Multigraded Betti numbers of some path ideals

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Abstract

We determine (multi)graded Betti numbers of path ideals of lines and star graphs.

1 Introduction

The path ideal of a directed graph was introduced by Conca and De Negri [7] and recently these ideals have been studied by many authors, see [1, 2, 3, 5, 6, 9, 11, 12]. In this paper we consider the path ideals of undirected graphs. In particular, we give formulas for Betti numbers of path ideals of lines and stars extending the work of [1].

2 Preliminaries

2.1 Simplicial complexes and homology

An abstract simplicial complex \( \Delta \) on a set of vertices \( V(\Delta) = \{v_1, ..., v_n\} \) is a collection of subsets of \( V \) such that \( \{v_i\} \in \Delta \) for all \( i \) and, \( F \in \Delta \) implies that all subsets of \( F \) are also in \( \Delta \). The elements of \( \Delta \) are called faces and the maximal faces under inclusion are called facets. If the facets \( F_1, ..., F_q \) generate \( \Delta \), we write \( \Delta = \langle F_1, ..., F_q \rangle \) or \( \text{Facets}(\Delta) = \{F_1, ..., F_q\} \).

A face \( \{v_1, v_2, ..., v_n\} - \{v_{i_1}, ..., v_{i_s}\} \) will be denoted by \( \{v_1, ..., \hat{v}_{i_1}, ..., \hat{v}_{i_s}, ..., v_n\} \) for \( i_1 < i_2 < ... < i_s \).

Two simplicial complexes \( \Delta \) and \( \Gamma \) are isomorphic if there is a bijection \( \varphi : V(\Delta) \to V(\Gamma) \) between their vertex sets such that \( F \) is a face of \( \Delta \) iff \( \varphi(F) \) is a face of \( \Gamma \).

Let \( \Delta \) and \( \Gamma \) be simplicial complexes which has no common vertices. Then the join of \( \Delta \) and \( \Gamma \) is the simplicial complex given by

\[
\Delta \ast \Gamma = \{ \delta \cup \gamma : \delta \in \Delta, \gamma \in \Gamma \}.
\]

A cone with apex \( v \) is a special join obtained by joining a simplicial complex \( \Delta \) with \( \{\emptyset, v\} \) where \( v \) is an element which is not in the vertex set of \( \Delta \). Equivalently, a simplicial complex is a cone with apex \( v \) if \( v \) is a member of every facet.

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For each integer $i$, the $k$-vector space $\tilde{H}_i(\Delta, k)$ is the $i^{th}$ reduced homology of $\Delta$ over $k$. For the sake of simplicity, we drop $k$ and write $\tilde{H}_i(\Delta)$ whenever we work on a fixed ground field $k$.

A simplex is a simplicial complex that contains all subsets of its nonempty vertex set. The boundary $\Sigma$ of a simplex $\Delta = \langle \{v_1, \ldots, v_n\} \rangle$ is obtained from $\Delta$ by removing the maximal face of $\Delta$. And, the homology groups of $\Sigma$ are given by

$$\tilde{H}_p(\Sigma, k) \cong \begin{cases} k & \text{if } p = n-2 \\ 0 & \text{otherwise} \end{cases} \quad (1)$$

The irrelevant complex $\{\emptyset\}$ has the homology groups

$$\tilde{H}_p(\{\emptyset\}, k) \cong \begin{cases} k & \text{if } p = -1 \\ 0 & \text{otherwise} \end{cases} \quad (2)$$

whereas the void complex $\{}$ has trivial reduced homology in all degrees.

A simplicial complex $\Delta$ is acyclic (over $k$) if $\tilde{H}_i(\Delta, k)$ is trivial for all $i$. Examples of acyclic complexes include cones and simplices.

The homology of two simplicial complexes is related to homology of their union and intersection by the Mayer–Vietoris long exact sequence:

**Theorem 2.1** (Corollary 6.4, [13]). Let $\Delta_1$ and $\Delta_2$ be two simplicial complexes. Then there is a long exact sequence

$$\cdots \rightarrow \tilde{H}_p(\Delta_1) \oplus \tilde{H}_p(\Delta_2) \rightarrow \tilde{H}_p(\Delta_1 \cup \Delta_2) \rightarrow \tilde{H}_{p-1}(\Delta_1 \cap \Delta_2) \rightarrow \tilde{H}_{p-1}(\Delta_1) \oplus \tilde{H}_{p-1}(\Delta_2) \rightarrow \cdots \quad (3)$$

where the homology can be taken over any field.

A particular case of Theorem 2.1 occurs when a simplicial complex $\Delta = \Delta_1 \cup \Delta_2$ is a union of two acyclic subcomplexes $\Delta_1$ and $\Delta_2$. In that case, the sequence (3) becomes

$$\cdots \rightarrow 0 \rightarrow \tilde{H}_p(\Delta_1 \cup \Delta_2) \rightarrow \tilde{H}_{p-1}(\Delta_1 \cap \Delta_2) \rightarrow 0 \rightarrow \cdots$$

whence $\tilde{H}_p(\Delta_1 \cup \Delta_2)$ and $\tilde{H}_{p-1}(\Delta_1 \cap \Delta_2)$ are isomorphic for all $p$. Since we will make frequent use of this specific case we state it separately as an immediate Corollary.

**Corollary 2.2.** If $\Delta_1$ and $\Delta_2$ are acyclic simplicial complexes then

$$\tilde{H}_p(\Delta_1 \cup \Delta_2) \cong \tilde{H}_{p-1}(\Delta_1 \cap \Delta_2)$$

for every $p$ and the homology can be taken over any field.
2.2 Graphs and resolutions

Let \( S = \mathbb{k}[x_1, \ldots, x_n] \) be the polynomial ring in \( n \) variables over a field \( \mathbb{k} \). Given a minimal multigraded free resolution

\[
0 \rightarrow \bigoplus_{m \in \mathbb{N}^n} S(-m)^{b_{i,m}(I)} \xrightarrow{\partial_i} \bigoplus_{m \in \mathbb{N}^n} S(-m)^{b_{i+1,m}(I)} \xrightarrow{\partial_i} \bigoplus_{m \in \mathbb{N}^n} S(-m)^{b_{i+2,m}(I)} \rightarrow \cdots \rightarrow \bigoplus_{m \in \mathbb{N}^n} S(-m)^{b_{1,m}(I)} \xrightarrow{\partial_1} \bigoplus_{m \in \mathbb{N}^n} S(-m)^{b_{0,m}(I)} \xrightarrow{\partial_0} I \rightarrow 0
\]

of \( I \), the associated ranks \( b_{i,m}(I) \) are called **multigraded Betti numbers** of \( I \). Graded and multigraded Betti numbers are related by the equation

\[
b_{i,j}(I) = \sum_{\deg(m) = j} b_{i,m}(I)
\]

where \( \deg(m) \) stands for the standard degree of \( m \) (i.e. \( \deg(x_1^{a_1} \cdots x_n^{a_n}) = a_1 + \cdots + a_n \)).

For a graph \( G \), the vertex and edge sets are denoted by \( V(G) \) and \( E(G) \) respectively. All graphs in this paper should be assumed simple meaning that loopless and without multiple edges. Two vertices \( u \) and \( v \) are adjacent to one another if \( \{u, v\} \) is an edge of \( G \). A vertex \( u \) of \( G \) is called an **isolated vertex** if it is not adjacent to any vertex of \( G \). We will say that \( G \) is of **size** \( e \) and of **order** \( n \) if it has \( e \) edges and \( n \) vertices. For two vertices \( u \) and \( v \) of \( G \), a **path** of length \( t - 1 \) from \( u \) to \( v \) is a sequence of \( t \geq 2 \) distinct vertices \( u = z_1, \ldots, z_t = v \) such that \( \{z_i, z_{i+1}\} \in E(G) \) for all \( i = 1, \ldots, t - 1 \). We will denote by \( L_n \) a line of order \( n \). Also \( C_n \) and \( S_n \) will be a cycle and a star graph of size \( n \) respectively.

**Example 2.3.** A graph \( G \) of order 4 which has the path ideals \( I_4(G) = (x_2x_1x_4x_3), I_3(G) = (x_2x_1x_4, x_2x_1x_3, x_1x_3x_4), I_2(G) = (x_1x_2, x_1x_3, x_1x_4, x_3x_4), I_1(G) = (x_1, x_2, x_3, x_4) \).
For a square-free monomial $m$ we denote by $G_m$ the induced subgraph of $G$ on the set of vertices that divide $m$.

Let $I = (m_1, ..., m_s)$ be a monomial ideal of $S$ which is minimally generated by the set of monomials $M = \{m_1, ..., m_s\}$. The Taylor simplex $\Theta$ of $I$ is a simplex on $s$ vertices which are labelled with the minimal generators of $I$. If $\tau = \{m_{i_1}, ..., m_{i_r}\}$ is a face of $\Theta$, then by $\text{lcm}(\tau)$ we mean $\text{lcm}(m_{i_1}, ..., m_{i_r})$. For any monomial $m$ in $S$,

$$\Theta \leq m = \{ \tau \in \Theta \mid \text{lcm}(\tau) \text{ divides } m \}$$

and

$$\Theta < m = \{ \tau \in \Theta \mid \text{lcm}(\tau) \text{ strictly divides } m \}$$

are subcomplexes of $\Theta$. Clearly we have the equation

$$\Theta < m = \bigcup_{x_i|m} \Theta \leq m$$

and every facet of $\Theta \leq m$ is of the form

$$F_i := V(\Theta \leq m) - \{ u \in M \mid x_i \text{ does not divide } u \}.$$

Therefore we have

$$F_i \in \text{Facets}(\Theta < m) \iff F_i \text{ is maximal in } \{ F_j \mid x_j \text{ divides } m \}. \quad (5)$$

The following Theorem will be our main tool to calculate Betti numbers in this paper.

**Theorem 2.4 (3).** Let $I$ be a monomial ideal of $S$ which is minimally generated by the monomials $m_1, ..., m_s$. Denote by $\Theta$ the Taylor simplex of $I$. For $i \geq 1$, the multigraded Betti numbers of $S/I$ are given by

$$b_{i,m}(S/I) = \begin{cases} \dim_k \overline{H}_{i-2}(\Theta \leq m; k) & \text{if } m \text{ divides } \text{lcm}(m_1, ..., m_s) \\ 0 & \text{otherwise.} \end{cases} \quad (6)$$

**Remark 2.5.** If $I = (m_1, ..., m_s)$ and $q = \deg \text{lcm}(m_1, ..., m_s)$ then for any $r > q$ we have $b_{i,r}(I) = 0$ for all $i$. Therefore we call the numbers $b_{i,q}(I), i \in \mathbb{Z}$ as the top grade Betti numbers.

**Remark 2.6.** Suppose that $\Delta$ is the Taylor simplex of $I(G)$ for some graph $G$. If the induced graph $G_m$ contains an isolated vertex, then $\Delta < m = \Delta$ is a simplex. So $b_{i,m}(S/I(G)) = 0$ for all $i$ by Theorem 2.4.

**Lemma 2.7 (3).** If $I_1, I_2, ..., I_N$ are square-free monomial ideals whose minimal generators contain no common variable and each $I_i$ has minimal generators whose least common multiple is of degree $q_i$, then

$$b_{i,q_1+...+q_N}(S/(I_1 + I_2 + ... + I_N)) = \sum_{u_1 + ... + u_N = i} b_{u_1,q_1}(S/I_1) ... b_{u_N,q_N}(S/I_N). \quad (7)$$

**Lemma 2.8.** If $m$ is a square-free monomial of degree $j$ and $t \geq 2$, then $b_{i,m}(S/I_t(G_m)) = b_{i,j}(S/I_t(G_m))$.

**Proof.** Proof is similar to Lemma 3.1 in [3].
3 Betti numbers of some path ideals

Definition 3.1. For any $n \geq t \geq 1$ the simplicial complex $\Omega^t_n$ on the set of vertices $\{1, ..., n\}$ is defined by

$$\text{Facets}(\Omega^t_n) = \{\{1, ..., i, i+1, ..., i+t-1, i+t, ..., n\} | i = 1, ..., n-t+1\}.$$

Example 3.2. For $n = 5$ and $t = 2$ the simplicial complex $\Omega^5_2$ has facets $\{1, 2, 3, 4, 5\}$, $\{1, 2, 3, 4, 5\}$, $\{1, 2, 3, 4, 5\}$ and, $\{1, 2, 3, 4, 5\}$.

Remark 3.3. For $n = t$ the simplicial complex $\Omega^t_t$ is the irrelevant complex $\{\emptyset\}$. If $t = 1$ then $\Omega^t_t$ coincides with the boundary of an $n-1$ dimensional simplex.

As the simplicial complex $\Omega^t_n$ will come up in the next sections, we study its homology groups.

Lemma 3.4. For $n \geq 2t+1$ we have $\tilde{H}_p(\Omega^t_n) \cong \tilde{H}_{p-2}(\Omega^{n-t-1}_t)$ Otherwise,

$$\tilde{H}_p(\Omega^t_n) \cong \begin{cases} \tilde{H}_p(\{\emptyset\}) & \text{if } n = t \\ \tilde{H}_{p-1}(\{\emptyset\}) & \text{if } n = t + 1 \\ 0 & \text{if } t + 2 \leq n \leq 2t. \end{cases}$$

Proof. The case $n = t$ is clear as $\Omega^t_t = \{\emptyset\}$. So we assume that $n > t$ and fix an index $p$. We write $\Omega^t_n = S \cup C$ where $S$ is the simplex on vertices $\{t+1, ..., n\}$ and $C$ is the cone generated by the facets of $\Omega^t_n$ that contain the vertex 1. Note that by Corollary 2.2 we have

$$\tilde{H}_p(\Omega^t_n) \cong \tilde{H}_{p-1}(S \cap C).$$

We consider the three cases left:

Case 1: If $n = t+1$ then $S \cap C$ is the irrelevant complex and we are done.

Case 2: If $t+2 \leq n \leq 2t$ then $S \cap C$ is a simplex whose maximal face is $\{t+2, ..., n\}$.

Case 3: If $n \geq 2t + 1$ then it is not hard to check that $S \cap C$ can be written as a union $S \cap C = S_1 \cup C_1$ where $S_1 = \{\{t+2, ..., n\}\}$ and, $C_1$ is the cone with apex $t+1$ such that

$$\text{Facets}(C_1) = \{\{t+1, ..., n\} - \{i, i+1, ..., i+t-1\} | i = t+2, ..., n-t+1\}.$$

Now observe that $S_1 \cap C_1 \cong \Omega^{n-t-1}_t$ and again by Corollary 2.2 we get $\tilde{H}_{p-1}(S \cap C) \cong \tilde{H}_{p-2}(S_1 \cap C_1) \cong \tilde{H}_{p-2}(\Omega^{n-t-1}_t)$ which completes the proof.

Theorem 3.5. The homology groups of $\Omega^t_n$ are given by

$$\tilde{H}_p(\Omega^t_n) \cong \begin{cases} \tilde{H}_{p+1-\frac{n}{t+1}}(\{\emptyset\}) & \text{if } n \equiv 0 \mod t + 1 \\ \tilde{H}_{p+2-\frac{2(n+1)}{t+1}}(\{\emptyset\}) & \text{if } n \equiv t \mod t + 1 \\ 0 & \text{otherwise.} \end{cases}$$
Proof. Follows by a straightforward induction using Lemma 3.4.

**Corollary 3.6.** The dimensions of reduced homologies of $\Omega^\mathbf{n}_t$ are independent of the ground field. And they are given by

$$\dim \widetilde{H}_p(\Omega^\mathbf{n}_t) = \begin{cases} 
\delta_{p+2, \frac{2n}{t+1}} & \text{if } n \equiv 0 \mod t + 1 \\
\delta_{p+3, \frac{2(n+1)}{t+1}} & \text{if } n \equiv t \mod t + 1 \\
0 & \text{otherwise.}
\end{cases} \quad (10)$$

Proof. Follows by Theorem 3.5 and Equation (2).

3.1 Lines and Cycles

Throughout this section let $\Delta$ be the Taylor simplex of $I_t(L_n)$ where $L_n$ is a line on vertices $x_1, ..., x_n$. If $n < t$ then there is no path on $L_n$ of length $t - 1$, and therefore $I_t(L_n) = 0$.

Let us assume $n \geq t$ then we have

$$\Delta = \langle x_i x_{i+1} ... x_{i+t-1} \mid i = 1, ..., n - t + 1 \rangle.$$  

For simplicity, we replace the label of a vertex $x_i x_{i+1} ... x_{i+t-1}$ with $i$ for all $i = 1, ..., n - t + 1$. Hence $\Delta$ can be viewed as a simplex with maximal face $\{1, 2, ..., n - t + 1\}$. Now we want to find $\Delta_{<m}$. Following the Equation (5), the maximal elements of

$$\{\{\hat{1}, 2, ..., n - t + 1\}, \{1, ..., n - t, n - \hat{t} + 1\}\}$$

$$\cup \{\{\hat{1}, ..., \hat{i}, \hat{i} + 1, ..., n - t + 1\} \mid i = 2, ..., t\}$$

$$\cup \{\{1, ..., i - 1, \hat{i}, \hat{i} + 1, ..., i + t - 1, i + t, ..., n - t + 1\} \mid i = 2, ..., n - 2t + 2\}$$

$$\cup \{\{1, ..., i - 1, \hat{i}, ..., n - \hat{t} + 1\} \mid i = n - 2t + 3, ..., n - t\}$$

give the facets of $\Delta_{<m}$. Therefore, if $n < 2t + 1$ then

$$\Delta_{<m} = \langle\{1, 2, ..., n - t + 1\}, \{1, ..., n - t, n - \hat{t} + 1\}\rangle. \quad (11)$$

And, if $n \geq 2t + 1$ we have the following equation.

$$\text{Facets}(\Delta_{<m}) = \{\{\hat{1}, 2, ..., n - t + 1\}, \{1, ..., n - t, n - \hat{t} + 1\}\}$$

$$\cup \{\{1, ..., i - 1, \hat{i}, \hat{i} + 1, ..., i + t - 1, i + t, ..., n - t + 1\} \mid i = 2, ..., n - 2t + 1\} \quad (12)$$

**Theorem 3.7** (Top grade Betti numbers of path ideals of lines). For all $i \geq 1$, and $n \geq 1$, we have

$$b_{i,n}(S/I_t(L_n)) = \begin{cases} 
\delta_{i, \frac{2n}{t+1}} & \text{if } n \equiv 0 \mod t + 1 \\
\delta_{i+1, \frac{2(n+2)}{t+1}} & \text{if } n \equiv t \mod t + 1 \\
0 & \text{otherwise.}
\end{cases} \quad (13)$$
Proof. First suppose that \( n < t \), then we know that \( I_t(L_n) = 0 \). As \( n \) cannot be 0 or \( t \mod t + 1 \) in this case we are done.

Now we assume that \( n \geq t \). Let \( m \) be the product of vertices of \( L_n \). By Equation (11) and Theorem 2.4 we have

\[
\dim_k H_{i-2}(\Delta_{<m}, k) = \dim_k H_{i-2}(\Delta_{<m}, k).
\]

We consider two cases:

Case 1: Suppose that \( n < 2t + 1 \). By Equation (11) we have

\[
\Delta_{<m} = \langle \{\hat{1}, 2, ..., n - t + 1\}, \{1, ..., n - t, n - t + 1\} \rangle.
\]

Then we have three cases to prove. If \( n = t \), then \( \Delta_{<m} = \{\emptyset\} \) and so that

\[
\dim_k H_{i-2}(\Delta_{<m}, k) = \dim_k H_{i-2}(\{\emptyset\}, k) = \delta_{i-2,-1}.
\]

Observe that \( \delta_{i-2,-1} = \delta_{i+1,1} \) for \( n = t \) which proves Equation (13) for this case. Now observe that if \( n > t \) then \( \Delta_{<m} \) is a union of two simplices

\[
\Delta_{<m} = \langle \{\hat{1}, 2, ..., n - t + 1\} \rangle \cup \langle \{1, ..., n - t, n - t + 1\} \rangle = S_1 \cup S_2.
\]

Hence by Corollary 2.2 \( \dim_k H_{i-2}(\Delta_{<m}, k) \cong \dim_k H_{i-3}(S_1 \cap S_2, k) \). If \( n = t + 1 \) then \( S_1 \cap S_2 \) is the irrelevant complex. Therefore we have

\[
\dim_k H_{i-3}(S_1 \cap S_2, k) \cong \dim_k H_{i-3}(\{\emptyset\}, k) = \delta_{i-3,-1}.
\]

Now we check that indeed \( \delta_{i-3,-1} = \delta_{i+1,1} \) for \( n = t + 1 \) and the proof follows for this case.

Next, if \( n \geq t + 1 \) then \( S_1 \cap S_2 \) is a simplex and has trivial reduced homology in all degrees.

Case 2: Suppose that \( n \geq 2t + 1 \). Then by Equation (12) we have \( \Delta_{<m} = S_1 \cup S_2 \cup \tilde{\Upsilon} \)

where \( \tilde{\Upsilon} = \langle \{1, ..., i - 1, \hat{i}, i + 1, ..., i + t - 1, i + t, ..., n - t + 1\} \mid i = 2, ..., n - 2t + 1 \rangle \), \( S_1 = \langle \{\hat{1}, 2, ..., n - t + 1\} \rangle \) and \( S_2 = \langle \{1, ..., n - t, n - t + 1\} \rangle \). Now we write \( \Delta_{<m} \) as a union \( \Delta_{<m} = S_1 \cup (S_2 \cup \tilde{\Upsilon}) \) where \( S_2 \cup \tilde{\Upsilon} \) is a cone with apex 1. By virtue of Corollary 2.2 we have

\[
\dim_k H_{i-2}(\Delta_{<m}, k) \cong \dim_k H_{i-3}(S_1 \cap (S_2 \cup \tilde{\Upsilon}), k).
\]

Now observe that \( S_1 \cap (S_2 \cup \tilde{\Upsilon}) = C \cup S_2 \) where \( C \) is the cone generated by the facets of \( S_1 \cap (S_2 \cup \tilde{\Upsilon}) \) that contain the vertex \( n - t + 1 \). Again by Corollary 2.2 we get

\[
\dim_k H_{i-3}(S_1 \cap (S_2 \cup \tilde{\Upsilon}), k) \cong \dim_k H_{i-4}(C \cap S_2, k).
\]

Note that \( C \cap S_2 \) is isomorphic to the simplicial complex \( \Omega^{|n-t|}_t \) and by Corollary 3.6 we have

\[
\dim H_{i-4}(\Omega^{|n-t|}_t) = \begin{cases} 
\delta_{i-2,\frac{2(n-t-1)}{2t+1}} & \text{if } n \equiv 0 \mod t + 1 \\
\delta_{i-1,\frac{2(n-t)}{2t+1}} & \text{if } n \equiv t \mod t + 1 \\
0 & \text{otherwise}
\end{cases}
\]

which agrees with the formula given in Equation (13).
Theorem 3.8 (Multigraded Betti numbers of path ideals of lines). Let \( t \geq 2 \) and \( m \) be a squarefree monomial of degree \( j \). Then the multigraded Betti number \( b_{i,m}(S/I_t(L_n)) = 1 \) if the induced graph \( (L_n)_m \) consists of a collection of disjoint lines that satisfy the following conditions:

(i) Each line is of order 0 or \( t \mod t + 1 \)

(ii) The number of lines of order \( t \mod t + 1 \) is equal to \( \frac{i(t+1)-2j}{1-t} \).

Otherwise, \( b_{i,m}(S/I_t(L_n)) = 0 \).

Proof. Let \((L_n)_m = \bigsqcup_{l=1}^{p} Q_l\) be a disjoint union of lines where each \( Q_l \) is a line of order \( v_l \).

We have

\[
b_{i,m}(S/I_t(L_n)) = b_{i,j}(S/I_t((L_n)_m)) \text{ by Lemma 2.8} = \sum_{u_1, \ldots + u_p = i} b_{u_1,v_1}(S/I_t(Q_1)) \ldots b_{u_p,v_p}(S/I_t(Q_p)) \text{ by Equation (7)}.\]

By Theorem 3.7 if one of \( Q_l \) is not of order 0 or \( t \mod t + 1 \) then the sum above is zero. So without loss of generality let us assume that \( Q_1, \ldots, Q_z \) are of order 0 mod \( t + 1 \) and \( Q_{z+1}, \ldots, Q_p \) are of order \( t \mod t + 1 \) for some \( 0 \leq z \leq p \). Again by Theorem 3.7 the sum above is equal to 1 if

\[
\sum_{l=1}^{z} \frac{2v_l}{t+1} + \sum_{l=z+1}^{p} \left( \frac{2v_l + 2}{l+1} - 1 \right) = i \tag{15}
\]

and zero otherwise. Observe that (15) holds iff \( p - z = \frac{i(t+1)-2j}{1-t} \) since \( v_1 + \ldots + v_p = j \).

Hence the result follows. \( \square \)

Corollary 3.9. If \( L \) is a line, \( b_{i,j}(S/I_t(L)) \) is the number of ways of choosing a collection of disjoint induced lines of \( L \) that satisfy the following conditions:

(i) The orders of the lines add up to \( j \)

(ii) Each line is of order 0 or \( t \mod t + 1 \)

(iii) The number of lines of order \( t \mod t + 1 \) is equal to \( \frac{i(t+1)-2j}{1-t} \).

Proof. Immediately follows by Theorem 3.8 and Equation (4). \( \square \)

Using the multigraded Betti numbers, we can calculate graded Betti numbers. The following result was also proved in [2].

Theorem 3.10 (Graded Betti numbers of path ideals of lines). For \( t \geq 2 \), the nonzero graded Betti numbers of \( S/I_t(L_n) \) are given by

\[
b_{i,j}(S/I_t(L_n)) = \binom{n-j+1}{\frac{i(t+1)-2j}{1-t}} \binom{n-j+\frac{j-t}{1-t}}{n-j}
\]

provided that \( n, i \) and \( j \) satisfy the following relations.
(i) \( n \geq j \)

(ii) \( j \geq t \left( \frac{i(t+1)-2j}{1-t} \right) \geq 0 \)

(iii) \( n - j \geq \frac{i(t+1)-2j}{1-t} - 1 \)

Otherwise, the graded Betti numbers are zero.

Proof. By Lemma 3.9 it is clear that \( b_{i,j}(S/I_c(L_n)) = 0 \) if the condition (i) or (ii) fails. So let us assume that (i) and (ii) hold.

Now suppose that we have chosen a collection of disjoint induced lines \( Q_1, ..., Q_p \) of \( L_n \) as in Lemma 3.9. Since the orders of \( Q_1, ..., Q_p \) add up to \( j \), we have \( j = |\cup_{k=1}^p V(Q_k)| \). Also as the number of lines of order \( t \mod t + 1 \) is equal to \( \frac{i(t+1)-2j}{1-t} \), at least \( t \left( \frac{i(t+1)-2j}{1-t} \right) \) vertices of \( \cup_{k=1}^p V(Q_k) \) belong to a line of order \( t \mod t + 1 \). Therefore at most \( j - t \left( \frac{i(t+1)-2j}{1-t} \right) = (1 + t) \left( \frac{i(t+1)-2j}{1-t} \right) \) vertices of \( \cup_{k=1}^p V(Q_k) \) belong to a line of order \( 0 \mod t + 1 \). Now it becomes clear that the problem of choosing a collection of disjoint induced lines of \( L_n \) that is described in Lemma 3.9 corresponds to the problem of ordering \( \frac{i(t+1)-2j}{1-t} \) many “\( t \)”s, \( \frac{i(t+1)-2j}{1-t} \) many “\( 1+t \)”s and “\( n-j \)” many points on a row such that there is a point between any “\( t \)”s and the order of “\( t \)”s and “\( t+1 \)”s between two points is ignored. (Note that for example, in the latter interpretation the orderings

\[
\cdot t \cdot (t+1) \cdot (t+1) \cdot \cdot (t+1) \cdot t \cdot t \quad \text{and} \quad (t+1) \cdot t \cdot (t+1) \cdot \cdot t \cdot (t+1) \cdot t
\]

are considered as the same since they both correspond to the collection \( L_{t+2(t+1)}, L_{(t+1)+t}, L_t \)

where

\[
L_n = L_{3(t+1)+3t+4} = L_1 \cup L_{t+2(t+1)} \cup L_1 \cup L_1 \cup L_{(t+1)+t} \cup L_1 \cup L_t.
\]

Now to count the number of solutions to this problem we spread \( \frac{i(t+1)-2j}{1-t} \) many “\( t \)”s on a row and put one point between any two:

\[
t \cdot t \cdot t \cdot ... \cdot t
\]

Observe that to achieve this there must be at least \( \frac{i(t+1)-2j}{1-t} - 1 \) many points, i.e. \( n - j \geq \frac{i(t+1)-2j}{1-t} - 1 \) which is condition (iii).

Now we are allowed to insert the remaining \( n - j - \left( \frac{i(t+1)-2j}{1-t} - 1 \right) \) points. Observe that we have \( \frac{i(t+1)-2j}{1-t} + 1 \) many places to put each of them as indicated with \( - \) below.

\[
- \cdot t \cdot - \cdot t \cdot - ... \cdot - \cdot t -
\]

This is equivalent to finding the number of integer solutions to the equation

\[
A_1 + A_2 + ... + A_{\frac{i(t+1)-2j}{1-t}+1} = n - j - \left( \frac{i(t+1)-2j}{1-t} - 1 \right)
\]

with \( A_i \geq 0 \), which is \( \left( \frac{n-j+1}{\frac{i(t+1)-2j}{1-t}} \right) \).
Finally we insert the “$t+1$”s. Since the order of “$t$”s and “$t+1$”s between two points is ignored there are $n-j+1$ places (spaces between two points plus endpoints) to insert each $t+1$. But the number of ways of doing this is equal to the number of integer solutions of the equation

$$A_1 + A_2 + \ldots + A_{n-j+1} = \frac{j-ti}{1-t}$$

with $A_i \geq 0$, which is $\binom{n-j+\frac{j-ti}{1-t}}{n-j}$. Hence the number of all possible collections is equal to

$$\binom{n-j+1}{\frac{i(t+1)-2j}{1-t}} \binom{n-j+\frac{j-ti}{1-t}}{n-j}$$

and the proof is completed. □

**Corollary 3.11 (Multigraded Betti numbers of path ideals of cycles).** Let $t \geq 2$ and $m$ be a squarefree monomial of degree $j < n$. Then the multigraded Betti number $b_{i,m}(S/I_t(C_n)) = 1$ if the induced graph $(C_n)_m$ consists of a collection of disjoint lines that satisfy the following conditions:

(i) Each line is of order 0 or $t$ mod $t+1$

(ii) The number of lines of order $t$ mod $t+1$ is equal to $\frac{i(t+1)-2j}{1-t}$.

Otherwise, $b_{i,m}(S/I_t(L_n)) = 0$.

*Proof.* By Lemma 2.8 we have $b_{i,m}(S/I_t(C_n)) = b_{i,j}(S/I_t((C_n)_m))$. Since $(C_n)_m$ is a disjoint union of lines the proof follows by Theorem 3.8. □

In the next Corollary, we will give a formula for the graded Betti numbers $b_{i,j}(S/I_t(C_n))$ when $j < n$. Note that Theorem 4.13 of [1] gives a complete formula for all Betti numbers.

**Corollary 3.12 (Graded Betti numbers of path ideals of cycles).** For $j < n$ and $t \geq 2$ the graded Betti numbers of $S/I_t(C_n)$ are given by

$$b_{i,j}(S/I_t(C_n)) = \frac{n}{n-j} \binom{n-j}{\frac{i(t+1)-2j}{1-t}} \binom{n-j-1+\frac{j-ti}{1-t}}{n-j-1}$$

provided that

(i) $n-1 \geq j$

(ii) $j \geq t(\frac{i(t+1)-2j}{1-t}) \geq 0$

(iii) $n-j \geq \frac{i(t+1)-2j}{1-t}$.

Otherwise, the graded Betti numbers are zero.
Proof. By Corollary 3.11 and Equation (4), \( b_{i,j}(S/I_t(C_n)) \) is the number of ways one can choose a collection of disjoint induced lines on \( C_n \) such that the orders of the lines add up to \( j \), each line is of order \( 0 \) or \( t \mod t+1 \) and, the number of lines of order \( t \mod t+1 \) is equal to \( \frac{i(t+1)-2j}{1-t} \). Then this is a problem of ordering \( \frac{i(t+1)-2j}{1-t} \) many “\( t \)”s, \( i-t \) many “\( t+1 \)”s and \( n-j \) many points around a circle such that there is at least one point between any “\( t \)”s and the order of “\( t \)”s and “\( t+1 \)”s between two points is ignored. Any such ordering can be obtained by first fixing a point on the cycle and ordering the remaining \( n-j-1 \) points, \( \frac{i(t+1)-2j}{1-t} \) many “\( t \)”s, \( i-t \) many “\( t+1 \)”s on a row with the same conditions. By Theorem 3.10, there are \( \left( \frac{n-j}{i(t+1)-2j} \right)^{n-j-1+\frac{i-t}{n-j-1}} \) ways to do it. Also there are \( n \) choices to fix a vertex on the cycle. However it is clear that \( n\left( \frac{n-j}{i(t+1)-2j} \right)^{n-j-1+\frac{i-t}{n-j-1}} \) will give an overcount since fixing different points may yield the same ordering. To overcome this problem, consider a circle with a desired ordering. It has \( n-j \) points and this ordering was counted once for fixing each of these points. Hence the result follows.

3.2 Stars

Throughout this section \( S_n \) will be a star graph of size \( n \).

Lemma 3.13. Let \( G \) be a connected graph, let \( \Delta \) be the Taylor simplex of \( I_2(G) \). If \( m \) is the product of the vertices of \( G \) then the simplicial complex \( \Delta_m \) is the boundary of \( \Delta \) iff \( G \) is a star.

Proof. Suppose that \( e_1, ..., e_q \) are the edges of the graph \( G \). Then, \( \Delta_m \) is the boundary of \( \Delta \) iff \( F_i = \{e_1, ..., e_q\} - \{e_i\} \) is a facet of \( \Delta_m \) for each \( i = 1, ..., q \). The latter holds only if multidegree of \( F_i \) properly divides \( m \) for every \( i \). Or, equivalently each \( e_i \) contains a vertex \( x_i \) such that \( x_i \notin \bigcup_{j \neq i} e_j \). But this happens only if \( G \) is a star since \( G \) is connected.

Corollary 3.14. Let \( G \) be a star on \( d+1 \) vertices. Then

\[
b_{i,d+1}(S/I(G)) = \begin{cases} 1 & \text{if } i = d \\ 0 & \text{otherwise} \end{cases}
\]

Proof. Follows by combining Lemma 3.13, Theorem 2.4 and Equation (1).

Corollary 3.15. Let \( G \) be a star on \( d+1 \) vertices. Then the graded Betti numbers of \( I(G) \) are given by

\[
b_{i,d+1-j}(S/I(G)) = \begin{cases} \binom{d}{i-j} & \text{if } i = d-j \\ 0 & \text{otherwise} \end{cases}
\]

Proof. Fix \( j \) and recall Equation (4) and Lemma 2.8. Any induced subgraph of \( G \) is either a star or contains isolated vertex. If it contains an isolated vertex then by Remark 2.6, the multigraded Betti number for such induced subgraph is zero. Hence by Corollary 3.14 we see that \( b_{i,d+1-j}(S/I(G)) \) is the number of induced star subgraphs of \( G \) of order \( d+1-j \) if \( i = d-j \) and zero otherwise.
**Proposition 3.16.** Let $\Gamma$ be a simplicial complex which is not a cone. Suppose that $\langle F_1, ..., F_q \rangle = \Gamma$ and there exists a sequence of distinct vertices $v_1, ..., v_q$ of $\Gamma$ such that $v_i \notin F_j$ iff $i = j$. Then $\widetilde{H}_p(\Gamma, k) \cong \widetilde{H}_{p-q+1}(\{\emptyset\}, k)$ for any field $k$.

**Proof.** We induct on $q$, the number of facets. Since there is no simplex which satisfies the assumptions of the given Proposition, the basis step starts at $q = 2$.

Suppose that $\Gamma = \langle F_1, F_2 \rangle$ is not a cone and it has two vertices $v_1, v_2$ such that $v_i \notin F_j \iff i = j$. Then $\langle F_1 \rangle \cap \langle F_2 \rangle \cong \{\emptyset\}$. Since $\Gamma = \langle F_1 \rangle \cup \langle F_2 \rangle$ and $\langle F_1 \rangle, \langle F_2 \rangle$ are acyclic, by virtue of Corollary 2.2 we have $\widetilde{H}_p(\Gamma) \cong \widetilde{H}_{p-1}(\langle F_1 \rangle \cap \langle F_2 \rangle) = \widetilde{H}_{p-1}(\{\emptyset\})$ as desired.

Now let $\Gamma = \langle F_1, ..., F_q \rangle, q \geq 3$ be a simplicial complex as in the statement of the Proposition. We write

$$\Gamma = \langle F_1, ..., F_{q-1} \rangle \cup \langle F_q \rangle$$

where $\langle F_q \rangle$ is a simplex and $\langle F_1, ..., F_{q-1} \rangle$ is a cone with apex $v_q$. By Corollary 2.2 we have $\widetilde{H}_p(\Gamma) \cong \widetilde{H}_{p-1}(\langle F_1, ..., F_{q-1} \rangle \cap \langle F_q \rangle)$. But observe that

$$\langle F_1, ..., F_{q-1} \rangle \cap \langle F_q \rangle = \langle F_1 \cap F_q, ..., F_{q-1} \cap F_q \rangle$$

as $v_j \in F_i \cap F_q, v_j \notin F_j \cap F_q$ so that $F_i \cap F_q \notin F_j \cap F_q$ for all $1 \leq i \neq j \leq q - 1$. Clearly, the simplicial complex $\langle F_1 \cap F_q, ..., F_{q-1} \cap F_q \rangle$ is not a cone and moreover

$$v_i \notin F_j \cap F_q \iff i = j$$

for all $1 \leq i, j \leq q - 1$. Hence it satisfies the inductive hypothesis and we get

$$\widetilde{H}_{p-1}(\langle F_1 \cap F_q, ..., F_{q-1} \cap F_q \rangle) \cong \widetilde{H}_{p-q+1}(\{\emptyset\})$$

which completes the proof. \hfill \Box

**Theorem 3.17.** Let $S_n$ be a star graph of size $n \geq 2$. Then for all $i \geq 1$

$$b_{i,n+1}(S/I_3(S_n)) = \begin{cases} i & \text{if } n + 1 = i + 2 \\
0 & \text{otherwise.} \end{cases} \quad (16)$$

**Proof.** Let $S_n$ be a star graph of size $n$ with the edge set $E(S_n) = \{\{x_0, x_i\} \mid i = 1, ..., n\}$. Suppose that $\Delta(S_n)$ is the Taylor simplex of $I_3(S_n)$. We prove the given statement by induction on $n$ and using Theorem 2.4 so that for all $i \geq 1$

$$b_{i,n+1}(S/I_3(S_n)) = \dim_k \widetilde{H}_{i-2}(\Delta(S_n)_{<x_0...x_n}, k).$$

For $n = 2$, we have $\Delta(S_2)_{<x_0x_1x_2} \cong \{\emptyset\}$ so the basis step is settled by (2).

Next we consider $\Delta(S_n)_{<x_0x_1...x_n}$ for some $n \geq 3$. We have a decomposition

$$\Delta(S_n)_{<x_0x_1...x_n} = \Delta(S_n)_{\leq x_0x_1...x_{n-1}} \bigcup \bigcup_{i=1}^{n-1} \Delta(S_n)_{\leq x_0x_1...x_n}$$

(17)

$$12$$
since \( \Delta(S_n)_{\leq x_1...x_n} \) is isomorphic to the irrelevant complex. For \( i \geq 1 \) we set
\[
\Delta(S_n)_{x_0...x_i} = \langle F_i \rangle := \langle \{x_0x_jx_k \mid j, k \in \{1, \ldots, n\} - \{i\} \text{ and } j < k\} \rangle
\]
and note that every element of \( \{F_1, \ldots, F_n\} \) is maximal with respect to inclusion because of the symmetry of star graphs. By Equation (5) we get \( \Delta(S_n)_{<x_0...x_n} = \langle F_1, \ldots, F_n \rangle \). Observe that (17) becomes
\[
\Delta(S_n)_{<x_0...x_n} = \langle F_n \rangle \cup \langle F_1, \ldots, F_{n-1} \rangle
\]
by definition of \( F_i \). Now we claim the followings:

(i) \( \langle F_n \rangle \cap \langle F_1, \ldots, F_{n-1} \rangle \cong \Delta(S_{n-1})_{<x_0...x_{n-1}} \)

(ii) \( \widetilde{H}_p(\langle F_1, \ldots, F_{n-1} \rangle) \cong \widetilde{H}_{p-n+2}(\{0\}) \).

Claim (i) is trivial as \( F \) is a facet of \( \langle F_n \rangle \cap \langle F_1, \ldots, F_{n-1} \rangle \) iff \( F = F_n \cap F_i \) for some \( 1 \leq i \leq n - 1 \). But the latter means that \( F \) consists of all paths of the form \( x_0x_jx_k \) where \( j, k \in \{x_1, \ldots, x_n\} - \{x_i, x_n\} \) and \( j \neq k \).

For claim (ii) we show that Proposition 3.16 applies to the simplicial complex \( \langle F_1, \ldots, F_{n-1} \rangle \). To this end, we first check that \( \langle F_1, \ldots, F_{n-1} \rangle \) is not a cone. Assume for a contradiction it is a cone with apex \( x_0x_jx_j \). Then \( x_0x_jx_k \in F_1 \cap \ldots \cap F_{n-1} \) and \( i, j \in \{1, \ldots, n\} - \{1, \ldots, n-1\} \). Thus \( i = j = n \) which is a contradiction. Now let \( v_1 = x_0x_1x_n, \ldots, v_{n-1} = x_0x_{n-1}x_n \) be a sequence of vertices of \( \langle F_1, \ldots, F_{n-1} \rangle \). Clearly we have \( v_i \notin F_j \Leftrightarrow i = j \) which proves claim (ii). Therefore (2) yields
\[
\dim \widetilde{H}_p(\langle F_1, \ldots, F_{n-1} \rangle) = \begin{cases} 1 & \text{if } p = n - 3 \\ 0 & \text{otherwise} \end{cases}
\]
Also by inductive assumption and (i) we have
\[
\dim \widetilde{H}_p(\langle F_n \rangle \cap \langle F_1, \ldots, F_{n-1} \rangle) = \begin{cases} p + 2 & \text{if } p = n - 4 \\ 0 & \text{otherwise} \end{cases}
\]
Therefore Mayer-Vietoris sequence for (19) is
\[
\ldots \to 0 \to 0 \to \widetilde{H}_{n-1}(\Delta(S_n)_{<x_0...x_n}) \to 0 \to 0 \to \widetilde{H}_{n-2}(\Delta(S_n)_{<x_0...x_n}) \to 0 \to \widetilde{H}_{n-3}(\langle F_1, \ldots, F_{n-1} \rangle) \to \widetilde{H}_{n-4}(\langle F_n \rangle \cap \langle F_1, \ldots, F_{n-1} \rangle) \to 0 \to 0 \to \widetilde{H}_{n-5}(\Delta(S_n)_{<x_0...x_n}) \to 0 \to 0 \to \ldots
\]
For \( i \leq n - 5 \) and \( i \geq n - 2 \) we have the sequence
\[
0 \to 0 \to \widetilde{H}_i(\Delta(S_n)_{<x_0...x_n}) \to 0
\]
which implies that \( \widetilde{H}_i(\Delta(S_n)_{<x_0...x_n}) = 0 \). Hence Mayer-Vietoris sequence above becomes
\[
\ldots \to 0 \to 0 \to 0 \to \widetilde{H}_{n-3}(\langle F_1, \ldots, F_{n-1} \rangle) \to \widetilde{H}_{n-4}(\Delta(S_n)_{<x_0...x_n}) \to 0 \to 0 \to \ldots
\]
Therefore Mayer-Vietoris sequence above becomes
\[
\ldots \to 0 \to \widetilde{H}_{n-3}(\langle F_1, \ldots, F_{n-1} \rangle) \to \widetilde{H}_{n-4}(\Delta(S_n)_{<x_0...x_n}) \to 0 \to 0 \to 0 \to \ldots
\]
and by (23) we see that \( \tilde{H}_{n-4}(\Delta(S_n)_{<x_0\ldots x_n}) = 0 \). Hence (22) and (23) turn into

\[
\ldots \to 0 \to \tilde{H}_{n-3}(\langle F_1, \ldots, F_{n-1} \rangle) \to \tilde{H}_{n-3}(\Delta(S_n)_{<x_0\ldots x_n}) \to \tilde{H}_{n-4}(\langle F_n \rangle \cap \langle F_1, \ldots, F_{n-1} \rangle) \to 0 \to \ldots
\]

which gives that

\[
\dim \tilde{H}_{n-3}(\Delta(S_n)_{<x_0\ldots x_n}) = \dim \tilde{H}_{n-3}(\langle F_1, \ldots, F_{n-1} \rangle) + \dim \tilde{H}_{n-4}(\langle F_n \rangle \cap \langle F_1, \ldots, F_{n-1} \rangle)
\]

\[
= 1 + (n - 2) \text{ by (20) and (21)}
\]

\[
= n - 1
\]

and, the proof is completed. \( \square \)

**Theorem 3.18 (Graded Betti numbers of path ideals of stars).** Let \( S_n \) be a star graph of size \( n \geq 2 \). For all \( i \geq 1 \) the nonzero graded Betti numbers of \( S/I_2(S_n) \) and \( S/I_3(S_n) \) are given by

\[
b_{i,j}(S/I_2(S_n)) = \begin{cases} 
\binom{n}{j-1} & \text{if } i = j - 1 \\
0 & \text{otherwise}
\end{cases}
\]

and,

\[
b_{i,j}(S/I_3(S_n)) = \begin{cases} 
i \binom{n}{j-1} & \text{if } i = j - 2 \\
0 & \text{otherwise}
\end{cases}
\]

where \( j \leq n + 1 \). In particular, \( S/I_t(S_n) \) has a \((t - 1)\)-linear resolution.

**Proof.** Similar to proof of Corollary 3.15 \( \square \)

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