A Systematic Analysis of Perturbations for Hexagonal Mixing Matrix

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ABSTRACT: We present a systematic analysis of perturbative Hexagonal(HG) mixing for describing recent global fit neutrino mixing data with normal and inverted hierarchy. The corrections to unperturbed mixing are parameterized in terms of small orthogonal rotations \(R_{\alpha\beta}\) with modified PMNS matrix of the forms \(R_{\alpha\beta} \cdot V_{HG}, V_{HG} \cdot R_{\alpha\beta}, R_{\alpha\beta} \cdot R_{\gamma\delta} \cdot V_{HG}, R_{\alpha\beta} \cdot V_{HG} \cdot R_{\gamma\delta}\). Here \(R_{\alpha\beta}\) is rotation in \(ij\) sector and \(V_{HG}\) is unperturbed Hexagonal mixing matrix. The detailed numerical investigation of all possible cases is performed with scanning of parameter space using \(\chi^2\) approach. We found that the perturbative schemes governed by single rotation are unable to fit the mixing angle data even at 3\(\sigma\) level. The mixing schemes which involves two rotation matrices, only \((R_{12} \cdot V_{HG}, R_{13} \cdot V_{HG}, R_{12} \cdot V_{HG} \cdot R_{12}, R_{12} \cdot V_{HG} \cdot R_{13}, R_{13} \cdot V_{HG} \cdot R_{13})\) are successful in fitting all neutrino mixing angles within 1\(\sigma\) range for normal hierarchy(NH). However for inverted hierarchy(IH), only \(R_{13} \cdot V_{HG} \cdot R_{13}\) is most preferable as it can fit all mixing angles at 1\(\sigma\) level. The remaining perturbative cases are either excluded at 3\(\sigma\) level or successful in producing mixing angles only at 2 – 3\(\sigma\) level. The corresponding results for all cases are reported.
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1 Introduction

Neutrinos are light elementary particles which offers to reveal various secrets of nature through their weak interactions with matter. The discovery of neutrino oscillations [1–5] is a major milestone in particle physics which established the fact that neutrino switches flavor while traveling because of their extremely tiny mass and flavor mixing among different weak eigenstates. However Standard Model of particle physics contains massless neutrinos and thus it is a clear hint of physics which is operating beyond the ambit of Standard Model in nature. As far neutrino mixing is concerned, it divulge interesting pattern in which two mixing angles of a three flavor scenario seems to be large while third turns out to be small. Among various proposed mixing schemes [6–8] for explaining neutrino mixing, Hexagonal(HG) mixing [9] stands out one of the interesting possibility with novel predictions of $\theta_{23} = 45^\circ$, $\theta_{12} = 30^\circ$ and $\theta_{13} = 0^\circ$.

However the data from reactor based Chinese Daya Bay [1] experiment presented first confirmed result of non zero 1-3 mixing angle with corresponding statistical significance of $5.2\sigma$. The value of $\theta_{13}$ was reported to be in the range $\sin^2 2\theta_{13} = 0.092 \pm 0.016(stat) \pm 0.05(syst)$ at 90% CL. Earlier Japanese T2K experiment [2] which is a long baseline neutrino oscillation experiment reported $\nu_\mu \rightarrow \nu_e$ events which is consistent with non zero $\theta_{13}$ in a three flavor scenario. The value of 1-3 mixing angle consistent with data at 90% CL is reported to be in the range $5^\circ(5.8^\circ) < \theta_{13} < 16^\circ(17.8^\circ)$ for Normal (Inverted) neutrino mass hierarchy. This non vanishing value of $\theta_{13}$ is also supported by other oscillation experiments like Double Chooz [3], Minos [4] and RENO [5]. Moreover it is evident from recent global fit [10–12] for neutrino masses and mixing angles (given in Table 1) that these mixing scenarios can only provide main structure of the consistent neutrino matrix at leading order and thus should be investigated for possible perturbations [13–17]. These corrections which claim to explain neutrino mixing data are often being parametrized in terms of rotation matrices [17–19] which acts on 12, 23 or 13 sector of these special matrices. This simpler way of parameterizing the corrections is useful to understand the nature of corrections which a particular sector of these special matrices will get in order to be consistent with neutrino mixing data. With similar motivation in mind, we looked into possible perturbations [20]
for Hexagonal(HG) mixing which are parameterized by one and two rotation matrices and thus are of the forms \((R_{ij}^l V_{HG}, V_{HG} R_{ij}^l, R_{ij}^l R_{kl}^l V_{HG}, V_{HG} R_{ij}^l R_{kl}^l, R_{ij}^l V_{HG} R_{kl}^l)\). These corrections show strong correlations among neutrino mixing angles which are weakened with full perturbation matrix. Since the form of PMNS matrix is given by \(U_{PMNS} = U_l^\dagger U_\nu\) so these modifications may originate from charged lepton [21], neutrino [22] or from both sectors [23]. We did numerical analysis with keeping all such possibilities in mind. The salient features of our detailed investigation are:

(i) We performed a systematic analysis of all possible perturbation cases expressed in terms of rotation matrices for Hexagonal(HG) mixing scheme with recent neutrino mixing data.

(ii) Here we followed \(\chi^2\) approach [18, 19] for scanning the parameter space with varying corresponding perturbation parameters. This will reveal overall picture of mixing angle fitting in parameter space along with capturing important information about magnitude and sign of correction parameters. It will also help in comparing different perturbative cases with \(\chi^2\) level of fitting achieved.

(iii) All mixing angles are varied in their permissible limits for studying the correlations among themselves instead of fixing one of them at a particular value for studying the correlation between remaining two mixing angles. This will show a complete picture and thus we present our results in terms of 2 dimensional scatter plots instead of line plots.

(iv) We worked in small rotation limit for our numerical investigation. This in turn justify to pronounce these modifications as perturbative corrections.

Here for our numerical investigation, we works in CP conserving limit i.e. all phases are assumed to be zero. Regarding CP Dirac phase, although there are some initial hints of preference for maximal CP violation but the data from long-baseline accelerator, solar and KamLAND is still consistent at 2\(\sigma\) or less in CP conserving limit [12] for both NH as well as for IH. Moreover recent global fits [10–12] also allow full \([0, 2\pi]\) range of CP violating phase(\(\delta_{CP}\)) at 3\(\sigma\). Thus the situation with CP violation is not conclusive so far. However it is imperative to check for the predictions of CP violating phase in this scenario. The corresponding effects for single rotation case are discussed in our another paper [24]. Thus this study along with our other studies [19, 24] completes the discussion about bimaximal, tribimaximal, Hexagonal and Democratic mixing scenarios for CP conserving as well as in CP violating scenario. These results by providing the sign and magnitude of correction parameters can help in understanding the structure of corrections that these well known mixing scenarios require in order to be consistent with neutrino mixing data. Hence this investigation might be useful for checking the viability of large number of possible models which offers different corrections to this mixing scheme in neutrino model building physics. It would also be fruitful to inspect the origin of these perturbations in a model dependent framework. However the discussion about all such objectives is left for future consideration.

The main outline of the paper is as follows. In section 2, we give detailed description about general setup of our work. In sections 3-9, we present results of our numerical investigation for perturbed HG mixing under various possible cases. Finally in section 10, we give brief summary and conclusions of our analysis.
2 General Setup

The neutrino mixing is described by $3 \times 3$ Unitary matrix which can be parametrized in terms of 3 mixing angles and 6 phases. However 5 phases are redundant and can be rotated away leaving behind only 1 physical phase. The neutrino mixing is given in standard form as [25]

$$
U = 
\begin{pmatrix}
{c_{12}c_{13}} & {s_{12}c_{13}e^{-i\delta}} & {s_{13}e^{-i\delta}} \\
-s_{12}c_{23} - c_{12}s_{23}s_{13}e^{i\delta} & {c_{12}c_{23} - s_{12}s_{23}s_{13}e^{i\delta}} & {s_{23}c_{13}} \\
{s_{12}s_{23} - c_{12}c_{23}s_{13}e^{i\delta}} & {-c_{12}s_{23} - s_{12}c_{23}s_{13}e^{i\delta}} & {c_{23}c_{13}}
\end{pmatrix}
\begin{pmatrix}
1 & 0 & 0 \\
0 & e^{i\rho} & 0 \\
0 & 0 & e^{i\sigma}
\end{pmatrix},
(2.1)
$$

where $c_{ij} \equiv \cos \theta_{ij}$, $s_{ij} \equiv \sin \theta_{ij}$ and $\delta$ is the Dirac CP violating phase. Here $\rho$ and $\sigma$ are Majorana phases which do not affect the neutrino oscillations and thus are not relevant for our discussion. In this study, we are investigating CP conserving case i.e. all the CP violating phases $\delta, \rho, \sigma$ are set to be zero.

The Hexagonal mixing matrix under consideration has following form:

$$
V_{HM} = 
\begin{pmatrix}
\frac{\sqrt{3}}{2} & \frac{1}{2} & 0 \\
-\frac{1}{2\sqrt{2}} & \frac{\sqrt{3}}{2\sqrt{2}} & -\frac{\sqrt{1}}{2} \\
-\frac{1}{2\sqrt{2}} & \frac{\sqrt{3}}{2\sqrt{2}} & \frac{\sqrt{1}}{2}
\end{pmatrix}.
$$

![Table 1](image)

**Table 1:** Three-flavor oscillation neutrino mixing angles from fit to global data [12].

This mixing scheme gives vanishing reactor mixing angle i.e. $\theta_{13} = 0^\circ$ with maximal value of atmospheric mixing angle i.e. $\theta_{23} = 45^\circ$ and lower value of solar mixing angle, $\theta_{12} = 30^\circ$. However recent experimental observations keeps best fitted values of mixing angles to be $\theta_{13} \sim 8^\circ$, $\theta_{12} \sim 33^\circ$ and $\theta_{23} \sim 41^\circ$ which in conflict with predictions of values obtained from considered mixing scheme. This in turn implies that departure of predicted values of mixing angles from best fit values should be tested for the possible perturbations around this mixing scheme.

However from theoretical point of view, neutrino mixing matrix $U$ is given as

$$
U = U_l^l U_\nu
$$
where $U_l$ and $U_\nu$ are the unitary matrices that diagonalizes the charged lepton ($M_l$) and neutrino mass matrix ($M_\nu$). Thus perturbations for discussed mixing scheme can originate from following sources:

(i) Leptonic sector i.e. $U_{PMNS}^l = U_{Pertub}^l \cdot U_{PMNS}$
(ii) Neutrino sector i.e. $U_{PMNS}^\nu = U_{PMNS} \cdot U_{Pertub}^\nu$
(iii) Leptonic and neutrino sector i.e. $U_{PMNS}^l = U_{Pertub}^l \cdot U_{PMNS} \cdot U_{Pertub}^\nu$

where, $U_{Pertub}^l$ and $U_{Pertub}^\nu$ are the real orthogonal matrices which can be described in terms of 3 mixing angles as elaborated earlier. Here we are testing all three possibilities that are governed by either one or two mixing angles with resultant PMNS matrix of the forms $R_X \cdot V_{HG}$, $V_{HG} \cdot R_X$, $R_X \cdot R_Y \cdot V_{HG}$, $V_{HG} \cdot R_X \cdot R_Y$ and $R_X \cdot V \cdot R_Y$ where $R_X$ and $R_Y$ denote generic perturbation matrices. The perturbation matrices $R_X$ and $R_Y$ are given by

$$R_{12} = \begin{pmatrix} \cos \mu & \sin \mu & 0 \\ -\sin \mu & \cos \mu & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad R_{23} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos \nu & \sin \nu \\ 0 & -\sin \nu & \cos \nu \end{pmatrix}, \quad R_{13} = \begin{pmatrix} \cos \lambda & 0 & \sin \lambda \\ 0 & 1 & 0 \\ -\sin \lambda & 0 & \cos \lambda \end{pmatrix}$$

where $\mu, \nu, \lambda$ denote rotation angles. Here $R_{23}$, $R_{13}$ and $R_{12}$ represent the rotations in 23, 13 and 12 sector respectively. The PMNS matrix for single rotation case is given by:

$$V_{\alpha\beta}^{HML} = R_{\alpha\beta}^l \cdot V_{HM}, \quad (2.2)$$
$$V_{\alpha\beta}^{HMR} = V_{HM} \cdot R_{\alpha\beta}^r, \quad (2.3)$$

where $(\alpha\beta) = (12), (13), (23)$ respectively. The corresponding PMNS matrix for two rotation matrices thus becomes:

$$V_{\alpha\beta\gamma\delta}^{HML} = R_{\alpha\beta}^l \cdot R_{\gamma\delta}^l \cdot V_{HM}, \quad (2.4)$$
$$V_{\alpha\beta\gamma\delta}^{HMR} = V_{HM} \cdot R_{\alpha\beta}^r \cdot R_{\gamma\delta}^r, \quad (2.5)$$
$$V_{\alpha\beta\gamma\delta}^{HMLR} = R_{\alpha\beta}^l \cdot V_{HM} \cdot R_{\gamma\delta}^r, \quad (2.6)$$
$$V_{\alpha\beta\gamma\delta}^{HMLR} = R_{\alpha\beta}^l \cdot V_{HM} \cdot R_{\alpha\beta}^r, \quad (2.7)$$

where $\alpha \beta \neq \gamma \delta$ and $(\alpha\beta), (\gamma\delta) = (12), (13), (23)$ respectively. The neutrino mixing angles from these perturbed matrices are obtained by comparing them with the standard PMNS matrix.

Here we are adopting $\chi^2$ approach for numerically investigating the effect of these perturbations in parameter space. We define a $\chi^2$ function given by:

$$\chi^2 = \sum_{i=1}^{3} \left( \frac{\theta_i^P - \theta_i^{exp}}{\delta \theta_i^{exp}} \right)^2 \quad (2.8)$$

with $\theta_i^P$ are the theoretical value of mixing angles obtained from perturbed mixing matrix and thus are functions of perturbation parameters $(\mu, \nu, \lambda)$. $\theta_i^{exp}$ are the experimental value of neutrino mixing angles with corresponding 1σ deviation $\delta \theta_i^{exp}$. The unperturbed value of $\chi^2$ in this mixing scheme is 732.8(868.0) for NH(IH) case. In this study, we investigated the role of these perturbations for bringing down $\chi^2$ value in parameter space.
3 Numerical Findings

Here we discuss numerical results of our investigation for perturbed HG mixing with Normal and Inverted Hierarchy case. The role of perturbation parameters is studied in producing large $\theta_{13}$ [13] and fitting other two mixing angles. We used exact expressions of modified mixing angles in terms of correction parameters for performing numerical investigations. However in relevant sections we present approximate form of these expressions that will give some insight about the nature of corrections. This in turn will be useful for determining the size of deviation a mixing angle can have in parameter space from its unperturbed value. The parameter space is scanned by randomly picking numerical value of correction parameters $\mu$, $\nu$ and $\lambda$ in the range [-0.5, 0.5]. This range will ensure that these parameters remains under perturbative limits. The plotting data points are being taken by putting the condition $\chi^2 < \chi^2_{\text{unpert}}$ during search of best fit. However $\chi^2_{\text{min}}$ value from parameter space is chosen in such a way that it corresponds to best level of fitting for all three mixing angles.

In Figs. 1-24, we present our numerical findings in terms of $\chi^2$ over perturbation parameters and $\theta_{13}$ over $\theta_{12} - \theta_{23}$ plane for various possible cases with NH. Since IH case show similar kind of dependance, so we skipped their plots here although we mention the interesting features if any along with quoting best fit $\chi^2$ and level of fitting for each case. In double rotation plots of $\chi^2$ vs perturbation parameters ($\theta_1, \theta_2$) red, blue and light green color regions corresponds to $\chi^2$ value in the interval $[0, 3]$, $[3, 10]$ and $> 10$ respectively. However white part of plot corresponds to completely disallowed region having $\chi^2 > \chi^2_{\text{unpert}}$. In figures of neutrino mixing angles, light green band corresponds to 1$\sigma$ and full color band to 3$\sigma$ values of $\theta_{13}$. Also ‘×’ refers to the case which is unsuccessful in fitting mixing angles even at 3$\sigma$ level while ‘-’ points to the situation where $\theta_{13}$ doesn’t receives any corrections i.e. $\theta_{13} = 0$. For showing the mapping between left and right figures we highlighted the $\chi^2 < 3$, $[3, 10]$ regions in neutrino mixing angle plots with different colors. The white region corresponds to $3 < \chi^2 < 10$ while yellow region belongs to $\chi^2 < 3$. Horizontal and vertical dashed black, dashed pink and thick black lines corresponds to 1$\sigma$, 2$\sigma$ and 3$\sigma$ ranges of the other two mixing angles. Now in subsequent sections we present our detailed analysis for various perturbative cases.

4 Rotations-\textbf{R}_\text{12}.\textit{V}_{HG}

Here we first consider the perturbations for which the form of modified PMNS matrix is given by $U_{PMNS} = R^l_{\alpha\beta}.V_{HG}$.

4.1 12 Rotation

This case pertains to rotation in 12 sector of HG mixing matrix. Since in small rotation limit, we can take $\sin \mu \approx \mu$ and $\cos \mu \approx 1 - \mu^2$, so the neutrino mixing angles up to order $O(\mu^2)$ are given by
\[ \sin \theta_{13} \approx |\mu V_{23}|, \]  
\[ \sin \theta_{23} \approx \left| \frac{(\mu^2 - 1)V_{23}}{\cos \theta_{13}} \right|, \]  
\[ \sin \theta_{12} \approx \left| \frac{V_{12} + \mu V_{22} - \mu^2 V_{12}}{\cos \theta_{13}} \right|. \]  

Fig. 1 shows our numerical results corresponding to this rotation scheme. The salient features of this perturbed 12 mixing are:

(i) Here atmospheric mixing angle(\(\theta_{23}\)) remains near to its unperturbed value since it receives corrections only of \(O(\mu^2)\).

(ii) The fitting of \(\theta_{13}\) under its 3\(\sigma\) domain constraints the magnitude of perturbation parameter |\(\mu\)| \(\in [0.1962(0.1988), 0.2204(0.2223)]\) which in turn fixes \(\theta_{12} \in [38.0^\circ(38.10^\circ), 39.0^\circ(39.08^\circ)]\) for its positive and \(\theta_{12} \in [20.99^\circ(20.91^\circ), 21.99^\circ(21.89^\circ)]\) for negative \(\mu\) values. The atmospheric angle(\(\theta_{23}\)) remain confined to a very narrow range of \(\theta_{23} \in [44.29^\circ(44.28^\circ), 44.44^\circ(44.43^\circ)]\) for this domain of \(\mu\).

(iii) The minimum value of \(\chi^2 \sim 38.6(49.9)\) for this case which gives \(\theta_{12} \sim 38.29^\circ(38.39^\circ)\), \(\theta_{23} \sim 44.40^\circ(44.39^\circ)\) and \(\theta_{13} \sim 8.20^\circ(8.30^\circ)\).

(iv) In this mixing scheme, \(\theta_{12}\) remains outside its 3\(\sigma\) range so it is is not consistent.

4.2 13 Rotation

This case corresponds to rotation in 13 sector of HG mixing matrix. The neutrino mixing angles for small perturbation parameter \(\lambda\) are given by
\[ \sin \theta_{13} \approx |\lambda V_{23}|, \] (4.4)
\[ \sin \theta_{23} \approx \left| \frac{V_{23}}{\cos \theta_{13}} \right|, \] (4.5)
\[ \sin \theta_{12} \approx \left| \frac{V_{12} + \lambda V_{22} - \lambda^2 V_{12}}{\cos \theta_{13}} \right|. \] (4.6)

Fig. 2 show the numerical results pertaining to this mixing scheme. The main features of this perturbative case are:

(i) Here \( \theta_{23} \) receives very minor corrections which comes through \( \sin \theta_{13} \) and thus its value remains close its original prediction.

(ii) The fitting of \( \theta_{13} \) in its 3\( \sigma \) domain constraints the magnitude of correction parameter \( |\lambda| \in [0.1962(0.1988), 0.2204(0.2223)] \) which in turn fixes \( \theta_{12} \in [38.0^\circ(38.10^\circ), 39.0^\circ(39.08^\circ)] \) for its positive and \( \theta_{12} \in [20.99^\circ(20.91^\circ), 21.99^\circ(21.89^\circ)] \) for negative \( \lambda \) values. The corresponding \( \theta_{23} \) remains quite close to its original prediction in the following range \( \theta_{23} \in [45.55^\circ(45.56^\circ), 45.70^\circ(45.71^\circ)] \).

(iii) The minimum value of \( \chi^2 \approx 37.5(44.2) \) which produces \( \theta_{12} \approx 38.29^\circ(38.40^\circ) \), \( \theta_{23} \approx 45.59^\circ(45.61^\circ) \) and \( \theta_{13} \approx 8.20^\circ(8.31^\circ) \) for its corresponding best fit.

(iv) Like previous case, it also produces the values of \( \theta_{12} \) which is outside its 3\( \sigma \) domain. Thus this case is not viable.

4.3 23 Rotation

For this rotation, \( \theta_{13} \) doesn’t get any corrections from perturbation matrix (i.e. \( \theta_{13} = 0 \)) and the minimum value of \( \chi^2 \approx 951.4 \). Thus we left this case for any further discussion.

5 Rotations-\( V_{HG} \cdot R_{\alpha\beta} \)

Here we consider the perturbations for which corrected PMNS matrix is given by the expression \( U_{PMNS} = R_{\alpha\beta} U \).
5.1 12 Rotation

Here $\theta_{13}$ doesn’t receive any corrections from perturbation matrix (i.e. $\theta_{13} = 0$) and the minimum value of $\chi^2 \sim 960.7$. Thus this case is also left out for any further discussion.

5.2 13 Rotation

This case corresponds to rotation in 13 sector of HG mixing matrix. The mixing angles for small perturbation parameter $\lambda$ are given by

$$\sin \theta_{13} \approx |\lambda V_{11}|,$$

$$\sin \theta_{23} \approx \left| \frac{V_{23} + \lambda V_{21} - \lambda^2 V_{23}}{\cos \theta_{13}} \right|,$$

$$\sin \theta_{12} \approx \left| \frac{-V_{12}}{\cos \theta_{13}} \right|.$$

Fig. 3 show our numerical results corresponding to perturbed HG case. The main features of these corrections are:

(i) Here solar mixing angle($\theta_{12}$) receives very minor corrections which comes through $\sin \theta_{13}$ and thus its value remains near to its unperturbed prediction.

(ii) For fitting $\theta_{13}$ in its $3\sigma$ domain constraints the magnitude of perturbation parameter $|\lambda| \in [0.1598(0.1620), 0.1795(0.1810)]$ which in turn fixes $\theta_{23} \in [49.60^\circ(49.67^\circ), 50.18^\circ(50.22^\circ)]$ for its positive and $\theta_{23} \in [39.81^\circ(39.77^\circ), 40.39^\circ(40.32^\circ)]$ for negative $\lambda$ values. The solar mixing angle($\theta_{12}$) remains confined in the narrow range $\theta_{12} \in [30.31^\circ(30.32^\circ), 30.40^\circ(30.41^\circ)]$. Thus both regions of $\lambda$ are allowed although its negative range is much preferable as it brings $\theta_{23}$ much closer to its central value.

(iii) The minimum value of $\chi^2 \sim 13.5(14.3)$ for this case and produces $\theta_{12} \sim 30.36^\circ(30.36^\circ)$, $\theta_{23} \sim 49.90^\circ(49.94^\circ)$ and $\theta_{13} \sim 8.41^\circ(8.48^\circ)$ for its best fit.

(iv) This mixing scheme produces low value of $\theta_{12}$ which just remains outside its $3\sigma$ range. Thus this case is not allowed.

5.3 23 Rotation

This case corresponds to rotation in 23 sector of HG mixing matrix. The neutrino mixing angles for small perturbation parameter $\nu$ are given by

$$\sin \theta_{13} \approx |\nu V_{12}|,$$

$$\sin \theta_{23} \approx \left| \frac{V_{23} + \nu V_{22} - \nu^2 V_{23}}{\cos \theta_{13}} \right|,$$

$$\sin \theta_{12} \approx \left| \frac{(\nu^2 - 1)V_{12}}{\cos \theta_{13}} \right|.$$

Fig. 4 show the numerical results corresponding to this rotation. The salient features in this perturbative scheme are:

(i) Here atmospheric mixing angle($\theta_{12}$) remains quite close to its unperturbed value since
it receives corrections only of $O(\nu^2)$.

(ii) The fitting of $\theta_{13}$ in its 3σ range constraints the magnitude of correction parameter $|\nu| \in [0.2793(0.2831), 0.3143(0.3171)]$ which in turn fixes $\theta_{23} \in [29.27^\circ (29.13^\circ), 31.05^\circ (30.85^\circ)]$ for its positive and $\theta_{23} \in [58.94^\circ (59.14^\circ), 60.72^\circ (60.86^\circ)]$ for negative values. However solar mixing angle($\theta_{12}$) remains near to its original prediction $\theta_{12} \in [28.76^\circ (28.74^\circ), 29.02^\circ (29.0^\circ)]$.

(iii) The minimum value of $\chi^2 \sim 46.2(110.7)$ for this case and produces $\theta_{12} \sim 28.93^\circ (28.96^\circ)$, $\theta_{23} \sim 59.65^\circ (59.41^\circ)$ and $\theta_{13} \sim 8.30^\circ (8.17^\circ)$ for its best fit.

(iii) This perturbed scheme is unable to fit $\theta_{12}$ and $\theta_{23}$ in their permissible range so this mixing case is not consistent.

6 Rotations $-R_{\alpha\beta}^l,R_{\gamma\delta}^l,V_{HG}$

Here we take up the perturbative cases for which modified PMNS matrix is given by $U_{PMNS} = R_{ij}^l R_{kl}^l V_{HG}$. The role of these corrections is studied in parameter space for
fitting 3 flavor mixing angles.

6.1 12-13 Rotation

This pertubative case pertains to rotations in 12 and 13 sector of HG mixing matrix. Under small rotation limit, $\sin x \approx x$ and $\cos x \approx 1 - x^2$, so the neutrino mixing angles truncated at order $O(\theta^2)$ in this scheme are given by

$$\sin \theta_{13} \approx |(\mu - \lambda)V_{23}|,$$

$$\sin \theta_{23} \approx \frac{1 - \mu^2 + \mu \lambda}{\cos \theta_{13}} \frac{V_{23}}{|V_{23}|},$$

$$\sin \theta_{12} \approx \frac{1 - \mu^2 - \lambda^2}{\cos \theta_{13}} \frac{V_{12} + (\mu + \lambda)V_{22}}{|V_{12} + (\mu + \lambda)V_{22}|}. $$

Fig. 5 show our numerical results with $\theta_1 = \lambda$ and $\theta_2 = \mu$. The main features of this perturbative matrix are:

(i) In this case, $\theta_{23}$ remains close to its unperturbed value since it receives corrections of only at $O(\theta^2)$ from perturbation parameters. However $\theta_{12}$ can have wide range of values in allowed parameter space.

(ii) The minimum value of $\chi^2 \sim 1.52(7.78)$ which produces $\theta_{12} \sim 33.42^\circ(33.42^\circ)$, $\theta_{23} \sim 44.99^\circ(44.99^\circ)$ and $\theta_{13} \sim 8.48^\circ(8.48^\circ)$.

(iii) This mixing scheme can fit all mixing angles at $1\sigma$ level for NH. However $1\sigma$ range of $\theta_{23}$ is much constrained in IH and thus same best fitted $1\sigma$ values of $\theta_{23}$ in NH belongs to $2\sigma$ range of IH in this mixing. Thus this case is consistent at $1\sigma(2\sigma)$ for NH(IH).

6.2 12-23 Rotation

This case corresponds to rotations in 12 and 23 sector of HG mixing matrix. The neutrino mixing angles for small perturbation parameters $\mu$ and $\nu$ are given by

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**Figure 5**: $U^{HGL}_{1213}$ scatter plot of $\chi^2$ (left fig.) over $\mu - \lambda$ (in radians) plane and $\theta_{13}$ (right fig.) over $\theta_{23} - \theta_{12}$ (in degrees) plane. The information about color coding and various horizontal, vertical lines in right fig. is given in text.
Figure 6: $U^{HGL}_{1223}$ scatter plot of $\chi^2$ (left fig.) over $\mu - \nu$ (in radians) plane and $\theta_{13}$ (right fig.) over $\theta_{23} - \theta_{12}$ (in degrees) plane.

$$\sin \theta_{13} \approx |\mu(1 - \nu)V_{23}|,$$

$$\sin \theta_{23} \approx \frac{|(1 - \nu - \mu^2 - \nu^2)V_{23}|}{\cos \theta_{13}},$$  

$$\sin \theta_{12} \approx \frac{|(1 - \mu^2)V_{12} + \mu(1 + \nu)V_{22}|}{\cos \theta_{13}}. \quad (6.6)$$

In Fig. 6 we present our numerical results for this case with $\theta_1 = \nu$ and $\theta_2 = \mu$. The main features in this mixing scheme are:

(i) Here all mixing angles receives corrections at leading order from perturbation parameters and thus show interesting correlations among themselves.

(ii) The minimum value of $\chi^2 \sim 15.2(21.6)$ which produces $\theta_{12} \sim 36.0^0(37.0^0)$, $\theta_{23} \sim 54.14^0(49.87^0)$ and $\theta_{13} \sim 8.36^0(8.38^0)$.

(iii) This perturbative case fails to fit all mixing angles even at 3$\sigma$ for IH and NH. Thus its not viable.

6.3 13-12 Rotation

This perturbative case corresponds to rotations in 13 and 12 sector of HG mixing matrix. The neutrino mixing angles for small perturbation parameters $\mu$ and $\lambda$ are given by

$$\sin \theta_{13} \approx |(\mu - \lambda)V_{23}|,$$

$$\sin \theta_{23} \approx \frac{|(1 - \mu^2)V_{23}|}{\cos \theta_{13}},$$  

$$\sin \theta_{12} \approx \frac{|(1 - \mu^2 - \lambda^2)V_{12} + (\mu + \lambda)V_{22}|}{\cos \theta_{13}}. \quad (6.9)$$

In Fig. 7 we present our numerical results with $\theta_1 = \lambda$ and $\theta_2 = \mu$. The main features of this rotation scheme are:
Figure 7: $U_{1312}^{HGL}$ scatter plot of $\chi^2$ (left fig.) over $\lambda - \mu$ (in radians) plane and $\theta_{13}$ (right fig.) over $\theta_{23} - \theta_{12}$ (in degrees) plane.

(i) The case is quite similar to 12-13 rotation except for $\theta_{23}$ where in previous case it got additional $O(\theta^2)$ correction term. Here also $\theta_{23}$ remains close to its unperturbed value while $\theta_{12}$ can have much wide range of values.

(ii) The minimum value of $\chi^2 \sim 1.0(5.49)$ which produces $\theta_{12} \sim 33.46^\circ(33.46^\circ)$, $\theta_{23} \sim 45.52^\circ(45.52^\circ)$ and $\theta_{13} \sim 8.46^\circ(8.46^\circ)$.

(iii) Like 12-13L rotation, this mixing case can fit all angles at 1$\sigma$ level for NH. However the same fitted value of $\theta_{23}$ belongs to 2$\sigma$ region in IH. Thus this mixing case is consistent at 1$\sigma$(2$\sigma$) level.

6.4 13-23 Rotation

This perturbative scheme corresponds to rotations in 13 and 23 sector of HG mixing matrix. The neutrino mixing angles for small perturbation parameters $\lambda$ and $\nu$ are given by

$$\sin \theta_{13} \approx |\lambda(1 + \nu)V_{23}|,$$

$$\sin \theta_{23} \approx \left(1 - \nu - \nu^2 \right) \frac{V_{23}}{\cos \theta_{13}},$$

$$\sin \theta_{12} \approx \frac{(1 - \lambda^2)V_{12} + \lambda(1 - \nu)V_{22}}{\cos \theta_{13}}.$$

In Fig. 8 we present our numerical results for this case with $\theta_1 = \lambda$ and $\theta_2 = \nu$. The following features define this perturbation scheme:

(i) Here perturbation parameters $\nu$ and $\lambda$ enters into the expressions of all mixing angles at leading order and thus show good correlations among themselves.

(ii) The best fit case have $\chi^2 \sim 27.6(44.1)$ and produces $\theta_{12} \sim 36.88^\circ(38.37^\circ)$, $\theta_{23} \sim 39.86^\circ(45.39^\circ)$ and $\theta_{13} \sim 8.30^\circ(8.35^\circ)$.

(iii) This case is quite similar to 12-23L rotation. It fails to fit all mixing angles even at 3$\sigma$ level. Hence this mixing case is not viable.
6.5 23-12 Rotation

This case corresponds to rotations in 23 and 12 sector of HG mixing matrix.

\[
\sin \theta_{13} \approx |\mu V_{23}|, \\
\sin \theta_{23} \approx \left| \frac{(1 - \nu - \mu^2 - \nu^2)V_{23}}{\cos \theta_{13}} \right|, \\
\sin \theta_{12} \approx \left| \frac{(1 - \mu^2)V_{12} + \mu V_{22}}{\cos \theta_{13}} \right|. 
\]

In Fig. 9 we show our results for this scheme with \( \theta_1 = \nu \) and \( \theta_2 = \mu \). The following are the main characteristics of this pertubative scheme:

(i) Here modifications to mixing angle \( \theta_{12} \) and \( \theta_{13} \) is only dictated by perturbation parameter \( \mu \). Since \( |\mu| \) is tightly constrained from fitting of \( \theta_{13} \) which in turn allows very narrow ranges for \( \theta_{12} \) corresponding to negative and positive values of \( \mu \) in parameter space. Since at leading order \( \theta_{23} \) contains parameter \( \nu \) and thus it can possess wide range of values in parameter space.

(ii) The minimum value of \( \chi^2 \) ∼ 36.7(39.1) which produces \( \theta_{12} \sim 38.27^\circ (38.42^\circ) \), \( \theta_{23} \sim 48.06^\circ (48.2^\circ) \) and \( \theta_{13} \sim 8.18^\circ (8.33^\circ) \) for its best fit.

(iii) This case produces values of \( \theta_{12} \) which lies outside its 3σ boundary for NH and IH. Thus this rotation case is not consistent.

6.6 23-13 Rotation

In this perturbative scheme, the neutrino mixing angles pertaining to small rotation parameters are given by
\[ \sin \theta_{13} \approx |\lambda V_{23}|, \]  
\[ \sin \theta_{23} \approx \left| \frac{(1 - \nu - \nu^2) V_{23}}{\cos \theta_{13}} \right|, \]  
\[ \sin \theta_{12} \approx \left| \frac{(1 - \lambda^2) V_{12} + \lambda V_{22}}{\cos \theta_{13}} \right|. \]

In Fig. 10 we present our numerical results for this case with \( \theta_1 = \lambda \) and \( \theta_2 = \nu \).

The main characteristic features of this scheme are:

(i) It is clear from the expressions of \( \theta_{13} \) and \( \theta_{12} \) that only perturbation parameter \( \lambda \) imparts corrections to them. Thus \( |\lambda| \) is tightly constrained from fitting of \( \theta_{13} \) which in turn allows very narrow ranges for \( \theta_{12} \) corresponding to negative and positive values of \( \lambda \) in parameter space. However at leading order \( \theta_{23} \) solely depends on \( \nu \) and thus can have wide range of values in parameter space.

(ii) The minimum value of \( \chi^2 \sim 36.8(39.1) \) which gives \( \theta_{12} \sim 38.25^\circ(38.41^\circ), \theta_{23} \sim 48.23^\circ(48.32^\circ) \) and \( \theta_{13} \sim 8.16^\circ(8.32^\circ) \).

(iii) This case is very much similar to 23-12L rotation apart from an additional correction of \( O(\theta^2) \) for \( \theta_{23} \) in previous case. It also produces values of \( \theta_{12} \) which lies outside its 3σ boundary in parameter space. Hence it is also not viable.

7 Rotations-\( V_{HG}.R'_{\alpha\beta}.R'_{\gamma\delta} \)

Now we will discuss the role of perturbations for which modified PMNS matrix is given by \( U_{PMNS} = U.R'_{ij}.R'_{kl} \).

7.1 12-13 Rotation

This case pertains to rotations in 12 and 13 sector of HG mixing matrix. The expressions for neutrino mixing angles truncated at order \( O(\theta^2) \) for this mixing scheme are given by
\begin{align*}
\sin \theta_{13} & \approx |\lambda V_{11} + \mu V_{12}|, \\
\sin \theta_{23} & \approx \left|\frac{(1 - \lambda^2)V_{23} + \lambda V_{21} - \mu \lambda V_{22}}{\cos \theta_{13}}\right|, \\
\sin \theta_{12} & \approx \left|\frac{(1 - \mu^2)V_{12} + \mu V_{11}}{\cos \theta_{13}}\right|.
\end{align*}

In Fig. 11, we present the numerical results corresponding to this mixing case with \( \theta_1 = \lambda \) and \( \theta_2 = \mu \). The main features of this perturbative scheme are:

(i) The correction parameters (\( \mu, \lambda \)) enters into these mixing angles at leading order and thus they show good correlations among themselves.

(ii) Here parameter space prefers two regions for \( \theta_{23} \) mixing angle. The first gives \( \theta_{23} \approx 36^\circ - 42^\circ \) while for second \( \theta_{23} \approx 48^\circ - 54^\circ \). However \( \theta_{12} \) can have wide range of values for these regions.

(iii) In this case, we can get \( \chi^2 < 3 \) for a very minute region of parameter space which can fit \( \theta_{12} \) and \( \theta_{13} \) in its 1\( \sigma \) domain while \( \theta_{23} \) stays in its 3\( \sigma \) range. However all angles can be fitted at 2\( \sigma \) level with \( \chi^2_{\text{min}} \approx 6.37(8.70) \) which gives \( \theta_{12} \approx 31.89^\circ(31.91^\circ) \), \( \theta_{23} \approx 50.05^\circ(50.08^\circ) \) and \( \theta_{13} \approx 8.16^\circ(8.21^\circ) \).

(iii) This mixing case is consistent at 2\( \sigma \) level for NH and IH.

### 7.2 12-23 Rotation

This case corresponds to rotations in 12 and 23 sector of HG mixing matrix. The neutrino mixing angles for small perturbation parameters \( \mu \) and \( \nu \) are given by
Figure 11: $U_{1213}^{HGR}$ scatter plot of $\chi^2$ (left fig.) over $\mu - \lambda$ (in radians) plane and $\theta_{13}$ (right fig.) over $\theta_{23} - \theta_{12}$ (in degrees) plane.

\[
\sin \theta_{13} \approx |\nu V_{12} + \mu V_{11}|, \quad (7.4)
\]
\[
\sin \theta_{23} \approx \left| \frac{(1 - \nu^2)V_{23} + \nu V_{22} + \mu V_{21}}{\cos \theta_{13}} \right|, \quad (7.5)
\]
\[
\sin \theta_{12} \approx \left| \frac{(1 - \mu^2 - \nu^2)V_{12} + \mu V_{11}}{\cos \theta_{13}} \right|. \quad (7.6)
\]

In Fig. 12, we present our numerical findings for this rotation scheme with $\theta_1 = \nu$ and $\theta_2 = \mu$. The main features of this perturbative case are:

(i) Since mixing angles receives leading order corrections from perturbation parameters so these angles exhibit interesting correlations among themselves.

(ii) The minimum value of $\chi^2 \sim 12.6(52.5)$ which produces $\theta_{12} \sim 33.8^\circ(35.24^\circ)$, $\theta_{23} \sim 57.23^\circ(56.58^\circ)$ and $\theta_{13} \sim 8.36^\circ(8.32^\circ)$ for its respective best fit.

(iii) This mixing case fails to fit all mixing angles even at $3\sigma$ level. Thus it is not consistent.

7.3 13-12 Rotation

This perturbative scheme corresponds to rotations in 13 and 12 sector of HG mixing matrix.

The neutrino mixing angles under small rotation limit are given by

\[
\sin \theta_{13} \approx |\lambda V_{11}|, \quad (7.7)
\]
\[
\sin \theta_{23} \approx \left| \frac{(1 - \lambda^2)V_{23} + \lambda V_{21}}{\cos \theta_{13}} \right|, \quad (7.8)
\]
\[
\sin \theta_{12} \approx \left| \frac{(1 - \mu^2)V_{12} + \mu V_{11}}{\cos \theta_{13}} \right|. \quad (7.9)
\]

In Fig. 13, we present our numerical results for this case with $\theta_1 = \lambda$ and $\theta_2 = \mu$.

(i) It is clear from above expressions that modifications to mixing angle $\theta_{13}$ and $\theta_{23}$ is only dictated by perturbation parameter $\lambda$. Thus $|\lambda|$ is tightly constrained from fitting of $\theta_{13}$.
which in turn allows narrow ranges for $\theta_{23}$ corresponding to its negative and positive values of $\lambda$ in parameter space. However $\theta_{12}$ gets leading order correction from parameter $\mu$ and thus can have wide range of values in parameter space.

(ii) The minimum value of $\chi^2 \sim 0.60(2.03)$ which gives $\theta_{12} \sim 33.53^\circ (33.24^\circ)$, $\theta_{23} \sim 49.88^\circ (49.94^\circ)$ and $\theta_{13} \sim 8.38^\circ (8.49^\circ)$ for this best fit.

(iii) This case quite accurately fit $\theta_{12}$ and $\theta_{13}$ but allowed range of $\theta_{23}$ remains at $2\sigma$ level. Thus this rotation case is allowed at $2\sigma$ for NH and IH.

### 7.4 13-23 Rotation

This case corresponds to rotations in 13 and 23 sector of HG mixing matrix. The neutrino mixing angles for small perturbation parameters $\lambda$ and $\nu$ are given by

---

**Figure 12:** $U_{123}^{HGR}$ scatter plot of $\chi^2$ (left fig.) over $\mu - \nu$ (in radians) plane and $\theta_{13}$ (right fig.) over $\theta_{23} - \theta_{12}$ (in degrees) plane.

**Figure 13:** $U_{132}^{HGR}$ scatter plot of $\chi^2$ (left fig.) over $\lambda - \mu$ (in radians) plane and $\theta_{13}$ (right fig.) over $\theta_{23} - \theta_{12}$ (in degrees) plane.
Figure 14: $U_{1323}^{HGR}$ scatter plot of $\chi^2$ (left fig.) over $\lambda - \nu$ (in radians) plane and $\theta_{13}$ (right fig.) over $\theta_{23} - \theta_{12}$ (in degrees) plane.

\[
\sin \theta_{13} \approx |\nu V_{12} + \lambda V_{11}|, \\
\sin \theta_{23} \approx \left| \frac{(1 - \nu^2 - \lambda^2)V_{23} + \nu V_{22} + \lambda V_{21}}{\cos \theta_{13}} \right|, \\
\sin \theta_{12} \approx \left| \frac{(1 - \nu^2)V_{12} - \nu \lambda V_{11}}{\cos \theta_{13}} \right|.
\]

In Fig. 14, we show our investigation results for this case with $\theta_1 = \lambda$ and $\theta_2 = \nu$.

(i) For this rotation scheme, $\theta_{12}$ receives corrections only at $O(\theta^2)$ and thus its value remains close to its unperturbed value.

(ii) The minimum value of $\chi^2 \sim 12.8(13.8)$ which produces $\theta_{12} \sim 30.67^\circ(30.27^\circ)$, $\theta_{23} \sim 51.91^\circ(49.32^\circ)$ and $\theta_{13} \sim 8.40^\circ(8.46^\circ)$ for this fit.

(iii) This perturbative case fails to fit all mixing angles even at 3$\sigma$ level. Hence it is not viable.

7.5 23-12 Rotation

This case pertains to rotations in 23 and 12 sector of HG mixing matrix. The expressions for neutrino mixing angles in this perturbed scheme are given by

\[
\sin \theta_{13} \approx |\nu V_{12}|, \\
\sin \theta_{23} \approx \left| \frac{(1 - \nu^2)V_{23} + \nu V_{22}}{\cos \theta_{13}} \right|, \\
\sin \theta_{12} \approx \left| \frac{(1 - \mu^2 - \nu^2)V_{12} + \mu \lambda V_{11}}{\cos \theta_{13}} \right|.
\]

In Fig. 15, we present our numerical results for this case with $\theta_1 = \nu$ and $\theta_2 = \mu$.

(i) The corrections to mixing angle $\theta_{13}$ and $\theta_{23}$ is only governed by perturbation parameter $\nu$. Thus magnitude of parameter $\nu$ is tightly constrained from fitting of $\theta_{13}$. This in turn
Figure 15: $U^{HGR}_{2312}$ scatter plot of $\chi^2$ (left fig.) over $\nu - \mu$ (in radians) plane and $\theta_{13}$ (right fig.) over $\theta_{23} - \theta_{12}$ (in degrees) plane.

allows only very narrow ranges for $\theta_{23}$ corresponding to negative and positive values of $\nu$ in parameter space. However $\theta_{12}$ solely depends on $\mu$ and thus can have wide range of possible values in parameter space.

(ii) The minimum value of $\chi^2 \sim 19.4(84.9)$ which produces $\theta_{12} \sim 33.35^\circ(33.41^\circ)$, $\theta_{23} \sim 59.62^\circ(59.49^\circ)$ and $\theta_{13} \sim 8.29^\circ(8.22^\circ)$ for its respective best fit.

(iii) This mixing case produces values of $\theta_{23}$ which is quite far away from its $3\sigma$ range. Hence this mixing case is not allowed.

7.6 23-13 Rotation

This perturbative scheme is quite similar to 13-12 rotation with interchange of expressions for $\theta_{12}$ and $\theta_{23}$ mixing angles. The neutrino mixing angles under small rotation limit are given by

$$
\sin \theta_{13} \approx |\nu V_{12} + \lambda V_{11}|, \quad (7.16)
$$

$$
\sin \theta_{23} \approx \frac{(1 - \nu^2 - \lambda^2)V_{23} + \nu V_{22} + \lambda V_{21}}{|\cos \theta_{13}|}, \quad (7.17)
$$

$$
\sin \theta_{12} \approx \frac{(\nu^2 - 1)V_{12}}{|\cos \theta_{13}|}. \quad (7.18)
$$

In Fig. 16, we present our numerical findings for this case with with $\theta_1 = \lambda$ and $\theta_2 = \nu$.

The salient features of this perturbative scheme are:

(i) Here $\theta_{12}$ mixing angle receives corrections of $O(\nu^2)$ and thus its value remain near to its unperturbed value. However $\theta_{23}$ can have wide range of values in parameter space since it got leading order correction from parameter $\nu$ and $\lambda$.

(ii) The minimum value of $\chi^2 \sim 13.0(12.4)$ and it produces $\theta_{12} \sim 30.34^\circ(30.35^\circ)$, $\theta_{23} \sim 48.02^\circ(48.18^\circ)$ and $\theta_{13} \sim 8.44^\circ(8.49^\circ)$.

(iii) This mixing case produces low value of $\theta_{12}$ which remains outside its $3\sigma$ range. Hence this case is not viable.
Figure 16: $U^{HGR}_{2313}$ scatter plot of $\chi^2$ (left fig.) over $\nu - \lambda$ (in radians) plane and $\theta_{13}$ (right fig.) over $\theta_{23} - \theta_{12}$ (in degrees) plane.

8 Rotations–$R_{\alpha\beta}^lV_{HG}.R_\gamma^r(\alpha\beta \neq \gamma\delta)$

Here we first discuss the perturbative schemes for which $\alpha\beta \neq \gamma\delta$ and investigate their role for fitting the neutrino mixing data in parameter space.

8.1 12-13 Rotation

This correction scheme pertains to rotation in 12 and 13 sector of HG mixing matrix. Under small rotation limit, we have $\sin \theta \approx \theta$ and $\cos \theta \approx 1 - \theta^2$, so the expressions for neutrino mixing angles truncated at order $O(\theta^2)$ are given as

$$\sin \theta_{13} \approx |\mu V_{23} + \lambda V_{11} + \mu \lambda V_{21}|,$$
$$\sin \theta_{23} \approx \left|\frac{(1 - \mu^2 - \lambda^2)V_{23} + \lambda V_{21} - \mu \lambda V_{11}}{\cos \theta_{13}}\right|,$$
$$\sin \theta_{12} \approx \left|\frac{(1 - \mu^2)V_{12} + \mu V_{22}}{\cos \theta_{13}}\right|.$$

In Fig. 17, we present our investigation results with $\theta_1 = \lambda$ and $\theta_2 = \mu$. The salient features of this mixing scheme are:

(i) Here mixing angles exhibit good correlations among themselves since perturbation parameters ($\mu, \lambda$) enters into all mixing angles at leading order.

(ii) The minimum value of $\chi^2 \sim 7.66(9.46)$ which produces $\theta_{12} \sim 34.73^\circ (31.55^\circ)$, $\theta_{23} \sim 42.09^\circ (50.85^\circ)$ and $\theta_{13} \sim 8.36^\circ (8.44^\circ)$.

(iii) This mixing case can fit all mixing angles at 2$\sigma$ for NH while it is consistent only at 3$\sigma$ for IH case.

8.2 12-23 Rotation

This mixing case pertains to rotation in 12 and 23 sector of HG mixing matrix. The expressions for neutrino mixing angles under small rotation limit are given as
Figure 17: $U_{HGLR}^{1213}$ scatter plot of $\chi^2$ (left fig.) over $\mu - \lambda$ (in radians) plane and $\theta_{13}$ (right fig.) over $\theta_{23} - \theta_{12}$ (in degrees) plane. The information about other color coding and various horizontal, vertical lines in right fig. is given in text.

\[
\sin \theta_{13} \approx \left| \mu V_{23} + \nu V_{12} + \mu \nu V_{22} \right|, \quad (8.4)
\]
\[
\sin \theta_{23} \approx \left| \frac{(1 - \mu^2 - \nu^2) V_{23} + \nu V_{22} - \mu \nu V_{12}}{\cos \theta_{13}} \right|, \quad (8.5)
\]
\[
\sin \theta_{12} \approx \left| \frac{(1 - \mu^2 - \nu^2) V_{12} + \mu V_{22} - \mu \nu V_{23}}{\cos \theta_{13}} \right|. \quad (8.6)
\]

In Fig. 18, we present our numerical results for this case with $\theta_1 = \nu$ and $\theta_2 = \mu$. The main features of this mixing scheme are:

(i) As like previous case, correction parameters enters at leading order into the expressions of these mixing angles and hence show interesting correlations among themselves.

(ii) The minimum value of $\chi^2 \sim 1.59(3.04)$ and it produces $\theta_{12} \sim 34.27^\circ (34.36^\circ)$, $\theta_{23} \sim 49.94^\circ (49.83^\circ)$ and $\theta_{13} \sim 8.38^\circ (8.40^\circ)$.

(iii) This mixing scheme can fit all mixing angles at $2\sigma$ level for NH and IH.

8.3 13-12 Rotation

This mixing case pertains to rotation in 13 and 12 sector of HG mixing matrix. The expressions for neutrino mixing angles with small perturbation parameters $\mu$ and $\lambda$ are given as

\[
\sin \theta_{13} \approx |\lambda V_{33}|, \quad (8.7)
\]
\[
\sin \theta_{23} \approx \left| \frac{V_{23}}{\cos \theta_{13}} \right|, \quad (8.8)
\]
\[
\sin \theta_{12} \approx \left| \frac{(1 - \mu^2 - \lambda^2) V_{12} + \mu V_{11} + \lambda V_{22} + \mu \lambda V_{21}}{\cos \theta_{13}} \right|. \quad (8.9)
\]

In Fig. 19, we show our numerical findings for this case with $\theta_1 = \lambda$ and $\theta_2 = \mu$. The main characteristics features of this mixing scheme are:
Figure 18: $U_{HGLR}^{1223}$ scatter plot of $\chi^2$ (left fig.) over $\mu - \nu$ (in radians) plane and $\theta_{13}$ (right fig.) over $\theta_{23} - \theta_{12}$ (in degrees) plane. The information about color coding and various horizontal, vertical lines in right fig. is given in text.

Figure 19: $U_{HGLR}^{1312}$ scatter plot of $\chi^2$ (left fig.) over $\lambda - \mu$ (in radians) plane and $\theta_{13}$ (right fig.) over $\theta_{23} - \theta_{12}$ (in degrees) plane. The information about color coding and various horizontal, vertical lines in right fig. is given in text.

(i) For mixing angle $\theta_{23}$, correction parameter $\lambda$ enters only through $\sin \theta_{13}$ and hence its value remains quite close to its unperturbed prediction.

(ii) The minimum value of $\chi^2 \sim 0.82(5.04)$ and it produces $\theta_{12} \sim 33.54^\circ(33.28^\circ)$, $\theta_{23} \sim 45.62^\circ(45.63^\circ)$ and $\theta_{13} \sim 8.41^\circ(8.47^\circ)$.

(iii) This perturbative case can fit all mixing angles at 1$\sigma$ for NH. However for IH same fitted range of $\theta_{23}$ corresponds to 2$\sigma$ level. Hence it is allowed at 1$\sigma$(2$\sigma$) level.

8.4 13-23 Rotation

This case refers to rotation in 13 and 23 sector of HG mixing matrix. The expressions for neutrino mixing angles under small rotation limit are given as
Figure 20: $U_{HGLR}^{1323}$ scatter plot of $\chi^2$ (left fig.) over $\lambda - \nu$ (in radians) plane and $\theta_{13}$ (right fig.) over $\theta_{23} - \theta_{12}$ (in degrees) plane. The information about color coding and various horizontal, vertical lines in right fig. is given in text.

$$\sin \theta_{13} \approx |\nu V_{12} - \lambda V_{23} + \nu \lambda V_{22}|,$$

$$\sin \theta_{23} \approx \left|\frac{(1 - \nu^2)V_{23} + \nu V_{22}}{\cos \theta_{13}}\right|,$$

$$\sin \theta_{12} \approx \left|\frac{(1 - \lambda^2 - \nu^2)V_{12} + \lambda V_{22} + \nu \lambda V_{23}}{\cos \theta_{13}}\right|.$$  

Fig. 20 corresponds to perturbed HG with $\theta_1 = \lambda$ and $\theta_2 = \nu$. The main characteristics of this scheme are:

(i) The correction parameters $\lambda$ and $\nu$ enters into the expressions of mixing angles at leading order and thus they show good correlations among themselves.

(ii) The minimum value of $\chi^2 \sim 10.5(34.5)$ and it produces $\theta_{12} \sim 34.99^\circ(35.96^\circ)$, $\theta_{23} \sim 41.08^\circ(42.45^\circ)$ and $\theta_{13} \sim 8.40^\circ(8.39^\circ)$.

(iii) This mixing case is able to fit all mixing angles in the region which lies quite close to $2\sigma$ boundaries. However it is only consistent at $3\sigma$ level.

8.5 23-12 Rotation

This perturbative scheme pertains to rotation in 23 and 12 sector of HG mixing matrix. However in this case $\theta_{13}$ doesn’t get any corrections from perturbation matrix i.e. $\theta_{13} = 0$. So we left any further discussion of this mixing case.

8.6 23-13 Rotation

This correction case is much similar to 13-12 rotation with interchange of expressions for $\theta_{12}$ and $\theta_{23}$ mixing angles. The expressions of neutrino mixing angles for small perturbation parameters $\nu$ and $\lambda$ are given as
Figure 21: $U_{HGLR}^{213}$ scatter plot of $\chi^2$ (left fig.) over $\nu - \lambda$ (in radians) plane and $\theta_{13}$ (right fig.) over $\theta_{23} - \theta_{12}$ (in degrees) plane. The information about color coding and various horizontal, vertical lines in right fig. is given in text.

\[
\sin \theta_{13} \approx |\lambda V_{11}|, \quad (8.13) \\
\sin \theta_{23} \approx |\frac{(1 - \nu - \nu^2 - \lambda^2) V_{23} + \lambda (1 + \nu) V_{21}}{\cos \theta_{13}}|, \quad (8.14) \\
\sin \theta_{12} \approx \frac{|V_{12}|}{\cos \theta_{13}}. \quad (8.15)
\]

In Fig. 21, we present our investigation results for this case with $\theta_1 = \lambda$ and $\theta_2 = \nu$. The main characteristics of this mixing scheme are:

(i) Here corrections to mixing angle $\theta_{12}$ enters through only $\sin \theta_{13}$ so its value remain near to its unperturbed value. However $\theta_{23}$ can have wide range of values since it get leading order correction from both perturbation parameters.

(ii) The minimum value of $\chi^2 \sim 12.9(12.3)$ for this case which gives $\theta_{12} \sim 30.36^\circ(30.36^\circ)$, $\theta_{23} \sim 47.89^\circ(48.37^\circ)$ and $\theta_{13} \sim 8.42^\circ(8.51^\circ)$.

(iii) This scenario is unable to bring $\theta_{12}$ to its allowed range. So it is not consistent even at $3\sigma$ level.

9 Rotations - $R_{\alpha\beta} V_{HG} R_{\gamma\delta}(\alpha\beta = \gamma\delta)$

Now we will take up the rotation schemes where $\alpha\beta = \gamma\delta$ and investigate their significance in fitting the neutrino mixing data.

9.1 12-12 Rotation

This perturbative scheme pertains to rotation in 12 sector of HG mixing matrix. Here 12 rotation matrix operates from left as well as right side of unperturbed matrix. The expressions for neutrino mixing angles under small rotation limit are given by
$$\sin \theta_{13} \approx |\mu_1 V_{23}|,$$  \hspace{1cm} (9.1)

$$\sin \theta_{23} \approx \left| \frac{(\mu_1^2 - 1)V_{23}}{\cos \theta_{13}} \right|,$$  \hspace{1cm} (9.2)

$$\sin \theta_{12} \approx \left| \frac{(1 - \mu_1^2 - \mu_2^2)V_{12} + \mu_1 V_{22} + \mu_2 V_{11} + \mu_1 \mu_2 V_{21}}{\cos \theta_{13}} \right|. \hspace{1cm} (9.3)$$

In Fig. 22, we present our numerical results corresponding to this perturbative case with $\theta_1 = \mu_2$ and $\theta_2 = \mu_1$. The main characteristics of this mixing are:

(i) Here atmospheric mixing angle($\theta_{23}$) remains quite close to its original value since it gets correction only of $O(\theta^2)$ from perturbation matrix. However $\theta_{12}$ receives leading order correction from parameter $\mu_1$ and $\mu_2$ and thus possess wide range of values in parameter space.

(ii) The minimum value of $\chi^2 \sim 1.94(11.0)$ for this case which pertains to $\theta_{12} \sim 33.45^\circ (33.39^\circ)$, $\theta_{23} \sim 44.37^\circ (44.36^\circ)$ and $\theta_{13} \sim 8.39^\circ (8.49^\circ)$.

(iii) This rotation case can fit all mixing angles at 1$\sigma$ level for NH. However the same best fitted range of $\theta_{23}$ corresponds to 2$\sigma$ level in IH. This this mixing case is allowed at 1$\sigma$ (2$\sigma$) level for NH(IH).

### 9.2 13-13 Rotation

In this correction scheme, 13 rotation matrix acts from left as well as right side of HG mixing matrix. The expressions for neutrino mixing angles under small rotation limit are given as
\[ \sin \theta_{13} \approx |\lambda_1 V_{23} - \lambda_2 V_{11} - \lambda_1 \lambda_2 V_{21}|, \]  
\[ \sin \theta_{23} \approx \left| \frac{(1 - \lambda_2^2) V_{23} + \lambda_2 V_{21}}{\cos \theta_{13}} \right|, \]  
\[ \sin \theta_{12} \approx \left| \frac{(1 - \lambda_1^2) V_{12} + \lambda_1 V_{22}}{\cos \theta_{13}} \right|. \]  

In Fig. 23, we show our results pertaining to this case with $\theta_1 = \lambda_2$ and $\theta_2 = \lambda_1$. The salient features of this mixing scheme are:

(i) Here perturbation parameters go into all mixing angles at leading order so this mixing scheme present nice correlations among themselves.

(ii) The minimum value of $\chi^2 \approx 0.07(0.15)$ for this case which gives $\theta_{12} \sim 33.43^\circ (33.67^\circ)$, $\theta_{23} \sim 48.60^\circ (48.53^\circ)$ and $\theta_{13} \sim 8.42^\circ (8.48^\circ)$.

(iii) This mixing case is most favorable among all discussed cases as it can fit all mixing angles at $1\sigma$ level with much lower value of $\chi^2$ for NH and IH.

### 9.3 23-23 Rotation

In this perturbation scheme, 23 rotation matrix operates on left as well as right side of HG mixing matrix. The expressions for neutrino mixing angles under small rotation limit are given as

\[ \sin \theta_{13} \approx |\nu_2 V_{12}|, \]  
\[ \sin \theta_{23} \approx \left| \frac{(1 - \nu_1 - \nu_1^2 - \nu_2^2) V_{23} + \nu_2 (1 + \nu_1) V_{22}}{\cos \theta_{13}} \right|, \]  
\[ \sin \theta_{12} \approx \left| \frac{(\nu_2^2 - 1) V_{12}}{\cos \theta_{13}} \right|. \]
In Fig. 24, we present our numerical investigation results for this case with $\theta_1 = \nu_2$ and $\theta_2 = \nu_1$. The main characteristics of this mixing are:

(i) Here solar mixing angle ($\theta_{12}$) remains quite near to its original value since it receives corrections only of $O(\theta^2)$ from perturbation matrix. However $\theta_{23}$ gets corrected at leading order from $\nu_1$ and $\nu_2$ and thus can possess wide range of values in parameter space.

(ii) The minimum value of $\chi^2 \sim 27.1(26.6)$ for this case which gives $\theta_{12} \sim 28.91^\circ(28.89^\circ)$, $\theta_{23} \sim 47.84^\circ(48.11^\circ)$ and $\theta_{13} \sim 8.37^\circ(8.44^\circ)$.

(iii) This mixing case fails to bring $\theta_{12}$ in its $3\sigma$ domain for NH and IH. Thus it is not viable.

10 Summary and Conclusions

Hexagonal mixing is one of the interesting possibility among various proposed mixing schemes to explain neutrino mixing data with a common prediction of vanishing reactor mixing angle. The atmospheric mixing angle ($\theta_{23}$) is maximal and solar mixing angle ($\theta_{12}$) value is $30^\circ$ in this scenario. However neutrino flavor mixing data point towards non zero reactor mixing angle ($\theta_{13} \approx 8^\circ$) with departure of other two mixing angles from predicted values. Thus this mixing scheme should be checked for its consistency under various perturbative schemes. In this work, we presented a systematic analysis of perturbations around this mixing scenario. The corrections are parametrized in terms of three orthogonal rotation matrices $R_{12}$, $R_{13}$ and $R_{23}$ which acts on 12, 13 and 23 sector of unperturbed PMNS matrix respectively. We performed numerical investigation of possible cases for which perturbation matrix is governed by one and two rotation matrices. Thus corresponding modified PMNS matrix is of the forms $\left( R_{ij} \cdot V_{HG}, R_{ij} \cdot R_{kl} \cdot V_{HG}, V_{HG} \cdot R_{ij}, V_{HG} \cdot R_{ij} \cdot R_{kl}, R_{ij} \cdot V_{HG} \cdot R_{kl} \right)$ where $V_{HG}$ is unperturbed HG matrix. From a theoretical point of view, PMNS matrix is given by $U_{PMNS} = U^\dagger \cdot U_v$ so these corrections might originate from charged lepton, neutrino or from both sectors. In this work, we assumed all CP violating phases to be zero.
The effects of Dirac CP violation on current analysis for single rotation case is already discussed in our another paper. For our investigation, we constructed a $\chi^2$ function which is a combined measure of deviation of three mixing angles coming from perturbed mixing matrix to that from experimental best fit values. We performed the scanning of parameter space with varying correction parameters in perturbative limits to find best fit in each case. The numerical results of our study are presented in terms of $\chi^2$ vs perturbation parameters and as correlations among different neutrino mixing angles.

The rotation $R_{ij}^l \cdot V_{HG}$, provides negligible corrections to $\theta_{23}$ and thus its value remain close to its unperturbed prediction. Moreover it is unable to bring $\theta_{12}$ in its $3\sigma$ range and thus this case is ruled out completely. Much like previous case, $\theta_{23}$ receives very minor corrections that comes through $\theta_{13}$ in rotation $R_{13}^l \cdot V_{HG}$ and thus its value remains almost close to its unperturbed prediction. This case also fails to bring $\theta_{12}$ under its $3\sigma$ periphery and thus it is not viable. For $R_{23}^l \cdot V_{HG}$ and $V_{HG} \cdot R_{ij}^l$ rotation, $\theta_{13}$ doesn’t receive any perturbative corrections and thus its value remains quite close to its unperturbed prediction which is outside its $3\sigma$ range. The $V_{HG} \cdot R_{23}^l$ rotation case provides $O(\theta^2)$ corrections to $\theta_{12}$ and hence like previous case its values lies close to its original prediction. Thus this case is also not consistent with mixing data.

The rotation $R_{ij}^l \cdot R_{13}^l \cdot V_{HG}$ and $R_{ij}^l \cdot R_{12}^l \cdot V_{HG}$ provides $O(\theta^2)$ corrections to $\theta_{23}$ and thus its value remain close to its original prediction which lies in $1\sigma$ and $2\sigma$ interval for NH and IH respectively. The solar mixing angle, $\theta_{12}$ can have wide range of values in parameter space since it gets leading order corrections from perturbation parameters. Thus both cases are consistent at $1\sigma(2\sigma)$ for NH(IH). The all other cases in mixing scheme $R_{ij}^l \cdot R_{kl}^l \cdot V_{HG}$ are not viable as they are unable to fit all mixing angles even at $3\sigma$ level.

The perturbation case $V_{HG} \cdot R_{ij}^l \cdot R_{13}^l$ prefers two regions for atmospheric mixing angle of $\theta_{23} \sim 36^0 - 42^0$ and $\theta_{23} \sim 48^0 - 54^0$. The later region is able to provide required range of $\theta_{23}$ in tiny parameter space for NH(IH) at $2\sigma$ level. However $\theta_{12}$ can have wide range of values in parameter space and this case is allowed at $2\sigma$ level for NH(IH). The rotation $V_{HG} \cdot R_{ij}^l \cdot R_{12}^l$, much like previous case, prefers $\theta_{23} \sim 40^0$ and $\theta_{23} \sim 50^0$. The later part falls under required range of $\theta_{23}$ at $2\sigma$ level for NH(IH). The $\theta_{12}$ possess much wider range of values in parameter space and this case is also allowed at $2\sigma$ level for NH(IH). The all other cases in rotation scheme $V_{HG} \cdot R_{ij}^l \cdot R_{kl}^l$ are not consistent at they failed to fit all mixing angles within required range.

The rotation $R_{ij}^l \cdot V_{HG} \cdot R_{12}^l$ and $R_{13}^l \cdot V_{HG} \cdot R_{12}^l$ provides minor corrections to $\theta_{23}$ and thus its value remains quite close to its unperturbed value. However $\theta_{12}$ can have wide range of values in parameter space since it gets leading order correction from perturbation parameters. The fitted value of $\theta_{23}$ remains in $1\sigma$ limit for NH and $2\sigma$ for IH. Thus both cases are consistent at $1\sigma(2\sigma)$ for NH(IH). The pertubative case $R_{13}^l \cdot V_{HG} \cdot R_{13}^l$ is most preferable as it can fit all mixing angles at $1\sigma$ level for NH(IH) with lowest value of $\chi^2$ among all cases discussed here. The reported value is $\chi^2_{min} \sim 0.07(0.15)$ in parameter space of this case for NH(IH). The $R_{13}^l \cdot V_{HG} \cdot R_{23}^l$ is only able to fit all mixing angles for a very small region of parameter space. This case is consistent at $3\sigma$ level for NH(IH). The
rotation $R_{12}^{l} \cdot V_{HG} \cdot R_{13}^{r}$ and $R_{12}^{l} \cdot V_{HG} \cdot R_{23}^{r}$ comes under allowed region for a small region of parameter space. They are consistent at $2\sigma(3\sigma)$ and $2\sigma(2\sigma)$ respectively. The mixing case $R_{23}^{l} \cdot V_{HG} \cdot R_{23}^{r}$ and $R_{23}^{l} \cdot V_{HG} \cdot R_{13}^{r}$ imparts negligible corrections to $\theta_{12}$ and thus its value remains near to its original prediction in parameter space. This fitted values lies outside its $3\sigma$ range and thus both cases are not viable for NH as well as for IH. The rotation $R_{23}^{l} \cdot V_{HG} \cdot R_{12}^{r}$ doesn’t impart corrections to $\theta_{13}$ and hence we left out its discussion any further.

This finishes our discussion on checking the consistency of Perturbed HG mixing for various perturbative schemes with latest neutrino mixing data. This analysis might be useful in restricting large number of possible models which offers different corrections to this mixing scheme. It thus can serve as a guideline for neutrino model building in this scenario. However all such issues including the origin of these perturbations are left for future studies.

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A Results: Summary

In Table 2, we present our results in form ($\chi^2_{\text{min}}$, Best fit) for all considered mixing schemes.
(\chi^2_{\text{min}}, \text{Best fit level}) \text{ for NH and IH from Mixing angles fitting}

| Rotation-NH | HG | $R'_{ij} \cdot R'_{ik} \cdot U$ | HG | $U \cdot R'^{r}_{ij} \cdot R'^{r}_{ik}$ | HG | $R'^{r}_{ij} \cdot U \cdot R'^{r}_{ik}$ | HG |
|-------------|----|-------------------------------|----|---------------------------------|----|---------------------------------|----|
| $U^{l}_{12}$ | (38.6, ×) | $R'^{l}_{12} \cdot R'^{l}_{13}$ | (1.52, 1σ) | $R'^{l}_{12} \cdot R'^{l}_{13}$ | (6.37, 2σ) | $R'^{l}_{12} \cdot R'^{l}_{13}$ | (7.66, 2σ) |
| $U^{l}_{13}$ | (37.5, ×) | $R'^{l}_{12} \cdot R'^{l}_{23}$ | (15.2, ×) | $R'^{l}_{12} \cdot R'^{l}_{23}$ | (12.6, ×) | $R'^{l}_{12} \cdot R'^{l}_{23}$ | (1.59, 2σ) |
| $U^{l}_{23}$ | (731.5, −) | $R'^{l}_{13} \cdot R'^{l}_{12}$ | (1.0, 1σ) | $R'^{l}_{13} \cdot R'^{l}_{12}$ | (0.60, 2σ) | $R'^{l}_{13} \cdot R'^{l}_{12}$ | (0.82, 1σ) |
| $U^{l}_{12}$ | (716.8, −) | $R'^{l}_{13} \cdot R'^{l}_{23}$ | (27.6, ×) | $R'^{l}_{13} \cdot R'^{l}_{23}$ | (12.8, ×) | $R'^{l}_{13} \cdot R'^{l}_{23}$ | (10.5, 3σ) |
| $U^{l}_{13}$ | (13.5, ×) | $R'^{l}_{23} \cdot R'^{l}_{12}$ | (36.7, ×) | $R'^{l}_{23} \cdot R'^{l}_{12}$ | (19.4, ×) | $R'^{l}_{23} \cdot R'^{l}_{12}$ | (715.5, −) |
| $U^{l}_{23}$ | (46.2, ×) | $R'^{l}_{23} \cdot R'^{l}_{13}$ | (36.8, ×) | $R'^{l}_{23} \cdot R'^{l}_{13}$ | (13.0, ×) | $R'^{l}_{23} \cdot R'^{l}_{13}$ | (12.9, ×) |

| Rotation-III |
|-------------|----|-------------------------------|----|---------------------------------|----|---------------------------------|----|
| $U^{l}_{12}$ | (49.9, ×) | $R'^{l}_{12} \cdot R'^{l}_{13}$ | (7.78, 2σ) | $R'^{l}_{12} \cdot R'^{l}_{13}$ | (8.70, 2σ) | $R'^{l}_{12} \cdot R'^{l}_{13}$ | (9.46, 3σ) |
| $U^{l}_{13}$ | (44.2, ×) | $R'^{l}_{12} \cdot R'^{l}_{23}$ | (21.6, ×) | $R'^{l}_{12} \cdot R'^{l}_{23}$ | (52.5, ×) | $R'^{l}_{12} \cdot R'^{l}_{23}$ | (3.04, 2σ) |
| $U^{l}_{23}$ | (860.2, −) | $R'^{l}_{13} \cdot R'^{l}_{12}$ | (5.49, 2σ) | $R'^{l}_{13} \cdot R'^{l}_{12}$ | (2.03, 2σ) | $R'^{l}_{13} \cdot R'^{l}_{12}$ | (5.04, 2σ) |
| $U^{l}_{12}$ | (852.6, −) | $R'^{l}_{13} \cdot R'^{l}_{23}$ | (44.1, ×) | $R'^{l}_{13} \cdot R'^{l}_{23}$ | (13.8, ×) | $R'^{l}_{13} \cdot R'^{l}_{23}$ | (34.5, 3σ) |
| $U^{l}_{13}$ | (14.3, ×) | $R'^{l}_{23} \cdot R'^{l}_{12}$ | (39.1, ×) | $R'^{l}_{23} \cdot R'^{l}_{12}$ | (84.9, ×) | $R'^{l}_{23} \cdot R'^{l}_{12}$ | (844.8, −) |
| $U^{l}_{23}$ | (110.7, ×) | $R'^{l}_{23} \cdot R'^{l}_{13}$ | (39.1, ×) | $R'^{l}_{23} \cdot R'^{l}_{13}$ | (12.4, ×) | $R'^{l}_{23} \cdot R'^{l}_{13}$ | (12.3, ×) |

**Table 2:** Here ‘×’ refers to the case which is unable to fit mixing angles even at 3σ level.
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