On the Connection Between Deutsch-Jozsa Algorithm and Bent Functions

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Abstract. It is presently shown that the Deutsch-Jozsa algorithm is connected to the concept of bent function. Particularly, it is noticeable that the quantum circuit used to denote the well-known quantum algorithm is by itself the quantum computer that performs the Walsh transform of a Boolean function. Consequently, the output from the Deutsch-Jozsa algorithm when the hidden function is bent corresponds to a flat spectrum of quantum states.

Key words: quantum circuits, quantum algorithms, Boolean functions, Walsh transform, bent function

1. Introduction
The concept of bent function was originally introduced by [4], for cryptography purposes, and is intuitively the Boolean function that most distances from linear, \( f(x) = k \cdot x \), and affine, \( f(x) = k \cdot x \oplus 1 \) cases. The dot-product notation corresponds to the Boolean inner product, namely

\[
k \cdot x = \sum_{j=0}^{n-1} k_j x_j \mod 2 = \bigoplus_{j=0}^{n-1} k_j x_j
\]

What is interesting, and is shown in this article, is that the general form of the Deutsch-Jozsa [1] algorithm coincides with the Walsh transform. This in turn has constant absolute value in the case of bent functions. Consequently, the famous quantum algorithm reproduces a flat spectrum in its output if the hidden function is a bent function; which is not the aim of the original application for the algorithm [1]. Thus, such peculiarity of the Deutsch–Jozsa algorithm provides a fast \( O(1) \) check if an unknown function is or not a bent function.

In Section 2, we will briefly summarize the Walsh transform on \( n \)-bit Boolean functions. In Section 3, it will be formally shown that the definition of bent functions reproduces a flat response spectrum of the Deutsch–Jozsa algorithm. In Section 4, we will show some examples obtained from a simple classic simulator of the famous quantum algorithm, for the bent and non-bent case of Boolean functions over a 4-bit space. Some conclusions and perspectives will be discussed in Section 5.
2. The Deutsch–Jozsa algorithm and the Walsh transform of a Boolean function

2.1. The Deutsch–Jozsa algorithm

To recall, if the test function \( f : \{0, 1\}^n \rightarrow \{0, 1\} \) is constant, either \( f = 0 \) or \( f = 1 \), the Deutsch–Jozsa algorithm responds with the monochromatic state \( |\psi\rangle = |0^n\rangle \equiv |0\rangle^{\otimes n} \). On the other hand, if the output state is \( |\psi\rangle \neq |0^n\rangle \), the algorithm is indicating that the function is balanced. However, such conclusion is a question of faith once it requires the belief that the function has been programmed to behave as a constant otherwise as a Boolean balanced function.

Particularly, [2] pointed out that the algorithm responds with monochromatic solutions, other than \( |\psi_{\text{out}}\rangle = |0^n\rangle \), namely \( |\psi_{\text{out}}\rangle = |k\rangle \), with \( 1 \leq k \leq 2^n - 1 \), if and only if the test function \( f \) is either linear or affine, namely \( f(x) = k \cdot x \oplus c \), with \( c \in \{0, 1\} \).

The general output from the Deutsch-Jozsa algorithm can be written as the following linear combination in the \( \mathcal{H}^{\otimes n} \) Hilbert space, namely:

\[
|\psi_{\text{out}}\rangle = \sum_{p \in \{0, 1\}^n} \psi_f(p) |p\rangle
\]

where the probability amplitude \( \psi_f(p) \) of having the pure \( n \)-qubit state \( |p\rangle \) as response (measuring) is given by the following transform [3]

\[
\psi_f(p) = \frac{1}{2^n} \sum_{x \in \{0, 1\}^n} (-1)^{f(x) \oplus p \cdot x}
\]

2.2. Walsh transform

Recalling that the Walsh transform of a Boolean function \( f : \{0, 1\}^n \rightarrow \{0, 1\} \) is another Boolean function, \( \hat{f} : \{0, 1\}^n \rightarrow \{0, 1\} \), which transforms a bit string \( p \in \{0, 1\}^n \) into the Boolean scalar

\[
\hat{f}(p) = \sum_{x \in \{0, 1\}^n} (-1)^{f(x) \oplus p \cdot x}
\]

Confronting equations (3) and (4) one finds \( \psi_f(p) \) in terms of the Walsh transform as follows

\[
\psi_f(p) = \frac{1}{2^n} \hat{f}(p)
\]

3. Bent function and its response to the Deutsch–Jozsa algorithm

A bent function is a very particular case of Boolean function which can be defined as follows:

**Definition 1** A bent function is the Boolean function \( f : \{0, 1\}^n \rightarrow \{0, 1\} \) whose Walsh transform \( \hat{f} : \{0, 1\}^n \rightarrow \{0, 1\} \) has constant absolute value.

Rothaus [4] proved that a bent function has the following metric signature:

\[
|\hat{f}(p)| = 2^{n/2}, \quad \forall p \in \{0, 1\}^n
\]

and that \( n \) must be is even.

An example in algebraic normal form of bent function in the two-bit domain \( \{0, 1\}^2 \) is \( f(x_1, x_2) = x_1 \land x_2 \). In this particular, \( |\hat{f}(p)| = 2, \quad \forall p \in \{0, 1\}^2 \).

Therefore, if \( f \) is a bent function, the spectrum of probability amplitude \( |\psi_f(p)| \) is flat through the \( \{0, 2^n - 1\} \equiv \{0, 1\}^n \) integer interval. Thus, according to equations (5) and (6), one finds that

\[
|\psi_f(p)| = \frac{1}{\sqrt{2^n}}
\]

which is consistent with the normalization principle of quantum mechanics, namely

\[
\sum_{p \in \{0, 1\}^n} |\psi_f(p)|^2 = 1
\]
4. Computer simulation of bent functions exercised by the Deutsch–Jozsa algorithm

![Figure 1](image1.png)

**Figure 1.** Left: the linear case for \( k = 9 \) (see main text). Right: The Deutsch-Jozsa output spectrum for the linear case shown in the left

To illustrate the behavior of the Deutsch-Jozsa algorithm, a C-language program was implemented to simulate Boolean functions, for linear, arbitrary, unbent, simplest bent, and random bent. The output of the quantum algorithm is simulated using Equation (3) to obtain the probability distribution function as the square of the amplitudes in the equation.

The simulations were made considering only 4 bits, which is enough for a good visualization. The first simulated case was the linear case, say \( f(x) = k \cdot x \) [2], with \( k = 9 \). The result is shown in Figure 1, the latter being the simulation of what would be observed in the output of the Deutsch–Jozsa algorithm after a large number of runs.

![Figure 2](image2.png)

**Figure 2.** Left: an arbitrary, non-bent Boolean function. Right: the Deutsch-Jozsa response for the non-bent case shown in the left

Figure 2 shows the result of a simulation for an arbitrary, non-bent function. Compare the difference between the output spectra, Figures 1 and 2, between the linear case and the arbitrary case. Also note that the non-bent case has an arbitrary probability distribution.

Figure 3 illustrates the simplest case of bent function, namely \( f(x_3,x_2,x_1,x_0) = x_0 \land x_1 + x_2 \land x_3 \), where \( \land \) means the bitwise AND operator. In this case, the probability spectrum is flat.

Finally, Figure 4 illustrate a case of randomly chosen bent function. The program has the option of random shuffling, and after a few runs, the one that returns a flat probability response is chosen.
5. Conclusion

It was shown in the previous section that the Deutsch–Jozsa algorithm responds with a flat spectrum of solutions if the test function is a bent function.

Particularly important is that the original purpose of the Deutsch-Jozsa algorithm was to decide whether a Boolean function was constant or balanced, and now we are showing that the same quantum algorithm works on deciding if a function is bent or not.

References

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