Primordial blackholes and gravitational waves for an inflection-point model of inflation

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In this article we provide a new closed relationship between cosmic abundance of primordial gravitational waves and primordial blackholes originated from initial inflationary perturbations for inflection-point models of inflation where inflation occurs below the Planck scale. The current Planck constraint on tensor-to-scalar ratio, running of the spectral tilt, and from the abundance of dark matter content in the universe, we can deduce a strict bound on the current abundance of primordial blackholes to be within a range, $9.99712 \times 10^{-3} < \Omega_{PBH}h^2 < 9.99736 \times 10^{-3}$.

I. INTRODUCTION

In the Einstein’s general relativity (GR) the primordial blackholes (PBHs) with a small mass can be created during the radiation epoch due to over density on length scales $R \sim 1/k_{PBH}$, which is typically much smaller than the pivot scale at which the relevant perturbations re-enter the Hubble patch for the large scale structures, $k_\ast$ [1–3]. Typically the regions with a mass less than the size of the Hubble radius can collapse to form PBHs, i.e., $M \leq \gamma M_H \sim \gamma (4\pi/3) \rho H^{-3}(t) \approx 2 \times 10^5 \gamma (t/1 \text{ s}) M_\odot$, where $\rho$ is the energy density of the radiation epoch, $H$ is the Hubble radius, $M_\odot \sim 10^{33}$.g, and $\gamma \leq 0.2$ is the numerical factor during the radiation era which depends on the dynamics of gravitational collapse [2]. For instance, an economical way would be to create PBH abundance from an initial primordial inflationary fluctuations which had already entered the Hubble patch during the radiation era, but whose amplitude had increased on small scales due to the running in the spectral index, $n_s$ [4, 5] 1.

An interesting observation was first made in Ref. [8] and in Refs. [9–14], that a sub-Planckian inflaton field can create a significant primordial gravitational waves (PGWs) provided the last $50 - 60$ e-foldings of inflation is driven through the inflection-point, such that the tensor to scalar ratio saturates the Planck constraint, $r \leq 0.12$ [15]. One requires a marginal running in the power spectrum which is now well constrained by the Planck+WMAP9 combined data. A valid particle physics model of inflation can only occur below the cut-off scale of gravity, see for a review on particle physics models of inflation [16], it would be interesting to study the implications of the running of the spectral tilt, $\alpha_s$, for both PGWs and PBHs.

Formation of the significant amount of PBHs on a specific mass scale is realized iff the power spectrum of primordial fluctuations has amplitude $10^{-2} - 10^{-1}$ on the corresponding scales [17]. In such a physical situation the second-order effects in the cosmological perturbation are expected to play an significant role in the present set up. Also such non-negligible effects also generate tensor fluctuations to produce PGWs from scalar-tensor modes via terms quadratic in the first-order matter and metric perturbations [18–20]. Most importantly, their amplitude may well exceed the first-order tensor perturbation generated by quantum fluctuation during inflation in the present set up as the amplitude of density fluctuations required to produce PBHs is large.

The aim of this paper is to provide an unique link between the current abundance of PBHs, $\Omega_{PBH}(\eta_0) = \rho_{PBH}/\rho_c$, and the abundance of primordial gravitational waves $\Omega_{GW} = \rho_{GW}/\rho_c$ in our universe originated from the primordial fluctuations, where $\eta_0$ is the present conformal time and $\rho_c$ denotes the critical energy density of the universe. With the help of Planck data, we will be able to constrain a concrete bound on $\Omega_{PBH}h^2$.

II. PBH FORMATION

Let us first start the discussion with the amplitude of the scalar power spectrum, which is defined at any arbitrary momentum scale lying within the window, $k_L < k < k_A$, by:

$$P_s(k) = P_s(k_s) \left( \frac{k}{k_s} \right)^{n_s-1+\frac{\alpha_s}{2}} \ln \left( \frac{k}{k_s} \right) + \left( \frac{k}{k_s} \right)^{n_s-1+\frac{\alpha_s}{2}} \ln^2 \left( \frac{k}{k_s} \right) + \cdots$$

(1)

where the parameters $n_s$, $\alpha_s$ and $k_s$ are spectral tilt, running and running of running of the tilt of the scalar perturbations defined in the momentum pivot scale $k_s$. Also note that the PBH formation scale is lying within the window, $k_L < k_{PBH} < k_s < k_A$. In the realistic situations the upper and lower bound of the momentum scale is fixed at, $k_A = 0.056 \text{ Mpc}^{-1}$ for $l_A = 2500$ and $k_L = 4.488 \times 10^{-5} \text{ Mpc}^{-1}$ for $l_L = 2$.

Within this window, we need to modify the power law parameterization of the power spectrum by incorporating the effects of higher order Logarithmic corrections in terms of the non-negligible running, and running of the running of the spectral tilt as shown in

1 A word of caution - GR is not an ultraviolet (UV) complete theory. An UV completion of gravity may naturally lead to ghost free and asymptotically free theory of gravity, as recently proposed in Ref. [6, 7]. In such a class of theory it has been shown that mini-blackhole with a mass less than the Planck mass, i.e., $10^{-7} \text{ g}$ does not have a singularity and nor does it have a horizon [6].
Eq (1), which involves higher order slow-roll corrections in the next to leading order of effective field theory of inflation. This is important to consider since Planck-WMAP9 combined data have already placed interesting constraints on $n_s(k_*) = 0.9603 \pm 0.0073$ (within 2σ C.L.), $\alpha_s(k_*) = 0.0134 \pm 0.0090$, and $\kappa_s(k_*) = 0.020^{+0.016}_{-0.015}$ (within 1σ - 1.5σ C.L.) [15].

Further using Eq (1), spectral tilt, running of the tilt, and running of the running of the tilt for the scalar perturbations can be written at any arbitrary momentum scale within the widow, $k_{i=2} < k < k_{i=2500}$, as:

$$n_s(k) = 1 + \frac{d \ln P_s(k)}{d \ln k} = n_s(k_*) + \alpha_s(k_*) \ln \left( \frac{k}{k_*} \right) + \frac{\kappa_s(k_*)}{2} \ln^2 \left( \frac{k}{k_*} \right) + \cdots$$

$$(2)$$

$$\alpha_s(k) = \frac{dn_s(k)}{d \ln k} = \alpha_s(k_*) + \kappa_S(k_*) \ln \left( \frac{k}{k_*} \right) + \cdots$$

$$(3)$$

$$\kappa_s(k) = \frac{d\alpha_s(k)}{d \ln k} = \frac{d^2 n_s(k)}{d \ln k^2} \approx \kappa_s(k_*) + \cdots$$

$$(4)$$

At the scale of PBH formation, $k = k_{PBH}$, the value of the tilt, and running of the running of tilt for the scalar perturbations can be expanded around the pivot scale ($k_*$) as:

$$n_s(k_{PBH}) = n_s(k_*) + \alpha_s(k_*) \ln \left( \frac{k_{PBH}}{k_*} \right)$$

$$+ \frac{\kappa_s(k_*)}{2} \ln^2 \left( \frac{k_{PBH}}{k_*} \right) + \cdots$$

$$(5)$$

$$\alpha_s(k_{PBH}) = \alpha_s(k_*) + \kappa_S(k_*) \ln \left( \frac{k_{PBH}}{k_*} \right) + \cdots$$

$$(6)$$

$$\kappa_s(k_{PBH}) \approx \kappa_s(k_*) + \cdots$$

$$(7)$$

provided the expansion is valid when, $k_{PBH} < k_*$. Throughout the article we will fix the pivot scale to be the same as that of the Planck, $k_* \sim 0.002 \text{ Mpc}^{-1}$. However, it is important to note that the physics is independent of the choice of the numerical value of $k_*$. The $\cdots$ represent higher order slow-roll corrections appearing in the expansion. Here the pivot scale of momentum $k_*$ is a normalization scale and is of the order of the UV regularized scale of the momentum cut-off of the power spectrum beyond which the logarithmically corrected power law parameterization of the primordial power spectrum for scalar modes does not hold good. More precisely, $k_*$ be a floating momenta in the present context. Additionally, we have used another restriction on the momentum scale, $k_{i=2} < k_{PBH} < k_* < k_{i=2500}$. For more details see Eq (1) mentioned later. The initial PBHs mass, $M_{PBH}$, is related to the Hubble mass, $M$, by:

$$M_{PBH} = M \gamma = \frac{4\pi}{3} \gamma \rho \mathcal{H}^{-3},$$

$$(8)$$
at the Hubble entry, where the Hubble parameter $H$ is defined in terms of the conformal time, $\eta$. The PBH is formed when the density fluctuation exceeds the threshold for PBH formation given by the Press–Schechter theory [21]

$$f(\geq M) = 2\gamma \int_{\theta_{cb}}^{\infty} d\theta \mathcal{P}(\theta; M(k_{PBH})).$$

Here $\mathcal{P}(\theta; M(k_{PBH}))$ is the Gaussian probability distribution function of the linearized density field $\theta$ smoothed on a scale, $k_{PBH} = 1/R$, by [22]:

$$\mathcal{P}(\theta; k_{PBH}) = \frac{1}{\sqrt{2\pi \Sigma_{\theta}(k_{PBH})}} \exp\left(-\frac{\theta^2}{2 \Sigma_{\theta}^2(k_{PBH})}\right)$$

where the standard deviation is given by

$$\Sigma_{\theta}(k_{PBH}) = \sqrt{\int_{0}^{\infty} \frac{dk}{k} \exp\left(-\frac{k^2}{k_{PBH}^2}\right) P_\theta(k)}.$$ 

Now substituting Eq. (12) and Eq. (1) in Eq. (11), for $k_{PBH} = 1/R$, we can express $\Sigma_{\theta}(k_{PBH})$ as:

$$\Sigma_{\theta}(k_{PBH}) = \frac{2}{5} \frac{(1+w)\sqrt{P_S(k_*)}}{(1+\frac{3}{5}w)} \left(\frac{k_*}{aH}\right)^2 \sqrt{\int_{k_L}^{k_*} \frac{dk}{k} \exp\left(-\frac{k^2}{k_{PBH}^2}\right) P_\theta(k)}.$$ 

where we have reparametrized the integral in terms of the regulated UV (high) and IR (low) momentum scales. The cut-offs ($k_A$ and $k_L$) are floating momenta to collect only the finite contributions. The technique we employ here has a similarity to the cut-off regularization scheme, which is being introduced in such a fashion that after taking the physical limit, ($k_A \to \infty$, $k_L \to 0$), the result returns to the original infinite integral.

Here the UV and the IR cut-offs must satisfy the constraint condition, $k_L \ll k_{PBH} \ll k_* \ll k_A$, for which the integral appearing in the expression for the standard deviation can be regularized. In Eq. (13), $A$, $B$, $C$ and $D$ are all momentum dependent coefficients which are explicitly mentioned in the appendix, see Eq. (23). Moreover, at the Hubble exit an additional constraint, $k_* = aH$, will have to be satisfied in order to do the matching of the long and short wavelength perturbations.

Hence, substituting the explicit expressions for $P_S$, $n_s$, $\alpha_s$ and $\kappa_s$ in presence of the higher order corrections at the pivot scale $k_*$, the simplified expression for the regularized standard deviation in terms of the leading order slow-roll parameters can be written as:

$$\Sigma_{\theta}(k_{PBH}) = \frac{(1+w)A_{\nu}}{8\sqrt{6\pi M_{Pl}^2}} \left\{ 1 + \frac{B}{A} \left(C_E + \frac{2}{5}\right) \eta_\nu(k_*) - \frac{B}{A} \left(2C_E + \frac{11}{5}\right) \epsilon_\nu(k_*) - \frac{C}{5A} \xi_\nu^2(k_*) + \frac{2D}{5A} \sigma_\nu^3(k_*) + \cdots \right\}$$

where $C_E = 4(\ln 2 + 2\gamma_E) - 5$, and $\gamma_E = 0.5772$ is the Euler-Mascheroni constant. Here the $(\epsilon_\nu, \eta_\nu, \xi_\nu^2, \sigma_\nu^3)$ are slow roll parameters for a given inflationary potential $V(\phi)$. It is important to mention that the results obtained in this paper are inflation centric - true only for inflection point models of inflation.

### III. PBH AND GW FOR SUB-PLANCKIAN MODEL OF INFLATION

For a successful inflation, the potential should be flat enough, and for a generic inflationary potential around the vicinity of the VEV $\phi_0$, where inflation occurs, we
may impose the flatness condition such that, \( V''(\phi_0) \approx 0 \). This yields a simple flat potential which has been imposed in many well motivated particle physics models of inflation with an inflection-point [27]:

\[
V(\phi) = \alpha + \beta (\phi - \phi_0) + \gamma (\phi - \phi_0)^2 + \kappa (\phi - \phi_0)^4 + \cdots ,
\]  

\( \text{(15)} \)

where \( \alpha \) denotes the height of the potential, and the coefficients \( \beta, \gamma, \kappa \) determine the shape of the potential in terms of the model parameters. Note that at this point, we do not need to specify any particular model of inflation for the above expansion of \( V(\phi) \). However, not all of the coefficients are independent once we prescribe the model of inflation here. This only happens if the VEV of the inflaton \( \phi_0 < M_p \) must be bounded by the cut-off of the particle theory, where the reduced Planck mass \( M_p = 2.4 \times 10^{18} \text{ GeV} \).

We are assuming that in 4 dimensions \( M_p \) puts a natural cut-off here for any physics beyond the Standard Model. The another assumption we have made here is that the range of flatness of the potential of inflation \( |\Delta \phi| = |\phi_{cmb} - \phi_0| \sim O(10^{-1} M_p) < M_p \), for which the model is fully embedded within a particle theory such as that of gauge invariant flat directions of minimal supersymmetric Standard Model (MSSM), or MSSM \( \times U(1)_{B-L} \) [27]. Here \( \phi_{cmb} \) and \( \phi_0 \) are the inflaton field at the Hubble crossing and the end of inflation respectively.

Both these sub-Planckian constraints leads to the observed tension of the low power of the CMB at low-\( l \) with the high-\( l \), which leads to a negative curvature of the power spectrum at small \( k \) and a positive curvature of the power spectrum at large \( k \), see [28]. For an inflexion-point model of inflation this will provide an improved constraint on PBH formation and GW waves via running in the power spectrum (see fig (2) for the details.).

The fractional density of PBH formation can be calculated as:

\[
f(\geq M) = \gamma \frac{\Theta_{th}}{\sqrt{2} \Sigma_G(k_{PBH})},
\]  

\( \text{(16)} \)

In general the mass of PBHs is expected to depend on the amplitude and the shape of the primordial perturbations. The relation between the PBH formation scale \( k_{PBH} = 1/R \) and the PBH mass can be expressed as:

\[
k_{PBH} = \frac{\sqrt{7}}{5.54 \times 10^{-24}} \left( \frac{M_{PBH}}{1 \text{ g}} \right)^{-\frac{1}{2}} \left( \frac{g_*}{3.36} \right)^{-\frac{1}{6}} Mpc^{-1}.
\]  

\( \text{(17)} \)

Moreover, we can express the fractional density of PBH formation in terms of the PBH abundance at the present epoch, \( \eta_0 \), as [18]:

\[
\Omega_{PBH} h^2(\eta_0) = 10^{14} \times \frac{M_{PBH}}{10^{20} \text{ g}} \left( \frac{g_*}{3.36} \right)^{-\frac{1}{2}}.
\]  

\( \text{(18)} \)

The recent observations from Planck puts an upper bound on the amplitude of primordial gravitational waves via tensor-to-scalar ratio, \( r(k_*) = P_T/P_S \). This bounds the potential energy stored in the inflationary potential, i.e., \( V_* \leq (1.96 \times 10^{18} \text{GeV})^4 (r(k_*)/0.12) \) [15].

With the help of Eqs. (10, 1, 14, 17, 18), we can link the GW abundance at the present time:

\[
\Omega_{GW} h^2(\eta_0) = \left( \frac{\Sigma_G(k_{PBH})}{10^{-2}} \right)^2 \left( \frac{M_{PBH}}{10^{20} \text{ g}} \right)^{1/2} \frac{\Omega_{PBH} h^2 f^{-1}}{1.7 \times 10^{21}}.
\]  

\( \text{(19)} \)

In order to realize inflation below the Planck scale, i.e. \( M_p \), we need to observe the constraint on the flatness of the potential, i.e. \( \Delta \phi \), via the tensor-to-scalar ratio as recently obtained in Refs. [14] 2:

\[
\frac{\Delta \phi}{M_p} \approx 6 \times \frac{\sqrt{25}}{25} \sqrt{\frac{r(k_*)}{0.12}} - \eta \frac{r(k_*)}{2} - 1 + \cdots.
\]  

\( \text{(20)} \)

Note that it is possible to saturate the Planck upper bound on tensor- scalar ratio, i.e. \( r(k_*) \leq 0.12 \) for small field excursion characterized by, \( \phi_0 < M_p \) and \( |\Delta \phi| < M_p \). Additionally, \( \cdots \) contain the higher order terms in the slow roll parameters which only dominate in the next to leading order.

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2 There was a very minor numerical error in earlier computation in Ref. [10, 13], which we have now corrected it in Ref. [14]. But the final conclusion remains unchanged due to such correction.
Collecting the real root of the tensor-to-scalar ratio, $r$, in terms of the field displacement $|\Delta \phi|$ from Eq. (20), at the leading order, we can derive a closed constraint relationship between $\Omega_{GW}$ and $\Omega_{PBH}$ at the present epoch, for inflection point inflationary models:

$$\begin{align*}
\Omega_{GW} h^2 &\leq \frac{6 \times 10^{-18}}{\gamma} \left( \frac{M_{PBH}}{10^{20} \text{ g}} \right)^{1/2} \frac{\mathcal{O}_{PBH}^{2} \Omega_{PBH} h^2}{\sigma_{\phi} \sqrt{2 \Phi_{PBH}}},
\end{align*}$$

(21)

where we introduce a new parameter $\mathcal{O}_{PBH}$, which can be expressed in terms of the inflationary observables as mentioned in the appendix, see Eq. (23). For definiteness, we also identify the PBH mass with the horizon mass when the peak scale is within the sub-horizon region. An interesting observation can be made here; typically the PBH mass can also be related to the peak frequency of the GWs produced from the collapse of an over-densed region of the space-time to form PBHs, see [18], by following the linearized gravitational wave fluctuations up to the second order [19]. The peak frequency is given by:

$$f_{GW}^{\text{peak}} = k_p / \sqrt{3 \pi a_0},$$

where $k_p$ is the peak value of the momentum scale, and $a_0$ is the scale factor at the present epoch. Therefore, the final expression yields for frequency and the amplitude [18]:

$$\begin{align*}
f_{GW}^{\text{peak}} & = 0.03 \text{ Hz} \\
A_{GW}^{\text{peak}} & = 6 \times 10^{-8} \left( \frac{g_\ast}{3.36} \right)^{-1/3} \left( \frac{\sigma_{\phi}(k_{PBH} \approx k_p)}{10^{-2}} \right)^2,
\end{align*}$$

(22)

where $\Sigma_{\phi}(k_{PBH} \approx k_p)$ is explicitly defined in Eq (14), see Ref. [18].

It is important to note that the space-based laser interferometers are sensitive to GWs with frequency range $10^{-5} H z \lesssim f \lesssim 10 H z$, which covers the entire mass range of the PBHs, $10^{20} \text{ g} < M_{PBH} < 10^{24} \text{ g}$, which comes from the stringent Dark Matter constraint, $\Omega_{DM} h^2 = 0.1196 \pm 0.0031$ from Planck + WMAP9 combined dataset within $1.5 \sigma$ C.L [28].

LISA [29] can probe up to its best sensitivity $\Omega_{GW} h^2 \sim 10^{-11}$ at GW frequency $f \sim 10^{-2} H z$ corresponding to the PBH mass $M_{PBH} \sim 10^{21} \text{ g}$. DECIGO/BBO [30] and the ultimate-DECIGO [31] are designed to detect primordial GWs with effective GW frequency $f_{GW} > 10 H z$ which cannot be observed at the present epoch.

Constraints from all of these GW detectors represented by convex lines with different color codes in Logarithmic scale in Fig. (2). We have also shown the variation of GW abundance for low (green) and high (yellow) scale sub-Planckian models by varying PBH mass ($M_{PBH}$) and tensor-to-scalar ratio ($r$) using Eq. (21) and Eq. (19) in Fig. (2). Additionally, we have shown the two wedge-shaped curves shown in red represented by ($\Omega_{PBH} h^2, M_{PBH}$)= ($10^{-5}, 30 M_\odot$) (left) and ($10^{-1}, 10^{22} \text{ g}$) (right) for relativistic degrees of freedom $g_\ast = 228.75$. The appearance of the sharp peaks in the left and right wedge shaped red curves signify the presence of maximum value of the GW abundances at the present epoch corresponding to the peak frequency given by Eq. (22). Each wedge shaped curves accompany smooth peaks, this is due to the resonant amplification procedure when the peak width for fluctuation, $\Delta_p < < k_p/2$. If the peak width exceeds such a limit then the frequency of the fluctuations will increase and we get back the peak for sharp fluctuation in the right side for each of the wedge shaped curve.

In Fig. (3), we have shown the behaviour of the PBH abundance with running of the spectral tilt within the Planck 2$\sigma$ C.L. (black region) of spectral-tilt [15]. We have explicitly shown the 1$\sigma$ allowed constraint on the running of the spectral tilt by the green shaded region which additionally puts a stringent constraint on the PBH abundance within a tiny region $9.99712 \times 10^{-3} < \Omega_{PBH} h^2 < 9.99736 \times 10^{-3}$. Note that if we incorporate Planck + WMAP9 + high L + BICEP2 data [35], our results would modify, although the physics behind the mechanism would remain be the same. We would like to revisit the problem in future with a more detailed study.
IV. CONCLUSION

To summarize, we have shown that it is possible to establish a generic relationship between PBH and GW abundance for a sub-Planckian model of inflation with a flat potential, where inflation is driven near an inflection-point. For such a class of model it is possible to predict $\Omega_{GW} h^2$ and $\Omega_{PBH} h^2$ with the help of this new expression given by Eq. (21). We have used important constraints arising from various GW detectors, which we have shown in Fig. (2), and the PBH abundance with running of the spectral tilt in Fig. (3).

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VI. APPENDIX

The momentum dependent co-efficients appearing in Eq (13) and $\mathcal{O}_{PBH}$ appearing in Eq (21) are given by:

\[
A = \left[ \frac{\sqrt{\pi} k_{PBH}}{2 k_\ast} \right] \text{erf} \left[ \frac{k}{k_{PBH}} \right] \left( 1 + \ln \left( \frac{k}{k_\ast} \right) \right) \frac{k_\Lambda}{k_L} + B, \\
B = \left[ \frac{\sqrt{\pi} k_{PBH}^2}{2 k_\ast} \right] \text{erf} \left[ \frac{k}{k_{PBH}} \right] \ln \left( \frac{k}{k_\ast} \right) - 2 \left( \frac{k}{k_\ast} \right) \left( \frac{k_{PBH}}{k_\ast} \right)^2 \mathcal{F}_{Q} \left[ \left\{ \frac{1}{2}, \frac{1}{2} \right\} ; \left\{ \frac{3}{2}, \frac{3}{2} \right\} ; - \left[ \frac{k_{PBH}^2}{k_\ast} \right] \right] \frac{k_\Lambda}{k_L}, \\
C = \left[ \frac{\sqrt{\pi} k_{PBH}}{4 k_\ast} \right] \text{erf} \left[ \frac{k}{k_{PBH}} \right] \ln^2 \left( \frac{k}{k_\ast} \right) - \left( \frac{k}{k_\ast} \right) \ln \left( \frac{k}{k_\ast} \right) \left( \frac{k}{k_\ast} \right) \left( \frac{k_{PBH}}{k_\ast} \right)^2 \mathcal{F}_{Q} \left[ \left\{ \frac{1}{2}, \frac{1}{2} \right\} ; \left\{ \frac{3}{2}, \frac{3}{2} \right\} ; - \left[ \frac{k_{PBH}^2}{k_\ast} \right] \right] \frac{k_\Lambda}{k_L}, \\
D = \left[ \frac{\sqrt{\pi} k_{PBH}}{12 k_\ast} \right] \text{erf} \left[ \frac{k}{k_{PBH}} \right] \ln^3 \left( \frac{k}{k_\ast} \right) - \left( \frac{k}{k_\ast} \right) \ln^2 \left( \frac{k}{k_\ast} \right) \mathcal{F}_{Q} \left[ \left\{ \frac{1}{2}, \frac{1}{2} \right\} ; \left\{ \frac{3}{2}, \frac{3}{2} \right\} ; - \left[ \frac{k_{PBH}^2}{k_\ast} \right] \right] \frac{k_\Lambda}{k_L}, \\
\mathcal{O}_{PBH} = \frac{5 \sqrt{A}(1 + w)(8.17 \times 10^{-3})^2}{12 \sqrt{2\pi} (1 + 3w)} \left\{ 1 + \frac{2 B \nu(k_\ast)}{5A} + \frac{B}{500A} \left( \frac{r(k_\ast)}{0.12} - \frac{3B}{A} \epsilon_V(k_\ast) - \frac{C}{5A} \xi_V(k_\ast) + \frac{2D}{5A} \sigma_V^2(k_\ast) + \cdots \right) \right\} \\
\tag{23}
\]

where $\mathcal{F}_Q$ represents generalized Hypergeometric function.

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