Resummation for $2 \rightarrow n$ processes in single-particle-inclusive kinematics

Matthew Forslund$^a$ and Nikolaos Kidonakis$^b$

$^a$Department of Physics and Astronomy, Stony Brook University, Stony Brook, NY 11794, USA
$^b$Department of Physics, Kennesaw State University, Kennesaw, GA 30144, USA

Abstract

We present a formalism and detailed analytical results for soft-gluon resummation for $2 \rightarrow n$ processes in single-particle-inclusive (1PI) kinematics. This generalizes previous work on resummation for $2 \rightarrow 2$ processes in 1PI kinematics. We also present soft anomalous dimensions at one and two loops for certain $2 \rightarrow 3$ processes involving top quarks and Higgs or $Z$ bosons, and we provide some brief numerical results.

1 Introduction

In theoretical calculations of hard-scattering cross sections of relevance to hadron colliders, the state-of-the-art has been moving steadily towards higher orders, more loops, and resummations at higher logarithmic accuracy; it has also been gradually expanded to processes with larger numbers of final-state particles. In particular, soft-gluon resummations have become a very useful tool in making predictions for additional corrections beyond complete fixed-order results. In many cases these soft-gluon corrections are large, and in fact they numerically dominate the complete corrections and can be thought of as very good approximations to complete results.

Soft-gluon resummation follows from factorization properties of the cross section [1–5] and it has been applied to a large number of processes in hadron collisions. Most of the applications for total cross sections and differential distributions have been done for $2 \rightarrow 2$ processes in single-particle-inclusive (1PI) as well as pair-invariant-mass (PIM) kinematics, most notably for top-quark production (see [6] for a review) but also many other processes. Applications to $2 \rightarrow 3$ processes using extensions of the PIM formalism, e.g. three-particle-invariant-mass kinematics, have also been made [7–14]. In this paper, we generalize resummation to processes with $n$ particles in the final state in 1PI kinematics. We also give more details for $2 \rightarrow 3$ processes with top quarks and Higgs or $Z$ bosons.

We begin in Section 2 with the development of the formalism, starting with elementary considerations and kinematics for $2 \rightarrow 2$ processes, and then for $2 \rightarrow 3$ processes, before moving on to the generalization to $2 \rightarrow n$ processes and the derivation of the resummed cross section in the general case. We also provide results for the expansion of the cross section to fixed order, in particular next-to-leading order (NLO) and next-to-next-to-leading order (NNLO). In Section 3, we provide details about the cross section calculation at the partonic and hadronic levels. Section 4 has details about the soft anomalous dimensions through two loops for $2 \rightarrow 3$ processes involving a top quark and a Higgs or $Z$ boson, and a brief numerical application
to $t$-channel $tgH$ and $tgZ$ production which shows the power of the formalism. We conclude in Section 5 and include an appendix with an alternate kinematical calculation of the cross section.

2 Resummation for $2 \rightarrow n$ processes

In this section we develop the formalism for resummation in 1PI kinematics with multi-particle final states. We begin with some simple considerations and definitions for $2 \rightarrow 2$ processes in the next subsection, and extend them to $2 \rightarrow 3$ processes in subsection 2.2 and to $2 \rightarrow n$ processes in subsection 2.3. The complete resummation formalism for $2 \rightarrow n$ processes is given in subsection 2.4. Fixed-order expansions of the resummed cross section are provided in subsection 2.5.

2.1 Kinematics and threshold for $2 \rightarrow 2$ processes

We first consider processes that are $2 \rightarrow 2$ at lowest order, $p_a + p_b \rightarrow p_1 + p_2$ (e.g. $q\bar{q} \rightarrow t\bar{t}$). We define the usual kinematical variables $s = (p_a + p_b)^2$, $t = (p_a - p_1)^2$, and $u = (p_b - p_1)^2$. We also define the threshold variable $s_{th} = s + t + u - p_1^2 - p_2^2$. Of course $p_1^2 = m_1^2$ and $p_2^2 = m_2^2$ where, depending on the process, the masses $m_1$ and $m_2$ can be zero or finite. As we approach partonic threshold, $s_{th} \rightarrow 0$ and there is vanishing energy for any additional radiation.

If we have an additional gluon with momentum $p_g$ being emitted in the final state, then by using momentum conservation, $p_a + p_b = p_1 + p_2 + p_g$, it is straightforward to show that the above definition of $s_{th}$ is equivalent to $s_{th} = (p_2 + p_g)^2 - p_2^2$. It is clear that $s_{th}$ goes to 0 as $p_g$ goes to 0 (soft gluon). The physical meaning is also more clear from this way of writing $s_{th}$: it is the invariant mass squared of the “particle 2 + gluon” system minus the invariant mass squared of particle 2, i.e. it describes the extra energy in the soft emission. Note that particle 1 is the observed particle in this single-particle-inclusive kinematics.

If the incoming partons $a$ and $b$ come from hadrons $A$ and $B$, then we also define the hadron-level variables $S = (p_A + p_B)^2$, $T = (p_A - p_1)^2$, $U = (p_B - p_1)^2$, and $S_{th} = S + T + U - p_1^2 - p_2^2$. Assuming that $p_a = x_a p_A$ and $p_b = x_b p_B$, where $x_a$ and $x_b$ denote the fraction of the momentum carried by partons $a$ and $b$ in hadrons $A$ and $B$, respectively, then we have the relations $s = x_a x_b S$, $t = x_a T + (1 - x_a) p_1^2$, and $u = x_b U + (1 - x_b) p_1^2$.

Then, using the above relations and after some algebra, we find that

$$
\frac{S_{th}}{S} = \frac{s_{th}}{s} - (1 - x_a) \frac{(u - p_2^2)}{s} - (1 - x_b) \frac{(t - p_2^2)}{s} + (1 - x_a)(1 - x_b) \frac{(p_1^2 - p_2^2)}{s}.
$$

(2.1)

The last term, involving $(1 - x_a)(1 - x_b)$, is higher order and can be ignored near threshold, as $x_a \rightarrow 1$ and $x_b \rightarrow 1$.

2.2 Kinematics and threshold for $2 \rightarrow 3$ processes

We next consider processes that are $2 \rightarrow 3$ at lowest order, $p_a + p_b \rightarrow p_1 + p_2 + p_3$ (e.g. $bq \rightarrow tq'H$). We define the parton-level variables $s$, $t$, $u$, and the hadron-level variables $S$,
kinematics is

The factorized form of the double-differential cross section in proton-proton collisions in 1PI

2.4 Resummation

1, the observed particle is

where

momentum conservation is

\( p_a + p_b = p_1 + p_2 + p_3 + p_g \).

We can define the threshold variable as

\[ s_{th} = (p_2 + p_3 + p_g)^2 - (p_2 + p_3)^2. \]

This clearly gives

the same physical meaning as extra energy from gluon emission and clearly vanishes as

One can also show after some work that this is equivalent to

\[ s_{th} = s + t + u - p_1^2 - (p_2 + p_3)^2. \]

We also define

\[ S_{th} = S + T + U - p_1^2 - (p_2 + p_3)^2, \]

and find, after some algebra, the relation

\[
\frac{S_{th}}{S} = \frac{s_{th}}{s} - (1 - x_a) \left( \frac{u - (p_2 + p_3)^2}{s} \right) - (1 - x_b) \left( \frac{t - (p_2 + p_3)^2}{s} \right) + (1 - x_a)(1 - x_b) \left( \frac{p_1^2 - (p_2 + p_3)^2}{s} \right). \tag{2.2}
\]

The last term, involving \((1 - x_a)(1 - x_b)\), can be ignored in the threshold limit, as \( x_a \to 1 \)

and \( x_b \to 1 \). We see that our results here are a natural extension of the relations for

2 \to 2

kinematics.

2.3 Kinematics and threshold for \( 2 \to n \) processes

These relations can be extended to an arbitrary number of particles: we consider processes that

are \( 2 \to n \) at lowest order, \( p_a + p_b \to p_1 + p_2 + \cdots + p_n \). Again, we define the parton-level

variables \( s, t, u \), and the hadron-level variables \( S, T, U \), as before. With an additional gluon

with momentum \( p_g \) in the final state, momentum conservation is

\( p_a + p_b = p_1 + p_2 + \cdots + p_n + p_g \).

Then the threshold variable is

\[ s_{th} = (p_2 + \cdots + p_n + p_g)^2 - (p_2 + \cdots + p_n)^2 \]

with the same physical meaning as before, and vanishing as \( p_g \to 0 \). Using the abbreviation

\( p_{2..n} = p_2 + \cdots + p_n \), we can rewrite the threshold variable as

\[ s_{th} = (p_{2..n} + p_g)^2 - p_{2..n}^2. \]

We can also show that this variable can also be written as

\[ s_{th} = s + t + u - p_1^2 - p_{2..n}^2. \]

We also define

\[ S_{th} = S + T + U - p_1^2 - p_{2..n}^2, \]

and find that

\[
\frac{S_{th}}{S} = \frac{s_{th}}{s} - (1 - x_a) \left( \frac{u - p_{2..n}^2}{s} \right) - (1 - x_b) \left( \frac{t - p_{2..n}^2}{s} \right) + (1 - x_a)(1 - x_b) \left( \frac{p_1^2 - p_{2..n}^2}{s} \right). \tag{2.3}
\]

Again, the last term, involving \((1 - x_a)(1 - x_b)\), can be ignored as \( x_a \to 1 \) and \( x_b \to 1 \).

Finally, we note that one can appropriately redefine the above relations if, instead of particle

1, the observed particle is \( n \) or any of the other particles.

2.4 Resummation

The factorized form of the double-differential cross section in proton-proton collisions in 1PI

kinematics is

\[
E_1 \frac{d\sigma^{A\to1\cdots n}}{d^3p_1} = \sum_{a,b} \int dx_a dx_b \phi_{a/A}(x_a) \phi_{b/B}(x_b) E_1 \frac{d\hat{\sigma}^{ab\to1\cdots n}(s_{th})}{d^3p_1}, \tag{2.4}
\]

where \( E_1 \) is the energy of the observed particle 1, \( \phi_{a/A} (\phi_{b/B}) \) are parton distribution functions

(pdf) for parton \( a (b) \) in hadron \( A (B) \), and \( \hat{\sigma}^{ab\to1\cdots n} \) is the hard-scattering partonic cross

section. For simplicity we do not explicitly show in the above equation the dependence on

\( \mu_F \) and \( \mu_R \), the factorization and renormalization scales.
The resummation of soft-gluon corrections follows from the factorization of the cross section in integral transform space \([1, 3]\). We define Laplace transforms (indicated by a tilde) of the partonic cross section as \(\tilde{\sigma}(N) = \int_0^s (s_{th}/s) e^{-N s_{th}/s} \tilde{\sigma}(s_{th})\), where \(N\) is the transform variable, and note that logarithms of \(s_{th}\) transform into logarithms of \(N\), with the latter exponentiating. We also define transforms of the pdf of \(\tilde{\sigma}(N) = \int_0^1 e^{-N(1-x)} \phi(x) dx\). We also consider the parton-parton cross section \(E_1 \frac{d\sigma^{a\rightarrow 1-n}}{d^3 p_1}\), of the same form as Eq. (2.4) but with the incoming hadrons replaced by partons \([1–5]\).

\[
E_1 \frac{d\sigma^{a\rightarrow 1-n}(S_{th})}{d^3 p_1} = \int dx_a dx_b \phi_{a/a}(x_a) \phi_{b/b}(x_b) E_1 \frac{d\tilde{\sigma}^{a\rightarrow 1-n}(s_{th})}{d^3 p_1}, \tag{2.5}
\]

and define its transform (again indicated by a tilde) as

\[
E_1 \frac{d\tilde{\sigma}^{a\rightarrow 1-n}(N)}{d^3 p_1} = \int_0^S \frac{dS_{th}}{S} e^{-N s_{th}/s} E_1 \frac{d\sigma^{a\rightarrow 1-n}(S_{th})}{d^3 p_1}. \tag{2.6}
\]

Taking a transform of Eq. (2.5), as defined in Eq. (2.6) above, and using Eq. (2.3) (ignoring the higher-order terms), we have

\[
E_1 \frac{d\tilde{\sigma}^{a\rightarrow 1-n}(N)}{d^3 p_1} = \int_0^1 dx_a e^{-N_a(1-x_a)} \phi_{a/a}(x_a) \int_0^1 dx_b e^{-N_b(1-x_b)} \phi_{b/b}(x_b)
\times \int_0^{s_{th}} \frac{dS_{th}}{s} e^{-N s_{th}/s} E_1 \frac{d\tilde{\sigma}^{a\rightarrow 1-n}(s_{th})}{d^3 p_1}
= \tilde{\phi}_{a/a}(N_a) \tilde{\phi}_{b/b}(N_b) E_1 \frac{d\tilde{\sigma}^{ab\rightarrow 1-n}(N)}{d^3 p_1}, \tag{2.7}
\]

where \(N_a = N(p_{2\ldots n}^2 - u)/s\) and \(N_b = N(p_{2\ldots n}^2 - t)/s\).

Next, we proceed with a refactorization of the cross section in terms of a new set of functions \([1, 5]\). We first rewrite Eq. (2.3) as

\[
S_{th}/S = -(1-x_a)\frac{(u-p_{2\ldots n}^2)}{s} - (1-x_b)\frac{(t-p_{2\ldots n}^2)}{s} + s_{th}/s
= -w_a\frac{(u-p_{2\ldots n}^2)}{s} - w_b\frac{(t-p_{2\ldots n}^2)}{s} + w_S + \sum_{i=1}^n w_i, \tag{2.8}
\]

where the \(w\)'s denote dimensionless weights. Note that \(w_a \neq 1-x_a\) and \(w_b \neq 1-x_b\) since they refer to different functions.

Then, a refactorized form of this cross section \([1, 3, 5]\) is

\[
E_1 \frac{d\tilde{\sigma}^{ab\rightarrow 1-n}(N)}{d^3 p_1} = \int dw_a dw_b \left( \prod_{i=1}^n dw_i \right) dw_S \psi_{a/a}(w_a) \psi_{b/b}(w_b) \left( \prod_{i=1}^n J_i(w_i) \right)
\times \text{tr} \left\{ H^{ab\rightarrow 1-n}(\alpha_s(\mu_F)) S^{ab\rightarrow 1-n} \left( \frac{w_S}{\mu_F} \right) \right\}
\times \delta \left( \frac{S_{th}}{S} + \frac{w_a(u-p_{2\ldots n}^2)}{s} + \frac{w_b(t-p_{2\ldots n}^2)}{s} - w_S - \sum_{i=1}^n w_i \right). \tag{2.9}
\]
The infrared-safe hard function \( H^{ab \rightarrow 1 \cdots n} \) does not depend on \( N \), and it describes contributions from the amplitude and from the complex conjugate of the amplitude. The soft function \( S^{ab \rightarrow 1 \cdots n} \) describes the emission of noncollinear soft gluons in the \( 2 \rightarrow n \) process. Both the hard and the soft functions are process-dependent matrices in color space in the partonic scattering, and the trace of their product is explicit in the above result. The functions \( \psi \) are distributions for incoming partons at fixed value of momentum, that describe the dynamics of collinear emission from those partons. The \( J_i \) denote functions that describe collinear emission from final-state colored particles.

Taking a transform of Eq. (2.9), of the form defined in Eq. (2.6), and using Eq. (2.8), we then have

\[
E_1 \frac{d\sigma^{ab \rightarrow 1 \cdots n}(N)}{d^3p_1} = \int_0^1 dw_a e^{-N_a w_a \psi_{a/a}(w_a)} \int_0^1 dw_b e^{-N_b w_b \psi_{b/b}(w_b)} \times \left( \prod_{i=1}^n \int_0^1 dw_i e^{-N_i w_i J_i(w_i)} \right) \text{tr} \left\{ H^{ab \rightarrow 1 \cdots n}(\alpha_s(\mu_R)) \int_0^1 dw_s e^{-N_w s} S^{ab \rightarrow 1 \cdots n}(\frac{\sqrt{s}}{N_{\mu_F}}) \right\} \\
= \tilde{\psi}_{a/a}(N_a) \tilde{\psi}_{b/b}(N_b) \left( \prod_{i=1}^n \tilde{J}_i(N) \right) \text{tr} \left\{ H^{ab \rightarrow 1 \cdots n}(\alpha_s(\mu_R)) \tilde{S}^{ab \rightarrow 1 \cdots n}(\frac{\sqrt{s}}{N_{\mu_F}}) \right\}.
\]

(2.10)

Comparing Eqs. (2.7) and (2.10), we get the following expression for the transform-space hard-scattering partonic cross section,

\[
E_1 \frac{d\tilde{\sigma}^{ab \rightarrow 1 \cdots n}(N)}{d^3p_1} = \frac{\tilde{\psi}_a(N_a) \tilde{\psi}_b(N_b)}{\psi_{a/a}(N_a) \psi_{b/b}(N_b)} \left( \prod_{i=1}^n \tilde{J}_i(N) \right) \text{tr} \left\{ H^{ab \rightarrow 1 \cdots n}(\alpha_s(\mu_R)) \tilde{S}^{ab \rightarrow 1 \cdots n}(\frac{\sqrt{s}}{N_{\mu_F}}) \right\}.
\]

(2.11)

The \( N \)-dependence of the soft matrix \( \tilde{S}^{ab \rightarrow 1 \cdots n} \) is resummed via renormalization group evolution \[1\]. We have

\[
\tilde{S}^{ab \rightarrow 1 \cdots n} = (Z_S^{ab \rightarrow 1 \cdots n})^\uparrow \tilde{S}^{ab \rightarrow 1 \cdots n} Z_S^{ab \rightarrow 1 \cdots n}
\]

(2.12)

where \( \tilde{S}^{ab \rightarrow 1 \cdots n} \) is the unrenormalized quantity and \( Z_S^{ab \rightarrow 1 \cdots n} \) is a matrix of renormalization constants. Thus, \( \tilde{S}^{ab \rightarrow 1 \cdots n} \) obeys the renormalization group equation

\[
\left( \mu \frac{\partial}{\partial \mu} + \beta(g_s) \frac{\partial}{\partial g_s} \right) \tilde{S}^{ab \rightarrow 1 \cdots n} = - (\Gamma_S^{ab \rightarrow 1 \cdots n})^\uparrow \tilde{S}^{ab \rightarrow 1 \cdots n} - \tilde{S}^{ab \rightarrow 1 \cdots n} \Gamma_S^{ab \rightarrow 1 \cdots n}
\]

(2.13)

where \( g_s^2 = 4\pi \alpha_s \) and \( \beta \) is the QCD beta function,

\[
\beta(\alpha_s) = \frac{d \ln \alpha_s}{d (\ln \mu)^2} = - \sum_{n=0}^{\infty} \beta_n \left( \frac{\alpha_s}{4\pi} \right)^{n+1}.
\]

(2.14)

The lowest-order term in the above series for the beta function is given by \( \beta_0 = (11C_A - 2n_f)/3 \) where \( C_A = N_c \), with \( N_c \) the number of colors, and \( n_f \) is the number of light quark flavors. The evolution of the soft function is controlled by the soft anomalous dimension matrix, \( \Gamma_S^{ab \rightarrow 1 \cdots n} \), which is calculated from the coefficients of the ultraviolet poles of eikonal diagrams \[1\,4\,15\,16\].
The transform-space resummed cross section is derived from the renormalization-group evolution of the soft function and the other $N$-dependent functions in Eq. (2.11), and it is given by [1,3,6]

$$E_1 \frac{d\hat{s}_{\text{resum}}}{d^3p_1} (N) = \exp \left[ \sum_{i=a,b} E_i(N_i) \right] \exp \left[ \sum_{i=a,b} \int_{\mu_F}^{\gamma_i/N} \frac{d\mu}{\mu} \gamma_i/\gamma_i(N_i) \right] \exp \left[ \sum_{i=1}^{N} E_i'(N) \right]$$

$$\times \text{tr} \left\{ H^{ab\rightarrow1\cdots n} (\alpha_s(\sqrt{s})) \exp \left[ \int_{\sqrt{s}}^{\sqrt{N}} \frac{d\mu}{\mu} \Gamma^{ab\rightarrow1\cdots n}_{S} (\alpha_s(\mu)) \right] \right\}.$$  

(2.15)

The first exponential resums universal soft and collinear contributions from the incoming partons [17,18].

$$E_i(N_i) = \int_0^1 dz \frac{z^N - 1}{1 - z} \left\{ \int_1^{(1-z)^2} \frac{d\lambda}{\lambda} A_i (\alpha_s(\lambda)) + D_i [\alpha_s((1-z)^2 s)] \right\},$$  

(2.16)

with $A_i = \sum_{n=1}^{\infty} (\alpha_s/\pi)^n A_i^{(n)}$, where $A_i^{(1)} = C_i$ with $C_i = C_F = (N_c^2 - 1)/(2N_c)$ for a quark or antiquark and $C_i = C_A$ for a gluon, while $A_i^{(2)} = C_F K/2$ with $K = C_A (67/18 - \pi^2/6) - 5n_f/9$. Also $D_i = \sum_{n=1}^{\infty} (\alpha_s/\pi)^n D_i^{(n)}$, with $D_i^{(1)} = 0$ in Feynman gauge ($D_i^{(1)} = -A_i^{(1)}$ in axial gauge). The second exponential gives the scale evolution in terms of the parton anomalous dimensions $\gamma_i/\gamma_i = -A_i (\ln N_i + \gamma_E) + \gamma_i$, where $\gamma_E$ is the Euler constant and $\gamma_i = \sum_{n=1}^{\infty} (\alpha_s/\pi)^n \gamma_i^{(n)}$, with $\gamma_q^{(1)} = 3C_F/4$ for quarks and $\gamma_g^{(1)} = \beta_0/4$ for gluons.

The expression for the final-state exponential, involving $E_i'$, depends on whether we have massless or massive particles or jets. For massive particles or for colorless particles it is 1. For massless quarks or gluons we have

$$E_i'(N) = \int_0^1 dz \frac{z^N - 1}{1 - z} \left\{ \int_1^{(1-z)^2} \frac{d\lambda}{\lambda} A_i (\alpha_s(\lambda)) + B_i [\alpha_s((1-z) s)] + D_i [\alpha_s((1-z)^2 s)] \right\},$$  

(2.17)

where $B_i = \sum_{n=1}^{\infty} (\alpha_s/\pi)^n B_i^{(n)}$, with $B_q^{(1)} = -3C_F/4$ for quarks and $B_g^{(1)} = -\beta_0/4$ for gluons.

The process-dependent hard and soft functions (matrices) have the perturbative expansions $H^{ab\rightarrow1\cdots n} = \sum_{n=0}^{\infty} (\alpha_s^{d+n}/\pi^n) H^{(n)}$, where the power $d$ depends on the partonic process, and $\tilde{S}^{ab\rightarrow1\cdots n} = \sum_{n=0}^{\infty} (\alpha_s/\pi)^n S^{(n)}$. Finally the soft anomalous dimension has the expansion $\Gamma_S^{ab\rightarrow1\cdots n} = \sum_{n=1}^{\infty} (\alpha_s/\pi)^n \Gamma_S^{(n)}$.

### 2.5 Fixed-order expansions

We can expand the formula for the resummed cross section, Eq. (2.15), to any fixed order [6,19] and invert it back to momentum space. Below we provide explicit results for the soft-gluon corrections at NLO and NNLO.
The NLO soft-gluon corrections are

\[ E_1 \frac{d\hat{\sigma}^{(1)}}{d^3p_1} = F_{LO} \frac{\alpha_s(\mu_R)}{\pi} \left\{ c_3 D_1(s_{th}) + c_2 D_0(s_{th}) + c_1 \delta(s_{th}) \right\} + \frac{\alpha_s^{d+1}(\mu_R^2)}{\pi} \left[ A_c D_0(s_{th}) + T_1^c \delta(s_{th}) \right], \]

(2.18)

where

\[ D_k(s_{th}) = \left[ \ln \left( \frac{k}{s_{th}/s} \right) \right]_+, \]

(2.19)

\[ F_{LO} = \alpha_s^d \text{tr}\{H^{(0)}S^{(0)}\} \]

denotes the leading-order (LO) coefficient,

\[ c_3 = 2(A_a^{(1)} + A_b^{(1)}) - \sum_{i=1}^n A_i^{(1)}, \]

(2.20)

and \( c_2 \) is given by \( c_2 = c_2^\mu + T_2 \), with

\[ c_2^\mu = -(A_a^{(1)} + A_b^{(1)}) \ln \left( \frac{\mu_F^2}{s} \right) \]

(2.21)

denoting the terms involving logarithms of the scale, and

\[ T_2 = -2 A_a^{(1)} \ln \left( \frac{-u + p_{2-n}^2}{s} \right) - 2 A_b^{(1)} \ln \left( \frac{-t + p_{2-n}^2}{s} \right) + D_a^{(1)} + D_b^{(1)} + \sum_{i=1}^n \left[ B_i^{(1)} + D_i^{(1)} \right] \]

(2.22)

denoting the scale-independent terms. Also,

\[ A_c^c = \text{tr} \left( H^{(0)} \Gamma_S^{(1)} S^{(0)} + H^{(0)} S^{(0)} \Gamma_S^{(1)} \right). \]

(2.23)

With regard to the \( \delta(s_4) \) terms, we split them into a term \( c_1 \), that is proportional to the Born cross section, and a term \( T_1^c \) that is not. We write \( c_1 = c_1^\mu + T_1^c \), with

\[ c_1^\mu = \left[ A_a^{(1)} \ln \left( \frac{-u + p_{2-n}^2}{s} \right) + A_b^{(1)} \ln \left( \frac{-t + p_{2-n}^2}{s} \right) - \gamma_a^{(1)} - \gamma_b^{(1)} \right] \ln \left( \frac{\mu_F^2}{s} \right) + \frac{d\beta_0}{4} \ln \left( \frac{\mu_R^2}{s} \right) \]

(2.24)

denoting the terms involving logarithms of the scale. We note that \( T_1 \) and \( T_1^c \) cannot be calculated from the resummation formalism but they can be determined from a comparison to a complete NLO calculation.
The NNLO soft-gluon corrections are

\[
E_1 \frac{d\hat{\sigma}^{(2)}}{d^{3}p_1} = F_{LO} \frac{\alpha_s^2(\mu_R)}{\pi^2} \left\{ \frac{1}{2} c_3 D_3(s_{th}) + \left[ \frac{3}{2} c_3 c_2 - \frac{\beta_0}{4} c_3 + \sum_{i=1}^{n} \frac{\beta_0}{8} A_i^{(1)} \right] D_2(s_{th}) 

+ \left[ c_3 c_1 + c_2^2 - \zeta c_2 c_3 - \frac{\beta_0}{2} T_2 + \frac{\beta_0}{4} c_3 \ln \left( \frac{\mu_R^2}{s} \right) + 2(A_a^{(2)} + A_b^{(2)}) - \sum_{i=1}^{n} A_i^{(2)} + \sum_{i=1}^{n} \frac{\beta_0}{4} B_i^{(1)} \right] D_1(s_{th}) 

+ \left[ c_2 c_1 - \zeta c_3 c_2 + \zeta c_3^2 + \frac{\beta_0}{4} c_2 \ln \left( \frac{\mu_R^2}{s} \right) - \frac{\beta_0}{2} A_a^{(1)} \ln^2 \left( \frac{-u + p_{2\ldots n}^2}{s} \right) - \frac{\beta_0}{2} A_b^{(1)} \ln^2 \left( \frac{-t + p_{2\ldots n}^2}{s} \right) \right. 

+ \left( -2A_a^{(2)} + \frac{\beta_0}{2} D_a^{(1)} \right) \ln \left( \frac{-u + p_{2\ldots n}^2}{s} \right) + \left( -2A_b^{(2)} + \frac{\beta_0}{2} D_b^{(1)} \right) \ln \left( \frac{-t + p_{2\ldots n}^2}{s} \right) 

+ D_a^{(2)} + D_b^{(2)} + \frac{\beta_0}{8} (A_a^{(1)} + A_b^{(1)}) \ln^2 \left( \frac{\mu_F^2}{s} \right) - (A_a^{(2)} + A_b^{(2)}) \ln \left( \frac{\mu_F^2}{s} \right) 

+ \left. \sum_{i=1}^{n} \left( B_i^{(2)} + D_i^{(2)} \right) \right\} D_0(s_{th}) \right\},
\]

where

\[
F^c = \text{tr} \left[ H^{(0)} \left( \Gamma_S^{(1)} \right)^\dagger S^{(0)} + H^{(0)} S^{(0)} \left( \Gamma_S^{(1)} \right)^2 + 2 H^{(0)} \Gamma_S^{(1)} S^{(0)} \Gamma_S^{(1)} \right]
\]

(2.26)

and

\[
G^c = \text{tr} \left[ H^{(1)} \Gamma_S^{(1)} S^{(0)} + H^{(1)} S^{(0)} \Gamma_S^{(1)} + H^{(0)} \Gamma_S^{(1)} S^{(1)} + H^{(0)} S^{(1)} \Gamma_S^{(1)} + H^{(0)} \Gamma_S^{(2)} S^{(0)} + H^{(0)} S^{(0)} \Gamma_S^{(2)} \right].
\]

(2.27)

We note that at next-to-next-to-leading-logarithm (NNLL) resummation accuracy for a given process, all soft-gluon terms in the expansion through NNLO can be fully calculated.

### 3 Cross section and kinematics

In this section we provide some formulas that are needed for the calculation of partonic and hadronic cross sections with multi-particle final states.

#### 3.1 Frame-invariant integration variables

It has been shown by Byczkling and Kajantie \[20][21] that one can write the expression for the phase space integration of a 2 \(\rightarrow n\) scattering process while integrating over only invariant
variables. For processes with massless initial states, we have the phase space integral

$$R_n(s) = \frac{1}{4s} \int dp_{1...n-1}^2 dt_{n-1} d\phi \int dp_{1...n-2}^2 dt_{n-2} ds_{n-1} \frac{\Theta(-\Delta_4(n-1))}{8 [-\Delta_4(n-1)]^{1/2}} \times \cdots \times \int dp_{1..2}^2 dt_{2} ds_3 \frac{\Theta(-\Delta_4(3))}{8 [-\Delta_4(3)]^{1/2}} \int dt_1 ds_2 \frac{\Theta(-\Delta_4(2))}{8 [-\Delta_4(2)]^{1/2}} |M|^2.$$  \hfill (3.1)

with $s = (p_a + p_b)^2$ and $p_{1...n} = (p_1 + \cdots + p_n)^2$. We define the generalised kinematic invariants $t_i = (p_a - p_1 - \cdots - p_i)^2$, $u_i = (p_b - p_1 - \cdots - p_i)^2$, and $s_i = (p_i + p_{i+1})^2$. $\Delta_4(i)$ is the four-dimensional Gram determinant which can be written as

$$\Delta_4(i) = \frac{1}{16} \left| \begin{array}{cccc} \lambda m_i & t_{i-1} - p_{1...i-1}^2 & t_i - p_{1...i}^2 & t_{i+1} - p_{1...i+1}^2 \\ t_i - p_{1...i-1}^2 & 2t_{i-1} & t_i + t_{i-1} - m_i^2 & t_{i-1} + t_{i+1} - s_i \\ t_{i+1} - p_{1...i+1}^2 & t_{i+1} - s_i & t_i + t_{i+1} - m_{i+1}^2 & 2t_{i+1} \\ \end{array} \right|. \hfill (3.2)$$

The limits of integration are given by

$$p_{1...i}^2 = (\sqrt{s} - m_n - \cdots - m_{i+1})^2,$$

$$p_{1...i}^2 = (m_1 + \cdots + m_i)^2,$$

$$t_{i-1}^\pm = p_{1...i-1}^2 + (2p_{1...i}^2)^{-1}[(-p_{1...i}^2 + t_i)(p_{1...i}^2 + p_{1...i-1}^2 - m_i^2)]$$

$$\pm \lambda^{1/2}(p_{1...i}^2, t_i, 0)\lambda^{1/2}(p_{1...i}^2, p_{1...i-1}^2, m_i^2),$$

$$s_i^\pm = p_{1...i-1}^2 + p_{1...i+1}^2 + \frac{2}{\lambda(p_{1...i}^2, t_i, 0)} \left[ 4V(i) \pm [G(i)G(i-1)]^{1/2} \right],$$

where $\lambda(x, y, z) = (x - y - z)^2 - 4yz$, and $G(i)$ and $V(i)$ are given by

$$G(i) = -\frac{1}{2} \left| \begin{array}{cccc} 0 & 1 & 1 & 1 \\ 1 & 0 & m_{i+1}^2 & t_i \\ 1 & m_{i+1}^2 & 0 & t_{i+1} \\ 1 & t_i & t_{i+1} & 0 \\ \end{array} \right|$$  \hfill (3.4)

and

$$V(i) = -\frac{1}{8} \left| \begin{array}{cccc} 0 & 2p_{1...i}^2 - t_i & p_{1...i}^2 - t_i & p_{1...i}^2 + p_{1...i-1}^2 - m_i^2 \\ 2p_{1...i}^2 - t_i & 0 & p_{1...i}^2 - t_i & p_{1...i-1}^2 - t_{i-1} \\ p_{1...i+1}^2 - m_{i+1}^2 & 0 & p_{1...i+1}^2 - t_{i+1} & 0 \\ p_{1...i+1}^2 - m_{i+1}^2 & 0 & p_{1...i+1}^2 - t_{i+1} & 0 \\ \end{array} \right|. \hfill (3.5)$$

The angle $\phi$ describes a rotation of the process around the beam axis and is trivial for our purposes. Integrating it out, including the flux factor and the matrix element $|M|$, and using the identity $p_{1...n-1}^2 = s + t_{n-1} + u_{n-1} - m_n^2$, we obtain the differential partonic cross section

$$s^2 \frac{d^2\sigma^{ab\rightarrow1...n}}{dt_{n-1}du_{n-1}} = \frac{1}{8} \frac{1}{(2\pi)^{3n-5}} \int dp_{1...n-2}^2 dt_{n-2} ds_{n-1} \frac{\Theta(-\Delta_4(n-1))}{8 [-\Delta_4(n-1)]^{1/2}} \times \cdots \times \int dp_{12}^2 dt_{2} ds_3 \frac{\Theta(-\Delta_4(3))}{8 [-\Delta_4(3)]^{1/2}} \int dt_1 ds_2 \frac{\Theta(-\Delta_4(2))}{8 [-\Delta_4(2)]^{1/2}} |M|^2. \hfill (3.6)$$
3.2 Hadronic cross section

The LO hadronic cross section is obtained by convoluting the differential partonic cross section with the appropriate parton distribution functions:

\[
S^2 \frac{d^2 \sigma^{pp \rightarrow 1\cdots n}}{dT_{n-1} dU_{n-1}} = \int_{x_n}^1 \frac{dx_a}{x_a} \int_{x_b}^1 \frac{dx_b}{x_b} \phi(x_a) \phi(x_b) s^2 \frac{d^2 \hat{\sigma}^{ab \rightarrow 1\cdots n}}{dt_{n-1} d\mu_{n-1}},
\]

(3.7)

where \( S, T_{n-1}, \) and \( U_{n-1} \) are the hadronic analogues of the partonic invariants. We extend 2 \( \rightarrow \) 3 particle kinematic definitions [22] to 2 \( \rightarrow \) \( n \) particle kinematics, giving the conditions

\[
t_{n-1} = x_b(T_{n-1} - m_n^2) + m_n^2, \quad u_{n-1} = x_a(U_{n-1} - m_n^2) + m_n^2, \quad s = x_ax_bS,
\]

\[
s + t_{n-1} + u_{n-1} - m_n^2 \geq \sum_{i=1}^{n-1} m_i^2, \quad 0 \leq x_a, x_b \leq 1,
\]

(3.8)

which yield the integration bounds for \( x_a \) and \( x_b \):

\[
x_a^− = \frac{-T_{n-1} + \sum_{i=1}^{n-1} m_i^2}{S + U_{n-1} - m_n^2}, \quad x_b^− = \frac{-m_n^2 - x_a(U_{n-1} - m_n^2) + \sum_{i=1}^{n-1} m_i^2}{x_aS + T_{n-1} - m_n^2}.
\]

(3.9)

For an arbitrary 2 \( \rightarrow \) \( n \) process, there are \( \frac{1}{2}(n-2)(n-3) \) relations between all possible kinematic invariants that are not fixed by momentum conservation. These must instead be fixed by the condition that any five or more vectors are always linearly dependent in four-dimensional space and their symmetric Gram determinant vanishes:

\[
\Delta_{l+1}(p_1, p_2, \cdots, p_l, -p_b) = 0, \quad 4 \leq l \leq n.
\]

(3.10)

The Gram determinant condition \( \Delta_{l+1} = 0 \) can be equivalently written as a Cayley determinant condition [20] as

\[
\Delta_{l+1}(p_1, p_2, \cdots, p_l, -p_b) = \begin{vmatrix}
0 & 1 & 1 & 1 & \cdots & 1 & 1 \\
1 & 0 & p_1^2 & p_{12} & \cdots & p_{12\ldots l} & 0 \\
1 & p_1^2 & 0 & p_2^2 & \cdots & p_{23\ldots l} & t_1 \\
1 & p_{12} & p_2^2 & 0 & \cdots & p_{34\ldots l} & t_2 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
1 & p_{12\ldots l} & p_{23\ldots l} & p_{34\ldots l} & \cdots & 0 & t_l \\
1 & 0 & t_1 & t_2 & \cdots & t_l & 0
\end{vmatrix} = 0.
\]

(3.11)

4 Soft-gluon corrections for 2 \( \rightarrow \) 3 processes with a top quark and a Higgs or Z boson

In this section we consider several processes involving a three-particle final state with a top quark and a Higgs boson, or a top quark and a Z boson. We present the soft anomalous dimension matrices for these processes at one and two loops. We also give some brief numerical results for \( tqH \) and \( tqZ \) production to illustrate the use of the formalism.
We begin with the s-channel processes \( q(p_a) + \bar{q}'(p_b) \rightarrow t(p_1) + \bar{b}(p_2) + H(p_3) \) and \( q(p_a) + \bar{q}'(p_b) \rightarrow t(p_1) + \bar{b}(p_2) + Z(p_3) \). We define \( s, t, \) and \( u \) as in Section 2, and further define \( s' = (p_1 + p_2)^2, t' = (p_b - p_2)^2, \) and \( u' = (p_a - p_2)^2 \). We choose the color basis \( c_1 = \delta_{ab}\delta_{12} \) and \( c_2 = T_{ba}^c T_{12}^c \). Then, at one loop, the four elements of the s-channel soft anomalous dimension matrix are given by

\[
\begin{align*}
\Gamma^{(1)}_{S11} &= C_F \left[ \ln \left( \frac{s' - m_t^2}{m_t \sqrt{s}} \right) - \frac{1}{2} \right], \\
\Gamma^{(1)}_{S12} &= \frac{C_F}{2N_c} \ln \left( \frac{t'(t - m_t^2)}{u'(u - m_t^2)} \right), \\
\Gamma^{(1)}_{S21} &= \ln \left( \frac{t'(t - m_t^2)}{u'(u - m_t^2)} \right), \\
\Gamma^{(1)}_{S22} &= C_F \left[ \ln \left( \frac{s' - m_t^2}{m_t \sqrt{s}} \right) - \frac{1}{2} \right] - \frac{1}{N_c} \ln \left( \frac{t'(t - m_t^2)}{u'(u - m_t^2)} \right) + \frac{N_c}{2} \ln \left( \frac{t'(t - m_t^2)}{s(s' - m_t^2)} \right), \quad (4.1)
\end{align*}
\]

where \( m_t \) is the top-quark mass.

We continue with the t-channel processes \( b(p_a) + q(p_b) \rightarrow t(p_1) + q'(p_2) + H(p_3) \) and \( b(p_a) + q(p_b) \rightarrow t(p_1) + q'(p_2) + Z(p_3) \). We define the kinematical variables as before and choose the color basis \( c_1 = \delta_{ab}\delta_{12} \) and \( c_2 = T_{1a}^c T_{2b}^c \).

The four elements of the t-channel soft anomalous dimension matrix at one loop for these processes are given by

\[
\begin{align*}
\Gamma^{(1)}_{T11} &= C_F \left[ \ln \left( \frac{t'(t - m_t^2)}{m_t s^{3/2}} \right) - \frac{1}{2} \right], \\
\Gamma^{(1)}_{T12} &= \frac{C_F}{2N_c} \ln \left( \frac{u'(u - m_t^2)}{s(s' - m_t^2)} \right), \\
\Gamma^{(1)}_{T21} &= \ln \left( \frac{u'(u - m_t^2)}{s(s' - m_t^2)} \right), \\
\Gamma^{(1)}_{T22} &= C_F \left[ \ln \left( \frac{t'(t - m_t^2)}{m_t s^{3/2}} \right) - \frac{1}{2} \right] - \frac{1}{N_c} \ln \left( \frac{u'(u - m_t^2)}{s(s' - m_t^2)} \right) + \frac{N_c}{2} \ln \left( \frac{u'(u - m_t^2)}{t'(t - m_t^2)} \right). \quad (4.2)
\end{align*}
\]

At two loops, the soft anomalous dimension matrices for each process can be written compactly in terms of the one-loop results. We have

\[
\begin{align*}
\Gamma^{(2)}_{S11} &= \frac{K}{2} \Gamma^{(1)}_{S11} + \frac{1}{4} C_F C_A (1 - \zeta_3), \\
\Gamma^{(2)}_{S12} &= \frac{K}{2} \Gamma^{(1)}_{S12}, \\
\Gamma^{(2)}_{S21} &= \frac{K}{2} \Gamma^{(1)}_{S21}, \\
\Gamma^{(2)}_{S22} &= \frac{K}{2} \Gamma^{(1)}_{S22} + \frac{1}{4} C_F C_A (1 - \zeta_3). \quad (4.3)
\end{align*}
\]

We also note that soft anomalous dimension matrices at one loop for processes with three colored particles in the final state have appeared in Refs. 23,24.
To illustrate the usefulness of our formalism, we now briefly apply our methods to the cross section for the $t$-channel processes $b(p_a) + q(p_b) \rightarrow t(p_1) + q'(p_2) + H(p_3)$ and $b(p_a) + q(p_b) \rightarrow t(p_1) + q'(p_2) + Z(p_3)$. NLO calculations for these processes have appeared in Refs. [25, 26]. We use a renormalisation and factorization scale of $\mu = m_t = 173.0$ GeV, and MMHT2014 pdf [27]. Our higher-order soft-gluon corrections are computed from resummation at next-to-leading-logarithm (NLL) accuracy. In our discussion below, we denote the sum of the LO cross section and the NLO soft-gluon corrections as approximate NLO (aNLO); and we denote the sum of the aNLO cross section and the NNLO soft-gluon corrections as approximate NNLO (aNNLO).

For $Z$ associated production, we find aNLO enhancements of the total top + antitop LO cross section of 13.4% at 8 TeV, 30.9% at 13 TeV, and 34.1% at 14 TeV. At aNNLO, we find enhancements over the aNLO cross section of 5.3% at 8 TeV, 5.6% at 13 TeV, and 5.7% at 14 TeV. Using MadGraph5\texttt{aMC@NLO} [28], we find a total NLO enhancement at 8 TeV of 13.4%, showing that our results at this energy approximate very well the total NLO cross section. This is also consistent with the results in Ref. [25]. However, at higher energies, our enhancements become larger than the full NLO corrections.

For Higgs associated production, we find aNLO enhancements of the total top + antitop LO cross section of 5.2% at 8 TeV, 14.9% at 13 TeV, and 16.4% at 14 TeV. At aNNLO, we find enhancements over aNLO of 5.0% at 8 TeV, 4.4% at 13 TeV, and 4.5% at 14 TeV. The NLO enhancements from MadGraph5\texttt{aMC@NLO} and from Refs. [25, 26] are higher than ours, but still the soft-gluon corrections are a significant and dominant portion of the full corrections.

A detailed phenomenological study of these processes is beyond the scope of this work. We plan to further study these and other processes in future work.

5 Conclusions

We have presented a soft-gluon resummation formalism for $2 \rightarrow n$ processes in 1PI kinematics, and provided analytical results for the resummed cross section and fixed-orders expansions. We also considered in particular $2 \rightarrow 3$ processes, involving a three-particle final state with a top quark and a Higgs boson, or a top quark and a $Z$ boson, and we provided explicit results for the soft anomalous dimension matrices at one and two loops as well as some brief numerical results for those processes. We foresee a large number of other applications to Standard Model and to Beyond the Standard Model processes.

Acknowledgements

We thank Marco Guzzi for useful discussions on multi-particle kinematics. This material is based upon work supported by the National Science Foundation under Grant No. PHY 1820795.
A Appendix: Frame-dependent integration variables

One can alternatively [21] do the phase space integration of a $2 \to n$ scattering process by breaking the process down into successive $1 \to 2$ decays and integrating over the relevant solid angle in each rest frame explicitly:

$$R_n(s) = \int dp_{1...n}^2 d\Omega_n \frac{\lambda_{1/2}(s,p_{1...n-1}^2,m_n^2)}{8s} \times \int dp_{1...n-2}^2 d\Omega_{n-1} \frac{\lambda_{1/2}(p_{1...n-1}^2,p_{1...n-2}^2,m_{n-1}^2)}{8p_{1...n-1}^2} \times \cdots \times \int dp_{12}^2 d\Omega_3 \frac{\lambda_{1/2}(p_{12}^2,m_1^2,m_2^2)}{8p_{12}^2}.$$  \hspace{1cm}\hspace{1cm}\hspace{1cm}\hspace{1cm}\hspace{1cm}\hspace{1cm}(A.1)

For each of the $1 \to 2$ decays, one takes the CM frame of the outgoing particles. For the $n^{th}$ particle, one takes the CM frame of the initial state:

$$p_a = (E_a^{(n)}, 0, 0, E_a^{(n)}) ,$$
$$p_b = (E_b^{(n)}, 0, 0, -E_b^{(n)}) ,$$
$$p_n = (E_n, 0, |p_n| \sin \alpha, |p_n| \cos \alpha) ,$$
$$p_{12...n-1} = (E_{12...n-1}, 0, -|p_n| \sin \alpha, -|p_n| \cos \alpha) .$$  \hspace{1cm}\hspace{1cm}\hspace{1cm}\hspace{1cm}\hspace{1cm}(A.2)

In this frame, we have

$$E_a^{(n)} = E_b^{(n)} = \frac{\sqrt{s}}{2} , \hspace{1cm} E_n = -\frac{t_{n-1} + u_{n-1} - 2m_1^2}{2\sqrt{s}} = \frac{s + m_1^2 - p_{1...n-1}^2}{2\sqrt{s}} ,$$
$$|p_n| = \frac{\lambda_{1/2}(s,p_{1...n-1}^2,m_n^2)}{2\sqrt{s}} , \hspace{1cm} \cos \alpha = \frac{u_{n-1} - t_{n-1}}{\lambda_{1/2}(s,p_{1...n-1}^2,m_n^2)} = \frac{s + 2u_{n-1} - m_n^2 - p_{1...n-1}^2}{\lambda_{1/2}(s,p_{1...n-1}^2,m_n^2)} ,$$  \hspace{1cm}\hspace{1cm}\hspace{1cm}\hspace{1cm}\hspace{1cm}\hspace{1cm}\hspace{1cm}\hspace{1cm}\hspace{1cm}\hspace{1cm}\hspace{1cm}\hspace{1cm}\hspace{1cm}(A.3)

and $d\Omega_n = d(\cos \alpha) d\phi_n$, where the integral over $\phi_n$ is trivial for our purposes as before. The first integration can therefore be converted to the frame-independent form

$$\int dp_{1...n-1}^2 d\Omega_n \frac{\lambda_{1/2}(s,p_{1...n-1}^2,m_n^2)}{8s} = \frac{1}{4s} \int dt_{n-1} du_{n-1} d\phi_n$$  \hspace{1cm}\hspace{1cm}\hspace{1cm}\hspace{1cm}\hspace{1cm}\hspace{1cm}(A.4)

to yield the differential partonic cross section

$$s^2 \frac{d^2 \hat{\sigma}^{ab \to 1...n}}{dt_{n-1} du_{n-1}} = \frac{1}{8(2\pi)^{3n-5}} \int dp_{1...n-2}^2 d\Omega_{n-1} \frac{\lambda_{1/2}(p_{1...n-1}^2,p_{1...n-2}^2,m_{n-1}^2)}{8p_{1...n-1}^2} \times \cdots \times \int dp_{12}^2 d\Omega_3 \frac{\lambda_{1/2}(p_{12}^2,m_1^2,m_2^2)}{8p_{12}^2} \int d\Omega_2 \frac{\lambda_{1/2}(p_{12}^2,m_1^2,m_2^2)}{8p_{12}^2} |M|^2 .$$  \hspace{1cm}\hspace{1cm}\hspace{1cm}\hspace{1cm}\hspace{1cm}\hspace{1cm}\hspace{1cm}\hspace{1cm}\hspace{1cm}\hspace{1cm}\hspace{1cm}\hspace{1cm}\hspace{1cm} (A.5)

In order to do these integrations, one must go into each $1 \to 2$ frame explicitly. For the $l^{th}$
particle, one takes the frame
\[ p_a = (E_a^{(l)}, 0, 0, E_a^{(l)}) , \]
\[ p_b = (E_b^{(l)}, 0, |p_{l+1...n}| \sin \psi^{(l)}, |p_{l+1...n}| \cos \psi^{(l)} - E_a^{(l)}) , \]
\[ p_{l+1...n} = (E_{l+1...n}, 0, |p_{l+1...n}| \sin \psi^{(l)}, |p_{l+1...n}| \cos \psi^{(l)}) , \]
\[ p_l = (E_l, 0, |p_l| \sin \theta_l \cos \phi_l, |p_l| \cos \theta_l) , \]
\[ p_{1...l-1} = (E_{1...l-1}, 0, -|p_l| \sin \theta_l \cos \phi_l, -|p_l| \cos \theta_l) , \]
(A.6)

with \( \int d\Omega_t = \int_0^\pi \sin \theta_l d\theta_l \int_0^{2\pi} d\phi_l \). Using the same definitions as above, conservation of momentum and on-mass-shell conditions yield

\[ E_a^{(l)} = \frac{s - u_l - p_{l+1...n}^2}{2 \sqrt{p_{1...l}^2}} , \]
\[ E_{l+1...n} = \frac{s - p_{l...l}^2 - p_{l+1...n}^2}{2 \sqrt{p_{1...l}^2}} , \]
\[ E_l = \frac{p_{l...l}^2 - p_{l...l-1}^2 + p_{l}^2}{2 \sqrt{p_{1...l}^2}} , \]
\[ |p_l| = \frac{\lambda^{1/2} (s, p_{l...l}^2, p_{l+1...n}^2)}{2 \sqrt{p_{1...l}^2}} , \]
\[ E_b^{(l)} = \frac{s - t_l - p_{l+1...n}^2}{2 \sqrt{p_{1...l}^2}} , \]
\[ |p_{l+1...n}| = \frac{\lambda^{1/2} (s, p_{l...l}^2, p_{l+1...n}^2)}{2 \sqrt{p_{1...l}^2}} , \]
\[ E_{1...l-1} = \frac{p_{l...l-1}^2 + p_{1...l-1} - p_l^2}{2 \sqrt{p_{1...l}^2}} , \]
\[ \cos \psi^{(l)} = \frac{(E_a^{(l)})^2 - (E_b^{(l)})^2 + |p_{l+1...n}|^2}{2|p_{l+1...n}|E_a^{(l)}} . \]
(A.7)

References

[1] N. Kidonakis and G. Sterman, Phys. Lett. B 387, 867 (1996); Nucl. Phys. B 505, 321 (1997) [hep-ph/9705234].

[2] H. Contopanagos, E. Laenen, and G. Sterman, Nucl. Phys. B 484, 303 (1997) [hep-ph/9604313].

[3] E. Laenen, G. Oderda, and G. Sterman, Phys. Lett. B 438, 173 (1998) [hep-ph/9806467].

[4] N. Kidonakis, G. Oderda, and G. Sterman, Nucl. Phys. B 525, 299 (1998) [hep-ph/9801268]; Nucl. Phys. B 531, 365 (1998) [hep-ph/9803241].

[5] N. Kidonakis and V. Del Duca, Phys. Lett. B 480, 87 (2000) [hep-ph/9911460].

[6] N. Kidonakis, Int. J. Mod. Phys. A 33, 1830021 (2018) [arXiv:1806.03336].

[7] H.T. Li, C.S. Li, and S.A. Li, Phys. Rev. D 90, 094009 (2014) [arXiv:1409.1460].

[8] A. Kulesza, L. Motyka, T. Stebel, and V. Theeuwes, JHEP 1603, 065 (2016) [arXiv:1509.02780].

[9] A. Broggio, A. Ferroglia, B.D. Pecjak, A. Signer, and L.L. Yang, JHEP 1603, 124 (2016) [arXiv:1510.01914].
[10] A. Broggio, A. Ferroglia, B.D. Pecjak, and L.L. Yang, JHEP 1702, 126 (2017) [arXiv:1611.00049].
[11] A. Broggio, A. Ferroglia, G. Ossola, B.D. Pecjak, and R.S. Sameshima, JHEP 1704, 105 (2017) [arXiv:1702.00800].
[12] A. Kulesza, L. Motyka, T. Stebel, and V. Theeuwes, Phys. Rev. D 97, 114007 (2018) [arXiv:1704.03363].
[13] A. Kulesza, L. Motyka, D. Schwartlander, T. Stebel, and V. Theeuwes, Eur. Phys. J. C 79, 249 (2019) [arXiv:1812.08622].
[14] A. Broggio, A. Ferroglia, R. Frederix, D. Pagani, B.D. Pecjak, and I. Tsinikos, JHEP 1908, 039 (2019) [arXiv:1907.04343].
[15] N. Kidonakis, Phys. Rev. Lett. 102, 232003 (2009) [arXiv:0903.2561]; Phys. Rev. D 82, 054018 (2010) [arXiv:1005.4451]; Phys. Rev. D 82, 114030 (2010) [arXiv:1009.4935].
[16] N. Kidonakis, Int. J. Mod. Phys. A 31, 1650076 (2016) [arXiv:1601.01666 [hep-ph]]; Phys. Rev. D 99, 074024 (2019) [arXiv:1901.09928].
[17] G. Sterman, Nucl. Phys. B 281, 310 (1987).
[18] S. Catani and L. Trentadue, Nucl. Phys. B 327, 323 (1989).
[19] N. Kidonakis, Phys. Rev. D 90, 014006 (2014) [arXiv:1405.7046]; D 91, 031501 (2015) [arXiv:1411.2633]; D 91, 071502 (2015) [arXiv:1501.01581]; D (in press) [arXiv:1912.10362].
[20] E. Byckling and K. Kajantie, “Particle Kinematics,” Wiley & Sons., London (1973)
[21] E. Byckling and K. Kajantie, Phys. Rev. 187, 2008 (1969).
[22] W. Beenakker, W. L. van Neerven, R. Meng, G. A. Schuler and J. Smith, Nucl. Phys. B 351, 507 (1991).
[23] M. Sjodahl, JHEP 0812, 083 (2008) [arXiv:0807.0555].
[24] E. Szarek, Acta Phys. Polon. B 49, 1839 (2018) [arXiv:1809.00384].
[25] J. Campbell, R.K. Ellis, and R. Rontsch, Phys. Rev. D 87, 114006 (2013) [arXiv:1302.3856].
[26] F. Demartin, F. Maltoni, K. Mawatari, and M. Zaro, Eur. Phys. J. C 75, 267 (2015) [arXiv:1504.00611].
[27] L.A. Harland-Lang, A.D. Martin, P. Molytinski, and R.S. Thorne, Eur. Phys. J. C 75, 204 (2015) [arXiv:1412.3989].
[28] J. Alwall et al., JHEP 1407, 079 (2014) [arXiv:1405.0301].