SOME NEW SOLUTIONS OF YANG-BAXTER EQUATION

by

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Abstract

We have found some new solutions of both rational and trigonometric types by rewriting Yang-Baxter equation as a triple product equation in a vector space of matrices.
The Yang-Baxter equation (YBE)

\[
\sum_{j,k,\ell=1}^{N} R_{a_1b_1}^{jk}(\theta) R_{k\ell}^{a_2}(\theta') R_{j\ell}^{b_2}(\theta'') = \sum_{j,k,\ell=1}^{N} R_{b_1c_1}^{\ell j}(\theta'') R_{a_1\ell}^{c_2k}(\theta') R_{kj}^{b_2a_2}(\theta)
\]  

\(\theta + \theta'' = \theta'\) (2)

appears in many subjects ranging from statistical physics [1], exactly solvable 2-dimensional field theories [1], and braid-group ([2] and [3]), as well as the quantum group ([1] and [4]). Let \(V\) be a \(N\)-dimensional vector space with a symmetric bilinear non-degenerate form \(<x|y> = <y|x>\). For a fixed basis \(e_1, e_2, \ldots, e_N\) of \(V\), we set

\[g_{jk} = g_{kj} = <e_j|e_k> \] (3)

with its inverse \(g^{jk}\). We raise and lower indices as usual in terms of these metric tensors as

\[e^j = \sum_{k=1}^{N} g^{jk} e_k\] (4)

We introduce [5] two \(\theta\)-dependent triple products by

\[ [e^c, e_a, e_b]_{\theta} = \sum_{d=1}^{N} e_d R_{a \theta}^{dc} (\theta) \] , (5a)

\[ [e^d, e_b, e_a]^* = \sum_{c=1}^{N} R_{a \theta}^{dc} (\theta) e_c \] (5b)

so that we have

\[ R_{a \theta}^{dc} (\theta) = <e^d| [e^c, e_a, e_b]_{\theta} > = <e^c| [e^d, e_b, e_a]^* > \] . (6)

Then as we noted in [5], YBE can be rewritten as a triple product equation

\[
\sum_{j=1}^{N} [v, [u, e_j, z]_{\theta'}^*, [e^j, x, y]_{\theta'}^*]_{\theta''}^* = \sum_{j=1}^{N} [u, [v, e_j, x]_{\theta'}^*, [e^j, z, y]_{\theta'}^*]_{\theta} .
\] (7)
As a matter of fact, if we identify $x = e_{a_1}$, $y = e_{b_1}$, $z = e_{c_1}$, $u = e^{a_2}$, and $v = e^{c_2}$ in Eq. (7), and if we note Eqs. (5), then we can readily verify that Eq. (7) will reproduce Eq. (1). Similarly, we have the validity of

$$< u | v, x, y > = < v | u, y, x >^*$$

for any $u, v, x, y \in V$ in view of Eq. (6).

In reference [5], some solutions of Eq. (7) for the case of $[x, y, z]^\theta = [x, y, z]_\theta$ have been found for some triple systems including the case of the octonionic solution of de Vega and Nicolai [6]. The purpose of this note is to present another simpler solutions in terms of $n \times n$ matrices. Let $V$ now be a vector space consisting of all $n \times n$ matrices with $N = n^2$, i.e.,

$$V = \{ x | x = n \times n \text{ matrix} \}$$

and set

$$< x | y > = \text{Tr} (xy) .$$

The completeness condition of the space $V$ can then be expressed as

$$\sum_{j=1}^{N} e_j x e_j = (\text{Tr} x) \textbf{1}$$

for any $x \in V$, where $\textbf{1}$ stands for the unit $n \times n$ matrix.

Following the reference [5], we seek a solution with the ansatz of

$$[x, y, z]_\theta = P_1(\theta)yzx + P_2(\theta)xzy + A(\theta) < y | z > x + C(\theta) < z | x > y , \quad (12a)$$

$$[x, y, z]^*_\theta = P_2(\theta)yzx + P_1(\theta)xzy + A(\theta) < y | z > x + C(\theta) < z | x > y , \quad (12b)$$

which satisfy the constraint Eq. (8). Here, $P_1(\theta)$, $P_2(\theta)$, $A(\theta)$, and $C(\theta)$ are some functions of $\theta$ to be determined. Also, the products $yzx$ and $xzy$ in Eqs. (12) represent the standard associative matrix products in $V$. We insert the expression Eqs. (12) into both
sides of Eq. (7) and note the validity of Eq. (11). This yields the following equation:

\[
O = \sum_{j=1}^{N} \left\{ [v, [u, e_j, z]_{\theta'}, [e^j, x, y]_{\theta'}^{*}] - [u, [v, e_j, x]_{\theta'}, [e^j, z, y]_{\theta'}^{*}] \right\}
\]

\[= K_0(uxyzv - vzyxu)
+ K_1(yzuix - vxuzy) - \hat{K}_1(yxzu - uzxv)
+ K_2\{< z|u > yxv - < v|x > uzv\} - \hat{K}_2\{< x|v > yzu - < u|z > vxy\}
+ K_3\{< x|y > uzx - < y|xzu > u\} - \hat{K}_3\{< z|y > vzu - < y|zux > v\}
+ K_4\{< y|x > vzu - < y|zvx > u\} - \hat{K}_4\{< y|z > uxv - < y|xuz > v\}
+ K_5 < x|y > < z|u > v, \hat{K}_5 < z|y > < x|v > u,
\]

where we have set for simplicity

\[
K_0 = P''_{n}A'P_1 - P''_{n}A'P_2, \\
K_1 = P''_{n}P'_1 C - P''_{n}P'_2 C, \\
\hat{K}_1 = C''P'_2 P_1 - C''P'_1 P_2,
\]

\[
K_2 = P''_{n}P'_1 P_2 + P''_{n}C'P_2 - C''P_2 P_2 - C''P_2 C + n P''_{n}C'P_2, \\
\hat{K}_2 = P''_{n}P'_2 P_1 + P''_{n}C'P_1 + P''_{n}P'_1 C - C''P'_1 C + n P''_{n}C'P_1, \\
K_3 = P''_{n}P'_2 P_1 + P''_{n}A' A + C''P'_2 A - C''A'P_2 + n P''_{n}P'_2 A, \\
\hat{K}_3 = P''_{n}P'_2 P_1 + A''A'P_1 + A''P'_1 C - P''_{n}A' C + n A''P'_1 P_1, \\
K_4 = P''_{n}P'_1 P_2 + P''_{n}A' A + C''P'_1 A - C''A'P_1 + n P''_{n}P'_1 A, \\
\hat{K}_4 = P''_{n}P'_1 P_2 + A''A'P_2 + A''P'_2 C - P''_2 A'C + n A''P'_2 P_2, \\
K_5 = P''_{n}P'_1 A + P''_{n}P'_2 A + A''P'_1 P_2 + A''P'_2 P_1 + P''_{n}C'P_1 + P''_{n}C'P_2
\]

\[+ n \left\{ P''_{n}C'A + P''_{n}C'A + A''P'_1 A + A''P'_2 A + A''C'P_1 + A''C'P_2 \right\}
+ C''C'A - C''A'C + A''C'C + A''A'C + n^2 A''C'A, \\
\hat{K}_5 = A''P'_2 P_1 + A''P'_1 P_2 + P''_{n}P'_2 A + P''_{n}P'_1 A + P''_{n}C'P_1 + P''_{n}C'P_2
\]

\[+ n \left\{ A''C'P_1 + A''C'P_2 + A''P'_2 A + A''P'_1 A + P''_{n}C'A + P''_{n}C'A \right\}
+ A''C'C - C''A'C + C''C'A + A''A'C + n^2 A''C'A.
\]
Here, $P''$, $P'$, and $P$ for example stand for

$$P = P(\theta) \quad , \quad P' = P(\theta') \quad , \quad P'' = P(\theta'') \quad .$$

(15)

We note that $\hat{K}_j$ ($j = 1, 2, 3, 4, 5$) is the same function as $K_j$ except for the interchanges of $\theta \leftrightarrow \theta''$ and $P_1 \leftrightarrow P_2$. The YBE can be satisfied, if we have

$$K_0 = K_1 = \hat{K}_1 = K_2 = \hat{K}_2 = K_3 = \hat{K}_3 = K_4 = \hat{K}_4 = K_5 = \hat{K}_5 = 0 \quad .$$

(16)

We can solve these eleven coupled function equations as in [5] and [6] to find the following trigonometric solutions, assuming that at least one of $P_1(\theta)$ and $P_2(\theta)$ is not identically zero:

**Solution (I)**

We have $P_1(\theta) = P_2(\theta)$. Setting

$$\lambda = \frac{1}{2} \left( n \pm \sqrt{n^2 - 4} \right) \quad ,$$

(17)

the solution is given by

$$\frac{A(\theta)}{P_1(\theta)} = -\frac{\lambda^2 e^{k\theta} - \beta}{\lambda(e^{k\theta} - \beta)} \quad ,$$

(18a)

$$\frac{C(\theta)}{P_1(\theta)} = -\frac{e^{k\theta} - \lambda^2}{\lambda(e^{k\theta} - 1)} \quad ,$$

(18b)

where $\beta$ can assume two possible values of $\lambda^2$ or $-\lambda^4$, and $k$ is an arbitrary constant including the value of $k = \pm\infty$.

**Solution (II)**

$$P_2(\theta) = 0 \quad , \quad \frac{C(\theta)}{P_1(\theta)} = \frac{n}{e^{k\theta} - 1} \quad , \quad \frac{A(\theta)}{P_1(\theta)} = \frac{ne^{k\theta}}{(n^2 - 1) - e^{k\theta}} \quad .$$

(19)

**Solution (III)**

$$P_1(\theta) = 0 \quad , \quad \frac{C(\theta)}{P_2(\theta)} = \frac{n}{e^{k\theta} - 1} \quad , \quad \frac{A(\theta)}{P_2(\theta)} = \frac{ne^{k\theta}}{(n^2 - 1) - e^{k\theta}} \quad .$$

(20)
In Eqs. (19) and (20), $k$ is again an arbitrary constant including the case of $k = \pm \infty$.

We remark that these solutions satisfy the so-called crossing relation ([2] and [7]) which can be expressed as

$$\frac{1}{P_1(\theta)} [y, x, z]_\theta = \frac{1}{P_1(\phi)} [x, y, z]_\phi$$

(21)

for example for solutions (I) and (II), where $\theta$ is related to $\phi$ by

$$k(\phi + \theta) = \begin{cases} \log \beta & \text{for (I)} \\ \log(n^2 - 1) & \text{for (II)} \end{cases}$$

(22)

They also satisfy the unitarity relation. To see it, we introduce $R(\theta)$ and $R^*(\theta) : V \otimes V \rightarrow V \otimes V$ by

$$R(\theta)e_a \otimes e_b = \sum_{j, k=1}^N R_{ab}^{kj}(\theta)e_j \otimes e_k$$

(23a)

$$R^*(\theta)e_a \otimes e_b = \sum_{j, k=1}^N R_{ba}^{jk}(\theta)e_j \otimes e_k$$

(23b)

The relationship between $R(\theta)$, $R^*(\theta)$, and triple products is given then by

$$R(\theta)x \otimes y = \sum_{j=1}^N e_j \otimes [e^j, x, y]_\theta = \sum_{j=1}^N [e^j, y, x]^*_\theta \otimes e_j$$

(24a)

$$R^*(\theta)x \otimes y = \sum_{j=1}^N e_j \otimes [e^j, x, y]^*_\theta = \sum_{j=1}^N [e^j, y, x]_\theta \otimes e_j$$

(24b)

Now, the unitarity relation for all solutions (I)-(III) is expressed in the form of

$$R^*(-\theta)R(\theta) = R(\theta)R^*(-\theta) = C(\theta)C(-\theta)I_d$$

(25)

where $I_d$ is the identity map in $V \otimes V$. Note especially that we have $R^*(\theta) = R(\theta)$ and $[x, y, z]^*_\theta = [x, y, z]_\theta$ for the solution (I).

We have also found another solution of YBE when we replace Eqs. (12) now by

$$[x, y, z]_\theta = [x, y, z]^*_\theta$$

$$= P_1(\theta)zxy + P_2(\theta)yxz + B(\theta) < x|y > z + C(\theta) < z|x > y$$
which is consistent with Eq. (8). Repeating the same procedure as before, the solutions are now found to be of rational type given by

**Solution (I)**

\[ P_2(\theta) = \alpha P_1(\theta) , \quad \frac{B(\theta)}{P_1(\theta)} = k \theta , \quad \frac{C(\theta)}{P_1(\theta)} = \frac{\alpha}{k \theta} \]  

(27a)

**Solution (II)**

\[ P_2(\theta) = C(\theta) = 0 \quad , \quad \frac{B(\theta)}{P_1(\theta)} = \beta + k \theta \]  

(27b)

**Solution (III)**

\[ P_1(\theta) = C(\theta) = 0 \quad , \quad \frac{B(\theta)}{P_2(\theta)} = \beta + k \theta \]  

(27c)

for arbitrary constants \( \alpha(\neq 0) , \beta , \) and \( k \). However, we will not go into details of the calculations.

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