String Network and 1/4 BPS States
in $\mathcal{N} = 4$ $SU(N)$ Supersymmetric Yang-Mills Theory

Teruhiko Kawano and Kazumi Okuyama

Department of Physics, University of Tokyo
Hongo, Tokyo 113-0033, Japan
kawano@hep-th.phys.s.u-tokyo.ac.jp
okuyama@hep-th.phys.s.u-tokyo.ac.jp

We construct the classical configurations of BPS states with 1/4 unbroken supersymmetries in four-dimensional $\mathcal{N} = 4$ $SU(N + 1)$ supersymmetric Yang-Mills theory, and discuss that these configurations correspond to string networks connecting $(N + 1)$ D3-branes in Type IIB string theory.

April, 1998
1. Introduction

The four-dimensional $\mathcal{N} = 4$ supersymmetric Yang-Mills theory (SYM) was conjectured to be invariant under $SL(2, \mathbb{Z})$ duality, which is the generalization of the Montonen-Olive duality $[1] [2]$. $SL(2, \mathbb{Z})$ duality predicts the existence of the dyon bound states with relatively prime magnetic and electric charges as BPS states with half unbroken supersymmetries in $\mathcal{N} = 4$ SYM. This prediction was checked for the two-monopole sector $[3]$. In contrast to the $1/2$ BPS states in the $\mathcal{N} = 4$ SYM, little is known about BPS states with $1/4$ unbroken supersymmetries, which we will call ‘Yon’ solutions. It sounds like dyon, and the letter ‘Y’ looks like a three-string junction which is a fundamental element of a string network. We will briefly explain the connection between the $1/4$ BPS state and the string junction soon below.

In string theory, the $d = 4$ $\mathcal{N} = 4$ SYM appears as the effective field theory on D3-branes. In the D-brane picture, $SL(2, \mathbb{Z})$ duality of the $d = 4 \mathcal{N} = 4$ SYM is that of the type IIB string theory. The $(p, q)$ dyon in the $\mathcal{N} = 4$ SYM corresponds to a $(p, q)$ string stretched between two D3-branes. In $[4]$, it was argued that the Yon solution in the $SU(3)$ SYM corresponds to a three-string junction $[5–8]$ which connects three D3-branes.

Recently, a classical Yon solution was constructed for the case of $SU(3)$ gauge group in $[9]$. In this paper, we generalize the result of $[9]$ to the gauge group $SU(N + 1)$. We construct the spherically symmetric solution of the $1/4$ BPS equation of the $d = 4 \mathcal{N} = 4 SU(N + 1)$ SYM. We discuss that these solutions can be interpreted in the string theory as the string networks with $N - 1$ junctions connecting $N + 1$ D3-branes.

2. $1/4$ BPS Configurations in $\mathcal{N} = 4$ $SU(N + 1)$ Supersymmetric Yang-Mills Theory

We begin with a summary on the strategy adopted by Hashimoto, Hata, and Sasakura $[4]$ to find $1/4$ BPS solution in $\mathcal{N} = 4 SU(3)$ supersymmetric Yang-Mills theory. But, here, we will extend the gauge group $SU(3)$ to $SU(N + 1)$.

In $\mathcal{N} = 4 SU(N + 1)$ super Yang-Mills theory, we have six real scalar fields, a gauge field, and a gaugino. To find our BPS solutions, we turn off the gaugino field and the four scalar fields. The two remaining real scalars $\tilde{X}, \tilde{Y}$ are expected to describe strings which
form a string network on the two-dimensional plane. In [10], the BPS condition for 1/4 unbroken supersymmetric solutions was given by

\[(D\tilde{X} D\tilde{Y}) = \left( \begin{array}{cc} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{array} \right) \left( \begin{array}{c} E \\ B \end{array} \right), \]

\[(D_0\tilde{X} D_0\tilde{Y}) = i[\tilde{X}, \tilde{Y}] \left( \begin{array}{c} -\sin \theta \\ \cos \theta \end{array} \right), \]

D \cdot E = i[\tilde{X}, D_0\tilde{X}] + i[\tilde{Y}, D_0\tilde{Y}]. \tag{2.1}\]

By the rotation on the two-dimensional plane \((\tilde{X}, \tilde{Y})\);

\[(\tilde{X} \tilde{Y}) = \left( \begin{array}{cc} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{array} \right) \left( \begin{array}{c} X \\ Y \end{array} \right), \tag{2.2}\]

we rewrite (2.1) into

\[(D\tilde{X} D\tilde{Y}) = \left( \begin{array}{c} E \\ B \end{array} \right), \]

\[(D_0\tilde{X} D_0\tilde{Y}) = i[X, Y] \left( \begin{array}{c} 0 \\ 1 \end{array} \right), \tag{2.3}\]

D \cdot E = i[X, D_0X] + i[Y, D_0Y].

Since we are interested in static solutions, we will drop the time-dependence of all the fields. Then, putting \(X = -A_0\), we can see that the two following equations remains to be solved:

\[DY = B, \tag{2.4}\]

\[D \cdot DX = -i [Y, [Y, X]]. \tag{2.5}\]

The first equation (2.4) is the well-known BPS condition for monopole solutions. As the monopole solution, we adopt spherically symmetric solutions given in [13], as the authors of [9] have done for \(SU(3)\) gauge group. The solution is given by

\[A(r) = (M(r) - T) \times \frac{r}{r^2}, \tag{2.6}\]

where \(T\) is the maximal \(SU(2)\) embedding in \(SU(N + 1)\) with \(T_3 = \text{diag}(1/2N, 1/2N - 1, \cdots, -1/2N + 1, -1/2N)\). On the \(z\)-axis, \(M\) and \(Y\) have the following forms:

\[M_+ = M_1 + iM_2 = \sum_{k=1}^{N} a_k(r)E_k, \tag{2.7}\]

\[Y = \frac{1}{2} \sum_{k=1}^{N} \phi_k(r)H_k, \]

\(^1\) Our condition here is more general than that in [10]. See appendix A.
which will be determined so as to satisfy the equation (2.4) on the z-axis

\[ r^2 Y' = \frac{1}{2} \left[ M_+, M_- \right] - T_3, \]
\[ M'_+ = \mp \left[ M_\pm, Y \right]. \]

(2.8)

Here \( H_k \) (\( k = 1, \cdots, N \)) is the generator in the Cartan subalgebra of \( SU(N+1) \). \( E_k \) (\( k = 1, \cdots, N+1 \)) is the generator which corresponds to a simple root. As a concrete representation, \((H_k)_{a,b} = (\delta_{a,k}\delta_{b,k} - \delta_{a,k+1}\delta_{b,k+1})\) and \((E_k)_{a,b} = (\delta_{a,k}\delta_{b,k+1}); a, b = 1, \cdots, N+1 \). Then, \([E_k, E_l^\dagger] = \delta_{k,l}H_k\) and \([H_k, E_l] = C_{l,k}E_l\), where \( C_{l,k} = 2\delta_{l,k} - \delta_{l,k+1} - \delta_{l,k-1} \) is the Cartan matrix of \( SU(N+1) \).

In particular, the functions \( a_k(r) \) and \( \phi_k(r) \) are expressed by \( N \) functions \( Q_k(r) \) (\( k = 1, \cdots, N \));

\[ a_k(r) = \frac{r}{Q_k} \left( k\tilde{k}Q_{k-1}Q_{k+1} \right)^{\frac{1}{2}}, \]
\[ \phi_k(r) = -\frac{d \ln Q_k}{dr} + \frac{k\tilde{k}}{r}, \]

(2.9)

with \( \tilde{k} = N+1 - k \). Here we assume \( Q_0 = Q_{N+1} = 1 \). These functions \( Q_k(r) \) satisfy the following differential equation:

\[ Q'_kQ'_k - Q_kQ''_k = k\tilde{k}Q_{k-1}Q_{k+1}, \]

(2.10)

for \( k = 1, 2, \cdots, N \). Here the prime denotes the differentiation with respect to the radial coordinate \( r \).

Although the general solution \( Q_k \) for the equation (2.10) are given in \[13\], it was difficult for the authors of this paper to solve the second equation (2.5) with the general solution \( Q_k \). Instead of it, we find a solution

\[ Q_k(r) = \left( \frac{\sinh cr}{c} \right)^{k\tilde{k}}, \]

(2.11)

which corresponds to the solution in \[13\] with a particular value of \( N \) parameters. We also notice that, for \( N+1 = 3 \), ours (2.11) is the same solution as that adopted in \[9\].

Therefore, inserting the solution (2.11) into (2.5) on the z-axis;

\[ (rX)'' - \frac{1}{2r^2} \left( \left[ M_+, [M_-, rX] \right] + \left[ M_-, [M_+, rX] \right] \right) = -i[Y, [Y, rX]], \]

(2.12)

and taking an ansatz for \( rX(r) \) as \( rX(r) = \sum_{k=1}^{N} f_kH_k \), we obtain

\[ f_k'' - \frac{c^2k\tilde{k}}{\sinh^2 rc} \left( 2f_k - f_{k+1} - f_{k-1} \right) = 0, \]

(2.13)
with \( f_0 = f_{N+1} = 0 \). In order to diagonalize this equation (2.13), we rewrite the functions \( f_k \) with \( N \) new functions \( \varphi_s \) \((s = 1, \cdots, N)\) by \( f_k = \sum_{s=1}^{N} v_{k}^{(s)} \varphi_s \), where \( v^{(s)} \) is the eigenvector of the matrix \( \tilde{C} = (\tilde{C}_{k,l}) := (k\bar{k}C_{k,l}) \) with the eigenvalue \( s(s+1) \). The generating function \( F^{(s)}(t) \) of \( v^{(s)} \) is given by

\[
F^{(s)}(t) := \sum_{k=1}^{N} v_{k}^{(s)} t^{k-1} = \sum_{n=0}^{N-s} \left( \frac{N+s+1}{N-s-n} \right) \left( \begin{array}{c} n+s \\ s \end{array} \right) (t-1)^{n+s-1},
\]

or the explicit form of \( v_{k}^{(s)} \) is

\[
v_{k}^{(s)} = \sum_{n=n_0}^{N-s} (-1)^{n+s-k} \left( \frac{N+s+1}{N-s-n} \right) \left( \begin{array}{c} n+s \\ s \end{array} \right) \left( \begin{array}{c} n+s-1 \\ k-1 \end{array} \right),
\]

where \( n_0 = \max\{0, k-s\} \). We can see that the equation (2.13) turns into

\[
\frac{d^2}{dx^2} \varphi_s(r) = \frac{1}{\sinh^2 x} s(s+1)\varphi_s(r)
\]

with \( x = rc \).

The equation (2.16) can be seen to be the Legendre differential equation, if we introduce a new variable \( \xi \) given by \( \xi = \coth x \) and rewrite (2.16) with \( \xi \).

Imposing the boundary condition on \( \varphi_s(r) \) by \( \varphi_s(r = 0) = \varphi'_s(r = 0) = 0 \), we can see that the solution \( \varphi_s(r) \) is given by \( \varphi_s(r) = c_s Q_s(\xi) \) with \( c_s \) any proportional constant, where \( Q_s(\xi) \) is Legendre function of the second kind \([14]\);

\[
Q_s(\xi) = \frac{1}{2s+1} \int_{-1}^{1} (1-t^2)^s(\xi-t)^{-s-1} dt.
\]

3. Discussion

We have constructed the 1/4 BPS configurations in \( \mathcal{N} = 4 \) SU\((N+1)\) SYM in the previous section. The configurations of two real scalars \( X, Y \) are

\[
X = \frac{1}{r} \sum_{s=1}^{N} \sum_{k=1}^{N} c_s Q_s(\coth cr) v_{k}^{(s)} H_k,
\]

\[
Y = -\frac{Q_1(\coth cr)}{r} T_3,
\]
Figure 1: The curved lines in the left(right) figure represent the configuration of Yon in $SU(4)(SU(5))$ gauge theory. The dashed straight lines are the corresponding string network, and the row vectors are the IIB- and D-string charges of each segment. The dots represent the asymptotic positions of D3-branes.

where $c_s$ is an integration constant. For $N = 2$, our solution is identical to that in [9]. For $N = 3, 4$, these configurations are drawn in Fig. 1.

From the asymptotic behaviour of $X$ and $Y$ in the limit $r \to \infty$,

\[
X \sim \text{diag} \left( x_1, \cdots, x_{N+1} \right) + \frac{1}{2r} \text{diag} \left( e_1, \cdots, e_{N+1} \right),
\]

\[
Y \sim \text{diag} \left( y_1, \cdots, y_{N+1} \right) + \frac{1}{2r} \text{diag} \left( g_1, \cdots, g_{N+1} \right),
\]

we can see the positions $(x_i, y_i)$ of D3-branes and the electric and the magnetic charges $(e_i, g_i)$ of strings with their ends on the branes [9], which are

\[
(x_i, y_i) = \left( |c| \sum_{s=1}^{N} (\text{sgn}(c))^{s-1} (v_i^{(s)} - v_{i-1}^{(s)})c_s , -\frac{|c|}{2} \left( N - 2(i - 1) \right) \right),
\]

\[
(e_i, g_i) = \left( -2 \sum_{s=1}^{N} (\text{sgn}(c))^{s-1} (v_i^{(s)} - v_{i-1}^{(s)})c_s \sum_{n=1}^{s} \frac{1}{n} , N - 2(i - 1) \right).
\]

Our solutions are not general one with arbitrary values of the parameters of the monopole solutions [3]. It was hard for us to obtain such general solutions, even for $SU(3)$ case. But it should be interesting to seek them, when we consider the correspondence
between them and the string network. Also, it is unclear whether there are any networks with closed loops among our solutions, although it is unlikely that our solutions include such loops. Then we are led to ask how we can obtain a Yon solution in SYM which corresponds to such a network with closed loops. We hope to return to these problems in the near future.

Acknowledgements

We are grateful to Kazuo Hosomichi and Masatoshi Sato for discussions. K.O. is supported in part by JSPS Research Fellowships for Young Scientists.

Note added: After submitting our paper to the e-print archive, we were informed that Hashimoto, Hata, and Sasakura have also obtained the Yon solutions[15].

Appendix A. The 1/4 BPS condition in $\mathcal{N} = 4$ Supersymmetric Yang-Mills Theory

The bosonic part of action of four-dimensional $\mathcal{N} = 4$ $SU(N)$ supersymmetric Yang-Mills theory is given by

$$ S = \int d^4x \, \text{tr} \left[ -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{1}{2} D_\mu X^I D^\mu X^I + \frac{1}{4} [X^I, X^J]^2 \right], \quad (A.1) $$

with $\mu, \nu = 0, 1, 2, 3$ and $I, J = 1, \cdots, 6$, where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - i[A_\mu, A_\nu]$ and $D_\mu X = \partial_\mu X - i[A_\mu, X]$. From this action, we obtain the Gauss law

$$ \mathbf{D} \cdot \mathbf{E} = i[X^I, D_0 X^I], \quad (A.2) $$

where $\mathbf{E}$ is the electric field, defined by $E_i = F_{0i}$.

The energy of this system is given by

$$ E = \int d^3x \, \frac{1}{2} \text{tr} \left[ \mathbf{E}^2 + \mathbf{B}^2 + (D_0 X^I)^2 + (D X^I)^2 - \frac{1}{2} [X^I, X^J]^2 \right], \quad (A.3) $$

where $\mathbf{B}$ is the magnetic field, defined by $B_i = \frac{1}{2} \epsilon_{ijk} F_{jk}$.
Henceforth, we keep the gauge field $A_\mu$ and only the two scalar fields $X^1 = X$, $X^2 = Y$ nonvanishing and, using the Gauss law (A.2), rewrite the energy (A.3) with a parameter $\theta$ into

$$E = \int d^3 x \frac{1}{2} \text{tr} \left[ \left( \cos \theta E - \sin \theta B - DX \right)^2 + \left( \cos \theta E + \sin \theta B - DY \right)^2 \right. $$

$$+ \left( D_0 X + i \sin \theta [X, Y] \right)^2 + \left( D_0 Y - i \cos \theta [X, Y] \right)^2 \left. \right]$$

$$+ (Q_X + M_Y) \cos \theta + (Q_Y - M_X) \sin \theta$$

$$\ge (Q_X + M_Y) \cos \theta + (Q_Y - M_X) \sin \theta$$

where $Q_X = \int dS \text{tr}[EX]$ and $M_X = \int dS \text{tr}[BX]$, and similarly for $Q_Y$ and $M_Y$.

Since the energy $E$ in the left hand side of (A.4) is independent of the parameter $\theta$, the inequality should be hold for any value of the parameter $\theta$. Thus, in order for the inequality to hold in such a way, the energy $E$ must satisfy an inequality $E \ge M$ with

$$M = \left[ (Q_X + M_Y)^2 + (Q_Y - M_X)^2 \right]^{1/2}.$$ 

When the value of the parameter $\theta$ is such that $\tan \theta = (Q_Y - M_X)/(Q_X + M_Y)$, the energy $E$ saturates the bound $M$, if the configuration of the fields satisfies the BPS condition (2.1), as you can see from the right hand side of the equation (A.4). Also, note that, since we use the Gauss law (A.2) to obtain (A.4), the configuration should satisfy the Gauss law (A.2), too.
References

[1] C. Montonen and D. Olive, “Magnetic Monopoles as Gauge Particles?” , Phys. Lett. 72B, 117-120 (1977).
[2] H. Osborn, “Topological Charges for $N = 4$ Supersymmetric Gauge Theories and Monopoles of Spin 1”, Phys. Lett. 83B, 321-326 (1979).
[3] A. Sen, “Dyon-Monopole Bound States, Self-dual Harmonic Forms on the Multi-Monopole Moduli Space, and $SL(2, \mathbb{Z})$ Invariance in String Theory”, Phys. Lett. B329, 217-221 (1994), hep-th/9402032.
[4] O. Bergman, “Three-Pronged Strings and 1/4 BPS States in $N = 4$ Super-Yang-Mills Theory”, hep-th/9712211.
[5] J. H. Schwarz, “Lectures on Superstring and M theory Dualities”, Nucl. Phys. Proc. Suppl. 55B, 1-32 (1997), hep-th/9607201.
[6] A. Sen, “String Network”, Phys. Lett. B329, 217-221 (1994), hep-th/9711130.
[7] K. Dasgupta and S. Mukhi, “BPS Nature of 3-String Junctions”, hep-th/9711094; S.-J. Rey and J.-T. Yee, “BPS Dynamics of Triple ($p, q$) String Junction”, hep-th/971202.
[8] K. Krogh and S. Lee, “String Network from M-Theory”, Nucl. Phys. B516, 241-254 (1998), hep-th/9712050; Y. Matsuo and K. Okuyama, “BPS Condition of String Junction from M theory”, hep-th/9712070; I. Kishimoto and N. Sasakura, “M-theory description of BPS string in 7-brane background”, hep-th/9712180.
[9] K. Hashimoto, H. Hata, and N. Sasakura, “3-String Junction and BPS Saturated Solutions in $SU(3)$ Supersymmetric Yang-Mills Theory”, hep-th/9803127.
[10] C. Fraser and T. J. Hollowood, “Semiclassical Quantization in $N = 4$ Supersymmetric Yang-Mills Theory and Duality”, Phys. Lett. B402, 106-112 (1997), hep-th/9704011.
[11] E. B. Bogomolny, “Stability of Classical Solutions”, Yad. Fiz. 24, 861-870 (1976), Sov. J. Nucl. Phys. 35, 449-454 (1976).
[12] M. K. Prasad and C. M. Sommerfield, “Exact classical solution for the ’t Hooft monopole and the Julia-Zee dyon”, Phys. Rev. Lett. 35, 760-762 (1975).
[13] D. Wilkinson and F. A. Bais, “Exact $SU(N)$ monopole solutions with spherical symmetry”, Phys. Rev. D19, 2410-2415 (1979).
[14] I. S. Gradshteyn and I. M. Ryzhik; A. Jeffrey ed., “Table of Integrals, Series, and Products (5th ed.)”, (Academic Press, 1994).
[15] K. Hashimoto, H. Hata, and N. Sasakura, to appear.