Geometro-Stochastically Quantized Fields
with Internal Spin Variables

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Abstract

The use of internal variables for the description of relativistic particles with arbitrary mass and spin in terms of scalar functions is reviewed and applied to the stochastic phase space formulation of quantum mechanics. Following Bacry and Kihlberg a four-dimensional internal spin space $\check{S}$ is chosen possessing an invariant measure and being able to represent integer as well as half integer spins. $\check{S}$ is a homogeneous space of the group $SL(2,\mathbb{C})$ parametrized in terms of spinors $\alpha \in \mathbb{C}^2$ and their complex conjugates $\bar{\alpha}$. The generalized scalar quantum mechanical wave functions may be reduced to yield irreducible components of definite physical mass and spin $[m, s]$, with $m \geq 0$ and $s = 0, \frac{1}{2}, 1, \frac{3}{2}, \ldots$, with spin described in terms of the usual $(2s + 1)$-component fields. Viewed from the internal space description of spin this reduction amounts to a restriction of the variable $\alpha$ to a compact subspace of $\check{S}$, i.e. a “spin shell” $S^2_{r=2s}$ of radius $r = 2s$ in $\mathbb{C}^2$. This formulation of single particles or single antiparticles of type $[m, s]$ is then used to study the geometro-stochastic (i.e. quantum) propagation of amplitudes for arbitrary spin on a curved background space-time possessing a metric and axial vector torsion treated as external fields. A Poincaré gauge covariant path integral-like representation for the probability amplitude (generalized wave function) of a particle with arbitrary spin is derived satisfying a second order wave equation on the Hilbert bundle constructed over curved space-time. The implications for the stochastic nature of polarization effects in the presence of gravitation are pointed out and the extension to Fock bundles of bosonic and fermionic type is briefly mentioned.
I. INTRODUCTION

Spin appeared in physics as a typical property of quantum mechanical states determining the multiplet structure of atomic spectra. The concept of a nonrelativistic quantum mechanical wave function had to be broadened to be able to account for the presence of spin yielding thereby a unified description for orbital as well as spin motion formulated with the help of group theoretical methods [1], or, more precisely, treated in terms of the representation theory of the rotation group $SO(3)$ [2,3]. Extending this nonrelativistic theory to a formulation in accord with special relativity, within a Lorentz and translation invariant formalism for free particles, leads to Wigner’s [4] identification of elementary particles observed in nature with the unitary irreducible representations (UIR) of the Poincaré group, $\mathcal{P} = ISO(3,1)$, characterized by mass and spin $[m, s]$, with real $m \geq 0$ and $s = 0, \frac{1}{2}, 1, \frac{3}{2}, 2, \ldots$.

Relativistic particles with definite mass and spin are in this scheme described in terms of multicomponent fields, $\psi_a(x); a = 1 \ldots n, x = (x^\mu; \mu = 0, 1, 2, 3)$, defined over Minkowski space-time $M_4$, which transform as vectors under $\mathcal{P}$ or, in the half integer spin cases, as spinors under the universal covering group $\bar{\mathcal{P}} = ISO(3,1) = T_4 \otimes_s SL(2,\mathbb{C})$, where $\otimes_s$ denotes the semidirect product. The interaction between various different fields is then usually formulated as a Lorentz invariant coupling of these multicomponent fields, for example, like $e \bar{\psi}(x) \gamma_\mu \psi(x) A^\mu(x)$ in the case of QED or like $g \bar{\psi}_N(x) \gamma_5 \vec{r} \psi_N(x) \bar{\phi}(x)$ as, for example, for the pseudoscalar pion nucleon interaction [5]. Writing the interaction between different elementary particle fields in this manner freezes the spin content of the fields to their free field values and thus forces the spin degrees of freedom to play an undynamical role in the theory.

When the Regge theory of strong interaction was en vogue in particle physics it was observed from the data on high energy collisions that the effective spin of the exchanged particle mediating the strong interaction, for example the $\rho$-meson or the pion, depended on the momentum transfer $\bar{t}$ between the collision partners taking part in the process and was not a constant given by the fixed spin value of the free field. There was a so-called...
trajectory relation involved connecting the scattering states for \( t < 0 \) and continuously variable effective spin with a family of Regge recurrences for \( t = m_s^2 > 0 \) and discrete physical integer or half integer spin values \( s \) of certain observed resonant states i.e. excited states of strong interaction. This showed that in going “off mass shell” with the invariant energy or momentum transfer variable of an analytic \( S \)-matrix element one had in strong interaction physics also to go “off spin shell” and analytically continue in the spin variable, i.e. one had to allow for a dynamical role of spin \( \pi \).

In this context one intended in the sixties to replace the elementary fields for particles of definite mass and spin by so-called spin-tower-fields with built-in trajectory relation between mass and spin (compare, for example, Bacry and Nuyts \( \pi \)). At the same time the question was asked whether it would be appropriate to represent a particle with spin not by a vector- or spinor-valued function over Minkowski space-time with fixed number of components but by a scalar function defined over a higher dimensional space, in particular, a homogeneous space of the underlying kinematic symmetry group \( \mathcal{P} \), i.e. to consider fields in particle physics as scalar-valued functions defined on \( \mathcal{P}/H \) where \( H \) is a closed subgroup of \( \mathcal{P} \). In order to consider homogeneous spaces of the Poincaré group which contain Minkowski space-time, i.e. being of the type \( M_4 \times S \), with \( S \) playing the role of a spin space, one regards \( H \) to be a subgroup of the Lorentz group contained in \( \mathcal{P} \) yielding thereby for the space \( S \) a homogeneous space of the Lorentz group.

All the homogeneous spaces of the Poincaré group of this type were listed by Finkelstein \( \pi \) and by Bacry and Kihlberg \( \pi \), and the existence of an invariant measure on \( \mathcal{P}/H \) as well as the suitability of these spaces for the description of half integer spins were investigated. The authors of Ref. \( \pi \) came to the conclusion that the lowest dimensional homogeneous space with invariant measure suitable for the description of half integer as well as integer spins is eight-dimensional with four internal variables for the representation of spin, \( S = “[4]” \) in Finkelstein’s notation, yielding thereby a generalized scalar wave function, \( \psi(X) \), for the description of a particle with spin, with \( X = (x^\mu, y^i) \in \mathcal{P}/H \) where \( x^\mu \in M_4 \) and \( y^i \in S, \) \( i = 1, 2, 3, 4. \)
If one wants to have fields with a fixed mass value $m$ and a definite spin $s$ it is required that the field $\psi(X)$ takes sharp values for the two Casimir operators of the Poincaré group, i.e.

\[
\hat{P}_\mu \hat{P}^\mu \psi(X) = m^2 \psi(X),
\]
\[
\hat{W}_\mu \hat{W}^\mu \psi(X) = -m^2 s(s + 1) \psi(X),
\]
with $\hat{P}_\mu$ denoting the momentum operator and with

\[
\hat{W}_\mu = \frac{1}{2} \varepsilon_{\mu\nu\rho\lambda} \hat{P}^\nu \hat{S}^{\rho\lambda}
\]

being the Pauli-Lubanski operator, where $\hat{S}^{\rho\lambda} = -\hat{S}^{\lambda\rho}$ is a set of spin operators satisfying the Lie algebra of $SO(3,1)$ which are expressed as differential operators in the additional internal spin variables, $y^i$, with the coordinates $y^i$ parametrizing the space $S = SO(3,1)/H$. [$SO(3,1)$ is used here to denote the proper orthochronous Lorentz group $O(3,1)^{++}$.]

However, it was pointed out by Bacry and Kihlberg [9] that in order to reduce the description completely to an irreducible one in the Wigner sense two additional conditions on the scalar wave functions for the quantized description of spin had to be introduced. These could most easily be expressed by using a two-dimensional spinor formulation of the internal spin space

\[
\bar{S} = SL(2,\mathbb{C})/\bar{H}
\]

where $\bar{H}$ (being the universal covering group of $H$) is a subgroup of $SL(2,\mathbb{C})$ determining the space defined by Eq.(1.4) [10]. The space $\bar{S}$ could thus be parametrized by spinor variables $\alpha^A, A = 1, 2$, given by [9]

\[
\alpha^1 = e^{\frac{1}{2} t} e^{i\frac{1}{2} \psi} \cos \frac{1}{2} \theta \ e^{i\frac{1}{2} \varphi},
\]
\[
\alpha^2 = e^{\frac{1}{2} t} e^{i\frac{1}{2} \psi} \sin \frac{1}{2} \theta \ e^{-i\frac{1}{2} \varphi},
\]

and their complex conjugates $\bar{\alpha}^A; \bar{A} = \bar{1}, \bar{2}$. Here $-\infty < t < \infty; 0 \leq \theta \leq \pi, 0 \leq \varphi < 2\pi$, and $-2\pi \leq \psi < 2\pi$. This shows that the internal spin space with the four real variables $t, \psi, \theta, \varphi$ and the Lorentz invariant measure
\[ d\mu_s = e^{2t} dt d\psi d\cos \theta d\varphi \quad (1.6) \]

is isomorphic to a two-dimensional complex space \( \mathcal{C}_2 \) with measure \( d\alpha d\bar{\alpha} = d\alpha^1 d\alpha^2 d\bar{\alpha}^1 d\bar{\alpha}^2 \) being invariant because of the unimodularity of \( SL(2, \mathbb{C}) \). From this point of view one would represent the wave function for a relativistic particle with arbitrary spin as a scalar function \( \psi(x, \alpha, \bar{\alpha}) \) with \( x = (x^\mu; \mu = 0, 1, 2, 3) \), denoting a point in Minkowski space, and with \( \alpha = (\alpha^A; A = 1, 2) \) and its complex conjugate \( \bar{\alpha} = (\bar{\alpha}^\dot{A}; \dot{A} = \dot{1}, \dot{2}) \) denoting a point in the internal spin space \( \mathcal{C}_2 \) and its complex conjugate, respectively. \( \psi(x, \alpha, \bar{\alpha}) \) would have to satisfy the Klein-Gordon equation in \( x \) which is essentially Eq. (1.1) with the velocity of light taken to be \( c = 1 \).

In parentheses we would like to remark already at this place that due to the impossibility of localizing a relativistic particle in Minkowski space in terms of projector-valued (PV) measures providing a system of imprimitivity of the Poincaré group in Mackey’s sense on the Hilbert space of states \([11,12]\), we shall present below a stochastic phase space description of relativistic particles, as advocated strongly by Prugovečki \([13–15]\), and construct a system of covariance of the Poincaré group in terms of positive operator-valued (POV) measures on a Hilbert space for particles with arbitrary spin. This yields a stochastic phase space description of relativistic particles as proposed by Prugovečki which is extended in this paper to nonzero spin by using a homogeneous space description of the Poincaré group for the spin degrees of freedom – or rather a \( \mathcal{C}_2 \)-description as mentioned above – leading to a fully covariant formalism for the kinematics and localization properties of free relativistic particles with definite mass and spin in terms of scalar functions. At a later stage of this investigation we shall study the implications of the internal space description of spin for the coupling of fields describing interactions among several particles with nonzero (dynamical) spin, i.e. couple several general spin fields together in using the internal spinorial variables introduced and investigated in the present paper.

Returning now to the scalar wave function \( \psi = \psi(x, \alpha, \bar{\alpha}) \) with internal spinorial variables \( \alpha \) and \( \bar{\alpha} \) transforming under the fundamental representations \( D^{(\frac{1}{2}, 0)}(\Lambda) \) and \( D^{(0, \frac{1}{2})}(\Lambda) \) of
$SL(2,\mathcal{C})$, respectively, one demands, following Bacry and Kihlberg [4], for the states of definite spin $s$ that the following two constraining equations are satisfied:

$$\hat{D}\psi = \alpha^A \frac{\partial \psi}{\partial \alpha^A} = 2s\psi,$$

(1.7)

and

$$\frac{\partial}{\partial \bar{\alpha}^{\dot{A}}} \psi = 0.$$

(1.8)

If Eqs. (1.7) and (1.8) are obeyed, $\psi$ is a homogeneous polynomial of degree $2s$ in the undotted spinor variables $\alpha^1$ and $\alpha^2$ fixing the spin to the integer or half integer value $s$ with no dependence of $\psi$ on the dotted spinor variables $\bar{\alpha}^{\dot{1}}$ and $\bar{\alpha}^{\dot{2}}$. We call Eq. (1.7) [with the summation convention used for the spinor indices] the homogeneity condition reducing the wave function $\psi$ for arbitrary spin to a particular spin value $s$; and we call Eq. (1.8) the holomorphicity condition yielding thus a spin description in terms of holomorphic functions of the variables $\alpha \in \mathcal{C}_2$.

It follows from Ref. [3] that if Eqs. (1.7), (1.8) and (1.1) are satisfied by $\psi$ also the Casimir operator appearing on the l.h.s. of Eq. (1.2) possesses a sharp eigenvalue for $\psi$ given by $-m^2s(s+1)$ and the description reduces to an irreducible one in the Wigner sense.

It is interesting to remark that Eqs. (1.7) and (1.8) are instructive also from another point of view. In the course of investigating the geometric quantization of constrained Hamiltonian systems describing particles with nonzero spin one proceeds by giving a classical phase space description of spin in extending the symplectic geometry, i.e. the phase space geometry, to include the spin degrees of freedom which – after quantization – yield discrete integer values for $2s$ and transform irreducibly under $SU(2)$, in the nonrelativistic case, or under $SL(2,\mathcal{C})$, in the relativistic case (compare N. Woodhouse [18]). Classically, the subspace of $\mathcal{C}_2$ appropriate for the description of a definite spin $s$ is the two-sphere, $S^2_{r=2s} = S^3_{r=2s}/U(1)$ obtained as a factorization by $U(1)$ of the three-sphere $S^3_{r=2s}$ of radius $r = 2s$, i.e. the “spin shell” given by

$$p_{\alpha^A \bar{\alpha}^{\dot{A}}} = \bar{r} = mcr,$$

(1.9)
where $p_{A\dot{A}} = (p^0 + i\vec{p})_{A\dot{A}}$, with the Pauli matrices $\vec{\sigma} = (\sigma_1, \sigma_2, \sigma_3)$, is the spinor form of the 4-momentum $p^\mu = (p^0, \vec{p})$. Here the $U(1)$ degree of freedom $(\alpha^A, \bar{\alpha}^{\dot{A}}) \to (e^{i\chi}\alpha^A, e^{-i\chi}\bar{\alpha}^{\dot{A}})$, with real $\chi$, defines an equivalence class for which Eq. (1.9) remains unchanged. After quantization this yields through Eqs. (1.7) and (1.8) a description in terms of a momentum and $\alpha$-dependent reduced wave function $\tilde{\psi}^{(s)}(p, \alpha)$, with $\alpha$ defined on a sphere $S^2_{r=2s}$ of radius $r = 2s$, with integer $r$, contained in $C_2$. The two-spheres $S^2_{r=2s}$ of non-zero integer radius $r = 2s$ define an integral orbit space representing the coadjoint orbits of the rotation group in Kirillov’s terminology [19].

It appears from Eqs. (1.7) and (1.8) that only half of the phase space variables $\alpha^1, \alpha^2$ and $\bar{\alpha}^{\dot{1}}, \bar{\alpha}^{\dot{2}}$ are relevant for a quantized description of physical spin and that the spin space for free noninteracting particles is essentially a two-sphere. A two-sphere $S^2$ may be regarded, on the one hand, as the homogeneous space $SU(2)/U(1)$ or, on the other hand, as the homogeneous space $SL(2, \mathbb{C})/\tilde{P}$ where $\tilde{P}$ is the subgroup of $SL(2, \mathbb{C})$ of complex triangular matrices of the form $\begin{pmatrix} \rho & 0 \\ \eta & \rho^{-1} \end{pmatrix}$ with $\rho, \eta \in \mathbb{C}$ (see [20]). The universal covering group $SL(2, \mathbb{C})$ of the Lorentz group acts transitively on the two-sphere in this latter form; it acts on $S^2 = SU(2)/U(1)$ through the Wigner rotation $\pm \tilde{R}(p, \Lambda) \in SU(2)$ related to the Lorentz transformation $\Lambda \in SO(3, 1)$ carrying a momentum eigenstate with momentum $p$ into one with momentum $\Lambda p$, i.e.

$$\Lambda = \Lambda_p \tilde{R}(p, \Lambda) \Lambda_p^{-1}$$

(1.10)

where $\Lambda_p$ is the boost, $p = \Lambda_p \tilde{p}$, taking the rest momentum $\tilde{p} = (mc, \vec{0})$ into $p = (p^0, \vec{p})$, and $\pm \tilde{R}(p, \Lambda)$ above is the element of $SU(2)$ corresponding to the $SO(3)$ rotation $R(p, \Lambda)$ in (1.10).

As viewed from the original space $C_2$, the reduction involved in the quantized description of arbitrary spin in terms of a function $\tilde{\psi}(p, \alpha, \bar{\alpha})$ depending $\alpha$ and its complex conjugate to a function $\tilde{\psi}^{(s)}(p, \alpha)$ for definite spin defined on a two-sphere in $C_2$ may be regarded also in the following way. One may view the space $C_2$ as a $GL(1, \mathbb{C})$-bundle over $S^2$,

$$C_2 = P(S^2, GL(1, \mathbb{C}))$$

(1.11)
with $GL(1, \mathbb{C}) = \mathbb{C}^* = \mathbb{C}\{0\}$ being isomorphic to the complex numbers without the origin. For $\lambda \in \mathbb{C}^*$ the $GL(1, \mathbb{C})$ transformations give rise to an equivalence relation provided by $(\alpha^A, \bar{\alpha}^A) \sim (\lambda \alpha^A, \bar{\lambda} \bar{\alpha}^A)$. The two-sphere $S^2$ may thus be regarded as the space $\mathcal{C}_2$ modulo this equivalence relation describing dilatations by $\lambda$ [21]. Hence, the reduction originating from the imposition of Eqs. (1.7) and (1.8) implies a corresponding reduction given by a projection in the principal bundle (1.11) from the bundle space to its base.

The key observation in the context of the present paper, however, is that in contradistinction to the vector-type representations for spin appearing in Refs. [22] and [17] based on Wigner rotations, spin can, indeed, be given a formulation in terms of scalar functions defined on a four-dimensional homogeneous space of the Lorentz group possessing an invariant measure with the internal spinorial variables transforming under $SL(2, \mathbb{C})$. This description reduces, as mentioned, to an irreducible one corresponding to a definite mass $m$ and spin $s$ for free physical particles if, besides (1.1), the constraints (1.7) and (1.8) are required to be satisfied. In this case results similar to those of Ali and Prugovečki [22] are obtained, where, in our presentation, a joint description for integer as well as half integer spins is given. This is due to the fact that even in the reduced case the internal spin variables may be considered to transform under $SL(2, \mathbb{C})$ although we know that the internal spin space, in fact, reduces to a subspace of $\mathcal{C}_2$, i.e. to a two-sphere (a compact space) and the transformation group may be considered to reduce to the group of Wigner rotations, i.e. to the transformations of the compact subgroup $SO(3)$ of the Lorentz group or its covering group $SU(2)$.

In the phase space framework which we are aiming at in this paper both the $(q, p)$ variables as well as the spin variables $(\alpha, \bar{\alpha})$ of the original internal spin space $\mathcal{C}_2$ and its complex conjugate are regarded as phase space variables transforming all, except for the translations affecting only $q$, in a similar manner under Poincaré transformations $(b, \Lambda)$. This property will be used in Sect. II to define a one-particle resolution kernel Hilbert space, $\mathcal{H}_{\bar{q}}^{(s)}$, for free particles of mass $m$ and arbitrary integer or half integer spin $s$ possessing physically reasonable relativistic localization properties, which are described in terms of scalar functions, $\psi^{(s)}(q, p, \alpha)$, obeying Eqs. (1.1), (1.7) and (1.8) and transforming irreducibly.
under a unitary phase space representation, \( U^{(s)}(b, \Lambda) \), of the Poincaré group. In Sect. III we extend this description to a first quantized soldered Hilbert bundle \( \mathcal{H}_{[m,s]} \) over a curved Riemann-Cartan space-time base \( U_4 \) in order to include and investigate influences due to gravity. The bundle \( \mathcal{H}_{[m,s]} \) is associated to the affine spin frame bundle \( P(U_4, \bar{G} = \bar{P}) \). In Sect. IV we then study the quantum propagation on \( \mathcal{H}_{[m,s]} \) and define a generalized path integral formula for particles with spin. Finally, Sect. V is devoted to some concluding remarks.

II. ONE-PARTICLE STOCHASTIC PHASE SPACE DESCRIPTION INCLUDING SPIN

In this section we develop the stochastic phase space representation of \( \mathcal{P} \) for particles of arbitrary spin by using the internal spinor variables \( \alpha \) and \( \bar{\alpha} \) introduced in the introduction. To define the notation, we begin by briefly reviewing the spin zero case treated in detail in [13], and then investigate an extended framework for the stochastic phase space description of one-particle states possessing arbitrary but unspecified spin. We then reduce this representation into irreducible components to yield a description for free relativistic particles of a definite physical spin, \( s = 0, \frac{1}{2}, 1, \frac{3}{2}, \ldots \), and a fixed mass value \( m \).

A) The spin-zero case

The aim is to construct a unitary irreducible phase space representation of the Poincaré group in terms of generalized quantum mechanical wave functions, \( \psi(q, p) \), which represent a relativistic spin-zero particle (or antiparticle) with stochastic localization properties in the configuration space variable \( q = (q^\mu; \mu = 0, 1, 2, 3) \) as well as in the momentum space variable \( p = (p^\mu; \mu = 0, 1, 2, 3) \), with \( p_\mu p^\mu = m^2 c^2 \), where \( m \) is the mass of the particle. The reason for introducing the stochastic phase space variables \( (q, p) \) for the description of particles in high energy physics is the impossibility to localize relativistic particles in
terms of PV-measures on Borel sets over configuration space alone, with the operators transforming under a unitary irreducible representation of \( \mathcal{P} \) acting in the respective Hilbert space of states carrying this system of imprimitivity. It is, however, possible to construct a \textit{generalized system of imprimitivity} (called a system of covariance) in terms of POV-measures on Borel sets over Minkowski space \textit{and} momentum space, realized on a Hilbert space \( \mathcal{H}_{\tilde{\eta}} \) of states transforming under a stochastic phase space representation \( U(b, \Lambda) \) of \( \mathcal{P} \). This is achieved by defining irreducibly transforming one-particle states over \textit{relativistic stochastic phase space}, constructed in terms of a resolution generator \( \tilde{\eta} = \tilde{\eta}_l \) for the particle in question, with \( \tilde{\eta}_l \) being parametrized by an elementary length parameter \( l \). Thereby Wigner’s 1932 phase space formulation of quantum mechanics \cite{23} is turned into (i) a fully relativistic formulation, and (ii) a formalism possessing a probability interpretation for the description based on the stochastic variables \( q \) and \( p \). The outcome of this endeavour is the construction of a resolution kernel Hilbert space, \( \mathcal{H}_{\tilde{\eta}} \), carrying a unitary irreducible spin-zero phase space representation, \( U(b, \Lambda) \), of the Poincaré group, and containing states \( \Psi \) with physically reasonable (stochastic) localization properties in the variables \( q \) and \( p \). For a detailed description of this whole approach we refer to the extended work of E. Prugovečki (see Refs. \cite{13,14} and the references quoted there) as well as to the review paper by S.T. Ali \cite{17}.

To define the notation we introduce the relativistic one-particle phase space

\[
\mathcal{M}_m^\pm = M_4 \times V_m^\pm, \tag{2.1}
\]

where \( M_4 = R_{1,3} \) denotes Minkowski space-time having the metric tensor \( \eta_{\mu\nu} = \text{diag}(1, -1, -1, -1) \), and \( V_m^\pm \) is the positive energy (\( V_m^+ \) with \( p^0 > 0 \)) or negative energy (\( V_m^- \) with \( p^0 < 0 \)) hyperboloid in momentum space, \( p_\mu p^\mu = m^2 c^2 \), possessing the Lorentz invariant measure

\[
d\Omega_m(p) = \frac{d^3 p}{2 p^0}. \tag{2.2}
\]

The momentum space wave function for a spin-zero particle of mass \( m \) will be denoted by \( \hat{\psi}(k) \), with \( \hat{\psi}(k) \in L^2(V_m^\pm), k \in V_m^\pm \), being square integrable with respect to the measure
The space $L^2(V_m^\pm)$ carries a unitary irreducible representation $\hat U(b,\Lambda)$ of the Poincaré group:

$$\hat U(b,\Lambda)\hat \psi(k) = e^{i\hat b \cdot k} \hat \psi(\Lambda^{-1} k)$$

(2.3)

leaving the scalar product

$$\langle \hat \psi_1 | \hat \psi_2 \rangle_{V_m^\pm} = \int_{V_m^\pm} \hat \psi_1^*(k) \hat \psi_2(k) d\Omega_m(k)$$

(2.4)

invariant. The notation used implies that both states with momentum wave functions $\hat \psi_1(k)$ and $\hat \psi_2(k)$ refer either to a particle (integration over $V_m^+$) or to an antiparticle (integration over $V_m^-$).

The next step is the construction of an isometric map, called $\mathcal{W}_\tilde{\eta}$, between the Hilbert space $L^2(V_m^\pm)$ and a Hilbert space $L^2(\Sigma_m^\pm)$ defined over relativistic phase space, where $\Sigma_m^\pm = \sigma \times V_m^\pm \subset \mathcal{M}_m^\pm$, with $\sigma$ being a space-like hypersurface in Minkowski space, and with the Poincaré invariant measure on $\Sigma_m^\pm$ being given by

$$d\Sigma_m(q,p) = 2\epsilon(p^0)p^\mu d\sigma_m \delta(p^2 - m^2 c^2) d^4 p.$$  

(2.5)

This is achieved with the help of the map

$$\mathcal{W}_{\tilde{\eta}} : L^2(V_m^\pm) \to L^2(\Sigma_m^\pm)$$

(2.6)

defined by the following integral transform:

$$\hat \psi(q,p) = (\mathcal{W}_{\tilde{\eta}}\hat \psi)(q,p) = \int_{V_m^\pm} \tilde{\eta}_{q,p}^*(k) \hat \psi(k) d\Omega_m(k).$$

(2.7)

where the $\tilde{\eta}_{q,p}(k)$ denote a set of coherent states, obtained from the resolution generators $\tilde{\eta}(k) \in L^2(V_m^\pm)$ with the help of the Poincaré transformation $\hat U(q,\Lambda_p)$, involving a translation by $q$ and a Lorentz boost with $v = p/m$ denoted by $\Lambda_p$. Using (2.3) one has

$$\tilde{\eta}_{q,p}(k) = (\hat U(q,\Lambda_p)\tilde{\eta})(k) = e^{i\hat b \cdot k} \tilde{\eta}(\Lambda_p^{-1} k).$$

(2.8)

Here $\tilde{\eta}(k)$ is the resolution generator, being $SO(3)$ invariant, i.e. obeying, with $R \in SO(3)$ and $\Lambda(R) = \begin{pmatrix} 1 & 0 \\ 0 & R \end{pmatrix}$,
\[ \bar{\eta}(\Lambda(R)k) = \bar{\eta}(k). \]  

(2.9)

It is easy to show that (2.9) implies that \( \bar{\eta}(\Lambda_{p}^{-1}k) = \eta(p \cdot k) \) with real \( \eta = \eta_{l} \) (compare Ref. [13], Chapter 2; we suppress the suffix \( l \) in the sequel).

Eq. (2.8) defines a set of generalized coherent states in \( L^{2}(V_{m}^{\pm}) \) parametrized in terms of the coset \( \mathcal{P}/SO(3) \). So we may, finally, write the integral transform \( \mathcal{W}_{\bar{\eta}} \) introduced in (2.7) as

\[
\psi(q, p) = (\mathcal{W}_{\bar{\eta}}\hat{\psi})(q, p) = \int_{V_{m}^{\pm}} e^{-\frac{i}{\hbar}q \cdot k} \bar{\eta}^* (p \cdot k) \hat{\psi}(k) d\Omega_{m}(k)
\]

(2.10)

where the complex conjugation symbol on \( \eta \) is, actually, unnecessary but we keep it for the later generalization of Eq. (2.10). By construction the right-hand side of (2.10) satisfies the Klein-Gordon equation in the variable \( q \).

Mapping the coherent states (2.8) into \( L^{2}(\Sigma_{m}^{\pm}) \), i.e. defining

\[
\phi_{q, p}(q', p') = (\mathcal{W}_{\bar{\eta}}\bar{\eta}_{q, p})(q', p') = \int_{V_{m}^{\pm}} \bar{\eta}_{q', p'}(k) \bar{\eta}_{q, p}(k) d\Omega_{m}(k) = \langle \bar{\eta}_{q', p'} | \bar{\eta}_{q, p} \rangle_{V_{m}^{\pm}},
\]

(2.11)

yields the result, because of the isometry property of the map \( \mathcal{W}_{\bar{\eta}} \), that the overlap between the coherent states computed in \( L^{2}(V_{m}^{\pm}) \) may be expressed as

\[
\phi_{q, p}(q', p') = \langle \bar{\eta}_{q', p'} | \bar{\eta}_{q, p} \rangle_{V_{m}^{\pm}} = \langle \phi_{q', p'} | \phi_{q, p} \rangle_{\Sigma_{m}^{\pm}},
\]

(2.12)

which is identical to the propagator in \( L^{2}(\Sigma_{m}^{\pm}) \) given by

\[
K_{\bar{\eta}}(q', p'; q, p) = \langle \phi_{q', p'} | \phi_{q, p} \rangle_{\Sigma_{m}^{\pm}}
= \int_{\Sigma_{m}^{\pm}} \hat{\phi}_{q', p'}(q'', p'') \phi_{q, p}(q'', p'') d\Sigma_{m}(q'', p''),
\]

(2.13)

where the second equality defines the scalar product in \( L^{2}(\Sigma_{m}^{\pm}) \). Using the fact that the states \( \phi_{q, p} \) allow the following resolution of the identity in \( L^{2}(\Sigma_{m}^{\pm}) \)

\[
\int_{\Sigma_{m}^{\pm}} |\phi_{q, p}| d\Sigma_{m}(q, p) \langle \phi_{q, p} | = 1^{\pm},
\]

(2.14)
we see that $K_{\tilde{\eta}}(q', p'; q, p)$ obeys the following reproducing and reality relations implied by (2.13):

$$K_{\tilde{\eta}}(q', p'; q, p) = \int_{\Sigma_m^\pm} K_{\tilde{\eta}}(q', p'; q'', p'')K_{\tilde{\eta}}(q'', p''; q, p)d\Sigma_m(q'', p''),$$  \hspace{1cm} (2.15)

and

$$K_{\tilde{\eta}}^*(q', p'; q, p) = K_{\tilde{\eta}}(q, p; q', p') = \phi_{q', p'}^*(q, p).$$  \hspace{1cm} (2.16)

Any state $\Psi \in \mathcal{H}_{\tilde{\eta}} \equiv L^2(\Sigma_m^\pm)$ may now be decomposed in terms of the states $\phi_{q, p}$ providing a coherent state basis in $\mathcal{H}_{\tilde{\eta}}$. The result is

$$\Psi = \int_{\Sigma_m^\pm} \psi(q, p)\phi_{q, p}d\Sigma_m(q, p),$$  \hspace{1cm} (2.17)

where $\psi(q, p) = \langle \phi_{q, p} | \Psi \rangle_{\Sigma_m^\pm}$ is a generalized one-particle relativistic quantum mechanical wave function (a scalar field on stochastic phase space) transforming under Poincaré transformations $(b, \Lambda)$ in the following manner:

$$(U(b, \Lambda)\psi)(q, p) = \psi(\Lambda^{-1}(q - b), \Lambda^{-1}p).$$  \hspace{1cm} (2.18)

Eq. (2.18) is easily proven by applying (2.3) in (2.10) and making use of the invariance of the measure (2.2). Thus $\mathcal{W}_{\tilde{\eta}}$ is an intertwining operator for the representations $\hat{U}(b, \Lambda)$ and $U(b, \Lambda)$ obeying

$$U(b, \Lambda)\mathcal{W}_{\tilde{\eta}} = \mathcal{W}_{\tilde{\eta}}\hat{U}(b, \Lambda).$$  \hspace{1cm} (2.19)

Using (2.13) and (2.14) in the definition of $\psi(q, p)$ given above one easily derives the following propagation formula:

$$\psi(q', p') = \int_{\Sigma_m^\pm} K_{\tilde{\eta}}(q', p'; q, p)\psi(q, p)d\Sigma_m(q, p).$$  \hspace{1cm} (2.20)

The phase space representation $U(b, \Lambda)$ of $\mathcal{P}$ defined by (2.10) and (2.18) leaves invariant the following scalar product in $\mathcal{H}_{\tilde{\eta}}$ obtained from (2.13) and (2.17):
\[ \langle \psi_1 \mid \psi_2 \rangle_{\Sigma_m^\pm} = \int_{\Sigma_m^\pm = \sigma \times V_m^\pm} \psi_1^*(q,p)\psi_2(q,p)d\Sigma_m(q,p). \tag{2.21} \]

Eq. (2.21) may also be written as
\[ \langle \psi_1 \mid \psi_2 \rangle_{\Sigma_m^\pm} = \frac{i\hbar}{\Sigma_\eta} \int \int \psi_1^*(q,p) \frac{\partial}{\partial q^\mu} \psi_2(q,p)d\sigma^\mu(q)d\Omega_m(p) \tag{2.22} \]
with \( \partial_\mu = \partial/\partial q^\mu \), and with
\[ Z_\eta = (2\pi\hbar)^3 \int_{V_m^\pm} |\eta(p \cdot k)|^2 d\Omega_m(p) \tag{2.23} \]
being a constant independent of \( k \). For a particular choice of the resolution generator \( \eta \) this yields an irreducible unitary representation \( U(b, \Lambda) \) on \( \mathcal{H}_\eta \) describing spin-zero particles of mass \( m \) (see Refs. \[13\] and \[14\]). In our context it is essential to observe that the resolution generator introduces a particular smearing in the variables \( q \) and \( p \) (in accordance with Heisenberg’s uncertainty relations) which is parametrized here in terms of a fundamental length parameter \( l \) for the particular type of particles involved with \([m, s] = [m, 0]\). The actual value of \( l \), i.e. whether it is of order of \( 10^{-16} \) cm, i.e. well below the charge radius of a nucleon, or even equal to the Planck length \( \sim 10^{-32} \) cm is not essential in the present context. The main point is the regularizing effect this length parameter has in the stochastic phase space formalism. Taking, however, the sharp point limit \( l \to 0 \) leads to the appearance of singularities in Eqs. (2.22) and (2.23) (compare Refs. \[13\] and \[14\]). This is reminiscent of the situation prevailing in the conventional relativistic quantum field theory based on \( q \)-space fields obtained by ordinary Fourier transformation from the \( p \)-space fields. The stochastic phase space description introduced in Refs. \[13\] \[14\] and \[17\] was just proposed in order to avoid the singularities of the conventional formalism. We shall thus assume the fundamental length parameter \( l \) to have a small but finite fixed value.

Our task now is to extend the spin-zero formalism reviewed above to the case of a particle with arbitrary spin \( s \). This will be done in the following subsection by using the internal spin space variables \( \alpha \) and \( \bar{\alpha} \) for a homogeneous space description of spin as described in the introduction.
B) The non-zero spin case

In view of the discussion presented in the introduction we represent a relativistic particle with arbitrary but unspecified spin and definite mass \( m \) by a scalar wave function \( \hat{\psi}(k, \alpha, \bar{\alpha}) \) defined on momentum and spin space, \( V_m^\pm \times C_2 \), with the invariant measure (2.2) on \( V_m^\pm \) and the invariant measure \( dad\bar{\alpha} \) on \( C_2 \) [24] (compare the remarks made after (1.6) above).

As regards the transformation rule for the spinor variables \( \alpha = (\alpha^1, \alpha^2) \) and \( \bar{\alpha} = (\bar{\alpha}^{\dot{1}}, \bar{\alpha}^{\dot{2}}) \) introduced in Sect. I we observe that, conventionally, a spinor with a lower undotted spinor index \( A \) is taken to transform with the \( SL(2, \mathbb{C}) \) matrix \( D(1/2, 0)(\Lambda) \), while an upper dotted spinor index \( \dot{A} \) transforms with the matrix \( [D(1/2, 0)(\Lambda^{-1})]^\dagger = D(0, 1/2)(\Lambda) \) (compare, for example, Carruthers [25]). This leads for our spinor variables \( \alpha, \bar{\alpha} \) with only upper indices to the following transformation rules (written as matrix operation from the left and with \( T \) denoting the transpose):

\[
\alpha' = D(\Lambda)\alpha \quad \text{with} \quad D(\Lambda) = [D(1/2, 0)(\Lambda^{-1})]^T \tag{2.24}
\]
\[
\bar{\alpha}' = \bar{D}(\Lambda)\bar{\alpha} \quad \text{with} \quad \bar{D}(\Lambda) = D(0, 1/2)(\Lambda) \tag{2.25}
\]

The generalization of Eq. (2.3) in the presence of spin described through the internal spinor variables \( \alpha, \bar{\alpha} \) now reads

\[
(\hat{U}(b, \Lambda)\hat{\psi})(k, \alpha, \bar{\alpha}) = e^{\frac{i}{\hbar}b \cdot k} \hat{\psi}(\Lambda^{-1}k, D(\Lambda^{-1})\alpha, \bar{D}(\Lambda^{-1})\bar{\alpha}) \tag{2.26}
\]

where, of course, \( \hat{U}(b, \Lambda) \) is not an irreducible representation here. Before we discuss the reduction of the function \( \hat{\psi}(k, \alpha, \bar{\alpha}) \) to one describing a particle with definite spin value \( s \) we also generalize the scalar product defined in (2.4) to functions defined on \( V_m^\pm \times C_2 \) [24]:

\[
\langle \hat{\psi}_1 | \hat{\psi}_2 \rangle_{V_m^\pm \times C_2} = \int_{V_m^\pm \times C_2} \hat{\psi}_1^*(k, \alpha, \bar{\alpha})\hat{\psi}_2(k, \alpha, \bar{\alpha})d\Omega_m(k)d\alpha d\bar{\alpha} . \tag{2.27}
\]

Due to (2.26) and the invariance of the measure on \( V_m^\pm \times C_2 \) this is a Poincaré invariant scalar product if it exists. We shall assume here that the momentum space wave function \( \hat{\psi}(k, \alpha, \bar{\alpha}) \) of a spinning particle is, indeed, square-integrable over \( V_m^\pm \) and \( C_2 \) requiring that the high spin states contained in \( \hat{\psi} \) (see the reduction described below) are sufficiently
damped to compensate for the exponential factor $e^{2t}$ in the measure (1.6) when expressed in the variables $(t, \psi, \theta, \varphi)$. In the conventional formulation (compare Ref. [22]) a decomposition of the original Hilbert space $\hat{H}$ into an infinite direct sum of irreducible subspaces
$$\hat{H} = \sum_{s=0}^{\infty} \oplus \hat{H}^{(s)} ,$$
where arbitrary large spin values $s$ are involved. In the present context we shall identify $\hat{H}$ with the Hilbert space $L^2(V_m^\pm \times C_2)$ defined by (2.26) and (2.27) assuming $\hat{\psi}(k, \alpha, \bar{\alpha})$ to be square-integrable with respect to the measure $d\Omega_m(k)d\alpha d\bar{\alpha}$.

Before we go on to construct the analogue of the map $\mathcal{W}_{\hat{n}}$ in the present case yielding an irreducible subspace $H^{(s)}_{\hat{n}}$ of the Hilbert space $L^2(\Sigma_m^\pm \times C_2)$ introduced below, let us reduce the general $C_2$-description of spin by demanding homogeneity and holomorphicity in the spinor variables as expressed by Eqs. (1.7) and (1.8). As described in the introduction, this amounts to the following restrictions:

$$\hat{\psi}(k, \alpha, \bar{\alpha}) \xrightarrow{(1.7),(1.8)} \hat{\psi}^{(s)}(k, \alpha),$$

where $s$ is a definite spin value. Here $\hat{D} = \alpha^A \partial/\partial \alpha^A$ represents an invariant operator which commutes with $\hat{U}(b, \Lambda)$ as defined in (2.26); the same is true for the invariant operator appearing on the l.-h. side of (2.32). Using Eqs. (1.5) and their complex conjugates, Eq. (2.32) may be written as

$$e^t(k^\phi + \vec{n} \cdot \vec{k}) = mcr ,$$

A scalar momentum space wave function satisfying (2.30)-(2.32) will be denoted, according to (2.28), by $\hat{\psi}^{(s)}(k, \alpha)$. Here $\hat{D} = \alpha^A \partial/\partial \alpha^A$ represents an invariant operator which commutes with $\hat{U}(b, \Lambda)$ as defined in (2.26); the same is true for the invariant operator appearing on the l.-h. side of (2.32). Using Eqs. (1.5) and their complex conjugates, Eq. (2.32) may be written as

$$e^t(k^\phi + \vec{n} \cdot \vec{k}) = mcr ,$$

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where $\vec{n}$ is a unit vector given by

$$
\vec{n} = (\sin \theta \cos \varphi, \sin \theta \sin \varphi, \cos \theta).
$$

(2.34)

In the derivation of (2.33) the angle $\psi$ has disappeared in accordance with the remarks concerning the $U(1)$ degree of freedom made around Eq. (1.9) in the introduction. In the rest system of the particle we thus have in the reduced case from (2.33) that

$$
e^t = r = 2s = \text{fixed integer}.
$$

(2.35)

This constrains the integration over the $C_2$ variables, for example, in (2.27) and in analogous equations derived below in the reduced case where Eqs. (2.30)-(2.32) are to be satisfied. Let us now decompose $\hat{\psi}(s)(k, \alpha)$ in view of (2.30) into a homogeneous polynomial of degree $2s$ in $\alpha^1$ and $\alpha^2$ yielding (compare Eqs. (1.5))

$$
\hat{\psi}(s)(k, \alpha) = \sum_{s_3} \hat{\psi}_{s_3}(s)(s) r^s \frac{(\alpha^1)^{s+s_3}(\alpha^2)^{s-s_3}}{(s + s_3)! (s - s_3)!} \sqrt{(s + s_3)! (s - s_3)!}
$$

(2.36)

where the sum over $s_3$ runs from $s_3 = -s$ to $s_3 = +s$. Moreover, we have written the $t$-dependence explicitly as $(e^t)^s = r^s$ and denoted the angular part depending on $\psi, \theta,$ and $\varphi$ by underlined spinor components. Following Edmonds [3] we call the appearing normalized homogeneous polynomials $u(s, s_3)$, i.e.

$$
u(s, s_3) = \frac{(\alpha^1)^{s+s_3}(\alpha^2)^{s-s_3}}{(s + s_3)! (s - s_3)!} \sqrt{(s + s_3)! (s - s_3)!},
$$

(2.37)

possessing the following well-known behaviour under SO(3)-rotations $R_{\alpha, \beta, \gamma}$ (compare also Ref. [2] p. 163 - 165)

$$
U(R_{\alpha, \beta, \gamma})u(s, s_3) = u'(s, s_3) = \sum_{s_3'} u(s, s_3') D_{s'_3, s_3}^{s}(\alpha, \beta, \gamma)
$$

(2.38)

where $u'(s, s_3)$ is the expression (2.37) computed with the rotated spinor components $\alpha'^{A}$; $A = 1, 2$, and $D_{s'_3, s_3}^{s}(\alpha, \beta, \gamma)$ are the Wigner $D^{s}$-functions.

We now consider Eq. (2.26) for the reduced function $\hat{\psi}(s)(k, \alpha)$. Using the decomposition (2.36) yields
\[
\sum_{s_3} (\hat{U}(b, \Lambda)\hat{\psi}_{s_3}^{(s)})(k) r^s u(s, s_3) = \sum_{s_3} e^{i\hat{h} b^k} \hat{\psi}_{s_3}^{(s)}(\Lambda^{-1}k) r^s u'(s, s_3)
\]  
(2.39)

where here \(u'(s, s_3)\) is constructed, according to (2.24) and (2.33), in terms of the spinor

\[
\alpha' = D(\Lambda^{-1})\alpha, \quad \alpha' = e^{-\frac{t}{2}}\alpha.
\]
(2.40)

Thus \(u'(s, s_3)\) differs from \(u(s, s_3)\) by a rotation associated with the transition from \(k\) to \(\Lambda^{-1}k\). This is the Wigner rotation (compare (1.10))

\[
R^{-1}(k, \Lambda^{-1}) = \Lambda^{-1}k \Lambda \Lambda^{-1}k.
\]
(2.41)

Using this rotation in (2.38) and inserting the resulting expression for \(u'(s, s_3)\) into the r.-h. side of (2.39) we conclude – taking, moreover, the orthonormality of the \(u(s, s_3)\) into account – that the following transformation rule for \(\hat{\psi}_{s_3}^{(s)}(k)\) must hold:

\[
\left(\hat{U}^{(s)}(b, \Lambda)\hat{\psi}_{s_3}^{(s)}\right)(k) = e^{i\hat{h} b^k} \sum_{s_3'} D_{s_3, s_3'}^s(\Lambda^{-1}k) \hat{\psi}_{s_3}^{(s)}(\Lambda^{-1}k).
\]
(2.42)

This is the typical transformation law for the momentum eigenstate of a free particle of spin \(s\) and spin projection \(s_3\), derived here from Eq. (2.26) for a definite integer or half integer spin value \(s\). Moreover, in (2.42) we have denoted by \(\hat{U}^{(s)}(b, \Lambda)\) the irreducible action of \(\hat{U}(a, \Lambda)\) in the \((2s + 1)\)-dimensional vector space defined by \(\hat{\psi}_{s_3}^{(s)}(k)\).

Considering now the square of the wave function \(\hat{\psi}^{(s)}(k, \alpha)\) according to (2.27), and taking recognition of the constraint (2.32) by introducing a \(\delta\)-function

\[
\delta\left(\frac{1}{mc} k_{\dot{A}} A^\dot{A} \alpha^\dot{A} \overline{\alpha}^\dot{A} - r\right), \quad \text{with integer } r = 2s
\]
(2.43)

into the \(C_2^\pm\)-integration, yields with the help of (2.33)

\[
\langle \hat{\psi}^{(s)} | \hat{\psi}^{(s)} \rangle_{V_m^+ \times C_2} = \int_{V_m^+ \times C_2} [\hat{\psi}^{(s)}(k, \alpha)]^* \hat{\psi}^{(s)}(k, \alpha) \delta\left(\frac{1}{mc} k_{\dot{A}} A^\dot{A} \alpha^\dot{A} \overline{\alpha}^\dot{A} - r\right) d\Omega_m(k) d\alpha d\overline{\alpha}.
\]
(2.44)

This may be written in terms of the coordinates \((t, \psi, \theta, \varphi)\) in using (1.3), (1.6) together with (2.36) as

\[
\langle \hat{\psi}^{(s)} | \hat{\psi}^{(s)} \rangle_{V_m^+ \times C_2} = \int_{V_m^+} d\Omega_m(k) \sum_{s_3 s_3'} [\hat{\psi}_{s_3}^{(s)}(k)]^* \psi_{s_3}^{(s)}(k) r^{2s} |u(s, s_3)|^2 u(s, s_3) \delta(e^t - r) e^{2t} dt d\psi \sin \theta d\theta d\varphi.
\]
(2.45)
Here \( u(s, s_3) \), defined in (2.37), is expressed in terms of \( \alpha^A(\psi, \theta, \varphi); A = 1, 2 \). We can use Eq. (2.38) to write \( u(s, s_3) \) in terms of \( D^s \)-functions, i.e.

\[
u(s, s_3) = \sum_{s_3'} \hat{u}(s, s_3') D^s_{s_3 s_3'}(\psi, \theta, \varphi) \tag{2.46}
\]

where \( \hat{u}(s, s_3) \) is the homogeneous polynomial constructed with \( \alpha^1 = 1 \) and \( \alpha^2 = 0 \) being different from zero only for \( s = s_3 \). With the help of the familiar result \[3\]

\[
\frac{1}{8\pi^2} \int_0^{2\pi} d\varphi \int_0^{2\pi} \int_0^{2\pi} D^{j_1}_{m_1 m_1} (\psi, \theta, \varphi) D^{j_2}_{m_2 m_2} (\psi, \theta, \varphi) d\psi \sin \theta d\theta d\varphi = \delta_{m_1 m_2} \delta_{m_1 m_2} \delta_{j_1 j_2} \frac{1}{2j_1 + 1} \tag{2.47}
\]
as well as the normalization of the \( \hat{u}(s, s_3) \), one finally obtains

\[
\langle \hat{\psi}^{(s)} | \hat{\psi}^{(s)} \rangle_{V_m^\pm \times C_2} = \frac{8\pi^2}{2s + 1} \int_{V_m^\pm} |\hat{\psi}^{(s)}(k)|^2 d\Omega_m(k). \tag{2.48}
\]

Here we could now absorb the constants appearing in front of the integral into the normalization of the wave functions \( \hat{\psi}^{(s)}(k) \); \( s_3 = -s \ldots + s \), for each particular spin value \( s = \frac{1}{2}, 1, \frac{3}{2}, 2 \ldots \) Summing up we may say that the reduction to a definite spin value \( s \) governed by Eqs. (2.29)-(2.32) thus leads to wave functions \( \hat{\psi}^{(s)}(k) \) being elements of the Hilbert space \( \hat{\mathcal{H}}^{(s)} = L^2(V_m^\pm) \otimes K_s \). As regards spin they transform irreducibly under the representation of \( SL(2, \mathbb{C}) \) usually denoted by \( D^{(s,0)} \).

In defining now, in the presence of spin, a map \( \hat{\mathcal{W}} \) from \( L^2(V_m^\pm \times C_2) \) to \( L^2(\Sigma_m^\pm \times C_2) \), and constructing a unitary reducible phase space representation for particles with arbitrary spin in terms of scalar functions on generalized phase space, we make the following observation concerning the variables \( \alpha \) and \( \bar{\alpha} \). In fact, the pair \((\alpha, \bar{\alpha})\) may be regarded as phase space variables for spin in analogy to \((q, p)\) being the phase space variables for the kinematic localization of spin-zero particles. Quantum mechanically the momentum operators (for the phase space representation) are constructed with the operators \( i\hbar \partial/\partial q^\mu \) producing the eigenvalue \( p_\mu \), while the spin operator (or “spin measuring operator”) \( \hbar \hat{D} \), producing the eigenvalue \( 2s\hbar \), is constructed in terms of \( \hbar \partial/\partial \alpha^A \). So one could regard \( \alpha^A \) as a position-type variable for spin and \( \bar{\alpha}^A \) as the corresponding conjugate momentum-type variable for
spin. (Of course, we already know from the discussion presented above that in a quantized theory describing free particles of definite spin the $\bar{\alpha}$ variables disappear and only half the spin variables remain to describe a particle of definite spin.) Hence, in defining the integral transform $\tilde{W}_\eta$ in the presence of spin (and prior to the reduction), an invariant integration over momentum-type variables must be involved – with spin included! One would therefore expect – provided the mentioned analogy between ordinary phase space variables and spin variables is indeed correct – that an integration over $\bar{\alpha}$ is involved in generalizing Eq. (2.7) to the non-zero spin case. We thus propose the following integral transform to yield $\psi(p, q; \alpha, \bar{\alpha})$:

$$\psi(q, p; \alpha, \bar{\alpha}) = (\tilde{W}_\eta \hat{\psi})(q, p; \alpha, \bar{\alpha})$$

$$= \frac{1}{N} \int \int [\tilde{\eta}_{\bar{\alpha}, \bar{\alpha}}(k, \alpha, \bar{\alpha}')]^{*} \psi(k, \alpha, \bar{\alpha}') d\Omega_m(k) d\bar{\alpha}' ,$$

where the coherent state basis for non-zero spin is, in analogy to (2.8) and in view of (2.26),

$$\tilde{\eta}_{q,p}(k, \alpha, \bar{\alpha}) = (\hat{U}(q, \Lambda_p) \tilde{\eta})(k, \alpha, \bar{\alpha})$$

$$= e^{i\bar{\eta}^k} \tilde{\eta}(\Lambda_p^{-1}k, D(\Lambda_p^{-1})\alpha, \bar{\Lambda}_p^{-1}D(\Lambda_p^{-1})\bar{\alpha}).$$

The measure $d\bar{\alpha}'$ in (2.49) is $P$-invariant due to the unimodularity of the group $SL(2, \mathbb{C})$. $N$ in (2.49) is a normalization constant associated with the integration over $\bar{\alpha}'$.

It is now essential to remark that the resolution generator $\tilde{\eta}(k, \alpha, \bar{\alpha})$ in (2.50) is again assumed to be $SO(3)$-invariant, i.e. generalizing Eq. (2.9) the following relation holds

$$\tilde{\eta}(\Lambda(R)k, D(R)\alpha, \bar{D}(R)\bar{\alpha}) = \tilde{\eta}(k, \alpha, \bar{\alpha}),$$

(2.51)

where $D(R)$ denotes the $SL(2, \mathbb{C})$ matrix corresponding to a rotation $R \in SO(3)$, with $D(R)$ and $\bar{D}(R)$ denoting thus equivalent representations of $SU(2)$ as is well-known.

It is easy to show using (2.26) and the rotation invariance (2.51) of $\tilde{\eta}$ that the intertwining relation (2.19) for $\tilde{W}_\eta$ is again valid with $U(b, \Lambda)$ acting on the states $\psi(q, p; \alpha, \bar{\alpha})$ in the following way:

$$(U(b, \Lambda)\psi)(q, p; \alpha, \bar{\alpha}) = \psi(\Lambda^{-1}(q - b), \Lambda^{-1}p; D(\Lambda^{-1})\alpha, \bar{D}(\Lambda^{-1})\bar{\alpha}).$$

(2.52)
In order to establish (2.52), using (2.49), the spinor $D(\Lambda_p^{-1})\alpha$ appearing in the argument of $\tilde{\eta}$ in $\tilde{W}_\eta \hat{U}(b, \Lambda) \hat{\psi}$ is written as

$$D(\Lambda_p^{-1})\alpha = D(R^{-1}(p, \Lambda^{-1}))D(\Lambda_{A^{-1}_p})D(\Lambda^{-1})\alpha$$

(2.53)

with $R^{-1}(p, \Lambda^{-1})$ as given by (2.41). In the argument of $\tilde{\eta}$ the Wigner rotation may however be dropped due to the $SO(3)$-invariance (2.51). This has the consequence that the spinor variables of $\tilde{\eta}$ “feel only the boosts”, acting in inequivalent ways on $\alpha$ and $\bar{\alpha}$, establishing thus, finally, Eq. (2.52).

A Poincaré invariant scalar product for the phase space wave functions $\psi(q, p; \alpha, \bar{\alpha})$ satisfying (2.52) may now be written down generalizing Eq. (2.21):

$$\langle \psi_1 | \psi_2 \rangle_{\Sigma^\pm \times C_2} = \int_{\Sigma^\pm \times C_2} [\psi_1(q, p; \alpha, \bar{\alpha})]^* \psi_2(q, p; \alpha, \bar{\alpha}) d\Sigma_m(q, p) d\alpha d\bar{\alpha}$$

(2.54)

Let us next investigate Eqs. (2.49) and (2.50) in the reduced case assuming Eqs. (2.29)-(2.32) to hold true. In this case the $\bar{\alpha}'$ dependence of $\tilde{\eta}$ and $\hat{\psi}$ disappears and the integration over $d\bar{\alpha}' = e^{\frac{1}{2}i} d\bar{\alpha}'$ represents, in view of (2.33), an angular integration which can be carried out. Adjusting the constant $N$ appropriately this yields

$$\psi^{(s)}(q, p; \alpha, \bar{\alpha}) = \left( \tilde{W}_\eta \hat{\psi}^{(s)} \right)(q, p; \alpha, \bar{\alpha})$$

$$= \int_{V_{\pm}} e^{-\frac{1}{2}i \hat{\psi}^{(s)}(\Lambda_p^{-1}k, D(\Lambda_p^{-1})\alpha)}^* \psi^{(s)}(k, \alpha) d\Omega_m(k) ,$$

(2.55)

where $\hat{\psi}^{(s)}(k, \alpha)$ is the homogeneous polynomial in $\alpha$ given in (2.30). Let us immediately remark that $\psi^{(s)}(q, p; \alpha, \bar{\alpha})$ defined by (2.53) does seem to develop now an $\bar{\alpha}$-dependence through the complex conjugation of the expression in the square brackets under the integral provided the resolution generator $\tilde{\eta}^{(s)}$ does, indeed, depend on the spinor variables $\alpha$. This is, however, not the case, and the r.-h. side of (2.55) will, in fact, define a quantity $\psi^{(s)}(q, p; \alpha)$ independent of $\bar{\alpha}$.

To see that $\tilde{\eta}^{(s)}(\Lambda_p^{-1}k, D(\Lambda_p^{-1})\alpha)$ cannot depend on $\alpha$ we note that the $SL(2, \mathbb{C})$ matrix for a rotation free boost,
\[ D^{(\frac{1}{2},0)}(\Lambda_p) = \frac{1}{\sqrt{2mc(p^0 + mc)}} [mc(1 + p^0 1 + \vec{\sigma} \cdot \vec{p})], \quad (2.56) \]

changes the real length factor \( e^{\frac{p^0}{2}} \) of \( \alpha \) (compare (1.7)). On the other hand, fixing the spin to the value \( s \) restricts this factor to \( \sqrt{r} = \sqrt{2s} \) (integrality condition (2.33)). This contradiction can, in view of the rotation invariance (2.51), only be avoided if \( \bar{\eta}^{(s)} \) does not depend on \( \alpha \) at all. Since in this case \( \hat{D}\bar{\eta}^{(s)} = 0 \), it does not depend on \( s \) either, and we can replace the resolution generator appearing in (2.55) by the one describing the spin zero case in Eq. (2.7), i.e.

\[ \bar{\eta}^{(s)}(\Lambda_p^{-1}k) = \eta(p \cdot k). \quad (2.57) \]

We may thus, finally, rewrite (2.55), remembering that \( \eta(p \cdot k) \) is real, as

\[
\psi^{(s)}(q, p; \alpha) = (\mathcal{W}_{\bar{\eta}} \hat{\psi}^{(s)}) (q, p; \alpha) \\
= \int_{V_m^\pm} e^{-\frac{i}{\hbar} q \cdot k} \eta(p \cdot k) \hat{\psi}^{(s)}(k, \alpha) d\Omega_m(k), \quad (2.58)
\]

where we have denoted the integral transform \( \mathcal{W}_{\bar{\eta}} \) by the same symbol as in the spinless case in Eq. (2.10) above. Applying the operator \( \hat{D} \) to both sides of this equation it is clear that \( \hat{D} \) commutes with the map \( \mathcal{W}_{\bar{\eta}} \), i.e.

\[ \hat{D}\mathcal{W}_{\bar{\eta}} = \mathcal{W}_{\eta}\hat{D}. \quad (2.59) \]

One can now again decompose \( \hat{\psi}^{(s)}(k, \alpha) \) according to Eq. (2.36) and define \((2s + 1)\)-dimensional vector-valued phase space functions \( \hat{\psi}^{(s)}_{s_3}(q, p) \) in terms of momentum space wave functions \( \hat{\psi}^{(s)}_{s_3}(k) \), with \((s, s_3)\); \( s_3 = -s \ldots + s \), taking values in the spin space \( K_s \) of dimension \( 2s + 1 \). We thus see that the correspondence (2.49) yields, for the reduced states of definite spin \( s \), an isometric map (2.58) – constructed in the same manner as in the spin-zero case – relating the Hilbert spaces \( L^2(V_m^\pm) \otimes K_s \equiv \hat{\mathcal{H}}^{(s)} \) [compare (2.48)] and \( L^2(\Sigma_m^\pm) \otimes K_s \equiv \mathcal{H}^{(s)}_\eta \). To have a condensed notation at our disposal one can, however, express the relations under study at first in terms of the scalar fields \( \psi^{(s)}(q, p; \alpha) \) and \( \hat{\psi}^{(s)}(k, \alpha) \) and
then go over at a later stage to the \((2s+1)\)-dimensional vector-valued fields by making an \(\alpha\)-expansion in terms of homogeneous polynomials of degree \(2s\).

Mapping the coherent state basis for the case of definite spin \(s\),

\[
\tilde{\eta}^{(s)}(k) = \tilde{\eta}_{q,p}(k) = e^{ikqk}\eta(q \cdot k),
\]

into \(L^2(\Sigma_m^\pm)\) as in Eq. (2.11) yields again

\[
\phi_{q,p}(q',p') = \langle \mathcal{W}_{\tilde{\eta}}\tilde{\eta}^{(s)}(q',p') \rangle = \langle \tilde{\eta}_{q',p'} | \tilde{\eta}_{q,p} \rangle_{\Sigma_m^\pm} = \langle \phi_{q',p'} | \phi_{q,p} \rangle_{\Sigma_m^\pm} = K_{\tilde{\eta}}(q',p';q,p).
\]

This implies that the stochastic phase space propagator \(K_{\tilde{\eta}}^{(s)}(q',p';q,p)\) for a free particle of spin \(s\) is the same as that for a spin-zero particle defined in (2.13). Hence, freezing the spin content of the fields to any physical value \(s\) does not alter the phase space kinematics of free stochastic propagation.

Introducing the resolution of the identity for the subspace of definite spin \(s = 0, \frac{1}{2}, 1, \frac{3}{2}, \ldots\) in \(L^2(\Sigma_m^\pm \times C_2)\) by (2.14) – derived above for \(s = 0\) only but due to (2.60) and (2.61) being valid generally – one can write down the following expansion for a state vector \(\Psi^{(s)}(\alpha)\) of definite integer or half integer spin belonging to the Hilbert space \(\mathcal{H}_{\tilde{\eta}}^{(s)}\):

\[
\Psi^{(s)}(\alpha) = \int_{\Sigma_m^\pm} \psi^{(s)}(q,p;\alpha)\phi_{q,p}d\Sigma_m(q,p)
\]

with

\[
\psi^{(s)}(q,p;\alpha) = \langle \phi_{q,p} | \Psi^{(s)}(\alpha) \rangle_{\Sigma_m^\pm}.
\]

The wave function \(\psi^{(s)}(q,p;\alpha)\) obeys the same propagation rule on stochastic phase space as does the spin zero wave function \(\psi(q,p)\) which is expressed by Eq. (2.20).

By construction \(\psi^{(s)}(q,p;\alpha)\) is a solution of the equations (1.1), (1.2), (1.7) and (1.8) characterized by \([m,s]\). \(\mathcal{H}_{\tilde{\eta}}^{(s)}\) carries the UIR of the Poincaré group denoted by \(U^{(s)}(b,\Lambda)\), and the generalized scalar one-particle phase space function \(\psi^{(s)}(q,p;\alpha)\) for a free particle (or antiparticle) of mass \(m\) and definite spin \(s\) transforms irreducibly under Poincaré transformations \(U^{(s)}(b,\Lambda)\) according to (compare (2.52)):
\[(U^{(s)}(b, \Lambda)\psi^{(s)})(q, p; \alpha) = \psi^{(s)}(\Lambda^{-1}(q - b), \Lambda^{-1}p; D(\Lambda^{-1})\alpha)\]  \hspace{1cm} (2.64)\]

with the invariant scalar product in \(\mathcal{H}^{(s)}_{\tilde{\eta}}\) given by (compare (2.44))

\[\langle \psi^{(s)}_1 | \psi^{(s)}_2 \rangle_{\Sigma_m \times \mathbb{C}_2} = \int_{\Sigma_m \times \mathbb{C}_2} \left[ \psi^{(s)}_1(q, p; \alpha) \right]^* \psi^{(s)}_2(q, p; \alpha) \delta(\frac{1}{mc}p_{\Lambda\hat{A}}\alpha^\dagger \hat{A}^\dagger - r) d\Sigma_m(q, p) d\alpha d\bar{\alpha}.\]  \hspace{1cm} (2.65)\]

Decomposing \(\psi^{(s)}_1(q, r; \alpha)\) and \(\psi^{(s)}_2(q, r; \alpha)\) into homogeneous polynomials in \(\alpha^1\) and \(\alpha^2\) as in Eq. (2.36) and carrying out the \(\alpha\)-integrations yields [compare (2.48)]

\[\langle \psi^{(s)}_1 | \psi^{(s)}_2 \rangle_{\Sigma_m \times \mathbb{C}_2} = N_s \int_{\Sigma_m} \sum_{s_3} [\psi^{(s)}_{1,s_3}(q, p)]^* \psi^{(s)}_{2,s_3}(q, p) d\Sigma_m(q, p)\]  \hspace{1cm} (2.66)\]

with \(N_s = 8\pi^2 r^{2s+1}/(2s + 1)\) and \(r = 2s\) according to (2.33).

To conclude this section we define a system of covariance of the Poincaré group (a generalized system of imprimitivity) for free particles of definite mass \(m\) and arbitrary integer or half integer spin \(s\) described by the wave function \(\psi^{(s)}(q, p; \alpha)\), defined in (2.58), transforming under the irreducible unitary phase space representation \(U^{(s)}(b, \Lambda)\) of \(\mathcal{P}\) [or rather \(\bar{\mathcal{P}}\) as far as the \(\alpha\)-variable is concerned; see Eq. (2.64)] realized on the Hilbert space \(\mathcal{H}^{(s)}_{\tilde{\eta}}\) constructed above.

Due to the independence on the spin variable \(s\) of Eqs. (2.60), (2.61) as well as the resolution of the identity obtained after the reduction to a definite integer or half integer spin [compare (2.14)], one has for any \(\Delta_j\) belonging to a family of Borel sets \(\mathcal{B}\) on \textit{relativistic stochastic phase space} \cite{16} the following operators: A positive operator-valued (POV) measure, \(E(\Delta_j)\), on \(\mathcal{H}^{(s)}_{\tilde{\eta}}\) together with a UIR of the Poincaré group, \(U^{(s)}(b, \Lambda) = U^{(s)}(g)\), on \(\mathcal{H}^{(s)}_{\tilde{\eta}}\) obeying

\[E(\Delta_j) = E^*(\Delta_j) \geq 0; \quad \text{with} \quad E(\emptyset) = 0,\]  \hspace{1cm} (2.67)\]

\[E\left(\bigcup_{j=1}^{\infty} \Delta_j\right) = \sum_{j=1}^{\infty} E(\Delta_j) \quad \text{for} \quad \Delta_i \cap \Delta_j = \emptyset; i \neq j,\]  \hspace{1cm} (2.68)\]

and

\[U^{(s)}(g)E(\Delta_j)U^{(s)}(g)^\dagger = E(g\Delta_j),\]  \hspace{1cm} (2.69)\]

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where, for brevity, we have denoted the element \((b, \Lambda)\) of \(\mathcal{P}\) by \(g\). The operator \(E(\Delta_j)\) in (2.67)-(2.69) is given by

\[
E(\Delta_j) = \int_{\Delta_j} |\phi_{q,p}\rangle d\Sigma_m(q,p)\langle\phi_{q,p}|
\]

where \(\Delta_j \subset \Sigma_m^\pm = \sigma \times V_m^\pm\) with \(E(\Sigma_m^\pm) = 1^\pm\).

The POV property (2.67) implies that for every normalized state \(\Psi(s) \in \mathcal{H}_{\tilde{\eta}}^{(s)}\) the expression

\[
P_\psi(\Delta_j) = \langle \Psi(s) | E(\Delta_j) \Psi(s) \rangle
= \int_{\Delta_j \times \mathbb{C}_2} |\psi^{(s)}(q,p;\alpha)|^2 \delta(\frac{1}{mc} p_{A\dot{A}} \alpha^A \dot{\alpha}^\dot{A} - r) d\Sigma_m(p,q) d\alpha d\dot{\alpha}
\]

computed according to (2.65) yields the probability of finding the (free) particle (or antiparticle) with mass \(m\) and spin \(s\) within the domain \(\Delta_j \in \mathcal{B}\) of stochastic phase space. (For a detailed discussion we refer to Ref. [16].) The last equality in (2.71) is obtained from (2.70) together with (2.63). The r.-h. side of (2.71) finally yields, remembering (2.66) for a normalized state,

\[
P_\psi(\Delta_j) = N_s \int_{\Delta_j} \sum_{s_3} |\psi_{s_3}^{(s)}(q,p)|^2 d\Sigma_m(q,p),
\]

with

\[
N_s = \left[ \int_{\Sigma_m^\pm} \sum_{s_3} |\psi_{s_3}^{(s)}(q,p)|^2 d\Sigma_m(q,p) \right]^{-1}.
\]
s in terms of scalar wave functions, \( \psi^{(s)}(q, p; \alpha) \), realized on a one-particle Hilbert space \( \mathcal{H}^{(s)}_{\tilde{\eta}} \) carrying a unitary irreducible phase space representation of the Poincaré group. From \( \psi^{(s)}(q, p; \alpha) \) the usual \((2s + 1)\)-dimensional vector representation of the spin degrees of freedom may easily be recovered. However, for many investigations it is simpler to use the scalar functions \( \psi^{(s)}(q, p; \alpha) \) directly together with their Poincaré transformation rule (2.64) and their invariant scalar product (2.65) characterizing the irreducible resolution kernel Hilbert space \( \mathcal{H}^{(s)}_{\tilde{\eta}} \).

In this section we would like to describe free quantum particles of mass \( m \) and arbitrary but definite physical integer or half integer spin \( s \) in the presence of gravitation. We aim at a formulation in terms of generalized wave functions defined on a first quantized (i.e. one-particle or one-antiparticle) Hilbert bundel, \( \mathcal{H}_{[m, s]} \), raised over a curved Riemann-Cartan space-time \( U_4 \) possessing a pseudo-Riemannian metric and a metric compatible torsion. The standard fiber of \( \mathcal{H}_{[m, s]} \) is the one-particle Hilbert space \( \mathcal{H}^{(s)}_{\tilde{\eta}} \) constructed above carrying an irreducible phase space representation, \( U^{(s)}(b, \Lambda) \), of the Poincaré group characterized in the Wigner sense by \([m, s]\). The group action on \( \mathcal{H}_{[m, s]} \) is given in terms of \( U^{(s)}(b, \Lambda) = U^{(s)}(g) \). The basic properties of the Hilbert space \( \mathcal{H}^{(s)}_{\tilde{\eta}} \) (for fixed physical \([m, s]\)) as resolution kernel Hilbert space with resolution generator \( \tilde{\eta} \) are determined by the Hilbert space \( \mathcal{H}_{\tilde{\eta}} = \mathcal{H}^{(s=0)}_{\tilde{\eta}} \). The spin description for free noninteracting particles of mass \( m \) and nonvanishing spin \( s \) in flat space adds only an “inessential complication” described, as mentioned, by \((2s + 1)\)-component fields \( \psi^{(s)}_{s_3}(q, p) \), or – more concisely and before performing a decomposition in terms of homogeneous polynomials in the variable \( \alpha \) – described by the scalar field \( \psi^{(s)}(q, p; \alpha) \) depending on the internal spinor variable \( \alpha = (\alpha^1, \alpha^2) \), obeying, because of (2.58) and (2.59),

\[
\hat{D}\psi^{(s)}(q, p; \alpha) = 2s\psi^{(s)}(q, p; \alpha).
\]

(3.1)

The use of the \( C_2 \)-variables \( \alpha, \bar{\alpha} \) for a general description of spin prior to the reduction to a particular value \( s = 0, 1, 2, 3, \ldots \) for free (asymptotic) physical particles, which would be considered in connection with a dynamical coupling of several general spin fields in a
theory incorporating a dynamical role of spin, will be investigated elsewhere. Here we want to concentrate on the formulation of the *kinematics of free quantum particles* with definite mass and arbitrary (but specified) physical spin, i.e. \([m, s]\) fixed, in terms of *reduced* fields transforming as phase space representations of the Poincaré group which is realized in the local fibers of a bundle over space-time in the presence of gravitational and possibly torsion fields, i.e. being given as sections on the Hilbert bundle \(\mathcal{H}_{[m, s]}\) over a Riemann-Cartan space-time \(U_4\) defined by

\[
\mathcal{H}_{[m, s]} = \mathcal{H}(U_4, \mathcal{F} = \mathcal{H}^{(s)}_{\eta}, U^{(s)}(g)).
\]

(3.2)

The bundle \(\mathcal{H}_{[m, s]}\) is associated to the Poincaré frame bundle over \(U_4\) with structural group \(G = ISO(3, 1) \equiv \mathcal{P}\), i.e.

\[
P = P(U_4, \mathcal{P}),
\]

(3.3)

or, rather, to the corresponding spin frame bundle \(\tilde{P} = \tilde{P}(U_4, \tilde{\mathcal{P}})\) with structural group \(\tilde{\mathcal{P}}\) as far as the transformation of the internal spinor variables \(\alpha^1, \alpha^2\) are concerned. We add in parenthesis that we shall assume a spin structure to exist on space-time, i.e. we shall assume that \(P\) is a trivial bundle possessing global sections so that the homomorphism between \(\tilde{\mathcal{P}}\) and \(\mathcal{P}\) carries over to a corresponding homomorphism between the bundles \(\tilde{P}\) and \(P\).

A further bundle associated to \(P\) is the one-particle phase space bundle for zero spin

\[
\tilde{E} = \tilde{E}_{s=0} = \tilde{E}(U_4, \tilde{F} = M^\pm_m, \mathcal{P})
\]

(3.4)

with structural group \(\mathcal{P}\) and standard fiber \(M^\pm_m = M_4 \times V^\pm_m\). \(\tilde{E}\) is a soldered bundle [26] with first order contact of the space-time base and the fiber over \(x\), \(M^\pm_m(x)\), for each \(x \in U_4\). The contact between base space and fiber is made in \(\tilde{E}\) through the local subspace \(M_4 \simeq T_x M^\pm_m(x)\). The affine tangent bundle \(T_A(U_4, F = M_4, \mathcal{P})\), with the Minkowski fiber viewed as an affine space with group of motion \(\mathcal{P}\), is in a natural manner submanifold of \(\tilde{E}\). Disregarding spin, an atlas on the bundle \(\tilde{E}\) provides the concrete kinematical localization and momentum variables \((x; q, p)\) on which the generalized wave functions, defined on \(\mathcal{H}_{[m, s]}\),
depend. Here \( x \in U_4 \) is a classical space-time variable and \((q, p) \in M^\pm_m\) are local stochastic phase space variables corresponding to mean position \( q \in T_x(U_4) \) and mean momentum \( p \in V^\pm_m \). (For a detailed account of the geometro-stochastic formalism and the basic fuzziness encoded in this description in the fiber variables \((q, p)\) and the corresponding resolution generator depending on a length parameter \( l \) we refer to Refs. \[13\], \[14\] and \[16\].)

We now want to extend the geometro-stochastic (g-s) description for quantized one-particle states on curved space-time to the bundle \( H_{[m,s]} \) for arbitrary physical spin by including the spinor variables \( \alpha^A \). These latter variables will, however, at first not play the role of stochastic variables for the description of spin. In this respect \( \alpha \) is different from the pair \((q, p)\) in the geometro-stochastic formalism. From the later discussion of the quantum propagation on \( H_{[m,s]} \) in the presence of gravitation, which is discussed in Sect. IV below, we shall however find that the spin polarization of states does finally also acquire a stochastic nature. We thus first generalize (3.4) to the classical phase space bundle \( \tilde{E}_s \) associated to \( \bar{P} \) for single particles of type \([m, s]\), i.e. define

\[
\tilde{E}_s = \tilde{E}_s(U_4, \tilde{F}_s = M^\pm_m \times S^2_{r=2s}, \bar{P}),
\]

where the \( SL(2, \mathbb{C}) \) part contained in \( \bar{P} \) acts on the two-sphere, \( S^2_{r=2s} \), as described in the introduction, and with \( \mathcal{P} \) (the homomorphic image of \( \bar{P} \)) acting on \( M^\pm_m \) in the usual way as in (3.4), i.e. for \( g = (b, \Lambda) \in \mathcal{P} \) : \( g(q, p) = (\Lambda q - b, \Lambda p) \). [Compare Eqs. (3.8) and (3.9) below.] The soldered bundle \( \tilde{E}_s \) provides the local variables \((q, p; \alpha)\) at the point \( x \) of the base on which the generalized geometro-stochastic wave functions for arbitrary physical mass and spin, which are to be defined on \( H_{[m,s]} \), will depend.

Let us now first choose a (global) gauge on \( P \) and denote it by \( \sigma_P(x) \). The corresponding coherent state base in the local fiber \( H_{q}^{(s)}(x) \) of \( H_{[m,s]} \) is, for any spin and for arbitrary \( x \in U_4 \), given by

\[
\tilde{\sigma}(x) : \Phi_{q,p}^{\sigma(x)} \rightarrow \phi_{q,p},
\]

where we have denoted the map in \( H_{[m,s]} \) corresponding to \( \sigma_P \) by the symbol \( \tilde{\sigma} \), i.e. with a bar in order to discriminate it from the space-like surfaces in \( M^\pm_m(x) \) which are denoted
by \( \sigma(x) \) [compare (2.5) and (2.21)]. The states denoted by \( \Phi^{\sigma(x)}_{q,p} \) provide a \textit{local coherent quantum frame basis} of \( \mathcal{H}^{(s)}_\eta(x) \) and yield a corresponding resolution of the identity in the fiber over \( x \in U_4 \) in \( \mathcal{H}_{[m,s]} \) independent of the value for \( s \):

\[
\int_{\Sigma^\pm_m(x)} |\Phi^{\sigma(x)}_{q,p}\rangle d\Sigma_m(q,p) \langle \Phi^{\sigma(x)}_{q,p}| = 1^\pm_x .
\]  

(3.7)

Here \( \Sigma^\pm_m(x) \) denotes a subspace of \( \mathcal{M}^\pm_m(x) \) given by the direct product of a space-like hypersurface \( \sigma(x) \) in \( T_x(U_4) \) and the hyperboloid \( V^\pm_m(x) \). It is easy to show that a change of section \( \sigma_P(x) \rightarrow \sigma'_P(x) \) on \( P \) corresponds to an \( x \)-dependent Poincaré transformation for sections on \( \mathcal{H}_{[m,s]} \) in the following manner (compare Ref. \[27\])

\[
\bar{\sigma}'(x) = U^{(s)}(g(x))\bar{\sigma}(x) = U^{(s)}(b(x), \Lambda(x))\bar{\sigma}(x)
\]  

(3.8)

with \( g(x) = (b(x), \Lambda(x)) \) acting on the local affine frame \( (a(x), e_j(x)) \) in the gauge \( \sigma_P(x) \) on \( P \), with \( a(x) = -a^k(x)e_k(x) \) denoting its origin, yielding the local affine frame \( (a'(x), e'_k(x)) \) in the gauge \( \sigma'_P(x) \) on \( P \) with \( a'(x) = -a'^k(x)e'_k(x) \) denoting the new origin. The relations between the primed and unprimed frames are given in components by

\[
a'^k(x) = [\Lambda(x)]^k_j a^j(x) + b^j(x); \quad e'_k(x) = e_j(x) [\Lambda^{-1}(x)]^j_k ,
\]  

(3.9)

where repeated local Lorentz indices are summed over 0, 1, 2, 3. Moreover, \( U^{(s)}(g(x)) \) leaves (3.7) invariant, where, in fact, only the \( s = 0 \) part is involved as mentioned.

A state of a particle of type \([m, s]\) in the Hilbert bundle description on \( \mathcal{H}_{[m,s]} \) is represented by a smooth section

\[
x \rightarrow \Psi^{(s)}_x(\alpha) \in \mathcal{H}^{(s)}_{\eta}(x)
\]  

(3.10)

involving a state vector \( \Psi^{(s)}_x(\alpha) \) to be defined in each local fiber of \( \mathcal{H}_{[m,s]} \) above the base point \( x \in U_4 \). In analogy to Eq. (2.62) the state vector \( \Psi^{(s)}_x(\alpha) \) may be decomposed with respect to the local quantum frame basis \( \Phi^{\sigma(x)}_{q,p} \) according to

\[
\Psi^{(s)}_x(\alpha) = \int_{\Sigma^\pm_m(x)} \psi^{(s)}_x(q, p; \alpha) \Phi^{\sigma(x)}_{q,p} d\Sigma_m(q, p)
\]  

(3.11)
where
\[
\psi^{(s)}_x(q, p; \alpha) = \langle \Phi^{(s)}_{q,p} | \Psi^{(s)}_x(\alpha) \rangle_{\Sigma^+_{m}(x)}
\] (3.12)
is the corresponding gauge dependent \textit{generalized one-particle geometro-stochastic (g-s) wave function} defined on \( \mathcal{H}_{[m,s]} \) which transforms under a change of section (3.8), i.e. under Poincaré gauge transformations, as
\[
\left[ \psi^{(s)}_x(q, p; \alpha) \right]' = \left( U^{(s)}(b(x), \Lambda(x)) \psi^{(s)}_x \right)(q, p; \alpha)
= \psi^{(s)}_x \left( \Lambda^{-1}(x)(q - b(x)), \Lambda^{-1}(x)p, D(\Lambda^{-1}(x))\alpha \right).
\] (3.13)

For ease of writing we have suppressed a label \( \bar{\sigma}(x) \) on \( \psi^{(s)}_x \) in the equations above.

There is again a Poincaré gauge invariant scalar product defined on \( \mathcal{H}_{[m,s]} \) constructed as in Eq. (2.63), however now written with the smooth \( x \)-dependent sections \( \psi^{(s)}_{1,x}(q, p; \alpha) \) and \( \psi^{(s)}_{2,x}(q, p; \alpha) \) defined on \( \mathcal{H}_{[m,s]} \) and involving invariant integration over \( \Sigma^\pm_{m} \times C_2 \) at the point \( x \) of the space-time base.

The internal spinor variable \( \alpha \) is now a local \( SL(2,\mathbb{C}) \) gauge variable comparable to the local kinematic Poincaré variables (i.e. the stochastic variables) \( q \) and \( p \). However, the reducing property of the generalized wave functions \( \psi^{(s)}_x(q, p; \alpha) \) for a free particle of type \([m, s]\) is independent upon \( x \), i.e. Eq. (3.1) is valid for \( \psi^{(s)}_x(q, p; \alpha) \) at any point \( x \) on the base of \( \mathcal{H}_{[m,s]} \) with the same relation (3.1), being satisfied also by the state vector \( \Psi^{(s)}_x(\alpha) \).

The covariant derivative of the generalized scalar g-s wave function \( \psi^{(s)}_x(q, p; \alpha) \) is given by
\[
D\psi^{(s)}_x(q, p; \alpha) = [d + i\Gamma(x)] \psi^{(s)}_x(q, p; \alpha)
\] (3.14)
with \( d = \theta^k \partial_k \), where \( \theta^k; k = 0, 1, 2, 3 \) is a base of the cotangent space \( T^*_x(U_4) \) at \( x \in U_4 \) and, correspondingly, \( D = \theta^k D_k \). Furthermore, we denote by \( (\tilde{\theta}^k(x), \tilde{\omega}_{ij}(x)) \) a connection on \( P \) pulled back to the base under the map \( \sigma_P \), where
\[
\tilde{\theta}^k(x) = \theta^k + \nabla a^k(x)
\] (3.15)
are the soldering forms defining the translational part of the connection on the Poincaré bundle (3.3) with \( \nabla a^k(x) \) denoting the covariant derivative of the \( k \)-th component of the translational part of the affine frame field \((a(x), e_j(x))\) taken with respect to the Lorentz part of the connection on (3.3) given by

\[
\tilde{\omega}_{ij}(x) = -\tilde{\omega}_{ji}(x) = \theta^k \tilde{\Gamma}_{kij}(x)
\]  

(3.16)

with coefficients \( \tilde{\Gamma}_{kij}(x) \). In (3.14) \( \Gamma(x) \) may now be defined as (compare Refs. [26] and [28])

\[
\Gamma(x) = -\tilde{\theta}^k(x) \tilde{P}_k + \frac{1}{2} \tilde{\omega}_{ij}(x) M^{ij} + \frac{1}{2} \tilde{\omega}_{ij}(x) \tilde{S}^{ij}.
\]  

(3.17)

Here \( \tilde{P}_k, \tilde{M}_{ij} \) (with the indices \( i, j \) lowered using the Minkowski metric \( \eta_{ik} \)) are the generators of the phase space representation \( U^{(s=0)}(b, \Lambda) \) constructed in terms of differential operators in the variables \( q_k \) and \( p_k \), and \( \tilde{S}_{ij} \) are the corresponding operators of the spin dependent part of the representation \( U^{(s)}(b, \Lambda) \), for \( s \neq 0 \), given as differential operators in the spin space variables \( \alpha^A, \bar{\alpha}^{\dot{A}} \) for definite \( s \) which are related to the generators of the \( SL(2, \mathcal{C}) \) transformations \( D(\Lambda) \) and \( \bar{D}(\Lambda) \) in Eqs. (2.24) and (2.25) (compare [9]).

IV. QUANTUM PROPAGATION ON \( \mathcal{H}_{[m,s]} \)

We are interested in the geometro-stochastic propagation, called quantum propagation, on the bundle \( \mathcal{H}_{[m,s]} \) of a generalized reduced wave function (section) \( \psi^{(s)}_x(q,p;\alpha) \), defined in (3.12) and (3.13), describing a single particle (or antiparticle) of definite physical mass and spin \([m,s]\). The phase space probability amplitude associated with the section \( \psi^{(s)}_x(q,p;\alpha) \) on \( \mathcal{H}_{[m,s]} \) is given by (compare the discussion presented in [27], [29] and [30]):

\[
\psi^{(s)}(x,p;\alpha) = \psi^{(s)}_x(q = -a(x),p;\alpha) .
\]  

(4.1)

Here \( q = -a(x) \) denotes the point of contact of \( T_x(B) \subset \mathcal{M}_m^\pm(x) \) with the space-time base \( B = U_4 \) on the bundle \( \tilde{E}_a \) in any Poincaré gauge on \( P \). This point will be identified with the point \( x \) of the base. Furthermore, \( p \in V_m^\pm(x) \) and \( \alpha \in S_{r=2a}^2(x) \) in (3.14) [compare Eq. (3.3)].
The particle is described quantum mechanically and is considered to be free except for influences of gravity described through the curvature of the base of \( \mathcal{H}_{[m,s]} \) which is treated as an external field. No back reaction of the quantum particle onto the underlying geometry is thus considered. Clearly, the propagation on \( \mathcal{H}_{[m,s]} \) conserves the mass and spin value; hence the quantum propagator for \( \psi_x^{(s)}(q,p;\alpha) \) and the associated probability amplitude (4.1) has to commute with \( \hat{D} \) and with the Casimir operators \( \hat{P}_\mu \hat{P}^\mu \) and \( \hat{W}_\mu \hat{W}^\mu \). Moreover, we found in Sect. III B above that the stochastic phase space propagator \( K_{\eta}^{(s)}(q',p';q,p) \) describing the propagation of a particle of spin \( s \) in the local fibers of \( \mathcal{H}_{[m,s]} \) is independent of \( s \).

On \( \mathcal{H}_{[m,s]} \) the generalized one-particle wave function \( \psi_x^{(s)}(q,p;\alpha) \) should be a solution of a second order wave equation,

\[
\left( \Box_{\mathcal{H}_{[m,s]}} + \beta \right) \psi_x^{(s)}(q,p;\alpha) = 0 ,
\]

where the invariant second order differential operator is

\[
\Box_{\mathcal{H}_{[m,s]}} = g^{\mu\nu} \bar{D}_\mu D_\nu = \frac{1}{\sqrt{-g}} D_\nu \sqrt{-g} g^{\mu\nu} D_\nu - g^{\mu\nu} K_{\mu\nu}^{\rho} D_\rho ,
\]

with \( D_\nu \) as defined in (3.14), using \( D_\nu = \lambda^k_{\nu}(x)D_k \), where the \( \lambda^k_{\nu}(x) \) are the vierbein fields, and with \( g_{\mu\nu}(x) = \lambda^k_{\mu}(x)\lambda^k_{\nu}(x)\eta_{kk} \) being the covariant metric tensor in the base of \( \mathcal{H}_{[m,s]} \), and correspondingly for the contravariant metric tensor \( g^{\mu\nu} \). [Tensor components referring to a natural basis, \( \partial_\mu; \mu = 0, 1, 2, 3 \), are labeled with Greek indices.] \( \bar{D}_\mu \) in (4.3) is the Poincaré gauge covariant and \( U_4 \)-covariant derivative, and \( K_{\mu\nu}^{\rho} \) denotes the torsion tensor. [For axial vector torsion, considered below, the last term on the r.-h. side of Eq. (4.3) is absent due to the antisymmetry of the \( K_{\mu\nu}^{\rho} \) in this case.] In Eq. (4.2) \( \beta \) is an invariant of dimension \( L^{-2} \) (\( L = \) length), depending on \( m \) and possibly on \( s \), which characterizes the wave motion on \( \mathcal{H}_{[m,s]} \). To what extent \( \beta \) contains a \( U_4 \)– (or, in the absence of torsion a \( V_4 \)-) curvature invariant characterizing the geometry of the base, as discussed in conformally invariant theories [31], will not be made explicite here; compare, however, in this context the work of Buchdahl for higher spin fields in Riemannian spaces [32] and the remarks made in Sect. V below.
We are aiming at a path integral-like solution of (4.2), valid for arbitrary integer or half integer spin, which is constructed in analogy to Feynman’s path integral representation of a nonrelativistic wave function satisfying the Schrödinger equation [33].

In [27] a careful study was undertaken to show that a formula conjectured by Prugovecki (compare Ref. [29] as well as [30]) for the quantum propagation on $\mathcal{H}_{[m,s]}$ for spinless particles is indeed Poincaré gauge covariant (i.e. is Poincaré gauge invariant except for endpoint transformations), it is curvature and hence path dependent (i.e. is sensitive to the metric curvature of the base), and yields the correct special relativistic expression in the flat space limit. In this path integral-like formula for the propagation on the Hilbert bundle one considers a particular foliation of the space-time base into space-like hypersurfaces $\sigma(\tau)$ with evolution parameter $\tau$ and regards the surfaces $\sigma(\tau)$ through the point $x_0 \in B$ for $\tau = \tau_0$ and $x = x_N \in B$ for $\tau = \tau_N$ after $N$ iterations, $n = 1...N$. The geometro-stochastic propagator for the probability amplitude of a spinless massive particle is now defined by considering all polygonal paths between $x_0$ and $x$ composed of free-fall segments, i.e. constructed with geodesic arcs of the underlying metric between points on two adjacent foiles $\sigma(\tau_{n-1})$ and $\sigma(\tau_n)$ of the foliation. One considers thus parallel transport on $\mathcal{H}_{[m,s]}$ between adjacent points $x_{n-1} \in \sigma(\tau_{n-1})$ and $x_n \in \sigma(\tau_n)$ using different starting conditions regarding the stochastic momentum variable in each step. The computation – assumed to apply to small space-time distances – is unrestricted by classical causality arguments, and integration in a Poincaré gauge invariant manner over the full intermediate space-like surfaces of the foliation is carried out like in relativistic Feynman path integral formulations in Minkowski space [34][35], i.e. without restricting the construction to the propagation along broken paths composed of time-like segments only. The quantum propagator for the amplitude, finally, results in the limit $N \to \infty$, i.e. by making the geometro-stochastic averaging involving broken polygonal paths finer and finer.

Before we continue our construction of a g-s propagator in the presence of gravitation, let us inject here some brief remarks concerning the so-called Einstein causality, observed to hold for macroscopic distances in space-time, arising in the present context as the re-
sult of the superposition and destructive interference of probability amplitudes originating from classically forbidden space-time regions. The property of stochastic microcausality in the framework of the stochastic phase space formulation of quantum mechanics has been investigated in detail by Greenwood and Prugovečki [36] using the concept of “asymptotic causality” [37], i.e. the causal features arising in the limit $\tau \to \infty$. Let us, however, first mention that the stochastic phase space propagator $K_{\tilde{\eta}} = K_{\tilde{\eta}}(q', p'; q, p)$ in flat space – or, more exactly, in $\mathcal{M}_m^\pm = M_4 \times V_m^\pm$ – which was defined in Eq. (2.13), is, for small stochastic smearing characterized by the fundamental length parameter $l$ [see Sect. II A], indeed “close” to the Feynman propagator $i\Delta_F(q' - q)$ which is known to be nonzero for spacelike separation of the points $q'$ and $q$ (compare the discussion presented in [36]). For finite (small) nonzero $l$ the stochastic phase space description using generalized wave functions is formulated in terms of spread out quantum events (at the scale of $l$) and, correspondingly, the propagation of wave functions describing such events is only “stochastically causal” and not deterministically causal in the strict sense as in the yes-no manner realized in classical relativistic physics with strictly zero influences on points outside the future light cone of an idealized pointlike event localized at $q$. In the stochastic setting used here one has the result, obtained first for the flat space case in [36], that the probability for a particle of propagating outside the future light cone of a certain point tends to zero with $\tau \to \infty$. Thus no events violating Einstein causality do occur in the infinite future in this stochastic formalism. This property has been called asymptotic stochastic microcausality. Hence also in the presence of gravitation, i.e. for a curved base $B$, the causal features of quantum propagation will be stochastic in nature with Einstein causality being approached for infinitely separated (stochastic) events.

Continuing now our construction of a quantum propagator on $\mathcal{H}_{[m, s]}$ by means of parallel transport along broken paths composed of geodesic segments, we may also consider that the quantum particle would be measured by a certain localization device with a given resolution in between the initial and final points $x_0$ and $x$, respectively, and with their stochastic localization given at these and at the intermediate points in terms of the respective fiber
variables. The class of possible broken paths composed of geodesic arcs would then have to be narrowed to a certain corridor in the sense of Mensky [38]. We shall, however, not discuss problems of this kind in the present paper and sum over all intermediate broken trajectories. But even if the quantum particle is not followed by continuous measurement with a certain resolution it is assumed that it keeps its identity with respect to its mass and spin value. Hence one has to postulate, as mentioned above, that the quantum propagator, which is path dependent for a curved base, does commute with the Casimir operators defined in (1.1) and (1.2) and with the spin operator \( \hat{D} \). While g-s propagation is required to conserve the spin value \( s \) it will, however, affect the spin projection \( s_3 \) i.e. the polarization of the state considered (see below).

In the works cited above the geodesic arcs and path dependences were computed using the Levi-Civita connection embedded into the Poincaré framework used here by putting the affine vector field \( a^k(x) \) in (3.15) equal to zero and considering the pull back of a connection on \( P \) given by the one-forms \( (\theta^k, \tilde{\omega}_{ij}(x)) \). However, now \( \tilde{\omega}_{ij}(x) \) may contain torsion effects for the base being a Riemann-Cartan space-time \( U_4 \), i.e. \( \tilde{\omega}_{ij}(x) = \bar{\omega}_{ij}(x) + \tau_{ij}(x) \), where \( \bar{\omega}_{ij}(x) \) is the purely metric part and \( \tau_{ij}(x) \) is the torsion addition with \( \tau_{ij}(x) = \theta^k K_{kij}(x) \). For axial vector torsion [ i.e. for a completely antisymmetric torsion tensor field \( K_{kij}(x) \) ] no effects on the geodesics would be possible. Since the role of torsion in this whole context is not yet clear and since no source equations for the determination of torsion – supposed to be induced in the underlying geometry by a feedback mechanism involving the quantum fields – has been discussed in this paper [39], we shall assume that torsion is not affecting the geodesics entering the definition of the quantum propagator, i.e. we shall use \( (\theta^k, \bar{\omega}_{ij}(x)) \) as connection one-forms in a particular gauge on \( P \) and regard the background metric \( g_{\mu\nu}(x) \) as determined by the solution of Einstein’s equations with given classical sources [41]. The quantum propagator for the \( \psi \)-field is then the “free” g-s propagator for a quantized test particle field under the influence of a classical background metric and possibly also in the presence of an external axial vector torsion field. Our aim here is to extend this kinematic description to quantum particles of arbitrary integer or half integer spin \( s \) by using the
internal spin variables investigated in the previous sections.

To this end we first quote the result for $s = 0$ using (with a slight change) the notation of Ref. [27] for the operator $K^\phi(x', q', p'; x, q, p)$ of quantum propagation on $H_{[m,0]}$ (compare also Refs. [29] and [30]):

$$K^\phi(x', q', p'; x, q, p) = \lim_{N \to \infty} \int K^\phi(x', x_{N-1} ; x_{N-1}, \hat{q}_{N-1} , \hat{p}_{N-1}) \prod_{n=N-1} \frac{1}{K^\phi(x_n, x_{n-1})} (x_n, \hat{q}_n, \hat{p}_n; x_{n-1}, \hat{q}_{n-1}, \hat{p}_{n-1}) \ d\Sigma_m(x_n, \hat{p}_n) . \quad (4.4)$$

Here we have replaced the complex variable $\zeta$ of [27] by the pair $(q, p)$ and denoted the gauge by $\bar{\sigma}$ instead of $s$ in order not to confuse it with the spin variable. $K^\phi(x_n, x_{n-1})$ represents the parallel transport operator (in the gauge $\bar{\sigma}$) for parallel transport from the point $x_{n-1}$ to the point $x_n$ along the geodesic arc $\gamma(x_n, x_{n-1})$ in the base and $d\Sigma_m(x, \hat{p})$ is the “contact point phase space measure” (compare [27]) given by the measure defined in the local fiber of the bundle $\tilde{E}$, introduced in (3.4), restricted to the point of contact of base space and fiber, i.e. evaluated for $q(x) = -a(x) \in T_x(B)$, which is identified with the point $x$ of the space-time base $B$. This procedure allows a Poincaré gauge invariant measure to be associated with the leaves of a foliation of the space-time base. Correspondingly, the intermediate fiber variables $(\hat{q}_n, \hat{p}_n)$, for $n = 1, \ldots (N - 1)$, are given by

$$\hat{q}_n = -a(x_n) \quad \text{identified with } x_n \in \sigma(\tau_n) \subset B ,$$
$$\hat{p}_n = p(x_n) \quad \in V_m^\pm(x_n) . \quad (4.5)$$

In (1.4), moreover, $x_0 = x \in \sigma(\tau_0)$ with $(\hat{q}_0, \hat{p}_0) = (q, p) \in M_m^\pm(x_0)$, and $x_N = x' \in \sigma(\tau_N)$ with $(\hat{q}_N, \hat{p}_N) = (q', p') \in M_m^\pm(x')$.

It was shown in [27] that with this interpretation of the measure $d\Sigma_m(x_n, \hat{p}_n) = d\Sigma_m(-a(x_n), p(x_n))$ and integration over the intermediate variables $(x_n, p(x_n)) \in \sigma(\tau_n) \times V_m^\pm(x_n)$ for $n = 1, \ldots (N - 1)$ the definition (1.4) of a spin zero quantum propagator $K^\phi(x', q', p'; x, q, p)$ is, indeed, Poincaré gauge covariant (i.e. is Poincaré gauge invariant except for transformations at the endpoints $x$ and $x'$ of the paths) and has the correct flat
space limit, where in a global Lorentz gauge existing in that limit one can identify \( K^\bar{\sigma} \) (being path-independent in the flat space case) with the stochastic phase space propagator \( K_\bar{q} \) defined in (2.13). As a short-hand notation we shall denote the domain of integration for \( \tau = \tau_n \), i.e. the hypersurface \( \sigma(\tau_n) \times V^+_m(x_n) \), by \( \Sigma^+_m(x_n) \).

It is now straightforward to generalize the expression (1.4) for spin zero to arbitrary physical values of spin by introducing the internal spin variables \( \alpha = \alpha(x) \) characterizing – together with the pair \((q,p)\) – a point in the local fiber at \( x \in B \) in the bundle \( \tilde{E}_s \) defined in (3.5) and restrict for the associated probability amplitude (1.1) in the description on \( \mathcal{H}_{[m,s]} \) the \( q \)-value in the local tangent space \( T_x(B) \subset \mathcal{M}^\pm_m(x) \) to the point of contact, given in an arbitrary Poincaré gauge \( \sigma_P \) on \( P \), by \( q = -a(x) \), and identify, as mentioned, this point with the base point \( x \).

However, before we discuss the generalization of Eq. (1.4) let us remark that, since the parallel transport on \( \mathcal{H}_{[m,s]} \) is path dependent for a curved space-time base affecting for \( s \neq 0 \) also the spin variable \( \alpha \) [compare Eqs. (3.14) and (3.17)], one would expect that starting with a generalized decomposed wave function \( \psi^{(s)}_{s_3,x}(q,p) \), obtained from \( \psi^{(s)}_x(q,p;\alpha) \) in analogy to (2.36), having a sharp spin projection value \( s_3 \) at a certain point \( x \in \sigma(\tau_0) \subset B \), there will appear – as a result of the g-s propagation involving different intermediate paths – a spread in the \( s_3 \)-distribution of the spin projection value at the end point of the paths. Hence, as the result of the quantum propagation in the presence of gravitation, i.e. for a metrically curved base, the spin projection \( s_3 \) of a certain state will become unsharp and develop a distribution of values corresponding to a mixed state with unsharp (stochastic) spin polarization. Such an effect of gravity on polarized spin states should in principle be measurable at particle accelerators provided it can be disentangled from electromagnetic effects.

We now define the probability amplitude (1.1) for definite physical mass and spin at \( x' \in B \) [corresponding to \( q(x') = -a(x') \)] which results from the quantum propagation on \( \mathcal{H}_{[m,s]} \) from the amplitude prepared for \( \tau = \tau_0 \) on the hypersurface \( \sigma(\tau_0) \subset B \) [with \( x \in \sigma(\tau_0) \) corresponding to \( q(x) = -a(x) \)] by
\[
\psi_{K_{x'}}^{(s)}(x', p'; \alpha') = \int_{\tilde{\Sigma}_m(x) \times C_2(x)} K^{\sigma, (s)}(x', q', p'; \alpha'; x, q, p, \alpha) \psi^{(s)}(x, p; \alpha) \times \delta\left(\frac{1}{mc}p_{\dot{A}\dot{\alpha}}\alpha^{\dot{A}\dot{\alpha}} - r\right)d\Sigma_m(x, p)d\alpha \dot{\alpha}, \tag{4.6}
\]

where [compare (4.5)] \( p' = p(x') \in V_m^\pm(x') \) and \( p = p(x) \in V_m^\pm(x); \) \( q' = q' = -a(x') \) [identified with \( x' \in B \)] and \( a = q = -a(x) \) [identified with \( x \in B \)]; \( \alpha' = \alpha(x') \in C_2(x') \) and \( \alpha = \alpha(x) \in C_2(x). \) The \( \delta \)-function in (4.6) guarantees, as in Eq. (2.65), that the integrations over the internal spin spaces, i.e. here the \( C_2 \)-fibers at \( x \) for all \( x \in \sigma(\tau_0) \), are restricted to the sphere \( S_{\tau = 2s}^2 \) with radius \( r = 2s \) corresponding to the spin value \( s \) of the reduced probability amplitude \( \psi^{(s)}(x, p, \alpha) \) at \( x \). We denote the amplitude for spin \( s \) of a state prepared at \( x \) on the hypersurface \( \sigma(\tau_0) \) and propagated to the point \( x' \) on the hypersurface \( \sigma(\tau') \), for \( \tau' = \tau N \), by \( \psi_{K_{x'}}^{(s)}(x', p'; \alpha'). \) Since we intend to construct a solution of Eq. (4.2) we may later drop the suffix \( K_{x'x} \). Finally, \( K^{\sigma, (s)}(x', q', p', \alpha'; x, q, p, \alpha) \) in (4.6) is the quantum propagator for the probability amplitude in the presence of spin given by the following expression:

\[
K^{\sigma, (s)}(x', q', p', \alpha'; x, q, p, \alpha) = \lim_{N \to \infty} \int K^{\sigma, (s)}_{\tilde{\Sigma}_m(x, x_{N-1})}(x', q', p', \alpha'; x_{N-1}, \dot{q}_{N-1}, \dot{p}_{N-1}, \alpha_{N-1}) \times \prod_{n=N-1}^{1} K^{\sigma, (s)}_{\tilde{\Sigma}_m(x_n, x_{n-1})}(x_n, \dot{q}_n, \dot{p}_n, \alpha_n; x_{n-1}, \dot{q}_{n-1}, \dot{p}_{n-1}, \alpha_{n-1}) \times \delta\left(\frac{1}{mc}[\dot{p}_n]_{\dot{A}'\dot{\alpha}_n}^{\dot{A}\dot{\alpha}} - r\right)d\Sigma_m(x_n, \dot{p}_n)d\alpha nd\dot{\alpha}_n. \tag{4.7}
\]

Eq. (4.7) is analogous to (4.4) and the same notation is used for the variables \( (q, p) \) as given in (4.5). Moreover, \( \alpha = \alpha_0 = \alpha(x_0) \in C_2(x_0) \) and \( \alpha' = \alpha(x') \in C_2(x') \) and analogously for the intermediate internal spin variables \( \alpha_n = \alpha(x_n) \in C_2(x_n); \) \( n = 1 \ldots (N-1) \), and their complex conjugates. The intermediate integrations in (4.7) run over \( \tilde{\Sigma}_m^{\pm}(x_n) \times C_2(x_n); \) \( n = 1 \ldots (N-1) \), i.e. involve, due to the \( \delta \)-functions, the hypersurfaces \( \sigma(\tau_n) \times V_m^\pm(x_n) \times S_{r = 2s}^2(x_n). \)

Clearly, \( K^{\sigma, (s)} \) is \( s \)-dependent and is defined in a Poincaré gauge invariant manner except for endpoint transformations of the variables \( [q(x) = -a(x), p(x), \alpha(x)] \) at the endpoints \( x \) and \( x' \) of the paths composed of geodesic arcs in the base. Here the point \( q(x) = -a(x) \), which is identified with the point \( x \) in the base, and analogously the point \( q(x') = -a(x') \), identified with \( x' \), remain unaffected by the gauge transformations. The arguments proving
the Poincaré gauge covariance of the expression for \( K_{\bar{\sigma},(s)} \) are the same as those presented in [27] except for the additional internal spin variables \( \alpha \) appearing in (4.7) together with their Poincaré gauge invariant integrations with the measure \( da d\bar{\alpha} \) constrained by the \( \delta \)-functions.

The propagator \( K_{\gamma(x_n,x_{n-1})} \) in Eq. (4.7), finally, is the free fall propagator on \( H_{[m,s]} \) for the motion of a particle along the geodesic arc \( \gamma(x_n,x_{n-1}) \) from \( x_{n-1} \) to \( x_n \), which is obtained as the solution of the differential equation \( D\psi^{(s)}(q,p,\alpha) = 0 \) [compare (3.14) and (3.17)] for parallel transport in \( H_{[m,s]} \) along \( \gamma(x_n,x_{n-1}) \) determining thus the propagator on \( H_{[m,s]} \) for the infinitesimal step from \( x_{n-1} \in \sigma(\tau_{n-1}) \) to \( x_n \in \sigma(\tau_n) \) of the motion along the geodesic arc \( \gamma(x_n,x_{n-1}) \) in the base.

The path integral expression for \( \psi^{(s)}_{K_{x',x}(x',p';\alpha)} \) constructed with “free fall segments”, i.e. with parallel shift along geodesic arcs of the underlying metric, as defined by Eqs. (4.6) and (4.7), obeying \( D_\mu \psi^{(s)} = 0 \) for any segment, is a solution of the second order wave equation (4.2) provided the \( \beta \)-term in (4.2) is zero by itself. This requirement has the consequence that the mass and spin dependent terms (if the latter is really there) must appear in such a way that they compensate the curvature terms which might also be present in the \( \beta \)-term. Hence a phenomenon which may be called the “Archimedes’ principle” must be at work setting mass and spin into correspondence with an invariant curvature expression balancing thus these two effects against one another: matter properties \([m,s]\), on the one side, and properties of the embedding geometry, on the other side. The role played by torsion in this context is still unclear and needs further study. However, it is apparent that torsion must play the main part in this balancing since it is torsion which is – ultimately – considered to be induced in the underlying geometry as the “footprint” of the quantum fields. In the present paper, however, we investigate only the kinematics of supposedly free (except for gravitation) spinning quantum particles and regard torsion as an external field. We thus cannot see this effect in detail without discussing field equations for torsion at the same time. Moreover, we should remember that torsion has been severely restricted considering only axial vector type (totally antisymmetric \( K_{\mu\nu\rho} \)).
V. DISCUSSION AND CONCLUSION

Following an idea proposed by Lurçat [3] several decades ago, we discussed in this paper the use of internal spin variables for a quantum mechanical description of particles with real positive mass $m$ and arbitrary integer or half-integer spin $s = 0, \frac{1}{2}, 1, \frac{3}{2}, 2, \ldots$ in terms of scalar functions. These generalized wave functions, $\psi(q, p; \alpha, \bar{\alpha})$, are defined over an extended phase space $\mathcal{M}_m^\pm \times \mathcal{C}_2$ in order to describe particles of mass $m$ but arbitrary unspecified spin, where $\mathcal{M}_m^\pm = M_4 \times V_m^\pm$ is the one-particle (+) or one-antiparticle (−) phase space with $q \in M_4$ and $p \in V_m^\pm$, $p^2 = m^2c^2$, $\pm = \text{sign}\, p_0$, and $\alpha$ denotes a point in the internal spin space being a homogenous space of the Lorentz group of the type $\tilde{S} = SL(2, \mathcal{C})/\tilde{H}$ characterized by the subgroup $\tilde{H}$ of $SL(2, \mathcal{C})$ as explained in the introduction. Following Bacry and Kihlberg [4] in choosing the lowest dimensional internal spin space possessing a measure and being capable of representing integer as well as half integer spins, we used a four-dimensional internal spin space parametrized in terms of spinor variables $\alpha \in \mathcal{C}_2$ with $\bar{\alpha}$ denoting the corresponding complex conjugate spinor (dotted spinor) varying in the complex conjugate spin space.

The one-particle wave function $\psi^{(s)}(q, p; \alpha)$ representing a particle ($p_0 > 0$) or antiparticle ($p_0 < 0$) of definite mass and fixed integer or half-integer spin, $[m, s]$, are then obtained by requiring that the Eqs. (1.1), (1.7) and (1.8) be satisfied yielding thereby – as far as the variables $(q, p)$ are concerned [playing the role of stochastic variables in this context] – an irreducible element of a resolution kernel Hilbert space with resolution generator $\tilde{\eta} = \tilde{\eta}_l$, and – as far as the spin variables $(\alpha, \bar{\alpha})$ are concerned – an irreducible element depending on $\alpha$ only (without dependence on $\bar{\alpha}$) with $\alpha$ varying on a two-sphere $S^2_{r-2s}$ implying, as a consequence of demanding the homogeneity condition (1.7), that $\psi^{(s)}(q, p; \alpha)$ is a homogenous polynomial of degree $2s$ in the undotted spinor variables $\alpha^A; A = 1, 2$ with no dependence on the dotted spinor variables $\bar{\alpha}^\dot{A}; \dot{A} = \dot{1}, \dot{2}$. The function $\psi^{(s)}(q, p; \alpha)$ may be decomposed with respect to a basis transforming under the representation $D^{(s,0)}$ of $SL(2, \mathcal{C})$ to yield the familiar $(2s + 1)$-dimensional vector representation of spin, $\psi^{(s)}_{s_3}(q, p); s_3 = -s \ldots + s.$

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leading thus, ultimately, to a stochastic phase space description for free particles (or antiparticles) of definite mass and physical spin, \([m, s]\), transforming irreducibly under the Poincaré group [compare Eqs. (2.42), (2.58) and (2.64)] and possessing stochastic localization properties as far as the variables \((q, p)\) are concerned. These functions are elements of the Hilbert space \(\mathcal{H}_{\eta}^{(s)} = L^2(\Sigma_{m}^{\pm}) \times K_s\) carrying an irreducible representation of the covering group of the Poincaré group, \(\bar{\mathcal{P}}\), characterized by \(m\) and \(s\). This one-particle (or one-antiparticle) stochastic phase space formulation for free particles of type \([m, s]\) in flat space was then generalized to a formulation on a Hilbert bundle \(\mathcal{H}_{[m,s]}\) with fiber \(\mathcal{H}_{\eta}^{(s)}\), being associated to the Poincaré spin frame bundle over a curved Riemann-Cartan space-time base possessing metric and torsion (the latter restricted to axial vector type) both treated as external fields. The aim was to derive a path integral-type expression for the geometro-stochastic propagation of fields for arbitrary physical mass and spin defined on the soldered Hilbert bundle \(\mathcal{H}_{[m,s]}\) constructed over a curved classical space-time base. Our result for the quantum propagation is given by Eqs. (4.6) and (4.7) containing besides the Poincaré gauge invariant integrations over the intermediate phase space variables \((x, p)\) with measure \(d\Sigma_{m}(x, p(x)) = d\Sigma_{m}(q(x) = -a(x), p(x))\) the Poincaré gauge invariant integrations over the intermediate internal spin variables with measure \(d\alpha d\bar{\alpha}\). The integrations over the spin variables are constrained by delta functions – coupling momentum and spin variables – restricting the integrations to a particular “spin shell”, \(S^2_{r=2s}\), for a particle with spin \(s\) in analogy to the momentum integrations restricted to the “mass shell”, \(V^\pm_m\), for a particle of mass \(m\). In this framework the stochastic localization properties as well as the spin properties are described by means of the local fibers of the bundle \(\mathcal{H}_{[m,s]}\).

It was pointed out in Sect. IV that, although in the beginning only \((q, p)\) were stochastic variables while the spin variables \((\alpha, \bar{\alpha})\) were not of this type, with \(s\) and \(s_3\) taking sharp values for a certain quantum state describing a particle of spin \(s\) and spin projection (polarization) \(s_3\), the quantum propagation of such states on a curved space-time background (i.e. in the presence of gravitation) leads, according to Eqs. (4.6) and (4.7), to a stochastic
nature also for the polarization of the states $\psi^{(s)}_{K_{x'}}$ at $x'$ when decomposed at that point, i.e. leads to a stochastic nature of the spin projection $s_3$. To investigate this result, let us use an analyzing (or detection) field at $x' \in \sigma(\tau') \subset B$ and denote it by $\psi^{(s)}_{D}(x', p'; \alpha')$ corresponding to a certain sharp $s_3$-value when decomposed (representing, say, a state filtered by a Stern-Gerlach magnet which is oriented in a certain way). Then the invariant matrix element measuring the overlap at $x'$ between the originally prepared sharp spin state on $\sigma(\tau)$ with, say, $s = s_3$, propagated to $x'$, and a sharp detection field with various settings of $s_3$ at $x'$ is given by

$$\langle \psi^{(s)}_{D}(x', p'; \alpha') \mid \psi^{(s)}_{K_{x'}}(x', p'; \alpha') \rangle \Sigma^{(s)}_{\tau} x \times C_{2}(x') \quad (5.1)$$

Sloppily stated the measuring procedure is the following: Produce a pure measuring or detection state at $x'$ on the hypersurface $\sigma(\tau')$ and let it interfere with the state propagated to $x'$ from all $x$ on the hypersurface $\sigma(\tau)$. The analyzing or detection field $\psi^{(s)}_{D}(x', p; \alpha')$ may then be varied with respect to the $s_3$-polarization involved and the $s_3$-spectrum of $\psi^{(s)}_{K_{x'}}(x', p'; \alpha')$ be measured in this way in order to determine what effect the quantum propagation on a curved base had on an originally pure spin state, i.e. how gravitation affected the propagation of the pure state prepared on the hypersurface $\sigma(\tau)$.

As was also discussed in the previous section, the quantum propagation on $H_{[m, s]}$ is not causal in a classical sense (Einstein causality) but is “stochastically causal” [36]. Furthermore, we remarked at the end of the section that the path integral representation of the probability amplitude associated with a section on $H_{[m, s]}$ (a generalized geometro-stochastic one-particle or one-antiparticle wave function) satisfies a certain invariant second order wave equation on $H_{[m, s]}$ with certain restrictions imposed on the curvature invariants appearing in the term denoted by $\beta$: The $\beta$-term in (4.2) had to vanish by itself compensating thus mass and possibly spin dependent terms against invariant curvature contributions. This we called “Archimedes’ principle” expecting that torsion plays the dominant role in it. Let us point out again that torsion was severely restricted in this context by allowing only axial vector torsion from the beginning.
We did not discuss coupled first order spinor equations for arbitrary spin which, historically, are known to develop inconsistencies for $s \geq \frac{3}{2}$ when the minimal electromagnetic coupling is introduced \cite{12} or when these equations are generalized from flat to curved space-time (possibly with torsion). To make these equations consistent usually various supplementary conditions have to be imposed, i.e. auxiliary fields must be introduced which render the resulting expressions very complicated and difficult to handle. Instead we give here an analytic description of spin in terms of internal variables for scalar functions based, as mentioned, on Lurçat’s idea that spin should be described in terms of variables defined on a homogeneous space of the underlying kinematic symmetry group i.e. the Poincaré group. In fact, also in curved space-time the Poincaré group may be considered, namely as gauge or structural group of a soldered bundle raised over space-time, acting there on the local phase space fiber variables (used there to describe the [stochastic] localization of quantized states) as well as on the internal spin variables. It is thus indeed possible to give a general formulation of one-particle states for arbitrary mass and spin in terms of scalar functions and project out the conventional $(2s + 1)$-component vector states whenever necessary. However, for the understanding of the quantum propagation of such fields in the presence of gravitation it may be preferable to use the original generalized scalar wave functions.

In concluding we remark that the stochastic phase space description for single free relativistic particles of arbitrary spin on a Hilbert bundle over curved space-time $B$, which we studied in this paper, may easily be generalized to the many-particle case by considering Fock bundles over space-time for particles of type $[m, s]$. The standard fiber of these bundles are tensor products of one-particle and one-antiparticle Hilbert spaces $\mathcal{H}_{\tilde{q}}^{(s)(+)}$ and $\mathcal{H}_{\tilde{q}}^{(s)(-)}$, respectively, where $\mathcal{H}_{\tilde{q}}^{(s)} = \mathcal{H}_{\tilde{q}}^{(s)(+)} \oplus \mathcal{H}_{\tilde{q}}^{(s)(-)}$ with $(\pm)$ denoting the sign of the energy. In order to be in accord with the Pauli principle one has to introduce Fock bundles $\mathcal{F}_{[m, s]}^{(s)}$ possessing a fiber which is a sum of symmetrized products for integer spin (bosonic case), i.e.

$$\mathcal{F}_{sym}^{(s)} = \left( \sum_{n=1}^{\infty} \otimes_{sym}^{n} \mathcal{H}_{\tilde{q}}^{(s)(+)} \right) \otimes \left( \sum_{n'=1}^{\infty} \otimes_{sym}^{n'} \mathcal{H}_{\tilde{q}}^{(s)(-)} \right)$$

for $s = 0, 1, 2 \ldots , \quad (5.2)$

and which is a sum of antisymmetrized products for half integer spin (fermionic case), i.e.
\[ \mathcal{F}_{anti}^{(s)} = \left( \sum_{n=1}^{\infty} \otimes_{anti} \mathcal{H}_{\tilde{\eta}}^{(s)(+)} \right) \otimes \left( \sum_{n'=1}^{\infty} \otimes_{anti} \mathcal{H}_{\tilde{\eta}}^{(s)(-)} \right) \text{ for } s = \frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \ldots \] (5.3)

The Fock bundle of type \([m, s]\) associated to \( \mathcal{P} \) [compare Eq. (3.3)] is thus

\[ \mathcal{F}_{[m, s]} = \mathcal{F}(B = U_4, \mathcal{F}_{sym/anti}^{(s)}, U^{(s)}(g)) \] (5.4)

with standard fiber (5.2) for \(2s\) being even, and with standard fiber (5.3) for \(2s\) being odd.

It is implied here that there exists a local vacuum state \(|O_x\rangle\), for every \(x \in B\), which is invariant under changes of sections on \(\mathcal{F}_{[m, s]}\) provided by Poincaré gauge transformations.

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spin space would actually be \( \mathcal{C}_2 \times \mathcal{C}_2 \), where \( \mathcal{C}_2 \) represents the complex conjugate space

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