On solutions to the Ginzburg-Landau equations in higher dimensions

Simon Brendle

August 13, 2003

1 Introduction

Let $M$ be a Riemannian manifold of dimension $n \geq 2$. Consider the semi-linear elliptic equation

$$d^* d\phi = \frac{1}{2\varepsilon^2} (1 - |\phi|^2) \phi,$$

where $\phi$ is a complex-valued function on $M$. This equation is the Euler-Lagrange equation for the functional

$$E_\varepsilon(\phi) = \int_M \left( |d\phi|^2 + \frac{1}{4\varepsilon^2} (1 - |\phi|^2)^2 \right).$$

The equation (1) has been studied by many authors, including F. Bethuel, H. Brezis and F. Hélein [2], F.-H. Lin [16, 17], and R. Jerrard and H. M. Soner [10, 11]. The corresponding Schrödinger and wave equation were studied by J. Colliander and R. Jerrard [6, 9] and by F.-H. Lin, J. Xin, and P. Zhang [19, 20]. While these results are mainly devoted to the case $n = 2$, the higher-dimensional situation has been studied by F.-H. Lin and T. Riviére [18] and F. Bethuel, H. Brezis and G. Orlandi [3].

An important problem is to describe the behavior of the solutions as $\varepsilon \to 0$. Suppose that $\phi_j$ is a sequence of complex-valued functions on $M$ such that

$$d^* d\phi_j = \frac{1}{2\varepsilon_j^2} (1 - |\phi_j|^2) \phi_j$$

where $\varepsilon_j \to 0$. Then there exists a closed set $S$ of Hausdorff codimension 2 and a harmonic map $\phi_{\infty} : M \setminus S \to S^1$ such that $\phi_j \to \phi_{\infty}$ on $M \setminus S$. In particular, if $M$ has dimension 2, then the set $S$ is finite, and its cardinality is given by the degree of $\phi_j$.

In higher dimensions, it follows from results of F.-H. Lin and T. Riviére [18] and F. Bethuel, H. Brezis, and G. Orlandi [3] that the vortex submanifold
S is stationary in the sense that its generalized mean curvature is equal to 0.

In this paper, we study the converse problem. To this end, we consider a smooth minimal submanifold $S$ of codimension 2. Our aim is to construct solutions of the Ginzburg-Landau equations

$$d^* F_A = \frac{1}{2\varepsilon^2} (\phi \overline{D_A \phi} - \overline{\phi} D_A \phi)$$

(2)

and

$$D_A^* D_A \phi = \frac{1}{2\varepsilon^2} (1 - |\phi|^2) \phi.$$

(3)

Here, $A$ is a connection on a complex line bundle $L$ over $M$, and $\phi$ is a section of $L$. A pair $(A, \phi)$ satisfies (2), (3) if and only if $(A, \phi)$ is a critical point of the Ginzburg-Landau functional

$$E_{\varepsilon}(A, \phi) = \int_M \left( \varepsilon^2 |F_A|^2 + |D_A \phi|^2 + \frac{1}{4\varepsilon^2} (1 - |\phi|^2)^2 \right).$$

In dimension 2, Bogomol’nyi observed that the Ginzburg-Landau functional has a lower bound which depends only on the degree of the line bundle $L$. This is a consequence of the identity

$$E_{\varepsilon}(A, \phi) = \int_{\mathbb{R}^2} \left( \varepsilon \ast (iF_A) - \frac{1}{2\varepsilon} (1 - |\phi|^2) \right)^2 + 2 \int_{\mathbb{R}^2} |\bar{\partial}_A \phi|^2 + 2\pi c_1(L).$$

From this it follows that

$$E_{\varepsilon}(A, \phi) \geq 2\pi c_1(L)$$

with equality if and only if $(A, \phi)$ is a solution of the vortex equations

$$\varepsilon \ast (iF_A) = \frac{1}{2\varepsilon} (1 - |\phi|^2)$$

(4)

and

$$\bar{\partial}_A \phi = 0.$$

(5)

In particular, if $(A, \phi)$ satisfies the vortex equations, then we have the identity

$$\varepsilon^2 |F_A|^2 = \frac{1}{4\varepsilon^2} (1 - |\phi|^2).$$

This relation will play an important role in our subsequent arguments.
The equation (1) can be viewed as a simplified version of the Ginzburg-Landau equations (2), (3). The Ginzburg-Landau equations play an important role in mathematical physics, where they arise in the mathematical description of superconductivity. They have been studied intensively, in particular by A. Jaffe and C. H. Taubes [8, 27]. S. Bradlow [4, 5] generalized the vortex equations (4), (5) to holomorphic vector bundles over Kähler manifolds.

The following result shows that every nondegenerate minimal submanifold of $M$ can be obtained as the limit of a family of solutions of the Ginzburg-Landau equations.

**Theorem 1.1.** Let $M$ be a Riemannian manifold of dimension $n$, and let $S$ be a nondegenerate minimal submanifold of dimension $n - 2$. Then the Ginzburg-Landau equation has a solution for all $0 < \varepsilon < \varepsilon_0$. The solutions satisfy

$$
\varepsilon^2 |F_A|^2 + |D_A\phi|^2 + \frac{1}{4\varepsilon^2} (1 - |\phi|^2)^2 \to dH^{n-2}|S
$$

as $\varepsilon \to 0$. Moreover, the first Chern class of $L$ is the Poincaré dual of the homology class of $S$.

A related result was established by C. H. Taubes [28, 29] for the Seiberg-Witten equations on symplectic 4-manifolds. In this case, every pseudo-holomorphic curve can be approximated by a sequence of Seiberg-Witten solutions with parameters $r_j \to \infty$.

A similar approach is used in a recent work of F. Pacard and M. Ritoré [24] which relates constant mean curvature hypersurfaces to the theory of phase transitions.

Moreover, T. Ilmanen [7] proved that Brakke’s motion by mean curvature is given by the limit of a sequence of solutions to the Allen-Cahn equation.

In Section 2, we recall some results about the linearized operator on $\mathbb{R}^2$. In particular, the kernel of the linearized operator on $\mathbb{R}^2$ is isomorphic to the space of parallel vector fields on $\mathbb{R}^2$ (see [8] and [29] for details).

In Section 3, we study the mapping properties of a model operator on the product manifold $\mathbb{R}^{n-2} \times \mathbb{R}^2$.

In Section 4, we construct a family of approximate solutions of the Ginzburg-Landau equations. More precisely, given any normal vector field $v$ satisfying

$$
\|v\|_{C^{2,\gamma}(S)} \leq \varepsilon,
$$

3
we construct a pair \((A, \phi)\) such that
\[
\left\| \left( d^* F_A - \frac{1}{2 \varepsilon^2} \left( \bar{\phi} D_A \phi - \phi D_A \bar{\phi} \right), D_A^* D_A \phi - \frac{1}{2 \varepsilon^2} \left( 1 - |\phi|^2 \right) \phi \right) \right\|_{C^\gamma_{\mu, \varepsilon}(M)} \leq C
\]
for some \(\mu > 0\). Here, the weighted Hölder space \(C^\gamma_{\mu, \varepsilon}(M)\) is defined by
\[
\| u \|_{C^\gamma_{\mu, \varepsilon}(M)} = \sup e^\mu \varepsilon^{- \frac{\text{dist}(p, S)}{2}} |u(p)| + \sup_{\text{dist}(p_1, p_2) \leq \varepsilon} \varepsilon^{\gamma} e^{\mu \left( \frac{\text{dist}(p_1, S) + \text{dist}(p_2, S)}{2 \varepsilon} \right)} \frac{|u(p_1) - u(p_2)|}{\text{dist}(p_1, p_2)^\gamma}.
\]
Moreover, we define
\[
\| (a, f) \|_{C^\gamma_{\mu, \varepsilon}(M)} = \varepsilon \| a \|_{C^\gamma_{\mu, \varepsilon}(M)} + \| f \|_{C^\gamma_{\mu, \varepsilon}(M)}.
\]
In Section 5, we derive uniform estimates for the operator \(L_{\varepsilon} = L_e + T_{\varepsilon} T^*_{\varepsilon}\). Here, \(L_e\) is the linearization of the Ginzburg-Landau equations at an approximate solution \((A, \phi)\). Moreover, the operator \(T_{\varepsilon}\) is defined as
\[
T_{\varepsilon} u = \left( \frac{1}{\varepsilon} du, - \frac{1}{\varepsilon} \phi u \right)
\]
for \(\bar{\eta} = -u\). Its adjoint is given by
\[
T^*_{\varepsilon} (a, f) = \varepsilon d^* a + \frac{1}{2 \varepsilon} (\phi \bar{f} - \bar{\phi} f).
\]
The additional term \(T_{\varepsilon} T^*_{\varepsilon}\) is necessary, because \(L_{\varepsilon}\) is not an elliptic operator.

To derive uniform estimates independent of \(\varepsilon\), we need to restrict the operator \(L_{\varepsilon}\) to a subspace \(E^\gamma_{\mu, \varepsilon}(M) \subset C^\gamma_{\mu, \varepsilon}(M)\). A pair \((a, f)\) belongs to \(E^\gamma_{\mu, \varepsilon}(M)\) if
\[
\int_{NS_x} \varepsilon^2 \sum_{\alpha=1}^4 \langle a(e^\perp_{\alpha}), F_A(w, e^\perp_{\alpha}) \rangle + \int_{NS_x} \langle f, D_A w \phi \rangle = 0
\]
for all \(x \in S\) and \(w \in NS_x\).

In Section 6, we apply the contraction mapping principle to deform the approximate solution \((A, \phi)\) to a nearby pair \((\tilde{A}, \tilde{\phi})\) such that
\[
(1 - \mathbb{P}) \left( d^* F_{\tilde{A}} - \frac{1}{2 \varepsilon^2} \left( \bar{\tilde{\phi}} D_{\tilde{A}} \tilde{\phi} - \tilde{\phi} D_{\tilde{A}} \bar{\tilde{\phi}} \right), D_{\tilde{A}}^* D_{\tilde{A}} \tilde{\phi} - \frac{1}{2 \varepsilon^2} \left( 1 - |\tilde{\phi}|^2 \right) \tilde{\phi} - \frac{1}{\varepsilon} \bar{\tilde{\phi}} u \right) = 0.
\]
Here, \((1 - \mathbb{P})\) is the fibrewise projection from \(C^\gamma_{\mu, \varepsilon}(M)\) to the subspace \(E^\gamma_{\mu, \varepsilon}(M)\).
In Section 7, we show that the glueing data can be chosen such that the corresponding pair \( (\tilde{\hat{A}}, \tilde{\phi}) \) satisfies the balancing condition

\[
\mathbb{P} \left( d^* F_{\tilde{\hat{A}}} - \frac{1}{2\varepsilon^2} (\tilde{\phi}^* D_{\tilde{\hat{A}}} D_{\tilde{\phi}} - \frac{1}{2} \varepsilon^2 (1 - |\tilde{\phi}|^2) \tilde{\phi} - \frac{1}{\varepsilon} \tilde{\phi} u) \right) = 0.
\]

This last step of the proof uses the invertibility of the Jacobi operator of \( S \).

2 The kernel of the linearized operator on \( \mathbb{R}^2 \)

In this section, we study the linearized operator on \( \mathbb{R}^2 \), in particular its kernel. To this end, we define an inner product on the space of pairs \((a, f)\) by

\[
\langle (a_1, f_1), (a_2, f_2) \rangle = \int_{\mathbb{R}^2} \varepsilon^2 \langle a_1, a_2 \rangle + \int_{\mathbb{R}^2} \langle f_1, f_2 \rangle.
\]

Then the linearized operator on \( \mathbb{R}^2 \) satisfies

\[
\langle L_\varepsilon (a, f), (a, f) \rangle = \int_{\mathbb{R}^2} |\varepsilon * d(a) + \frac{1}{2\varepsilon} (\psi f + \overline{\psi} f)|^2 + 2 \int_{\mathbb{R}^2} |\partial_\alpha f + \frac{1}{2} \psi a|^2,
\]

where

\[
\alpha = a_1 + \imath a_2
\]

denotes the \((0, 1)\)-parts of \( a \). This implies

\[
\partial \alpha = \frac{1}{2} (\partial_1 - \imath \partial_2) (a_1 + \imath a_2) = \frac{1}{2} (\partial_1 a_2 - \partial_2 a_1) + \frac{1}{2} (\partial_1 a_1 + \partial_2 a_2) = \frac{1}{2} * d(a) - \frac{1}{2} d^* a.
\]

We define an operator \( T_\varepsilon : \Omega^0(\mathbb{R}^2, i\mathbb{R}) \to \Omega^1(\mathbb{R}^2, i\mathbb{R}) \oplus \Omega^0(\mathbb{R}^2, L) \) by

\[
T_\varepsilon u = \left( \frac{1}{\varepsilon} du, -\frac{1}{\varepsilon} \psi u \right)
\]

for \( \overline{u} = -u \). Its adjoint is given by

\[
T_\varepsilon^*(a, f) = \varepsilon d^* a + \frac{1}{2\varepsilon} (\psi f - \overline{\psi} f).
\]

This implies

\[
\langle T_\varepsilon T_\varepsilon^*(a, f), (a, f) \rangle = \int_{\mathbb{R}^2} |\varepsilon d^* a + \frac{1}{2\varepsilon} (\psi f - \overline{\psi} f)|^2.
\]
Thus, we conclude that
\[
\langle L_\epsilon(a, f) + T_\epsilon T_\epsilon^*(a, f), (a, f) \rangle \\
= \int_{\mathbb{R}^2} \left| \epsilon \star (i a) - \epsilon d^* a + \frac{1}{2\epsilon} (\psi \overline{f} + \overline{\psi} f) - \frac{1}{2\epsilon} (\psi \overline{f} - \overline{\psi} f) \right|^2 + 2 \int_{\mathbb{R}^2} |\bar{\partial}_B f + \frac{1}{2} \psi \alpha|^2 \\
= 4 \int_{\mathbb{R}^2} \left| \epsilon \partial_\alpha + \frac{1}{2\epsilon} \overline{\psi} f \right|^2 + 2 \int_{\mathbb{R}^2} |\bar{\partial}_B f + \frac{1}{2} \psi \alpha|^2.
\]

In particular, the sum
\[
L_\epsilon + T_\epsilon T_\epsilon^* : \Omega^1(\mathbb{R}^2, i\mathbb{R}) \oplus \Omega^0(\mathbb{R}^2, L) \to \Omega^1(\mathbb{R}^2, i\mathbb{R}) \oplus \Omega^0(\mathbb{R}^2, L)
\]
is an elliptic operator.

**Proposition 2.1.** The kernel of the operator $L_\epsilon + T_\epsilon T_\epsilon^*$ is a vector space of real dimension 2. It consists of all pairs of the form $(F_B(w, \cdot), D_B, w \psi)$, where $w$ is a fixed vector in $\mathbb{R}^2$.

**Proof.** Suppose that $(a, f)$ satisfies the equation
\[
L_\epsilon(a, f) + T_\epsilon T_\epsilon^*(a, f) = 0.
\]

This implies
\[
\epsilon \partial_\alpha + \frac{1}{2\epsilon} \overline{\psi} f = 0
\]
and
\[
\bar{\partial}_B f + \frac{1}{2} \psi \alpha = 0.
\]

The set $V$ of pairs $(a, f)$ satisfying these conditions is a vector space of real dimension 2 (see, for example, [8]). We claim that the pair
\[
a = F_B(w, \cdot)
\]
and
\[
f = D_{B, w} \psi
\]
belongs to $V$. Using the identity
\[
\epsilon \star (i F_B) = \frac{1}{2\epsilon} (1 - |\psi|^2),
\]
we obtain
\[
\epsilon \star (i a) + \frac{1}{2\epsilon} (\psi \overline{f} + \overline{\psi} f) = \epsilon \star (i F_B(w, \cdot)) + \frac{1}{2\epsilon} (\psi \overline{D_{B, w} \psi} + \overline{\psi} D_{B, w} \psi)
\]
\[
= \epsilon \partial_w \star (i F_B) + \frac{1}{2\epsilon} \partial_w |\psi|^2
\]
\[
= 0
\]
and
\[ \varepsilon d^*a + \frac{1}{2\varepsilon} (\psi \overline{f} - \overline{\psi} f) = -\varepsilon (d^* F_B)(w) + \frac{1}{2\varepsilon} (\psi \overline{D_B w \psi} - \overline{\psi} D_B w \psi) \]
\[ = 0. \]

Moreover, we have
\[ \bar{\partial}_B f = \bar{\partial}_B D_B w \psi = \bar{\partial}_B D_B w \psi - D_B w \bar{\partial}_B \psi = -\frac{1}{2} \alpha \psi. \]
This proves the assertion.

3 The model problem on \( \mathbb{R}^{n-2} \times \mathbb{R}^2 \)

In this section, we consider a complex line bundle \( L \) over the product \( \mathbb{R}^{n-2} \times \mathbb{R}^2 \). Let \( B \) be a connection on \( L \) and let \( \psi \) be a section of \( L \). We assume that the pair \((B,\psi)\) is invariant under translations along the \( \mathbb{R}^{n-2} \) factor and agrees with the one-vortex solution along the \( \mathbb{R}^2 \) factor.

As in Section 2, we consider the inner product
\[ \langle (a_1, f_1), (a_2, f_2) \rangle = \int_{\mathbb{R}^{n-2} \times \mathbb{R}^2} \varepsilon^2 \langle a_1, a_2 \rangle + \int_{\mathbb{R}^{n-2} \times \mathbb{R}^2} \langle f_1, f_2 \rangle. \]
Then the linearized operator \( L_\varepsilon \) satisfies
\[ \langle L_\varepsilon (a, f), (a, f) \rangle = \int_{\mathbb{R}^{n-2} \times \mathbb{R}^2} (\varepsilon^2 |da|^2 + |D_B f|^2) \]
\[ + 2 \int_{\mathbb{R}^{n-2} \times \mathbb{R}^2} ((D_B \psi, a f) + (a \psi, D_B f)) \]
\[ + \int_{\mathbb{R}^{n-2} \times \mathbb{R}^2} \left( |\psi|^2 |a|^2 + \frac{1}{\varepsilon^2} \text{Re} (\overline{\psi} f)^2 - \frac{1}{2\varepsilon^2} (1 - |\psi|^2) |f|^2 \right). \]

As in Section 2, we define an operator \( T_\varepsilon : \Omega^0(\mathbb{R}^{n-2} \times \mathbb{R}^2, i\mathbb{R}) \to \Omega^1(\mathbb{R}^{n-2} \times \mathbb{R}^2, i\mathbb{R}) \oplus \Omega^0(\mathbb{R}^{n-2} \times \mathbb{R}^2, L) \) by
\[ T_\varepsilon u = \left( \frac{1}{\varepsilon} du, -\frac{1}{\varepsilon} \psi u \right) \]
for \( \overline{u} = -u \). Its adjoint is given by
\[ T_\varepsilon^*(a, f) = \varepsilon d^* a + \frac{1}{2\varepsilon} (\psi \overline{f} - \overline{\psi} f). \]

This implies
\[ \langle T_\varepsilon T_\varepsilon^*(a, f), (a, f) \rangle = \int_{\mathbb{R}^{n-2} \times \mathbb{R}^2} \left| \varepsilon d^* a + \frac{1}{2\varepsilon} (\psi \overline{f} - \overline{\psi} f) \right|^2. \]
Therefore, we obtain
\[
\langle L_\varepsilon(a, f), (a, f) \rangle + \langle T_\varepsilon T_\varepsilon^*(a, f), (a, f) \rangle
\]
\[
= \int_{\mathbb{R}^{n-2} \times \mathbb{R}^2} (\varepsilon^2 |da|^2 + \varepsilon^2 |d^* a|^2 + |D_B f|^2)
\]
\[
+ 2 \int_{\mathbb{R}^{n-2} \times \mathbb{R}^2} \langle (D_B \psi, a f) + (a \psi, D_B f) \rangle - 2 \int_{\mathbb{R}^{n-2} \times \mathbb{R}^2} \langle d^* a \psi, f \rangle
\]
\[
+ \int_{\mathbb{R}^{n-2} \times \mathbb{R}^2} \left( |\psi|^2 |a|^2 + \frac{1}{\varepsilon^2} \text{Re} (\bar{\psi} f^2) + \frac{1}{\varepsilon^2} \text{Im} (\bar{\psi} f)^2 - \frac{1}{2\varepsilon^2} (1 - |\psi|^2) |f|^2 \right)
\]
\[
= \int_{\mathbb{R}^{n-2} \times \mathbb{R}^2} (\varepsilon^2 |da|^2 + \varepsilon^2 |d^* a|^2 + |D_B f|^2)
\]
\[
+ 4 \int_{\mathbb{R}^{n-2} \times \mathbb{R}^2} \langle D_B \psi, a f \rangle
\]
\[
+ \int_{\mathbb{R}^{n-2} \times \mathbb{R}^2} \left( |\psi|^2 |a|^2 + \frac{1}{\varepsilon^2} |\psi|^2 |f|^2 - \frac{1}{2\varepsilon^2} (1 - |\psi|^2) |f|^2 \right).
\]

From this it follows that
\[
L_\varepsilon(a, f) + T_\varepsilon T_\varepsilon^*(a, f) = \left( \nabla^* \nabla a - \frac{1}{\varepsilon^2} (D_B \bar{\psi} f - D_B \bar{\psi} \bar{f}) + \frac{1}{\varepsilon^2} |\psi|^2 a, \right.
\]
\[
D_B^* D_B f - 2 * (a \wedge * D_B \psi) + \frac{1}{\varepsilon^2} |\psi|^2 f - \frac{1}{2\varepsilon^2} (1 - |\psi|^2) f \right)
\]
for \(a \in \Omega^1(\mathbb{R}^{n-2} \times \mathbb{R}^2, i\mathbb{R})\) and \(f \in \Omega^0(\mathbb{R}^{n-2} \times \mathbb{R}^2, L)\). For abbreviation, let
\[
L_\varepsilon = L_\varepsilon + T_\varepsilon T_\varepsilon^*. \text{ Note that}
\]
\[
L_\varepsilon : \Omega^1(\mathbb{R}^{n-2} \times \mathbb{R}^2, i\mathbb{R}) \oplus \Omega^0(\mathbb{R}^{n-2} \times \mathbb{R}^2, L)
\]
\[
\rightarrow \Omega^1(\mathbb{R}^{n-2} \times \mathbb{R}^2, i\mathbb{R}) \oplus \Omega^0(\mathbb{R}^{n-2} \times \mathbb{R}^2, L)
\]
is an elliptic operator.

We define the weighted Hölder space \(C^{\gamma}_{\mu, \varepsilon}(\mathbb{R}^{n-2} \times \mathbb{R}^2)\) by
\[
\|u\|_{C^{\gamma}_{\mu, \varepsilon}(\mathbb{R}^{n-2} \times \mathbb{R}^2)} = \sup_{|x - y| \leq \varepsilon} \varepsilon^{\gamma} \left[ \frac{|\nabla u(x, y)|}{\mu(|x| + |y|)} \right]^{\gamma}
\]
More generally, let
\[
\|u\|_{C^{k,\gamma}_{\mu, \varepsilon}(\mathbb{R}^{n-2} \times \mathbb{R}^2)} = \sum_{l=0}^{k} \varepsilon^{l} \|\nabla^l u\|_{C^{\gamma}_{\mu, \varepsilon}(\mathbb{R}^{n-2} \times \mathbb{R}^2)}.
\]
Furthermore, we define
\[
\|(a, f)\|_{C^{k,\gamma}_{\mu, \varepsilon}(\mathbb{R}^{n-2} \times \mathbb{R}^2)} = \varepsilon \|a\|_{C^{k,\gamma}_{\mu, \varepsilon}(\mathbb{R}^{n-2} \times \mathbb{R}^2)} + \|f\|_{C^{k,\gamma}_{\mu, \varepsilon}(\mathbb{R}^{n-2} \times \mathbb{R}^2)}.
\]
Let $\mathcal{E}_{\mu,\varepsilon}^{k,\gamma}(\mathbb{R}^{n-2} \times \mathbb{R}^2)$ be the set of all pairs $(a, f) \in \Omega^1(\mathbb{R}^{n-2} \times \mathbb{R}^2, i\mathbb{R}) \oplus \Omega^0(\mathbb{R}^{n-2} \times \mathbb{R}^2, L)$ such that $(a, f) \in C_{\mu,\varepsilon}^{k,\gamma}(\mathbb{R}^{n-2} \times \mathbb{R}^2)$ and

$$\int_{\{x\} \times \mathbb{R}^2} \varepsilon^2 \sum_{\alpha=1}^2 (a(e^{\perp}_\alpha), F_B(w, e^{\perp}_\alpha)) + \int_{\{x\} \times \mathbb{R}^2} \langle f, D_B,w\rangle = 0$$

for all $x \in \mathbb{R}^{n-2}$ and all $w \in \mathbb{R}^2$.

Proposition 3.1. The operator $L_\varepsilon$ maps $C_{\mu,\varepsilon}^{2,\gamma}(\mathbb{R}^{n-2} \times \mathbb{R}^2)$ into $C_{\mu,\varepsilon}^{\gamma}(\mathbb{R}^{n-2} \times \mathbb{R}^2)$.

Proof. It is obvious from the definition that $L_\varepsilon$ maps $C_{\mu,\varepsilon}^{2,\gamma}(\mathbb{R}^{n-2} \times \mathbb{R}^2)$ into $C_{\mu,\varepsilon}^{\gamma}(\mathbb{R}^{n-2} \times \mathbb{R}^2)$. We now assume that $(a, f) \in C_{\mu,\varepsilon}^{2,\gamma}(\mathbb{R}^{n-2} \times \mathbb{R}^2)$ satisfies

$$\int_{\mathbb{R}^2} \varepsilon^2 \langle a, F_B(w, \cdot) \rangle + \int_{\mathbb{R}^2} \langle f, D_B,w\rangle = 0$$

for all $x \in \mathbb{R}^{n-2}$ and all $w \in \mathbb{R}^2$. Taking derivatives in horizontal direction, we obtain

$$\int_{\{x\} \times \mathbb{R}^2} \varepsilon^2 \sum_{\alpha=1}^2 \sum_{j=1}^{n-2} \langle \partial_j \partial_j a(e^{\perp}_\alpha), F_B(w, e^{\perp}_\alpha) \rangle + \int_{\mathbb{R}^2} \langle f, D_B,w\rangle = 0.$$ 

Furthermore, integration by parts gives

$$\int_{\{x\} \times \mathbb{R}^2} \sum_{\alpha=1}^2 \sum_{j=1}^{n-2} \varepsilon^2 \langle \nabla e^{\perp}_\alpha \nabla e^{\perp}_\rho a(e^{\perp}_\alpha), F_B(w, e^{\perp}_\alpha) \rangle$$

$$+ \int_{\{x\} \times \mathbb{R}^2} \sum_{\rho=1}^2 \langle D_B,e^{\perp}_\rho D_B,e^{\perp}_\rho f, D_B,w\rangle$$

$$- 2 \int_{\{x\} \times \mathbb{R}^2} \langle D_B\psi, a D_B,w\psi \rangle - 2 \int_{\{x\} \times \mathbb{R}^2} \langle D_B\psi, F_B(w, \cdot) f \rangle$$

$$- \int_{\{x\} \times \mathbb{R}^2} (|\psi|^2 \langle a, F_B(w, \cdot) \rangle + \frac{1}{\varepsilon^2} |\psi|^2 \langle f, D_B,w\rangle - \frac{1}{2\varepsilon^2} (1 - |\psi|^2) \langle f, D_B,w\rangle \rangle$$

$$= \int_{\{x\} \times \mathbb{R}^2} \sum_{\alpha=1}^2 \sum_{\rho=1}^2 \varepsilon^2 \langle a(e^{\perp}_\alpha), \nabla e^{\perp}_\rho \nabla e^{\perp}_\rho F_B(w, e^{\perp}_\alpha) \rangle$$

$$+ \int_{\{x\} \times \mathbb{R}^2} \sum_{\rho=1}^2 \langle f, D_B,e^{\perp}_\rho D_B,e^{\perp}_\rho D_B,w\rangle$$

$$- 2 \int_{\{x\} \times \mathbb{R}^2} \langle D_B\psi, a D_B,w\psi \rangle - 2 \int_{\{x\} \times \mathbb{R}^2} \langle D_B\psi, F_B(w, \cdot) f \rangle$$

$$- \int_{\{x\} \times \mathbb{R}^2} (|\psi|^2 \langle a, F_B(w, \cdot) \rangle + \frac{1}{\varepsilon^2} |\psi|^2 \langle f, D_B,w\rangle - \frac{1}{2\varepsilon^2} (1 - |\psi|^2) \langle f, D_B,w\rangle \rangle$$

$$= 0.$$
Hence, if we define \((b, h) = \mathbb{L}_\varepsilon(a, f)\), then we obtain
\[
\int_{\{x\} \times \mathbb{R}^2} \varepsilon^2 \sum_{\alpha=1}^{2} (b(e_\alpha^\perp), F_B(w, e_\alpha^\perp)) + \int_{\{x\} \times \mathbb{R}^2} \langle h, D_B w \psi \rangle = 0
\]
for all \(x \in \mathbb{R}^{n-2}\) and \(w \in \mathbb{R}^2\).

**Proposition 3.2.** Let \(0 < \nu < 1\), \((b, h) \in C^{\gamma}_\mu,\varepsilon(\mathbb{R}^2),\) and \(\eta \in \mathcal{S}(\mathbb{R}^{n-2})\). Moreover, assume that the Fourier transform of \(\eta\) satisfies \(\hat{\eta}(\xi) = 0\) for \(|\xi| \leq \delta\) for some \(\delta > 0\). Then there exists a pair \((a, f) \in C^{2,\gamma}_\mu,\varepsilon(\mathbb{R}^{n-2} \times \mathbb{R}^2)\) such that
\[
\mathbb{L}_\varepsilon(a, f) = (\eta(x) b(y), \eta(x) h(y)).
\]

**Proof.** We perform a Fourier transformation in the \(\mathbb{R}^{n-2}\) variables. Let
\[
\eta(x) = \int_{\mathbb{R}^{n-2}} e^{ix\xi} \hat{\eta}(\xi) d\xi.
\]
For every \(\xi \in \mathbb{R}^{n-2}\), there exists a pair \((\hat{a}(\xi, \cdot), \hat{f}(\xi, \cdot)) \in C^{2,\gamma}_\mu,\varepsilon(\mathbb{R}^2)\) such that
\[
\sum_{\rho=1}^{2} \partial_\rho \partial_\rho \hat{a}(\xi, y) + \frac{1}{\varepsilon^2} (D_B \psi f) \hat{f}(\xi, y) - D_B \psi \hat{f}(\xi, y) - \frac{1}{\varepsilon^2} |\psi(\xi, y)|^2 \hat{a}(\xi, y) = -b(y)
\]
and
\[
\sum_{\rho=1}^{2} D_{B,\rho} D_{B,\rho} \hat{f}(\xi, y) + 2 \sum_{\rho=1}^{2} D_{B,\rho} \psi(\xi, y) \hat{a}_\rho(\xi, y)
\]
\[
- \frac{1}{\varepsilon^2} |\psi(\xi, y)|^2 \hat{f}(\xi, y) + \frac{1}{2\varepsilon^2} (1 - |\psi(\xi, y)|^2) \hat{f}(\xi, y) - |\xi|^2 \hat{f}(\xi, y) = -h(y).
\]
We now define a pair \((a, f)\) by
\[
a(x, y) = \int_{\mathbb{R}^{n-2}} e^{ix\xi} \hat{\eta}(\xi) \hat{a}(\xi, y) d\xi
\]
and
\[
f(x, y) = \int_{\mathbb{R}^{n-2}} e^{ix\xi} \hat{\eta}(\xi) \hat{f}(\xi, y) d\xi.
\]
Then the pair \((a, f)\) satisfies
\[
\sum_{i=1}^{n-2} \partial_i \partial_i a + \sum_{\rho=1}^{2} \partial_\rho \partial_\rho a + \frac{1}{\varepsilon^2} (D_B \psi f - D_B \psi f) - \frac{1}{\varepsilon^2} |\psi|^2 a = -\eta(x) b(y)
\]
and
\[
\sum_{i=1}^{n-2} \partial_i \partial_i f + \sum_{\rho=1}^{2} D_{B,\rho} D_{B,\rho} f + 2 \sum_{\rho=1}^{2} D_{B,\rho} \psi a_\rho - \frac{1}{\varepsilon^2} |\psi|^2 f + \frac{1}{2 \varepsilon} (1 - |\psi|^2) \hat{f} = -\eta(x) h(y).
\]

From this we deduce that
\[
\mathbb{L}_\varepsilon(a, f) = (\eta(x) b(y), \eta(x) h(y)).
\]

This proves the assertion.

**Proposition 3.3.** Let \(0 < \nu < 1\), and suppose that \((a, f) \in \mathcal{E}^{2,\gamma}_{\mu,\varepsilon}(\mathbb{R}^{n-2} \times \mathbb{R}^2)\) satisfies \(\mathbb{L}_\varepsilon(a, f) = 0\). Then \((a, f) = 0\).

**Proof.** Let \((b, h) \in C^{\gamma}_{\mu,\varepsilon}(\mathbb{R}^2)\) and \(\zeta \in \mathcal{S}(\mathbb{R}^{n-2})\) be given. We define a function \(\eta \in \mathcal{S}(\mathbb{R}^{n-2})\) by \(\eta(x) = \zeta(x + x_0) - \zeta(x)\). Then the Fourier transform of \(\eta\) satisfies \(\hat{\eta}(0) = 0\). We approximate \(\eta\) by functions \(\eta_\delta\) such that
\[
\| \hat{\eta} - \hat{\eta}_\delta \|_{L^p(\mathbb{R}^{n-2})} \leq C \delta^{n-2} \| \eta \|_{L^p(\mathbb{R}^{n-2})}.
\]

for all \( \delta > 0 \). From this it follows that
\[
\| \eta - \eta_\delta \|_{L^1(\mathbb{R}^{n-2})} \leq C \delta^{1 - \frac{n-2}{p}}
\]

for all \( p \geq 2 \). This implies
\[
\| \eta - \eta_\delta \|_{L^1(\mathbb{R}^{n-2})} \to 0
\]
as \( \delta \to 0 \).

For each \( \delta > 0 \), the pair \((\eta_\delta(x) b(y), \eta_\delta(x) h(y))\) belongs to the image of \(\mathbb{L}_\varepsilon\). Since \((a, f)\) belongs to the kernel of \(\mathbb{L}_\varepsilon\), we obtain
\[
\int_{\mathbb{R}^{n-2} \times \mathbb{R}^2} \varepsilon^2 \langle a(x, y), \eta_\delta(x) b(y) \rangle + \int_{\mathbb{R}^{n-2} \times \mathbb{R}^2} \langle f(x, y), \eta_\delta(x) h(y) \rangle = 0.
\]
Letting \( \delta \to 0 \), we obtain
\[
\int_{\mathbb{R}^{n-2} \times \mathbb{R}^2} \varepsilon^2 \langle a(x, y), \eta(x) b(y) \rangle + \int_{\mathbb{R}^{n-2} \times \mathbb{R}^2} \langle f(x, y), \eta(x) h(y) \rangle = 0.
\]
From this it follows that
\[
\int_{\mathbb{R}^{n-2} \times \mathbb{R}^2} \varepsilon^2 \langle a(x, y), \zeta(x) b(y) \rangle + \int_{\mathbb{R}^{n-2} \times \mathbb{R}^2} \langle f(x, y), \zeta(x) h(y) \rangle
= \int_{\mathbb{R}^{n-2} \times \mathbb{R}^2} \varepsilon^2 \langle a(x - x_0, y), \zeta(x) b(y) \rangle + \int_{\mathbb{R}^{n-2} \times \mathbb{R}^2} \langle f(x - x_0, y), \zeta(x) h(y) \rangle.
\]
Since \((b, h)\) and \(\zeta\) are arbitrary, we conclude that \(a(x, y) = a(x - x_0, y)\) and \(f(x, y) = f(x - x_0, y)\). Therefore, \(a(x, y)\) and \(f(x, y)\) are constant in \(x\).
Using Proposition 2.1, we obtain
\[
(a, f) = (F_B(w, \cdot), D_B,w\psi)
\]
for some \(w \in \mathbb{R}^2\). This proves the assertion.

**Proposition 3.4.** Let \(0 < \nu < 1\). Then we have the estimate
\[
||(a, f)||_{C^{2, \gamma}_{\mu, \varepsilon}(\mathbb{R}^{n-2} \times \mathbb{R}^2)} \leq C \varepsilon^2 \|L_\varepsilon(a, f)\|_{C^0_{\mu, \varepsilon}(\mathbb{R}^{n-2} \times \mathbb{R}^2)}
\]
for all \((a, f) \in \mathcal{E}_{\mu, \varepsilon}^2(\mathbb{R}^{n-2} \times \mathbb{R}^2)\).

**Proof.** By Schauder estimates, it suffices to show that
\[
\sup e^{\frac{\varepsilon |a|}{2}} (\varepsilon |a(x, y)| + |f(x, y)|) \leq C \sup e^{\frac{\varepsilon |b|}{2}} (\varepsilon |b(x, y)| + |h(x, y)|),
\]
where \((b, h) = L_\varepsilon(a, f)\). To prove this estimate, we argue by contradiction. Let \((a^{(j)}, f^{(j)})\) be a sequence of pairs such that
\[
\sup e^{\frac{\varepsilon |a^{(j)}|}{2}} (\varepsilon |a^{(j)}(x, y)| + |f^{(j)}(x, y)|) = 1
\]
and
\[
\sup \varepsilon^2 e^{-\frac{\varepsilon |b^{(j)}|}{2}} (\varepsilon |b^{(j)}(x, y)| + |h^{(j)}(x, y)|) \to 0,
\]
where \((b^{(j)}, h^{(j)}) = L_\varepsilon(a^{(j)}, f^{(j)})\). We choose a sequence of points \((x_j, y_j) \in \mathbb{R}^{n-2} \times \mathbb{R}^2\) such that
\[
e^{\frac{\varepsilon |a^{(j)}(x_j, y_j)|}{2}} (\varepsilon |a^{(j)}(x_j, y_j)| + |f^{(j)}(x_j, y_j)|) \geq \frac{1}{2}
\]
for all \(j\). There are two possibilities:
(i) Suppose that the sequence $|y_j|$ is bounded. After passing to a subsequence, we may assume that the sequence $(a^{(j)}, f^{(j)})$ converges to a pair $(a, f) \in \mathcal{E}_{\mu, \varepsilon}^{2, \gamma}(\mathbb{R}^{n-2} \times \mathbb{R}^2)$ such that
\[
\sup e^{-\frac{\mu|y|}{\varepsilon}} (\varepsilon |a(x, y)| + |f(x, y)|) \leq 1
\]
and
\[
\mathbb{L}_\varepsilon(a, f) = 0.
\]
Using Proposition 3.3, we conclude that $(a, f) = 0$. This is a contradiction.

(ii) We now assume that $|y_j| \to \infty$. We define a sequence of pairs $(\tilde{a}_j, \tilde{f}_j)$ by
\[
\tilde{a}^{(j)}(x, y) = e^{\frac{\mu|y|}{\varepsilon}} a^{(j)}(x + x_j, y + y_j)
\]
and
\[
\tilde{f}^{(j)}(x, y) = e^{\frac{\mu|y|}{\varepsilon}} f^{(j)}(x + x_j, y + y_j).
\]
After passing to a subsequence, we may assume that the sequence $(\tilde{a}_j, \tilde{f}_j)$ converges to a pair $(\tilde{a}, \tilde{f})$. The pair $(\tilde{a}, \tilde{f})$ is defined on $\mathbb{R}^{n-2} \times \mathbb{R}^2$ and satisfies
\[
\sup e^{-\frac{\mu|y|}{\varepsilon}} (\varepsilon |\tilde{a}(x, y)| + |\tilde{f}(x, y)|) \leq 1
\]
and
\[
\left(\nabla^* \nabla \tilde{a} + \frac{1}{\varepsilon^2} \tilde{a}, \nabla^* \nabla \tilde{f} + \frac{1}{\varepsilon^2} \tilde{f}\right) = 0.
\]
If $\mu$ is sufficiently small, it follows that $(\tilde{a}, \tilde{f}) = 0$. This is a contradiction.

**Proposition 3.5.** Let $0 < \nu < 1$. Assume that $(b, h) \in C_\Gamma^\gamma(\mathbb{R}^2)$ satisfies
\[
\int_{\mathbb{R}^2} \varepsilon^2 \langle b, F_B(w, \cdot) \rangle + \int_{\mathbb{R}^2} \langle h, D_B w \psi \rangle = 0
\]
for all $x \in \mathbb{R}^{n-2}$ and all $w \in \mathbb{R}^2$. Moreover, let $\eta \in \mathcal{S}(\mathbb{R}^{n-2})$. Then there exists a pair $(a, f) \in \mathcal{E}_{\mu, \varepsilon}^{2, \gamma}(\mathbb{R}^{n-2} \times \mathbb{R}^2)$ such that
\[
\mathbb{L}_\varepsilon(a, f) = (\eta(x) b(y), \eta(x) h(y)).
\]
Proof. Let
\[ \eta(x) = \int_{\mathbb{R}^{n-2}} e^{ix\xi} \hat{\eta}(\xi) \, d\xi. \]

For every \( \xi \in \mathbb{R}^{n-2} \), there exists a 1-form \( \hat{a}(\xi, \cdot) \in \mathcal{C}^2_{\mu, \gamma}(\mathbb{R}^2) \) such that
\[
\sum_{\rho=1}^{2} \partial_{\rho} \partial_{\rho} \hat{a}(\xi, y) + 2 \sum_{\rho=1}^{2} D_{B, \rho} \psi(y) \hat{a}_{\rho}(\xi, y) - \frac{1}{\varepsilon^2} |\psi(y)|^2 \hat{f}(\xi, y) - |\xi|^2 \hat{\eta}(\xi, y) = -b(y)
\]
and
\[
\sum_{\rho=1}^{2} D_{B, \rho} D_{B, \rho} \hat{f}(\xi, y) + 2 \sum_{\rho=1}^{2} D_{B, \rho} \psi(y) \hat{a}_{\rho}(\xi, y) - \frac{1}{2\varepsilon^2} (1 - |\psi(y)|^2) \hat{f}(\xi, y) - |\xi|^2 \hat{f}(\xi, y) = -h(y).
\]

Furthermore, the pair \( (\hat{a}(\xi, \cdot), \hat{f}(\xi, \cdot)) \) satisfies
\[
\int_{\mathbb{R}^2} \varepsilon^2 \langle \hat{a}(\xi, \cdot), F_B(w, \cdot) \rangle + \int_{\mathbb{R}^2} \langle \hat{f}(\xi, \cdot), D_B w \psi \rangle = 0
\]
for all \( x \in \mathbb{R}^{n-2} \) and all \( w \in \mathbb{R}^2 \). We now define a pair \( (a, f) \in \mathcal{E}^2_{\mu, \gamma}(\mathbb{R}^{n-2} \times \mathbb{R}^2) \) by
\[
a(x, y) = \int_{\mathbb{R}^{n-2}} e^{ix\xi} \hat{\eta}(\xi) \hat{a}(\xi, y) \, d\xi
\]
and
\[
f(x, y) = \int_{\mathbb{R}^{n-2}} e^{ix\xi} \hat{\eta}(\xi) \hat{f}(\xi, y) \, d\xi.
\]
Then the pair \( (a, f) \) satisfies
\[
\sum_{i=1}^{n-2} \partial_{i} \partial_{i} a + \sum_{\rho=1}^{2} \partial_{\rho} \partial_{\rho} a + \frac{1}{\varepsilon^2} (D_B \psi f - D_B \psi \hat{f}) - \frac{1}{\varepsilon^2} |\psi|^2 a = -\eta(x) b(y)
\]
and
\[
\sum_{i=1}^{n-2} \partial_{i} \partial_{i} f + \sum_{\rho=1}^{2} D_{B, \rho} D_{B, \rho} f + 2 \sum_{\rho=1}^{2} D_{B, \rho} \psi a_{\rho} - \frac{1}{\varepsilon^2} |\psi|^2 f + \frac{1}{2\varepsilon^2} (1 - |\psi|^2) \hat{f} = -\eta(x) h(y).
\]
Thus, we conclude that
\[
\mathbb{L}_\varepsilon(a, f) = (\eta(x) b(y), \eta(x) h(y)).
\]
This proves the assertion.
Corollary 3.6. Let $0 < \nu < 1$, and suppose that $(b, h) \in \mathcal{E}_{\mu, \varepsilon}^\gamma(\mathbb{R}^{n-2} \times \mathbb{R}^2)$ has compact support. Then there exists a pair $(a, f) \in \mathcal{E}_{\mu, \varepsilon}^{2, \gamma}(\mathbb{R}^{n-2} \times \mathbb{R}^2)$ such that

$$
\| (a, f) \|_{\mathcal{C}_{\mu, \varepsilon}^{2, \gamma}(\mathbb{R}^{n-2} \times \mathbb{R}^2)} \leq C \varepsilon^2 \| (b, h) \|_{\mathcal{C}_{\mu, \varepsilon}^\gamma(\mathbb{R}^{n-2} \times \mathbb{R}^2)}
$$

and

$$
\mathbb{L}_\varepsilon(a, f) = (b, h).
$$

Proof. It follows from Proposition 3.4 that the range of the operator $\mathbb{L}_\varepsilon : \mathcal{E}_{\mu, \varepsilon}^{2, \gamma}(\mathbb{R}^{n-2} \times \mathbb{R}^2) \to \mathcal{E}_{\mu, \varepsilon}^\gamma(\mathbb{R}^{n-2} \times \mathbb{R}^2)$ is a closed subspace of the Banach space $\mathcal{E}_{\mu, \varepsilon}^\gamma(\mathbb{R}^{n-2} \times \mathbb{R}^2)$. By Proposition 3.5, it contains all pairs of the form $(\eta(x) b(y), \eta(x) h(y))$, where $\eta \in \mathcal{S}(\mathbb{R}^{n-2})$ and $(b, h) \in \mathcal{C}_{\mu, \varepsilon}^\gamma(\mathbb{R}^2)$ satisfies

$$
\int_{\mathbb{R}^2} \varepsilon^2 \langle b, F_B(w, \cdot) \rangle + \int_{\mathbb{R}^2} \langle h, D_B(w, \psi) \rangle = 0
$$

for all $x \in \mathbb{R}^{n-2}$ and all $w \in \mathbb{R}^2$. The assertion follows now by approximation.

Proposition 3.7. Let $0 < \nu < 1$. Suppose that $(b, h) \in \mathcal{E}_{\mu, \varepsilon}^\gamma(\mathbb{R}^{n-2} \times \mathbb{R}^2)$ is supported in the set $\{(x, y) \in \mathbb{R}^{n-2} \times \mathbb{R}^2 : |x| \leq \delta, |y| \leq 2\delta\}$. Then there exists a pair $(a, f) \in \mathcal{C}_{\mu, \varepsilon}^{2, \gamma}(\mathbb{R}^{n-2} \times \mathbb{R}^2)$ such that $(a, f)$ is supported in $\{(x, y) \in \mathbb{R}^{n-2} \times \mathbb{R}^2 : |x| \leq 2\delta, |y| \leq 4\delta\}$,

$$
\| (a, f) \|_{\mathcal{C}_{\mu, \varepsilon}^{2, \gamma}(\mathbb{R}^{n-2} \times \mathbb{R}^2)} \leq C \varepsilon^2 \| (b, h) \|_{\mathcal{C}_{\mu, \varepsilon}^\gamma(\mathbb{R}^{n-2} \times \mathbb{R}^2)}
$$

and

$$
\| \mathbb{L}_\varepsilon(a, f) - (b, h) \|_{\mathcal{C}_{\mu, \varepsilon}^\gamma(\mathbb{R}^{n-2} \times \mathbb{R}^2)} \leq C \left( \frac{\varepsilon}{\delta} + \frac{\varepsilon^2}{\delta^2} \right) \| b \|_{\mathcal{C}_{\mu, \varepsilon}^\gamma(\mathbb{R}^{n-2} \times \mathbb{R}^2)}.
$$

Proof. By Corollary 3.6, there exists a pair $(a, f) \in \mathcal{E}_{\mu, \varepsilon}^{2, \gamma}(\mathbb{R}^{n-2} \times \mathbb{R}^2)$ such that

$$
\| (a, f) \|_{\mathcal{C}_{\mu, \varepsilon}^{2, \gamma}(\mathbb{R}^{n-2} \times \mathbb{R}^2)} \leq C \varepsilon^2 \| (b, h) \|_{\mathcal{C}_{\mu, \varepsilon}^\gamma(\mathbb{R}^{n-2} \times \mathbb{R}^2)}
$$

and

$$
\mathbb{L}_\varepsilon(a, f) = (b, h).
$$

Let $\zeta$ be a cut-off function on $\mathbb{R}^{n-2}$ such that $\zeta(x) = 1$ for $|x| \leq \delta$, $\zeta(x) = 0$ for $|x| \geq 2\delta$, and

$$
\sup \delta |\nabla \zeta| + \sup \delta^2 |\nabla^2 \zeta| \leq C.
$$
Furthermore, let $\eta$ be a cut-off function on $\mathbb{R}^{n-2}$ satisfying $\eta(y) = 1$ for $|y| \leq 2\delta$, $\eta(y) = 0$ for $|y| \geq 4\delta$, and 

$$\sup \delta |\nabla \eta| + \sup \delta^2 |\nabla^2 \eta| \leq C.$$ 

Then we have the estimates 

$$\| (\eta \zeta a, \eta \zeta f) \|_{C^{2,\gamma}_{\mu,\varepsilon}(\mathbb{R}^{n-2} \times \mathbb{R}^2)} \leq C \varepsilon^2 \| (b, h) \|_{C^{1,\gamma}_{\mu,\varepsilon}(\mathbb{R}^{n-2} \times \mathbb{R}^2)}$$ 

and 

$$\| L_\varepsilon (\eta \zeta a, \eta \zeta f) - (b, h) \|_{C^{1,\gamma}_{\mu,\varepsilon}(\mathbb{R}^{n-2} \times \mathbb{R}^2)} = \| L_\varepsilon (\eta \zeta a, \eta \zeta f) - \eta \zeta L_\varepsilon a \|_{C^{1,\gamma}_{\mu,\varepsilon}(\mathbb{R}^{n-2} \times \mathbb{R}^2)}$$ 

$$\leq C \left( \frac{\varepsilon}{\delta} + \frac{\varepsilon^2}{\delta^2} \right) \| (a, f) \|_{C^{1,\gamma}_{\mu,\varepsilon}(\mathbb{R}^{n-2} \times \mathbb{R}^2)}$$ 

$$\leq C \left( \frac{\varepsilon}{\delta} + \frac{\varepsilon^2}{\delta^2} \right) \| (b, h) \|_{C^{1,\gamma}_{\mu,\varepsilon}(\mathbb{R}^{n-2} \times \mathbb{R}^2)}.$$ 

From this the assertion follows.

### 4 Construction of the approximate solutions

In this section, we define a family of approximate solutions which concentrate near $S$ as $\varepsilon \to 0$. To this end, we identify the total space of the normal bundle $NS$ with a neighborhood of the submanifold $S$ by means of the exponential map 

$$\exp : NS \to M.$$ 

Note that this identification is not isometric. To see this, we choose an orthonormal basis $\{ e_i : 1 \leq i \leq n-2 \}$ for the horizontal subspace, and an orthonormal basis $\{ e_\alpha^\perp : 1 \leq \alpha \leq 2 \}$ for the vertical subspace. Using Jacobi’s equation, we obtain 

$$\exp_\varepsilon (e_i) = \pi_\varepsilon (e_i) + \sum_{j=1}^{n-2} \sum_{\rho=1}^{2} h_{ij,\rho} y_{\rho} \pi_\varepsilon (e_j)$$ 

$$- \frac{1}{2} \sum_{j=1}^{n-2} \sum_{\rho,\sigma=1}^{2} R_{\rho j,\sigma} y_{\rho} y_{\sigma} \pi_\varepsilon (e_j)$$ 

$$- \frac{1}{2} \sum_{\beta,\rho,\sigma=1}^{2} R_{\beta,\rho,\sigma} y_{\rho} y_{\sigma} e_\beta^\perp + O(|y|^3).$$
This implies

\[
\langle \exp_*(e_i), \exp_*(e_j) \rangle = \delta_{ij} + 2 \sum_{\rho=1}^{2} h_{ij,\rho} y_\rho \\
+ \sum_{k=1}^{n-2} \sum_{\rho,\sigma=1}^{2} h_{ik,\rho} h_{jk,\sigma} y_\rho y_\sigma \\
- \sum_{\rho,\sigma=1}^{2} R_{i\rho\sigma j} y_\rho y_\sigma + O(|y|^3).
\]

Furthermore, we have

\[
\langle \exp_*(e_i), \exp_*(e^\perp_\alpha) \rangle = O(|y|^2)
\]

and

\[
\langle \exp_*(e^\perp_\alpha), \exp_*(e^\perp_\beta) \rangle = \delta_{\alpha\beta} - \frac{1}{3} \sum_{\rho,\sigma=1}^{2} R_{\alpha\rho\sigma\beta} y_\rho y_\sigma + O(|y|^3).
\]

Let \( v \) be a section of the normal bundle \( NS \). For every point \( x \in S \), there is a unique one-vortex solution \( (B, \psi) \) on \( NS_x \) with center \( v_x \). We now define a pair \( (A, \phi) \) by

\[
A(e^\perp_\alpha) = B(e^\perp_\alpha), \\
A(e_i) = -\nabla_i v_\rho B(e^\perp_\rho),
\]

and

\[
\phi = \psi.
\]

**Proposition 4.1.** The curvature of \( A \) is given by

\[
F_A(e^\perp_\alpha, e^\perp_\beta) = F_B(e^\perp_\alpha, e^\perp_\beta),
\]

\[
F_A(e_i, e^\perp_\alpha) = -\nabla_i v_\rho F_B(e^\perp_\rho, e^\perp_\alpha),
\]

and

\[
F_A(e_i, e_j) = \nabla_i v_\rho \nabla_j v_\sigma F_A(e^\perp_\rho, e^\perp_\sigma) + C_{ij} + A(C_{ij} (y - v)),
\]

where \( C_{ij} \in \Lambda^2 NS \) is the curvature of the normal bundle. Furthermore, the covariant derivative of the section \( \phi \) is given by

\[
D_{A,e^\perp_\alpha} \phi = D_{B,e^\perp_\alpha} \psi
\]

and

\[
D_{A,e_i} \phi = -\nabla_i v_\rho D_{B,e^\perp_\rho} \psi.
\]
Proof. Using the identity

\[ [e_i, e_j] = -C_{ij} y, \]

we obtain

\[
F_A(e_i, e_j) = C_{ij} + \nabla_{e_i} A(e_j) - \nabla_{e_j} A(e_i) - A([e_i, e_j])
\]

\[
= C_{ij} - (\nabla_i \nabla_j v \rho - \nabla_j \nabla_i v \rho) A(e^+_{\rho}) + \nabla_i v \rho \nabla_j v \sigma (\nabla_{e_{\rho}} A(e^+_{\sigma}) - \nabla_{e_{\sigma}} A(e^+_{\rho})) + A(C_{ij} y)
\]

\[
= C_{ij} - A(C_{ij} v)
\]

\[
= \nabla_i v \rho \nabla_j v \sigma (\nabla_{e_{\rho}} A(e^+_{\sigma}) - \nabla_{e_{\sigma}} A(e^+_{\rho})) + A(C_{ij} y)
\]

\[
= \nabla_i v \rho \nabla_j v \sigma F_A(e^+_{\rho}, e^+_{\sigma}) + A(C_{ij} (y - v)).
\]

This proves the assertion.

Note that the pair \((B, \psi)\) satisfies the estimates

\[
|F_B| \leq C \varepsilon e^{-\frac{\mu}{\varepsilon}},
\]

\[
|D_B \psi| \leq C \varepsilon^{-1} e^{-\frac{\mu}{\varepsilon}},
\]

\[
0 \leq 1 - |\psi|^2 \leq C e^{-\frac{\mu}{\varepsilon}}
\]

for suitable constant \(\mu > 0\).

Throughout this paper, we will assume that the normal vector field \(v\) satisfies the estimate \(\|v\|_{C^{2,\gamma}(S)} \leq \varepsilon\).

**Proposition 4.2.** If the normal vector field \(v\) satisfies the estimate \(\|v\|_{C^{2,\gamma}(S)} \leq \varepsilon\), then the error term verifies the estimate

\[
\left\| \left( d^* F_A - \frac{1}{2\varepsilon^2} (\phi D_A \phi - \overline{\phi} D_A \overline{\phi}), D_A^* D_A \phi - \frac{1}{2\varepsilon^2} (1 - |\phi|^2) \phi \right) \right\|_{C^{\gamma}_{\mu,\varepsilon}(M)} \leq C
\]

for some \(\mu > 0\).

Proof. Since \((B, \psi)\) is a solution of the vortex equations on \(\mathbb{R}^2\), we have

\[
\sum_{\rho=1}^{2} \nabla_{e^+_{\rho}} F_A(e^+_{\rho}, e^+_{\alpha}) + \frac{1}{2\varepsilon^2} (\phi D_A e^+_{\rho} \phi - \overline{\phi} D_A^* e^+_{\rho} \phi) = 0
\]

and

\[
\sum_{\rho=1}^{2} D_{A, e^+_{\rho}} D_{A, e^+_{\rho}} \phi + \frac{1}{2\varepsilon^2} (1 - |\phi|^2) \phi = 0.
\]
Furthermore, we have
\[
\sum_{\rho=1}^{2} \nabla_{e_{\rho}} F_A(e_{\rho}^1, e_i) + \frac{1}{2\varepsilon^2} (\phi \overline{D_A e_i} \phi - \overline{\phi} D_A e_i \phi)
\]
\[
= -\sum_{\rho=1}^{2} \nabla_{i} v_{\rho} \left( \sum_{\beta=1}^{2} \nabla_{e_{\beta}} F_A(e_{\beta}^1, e_{\rho}^1) + \frac{1}{2\varepsilon^2} (\phi \overline{D_A e_{\rho}^1} \phi - \overline{\phi} D_A e_{\rho}^1 \phi) \right)
\]
\[
= 0.
\]

From this it follows that
\[
\left\| \left( d^{*}g_{0} F_A - \frac{1}{2\varepsilon^2} (\phi \overline{D_A \phi} - \overline{\phi} D_A \phi), D_A^{*} D_A \phi - \frac{1}{2\varepsilon^2} (1 - |\phi|^2) \phi \right) \right\|_{C_{N,\varepsilon}(M)} \leq C.
\]

Here, \(g_0\) denotes the product metric on \(NS\), i.e.
\[
g_0(e_i, e_j) = \delta_{ij}
\]
\[
g_0(e_i, e_{j}^1) = 0
\]
\[
g_0(e_{\alpha}^1, e_{\beta}^1) = \delta_{\alpha\beta}.
\]

Let \(g\) be the pull-back of the Riemannian metric on \(M\) under the exponential map \(\exp : NS \rightarrow M\). Then the metric \(g\) satisfies an asymptotic expansion of the form
\[
g(e_i, e_j) = \delta_{ij} + 2 \sum_{\rho=1}^{2} h_{ij, \rho} y_{\rho} + O(|y|^2)
\]
\[
g(e_i, e_{j}^1) = O(|y|^2)
\]
\[
g(e_{\alpha}^1, e_{\beta}^1) = \delta_{\alpha\beta} + O(|y|^2),
\]

where \(h\) denotes the second fundamental form of \(S\). In particular, the volume form of \(g\) is related to the volume form of \(g_0\) by
\[
\left( \frac{\det g}{\det g_0} \right)^{\frac{1}{2}} = 1 + H_{\rho} y_{\rho} + O(|y|^2),
\]
where \(H\) is the mean curvature vector of \(S\). Since the mean curvature of \(S\) is 0, we obtain
\[
\left( \frac{\det g}{\det g_0} \right)^{\frac{1}{2}} = 1 + O(|y|^2).
\]

Thus, we conclude that
\[
\left\| \left( d^{*}F_A - \frac{1}{2\varepsilon^2} (\phi \overline{D_A \phi} - \overline{\phi} D_A \phi), D_A^{*} D_A \phi - \frac{1}{2\varepsilon^2} (1 - |\phi|^2) \phi \right) \right\|_{C_{N,\varepsilon}(M)} \leq C.
\]
This proves the assertion.
5 Estimates for the operator \( \mathbb{L}_\varepsilon = L_\varepsilon + T_\varepsilon T_\varepsilon^* \) in weighted Hölder spaces

Our aim in this section is to analyze the mapping properties of the linearized operator

\[
\mathbb{L}_\varepsilon : \Omega^1(\mathbb{R}^{n-2} \times \mathbb{R}^2, i\mathbb{R}) \oplus \Omega^0(\mathbb{R}^{n-2} \times \mathbb{R}^2, L) \rightarrow \Omega^1(\mathbb{R}^{n-2} \times \mathbb{R}^2, i\mathbb{R}) \oplus \Omega^0(\mathbb{R}^{n-2} \times \mathbb{R}^2, L).
\]

**Proposition 5.1.** Suppose that \((b,h) \in C_{\mu,\varepsilon}(M)\) is supported in the set \(\{p \in M : \text{dist}(p,S) \leq 2\delta\}\) and satisfies

\[
\int_{NS_x} \varepsilon^2 \sum_{\alpha=1}^4 \langle b(e^\perp_\alpha), A(w, e^\perp_\alpha) \rangle + \int_{NS_x} \langle h, D A, w \phi \rangle = 0
\]

for all \(x \in S\) and all \(w \in NS\). Then there exists a pair \((a,f) \in C_{\mu,\varepsilon}(M)\) which is supported in the region \(\{p \in M : \text{dist}(p,S) \leq 4\delta\}\) such that

\[
\| (a,f) \|_{C_{\mu,\varepsilon}^2(M)} \leq C \varepsilon^2 \| b \|_{C_{\mu,\varepsilon}(M)}
\]

and

\[
\| \mathbb{L}_\varepsilon(a,f) - (b,h) \|_{C_{\mu,\varepsilon}(M)} \leq C \left( \delta + \frac{\varepsilon}{\delta} + \frac{\varepsilon^2}{\delta^2} \right) \| (b,h) \|_{C_{\mu,\varepsilon}(M)}.
\]

**Proof.** Let \(\{\zeta^{(j)} : 1 \leq j \leq j_0\}\) be a partition of unity on \(S\) such that each function \(\zeta^{(j)}\) is supported in a ball \(B_{\delta}(p_j)\), and

\[
|\{1 \leq j \leq j_0 : x \in B_{4\delta}(p_j)\}| \leq C
\]

for all \(x \in S\) and some uniform constant \(C\). For each \(1 \leq j \leq j_0\), there exists a pair \((a^{(j)}, f^{(j)}) \in C_{\mu,\varepsilon}^2(M)\) which is supported in the region \(\{(x,y) \in NS : x \in B_{2\delta}(p_j), \|y\| \leq 4\delta\}\) such that

\[
\| (a^{(j)}, f^{(j)}) \|_{C_{\mu,\varepsilon}^2(M)} \leq C \varepsilon^2 \| (\zeta^{(j)} b, \zeta^{(j)} h) \|_{C_{\mu,\varepsilon}(M)}
\]

and

\[
\| \mathbb{L}_\varepsilon(a^{(j)}, f^{(j)}) - (\zeta^{(j)} b, \zeta^{(j)} h) \|_{C_{\mu,\varepsilon}(M)} \leq C \left( \delta + \frac{\varepsilon}{\delta} + \frac{\varepsilon^2}{\delta^2} \right) \| (\zeta^{(j)} b, \zeta^{(j)} h) \|_{C_{\mu,\varepsilon}(M)}.
\]

We now define

\[
(a, f) = \sum_{j=1}^{j_0} (a^{(j)}, f^{(j)}).
\]
Then we have the estimates
\[
\| (a, f) \|_{C^{2,\gamma}_{\mu,\epsilon}(M)} \leq C \sup_{1 \leq j \leq j_0} \| (a^{(j)}, f^{(j)}) \|_{C^{2,\gamma}_{\mu,\epsilon}(M)} \\
\leq C \varepsilon^2 \sup_{1 \leq j \leq j_0} \| (\zeta^{(j)} b, \zeta^{(j)} h) \|_{C^{\gamma}_{\mu,\epsilon}(M)} \\
\leq C \varepsilon^2 \| (b, h) \|_{C^{\gamma}_{\mu,\epsilon}(M)}
\]
and
\[
\| L_\varepsilon (a, f) - (b, h) \|_{C^{\gamma}_{\mu,\epsilon}(M)} \leq C \sup_{1 \leq j \leq j_0} \| L_\varepsilon (a^{(j)}, f^{(j)}) - (\zeta^{(j)} b, \zeta^{(j)} h) \|_{C^{\gamma}_{\mu,\epsilon}(M)} \\
\leq C \left( \delta + \frac{\varepsilon}{\delta} + \frac{\varepsilon^2}{\delta^2} \right) \sup_{1 \leq j \leq j_0} \| (\zeta^{(j)} b, \zeta^{(j)} h) \|_{C^{\gamma}_{\mu,\epsilon}(M)} \\
\leq C \left( \delta + \frac{\varepsilon}{\delta} + \frac{\varepsilon^2}{\delta^2} \right) \| (b, h) \|_{C^{\gamma}_{\mu,\epsilon}(M)}.
\]
This proves the assertion.

**Proposition 5.2.** For every \((b, h) \in C^{\gamma}_{\mu,\epsilon}(M)\), there exists a pair \((a, f) \in C^{2,\gamma}_{\mu,\epsilon}(M)\) such that
\[
\| (a, f) \|_{C^{2,\gamma}_{\mu,\epsilon}(M)} \leq C \varepsilon^2 \| (b, h) \|_{C^{\gamma}_{\mu,\epsilon}(M)}
\]
and
\[
\left( \nabla^* \nabla a + \frac{1}{\varepsilon^2} a, \nabla^* \nabla f + \frac{1}{\varepsilon^2} f \right) = (b, h).
\]

**Proof.** By Schauder estimates, it suffices to show that
\[
\sup e^{\frac{\mu \dist(p, \bar{S})}{\varepsilon}} (\varepsilon |a| + |f|) \leq C \sup e^{\frac{\mu \dist(p, \bar{S})}{\varepsilon^2}} \left( \varepsilon |\nabla^* \nabla a + \frac{1}{\varepsilon^2} a| + |\nabla^* \nabla f + \frac{1}{\varepsilon^2} f| \right).
\]
Suppose that there exists a sequence of positive real numbers \(\varepsilon_j\) and a sequence of pairs \((a^{(j)}, f^{(j)}) \in C^{2,\gamma}_{\mu,\epsilon}(M)\) such that
\[
\sup e^{\frac{\mu \dist(p, \bar{S})}{\varepsilon_j}} (\varepsilon_j |a^{(j)}| + |f^{(j)}|) = 1
\]
and
\[
\sup e^{\frac{\mu \dist(p, \bar{S})}{\varepsilon_j}} \left( \varepsilon_j |\nabla^* \nabla a^{(j)} + \frac{1}{\varepsilon_j^2} a^{(j)}| + |\nabla^* \nabla f^{(j)} + \frac{1}{\varepsilon_j^2} f^{(j)}| \right) \to 0.
\]
Then there exists a sequence of points \(p_j \in M\) such that
\[
e^{\frac{\mu \dist(p_j, \bar{S})}{\varepsilon_j}} (\varepsilon_j |a^{(j)}(p_j)| + |f^{(j)}(p_j)|) \geq \frac{1}{2}.
\]
After rescaling, we obtain a sequence of pairs \((\tilde{a}^{(j)}, \tilde{f}^{(j)})\) such that
\[
\sup e^{-\mu \text{dist}(p,p_j)} (|\tilde{a}^{(j)}| + |\tilde{f}^{(j)}|) \leq 1
\]
and
\[
\sup e^{-\mu \text{dist}(p,p_j)} (|\nabla^* \nabla \tilde{a}^{(j)} + \tilde{a}^{(j)}| + |\nabla^* \nabla \tilde{f}^{(j)} + \tilde{f}^{(j)}|) \to 0.
\]
Moreover, we have
\[
|\tilde{a}^{(j)}(p_j)| + |\tilde{f}^{(j)}(p_j)| \geq \frac{1}{2}.
\]
Taking the limit as \(j \to \infty\), we obtain a pair \((\tilde{a}, \tilde{f})\) such that
\[
\sup e^{-\mu \text{dist}(p,p_0)} (|\tilde{a}| + |\tilde{f}|) \leq 1
\]
and
\[
(\nabla^* \nabla \tilde{a} + \tilde{a}, \nabla^* \nabla \tilde{f} + \tilde{f}) = (b,h).
\]
If \(\mu\) is sufficiently small, we conclude that \((\tilde{a}, \tilde{f}) = 0\). This is a contradiction.

**Proposition 5.3.** Suppose that \((b,h) \in C_{\mu,\varepsilon}^{\gamma}(M)\) is supported in the region \(\{p \in M : \text{dist}(p,S) \geq \delta\}\). Then there exists a pair \((a,f) \in C_{\mu,\varepsilon}^{2,\gamma}(M)\) which is supported in the region \(\{p \in M : \text{dist}(p,S) \geq \frac{\delta}{2}\}\) such that
\[
\| (a,f) \|_{C_{\mu,\varepsilon}^{2,\gamma}(M)} \leq C \varepsilon^2 \| (b,h) \|_{C_{\mu,\varepsilon}^{\gamma}(M)}
\]
and
\[
\| L_\varepsilon(a,f) - (b,h) \|_{C_{\mu,\varepsilon}^{\gamma}(M)} \leq C \left( \frac{\varepsilon}{\delta} + \frac{\varepsilon^2}{\delta^2} + e^{-\frac{4\delta}{\varepsilon}} \right) \| (b,h) \|_{C_{\mu,\varepsilon}^{\gamma}(M)}.
\]
**Proof.** By Proposition 5.2, we can find a pair \((a,f)\) such that
\[
\| (a,f) \|_{C_{\mu,\varepsilon}^{2,\gamma}(M)} \leq C \varepsilon^2 \| (b,h) \|_{C_{\mu,\varepsilon}^{\gamma}(M)}
\]
and
\[
(\nabla^* \nabla a + \frac{1}{\varepsilon^2} a, \nabla^* \nabla f + \frac{1}{\varepsilon^2} f) = (b,h).
\]
Let \(\eta\) be a cut-off function such that \(\eta(p) = 0\) for \(\text{dist}(p,S) \leq \frac{\delta}{2}\), \(\eta(p) = 1\) for \(\text{dist}(p,S) \geq \delta\) and
\[
\sup \delta |\nabla \eta| + \sup \delta^2 |\nabla^2 \eta| \leq C.
\]
Then the pair \((\eta a, \eta f)\) is supported in the region \(\{ p \in M : \text{dist}(p, S) \geq \frac{\delta}{2} \}\) and satisfies

\[
\| L_{\varepsilon}(\eta a, \eta f) - (b, h) \|_{C^\gamma_{\mu, \varepsilon}(M)} \\
\leq \left\| L_{\varepsilon}(\eta a, \eta f) - \left( \nabla^* \nabla(\eta a) + \frac{1}{\varepsilon^2} \eta a, \nabla^* \nabla(\eta f) + \frac{1}{\varepsilon^2} \eta f \right) \right\|_{C^\gamma_{\mu, \varepsilon}(M)} \\
+ \left\| (\nabla^* \nabla(\eta a) - \eta \nabla^* \nabla a, \nabla^* \nabla(\eta f) - \eta \nabla^* \nabla f) \right\|_{C^\gamma_{\mu, \varepsilon}(M)} \\
\leq C \left( \frac{1}{\varepsilon^2} (e^{-\frac{\mu}{\delta}} + \frac{\varepsilon}{\delta} + \frac{\varepsilon^2}{\delta^2}) \right) \| (a, f) \|_{C^\gamma_{\mu, \varepsilon}(M)} \\
\leq C \left( e^{-\frac{\mu}{\delta}} + \frac{\varepsilon}{\delta} + \frac{\varepsilon^2}{\delta^2} \right) \| (b, h) \|_{C^\gamma_{\mu, \varepsilon}(M)}.
\]

This proves the assertion.

In the following, we will choose \(\delta = \varepsilon^{1\over 2\gamma} \). Let \(\kappa\) be a cut-off function such that \(\kappa(p) = 1\) for \(\text{dist}(p, S) \leq \varepsilon^{1\over 2\gamma}\) and \(\kappa(p) = 0\) for \(\text{dist}(p, S) \geq 2\varepsilon^{1\over 2\gamma}\).

Let \(E^{k, \gamma}_{\mu, \varepsilon}(M)\) be the set of all pairs \((b, h) \in \Omega^1(M, i\mathbb{R}) \oplus \Omega^0(M, L)\) such that \((b, h) \in C^{k, \gamma}_{\mu, \varepsilon}(M)\) and

\[
\int_{NS_x} \varepsilon^2 \kappa \sum_{\alpha=1}^4 \langle b(e_\alpha^\perp), F_A(w, e_\alpha^\perp) \rangle + \int_{NS_x} \kappa \langle h, D_{A,w} \phi \rangle = 0
\]

for all \(x \in S\) and \(w \in NS_x\).

We denote by \(I - P\) the fibrewise projection from \(C^\gamma_{\mu, \varepsilon}(M)\) to the subspace \(E^{0}_\gamma(M)\). Hence, for every pair \((b, h)\) there exists a normal vector field \(w\) such that

\[
P(b, h) = (F_A(w, \cdot), D_{A,w} \phi).
\]

Let \(\Pi\) be the linear operator which assigns to every pair \((b, h)\) the vector field

\[
\Pi(b, h) = w.
\]

It is not difficult to show that

\[
\| \Pi(b, h) \|_{C^\gamma(S)} \leq C \varepsilon^{1-\gamma} \| (b, h) \|_{C^\gamma_{\mu, \varepsilon}(M)}
\]

and

\[
\| P(b, h) \|_{C^\gamma_{\mu, \varepsilon}(M)} \leq C \| (b, h) \|_{C^\gamma_{\mu, \varepsilon}(M)}.
\]
Proposition 5.4. For every pair \((b,h) \in \mathcal{E}_{\mu,\varepsilon}(M)\) there exists a pair \((a,f) \in \mathcal{C}_{\mu,\varepsilon}^{2,\gamma}(M)\) such that
\[
\|(a,f)\|_{\mathcal{C}_{\mu,\varepsilon}^{2,\gamma}(M)} \leq C \varepsilon^2 \|(b,h)\|_{\mathcal{C}_{\mu,\varepsilon}^{\gamma}(M)}
\]
and
\[
\|\mathbb{L}_\varepsilon(a,f) - (b,h)\|_{\mathcal{C}_{\mu,\varepsilon}^{\gamma}(M)} \leq C \varepsilon^{\frac{1}{2}} \|(b,h)\|_{\mathcal{C}_{\mu,\varepsilon}^{\gamma}(M)}.
\]

Proof. Apply Proposition 5.1 to \((\kappa b,\kappa h)\) and Proposition 5.3 to \(((1 - \kappa) b, (1 - \kappa) h)\).

Proposition 5.5. For every \((b,h) \in \mathcal{E}_{\mu,\varepsilon}(M)\) there exists a pair \((a,f) \in \mathcal{C}_{\mu,\varepsilon}^{2,\gamma}(M)\) such that
\[
\|(a,f)\|_{\mathcal{C}_{\mu,\varepsilon}^{2,\gamma}(M)} \leq C \varepsilon^2 \|(b,h)\|_{\mathcal{C}_{\mu,\varepsilon}^{\gamma}(M)}
\]
and
\[
(I - P) \mathbb{L}_\varepsilon(a,f) = (b,h).
\]
Furthermore, the pair \((a,f)\) satisfies the estimate
\[
\|\Pi \mathbb{L}_\varepsilon(a,f)\|_{\mathcal{C}_{\gamma}(S)} \leq C \varepsilon^{\frac{3}{2}} \|(b,h)\|_{\mathcal{C}_{\mu,\varepsilon}^{\gamma}(M)}.
\]

Proof. By Proposition 5.4, there exists an operator \(S : \mathcal{E}_{\mu,\varepsilon}^{\gamma}(M) \to \mathcal{C}_{\mu,\varepsilon}^{2,\gamma}(M)\) such that
\[
\|S(b,h)\|_{\mathcal{C}_{\mu,\varepsilon}^{2,\gamma}(M)} \leq C \varepsilon^2 \|(b,h)\|_{\mathcal{C}_{\mu,\varepsilon}^{\gamma}(M)}
\]
and
\[
\|\mathbb{L}_\varepsilon S(b,h) - (b,h)\|_{\mathcal{C}_{\mu,\varepsilon}^{\gamma}(M)} \leq C \varepsilon^{\frac{3}{2}} \|(b,h)\|_{\mathcal{C}_{\mu,\varepsilon}^{\gamma}(M)}.
\]
This implies
\[
\|\Pi \mathbb{L}_\varepsilon S(b,h)\|_{\mathcal{C}_{\gamma}(S)} = \|\Pi (\mathbb{L}_\varepsilon S(b,h) - (b,h))\|_{\mathcal{C}_{\gamma}(S)} \leq C \varepsilon^{\frac{3}{2} - \gamma} \|(b,h)\|_{\mathcal{C}_{\mu,\varepsilon}^{\gamma}(M)}.
\]
From this it follows that
\[
\|(I - P) \mathbb{L}_\varepsilon S(b,h) - (b,h)\|_{\mathcal{C}_{\mu,\varepsilon}^{\gamma}(M)} \leq C \varepsilon^{\frac{3}{2}} \|(b,h)\|_{\mathcal{C}_{\mu,\varepsilon}^{\gamma}(M)}.
\]
Therefore, the operator \((I - P) \mathbb{L}_\varepsilon S : \mathcal{E}_{\mu,\varepsilon}^{\gamma}(M) \to \mathcal{E}_{\mu,\varepsilon}^{\gamma}(M)\) is invertible. Hence, if we define \((a,f) = S \left[(I - P) \mathbb{L}_\varepsilon S\right]^{-1}(b,h)\), then \((a,f)\) satisfies
\[
\|(a,f)\|_{\mathcal{C}_{\mu,\varepsilon}^{2,\gamma}(M)} \leq C \varepsilon^2 \|b\|_{\mathcal{C}_{\mu,\varepsilon}^{\gamma}(M)}
\]
and
\[
(I - P) \mathbb{L}_\varepsilon(a,f) = (b,h).
\]
This proves the assertion.
6 From approximate to exact solutions

Proposition 6.1. Let \( v \) be a normal vector field along \( S \), and let \( A \) be the approximate solution associated to \( v \). Then there exists a pair \((\tilde{A}, \tilde{\phi}) = (A + a, \phi + f)\) such that \((a, f) \in C^{2, \gamma}_{\mu, \varepsilon}(M)\),

\[
\|(a, f)\|_{C^{2, \gamma}_{\mu, \varepsilon}(M)} \leq C \varepsilon^2
\]

and

\[
(1 - \mathcal{P})(d^* F_{\tilde{A}} - \frac{1}{2\varepsilon^2} (\tilde{\phi} D_A \tilde{\phi} - \tilde{\phi} D_A \tilde{\phi}) + \frac{1}{\varepsilon} du, D_A^* D_A \tilde{\phi} - \frac{1}{2\varepsilon^2} (1 - |\tilde{\phi}|^2) \tilde{\phi} - \frac{1}{\varepsilon} \tilde{\phi} u) = 0,
\]

where

\[
u = \varepsilon d^* a + \frac{1}{2\varepsilon} (\phi \tilde{f} - \tilde{\phi} f).
\]

Furthermore, the pair \((a, f)\) satisfies the estimate

\[
\|\Pi L_{\varepsilon}(a, f)\|_{C^{1}(S)} \leq C \varepsilon^{\frac{1}{4}}.
\]

Proof. We use the identity

\[
\left(d^* F_{\tilde{A}} - \frac{1}{2\varepsilon^2} (\tilde{\phi} D_A \tilde{\phi} - \tilde{\phi} D_A \tilde{\phi}) + \frac{1}{\varepsilon} du, D_A^* D_A \tilde{\phi} - \frac{1}{2\varepsilon^2} (1 - |\tilde{\phi}|^2) \tilde{\phi} - \frac{1}{\varepsilon} \tilde{\phi} u\right) = 0,
\]

Here, the remainder term \(Q(a, f)\) is at least quadratic in \((a, f)\). This implies

\[
\|Q(a, f)\|_{C^{2, \gamma}_{\mu, \varepsilon}(M)} \leq C \varepsilon^{-2} \|(a, f)\|^2_{C^{2, \gamma}_{\mu, \varepsilon}(M)}.
\]

Hence, if we define

\[
u = T_{\varepsilon}^*(a, f),
\]

then we obtain

\[
\left(d^* F_{\tilde{A}} - \frac{1}{2\varepsilon^2} (\tilde{\phi} D_A \tilde{\phi} - \tilde{\phi} D_A \tilde{\phi}) + \frac{1}{\varepsilon} du, D_A^* D_A \tilde{\phi} - \frac{1}{2\varepsilon^2} (1 - |\tilde{\phi}|^2) \tilde{\phi} - \frac{1}{\varepsilon} \tilde{\phi} u\right) = 0,
\]

According to Proposition 5.5, there exists an operator \(G : E^{\gamma}_{\mu, \varepsilon}(M) \to C^{2, \gamma}_{\mu, \varepsilon}(M)\) such that

\[
\|G(b, h)\|_{C^{2, \gamma}_{\mu, \varepsilon}(M)} \leq C \varepsilon^2 \|(b, h)\|_{C^{2, \gamma}_{\mu, \varepsilon}(M)}
\]
and

\[(I - \mathcal{P}) \mathbb{L}_\varepsilon \mathcal{G} = I.\]

We now define a mapping \(\Phi : C^{2,\gamma}_{\mu,\varepsilon}(M) \to C^{2,\gamma}_{\mu,\varepsilon}(M)\) by

\[
\Phi(a, f) = -G(I - \mathcal{P}) \left( d^* F_A - \frac{1}{2 \varepsilon^2} (\phi D_A \phi - \bar{\phi} D_A \phi), D_A^* D_A \phi - \frac{1}{2 \varepsilon^2} (1 - |\phi|^2) \phi \right) - G(I - \mathcal{P}) Q(a, f).
\]

Then we have the estimate

\[
\|\Phi(a, f)\|_{C^{2,\gamma}_{\mu,\varepsilon}(M)} \leq C \varepsilon^2 \left\| (I - \mathcal{P}) \left( d^* F_A - \frac{1}{2 \varepsilon^2} (\phi D_A \phi - \bar{\phi} D_A \phi), D_A^* D_A \phi - \frac{1}{2 \varepsilon^2} (1 - |\phi|^2) \phi \right) \right\|_{C^{2,\gamma}_{\mu,\varepsilon}(M)}
\]

\[
+ C \varepsilon^2 \| (I - \mathcal{P}) Q(a, f) \|_{C^{2,\gamma}_{\mu,\varepsilon}(M)}
\]

\[
\leq C \varepsilon^2 \left\| (d^* F_A - \frac{1}{2 \varepsilon^2} (\phi D_A \phi - \bar{\phi} D_A \phi), D_A^* D_A \phi - \frac{1}{2 \varepsilon^2} (1 - |\phi|^2) \phi) \right\|_{C^{2,\gamma}_{\mu,\varepsilon}(M)}
\]

\[
+ C \varepsilon^2 \| Q(a, f) \|_{C^{2,\gamma}_{\mu,\varepsilon}(M)}
\]

\[
\leq C \varepsilon^2
\]

for all \((a, f) \in C^{2,\gamma}_{\mu,\varepsilon}(M)\) satisfying

\[
\|a\|_{C^{2,\gamma}_{\mu,\varepsilon}(M)} \leq \varepsilon^3.
\]

Moreover, we have

\[
\|
\Phi(a, f) - \Phi(a', f')\|_{C^{2,\gamma}_{\mu,\varepsilon}(M)} \leq C \varepsilon^2 \| Q(a, f) - Q(a', f') \|_{C^{2,\gamma}_{\mu,\varepsilon}(M)}
\]

\[
\leq C \varepsilon^2 \|(a, f) - (a', f')\|_{C^{2,\gamma}_{\mu,\varepsilon}(M)}
\]

for all \((a, f), (a', f') \in C^{2,\gamma}_{\mu,\varepsilon}(M)\) satisfying

\[
\|(a, f)\|_{C^{2,\gamma}_{\mu,\varepsilon}(M)}, \|(a', f')\|_{C^{2,\gamma}_{\mu,\varepsilon}(M)} \leq \varepsilon^3.
\]

Hence, it follows from the contraction mapping principle that there exists a pair \((a, f) \in C^{2,\gamma}_{\mu,\varepsilon}(M)\) such that

\[
\|(a, f)\|_{C^{2,\gamma}_{\mu,\varepsilon}(M)} \leq C
\]

and

\[
\Phi(a, f) = (a, f).
\]
From this it follows that
\[ G(I-P) \left( d^* F_A - \frac{1}{2\varepsilon^2} (\phi D_A \phi - \overline{\phi} D_A \phi), D_A^* D_A \phi - \frac{1}{2\varepsilon^2} (1 - |\phi|^2) \phi \right) \]
\[ + (a, f) + G(I-P) Q(a, f) = 0, \]

hence
\[ (I-P) \left( d^* F_A - \frac{1}{2\varepsilon^2} (\phi D_A \phi - \overline{\phi} D_A \phi), D_A^* D_A \phi - \frac{1}{2\varepsilon^2} (1 - |\phi|^2) \phi \right) \]
\[ + (I-P) L_\varepsilon(a, f) + (I-P) Q(a, f) = 0. \]

Thus, we conclude that
\[ (I-P) \left( d^* F_{\tilde{A}} - \frac{1}{2\varepsilon^2} (\phi D_{\tilde{A}} \phi - \overline{\phi} D_{\tilde{A}} \phi) + \frac{1}{\varepsilon} du, D_{\tilde{A}}^* D_{\tilde{A}} \tilde{\phi} - \frac{1}{2\varepsilon^2} (1 - |\tilde{\phi}|^2) \tilde{\phi} - \frac{1}{\varepsilon} \tilde{\phi} u \right) \]
\[ = 0. \]

This proves the assertion.

**Proposition 6.2.** If
\[ \mathcal{P} \left( d^* F_{\tilde{A}} - \frac{1}{2\varepsilon^2} (\phi D_{\tilde{A}} \phi - \overline{\phi} D_{\tilde{A}} \phi) + \frac{1}{\varepsilon} du, D_{\tilde{A}}^* D_{\tilde{A}} \tilde{\phi} - \frac{1}{2\varepsilon^2} (1 - |\tilde{\phi}|^2) \tilde{\phi} - \frac{1}{\varepsilon} \tilde{\phi} u \right) = 0, \]

then \((\tilde{A}, \tilde{\phi})\) is a solution of the Ginzburg-Landau equations.

**Proof.** It follows from the definition of the pair \((\tilde{A}, \tilde{\phi})\) that
\[ d^* F_{\tilde{A}} = \frac{1}{2\varepsilon^2} (\phi D_{\tilde{A}} \phi - \overline{\phi} D_{\tilde{A}} \phi) - \frac{1}{\varepsilon} du \]

and
\[ D_{\tilde{A}}^* D_{\tilde{A}} \tilde{\phi} = \frac{1}{2\varepsilon^2} (1 - |\tilde{\phi}|^2) \tilde{\phi} + \frac{1}{\varepsilon} \tilde{\phi} u, \]

where \(u\) satisfies \(\overline{u} = -u\). From this it follows that
\[ 0 = -\varepsilon d^* d^* F_{\tilde{A}} \]
\[ = -\frac{1}{2\varepsilon} (\phi D_{\tilde{A}}^* D_{\tilde{A}} \phi - \overline{\phi} D_{\tilde{A}}^* D_{\tilde{A}} \phi) + d^* du \]
\[ = -\frac{1}{2\varepsilon^2} (\phi \phi u - \overline{\phi} \phi u) + d^* du \]
\[ = \frac{1}{\varepsilon^2} |\tilde{\phi}|^2 u + d^* du. \]

Thus, we conclude that \(u = 0\). Hence, \((\tilde{A}, \tilde{\phi})\) is a solution of the Ginzburg-Landau equations.
7 The balancing condition

**Proposition 7.1.** Let $g_0$ be the product metric on the normal bundle $NS$ (cf. Section 4). Then we have the identity

\[
\Pi \left( d^{*g_0} F_A - \frac{1}{2\varepsilon^2} (\phi \bar{D}_A \phi - \bar{\phi} D_A \phi), D^{*g_0}_A \phi - \frac{1}{2\varepsilon^2} (1 - |\phi|^2) \phi \right) = \Delta v.
\]

**Proof.** Using the results from Section 4, we obtain

\[
\sum_{\beta=1}^{2} \nabla_{e^{-}} F_A(e_{\beta}, e_{\alpha}) + \frac{1}{2\varepsilon^2} (\phi \bar{D}_A e_{\beta} \phi - \bar{\phi} D_A e_{\beta} \phi)
\]

\[
= -\sum_{\rho=1}^{2} \nabla_i v_{\rho} \left( \sum_{\beta=1}^{2} \nabla_{e^{-}} F_A(e_{\beta}, e_{\rho}) + \frac{1}{2\varepsilon^2} (\phi \bar{D}_A e_{\rho} \phi - \bar{\phi} D_A e_{\rho} \phi) \right)
\]

\[
= 0.
\]

The Bianchi identity implies that

\[
\nabla_{e_{\beta}} F_A(e_{i}, e_{\alpha}) - \nabla_{e_{\alpha}} F_A(e_{i}, e_{\beta}) + \nabla_{e_{i}} F_A(e_{\alpha}, e_{\beta}) = 0.
\]

Furthermore, we have

\[
D_{A,e_{i}} D_{A,e_{\beta}} \phi - D_{A,e_{\alpha}} D_{A,e_{i}} \phi = F_A(e_{i}, e_{\alpha}^{-}) \phi.
\]

From this it follows that

\[
\sum_{i=1}^{n-2} \sum_{\alpha,\beta=1}^{2} \varepsilon^2 \left( \nabla_{e_{\beta}} F_A(e_{i}, e_{\beta}^{-}), F_A(e_{i}, e_{\alpha}^{-}) \right) - \frac{1}{2} \nabla_{e_{\beta}} F_A(e_{i}, e_{\beta}^{-}, F_A(e_{i}, e_{\beta}^{-})) w^\alpha
\]

\[
+ \nabla_{e_{i}} F_A(e_{i}, e_{\beta}^{-}), F_A(e_{\alpha}^{-}, e_{\beta}^{-})) w^\alpha
\]

\[
= \sum_{i=1}^{n-2} \sum_{\alpha=1}^{2} \left( -\frac{1}{2} \nabla_{e_{\alpha}} (D_{A,e_{i}} \phi, D_{A,e_{i}} \phi) + \nabla_{e_{i}} (D_{A,e_{i}} \phi, D_{A,e_{\alpha}} \phi) \right) w^\alpha
\]

\[
= \sum_{i=1}^{n-2} \sum_{\alpha,\beta=1}^{2} \varepsilon^2 \left( D_{A,e_{i}} F_A(e_{i}, e_{\beta}^{-}), F_A(e_{\alpha}^{-}, e_{\beta}^{-}) \right) w^\alpha
\]

\[
+ \sum_{i=1}^{n-2} \sum_{\alpha=1}^{2} \left( D_{A,e_{i}} D_{A,e_{i}} \phi, D_{A,e_{\alpha}} \phi \right) w^\alpha
\]

\[
= \varepsilon^2 \left( d^{*g_0} F_A - \frac{1}{2\varepsilon^2} (\phi \bar{D}_A \phi - \bar{\phi} D_A \phi), F_A(w, \cdot) \right)
\]

\[
+ \left( D^{*g_0}_A D_A \phi - \frac{1}{2\varepsilon^2} (1 - |\phi|^2) \phi, D_A w \phi \right).
\]
We now take advantage of the identity
\[
\int_{N^2} \sum_{\alpha, \beta=1}^2 \varepsilon^2 \langle F_A(e_i, e_{i_2}), F_A(e_{i_1}, e_{j_2}) \rangle w^\alpha
\]
\[+ \int_{N^2} \sum_{\alpha=1}^2 \langle D_A e_i \phi, D_A e_i \phi \rangle w^\alpha \]
\[= -\frac{1}{2} \langle \nabla_i v, w \rangle \left( 2 \int_{\mathbb{R}^2} \varepsilon^2 |F_B|^2 + \int_{\mathbb{R}^2} |D_B \psi|^2 \right) \]
\[= -\frac{1}{2} \langle \nabla_i v, w \rangle \left( \int_{\mathbb{R}^2} \varepsilon^2 |F_B|^2 + \int_{\mathbb{R}^2} |D_B \psi|^2 + \int_{\mathbb{R}^2} \frac{1}{4\varepsilon^2} (1 - |\psi|^2)^2 \right) \]
\[= -\pi \langle \nabla_i v, w \rangle. \]

Differentiating this identity, we obtain
\[
\int_{N^2} \sum_{i=1}^{n-2} \sum_{\alpha, \beta=1}^2 \varepsilon^2 \nabla_i \langle F_A(e_i, e_{i_2}), F_A(e_{i_1}, e_{j_2}) \rangle w^\alpha
\]
\[+ \int_{N^2} \sum_{i=1}^{n-2} \sum_{\alpha=1}^2 \nabla_i \langle D_A e_i \phi, D_A e_i \phi \rangle w^\alpha \]
\[= \sum_{i=1}^{n-2} \nabla_i \int_{N^2} \sum_{\alpha, \beta=1}^2 \varepsilon^2 \langle F_A(e_i, e_{i_2}), F_A(e_{i_1}, e_{j_2}) \rangle w^\alpha
\]
\[+ \int_{N^2} \sum_{i=1}^{n-2} \sum_{\alpha=1}^2 \nabla_i \langle D_A e_i \phi, D_A e_i \phi \rangle w^\alpha \]
\[= \sum_{i=1}^{n-2} \nabla_i \langle \Delta v, w \rangle. \]

Thus, we conclude that
\[
\int_{N^2} \varepsilon^2 \left\langle d^{* g_0} F_A - \frac{1}{2\varepsilon^2} (\phi \overline{D_A \phi} - \overline{\phi} D_A \phi), F_A(w, \cdot) \right\rangle
\]
\[+ \int_{N^2} \left\langle D^{* g_0} D_A \phi - \frac{1}{2\varepsilon^2} (1 - |\phi|^2) \phi, D_A w \phi \right\rangle \]
\[= -\pi \langle \Delta v, w \rangle. \]
Proposition 7.2. The fibrewise projection of the error term to the obstruction space satisfies the estimate

\[ \left\| \Pi \left( d^* F_A - \frac{1}{2\varepsilon^2} (\phi \overline{D_A \phi} - \overline{\phi D_A \phi}), D_A^* D_A \phi - \frac{1}{2\varepsilon^2} (1 - |\phi|^2) \phi \right) - \left( \Delta v_\rho + \sum_{i,j=1}^{n-2} \sum_{\rho,\sigma=1}^{2} h_{ij,\rho} h_{ij,\sigma} v_\sigma + \sum_{i=1}^{n-2} \sum_{\rho,\sigma=1}^{2} R_{i\rho\sigma i} v_\sigma \right) \right\|_{C^\gamma(S)} \leq C \varepsilon^2. \]

Proof. The Riemannian metric satisfies the asymptotic expansion

\[ g(e_i, e_j) = \delta_{ij} + 2 \sum_{\rho=1}^{n-2} h_{ij,\rho} y_\rho + \sum_{k=1}^{n-2} \sum_{\rho,\sigma=1}^{2} h_{ik,\rho} h_{jk,\sigma} y_\rho y_\sigma - \sum_{\rho,\sigma=1}^{2} R_{i\rho\sigma j} y_\rho y_\sigma + O(|y|^3) \]

Using this asymptotic expansion, one can deduce Proposition 7.2 from Proposition 7.1. The details are left to the reader.

By Proposition 6.2, it suffices to find a normal vector field \( v \) such that

\[ \Pi \left( d^* F_{\tilde{A}} - \frac{1}{2\varepsilon^2} (\phi \overline{D_{\tilde{A}} \phi} - \overline{\phi D_{\tilde{A}} \phi} + \frac{1}{\varepsilon} du, D_{\tilde{A}}^* D_{\tilde{A}} \tilde{\phi} - \frac{1}{2\varepsilon^2} (1 - |\tilde{\phi}|^2) \tilde{\phi} - \frac{1}{\varepsilon} \tilde{\phi} u) \right) = 0. \]

To this end, we need the following result.

Proposition 7.3. The pair \( (\tilde{A}, \tilde{\phi}) \) satisfies

\[ \left\| \Pi \left( d^* F_{\tilde{A}} - \frac{1}{2\varepsilon^2} (\overline{\phi D_{\tilde{A}} \phi} - \phi \overline{D_{\tilde{A}} \phi}) + \frac{1}{\varepsilon} du, D_{\tilde{A}}^* D_{\tilde{A}} \tilde{\phi} - \frac{1}{2\varepsilon^2} (1 - |\tilde{\phi}|^2) \tilde{\phi} - \frac{1}{\varepsilon} \tilde{\phi} u \right) \right\|_{C^\gamma(S)} \leq C \varepsilon^2. \]

Proof. Using the estimate

\[ \|(a, f)\|_{C^2_{\gamma,\beta}(M)} \leq C \varepsilon^2, \]
we obtain
\[ \| \Pi Q(a, f) \|_{C^\gamma(S)} \leq C \varepsilon^{1-\gamma} \| Q(a, f) \|_{C^\beta_\gamma(M)} \leq C \varepsilon^{3-\gamma}. \]
Moreover, we have
\[ \| \Pi L_\varepsilon(a, f) \|_{C^\gamma(S)} \leq C \varepsilon^\delta. \]
The assertion follows now from Proposition 7.2.

**Proposition 7.4.** If \( S \) is non-degenerate, then we can choose the normal vector field \( v \) such that
\[
\Pi \left( d^* F_{\hat{A}} - \frac{1}{2\varepsilon^2} \left( \hat{\phi} D_A \hat{\phi} - \phi D_A \phi \right) + \frac{1}{\varepsilon} du, D_A^* D_A \hat{\phi} - \frac{1}{2\varepsilon^2} (1 - |\hat{\phi}|^2) \hat{\phi} - \frac{1}{\varepsilon} \hat{\phi} u \right) = 0.
\]

Therefore, the pair \( (\hat{A}, \hat{\phi}) \) corresponding to this normal vector field \( v \) is a solution of the Ginzburg-Landau equations.

**Proof.** Let \( J \) be the Jacobi-operator of the submanifold \( S \). According to Proposition 7.3, we may write
\[
\Pi \left( d^* F_{\hat{A}} - \frac{1}{2\varepsilon^2} \left( \hat{\phi} D_A \hat{\phi} - \phi D_A \phi \right) + \frac{1}{\varepsilon} du, D_A^* D_A \hat{\phi} - \frac{1}{2\varepsilon^2} (1 - |\hat{\phi}|^2) \hat{\phi} - \frac{1}{\varepsilon} \hat{\phi} u \right) = Jv + R(v),
\]
where \( \| R(v) \|_{C^\gamma(S)} \leq C \varepsilon^\delta \) for \( \| v \|_{C^{2,\gamma}(S)} \leq \varepsilon \). Hence, the mapping \( -J^{-1} R \) maps a ball of radius \( \varepsilon \) in the Banach space \( C^{2,\gamma}(S) \) into a ball of radius \( C \varepsilon^\delta \) in \( C^{2,\gamma}(S) \). Unfortunately, Schauder’s fixed point theorem cannot be applied, since the mapping \( -J^{-1} R \) need not be compact. To overcome this problem, we use an idea of F. Pacard and M. Ritoré [25]. Using an appropriate sequence of smoothing operators, we may approximate the mapping \( -J^{-1} R \) by a sequence of compact mappings. Each of these mappings has a fixed point in \( C^{2,\gamma}(S) \) by Schauder’s fixed point theorem. Taking limits, we obtain a fixed point of the original mapping \( -J^{-1} R \) in the Banach space \( C^{2,\gamma}(S) \). Hence, there exists a normal vector field \( v \in C^{2,\gamma}(S) \) such that \( Jv + R(v) = 0 \). Hence, the pair \( (\hat{A}, \hat{\phi}) \) corresponding to that choice of the vector field \( v \) satisfies
\[
\Pi \left( d^* F_{\hat{A}} - \frac{1}{2\varepsilon^2} \left( \hat{\phi} D_A \hat{\phi} - \phi D_A \phi \right) + \frac{1}{\varepsilon} du, D_A^* D_A \hat{\phi} - \frac{1}{2\varepsilon^2} (1 - |\hat{\phi}|^2) \hat{\phi} - \frac{1}{\varepsilon} \hat{\phi} u \right) = 0.
\]
This concludes the proof.
References

[1] A. Bahri and J.M. Coron, On a nonlinear elliptic equation involving the critical Sobolev exponent: The effect of the topology of the domain, Comm. Pure Appl. Math. 41, 253-290 (1988)

[2] F. Bethuel, H. Brezis, and F. Hélein, Ginzburg-Landau vortices, Progress in Nonlinear Differential Equations and their Applications, vol. 13, Birkhäuser, Boston (1994)

[3] F. Bethuel, H. Brezis, and G. Orlandi, Asymptotics for the Ginzburg-Landau equation in arbitrary dimensions, J. Funct. Anal. 186, 432-520 (2001)

[4] S. Bradlow, Vortices in holomorphic line bundles over closed Kähler manifolds, Comm. Math. Phys. 135, 1-17 (1990)

[5] S. Bradlow, Special metrics and stability for holomorphic bundles with global sections, J. Diff. Geom. 33, 169-213 (1991)

[6] J. Colliander and R. Jerrard, Vortex dynamics for the Ginzburg-Landau-Schrödinger equation, IMRN 7, 333-358 (1998)

[7] T. Ilmanen, Convergence of the Allen-Cahn equation to Brakke’s motion by mean curvature, J. Diff. Geom. 38, 417-461 (1993)

[8] A. Jaffe and C. H. Taubes, Vortices and monopoles, Progress in Physics, vol. 2, Birkhäuser, Boston (1980)

[9] R. Jerrard, Vortex dynamics for the Ginzburg-Landau wave equation, Calc. Var. 9, 1-30 (1999)

[10] R. Jerrard and H. M. Soner, The Jacobian and the Ginzburg-Landau energy, Calc. Var. 14, 151-191 (2002)

[11] R. Jerrard and H. M. Soner, Limiting behavior of the Ginzburg-Landau functional, J. Funct. Anal. 192, 524-561 (2002)

[12] N. Kapouleas, Complete constant mean curvature surfaces in Euclidean three-space, Ann. of Math. 131, 239-330 (1990)

[13] N. Kapouleas, Compact constant mean curvature surfaces in Euclidean three-space, J. Diff. Geom. 33, 683-715 (1991)

[14] N. Korevaar, R. Mazzeo, F. Pacard, and R. Schoen, Refined asymptotics for constant scalar curvature metrics with isolated singularities, Invent. Math. 135, 233-272 (1999)
[15] R. Kusner, R. Mazzeo, and D. Pollack, *The moduli space of complete embedded constant mean curvature surfaces*, Geom. Funct. Anal. 6, 120-137 (1996)

[16] F.-H. Lin, *Some dynamical properties of Ginzburg-Landau vortices*, Comm. Pure Appl. Math. 49, 323-359 (1996)

[17] F.-H. Lin, *Complex Ginzburg-Landau equations and dynamics of vortices, filaments, and codimension-2 submanifolds*, Comm. Pure Appl. Math. 51, 385-441 (1998)

[18] F.-H. Lin and T. Rivière, *Complex Ginzburg-Landau equations in high dimensions and codimension two area minimizing currents*, J. Eur. Math. Soc. 1, 237-311 (1999)

[19] F.-H. Lin and J. Xin, *On the incompressible fluid limit and the vortex motion law of the nonlinear Schrödinger equations*, Comm. Math. Phys. 200, 249-274 (1999)

[20] F.-H. Lin and P. Zhang, *On the hydrodynamic limit of Ginzburg-Landau wave vortices*, Comm. Pure Appl. Math. 55, 831-856 (2002)

[21] R. Mazzeo and F. Pacard, *A construction of singular solutions for a semilinear elliptic equation using asymptotic analysis*, J. Diff. Geom. 44, 331-370 (1996)

[22] R. Mazzeo and F. Pacard, *Constant mean curvature surfaces with Delaunay ends*, Comm. Anal. Geom. 9, 169-237 (2001)

[23] R. Mazzeo, F. Pacard and D. Pollack, *Connected sums of constant mean curvature surfaces in Euclidean 3-space*, J. Reine Angew. Math. 536, 115-165 (2001)

[24] R. Mazzeo and N. Smale, *Conformally flat metrics of constant positive scalar curvature on subdomains of the sphere*, J. Diff. Geom. 34, 581-621 (1991)

[25] F. Pacard and M. Ritoré, *From constant mean curvature hypersurfaces to the gradient theory of phase transitions*, preprint (2003)

[26] F. Pacard and T. Rivière, *Linear and Nonlinear Aspects of Vortices. The Ginzburg-Landau Model*, Progress in Nonlinear Differential Equations and their Applications, vol. 39, Birkhäuser, Boston (2000)

[27] C. H. Taubes, *Arbitrary N-vortex solutions to the first order Ginzburg-Landau equations*, Comm. Math. Phys. 72, 277-292 (1980)

[28] C. H. Taubes, *SW → Gr: from the Seiberg-Witten equations to pseudo-holomorphic curves*, J. Amer. Math. Soc. 9, 845-918 (1996)
[29] C. H. Taubes, \(Gr \Rightarrow SW: \) from pseudo-holomorphic curves to Seiberg-Witten solutions, J. Diff. Geom. 51, 203-334 (1999)

[30] G. Tian, Gauge theory and calibrated geometry, Ann. of Math. 151, 193-268 (2000)