Group-Algebraic Characterization of Spin Particles: Semi-Simplicity, SO(2N) Structure and Iwasawa Decomposition

Mahouton Norbert Hounkonnou, Francis Atta Howard* and Kinvi Kangni

Abstract. In this paper, we focus on the characterization of Lie algebras of fermionic, bosonic and parastatistic operators of spin particles. We provide a method to construct a Lie group structure for the quantum spin particles. We show the semi-simplicity of the Lie algebra for a quantum spin particle, and extend the results to the Lie group level. Besides, we perform the Iwasawa decomposition for spin particles at both the Lie algebra and the Lie group levels. Then, we give a general decomposition for spin particles. Finally, we investigate the coupling of angular momenta of spin half particles, and give a general construction for such a study.

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1. Introduction

Certain authors, including, David Hestenes [10], Garret Sobczyk [26] and Doran et al. [8] showed that every linear transformation can be represented as a monomial of vectors in geometric algebra, every Lie algebra as a bivector algebra, and every Lie group as a spin group. Schwinger’s realization of $\mathfrak{su}(1,1)$ Lie algebra with creation and annihilation operators [24] was defined with spatial reference in the Pauli matrix representation [33]. Several relations as well as connections were observed in spin particles such as fermionic, bosonic,
parastatistic Lie algebras, and in geometric algebras such as the Clifford algebra, Grassmannian algebra and so on [25]. Sobczyk [25] also proved that the spin half particles can be represented by geometric algebras. Palev [19] highlighted that a semi-simple Lie algebra can be generated by the creation and annihilation operators. In all the above mentioned works, the classical algebras such as $B_n$ and $D_n$ play a crucial role in the spin particle Lie algebra. Moreover, several evidences from particle and theoretical physics showed the connection between quantum spin particle Lie algebra and Clifford algebra [8]. The spin of elementary particles obeying Fermi-Dirac statistics, Bose-Einstein statistics, and the quantization operators of parastatistics such as parafermions and parabosons, also gained much attention in the literature [2,8,24,25,28,33].

Moreover, Palev [19] studied in detail the Lie-algebraical properties of the quantization for spinor fields. However, these results were not extended to the Lie group structure of the spin particles such as the classical groups $SO(2n+1, \mathbb{C})$ and $SO(2n, \mathbb{C})$. Doran et al. [8] also showed that every Lie group can be represented by a spin group and they further analyzed the spin version of the general linear group. In the opposite, exhaustive investigations on spin particle creation and annihilation and their angular momentum in connection with Lie groups, Lie algebras, Clifford algebras, their representations and the connections with classical groups like $SL(2, \mathbb{R})$ and $SU(1, 1)$ are still lacking. This study aims at filling this gap.

A semi-simple Lie group such as $SU(1, 1)$ can be decomposed into compact (K), abelian (A), and nilpotent (N) subgroups. This is the Iwasawa decomposition which was introduced by Japanese mathematician Kenkichi Iwasawa. This decomposition generalizes the Gram-Schmidt orthogonalization process from linear algebra [15].

Motivated by all of the above mentioned work, we prove, in this paper, that the spin particles admit a Lie group structure, show its connectedness and semi-simplicity, and construct the Iwasawa decomposition at both the Lie algebra and the Lie group levels.

The paper is organized as follows. In Sect. 2, we recall main definitions and known results useful in the sequel, and set the notation. Section 3 deals with the semi-simplicity of the Lie structures related to a spin particle. In Sect. 4, we develop the real Lie algebra of a spin particle. We construct the Iwasawa decomposition which we then generalize in Sect. 5. Finally, we end with some concluding remarks in Sect. 6.

2. Preliminaries, Definitions and Notation

2.1. Para-Fermionic Algebra

Let $a^+_1, \ldots, a^+_n$ be the creation and annihilation operators for a system consisting of $n$-fermions with commutator relations [12]:

\[
[a^-_i, a^+_j] = \delta_{ij}, \quad (2.1)
\]

\[
[a^-_i, a^-_j] = [a^+_i, a^+_j] = 0, \quad (2.2)
\]
or, of $n$-parafermions, with
\[
[a_i^-, a_j^+] = \pm 2\delta_{ij} a_j^+, 
\]
(2.3)
Let $\mathbb{K}$ be a field. A vector space $\mathfrak{g}$ over $\mathbb{K}$ is called a Lie algebra over $\mathbb{K}$ if a bilinear map
\[
[\cdot, \cdot] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g},
(x, y) \mapsto [x, y],
\]
satisfies the following axioms:
(i) Bilinearity,
\[
[a x + b y, z] = a [x, z] + b [y, z], [z, a x + b y] = a [z, x] + b [z, y],
\]
for all scalars $a, b \in \mathbb{K}$ and all elements $x, y, z \in \mathfrak{g}$.
(ii) Skew-symmetric, $[x, x] = 0$, for all $x \in \mathfrak{g}$.
(iii) The Jacobi identity
\[
[x, [y, z]] + [z, [x, y]] + [y, [z, x]] = 0
\]
for all $x, y, z \in \mathfrak{g}$.
The vector space $\otimes V$ endowed with a unital associative algebra structure via
the tensor product $\otimes$, is called the tensor algebra of $V$, denoted by $T$.
For an associative algebra $A$, we consider a linear map $\phi : \mathfrak{g} \to A$ such that
for all $x, y \in \mathfrak{g}$,
\[
[\phi(x), \phi(y)] = \phi(x)\phi(y) - \phi(y)\phi(x).
\]
The universal enveloping algebra $U_\mathfrak{g}$ of $\mathfrak{g}$ is an associative algebra with a Lie
map $i : \mathfrak{g} \to U_\mathfrak{g}$ extending any Lie map $\phi : \mathfrak{g} \to A_L$ uniquely, in particular
there is a unique associative algebra morphism $\psi : U_\mathfrak{g} \to A_L$, such that the
diagram commutes:
```
g \quad \phi \quad U_\mathfrak{g} \quad \psi \quad A_L
```
where $A_L$ denote the Lie algebra obtained from $A$ per the bracket (2.4). Let
$T$ be the associative free algebra of $a_i^-, a_j^+; i, j \in N = \{1, 2, \ldots, n\}$, and $I$
be the two sided ideal in $T$ containing all elements of the form
\[
x \otimes y - y \otimes x - [x, y], \quad x, y \in \mathfrak{g},
\]
where
\[
[x, y] := xy - yx.
\]
(2.4)
The quotient (factor) algebra
\[
Q = \frac{T}{I} \cong U_\mathfrak{g}
\]
(2.5)
is called para-Fermi algebra (universal enveloping algebra) [31]. This is an
infinite dimensional Lie algebra with respect to the bracket defined by (2.4).
2.2. Semi-Simple Lie Algebra Generated by Creation and Annihilation Operators

In this subsection, we quickly review main Lie algebraic properties retrieved from the work by Palev [19], which are useful for our construction performed using the same notation.

Let \( g \) be a semi-simple Lie algebra generated by \( n \) pairs \( a_i^+, \ldots, a_i^- \) of creation and annihilation operators. The elements

\[
h_i = \frac{1}{2}[a_i^-, a_i^+], \quad i = 1, \ldots, n,
\]

are contained in a Cartan subalgebra \( H \) of \( g \) with rank of \( g \) \( \geq n \). If the semi-simple Lie algebra \( g \) of rank \( n \) is generated by \( n \) pairs of creation and annihilation operators, then, with respect to the basis of the Cartan subalgebra, the creation (resp. annihilation) operators are negative (resp. positive) root vectors. The correspondence with their roots is:

\[
a_i^\pm \longleftrightarrow \pm h^* \iota.
\]

where \( h^* \iota \) is a basis in the space dual to the Cartan subalgebra [19].

The semi-simple Lie algebra \( g \) of rank \( n \) is generated by \( n \) pairs of creation and annihilation operators if and only if it contains a complete system of roots \( \Phi \) orthogonal with respect to the Cartan-Killing form. The semi-simple Lie algebra \( g \) of rank \( n \) is generated by \( n \) pairs of creation and annihilation operators if and only if it is a direct sum of classical Lie algebras

\[
g = B_{m_1} \oplus \cdots \oplus B_{m_k}
\]

where \( m_1 + \cdots + m_k = n \).

For an example of a semi-simple Lie algebra, we adapt Schwinger notation for the Lie algebra \( \text{su}(1,1) \): Let \( a_r^+ = (a_r^+, a_r^-) \) and \( a_r = (a_r^+, a_r^-) \) be the spin creation and annihilation operators which obey the following commutation relations:

\[
[a_r, a_{r'}] = 0, \quad [a_r, a_{r'}^+] = \delta_{rr'}, \quad \text{and} \quad [a_r^+, a_{r'}^-] = 0.
\]

The number of spins and the resultant angular momentum are, respectively, given by (see [24]):

\[
n = \sum_r a_r^+ a_r = n_+ + n_-, \quad J = \sum_{r,r'} a_r^+ \left(r \mid \frac{1}{2} \sigma \mid r'\right) a_r,
\]

with the conventional Pauli’s matrix representation for \( \sigma \), the components of total angular momentum \((J)\) as [24,28]:

\[
J_+ = a_r^+ a_r, \quad J_- = a_r^- a_r, \quad J_z = \frac{1}{2}(n_+ - n_-), \quad J_x = \frac{J_+ + J_-}{2}, \quad J_y = \frac{J_+ - J_-}{2i}, \quad m = \frac{1}{2}(n_+ - n_-),
\]

\[
j = \frac{1}{2}(n_+ + n_-), \quad J^2 = \frac{1}{2} n \left(\frac{1}{2} n + 1\right),
\]

\[
j = \frac{1}{2}(n_+ + n_-), \quad J^2 = \frac{1}{2} n \left(\frac{1}{2} n + 1\right).
\]
where $J_z$ is the angular momentum measured along any given direction and $J_+, J_-$ are the raising and lowering operators, respectively. The definite magnetic quantum numbers $j$ and $m$ have a positive or negative spin number $n$. The operators $J, J_z, J_+, J_-$ and $J^2$ satisfy the Schwinger boson Lie algebra $\mathfrak{su}(1,1)$ (see [24]) for a one-dimensional harmonic oscillator, the $a$ and $a^+$ boson creation and annihilation operators can be defined in the usual way:

$$a = \frac{1}{\sqrt{2}} \left( x + \frac{\partial}{\partial x} \right) \quad \text{and} \quad a^+ = \frac{1}{\sqrt{2}} \left( x - \frac{\partial}{\partial x} \right).$$

Introduction the operators $K_+, K_-$ and $K_z$ through (see [24])

$$K_+ = \frac{1}{2} a^+ a_+, \quad K_- = \frac{1}{2} a_+ a^-, \quad K_z = \frac{1}{2} (n_+ + n_- + 1),$$

$$K_+ = K_x + i K_y, \quad K_- = K_x - i K_y,$$

and the Casimir operator $C = -K_x^2 - K_y^2 + K_z^2$ while the operators $K_+, K_-$ and $K_z$ obey the commutation relations:

$$[K_z, K_+] = +K_z, \quad [K_z, K_-] = -K_z, \quad [K_+, K_-] = -2K_z$$

which define a Lie algebra of the SU(1,1) group. One can construct a monomial using the component of arbitrary spinors $x_r = (x_+, x_-)$:

$$\phi_{jm}(x) = \frac{x_{j+m}^j x_{j-m}^j}{[(j+m)!(j-m)!]\frac{1}{2}},$$

and sum with respect to $m$ and then to $j$,

$$\sum_{m=-j}^{j} \phi_{jm}(x) \psi_{jm} = \frac{(xa^+)^2j}{(2j)!} \psi_o$$

and

$$\sum_{jm} \phi_{jm}(x) \psi_{jm} = \exp(xa^+) \psi_o$$

in which $xa^+ = \sum_r x_r a_r^+$ and

$$\psi_{jm}(x) = \frac{(a_+^r)^{j+m} (a^r_+)^{j-m}}{[(j+m)!(j-m)!]\frac{1}{2}} \psi_o.$$  

We introduce the operators

$$a_{r+}^+ = xa^+, \quad a_+^r = x^r a, \quad a_{r+}^r = [x^r a^r^+], \quad a^-_r = [x]$$

where $[xy] := x_+ y_- - x_- y_+$ with the restriction $(x^r x) = 1$. These operators constitute the spin creation and annihilation operators associated with an altered spatial reference system. When the state $m = j$ in a rotated coordinate system, (2.6) can be regarded as a linear combination of the eigenvectors in a fixed coordinate system is:

$$\psi_{jj}^j = \frac{(a_{r+}^r)^{2j}}{[(2j)!]\frac{1}{2}} \psi_o = (2j)!^{\frac{1}{2}} \sum_{m=-j}^{j} \phi_{jm}(x) \psi_{jm}.$$
The unitary nature of this transformation is verified by:

$$(2j)! \sum_m \phi_{jm}(x^*) \phi_{jm}(x) = (x^* x)^{2j} = 1.$$ 

Put

$$x_+ = \exp \left( i \frac{\varphi + \psi}{2} \right) \cosh \left( \frac{t}{2} \right)$$

and

$$x_- = \exp \left( i \frac{\varphi - \psi}{2} \right) \sinh \left( \frac{t}{2} \right),$$

which give rise to the $U$-matrix:

$$U = \begin{pmatrix} x_+ & x_- \\ x_+^* & x_-^* \end{pmatrix}$$

such that $|x_+|^2 - |x_-|^2 = 1$. Letting $x_+ = \alpha$ and $x_- = \beta$, the $U$-matrix is:

$$U = \begin{pmatrix} \alpha & \beta \\ \beta^* & \alpha^* \end{pmatrix} \in \text{SU}(1,1).$$

The group SU(1,1) is the group of two-dimensional unitary unimodular matrices which leave the form $|x_1|^2 - |x_2|^2$ invariant [1,3,13]. Its Lie algebra is the complex extension of that of the SU(2) and it is composed of the generators $K_1, K_2, K_3$ which obey the following commutation relations:

$$[K_1, K_2] = -iK_3, \quad [K_3, K_1] = iK_2, \quad [K_2, K_3] = iK_1.$$ 

These operators are Hermitian but indefinite, and have an explicit realization in terms of Pauli matrices:

$$k_1 \rightarrow \frac{i}{2} \sigma_1, \quad k_2 \rightarrow \frac{i}{2} \sigma_2, \quad k_3 \rightarrow \frac{1}{2} \sigma_3.$$ 

The latter are the generators of a 2-dimensional non-unitary representation of SU(1,1), where $\sigma_1, \sigma_2, \sigma_3$ are $\sigma_x, \sigma_y, \sigma_z$, respectively, which correspond to $k_x, k_y, k_z$. Now for a fixed choice of the phase, a $2 \times 2$ matrix representation ($d$-function) of exp($-itK_y$) will be:

$$\exp(-itK_y) = \exp \left[ -i \frac{t}{2} \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \right] = \exp \left[ \frac{t}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right] = \exp \left( \frac{t}{2} \sigma_x \right) = \begin{pmatrix} \cosh \frac{t}{2} & \sinh \frac{t}{2} \\ \sinh \frac{t}{2} & \cosh \frac{t}{2} \end{pmatrix} = d_t^1.$$ 

Let us consider the Clifford algebra $C\ell_{1,1}$. The group Spin$_+(1,1)$ is defined by the elements $R \in C\ell_{1,1}^+$ such that $R \widetilde{R} = \widetilde{R} R = 1$, where $\widetilde{R}$ denotes the reversion anti-automorphism. Hence, 

$$\text{Spin}_+(1,1) = \left\{ a + be_1 e_2 \mid a, b \in \mathbb{R}, a^2 - b^2 = 1 \right\},$$

that is, $\text{Spin}_+(1,1) \cong \mathbb{R} \otimes \mathbb{R}$. This group has two components, with representatives $R = \cosh \alpha + \sinh \alpha e_1 e_2$, and $\widetilde{R} = - \cosh \alpha + \sinh \alpha e_1 e_2$, respectively, $-\infty < \alpha < \infty$. Therefore, the two-fold covering Spin$_+(1,1)$ is not connected, although SO$_+(1,1)$ is connected. Indeed, the groups Spin$_+(1,1)$ with $p + q \geq 2$ are connected, and the exception is precisely Spin$_+(1,1)$ [30].
Remark 2.1. The $\mathfrak{su}(1, 1)$ quasi-boson Lie algebra is semi-simple.

2.3. The Lie Algebra of the Spin Group

Let $V$ be an $n$-dimensional oriented real vector space with an inner product $\langle , \rangle$. We define the Clifford algebra [6] $\text{Cl}(V)$ over $V$ as the quotient $T(V)/I$ where $T(V)$ is the tensor algebra over $V$ and $I$ is the ideal generated by all elements $v \otimes v + \langle v, v \rangle 1, v \in V$. The multiplication in $\text{Cl}(V)$ will be denoted by $xy$. Let $p : T(V) \to \text{Cl}(V)$ be the canonical projection. Then, $\text{Cl}(V)$ is decomposed into the direct sum $\text{Cl}^+(V) \oplus \text{Cl}^-(V)$ of the $p$-images of the elements of even and odd degrees of $T(V)$, and $m$ is identified with the subspace of $\text{Cl}(V)$ through the projection $p$. Let $e_1, e_2, \ldots, e_n$ be an oriented orthonormal basis of $V$. The map: $e_i e_i \cdots e_i \mapsto (-1)^p e_i e_i \cdots e_i e_i$ defines a linear map of $\text{Cl}(V)$ and the image of $x \in \text{Cl}(V)$ by this linear map is denoted by $x$. The spin group is the multiplicative group $\text{Spin}(V)$ of all invertible elements $x \in \text{Cl}^+(V)$. More precisely:

$$\text{Spin}(V) = \{ x \in \text{Cl}^+(V) : xVx^{-1} \subset V \text{ and } x\bar{x} = 1 \}.$$ 

Moreover, the subspace $\text{spin}(V)$ of $\text{Cl}(V)$ spanned by $\{e_i e_j\}_{i<j}$ is a Lie algebra of $\text{Spin}(V)$ with the mapping $\exp : \text{spin}(V) \to \text{Spin}(V)$.

Let $\text{Cl}(V) = \text{Cl}_{p,q} = \mathbb{R}_{p,q}$ be the real Clifford algebra generated by the vector space $\mathbb{R}^{p,q}$ over $\mathbb{R}$ of signature $(p,q)$. Consider $\mathbb{R}^*_{p,q} = \text{Cl}_{p,q}^*$ the group of invertible elements of $\mathbb{R}^*_{p,q}$. The exponential of $y \in \text{Cl}_{p,q}$ is defined by:

$$\exp(y) = \sum_{n=0}^{\infty} \frac{1}{n!} y^n.$$ 

Since $\exp(-y) = (\exp y)^{-1}$, the exponential mapping links an element of $\text{Cl}_{p,q}$ with an element from the group of invertible elements $\text{Cl}_{p,q}^*$; namely, $\exp : \text{Cl}_{p,q} \to \text{Cl}_{p,q}^*$. Let $\pi : \text{Spin}(V) \to \text{SO}(V)$ be defined by $\pi(x)v = xvx^{-1}$.

Then, the differential $\dot{\pi}$ of $\pi$ is given by:

$$\dot{\pi}(x)v = xv - vx,$$

for $x \in \text{spin}(V)$ and $v \in V$.

$\mathbb{R}_{p,q}^*$ acts naturally on $\mathbb{R}_{p,q}$ as an algebra of homomorphisms through its adjoint representation

$$\text{Ad} : \mathbb{R}_{p,q}^* \to \text{Aut}(\mathbb{R}_{p,q}), \quad u \mapsto \text{Ad}_u, \quad \text{with } \text{Ad}_u \longmapsto ux\bar{u}^{-1}.$$ 

The Clifford-Lipschitz group is the set

$$\Gamma(p,q) = \{ u \in \mathbb{R}_{p,q}^* | \forall x \in \mathbb{R}^{p,q}, ux\bar{u}^{-1} \in \mathbb{R}^{p,q} \}.$$ 

(2.7)

The set $\Gamma_{p,q}^* = \Gamma_{p,q} \cap \mathbb{R}_{p,q}$ is called special Clifford-Lipschitz group [7,11,22]. The Pinor group $\text{Pin}_{p,q}$ is the subgroup of $\Gamma_{p,q}$ such that

$$\text{Pin}_{p,q} = \{ u \in \Gamma_{p,q} | N(u) = \pm 1 \} \quad \text{and} \quad N : \mathbb{R}_{p,q} \to \mathbb{R}_{p,q}, \quad N(x) = \frac{<x,x>}{2}.$$

Example 1. The Clifford-Lipschitz group $\Gamma_{p,q}$ is a Lie subgroup of the Lie group $\text{Cl}_{p,q}^*$. Its associated Lie algebra is a vector subspace of $\text{Cl}_{p,q}$ [5,30].
Let us suppose that \( X \) is an element in the Lie algebra of \( \Gamma_{p,q} \) in such a way that \( \exp(tX) \) is an element of \( \Gamma_{p,q} \), that is,

\[
f(t) = Ad \exp(tX)(v) = \exp(tX)v \exp(-tX) \in \mathbb{R}_{p,q},
\]

for all \( v \in \mathbb{R}_{p,q} \). Defining \((ad(X))(v) = [X, v] = Xv - vX\), and using the well known results

\[
Ad(\exp(tX)) = \exp(ad(tX)),
\]

we have that \( f(t) \in \mathbb{R}_{p,q} \) if and only if \( ad(X)(v) = Xv - vX \in \mathbb{R}_{p,q} \).

It can be proved that \( X \in \Gamma_{p,q} \) is written as \( X \in \text{Cen}(\mathbb{C}U_{p,q}) \oplus \wedge^2(\mathbb{R}_{p,q}) \). In this way, \( \exp(tX) \in \Gamma_{p,q} \), \( R \in \text{Spin}^+(p, q) \) if and only if \( \tilde{R} = R \), and for \( R = \exp(tX) \), \( X \) must be written as \( X = a + B \), where \( a \in \mathbb{R}, B \in \wedge^2(\mathbb{R}_{p,q}) \). Besides, the condition \( R\tilde{R} = 1 \) implies that

\[
1 = \exp(t\tilde{X}) \exp(tX) = \exp(2ta),
\]

that is, \( a = 0 \). Then,

\[
\exp(tB) = R \in \text{Spin}^+(p, q), B \in \wedge^2(\mathbb{R}_{p,q}).
\]

The Lie algebra of \( \text{Spin}^+(p, q) \), denoted by \( \text{spin}^+(p, q) \), is generated by the space of two vectors endowed with the commutator. Indeed, if \( B \) and \( C \) are bivectors, then

\[
BC = \langle BC \rangle_0 + \langle BC \rangle_2 + \langle BC \rangle_4,
\]

and since \( \tilde{B} = -B, \tilde{C} = -C \), it follows that \( \tilde{B}C = \tilde{C}B = CB \) and so \( BC = CB = \langle BC \rangle_0 + \langle BC \rangle_2 + \langle BC \rangle_4 \), from which we obtain

\[
BC - CB = [B, C] = 2\langle BC \rangle_2,
\]

that is, \( (\wedge^2(\mathbb{R}_{p,q}), [\cdot, \cdot]) = \text{Spin}^+(p, q) \).

2.4. Root System for Semi-Simple Lie Group

Eugene Dynkin [9], based on a geometric method of classifying all simple Lie groups, proved that the semi-simple Lie group is determined by its system of simple roots. Let \( G \) be a semi-simple Lie group and let \( \sum(G) \) be the system of its roots. The root system \( \sum = \sum(G) \) satisfies the following:

1. If \( a \in \sum \), then \(-a \in \sum \), but \( ka \notin \sum \) for \( k = 2, 3, \ldots \).
2. Let \( a \) and \( b \) to be two different roots. If \( b + ia \in \sum \) for \(-p \leq i \leq q \), while \( b - (p + 1)a \notin \sum \) and \( b + (q + 1)a \notin \sum \), then

\[
p - q = \frac{2(b, a)}{(a, a)}.
\]

3. If two systems \( \sum(G_1) \) and \( \sum(G_2) \) are similar, that is, are transformed one into another by a homothety of \( \mathbb{R}^n \), then they coincide.
4. If, in particular, \( G \) is a simple group, then \( \sum(G) \) cannot be split into two orthogonal subsets \( \sum_1 \) and \( \sum_2 \).
Finite-dimensional Lie algebras are classified into \( A_n = su(n + 1), B_n = so(2n + 1), C_n, D_n = so(2n), E_n(n = 6, 7, 8), F_4, \) and \( G_2. \) We consider the following classical Lie algebras \( B_n = so(2n + 1) \) and \( D_n = so(2n) (n \geq 1) \) and their corresponding classical Lie groups \([4,21]\), in particular the compact groups \( SO(2n + 1) \) and \( SO(2n) \), respectively. The group \( SO(2n + 1) \) is the group of orthogonal transformations of a \((2n+1)\)-dimensional complex vector space \([9]\). The root system

\[
\sum(B_n) = \{ \pm e_p, \pm e_p \pm e_q \}^{n}_{p,q} = 1 \quad (p \neq q),
\]

(2.8)

where \( \{e_1, \ldots, e_{2n+1}\} \) is an orthonormal basis of \( \mathbb{R}^{2n+1} \).

\( SO(2n) \) is the group of orthogonal transformations of a \( 2n \)-dimensional complex vector space \([9]\). The root system

\[
\sum(D_n) = \{ \pm e_p \pm e_q \}^{n}_{p,q} = 1 \quad (p \neq q),
\]

(2.9)

where \( \{e_1, \ldots, e_{2n}\} \) is an orthonormal basis of \( \mathbb{R}^{2n} \). Here, \( \sum(G) \) is a finite subset of an \( n \)-dimensional real Euclidean vector space of \( \mathbb{R}^n \) as defined in \([9]\). The systems \( \sum(B_n) \) and \( \sum(D_n) \) of its roots completely determine \( SO(2n+1) \) and \( SO(2n) \). \( B_n \) and \( D_n \) as well as their associated classical groups \( SO(2n+1) \) and \( SO(2n) \) are of great importance in particle physics \([9,20]\).

Suppose \( G \) is a connected Lie group. A semi-simple Lie group \( G \) is completely determined by the system \( \Pi(G) \) of its simple roots.

### 2.5. Spin Lie Group and its Lie Algebra

Let \((M, g, \nabla, \tau_g, \uparrow)\) be the spacetime structure (see \([23]\)) where:

- \( M \) denotes a \( n \)-dimensional manifold that we assume is a compact, para-compact, pseudo-Riemannian manifold which admits spinor fields;
- \( g \) is metric;
- \( \nabla \) denotes the connection associated to \( g; \)
- \( \tau_g \) defines a spacetime orientation and \( \uparrow \) refers to a time orientation;
- \( T^*M \) (resp. \( TM \)) denotes the co-tangent (resp. tangent) bundle over \( M; \)
- \( F(M) \) denotes the principal bundle of frames;
- \( P_{SO_{p,q}}(M) \) denotes the orthonormal co-frame bundle. Such bundles do exist on spin manifolds. Sections of \( P_{SO_{p,q}}(M) \) are orthonormal co-frames, and sections of \( P_{Spin_{p,q}}(M) \) are also orthonormal co-frames such that although two co-frames differing by \( 2\pi \) rotation are distinct, two co-frames differing by \( 4\pi \) are identified.

**Definition 2.2.** \([18,22,23]\) A spin structure on \( M \) consists of a principal fiber bundle \( \pi : P_{Spin_{p,q}}(M) \longrightarrow M \) with a group \( P_{Spin_{p,q}} \) and the fundamental map (two fold cover)

\[
s : P_{Spin_{p,q}}(M) \longrightarrow P_{SO_{p,q}}(M),
\]

satisfying the following conditions:

1. \( \pi(s(p)) = \pi_s(p) \) for every \( p \in P_{Spin_{p,q}}(M) \); \( \pi \) is the projection map of the bundle \( P_{SO_{p,q}}(M) \).
(ii) \( s(pu) = s(p)Ad_u \) for every \( p \in P_{\text{Spin}_{p,q}}(M) \) and

\[
Ad : \text{Spin}_{p,q} \rightarrow \text{Aut}(\mathcal{C}_{p,q}), \quad Ad_u : \mathbb{R}_{p,q} \mapsto uxu^{-1} \in \mathcal{C}_{p,q}.
\]

The following diagram commutes:

\[
\begin{array}{ccc}
P_{\text{Spin}_{p,q}}(M) & \xrightarrow{s} & P_{\text{SO}_{p,q}}(M) \\
\downarrow{\pi_s} & & \downarrow{\pi} \\
M & &
\end{array}
\]

**Definition 2.3.** [18, 23] A spin manifold is an orientable manifold \( M \) together with a spin structure on the tangent bundle of \( M \).

An oriented Riemannian manifold \( M \) is a spin manifold if and only if the second Steifel-Whitney class of its tangent bundle vanishes. If this condition is fulfilled, then the set of spin structure on \( (M, g) \) stands in one-to-one correspondence with \( H^1(M, \mathbb{Z}_2) \).

The Lie algebra \( \text{spin}(j) \) of spin particles can be represented by classical matrices, which makes it easier to see their algebraic nature [19, 25, 33]:

\[
\text{spin}(j) = \begin{cases} 
\text{higgs} & j = 0; \\
\text{fermions} & j = \frac{1}{2}\mathbb{Z} \text{ when odd integer spins are considered;} \\
\text{bosons} & j = \mathbb{Z} \text{ when positive integer spins are considered.}
\end{cases}
\]

The Lie algebra \( \mathfrak{sl}(2n, \mathbb{C}) \) can represent the fermion spin Lie algebra of elementary particles in quantum physics [28]. As indicated in the mapping below. We define \( \frac{1}{2}\mathbb{Z} \) as fraction of the form \( \frac{2k+1}{2}, k = 0, 1, 2, 3, \ldots \) [12, 28, 33]:

\[
\mathfrak{sl}(2n, \mathbb{C}) \longrightarrow \text{spin} \left( \frac{1}{2}\mathbb{Z} \right) \longrightarrow \text{fermions}.
\]

The Lie algebra \( \mathfrak{sl}(2n+1, \mathbb{C}) \) can represent the boson spin Lie algebra of elementary particles. The map below gives a clear view with \( \mathbb{Z} \) as the set of positive integers:

\[
\mathfrak{sl}(2n+1, \mathbb{C}) \longrightarrow \text{spin}(\mathbb{Z}) \longrightarrow \text{bosons}.
\]

Parafermions and parabosons have creation and annihilation operators that correspond to the Dynkin’s root \( B_n \). We can lift the results from the Lie algebra level to the Lie group level. The classical Lie algebra matrices have corresponding Lie group analogues [4, 21].

The groups \( \text{GL}(n, \mathbb{C}), \text{SL}(n, \mathbb{C}), \text{SL}(n, \mathbb{C}), \text{SU}(p, q), \text{SU}^*(2n), \text{SU}(n), \text{U}(n), \text{SO}(n, \mathbb{C}), \text{SO}(n), \text{SO}^*(2n), \text{Sp}(n, \mathbb{C}), \text{Sp}(n), \text{Sp}(2, \mathbb{R}), \text{Sp}(p, q) \) are all connected. For more details see [13].

The groups \( \text{SL}(n, \mathbb{C}) \) and \( \text{SU}(n) \) are simply connected.

The groups \( \text{GL}(n, \mathbb{R}) \) and \( \text{SO}(p, q) \) \((0 < p < p + q)\) have two connected components.

The group \( \text{SO}(2n+1, \mathbb{C}) \) is doubly connected and \( \text{SO}(2n, \mathbb{C}) \) is fourfold connected [13].
SU\(^*(2n)\) is the group of matrices in SL(2\(n\), C) which commute with the transformations \(\psi\) of \(\mathbb{C}^{2n}\) given by

\[
(\tilde{z}_1, \ldots, \tilde{z}_n, \tilde{z}_{n+1}, \ldots, \tilde{z}_{2n}) \mapsto (\tilde{z}_{n+1}, \ldots, \tilde{z}_{2n}, -\tilde{z}_1, \ldots, -\tilde{z}_n)
\]

and SO\(^*(2n)\) is the group of matrices in SO(2\(n\), C) which leave invariant the skew Hermitian form

\[
-z_1 \bar{z}_{n+1} + z_{n+1} \bar{z}_1 - z_2 \bar{z}_{n+2} + z_{n+2} \bar{z}_2 - \cdots - z_n \bar{z}_{2n} + z_{2n} \bar{z}_n,
\]

where \((z_1, \ldots, z_n) \in \mathbb{C}^n\). We denote by Spin\((J)\) the spin Lie group of a quantum spin particle as follows:

\[
\text{Spin}(J) = \begin{cases} 
\text{Higgs} & J = 0; \\
\text{Fermions} & J = \frac{1}{2} \mathbb{Z} \text{ when odd integer spins are considered;} \\
\text{Bosons} & J = \mathbb{Z} \text{ when positive integer spins are considered.}
\end{cases}
\]

The Lie group SL\((2n, \mathbb{C})\) structure can represent the fermion Spin Lie group analog [12,28,33]:

\[
\text{SL}(2n, \mathbb{C}) \longrightarrow \text{Spin}\left(\frac{1}{2} \mathbb{Z}\right) \longrightarrow \text{fermions},
\]

while the Lie group SL\((2n+1, \mathbb{C})\) represents the boson Spin Lie group analog [12,28,33]:

\[
\text{SL}(2n+1, \mathbb{C}) \longrightarrow \text{Spin}(\mathbb{Z}) \longrightarrow \text{bosons}.
\]

For integer \(n \geq 1\) of the quantum state of fermions and bosons, one may easily detect the following using the angular momentum coupling of spin particles [12,25,28,33]:

\[
\text{Spin}\left(\frac{1}{2}\right) = \{\text{quantum spanned by 2 states with 2 } \times \text{ 2 matrix basis}\},
\]

\[
\text{Spin}\left(\frac{2n-1}{2}\right) = \{\text{quantum spanned by 2n states with 2n } \times \text{ 2n matrix basis}\},
\]

\[
\text{Spin}(1) = \{\text{quantum spanned by 3 states with 3 } \times \text{ 3 matrix basis}\},
\]

\[
\text{Spin}(n) = \{\text{quantum spanned by 2n + 1 states with (2n + 1) } \times \text{ (2n + 1) matrix basis}\}.
\]

Spin half particles are fermions described by Fermi-Dirac statistics and have quantum numbers described by Pauli exclusion principle [33]. They include the electron, proton, neutron, quarks and leptons. In particle physics, all these particles have symmetry and matrix representations.

2.6. SU(2) and Wigner Coefficients

Certain part of the work will follow known results of the SU(2) group. It will be useful to show the SU(2) model explicitly as a review to our discussion of the group SU(1, 1). We follow the same notation as in Holman and Biedenharn [14]. SU(2) is the group of transformations in a 2-dimensional unitary space, that is, the group of transformations leaving the form \(|x_1|^2 + |x_2|^2\) invariant with \(x_1, x_2 \in \mathbb{C}\). This group is simply compact and its Lie algebra is spanned by three generators \(J_1, J_2, \) and \(J_3\) obeying the commutation rule [14]:

\[
[J_i, J_j] = i\epsilon_{ijk}J_k.
\]
Next, we define a set of creation and annihilation operators \(a_i^*, a_j, a_i, a_j^*\), which obey the commutation rule

\[
[a_i, a_j^*] = \delta_{ij}, \quad i, j = 1, 2,
\]

while all other commutators vanish. The vacuum is defined by

\[
a_i |0\rangle = 0,
\]

and the states by

\[
|j, m\rangle = (a_1^*)^j (a_2^*)^m |0\rangle,
\]

where \(j\) is defined as the negative of the minimum of \(m\) or positive of the maximum of \(m\). Since SU(2) is compact, the representation must be finite dimensional [14]:

\[
J_+ = a_1^* a_2^*; \quad J_- = a_2^* a_1; \quad J_z = \frac{1}{2} (a_1^* a_1 - a_2^* a_2).
\]

We can find familiar results in [14]

\[
J_z |J, M\rangle = M\hbar |J, M\rangle,
\]

\[
J_\pm |J, M\rangle = \hbar \sqrt{J(J+1) - M(M \pm 1)} |J, M \pm 1\rangle,
\]

where \(\hbar = \frac{\hbar}{2\pi}\) is the reduced Plank’s constant. Furthermore, the operators \(e_i\) defined as

\[
e_1 = \frac{J_+ + J_-}{2}, \quad e_2 = \frac{J_+ - J_-}{2i}, \quad e_3 = J_z,
\]

obey the commutation relations (2.10). This gives a Lie algebra realization of SU(2) since equation (2.10) constitutes a necessary and sufficient condition; thus we have established a mapping from the generators of the group onto the operators \(e_i\) [2,14]:

\[
J_i \rightarrow e_i, \quad i = 1, 2, 3,
\]

which provides representation states for the group algebra on the states \(|j, m\rangle\).

We now derive the Wigner coefficients of the group, that is, the coefficients coupling two states(see [14,28,32]):

\[
|J, M\rangle = \sum_{m_1, m_2} C_{m_1 m_2 M}^{J_1 J_2 J} |j_1, m_1\rangle |j_2, m_2\rangle \delta_{m_1 + m_2, M}.
\]

To do this, we consider the coupling of two spins one-half. Denoting by \(S_1\) and \(S_2\) two spin \((\frac{1}{2}\)) angular momenta, we define the total spin

\[
S = S_1 + S_2.
\]

The eigenstates of \(S_1^2, S_{1z}, S_2^2\) and \(S_{2z}\) satisfy

\[
S_1^2 |s_1, m_1\rangle = s_1 (s_1 + 1) \hbar^2 |s_1, m_1\rangle,
\]

\[
S_2^2 |s_2, m_2\rangle = s_2 (s_2 + 1) \hbar^2 |s_2, m_2\rangle,
\]

\[
S_{1z} |s_1, m_1\rangle = m \hbar |s_1, m_1\rangle,
\]

\[
S_{2z} |s_2, m_2\rangle = m \hbar |s_2, m_2\rangle.
\]
We define the raising and lowering operators as follows [14,28]:
\[
S_{1\pm} |s_1, m_1 \rangle = \hbar \sqrt{s_1(s_1 + 1) - m_1(m_1 + 1)} |s_1, m_1 \pm 1 \rangle,
\]
\[
S_{2\pm} |s_2, m_2 \rangle = \hbar \sqrt{s_2(s_2 + 1) - m_2(m_2 + 1)} |s_2, m_2 \pm 1 \rangle,
\]
and the tensor product basis vectors as:
\[
|s_1, m_1 : s_2, m_2 \rangle = |s_1, m_1 \rangle \otimes |s_2, m_2 \rangle,
\]
where \(-s_1 \leq m_1 \leq s_1\) and \(-s_2 \leq m_2 \leq s_2\). We seek a transformation to a basis, denoted by \(|S, M\rangle\), which obeys [2,14,28]:
\[
(2.11) \quad S^2 |S, M\rangle = S(S+1)\hbar^2 |S, M\rangle,
\]
\[
(2.12) \quad S_z |S, M\rangle = M\hbar |S, M\rangle,
\]
\[
(2.13) \quad S_\pm |S, M\rangle = \hbar \sqrt{S(S+1) - M(M \pm 1)} |S, M \pm 1\rangle.
\]
In relation to the unitary transformation
\[
|s, m\rangle = \sum_{m_1, m_2} U^{s_1s_2}_{m_1m_2;sm} |m_1m_2\rangle,
\]
where \(U^{s_1s_2}_{ij}\) is the \((ij)\)-th element of the unitary matrix \(U^{s_1s_2}\) that transforms the basis \(|m_1m_2\rangle\) to the basis \(|s, m\rangle\) [28]. Using the closure property of the basis \(|m_1m_2\rangle\),
\[
|s, m\rangle = \sum_{m_1, m_2} |s_1s_2m_1m_2\rangle \langle s_1s_2m_1m_2 | s, m \rangle,
\]
and comparing equations (2.14) and (2.15) leads to:
\[
U^{s_1s_2}_{m_1m_2;sm} \equiv \langle s_1s_2m_1m_2 | s, m \rangle.
\]
The Clebsch-Gordan coefficients (CG) are obtained as:
\[
U^{s_1s_2}_{m_1m_2;sm} := C^{s_1s_2s}_{m_1m_2m}.
\]
The Wigner coefficients [32] of SU(2) are then derived as:
\[
C^{s_1s_2S}_{m_1m_2M} = [2S+1]^{1/2} (-1)^{s_2+m_2} \left[ \frac{(S+s_1-s_2)!(S-s_1+s_2)!(s_1+s_2-S)!}{(S+s_1+s_2+1)!(s_1-m_1)!(s_1+m_1)!} \right]^{1/2}
\times \frac{(S+M)!(S-M)!}{(s_2+m_2)!(s_2-m_2)!} 
\times \sum_k (-1)^k \frac{(S+s_2+m_1-k)!(s_1-m_1+k)!}{k!(S-s_1+s_2-k)!(M+S+k)!(s_1-s_2-M+k)}.
\]
Details can be found in [2,14,28].

3. Spin Semi-Simplicity

While investigating the general semi-simple Lie group structure, one can examine a similar structure in its Lie algebra. In this section, we prove some lemmas, which help us lift the notion of spin particle Lie algebra to the Lie group level, and prove a statement giving a clear picture of spin particles as Lie groups. Finally, we prove a theorem on its semi-simplicity [8].
Lemma 3.1. Any Lie algebra of a spin particle admits a Clifford algebra and a spin group structure.

Proof. Consider any spin$(j)$ with $j = 0, \frac{1}{2}, 1, \ldots$, satisfying the spin particle commutator and anticommutator relations (2.1), (2.2), (2.3) as well as the spin Lie algebra commutation bracket rule. From (2.5), (2.7), and example (1) it is obvious that the Lie algebra spin$(j)$ is a Clifford algebra. Thus, the spin$(j)$ exponential is just

$$\exp : \text{spin}(j) \to \text{Spin}(J)$$

where Spin$(J)$ is the spin group. Hence, any spin particle admits a spin group. □

Lemma 3.2. Any spin group of a spin particle admits an almost complex spin manifold (Riemannian manifold) and a spin Lie group structure.

Proof. From Lemma 3.1, any spin particle admits a spin group. Also, from Definition 2.2, the spin group, say Spin$(J)$, has a group structure with an almost complex manifold. Thus, from Definition 2.3, the spin particle, say Spin$(J)$ with $J = 0, \frac{1}{2}, \ldots$, admits a spin manifold. Next, we see that any spin particle has a spin group, say Spin$(J)$. Since any spin particle has a spin manifold, we observe that Spin$(J)$ is a spin group and, hence, a spin Lie group. □

Proposition 3.3. Any spin half odd integer (resp. integer spin) Lie group is a fourfold cover of the compact Lie group SO$(2n)$ (resp. a double cover of SO$(2n + 1)$).

Proof. The fermion quantum structure can be given as:

$$\text{Spin}(J) \longrightarrow \text{Spin}(\frac{1}{2}\mathbb{Z}).$$

The map

$$\text{SL}(2n, \mathbb{C}) \longrightarrow \text{Spin}(\frac{1}{2}\mathbb{Z}) \longrightarrow \text{SO}(2n),$$

where SO$(2n)$ conserves the quadratic form in $\mathbb{C}^{2n}$. The compact simple Lie group SO$(2n)$ is fourfold connected and its center $Z(G)$ is $\mathbb{Z}_4$ when $n$ is odd or $\mathbb{Z}_2 \times \mathbb{Z}_2$ when $n$ is even. Since Spin$(\frac{1}{2}\mathbb{Z})$ is a fermion with $\mathbb{Z}$ as an odd integer, the diagram

$$\text{SL}(2n, \mathbb{C}) \longrightarrow \text{Spin}(\frac{1}{2}\mathbb{Z})$$

must commute. Thus, the fermion Spin Lie group is a fourfold cover of SO$(2n)$. Similarly, the boson quantum structure can be given as:

$$\text{Spin}(J) \longrightarrow \text{Spin}(\mathbb{Z}).$$
where $Z$ is an integer. The map
\[ \text{SL}(2n + 1, \mathbb{C}) \rightarrow \text{Spin}(Z) \rightarrow \text{SO}(2n + 1), \]
where $\text{SO}(2n + 1)$ conserves the quadratic form in $\mathbb{C}^{2n+1}$. The compact simple Lie group $\text{SO}(2n + 1)$ is doubly connected and its center $Z(G)$ is $\mathbb{Z}_2$. Since $\text{Spin}(Z)$ is a boson, where $Z$ is an integer, the diagram
\[ \text{SL}(2n + 1, \mathbb{C}) \rightarrow \text{Spin}(Z) \rightarrow \text{SO}(2n + 1) \rightarrow \text{boson} \]
must commute. We conclude that the boson spin Lie group is the double cover of $\text{SO}(2n + 1)$. See [13] for more details. \qed

**Theorem 3.4.** Any spin Lie group $\text{Spin}(J)$ of a spin particle is:

(i) connected;
(ii) semi-simple if and only if its simple roots are one of the Dynkin’s root systems $\Pi(B_n)$ or $\Pi(D_n)$ associated with the classical groups $\text{SO}(2n+1)$ and $\text{SO}(2n)$, respectively.

**Proof of (i).** We let $\text{Spin}(J)$ be a spin Lie group with $J = 0, \frac{1}{2}, 1, \ldots$. For $\text{Spin}(0)$, $\text{Spin}(\frac{1}{2})$, and $\text{Spin}(1)$, we have, respectively, the diagram:

\[ \text{SL}(1, \mathbb{C}) \rightarrow \text{Spin}(0) \rightarrow \text{SO}(1) \rightarrow \text{boson (Higgs)}, \]
\[ \text{SL}(2, \mathbb{C}) \rightarrow \text{Spin}(\frac{1}{2}) \rightarrow \text{SO}(2) \rightarrow \text{fermion}, \]
\[ \text{SL}(3, \mathbb{C}) \rightarrow \text{Spin}(1) \rightarrow \text{SO}(3) \rightarrow \text{boson}. \]

From Section 2, the Lie groups $\text{SO}(1)$, $\text{SO}(2)$ and $\text{SO}(3)$ are connected [13]. The results can be extended to all spin Lie groups of elementary particles as shown in Proposition 3.3. Fermions and bosons spin Lie groups are fourfold connected and double connected, respectively. More specifically, the spin Lie groups such as $\text{Spin}(\frac{1}{2})$ fourfold covers the compact Lie group $\text{SO}(2)$, while $\text{Spin}(1)$ double covers $\text{SO}(3)$. \qed

**Proof of (ii).** We know that the creation and annihilation operators generate a semi-simple Lie algebra $\mathfrak{g}$ of rank $n$ which is a direct sum of classical Lie
algebras
\[ g = B_{m_1} \oplus \cdots \oplus B_{m_k}, \]
where \( m_1 + \cdots + m_k = n \). Therefore, the creation and annihilation operators of spin particles generate simple Lie algebra \( g \) of rank \( n \) isomorphic to the classical algebra \( B_n \) with a complete system \( \Phi \) of roots orthogonal with respect to the Killing form \([19]\). Also, from equations (2.1) and (2.2), when we compare the bracket relation to that of the Dynkin’s root \( \sum D_n \), see equation (2.9), we observe that there is a correspondence.

In Lemma 3.2, we showed that every spin group of a spin particle is a spin Lie group. We can determine the system of simple roots \( \Pi(B_n) \) and \( \Pi(D_n) \) associated with the classical groups \( \text{SO}(2n) \) and \( \text{SO}(2n + 1) \). A semi-simple Lie group \( G \) is completely determined by the system \( \Pi(G) \) of its simple roots \([9]\). Thus, the spin Lie group of a spin particle is completely determined by \( \Pi(G) \) of its simple roots. The converse is trivial since the classical groups \([4,21]\) \( \text{SO}(2n) \) and \( \text{SO}(2n + 1) \) correspond with the \( \Pi(B_n) \) and \( \Pi(D_n) \) (Dynkin’s root system), which are the operators of the quantum spin particles generated by the creation and annihilation operators of rank \( n \), since the spin Lie group is connected and its Lie algebra is semi-simple. Thus, the spin Lie group is semi-simple. \( \square \)

4. Real Lie Algebra of Spin Particle

The Lie algebra \( \mathfrak{sl}(2, \mathbb{C}) \) can be decomposed into the compact real form \( \mathfrak{su}(2) \) and the imaginary form \( i\mathfrak{su}(2) \), or \( \mathfrak{sl}(2, \mathbb{R}) \) and \( i\mathfrak{sl}(2, \mathbb{R}) \). It is only natural to seek the real form of the spin half particle Lie algebra in terms of Pauli matrices \([33]\), which are \( \mathfrak{sl}(2, \mathbb{C}) \) matrix basis elements.

**Proposition 4.1.** The real Lie algebra \( \mathfrak{spin}_R(\frac{1}{2}) \) of spin half particles is given by
\[ \mathfrak{spin}_R(\frac{1}{2}) = \{ S \in M_2(\mathbb{R}) \mid \text{Tr} S = 0 \}. \]

1. The elements
\[ S_k = \frac{\hbar}{2} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad S_z = \frac{\hbar}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad S_+ = \hbar \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \]
form a basis of \( \mathfrak{spin}_R(\frac{1}{2}) \).

2. The commutation relations are given by:
\[ [S_k, S_z] = -\hbar S_x, \quad [S_k, S_+] = \hbar S_z, \quad [S_z, S_+] = \hbar S_. \]

Take an arbitrary angular momentum spin \( (\frac{1}{2}) \) with spinors
\[ \chi = \begin{pmatrix} a \\ b \end{pmatrix} = a\chi_{\frac{1}{2}} + b\chi_{-\frac{1}{2}}, \quad \chi_{\frac{1}{2}} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \text{and} \quad \chi_{-\frac{1}{2}} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \]

Let
\[ S_x = \frac{S_+ + S_-}{2} \quad \text{and} \quad S_y = \frac{S_+ - S_-}{2i}. \]

From equations (2.11) and (2.12), we find:
\[ S^2 \chi_{\frac{1}{2}} = \hbar^2 \frac{1}{2} \left( \frac{1}{2} + 1 \right) \left| \chi_{\frac{1}{2}} \right\rangle = \frac{3}{4} \hbar^2 \chi_{\frac{1}{2}}, \quad S^2 \chi_{-\frac{1}{2}} = \hbar^2 \frac{3}{4} \chi_{-\frac{1}{2}}. \quad (4.1) \]
From (4.1), we deduce that
\[ S^2 = \frac{3}{4} \hbar^2 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \frac{3}{4} \hbar^2 I, \]
where \( I \) is the 2 \( \times \) 2 identity matrix. Similarly,
\[ S_z \chi_\frac{1}{2} = \frac{\hbar}{2} \chi_\frac{1}{2} \quad \text{and} \quad S_z \chi_{-\frac{1}{2}} = -\frac{\hbar}{2} \chi_{-\frac{1}{2}}. \]
Therefore,
\[ S_z = \frac{\hbar}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \frac{\hbar}{2} \sigma_z. \]
By analogous computation, we find:
\[ S_x = \frac{S_+ + S_-}{2} = \frac{\hbar}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \frac{\hbar}{2} \sigma_x \] (4.2)
and
\[ S_y = \frac{S_+ - S_-}{2i} = -iS_k = \frac{-i\hbar}{2} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \]
\[ = -\frac{i\hbar}{2} \sigma_k = \frac{\hbar}{2} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = \frac{\hbar}{2} \sigma_y. \] (4.3)
From the commutation relations (2.4) we observe that:
(i) \([x, y] \in \text{spin}_R(\frac{1}{2})\) if \(x, y \in \text{spin}_R(\frac{1}{2})\);
(ii) \([x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0\) for \(x, y, z \in \text{spin}_R(\frac{1}{2})\);
(iii) \([x, y] = -[y, x]\) if \(x, y \in \text{spin}_R(\frac{1}{2})\).
Thus, \(\text{spin}_R(\frac{1}{2})\) is a real Lie algebra. It is trivial that
\[ [S_k, S_z] = -\hbar S_x, \quad [S_k, S_+] = \hbar S_z, \quad \text{and} \quad [S_z, S_+] = \hbar S_. \]

**Lemma 4.2.** For a \(\text{spin}(\frac{1}{2})\) there exists an orthogonal (skew symmetric) matrix element \(S_k\) (with \(\hbar = 1\)), which can be transformed into an \(\text{SO}(2)\) compact Lie group. For \(g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}(2, \mathbb{R}) = G\), the stabilizer of \(i \in \mathbb{C}\) under the action of \(g\) is the subgroup \(K = \text{SO}(2)\).

**Proof.** From (4.3), we have:
\[ S_y = \frac{S_+ - S_-}{2i} = -iS_k = \frac{-i\hbar}{2} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \frac{-i\hbar}{2} \sigma_k = \frac{\hbar}{2} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = \frac{\hbar}{2} \sigma_y, \]
where \(S_k\) is a matrix in the basis of \(\text{spin}(\frac{1}{2})\) from Proposition 4.1. For skew symmetric matrix, we have \(\sigma_k^T = -\sigma_k\) while for an orthogonal matrix, we have \(\sigma_k^{-1} = \sigma_k^T\). Also, we see that for any \(t \in \mathbb{R}\),
\[ \exp(t\sigma_k) = \begin{pmatrix} \cos \frac{t}{2} & \sin \frac{t}{2} \\ -\sin \frac{t}{2} & \cos \frac{t}{2} \end{pmatrix} = k_t. \]
\[ \det k_t = 1, \text{ and thus, } k_t \text{ is compact.} \]

Next, we show that \( St(i) = \{ g \in G \mid g \cdot i = i \} \) is the stabilizer of \( i \). Let \( g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}(2, \mathbb{R}) \), we can see that \( G = \text{SL}(2, \mathbb{R}) \) acts on the upper half plane \( P \). It is easy to see that for \( P = \{ z = x + iy \in \mathbb{C} \mid y > 0 \} \) and \( g, g' \in G \), we have \( 1 \cdot z = z \) and \((gg') \cdot z = g \cdot (g' \cdot z) \). Indeed,

\[
g \cdot i = i \iff \frac{ai + b}{ci + d} = i \iff ai + b = i(ci + d) = id - c \\
\iff a = d \text{ and } c = -b.
\]

Then,

\[
St(i) = \left\{ g = \begin{pmatrix} a & b \\ -b & a \end{pmatrix} \in G \mid a^2 + b^2 = 1 \right\}.
\]

The relation \( a^2 + b^2 = 1 \) implies that there exists \( \theta \in [0, 4\pi] \) such that \( a = \cos \frac{\theta}{2} \) and \( b = \sin \frac{\theta}{2} \), where

\[
St(i) = \left\{ g = \begin{pmatrix} \cos \frac{\theta}{2} & \sin \frac{\theta}{2} \\ -\sin \frac{\theta}{2} & \cos \frac{\theta}{2} \end{pmatrix} \mid 0 \leq \theta \leq 4\pi \right\} = K.
\]

Thus, the subgroup \( K = \text{SO}(2) \) is the compact Lie group. Therefore, \( S_k \) is matrix element in the \( \text{spin} \left( \frac{1}{2} \right) \) basis. \( \square \)

**Remark 4.3.** Pauli matrices \cite{33}, as seen in Proposition 4.1, are just the basis of \( \text{sl}(2, \mathbb{C}) \). Moreover, \( \text{sl}(2, \mathbb{C}) = \text{sl}(2, \mathbb{R}) \oplus i \text{sl}(2, \mathbb{R}) = \text{su}(2) \oplus i \text{su}(2) \), where \( \text{sl}(2, \mathbb{R}) \) and \( \text{su}(2) \) are the real forms of the complex Lie algebras \( \text{sl}(2, \mathbb{C}) \) and \( \text{su}(2, \mathbb{C}) \) \cite{16}, respectively. Similarly,

\[
\text{spin} \left( \frac{1}{2}, \mathbb{C} \right) = \text{spin}_\mathbb{R} \left( \frac{1}{2} \right) \oplus i \text{spin}_\mathbb{R} \left( \frac{1}{2} \right),
\]

where \( \text{spin}_\mathbb{R} \left( \frac{1}{2} \right) \) is the real form of the spin half Lie algebra and for \( \hbar = 1 \) \( \text{spin}_\mathbb{R} \left( \frac{1}{2} \right) \subset \text{sl}(2, \mathbb{R}) \). Thus, it is complex, and for good notation, we write \( \text{spin} \left( \frac{1}{2}, \mathbb{C} \right) \subset \text{sl}(2, \mathbb{C}) \). For the real form, we write \( \text{spin}_\mathbb{R} \left( \frac{1}{2} \right) \subset \text{sl}(2, \mathbb{R}) \). Finally,

\[
\text{spin} \left( \frac{1}{2}, \mathbb{C} \right) = \text{spin}_\mathbb{R} \left( \frac{1}{2} \right) \oplus i \text{spin}_\mathbb{R} \left( \frac{1}{2} \right).
\]

For simplicity, in the next section, we use the usual notation \( \text{spin}_\mathbb{R} \left( \frac{1}{2} \right) \) to be the real form \( \text{spin} \left( \frac{1}{2}, \mathbb{R} \right) \) of the spin half particle. Note, when \( \hbar = 1 \), \( \text{spin}_\mathbb{R} \left( \frac{1}{2} \right) = \text{su}(2) \) and \( \text{spin} \left( \frac{1}{2}, \mathbb{C} \right) \subset \text{sl}(2, \mathbb{C}) \).

5. Iwasawa Decomposition of Lie Algebra and Lie Group Levels

Following the Iwasawa decomposition, we can uniquely decompose any semi-simple \( \text{spin} \left( \frac{1}{2} \right) \) particle Lie algebra as follows:

\[
g = < S_k > \oplus < S_z > \oplus < S_+ >,
\]
where \(g\) is the Lie algebra of \(\text{Spin}(\frac{1}{2})\); \(S_k\) is the set of skew symmetric \(2 \times 2\) matrices; \(S_z\) is a set of \(2 \times 2\) real diagonal trace zero matrices; and \(S_+\) is a set of upper triangular \(2 \times 2\) matrices with zeros on the diagonal. It is just like the Iwasawa decomposition of the Lie algebra \(\mathfrak{sl}(2, \mathbb{R})\) when the value of \(\hbar = 1\) [15,16].

**Theorem 5.1.** (Iwasawa Decomposition of \(\text{Spin}(\frac{1}{2})\) Particle [27])

(i) Let \(\theta, t, \xi\) be arbitrary real numbers and let

\[
hk\theta = h\exp(\theta \sigma_k), \quad hd^\frac{1}{2}_t = h\exp(t \sigma_z), \quad hn\xi = h\exp(\xi \sigma_+). \tag{5.1}\]

Then, for \(h = 1\), the subgroups \(K, D,\) and \(N\) of \(\text{Spin}(\frac{1}{2})\) are defined as:

\[
K = \{k_\theta \mid \theta \in \mathbb{R}\}, \quad D = \{d^\frac{1}{2}_t \mid t \in \mathbb{R}\}, \quad N = \{n_\xi \mid \xi \in \mathbb{R}\}.
\]

We have:

\[
k_\theta = \begin{pmatrix}
\cos \frac{\theta}{2} & \sin \frac{\theta}{2} \\
-\sin \frac{\theta}{2} \cos \frac{\theta}{2}
\end{pmatrix}, \quad d^\frac{1}{2}_t = \begin{pmatrix}
\exp(\frac{t}{2}) & 0 \\
0 & \exp(-\frac{t}{2})
\end{pmatrix}, \quad n_\xi = \begin{pmatrix}
1 & \xi \\
0 & 1
\end{pmatrix},
\]

\[
K \cong \mathbb{R}_{\frac{4\pi \mathbb{Z}}{\mathbb{Z}}}, \quad D \cong \mathbb{R}, \quad N \cong \mathbb{R}. \tag{5.5}
\]

(ii) Any element \(g \in \text{Spin}_\mathbb{R}(\frac{1}{2})\) particle is uniquely decomposable in the form:

\[
g = k_\theta d^\frac{1}{2}_t n_\xi = \exp(\theta \langle s, m | \sigma_k | s, m \rangle) \exp(t \langle s, m | \sigma_z | s, m \rangle) \exp(\xi \langle s, m | \sigma_+ | s, m \rangle). \tag{5.1}\]

If \(\begin{pmatrix}a & b \\ c & d\end{pmatrix} \in \text{SL}(2, \mathbb{R})\), then, \(\theta, t, \xi\) in Theorem 5.1(i) are given by the relations:

\[
\exp \left(\frac{i\theta}{2}\right) = \frac{a - ic}{\sqrt{a^2 + c^2}}, \quad \exp(t) = a^2 + c^2, \tag{5.2, 5.3}
\]

and

\[
\xi = \frac{ab + cd}{a^2 + c^2}. \tag{5.4}
\]

**Proof.** Since

\[
k_\theta = \exp(\theta \sigma_k) = \exp(\theta \langle s, m | \sigma_k | s, m \rangle) = \exp \left(\frac{\theta}{2} \begin{pmatrix}0 & 1 \\ -1 & 0\end{pmatrix} \right) \tag{5.5}
\]

and

\[
k_\theta = \left[ \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} \left(\frac{\theta}{2}\right)^{2n} I + \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} \left(\frac{\theta}{2}\right)^{2n+1} \sigma_k \right]
\]

\[
= \begin{pmatrix}
\cos \frac{\theta}{2} & \sin \frac{\theta}{2} \\
-\sin \frac{\theta}{2} \cos \frac{\theta}{2}
\end{pmatrix},
\]

\[
\exp \left(\frac{t}{2}\right) = \begin{pmatrix}
\exp(\frac{t}{2}) & 0 \\
0 & \exp(-\frac{t}{2})
\end{pmatrix}
\]

\[
\exp(\xi \langle s, m | \sigma_+ | s, m \rangle) = \begin{pmatrix}
1 & \xi \\
0 & 1
\end{pmatrix},
\]

\[
K \cong \mathbb{R}_{\frac{4\pi \mathbb{Z}}{\mathbb{Z}}}, \quad D \cong \mathbb{R}, \quad N \cong \mathbb{R}. \tag{5.5}
\]
by the isomorphism \( \theta \mapsto k_\theta \), we obtain \( K \cong \frac{R}{4\pi\mathbb{Z}} \cong T \). Moreover,

\[
d_T^\frac{1}{2} = \exp(t\sigma_z) = \exp(t \langle s, m | \sigma_z | s, m \rangle) = \sum_{n=0}^{\infty} \frac{1}{n!}(t\sigma_z)^n
\]

\[
= \sum_{n=0}^{\infty} \frac{1}{n!}(t\sigma_z)^n = \left( \begin{array}{cc}
\exp\left(\frac{t}{2}\right) & 0 \\
0 & \exp\left(-\frac{t}{2}\right)
\end{array} \right)
\]

By the isomorphism \( t \mapsto d_T^\frac{1}{2} \), we also have \( D \cong \mathbb{R} \). Now, since \((\sigma_+)^2 = 0\),

\[
n_\xi = \exp(\xi\sigma_+) = \exp(\xi \langle s, m | \sigma_+ | s, m \rangle) = \left( \begin{array}{cc} 1 & \xi \\
0 & 1 \end{array} \right)
\]

By the isomorphism \( \xi \mapsto n_\xi \), we have \( N \cong \mathbb{R} \). By matrix multiplication, we have:

\[
\begin{pmatrix} a & b \\
c & d \end{pmatrix} = k_\theta d_T^\frac{1}{2} n_\xi
\]

\[
= \exp(\theta \langle s, m | \sigma_k | s, m \rangle) \exp(t \langle s, m | \sigma_z | s, m \rangle) \exp(\xi \langle s, m | \sigma_+ | s, m \rangle)
\]

\[
= \left( \begin{array}{cc}
\exp\left(\frac{t}{2}\right) \cos\frac{\theta}{2} & \cos\frac{\theta}{2} \exp\left(\frac{t}{2}\right) \xi + \sin\frac{\theta}{2} \exp\left(-\frac{t}{2}\right) \\
-\exp\left(\frac{t}{2}\right) \sin\frac{\theta}{2} - \sin\frac{\theta}{2} \exp\left(\frac{t}{2}\right) \xi + \cos\frac{\theta}{2} \exp\left(-\frac{t}{2}\right)
\end{array} \right)
\]

yielding

\[
a = \exp\left(\frac{t}{2}\right) \cos\left(\frac{\theta}{2}\right), \quad c = -\exp\left(\frac{t}{2}\right) \sin\left(\frac{\theta}{2}\right),
\]

and

\[
a - ic = \exp\left(\frac{t}{2} + i\frac{\theta}{2}\right).
\]

Hence \( |a - ic| = \exp\left(\frac{t}{2}\right) \). Moreover, we easily obtain equations (5.2) and (5.3), and

\[
ab + cd = \exp(t\xi)
\]

from which we can clearly obtain equation (5.4). \( \square \)

Recall that \( \text{spin}(\frac{1}{2}) \) is spanned by two states: \( \{|\frac{1}{2}, \frac{1}{2}\}, |\frac{1}{2}, -\frac{1}{2}\} \). From equations (2.11), (2.12), and (2.13) we can calculate the angular momentum for spin half integers such as \( \frac{1}{2}, \frac{3}{2}, \frac{5}{2} \), and so on [28].

A question arises: What can the general term (last term) of a spin half integer be? From a theoretical point of view, this can be useful in the study of particle rotational forms. We have the following result:

**Theorem 5.2.** For any \( \text{spin}(\frac{2n-1}{2}) \) (fermions) quantum state, where \( n = 1, 2, 3, \ldots \), we have:

(i) \( S^2 |S, M\rangle_n = \left( \frac{4n^2-1}{4} \right) \hbar^2 |S, M\rangle \).

(ii) \( S_z |S, M\rangle_n = \pm \left( \frac{2n-k}{2} \right) \hbar |S, M\rangle \) where \( k \leq 2n \), and \( n = 1, 2, \ldots \), with \( k = 1, 3, 5, \ldots \).
(iii) The $n^{th}$ possible state of a spin half particle is given by:

$$M_n = 2S_n + 1 = 2n,$$

where $n = 1, 2, 3, \ldots$. The quantum state of the fermion is spanned by $2n$ states:

$$\left| \left( \frac{2n-1}{2} \right), \pm \left( \frac{2n-1}{2} \right) \right>, \ldots, \left| \left( \frac{2n-1}{2} \right), \pm \left( \frac{2n-k}{2} \right) \right>,$$

where $k = 1, 3, 5, \ldots$, with $k \leq 2n$.

(iv) The ladder operators act as follows:

\[
\begin{align*}
S_+^n & \left| \left( \frac{2n-1}{2} \right), \left( \frac{2n-k}{2} \right) \right> = \hbar \sqrt{(k-1)n - \left( \frac{(k-1)(k-1)}{4} \right)} \left| S, M+1 \right>, \\
S_+^n & \left| \left( \frac{2n-1}{2} \right), -\left( \frac{2n-k}{2} \right) \right> = \hbar \sqrt{(k+1)n - \left( \frac{(k+1)(k+1)}{4} \right)} \left| S, M+1 \right>, \\
S_-^n & \left| \left( \frac{2n-1}{2} \right), \left( \frac{2n-k}{2} \right) \right> = \hbar \sqrt{(k+1)n - \left( \frac{(k+1)(k+1)}{4} \right)} \left| S, M-1 \right>, \\
S_-^n & \left| \left( \frac{2n-1}{2} \right), \left( \frac{k-2n}{2} \right) \right> = \hbar \sqrt{(k-1)n - \left( \frac{(k-1)(k-1)}{4} \right)} \left| S, M-1 \right>.
\end{align*}
\]

(v) The ladder operators can be split as:

$$S_{\pm}^n = S_x^n \pm S_{k^n}.$$

Proof of (i). For the spin half integer, we have the sequence:

$$\frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \ldots, \frac{2n-1}{2},$$

where $n = 1, 2, 3, \ldots$. Similarly, we can use the sequence

$$\frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \ldots, \frac{2n+1}{2},$$

where $n = 0, 1, 2, \ldots$. However, we will use that of equation (5.7).

From equation (2.11), we have $S^2 |S, M\rangle = S(S+1)\hbar^2 |S, M\rangle$.

For the spin one-half, $S = \frac{2n-1}{2}$, we have:

$$S^2 |S, M\rangle = \frac{2n-1}{2} \left( \frac{2n-1}{2} + 1 \right) \hbar^2 |S, M\rangle.$$
By simple computation, we arrive at:

$$S^2 |S, M\rangle_n = \left(\frac{4n^2 - 1}{4}\right) \hbar^2 |S, M\rangle.$$

□

Proof of (ii). We check the $n^{th}$ term of spin $S_n$:

$$S_n |S, M\rangle = M\hbar |S, M\rangle, \quad S_n |S, M\rangle_n = \pm \left(\frac{2n - k}{2}\right) \hbar |S, M\rangle,$$

where $k \leq 2n$ and $n = 1, 2, 3, \ldots$, with $k = 1, 3, 5, \ldots$, as required. □

Proof of (iii). The $n^{th}$ possible states of a spin half particle are given by:

$$M_{s_n} = 2S_n + 1 = 2\left(\frac{2n - 1}{2}\right) + 1 = 2n,$$

where $n = 1, 2, 3, \ldots$. For spin $\left(\frac{1}{2}\right)$, we have:

$$M_s = 2n = 2(1) = 2 \text{ states, i.e., } \left|\frac{1}{2}, \frac{1}{2}\right\rangle \text{ and } \left|\frac{1}{2}, -\frac{1}{2}\right\rangle;$$

since $\frac{2n - 1}{2} = \frac{2(1) - 1}{2} = \frac{1}{2}$. The case $n = 1$ corresponds to spin $\left(\frac{1}{2}\right)$.

Thus, for spin $\left(\frac{2n - 1}{2}\right)$ particles, the spin quantum states is spanned by $2n$ states. It is easy to check that:

$$\left|\left(\frac{2n - 1}{2}\right), \pm \left(\frac{2n - k}{2}\right)\right\rangle, \ldots, \left|\left(\frac{2n - 1}{2}\right), \pm \left(\frac{2n - k}{2}\right)\right\rangle,$$

where $k = 1, 3, 5, \ldots$, with $k \leq 2n$ as required. □

Proof of (iv). We define the ladder operators for spin $\left(\frac{2n - 1}{2}\right)$:

$$S_{+n} \left|\left(\frac{2n - 1}{2}\right), \pm \left(\frac{2n - k}{2}\right)\right\rangle.$$

From equation (2.13), we have:

$$S_\pm |S, M\rangle = \hbar \sqrt{(S \mp M)(S \mp M + 1)} |S, M \pm 1\rangle \quad (5.8)$$

and the computation provides the actions of the raising operator as given by these relations:

$$S_{+n} \left|\left(\frac{2n - 1}{2}\right), \left(\frac{2n - k}{2}\right)\right\rangle = \hbar \sqrt{\left(\frac{2n - 1}{2} - \frac{2n - k}{2}\right) \left(\frac{2n - 1}{2} + \frac{2n - k}{2} + 1\right)} |S, M + 1\rangle$$

$$= \hbar \sqrt{\left(\frac{k - 1}{2}\right) \left(\frac{4n - k + 1}{2}\right)} |S, M + 1\rangle$$

$$= \hbar \sqrt{(k - 1)n - \left(\frac{(k - 1)(k - 1)}{4}\right)} |S, M + 1\rangle,$$
and
\[ S_{+n} \left| \left( \frac{2n-1}{2} \right), \left( \frac{2n-k}{2} \right) \right\rangle = S_{+n} \left| \left( \frac{2n-1}{2} \right), \left( \frac{k-2n}{2} \right) \right\rangle = \hbar \sqrt{\left( \frac{2n-1}{2} - \frac{k-2n}{2} \right) \left( \frac{2n-1}{2} + \frac{k-2n}{2} + 1 \right)} |S, M + 1\rangle \]
\[ = \hbar \sqrt{\frac{4n-k-1}{2}} \left( \frac{k+1}{2} \right) |S, M + 1\rangle \]
\[ = \hbar \sqrt{(k+1)n - \left( \frac{(k+1)(k+1)}{4} \right)} |S, M + 1\rangle . \]

Similarly, for the lowering operator \( S_{-n} \), it is easy to obtain:
\[ S_{-n} \left| \left( \frac{2n-1}{2} \right), \left( \frac{2n-k}{2} \right) \right\rangle = \hbar \sqrt{(k+1)n - \left( \frac{(k+1)(k+1)}{4} \right)} |S, M - 1\rangle \]
and
\[ S_{-n} \left| \left( \frac{2n-1}{2} \right), \left( \frac{k-2n}{2} \right) \right\rangle = \hbar \sqrt{(k-1)n - \left( \frac{(k-1)(k-1)}{4} \right)} |S, M - 1\rangle . \]

**Proof of (v).** We recall equations (4.2) and (4.3) and apply these operators to \( \text{spin}(\frac{2n-1}{2}) \) to obtain:
\[ S_{\pm n} = S_{x_n} \pm iS_{y_n} = S_{x_n} \pm i(-iS_{k_n}) = S_{x_n} \pm S_{k_n}. \]
Hence, the proof is complete. \( \square \)

**Remark 5.3.** Note that \( S_{T n}^T = S_{-n} \). From equation (4.2), we have:
\[ S_{x_n} = \frac{S_{+n} + S_{-n}}{2}. \]
Similarly, from equation (4.3), we obtain:
\[ S_{y_n} = \frac{S_{+n} - S_{-n}}{2i} = \frac{\hbar \sigma_{k_n}}{2i} = -iS_{k_n}. \] (5.9)

For \( k \leq 2n \), the above ladder operators \( S_{+n} \) and \( S_{-n} \) act as:
\[ S_{+n} \left| \left( \frac{2n-1}{2} \right), \pm \left( \frac{2n-k}{2} \right) \right\rangle \text{ and } S_{-n} \left| \left( \frac{2n-1}{2} \right), \pm \left( \frac{2n-k}{2} \right) \right\rangle . \]
We observe that
\[ S_{+n} \left| \left( \frac{2n-1}{2} \right), \left( \frac{2n-k}{2} \right) \right\rangle = 0 \]
if and only if \( k = 1 \). Similarly,
\[ S_{-n} \left| \left( \frac{2n-1}{2} \right), \left( \frac{k-2n}{2} \right) \right\rangle = 0, \]
where \( n = 1, 2, 3, \ldots \).

**Theorem 5.4.** For any \( \text{spin}(\frac{2n-1}{2}) \), the quantum state of the particle is spanned by \( 2n \) states and there exists orthogonal matrix element \( S_{kn} \) in the \( S_{yn} \) matrix which can be transformed into the classical group \( \text{SO}(2n) \) with natural numbers \( n = 1, 2, 3, \ldots \). This compact Lie group \( \text{SO}(2n) \) corresponds to the Dynkin’s root \( \Pi(D_n) \).

**Proof.** From Lemma 4.2 we observe that the theorem is true for \( n = 1 \). For \( \text{spin}(\frac{2n-1}{2}) \) particle quantum state spanned by \( 2n \) states, we consider similar arguments for Theorem 5.4, replacing the \( S_k \) matrix by the \( n^{th} \) matrix \( S_{kn} \) and deducing in the same manner as in Lemma 4.2 to obtain the above Theorem 5.4. Specifically, from Theorem 5.2, there exists \( S_{kn} \) matrix in the \( S_{yn} \) matrix from equation (5.9). One can check that these matrices are orthogonal and deducing in the same manner as in Lemma 4.2 to obtain the above Theorem 5.4. Specifically, from Theorem 5.2, there exists \( S_{kn} \) matrix in the \( S_{yn} \) matrix from equation (5.9). One can check that these matrices are orthogonal and generate \( \text{SO}(2n) \) with \( n = 1, 2, 3, \ldots \). For \( n = 1 \) we have the compact Lie group \( \text{SO}(2) \) as in Lemma 4.2. Finally, the correspondence to the Dynkin’s roots \( \Pi(D_n) \), it is obvious from Proposition 3.3 and Theorem 3.4 [4, 21]. ☐

Theorem (5.1) shows that the mapping \((\theta, t, \xi) \mapsto k_{\theta}d_{t(\theta, \xi)}^{\frac{1}{2}}\) is an analytic diffeomorphism of \( K \times D \times N \) onto \( \text{SL}(2, \mathbb{R}) \). It is known that \( \text{SL}(2, \mathbb{R}) \) is connected. \( \text{Spin}(\frac{1}{2}) \) (Proposition 3.3) is also connected and for \( \hbar = 1 \), \( \text{Spin}(\frac{1}{2}) \subset \text{SL}(2, \mathbb{R}) \). The explicit formulae of Theorem 5.1 show that the mappings \((g, g') \mapsto gg' \) and \( g \mapsto g^{-1} \) express real analytic functions and particularly they are differentiable. Also, \( \text{Spin}(\frac{1}{2}) \) possesses a differentiable structure compatible with the algebraic operations of the group structure. We can speak of differentiable functions on the \( \text{Spin}(\frac{1}{2}) \) as well as the differential operators on \( \text{Spin}(\frac{1}{2}) \).

**Proposition 5.5.** [27] For an element \( g \in \text{Spin}_{\mathbb{R}}(\frac{1}{2}) \) and \( \theta \in \mathbb{R} \), let

\[
g_{k_{\theta}} = k_{\theta}d_{t(\theta, \xi)}^{\frac{1}{2}}
\]

be the Iwasawa decomposition of \( g_{k_{\theta}} \). If \( \hbar = 1 \), then, the following cocycle conditions hold, for \( g, g' \) which are projections in \( \text{SL}(2, \mathbb{R}) \supset \text{Spin}_{\mathbb{R}}(\frac{1}{2}) \):

1. (i) \( (gg') \cdot \theta \equiv g \cdot (g' \cdot \theta) \) (mod 4\( \pi \));
   (ii) \( t(gg', \theta) = t(g, g' \cdot \theta) + t(g', \theta) \);
   (iii) \( g \cdot (\theta + 2\pi) = g \cdot \theta + 2\pi \) (mod 4\( \pi \)), \( t(g, \theta + 2\pi) = t(g, \theta) \).

2. If \( g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \), then, we have:
   (i) \( \exp\left(\frac{i(\theta, \xi)}{2}\right) = \frac{(a - ic) \cos \frac{\theta}{2} + (-b + id) \sin \frac{\theta}{2}}{|(a - ic) \cos \frac{\theta}{2} + (-b + id) \sin \frac{\theta}{2}|^2};\)
   (ii) \( \exp(t(g, \theta)) = |(a - ic) \cos \frac{\theta}{2} + (-b + id) \sin \frac{\theta}{2}|^2;\)
   (iii) \( \xi(g, \theta) = \exp(-t(g, \theta)) \times [(ab + cd) \cos \theta + \frac{1}{2} (a^2 - b^2 + c^2 - d^2) \sin \theta];\)
   (iv) \( d\frac{(g, \theta)}{d\theta} = \exp(-t(g, \theta)).\)
Proof. Let \( g' k_\theta = k_\theta d^{\frac{1}{2}}_\xi n_\xi \), where \( \theta' = g' \cdot \theta \), \( t' = t(g', \theta) \) and \( \xi' = \xi(g', \theta) \). Then, for \( g, g' \) in \( \text{SL}(2, \mathbb{R}) \supset \text{Spin}_\mathbb{R} \left( \frac{1}{2} \right) \), we have:

\[
(gg') k_\theta = g(g' k_\theta) = (g k_{\theta'}) d^{\frac{1}{2}}_\xi n_\xi k_{\theta'} d^{\frac{1}{2}}_\xi n_\xi = k_{\theta'} d^{\frac{1}{2}}_\xi n_\xi d^{\frac{1}{2}}_\xi n_\xi = k_{\theta'} d^{\frac{1}{2}}_\xi n_\xi d^{\frac{1}{2}}_\xi n_\xi \in K DN.
\]

Using \( \theta'' = g \cdot \theta' \), \( t'' = t(g, \theta') \), and \( \xi'' = \xi(g, \theta') \), because \( N \) is normal in \( DN \), by the uniqueness of Iwasawa decomposition, we find:

\[
(gg') \cdot \theta \equiv \theta'' \equiv g \cdot \theta' \equiv g \cdot (g' \cdot \theta) \pmod{4 \pi}
\]
and

\[
t(gg', \theta) = t'' + t' = t(g, g' \cdot \theta) + t(g', \theta).
\]

We note that \( k_{\theta + 2\pi} = -k_\theta \) from which we obtain:

\[
gk_{\theta + 2\pi} = -gk_{\theta} = -k_{g \cdot \theta + 2\pi} d^{\frac{1}{2}}_{t(g, \theta + 2\pi)} n_{\xi(g, \theta + 2\pi)}.
\]

Thus, again by the uniqueness of Iwasawa decomposition, we have:

\[
g \cdot (\theta + 2\pi) = g \cdot \theta + 2\pi \pmod{4 \pi}
\]
and

\[
t(g, \theta + 2\pi) = t(g, \theta).
\]

We can express \( g \cdot \theta \) and \( t(g, \theta) \) as functions of the coefficients of \( g \):

\[
gk_{\theta} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \cos \frac{\theta}{2} & \sin \frac{\theta}{2} \\ -\sin \frac{\theta}{2} & \cos \frac{\theta}{2} \end{pmatrix} = \begin{pmatrix} a \cos \frac{\theta}{2} - b \sin \frac{\theta}{2} & a \sin \frac{\theta}{2} + b \cos \frac{\theta}{2} \\ c \cos \frac{\theta}{2} - d \sin \frac{\theta}{2} & c \sin \frac{\theta}{2} + d \cos \frac{\theta}{2} \end{pmatrix}.
\]

By direct computation as in Theorem 5.1, we find:

\[
\exp \left( i \frac{g \cdot \theta}{2} \right) \exp \left( t(g, \theta) \right) = (a - ic) \cos \frac{\theta}{2} + (-b + id) \sin \frac{\theta}{2}, \quad (5.10)
\]

\[
\exp (t(g, \theta)) = |(a - ic) \cos \frac{\theta}{2} + (-b + id) \sin \frac{\theta}{2}|^2
\]

\[
= (a^2 + c^2) \cos^2 \frac{\theta}{2} + (b^2 + d^2) \sin^2 \frac{\theta}{2} - 2(ab + cd) \cos \frac{\theta}{2} \sin \frac{\theta}{2}
\]

\[
= (a^2 + c^2) \frac{1 + \cos \theta}{2} + (b^2 + d^2) \frac{1 - \cos \theta}{2} - 2(ab + cd) \sin \theta
\]

\[
= \frac{1}{2} \left( a^2 + b^2 + c^2 + d^2 \right) + \frac{1}{2} \left( a^2 - b^2 + c^2 - d^2 \right) \cos \theta
\]

\[- (ab + cd) \sin \theta.
\]

From equation (5.10), we arrive at:

\[
\exp \left( i \frac{(g, \theta)}{2} \right) = \frac{(a - ic) \cos \frac{\theta}{2} + (-b + id) \sin \frac{\theta}{2}}{|(a - ic) \cos \frac{\theta}{2} + (-b + id) \sin \frac{\theta}{2}|}.
\]
Differentiating both sides of equation (5.10) with respect to $\theta$, we obtain:

$$
\frac{d}{d\theta}(g \cdot \theta) + \frac{dt(g, \theta)}{d\theta} = \left\{ -(a - ic) \sin \frac{\theta}{2} + (-b + id) \cos \frac{\theta}{2} \right\} \exp \left( \frac{i(g, \theta)}{2} \right) \exp \left( \frac{t(g, \theta)}{2} \right).
$$

By equating the imaginary parts of both sides of this relation, we find:

$$
\frac{d}{d\theta}(g \cdot \theta) = \frac{ad - bc}{\exp (t(g, \theta))} = \exp (-t(g, \theta)).
$$

Similarly, by direct computation as in Theorem 5.1, we have:

$$
\xi(g, \theta) = \exp (-t(g, \theta)) \left[ (ab + cd) \cos \theta + \frac{1}{2} \left( a^2 - b^2 + c^2 - d^2 \right) \sin \theta \right].
$$

**Theorem 5.6.** Any spin Lie group $G$ can be uniquely decomposed in the form:

$$
G = KKD^*N
$$

where $K$ is compact, $D^*$ is a rotational function ($d$-function), and $N$ is nilpotent (Ladder operators). We denote by $\kappa$ ($\alpha^{-1}$) the fine structure constant and all other translational energy of elementary spin particles.

**Proof.** We will give a physical interpretation to this theorem. Let us consider the Iwasawa decomposition of an electron which has a spin half. Suppose an electron is at rest in a homogeneous magnetic field. Its eigenfunctions do not depend upon its position. Given $\mu$, the magnitude of Bohr magneton and $H$, the Hamiltonian, one then obtains the system of equations for $(\psi_\alpha, \psi_\beta)$ as follows:

$$
\mu [(H_x - iH_y)\psi_\beta + H_z\psi_\alpha] = E\psi_\alpha
$$

$$
\mu [(H_x + iH_y)\psi_\alpha - H_z\psi_\beta] = E\psi_\beta
$$

from which we find that $E = \pm \mu |H|$. If we denote the angle between the field direction and the $z$-axis by $\theta$ and normalizes $(\psi_\alpha, \psi_\beta)$ by the way of

$$
|\psi_\alpha|^2 + |\psi_\beta|^2 = 1.
$$

(5.11)

This then corresponds to the determinant of the compact ($K$) function in the Iwasawa decomposition, that is, $\cos^2 \frac{\theta}{2} + \sin^2 \frac{\theta}{2} = 1$. If one suddenly rotates the external magnetic field in the $z$-direction; $\cos^2 \frac{\theta}{2}$ is then the fraction of the electron with moments that are directed parallel to the $z$-axis and $\sin^2 \frac{\theta}{2}$ is the fraction of electron with moments that are directed anti-parallel to the $z$-axis and vice versa [33].

Consider the subgroup $D^* = \{D^*_{mm'}(t) \mid t \in \mathbb{R}\}$, with

$$
D^*_{mm'}(t) = d^*_{mm'}(t) = d^*_{t}.
$$

$D^*$ is the rotational function ($d$-function) and $D^*_{mm'}(t)$ is the Clebsch-Gordon coefficient. Given the identity

$$
\exp(itK_y)K_z\exp(-itK_y) = \cosh t \cdot K_z - \frac{1}{2} \sinh t \cdot (K_+ + K_-)
$$

(5.12)
which can be defined using the Campbell-Hausdorff formula together with the SU(1, 1) commutation relation. Taking the matrix element of

\[
K_z \exp(-itK_y) = \cosh t \exp(-itK_y)K_z - \frac{1}{2} \sinh t \exp(-itK_y)(K_+ + K_-)
\]

(5.13)

between \(|s, m\rangle\) and \(|s, m'\rangle\), one can easily obtain the recurrence formula for the \(d\)-function. For the finite \((2S + 1)\)-dimensional case, we have:

\[
(m' - \cosh t \cdot m)d_{mm'}^s(t) + \frac{1}{2} \sinh t \left( -\sqrt{(s - m)(s + m + 1)}d_{mm'}^s(t) + \sqrt{(s + m)(s - m + 1)}d_{m'm-1}^s(t) \right) = 0
\]

(5.14)

which leads to the factorization of the \(d\)-function of SU(1, 1):

\[
d_{mm'}^s(t) = \frac{\Gamma(s - m + 1)\Gamma(s + m' + 1)}{\Gamma(s + m + 1)\Gamma(s - m' + 1)} \frac{1}{\Gamma(m' - m + 1)}
\]

\[
\times \left( \cosh \left( \frac{t}{2} \right) \right)^{2s+m-m'} \left( \sinh \left( \frac{t}{2} \right) \right)^{m-m'} \times_2 F_1 \left( m; \tanh^2 \left( \frac{t}{2} \right) \right)
\]

(5.15)

where \(m' \geq m\). Substituting formula (5.15) into (5.14), we obtain:

\[
(s + m)(s - m + 1) \tanh^2 \left( \frac{t}{2} \right) \times_2 F_1 \left( m - 1; \tanh^2 \left( \frac{t}{2} \right) \right)
\]

\[
+ \cosh^{-2} \left( \frac{t}{2} \right) (m' - \cosh t \cdot m) \times_2 F_1 \left( m; \tanh^2 \left( \frac{t}{2} \right) \right)
\]

\[
- (m' - m + 1)(m' - m) \cdot \times_2 F_1 \left( m + 1; \tanh^2 \left( \frac{t}{2} \right) \right) = 0.
\]

(5.16)

In order to solve equation (5.16) we use these identities:

\[
\frac{d}{dz} \times_2 F_1 (a, b; c; z) = \frac{ab}{c} \times_2 F_1 (a + 1, b + 1; c + 1; z)
\]

(5.17)

and

\[
\times_2 F_1 (a, b; c; z) = \frac{\Gamma(z)}{\Gamma(a)\Gamma(b)} \sum_k \frac{\Gamma(a + k)\Gamma(b + k)}{\Gamma(c + k)} \frac{z^k}{k!}.
\]

(5.18)

We can easily obtain

\[
(a + 1)(c - b + 1)z_2 \times_2 F_1 (a + 2, b; c + 2; z)
\]

\[
-(c + 1)(c + (a - b + 1)z) \times_2 F_1 (a + 1, b; c + 1; z)
\]

\[
+c(c + 1) \times_2 F_1 (a, b; c; z) = 0.
\]

(5.19)

From equation (5.19) we can adopt \(a = -m - s\), \(b = m' - s\), \(c = m' - m + 1\); \(z = \tanh^2 \frac{t}{2}\) to identify (5.16) with (5.19). Thus, we obtain:
\[ d_{m'm}(t) = \left[ \frac{\Gamma(s - m + 1)\Gamma(s + m' + 1)}{\Gamma(s + m + 1)\Gamma(s - m' + 1)} \right]^{\frac{1}{2}} \frac{1}{\Gamma(m' - m + 1)} \times \left( \cosh \left( \frac{t}{2} \right) \right)^{2s+m'-m'} \times \left( \sinh \left( \frac{t}{2} \right) \right)^{m-m'} \times _2F_1 \left( -m - s, m' - s; m' - m + 1; \tanh^2 \left( \frac{t}{2} \right) \right) \quad (m' \geq m). \] 

(5.20)

Using equation (5.18) one can write down explicit algebraic formulas of the \( d \)-functions. For the non-unitary representation, \( K_y \) has been taken to be anti-Hermitian. Therefore, \( \exp(-itK_y) \) is Hermitian. The hermiticity of the \( d \)-function implies:

\[ d_{m'm}(t) \equiv d_{mm'}(t), \]
\[ d_{m'm}(-t) \equiv (-1)^{m'-m}d_{m'm}(t), \]
\[ d_{m'm}(t) \equiv d_{-m'-m}(t), \]

complete symmetry relation of the \( d \)-function for the non-unitary finite dimensional representations

\[ d_{m'm}(t) = d_{mm'}(t) = d_{-m'-m}(t) = d_{-m'-m}(t). \]

(5.21)

We observe that since \( a = -m - s \) is always negative integer.

The hypergeometric functions appearing in equation (5.20) are Jacobi polynomials defined by:

\[ P_{n}^{\alpha,\beta}(x) = \frac{\Gamma(n + \alpha + 1)}{\Gamma(n + 1) \cdot (\alpha + 1)} \left( \frac{1 + x}{2} \right)^n \times _2F_1 \left( -n, -n - \beta; \alpha + 1; \frac{x - 1}{x + 1} \right). \]

(5.22)

Therefore,

\[ d_{m'm}(t) = \left[ \frac{\Gamma(s - m + 1)\Gamma(s + m' + 1)}{\Gamma(s - m' + 1)\Gamma(s + m' + 1)} \right]^{\frac{1}{2}} \left( \cosh \left( \frac{t}{2} \right) \right)^{-m-m'} \times \left( \sinh \left( \frac{t}{2} \right) \right)^{m-m'} \times _2F_1 \left( -m - m - s, m' - s; m' - m + 1; \tanh^2 \left( \frac{t}{2} \right) \right) \]

(5.23)

one can easily deduce the orthogonality of the Jacobi polynomials. The recurrence relations of the \( d \)-functions of SU(1, 1) are nothing but special cases of the Clebsch-Gordon formula of SU(1, 1) \([2,3,14]\). Similarly, the Clebsch-Gordon formula can now be obtained simply by taking the matrix element of the rotation of angle \( t \) around \( y \)-axis, \( R(t) = \exp(-itK_y) \), between the states \( |s, m\rangle \) and \( |s, m'\rangle \):

\[ d_{m'm}(t) = \sum_{m_1, m_1'} (s_1 s_2 m_1 m_2 | s, m)(s_1 s_2 m_1' m_2' | s, m')d^{s_1}_{m_1'm_1}(t) \cdot d^{s_2}_{m_2'm_2}(t) \]

(5.24)
from which we can calculate all the explicit forms of $d_t^s$ starting from $d_t^{\frac{1}{2}}$ (see [29]). We note that the form $D_{mm'}^s = d_{mm'}^{s}(t) = d_t^s$ and $d_t^{\frac{1}{2}}$ is the Abelian subgroup in the Iwasawa decomposition (see Theorem 5.1).

Finally, for the Nilpotent (N) function say, $K_+$ in the 2-dimensional non-unitary representation is the non-compact operator which generates elements of the parabolic subgroup of SU(1, 1) (see [17]). □

Remark 5.7. The Iwasawa decomposition of the spin half particle into compact, rotational (Abelian), and nilpotent functions (subgroups) can also be performed for integer spin particles as well as for isospins. From the Wigner representation of SU(2) in section 2.6, one can have an analog to the $d$-function of SU(2):

$$d_{m'm}(\psi) = \left[ \frac{\Gamma(s - m + 1)\Gamma(s + m' + 1)}{\Gamma(s + m + 1)\Gamma(s - m' + 1)} \right]^{\frac{1}{2}} \frac{1}{\Gamma(m' - m + 1)} \times \left( \cos\left(\frac{\psi}{2}\right) \right)^{2s+2m'} \left( -\sin\left(\frac{\psi}{2}\right) \right)^{m-m'} \times \ _2F_1\left(-m - s, m' - s; m' - m + 1; -\tan^2\left(\frac{\psi}{2}\right)\right) \quad (m' \geq m) \quad (5.25)$$

where $_2F_1$ is the hypergeometric function defined by the formal series (5.18).

We can also see from (5.20) that the Wigner coefficient of SU(1, 1) can be derived as follows:

$$C_{m_1m_2M}^{s_1s_2S} = C_{-s_1s_1-S-S}^{s_1s_1S} \left[ \frac{(M - S - 1)!(-2s_1 - 1)!(-2S - 1)!}{(s_1 - s_2 - S - 1)!(m_2 - s_1 - 1)!(s_1 + s_2 - S)!} \right] \times \left( \frac{(m_1 + s_1)!}{(m_2 + s_2)!(m_1 - s_1 - 1)!} \right)^{\frac{1}{2}} \times \sum_k (-1)^k \frac{(m_2 + s_2 + k)!(m_2 - s_2 - 1 + k)!}{k!(m_2 + S - s_1 + k)(m_1 + s_1 - k)!(m_2 - S - 1 - s_1 + k)!} \quad (5.26)$$

We observe that the Wigner coefficient (5.26) is an analog of that obtained from section 2.6, the Wigner coefficient of SU(2), by the replacements $s_1 \to -s_1$, $s_2 \to -s_2$, and $S \to -S$.

See [2,14,28] for details.

6. Concluding Remarks

In this paper, we provided an extension of the semi-simplicity of the spin particle Lie algebra to the Lie group level. We showed that a spin particle Lie algebra admits a Clifford algebra, an almost complex manifold (Riemannian manifold) and a spin Lie group structure. We demonstrated that any one-half (resp. integer) spin particle, spin Lie group is a fourfold, (resp. double), cover of SO($2n$), (resp. SO($2n+1$)). We also proved that any spin Lie group of a spin particle is connected and semi-simple. We constructed the real Lie algebra of
the Spin(\(\frac{1}{2}\)) particle. We also performed the Iwasawa decomposition of the one-half spin particle into \(KDN\) and extended it to a general decomposition of all spin particles into \(KKD^N\). Finally, we applied the angular momentum coupling to the quantum state of the spin(\(\frac{2n-1}{2}\)) and demonstrated that the orthogonal basis transforms into the SO(2n) one, which corresponds to the Dynkin’s root \(\Pi(D_n)\).

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Declarations

Conflict of Interest

The authors declare that they have no conflict of interest.

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Mahouton Norbert Hounkonnou and Francis Atta Howard
University of Abomey-Calavi, International Chair in Mathematical Physics and
Applications (ICMPA–UNESCO Chair)
072 B.P. 50 Cotonou
Republic of Benin
e-mail: Francis.att Howard@cipma.net;
hfrancis atta@ymail.com

Mahouton Norbert Hounkonnou
e-mail: nor bert.houkonnou@cipma.uac.bj;
houkonnou@yahoo.fr

Kinki Kangni
UFR-Mathematiques et Informatique
Université de Cocody
22 B.P. 1214 Abidjan 22
Côte d’Ivoire
e-mail: kinki.kangi@univ-fhb.edu.ci;
kangnikangi@yahoo.fr

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