Next-to-next-to-leading order gravitational spin-orbit coupling via the effective field theory for spinning objects in the post-Newtonian scheme

Michele Levi\textsuperscript{a,b} and Jan Steinhoff\textsuperscript{c,d}

\textsuperscript{a}Université Pierre et Marie Curie, CNRS-UMR 7095, Institut d’Astrophysique de Paris, 98 bis Boulevard Arago, 75014 Paris, France
\textsuperscript{b}Sorbonne Universités, Institut Lagrange de Paris, 98 bis Boulevard Arago, 75014 Paris, France
\textsuperscript{c}Max-Planck-Institute for Gravitational Physics (Albert-Einstein-Institute), Am Mühlenberg 1, 14476 Potsdam-Golm, Germany
\textsuperscript{d}Centro Multidisciplinar de Astrofisica, Instituto Superior Tecnico, Universidade de Lisboa, Avenida Rovisco Pais 1, 1049-001 Lisboa, Portugal

E-mail: michele.levi@upmc.fr, jan.steinhoff@aei.mpg.de

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Abstract. We implement the effective field theory for gravitating spinning objects in the post-Newtonian scheme at the next-to-next-to-leading order level to derive the gravitational spin-orbit interaction potential at the third and a half post-Newtonian order for rapidly rotating compact objects. From the next-to-next-to-leading order interaction potential, which we obtain here in a Lagrangian form for the first time, we derive straightforwardly the corresponding Hamiltonian. The spin-orbit sector constitutes the most elaborate spin dependent sector at each order, and accordingly we encounter a proliferation of the relevant Feynman diagrams, and a significant increase of the computational complexity. We present in detail the evaluation of the interaction potential, going over all contributing Feynman diagrams. The computation is carried out in terms of the “nonrelativistic gravitational” fields, which are advantageous also in spin dependent sectors, together with the various gauge choices included in the effective field theory for gravitating spinning objects, which also optimize the calculation. In addition, we automatize the effective field theory computations, and carry out the automated computations in parallel. Such automated effective field theory computations would be most useful to obtain higher order post-Newtonian corrections. We compare our Hamiltonian to the ADM Hamiltonian, and arrive at a complete agreement between the ADM and effective field theory results. Finally, we provide Hamiltonians in the center of mass frame, and complete gauge invariant relations among the binding energy, angular momentum, and orbital frequency of an inspiralling binary with generic compact spinning components to third and a half post-Newtonian order. The derivation presented here is essential to obtain further higher order post-Newtonian corrections, and to reach the accuracy level required for the successful detection of gravitational radiation.

Keywords: gravitational waves / theory, gravitational waves / sources, gravity, GR black holes

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1 Introduction

In light of the upcoming operation of second-generation ground-based interferometers worldwide, such as Advanced LIGO in the US [1], Advanced Virgo in Europe [2], and KAGRA in Japan [3], we may witness a direct detection of gravitational waves (GW) within the end of the decade, which will open a new era of observational gravitational wave astronomy. Inspiralling binaries of compact objects are of the most promising sources in the accessible frequency band of these experiments, where they can be treated analytically within the post-Newtonian (PN) approximation of General Relativity [4]. As the search for the GW signals employs the matched filtering technique, there is a pressing need to obtain accurate waveform templates, using the Effective-One-Body (EOB) formulation to model the continuous signal [5].

Recently the fourth PN (4PN) order correction has been completed for the non spinning case [6], and it is necessary to reach a similar accuracy level for the spin dependent case, as such compact objects are expected to be rapidly rotating. Of the spin dependent sectors the spin-orbit sector contributes the leading order (LO) spin-dependent PN correction, and represents the most dominant spin effects. The LO spin-orbit correction at the 1.5PN order has been obtained by Tulczyjew already at 1959 [7], see erratum therein, and later in the form of a higher-order Lagrangian used here by Damour [8]. Yet, the next-to-leading order (NLO) spin-orbit interaction has been approached much later, first at the level of the equations of motion (EOM) in [9], and [10], and then within the ADM Hamiltonian formalism [11]. The
next-to-next-to-leading order (NNLO) spin-orbit interaction was first derived in Hamiltonian form in [12, 13], building on [14, 15], and then at the level of the EOM in [16, 17].

In this work we apply the effective field theory (EFT) for gravitating spinning objects in the PN scheme [18] at the NNLO level, which was already tackled in the spin dependent sector using EFT techniques in [19]. We derive the NNLO gravitational spin-orbit interaction potential at the 3.5PN order for rapidly rotating compact objects. The EFT for gravitating spinning objects [18] builds on the novel, self-contained EFT approach for the binary inspiral, which was introduced in [20, 21]. The seminal works in [22, 23] already treated spin in an action approach in flat and curved spacetime, respectively. Then an extension to spinning objects of EFT techniques was approached in [24], which later adopted a Routhian approach from [25], and leaves the temporal components of the spin tensor in the final results. Yet, these components depend on field modes at the orbital scale, and they must be eliminated in order to obtain physical observables. The EFT for gravitating spinning objects [18] actually obtains an effective action at the orbital scale by integrating out all the field modes below this scale. Moreover, it actually enables the relation to physical observables: the physical EOM are directly obtained via a proper variation of the effective action [18, 26]. Furthermore, it also enables to obtain the corresponding Hamiltonians in a straightforward manner from the potentials derived via this formulation [18]. Indeed, from the potential, which we obtain here in a Lagrangian form, we derive the corresponding NNLO spin-orbit Hamiltonian, and then compare our result to the ADM Hamiltonian in [12, 13]. We arrive at a complete agreement between the ADM and EFT results.

The spin-orbit sector constitutes the most elaborate spin dependent sector at each order, see [18] for the LO and NLO levels, and [19, 27] for the other sectors at the NNLO level. Accordingly, we encounter here a proliferation of the relevant Feynman diagrams, where there are 132 diagrams contributing to this sector, and a significant increase of the computational complexity, e.g. there are 32 two-loop diagrams here. We also recall that as the spin is derivative-coupled, higher-order tensor expressions are required for all integrals involved in the calculations, compared to the non spinning case. Yet, the computation is carried out in terms of the “nonrelativistic gravitational” (NRG) fields [28, 29], which are advantageous also in spin dependent sectors, as was first shown in [30], and later also in [18, 19, 31, 32]. We also apply the various gauge choices included in the EFT for gravitating spinning objects [18], which also optimize the calculation. In addition, we automatize the EFT computations here, and carry out the automated computations in parallel, where we have used the suite of free packages xAct with the Mathematica software [33, 34]. Such automated EFT computations would be most useful to obtain higher order PN corrections. It should be stressed that in order to obtain further higher order results, all lower order results are required, consistently within one formalism, and so also in that respect the derivation presented in this work is essential. Finally, we provide Hamiltonians in the center of mass frame, and complete gauge invariant relations among the binding energy, angular momentum, and orbital frequency of an inspiralling binary with generic compact spinning components to 3.5PN order.

The outline of the paper is as follows. In section 2 we briefly review the EFT for gravitating spinning objects in the PN scheme, and present the Feynman rules required for the EFT computation. In section 3 we present the evaluation of the NNLO spin-orbit interaction potential, going over all contributing Feynman diagrams, and giving the value of each diagram. In section 4 we present the NNLO spin-orbit potential EFT result, and from it we obtain the corresponding EFT Hamiltonian. We then compare our result to the ADM Hamiltonian, where we resolve the difference between the Hamiltonians, using higher
order PN canonical transformations, and arrive at a complete agreement between the ADM and EFT results. We also present all relevant Hamiltonians to 3.5PN order in the center of mass frame. In section 5 we provide the complete gauge invariant relations among the binding energy, angular momentum, and orbital frequency of an inspiralling binary with generic compact spinning components to 3.5PN order. In section 6 we summarize our main conclusions. Finally, in appendix A we provide the additional irreducible two-loop tensor integrals required for this work.

2 The EFT for gravitating spinning objects in the PN scheme

In this section we present the effective action, and the Feynman rules, which are derived from it, and are required for the EFT computation of the NNLO spin-orbit interaction. We employ here the “NRG” fields, as applied with spin in [18, 19, 30–32]. Here, we briefly review and build on [18, 19, 26, 30, 31], following similar notations and conventions as those that were used there. Hence we use \( c = 1 \), \( \eta_{\mu\nu} \equiv \text{Diag}[1, -1, -1, -1] \), and the convention for the Riemann tensor is \( R^{\mu}_{\nu\alpha\beta} = \partial_\alpha \Gamma^{\mu}_{\nu\beta} - \partial_\beta \Gamma^{\mu}_{\nu\alpha} + \Gamma^{\mu}_{\lambda\alpha} \Gamma^{\lambda}_{\nu\beta} - \Gamma^{\mu}_{\lambda\beta} \Gamma^{\lambda}_{\nu\alpha} \). The scalar triple product appears here with no brackets, i.e. \( \vec{a} \times \vec{b} \cdot \vec{c} \equiv (\vec{a} \times \vec{b}) \cdot \vec{c} \). The notation \( \int \vec{k} = \int d^4 k / (2\pi)^4 \) is used for abbreviation. In fact, the generic \( d \)-dimensional dependence can be and is suppressed in what follows, and \( d \) can be set to 3, except for computations which involve loops, where only the \( d \) dependence from the generic \( d \)-dimensional Feynman integrals, see appendix A in [19], should be considered.

First, in terms of the “NRG” fields the metric reads

\[
g_{\mu\nu} = \left( \begin{array}{cc} e^{2\phi} & -e^{2\phi} A_j \\
-e^{2\phi} A_i & e^{2\phi} A_i A_j + e^{2\phi} A_i A_j \end{array} \right) \approx \left( \begin{array}{cc} 1 + 2\phi + 2\phi^2 + \frac{4}{3} \phi^3 & -A_j - 2A_j \phi - 2A_j \phi^2 \\
-A_i - 2A_i \phi - 2A_i \phi^2 & -\delta_{ij} + 2\phi \delta_{ij} - \sigma_{ij} - 2\phi^2 \delta_{ij} + 2\phi \sigma_{ij} + A_i A_j + \frac{4}{3} \phi^3 \delta_{ij} \end{array} \right),
\]

where we have written the approximate metric in the weak-field limit up to the orders in the fields that are required for this sector.

We recall that the effective action, describing the binary system, is given by

\[
S = S_g + \sum_{I=1}^{2} S_{(I)pp},
\]

where \( S_g \) is the pure gravitational action, and \( S_{(I)pp} \) is the worldline point particle action for each of the two particles in the binary. The gravitational action is the usual Einstein-Hilbert action plus a gauge-fixing term, which we choose as the fully harmonic gauge, such that we have

\[
S_g = S_{EH} + S_{GF} = -\frac{1}{16\pi G} \int d^4 x \sqrt{g} R + \frac{1}{32\pi G} \int d^4 x \sqrt{g} g_{\mu\nu} \Gamma^{\mu}_{\nu},
\]

where \( \Gamma^{\mu}_{\nu} \equiv \Gamma^{\mu}_{\rho\sigma} g^{\rho\sigma} \).

From the gravitational action we derive the propagators, and the self-gravitational vertices. The “NRG” scalar, vector, and tensor field propagators in the harmonic gauge are then given by

\[
\langle \phi(x_1) \phi(x_2) \rangle = 4\pi G \int \frac{e^{ik\cdot(x_1-x_2)}}{k^2} \delta(t_1 - t_2),
\]
\begin{equation}
\langle A_i(x_1) A_j(x_2) \rangle = -16\pi G \delta_{ij} \int \frac{e^{i\vec{k} \cdot (\vec{x}_1 - \vec{x}_2)}}{k^2} \delta(t_1 - t_2),
\end{equation}
\begin{equation}
\langle \sigma_{ij}(x_1) \sigma_{kl}(x_2) \rangle = 32\pi G P_{ijkl} \int \frac{e^{i\vec{k} \cdot (\vec{x}_1 - \vec{x}_2)}}{k^2} \delta(t_1 - t_2),
\end{equation}
where \( P_{ijkl} \equiv \frac{1}{2} (\delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk} - 2\delta_{ij} \delta_{kl}) \).

The Feynman rules for the propagator correction vertices are given by
\begin{equation}
\begin{aligned}
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\end{aligned}
= \frac{1}{8\pi G} \int d^4x \ (\partial_i \phi)^2,
\end{equation}
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= -\frac{1}{32\pi G} \int d^4x \ (\partial_i A_i)^2,
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\end{aligned}
= \frac{1}{128\pi G} \int d^4x \ [2(\partial_i \sigma_{ij})^2 - (\partial_i \sigma_{ii})^2],
\end{equation}
where the crosses represent the self-gravitational quadratic vertices, which contain two time derivatives.

The Feynman rules for the three-graviton vertices required for the NNLO of the spin-orbit interaction are given by
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\end{aligned}
= \frac{1}{8\pi G} \int d^4x \ \phi \left( \partial_i A_j (\partial_i A_j - \partial_j A_j) + (\partial_i A_j)^2 \right),
\end{equation}
\begin{equation}
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\end{aligned}
= -\frac{1}{64\pi G} \int d^4x \ \left[ 2\sigma_{ij} (\partial_i A_k \partial_j A_k + \partial_k A_i \partial_k A_j - 2\partial_k A_i \partial_j A_k + 2\partial_i A_j \partial_k A_k) - \sigma_{kk} \left( \partial_i A_j (\partial_i A_j - \partial_j A_i) + (\partial_i A_i)^2 \right) \right],
\end{equation}
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= \frac{1}{16\pi G} \int d^4x \ \left[ (2\sigma_{ij} \partial_i \phi \partial_j \phi - \sigma_{jj} \partial_j \phi \partial_i \phi) + (\sigma_{ii} (\partial_i \phi)^2 - 16\partial_i \sigma_{ii} \phi \partial_i \phi) \right],
\end{equation}
\begin{equation}
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\end{array}
\end{aligned}
\end{aligned}
= -\frac{1}{4\pi G} \int d^4x \ (A_i \partial_i \phi \partial_i \phi),
\end{equation}
\begin{equation}
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\end{aligned}
= \frac{1}{8\pi G} \int d^4x \ \left[ 2\sigma_{ij} (\partial_i \phi \partial_j A_j - \partial_j \phi \partial_i A_j) - \sigma_{jj} (\partial_j \phi \partial_i A_i - \partial_i \phi \partial_j A_i) \right],
\end{equation}
\begin{equation}
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\end{array}
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\end{aligned}
= \frac{1}{16\pi G} \int d^4x \ (A_i \partial_i A_j \partial_j A_j),
\end{equation}
\begin{equation}
\begin{aligned}
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\end{aligned}
\end{aligned}
= -\frac{1}{2\pi G} \int d^4x \ \left[ \phi (\partial_i \phi)^2 \right],
\end{equation}
where the first two vertices are stationary, and can be read off from the stationary Kaluza-Klein part of the gravitational action. The next five vertices are time dependent, and contain up to two time derivatives.

The Feynman rule for the four-graviton vertex required to the order considered here is given by
\begin{equation}
\begin{aligned}
\begin{array}{l}
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\end{aligned}
\end{aligned}
= \frac{1}{4\pi G} \int d^4x \ \phi^2 \left( \partial_i A_j (\partial_i A_j - \partial_j A_i) + (\partial_i A_i)^2 \right),
\end{equation}
where this vertex is stationary.

Next, we recall that the minimal coupling part of the point particle action of each of the particles with spins is given by

\[
S_{pp} = \int d\lambda \left[ -m\sqrt{u^2} - \frac{1}{2} \hat{S}_{\mu\nu} \dot{\Omega}^{\mu\nu} - \frac{\dot{S}_{\mu\nu} p^\nu}{p^2} \frac{Dp^\mu}{D\lambda} \right],
\]  

(2.18)

which is covariant, as well as invariant under rotational variables gauge transformations [18]. Here \(\lambda\) is the affine parameter, \(u^\mu \equiv dx^\mu / d\lambda\) is the 4-velocity, and \(\Omega^{\mu\nu}, S_{\mu\nu}\) are the angular velocity and spin tensors of the particle, respectively [18]. We parametrize the worldline using the coordinate time \(t = x^0\), i.e. \(\lambda = t\), so that we have for \(u^\mu \equiv dx^\mu / d\lambda\): \(u^0 = 1, u^i = dx^i / dt \equiv v^i\). Since the spin-orbit interaction is linear in the spins, only the minimal coupling part of the action, i.e. that which appears in eq. (2.18) is required. We stress that in the spin-orbit sector both mass and spin couplings play central roles in the interaction.

Let us then present first the mass couplings required for this sector. The Feynman rules of the one-graviton couplings to the worldline mass required for the NNLO spin-orbit interaction are given by

\[
= -m \int dt \phi \left[ 1 + \frac{3}{2} v^2 + \frac{7}{8} v^4 + \cdots \right],
\]  

(2.19)

\[
= m \int dt A_i v^i \left[ 1 + \frac{1}{2} v^2 + \frac{3}{8} v^4 + \cdots \right],
\]  

(2.20)

\[
= \frac{1}{2} m \int dt \sigma_{ij} v^i v^j \left[ 1 + \frac{1}{2} v^2 + \cdots \right],
\]  

(2.21)

where the heavy solid lines represent the worldlines, and the spherical black blobs represent the masses on the worldline. The ellipsis denotes higher orders in \(v\), beyond the order considered here.

For the two-graviton couplings to the worldline mass required here, we have the following Feynman rules:

\[
= \frac{1}{2} m \int dt \phi^2 \left[ 1 - \frac{9}{2} v^2 + \cdots \right],
\]  

(2.22)

\[
= m \int dt \phi A_i v^i \left[ 1 - \frac{3}{2} v^2 + \cdots \right],
\]  

(2.23)

\[
= \frac{3}{2} m \int dt \phi \sigma_{ij} v^i v^j \left[ 1 + \cdots \right].
\]  

(2.24)
Finally, for the three-graviton couplings to the worldline mass required here, we have the following Feynman rules:

\[ \frac{1}{6} m \int dt \phi^3 \left[ 1 + \cdots \right], \quad (2.25) \]

\[ \frac{1}{2} m \int dt \phi^2 A_i v^i \left[ 1 + \cdots \right]. \quad (2.26) \]

Let us go on to the spin couplings required for this sector. These are given here in terms of the physical spatial components of the local spin variable in the canonical gauge [18]. First, we have contributions from kinematic terms involving spin without field coupling [18]. To the order we are considering, these are given by

\[ L_{\text{kin}} = -\vec{S} \cdot \vec{\Omega} + \frac{1}{2} \vec{S} \cdot \vec{v} \times \vec{a} \left( 1 + \frac{3}{4} v^2 + \frac{5}{8} v^4 \right), \quad (2.27) \]

where \( S_{ij} = \epsilon_{ijk} S_k, \Omega_{ij} = \epsilon_{ijk} \Omega_k, \epsilon_{ijk} \) is the 3-dimensional Levi-Civita symbol, and \( a^i \equiv \dot{v}^i \).

We recall that all indices are Euclidean.

The required Feynman rules of the one-graviton couplings to the worldline spin are thus

\[ \text{---} = \int dt \left[ \epsilon_{ijk} S_k \left( \frac{1}{2} \partial_i A_j + \frac{1}{4} v^i v^j \left( \partial_i A_j - \partial_j A_i \right) \left( 3 + \frac{7}{4} v^2 \right) + v^i \partial_t A_j \left( 1 + \frac{1}{2} v^2 \right) \right. \right. \]

\[ \left. + v^i a^j A_l v^l \right) \right], \quad (2.28) \]

\[ \text{---} = \int dt \left[ \epsilon_{ijk} S_k v^j \left( \partial_j \phi \left( 2 + v^2 + \frac{3}{4} v^4 \right) - a^j \phi \left( 2 + 3 v^2 \right) \right) \right], \quad (2.29) \]

\[ \text{---} = \int dt \left[ \frac{1}{2} \epsilon_{ijk} S_k \left( \partial_i \sigma_{jl} v^l + \left( \frac{1}{2} \partial_i \sigma_{lm} - \partial_l \sigma_{im} \right) v^j v^l v^m - \frac{3}{2} \partial_t \sigma_{il} v^j v^l \right. \right. \]

\[ + \frac{1}{2} \sigma_{il} \left( v^j a^l - v^l a^j \right) \right], \quad (2.30) \]

where the (gray) oval blobs represent the spins on the worldlines.

For the two-graviton couplings to the worldline spin, the Feynman rules required here are:

\[ \text{---} = \int dt \left[ 2 \epsilon_{ijk} S_k \left( \partial_i A_j \phi + A_j \partial_i \phi v^i \right) \right], \quad (2.31) \]

\[ \text{---} = \int dt \left[ \frac{1}{4} \epsilon_{ijk} S_k \sigma_{il} \left( \partial_j A_l - \partial_l A_j \right) \right], \quad (2.32) \]
\[ = \int dt \left[ \frac{1}{2} \epsilon_{ijk} S_k \left( \partial_j A_i A_i v^l + A_i \partial_l A_j \right) \right], \quad (2.33) \]

\[ = \int dt \left[ 4 \epsilon_{ijk} S_k \phi \left( \partial_j \phi v^l + v^l a^j \phi \right) \right], \quad (2.34) \]

\[ = \int dt \left[ \epsilon_{ijk} S_k \sigma_{il} \left( \partial_j \phi v^l + \partial_l \phi v^j \right) \right]. \quad (2.35) \]

Finally, we also have to include three-graviton spin couplings. For three-graviton couplings to the worldline spin, the only Feynman rule required here is

\[ = \int dt \left[ 4 \epsilon_{ijk} S_k \partial_t A_j \phi^2 \right]. \quad (2.36) \]

Note that similarly to the NLO, where the two-scalar spin coupling is absent, here at NNLO the three-scalar spin coupling vanishes due to the use of the “NRG” fields, and our gauge of the rotational spin variables. Its appearance is then deferred to higher PN orders, which is advantageous.

3 Next-to-next-to-leading order spin-orbit interaction

In this section we evaluate the relevant two-body effective action by its diagrammatic expansion. As explained in [19], in the NNLO spin-orbit potential, which is evaluated at 3.5PN order, we have diagram contributions up to order \( G^3 \), coming from all 12 possible topologies appearing at these orders, as displayed in figures 1–6 below: one topology at \( O(G) \), two at \( O(G^2) \), and nine topologies at \( O(G^3) \). For the construction of the Feynman diagrams we use the Feynman rules presented in the previous section, see [19] for more detail. We have a total of 132 diagrams contributing to the NNLO spin-orbit interaction. Eventually, we will also have contribution from terms with higher order time derivatives coming from the point-mass 2PN order [35], and up to NLO spin-orbit [18] sectors.

We denote \( \vec{r} \equiv \vec{x}_1 - \vec{x}_2 \), \( r \equiv |\vec{r}| \), and \( \vec{n} \equiv \frac{\vec{r}}{r} \). Labels 1 and 2 are used for the left and right worldlines in the figures, respectively. All of the diagrams should be included together with their mirror images. Accordingly, the \((1 \leftrightarrow 2)\) notation stands for a term, whose value is obtained under the interchange of particles labels. Finally, a multiplicative factor of \( \int dt \) is omitted from all diagram values.

3.1 One-graviton exchange

For the NNLO spin-orbit interaction we have 8 one-graviton exchange diagrams as shown in figure 1. In addition to the one-graviton exchange diagrams, which already appeared in the NLO spin-orbit sector [18, 31], new diagrams are added here by inserting further propagator correction vertices. Tensor Fourier integrals of up to order 5 are required here, due to the derivative-coupling of spin, which makes the computations heavier, see [19], and appendix A there.
At the NNLO level we inevitably obtain terms with higher order time derivative, i.e. with accelerations and precessions, and these are all kept until they are treated rigorously in the resulting action \cite{18,26}, see section 4 below. Finally, we recall that there are several ways to evaluate diagrams with time derivatives, which differ only by total time derivatives. Our convention for their evaluation is, that time derivatives from the spin couplings are taken on their respective worldlines, whereas those from the propagator correction vertices are taken symmetrically on the worldlines.

The values of the one-graviton exchange diagrams are given as follows:

\[
\begin{align*}
\text{Fig. 1(a)} & = \frac{2Gm_2}{r^2} \vec{S}_1 \times \vec{n} \cdot \vec{v}_2 + \frac{Gm_2}{r^2} \left[ 3 \vec{S}_1 \times \vec{n} \cdot \vec{v}_1 \cdot \vec{v}_2 + \vec{S}_1 \times \vec{n} \cdot \vec{v}_2 \cdot \vec{v}_2^2 - \vec{S}_1 \times \vec{v}_1 \cdot \vec{v}_2 \cdot \vec{n} \right] \\
& + \frac{4Gm_2}{r} \vec{S}_1 \times \vec{a}_1 \cdot \vec{v}_2 + \frac{4Gm_2}{r} \vec{S}_1 \times \vec{v}_1 \cdot \vec{v}_2 + \frac{Gm_2}{4r^2} \left[ \vec{S}_1 \times \vec{n} \cdot \vec{v}_1 \left( 7\vec{v}_2^4 \vec{v}_1 \cdot \vec{v}_2 \right) \right] \\
& + 6\vec{v}_1 \cdot \vec{v}_2 \vec{v}_2^2 + 3\vec{S}_1 \times \vec{n} \cdot \vec{v}_2 \vec{v}_2^4 - \vec{S}_1 \times \vec{v}_1 \cdot \vec{v}_2 \left( \vec{v}_1 \cdot \vec{n} \vec{v}_1^2 + 2\vec{v}_1 \cdot \vec{n} \vec{v}_2^2 \right) \\
& - \frac{2Gm_2}{r} \left[ 2\vec{S}_1 \times \vec{v}_1 \cdot \vec{a}_1 \cdot \vec{v}_2 - 2\vec{S}_1 \times \vec{v}_1 \cdot \vec{v}_2 \vec{v}_1 \cdot \vec{a}_1 - \vec{S}_1 \times \vec{a}_1 \cdot \vec{v}_2 \left( \vec{v}_1^2 + \vec{v}_2^2 \right) \right] \\
& + \frac{2Gm_2}{r} \vec{S}_1 \times \vec{v}_1 \cdot \vec{v}_2 \left( \vec{v}_1^2 + \vec{v}_2^2 \right) \\
\text{Fig. 1(b)} & = \frac{Gm_2}{2r} \left[ \vec{S}_1 \times \vec{n} \cdot \vec{v}_2 \left( \vec{v}_1 \cdot \vec{v}_2 - 3\vec{v}_1 \cdot \vec{n} \vec{v}_2 \cdot \vec{n} \right) + \vec{S}_1 \times \vec{v}_1 \cdot \vec{v}_2 \vec{v}_2 \cdot \vec{n} \right] \\
& - \frac{Gm_2}{r} \left[ \vec{S}_1 \times \vec{n} \cdot \vec{a}_2 \vec{v}_1 \cdot \vec{n} - \vec{S}_1 \times \vec{v}_1 \cdot \vec{a}_2 \right] + \frac{Gm_2}{r} \vec{v}_2 \cdot \vec{n} \vec{S}_1 \times \vec{n} \cdot \vec{v}_2 \\
& + Gm_2 \vec{S}_1 \times \vec{n} \cdot \vec{a}_2 + \frac{Gm_2}{2r^2} \left[ 3\vec{S}_1 \times \vec{n} \cdot \vec{v}_1 \left( \left( \vec{v}_1 \cdot \vec{v}_2 \right)^2 - 3\vec{v}_1 \cdot \vec{n} \vec{v}_2 \cdot \vec{n} \vec{v}_1 \cdot \vec{v}_2 \right) \right] \\
& + \vec{S}_1 \times \vec{n} \cdot \vec{v}_2 \left( \vec{v}_1 \vec{v}_2 \vec{v}_2^2 - 3\vec{v}_1 \cdot \vec{n} \vec{v}_2 \cdot \vec{n} \vec{v}_2^2 \right) - \vec{S}_1 \times \vec{v}_1 \cdot \vec{v}_2 \left( \vec{v}_1^2 \vec{v}_2 \cdot \vec{n} + 5\vec{v}_1 \cdot \vec{n} \vec{v}_1 \cdot \vec{v}_2 \right) \\
& - \vec{v}_2 \cdot \vec{n} \vec{v}_2 \left( 3\vec{v}_2 \cdot \vec{n} \left( \vec{v}_1 \cdot \vec{n} \right)^2 \right) + \frac{Gm_2}{2r^2} \left[ 3\vec{S}_1 \times \vec{n} \cdot \vec{v}_1 \left( \vec{v}_2 \cdot \vec{n} \vec{a}_1 \cdot \vec{v}_2 - \vec{v}_1 \cdot \vec{n} \vec{v}_1 \cdot \vec{a}_2 \right) \right] \\
& + 3\vec{S}_1 \times \vec{n} \cdot \vec{a}_1 \vec{v}_2 \cdot \vec{n} \vec{v}_1 \cdot \vec{v}_2 - 2\vec{S}_1 \times \vec{n} \cdot \vec{v}_2 \vec{v}_1 \cdot \vec{n} \vec{v}_2 \cdot \vec{a}_2 + \vec{S}_1 \times \vec{n} \cdot \vec{v}_2 \left( 4\vec{a}_1 \cdot \vec{v}_2 \right) \\
& + 2\vec{v}_2 \cdot \vec{a}_2 - \vec{a}_1 \cdot \vec{n} \vec{v}_2 \cdot \vec{n} + \vec{S}_1 \times \vec{a}_1 \cdot \vec{v}_2 \left( 8\vec{v}_1 \cdot \vec{v}_2 - 5\vec{v}_1 \cdot \vec{n} \vec{v}_2 \cdot \vec{n} \right) \\
& - \vec{S}_1 \times \vec{n} \cdot \vec{a}_2 \vec{v}_1 \cdot \vec{n} \vec{v}_2 - \vec{S}_1 \times \vec{v}_1 \cdot \vec{a}_2 \left( \vec{v}_1^2 - \vec{v}_2^2 - \left( \vec{v}_1 \cdot \vec{n} \right)^2 \right) \right] \\
& + \frac{Gm_2}{2r^2} \left[ 3\vec{S}_1 \times \vec{n} \cdot \vec{v}_1 \vec{v}_2 \cdot \vec{n} \vec{v}_1 \cdot \vec{v}_2 + \vec{S}_1 \vec{n} \cdot \vec{v}_2 \vec{v}_2 \cdot \vec{n} \vec{v}_2^2 + \vec{S}_1 \vec{n} \cdot \vec{v}_1 \vec{v}_2 \left( 8\vec{v}_1 \cdot \vec{v}_2 \right) - 5\vec{v}_1 \cdot \vec{n} \vec{v}_2 \cdot \vec{n} \right] + \frac{1}{2} Gm_2 \left[ 4\vec{S}_1 \times \vec{a}_1 \cdot \vec{v}_2 \vec{v}_2 \cdot \vec{n} + 4\vec{S}_1 \vec{n} \cdot \vec{v}_1 \vec{v}_2 \cdot \vec{n} \vec{v}_2 \vec{v}_2 \cdot \vec{n} \right] \\
& + \left( 3\vec{S}_1 \times \vec{n} \cdot \vec{v}_1 \vec{a}_1 \cdot \vec{a}_2 + 3\vec{S}_1 \vec{n} \cdot \vec{a}_1 \vec{v}_1 \cdot \vec{a}_2 - \vec{S}_1 \vec{n} \cdot \vec{a}_2 \vec{a}_1 \cdot \vec{n} \right) \\
& - 5\vec{S}_1 \times \vec{a}_1 \cdot \vec{a}_2 \vec{v}_1 \cdot \vec{n} \right) + \left( 3\vec{S}_1 \times \vec{n} \cdot \vec{v}_1 \vec{v}_1 \cdot \vec{a}_2 + 2\vec{S}_1 \vec{n} \cdot \vec{v}_2 \vec{v}_2 \cdot \vec{a}_2 \right)
\end{align*}
\]
\[ + 8 \mathbf{\hat{S}}_1 \times \mathbf{\hat{v}}_2 \mathbf{\hat{v}}_2 \cdot \mathbf{n} + \mathbf{\hat{S}}_1 \times \mathbf{\hat{v}}_1 \cdot \mathbf{\hat{a}}_2 \mathbf{v}_2^2 \mathbf{v}_2^2 - 5 \mathbf{\hat{S}}_1 \times \mathbf{\hat{v}}_1 \cdot \mathbf{\hat{a}}_2 \mathbf{v}_1 \cdot \mathbf{n} \]\[ - 2Gm_2 r \left[ \mathbf{\hat{S}}_1 \times \mathbf{\hat{a}}_1 \cdot \mathbf{\hat{a}}_2 + \mathbf{\hat{S}}_1 \times \mathbf{\hat{v}}_1 \cdot \mathbf{\hat{a}}_2 + 2 \mathbf{\hat{S}}_1 \times \mathbf{\hat{a}}_1 \cdot \mathbf{\hat{a}}_2 \right] \]

Fig. 1(c) = \[ \frac{Gm_2}{4r^2} \left[ \mathbf{\hat{S}}_1 \times \mathbf{\hat{n}} \cdot \mathbf{\hat{v}}_2 \left( v_1^2 v_2^2 + 2v_1 \cdot \mathbf{\hat{v}}_2 \right)^2 - 12 \mathbf{\hat{v}}_1 \cdot \mathbf{\hat{n}} \mathbf{\hat{v}}_2 \cdot \mathbf{\hat{n}} \mathbf{\hat{v}}_1 \cdot \mathbf{\hat{v}}_2 - 3 \mathbf{\hat{v}}_1 \cdot \mathbf{n} \mathbf{\hat{v}}_2 \cdot \mathbf{n} \right] \]

\[ - 3 \mathbf{\hat{v}}_1 \cdot \mathbf{n} \mathbf{\hat{v}}_2 \cdot \mathbf{n} \left( v_2 \cdot \mathbf{n} \right)^2 \right] + \frac{Gm_2}{4r} \left[ \mathbf{\hat{S}}_1 \times \mathbf{\hat{n}} \cdot \mathbf{\hat{v}}_2 \left( \mathbf{\hat{v}}_2 \cdot \mathbf{n} \right)^2 + \mathbf{\hat{v}}_1 \cdot \mathbf{\hat{v}}_2 \left( \mathbf{\hat{v}}_2 \cdot \mathbf{n} \right)^2 \right] - 2 \mathbf{\hat{S}}_1 \times \mathbf{\hat{v}}_1 \cdot \mathbf{\hat{v}}_2 \left( \mathbf{\hat{v}}_2 \cdot \mathbf{n} \right)^2 \]

\[ - 3 \mathbf{\hat{v}}_1 \cdot \mathbf{n} \mathbf{\hat{v}}_2 \cdot \mathbf{n} \left( v_2 \cdot \mathbf{n} \right)^2 \right] + \frac{Gm_2}{4r} \left[ \mathbf{\hat{S}}_1 \times \mathbf{\hat{n}} \cdot \mathbf{\hat{v}}_2 \left( \mathbf{\hat{v}}_2 \cdot \mathbf{n} \right)^2 + \mathbf{\hat{v}}_1 \cdot \mathbf{\hat{v}}_2 \left( \mathbf{\hat{v}}_2 \cdot \mathbf{n} \right)^2 \right] - 2 \mathbf{\hat{S}}_1 \times \mathbf{\hat{v}}_1 \cdot \mathbf{\hat{v}}_2 \left( \mathbf{\hat{v}}_2 \cdot \mathbf{n} \right)^2 \]

\[ - 3 \mathbf{\hat{v}}_1 \cdot \mathbf{n} \mathbf{\hat{v}}_2 \cdot \mathbf{n} \left( v_2 \cdot \mathbf{n} \right)^2 \right] + \frac{Gm_2}{4r} \left[ \mathbf{\hat{S}}_1 \times \mathbf{\hat{n}} \cdot \mathbf{\hat{v}}_2 \left( \mathbf{\hat{v}}_2 \cdot \mathbf{n} \right)^2 + \mathbf{\hat{v}}_1 \cdot \mathbf{\hat{v}}_2 \left( \mathbf{\hat{v}}_2 \cdot \mathbf{n} \right)^2 \right] - 2 \mathbf{\hat{S}}_1 \times \mathbf{\hat{v}}_1 \cdot \mathbf{\hat{v}}_2 \left( \mathbf{\hat{v}}_2 \cdot \mathbf{n} \right)^2 \]

\[ - 3 \mathbf{\hat{v}}_1 \cdot \mathbf{n} \mathbf{\hat{v}}_2 \cdot \mathbf{n} \left( v_2 \cdot \mathbf{n} \right)^2 \right] + \frac{Gm_2}{4r} \left[ \mathbf{\hat{S}}_1 \times \mathbf{\hat{n}} \cdot \mathbf{\hat{v}}_2 \left( \mathbf{\hat{v}}_2 \cdot \mathbf{n} \right)^2 + \mathbf{\hat{v}}_1 \cdot \mathbf{\hat{v}}_2 \left( \mathbf{\hat{v}}_2 \cdot \mathbf{n} \right)^2 \right] - 2 \mathbf{\hat{S}}_1 \times \mathbf{\hat{v}}_1 \cdot \mathbf{\hat{v}}_2 \left( \mathbf{\hat{v}}_2 \cdot \mathbf{n} \right)^2 \]

\[ - 3 \mathbf{\hat{v}}_1 \cdot \mathbf{n} \mathbf{\hat{v}}_2 \cdot \mathbf{n} \left( v_2 \cdot \mathbf{n} \right)^2 \right] + \frac{Gm_2}{4r} \left[ \mathbf{\hat{S}}_1 \times \mathbf{\hat{n}} \cdot \mathbf{\hat{v}}_2 \left( \mathbf{\hat{v}}_2 \cdot \mathbf{n} \right)^2 + \mathbf{\hat{v}}_1 \cdot \mathbf{\hat{v}}_2 \left( \mathbf{\hat{v}}_2 \cdot \mathbf{n} \right)^2 \right] - 2 \mathbf{\hat{S}}_1 \times \mathbf{\hat{v}}_1 \cdot \mathbf{\hat{v}}_2 \left( \mathbf{\hat{v}}_2 \cdot \mathbf{n} \right)^2 \]

\[ - 3 \mathbf{\hat{v}}_1 \cdot \mathbf{n} \mathbf{\hat{v}}_2 \cdot \mathbf{n} \left( v_2 \cdot \mathbf{n} \right)^2 \right] + \frac{Gm_2}{4r} \left[ \mathbf{\hat{S}}_1 \times \mathbf{\hat{n}} \cdot \mathbf{\hat{v}}_2 \left( \mathbf{\hat{v}}_2 \cdot \mathbf{n} \right)^2 + \mathbf{\hat{v}}_1 \cdot \mathbf{\hat{v}}_2 \left( \mathbf{\hat{v}}_2 \cdot \mathbf{n} \right)^2 \right] - 2 \mathbf{\hat{S}}_1 \times \mathbf{\hat{v}}_1 \cdot \mathbf{\hat{v}}_2 \left( \mathbf{\hat{v}}_2 \cdot \mathbf{n} \right)^2 \]

\[ - 3 \mathbf{\hat{v}}_1 \cdot \mathbf{n} \mathbf{\hat{v}}_2 \cdot \mathbf{n} \left( v_2 \cdot \mathbf{n} \right)^2 \right] + \frac{Gm_2}{4r} \left[ \mathbf{\hat{S}}_1 \times \mathbf{\hat{n}} \cdot \mathbf{\hat{v}}_2 \left( \mathbf{\hat{v}}_2 \cdot \mathbf{n} \right)^2 + \mathbf{\hat{v}}_1 \cdot \mathbf{\hat{v}}_2 \left( \mathbf{\hat{v}}_2 \cdot \mathbf{n} \right)^2 \right] - 2 \mathbf{\hat{S}}_1 \times \mathbf{\hat{v}}_1 \cdot \mathbf{\hat{v}}_2 \left( \mathbf{\hat{v}}_2 \cdot \mathbf{n} \right)^2 \]

\[ - 3 \mathbf{\hat{v}}_1 \cdot \mathbf{n} \mathbf{\hat{v}}_2 \cdot \mathbf{n} \left( v_2 \cdot \mathbf{n} \right)^2 \right] + \frac{Gm_2}{4r} \left[ \mathbf{\hat{S}}_1 \times \mathbf{\hat{n}} \cdot \mathbf{\hat{v}}_2 \left( \mathbf{\hat{v}}_2 \cdot \mathbf{n} \right)^2 + \mathbf{\hat{v}}_1 \cdot \mathbf{\hat{v}}_2 \left( \mathbf{\hat{v}}_2 \cdot \mathbf{n} \right)^2 \right] - 2 \mathbf{\hat{S}}_1 \times \mathbf{\hat{v}}_1 \cdot \mathbf{\hat{v}}_2 \left( \mathbf{\hat{v}}_2 \cdot \mathbf{n} \right)^2 \]
\[-\vec{S}_1 \times \vec{v}_1 \cdot \hat{a}_1 \vec{v}_2 \cdot \vec{n} \]  \hspace{1cm} (3.5)

Fig. 1(f) = \[-\frac{Gm_2}{4r^2} \left[ \vec{S}_1 \times \vec{n} \cdot \vec{v}_1 (v_1^2 v_2^2 + 2(\vec{v}_1 \cdot \vec{v}_2)^2 - 12\vec{v}_1 \cdot \vec{n} \vec{v}_2 \cdot \vec{n} \vec{v}_1 \cdot \vec{v}_2 - 3v_1^2(\vec{v}_1 \cdot \vec{n})^2 - 3v_1^2(\vec{v}_2 \cdot \vec{n})^2 + 15(\vec{v}_1 \cdot \vec{n})(\vec{v}_2 \cdot \vec{n})^2 \right] - \frac{Gm_2}{4r} \left[ \vec{S}_1 \times \vec{n} \cdot \vec{v}_1 (2\vec{v}_2 \cdot \vec{n} \vec{a}_1 \vec{v}_2 \cdot \vec{v}_2 + \vec{a}_1 \cdot \vec{n} v_2^2 - v_1^2 \vec{a}_2 \cdot \vec{n} \right.
\]
\[-2\vec{v}_1 \cdot \vec{n} \vec{v}_1 \cdot \vec{a}_2 + 3\vec{a}_2 \cdot \vec{n}(\vec{v}_1 \cdot \vec{n})^2 - 3\vec{a}_1 \cdot \vec{n}(\vec{v}_2 \cdot \vec{n})^2 + 2\vec{S}_1 \times \vec{n} \cdot \vec{a}_1 (2\vec{v}_2 \cdot \vec{n} \vec{v}_1 \cdot \vec{v}_2 + \vec{v}_1 \cdot \vec{n} v_2^2 - 3\vec{v}_1 \cdot \vec{n}(\vec{v}_2 \cdot \vec{n})^2) - \vec{S}_1 \times \vec{v}_1 \cdot \vec{a}_1 (v_2^2 - (\vec{v}_2 \cdot \vec{n})^2) + 2\vec{S}_1 \times \vec{n} \cdot \vec{v}_2 (\vec{a}_1 \cdot \vec{v}_2 - \vec{a}_1 \vec{v}_2 \cdot \vec{n}) + 4\vec{S}_1 \times \vec{a}_1 \cdot \vec{v}_2 (\vec{v}_1 \cdot \vec{v}_2 - \vec{v}_1 \cdot \vec{n} \vec{v}_2 \cdot \vec{n}) - \vec{S}_1 \times \vec{v}_1 \cdot \vec{a}_2 (v_1^2 - (\vec{v}_1 \cdot \vec{n})^2) \]
\[-\frac{Gm_2}{2r} \left[ \vec{S}_1 \times \vec{n} \cdot \vec{v}_1 (2\vec{v}_2 \cdot \vec{n} \vec{v}_1 \cdot \vec{v}_2 + \vec{v}_1 \cdot \vec{n} v_2^2 - 3\vec{v}_1 \cdot \vec{n}(\vec{v}_2 \cdot \vec{n})^2) \right.
\]
\[+ 2\vec{S}_1 \times \vec{v}_1 \cdot \vec{v}_2 (\vec{v}_1 \cdot \vec{v}_2 - \vec{v}_1 \cdot \vec{n} \vec{v}_2 \cdot \vec{n}) \]  \hspace{1cm} (3.6)

Fig. 1(g) = \[\frac{2Gm_2}{r^2} \left[ \vec{S}_1 \times \vec{n} \cdot \vec{v}_1 v_2^2 - \vec{S}_1 \times \vec{n} \cdot \vec{v}_2 \vec{v}_1 \cdot \vec{v}_2 \right] + \frac{Gm_2}{r^2} \left[ \vec{S}_1 \times \vec{n} \cdot \vec{v}_1 (v_1^2 v_2^2 - (\vec{v}_1 \cdot \vec{v}_2)^2 + v_1^2) - \vec{S}_1 \times \vec{n} \cdot \vec{v}_2 \vec{v}_1 \cdot \vec{v}_2 + \vec{S}_1 \times \vec{n} \cdot \vec{v}_2 \vec{v}_1 \cdot \vec{v}_2 \right] - \frac{2Gm_2}{r} \left[ \vec{S}_1 \times \vec{n} \cdot \vec{a}_1 v_2^2 + \vec{S}_1 \times \vec{n} \cdot \vec{v}_2 \vec{a}_1 \cdot \vec{n} + 2\vec{S}_1 \times \vec{n} \cdot \vec{a}_1 (v_2^2 - (\vec{v}_2 \cdot \vec{n})^2) + 4\vec{S}_1 \times \vec{a}_1 \cdot \vec{v}_2 \vec{v}_2 \cdot \vec{n} - 2\vec{S}_1 \times \vec{v}_1 \cdot \vec{a}_2 \vec{v}_1 \cdot \vec{n} \right) \hspace{1cm} (3.7)

Fig. 1(h) = \[\frac{Gm_2}{r^2} \left[ \vec{S}_1 \times \vec{n} \cdot \vec{v}_1 (v_1^2 v_2^2 - 3\vec{v}_1 \cdot \vec{n} \vec{v}_2 \cdot \vec{n} v_2^2) \right] - \vec{S}_1 \times \vec{n} \cdot \vec{v}_2 (\vec{v}_1 \cdot \vec{v}_2)^2 \]
\[-\frac{Gm_2}{r} \left[ \vec{S}_1 \times \vec{n} \cdot \vec{v}_2 \vec{v}_1 \cdot \vec{a}_2 - \vec{S}_1 \times \vec{n} \cdot \vec{a}_1 \vec{v}_2 \cdot \vec{n} v_2^2 + 2\vec{S}_1 \times \vec{n} \cdot \vec{a}_1 \vec{v}_2 \cdot \vec{n} \vec{v}_1 \cdot \vec{v}_2 \right] \hspace{1cm} (3.8)
\]
3.2 Two-graviton exchange and cubic self-interaction

3.2.1 Two-graviton exchange

For the NNLO spin-orbit interaction we have 15 two-graviton exchange diagrams, as shown in figure 2, where they either contain a two-graviton spin or mass coupling.

The two-graviton exchange diagrams also require only tensor Fourier integrals. We encounter here two-graviton exchange diagrams, which involve time derivatives, either from the spin couplings or from propagator correction vertices.

The values of the two-graviton exchange diagrams are given in the following:

\[
\text{Fig. 2(a1)} = -\frac{8G^2m_2^2}{r^3}S_1 \times \vec{n} \cdot \vec{v}_2 - \frac{8G^2m_2^2}{r^3} \left[ 2S_1 \times \vec{n} \cdot \vec{v}_2 v_2^2 - S_1 \times \vec{v}_1 \cdot \vec{v}_2 \vec{v}_2 \cdot \vec{n} \right] \quad (3.9)
\]

\[
\text{Fig. 2(a2)} = -\frac{4G^2m_2^2}{r^5} \left[ S_1 \times \vec{n} \cdot \vec{v}_2 (\vec{v}_1 \cdot \vec{v}_2 - 4\vec{v}_1 \cdot \vec{v}_2 \vec{v}_2 \cdot \vec{n}) + (\vec{v}_2 \cdot \vec{n})^2 \right] + \frac{4G^2m_2^2}{r^5} \left[ S_1 \times \vec{n} \cdot \vec{a}_2 (2\vec{v}_1 \cdot \vec{n} - \vec{v}_2 \cdot \vec{n}) - \vec{S}_1 \times \vec{v}_1 \cdot \vec{a}_2 \right] - \frac{4G^2m_2^2}{r^5} \vec{v}_2 \cdot \vec{n}\vec{S}_1 \times \vec{n} \cdot \vec{v}_2
\]

\[
\text{Fig. 2(a3)} = -\frac{4G^2m_2^2}{r^5} \left[ S_1 \times \vec{n} \cdot \vec{v}_2 (\vec{v}_1 \cdot \vec{v}_2 - 4\vec{v}_1 \cdot \vec{v}_2 \vec{v}_2 \cdot \vec{n}) + (\vec{v}_2 \cdot \vec{n})^2 \right] + \frac{4G^2m_2^2}{r^5} \vec{v}_2 \cdot \vec{n}\vec{S}_1 \times \vec{n} \cdot \vec{v}_2 \quad (3.11)
\]

\[
\text{Fig. 2(b)} = \frac{4G^2m_2^2}{r^3} v_2^2 \vec{S}_1 \times \vec{n} \cdot \vec{v}_2 \quad (3.12)
\]

\[
\text{Fig. 2(c)} = -\frac{4G^2m_2^2}{r^3} v_1^2 \vec{S}_1 \times \vec{n} \cdot \vec{v}_1 + \frac{4G^2m_2^2}{r^2} \vec{S}_1 \times \vec{v}_1 \cdot \vec{a}_1 \quad (3.13)
\]

\[
\text{Fig. 2(d)} = -\frac{4G^2m_2^2}{r^3} \left[ \vec{S}_1 \times \vec{n} \cdot \vec{v}_2 \vec{v}_1 \cdot \vec{v}_2 + \vec{S}_1 \times \vec{v}_1 \cdot \vec{v}_2 \vec{v}_2 \cdot \vec{n} \right] \quad (3.14)
\]

\[
\text{Fig. 2(e)} = \frac{8G^2m_2^2}{r^3} \vec{v}_1 \cdot \vec{v}_2 \vec{S}_1 \times \vec{n} \cdot \vec{v}_2 + \frac{8G^2m_2^2}{r^2} \vec{S}_1 \times \vec{v}_2 \cdot \vec{a}_2 \quad (3.15)
\]

\[
\text{Fig. 2(f1)} = -\frac{2G^2m_1m_2}{r^3} \vec{S}_1 \times \vec{n} \cdot \vec{v}_2 - \frac{G^2m_1m_2}{r^3} \left[ 3\vec{S}_1 \times \vec{n} \cdot \vec{v}_1 \vec{v}_1 \cdot \vec{v}_2 + 3\vec{S}_1 \times \vec{n} \cdot \vec{v}_2 (v_1^2 - v_2^2) \right]
\]
The cubic self-gravitational interaction diagrams require first the application of one-loop \[ G \]. Here, we encounter time derivatives from the spin couplings, the propagator correction \[ \frac{G^{m_1m_2}}{r^2} \hat{S}_1 \times \tilde{n} \cdot \tilde{a}_2 \tilde{v}_1 \cdot \tilde{n} - \frac{G^{m_1m_2}}{r^2} \hat{S}_1 \times \tilde{n} \cdot \tilde{a}_2 \]

Fig. 2(g2) = \[ - \frac{G^{m_1m_2}}{r^3} \left[ \tilde{S}_1 \times \tilde{n} \cdot \tilde{v}_2 (\tilde{v}_1 \cdot \tilde{v}_2 - 4\tilde{v}_1 \cdot \tilde{n} \tilde{v}_2 \cdot \tilde{n} + (\tilde{v}_1 \cdot \tilde{n})^2) - \tilde{S}_1 \times \tilde{v}_1 \cdot \tilde{v}_2 (\tilde{v}_1 \cdot \tilde{n} - 2\tilde{v}_2 \cdot \tilde{n}) \right] + \frac{G^{m_1m_2}}{r^2} \tilde{S}_1 \times \tilde{n} \cdot \tilde{a}_2 \tilde{v}_1 \cdot \tilde{n} - \tilde{S}_1 \times \tilde{v}_1 \cdot \tilde{v}_2 \]

Fig. 2(f2) = \[ - \frac{G^{m_1m_2}}{r^3} \left[ \tilde{S}_1 \times \tilde{n} \cdot \tilde{v}_2 (\tilde{v}_1 \cdot \tilde{v}_2 - 4\tilde{v}_1 \cdot \tilde{n} \tilde{v}_2 \cdot \tilde{n} + (\tilde{v}_1 \cdot \tilde{n})^2) - \tilde{S}_1 \times \tilde{v}_1 \cdot \tilde{v}_2 (\tilde{v}_1 \cdot \tilde{n} - 2\tilde{v}_2 \cdot \tilde{n}) \right] + \frac{G^{m_1m_2}}{r^2} \tilde{S}_1 \times \tilde{n} \cdot \tilde{a}_2 \tilde{v}_1 \cdot \tilde{n} - \tilde{S}_1 \times \tilde{v}_1 \cdot \tilde{v}_2 \]

Fig. 2(f3) = \[ - \frac{G^{m_1m_2}}{r^3} \left[ \tilde{S}_1 \times \tilde{n} \cdot \tilde{v}_2 (\tilde{v}_1 \cdot \tilde{v}_2 - 4\tilde{v}_1 \cdot \tilde{n} \tilde{v}_2 \cdot \tilde{n} + (\tilde{v}_1 \cdot \tilde{n})^2) - \tilde{S}_1 \times \tilde{v}_1 \cdot \tilde{v}_2 (\tilde{v}_1 \cdot \tilde{n} - 2\tilde{v}_2 \cdot \tilde{n}) \right] + \frac{G^{m_1m_2}}{r^2} \tilde{S}_1 \times \tilde{n} \cdot \tilde{a}_2 \tilde{v}_1 \cdot \tilde{n} - \tilde{S}_1 \times \tilde{v}_1 \cdot \tilde{v}_2 \]

Fig. 2(f4) = \[ - \frac{8G^{m_1m_2}m_3}{r^3} \tilde{S}_1 \times \tilde{n} \cdot \tilde{v}_1 \cdot \tilde{v}_2 \]

Fig. 2(g1) = \[ \frac{2G^{m_1m_2}}{r^3} \tilde{S}_1 \times \tilde{n} \cdot \tilde{v}_1 + \frac{2G^{m_1m_2}}{r^3} \tilde{S}_1 \times \tilde{n} \cdot \tilde{v}_1 (4\tilde{v}_1^2 - 9\tilde{v}_2^2) - \frac{2G^{m_1m_2}}{r^2} \tilde{S}_1 \times \tilde{n} \cdot \tilde{a}_1 \]

Fig. 2(g2) = \[ \frac{G^{m_1m_2}}{r^3} \left[ \tilde{S}_1 \times \tilde{n} \cdot \tilde{v}_1 (\tilde{v}_1 \cdot \tilde{v}_2 - 4\tilde{v}_1 \cdot \tilde{n} \tilde{v}_2 \cdot \tilde{n} + (\tilde{v}_1 \cdot \tilde{n})^2) - \tilde{S}_1 \times \tilde{v}_1 \cdot \tilde{v}_2 \tilde{v}_1 \cdot \tilde{n} \right] - \frac{G^{m_1m_2}}{r^2} \left[ \tilde{S}_1 \times \tilde{n} \cdot \tilde{a}_1 (\tilde{v}_1 \cdot \tilde{n} - 2\tilde{v}_2 \cdot \tilde{n}) - \tilde{S}_1 \times \tilde{a}_1 \cdot \tilde{v}_1 \cdot \tilde{v}_2 \right]

Fig. 2(g3) = \[ \frac{2G^{m_1m_2}}{r^3} \left[ \tilde{S}_1 \times \tilde{n} \cdot \tilde{v}_1 (\tilde{v}_1 \cdot \tilde{v}_2 - 4\tilde{v}_1 \cdot \tilde{n} \tilde{v}_2 \cdot \tilde{n} + (\tilde{v}_1 \cdot \tilde{n})^2) - \tilde{S}_1 \times \tilde{v}_1 \cdot \tilde{v}_2 \tilde{v}_1 \cdot \tilde{n} \right]

3.2.2 Cubic self-interaction

For the NNLO spin-orbit interaction we have 49 cubic self-interaction diagrams, as shown in figure 3, where the cubic vertices contain up to two time derivatives.

The cubic self-gravitational interaction diagrams require first the application of one-loop tensor integrals up to order 3, in addition to the Fourier tensor integrals, see appendix A in [19]. Here, we encounter time derivatives from the spin couplings, the propagator correction vertices, and the time dependent cubic self-gravitational vertices.

The values of the cubic self-interaction diagrams are given as follows:

Fig. 3(a1) = \[ \frac{8G^{m_1m_2}m_3}{r^3} \tilde{S}_1 \times \tilde{n} \cdot \tilde{v}_2 + \frac{4G^{m_1m_2}m_2}{r^3} \left[ 3\tilde{S}_1 \times \tilde{n} \cdot \tilde{v}_1 \tilde{v}_1 \cdot \tilde{v}_2 + 4\tilde{S}_1 \times \tilde{n} \cdot \tilde{v}_2 \tilde{v}_2 \right]

- \tilde{S}_1 \times \tilde{v}_1 \cdot \tilde{v}_2 \tilde{v}_1 \cdot \tilde{n} ] + \frac{8G^{m_1m_2}m_3}{r^2} \tilde{S}_1 \times \tilde{a}_1 \cdot \tilde{v}_2 + \frac{8G^{m_1m_2}m_2}{r^2} \tilde{S}_1 \times \tilde{v}_1 \cdot \tilde{v}_2 \]

Fig. 3(a2) = \[ \frac{2G^{m_1m_2}}{r^3} \tilde{S}_1 \times \tilde{n} \cdot \tilde{v}_2 + \frac{G^{m_1m_2}}{r^3} \left[ 3\tilde{S}_1 \times \tilde{n} \cdot \tilde{v}_1 \tilde{v}_1 \cdot \tilde{v}_2 + \tilde{S}_1 \times \tilde{n} \cdot \tilde{v}_2 \tilde{v}_2 \right]

- 5\tilde{S}_1 \times \tilde{v}_1 \cdot \tilde{v}_2 \tilde{v}_1 \cdot \tilde{n} ] + \frac{8G^{m_1m_2}m_2}{r^2} \tilde{S}_1 \times \tilde{a}_1 \cdot \tilde{v}_2 + \frac{8G^{m_1m_2}m_2}{r^2} \tilde{S}_1 \times \tilde{v}_1 \cdot \tilde{v}_2 \]

(3.24)

(3.25)
Figure 3. NNLO spin-orbit Feynman diagrams of cubic self-interaction.
Fig. 3(a3) = \(-\frac{2G^2m_1m_2}{r^3}\vec{S}_1 \times \vec{n} \cdot \vec{v}_1 - \frac{G^2m_1m_2}{r^3}\vec{S}_1 \times \vec{n} \cdot \vec{v}_1(8\vec{v}_1^2 + 3\vec{v}_2^2)\)
+ \(8G^2m_1m_2\vec{S}_1 \times \vec{v}_1 \cdot \vec{a}_1\) (3.26)

Fig. 3(a4) = \(\frac{8G^2m_2^2}{r^3}\left[\vec{S}_1 \times \vec{n} \cdot \vec{v}_2(\vec{v}_1 \cdot \vec{v}_2 - 4\vec{v}_1 \cdot \vec{n}\vec{v}_2 \cdot \vec{n}) + \vec{S}_1 \times \vec{v}_1 \cdot \vec{v}_2\vec{v}_2 \cdot \vec{n}\right]\)
- \(\frac{4G^2m_2^2}{r^2}\left[2\vec{S}_1 \times \vec{n} \cdot \vec{a}_2\vec{v}_1 \cdot \vec{n} - \vec{S}_1 \times \vec{v}_1 \cdot \vec{a}_2\right] + \frac{8G^2m_2^2}{r^2}\vec{v}_2 \cdot \vec{n}\vec{S}_1 \times \vec{n} \cdot \vec{v}_2\)
+ \(\frac{4G^2m_2^2}{r}\vec{S}_1 \times \vec{n} \cdot \vec{a}_2\) (3.27)

Fig. 3(a5) = \(\frac{4G^2m_2^2}{r^3}\left[\vec{S}_1 \times \vec{n} \cdot \vec{v}_2(\vec{v}_1 \cdot \vec{v}_2 - 4\vec{v}_1 \cdot \vec{n}\vec{v}_2 \cdot \vec{n} + (\vec{v}_2 \cdot \vec{n})^2) + \vec{S}_1 \times \vec{v}_1 \cdot \vec{v}_2\vec{v}_2 \cdot \vec{n}\right]\)
+ \(\frac{2G^2m_2^2}{r^2}\left[2\vec{S}_1 \times \vec{n} \cdot \vec{v}_2\vec{a}_2 \cdot \vec{n} - 4\vec{S}_1 \times \vec{n} \cdot \vec{a}_2\vec{v}_1 \cdot \vec{n} + 2\vec{S}_1 \times \vec{v}_1 \cdot \vec{a}_2\right] + \frac{4G^2m_2^2}{r}\vec{v}_2 \cdot \vec{n}\vec{S}_1 \times \vec{n} \cdot \vec{v}_2\) (3.28)

Fig. 3(a6) = \(\frac{4G^2m_2^2}{r^3}\left[\vec{S}_1 \times \vec{n} \cdot \vec{v}_2(\vec{v}_1 \cdot \vec{v}_2 - 4\vec{v}_1 \cdot \vec{n}\vec{v}_2 \cdot \vec{n} + 3(\vec{v}_2 \cdot \vec{n})^2) + \vec{S}_1 \times \vec{v}_1 \cdot \vec{v}_2\vec{v}_2 \cdot \vec{n}\right]\)
+ \(\frac{2G^2m_2^2}{r^2}\left[2\vec{S}_1 \times \vec{n} \cdot \vec{v}_2\vec{a}_2 \cdot \vec{n} + \vec{S}_1 \times \vec{v}_2 \cdot \vec{a}_2\right] + \frac{4G^2m_2^2}{r}\vec{v}_2 \cdot \vec{n}\vec{S}_1 \times \vec{n} \cdot \vec{v}_2\) (3.29)

Fig. 3(a7) = \(-\frac{G^2m_1m_2}{r^3}\left[\vec{S}_1 \times \vec{n} \cdot \vec{v}_1(\vec{v}_1 \cdot \vec{v}_2 - \vec{v}_2^2 + 4(\vec{v}_2 \cdot \vec{n})^2) + 2\vec{S}_1 \times \vec{n} \cdot \vec{v}_2(\vec{v}_1 \cdot \vec{n} - 5\vec{v}_2 \cdot \vec{n})\right]\)
- \(\frac{G^2m_1m_2}{r^2}\left[\vec{S}_1 \times \vec{n} \cdot \vec{v}_1\vec{a}_2 \cdot \vec{n} + 6\vec{S}_1 \times \vec{n} \cdot \vec{a}_2\vec{v}_1 \cdot \vec{n} - 3\vec{S}_1 \times \vec{v}_1 \cdot \vec{a}_2\right]\)
+ \(\frac{G^2m_1m_2}{r}\vec{S}_1 \times \vec{n} \cdot \vec{v}_1\vec{v}_2 \cdot \vec{n} - 6\vec{S}_1 \times \vec{n} \cdot \vec{v}_2(\vec{v}_1 \cdot \vec{n} - 2\vec{v}_2 \cdot \vec{n}) + 3\vec{S}_1 \times \vec{v}_1 \cdot \vec{v}_2\) (3.30)

Fig. 3(a8) = \(\frac{G^2m_1m_2}{r^3}\left[\vec{S}_1 \times \vec{n} \cdot \vec{v}_1(\vec{v}_1 \cdot \vec{v}_2 - \vec{v}_2^2 - 8\vec{v}_1 \cdot \vec{n}\vec{v}_2 \cdot \vec{n} + 12(\vec{v}_2 \cdot \vec{n})^2)\right]\)
+ 2\vec{S}_1 \times \vec{n} \cdot \vec{v}_2(\vec{v}_1 \cdot \vec{v}_2 - 4\vec{v}_1 \cdot \vec{n}\vec{v}_2 \cdot \vec{n} + 3(\vec{v}_2 \cdot \vec{n})^2) - \vec{S}_1 \times \vec{v}_1 \cdot \vec{v}_2(4\vec{v}_1 \cdot \vec{n}
- 5\vec{v}_2 \cdot \vec{n})\]
+ \(\frac{G^2m_1m_2}{r^2}\left[3\vec{S}_1 \times \vec{n} \cdot \vec{v}_1\vec{a}_2 \cdot \vec{n} - 2\vec{S}_1 \times \vec{n} \cdot \vec{a}_2\vec{v}_1 \cdot \vec{n} + \vec{S}_1 \times \vec{v}_1 \cdot \vec{a}_2\right] + \frac{G^2m_1m_2}{r}\vec{v}_2 \cdot \vec{n}\vec{S}_1 \times \vec{n} \cdot \vec{v}_2\) (3.31)

Fig. 3(a9) = \(\frac{2G^2m_1m_2}{r^3}\left[\vec{S}_1 \times \vec{n} \cdot \vec{v}_2(\vec{v}_1 \cdot \vec{v}_2 - 4\vec{v}_1 \cdot \vec{n}\vec{v}_2 \cdot \vec{n}) + \vec{S}_1 \times \vec{v}_1 \cdot \vec{v}_2\vec{v}_2 \cdot \vec{n}\right]\)
- \(\frac{G^2m_1m_2}{r^2}\left[2\vec{S}_1 \times \vec{n} \cdot \vec{a}_2\vec{v}_1 \cdot \vec{n} - \vec{S}_1 \times \vec{v}_1 \cdot \vec{a}_2\right] + \frac{2G^2m_1m_2}{r^2}\vec{v}_2 \cdot \vec{n}\vec{S}_1 \times \vec{n} \cdot \vec{v}_2\)
+ \(\frac{G^2m_1m_2}{r}\vec{S}_1 \times \vec{n} \cdot \vec{a}_2\) (3.32)

Fig. 3(a10) = \(\frac{G^2m_1m_2}{r^3}\left[\vec{S}_1 \times \vec{n} \cdot \vec{v}_1(\vec{v}_1^2 - \vec{v}_1 \cdot \vec{v}_2 + 4\vec{v}_1 \cdot \vec{n}\vec{v}_2 \cdot \vec{n} - 2(\vec{v}_1 \cdot \vec{n})^2)\right]\)
\[ + \vec{S}_1 \times \vec{v}_1 \cdot \vec{v}_2 \vec{v}_1 \cdot \vec{n} + \frac{G^2 m_1 m_2}{r^2} \left[ \vec{S}_1 \times \vec{n} \cdot \vec{v}_1 \vec{v}_1 \cdot \vec{n} - \vec{S}_1 \times \vec{v}_1 \cdot \vec{n} \right] \\
+ \frac{G^2 m_1 m_2}{r^2} \left[ \vec{S}_1 \times \vec{n} \cdot \vec{v}_1 (3\vec{v}_1 \cdot \vec{n} - 4\vec{v}_2 \cdot \vec{n}) - 2\vec{S}_1 \times \vec{v}_1 \cdot \vec{v}_2 \right] \\
- 2G^2 m_1 m_2 \frac{\vec{S}_1 \times \vec{n} \cdot \vec{a}_1}{r} \tag{3.33} \]

Fig. 3(a11) = \[ - \frac{G^2 m_1 m_2}{r^3} \left[ \vec{S}_1 \times \vec{n} \cdot \vec{v}_1 (\vec{v}_1^2 - \vec{v}_1 \cdot \vec{v}_2 + 4\vec{v}_1 \cdot \vec{v}_2 \vec{v}_2 \cdot \vec{n} - 2(\vec{v}_1 \cdot \vec{n})^2) \\
+ \vec{S}_1 \times \vec{v}_1 \cdot \vec{v}_2 \vec{v}_1 \cdot \vec{n} \right] \]

\[ + \frac{G^2 m_1 m_2}{r^2} \left[ 3\vec{S}_1 \times \vec{n} \cdot \vec{a}_1 \vec{v}_1 \cdot \vec{n} - 2\vec{S}_1 \times \vec{n} \cdot \vec{a}_1 (3\vec{v}_1 \cdot \vec{n} \\
- 2\vec{v}_2 \cdot \vec{n}) + 3\vec{S}_1 \times \vec{v}_1 \cdot \vec{v}_1 + 2\vec{S}_1 \times \vec{a}_1 \cdot \vec{v}_2 \right] \]

\[ - \frac{G^2 m_1 m_2}{r^2} \vec{v}_1 \cdot \vec{n} \vec{S}_1 \times \vec{n} \cdot \vec{v}_1 \vec{v}_1 \cdot \vec{v}_2 \vec{v}_1 \cdot \vec{n} \\
+ \frac{2G^2 m_1 m_2}{r^3} \vec{S}_1 \times \vec{n} \cdot \vec{a}_1 \vec{v}_2 \cdot \vec{n} \tag{3.34} \]

Fig. 3(a12) = \[ - \frac{8G^2 m_1 m_2}{r^3} \left[ 2\vec{S}_1 \times \vec{n} \cdot \vec{v}_1 \vec{v}_1 \cdot \vec{v}_2 + \vec{S}_1 \times \vec{v}_1 \cdot \vec{v}_2 \vec{v}_1 \cdot \vec{n} \right] \tag{3.37} \]

Fig. 3(b1) = \[ \frac{G^2 m_2^2}{2r^3} \vec{S}_1 \times \vec{n} \cdot \vec{v}_1 + \frac{G^2 m_2^2}{4r^3} \left[ \vec{S}_1 \times \vec{n} \cdot \vec{v}_1 (3\vec{v}_1^2 + 2\vec{v}_2 - 8(\vec{v}_1 \cdot \vec{n})^2 + 8(\vec{v}_2 \cdot \vec{n})^2) \\
+ 4\vec{S}_1 \times \vec{v}_1 \cdot \vec{v}_2 \vec{v}_2 \cdot \vec{n} \right] + \frac{G^2 m_2^2}{2r^2} \left[ \vec{S}_1 \times \vec{n} \cdot \vec{v}_1 \vec{a}_1 \cdot \vec{n} + 2\vec{S}_1 \times \vec{n} \cdot \vec{a}_1 \vec{v}_1 \cdot \vec{n} \\
- \vec{S}_1 \times \vec{v}_1 \cdot \vec{a}_1 \right] + \frac{3G^2 m_2^2}{4r^2} \vec{v}_1 \cdot \vec{n} \vec{S}_1 \times \vec{n} \cdot \vec{v}_1 \vec{n} \tag{3.38} \]

Fig. 3(b2) = \[ \frac{G^2 m_1 m_2}{2r^3} \vec{S}_1 \times \vec{n} \cdot \vec{v}_1 - \frac{G^2 m_1 m_2}{4r^3} \left[ \vec{S}_1 \times \vec{n} \cdot \vec{v}_1 (2\vec{v}_1^2 - 4\vec{v}_1 \cdot \vec{v}_2 - 3\vec{v}_2^2) \\
+ 16\vec{v}_1 \cdot \vec{n} \vec{v}_2 \cdot \vec{n} - 16(\vec{v}_1 \cdot \vec{n})^2 \right] + \frac{G^2 m_1 m_2}{2r^2} \left[ 2\vec{S}_1 \times \vec{n} \cdot \vec{v}_1 \vec{a}_1 \cdot \vec{n} + 4\vec{S}_1 \times \vec{n} \cdot \vec{a}_1 \vec{v}_1 \cdot \vec{n} - \vec{S}_1 \times \vec{n} \cdot \vec{a}_1 \right] \]

\[ - \frac{6G^2 m_1 m_2}{2r^2} \vec{v}_1 \cdot \vec{n} \vec{S}_1 \times \vec{n} \cdot \vec{v}_1 \vec{n} \tag{3.39} \]

Fig. 3(b3) = \[ \frac{G^2 m_2^2}{2r^3} \left[ \vec{S}_1 \times \vec{n} \cdot \vec{v}_1 (\vec{v}_1 \cdot \vec{v}_2 - 4\vec{v}_1 \cdot \vec{n} \vec{v}_2 \cdot \vec{n}) - \vec{S}_1 \times \vec{n} \cdot \vec{v}_2 \vec{v}_1 \cdot \vec{n} \right] \\
+ \frac{G^2 m_2^2}{4r^2} \left[ 2\vec{S}_1 \times \vec{n} \cdot \vec{a}_1 \vec{v}_2 \cdot \vec{n} + \vec{S}_1 \times \vec{a}_1 \cdot \vec{v}_2 \right] \]

\[ + \frac{G^2 m_2^2}{4r^2} \left[ 2\vec{S}_1 \times \vec{n} \cdot \vec{v}_1 \vec{v}_2 \vec{v}_1 \cdot \vec{n} + \vec{S}_1 \times \vec{n} \cdot \vec{v}_2 \vec{v}_1 \cdot \vec{n} \right] \tag{3.40} \]

Fig. 3(b4) = \[ - \frac{G^2 m_2^2}{2r^3} \left[ \vec{S}_1 \times \vec{n} \cdot \vec{v}_1 (\vec{v}_1 \cdot \vec{v}_2 - \vec{v}_2^2 - 4\vec{v}_1 \cdot \vec{n} \vec{v}_2 \cdot \vec{n} + 2(\vec{v}_2 \cdot \vec{n})^2) - \vec{S}_1 \times \vec{n} \cdot \vec{v}_2 (\vec{v}_1^2 \right] \]
\begin{align}
&+ 4\vec{v}_1 \cdot \vec{n}\vec{v}_2 \cdot \vec{n} - 4(\vec{v}_1 \cdot \vec{n})^2) - \vec{S}_1 \times \vec{v}_1 \cdot \vec{v}_2 (3\vec{v}_1 \cdot \vec{n} - 2\vec{v}_2 \cdot \vec{n}) \\
&- \frac{G^2 m_2^2}{2r^2} \left[ \vec{S}_1 \times \vec{n} \cdot \vec{a}_1 \vec{v}_2 \cdot \vec{n} - \vec{S}_1 \times \vec{n} \cdot \vec{v}_2 \vec{a}_1 \cdot \vec{n} + \vec{S}_1 \times \vec{a}_1 \cdot \vec{v}_2 \right] \\
&- \frac{G^2 m_2^2}{2r^2} \left[ \vec{S}_1 \times \vec{n} \cdot \vec{v}_1 \vec{v}_2 \cdot \vec{n} - \vec{S}_1 \times \vec{n} \cdot \vec{v}_2 \vec{v}_1 \cdot \vec{n} + \vec{S}_1 \times \vec{v}_1 \cdot \vec{v}_2 \right]
\end{align}

\text{Fig. 3(b5) = } - \frac{G^2 m_2 m_2}{2r^3} \left[ \vec{S}_1 \times \vec{n} \cdot \vec{v}_1 (v_1^2 - 3\vec{v}_1 \cdot \vec{v}_2 + 12\vec{v}_1 \cdot \vec{n}\vec{v}_2 \cdot \vec{n} - 7(\vec{v}_1 \cdot \vec{n})^2) \\
+ 3\vec{S}_1 \times \vec{v}_1 \cdot \vec{v}_2 \vec{v}_1 \cdot \vec{n} \right] + \frac{G^2 m_1 m_2}{4r^2} \left[ 7\vec{S}_1 \times \vec{n} \cdot \vec{v}_1 \vec{a}_1 \cdot \vec{n} + \vec{S}_1 \times \vec{n} \cdot \vec{a}_1 (3\vec{v}_1 \cdot \vec{n}) \\
- 4\vec{v}_2 \cdot \vec{n} + 2\vec{S}_1 \times \vec{v}_1 \cdot \vec{a}_1 - 2\vec{S}_1 \times \vec{a}_1 \cdot \vec{v}_2 \right] + \frac{G^2 m_1 m_2}{2r^2} \left[ \vec{S}_1 \times \vec{n} \cdot \vec{v}_1 (5\vec{v}_1 \cdot \vec{n} \\
- 2\vec{v}_2 \cdot \vec{n}) - \vec{S}_1 \times \vec{v}_1 \cdot \vec{v}_2 \right]
\end{align}

\text{Fig. 3(b6) = } \frac{G^2 m_1 m_2}{2r^3} \left[ \vec{S}_1 \times \vec{n} \cdot \vec{v}_1 (2v_1^2 - 3\vec{v}_1 \cdot \vec{v}_2 + 12\vec{v}_1 \cdot \vec{n}\vec{v}_2 \cdot \vec{n} - 5(\vec{v}_1 \cdot \vec{n})^2) \\
+ 3\vec{S}_1 \times \vec{v}_1 \cdot \vec{v}_2 \vec{v}_1 \cdot \vec{n} \right] + \frac{G^2 m_1 m_2}{4r^2} \left[ 5\vec{S}_1 \times \vec{n} \cdot \vec{v}_1 \vec{a}_1 \cdot \vec{n} + \vec{S}_1 \times \vec{n} \cdot \vec{a}_1 \vec{v}_1 \cdot \vec{n} \\
+ 2\vec{S}_1 \times \vec{v}_1 \cdot \vec{a}_1 \right] + \frac{3G^2 m_1 m_2}{2r^2} \vec{v}_1 \cdot \vec{n} \vec{S}_1 \times \vec{n} \cdot \vec{v}_1
\end{align}

\text{Fig. 3(b7) = } \frac{G^2 m_1 m_2}{2r^3} \left[ \vec{S}_1 \times \vec{n} \cdot \vec{v}_1 (v_1^2 - 4(\vec{v}_2 \cdot \vec{n})^2) - 2\vec{S}_1 \times \vec{v}_1 \cdot \vec{v}_2 \vec{v}_2 \cdot \vec{n} \right]
\end{align}

\text{Fig. 3(b8) = } - \frac{8G^2 m_2^2}{r^3} \left[ \vec{S}_1 \times \vec{n} \cdot \vec{v}_1 (v_2^2 - 4(\vec{v}_2 \cdot \vec{n})^2) \right] - 2\vec{S}_1 \times \vec{v}_1 \cdot \vec{v}_2 \vec{v}_2 \cdot \vec{n}
\end{align}

\text{Fig. 3(b9) = } - \frac{6G^2 m_1 m_2}{r^3} \vec{S}_1 \times \vec{n} \cdot \vec{v}_1 (v_1^2 - 4(\vec{v}_1 \cdot \vec{n})^2)
\end{align}

\text{Fig. 3(b10) = } \frac{2G^2 m_1 m_2}{2r^3} \left[ \vec{S}_1 \times \vec{n} \cdot \vec{v}_1 (v_2^2 - 2(\vec{v}_2 \cdot \vec{n})^2) - \vec{S}_1 \times \vec{v}_1 \cdot \vec{v}_2 \vec{v}_2 \cdot \vec{n} \right]
\end{align}

\text{Fig. 3(c1) = } - \frac{G^2 m_2^2}{2r^3} \vec{S}_1 \times \vec{n} \cdot \vec{v}_2 - \frac{G^2 m_2^2}{4r^3} \left[ \vec{S}_1 \times \vec{n} \cdot \vec{v}_1 (7\vec{v}_1 \cdot \vec{v}_2 - 16\vec{v}_1 \cdot \vec{n}\vec{v}_2 \cdot \vec{n}) \\
+ 6\vec{S}_1 \times \vec{n} \cdot \vec{v}_2 \vec{v}_2 \cdot \vec{n} - 5\vec{S}_1 \times \vec{v}_1 \cdot \vec{v}_2 \vec{v}_1 \cdot \vec{n} \right] - \frac{G^2 m_2^2}{r^2} \left[ \vec{S}_1 \times \vec{n} \cdot \vec{a}_1 \vec{v}_2 \cdot \vec{n} \\
+ \vec{S}_1 \times \vec{a}_1 \cdot \vec{v}_2 \right] - \frac{G^2 m_2^2}{r^2} \left[ \vec{S}_1 \times \vec{n} \cdot \vec{v}_1 \vec{v}_2 \cdot \vec{n} + \vec{S}_1 \times \vec{v}_1 \cdot \vec{v}_2 \right]
\end{align}

\text{Fig. 3(c2) = } - \frac{G^2 m_1 m_2}{2r^3} \vec{S}_1 \times \vec{n} \cdot \vec{v}_1 + \frac{2G^2 m_1 m_2}{4r^3} \left[ \vec{S}_1 \times \vec{n} \cdot \vec{v}_1 (2v_1^2 + 4\vec{v}_1 \cdot \vec{v}_2 - 3v_2^2 \\
- 16\vec{v}_1 \cdot \vec{n}\vec{v}_2 \cdot \vec{n} - 16(\vec{v}_1 \cdot \vec{n})^2) - 4\vec{S}_1 \times \vec{v}_1 \cdot \vec{v}_2 \vec{v}_1 \cdot \vec{n} \right] + \frac{2G^2 m_1 m_2}{r^2} \left[ 2\vec{S}_1 \times \vec{n} \cdot \vec{a}_1 (\vec{v}_1 \cdot \vec{n} + \vec{v}_2 \cdot \vec{n}) - \vec{S}_1 \times \vec{v}_1 \cdot \vec{a}_1 + \vec{S}_1 \times \vec{a}_1 \cdot \vec{v}_2 \right] + \frac{2G^2 m_1 m_2}{r^2} \left[ 2\vec{S}_1 \times \vec{n} \cdot \vec{v}_1 (\vec{v}_1 \cdot \vec{n} + \vec{v}_2 \cdot \vec{n}) + \vec{S}_1 \times \vec{v}_1 \cdot \vec{v}_2 \right]
\end{align}
Fig. 3(c3) = \(- \frac{G^2 m_2^2}{2r^3} \left[ S_1 \times \vec{n} \cdot \vec{v}_2 (\vec{v}_1 \cdot \vec{v}_2 - 4\vec{v}_1 \cdot \vec{n}\vec{v}_2 \cdot \vec{n}) + S_1 \times \vec{v}_1 \cdot \vec{v}_2\vec{v}_2 \cdot \vec{n} \right] + \frac{G^2 m_2^2}{4r^2} \left[ 2\vec{S}_1 \times \vec{n} \cdot \vec{a}_2 \vec{v}_1 \cdot \vec{n} - S_1 \times \vec{v}_1 \cdot \vec{a}_2 \right] - \frac{G^2 m_2^2}{2r^2} \vec{v}_2 \cdot \vec{n} \vec{S}_1 \times \vec{n} \cdot \vec{v}_2 \right) - \frac{G^2 m_2^2}{4r^2} \vec{S}_1 \times \vec{n} \cdot \vec{a}_2 \right)
\tag{3.50}

Fig. 3(c4) = \(- \frac{G^2 m_1 m_2}{2r^3} S_1 \times \vec{n} \cdot \vec{v}_2 (v_2^2 + 2(\vec{v}_2 \cdot \vec{n})^2) - \frac{G^2 m_2^2}{2r^2} \left[ 2\vec{S}_1 \times \vec{n} \cdot \vec{v}_2\vec{a}_2 \cdot \vec{n} + 2\vec{S}_1 \times \vec{n} \cdot \vec{a}_2 (\vec{v}_1 \cdot \vec{n} - \vec{v}_2 \cdot \vec{n}) - \vec{S}_1 \times \vec{v}_1 \cdot \vec{a}_2 + 2\vec{S}_1 \times \vec{v}_2 \cdot \vec{a}_2 \right] + \frac{G^2 m_2^2}{2r} \vec{S}_1 \times \vec{n} \cdot \vec{a}_2 \right)
\tag{3.51}

Fig. 3(c5) = \(- \frac{G^2 m_1 m_2}{2r^3} \left[ S_1 \times \vec{n} \cdot \vec{v}_1 (2v_2^2 + 3(\vec{v}_1 \cdot \vec{n})^2 - 8(\vec{v}_2 \cdot \vec{n})^2) - 4\vec{S}_1 \times \vec{v}_1 \cdot \vec{v}_2\vec{v}_2 \cdot \vec{n} \right] + \frac{G^2 m_1 m_2}{4r^2} \left[ S_1 \times \vec{n} \cdot \vec{v}_1 (3\vec{a}_1 \cdot \vec{n} + 4\vec{a}_2 \cdot \vec{n}) - \vec{S}_1 \times \vec{n} \cdot \vec{a}_1 \vec{v}_1 \cdot \vec{n} + 2\vec{S}_1 \times \vec{v}_1 \cdot \vec{a}_1 \right] + \frac{3G^2 m_1 m_2}{2r^2} \vec{v}_1 \cdot \vec{n} \vec{S}_1 \times \vec{n} \cdot \vec{v}_1 + \frac{G^2 m_1 m_2}{2r} \vec{S}_1 \times \vec{n} \cdot \vec{a}_1 \right)
\tag{3.52}

Fig. 3(c6) = \(- \frac{G^2 m_1 m_2}{2r^3} \left[ S_1 \times \vec{n} \cdot \vec{v}_1 (v_1^2 - 2v_2^2 - (\vec{v}_1 \cdot \vec{n})^2 + 8(\vec{v}_2 \cdot \vec{n})^2) + 4\vec{S}_1 \times \vec{v}_1 \cdot \vec{v}_2\vec{v}_2 \cdot \vec{n} \right] + \frac{G^2 m_1 m_2}{4r^2} \left[ S_1 \times \vec{n} \cdot \vec{a}_1 \cdot \vec{n} - 3\vec{a}_2 \cdot \vec{n} - \vec{S}_1 \times \vec{n} \cdot \vec{a}_1 (3\vec{v}_1 \cdot \vec{n} - 4\vec{v}_2 \cdot \vec{n}) - 2\vec{S}_1 \times \vec{v}_1 \cdot \vec{v}_2 - 2\vec{S}_1 \times \vec{v}_1 \cdot \vec{a}_2 \right] - \frac{G^2 m_1 m_2}{2r^2} \left[ \vec{S}_1 \times \vec{n} \cdot \vec{v}_1 (\vec{v}_1 \cdot \vec{n}) + 2\vec{v}_2 \cdot \vec{n} + \vec{S}_1 \times \vec{v}_1 \cdot \vec{v}_2 \right] + \frac{G^2 m_1 m_2}{2r} \vec{S}_1 \times \vec{n} \cdot \vec{a}_1 \right)
\tag{3.53}

Fig. 3(c7) = \(- \frac{G^2 m_1 m_2}{2r^3} \left[ S_1 \times \vec{n} \cdot \vec{v}_1 (\vec{v}_1 \cdot \vec{v}_2 - 4\vec{v}_1 \cdot \vec{n}\vec{v}_2 \cdot \vec{n}) - \vec{S}_1 \times \vec{v}_1 \cdot \vec{v}_2\vec{v}_2 \cdot \vec{n} \right] - \frac{G^2 m_1 m_2}{4r^2} \left[ 2\vec{S}_1 \times \vec{n} \cdot \vec{a}_2 \vec{v}_1 \cdot \vec{n} + \vec{S}_1 \times \vec{a}_1 \cdot \vec{v}_2 \right] - \frac{G^2 m_1 m_2}{4r^2} \left[ 2\vec{S}_1 \times \vec{n} \cdot \vec{v}_1\vec{v}_2 \cdot \vec{n} + \vec{S}_1 \times \vec{v}_1 \cdot \vec{v}_2 \right] \right)
\tag{3.54}

Fig. 3(c8) = \frac{8G^2 m_2^2}{r^3} \left[ S_1 \times \vec{n} \cdot \vec{v}_1 (\vec{v}_1 \cdot \vec{v}_2 + v_2^2 - 4\vec{v}_1 \cdot \vec{n}\vec{v}_2 \cdot \vec{n} - 4(\vec{v}_2 \cdot \vec{n})^2) - \vec{S}_1 \times \vec{v}_1 \cdot \vec{v}_2 (\vec{v}_1 \cdot \vec{n}) + 2\vec{v}_2 \cdot \vec{n} \right] + \frac{4G^2 m_2^2}{r^2} \left[ 2\vec{S}_1 \times \vec{n} \cdot \vec{a}_1 \vec{v}_2 \cdot \vec{n} + \vec{S}_1 \times \vec{a}_1 \cdot \vec{v}_2 \right] + \frac{4G^2 m_2^2}{r^2} \left[ 2\vec{S}_1 \times \vec{n} \cdot \vec{v}_1\vec{v}_2 \cdot \vec{n} + \vec{S}_1 \times \vec{v}_1 \cdot \vec{v}_2 \right]
\tag{3.55}

Fig. 3(c9) = \frac{6G^2 m_1 m_2}{r^3} \left[ S_1 \times \vec{n} \cdot \vec{v}_1 (v_1^2 + \vec{v}_1 \cdot \vec{v}_2 - 4\vec{v}_1 \cdot \vec{n}\vec{v}_2 \cdot \vec{n} - 4(\vec{v}_1 \cdot \vec{n})^2) - \vec{S}_1 \times \vec{v}_1 \cdot \vec{v}_2\vec{v}_2 \cdot \vec{n} \right] + \frac{4G^2 m_1 m_2}{r^2} \left[ 2\vec{S}_1 \times \vec{n} \cdot \vec{a}_1 \vec{v}_1 \cdot \vec{n} - \vec{S}_1 \times \vec{v}_1 \cdot \vec{a}_1 \right] + \frac{8G^2 m_1 m_2}{r^2} \vec{v}_1 \cdot \vec{n} \vec{S}_1 \times \vec{n} \cdot \vec{v}_1 \right)
\tag{3.56}

Fig. 3(c10) = \frac{2G^2 m_1 m_2}{r^3} \left[ 2\vec{S}_1 \times \vec{n} \cdot \vec{v}_1 (\vec{v}_1 \cdot \vec{v}_2 - 2\vec{v}_1 \cdot \vec{n}\vec{v}_2 \cdot \vec{n}) - \vec{S}_1 \times \vec{v}_1 \cdot \vec{v}_2\vec{v}_2 \cdot \vec{n} \right]
\tag{3.57}
\[-2G^2 m_1 m_2 \rho \left[ \vec{S}_1 \times \vec{n} \cdot \vec{v}_2 \cdot \vec{n} + \vec{S}_1 \times \vec{a}_1 \cdot \vec{v}_2 \right] - \frac{2G^2 m_1 m_2}{\rho^2} \left[ \vec{S}_1 \times \vec{n} \cdot \vec{v}_1 \vec{v}_2 \cdot \vec{n} \right]
+ \vec{S}_1 \times \vec{v}_1 \cdot \vec{v}_2 \vec{v}_2 \cdot \vec{n} \right] - \frac{2G^2 m_1 m_2}{\rho^2} \left[ \vec{S}_1 \times \vec{n} \cdot \vec{v}_1 \vec{v}_2 \cdot \vec{n} \right]
+ \vec{S}_1 \times \vec{v}_1 \cdot \vec{v}_2 \vec{v}_2 \cdot \vec{n} \right] + \frac{8G^2 m_2^3}{\rho^2} \vec{v}_2 \cdot \vec{n} \vec{S}_1 \times \vec{n} \cdot \vec{v}_2 \right)\]  
(3.57)

Fig. 3(d1) = \frac{8G^2 m_2^3}{\rho^3} \left[ \vec{S}_1 \times \vec{n} \cdot \vec{v}_2 (\vec{v}_1 \cdot \vec{v}_2 - \vec{v}_2^2 - 4\vec{v}_1 \cdot \vec{n} \vec{v}_2 \cdot \vec{n} + 4(\vec{v}_2 \cdot \vec{n})^2)\right]
+ \vec{S}_1 \times \vec{v}_1 \cdot \vec{v}_2 \vec{v}_2 \cdot \vec{n} \right] + \frac{8G^2 m_2^3}{\rho^2} \vec{v}_2 \cdot \vec{n} \vec{S}_1 \times \vec{n} \cdot \vec{v}_2 \right)\]  
(3.58)

Fig. 3(d2) = \frac{6G^2 m_1 m_2}{\rho^3} \left[ \vec{S}_1 \times \vec{n} \cdot \vec{v}_1 (\vec{v}_1^2 - \vec{v}_1 \cdot \vec{v}_2 + 4\vec{v}_1 \cdot \vec{n} \vec{v}_2 \cdot \vec{n} - 4(\vec{v}_1 \cdot \vec{n})^2)\right]
+ \vec{S}_1 \times \vec{v}_1 \cdot \vec{v}_2 \vec{v}_1 \cdot \vec{n} \right] + \frac{8G^2 m_1 m_2}{\rho^2} \vec{v}_1 \cdot \vec{n} \vec{S}_1 \times \vec{n} \cdot \vec{v}_1 \right)\]  
(3.59)

Fig. 3(d3) = \frac{4G^2 m_2^3}{\rho^2} \left[ 2\vec{S}_1 \times \vec{n} \cdot \vec{v}_2 \vec{v}_1 \cdot \vec{n} - \vec{S}_1 \times \vec{v}_1 \cdot \vec{v}_2 \right]\]  
(3.60)

Fig. 3(d4) = \frac{4G^2 m_1 m_2}{\rho^3} \left[ 2\vec{S}_1 \times \vec{n} \cdot \vec{v}_2 \vec{v}_1 \cdot \vec{n} - \vec{S}_1 \times \vec{v}_1 \cdot \vec{v}_2 \right]\]  
(3.61)

Fig. 3(d5) = \frac{2G^2 m_2^3}{\rho^3} \left[ \vec{S}_1 \times \vec{n} \cdot \vec{v}_1 (\vec{v}_1 \cdot \vec{v}_2 - 4\vec{v}_1 \cdot \vec{n} \vec{v}_2 \cdot \vec{n} + 4(\vec{v}_1 \cdot \vec{n})^2)\right]
+ \vec{S}_1 \times \vec{v}_1 \cdot \vec{v}_2 \vec{v}_1 \cdot \vec{n} \right] + \frac{4G^2 m_1 m_2}{\rho^2} \left[ 2\vec{S}_1 \times \vec{n} \cdot \vec{v}_2 \vec{v}_1 \cdot \vec{n} - \vec{S}_1 \times \vec{v}_1 \cdot \vec{v}_2 \right]\]  
(3.62)

Fig. 3(d6) = \frac{2G^2 m_1 m_2}{\rho^3} \left[ \vec{S}_1 \times \vec{n} \cdot \vec{v}_1 \vec{v}_2 \cdot \vec{n} - \vec{S}_1 \times \vec{n} \cdot \vec{v}_1 \vec{v}_2 \cdot \vec{n} \right]
+ \vec{S}_1 \times \vec{v}_1 \cdot \vec{v}_2 \vec{v}_1 \cdot \vec{n} \right] + \frac{4G^2 m_1 m_2}{\rho^2} \left[ 2\vec{S}_1 \times \vec{n} \cdot \vec{v}_2 \vec{v}_1 \cdot \vec{n} - \vec{S}_1 \times \vec{v}_1 \cdot \vec{v}_2 \right]\]  
(3.63)

Fig. 3(e1) = \frac{8G^2 m_2^3}{\rho^3} \left[ \vec{S}_1 \times \vec{n} \cdot \vec{v}_2 \vec{v}_1 \cdot \vec{n} - 4(\vec{v}_2 \cdot \vec{n})^2)\right]\]  
(3.64)

Fig. 3(e2) = \frac{2G^2 m_1 m_2}{\rho^3} \left[ 2\vec{S}_1 \times \vec{n} \cdot \vec{v}_1 \vec{v}_2 \vec{v}_1 \cdot \vec{n} \right]\]  
(3.65)

Fig. 3(e3) = \frac{2G^2 m_1 m_2}{\rho^3} \left[ \vec{S}_1 \times \vec{n} \cdot \vec{v}_1 \vec{v}_2 \vec{v}_1 \cdot \vec{n} + 2\vec{S}_1 \times \vec{n} \cdot \vec{v}_1 \vec{v}_2 \vec{v}_1 \cdot \vec{n} \right]
+ \frac{4G^2 m_1 m_2}{\rho^2} \vec{v}_2 \vec{v}_1 \cdot \vec{n} \right]\]  
(3.66)

Fig. 3(e4) = \frac{2G^2 m_2^3}{\rho^3} \left[ 3\vec{S}_1 \times \vec{n} \cdot \vec{v}_2 \vec{v}_1 \cdot \vec{n} - 4\vec{S}_1 \times \vec{n} \cdot \vec{v}_2 \vec{v}_1 \cdot \vec{n} \right]\]  
(3.67)

Fig. 3(e5) = \frac{2G^2 m_1 m_2}{\rho^3} \left[ 4\vec{S}_1 \times \vec{n} \cdot \vec{v}_1 \vec{v}_2 \vec{v}_1 \cdot \vec{n} - 3\vec{S}_1 \times \vec{n} \cdot \vec{v}_2 \vec{v}_1 \cdot \vec{n} + \vec{S}_1 \times \vec{n} \cdot \vec{v}_2 \vec{v}_1 \cdot \vec{n} \right]\]  
(3.68)

Fig. 3(f1) = \frac{2G^2 m_2^3}{\rho^3} \left[ \vec{S}_1 \times \vec{n} \cdot \vec{v}_2 (4\vec{v}_1 \cdot \vec{v}_2 + 3\vec{v}_2^2 - 16\vec{v}_1 \cdot \vec{n} \vec{v}_2 \cdot \vec{n} - 16(\vec{v}_2 \cdot \vec{n})^2)\right]
+ \frac{8G^2 m_2^3}{\rho^2} \vec{v}_2 \vec{v}_1 \cdot \vec{n} \right] + \frac{8G^2 m_2^3}{\rho^2} \vec{v}_2 \vec{v}_1 \cdot \vec{n} \right] - \frac{8G^2 m_2^3}{\rho^2} \vec{v}_2 \vec{v}_1 \cdot \vec{n} \right] + \frac{8G^2 m_2^3}{\rho^2} \vec{v}_2 \vec{v}_1 \cdot \vec{n} \right]\]  
(3.69)

Fig. 3(f2) = \frac{2G^2 m_1 m_2}{\rho^3} \left[ \vec{S}_1 \times \vec{n} \cdot \vec{v}_1 \vec{v}_2 (3\vec{v}_1 \cdot \vec{v}_2 - 4\vec{v}_1 \cdot \vec{n} \vec{v}_2 \cdot \vec{n} - 3\vec{S}_1 \times \vec{n} \cdot \vec{v}_2 \vec{v}_1 \cdot \vec{n} \right]\]  
(3.69)
In figure 3, we illustrate the three-graviton exchange diagrams. These diagrams involve cubic in G interactions and are depicted as follows:

\[ -4 \vec{v}_1 \cdot \vec{n} \times \vec{v}_2 - 4(\vec{v}_1 \cdot \vec{n})^2 \]
\[ + 2G^2 m_1 m_2 \frac{r^2}{r^2} \left[ 3 \vec{S}_1 \times \vec{n} \times \vec{a}_1 \vec{v}_2 - \vec{n} + 3 \vec{S}_1 \times \vec{n} \times \vec{a}_2 \vec{v}_1 - \vec{n} + \vec{S}_1 \times \vec{n} \times \vec{a}_2 \vec{v}_1 - \vec{n} \right] \]
\[ - \vec{S}_1 \times \vec{v}_1 \cdot \vec{a}_2 \]
\[ + 2G^2 m_1 m_2 \frac{r^2}{r^2} \left[ \vec{S}_1 \times \vec{n} \times \vec{v}_1 \vec{v}_2 - \vec{n} - 4\vec{S}_1 \times \vec{n} \times \vec{v}_2 \vec{v}_1 - \vec{n} \right] \]
\[ + 3 \vec{S}_1 \times \vec{v}_1 \cdot \vec{v}_2 \]
\[ (3.70) \]

For the NNLO spin-orbit interaction, we have 7 diagrams at order $G^3$ with no loops, as shown in figure 4. These three-graviton exchange diagrams are constructed with either one-, two-, or three-graviton spin couplings.

The values of these diagrams are given by

\[ \text{Fig. 4(a1)} = -8 \frac{G^3 m_1 m_2}{r^4} \vec{S}_1 \times \vec{v}_2 \times \vec{n}, \]
\[ \text{Fig. 4(a2)} = -8 \frac{G^3 m_1 m_2}{r^4} \vec{S}_1 \times \vec{v}_2 \times \vec{n}, \]
\[ \text{Fig. 4(a3)} = -2 \frac{G^3 m_1 m_2}{r^4} \vec{S}_1 \times \vec{v}_2 \times \vec{n}, \]
\[ \text{Fig. 4(a4)} = 2 \frac{G^3 m_1 m_2}{r^4} \vec{S}_1 \times \vec{v}_2 \times \vec{n}, \]
\[ \text{Fig. 4(b1)} = -16 \frac{G^3 m_3^2}{r^4} \vec{S}_1 \times \vec{v}_2 \times \vec{n}, \]
\[ \text{Fig. 4(b2)} = \text{Fig. 4(b3)} \]

3.3 Cubic in G interaction

3.3.1 Three-graviton exchange

For the NNLO spin-orbit interaction, we have 7 diagrams at order $G^3$ with no loops, as shown in figure 4. These three-graviton exchange diagrams are constructed with either one-, two-, or three-graviton spin couplings.

The values of these diagrams are given by

\[ \text{Fig. 4(a1)} = -8 \frac{G^3 m_1 m_2}{r^4} \vec{S}_1 \times \vec{v}_2 \times \vec{n}, \]
\[ \text{Fig. 4(a2)} = -8 \frac{G^3 m_1 m_2}{r^4} \vec{S}_1 \times \vec{v}_2 \times \vec{n}, \]
\[ \text{Fig. 4(a3)} = -2 \frac{G^3 m_1 m_2}{r^4} \vec{S}_1 \times \vec{v}_2 \times \vec{n}, \]
\[ \text{Fig. 4(a4)} = 2 \frac{G^3 m_1 m_2}{r^4} \vec{S}_1 \times \vec{v}_2 \times \vec{n}, \]
\[ \text{Fig. 4(b1)} = -16 \frac{G^3 m_3^2}{r^4} \vec{S}_1 \times \vec{v}_2 \times \vec{n}, \]
For the NNLO spin-orbit interaction we have 21 diagrams at order $G^3$ with one loop, as shown in figure 5. These diagrams contain both cubic self-interaction and two-graviton worldline couplings.

The values of these diagrams are given by

\begin{align}
\text{Fig. 5(a1)} &= 32 \frac{G^3 m_1^2 m_2^3}{r^4} \vec{S}_1 \cdot \vec{v}_2 \times \vec{n} \\
\text{Fig. 5(a2)} &= 8 \frac{G^3 m_1^2 m_2^2}{r^4} \vec{S}_1 \cdot \vec{v}_2 \times \vec{n} \\
\text{Fig. 5(a3)} &= 8 \frac{G^3 m_1^2 m_2^2}{r^4} \vec{S}_1 \cdot \vec{v}_2 \times \vec{n} \\
\text{Fig. 5(a4)} &= 8 \frac{G^3 m_1^2 m_2^2}{r^4} \vec{S}_1 \cdot \vec{v}_2 \times \vec{n} \\
\text{Fig. 5(a5)} &= 2 \frac{G^3 m_1^2 m_2^2}{r^4} \vec{S}_1 \cdot \vec{v}_2 \times \vec{n} \\
\text{Fig. 5(a6)} &= 2 \frac{G^3 m_1^2 m_2^2}{r^4} \vec{S}_1 \cdot \vec{v}_2 \times \vec{n} \\
\text{Fig. 5(a7)} &= -8 \frac{G^3 m_1^2 m_2^2}{r^4} \vec{S}_1 \cdot \vec{v}_1 \times \vec{n},
\end{align}

3.3.2 Cubic self-interaction with two-graviton exchange

Fig. 4(b2) = $-\frac{G^3 m_1^2 m_2^3}{r^4} \vec{S}_1 \cdot \vec{v}_2 \times \vec{n}$, \hfill (3.78)

Fig. 4(b3) = $\frac{G^3 m_1^2 m_2^3}{r^4} \vec{S}_1 \cdot \vec{v}_1 \times \vec{n}$. \hfill (3.79)
Note that the total value of the diagram in figure 5(b4) equals 0, although it does not stand for a short distance contribution.

### 3.3.3 Two-loop interaction

For the NNLO spin-orbit interaction we have 32 two-loop diagrams at order $G^3$, as shown in figure 6. These diagrams contain two cubic vertices or one quartic vertex, and even include cubic vertices with time dependence. As explained in [19], they contain two-loop Feynman integrals of three kinds: factorizable, nested, and irreducible. The factorizable two-loop diagrams do not contribute at the NNLO level, and they yield here purely short distance contributions, of the form $\delta^{(1)}(\vec{r})$, which are contact interaction terms. For other two-loop diagrams calculations should be made, keeping the dimension $d$ general, and the limit $d \to 3$ is only taken in the end.

For the irreducible two-loop diagrams, which are the most complicated, irreducible two-loop tensor integrals of order 3 are encountered here. These are reduced using the integration by parts method to a sum of factorizable and nested two-loop integrals, as explained in [19], and see appendix A there. In addition to the irreducible two-loop tensor integrals, which were given in appendix A of [19], eqs. (A11), (A12) there, two further irreducible tensor integrals are required here, and we provide them in appendix A below. The values of the two-loop diagrams are given in the following:
Figure 6. NNLO spin-orbit Feynman diagrams of order $G^3$ with two loops.
Fig. 6(a1) = 0, (3.101)
Fig. 6(a2) = 0, (3.102)
Fig. 6(a3) = 0, (3.103)
Fig. 6(a4) = 0, (3.104)
Fig. 6(a5) = 0, (3.105)
Fig. 6(b1) = \(-32 \frac{G^3 m_2^3 S_1 \cdot \vec{v}_2 \times \vec{n}}{r^4}\), (3.106)
Fig. 6(b2) = \(-\frac{24}{5} \frac{G^3 m_2^3 m_2 S_1 \cdot \vec{v}_2 \times \vec{n}}{r^4}\), (3.107)
Fig. 6(b3) = \(\frac{16}{5} \frac{G^3 m_2^3 m_2 S_1 \cdot \vec{v}_1 \times \vec{n}}{r^4}\), (3.108)
Fig. 6(b4) = \(\frac{8}{5} \frac{G^3 m_2^3 S_1 \cdot \vec{v}_2 \times \vec{n}}{r^4}\), (3.109)
Fig. 6(b5) = \(2 \frac{G^3 m_2^3 S_1 \cdot \vec{v}_1 \times \vec{n}}{r^4}\), (3.110)
Fig. 6(b6) = \(\frac{6}{5} \frac{G^3 m_2^3 m_2 S_1 \cdot \vec{v}_1 \times \vec{n}}{r^4}\), (3.111)
Fig. 6(b7) = \(\frac{4}{5} \frac{G^3 m_2^3 m_2 S_1 \cdot \vec{v}_1 \times \vec{n}}{r^4}\), (3.112)
Fig. 6(b8) = \(-\frac{2}{5} \frac{G^3 m_2^3 S_1 \cdot \vec{v}_2 \times \vec{n}}{r^4}\), (3.113)
Fig. 6(b9) = \(-\frac{2}{5} \frac{G^2 m_2^3 m_2 S_1 \cdot \vec{v}_2 \times \vec{n}}{r^4}\), (3.114)
Fig. 6(b10) = \(-\frac{2}{5} \frac{G^3 m_2^3 m_2 S_1 \cdot \vec{v}_2 \times \vec{n}}{r^4}\), (3.115)
Fig. 6(b11) = \(\frac{2}{5} \frac{G^3 m_2^3 S_1 \cdot \vec{v}_1 \times \vec{n}}{r^4}\), (3.116)
Fig. 6(b12) = \(-\frac{1}{5} \frac{G^3 m_2^3 m_2 S_1 \cdot \vec{v}_1 \times \vec{n}}{r^4}\), (3.117)
Fig. 6(b13) = \(\frac{1}{5} \frac{G^3 m_2^3 m_2 S_1 \cdot \vec{v}_1 \times \vec{n}}{r^4}\), (3.118)
Fig. 6(b14) = \(\frac{2}{5} \frac{G^3 m_2^3 S_1 \cdot \vec{v}_2 \times \vec{n}}{r^4}\), (3.119)
Fig. 6(b15) = \(-\frac{3}{5} \frac{G^3 m_2^3 m_2 S_1 \cdot \vec{v}_1 \times \vec{n}}{r^4}\), (3.120)
Fig. 6(b16) = \(\frac{G^3 m_2^3 m_2 S_1 \cdot \vec{v}_1 \times \vec{n}}{r^4}\), (3.121)
Fig. 6(c1) = \(-\frac{4}{5} \frac{G^3 m_1 m_2^2 S_1 \cdot \vec{v}_2 \times \vec{n}}{r^4}\), (3.122)
Fig. 6(c2) = \(\frac{4}{5} \frac{G^3 m_1 m_2^2 S_1 \cdot \vec{v}_1 \times \vec{n}}{r^4}\), (3.123)
Fig. 6(c3) = \(\frac{12}{5} \frac{G^3 m_1 m_2^2 S_1 \cdot \vec{v}_1 \times \vec{n}}{r^4}\), (3.124)
4 Next-to-next-to-leading order spin-orbit potential and Hamiltonian

Summing up all of the Feynman diagrams from the previous section, we obtain the NNLO spin-orbit interaction potential for a binary system of compact spinning objects. We split the potential into several pieces according to the number and order of higher-order time derivatives as follows:

\[ V^{SO}_{NNLO} = V^{(0)} + V^{(1)} + V^{(2)} + V^{(3)} + V^{(4)}. \]  

The ordinary part of the potential, which does not contain higher-order time derivatives, reads

\[
V^{(0)} = -\frac{G m_2}{4r^2} \left[ \mathcal{S}_1 \times \vec{n} \cdot \vec{v}_1 \left(5v_1^2 \vec{v}_1 \cdot \vec{v}_2 - 3v_1^2 v_2^2 + 4\vec{v}_1 \cdot \vec{v}_2 v_2^2 - 3v_1^4 - 3v_2^4 + 6\vec{v}_1 \cdot \vec{n}v_1^2 \vec{v}_2 \cdot \vec{n} 
\right.
\right.
\]
\[
- 6\vec{v}_1 \cdot \vec{n}v_2 \cdot \vec{n}v_1 \cdot \vec{v}_2 + 6\vec{v}_1 \cdot \vec{n}v_2 \cdot \vec{n}v_2 + 3v_2^2(\vec{v}_1 \cdot \vec{n})^2 + 3v_1^2(\vec{v}_2 \cdot \vec{n})^2 - 15(\vec{v}_1 \cdot \vec{n})(\vec{v}_2 \cdot \vec{n})^2
\left.\right]
\]
\[
+ \mathcal{S}_1 \times \vec{n} \cdot \vec{v}_2 \left(v_1^2 v_2^2 - 2\vec{v}_1 \cdot \vec{v}_2 v_2^2 - 2(\vec{v}_1 \cdot \vec{v}_2)^2 + 3v_2^4 - 6\vec{v}_1 \cdot \vec{n}v_2 \cdot \vec{n}v_2 - 3v_2^2(\vec{v}_1 \cdot \vec{n})^2 
\right]
\]
\[
- 3v_2^4(\vec{v}_2 \cdot \vec{n})^2 + 15(\vec{v}_1 \cdot \vec{n})(\vec{v}_2 \cdot \vec{n})^2 \right] + \mathcal{S}_1 \times \vec{n} \cdot \vec{v}_2 \left(\vec{v}_1 \cdot \vec{n}v_1^2 - 2\vec{v}_1 \cdot \vec{n}v_2 \cdot \vec{v}_2 + 2\vec{v}_1 \cdot \vec{n}v_2^2 
\right)
\]
\[
+ 2\vec{v}_2 \cdot \vec{n}v_2^2 - 6\vec{v}_1 \cdot \vec{n}(\vec{v}_2 \cdot \vec{n})^2) \right]
\]
\[
+ \frac{G^2 m_2}{4r^3} \left[ \mathcal{S}_1 \times \vec{n} \cdot \vec{v}_1 \left(13v_1^2 - 41\vec{v}_1 \cdot \vec{v}_2 + 28v_2^2 - 16\vec{v}_1 \cdot \vec{n}v_2 \cdot \vec{n} + 8(\vec{v}_1 \cdot \vec{n})^2 + 12(\vec{v}_2 \cdot \vec{n})^2) 
\right.
\right.
\]
\[
- 2\mathcal{S}_1 \times \vec{n} \cdot \vec{v}_2 \left(v_1^2 + 7\vec{v}_1 \cdot \vec{v}_2 - 8v_2^2 - 56\vec{v}_1 \cdot \vec{n}v_2 \cdot \vec{n} - 4(\vec{v}_1 \cdot \vec{n})^2 + 62(\vec{v}_2 \cdot \vec{n})^2 
\right]
\]
\[
+ \mathcal{S}_1 \times \vec{n} \cdot \vec{v}_2 \left(7\vec{v}_1 \cdot \vec{n} - 38\vec{n} \cdot \vec{n} \right) \right] - \frac{2G^2 m_1 m_2}{r^3} \left[ \mathcal{S}_1 \times \vec{n} \cdot \vec{v}_1 \left(2v_1^2 + \vec{v}_1 \cdot \vec{v}_2 - 3v_2^2 
\right.
\right.
\]
\[
+ 12\vec{v}_1 \cdot \vec{n}v_2 \cdot \vec{n} - 16(\vec{v}_1 \cdot \vec{n})^2 + 4(\vec{v}_2 \cdot \vec{n})^2 \right) - 2\mathcal{S}_1 \times \vec{n} \cdot \vec{v}_2 \left(\vec{v}_1 \cdot \vec{v}_2 - v_2^2 
\right.
\right.
\]
\[
+ 4\vec{v}_1 \cdot \vec{n}v_2 \cdot \vec{n} - 4(\vec{v}_1 \cdot \vec{n})^2 \right) - \mathcal{S}_1 \times \vec{n} \cdot \vec{v}_2 \left(2\vec{v}_1 \cdot \vec{n} - 5\vec{n}v_2 \cdot \vec{n} \right)
\left.\right]
\]
\[
+ \frac{G^3 m_1 m_2}{r^4} \left[ \mathcal{S}_1 \times \vec{n} \cdot \vec{v}_1 - \mathcal{S}_1 \times \vec{n} \cdot \vec{v}_2 \right] + \frac{5G^3 m_1 m_2}{r^4} \left[ \mathcal{S}_1 \times \vec{n} \cdot \vec{v}_1 - \mathcal{S}_1 \times \vec{n} \cdot \vec{v}_2 \right]
\left.\right]
\]
\[
+ \frac{G^3 m_2^2}{r^4} \left[ \mathcal{S}_1 \times \vec{n} \cdot \vec{v}_1 - \mathcal{S}_1 \times \vec{n} \cdot \vec{v}_2 \right] + (1 \leftrightarrow 2). \]
Due to the large number of terms, it makes sense to sort the terms with a single higher-order time derivative into

\[ V = V_a + V_S, \tag{4.3} \]

where

\[
V_a = -\frac{5}{16} \vec{S}_1 \times \vec{v}_1 \cdot \vec{a}_1 v_1^4 + \frac{G m_2}{4r} \left[ \vec{S}_1 \times \vec{n} \cdot \vec{v}_1 \left( 4\vec{v}_1 \cdot \vec{a}_1 \vec{v}_2 \cdot \vec{n} - 4\vec{v}_2 \cdot \vec{n}\vec{a}_1 \cdot \vec{v}_2 + \vec{a}_1 \cdot \vec{n}\vec{v}_2^2 - v_1^2 \vec{a}_2 \cdot \vec{n} + 4\vec{v}_1 \cdot \vec{n}\vec{v}_1 \cdot \vec{a}_2 \right) - 4\vec{v}_1 \cdot \vec{n}\vec{v}_2 \cdot \vec{a}_2 + 3\vec{a}_2 \cdot \vec{n}(\vec{v}_1 \cdot \vec{n})^2 - 3\vec{a}_1 \cdot \vec{n}((\vec{v}_2 \cdot \vec{n})^2) + 2\vec{S}_1 \times \vec{n} \cdot \vec{a}_1 (v_1^2 \vec{v}_2 \cdot \vec{n} - \vec{v}_2 \cdot \vec{n}\vec{v}_1 \cdot \vec{v}_2 + \vec{v}_1 \cdot \vec{n}\vec{v}_2^2 + 3\vec{v}_1 \cdot \vec{n}(\vec{v}_2 \cdot \vec{n})^2 - \vec{S}_1 \times \vec{v}_1 \cdot \vec{a}_1 \left( 12v_1^2 - 12\vec{v}_1 \cdot \vec{v}_2 + 5v_2^2 \right) - 4\vec{v}_1 \cdot \vec{n}\vec{v}_2 \cdot \vec{n} - (\vec{v}_2 \cdot \vec{n})^2) + \vec{S}_1 \times \vec{v}_2 \left( 2\vec{v}_2 \cdot \vec{n}\vec{a}_1 \cdot \vec{v}_2 - \vec{a}_1 \cdot \vec{n}\vec{v}_2^2 + v_1^2 \vec{a}_2 \cdot \vec{n} - 2\vec{v}_1 \cdot \vec{n}\vec{v}_2 \cdot \vec{a}_2 + 4\vec{v}_1 \cdot \vec{n}\vec{v}_2 - 3\vec{a}_2 \cdot \vec{n}(\vec{v}_1 \cdot \vec{n})^2 + 3\vec{a}_1 \cdot \vec{n}((\vec{v}_2 \cdot \vec{n})^2) - 2\vec{S}_1 \times \vec{v}_1 \cdot \vec{v}_2 (6\vec{v}_1 \cdot \vec{a}_1 - \vec{a}_1 \cdot \vec{v}_2 - \vec{v}_1 \cdot \vec{a}_2 + 2\vec{v}_2 \cdot \vec{a}_2 - \vec{v}_1 \cdot \vec{n}\vec{a}_2 \cdot \vec{n}) - \vec{S}_1 \times \vec{v}_2 \cdot \vec{a}_2 (v_1^2 - (\vec{v}_1 \cdot \vec{n})^2) \right] - \frac{2G^2 m_1 m_2}{r^2} \left[ \vec{S}_1 \times \vec{n} \cdot \vec{v}_1 \left( 2\vec{a}_1 \cdot \vec{v}_1 + \vec{a}_2 \cdot \vec{n} \right) + \vec{S}_1 \times \vec{n} \cdot \vec{a}_1 (3\vec{v}_1 \cdot \vec{n} + 2\vec{v}_2 \cdot \vec{n}) - 2\vec{S}_1 \times \vec{n} \cdot \vec{v}_1 \vec{a}_1 \cdot \vec{n} + 4\vec{S}_1 \times \vec{n} \cdot \vec{v}_2 (\vec{a}_1 \cdot \vec{n} + 14\vec{a}_2 \cdot \vec{n}) - 2\vec{S}_1 \times \vec{n} \cdot \vec{a}_2 (\vec{v}_1 \cdot \vec{n} - 18\vec{v}_2 \cdot \vec{n}) = 14\vec{S}_1 \times \vec{v}_1 \cdot \vec{a}_1 + 27\vec{S}_1 \times \vec{a}_1 \cdot \vec{v}_2 + \vec{S}_1 \times \vec{v}_1 \cdot \vec{a}_2 + 12\vec{S}_1 \times \vec{v}_2 \cdot \vec{a}_2 \right] + (1 \leftrightarrow 2), \tag{4.4} \]

and

\[
V_S = \frac{G m_2}{2r} \left[ \vec{S}_1 \times \vec{n} \cdot \vec{v}_1 (v_1^2 \vec{v}_2 \cdot \vec{n} - \vec{v}_2 \cdot \vec{n}\vec{v}_1 \cdot \vec{v}_2 + \vec{v}_1 \cdot \vec{n}\vec{v}_2^2 + \vec{v}_2 \cdot \vec{n}\vec{v}_1^2 - 3\vec{v}_1 \cdot \vec{n}((\vec{v}_2 \cdot \vec{n})^2) - \vec{S}_1 \times \vec{n} \cdot \vec{v}_2 \vec{v}_2 \cdot \vec{n} - \vec{v}_2 \cdot \vec{n}\vec{v}_1^2 + \vec{v}_1 \cdot \vec{n}\vec{v}_2^2 ) - 3\vec{v}_1 \cdot \vec{n}\vec{v}_2^2 \cdot \vec{n} + (\vec{v}_2 \cdot \vec{n})^2) \right] - \frac{2G^2 m_1 m_2}{r^2} \left[ 6\vec{S}_1 \times \vec{n} \cdot \vec{v}_1 (3\vec{v}_1 \cdot \vec{n} - \vec{v}_2 \cdot \vec{n}) - 2\vec{S}_1 \times \vec{n} \cdot \vec{v}_2 (7\vec{v}_1 \cdot \vec{n} - 5\vec{v}_2 \cdot \vec{n}) + 7\vec{S}_1 \times \vec{v}_1 \cdot \vec{v}_2 \right] - \frac{G^2 m_1 m_2}{4r^2} \left[ \vec{S}_1 \times \vec{n} \cdot \vec{v}_1 (3v_1 \cdot \vec{n} - 4\vec{v}_2 \cdot \vec{n}) + 2\vec{S}_1 \times \vec{v}_1 \cdot \vec{v}_2 (\vec{v}_1 \cdot \vec{n} - 15\vec{v}_2 \cdot \vec{n}) + 27\vec{S}_1 \times \vec{v}_1 \cdot \vec{v}_2 \right] + (1 \leftrightarrow 2). \tag{4.5} \]

The piece with two higher-order time derivatives is given by

\[
V = \frac{1}{4} G m_2 \left[ \vec{S}_1 \times \vec{n} \cdot \vec{a}_1 (v_2^2 - (\vec{v}_2 \cdot \vec{n})^2) + 4\vec{S}_1 \times \vec{n} \cdot \vec{a}_1 \vec{v}_2 \cdot \vec{n} + 6\vec{S}_1 \times \vec{a}_1 \cdot \vec{v}_2 \vec{v}_2 \cdot \vec{n} - \vec{S}_1 \times \vec{n} \cdot \vec{a}_2 (v_1^2 - (\vec{v}_1 \cdot \vec{n})^2) - 2\vec{S}_1 \times \vec{v}_1 \cdot \vec{a}_2 \vec{v}_2 \cdot \vec{n} \right] \tag{4.2} \]

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The contribution with three higher-order time derivatives reads

\[- \frac{1}{4} Gm_2 \left[ \ddot{S}_1 \times \dddot{v}_1 (v_2^2 - (\ddot{v}_2 \cdot \dddot{n})) - \dddot{S}_1 \times \dddot{v}_1 (v_2^2 - (\ddot{v}_2 \cdot \dddot{n})) + 6 \dddot{S}_1 \times \dddot{v}_1 \dddot{v}_2 \cdot \dddot{n} \right] \]

\[- \frac{1}{4} Gm_2 \left[ \dddot{S}_1 \times \dddot{v}_1 (5\dddot{a}_1 \cdot \dddot{a}_2 + \dddot{a}_1 \cdot \dddot{n} \dddot{a}_2 \cdot \dddot{n}) + 2 \dddot{S}_1 \times \dddot{v}_1 \dddot{a}_1 (2\dddot{v}_1 \cdot \dddot{a}_2 - 2\dddot{v}_2 \cdot \dddot{a}_2 \\
+ \dddot{v}_1 \cdot \dddot{n} \dddot{a}_2 \cdot \dddot{n}) - \dddot{S}_1 \times \dddot{v}_1 \dddot{a}_2 + \dddot{S}_1 \times \dddot{v}_2 (3\dddot{a}_1 \cdot \dddot{a}_2 + \dddot{a}_1 \cdot \dddot{n} \dddot{a}_2 \cdot \dddot{n}) \right] \]

\[- 2 \dddot{S}_1 \times \dddot{a}_1 \dddot{a}_2 (4\dddot{v}_1 \cdot \dddot{n} - \dddot{v}_2 \cdot \dddot{n}) + \dddot{S}_1 \times \dddot{v}_2 \dddot{a}_2 \dddot{a}_1 \cdot \dddot{n} \right] \]

\[- \frac{1}{4} Gm_2 \left[ 2 \dddot{S}_1 \times \dddot{v}_1 (2\dddot{v}_1 \cdot \dddot{a}_2 - 2\dddot{v}_2 \cdot \dddot{a}_2 + \dddot{v}_1 \cdot \dddot{n} \dddot{a}_2 \cdot \dddot{n}) + 2 \dddot{S}_1 \times \dddot{v}_1 \dddot{a}_1 \left( v_2^2 - (\ddot{v}_2 \cdot \dddot{n})^2 \right) \\
+ 4 \dddot{S}_1 \times \dddot{v}_1 \dddot{a}_1 \dddot{v}_2 \cdot \dddot{n} - 2 \dddot{S}_1 \times \dddot{v}_2 (\dddot{v}_1 \cdot \dddot{a}_2 - 2\dddot{v}_2 \cdot \dddot{a}_2 + \dddot{v}_1 \cdot \dddot{n} \dddot{a}_2 \cdot \dddot{n}) + 2 \dddot{S}_1 \times \dddot{v}_1 \dddot{v}_2 \dddot{a}_2 \cdot \dddot{n} \\
+ 12 \dddot{S}_1 \times \dddot{a}_1 \dddot{v}_2 \dddot{v}_2 \cdot \dddot{n} + 2 \dddot{S}_1 \times \dddot{v}_2 \dddot{a}_2 \left( v_2^2 - 2\dddot{v}_1 \cdot \dddot{n} \dddot{v}_2 \cdot \dddot{n} \right) - 4 \dddot{S}_1 \times \dddot{v}_1 \dddot{a}_2 (2\dddot{v}_1 \cdot \dddot{n} - \dddot{v}_2 \cdot \dddot{n}) \\
+ 2 \dddot{S}_1 \times \dddot{v}_2 \dddot{a}_2 \dddot{v}_1 \cdot \dddot{n} \right] \]

\[- \frac{G^2 m_1 m_2}{r} \left[ \dddot{S}_1 \times \dddot{n} \dddot{a}_1 + 6 \dddot{S}_1 \times \dddot{n} \dddot{a}_2 \right] - \frac{17G^2 m_1 m_2^2}{4r} \dddot{S}_1 \times \dddot{n} \dddot{a}_2 + (1 \leftrightarrow 2). \tag{4.6} \]

The contribution with three higher-order time derivatives of the velocity and spin blow the potential. These can be handled at the level of the EOM through a substitution of lower order EOM. However, it is often more useful to perform the elimination of higher order time derivatives at the level of the potential, and to also transform to a Hamiltonian. This will make the result considerably more compact.

For the reduction of higher order time derivatives we follow the procedure outlined in [26, 36], and its explicit extension for spin variables in [26]. It should be stressed that this procedure is in general not equivalent to a substitution of EOM at the level of the potential, but rather to a redefinition of variables, which combines with the higher order time derivatives to a total time derivative. Yet, as long as this redefinition contributes only linearly to the level of approximation, its result is equivalent to an insertion of the lower order EOM, up to a total time derivative. This total time derivative is the same that arises for the derivation of equations of motion through linear variation of variables. Variation and redefinition of variables are essentially the same.

We are eliminating the higher order time derivatives successively at each PN order, starting with the LO spin-orbit potential at 1.5PN order. As in [7], also see [8, 37], and also
discussed in [18], the variable shift reads
\[ y_1 \rightarrow \bar{y}_1 + \frac{1}{2m_1} \bar{S}_1 \times \bar{v}_1, \] (4.9)
and similar for \( \bar{y}_2 \). This shift corresponds to an insertion of EOM into the LO spin-orbit potential, where the EOM was derived from the complete 3.5PN order potential. The redefinition in eq. (4.9) is linear in the spin, so that its square does not contribute to the spin-orbit sector. After this step the potential, and thus the EOM are changed. The higher order time derivatives at the next PN orders are then eliminated using these modified EOM (the corresponding redefinitions of variables are omitted here due to their length). The remaining higher order time derivatives start to appear at 2PN order, so that the redefinitions would contribute quadratically, and thus the insertion of EOM breaks down, only at the 4PN order. That is, this successive insertion of EOM to remove higher order time derivatives is still valid to linear in spin and 3.5PN order.

It should be noted that this procedure is different from inserting the EOM in the complete 3.5PN order potential in a single step. This would correspond to a redefinition of the position, which contains among others a contribution from the LO spin-orbit and the 2PN order point-mass potentials. The quadratic contribution of this redefinition therefore leads to NNLO spin-orbit terms. That is, the method of inserting EOM breaks down in this case, in contrast to the successive elimination discussed above. Furthermore, both procedures can lead to different (yet equivalent) results, since the total time derivatives generated by the successive linear redefinitions and by the corresponding single quadratic redefinition are in general different.

Next, we can perform a Legendre transformation to obtain a Hamiltonian. For that we need to replace the velocity in terms of canonical momenta, which reads
\[
v_1 = \dot{\bar{p}}_1 - \frac{1}{2} \bar{p}_1 \ddot{\bar{p}}_1^2 + \frac{G m_2}{2 r} \left[ -6 \bar{p}_1 + 7 \bar{p}_2 + n^i \bar{n} \cdot \bar{p}_2 \right] - \frac{2 G}{r^2} \epsilon_{ijk} n^j S^k_2 - \frac{3 G m_2}{2 m_1 r^2} \epsilon_{ijk} n^k S^j_1,
\]
\[
+ \frac{G}{2 r^2} \left[ -5 \epsilon_{ijk} p^k_2 S^j_2 \bar{n} \cdot \bar{p}_1 + 4 \epsilon_{ijk} p^k_2 S^j_2 \bar{n} \cdot \bar{p}_2 + 6 \epsilon_{ijk} n^k S^j_2 \bar{n} \cdot \bar{p}_1 \bar{n} \cdot \bar{p}_2 \right] + 2 \epsilon_{ijk} n^k S^j_2 \bar{n} \cdot \bar{p}_2 - 2 \bar{p}_1 \bar{S}^2_1 \times \bar{n} \cdot \bar{p}_1 - 6 n^i \bar{n} \cdot \bar{p}_2 \bar{S}^2_1 \times \bar{n} \cdot \bar{p}_1 - 3 n^i \bar{n} \cdot \bar{S}_1 \bar{S}_2 \times \bar{n} \cdot \bar{p}_2
\]
\[
+ 6 n^i \bar{n} \cdot \bar{p}_2 \bar{S}_2 \times \bar{n} \cdot \bar{p}_1 - 5 n^i \bar{S}_2 \times \bar{n} \cdot \bar{p}_1 + 2 \bar{p}_2 \right] + \frac{35 G^2 m_1}{4 r^3} \epsilon_{ijk} n^k S^j_2 + \frac{6 G^2 m_2}{2 r^3} \epsilon_{ijk} n^k S^j_2
\]
\[
+ \frac{G m_2}{8 m_1 r^2} \left[ 16 \epsilon_{ijk} p^j_2 S^k_1 \bar{n} \cdot \bar{p}_1 + 5 \epsilon_{ijk} n^j S^k_1 \bar{p}_1^2 - 20 \epsilon_{ijk} \bar{p}^j_2 S^k_1 \bar{n} \cdot \bar{p}_2 \right]
\]
\[
+ 24 \epsilon_{ijk} n^j S^k_1 \bar{n} \cdot \bar{p}_1 \bar{n} \cdot \bar{p}_2 - 10 \bar{p}_1 \bar{S}^2_1 \times \bar{n} \cdot \bar{p}_1 - 24 n^i \bar{n} \cdot \bar{p}_2 \bar{S}^2_1 \times \bar{n} \cdot \bar{p}_1 + 8 \bar{p}_2 \bar{S}^1_1 \times \bar{n} \cdot \bar{p}_2
\]
\[
+ 24 n^i \bar{n} \cdot \bar{p}_2 \bar{S}^1_1 \times \bar{n} \cdot \bar{p}_2 + 16 n^i \bar{S}^2_1 \times \bar{n} \cdot \bar{p}_1 + 2 \bar{p}_2 - 6 \epsilon_{ijk} n^j S^k_1 (\bar{n} \cdot \bar{p}_2)^2] + \frac{7 G^2 m_2}{2 r^3} \epsilon_{ijk} n^j S^k_2
\]
\[
+ \frac{5 G^2 m_2}{m_1 r^3} \epsilon_{ijk} n^j S^k_2,
\]
where we have used the abbreviation \( \bar{p}_m \equiv \bar{p}_m / m_a \). This results in a compact Hamiltonian:
\[
H^{S_{SO}}_{NNLO} = \frac{G m_2}{16 \pi^2} \left[ \bar{S}_1 \times \bar{n} \cdot \bar{p}_1 (4 \bar{p}_1 \bar{n} \times \bar{p}_2 + 16 \bar{p}_1 \bar{n} \cdot \bar{p}_2) - 16 (\bar{p}_1 \cdot \bar{p}_2) \right] + 7 \bar{p}_1^4 - 4 \bar{p}_2^4
\]
\[
+ 24 \bar{p}_1 \cdot \bar{n} \bar{p}_2^2 \bar{n} \cdot \bar{p}_1 + 24 \bar{p}_1 \bar{n} \bar{p}_2 \bar{n} \bar{p}_1 \bar{p}_2 - 12 \bar{p}_2 (\bar{p}_1 \cdot \bar{n})^2 + 3 \bar{p}_1^2 (\bar{p}_2 \cdot \bar{n})^2
\]
\[
- 48 \bar{p}_1 \cdot \bar{p}_2 (\bar{p}_2 \cdot \bar{n})^2 + 24 \bar{p}_2^2 (\bar{p}_2 \cdot \bar{n})^2 + 60 (\bar{p}_1 \cdot \bar{n})^2 (\bar{p}_2 \cdot \bar{n})^2 - 15 (\bar{p}_2 \cdot \bar{n})^4
\]
If the EFT Hamiltonian obtained here in eq. (4.11) is physically equivalent to that of [12, 13], then there exists an infinitesimal generator of g of a canonical transformation such that

$$\Delta H = \{H, g\} = \{H_N + H_{1PN} + H_{2PN}, g_{NNLO} + g_{NLO} + g_{2PN}\}$$

$$\Delta H = \Delta H_{2PN} + \Delta H_{3PN} + \Delta H_{SO}^{NLO} + \Delta H_{SO}^{NNLO},$$

where here we have dropped contributions to sectors beyond linear in spin and beyond NNLO, and where

$$\Delta H = H_{\text{EFT}} - H_{\text{ADM}}.$$  \hfill (4.13)$$

Thus, the contribution to the NNLO spin-orbit sector comprises

$$\Delta H_{SO}^{NNLO} = \{H_N, g_{NNLO}\} + \{H_{1PN}, g_{NLO}\} + \{H_{2PN}, g_{2PN}\},$$

and we also have here contributions to lower orders, given by

$$\Delta H_{SO}^{NLO} = \{H_N, g_{NLO}\},$$

$$\Delta H_{2PN} = \{H_N, g_{2PN}\},$$  \hfill (4.15)$$

so we also require that the canonical transformation is consistent with the equivalence at NLO of the spin-orbit, and 2PN non spinning Hamiltonians.

Similarly to the construction considerations in [26, 31], we find for the infinitesimal generator of PN canonical transformations for the NNLO spin-orbit sector, $g_{NNLO}^{SO}$, the following general form:

$$g_{NNLO}^{SO} = \frac{G m_2}{r} \vec{\gamma}_1 \cdot \vec{p}_1 \times \vec{p}_2 \left( g_1 \vec{p}_1 + g_2 \vec{p}_1 \cdot \vec{p}_2 + g_3 \vec{p}_2 + g_4 \left( \vec{p}_1 \cdot \vec{n} \right)^2 + g_5 \vec{p}_1 \cdot \vec{n} \vec{p}_2 \cdot \vec{n} \right)$$
\begin{align*}
+ g_6 \left( \vec{p}_2 \cdot \vec{n} \right)^2 + \vec{p}_1 \times \vec{n} \left( g_7 \vec{p}_1^2 \vec{p}_1 \cdot \vec{n} + g_8 \vec{p}_1^2 \vec{p}_2 \cdot \vec{n} + g_9 \vec{p}_1 \cdot \vec{p}_2 \vec{n} \cdot \vec{n} \right)
+ g_{10} \vec{p}_1 \cdot \vec{p}_2 \vec{n} + g_{11} \vec{p}_2 \vec{p}_1 \cdot \vec{n} + g_{12} \vec{p}_2 \vec{p}_2 \cdot \vec{n} + g_{13} \vec{n} \left( \vec{p}_2 \cdot \vec{n} \right)^2 \\
+ g_{14} \left( \vec{p}_1 \cdot \vec{n} \right)^2 \vec{p}_2 \cdot \vec{n} + g_{15} \left( \vec{p}_1 \cdot \vec{n} \right)^3 + g_{16} \left( \vec{p}_2 \cdot \vec{n} \right)^3 + \vec{p}_2 \times \vec{n} \left( g_{17} \vec{p}_1^2 \vec{p}_1 \cdot \vec{n} + g_{18} \vec{p}_2^2 \vec{p}_1 \cdot \vec{n} + g_{19} \vec{p}_1 \cdot \vec{p}_2 \vec{n} \cdot \vec{n} + g_{20} \vec{p}_1 \cdot \vec{p}_2 \vec{p}_2 \cdot \vec{n} + g_{21} \vec{p}_2^2 \vec{p}_1 \cdot \vec{n} + g_{22} \vec{p}_2^2 \vec{p}_2 \cdot \vec{n} \\
+ g_{23} \vec{n} \left( \vec{p}_2 \cdot \vec{n} \right)^2 + g_{24} \left( \vec{p}_1 \cdot \vec{n} \right)^2 \vec{p}_2 \cdot \vec{n} + g_{25} \left( \vec{p}_1 \cdot \vec{n} \right)^3 + g_{26} \left( \vec{p}_2 \cdot \vec{n} \right)^3 \right) \\
+ \frac{G^2 m_2}{r^2} \vec{S}_1 \cdot \left[ \vec{p}_1 \times \vec{p}_2 \left( g_{27} m_1 + g_{28} m_2 \right) \right. \\
+ \vec{p}_1 \times \vec{n} \left( \vec{p}_1 \cdot \vec{n} \left( g_{29} m_1 + g_{30} m_2 \right) + \vec{p}_2 \cdot \vec{n} \left( g_{31} m_1 + g_{32} m_2 \right) \right) \\
+ \vec{p}_2 \times \vec{n} \left( \vec{p}_1 \cdot \vec{n} \left( g_{33} m_1 + g_{34} m_2 \right) + \vec{p}_2 \cdot \vec{n} \left( g_{35} m_1 + g_{36} m_2 \right) \right) \right].
\end{align*}

(4.16)

We should also have the generators contributing first to lower orders, as noted in eq. (4.14), so that their coefficients are already set from eq. (4.15). For the NLO spin-orbit sector we have the generator from eq. (7.8) in [26, 31] with its coefficients set to the values
\begin{align}
g_1 = -\frac{1}{2}, \quad g_2 = 0, \quad g_3 = \frac{1}{2}, \quad g_4 = 0, \quad g_5 = 0.
\end{align}

(4.17)

We should also take into account the generator, which contributes first at the 2PN nonspinning sector, from eq. (7.10) in [26] with its coefficients set to
\begin{align}
g_1 = 0, \quad g_2 = -\frac{1}{2}, \quad g_3 = 0, \quad g_4 = 0, \quad g_5 = 0, \quad g_6 = 0, \quad g_7 = -\frac{1}{4}.
\end{align}

(4.18)

Thus, we plug in eq. (4.14) our ansatz for \( g_{\text{NLO}}^{\text{SO}} \) from eq. (4.16), together with the fixed generators in eqs. (4.17), (4.18), and we compare that to eq. (4.13). Comparing \( O(G) \) terms fixes the \( O(G) \) coefficients of \( g_{\text{NLO}}^{\text{SO}} \) to the values
\begin{align*}
g_1 &= \frac{7}{16}, \quad g_2 = -\frac{3}{2}, \quad g_3 = \frac{9}{16}, \quad g_4 = 0, \quad g_5 = 0, \quad g_6 = -\frac{13}{16}, \quad g_7 = 0, \\
g_8 &= \frac{1}{16}, \quad g_9 = 0, \quad g_{10} = -1, \quad g_{11} = \frac{1}{4}, \quad g_{12} = \frac{7}{16}, \quad g_{13} = -\frac{3}{4}, \quad g_{14} = 0, \\
g_{15} = 0, \quad g_{16} = -\frac{3}{16}, \quad g_{17} = 0, \quad g_{18} = 0, \quad g_{19} = 0, \quad g_{20} = 0, \quad g_{21} = 0, \\
g_{22} = 0, \quad g_{23} = 0, \quad g_{24} = 0, \quad g_{25} = 0, \quad g_{26} = 0.
\end{align*}

(4.19)

This eliminates all of the \( O(G) \) terms in the difference. Comparing the remaining \( O(G^2) \) terms in the difference fixes the \( O(G^2) \) coefficients of \( g_{\text{NLO}}^{\text{SO}} \) to the values
\begin{align*}
g_{27} &= -\frac{19}{16}, \quad g_{28} = -\frac{1}{2}, \quad g_{29} = -\frac{53}{8}, \quad g_{30} = -\frac{3}{4}, \quad g_{31} = -3, \\
g_{32} &= -\frac{17}{4}, \quad g_{33} = 2, \quad g_{34} = -\frac{1}{2}, \quad g_{35} = 0, \quad g_{36} = -\frac{11}{2}.
\end{align*}

(4.20)

This eliminates all of the \( O(G^2) \) terms, as well as all terms at \( O(G^3) \) in the difference. Hence, we have shown that the ADM Hamiltonian and the EFT Potential at NNLO spin-orbit are completely equivalent.
4.2 Hamiltonians in the center of mass frame

Since the center of mass of a binary moves uniformly along a straight line, it makes sense to separate this motion and reduce the number of orbital variables. This is achieved by a specialization of the Hamiltonian to the center of mass frame, where the total linear momentum vanishes. Here the total linear momentum is just the sum of the individual momenta, \( \vec{p}_1 + \vec{p}_2 = 0 \), since we are considering the conservative sector where the recoil due to gravitational waves vanishes. The orbital dynamics is then governed by a single canonical momentum \( \vec{p} \equiv \vec{p}_1 = -\vec{p}_2 \) and its conjugate \( \vec{r} = \vec{y}_1 - \vec{y}_2 \), the binary separation vector.

We further introduce the tetrad basis \( \vec{n}, \vec{\lambda}, \vec{l} \), where \( \vec{l} = \vec{L}/L, \vec{\lambda} = \vec{l} \times \vec{n} \), and the orbital angular momentum is \( \vec{L} = r\vec{n} \times \vec{p} \), where a similar tetrad was used in [38] for an approximate solution of the EOM with spin. This is a sensible choice for the description of a spinning binary, as it allows, for instance, a separation of the spin-dependent Hamiltonians to 2PN order in spin from [39], see also [38] for an approximate black hole case, and [41]. All expressions here are then complete to the 3.5PN order with spins. Before we present the spin-dependent Hamiltonians, we provide for completeness the point-mass Hamiltonians to 2PN order in the center of mass frame, reading

\[
\vec{p} = p_r \vec{n} + \frac{L}{r} \vec{\lambda},
\]

where \( p_r = \vec{p} \cdot \vec{n} \). However, we will not expand the spin products as in

\[
\vec{S}_A \cdot \vec{S}_B = \vec{S}_A \cdot \vec{n} \vec{n} \cdot \vec{S}_B + \vec{S}_A \cdot \vec{\lambda} \vec{\lambda} \cdot \vec{S}_B + \vec{S}_A \cdot \vec{l} \vec{l} \cdot \vec{S}_B,
\]

since these spin products better highlight the structure of the interaction and keep expressions more compact. Furthermore, the products \( \vec{S}_A \cdot \vec{S}_A \) are constants in this work. As a convention, we then replace multiple products involving \( \vec{L} \) by the products \( \vec{S}_A \cdot \vec{S}_B \), so with this prescription at most one scalar product containing \( \vec{L} \) remains in each term.

As in [26] we are expressing the results in terms of dimensionless variables, where dimensions of length are rescaled by the total mass \( Gm = G(m_1 + m_2) \) and dimensions of mass are rescaled by the reduced mass \( \mu = m_1 m_2/m \). This rescaling is denoted by a tilde, e.g., \( \tilde{L} = \frac{L}{Gm\mu} \). The dependence on the masses is then reduced to the mass ratio \( q = m_1/m_2 \), or the symmetric mass ratio \( \nu \equiv \mu/m \).

We use here the Hamiltonians arising from the formalism recently set up in [18], and hence within the corresponding gauge choices. We include also the results from the cubic order in spin from [32], see also [39, 40] for the black hole case, and [41]. All expressions here are then complete to the 3.5PN order with spins. Before we present the spin-dependent Hamiltonians, we provide for completeness the point-mass Hamiltonians to 2PN order in the same formalism [35] and center of mass frame, reading

\[
H_N = \frac{\tilde{L}^2}{2r^2} - \frac{1}{\tilde{r}} + \frac{\tilde{p}_r^2}{2},
\]

\[
H_{1\text{PN}} = \frac{1}{8} \left[ \frac{\tilde{L}^2}{\tilde{r}^2} + \tilde{p}_r^2 \right]^2 (3\nu - 1) - \frac{\tilde{L}^2}{2\tilde{r}^2} (3 + \nu) + \frac{1}{2\tilde{r}^2} \tilde{p}_r^2 \left( 3 - 2\nu \right),
\]

\[
H_{2\text{PN}} = \frac{\tilde{L}^6}{16\tilde{r}^6} (1 - 5\nu + 5\nu^2) + \frac{\tilde{L}^4}{8\tilde{r}^4} (5 - 16\nu - 3\nu^2) + \frac{\tilde{L}^2}{4\tilde{r}^2} (11 + 12\nu) - \frac{1}{4\tilde{r}^4} (2 + \nu)
\]

\[
+ \frac{3\tilde{L}^4\tilde{p}_r^2}{16\tilde{r}^6} (1 - 5\nu + 5\nu^2) + \frac{\tilde{L}^2\tilde{p}_r^2}{4\tilde{r}^4} (5 - 18\nu - 4\nu^2) + \frac{9\tilde{p}_r^4}{4\tilde{r}^6} (1 + 2\nu)
\]

\[
+ \frac{3\tilde{L}^2\tilde{p}_r^2}{16\tilde{r}^4} (1 - 5\nu + 5\nu^2) + \frac{\tilde{p}_r^4}{8\tilde{r}^2} (5 - 20\nu - 8\nu^2) + \frac{\tilde{p}_r^6}{16} (1 - 5\nu + 5\nu^2).
\]

The 2PN order in the point-mass sector is sufficient for deriving the spin-dependent part of the binding energy to 4PN order, or to transform the spin-dependent Hamiltonians to EOB gauge up to 4PN order.
The center of mass spin-orbit Hamiltonians to 3.5PN order read

\begin{align}
H_{\text{LO}}^{\text{SO}} &= \frac{\nu \tilde{L} \cdot \tilde{S}_1}{2r^3} (4 + 3q^{-1}) + [1 \leftrightarrow 2], \\
H_{\text{NLO}}^{\text{SO}} &= \left[ \frac{13\nu^2 \tilde{L}^2}{8r^5} - \frac{\nu}{4r^4} (24 + 5\nu) + \frac{43\nu^2 \tilde{L}^2}{8r^3} \right] \tilde{L} \cdot \tilde{S}_1 \\
&\quad + \left[ \frac{5\nu \tilde{L}^2}{8r^5} (2\nu - 1) - \frac{5\nu}{4r^4} (4 + \nu) + \frac{\nu \tilde{L}^2}{8r^3} (34\nu - 5) \right] \frac{\tilde{L} \cdot \tilde{S}_1}{q} + [1 \leftrightarrow 2], \\
H_{\text{NNLO}}^{\text{SO}} &= \left[ \frac{\nu^2 \tilde{L}^4}{16r^7} (22\nu - 19) - \frac{\nu^2 \tilde{L}^2}{16r^5} (265 + 17\nu) + \frac{3\nu}{2r^4} (8 + 7\nu) + \frac{\nu \tilde{L}^2 \tilde{p}_r^2}{8r^3} (46\nu - 19) \\
&\quad - \frac{9\nu^2 \tilde{p}_r^2}{16r^5} (32 + 7\nu) + \frac{\nu \tilde{p}_r^2}{4r^3} (80\nu - 17) \right] \tilde{L} \cdot \tilde{S}_1 + \left[ \frac{\nu \tilde{L}^4}{16r^4} (7 - 32\nu + 17\nu^2) \\
&\quad + \frac{\nu \tilde{L}^2 \tilde{p}_r^2}{16r^3} (42 - 173\nu - 17\nu^2) + \frac{\nu \tilde{p}_r^2}{8r^3} (90 + 61\nu) + \frac{\nu \tilde{L}^2 \tilde{p}_r^2}{16r^5} (14 - 88\nu + 73\nu^2) \\
&\quad + \frac{9\nu^2 \tilde{p}_r^2}{16r^5} (10 - 31\nu - 7\nu^2) + \frac{\nu \tilde{p}_r^2}{16r^3} (7 - 56\nu + 131\nu^2) \right] \frac{\tilde{L} \cdot \tilde{S}_1}{q} + [1 \leftrightarrow 2],
\end{align}

where the NNLO Hamiltonian was derived in the present paper and the lower order Hamiltonians are taken from [18]. The relevant Hamiltonians at quadratic order in spin are also given in [18]. After transformation to the center of mass form, they read

\begin{align}
H_{\text{LO}}^{S_1S_2} &= \frac{\nu}{r^2} \left[ 3\tilde{n} \cdot \tilde{S}_1 \tilde{n} \cdot \tilde{S}_2 - \tilde{S}_1 \cdot \tilde{S}_2 \right], \\
H_{\text{NLO}}^{S_1S_2} &= -\left[ \frac{\nu \tilde{L}^2}{2r^5} (5 + 2\nu) - \frac{7\nu}{r^4} + \frac{7\nu \tilde{p}_r^2}{4r^3} (-2 + \nu) \right] \tilde{S}_1 \cdot \tilde{S}_2 + \frac{\nu \tilde{L}^2}{2r^5} (5 + \nu) \tilde{S}_1 \cdot \tilde{S}_2 \cdot \tilde{\lambda} \\
&\quad + \left[ \frac{3\nu \tilde{L}^2}{4r^5} (2 + 3\nu) - \frac{13\nu}{r^4} + \frac{\nu \tilde{p}_r^2}{4r^3} (-14 + 23\nu) \right] \tilde{n} \cdot \tilde{S}_1 \tilde{n} \cdot \tilde{S}_2 + \frac{\nu \tilde{L} \tilde{p}_r}{4r^4} (4 - 25\nu) \\
&\quad \times \left[ \tilde{n} \cdot \tilde{S}_2 \tilde{S}_1 \cdot \tilde{\lambda} + \tilde{n} \cdot \tilde{S}_1 \tilde{S}_2 \cdot \tilde{\lambda} \right] - \frac{9\nu^2 \tilde{L} \tilde{p}_r}{2r^4} \left[ q \tilde{n} \cdot \tilde{S}_2 \tilde{S}_1 \cdot \tilde{\lambda} + \frac{1}{q} \tilde{n} \cdot \tilde{S}_1 \tilde{S}_2 \cdot \tilde{\lambda} \right], \\
H_{\text{LO}}^{S_2^2} &= \frac{\nu C_1(\text{ES}^2)}{2r^3} \left[ 3(\tilde{n} \cdot \tilde{S}_1)^2 - \tilde{S}_2^2 \right] + [1 \leftrightarrow 2], \\
H_{\text{NLO}}^{S_2^2} &= \nu^2 \left[ \frac{5\tilde{L}^2}{4r^5} - \frac{1}{r^4} + \frac{\tilde{p}_r^2}{8r^3} \right] \tilde{S}_1^2 + \left[ \frac{21\tilde{L}^2}{8r^5} + \frac{\tilde{p}_r^2}{8r^3} \right] (\tilde{n} \cdot \tilde{S}_1)^2 \\
&\quad + \frac{5\tilde{L}^2}{4r^5} \tilde{S}_1 \cdot \tilde{\lambda} \left[ \tilde{S}_1 \cdot \tilde{\lambda} - \frac{\tilde{p}_r}{L} \tilde{n} \cdot \tilde{S}_1 \right] + \nu^2 C_1(\text{ES}^2) \left\{ \left[ \frac{\tilde{L}^2}{2r^5} - \frac{3}{2r^4} - \frac{\tilde{p}_r^2}{4r^3} \right] \tilde{S}_1^2 \\
&\quad + \left[ \frac{7}{2r^4} + \frac{\tilde{p}_r^2}{r^3} \right] (\tilde{n} \cdot \tilde{S}_1)^2 - \frac{\tilde{L}}{2r^4} \tilde{S}_1 \cdot \tilde{\lambda} \left[ \tilde{S}_1 \cdot \tilde{\lambda} - \frac{\tilde{p}_r}{L} \tilde{n} \cdot \tilde{S}_1 \right] \right\} \\
&\quad + \frac{\nu}{q} \left\{ \left[ \frac{1}{r^4} (1 + \nu) + \frac{\tilde{p}_r^2}{8r^3} (1 - 2\nu) \right] \tilde{S}_1^2 + \left[ \frac{\tilde{L}^2}{4r^5} (5 - 4\nu) \right] (\tilde{S}_1^2 - (\tilde{S}_1 \cdot \tilde{\lambda})^2 + \frac{\tilde{p}_r}{L} \tilde{n} \cdot \tilde{S}_1 \tilde{S}_2 \cdot \tilde{\lambda}) \\
&\quad - \left[ \frac{3\tilde{L}^2}{8r^5} (7 - 6\nu) + \frac{1}{r^4} (1 + 2\nu) + \frac{\tilde{p}_r^2}{8r^3} (1 - 2\nu) \right] (\tilde{n} \cdot \tilde{S}_1)^2 \right\} + \frac{\nu C_1(\text{ES}^2)}{q} \left\{ \left[ \frac{\tilde{L}^2}{4r^5} (\nu - 5) \\
&\quad + \frac{1}{2r^4} (4 - 3\nu) + \frac{\tilde{p}_r^2}{4r^3} (1 - 8\nu) \right] \tilde{S}_1^2 - \frac{\tilde{L} \tilde{p}_r}{2r^4} (1 + 2\nu) \tilde{n} \cdot \tilde{S}_1 \tilde{S}_2 \cdot \tilde{\lambda} + \frac{\tilde{L}^2}{2r^5} (1 - \nu) (\tilde{S}_1 \cdot \tilde{\lambda})^2 \right\}
\end{align}
The LO cubic in spin Hamiltonian at 3.5PN order from [32] reads in the center of mass frame

\begin{equation}
H_{\text{LO}}^{S_3} = \frac{3\nu^2}{2r^5} \frac{\tilde{z} \tilde{z}}{L} \left\{ \frac{1}{6q^2} \left[ \tilde{S}_1^2 - 5(\tilde{n} \cdot \tilde{S}_1)^2 \right] \left[ 4C_{1(\text{BS}^3)}(1 + q) - 3C_{1(\text{ES}^3)} \right] + q \left[ \tilde{S}_2 - (\tilde{S}_2 \cdot \tilde{\lambda}) \tilde{S}_2 \right] \right\} \frac{1}{q} \left[ \tilde{S}_1 - 2\tilde{S}_1 \tilde{n} \cdot \tilde{S}_1 \tilde{n} \cdot \tilde{S}_1 \tilde{n} \right] + \frac{C_{2(\text{ES}^3)}}{2} \left[ 3\tilde{S}_2 - 5(\tilde{n} \cdot \tilde{S}_2)^2 \right] - \frac{1}{q} \left[ 2\tilde{S}_1 \tilde{S}_2 - 5\tilde{n} \cdot \tilde{S}_1 \tilde{n} \cdot \tilde{S}_2 - \tilde{S}_1 \tilde{S}_2 \cdot \tilde{\lambda} \right] + \frac{C_{2(\text{ES}^3)}}{2} (3 + 4q) \left[ 3\tilde{S}_2 - 5(\tilde{n} \cdot \tilde{S}_2)^2 \right] - 2(\tilde{S}_2 \cdot \tilde{\lambda})^2 - \frac{2\tilde{p}_r}{L} \tilde{n} \cdot \tilde{S}_2 \tilde{n} \cdot \tilde{S}_2 \cdot \tilde{\lambda} - C_{1(\text{ES}^3)} (3 + 4q^{-1}) \left[ \tilde{S}_2 \cdot \tilde{S}_2 - \tilde{S}_1 \tilde{S}_2 \cdot \tilde{\lambda} \tilde{S}_2 \cdot \tilde{\lambda} \right] - \frac{\tilde{p}_r}{L} \tilde{n} \cdot \tilde{S}_1 \tilde{n} \cdot \tilde{\lambda} \tilde{S}_2 \cdot \tilde{\lambda} \left[ \frac{\tilde{S}_1}{q} - q\tilde{S}_2 \right] \right\} + [1 \leftrightarrow 2].
\end{equation}

Recall that in the black hole case it holds $C_{A(\text{ES}^3)} = 1 = C_{A(\text{BS}^3)}$, and that $[1 \leftrightarrow 2]$ implies $q \leftrightarrow q^{-1}$. It is nice to note that in the representation that we picked, the Hamiltonian $H_{\text{LO}}^{S_3}$ appears as a correction to the spin-orbit interaction at 3.5PN order, which is nonlinear in the spins. It should also be noted though that the representation is not unique. One can expand a scalar product of spins with eq. (4.22), which produces two products involving $\tilde{L}$, and then absorb two different products involving $\tilde{L}$ by rewriting them in terms of a spin product.

All of the results in the present section are still gauge dependent. We are going to work out gauge invariant relations in the following section, which however require restrictions on the orbit and spin orientations.

5 Complete gauge invariant relations to 3.5PN order with spins

Starting from the Hamiltonians given in the last section, it is straightforward to obtain gauge invariant relations between the binding energy and the orbital frequency or orbital angular momentum for circular orbits. Circular orbits are defined by $\tilde{r} = \text{const}$. This holds if the radial momentum vanishes, $\tilde{p}_r = 0$. Since there is no obvious way to define spin orientations in a gauge invariant manner, we further restrict to aligned spins, $\tilde{S}_A \cdot \tilde{n} = 0 = \tilde{S}_A \cdot \tilde{\lambda}$, and $\tilde{S}_A \cdot \tilde{L} = S_AL$. The relations presented here were given in part already in [26]. Here we complete these relations to the 3.5PN order with spins by including recent results for the cubic order in spin [32]. The computation follows that in [26].

The circular orbit relation between $\tilde{L}$ and $\tilde{r}$ takes the form

\begin{equation}
\frac{1}{\tilde{r}} = \frac{1}{\tilde{L}^2} + \frac{4}{\tilde{L}^4} + \frac{1}{\tilde{L}^6} \left[ \frac{25}{8} - \frac{43\nu}{8} \right] + \left[ \frac{3\nu}{L^6} + \frac{\nu}{2L^8} (290 + 19\nu) \right] \tilde{S}_1 \tilde{S}_2 + \left\{ - \frac{6\nu}{L^5} + \frac{\nu}{8L^7} (648 - 47\nu) + \frac{\nu}{16L^9} (15348 - 4508\nu + 13\nu^2) \right\} \tilde{S}_1
\end{equation}

\begin{equation}
- \left[ \frac{9\nu}{2L^5} + \frac{\nu}{8L^7} (445 - 44\nu) + \frac{\nu}{16L^9} (10029 - 3614\nu + 14\nu^2) \right] \tilde{S}_1 \frac{1}{q} + \left[ \frac{3\nu}{2L^6} C_{1(\text{ES}^3)} + \frac{\nu}{4L^8} (121 + 112\nu + 2(49 + 5\nu) C_{1(\text{ES}^3)}) \right] \tilde{S}_1^2
\end{equation}

\begin{equation}
- \frac{\nu^2}{L^9} \left[ 5q(1 + q) C_{2(\text{BS}^3)} + \frac{3}{4} q^2 (48 + 31q) C_{2(\text{ES}^3)} \right] \tilde{S}_2^2
\end{equation}
The point-mass gauge invariant relations to 4PN order can be found in [26]. A gauge invariant quantity is the orbital frequency $\omega$, which is commonly expressed in terms of $x = \omega^{2/3}$ in the binding energy. The gauge invariant relation between $\tilde{L}$ and $x$ has the following addition compared to eq. (8.24) in [26]:

$$\frac{1}{L^2} = x + \cdots + x^{9/2}\nu^2 \left[ [6C_{2(BS^3)} - 26C_{2(E5^2)} + 6q(C_{2(BS^3)} - 4C_{2(E5^2)})] q\tilde{S}_2^3 \right. $$

$$ - \left. [52 + 6C_{2(E5^2)} + 8q(6 + C_{2(E5^2)})] \tilde{S}_1\tilde{S}_2^2 + [1 \leftrightarrow 2] \right] . \tag{5.2}$$

Finally, the gauge invariant relation for the binding energy as a function of the orbital angular momentum reads

$$e_{\text{spin}}(L) = \left( \tilde{S}_1 + \tilde{S}_2 \right) \frac{\nu}{L^5} \left[ 2 + \frac{1}{L^2} \left( \frac{3\nu}{8} + 18 \right) + \frac{1}{L^4} \left( \frac{5\nu^2}{16} - 27\nu + 162 \right) \right]$$

$$+ \left( \tilde{S}_1/q + \tilde{S}_2 q \right) \frac{\nu}{L^5} \left[ \frac{3}{2} + \frac{99}{8L^2} - \frac{1}{L^4} \left( \frac{195\nu}{8} - \frac{1701}{16} \right) \right]$$

$$- \frac{\nu \tilde{S}_1 \tilde{S}_2}{L^6} \left[ 1 + \frac{1}{L^2} \left( \frac{13\nu}{4} + \frac{69}{2} \right) \right] - \frac{\nu \tilde{S}_2^3}{L^5} \left[ q \frac{C_{2(E5^2)}}{2} + \frac{1}{8L^2} (54\nu + 63) \right]$$

$$+ \frac{C_{2(E5^2)}}{4L^2} (5\nu + 21) + \frac{\nu}{L^2} \left( \frac{65}{8} + C_{2(E5^2)} \right) \right] - \frac{\nu^2}{L^9} \left[ q\tilde{S}_2^3 [q(6C_{2(E5^2)} + C_{2(BS^3)})$$

$$+ 9C_{2(E5^2)} + C_{2(BS^3)}] + \tilde{S}_1\tilde{S}_2^2 [12(1 + C_{2(E5^2)} q + 9(2 + C_{2(E5^2)}))] + [1 \leftrightarrow 2] \right] , \tag{5.3}$$

and as a function of $x$ it reads

$$e_{\text{spin}}(x) = \nu x^{5/2} \left( \tilde{S}_1 + \tilde{S}_2 \right) \left[ -\frac{4}{3} + x \left( \frac{31\nu}{18} - 4 \right) - x^2 \left( \frac{7\nu^2}{12} - \frac{211\nu}{8} + \frac{27}{2} \right) \right]$$

$$+ \nu x^{5/2} \left( \tilde{S}_1/q + \tilde{S}_2 q \right) \left[ -1 + x \left( \frac{5\nu}{3} - \frac{3}{2} \right) - x^2 \left( \frac{5\nu^2}{8} - \frac{39\nu}{2} + \frac{27}{8} \right) \right]$$

$$+ \nu x^{3/2} \tilde{S}_1 \tilde{S}_2 \left[ 1 + x \left( \frac{5\nu}{18} + \frac{5}{6} \right) + \nu x^{3/2} \tilde{S}_2^2 \left[ q \frac{C_{2(E5^2)}}{2} + \frac{5x}{6} \nu - 3 \right]$$

$$+ \frac{5x C_{2(E5^2)}}{4} \nu + 1 \right) + \nu x \left( \frac{25}{18} + \frac{5C_{2(E5^2)}}{3} \right) \right] + \nu^2 x^{9/2} \left[ q\tilde{S}_2^3 [2(C_{2(E5^2)} - C_{2(BS^3)})$$

$$+ q(3C_{2(E5^2)} - 2C_{2(BS^3)})] + \tilde{S}_1\tilde{S}_2^2 [q(6 - 4C_{2(E5^2)}) + (4 - 3C_{2(E5^2)})] \right] + [1 \leftrightarrow 2] \right] . \tag{5.4}$$

The point-mass gauge invariant relations to 4PN order can be found in [6].

6 Conclusions

In this work we implemented the EFT for gravitating spinning objects in the PN scheme [18] at the NNLO level, which was first treated in [19]. We derived the NNLO spin-orbit interaction potential at the 3.5PN order for rapidly rotating compact objects. Such high PN orders
are required for the successful detection of gravitational radiation, as the EOB Hamiltonian, e.g., requires parameters for the even higher 5PN and 6PN orders in the point-mass case, in order to produce good waveforms. From the NNLO spin-orbit interaction potential, which we obtain here in a Lagrangian form for the first time, we directly derived the corresponding Hamiltonian. We then compared our result to the ADM Hamiltonian result [12], and arrived at a complete agreement between the ADM and EFT results. Therefore, in order to complete the spin dependent conservative sector to 4PN order, it remains to apply the EFT for gravitating spinning objects [18] at NNLO to quadratic level in the spin, and for higher order in spin finite size effects, as was already done in [32] for cubic and quartic orders in spin. The NNLO spin-squared result is indeed presented in another recent paper [27], which then completes the conservative sector to 4PN order. Finally, we provide the relevant Hamiltonians in the center of mass frame, and the complete gauge invariant relations among the binding energy, angular momentum, and orbital frequency of an inspiralling binary with generic compact spinning components to 3.5PN order.

The spin-orbit sector constitutes the most elaborate spin dependent sector at each order, and accordingly we encountered here a proliferation of the relevant Feynman diagrams, where there are 132 diagrams contributing to this sector, and a significant increase of the computational complexity, e.g. there are 32 two-loop diagrams here. We also recall that as the spin is derivative-coupled, higher-order tensor expressions are required for all integrals involved in the calculations, compared to the non spinning case. However, the computation is made efficient through the use of the “NRG” fields, which are advantageous also in the spin dependent sectors, together with the various gauge choices included in the EFT for gravitating spinning objects [18]. In addition, we automatized the EFT computations here, and carried out the automated computations in parallel. Hence, it is clear that for higher order corrections automated EFT computations, utilizing the “NRG” fields, should be implemented, and are most powerful and efficient. It should be stressed that in order to obtain such higher order results, all lower order results are required consistently within one formalism, and so also for that the derivation presented in this work is essential. This work then paves the way for the obtainment of the next-to-NNLO spin-orbit interaction potential at 4.5PN order for rapidly rotating compact objects, once this level of accuracy would be approached.

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A Irreducible two-loop tensor integrals

In the evaluation of the irreducible two-loop diagrams we encounter irreducible two-loop tensor integrals up to order 3. These are reduced using the integration by parts method to a sum of factorizable and nested two-loop integrals, as explained in [19], and see appendix A there. In addition to the irreducible two-loop tensor integrals, which were given in appendix A

\[ \text{Irreducible two-loop tensor integrals} \]

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of [19], the two following reductions are also required here:

\[
\int_{k_1, k_2} \frac{k_1^i}{k_1^2 (p - k_1)^2 k_2^2 (p - k_2)^2 (k_1 - k_2)^2} = \frac{1}{d-4} \int_{k_1, k_2} \left[ \frac{1}{k_1^2 (p - k_1)^2 k_2^2 (p - k_2)^4} - \frac{1}{k_1^2 (k_1 - k_2)^2 k_2^2 (p - k_2)^4} \right], \tag{A.1}
\]

\[
\int_{k_1, k_2} \frac{k_1^i k_1^j}{k_1^2 (p - k_1)^2 k_2^2 (p - k_2)^2 (k_1 - k_2)^2} = \frac{1}{d-4} \int_{k_1, k_2} \left[ \frac{p^i p^j - p^i k_1^j - p^j k_1^i + 2 k_1^i k_1^j}{k_1^2 (p - k_1)^2 k_2^2 (p - k_2)^4} \right. \\
\left. - \frac{p^i p^j - p^i k_1^j - p^j k_1^i + 2 k_1^i k_1^j}{k_1^2 (k_1 - k_2)^2 k_2^2 (p - k_2)^4} \right]. \tag{A.2}
\]

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