Hadamard Logarithmic Series and Inequalities on The parameters of a Strongly Regular Graph

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Abstract: Let G be a primitive strongly regular graph of order n and A its adjacency matrix. In this paper, we first associate an Euclidean Jordan algebra V to G considering the real Euclidean Jordan algebra spanned by the identity of order n and the natural powers of A. Next, by the analysis of the spectra of an Hadamard logarithmic series of V we establish new admissibility conditions on the parameters of the strongly regular graph G.

INTRODUCTION

Euclidean Jordan algebras become a good tool for the analysis of primal dual interior point methods, see [1], [2] and [3], and had many applications on various areas of Mathematics. Namely, on the formalism of quantum mechanics [4], on combinatorics, see [5], [6], [7], [8] and [9], and on developing applications on statistics [11]. Presently, several generalizations of the properties of symmetric matrices to Euclidean Jordan algebras were established, see [12] and [13].

In this paper we deduce admissibility conditions on the parameters of a strongly regular graph, see [10], in the environment of Euclidean Jordan algebras.

The organization of the paper is as follows. In next section, we present the more relevant definitions and results about Euclidean Jordan algebras necessary for a clear understanding of this paper. In the third section, we present some definitions and results about strongly regular graphs. Finally, in the last section we establish some admissibility conditions on the spectra and over the parameters of a strongly regular graph.

SOME NOTIONS ON EUCLIDEAN JORDAN ALGEBRAS

One finds a clear and simple exposition about Euclidean Jordan algebras on the publication of Farid Alizahed [14]. More detailed surveys about Euclidean Jordan algebras can be encountered in the book Analysis on symmetric cones of Faraut and Korányi [15] and in the book Structure and Representations of Jordan Algebras of Nathan Jacobson [16]. But, we must also say that the monograph of McCrimmon, Taste of Jordan algebras [17] is also a good textbook about Euclidean Jordan Algebras.

Let A be a n-dimensional vector space over a field K with a bilinear map (x, y) \mapsto x \cdot y from A \times A to A. A is a Jordan algebra if x \cdot y = y \cdot x and x \cdot (x \cdot y) = x^2 \cdot (x \cdot y), where x^2 = x \cdot x.

REMARK 1. We must say herein that if V is an associative algebra over a field K with characteristic not equal to 2 and with the operation of multiplication of x by y denoted by x \cdot y, that from now on we will denote by xy then we can obtain a Jordan algebra considering on V a new operation \cdot defined in the following way x \cdot y = (xy + yx)/2. Indeed, we have y \cdot x = (yx + xy)/2 = (xy + yx)/2 = x \cdot y and we have that:
Hence we deduced that \( x \ast (x^2 \ast y) = x^3 \ast (x \ast y) \) and therefore we conclude that \( V \) is a Jordan algebra when equipped with the product \( \ast \).

**REMARK 2.** We will suppose, from now on that when we say let \( \mathcal{A} \) be a Jordan algebra we mean that \( \mathcal{A} \) is a real finite dimensional Jordan algebra and has a unit element denoted by \( e \).

Let \( \mathcal{A} \) be a Jordan algebra. Then \( \mathcal{A} \) is power associative, this is an algebra such that for any \( x \) in \( \mathcal{A} \) the algebra spanned by \( x \) and \( e \) is associative.

The rank of \( x \) in \( \mathcal{A} \) is the least natural number \( k \) such that \( \{e, x, \ldots, x^k\} \) is linearly dependent and we write \( \text{rank}(x) = k \). Since \( \text{rank}(x) \leq n \) the rank of \( \mathcal{A} \) is defined as being the natural number \( \text{rank}(\mathcal{A}) = \max(\text{rank}(x) : x \in \mathcal{A}) \). An element \( x \) in \( \mathcal{A} \) is regular if \( \text{rank}(x) = n \). Let \( x \) be a regular element of \( \mathcal{A} \) and \( r = \text{ran}(x) \).

Then, there exist real scalars \( a_1(x), a_2(x), \ldots, a_{r-1}(x) \) and \( a_r(x) \) such that:

\[
x' = a_1(x)x'^{-1} + \cdots + (-1)^r a_r(x)e = 0
\]

(1)

Where \( 0 \) is the zero vector of \( \mathcal{A} \). Taking in account (1) we conclude that the polynomial \( p \) such that

\[
p(x, \lambda) = \lambda^r - a_1(x)\lambda^{r-1} + \cdots + (-1)^r a_r(x)
\]

(2)

is the minimal polynomial of \( x \). When \( x \) is not regular the minimal polynomial of \( x \) has a degree less that \( r \). The roots of the minimal polynomial of \( x \) are the eigenvalues of \( x \).

A real Euclidean Jordan algebra \( \mathcal{A} \) is a Jordan algebra with an inner product \( \langle - , - \rangle \) such that \( \langle x \ast y, z \rangle = \langle y, x \ast z \rangle \) for all \( x, y \) and \( z \) in \( \mathcal{A} \). The real vector space of real symmetric matrices of order \( n, \text{Sym}(n, \mathbb{R}) \), is a real Euclidean Jordan algebra when \( \text{Sym}(n, \mathbb{R}) \) is equipped with the Jordan product \( \ast \) defined in the following way \( x \ast y = (xy + yx)/2 \) where \( xy \) and \( yx \) denotes the usual product of the matrices \( x \) and \( y \) respectively, and the inner product \( \langle - , - \rangle \) defined by \( \langle x, y \rangle = \text{tr}(x \ast y) \), where \( \text{tr} \) denotes the usual trace of matrices. The unit element of this Euclidean Jordan algebra is the identity matrix of order \( n \), \( I_n \).

Let \( \mathcal{A} \) be a real Euclidean Jordan algebra with unit element \( e \). An element \( c \) in \( \mathcal{A} \) is an idempotent if \( c^2 = c \).

Two idempotents \( c \) and \( d \) are orthogonal if \( c \ast d = 0 \). Let \( l \) be a natural number. The set \( \{f_1, f_2, \ldots, f_{l-1}, f_l\} \) is a complete system of orthogonal idempotents if the following three conditions hold: (i) \( f_i^2 = f_i \), for \( i = 1, \ldots, l \), (ii) \( f_i \ast f_j = 0 \) if \( i \neq j \), and (iii) \( \sum_{i=1}^{l} f_i = e \). An idempotent \( c \) of \( \mathcal{A} \) is primitive if it is a nonzero idempotent of \( \mathcal{A} \) and if it can not be written as a sum of two nonzero orthogonal idempotents. We say that \( \{f_1, f_2, \ldots, f_{l-1}, f_l\} \) is a Jordan frame if \( \{f_1, f_2, \ldots, f_{l-1}, f_l\} \) is a complete system of orthogonal idempotents such that each idempotent is primitive.
EXAMPLE 1. Let consider the Euclidean Jordan algebra \( A = \text{Sym}(3, \mathbb{R}) \) then \( S = \{ c, d \} \)
\[
\begin{bmatrix}
-1 & 0 & 0 \\
0 & -1 & 0 \\
0 & 0 & -1
\end{bmatrix}
\]
is a complete system of orthogonal idempotents of \( A \). But is not a Jordan frame since
\[
d = f + g
\]
where
\[
f = \begin{bmatrix}
0 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 0
\end{bmatrix}
\]
and
\[
g = \begin{bmatrix}
0 & 0 & 0 \\
0 & 0 & 1 \\
0 & 0 & 0
\end{bmatrix}
\]
is a Jordan frame of the Euclidean Jordan algebra \( A \). More generally, let
\[
S = \{ E_{11}, E_{22}, E_{32}, \ldots, E_{nn} \} \subset A = \text{Sym}(n, \mathbb{R})
\]
where \( E_{ii} \) is the elementary matrix with all entries null except the entry \( (E_{ii})_{ii} = 1 \). Then \( S \) is a Jordan frame of \( A \).

PROPOSITION 1. ([15], pp.43).

Let be a real Euclidean Jordan algebra. Then for \( x \) in \( V \) there exist unique real numbers \( \lambda_1, \lambda_2, \ldots, \lambda_{r-1}, \lambda_r \) all distinct, and a unique complete system of orthogonal idempotents \( \{ f_1, f_2, \ldots, f_{r-1}, f_r \} \) such that
\[
x = \lambda_1 f_1 + \lambda_2 f_2 + \cdots + \lambda_{r-1} f_{r-1} + \lambda_r f_r
\]
The numbers \( \lambda_j \)'s for \( j = 1, \ldots, l \) of (3) are the eigenvalues of \( x \) and the decomposition (3) is called the first spectral decomposition of \( x \).

EXAMPLE 2.

Let consider the Euclidean Jordan algebra \( A = \text{Sym}(n, \mathbb{R}) \) and \( B \in A \) a matrix with the distinct eigenvalues \( \lambda_1, \lambda_2, \ldots, \lambda_r, \lambda_{r+1} \) and \( \lambda_r \). Then the set \( S = \{ P_1, P_2, \ldots, P_{r}, P_{r+1} \} \) with
\[
P_i = \prod_{j \neq i} (B - \lambda_j I) / (\lambda_i - \lambda_j)
\]
is a complete system of orthogonal idempotents of \( A \) and the first spectral decomposition of \( B \) is
\[
B = \lambda_1 P_1 + \lambda_2 P_2 + \cdots + \lambda_r P_r.
\]

REMARK 3.

Herein, we must refer the paper of Alizadeh Schmieta, An introduction to Formally Real Jordan Algebras and Their Applications on Optimization in [14], where the author makes a clear presentation of Euclidean Jordan algebras.

PROPOSITION 2 ([15], pp.44)

Let be a real Euclidean Jordan algebra with \( \text{rank}(V) = r \). Then for each \( x \) in \( V \) there exists a Jordan frame \( \{ f_1, f_2, \ldots, f_r \} \) and real numbers \( \lambda_1, \ldots, \lambda_{r-1}, \lambda_r \) such that
\[
x = \lambda_1 f_1 + \lambda_2 f_2 + \cdots + \lambda_r f_r.
\]
The decomposition (4) is called the second spectral decomposition of \( x \).
EXAMPLE 3.

Let consider the Euclidean Jordan algebra \( \mathcal{A}_1 = \mathbb{R}^{n+1} \) equipped with the product \( \cdot \) defined such that
\[
x \cdot y = \begin{bmatrix} x^T y \\ x_1 y + y_1 x \end{bmatrix}
\]
where \( x = \begin{bmatrix} x_1 \\ \vdots \\ x_n \\ x_{n+1} \end{bmatrix} \) and \( y = \begin{bmatrix} y_1 \\ \vdots \\ y_n \\ y_{n+1} \end{bmatrix} \), and with the inner product \( \langle \cdot, \cdot \rangle \) defined in the usual way \( \langle x, y \rangle = x^T y \). We must say that \( \mathcal{A}_1 \) is an Euclidean Jordan algebra such that \( \text{rank}(\mathcal{A}_1) = 2 \) and such that the unit element of \( \mathcal{A}_1 \) is \( e = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \) where \( 0_n \) represents the null vector of \( \mathbb{R}^n \). Now, we will show that \( \{f_1, f_2\} = \left\{ \begin{bmatrix} 1 \\ 1/\|x\| \\ 1/\|x\| \\ \vdots \\ 1/\|x\| \end{bmatrix}, \begin{bmatrix} 1 \\ -x/\|x\| \\ -x/\|x\| \\ \vdots \\ -x/\|x\| \end{bmatrix} \right\} \) is a Jordan frame of \( \mathcal{A}_1 \).

Indeed, \( f_1 \) and \( f_2 \) are idempotent, since
\[
f_1 \cdot f_1 = \frac{1}{2} \begin{bmatrix} 1 \\ -x/\|x\| \end{bmatrix} \cdot \frac{1}{2} \begin{bmatrix} 1 \\ -x/\|x\| \end{bmatrix} = \begin{bmatrix} 1 + x x^T x \\ -1/\|x\| - 1/\|x\| \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 1 \\ x/\|x\|^2 \end{bmatrix} = f_1,
\]
\[
f_2 \cdot f_2 = \frac{1}{2} \begin{bmatrix} 1 \\ x/\|x\| \end{bmatrix} \cdot \frac{1}{2} \begin{bmatrix} 1 \\ x/\|x\| \end{bmatrix} = \begin{bmatrix} 1 + x x^T x \\ 1/\|x\| + 1/\|x\| \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 1 \\ x/\|x\|^2 \end{bmatrix} = f_2.
\]

Now, we can clearly say that the idempotents \( f_1 \) and \( f_2 \) are orthogonal relatively to the product \( \cdot \) of \( \mathcal{A}_1 \), this is by the definition of this operation we deduce that \( f_1 \cdot f_2 = 0_{n+1} \) where \( 0_{n+1} \) is the zero vector of \( \mathcal{A}_1 \).

We also have that:
\[
f_1 + f_2 = f_1 + f_2 = \frac{1}{2} \begin{bmatrix} 1 \\ x/\|x\| \end{bmatrix} + \frac{1}{2} \begin{bmatrix} 1 \\ x/\|x\| \end{bmatrix} = \begin{bmatrix} 1 \\ x/\|x\| \end{bmatrix} = e.
\]

Therefore \( S = \{f_1, f_2\} \) is a complete orthogonal system of idempotents of the Euclidean Jordan algebra \( \mathcal{A}_1 \).

Now since \( \text{rank}(\mathcal{A}_1) = 2 \) then \( S \) is a Jordan frame of \( \mathcal{A}_1 \). And, we have for \( x = \begin{bmatrix} x_1 \\ \bar{x} \end{bmatrix} \), with \( \bar{x} \neq 0_n \) and \( \bar{x} \in \mathbb{R}^n \), the second spectral decomposition of \( x \) is
\[
x = (x_1 - \|\bar{x}\|) f_1 + (x_1 + \|\bar{x}\|) f_2.
\]
EXAMPLE 4.

Let \( \mathcal{A} \) be the Euclidean Jordan algebra such \( \mathcal{A} = \text{Sym}(n, \mathbb{R}) \) equipped the Jordan product \( \bullet \) and with the inner product \( \langle x, y \rangle = \text{tr}(x \bullet y) \) and let \( B \) be a symmetric matrix of \( \mathcal{A} \), and \( S = \{c_1, c_2, \ldots, c_n\} \) be an orthonormal basis of \( \mathbb{R}^n \) of eigenvectors of \( B \) such that \( Bc_i = \lambda_i c_i, \forall i = 1, \ldots, n \). We suppose that we are using the column notation, this is we consider the notation \( c_i = \begin{bmatrix} c_{i1} \\ c_{i2} \\ \vdots \\ c_{in} \end{bmatrix} \), \( \forall i = 1, \ldots, n \).

Let consider \( f_i = c_i c_i^T \) for \( i = 1, \ldots, n \). Then \( \{f_1, f_2, \ldots, f_n\} \) is a Jordan frame of \( \mathcal{A} \). Indeed, let \( i \) be a natural number such that \( 1 \leq i \leq n \) then we have \( f_i^2 = f_i \bullet f_i = f_i = f_i c_i c_i^T = f_i \), \( \forall i = 1, \ldots, n \).

Let \( i \) and \( j \) be two natural numbers such that \( 1 \leq i, j \leq n \) and such that \( i \neq j \). Then we have:

\[
f_i \bullet f_j = \frac{f_i f_j + f_j f_i}{2} = c_i c_j + c_j c_i = \frac{1}{2} \left( c_i^T c_j + c_j^T c_i \right) = O_n + O_n = O_n,\]

where \( O_n \) is the null matrix of order \( n \). So we have proved that the idempotents \( c_i \)'s are orthogonal between each other. Finally, since \( \{c_1, c_2, \ldots, c_n\} \) is an orthonormal basis of \( \mathbb{R}^n \) then we have \( c_1 c_1^T + c_2 c_2^T + \ldots + c_n c_n^T = I_n \), this is we have \( f_1 + f_2 + \ldots + f_n = I_n \). So, we conclude that \( \{f_1, f_2, \ldots, f_n\} \) is a Jordan frame of \( \mathcal{A} \).

Now, let obtain the spectral decomposition of the matrix \( B \). Now since:

\[
B = B|_n = B(f_1 + f_2 + \ldots + f_n) = B(c_1 c_1^T + c_2 c_2^T + \ldots + c_n c_n^T) = (Bc_1) c_1^T + (Bc_2) c_2^T + \ldots + (Bc_n) c_n^T = \lambda_1 c_1 c_1^T + \lambda_2 c_2 c_2^T + \ldots + \lambda_n c_n c_n^T
\]

Then we have:

\[
B = B|_n = B(f_1 + f_2 + \ldots + f_n) = B(c_1 c_1^T + c_2 c_2^T + \ldots + c_n c_n^T) = (Bc_1) c_1^T + (Bc_2) c_2^T + \ldots + (Bc_n) c_n^T = \lambda_1 c_1 c_1^T + \lambda_2 c_2 c_2^T + \ldots + \lambda_n c_n c_n^T
\]

Therefore the second spectral decomposition of the matrix \( B \) is:

\[
B = \lambda_1 f_1 + \lambda_2 f_2 + \ldots + \lambda_n f_n.
\]

RESULTS ON STRONGLY REGULAR GRAPHS

A graph \( G \) non-null and non-complete, is a strongly regular graph if there are integers \( k, \lambda, \mu \) such that \( G \) is \( k \)-regular \((k \geq 1)\) and of order \( n \geq 3 \) and any two adjacent vertices of \( G \) have exactly \( \lambda \) common neighbors and any two non-adjacent vertices have \( \mu \) common neighbors.

From now on we will say that a graph \( G \) is a \((n, k; \lambda, \mu)\)-strongly regular graph if \( G \) is a strongly regular graph of order \( n \) and have the parameters \( k, \lambda, \mu \).
Let \( G \) be a \((n,k;\lambda,\mu)\)-strongly regular graph and \( A \) its adjacency matrix. Then
\[
A^2 = kI_n + \lambda A + \mu(J_n - A - I_n),
\]
where \( J_n \) is the all one matrix of order \( n \).

It is well known (see, for instance, [18]) that the eigenvalues of \( G \) are \( k, \theta \) and \( \tau \), where \( \theta \) and \( \tau \) are given by
\[
\Theta = (\theta - \mu + \sqrt{(\lambda - \mu)^2 + 4(k - \mu)}) / 2 \quad \text{and} \quad \Theta = (\theta - \mu - \sqrt{(\lambda - \mu)^2 + 4(k - \mu)}) / 2.
\]

If \( G \) is a \((n,k;\lambda,\mu)\)-strongly regular graph then \( k(k - \lambda - 1) = \mu(n - k - 1) \) and the eigenvalues \( \theta \) and \( \tau \) of \( G \) satisfy, the Krein admissibility conditions established by Jr. L. L. Scott in [19], the inequalities
\[
(k + \theta + 2\Theta \tau)(\Theta + 1) \leq (k + \theta)(\Theta + 1)^2 \quad \text{and} \quad (k + \tau + 2\Theta \tau)(\Theta + 1) \leq (k + \tau)(\Theta + 1)^2.
\]

Now, we present the admissibility conditions introduced by Delsarte, Goethals and Seidel [20], which establish if \( G \) is a \((n,k;\lambda,\mu)\)- strongly regular graph then \( n \leq (f_\theta(f_\theta + 3)) / 2 \) and \( n \leq (f_\tau(f_\tau + 3)) / 2 \) where \( f_\theta \) and \( f_\tau \) are the multiplicities of the eigenvalues \( \theta \) and \( \tau \) of \( G \).

A \((n,k;\lambda,\mu)\)-strongly regular graph \( G \) is primitive if and only if \( G \) is connect and its complement \( \overline{G} \) is also connected. A strongly regular graph that is not primitive is called imprimitive.

Finally, we must present another property of a strongly regular graph that is: a \((n,k;\lambda,\mu)\)-strongly regular graph is a primitive strongly regular graph if and only if \( \mu = 0 \) or \( \mu = k \).

\section*{INEQUALITIES ON THE PARAMETERS OF A STRONGLY REGULAR GRAPH}

Herein, we present some admissibility conditions on the spectra and on the parameters of a primitive strongly regular graph but obtained on na asymptotic algebraic way.

Let \( G \) be a \((n,k;\lambda,\mu)\) strongly regular graph with \( 0 < \mu < k < n - 1 \) and \( \lambda > \mu \) and let \( A \) be its adjacency matrix with the distinct eigenvalues, namely \( k, \theta \) and \( \tau \), and let \( \mathcal{A} \) be the 3 dimensional real Euclidean subalgebra, with \( \text{rank}(\mathcal{A}) = 3 \), of the Euclidean Jordan algebra \( \mathcal{V} = \text{Sym}(n, \mathbb{R}) \) spanned by \( I_n \) and the natural powers of \( A \).

Let \( S = \{E_1,E_2,E_3\} \) be the complete system of orthogonal idempotents of \( \mathcal{A} \) associated to \( A \), where
\[
E_1 = 1/nI_n + 1/nA + 1/n(J_n - A - I_n), \quad E_2 = (\theta n + \tau - k) / (n(\theta - \tau))I_n + (n + \tau - k) / (n(\theta - \tau))A + (\tau - k) / (n(\theta - \tau))(J_n - A - I_n) \quad \text{and} \quad E_3 = (\theta n + k - \theta) / (n(\theta - \tau))I_n + (n + k - \theta) / (n(\theta - \tau))A + (k - \tau) / (n(\theta - \tau))(J_n - A - I_n).
\]

Let \( ij \) be natural numbers such that \( 1 \leq i,j \leq 3 \) and \( i \neq j \). So, since the idempotents \( E_i \) and \( E_j \) are orthogonal relatively to the Jordan product of matrices, then they are orthogonal relatively to the inner product \( <X,Y> = \text{tr}(X^\ast Y) \), \( \forall X,Y \in \mathcal{A} \). Therefore, we conclude that \( S = \{E_1, E_2, E_3\} \) is a basis of \( \mathcal{A} \).

Now, we consider some notation for defining the Hadamard product of two matrices. We denote the space of real square matrices of order \( n \) by \( M_n(\mathbb{R}) \) and we consider the Hadamard product and the Kronecker product of two matrices of order \( n \) \( E \) and \( F \) of \( M_n(\mathbb{R}) \) in the following way: if \( E = [e_{ij}] \) and \( F = [f_{ij}] \), then we define \( E \circ F = [e_{ij}f_{ij}] \) for all \( i,j \in [1,\ldots,n] \). For any natural number \( l \) and for any matrix \( H \in M_n(\mathbb{R}) \) we define \( H^{\Theta} = H^{\Theta}_{n} \) and \( H^{\Theta} = H^{\Theta}_{n} \), \( \forall \Theta \in \mathcal{N} \) where \( H^{\Theta}_{n} = J_n, H^{\Theta} = H \), see [21].
Let’s suppose that $\lambda > \mu$. Now, we will analyze the spectra of an Hadamard series associated to the matrix $X = \frac{A^2 - |\tau|^2 I_n}{k^2 + k}$ where $A$ is the adjacency matrix of $G$. From now on, we will denote the matrix $X$ by $x$. So, let’s consider the Hadamard series $S_x = \sum_{j=1}^{+\infty} \left( \frac{A^2 - |\tau|^2 I_n}{k^2 + k} \right)^j$.

Now, we consider the notation: $S_x = \sum_{i=1}^{3} q_{i} x_i$ is the spectral decomposition of $S_x$ respectively to the Jordan frame $S = \{E_1, E_2, E_3\}$ of $A$.

In the following text, we will explain that the eigenvalues $q_{i} x$ s of $S_x$ are positive.

Let consider the following notation: $S_{nx} = \sum_{j=1}^{n} \left( (A^2 - |\tau|^2 I_n)/(k^2 + k) \right)^j$ and $S_{nx} = q_{1nx} E_1 + q_{2nx} E_2 + q_{3nx} E_3$.

Since for any two real matrices $E$ and $F$ of order $n$ we have $\lambda_{\min}(E)\lambda_{\min}(F) \leq \lambda_{\min}(E \circ F)$ and $S$ is a Jordan frame of the Euclidean Jordan algebra $A$ that is a basis of $A$ and this Euclidean Jordan algebra is closed for the Hadamard product then we conclude that the eigenvalues of $S_{nx}$ are all positive. Now, we must note that $q_{1x} = \lim_{n \to +\infty} q_{1nx}$, $q_{2x} = \lim_{n \to +\infty} q_{2nx}$, $q_{3x} = \lim_{n \to +\infty} q_{3nx}$ then we have $q_{1x} \geq 0$, $q_{2x} \geq 0$ and $q_{3x} \geq 0$. We have $S_x E_1 = q_{1x} E_1, S_x E_2 = q_{2x} E_2$ and $S_x E_3 = q_{3x} E_3$. Therefore we have the following expressions for $q_{i}$: $q_{1x} = -\ln\left(1 - \frac{1}{(k^2 + k)}\right) - \ln\left(1 - \frac{1}{(k^2 + k)}\right)$, $q_{2x} = -\ln(1 - \frac{1}{(k^2 + k)})\tau - \ln(1 - \frac{1}{(k^2 + k)})\tau$, $q_{3x} = -\ln\left(1 - \frac{1}{(k^2 + k)}\right)\tau - \ln(1 - \frac{1}{(k^2 + k)})\tau$.

Now let consider the element $E_3 \circ S_x$ of $A$, that has positive eigenvalues since $E_2$ is an idempotent matrix and the eigenvalues $q_{i}$ s of $S_x$ are all positive.

Now let consider the notation $E_3 \circ S_x = q_{1x} E_1 + q_{2x} E_2 + q_{3x} E_3$.

We have $q_{1x} = -\ln(1 - \frac{1}{(k^2 + k)})\tau - \ln(1 - \frac{1}{(k^2 + k)})\tau$, $q_{2x} = -\ln(1 - \frac{1}{(k^2 + k)})\tau - \ln(1 - \frac{1}{(k^2 + k)})\tau$, $q_{3x} = -\ln(1 - \frac{1}{(k^2 + k)})\tau - \ln(1 - \frac{1}{(k^2 + k)})\tau$.

By an asymptotical analysis of the eigenvalues $q_{1x}$ we establish the inequality (5) and of Proposition 3.
PROPOSITION 3.
Let \( G \) be a \((n, k; \lambda, \mu)\)-strongly regular graph with \( 0 < \mu < k < n-1 \) and with the distinct eigenvalues \( k, \theta \) and \( \tau \). If \( \lambda > \mu \) and \( k < n/2 \) then
\[
\left( \frac{k^2 + k - \mu}{k^2 + \tau^2} \right)^{2\theta + 1} \geq \left( \frac{k^2 + k - \mu}{k^2 + k - \lambda} \right)^k.
\] (5)

PROOF
Since \( q_{1X}^j \geq 0 \) then we obtain (6)
\[
-\frac{\theta n + k - \theta}{n(\theta - \tau)} \ln \left( 1 - \frac{k - |\tau|^2}{k^2 + k} \right) - \frac{n + k - \theta}{n(\theta - \tau)} \ln \left( 1 - \frac{\lambda}{k^2 + k} \right) - \frac{k - \theta}{n(\theta - \tau)} \ln \left( 1 - \frac{\mu}{k^2 + k} \right) (n - k - 1) \geq 0.
\] (6)

Now, since we have \((\theta n + k - \theta)/(n(\theta - \tau)) + (-n + k - \theta)/(n(\theta - \tau))k + (k - \theta)/(n(\theta - \tau))(n - k - 1) = 0\) then from (6) we obtain (7).
\[
-\frac{\theta n + k - \theta}{n(\theta - \tau)} \ln \left( 1 - \frac{k - |\tau|^2}{k^2 + k} \right) - \frac{n + k - \theta}{n(\theta - \tau)} \ln \left( 1 - \frac{\lambda}{k^2 + k} \right) - \frac{\theta n + k - \theta}{n(\theta - \tau)} \ln \left( 1 - \frac{\mu}{k^2 + k} \right) \geq 0.
\] (7)

Then after rewriting (7) we deduce (8).
\[
-\frac{\theta n + k - \theta}{n(\theta - \tau)} \ln \left( 1 - \frac{k - |\tau|^2}{k^2 + k} \right) - \frac{n + k - \theta}{n(\theta - \tau)} \ln \left( 1 - \frac{\mu}{k^2 + k} \right) \geq 0.
\] (8)

Now, multiplying both hand sides of (8) by \((n(\theta - \tau))/(\theta n + k - \theta)\) we obtain the inequality (9).
\[
\ln \left( \frac{k^2 + k - \mu}{k^2 + \tau^2} \right) + \frac{n + k - \theta}{\theta n + k - \theta} \ln \left( \frac{k^2 + k - \mu}{k^2 + k - \lambda} \right) \geq 0.
\] (9)

Now, after a rewriting (9) we obtain the inequality (10).
\[
\ln \left( \frac{k^2 + k - \mu}{k^2 + \tau^2} \right) \geq \frac{n + k + \theta}{\theta n + k - \theta} \ln \left( \frac{k^2 + k - \mu}{k^2 + k - \lambda} \right).
\] (10)

Now, since \( k < n/2 \) then we have the inequality (10).
\[ \frac{n-k+\theta}{\theta n+k-\theta} \geq 1 \quad 2^{\theta n+1}. \] (11)

Therefore by (10) and (11) we conclude (12).

\[ \ln \left( \frac{k^2+k-\mu}{k^2+|\tau|^2} \right) \geq \frac{1}{2^{\theta n+1}} \ln \left( \frac{k^2+k-\mu}{k^2+k-\lambda} \right). \] (12)

Finally, from (12) we obtain (13).

\[ \left( \frac{k^2+k-\mu}{k^2+|\tau|^2} \right)^{2\theta+1} \geq \left( \frac{k^2+k-\mu}{k^2+k-\lambda} \right)^k. \] (13)

Let suppose that \( \mu > \lambda \) herein we will analyze the spectra of an Hadamard series associated to the matrix \( X = (A^2 - \theta^2 I_n)/(k^2 + k) \) where \( A \) is the adjacency matrix of \( G \). From now on, we will denote the matrix \( X \) by \( x \). Let’s consider the Hadamard series \( S_x = \sum_{j=0}^{+\infty} ((A^2 - \theta^2 I_n)/(k^2 + k))^j \).

We have \( S_x = \sum_{i=1}^{3} q_i x E_i \) is the spectral decompositions of \( S_x \) respectively to the Jordan frame \( B = \{E_1, E_2, E_3\} \) of \( A \).

Now, we will show that the eigenvalues \( q_i x \) of \( S_x \) are positive. Let consider the notation: \( S_{nx} = \sum_{j=0}^{n} ((A^2 - \theta^2 I_n)/(k^2 + k))^j \) and the second spectral decomposition of \( S_{nx} \cdot S_{nx} = q_{nx} x E_1^x + q_{nx} E_2^x + q_{nx} E_3^x \).

Now since \( A^2 = k I_n + \lambda A + \mu (J_n - A - I_n) \) then we have:

\[ S_{nx} = \sum_{j=1}^{n} \left\{ \left( \frac{k-\theta^2}{k^2+k} \right)^j I_n + \sum_{j=0}^{n} \left( \frac{\lambda}{k^2+k} \right)^j A + \sum_{j=0}^{n} \left( \frac{\mu}{k^2+k} \right)^j (J_n - A - I_n) \right\} \]

Hence, we have:

\[ S_n = \sum_{j=1}^{n} \left\{ \left( \frac{k-\theta^2}{k^2+k} \right)^j I_n + \sum_{j=0}^{n} \left( \frac{\lambda}{k^2+k} \right)^j A + \sum_{j=0}^{n} \left( \frac{\mu}{k^2+k} \right)^j (J_n - A - I_n) \right\} \]

The series \( \sum_{j=1}^{+\infty} ((A^2 - \theta^2 I_n)/(k^2 + k))^j \) is convergent and the sum of this series, that we denote by \( S_x \), is such that:
\[ S_x = -\ln \left( 1 - \frac{k^2}{k^2 + k} \right) I_n - \ln \left( 1 - \frac{\lambda}{k^2 + k} \right) A - \ln \left( 1 - \frac{\mu}{k^2 + k} \right) (J_n - A - I_n). \]

Since \( S_{nx} = \sum_{j=1}^{n} \left( (A^2 - \theta^2 I_n) / (k^2 + k) \right) \) then \( S_{nx} \) has positive eigenvalues.

Since, for any two real matrices \( E \) and \( F \) of order \( n \) we have \( \lambda_{\min}(E) \lambda_{\min}(F) \leq \lambda_{\min}(E \circ F) \), and we must also observe that since \( B \) is a Jordan frame of the Euclidean Jordan algebra \( A \) that is a basis of \( A \) and this Euclidean Jordan algebra is closed for the Hadamard product then we conclude that the eigenvalues of \( S_{nx} \) are all positive. And therefore, since \( q_{1x} = \lim_{n \to +\infty} q_{11x}, q_{2x} = \lim_{n \to +\infty} q_{12x} \) and \( q_{3x} = \lim_{n \to +\infty} q_{13x} \), then we have \( q_{1x} \geq 0, q_{2x} \geq 0 \) and \( q_{3x} \geq 0 \).

We have \( S_x E_1 = q_{1x} E_1, S_x E_2 = q_{2x} E_2 \) and \( S_x E_3 = q_{3x} E_3 \), and therefore we conclude that:

\[
q_{1x} = -\ln \left( 1 - \frac{k^2}{k^2 + k} \right) - \ln \left( 1 - \frac{\lambda}{k^2 + k} \right) - \ln \left( 1 - \frac{\mu}{k^2 + k} \right) (n-k-1), \tag{14}
\]

\[
q_{2x} = -\ln \left( 1 - \frac{k^2}{k^2 + k} \right) - \ln \left( 1 - \frac{\lambda}{k^2 + k} \right) \theta - \ln \left( 1 - \frac{\mu}{k^2 + k} \right) (-\theta-1), \tag{15}
\]

\[
q_{3x} = -\ln \left( 1 - \frac{k^2}{k^2 + k} \right) - \ln \left( 1 - \frac{\lambda}{k^2 + k} \right) \tau - \ln \left( 1 - \frac{\mu}{k^2 + k} \right) (-\tau-1). \tag{16}
\]

Let consider \( E_2 \circ S_x \) of \( A \). Then \( E_2 \circ S_x \) has positive eigenvalues since \( E_3 \) is an idempotent matrix and the eigenvalues \( q_{1x} \)s of \( S_x \) are all positive. We have the following notation: \( E_2 \circ S_x = q_{1x}^2 E_1 + q_{2x}^2 E_2 + q_{3x}^2 E_3 \). The expressions of the \( q_{1x}^2 \)s are presented on the equalities (17),(18) and (19).

\[
q_{1x}^2 = -\frac{\tau}{n(\theta - \tau)} n + \tau - k \ln \left( 1 - \frac{k^2}{k^2 + k} \right)^n + \tau - k \ln \left( 1 - \frac{\lambda}{k^2 + k} \right)^{n-k-1} \ln \left( 1 - \frac{\mu}{k^2 + k} \right) \tag{17}
\]

\[
q_{2x}^2 = -\frac{\tau}{n(\theta - \tau)} n + \tau - k \ln \left( 1 - \frac{k^2}{k^2 + k} \right)^n + \tau - k \ln \left( 1 - \frac{\lambda}{k^2 + k} \right)^{n-k-1} \ln \left( 1 - \frac{\mu}{k^2 + k} \right) \tag{18}
\]

\[
q_{3x}^2 = -\frac{\tau}{n(\theta - \tau)} n + \tau - k \ln \left( 1 - \frac{k^2}{k^2 + k} \right)^n + \tau - k \ln \left( 1 - \frac{\lambda}{k^2 + k} \right)^{n-k-1} \ln \left( 1 - \frac{\mu}{k^2 + k} \right) \tag{19}
\]

Doing a similar algebraic analysis of the parameter \( q_{1x}^2 \) has we have done for proving the inequality (5) of Proposition 3 we establish the inequality (20) of Proposition 4, for that reason we don’t present the proof of Proposition 4.
PROPOSITION 4

Let $G$ be a $(n,k;\lambda,\mu)$-strongly regular graph with $0 < \mu < k < n-1$ and with the distinct eigenvalues $k, \theta$ and $\tau$. If $\mu > \lambda$ and $k > \frac{n}{2}$ then

$$\left( \frac{k^2 + k - \lambda}{k^2 + \theta^2} \right)^{\frac{2|\tau|}{2}} \geq \left( \frac{k^2 + k - \lambda}{k^2 + k - \mu} \right)^{n-k-1}.$$ (20)

CONCLUSIONS

In this paper we had established new inequalities on the parameters and on the spectra of a strongly regular graph very different from the well known conditions on the parameters of a strongly regular, like for instance the Krein conditions or the absolute bounds admissibility conditions. Further investigations of the other parameters of the Logarithmic Hadamard series analyzed in this paper will be studied in future research.

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