Positive Solutions to Periodic Boundary Value Problems of Nonlinear Fractional Differential Equations at Resonance

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By Leggett-Williams norm-type theorem for coincidences due to O’Regan and Zima, we discuss the existence of positive solutions to fractional order with periodic boundary conditions at resonance. At last, an example is presented to demonstrate the main results.

1. Introduction

Fractional differential equations are generalizations of ordinary differential equations to an arbitrary order. It has played a significant role in many fields, such as viscoelasticity, engineering, physics, and economics; see [1–5]. During the last ten years, there have been a large number of papers dealing with the existence of solutions boundary value problem for fractional differential equations; see [6–10].

Recently, there is an increasing tendency on discussion for the existence of positive solutions to boundary value problems of fractional differential equations which enriched many previous results.

In [9], Yang and Wang considered the existence of solutions for the following two-point boundary value problems for fractional differential equations:

\[ D_0^\alpha u(t) = f(t, u(t), u'(t)), \quad 0 < t < 1, \]
\[ u(0) = 0, \]
\[ u'(0) = u'(1), \quad 1 < \alpha < 2, \]

where \( 0 < t < 1 \), \( D_0^\alpha \) denoting the Caputo fractional derivative. By using the Leggett-Williams norm-type theorem for coincidences due to O’Regan and Zima, the authors obtained the existence of positive solutions to the above problem.

In [10], Hu discussed the existence of positive solutions for a boundary value problem of fractional differential inclusions with resonant boundary conditions:

\[ D_0^\alpha u(t) \in f(t, u(t)), \quad 0 \leq t \leq 1, \]
\[ u^{(i)}(0) = 0, \quad i = 1, 2, \ldots, n - 1, \]
\[ u(0) = u(1), \quad n - 1 < \alpha < n, \quad n \geq 2, \]

where \( n - 1 < \alpha < n, n \geq 2, D_0^\alpha \) denotes the Caputo fractional derivative, \( f : [0, 1] \times \mathbb{R} \to \mathcal{F} (\mathbb{R}) \), and \( \mathcal{F} (\mathbb{R}) \) donates the family of nonempty compact and convex subsets of \( \mathbb{R} \). By using the Leggett-Williams theorem for coincidences of multivalued operators due to O’Regan and Zima, results on the existence of positive solutions were established.

Periodic boundary value problems have profound practical background and wide range of applications, such as mechanics, biology, and engineering; see [11–14]. Recently, periodic boundary value conditions of fractional order have been studied by some authors, such as [15–17].

In [16], Chen et al. studied the following periodic boundary value problem for fractional \( p \)-Laplacian equation:

\[ D_0^\beta \phi_p (D_0^\alpha x(t)) = f(t, x(t), D_0^\alpha x(t)), \quad t \in [0, T], \]
\[ x(0) = x(T), \]
\[ D_0^\alpha x(0) = D_0^\alpha x(T), \]

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where \(0 < \alpha, \beta \leq 1, D_0^\alpha\) is a Caputo fractional derivative, and \(f : [0, T] \times \mathbb{R}^2 \rightarrow \mathbb{R}\) is continuous.

In [17], Hu et al. considered the existence of solutions for the following periodic boundary value problem for fractional differential equation:

\[
D_0^\alpha u(t) = f(t, u(t), u'(t)), \quad 0 < t < 1, \\
u(0) = u(1), \\
u'(0) = u'(1),
\]

(4)

where \(1 < \alpha < 2, D_0^\alpha\) denotes the Caputo fractional derivative, and \(f : [0, 1] \times \mathbb{R} \rightarrow \mathbb{R}\) is continuous. By using the coincidence degree theory, the authors obtained the existence of solutions.

From the above works, we notice that the study of positive solutions to periodic boundary value problems of fractional order at resonance is poor. Now, the question is as follows: though the existence of solutions to (4) is obtained, how can we get the positive solutions of it? The aim of this paper is to fill the gap in the relevant literature. Our main tool is the recent Leggett-Williams norm-type theorem for coincidences due to O’Regan and Zima [18].

The rest of this paper is organized as follows. Section 2, we obtain the existence of positive solutions of (4) by Theorem 9. Finally, an example is given to illustrate our results in Section 4.

2. Preliminaries

First of all, we present the necessary definitions and lemmas from fractional calculus theory. For more details, see [1].

**Definition 1** (see [1]). The Riemann-Liouville fractional integral of order \(\alpha > 0\) of a function \(f : (0, \infty) \rightarrow \mathbb{R}\) is given by

\[
I_0^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s) \, ds,
\]

(5)

provided that the right-hand side is pointwise defined on \((0, \infty)\).

**Definition 2** (see [1]). The Caputo fractional derivative of order \(\alpha > 0\) of a continuous function \(f : (0, \infty) \rightarrow \mathbb{R}\) is given by

\[
D_0^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-s)^{n-\alpha-1} f^{(n)}(s) \, ds,
\]

(6)

where \(n-1 < \alpha \leq n\), provided that the right-hand side is pointwise defined on \((0, \infty)\).

**Lemma 3** (see [1]). The fractional differential equation

\[
D_0^\alpha y(t) = 0
\]

has solution \(y(t) = c_0 + c_1 t + \cdots + c_{n-1} t^{n-1}\), \(c_i \in \mathbb{R}\), \(i = 0, 1, \ldots, n-1\), and \(n = [\alpha] + 1\).

Furthermore, for \(y \in AC^n[0, 1]\),

\[
(I_0^\alpha D_0^\alpha y)(t) = y(t) - \sum_{k=0}^{n-1} \frac{t^k}{k!} y^{(k)}(0),
\]

(8)

\[
(D_0^\alpha I_0^\alpha y)(t) = y(t).
\]

**Lemma 4** (see [1]). The relation

\[
I_0^\alpha I_0^\beta f(x) = I_0^{\alpha+\beta} f(x)
\]

(9)

is valid in the following case: \(\beta > 0, \alpha + \beta > 0\), and \(f(x) \in L_1(a, b)\).

In the following, let us recall some definitions on Fredholm operators and cones in Banach space (see [19]).

Let \(X, Y\) be real Banach spaces. Consider a linear mapping \(L : dom L \subset X \rightarrow Y\) and a nonlinear operator \(N : X \rightarrow Y\). Assume that

\((A1)\ L\ is\ a\ Fredholm\ operator\ of\ index\ zero;\ that\ is,\ Im\ L\ is\ closed\ and\ \dim ker L = \text{codim Im L} < \infty.\)

This assumption implies that there exist continuous projections \(P : X \rightarrow X\) and \(Q : Y \rightarrow Y\) such that\(Im P = ker L\) and \(ker Q = \text{Im L}\). Moreover, since \(\dim \text{Im Q} = \text{codim Im L}\), there exists an isomorphism \(J : \text{Im Q} \rightarrow \ker L\). Denote by \(L_P\) the restriction of \(L\) to \(ker P \cap dom L\). Clearly, \(L_P\) is an isomorphism from \(ker P \cap dom L\) to \(\text{Im L}\); we denote its inverse by \(K_P : \text{Im L} \rightarrow ker P \cap dom L\). It is known that the coincidence equation \(Lx = Nx\) is equivalent to

\[
x = (P + JQN)x + K_P(I - Q)Nx.
\]

(10)

Let \(C\) be a cone in \(X\) such that

\((i)\ \mu x \in C\ for\ all\ x \in C\ and\ \mu \geq 0,\)

\((ii)\ x, -x \in C\ implies\ x = \theta.\)

It is well known that \(C\) induces a partial order in \(X\) by

\[
x \preceq y\ \text{iff}\ y - x \in C.
\]

(11)

The following property is valid for every cone in a Banach space \(X\).

**Lemma 5** (see [18]). Let \(C\ be\ a\ cone\ in\ X.\ Then\ for\ every\ u \in C \setminus \{0\}\ there\ exists\ a\ positive\ number\ \sigma(u)\ such\ that\)

\[
\|x + u\| \geq \sigma(u) \|u\| \quad \forall x \in C.
\]

(12)

Let \(\gamma : X \rightarrow C\ be\ a\ retraction,\ that\ is,\ a\ continuous\ mapping\ such\ that\ \gamma(x) = x\ for\ all\ x \in C.\ Set\)

\[
\Psi = P + JQN + K_P(I - Q)N,
\]

\[
\Psi = \Psi \ast \gamma.
\]

(13)

We use the following result due to O’Regan and Zima.
Theorem 6 (see [18]). Let $C$ be a cone in $X$ and let $\Omega_1, \Omega_2$ be open bounded subsets of $X$ with $\overline{\Omega}_1 \subset \Omega_2$ and $C \cap (\overline{\Omega}_2 \setminus \Omega_1) \neq \emptyset$. Assume (A1) and the following assumptions hold:

(A2) $QN : X \to Y$ is continuous and bounded and $K_P(I - Q)N : X \to X$ is compact on every bounded subset of $X$.

(A3) $\lambda N x \neq x$ for all $x \in C \cap \partial \Omega_2 \cap \text{Im} L$ and $\lambda \in (0, 1)$.

(A4) $\gamma$ maps subsets of $\overline{\Omega}_2$ into bounded subsets of $C$.

(A5) $\deg \{ (I - (P + J Q) \gamma) | \ker L, \ker L \cap \Omega_2, 0 \} \neq 0$.

(A6) $\exists u_0 \in C \setminus \{ 0 \}$ such that $\| x \| \leq \sigma(u_0) \| \Psi x \|$ for $x \in C(u_0) \cap \partial \Omega_2$ and $\sigma(u_0) > 0$.

(A7) $(P + J Q) \gamma(\partial \Omega_2) \subset C$.

(A8) $\Psi \gamma(\overline{\Omega}_2 \setminus \Omega_1) \subset C$.

Then the equation $L x = N x$ has a solution in the set $C \cap (\overline{\Omega}_2 \setminus \Omega_1)$.

3. Main Results

In this section, we give the existence theorems for problem (4). We write Banach space $X = Y = C[0, 1]$ with the norm $\| x \| = \max_{t \in [0, 1]} |x(t)|$.

Define the operator $L : \text{dom} L \to X$ by

$$Lu = D_0^\alpha u,$$

where

$$\text{dom} L = \{ x \in X : D_0^\alpha u(t) \in Y, u(0) = u(1), u'(0) = u'(1) \}.$$  

Define the operator

$$N : X \to Y$$

by

$$Nu(t) = f(t, u(t), u'(t)).$$

Then problem (4) can be written by $L u = N u, u \in \text{dom} L$.

Let $\kappa$ be a constant, which is in $(0, 1)$ and satisfies

$$\kappa G(t, s) < 1.$$  

Lemma 7. The mapping $L : \text{dom} L \subset Z$ is a Fredholm operator of index zero. Furthermore, the operator $K_P : \text{Im} L \to \text{dom} L \cap \ker P$ can be written as

$$K_P y(t) = \int_0^1 k(t, s) y(s) ds, \quad t \in [0, 1],$$

where

$$k(t, s) = \begin{cases} 
\frac{(t - s)^{a-1}}{\Gamma(\alpha)} - \frac{\Gamma(\alpha)}{\Gamma(2\alpha - 1)} (1 - s)^{2a - 2} + \frac{1}{\alpha \Gamma(\alpha)} (1 - s)^{a-1} - \frac{1}{\Gamma(\alpha)} (1 - s)^{a-1}, & 0 \leq s \leq t \leq 1, \\
- \frac{\Gamma(\alpha)}{\Gamma(2\alpha - 1)} (1 - s)^{2a - 2} + \frac{1}{\alpha \Gamma(\alpha)} (1 - s)^{a-1} - \frac{1}{\Gamma(\alpha)} (1 - s)^{a-1}, & 0 \leq t < s \leq 1.
\end{cases}$$

Proof. By Lemma 3, $D_0^\alpha u(t) = 0$ has solution

$$u(t) = c_0 + c_1 t, \quad c_0, c_1 \in \mathbb{R}.$$  

According to the boundary value conditions of (4), we have

$$\ker L = \{ c, c \in \mathbb{R} \} \equiv \mathbb{R}^1.$$
Let \( u(t) = I_t^a y(t) - I_t^a y(t)|_{t=1} \cdot t \). We can easily prove \( u(t) \in \text{dom} \, L \). Thus, we conclude that

\[
\text{Im} \, L = \left\{ y \in Y : \int_0^1 (1 - s)^{\alpha-2} y(s) \, ds = 0 \right\}. \tag{25}
\]

Consider the linear operator \( P : X \to Y \) defined by

\[
P x(t) = (\alpha - 1) \int_0^1 (1 - s)^{\alpha-2} x(s) \, ds \quad t \in [0, 1]. \tag{26}
\]

Define the operator \( Q : Y \to Y \) by

\[
Q y(t) = (\alpha - 1) \int_0^1 (1 - s)^{\alpha-2} y(s) \, ds, \quad t \in [0, 1]. \tag{27}
\]

For \( u(t) \in X \), we get

\[
P (Pu) = P \left( (\alpha - 1) \int_0^1 (1 - s)^{\alpha-2} u(s) \, ds \right)
= (\alpha - 1) \int_0^1 (1 - s)^{\alpha-2} u(s) \, ds = Pu.
\]

So we have \( P^2 = P \). In the same way, \( Q^2 = Q \). We notice that \( \text{Im} \, P = \ker \, L \) and \( \ker \, Q = \text{Im} \, L \). It follows from \( \text{Ind} \, L = \dim \, \ker \, L - \text{codim} \, \text{Im} \, L = 0 \) that \( L \) is a Fredholm mapping of index zero.

Next, we will prove that the operator \( K_p \) is the inverse of \( L |_{\text{dom} \, L \cap \ker \, P} \).

In fact, for \( u(t) \in \text{dom} \, L \cap \ker \, P \), we have \( D_0^\alpha u(t) = y(t) \).

By Lemma 3, we have \( u(t) = I_t^a y(t) + c_0 + c_1t \). According to \( u(0) = u(1) \), we get

\[
c_1 = -I_0^a y(t) \quad \text{or} \quad \alpha = -I_0^a y(t) + c_1t. \tag{29}
\]

By \( u(t) \in \ker \, P \), that is, \((\alpha - 1) \int_0^1 (1 - s)^{\alpha-2} u(s) \, ds = 0 \), we have

\[
c_0 = -I_0^a y(t) - \frac{c_1}{\alpha}. \tag{30}
\]

Then, we have

\[
K_p y(t) = u(t) = I_t^a y(t) + c_0 + c_1t
= I_t^a y(t) - (\alpha) I_t^a y(t) + \frac{1}{\alpha} I_t^a y(t)
= \frac{1}{\Gamma(\alpha)} \int_0^t (t - s)^{\alpha-1} y(s) \, ds
- \frac{\Gamma(\alpha)}{2 \alpha - 1} \left[ \int_0^1 (1 - s)^{2\alpha-2} y(s) \, ds \right]
+ \frac{1}{\alpha \Gamma(\alpha)} \int_0^1 (1 - s)^{\alpha-1} y(s) \, ds
+ \frac{1}{\Gamma(\alpha)} \int_0^1 (1 - s)^{\alpha-1} ty(s) \, ds
= \int_0^t k(t, s) y(s) \, ds. \tag{31}
\]

It is easy to see that \( L K_p y(t) = y(t) \). Hence, \( K_p = (L |_{\text{dom} \, L \cap \ker \, P})^{-1} \). This completes the proof of Lemma 7.

**Lemma 8.** Assume \( \Omega \subset X \) is an open bounded set such that \( \text{dom}(L) \cap \Omega \neq \emptyset \); then \( N \) is \( L \)-compact on \( \Omega \).

**Proof.** By the continuity of \( f \), we can obtain that \( QN(\Omega) \) and \( K_p (I - Q) N(\Omega) \) are bounded. Hence, for \( u(t) \in \Omega, \, t \in [0, 1] \), there exists a positive constant \( T \) such that

\[
|(I - Q) Nu(t)| \leq T,
\]

\[
|Nu(t)| \leq T. \tag{32}
\]

Thus, in the view of Arzela-Ascoli theorem, we need only to prove that \( K_p (I - Q) N(\Omega) \) is equicontinuous.

For \( 0 \leq t_1 < t_2 \leq 1, \, u \in \Omega \), we have

\[
|K_p (I - Q) Nu(t_2) - K_p (I - Q) Nu(t_1)|
= \left| \left[ I_0^a y(t_1) - I_0^a y(t_2) \right] - \Gamma(\alpha) I_0^a (I - Q) \right|.
\]

\[
\cdot \frac{t_1}{\Gamma(\alpha)} \left| t_2 - t_1 \right| \leq \frac{t_1}{\Gamma(\alpha)} \left| t_2 - t_1 \right| \tag{33}
\]

\[
\cdot \left| Nu(s) \right| \cdot \left| t_2 - t_1 \right| \leq \frac{T}{\Gamma(\alpha + 1)} \left| t_2 - t_1 \right| \leq \frac{T}{\Gamma(\alpha + 1)} \left[ t_2^\alpha - t_1^\alpha \right].
\]

Notice that \( \ell^\alpha, \, t \) are uniformly continuous on \([0, 1] \). Thus, we have \( K_p (I - Q) N(\Omega) \) is equicontinuous on \([0, 1] \). The proof is completed.

**Theorem 9.** Assume that\( (H1) \) there exist positive constants \( b_1, b_2, b_3, c_1, c_2, \) and \( B \) with

\[
B > \frac{c_2}{c_1} + \frac{3b_2c_3}{(\alpha - 1) b_1 c_1} + \frac{3b_3}{(\alpha - 1) b_1}. \tag{34}
\]


For all \( t \in [0, 1] \) and \( u(t) \in [0, B] \), one has
\[
-\kappa u(t) \leq f\left(t, u(t), u'(t)\right) \leq -c_1 u(t) + c_2,
\]
\[
f\left(t, u(t), u'(t)\right) \leq -b_1 \left| f\left(t, u(t), u'(t)\right) \right| + b_2 u(t) \tag{35}
+ b_3;
\]
\[
\text{(H2)} \text{ there exist } b \in (0, B), t_0 \in [0, 1], \rho \in (0, 1), \delta \in (0, 1), \text{ and } q(t) \in L^1[0, 1], \rho(t) \geq 0 \text{ on } [0, 1], \text{ and } h(x) \in C([0, b], \mathbb{R}^+), \text{ such that } G(t, s) > 0 \text{ and } f(t, u, u') \geq q(t) h(u) \text{ for } (t, u, u') \in [0, 1] \times (0, B) \times \mathbb{R}. \text{ Moreover, } h(u)/u^\alpha \text{ is nonincreasing on } (0, b] \text{ and }
\]
\[
(\alpha - 1) \frac{h(b)}{b} \int_0^1 G(t, s) (1-s)^{\alpha-2} q(s) \, ds \geq \frac{1 - \delta}{\delta^\rho}. \tag{36}
\]

Then problem (4) has at least one positive solution on \([0, 1]\).

**Proof.** According to Lemmas 7 and 8, we have that conditions (A1) and (A2) of Theorem 6 are satisfied.

Consider the cone
\[
C = \{ x \in X : x(t) \geq 0, \ t \in [0, 1] \}. \tag{37}
\]

Let
\[
\Omega_1 = \{ x \in X : \delta \|x\| < |x(t)| < b, \ t \in [0, 1] \}, \tag{38}
\]
\[
\Omega_2 = \{ x \in X : |x(t)| < B, \ t \in [0, 1] \}.
\]

Obviously, \( \Omega_1 \) and \( \Omega_2 \) are bounded and
\[
\overline{\Omega}_1 = \{ x \in X : \delta \|x\| \leq |x(t)| < b, \ t \in [0, 1] \} \subset \Omega_2. \tag{39}
\]

Moreover, \( C \cap \overline{\Omega}_2 \cap \Omega_1 \neq \emptyset \). Let \( I \) and \( (\gamma x)(t) = |x(t)| \) for \( x \in X \); then \( \gamma \) is a retraction and maps subsets of \( \overline{\Omega}_2 \) into bounded subsets of \( C \), which means that (A4) holds.

Next, we will show that (A3) holds. Suppose that there exist \( u_0 \in \partial \Omega_2 \cap C \cap \text{ dom } L \) and \( \lambda_3 \in (0, 1) \) such that \( Lu_0 = \lambda_3 Nu_0 \); that is, \( D_0^\alpha u_0(t) = \lambda_3 f(t, u_0(t), u'_0(t)), t \in [0, 1] \). In view of (H1), we get
\[
D_0^\alpha u_0(t) = \lambda_3 f\left(t, u_0(t), u'_0(t)\right)
\]
\[
\leq -\lambda_3 b_1 \left| f\left(t, u_0(t), u'_0(t)\right) \right| + \lambda_3 b_2 u_0(t)
+ \lambda_3 b_3
= -b_1 \left| \lambda_3 f\left(t, u_0(t), u'_0(t)\right) \right| + \lambda_3 b_2 u_0(t)
+ \lambda_3 b_3
\leq -b_1 \left| D_0^\alpha u_0(t) \right| + \lambda_3 b_2 u_0(t) + b_3,
\]
\[
D_0^\alpha u_0(t) = \lambda_3 f\left(t, u_0(t), u'_0(t)\right)
\]
\[
\leq -\lambda_3 c_1 u_0(t) + \lambda_3 c_2. \tag{41}
\]

In view of \( D_0^\alpha u_0(t) = \lambda_3 f(t, u_0(t), u'_0(t)) \in \text{ Im } L \), from the definition of \( \text{ Im } L \) and (41), we obtain
\[
0 = \int_0^1 (1-s)^{\alpha-2} D_0^\alpha u_0(s) \, ds \tag{42}
\]
\[
\leq \int_0^1 (1-s)^{\alpha-2} \left( -\lambda_3 c_1 u_0(s) + \lambda_3 c_2 \right) \, ds
\]
which gives
\[
\int_0^1 (1-s)^{\alpha-2} u_0(s) \, ds \leq \frac{c_2}{(\alpha - 1) c_1}. \tag{43}
\]

From (40), we have
\[
0 = \int_0^1 (1-s)^{\alpha-2} D_0^\alpha u_0(s) \, ds
\]
\[
\leq \int_0^1 (1-s)^{\alpha-2} \left[ -b_1 \left| D_0^\alpha u_0(t) \right| + b_2 u_0(t) + b_3 \right] \, ds
\]
\[
= -b_1 \int_0^1 (1-s)^{\alpha-2} \left| D_0^\alpha u_0(s) \right| \, ds
+ b_2 \int_0^1 (1-s)^{\alpha-2} u_0(s) \, ds + \frac{b_3}{\alpha - 1}. \tag{44}
\]

By (43), we have
\[
\int_0^1 (1-s)^{\alpha-2} \left| D_0^\alpha u_0(s) \right| \, ds
\]
\[
\leq \frac{b_2}{b_1} \int_0^1 (1-s)^{\alpha-2} u_0(s) \, ds + \frac{b_3}{\alpha - 1} \tag{45}
\]
\[
\leq \frac{b_2 c_2}{\alpha - 1} b_1 c_1 + \frac{b_3}{\alpha - 1} b_1.
\]

According to the function expression of \( k(t, s) \), it is easy to see that \( |k(t, s)| \leq 3 \left( 1 - s \right)^{\alpha - 2}, s, t \in [0, 1] \). From (43) and the equation \( u_0 = (I - P) u_0 + P u_0 = K_p L(I - P) u_0 + Pu_0 = Pu_0 + K_p P u_0 \), we can get
\[
u_0 = (\alpha - 1) \int_0^1 (1-s)^{\alpha-2} u_0(s) \, ds
+ \int_0^1 k(t, s) D_0^\alpha u_0(s) \, ds \tag{46}
\]
\[
\leq (\alpha - 1) \cdot \frac{c_2}{c_1} + \int_0^1 |k(t, s)| \cdot \left| D_0^\alpha u_0(s) \right| \, ds
= \frac{c_2}{c_1} + \int_0^1 \left( \frac{|k(t, s)|}{(1-s)^{\alpha-2}} \cdot (1-s)^{\alpha-2} \left| D_0^\alpha u_0(s) \right| \right) \, ds
\]
\[
\leq \frac{c_2}{c_1} + 3 \int_0^1 (1-s)^{\alpha-2} \left| D_0^\alpha u_0(s) \right| \, ds
\]
\[
\leq \frac{c_2}{c_1} + \frac{3 b_2 c_2}{(\alpha - 1) b_1 c_1} + \frac{3 b_3}{(\alpha - 1) b_1}.
\]
Then, we have

\[ B = \|u_0\| \leq \frac{c_2}{c_1} + \frac{3b_2 c_2}{(\alpha - 1) b_1 c_1} + \frac{3b_3}{(\alpha - 1) b_1}, \tag{47} \]

which contradicts (H1). Hence (A3) holds.

To prove (A5), consider \( u(t) \in \text{ker} L \cap \Omega \); then \( u(t) \equiv c \).

For \( c \in [-B, B] \) and \( \lambda \in [0, 1] \), we have

\[ H(c, \lambda) = [I - \lambda (P + JQN)] c \]

\[ = c - \lambda (\alpha - 1) \int_0^1 (1 - s)^{\alpha - 2} f(s, |c|, 0) \, ds \]

\[ = c - \lambda |c| - \lambda (\alpha - 1) \int_0^1 (1 - s)^{\alpha - 2} f(s, |c|, 0) \, ds \]

\[ = c - \lambda (\alpha - 1) \int_0^1 (1 - s)^{\alpha - 2} f(s, |c|, 0) + |c| \, ds. \tag{48} \]

By use of proof by contradiction, it is easy to show that \( H(c, \lambda) = 0 \) implies \( \lambda \geq 0 \). Suppose \( H(B, \lambda) = 0 \) for some \( \lambda \in (0, 1) \); then we have

\[ 0 = B - \lambda B - \lambda (\alpha - 1) \int_0^1 (1 - s)^{\alpha - 2} f(s, B) \, ds. \tag{49} \]

According to (H1), we have

\[ 0 \leq B(1 - \lambda) = \lambda (\alpha - 1) \int_0^1 (1 - s)^{\alpha - 2} f(s, B, 0) \, ds \]

\[ \leq \lambda (-c_1 B + c_2) < 0, \tag{50} \]

which is a contradiction. In addition, if \( \lambda = 0 \), then \( B = 0 \), which is impossible. As a result, for \( x \in \text{ker} L \cap \partial \Omega_2 \) and \( \lambda \in [0, 1] \), we have \( H(x, \lambda) \neq 0 \). Thus,

\[ \text{deg} \{[I - (P + JQN) \gamma]_{\text{ker} L \cap \partial \Omega_2, 0} \} \]

\[ = \text{deg} \{H(\cdot, 1), \text{ker} L \cap \partial \Omega_2, 0\} \]

\[ = \text{deg} \{H(\cdot, 0), \text{ker} L \cap \partial \Omega_2, 0\} \]

\[ = \text{deg} \{I, \text{ker} L \cap \partial \Omega_2, 0\} = 1 \neq 0. \tag{51} \]

So (A5) holds.

Next, we prove (A6). Let \( u_0(t) \equiv 1, t \in [0, 1] \); then \( u_0 \in C_C \setminus \{0\}, C_C(u_0) = \{x \in C : x(t) > 0, t \in [0, 1]\} \). We take \( \sigma(u_0) = 1 \). Let \( x \in C_C(u_0) \cap \partial \Omega_1 \); then \( 0 < \|x\| \leq b \) and \( x(t) \geq \delta\|x\| \) on \([0, 1] \).

By (H2), for every \( x \in C_C(u_0) \cap \partial \Omega_1 \), we have

\[ (\Psi) x(t_0) = [(P + JQN + K_P(I - Q)N)x(t)|_{t=t_0} \]

\[ = (\alpha - 1) \int_0^1 (1 - s)^{\alpha - 2} x(s) \, ds + (\alpha - 1) \]

\[ \cdot \int_0^1 G(t_0, s) (1 - s)^{\alpha - 2} f(s, x(s), x'(s)) \, ds \]

\[ \geq (\alpha - 1) \int_0^1 (1 - s)^{\alpha - 2} \delta\|x\| \, ds + (\alpha - 1) \]

\[ \cdot \int_0^1 G(t_0, s) (1 - s)^{\alpha - 2} f(s, x(s), x'(s)) \, ds \]

\[ \geq \delta\|x\| + (\alpha - 1) \]

\[ \cdot \int_0^1 G(t_0, s) (1 - s)^{\alpha - 2} q(s) h(x(s)) \, ds = \delta\|x\| + (\alpha - 1) \]

\[ \cdot \int_0^1 G(t_0, s) (1 - s)^{\alpha - 2} q(s) \cdot \frac{h(x(s))}{x^\rho(s)} x^\rho(s) \, ds \]

\[ \geq \delta\|x\| + (\alpha - 1) \]

\[ \cdot \int_0^1 G(t_0, s) (1 - s)^{\alpha - 2} q(s) \cdot \frac{h(x(s))}{x^\rho(s)} \cdot \delta^\rho \|x\|^\rho \, ds \]

\[ \geq \delta\|x\| + \delta^\rho \|x\|^\rho (\alpha - 1) \]

\[ \cdot \int_0^1 G(t_0, s) (1 - s)^{\alpha - 2} q(s) \cdot \frac{h(b)}{b} \, ds \geq \delta\|x\| \]

\[ + \delta^\rho \|x\| \cdot \frac{1 - \delta}{\delta^\rho} \|x\|^\rho = \|x\|. \tag{52} \]

Thus, for all \( x \in C_C(u_0) \cap \partial \Omega_1 \), we have \( \|x\| \leq \sigma(u_0)\|\Psi x\| \); that is, (A6) holds.

For \( x \in \partial \Omega_2 \), by (H2) and (19), we have

\[ [(P + JQN) \gamma] x(t) = P([x(t)]) + JQN([x(t)]) \]

\[ = (\alpha - 1) \int_0^1 (1 - s)^{\alpha - 2} |x(s)| \, ds \]

\[ + (\alpha - 1) \int_0^1 (1 - s)^{\alpha - 2} \left( f(s, |x(s)|, (|x(s)|)') \right) \, ds \tag{53} \]

\[ \geq (\alpha - 1) \int_0^1 (1 - s)^{\alpha - 2} (1 - \kappa)|x(s)| \, ds \geq 0. \]

Thus, \([P + JQN] \gamma] x(t) \in C \) for \( x \in \partial \Omega_2 \). Then (A7) holds.
Next, we prove (A8). For \( x(t) \in \Omega_2 \setminus \Omega_1 \), by (H2), we have
\[
\Psi_x(t) = \left[ (P + JQN + K_P(I - Q)N) \circ \gamma \right] x(t)
\]
\[
= (P + JQN + K_P (I - Q)N) |x(t)| = P (|x(t)|)
\]
\[
+ (JQN + K_P(I - Q)N) |x(t)| = (\alpha - 1) \int_0^1 (1-s)^{\alpha-2} |x(s)| \, ds + (\alpha - 1)
\]
\[
\cdot \int_0^1 (1-s)^{\alpha-2} G(t,s) f(s,|x(s)|,|x(s)|') \, ds
\]
\[
> (\alpha - 1) \int_0^1 (1-s)^{\alpha-2} |x(s)| \, ds + (\alpha - 1)
\]
\[
\cdot \int_0^1 (1-s)^{\alpha-2} G(t,s) (-\kappa |x(s)|) \, ds + (\alpha - 1)
\]
\[
\cdot \int_0^1 (1-s)^{\alpha-2} |x(s)| (1 - \kappa G(t,s)) \, ds \geq 0.
\]
Hence, \( \Psi_x(\Omega_2 \setminus \Omega_1) \subset C \); that is, (A8) holds.

Hence, applying Theorem 6, BVP (4) has a positive solution \( u(t) \) on \([0,1]\) with \( b \leq \|u(t)\| \leq B \). This completes the proof. \( \square \)

4. Example

To illustrate how our main result can be used in practice, we present here an example.

Let us consider the following fractional differential equation at resonance:
\[
D_{0+}^{1.5} u(t) = f\left(t, u(t), u'(t)\right), \quad 0 < t < 1,
\]
\[
u(0) = u(1),
\]
\[
u'(0) = u'(1),
\]
(55)

where
\[
f\left(t, u(t), u'(t)\right) = \frac{1}{400} (1 + t - t^2) (u^2(t) - 8u(t) + 12) u(t) + \frac{1}{30} e^{-|u'(t)|}.
\]
(56)

Corresponding to BVP (4), we have that \( \alpha = 1.5 \) and
\[
G(t,s) = \begin{cases}
1 + \frac{(t-s)^{0.5} (1-s)^{0.5}}{0.5 \Gamma(1.5)} - \frac{\Gamma(1.5) (1-s)^{1.5}}{0.5 \Gamma(2)} + \frac{(1-s) (1-1.5t)}{0.5 \Gamma(2.5)} - \frac{t^{1.5}}{\Gamma(2.5)} + \frac{\Gamma(1.5)}{\Gamma(3)} - \frac{1}{1.5 \Gamma(2.5)} + \frac{t}{\Gamma(2.5)}, & 0 \leq s < t \leq 1, \\
1 - \frac{\Gamma(1.5) (1-s)^{1.5}}{0.5 \Gamma(2)} + \frac{(1-s) (1-1.5t)}{0.5 \Gamma(2.5)} - \frac{t^{1.5}}{\Gamma(2.5)} + \frac{\Gamma(1.5)}{\Gamma(3)} - \frac{1}{1.5 \Gamma(2.5)} + \frac{t}{\Gamma(2.5)}, & 0 \leq t < s \leq 1.
\end{cases}
\]
(57)

By simple calculation, we can get that if \( t \in [0,1] \) and \( u \in [0,1] \), one has
\[
-\frac{1}{15} u(t) \leq f\left(t, u(t), u'(t)\right) \leq \frac{1}{20} u(t) + \frac{1}{10},
\]
(58)

\[
f\left(t, u(t), u'(t)\right) \leq -39 \left| f\left(t, u(t), u'(t)\right) \right| + \frac{1}{2} u(t)
\]
\[
+ \frac{8}{5}.
\]
So, we can choose \( \kappa = 1/15, B = 1, c_1 = 1/20, c_2 = 1/10, b_1 = 39, b_2 = 1/2, \) and \( b_3 = 8/5 \). Furthermore, we can verify
\[
B > \frac{c_2}{c_1} + \frac{3b_3 c_2}{(\alpha - 1) b_1 c_1} + \frac{3b_3}{(\alpha - 1) b_1}.
\]
(59)

So, (H1) is satisfied.
\[ \frac{h(u)}{u'} = \frac{(1/20) u(t)}{u'(t)} = \frac{1}{20} \] 

(62)

is nonincreasing on \((0, 1/2]\) \(\delta = 0.9995\). So, conditions (H1)-(H2) of Theorem 9 are satisfied; then BVP (55) has a positive solution on \([0, 1]\).

**Competing Interests**

The author declares that he has no competing interests.

**Authors’ Contributions**

The author contributed to the writing of this paper. The author read and approved the final manuscript.

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