Open $N=2$ Strings in a B-Field Background and Noncommutative Self-Dual Yang-Mills

Olaf Lechtenfeld, Alexander D. Popov* and Bernd Spendig

Institut für Theoretische Physik
Universität Hannover
Appelstraße 2, 30167 Hannover, Germany
Email: lechtenf, popov, spendig@itp.uni-hannover.de

Abstract

In the presence of $D$-branes, fermionic $N=2$ strings in 2+2 dimensions can be coupled to a Kähler NS-NS two-form $B$. We present the corresponding action which produces $N=2$ supersymmetric boundary conditions and discuss the Seiberg-Witten zero-slope limit. After recalling the constraints on the Chan-Paton gauge group, we demonstrate for $U(n)$ groups that the open $N=2$ string with a nonzero $B$-field coincides on tree level with noncommutative self-dual Yang-Mills. Several misconceptions of hep-th/0011206 are corrected.

* On leave from Bogoliubov Laboratory of Theoretical Physics, JINR, Dubna, Russia
1 Introduction and results

A constant NS-NS two-form background modifies string dynamics nontrivially if D-branes are present \([1, 2, 3, 4, 5]\). In particular, open strings ending on \(n\) coincident \(D\)-branes see a deformed space-time metric \(G_{\mu\nu}\) and acquire a noncommutativity parameter \(\theta^{\mu\nu}\). The latter means that the \(D\)-brane world volume carrying the \(U(n)\) Yang-Mills fields becomes noncommutative. In an \(\alpha' \to 0\) limit which keeps the above open-string parameters finite the string indeed reduces to noncommutative \(U(n)\) gauge theory on the brane \([1, 2, 3, 4, 5]\). A restriction to \(SO(n)\) or \(Sp(n)\) subgroups is nontrivial but emerges via orientifold projection \([7]\).

In the present letter we apply this analysis to \(N=2\) strings. Twenty years ago it was discovered \([8]\) that the open \(N=2\) fermionic string at tree level is identical to self-dual Yang-Mills field theory in \(2+2\) dimensions. The complete absence of a massive physical spectrum ties in with the vanishing of all string amplitudes beyond three-point although the one-loop structure \([3]\) seems to be anomalous \([10]\).

It is expected that switching on a constant \(B\)-field background renders the self-dual gauge theory noncommutative. However, it also interferes with the global world-sheet supersymmetry of fermionic strings in the superconformal gauge \([11]\). We shall show that a boundary term must be added to the \(N=2\) string action to preserve its supersymmetries for \(B_{\mu\nu} \neq 0\). In addition, it turns out that the two-form \(B_{\mu\nu}dx^\mu \wedge dx^\nu\) must be Kähler.

In order to verify the identity of the open \(N=2\) string in a constant \(B\)-field background with noncommutative self-dual Yang-Mills theory, we shall discuss the factorization of open-string trees in this context and prove that the four-point amplitude of noncommutative \(U(n)\) self-dual gauge theory vanishes in accord with the string result. As is demonstrated for the example of \(U(2)\), the restriction of \(U(n)\) to \(SU(n)\) is not admissible \([12, 13, 14, 7]\). This explains the erroneous result of a recent paper \([13]\) claiming inconsistency for non-abelian gauge groups.

2 Open \(N=2\) strings

The critical \(N=2\) string lives in 2+2 real or equivalently 1+1 complex dimensions. Put differently, the string world sheet \(\Sigma\) is embedded into a four-dimensional target space with signature \((2, 2)\). The Brink-Schwarz action \([13]\) for the \(N=2\) string in \(\mathbb{R}^{2,2}\) is given by

\[
S = -\frac{1}{2\pi\alpha'} \int_{\Sigma} d^2\xi \ e \left\{ \frac{1}{2} h^{\alpha\beta} \partial_\alpha X^{-a} \partial_\beta X^{+a} + \frac{i}{2} \bar{\psi}^{-\bar{a}} \rho^\alpha \bar{D}_{\bar{\alpha}} + A_\alpha \bar{\psi}^{-\bar{a}} \rho^\alpha \psi^{+a} + \left( \partial_\alpha X^{+a} + \bar{X}_{\bar{\alpha}} \psi^{+a} \right) \bar{\psi}^{-\bar{a}} \rho^\beta \rho^\gamma X^\beta_{\gamma} + \left( \partial_\alpha X^{-a} + \bar{X}_{\bar{\alpha}} \psi^{-a} \right) \psi^{+a} \bar{X}^\beta_{\gamma} \rho^\beta \rho^\gamma \right\} \eta_{a\bar{a}}. \tag{1}
\]

The matter fields \(X^{+a}\) and \(\psi^{+a}\) are complex valued \((X^{-\bar{a}}=(X^{+a})^*, \ \psi^{-\bar{a}}=(\psi^{+a})^*)\), so that the space-time indices \(a, \bar{a} = 1, 2\) run over two values only. The fields are coupled to the \(N=2\) supergravity multiplet consisting of the zweibein \(e_\alpha^m\) (related to the world-sheet metric \(h_{\alpha\beta}\) via \(h_{\alpha\beta} = \eta_{mn} e_\alpha^m e_\beta^n\)), the complex gravitino \(\chi_\alpha^a\) and the \(U(1)\) connection \(A_\alpha\). Using symmetries of the action (see e.g. \([7]\) for a discussion) one can locally gauge away all gravitational degrees of freedom. In this superconformal gauge the action becomes

\[
S = -\frac{1}{4\pi\alpha'} \int_{\Sigma} d^2\xi \ h^{\alpha\bar{\beta}} \left( \partial_\alpha X^{-a} \partial_\beta X^{+a} + \frac{i}{2} \bar{\psi}^{-\bar{a}} \rho_\alpha \partial_\beta \psi^{+a} + \frac{i}{2} \bar{\psi}^{+a} \rho_\alpha \partial_\bar{\beta} \psi^{-\bar{a}} \right) \eta_{a\bar{a}}. \tag{2}
\]

\[1\] We use \(\rho^\alpha = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \ \rho_\beta = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \ \{\rho^m, \rho^n\} = 2 \eta^{mn}, \ (\eta^{mn}) = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \ \xi^a = \tau, \ \xi^{\bar{a}} = \sigma\). The space-time metric is \((\eta_{a\bar{a}}) = \zeta \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\), where \(\zeta > 0\) is a real scaling parameter. For the definition of \(D_\alpha, \bar{\psi}, \chi, e\) see \([17]\).
We now switch to a real notation via \((\pm a) \to (\mu), \) with \(\mu, \nu, \ldots = 1, 2, 3, 4.\) To be more explicit,

\[
X^1 := \frac{1}{2}(X^1 + X^{-1}) \quad , \quad X^2 := \frac{1}{2i}(X^1 - X^{-1}) \quad ,
\]
\[
X^3 := \frac{1}{2}(X^2 + X^{-2}) \quad , \quad X^4 := \frac{1}{2i}(X^2 - X^{-2}) \quad ,
\]
and analogously for the fermionic fields. The action functional then takes the form

\[
S = -\frac{1}{4\pi\alpha'} \int d^2\xi \left( \eta^{\alpha\beta}(\partial_\alpha X^\mu \partial_\beta X^\nu + i \bar{\psi}^\mu \rho_\alpha \partial_\beta \psi^\nu) \right) g_{\mu\nu} \quad ,
\]
with \((g_{\mu\nu}) = \zeta \text{ diag}(+1, +1, -1, -1).\) This action enjoys a residual gauge invariance under \(N=2\) superconformal transformations. In particular, the rigid \(N=2\) supersymmetry transformations have the form \([18, 19]\)

\[
\delta X^\mu = \bar{\sigma}_1 \psi^\mu + J^\mu_\nu \bar{\sigma}_2 \psi^\nu \quad ,
\]
\[
\delta \psi^\mu = -i \rho_\alpha \partial_\alpha X^\mu \varepsilon_1 + i J^\mu_\nu \rho_\alpha \partial_\alpha X^\nu \varepsilon_2 \quad .
\]

Here \((J^\mu_\nu)\) is a constant complex structure compatible with our metric, i.e. \(g_{\mu\nu}J^\mu_\lambda + J^\mu_\nu g_{\lambda\nu} = 0.\)

### 3 \(N=2\) supersymmetric boundary conditions

We now turn our attention to \(N=2\) open strings in a \(B\)-field background. Since \(B\)-field components not parallel to a \(D\)-brane world volume can be gauged away, we shall consider \(n\) coincident \(D3\)-branes in order to allow for the most general \(B\)-field configuration. Let us investigate how a \(B\)-field can be coupled to supersymmetric \(2d\) matter fields so that the action is still globally supersymmetric.

The gauge-fixed action functional derived from the standard superfield action is

\[
S = -\frac{1}{4\pi\alpha'} \int d^2\xi \left( \eta^{\alpha\beta} g_{\mu\nu} + \varepsilon^{\alpha\beta} 2\pi\alpha' B_{\mu\nu} \right) \left( \partial_\alpha X^\mu \partial_\beta X^\nu + i \bar{\psi}^\mu \rho_\alpha \partial_\beta \psi^\nu \right) \quad .
\]

The boundary conditions for \(X^\mu\) following from this action read

\[
(E_{\mu\nu} \partial_+ X^\nu - E_{\mu\nu} \partial_- X^\nu)|_{\partial\Sigma} = 0 \quad ,
\]

where

\[
E_{\mu\nu} := g_{\mu\nu} + 2\pi\alpha' B_{\mu\nu} \quad ,
\]
and \(\partial\Sigma = \{\xi^1=0, \pi\},\) while \(\partial_+ = \partial_0 + \partial_1\) and \(\partial_- = \partial_0 - \partial_1.\) The boundary conditions for \(\psi^\mu\) must get mapped to \([3]\) under supersymmetry. The appropriate fermionic boundary conditions are (see e.g. \([3, 11]\))

\[
(E_{\mu\nu} \psi^\nu_+ - \gamma E_{\mu\nu} \psi^\nu_-)|_{\partial\Sigma} = 0 \quad ,
\]

where we use the fact\(^2\) that \(\varepsilon^-_i = \varepsilon^+_i\) at \(\sigma=0\) and \(\varepsilon^-_i = \gamma \varepsilon^+_i\) at \(\sigma=\pi\) \((\gamma=+1\) for the Ramond sector, \(\gamma=-1\) for the Neveu-Schwarz sector).

A straightforward calculation shows that the fermionic boundary conditions derived from \([3]\) are inconsistent with \([3].\) It was shown in \([11]\) that the \(N=1\) fermionic string requires adding two

\(^2\) Recall that a Majorana spinor \(\varphi\) (in \(1+1\) dimensions) has two components \(\varphi^\pm = \frac{1}{2}(1 \pm \rho^0 \rho^0)i\varphi.\) Furthermore, \(\bar{\varphi} \equiv (\varphi_+, \varphi_-) = \varphi^1 \rho^0.\)
$B$-dependent boundary terms to restore supersymmetry. This leads to the following expression for the $N=1$ string action:

$$ S = -\frac{1}{4\pi\alpha'} \int d^2\xi \left[ (\eta^{\alpha\beta} g_{\mu\nu} + \epsilon^{\alpha\beta} 2\pi\alpha' B_{\mu\nu}) \partial_\alpha X^\mu \partial_\beta X^\nu + i E_{\mu\nu} \bar{\psi}^\mu \rho^\alpha \partial_\alpha \psi^\nu \right] . \quad (10) $$

For the $N=2$ string we find the same result. Furthermore, the second supersymmetry applied to the action leads to additional equations,

$$(E_{\nu\mu} J^\nu_\lambda \partial_\mu X^\lambda - E_{\mu\nu} J^\mu_\lambda \partial_\nu X^\lambda)|_{\partial \Sigma} = 0 . \quad (11)$$

These conditions are equivalent to (7) and thus pose no further constraint only if we demand that

$$ g_{\mu\nu} J^\mu_\lambda + J^\nu_\rho g_{\lambda\nu} = 0 \quad \text{and} \quad B_{\mu\nu} J^\mu_\lambda - J^\nu_\rho B_{\lambda\nu} = 0 . \quad (12) $$

These relations mean that $(g_{\mu\nu})$ is a hermitian metric and $B_{\mu\nu} dx^\mu \wedge dx^\nu$ has to be a Kähler two-form on $\mathbb{R}^{2,2}$, i.e. a closed two-form compatible with the complex structure $J=(J^\mu_\nu)$. It is important to notice that the action functional (10) cannot be written in terms of superfields. In particular, the action used in [15] is not $N=2$ supersymmetric without adding boundary terms.

4 Seiberg-Witten limit

We now want to investigate the effects of background $B$-fields on open $N=2$ strings and exhibit their effective field theory. The starting point is the form of the open-string correlators [4, 6],

$$ \langle X^\mu(\tau) X^\nu(\tau') \rangle = -\alpha' G^{\mu\nu} \ln(\tau - \tau')^2 + \frac{i}{2} \theta^{\mu\nu} \varepsilon(\tau - \tau') , \quad (13) $$

$$ \langle \psi^\mu(\tau) \psi^\nu(\tau') \rangle = \frac{G^{\mu\nu}}{\tau - \tau'} , \quad (14) $$

for $\tau, \tau' \in \partial \Sigma$. Here, $[E^{-1}]^{\mu\nu} \equiv [(g + 2\pi\alpha' B)^{-1}]^{\mu\nu} = G^{\mu\nu} + \frac{1}{2\pi\alpha'} \theta^{\mu\nu}$ yields the effective metric $G_{\mu\nu}$ seen by the open string and gives rise to the noncommutativity parameter $\theta^{\mu\nu}$ appearing in $[X^\mu(\tau), X^\nu(\tau)] = i\theta^{\mu\nu}$. With an appropriate choice of the $SO(2,2)$ generators [20], the matrices $J$ and $B$ can be written in terms of the generators of a $U(1) \times U(1)$ subgroup of $SO(2,2)$. Then, the complex structure $J$ and the most general ‘magnetic’ $B$-field are expressed as

$$ J = (J^\mu_\nu) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix} \quad \text{and} \quad B = (B_{\mu\nu}) = \begin{pmatrix} 0 & B_1 & 0 & 0 \\ -B_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & B_2 \\ 0 & 0 & -B_2 & 0 \end{pmatrix} . \quad (15) $$

In this basis we obtain

$$ (G^{\mu\nu}) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \zeta & 0 \\ 0 & 0 & 0 & -\zeta \end{pmatrix} , \quad (16) $$

$$ (\theta^{\mu\nu}) = \begin{pmatrix} 0 & -\frac{(2\pi\alpha')^2 B_1}{\zeta^2 + (2\pi\alpha' B_1)^2} & 0 & 0 \\ \frac{(2\pi\alpha')^2 B_1}{\zeta^2 + (2\pi\alpha' B_1)^2} & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{(2\pi\alpha')^2 B_2}{\zeta^2 + (2\pi\alpha' B_2)^2} \\ 0 & 0 & \frac{(2\pi\alpha')^2 B_2}{\zeta^2 + (2\pi\alpha' B_2)^2} & 0 \end{pmatrix} . \quad (17) $$
Note that for $B_2=-B_1$ the background will be self-dual, and the action (14) will have $N=4$ supersymmetry [19].

We also introduce a tetrad $e_{\hat{\mu}} = (e_{\hat{\mu}}^\nu)$ related to the metric $G$ by the formula

$$G^\mu_\nu = e^\mu_{\hat{\alpha}} e^{\nu}_{\hat{\lambda}} \eta^{\hat{\alpha} \hat{\lambda}},$$

where $(\eta^{\hat{\mu} \hat{\nu}}) = \text{diag}(+1,+1,-1,-1)$ is the metric in the orthonormal frame and $\hat{\mu}, \hat{\nu} = 1, \ldots, 4$ are Lorentz indices.

Next we calculate the effective open-string coupling $G_s$ which is related to the closed-string coupling $g_s$ via $G_s = g_s[\det G/\det(g + 2\pi \alpha' B)]^{1/2}$ and obtain

$$G_s = g_s \left[ (1 + (2\pi \alpha' / \zeta B_1)^2) (1 + (2\pi \alpha') / \zeta B_2)^2 \right]^{1/2}.$$  

The Seiberg-Witten limit consists of taking $\alpha' \to 0$ while sending $\zeta \sim (\alpha')^2 \to 0$ (and therefore $g_{\mu \nu} \to 0$) so that $G$, $G^{-1}$, and $\theta$ remain finite. This $\alpha' \sim \zeta^{1/2} \to 0$ limit is equivalent to the limit $B \to \infty$. We arrive at the following effective open-string coupling,

$$G_s \to \frac{g_{SM}^2}{2\pi} \equiv 4\pi^2 |B_1 B_2| = \text{const},$$

since $g_s \sim \zeta \sim (\alpha')^2$. The inverse open string metric $(G^{\mu \nu})$ and the matrix $(\theta^{\mu \nu})$ become

$$(G^{\mu \nu}) \to \begin{pmatrix} \frac{1}{(2\pi B_1)^2} & 0 & 0 & 0 \\ 0 & \frac{1}{(2\pi B_1)^2} & 0 & 0 \\ 0 & 0 & \frac{-1}{(2\pi B_2)^2} & 0 \\ 0 & 0 & 0 & \frac{-1}{(2\pi B_2)^2} \end{pmatrix} \quad \text{and} \quad (\theta^{\mu \nu}) \to \begin{pmatrix} 0 & -\frac{1}{B_1} & 0 & 0 \\ \frac{1}{B_1} & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{1}{B_2} \\ 0 & 0 & \frac{1}{B_2} & 0 \end{pmatrix}.$$  

A remark is in order. Kumar et al. [15] use a notation similar to ours. Their choice of $B$-field, though, leads to the ‘electric’ type of two-form, i.e. their $B$-field has non-vanishing components simultaneously in space and time direction. In [21] this type of field has been considered in much detail, and it was shown that it does not admit a zero-slope limit which produces a field theory on a noncommutative space-time. It is not clear to us how a $\ast$ product and noncommutative gauge field theory can appear in [15] without the $\alpha' \to 0$ limit.

5 Factorization of open-string trees

In the absence of a $B$-field background ($\theta=0$) it is well known [22] that the factorization properties of open string amplitudes (as required for unitarity) restrict the possible Chan-Paton gauge groups to $U(n)$, $SO(n)$, and $Sp(n)$. In addition, one observes [22] that the $U(1)$ part of $U(n)$ decouples from all amplitudes; hence, $SU(n)$ is admissible as well.

It is natural to ask whether turning on a non-vanishing constant $B$-field background further constrains the set of allowed Chan-Paton labels. This question has been answered in the appendix of [12] (correctly for $U(n)$) and in [7] (for $SO(n)$ and $Sp(n)$): all these classical Lie groups are still allowed. However, the analysis of [12] immediately shows and ref. [7] explains that the restriction $U(n) \to SU(n)$ is no longer permitted because the $U(1)$ degree of freedom ceases to decouple. Let us briefly review the argument for $U(n)$.
The full $M$-particle open-string tree-level scattering amplitude reads

$$T(1, 2, \ldots, M) = A(1, 2, \ldots, M) \text{tr}(\lambda_1 \lambda_2 \ldots \lambda_M) E(1, 2, \ldots, M) + \text{non-cyclic permutations},$$

where $A(1, 2, \ldots, M)$ denotes the uncharged $\theta=0$ primitive amplitude obtained from the disk diagram with external leg ordering $(1, 2, \ldots, M)$, and the anti-hermitian matrix $\lambda_i \in u(n)$ describes the group quantum number of the $i$th external particle, $i=1, 2, \ldots, M$. The only effect of the $B$-field background consists in multiplying each primitive amplitude with a phase $\theta$.

$$E(1, 2, \ldots, M) := \prod_{1 \leq j < \ell \leq M} e^{-\frac{\theta}{2}k_{j\mu}k_{\ell\nu}k_{i\nu}},$$

which, due to momentum conservation, is cyclically invariant just like the two factors it multiplies.

Let us focus on factorization. Whenever a partial sum of external momenta goes on-shell, the amplitude $T$ develops a pole whose residue should factorize into the $T$ amplitudes for the two halves of the cut diagram. For a given pole, a subset of the permutations in (22) contributes. Generically,

$$A(1, 2, \ldots, M) \sim \frac{1}{m^2 - s} \sum_X A(1, 2, \ldots, P, X) A(X, P+1, \ldots, M),$$

where $s = -(k_1 + k_2 + \ldots + k_P)^2$, and $X$ runs over all states in the spectrum with mass $m$. Similarly, for $u(n)$ (but not for $su(n)$!) one has

$$\text{tr}(\lambda_1 \lambda_2 \ldots \lambda_M) = -2 \sum_x \text{tr}(\lambda_1 \lambda_2 \ldots \lambda_P \lambda_x) \text{tr}(\lambda_x \lambda_{P+1} \ldots \lambda_M),$$

where $x$ labels a basis of anti-hermitian $u(n)$ generators normalized to $\text{tr}(\lambda_x \lambda_y) = -\frac{1}{2}\delta_{xy}$. Note that the product $\lambda_1 \lambda_2 \ldots \lambda_L \notin u(n)$ but lies in the universal enveloping algebra. Finally, momentum conservation yields the factorization

$$E(1, 2, \ldots, M) = E(1, 2, \ldots, P) E(P+1, \ldots, M).$$

Taken together, one sees that each term in (22) factorizes correctly by itself in case of a $U(n)$ Chan-Paton group. Yet, $T$ amplitude factorization functions under a somewhat weaker requirement. Since the primitive amplitudes $A$ at different leg orderings are further related by

$$A(L, \ldots, 2, 1) = (-1)^L A(1, 2, \ldots, L)$$

for $L$ massless external states, we may group the permutations in (22) in quartets. With the help of

$$E(L, \ldots, 2, 1) = E(1, 2, \ldots, L)^*,$$

the generic combination

$$(1, \ldots, P, P+1, \ldots, M) + (P, \ldots, 1, P+1, \ldots, M) + (1, \ldots, P, M, \ldots, P+1) + (P, \ldots, 1, M, \ldots, P+1)$$

produces a factor of $\text{tr}[\Lambda(1, \ldots, P) \Lambda(P+1, \ldots, M)]$ with

$$\Lambda(1, 2, \ldots, L) := \lambda_1 \lambda_2 \ldots \lambda_L E(1, 2, \ldots, L) - \lambda_L \ldots \lambda_2 \lambda_1 E(1, 2, \ldots, L)^* \in u(n),$$

5
multiplying the right-hand side of (23). Hence, a subgroup \( G \subset U(n) \) is compatible with factorization, if \( \lambda_1 \in \text{Lie}(G) \) implies \( \Lambda(1, 2, \ldots, L) \in \text{Lie}(G) \) so that the trace may be split by inserting a complete basis \( \{ \lambda_i \} \) for \( \text{Lie}(G) \).

When \( \theta=0 \), one has \( E=1 \), and the condition above holds for the classical groups \( U(n) \), \( SO(n) \), and \( Sp(n) \). In a \( B \)-field background the condition becomes nontrivial already at \( L=2 \),

\[
\lambda_1, \lambda_2 \in \text{Lie}(G) \quad \implies \quad \Lambda(1, 2) \equiv \lambda_1 \lambda_2 E(1, 2) - \lambda_2 \lambda_1 E(1, 2)^* \in \text{Lie}(G) \quad . \tag{30}
\]

This seems to exclude \( SO(n) \) and \( Sp(n) \) groups \[12\]. However, a refined analysis employing the orientifold construction for non-oriented open strings leads to a modified factorization condition, which is indeed fulfilled by \( SO(n) \) and \( Sp(n) \) \[7\]. In contrast, a reduction of \( U(n) \) to \( SU(n) \) in the orientable case is no longer possible because already \( \text{tr}[\Lambda(1, 2)] \neq 0 \), indicating the fusion of two \( SU(n) \)-charged states to a \( U(1) \)-charged one.

As mentioned before, the Seiberg-Witten limit \((\alpha' \to 0 \text{ but keeping the open-string parameters finite})\) reduces the string to a noncommutative quantum field theory. It is therefore not surprising that the list of admissible open-string gauge groups for \( \theta \neq 0 \) is in perfect agreement with the list of possible noncommutative Yang-Mills theories. In particular, the failure of the Moyal commutator \( f \ast g - g \ast f \) to close in \( su(n) \) signals the necessity for the coupling of an additional \( U(1) \) gauge boson enlarging \( SU(n) \) to \( U(n) \) \[13, 14\].

6 Noncommutative self-dual Yang-Mills

We have already stated that beyond three-point all tree-level amplitudes of the \( N=2 \) fermionic string are known to vanish. It is not always appreciated \[13\] that complete \( N=2 \) string amplitudes even at tree level include a sum over world-sheet instanton sectors labeled by the first Chern number of the gauged R-symmetry \( U(1) \) bundle, which turns each primitive amplitude into a function \( A(G_s, \vartheta) \) not only of the open-string coupling \( G_s \) but also of an instanton (theta) angle \( \vartheta \) \[24\]. Surprisingly, \( SO(2, 2) \) ‘Lorentz’ transformations treat \( \sqrt{G_s} (\cos \frac{\vartheta}{2}, \sin \frac{\vartheta}{2}) \) as a \((\frac{1}{2}, 0)\) spinor, so that we may put \( G_s=1 \) and \( \vartheta=0 \) in a suitable Lorentz frame \[24\]. The resulting three-string amplitude \[3\] (in flat \( \mathbb{R}^{2,2} \) with \( B=0 \)),

\[
T_3(1, 2, 3) = A_3(1, 2, 3) \text{tr}(\lambda_1 \lambda_2 \lambda_3) + A_3(2, 1, 3) \text{tr}(\lambda_2 \lambda_1 \lambda_3) = k_1^+ \wedge k_2^+ \text{tr}(\lambda_1 [\lambda_2, \lambda_3]) \quad , \tag{31}
\]

represents the totally symmetric cubic interaction of the Leznov \[23\] prepotential \( \phi \) for self-dual Yang-Mills theory \[24\]. For more than three external legs, any tree-level \( N=2 \) string scattering vanishes already on the level of the primitive amplitudes, \( A(1, 2, \ldots, L>3) = 0 \), thanks to the kinematical identity

\[
k_1^+ \wedge k_2^+ \frac{1}{s_{12}} k_3^+ \wedge k_4^+ + k_2^+ \wedge k_3^+ \frac{1}{s_{23}} k_1^+ \wedge k_4^+ + k_3^+ \wedge k_4^+ \frac{1}{s_{31}} k_1^+ \wedge k_2^+ = 0 \quad \tag{32}
\]

valid only in \( 2+2 \) dimensions \[8, 20\]. It is very useful to note that this identity renders

\[
\tilde{A}_4(1, 2, 3, 4) := k_1^+ \wedge k_2^+ \frac{1}{s_{12}} k_3^+ \wedge k_4^+ \quad , \tag{33}
\]

totally antisymmetric in all labels.

The vanishing of amplitudes implies the existence of symmetries and vice versa. For the \( N=2 \) string an infinite number of tree-level scattering amplitudes vanishes and therefore an infinite

\[ k_i^+ \wedge k_j^+ := k_{i4} k_{j1} - k_{i4} k_{j3} + k_{i2} k_{j1} - k_{i2} k_{j3} - (i \leftrightarrow j) \]
number of symmetries is to be expected. For open \( N=2 \) strings these symmetries have been described in \[27, 28\].

The more commonly used Yang gauge \[29\] may also be obtained, by restricting oneself to the zero-instanton sector (or, equivalently, by averaging over \( \vartheta \)). Full \( N=2 \) string amplitudes, however, produce self-dual Yang-Mills in the Leznov gauge. The latter is also preferred by the simplicity of a merely quadratic field equation, leading to no further field-theory vertices beyond the cubic one corresponding to \[31\]). Indeed, using again the ‘magical’ identity \[32\] it is easy to prove that the sum of the s-, t-, and u-channel diagram for the field-theory four-point function already vanishes, leaving no room for a quartic vertex \[30\]. This result has been extended to all tree amplitudes \[31\].

The generalization to a non-vanishing constant \( B \)-field background is straightforward. We switch on only ‘magnetic’ components of the \( B \) field (see section 4), in order to allow for a Seiberg-Witten limit to noncommutative gauge theory. The three-string (Leznov) amplitude is modified to

\[
T_3(1, 2, 3) = A_3(1, 2, 3) \left[ \text{tr}(\lambda_1 \lambda_2 \lambda_3)E(1, 2, 3) - \text{tr}(\lambda_2 \lambda_1 \lambda_3)E(2, 1, 3) \right]
\]  

while the dressing of the primitive amplitudes \( A \) by phase factors \( E \) does not alter their vanishing. These amplitudes lead to the cubic Lagrangian

\[
\mathcal{L} = \frac{1}{2} G^{\mu\nu} \partial_\mu \phi \ast \partial_\nu \phi + \frac{1}{3} \epsilon^{\alpha\beta\gamma} \partial_\alpha \phi \ast \partial_\beta \phi \ast \partial_\gamma \phi
\]

with noncommutative \( \ast \) product. Here, \( \partial_\alpha := \hat{\partial}_2 + \hat{\partial}_4 \) and \( \partial_\beta := \hat{\partial}_1 - \hat{\partial}_3 \), where \( \hat{\partial}_\mu := \epsilon_{\mu\nu} \partial_\nu \) is defined with the help of the tetrad \[13\].

In accord with the general discussion \[6\] we expect the \( N=2 \) string in a constant \( B \)-field background to be identical to noncommutative self-dual Yang-Mills theory \[34, 35\] in the Leznov gauge, as described by the Lagrangian \[35\]. Since the latter has only a cubic interaction vertex (for the Leznov prepotential \( \phi \)), all tree-level field-theory amplitudes (with more than three external legs) obtained using the Feynman rules based on \[34\] should be zero. As a nontrivial check, we should be able to reproduce the vanishing of the four-point function for noncommutative \( U(n) \) self-dual Yang-Mills.

The field-theory four-point function \( T_4^{\text{Leznov}} \) for the Leznov prepotential \( \phi \in u(n) \) is a sum over 24 permutations of

\[
T_3(1, 2, 3) \frac{1}{s} T_3(3, 4) = \frac{1}{s} A_3(1, 2, 3) A_3(3, 4) \sum_x \text{tr}[\Lambda(1, 2) \lambda_x] \text{tr}[\lambda_x \Lambda(3, 4)]
\]

\[
= \frac{1}{2} \tilde{A}_4(1, 2, 3, 4) \text{tr}[\Lambda(1, 2) \Lambda(3, 4)]
\]

where the last equation makes use of \[25\]. Due to its total antisymmetry \( \tilde{A}_4 \) may be pulled out of the permutation sum, which reduces to

\[
\sum_{\pi \in S_4} (-)^\pi \text{tr}[\Lambda(\pi_1, \pi_2) \Lambda(\pi_3, \pi_4)] = 4 \sum_{\pi \in S_4} (-)^\pi \text{tr}[\lambda_{\pi_1} \lambda_{\pi_2} \lambda_{\pi_3} \lambda_{\pi_4}] E(\pi_1, \pi_2, \pi_3, \pi_4)
\]

However, since each term under the sum is cyclically invariant, the four contributions to any cycle cancel each other in pairs, leaving us with

\[
T_4^{\text{Leznov}}(1, 2, 3, 4) = 0
\]

Of course, the Yang gauge produces the same result. Because the gradient of \( \phi \) yields the Yang-Mills gauge potential, the same-helicity four-gluon amplitude emerges from \( T_4^{\text{Leznov}} \) by multiplication of leg factors and thus continues to vanish in the noncommutative case.
7 Four-point amplitude: an example

It has recently been claimed \[15\] that noncommutative self-dual Yang-Mills (in the Yang gauge) descends from the $N=2$ string only for abelian gauge groups, because the field-theory four-point function (for the Yang prepotential) allegedly fails to vanish otherwise. Although we have demonstrated in generality that proper factorization guarantees the agreement of string with field-theory amplitudes, let us elucidate the error of \[15\] for the simplest non-abelian gauge group admitted, $G = U(2)$. In this case, the Leznov prepotential consists of an $su(2)$ triplet $\phi^T$ plus an $su(2)$ singlet $\phi^S$ stemming from the $U(1)$ gauge boson. As generators we take $\lambda_a = \frac{i}{2} \sigma_a$, $a=1,2,3$, and $\lambda_0 = \frac{i}{2} 1$.

The vertices involving triplet states are

\[
T^{TTT}_3(1,2,3) = \frac{1}{2} k^+_1 \wedge k^+_2 c_{123} \quad \text{and} \quad T^{TTS}_3(1,2,3) = -\frac{1}{2} k^+_1 \wedge k^+_2 s_{12} \delta_{12}, \quad (39)
\]

where

\[
c_{ij} := \cos\left(\frac{1}{2} k_i \theta k_j\right) \quad \text{and} \quad s_{ij} := \sin\left(\frac{1}{2} k_i \theta k_j\right). \quad (40)
\]

Let us compose the four-triplet amplitude $T^{\text{Leznov}}_4$. Triplet exchange in $s$-, $t$-, and $u$-channel yields

\[
T^{(T)}_4 = -\frac{1}{4} \tilde{A}_4(1,2,3,4) \sum_{x=1,2,3} [c_{12} c_{34} \epsilon_{12x} \epsilon_{x34} + c_{23} c_{14} \epsilon_{23x} \epsilon_{x14} + c_{31} c_{24} \epsilon_{31x} \epsilon_{x24}] \quad (41)
\]

\[
= -\frac{1}{4} \tilde{A}_4(1,2,3,4) [\delta_{12} \delta_{34} (c_{23} c_{14} - c_{31} c_{24}) + \delta_{23} \delta_{14} (c_{31} c_{24} - c_{12} c_{34}) + \delta_{31} \delta_{24} (c_{12} c_{34} - c_{23} c_{14})],
\]

while singlet exchange produces

\[
T^{(S)}_4 = -\frac{1}{4} \tilde{A}_4(1,2,3,4) [s_{12} s_{34} \delta_{12} \delta_{34} + s_{23} s_{14} \delta_{23} \delta_{14} + s_{31} s_{24} \delta_{31} \delta_{24}] \quad . \quad (42)
\]

Even though the partial amplitudes do not vanish (for $\theta \neq 0$), their sum $T^{\text{Leznov}}_4 = T^{(T)}_4 + T^{(S)}_4$ does, as may be verified from

\[
c_{23} c_{14} - c_{31} c_{24} + s_{12} s_{34} = 0 \quad (43)
\]

by employing momentum conservation.

For $G=U(n)$, $\epsilon_{abc} \to f_{abc}$, but additional cubic couplings appear due to the non-vanishing of the symmetric $SU(n)$ rank-three tensor $d_{abc}$. In conclusion, noncommutative self-dual Yang-Mills (at tree level) is identical to the $N=2$ string in a constant $B$-field, as long as one does not attempt to use pure $SU(n)$ or an exceptional gauge group.

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