Banach spaces with no proximinal subspaces of
codimension 2

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Abstract

The classical theorem of Bishop-Phelps asserts that, for a Banach
space $X$, the norm-achieving functionals in $X^*$ are dense in $X^*$. Béla
Bollobás’s extension of the theorem gives a quantitative descriptio
just how dense the norm-achieving functionals have to be: if $(x, \varphi) \in X \times X^*$
with $\|x\| = \|\varphi\| = 1$ and $|1 - \varphi(x)| < \varepsilon^2/4$ then there are $(x', \varphi') \in X \times X^*$
with $\|x'\| = \|\varphi'\| = 1$, $\|x - x'\| \vee \|\varphi - \varphi'\| < \varepsilon$ and
$\varphi'(x') = 1$.

This means that there are always “proximinal” hyperplanes $H \subset X$
(a nonempty subset $E$ of a metric space is said to be “proximinal” if,
for $x \notin E$, the distance $d(x, E)$ is always achieved - there is always an
e $e \in E$ with $d(x, E) = d(x, e)$); for if $H = \ker \varphi$ ($\varphi \in X^*$) then it is easy
to see that $H$ is proximinal if and only if $\varphi$ is norm-achieving. Indeed
the set of proximinal hyperplanes $H$ is, in the appropriate sense, dense
in the set of all closed hyperplanes $H \subset X$.

Quite a long time ago [Problem 2.1 in his monograph “The Theory
of Best approximation and Functional Analysis” Regional Conference
series in Applied Mathematics, SIAM, 1974], Ivan Singer asked if this
result generalized to closed subspaces of finite codimension - if every
Banach space has a proximinal subspace of codimension 2, for example.
In this paper I will show that there is a Banach space $X$ such that
$X$ has no proximinal subspace of finite codimension $n \geq 2$. So we
have a converse to Bishop-Phelps-Bollobás: a dense set of proximinal
hyperplanes can always be found, but proximinal subspaces of larger,
finite codimension need not be.
1 Introduction.

I’m grateful to David Blecher for awakening me to the joys of proximinality in the context of operator algebras (norm-closed subalgebras of $B(H)$), and to Gilles Godefroy for alerting me to this particular problem.

The original Bishop-Phelps theorem is [1], and Bollobás’ improved version of the theorem is [3]. The place where the problem solved in this paper was originally posed is in Ivan Singer [6]. Gilles Godefroy’s exhaustive survey article on isometric preduals in Banach spaces, which discusses this problem among many others, is [5]. Our work with David Blecher involving proximinality of ideals in operator algebras is [2]. This is a successful attempt to generalize, to a noncommutative setting, the classical Glicksberg peak set theorem in uniform algebras (Theorem 12.7 in Gamelin [4]).

All the Banach spaces in this paper are over the real field. At risk of stating the obvious, a proximinal subset is necessarily closed; so we lose no generality later on by assuming that a (hypothetical) proximinal subspace of finite codimension is the intersection of the kernels of finitely many continuous linear functionals.

Let $c_00(Q)$ denote the terminating sequences with rational coefficients (a much-loved countable set), and let $(u_k)_{k=1}^\infty$ be a sequence of elements of $c_00(Q)$ which lists every element infinitely many times. For $x \in c_00(Q)$, write $u^{-1}(x)$ for the infinite set $\{k \in \mathbb{N} : u_k = x\}$.

Let $(a_k)_{k=1}^\infty$ be a strictly increasing sequence of positive integers. We impose a growth condition: if $u_k \neq 0$, we demand that

$$a_k > \max \text{supp} u_k, \quad a_k \geq \|u_k\|_1, \quad (1)$$

where $\text{supp} u$ denotes the (finite) support of $u \in c_00(Q)$, and $\|u\|_1$ denotes the $l^1$ norm.

For $E \subseteq \mathbb{N}$ we write $A_E$ for the set $\{a_k : k \in E\}$; for $x \in c_00(Q)$ we write $A_x$ for $A_{u^{-1}(x)}$. $A_x$ is an infinite set, and in view of (1), for each $x \in c_00(Q) \setminus \{0\}$ we have

$$\min A_x > \max \text{supp} x, \quad \min A_x \geq \|x\|_1. \quad (2)$$

Given sequences $(u_k), (a_k)$ as described above, we define a new norm $\|\cdot\|$ on $c_0$ as follows:

$$\|x\| = \|x\|_0 + \sum_{k=1}^\infty 2^{-a_k^2} |\langle x, u_k - e_{a_k} \rangle|. \quad (3)$$

Here $\|x\|_0 = \sup_n |x_n|$ is the usual norm on $c_0$; $(e_j)$ are the unit vectors; and the duality $\langle x, u_k - e_{a_k} \rangle$ is the $\langle c_0, l^1 \rangle$ duality. Now in view of (1), we have $\|u_k - e_{a_k}\|_1 = 1 + \|u_k\|_1 \leq 1 + a_k$, so

$$\sum_{k=1}^\infty 2^{-a_k^2} \|u_k - e_{a_k}\|_1 \leq \sum_{k=1}^\infty 2^{-a_k^2} (1 + a_k) \leq \sum_{n=1}^\infty (1 + n) \cdot 2^{-n^2} < 2.$$  Accordingly, we have

$$\|x\|_0 \leq \|x\| \leq 3\|x\|_0 \quad (4)$$

for all $x \in c_0$. For our main theorem in this paper, we shall show:

Theorem 1.1. The Banach space $(c_0, \|\cdot\|)$ has no proximinal subspace $H$ of finite codimension $n \geq 2$.  

2
2 Gâteaux derivatives

Recall that if \( X \) is a real vectorspace, \( u, x \in X \) and \( f : X \to \mathbb{R} \), then the Gâteaux derivative (of \( f \), at \( x \), in direction \( u \)) is defined as

\[
df(x; u) = \lim_{h \to 0} \frac{f(x + hu) - f(x)}{h},
\]

when that limit exists. We will make use of the one-sided forms of this derivative:

\[
df^+(x; u) = \lim_{h \to 0^+} \frac{f(x + hu) - f(x)}{h},
\]

and

\[
df^-(x; u) = \lim_{h \to 0^-} \frac{f(x + hu) - f(x)}{h}.
\]

Obviously \( df^-(x; u) = -df^+(x; -u) \) for all \( f, x \) and \( u \) such that either derivative exists. Of particular interest to us is when \( X = c_0 \) and \( f(x) = \|x\| \) as defined in (3) (the “usual” norm on \( c_0 \) will always be referred to as \( \|\cdot\|_0 \) in this paper). The derivatives \( d^\pm f(x; u) \) for this function \( f \) will be written \( d^\pm \|x; u\| \). Now it is a fact that the derivative \( d^\pm \|x; u\| \) exists everywhere. To see this, let us prove some small lemmas.

**Lemma 2.1** If \( \|x\|_0 \) denotes the \( c_0 \)-norm, the derivative \( d^+ \|x; u\|_0 \) exists at all points \( x, u \in c_0 \). In fact, if \( x = 0 \) then the derivative is \( \|u\|_0 \); whereas if \( x \neq 0 \), we may write \( E_+ = \{n \in \mathbb{N} : |x_n| = \|x\|_0, u_n x_n > 0\} \) and \( E_- = \{n \in \mathbb{N} : |x_n| = \|x\|_0, u_n x_n \leq 0\} \), and we have

\[
d^+ \|x; u\|_0 = \begin{cases} \max\{|u_n| : n \in E_+\}, & \text{if } E_+ \neq \emptyset; \\ -\min\{|u_n| : n \in E_-\}, & \text{if } E_+ = \emptyset, E_- \neq \emptyset. \end{cases}
\]

**Proof.** This is an easy calculation which we omit (note that \( E_+ \) and \( E_- \) cannot both be empty!).

**Lemma 2.2** Let \( X \) be a Banach space and \( \varphi \in X^* \). Then the Gâteaux derivative of \( f(x) = |\varphi(x)| \) exists at all points \( (x; u) \in X \times X \). We have

\[
d^+ f(x; u) = f(u)\sigma(u\varphi(x)),
\]

where the sign

\[
\sigma(t) = \begin{cases} +1, & \text{if } t \geq 0; \\ -1, & \text{if } t < 0. \end{cases}
\]

**Proof.** This is an even simpler calculation, which we also omit.

**Definition 2.3** For a real normed space \( X \) and a function \( f : X \to \mathbb{R} \), define the Lipschitz constant

\[
\text{Lip}_1 f = \sup\{ \frac{|f(x) - f(y)|}{\|x - y\|} : x, y \in X, x \neq y \}.
\]
Lemma 2.4 Let $X$ be a real normed space, and $(f_n)_{n=0}^\infty$ a sequence of functions from $X$ to $\mathbb{R}$, such that $d_+f_n(x;u)$ exists at each $(x;u) \in X \times X$. Suppose $\sum_{n=0}^\infty Lip_1f_n < \infty$, and $\sum_{n=0}^\infty f_n(0)$ converges. Then the function $f = \sum_{n=0}^\infty f_n$ exists everywhere on $X$, and $d_+f = \sum_{n=0}^\infty d_+f_n$ exists everywhere also.

Proof. The sum $\sum_{n=0}^\infty f_n(x)$ converges because $\sum_{n=0}^\infty f_n(0)$ converges, and $|f_n(x) - f_n(0)| \leq \|x\|\cdot Lip_1f_n$ so $\sum_{n=0}^\infty f_n(x) - f_n(0)$ converges also. Since $|d_+f_n(x;u)| \leq \|u\|\cdot Lip_1f_n$, we find that the sum $\sum_{n=0}^\infty d_+f_n(x;u)$ converges; we claim the sum is $d_+f(x;u)$. For given $x,u \neq 0$, and $\varepsilon > 0$, we can choose $N$ so large that $\|u\| \cdot \sum_{n=N+1}^\infty Lip_1f_n < \varepsilon/3$, so
\[
|\sum_{n=N+1}^\infty d_+f_n(x;u)| < \varepsilon/3
\] and for every $h > 0$,
\[
|\sum_{n=N+1}^\infty (f_n(x + hu) - f_n(x))/h| \leq \|u\| \cdot \sum_{n=N+1}^\infty Lip_1f_n < \varepsilon/3
\]
also. As $h \to 0^+$, we know $(\sum_{n=1}^N f_n(x + hu) - \sum_{n=1}^N f_n(x))/h \to \sum_{n=1}^N d_+f_n(x;u)$, so we can choose $\delta > 0$ such that whenever $0 < h < \delta$, we have
\[
|\sum_{n=1}^N f_n(x + hu) - \sum_{n=1}^N f_n(x))/h - \sum_{n=1}^N d_+f_n(x;u)| < \varepsilon/3.
\]
Adding up (11), (12) and (13), we find that whenever $0 < h < \delta$, we have $|(f(x + hu) - f(x))/h - \sum_{n=1}^\infty d_+f_n(x;u)| < \varepsilon$. This completes the proof. \hfill \Box

Corollary 2.5 The new norm $\|\cdot\|$ on $\ell_0$ has a one-sided derivative $d_+\|x;u\|$ everywhere. Furthermore,
\[
d_+\|x;u\| = d_+\|x;u\|_0 + \sum_{k=1}^\infty 2^{-a_k^2} \sigma_k[(u,u_k - e_{a_k})],
\] where $\sigma_k = \sigma_k(x;u) = \sigma((u,u_k - e_{a_k}),(x,u_k - e_{a_k}))$, and the function $\sigma$ is as in (7).

Proof. If we write $f_0(x) = \|x\|_0$ and $f_k(x) = 2^{-a_k^2}\|x,u_k - e_{a_k}\|$, then the Lipschitz constants for $f_k$ are 1 (if $k = 0$) or $2^{-a_k^2}\|u_k - e_{a_k}\|_1 \leq (1 + a_k)\cdot 2^{-a_k^2}$ for $k > 0$. Accordingly $\sum_{k=0}^\infty Lip_1f_k < \infty$, and the derivatives $d_+f_k$ are given by Lemma 2.4 and Lemma 2.2. We have $\|x\| = \sum_{k=0}^\infty f_k(x)$ so $d_+\|x;u\| = \sum_{n=0}^\infty d_+f_n(x;u)$ by Lemma 2.4. This sum works out to expression (14). \hfill \Box

The key link between Gâteaux derivatives and proximinality is as follows:
Lemma 2.6 Suppose $(X, \|\cdot\|)$ is a Banach space, $H \subset X$ a subspace, and suppose that for some $x \in X \setminus H$, and $v \in H$, the Gâteaux derivatives $d_\pm \|x; v\|$ both exist, are nonzero, and have the same sign. Then $\|v\| = \inf\{\|y\| : y \in x + H\}$. $x$ is not a closest point to zero in the coset $x + H$.

Proof. We may consider $y = x + hv$ for small nonzero $h \in \mathbb{R}$. Depending on the sign of $h$, the norm $\|y\|$ is roughly $\|x\| + h \cdot d_\pm \|x; v\|$. But the signs of $d_\pm \|x; v\|$ are the same, so if $h$ is chosen correctly, we get $\|y\| < \|x\|$. □

Corollary 2.7 Suppose $H \subset X$ as in Lemma 2.6 and there is an $x \in X \setminus H$ such that for every $z \in H$, there is a $v \in H$ such that the Gâteaux derivatives $d_\pm \|x; zv\|$ exist, are nonzero, and have the same sign. Then $H$ is not proximinal in $X$.

Proof. For in this case, there is no element $x + z \in x + H$ which achieves the minimum distance from that coset to zero. Equivalently, there is no element $z \in H$ which achieves the minimum distance from $H$ to $-x$. $H$ is not proximinal. □

3 Approximate linearity of $d_\pm$

It is a feature of the Gâteaux derivative $df(x; u)$ that it does not have to be linear in $u$. This is of course also true of the single-sided derivatives $df_\pm$. So, in this section we develop a result asserting “approximate linearity” of $d_\pm \|x; v\|$ for $x, v \in c_0$.

Definition 3.1 Let $f : c_0 \to \mathbb{R}$ be such that $d_\pm f(x; v)$ exists for all $x, v \in c_0$. Let $x \in c_0$, and let $\gamma \in l^1$ be such that the support $E = \{i : \langle e_i, \gamma \rangle \neq 0\}$ is infinite. We shall say $d_\pm f(x)$ is approximately linear on $E$ (and approximately equal to $\gamma$) if there is an “error sequence” $(\varepsilon_i)_{i \in E}$ with $\varepsilon_i > 0$, $\varepsilon_i \to 0$, such that for all $v \in c_0$ with $supp v \subset E$, we have

$$|d_\pm f(x; v) - \langle v, \gamma \rangle| \leq \sum_{i \in E} \varepsilon_i |v_i \gamma_i|.$$  \hfill (15)

Note that if $v$ is chosen so that $|\langle v, \gamma \rangle| > \sum \varepsilon_i |v_i \gamma_i|$, then (15) implies that $d_\pm f(x; v)$ is nonzero, and has the same sign as $d_- f(x; v) = -d_+ f(x; v)$.

Lemma 3.2 Let $x \in c_0$ be given, and $z_1, \ldots, z_m \in c_0$ such that $\langle x, z_j \rangle \neq 0$ for any $j = 1, \ldots, m$. Let $A_{z_j} = A_{u^{-1}(z_j)}$ as in (17) and let $A = \bigcup_{j=1}^m A_{z_j}$. Then $d_\pm \|x\|$ is approximately linear on a cofinite subset $A_0 \subset A$, the derivative being approximately equal to $\gamma = \sum_{i=1}^\infty \gamma_i e_i^*$, where

$$\gamma_i = \begin{cases} -2^{-a_i^2} \sigma(\langle x, z_j \rangle) & \text{if } i = a_k \in A_0 \cap A_{z_j} \\ 0 & \text{otherwise.} \end{cases}$$  \hfill (16)
The error sequence \((\varepsilon_i)_{i \in A_0}\) can be taken to be
\[
\varepsilon_i = 2^{-a_i^2} \sum_{l=k+1}^{\infty} 2^{-a_l^2}, \quad i = a_k \in A_0.
\tag{17}
\]

**Proof.** Let \(\mathbf{v}\) be any vector supported on \(A\). The error \(\delta = d_+ \|\mathbf{x}; \mathbf{v}\|_0 - \langle \mathbf{v}, \gamma \rangle\) is given by (14) and (15); we have
\[
\delta = d_+ \|\mathbf{x}; \mathbf{v}\|_0 + \sum_{k \in \mathbb{N} \cup \{m\}} 2^{-a_k^2} \sigma_k \langle \mathbf{v}, \mathbf{u}_k - e_{a_k} \rangle
\]
\[
+ \sum_{j=1}^{m} \sum_{k \in u^{-1}\{z_j\}} 2^{-a_k^2} \sigma_k \langle \mathbf{v}, \mathbf{u}_k - e_{a_k} \rangle + v_{a_k} \sigma(\langle \mathbf{x}, \mathbf{z}_j \rangle) \tag{18}
\]
where \(\sigma_k = \sigma(\langle \mathbf{v}, \mathbf{u}_k - e_{a_k} \rangle, \langle \mathbf{x}, \mathbf{u}_k - e_{a_k} \rangle)\).

Now by Lemma 2.1, the derivative \(d_+ \|\mathbf{x}; \mathbf{v}\|_0\) is zero unless \(v_i \neq 0\) for some \(i \in E = \{n : |x_n| = \|\mathbf{x}\|_0\}\). This set \(E\) is finite, and \(\mathbf{v}\) will be supported on \(A_0\); so if we choose our cofinite set \(A_0 \subset A\) so that \(A_0 \cap E = \emptyset\), we have \(d_+ \|\mathbf{x}; \mathbf{v}\|_0 = 0\). If we also ensure that \(A_0 \cap \text{supp} \mathbf{z}_j = \emptyset\) for each \(j = 1, \ldots, m\), we find that when \(\mathbf{v}\) is supported on \(A_0\), and \(k \in u^{-1}\{z_j\}\), we have
\[
\langle \mathbf{v}, \mathbf{u}_k - e_{a_k} \rangle = \langle \mathbf{v}, \mathbf{z}_j - e_{a_k} \rangle = -\langle \mathbf{v}, e_{a_k} \rangle = -v_{a_k}. \tag{19}
\]
So if we choose \(A_0\) so that \(A_0 \cap (E \cup \{z_j\}) = \emptyset\), the expression (18) simplifies somewhat to
\[
\delta = \sum_{k \in \mathbb{N} \cup \{m\}} 2^{-a_k^2} \sigma_k \langle \mathbf{v}, \mathbf{u}_k - e_{a_k} \rangle
\]
\[
+ \sum_{j=1}^{m} \sum_{k \in u^{-1}\{z_j\}} \sigma_k |v_{a_k}| + v_{a_k} \sigma(\langle \mathbf{x}, \mathbf{z}_j \rangle) \tag{20}
\]
and \(\sigma_k\) itself simplifies to \(\sigma_k = \sigma(-\langle \mathbf{x}, \mathbf{z}_j - e_{a_k} \rangle, v_{a_k})\). Now \(F_j = \{k \in \mathbb{N} : |\langle \mathbf{x}, e_{a_k} \rangle| \geq |\langle \mathbf{x}, \mathbf{z}_j \rangle|\}\) is a finite set; we may thus also assume that \(A_0\) does not meet any \(F_j\). In that case, \(\langle \mathbf{x}, \mathbf{z}_j - e_{a_k} \rangle\) is nonzero and has the same sign as \(\langle \mathbf{x}, \mathbf{z}_j \rangle\), so \(\sigma_k = -v_{a_k} \sigma(\langle \mathbf{x}, \mathbf{z}_j \rangle)\). So the second term in (20) disappears, and we have
\[
\delta = \sum_{k \in \mathbb{N} \cup \{m\}} 2^{-a_k^2} \sigma_k \langle \mathbf{v}, \mathbf{u}_k - e_{a_k} \rangle \tag{21}
\]
Even better, \(\mathbf{v}\) is supported on \(A = a \cdot \cup_{j=1}^{m} u^{-1}\{z_j\}\), so all terms \(\langle \mathbf{v}, e_{a_k} \rangle\) are zero in (21), and we have
\[
\delta = \sum_{l \in \mathbb{N} \cup \{m\}} 2^{-a_l^2} \sigma_l |\langle \mathbf{v}, \mathbf{u}_l \rangle| \tag{22}
\]
\[
|\delta| \leq \sum_{i \in A_0} \sum_{l \in \mathbb{N} \cup \{m\}} 2^{-a_l^2} |v_i| \cdot |\langle \epsilon_i, \mathbf{u}_l \rangle|. \tag{22}
\]
Now in every case when \( i \in A_0 \) we have \( i = a_k \) for some \( k \in \mathbb{N} \). If \( \langle e_{a_k}, u_i \rangle \neq 0 \) then the support \( \text{supp} \, u_i \) is not contained in \([0,a_k)\). But if \( l \leq k \) then the support of \( u_l \) is contained in \([0,a_k)\) by (11). So \( k < l \) in every case when \( \langle e_{a_k}, u_i \rangle \neq 0 \). Accordingly,

\[
|\delta| \leq \sum_{i = a_k \in A_0} \sum_{l > k} 2^{-a_i^2} |e_i| \cdot |\langle e_i, u_i \rangle| \leq \sum_{i = a_k \in A_0} \sum_{l > k} 2^{-a_i^2} \cdot a_l \cdot |v_i|
\]

because \( \|u_i\|_1 \leq a_l \) by (11) again. Now for \( i = a_k \in A_0 \), we have \( |\gamma_i| = 2^{-a_i^2} \) by (16); so writing \( \varepsilon_i = 2^{-a_i^2} \cdot \sum_{l = k+1} \|2^{-a_l^2} \) as in (17), we have \( \varepsilon_i \to 0 \) and

\[
|\delta| = |d_+ \|x; v\| - \langle v, \gamma \rangle| \leq \sum_{i \in A_0} \varepsilon_i |v_i \gamma_i|
\]

exactly as in (15). So \( d_+ \|x\| \) is approximately linear on a cofinite subset \( A_0 \subset A \), with the derivative \( \gamma \in L^1 \) given by (19), and the error sequence \( \varepsilon_i \) given by (17). \( \square \)

4 Using the Hahn-Banach Theorem.

**Lemma 4.1** Let \( H \subset c_0 \) be a closed subspace of finite codimension, say \( H = \cap_{i=1}^N \ker \varphi_i \), where each \( \varphi_i \in L^1 \). Let \( x \notin H \) be an element of minimum norm in the coset \( x + H \), and let \( z_1, \ldots, z_m \in c_0 \) be such that \( \langle x, z_j \rangle \neq 0 \) for any \( j = 1, \ldots, m \). Let \( A_0 \subset A = \bigcup_{j=1}^m A_{z_j} \) be a cofinite subset satisfying the conditions of Lemma 3.3 and let \( \gamma \in L^1 \) be the approximate derivative as in Lemma 3.3 \( (\varepsilon_i)_{i \in A_0} \) the error sequence as in (17). Then there is a \( \varphi \in \text{lin}\{\varphi_j : i \leq j \leq m\} \) such that for every \( i \in A_0 \), we have

\[
|\langle e_i, \varphi \rangle - \gamma_i| \leq \varepsilon_i |\gamma_i|.
\]  

**Proof.** We consider the weak-* topology on \( l^1 \) with respect to its usual predual, \( c_0 \). The set \( G = \{\varphi \in L^1 : |\langle e_i, \varphi \rangle - \gamma_i| \leq \varepsilon_i |\gamma_i| \text{ (all } i \in A_0)\}, \) and \( \langle e_i, \varphi \rangle = 0 \) (all \( i \notin A_0 \)) is a weak-* compact convex set. The set \( \Phi = \text{lin}\{\varphi_i, i = 1, \ldots, N\} + \text{lin}\{e_j : j \in \mathbb{N} \setminus A_0\} \subset L^1 \) is a weak-* closed subspace, because it is \( \{\varphi \in L^1 : \varphi(u) = 0 \text{ for every } u \in c_0 \text{ supported on } A_0, \text{ such that } \varphi(u) = 0 \text{ (i = 1, \ldots, N)}\} \).

If \( \Phi \cap G \neq \emptyset \), then the assertion of the Lemma is satisfied. If \( \Phi \cap G = \emptyset \), then the Hahn-Banach Separation Lemma tells us that there is a weak-* continuous \( v \in L^\infty \) separating them; of course the weak-* continuity means that \( v \in c_0 \). We may assume \( \langle \varphi, v \rangle = 0 \) for \( \varphi \in \Phi \), but \( \langle \varphi, v \rangle \geq 1 \) whenever \( \varphi \in G \). Since \( v \) annihilates \( \Phi \), the support of \( v \) is contained in \( A_0 \). By approximate linearity of \( d_+ \|x\| \), from (15) we have

\[
|d_+ \|x; v\| - \langle v, \gamma \rangle| \leq \sum_{i \in A_0} \varepsilon_i |v_i \gamma_i|;
\]

and the same is true with \( d_+ \) replaced by \( d_- \). We cannot have \( d_+ \|x; v\| \) and \( d_- \|x; v\| \) the same sign, or Lemma 2.3 would tell us \( x \) does not
have minimum norm in the coset \( x + H \). So, as observed after (15), we must have

\[ |\langle v, \gamma \rangle| \leq \sum_{i \in A_0} \varepsilon_i|v_i \gamma_i|. \]  

(25)

Let us write \( \eta = \langle v, \gamma \rangle/\sum_{i \in A_0} \varepsilon_i|v_i \gamma_i| \in [-1, 1] \) (noting that the denominator cannot be zero since \( \varepsilon_i, \gamma_i \) are never zero for \( i \in A_0 \), and \( v \neq 0 \) is supported on \( A_0 \)). Define a new \( \varphi \in l^1 \) by

\[ \langle \varepsilon_i, \varphi \rangle = \begin{cases} 
\gamma_i(1 - \eta \varepsilon_i \sigma(v_i \gamma_i)), & \text{if } i \in A_0; \\
0, & \text{otherwise.} 
\end{cases} \]  

(26)

We then have \( |\langle \varepsilon_i, \varphi \rangle - \gamma_i| \leq \varepsilon_i|\gamma_i| \) (\( i \in A_0 \)), so \( \varphi \in G \), yet

\[ \langle v, \varphi \rangle = \langle v, \gamma \rangle - \eta \cdot \sum_{i \in A_0} \varepsilon_i v_i \gamma_i \sigma(v_i \gamma_i) = 0. \]  

(27)

This contradicts the Hahn-Banach separation of \( v \), which asserts that for such \( \varphi \) we should have \( \langle v, \varphi \rangle \geq 1 \). Thus the Lemma is proved. \( \square \)

Let us now begin to use our information to investigate proximinal subspaces. If \( i = a_l \) for some \( l \in \mathbb{N} \), we shall write \( \alpha_l = 2^{-s_l} \).

**Theorem 4.2** Let \( H \subset (c_0, \| \cdot \|) \) be a proximinal subspace of finite codimension, say \( H = \cap_{i=1}^N \ker \varphi_i \), where \( \varphi_i \in l^1 \). Let \( z_j \in c_0 \) \( (j = 1, \ldots, m) \), and \( A = \cup_{i=1}^m A_{z_i} \). Write \( \Phi = \text{lin}\{\varphi_i : i = 1, \ldots, N\} \), and let

\[ \Phi_0 = \{ \varphi \in \Phi : \sup\{\alpha_i^{-1}|\langle e_i, \varphi \rangle| : i \in A\} < \infty \}. \]  

(28)

Let \( \theta_0 : \Phi_0 \to l^\infty(A) \) be the linear map such that

\[ \langle \theta_0 \varphi_i \rangle = \alpha_i^{-1}\langle e_i, \varphi \rangle \ (i \in A); \]  

(29)

and let \( q : l^\infty(A) \to l^\infty(A)/c_0(A) \) be the quotient map. Write \( \theta = q \theta_0 \). Let \( x \in H \) be an element such that \( \|x\| \) is minimal in the coset \( x + H \), and suppose \( \langle x, z_j \rangle \neq 0 \) for any \( i = 1, \ldots, m \). Then the image \( \theta \Phi_0 \) includes the vector \( \sigma_x + c_0(A) \) in \( l^\infty(A)/c_0(A) \), where

\[ \langle \sigma_{x_i} \rangle = \sigma(\langle x, z_j \rangle) \text{ if } i \in A_{z_i}, j = 1, \ldots, m. \]  

(30)

**Proof.** By Lemma 1.1 there is a \( \varphi \in \Phi \) such that for all but finitely many \( i \in A \), we have \( |\langle e_i, \varphi \rangle - \gamma_i| \leq \varepsilon_i|\gamma_i| \), where for \( i = a_l \in A_{z_j} \) we define \( \gamma_i = -2^{-s_l} \sigma(\langle x, z_j \rangle) = -\alpha_l \sigma(\langle x, z_j \rangle) \). Since \( \varepsilon_i \to 0 \) it is clear that \( \sup\{\alpha_i^{-1}|\langle e_i, \varphi \rangle| \} < \infty \), so \( \varphi \in \Phi_0 \), and the image \( \theta \varphi \) is the vector \( -\sigma_x + c_0(A) \), because for \( i = a_l \in A_{z_j} \) we have \( \langle \theta_0 \varphi_i \rangle = \alpha_l^{-1}\langle e_i, \varphi \rangle \in -\sigma(\langle x, z_j \rangle) + [-\varepsilon_i, \varepsilon_i] \). So, \( \theta(-\varphi) = \sigma_x + c_0(A) \). \( \square \)

5 Proof of Theorem 1.1

Suppose towards a contradiction that \( H \subset (c_0, \| \cdot \|) \) is a proximinal subspace of finite codimension \( N \geq 2 \). Any proximinal subspace must be closed, so let us say \( H = \cap_{i=1}^N \ker \varphi_i \), where the \( \varphi_i \in l^1 \) are linearly independent. For \( r = 0, \ldots, N + 1 \), let us write \( \beta_r = r\pi/(2N + 2) \), and
for \( r = 1, \ldots, N + 1 \) let us pick \( x^{(r)} \in c_0 \) such that \( \langle x^{(r)}, \varphi_2 \rangle = \cos \beta_r \), and \( \langle x^{(r)}, \varphi_2 \rangle = \sin \beta_r \). Perturbing each \( x^{(r)} \) by an element of \( H \) as necessary, we can assume that each \( \|x^{(r)}\| \) is minimal in the coset \( x^{(r)} + H \). Writing \( \zeta_r = (\beta_r + \beta_{r-1})/2 \) (\( r = 1, \ldots, N + 1 \)), we define the linear functional \( \psi_r = \sin \zeta_r \cdot \varphi_1 - \cos \zeta_r \cdot \varphi_2 \), so

\[
\langle x^{(r)}, \psi_s \rangle = \sin \zeta_s \cos \beta_r - \cos \zeta_s \sin \beta_r = \sin(\zeta_s - \beta_r);
\]

thus \( \langle x^{(r)}, \psi_s \rangle > 0 \) if \( s > r \), but \( \langle x^{(r)}, \psi_s \rangle < 0 \) if \( s \leq r \).

Pick a finite sequence \( (z_r)_{r=1}^{N+1} \subset c_{00} \) with \( \|z_r - \psi_r\|_1 \) sufficiently small, and we will also find that \( \langle x^{(r)}, z_s \rangle > 0 \) if \( s > r \), but \( \langle x^{(r)}, z_s \rangle < 0 \) if \( s \leq r \). We find that the sequence \( (\sigma((x^{(r)}, z_s)))_{s=1}^{N+1} \) is the vector \( y_r = (-1, -1, \ldots, -1, 1, 1, \ldots, 1) \), where there are \( r \) entries \(-1\) followed by \( N + 1 - r \) entries \(+1\). It is a fact that the \( y_s \) span \( \mathbb{R}^{N+1} \) - they are linearly independent.

We can apply Theorem \([4, 2]\) with the sequence \( z_1, \ldots, z_{N+1} \), and \( x \) can be any of the vectors \( x^{(1)}, \ldots, x^{(N+1)} \). The map \( \theta \) is the same for each \( x^{(r)} \) (because the sequence \( \alpha_i \) doesn’t change, only the signs \( \sigma((x^{(r)}, z_s))) \)). Writing \( A = \bigcup_{j=1}^{N+1} A_{x_j} \), we find that the image \( \theta \Phi_0 \) must contain, for each \( r = 1, \ldots, N + 1 \), the vector \( \sigma_{x^{(r)}} + c_0(A) \) with

\[
(\sigma_{x^{(r)}})_i = \sigma(\langle x^{(r)}, z_j \rangle) = (y_r, e_j) \text{ for all } i \in A_{x_j},
\]

(where here \( (e_j)_{j=1}^{N+1} \) denote the unit vector basis of \( \mathbb{R}^{N+1} \)). Because the vectors \( y_s \) are independent, the dimension of \( \theta \Phi_0 \) must at least \( N + 1 \). However \( \Phi_0 \subset \Phi \), and \( \dim \Phi = N \). This contradiction implies that \( H \) is not proximinal. \( \Box \)

References

[1] Bishop, E. and Phelps, R. R., A proof that every Banach space is subreflexive, Bull. Amer. Math. Soc. 67 (1961), 97-98.

[2] Blecher, David P. and Read, Charles John, Operator algebras with contractive approximate identities, II. J. Funct. Anal. 264 (2013), no. 4, 1049-1067.

[3] Bollobás, B., An extension to the theorem of Bishop and Phelps, Bull. London Math. Soc. 2 (1970) 181-182.

[4] Gamelin, T.W., Uniform Algebras, second edition, Chelsea, New York, 1984.

[5] Godefroy, Gilles, Existence and Uniqueness of isometric preduals: a survey. Contemporary Mathematics 85 (1989) 131-193.

[6] Singer, Ivan The theory of best approximation and functional analysis. Conference Board of the Mathematical Sciences Regional Conference Series in Applied Mathematics 13. Society for Industrial and Applied Mathematics, Philadelphia, Pa., 1974.