Multiple gcd-closed sets and determinants of matrices associated with arithmetic functions

Abstract: Let $f$ be an arithmetic function and $S = \{x_1, \ldots, x_n\}$ be a set of $n$ distinct positive integers. By $(f(x_i, x_j))$ (resp. $(f[x_i, x_j])$) we denote the $n \times n$ matrix having $f$ evaluated at the greatest common divisor $(x_i, x_j)$ (resp. the least common multiple $[x_i, x_j]$) of $x_i$ and $x_j$ as its $(i, j)$-entry, respectively. The set $S$ is said to be gcd closed if $(x_i, x_j) \in S$ for $1 \leq i, j \leq n$. In this paper, we give formulas for the determinants of the matrices $(f(x_i, x_j))$ and $(f[x_i, x_j])$ if $S$ consists of multiple coprime gcd-closed sets (i.e., $S$ equals the union of $S_1, \ldots, S_k$ with $k \geq 1$ being an integer and $S_1, \ldots, S_k$ being gcd-closed sets such that $(\text{lcm}(S_i), \text{lcm}(S_j)) = 1$ for all $1 \leq i \neq j \leq k$). This extends the Bourque-Ligh, Hong’s and the Hong-Loewy formulas obtained in 1993, 2002 and 2011, respectively. It also generalizes the famous Smith’s determinant.

Keywords: Matrix associated with arithmetic function, Determinant, Multiple coprime gcd-closed sets, Smith’s determinant

MSC: 11C20, 11A05, 15B36

1 Introduction and statements of main results

Let $n$ be a positive integer and $f$ be an arithmetic function. Let $S = \{x_1, \ldots, x_n\}$ be a set of $n$ distinct positive integers. We denote by $(f(S)) = (f(x_i, x_j))$ and $(f(S)) = (f[x_i, x_j])$ the $n \times n$ matrices having $f$ evaluated at the greatest common divisor $(x_i, x_j)$ and the least common multiple $[x_i, x_j]$ of $x_i$ and $x_j$ as their $(i, j)$-entries, respectively. In 1875, Smith [25] published his famous result stating that det$(f(S)) = \prod_{i=1}^{n} (f \ast \mu)(x_i)$ if $S$ is factor closed (i.e., $d \in S$ if $x \in S$ and $d \mid x$), where $f \ast \mu$ is the Dirichlet convolution of $f$ and the Möbius function $\mu$. Since then this topic has received a lot of attention from many authors and particularly became extremely active in the past decades (see, for example, [1]-[7], [9]-[23] and [26]-[28]).

In 1989, Beslin and Ligh [3] extended Smith’s result by showing that det$(S) = \prod_{l=1}^{n} \sum_{d \mid x_l} \varphi(d)$ if $S$ is gcd closed (i.e., $(x_i, x_j) \in S$ for all integers $i$ and $j$ with $1 \leq i, j \leq n$). In 1992, Bourque and Ligh [4] proved that if $S$ is gcd closed, then det$[S] = \prod_{l=1}^{n} x_l^2 \sum_{d \mid x_l} g(d)$ with $g$ being the multiplicative function defined by $g(m) := \frac{1}{m} \sum_{d \mid m} d \mu(d)$. 

Siao Hong: Center for Combinatorics, Nankai University, Tianjin 300071, China, E-mail: sahongnk@gmail.com
Shuangnian Hu: School of Mathematics and Statistics, Nanyang Institute of Technology, Nanyang 473004, China, E-mail: hushuangnian@163.com
*Corresponding Author: Shaofang Hong: Mathematical College, Sichuan University, Chengdu 610064, China, E-mail: sfhong@scu.edu.cn, s-lhong@tom.com, hongsf02@yahoo.com

© 2016 Hong et al., published by De Gruyter Open. This work is licensed under the Creative Commons Attribution-NonCommercial-NoDerivs 3.0 License.
In 1993, Bourque and Ligh [5] generalized Smith’s determinant and Beslin and Ligh’s result [3] by proving that if $S$ is gcd closed, then $\det(f(S)) = \prod_{x \in S} \alpha_{S,f}(x)$, where

$$\alpha_{S,f}(x) := \sum_{d \mid x \land y \in S} (f * \mu)(d).$$

In 2002, Hong [13] extended the Bourque-Ligh result by showing that $\det(f[S]) = \prod_{i=1}^{n} f(x_i)^2 \alpha_{S,f}(x_i)$ if $S$ is gcd closed, where $\frac{1}{f(x)} := \frac{1}{f(x)}$, $f$ is multiplicative and $f(x) \neq 0$ for all $x \in S$.

We say that $S$ consists of multiple coprime gcd-closed sets if there is a positive integer $h$ and $h$ distinct gcd-closed sets $S_1, \ldots, S_h$ with $(\text{lcm}(S_i), \text{lcm}(S_j)) = 1$ for all integers $i$ and $j$ with $1 \leq i \neq j \leq h$ such that $S$ can be partitioned as the union of $S_1, \ldots, S_h$ (see, for instance, [15]). Clearly, if $S$ consists of multiple coprime gcd-closed sets, then either we have $1 \in S$ or $1 \notin S$. For the former case $1 \in S$, $S$ is gcd closed and the formulas for determinants of the matrices $(f(S))$ and $(f[S])$ were given by Bourque and Ligh [5] and Hong [13], respectively. For the latter case $1 \notin S$, the formulas for determinants of the matrices $(f(S))$ and $(f[S])$ are unknown. This problem is still kept open so far.

In this paper, our main goal is to introduce a new method to investigate the above problem. Actually, we first give the formula for the determinant of $(f(S))$ on any positive integers set $S$. Then we present formulas for the determinants of the matrices $(f(S))$ and $(f[S])$ on the multiple coprime gcd-closed sets $S$. Evidently, any rearrangement of the elements of $S$ yields matrices similar to the matrices $(f(S))$ and $(f[S])$. So we can rearrange the elements of $S$ in any case of necessity. To give the main result, we need two concepts as follows.

**Definition 1.1.** Let $f$ be an arithmetic function and $T$ be a set of distinct positive integers. Then associated to $f$, we define the functions $s^{(1)}_f$ and $s^{(2)}_f$ on the set $T$ as follows:

$$s^{(1)}_f(T) := f(1) \sum_{y \in T} \prod_{z \in T \setminus \{y\}} (f(z) - f(1)) + \prod_{y \in T} (f(y) - f(1)).$$

$$s^{(2)}_f(T) := (-1)^{|T|+1} \left( \sum_{y \in T} \prod_{z \in T \setminus \{y\}} (f(z) - 1) + (|T| - 1) \prod_{y \in T} (f(y) - 1) \right) \prod_{y \in T} f(y).$$

**Definition 1.2.** Let $S$ consist of $h$ coprime gcd-closed sets $S_1, \ldots, S_h$. Then we define the set of minimal elements of $S$, denoted by $M(S)$, to be $M(S) := \{\min(S_i)\}^{h}_{i=1}$, where $\min(S_i)$ stands for the smallest element of $S_i$.

For example, if $S = \{2, 5, 6, 8, 11, 35, 143\}$, then $S$ consists of three coprime gcd-closed sets and the set $M(S)$ of minimal elements of $S$ is equal to $\{2, 5, 11\}$.

Now we can state the main result of this paper.

**Theorem 1.3.** Let $f$ be an arithmetic function. Let $S$ consist of multiple coprime gcd-closed sets such that $1 \notin S$ and $M(S)$ denote the set of minimal elements of $S$. Then

$$\det(f(S)) = s^{(1)}_f(M(S)) \prod_{x \in S \setminus M(S)} \alpha_{S,f}(x).$$

Furthermore, if $f$ is a multiplicative function and $f(x) \neq 0$ for all $x \in S$, then

$$\det(f[S]) = s^{(2)}_f(M(S)) \prod_{x \in S \setminus M(S)} f(x)^2 \alpha_{S,f}(x).$$

If letting $S$ be a gcd-closed set, then Theorem 1.3 reduces to the Bourque-Ligh theorem [5] and Hong’s theorem [13]. If $S$ consists of coprime divisor chains, then Theorem 1.3 becomes the main result of [18]. From Theorem 1.3, one can easily deduce the following interesting consequence.
Corollary 1.4. Let $S$ consist of multiple coprime gcd-closed sets and $M(S)$ denote the set of minimal elements of $S$. Then

$$\det((S)) = s_{(1)}^{(1)}(M(S)) \prod_{x \in S \setminus M(S)} \sum_{d \mid x, d \not\mid y} \psi(d)$$

and

$$\det([S]) = s_{(2)}^{(1)}(M(S)) \prod_{x \in S \setminus M(S)} x^2 \sum_{d \mid x, d \not\mid y} g(d),$$

where $I$ is the arithmetic function defined by $I(n) := n$.

Obviously, picking $S$ to be a gcd-closed set in Corollary 1.4 gives us the Beslin-Ligh result [3] and the Bourque-Ligh result [4]. If $S = M(S)$, then Corollary 1.4 is Lemma 2.1 of [17].

We organize the paper as follows. In Section 2, we present some lemmas which are needed in the proof of Theorem 1.3. In Section 3, we prove Theorem 1.3 and Corollary 1.4.

2 Several lemmas

In this section, we present some useful lemmas that are needed in the next section. The first two lemmas are well known.

Lemma 2.1 ([13]). Let $f$ be any arithmetic function and $n$ be a positive integer. Then $\sum_{d \mid n} (f * \mu)(d) = f(n)$.

Lemma 2.2 ([24]). Let $m, n$ be any positive integers and $f$ be a multiplicative function. Then $f(m)f(n) = f((m, n))f(\lcm(m, n))$.

Lemma 2.3. Let $g$ be any arithmetic function and $S$ be gcd closed. Then for any $x \in S$, we have

$$\sum_{\substack{y \mid x \setminus \sum_{d \mid x, d \not\mid y, z \in S}} g(d) = \sum_{d \mid x} g(d). \tag{2}$$

Proof. Clearly, the terms in the sum of the right-hand side of (2) are non-repetitive. Now we show that the terms in the sum of the left-hand side of (2) are non-repetitive. For this purpose, for any $y \in S$ with $y \mid x$, we let $D(y) = \{d \in \mathbb{Z}^+ : d \mid y, d \not\mid z, z < y, z \in S\}$. Claim that $D(y_1) \cap D(y_2) = \phi$ for any distinct elements $y_1$ and $y_2$ in the set $S$ satisfying $y_1 \mid x$ and $y_2 \mid x$. Otherwise, we may let $d \in D(y_1) \cap D(y_2)$. Then $d \mid y_1$ and $d \mid y_2$. So $d \mid (y_1, y_2)$. But the assumption that $S$ being gcd closed tells us that $(y_1, y_2) = y_3$ for some $y_3 \in S$. Hence $d \mid y_3$. On the other hand, we have $y_3 \mid y_1$ and $y_3 \mid y_2$ since $y_1 \neq y_2$. It then follows from $d \in D(y_1)$ that $d \nmid y_3$. We arrive at a contradiction. The claim is proved. By the claim we know immediately that the terms in the sum of the left-hand side of (2) are non-repetitive.

For any term $g(d)$ in the sum of the left-hand side of (2), one has $d \mid y, y \mid x$ and $y \in S$. Thus $d \mid x$. This implies that $g(d)$ is a term in the sum of the right-hand side of (2). To show that the converse is true, for any given positive integer $d$ and $x \in S$ with $d \mid x$, let $I(d, x) = \{u : d \mid u, u \mid x, u \in S\}$. Then $I(d, x) \neq \phi$ since $x \in I(d, x)$ and $I(d, x)$ is finite. Let $v = \min(I(d, x))$. Then $v \mid x, v \in S$ and $d \mid v$ and $d \not\mid z$ for any $z \in S$ with $z < v$. It infers that the term $g(d)$ in the sum of the right-hand side of (2) is also a term in the sum of the left-hand side of (2). So (2) is proved.

This ends the proof of Lemma 2.3.

Note that a special case of Lemma 2.3 is due to Beslin and Ligh [3] and a more general form is given in (3.4) of [10].

Lemma 2.4. Let $S$ be gcd closed. Then for any $x \in S$, $\sum_{\substack{y \mid x \setminus \sum_{y \in S}} f(y) = f(x)$. 

Proof. Letting \( g = f * \mu \) in Lemma 2.3 gives us that

\[
\sum_{y \in S} \sum_{d \mid y} \sum_{z \leq y, z \in \mathbb{N}} (f \ast \mu)(d) = \sum_{d \mid x} (f \ast \mu)(d).
\]

Then the desired result follows from the definition of \( \alpha_{S,f}(d) \) and Lemma 2.1. This completes the proof of Lemma 2.4.

We need the following definition to state Lemma 2.6 below.

**Definition 2.5.** Let \( S = \{x_1, \ldots, x_n\} \) be a set of positive integers and \( \tilde{S} = \{y_1, \ldots, y_m\} \) be the minimal gcd-closed set containing \( S \). Then we define the \( n \times m \) matrix \( E(S) = (e_{ij}) \) by

\[
e_{ij} = \begin{cases} 1, & \text{if } y_j \mid x_i, \\ 0, & \text{otherwise}. \end{cases}
\]

For \( 1 \leq l \leq m \), we define \( E_l(S) \) to be the \( n \times (m - 1) \) matrix obtained from \( E(S) \) by deleting its \( l \)th column.

We can now use the gcd-closed set to describe the structure of the matrix \((f(S))\) on any set \( S \) of positive integers.

**Lemma 2.6.** Let \( f \) be an arithmetic function and \( S = \{x_1, \ldots, x_n\} \) be a set of distinct positive integers and \( \tilde{S} = \{y_1, \ldots, y_m\} \) be the minimal gcd-closed set containing \( S \). Then \( (f(S)) = E(S) \cdot \text{diag}(\alpha_{\tilde{S},f}(y_1), \ldots, \alpha_{\tilde{S},f}(y_m)) \cdot E(S)^T \).

**Proof.** Let \( S = \{x_1, \ldots, x_n\} \) and \( \Delta = \text{diag}(\alpha_{\tilde{S},f}(y_1), \ldots, \alpha_{\tilde{S},f}(y_m)) \). Then for any integers \( i \) and \( j \) with \( 1 \leq i, j \leq n \), we have

\[
(E(S) \cap E(S)^T)_{ij} = \sum_{k=1}^{m} e_{ik} \alpha_{\tilde{S},f}(y_k) e_{jk} = \sum_{y_k \mid y_j} \sum_{y_k \mid y_i} \alpha_{\tilde{S},f}(y_k) = \sum_{y_k \mid \{x_i, x_j\}} \alpha_{\tilde{S},f}(y_k).
\]

Since \( \tilde{S} \) is the minimal gcd-closed set containing \( S \), one has \((x_i, x_j) \in \tilde{S}\). Then there exists one element \( y_h \in \tilde{S} \) such that \( y_h = (x_i, x_j) \). It follows that

\[
(E(S) \cap E(S)^T)_{ij} = \sum_{y_k \mid y_h} \alpha_{\tilde{S},f}(y_k)
\]

(3)

But Lemma 2.4 together with the fact that \( \tilde{S} \) being gcd closed implies that

\[
\sum_{y_k \mid y_h} \alpha_{\tilde{S},f}(y_k) = f(y_h) = f(x_i, x_j).
\]

(4)

Thus by (3) and (4), one has \((E(S) \cap E(S)^T)_{ij} = (f(S))_{ij}\) as desired. This completes the proof of Lemma 2.6.

Li [21], Hong [12] and Mattila and Haukkanan [22] made use of the Cauchy-Binet formula to the Smith’s matrices. Now we use this renowned formula to show the following lemma.

**Lemma 2.7.** Let \( f \) be an arithmetic function and \( S = \{x_1, \ldots, x_n\} \) be a set of \( n \) distinct positive integers, and \( \tilde{S} = \{y_1, \ldots, y_m\} \) be the minimal gcd-closed set containing \( S \). Then

\[
\det(f(S)) = \sum_{1 \leq k_1 < \ldots < k_n \leq m} \left( \det(E(S)(k_1, \ldots, k_n)) \right)^2 \prod_{i=1}^{n} \alpha_{\tilde{S},f}(y_{k_i}).
\]

(5)

with \( E(S)(k_1, \ldots, k_n) \) being the \( n \times n \) matrix whose columns are the \( k_1 \)th, \ldots, \( k_n \)th columns of \( E(S) \).
Proof. Let \( A = E(S) \cdot \text{diag}\left(\sqrt{\alpha_{S,f}(y_1)}, \ldots, \sqrt{\alpha_{S,f}(y_m)}\right) \). Then by Lemma 2.6, one has \((f(S)) = AA^T\). Using the Cauchy-Binet formula [8] we get
\[
\det(f(S)) = \sum_{1 \leq k_1 \leq \ldots \leq k_n \leq m} \det A_{(k_1, \ldots, k_n)} \cdot \det A_{(k_1, \ldots, k_n)}^T = \sum_{1 \leq k_1 \leq \ldots \leq k_n \leq m} (\det A_{(k_1, \ldots, k_n)})^2,
\]
where \( A_{(k_1, \ldots, k_n)} \) is the \( n \times n \) matrix whose columns are the \( k_1 \)th, \( k_2 \)th, ..., \( k_n \)th column of \( A \). One can easily check that
\[
A_{(k_1, \ldots, k_n)} = E(S)_{(k_1, \ldots, k_n)} \cdot \text{diag}\left(\sqrt{\alpha_{S,f}(y_{k_1})}, \ldots, \sqrt{\alpha_{S,f}(y_{k_n})}\right).
\]
It then follows that
\[
\det(A_{(k_1, \ldots, k_n)}) = \prod_{i=1}^{n} \alpha_{S,f}(y_{k_i}) \cdot \det(E(S)_{(k_1, \ldots, k_n)}).
\]
So the desired formula (5) follows immediately. This finishes the proof of Lemma 2.7.

In what follows, we write \( S = \bigcup_{i=1}^{h} S_i \) with \( S_i = \{x_{i1}, \ldots, x_{in_i}\}(1 \leq i \leq h) \) being gcd closed and \( 1 < x_{i1} < \ldots < x_{in_i} \) and \( \text{gcd}(\text{lcm}(S_i), \text{lcm}(S_j)) = 1 \) for all integers \( i \) and \( j \) with \( 1 \leq i \neq j \leq h \). That is,
\[
S = \{x_{11}, \ldots, x_{1n_1}, \ldots, x_{h1}, \ldots, x_{hn_h}\}.
\]
Let \( \tilde{S} := S \cup \{1\} = \{x_{11}, \ldots, x_{1n_1}, \ldots, x_{h1}, \ldots, x_{hn_h}, 1\} \). Clearly \( \tilde{S} \) is the minimal gcd-closed set containing \( S \).

**Lemma 2.8.** Let \( S \) be as in (6) and \( t \) be a given integer such that \( 1 \leq t \leq h \). Let \( l_t = n_1 + \ldots + n_t \). Let \( n_t \geq 2 \). Then each of the following is true.

(i) If \( x_{t,n_t-1} \) does not divide \( x_{t,n_t} \), then \( \det(E_{l_t}(S)) = \det(E_{l_{t-1}}(S \setminus \{x_{t,n_t-1}\})) \).

(ii) If \( x_{t,n_t-1} \) divides \( x_{t,n_t} \), then
\[
\det(E_{l_t}(S)) = \det(E_{l_{t-1}}(S \setminus \{x_{t,n_t-1}\})) - \det(E_{l_{t-1}}(S \setminus \{x_{t,n_t}\})).
\]

**Proof.** Since \( S \) is as in (6), by the definition of \( E(S) \) we have
\[
E(S) = \begin{pmatrix}
E_1 & 0 & \cdots & 0 & 1 \\
0 & E_2 & \cdots & 0 & 1 \\
\cdots & \cdots & \cdots & \cdots & \cdots \\
0 & 0 & \cdots & E_h & 1
\end{pmatrix},
\]
where for \( 1 \leq l \leq h \), one has
\[
E_l = \begin{pmatrix}
1 & 0 & 0 & \cdots & 0 & 0 \\
1 & 1 & 0 & \cdots & 0 & 0 \\
1 & e'_{l,2} & 1 & \cdots & 0 & 0 \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
1 & e'_{l-1,2} & e'_{l-1,3} & \cdots & 1 & 0 \\
1 & e'_{l,2} & e'_{l,3} & \cdots & e'_{l,n_l-1} & 1
\end{pmatrix}
\]
with \( e'_{ij} \) (\( 1 \leq i, j \leq n_l \)) being defined as
\[
e'_{ij} = \begin{cases}
1, & \text{if } x_{lj} | x_{li}, \\
0, & \text{otherwise}.
\end{cases}
\]
But \( E_{l_t}(S) \) is the \( l_h \times l_h \) matrix obtained from \( E(S) \) by deleting its \( l_t \)th column. So one has
\[
E_{l_t}(S) = \begin{pmatrix}
E_1 & \cdots & 0 & \cdots & 0 & 1 \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
0 & \cdots & E'_t & \cdots & 0 & 1 \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
0 & 0 & \cdots & 0 & 1 \\
0 & 0 & \cdots & E_h & 1
\end{pmatrix}.
\]
where

\[
E'_t = \begin{pmatrix}
1 & 0 & 0 & \cdots & 0 \\
1 & 1 & 0 & \cdots & 0 \\
1 & e'_{32} & 1 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
1 & e'_{n_t-2} & e'_{n_t-1} & \cdots & e'_{n_t,n_t-1}
\end{pmatrix}
\]

(i). \( x_{t,n_t-1} \nmid x_{t,n_t} \). Then one has that \( e'_{n_t,n_t-1} = 0 \). Thus the \((l_t - 1)\)th column of \( E_{l_t}(S) \) is \((0, \ldots, 0, 1, 0, \ldots, 0)^T \).

Then using the Laplace expansion theorem, we obtain that

\[
\det(E_{l_t}(S)) = \det(\begin{pmatrix}
E_1 & \cdots & 0 & \cdots & 0 & 1 \\
\vdots & & \vdots & \ddots & \vdots & \vdots \\
0 & \cdots & E''_{l_t-1} & \cdots & 0 & 1 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & \cdots & 0 & \cdots & 0 & 1 \\
0 & \cdots & 0 & \cdots & 0 & E_h
\end{pmatrix}),
\]

(8)

with

\[
E''_t = \begin{pmatrix}
1 & 0 & 0 & \cdots & 0 \\
1 & 1 & 0 & \cdots & 0 \\
1 & e'_{32} & 1 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
1 & e'_{n_t-2} & e'_{n_t-2,3} & \cdots & 1 \\
1 & e'_{n_t-1,2} & e'_{n_t-1,3} & \cdots & e'_{n_t-1,n_t-2}
\end{pmatrix}
\]

On the other hand, by the definition of \( S \), one can easily deduce that \( S \setminus \{x_{t,n_t-1}\} \) consists of multiple coprime gcd-closed sets and \( \tilde{S} \setminus \{x_{t,n_t-1}\} \) is the minimal gcd-closed set containing the set \( S \setminus \{x_{t,n_t-1}\} \). Hence by the definition of \( E_{l_t-1}(S \setminus \{x_{t,n_t-1}\}) \), one knows that the right-hand side of (8) is equal to \( \det(E_{l_t-1}(S \setminus \{x_{t,n_t-1}\})) \). So the desired result follows. Part (i) is proved.

(ii). \( x_{t,n_t-1} \nmid x_{t,n_t} \). Thus \( e'_{n_t,n_t-1} = 1 \). Clearly the \((l_t - 1)\)th column of \( E_{l_t}(S) \) is \((0, \ldots, 0, 1, 0, \ldots, 0)^T \).

Applying the Laplace expansion theorem gives us that

\[
\det(E_{l_t}(S)) = \det(\begin{pmatrix}
E_1 & \cdots & 0 & \cdots & 0 & 1 \\
\vdots & & \vdots & \ddots & \vdots & \vdots \\
0 & \cdots & E''_{l_t-1} & \cdots & 0 & 1 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & \cdots & 0 & \cdots & 0 & 1 \\
0 & \cdots & 0 & \cdots & 0 & E_h
\end{pmatrix}) - \det(\begin{pmatrix}
E_1 & \cdots & 0 & \cdots & 0 & 1 \\
\vdots & & \vdots & \ddots & \vdots & \vdots \\
0 & \cdots & E''_{l_t-1} & \cdots & 0 & 1 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & \cdots & 0 & \cdots & 0 & 1 \\
0 & \cdots & 0 & \cdots & 0 & E_h
\end{pmatrix}),
\]

(9)

with

\[
E'''_t = \begin{pmatrix}
1 & 0 & 0 & \cdots & 0 \\
1 & 1 & 0 & \cdots & 0 \\
1 & e'_{32} & 1 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
1 & e'_{n_t-2} & e'_{n_t-2,3} & \cdots & 1 \\
1 & e'_{n_t-1,2} & e'_{n_t-1,3} & \cdots & e'_{n_t-1,n_t-2}
\end{pmatrix}
\]

Clearly \( S \setminus \{x_{t,n_t}\} \) consists of multiple coprime gcd-closed sets and \( \tilde{S} \setminus \{x_{t,n_t}\} \) is the minimal gcd-closed set containing \( S \setminus \{x_{t,n_t}\} \). Thus by the definition of \( E_{l_t-1}(S \setminus \{x_{t,n_t}\}) \), we know that the right-hand side of (9) is equal to \( \det(E_{l_t-1}(S \setminus \{x_{t,n_t}\})) - \det(E_{l_t-1}(S \setminus \{x_{t,n_t}\})) \). So part (ii) is true.

This concludes the proof of Lemma 2.8. \( \square \)
In ending this section, we show the following relation between $S_f^{(1)}(T)$ and $S_f^{(2)}(T)$ which is also needed in the proof of Theorem 1.3.

Lemma 2.9. Let $f$ be an arithmetic function and $T$ be a set of distinct positive integers. If $f(x) \neq 0$ for any $x \in T$ and $f(1) = 1$, then one has that $S_f^{(1)}(T) \prod_{x \in T} f(x)^2 = S_f^{(2)}(T)$.

Proof. Since $f(x) \neq 0$ for any $x \in T$ and $f(1) = 1$, it follows that

$$S_f^{(1)}(T) \prod_{x \in T} f(x)^2 = \left( \sum_{y \in T} \prod_{x \in T \setminus \{y\}} \left( \frac{1}{f(x)} - 1 \right) \prod_{x \in T} f(x)^2 \right) \prod_{x \in T} f(x)^2$$

$$= \left( \sum_{y \in T} \prod_{x \in T \setminus \{y\}} \left( 1 - f(x) \right) + \prod_{x \in T} (1 - f(x)) \right) \prod_{x \in T} f(x)^2$$

$$= \left( \prod_{y \in T} (f(x) - 1) + \prod_{x \in T} f(x) \right) \prod_{x \in T} f(x)^2$$

$$= S_f^{(2)}(T)$$

as desired. So Lemma 2.9 is proved.

\[\square\]

## 3 Proofs of Theorem 1.3 and Corollary 1.4

In this section, we prove Theorem 1.3 and Corollary 1.4. We begin with the proof of Theorem 1.3.

Proof of Theorem 1.3. Since $S$ consists of multiple coprime gcd-closed sets such that $1 \notin S$, one may write $S$ as in the form of (6). For $1 \leq i \leq h$, let $l_i = n_1 + \ldots + n_i$. Then the $l_h \times (l_h + 1)$ matrix $E(S)$ is of the form (7).

Let’s first deal with $\det(f(S))$. Define $x_1 = x_{11}, \ldots, x_{l_1} = x_{11}, \ldots, x_{l_i-1} = x_{h1}, \ldots, x_{l_i} = x_{hnh}$, $x_{l_i+1} = 1$ and $l_0 = 0$. Lemma 2.7 tells us that

$$\det(f(S)) = \sum_{1 \leq k_1 < \ldots < k_{l_h} \leq l_h + 1} (\det(E(S)_{k_1, \ldots, k_{l_h}}))^2 \prod_{i=1}^{l_h} a_{S, f}(x_{k_i}). \quad (10)$$

For any $\{k_1, \ldots, k_{l_h}\}$ with $1 \leq k_1 < \ldots < k_{l_h} \leq l_h + 1$, write $\{k_1, \ldots, k_{l_h}\} := \{1, 2, \ldots, l_h + 1\} \setminus \{k\}$. Then

$$\det(E(S)_{k_1, \ldots, k_{l_h}}) = \det(E_k(S)).$$

We claim that $(\det(E(S)_{k_1, \ldots, k_{l_h}}))^2 = 1$ if $k = l_i + 1$ for some $i \in \{0, 1, \ldots, h\}$, and $\det(E(S)_{k_1, \ldots, k_{l_h}}) = 0$ if $k \notin \{l_i + 1\}_{i=0}^{h}$.

First, let $k = l_i + 1$ for some integer $i$ with $0 \leq i \leq h$. If $0 \leq i \leq h - 1$, then the $l_i + 1$ row of $E_{l_i+1}(S)$ is $(0, \ldots, 0, 1)$. So the Laplace expansion applied to $\det(E_{l_i+1}(S))$ gives us a lower triangular determinant with all the diagonal elements being 1. It follows that

$$(\det(E(S)_{k_1, \ldots, k_{l_h}}))^2 = (\det(E_{l_i+1}(S)))^2 = 1.$$
If \( i = h \), then \( E_{l_i+1}(S) \) is a lower triangular matrix with all the diagonal elements being 1. Hence 
\[
\det(E(S)_{(k_1, \ldots, k_{l_i})}) = \det(E_{l_i+1}(S)) = 1.\]

Therefore the first part of the claim is true.

Obviously, the second part of the claim is equivalent to the statement that \( \det(E_T(S)) = 0 \) for all integers \( t \) with \( l_t-1 + 2 \leq t \leq l_t \) and \( 1 \leq i \leq h \), which will be proved in the following.

Given any integer \( i \) with \( 1 \leq i \leq h \). We prove the claim by using induction on \( n_i \). If \( n_i = 2 \), then \( t = l_i = l_i-1+2 \). Further, \( x_{i1} \mid x_{i2} \). Using Lemma 2.8, we have \( \det(E_{l_i}(S)) = \det(E_{l_i-1}(S) \setminus \{x_{i1}\}) - \det(E_{l_i-1}(S) \setminus \{x_{i2}\}) \).

Since \( n_i = 2 \), we derive that \( E_{l_i-1}(S) \setminus \{x_{i1}\}) = E_{l_i-1}(S) \setminus \{x_{i2}\}) \). This implies immediately that \( \det(E_{l_i}(S)) = 0 \). So the claim is true if \( n_i = 2 \).

Let \( n_i \geq 3 \) and assume that the claim is true for the \( n_i - 1 \) case. In what follows we consider the \( n_i \) case. For \( l_{i-1} + 2 \leq t \leq l_i - 1 \), noting that all elements of the \((l_i - 1)\)-th column of \( E_T(S) \) are zero except for its \((l_i, l_i-1)\)-entry is 1, applying the Laplace theorem to \( \det(E_T(S)) \) gives that \( \det(E_T(S)) = -\det(E_T(S) \setminus \{x_{i,n_i}\}) \). But the inductive assumption implies that \( \det(E_T(S) \setminus \{x_{i,n_i}\}) = 0 \). Thus \( \det(E_T(S)) = 0 \) as claimed. Now let \( t = l_i \). If \( x_{i,n_i-1} \nmid x_{i,n_i} \), then Lemma 2.8 infers that \( \det(E_{l_i}(S)) = \det(E_{l_i-1}(S) \setminus \{x_{i,n_i-1}\}) \). However, by the induction assumption we have \( \det(E_{l_i-1}(S) \setminus \{x_{i,n_i-1}\}) = 0 \). Hence \( \det(E_{l_i}(S)) = 0 \) as required. If \( x_{i,n_i-1} \mid x_{i,n_i} \), by Lemma 2.8 we have
\[
\det(E_{l_i}(S)) = \det(E_{l_i-1}(S) \setminus \{x_{i,n_i-1}\}) - \det(E_{l_i-1}(S) \setminus \{x_{i,n_i}\}).
\]

But the induction assumption tells us that
\[
\det(E_{l_i-1}(S) \setminus \{x_{i,n_i-1}\}) = \det(E_{l_i-1}(S) \setminus \{x_{i,n_i}\}).
\]

So \( \det(E_{l_i}(S)) = 0 \) and the claim is proved.

Now, by (10) and the claim, one deduces that
\[
\det(f(S)) = \sum_{i=0}^{h-1} \prod_{j=0}^{l_h} \alpha_{S,f}(x_j) + \prod_{i=1}^{l_h} \alpha_{S,f}(x_i).
\]

From (1), we deduce that \( \alpha_{S,f}(1) = f(1) \) and \( \alpha_{S,f}(x_{i1}) = (f(x_{i1}) - f(1)) \). It then follows from (11) that
\[
\det(f(S)) = f(1) \left( \sum_{i=0}^{h-1} \prod_{j=0}^{l_h} \alpha_{S,f}(x_j) \right) + \prod_{i=1}^{l_h} \alpha_{S,f}(x_i)
\]
\[
= \left( f(1) \sum_{i=1}^{h} \prod_{j=0}^{l_h} \alpha_{S,f}(x_{i1}) \right) + \prod_{i=1}^{h} \alpha_{S,f}(x_{i1}) \prod_{i=1}^{h} \alpha_{S,f}(x_{ij})
\]
\[
= \left( f(1) \sum_{i=1}^{h} \prod_{j=0}^{l_h} (f(x_{i1}) - f(1)) \right) + \prod_{i=1}^{h} \prod_{j=1}^{h} \alpha_{S,f}(x_{ij})
\]
\[
= s_f^{(1)}(M(S)) \prod_{x \in S \setminus M(S)} \alpha_{S,f}(x)
\]
as desired.

Finally, we turn our attention to \( \det(f[S]) \). Let \( f \) be a multiplicative function and \( f(x) \neq 0 \) for all \( x \in S \). Then \( f(1) = 1 \) and from Lemma 2.2 we derive that
\[
(f[S]) = \text{diag}(f(x_{i1}), \ldots, f(x_{ih}))(\frac{1}{f(x_i, x_j)} \text{diag}(f(x_{i1}), \ldots, f(x_{ih})).
\]

Therefore \( \det(f[S]) = \det \left( \frac{1}{f(S)} \right) \prod_{i=1}^{l_h} f(x_{i1})^2 \). Thus the formula for \( \det(f(S)) \) applied to \( \det(\frac{1}{f(S)}) \) and Lemma 2.9 applied to the set \( T = M(S) \) give us that
\[
\det(f[S]) = s_T^{(1)}(M(S)) \left( \prod_{x \in M(S)} \alpha_{S,f}(x) \right) \prod_{x \in S} f(x)^2
\]
\[ s^{(1)}_{I}(M(S)) = \prod_{x \in M(S)} f(x)^2 \prod_{x \in S \setminus M(S)} f(x)^2 \alpha_{S \setminus I}(x) \]
\[ s^{(2)}_{I}(M(S)) = \prod_{x \in S \setminus M(S)} f(x)^2 \alpha_{S \setminus I}(x) \]
as required. This finishes the proof of Theorem 1.3.

We are now ready to prove Corollary 1.4 as the conclusion of this paper.

**Proof of Corollary 1.4.** Let \( f = I \). Then Theorem 1.3 tells us that
\[ \det(\cdot) = s^{(1)}_{I}(M(S)) \prod_{x \in S \setminus M(S)} \alpha_{S \setminus I}(x). \]
By the definition of \( \alpha_{S \setminus I}(x) \) and \( I \ast \mu = \varphi \), we have
\[ \alpha_{S \setminus I}(x) = \sum_{d \mid x, d \in S} (I \ast \mu)(d) = \sum_{d \mid x, d \in S} \varphi(d). \]
Thus the desired result follows immediately. By Theorem 1.3, one has
\[ \det(\cdot) = s^{(2)}_{I}(M(S)) \prod_{x \in S \setminus M(S)} x^2 \alpha_{S \setminus I}(x) = s^{(2)}_{I}(M(S)) \prod_{x \in S \setminus M(S)} x^2 \sum_{d \mid x, d \in S} (1 \ast \mu)(d). \]
Since
\[ (1 \ast \mu)(d) = \sum_{d' \mid d} d' \mu(d) = \frac{1}{d} \sum_{d' \mid d} d' \mu(d) = g(d), \]
the desired result then follows immediately. This completes the proof of Corollary 1.4.

**Remark 3.1.** If \( S \) consists of multiple coprime gcd-closed sets such that \( 1 \notin S \), then Theorem 1.3 gives us formulas for \( \det(f(S)) \) and \( \det(f[S]) \). One can easily see that Theorem 1.3 is not true if \( S \) consists of multiple gcd-closed sets which are not coprime. If \( S \) consists of multiple gcd-closed sets such that \( \gcd(S) = 1 \notin S \) and these gcd-closed sets are not coprime, then what are the formulas for \( \det(f(S)) \) and \( \det(f[S]) \)? This interesting problem keeps open.

**Acknowledgement:** This research was supported partially by National Science Foundation of China Grant # 11371260. The authors thank the anonymous referees for helpful comments and suggestions.

**References**

[1] Apostol T.M., Arithmetical properties of generalized Ramanujan sums, Pacific J. Math., 1972, 41, 281-293
[2] Bege A., Generalized LCM matrices, Publ. Math. Debrecen 2011, 79, 309-315
[3] Beslin S., Ligh S., Another generalization of Smith’s determinant, Bull. Aust. Math. Soc., 1989, 40, 413-415
[4] Bourque K., Ligh S., On GCD and LCM matrices, Linear Algebra Appl., 1992, 174, 65-74
[5] Bourque K., Ligh S., Matrices associated with classes of arithmetical functions, J. Number Theory, 1993, 45, 367-376
[6] Bourque K., Ligh S., Matrices associated with arithmetical functions, Linear Multilinear Algebra, 1993, 34, 261-267
[7] Codecá P., Nair M., Calculating a determinant associated with multiplicative functions, Boll. Unione Mat. Ital. Sez. B Artic. Ric. Mat., 2002, 5, 545-555
[8] Gantmacher F.R., Matrix theory, Vol. I, Chelsea, New York, 1960
[9] Haukkanen P., Higher-dimensional GCD matrices, Linear Algebra Appl., 1992, 170, 53-63
[10] Haukkanen P., On meet matrices on posets, Linear Algebra Appl., 1996, 249, 111-123
[11] Hilberdink T., Determinants of multiplicative Toeplitz matrices, Acta Arith., 2006, 125, 265-284
[12] Hong S., The lower bound of the determinants of matrices associated with a class of arithmetical functions, Chinese Ann. Math., Ser. A, 2000, 21, 377-382 (in Chinese)
[13] Hong S., Gcd-closed sets and determinants of matrices associated with arithmetical functions, Acta Arith., 2002, 101, 321-332
[14] Hong S., Lee K.S.E., Asymptotic behavior of eigenvalues of reciprocal power LCM matrices, Glasgow Math. J., 2008, 50, 163-174
[15] Hong S., Li M., Wang B., Hyperdeterminants associated with multiple even functions, Ramanujan J., 2014, 34, 265-281
[16] Hong S., Loewy R., Asymptotic behavior of eigenvalues of greatest common divisor matrices, Glasgow Math. J., 2004, 46, 551-569.
[17] Hong S., Loewy R., Asymptotic behavior of the smallest eigenvalue of matrices associated with completely even functions (mod $r$), Int. J. Number Theory, 2011, 7, 1681-1704
[18] Hu S., Hong S., Multiple divisor chains and determinants of matrices associated with completely even functions (mod $r$), Linear Multilinear Algebra, 2014, 62, 1240-1257.
[19] Korkee I., Haukkanen P., On a general form of join matrices associated with incidence functions, Aequationes Math., 2008, 75, 29-42
[20] Li M., Tan Q., Divisibility of matrices associated with multiplicative functions, Discrete Math., 2011, 311, 2276-2282
[21] Li Z., The determinants of GCD matrices, Linear Algebra Appl., 1990, 134, 137-143
[22] Mattila M., Haukkanen P., Determinant and inverse of join matrices on two sets, Linear Algebra Appl., 2013, 438, 3891-3904
[23] McCarthy P.J., A generalization of Smith's determinant, Canad. Math. Bull., 1986, 29, 109-113
[24] Nathanson M.B., Elementary methods in number theory, Grad. Texts in Math., 195, Springer, Berlin-New York-Heidelberg, 2000
[25] Smith H.J.S., On the value of a certain arithmetical determinant, Proc. London Math. Soc., 1875-1876, 7, 208-212
[26] Tan Q., Li M., Divisibility among power GCD matrices and among power LCM matrices on finitely many coprime divisor chains, Linear Algebra Appl., 2013, 438, 1454-1466
[27] Tan Q., Luo M., Lin Z., Determinants and divisibility of power GCD and power LCM matrices on finitely many coprime divisor chains, Appl. Math. Comput., 2013, 219, 8112-8120
[28] Yamasaki Y., Arithmetical properties of multiple Ramanujan sums, Ramanujan J., 2010, 21, 241-261