Sharing values of $q$-difference-differential polynomials

Jian Li¹ and Kai Liu*¹

*Correspondence: liukai418@126.com; liukai@ncu.edu.cn
¹Department of Mathematics, Nanchang University, Nanchang, Jiangxi, China

Abstract
This paper is devoted to the uniqueness of $q$-difference-differential polynomials of different types. Using the idea of common zeros and common poles (Chin. Ann. Math., Ser. A 35:675–684, 2014), we improve the conditions of the former theorems and obtain some new results on the uniqueness of $q$-difference-differential polynomials of meromorphic functions.

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1 Introduction and main results
In this paper, a meromorphic function is assumed meromorphic in the whole complex plane. We assume that the reader is familiar with the basic symbols and fundamental results of Nevanlinna theory; see, for example, [2, 3, 10]. We say that two meromorphic functions $f$ and $g$ share a point $a$ CM (IM) if $f(z) - a$ and $g(z) - a$ have the same zeros counting multiplicities (ignoring multiplicities). The logarithmic density of the set $E$ is defined by

$$\limsup_{r \to \infty} \frac{1}{\log r} \int_{|t| \leq r} \frac{1}{t} \, dt.$$ 

Denote by $S(r,f)$ a quantity of $o(T(r,f))$ as $r \to \infty$ outside a possible exceptional set $E$ of logarithmic density 0.

Yang and Hua [9] obtained an important result on the uniqueness when the differential polynomials $f^nf'$ and $g^mg'$ share one value CM. Recently, many studies are devoted to the uniqueness of difference and $q$-difference polynomials; see [4–6, 11–14]. Zhang [12] obtained the following result.

Theorem A ([12]) Let $f(z)$ and $g(z)$ be transcendental entire functions of zero order, and let $n, m, d$ be positive integers. If $n \geq m + 5d$ and $f(z)^n(f(z)^m - 1) \prod_{i=1}^d f(qiz)$ and $g(z)^m \times (g(z)^m - 1) \prod_{i=1}^d g(qiz)$ share 1 CM, then $f(z) \equiv tg(z)$, $t^{n+d} = t^m = 1$. 

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Liu, Liu, and Cao [4] and Zhang and Korhonen [11] obtained the following two theorems.

**Theorem B** ([4, Theorem 1.5]) Let \( f(z) \) and \( g(z) \) be transcendental zero-order entire functions, and let \( m \) be a positive integer. If \( n \geq m + 5 \) and \( f(z)^n(f(z)^m - a)f(qz + c) \) and \( g(z)^n(g(z)^m - a)g(qz + c) \) share a nonzero polynomial \( p(z) \) CM, then \( f(z) \equiv g(z) \).

**Theorem C** ([11, Theorem 5.1]) Let \( f(z) \) and \( g(z) \) be transcendental zero-order meromorphic functions. If \( n \geq 8 \) and \( f(z)^nf(qz) \) and \( g(z)^ng(qz) \) share \( 1 \) and \( \infty \) CM, then \( f(z) \equiv tg(z) \), \( t^{n+1} = 1 \).

Zhao and Zhang [13] proved the following theorem.

**Theorem D** ([13, Theorem 1.4]) Let \( f(z) \) and \( g(z) \) be transcendental zero-order entire functions, and let \( k \) be a positive integer. If \( n \geq 2k + 6 \) and \( f(z)^nf(qz) \) and \( g(z)^ng(qz) \) share \( 1 \) and \( \infty \) CM, then \( f(z) \equiv tg(z) \), where \( t^{n+1} = 1 \).

Wang and Ye [8] improved the conditions of Theorems B and C to \( n \geq m + 4 \) and \( n \geq 6 \), respectively, by using the idea of common zeros and common poles. Here we give the main idea of common zeros and common poles. Let \( f, g \) be two nonconstant meromorphic functions. Denote by \( \overline{\pi}_0(r) \) or \( \overline{\pi}_1(r) \) the numbers of common zeros or poles of \( fg \) and \( g \), ignoring multiplicities. Let \( p, q \) be positive integers. We assume that the Laurent series of \( f \) and \( g \) at \( z_0 \) are as follows:

\[
f(z) = \frac{1}{(z-z_0)^p}f_1(z), \quad g(z) = (z-z_0)^qg_1(z),
\]

where \( f_1(z) \) and \( g_1(z) \) are analytic functions at \( z_0 \), and \( f_1(z_0) \neq 0, g_1(z_0) \neq 0 \); the other cases can be discussed in a similar way. So \( z_0 \) is a zero of \( g(z) \) with multiplicities \( q \). If \( q > p \), then \( z_0 \) is a zero of \( f(z)g(z) \) with multiplicity \( q - p \), and thus the contribution to \( \overline{\pi}_0(r) \) is 1 at \( z_0 \). If \( q \leq p \), then \( z_0 \) is a pole of \( f(z)g(z) \) with multiplicity \( p - q \) or an analytic point of \( f(z)g(z) \), and thus the contribution to \( \overline{\pi}_0(r) \) is 0 at \( z_0 \). A similar method can be discussed for \( \overline{\pi}_1(r) \). As usual, denote by \( \overline{N}_0(r) \) or \( \overline{N}_1(r) \) the counting functions of the common zeros or poles of \( fg \) and \( g \), ignoring multiplicities. Thus we have \( \overline{N}(r, \frac{1}{k}) \leq \overline{N}(r, \frac{1}{j}) + \overline{N}_0(r) \) and \( \overline{N}(r, \frac{1}{j}) \leq \overline{N}(r, f) + \overline{N}_1(r) \). In this paper, we continue to consider the uniqueness of \( q \)-difference-differential polynomials. Firstly, we improve the condition \( n \geq m + 5d \) in Theorem A to \( n \geq m + d + 3 \) in Theorem 1.1. Set

\[
L(z,f) = \prod_{i=1}^{d} f(q_iz + c_i),
\]

where \( c_i \) and \( q_i \neq 0 (i = 1, \ldots, d) \) are constants, and \( d \) is a positive integer.

**Theorem 1.1** Let \( f(z) \) and \( g(z) \) be transcendental zero-order entire functions, and let \( m \) be a positive integer. If \( n \geq m + d + 3 \) and \( f(z)^n(f(z)^m - 1)L(z,f) \) and \( g(z)^n(g(z)^m - 1)L(z,g) \) share 1 CM, then \( f(z) \equiv cg(z), c_1^{eqd} = c_1^n = 1 \).

In the following theorem, we improve the condition \( n \geq 2k + 6 \) in Theorem D to \( n \geq 6 \).
Theorem 1.2 Let \( f(z) \) and \( g(z) \) be transcendental zero-order meromorphic functions, and let \( k \) be a positive integer. If \( n \geq 6 \) and \( (f(z))^n(f(qz + c))^k \) and \( (g(z))^n(g(qz + c))^k \) share 1 and \( \infty \) CM, then \( f(z) \equiv cg(z) \), \( c_1^{k+1} = 1 \).

We also consider the following theorems for \( q \)-difference polynomials of different types. The following theorem is also an improvement of Theorem C.

Theorem 1.3 Let \( f(z) \) and \( g(z) \) be transcendental zero-order meromorphic functions, and let \( s \) be a positive integer. If \( n \geq (d + 1)s + 4 \) and \( f(z)^sL(z,f)^s \) and \( g(z)^sL(z,g)^s \) share 1 and \( \infty \) CM, then \( f(z) \equiv cg(z) \), \( c_1^{s+1} = 1 \).

Theorem 1.4 Let \( f(z) \) and \( g(z) \) be transcendental zero-order meromorphic functions, \( q, c \in \mathbb{C} \), and \( q \neq 0 \). If \( n \geq 7 \), and \( f(z)^n(f(qz + c) - f(z)) \) and \( g(z)^n(g(qz + c) - g(z)) \) share 1 and \( \infty \) CM, then

\[
f(z)^n(f(qz + c) - f(z)) = g(z)^n(g(qz + c) - g(z)).
\]

If \( g(z) \) is transcendental with only finitely many zeros, then \( f(z) \equiv cg(z) \), where \( c_1^{n+1} = 1 \).

2 Lemmas
Combining [11, Theorem 1.1] and [1, Theorem 2.1], we easily get the following lemma.

Lemma 2.1 Let \( f(z) \) be a transcendental zero-order meromorphic function, \( q, c \in \mathbb{C} \), and \( q \neq 0 \). Then

\[
T(r,f(qz + c)) = T(r,f) + S(r,f)
\]
on a set of logarithmic density 1.

Lemma 2.2 ([7]) Let \( f(z) \) be a zero-order meromorphic function \( q, c \in \mathbb{C} \), and \( q \neq 0 \). Then

\[
m\left(r, \frac{f(qz + c)}{f(z)}\right) = S(r,f)
\]
on a set of logarithmic density 1.

Lemma 2.3 If \( f \) is a transcendental zero-order entire function, then

\[
T(r,f(z)^m(f(z)^m - 1)L(z,f)) = (m + n + d)T(r,f) + S(r,f)
\]
on a set of logarithmic density 1.

Proof Set \( F(z) = f(z)^m(f(z)^m - 1)L(z,f) \). By Lemma 2.2 and the standard Valiron–Mohon’ko theorem, if \( f \) is a transcendental zero-order entire function, then

\[
(n + m + d)T(r,f) = T(r, f^{m+1}(f^m - 1)) + S(r,f)
\]

\[
= m(r, f^{m+1}(f^m - 1)) + S(r,f)
\]
\[
\leq m\left(r, \frac{f^{n}(f^{m} - 1)}{f^{n}(f^{m} - 1) L(z, f)}\right) + m(r, F) + S(r, f)
\]
\[
\leq m\left(r, \frac{f^{d}}{L(z, f)}\right) + m(r, F) + S(r, f)
\]
\[
\leq T(r, F) + S(r, f)
\]
on a set of logarithmic density 1. On the other hand, combining Lemma 2.1 with the fact that \( f \) is a transcendental zero-order function, we have

\[
T(r, F) \leq T(r, f^{n}(f^{m} - 1)) + T(r, L(z, f))
\]
\[
\leq (n + m + d) T(r, f) + S(r, f)
\]
on a set of logarithmic density 1.

\[\square\]

### 3 Proofs of theorems

**Proof of Theorem 1.1** Let \( F(z) = f^{n}(f^{m} - 1)L(z, f) \) and \( G(z) = g^{n}(g^{m} - 1)L(z, g) \). Since \( F(z) \) and \( G(z) \) share 1 and \( \infty \) CM, we have that \( \frac{F'}{G'} = B \), that is,

\[
F = BG + 1 - B, \tag{1}
\]

where \( B \) is a nonzero constant.

If \( B \neq 1 \), then from the second main theorem of Nevanlinna theory, Lemma 2.1, and Lemma 2.3 we obtain

\[
(n + m + d)T(r, f) = T(r, F) + S(r, f)
\]
\[
\leq N(r, F) + N\left(r, \frac{1}{F}\right) + N\left(r, \frac{1}{F - 1 + B}\right) + S(r, f)
\]
\[
\leq N\left(r, \frac{1}{f^{n}}\right) + N\left(r, \frac{1}{f^{m} - 1}\right) + N\left(r, \frac{1}{L(z, f)}\right) + N\left(r, \frac{1}{G}\right) + S(r, f)
\]
\[
\leq (m + d + 1)(T(r, f) + T(r, g)) + S(r, f) + S(r, g). \tag{2}
\]

Using the same method, we have

\[
(n + m + d)T(r, g) \leq (m + d + 1)(T(r, f) + T(r, g)) + S(r, f) + S(r, g). \tag{3}
\]

Combining (2) with (3), we have

\[
(n - m - d - 2)(T(r, f) + T(r, g)) \leq S(r, f) + S(r, g),
\]

which contradicts to \( n \geq m + d + 3 \). Thus \( B = 1 \), and from (1) we have

\[
f^{n}(f^{m} - 1)L(z, f) = g^{n}(g^{m} - 1)L(z, g). \tag{4}
\]
Let \( h(z) = \frac{f(z)}{g(z)} \). So \( \frac{L(zf)}{L( zg)} = \prod_{i=1}^{d} \frac{L(q_{i}z + c_{i})}{L(q_{i}z + c_{i})} = \prod_{i=1}^{d} h(q_{i}z + c_{i}) = L(z, h) \), and then (4) can be written as

\[
g^{m}(h^{n}L(z, h) - 1) = h^{m}L(z, h) - 1. \quad (5)
\]

Next, we prove that \( h(z) \equiv c_{1} \) and \( c_{1}^{n+d} = c_{1}^{m} = 1 \), where \( c_{1} \) is a constant. Assume on the contrary that \( h(z) \) is not a constant. From Lemma 2.1 we have

\[
T(r, h^{n}L(z, h)) \leq (n + m + d)T(r, h) + S(r, h).
\]

We also have

\[
(n + m)T(r, h) = T(r, h^{n}m)
\]

\[
\leq T\left(r, \frac{1}{h^{n}mL(z, h)}\right) + T(r, L(z, h)) + O(1)
\]

\[
\leq T(r, h^{n}mL(z, h)) + dT(r, h) + S(r, h).
\]

Since \( n \geq m + d + 3 \), from the last two inequalities it follows that \( S(r, h^{n}mL(z, h)) = S(r, h) \). Denote by \( \overline{N}(r) \) the counting function of the common poles of \( h^{n}mL(z, h) \) and \( L(z, h) \) ignoring multiplicities. Then

\[
\overline{N}(r, h^{n}mL(z, h)) \leq \overline{N}(r, h) + \overline{N}_{1}(r).
\]

Here we should remark that the poles of \( L(z, h) \) may be the zeros of \( h \) and the zeros of \( L(z, h) \) may be the poles of \( h \). Similarly, denote by \( \overline{N}_{0}(r) \) the counting function of the common zeros of \( h^{n}mL(z, h) \) and \( L(z, h) \) ignoring multiplicities, and then

\[
\overline{N}\left(r, \frac{1}{h^{n}mL(z, h)}\right) \leq \overline{N}\left(r, \frac{1}{h}\right) + \overline{N}_{0}(r).
\]

From the second main theorem of Nevanlinna theory and the last two inequalities we have

\[
T(r, h^{n}mL(z, h)) \leq \overline{N}(r, h^{n}mL(z, h)) + \overline{N}\left(r, \frac{1}{h^{n}mL(z, h)}\right)
\]

\[
+ \overline{N}\left(r, \frac{1}{h^{n}mL(z, h) - 1}\right) + S(r, h^{n}mL(z, h))
\]

\[
\leq \overline{N}(r, h) + \overline{N}_{1}(r) + \overline{N}\left(r, \frac{1}{h}\right) + \overline{N}_{0}(r)
\]

\[
+ \overline{N}\left(r, \frac{1}{h^{n}mL(z, h) - 1}\right) + S(r, h). \quad (6)
\]

On the other hand,

\[
(n + m)m(r, h) = m(r, h^{n}m) \leq m(r, h^{n}mL(z, h)) + m\left(r, \frac{1}{L(z, h)}\right) + O(1), \quad (7)
\]
\[(n + m)N(r, h) = N(r, h^{n+m}) = N\left(\frac{h^{n+m}L(z, h)}{L(z, h)}\right) \leq N(r, h^{n+m}L(z, h)) + N\left(r, \frac{1}{L(z, h)}\right) - \overline{N}_1(r) - \overline{N}_0(r). \tag{8}\]

From (7) and (8) we get

\[(n + m)T(r, h) \leq T(r, h^{n+m}L(z, h)) + T\left(r, \frac{1}{L(z, h)}\right) - \overline{N}_1(r) - \overline{N}_0(r) + O(1). \tag{9}\]

From (6) and (9) we get

\[(n + m)T(r, h) \leq \overline{N}(r, h) + \overline{N}\left(r, \frac{1}{h}\right) + \overline{N}\left(r, \frac{1}{h^{n+m}L(z, h) - 1}\right) + T\left(r, \frac{1}{L(z, h)}\right) + S(r, h) \leq (d + 2)T(r, h) + \overline{N}\left(r, \frac{1}{h^{n+m}L(z, h) - 1}\right) + S(r, h). \tag{10}\]

Since \(n \geq m + d + 3\), the value 1 is not the Picard exceptional value of \(h^{n+m}L(z, h)\) from (10). Furthermore, we prove \(h^{n+m}L(z, h) \equiv 1\) and \(h(z) \equiv c_1\) is a nonzero constant. If \(h^{n+m}L(z, h) \not\equiv 1\), then since 1 is not the Picard exceptional value of \(h^{n+m}L(z, h)\), there exists a point \(z_0\) satisfying \(h(z_0)^{n+m}L(z, h(z_0)) = 1\). From the condition that \(g(z)\) is an entire function and (5) we have \(h(z_0)^m = 1\) and

\[\overline{N}\left(r, \frac{1}{h^{n+m}L(z, h) - 1}\right) \leq \overline{N}\left(r, \frac{1}{h^m - 1}\right) \leq mT(r, h) + S(r, h). \tag{11}\]

Substituting (11) into (10), we get a contradiction to \(n \geq m + d + 3\), so \(h^{n+m}L(z, h) \equiv 1\), that is, \(h^{n+m} = \frac{1}{L(z, h)}\). From Lemma 2.1 we have

\[(n + m)T(r, h) = T\left(r, L(z, h)\right) \leq dT(r, h) + S(r, h), \]

which also contradicts to \(n \geq m + d + 3\), so \(h(z) \equiv c_1\), where \(c_1\) is a nonzero constant, that is, \(f(z) \equiv c_1g(z)\), and \(L(z, h) = c_1^d\). From (5) we can get \(c_1^n = c_1^{n+d} = 1\). Thus the theorem is proved. \( \square \)

**Proof of Theorem 1.2** Let \(F(z) = f(z)^nf(qz + c)\) and \(G(z) = g(z)^ng(qz + c)\). From the condition in Theorem 1.2 we know that \(F^{(k)}\) and \(G^{(k)}\) share 1 and \(\infty\) CM, so

\[\frac{F^{(k)} - 1}{G^{(k)} - 1} = C,\]

where \(C\) is a nonzero constant, that is,

\[F^{(k)} = CG^{(k)} - C + 1. \tag{12}\]
Integrating both sides of (12), we have

\[ F = CG + \frac{1 - C}{k!} z^k + p_1(z), \]  

(13)

where \( p_1(z) \) is a polynomial of degree at most \( k - 1 \). Denote \( \frac{1 - C}{k!} z^k \) by \( p(z) \). If \( p(z) \not\equiv 0 \), then by the second main theorem of Nevanlinna theory, Lemma 2.1, and (13) we obtain

\[
T(r,F) \leq \overline{N}(r,F) + \overline{N} \left( r, \frac{1}{F} \right) + \overline{N} \left( r, \frac{1}{F - p} \right) + S(r,f) \\
\leq \overline{N}(r,f) + \overline{N}_1(r) + \overline{N} \left( r, \frac{1}{f} \right) + \overline{N}_0(r) + \overline{N} \left( r, \frac{1}{G} \right) + S(r,f) \\
\leq 2T(r,f) + 2T(r,g) + \overline{N}_1(r) + \overline{N}_0(r) + S(r,f) + S(r,g),
\]

(14)

where \( \overline{N}_0(r) \) denotes the counting function ignoring multiplicities of the common zeros of \( F(z) \) and \( f(qz + c) \), and \( \overline{N}_1(r) \) denotes the counting function ignoring multiplicities of the common poles of \( F(z) \) and \( f(qz + c) \). On the other hand,

\[
nm(r,f) = m(r,F) + m \left( r, \frac{1}{f(qz + c)} \right) + O(1).
\]

(15)

\[
nN(r,f) = N(r,f^n) = N \left( \frac{F(z)}{f(qz + c)} \right) \\
\leq N(r,F) + N \left( r, \frac{1}{f(qz + c)} \right) - \overline{N}_1(r) - \overline{N}_0(r).
\]

(16)

From (15), (16), and Lemma 2.1 we have

\[
(n - 1)T(r,f) \leq T(r,F) - \overline{N}_1(r) - \overline{N}_0(r) + O(1).
\]

(17)

Substituting (14) into (17), we obtain

\[
(n - 3)T(r,f) \leq 2T(r,g) + S(r,f) + S(r,g).
\]

(18)

Using the same method, we also get

\[
(n - 3)T(r,g) \leq 2T(r,f) + S(r,f) + S(r,g).
\]

(19)

Combining (18) with (19), we have

\[
(n - 5)(T(r,g) + T(r,f)) \leq S(r,f) + S(r,g),
\]

which contradicts to \( n \geq 6 \), and thus \( p(z) \equiv 0 \). Since the degree of \( p_1(z) \) is at most \( k - 1 \), we have \( C = 1 \) and \( p_1(z) \equiv 0 \). From (13) we get

\[ f^n f(qz + c) = g^n g(qz + c). \]
Assume that $h(z) = \frac{g(z)}{g'(z)}$. Then $h(qz + c)h(z)^n = 1$, that is, $h(z)^n = \frac{1}{h(qz + c)}$, and from Lemma 2.1 we have

$$nT(r, h) = T(r, h(qz + c)) \leq T(r, h) + S(r, h),$$

which also contradicts to $n \geq 6$, so $h(z)$ is a nonzero constant, say $c_2$. So $f(z) \equiv c_2g(z)$, and $c_2^n = 1$. Thus the theorem is proved. \hfill \Box

**Proof of Theorem 1.3** Since $f(z)$ and $g(z)$ are transcendental zero-order meromorphic functions and $f(z)^nL(z, f)^{s}$ and $g(z)^nL(z, g)^{s}$ share $1$ and $\infty$ CM, we have

$$\frac{f(z)^nL(z, f)^{s} - 1}{g(z)^nL(z, g)^{s} - 1} = E,$$  \hspace{1cm} (20)

where $E$ is a nonzero constant. Then (20) can be rewritten as

$$Eg(z)^nL(z, g)^{s} = f(z)^nL(z, f)^{s} - 1 + E.$$  \hspace{1cm} (21)

Let $F(z) = f(z)^nL(z, f)^{s}$ and $G(z) = g(z)^nL(z, g)^{s}$. We affirm that $E = 1$. On the contrary, assume that $E \neq 1$. Using the second main theorem of Nevanlinna theory and Lemma 2.1 for (21), we get

$$T(r, F) \leq N(r, F) + N \left( r, \frac{1}{F} \right) + N \left( r, \frac{1}{F - 1 + E} \right) + S(r, f)$$

$$\leq N(r, f) + N \left( r, \frac{1}{f} \right) + N \left( r, \frac{1}{g} \right) + N \left( r, \frac{1}{G} \right) + S(r, f)$$

$$\leq 2T(r, f) + N \left( r, \frac{1}{g} \right) + N \left( r, \frac{1}{L(z, g)} \right) + N \left( r, \frac{1}{L(z, f)} \right) + S(r, f)$$

$$\leq 2T(r, f) + (d + 1)T(r, g) + N \left( r, \frac{1}{L(z, f)} \right) + N \left( r, \frac{1}{L(z, g)} \right) + S(r, f) + S(r, g),$$  \hspace{1cm} (22)

where $N_0(r)$ denotes the counting function ignoring multiplicities of the common zeros of $F(z)$ and $L(z, f)$, and $N_1(r)$ denotes the counting function ignoring multiplicities of the common poles of $F(z)$ and $L(z, f)$.

Since $F(z) = f(z)^nL(z, f)^{s}$, we have

$$nm(r, f) = m(r, F) + sm \left( r, \frac{1}{L(z, f)} \right) + O(1),$$

$$nN(r, f) = N(r, f^n) = N \left( r, F(z) \left[ \frac{1}{L(z, f)} \right]^s \right)$$

$$\leq N(r, F) + sN \left( r, \frac{1}{L(z, f)} \right) - N_1(r) - N_0(r).$$

So

$$nT(r, f) \leq T(r, F) + sT \left( r, \frac{1}{L(z, f)} \right) - N_1(r) - N_0(r) + O(1)$$

$$\leq T(r, F) + dsT(r, f) - N_1(r) - N_0(r) + S(r, f).$$  \hspace{1cm} (23)
Substituting (22) into (23), we obtain

\[(n - ds - 2)T(r,f) \leq (d + 1)T(r,g) + S(r,f) + S(r,g).\]  
(24)

Using the same method, we can get

\[(n - ds - 2)T(r,g) \leq (d + 1)T(r,f) + S(r,f) + S(r,g).\]  
(25)

Combining (24) with (25), it follows

\[(n - d - ds - 3)(T(r,g) + T(r,f)) \leq S(r,f) + S(r,g),\]

which contradicts to \(n \geq (d + 1)s + 4\), and thus \(E = 1\). From (21) we get

\[f(z)^n L(z,f) = g(z)^n L(z,g).\]

Let \(h(z) = \frac{f(z)}{g(z)}\). So \(L(z,f) = L(z,h)\). Then

\[h(z)^n[L(z,h)]^\prime = 1.\]  
(26)

So \(nT(r,h) = sT(r,L(z,h)) \leq sdT(r,h) + S(r,h)\). Since \(n \geq (d + 1)s + 4\), \(h(z)\) must be a constant, say \(c_3\), that is, \(f(z) \equiv c_3g(z)\). Then from (26) it follows that \(c_3^{n + sd} = 1\). \(\square\)

**Proof of Theorem 1.4** Letting \(L(z,f) = f(qz + c) - f(z)\) and \(L(z,g) = g(qz + c) - g(z), s = 1\) in Theorem 1.3, we obtain that if \(n \geq 7\), then

\[f(z)^n(f(qz + c) - f(z)) = g(z)^n(g(qz + c) - g(z)).\]

Let \(h(z) = \frac{f(z)}{g(z)}\) and \(H(z) = h(qz + c)h(z)^n\). The last equation implies that

\[g(qz + c)(H(z) - 1) = g(z)(h(z)^{n+1} - 1).\]  
(27)

We know that \(T(r,H) \leq (n + 1)T(r,h) + S(r,h)\) from the expression of \(H(z)\) and Lemma 2.1. Thus \(S(r,H) = S(r,h)\). Next, we prove that \(h(z) \equiv c_4\), where \(c_4^{n+1} = 1\), when \(\frac{g(z)}{g(qz + c)}\) is transcendental with only finitely many zeros. Obviously, \(h(z)\) is neither a constant except \(c_4\) nor a rational function from (27) since \(\frac{g(z)}{g(qz + c)}\) is transcendental. Thus we assume that \(h(z)\) is a transcendental meromorphic function.

First, we affirm that \(H(z) - 1\) has infinitely many zeros. Otherwise, by the second main theorem of Nevanlinna theory and Lemma 2.1

\[T(r,H(z)) \leq \mathcal{N}(r,H(z)) + \mathcal{N}\left(r, \frac{1}{H(z)}\right) + \mathcal{N}\left(r, \frac{1}{H(z) - 1}\right) + S(r,H)\]
\[\leq 2T(r,h(z)) + 2T(r,h(qz + c)) + S(r,h)\]
\[\leq 4T(r,h(z)) + S(r,h).\]  
(28)
From the Valiron–Mohon’ko theorem, Lemma 2.1, and (28) we obtain

\[ nT(r, h) = T(r, h(z)^n) \leq T(r, H(z)) + T\left( r, \frac{1}{h(qz + c)} \right) + S(r, h) \]

\[ \leq 5T(r, h(z)) + S(r, h), \]

which contradicts to \( n \geq 7 \). Thus \( H(z) - 1 \) has infinitely many zeros.

Then we prove that \( H(z) \equiv 1 \) and \( h(z) \equiv c_4 \) is a nonzero constant, that is, \( c_4^{n+1} = 1 \). If \( H(z) \not\equiv 1 \), then since \( H(z) - 1 \) has infinitely many zeros, we can choose a point \( z_0 \) satisfying \( H(z_0) = 1 \), and \( z_0 \) is not the zero of \( \frac{g(z)}{h(z+1)} \). From (27) we have \( h(qz_0 + c) = h(z_0) \). By Lemma 2.1

\[ \overline{N}(r, \frac{1}{H(z) - 1}) \leq \overline{N}(r, \frac{1}{h(qz + c) - h(z)}) \leq 2T(r, h(z)) + S(r, h). \]  

(29)

Using the second main theorem of Nevanlinna theory, Lemma 2.1, and (29), we obtain

\[ T(r, H(z)) \leq \overline{N}(r, H(z)) + \overline{N} \left( r, \frac{1}{H(z)} \right) + \overline{N} \left( r, \frac{1}{H(z) - 1} \right) + S(r, h) \]

\[ \leq \overline{N}(r, h) + \overline{N}_1(r) + \overline{N} \left( r, \frac{1}{h} \right) + \overline{N}_0(r) + \overline{N} \left( r, \frac{1}{H(z) - 1} \right) + S(r, h) \]

\[ \leq 4T(r, h) + \overline{N}_1(r) + \overline{N}_0(r) + S(r, h), \]  

(30)

where \( \overline{N}_0(r) \) denotes the counting function ignoring multiplicities of the common zeros of \( H(z) \) and \( h(qz + c) \), and \( \overline{N}_1(r) \) denotes the counting function ignoring multiplicities of the common poles of \( H(z) \) and \( h(qz + c) \).

On the other hand, from \( H(z) = h(qz + c)h(z)^n \) we have

\[ nM(r, h) = m(r, h^n) \leq m(r, H(z)) + m \left( r, \frac{1}{h(qz + c)} \right) + O(1), \]  

(31)

\[ nN(r, h) = N(r, h^n) \leq N(r, H(z)) + N \left( r, \frac{1}{h(qz + c)} \right) - \overline{N}_1(r) - \overline{N}_0(r). \]  

(32)

From (31), (32), and Lemma 2.1 we have

\[ nT(r, h) \leq T(r, H(z)) + T \left( r, \frac{1}{h(qz + c)} \right) - \overline{N}_1(r) - \overline{N}_0(r) + O(1) \]

\[ \leq T(r, H(z)) + T(r, h) - \overline{N}_1(r) - \overline{N}_0(r) + S(r, h). \]  

(33)

Substituting (30) into (33), we obtain

\[ nT(r, h) \leq 5T(r, h) + S(r, h), \]

which contradicts to \( n \geq 7 \), so \( H(z) = h(qz + c)h(z)^n \equiv 1 \), that is, \( h(z)^n = \frac{1}{h(qz + c)} \). From Lemma 2.1 we have

\[ nT(r, h) = T(r, h(qz + c)) \leq T(r, h) + S(r, h), \]
which also contradicts to \( n \geq 7 \), so \( h(z) \) is a nonzero constant, say \( c_4 \), and from (27) we get \( c_4^{n+1} = 1 \). Thus the theorem is proved. □

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References
1. Chiang, Y.M., Feng, S.J.: On the Nevanlinna characteristic of \( f(z + \eta) \) and difference equations in the complex plane. Ramanujan J. 16(1), 105–129 (2008)
2. Hayman, W.K.: Meromorphic Functions. Clarendon Press, Oxford (1964)
3. Laine, I.: Nevanlinna Theory and Complex Differential Equations. de Gruyter, Berlin (1993)
4. Liu, K., Liu, X.L., Cao, T.B.: Uniqueness and zeros of \( q \)-shift difference polynomials. Proc. Indian Acad. Sci. Math. Sci. 121(3), 301–310 (2011)
5. Liu, K., Liu, X.L., Cao, T.B.: Value distribution and uniqueness of difference polynomials. Adv. Differ. Equ. 2011, Article ID 234215 (2011)
6. Liu, K., Liu, X.L., Cao, T.B.: Some difference results on Hayman conjecture and uniqueness. Bull. Iran. Math. Soc. 38(4), 1007–1020 (2012)
7. Liu, K., Qi, X.G.: Meromorphic solutions of \( q \)-shift difference equations. Ann. Pol. Math. 101, 215–225 (2011)
8. Wang, Q.Y., Ye, Y.S.: Value distribution and uniqueness of difference polynomials for meromorphic functions. Chin. Ann. Math., Ser. A 35, 675–684 (2014) (in Chinese)
9. Yang, C.C., Hua, X.H.: Uniqueness and value sharing of meromorphic functions. Ann. Acad. Sci. Fenn., Math. 22(2), 395–406 (1997)
10. Yang, C.C., Yi, H.X.: Uniqueness Theory of Meromorphic Functions. Kluwer Academic, Dordrecht (2003)
11. Zhang, J.L., Korhonen, R.J.: On the Nevanlinna characteristic of \( f(qz) \) and its applications. J. Math. Anal. Appl. 369, 537–544 (2010)
12. Zhang, K.Y.: Uniqueness of \( q \)-difference polynomials of meromorphic functions. Adv. Mater. Res. 756–759, 2948–2951 (2013)
13. Zhao, Q.X., Zhang, J.L.: Zeros and shared one value of \( q \)-shift difference polynomials. J. Contemp. Math. Anal. 50(2), 63–69 (2015)
14. Zheng, X.M., Xu, H.Y.: On value distribution and uniqueness of meromorphic function with finite logarithmic order concerning its derivative and \( q \)-shift difference. J. Inequal. Appl. 2014, Article ID 295 (2014)