-, \beta_+)$ depending on the two parameters $(l, d)$ that parametrize the group of affine transformations $\{t \to lt + d\}$ of $\mathbb{R}^1$ defined as follows.

Rename $\alpha$ by $\alpha_-$ and $1 - \alpha$ by $\alpha_+$. Fix $l_0 > 1$ and $d_0 > 1$. Then $\beta_\pm = \{\beta_\pm, l_0, d_0\} : \mathbb{R}^1 \times [l_0, \infty) \times [d_0, \infty) \to [0, 1]$ defined by $\beta_\pm(t, l, d) = \alpha_\pm((t \pm d)/l)$, or $\beta_\pm, l_0, d_0 = \rho_l \circ \tau_\pm d \alpha$. Here the translation and multiplication operators are defined by $\tau_d(\xi)(t) = \xi(t + d)$ and $\rho_l(\xi)(t) = \xi(t/l)$ respectively.

The pair $(l, d)$ will be the functions on $R$, $(l = L(R), d = d(R))$ defined above with $d \geq 3l$. To be specific, set $d = 3l$. Clearly $\beta_\pm$ is a smooth cut-off function with the following two properties:
\( P_1 \): for \( k \leq k_0 \) the \( C^0 \)-norm of the \( k \)-th derivative \( \| \beta^{(k)}_\pm \|_{C^0} \leq C/l^k \), where \( C = \| \alpha \|_{C^{k_0}} \), which is independent of \( l \);

\( P_2 \): under the assumption that \( d \geq 3l \), the support of \( \beta'_- \) is contained in the interval \((d-l, d+l)\) with \( \beta_- = 1 \) on \((-\infty, d-l]\) and \( \beta_- = 0 \) on \([d+l, \infty)\); and \( \beta'_+ \) is contained in the interval \((-d-l, -d+l)\) with \( \beta_+ = 1 \) on \([-d+l, \infty)\) and \( \beta_+ = 0 \) on \((-\infty, -d-l]\).

Thus comparing with \( \alpha_- \) and \( \alpha_+ \), whose supports are in the common interval \([-1, 1]\), the supports of \( \beta'_- \) and \( \beta'_+ \) are in the two intervals \([-d-l, -d+l]\) and \([d-l, d+l]\) without overlaps. Those intervals are corresponding to the regions where the splicing takes place. The splicing matrix then is defined by

\[
T_\beta = \begin{bmatrix}
\beta_- & -\beta_+ \\
\beta_+ & \beta_-
\end{bmatrix}.
\]

Note that from \( P_2 \), on \((-d+l, d-l)\), \( \beta_- = \beta_+ = 1 \). Then for \( t \) in the three intervals \((-\infty, -d-l), (-d+l, d-l) \) and \((d+l, \infty)\), \( T_\beta(t) \) are the following constant matrices

\[
M_1 = Id = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, M_2 = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}, \text{ and } M_3 = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.
\]

The length \( 2(l+d) \) of the interval \((-l-d, l+d)\) with \( l = L(R) \) is defined to be the length of the splicing matrix \( T_\beta \).

Note that \( \beta_\pm(t) < 1 \) implies that \( \beta_\pm(t) = 1 \) so that the determinant

\[
1 \leq D = \det \begin{bmatrix}
\beta_- & -\beta_+ \\
\beta_+ & \beta_-
\end{bmatrix} = \beta_-^2 + \beta_+^2 \leq 2.
\]

This implies that \( T^a = (\ominus_a, \oplus_a) \) defined below is invertible uniformly.

### 2.4 Total gluing \( T^a \) of maps and sections

Let \( C_c^\infty(C_\pm, E) \) be the set of \( E \)-valued \( C^\infty \) functions on \( C_\pm \) with compact support, where \( E = \mathbb{C}^n \); similarly for \( C_c^\infty(S^a_\pm, E) \). Note that for the definitions here the surfaces \( C_\pm \) and \( S^a_\pm \) are considered as the cylinders with boundary.

Then \( T^a = (T^a_-, T^a_+) : C_c^\infty(C_-E) \times C_c^\infty(C_+E) \to C_c^\infty(S^a_-E) \times C^\infty(S^a_+, E) \) is defined as follows.
In matrix notation, for each \((\xi_-, \xi_+) \in C^\infty(C_-, E) \times C^\infty(C_+, E)\) considered as a column vector,
\[
T^a((\xi_-, \xi_+)) = (T^a_-(\xi_-, \xi_+), T^a_+(\xi_-, \xi_+)) = (\xi_+ \ominus_a \xi_+, \xi_- \ominus_a \xi_+) = \begin{bmatrix} \beta_- - \beta_+ \\ \beta_+ \beta_- \end{bmatrix} \begin{bmatrix} \tau_{-a}\xi_- \\ \tau_{a}\xi_+ \end{bmatrix}.
\]

The inverse of the total gluing, \((T^a)^{-1} = (T^a_-, T^a_+)^{-1} : C^\infty(S^a_-, E) \times C^\infty(S^a_+, E) \to C^\infty(S^a_-, E) \times C^\infty(S^a_+, E)\) is defined as following: for a pair of the \(E\)-valued functions \((\eta_-, \eta_+) \in C^\infty(S^a_-, E) \times C^\infty(S^a_+, E)\),
\[
(T^a)^{-1}(\eta_-, \eta_+) = (\oplus_a \ominus_a)^{-1}(\eta_-, \eta_+) = \begin{bmatrix} \tau_a & 0 \\ 0 & \tau_{-a} \end{bmatrix} \cdot \frac{1}{D} \cdot \begin{bmatrix} \beta_- & \beta_+ \\ -\beta_+ & \beta_- \end{bmatrix} \begin{bmatrix} \eta_- \\ \eta_+ \end{bmatrix}.
\]

### 2.5 Complex total gluing

Denote \(\tau_{-a}\xi_{\pm}\) in the previous subsection by \(\eta_{\pm}\). Then the above formula for the total gluing can be express in term of complex notations as follows.

Let \(\eta = \eta_- + \eta_+ i\) and \(\beta = \beta_- + \beta_+ i\). Then \(\beta \cdot \eta = (\beta_- \eta_- - \beta_+ \eta_+) + (\beta_- \eta_+ + \beta_+ \eta_-) i\),

which is the same as
\[
\begin{bmatrix} \beta_- & -\beta_+ \\ \beta_+ & \beta_- \end{bmatrix} \begin{bmatrix} \eta_- \\ \eta_+ \end{bmatrix}
\]

by matrix notations.

### 3 The linear part \(\Psi_L\) of the filled-section

The linear part \(\Psi^R_L\) and \(\Psi^a_L\):

Let \(T_\beta\) be the matrix associated to the total gluing map \(T^a\) and

\[
\omega(t) = -\begin{bmatrix} \beta_-^t & -\beta_+^t \\ 0 & 0 \end{bmatrix} \cdot T_\beta^{-1} = -1/D \begin{bmatrix} \beta_-^t \beta_- + \beta_+^t \beta_+ & \beta_-^t \beta_+ - \beta_+^t \beta_- \\ 0 & 0 \end{bmatrix}
\]

denoted by
\[
\begin{bmatrix} e & f \\ 0 & 0 \end{bmatrix}.
\]
Lemma 3.1  For \( R \neq \infty \), \((\Phi^a_L)_+(u_- \oplus u_+) = \partial_t (u_- \oplus u_+)\) as required, and \((\Phi^a_L)_-(u_- \ominus u_+) = \partial_t (u_- \ominus u_+) + \omega(t)(u_- \ominus u_+, u_- \oplus u_+).

Lemma 3.2

\[
\Psi^a_L(u_-, u_+) = \partial_t \begin{bmatrix} u_- \\ u_+ \end{bmatrix} + \left[ \frac{\tau_a(\beta+\beta'_+/D)}{\tau-a} u_- + \frac{\tau_a(\beta+\beta'_+/D)}{\tau-2a} u_- + \frac{\tau_a(\beta-\beta'_-/D)}{\tau-2a} u_- + \frac{\tau_a(\beta-\beta'_-/D)}{\tau-2a} u_- \right] .
\]

There is no loss of differentiability in \( \Psi^a_L(u_-, u_+) \). For \( t \notin (-d-l+1, d+l-1) \), \( \Psi^a_L(u_-, u_+) = \partial_t (u_-, u_+). \)

Proof:

By definition \( \Psi^a_L = (T^a)^{-1} \circ \Phi^a_L \circ T^a \), and

\[
\Psi^a_L(u_-, u_+) = \begin{bmatrix} \tau_a & 0 \\ 0 & \tau-a \end{bmatrix} \cdot T^a_\beta^{-1} \cdot \left( \partial_t + \omega(t) \right) \left\{ T^a_\beta \cdot \begin{bmatrix} \tau_a u_- \\ \tau_a u_+ \end{bmatrix} \right\} \\
= \begin{bmatrix} \tau_a & 0 \\ 0 & \tau-a \end{bmatrix} \cdot T^a_\beta^{-1} \cdot \left\{ \partial_t T^a_\beta \cdot \begin{bmatrix} \tau_a u_- \\ \tau_a u_+ \end{bmatrix} + \omega(t) \cdot T^a_\beta \cdot \begin{bmatrix} \tau_a u_- \\ \tau_a u_+ \end{bmatrix} + T^a_\beta \cdot \begin{bmatrix} \tau_a \partial_t u_- \\ \tau_a \partial_t u_+ \end{bmatrix} \right\} \\
= \begin{bmatrix} \tau_a & 0 \\ 0 & \tau-a \end{bmatrix} \cdot T^a_\beta^{-1} \cdot \left\{ \begin{bmatrix} \beta' - \beta'_+ \\ \beta'_+ \beta'_- \end{bmatrix} \cdot \begin{bmatrix} \tau_a u_- \\ \tau_a u_+ \end{bmatrix} - \begin{bmatrix} \beta' - \beta'_+ \\ \beta'_+ \beta'_- \end{bmatrix} \cdot \begin{bmatrix} \tau_a u_- \\ \tau_a u_+ \end{bmatrix} + T^a_\beta \cdot \begin{bmatrix} \tau_a \partial_t u_- \\ \tau_a \partial_t u_+ \end{bmatrix} \right\} \\
= \partial_t \begin{bmatrix} u_- \\ u_+ \end{bmatrix} + \begin{bmatrix} \tau_a & 0 \\ 0 & \tau-a \end{bmatrix} \cdot T^a_\beta^{-1} \cdot \left\{ \begin{bmatrix} 0 & 0 \\ \beta'_+ \beta'_- \end{bmatrix} \cdot \begin{bmatrix} \tau_a u_- \\ \tau_a u_+ \end{bmatrix} \right\} .
\]
\[
= \partial_t \left[ \begin{array}{c}
u_- \\ u_+ \end{array} \right] + \left[ \begin{array}{cc}
\tau_a & 0 \\
0 & \tau_a \end{array} \right] \cdot \frac{1}{D} \left[ \begin{array}{cc}
\beta_- & \beta_+ \\
-\beta_+ & \beta_- \end{array} \right] \cdot \left[ \begin{array}{cc}
0 & 0 \\
\beta_+ & \beta_- \end{array} \right] \cdot \frac{\tau_a u_-}{\tau_a u_+} \\
\right]
= \partial_t \left[ \begin{array}{c}
u_- \\ u_+ \end{array} \right] + \left[ \begin{array}{cc}
\tau_a & 0 \\
0 & \tau_a \end{array} \right] \cdot \frac{1}{D} \left[ \begin{array}{cc}
\beta_+ & \beta_- \\
-\beta_- & \beta_+ \end{array} \right] \cdot \left[ \begin{array}{cc}
\beta_+ & \beta_- \\
-\beta_- & \beta_+ \end{array} \right] \cdot \frac{\tau_a u_-}{\tau_a u_+} \\
= \partial_t \left[ \begin{array}{c}
u_- \\ u_+ \end{array} \right] + \frac{\tau_a (\beta_+\beta_-/D) u_- + \tau_a (\beta_-\beta_+/D) \tau_2 u_+}{\tau_a (\beta_-\beta_+/D) \tau_2 u_- + \tau_2 (\beta_-\beta_+/D) u_+}.
\]

For \( t \not\in (-d - l + 1, d + l - 1) \), \( \beta_\pm \) is independent of \( t \) so that \( \beta_\pm' = 0 \).

\[
= \partial_t \left[ \begin{array}{c}
u_- \\ u_+ \end{array} \right] + \left[ \begin{array}{cc}
\tau_a (\beta_+\beta_-/D) u_- + \tau_a (\beta_-\beta_+/D) \tau_2 u_+ \\
\tau_a (\beta_-\beta_+/D) \tau_2 u_- + \tau_2 (\beta_-\beta_+/D) u_+ 
\right]
\]

Since the splicing matrix \( \beta \) is \( s \)-independent and the translation operator \( \tau_\theta \) appeared in the total gluing map \( T^a \) commutes with both \( \partial_t \) and \( \partial_s \), \( \tau_\theta \) does not affect analysis here in any essential way. In the most part of the rest of this section we will only give the details for the results using \( T^R \) and state the corresponding ones using \( T^a \).

Denote the error term

\[
\left[ \begin{array}{cc}
\tau_R (\beta_+\beta_-/D) u_- + \tau_R (\beta_-\beta_+/D) \tau_2 u_+ \\
\tau_R (\beta_-\beta_+/D) \tau_2 u_- + \tau_R (\beta_-\beta_+/D) u_+ 
\right]
\]

by \( E^R(u_-, u_+) = (E^R_-(u_-, u_+), E^R_+(u_-, u_+)) \).

Then \( \Psi^R_L(u_-, u_+) = \partial_t(u_-, u_+) + E(u_-, u_+) \).

\* Exponentially weighted \( L^p_k \)-maps/sections

Starting from next lemma, we will consider the spaces of \( L^p_k,\delta \)-maps/sections. Throughout this paper, we assume that \( k \geq 1 \) and \( p > 2 \).

The spaces of such maps used in this paper are defined as follows.

\[
L^p_k,\delta(C_\pm, E) = \{ u_\pm : C_\pm \simeq (\pm R_0, \pm \infty) \times S^1 \to E \mid \| u_\pm \|_{k,p,\delta} < \infty \}.
\]

Here \( \| u_\pm \|_{k,p,\delta} = \| e_\pm \cdot u_\pm \|_{k,p} \) and the weight function \( e_\pm(t, s) = e^{\delta |t|} \) where \( 0 < \delta < 1 \) and 1 (or \( 2\pi \) depending on how to parametrize \( S^1 \)) is the smallest positive eigenvalues of the self-adjoint operator \( i\partial_s \) acting on complex valued functions on \( S^1 \).

Thus for each \( u_\pm \in L^p_k,\delta(C_\pm, E) \) by Sobolev embedding theorem applying to sub-cylinders \([\pm R, \pm R \pm 1] \times S^1 \), there exists a constant \( C \) such that
\[ |u_+(\pm t, s)| \leq C e^{-\delta |\pm t|}. \] In particular \( \lim_{t \to \pm \infty} u_+(\pm t, s) = A_0 = 0 \). In other words, we only consider the case of \( L^p_{k,\delta} \)-maps with fixed end (= 0 of \( E \)) in this paper.

**Lemma 3.3** Considered as a map \( E = \{ E^R, R \in [R^0, \infty] \} : L^p_{k,\delta}(S, E) \times [R^0, \infty) \to L^p_{k-1,\delta}(S, E) \), hence a map \( L^p_{k,\delta}(S, E) \times [0, r_0] \to L^p_{k-1,\delta}(S, E) \), \( E \) is continuous. So is \( \Psi_L : L^p_{k,\delta}(S, E) \times [0, r_0] \to L^p_{k-1,\delta}(S, E) \).

**Proof:**

For \( R \neq \infty \), this is reduced to show that \( F : L^p_{k,\delta}(S, E) \times [0, r_0] \to L^p_{k-1,\delta}(S, E) \) give by \( F(u, R) = u \circ \tau_R \) is continuous. The result is well-known. A proof of this is given in the last section.

Note that for \( R = \infty \), \( \Psi^\infty = \Phi^\infty = \partial_t \) which is consistent with \( \lim_{R \to \infty} \omega^R = 0 \). Hence by definition \( E^\infty = 0 \).

To see the continuity of \( E \) at \( R = \infty \), note that

\[
\|E_+(u, R)\|_{k-1,p,\delta} \leq \|\tau_R(\beta_-\beta'_- / D)\tau_{-2R}u_-\|_{k-1,p,\delta} + \|\tau_R(\beta_-\beta'_- / D)u_+\|_{k-1,p,\delta}
\]

\[
\leq C' \{\|\tau_R\beta'_+\tau_{-2R}u_-\|_{k-1,p,\delta} + \|\beta'_-\|_{C^k-1} \cdot \|\tau_R(\beta_- / D)u_+\|_{k-1,p,\delta}\}
\]

\[
\sim e^{2\delta(-d+l)}\|\beta'_+\|_{C^k-1} \cdot \|u_-\|_{k-1,p,\delta} + 1/L(R)\|u_+\|_{k-1,p,\delta},
\]

which goes to zero as \( R \to \infty \) as long as \( \|u\|_{k,p,\delta} \leq M \) for some fixed \( M >> 0 \). Similarly for \( \|E_-(u, R)\|_{k-1,p,\delta} \) so that \( \|E(u, R)\|_{k-1,p,\delta} \leq 1/L(R)\|u\|_{k,p,\delta} \).

Consequently,

\[
\|E(u, R) - E(u_0, \infty)\|_{k-1,p,\delta} = \|E(u, R)\|_{k-1,p,\delta} \leq 1/L(R)\|u\|_{k,p,\delta}
\]

\[
\leq 1/L(R)\{\|u - u_0\|_{k,p,\delta} + \|u_0\|_{k,p,\delta}\}.
\]

This implies the continuity of \( E \) at \( R = \infty \).

Here we have used the estimate

\[
\|\tau_R\beta'_+\tau_{-2R}u_-\|_{k-1,p,\delta}
\]

\[
\leq e^{2\delta(-d+l)}\|\beta'_+\|_{C^k-1} \cdot \|u_-\|_{k-1,p,\delta}
\]

proved below in this section. \( \square \)
Let $W = W_{k,\delta}^p$ be a small neighborhood of the space of $L_{k,\delta}^p$-maps with the domain $S$ near the initial map $f$. To compute the derivative $(D_W \Psi_R^L)_{(u_-,u_+)}((\xi_-,\xi_+))$ at $(u_-,u_+)$ of $\Psi_R^L$ at $u = (u_-,u_+)$ in the direction $\xi = (\xi_-,\xi_+)$ for $\xi \in T_{(u_-,u_+)}W$, we identify the tangent space $T_{(u_-,u_+)}W = L_{k,\delta}^p(S,u^*TM)$ at $u = (u_-,u_+)$ with $T_{(f_-,f_+)}W = L_{k,\delta}^p(S,f^*TM)$ of the initial map $f$ by the usual trivialization of $TW$. In fact in our case, $W$ is a small neighborhood of the space of $L_{k,\delta}^p$-maps with the domain $S$ near $f$. Since the images of all such maps lying a small neighborhood $U(f(d))$ the double point $f(d)$, which can be identified with a small ball $B \subset E = C^n$, we may consider $W$ as an open subset of $L_{k,\delta}^p(S,E)$ so that $TW \simeq W \times L_{k,\delta}^p(S,E)$. Thus $\xi = (\xi_-,\xi_+)$ can be thought as an element in $L_{k,\delta}^p(S,E)$ independent of $u$.

Then we have

**Lemma 3.4**

$$(D_W \Psi_R^L)_{(u_-,u_+)}((\xi_-,\xi_+)) = \partial_t(\xi_-,\xi_+) + (D_W E)_{(u_-,u_+)}((\xi_-,\xi_+))$$

$$= \partial_t(\xi_-,\xi_+) + (\tau_R(\beta_+\beta'_+/D)\xi_- + \tau_R(\beta_+\beta'_+/D)\tau_2R\xi_+ + \tau_R(\beta_+\beta'_+/D)\tau_-R(\beta_+\beta'_+/D)\xi_+).$$

$$(\partial_R \Psi_R^L)_{(u_-,u_+)} = (\partial_R E)_{(u_-,u_+)} =$$

$$\{\partial_R \tau_R(\beta_+\beta'_+/D)\}u_- + \{\partial_R \tau_R(\beta_+\beta'_+/D)\} \tau_2R u_+ + \tau_R(\beta_+\beta'_+/D) \partial_R \tau_2R u_+,$$

$$\{\partial_R \tau_-R(\beta_+\beta'_+/D)\} \tau_-2R u_- + \tau_-R(\beta_+\beta'_+/D) \partial_R \tau_-2R u_- + \{\partial_R \tau_-R(\beta_+\beta'_+/D)\} u_+).$$

$$(\partial_\theta \Psi_R^L)_{(u_-,u_+)} = (\partial_\theta E)_{(u_-,u_+)} =$$

$$(\tau_R(\beta_+\beta'_+/D) \partial_\theta \tau_2R u_+ + \tau_-R(\beta_+\beta'_+/D) \partial_\theta \tau_-2R u_-).$$

Here $\pm R_\theta = (\pm R, \pm \theta)$ and the action of $\tau_{\pm R_\theta}$ on $[0,\pm \infty) \times S^1$ is $(t,s) \to (t \pm R, s \pm \theta)$. We will use this kind of notations in the rest of the paper.

The main theorem is

**Theorem 3.1** The section $\Psi_L$ is of class $C^1$. 

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The proof of the theorem will be divided into two parts according to if $R = \infty$.

In order to prove next lemma we list a few general facts that will be used repeatedly:

(A) \( F_+ : W \times [0, \infty) \to L(L^p_{k,\delta}(C_+, E), L^p_{k-1,\delta}(C_+, E)) \) defined by \( F(u, R)(\xi) = \tau_R \xi \) is continuous. There is a corresponding statement for the function \( F_- \).

(B) Any smooth function \( f \) such as \( f = \beta'_\pm : C_\pm \to \mathbb{R}^1 \) gives rise a \( C^\infty \)-map \( F_\pm : \mathbb{R}^1 \to C^m(C_\pm, \mathbb{R}^1) \) defined by \( F_\pm(R) = f \circ \tau_R \) for any \( m \). In particular, we may assume that \( m >> k \).

(C) The paring \( L_k^p(C_\pm, E) \times L(L_k^p(C_\pm, E), L^p_{k,\delta}(C_\pm, E)) \to L_k^p(C_\pm, E) \) is bounded bilinear and hence smooth as long as the space \( L_k^p(C_\pm, E) \) forms Banach algebra.

(D) For \( m >> k \), \( L(L_k^p(C_\pm, E), L^p_{k,\delta}(C_\pm, E)) \) is a \( C^m(C_\pm, \mathbb{R}^1) \)-module and the multiplication map

\[
C^m(C_\pm, \mathbb{R}^1) \times L(L_k^p(C_\pm, E), L^p_{k,\delta}(C_\pm, E)) \to L(L_k^p(C_\pm, E), L^p_{k,\delta}(C_\pm, E))
\]

is bounded bilinear and hence smooth.

The proofs for (B) and (D) are straightforward and (C) is stated in the first chapter of Lang’s book [4]. The property (A) was proved in the last section of this paper.

**Lemma 3.5** For \( R \neq \infty \), \( \Psi_L \) is of class \( C^1 \). Moreover for \( \Psi_L = M + E \) with \( M = \partial_t \), \( D_WM \) is continuous as a map \( D_WM : W \times (R_0, \infty] \to L(L_k^p(S, E), L^p_{k-1,\delta}(S, E)) \). Here \( W \subset L_k^p(S, E) \) is a small neighborhood of the initial map \( f \) in \( L_k^p(S, E) \).

**Proof:**

The second statement is clear since \( M = \partial_t \) is linear and bounded so that \( D_WM_{(u_-, u_+, R)}(\xi, \xi) = \partial_t(\xi, \xi) \). Then considered as a map, \( D_WM : W \times (R_0, \infty] \to L(L_k^p(S, E), L^p_{k-1,\delta}(S, E)) \) is a constant map, hence continuous. Here \( L(L_k^p(S, E), L^p_{k-1,\delta}(S, E)) \) is the space of bounded, hence continuous linear maps between the Banach spaces \( L_k^p(S, E) \) and \( L^p_{k-1,\delta}(S, E) \) under the operator norm.

By the formulas for \( D\Psi_L \) above and the properties listed above, the first statement can be proved by a modified version of (A) stated in Lemma 4.4. Note that the second component of the error term of \( (D_WM_{(u_-, u_+)}(\xi, \xi) \) is
\[ \tau_R(\beta_\beta'/D)\tau_{-2R}\xi_{-} + \tau_R(\beta_\beta'/D)\xi_{+} \]
with the domain with \(t\)-range \([0, \infty)\) (in the natural coordinate \((t_+, s_+)\)). We only need to deal with the first term \(\tau_R(\beta_\beta'/D)\tau_{-2R}\xi_{-}\). Then the \(t\)-range of \(\xi_{-}\) is \((-\infty, 0]\) so that \(\tau_{-2R}\xi_{-} = \xi \circ \tau_{-2R}\), which is supposed to have positive \(t\)-range of the domain, can only defined on \([0, 2R]\). However, since the support of \(\beta_\beta'\) is \([-d - l, -d + l]\) the support of \(\tau_{-R}\beta_\beta' = \beta_\beta' \circ \tau_{-R}\) is \([R - d - l, R - d + l]\) so that \(\tau_{-R}(\beta_\beta'/D)\tau_{-2R}\xi_{-}\) becomes a well-defined function on \([0, \infty) \times S^1\). Thus we consider the function \(F : [0, \infty) \rightarrow L(L_{k,\delta}^p(C_-, E), L_{k-1,\delta}^p(C_+, E))\) defined by \(F(R)(\xi) = \tau_{-R}\beta_\beta' \cdot \tau_{-2R}\xi_{-}\). This is a special case considered in Lemma 4.4 with \(f_R\) there being \(\beta_\beta'\) so that \(F\) is continuous by Lemma 4.4.

\[ \square \]

Next we need to show that \(DE\) and hence \(D\Psi_L\) can be extended continuously over \(R = \infty\). This will be established by the estimates in the rest of this section.

For \(E_R^R(u_-, u_+) = (E_R^R(u_-, u_+), E_R^R(u_-, u_+))\), the two components are of the same natural. We will only consider \(E_R^R = (\tau_{-R}\beta_\beta'/D)\tau_{-2Ru_} - \tau_{-R}(\beta_\beta'/D)u_+)\).

The second term \(\tau_{-R}(\beta_\beta'/D)u_+\) without involving the actions of the translation operators behaves as expected. The main technique reason for intruding the connection matrix \(\omega\) is to get a well-behaved term \(\tau_{-R}(\beta_\beta'/D)\tau_{-2Ru_}\) here.

**Lemma 3.6** On the interval \((R - d - l, R - d + l)\) of length \(2l\) where \(\tau_{-R}\beta_\beta' \neq 0\), the weight function \(e(t)\) of \(\tau_{-2Ru_}\) satisfies the bounds \(e^{\delta(R - d - l)} \leq e(t) \leq e^{\delta(R - d + l)}\) so that

\[ \|\tau_{-R}\beta_\beta' \tau_{-2Ru_}\|_{k-1,p,\delta} \]

\[ \leq e^{2\delta(-d + l)}\|\beta_\beta'\|_{C_{k-1}} \cdot \|u_\|_{k-1,p,\delta}. \]

Similarly,

\[ \|\tau_{-R}\beta_\beta' \partial_R(\tau_{-2Ru_})\|_{k-1,p,\delta} = \|\tau_{-R}\beta_\beta' \tau_{-2Ru_}\|_{k-1,p,\delta} \]

\[ \leq e^{2\delta(-d + l)}\|\beta_\beta'\|_{C_{k-1}} \cdot \|u_\|_{k,p,\delta}. \]

and

\[ \|\tau_{-R}\beta_\beta'(\tau_{-2R}\xi_{-})\|_{k-1,p,\delta} \]

\[ \leq e^{2\delta(-d + l)}\|\beta_\beta'\|_{C_{k-1}} \cdot \|\xi_{-}\|_{k-1,p,\delta}. \]
Proof:

\[
\| \tau_R \beta'_+ \tau_{-2R}u_- \|_{k-1,p,\delta} = \| (\tau_R \beta'_+ \tau_{-2R}u_-) (R-d-l, R-d+l) \|_{k-1,p,\delta}
\]
\[
\leq e^{\delta(R-d+l)} \| \beta'_+ \|_{C^{k-1}} \cdot \| u_- (t - 2R) \|_{t \in (R-d-l, R-d+l)} \|_{k-1,p}
\]
\[
\leq e^{\delta(R-d+l)} \| \beta'_+ \|_{C^{k-1}} \cdot \| u_-(v) \|_{v \in (-R-d-l, -R-d+l)} \|_{k-1,p}.
\]
\[
\leq e^{\delta(R-d+l)} \cdot e^{(\delta(-R-d+l))} \| \beta'_+ \|_{C^{k-1}} \cdot \| u_- \|_{k-1,p,\delta}.
\]

\[\square\]

**Lemma 3.7** The interval where \( \partial_R \tau_R (\beta_- \beta'_+ / D) \tau_{-2R}u_- \neq 0 \) is the same \((R-d-l, R-d+l)\) so that

\[
\| \{ \partial_R \tau_R (\beta_- \beta'_+ / D) \} \tau_{-2R}u_- \|_{k-1,p,\delta}
\]
\[
\leq e^{2\delta(-d+l)} \| \beta'_+ \|_{C^{k-1}} \cdot \| u_- \|_{k-1,p,\delta}.
\]

**Proof:**

By the proof of lemma before, we only need to prove the first statement.

\[
\partial_R \tau_R (\beta_- \beta'_+ / D) = \partial_R \{ (\beta_- \beta'_+ / D) \circ \tau_R \} = \partial_R \{ (\beta_- \beta'_+ / D) (t - R) \}
\]
\[
= -((\beta_- \beta'_+ / D)' \circ \tau_R = -( (\beta_- / D)' \beta'_+ + (\beta_- / D) \beta''_+) \circ \tau_R
\]
\[
= \tau_R (\beta_- / D)' \tau_R \beta'_+ + \tau_R (\beta_- / D) \cdot \tau_R \beta''_+.
\]

Since outside \((R-d-l, R-d+l)\), \( \tau_R \beta'_+ = \tau_R \beta''_+ = 0 \), so is \( \partial_R \tau_R (\beta_- \beta'_+ / D) \). \(\square\)

**Lemma 3.8** \( \| (D_W \Psi_L)_{+}(u_-, u_+, R) - \partial_t \|_o \to 0 \) as \( R \to \infty \) uniformly in \( u \).

**Proof:**

\[
\| (D_W \Psi_L)_{+}(u_-, u_+, R) - \partial_t \|_o = \sup_{\| \xi \|_{k,p,\delta} \leq 1} \| (D_W \Psi_L)_{+}(u_-, u_+, R)(\xi) - \partial_t (\xi_+) \|_{k-1,p,\delta}
\]

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Lemma 3.9
\[ \| \tau_{-R}(\beta_-\beta'_+/D)\tau_{-2R}\xi_- \|_{k-1,p,\delta} + \| \tau_{-R}(\beta_-\beta'_-/D)\xi_+ \|_{k-1,p,\delta} \]
\[ \leq \sup_{\|\xi\|_{k,p,\delta} \leq 1} \{ \| \beta_-/D \|_{C^{k-1}} \| \tau_{-R}(\beta'_+)/\tau_{-2R}\xi_- \|_{k-1,p,\delta} + \| \beta_-/D \|_{C^{k-1}} \| \xi_+ \|_{k-1,p,\delta} \} \]
\[ \leq \sup_{\|\xi\|_{k,p,\delta} \leq 1} \{ \| \beta_-/D \|_{C^{k-1}} \| \beta'_+/\beta'_+ \|_{C^{k-1}} \| \xi_- \|_{k-1,p,\delta} + \| \beta_-/D \|_{C^{k-1}} \| \xi_+ \|_{k-1,p,\delta} \} \]

Now the key point is
\[ \| \beta_-\beta'_-/D \|_{C^{k-1}} \sim \| \beta_-\beta'/D \|_{C^0} \sim \| \beta_-/D \|_{C^0} \| \beta'_- \|_{C^0} \sim 1/L(R) \rightarrow 0 \]
as \( R \rightarrow \infty \).

The first "\( \sim \)" above can be proved inductively. Indeed since
\[
\left( \frac{\beta_-\beta'_-/D}{D} \right)' = \left\{ \left( \frac{(\beta_-')^2 + \beta_-\beta''_+}{\beta_-\beta'_-} \right) - \left( \frac{\beta_-\beta'_-}{D} \right)' \right\}/D^2
\]
\[
= \left\{ \left( \frac{(\beta_-')^2 + \beta_-\beta''_+}{\beta_-\beta'_-} \right) - 2(\beta_-\beta'_-)(\beta_-\beta'_+ + \beta_-\beta'_+) \right\}/D^2,
\]
it is easy to see that for any \( i > 0 \), each term of \( \nabla^i\{(\beta_-\beta'_-/D)\} \) is a product of the terms of the form containing at least one of \( \beta'_- \) and \( \beta'_+ \), or their derivatives. Hence \( \| \{ \{(\beta_-\beta'_-/D)\} \|_{C^0} \) is the lowest order term in \( 1/L(R) \) inside \( \| \{ (\beta_-\beta'_-/D) \|_{C^{k-1}} \)

This proves that
\[ \| (D_W \Psi_L^+)(u_-, u_+, R) - \partial_t \|_0 \leq \| \beta_-/D \|_{C^{k-1}} \{ e^{2\delta(-d+1)} \| \beta'_+ \|_{C^{k-1}} + \| \beta_-/D \|_{C^0} \} \sim 1/L(R) \rightarrow 0 \]
as \( R \rightarrow \infty \).

\[ \square \]

Lemma 3.9
\[ \| (\partial_R(\Psi_L^R)+)(u_-, u_+) \|_{k-1,p,\delta} \sim 1/(\ln^2 R) \{ \| u_- \|_{k,p,\delta} + \| u_+ \|_{k,p,\delta} \} \]

Proof:
\[ \| (\partial_R(\Psi_L^R)+)(u_-, u_+) \|_{k-1,p,\delta} = \| (\partial_R E_+)(u_-, u_+) \|_{k-1,p,\delta} \leq \]

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\[
\| \{ \partial_R \tau_R (\beta_- \beta'_+ / D) \} \tau_{-2} R u_- \|_{k-1,p,\delta} + \| \tau_R (\beta_- \beta'_+ / D) \partial_R \tau_{-2} R u_- \|_{k-1,p,\delta} + \\
\| \{ \partial_R \tau_R (\beta_- \beta'_- / D) u_+ \} \|_{k-1,p,\delta}
\]

\[
\leq e^{2\delta(-d+l)} \cdot \| (\beta_- \beta'_+ / D) \|_{C^k} \| u_- \|_{k-1,p,\delta} + e^{2\delta(-d+l)} \cdot \| \beta_- / D \|_{C^{k-1}} \| \beta'_+ \|_{C^{k-1}} \cdot \| u_- \|_{k,p,\delta}
\]

\[
+ \| \{ \partial_R \tau_R (\beta_- \beta'_- / D) \} \|_{C^{k-1}} \cdot \| u_+ \|_{k-1,p,\delta}
\]

\[
\leq e^{2\delta(-d+l)} \cdot C(\| \beta \|_{C^{k+1}}) \| u_- \|_{k,p,\delta} + \| (\beta_- \beta'_- / D) \| \tau_R \|_{C^{k-1}} \cdot \| u_+ \|_{k-1,p,\delta}
\]

Now \| (\beta_- \beta'_- / D) \|_{C^{k-1}} \sim \| (\beta_- \beta'_- / D) \|_{C^0} \] and the key point is

\[
\| (\beta_- \beta'_- / D) \| \tau_R \|_{C^{k-1}} = \| (\beta_- \beta'_- / D) \|_{C^k-1}
\]

\[
\sim \| (\beta_- \beta'_- / D) \|_{C^0} \sim \| (\beta'_- \|_{D} / D^2) \|_{C^0}
\]

\[
\sim \| (\beta'_- \|_{D} / D^2) \|_{C^0}
\]

\[
\sim 1/L(R)^2 = 1/(R\{\ln R\})^4;
\]

This first statement can be proved inductively. Indeed it is easy to see from above that for any \( i > 0 \), each term of \( \nabla^i \{ (\beta_- \beta'_- / D) \} \) is a product of the terms containing at least one of the terms (\( \beta'_- \), \( \beta''_- \), \( \beta'_- \beta'_+ \) or their derivatives so that for \( i > 0 \), \| \nabla^i \{ (\beta_- / D') \} \|_{C^0} \) is a lower order term comparing with \| (\beta_- \beta'_- / D) \|_{C^0} \].

Thus

\[
\| (\partial_R (\Psi_L^R)_{+}(u_-,u_+)) \|_{k-1,p,\delta} \sim 1/(R\{\ln R\})^4 \| u_- \|_{k,p,\delta} + \| u_+ \|_{k,p,\delta}
\]

so that

\[
\| (\partial_r (\Psi_L^R)_{+}(u_-,u_+)) \|_{k-1,p,\delta} = \| (\partial_R (\Psi_L^R)_{+}(u_-,u_+)) \|_{k-1,p,\delta} dR/dr
\]

\[
\sim 1/(R\{\ln R\})^4 \| u_- \|_{k,p,\delta} + \| u_+ \|_{k,p,\delta} \cdot dR/dr
\]

\[
= 1/(R\{\ln R\})^4 \| u_- \|_{k,p,\delta} + \| u_+ \|_{k,p,\delta} R (\ln R)^2 = 1/(\ln^2 R) \| u_- \|_{k,p,\delta} + \| u_+ \|_{k,p,\delta}
\]

since \( R = e^{1/r} \) and \( \frac{dR}{dr} = \frac{de^{1/r}}{dr} = e^{1/r} \cdot (-1/r^2) = -R \cdot (\ln R)^2 \).
Lemma 3.10

$$(\partial_\theta (\Psi^R_{L+})_{(u_- u_+)} \sim C e^{\delta(-2d+2l)} \cdot \| u_- \|_{k,p,\delta}.$$  

Hence

$$1/r (\partial_\theta (\Psi^R_{L+})_{(u_- u_+)} << e^{-1/r} \| u_- \|_{k,p,\delta}\)$$

Proof:

$$(\partial_\theta (\Psi^R_{L+})_{(u_- u_+)} = \| (\partial_\theta E)_{(u_- u_+)} \|_{k-1,p,\delta} =$$

$$\| \tau_{-R_\theta}(\beta_{-\beta_+}' / D) \partial_\theta \tau_{-2R_\theta} u_- \|_{k-1,p,\delta}.$$  

Recall that $\tau_{-R_\theta}(\beta_{-\beta_+}' / D) \neq 0$ only when $t \in (R - d - l, R - d + l)$ and on this interval the weight function $e(t)$ of $\tau_{-2R} u_-$ satisfies the bounds $e^\delta(R-d-l) \leq e(t) \leq e^\delta(R-d+l).$

Thus on this interval

$$\| \tau_{-R_\theta}(\beta_{-\beta_+}' / D) \partial_\theta \tau_{-2R_\theta} u_- \|_{k-1,p,\delta} \sim C \| \partial_\theta \tau_{-2R_\theta} u_- |_{[-R-d-l, R-d+l]} \|_{k-1,p,\delta}$$

$$\sim C e^{\delta(R-d+l)} \cdot \| \partial_\delta u_-(t - 2R, s - 2\theta) |_{[-R-d-l, R-d+l]} \|_{k-1,p}$$

$$= C e^{\delta(R-d+l)} \cdot \| \partial_\delta u_-(t, s) |_{[-R-d-l, R-d+l]} \|_{k-1,p}$$

$$= C e^{\delta(R-d+l)} \cdot e^\delta(-R-d+l) \cdot e(-t) \partial \delta u_-(t, s) |_{[-R-d-l, R-d+l]} \|_{k-1,p}$$

$$= C e^{\delta(-2d+2l)} \cdot \| \partial \delta u_- |_{k-1,p,\delta} = C e^{\delta(-2d+2l)} \cdot \| u_- \|_{k,p,\delta}.$$  

Hence

$$1/r (\partial_\theta (\Psi^R_{L+})_{(u_- u_+)} \sim C \ln R \cdot e^{\delta(-d)} \cdot \| u_- \|_{k,p,\delta}.$$  

Recall $d \sim l \sim R^{1/2}$ so that $\ln R \cdot e^{\delta(-d)} \| u_- \|_{k,p,\delta} \sim e^{-\delta R^{1/2}} \| u_- \|_{k,p,\delta} << 1/R \| u_- \|_{k,p,\delta} = e^{-1/r} \| u_- \|_{k,p,\delta}.$  

\[ \square \]

Proposition 3.1 Away from $r = 0$, the partial derivatives $\partial_x \Psi_L$ and $\partial_y \Psi_L$ exits and continuous, and they can be extended continuously over $r = 0$. 
Proof:
This follows from the estimates established so far together with the following elementary formula:

\[
\begin{bmatrix}
\frac{\partial}{\partial x} \\
\frac{\partial}{\partial y}
\end{bmatrix} = \begin{bmatrix}
\cos \theta & -\sin \theta \\
\sin \theta & \cos \theta
\end{bmatrix} \begin{bmatrix}
\frac{\partial}{\partial r} \\
1/r \cdot \frac{\partial}{\partial \theta}
\end{bmatrix}.
\]

Indeed, by this formula \( \partial_x \Psi_L : L^p_k(C_-, E) \times L^p_k(C_+, E) \times D^*_r \to L^p_{k-1, \delta}(C_-, E) \times L^p_{k-1, \delta}(C_+, E) \) is given by

\[
\partial_x \Psi_L(u, (x, y)) = \partial_x \Psi_L(u, (R, \theta)) = \cos \theta \partial_r \Psi_L(u, (R, \theta)) - \sin \theta \cdot 1/r \partial_\theta \Psi_L(u, (R, \theta)).
\]

The proof for the case \( k = 1 \) is clear. For the general \( k \), note that For \( i + j \leq k - 1 \),

\[
D^i_s D^j_t \{ \partial_x \Psi_L(u, (R, \theta)) \}
\]

\[
= \cos \theta \{ D^i_s D^j_t \partial_r \Psi_L(u, (R, \theta)) \} - \sin \theta \cdot \{ D^i_s D^j_t 1/r \partial_\theta \Psi_L(u, (R, \theta)) \}
\]

so that

\[
\| \partial_x \Psi_L(u, (R, \theta)) \|_{k-1, p, \delta}
\]

\[
\leq \| \partial_r \Psi_L(u, (R, \theta)) \|_{k-1, p, \delta} + \|1/r \partial_\theta \Psi_L(u, (R, \theta)) \|_{k-1, p, \delta}.
\]

Then the conclusion follows.

\[\square\]

4 Some basic estimates

In this last section, we collect some basic estimates used in paper first, then use them to finish the proof of the main theorem of this paper. The proofs are only given in \( L^p \)-norm with \( p > 1 \). The corresponding results for \( L^p_k \)-norm can be derived from this by replacing a \( L^p_k \)-function \( f \) by a \( L^p \)-function \( (f, Df, \cdots, D^k f) \). For the purpose of this paper, we assume in addition that \( 1 - 2/p > 0 \).

The following inequality will be used repeatedly.

**Lemma 4.1** Let \( F(x, t) \) be a smooth function for \((x, t) \in \Sigma \times [0, 1] \) with compact support. Then \( \int_{\Sigma} \left| \int_{[0,1]} F(x, t) dt \right|^p dvol_{\Sigma} \leq \int_{[0,1]} \int_{\Sigma} |F(x, t)|^p dvol_{\Sigma} dt. \)
Proof:

Since the functions $f_1(x) = |x|^p$ is convex, for each fixed $x$ and $a$ and $b$, $|tF(x, a) + (1-t)F(x, b)|^p \leq t|F(x, a)|^p + (1-t)|F(x, b)|^p$ so that $|\sum_{i=1}^n F(x, i/n)|^p \leq \sum_{i=1}^n |F(x, i/n)|^p / n$.

Hence

$$\left| \int_{[0,1]} F(x, t) dt \right|^p \leq \int_{[0,1]} |F(x, t)|^p dt$$

and

$$\int_{\Sigma} \left| \int_{[0,1]} F(x, t) dt \right|^p dvol_{\Sigma} \leq \int_{[0,1]} |F(x, t)|^p dt dvol_{\Sigma} \leq \int_{[0,1]} \int_{\Sigma} |F(x, t)|^p dvol_{\Sigma} dt.$$

□

In our case, $\Sigma = \mathbb{R}^1 \times S^1$ which is not compact. Then the set of smooth functions with compact support is dense in the space of $L_k^p$ or $L_{k,\delta}^p$-functions so that above is applicable.

Lemma 4.2

$$\|\tau_R \xi\|_{0,p,\delta} = e^{-\delta R} \|\xi\|_{0,p,\delta}$$

for $\xi : [0, \infty) \to E$.

Proof:

$$\|\tau_R \xi\|_{0,p,\delta} = \|\xi(t + R)\|_{0,p,\delta} = \|e(t) \xi(t + R)\|_{0,p} = e^{-\delta R} \|e(u) \xi(u)\|_{u \in [R, \infty]} \|_{0,p}$$

$$\leq e^{-\delta R} \|\xi\|_{0,p,\delta}.$$

□

Let $F : L_{0,\delta}^p(C_+, E) \times [0, \infty) \to L_{0,\delta}^p(C_+, E)$ defined by $F(\xi, R) = \xi \circ \tau_R$. It is proved blow in this section that $\partial R F = \xi' \circ \tau_R$. Denote $\partial R F$ by $G$ and $\xi'$ by $\eta$.

Consider $G : L_{0,\delta}^p(C_+, E) \times [0, \infty) \to L_{0,\delta}^p(C_+, E)$, $G(\eta, R) = \eta \circ \phi_R$.

Above functions $F$ and $G$ are repeatedly used in this paper.

Lemma 4.3 The function $G$ is continuous.
Proof:

\[ \|G(\eta_1, R_1) - G(\eta_2, R_2)\| \leq \|G(\eta_1, R_1) - G(\eta_1, R_2)\| + \|G(\eta_1, R_2) - G(\eta_2, R_2)\| \]

\[ \leq \|\eta_1 \circ \tau_{R_1} - \eta_1 \circ \tau_{R_2}\| + \|\eta_1 \circ \tau_{R_2} - \eta_2 \circ \tau_{R_2}\| \]

\[ \leq \|\eta_1 \circ \tau_{R_1} - \hat{\eta} \circ \tau_{R_1}\| + \|\hat{\eta} \circ \tau_{R_1} - \hat{\eta} \circ \tau_{R_2}\| + \|\hat{\eta} \circ \tau_{R_2} - \eta_1 \circ \tau_{R_2}\| + \|\eta_1 - \hat{\eta}\| \circ \tau_{R_2}\| \]

\[ \leq (e^{-\delta R_1} + e^{-\delta R_2})(\|\eta_1 - \tilde{\eta}\| + \|\eta_1 - \eta_2\|) + \|\hat{\eta} \circ \tau_{R_1} - \hat{\eta} \circ \tau_{R_2}\| . \]

The last term

\[ \|\hat{\eta} \circ \tau_{R_1} - \hat{\eta} \circ \tau_{R_2}\| \]

\[ = \|\hat{\eta} \circ \tau_{R_1} - \hat{\eta} \circ \tau_{R_2}\|_{0, p, \delta} = |R_2 - R_1| \cdot \| \hat{\eta}'((1-s)(t+R_1) + s(t+R_2))ds\|_{0, p, \delta} \]

\[ \leq |R_2 - R_1| \cdot \int_{[0,1]} \| \hat{\eta}'(t + (1-s)R_1 + sR_2)\|_{0, p, \delta} ds \]

\[ \leq |R_2 - R_1| \cdot \| \hat{\eta}'\|_{0, p, \delta} \cdot \int_{[0,1]} e^{-\delta((1-s)R_1 + sR_2)} ds \]

\[ = |R_2 - R_1| \cdot \{ e^{-\delta R_2} - e^{-\delta R_1} \} / \{ \delta(R_2 - R_1) \} \cdot \| \hat{\eta}'\|_{0, p, \delta} = e^{-\delta R_2} - e^{-\delta R_1} / \delta \cdot \| \hat{\eta}'\|_{0, p, \delta} \]

\[ = e^{-\delta R_2} - e^{-\delta R_1} / \delta \cdot \| \hat{\eta}'\|_{1, p, \delta} . \]

Given any \( \epsilon > 0 \), we may chose a smooth \( \hat{\eta} \) with compact support such that \( \| \hat{\eta} - \eta_1 \|_{0, p, \delta} < \epsilon \). Then above estimate proves the continuity of \( G \). \( \Box \)

For the function \( F(u, R) = u \circ \tau_R \) with \( u \in L^p_{1, \delta}(C_+, E) \) above, it is proved below that \( \partial_u F = \partial_1 F : L^p_{1, \delta}(C_+, E) \times \mathbb{R}^1 \rightarrow L(L^p_{1, \delta}(C_+, E), L^p_{0, \delta}(C_+, E)) \) given by \( (\partial_1 F)(u, R)(\eta) = \eta \circ \tau_R \). It is \( u \)-independent, and hence becomes a map, denoted by \( D : \mathbb{R}^1 \rightarrow L(L^p_{1, \delta}(C_+, E), L^p_{0, \delta}(C_+, E)) \).

For the applications in this paper and its companion, we need a few functions similar to above. We define the following function that is general enough for these applications.

Let \( H : \mathbb{R}^1 \rightarrow L(L^p_{1, \delta}(C_-, E), L^p_{0, \delta}(C_+, E)), \) given by \( H(R)(\xi) = \tau_{-R} f_R \tau_{-2R} \xi, \) where \( f_R : \mathbb{R}^1 \rightarrow \mathbb{R}^4 \) is a \( C^\infty \)-function smoothly depending on \( R \in [R_0, \infty) \) such that (i) the support of \( f_R \) is in \( [A(R), B(R)] = [a_1 d(R) + a_2 l(R), b_1(R) + b_2 l(R)] \) with \( A(R) < B(R) \) and \( |A(R)|, |B(R)| << R; \) (ii) For \( i \geq 0, \| \partial_1^i f_R \|_{C^k} < C_k \) independent of \( R \). Here \( \xi \in L^p_{1, \delta}(C_-, E) \) with \( C_- = (-\infty, 0) \times S^1 \). Hence the \( t \)-range of \( \xi \) is \( (-\infty, 0] \) so that \( \tau_{-2R} \xi = \xi \circ \tau_{-2R} \) can only have positive
Lemma 4.4 Let \( D : [0, \infty) \to L(\mathbb{L}_{1,p,\delta}(C_+, E), \mathbb{L}_{0,p,\delta}(C_+, E)) \) and \( H : [0, \infty) \to L(\mathbb{L}_{1,p,\delta}(C_-, E), \mathbb{L}_{0,p,\delta}(C_+, E)) \) be the functions defined above.

Then
\[
\| D(R_1) - D(R_2) \|_o = \sup_{\| \xi \|_{1,p,\delta} \leq 1} \| \tau_{-R_1} \xi - \tau_{-R_2} \xi \|_{0,p,\delta} \leq |e^{-\delta R_1} - e^{-\delta R_2}|/\delta.
\]

and
\[
\| H(R_1) - H(R_2) \|_o \leq C_0 \{ |R_1 - R_2| e^{2\delta |A(R_1) + B(R_1)|} + |e^{2\delta R_1} - e^{2\delta R_2}|/\delta \cdot e^{-\delta R_1} \}.
\]

Hence \( D \) and \( H \) are continuous. Here we denote the operator norms by \( \| \cdot \|_o \).

Proof:

We only prove the lemma for \( H \) since the proof for \( D \) is easier.

\[
\| H(R_1) - H(R_2) \|_o = \sup_{\| \xi \|_{1,p,\delta} \leq 1} \| \tau_{-R_1} f_{R_1} \tau_{-2R_1} \xi - \tau_{-R_2} f_{R_2} \tau_{-2R_2} \xi \|_{0,p,\delta}
\]

\[
= \sup_{\| \xi \|_{1,p,\delta} \leq 1} \| \tau_{-R_1} f_{R_1} \tau_{-2R_1} \hat{\xi} - \tau_{-R_2} f_{R_2} \tau_{-2R_2} \hat{\xi} \|_{0,p,\delta}
\]

\[
= \sup_{\| \hat{\xi} \|_{1,p,\delta} \leq 1} \| \tau_{R_1} f_{R_1} \hat{\xi} \circ \tau_{-2R_1} - [\tau_{R_2} f_{R_2} \hat{\xi}] \circ \tau_{-2R_2} \|_{0,p,\delta}.
\]

with \( \hat{\xi} \) being smooth with compact support with domain containing in \( C_- = (-\infty, 0] \times S^1 \).

Then
\[
\| \tau_{R_1} f_{R_1} \hat{\xi} \circ \tau_{-2R_1} - [\tau_{R_2} f_{R_2} \hat{\xi}] \circ \tau_{-2R_2} \|_{0,p,\delta}
\]

\[
\leq \| \tau_{R_1} f_{R_1} \hat{\xi} \circ \tau_{-2R_1} - [\tau_{R_1} f_{R_1} \hat{\xi}] \circ \tau_{-2R_2} \|_{0,p,\delta} + \| [\tau_{R_1} f_{R_1} \hat{\xi}] \circ \tau_{-2R_2} - [\tau_{R_2} f_{R_2} \hat{\xi}] \circ \tau_{-2R_2} \|_{0,p,\delta}.
\]

Here the \( L^p_{0,\delta} \)-norm is taken over \( [0, \infty) \times S^1 \).

Assume that \( R_1 \leq R_2 \) with \( |R_2 - R_1| \leq 1 \).
Then a similar estimate to the proof of the last lemma gives

\[ \| [\tau_{R_1} f R_1 \hat{\xi}] \circ \tau_{-2R_1} - [\tau_{R_1} f R_1 \hat{\xi}] \circ \tau_{-2R_2} \|_{0,p,\delta} \]

\[ \leq 2 |R_2 - R_1| \cdot \int_{[0,1]} \| [\tau_{R_1} f R_1 \hat{\xi}]'((1 - s)(t - 2R_1) + s(t - 2R_2))ds \|_{0,p,\delta} \]

\[ \leq 2 |R_2 - R_1| \cdot \int_{[0,1]} \| [\tau_{R_1} f R_1 \hat{\xi}]'(t - 2(1 - s)R_1 - 2sR_2) \|_{0,p,\delta}ds \]

\[ \leq 2 |R_2 - R_1| \cdot \int_{[0,1]} \| e^{\delta t}[\tau_{R_1} f R_1 \hat{\xi}]'(t - 2(1 - s)R_1 - 2sR_2) \|_{0,p}ds \]

\[ ( \text{with } u = t - 2(1 - s)R_1 - 2sR_2 \in [-R_1 + A(R_1), -R_1 + B(R_1)]) \]

\[ = 2 |R_2 - R_1| \cdot \int_{[0,1]} e^{\delta[2(1-s)R_1+2sR_2]} \cdot \| e^{\delta u}[\tau_{R_1} f R_1 \hat{\xi}]'(u) \|_{[-R_1+A(R_1), -R_1+B(R_1)] \times S^1} ds \]

\[ = | e^{2\delta R_1} - e^{2\delta R_2} | / \delta \cdot \| e^{\delta u} \cdot \tau_{R_1} [f R_1 \tau_{-R_1} \hat{\xi}]'(u) \|_{[-R_1+A(R_1), -R_1+B(R_1)] \times S^1} \|_{0,p} \]

\[ = | e^{2\delta R_1} - e^{2\delta R_2} | / \delta \cdot \| f R_1 \|_{C^0} \| \tau_{-R_1} e^{\delta v} \cdot \tau_{-R_1} \hat{\xi}(v) \|_{[A(R_1), B(R_1)] \times S^1} \|_{1,p} \]

\[ \leq | e^{2\delta R_1} - e^{2\delta R_2} | / \delta \cdot \| f R_1 \|_{C^0} \cdot \| e^{\delta v} \cdot \hat{\xi}(v) \|_{[-R_1+A(R_1), -R_1+B(R_1)] \times S^1} \|_{1,p} \]

\[ \leq | e^{2\delta R_1} - e^{2\delta R_2} | / \delta \cdot \| f R_1 \|_{C^0} \cdot \| e^{\delta(1+A(R_1))} \cdot \hat{\xi}(t) \|_{[-R_1+A(R_1), -R_1+B(R_1)] \times S^1} \|_{1,p} \]

\[ \leq | e^{2\delta R_1} - e^{2\delta R_2} | / \delta \cdot \| f R_1 \|_{C^0} \cdot \| e^{\delta(1+A(R_1))} \cdot \hat{\xi}(t) \|_{[-R_1+A(R_1), -R_1+B(R_1)] \times S^1} \|_{1,p} \]

\[ = | e^{2\delta R_1} - e^{2\delta R_2} | / \delta \cdot \| f R_1 \|_{C^0} \cdot \| e^{\delta(1+A(R_1))} \cdot \hat{\xi}(t) \|_{[-R_1+A(R_1), -R_1+B(R_1)] \times S^1} \|_{1,p} \]

\[ \leq | e^{2\delta R_1} - e^{2\delta R_2} | / \delta \cdot \| f R_1 \|_{C^0} \cdot \| e^{\delta(1+A(R_1))} \cdot \hat{\xi}(t) \|_{[-R_1+A(R_1), -R_1+B(R_1)] \times S^1} \|_{1,p} \]

\[ \leq | e^{2\delta R_1} - e^{2\delta R_2} | / \delta \cdot \| f R_1 \|_{C^0} \cdot \| e^{\delta(1+A(R_1))} \cdot \hat{\xi}(t) \|_{[-R_1+A(R_1), -R_1+B(R_1)] \times S^1} \|_{1,p} \]
and
\[\|\tau_{R_1} f_{R_1} \hat{\xi} \circ \tau_{-2R_2} - [\tau_{R_2} f_{R_2} \hat{\xi}] \circ \tau_{-2R_2} \|_{0,p,\delta}\]
\[= \| e_+(t) \{ [\tau_{R_1} f_{R_1} \hat{\xi}] - [\tau_{R_2} f_{R_2} \hat{\xi}] \} \circ \tau_{-2R_2} \|_{0,p}.\]

( with \( u = t - 2R_2 \in [-R_1 + A(R_1) - 1, -R_1 + B(R_1) + 1] \), noting that \(|R_2 - R_1| \leq 1\))

\[= e^{2\delta R_2} \| e^{\delta u} [\tau_{R_1} f_{R_1} - \tau_{R_2} f_{R_2}] \hat{\xi}(u) \|_{[-R_1 + A(R_1) - 1, -R_1 + B(R_1) + 1] \times S^1} \|_{0,p} \]
\[\leq e^{2\delta R_2} e^{\delta(-R_1 + |A(R_1)| + |B(R_1)| + 2)} \| [\tau_{R_1} f_{R_1} - \tau_{R_2} f_{R_2}] \hat{\xi}(u) \|_{[-R_1 + A(R_1) - 1, -R_1 + B(R_1) + 1] \times S^1} \|_{0,p} \]
\[\leq e^{2\delta R_2} e^{\delta(-R_1 + |A(R_1)| + |B(R_1)| + 2)} \| \tau_{R_1} f_{R_1} - \tau_{R_2} f_{R_2} \|_{C_0} \cdot \| \hat{\xi} \|_{0,p,\delta} \]

Write \( f(R, t) = f_R(t) \).

Then

\[|f_{R_1}(t) - f_{R_2}(t)| = |R_1 - R_2| \cdot \left| \int_{[0,1]} \partial_R f((1 - s)R_1 + sR_2, t) ds \right| \]
\[\leq |R_1 - R_2| \cdot \int_{[0,1]} |\partial_R f((1 - s)R_1 + sR_2, t)| ds \]
\[\leq |R_1 - R_2| \cdot |\partial_R f|_{C^0} = C_0 |R_1 - R_2| \]

so that \( \|\tau_{R_1} f_{R_1} - \tau_{R_2} f_{R_2}\|_{C_0} \leq C_0 |R_1 - R_2| \).

Hence

\[\| [\tau_{R_1} f_{R_1} \hat{\xi}] \circ \tau_{-2R_2} - [\tau_{R_2} f_{R_2} \hat{\xi}] \circ \tau_{-2R_2} \|_{0,p,\delta} \leq C_0 |R_1 - R_2| e^{2\delta(|A(R_1)| + |B(R_1)|)} \cdot \| \hat{\xi} \|_{0,p,\delta}.\]

Therefore

\[\| [\tau_{R_1} f_{R_1} \hat{\xi}] \circ \tau_{-2R_2} - [\tau_{R_2} f_{R_2} \hat{\xi}] \circ \tau_{-2R_2} \|_{0,p,\delta} \leq C_0 |R_1 - R_2| e^{2\delta(|A(R_1)| + |B(R_1)|)} \cdot \| \hat{\xi} \|_{0,p,\delta} \]

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\[ + \|e^{2\delta R_1} - e^{2\delta R_2}\|/\delta \cdot C_0 \cdot e^{-\delta(R_1)}\|\hat{\xi}(t)\|_{1,p,\delta}. \]

So that
\[ \|H(R_1) - H(R_2)\|_o \leq C_0 \{|R_1 - R_2|e^{2\delta(|A(R_1)|+|B(R_1)|)} + |e^{2\delta R_1} - e^{2\delta R_2}|/\delta \cdot e^{-\delta(R_1)}\}. \]

\[ \square \]

Corollary 4.1 \(\|D(R)\|_o \leq e^{-\delta R}.\)

Proof:

\[
\|D(R)\|_o \leq \|D(R)-D(R_1)\|_o + \|D(R_1)\|_o \leq |e^{-\delta R} - e^{-\delta R_1}|/\delta + \sup_{\|\xi\|_{1,p,\delta} \leq 1} \|\xi \tau_{R_1}\|_{0,p,\delta}
\]

\[ \leq |e^{-\delta R} - e^{-\delta R_1}|/\delta + \sup_{\|\xi\|_{1,p,\delta} \leq 1} e^{-\delta R_1} \|\xi\|_{0,p,\delta} = |e^{-\delta R} - e^{-\delta R_1}|/\delta + e^{-\delta R_1}. \]

Now let \(R_1 \to \infty\), we get \(\|D(R)\|_o \leq e^{-\delta R}.\)

\[ \square \]

Corollary 4.2 \(D(R)\) extends continuous over \([R_0, \infty]\) with \(D(\infty) = 0.\)

Back to the function \(F : L_{1,p,\delta}(C_+, E) \times [0, \infty) \to L_{0,p,\delta}(C_+, E)\) or \(F : L_{1,p,\delta}(\mathbb{R}^1 \times S^1, E) \times \mathbb{R}^1 \to L_{0,p,\delta}(\mathbb{R}^1 \times S^1, E)\) defined by \(F(u, R) = u \circ \tau_R.\) For each fixed \(R,\) \(F\) is linear in \(u\) and bounded even in \(L_{1,p,\delta}\)-norm of the target: \(F(au_1 + bu_2) = (au_1 + bu_2) \circ \tau = au_1 \circ \tau + bu_2 \circ \tau = aF(u_1) + bF(u_2);\) and \(\|F(u)\|_{1,p,\delta} = \|u \circ \tau\|_{1,p,\delta} = e^{\pm\delta(R)}\|u\|_{1,p,\delta}.\) This proves the following

Lemma 4.5 The partial derivative \((D_1F)_{u,R} =: (D_uF)_{u,R}\) is given by \((D_1F)_{u,R}(\xi) = \xi \circ \tau_R.\)

Next consider the partial derivative of \(F\) along \(R\)-direction.

Lemma 4.6 The partial derivative \((D_RF)_{u,R}\) is given by \((D_RF)_{u,R}(\partial \tau_R) = \partial_u u \circ \tau_R.\)
Proof:

Assume that $u$ is smooth first. Then \[ \|F(u, R + s) - F(u, R) - s \cdot \partial_t u \circ \tau_R\|_{0,p,\delta} = \|u \circ \tau_{R+s} - u \circ \tau_R - s \cdot \partial_t u \circ \tau_R\|_{0,p,\delta} = |s| \cdot \| \int_{[0,1]} \{ \partial_t u(v(R + s) + (1 - v)R) - \partial_t u \circ \tau_R \} dv\|_{0,p,\delta} \]

We will prove in next lemma that above inequality still true for $u \in L^p_{1,\delta}$.

In other word, the inequality before for smooth $u$ still holds for $u \in L^p_{1,\delta}$.

Then for $u \in L^p_{1,\delta}$ and $R \in [R_0, \infty)$, \[ \lim_{s \to 0} \int_{[0,1]} \| \{ \partial_t u(vs + R) - \partial_t u \circ \tau_R \}\|_{0,p,\delta} dv = \lim_{s \to 0} \int_{[0,1]} \| \{ \partial_t u_n(vs + R) - \partial_t u_n \circ \tau_R \}\|_{0,p,\delta} dv = 0. \] Here the identity before the last one follows from the dominated convergence theorem.

\[\square\]

**Lemma 4.7** The inequality for smooth function $u$ in the proof of the above lemma still holds for $u \in L^p_{1,\delta}$.

**Proof:**

The result is well-known. For completeness, we include a proof here. For general $u = \lim_{n \to \infty} u_n$ in $L^p_{1,\delta}$-norm with $u_n$ being smooth, since the functions $u \to \|u\|_{1,p,\delta}$ and $(u, R) \to u \circ \tau_R$ are continuous,

\[ \|F(u, R + s) - F(u, R) - s \cdot \partial_t u \circ \tau_R\|_{0,p,\delta} = \| \lim_{n \to \infty} u_n \circ \tau_{R+s} - \lim_{n \to \infty} u_n \circ \tau_R - s \cdot \lim_{n \to \infty} \partial_t u_n \circ \tau_R\|_{0,p,\delta} \]

\[ = \lim_{n \to \infty} \int_{[0,1]} \{ \partial_t u_n(v(R + s) + (1 - v)R) - \partial_t u_n \circ \tau_R \} dv\|_{0,p,\delta} \]

\[ \leq |s| \lim_{n \to \infty} \int_{[0,1]} \| \{ \partial_t u_n(vs + R) - \partial_t u_n \circ \tau_R \}\|_{0,p,\delta} dv \]

\[ = |s| \int_{[0,1]} \lim_{n \to \infty} \| \{ \partial_t u_n(vs + R) - \partial_t u_n \circ \tau_R \}\|_{0,p,\delta} dv \]

\[ = |s| \int_{[0,1]} \| \lim_{n \to \infty} \{ \partial_t u_n(vs + R) - \partial_t u_n \circ \tau_R \}\|_{0,p,\delta} dv \]

Here the identity interchanging the integral and limit follows from the theorem on dominated convergence. Indeed, the function $f_n(v) =: \| \{ \partial_t u_n(vs +
$R) - \partial_t u_n \circ \tau_R \rvert_{0,p,\delta}$ is continuous when $u_n$ is smooth and $|f_n(v)| = f_n(v) \leq e^{\delta(R+|s|)} \{\|u\|_{1,p,\delta} + 1\}$ so that $\lim_{n \to \infty} \int_{[0,1]} \{\|\partial_t u_n(vs + R) - \partial_t u_n \circ \tau_R\|_{0,p,\delta} \}dv = \int_{[0,1]} \lim_{n \to \infty} \{\|\partial_t u_n(vs + R) - \partial_t u_n \circ \tau_R\|_{0,p,\delta} \}dv.

By the second lemma of this section, $\| (D_R F)_{u,R}(\frac{\partial}{\partial \theta}) \|_{0,p,\delta} = \| \partial_x u \circ \tau_{R_0} \|_{0,p,\delta} \leq e^{-\delta R} \|u\|_{1,p,\delta}$. This implies that $D_R F$ can be extended continuously over $R = \infty$ with the value equal to zero over $R = \infty$.

Similarly, $(D_\theta F)_{u,R_0}(\frac{\partial}{\partial \theta}) = \partial_{s} u \circ \tau_{R_0}$. Then

$$\|1/r \cdot (D_\theta F)_{u, R_0}(\frac{\partial}{\partial \theta})\|_{k-1,p,\delta} = 1/r \cdot \|\partial_x u \circ \tau_{R_0}\|_{k-1,p,\delta} = 1/r \cdot \|\partial_x u \circ \tau_{R}\|_{k-1,p,\delta} \sim 1/r \cdot e^{-\delta R} \|u\|_{k,p,\delta} \sim 1 \cdot e^{-\delta R/2} \|u\|_{k,p,\delta}.$$ Combing all the lemmas above, this proves the following

**Corollary 4.3** The function $F$ is of class $C^1$ for $R \neq \infty$ and $DF$ can be extended continuously over $R = \infty$.

In fact, the formula for $\partial_{u} F$ is already valid even for $R = \infty$.

To prove $\partial_x \Psi_L$ and $\partial_y \Psi_L$ in last section or $\partial_x \Psi_N$ and $\partial_y \Psi_N$ in the sequel of this paper are the real derivatives, we need the following lemma. Note that both $\Psi_L$ and $\Psi_N$ are already continuous including $r = 0$.

**Lemma 4.8** Let $K : W \times I = L_{p,\delta}^k \times (-x_0, x_0) \to L = L_{k-1,p,\delta}$ be a continuous function such that $\partial_x K : W \times (I \setminus \{0\}) \to L$ is continuous and extended continuously over $x = 0$, with the extension denoted by $\tilde{\partial}_x K : W \times I \to L$. Then $\partial_x K$ exits over $W \times \{x = 0\}$.

**Proof:**

For $s > 0,$

$$\|K(u, s) - K(u, 0) - \tilde{\partial}_x K(u, 0)s\|_{k-1,p,\delta} = \|K(u, s) - \lim_{\mu > 0, \mu \to 0} K(u, \mu) - \tilde{\partial}_x K(u, 0)s\|_{k-1,p,\delta}$$

$$= s \cdot \lim_{\mu > 0, \mu \to 0} \int_{\mu}^{1} \{\partial_x K(u, vs + (1 - v)\mu) - \tilde{\partial}_x K(u, 0)\}dv \|_{k-1,p,\delta}$$

$$\leq s \cdot \lim_{\mu > 0, \mu \to 0} \int_{\mu}^{1} \|\tilde{\partial}_x K(u, vs + (1 - v)\mu) - \tilde{\partial}_x K(u, 0)\|_{k-1,p,\delta} dv.$$
For any given $\epsilon > 0$, the continuity of $\tilde{\partial}_x K$ implies that there exists a $\rho > 0$ such that when $|v s + (1 - v) \mu| < \rho$,

$$||\tilde{\partial}_x K(u, v s + (1 - v) \mu) - \tilde{\partial}_x K(u, 0)||_{k-1,p,\delta} < \epsilon$$

for fixed $u$. Now for fixed $s > 0$, if $0 < \mu < s$, the condition that $s < \rho$ implies that $|v s + (1 - v) \mu| < \rho$ so that

$$\lim_{\mu \to 0, \mu > 0} \int_{\mu}^{1} ||\tilde{\partial}_x K(u, v s + (1 - v) \mu) - \tilde{\partial}_x K(u, 0)||_{k-1,p,\delta} dv$$

$$= \lim_{0 < \mu < s, \mu \to 0} \int_{\mu}^{1} ||\tilde{\partial}_x K(u, v s + (1 - v) \mu) - \tilde{\partial}_x K(u, 0)||_{k-1,p,\delta} dv$$

$$\leq \epsilon.$$  

In other words, the function $E(u, s) =: \lim_{\mu \to 0, \mu > 0} \int_{\mu}^{1} ||\tilde{\partial}_x K(u, v s + (1 - v) \mu) - \tilde{\partial}_x K(u, 0)||_{k-1,p,\delta} dv$ has the property that $\lim_{s \to 0} E(u, s) = 0$. Similar result holds for $s < 0$ so that $||K(u, s) - K(u, 0) - \tilde{\partial}_x K(u, 0) s||_{k-1,p,\delta} \sim |s| o(|s|)$.

Combining all the results so far, we have proved the main theorem of this paper.

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