Galaxy Redshifts: improved techniques

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ABSTRACT
This paper analyses the effects of random noise in determining errors and confidence levels for galaxy redshifts obtained by cross-correlation techniques. The main finding is that confidence levels have previously been overestimated, and errors inaccurately calculated in certain applications. New formulæ are presented.

KEYWORDS: Methods: data analysis, statistical. Galaxies: distances & redshifts.

1 INTRODUCTION
It is becoming increasingly possible to obtain very large numbers of galaxy spectra through the use of systems such as those involving optical fibres. With such large datasets, automated procedures for obtaining redshifts become desirable and necessary for efficiency. It is important to have automatic and objective measures of the confidence levels and errors in the results, so that suspect redshifts may be investigated individually by manual inspection of the data. Clearly, it is important that the initial automatic quality assessment is founded on a firm theoretical basis.

Obtaining redshifts from galaxy spectra is commonly effected by cross-correlating the spectrum with the spectrum of a ‘template’ galaxy of known redshift. The relative shift of the two spectra when the cross-correlation function (ccf) has its highest peak is then used to estimate the redshift of the galaxy. The procedure for doing this was detailed and analysed by Tonry & Davis (1979; hereafter TD). In the case of identical galaxy and template spectra, the technique is excellent, but in practice there are differences – either intrinsic spectral differences, or the presence of noise. These differences can lead to an error in the derived redshifts, or an entirely spurious redshift being picked up. This paper puts the analysis of noise in ccf techniques on a firm footing, and presents improved formulæ for calculating confidence levels and errors.

2 METHOD
The method of obtaining redshifts from ccf techniques is presented in TD. One bins the galaxy spectrum \( g_n \) and template \( t_n \) into \( N \) bins \( (n = 0, \ldots, N - 1) \) logarithmically-spaced in wavelength (initially we follow almost the same notation as TD: \( g(n) \) in TD is written here as \( g_n \)). The spectra are continuum-subtracted and may be filtered to remove long-wavelength components (e.g. from incomplete continuum subtraction) and/or short wavelength contributions on sub-resolution scales. A cross-correlation function is then made from the galaxy and template spectra, and the highest peak in the ccf is used to calculate the relative shift between the two. This is then used to deduce the galaxy redshift, assuming the template redshift is known (see TD for more details).

Since the ccf technique is most conveniently implemented via Fourier methods, the spectra are operationally assumed to be periodic, so are apodised to remove spurious short-wavelength components coming from a mismatch at the two ends of the spectrum.

Let the discrete Fourier transforms (FTs) of \( g_n \) and \( t_n \) be \( G_k \) and \( T_k \), where, for example

\[
G_k = \sum_{n=0}^{N-1} g_n e^{-2\pi i nk/N} \quad k = 0, \ldots, N - 1. \tag{1}
\]

With this notation, the inverse transform is \( g_n = (1/N) \sum_k G_k e^{2\pi i nk/N} \) and the variance of \( g \) is \( \sigma_g^2 = (1/N) \sum_n g_n^2 = (1/N^2) \sum_k |G_k|^2 \) and all sums run from 0 to \( N - 1 \) unless otherwise stated. The normalised ccf is formed:

\[
c_n \equiv \frac{1}{N\sigma_g\sigma_t} (g \times t)_n \equiv \frac{1}{N\sigma_g\sigma_t} \sum_m g_m t_{m-n}. \tag{2}
\]

We assume that \( g_n \) is identical to \( t_n \) except it is:

1. different by addition of a random noise component \( e_n \)
2. multiplied by \( \alpha \)
3. shifted by \( \delta \) (in bins)
4. broadened by convolution with a symmetric function \( b_n \)
\[ g_n = \alpha [(t * b)_{n-\delta} + e_n] \] (3)

where the \(*\) indicates convolution. Specifically:
\[ (t * b)_n = \sum_m t_m b_{n-m}. \] (4)

Transforming,
\[ G_k = \alpha \left( T_k B_k e^{-2\pi i k/N} + E_k \right) \] (5)
where \(B_k\) and \(E_k\) are the Fourier transforms of the broadening function \(b_n\) and the noise \(e_n\). The normalised ccf is
\[ c_n = \frac{1}{N\sigma_g^2} \sum_m \alpha \left( \sum_p t_p b_{m-\delta-p} + e_m \right) t_{m-n} \] (6)
or
\[ c_n = \alpha [(t * b) \times t]_{n-\delta} + [e \times t]_n. \] (8)

In Fourier space, we assume the noise and the template are the power of the cross-correlation technique. We want the probability that a peak in the ccf has been caused by noise. It is here that we may improve substantially on the analysis of TD, using the theory of peaks in gaussian noise originally developed by Rice (1954). Let the range of the ccf be \(L\) bins, and let the believable fraction of this be \(f\) (one might wish, for example, to exclude negative redshifts). With the Fourier technique, \(L = N\). The expected number of noise peaks, in a range \(fL\), higher than the observed peak \(c_d\) is
\[ \hat{N} = n_{pkf}(> \nu) fL \] (12)
where \(\nu = c_j/(\sqrt{2}\sigma_n)\) is the height of the ccf peak in units of the r.m.s. noise. \(n_{pkf}(> \nu)\) is the number density of peaks above \(\nu\), and is given by (Williams et al. 1991):
\[ n_{pkf}(> \nu) = \frac{1}{4\pi R} \left\{ \text{erfc} \left( \frac{\nu}{\sqrt{2(1-\gamma^2)}} \right) + \gamma e^{-\nu^2/2} \left[ 1 + \text{erf} \left( \frac{\gamma\nu}{\sqrt{2(1-\gamma^2)}} \right) \right] \right\}. \] (13)

Note that \(\nu\) here is identical to \(r\) in TD. The quantities \(R \equiv \sigma_1/\sigma_2\) and \(\gamma \equiv \sigma_2/\sigma_1\) characterise the noise in terms of moments of its power spectrum:
\[ \sigma_j^2 = \frac{2}{N^2} \sum_k \left| N_k \right|^2 \left( \frac{2\pi k}{N} \right)^{2j}. \] (14)

Note that the sum extends over half the space. Reality of the noise ensures each half contributes equally. \(R\) is a length which characterises the coherence properties of the noise, and \(\gamma\) is a dimensionless parameter which measures the relative contribution from short and long wavelengths.

To obtain the confidence level \(C\) for the observed peak in the ccf, we need the probability that there are no higher noise peaks in the interval. This is a non-trivial task, and the theory of peaks cannot currently provide an answer.
However, a good approximation for high peaks is to assume that the peaks are uncorrelated (Williams et al. 1991), in which case

$$C \simeq e^{-\bar{N}} = e^{-n_{pks}(>0) f L}.$$  

(15)

This should be very accurate if $\bar{N} \ll 1$. If not, the confidence is low, and the redshift obtained will be suspect anyway.

To obtain the confidence level, we need to estimate the parameters $R$ and $\gamma$ from the noise. This may be done either by performing an FT on the antisymmetric part of the ccf, and using (14), or by making use of the following results, which follow from (13):

$$n_{pks}(> - \infty) = \frac{1}{2\pi R}; \quad n_{pks}(> 0) = \frac{1 + \gamma}{4\pi R}. \quad (16)$$

The total number of peaks $N_{pks}$, and the number of positive peaks $N_{pks}(> 0)$ may therefore be used more straightforwardly to estimate $R$ and $\gamma$:

$$\gamma \simeq 2 N_{pks}(> 0) / N_{pks}; \quad R \simeq \frac{L}{2\pi N_{pks}}. \quad (17)$$

The two methods give, in tests involving nearly a thousand ccf's, the same answers to an r.m.s. accuracy of 3% for $R$. The accuracy in $\gamma$ is not so high, 7% to 24% depending on whether the spectra are filtered or not. The Fourier method is to be preferred.

For high peaks ($\nu \gg 1$), one may make use of the asymptotic expansion $\text{erfc}(z) \rightarrow e^{-z^2}/(\sqrt{\pi z})$ to get an answer good to 3% for $\nu > 2$ and $\gamma > 0.5$:

$$C \simeq 1 - 2f \left( N_{pks}(> 0) - \frac{1}{2} N_{pks} \right) e^{-\nu^2/2} \quad (\nu \gg 1). \quad (18)$$

This may be compared with TD’s high-peak limit $C_{TD} = 1 - \sqrt{2/\pi} f N_{pks}(> 0) e^{-\nu^2/2}/\nu$. TD’s analysis thus overestimates the confidence by an amount

$$\frac{1 - C}{1 - C_{TD}} = \sqrt{2\pi} \frac{\nu\gamma}{1 + \gamma} \quad (\nu \gg 1). \quad \quad (19)$$

These confidence estimates have been compared with TD’s for a sample of around 80 galaxies with redshifts $\lesssim 0.05$, each correlated with 8 radial velocity standard stars and 2 nearby galaxies. The signal-to-noise of the ccf's is generally high, with only one with $\nu < 2.5$ and most with $\nu > 5$. For this test sample, the Tonry and Davis method systematically overestimates the confidence level (see Fig. 2). As an indicator, a 95% confidence level under the previous analysis corresponds roughly to a true confidence of 84%.

4 ERRORS

If the correct peak has been selected, the redshift may still have an error as the noise can move the position of the maximum. The function

$$c_n = \frac{\lambda_n}{N} \sum \left( |T_k| B_k e^{-2\pi i k \delta}/N + T_k^* E_k \right) e^{2\pi i k n/N} \quad (20)$$

is maximised. The first term ($\equiv \tilde{c}$) is symmetric, peaking at $n = \delta$. Expanding this term in a Taylor expansion, and writing the second term as $\epsilon$, the peak is shifted to $n - \delta = -\epsilon' / \sigma'^2$, the primes indicating derivatives. Thus, for an ensemble of ccf's with the same template but different noise realisations, the maxima are shifted by an r.m.s. amount

$$\Delta x = \sqrt{2} \sigma_{1a}/(-\epsilon''_3). \quad (21)$$

where we have used the antisymmetric part of the ccf to write $\sigma''_3 = 2\bar{\sigma}'_3$. The second derivative at the peak may be written

$$-\epsilon''_3 = \lambda_\alpha (2\pi)^2 \sum_{k=1}^{N/2} k^2 |T_k|^2 B_k \quad (22)$$

To make further progress requires specific assumptions about the broadening function. Broadening by convolution with a gaussian is considered in Section 5, which demonstrates that the error is rather insensitive to broadening. Here therefore I consider no broadening ($B_k = 1$), in which case,

$$-\epsilon''_3 = \lambda_\alpha N \sigma'^2 = \frac{\alpha \sigma'^2_{1t}}{\sigma_g \sigma_t} \quad (23)$$

where $\sigma'^2_{1t}$ is the second moment of the filtered template spectrum. Calculation of $\sigma_g$ requires $\sigma_x$, which may be calculated by noting that $\lambda_\alpha|E_k| = \sqrt{2}|A_k|/|T_k|$, from which we get

$$\sigma'^2_\epsilon + \sigma'^2_t = \frac{\sigma'^2_{1t}}{1 - 2\sigma'^2_t} \sum \left( |A_k|^2/|T_k|^2 \right) \quad (24)$$

So we obtain the final expression for the r.m.s. error:

$$\Delta x = \sqrt{2 \sigma_{1a} \sigma'^2_{1t}} \left[ 1 - 2 \sigma'^2_t \sum \left( |A_k|^2/|T_k|^2 \right) \right]^{-1/2}. \quad (25)$$

This can be tested in two ways. The first test is to do exactly what this analysis assumes: a template spectrum has noise added, and is shifted. The ccf technique is then used to estimate the shift. Fig. 3a shows the distribution of errors, normalised to the error estimate (25), along with the expected gaussian of unit variance, for white noise. Slight deviations are expected due to the discrete sampling of the spectra, but the agreement is remarkably good. Differences may arise because, although the peak of the ccf is calculated to sub-pixel accuracy, the antisymmetric part is calculated assuming $\delta$ is an integer.

A further weak test of the method is shown in Fig. 3b, which is based on real data; if the differences between galaxy spectra and templates can be characterised in the way suggested in this paper, the errors should be distributed normally. The figure shows the distribution of errors, each normalised to the r.m.s. for each galaxy cross-correlated to the ten templates), and the ideal case of a gaussian of unit variance. Cases where the wrong ccf peak has been selected are almost always easy to spot
and these have been removed. Note that it is incorrect to use the standard error for the mean of the $n$ redshifts (r.m.s./$\sqrt{n}$ if all errors are the same), as the noise in the galaxy spectrum may shift the derived redshift systematically for all templates.

4.1 Filtering

The results shown so far have used a FIGARO routine SCROSS, which has been generalised to include errors (available on some STARLINK sites as XCORR). SCROSS filters out low-frequency components after continuum subtraction, whereas XCORR offers the user the option not to do this. In XCORR, the errors are estimated from the actual full-width at half-maximum $W$ of the peak in the ccf: $\text{error} = 0.2833W/(1 + \nu)$. A similar routine, XCSAO, written for IRAF (Kurtz et al. 1992) uses the formula $0.375W/(1 + \nu)$. These work reasonably well in practice, but there can be cases where noise broadens the wings of the peak substantially, while the peak remains locally sharp, as illustrated in Fig. 4. The error in such cases can be grossly overestimated. Fig. 1 shows the filtered case, for the same spectra. Here the actual width is a better guide, but sometimes provides a poor error estimate, usually an overestimate. The formula (25) is based on local properties at the peak itself, so is insensitive to any broad wings.

5 GAUSSIAN POWER SPECTRA

In the ideal case of no noise and no broadening, the normalised ccf has a peak of 1. In practice, the height of the peak falls below this, and one ought to be able to use the actual height to estimate the significance of the redshift obtained. This is indeed the case, and one can get a rule-of-thumb estimate by making a specific assumption that the power spectra involved are gaussian. If the spectra are filtered, this approximation is usually quite good.

We assume the broadening function is a gaussian, with width $\sigma$, that the template and noise have gaussian power spectra with characteristic widths $\tau$ and $\eta$, i.e.

$$B_k = \exp \left( -\frac{2\pi^2 k^2 \sigma^2}{N^2} \right)$$

$$|T_k|^2 = 2\sqrt{\pi} N \tau \sigma_\eta^2 \exp \left( -\frac{2\pi k \tau}{N} \right)^2$$

$$|E_k|^2 = 2\sqrt{\pi} N \eta \sigma_\tau^2 \exp \left( -\frac{2\pi k \eta}{N} \right)^2.$$  

A result useful for the estimators below is

$$2 \sum_{k=0}^{N/2} \frac{k^n}{2} \exp(-\beta k^2) \simeq \int_{-\infty}^{\infty} \frac{k^n}{2} \exp(-\beta k^2) dk$$

$$= \Gamma \left( \frac{n+1}{2} \right) \beta^{-(n+1)/2}.$$ Evaluating the sums in sections 2 and 3, we find

$$\sigma^2 = \alpha^2 \left( \frac{\tau}{\sqrt{\tau^2 + \sigma^2}} + \frac{\eta}{\sqrt{\tau^2 + \eta^2}} \right).$$  

The mean value of the second derivative obeys

$$< -c^2_0 >= \frac{\sqrt{2}}{(2\tau^2 + \sigma^2)^{3/2}} \sqrt{\frac{\tau^2}{\tau^2 + \sigma^2}} + \frac{\eta^2}{\sqrt{\tau^2 + \eta^2}}.$$ and if we approximate the ccf peak as a gaussian, $c_n \simeq c(\delta \exp \left[ -(n - \delta)^2/(2\mu^2) \right)]$, matching second derivatives at the peak gives

$$\sigma^2 = \mu^2 - 2\tau^2$$

in agreement with the maximum likelihood method of TD. From this, one can obtain the velocity broadening in the galaxy provided any additional filtering is accounted for.

For gaussian power spectra, the error may be written

$$\Delta x = \frac{2\pi^{1/4} \eta \tau^2}{N^{1/2} (\tau^2 + \eta^2)^{3/4}} \sigma_\tau.$$ To evaluate this, we note that the noise term in the ccf has a variance

$$\sigma_\tau^2 = \frac{2\sqrt{\pi} \tau \eta \sigma_\tau^2}{N \left( \frac{\tau}{\sqrt{\tau^2 + \sigma^2}} + \frac{\eta}{\sqrt{\tau^2 + \eta^2}} \right)} \tau^2 + \eta^2.$$ and we may use this to eliminate $\sigma_\tau/\sigma_\tau$ from (25). After some algebra, (25) becomes

$$\Delta x = \frac{2\pi^{1/4} \eta \tau^2}{N^{1/2} (\tau^2 + \eta^2)^{3/4}} \frac{\tau^2 + \sigma^2}{1} \left( \frac{\tau^2}{\tau^2 + \sigma^2} \right)^{1/4} \tau \sigma_\tau.$$ For high peaks this reduces to

$$\Delta x = \frac{\tau \eta}{\tau^2 + \eta^2} \frac{\tau^2}{\tau^2 + \sigma^2} \left( \frac{\tau^2}{\tau^2 + \sigma^2} \right)^{1/4} \tau \sigma_\tau$$

which demonstrates the insensitivity of the error to the broadening. One can write the error in terms of the number of peaks, with the hope that the result can be used generally. If one assumes $\eta \simeq \tau \gg \sigma$ and uses

$$N_{\text{pks}} = \frac{\sqrt{3}}{2\sqrt{\pi}} \frac{N}{\sqrt{\tau^2 + \eta^2}},$$

then the error may be written

$$\Delta x \simeq \frac{\sqrt{3} N \sigma_\tau}{4\pi N_{\text{pks}}}.$$

Unfortunately, in our sample, this turns out to be a poor estimator, underestimating errors by a factor of about 3.
6 CONCLUSIONS

This paper has analysed the effects of noise on the confidence levels and errors in the cross-correlation technique widely used to obtain galaxy redshifts. This improved treatment finds somewhat lower confidence levels than previously found, and also provides a formally more correct error assessment for the redshifts obtained. The principal results are the confidence level (equation 15 with 13, or the approximation, equation 18), and the error estimate (equation 25).

The results should be particularly useful in identifying questionable redshifts in programmes which obtain large numbers of galaxy spectra. In particular, a low confidence level would indicate that the object should be checked carefully before the redshift is accepted. It should be borne in mind that errors may well arise from factors which are not treated in this paper, principally wavelength calibration errors and spectral differences between galaxies and templates which cannot be described by noise (even when filtered).

It is hoped to provide a FIGARO version of the cross-correlation program SCROSS to provide error and confidence analysis.

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Figure Captions
Figure 1. Sample spectra of a galaxy and template star, and their cross-correlation function. In the cross-correlation process, a Fourier filter has been applied to remove long-wavelength components.
Figure 2. Confidence assessments compared with the method of Tonry and Davis for about 800 filtered ccfs, based on around 80 galaxies with redshifts \( \lesssim 0.05 \), and 10 templates, 8 of which are radial velocity standard stars and the other two nearby galaxies. The spectra are mostly good signal-to-noise, with all ccf peak heights except one above \( \nu = 2.5 \), and about 2/3 above \( \nu = 5 \).
Figure 3. The distribution of errors, normalised to the sample standard deviation by a) adding white noise to a template and cross-correlating in the normal way. The error plotted is divided by the program error estimate (25) and the distribution should follow the solid curve b) cross-correlating real galaxies with 10 templates. The error estimate in this case is the r.m.s. deviation from the mean of the templates. If the assumptions in the analysis were strictly realised in practice, the distribution should be normal (solid curve).
Figure 4. The ccf for the galaxy-template pair of figure 1, but with no long-wavelength filtering applied. The actual FWHM of the peak is a poor estimate of the error.