Origin of structure: Primordial Bispectrum without non-Gaussianities

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The primordial bispectrum has been considered in the past decade as a powerful probe of the physical processes taking place in the early Universe. Within the inflationary paradigm, the properties of the bispectrum are one of the keys that serve to discriminate among competing scenarios concerning the details of the origin of cosmological perturbations. However, all of the scenarios, based on the conventional approach to the so-called “quantum-to-classical transition” during inflation, lack the ability to point out the precise physical mechanism responsible for generating the inhomogeneity and anisotropy of our Universe starting from and exactly homogeneous and isotropic vacuum state associated with the early inflationary regime. In past works, we have shown that the proposals involving a spontaneous dynamical reduction of the quantum state provide plausible explanations for the birth of said primordial inhomogeneities and anisotropies. In the present letter, we show that, when considering single-field slow-roll inflation within the context of such proposals, the expected characteristics of the bispectrum turn out to be quite different from those found in the traditional approach. In particular, the statistical features corresponding to the primordial perturbations, which are normally associated with the bispectrum, are treated here in a novel way leading to rather different conclusions.
I. INTRODUCTION

Recent advances in observational cosmology are allowing a detailed testing of various models regarding the early Universe. In particular, the inflationary paradigm, considered one of the most promising models for the primordial Universe, is considered as providing an explanation for the origin of cosmological perturbations. Indeed, recent observational data (e.g., WMAP, SDSS, Planck), is in rather broad of agreement with the theoretical predictions offered by the inflationary paradigm. According to this theory, the idea behind the generation of the primordial inhomogeneities, “the seeds of galaxies,” is also rather image-evoking: The perturbations start as quantum fluctuations of the inflaton field, as the Universe experiments a phase of accelerated expansion, the physical wavelength associated with the perturbations is stretched out reaching cosmological scales. At this point one is invited to treat the quantum fluctuations as classical density perturbations. Then, the argument goes, at later cosmological epochs, these perturbations continue evolving into the cosmic structure responsible for galaxy formation, stars, planets and eventually life and human beings.

However, as has been discussed at length in previous works, the complete theory must not only allow one to find expressions that are in agreement with observations, but also, be able to provide an explanation of the precise physical mechanism behind its predictions. As originally discussed in [10], there is a conceptual difficulty in the standard explanation for the birth of cosmic structure provided by inflation, this is, from a highly homogeneous and isotropic state that characterizes both: the initial state of the so-called quantum perturbations and the classical background inflaton and space-time, the Universe ends in a state with “actual” inhomogeneities and anisotropies. In other words, if we consider quantum mechanics as a fundamental theory applicable in particular to the Universe as a whole; then any classical descriptions must be regarded as imprecise characterizations of complicated quantum mechanical states. The Universe that we observe today is clearly well described by an inhomogeneous and anisotropic classical state; therefore, such description must be considered as an imperfect description of an equally inhomogeneous and anisotropic quantum state. Consequently, if we want to consider the inflationary account as providing the physical mechanism for the generation of the seeds of structure, such account must contain an explanation for why the quantum state that describes our actual Universe does not possess the same symmetries as the early quantum state of the Universe, which happened to be perfectly symmetric (the symmetry being the homogeneity and isotropy). Since there is nothing in the dynamical evolution (as given by the standard inflationary approach) of the quantum state that can break those symmetries, then we are left with an incomplete theory. In fact, this and other shortcomings have been recognized by others in the literature [11-14].

The detailed discussion of the conceptual problems associated with the inflationary paradigm, and its possible solutions following the standard rules provided by Quantum Mechanics (e.g., the decoherence program, many-worlds interpretation and the consistent histories approach), have been presented by some of us and by others in [10,15-17]. We will not reproduce those arguments here and invite the interested reader to consult those references; the above paragraph is meant only to provide the reader to a small indication of the kinds of issues that the detailed analysis of such questions involves.

The idea that has been presented in previous works [10,17,20], as a possibility to deal with the aforementioned problem, involves supplementing the standard inflationary model with an hypothesis concerning the modification of quantum theory including a spontaneous dynamical reduction of the quantum state (sometimes referred as the self-induced collapse of the wave function) and consider it as an actual physical process; taking place independently of observers or measuring devices. Regarding the situation at hand, the basic scheme is the following: A few e-folds after inflation has started, the Universe finds itself in an homogeneous and isotropic quantum state, then during the inflationary regime a quantum collapse of the wave function is triggered (by novel physics possibly related with quantum gravitational effects), breaking in the process the unitary evolution of quantum mechanics and also, in general, breaking the symmetries of the original state. The post-collapse state continues to evolve leading to one state that is not isotropic or homogeneous and, moreover, it is susceptible to an approximate classical characterization describing a Universe, which includes the inhomogeneities and anisotropies, that will unfold, due to standard physical processes, into what we observe today.

The hypothesis regarding the self-induced collapse is not a new idea and there has been a considerably amount of research along this lines: The continuous spontaneous localization (CSL) model [23], representing a continuous version of the Ghirardi-Rimini-Weber (GRW) model [24], and the ideas of Penrose [25] and Diósi [26] regarding gravity as the main agent responsible for the collapse, are among the main programs proposed to describe the physical mechanism of a self-induced collapse of the wave function. For more recent examples see Refs. [21,22]. In fact, the implications of

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1 At the level of one part in $e^N$ where $N$ is the number of e-folds of inflation.
2 In fact, even if one wants to adopt, say, the many-worlds interpretational posture, the issue at hand can be rephrased by asking questions about the precise nature of the quantum state that can be taken as representing our specific branch of the many worlds. One can also focus on the issue of characterizing in a precise mathematical way the quantities that encode the so-called stochastic aspects, which are often only vaguely referred to. We will see this in a very concrete way in the the following.
applying the CSL model to the inflationary scenario have been studied in Refs. [27, 28] and [29] leading to interesting results constraining the parameters of the model in terms of the parameters characterizing the inflationary model.

On the other hand, the statistical analyses in Refs. [32, 33] show how the predictions of the simple collapse schemes, used in previous works, can be confronted with recent data from the Cosmic Microwave Background (CMB), including the 7-yr release of WMAP [34] and the matter power spectrum measured using LRGs by the Sloan Digital Sky Survey [35]. In fact, results from those analyses indicate that while several schemes or “recipes” for the collapse are compatible with the observational data, others are not, allowing to establish constraints on the free parameters of the schemes. Those works serve to underscore the point that, in addressing the conceptual issues of the inflationary paradigm, one is not only dealing with “philosophical issues,” but that these have impact in the theoretical predictions. While, at the same time, the conclusions drawn can lead to important insights, as well as a better understanding of the nature of its predictions and a novel way to consider the relation of these predictions with the observations.

In this work, we will be primarily concerned with the characteristics of the “primordial bispectrum.” In the inflationary paradigm, the primordial bispectrum has been regarded as one of the indicators characterizing any primordial “non-Gaussianities.” By non-Gaussianity, one generically refers to any deviations in the observed fluctuations from the random field of linear, Gaussian, curvature perturbation. It is commonly believed that the study of non-Gaussianities will play a leading role in furthering our understanding of the physics of the very early Universe that created the primordial seeds for large-scale structures [36]. As a matter of fact, the shape and amplitude of the bispectrum is involved (quantum averages, ensemble averages, time-space averages and orientation averages) yields a different set of the bispectrum. Second, it serves to illustrate that maintaining clear distinctions between each of the averages involved (quantum averages, ensemble averages, time-space averages and orientation averages) yields a different set of predictions for the observational quantities. As we will find, in the traditional picture the expected value for the new quantity vanishes exactly, while working in the approach based on collapse framework, one is lead to expect a non-vanishing value for this quantity.

The rest of the paper is organized as follows: In Section II we start by reviewing the ideas and technical aspects of our proposal, in particular we focus on how to implement the collapse of the wave function during the inflationary Universe. Then, in Section III we derive an analytic estimate of the expected value of the observed primordial bispectrum within our approach. Afterwards, in Section IV we discuss the main differences between the collapse proposal and the standard approach regarding the estimates of the primordial bispectrum and its statistical aspects. Section V contains a detailed analysis for a novel observational quantity that allows us to differentiate between the two approaches, both, at the quantitative and qualitative level. In Section VI we present a discussion regarding the comparisons between our theoretical prediction and the observational data. Finally, in Section VII we end with a...
II. BRIEF REVIEW OF THE COLLAPSE PROPOSAL WITHIN THE INFLATIONARY UNIVERSE

In this section we will present a brief review of the collapse proposal which has been exposed in great detail in previous works \[10, 32, 48, 49\]. The main purpose is to present the central ideas behind the proposal in order to make the presentation as self-contained as possible. The full self-consistent formalism was developed in \[50\]; however, we will not use here such full fledged formal treatment. The inflationary period is assumed to start at energy scales smaller than the Planck mass \((\sim 10^{-2} M_P)\), thus, the semi-classical approach is a suitable approximation for something that, in principle, ought to be treated in a precise fashion within a quantum gravity theory. The semi-classical framework is characterized by Einstein semi-classical equations \(G_{ab} = 8\pi G \langle T_{ab} \rangle\), which allow to relate the quantum degrees of freedom of the matter fields with the classical description of gravity in terms of the metric \(g_{ab}\). The use of such semi-classical picture has two main conceptual advantages:

First, the description and treatment of the metric is always “classical.” As consequence there is no issue with the “quantum-to-classical transition” in the sense that one needs to justify going from “metric operators” (e.g. \(\hat{\Psi}\)) to classical metric variables (such as \(\Psi\)). The fact that the space-time remains classical is particularly important in the context of models involving dynamical reduction of the wave function, as such “collapse or reduction” is regarded as a physical process taking place in time and, therefore, it is clear that a setting allowing consideration of full space-time notions is preferred over, say, the “timeless” settings usually encountered in canonical approaches to quantum gravity (for some basic references on “the problem of time on quantum gravity” see Ref. \[51\]).

Second, it allows to present a transparent picture of how the inhomogeneities and anisotropies are born from the quantum collapse: the initial state of the Universe (i.e. the one characterized by a few e-folds after inflation has started) is described by the homogeneous and isotropic Bunch-Davies vacuum, and the equally homogeneous and isotropic classical Friedmann-Robertson-Walker space-time. Then, at a later stage, the quantum state of the matter fields reaches a phase whereby the corresponding state for the gravitational degrees of freedom are forbidden, and a quantum collapse of the matter field wave function is triggered by some unknown physical mechanism. In this manner, the state resulting from the collapse needs not to share the same symmetries as the initial state. After the collapse, the gravitational degrees of freedom are assumed to be, once more, accurately described by Einstein semi-classical equation. However, as \(\langle T_{ab} \rangle\) for the new state needs not to have the symmetries of the pre-collapse state, we are led to a geometry that generically will no longer be homogeneous and isotropic.

We proceed now to introduce the details of the simplest collapse proposal. The starting point in our approach is the same as the standard slow-roll inflationary model; this is one writes the action of a scalar field \((\text{the inflaton})\) minimally coupled to gravity:

\[
S[\phi, g_{ab}] = \int d^4 x \sqrt{-g} \left( \frac{1}{16\pi G} R[g] - \frac{1}{2} \nabla_a \phi \nabla_b \phi g^{ab} - V[\phi] \right). \tag{1}
\]

During the collapse, the semi-classical approximation will not remain 100% valid; this is because the quantum collapse would induce sudden changes or “state jumps” to the initial quantum state, thus the divergence \(\nabla_a \langle T^{ab} \rangle \neq 0\) while \(\nabla_a G^{ab} = 0\). However, as we will be only interested in the states before and after the collapse, this breakdown of the semi-classical approximation would not be important for our present work.
Einstein’s field equations $G_{ab} = 8\pi GT_{ab}$ are derived from (1), with $T^\mu_\nu$ given by

$$T^\mu_\nu = g^{\alpha\mu}\partial_\alpha\phi\partial_\nu\phi + \delta^\mu_\nu \left( -\frac{1}{2}g^{\alpha\beta}\partial_\alpha\phi\partial_\beta\phi - V[\phi] \right).$$

The next step is to split the metric and the scalar field into a background plus perturbations $g_{ab} = g_{ab}^{(0)} + \delta g_{ab}$, $\phi = \phi_0 + \delta\phi$. The background is represented by a spatially flat FRW space-time with line element $ds^2 = a(\eta)[d\eta^2 + \delta_{ij}dx^i dx^j]$ and the homogeneous part of the scalar field $\phi_0(\eta)$. We will choose $a = 1$ at the present cosmological time; while we assume that the inflationary period ends at a conformal time $\eta_\ast \simeq -10^{-23}$ Mpc.

The scale factor corresponding to the inflationary era is $a(\eta) \simeq -1/(H\eta)$ with $H$ the Hubble factor defined as $H \equiv \partial_\eta a/a$, thus $H \simeq \text{const}$. During inflation $H$ is related to the inflaton potential as $H^2 \simeq (8\pi G/3)V$. The scalar field $\phi_0(\eta)$ is in the slow-roll regime, which means that $\phi_0' \simeq -(a^3/3a')\partial_\eta V$. The slow-roll parameter defined by $\epsilon \equiv \frac{1}{2}M^2_P(\partial_\eta V/V)^2$ is considered to be $\epsilon \ll 1$; $M_P$ is the reduced Planck mass defined as $M^2_P \equiv 1/(8\pi G)$.

Next, we consider the quantization of the theory. As mentioned above, we will work within the collapse-modified semi-classical gravity setting (for a detailed discussion of the self-consistent formalism see Ref. [50]). In particular, we will quantize the fluctuation of the inflaton field $\delta\phi(x, \eta)$, but not the metric perturbations. For simplicity, we will work with the rescaled field variable $y = a\delta\phi$. One then proceeds to expand the action (1) up to second order in the rescaled variable (i.e. up to second order in the scalar field fluctuations)

$$\delta S^{(2)} = \int d^4x \delta L^{(2)} = \int d^4x \frac{1}{2} \left[ y'^2 - (\nabla y)^2 + \left( \frac{a'}{a} \right)^2 y^2 - 2 \left( \frac{a'}{a} \right) y y' \right].$$

The canonical momentum conjugated to $y$ is $\pi \equiv \partial \delta L^{(2)}/\partial y' = y' - (a'/a)y = a\delta\phi'$. In order to avoid distracting infrared divergences, we set the problem in a finite box of side $L$. At the end of the calculations we can take the continuum limit by taking $L \to \infty$. The field and momentum operators are decomposed in plane waves

$$\hat{y}(\eta, \vec{k}) = \frac{1}{L^3} \sum_\vec{k} \hat{y}_\vec{k}(\eta)e^{i\vec{k}\cdot\vec{x}} \quad \hat{\pi}(\eta, \vec{x}) = \frac{1}{L^3} \sum_\vec{k} \hat{\pi}_\vec{k}(\eta)e^{i\vec{k}\cdot\vec{x}},$$

where the sum is over the wave vectors $\vec{k}$ satisfying $k_i L = 2\pi n_i$ for $i = 1, 2, 3$ with $n_i$ integer and $\hat{y}_\vec{k}(\eta) \equiv y_k(\eta)\hat{a}_\vec{k} + y^*_\vec{k}(\eta)\hat{a}^{\dagger}_{-\vec{k}}$ and $\hat{\pi}_\vec{k}(\eta) \equiv g_k(\eta)\hat{a}^{\dagger}_\vec{k} + g^*_k(\eta)\hat{a}_\vec{k}$. The function $y_k(\eta)$ satisfies the equation:

$$y_k''(\eta) + \left( k^2 - \frac{a''}{a} \right) y_k(\eta) = 0.$$  

To complete the quantization, we have to specify the mode solutions of (1). The canonical commutation relations between $\hat{y}$ and $\hat{\pi}$, will give $[\hat{a}_{\vec{k}}, \hat{a}^{\dagger}_{\vec{l}}] = L^3 \delta_{\vec{k}, \vec{l}}$, when $y_k(\eta)$ is chosen to satisfy $y_k g_k' - y_k' g_k = i$ for all $k$ at some time $\eta$.

The remainder of the choice of $y_k(\eta)$ corresponds to the so-called Bunch-Davies (BD) vacuum, which is characterized by
\[ y_k(\eta) = \frac{1}{\sqrt{2k}} \left( 1 - \frac{i}{\eta_k} \right) e^{-ik\eta}, \quad g_k(\eta) = -i\sqrt{\frac{k}{2}} e^{-ik\eta}. \] (8)

There is certainly some arbitrariness in selection of a natural vacuum state, but it seems clear that any such natural choice would be spatially a homogeneous and isotropic state. The BD vacuum certainly is a homogeneous and isotropic state as can be seen by evaluating directly the action of a translation or rotation operator on the state.

From \( G_{ab} = 8\pi G\langle T_{ab} \rangle \) and \( 4 \) it follows that

\[ \Psi_k(\eta) = \sqrt{\frac{e}{2M\eta R^2}} \langle \pi_k(\eta) \rangle. \] (9)

It is clear from Eq. 9 that if the state of the field is the vacuum state, the metric perturbations vanish, and, thus the space-time is homogeneous and isotropic.

The self-induced collapse model is based on considering that the collapse operates very similar to a kind of self-induced “measurement” (evidently, there is no external observer or detector involved). In considering the operators used to characterize the post-collapse states, it seems natural therefore to focus on Hermitian operators, which in ordinary quantum mechanics are the ones susceptible of direct measurement. We thus separate \( \hat{y}_k(\eta) \) and \( \hat{\pi}_k(\eta) \) into their “real and imaginary parts” \( \hat{y}_R^k(\eta) + i\hat{y}_I^k(\eta) \) and \( \hat{\pi}_R^k(\eta) + i\hat{\pi}_I^k(\eta) \). The point is that the operators \( \hat{y}_R^k(\eta) \) and \( \hat{\pi}_R^k(\eta) \) are hermitian. Thus

\[ \hat{y}_R^k(\eta) = \sqrt{2R} \langle \eta_k | \hat{a}_R^{\dagger} \rangle, \quad \hat{\pi}_R^k(\eta) = \sqrt{2R} \langle \eta_k | \hat{a}_k \rangle. \]

The commutation relations for the \( \hat{a}_k \) with all other commutators vanishing.

Following the a line of thought described above, we assume that the collapse is somehow analogous to an imprecise measurement of the operators \( \hat{y}_R^k(\eta) \) and \( \hat{\pi}_R^k(\eta) \). The rules according to which the collapse is assumed to happen are guided by simplicity and naturalness.

In particular, as we are taking the view that a collapse effect on a state is analogous to some sort of approximate measurement, we will postulate that after the collapse, the expectation values of the field and momentum operators in each mode will be related to the uncertainties of the initial state. For the purpose of this work we will work with a particular collapse scheme called the Newtonian collapse scheme which is given by

\[ \langle \hat{\pi}_R^k(\eta_k^c) \rangle = 0 \] (11)

\[ \langle \hat{y}_R^k(\eta_k^c) \rangle = x_R^k \sqrt{\Delta \hat{\pi}_R^k(\Delta \eta_k^c)^2}, \]

where \( \eta_k^c \) represents the time of collapse for each mode. In the vacuum state, \( \hat{\pi}_k \) is distributed according to a Gaussian wave function centered at 0 with spread \( (\Delta \hat{\pi}_k)^2 \). The motivation for choosing such scheme is two-folded. First, the calculations performed for this scheme are relatively easier to handle. Second, in Eq. 9 the variable that is directly related with the Newtonian Potential \( \Psi \) is the expectation value of \( \hat{\pi} \); therefore, it seems natural to consider that the variable affected at the time of collapse is \( \langle \hat{\pi}_k(\eta_k^c) \rangle \) while \( \langle \hat{y}_k(\eta_k^c) \rangle = 0 \).

The random variables \( x_R^k \) represent values selected randomly from a Gaussian distribution with unit dispersion. At this point, we must emphasize that our Universe corresponds to a single realization of these random variables, and thus each of these quantities \( x_R^k, x_I^k \) has a single specific value. It is clear that even though we will not do that here, one could also investigate how the statistics of \( x_R^k, x_I^k \) might be affected by the physical process of the collapse.

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6 \( \Re \{z\} \) denotes the real part of \( z \in \mathbb{C} \)

7 An imprecise measurement of an observable is one in which one does not end with an exact eigenstate of that observable, but rather with a state that is only peaked around the eigenvalue. Thus, we could consider measuring a certain particle’s position and momentum so as to end up with a state that is a wave packet with both position and momentum defined to a limited extent and, which certainly, does not entail a conflict with Heisenberg’s uncertainty bound.

8 In previous works, we have analyzed other collapse schemes such as the independent scheme and the Wigner scheme. See Refs. [10, 32, 33] for detailed analyses of the collapse schemes.
(e.g. see Ref. [12]). The statistics of these quantities can be studied using as a tool an imaginary ensemble of “possible Universes,” but we should in principle distinguish those from the statistics of such quantities for the particular Universe we inhabit; we will discuss these and other aspects in the next sections.

The next step is to find an expression for the evolution of the expectation values of the field operators at all times. This can be done in various ways but the simplest invokes using Ehrenfest’s theorem to obtain the expectation values of the field operators at any later time in terms of the expectation values at the time of collapse. The result is:

\[
\langle \hat{y}^{R,I}_{\vec{k}}(\eta) \rangle = \frac{\sin(k\eta - z_k)}{k} \left( \frac{1}{k\eta z_k} + 1 \right) \langle \hat{\pi}^{R,I}_{\vec{k}}(\eta_{\vec{k}}^c) \rangle, \tag{13}
\]

\[
\langle \hat{\pi}^{R,I}_{\vec{k}}(\eta) \rangle = \left( \cos(k\eta - z_k) + \frac{\sin(k\eta - z_k)}{z_k} \right) \langle \hat{\pi}^{R,I}_{\vec{k}}(\eta_{\vec{k}}^c) \rangle, \tag{14}
\]

with \( z_k = k\eta_{\vec{k}}^c \). This calculation is explicitly done in Refs. [10, 49].

Finally, using (9), (12) and (14) we find an expression for the Newtonian potential in terms of the random variables and the time of collapse

\[
\Psi_{\vec{k}}(\eta) = \sqrt{\frac{c}{H}} M_p \left( \frac{L}{2k} \right)^{3/2} \left( \cos(k\eta - z_k) + \frac{\sin(k\eta - z_k)}{z_k} \right) X_{\vec{k}}, \tag{15}
\]

where \( X_{\vec{k}} \equiv x_{\vec{k}}^R + ix_{\vec{k}}^I \). This last expression is the main result of the present section. It relates the Newtonian potential during inflation to the parameters describing the collapse (i.e. the random variables and the time of collapse). It is worth noting that all the quantities occurring in (15) are all classical quantities and no quantum operators appear in the expression. This is an important difference between our approach and the standard treatment of perturbations during inflation. That is, in the latter approach, the Newtonian potential is strictly a quantum operator and then one needs to invoke various kinds of arguments that are often vague and do not lead to clear connections with the quantities found in the observations; in particular, there is often an appeal to quantum randomness that is, however, left completely unspecified. Thus, the standard approach suffers from the lack of opportunity for clear characterization of the stochastic aspects of the situation (as well as from other conceptual deficiencies that have been have discussed in [13]). In our approach, we will not rely on arguments involving horizon-crossing of the modes, decoherence or many worlds interpretation of quantum mechanics, to justify the transition from a quantum object \( \hat{\Psi} \) to a classical stochastic field \( \Psi \), which result in a rather vague connection of the mathematical expressions used and the objects that emerge from observations. One of the advantages of the approach we favor is that, as a result of the collapse postulate, such connection become transparent and specific: we have the variables \( X_{\vec{k}} \) characterizing, once and for all, every stochasticity we will need to deal with.

As is well known, the Newtonian potential is closely related with the the temperature anisotropies whose origins can be traced back (in the specific gauge) with the extra red/blue shift photons suffered when emerging from the local potential wells/hills. As the values of the two random variables associated to each mode, \( x_{\vec{k}}^R \) and \( x_{\vec{k}}^I \), are fixed for our Universe, it follows from expression (15) that these values determine the value of the Newtonian potential Fourier components corresponding to our Universe, which in turn fix the value of the observed temperature anisotropies.

The statistic nature of the prescribed distribution of the random variable \( X_{\vec{k}} \equiv x_{\vec{k}}^R + ix_{\vec{k}}^I \) gets transfered to the Newtonian potential \( \Psi_{\vec{k}} \); if the random variable is Gaussian, then \( \Psi_{\vec{k}} \) is also Gaussian. It is clear that we cannot give a definite prediction for the values that these random variables take in our Universe, given the intrinsic randomness of the collapse. However, as we will show next, the fact that we have a large number of modes \( \vec{k} \) contributing to each of the observed quantities, will allow us to perform a statistical analysis and obtain theoretical estimates for the observational quantities.

### III. CHARACTERIZING THE PRIMORDIAL CMB BISPECTRUM

In the first subsection, we will study the connection between the Newtonian potential at the end of inflation and the observational quantities obtained from the temperature anisotropies of the CMB; in particular, we will show how
the temperature fluctuations are related with the collapse parameters. In the second subsection, we will provide the connection between the parameters characterizing the collapse and the primordial bispectrum.

A. Observational quantities

The observational quantity of interest corresponds to the temperature fluctuations of the CMB observed today on the celestial two-sphere. The temperature anisotropies are expanded using the spherical harmonics $\frac{\delta T}{T_0}(\theta, \varphi) = \sum_{l,m} a_{lm} Y_{lm}(\theta, \varphi)$, which means that the coefficients $a_{lm}$ can be expressed as

$$ a_{lm} = \int \frac{\delta T}{T_0}(\theta, \varphi) Y^*_{lm}(\theta, \varphi) d\Omega, \quad (16) $$

here $\theta$ and $\varphi$ are the coordinates on the celestial two-sphere, with $Y_{lm}(\theta, \varphi)$ the spherical harmonics ($l = 0, 1, 2...$ and $-l \leq m \leq l$), and $T_0 \approx 2.725$ K the temperature average.

The different multipole numbers $l$ correspond to different angular scales; low $l$ to large scales and high $l$ to small scales. At large angular scales ($l \leq 20$), the Sachs-Wolfe effect is the predominant source to the temperature fluctuations in the CMB. That effect relates the anisotropies in the temperature observed today on the celestial two-sphere to the inhomogeneities in the last scattering surface,

$$ \frac{\delta T}{T_0}(\theta, \varphi) \simeq \frac{1}{3} \Psi_{\text{matt}}(\eta_D, \vec{x}_D), \quad (17) $$

where $\vec{x}_D = R_D (\sin \theta \sin \varphi, \sin \theta \cos \varphi, \cos \theta)$, with $R_D$ the radius of the last scattering surface and $\eta_D$ is the conformal time of decoupling ($R_D \simeq 4000$ Mpc, $\eta_D \simeq 100$ Mpc). The Newtonian potential can be expanded in Fourier modes, $\Psi_{\text{matt}}(\eta_D, \vec{x}_D) = \sum_{k} \Psi^\text{matt}_k(\eta_D) e^{i \vec{k} \cdot \vec{x}_D} / L^3$. Furthermore, using that $e^{i \vec{k} \cdot \vec{x}_D} = 4\pi \sum_{lm} i^l j_l(k R_D) Y_{lm}(\theta, \varphi) Y^*_{lm}(\hat{k})$, expression (17) can be rewritten as

$$ a_{lm} = \frac{4\pi i^l}{3L^3} \sum_{k} j_l(k R_D) Y^*_{lm}(\hat{k}) \Psi^\text{matt}_k(\eta_D), \quad (18) $$

with $j_l(k R_D)$ the spherical Bessel function of order $l$.

The Newtonian potential $\Psi^\text{matt}$ appearing in (18) is evaluated at the time of decoupling which corresponds to the matter dominated cosmological epoch. Traditionally, the relation between $\Psi^\text{matt}$ and the Newtonian potential at the end of inflation is made by making use of the so-called transfer functions $T(k)$; the transfer functions contain all relevant physics from the end of inflation to the latter matter dominated epoch, which includes among others the acoustic oscillations of the plasma. Thus, $\Psi^\text{matt}_k(\eta_D) = T(k) \Psi^{\text{FK}}_k$, where $\Psi_k$ corresponds to the Newtonian potential during inflation and, since one is interested in the modes with scales of observational interest, $\Psi^{\text{FK}}_k$ correspond to the limit $-k \eta \to 0$ of $\Psi^{\text{FK}}_k(\eta)$ (or, as commonly referred, its scale should be “well outside the horizon” during inflation). This is, the coefficients $a_{lm}$ are rewritten as

$$ a_{lm} = \frac{4\pi i^l}{3L^3} \sum_{k} j_l(k R_D) Y^*_{lm}(\hat{k}) T(k) \Psi^{\text{FK}}_k. \quad (19) $$

At this point the traditional approach would proceed to calculate averages and higher-correlation functions of the coefficients $a_{lm}$. Nevertheless, within our model we can make a further step, by substituting Eq. (15) (and taking the limit $-k \eta \to 0$), which gives an explicit expression for $\Psi^{\text{FK}}_k$ in terms of the parameters of the collapse, in Eq. (19) one obtains\footnote{Note that we have multiplied by a factor of $3/(5\epsilon)$ the $\Psi^{\text{FK}}_k$ we obtained during inflation, Eq. (15). This is because, while $\Psi^{\text{FK}}_k(\eta)$ is constant for modes $-k \eta \ll 1$ during any cosmological epoch, its behavior changes substantially during a change in the equation of state for the dominant type of matter in the Universe. In particular, during the change from inflation to radiation epochs, $\Psi$ is amplified by a factor of approximately $1/\epsilon$. For a detailed discussion regarding the amplitude within the collapse framework see Ref. [13].}
Several kinds of arguments would normally be invoked at this point in defense of the standard treatments. For a detailed discussion of which upon using the normalized gaussian assumption for fiduciary ensemble \( \langle \hat{\Psi} \rangle \), expectation values with classical quantities, the prediction given by the standard inflationary paradigm would be Eq. (20) in the standard approach. As a matter of fact, if we follow the conventional way of identifying quantum discussed in terms of the statistics of the random variables \( X \). We note that there is no analog expression of Eq. (20) in the standard approach. As a matter of fact, if we follow the conventional way of identifying quantum expectation values with classical quantities, the prediction given by the standard inflationary paradigm would be \( \langle 0 | \hat{\Psi}_k | 0 \rangle = \Psi_k \); thus, we would be led, by Eq. (19), to conclude that \( a_{lm} = 0 \); this is, the theoretical prediction for the temperature fluctuations would be exactly zero in an evident contradiction\(^{11}\) with the observations (see Ref. [17]).

One key aspect that in our treatment differs, from those followed in the standard approaches, is the manner in which the results from the formalism are connected to observations. This is most clearly exhibited by our result regarding the quantity \( a_{lm} \) in Eq. (20). Despite the fact that we have in principle a close expression for the quantity of interest, we cannot use Eq. (20) to make a definite prediction because the expression involves the numbers \( a_{lm} \), but one might estimate the most likely value of the magnitude of such displacement. Thus, we focus precisely on the random walk, i.e. the sum of complex numbers depending on random choices (characterized by the \( X_F \)). As is well known, for a random walk, one cannot predict the final displacement (which would correspond to the complex quantity \( a_{lm} \)), but one might estimate the most likely value of the magnitude of such displacement. Thus, we focus precisely on the most likely value of \( |a_{lm}| \), which we denote by \( |a_{lm}|_{\text{M.L.}} \). In order to compute that quantity, we make use of a fiducial (imaginary) ensemble of realizations of the random walk and compute the ensemble average value over of the total displacement. Thus we identify:

\[
|a_{lm}|_{\text{M.L.}} = |\overline{a_{lm}}|.
\]

The overline appearing denotes average over the fiducial ensemble of realizations, which would correspond to an imaginary “ensemble of universes.”

The estimate is done now in the standard way in which one deals with such random walks:

\[
|a_{lm}|^2_{\text{M.L.}} = |\overline{a_{lm}}|^2 = \frac{1}{L^3} \sum_{k,k'} g(z_k) j_l(kR_D) Y_{lm}^*(\hat{k}) T(k) g(z_{k'}) j_l(k'R_D) Y_{lm}(\hat{k}') T(k') \overline{X_F X_{F}^*},
\]

which upon using the normalized gaussian assumption for fiduciary ensemble \( (X_F X_{F}^* = 2\delta_{F,F}) \), leads to

\[
|a_{lm}|^2_{\text{M.L.}} = \frac{2}{L^3} \sum_{k} k^{-3} j_l(kR_D)^2 |Y_{lm}(\hat{k})|^2 T(k)^2 g(z_k)^2.
\]

Finally, we can remove the fiducial box of side \( L \) and pass to the continuum

\[
|a_{lm}|^2_{\text{M.L.}} = \int \frac{d^3k}{4\pi^2 k^3} j_l(kR_D)^2 |Y_{lm}(\hat{k})|^2 T(k)^2 g(z_k)^2.
\]

\(^{11}\) Several kinds of arguments would normally be invoked at this point in defense of the standard treatments. For a detailed discussion of their merits and shortcomings see Ref. [17].
At this point, one could focus on the quantity that is most often studied in this context, namely

\[
C_l \equiv \frac{1}{2l+1} \sum_m |a_{lm}|^2
\]  

for which we would have the estimate

\[
C_l^{\text{M.L.}} = \frac{1}{2l+1} \sum_m |a_{lm}|^2_{\text{M.L.}} = \frac{1}{4\pi^3} \int_0^\infty \frac{dk}{k} j_1(kR_D)^2 T(k)^2 g(z_k)^2.
\]  

where in the last step, we used the fact that the most likely value estimate in Eq. (25) is independent of \(m\). Furthermore, if we consider the time of collapse as \(\eta^c_k \propto k^{-1}\), i.e. \(z_k = z\) independent of \(k\) and take \(T(k) = 1\), which is a valid approximation for \(l \ll 20\), we recover an exact scale-invariant spectrum, this is,

\[
l(l+1)C_{l}^{\text{M.L.}} = \frac{g(z)^2}{(2\pi)^3} = \frac{H^2}{10^5Mpc^3} \left(\cos z - \frac{\sin z}{z}\right)^2 \equiv A
\]  

The quantity \(A\) is fixed by the observational data to be \(A \simeq 10^{-10}\). The fact that \(\eta^c_k \propto k^{-1}\) is also motivated by the results in previous works \([10, 32, 48, 49, 55]\).

If we would like to recover the full angular spectrum, one should then use expression (27) including the transfer functions, which can be obtained using numerical codes, and assume a particular form for the time of collapse \(\eta^c_k\) in terms of \(k\). This type of studies have been done, and the results can be consulted in Ref. \([33]\).

The expression above is the theoretical estimate to be compared with the observational data, and as should be clear from the discussion, the fact that we have to rely on most likely values, for what are in effect the mathematical equivalent of random walks, leads us to expect that there should be a general and rough agreement between our estimates and observations (assuming the theory is correct). However, we do not really expect a detailed and precise match simply due to the intrinsic randomness involved. In the standard approach, similar considerations involving the randomness of the fluctuations and the uncertainties tied to stochasticity, and with the limited region of the Universe one is observing, also leads to people in the community to expect small differences in predictions and observations. Nevertheless, in general, such discussions are based on heuristic arguments; therefore, are limited both in scope and precession. The essential difficulty is that, in the standard analysis, the precise stochastic elements are not identified and have no mathematical representation in the formalism. We believe that, in our approach, the stochastic elements are clearly identifiable (i.e. the \(X_E\)). This represents a great advantage providing us, for instance, with an explicit expression for the quantity \(a_{lm}\), such as in Eq. (20), and, thus, allowing us to study in great detail the precise nature of higher order statistical estimates as we will do in the following.

B. The primordial bispectrum

The usual path to look for non-trivial statistical features (e.g. possible non-Gaussianities) in the CMB is to study the \textit{bispectrum}, which is considered as related with the three-point function of the temperature anisotropies in harmonic space. The CMB angular bispectrum is defined as

\[
B_{l_1l_2l_3}^{m_1m_2m_3} \equiv a_{l_1m_1}a_{l_2m_2}a_{l_3m_3}.
\]  

The overline appearing in (29) denotes average over an ensemble of universes, but in practice it is taken as an average over orientations in our own Universe; the relation between the two types of averages is not clear and direct (this fact has been discussed in great detail in Ref. \([13]\)). In the following, we will show how our approach helps to clarify certain issues that emerge when dealing with the statistical aspects of the spectrum and when comparing theoretical estimates and observations.

Given the definition of the CMB bispectrum and considering a rotational invariant sky, one finds in the literature \([36, 44]\) another object called the “angle-averaged bispectrum” defined by

\[
B_{l_1l_2l_3} \equiv \sum_{m_1} \binom{l_1}{m_1} \binom{l_2}{m_2} \binom{l_3}{m_3} B_{l_1l_2l_3}^{m_1m_2m_3} = \sum_{m_1} \binom{l_1}{m_1} \binom{l_2}{m_2} \binom{l_3}{m_3} a_{l_1m_1}a_{l_2m_2}a_{l_3m_3}.
\]
The object \( \left( \begin{array}{ccc} l_1 & l_2 & l_3 \\ m_1 & m_2 & m_3 \end{array} \right) \) is called the Wigner 3-j symbol (see Ref. [47] for more details and properties for these functions) and is non-vanishing for the values of \( l, m \) satisfying the following conditions:

1. \( m_1 + m_2 + m_3 = 0 \).
2. \( l_1 + l_2 + l_3 \) is an integer, (or an even integer if \( m_1 = m_2 = m_3 = 0 \)).
3. \( |l_i - l_j| \leq k \leq l_i + l_j \) for all permutations of indices.

These conditions are called “the triangle conditions” as \( l_1, l_2, l_3 \) must correspond to the sides of a triangle. As a matter of fact, in the standard approach, one intends to estimate \( B_{l_1l_2l_3} \) from the observational data (e.g. see Sec. 3.1 of Ref. [44]) by testing different configurations for such “triangles.”

Motivated by the fact that within the collapse proposal we can obtain a direct relation between the coefficients \( a_{im} \) and the random variables characterizing the collapse [Eq. (20)] we will be focussing on the expression for the “observational” bispectrum:

\[
B_{l_1l_2l_3}^{\text{obs}} = \sum_{m_i} \left( \begin{array}{ccc} l_1 & l_2 & l_3 \\ m_1 & m_2 & m_3 \end{array} \right) a_{1m_1} a_{2m_2} a_{3m_3} .
\]

(31)

Note that this object also contains the Wigner 3-j symbol, therefore, unless \( l_1, l_2, l_3 \) satisfy the three previous conditions, the observational bispectrum will vanish. The difference between \( B_{l_1l_2l_3} \) and \( B_{l_1l_2l_3}^{\text{obs}} \) is a subtle but important one. While in the definition of \( B_{l_1l_2l_3} \), one should perform an average over an ensemble of universes [as is explicitly stated in the definition (30)], the object \( B_{l_1l_2l_3}^{\text{obs}} \) involves no averages over idealized ensembles whatsoever.

The only average that is being performed in \( B_{l_1l_2l_3}^{\text{obs}} \) is an average over orientations (i.e. a sum over \( m_i \) with a weight given by the Wigner 3-j symbols).

Explicitly \( B_{l_1l_2l_3}^{\text{obs}} \) is given by substituting (20) in (31) which yields

\[
B_{l_1l_2l_3}^{\text{obs}} = \sum_{m_i} \left( \begin{array}{ccc} l_1 & l_2 & l_3 \\ m_1 & m_2 & m_3 \end{array} \right) \frac{1}{L^{9/2}} \sum_{\hat{k}_1, \hat{k}_2, \hat{k}_3} g(z_{k_1}) g(z_{k_2}) g(z_{k_3}) j_{l_1}(k_1 R_D) j_{l_2}(k_2 R_D) j_{l_3}(k_3 R_D) \times Y_{l_1 m_1}^{*}(\hat{k}_1) Y_{l_2 m_2}^{*}(\hat{k}_2) Y_{l_3 m_3}^{*}(\hat{k}_3) T(k_1) T(k_2) T(k_3) \left| X_{\hat{k}_1} \right| X_{\hat{k}_2} X_{\hat{k}_3}.
\]

(32)

As is clear from Eq. (32), the collapse bispectrum is in effect a sum of random complex numbers (i.e. a sum where each term is characterized by the product \( X_{\hat{k}_1} X_{\hat{k}_2} X_{\hat{k}_3} \), which is itself a complex random number), leading to what can be considered effectively as a two-dimensional (i.e. a complex plane) random walk. As is well known, one cannot give a perfect estimate for the direction of the final displacement resulting from the random walk.

Similarly as \( B_{l_1l_2l_3}^{\text{obs}} \) is characterized by the sum of random variables we cannot give a specific value for its outcome. However, as we will see, in complete analogy of our analysis of the quantities \( a_{im} \), by focusing on the most likely value of the magnitude \( B_{l_1l_2l_3}^{\text{obs}} \) we will obtain a reasonable prediction.

To recapitulate, the original situation corresponds to the homogeneous and isotropic vacuum state. When a sudden change of the initial state takes place due to the collapse (one for each mode), the mode becomes characterized by a fixed value of the corresponding random variables; the collection of all the values of such random variables associated to all the modes characterizes, therefore, our single and unique Universe (which in consequence fixes \( B_{l_1l_2l_3}^{\text{obs}} \)); let us denote this set by

\[
U = \{ X_{\hat{k}_1}, X_{\hat{k}_2}, \ldots \}.
\]

(33)

Nevertheless, given the stochastic nature of the collapse, we can consider that the Universe could have corresponded to different set of values for the random variables characterizing the Universe in a different manner \( \hat{U} = \{ \hat{X}_{\hat{k}_1}, \hat{X}_{\hat{k}_2}, \ldots \} \).

The collection of different sets \( \{ U, \hat{U}, \ldots \} \) thus describe an hypothetical ensemble of universes. In making an estimate, we will be assuming that our Universe is a typical member of this hypothetical ensemble. Furthermore, we will make

\footnote{One should not confuse the fact that when obtaining the specific values of \( a_{im} \), which result from observations, one needs to perform an integral over the CMB sky [as indicated in Eq. (48)], with taking averages over ensembles of universes as considered above.}
the assumption that the most likely (M.L.) value of the magnitude $|\mathcal{B}_{l_1 I_1 l_2 I_2 l_3 I_3}^{\text{obs}}|^2$ in such ensemble comes very close to the corresponding one for our own Universe, that is

$$|\mathcal{B}_{l_1 I_1 l_2 I_2 l_3 I_3}^{\text{obs}}|^2 \simeq |\mathcal{B}_{l_1 I_1 l_2 I_2 l_3 I_3}^{\text{obs}}|_{\text{M.L.}}^2 \simeq |\mathcal{B}_{l_1 I_1 l_2 I_2 l_3 I_3}^{\text{obs}}|_{\text{M.L.}}^2.$$  \hspace{1cm} (34)

Moreover, we can simplify the estimate by taking the ensemble average $|\mathcal{B}_{l_1 I_1 l_2 I_2 l_3 I_3}^{\text{obs}}|^2$ (the bar indicates that we are taking the ensemble average) and identify it with the most likely $|\mathcal{B}_{l_1 I_1 l_2 I_2 l_3 I_3}^{\text{obs}}|_{\text{M.L.}}^2$. It is needless to say that these two notions are not exactly the same for arbitrary kinds of ensembles; as a matter of fact, the relation between the two concepts depends on the probability distribution function (PDF) of the random variables. In principle, we do not know the exact PDF, as we have only access to a single realization–our own Universe–, but a natural way to proceed is to assume a normal (Gaussian) distribution for the random variable $X_\vec{k}$. In such case, we can relate

$$|\mathcal{B}_{l_1 I_1 l_2 I_2 l_3 I_3}^{\text{obs}}|^2 = |\mathcal{B}_{l_1 I_1 l_2 I_2 l_3 I_3}^{\text{obs}}|_{\text{M.L.}}^2.$$ \hspace{1cm} (35)

which implies that

$$|\mathcal{B}_{l_1 I_1 l_2 I_2 l_3 I_3}^{\text{obs}}|^2 = |\mathcal{B}_{l_1 I_1 l_2 I_2 l_3 I_3}^{\text{obs}}|_{\text{M.L.}}^2 = |\mathcal{B}_{l_1 I_1 l_2 I_2 l_3 I_3}^{\text{obs}}|^2.$$ \hspace{1cm} (36)

In the reminder of this section, we will focus on computing $|\mathcal{B}_{l_1 I_1 l_2 I_2 l_3 I_3}^{\text{obs}}|^2$.

Considering a Gaussian PDF for the random variables $X_\vec{k}$ implies taking a Gaussian PDF for $x_\vec{k}^R$ and $x_\vec{k}^I$ (i.e., the real an imaginary parts of the complex random number $X_\vec{k}$). This is, the ensemble average of the products $x_\vec{k}^R x_\vec{k}'^R$ and $x_\vec{k}^I x_\vec{k}'^I$ is characterized by

$$\bar{x_\vec{k}^R x_\vec{k}'^R} = \delta_{\vec{k},-\vec{k}'} + \delta_{\vec{k},\vec{k}'} \quad \text{and} \quad \bar{x_\vec{k}^I x_\vec{k}'^I} = \delta_{\vec{k},-\vec{k}'} - \delta_{\vec{k},\vec{k}'}.$$ \hspace{1cm} (37)

Note that we have taken into account that the variables $x_\vec{k}^R$ and $x_\vec{k}^I$ are independent. Additionally, we have considered the correlation between the modes $\vec{k}$ and $-\vec{k}$ in accordance with the commutation relation given by $[\hat{a}_{\vec{k}}^R, \hat{a}_{\vec{k}'}^R]$ and $[\hat{a}_{\vec{k}}^I, \hat{a}_{\vec{k}'}^I]$ [see Eq. (10)]. Given the relations (37), the average for the product of two random variables $X_{\vec{k}}$ (over the imaginary ensemble of universes) yields

$$\bar{X_{\vec{k}} X_{\vec{k}'}} = (x_\vec{k}^R + ix_\vec{k}^I)(x_\vec{k}'^R + ix_\vec{k}'^I) = 2\delta_{\vec{k},-\vec{k}'}.$$ \hspace{1cm} (38)

Furthermore, it is easy to check that

$$\bar{X_{\vec{k}}^* X_{\vec{k}'}} = \bar{X_{\vec{k}'}} X_{\vec{k}}^* = 2\delta_{\vec{k},-\vec{k}'} \quad \text{and} \quad \bar{X_{\vec{k}} X_{\vec{k}'}}^* = 2\delta_{\vec{k},\vec{k}'}.$$ \hspace{1cm} (39)

Given the previous discussion and using Eq. (29), we can perform the average $|\mathcal{B}_{l_1 I_1 l_2 I_2 l_3 I_3}^{\text{obs}}|^2$

\[
|\mathcal{B}_{l_1 I_1 l_2 I_2 l_3 I_3}^{\text{obs}}|^2 = \frac{g(z)}{L^3} \sum_{m_1, \ldots, m_6} \left( \frac{l_1}{m_1} \frac{l_2}{m_2} \frac{l_3}{m_3} \right) \left( \frac{l_1}{m_4} \frac{l_2}{m_5} \frac{l_3}{m_6} \right) \times \sum_{\vec{k}_1, \ldots, \vec{k}_6} \frac{j_{l_1}(k_1 R D) j_{l_2}(k_2 R D) j_{l_3}(k_3 R D)}{(k_1 k_2 k_3 k_4 k_5 k_6)^{3/2}} j_{l_4}(k_4 R D) j_{l_5}(k_5 R D) j_{l_6}(k_6 R D) Y_{l_1 m_1}(\vec{k}_1) \times Y_{l_2 m_2}^*(\vec{k}_2) Y_{l_3 m_3}^*(\vec{k}_3) Y_{l_4 m_4}(\vec{k}_4) Y_{l_5 m_5}(\vec{k}_5) Y_{l_6 m_6}(\vec{k}_6) T(k_1) T(k_2) T(k_3) T(k_4) T(k_5) T(k_6) \times \bar{X_{\vec{k}_1}} X_{\vec{k}_2}^* X_{\vec{k}_3}^* X_{\vec{k}_4}^* X_{\vec{k}_5}^* X_{\vec{k}_6}^* .
\] \hspace{1cm} (40)

where in obtaining Eq. (10) we have assumed $z_k$ independent of $k$ (see discussion bellow Eq. (27)).

As the random variables $X_{\vec{k}}$ are taken to be distributed according to a Gaussian PDF, we can use the following relation
\[
\frac{X_{k_1}X_{k_2}X_{k_3}X_{k_4}X_{k_5}X_{k_6}}{X_{k_1}X_{k_2}X_{k_3}X_{k_4}X_{k_5}X_{k_6}} + \frac{X_{k_1}X_{k_2}X_{k_3}X_{k_4}X_{k_5}X_{k_6}}{X_{k_1}X_{k_2}X_{k_3}X_{k_4}X_{k_5}X_{k_6}} + \frac{X_{k_1}X_{k_2}X_{k_3}X_{k_4}X_{k_5}X_{k_6}}{X_{k_1}X_{k_2}X_{k_3}X_{k_4}X_{k_5}X_{k_6}} + \frac{X_{k_1}X_{k_2}X_{k_3}X_{k_4}X_{k_5}X_{k_6}}{X_{k_1}X_{k_2}X_{k_3}X_{k_4}X_{k_5}X_{k_6}} + \frac{X_{k_1}X_{k_2}X_{k_3}X_{k_4}X_{k_5}X_{k_6}}{X_{k_1}X_{k_2}X_{k_3}X_{k_4}X_{k_5}X_{k_6}} + \frac{X_{k_1}X_{k_2}X_{k_3}X_{k_4}X_{k_5}X_{k_6}}{X_{k_1}X_{k_2}X_{k_3}X_{k_4}X_{k_5}X_{k_6}}.
\]

(41)

Using Eqs. (39), we find

\[
\frac{X_{k_1}X_{k_2}X_{k_3}X_{k_4}X_{k_5}X_{k_6}}{X_{k_1}X_{k_2}X_{k_3}X_{k_4}X_{k_5}X_{k_6}} = \frac{2^3}{2} \delta_{k_1,-k_2} \delta_{k_3,k_4} \delta_{k_5,-k_6} + \delta_{k_1,-k_2} \delta_{k_3,k_4} \delta_{k_5,-k_6} + \delta_{k_1,-k_2} \delta_{k_3,k_4} \delta_{k_5,-k_6} + \delta_{k_1,-k_2} \delta_{k_3,k_4} \delta_{k_5,-k_6} + \delta_{k_1,-k_2} \delta_{k_3,k_4} \delta_{k_5,-k_6} + \delta_{k_1,-k_2} \delta_{k_3,k_4} \delta_{k_5,-k_6} + \delta_{k_1,-k_2} \delta_{k_3,k_4} \delta_{k_5,-k_6}.
\]

(42)

The following steps are straightforward: First, substitute (42) in (40) which will convert the sum over \( \vec{k}_1, ..., \vec{k}_6 \) into a sum over \( \vec{k}_1, \vec{k}_2, \vec{k}_3 \). Then, take the limit \( L \to \infty \) and \( k \to \) continuum in order to switch from sums to integrals over \( \vec{k} \). Finally, perform the remaining integrals (note that we have, once again, assumed \( T(k) = 1 \), which should be a good approximation for \( l \lesssim 20 \)). The interested reader can consult the details of the calculation in Appendix A. Here, we will just present the result which is

\[
|\mathcal{B}_{l_1,l_2,l_3}|^2 = \frac{g(z)^6(1 + \Delta_{l_1,l_2,l_3})}{(2\pi)^9l_1(l_1 + 1)l_2(l_2 + 1)l_3(l_3 + 1)},
\]

(43)

where

\[
\Delta_{l_1,l_2,l_3} \equiv \delta_{l_1,l_2} \left[ (2l_3 + 1) \left( \sum_{m_1} \left( \begin{array}{ccc} l_1 & l_3 & 0 \\ m_1 & -m_1 & 0 \end{array} \right) \right)^2 + (-1)^{l_3+2l_1} \right] \\
+ \delta_{l_3,l_3} \left[ (2l_1 + 1) \left( \sum_{m_2} \left( \begin{array}{ccc} l_2 & l_1 & 0 \\ m_2 & -m_2 & 0 \end{array} \right) \right)^2 + (-1)^{l_1+2l_3} \right] \\
+ \delta_{l_1,l_1} \left[ (2l_2 + 1) \left( \sum_{m_3} \left( \begin{array}{ccc} l_3 & l_2 & 0 \\ m_3 & -m_3 & 0 \end{array} \right) \right)^2 + (-1)^{l_2+2l_3} \right] \\
+ \delta_{l_1,l_2} \delta_{l_2,l_3} \left[ 2 + 3(1 + (-1)^{3l_1})(2l_1 + 1) \left( \sum_{m_1} \left( \begin{array}{ccc} l_1 & l_1 & l_1 \\ m_1 & -m_1 & 0 \end{array} \right) \right)(-1)^{m_1} \right]^2 \right],
\]

(44)

with \( l = 1, 2, \ldots \) and \(-l \leq m \leq l\). Consequently, the most likely value for the magnitude of the collapse bispectrum is \( |\mathcal{B}_{l_1,l_2,l_3}|_{\text{M.L.}} = \left( |\mathcal{B}_{l_1,l_2,l_3}|^2 \right)^{1/2}, \) i.e.
\[ |B_{l_1l_2l_3}|_{\text{M.L.}} = \frac{1}{\pi^{3/2}} \left( \frac{H}{10 \mathcal{M} P \xi^{1/2}} \right)^3 \cos z - \sin z \left( \frac{1 + \Delta_{l_1l_2l_3}}{l_1(l_1 + 1)l_2(l_2 + 1)l_3(l_3 + 1)} \right)^{1/2}, \tag{45} \]

where we used the definition of \( g(z) \) given in Eq. (21).

Equation (45) is the main result of this section. Given the definition of the collapse bispectrum [see, Eq. (31)], we note that \( l_1, l_2, l_3 \) must correspond to the sides of a triangle, otherwise \( |B_{l_1l_2l_3}|_{\text{M.L.}} = 0 \). Furthermore, if such “triangle” has different side lengths \( (l_1 \neq l_2 \neq l_3) \), then \( \Delta_{l_1l_2l_3} \) vanishes exactly (but not \( |B_{l_1l_2l_3}^{\text{obs}}|_{\text{M.L.}} \)). However, if \( l_1, l_2, l_3 \) are associated with the sides of an isosceles \( (l_1 = l_2 \neq l_3) \) or an equilateral triangle \( (l_1 = l_2 = l_3) \) the terms appearing in \( \Delta_{l_1l_2l_3} \), contribute to the collapse bispectrum, which generically does not vanishes (e.g., \( |B_{222}^{\text{obs}}|_{\text{M.L.}} = g(z)^6/((2\pi)^9 \delta^2) \)).

We must emphasize that \( |B_{l_1l_2l_3}^{\text{obs}}| \) is not exactly the magnitude of the traditional theoretical angle-averaged-bispectrum \( |B_{l_1l_2l_3}| \) as the latter would correspond to take the magnitude of the object defined in Eq. (30). In fact, in the conventional approach, one would relate an average over an ensemble of universes with a certain quantum three-point function and that would vanish in the absence of “non-Gaussianities.” Then, the average over an ensemble of universes would be somehow related with the angle-averaged-bispectrum \( B_{l_1l_2l_3} \), which is a sort of orientation average in our Universe, connected with suitable averages over \( m \)'s of the quantity \( a_{l_1m_1}a_{l_2m_2}a_{l_3m_3} \) measured in our own (single) Universe.

Our point is, thus, connected with the fact that these series of identifications have an unequivocal no clear justification (see Ref. [15]). On the other hand, within the collapse model, we find a prediction for the most likely value of \( |B_{l_1l_2l_3}^{\text{obs}}|_{\text{M.L.}} \) which can be related directly with the actual observational quantity:

\[ |B_{l_1l_2l_3}|_{\text{Actual obs}} \equiv \sum_{m_1m_2m_3} \left( \frac{l_1}{m_1} \frac{l_2}{m_2} \frac{l_3}{m_3} \right) (a_{l_1m_1}a_{l_2m_2}a_{l_3m_3})_{\text{Actual obs}}, \tag{46} \]

in a direct and transparent manner. Note, however, that we make a distinction between the theory’s prediction for the most likely value of the observational quantity \( |B_{l_1l_2l_3}^{\text{obs}}|_{\text{M.L.}} \) and the actually observed quantity itself \( |B_{l_1l_2l_3}|_{\text{Actual obs}} \).

In the next section, we will extend the discussion about the relation between the statistical aspects of the traditional bispectrum and the collapse bispectrum.

### IV. MAIN DIFFERENCES BETWEEN THE STANDARD AND THE COLLAPSE APPROACH REGARDING THE PRIMORDIAL BISPECTRUM

#### A. The standard approach to the CMB bispectrum

We begin this section by giving a rather brief review of the conventional approach to the primordial bispectrum, its amplitude and the usual arguments given to relate it with possible non-Gaussian features in the perturbations; extended reviews can be found in Refs. [13, 47].

According to the standard approach, if \( \Psi(\vec{x}, \eta) \) is taken to be characterized by a Gaussian distribution all its statistical properties are codified in the two-point correlation function. Otherwise, one needs to consider higher order correlation functions, e.g. the three-point correlation function \( \Psi(\vec{x}, \eta)\Psi(\vec{y}, \eta)\Psi(\vec{z}, \eta) \). The Fourier transform is commonly referred as the bispectrum, defined by

\[ \Psi_{k_1}\Psi_{k_2}\Psi_{k_3} = (2\pi)^3\delta(\vec{k}_1 + \vec{k}_2 + \vec{k}_3)B_{\Psi}(k_1, k_2, k_3). \tag{47} \]

The Dirac delta appearing in Eq. (47) is said to indicate that the ensemble average \( \Psi(\vec{x}, \eta)\Psi(\vec{y}, \eta)\Psi(\vec{z}, \eta) \) is invariant under spatial translations; in addition, the dependence only on the magnitudes \( k \), appearing in the function \( B_{\Psi}(k_1, k_2, k_3) \), is tied to the rotational invariance of such ensemble. The Dirac delta constrains the three modes involved; this is, the modes must satisfy \( \vec{k}_1 + \vec{k}_2 + \vec{k}_3 = 0 \), which is known as the triangle condition. Thus, according to the direct relation of the Newtonian potential with the source of the observed anisotropies, the assumption about the rotational and translational invariance of the ensemble and its impact on \( \Psi(\vec{x}, \eta)\Psi(\vec{y}, \eta)\Psi(\vec{z}, \eta) \) should be reflected in the invariance of the ensemble average of the temperature fluctuations \( \frac{1}{l_0^2} \theta_{l_0}^2 \theta_{l_0}^2(\theta_1, \varphi_1) \frac{1}{l_0^2} \theta_{l_0}^2(\theta_2, \varphi_2) \frac{1}{l_0^2} \theta_{l_0}^2(\theta_3, \varphi_3) \). However, we should once again emphasize that strictly speaking the discussion above and, thus, the average indicated by the overline in the latter expressions, strictly refers to an average over an ensemble of Universes and not averages over orientations in one Universe. That distinction is often ignored in the traditional treatments of the subject.
One of the first (and most popular ways) to parameterize the traditional non-Gaussianity phenomenologically is via the introduction of a non-linear correction to the linear Gaussian curvature perturbation \(^{37, 38}\),

\[
\Psi(\vec{x}, \eta) = \Psi_g(\vec{x}, \eta) + f_{NL}^2[\Psi_g^2(\vec{x}, \eta) - \Psi_g^2(\vec{x}, \eta)],
\]

(48)

where \(\Psi_g(\vec{x}, \eta)\) denotes a linear Gaussian part of the perturbation and \(\Psi_g^2(\vec{x}, \eta)\) is the variance of the of the Gaussian part\(^{13}\). The parameter \(f_{NL}^2\) is called the “local non-linear coupling parameter” and determines the “strength” of the primordial non-Gaussianity. This parametrization of non-Gaussianity is local in real space and therefore is called “local non-Gaussianity.” Using Eqs. (47) and (48), the bispectrum of local non-Gaussianity may be derived:

\[
B_{\Psi}(\vec{k}_1, \vec{k}_2, \vec{k}_3) = 2f_{NL}^2[P_{\Psi}(\vec{k}_1)P_{\Psi}(\vec{k}_2) + P_{\Psi}(\vec{k}_2)P_{\Psi}(\vec{k}_3) + P_{\Psi}(\vec{k}_3)P_{\Psi}(\vec{k}_1)],
\]

\[
= 2f_{NL}^2 \frac{A_{\Psi}^2}{(k_1 k_2 k_3)^3} \left( \frac{k_1^2}{k_1 k_2 k_3} + \frac{k_2^2}{k_1 k_2 k_3} + \frac{k_3^2}{k_1 k_2 k_3} \right),
\]

(49)

with \(P_{\Psi}(\vec{k})\) the power spectrum of the Newtonian potential defined as \(\Psi(\vec{x}, \eta)\), the variance of the Gaussian part, for the delta function that appears in the bispectrum definition (47). The integral over the angular part of \(k_1 l_1 m_1 (\vec{k}_1)\), in the last line, is known as the Gaunt integral, i.e.,

\[
B^{l_1 l_2 l_3}_{m_1 m_2 m_3} = \frac{4\pi}{3} \int d^3k_1 d^3k_2 d^3k_3 \int d\Omega_2 Y_{l_1 m_1}(\vec{k}_1) Y_{l_2 m_2}(\vec{k}_2) Y_{l_3 m_3}(\vec{k}_3)
\]

where in the last line, one performs the integral over the angular parts of the three \(k_i\) and use the exponential integral form for the delta function that appears in the bispectrum definition (47). The integral over the angular part of \(\vec{x}\), in the last line, is known as the Gaunt integral, i.e.,

\[
G^{l_1 l_2 l_3}_{m_1 m_2 m_3} = \int d\Omega_2 Y_{l_1 m_1}(\vec{x}) Y_{l_2 m_2}(\vec{x}) Y_{l_3 m_3}(\vec{x})
\]

\[
= \sqrt{\frac{(2l_1 + 1)(2l_2 + 1)(2l_3 + 1)}{4\pi}} \begin{pmatrix} l_1 & l_2 & l_3 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} l_1 & l_2 & l_3 \\ m_1 & m_2 & m_3 \end{pmatrix}.
\]

(51)

The fact that the bispectrum \(B^{l_1 l_2 l_3}_{m_1 m_2 m_3}\) turns out to be proportional of the Gaunt integral, \(G^{l_1 l_2 l_3}_{m_1 m_2 m_3}\), implies that the values of \(l, m\), corresponding to the non-vanishing components of the bispectrum, must satisfy “the triangle conditions” and also reflect the fact that in the definition of the bispectrum there is a Dirac delta \([47]\) which guarantees the rotational invariance for the ensemble average \(\Psi_{\vec{k}_1} \Psi_{\vec{k}_2} \Psi_{\vec{k}_3}\).

One can find a close form for the bispectrum by choosing to work in the Sach-Wolfe approximation, where the transfer function \(T(k) = 1\); thus, substituting Eq. (49) into Eq. (50) and evaluating the remaining integrals one obtains (See Ref. [57]):

---

\(^{13}\) Let us recall that the variance \(\Psi_g^2(\vec{x}, \eta) = \int_0^\infty dk k^2 P_{\Psi}(k, \eta)\) diverges logarithmically for a power spectrum such that \(P_{\Psi}(k, \eta) \propto k^{-3}\) unless one introduce an \(ad \ hoc\) cutoff for \(k\). For a detailed discussion of this and other related issues see Ref. [12] and Appendix B of the same Ref.
Finally, using the definition of the angle-averaged bispectrum [Eq. (50)], one has

\[
B_{l_1l_2l_3} = \frac{1}{4\pi} \left( \begin{array}{ccc}
l_1 & l_2 & l_3 \\
0 & 0 & 0
\end{array} \right) f_{NL} \left( \begin{array}{c}
2A_{f}^2 \\
27\pi^2
\end{array} \right) l_1(l_1+1)l_2(l_2+1) + \frac{1}{l_2(l_2+1)l_3(l_3+1)} + \frac{1}{l_1(l_1+1)l_3(l_3+1)}.
\]

(53)

When working within the standard scenario for slow-roll inflation (and thus a “nearly” scale-invariant spectrum), Maldacena found that the estimate for the amplitude of non-Gaussianities of the local form, is \( f_{NL} \approx \epsilon \) (this is assumed to be in the limit when \( k_1 \ll k_2 \approx k_3 \), i.e. in the so-called “squeezed” configuration). As seen from the previous discussion, in the conventional approach, the amplitude of the primordial bispectrum and the non-Gaussian statistics for the curvature perturbation are intrinsically related.

### B. Comparing the magnitude of the collapse and the traditional bispectrum

The magnitude of \(|B_{l_1l_2l_3}|\) is obtained from Eq. (53)

\[
|B_{l_1l_2l_3}| = \sqrt{\frac{(2l_1+1)(2l_2+1)(2l_3+1)}{4\pi}} \left| \begin{array}{ccc}
l_1 & l_2 & l_3 \\
0 & 0 & 0
\end{array} \right| f_{NL} \left( \begin{array}{c}
2H^4 \\
27\pi^2 M_*^2 \epsilon^2
\end{array} \right) \times \left( \frac{1}{l_1(l_1+1)l_2(l_2+1)} + \frac{1}{l_2(l_2+1)l_3(l_3+1)} + \frac{1}{l_1(l_1+1)l_3(l_3+1)} \right);
\]

(54)

where we have used the estimate for the amplitude of the power spectrum for a single scalar field in the slow-roll scenario, given by \( A_f \approx H^2/(M_*^2 \epsilon) \).

On the other hand, the magnitude of the collapse bispectrum is given by Eq. (45), which we will write again

\[
|\mathcal{B}^{\text{obs}}_{l_1l_2l_3}|_{\text{M.L.}} = \frac{1}{\pi^{3/2}} \left( \frac{H}{10M_*\epsilon^{1/2}} \right)^3 \left| \cos z - \frac{\sin z}{z} \right|^3 \left( \frac{1 + \Delta_{l_1l_2l_3}}{l_1(l_1+1)l_2(l_2+1)l_3(l_3+1)} \right)^{1/2}.
\]

Therefore we have two distinct theoretical predictions for the actual observed bispectrum \(|B_{l_1l_2l_3}|_{\text{Actual obs}} \) [see Def. 40]. In the standard slow-roll inflationary scenario the prediction is given by \(|B_{l_1l_2l_3}|\) Eq. \( (44) \); meanwhile, by considering the collapse hypothesis the prediction is \(|\mathcal{B}^{\text{obs}}_{l_1l_2l_3}|_{\text{M.L.}} \) Eq. \( (45) \).

The first and most important difference is the fact that Eq. \( (53) \) vanishes unless \( f_{NL} \neq 0 \), while no such “primordial non-Gaussianity” is required for a non-vanishing value of Eq. \( (45) \).

The second difference is that the shape of the bispectrum, i.e. its dependence on \( l \), is not the same: \(|\mathcal{B}^{\text{obs}}_{l_1l_2l_3}|_{\text{M.L.}} \) scales roughly as \( \sim |l(l+1)|^{-3/2} \) while \(|B_{l_1l_2l_3}| \) as \( \sim (2l+1)^{3/2}/[l_3(l+1)]^2 \). Unfortunately, the existing analysis of the observational data do not focus the exact shape of the bispectrum, but rather on a generic measure of its amplitude. However, the reported observational amplitude of the bispectrum, which in the standard picture of slow-roll inflation corresponds to the non-linear parameter \( f_{NL}^{\text{loc}} \), depends on the expected shape that emerges from the theoretical estimates of bispectrum \( (43), (46) \). In other words, in the standard approach, in order to obtain an estimate for the amplitude of the bispectrum from the observational data, one requires a theoretical motivated shape for the bispectrum. The observational data and theory are strongly interdependent. In this way, and relying on such theoretical considerations, the latest results from Planck mission \( (50) \) lead to an estimate for the amplitudes of the bispectrum for the local, equilateral, and orthogonal models given by \( f_{NL}^{\text{loc}} = 2.7 \pm 5.8, f_{NL}^{\text{equil}} = -42 \pm 75 \), and \( f_{NL}^{\text{ortho}} = -25 \pm 39 \) (68% CL statistical). On the other hand, none of the previously considered shapes, namely the local, equilateral or orthogonal, correspond to the one given by the collapse bispectrum (in Sec VI we will deepen this discussion).
It is also interesting to note that, in the traditional framework, there is a well known result when considering single-field inflation named the “consistency relation.” This is, the value predicted for the local non-linear parameter is given by \( f_{NL}^{\text{loc}} \approx n_s - 1 \), with \( n_s \) the “spectral index” of the primordial scalar fluctuations. This prediction is obtained by assuming a single scalar field and no other assumptions (within the standard approach); in particular, it is independent of: the form of the potential, the form of the kinetic term and the initial vacuum state. It is clear obtained assuming a single scalar field and no other assumptions (within the standard approach); in particular, assuming a perfect scale-independent spectrum does not imply that the predicted value should be \( |B_{l_1l_2l_3}|_{\text{M.L.}} = 0 \); in some sense we have given a counter-example for the “consistency relation.”

We conclude this section by noting a fundamental implication that comes from the difference between the statistical treatment in the collapse bispectrum \( |B_{l_1l_2l_3}|_{\text{M.L.}} \) and the common bispectrum \( |B_{l_1l_2l_3}| \). The collapse bispectrum was obtained assuming a Gaussian PDF for the random variable \( X_F \) and, as can be seen from Eq. (15), this translates into a Gaussian distribution for the Newtonian potential \( \Psi^F \). In other words, we have taken a Gaussian curvature perturbation and obtained a non-vanishing prediction for the observed bispectrum \( |B_{l_1l_2l_3}|_{\text{Actual obs.}} \). On the other hand, assuming a Gaussian metric perturbation—with the traditional inflationary paradigm—would have yield \( |B_{l_1l_2l_3}| = 0 \), since, in this approach, one is led to conclude that an observed non-vanishing bispectrum is an irrefutable proof of non-Gaussian statistics for the primordial perturbations. From the collapse hypothesis point of view, the observed \( |B_{l_1l_2l_3}|_{\text{Actual obs.}} \) corresponds to just one particular realization of a random quantum process (the self-induced collapse of the wave function). Since we do not have access to other realizations (i.e. we do not have observational access to other universes) we cannot say anything conclusive as to whether the underlying PDF is Gaussian or not. In other words, by measuring a non-vanishing \( |B_{l_1l_2l_3}|_{\text{Actual obs.}} \) in our own Universe, which corresponds to a single realization of the physical process, does not necessary mean that the ensemble average \( \left\langle \Psi^F_{k_1} \Psi^F_{k_2} \Psi^F_{k_3} \right\rangle \) is also non-vanishing and would consequently proving non-Gaussianity statistics for the ensemble.

In fact, just as we do not expect the actual value of the one available realization of \( a_{lm} \) (for a fixed value of \( l \) and \( m \)) to vanish identically, even if somehow the ensemble average of such quantity (sorting to which we would have no access even if an ensemble of universes did exist) would vanish, we should not expect the single realization of a bispectrum, corresponding to our observations of the Universe, to vanish identically, even if the average value would vanish.

V. A NEW OBSERVATIONAL QUANTITY OBTAINED FROM THE BISPECTRUM

In this section, we will give an explicit expression for a new quantity that in principle can be measured directly. The proposed quantity was introduced first in Ref. [15]. The main feature of this new quantity is that there is no mixing between theory and observational data. The motivation for presenting here the explicit form of this quantity is to illustrate the different kinds of considerations that are natural within different approaches to the subject (the traditional one or with the collapse hypothesis). In fact, in the conventional case, by handling all averages involved (quantum averages, ensemble averages, space average, orientation averages) as essentially the same average, one is lead to theoretical predictions that cannot be clearly and directly connected with the observational quantities and, thus, the discussions about issues such as non-Gaussianities tend to become obscure and are prone to lead to confusion. In contrast with the one given by considering the collapse proposal, the separate issues can be discussed in a clearly independent manner; also, the various kinds of statistical considerations become disentangled and their treatments more transparent.

We begin by recalling that, within the traditional picture, in order to obtain a theoretical estimate of \( B_{l_1l_2l_3} \), one needs to compute \( \left\langle \Psi^F_{k_1} \Psi^F_{k_2} \Psi^F_{k_3} \right\rangle \), i.e. a quantum three-point function that involves quantum field operators and initial quantum states.\(^{14}\) After such calculation, one is required to accept the identification:

\[
\left\langle \Psi^F_{k_1} \Psi^F_{k_2} \Psi^F_{k_3} \right\rangle = \frac{1}{|k_1| |k_2| |k_3|} \left( \sigma_{k_1} \sigma_{k_2} \sigma_{k_3} \right),
\]

with the understanding that the right hand side of Eq. (55) is now an average over an ensemble of universes. Such identification is not clearly justified in the conventional approach; the reason is that the situation in the cosmological context is quite distinct from the laboratory setting, in which one can clearly identify the observer, the measuring

\(^{14}\) Much has been written in the literature on this subject; nevertheless, the goal has been always the same, that is, to calculate the three-point correlation function by using the so-called in-in formalism. Probably the most known works were produced by Maldacena and Weinberg, additional known works are in Refs. [52] and [6].
device and the physical observables. Additionally, some of the usual lines of the argument continues by invoking ergodic considerations to make a further connection between ensemble averages and time averages; likewise, with other imprecise arguments, one is asked to accept replacing the time averages with spatial averages and often turning, in practice, to orientation averages.

To be more precise, in the traditional approach, once the identification given by Eq. (55) is made, one proceeds to calculate \( B_{m_1 m_2 m_3}^{l_1 l_2 l_3} = \bar{a}_{l_1 m_1} a_{l_2 m_2} a_{l_3 m_3} \) and that allows one to obtain \( B_{l_1 l_2 l_3} \) [see Eqs. (20) and (30)].

Furthermore, if one considers the geometrical factors associated with the Gaunt integral, which comes from considering a rotationally invariant ensemble of universes (which is often mistakenly considered equivalently as a rotationally invariant CMB sky), then the angular bispectrum can be factorized as follows:

\[
B_{m_1 m_2 m_3}^{l_1 l_2 l_3} = \mathcal{G}_{l_1 l_2 l_3}^{m_1 m_2 m_3} b_{l_1 l_2 l_3},
\]  

(56)

where \( b_{l_1 l_2 l_3} \) is the so-called “reduced bispectrum.” Note that if we were to consider some, in principle, arbitrary values of \( B_{m_1 m_2 m_3}^{l_1 l_2 l_3} \), then, in general, it will not be the case that the quantity \( b_{l_1 l_2 l_3} \) would contain all information about the former and, thus; it will not be the case that Eq. (56) would reproduce the original values.

The relation between the reduced bispectrum and the angle-averaged-bispectrum is implicitly taken to be

\[
B_{l_1 l_2 l_3} = \sqrt{(2 l_1 + 1)(2 l_2 + 1)(2 l_3 + 1)} \times \begin{pmatrix} l_1 & l_2 & l_3 \\ 0 & 0 & 0 \end{pmatrix} b_{l_1 l_2 l_3}.
\]  

(57)

As \( B_{m_1 m_2 m_3}^{l_1 l_2 l_3} \) takes explicitly into account the conditions over \( l \) and \( m \) codified in the Gaunt integral, and as, in the standard approach all the various types of averages are identified, it is commonly stated that the reduced bispectrum contains all the relevant physical information of an inflationary model. In particular, the amplitude and the shape of the bispectrum are taken as encoded in this object. In fact, from the previous definitions, it is straightforward to conclude that the prediction for the observational quantity \( a_{l_1 m_1} a_{l_2 m_2} a_{l_3 m_3} \), in the standard picture, is simply

\[
a_{l_1 m_1} a_{l_2 m_2} a_{l_3 m_3} = \mathcal{G}_{l_1 l_2 l_3}^{m_1 m_2 m_3} b_{l_1 l_2 l_3},
\]  

(58)

which however is completely inaccessible to us, because we have information only about the realization that occurs in our Universe.

The development of the collapse approach and the fact that within it we have expressions such as that occurring in Eq. (20), has motivated us to introduce the previous definition (30), but also the definition of a new object called the “reduced bispectrum” as the quantity

\[
\hat{b}_{l_1 l_2 l_3} = \left[ \frac{(2 l_1 + 1)(2 l_2 + 1)(2 l_3 + 1)}{4\pi} \begin{pmatrix} l_1 & l_2 & l_3 \\ 0 & 0 & 0 \end{pmatrix} \right]^{-1} B_{l_1 l_2 l_3}^{\text{obs}}.
\]  

(59)

We want to remark that the quantities \( B_{l_1 l_2 l_3}^{\text{obs}} \) and \( \hat{b}_{l_1 l_2 l_3} \) are not exactly the same as \( B_{l_1 l_2 l_3} \) and \( b_{l_1 l_2 l_3} \) since in the former we are not performing any average over any ensemble of universes. The distinction is again subtle but important.

Finally, the new quantity, which was introduced in Ref. [15], is called the “magnitude of the bispectral fluctuations” and is defined as

\[
\mathcal{F}_{l_1 l_2 l_3} = \frac{1}{(2 l_1 + 1)(2 l_2 + 1)(2 l_3 + 1)} \sum_{m_i} a_{l_1 m_1} a_{l_2 m_2} a_{l_3 m_3} - \mathcal{G}_{l_1 l_2 l_3}^{m_1 m_2 m_3} \hat{b}_{l_1 l_2 l_3}^2.
\]  

(60)

As mentioned in Ref. [15], such quantity could be evaluated in principle from a new type of analysis of the existing data. We will focus next in finding the theoretical predicted value for \( \mathcal{F}_{l_1 l_2 l_3} \) in the traditional approach, as well as, when considering the collapse hypothesis.

In the conventional approach, the prediction for the observed \( a_{l_1 m_1} a_{l_2 m_2} a_{l_3 m_3} \) would be identified with the ensemble average \( \bar{a}_{l_1 m_1} a_{l_2 m_2} a_{l_3 m_3} \) which is the definition of \( B_{l_1 l_2 l_3}^{m_1 m_2 m_3} \). Furthermore, by making such prediction for the observed value (i.e. \( a_{l_1 m_1} a_{l_2 m_2} a_{l_3 m_3} = \bar{a}_{l_1 m_1} a_{l_2 m_2} a_{l_3 m_3} \)), then \( \hat{b}_{l_1 l_2 l_3} = b_{l_1 l_2 l_3} \). Consequently, by Eq. (58), the prediction for the observed value of \( \mathcal{F}_{l_1 l_2 l_3} \), in the traditional approach, would be exactly zero. The reason behind such result, is that all the averages involved are essentially taken to be the same; the predicted value for the observed \( a_{l_1 m_1} a_{l_2 m_2} a_{l_3 m_3} \) is the ensemble average \( \bar{a}_{l_1 m_1} a_{l_2 m_2} a_{l_3 m_3} \), then this last quantity is made equal to \( \mathcal{G}_{m_1 m_2 m_3} b_{l_1 l_2 l_3} \) which is in practice obtained from a further averaging over different orientations in our own Universe.
On the other hand, the prediction within the collapse proposal is a non-vanishing value. For a better understanding of the analysis we are lead to in our approach, we will recapitulate some of the arguments mentioned in Sec. [III.B] when dealing with the basic statistical aspects in our proposal. First, we are interested in the most likely value for \( \mathcal{F}_{l_1 l_2 l_3} \). The most likely value will be the theoretical estimate for the observed value of \( \mathcal{F}_{l_1 l_2 l_3} \). Assuming again a Gaussian PDF for the random variables \( X_{g_i} \), and thus, a Gaussian PDF for each \( a_{lm} \) characterizing a single Universe in the imaginary ensemble, we identify the ensemble average \( \mathcal{F}_{l_1 l_2 l_3} \) with the most likely value.

\[
\mathcal{F}_{l_1 l_2 l_3}^{\text{M.L.}} = \frac{1}{(2l_1 + 1)(2l_2 + 1)(2l_3 + 1)} \sum_{m_i} |a_{l_1 m_1}a_{l_2 m_2}a_{l_3 m_3} - G_{l_1 l_2 l_3}^{m_1 m_2 m_3}b_{l_1 l_2 l_3}|^2. \tag{61}
\]

Once again, we will present only the final result of the calculation to not deviate the attention of the reader from the main point of this section. The interested reader can consult Appendices [A] and [B] for more details. Hence, the prediction for the observed value of \( \mathcal{F}_{l_1 l_2 l_3} \), in our approach, is

\[
\mathcal{F}_{l_1 l_2 l_3}^{\text{M.L.}} = \frac{(2\pi)^6 l_1 l_2 l_3}{(2l_1 + 1)(2l_2 + 1)(2l_3 + 1)} \left( 1 - \frac{1}{(2l_1 + 1)(2l_2 + 1)(2l_3 + 1)} + \frac{2\delta_{l_1 l_2}}{2l_1 + 1} + \frac{2\delta_{l_2 l_3}}{2l_2 + 1} + \frac{8\delta_{l_1 l_2}\delta_{l_2 l_3}}{(2l_1 + 1)(2l_2 + 1)(2l_3 + 1)} \right), \tag{62}
\]

with \( \Delta_{l_1 l_2 l_3} \) the object defined in expression [44]. Equation (62) is the main result for this section. Such expression is valid as long as \( l_1, l_2, l_3 \gg 1 \), the rest of the terms inside the parenthesis of Eq. (62) are negligible respect to the first, thus,

\[
l_1(l_1 + 1)l_2(l_2 + 1)l_3(l_3 + 1)\mathcal{F}_{l_1 l_2 l_3} \simeq \left( \frac{H}{10\pi^{1/2}M_p\ell^{1/2}} \right)^6 \left( \cos z - \frac{\sin z}{z} \right)^6, \tag{63}
\]

where we have used the definition of \( g(z) \) [Eq (21)].

The result in Eq. (63) possess a similar structure as the one given by the observed angular spectrum when the Sachs-Wolfe effect is dominant, i.e.

\[
l(l + 1)C_l \simeq \text{constant}. \tag{64}
\]

Thus, having a well defined notion over which elements one is performing the average, the predictions for the observational quantities, in this case \( \mathcal{F}_{l_1 l_2 l_3} \), change substantially.

The complete analysis of estimators like the one introduced in this section will be object of future research; however, we wished to present it as an example of the kind of studies that can be done in our approach, where one can clearly identify the statistical aspects of the problem at hand.

**VI. ESTIMATES AND COMPARISONS WITH OBSERVATIONS**

Let’s recall that, in the standard approaches, the study of the bispectrum is tied to the search for non-Gaussianities and is, thus, based primarily on the quantity

\[
\langle \tilde{\Psi}(\vec{k}_1)\tilde{\Psi}(\vec{k}_2)\tilde{\Psi}(\vec{k}_3) \rangle = \Psi(\vec{k}_1)\Psi(\vec{k}_2)\Psi(\vec{k}_3) \equiv (2\pi)^3\delta(\vec{k}_1 + \vec{k}_2 + \vec{k}_3)B(k_1, k_2, k_3). \tag{65}
\]

The connection with observations is made by relating \( B(k_1, k_2, k_3) \) with the angle-averaged bispectrum, for which we have expression (50), this is
The quantities $B_{1123}$ need not concern us in this discussion. We note that the dependence on $A$ is because, when searching for a signal, it would be done by taking one particular model that could be very different from the Bunch-Davies vacuum and might lead to strong deviations from slow-roll. The quantities arising from considering local non-linear effects in the curvature perturbation, originated from initial states other than the Bunch-Davies vacuum and $B(k_1, k_2, k_3)$ is obtained by considering inflationary potentials with a “feature” that can lead to strong deviations from slow-roll. The quantities, $k_1$ and $\phi$ are constant parameters of the models that we need not consider in this discussion. We note that the dependence on $k_1$, $k_2$, $k_3$ differs sharply from case to case. The quantity $A_{\Psi}^2$ is the amplitude of the traditional power spectrum, the observed angular spectrum $\ell(\ell+1)C_\ell$ fixes it to $A_{\Psi} \approx 10^{-10}$. $f_{NL}$ is what is known as the bispectrum amplitude and it varies depending on the model. It is clear that different forms of $B(k_1, k_2, k_3)$ lead to different $l$ dependences for the expected observational angular bispectrum.

Unfortunately, this means that one cannot estimate directly a generic amplitude for all different kind of models; this is because, when searching for a signal, it would be done by taking one particular model that could be very different from another.

For instance, the analysis of the data from Planck, WMAP, etc. do not report the observed values for $B_{1123}$ (or the values for $B_{m1(2m3)}$ for that matter), but rather concentrate on the estimates of the value $f_{NL}$ for various models that have been proposed. In fact, in one of the recent articles presenting the results from the Planck satellite [50], one reads (see pg. 32 of that reference):

"The full bispectrum for a high-resolution map cannot be evaluated explicitly because of the sheer number of operations involved, $O(l_{max}^5)$, as well as the fact that the signal will be too weak to measure in individual multipoles with any significance. Instead, we essentially use a least-squares fit to compare the bispectrum of the observed CMB multipoles with a particular theoretical bispectrum $b_{1123}$. We then extract an overall "amplitude parameter" $f_{NL}$ for that specific template, after defining a suitable normalization convention so that we can write $b_{1123} = f_{NL} b_{1123}^{th}$, where $b_{1123}^{th}$ is defined as the value of the theoretical bispectrum ansatz for $f_{NL} = 1$.

There are various schemes for estimating $f_{NL}$ but one of the most general methods for evaluating $f_{NL}$ (and one employed by the studies of Planck’s data) relies on the following:
\[ \hat{f}_{NL} = \frac{1}{N^2} \sum_{l,m_i} \left( \frac{l_1}{m_1} \frac{l_2}{m_2} \frac{l_3}{m_3} \right) \frac{B_{lm_1}^{\text{th}}}{(C_{l_1} C_{l_2} C_{l_3})_{\text{obs}}} (a_{l_1 m_1} a_{l_2 m_2} a_{l_3 m_3})_{\text{obs}} \]  

(70)

where \( \hat{f}_{NL} \) represents the estimated value; \( B_{lm_1}^{\text{th}} \) is the input characterizing the theoretical model under consideration by setting \( f_{NL} = 1 \), \( N \) is a normalization constant and \( (a_{l_1 m_1} a_{l_2 m_2} a_{l_3 m_3})_{\text{obs}} \) are the quantities extracted directly from observations (and so are the \( C_l \)).

For the models previously mentioned, the estimates from Planck’s data, using the above scheme, lead to:

(i) \( f_{NL}^{\text{loc}} = 2.7 \pm 5.8 \) (68% CL), thus, \( |f_{NL}^{\text{loc}}| \approx 10 \).

(ii) \( f_{NL}^{\text{flat}} = 37 \pm 77 \) (0.9σ). Actually, in some other models considering non-Bunch Davies vacuum states, the form of \( B(k_1, k_2, k_3)^{\text{NBD}} \), as given in Eqs. (6.2) and (6.3) of Ref. [61], Planck’s data estimate \( f_{NL}^{\text{NBD}} = 155 \pm 78 \) at (2.2σ) (see Table 11 of Ref. [56] for other estimated values of \( f_{NL} \) within these type of models). Therefore, \( |f_{NL}^{\text{NBD}}| \approx 10^2 \).

(iii) \( f_{NL}^{\text{feat}} = 434 \pm 170 \) at (2.6σ) for \( k_c = 0.0125 \) and \( \phi = 0 \). Another estimate for the same “feature” model, but considering an envelope decay function of the form \( \exp[-(k_1 + k_2 + k_3)/mk_c] \) (where \( m \) is a model-dependent parameter) with a width \( \Delta k = 0.015 \), yields \( f_{NL}^{\text{feat}} = 765 \pm 275 \) at (2.8σ) for \( k_c = 0.01125 \) and \( \phi = 0 \) (see Tables 12 and 13 of Ref. [56] for other estimated values of \( f_{NL} \) within these type of models). Consequently, \( |f_{NL}^{\text{feat}}| \approx 10^3 \).

Items above clearly show that the various functional dependencies, assumed for the bispectrum, lead to dramatically different estimates for the bispectrum amplitude.

The prediction for the bispectrum arising from the collapse model is:

\[ |B_{l_1 l_2 l_3}^{\text{obs}}|_{\text{M.L.}} = \frac{1}{\pi^{3/2}} \left( \frac{H}{10 M_P c^{1/2}} \right)^3 \cos z - \left( \sin z \right)^3 \left( 1 + \Delta_{l_1 l_2 l_3} \right)^{1/2} \left( \frac{l_1 (l_1 + 1) l_2 (l_2 + 1) l_3 (l_3 + 1)}{l_1 + 1} \right)^{1/2}. \]  

(71)

The most noteworthy feature in this prediction is the absence of any new and unknown parameter characterizing a novel element (i.e the model does not introduce an analogue free parameter \( f_{NL} \) characterizing non-Gaussianities). In fact, we can rewrite Eq. (71) as

\[ |B_{l_1 l_2 l_3}^{\text{obs}}|_{\text{M.L.}} = A^2 \hat{f} \left( \frac{1 + \Delta_{l_1 l_2 l_3}}{l_1 (l_1 + 1) l_2 (l_2 + 1) l_3 (l_3 + 1)} \right)^{1/2}. \]  

(72)

where \( A \) was defined in Eq. (28) and

\[ \hat{f} \equiv A^{-1/2}. \]  

(73)

As we have shown in Sec. IIIA, the observed angular spectrum fixes \( A \approx 10^{-10} \), so the prediction of the collapse model is \( \hat{f} \approx 10^5 \). The quantity \( \hat{f} \) is what replaces, in the collapse model, the quantity \( f_{NL} \) of the standard analyses, and as indicated before, there is absolutely no adjustable parameter to be fixed by observations.

Nevertheless, we should, at this point, offer a strong warning regarding the way one might make use of our estimates in comparisons with observations. As we have argued at the end of Sec. IIIB in our approach, we can only make estimates regarding the most likely value of the magnitude of the \( a_{lm} \) and, thus, of the spectrum, as well as that of the bispectrum; while in general, the actual values of such quantities would be complex. Thus, one cannot simply use expression (71), together with data on the \( a_{lm} \), substitute it in the formula (70) and take that to represent our model’s estimate for the quantity \( f_{NL} \). The expected complex nature of the quantities would, without doubt, play an important role that cannot be ignored.

As we have seen, different shapes of the bispectrum lead to rather distinct estimates for its amplitude and given that the shape of the bispectrum in our model is very different as, say, the local model, one cannot state that \( f_{NL}^{\text{loc}} \) should be of the same order of magnitude as \( \hat{f} \). Instead, one should perform a similar estimate for \( \hat{f} \), as the one made with the other models using the observational data, and compare it with the prediction \( \hat{f} \approx 10^5 \). This means that, on the one hand the collapse model enjoys a stronger predictive power, and, on the other, that is completely susceptible to falsification by observational data.
VII. CONCLUSIONS

We have presented a detailed discussion on the manner, in which the study of essential statistical features on the CMB spectrum, must be studied in the context of the collapse models. We have shown the important differences that arise between the analysis in this approach and those tied to the standard approaches. We have seen, for instance, that the collapse models lead to explicit expressions for the quantities that are rather directly observable, the $a_{lm}$ [see Eq. (20)], which have no counterpart in the usual analyses. Among other advantages, the expression for the coefficients $a_{lm}$ [Eq. (20)] exhibits directly the source of the randomness involved, aspects that in the standard approach can only be discussed heuristically (simply because there, the random variables are not clearly identified and named as in this approach). We have seen that this leads to a rather different analysis of the higher order statistical features, such as the bispectrum, which is usually associated with non-Gaussianities. We have seen that in the collapse scheme, one is lead to a non-vanishing expected bispectrum, even if there are no primordial non-Gaussianities (i.e. in the statistical analysis of Sec. III B all statistical correlations are encoded in the two variable correlation functions of the fiducial ensemble of universes, which are taken as standard). In fact, we have seen that the magnitude of such bispectrum is predicted and it involves no adjustable parameters. This makes the model highly predictive, and by the same token, highly susceptible to falsification.

One might be tempted to use the estimates for the bispectrum amplitude obtained in the analysis of the data for the other models, but as explained in Sec. VI, the fact that the functional form of the expected bispectrum is very different in our model and the models that have been used as for the comparison with observations, invalidates from the start that program. One can confirm this fact simply by noting the sharp differences in the estimates of $f_{NL}$ that are extracted from the same data for the various models mentioned above. However, the data are in principle available and, thus, testing the prediction of the collapse models seems to be well within reach.

Nevertheless, before even proceeding to do this, we need to reevaluate the predicted bispectrum taking into account the effect of the transfer functions that we have ignored in performing the calculation leading to the expression (45). That is, in such calculation, we replaced the transfer functions by the number 1 so that the integrals could be evaluated in closed form.

Thus, the actual comparison of the prediction of this model with data will require the reintroduction of the transfer functions in the evaluation. Carrying this out would involve numerical calculations in order to perform the desired integrals that will result in the specific form of the $F_{l_{1}l_{2}l_{3}}^{obs}$, which will be suitable for comparison with observations. We plan to carry this analysis in the near future and to obtain the data to contrast with the model’s predictions. We would view a reasonable match between observations and the model as a strong indication we are on the right track. We would certainly not expect a complete and precise agreement, simply due to the fact that, our model, allows us to obtain only a most likely value for the quantities controlled by the random numbers associated with the collapse processes. However, the fact that the scheme for evaluating the most likely value of the bispectrum is essentially the same scheme we used too evaluate the spectrum, as well as the most likely values of the $C_{l}$ [see Eq. (27)], and that we found a very good match between theory and observations, would make it very difficult to understand such agreement in one case and any strong departure in the second.

Finally, in Sec. V we have proposed a new quantity for use in the study of higher order statistical features of the CMB data: $F_{l_{1}l_{2}l_{3}}$ [see Eq. (60)]. The estimate for this quantity in any of the standard scenarios would vanish, independently of the functional form of the bispectrum, simply because the standard schemes offer no mathematical characterization of the randomness, and that should be a key aspect of any description of something like the distribution of the seeds of cosmic structure. In the collapse scheme, we have a specific expression for the most likely value of this quantity. Therefore, a comparison with the observation would be a nice empirical test of the ideas tied to our proposals.

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Appendix A: Details concerning the derivation of the collapse bispectrum

In this appendix we will extend the details involved in computing Eq. (13). We start by rewriting the coefficient $a_{lm}$, Eq. (20), as
where \( g(z) \) corresponds to the definition in Eq. (A1) (we have also assumed that \( \eta^2 = z/k \), i.e. \( z \) independent of \( k \) and also set \( T(k) = 1 \), and

\[
F_{lm}(\hat{k}) = \frac{j_i(kR_D)Y_{lm}^*(\hat{k})}{k^{3/2}}. 
\]  

(A2)

Therefore, the ensemble average of the squared magnitude of the collapse bispectrum is

\[
\left\langle B_{\ell_1\ell_2\ell_3}^{\text{obs}} \right\rangle^2 = \sum_{m_1, \ldots, m_6} \left( \frac{l_1}{m_1} \frac{l_2}{m_2} \frac{l_3}{m_3} \right) \left( \frac{l_1}{m_4} \frac{l_2}{m_5} \frac{l_3}{m_6} \right) a_{l_1m_1}a_{l_2m_2}a_{l_3m_3}a^*_{l_4m_4}a^*_{l_5m_5}a^*_{l_6m_6}. 
\]  

(A3)

Let us focus on the term \( a_{l_1m_1}a_{l_2m_2}a_{l_3m_3}a^*_{l_4m_4}a^*_{l_5m_5}a^*_{l_6m_6} \) as its value will be of use in the calculations of Appendix B. By making use of Eqs. (A1) and (A2) we have

\[
a_{l_1m_1}a_{l_2m_2}a_{l_3m_3}a^*_{l_4m_4}a^*_{l_5m_5}a^*_{l_6m_6} = \frac{g(z)^6}{L^9} \sum_{\hat{k}_1, \ldots, \hat{k}_6} F_{l_1m_1}(\hat{k}_1)F_{l_2m_2}(\hat{k}_2)F_{l_3m_3}(\hat{k}_3)F^*_{l_4m_4}(\hat{k}_4)F^*_{l_5m_5}(\hat{k}_5)F^*_{l_6m_6}(\hat{k}_6) 
\]  

\[
\times F^*_{l_1m_1}(\hat{k}_1)F^*_{l_2m_2}(\hat{k}_2)F^*_{l_3m_3}(\hat{k}_3)F^*_{l_4m_4}(\hat{k}_4)F^*_{l_5m_5}(\hat{k}_5)F^*_{l_6m_6}(\hat{k}_6). 
\]  

(A4)

The next step is to substitute (A2) in (A4). The expression obtained from such substitution will contain 15 terms of triple products of Kronecker deltas

\[
a_{l_1m_1}a_{l_2m_2}a_{l_3m_3}a^*_{l_4m_4}a^*_{l_5m_5}a^*_{l_6m_6} = \frac{2^3g(z)^6}{L^9} \sum_{\hat{k}_1, \ldots, \hat{k}_6} F_{l_1m_1}(\hat{k}_1)F_{l_2m_2}(\hat{k}_2)F_{l_3m_3}(\hat{k}_3)F^*_{l_4m_4}(\hat{k}_4)F^*_{l_5m_5}(\hat{k}_5)F^*_{l_6m_6}(\hat{k}_6) 
\]  

\[
\times \delta_{\hat{k}_1,-\hat{k}_2} \cdot \delta_{\hat{k}_3,-\hat{k}_4} \cdot \delta_{\hat{k}_5,-\hat{k}_6} + \delta_{\hat{k}_1,-\hat{k}_2} \cdot \delta_{\hat{k}_3,-\hat{k}_5} \cdot \delta_{\hat{k}_4,-\hat{k}_6} + \delta_{\hat{k}_1,-\hat{k}_3} \cdot \delta_{\hat{k}_4,-\hat{k}_5} \cdot \delta_{\hat{k}_2,-\hat{k}_6} 
\]  

\[
+ \delta_{\hat{k}_1,-\hat{k}_4} \cdot \delta_{\hat{k}_2,-\hat{k}_3} \cdot \delta_{\hat{k}_5,-\hat{k}_6} \cdot \delta_{\hat{k}_3,-\hat{k}_5} \cdot \delta_{\hat{k}_4,-\hat{k}_6} \cdot \delta_{\hat{k}_1,-\hat{k}_5} \cdot \delta_{\hat{k}_2,-\hat{k}_6} \cdot \delta_{\hat{k}_3,-\hat{k}_4} \cdot \delta_{\hat{k}_5,-\hat{k}_6} 
\]  

\[
+ \delta_{\hat{k}_1,-\hat{k}_6} \cdot \delta_{\hat{k}_2,-\hat{k}_3} \cdot \delta_{\hat{k}_4,-\hat{k}_5} \cdot \delta_{\hat{k}_1,-\hat{k}_5} \cdot \delta_{\hat{k}_2,-\hat{k}_6} \cdot \delta_{\hat{k}_3,-\hat{k}_4} \cdot \delta_{\hat{k}_5,-\hat{k}_6} 
\]  

\[
+ \delta_{\hat{k}_1,-\hat{k}_3} \cdot \delta_{\hat{k}_2,-\hat{k}_4} \cdot \delta_{\hat{k}_5,-\hat{k}_6} \cdot \delta_{\hat{k}_1,-\hat{k}_5} \cdot \delta_{\hat{k}_2,-\hat{k}_6} \cdot \delta_{\hat{k}_3,-\hat{k}_4} \cdot \delta_{\hat{k}_5,-\hat{k}_6}. 
\]  

(A5)

Let us focus on the first term

\[
\frac{2^3g(z)^6}{L^9} \sum_{\hat{k}_1, \ldots, \hat{k}_6} F_{l_1m_1}(\hat{k}_1)F_{l_2m_2}(\hat{k}_2)F_{l_3m_3}(\hat{k}_3)F^*_{l_4m_4}(\hat{k}_4)F^*_{l_5m_5}(\hat{k}_5)F^*_{l_6m_6}(\hat{k}_6) \delta_{\hat{k}_1,-\hat{k}_2} \cdot \delta_{\hat{k}_3,-\hat{k}_4} \cdot \delta_{\hat{k}_5,-\hat{k}_6}. 
\]

The Kronecker deltas in expression (A6), will turn the sum over \( \hat{k}_1, \ldots, \hat{k}_6 \) into a sum over \( \hat{k}_1, \hat{k}_3, \hat{k}_5, \) that is

\[
\frac{2^3g(z)^6}{L^9} \sum_{\hat{k}_1, \hat{k}_3, \hat{k}_5} F_{l_1m_1}(\hat{k}_1)F_{l_2m_2}(\hat{k}_2)F_{l_3m_3}(\hat{k}_3)F^*_{l_4m_4}(\hat{k}_4)F^*_{l_5m_5}(\hat{k}_5)F^*_{l_6m_6}(\hat{k}_6). 
\]  

(A6)

Using the parity of the spherical harmonics \( Y_{lm}(-\hat{k}) = (-1)^l Y_{lm}(\hat{k}) \) gives the relation \( F_{lm}(-\hat{k}) = (-1)^l F_{lm}(\hat{k}) \); therefore expression (A6) becomes
\[ \frac{2^3 g(z)^6 (-1)^{l_2 + l_3}}{L^9} \sum_{\vec{k}_1, \vec{k}_3, \vec{k}_5} F_{l_1 m_1} (\vec{k}_1) F_{l_2 m_2} (\vec{k}_2) F_{l_3 m_3} (\vec{k}_3) F_{l_4 m_4} (\vec{k}_4) F_{l_5 m_5} (\vec{k}_5) F_{l_6 m_6} (\vec{k}_6). \]  

(A7)

Taking the continuum limit \( (L \to \infty \text{ and } \vec{k} \text{ discrete } \to \vec{k} \text{ continuous}) \) in expression (A7) yields

\[ \frac{2^3 g(z)^6 (-1)^{l_2 + l_3}}{(2\pi)^9} \left( \int d^3 k_1 F_{l_1 m_1} (\vec{k}_1) F_{l_2 m_2} (\vec{k}_1) \right) \left( \int d^3 k_3 F_{l_4 m_4} (\vec{k}_3) F_{l_5 m_5} (\vec{k}_3) \right) \times \left( \int d^3 k_5 F_{l_6 m_6} (\vec{k}_5) F_{l_6 m_6} (\vec{k}_5) \right). \]  

(A8)

Focusing now on the first integral appearing in the product of expression (A8) we have

\[ \int d^3 k F_{l_1 m_1} (\vec{k}) F_{l_2 m_2} (\vec{k}) = \int d^3 k \frac{j_{l_1} (k R_D) j_{l_2} (k R_D)}{k} Y_{l_1 m_1} (\vec{k}) Y_{l_2 m_2} (\vec{k}) \]

\[ = \int_0^\infty \frac{dk}{k} j_{l_1} (k R_D) j_{l_2} (k R_D) \int d\Omega Y_{l_1 m_1} (\vec{k}) Y_{l_2 m_2} (\vec{k}) \]

\[ = \int_0^\infty \frac{dk}{k} j_{l_1} (k R_D) j_{l_2} (k R_D) \int d\Omega Y_{l_1 m_1} (\vec{k}) Y_{l_2, m_2} (\vec{k}) (-1)^{m_2} \]

\[ = (-1)^{m_2} \int_0^\infty \frac{dk}{k} j^2_{l_1} (k R_D) \delta_{l_1, l_2} \delta_{m_1, -m_2} \]

\[ = (-1)^{m_2} \frac{\delta_{l_1, l_2} \delta_{m_1, -m_2}}{2l_1(l_1 + 1)}. \]  

(A9)

In the same way, one can easily check that

\[ \int d^3 k F_{l_1 m_1}^* (\vec{k}) F_{l_2 m_2}^* (\vec{k}) = \int d^3 k F_{l_1 m_1} (\vec{k}) F_{l_2 m_2} (\vec{k}) = (-1)^{m_2} \frac{\delta_{l_1, l_2} \delta_{m_1, -m_2}}{2l_1(l_1 + 1)} \]  

(A10)

and

\[ \int d^3 k F_{l_1 m_1} (\vec{k}) F_{l_2 m_2}^* (\vec{k}) = \frac{\delta_{l_1, l_2} \delta_{m_1, m_2}}{2l_1(l_1 + 1)}. \]  

(A11)

Therefore, using Eqs. (A10) and (A11), the expression (A8) is

\[ g(z)^6 (-1)^{l_2 + l_3} \left( \frac{(-1)^{m_2} \delta_{l_1, l_2} \delta_{m_1, -m_2}}{l_1(l_1 + 1)} \right) \left( \frac{\delta_{l_3, l_4} \delta_{m_3, m_4}}{l_3(l_3 + 1)} \right) \left( \frac{(-1)^{m_6} \delta_{l_5, l_6} \delta_{m_5, -m_6}}{l_2(l_2 + 1)} \right) \]

\[ = \frac{g(z)^6 (-1)^{m_2 + m_6} \delta_{l_1, l_2} \delta_{l_3, l_4} \delta_{m_1, -m_2} \delta_{m_3, m_4} \delta_{m_5, -m_6}}{(2\pi)^9 l_1(l_1 + 1)l_2(l_2 + 1)l_3(l_3 + 1)} \]  

(A12)

One can perform similar calculations for the rest of the 14 terms contained in Eq. (A5). One special term is the one involving the product \( \delta_{\vec{k}_1 \vec{k}_4} \delta_{\vec{k}_2 \vec{k}_5} \delta_{\vec{k}_3 \vec{k}_6} \); explicitly such term is

\[ \frac{2^3 g(z)^6}{L^9} \sum_{\vec{k}_1, \ldots, \vec{k}_6} F_{l_1 m_1} (\vec{k}_1) F_{l_2 m_2} (\vec{k}_2) F_{l_3 m_3} (\vec{k}_3) F_{l_4 m_4} (\vec{k}_4) F_{l_5 m_5} (\vec{k}_5) F_{l_6 m_6} (\vec{k}_6) \delta_{\vec{k}_1 \vec{k}_4} \delta_{\vec{k}_2 \vec{k}_5} \delta_{\vec{k}_3 \vec{k}_6}. \]  

(A13)

In the continuum limit, expression (A13) takes the form
\[
\frac{2^3 \langle z \rangle^6}{(2\pi)^9} \left( \int d^3k_1 F_{1i1i}(\vec{k}_1) F_{1i1i}^*(\vec{k}_1) \right) \left( \int d^3k_2 F_{2i2i}(\vec{k}_2) F_{2i2i}^*(\vec{k}_2) \right) \\
\times \left( \int d^3k_3 F_{3i3i}(\vec{k}_3) F_{3i3i}^*(\vec{k}_3) \right). \tag{A14}
\]

Using Eqs. (A11) for the integrals, expression (A14) is simply

\[
\frac{\langle z \rangle^6}{(2\pi)^9 l_1(l_1+1)l_2(l_2+1)l_3(l_3+1)} \delta_{m_1,m_4} \delta_{m_2,m_5} \delta_{m_3,m_6}. \tag{A15}
\]

The term given by expression (A15) is special because there appear no Kronecker deltas that depend on \( l \) (which means that if \( l_1 \neq l_2 \neq l_3 \) this is the only term that is not vanishing). Computing the rest of the terms of Eq. (A15) yields

\[
\frac{a_{l_1,m_1} a_{l_2,m_2} a_{l_3,m_3} a_{l_4,m_4} a_{l_5,m_5} a_{l_6,m_6}}{(2\pi)^9 l_1(l_1+1)l_2(l_2+1)l_3(l_3+1)}
\times \left[ \delta_{m_1,m_4} \delta_{m_2,m_5} \delta_{m_3,m_6} + \delta_{l_1,l_2} \left( -1 \right)^{m_2+m_3,m_1-m_2} \delta_{m_4,m_5-m_3,m_6} + \delta_{m_1,m_2} \delta_{m_3,m_4} \delta_{m_5,m_6} \right]
\]

\[
+ \delta_{l_1,l_3} \left( -1 \right)^{m_3+m_4,m_1} \delta_{m_2,m_3} \delta_{m_4,m_5} \delta_{m_6,m_6} + \delta_{m_1,m_4} \delta_{m_2,m_5} \delta_{m_3,m_6} \right]
\]

\[
+ \left( -1 \right)^{m_2+m_3,m_1} \delta_{m_1,m_2} \delta_{m_3,m_4} \delta_{m_5,m_6} + \left( -1 \right)^{m_3+m_4,m_1} \delta_{m_2,m_3} \delta_{m_4,m_5} \delta_{m_6,m_6} \right]
\]

\[
+ \left( -1 \right)^{m_3+m_4,m_1} \delta_{m_2,m_3} \delta_{m_4,m_5} \delta_{m_6,m_6} + \delta_{l_1,l_3} \left( -1 \right)^{m_1+m_3,m_2} \delta_{m_1,m_2} \delta_{m_3,m_4} \delta_{m_5,m_6} \right]
\]

\[
+ \left( -1 \right)^{m_2+m_3,m_1} \delta_{m_1,m_2} \delta_{m_3,m_4} \delta_{m_5,m_6} + \delta_{l_1,l_3} \left( -1 \right)^{m_1+m_3,m_2} \delta_{m_1,m_2} \delta_{m_3,m_4} \delta_{m_5,m_6} \right] \tag{A16}
\]

Substituting Eq. (A16) in Eq. (A13) we obtain Eq. (43)

\[
|R_{\text{obj}}|_{l_1,l_2,l_3}^2 = \sum_{m_1,...,m_6} \left( \frac{l_1}{m_1} \frac{l_2}{m_2} \frac{l_3}{m_3} \right) \left( \frac{l_1}{m_4} \frac{l_2}{m_5} \frac{l_3}{m_6} \right) a_{l_1,m_1} a_{l_2,m_2} a_{l_3,m_3} a_{l_4,m_4} a_{l_5,m_5} a_{l_6,m_6}
\]

\[
= \frac{g(z)^6 (1 + \Delta_{l_1,l_2,l_3})}{(2\pi)^9 l_1(l_1+1)l_2(l_2+1)l_3(l_3+1)} \tag{A17}
\]

whith \( \Delta_{l_1,l_2,l_3} \) as defined by Eq. (44).

### Appendix B: General derivation of the object \( F_{l_1,l_2,l_3}^{M.L.} \)

In this appendix we will extend the details involved in computing the object \( F_{l_1,l_2,l_3}^{M.L.} \) as expressed in Eq. (61) that is

\[
F_{l_1,l_2,l_3}^{M.L.} = \frac{1}{(2l_1+1)(2l_2+1)(2l_3+1)} \sum_{m_1} |a_{l_1,m_1} a_{l_2,m_2} a_{l_3,m_3} - G_{l_1,l_2,l_3}^{m_1,m_2,m_3} |^2. \tag{B1}
\]

Using the defintion of \( b_{l_1,l_2,l_3} \) [Eq. (59)], \( F_{l_1,l_2,l_3}^{M.L.} \) takes the following form:

\[
F_{l_1,l_2,l_3}^{M.L.} = \frac{1}{(2l_1+1)(2l_2+1)(2l_3+1)} \left[ \sum_{m_1,m_2,m_3} |a_{l_1,m_1} a_{l_2,m_2} a_{l_3,m_3}|^2 
\right]
\]

\[
- \sum_{m_1,...,m_6} \left( \frac{l_1}{m_1} \frac{l_2}{m_2} \frac{l_3}{m_3} \right) \left( \frac{l_1}{m_4} \frac{l_2}{m_5} \frac{l_3}{m_6} \right) a_{l_1,m_1} a_{l_2,m_2} a_{l_3,m_3} a_{l_4,m_4} a_{l_5,m_5} a_{l_6,m_6}. \tag{B2}
\]
The first term in the square brackets of Eq. (B2) can be obtained from Eq. (A16) by taking $m_4 = m_1, m_5 = m_2, m_3 = m_6$; this is,

\[
\sum_{m_1, m_2, m_3} |a_{l_1 m_1} a_{l_2 m_2} a_{l_3 m_3}|^2 = \frac{g(z)^6}{(2\pi)^3 l_1 (l_1 + 1) l_2 (l_2 + 1) l_3 (l_3 + 1)} \times \sum_{m_1, m_2, m_3} \left[ 1 + \delta_{l_1, l_2} (\delta_{m_1, m_2} + \delta_{m_1, -m_2}) + \delta_{l_1, l_3} (\delta_{m_2, m_3} + \delta_{m_2, -m_3}) + \delta_{l_2, l_3} (\delta_{m_1, m_3} + \delta_{m_1, -m_3}) + 2\delta_{l_1, l_2} \delta_{l_2, l_3} \delta_{m_1, m_2} (\delta_{m_2, m_3} + \delta_{m_2, -m_3}) + \delta_{m_1, -m_2} (\delta_{m_2, m_3} + \delta_{m_2, -m_3}) \right] \\
= \frac{g(z)^6}{(2\pi)^3 l_1 (l_1 + 1) l_2 (l_2 + 1) l_3 (l_3 + 1)} \left[ (2l_1 + 1)(2l_2 + 1)(2l_3 + 1) + 2\delta_{l_1, l_2} (2l_1 + 1)(2l_1 + 2) + 2\delta_{l_2, l_3} (2l_2 + 1)(2l_2 + 2) \right] \left( 1 + \frac{1}{(2l_1 + 1)(2l_2 + 1)(2l_3 + 1)} + \frac{2\delta_{l_1, l_2}}{2l_1 + 1} + \frac{2\delta_{l_2, l_3}}{2l_2 + 1} + \frac{2\delta_{l_1, l_3}}{2l_1 + 1} \right) \\
+ \frac{8\delta_{l_1, l_2} \delta_{l_2, l_3}}{(2l_1 + 1)^2} \frac{\Delta_{l_1, l_2, l_3}}{(2l_1 + 1)(2l_2 + 1)(2l_3 + 1)},
\]

which is Eq. (B2).

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