**ML(θC)-Space in Topological Spaces**

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**ABSTRACT**

The purpose of this paper is to introduce a new concept of spaces which is called minimal $L(\theta C)$-space, namely $M\text{in}L(\theta C)$-space or $ML(\theta C)$-space, also given some properties, examples, theorems and the topological property of $M\text{in}L(\theta C)$-space are discussed.

**KEYWORDS:** $\theta$-closed set, Lindelof, $L(\theta C)$-space, $Min\theta T_2$-space, $Min\mathcal{K}(\theta C)$-space

**INTRODUCTION**

The concept of Lindelof space was introduced in 1929 by Alexandroff and Urysohn [2], this space is an important in a topological space. Later, in 1979 Mukherji and Sarkar [10], provide the concept of $LC$-space. (A topological space $X$ is called $LC$-space, if every Lindelof subset of a space $X$ is a closed set). $LC$-space studied by many researchers such as [3]. Notices that $LC$-space is also known under the name $L$-closed such as [6, 9, 13].

The concept of $\theta$-closed and $\theta$-open set were first introduced by Velicko [16] in 1969. (Let $(X,\mathcal{T})$ be a topological space, $\mathcal{F}$ be a subset of $X$ and $x \in X$. A point $x$ is called $\theta$-interior point of $\mathcal{F}$, if there is $\mathcal{C} \in \mathcal{T}$, such that $x \in \mathcal{C}$ and $x \in \bar{\mathcal{C}} \subseteq \mathcal{F}$. $\theta$-interior set which denoted by $Int_{\theta}(\mathcal{F})$ is the set of all $\theta$-interior points. A subset $\mathcal{F}$ of $X$ is called $\theta$-open set if $Int_{\theta}(\mathcal{F}) = \mathcal{F}$. And (Let $(X,\mathcal{T})$ be topological space, $H \subseteq X$, a point $b \in X$ is said to be $\theta$-adherent point for a subset $H$ of $X$, if $H \cap G \neq \emptyset$ for any open set $G$ of $X$ and $b \in G$. The set of $\theta$-adherent points is said to be $\theta$-closure of $H$ which denoted by $Cl_{\theta}(H)$. A subset $H$ of $X$ is called $\theta$-closed set if $H = Cl_{\theta}(H))$. These concepts have been studied by many authors such as [8, 12]. In 2011, Al-Taai and Haider [4], study the new term called $L(\theta C)$-space. (A topological space $X$ is called $L(\theta C)$-space, if every Lindelof subset of a space $X$ is $\theta$-closed set), which is a strong than $LC$-space. And since the union of $\theta$-closed set may be not $\theta$-closed set. Encourage the author to define $\mathcal{T}_{\theta}$-$\theta$-closed set which is a countable union many $\theta$-closed sets.

In 2005, H. J. Ali [3], introduce Minimal $LC$-space, any $LC$-space $(X,\mathcal{T})$ is $MinLC$-space, if $T^* \cap \mathcal{T}$ on $X$ is not $LC$-space studied by [14, 15]. The aim of this paper is to introduce a minimal $L(\theta C)$-space (denoted by $M\text{in}L(\theta C)$-space), that is a space $X$ which is $L(\theta C)$-space is called $M\text{in}L(\theta C)$-space, if $T^* \cap \mathcal{T}$ on $X$ is not $L(\theta C)$-space. Note that every $M\text{in}L(\theta C)$-space is $L(\theta C)$-space, and study some properties of this space, also study the relation between this concept with $M\text{in}\mathcal{K}(\theta C)$-space and $M\theta T_2$-space. Also study some important property such as, a topological property of $M\text{in}L(\theta C)$-space.

**PRELIMINARIES**

**Definition (2.1) [5]:** A space $X$ is called $R_1$-space, if $e$ and $d$ have a disjoint neighborhoods, whenever $Cl(e) \neq Cl(d)$.

**Remark (2.2) [5]:** A space $X$ is $T_2$-space iff $X$ is $R_1$ and $T_1$-space.
Definition (2.3) [16]: Let \((X, T)\) be a topological space, \(F\) be a subset of \(X\) and \(x \in X\). A point \(x\) is called a \(\theta\)-interior point of \(F\), if there is \(C \in T\) such that \(x \in C\) and \(x \in \overline{C} \subseteq F\). A \(\theta\)-interior set which denoted by \(\text{Int}_\theta(F)\) is the set of all \(\theta\)-interior points. A subset \(F\) of \(X\) is called \(\theta\)-open set iff \(\text{Int}_\theta(F) = F\).

Definition (2.4) [16]: Let \((X, T)\) be topological space, \(H \subseteq X\), a point \(b \in X\) is said to be \(\theta\)-adherent point for a subset \(H\) of \(X\), if \(H \cap \overline{G} \neq \emptyset\) for any open set \(G\) of \(X\) and \(b \in G\). The set of \(\theta\)-adherent points is said to be \(\theta\)-closure of \(H\) which denoted by \(\text{Cl}_\theta(H)\). A subset \(H\) of \(X\) is called \(\theta\)-closed set iff \(H = \text{Cl}_\theta(H)\).

Example (2.5): Any subset of a discrete space \((\mathbb{R}, D)\) on a real numbers \(\mathbb{R}\) is \(\theta\)-closed set and \(\theta\)-open set.

Remark (2.6) [16]: Every \(\theta\)-closed (resp. \(\theta\)-open) set is a closed (resp. open) set.

Lemma (2.7) [3]: Let \(Y\) be a subspace of a space \(X\). If \(P\) is \(\theta\)-closed in \(X\) then \(P\) is \(\theta\)-closed in \(Y\), whenever \(P \subseteq Y\).

Definition (2.8) [1, 4]: A subset \(F\) of a space \(X\) is said to be \(\mathcal{F}_\sigma\)-\(\theta\)-closed, if it is a countable union of \(\theta\)-closed sets. The complement of \(\mathcal{F}_\sigma\)-\(\theta\)-closed is said to be \(G_\sigma\)-\(\theta\)-open set.

Remark (2.9) [1]: Every \(\theta\)-closed set is \(\mathcal{F}_\sigma\)-\(\theta\)-closed set. But the converse need not be true.

Example (2.10): Let \((\mathbb{R}, T_u)\) be a usual topology on a real line \(\mathbb{R}\), and \(G_n = [1/n, 1]\), where \((n = 2, 3, 4, ...)\), be a \(\theta\)-closed sets, then \(U_n G_n = (0, 1] \subseteq \mathcal{F}_\sigma\)-\(\theta\)-closed, but neither closed nor \(\theta\)-closed.

Definition (2.11) [1, 3, 4]: A space \(X\) is said to be:

1. A \(\theta\)-space, if every \(\mathcal{F}_\sigma\)-\(\theta\)-closed is \(\theta\)-closed.
2. A \(\mathcal{K}(\theta C)\)-space, if every \(\mathcal{K}\) compact subset of \(X\) is \(\theta\)-closed set.
3. A \(\mathcal{L}(\theta C)\)-space, if every Lindelof subset of \(X\) is \(\theta\)-closed set.

Example (2.12): Let \((Z, T_\emptyset)\) be a topological space where \(T_\emptyset\) be a discrete topology on an integer numbers \(\mathbb{Z}\), \((Z, T_\emptyset)\) is \(\mathcal{L}(\theta C)\)-space.

Definition (2.13) [4]: A subset \(A\) of a space \(X\) is said to be \(\theta\)-dense, if \(\text{Cl}_\theta(A) = X\).

Proposition (2.14) [3]: The property of \(\mathcal{L}(\theta C)\)-space is a topological property.

Proposition (2.15) [3]: The property of \(\mathcal{L}(\theta C)\)-space is a hereditary property.

Theorem (2.16) [4]:

1. If a space \(X\) is \(\theta\)\(L_1\)-space and \(\theta\)\(L_2\)-space, then \(X\) is \(\mathcal{L}(\theta C)\)-space.
2. Every \(\theta\)\(P\)-space is \(\theta\)\(L_1\)-space.

Definition (2.17) [7]: A space \(X\) is called \(\theta\)\(T_1\) (resp. \(\theta\)\(T_2\))-space, if every two distinct points \(a, b\) belong to \(X\), there are two \(\theta\)-open (resp. open) sets each one contain one point but not contain the other.

Theorem (2.18) [7]: A space \(X\) is called \(\theta\)\(T_1\)-space if and only if every singleton set is \(\theta\)-closed set.

Definition (2.19) [7]: A space \(X\) is called \(\theta\)\(T_2\)-space (resp. \(\theta\)\(T_2\)-space), if every two points \(a, b\) belong to \(X\), \(a \neq b\) there is two disjoint \(\theta\)-open (resp. open) sets \(M\) and \(N\) containing \(a\) and \(b\) respectively.

Remarks (2.20):

1. Every \(\mathcal{L}(\theta C)\)-space is \(\theta\)\(T_1\)-space.
2. Every \(\theta\)\(T_1\)-space is \(\theta\)\(T_1\)-space.
3. Every \(\mathcal{L}(\theta C)\)-space is \(\theta\)\(T_1\)-space.

Proof:

1. Let \(\{x\}\) be a Lindelof subset of a space \(X\), for each \(x \in X\), which is \(\mathcal{L}(\theta C)\)-space, so \(\{x\}\) is \(\theta\)-closed set, then from Theorem (2.37), a space \(X\) is \(\theta\)\(T_1\)-space.

2. Let \(a, b\) be two distinct point in a space \(X\) which is \(\theta\)\(T_1\)-space, so there exist two \(\theta\)-open sets \(G\) and \(H\) containing \(a, b\) respectively with \(a \notin H\) and \(b \notin G\), from Remark 2.21, \(G\) and \(H\) are open set in \(X\), containing \(a, b\) respectively with \(a \notin H\) and \(b \notin G\), that means \(X\) is \(\theta\)\(T_2\)-space.

3. Let a space \(X\) be \(\mathcal{L}(\theta C)\)-space, from part (1) of this Remark, \(X\) is \(\theta\)\(T_1\)-space and from part (2), \(X\) is \(\theta\)\(T_1\)-space.

Definition (2.21) [11]: A space \(X\) is called \(\theta\)\(R_1\)-space, if \(e\) and \(d\) have a disjoint \(\theta\)-neighbourhood, whenever \(\text{Cl}_\theta(e) \neq \text{Cl}_\theta(d)\).

Remark (2.22) [11]: A space \(X\) is \(\theta\)\(R_2\)-space iff \(X\) is \(\theta\)\(R_1\) and \(\theta\)\(R_2\)-space.

Definition (2.23) [8]: Let \((X, T)\) and \((Y, T')\) be two topological space and \(f: (X, T) \rightarrow (Y, T')\) be a function. Then \(f\) is called:

1. \(\theta\)-closed function [1], if \(f(F)\) is \(\theta\)-closed in \(Y\) for each closed subset \(F\) of \(X\).

2. Closed function [10], if \(f(F)\) is closed set in \(Y\) for each closed subset \(F\) of \(X\).
Remark (2.24) [1]: Every $\theta$-closed function is closed function.

Definition (2.25) [11]: Let $(Χ, T)$ be $K(\theta C)$-space, a space $Χ$ is said to be $\text{Min} K(\theta C)$-space, if $T^* \subseteq T$ on $Χ$, then $(Χ, T^*)$ is $\text{Min} K(\theta C)$-space.

Example (2.26): Let $(R, T_u)$ be a usual topology defined on the real numbers, $(R, T_u)$ is $\text{Min} K(\theta C)$-space.

Theorem (2.27) [11]: If a space $Χ$ is compact $K(\theta C)$-space, then it is $\text{Min} K(\theta C)$-space.

Proposition (2.28) [11]: If a space $Χ$ is Locally compact, $K(\theta C)$-space then $Χ$ is $\theta T_2$-space.

Definition (2.29) [11]: A space $Χ$ is $\theta T_2$-space, we say that $Χ$ is $\text{Min} T_2$-space, if there is $T^* \subseteq T$ on $Χ$, then $(Χ, T^*)$ is not $\theta T_2$-space.

Theorem (2.30) [11]: If a space $Χ$ is $\theta T_2$ and $\text{Min} K(\theta C)$-space then $Χ$ is $\text{Min} \theta T_2$-space.

$\text{Min} L(\theta C)$-Spaces

Definition (3.1): Let $(Χ, T)$ be $L(\theta C)$-space, a space $Χ$ is said to be $\text{Min} L(\theta C)$-space, if $T^* \subseteq T$ on $Χ$, then $(Χ, T^*)$ is not $L(\theta C)$-space.

Example (3.2): Let $(Χ, T_D)$ be a discrete topology defined on countable set $Χ$, $(Χ, T_D)$ is $\text{Min} L(\theta C)$-space, since, if we take any subset $H$ of a space $Χ$, which is countable then $H$ is countable, so $H$ is Lindelof, let $x \notin H$, also $\{x\}$ is open set containing $x$, also $\{x\} \cap H = \emptyset$, so $H$ is $\theta$-closed set and then $Χ$ is $L(\theta C)$-space, also since $T_{ind} \subseteq T_D$, but $(Χ, T_{ind})$ is not $L(\theta C)$-space. Therefore $Χ$ is $\text{Min} L(\theta C)$-space.

Theorem (3.3): If a space $Χ$ is Lindelof $L(\theta C)$-space, then it is $\text{Min} L(\theta C)$-space.

Proof: Let $(Χ, T)$ be $L(\theta C)$-space and suppose $Χ$ is not $\text{Min} L(\theta C)$-space, that is there is a topology $T^* \subseteq T$ on $Χ$ and $(Χ, T^*)$ is $L(\theta C)$-space. Let $I_x: (Χ, T) \rightarrow (Χ, T^*)$ be the identity function on $Χ$. Now $I_x$ is continuous, bijective and $\theta$-closed function since (if $N$ is a closed subset of $Χ$, and $Χ$ is Lindelof, so $N$ is Lindelof), also $I_x$ is continuous, then $I_x(N)$ is Lindelof subset of $(Χ, T^*)$ which is $L(\theta C)$-space, hence $I_x(N)$ is $\theta$-closed and then $I_x$ is $\theta$-closed function, by Remark 2.24, $I_x$ is a closed function that is $I_x$ is homeomorphism function, so $T^* \cong T$ and this is contradiction, so $Χ$ is $\text{Min} L(\theta C)$-space.

Example (3.4): Let $Χ = \mathbb{R}$ be a real numbers, and $T_{\text{Exc}} = \{ U \subseteq \mathbb{R}: x \notin U, \text{for some } x \in \mathbb{R} \} \cup \{ R \}$, be excluded point topology, $(R, T_{\text{Exc}})$ is not $\text{Min} K(\theta C)$-space, since $(R, T_{\text{Exc}})$ is compact, so $(R, T_{\text{Exc}})$ is Lindelof, but not $L(\theta C)$-space because, if we take $x = 5$ and $C = \{ \{x\} \}_{x \in \mathbb{R}} \cup R$ is an open cover to $R$, then we can reduce to just $R$ that is $(R, T_{\text{Exc}})$ is Lindelof, also $\{1, 5\}$ is finite set, then it is countable, so it is Lindelof set and $\emptyset \not\in \{1, 5\}$, therefore $2 \in \theta$-adherent point, that is $\{1, 5\}$ is not $\theta$-closed set, hence $(R, T_{\text{Exc}})$ is not $L(\theta C)$-space and from Theorem 3.3, this topological space is not $\text{Min} L(\theta C)$-space.

Corollary (3.5): Every compact and $L(\theta C)$-space is $\text{Min} L(\theta C)$-space.

Proof: From Theorem 3.3. And every compact space is Lindelof space.

Remark (3.6): The continuous image of $\text{Min} L(\theta C)$ is not necessarily $\text{Min} L(\theta C)$, the following example explain this Remark:

Example (3.7): Let $f: (R, T_D) \rightarrow (R, T_{\text{ind}})$ be a function from a discrete topology $T_D$ into indiscrete topology $T_{\text{ind}}$, defined by $f(x) = x, \forall x \in R$, so $f$ is continuous and $(R, T_D)$ is $L(\theta C)$, also $T_{\text{ind}} \subseteq T_D$, but $(R, T_{\text{ind}})$ is not $L(\theta C)$, and from Proposition 2.14, $(Y, T')$ is $L(\theta C)$-space, suppose $(Y, T')$ is not $\text{Min} L(\theta C)$-space, then there is a topology $T'^* \subseteq T'$ on $Y$, implies $(Y, T'^*)$ is $L(\theta C)$-space.

Proposition (3.8): The property of being $\text{Min} L(\theta C)$-space is a topological property.

Proof: Let $(Χ, T)$ be $\text{Min} L(\theta C)$-space, $f: (Χ, T) \rightarrow (Y, T')$ is a homeomorphism function, to prove $(Y, T')$ is $\text{Min} L(\theta C)$-space. Now from Proposition 2.14, $(Y, T')$ is $L(\theta C)$-space, suppose $(Y, T')$ is not $\text{Min} L(\theta C)$-space, then there is a topology $T'^* \subseteq T'$ on $Y$, implies $(Y, T'^*)$ is $L(\theta C)$-space.

Define $T_{\phi} = \{ f^{-1}(U): U \subseteq T' \}$, so $(Χ, T_{\phi})$ is a topology on $(Χ, T)$ and $T_{\phi} \subseteq T$ and $(Χ, T_{\phi})$ is $L(\theta C)$-space, (let $S$ be a Lindelof subset of $Χ$, then $S$ is $\theta$-closed in $Χ$, since $f$ is continuous and then we have $f(S)$ is Lindelof set in $Y$ which is $L(\theta C)$-space, then $f(S)$ is $\theta$-closed in $(Y, T'^*)$, to show $S$ is $\theta$-closed set, that is to show $S = C_{\theta}(S)$, since $S \subseteq C_{\theta}(S)$, let $s \in C_{\theta}(S)$ and $s \notin S$, since $f$ is injective, then $f(s) \notin f(S)$ and $f$ is surjective, so $w \notin f(S)$ where $w = f(s)$, but $f(S)$ is $\theta$-closed in $Y$, then there is open set $W$ in $Y$ with $w \in W$ and $\overline{W} \cap f(S) = \phi$, so $f^{-1}(\overline{W}) \cap f(S) = f^{-1}(\phi) = \phi$, and $f^{-1}(\overline{W}) \cap f^{-1}(f(S)) = \phi$, then $f^{-1}(\overline{W}) \cap S = \phi$, since $f$ is
homeomorphism, then $f^{-1}(\overline{W}) \cap S = \phi$, we have 
$s$ is not $\theta$-adherent point to $S$. Therefore $S$ is $\theta$-
closed in $X$). which is contradiction, since $X$ is 
$\text{MinL} L(\theta C)$-space. Hence $(Y,T^*)$ is 
$\text{MinL} L(\theta C)$-space.

**Lemma (3.9):** In Lindelof space, any $\theta$-closed set 
is Lindelof set.

**Proof:** Let $X$ be a Lindelof space and $A$ be 
$\theta$-closed subset of $X$. From Remark 2.6, $A$ is a 
closed subset of $X$. Then $A$ is Lindelof set.

**Proposition (3.10):** Let $(Y,T)$ be a subspace of a 
Lindelof $L(\theta C)$-space $(X,T)$, $Y$ is Lindelof iff $Y$ 
is $\theta$-closed.

**Proof:** Suppose $Y$ is Lindelof subspace of $X$, 
since $X$ is $L(\theta C)$-space, then $Y$ is $\theta$-closed. 
Conversely, suppose $Y$ is $\theta$-closed in $X$, which is 
Lindelof, then by Lemma 3.9, $Y$ is Lindelof.

**Example (3.11):** The discrete topology $T_D$ on an 
integer numbers $Z$, $(Z,T_D)$ is Lindelof $L(\theta C)$-
space, also subspace $(N,T_D)$ is Lindelof and $\theta$-
closed, where $N$ is a natural number.

**Proposition (3.12):** If $(X,T)$ is a Lindelof 
$L(\theta C)$-space, then every $\theta$-closed subspace of $X$ 
is $\text{MinL} L(\theta C)$-space.

**Proof:** Let $Y$ be $\theta$-closed in $X$, but $X$ is Lindelof, 
then by Proposition 3.10, $Y$ is Lindelof. Now let 
$N$ be a Lindelof subset of $Y$, then $N$ is Lindelof in 
$L(\theta C)$-space, so $N$ is $\theta$-closed in $X$. Now $N = N \cap Y$, since $N \subseteq Y$, by Lemma 
2.7, $N$ is $\theta$-closed in $Y$, hence $Y$ is $L(\theta C)$-space 
and by Theorem 3.3, $Y$ is $\text{MinL} L(\theta C)$-space.

**Lemma (3.13):** A subset $H$ of a space $X$ is $G_\delta$-
$\theta$-open set if and only if every point in $H$ is $G_\delta$-
$\theta$-interior point to $H$.

**Proof:** Suppose $H$ is $G_\delta$-$\theta$-open set and $x \in H$, 
then there exists $A = H$ which is $G_\delta$-$\theta$-open set 
and $x \in A = H \subseteq H$, so $x$ is $G_\delta$-$\theta$-interior point 
to $H$, but $x$ is an arbitrary point, so any point in $A$ 
is $G_\delta$-$\theta$-interior point to $A$. Conversely, suppose 
any point in $H$ is $G_\delta$-$\theta$-interior point to $H$, that is, 
for each $x_i \in H$, there is $A_{x_i}$ is $G_\delta$-$\theta$-open 
subset of $H$, we get $H = \bigcup_{x_i \in H} A_{x_i}$, then $H$ is $G_\delta$-$\theta$-
open set.

**Proposition (3.14):** Every Lindelof set in $\theta T_2$-
space is $F_\sigma$-$\theta$-closed set.

**Proof:** Let $A$ be a Lindelof subset of a space $X$, 
and $p \in A$, then for each $q \in A$, $p \neq q$ and 
p, $q \in X$, since $X$ is $\theta T_2$-space, then there exist 
two $\theta$-open sets $U$ and $V$, with $q \in U$, $p \in V$ and 
$U \cap V = \emptyset$. Let $U_{q \in A} U_q$ is $\theta$-open cover to $A$, 
then it is open cover to $A$ which is Lindelof, so 
$A \subseteq \bigcup_{i \in \mathbb{N}} U_{q_i}$, then $U^* = \bigcap_{i \in \mathbb{N}} U_{q_i}(p)$, since $V^*$ is the intersection of 
countable many $\theta$-open set, then $V^*$ is $G_\delta$-$\theta$-open 
set and $V^* \cap U^* = \emptyset$, so $p \in V^* \subseteq \mathcal{A}^c$, then $p$ is 
$G_\delta$-$\theta$-interior point to $A^c$, from Lemma 3.13, $\mathcal{A}^c$ 
is $G_\delta$-$\theta$-open set. Therefore $A$ is $F_\sigma$-$\theta$-closed set.

**Proposition (3.15):** Every $F_\sigma$-$\theta$-closed set in 
Lindelof space is Lindelof.

**Proof:** Let $H$ be $F_\sigma$-$\theta$-closed subset of a space $X$, 
that is $H = \bigcup_{i \in \mathbb{N}} F_i$, where $F_i$ is $\theta$-closed set in 
$X$, but $X$ is Lindelof space, so by Lemma 3.9, $F_i$ , 
i \in \mathbb{N}, is Lindelof. Now, $\bigcup_{i \in \mathbb{N}} F_i$ is Lindelof and 
$H = \bigcup_{i \in \mathbb{N}} F_i$, so $H$ is Lindelof.

**Remark (3.16):** Let $(\mathcal{R}, T_D)$ be a discrete 
topology on a real numbers $\mathcal{R}$. Every singleton 
set is $\theta$-closed, then it is $F_\sigma$-$\theta$-closed set and 
Lindelof, but $(\mathcal{R}, T_D)$ is not Lindelof.

**Theorem (3.17):** Let a space $X$ is $\theta T_2$, Lindelof 
space, then $X$ is $\text{MinL} L(\theta C)$-space iff $X$ is $\theta P$-
space.

**Proof:** Let $X$ be $\text{MinL} L(\theta C)$-space, to prove 
$X$ is $\theta P$-space. Let $A$ be $F_\sigma$-$\theta$-closed subset in $X$, 
which is Lindelof, by Proposition 3.15, $A$ is 
Lindelof subset of $X$, which is $L(\theta C)$-space, then 
$A$ is $\theta$-closed set in $X$. Therefore $X$ is $\theta P$-space.

Conversely, suppose $X$ is $\theta P$-space, to prove $X$ is 
$\text{MinL} L(\theta C)$-space, let $H$ be a Lindelof subset of $X$, 
but $X$ is $\theta T_2$-space, then by Proposition 3.14 , 
$H$ is $F_\sigma$-$\theta$-closed set, also $X$ is $\theta P$-space, then $H$ 
is $\theta$-closed subset of $X$, that means $X$ is $L(\theta C)$-
space and it is Lindelof, so from Theorem 3.3, $X$ is 
$\text{MinL} L(\theta C)$-space.

**Proposition (3.18):** Every $\theta T_2$-space and $\theta P$-
space is $L(\theta C)$-space.

**Proof:** Let $M$ be a Lindelof subset of $X$, but $X$ is 
$\theta T_2$-space, so by Proposition 3.14, $M$ is $F_\sigma$-$\theta$-
closed set in $X$, which is $\theta P$-space, hence $M$ is $\theta$-
closed set in $X$, therefore $X$ is $L(\theta C)$-space.

**Theorem (3.19):** Every Lindelof $\theta T_2$ and $\theta P$-
space is $\text{MinL} L(\theta C)$-space.

**Proof:** Let a space $X$ be $\theta T_2$ and $\theta P$-space, by 
Proposition 3.18, $X$ is $L(\theta C)$-space and it is 
Lindelof, so by Theorem 3.3, $X$ is $\text{MinL} L(\theta C)$-
space.

**Proposition (3.20):** Every $L(\theta C)$-space is 
$K(\theta C)$-space.

**Proof:** Let $B$ be a compact subset of a space $X$, 
then $B$ is Lindelof in $X$, but $X$ is $L(\theta C)$-space, so 
$B$ is $\theta$-closed. Hence $X$ is $K(\theta C)$-space.

The convers of Proposition 3.20, is not true as 
shown by the following example.
Example (3.21): Let \((\mathcal{R}, \mathcal{T}_\mathcal{U})\) be a usual topology on a real numbers \(\mathcal{R}\). The compact subset of this space is only finite sets or closed interval, also they are \(\theta\)-closed. Therefore, \((\mathcal{R}, \mathcal{T}_\mathcal{U})\) is \(\mathcal{K}(\mathcal{C})\)-space. Also, the rational numbers \(\mathbb{Q}\) is Lindelof but not \(\theta\)-closed. Hence \((\mathcal{R}, \mathcal{T}_\mathcal{U})\) is not \(L(\mathcal{C})\)-space.

Theorem (3.22): If a space \(\mathcal{X}\) is compact and \(\theta\)-P-space, then \(\mathcal{X}\) is \(Min\theta\mathcal{T}_2\)-space iff \(\mathcal{X}\) is \(\mathcal{T}_2\)-space and \(MinL(\mathcal{C})\)-space.

Proof: Suppose a space \(\mathcal{X}\) is \(Min\theta\mathcal{T}_2\)-space, then \(\mathcal{X}\) is \(\theta\mathcal{T}_2\)-space, by Proposition 3.18, \(\mathcal{X}\) is \(L(\mathcal{C})\)-space. Also \(\mathcal{X}\) is compact, then \(\mathcal{X}\) is Lindelof, hence by Theorem 3.3, \(\mathcal{X}\) is \(MinL(\mathcal{C})\)-space. Conversely, suppose \(\mathcal{X}\) is \(\theta\mathcal{T}_2\)-space and \(MinL(\mathcal{C})\)-space, so \(\mathcal{X}\) is \(\theta\mathcal{T}_2\)-space and \(L(\mathcal{C})\)-space, by Proposition 3.20, \(\mathcal{X}\) is \(\mathcal{T}_2\)-space and \(\mathcal{K}(\mathcal{C})\)-space, and since \(\mathcal{X}\) is compact \(\mathcal{K}(\mathcal{C})\)-space, so from Theorem 2.27, \(\mathcal{X}\) is \(MinK(\mathcal{C})\)-space, and by Theorem 2.30, \(\mathcal{X}\) is \(Min\theta\mathcal{T}_2\)-space.

Corollary (3.23): If a space \(\mathcal{X}\) is compact and \(MinL(\mathcal{C})\)-space, then \(\mathcal{X}\) is \(MinK(\mathcal{C})\)-space.

Proof: Suppose \(\mathcal{X}\) is compact and \(MinL(\mathcal{C})\)-space, so \(\mathcal{X}\) is compact and \(L(\mathcal{C})\)-space, by Proposition 3.20, \(\mathcal{X}\) is compact and \(\mathcal{K}(\mathcal{C})\)-space, so from Theorem 2.27, we have \(\mathcal{X}\) is \(MinK(\mathcal{C})\)-space.

Corollary (3.24): If a space \(\mathcal{X}\) is compact and \(L(\mathcal{C})\)-space, then \(\mathcal{X}\) is \(MinK(\mathcal{C})\)-space.

Proof: Suppose \(\mathcal{X}\) is compact and \(L(\mathcal{C})\)-space, by Proposition 3.20, \(\mathcal{X}\) is compact and \(\mathcal{K}(\mathcal{C})\)-space, so from Theorem 2.27, \(\mathcal{X}\) is \(MinK(\mathcal{C})\)-space.

Corollary (3.25): Every countably compact, Lindelof and \(L(\mathcal{C})\)-space is \(MinK(\mathcal{C})\)-space.

Proof: Suppose \(\mathcal{X}\) is countably compact and Lindelof space, then \(\mathcal{X}\) is compact, from Corollary 3.24, \(\mathcal{X}\) is \(MinK(\mathcal{C})\)-space.

Theorem (3.26): If a space \(\mathcal{X}\) is compact and \(L(\mathcal{C})\)-space, then a closed subspace of \(\mathcal{X}\) is \(MinL(\mathcal{C})\)-space and \(MinK(\mathcal{C})\)-space.

Proof: Let \(\mathcal{Y}\) be a closed subspace of a compact space \(\mathcal{X}\), so \(\mathcal{Y}\) is compact set in \(\mathcal{X}\), then \(\mathcal{Y}\) is Lindelof. Also, \(\mathcal{X}\) is \(L(\mathcal{C})\)-space, so by Proposition 2.15, \(\mathcal{Y}\) is \(L(\mathcal{C})\)-space. Hence from Theorem 3.3, \(\mathcal{Y}\) is \(MinL(\mathcal{C})\)-space. Now, from Proposition 3.20, \(\mathcal{X}\) is \(\mathcal{K}(\mathcal{C})\)-space, so from Theorem 2.27, \(\mathcal{X}\) is \(MinK(\mathcal{C})\)-space.

Corollary (3.27): If a space \(\mathcal{X}\) is Lindelof and \(L(\mathcal{C})\)-space, then a closed subspace of \(\mathcal{X}\) is \(MinL(\mathcal{C})\)-space.

Proof: Let \(\mathcal{Y}\) be a closed subset of a Lindelof space \(\mathcal{X}\), then \(\mathcal{Y}\) is Lindelof in \(\mathcal{X}\), and then by Proposition 2.15, \(\mathcal{Y}\) is \(L(\mathcal{C})\)-space, from Theorem 3.3, \(\mathcal{X}\) is \(MinL(\mathcal{C})\)-space.

Corollary (3.28): If a space \(\mathcal{X}\) is Lindelof and \(L(\mathcal{C})\)-space, then a \(\theta\)-closed subspace of \(\mathcal{X}\) is \(MinL(\mathcal{C})\)-space.

Corollary (3.29): If a space \(\mathcal{X}\) is hereditarily Lindelof and \(L(\mathcal{C})\)-space, then any subspace of \(\mathcal{X}\) is \(MinL(\mathcal{C})\)-space.

Proof: Let \(\mathcal{Y}\) be a subspace of a space \(\mathcal{X}\), since \(\mathcal{X}\) is hereditarily Lindelof, so \(\mathcal{Y}\) is Lindelof, also by Proposition 2.15, \(\mathcal{Y}\) is \(L(\mathcal{C})\)-space, from Theorem 3.3, \(\mathcal{X}\) is \(MinL(\mathcal{C})\)-space.

Theorem (3.30): If a space \(\mathcal{X}\) is compact \(\theta\)-P-space, then \(\mathcal{X}\) is \(MinK(\mathcal{C})\)-space if and only if \(\mathcal{X}\) is \(MinL(\mathcal{C})\)-space.

Proof: Suppose \(\mathcal{X}\) is \(MinK(\mathcal{C})\)-space, that means \(\mathcal{X}\) is \(\mathcal{K}(\mathcal{C})\)-space and by hypothesis \(\mathcal{X}\) is compact, so \(\mathcal{X}\) is locally compact space and then from Proposition 2.28, \(\mathcal{X}\) is \(\theta\mathcal{T}_2\)-space, also \(\mathcal{X}\) is Lindelof. Therefore, by Theorem 3.19, \(\mathcal{X}\) is \(MinL(\mathcal{C})\)-space. Conversely, suppose \(\mathcal{X}\) is \(MinL(\mathcal{C})\)-space, so \(\mathcal{X}\) is \(L(\mathcal{C})\)-space, by Proposition 3.20, \(\mathcal{X}\) is \(\mathcal{K}(\mathcal{C})\)-space and it is compact, hence from Theorem 2.27, \(\mathcal{X}\) is \(MinK(\mathcal{C})\)-space.

Definition (3.31): A space \(\mathcal{X}\) is said to be \(\theta\mathbb{Q}\)-set space, if any subset of \(\mathcal{X}\) is \(\mathcal{T}_\sigma\)-\(\theta\)-closed set in \(\mathcal{X}\).

Proposition (3.32):
1. Every \(\theta\mathbb{Q}\)-set space is \(L_{L_3}\)-space.
2. Every \(\theta\mathbb{Q}\)-set space and \(\theta\mathcal{L}_1\)-space is \(L(\mathcal{C})\)-space.
3. Every \(\theta\mathbb{Q}\)-set space and \(\theta\mathcal{P}\)-space is \(L(\mathcal{C})\)-space.
4. Every Lindelof \(\theta\mathcal{L}_1\)-space is \(\theta\mathcal{P}\)-space.
5. Every \(\theta\mathcal{P}\)-space and \(\theta\mathcal{L}_3\)-space is \(L(\mathcal{C})\)-space.

Proof:
1. Let \(\mathcal{H}\) be a Lindelof subset of \(\theta\mathbb{Q}\)-set space \(\mathcal{X}\), then \(\mathcal{H}\) is \(\mathcal{T}_\sigma\)-\(\theta\)-closed set in \(\mathcal{X}\). Therefore, \(\mathcal{X}\) is \(L_{L_3}\)-space.
2. Let \(\mathcal{X}\) be a \(\theta\mathbb{Q}\)-set space \(\mathcal{X}\), by part(1), \(\mathcal{X}\) is \(\theta\mathcal{L}_3\)-space, and from Theorem 2.16 part (1), \(\mathcal{X}\) is \(L(\mathcal{C})\)-space.
3. Let $L$ be a Lindelof subset of $\theta Q$-set space $X$, then $L$ is $\mathcal{F}_\alpha$-$\theta$-closed set in $X$ which is $\theta P$-space, then $L$ is $\theta$-closed set in $X$. Therefore, $X$ is $L(\theta C)$-space.

4. Let $\mathcal{K}$ be $\mathcal{F}_\alpha$-$\theta$-closed set in a Lindelof space $X$, then $\mathcal{K} = \bigcup_{i \in \mathbb{N}} \mathcal{H}_i$, where $\mathcal{H}_i$ is $\theta$-closed set in a space $X$, for each $i \in \mathbb{N}$, by Lemma 3.9, $\mathcal{H}_i$ is Lindelof, so $\mathcal{K}$ is Lindelof and $\mathcal{F}_\alpha$-$\theta$-closed set, since $X$ is $\theta L_1$-space, then $\mathcal{K}$ is $\theta$-closed set. Therefore, $X$ is $\theta P$-space.

5. Suppose $X$ is $\theta P$-space, by Theorem 2.16, part(2), $X$ is $\theta L_1$-space, and it is $\theta L_3$-space, so by Theorem 2.16, part(1), $X$ is $L(\theta C)$-space.

**Proposition (3.33):**

1. Every Lindelof $\theta L_1$-space and $\theta L_3$-space is $\text{Min} L(\theta C)$-space.
2. Every Lindelof $\theta L_1$-space and $\theta T_2$-space is $\text{Min} L(\theta C)$-space.
3. Every Lindelof $\theta Q$-set and $\theta L_1$-space is $\text{Min} L(\theta C)$-space.

**Proof:**

1. Let a space $X$ is $\theta L_1$-space and $\theta L_3$-space, the by Theorem 2.16, part(1), $X$ is $L(\theta C)$-space and it is Lindelof, so from Theorem 3.3, $X$ is $\text{Min} L(\theta C)$-space.

2. Let a space $X$ is Lindelof $\theta L_1$-space, the by Proposition 3.32 part(4), $X$ is $\theta P$-space, and from Proposition 3.18, $X$ is $L(\theta C)$-space, also from Theorem 3.3, $X$ is $\text{Min} L(\theta C)$-space.

3. Let a space $X$ is Lindelof $\theta L_1$-space, the by Proposition 3.32, part (4), $X$ is $\theta P$-space, and from Proposition 3.32, part (3), $X$ is $L(\theta C)$-space, also from Theorem 3.3, $X$ is $\text{Min} L(\theta C)$-space.

**Theorem (3.34):** Every $L(\theta C)$-space having $\theta$-dense Lindelof subset is $\text{Min} L(\theta C)$-space.

**Proof:** Let $\mathcal{A}$ be a $\theta$-dense Lindelof subset of a space $X$, but $X$ is $L(\theta C)$-space, then $\mathcal{A}$ is $\theta$-closed, then $\mathcal{A} = Cl_\theta(\mathcal{A}) = X$, hence $X$ is Lindelof and it is $L(\theta C)$-space, so from Theorem 3.3, $X$ is $\text{Min} L(\theta C)$-space.

**Proposition (3.35):** Every Lindelof $\theta Q$-set space and $\theta P$-space is $\text{Min} L(\theta C)$-space.

**Proof:** From Proposition 3.32, part(3) and Theorem 3.3.

**Proposition (3.36):** Every compact $\theta Q$-set space and $\theta P$-space is $\text{Min} K(\theta C)$-space.

**Proof:** Let a space $X$ be $\theta Q$-set space and $\theta P$-space, then from Proposition 3.32, part(3), $X$ is $L(\theta C)$-space and by Proposition 3.20, $X$ is $K(\theta C)$-space, since $X$ is compact and $K(\theta C)$-space so by Theorem 2.27, $X$ is $\text{Min} K(\theta C)$-space.

**Theorem (3.37):** Every compact $\theta Q$-set space and $\theta L_1$-space is $\text{Min} K(\theta C)$-space.

**Proof:** Let a space $X$ be $\theta Q$-set space and $\theta L_1$-space, so by Proposition 3.32 part (3), $X$ is $L(\theta C)$-space, and from Proposition 3.20, $X$ is $K(\theta C)$-space, so we have a space $X$ is compact $K(\theta C)$-space, hence by Theorem 2.27, $X$ is $\text{Min} K(\theta C)$-space.

**Corollary (3.38):** Every compact $\theta L_1$-space and $\theta L_3$-space is $\text{Min} K(\theta C)$-space.

**Proof:** Let $X$ be $\theta L_1$ and $\theta L_3$-space, from Proposition 2.16, part(1), $X$ is $L(\theta C)$-space, also by Proposition 3.20, $X$ is $K(\theta C)$-space and from Theorem 2.27, $X$ is $\text{Min} K(\theta C)$-space.

**Corollary (3.39):** Every compact $\theta P$-space and $\theta L_3$-space is $\text{Min} K(\theta C)$-space.

**Proof:** Let $X$ be $\theta P$-space, from Theorem 2.16, part(2), $X$ is $\theta L_1$-space and from Corollary 3.38, $X$ is $\text{Min} K(\theta C)$-space.

**Theorem (3.40):** If $X$ and $Y$ are $\mathcal{T}_2$-spaces, $L(\theta C)$-spaces, then $X \times Y$ is $L(\theta C)$-space.

**Proof:** Let $L$ be a Lindelof subset of $X \times Y$, and let $(x_0, y_0) \notin L$, for each $(x, y) \in L$, then there exists open neighbourhoods $U_x$ and $V_y$ of $x$ and $y$ respectively, such that $(x_0, y_0) \notin \overline{U_x \times V_y}$, since $L \subseteq \bigcup \{U_x \times V_y : (x, y) \in L\}$, we have $L \subseteq \bigcup \{U_{x_n} \times V_{y_n} : n \in \mathbb{Z}^+ \}$. Now, let $E_1 = \{n \in \mathbb{Z}^+ : x_0 \notin \overline{U_{x_n}}\}$ and $E_2 = \{n \in \mathbb{Z}^+ : y_0 \notin \overline{V_{y_n}}\}$, then $E_1 \cup E_2 = \mathbb{Z}^+$. And, if $L_1 = \{L \cap (\overline{U_{x_n}} \times \overline{V_{y_n}}) : n \in E_1\}$ and $L_2 = \{L \cap (\overline{U_{x_n}} \times \overline{V_{y_n}}) : n \in E_2\}$, then $L_1$ and $L_2$ are Lindelof subset of $X \times Y$, such that $L_1 \cup L_2 = L$. Clearly $x_0 \notin \pi_1(L_1)$ and since $L_1$ is Lindelof and $\pi_1$ is continuous, then $\pi_1(L_1)$ is Lindelof in $X$, and since $X$ is $L(\theta C)$-space, then $\pi_1(L_1)$ is $\theta$-closed, by Remark 2.6, $\pi_1(L_1)$ is closed in $X$, so there is an open neighbourhood $G \subseteq X$ of $x_0$, with $G \cap \pi_1(L_1) = \emptyset$. In the same way, since $y_0 \notin \pi_2(L_2)$ and $L_2$ is Lindelof in $Y$, with $\pi_2$ is continuous, so $\pi_2(L_2)$ is Lindelof in $Y$, and since $Y$ is $L(\theta C)$-space, then $\pi_2(L_2)$ is $\theta$-closed, so $\pi_2(L_2)$ is closed in $Y$, so there is an
open neighbourhood \( H \subseteq Y \) of \( x_0 \), with \( H \cap \pi_2(L_2) = \emptyset \), we now claim \((G \times H) \cap L = \emptyset\), since\((x,y) \in L\), suppose\((x,y) \in (G \times H)\), then \( x \in G \), but \( G \cap \pi_1(L_1) = \emptyset \), then \( x \notin \pi_1(L_1) \), so \((x,y) \notin L_1\), also \((x,y) \notin L_2\), hence \((x,y) \notin L\), since \( L_1 \cup L_2 = L \) That is \((x,y) \notin L\) and this is contradiction, so \( X \times Y \) is \( L(\theta C) \)-space.

**Corollary (3.41):** If \( X \) and \( Y \) are compact \( T_2 \)spaces and \( L(\theta C) \)-spaces, then \( X \times Y \) is \( MinL(\theta C) \)-space and \( MinK(\theta C) \)-space.

**Proof:** Let \( X \) and \( Y \) are compact \( T_2 \)-spaces and \( L(\theta C) \)-space, then by Theorem 3.40, \( X \times Y \) is \( L(\theta C) \)-space and then \( X \times Y \) is compact, also \( X \times Y \) is Lindelof and \( X \times Y \) is \( L(\theta C) \)-space, by Theorem 3.3, \( X \times Y \) is \( MinL(\theta C) \)-space. Now, by Proposition 3.20, \( X \times Y \) is \( K(\theta C) \)-space and it is compact, then by Theorem 2.27, \( X \times Y \) is \( MinK(\theta C) \)-space.

**Corollary (3.42):** If \( X \) and \( Y \) are Lindelof \( T_2 \)-spaces and \( L(\theta C) \)-space, then \( X \times Y \) is \( Min(\theta C) \)-space.

**Proof:** Let \( X \) and \( Y \) are compact \( T_2 \)-spaces and \( L(\theta C) \)-spaces, then by Theorem 3.40, \( X \times Y \) is \( L(\theta C) \)-space and from hypothesis \( X \times Y \) is Lindelof, and the by Theorem 3.3, \( X \times Y \) is \( MinL(\theta C) \)-space.

**Proposition (3.43):** If \( X \) and \( Y \) are \( R_1 \), \( L(\theta C) \)-spaces, then \( X \times Y \) is \( L(\theta C) \)-space.

**Proof:** Let \( X \) and \( Y \) are \( L(\theta C) \)-spaces, by Remarks 2.20, part (3), \( X \) and \( Y \) are \( T_1 \)-spaces, but \( X \) and \( Y \) are \( R_1 \)-spaces, then \( X \) and \( Y \) are \( T_2 \)-space and by Theorem 3.40, \( X \times Y \) is \( L(\theta C) \)-space.

**Theorem (3.44):** If \( X \) and \( Y \) are compact \( R_1 \) and \( L(\theta C) \)-space, then \( X \times Y \) is \( MinL(\theta C) \)-space and \( MinK(\theta C) \)-space.

**Proof:** Let \( X \) and \( Y \) are \( R_1 \), \( L(\theta C) \)-space, then by Proposition 3.43, \( X \times Y \) is \( L(\theta C) \)-space. Also, \( X \) and \( Y \) are compact spaces, so \( X \times Y \) is compact and then \( X \times Y \) is Lindelof. Therefor from Theorem 3.3, \( X \times Y \) is \( MinL(\theta C) \)-space. Now, from Proposition 3.20, \( X \times Y \) is \( K(\theta C) \)-space and it is compact, then by Theorem 2.27, \( X \times Y \) is \( MinK(\theta C) \)-space.

**Theorem (3.45):** If \( X \) and \( Y \) are Lindelof \( R_1 \) and \( L(\theta C) \)-spaces, then \( X \times Y \) is \( MinL(\theta C) \)-space

**Proof:** Let \( X \) and \( Y \) are \( R_1 \), \( L(\theta C) \)-spaces, then by Proposition 3.43, \( X \times Y \) is \( L(\theta C) \)-space, also \( X \) and \( Y \) are Lindelof spaces, so \( X \times Y \) is Lindelof. Hence from Theorem 3.3, \( X \times Y \) is \( MinL(\theta C) \)-space.

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