CONFINEMENT-DECONFINEMENT TRANSITIONS FOR TWO-DIMENSIONAL DIRAC PARTICLES

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ABSTRACT. We consider a two-dimensional massless Dirac operator coupled to a magnetic field $B$ and an electric potential $V$ growing at infinity. We find a characterization of the spectrum of the resulting operator $H$ in terms of the relation between $B$ and $V$ at infinity. In particular, we give a sharp condition for the discreteness of the spectrum of $H$ beyond which we find dense pure point spectrum.

1. Introduction

Graphene, a two dimensional lattice of carbon atoms arranged in a honeycomb structure, has attracted great attention in the last few years due to its unusual properties [1, 13]. The dynamics of its low-energy excitations (the charge carriers) can be described by a two-dimensional massless Dirac operator $D_0$ [23, 6], where the speed of light $c$ is replaced by the Fermi velocity, $v_F \sim 10^{-2} c$. A remarkable property of these Dirac particles is their lack of localization in the presence of electric potential walls (i.e., potentials $V$ with $V(x) \to \infty$ as $|x| \to \infty$). Indeed, if we assume that $V$ is rotationally symmetric and of ‘regular growth’ the spectrum of the operator $D_0 + V$ equals the whole real line and is absolutely continuous [21, 15, 8, 17]. It is also known, at least in three dimensions, that a much larger class of potentials growing at infinity do not produce eigenvalues [22, 9].

One way to localize Dirac particles (in the sense that the Hamiltonian has non-trivial discrete spectrum) is through inhomogeneous magnetic fields when they are asymptotically constant [8, 11] as well as when they grow to infinity. Consider, for instance, a magnetic field $B = \text{curl} A$ with $B(x) \to \infty$ as $|x| \to \infty$ and denote by $D_A$ the corresponding Dirac operator coupled to $B$. It is known that the spectrum of $D_A$ is discrete away from zero and zero is an isolated eigenvalue of infinite multiplicity (see Section 5).

In this article we consider two-dimensional massless Dirac operators coupled to both an electric potential $V$ and a magnetic field $B$. We study the combination of the two effects described above: The deconfinement effect associated to $V$ and the confinement one associated to $B$.

Before presenting our main results let us first discuss this problem assuming that $V$ and $B$ are sufficiently regular positive rotationally symmetric functions. In this case, the Dirac operator admits an angular momentum decomposition $D_A + V =$
\[ \bigoplus_{j \in \mathbb{Z}} h_j \] and its spectrum, \( \sigma(D_A + V) \), satisfies
\[ \sigma(D_A + V) = \bigcup_{j \in \mathbb{Z}} \sigma(h_j). \]

Let \( A(r) = \frac{1}{2} \int_0^r B(s)ds \) be the modulus of the magnetic vector potential in the rotational gauge. It is easy to show, on the one hand, that if \( A(r) \to \infty \) as \( r \to \infty \) and
\[ \lim_{|x| \to \infty} \frac{V(|x|)}{A(|x|)} < 1, \]
the spectrum of \( h_j \) is discrete, for each \( j \in \mathbb{Z} \) (see Proposition 1 in the appendix for the precise statement). On the other hand, as opposed to the non relativistic case, it is known [18, Proposition 2] that if \( V(r) \to \infty \) as \( r \to \infty \) and
\[ \lim_{|x| \to \infty} \frac{V(|x|)}{A(|x|)} > 1, \]
the spectrum of each \( h_j \) equals the whole real line and is purely absolutely continuous. This phenomenon was recently discussed from the physical point of view in [7]. In that article a device was proposed to control the localization properties of particles in graphene by manipulating the electro-magnetic field at infinity, i.e., far away from the sample.

Clearly, if condition (2) is satisfied the spectrum of the full operator \( H := D_A + V \) is also absolutely continuous and equals \( \mathbb{R} \). Conversely, one has pure point spectrum if condition (1) holds. It is, however, unclear whether the eigenvalues of \( H \) accumulate. Assume that \( V(x), B(x) \to \infty \) as \( |x| \to \infty \). One may expect that when the quotient \( |V(|x|)/A(|x|)| \) is sufficiently small, for large \( |x| \), the main effect of the electric potential is to remove the zero modes of \( D_A \) yielding purely discrete spectrum for \( H \). However, as this quotient grows the eigenvalues of the \( h_j \) might accumulate creating points in the essential spectrum of \( H \).

The aim of our work is to shed some light on the spectrum of \( H \) in terms of the relation between \( B \) and \( V \) at infinity. We emphasize that most of our results do not assume rotational symmetry. In fact, besides some regularity conditions on \( B \) and \( V \) we only require that the potential \( V \) grows subexponentially fast.

Let us describe our results disregarding technical assumptions (for the precise statements see Section 2). We show (see Theorem 1) that the spectrum of \( H \) is discrete if
\[ \limsup_{|x| \to \infty} \frac{V^2(x)}{2|B(x)|} < 1 \]
and moreover that this condition is sharp in the following sense: If the quotient in (3) converges to 1 along a sequence \( (x_n)_{n \in \mathbb{N}} \subset \mathbb{R}^2 \) with \( \lim_{n \to \infty} |x_n| = \infty \) the essential spectrum of \( H, \sigma_{\text{ess}}(H) \), is not empty. In fact, we prove (see Theorem 3) that zero belongs to the essential spectrum of \( H \) if, for some natural number \( k \), the quotient \( V^2(x_n)/2\Gamma|B(x_n)| \) converges to \( k \) sufficiently fast, as \( n \to \infty \). In addition, we find (see Corollary 1) that the essential spectrum of \( H \) covers the whole real line if there is a continuous path \( \gamma : \mathbb{R}^+ \to \mathbb{R}^2 \) with \( |\gamma(t)| \to \infty \) as \( t \to \infty \) such that \( V^2(\gamma(t))/2B(\gamma(t)) \) converges to infinity with moderate speed (see Remark 5), as \( t \to \infty \).

In order to get a better picture of our results consider the example when \( V(x) = V_0|x|^t \) and \( B(x) = B_0|x|^s \) for some constants \( V_0, B_0, t > 0 \) and \( s \geq 0 \). In this case
Thus, according to Proposition 1 and [18, Proposition 5.1] (see also (1) and (2)) we have that

(a) \( \sigma(H) \) is pure point if either \( t < s + 1 \), or \( t = s + 1 \) and \( V_0 < B_0/(s + 2) \).

(b) \( \sigma(H) = \mathbb{R} \) is purely absolutely continuous if either \( t > s + 1 \), or \( t = s + 1 \) and \( V_0 > B_0/(s + 2) \).

From Theorems 1 and 3 and Corollary 1 (see Example 1) we conclude that

(c) \( \sigma_{\text{ess}}(H) = \emptyset \) if either \( t < s/2 \), or \( t = s/2 \) and \( V_0^2 < 2B_0 \).

(d) \( 0 \in \sigma_{\text{ess}}(H) \) if \( t = s/2 \) and \( V_0^2 = 2kB_0 \) for some \( k \in \mathbb{N} \).

(e) \( \sigma(H) = \mathbb{R} \) is pure point if \( t \in (s/2, (s + 1)/2) \).

Hence, if we fix \( B \) and increase the strength of \( V \) (at infinity), we observe a transition from purely discrete (c) to purely absolutely continuous spectrum (b) passing through dense pure point spectrum (e). In other words, we observe a transition between a strongly confined system, in the sense that the energy required to ‘bring a particle to infinity’ is not finite, to a completely deconfined system. Between these two regimes there is another one (e) belonging to a (presumably) weaker form of confinement.

The organization of this article is as follows: In the next section we state our main results precisely. We recall some basic facts about magnetic Dirac operators in Section 3. In Section 4 we prove some useful commutator estimates used in the proof of Theorem 1 which is given in Section 5. Theorem 2 is proven in Section 6 and the proofs of Theorems 3 and 4 are given in Section 7. The main text is followed by an appendix containing some auxiliary results.

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2. Main results

Assume that \( V \) and \( B \) are continuous functions on \( \mathbb{R}^2 \). We define the two-dimensional massless Dirac operator coupled to a magnetic field \( B \) on \( \mathcal{H} := L^2(\mathbb{R}^2, \mathbb{C}^2) \) a priori as

\[
D_A \varphi := \sigma \cdot (-i \nabla - A) \varphi, \quad \varphi \in C_0^\infty(\mathbb{R}^2, \mathbb{C}^2),
\]

where \( A = (A_1, A_2) \in C^1(\mathbb{R}^2, \mathbb{R}^2) \) satisfies \( B = \text{curl} A := \partial_1 A_2 - \partial_2 A_1 \) and \( \sigma = (\sigma_1, \sigma_2) \) with

\[
\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}.
\]

Similarly we define

\[
H \varphi := (D_A + V) \varphi, \quad \varphi \in C_0^\infty(\mathbb{R}^2, \mathbb{C}^2).
\]

In view of (2) \( D_A \) and \( H \) are essentially self-adjoint. We denote the self-adjoint extensions of \( D_A \) and \( H \) by the same symbols and their domains by \( \mathcal{D}(D_A) \) and \( \mathcal{D}(H) \) respectively.
Theorem 1. Let $V \in C^1(\mathbb{R}^2, \mathbb{R})$, $B \in C(\mathbb{R}^2, \mathbb{R})$, and $A \in C^1(\mathbb{R}^2, \mathbb{R}^2)$ such that $B = \text{curl} A$ on $\mathbb{R}^2$. Assume that

$$|V(x)| \to \infty \quad \text{as} \quad |x| \to \infty,$$

$$\left| \frac{\nabla V(x)}{V(x)} \right| \to 0 \quad \text{as} \quad |x| \to \infty,$$

$$\limsup_{|x| \to \infty} \frac{V^2(x)}{2|B(x)|} < 1.$$ 

Then $(H + i)^{-1}$ is compact in $\mathcal{H}$, i.e., $\sigma_{\text{ess}}(H) = \emptyset$.

Remark 1. A similar result was obtained in [19]. However, there the statement was only proved when the limit superior in [9] is strictly smaller than $1/4$ instead of $1$. Our proof is quite different from the one given in [19]. We split the analysis on the spaces $\ker D_A$ and $\ker D_{\overline{A}}$ and estimate the cross terms with the commutator bounds derived in Section 4.

The next two theorems state that the constant 1 in (8) above is in fact sharp.

Theorem 2. Let $V \in C^1(\mathbb{R}^2, \mathbb{R})$, $B \in C^2(\mathbb{R}^2, \mathbb{R})$, with $|B| > B_0$ for some $B_0 > 0$ and $A \in C^3(\mathbb{R}^2, \mathbb{R}^2)$ such that $B = \text{curl} A$ on $\mathbb{R}^2$. Assume that

$$|V(x)| \to \infty \quad \text{as} \quad |x| \to \infty,$$

$$\|\nabla V/V\|_\infty < \infty \quad \text{and} \quad \|\Delta B/B\|_\infty < \infty,$$

$$\|\nabla^2 - |B|/V\|_\infty < \infty.$$ 

Then $\sigma_{\text{ess}}(H) \neq \emptyset$.

Remark 2. Note that conditions [19] and [11] are equivalent to

$$\left| \frac{V^2(x)}{2|B(x)|} - 1 \right| \leq \frac{c}{|V(x)|} \to 0 \quad \text{as} \quad |x| \to \infty,$$

for some constant $c > 0$.

Remark 3. The statement of Theorem 2 can also be obtained assuming that

$$\liminf_{|x| \to \infty} |x|^2(B(x) - \frac{1}{2}\Delta B(x)/B(x)) > 0 \quad \text{instead of [19]} \quad \text{and only that} \|\nabla V/V\|_\infty < \infty \quad \text{instead of [11]}.$$

In order to state the next two results we use the following definition: We say that a function $f : \mathbb{R}^2 \to \mathbb{R}$ varies with rate $\nu \in [0, 1]$ at infinity if there are constants $R > 1$ and $C > 0$ such that for any $\alpha : \mathbb{R}^2 \to \mathbb{R}^2$ with $\alpha(x) = o(|x|^{\nu})$ as $|x| \to \infty$ the function $f$ satisfies the bound

$$|f(x + \alpha(x))| \leq C|f(x)|, \quad \text{for all} \quad |x| > R.$$ 

Clearly, if $f$ varies with rate $\nu \in [0, 1]$ then it also varies with rate $\nu'$ at infinity for all $\nu' \in [0, \nu]$. Note also that power functions of $|x|$ with positive power vary with any rate $\nu \in [0, 1]$.

Theorem 3. Let $B, V \in C^1(\mathbb{R}^2, \mathbb{C})$ such that $|\nabla V|, |\nabla B|$ vary with rate 0 at infinity. Let $A \in C^2(\mathbb{R}^2, \mathbb{R}^2)$ such that $B = \text{curl} A$ on $\mathbb{R}^2$. Assume that there is a sequence $(x_n)_{n \in \mathbb{N}}$ with $|x_n| \to \infty$ as $n \to \infty$ and constants $k \in \mathbb{N}$ and $\varepsilon \in (0, 1)$
such that, as $n \to \infty$,

\begin{align}
\frac{\nabla B(x_n)}{|B(x_n)|^{1-\varepsilon}} & \to 0, \\
\frac{\nabla V(x_n)}{|V(x_n)|^{1-\varepsilon}} & \to 0, \\
\frac{V^2(x_n) - 2k|B(x_n)|}{V(x_n)} & \to 0.
\end{align}

Then $0 \in \sigma_{\text{ess}}(H)$.

**Remark 4.** From (13) and (15) it follows that $|B(x_n)| \to \infty$ as well as

$$\left| \frac{V^2(x_n)}{2|B(x_n)|} - k \right| \leq \frac{c}{|V(x_n)|} \to 0 \quad \text{as} \quad |x| \to \infty,$$

for some constant $c > 0$. Note, however, that the converse is not in general true.

We also remark that, under somewhat different assumptions, Theorem 3 (with $k = 1$) improves the statement of Theorem 2. The proof of Theorem 3 is based on the construction of an infinite dimensional subspace of the operator domain on which $H$ stays bounded. This space is constructed using the zero-modes of the operator $D_A$. The proof of Theorem 3 is, in contrast, based on the construction of a Weyl sequence of functions localized around points $(x_n)_{n \in \mathbb{N}}$ where $|V(x_n)| \approx \sqrt{k/B(x_n)}$, i.e., the points where the potential $V$ has the same value as the $k$-th Landau-level of the magnetic Dirac operator with constant field $B(x_n)$. This idea of $V$ crossing through the (local) Landau-levels can also be used in the case when $V^2(x)/2B(x) \to \infty$ as $|x| \to \infty$ (along some sequence) to obtain the following result.

**Theorem 4.** Let $B, V \in C^1(\mathbb{R}^2, \mathbb{C})$ such that $|\nabla V|$ and $|\nabla B|$ vary with rate $\nu \in [0, 1]$ at infinity. Let $A \in C^1(\mathbb{R}^2, \mathbb{R}^2)$ such that $B = \text{curl} A$ on $\mathbb{R}^2$. Assume that there is a sequence $(x_n)_{n \in \mathbb{N}}$ with $|x_n| \to \infty$ as $n \to \infty$ and constants $\varepsilon, \alpha, \kappa, B_0 > 0$ such that, as $n \to \infty$,

\begin{align}
\frac{V^2(x_n) - 2n|B(x_n)|}{V(x_n)} & \to 0, \\
\frac{\nabla B(x_n)}{B(x_n)} \left( \frac{V^2(x_n)}{2|B(x_n)|} \right)^{1+\varepsilon} & \to 0, \\
\frac{\nabla V(x_n)}{V(x_n)} \left( \frac{V^2(x_n)}{2|B(x_n)|} \right)^{1+\varepsilon} & \to 0,
\end{align}

and furthermore, for all $n \in \mathbb{N}$,

\begin{align}
B_0 \leq |B(x_n)| & \leq \alpha \left( \frac{V^2(x_n)}{2|B(x_n)|} \right) \kappa.
\end{align}

Then $0 \in \sigma_{\text{ess}}(H)$.

**Remark 5.** Due to (16) and (19) we have that $|V(x_n)| \to \infty$ and moreover

$$\left| \frac{V^2(x_n)}{2n|B(x_n)|} - 1 \right| \leq \frac{c}{|V(x_n)|} \to 0 \quad \text{as} \quad |x| \to \infty,$$

for some constant $c > 0$. Observe that conditions (17) and (18) give an upper bound for the growth of the ratio $V^2(x_n)/|2B(x_n)|$. 


Note in addition that the theorem is also applicable for bounded magnetic field. In this case the condition \((15)\) can only be satisfied for \(\nu > 0\).

**Remark 6.** It is easy to see that the regularity conditions on \(V\) and \(B\) in theorems \((8)\) and \((9)\) can be weakened to hold only outside some compact set \(K \subset \mathbb{R}^2\). Inside \(K\) it is sufficient that these functions are bounded. The same holds true for Theorem \((7)\) (compare with Lemma \((7)\).

The theorem above can also be used to find other points in the essential spectrum of \(H\). To this end it suffices to find a sequence \((x_n)_{n \in \mathbb{N}}\) satisfying the conditions of the theorem for \(V-E\) instead of \(V\). As an example we get the following result.

**Corollary 1.** Let \(B, V \in C^1(\mathbb{R}^2, \mathbb{R})\) such that \(|\nabla V|\) and \(|\nabla B|\) vary with rate \(s \in [0, 1]\) at infinity. Let \(A \in C^1(\mathbb{R}^2, \mathbb{R}^2)\) such that \(B = \text{curl} A\) on \(\mathbb{R}^2\). Assume that there is a continuous path \(\gamma : \mathbb{R}^+ \to \mathbb{R}^2\) with \(|\gamma(t)| \to \infty, as \ t \to \infty, such that\)

\[
\frac{V^2(\gamma(t))}{2B(\gamma(t))} \to \infty \quad as \quad t \to \infty.
\]

Moreover, assume that the conditions \((17)\), \((18)\), and \((19)\) are satisfied for any sequence \((x_n)_{n \in \mathbb{N}}\) in the range of \(\gamma\) with \(|x_n| \to \infty\). Then \(\sigma(H) = \mathbb{R}\).

For completeness we give a proof of this corollary.

**Proof.** Let \(E \in \mathbb{R}\). Due to \((20)\) and the continuity of \(V^2/B\) along \(\gamma\) we find a constant \(N > 0\) and a sequence \((x_n)_{n \in \mathbb{N}}\), on the range of \(\gamma\) with \(|x_n| \to \infty\), such that

\[
(V-E)^2(x_n) = 2nB(x_n), \quad \text{for all} \quad n > N.
\]

Since \(|V(x_n)| \to \infty\) as \(n \to \infty\) it is clear that the conditions \((17)\), \((18)\), and \((19)\) are also satisfied for \(V(x_n) - E\) instead of \(V(x_n)\). This implies the claim. \(\square\)

**Remark 7.** As we already mentioned in the Introduction one can combine Corollary \((7)\) and Proposition \((7)\) from the Appendix for functions \(V\) and \(B\) that are rotationally symmetric. In this case one obtains that \(\sigma(H) = \mathbb{R}\) is pure point, i.e., \(H\) has dense pure point spectrum (see Example \((7)\) below). Note that the same type of spectral phenomenon occurs for \(\sigma(D_A)\) when \(B\) is rotationally symmetric and decays at infinity but \(B(x)|x| \to \infty\) as \(|x| \to \infty\). (see also \((20)\) pp. 208).

Let us now apply Corollary \((7)\) to some particular cases.

**Example 1.** Let \(V(x) = V_0|x|^{t}\) and \(B(x) = B_0|x|^s\) for some constants \(V_0, B_0 > 0\) and \(0 \leq s < 2t\). Then \(|\nabla V|\) and \(|\nabla B|\) vary with any rate \(\nu \in [0, 1]\) at infinity and we have that \(V^2(x)/(2B(x)) \to \infty\) as \(|x| \to \infty\). Condition \((17)\) is satisfied for any sequence if and only if

\[
\frac{1}{|x|} \left( \frac{V^2(x)}{2|B(x)|} \right)^{1+\varepsilon} \to 0 \quad as \quad |x| \to \infty,
\]

which is the case whenever \(2t < s+1\). For these exponents \((18)\) and \((19)\) are clearly fulfilled for any sequence which tends to infinity. Hence, Corollary \((7)\) states that for \(0 \leq s < 2t < s+1\) we have \(\sigma(H) = \mathbb{R}\). Furthermore, in view of Proposition \((7)\) we get that the spectrum in this case is pure point.


3. Supersymmetry and zero modes

In this section we recall some basic facts about magnetic Dirac operator which are going to be useful in our proofs. As we mentioned in the Section 2 the operator $D_A$ defined in (1) is essentially self-adjoint on $C_0^\infty(\mathbb{R}^2, \mathbb{C})$. We write

\begin{equation}
D_A := \begin{pmatrix} 0 & d^* \\ d & 0 \end{pmatrix},
\end{equation}

where

\[ d := \frac{p_1 - A_1 + i(p_2 - A_2)}{\sqrt{2}} \big|_{C_0^\infty(\mathbb{R}^2, \mathbb{C})} \]

and $p_j = -i \partial_j$, $j = 1, 2$, is the momentum operator in the $j$-th direction. The operators $d, d^*$ satisfy the commutation relation

\begin{equation}
[d, d^*] := (dd^* - d^*d)\varphi = 2B\varphi, \quad \varphi \in \mathcal{D}(d^*d) \cap \mathcal{D}(dd^*).
\end{equation}

We now investigate further the relation between $dd^*$ and $d^*d$. We note that the of kernel $D_A$ fulfills $\ker(D_A) = \ker(d) \oplus \ker(d^*)$. Due to the matrix structure of $D_A$ we have that

\begin{equation}
\text{sgn}(D_A) := \frac{D_A}{|D_A|} = \begin{pmatrix} 0 & s^* \\ s & 0 \end{pmatrix}
\end{equation}

on $\ker(D_A)^\perp$,

with the unitary operators

\begin{equation}
s : \ker(d)^\perp \to \ker(d^*)^\perp, \quad s^* : \ker(d^*)^\perp \to \ker(d)^\perp
\end{equation}

(see [20] Section 5.2.3 for a related discussion). Using the operator identity $D_A^2 = \text{sgn}(D_A)D_A sgn(D_A)$ we find, for any $\varphi = (\varphi_1, \varphi_2)^T$ with $\varphi_1 \in \mathcal{D}(d^*d) \cap \ker(d)^\perp$ and $\varphi_2 \in \mathcal{D}(dd^*) \cap \ker(d^*)^\perp$, that

\begin{equation}
\begin{pmatrix} d^*d & 0 \\ 0 & dd^* \end{pmatrix} \varphi = \begin{pmatrix} s^*dd^*s & 0 \\ 0 & sd^*s^* \end{pmatrix} \varphi.
\end{equation}

We now recall some results concerning the structure of the kernel of $D_A$. For a given $B \in C(\mathbb{R}^2, \mathbb{R})$ one finds a weak solution $\phi \in C^1(\mathbb{R}^2, \mathbb{R})$ of the equation

\[ \Delta \phi = B, \]

see, e.g. [9] where a much larger class of fields $B$ is considered. (Using standard elliptic regularity it is easy to see that if $B$ belongs to some Hölder class the solution $\phi \in C^2(\mathbb{R}^2, \mathbb{R})$). A direct computation yields

\begin{equation}
\ker(d^*d) = \{ \omega e^{-\phi} \in L^2(\mathbb{R}^2, \mathbb{C}) \mid \omega \text{ is entire in } x_1 + i x_2 \}.
\end{equation}

Moreover, it is known [10] that whenever

\begin{equation}
\int_{\mathbb{R}^2} [B(x)]_+ dx = \infty, \quad \int_{\mathbb{R}^2} [B(x)]_- dx < \infty,
\end{equation}

zero is an eigenvalue of infinite multiplicity of $d^*d$, i.e., $\dim \ker(d^*d) = \infty$.

With these observations we can easily go through the following example that will be useful later on.

**Lemma 1.** Let $B \in C(\mathbb{R}^2, \mathbb{R})$ such that $B \geq B_0 > 0$ and $A \in C^1(\mathbb{R}^2, \mathbb{R}^2)$ with $B = \text{curl} A$. Then 0 is an isolated eigenvalue of infinite multiplicity of $D_A$. In addition,

\begin{equation}
\ker(D_A) = \left\{ \begin{pmatrix} \Omega \\ 0 \end{pmatrix} \in L^2(\mathbb{R}^2, \mathbb{C}^2) \mid \Omega \in \ker(d^*d) \right\}.
\end{equation}
Moreover,
\[ (-\sqrt{2B_0}, 0) \cup (0, \sqrt{2B_0}) \subset \varrho(D_A), \]
where \( \varrho(D_A) \) denotes the resolvent set of \( D_A \).

**Proof.** Due to (23) we have the operator inequality
\[ \dd^* \geq 2B \geq 2B_0. \]
Hence, \( \ker(d^*) = \ker(\dd^*) = \{0\} \) and we find
\[ \ker(D_A) = \ker(d) \oplus \ker(d^*) = \ker(d^*d) \oplus \{0\}. \]
This yields (29). Moreover, using again (30), the isospectrality
\[ \sigma(d^*d) \setminus \{0\} = \sigma(\dd^*) \setminus \{0\}, \]
together with
\[ D_A^2 \varphi = \begin{pmatrix} d^*d & 0 \\ 0 & dd^* \end{pmatrix} \varphi, \quad \varphi \in C^\infty_0(\mathbb{R}^2, \mathbb{C}^2), \]
imply that \( (0, 2B_0) \subset \varrho(D_A^2) \). Therefore, by the spectral theorem, we find the desired spectral gap. \( \square \)

**Remark 8.** If we assume further that \( B(x) \to \infty \) as \( |x| \to \infty \) we have by \( [8] \) that
the spectrum of \( D_A \) is discrete away from zero. Since \( B \) fulfills in this case (28)
this implies that \( \sigma_{\text{ess}}(D_A) = \{0\} \).

### 4. Useful commutator estimates

We denote by \( P_0 \) the orthogonal projection onto \( \ker(D_A) \) and set \( P_0^\perp := 1 - P_0 \).
In this section we show some commutator bounds between the electric potential \( V \), \( P_0 \), and the sign of \( D_A \) denoted by \( \text{sgn}(D_A) \). We use these bounds in Section 5 to show Theorem 1.

Throughout this section we use the following notation: For \( 0 < V \in C^1(\mathbb{R}^2, \mathbb{R}) \)
such that \( \|\nabla V/V\|_\infty < \infty \) we set
\[ T := \frac{i\sigma \cdot \nabla V}{V}. \]
Note that \( T \) formally equals \([D_A, V^{-1}]V\), where \([,] \) is the symbol for the commutator. We define for \( z \in \varrho(D_A) \)
\[ R_A(z) := (D_A - z)^{-1}. \]

**Lemma 2.** Let \( 0 < V \in C^1(\mathbb{R}^2, \mathbb{R}), B \in C(\mathbb{R}^2, \mathbb{R}), \) and \( A \in C^1(\mathbb{R}^2, \mathbb{R}^2) \) with \( B = \text{curl}A \). Assume further that \( \|V^{-1}\|_\infty, \|\nabla V/V\|_\infty < \infty \) and that \( z \in \varrho(D_A) \) is such that \( \|TR_A(z)\| < 1 \). Then
\[ [R_A(z), V^{-1}] = -V^{-1}R_A(z)TR_A(z)\Theta_r(z) \]
\[ = \Theta_l(z)R_A(z)TR_A(z)V^{-1}, \]
where
\[ \Theta_r(z) := (1 + TR_A(z))^{-1}, \]
\[ \Theta_l(z) := (1 - R_A(z)T)^{-1}, \]
and
\[
\|\Theta_r(z)\|, \|\Theta_l(z)\| \leq (1 - \|TR_A(z)\|)^{-1}.
\]

**Proof.** For \( z \in \varrho(D_A) \) we get the following relation on \( \mathcal{H} \)
\[
\left[ R_A(z), V^{-1} \right] = R_A(z) \left[ V^{-1}, D_A \right] R_A(z)
= -R_A(z) V^{-1} TR_A(z)
= -V^{-1} R_A(z) TR_A(z) - \left[ R_A(z), V^{-1} \right] TR_A(z).
\]
From this follows that
\[
\left[ R_A(z), V^{-1} \right] (1 + TR_A(z)) = -V^{-1} R_A(z) TR_A(z).
\]
Since \( \|TR_A(z)\| < 1 \) we get that \( \pm 1 \in \varrho(TR_A(z)) \) by the Neumann series. Therefore, we get the desired expression (32) and the estimate on \( \Theta \).

As stated in Lemma 7 in the appendix we can reduce the proof of Theorem 1 for potentials \( V \in C^1(\mathbb{R}^2, \mathbb{R}) \) and the magnetic fields \( B \in C(\mathbb{R}^2, \mathbb{R}) \) satisfying the following conditions: There exist constants \( \delta, \eta \in (0, 1) \) such that, for all \( x \in \mathbb{R} \),
\[
V(x) \geq 1/\delta,
\]
\[
|\nabla V(x)| \leq 6V(x),
\]
\[
V^2(x) \leq 2(1 - \eta)B(x).
\]
Due to Lemma 4 we see that under these assumptions 0 is an isolated eigenvalue of \( D_A \) of infinite multiplicity and that \( \sigma(D_A) \setminus \{0\} \) is discrete. Moreover, we find a spectral gap \( (-2\beta_0, 0) \cup (0, 2\beta_0) \subset \varrho(D_A) \), where
\[
\beta_0 := (2\delta\sqrt{1 - \eta})^{-1}.
\]

**Lemma 3.** Let \( V \in C^1(\mathbb{R}^2, \mathbb{R}) \), \( B \in C(\mathbb{R}^2, \mathbb{R}) \), and \( A \in C^1(\mathbb{R}^2, \mathbb{R}^2) \) with \( B = \text{curl} A \). Assume further that the conditions (35), (36), (37) are fulfilled for \( \delta \in (0, \frac{1}{2}) \) and \( \eta \in (0, 1) \). Then, we have
\[
(a) \quad \text{The operators } \left[ P_0^\perp, V^{-1} \right] \text{ and } [P_0^\perp, V] \text{ are well-defined on } C_0^\infty(\mathbb{R}^2, \mathbb{C}^2) \text{ and extend to bounded operators on } \mathcal{H} \text{ with}
\]
\[
\| V \left[ P_0^\perp, V^{-1} \right] \|, \| [P_0^\perp, V] \| \leq 4\delta^2.
\]
The same holds true if we replace \( P_0^\perp \) above by \( P_0 \).
\[
(b) \quad P_0 \mathcal{D}(V), P_0^\perp \mathcal{D}(V) \subset \mathcal{D}(V).
\]
\[
(c) \quad \text{The operator } V \left[ P_0^\perp, V^{-1} \right] \text{ maps } C_0^\infty(\mathbb{R}^2, \mathbb{C}^2) \text{ in } \mathcal{D}(D_A). \text{ Moreover, we have}
\]
\[
\| D_A V \left[ P_0^\perp, V^{-1} \right] \| \leq 4\delta.
\]
**Proof.** Part (a): Let \( \varphi, \psi \in C_0^\infty(\mathbb{R}^2, \mathbb{C}^2) \) with \( \|\varphi\| = \|\psi\| = 1 \). Using the representation formula for the spectral projection and that \( (-2\beta_0, 0) \cup (0, 2\beta_0) \subset \varrho(D_A) \) we have
\[
P_0 = -\frac{1}{2\pi i} \int_{|z|=\beta_0} R_A(z)dz.
\]
Thus, applying Lemma 2 we get
\[ |⟨V ϕ, [P_0, V^{-1}] ψ⟩| ≤ \frac{1}{2π} \int |⟨V ϕ, [R_A(z), V^{-1}] ψ⟩| \, dz \]
with
\[ \eta \quad \text{and} \quad \lambda \]
we have
\[ |⟨V ϕ, [P_0, V^{-1}] ψ⟩| = |⟨V ϕ, 0⟩| \]
\[ \leq \frac{1}{2π} \int |⟨V ϕ, [R_A(z), V^{-1}] ψ⟩| \, dz \]
\[ = \frac{1}{2π} \int |⟨ϕ, R_A(z)TR_A(z)Θ_r(z)ψ⟩| \, dz \]
\[ \leq \frac{1}{2π} \int \frac{∥T∥∥R_A(z)∥^2}{1 - ∥T∥∥R_A(z)∥} \, dz \leq 4δ^2. \]

The estimate for \([P_0^+, V^{-1}] V\) follows analogously.

**Part (b):** From the previous computation we see that \([P_0^+, V^{-1}]\) maps \(H\) on \(D(V)\). The claim is therefore an immediate consequence of the identity
\[ P_0ϕ = V^{-1}P_0Vϕ + [P_0, V^{-1}] φ, \quad φ ∈ D(V). \]

**Part (c):** By the spectral theorem we get, for \(|z| = β_0\),
\[ ∥D_A R_A(z)∥ = \sup_{λ ∈ σ(D_A)} \frac{∥ϕ, D_A R_A(z)TR_A(z)Θ_r(z)ψ⟩|}{∥ϕ∥∥ψ∥} \leq 1. \]

Proceeding similarly as in the proof of part (a) we have, for \(φ, ψ ∈ C_0^∞(R^2, C^2)\) with \(∥φ∥ = ∥ψ∥ = 1\),
\[ |⟨D_A φ, [P_0^+, V^{-1}] ψ⟩| ≤ \frac{1}{2π} \int \frac{∥D_A φ, V[P_0, V^{-1}] ψ⟩|}{∥D_A φ∥} \, dz \]
\[ ≤ \frac{1}{2π} \int \frac{∥D_A φ, V[P_0, V^{-1}] ψ⟩|}{∥D_A φ∥} 4δ. \]

\[ \square \]

**Lemma 4.** Let \(V ∈ C^1(R^2, R)\), \(B ∈ C(R^2, R)\), and \(A ∈ C^1(R^2, R^2)\) with \(B = \text{curl} A\). Assume further that the conditions \([26] - [30]\) are fulfilled for \(δ ∈ (0, \frac{1}{2})\) and \(η ∈ (0, 1)\). Then \(\text{sgn}(D_A)P_0^+, V^{-1}\) maps \(L^2(R^2, C^2)\) in \(D(V)\) and
\[ ∥V \, [\text{sgn}(D_A)P_0^+, V^{-1}]∥ ≤ 4δ^2. \]

**Proof.** Let \(P_I\) denote the spectral projection of \(D_A\) on the interval \(I ⊂ R\). Then, we have
\[ \text{sgn}(D_A)P_0^+ = P_{(β_0, ∞)} - P_{(-∞, -β_0)} \]
with \(β_0\) as in \([30]\). Using \([10]\) Lemma VI-5.6] we find the representations
\[ P_{(-∞, -β_0)} = \frac{1}{2} (1 - U(-β_0)), \]
\[ P_{(β_0, ∞)} = \frac{1}{2} (1 + U(β_0)), \]
where, for \(λ ∈ σ(D_A)\),
\[ U(λ) = s - \lim_{R → ∞} \int_{-R}^{R} R_A(λ + it) \frac{dt}{π} = \int_{-∞}^{∞} R_A(λ + it) \frac{dt}{π}. \]

This yields the commutator identity
\[ [\text{sgn}(D_A)P_0^+, V^{-1}] = \frac{1}{2} [U(-β_0), V^{-1}] + \frac{1}{2} [U(β_0), V^{-1}]. \]
Pick \( \varphi, \psi \in C_0^\infty(\mathbb{R}^2, \mathbb{C}^2) \) with \( \|\varphi\| = \|\psi\| = 1 \). Applying Lemma 2 we get
\[
|\langle V \varphi, [U(\beta_0), V^{-1}] \psi \rangle| = \frac{1}{\pi} \int_{-\infty}^{\infty} \langle V \varphi, [R_A(\beta_0 + i t), V^{-1}] \psi \rangle \, dt
\leq \frac{1}{\pi} \int_{-\infty}^{\infty} \|\varphi\| \|V [R_A(\beta_0 + i t), V^{-1}] \psi\| \, dt
= \frac{1}{\pi} \int_{-\infty}^{\infty} \|R_A(\beta_0 + i t) T R_A(\beta_0 + i t) \Theta_r(\beta_0 + i t) \psi\| \, dt
\leq \delta \frac{1}{\pi} \int_{-\infty}^{\infty} \|R_A(\beta_0 + i t)\|^2 \|\Theta_r(\beta_0 + i t)\| \, dt
\leq \frac{2\delta}{\pi} \int_{-\infty}^{\infty} \frac{1}{\beta_0^2 + t^2} \, dt = \frac{2\delta}{\beta_0} \leq 4\delta^2.
\]
In the estimate above we use that, for \( t \in \mathbb{R} \),
\[
\|\Theta_r(\beta_0 + i t)\| \leq (1 - \|R_A(\beta_0 + i t)\| \|T\|)^{-1} \leq (1 - \delta \|R_A(\beta_0)\|)^{-1} \leq 2.
\]
Similarly, the same inequality holds for \( [U(-\beta_0), V^{-1}] \). Hence, we find that
\[
|\langle V \varphi, [P_0^+ \text{sgn}(D_A) P_0^+, V^{-1}] \psi \rangle| \leq 4\delta^2,
\]
From this follows the proof of the lemma.

5. Proof of Theorem 1

We note that the assumptions in Theorem 1 imply either that \( V(x) \to \infty \) as \( |x| \to \infty \) or \( V(x) \to -\infty \) as \( |x| \to \infty \) by using the continuity of \( V \). We may assume without loss of generality that \( V \) is positive at infinity. Similarly, it suffices to consider the case \( B(x) \to \infty \) as \( |x| \to \infty \) since otherwise we just have to change the roles of \( d \) and \( d^* \) in the proof.

In order to prove Theorem 1 it suffices to find a constant \( c > 0 \) such that
\[
\| (D_A + V) \varphi \| \geq c \| V \varphi \|, \quad \varphi \in C_0^\infty(\mathbb{R}^2, \mathbb{C}^2), \tag{42}
\]
holds; see, e.g. [19]. Moreover, according to Lemma 4 we may assume that \( V \) and \( B \) fulfill the conditions [35]–[37], where \( \eta \in (0, 1) \) is some fix constant and \( \delta \in (0, 1) \) can be chosen arbitrarily small.

Since \( \sigma_{\text{ess}}(D_A) = \{0\} \) it is convenient to show (42) by splitting \( \varphi \) as the sum of \( P_0^\varphi \) and \( P_0^\perp \varphi \). Using the bounds derived in Section 4 we can then estimate the cross terms.

**Proof of Theorem 1.** Let \( \varphi \in C_0^\infty(\mathbb{R}^2, \mathbb{C}^2) \). We compute, using Lemma 3
\[
\| (D_A + V) \varphi \|^2 = \| (D_A + V) (P_0 + P_0^\perp) \varphi \|^2
= \| (VP_0 + (D_A + V) P_0^\perp) \varphi \|^2
= \| (D_A + V) P_0^\perp \varphi \|^2 + 2 \text{Re} \langle (D_A + V) P_0^\perp \varphi, VP_0 \varphi \rangle + \| VP_0 \varphi \|^2
= \| (D_A + V) P_0^\perp \varphi \|^2 - \delta \| VP_0^\perp \varphi \|^2
+ 2 \text{Re} \langle VP_0 \varphi, D_A P_0^\perp \varphi \rangle + \| V \varphi \|^2 - (1 - \delta) \| VP_0^\perp \varphi \|^2.
\]
We estimate each of the terms above separately. Observe that for any ε ∈ (0, 1) we have that

\[
\|(D_\mathbf{A} + V)P_0^{\perp}\varphi\|^2 - \delta \|VP_0^{\perp}\varphi\|^2 \\
\geq (1 - \epsilon)\|D_\mathbf{A}P_0^{\perp}\varphi\|^2 + (1 - \epsilon^{-1} - \delta)\|VP_0^{\perp}\varphi\|^2.
\]

(43)

An application of Lemma 3 yields

\[
\|(VP_0\varphi, D_\mathbf{A}P_0^{\perp}\varphi)| = |(VP_0V^{-1}V\varphi, D_\mathbf{A}P_0^{\perp}\varphi)| \\
= |(P_0V\varphi, D_\mathbf{A}P_0^{\perp}\varphi) + (V [P_0, V^{-1}] V\varphi, D_\mathbf{A}P_0^{\perp}\varphi)| \\
\leq 4\delta \|\varphi\| \|\varphi\| \\
\leq 4\delta^2 \|\varphi\|^2,
\]

where in the last equality we use \(35\). Further, by Lemma 3 (a), we obtain

\[
\|VP_0^{\perp}\varphi\| \leq \|V\varphi\| + \|V [P_0^{\perp}, V^{-1}] V\varphi\| \leq (1 + 4\delta^2)\|\varphi\|.
\]

Thus

\[
\|V\varphi\|^2 - (1 - \delta) \|VP_0^{\perp}\varphi\|^2 \geq (\delta - 4\delta^2)\|\varphi\|^2.
\]

(45)

Therefore, in view of \((13), (13)\), and \((15)\) it suffices to show that

\[
\|D_\mathbf{A}P_0^{\perp}\varphi\|^2 + \frac{(1 - \epsilon^{-1} - \delta)}{(1 - \epsilon)} \|VP_0^{\perp}\varphi\|^2 \geq 0,
\]

(46)

for δ > 0 small enough and some ε ∈ (0, 1). We set

\[
c_{\epsilon, \delta} := -\frac{(1 - \epsilon^{-1} - \delta)}{(1 - \epsilon)} > 0.
\]

In view of Lemma 1 we have that

\[
P_0^{\perp} = \begin{pmatrix} \hat{\pi} & 0 \\ 0 & 1 \end{pmatrix},
\]

where \(\hat{\pi}\) denotes the orthogonal projection onto \(\text{ker}(d)^\perp\). Using this, \((22)\), and writing \(\varphi = (\varphi_1, \varphi_2)^T\) we get

\[
\|D_\mathbf{A}P_0^{\perp}\varphi\|^2 - c_{\epsilon, \delta} \|VP_0^{\perp}\varphi\|^2 = \|d\hat{\pi}\varphi_1\|^2 - c_{\epsilon, \delta}\|V\hat{\pi}\varphi_1\|^2 + \|d^*\varphi_2\|^2 - c_{\epsilon, \delta}\|V\varphi_2\|^2.
\]

According to condition \((37)\) and \((23)\) we have that

\[
\|d^*\varphi_2\|^2 - c_{\epsilon, \delta}\|V\varphi_2\|^2 = \langle \varphi_2, dd^*\varphi_2 \rangle - c_{\epsilon, \delta}\langle \varphi_2, V^2\varphi_2 \rangle \\
\geq [1 - c_{\epsilon, \delta}(1 - \eta)]\langle \varphi_2, 2B\varphi_2 \rangle.
\]

(48)

In order to give a lower bound to \(\|d\hat{\pi}\varphi_1\|^2 - c_{\epsilon, \delta}\|V\hat{\pi}\varphi_1\|^2\) we will use that

\[
(\hat{\pi})^2 = sd^*s^* \quad \text{on} \quad \text{Ran}(\hat{\pi}) \cap \mathcal{D}(dd^*),
\]

where \(s, s^*\) are the isometries given in \((23)\). A simple computation yields

\[
\|d\hat{\pi}\varphi_1\|^2 - c_{\epsilon, \delta}\|V\hat{\pi}\varphi_1\|^2 = \|s^*s\hat{\pi}\varphi_1\|^2 - c_{\epsilon, \delta}\|Vs^*s\hat{\pi}\varphi_1\|^2 \\
= \|s^*s\hat{\pi}\varphi_1\|^2 - c_{\epsilon, \delta}\|Vs^*V^{-1}Vs\hat{\pi}\varphi_1\|^2 \\
= \|d^*s\hat{\pi}\varphi_1\|^2 - c_{\epsilon, \delta}\|s^* + V \left[ s^*, V^{-1} \right] V s\hat{\pi}\varphi_1\|^2.
\]

(49)
We note that $V\left[s^*, V^{-1}\right]$ is one of the components of the operator
\[
V\left[\text{sgn}(D_A)P_0^+, V^{-1}\right] = \begin{pmatrix} 0 & V\left[s^*, V^{-1}\right] \\ V\left[s^*, V^{-1}\right] & 0 \end{pmatrix}.
\]
Using the definition of the operator norm, we obtain that
\[
\left\|V\left[s^*, V^{-1}\right]\right\| = \sup_{\varphi = (0, \varphi_2)^T, \|\varphi_2\| = 1} \left\|V\left[\text{sgn}(D_A)P_0^+, V^{-1}\right] \varphi\right\|
\leq \left\|V\left[\text{sgn}(D_A)P_0^+, V^{-1}\right]\right\| \leq 4\delta^2,
\]
where in the last bound we use Lemma 5. Combining this with (19) and proceeding as in (18) we obtain
\[
\|d\tilde{\pi}\varphi_1\|^2 - c_{e, \delta}\|V\tilde{\pi}\varphi_1\|^2 \geq \left[1 - c_{e, \delta}(1 - \eta)(1 + 12\delta^2)\right] \langle s\tilde{\pi}\varphi_1, 2B s\tilde{\pi}\varphi_1\rangle.
\]
Choosing $\varepsilon = 1 - \delta^{1/2}$ we get that $c_{e, \delta} = 1 + O(\delta^{1/2})$ as $\delta \to 0$. This implies that, for $\delta > 0$ sufficiently small, the terms in (18) and (50) are positive. This concludes the proof of the theorem.

6. Proof of Theorem 2

In this section we prove Theorem 2 for $B > 0$. The case $B < 0$ can be done similarly. As we mention in Section 3 there is a function $\phi \in C^2(\mathbb{R}^2, \mathbb{R})$ satisfying $\Delta\phi = B$. We choose $A(x) = (-\partial_2\phi, \partial_1\phi)$ as the vector potential in the Hamiltonian $D_A$. A key element in our proof is that the space
\[
X := \{\omega e^{-\phi} \in L^2(\mathbb{R}^2, \mathbb{C}) \mid \omega \text{ is entire in } x_1 + i x_2\},
\]
is infinite dimensional. Since $B > B_0$ we see that $X$ is a subspace of ker($d$) (see (27)). Let us first state a technical result concerning this space whose proof is given at the end of this section.

Lemma 5. Assume that the conditions of Theorem 2 are fulfilled. Let $\phi \in C^2(\mathbb{R}^2, \mathbb{R})$ be such that $B = \Delta\phi$. Then we have, for $\Omega \in X$,
\[
\text{a) } \Omega \in D(d^*) \cap D(V) \text{ and } \|d^*\Omega\| = \|\sqrt{2B}\Omega\|
\]
\[
\text{b) } (d^*\Omega, -V\Omega)^T \in D(H)
\]
Proof of Theorem 2. We first show that the dimension of $X$ (defined in (51)) is infinite. To this end define $\bar{\phi} = \phi - \frac{1}{2}\ln(B)$ and note that
\[
\Delta\bar{\phi} = B + \frac{(\nabla B)^2}{2B^2} - \frac{\Delta B}{2B} > B - \frac{\Delta B}{2B}.
\]
Thus, by the discussion in Section 3 (see (28)) the space
\[
Y := \{\omega e^{-\phi} \in L^2(\mathbb{R}^2, \mathbb{C}) \mid \omega \text{ is entire in } x_1 + i x_2\},
\]
is infinite dimensional. The claim now follows since simply $X$ and $Y$ are isomorphic. Let us define the subspace
\[
W := \left\{ \begin{pmatrix} d^*\Omega \\ -V\Omega \end{pmatrix} \in L^2(\mathbb{R}^2, \mathbb{C}^2) \mid \Omega \in X \right\} \subset D(H).
\]
By Lemma 5 we have, for any $\Omega \in X$,
\[
\left\| \begin{pmatrix} d^*\Omega \\ -V\Omega \end{pmatrix} \right\| = \|d^*\Omega\| + \|V\Omega\| \geq \sqrt{2B_0}\|\Omega\|.
\]
Therefore, the map $X \ni \Omega \mapsto (d^*\Omega, -V\Omega)^T \in W$ is bijective and we conclude that $\dim W = \infty$.

Now pick $\psi = (d^*\Omega, -V\Omega)^T \in W$, then

$$
\left\| H\psi \right\| = \left\| \begin{pmatrix} V & d^* \\ d & V \end{pmatrix} \begin{pmatrix} d^* \Omega \\ -V\Omega \end{pmatrix} \right\| = \left\| \begin{pmatrix} (i\partial_1 V + \partial_2 V)\Omega \\ (2B - V^2)\Omega \end{pmatrix} \right\| \\
\leq (\|\nabla V/V\|_\infty + (2B - V^2)/V\|_\infty)\|\psi\|.
$$

Since $\dim W = \infty$ the inequality above implies that $\sigma_{ess}(H) \neq \emptyset$. \hfill \Box

**Proof of Lemma 5.** Note first that from Remark 2 follows that $X \subset D(V)$. Let $\Omega \in X$ and $\chi \in C_0^\infty(\mathbb{R}^2, [0, 1])$ be such that

$$
\chi(x) = \begin{cases} 1, & \text{for } |x| \leq 1 \\ 0, & \text{for } |x| \geq 2. \end{cases}
$$

We define $\Omega_n := \chi(\frac{n}{n})\Omega, n \in \mathbb{N}$. Clearly, $\Omega_n \in C_0^\infty(\mathbb{R}^2, \mathbb{C})$ holds and moreover $\Omega_n \to \Omega$ as $n \to \infty$ in $L^2(\mathbb{R}^2, \mathbb{C})$. Since $\Omega \in \ker(d)$ we have, using the commutator identity (23), that

$$
\|d^*\Omega_n\|^2 = \|d\Omega_n\|^2 + 2\|\sqrt{B}\Omega_n\|^2 = \frac{1}{n^2}\left\| (i\partial_1 \chi - \partial_2 \chi) \left( \frac{\cdot}{n} \right) \Omega \right\|^2 + 2\|\sqrt{B}\Omega_n\|^2
$$

holds, for any $n \geq 1$. Thus, $(d^*\Omega_n)_{n \in \mathbb{N}}$ is a Cauchy sequence in $L^2(\mathbb{R}^2, \mathbb{C})$. By the closedness of $d^*$ we conclude that $\Omega \in D(d^*)$ with $\|d^*\Omega\| = \|\sqrt{B}\Omega\|$.

For $\Omega \in X$ define $\psi := (d^*\Omega, -V\Omega)^T$. A direct computation, using integration by parts, shows that one finds $\phi \in L^2(\mathbb{R}^2, \mathbb{C})$ such that for all $\varphi \in C_0^\infty(\mathbb{R}^2, \mathbb{C}^2)$ holds

$$
\langle H\varphi, \psi \rangle = \langle \varphi, \phi \rangle.
$$

This implies the claim by the definition of the adjoint operator. \hfill \Box

7. Proof of Theorems 3 and 4

Note first that it suffices to construct a sequence $(\psi_n)_{n \in \mathbb{N}} \subset D(H)$ which converges weakly to zero such that $\|H\psi_n\|/\|\psi_n\| \to 0$ as $n \to \infty$ (Weyl sequence). For simplicity we consider the case when $B(\infty) = B_0$. Recall that the corresponding magnetic Dirac operator $D_\Lambda$ has the (Landau) eigenvalues $l_n := \text{sgn}(n)\sqrt{2|n|B_0}, n \in \mathbb{Z}$. Now assume that there is a sequence $(x_n)_{n \in \mathbb{N}}$ such that $V(x_n) = \sqrt{2|n|B_0} = l_n$ for all $n \in \mathbb{N}$ (this is a simplification of condition (13)). The bulk of the Weyl sequence consists of eigenfunction $(\varphi_n)_{n \in \mathbb{N}}$ of $D_\Lambda$ centered around $x_n$ and with eigenenergies $-l_n$. It is well known that these eigenfunctions are almost Gaussian-like localized. Due to this strong localization one has that $(V\varphi_n)(x) \approx V(x_n)\varphi_n(x) = l_n\varphi_n(x)$ in the sense of $L^2$. Therefore, we get that

$$
\|(D_\Lambda + V)\varphi_n\| \approx 0,
$$

where the error terms will be controlled using the remaining assumptions of the theorems.

This consideration can be also extended to non-constant magnetic fields. In this case we construct the Weyl sequence using eigenfunctions of the magnetic Dirac operator with constant magnetic field $B(x_n)$. Such a local linearization can be well controlled using again that the Landau eigenfunction are strongly localized.
Throughout this section we will assume without lost of generality that $B(x_n) \geq B_0$ is positive. In order to prepare the proof of Theorems 3 and 4 we introduce some notation: For a sequence $(x_n)_{n \in \mathbb{N}} \subset \mathbb{R}^2$ we set $V_n := V(x_n)$ and $B_n := B(x_n)$ and define the magnetic vector potentials

$$A_n(x) := \int_0^1 B_n \wedge (x - x_n)ds = \frac{1}{2} B_n \wedge (x - x_n),$$

$$\tilde{A}_n(x) := \int_0^1 B(x_n + s(x - x_n)) \wedge (x - x_n)ds,$$

where $a \wedge v := a(\nu_2, v_1)$, for $a \in \mathbb{R}$ and $v = (v_1, v_2) \in \mathbb{R}^2$. Clearly, $\text{curl } A_n = B_n$ and $\text{curl } \tilde{A}_n = B$ on $\mathbb{R}^2$.

We define the operators $d_n$ and $d_n^*$ through the relation

$$D_{A_n} = \begin{pmatrix} 0 & d_n^* \\ d_n & 0 \end{pmatrix}.$$

Let $(p_n)_{n \in \mathbb{N}}$ be a sequence of natural numbers. As already mentioned above an important ingredient in our proof is that $2p_n B_n$ is the square of the $p_n$-th Landau level of the Dirac operator $D_{A_n}$. For all $n \in \mathbb{N}$, we define the functions

$$\varphi_n(x) := \begin{pmatrix} (iB_n(x_1 - i x_2))^p_n e^{-B_n|x|^2/4} \\ -V_n (iB_n(x_1 - i x_2))^p_n - 1 e^{-B_n|x|^2/4} \end{pmatrix}$$

and observe that

$$\tilde{\varphi}_n(x) := \varphi_n(x - x_n) = \begin{pmatrix} (d_n^* p_n e^{-B_n|x-x_n|^2/4} \\ -V_n (d_n^* p_n - 1 e^{-B_n|x-x_n|^2/4} \end{pmatrix}.$$

We have, for any $k \in \mathbb{N}$ (see e.g., [20, Section 7.1.3]),

$$d_n d_n^* [(d_n^* k - 1 e^{-B_n|x-x_n|^2/4}] = 2k B_n [(d_n^* k - 1 e^{-B_n|x-x_n|^2/4}].$$

Next we define the localization functions. Let $\chi \in C_0^\infty(\mathbb{R}^2, [0, 1])$ be such that $\chi(x) = 1$ for $|x| \leq 1$ and $\chi(x) = 0$ for $|x| \geq 2$. We set

$$\chi_n(x) := \chi\left(\frac{x-x_n}{r_n}\right),$$

where $r_n > 0$ will be chosen in the proofs later on.

Finally, we observe that since $\text{curl } (A - \tilde{A}_n) = 0$ there exists a function $g_n \in C^2(\mathbb{R}^2, \mathbb{R})$ such that $\nabla g_n = A - \tilde{A}_n$ on $\mathbb{R}^2$.

We define the Weyl functions to be given, for all $n \in \mathbb{N}$, by

$$\psi_n(x) := e^{ig_n(x)} \chi_n(x) \tilde{\varphi}_n(x), \quad x \in \mathbb{R}^2.$$

Clearly, we have

$$e^{-ig_n} (D_A + V) \psi_n = (D_{\tilde{A}_n} + V) \chi_n \tilde{\varphi}_n$$

$$= \chi_n(D_{\tilde{A}_n} + V) \tilde{\varphi}_n - i (\sigma \cdot \nabla \chi_n) \tilde{\varphi}_n$$

$$+ \sigma \cdot (A_n - \tilde{A}_n) \chi_n \tilde{\varphi}_n + (V - V_n) \chi_n \tilde{\varphi}_n.$$

In order to prove Theorems 3 and 4 we will choose $r_n > 0$ such that $\psi_n$ are linear independent and each of the four terms above tend to zero uniformly in $\|\psi_n\|$ as $n \to \infty$. In the next lemma we estimate the norms $\|\psi_n\|$. 
Lemma 6. Assume that $V_n^2/(2p_nB_n) \to 1$ as $n \to \infty$. Then, for all $n \in \mathbb{N}$ large enough, we have

$$\|\varphi_n\|^2 \geq 2^{p_n+1} \pi B_n^{p_n-1} p_n!$$  \hspace{1cm} (57)$$

$$\|\psi_n\|^2 \geq \frac{1}{4} \|\varphi_n\|^2 \left(1 - \frac{1}{p_n!} \int_{B_n r_n^2/2}^{\infty} s^{p_n} e^{-s} ds \right).$$  \hspace{1cm} (58)

Proof. For $\psi_n = (\psi_{n,1}, \psi_{n,2})^T$ we compute for $n \in \mathbb{N}$ so large that $V_n^2/(2p_nB_n) \leq 3$

$$\|\psi_n\|^2 \leq \|\varphi_n\|^2 = \|\varphi_{n,1}\|^2 + \|\varphi_{n,2}\|^2$$

$$= 2B_n^{2p_n \pi} \int_0^\infty s^{2p_n} e^{-B ns^2/2} ds + 2V_n^2 B_n^{2(p_n-1)} \pi \int_0^\infty s^{2(p_n-1)} e^{-B ns^2/2} ds$$

$$= 2^{p_n+1} \pi B_n^{p_n-1} \left(\int_0^\infty s^{p_n} e^{-s} ds + \frac{V_n^2}{2B_n} \int_0^\infty s^{p_n-1} e^{-s} ds\right)$$

$$= 2^{p_n+1} \pi B_n^{p_n-1} \left(1 + \frac{V_n^2}{2B_n p_n}\right) p_n!.$$  

Noting that

$$2^{p_n+1} \pi B_n^{p_n-1} p_n! \leq 2^{p_n+1} \pi B_n^{p_n-1} \left(1 + \frac{V_n^2}{2B_n p_n}\right) p_n! \leq 2^{p_n+3} \pi B_n^{p_n-1} p_n!,$$

we get an upper and lower bound for $\|\varphi_n\|^2$. In particular, we obtain (57). Moreover,

$$\|\psi_n\|^2 \geq \|\varphi_{n,1}\|^2 \geq 2B_n^{2p_n \pi} \int_0^\infty \left(s^{2}\right)^{p_n} e^{-B ns^2/2} ds$$

$$= 2^{p_n+1} B_n^{p_n-1} \pi \left(p_n! - \int_{B_n r_n^2/2}^{\infty} s^{p_n} e^{-s} ds\right)$$

$$\geq \frac{1}{4} \|\varphi_n\|^2 \left(1 - \frac{1}{p_n!} \int_{B_n r_n^2/2}^{\infty} s^{p_n} e^{-s} ds \right).$$

This finishes the proof.

Proof of Theorem 3. In this proof we use Lemma 6 for $p_n = k$ ($n \in \mathbb{N}$), where $k$ is some fixed natural number. We choose the localization radii to be given by

$$r_n = B_n^{(\varepsilon-1)/2}.$$  \hspace{1cm} (59)

Since $r_n \to 0$ as $n \to \infty$ we can assume that the $\psi_n$ have disjoint support, for otherwise we can extract a subsequence satisfying this property. In view of Remark 4 we see that, as $n \to \infty$,

$$\frac{1}{p_n!} \int_{B_n r_n^2/2}^{\infty} s^{p_n} e^{-s} ds = \frac{1}{k!} \int_{B_n r_n^2/2}^{\infty} s^{k} e^{-s} ds \to 0.$$  \hspace{1cm} (60)

Then, according to Lemma 5 there exists an $N > 0$ such that, for all $n > N$,

$$\|\varphi_n\| \leq 8\|\psi_n\|.$$  \hspace{1cm} (61)

Next we estimate the corresponding terms of (56). A simple calculation shows that

$$\|(DA_n + V_n)\varphi_n\|^2 \leq \|(V_n^2 - 2kB_n)/V_n\|\|\varphi_n,2\|^2 \leq 8(V_n^2 - 2kB_n)/V_n\|\psi_n\|^2.$$
Thus, by (15), \( \|(D_{\mathcal{A}} + V_n)\hat{\varphi}_n^2/\|\psi_n\|^2 \) converges to 0 as \( n \to \infty \). Moreover, using (57), we get for \( n \) sufficiently large

\[
\|\sigma \cdot \nabla \chi_n \hat{\varphi}_n\|^2 \leq r_n^{-2} \|\nabla \chi\|_{\infty} \int_{r_n \leq |x| \leq 2r_n} |\varphi_n(x)|^2 \, dx
\]

\[
\leq B_n^{1-\epsilon} \|\nabla \chi\|_{\infty} 2^{(k+3)} \pi B_n^{k-1} \int_{B_n/2}^{\infty} \frac{s^k e^{-s}}{s} \, ds
\]

\[
\leq 2^5 \|\psi_n\|^2 \|\nabla \chi\|_{\infty} B_n^{1-\epsilon} \frac{1}{k!} \int_{B_n/2}^{\infty} s^k e^{-s} \, ds.
\]

Thus, \( \|\sigma \cdot \nabla \chi_n \hat{\varphi}_n\| / \|\psi_n\| \to 0 \) as \( n \to \infty \). For the last two terms of (60) we recall that \( |\nabla B| \) and \( |\nabla V| \) grow with rate 0 at infinity. Therefore, applying the mean value theorem we find a constant \( C > 0 \) such that

\[
\|\sigma \cdot (\mathcal{A}_n - \tilde{\mathcal{A}})_n \hat{\varphi}_n\|^2 \leq C^2 |\nabla B(x_n)|^2 r_n^4 \|\hat{\varphi}_n\|^2 \leq 8C^2 \left| \frac{\nabla B(x_n)}{B_n^{1-\epsilon}} \right|^2 \|\psi_n\|^2
\]

and

\[
\|\hat{V}(x_n)\hat{\varphi}_n\|^2 \leq 8C^2 \left| \frac{\nabla V(x_n)}{|V_n|^{1-\epsilon}} \right|^2 \left( \frac{V_n^2 B_n}{B_n^2} \right)^{1-\epsilon} \|\psi_n\|^2.
\]

Therefore, we obtain the desired result in view of (13) and since \( V^2/B_n \) is uniformly bounded for large \( n \).

**Proof of Theorem 4.** In this case we apply Lemma 6 for \( p_n = n \) \((n \in \mathbb{N})\). We set

\[
r_n := \sqrt{2n^{(1+\epsilon)}} / B_n.
\]

Using that \( s^ne^{-s/2} \leq \(2n\)^n e^{-n} \) we get

\[
\frac{1}{n!} \int_{n^{(1+\epsilon)}}^{\infty} s^n e^{-s} \, ds \leq 2 \frac{2n \cdot n^{-n^{-1+\epsilon}/2-n}}{n!} \leq \exp(n \ln(2n) - n^{(1+\epsilon)/2-n}).
\]

The last term tends to zero as \( n \to \infty \). Hence, we find \( N > 0 \) so large that

\[
\exp(n \ln(2n) - n^{(1+\epsilon)/2-n}) < 1/2, \quad \text{for all } n > N.
\]

As a consequence we get in view of Lemma 6 for all \( n > N \),

\[
\|\varphi_n\|^2 \leq 8\|\psi_n\|^2.
\]

We now estimate the radii. In view of Remark 5 we may assume that, for all \( n > N \),

\[
\frac{1}{2} \leq \frac{V_n^2}{2nB_n} \leq 2.
\]

This together with (19) implies that

\[
r_n^{-2} \leq 2^{\kappa-1} \alpha n^{\kappa-1-\epsilon} \leq n^\kappa,
\]

for \( n \in \mathbb{N} \) large enough. Therefore, using (57), we get

\[
\|\sigma \cdot \nabla \chi_n \hat{\varphi}_n\|^2 \leq \|\varphi_n\| \|\nabla \chi\|_{\infty} 2^{(k+3)} \pi B_n^{k-1} \int_{n^{1+\epsilon}}^{\infty} s^n e^{-s} \, ds
\]

\[
\leq 4 \|\nabla \chi\|_{\infty} \|\varphi_n\|^2 n^{\kappa} \exp(n \ln(2n) - n^{(1+\epsilon)/2-n})
\]

\[
\leq 2^5 \|\nabla \chi\|_{\infty} \|\psi_n\|^2 n^{\kappa} \exp(n \ln(2n) - n^{(1+\epsilon)/2-n}).
\]

Thus, \( \|\sigma \cdot \nabla \chi_n \hat{\varphi}_n\|^2 / \|\psi_n\|^2 \to 0 \) as \( n \to \infty \).
For the last two terms in (64) we use (62) and (65) to get, for \( \nu \in [0, 1] \),
\[
\frac{r_n^2}{|x|^{2\nu}} = \frac{2^{\nu+\epsilon} n^{1+\epsilon}}{B_n |x_n|^{2\nu}} \leq \frac{2^{\nu+\epsilon}}{B_n |x_n|^{2\nu}} \left( \frac{V_n^2}{2B_n} \right)^{1+\epsilon}.
\]
This implies by (13) that for \( \nu \in [0, 1] \) the ratio \( r_n/|x_n|^\nu \to 0 \) as \( n \to \infty \). Hence, on
the one hand, we see that the support of the \( \psi_n \) are mutually disjoint (at least for a subsequence). On the other hand, since \( |\nabla V| \) and \( |\nabla B| \) vary with rate \( \nu \in [0, 1] \),
this enable us to use the mean value theorem to obtain that
\[
\| \sigma \cdot (A_n - \tilde{A}_n) x_n \hat{\psi}_n \|^2 \leq C^2 |\nabla B(x_n)|^2 r_n^2 \| \hat{\psi}_n \|^2
\]
\[
\leq 2^{7+2\epsilon} C^2 \left[ \frac{|\nabla B(x_n)|}{B_n} \left( \frac{V_n^2}{2B_n} \right)^{1+\epsilon} \right]^2 \| \psi_n \|^2,
\]
where we use again (62) combined with (65) and (64). Analogously, we get
\[
\|(V - V_n) x_n \hat{\psi}_n \|^2 \leq C^2 |\nabla V(x_n)|^2 r_n^2 \| \hat{\psi}_n \|^2
\]
\[
\leq 2^{6+2\epsilon} C^2 \left[ \frac{|\nabla V(x_n)|}{V_n} \left( \frac{V_n^2}{2B_n} \right)^{1+\epsilon} \right]^2 \| \psi_n \|^2.
\]
Hence, in view of (17) we get that \( \|(DA + V)\psi_n\|/\|\psi_n\| \to 0 \) as \( n \to \infty \) which
proves the theorem. \( \square \)

**Appendix A. Results for rotationally symmetric potentials**

In this appendix we study some properties of the Dirac operator \( H = DA + V \)
when \( V \) and \( B \) are rotationally symmetric. Let \( A \) be given by the rotational gauge
\[
A(x) := \frac{A(r)}{r} \left( \begin{array}{c} -x_2 \\ x_1 \end{array} \right), \quad A(r) = \frac{1}{r} \int_0^r B(s) \, ds,
\]
where \( r = |x| \). We can decompose \( H \) into a direct sum of operators on the \( j \)-th
angular momentum eigenspace, i.e., there is a unitary map
\[
U : L^2(\mathbb{R}^2, \mathbb{C}^2) \to \bigoplus_{j \in \mathbb{Z}} L^2(\mathbb{R}^+, \mathbb{C}^2; dr)
\]
such that \( UHU^* = \bigoplus_{j \in \mathbb{Z}} h_j \) with
\[
h_j := -i \sigma_2 \partial_r + \sigma_1 \left( A(r) - \frac{m_j}{r} \right) + v(r) \quad \text{on} \quad L^2(\mathbb{R}^+, \mathbb{C}^2; dr),
\]
where \( v(|x|) := V(x) \) and \( m_j = j + 1/2 \), \( j \in \mathbb{Z} \) (see e.g. [20]). Since \( H \) is essentially
self-adjoint on \( C_0^\infty(\mathbb{R}^2, \mathbb{C}^2) \) we deduce that \( h_j \) is also essentially self-adjoint on
\( (UC_0^\infty(\mathbb{R}^2, \mathbb{C}^2))_j \subset L^2(\mathbb{R}^+, \mathbb{C}^2; dr) \) for any \( j \in \mathbb{Z} \).

**Proposition 1.** Assume that \( A \in C^1(\mathbb{R}^+, \mathbb{R}) \) and \( v \in C(\mathbb{R}^+, \mathbb{R}) \) are such that the
conditions
\[
|A(r)| \to \infty \quad \text{as} \quad r \to \infty,
\]
\[
\frac{A'(r)}{A^2(r)} \to 0 \quad \text{as} \quad r \to \infty,
\]
\[
\limsup_{r \to \infty} \left| \frac{v(r)}{A(r)} \right| < 1,
\]
are fulfilled. Then, for all \( j \in \mathbb{Z} \), the spectrum of \( h_j \) is purely discrete. In particular, the spectrum of \( D_A + V \) is pure point.

**Proof.** Since \( \limsup_{r \to \infty} v^2(r)/A^2(r) < 1 \) we find constants \( \mu > 1 \) and \( r_0 > 0 \) such that \( A^2(r) > \mu v^2(r) \) for \( r > r_0 \). We pick \( \lambda \in (\mu^{-1}, 1) \) and define, for \( j \in \mathbb{Z} \), the potential function \( w_j \) on \( \mathbb{R}^+ \) as

\[
w_j(r) := \left(A^2(r) - \frac{2m_j}{r} A(r) - \lambda^{-1} v^2(r)\right) + \begin{pmatrix} -A'(r) & 0 \\ 0 & A'(r) \end{pmatrix}.
\]

For any \( j \in \mathbb{Z} \) the matrix valued potential \( w_j \) is real-valued and diagonal. Moreover, the diagonal entries of \( w_j \) are fulfilled. Then, for all \( j \in \mathbb{Z} \), the potential function \( w_j \) on \( \mathbb{R}^+ \) is discrete. In particular, there is a constant \( \eta \) such that \( \limsup_{r \to \infty} |\psi| \to \infty \) as \( |x| \to \infty \). An application of the min-max principle (see [13, Section XIII.1]) gives that \( h_j^2 \) has purely discrete spectrum. This implies the claim for \( h_j \).

**APPENDIX B. SOME TECHNICAL TOOLS**

**Lemma 7.** Assume \( V \in C^1(\mathbb{R}^2, \mathbb{R}) \), \( B \in C(\mathbb{R}^2, \mathbb{R}) \) and \( A \in C^1(\mathbb{R}^2, \mathbb{R}^2) \) with \( B = \text{curl} A \) on \( \mathbb{R}^2 \), such that the conditions

\[
V(x), B(x) \to \infty \quad \text{as} \quad |x| \to \infty
\]

(71)

\[
\frac{\nabla V(x)}{|V(x)|} \to 0 \quad \text{as} \quad |x| \to \infty
\]

(72)

\[
\limsup_{|x| \to \infty} \frac{V^2(x)}{2B(x)} < 1
\]

(73)

are fulfilled. Then, there is a constant \( \eta \in (0, 1) \) such that for any \( \delta \in (0, 1) \) we can find an electric potential \( \hat{V} \in C^1(\mathbb{R}^2, \mathbb{R}) \) and a magnetic field \( \hat{B} \in C(\mathbb{R}^2, \mathbb{R}) \) satisfying conditions \([35, 37]\) such that

\[
\sigma(D_A + V) \text{ is discrete if and only if } \sigma(D_{\hat{A}} + \hat{V}) \text{ is discrete.}
\]

Here \( \hat{A} \in C^1(\mathbb{R}^2, \mathbb{R}^2) \) is a magnetic vector potential satisfying \( \hat{B} = \text{curl} \hat{A} \) on \( \mathbb{R}^2 \).

**Remark 9.** An analogue statement holds true for potentials \( V \) with \( V(x) \to -\infty \) as \( |x| \to \infty \) and magnetic fields \( B \) with \( B(x) \to -\infty \) as \( |x| \to \infty \).
Proof. We first construct the potential $\hat{V}$. Without lost of generality we assume that $V(x) \geq 0$ for all $x \in \mathbb{R}^2$ (otherwise start with $V + c$ for some constant $c$). Due to assumption (73) we find $R_1 \geq 1$ and $\eta \in (0, 1)$ such that

$$V^2(x) < 2(1 - \eta)B(x), \quad |x| \geq R_1. \quad (74)$$

Similarly by (71) and (72) we find for any $\delta \in (0, 1)$ a constant $R_2 > R_1$ such that

$$V(x) > 2\delta^{-1} \quad \text{and} \quad |\nabla V(x)| < \frac{\delta}{4} V(x), \quad |x| \geq R_2. \quad (75)$$

Pick $\xi \in C^\infty(\mathbb{R}^2, [\frac{1}{2}, 1])$ such that $\xi(x) = 1/2$ for $|x| \leq 1$ and $\xi(x) = 1$ for $|x| \geq 2$. For $r > R_2$ we set $\xi_r(x) := \xi(\frac{x}{r})$, $x \in \mathbb{R}^2$. Define for $x \in \mathbb{R}^2$

$$\hat{V}_r(x) := \xi_r(x)V(x) + (1 - \xi_r(x))4\delta^{-1}(1 + M),$$

where $M := \sup \{|\nabla V(x)| \mid |x| \leq R_2\}$. Clearly

$$\hat{V}_r(x) = V(x), \quad |x| > 2r. \quad (76)$$

Furthermore, we find that

$$\hat{V}_r(x) \geq \delta^{-1}, \quad 2\hat{V}_r(x) \geq V(x), \quad x \in \mathbb{R}^2. \quad (77)$$

A simple computation yields, for any $x \in \mathbb{R}^2$,

$$|\nabla \hat{V}_r(x)| \leq |\nabla V(x)| + \frac{1}{r} \left(2\|\nabla \xi\|_\infty + 4\|\nabla \xi\|_\infty \delta^{-1}(1 + M)\right)|\nabla V(x)|. \quad (78)$$

Note that $|\nabla V(x)| \leq \max\{M, \delta V(x)/4\}, x \in \mathbb{R}$. Hence, using (78) and the definition of $\hat{V}_r$ we get that $|\nabla V(x)| \leq \delta\hat{V}_r(x)/2$, for all $x \in \mathbb{R}^2$. Thus, we find a constant $r_0 > R_2$ so large that

$$|\nabla \hat{V}_{r_0}(x)| \leq \delta\hat{V}_{r_0}(x), \quad x \in \mathbb{R}^2. \quad (79)$$

Define $\hat{V} := \hat{V}_{r_0}$ and note that $\hat{V}$ fulfills the desired properties in view of (77) and (79).

Next we define the magnetic field as

$$\hat{B}(x) := 2\left(\xi_{2r_0}(x) - \frac{1}{2}\right)B(x) + (1 - \xi_{2r_0}(x))\frac{\hat{V}_r^2(x)}{(1 - \eta)}, \quad x \in \mathbb{R}^2. \quad (80)$$

In view of (70) and (74) we have

$$\hat{B}(x) \geq \frac{\hat{V}_r^2(x)}{2(1 - \eta)}, \quad x \in \mathbb{R}^2. \quad (81)$$

Hence, $\hat{V}$ and $\hat{B}$ satisfy conditions (73)–(74).

Since the function $B - \hat{B}$ has compact support in $\mathbb{R}^2$, we find a function $G \in C^1(\mathbb{R}^2, \mathbb{R}^2)$ such that $||G||_\infty < \infty$ and $B - \hat{B} = \text{curl} G$ on $\mathbb{R}^2$. We define $A(x) := A(x) - G(x)$ for $x \in \mathbb{R}^2$. By construction we know that $(D_A + V) - (D_{\hat{A}} + \hat{V})$ is bounded on $L^2(\mathbb{R}^2, \mathbb{C}^2)$. Hence, the resolvent difference of $D_A + V$ and $D_{\hat{A}} + \hat{V}$ is compact if one of the resolvents is itself compact. From this follows the claim. $\square$
CONFINEMENT-DECONFINEMENT TRANSITIONS FOR DIRAC PARTICLES

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