New ground state for quantum gravity

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In this paper we conjecture the existence of a new “ground” state in quantum gravity, supplying a wave function for the inflationary Universe. We present its explicit perturbative expression in the connection representation, exhibiting the associated inner product. The state is chiral, dependent on the Immirzi parameter, and is the vacuum of a second quantized theory of graviton particles. We identify the physical and unphysical Hilbert sub-spaces. We then contrast this state with the perturbed Kodama state and explain why the latter can never describe gravitons in a de Sitter background. Instead, it describes self-dual excitations, which are composites of the positive frequencies of the right-handed graviton and the negative frequencies of the left-handed graviton. These excitations are shown to be unphysical under the inner product we have identified. Our rejection of the Kodama state has a moral tale to it: the semi-classical limit of quantum gravity can be the wrong path for making contact with reality (which may sometimes be perturbative but nonetheless fully quantum). Our results point towards a non-perturbative extension, and we present some conjectures on the nature of this hypothetical state.

I. INTRODUCTION

Although the modern canonical approach to quantum gravity\(^1\,\,3\) has in many ways been a formal success, it has not always made easy contact with the real world (see, however, \([4–8]\)). This is often blamed not on the formalism, but on the difficulty in finding solutions representing something akin to the reality we observe, possibly corrected by new effects still beyond the reach of experiment. The matter is closely related to the identification of the ground state (or “base” state, a terminology intended to skirt the issue of energy) around which small excitations would form the reality we probe. In some approximation the theory should produce solutions representing classical space-times satisfying Einstein’s equations, subject to small quantum fluctuations or semi-classical corrections. In a cosmological setting the theory should supply a wave-function for the early Universe, for example in a de Sitter or inflationary primordial phase. This wave function should encode predictions for the vacuum quantum fluctuations that seed the structure of our Universe or make up a cosmic gravitational wave background. Ultimately quantum gravity should corroborate—or contradict—the textbook treatment\(^9\,\,10\), based on second quantized effective field theory and the Bunch-Davis vacuum.

One of the earliest proposals for a “base state” describing de Sitter space-time was the Kodama state\(^11\), sometimes referred to as the Chern-Simons wave function. This state solves the Hamiltonian constraint in a diagonal representation for the Ashtekar connection, with “EEF” factor ordering, cosmological constant \(\Lambda \neq 0\), and Immirzi parameter \(\gamma = \pm i\). As such, the Kodama state should be the ideal candidate for the wave function of the inflationary Universe, expressed in terms of the Ashtekar self-dual (or anti-self-dual) connection, instead of the more traditional metric representation. Furthermore, the Chern-Simons functional is the Hamilton-Jacobi function of the theory, so that the Kodama state is a semi-classical or WKB solution even if the space-time is only approximately de Sitter. It should therefore provide the vacuum for inflationary fluctuations, placing existing heuristic calculations on safe grounds.

Regrettably, in spite of earlier claims to that effect\(^12\,\,13\), it has proved elusive and even self-contradictory\(^13\,\,14\) finding a second quantized theory of linearized gravitons inside the perturbed Kodama state. Tensor modes around de Sitter space-time simply do not fit into the perturbed Kodama state. In this paper we explain the reasons for this failure and exhibit the true perturbative state representing gravitons in the same representation used for the Kodama state. We will do this for the same \(\gamma\) and ordering of the Hamiltonian constraint as that implicit in the construction of the Kodama state, but also more generally.

In recent work\(^15–17\) we derived explicit perturbative expressions for the Fock vacuum and its graviton states, using perturbative expansions in Ashtekar gravity. These papers built on earlier research\(^18\,\,19\), and lead to the conclusion that, even though the graviton particle spectrum is identical for right and left gravitons (in contrast with the findings of\(^14\)), their vacuum energy and fluctuations have a chiral signature (dependent on \(\gamma\) and factor ordering issues). The observational implications of this conclusion are striking, as discussed in\(^20\). The first task of the present paper is to express these results in the same representation as that used for the Kodama state, i.e. a holomorphic connection representation. A number of technical issues, not evident in the Bargmann representation used in\(^13\,\,16\), are discussed in Section\(^11\) following which explicit forms for the wave functions are found (Section\(^11\)). Their norms are evaluated in Section\(^14\) selecting the physical and unphysical sub-spaces.

Direct comparison of these states with the perturbed Kodama state reveals blatant antagonism. In Section\(^15\) we identify its origin. We show that even before issues related to the inner product are considered (normaliz-
ability, the physical sub-space, etc), it is obvious that the perturbed Kodama state is not an eigenstate of the perturbative Hamiltonian, and so can never represent gravitons or their vacuum in a second quantized theory of tensor perturbations around de Sitter space-time. This arises from the simple algebraic fact that the perturbed self-dual operator does not factor out in the perturbed Hamiltonian, even though it does so non-perturbatively. This is far from an oddity and results from the “braided” fashion in which perturbative expansions contribute to a non-linear expression at a given order, as spelled out in Section IV A.

Our conclusion is far from surprising. The perturbed Kodama state represents solutions to the perturbed self-dual operator, i.e. self-dual excitations. These have long been known \[18\] to combine the positive frequency of the right-handed graviton and the negative frequency of the left-handed anti-graviton (see \[16\] for further discussion). As in the age-old adage, two halves don’t make a whole: not only are these composites of half-particles not gravitons, but they are unphysical, as can be checked by evaluating their norm using the same criteria employed to select physical graviton states. This conclusion is also true for the Chern-Simons state in Yang-Mills theories, although the reason there is more subtle in this case.

Having presented a solution for the base state and discredited its competition we close this paper by speculating on possible non-perturbative generalizations for our construction (Section VI).

II. THE CONNECTION REPRESENTATION OF GRAVITONS

In \[15, 16\] we obtained an expression for the perturbative Hamiltonian in the Ashtekar formalism by expanding the Hamiltonian constraint to second order, retaining only terms quadratic in the first order expansions. We refer the reader to \[16\] for notation, definitions and all the subtleties. The upshot is the identification of a set of particle creation and annihilation graviton operators, \(\hat{G}^r_p, \hat{G}_r\), combining metric and connection variables. The theory contains gravitons and anti-gravitons before the reality conditions are imposed and, due to their second class nature, these conditions are imposed via the choice of the inner product with which the Hilbert space is endowed. The inner product then flags half the particles as unphysical, restoring the correct number of physical degrees of freedom in the theory. In \[13, 16\] we performed this exercise in representations which diagonalize the connection operator—the so-called Bargmann representation \[21, 22\]. For each value of \(\gamma\) we obtain a different representation. Here we recast these results in the connection representation, mimicking the format in which the Kodama state is expressed.

It is not immediately obvious that a holomorphic representation diagonalizing the connection exists at all. Indeed, in a strict mathematical sense, such representations do not exist for \(\gamma = \pm i\), as we shall see. Yet, wave functions in the Ashtekar approach are usually expressed as functions of the connection or its holonomies. In this Section we show how to get as close as possible to a holomorphic representation diagonalizing the connection, adapting the usual construction.

There are different terminologies in use, so to be completely clear let us define as holomorphic a representation in terms of complex functions \(\Psi(z)\) which depend on a set of complex variables \(z\) but not their complex conjugates \(\bar{z}\) (\(z\) and \(\bar{z}\) are to be seen as independent variables when differentiating or integrating). If in addition these functions are analytic functions over the whole complex domain (i.e. they can be expressed as a power series over the whole domain) we call them entire functions. This distinction will be essential later.

Using this terminology, the functions forming the basis for the Bargmann representation are entire functions. The problems encountered searching for the dual of this representation (diagonalizing the annihilation operator, instead of the creation operator) whilst keeping the functions entire have been discussed, for example in \[24, 25\]. The matter is relevant for discussing wave-functions when \(\gamma = \pm i\), (and more generally for the definition of a delta function in the complex domain), and it will be discussed in Section III B. More generally, even for \(\gamma \neq \pm i\), finding a holomorphic representation (in the sense defined above) diagonalizing the connection proves problematic. Since our proposal will not be the obvious one, we start by spelling out the problem before providing the alternative.

A. The apparent obstruction

As in \[16\] (where we focused on a purely imaginary \(\gamma\)) we expand connection and metric perturbations into Fourier modes \(\tilde{a}_{r p}(k)\) and \(\tilde{e}_{r p}(k)\), where \(r = \pm\) represents right and left polarizations, and \(p = \pm\) indexes the graviton and anti-graviton modes (needed to expand variables which, a priori, are complex). We then find, from their Poisson bracket, that upon quantization the operators to which they are promoted should satisfy:

\[ [\tilde{a}_{r p}(k), \tilde{e}_{r q}^\dagger(k')] = -i\gamma p \frac{\bar{p}}{2} \delta_{r, s} \delta_{pq} \delta(k - k'), \]

with all other commutators set to zero. We want to represent this algebra. Since \(\tilde{a}_{r p}\) commutes with \(\tilde{a}_{r p}^\dagger\), we might be tempted to think that if we diagonalize one of them we diagonalize the other, a statement which relies on the crucial assumption that they act on the same space of functions. This is a trap leading to serious problems, which we spell out here in order to motivate the alternative construction presented in Section II B.

Let us then accept that \(\tilde{a}_{r p}\) and \(\tilde{a}_{r p}^\dagger\) act on the same space of functions. Since they commute, they can be simultaneously diagonalized, so that \(\hat{a}\Psi = \lambda \Psi\) and \(\hat{a}^\dagger \Psi = \mu \Psi\). From \(\langle \Psi | \hat{a}^\dagger \hat{a} | \Psi \rangle = \langle \Psi | \hat{a}^\dagger \hat{a} | \Psi \rangle\) we can conclude
\[ \mu = \lambda, \text{ i.e. the eigenvalues of } a_{rp}^\dagger \text{ must be complex conjugate to those of } a_{rp}. \] It is therefore inevitable that functions \( \Phi \) must mix variables and their conjugates, i.e. \( \Phi = \Phi(a_{rp}, \overline{a}_{rp}) \). The representation can no longer be holomorphic, and we have instead:

\[ \tilde{a}_{rp}\Phi(a_{rp}, \overline{a}_{rp}) = a_{rp}\Phi(a_{rp}, \overline{a}_{rp}) \]  
\[ \tilde{a}_{rp}\Phi(a_{rp}, \overline{a}_{rp}) = \overline{a}_{rp}\Phi(a_{rp}, \overline{a}_{rp}) \]  
with commutation relations (11) implying:

\[ \tilde{e}_{rp}\Phi(a_{rp}, \overline{a}_{rp}) = -i\gamma P \frac{\delta}{\delta a_{rp}} \Phi(a_{rp}, \overline{a}_{rp}) \]  
\[ \tilde{e}_{rp}\Phi(a_{rp}, \overline{a}_{rp}) = i\gamma P \frac{\delta}{\delta a_{rp}} \Phi(a_{rp}, \overline{a}_{rp}) \]  
The fact that the representation is no longer holomorphic leads to the breakdown of several results, namely the standard derivation of the inner product. Indeed, the anzatz for the inner product would now have to be:

\[ \langle \Phi_1|\Phi_2 \rangle = \int da \, \Phi_1(\tilde{a}, a) \overline{\Phi_2(a, \tilde{a})}, \]  

(note the dependence on \( a \) and \( \tilde{a} \) of both functions). The reality conditions are:

\[ \tilde{a}_{rp} + a_{rp} = 2rk \tilde{e}_{rp} \]  
\[ \tilde{a}_{rp}^\dagger + a_{rp}^\dagger = 2rk \tilde{e}_{rp}^\dagger \]  
\[ \tilde{e}_{rp} = \tilde{e}_{rp}^\dagger \]  
and these are formally valid only when sandwiched between a generic bra and a ket. In the standard treatment (10, 26, 27) this fully fixes \( \mu \), but in the derivation one needs to perform an integration by parts which assumes that the functions are holomorphic. With non-holomorphic functions this can no longer be done (since both \( a \) and \( \tilde{a} \) appear in each of \( \Phi_1 \) and \( \Phi_2 \)). It is straightforward to show that erroneously neglecting this detail leads to contradictory conditions regarding the sign of \( \mu \).

### B. The resolution

It is possible to define a holomorphic representation in connection space, but as the previous sub-Section showed, one must drop the assumption that \( \tilde{a}_{rp} \) and \( a_{rp}^\dagger \) act on the same space of functions. In fact this feature is suggested by the formalism. By direct inspection (see 16) we can check that the only operators that appear in the Hamiltonian (on- and off-shell) are \( \tilde{a}_{rp}, \tilde{a}_{rp}^\dagger, \tilde{e}_{rp}, \tilde{e}_{rp}^\dagger \) a feature which propagates into the definition of creation and annihilation operators, \( G_{rp}^\dagger \) and \( G_{rp} \). This suggests restricting our ket functions to be functions of \( a_{rp} \) and \( \overline{a}_{rp} \) only, noting that \( \tilde{e}_{rp} \) and \( \tilde{e}_{rp}^\dagger \) will naturally act on them, given commutation relations (11). The operators \( \tilde{a}_{rp}, \tilde{a}_{rp}^\dagger, \tilde{e}_{rp} \) and \( \tilde{e}_{rp}^\dagger \) therefore act on the right upon functions \( \Phi = \Phi(a_{rp}, \overline{a}_{rp}) \). The dual space of functions (“bra” functions) will then be functions of the conjugate variables, \( \Phi = \Phi(\overline{a}_{rp}, a_{rp}) \), with the conjugate operators, \( \tilde{a}_{rp}^\dagger, \tilde{a}_{rp}, \tilde{e}_{rp}^\dagger \) and \( \tilde{e}_{rp} \), acting on the left upon them.

Spelling this out¹, we postulate a Hilbert space of (ket) wavefunctions holomorphic in \( a_{rp} \) and \( \tilde{a}_{rp} \):

\[ \Phi(a_{rp}, \overline{a}_{rp}) = \langle a_{rp}|\Phi \rangle \]  

and a dual space of (bra) functions holomorphic in \( \tilde{a}_{rp} \) and \( a_{rp} \):

\[ \Phi(\tilde{a}_{rp}, a_{rp}) = \langle \Phi|a_{rp} \rangle. \]

Instead of (2)-5, we have:

\[ \tilde{a}_{rp} + a_{rp} = 2rk \tilde{e}_{rp} \]  
\[ \tilde{a}_{rp}^\dagger + a_{rp}^\dagger = 2rk \tilde{e}_{rp}^\dagger \]  
\[ \tilde{e}_{rp} = \tilde{e}_{rp}^\dagger \]  
for operators acting on the right upon ket functions \( \Phi = \Phi(a_{rp}, \overline{a}_{rp}) \). Reciprocally, we have:

\[ \Phi(\overline{a}_{rp}, a_{rp})\tilde{a}_{rp}^\dagger = \Phi(\overline{a}_{rp}, a_{rp})\tilde{a}_{rp} \]  
\[ \Phi(\overline{a}_{rp}, a_{rp})a_{rp}^\dagger = \Phi(\overline{a}_{rp}, a_{rp})a_{rp} \]  
\[ \Phi(\overline{a}_{rp}, a_{rp})\tilde{e}_{rp} = \Phi(\overline{a}_{rp}, a_{rp})\tilde{e}_{rp} \]  
\[ \Phi(\overline{a}_{rp}, a_{rp})\tilde{e}_{rp}^\dagger = \Phi(\overline{a}_{rp}, a_{rp})\tilde{e}_{rp}^\dagger \]

for operators acting on the left, upon dual functions \( \Phi = \Phi(\overline{a}_{rp}, a_{rp}) \). In both cases the algebra (11) is realized, but notice that there is a minus sign in the last two formulae, (18) and (19), with respect to (4) and (5), resulting from the fact that the operators act on the left, not on the right. With this prescription operators never map a holomorphic function into a non-holomorphic function, and since operators and their conjugates never act upon the same space, we evade the “trap” highlighted in the previous Section, which forced the functions to be non-holomorphic.

Within this set up we can now determine the inner product from the reality conditions, following the usual

¹ In this paper we adopt the convention \( \Psi(\lambda) = \langle \lambda|\Psi \rangle \) for “ket” functions, and \( \Psi(\lambda) = \langle \Psi|\lambda \rangle \) for the dual “bra” functions. Other conventions are possible, e.g. 24.
procedure. In contrast to [1] we consider ansatz:

\[ \langle \Phi_1|\Phi_2 \rangle = \int da \bar{e}^{\mu(a, \bar{a})} \bar{\Phi}_1(a_{r+}, a_{r-}) \Phi_2(a_{r+}, a_{r-}) \]

where the integration measure mixes variables and their conjugates, but the wavefunctions and their duals do not. The reality conditions are still \( \gamma \) and only make sense in that context. Condition \( \bar{\gamma}_{r+} = \bar{\gamma}_{r-} \), for example, does not make sense as an operator condition (it’s like imposing an identity between different types of objects), but \( \langle \Phi|\bar{\gamma}_{r+}|\Psi \rangle = \langle \Phi|\bar{\gamma}_{r-}|\Psi \rangle \) does.

Following the standard argument, conditions (7) and (8) lead to:

\[ a_{r+} + a_{r-} = -i \gamma l^2 kr \frac{\delta \mu}{\delta a_{r-}} \tag{20} \]
\[ \bar{a}_{r+} + \bar{a}_{r-} = -i \gamma l^2 kr \frac{\delta \mu}{\delta a_{r+}} \tag{21} \]

with solution:

\[ \mu = \int d^3k \sum_r \frac{-1}{\gamma l^2 kr} (a_{r+} + a_{r-})(\bar{a}_{r+} + \bar{a}_{r-}). \tag{22} \]

In contrast to the apparent sign contradiction mentioned at the end of Section II.A identities \( \bar{e}_{r+} = \bar{e}_{r-} \) and \( \bar{e}_{r+} = \bar{e}_{r-} \) now merely signify:

\[ \frac{\delta \mu}{\delta a_{r-}} = \frac{\delta \mu}{\delta a_{r+}} \tag{23} \]
\[ \frac{\delta \mu}{\delta \bar{a}_{r-}} = \frac{\delta \mu}{\delta \bar{a}_{r+}} \tag{24} \]

satisfied by solution (22). We note that \( \mu \) is real, as it should be.

C. Result for a general complex \( \gamma \)

The results shown thus far (as well as those in [12]) are valid for a purely imaginary \( \gamma \). For a generally complex \( \gamma \) (investigated in [14]), the reality conditions \( \gamma \), (20)–(21) should be replaced by:

\[ i \gamma^* \bar{a}_{r+} - i \gamma \bar{a}_{r-} = 2 r k \gamma_l \bar{e}_{r+} \tag{25} \]
\[ -i \gamma \bar{a}_{r+}^\dagger + i \gamma^* \bar{a}_{r-}^\dagger = 2 r k \gamma_l \bar{e}_{r-}^\dagger \tag{26} \]
\[ \bar{e}_{r+} = \bar{e}_{r-}. \tag{27} \]

Additionally, the commutation relations (11) become:

\[ [\bar{a}_{rp}(k), \bar{e}_{sq}^\dagger(k')] = -i (\gamma_{rl} + pi \gamma_l) \frac{l^2}{2} \delta_{r,s} \delta_{p,q} \delta(k-k'). \tag{28} \]

In a holomorphic representation that diagonalizes the connection, as above, this leaves equations (12) to (17) unmodified, but (18) and (19) change to:

\[ \bar{\Phi}^\dagger \bar{\gamma}_{r+} = \bar{\Phi} \left( -i \gamma^* \frac{l^2}{2} \frac{\tilde{\delta}}{\delta \bar{a}_{r-}} \right) \tag{29} \]
\[ \bar{\Phi} \bar{\gamma}_{r-} = \bar{\Phi} \left( -i \gamma \frac{l^2}{2} \frac{\tilde{\delta}}{\delta a_{r+}} \right). \tag{30} \]

Using the standard approach, we can find functional differential equations for the measure:

\[ i \gamma^* a_{r+} - i \gamma a_{r-} = -i \gamma l^2 kr \gamma_l \frac{\delta \mu}{\delta a_{r-}} \tag{31} \]
\[ -i \gamma \bar{a}_{r+} + i \gamma^* \bar{a}_{r-} = -i \gamma l^2 kr \gamma_l \frac{\delta \mu}{\delta a_{r+}} \tag{32} \]

with the reality of the metric implying:

\[ -i \gamma \frac{\delta \mu}{\delta a_{r+}} = i \gamma^* \frac{\delta \mu}{\delta a_{r-}}. \tag{33} \]

The measure for a generally complex \( \gamma \) is therefore:

\[ \mu = \int d^3k \sum_r \frac{1}{l^2 kr \gamma_l} \left[ a_{r+} \bar{a}_{r+} + a_{r-} \bar{a}_{r-} - \left( \gamma^* a_{r+} \bar{a}_{r-} + \gamma a_{r-} \bar{a}_{r+} \right) \right] \tag{34} \]

where we’ve assumed that \( \gamma_l \neq 0 \).

This reduces to (22) in the limit \( \gamma_l \to 0 \) as required. Note, however, that the limit \( \gamma_l \to 0 \) is ill-defined. This is because in the case of a purely real \( \gamma \) the representation is no longer holomorphic. For \( \gamma_l = 0 \), Eq. (31) becomes:

\[ a_{r+} = a_{r-} \tag{34} \]

which makes sense: since the theory is real we do not need the index \( p \). This precludes the segregation of variables between kets \( (a_{r+} \text{ and } \bar{a}_{r-}) \) and bras \( (\bar{a}_{r+} \text{ and } a_{r-}) \). If \( \Phi(a_{r+}, \bar{a}_{r-}) \), then, dropping the redundant second index, this means \( \Phi(a_{r+}, \bar{a}) \), i.e. the function is no longer holomorphic. The procedure introduced in this paper to find a connection holomorphic representation for the graviton states therefore does not work for real \( \gamma \).

D. Absence of a metric holomorphic representation

Another context where the prescription in Section II.B breaks down, for the same reasons as in the previous subsection, is a representation diagonalizing the metric. This is very interesting: not only does the connection take precedence over the metric in this formalism, but it appears that some results do not have counterparts expressed in terms of metric variables.

It may seem at first that setting up a representation diagonalizing the metric is a trivial extension of our
method. In analogy with Section 1113, we should define kets such that \( \Phi = \Phi(e_{r+}, \bar{e}_{r-}) \) and bras such that \( \bar{\Phi} = \bar{\Phi}(\bar{e}_{r+}, e_{r-}) \) with the following diagonal operators acting on the right:

\[
\bar{e}_{r+}\Phi = e_{r+}\Phi \quad \text{(35)}
\]

\[
\bar{e}_{r-}\Phi = e_{r-}\Phi \quad \text{(36)}
\]

and the remaining ones acting on the left:

\[
\bar{\Phi}\bar{e}_{r+} = \bar{\Phi}\bar{e}_{r+} \quad \text{(37)}
\]

\[
\bar{\Phi}\bar{e}_{r-} = \bar{\Phi}e_{r-} . \quad \text{(38)}
\]

The commutation relations then lead to

\[
\hat{a}_{r+}\Phi = -i\gamma \frac{\hat{P}_r}{2} \frac{\delta}{\delta e_{r-}} \Phi \quad \text{(39)}
\]

\[
\hat{a}_{r-}\Phi = -i\gamma \frac{\hat{P}_r}{2} \frac{\delta}{\delta e_{r+}} \Phi \quad \text{(40)}
\]

for right-acting operators, and

\[
\bar{\Phi}\bar{a}_{r+} = \bar{\Phi} \left(-i\gamma \frac{\hat{P}_r}{2} \frac{\delta}{\delta e_{r-}} \right) \quad \text{(41)}
\]

\[
\bar{\Phi}\bar{a}_{r-} = \bar{\Phi} \left(-i\gamma \frac{\hat{P}_r}{2} \frac{\delta}{\delta e_{r+}} \right) \quad \text{(42)}
\]

for left-acting operators. However, when one tries to find the conditions upon the inner product imposed by the reality condition for the metric:

\[
\langle \Phi | \bar{e}_{r+} | \Psi \rangle = \langle \Phi | \bar{e}_{r-} | \Psi \rangle \quad \text{(43)}
\]

this leads to:

\[
e_{r+} = e_{r-} \quad \text{(44)}
\]

contradicting the assumption that the functions are holomorphic, as initially stated.

### III. THE GROUND STATE

Having established the formalism we now derive the wave functions for the ground and particle states of gravitons in the connection representation. In the Bargmann representation (diagonalizing operators \( G^l_{r+} \)) the vacuum wave functions are just \( \Psi = 1 \), whereas particle states are monomials in their respective variables \( 16 \). In this Section we will rederive these wave functions in the holomorphic connection representation defined in the last Section. Ordering issues affect the vacuum energy and fluctuations, but not the form of the wave functions. The cases \( \gamma \neq \pm i \) and \( \gamma = \pm i \) are very different and will be discussed separately. While \( \gamma \neq \pm i \) leads to straightforward Gaussian wavefunctions, the case \( \gamma = \pm i \) requires the introduction of a new mathematical tool.

#### A. Wave functions for \( \gamma \neq \pm i \)

In \( 16 \) we found a set of annihilation operators \( G_{r+} \) in terms of metric and connection operators. To obtain the physical vacuum we must solve \( G_{r+} \Psi_0 = 0 \), which translates into:

\[
(\hat{a}_{r+} - k(r + i\gamma)\bar{e}_{r+})\Psi_0 = 0 . \quad \text{(45)}
\]

Using \( 12 \) and \( 13 \) this equation becomes, in the connection representation:

\[
(\hat{a}_{r+} - k(r + i\gamma)i\gamma \frac{\hat{P}_r}{2} \frac{\delta}{\delta a_{r-}})\Psi_0 = 0 . \quad \text{(46)}
\]

This has solution:

\[
\Psi_0(a_{r+}, \bar{a}_{r-}) = N \exp \left[ \frac{2}{i\gamma(r + i\gamma)k\hat{P}_r} a_{r+}\bar{a}_{r-} \right] \quad \text{(47)}
\]

where \( N \) is a normalization constant (which is finite, as discussed in the next Section).

The counterpart vacuum condition for unphysical modes follows from \( G_{r+} \Psi = 0 \), leading to:

\[
(\hat{a}_{r+} - k(r - i\gamma)i\gamma \frac{\hat{P}_r}{2} \frac{\delta}{\delta a_{r-}})\Psi_0^{unph} = 0 , \quad \text{(48)}
\]

with solution:

\[
\Psi_0^{unph}(a_{r+}, \bar{a}_{r-}) = N \exp \left[ \frac{2}{i\gamma(r - i\gamma)k\hat{P}_r} a_{r+}\bar{a}_{r-} \right] . \quad \text{(49)}
\]

Unsurprisingly, the two conditions are inconsistent, i.e. we cannot find a vacuum simultaneously for physical and unphysical modes. Weeding out the unphysical modes does not amount to setting them to the vacuum state, but instead to factoring them out of the Hilbert space. We will discuss this matter further in the next Section, where we will also explicitly show that the unphysical wave functions are not normalizable.

Particle states can be constructed by acting with creation operators upon the vacuum. These operators are \( 10 \):

\[
G^l_{r+} = \frac{r}{i\gamma} (\hat{a}_{r-}^l - k(r - \mathcal{P}i\gamma)\hat{a}_{r-}^l) . \quad \text{(50)}
\]

Therefore we have:

\[
\Psi_n \propto \bar{a}_{r-}^l \Psi_0 \quad \text{(51)}
\]

for physical gravitons, and

\[
\Psi_n^{unph} \propto \bar{a}_{r-}^l \Psi_0^{unph} \quad \text{(52)}
\]

for the unphysical modes.
B. The singular case \( \gamma = \pm i \)

For the SD and ASD connections we see that two of the four Gaussian wave functions derived above become ill-defined (the denominator in the exponent is zero). The origin of this singularity is interesting. We recall that the graviton operators are now:

\[
\begin{align*}
\tilde{g}_{r+}(k) &= \tilde{a}_{r+}(k) \quad (53) \\
\tilde{g}_{r-}(k) &= -\tilde{a}_{r-}(k) + 2kr\tilde{c}_{r-}(k) \quad (54) \\
\tilde{g}_{r+}(k) &= -\tilde{a}_{r+}(k) + 2kr\tilde{c}_{r+}(k) \quad (55) \\
\tilde{g}_{r-}(k) &= \tilde{a}_{r-}(k) \quad (56)
\end{align*}
\]

so that diagonalizing the connection entails diagonalizing the annihilation operator \( \tilde{g}_{r+} \). Note that since \( a_r, a^\dag_\gamma \) commute no variation of the prescription in Section II B would get us out of this conclusion. Therefore, by going to the connection representation, we are forced to face a notorious problem: that of defining the dual of the Bargmann representation, i.e. a representation diagonalizing annihilation instead of creation operators (see, e.g. [24, 25]).

1. The dual of the Bargmann representation for a harmonic oscillator

We illustrate the issues surrounding the definition of a dual to Bargmann’s representation resorting to the simple harmonic oscillator. Several attitudes can be adopted. In [24, 25] it was advocated that one should abandon the concept of a dual space of “bra” vectors, from which the dual functions are derived. It was shown that it was possible to define a set of holomorphic functions, \( \Psi(\bar{w}) \), conjugate or dual to the holomorphic Bargmann functions, \( \Psi(z) = \langle z|\Psi \rangle \), which are not \( \langle \Psi|\Psi \rangle \), since \( |w\rangle \) is left undefined. With this strategy the dual functions are still entire functions, i.e. analytic over the whole complex domain. Particle states are now inverse powers \( 1/|w|^n \) (cf. eqn (25) of [25]) whereas the vacuum is a constant. The associated inner product was identified for suitably chosen contours of integration (cf. eqn (46) in [24]). The result is somewhat cumbersome, but remains a distinct possibility.

An alternative is often used by the quantum gravity community, but its radical mathematical nature is not often spelled out (see [1], however). The idea is to work with a space of “functions” which are holomorphic in the sense defined in the previous Section (functions of “\( z \)” but not “\( \bar{z} \)”), whilst dropping the requirement that they be entire functions. In fact, as we shall see, we should not even require these “functions” to be distributions in the usual sense (or, at least, we should broaden the concept of distribution [1]). The matter is closely related to the definition of a “holomorphic delta function”. A number of mathematical identities valid in the Bargmann formalism will cease to be true. However there are also advantages with respect to the approach of [24].

2. Holomorphic delta function

Let us define the dual of the Bargmann representation by:

\[
a\Psi(w) = w\Psi(w) \quad , \tag{57}
\]

in contrast with the usual \( a^\dag \Psi(z) = z\Psi \) for which the physical vacuum is \( \Psi_0 = 1 \), particle states \( \Psi_n \propto z^n \), and the inner product measure is \( e^{-z\bar{z}} \). From \( [a, a^\dag] = 1 \) we have:

\[
a\Psi(w) = -\frac{d\Psi(w)}{dw} . \tag{58}
\]

Solving for the vacuum leads directly to a definition for the “holomorphic delta function”. The vacuum equation \( a\Psi_0 = 0 \) results in \( w\Psi_0(w) = 0 \), suggesting \( \Psi_0 = \delta(w) \) with the first defining property:

\[
w\delta(w) = 0 . \tag{59}
\]

The second defining property may be obtained from the fact that norms are independent of the representation. The inner product in the dual representation may be derived as usual [23] by formally requiring \( \langle \Phi|a^\dag|\Psi \rangle = \langle \Psi|a|\Phi \rangle \). However, this exercise leads to:

\[
\langle \Phi|\Psi \rangle = \int dw\,d\bar{w}\, e^{w\bar{w}}\Phi(\bar{w})\Psi(w) \tag{60}
\]

which differs by the sign in the exponent with regards to the measure for the Bargmann representation. Since:

\[
\langle 0|0 \rangle = \int dz\,d\bar{z}\, e^{-z\bar{z}} = \int dw\,d\bar{w}\, e^{w\bar{w}}\delta(\bar{w})\delta(w) , \tag{61}
\]

we should impose the second defining property for the holomorphic delta function:

\[
\int dw\,d\bar{w}\, e^{w\bar{w}}\delta(\bar{w})\delta(w) = 1 . \tag{62}
\]

This delta function is obviously an odd object. It is not analytical, or even a distribution in the usual sense. It only makes sense when integrated multiplied by its complex conjugate. In a sense, it is the “square root of a distribution”. It should not be confused with the “delta” function used in the Bargmann formalism [23], which is better described as a “Reproducing Kernel” or a “Principal Vector”, and is given by the entire function \( K(z; w) = e^{z\bar{w}} \).

3. Further properties of the complex delta function

We stress again that in the dual representation just defined, wave functions are no longer entire functions and so many commonly used identities in the Bargmann formalism (e.g. those involving the Reproducing Kernel)
are no longer valid. This is to be contrasted with the approach in [24, 25]. Instead one must become acquainted with the algebraic properties of the holomorphic delta function. From (63) we see that the particle states are now:

\[ \Psi_n(w) = \langle w | n \rangle = \frac{(-1)^n}{\sqrt{n!}} d^n \delta(w) \]  \hspace{1cm} (63)

It can then be proved (either by direct integration by parts, or using the invariance of the inner product with respect to the representation) that:

\[ \int dw \, d\bar{w} \, e^{w\bar{w}} \frac{d^n \delta(w)}{d\bar{w}^n} \frac{d^n \delta(w)}{dw^n} = \frac{\delta_{nm}}{n!} \]  \hspace{1cm} (64)

Likewise, considering unphysical modes in the Bargmann representation (those belonging to the “Dirac sea”, as in [14]), we can infer that:

\[ \int dw \, d\bar{w} \, e^{-w\bar{w}} \delta(w) \delta(w) = \infty \]  \hspace{1cm} (65)

In a similar fashion many identities of this sort may be derived, establishing the basic rules of calculus for the holomorphic delta function.

It is also possible to write the holomorphic delta function in a more explicit form. Consider the relationship in a more explicit form. Consider the relationship between the Bargmann representation with eigenstates \(|z\rangle\) (for which the measure is \(e^{-z\bar{z}}\)) and the dual we just defined. If we want to transform between the two, we can write a state \(|w\rangle\) in terms of \(|z\rangle\) as

\[ \langle w | \Psi \rangle = \int dzd\bar{z} e^{-z\bar{z}} \langle w | z \rangle \langle z | \Psi \rangle \]  \hspace{1cm} (66)

The ground state in the two representations is given by a constant and \(\delta(w)\), respectively, so that:

\[ \delta(w) = \int dzd\bar{z} e^{-z\bar{z}} \langle w | z \rangle \]  \hspace{1cm} (67)

The first excited state is \(-\frac{d}{dw} \delta(w)\) in the dual representation and \(z\) in the Bargmann representation, so:

\[ -\frac{d}{dw} \delta(w) = \int dzd\bar{z} e^{-z\bar{z}} \langle w | z \rangle z = -\int dzd\bar{z} e^{-z\bar{z}} \frac{d}{dw} \langle w | z \rangle \]  \hspace{1cm} (68)

where the second identity comes from differentiating \(67\) with respect to \(w\). Therefore we obtain a differential equation for the (non-analytic) inner product \(\langle w | z \rangle\):

\[ \langle w | z \rangle z = -\frac{d}{dw} \langle w | z \rangle \]  \hspace{1cm} (69)

This has solution \(\langle w | z \rangle = e^{-wz}\), implying:

\[ \delta(w) = \int dzd\bar{z} e^{-z(w+\bar{z})} \]  \hspace{1cm} (70)

4. Wave functions for \(\gamma = \pm i\)

We have illustrated our method with the simple harmonic oscillator, but what we’ve said so far transposes directly to the wave functions of gravitons in the connection representation, when \(\gamma = \pm i\). For definiteness we discuss the SD (\(\gamma = i\)) case (the wave functions don’t change if \(\gamma = -i\), but what one calls physical and unphysical modes does change, as shall be seen in the next Section).

With the definition for the holomorphic delta function just provided, the equations for the physical modes \(R+\) and \(L-\):

\[ \hat{g}_{R+} \Psi_0 = \hat{a}_{R+} \Psi_0 = a_{R+} \Psi_0 = 0 \]  \hspace{1cm} (71)

\[ \hat{g}_{L-} \Psi_0 = (-\hat{a}_{L+} + 2k r \hat{c}_{L+}) \Psi_0 \]  \hspace{1cm} (72)

can be solved as:

\[ \Psi_0 = \mathcal{N} \delta(a_{R+}) \exp \left[ \frac{a_{R+} \hat{a}_{L+}}{k^2} \right] \]  \hspace{1cm} (73)

The unphysical modes, on the other hand, can be obtained from:

\[ \hat{g}_{L+} \Psi_0 = \hat{a}_{L+} \Psi_0 = a_{L+} \Psi_0 = 0 \]  \hspace{1cm} (74)

\[ \hat{g}_{R-} \Psi_0 = (-\hat{a}_{R+} + 2k r \hat{c}_{R+}) \Psi_0 \]  \hspace{1cm} (75)

resulting in:

\[ \Psi_0 = \mathcal{N} \delta(a_{L+}) \exp \left[ \frac{a_{R+} a_{L-}}{k^2} \right] \]  \hspace{1cm} (76)

Particle states are products of exponentials and monomials, or derivatives of the delta function, as appropriate.

IV. SELECTION OF PHYSICAL STATES

As in the graviton representation, we need to show that the physical/unphysical states are normalizable/non-normalizable. This requires identifying the inner product. In the graviton representation, physical and unphysical graviton states depend on different variables (\(z_r, r_+\) and \(z_r, r_-\), respectively). Evaluating inner products as integrals, the measure splits into a \(z_r, r_+\)-dependent part belonging to physical states and a \(z_r, r_-\)-dependent factor corresponding to unphysical states. The same cannot be said for the connection representation, as all states depend on common variables \(a_r, r_+\) and \(a_r, r_-\). The measure therefore can’t be split in the same way.

A generalization of the usual procedure, applicable to the connection representation, consists of applying the torsion-free condition in the integration leading to the physical inner product. Note that in the graviton representation no torsion implies \(G_{r, r_-} \approx 0\), and therefore
When we factor out the $z_r p_-$ integration, to claim that the physical modes are normalizable, we are therefore applying this prescription. In the connection representation the torsion-free condition implies:

$$a_{r-} = \frac{r + i\gamma}{r - i\gamma} a_{r+}.$$  \hfill (77)

The physical inner product is obtained by inserting (77) into (72), eliminating $a_{r-}$. We can then use this inner product to check whether physical graviton states are normalizable.

For $\gamma \neq \pm i$ (applying Eq. (77) to the states as well) we find for the physical vacuum

$$\langle \Psi_0 | \Psi_0 \rangle = \int d a_{r+} d \bar{a}_{r+} \exp \left[ \sum_r \frac{-4a_{r+}\bar{a}_{r+}}{kl^2 p(r - i\gamma)^2} \right].$$

This converges for all values of $r$ and $\gamma$ as required. Using the same prescription for the unphysical ground state $\Psi_0^{unph}$ we obtain

$$\langle \Psi_0^{unph} | \Psi_0^{unph} \rangle = \int d a_{r+} d \bar{a}_{r+} \exp \left[ \sum_r \frac{4}{kl^2 p(r - i\gamma)^2} a_{r+}\bar{a}_{r+} \right],$$

so these states aren’t normalizable for any $\gamma$, as expected. We can also compute

$$\langle \Psi_0 | \Psi_0^{unph} \rangle = 0,$$

implying that physical and unphysical states are orthogonal.

For the SD connection ($\gamma = i$), the measure (22) reduces to

$$\mu = \int d^3k \left( \frac{1}{l^2 k} |a_{R+}|^2 - \frac{1}{l^2 k} |a_{L-}|^2 \right).$$  \hfill (81)

In this case the torsion-free condition is $a_{R-} = a_{L+} = 0$, so again we are factoring out of the integrals some of the variables. Applying the same prescription as above, the norm of the physical ground state (73) is given by:

$$\langle \Psi_0 | \Psi_0 \rangle = \int d a_{R+} d \bar{a}_{R+} \exp \left( \frac{1}{kl^2 p} |a_{R+}|^2 \right) \delta(a_{R+}) \bar{\delta}(\bar{a}_{R+})$$

$$\int d a_{L-} d \bar{a}_{L-} \exp \left( -\frac{1}{kl^2 p} |a_{L-}|^2 \right).$$  \hfill (82)

The second integral obviously converges and the first integral is also finite due to equation (82). As in the more general case, the unphysical vacuum (70) is non-normalizable and orthogonal to the physical vacuum state.

V. WHY THE KODAMA STATE CAN NEVER DESCRIBE GRAVITONS

In the preceding Sections we studied the connection representation for gravitons and their vacuum. It was suggested in [12] that the perturbed Kodama state could represent gravitons in a de Sitter background. The claim was further examined in [13], where evident contradictions began to surface. In this Section the conflict between the two is rendered explicit and explained. In Appendix A we derive the perturbed Kodama state using the same set of conventions we have used for deriving gravitons states. The outcome is

$$\Psi^{KOD} = N \exp \left( \frac{2i\gamma}{l^2 p H^2 a^2} \sum_r (kr - \gamma H a) a_{r+}\bar{a}_{r-} \right),$$  \hfill (83)

to be contrasted, in the limit $|k\eta| \to \infty$, with the wave functions presented in Section III.

The conflict is far from surprising. An algebraic argument is given in Section V A showing that the perturbed Kodama state isn’t even an eigenstate of the perturbative Hamiltonian, as gravitons are. It represents self-dual excitations, combining the positive frequency half of the right-handed graviton expansion, and the negative frequencies of the left graviton. Such self-dual states are not physical states, as we prove explicitly in Section V B.

A. A simple argument

There is a very simple algebraic reason why the perturbed Kodama state cannot represent gravitons or their ground state. The argument is valid even before issues such as the inner product and normalizability are brought into play. The argument concerns the relation between the Hamiltonian constraint and the self-dual operator when re-examined at the perturbative level.

Schematically (dropping integrals, summations, proportionality constants and contractions with $\epsilon$, irrelevant for the argument), the Hamiltonian constraint, when $\gamma = \pm i$ and in the presence of a cosmological constant, takes the form:

$$\mathcal{H} = EE S$$  \hfill (84)

where the “self dual” operator $^2$ is

$$S = B + H^2 E,$$  \hfill (85)

(see Eq. 44 in [10] with $\gamma = \pm i$, for example). Here $B$ is the magnetic field of $A$ and we have assumed an “EEF” ordering for the Hamiltonian constraint. The Kodama state

\footnote{The term self-dual is often used in two senses in this context. The connection $A$ is always self-dual when $\gamma = i$. However, $S$ resembles a “self-dual” operator if $H^2 = i$.}
(or Chern-Simons) wave function is annihilated by $\mathcal{S}$ and therefore it is a solution to the Hamiltonian constraint with this ordering.

It can be explicitly checked that this is true to zeroth order, considering a de Sitter solution (e.g. [16], Section II A). However, the relation between self-dual states and solutions to the Hamiltonian constraint breaks down in perturbation theory. We use a notation (already used in [13, 16]) where left-side superscripts denote the order of a quantity and left-side subscripts the order of the metric and connection quantities on which it depends. Thus, the second order Hamiltonian ($^2\mathcal{H}$; no left subscript specified) contains terms quadratic in second order variables ($^2\mathcal{H}$), and terms linear in second order variables ($^1\mathcal{H}$), that is:

\begin{equation}
^2\mathcal{H} = ^1\mathcal{H} + ^1\mathcal{H}.
\end{equation}

As explained in [16], perturbation theory is ruled by $^1\mathcal{H}$. Gravitons and their vacuum are eigenstates of $^1\mathcal{H}$. The term $^1\mathcal{H}$ is called by cosmologists the “backreaction”. Only the full $^2\mathcal{H}$ needs to be weakly zero; thus gravitons have dynamics, with the Hamiltonian constraint being enforced to second order by the backreaction.

The perturbed Kodama state ($^{1}\Psi$ (see Appendix A 2) is annihilated by $^{1}\mathcal{S}$. Such a state (or any eigenstate of $^{1}\mathcal{S}$) can never be an eigenstate of $^2\mathcal{H}$. This is obvious even without inspecting the complicated perturbative expressions for both. Schematically, again, perturbing $^{1}\mathcal{S}$ produces an expansion of the form:

\begin{equation}
^2\mathcal{H} = 2(^1\mathcal{E})(^1\mathcal{S}) + (^1\mathcal{S})(^0\mathcal{S}) + (^0\mathcal{E})(^0\mathcal{E})(^1\mathcal{S})
\end{equation}

and obviously $^1\mathcal{S}$ does not factorize on the right, as it does non-perturbatively. Therefore a state annihilated by $^{1}\mathcal{S}$ is not and an eigenstate of $^2\mathcal{H}$, and so the perturbed Kodama state cannot represent gravitons or their vacuum.

Even though this is a new result, it is hardly surprising. It was pointed out in [16] (Section II), reviving an old result [13], that gravitons are not self-dual or anti-self-dual. Once positive and negative frequencies are included in the expansions (an omission behind much confusion in the literature), one finds that self-dual states combine the positive frequency of the right handed graviton and the negative frequency of the left-handed anti-graviton, before reality conditions are imposed. These are the states described by the perturbed Kodama state: a composite of half gravitons, which can never be physical or normalizable, once the inner product is fixed by the reality conditions, as we’ll prove explicitly in Section V.B.

It is interesting to contrast Yang-Mills and gravity with regards to this argument. In Yang-Mills theories the Hamiltonian, again schematically, is of the form:

\begin{equation}
\mathcal{H} = \mathcal{S}^* \mathcal{S},
\end{equation}

with self-dual operator:

\begin{equation}
\mathcal{S} = E + i B.
\end{equation}

Perturbatively this becomes:

\begin{equation}
^2\mathcal{H} = (^1\mathcal{S})^* (^1\mathcal{S}),
\end{equation}

and since $^1\mathcal{S}$ factors on the right one might think that, unlike with gravitons, gauge bosons are indeed eigenstates of the perturbed self-dual operator. One might then be tempted to argue that self-dual and anti-self-dual states correspond to positive and negative helicities and that the Chern-Simons state (solving $\mathcal{H} \approx 0$) represents states with equal numbers of gravitons with positive and with negative energy, or a ground state devoid of vacuum energy. This is essentially the argument in [14].

It turns out that the argument is not only inapplicable to gravity (due to the more intricate nature of its Hamiltonian) but is also incorrect in Yang-Mills theories, due to quantum mechanical ordering issues. The ordering leading to (89) is non-locally distinct from the ordering leading to boson operators. The non-alignment of (anti-)self-dual states and graviton states applies to gauge bosons, too, and the perturbed Chern-Simons state describes non-physical composites of half gauge bosons.

\section{Non-normalizability of the Kodama state fluctuations}

Here we explicitly prove that the Kodama state,

\begin{equation}
\Psi^{KOD} = N \exp \left( \frac{2i\gamma}{l_p^2 H^2 a^2} \sum_r (kr - \gamma Ha) a_{r+} \bar{a}_{r-} \right),
\end{equation}

is non-normalisable, adopting the inside horizon limit ($k \gg Ha$). As it is only defined for $\gamma = \pm i$, only two components of the connection remain once the torsion-free condition is invoked. For $\gamma = i$, these are $\bar{a}_{L+}$ and $\bar{a}_{L-}$ with $\bar{a}_{R-} = \bar{a}_{R+} \approx 0$. Therefore, when computing the inner product of the Kodama state on-shell most of the terms vanish and we are left with

\begin{equation}
\langle \Psi^{KOD} | \Psi^{KOD} \rangle = \int da_{R+} d\bar{a}_{R+} \exp \left( \frac{|a_{R+}|^2}{kl_p^2} \right) \int da_{L-} d\bar{a}_{L-} \exp \left( -\frac{|a_{L-}|^2}{kl_p^2} \right).
\end{equation}

While the second term converges, the first term clearly diverges and the state is not normalisable. A similar result is obtained for $\gamma = -i$. Hence it can be seen that the perturbed Kodama state is not normalisable under the inner product previously identified, and our criterium implies that the excitations identified in the previous subsection belong to the unphysical Hilbert sub-space.

\begin{itemize}
\item We thank Abhay Ashtekar for pointing this out to us.
\end{itemize}
VI. CONCLUDING REMARKS

In this paper we have derived a perturbative quantum ground state for tensor fluctuations in de Sitter spacetime. We did so in the same set up leading to the Kodama state but also within the more general framework of any representation diagonalizing the connection. We identified perturbative wave functions for all values of $\gamma$ and factor orderings (factor ordering changes the vacuum energy and fluctuations but not the form of the wave functions; see [13]). We also identified the inner product, and the physical and non-physical modes. Two things are immediately evident about what these states are not. They’re not the Bunch-Davis vacuum: our vacua depend on $\gamma$ and for each value display a distinct chirality, reflected in the vacuum energy and fluctuations, in contrast with the Bunch-Davis vacuum. They’re also not the perturbed Kodama state, as can be seen by direct inspection. Our states represent new solutions which, we argue, are the physically correct ones for discussing phenomenology in quantum gravity. The perturbed Kodama state, as can be seen by direct inspection. Our states represent new solutions which, we argue, are the physically correct ones for discussing phenomenology in the Ashtekar formalism, at least in this representation.

Even though rejecting the Kodama state has by now become passé, our rejection has a strong object lesson to it, which is why we devoted Section V to contrasting our results and the perturbed Kodama state. Reality is often built from perturbation theory in a “Russian doll” set up where quantities order by order appear non-trivially in non-linear expressions (such as the Hamiltonian constraint) expanded to a certain order. For example, in [16] it was argued that the infamous “problem of time” in quantum gravity naturally fades away if one adopts a more practical perspective based on perturbation theory. The second order Hamiltonian is made up of terms quadratic in first order variables (to be used in perturbation theory) and terms linear in second order variables (the “backreaction”, to be ignored). The Hamiltonian constraint applies to the full second order Hamiltonian (including the backreaction), not to the Hamiltonian relevant for perturbation theory. Pragmatically, and somewhat ideosyncratically, it can be claimed that the problem of time is an illusion of non-perturbative theory, to be ignored in practice.

A similar sleight of hand in perturbation theory sees the self-dual operator factor non-perturbatively, but not perturbatively, in the Hamiltonian constraint. For this reason, the perturbed Kodama state is not a solution to the perturbative Hamiltonian. This convincingly disproves the Kodama state as the correct path to phenomenology in quantum gravity. The perturbed Kodama state cannot describe gravitons and its excitations are unphysical states. A theory of quantum gravity without gravitons is not a theory of quantum gravity at all. The implications are deep. The Kodama state is a semi-classical solution to the theory. Therefore it appears that (at least in this case) the semi-classical limit is the wrong path to reality, which may or may not be perturbative, but which is nonetheless fully quantum. This may well happen more generally, raising alarm bells about using WKB solutions to make contact with reality in quantum gravity theories. If nothing else, a moral lesson may be drawn from this paper: using the semi-classical approximation as a bridge to phenomenology may be tremendously unprofitable.

Obviously much work remains to be done. Somewhat trivially one may ask whether our construction could be extended to scalar and vector linear fluctuations, and with what implications for cosmology. More intriguing is the possibility that what we have uncovered in this paper is merely the tip of a non-perturbative iceberg, hinting at a new ground state in the full quantum gravity theory. We may conjecture that our perturbative solution is a linear approximation to a full non-perturbative solution. Whilst we have been unable to derive this state, we can infer some properties about it. It should be chiral. It should violate CPT, at least face value. The possibility remains that one might prove a no-go theorem regarding the existence of this non-perturbative state. Until that happens, it is an interesting challenge to work out its expression, as well as its associated inner product.

Acknowledgments

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Appendix A: The perturbed Kodama state in Fourier space

In this Appendix we derive an explicit expression for the perturbed Kodama state in Fourier space. We aim to do so with the same set of conventions we used for deriving graviton modes, and so start by reviewing these briefly (referring the reader to [16] for more details). We recall that the point of our expansions was the introduction of a priori independent positive and negative frequencies, and the introduction of boundary conditions ensuring that $k$ is indeed the direction of motion of a mode (rather than $-k$). The former point is essential in identifying all the modes in a theory which starts off complex. The latter point ensures that the correct physical polarizations are assigned to each mode ($R$ and $L$ don’t mean anything until we know the direction of motion); it also removes spurious pump terms inside the horizon (particle pair production). The implications are deep. The Kodama state is a semi-classical solution to the theory. Therefore it appears that (at least in this case) the semi-classical limit is the wrong path to reality, which may or may not be perturbative, but which is nonetheless fully quantum. This may well happen more generally, raising alarm bells about using WKB solutions to make contact with reality in quantum gravity theories. If nothing else, a moral lesson may be drawn from this paper: using the semi-classical approximation as a bridge to phenomenology may be tremendously unprofitable.

1. Hamilton’s equations in Fourier space

In writing the Kodama state in Fourier space there is an issue affecting the sympletic structure and Hamilton’s equations to that cannot be ignored when writing the perturbed self-dual equations in Fourier space. When
transitioning from position space to Fourier space, we use expansions \[16\]:

\[
\delta e_{ij} = \int \frac{d^3k}{(2\pi)^3} \sum_r e'_{ij}(k)\dot{\Psi}_r(k,\eta)e_{r+}(k)
+ e'_{ij}(k)\dot{\Psi}_r^+(k,\eta)e_{r-}(k)
\]

\[
a_{ij} = \int \frac{d^3k}{(2\pi)^3} \sum_r e'_{ij}(k)\dot{\Psi}_r^+(k,\eta)a_{r+}(k)
+ e'_{ij}(k)\dot{\Psi}_r^-\dot{\Psi}_r^+(k,\eta)a_{r-}(k) . \tag{A1}
\]

The virtues of these expansions were extolled in \[16\]. They avoid the embarrassment of finding that the reality conditions constrain the number of gravitons moving in opposite directions \[13\]. The central point in this respect is the use of independent positive and negative frequencies—i.e. ensuring that the theory initially contains gravitons and anti-gravitons. The amplitudes carry two indices: \(r\) for helicity and \(p\) for graviton/anti-graviton. All the reality conditions then do is to identify gravitons and anti-gravitons, mode by mode: \(k\) by \(k\), helicity by helicity.

It is also crucial that we require that \(k\) label the direction of motion (in the sense that \(-k\) should label the opposite direction of motion). This amounts to boundary condition:

\[
\Psi(k,\eta) \sim e^{-ik\eta} \tag{A2}
\]

when \(|\eta\eta| > 1\) for both \(+k\) and \(-k\) directions (where \(\tilde{\Psi}(k,\eta) = \Psi(k,\eta)e^{i\mathbf{k} \cdot \mathbf{x}}\)). However, when we insert expansions \([A1]\) into the Hamiltonian, it knows nothing about the boundary condition, so quite naturally its Hamiltonian’s equations try to spit out two types of modes for each \(r, p\):

\[
\Psi(k,\eta) \sim e^{\pm ik\eta}. \tag{A2}
\]

And obviously there is then a coupling between \(k\) and \(-k\) modes, because some of them are the same physical modes, only written down differently. Therefore we have to modify the Hamiltonian in Fourier space in order to account for the boundary condition implicit in \([A1]\).

To illustrate the issue more explicitly, let’s take Hamilton’s equations in position space for \(\gamma = \pm i\):

\[
\dot{a}_{ij}' = 2\gamma H^2a^2\delta e_{ij} - \gamma e_{imn}\partial_a a_{mj} \tag{A3}
\]

\[
\delta e_{ij}' = -\gamma(a_{ij} - e_{imn}\partial_a e_{mj}) . \tag{A4}
\]

If we Fourier transform them according to \([A1]\), assuming that all the modes are independent and satisfy the right boundary conditions, we obtain:

\[
\dot{a}_{rp}'(k) = \gamma p(-rka_{rp}(k) + 2H^2a^2\dot{e}_{rp}(k)) \tag{A5}
\]

\[
\dot{e}_{rp}'(k) = -\gamma(a_{rp}'(k) - rpk\dot{e}_{rp}(k)) . \tag{A6}
\]

Then, so that boundary condition \([A2]\) is met, we must have \(a_{rp} = 0\) when \(\gamma \gamma p = 1\). We also recover the results:

\[
\Psi'' + (k^2 - 2H^2a^2)\Psi = 0 \tag{A7}
\]

\[
\Psi_{rp} = \gamma p\Psi_{e} + r k\Psi_{e} \tag{A8}
\]

However, if we insert expansions \([A1]\) into the Hamiltonian (Eq. 17 of \([15]\)) we’ll find for modes inside the horizon:

\[
H_{eff} = \frac{1}{lp^2} \int d^3k \sum_r g_r-(k)g_{r+}(-k) + g_r+(k)g_{r-}^\dagger(-k)
+ g_{r+}^\dagger(k)g_{r-} + g_{r-}^\dagger(k)g_{r+}^\dagger(-k) , \tag{A9}
\]

with graviton operators defined as in \([53]-[50]\). With commutators \([1]\) we therefore obtain equations:

\[
\dot{a}_{r+}(k) = -\gamma rk(a_{r+}(k) + a_{r-}^\dagger(-k)) \tag{A10}
\]

\[
\dot{a}_{r-}^\dagger(k) = -\gamma rk(a_{r-}(k) + a_{r+}^\dagger(-k)) \tag{A11}
\]

that is, we get spurious couplings between the \(k\) and \(-k\) modes, which reflect the fact that nowhere in the formalism have we expressed the requirement that modes labeled by \(k\) are moving along \(k\), not \(-k\). This can be corrected by adopting the improved Hamiltonian:

\[
H_{eff} = \frac{1}{lp^2} \int d^3k \sum_r g_r-(k)g_{r-}^\dagger(-k) + g_{r+}^\dagger(k)g_{r+}(k) \tag{A9}
\]

which has been “told” the correct boundary condition.

2. The perturbed and the full Kodama state

With these conventions in mind we now write down the Kodama state non-perturbatively and perturbatively, first in position space then in Fourier space. A number of issues found with Hamilton’s equations are re-encountered here, and can be resolved in the same way.

The Kodama state is the solution to the self-dual equation \(\Phi = 0\) in the connection representation, that is with:

\[
\hat{A}_i^a(x)\Phi(A_i^a) = A_i^a(x)\Phi(A_i^a) \tag{A12}
\]

\[
\hat{E}_i^a(x)\Phi(A_i^a) = -i\gamma l_p^2\frac{\delta}{\delta A_i^a(x)}\Phi(A_i^a) \tag{A13}
\]

representing algebra:

\[
\left[\hat{A}_i^a(x), \hat{E}_j^b(y)\right] = i\gamma l_p^2\delta^a_b\delta_i^j(x - y) . \tag{A14}
\]

It’s easy to prove that the self-dual equation is satisfied by the Kodama, or Chern-Simons state:

\[
\Phi = N \exp\left(\frac{i\gamma}{2l_p H^2}S_{CS}\right) \tag{A15}
\]

with

\[
S_{CS} = \int \text{Tr}(A \wedge dA + \frac{2}{3}A \wedge A \wedge A)
+ \int d^3xe_{abc}A_i^a\partial_b A_i^c + \frac{1}{3}e_{ijk}A_i^aA_j^bA_k^c \tag{A16}
\]
This is because it can be easily checked that:

\[ \hat{E}_k^a S_{CS} = -2i\gamma l_p^2 B_k^a \]  

(A17)

so that:

\[ (\hat{B}^{ka} + H^2 \hat{E}^{ka}) \Phi = 0 \]  

(A18)

which is the self-dual equation.

These expressions result in equivalent ones for the perturbations in real space (see [14] for definitions). Specifically we have:

\[ 2 S_{CS} = \int \frac{d^3x}{a^3} (\epsilon_{ijk} a_{ni} \partial_j a_{kn} - \gamma H a a_{ij} a_{ij}) \]  

(A19)

and since now the algebra is:

\[ [a^a_i(x), \delta a^b_j(y)] = -i\gamma l_p^2 \delta a^b_j \delta(x - y) \]  

(A20)

we should have

\[ \delta a_{ij}(x) \Phi(a_{ij}) = a_{ij}(x) \Phi(a_{ij}) + \frac{\delta}{\delta a_{ij}(x)} \Phi(a_{nm}) \]  

(A21)

\[ \delta e_{ij}(x) \Phi(a_{nm}) = i\gamma l_p^2 \frac{\delta}{\delta a_{ij}(x)} \Phi(a_{nm}) \]  

(A22)

Therefore:

\[ aH^2 \delta e_{ij} \Phi = \Phi(\epsilon_{inn} \partial_n a_{mj} - \gamma H a a_{ij}) \]  

(A23)

and we still satisfy

\[ (\delta B_{ij} - H^2 \delta e_{ij})^2 \Phi = 0 \]  

(A24)

(which is the perturbed SD equation), because:

\[ \delta B_{ij} = \frac{1}{a} (\epsilon_{inn} \partial_n a_{mj} - \gamma H a a_{ij}) \]  

(A25)

If now we try to transpose this to Fourier modes we find a problem similar to that described in the previous subsection. If we expand the Chern-Simons action naively, we obtain for the second order terms quadratic in first order variables:

\[ \frac{1}{4} S_{CS} = \frac{1}{a^2} \int d^3k \sum_r 2(\kappa r - \gamma H a)[a_{r+}(k)a_{r+}(-k) + 2a_{r+}(k)\bar{a}_{r-}(k) + \bar{a}_{r-}(k)\bar{a}_{r-}(-k)] \]  

(A26)

We find that this represents

\[ \delta B_{ij} = \frac{i\gamma a}{2l_p} \frac{\delta}{\delta a_{ij}} 2 S_{CS} \]  

(A27)

but not mode by mode, independently. Indeed

\[ B_{rp} = (\kappa k - \gamma p H a) a_{rp} \]  

(A28)

and in order to get the required

\[ B_{rp} = \frac{a^2}{4} \frac{\delta}{\delta a_{rp}} 2 S_{CS} \]  

(A29)

we should discard the terms coupling k to -k. Therefore the perturbed Kodama state in Fourier space is:

\[ \hat{\delta \Phi} = N \exp \left( \frac{2i\gamma}{l_p^2 H^2 a^2} \int d^3k \sum_r (\kappa r - \gamma H a)[a_{r+}(k)\bar{a}_{r+}(-k) + 2a_{r+}(k)\bar{a}_{r-}(k) + \bar{a}_{r-}(k)\bar{a}_{r-}(-k)] \right) \]  

(A30)

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