T-Dual Formulation of Yang-Mills Theory

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ABSTRACT

We introduce a self-dual field strength which replaces the gauge field in spontaneously broken Yang-Mills theory, reformulating it as a Lorentz covariant non-linear sigma model. This dualized theory is in both a unitary and renormalizable gauge: The self-dual field strength has exactly the three components necessary to describe a massive vector field. In future work we shall utilize this new formulation as a calculational tool in spontaneously broken gauge theories.
1. Introduction

In the recent past there has been much progress in improving the technical aspects of perturbative gauge theory calculations.¹ New Feynman rules derived from string theory (i.e. Bern-Kosower rules) [2], unitarity techniques, analyticity constraints on S-matrix elements [3,4], spinor helicity techniques [5], and color ordering [6,7] have been indispensable tools in avoiding much of the algebraic complexity typically found in the amplitude calculations.

In a recent work [8] we furthered this program by introducing a chiral formulation of minimally coupled fermion fields, found by integrating out half of the components of the Dirac spinor in the usual formulation. The resulting gauge covariantized theory, in which only the self-dual field strength enters explicitly, is

\[ \mathcal{L} = \psi^{\alpha} (\Box - m^2) \psi_{\alpha} + \psi^{\alpha} F_{\alpha}^{\beta} \psi_{\beta} . \]  

(1.1)

The indices \( \alpha, \beta \) are Weyl spinor indices, in which only one chirality enters in eq. (1.1). This action leads to a number of technical advantages in the calculation of scattering amplitudes involving external fermions. Two noteworthy features of the new Feynman rules are the elimination of much of the gamma matrix algebra directly within the Lagrangian in (1.1) and the introduction of a reference momentum for the external fermion line. Gauge theory amplitudes with the polarizations of the external lines closer to being the same are simpler than usual as can be seen from the spin independence of the Lagrangian when truncating to \( F_{\alpha\beta} = 0 \). [8].

In this work we perform an analogous manipulation for the gauge field. Pure Yang-Mills theory may be written using a symmetric (anti-) self-dual field \( G_{\alpha\beta} \) as [9]

\[ \mathcal{L} = \text{Tr} \left[ \frac{1}{2} G_{\alpha\beta} G_{\alpha\beta} + G_{\alpha\beta} F_{\alpha\beta} \right] , \]  

(1.2)

where upon functionally integrating out \( G_{\alpha\beta} \) we obtain the usual formulation of Yang-Mills theory. However, we may alternatively integrate out the gauge field \( A_{\alpha\dot{\beta}} \) in the presence of a Higgs effect, and the new Lagrangian may be used as a starting point to find further simplifications in gauge theory calculations. For example, the propagator

² These developments are summarized in a relatively concise form in [1].
in this case will be shown in a simultaneously unitary (no ghosts) and renormalizable
(i.e. $1/(p^2 + m^2)$) gauge as $G_{\alpha\beta}$ has precisely the number of components to describe a
massive vector field (i.e. 3). (This distinguishes it from earlier field strength formulations
that used the full non-self-dual field strength [10]; further work along these lines has also
previously been performed in three dimensions [11].) The action in (1.2), when dropping
the $L = \frac{1}{2} \text{Tr} \int d^4x \, G_{\alpha\beta} G_{\alpha\beta}$ component of the Lagrangian, gives rise to a self-dual field
theory perturbatively solved in [12].

We outline this work as follows. In section 2 we illustrate our technique first with
the Stueckleberg model followed by the abelian Higgs model. In section 3 we examine the
dualization of the non-abelian theory with couplings to matter. We close in section 4 with
a discussion of related and future work.

2. Abelian Dualization

In this section we take as a simplest case the Abelian vector theory and demonstrate
the dualization by exchanging the vector field $A_{\alpha\dot{\alpha}}$ with a(n) (anti-) self-dual field strength
$G_{\alpha\beta}$ (which is symmetric under the interchange of indices). The modified action becomes
non-polynomial in the matter fields. The specification of the external line factors for the
dualized vector field $G_{\alpha\beta}$ is singular in the massless limit, but leads to similar simplifica-
tions as one obtains through the use of the reference momenta in [5].

2.1 Stueckelberg Theory

The simplest example to consider is that of the Stueckelberg theory,

$$ L = \frac{1}{2} G_{\alpha\beta} G_{\alpha\beta} + G_{\alpha\beta} F_{\alpha\beta} + \frac{m^2}{2} A_{\alpha\dot{\alpha}} A_{\alpha\dot{\alpha}}, \quad (2.1) $$

where the self-dual field strength is defined by $F_{\alpha\beta} = \frac{1}{2} \partial_{(\alpha} \partial_{\beta)} A_{\dot{\gamma}}$. Integrating out the field
$G_{\alpha\beta}$ gives the Stueckelberg model. In the following we will examine this model and in the
next section the abelian Higgs system.

We dualize the vector potential by first varying the Lagrangian in (2.1) with respect
to the gauge field,

$$ \frac{\delta L}{\delta A_{\alpha\dot{\alpha}}} = 0 \quad \Rightarrow \quad A_{\alpha\dot{\alpha}} = \frac{1}{m^2} \partial_{\gamma} A_{\alpha\dot{\gamma}}. \quad (2.2) $$
The field strength $G_{\alpha\beta}$ appears here in the same way as the Hertz potential [13] except for the fact that it is self-dual. We use eq. (2.2) to eliminate the gauge field $A^{\alpha\dot{\alpha}}$ appearing in eq. (2.1). Performing this gives our dualized theory,

$$L = \frac{1}{2m^2}G^{\alpha\beta}(\Box - m^2)G_{\alpha\beta}.$$  \hspace{1cm} (2.3)

The Lagrangian for the massive vector field in eq. (2.3) is in a renormalizable and ghost-free gauge.

The same theory coupled as well to fermions through

$$L = i\overline{\psi}^{\dot{\alpha}}\nabla_{\dot{\alpha}\alpha}\psi^{\alpha} + m_\psi (\psi^{\alpha}\psi_{\alpha} + \overline{\psi}^{\dot{\alpha}}\overline{\psi}_{\dot{\alpha}}) ,$$
i.e., massive QED, gives us the field equation

$$-\partial_{\gamma}\dot{G}^{\gamma\alpha} + m^2 A_{\alpha\dot{\alpha}} + ie\overline{\psi}_{\dot{\alpha}}\psi_{\alpha} = 0 ,$$

and dualized theory,

$$L = \frac{1}{2}G^{\alpha\beta}(\Box - m^2)G_{\alpha\beta} + \overline{\psi}^{\dot{\alpha}}i\partial_{\dot{\alpha}\alpha}\psi^{\alpha} + m_\psi (\psi^{\alpha}\psi_{\alpha} + \overline{\psi}^{\dot{\alpha}}\overline{\psi}_{\dot{\alpha}})$$

$$+ \frac{e}{m}\overline{\psi}^{\dot{\alpha}}(\partial_{\dot{\gamma}}G^{\gamma\alpha})\psi^{\alpha} + \left(\frac{e}{m}\right)^2 \overline{\psi}^{\dot{\alpha}}\overline{\psi}_{\dot{\alpha}}\psi^{\alpha}\psi_{\alpha} ,$$

where we have rescaled the mass parameter out of the $G$ field through $G^{\alpha\beta} \rightarrow mG^{\alpha\beta}$. The covariant derivatives are defined by $\nabla_{\alpha\dot{\alpha}} = \partial_{\alpha\dot{\alpha}} - ieA_{\alpha\dot{\alpha}}$. We may alternatively use the abelian Higgs model to derive the couplings to fermions, which will be presented in the next section. Furthermore, it is interesting that in the dualized theory the coupling constant appears only in the combination $e/m$.

### 2.2 Abelian Higgs Model

In this section we examine massive vector fields through the Higgs mechanism. The general (first-order) form of the gauge theory Lagrangian possessing spontaneous symmetry breaking is

$$L = -\frac{1}{2}G^{\alpha\beta}G_{\alpha\beta} + iG^{\alpha\beta}F_{\alpha\beta} + \overline{\psi}^{\dot{\alpha}}i\nabla_{\alpha\dot{\alpha}}\psi^{\alpha} + (\nabla_{\alpha\dot{\alpha}}\phi)^* (\nabla^{\alpha\dot{\alpha}}\phi) + \frac{\lambda}{4!}(|\phi|^2 - v^2)^2 ,$$

(2.5)
We have introduced the complex scalar together with a potential used to generate spontaneous symmetry breaking. Also, we have rescaled the field $G_{\alpha\beta}$ by a factor of $i$ to eliminate extraneous factors of $i$ from appearing in the Feynman rules. We use the first order field equation for $A_{\alpha\dot{\alpha}}$, i.e. $\delta \mathcal{L} / \delta A_{\alpha\dot{\alpha}} = 0$, to eliminate the gauge field:

$$-i\partial_\gamma \dot{\phi} G^{\alpha\dot{\alpha}} - ei\phi^* \nabla^{\alpha\dot{\alpha}} \phi + ei\phi \nabla^{\alpha\dot{\alpha}} \phi^* - e\tilde{\phi} \psi = 0 \quad (2.6)$$

Using eq. (2.6) we may express the Lagrangian in eq. (2.5) in terms of the fields $G_{\alpha\beta}$ (as well as the matter fields $\phi$ and $\psi$). We first choose the gauge where $\phi = \phi^*$. We find upon substituting the field $A_{\alpha\dot{\alpha}}$ into the components of the Lagrangian the terms,

$$\mathcal{L} = -\frac{1}{2} G^{\alpha\beta} G_{\alpha\beta} + \partial_{\alpha\dot{\alpha}} \phi \partial^{\alpha\dot{\alpha}} \phi + \tilde{\psi} i \partial_{\alpha\dot{\alpha}} \psi^\alpha - \frac{1}{\phi^2} V^{\beta\dot{\gamma}} V_{\beta\dot{\gamma}} \quad (2.7)$$

where

$$V^{\beta\dot{\gamma}} = \frac{1}{e} i \partial_\beta \dot{\phi} G^{\alpha\beta} + \tilde{\psi} \gamma \psi \beta \quad .$$

We last expand around the true vacuum, $\phi = \tilde{\phi} + v$ in order to generate the Feynman rules for the broken gauge theory. The dualized theory in eq.(2.7) derived from the original $G_{\alpha\beta} F^{\alpha\beta}$ coupling is suitable for a Taylor expansion around the vacuum $\langle \tilde{\phi} \rangle = 0$.

The unbroken abelian theory is naively recovered from the massive one after taking $v = 0$. The Lagrangian is, after dropping the potential for the scalar field,

$$\mathcal{L} = -\frac{1}{2} G^{\alpha\beta} G_{\alpha\beta} + i G^{\alpha\beta} F_{\alpha\beta} + \tilde{\psi} \gamma i \nabla_{\alpha\dot{\alpha}} \psi^\alpha + (\nabla_{\alpha\dot{\alpha}} \phi)^* (\nabla^{\alpha\dot{\alpha}} \phi) \quad . \quad (2.8)$$

We specify the gauge $\phi = \phi^*$; the dualized Lagrangian is then

$$\mathcal{L}' = \partial_{\alpha\dot{\alpha}} \phi \partial^{\alpha\dot{\alpha}} \phi - \frac{1}{2} G^{\alpha\beta} G_{\alpha\beta} + \tilde{\psi} \gamma i \partial_{\alpha\dot{\alpha}} \psi^\alpha - \frac{1}{2\phi^2} \left[ \frac{1}{e} i \partial_\beta \dot{\phi} G^{\alpha\beta} + \tilde{\psi} \gamma \psi \beta \right]^2 \quad . \quad (2.9)$$

Although one may dualize the vector potential in this manner, the theory in eq.(2.9) does not admit a perturbative expansion because the scalar field enters through the denominator in the last term, and the action is both nonpolynomial and singular. In the previous example of a spontaneously broken gauge theory the singularity in $\phi$ is cured. In the spontaneously broken theory, the three components of the symmetric field $G_{\alpha\beta}$ naturally describe the degrees of freedom of a massive vector field.
We may formally define the scattering amplitudes in the unbroken theory by taking the vacuum value \( \langle \phi \rangle \) to zero in the amplitudes derived from eq. (2.9). Of course, one must be careful in defining this limit because one component of the three states modeled by \( G_{\alpha \beta} \) decouples in the massless limit. The actual amplitude calculations derived from the couplings in (2.7) for the massive vector field should limit appropriately to the massless case upon taking the scalar vacuum value to zero. The free field equation for the \( G_{\alpha \beta} \) is

\[
\left[ \square - (ev)^2 \right] G_{\alpha \beta} = 0 ,
\]

and becomes in the massless limit simply \( \square G_{\alpha \beta} = 0 \), i.e. that for a massless photon field. The components of the gauge field strength \( G_{\alpha \beta} \) describe in the massive case the three independent spin components of the massive vector; in the massless limit, we may explicitly solve for the free-field solutions describing the helicity states of the photon: We have from eq. (2.6) the zeroth order equation for \( A_{\alpha \dot{\alpha}} \) in terms of \( G_{\alpha \beta} \),

\[
A_{\alpha \dot{\alpha}} = \frac{1}{v^2} i \phi^{\beta \dot{\beta}} G_{\beta \alpha} + O(\psi, \phi) .
\]

Using, for example, the + helicity solution to the gauge field we may find the corresponding free-field state for \( G_{\alpha \beta} \) from eq. (2.11):

\[
A_{\alpha \dot{\alpha}}^{(+)}(k) = \frac{i q_\alpha k_\dot{\alpha}}{\langle q k \rangle} \rightarrow G_{\alpha \beta}^{(+)} = \frac{q_\alpha q_\beta}{\langle q k \rangle^2} .
\]

The two remaining states are found by orthogonality, e.g. \( G_{\alpha \beta}^{(+)} G_{\alpha \beta}^{(-)} = -1 \), and give for the negative and scalar helicity components of the \( G_{\alpha \beta} \)-field,

\[
G_{\alpha \beta}^{(-)} = -k_\alpha k_\beta \quad G_{\alpha \beta}^{(0)} = \frac{q_\alpha k_\beta}{\langle q k \rangle} .
\]

In deriving the massless limit from the massive theory in eq. (2.7) we should use the solutions to the massive field equation \( \left[ \square - (ev)^2 \right] G_{\alpha \beta} = 0 \) that limit to the \( \pm \) helicity states described above.

2.3 Feynman Rules - Abelian

The Feynman rules may be directly found from the Lagrangian in eqs. (2.7) by expanding around \( \bar{\phi} = 0 \). To order \( \bar{\phi}^2 \) we have
\[ L_k = -\frac{1}{2} G^{\alpha\beta} \left[ \square - (ev)^2 \right] G_{\alpha\beta} + \partial^{\dot{\alpha}} \partial_{\alpha\dot{\alpha}} \phi + \bar{\psi}^{\dot{\alpha}} i \partial_{\alpha\dot{\alpha}} \psi^{\alpha} \]

\[ L_i = V(\tilde{\phi} + v) - \frac{1}{2} (\partial_{\alpha} \tilde{\phi} G^{\alpha\beta}) (\partial_{\beta} G_{\mu\beta}^{\mu}) \left[ 2 \frac{\tilde{\phi}}{v} - \left( \frac{\tilde{\phi}}{v} \right)^2 \right] \]

\[ + \left[ \frac{1}{v} (i \partial_{\alpha} \tilde{\phi} G^{\alpha\beta}) (\bar{\psi}^{\dot{\alpha}} \psi_{\beta}) + \frac{1}{v^2} \bar{\psi}^{\alpha} \psi_{\dot{\alpha}} \bar{\psi}^{\dot{\alpha}} \psi_{\alpha} \right] \left\{ 1 + 2 \frac{\tilde{\phi}}{v} - \left( \frac{\tilde{\phi}}{v} \right)^2 \right\} . \quad (2.14) \]

We have rescaled the field strength \( G^{\alpha\beta} \rightarrow evG^{\alpha\beta} \) in obtaining the Lagrangian in eq. (2.14). It may appear that there are terms of the form \( \partial_{\alpha} \tilde{\phi} G^{\alpha\beta} \partial_{\beta} \tilde{\phi} \) in the Taylor expansion of eq. (2.7); however, these are trivially zero after integrating by parts.

The dualized form above in (2.7) generates around the vacuum all of the interactions obtained from the theory in eq.(2.5), albeit through an infinite number of terms (as in a non-linear sigma model, or super Yang-Mills in superspace). The field redefinition in eq. (2.6) has shuffled many interactions into the definition of the \( G \)-field. The manipulations we have performed at the level of the action is similar to the ones performed in [12] for the case of fermions; to any given order there are fewer contributing terms to a Feynman graph and we expect the perturbative calculations of amplitudes to be simpler [14].

Several of the Feynman rules will be needed in the following sections; we list them in the following. The propagator for the \( G^{\alpha\beta} \) field gives,

\[ \langle G^{\alpha\beta}(p)G_{\mu\nu}(-p) \rangle = \frac{1}{p^2 + (ve)^2} (C^{\alpha\mu}C^{\beta\nu} + C^{\alpha\nu}C^{\beta\mu}) . \quad (2.15) \]

The three-point vertices from the fermion contribution in (2.7) and (2.14) are of the form:

\[ \langle \bar{\psi}^{\dot{\alpha}}(k_1)\psi_{\mu}(k_2)G_{\alpha\beta}(k_3) \rangle = \frac{v}{2} (k_1 - k_2) \bar{\psi}^{\dot{\alpha}}(\alpha C_{\beta})_{\mu} \]

\[ \langle \bar{\phi}(k_1)G_{\mu\nu}(k_2)G_{\alpha\beta}(k_3) \rangle = \frac{1}{v} k_{2,\alpha} \gamma_{\beta k_3,\gamma_{\mu}} . \quad (2.16) \]

We may absorb the factor of \( v \) into the field \( G_{\alpha\beta} \) giving a form for the couplings more similar to the ones derived from the Stuckelberg model in (2.1). The remaining four-point couplings are easily obtained from the expanded terms in eq. (2.14).
3. Non-Abelian Dualization

In this section we extend the previous analysis to a non-abelian Yang-Mills theory coupled to matter. For simplicity we will in this section consider only the case where all vector bosons acquire masses, hence we will dualize the entire set. The Lagrangian is

\[ \mathcal{L} = \text{Tr} \left( \frac{1}{2} G^{\alpha \beta} G_{\alpha \beta} + G^{\alpha \beta} F_{\alpha \beta} \right) + \bar{\psi}^\alpha i \nabla^\alpha \psi^\alpha + (\nabla^\alpha \phi)^\dagger (\nabla^\alpha \phi) + V(\phi) , \]  

where the self-dual field strength is defined by \( F^a_{\alpha \beta} = \partial^\beta (\partial^\alpha A^a_{\alpha \beta}) + ig f_{abc} A^b_{\beta \mu} T^c \) and \( \nabla^\psi = \partial^\alpha + g A^a_{\alpha \beta} T^a \), \( \nabla^\phi = \partial^\alpha + g A^a_{\alpha \beta} T^a \),

are the representation dependent covariant derivatives.

We consider the dual to the general form of the theory above in eq. (3.1). The field equation for \( A^a_{\alpha \beta} \) gives,

\[ \partial^\rho G^\alpha_{\rho \beta} + ig G^\alpha_{\beta \rho} A^a_{\rho \mu} f_{a}^{bc} + ig \bar{T}^\psi T^\psi \psi^\alpha + g(\nabla^\alpha \phi)^\dagger (\nabla^\alpha \phi) + g(\nabla^\alpha \phi)^\dagger (\nabla^\alpha \phi) = 0 . \]  

Using the field equation, or alternatively, directly integrating out the gauge field in the Lagrangian in eq. (3.1) gives the dual formulation

\[ \mathcal{L} = V^a_{\beta \gamma} X^{-1} X^{\gamma} \phi + \partial^\alpha \phi + g^2 G^{a, \alpha \beta} G_{a, \alpha \beta} + \bar{T}^\psi T^\phi \phi + g T^\phi \phi \]  

The matrix \( X \) is defined by

\[ X^{ab}_{\alpha \beta} = g f_{c}^{ab} G^c_{\alpha \beta} + g^2 T^a \phi \{ T^a , T^b \} \phi C_{\alpha \beta} , \]  

and the vector \( V \) by,

\[ V^a_{\beta \gamma} = \partial^\gamma G^a_{\beta \mu} + g \bar{T}^\psi T^\psi \phi + g T^\phi \phi \]  

with \( C^{\alpha \beta} \) and \( \eta^{ab} \) used to raise Weyl and group indices. We may power expand the dual theory in eq. (3.4) to obtain the Feynman rules describing the broken phase. The non-abelian spontaneously broken theory will be considered in detail in [14], where it will be used to generate in an efficient manner amplitudes containing massive vectors.
4. Discussion

In this work we have derived from a first-order formulation of a gauge theory its dualized form; the gauge potential is effectively replaced by a self-dual field strength. These dual theories are well-defined and admit perturbative expansions only in the case of spontaneous symmetry breaking; the dual to the unbroken gauge theory is singular. The dual formulation expresses Yang-Mills theory as a Lorentz covariant non-linear sigma model unlike the Yang formulation which expresses the original gauge field (after gauge fixing) in non-polynomial scalar form for a self-dual theory [15].

These dual theories have a number of interesting applications. First, the fact that the dual to the gauge theory lacks the gauge connection is interesting for problems involving non-trivial solutions that require more than one patch (i.e. Dirac monopole). Second, it is interesting that the dual description to Yang-Mills theory may be understood as a non-linear sigma model. In further work we shall use this dual formulation as a tool in doing perturbative calculations involving massive vector bosons [14].

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