RESONANCE INDEX AND SINGULAR $\mu$-INARIANT

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ABSTRACT. In this paper we give a direct proof of equality of the total resonance index and of singular part of the $\mu$-invariant under mild conditions which include $n$-dimensional Schrödinger operators. Previously it was proved for trace class perturbations that each of these two integer-valued functions were equal to the singular spectral shift function.

The proof is self-contained and is based on application of the Argument Principle from complex analysis to poles and zeros of analytic continuation of scattering matrix considered as a function of coupling constant.

INTRODUCTION

Let $\lambda$ be a real number. Let $H_0$ be a self-adjoint operator on a separable Hilbert space $\mathcal{H}$ and let $V$ be a $H_0$-compact self-adjoint operator on $\mathcal{H}$ such that

1. $V$ admits a factorization $V = F^* J F$, where $F: \mathcal{H} \to \mathcal{K}$ is a closed operator from $\mathcal{H}$ to another Hilbert space $\mathcal{K}$ and $J$ is a self-adjoint bounded operator on $\mathcal{K};$
2. The operator $FR_z(H_0)F^*$, where $R_z(H) = (H - z)^{-1}$ is the resolvent of $H$, is compact for some and thus for any complex number $z$ which does not belong to the spectrum of $H_0$;
3. The uniform limit
$$T_{\lambda + i0}(H_0) := FR_{\lambda + i0}(H_0)F^*$$
of the sandwiched resolvent
$$T_{\lambda + iy}(H_0) := FR_{\lambda + iy}(H_0)F^*$$
as $y$ approaches 0 exists, $y \in \mathbb{R}$.

Under these conditions one can consider the following operator, which can be interpreted as the scattering matrix for the pair of operators $(H_0, H_r)$, $r \in \mathbb{R}$, (see e.g. [Y] Chapter 5, §5)

$$S(\lambda + i0; H_r, H_0) = 1 - 2i \sqrt{\text{Im} \, T_{\lambda + i0}(H_0) J(1 + rT_{\lambda + i0}(H_0) J)^{-1} \sqrt{\text{Im} \, T_{\lambda + i0}(H_0)}},$$

where $H_r = H_0 + rV$, as long as the operator $1 + rT_{\lambda + i0}(H_0) J$ has bounded inverse. By the analytic Fredholm alternative, the set $R(\lambda; H_0, V; F)$ of real numbers $r$ for which the operator $1 + rT_{\lambda + i0}(H_0) J$ is not invertible is discrete; elements of this set were called in [Az3] [Az5] resonance points. It is a well-known fact which can also be verified by a simple calculation that for all non-resonant values of the coupling constant $r$ the operator $S(\lambda + i0; H_0, H_r)$ is unitary. Moreover, for all $y \in (0, \infty)$ and for all $r \in \mathbb{R}$ the operator

$$S(\lambda + iy; H_r, H_0) = 1 - 2i \sqrt{\text{Im} \, T_{\lambda + iy}(H_0) r J(1 + rT_{\lambda + iy}(H_0) J)^{-1} \sqrt{\text{Im} \, T_{\lambda + iy}(H_0)}}$$
is also unitary and depends analytically on $y$. Further, as $y \to +\infty$, the operator $S(\lambda + iy; H_0, H_r)$ converges in uniform topology to the identity operator 1 and it can be shown that this convergence is locally uniform with respect to $r$ in $\mathbb{R}$. If the value $r = 1$ of the coupling constant is
non-resonant then for a fixed number $e^{i\theta}$ on the unit circle, one can count the number of eigenvalues of the scattering matrix $S(\lambda + iy; H_1, H_0)$ which cross the point $e^{i\theta}$ in clockwise direction as $y$ goes from $0^+$ to $+\infty$. This number was denoted $\mu(\theta, \lambda; H_1, H_0)$ in [Pu] and was called $\mu$-invariant. The $\mu$-invariant measures spectral flow of eigenvalues of the scattering matrix. Pushnitski showed in [Pu] that for relatively trace-class perturbations $\xi$ and was called the absolutely continuous part of Pushnitski $\mu$-invariant. One to measure spectral flow of eigenvalues of the scattering matrix corresponding to the path in the stationary formula (0.1) for the scattering matrix is undefined. This circumstance allows several formulas for the spectral shift function $\xi$ are distribution by formula

$$\xi(\varphi) = \int_0^1 \text{Tr}(V \varphi(H_r)) \, dr$$

for a test function $\varphi$.

In [Az3, Az3] it was observed that there is another natural way to deform the scattering matrix $S(\lambda + i0; H_1, H_0)$ continuously to the identity operator: by sending the coupling constant $\tau$ from 1 to 0. Indeed, by analytic Fredholm alternative the operator $S(\lambda + i0; H_1, H_0)$ is a meromorphic function of the coupling constant $\tau$ considered as a complex variable. Since the operator-function $S(\lambda + i0; H_1, H_0)$ is unitary for real values of $\tau$ it cannot have poles on the real axis. Thus, it can be continued analytically to the resonant values of $\tau$ for which the factor $(1 + rT_{\lambda + i0}(H_0)J)^{-1}$ in the stationary formula (0.1) for the scattering matrix is undefined. This circumstance allows one to measure spectral flow of eigenvalues of the scattering matrix corresponding to the path $\{S(\lambda + i0; H_1, H_0) : r \in [0, 1]\}$. This spectral flow was denoted in [Az3, Az3] by $\mu(\theta, \lambda; H_1, H_0)$ and was called the absolutely continuous part of Pushnitski $\mu$-invariant. This terminology is justified as follows. For trace class perturbations $V$ it was shown in [Az3] that

$$\xi^{(a)}(\lambda; H_1, H_0) = -\frac{1}{2\pi} \int_0^{2\pi} \mu^{(a)}(\theta, \lambda; H_1, H_0) \, d\theta,$$

where $\xi^{(a)}(\lambda; H_1, H_0)$ is the absolutely continuous part of the spectral shift function which was introduced in [Az3] (see also [Az, Az2]) as a distribution by formula

$$\xi^{(a)}(\varphi) = \int_0^1 \text{Tr}(V \varphi(H^{(a)}_r)) \, dr, \quad \varphi \in C^\infty_c(\mathbb{R}).$$

Here $H^{(a)}_r$ is the absolutely continuous part of the self-adjoint operator $H_r$, and we shall soon use $H^{(s)}_r$ the singular part of the self-adjoint operator $H_r$. The two formulas connecting $\xi$ and $\xi^{(a)}$ with $\mu$ and $\mu^{(a)}$ respectively, imply that

$$\xi^{(s)}(\lambda; H_1, H_0) = -\frac{1}{2\pi} \int_0^{2\pi} \mu^{(s)}(\lambda; H_1, H_0) \, d\theta$$

$$= -\mu^{(s)}(\lambda; H_1, H_0),$$

where $\xi^{(s)}$ is the singular part of the spectral shift function defined as distribution by formula

$$\xi^{(s)}(\varphi) := \int_0^1 \text{Tr}(V \varphi(H^{(s)}_r)) \, dr$$

and where

$$\mu^{(s)}(\lambda; H_1, H_0) := \mu(\theta, \lambda; H_1, H_0) - \mu^{(a)}(\theta, \lambda; H_1, H_0)$$
is the singular part of $\mu$-invariant. Topological considerations imply that the function $\mu^{(s)}(\lambda; H_1, H_0)$ does not depend on the angle $\theta$ and so this variable is omitted in the list of arguments of $\mu^{(s)}$, see [Az3] Section 9]. This implies in particular that $\xi^{(s)}(\lambda; H_1, H_0)$ takes integer values for a.e. $\lambda \in \mathbb{R}$.

One can easily check that a real number $r_\lambda$ is a resonance point if and only if the real number $(s - r_\lambda)^{-1}$ is an eigenvalue of positive algebraic multiplicity $N$ for the compact operator $T_{\lambda+in\theta}(H_s)\mathbf{J}$. This definition is independent from $s \in \mathbb{R}$ as long as it is non-resonant. If we shift $\lambda + i0$ to $\lambda + iy$, the eigenvalue $(s - r_\lambda)^{-1}$ also changes and in general splits to eigenvalues $(s - r_{\lambda+iy})^{-1}, \ldots, (s - r_{\lambda+iy}^{N})^{-1}$ of $T_{\lambda+iy}(H_s)\mathbf{J}$, where the eigenvalues are listed according to their multiplicities. It is well-known and is not difficult to show that these shifted eigenvalues are all non-real. Let $N_+$ and $N_-$ be the numbers of the shifted eigenvalues in the upper $\mathbb{C}_+$ and the lower $\mathbb{C}_-$ complex half-planes respectively. The so-called resonance index of the triple $(\lambda; H_{r_\lambda}, V)$ is defined by formula

$$\text{ind}_{\text{res}}(\lambda; H_{r_\lambda}, V) = N_+ - N_-.$$ 

In [Az4] (see also [Az2] §6]) it was shown that $\xi^{(s)}(\lambda; H_1, H_0)$ is equal for a.e. $\lambda \in \mathbb{R}$ to the total resonance index

$$\sum_{r_\lambda \in [0,1]} \text{ind}_{\text{res}}(\lambda; H_{r_\lambda}, V),$$

where the sum is taken over all resonance points from $[0,1]$, of which there is a finite number.

Hence, for trace-class perturbations $V$ and for a.e. $\lambda$ we have the equality

$$(0.2) \quad -\mu^{(s)}(\lambda; H_1, H_0) = \sum_{r_\lambda \in [0,1]} \text{ind}_{\text{res}}(\lambda; H_{r_\lambda}, V).$$

The minus sign here appears because the $\mu$-invariant calculates the spectral flow of eigenvalues of the scattering matrix in clockwise direction.

In this paper we give a new and direct proof of this formula assuming only that the pair $H_0$ and $V$ satisfy the conditions (1)–(3) above.

**Theorem 1.** If $H_0$ is a self-adjoint operator on a separable Hilbert space $\mathcal{H}$ and if $V$ is a $H_0$-compact self-adjoint operator on $\mathcal{H}$ such that the conditions (1)–(3) hold, then the equality (0.2) holds.

The conditions (1)–(3) are well-known in scattering theory, see e.g. [BE] [KK] [Ag] [Ku] [Y]. Three classical examples for which the conditions (1)–(3) hold for a.e. $\lambda$ are:

1. arbitrary self-adjoint operator $H_0$ and a trace class self-adjoint operator $V$,
2. a Schrödinger operator $H_0 = -\Delta + V_0(x)$ on $L_2(\mathbb{R}^n)$ with bounded measurable real-valued function $V_0(x)$ and $V$ an operator of multiplication by a real-valued function $V(x)$ such that $|V(x)| \leq C(1 + |x|)^{-\nu - \varepsilon}$ for some $C, \varepsilon > 0$,
3. the operator $H_0$ is the Laplacian $-\Delta$ on $L_2(\mathbb{R}^n)$ and $V$ an operator of multiplication by a real-valued function $V(x)$ such that $|V(x)| \leq C(1 + |x|)^{-1 - \varepsilon}$ for some $C, \varepsilon > 0$.

Finally, we note that the equality (0.2) makes a nontrivial sense even for self-adjoint matrices $H_0$ and $V$ on a finite dimensional Hilbert space $\mathbb{C}^n$, in which case the equality can be tested in numerical experiments.

For more motivation for this work one may also look at the introduction of [Az5].

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1. Sketch and Idea of Proof of Theorem 1

In this section we give a brief sketch of the idea of proof of Theorem 1. Full details are given in the next section.

There are two ways to deform continuously the scattering matrix

\[ S(\lambda + iy; H_r, H_0) \mid_{y=0, r=1} = S(\lambda + i0; H_1, H_0) \]

to the identity operator 1: by sending the imaginary part \( y \) of energy from 0 to \( +\infty \) and by sending coupling constant \( r \) from 1 to 0. Why these paths are continuous was explained in the introduction. As the scattering matrix is deformed to the identity operator the eigenvalues of it are also continuously deformed to (the number) 1. The difference of spectral flows through a point \( e^{i\theta} \) on the unit circle distinct from 1 does not depend on \( \theta \). This difference is the singular \( \mu \)-invariant. Now we are going to deform continuously one of these two paths to another one but as we do so at one point we shall encounter an obstruction in the form of poles and zeros of the scattering matrix as a function of the coupling constant \( s \). Overcoming this obstruction gives the resonance index \( N_+ - N_- \) which is the difference of the number of poles and of the number of zeros from the upper complex half-plane, where poles and zeros are counted according to their multiplicities.

Firstly we deform the path \((y: 0 \rightarrow +\infty, r = 1)\) to the path \((y: 0 \rightarrow \varepsilon, r = 1 & y = \varepsilon, r: 1 \rightarrow 0)\), as shown in the figure below. The rectangle \([0, +\infty] \times [0, 1]\) is the domain of the operator-function \( f(y, r) = S(\lambda + iy; H_r, H_0) \) of real variables \( y \) and \( r \). (Contrary to the common tradition, in the figure the first variable \( y \) changes along the vertical axis). This deformation does not meet any obstructions since the function is continuous in the rectangle \([\varepsilon, +\infty] \times [0, 1]\). But the deformation cannot be pushed all the way to the lower rim \( y = 0 \) of the rectangle since the lower rim may have resonance points at which the function \( f(y, r) \) is not continuous. It is to be noted however that the function \( f(0, r) \) of variable \( r \) is continuous and deformation along this path gives the \( \mu^{(a)} \)-invariant. As we shall see, what happens here is cancellation of poles and zeros such as in the function \( \frac{\lambda + iy}{\lambda - iy} \) in the limit \( y \rightarrow 0 \).

\[ S(0, 1) = S(\lambda + i0; H_1, H_0) \]

The three points \( r_\lambda, r'_\lambda, r''_\lambda \) represent resonance points from \([0, 1]\).

So far we were interested in real values of the coupling constant \( s \). To stress on the fact that the coupling constant is considered as a complex variable we now denote it by \( s \). Now we fix a small value \( \varepsilon \) of \( y \) and consider the function \( f(\varepsilon, s) \) of the complex variable \( s \).

While for \( y = 0 \) the factor \((1 + sT_{\lambda+iy}J)^{-1}\) may have real singularities (which are by definition real resonance points), for non-real values of \( y \) the factor \((1 + sT_{\lambda+iy}J)^{-1}\) as a function of \( s \) is holomorphic on the real axis. A pole \( r_\lambda \) of the factor \((1 + sT_{\lambda+iy}J)^{-1}\) for small non-real values of \( y \) gets shifted out of the real axis. In general it may split into a group of several poles.
resonance points $r_\lambda^{1+iy}, \ldots, r_\lambda^{N+iy}$, which are listed here according to their multiplicities. We note that while for real values of $s$ the scattering matrix $f(y,s)$ is a unitary operator for any $y$, for complex values of $s$ this is not the case anymore. In particular, for some values of $s$ the scattering matrix $f(y,s)$ may have a non-trivial kernel. These values we shall call zeros of $f(y,s)$. It turns out that conjugates of the poles $r_\lambda^{1+iy}, \ldots, r_\lambda^{N+iy}$ are zeros of $f(y,s)$ and vice versa. Formally, this can be seen from the following formula for the scattering matrix proof of which is a simple algebraic transformation:

$$\sqrt{\text{Im} T_z(H_0)} S(z,s) = (1 + sT_z(H_0)J)(1 + sT_z(H_0)J)^{-1} \sqrt{\text{Im} T_z(H_0)}.$$ 

$s$-plane for $y = \varepsilon$. Black dots are resonance points, white dots are zeros (anti-resonance points).

As it was discussed above, the deformation of the $S$-matrix $f(y,s)|_{y=0,s=1}$ to the identity operator which corresponds to $\mu$-invariant is shown in the following figure. 

The deformation of the $S$-matrix $f(y,s)$ to the identity operator which corresponds to $\mu^{(a)}$-invariant is shown in the following figure. That is, to obtain $\mu^{(a)}$-invariant, we should circumvent resonant and anti-resonant points in the $s$-plane. Indeed, as we take the limit $y \to 0$ the $\mu$-invariant of the path shown below does not change for topological reasons since there are no obstructions, but at the same time all resonance and anti-resonance points start to converge to the corresponding real resonance point where they eventually get cancelled by each other (as it happens to the pole and the zero of the holomorphic function $\frac{i\varepsilon}{z+i\varepsilon}$ in the limit $\varepsilon \to 0$). Once we get to $y = 0$ the path below can be deformed to the straight path from $s = 1$ to $s = 0$ since the resulting function $f(0,s)$ has no singularities in a neighbourhood of the interval $[0, 1]$.

Hence, the singular $\mu$-invariant, that is the difference in winding numbers of eigenvalues of $f(y,s)$ corresponding to the two deformations, is equal to the total number of windings of eigenvalues of the $S$-matrix along the clockwise oriented closed contours enclosing those and
only those poles and zeros of the groups of real resonance points which belong to the upper half-plane, see the next figure.

It remains to apply the Argument Principle from complex analysis to eigenvalues of the S-matrix to infer that the singular $\mu$-invariant is equal to the number of poles minus the number of zeros inside of these contours in the upper complex half-plane.

We note that it is not essential to circumvent the poles and zeros in the upper half-plane. Of course, in this case the difference of the number of poles and zeros changes sign but this is compensated by the change of direction of circumvention of the contours from clockwise to anti-clockwise.

Finally, some care should be taken in the case there are degenerate resonance or anti-resonance points. This is done in the next section.

2. Proof of Theorem 1

2.1. The Argument Principle. The following theorem from complex analysis known as the Argument Principle can be found in e.g. [Sh].

**Theorem 2.1.** Let $f(z)$ be a meromorphic function in a simply connected domain $\Omega$ of the complex plane. Let $\gamma$ be a closed contour in $\Omega$ which does not contain poles and zeros of $f(z)$. As $z$ makes one round along the curve $\gamma$ the argument of $f(z)$ makes $N - P$ rounds around zero where $N$ is the number of zeros, counting multiplicities, of $f(z)$ inside $\gamma$ and $P$ is the number of poles, counting multiplicities, of $f(z)$ inside $\gamma$.

2.2. Zeros and poles of the scattering matrix. Let $H_0$, $F$ and $V$ be as in the introduction. Without loss of any generality we shall assume that the range of $F$ is dense in $\mathcal{K}$. Let $z$ be an arbitrary complex number which does not belong to the resolvent set of $H_0$, so in particular the operator $T_z(H_0)$ exists. We denote by $S(z; s, H_0)$ the operator function

\[
S(z; H_s, H_0) = 1 - 2is\sqrt{\text{Im} T_z(H_0)}J(1 + sT_z(H_0)J)^{-1}\sqrt{\text{Im} T_z(H_0)},
\]

where $s$ is treated as a complex variable and $\text{Im} z > 0$. The right hand side can also be written as

\[
S(z; H_s, H_0) = 1 - 2is\sqrt{\text{Im} T_z(H_0)(1 + sJT_z(H_0))^{-1}} J\sqrt{\text{Im} T_z(H_0)}.
\]

By definition, a point $s$ is a resonance point corresponding to $z$ if the operator

\[
1 + sJT_z(H_0)
\]

has a non-zero kernel. Dimension of this kernel will be called algebraic multiplicity of the resonance point. Non-zero vectors from the kernel will be called resonance vectors for given values of $z$ and $s$. A point $s$ is an anti-resonance point corresponding to $z$ if the operator

\[
1 + sJT_z(H_0)
\]

has a non-zero kernel. Dimension of this kernel will be called algebraic multiplicity of the anti-resonance point. Non-zero vectors from this kernel will be called anti-resonance vectors for given
Proof. It can be shown that anti-resonance points are complex conjugates of resonance points and algebraic multiplicities of a pair of resonance and anti-resonance points are equal, see e.g. [Az, Section 3]. Resonance points will usually be denoted by \( r_z \) and anti-resonance points by \( \bar{r}_z \).

**Lemma 2.2.** For any complex number \( z \) with positive imaginary part the equalities
\[
\sqrt{\text{Im} T_z(H_0)} S(z, s) = (1 + sT_z(H_0))^{-1} \sqrt{\text{Im} T_z(H_0)}
\]
and
\[
S(z, s) \sqrt{\text{Im} T_z(H_0)} = \sqrt{\text{Im} T_z(H_0)} (1 + sJT_z(H_0))^{-1} (1 + sJT_z(H_0))^{-1}
\]
hold as equalities between meromorphic functions of \( s \).

**Proof.** Using (2.1), we have
\[
\sqrt{\text{Im} T_z(H_0)} S(z, s) = \sqrt{\text{Im} T_z(H_0)} - 2is \text{Im} T_z(H_0) (1 + sT_z(H_0))^{-1} \sqrt{\text{Im} T_z(H_0)}
\]
\[
= \left[ 1 - sT_z(H_0) - T_z(H_0) \right] (1 + sT_z(H_0))^{-1} \sqrt{\text{Im} T_z(H_0)}
\]
\[
= \left[ 1 - sT_z(H_0) + sT_z(H_0)J^{-1} + sT_z(H_0)J^{-1} \right] \sqrt{\text{Im} T_z(H_0)}
\]
\[
= \left[ (1 + sT_z(H_0))^{-1} + sT_z(H_0)J^{-1} \right] \sqrt{\text{Im} T_z(H_0)}
\]
\[
= (1 + sT_z(H_0))^{-1} \sqrt{\text{Im} T_z(H_0)}.
\]
The equality (2.4) is proved similarly but instead of (2.1) one uses (2.2).

Given a non-real complex number \( z \), we say that \( s \) is non-critical if \( s \) is neither resonant nor anti-resonant for \( z \).

**Lemma 2.3.** Let \( \text{Im} z > 0 \). For any non-critical \( s \in \mathbb{C} \) the operator \( S(z, s) \) has a bounded inverse.

**Proof.** By analytic Fredholm alternative, the function \( S(z, s) \) is a meromorphic function of \( s \), which may have poles only at resonance points. Hence, for non-resonant \( s \) the operator \( S(z, s) \) is well-defined and bounded.

Let \( s \) be a point which is not resonant. If \( \varphi_j \) is an eigenvector of \( S(z, s) \) corresponding to eigenvalue zero, then (2.3) implies that the vector
\[
(1 + sT_z(H_0)J)^{-1} \sqrt{\text{Im} T_z(H_0)} \varphi_j
\]
is a non-zero anti-resonance vector for \( z \) and \( s \). Therefore, in this case \( s \) is an anti-resonance point corresponding to \( z \).

It follows that if \( s \) is neither resonant nor anti-resonant then \( S(z, s) \) is a bounded operator with zero kernel. Since \( S(z, s) - 1 \) is compact, by Fredholm alternative, this implies that in this case \( S(z, s) \) is a bounded invertible operator.

Let \( \mathfrak{h}_z \) be the range of the operator \( \sqrt{\text{Im} T_z(H_0)} \):
\[
\mathfrak{h}_z := \text{ran} \sqrt{\text{Im} T_z(H_0)}.
\]
The set \( \mathfrak{h}_z \) is a dense subspace of the Hilbert space \( \mathcal{K} \). If \( s \) is neither resonant nor anti-resonant point, then by Fredholm alternative the operator \( (1 + sJT_z(H_0))^{-1} (1 + sJT_z(H_0)) \) is invertible. Hence, the equality (2.4) implies that for such \( s \)
\[
S(z, s) \mathfrak{h}_z = \mathfrak{h}_z.
\]
Let
\[
M(z, s) := (1 + sT_z(H_0)J)(1 + sT_z(H_0)J)^{-1} = 1 - 2is \operatorname{Im} T_z(H_0)(1 + sT_z(H_0)J)^{-1}.
\]
(2.5)

The equality (2.3) can be rewritten as
\[
\sqrt{\operatorname{Im} T_z(H_0)} S(z, s) = M(z, s) \sqrt{\operatorname{Im} T_z(H_0)}.
\]
(2.6)

Since by Lemma 2.3 the operator \( S(z, s) \) is invertible for non-critical \( s \), for such \( s \) we have \( S(z, s)K = K \). Hence, (2.6) implies that for all non-critical \( s \)
\[
M(z, s)h_z = h_z.
\]

We have proved the following lemma.

**Lemma 2.4.** Given a non-real complex number \( z \), for any non-critical \( s \)
\[
S(z, s)h_z = h_z \quad \text{and} \quad M(z, s)h_z = h_z.
\]

For a given number \( z \) we say that a point \( s_0 \) is a zero of the meromorphic function \( S(z, s) \) if and only if 0 is an eigenvalue of \( S(z, s) \). Multiplicity of zero \( s_0 \) is the algebraic multiplicity of the eigenvalue 0. Geometrically, this means that if multiplicity of a zero \( s_0 \) is \( m \) then \( m \) eigenvalues (counting multiplicities) of \( S(z, s) \) approach zero as \( s \) approaches \( s_0 \).

**Lemma 2.5.** Let \( z \) be a non-real complex number and let \( s \) be a non-critical value. All eigenvectors of the operator \( S(z, s) \) corresponding to eigenvalues not equal to 1 belong to \( h_z \). All eigenvectors of the operator \( M(z, s) \) corresponding to eigenvalues not equal to 1 belong to \( \operatorname{ran} \operatorname{Im} z(H_0) \).

Further, a vector \( \varphi_j \) is an eigenvector (a generalized eigenvector) of \( S(z, s) \) corresponding to a non-identity eigenvalue if and only if \( \sqrt{\operatorname{Im} z(H_0)} \varphi_j \) is an eigenvector (respectively, a generalized eigenvector of the same order) of \( M(z, s) \) corresponding to the same non-identity eigenvalue.

**Proof.** The first assertion follows directly from (2.1). The second assertion follows from the second of the two equalities (2.5). The third assertion follows from (2.6), the previous two assertions and the fact that \( \sqrt{\operatorname{Im} z(H_0)} \) has trivial kernel for non-real \( z \). \(\square\)

**Corollary 2.6.** The operators \( S(z, s) \) and \( M(z, s) \) have identical spectra, including multiplicities of eigenvalues.

This corollary implies the following

**Theorem 2.7.** For any non-real complex number \( z \) the meromorphic functions \( S(z, s) \) and \( M(z, s) \) have the same sets of poles and zeros; moreover, multiplicities of these poles and zeros are also the same.

Indeed, \( s \) is a zero of multiplicity \( N \) for \( S(z, s) \) or \( M(z, s) \) iff zero is an eigenvalue of \( S(z, s) \) or \( M(z, s) \) of multiplicity \( N \).

This theorem allows us to work with either \( S(z, s) \) or \( M(z, s) \) as far as we are concerned only with eigenvalues of these operators.

**Corollary 2.8.** For any non-real \( z \) the meromorphic function \( S(z, s) \) of \( s \) has poles at resonance points \( r_z \) and zeros at antiresonance points \( \bar{r}_z \). There are no other poles and zeros of \( S(z, s) \).

Further, multiplicities of a pole \( r_z \) and of a zero \( \bar{r}_z \) coincide.
We are interested in the following question. Assume that \( r_z \) is a resonance point (respectively, anti-resonance point) of algebraic multiplicity \( N \). In this case \( s = r_z \) is a pole (respectively, zero) of \( S(z, r_z) \) of algebraic multiplicity \( N \). For values of \( s \) close to \( r_z \) the operator \( S(z, s) \) will have \( N \) eigenvalues close to infinity (respectively, to zero); these eigenvalues are called eigenvalues of the group infinity (respectively, zero). As the variable \( s \) makes one round around \( r_z \) the eigenvalues of the operator \( S(z, s) \) which belong to the group of infinity (respectively, zero) will undergo a permutation. We are interested in the total number of windings which these eigenvalues make. If we knew that all the eigenvalues of the group of infinity (respectively, zero) admit single-valued analytic continuation to a neighbourhood of \( r_z \) then the Argument Principle would imply that the total number of windings would be \( N \). But we don’t know whether the eigenvalues admit such single-valued analytic continuations. We shall call the total number of windings of the eigenvalues of \( S(z, r_z) \) about zero \( S\text{-index} \) of the resonance point \( r_z \). Our aim is to prove that \( S\text{-index} \) of a resonance point (respectively, anti-resonance point) of algebraic multiplicity \( N \) is equal to \(-N \) (respectively, \( N \)).

The following assertion is trivial.

**Lemma 2.9.** \( S\text{-index} \) of a resonance point \( r_z \) is equal to the sum of \( S\text{-indices} \) of resonance points into which \( r_z \) splits when \( z \) is slightly perturbed.

It is possible that a resonance point \( r_z \) corresponding to \( z \) is also an anti-resonance point. This creates some technical difficulties. The following lemma allows to avoid them.

**Lemma 2.10.** A resonance point \( r_z^1 \) as a function of \( z \) cannot coincide with an anti-resonance point \( r_z^2 \) as a function of \( z \). As a result, if \( r_z \) is both a resonance and anti-resonance point then if \( z \) is perturbed slightly \( r_z \) will split into resonance points none of which is an anti-resonance point.

**Proof.** Indeed, a resonance point \( r_z^1 \) is a holomorphic function of \( z \) while an anti-resonance point \( r_z^2 \) is an anti-holomorphic function of \( z \). So, they can coincide only if both of them are constants. But a resonance point \( r_z \) cannot be a constant function of \( z \). \( \square \)

This lemma implies that slightly perturbing \( z \) we can always achieve a situation where neither of a finite number of resonance points is an anti-resonance point.

**Lemma 2.11.** \( S\text{-index} \) of a resonance point of algebraic multiplicity \( N \) is equal to \(-N \).

**Proof.** According to Theorem 2.7 as far as the total number of windings of eigenvalues is concerned, instead of the operator \( S(z, s) \) we can consider the operator \( M(z, s) \). The operator \( 1 + sT_z(H_0)J \) for \( s = r_z \) has zero as an eigenvalue of multiplicity \( N \). Let \( \varepsilon_j(s)|_{s=r_z} \) be this eigenvalue. When \( s \) is perturbed to a value close to \( r_z \) the zero eigenvalue \( \varepsilon_j(r_z) \) shifts to \( \varepsilon_j(s) \) but does not split. By the Argument Principle (Theorem 2.11), when \( s \) makes one winding around \( r_z \) this eigenvalue \( \varepsilon_j(s) \) makes one winding around zero. But since this eigenvalue has multiplicity \( N \), it results in \( S\text{-index} \) being equal to \( N \). Hence, \( S\text{-index} \) of the operator

\[
(1 + sT_z(H_0)J)^{-1}
\]

is equal to \(-N \).

By Lemmas 2.9 and 2.10 we can assume that \( r_z \) is a stable (non-splitting) resonance point and that this resonance point is not an anti-resonance point. Thus, the operator \( 1 + sT_z(H_0)J \) is invertible. Since \( S\text{-index} \) is plainly stable under continuous deformations and since the invertible operator \( 1 + sT_z(H_0)J \) can be continuously deformed into the identity operator in the space of invertible operators, it follows that the \( S\text{-index} \) of the operator

\[
(1 + sT_z(H_0)J)(1 + sT_z(H_0)J)^{-1}
\]
is equal to $-N$. □

A similar argument proves the following

**Lemma 2.12.** \(S\)-index of an anti-resonance point of algebraic multiplicity \(N\) is equal to \(N\).

**Remark 1.** If zeros and poles of eigenvalues of the function \(S(z, s)\) of \(s\) for a fixed \(z\) were known to have single-valued analytic continuations in some neighbourhoods of resonance \(\bar{r}_z\) and anti-resonance points \(\bar{r}_z\) then, combined with the Argument Principle, this would directly imply Lemmas 2.11 and 2.12. But Lemmas 2.11 and 2.12 alone do not imply that the eigenvalues have the single-valued analytic continuations. For example, if a zero \(\bar{r}_z\) of an eigenvalue of \(S(z, s)\) has algebraic multiplicity four, then it is possible that as \(s\) makes one winding around \(\bar{r}_z\) two eigenvalues swap making half windings around zero and the other two eigenvalues swap making one and a half windings around zero, resulting in \(S\)-index four in agreement with Lemmas 2.11 and 2.12.

2.3. **Uniformity of spectrum of** \(S(z, s)\) **as** \(\text{Im } z \to +\infty\). As usual, we are assuming conditions (1)–(3) from the introduction.

The aim of this subsection is to prove the following theorem.

**Theorem 2.13.** For any \(\lambda \in \mathbb{R}\) the operator \(S(\lambda + iy, s)\) converges to the identity operator as \(y \to +\infty\) locally uniformly with respect to \(s \in \mathbb{R}\). Moreover, eigenvalues \(\varepsilon_j(\lambda + iy, s)\) of the operator \(S(\lambda + iy, s)\) also converge to 1 as \(y \to +\infty\) locally uniformly with respect to \(s \in \mathbb{R}\).

We shall use the following lemma, see e.g. [Y, Lemma 6.1.3].

**Lemma 2.14.** If a sequence of bounded operators \(A_n\) converges to zero in \(*\)-strong operator topology and if \(T\) is a compact operator then \(TA_n\) and \(A_nT\) converge to zero uniformly.

**Lemma 2.15.** Let \(z\) be any non-real complex number. The operators

\[
FE_{(-\infty, -M)}(H_0) \text{Im } R_z(H_0) F^* \quad \text{and} \quad FE_{(M, +\infty)}(H_0) \text{Im } R_z(H_0) F^*
\]

converge to zero in norm as \(M \to +\infty\).

If \(F\) is bounded then this assertion is an easy consequence of the spectral theorem.

**Proof.** We prove the assertion for the first operator. By Lemma 2.14 the operator

\[
FE_{(-\infty, -M)}(H_0) R_z(H_0)
\]

converges in norm to zero as \(M \to \infty\) since it is product of a compact operator \(FR_z(H_0)\) (for a proof that this operator is compact see e.g. [Az 5, Lemma 2.8]) and of an operator \(E_{(-\infty, -M)}(H_0)\) which converges \(*\)-strongly to 0 as \(M \to +\infty\). It follows that the operator

\[
FE_{(-\infty, -M)}(H_0) R_z(H_0) R_{\bar{z}}(H_0) E_{(-\infty, -M)}(H_0) F^*
\]

also converges to zero as \(\Delta \to \mathbb{R}\). By the first resolvent identity, this implies that

\[
FE_{(-\infty, -M)}(H_0) (R_z(H_0) - R_{\bar{z}}(H_0)) F^* \to 0.
\]

□
Lemma 2.16. Let \( z \) be any non-real complex number. The operators
\[
FE_{(-\infty,-M)}(H) \Re R_z(H) F^*
\]
and
\[
FE_{(M, +\infty)}(H) \Re R_z(H) F^*
\]
converge to zero in norm as \( M \to +\infty \).

Proof. We prove the assertion for the second operator. Without loss of generality we assume that \( z = i \). Then
\[
(2.7) \quad FE_{(M, +\infty)}(H) \Re R_z(H) F^* = FE_{(M, +\infty)}(H) \frac{H}{H^2 + 1} F^*.
\]
By Lemma 2.14, the compact operator \( FE_{(M, +\infty)}(H) \frac{H}{H^2 + 1} F^* \) converges to zero in norm as \( M \to +\infty \). Using this one can show that the operator \( (2.7) \) converges to zero strongly as \( M \to +\infty \).

Indeed, for \( M \in \mathbb{R} \) let \( T_M := FE_{(M, +\infty)}(H) \frac{H}{H^2 + 1} F^* \).

If \( f \in \text{dom} F^* \) then clearly \( T_M f \to 0 \) as \( M \to +\infty \). If \( f \notin \text{dom} F^* \), then for any \( \varepsilon > 0 \) we find \( g \in \text{dom} F^* \) such that \( \|f - g\| < \frac{\varepsilon}{4\|T_0\|} \). Let \( h = f - g \). Then we have for any \( M > 0 \)
\[
\|T_M h\| \leq \|(T_0 - T_M) h\| + \|T_0 h\| < \|T_0 - T_M\| \frac{\varepsilon}{4\|T_0\|} \left( \frac{\varepsilon}{4\|T_0\|} + \frac{\varepsilon}{4} \right).
\]
Since for any \( M > 0 \) we have \( 0 \leq T_0 - T_M \leq T_0 \), it follows that \( \|T_0 - T_M\| \leq \|T_0\| \), see e.g. [GK]. Hence, for any \( M > 0 \)
\[
\|T_M (f - g)\| = \|T_M h\| < \frac{\varepsilon}{2}.
\]
Since \( T_M g \to 0 \) as \( M \to +\infty \) and since \( \varepsilon > 0 \) is arbitrary, it follows from this that \( T_M f \to 0 \) as \( M \to +\infty \).

Thus, we have a decreasing directed family of positive compact operators which converge to zero operator in strong operator topology. It is a well-known result then that this family converges to zero in norm which can be deduced e.g. from [DDP, Lemma 3.5] and the fact that the norm \( \|A\| \) of a compact operator coincides with its largest singular value \( s_1(A) \). \( \square \)

Lemmas 2.15 and 2.16 imply the following corollary.

Corollary 2.17. Let \( z \) be any non-real complex number. The operators
\[
FE_{(-\infty,-M)}(H) R_z(H) F^*
\]
and
\[
FE_{(M, +\infty)}(H) R_z(H) F^*
\]
converge to zero in norm as \( M \to +\infty \).

In order to omit the condition “\( F \) is bounded” we need the following lemma.

Lemma 2.18. Let \( F \) be an arbitrary possibly unbounded rigging. Then
\[
\|T_{\lambda + iy}(H_0)\| \to 0
\]
as \( y \to +\infty \).
Proof of this lemma is obvious, if the rigging operator $F$ is bounded, since in this case we have an estimate
\[
\|T_{\lambda+iy}(H_0)\| = \|FR_{\lambda+iy}(H_0)F^*\| \leq \|F\| \|R_{\lambda+iy}(H_0)\| \|F^*\| \leq \frac{1}{y} \|F\|^2.
\]
Moreover, in this case the convergence is uniform with respect to $H_0$.

Proof. We write
\[
T_{\lambda+iy}(H_0) = T_{\Delta}^\lambda(H_0) + T_{\Delta}^{\lambda+iy}(H_0),
\]
where $\Delta$ is a finite open interval which contains $\lambda$, $\Delta^c$ is the complement of $\Delta$ and
\[
T_{\lambda+iy}(H_0) = :FR_{\Delta}^\lambda(H_0)F^* = :FE_\Delta(H_0)R_{\lambda+iy}(H_0)F^*.
\]
Since for a compact $\Delta$ the operator $FE_\Delta$ is bounded, the norm of the term $T_{\Delta}^{\lambda+iy}(H_0)$ can be estimated by
\[
\|FE_\Delta(H_0)R_{\lambda+iy}(H_0)E_\Delta(H_0)F^*\| \leq \|R_{\lambda+iy}(H_0)\| \|FE_\Delta\|^2 \leq \frac{1}{y} \|FE_\Delta\|^2.
\]
Hence, this term converges to 0 as $y \to +\infty$.

Now Corollary 2.17 implies that for any fixed $y > 0$ the term $T_{\lambda+iy}^{\lambda+iy}(H_0)$ converges to zero as $\Delta \to \mathbb{R}$. This completes the proof. \qed

Proof of Theorem 2.13. As can be seen from the formula (2.1), Lemma 2.18 implies that $S(z, s)$ converges locally uniformly with respect to $s \in \mathbb{R}$ to the identity operator. This fact combined with unitarity of $S(z, s)$ also proves the second part of Theorem 2.13. \qed

2.4. Proof of Theorem 1. Given a continuous path $\gamma$ of unitary operators of the class “identity + compact operator” which ends at the identity operator, we denote by $\mu(\theta, \gamma)$ the spectral flow of eigenvalues of the path through the point $e^{i\theta}$. A continuous deformation of such a path does not change the $\mu$-invariant of the path provided that ends stay fixed, see in this regard e.g. [ADT]. Since for $y > 0$ and real $s$ the operator $S(\lambda + iy, s)$ is well-defined (this is because for non-real $z$ the operator $S(z, s)$ has no real resonance points), it is continuous in the rectangle \{$(y, s) : (y, s) \in [y_0, Y_0] \times [0, 1]$\}. Theorem 2.13 in fact shows that $S(z, s)$ is continuous in the rectangle \{$(y, s) : (y, s) \in [y_0, +\infty] \times [0, 1]$\}. This implies that the $\mu$-invariant $\mu(\theta, \lambda; H_1, H_0)$ is equal to the $\mu$-invariant of the path $(0, 1) \to (y_0, 1) \to (y_0, 0)$ with very small $y_0$. The difference between the $\mu$-invariant of this path and of the one which circumvents resonance and antiresonance points $r_{\lambda+iy}^1, \ldots, r_{\lambda+iy}^N$ and $\bar{r}_{\lambda+iy}^1, \ldots, \bar{r}_{\lambda+iy}^N$ of the group of $r_{\lambda}$ from above is equal to the sum of $S$-indices of the points from the upper half-plane. By Lemmas 2.11 and 2.12 this sum of $S$-indices is equal to the resonance index of $r_{\lambda}$.

Further, the $\mu$-invariant of the path which circumvents resonance and antiresonance points $r_{\lambda+iy}^1, \ldots, r_{\lambda+iy}^N$ from above does not change as $y \to 0^+$ (the resonance and anti-resonance points of the group of $r_{\lambda}$ converge to $r_{\lambda}$ but this point is circumvented). The path thus obtained can further be continuously deformed to the straight line path from $s = 1$ to $s = 0$ in the complex plane, since when $y = 0$ the function $f(y, r)$ is holomorphic in a neighbourhood of the real axis. Hence, the $\mu$-invariant of this path is equal to the absolutely continuous part of the $\mu$-invariant.

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