SOME IDENTITIES OF EULERIAN POLYNOMIALS ARISING
FROM NONLINEAR DIFFERENTIAL EQUATIONS

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ABSTRACT. In this paper, we study nonlinear differential equations arising from Eulerian polynomials and their applications. From our study of nonlinear differential equations, we derive some new and explicit identities involving Eulerian and higher-order Eulerian polynomials.

1. INTRODUCTION

The Eulerian polynomials were introduced by L. Euler in his Remarques sur un beau rapport entre les sérises des puissances tant directes que réciproques in 1749 (first printed in 1765) where he describes a method of computing values of the zeta function at negative integers by a precursor of Abel’s theorem applied to a divergent series (see [3–5, 19, 22]).

As is well known, the Eulerian polynomials, $A_n(t)$, $(n \geq 0)$, are defined by the generating function

$\frac{1 - t}{e^{x(t-1)} - t} = e^{A(t)x} = \sum_{n=0}^{\infty} A_n(t) \frac{x^n}{n!},$ (see [11]),

with the usual convention about replacing $A^n(t)$ by $A_n(t)$.

From (1.1), we can derive the following recurrence relation for the Eulerian polynomials:

$(A(t) + (t-1))^n - tA_n(t) = (1 - t) \delta_{0,n}, \quad (n \geq 0), \quad (see \ [11]).$

By (1.2), we easily get

$A_0(t) = 1, \quad A_n(t) = \frac{1}{t - 1} \sum_{l=0}^{n-1} \binom{n}{l} A_l(t) (t-1)^{n-l}, \quad (n \geq 1).$

Furthermore,

$A_n(t)\frac{(1-t)^{n+1}}{(1-t)^n} = \sum_{j=0}^{\infty} t^j (j+1)^n, \quad (n \geq 0), \quad (see \ [2–6])$

The first few Eulerian polynomials are

$1 + t + t^2 + t^3 + \cdots = \frac{1}{1-t} = A_0(t)\frac{1}{1-t},$

$1 + 2t + 3t^2 + 4t^3 + \cdots = \frac{1}{(1-t)^2} = A_1(t)\frac{1}{(1-t)^2}.$

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$$1 + 2^2t + 3^2t^2 + 4^2t^3 + \cdots = \frac{1 + t}{(1 - t)^3} = \frac{A_2(t)}{(1 - t)^3}.$$  

Recently, several authors has studied some interesting extensions and modifications of Eulerian polynomials along with related combinatorial, probabilistic and statistical applications (see [1–22]).

In [15], Kim has studied nonlinear differential equations arising from Frobenius-Euler numbers and polynomials.

In this paper, we give some new and explicit identities on Eulerian and higher-order Eulerian polynomials which are derived from solutions of nonlinear differential equations.

## 2. Nonlinear differential equations arising from Eulerian polynomials

Let us put

$$F = F(t, x) = \frac{1}{e^{x(t-1)} - t}, \quad (t \neq 1).$$

Now, we consider the differentiation of $F$ with respect to $x$ while $t$ is being fixed.

$$F^{(1)} = \frac{d}{dx} F(t, x)$$

$$= \frac{(-1)^2 e^{x(t-1)}}{(e^{x(t-1)} - t)^2} (t - 1)$$

$$= (1 - t) \frac{1}{(e^{x(t-1)} - t)^2} \left( e^{x(t-1)} - t + t \right)$$

$$= (1 - t) \left( F + tF^2 \right).$$

Thus, by (2.2), we easily get

$$F^{(2)} = \frac{d}{dx} F^{(1)}$$

$$= (1 - t) \left( F^{(1)} + 2tF F^{(1)} \right)$$

$$= (1 - t) \left( 1 + 2tF \right) F^{(1)}$$

$$= (1 - t)^2 \left( F + tF^2 \right)$$

$$= (1 - t)^2 \left( F + 3tF^2 + 2t^2 F^3 \right),$$

and

$$F^{(3)} = \frac{d}{dx} F^{(2)}$$

$$= (1 - t)^2 \left( F^{(1)} + 6tF F^{(1)} + 6t^2 F^2 F^{(1)} \right)$$

$$= (1 - t)^2 \left( 1 + 6tF + 6t^2 F^2 \right) F^{(1)}$$

$$= (1 - t)^3 \left( F + 6tF^2 \right)$$

$$= (1 - t)^3 \left( F + 7tF^2 + 12t^2 F^3 + 6t^3 F^4 \right).$$
Continuing this process, we set

\[
F^{(N)} = \left( \frac{d}{dx} \right)^N F(t, x)
\]

\[
= \left( \frac{d}{dx} \right)^N \left( \frac{1}{e^{x(t-1)} - t} \right)
\]

\[
= (1 - t)^N \sum_{i=1}^{N+1} a_{i-1} (N, t) F^i, \quad (N \in \mathbb{N} \cup \{0\}).
\]

From (2.5), we can derive the following equation (2.6):

\[
F^{(N+1)} = \frac{d}{dx} F^{(N)}
\]

\[
= (1 - t)^N \sum_{i=1}^{N+1} a_{i-1} (N, t) i F^{i-1} (F + t F^2)
\]

\[
= (1 - t)^{N+1} \sum_{i=1}^{N+1} a_{i-1} (N, t) i F^{i-1} \left(F + t F^2\right)
\]

\[
= (1 - t)^{N+1} \left\{ \sum_{i=1}^{N+1} a_{i-1} (N, t) i F^i + \sum_{i=1}^{N+1} a_{i-1} (N, t) i^2 F^{i+1} \right\}
\]

\[
= (1 - t)^{N+1} \left\{ \sum_{i=1}^{N+1} a_{i-1} (N, t) i F^i + \sum_{i=2}^{N+2} a_{i-2} (N, t) (i - 1) t F^i \right\}
\]

\[
= (1 - t)^{N+1} \left\{ a_0 (N, t) F + (N + 1) t a_N (N, t) F^{N+2} + \sum_{i=2}^{N+1} \left( i a_{i-1} (N, t) + (i - 1) t a_{i-2} (N, t) \right) F^i \right\}.
\]

By replacing \( N \) by \( N + 1 \) in (2.5), we get

\[
F^{(N+1)} = (1 - t)^{N+1} \sum_{i=1}^{N+2} a_{i-1} (N + 1, t) F^i.
\]

From (2.6) and (2.7), we can derive the following recurrence relation for the coefficients \( a_i (N, t) \):

\[
a_0 (N + 1, t) = a_0 (N, t),
\]

\[
a_{N+1} (N + 1, t) = (N + 1) t a_N (N, t),
\]

and

\[
a_{i-1} (N + 1, t) = (i - 1) t a_{i-2} (N, t) + i a_{i-1} (N, t),
\]

where \( 2 \leq i \leq N + 1 \).

It is not difficult to show that

\[
F = F^{(0)} = a_0 (0, t) F.
\]

Thus, by (2.11), we have

\[
a_0 (0, t) = 1.
\]
From (2.3) and (2.5), we note that
\[
(1 - t) \left( F + tF^2 \right) = F^{(1)}
\]
\[
= (1 - t) \sum_{i=1}^{2} a_{i-1} (1, t) F^i
\]
\[
= (1 - t) \left\{ a_0 (1, t) F + a_1 (1, t) F^2 \right\}. \tag{2.13}
\]

By comparing the coefficients on both sides of (2.13), we have
\[
a_0 (1, t) = 1, \quad a_1 (1, t) = t. \tag{2.14}
\]

From (2.14), we note that
\[
a_{N+1} (N + 1, t) = a_{N+1} (N, t) = \cdots = a_0 (1, t) = a_0 (0, t) = 1,
\]
and
\[
a_{N+1} (N + 1, t) = (N + 1) t a_N (N, t)
\]
\[
= (N + 1) t N t a_{N-1} (N - 1, t)
\]
\[
= t^2 (N + 1) N a_{N-1} (N - 1, t)
\]
\[
\vdots
\]
\[
= t^N (N + 1) N \cdots 2 a_1 (1, t)
\]
\[
= t^{N+1} (N + 1)!. \tag{2.15}
\]

So, we have the matrix \((a_i (j, t))_{0 \leq i, j \leq N}\) as follows:
\[
\begin{bmatrix}
0 & 1 & 2 & 3 & N \\
0 & 1 & 1 & 1 & \cdots & 1 \\
1 & & 1! t & & \\
2 & & 2! t^2 & & \\
3 & & 3! t^3 & & \\
& & & & \ddots & \\
N & & & & & N! t^N
\end{bmatrix}
\]

From (2.10), we have
\[
a_1 (N + 1, t) = t a_0 (N, t) + 2 a_1 (N, t)
\]
\[
= t a_0 (N, t) + 2 \{ t a_0 (N - 1, t) + 2 a_1 (N - 1, t) \}
\]
\[
= t \{ a_0 (N, t) + 2 a_0 (N - 1, t) \} + 2^2 \{ t a_0 (N - 2, t) + 2 a_1 (N - 2, t) \}
\]
\[
= t \{ a_0 (N, t) + 2 a_0 (N - 1, t) + 2^2 a_0 (N - 2, t) \} + 2^3 a_1 (N - 2, t)
\]
\[
\vdots
\]
\[
= t \sum_{i=0}^{N-1} 2^i a_0 (N - i, t) + 2^N a_1 (1, t) \tag{2.17}
\]
\[= t \sum_{i=0}^{N} 2^i a_0(N - i, t),\]

(2.18)

\[a_2(N + 1, t) = 2ta_1(N, t) + 3a_2(N, t)\]
\[= 2ta_1(N, t) + 3 \{ 2ta_1(N - 1, t) + 3a_2(N - 1, t) \} = 2t \{ a_1(N, t) + 3a_1(N - 1, t) \} + 3^2 \{ 2ta_1(N - 2, t) + 3a_2(N - 2, t) \} = 2t \{ a_1(N, t) + 3a_1(N - 1, t) \} + 3^2a_2(N - 2, t)\]
\[\vdots\]
\[= 2t \sum_{i=0}^{N-2} 3^i a_1(N - i, t) + 3^{N-1} a_2(2, t)\]
\[= 2t \sum_{i=0}^{N-1} 3^i a_1(N - i, t),\]

and

(2.19)

\[a_3(N + 1, t) = 3ta_2(N, t) + 4a_3(N, t)\]
\[= 3ta_2(N, t) + 4 \{ 3ta_2(N - 1, t) + 4a_3(N - 1, t) \} = 3t \{ a_2(N, t) + 4a_2(N - 1, t) \} + 4^2 \{ 3ta_2(N - 2, t) + 4a_3(N - 2, t) \} = 3t \{ a_2(N, t) + 4a_2(N - 1, t) \} + 4^2a_3(N - 2, t)\]
\[\vdots\]
\[= 3t \sum_{i=0}^{N-3} 4^i a_2(N - i, t) + 4^{N-2} a_3(3, t)\]
\[= 3t \sum_{i=0}^{N-2} 4^i a_2(N - i, t).\]

Continuing this process, we get

(2.20) \[a_j(N + 1, t) = jt \sum_{i=0}^{N-j+1} (j + 1)^i a_{j-1}(N - i, t), \quad (1 \leq j \leq N + 1).\]

Therefore, by (2.20), we obtain the following theorem.

**Theorem 1.** For each fixed \(t \neq 1\) and \(N \in \mathbb{N} \cup \{0\}\), \(F = F(t, x) = \frac{1}{e^{(x-1) - t}}\) satisfies the nonlinear differential equation

(2.21) \(\left( \frac{d}{dx} \right)^{N} F = (1 - t)^{N} \sum_{i=1}^{N+1} a_{i-1}(N, t) F^{i},\)
where \(a_0(N, t) = a_0(N-1, t) = \cdots = a_0(1, t) = a_0(0, t) = 1,\)
\[a_i(N, t) = it \sum_{j=0}^{N-i} (i+1)^j a_{i-1} (N-j-1, t) \quad (1 \leq j \leq N).\]

Taking the \(N\)-th derivative with respect to \(x\) on both sides of (1.1), we obtain
\[
(\frac{d}{dx})^N \left( \frac{1 - t}{e^{x(t-1)} - t} \right) = \sum_{n=N}^{\infty} A_n(t) (n)_N \frac{x^{n-N}}{n!} \]
\[= \sum_{n=0}^{\infty} A_{n+N}(t) (n + N)_N \frac{x^n}{(n + N)!} \]
\[= \sum_{n=0}^{\infty} A_{n+N}(t) \frac{x^n}{n!}.
\]

On the other hand, from (2.21), we have
\[
(\frac{d}{dx})^N \left( \frac{1 - t}{e^{x(t-1)} - t} \right) = (1 - t)^{N+1} \sum_{i=1}^{N+1} a_{i-1} (N, t) (1-t)^{-i} \left( \frac{1 - t}{e^{x(t-1)} - t} \right)^i \]
\[= \sum_{n=0}^{\infty} (\sum_{i=1}^{N+1} a_{i-1} (N, t) (1-t)^{N+1-i} \sum_{n=0}^{\infty} A_n(t) \frac{x^n}{n!}) \frac{x^n}{n!},
\]
where \(A_n^{(i)}(t)\) are called the higher-order Eulerian polynomials and defined by the generating function
\[
(\frac{1 - t}{e^{x(t-1)} - t})^m = \sum_{n=0}^{\infty} A_n^{(m)}(t) \frac{x^n}{n!}.
\]

Therefore, by (2.22) and (2.23), we obtain the following theorem.

**Theorem 2.** For all \(t, \) and \(n, N \in \mathbb{N} \cup \{0\},\) we have
\[A_{n+N}(t) = \sum_{i=1}^{N+1} a_{i-1} (N, t) (1-t)^{N+1-i} A_n^{(i)}(t).
\]

Note here that, as both sides are polynomials and it holds for all \(t \neq 1),\) it is true as polynomials. Explicit expressions for \(a_i(N, t),\) \((1 \leq i \leq N),\) are given by
\[
a_1(N, t) = t \sum_{j=0}^{N-1} 2^j a_0(N-j-1, t) \]
\[= t \sum_{j=0}^{N-1} 2^j \]
\[= t (2^N - 1),
\]
Recall that

Continuing this process, we have

Thus, by \( (2.30) \), we get

and

Continuing this process, we have

Recall that

Thus, by \( (2.30) \), we get

\[
\sum_{j=0}^{\infty} t^j (j + 1)^{n+N} = \frac{A_{n+N}(t)}{(1-t)^{n+N+1}} = \sum_{i=1}^{N+1} a_{i-1}(N,t) (1-t)^{-n-i} A_n^{(i)}(t)
\]
\[
(1 - t)^{-n} \sum_{i=1}^{N+1} a_{i-1} (N, t) (1 - t)^{-i} A_n^{(i)} (t).
\]

Therefore, by (2.31), we obtain the following theorem.

**Theorem 3.** For \( n, N \in \mathbb{N} \cup \{0\} \), we have

\[
\sum_{j=0}^{\infty} t^j (j + 1)^{n+N} = (1 - t)^{-n} \sum_{i=1}^{N+1} a_{i-1} (N, t) (1 - t)^{-i} A_n^{(i)} (t).
\]

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