How strongly does diffusion or logistic-type degradation affect existence of global weak solutions in a chemotaxis-Navier–Stokes system?

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Abstract. This paper considers the chemotaxis-Navier–Stokes system with nonlinear diffusion and logistic-type degradation term

\[
\begin{align*}
  n_t + u \cdot \nabla n &= \nabla \cdot (D(n) \nabla n) - \nabla \cdot (n \chi(c) \nabla c) + \kappa n - \mu n^\alpha, \quad x \in \Omega, \ t > 0, \\
  c_t + u \cdot \nabla c &= \Delta c - n f(c), \quad x \in \Omega, \ t > 0, \\
  u_t + (u \cdot \nabla) u &= \Delta u + \nabla P + n \nabla \Phi + g, \quad \nabla \cdot u = 0, \quad x \in \Omega, \ t > 0,
\end{align*}
\]

where \(\Omega \subset \mathbb{R}^3\) is a bounded smooth domain; \(D \geq 0\) is a given smooth function such that \(D_1 s^{m-1} \leq D(s) \leq D_2 s^{m-1}\) for all \(s \geq 0\) with some \(D_2 \geq D_1 > 0\) and some \(m > 0\); \(\chi, f\) are given functions satisfying some conditions; \(\kappa \in \mathbb{R}, \mu \geq 0, \alpha > 1\) are constants. This paper shows existence of global weak solutions to the above system under the condition that

\[
m > \frac{2}{3}, \quad \mu \geq 0 \quad \text{and} \quad \alpha > 1
\]

hold, or that

\[
m > 0, \quad \mu > 0 \quad \text{and} \quad \alpha > \frac{4}{3}
\]

hold. This result asserts that “strong” diffusion effect or “strong” logistic damping derives existence of global weak solutions even though the other effect is “weak”, and can include previous works \([8, 10, 23, 26]\).
1. Introduction

This work deals with the chemotaxis-Navier–Stokes system with nonlinear diffusion and logistic-type degradation term

\[
\begin{aligned}
\frac{\partial n}{\partial t} + u \cdot \nabla n &= \Delta n^m - \chi \nabla \cdot (n \nabla c) + \kappa n - \mu n^\alpha, \\
\frac{\partial c}{\partial t} + u \cdot \nabla c &= \Delta c - n c, \\
\frac{\partial u}{\partial t} + (u \cdot \nabla)u &= \Delta u + \nabla P + n \nabla \Phi
\end{aligned}
\] (1.1)

for \( x \in \Omega \) and \( t > 0 \), where \( \Omega \subset \mathbb{R}^3 \) is a bounded domain with smooth boundary, \( m, \chi > 0, \kappa, \mu \geq 0 \) and \( \alpha > 1 \) are constants and \( \Phi \) is a given function, and consider the question:

How strongly does diffusion or logistic-type degradation affect existence of global weak solutions in a chemotaxis-Navier–Stokes system?

More precisely, the purpose of this work is to determine conditions for \( m \) and \( \alpha \) which derive global existence of weak solutions to the system (1.1). The system (1.1) is a generalization of a chemotaxis-Navier–Stokes system which is proposed by Tuval et al. [18] and describes the situation where a species in a drop of water moves towards higher concentration of oxygen according to a property called chemotaxis. Here chemotaxis is a property such that a species reacts on some chemical substance and moves towards or moves away from higher concentration of that substance. Chemotaxis is one of important properties in the animals’ life, e.g., movement of sperm, migrations of neurons and lymphocytes and tumor invasion. In (1.1), \( \mathbb{R} \)-valued unknown functions \( n = n(x, t), c = c(x, t), P = P(x, t) \) shows the density of species, the concentration of oxygen, the pressure of the fluid, respectively, and an \( \mathbb{R}^3 \)-valued unknown function \( u = u(x, t) \) describes the fluid velocity field.

In the study of the system (1.1), we often refer to the study of the chemotaxis system

\[
\begin{aligned}
\frac{\partial n}{\partial t} &= \Delta n^m - \chi \nabla \cdot (n \nabla c) + \kappa n - \mu n^\alpha, \\
\frac{\partial c}{\partial t} &= \Delta c - c + n
\end{aligned}
\] (1.2)

Thus we first introduce several known results about the chemotaxis system:

In the study of the Keller–Segel system, that is, the system (1.2) with \( m = 1 \) and \( \kappa = \mu = 0 \)

\[
\begin{aligned}
\frac{\partial n}{\partial t} &= \Delta n - \chi \nabla \cdot (n \nabla c), \\
\frac{\partial c}{\partial t} &= \Delta c - c + n
\end{aligned}
\]

the chemotaxis term \(-\chi \nabla \cdot (n \nabla c)\) derives blow-up phenomena in some cases; in the 2-dimensional setting it was shown that there exists \( \vartheta > 0 \) such that, if a mass of an initial data of \( n \) is less than \( \vartheta \) then global bounded classical solutions exist (see Nagai–Senba–Yoshida [12]), and for all \( M > \vartheta \) there is an initial data \( n_0 \) of \( n \) such that \( M = \int_{\Omega} n_0 \) and a corresponding solution blows up in finite/infinite time (see Horstmann–Wang [4] and Mizoguchi–Winkler [11]); in the 3-dimensional setting for all \( M > 0 \) there is an initial data \( n_0 \) of \( n \) such that \( M = \int_{\Omega} n_0 \) and a corresponding solution blows up in finite
time; related works about the Keller–Segel system can be found in \([1, 14, 19]\); blow-up phenomena is excluded in the 1-dimensional case \([14]\); global existence results in the higher-dimensional setting are in \([19, 1]\).

On the other hand, in the chemotaxis system with logistic term which is \((1.2)\) with \(m = 1\) and \(\alpha = 2\)

\[
\begin{align*}
  n_t &= \Delta n - \chi \nabla \cdot (n \nabla c) + \kappa n - \mu n^2, \\
  c_t &= \Delta c - c + n,
\end{align*}
\]

the logistic term \(\kappa n - \mu n^2\) suppresses blow-up phenomena; in the 2-dimensional setting Osaki et al. \([13]\) and Jin–Xiang \([7]\) derived that for all \(\mu > 0\) there exist global classical solutions; Winkler \([20]\) showed global existence of classical solutions under some largeness condition for \(\mu > 0\); recently, Xiang \([25]\) obtained a precise condition for \(\mu > 0\) to derive global existence of classical solutions; Lankeit \([9]\) established global existence of weak solutions for arbitrary \(\mu > 0\); more related works are in \([3, 22]\); Winkler \([22]\) and He–Zheng \([3]\) showed asymptotic behavior of global classical solutions.

However, the small logistic-type degradation damping may not suppress blow-up phenomena; in the parabolic–elliptic chemotaxis system with logistic-type degradation term

\[
\begin{align*}
  n_t &= \Delta n - \chi \nabla \cdot (n \nabla c) + \kappa n - \mu n^\alpha, \\
  0 &= \Delta c - c + n,
\end{align*}
\]

Winkler \([24]\) showed that, if \(\alpha < \frac{7}{6}\) in the 3,4-dimensional cases and if \(\alpha < 1 + \frac{1}{2(N-1)}\) in the \(N\)-dimensional case with \(N \geq 5\), then there exists an initial data such that a corresponding solution blows up in finite time.

Moreover, in the chemotaxis system with degenerate diffusion

\[
\begin{align*}
  n_t &= \Delta n^m - \chi \nabla \cdot (n^{q-1} \nabla c), \\
  c_t &= \Delta c - c + n
\end{align*}
\]

with some \(q \geq 2\), in the \(N\)-dimensional setting some smallness condition for \(m \geq 1\) yields existence of blow-up solutions to the system; Ishida–Yokota \([6]\) and Hashira–Ishida–Yokota \([2]\) obtained that the condition that \(m < q - \frac{2}{N}\) entails existence of an initial data such that a corresponding solution blows up in finite time; conversely, it was shown that the restriction of \(m > q - \frac{2}{N}\) enables us to find global weak solutions \(([5])\).

In summary, in the study of the chemotaxis system, some largeness condition for an effect of the logistic-type degradation or the nonlinear diffusion entails global existence, and some smallness condition for the effect derives existence of blow-up solutions. Does this happen also in the chemotaxis-Navier–Stokes system? In order to consider this question we next recall several related works about the chemotaxis-Navier–Stokes system \((1.1)\):

In the case that \(\kappa = \mu = 0\) and in the 3-dimensional setting, it was shown that the diffusion effect dominate the chemotactic interaction; in the case that \(m = 1\) Winkler \([23]\) showed global existence of weak solutions; in \((1.1)\) with \(\kappa = \mu = 0\), Zhang–Li \([26]\)
asserts that if \( m \geq \frac{2}{3} \) then global weak solutions exist; however, there seem to be several miscalculations in the proof, e.g., in the proof of \([26, (3.6)]\) they used the Young inequality
\[
a^{\frac{1}{3m-1}} \leq \varepsilon a + C(\varepsilon, m)
\]
with some \( C(\varepsilon, m) > 0 \) for all \( \varepsilon > 0 \) and for \( m \geq \frac{2}{3} \) even though this inequality does not hold when \( m = \frac{2}{3} \) (which implies that \( \frac{1}{3m-1} = 1 \)); also in the case that \( m > \frac{2}{3} \) there are still gaps in the proof (for more details, see Remarks 2.2 and 4.2 in this paper); although there are miscalculations, they constructed essential estimates for obtaining global existence of weak solutions; thus another purpose of this work is to correct arguments in \([26]\) and to establish some condition of \( m \) for deriving global existence of weak solutions.

Moreover, in the case that \( \mu > 0 \) and \( \alpha = 2 \), it was established that, for all \( \mu > 0 \), global weak solutions exist; Lankeit \([10]\) first obtained global existence of weak solutions to \((1.1)\) with \( m = 1 \) and \( \alpha = 2 \); recently, global existence of weak solutions to \((1.1)\) with \( m > 0 \) and \( \alpha = 2 \) was shown in \([8]\); however, a general case such as that \( m > 0 \) and \( \alpha > 1 \) has not considered yet. Thus the main purpose of this paper is to obtain some conditions for \( m > 0 \) and \( \alpha > 1 \) which derive existence of global weak solutions to the chemotaxis-Navier–Stokes system.

In order to attain the purposes of this paper:

- to obtain some conditions for deriving global existence of weak solutions,
- to correct arguments in \([26]\) for establishing global weak solutions,

we consider the following chemotaxis-Navier–Stokes system with nonlinear diffusion and logistic-type degradation term:

\[
\begin{aligned}
&n_t + u \cdot \nabla n = \nabla \cdot (D(n) \nabla n) - \nabla \cdot (n \chi(c) \nabla c) + \kappa n - \mu n^\alpha, \quad x \in \Omega, \ t > 0, \\
&c_t + u \cdot \nabla c = \Delta c - n f(c), \quad x \in \Omega, \ t > 0, \\
&u_t + (u \cdot \nabla) u = \Delta u + \nabla P + n \nabla \Phi + g, \quad \nabla \cdot u = 0, \quad x \in \Omega, \ t > 0, \\
&D(n) \partial_\nu n = \partial_\nu c = 0, \quad u = 0, \quad x \in \partial \Omega, \ t > 0, \\
n(x, 0) = n_0(x), \ c(x, 0) = c_0(x), \ u(x, 0) = u_0(x), \quad x \in \Omega,
\end{aligned}
\]

where \( \Omega \) is a bounded domain in \( \mathbb{R}^3 \) with smooth boundary \( \partial \Omega \) and \( \partial_\nu \) denotes differentiation with respect to the outward normal of \( \partial \Omega \); \( D \) is a function satisfying
\[
D \in C^{1+\gamma}_\text{loc}([0, \infty)), \quad D_1 s^{m-1} \leq D(s) \leq D_2 s^{m-1} \quad \text{for all } s \geq 0
\]  
with some \( \gamma > 0 \), \( D_2 \geq D_1 > 0 \) and \( m > 0 \); \( \kappa \in \mathbb{R}, \mu \geq 0, \alpha > 1 \) are constants; functions \( \chi \) and \( f \) are assumed that
\[
\chi \in C^2([0, \infty)), \quad \chi > 0 \quad \text{on } [0, \infty),
\]
\[
f \in C^1([0, \infty)), \quad f(0) = 0, \quad f > 0 \quad \text{on } [0, \infty),
\]
and moreover,
\[
\begin{align*}
\left( \frac{f}{\chi} \right)' & > 0, \quad \left( \frac{f}{\chi} \right)'' \leq 0, \quad (\chi f)' \geq 0 \quad \text{on } [0, \infty) \tag{1.7}
\end{align*}
\]
hold; \(n_0, c_0, u_0, \Phi, g\) are known functions satisfying
\[
0 < n_0 \in X := \begin{cases} 
L^{m-1}(\Omega) & \text{if } m > 2, \\
L \log L(\Omega) & \text{if } m \leq 2,
\end{cases} \tag{1.8}
\]
\[
0 \leq c_0 \in L^\infty(\Omega) \quad \text{such that} \quad \sqrt{c_0} \in W^{1,2}(\Omega), \quad u_0 \in L^2_\sigma(\Omega), \tag{1.9}
\]
\[
\Phi \in C^{1+\beta}(\Omega), \quad g \in L^2_\text{loc}([0, \infty); L^6(\Omega)) \tag{1.10}
\]
for some \(\beta > 0\).

The main result reads as follows. The following theorem gives existence of global weak solutions to (1.3).

**Theorem 1.1.** Let \(\Omega \subset \mathbb{R}^3\) be a bounded smooth domain and let \(\kappa \in \mathbb{R}, \mu \geq 0, \alpha > 1\). Assume that \(D\) satisfies (1.4) with some \(\gamma > 0\), \(D_2 \geq D_1 > 0\) and \(m > 0\), and that \(\chi, f\) satisfy (1.5)–(1.7) as well as that \(n_0, c_0, u_0, \Phi, g\) satisfy (1.8)–(1.10) with some \(\beta \in (0, 1)\).

Then, if
\[
m > \frac{2}{3}, \quad \mu \geq 0, \quad \alpha > 1, \quad \text{or} \quad m > 0, \quad \mu > 0, \quad \alpha > \frac{4}{3} \tag{1.11}
\]
hold, there exists a global weak solution \((n, c, u)\) of (1.3) in the sense of Definition 5.1, which can be approximated by a sequence of solutions \((n_\varepsilon, c_\varepsilon, u_\varepsilon)\) of an approximate problem (see Section 2) in a pointwise manner. Moreover, if \(1 \leq m \leq 2\), then \(\nabla \left( \int_0^n D(\sigma) \, d\sigma \right) = D(n) \nabla n\) holds.

As an application of this result, we can construct existence result of global weak solutions to (1.3) with \(\kappa = \mu = 0\), which is a correction of the result by Zhang–Li [26].

**Corollary 1.2.** Let \(\Omega \subset \mathbb{R}^3\) be a bounded smooth domain and let \(\kappa = \mu = 0\). Assume that \(D\) satisfies (1.4) with some \(\gamma > 0\), \(D_2 \geq D_1 > 0\) and \(m > 0\), and that \(\chi, f\) satisfy (1.5)–(1.7) as well as that \(n_0, c_0, u_0, \Phi, g\) satisfy (1.8)–(1.10) with some \(\beta \in (0, 1)\). Then, if \(m > \frac{2}{3}\) holds, there exists a global weak solution \((n, c, u)\) of (1.3) in the sense of Definition 5.1. Moreover, if \(1 \leq m \leq 2\), then \(\nabla \left( \int_0^n D(\sigma) \, d\sigma \right) = D(n) \nabla n\) holds.

**Remark 1.1.** In this result we could verify existence of at least one global weak solution under the condition that \(m > \frac{2}{3}\). Here we could not include the case that \(m = \frac{2}{3}\) because of several reasons (see Remarks 2.2 and 4.2).

The strategy of the proof of Theorem 1.1 is to consider approximate problem (see (2.1)) and to show convergences via using arguments similar to those in [23] and [26]. In Section 2 we introduce an approximate problem and show several useful properties for an approximate solution \((n_\varepsilon, c_\varepsilon, u_\varepsilon)\) by using an energy function \(\int_\Omega n_\varepsilon \log n_\varepsilon + \frac{1}{2} \int_\Omega |\nabla \Psi(c_\varepsilon)|^2 + K \int_\Omega |u_\varepsilon|^2\) with some function \(\Psi\) and some constant \(K > 0\), which is used in [23] and [26];
one of keys for a treatment of an energy function is to derive some estimate for \( \|n_\varepsilon\|_{L^\infty(\Omega)}^2 \) which comes from a derivative of \( \int_\Omega |u_\varepsilon|^2 \) (see Lemma 2.4); in the case that \( m > \frac{2}{3} \), by virtue of the Gagliardo–Nirenberg inequality, we correct arguments in [26] and show that
\[
\|n_\varepsilon\|_{L^\infty(\Omega)}^2 \leq \eta \|\nabla(n_\varepsilon + \varepsilon)^{\frac{\mu}{2}}\|_{L^2(\Omega)}^2 + C(\eta)
\]
holds with some \( C(\eta) > 0 \) for all \( \eta > 0 \) (see Lemma 2.5); on the other hand, in the case that \( \mu > 0 \) and \( \alpha > \frac{4}{3} \), from an interpolation argument we have the new estimate:
\[
\|n_\varepsilon\|_{L^\infty(\Omega)}^2 \leq C \left( \mu \int_\Omega n_\varepsilon^\alpha + 1 \right)
\]
with some \( C > 0 \) (see Lemma 2.6); then we can establish some differential inequality of an energy function, and obtain several important estimates. In Section 3 we verify global existence in the approximate problem. Then, aided by estimates obtained in Section 2, we can see uniform-in-parameter estimates in Section 4. Finally, in Section 5, we obtain convergences and establish existence of global weak solutions in (1.3).

2. An energy time inequality

We start by considering the following approximate problem with parameter \( \varepsilon \in (0, 1) \):
\[
\begin{aligned}
(n_\varepsilon)_t + u_\varepsilon \cdot \nabla n_\varepsilon &= \nabla \cdot D_\varepsilon(n_\varepsilon) \nabla n_\varepsilon - \nabla \cdot \left( \frac{n_\varepsilon}{1 + n_\varepsilon} \nabla c_\varepsilon \right) + \kappa n_\varepsilon - \mu n_\varepsilon^\alpha - \varepsilon n_\varepsilon^2, \\
(c_\varepsilon)_t + u_\varepsilon \cdot \nabla c_\varepsilon &= \Delta c_\varepsilon - f(c_\varepsilon)^\frac{1}{2} \log (1 + \varepsilon n_\varepsilon), \\
(u_\varepsilon)_t + (Y_\varepsilon u_\varepsilon \cdot \nabla)u_\varepsilon &= \Delta u_\varepsilon + \nabla P_\varepsilon + n_\varepsilon \nabla \Phi + g_\varepsilon, \quad \nabla \cdot u_\varepsilon = 0, \\
\partial \nu n_\varepsilon|_{\partial \Omega} = \partial \nu c_\varepsilon|_{\partial \Omega} = 0, &\quad u_\varepsilon|_{\partial \Omega} = 0, \\
n_\varepsilon(\cdot, 0) = n_{0\varepsilon}, &\quad c_\varepsilon(\cdot, 0) = c_{0\varepsilon}, \\ &\quad u_\varepsilon(\cdot, 0) = u_{0\varepsilon},
\end{aligned}
\tag{2.1}
\]
where
\[
D_\varepsilon(s) := D(s + \varepsilon) \quad \text{for all } s \geq 0, \quad Y_\varepsilon = (1 + \varepsilon A)^{-1}
\]
and \( n_{0\varepsilon}, c_{0\varepsilon}, u_{0\varepsilon}, g_\varepsilon \) are functions satisfying
\[
\begin{aligned}
n_{0\varepsilon} &\in C^\infty_0(\Omega), \quad \int_\Omega n_{0\varepsilon} = \int_\Omega n_0 \text{ in } X \text{ as } \varepsilon \searrow 0, \tag{2.2} \\
c_{0\varepsilon} &\in C^\infty_0(\Omega), \quad \|c_{0\varepsilon}\|_{L^\infty(\Omega)} \leq \|c_0\|_{L^\infty(\Omega)}, \quad \sqrt{c_{0\varepsilon}} \to \sqrt{c_0} \text{ in } L^2(\Omega) \text{ as } \varepsilon \searrow 0, \tag{2.3} \\
u_{0\varepsilon} &\in C^\infty_{0, \sigma}(\Omega), \quad \|u_{0\varepsilon}\|_{L^2(\Omega)} = \|u_0\|_{L^2(\Omega)}, \quad u_{0\varepsilon} \to u_0 \text{ in } L^2(\Omega) \text{ as } \varepsilon \searrow 0, \tag{2.4} \\
g_\varepsilon &\in C^\infty_0(\Omega), \quad \|g_\varepsilon\|_{L^2(0,T;L^\frac{4}{3}(\Omega))} \leq \|g\|_{L^2(0,T;L^\frac{4}{3}(\Omega))} \text{ for all } T > 0, \quad g_\varepsilon \to g \text{ in } L^2_{\text{loc}}([0, \infty); L^\frac{4}{3}(\Omega)) \text{ as } \varepsilon \searrow 0, \tag{2.5}
\end{aligned}
\]
where \( X \) is the space defined in (1.8). The first step for the proof of Theorem 1.1 is to show global existence of solutions to the approximate problem (2.1). Now we recall the following result concerned with local existence in (2.1).
Lemma 2.1. Let $D$ satisfy (1.4) with some $\gamma > 0$, $D_2 \geq D_1 > 0$ and $m > 0$. Assume that $\kappa \in \mathbb{R}$, $\mu \geq 0$, $\alpha > 1$, $f \in C^1([0, \infty))$, $\chi \in C^2([0, \infty))$, $\Phi \in C^{1+\beta}(\Omega)$ for some $\beta \in (0,1)$ and that $n_{0\varepsilon}, c_{0\varepsilon}, u_{0\varepsilon}, g\varepsilon$ satisfy (2.2)–(2.5). Then for each $\varepsilon > 0$ there exist $T_{\text{max},\varepsilon} \in (0, \infty]$ and uniquely determined functions:

\[
\begin{align*}
n_{\varepsilon} &\in C^0(\Omega \times (0, T_{\text{max},\varepsilon})) \cap C^{2,1}(\Omega \times (0, T_{\text{max},\varepsilon})), \\
c_{\varepsilon} &\in C^0(\Omega \times (0, T_{\text{max},\varepsilon})) \cap C^{2,1}(\Omega \times (0, T_{\text{max},\varepsilon})) \cap L^\infty(\Omega \times (0, T_{\text{max},\varepsilon}); W^{1,\infty}(\Omega)), \\
u_{\varepsilon} &\in C^0(\Omega \times (0, T_{\text{max},\varepsilon})) \cap C^{2,1}(\Omega \times (0, T_{\text{max},\varepsilon})),
\end{align*}
\]

which together with some $P_{\varepsilon} \in C^{1,0}(\Omega \times (0, T_{\text{max},\varepsilon}))$ solve (2.1) classically. Moreover, $n_{\varepsilon}$ and $c_{\varepsilon}$ are positive and the following alternative holds: $T_{\text{max},\varepsilon} = \infty$ or

\[
\|n_{\varepsilon}(\cdot, t)\|_{L^\infty(\Omega)} + \|c_{\varepsilon}(\cdot, t)\|_{W^{1,q}(\Omega)} + \|A^\theta u_{\varepsilon}(\cdot, t)\|_{L^2(\Omega)} \to \infty
\]

as $t \not\rightarrow T_{\text{max},\varepsilon}$ for all $q > 3$ and all $\theta \in (\frac{3}{4}, 1)$.

Proof. Combination of arguments in [16, Lemma 2.1] and [21, Lemma 2.1], which is based on a standard fixed point argument with a parabolic regularity theory, entails this lemma.

In the following for all $\varepsilon \in (0, 1)$ we denote by $(n_{\varepsilon}, c_{\varepsilon}, u_{\varepsilon})$ the corresponding solution of (2.1) given by Lemma 2.1 and by $T_{\text{max},\varepsilon}$ its maximal existence time. Then we shall see that $T_{\text{max},\varepsilon} = \infty$ for all $\varepsilon \in (0, 1)$ and useful estimates for the approximate solution. We first provide the following lemma which is obtained from the first and second equations in (2.1).

Lemma 2.2. For all $\varepsilon \in (0, 1)$,

\[
\int_\Omega n_{\varepsilon}(\cdot, t) \leq e^{\kappa t} \int_\Omega n_0 \quad \text{for all } t \in (0, T_{\text{max},\varepsilon})
\]

and

\[
\mu \int_0^t \int_\Omega n_{\varepsilon}^\alpha + \varepsilon \int_0^t \int_\Omega n_{\varepsilon}^2 \leq e^{\kappa t} \int_\Omega n_0 + \int_\Omega n_0 \quad \text{for all } t \in (0, T_{\text{max},\varepsilon})
\]

as well as

\[
\|c_{\varepsilon}(\cdot, t)\|_{L^\infty(\Omega)} \leq \|c_0\|_{L^\infty(\Omega)} \quad \text{for all } t \in (0, T_{\text{max},\varepsilon})
\]

hold.

Proof. This result is obtained from integrating the first equation in (1.3) on $(0, t)$ for all $t \in (0, T_{\text{max},\varepsilon})$, and from applying the maximum principle to the second equation.

Remark 2.1. In order to deal with the case that $\kappa > 0$ and $\mu = 0$, estimates for $n_{\varepsilon}$ in this lemma are local-in-time estimates which are not often used in the study of the chemotaxis system. In the case that $\kappa = \mu = 0$ or $\mu > 0$, from the well-known arguments we can establish a uniform-in-time estimate for $\int_\Omega n_{\varepsilon}$.
We then establish estimates for the approximate solution, which are useful not only to see $T_{\max,\varepsilon} = \infty$ for each $\varepsilon \in (0, 1)$ but also to obtain uniform-in-$\varepsilon$ estimates, by using an energy function defined as

$$\int_{\Omega} n_\varepsilon \log n_\varepsilon + \frac{1}{2} \int_{\Omega} |\nabla \Psi(c_\varepsilon)|^2 + K \int_{\Omega} |u_\varepsilon|^2$$

with some function $\Psi$ and some constant $K > 0$, which is the function same as that used in the previous works [23] and [26]. We first give some estimate for derivatives of the first and second summands in the energy function.

**Lemma 2.3.** There exists $K > 0$ such that for any $\varepsilon \in (0, 1)$,

$$\frac{d}{dt} \left( \int_{\Omega} n_\varepsilon \log n_\varepsilon + \frac{1}{2} \int_{\Omega} |\nabla \Psi(c_\varepsilon)|^2 \right) + \frac{1}{K} \left( \int_{\Omega} \frac{D_\varepsilon(n_\varepsilon)}{n_\varepsilon} |\nabla n_\varepsilon|^2 + \int_{\Omega} \frac{|D^2 c_\varepsilon|^2}{c_\varepsilon} + \int_{\Omega} \frac{|\nabla c_\varepsilon|^4}{c_\varepsilon^3} \right)$$

$$+ \int_{\Omega} \left( \frac{\mu}{2} n_\varepsilon^\alpha + \varepsilon n_\varepsilon^2 - \kappa \delta_{\mu,0} n_\varepsilon \right) \log n_\varepsilon \leq K \left( \int_{\Omega} |\nabla u_\varepsilon|^2 + 1 \right)$$

holds on $(0, T_{\max,\varepsilon})$, where $\Psi(s) := \int_1^s \frac{ds}{\sqrt{h(s)}}$ with $h(s) := \frac{f(s)}{\lambda(s)}$, and $\delta_{\mu,0} = 1$ when $\mu = 0$ and $\delta_{\mu,0} = 0$ when $\mu > 0$.

**Proof.** The proof of this lemma is similar to those of [26, Lemma 3.1] and [10, Lemmas 2.6 and 2.8]. Aided by arguments in the proof of [23, Lemma 3.1] and noting that $(\kappa s - \frac{\mu}{2} s^\alpha) \log s \leq \kappa \delta_{\mu,0} s \log s + C$ for all $s > 0$ with some $C > 0$, where $\delta_{\mu,0}$ is the constant defined in the statement of this lemma, from straightforward calculations of $\frac{d}{dt} \int_{\Omega} n_\varepsilon \log n_\varepsilon$ and $\frac{d}{dt} \int_{\Omega} |\nabla \Psi(c_\varepsilon)|^2$ we can verify this lemma. \qed

We next calculate a derivative of the third summand $\int_{\Omega} |u_\varepsilon|^2$ in the energy function.

**Lemma 2.4.** There is a constant $C > 0$ such that for all $\varepsilon \in (0, 1)$,

$$\frac{1}{2} \frac{d}{dt} \int_{\Omega} |u_\varepsilon|^2 + \int_{\Omega} |\nabla u_\varepsilon|^2 \leq C \left( \|n_\varepsilon\|^2_{L^4(\Omega)} + \|g_\varepsilon\|^2_{L^\infty(\Omega)} \right) + \frac{1}{4} \|\nabla u_\varepsilon\|^2_{L^2(\Omega)}$$

(2.6)

holds on $(0, T_{\max,\varepsilon})$.

**Proof.** Testing the third equation of (2.1) by $u_\varepsilon$, we obtain from the Hölder inequality, the continuous embedding $W^{1,2}(\Omega) \hookrightarrow L^6(\Omega)$ and the Young inequality that

$$\frac{1}{2} \frac{d}{dt} \int_{\Omega} |u_\varepsilon|^2 + \int_{\Omega} |\nabla u_\varepsilon|^2 = \int_{\Omega} \left( n_\varepsilon \nabla \Phi + g_\varepsilon \right) \cdot u_\varepsilon$$

$$\leq \left( \|\nabla \Phi\|_{L^\infty(\Omega)} \|n_\varepsilon\|_{L^4(\Omega)} + \|g_\varepsilon\|_{L^\infty(\Omega)} \right) \|u_\varepsilon\|_{L^6(\Omega)}$$

$$\leq C_1 \left( \|n_\varepsilon\|_{L^4(\Omega)} + \|g_\varepsilon\|_{L^\infty(\Omega)} \right) \|\nabla u_\varepsilon\|_{L^2(\Omega)}$$

$$\leq C_2 \left( \|n_\varepsilon\|^2_{L^4(\Omega)} + \|g_\varepsilon\|^2_{L^\infty(\Omega)} \right) + \frac{1}{4} \|\nabla u_\varepsilon\|^2_{L^2(\Omega)}$$

holds on $(0, T_{\max,\varepsilon})$ with some $C_1, C_2 > 0$. \qed
In order to derive some differential inequality of the energy function we have to deal with \( \| n_\varepsilon \|_{L^{\infty}(\Omega)} \) in (2.6). Now we divide arguments into the cases that \( m > \frac{2}{3}, \mu \geq 0, \alpha > 1 \) hold, and that \( m > 0, \mu > 0, \alpha > \frac{1}{3} \) hold. We first deal with the case that \( m > \frac{2}{3}, \mu \geq 0, \alpha > 1 \) hold. In this case we use the diffusion effect to control \( \| n_\varepsilon \|_{L^{\infty}(\Omega)} \). The proof of the following lemma is based on that of [26, Lemma 3.2]. However, in order to see the following lemma, we need the restriction of \( m > \frac{2}{3} \) instead of \( m \geq \frac{2}{3} \) which is assumed in [26] (see Remark 2.2).

**Lemma 2.5.** Assume that \( m > \frac{2}{3}, \mu \geq 0, \alpha > 1 \). Then for all \( T > 0 \) and all \( \eta > 0 \) there is \( C(T, \eta) > 0 \) such that

\[
\| n_\varepsilon \|_{L^{\infty}(\Omega)}^2 \leq \eta \| \nabla (n_\varepsilon + \varepsilon) \|_{L^2(\Omega)}^2 + C(T, \eta)
\]

for all \( t \in (0, \bar{T}) \) and all \( \varepsilon \in (0, 1) \), where \( \bar{T} := \min \{T, T_{\text{max}, \varepsilon}\} \).

**Proof.** The proof is similar to that of [26, Lemma 3.2]. Let \( T > 0 \) and put \( \bar{T} := \{T, T_{\text{max}, \varepsilon}\} \). Noting from Lemma 2.2 that

\[
\| (n_\varepsilon + \varepsilon) \|_{L^{\frac{3m}{2m-1}}(\Omega)} \leq \| n_\varepsilon + \varepsilon \|_{L^{\frac{3m}{2m-1}}(\Omega)}^2 \leq C(1) \]

for all \( t \in (0, \bar{T}) \) and all \( \varepsilon \in (0, 1) \) with some \( C_1(T) > 0 \), we use the Gagliardo–Nirenberg inequality to see that

\[
\| n_\varepsilon \|_{L^{\infty}(\Omega)}^2 \leq \| n_\varepsilon + \varepsilon \|_{L^{\infty}(\Omega)}^2
\]

\[
= \| (n_\varepsilon + \varepsilon) \|_{L^{\frac{3m}{2m-1}}(\Omega)}^2
\]

\[
\leq C_2 \| \nabla (n_\varepsilon + \varepsilon) \|_{L^2(\Omega)}^2 \| (n_\varepsilon + \varepsilon) \|_{L^{\frac{3m}{2m-1}}(\Omega)}^2
\]

\[
\leq C_3(T) \left( \| \nabla (n_\varepsilon + \varepsilon) \|_{L^2(\Omega)}^2 + 1 \right)
\]

for all \( t \in (0, \bar{T}) \) and all \( \varepsilon \in (0, 1) \) with some \( C_2 > 0 \) and some \( C_3(T) > 0 \). Now, since the condition \( m > \frac{2}{3} \) implies that \( \frac{2}{3m-1} < 2 \), we establish from the Young inequality that

\[
\| n_\varepsilon \|_{L^{\infty}(\Omega)}^2 \leq \eta \| \nabla (n_\varepsilon + \varepsilon) \|_{L^2(\Omega)}^2 + C_4(T, \eta)
\]

for all \( t \in (0, \bar{T}) \) and all \( \varepsilon \in (0, 1) \) with some \( C_4(T, \eta) > 0 \) for all \( \eta > 0 \). \( \square \)

**Remark 2.2.** In [26, (3.6)], they used the Young inequality as

\[
\left( \int_{\Omega} (n_\varepsilon + \varepsilon)^{m-2} |\nabla n_\varepsilon|^2 \right)^{\frac{1}{m-1}} \leq \eta \int_{\Omega} (n_\varepsilon + \varepsilon)^{m-2} |\nabla n_\varepsilon|^2 + C(\eta, m)
\]

with some \( C(\eta, m) > 0 \) for all \( \eta > 0 \) and for \( m \geq \frac{2}{3} \) even though this inequality does not hold when \( m = \frac{2}{3} \). Indeed, since \( \frac{1}{3m-1} = 1 \) holds when \( m = \frac{2}{3} \), we could not apply the Young inequality to \( \int_{\Omega} (n_\varepsilon + \varepsilon)^{m-2} |\nabla n_\varepsilon|^2 \). Thus we need to assume that \( m > \frac{2}{3} \) when we use this method.
Lemma 2.6. Assume that $m > 0$, $\mu > 0$, $\alpha \geq \frac{4}{3}$ hold. Then for all $T > 0$ there is $C(T) > 0$ such that
\[
\|n_\varepsilon\|_{L_1^\alpha(\Omega)}^2 \leq C(T) \left( \mu \int_\Omega n_\varepsilon^\alpha + 1 \right)
\]
holds for all $t \in (0, \tilde{T})$ and all $\varepsilon \in (0, 1)$, where $\tilde{T} := \min\{T, T_{\text{max}, \varepsilon}\}$.

Proof. Let $T > 0$ and put $\tilde{T} := \min\{T, T_{\text{max}, \varepsilon}\}$. We use an interpolation inequality and the Young inequality to obtain that
\[
\|n_\varepsilon\|_{L_1^\alpha(\Omega)}^2 \leq \|n_\varepsilon\|_{L_1^\alpha(\Omega)}^{\frac{2\alpha}{\alpha-1}} \|n_\varepsilon\|_{L_1^\alpha(\Omega)}^{\frac{\alpha-6}{6(\alpha-1)}} \leq C_1(T) \left( \mu \|n_\varepsilon\|_{L_1^\alpha(\Omega)}^\alpha + 1 \right)
\]
on $(0, \tilde{T})$ with some $C_1(T) > 0$, which with Lemma 2.2 and the fact $\frac{\alpha}{6} < \alpha$ (from $\alpha > \frac{4}{3}$) implies this lemma.

The lemmas obtained in this section yield the following estimate for a derivative of the energy function.

Lemma 2.7. Let $\Psi$ and $K > 0$ be given in Lemma 2.3 and assume that (1.11) holds. Then for all $T > 0$ there is $C(T) > 0$ such that for any $\varepsilon \in (0, 1)$,
\[
\frac{d}{dt} \left( \int_\Omega n_\varepsilon \log n_\varepsilon + \frac{1}{2} \int_\Omega |\nabla \Psi(c_\varepsilon)|^2 + K \int_\Omega |u_\varepsilon|^2 \right) + \int_\Omega \left( \frac{\mu n_\varepsilon^\alpha + \varepsilon n_\varepsilon^2}{2} \right) \log n_\varepsilon
\]
\[
+ \frac{1}{2K} \left( \int_\Omega D_\varepsilon(n_\varepsilon) |\nabla n_\varepsilon|^2 + \int_\Omega \frac{|D^2 c_\varepsilon|^2}{c_\varepsilon} + \int_\Omega \frac{|\nabla c_\varepsilon|^4}{c_\varepsilon^3} + \int_\Omega |\nabla u_\varepsilon|^2 \right)
\]
\[
\leq C(T) \left( 1 + \mu \int_\Omega n_\varepsilon^\alpha + \|g_\varepsilon(\cdot, t)\|_{L_1^\alpha(\Omega)}^2 \right) + \kappa \delta_{\mu, 0} \int_\Omega n_\varepsilon \log n_\varepsilon
\]
holds on $(0, \tilde{T})$ with $\tilde{T} := \min\{T, T_{\text{max}, \varepsilon}\}$.

Proof. Let $T > 0$ and put $\tilde{T} := \min\{T, T_{\text{max}, \varepsilon}\}$. Aided by Lemmas 2.4, 2.5 and 2.6, we can obtain that
\[
\frac{1}{2} \frac{d}{dt} \int_\Omega |u_\varepsilon|^2 + \frac{3}{4} \int_\Omega |\nabla u_\varepsilon|^2 \leq \frac{D_1}{4K^2} \int_\Omega (n_\varepsilon + \varepsilon)^{m-2} |\nabla n_\varepsilon|^2
\]
\[
+ C_1(T) \left( \|g_\varepsilon\|_{L_1^\alpha(\Omega)}^2 + \mu \int_\Omega n_\varepsilon^\alpha + 1 \right)
\]
for all $t \in (0, \tilde{T})$ and for all $\varepsilon \in (0, 1)$ with some $C_1(T) > 0$. Thus a combination of Lemma 2.3 and (2.7) derives this lemma.

In the end of this section we provide the following uniform-in-$\varepsilon$ estimates for the approximate solution which will be used later.
Lemma 2.8. Let \( \Psi \) be given in Lemma 2.3 and assume that (1.11) holds. Then for all \( T > 0 \) there exists \( C(T) > 0 \) such that

\[
\int_{\Omega} n_{\varepsilon} \log n_{\varepsilon} + \int_{\Omega} |\nabla \Psi(c_{\varepsilon})|^2 + \int_{\Omega} |u_{\varepsilon}|^2 \leq C(T) \quad \text{for all} \quad t \in (0, \bar{T})
\]

and

\[
\int_{0}^{\bar{T}} \int_{\Omega} \frac{D_{\varepsilon}(n_{\varepsilon})}{n_{\varepsilon}} |\nabla n_{\varepsilon}|^2 + \int_{0}^{\bar{T}} \int_{\Omega} (\mu n_{\varepsilon}^0 + \varepsilon n_{\varepsilon}^2) \log n_{\varepsilon} \leq C(T)
\]

as well as

\[
\int_{0}^{\bar{T}} \int_{\Omega} |D^2 c_{\varepsilon}|^2 + \int_{0}^{\bar{T}} \int_{\Omega} |\nabla c_{\varepsilon}|^4 + \int_{0}^{\bar{T}} \int_{\Omega} |\nabla u_{\varepsilon}|^2 \leq C(T)
\]

hold for all \( \varepsilon \in (0, 1) \) with \( \bar{T} := \min \{ T, T_{\text{max}, \varepsilon} \} \).

Proof. Let \( T > 0 \) and put \( \bar{T} := \min \{ T, T_{\text{max}, \varepsilon} \} \), and let \( K \) be given in Lemma 2.3. Putting

\[
y_{\varepsilon}(t) := \int_{\Omega} n_{\varepsilon}(\cdot, t) \log n_{\varepsilon}(\cdot, t) + \frac{1}{2} \int_{\Omega} |\nabla \Psi(c_{\varepsilon}(\cdot, t))|^2 + K \int_{\Omega} |u_{\varepsilon}(\cdot, t)|^2
\]

and

\[
z_{\varepsilon}(t) := \int_{\Omega} \frac{D_{\varepsilon}(n_{\varepsilon}(\cdot, t))}{n_{\varepsilon}(\cdot, t)} |\nabla n_{\varepsilon}(\cdot, t)|^2 + \int_{\Omega} |D^2 c_{\varepsilon}(\cdot, t)|^2 + \int_{\Omega} |\nabla c_{\varepsilon}(\cdot, t)|^4 + \int_{\Omega} |\nabla u_{\varepsilon}(\cdot, t)|^2 + 2K \int_{\Omega} \left( \frac{\mu}{2} n_{\varepsilon}^0(\cdot, t) + \varepsilon n_{\varepsilon}^2(\cdot, t) \right) \log n_{\varepsilon}(\cdot, t)
\]

for \( t \in (0, \bar{T}) \), we obtain from Lemma 2.7 that

\[
y_{\varepsilon}'(t) + \frac{1}{2K} z_{\varepsilon}(t) - \kappa \delta_{\mu,0} \int_{\Omega} n_{\varepsilon}(\cdot, t) \log n_{\varepsilon}(\cdot, t) \leq C_1(T) \left[ 1 + \mu \int_{\Omega} n_{\varepsilon}^0(\cdot, t) + \|g_{\varepsilon}(\cdot, t)\|_{L_2(\Omega)} \right] \quad \text{for all} \quad t \in (0, \bar{T})
\]

with some \( C_1(T) > 0 \). Then, in order to derive a differential inequality of \( y_{\varepsilon} \), we shall show that

\[
C y_{\varepsilon}(t) + \kappa \delta_{\mu,0} \int_{\Omega} n_{\varepsilon}(\cdot, t) \log n_{\varepsilon}(\cdot, t) \leq \frac{1}{2K} z_{\varepsilon}(t) + \bar{C} \quad \text{for all} \quad t \in (0, \bar{T})
\]

with some \( C, \bar{C} > 0 \). Now, in the case that \( m > \frac{2}{3} \), the inequality

\[
s \log s \leq \frac{3}{3m - 2} s^{m+\frac{1}{3}} \quad \text{for all} \quad s > 0
\]
and the Gagliardo–Nirenberg inequality entail that

\[
(\kappa \delta_{\mu,0} + 1) \int_{\Omega} n_\varepsilon \log n_\varepsilon \\
\leq \frac{3(\kappa \delta_{\mu,0} + 1)}{3m-2} \int_{\Omega} (n_\varepsilon + \varepsilon)^{m+\frac{1}{3}} \\
\leq C_2 \left( \| \nabla (n_\varepsilon + \varepsilon)^{\frac{m}{2}} \|_{L^2(\Omega)}^{\frac{2(3m-2)}{3m}} \| (n_\varepsilon + \varepsilon)^{\frac{m}{2}} \|_{L^\infty(\Omega)}^{\frac{2(6m-1)}{3m}} + \| (n_\varepsilon + \varepsilon)^{\frac{m}{2}} \|_{L^\infty(\Omega)}^{\frac{2(3m+1)}{3m}} \right) \\
\leq \frac{D_1}{2K} \int_{\Omega} (n_\varepsilon + \varepsilon)^{m-2} |\nabla n_\varepsilon|^2 + C_3(T) \\
\leq \frac{1}{2K} \int_{\Omega} \frac{D_\varepsilon(n_\varepsilon)}{n_\varepsilon} |\nabla n_\varepsilon|^2 + C_3(T)
\]

for all \( \varepsilon \in (0, 1) \) with some \( C_2, C_3(T) > 0 \). On the other hand, in the case that \( \mu > 0 \) (which means that \( \delta_{\mu,0} = 0 \) holds), the inequality

\[
s \log s \leq s^\alpha \log s \quad \text{for all } s > 0
\]

enables us to see that

\[
\int_{\Omega} n_\varepsilon \log n_\varepsilon \leq \int_{\Omega} n_\varepsilon^\alpha \log n_\varepsilon.
\]

Moreover, by putting \( M := \min \{ h'(s) \mid s \in [0, \|c_0\|_{L^\infty(\Omega)}] \} > 0 \) and using the inequality

\[
h(s) \geq Ms \quad \text{for all } s \in [0, \|c_0\|_{L^\infty(\Omega)}]
\]

(from the facts that \( h \in C^1([0, \infty)), h' > 0 \) on \([0, \|c_0\|_{L^\infty(\Omega)}]\) and \( h(0) = 0 \)), we can see that

\[
\frac{1}{2} \int_{\Omega} |\nabla \Psi(c_\varepsilon)|^2 = \frac{1}{2} \int_{\Omega} \frac{|\nabla c_\varepsilon|^2}{h(c_\varepsilon)} \\
\leq \frac{1}{4} \int_{\Omega} \frac{|\nabla c_\varepsilon|^4}{c_\varepsilon^3} + \frac{1}{4} \int_{\Omega} \frac{c_\varepsilon^3}{h^2(c_\varepsilon)} \\
\leq \frac{1}{4} \int_{\Omega} \frac{|\nabla c_\varepsilon|^4}{c_\varepsilon^3} + \frac{\|c_0\|_{L^\infty(\Omega)}^2|\Omega|}{4M^2}
\]

holds for all \( t \in (0, \tilde{T}) \). Therefore the Poincaré inequality

\[
K \int_{\Omega} |u_\varepsilon|^2 \leq C_4 \int_{\Omega} |\nabla u_\varepsilon|^2
\]

with some \( C_4 > 0 \) and the inequality \( -\frac{1}{\varepsilon} \leq \varepsilon s \log s \leq \varepsilon s^2 \log s \) for all \( s > 0 \) and all \( \varepsilon \in (0, 1) \) yield that

\[
C_5(T)y_\varepsilon(t) + \kappa \delta_{\mu,0} \int_{\Omega} n_\varepsilon(\cdot, t) \log n_\varepsilon(\cdot, t) \leq \frac{1}{2K} z_\varepsilon(t) + C_6(T) \quad \text{for all } t \in (0, \tilde{T}) \quad (2.10)
\]
with some $C_5(T), C_6(T) > 0$. Since (2.9) and (2.10) derive that
\[
y'_\varepsilon(t) + C_7(T)y_\varepsilon(t) \leq C_8(T) \left( 1 + \mu \int_{\Omega} n_\varepsilon(\cdot, t)^{\alpha} + \|g_\varepsilon(\cdot, t)\|_{L^6(\Omega)}^2 \right)
\]
for all $t \in (0, \tilde{T})$ with some $C_7(T), C_8(T) > 0$, the existence of $C_9(T) > 0$ satisfying
\[
\int_0^{\tilde{T}} \int_{\Omega} n_\varepsilon + \int_0^{\tilde{T}} \|g_\varepsilon\|_{L^6(\Omega)}^2 \leq C_9(T),
\]
which is obtained from (1.10) and (2.5) as well as Lemma 2.2, means that this lemma holds.

3. Global existence for the regularized problem (2.1)

In this section we show global existence in the approximate problem by using the estimates obtained in Lemma 2.8.

Lemma 3.1. Assume that (1.11) holds. Then for all $\varepsilon \in (0, 1)$, $T_{max, \varepsilon} = \infty$ holds.

Proof. Arguments similar to those in the proof of [8, Lemma 2.9] enable us to see this lemma; thus we only write a short proof. Assume that $T_{max, \varepsilon} < \infty$. We can obtain from Lemma 2.8 with $T = T_{max, \varepsilon}$ that
\[
\int_0^{T_{max, \varepsilon}} \int_{\Omega} |\nabla c_\varepsilon|^4 \leq \|c_0\|_{L^\infty(\Omega)}^3 \int_0^{T_{max, \varepsilon}} \int_{\Omega} |\nabla c_\varepsilon|^4 \leq C_2
\]
with some $C_1, C_2 > 0$. Now we let $p := \min\{3 + m, 4\} > 3$. Then, considering $\frac{d}{dt} \int_{\Omega} (n_\varepsilon + \varepsilon)^s$ together with the inequality $\frac{s}{1 + \varepsilon} \leq \frac{1}{\varepsilon}$ for all $s > 0$, we can verify an $L^p(\Omega \times (0, T_{max, \varepsilon}))$-estimate for $n_\varepsilon$. Then, through boundedness of $\sup_{t \in (0, T_{max, \varepsilon})} \|u_\varepsilon(\cdot, t)\|_{D(A^\theta)}$ with some $\theta \in (\frac{3}{4}, 1)$ and $\sup_{t \in (0, T_{max, \varepsilon})} \|\nabla c_\varepsilon(\cdot, t)\|_{L^6(\Omega)}$, a Moser–Alikakos-type procedure (see the proof of [17, Lemma A.1]) enables us to have an $L^\infty(\Omega \times (0, \infty))$-estimate for $n_\varepsilon$, which with the extensibility criterion implies that $T_{max, \varepsilon} = \infty$ for all $\varepsilon \in (0, 1)$.

4. Further $\varepsilon$-independent estimates for (2.1)

In this section we derive uniform-in-$\varepsilon$ estimates for the approximate solution which will be used in Section 5. We first give the following estimates.

Lemma 4.1. Assume that (1.11) holds. Then for all $T > 0$ there exists $C(T) > 0$ such that
\[
\int_0^{T} \int_{\Omega} |\nabla (n_\varepsilon + \varepsilon)^{\frac{4}{3}}|^2 + \int_0^{T} \int_{\Omega} |\nabla c_\varepsilon|^4 + \int_0^{T} \int_{\Omega} |u_\varepsilon|^4 \leq C(T)
\]
holds for all $\varepsilon \in (0, 1)$. 

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Proof. The proof is based on arguments in the proof of [26, Lemma 5.1]. Thus we only write a short proof. Let \( T > 0 \). Due to (2.8), we can find \( C_1(T) > 0 \) such that

\[
\int_0^T \int_\Omega |\nabla (n_{\varepsilon} + \varepsilon) \frac{\varepsilon}{m} |^2 = \frac{m^2}{4} \int_0^T \int_\Omega (n_{\varepsilon} + \varepsilon)^{m-2} |\nabla n_{\varepsilon}|^2 \\
\leq \frac{m^2}{4D_1} \int_0^T \int_\Omega \frac{D_\varepsilon(n_{\varepsilon})}{n_{\varepsilon}} |\nabla n_{\varepsilon}|^2 \\
\leq C_1(T)
\]

for all \( \varepsilon \in (0,1) \). On the other hand, in light of Lemma 2.2 and the Gagliardo–Nirenberg inequality, we infer from Lemma 2.8 that

\[
\int_0^T \int_\Omega |\nabla c_{\varepsilon}|^4 \leq \|c_0\|^3_{L^\infty(\Omega)} \int_0^T \int_\Omega |\nabla c_{\varepsilon}|^4 c_{\varepsilon}^3 \\
\leq C_2(T)
\]

and

\[
\int_0^T \int_\Omega |u_{\varepsilon}|^{10} \leq C_3 \int_0^T \left( \|\nabla u_{\varepsilon}\|_{L^2(\Omega)}^2 \|u_{\varepsilon}\|_{L^2(\Omega)}^{\frac{4}{3}} + \|u_{\varepsilon}\|_{L^2(\Omega)}^{10} \right) \\
\leq C_4(T)
\]

for all \( \varepsilon \in (0,1) \) with some \( C_2(T), C_3, C_4(T) > 0 \).

In the following, we only consider the case that

\[
m > \frac{2}{3}, \quad \mu \geq 0, \quad \alpha > 1
\]

(4.2) hold, or that

\[
m \in \left( 0, \frac{2}{3} \right], \quad \mu > 0, \quad \alpha > \frac{4}{3}
\]

(4.3) hold. Here we note that, since

\[
\left\{ (m, \mu, \alpha) \left| \frac{2}{3}, \mu > 0, \alpha > \frac{4}{3} \right\} \subset \left\{ (m, \mu, \alpha) \left| m > \frac{2}{3}, \mu \geq 0, \alpha > 1 \right\}
\]

holds, it is enough to consider the case that (4.2) or (4.3) holds when (1.11) holds.

4.1. Key estimates. Case 1: \( m > \frac{2}{3}, \mu \geq 0, \alpha > 1 \)

In this subsection we establish estimates for \( n_\varepsilon \) in the case that (4.2) holds. In this case by using the diffusion effect we can obtain the following estimates.
Lemma 4.2. Assume that (4.2) holds. Then for all $T > 0$ there exists $C(T) > 0$ such that
\[
\int_0^T \int_{\Omega} (n_\varepsilon + \varepsilon)^{3m+2 \over 6m+2} \leq C(T),
\]
and moreover, if ${2 \over 3} < m \leq 2$, then
\[
\int_0^T \int_{\Omega} \frac{3m+2}{3m+1} m \leq C(T)
\]
hold for all $\varepsilon \in (0, 1)$.

Proof. The proof is similar to that of [26, Lemma 5.1]. Thus again we only write a short proof. Let $T > 0$. The estimate (4.1) yields from the Gagliardo–Nirenberg inequality that
\[
\int_0^T \int_{\Omega} (n_\varepsilon + \varepsilon)^{3m+2 \over 6m+2} \leq C_1 \left( \int_0^T \int_{\Omega} \left( \| \nabla (n_\varepsilon + \varepsilon) \|^2_{L^2(\Omega)} \right) \right)^{3m+2 \over 6m+2} \leq C_2(T)
\]
for all $\varepsilon \in (0, 1)$ with some $C_1 > 0$ and $C_2(T) > 0$. Then, we use the Hölder inequality and (2.8) to confirm that
\[
\int_0^T \int_{\Omega} \left| D_\varepsilon (n_\varepsilon) \nabla n_\varepsilon \right|^{3m+2 \over 3m+1} \leq C_3 \left( \int_0^T \int_{\Omega} \frac{D_\varepsilon (n_\varepsilon)}{n_\varepsilon} \left| \nabla n_\varepsilon \right|^2 \right)^{3m+2 \over 6m+2} \left( \int_0^T \int_{\Omega} (n_\varepsilon + \varepsilon)^{3m+2 \over 3m+1} \right)^{3m+2 \over 6m+2}
\]
for all $\varepsilon \in (0, 1)$ with $C_3 := D_2^{3m+2 \over 6m+2} > 0$ and some $C_4(T) > 0$, which implies (4.4) holds. Moreover, if ${2 \over 3} < m \leq 2$, then a combination of the Young inequality and (4.1), along with (4.4) leads to existence of the constant $C_5(T) > 0$ such that for all $\varepsilon \in (0, 1)$,
\[
\int_0^T \int_{\Omega} \left| \nabla n_\varepsilon \right|^{3m+2 \over 4} \leq \int_0^T \int_{\Omega} (n_\varepsilon + \varepsilon)^{m-2 \over 2} \left| \nabla n_\varepsilon \right|^2 + \int_0^T \int_{\Omega} (n_\varepsilon + \varepsilon)^{3m+2 \over 3} \leq C_5(T),
\]
which completes the proof.

Remark 4.1. In this lemma we establish the boundedness of $\int_0^T \int_{\Omega} \left| \nabla (n_\varepsilon + \varepsilon)^{m \over 2} \right|^2$ instead of $\int_0^T \int_{\Omega} \left| \nabla n_\varepsilon^{m \over 2} \right|^2$ which was shown in [26, Lemma 5.1], because the inequality
\[
\int_0^T \int_{\Omega} n_\varepsilon^{m-2 \over 2} \left| \nabla n_\varepsilon \right|^2 \leq \int_0^T \int_{\Omega} (n_\varepsilon + \varepsilon)^{m-2 \over 2} \left| \nabla n_\varepsilon \right|^2,
\]
which was utilized to show boundedness of $\int_0^T \int_{\Omega} \left| \nabla n_\varepsilon^{m \over 2} \right|^2$ in the proof of [26, Lemma 5.1], seems to hold only when $m \geq 2$. 


4.2. Key estimates. Case 2: $m \in (0, \frac{2}{3}]$, $\mu > 0$, $\alpha > \frac{4}{3}$

We next deal with the case that (4.3) holds. In this case we can obtain important estimates for $n_\varepsilon$ from the logistic-type damping.

**Lemma 4.3.** Assume that (4.3) holds. Then for all $T > 0$ there exists $C(T) > 0$ such that

\[
\int_0^T \int_\Omega (n_\varepsilon + \varepsilon)^{\alpha} + \int_0^T \int_\Omega |D_\varepsilon(n_\varepsilon)\nabla n_\varepsilon|^{\frac{2\alpha}{\alpha + m}} + \int_0^T \int_\Omega |\nabla n_\varepsilon|^{\frac{2\alpha}{2 + \alpha - m}} \leq C(T)
\]

holds for all $\varepsilon \in (0, 1)$.

**Proof.** Let $T > 0$. Noticing that $\alpha > \frac{4}{3}(> 1)$ and $\varepsilon \in (0, 1)$, from Lemma 2.2 we can find $C_1(T) > 0$ such that

\[
\int_0^T \int_\Omega (n_\varepsilon + \varepsilon)^{\alpha} \leq 2^{\alpha - 1}\int_0^T \int_\Omega (n_\varepsilon^{\alpha} + 1) \leq C_1(T)
\]  

(4.5)

for all $\varepsilon \in (0, 1)$. Then the H"older inequality and (4.1) enable us to have that

\[
\int_0^T \int_\Omega |D_\varepsilon(n_\varepsilon)\nabla n_\varepsilon|^{\frac{2\alpha}{\alpha + m}} \leq \left( \int_0^T \int_\Omega \frac{D_\varepsilon(n_\varepsilon)|\nabla n_\varepsilon|^2}{n_\varepsilon^{\alpha}} \right)^{\frac{\alpha}{\alpha + \alpha}} \left( \int_0^T \int_\Omega \frac{D_\varepsilon^{\alpha}(n_\varepsilon) \cdot n_\varepsilon^{\alpha}}{n_\varepsilon^{\alpha + \alpha}} \right)^{\frac{\alpha + \alpha}{\alpha + m}} 
\]

\[
\leq C_2 \left( \int_0^T \int_\Omega \frac{D_\varepsilon(n_\varepsilon)|\nabla n_\varepsilon|^2}{n_\varepsilon^{\alpha}} \right)^{\frac{\alpha}{\alpha + \alpha}} \left( \int_0^T \int_\Omega (n_\varepsilon + \varepsilon)^{\alpha} \right)^{\frac{\alpha + \alpha}{\alpha + m}} 
\]

\[
\leq C_3(T)
\]

for all $\varepsilon \in (0, 1)$ with $C_2 := D_2^{\\frac{\alpha}{\alpha + m}} > 0$ and some $C_3(T) > 0$. Moreover, we establish from the Young inequality that

\[
\int_0^T \int_\Omega |\nabla n_\varepsilon|^{\frac{2\alpha}{2 + \alpha - m}} \leq \int_0^T \int_\Omega (n_\varepsilon + \varepsilon)^{\alpha - 2}|\nabla n_\varepsilon|^2 + \int_0^T \int_\Omega (n_\varepsilon + \varepsilon)^{\alpha},
\]

which with Lemma 4.1 and (4.5) concludes the proof. \hfill \square

4.3. An estimate for $n_\varepsilon u_\varepsilon$

In summary, in both cases that (4.2) holds and that (4.3) holds, we verified the following important estimates for $n_\varepsilon$, which is a cornerstone in the proof of Theorem 1.1.

**Lemma 4.4.** Assume that (4.2) or (4.3) holds. Then for all $T > 0$ there is $C(T) > 0$ such that

\[
\int_0^T \int_\Omega (n_\varepsilon + \varepsilon)^{p_1} + \int_0^T \int_\Omega |D_\varepsilon(n_\varepsilon)\nabla n_\varepsilon|^{p_2} \leq C(T),
\]  

(4.6)

where $p_1 > \frac{4}{3}$ and $p_2 \in (1, 2)$ are constants defined as

\[
p_1 := \begin{cases} 
\frac{3m + 2}{3} & \text{if (4.2) holds,} \\
\alpha & \text{if (4.3) holds}
\end{cases} \quad \text{and} \quad p_2 := \begin{cases} 
\frac{3m + 2}{3m + 1} & \text{if (4.2) holds,} \\
\frac{2\alpha}{\alpha + m} & \text{if (4.3) holds.}
\end{cases}
\]  

(4.7)
Moreover, if $0 < m \leq 2$, for all $T > 0$

\[
\int_{0}^{T} \int_{\Omega} |\nabla n_\varepsilon|^{p_3} \leq \tilde{C}(T)
\]

holds for all $\varepsilon \in (0, 1)$ with some $\tilde{C}(T) > 0$, where

\[
p_3 := \begin{cases} 
\frac{3m+2}{4} & \text{if } (4.2) \text{ holds}, \\
\frac{2\alpha}{2+\alpha-m} & \text{if } (4.3) \text{ holds}.
\end{cases}
\]

**Proof.** By virtue of Lemmas 4.2 and 4.3, we can obtain the estimates in the statement. □

**Remark 4.2.** In order to obtain estimates for $n_\varepsilon$ stated in Lemma 4.4 we need to assume that $m > \frac{2}{3}$ or that $\alpha > \frac{4}{3}$ (with $\mu > 0$). Indeed, if we assume that $m > \frac{2}{3}$ or that $\alpha > \frac{4}{3}$, then we can confirm that $p_1 > \frac{4}{3}$ holds, which is important when we consider convergences of the approximate solution (see Lemma 5.2 and its proof).

In the proof of Theorem 1.1 we also need to establish some estimate for $n_\varepsilon u_\varepsilon$ (see e.g., Proof of Lemma 4.7). Here, in the case that $m > \frac{2}{3}$, the previous work [26] asserts from Lemmas 4.1 and 4.2 that

\[
\int_{0}^{T} \int_{\Omega} |n_\varepsilon u_\varepsilon| \leq \int_{0}^{T} \int_{\Omega} \frac{n_\varepsilon}{10} + \int_{0}^{T} \int_{\Omega} |u_\varepsilon|^{\frac{10}{3}} \leq C(T)
\]

with some $C(T) > 0$; however, $\int_{0}^{T} \int_{\Omega} \frac{n_\varepsilon}{10}$ is bounded only when $(\frac{2}{3} < \frac{16}{27}) \leq m$ (from the relation $\frac{10}{3} \leq \frac{3m+2}{3}$). Therefore we need the following additional estimate to control the term $\int_{0}^{T} \int_{\Omega} n_\varepsilon u_\varepsilon$.

**Lemma 4.5.** Assume that (4.2) or (4.3) holds. Then for all $T > 0$ there exists $C(T) > 0$ such that

\[
\int_{0}^{T} \|u_\varepsilon\|_{L^r(\Omega)}^{2} + \int_{0}^{T} \|n_\varepsilon\|_{L^r(\Omega)}^{2} \leq C(T) \quad \text{for all } \varepsilon \in (0, 1)
\]

with some $r > \frac{6}{5}$. Moreover, there exists $q_0 > 1$ such that for all $T > 0$ and all $q \in [1, q_0]$

\[
\int_{0}^{T} \left( \int_{\Omega} |n_\varepsilon u_\varepsilon|^q \right)^{\frac{1}{q}} \leq \tilde{C}(q, T) \quad \text{for all } \varepsilon \in (0, 1)
\]

with some $\tilde{C}(q, T) > 0$.

**Proof.** Let $T > 0$. Aided by the continuous embedding $W^{1,2}(\Omega) \hookrightarrow L^6(\Omega)$, we first see from Lemma 2.8 that for all $\varepsilon \in (0, 1),$

\[
\int_{0}^{T} \|u_\varepsilon\|_{L^6(\Omega)}^{2} \leq C_1 \int_{0}^{T} \left( \|\nabla u_\varepsilon\|_{L^2(\Omega)}^{2} + \|u_\varepsilon\|_{L^2(\Omega)}^{2} \right) \leq C_2(T)
\]
with some $C_1 > 0$ and $C_2(T) > 0$. Now, since $\frac{3\epsilon - 2}{q} = \frac{4}{3} < p_1$, we can find $r \in \left(\frac{4}{3}, p_1\right)$ such that $\frac{3\epsilon - 2}{r} \leq p_1$, which implies that

$$2 \cdot \frac{r-1}{r} \cdot \frac{p_1}{p_1-1} \leq p_1$$

holds. Therefore an interpolation inequality with (4.9) entails from (4.6) that

$$\int_0^T \|n_\epsilon\|_{L^r(\Omega)}^2 \leq \int_0^T \|n_\epsilon\|_{L^{p_1}(\Omega)}^{2(p_1-r)} \|n_\epsilon\|_{L^r(\Omega)}^{2(r-p_1)}$$

$$\leq C_3(T) \int_0^T \left(\|n_\epsilon\|_{L^{p_1}(\Omega)} + 1\right)^{\frac{p_1}{p_1-1}} \leq C_4(T)$$

for all $\epsilon \in (0, 1)$ with some $C_3(T), C_4(T) > 0$. Moreover, letting $q_0 := \frac{6r}{6+r} \in (1, 6)$ (from $r > \frac{6}{9}$) and letting $q \in [1, q_0]$, we infer from $\frac{6q}{6-q} \leq \frac{6q_0}{6-q_0} = r$ that

$$\int_0^T \left(\int_\Omega |n_\epsilon u_\epsilon|^q\right)^{\frac{1}{q}} \leq \int_0^T \|n_\epsilon\|_{L^{\frac{6q}{6-q}}(\Omega)} \|u_\epsilon\|_{L^q(\Omega)} \leq C_5(q) \left(\int_0^T \|n_\epsilon\|_{L^q(\Omega)} + \int_0^T \|u_\epsilon\|_{L^q(\Omega)}^2\right)$$

with some $C_5(q) > 0$, which with (4.8) and (4.10) completes the proof. \hfill \box

### 4.4. Time regularities

One of strategies for establishing convergences of the approximate solution is to use an Aubin–Lions-type lemma (cf. [15, Corollary 4]). To apply an Aubin–Lions-type lemma to $(n_\epsilon)_{\epsilon \in (0, 1)}$ we desire some estimates for $\nabla n_\epsilon$ and $\partial_t n_\epsilon$; however, in view of Lemma 4.4, we could have some estimate for $\nabla n_\epsilon$ only in the case that $m \in (0, 2)$. Thus, in the case that $m > 2$, we need to use some different quantity e.g., $(n_\epsilon + \epsilon)^\gamma$ with some $\gamma > 0$. By virtue of Lemma 4.1, $(n_\epsilon + \epsilon)^\frac{m}{2}$ is one of candidates of this quantity. Now we show the following lemma which is utilized to obtain an estimate for $\partial_t (n_\epsilon + \epsilon)^\frac{m}{2}$ when $m > 2$.

**Lemma 4.6.** If $m > 2$, then for all $T > 0$ there exists $C(T) > 0$ such that

$$\int_\Omega (n_\epsilon + \epsilon)^{m-1} \leq C(T) \quad \text{for all } t \in (0, T) \text{ and all } \epsilon \in (0, 1)$$

and

$$\int_0^T \int_\Omega |\nabla (n_\epsilon + \epsilon)^{m-1}|^2 \leq C(T) \quad \text{for all } \epsilon \in (0, T).$$

**Proof.** The main strategy for the proof is based on that in the proof of [8, Lemma 3.3]. Since the inequality $(m-1)(m-2) > 0$ holds from the condition $m > 2$, the condition (1.4) and the Hölder inequality yield that

$$\frac{d}{dt} \int_\Omega (n_\epsilon + \epsilon)^{m-1} = -(m-1)(m-2) \int_\Omega (n_\epsilon + \epsilon)^{m-3} D_\epsilon(n_\epsilon) \left|\nabla n_\epsilon\right|^2$$

$$+ (m-1)(m-2) \int_\Omega \frac{(n_\epsilon + \epsilon)^{m-3} n_\epsilon}{1 + \epsilon n_\epsilon} \chi(c_\epsilon) \left|\nabla n_\epsilon \cdot \nabla c_\epsilon\right|$$

$$\leq -\frac{D_1(m-2)}{2(m-1)} \int_\Omega |\nabla (n_\epsilon + \epsilon)^{m-1}|^2 + \frac{(m-1)(m-2)}{2D_1} \int_\Omega |\nabla c_\epsilon|^2.$$
Then, for each \( T > 0 \) and all \( t \in (0, T) \), integrating it over \( (0, t) \) derives that

\[
\int_{\Omega} (n_{\varepsilon} + \varepsilon)^{m-1} + \frac{D_1(m-2)}{2(m-1)} \int_0^t \int_{\Omega} |\nabla(n_{\varepsilon} + \varepsilon)^{m-1}|^2
\leq \int_{\Omega} (n_{0,\varepsilon} + \varepsilon)^{m-1} + \frac{(m-1)(m-2)}{2D_1} \int_0^T \int_{\Omega} |\nabla c_{\varepsilon}|^2,
\]

which implies that this lemma holds. \( \square \)

**Remark 4.3.** This lemma is similar to [26, Lemma 5.2]; however, there is a mistake in the class of the initial data \( n_0 \in L \log L(\Omega) \) when \( m > 2 \); indeed, to estimate \( \int_{\Omega} n_0^{p} \) for all \( p \in [1,9(m-1)) \), we have to assume that \( n_0 \in \bigcap_{p \in [1,9(m-1))} L^p(\Omega) \). Moreover, in view of the fact \( \bigcap_{p \in [1,9(m-1))} L^p(\Omega) \subset L^{m-1}(\Omega) \), Lemma 4.6 is more suitable than the previous result.

Then, thanks to this lemma, we shall see some time regularity properties of \((n_{\varepsilon} + \varepsilon)^{\gamma}\) with some \( \gamma > 0 \).

**Lemma 4.7.** Assume that (4.2) or (4.3) holds. Then for all \( T > 0 \) there exists \( C(T) > 0 \) such that

\[
\int_0^T \| \partial_t (n_{\varepsilon} + \varepsilon)^{\gamma} \|_{W^{2,4}_0(\Omega)} \leq C(T) \quad \text{for all } \varepsilon \in (0,1),
\]

where

\[
\gamma := \begin{cases} 
1 & (0 < m \leq 2), \\
\frac{m}{2} & (m > 2).
\end{cases}
\]

**Proof.** Let \( T > 0 \) and let \( \psi \in W^{2,4}_0(\Omega) \) satisfy \( \|\psi\|_{W^{2,4}(\Omega)} \leq 1 \). Here we note from the continuous embedding \( W^{2,4}_0(\Omega) \hookrightarrow W^{1,\infty}(\Omega) \) that there exists \( C_1 > 0 \) such that \( \|\psi\|_{W^{1,\infty}(\Omega)} \leq C_1 \). Now straightforward calculations and integration by parts enable us to have that

\[
\int_{\Omega} (\partial_t (n_{\varepsilon} + \varepsilon)^{\gamma}) \psi = - \gamma (\gamma - 1) \int_{\Omega} (n_{\varepsilon} + \varepsilon)^{\gamma-2} D_\varepsilon(n_{\varepsilon}) |\nabla n_{\varepsilon}|^2 \psi
- \gamma \int_{\Omega} (n_{\varepsilon} + \varepsilon)^{\gamma-1} D_\varepsilon(n_{\varepsilon}) \nabla n_{\varepsilon} \cdot \nabla \psi
+ \gamma (\gamma - 1) \int_{\Omega} (n_{\varepsilon} + \varepsilon)^{\gamma-2} n_{\varepsilon} \chi(c_{\varepsilon}) (\nabla n_{\varepsilon} \cdot \nabla c_{\varepsilon}) \psi
+ \gamma \int_{\Omega} (n_{\varepsilon} + \varepsilon)^{\gamma-1} n_{\varepsilon} \chi(c_{\varepsilon}) \nabla c_{\varepsilon} \cdot \nabla \psi
- \gamma \int_{\Omega} (n_{\varepsilon} + \varepsilon)^{\gamma-1} (u_{\varepsilon} \cdot \nabla n_{\varepsilon}) \psi
=: I_1 + I_2 + I_3 + I_4 + I_5.
\]

We first deal with the case that \( 0 < m \leq 2 \), which means \( \gamma = 1 \). Then we note that \( I_1 = I_3 = 0 \). Furthermore, we use the Hölder inequality to obtain that

\[
|I_2| \leq C_2 \left( \int_{\Omega} (D_\varepsilon(n_{\varepsilon}) |\nabla n_{\varepsilon}|)^{p_2} + 1 \right)
\]

...
\[ |I_4| \leq C_3 \left( \int_{\Omega} (n_\varepsilon + \varepsilon)^{\frac{4}{3}} + \int_{\Omega} |\nabla c_\varepsilon|^4 \right) \]

as well as
\[ |I_5| = \gamma \left( \int_{\Omega} n_\varepsilon u_\varepsilon \cdot \nabla \psi \right) \leq C_1 \gamma \int_{\Omega} |n_\varepsilon u_\varepsilon| \]

with some \( C_2, C_3 > 0 \), which with the standard duality argument implies from Lemmas 4.4 and 4.5 and the fact \( p_1 > \frac{4}{3} \) that
\[
\int_0^T \| \partial_t (n_\varepsilon + \varepsilon) \|_{W^{2,4}(\Omega)} \ast \leq \int_0^T \sup \left\{ \left| \int_{\Omega} \partial_t (n_\varepsilon + \varepsilon) \psi \right| \mid \psi \in W^{2,4}_0(\Omega), \| \psi \|_{W^{2,4}(\Omega)} \leq 1 \right\} \\
\leq C_4 \int_0^T \int_{\Omega} ((D_\varepsilon(n_\varepsilon)|\nabla n_\varepsilon|)^p + 1 + (n_\varepsilon + \varepsilon)^p + |\nabla c_\varepsilon|^4 + |n_\varepsilon u_\varepsilon|) \\
\leq C_5(T)
\]

with some \( C_4 > 0 \) and \( C_5(T) > 0 \). On the other hand, in the case that \( m > 2 \), which implies \( \gamma = \frac{m}{2} \), we obtain from (1.4) that
\[
|I_1| \leq C_6 \int_{\Omega} |\nabla (n_\varepsilon + \varepsilon)^{\frac{m}{2}}| \cdot |\nabla (n_\varepsilon + \varepsilon)^{m-1}|, \\
|I_2| \leq C_7 \int_{\Omega} (n_\varepsilon + \varepsilon)^{\frac{m}{2}}|\nabla (n_\varepsilon + \varepsilon)^{m-1}|
\]
and
\[
|I_3| \leq C_8 \int_{\Omega} |\nabla (n_\varepsilon + \varepsilon)^{\frac{m}{2}}| \cdot |\nabla c_\varepsilon|, \\
|I_4| \leq C_9 \int_{\Omega} (n_\varepsilon + \varepsilon)^{\frac{m}{2}}|\nabla c_\varepsilon|
\]
as well as
\[
|I_5| \leq C_{10} \int_{\Omega} |\nabla (n_\varepsilon + \varepsilon)^{\frac{m}{2}}| \cdot |u_\varepsilon|
\]
with some \( C_6, C_7, C_8, C_9, C_{10} > 0 \). Then a combination of the Young inequality and Lemmas 4.1, 4.2, 4.6, along with the standard duality argument derives that
\[
\int_0^T \| \partial_t (n_\varepsilon + \varepsilon)^{\frac{m}{2}} \|_{W^{2,4}(\Omega)} \ast \leq C_{11}(T)
\]
holds with some \( C_{11}(T) > 0 \).

Similar arguments in the proof of [26, Lemma 5.3] can derive the following lemma; thus we only introduce the statement.

**Lemma 4.8.** Assume that (4.2) or (4.3) holds. Then for all \( T > 0 \) there exists \( C(T) > 0 \) such that
\[
\int_0^T \| \partial_t \sqrt{c_\varepsilon} \|_{W^{2,4}(\Omega)} \ast + \int_0^T \| \partial_t u_\varepsilon \|_{W^{2,4}_d(\Omega)} \ast \leq C(T) \quad \text{for all } \varepsilon \in (0, 1).
\]
5. Convergences: Proof of Theorem 1.1

Before stating convergences properties, we define weak solutions of (1.3).

Definition 5.1. A triplet \((n, c, u)\) is called a global weak solution of (1.3) if \(n, c, u\) satisfy
\[
n \in L^1_{\text{loc}}(\overline{\Omega} \times [0, \infty)), \quad c \in L^1_{\text{loc}}([0, \infty); W^{1,1}(\Omega)), \quad u \in L^1_{\text{loc}}([0, \infty); W^{1,1}_0(\Omega))
\]
and
\[
\int_0^n D(s) \, ds \in L^1_{\text{loc}}([0, \infty); W^{1,1}(\Omega)), \quad n^\alpha \in L^1_{\text{loc}}(\Omega \times [0, \infty))
\]
as well as
\[
nu, cu, n\chi(c)\nabla c, u \otimes u \in L^1_{\text{loc}}(\Omega \times [0, \infty))
\]
and the identities
\[
- \int_0^\infty \int_\Omega n \varphi_t - \int_\Omega n_0 \varphi(\cdot, 0) - \int_0^\infty \int_\Omega nu \cdot \nabla \varphi = - \int_0^\infty \int_\Omega n \varphi \nabla c \cdot \nabla \varphi + \int_0^\infty \int_\Omega (kn - \mu n^\alpha) \varphi,
\]
\[
- \int_0^\infty \int_\Omega c \varphi_t - \int_\Omega c_0 \varphi(\cdot, 0) - \int_0^\infty \int_\Omega cu \cdot \nabla \varphi = - \int_0^\infty \int_\Omega c \varphi \nabla c \cdot \nabla \varphi - \int_0^\infty \int_\Omega nc \varphi,
\]
\[
- \int_0^\infty \int_\Omega u \cdot \psi_t - \int_\Omega u_0 \cdot \psi(\cdot, 0) - \int_0^\infty \int_\Omega u \otimes u \cdot \nabla \psi = - \int_0^\infty \int_\Omega \nabla u \cdot \nabla \psi + \int_0^\infty \int_\Omega n \nabla \Phi \cdot \psi + \int_0^\infty \int_\Omega g \cdot \psi
\]
hold for all \(\varphi \in C_0^\infty(\Omega \times [0, \infty))\) and all \(\psi \in C_0^\infty(\Omega \times [0, \infty))\), respectively.

Collecting the boundedness properties obtained in Lemmas 2.8, 4.4, 4.5, 4.7 and 4.8, we establish the following convergences.

Lemma 5.1. Assume that (4.2) or (4.3) holds. Then there exist a subsequence \(\varepsilon \downarrow 0\) and functions \(n, c, u\) such that for all \(p \in [1, p_1)\) and \(q \in [1, \infty)\),
\[
n_\varepsilon \to n \quad \text{in } L^p_{\text{loc}}(\Omega \times [0, \infty)) \cap L^2_{\text{loc}}([0, \infty); L^q_\Sigma(\Omega)) \text{ and a.e. in } \Omega \times (0, \infty),
n_\varepsilon^\alpha \to n^\alpha \quad \text{in } L^1_{\text{loc}}(\Omega \times [0, \infty)),$n_\varepsilon^2 \to 0 \quad \text{in } L^1_{\text{loc}}(\Omega \times [0, \infty)),$c_\varepsilon \to c \quad \text{in } L^q_{\text{loc}}(\Omega \times [0, \infty)) \text{ and a.e. in } \Omega \times (0, \infty),c_\varepsilon^* \to c \quad \text{in } L^\infty(\Omega \times (0, \infty)),$\nabla c_\varepsilon \to \nabla c \quad \text{in } L^4_{\text{loc}}(\Omega \times [0, \infty)),$\nabla c_\varepsilon^\frac{1}{2} \to \nabla c^\frac{1}{2} \quad \text{in } L^4_{\text{loc}}(\Omega \times [0, \infty)),$u_\varepsilon \to u \quad \text{in } L^2_{\text{loc}}(\Omega \times [0, \infty)) \text{ and a.e. in } \Omega \times (0, T),$u_\varepsilon \to u \quad \text{in } L^2_{\text{loc}}([0, \infty); L^6(\Omega)),$\nabla u_\varepsilon \to \nabla u \quad \text{in } L^2_{\text{loc}}(\Omega \times [0, \infty))$

hold as \(\varepsilon = \varepsilon \downarrow 0\), where \(p_1 > \frac{4}{3}\) is the constant defined in (4.7).
Proof. This proof is based on [26, Proof of Theorem 1.1] and [9, Proposition 6.1]. Let \( T > 0 \) and let
\[
\beta := \begin{cases} 
  p_2 & (0 < m \leq 2), \\
  2 & (m > 2),
\end{cases}
\quad \text{and} \quad \gamma := \begin{cases} 
  1 & (0 < m \leq 2), \\
  \frac{m}{2} & (m > 2).
\end{cases}
\]
Then since Lemmas 4.4 and 4.7 derive that
\[
((n_\varepsilon + \varepsilon)^\gamma)_{\varepsilon \in (0,1)} \text{ is bounded in } L^\beta(0, T; W^{1, \beta}(\Omega))
\]
and
\[
(\partial_t (n_\varepsilon + \varepsilon)^\gamma)_{\varepsilon \in (0,1)} \text{ is bounded in } L^1(0, T; (W^{2,4}_0(\Omega))^*)
\]
hold, a combination of the compact embedding \( W^{1,\beta}(\Omega) \hookrightarrow L^\beta(\Omega) \) and the continuous embedding \( L^\beta(\Omega) \hookrightarrow (W^{2,4}_0(\Omega))^* \) together with an Aubin–Lions-type lemma (see [15, Corollary 4]) enables us to find a subsequence \( \varepsilon_j \downarrow 0 \) and a function \( n \) such that
\[
(n_\varepsilon + \varepsilon)^\gamma \to n^\gamma \text{ in } L^\beta(\Omega \times (0, T)) \text{ and a.e in } \Omega \times (0, T)
\]
as \( \varepsilon = \varepsilon_j \downarrow 0 \). Therefore, aided by Lemmas 4.4 and 4.5, we obtain from the Vitali convergence theorem that
\[
n_\varepsilon \to n \text{ in } L^p(\Omega \times (0, T)) \cap L^2(0, T; L^6(\Omega)) \text{ and a.e. in } \Omega \times (0, \infty)
\]
as \( \varepsilon = \varepsilon_j \downarrow 0 \) for all \( p \in [1, p_1] \), where we used the relation \( r > \frac{6}{5} \) (\( r \) is the constant obtained in Lemma 4.5). Moreover, since Lemma 2.8 yields that \((n_\varepsilon^\alpha)_{\varepsilon \in (0,1)} \) and \((\varepsilon n_\varepsilon^2)_{\varepsilon \in (0,1)} \) are weakly relatively precompact by the Dunford–Pettis theorem, we can find a further subsequence \( \varepsilon_j \downarrow 0 \) and a function \( n \) such that
\[
n_\varepsilon^\alpha \to z_1, \quad \varepsilon n_\varepsilon^2 \to z_2 \text{ in } L^1(\Omega \times (0, T))
\]
as \( \varepsilon = \varepsilon_j \downarrow 0 \), which with pointwise a.e. convergences implies that \( z_1 = n^\alpha \) and \( z_2 = 0 \) (for more details, see the proof of [9, Proposition 6.1]). On the other hand, noticing from Lemmas 2.8 and 4.8 that \((\sqrt{c_\varepsilon})_{\varepsilon \in (0,1)} \) and \((\partial_t \sqrt{c_\varepsilon})_{\varepsilon \in (0,1)} \) are bounded in \( L^2(0, T; W^{2,2}(\Omega)) \) and \( L^1(0, T; (W^{2,4}_0(\Omega))^*) \), respectively, as well as \((u_\varepsilon)_{\varepsilon \in (0,1)} \) and \((\partial_t u_\varepsilon)_{\varepsilon \in (0,1)} \) are bounded in \( L^2(0, T; W^{1,2}_{0,\sigma}(\Omega)) \) and \( L^2(0, T; (W^{1,2}_{0,\sigma}(\Omega))^*) \), respectively, we obtain from the Aubin–Lions-type lemma that there are a further subsequence (again denoted by \( \varepsilon_j \)) and functions \( c, u \) such that
\[
\sqrt{c_\varepsilon} \to \sqrt{c} \text{ in } L^2(0, T; W^{1,2}(\Omega)) \text{ and a.e. in } \Omega \times (0, \infty)
\]
and
\[
u_\varepsilon \to u \text{ in } L^2(\Omega \times (0, T)) \text{ and a.e. in } \Omega \times (0, \infty).
\]
These convergences and Lemmas 2.2, 2.8, 4.1 and 4.5 entail this lemma. \( \square \)

Then, aided by Lemma 5.1, we can obtain the following convergences which are needed for the proof of Theorem 1.1.
Lemma 5.2. Assume that (4.2) or (4.3) holds and let \( \varepsilon_j, n, c, u \) be obtained in Lemma 5.1. Then

\[
n\varepsilon u \varepsilon \rightarrow nu, \quad c\varepsilon u \varepsilon \rightarrow cu, \quad \frac{f(c\varepsilon)}{\varepsilon} \log(1 + \varepsilon n\varepsilon) \rightarrow nf(c) \quad \text{in} \quad L^1_{\text{loc}}(\Omega \times [0, \infty))
\]

and

\[
Y\varepsilon u \varepsilon \otimes u \varepsilon \rightarrow u \otimes u, \quad \frac{n\varepsilon}{1 + \varepsilon n\varepsilon} \chi(c\varepsilon) \nabla c\varepsilon \rightarrow n\chi(c) \nabla c \quad \text{in} \quad L^1_{\text{loc}}(\Omega \times [0, \infty)).
\]

Proof. Let \( T > 0 \). From Lemma 5.1 we can see that

\[
n\varepsilon u \varepsilon \rightarrow nu, \quad c\varepsilon u \varepsilon \rightarrow cu \quad \text{in} \quad L^1(\Omega \times (0, T))
\]

and

\[
\frac{f(c\varepsilon)}{\varepsilon} \log(1 + \varepsilon n\varepsilon) \rightarrow nf(c) \quad \text{a.e. in} \ \Omega \times (0, T)
\]

hold as \( \varepsilon = \varepsilon_j \searrow 0 \). This together with the estimate

\[
\int_0^T \int_{\Omega} \left| \frac{f(c\varepsilon)}{\varepsilon} \log(1 + \varepsilon n\varepsilon) \right|^{p_1} \leq \|f(s)\|_{L^{p_1}(0, \|c_0\|_{L^\infty(\Omega)})} \int_0^T \int_{\Omega} \left| n\varepsilon \right|^{p_1} \leq C_1(T)
\]

with some \( C_1(T) > 0 \) (from Lemma 4.4) implies from the Vitali convergence theorem that

\[
\frac{f(c\varepsilon)}{\varepsilon} \log(1 + \varepsilon n\varepsilon) \rightarrow nf(c) \quad \text{in} \quad L^1_{\text{loc}}(\Omega \times [0, \infty))
\]

as \( \varepsilon = \varepsilon_j \searrow 0 \). Moreover, a combination of Lemma 5.1 and arguments in the proof of [23, (4.26)] yields that

\[
Y\varepsilon u \varepsilon \otimes u \varepsilon \rightarrow u \otimes u \quad \text{in} \quad L^1(\Omega \times (0, T))
\]

as \( \varepsilon = \varepsilon_j \searrow 0 \). Now, in light of Lemmas 2.2 and 5.1 together with the condition for \( \chi \) (see (1.5)), the dominated convergence theorem implies that

\[
\frac{1}{1 + \varepsilon n\varepsilon} \chi(c\varepsilon) c^{\frac{4}{3}} \rightarrow \chi(c) c^{\frac{4}{3}} \quad \text{in} \quad L^q(\Omega \times (0, T))
\]

for all \( q \in [1, \infty) \). Therefore, noticing that \( p_1 > \frac{4}{3} \), we obtain from Lemma 5.1 that

\[
\frac{n\varepsilon}{1 + \varepsilon n\varepsilon} \chi(c\varepsilon) \nabla c\varepsilon = n\varepsilon \cdot \frac{4}{1 + \varepsilon n\varepsilon} \chi(c\varepsilon) c^{\frac{4}{3}} \cdot \nabla c^{\frac{1}{3}}
\]

\[
\rightarrow 4n \cdot \chi(c) c^{\frac{4}{3}} \cdot \nabla c^{\frac{1}{3}} = n\chi(c) \nabla c \quad \text{in} \quad L^1(\Omega \times (0, T))
\]

as \( \varepsilon = \varepsilon_j \searrow 0 \). \( \square \)

We also desire the following lemma to show Theorem 1.1.
Lemma 5.3. Assume that (4.2) or (4.3) holds. Then \( \int_0^n D(\sigma) \, d\sigma \in L^1_{\text{loc}}([0, \infty); W^{1,1}(\Omega)) \) and there is a further subsequence (again denoted by \( \varepsilon_j \)) such that

\[
\nabla \left( \int_{-\varepsilon}^{n_\varepsilon} D_\varepsilon(\sigma) \, d\sigma \right) \rightarrow \nabla \left( \int_{-\varepsilon}^{n_\varepsilon} D(\sigma) \, d\sigma \right) \quad \text{in } L^{p_2}_{\text{loc}}(\Omega \times [0, \infty))
\]

as \( \varepsilon = \varepsilon_j \searrow 0 \), where \( p_2 \in (1, 2) \) is the constant defined in (4.7). Moreover, if \( 1 \leq m \leq 2 \), then

\[
\nabla \left( \int_{-\varepsilon}^{n_\varepsilon} D_\varepsilon(\sigma) \, d\sigma \right) \rightarrow w \quad \text{in } L^{p_2}(\Omega \times (0, T))
\]

as \( \varepsilon = \varepsilon_j \searrow 0 \). In order to see \( w = \nabla (\int_0^n D(\sigma) \, d\sigma) \) we shall show that \( \int_{-\varepsilon}^{n_\varepsilon} D_\varepsilon(\sigma) \, d\sigma \rightarrow \int_{-\varepsilon}^{n_\varepsilon} D(\sigma) \, d\sigma \) in \( L^{p_2}(\Omega \times (0, T)) \). Since Lemma 5.1 asserts that \( n_\varepsilon + \varepsilon \rightarrow n \) a.e. in \( \Omega \times (0, T) \), we first establish that

\[
\int_{-\varepsilon}^{n_\varepsilon} D_\varepsilon(\sigma) \, d\sigma = \int_{0}^{n_\varepsilon} D(\sigma) \, d\sigma \rightarrow \int_{0}^{n} D(\sigma) \, d\sigma \quad \text{a.e. in } \Omega \times (0, T).
\]

On the other hand, the condition for \( D \) (see (1.4)) and Lemma 4.4 derive that

\[
\int_0^T \int_\Omega \left| \int_{-\varepsilon}^{n_\varepsilon} D_\varepsilon(\sigma) \, d\sigma \right|^\frac{p_1}{m} \leq D_2 \int_0^T \int_\Omega \left( \int_{-\varepsilon}^{n_\varepsilon} (\sigma + \varepsilon)^{m-1} \, d\sigma \right) \frac{p_1}{m} 
= \left( \frac{D_2}{m} \right) \frac{p_1}{m} \int_0^T \int_\Omega ((n_\varepsilon + \varepsilon)^m) \frac{p_1}{m} \leq C_1(T)
\]

with some \( C_1(T) > 0 \). Therefore the Vitali convergence theorem entails from the fact \( \frac{p_1}{m} > p_2 \) that

\[
\int_{-\varepsilon}^{n_\varepsilon} D_\varepsilon(\sigma) \, d\sigma \rightarrow \int_{-\varepsilon}^{n} D(\sigma) \, d\sigma \quad \text{in } L^{p_2}(\Omega \times (0, T)),
\]

which means that \( w \) coincides with \( \nabla (\int_0^n D(\sigma) \, d\sigma) \). Moreover, if \( 1 \leq m \leq 2 \), then from Lemma 4.2 and the Vitali convergence theorem we can find a further subsequence (again denoted by \( \varepsilon_j \)) such that

\[
D_\varepsilon(n_\varepsilon) \rightarrow D(n) \quad \text{in } L^{\frac{3m+2}{m-2}}(\Omega \times (0, T))
\]

and

\[
\nabla n_\varepsilon \rightarrow \nabla n \quad \text{in } L^{\frac{3m+2}{4m}}(\Omega \times (0, T))
\]

as \( \varepsilon = \varepsilon_j \searrow 0 \). Thus we see that

\[
\nabla \left( \int_{-\varepsilon}^{n_\varepsilon} D_\varepsilon(\sigma) \, d\sigma \right) = D_\varepsilon(n_\varepsilon) \nabla n_\varepsilon \rightarrow D(n) \nabla n \quad \text{in } L^1(\Omega \times (0, T)),
\]

which means that \( \nabla (\int_0^n D(\sigma) \, d\sigma) = D(n) \nabla n \) holds. \( \Box \)
Thanks to convergences established in Lemmas 5.1, 5.2 and 5.3, we can establish existence of a global weak solution in the sense of Definition 5.1.

**Lemma 5.4.** Assume that (4.2) or (4.3) holds. Then there exists at least one global weak solution of (1.3) in the sense of Definition 5.1.

**Proof.** Testing the first and second equations in (2.1) by \( \psi \in C_0^\infty(\Omega \times [0, \infty)) \) and testing the third equation in (2.1) by \( \psi \in C_0^\infty(\Omega \times [0, \infty)), \) we have that

\[
- \int_0^\infty \int_\Omega n_\varepsilon \varphi_t - \int_\Omega n_0 \varphi(\cdot, 0) - \int_0^\infty \int_\Omega n_\varepsilon u_\varepsilon \cdot \nabla \varphi \\
= - \int_0^\infty \int_\Omega \nabla \left( \int_0^{n_\varepsilon} D_\varepsilon(s) \, ds \right) \cdot \nabla \varphi + \int_0^\infty \int_\Omega \frac{n_\varepsilon \chi(c_\varepsilon)}{1 + \varepsilon n_\varepsilon} \nabla c_\varepsilon \cdot \nabla \varphi \\
+ \int_0^\infty \int_\Omega (\kappa n_\varepsilon - \mu n_\varepsilon^\alpha - \varepsilon n_\varepsilon^2) \varphi
\]

and

\[
- \int_0^\infty \int_\Omega c_\varepsilon \varphi_t - \int_\Omega c_0 \varphi(\cdot, 0) - \int_0^\infty \int_\Omega c_\varepsilon u_\varepsilon \cdot \nabla \varphi \\
= - \int_0^\infty \int_\Omega c_\varepsilon \varphi \nabla \varphi - \frac{1}{\varepsilon} \int_0^\infty \int_\Omega f(c_\varepsilon) \log(1 + \varepsilon n_\varepsilon) \varphi
\]

as well as

\[
- \int_0^\infty \int_\Omega u_\varepsilon \cdot \psi_t - \int_\Omega u_0 \cdot \psi(\cdot, 0) - \int_0^\infty \int_\Omega Y_\varepsilon u_\varepsilon \otimes u_\varepsilon \cdot \nabla \psi \\
= - \int_0^\infty \int_\Omega \nabla u_\varepsilon \cdot \nabla \psi + \int_0^\infty \int_\Omega n_\varepsilon \nabla \Phi \cdot \psi + \int_0^\infty \int_\Omega g_\varepsilon \cdot \psi
\]

for all \( \varepsilon \in (0,1). \) Thus, plugging (2.2)–(2.5) and Lemmas 5.1, 5.2, 5.3 into the above identities, by taking the limit as \( \varepsilon = \varepsilon_j \searrow 0 \) we can obtain this lemma. \( \square \)

**Proof of Theorem 1.1.** Assume that (1.11) holds. Then we verify that (4.2) or (4.3) holds. Thus from Lemmas 5.3 and 5.4 we can attain Theorem 1.1. \( \square \)

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