The Post-Newtonian Limit of $f(R)$-gravity in the Harmonic Gauge

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A general analytic procedure is developed for the post-Newtonian limit of $f(R)$-gravity with metric approach in the Jordan frame by using the harmonic gauge condition. In a pure perturbative framework and by using the Green function method a general scheme of solutions up to $(v/c)^4$ order is shown. Considering the Taylor expansion of a generic function $f$ it is possible to parameterize the solutions by derivatives of $f$. At Newtonian order, $(v/c)^2$, all more important topics about the Gauss and Birkhoff theorem are discussed. The corrections to ”standard” gravitational potential ($tt$-component of metric tensor) generated by an extended uniform mass ball-like source are calculated up to $(v/c)^4$ order. The corrections, Yukawa and oscillating-like, are found inside and outside the mass distribution. At last when the limit $f \to R$ is considered the $f(R)$-gravity converges in General Relativity at level of Lagrangian, field equations and their solutions.

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I. INTRODUCTION

The study of possible modifications of Einstein’s theory of gravity has a long history which reaches back to the early 1920s [1–3]. Corrections to the gravitational Lagrangian, leading to higher-order field equations, were already studied by several authors [4–6] shortly after General Relativity (GR) was proposed. Developments in the 1960s and 1970s [7–11], partly motivated by the quantization schemes proposed at that time, made clear that theories containing only a $R^2$ term in the Lagrangian were not viable with respect to their weak field behavior. Buchdahl, in 1962 [7], rejected pure $R^2$ theories because of the non-existence of asymptotically flat solutions.

In recent years, the effort to give a physical explanation to the today observed cosmic acceleration [12–14] has attracted a good amount of interest in $f(R)$-gravity, considered as a viable mechanism to explain the cosmic acceleration by extending the geometric sector of field equations without the introduction of dark matter and dark energy. There are several physical and mathematical motivations to enlarge GR by these theories. For comprehensive review, see [15–17].

Other issues as, for example, the observed Pioneer anomaly problem [18] can be framed into the same approach [19] and then, apart the cosmological dynamics, a systematic analysis of such theories urges at short scale and in the low energy limit.

While it is very natural to extend Einstein’s gravity to theories with additional geometric degrees of freedom, recent attempts focused on the old idea of modifying the gravitational Lagrangian in a purely metric framework, leading to higher-order field equations. Due to the increased complexity of the field equations in this framework, the main body of works dealt with some formally equivalent theories, in which a reduction of the order of the field equations was achieved by considering the metric and the connection as independent objects [20].

In addition, many authors exploited the formal relationship to scalar-tensor theories to make some statements about the weak field regime [21], which was already worked out for scalar-tensor theories [22]. Also a Post-Newtonian parameterization with metric approach in the Jordan Frame has been considered [23].

In this paper, we study the Post Newtonian limit of $f(R)$ in the harmonic gauge. We are going to focus on the small velocity and weak field limit within the metric approach. In principle, any alternative or extended theory of gravity should allow to recover positive results of General Relativity. It will be very important to check at any level of our modified theory we can cover the outcomes of GR.

The plan of the paper is the following. In the Sec. III we report the complete scheme of Newtonian and post-Newtonian limit of field equations for $f(R)$-gravity and their formal solutions in the harmonic gauge condition. General comments about the mathematical properties of equations, their relative solutions (Gauss and Birkhoff theorem) and Minkowskian behavior of metric tensor are reported. In the Sec. IV we show the complete solutions (Newtonian and post-Newtonian level) when an uniform mass ball-like source is considered. The point-like source limit of the newtonian solution is considered and the compatibility of $f(R)$-gravity with respect to GR is shown. Concluding remarks are drawn in Sec. V.

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II. THE FIELD EQUATIONS UP TO POST-NEWTONIAN LEVEL

Let us start with a general class of higher order theories given by the action

$$\mathcal{A} = \int d^4x \sqrt{-g}[f(R) + \mathcal{L}_m]$$

(1)

where $f$ is an unspecified function of curvature invariant $R$. The term $\mathcal{L}_m$ is the minimally coupled ordinary matter contribution. In the metric approach, the field equations are obtained by varying (1) with respect to $g_{\mu\nu}$. We get

$$H_{\mu\nu} = f'R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}f'' - f'g_{\mu\nu}\Box f' + g_{\mu\nu}\mathcal{X} T_{\mu\nu}$$

(2)

$$H = g^{\alpha\beta}H_{\alpha\beta} = 3\Box f' + f'R - 2f = \mathcal{X} T$$

(3)

Here, $T_{\mu\nu} = \frac{2}{\sqrt{-g}}\frac{\delta(\sqrt{-g}\mathcal{L}_m)}{\delta g_{\mu\nu}}$ is the energy-momentum tensor of matter, while $T = T^\sigma_\sigma$ is the trace, $f' = \frac{df(R)}{dR}$, $\Box = \Box_{\sigma}^\sigma$ and $\mathcal{X} = 8\pi G^1$. The conventions for Ricci’s tensor is $R_{\mu\nu} = R^\sigma_{\mu\sigma\nu}$ while for the Riemann tensor is $R^{\alpha}_{\beta\mu\nu} = \Gamma^\alpha_{\beta\nu\mu} + ...$. The affinities are the usual Christoffel’s symbols of the metric: $\Gamma^\mu_{\alpha\beta} = \frac{1}{2}g^{\mu\sigma}(g_{\alpha\sigma,\beta} + g_{\beta\sigma,\alpha} - g_{\alpha\beta,\sigma})$.

The adopted signature is $(+ − − −)$ (see for the details [24]).

The paradigm of post-Newtonian limit is starting from a develop of metric tensor (and of all additional fields in the theory) with respect to dimensionless quantity $v$. A system of moving bodies radiates gravitational waves and thus loses energy. This loss appears only in the fifth approximation in $v$. In the first four approximations, the energy of the system remains constant. From this it follows that a system of gravitating bodies can be described by a Lagrangian correctly to terms of order $v^4$ in the absence of an electromagnetic field, for which a Lagrangian exists in general only to terms of second order. We thus find the equations of motion of the system in the next approximation after the Newtonian.

To solve our problem we must start with the determination, in this same approximation of the weak gravitational field, of the metric tensor $g_{\mu\nu}$ (for details see [25])

$$g_{\mu\nu} \sim \left(1 + g_{tt}^{(2)}(t, x) + g_{tt}^{(4)}(t, x) + \ldots + g_{tt}^{(3)}(t, x) + \ldots - \delta_{ij} + g_{ij}^{(2)}(t, x) + \ldots\right)$$

(4)

The set of coordinates adopted is $x^\mu = (t, x^1, x^2, x^3)$. The Ricci scalar becomes

$$R \sim R^{(2)}(t, x) + R^{(4)}(t, x) + \ldots$$

(5)

The $n$-th derivative of Ricci function can be developed as

$$f^n(R) \sim f^n(R^{(2)} + R^{(4)} + \ldots) \sim f^n(0) + f^{n+1}(0)R^{(2)} + f^{n+1}(0)R^{(4)} + \frac{1}{2}f^{n+2}(0)R^{(2)}^2 + \ldots$$

(6)

From lowest order of field equations [2] we have

$$f(0) = 0$$

(7)

which trivially follows from the above assumption that the space-time is asymptotically Minkowskian. This result suggests a first consideration. If the Lagrangian is developable around a vanishing value of the Ricci scalar the relation

1. Here we use the convention $c = 1$.
2. The greek index runs between 0 and 3; the latin index between 1 and 3.
will imply that the cosmological constant contribution has to be zero whatever is the \( f(R) \)-gravity theory. This result appears quite obvious but sometime it is not considered in literature \[27\].

The Eqs. (2) and (3) at \( \mathcal{O}(2) \) - order (Newtonian level) become

\[
\begin{align*}
H^{(2)}_{tt} &= f'(0)R^{(2)}_{tt} - \frac{f''(0)}{2}R^{(2)} - f''(0)\triangle R^{(2)} = \mathcal{X} T^{(0)}_{tt} \\
H^{(2)}_{ij} &= f'(0)R^{(2)}_{ij} + \left[ \frac{f''(0)}{2}R^{(2)} + f''(0)\triangle R^{(2)} \right] \delta_{ij} - f''(0)R^{(2)}_{ij} = 0 \\
H^{(2)} &= -3f''(0)\triangle R^{(2)} - f'(0)R^{(2)} = \mathcal{X} T^{(0)}
\end{align*}
\]

where \( \triangle \) is the Laplacian in the flat space, while at \( \mathcal{O}(3) \) - order become

\[
H^{(3)}_{ti} = f'(0)R^{(3)}_{ti} - f''(0)R^{(2)}_{ti} = \mathcal{X} T^{(1)}_{ti}
\]

The solution for the gravitational potential \( g^{(2)}_{tt}/2 \) has a Yukawa-like behavior \[25\] depending by a characteristic length on which it evolves. Besides the Birkhoff theorem at Newtonian level is modified: the solution can be only factorized with a function space-depending and an arbitrary function time depending \[23\]. Still more the corrections to the gravitational potential and the gravito-magnetic effects \[9\] are depending on the only first two derivatives of \( f \) in \( R = 0 \). So different theories from the third derivative admit the same newtonian solution.

Remembering the expressions of Christoffel symbols and using the following approximation for the determinant of metric tensor \( \ln \sqrt{-g} \sim \frac{1}{2} [g^{(2)}_{tt} - g^{(2)}_{mn}] + \ldots \), at \( \mathcal{O}(4) \) - order we have,

\[
\begin{align*}
H^{(4)}_{tt} &= f'(0)R^{(4)}_{tt} + f''(0)R^{(2)}R^{(2)}_{tt} - \frac{f''(0)}{2}R^{(4)} - \frac{f''(0)}{2}g^{(2)}_{tt}R^{(2)} - \frac{f''(0)}{4}R^{(2)}^2 \\
&\quad - f''(0)\left[ g^{(2)}_{mn,m}R^{(2)}_{,n} + \triangle R^{(2)} + g^{(2)}_{tt}R^{(2)} + g^{(2)}_{mn}R^{(2)}_{,mn} - \frac{1}{2} \nabla g^{(2)}_{mn} \cdot \nabla R^{(2)} \right] \\
&\quad - f''(0)\left[ |\nabla R^{(2)}|^2 + R^{(2)}\triangle R^{(2)} \right] = \mathcal{X} T^{(2)}_{tt}
\end{align*}
\]

\[
\begin{align*}
H^{(4)} &= -3f''(0)\triangle R^{(4)} - f'(0)R^{(4)} - 3f''(0)\left[ |\nabla R^{(2)}|^2 + R^{(2)}\triangle R^{(2)} \right] \\
&\quad + 3f''(0)\left[ R^{(2)}_{tt} - g^{(2)}_{mn}R^{(2)}_{,mn} - \frac{1}{2} \nabla (g^{(2)}_{tt} - g^{(2)}_{mm}) \cdot \nabla R^{(2)} - g^{(2)}_{mn}R^{(2)}_{,mn} \right] = \mathcal{X} T^{(2)}
\end{align*}
\]

where \( \nabla \) is the gradient in the flat space. Note that the propagation of Ricci scalar \( R^{(4)} \) has the same dynamics of previous one. The complete knowledge of correction at fourth order for the \( tt \)-component of Ricci tensor fix the third derivative of \( f \) in \( R = 0 \). Also at this level there is a degeneracy of \( f(R) \)-theory: different theories for only the first three derivatives admit the same gravitational field without obviously radiation emission.

To complete the perturbative scheme and later find the solutions it needs to calculate the Ricci tensor components in \[8\], \[9\], \[10\]. After some calculus \[27, 28\] one obtains

\[
\begin{align*}
R^{(2)}_{tt} &= \frac{1}{2} g^{(2)}_{tt,mm} \\
R^{(4)}_{tt} &= \frac{1}{2} g^{(4)}_{tt,mm} + \frac{1}{4} g^{(2)}_{mm,gt_{,n}} + \frac{1}{4} g^{(2)}_{mn,g_{tt,mm}} + \frac{1}{4} g^{(2)}_{mm,tt} - \frac{1}{4} g^{(2)}_{tt,mm}g_{tt,nn} - \frac{1}{4} g^{(2)}_{mm,nn}g_{tt,tt} - g^{(2)}_{tt,nn} \\
R^{(3)}_{ti} &= \frac{1}{2} g^{(3)}_{ti,mm} - \frac{1}{2} g^{(2)}_{mm,m} + \frac{1}{2} g^{(3)}_{mt,mi} + \frac{1}{2} g^{(2)}_{mm,ti} \\
R^{(2)}_{ij} &= \frac{1}{2} g^{(2)}_{ij,mm} - \frac{1}{2} g^{(2)}_{mm,j} - \frac{1}{2} g^{(2)}_{mm,m} - \frac{1}{2} g^{(2)}_{tt,ij} + \frac{1}{2} g^{(2)}_{mm,ij}
\end{align*}
\]

which represent the most general expressions without assuming any gauge condition.
A. The Newtonian limit of $f(R)$-gravity

We want to rewrite and generalize the outcome of (23) by introducing the Green function method (we remember that the Newtonian limit corresponds also to linearization of field equations). Let us start from the trace equation. The solution for the Ricci scalar $R^{(2)}$ in the third line of (8) is

$$R^{(2)}(t, x) = \frac{m^2 \mathcal{X}}{f'(0)} \int d^3x' \mathcal{G}(x, x') T^{(0)}(t, x')$$

(12)

where $m^2 \equiv -\frac{f''(0)}{f'(0)}$ and $\mathcal{G}(x, x')$ is the Green function of field operator $\Delta - m^2$.

The solution for $g_{tt}^{(2)}$, from the first line of (8) by considering that $R^{(2)}_{tt} = \frac{1}{2} \Delta g_{tt}^{(2)}$, is

$$g_{tt}^{(2)}(t, x) = -\frac{\mathcal{X}}{2\pi f'(0)} \int d^3x' \frac{T^{(0)}_{tt}(t, x')}{|x - x'|} - \frac{1}{4\pi} \int d^3x' \frac{R^{(2)}_{tt}(t, x')}{|x - x'|} - \frac{2}{3m^2} R^{(2)}(t, x)$$

(13)

We can check immediately that when $f \rightarrow R$ we find $g_{tt}^{(2)}(t, x) \rightarrow -2G \int d^3x' \frac{\mathcal{G}(x', x)}{|x - x'|}$. The expression (13) is the "modified" gravitational potential (here we have a factor 2) for $f(R)$-gravity. A such solution which is the newtonian limit of $f(R)$-gravity is also gauge-free.

Since we have a linearized version of field equations a such limit corresponds to one of Einstein equation and the linear superposition is satisfied. So the $tt$-component of energy-momentum tensor is, in this limit, the sum of mass energy volume density of sources: $T^{(0)}_{tt} = \Sigma_a M_a \delta(x - x_a)$ where $\delta(x)$ is the delta function.

As it is evident the Gauss theorem is not valid since the force law is not $\propto \frac{1}{|x|}$. The equivalence between a spherically symmetric distribution and point-like distribution is not valid and how the matter is distributed in the space is very important (29).

In this limit the geodesic equation is the Lagrangian of material point-like embedded in the gravitational field and its dynamics follows the Newtonian law:

$$\frac{d^2x}{dt^2} = \frac{1}{2} \nabla g^{(2)}_{tt}(t, x)$$

(14)

B. The post-Newtonian limit of $f(R)$-gravity in the harmonic gauge

To simplify the expressions of the components of the Ricci tensor (11) we can use the condition so-called of harmonic gauge: $g^\mu_\nu T^{(\mu)}_{\nu\sigma} = 0$ (see 23). The Ricci tensor assumes the following simpler form:

$$\begin{align*}
R^{(2)}_{ij} &= \frac{1}{2} \Delta g^{(2)}_{ij} \\
R^{(3)}_{ii} &= \frac{1}{2} \Delta g^{(3)}_{ii} \\
R^{(4)}_{tt} &= \frac{1}{2} \Delta g^{(4)}_{tt} + \frac{1}{2} g^{(2)}_{mn} g^{(2)}_{tt, mn} - \frac{1}{2} g^{(2)}_{tt, tt} - \frac{1}{2} | \nabla g^{(2)}_{tt} |^2
\end{align*}$$

(15)

From the field equation (9) we find the general solution for $g^{(3)}_{ii}$

$$g^{(3)}_{ii}(t, x) = -\frac{\mathcal{X}}{2\pi f'(0)} \int d^3x' \frac{T^{(1)}_{ii}(t, x')}{|x - x'|} + \frac{1}{6\pi m^2} \frac{\partial}{\partial t} \int d^3x' \frac{\nabla' \cdot R^{(2)}_{tt}(t, x')}{|x - x'|}$$

(16)

The choice of harmonic gauge enable us to solve with facility the equation (9) but we lose potential information about the temporal dynamics of $g^{(2)}_{tt}(t, x)$. A such knowledge is very important to obtain at least in perturbative approach some information about the Birkhoff theorem. By hypothesizing a perturbative approach (newtonian-like) we relegated inevitably eventual temporal dynamics only on the temporal variation of matter source. In fact in a such hypothesis of work the motion of bodies embedded in gravitational fields develops very slow with respect to motion.
of matter. Then we have ever an instantaneous readjustment of spacetime. In other words the motion od bodies is
adiabatic and it enables us to factorize the solution and with a time transformation we get a static solution.

From second line of \([30]\) the solution for \(g^{(2)}_{ij}\) follows

\[
g^{(2)}_{ij}(t, x) = \int d^3x' \frac{R^{(2)}(t, x')}{|x - x'|} + \frac{2}{3m^2} R^{(2)}(t, x) - \frac{1}{6\pi m^2 |x|^3} \int d^3x' R^{(2)}(t, x') \delta_{ij}
\]

\[+ \frac{1}{2\pi m^2 |x|^3} \int d^3x' R^{(2)}(t, x') - \frac{2}{3m^2} R^{(2)}(t, x) \frac{x_i x_j}{|x|^2} \delta_{ij}
\]

(17)

where \(\Omega|\) represents the integration volume with radius \(|x|\) (for the details see \([30]\)). By the solutions \([13], [16], [17]\)
we can affirm that it is possible to have solution non-Ricci-flat in vacuum: Higher Order Gravity mimics a matter
source. It is evident from \([13]\) the Ricci scalar is a “matter source” which can curve the spacetime also in absence
of ordinary matter. Then it is clear also that the knowledge of behavior of Ricci scalar inside mass distribution is
fundamental to obtain the behavior of metric tensor outside the mass.

From the fourth order of field equation, we note also the Ricci scalar \((R^{(4)})\) propagates with the same \(m\) (the second
line of \([10]\)) and the solution at second order originates a supplementary matter source in r.h.s. of \([2]\). The solution is

\[
R^{(4)}(t, x) = \int d^3x' \frac{n^2 x}{f'(0)} T^{(2)}(t, x') - g^{(2)}_{mm(m)(t, x')} R^{(2)}_{mn}(t, x') - g^{(2)}_{mm(m)(t, x')} R^{(2)}_{mn}(t, x')
\]

\[+ R^{(2)}_{tt}(t, x') - \frac{m^2}{\mu^2} \left[|\nabla x' R^{(2)}(t, x')|^2 + R^{(2)}(t, x') \Delta x' R^{(2)}(t, x')\right]
\]

\[-\frac{1}{2} \nabla x' \left[ g^{(2)}_{tt}(t, x') - g^{(2)}_{mm(t, x')} \right] \cdot \nabla x' R^{(2)}(t, x') \}
\]

(18)

where \(\mu^4 \equiv - \frac{f'(0)}{3m^2(0)}\). Also in this case we can have a non-vanishing curvature in absence of matter. The solution for
\(g^{(4)}_{tt}\), from the first line of \([10]\), is

\[
g^{(4)}_{tt}(t, x) = \int d^3x' \frac{1}{|x - x'|} \left\{ \frac{\chi T^{(2)}_{tt}(t, x')}{2\pi f'(0)} + \frac{1}{6\pi \mu^4} \left[ |\nabla R^{(2)}(t, x')|^2 + R^{(2)}(t, x') \Delta R^{(2)}(t, x') \right]
\]

\[+ \frac{1}{4 \pi} \left[ g^{(2)}_{mm(t, x')} g^{(2)}_{tt(mm(t, x')} - g^{(2)}_{tt(t, x')} - |\nabla x' g^{(2)}_{tt(t, x')}|^2 - R^{(4)}(t, x') - g^{(2)}_{tt(t, x')} R^{(2)}(t, x') \right]
\]

\[+ \frac{1}{6\pi m^2} \left[ R^{(2)}_{tt}(t, x') \right] \Delta g^{(2)}_{tt}(t, x') + g^{(2)}_{mm(t, x')} R^{(2)}_{mn(t, x')} + \Delta R^{(4)}(t, x')
\]

\[+ g^{(2)}_{tt(t, x')} \Delta R^{(2)}(t, x') + g^{(2)}_{mm(t, x')} R^{(2)}_{mn(t, x')} - \frac{1}{2} \nabla g^{(2)}_{mm}(t, x') \cdot \nabla R^{(2)}(t, x') \}
\]

(19)

We conclude this paragraph by having shown the more general solution of field equations of \(f(R)\)-gravity in the
Newtonian and post-Newtonian limit assuming a coordinates transformation for the whose the gauge harmonic
condition is verified. In the next paragraph we shall apply a such scheme to obtain the explicit form of metric tensor for
a static and spherically symmetric matter source.

III. THE SPACETIME GENERATED BY AN EXTENDED UNIFORM MASS BALL-LIKE SOURCE

Let us consider a ball-like source with mass \(M\) and radius \(\xi\). The energy-momentum tensor is (we are not interesting
to the internal structure)

\[
\left\{ \begin{array}{ll}
T_{\mu\nu} &= \rho(x) u_{\mu} u_{\nu} \\
T &= \rho(x)
\end{array} \right.
\]

(20)
where \( \rho(x) \) is the mass density and \( u_i \) satisfies the condition \( g^{\mu\nu} u_i u_i = 1 \), \( u_i = 0 \). Since (1) the expression (20) becomes

\[
\begin{align*}
T_{tt}(t, x) &\sim \rho(x) + \rho(x) g^{(2)}_{tt}(t, x) = T_{tt}^{(0)}(t, x) + T_{tt}^{(2)}(t, x) \\
T &\sim \rho(x) = T^{(0)}(t, x)
\end{align*}
\]

(21)

The possible choices of Green function, for spherically symmetric systems (i.e. \( G(x, x') = G(|x - x'|) \)), are the following

\[
G(x, x') = \begin{cases} 
- \frac{1}{4\pi} e^{-m|x-x'|} & \text{if } m^2 > 0 \\
C_1 e^{-m|x-x'|} + C_2 e^{im|x-x'|} & \text{if } m^2 < 0
\end{cases}
\]

(22)

with \( C_1 + C_2 = -\frac{1}{4\pi} \). It notes, for any function of only modulus \( h(|x|) \), that

\[
I = \int d^3x' G(x, x') h(x') = -\frac{1}{4\pi} \int d|x'||x'^2| h(|x'|) \int_0^{2\pi} d\phi' \int_0^{\pi} d\theta' e^{-m\sqrt{|x'^2 + |x'|-2|x||}| \cos \alpha} (\sin \theta') (\cos \phi - \phi') \]

where \( \cos \alpha = \cos \theta \cos \theta' + \sin \theta \sin \theta' \cos (\phi - \phi') \) and \( \alpha \) is the angle between two vectors \( x, x' \). In the spherically symmetric case we can choose \( \theta = 0 \) without losing generality (the symmetry of system is independent by the angle). By making the angular integration we get

\[
I = -\frac{1}{2m|x|} \int d|x'||x'|^2 h(|x'|) \left[ e^{-m|x|-|x'|} - e^{-m(|x|+|x'|)} \right]
\]

(24)

An analogous relation is useful also for the Green function of Newtonian mechanics \( |x - x'|^{-1} \):

\[
\int d^3x' h(|x'|) = -\frac{2\pi}{|x|} \int d|x'||x'| \left[ ||x| - |x'|| - |x| - |x'|| \right] h(|x'|)
\]

(25)

A. Solutions at \( O(2) \) - and \( O(3) \) - order

Supposing that \( m^2 > 0 \) (i.e. \( \text{sign}[f'(0)] = -\text{sign}[f''(0)] \)) the Ricci scalar (12), if we hypothesize a matter density as \( \rho(x) = \frac{4M}{4\pi \xi^3} \Theta(\xi - |x|) \), is

\[
R^{(2)}(t, x) = \frac{3 \rho_g}{\xi^3} \left[ 1 - e^{-m\xi} \sinh m|\xi| \right] \Theta(\xi - |x|) - r_g m^2 F(\xi) \frac{e^{-m|\xi|}}{|x|} \Theta(|x| - \xi)
\]

(26)

where \( \Theta \) is the Heaviside function, \( F(x) \equiv \frac{3 \pi \xi e^{-m|\xi|} \sinh m\xi}{m\xi^3} \) and \( r_g = 2GM \) is the Schwarzschild radius\(^3\). The solutions of (13), (16) and (17), given the relations (24) and (25), respectively are

\[
g^{(2)}_{tt}(t, x) = -r_g \left[ 3 \frac{2\xi}{2\xi^3} + \frac{1}{m^2 \xi^3} - \frac{|x|^2}{2\xi^3} - \frac{e^{-m\xi}(1 + m\xi) \sinh m|\xi|}{m|\xi|} \right] \Theta(\xi - |x|) - r_g \left[ \frac{1}{|x|} + \frac{F(\xi) e^{-m|\xi|}}{3 |x|} \right] \Theta(|x| - \xi)
\]

(27)

\[
g^{(3)}_{tt}(t, x) = 0
\]

(28)

\(^3\) we have set for simplicity \( f'(0) = 1 \) (otherwise we have to renormalize the coupling constant \( X \) in the action (3)).
The spatial behavior (26) is shown in FIG. 1. The metric potentials are shown in FIGs. 2, 3 and 4. It is interesting to note as the function $\Phi$ assumes smaller value of its equivalent in GR, then in terms of gravitational attraction we have a potential well more deep. A such scheme can be interpretable or assuming a variation of the gravitational constant $G$ or requiring that there is a central greater mass. These two affirmations are compatible on the one hand with the tensor-scalar theories (in which we have a scaling of gravitational constant) and on the other hand with the theory of GR plus the hypothesis of the existence of the dark matter. In particular, if the mass distribution takes a bigger volume, the potential increases and vice versa.

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For fixed values of the distance $|x|$, the solutions $g_{tt}^{(2)}$ and $g_{ij}^{(2)}$ depend on the value of the radius $\xi$, then the Gauss theorem does not work also if the Bianchi identities hold \[11\]. In other words, since the Green function does not scale as the inverse of distance but has also an exponential behavior, the Gauss theorem is not verified. We can affirm: the potential does not depend only on the total mass but also on the mass - distribution in the space.

By introducing three metric potentials $\Phi(x)$, $\Psi(x)$ and $\Lambda(x)$ (the dimension is the inverse of length) we can rewrite (27) and (29) as follows

\[
\begin{align*}
  g_{tt}^{(2)}(t, x) &= r_g \Phi(x) \\
  g_{ij}^{(2)}(t, x) &= r_g \Psi(x) \delta_{ij} + r_g \Lambda(x) \frac{x_i x_j}{|x|^2}
\end{align*}
\] (30)

and with a fourth function, $\Xi(x)$, (the dimension is the cubic inverse of length) the Ricci scalar (26) is

\[
R^{(2)}(t, x) = r_g \Xi(x)
\] (31)

The spatial behavior (26) is shown in FIG. 1. The metric potentials are shown in FIGs. 2, 3 and 4. It is interesting to note as the function $\Phi$ assumes smaller value of its equivalent in GR, then in terms of gravitational attraction we have a potential well more deep. A such scheme can be interpretable or assuming a variation of the gravitational constant $G$ or requiring that there is a central greater mass. These two affirmations are compatible on the one hand with the tensor-scalar theories (in which we have a scaling of gravitational constant) and on the other hand with the theory of GR plus the hypothesis of the existence of the dark matter. In particular, if the mass distribution takes a bigger volume, the potential increases and vice versa.

![FIG. 1: Plot of dimensionless function $\zeta^4 m^{-3} \Xi$ for $\zeta = m \xi = .5$ representing the spatial behavior of Ricci scalar at second order. In GR we would have $\Xi(x) = \frac{1}{4} \Theta(\xi - |x|)$.](image)

In the limit of point-like source, \textit{i.e.} $\lim_{\xi \to 0} \frac{3M}{32\xi^5} \Theta(\xi - |x|) = M \delta(x)$ and $\lim_{x \to 0} F(x) = 1$, we get
FIG. 2: Plot of metric potential $\zeta m^{-1}\Phi$ vs distance from central mass with $\zeta = m\xi = .5$. The dashed line is the GR behavior:

$$\Phi = -\left[\frac{3}{2r^2} - \frac{|x|^2}{2r^4}\right] \Theta(\xi - x) - \frac{\Theta(\xi + x)}{|x|}.$$

FIG. 3: Plot of metric potential $\zeta m^{-1}\Psi$ vs distance from central mass with $\zeta = m\xi = .5$. The dashed line is the GR behavior (similar to metric potential $\Phi$).

\[
\begin{align*}
R^{(2)}_{tt}(t, x) &= -r_g m^2 \frac{e^{-m|x|}}{|x|} \\
g^{(2)}_{tt}(t, x) &= -r_g \left( \frac{1}{|x|} + \frac{1}{3} \frac{e^{-m|x|}}{|x|} \right) \\
g^{(2)}_{ij}(t, x) &= -r_g \left\{ \frac{1}{|x|} - \frac{2}{3m^2|x|^2} - \frac{2}{3m^2|x|^3} \frac{1}{|x|} - \frac{2}{m^2|x|^2} \right\} e^{-m|x|} \delta_{ij} \\
&= -r_g \left[ \frac{2}{m^2|x|^2} - \frac{2}{3} \left( \frac{1}{|x|} + \frac{3}{m^2|x|^2} \right) e^{-m|x|} \right] \frac{\partial^2}{|x|^2} \\
\end{align*}
\]

An important point has to be considered. The PPN-parameters $\gamma$ and $\beta$, in the GR context, are intended to parameterize the deviations from the Newtonian behavior of the gravitational potentials. They are defined according to the standard Eddington metric (in the vacuum)

\[
\begin{align*}
g_{tt} &= 1 - \frac{r_g}{|x|} + \frac{\beta r^2}{2 |x|^2} \\
g_{ij} &= -\delta_{ij} - \gamma \frac{r_g}{|x|} \delta_{ij} \\
\end{align*}
\]
FIG. 4: Plot of metric potential $\zeta m^{-1} \Lambda$ vs distance from central mass with $\zeta = m \xi = .5$. In GR a such behavior is missing.

In particular, the PPN parameter $\gamma$ is related with the second order correction to the gravitational potential while $\beta$ is linked with the fourth order perturbation in $v$. Since the Gauss theorem is not verified for $f(R)$-gravity, while the relations (33) satisfy it, we must consider the relations (32) and not (27), (29). After making the limit $f \to R$ we can compare the results with (33). Actually, if we consider this limit for (32), we have

$$
\begin{align*}
R^{(2)}(t, x) &= 0 \\
g^{(2)}_{tt}(t, x) &= -\frac{r_g}{|x|} \\
g^{(2)}_{ij}(t, x) &= -\frac{r_g}{|x|} \delta_{ij}
\end{align*}
$$

which suggest that the $f(R)$-gravity is compatible with respect to GR. At last it is interesting to note that also in the case of extended spherically symmetric distribution of matter when we perform the limit $f \to R$ the solutions (27) and (29) directly converge (in the vacuum) in solutions (34), demonstrating the validity of Gauss theorem in GR.

Other important consideration is about the asymptotic behavior of $f$-theory with respect to GR. In fact increasing the distance from the central mass the gravitational field brings near to one of GR. A such rejoining is a normal consequence of the hypothesis (spherically symmetric system and request of Minkowskian limit). At last in FIG. 1 we report the spatial behavior of Ricci scalar (26) approximating asymptotically the given value in GR. In fact hypothesizing a $f(R)$-theory the Ricci scalar acquires dynamics, and in the Newtonian limit, we find a characteristic scale length ($m^{-1}$) on the which distance the scalar massive field evolves. Only for distances more great than $m^{-1}$ we recover the outcome of GR: $R = 0$.

To conclude this section we show in FIG. 5 the comparison between gravitational forces induced in GR and in $f(R)$-theory in the Newtonian limit. Obviously also about the force we obtained an intensity stronger than in GR.

**B. The oscillating Newtonian Limit of $f(R)$-gravity**

If we consider $m^2 < 0$ (i.e. sign[$f'(0)$] = sign[$f''(0)$]) from (22) we can choose the "oscillating" Green function

$$
G(x, x') = -\frac{1}{4\pi} \frac{\cos m|x - x'| + \sin m|x - x'|}{|x - x'|}
$$

(35)

The Ricci scalar (12) and the $tt$-component of $g_{\mu\nu}$ at $O(2)$ order (13) become

$$
R^{(2)}(t, x) = -\frac{6r_g}{\xi^3} \left[ 1 - H(\xi) \frac{\sin m|x|}{m|x|} \right] \Theta(|x| - \xi) - 2 r_g m^2 G(\xi) \frac{\cos m|x| + \sin m|x|}{|x|} \Theta(|x| - |\xi|)
$$

(36)

$$
g^{(2)}_{tt}(t, x) = -r_g \left[ \frac{3}{2\xi} - \frac{2}{m^2 \xi^3} \frac{|x|^2}{2\xi^3} + \frac{H(\xi) \sin m|x|}{m^2 |x|^3} \right] \Theta(|x| - \xi) - r_g \left[ \frac{1}{|x|} - \frac{2G(\xi) \cos m|x| + \sin m|x|}{3} \right] \Theta(|x| - \xi)
$$

(37)
where \( G(\xi) = \frac{3}{m_\xi} \cos m_\xi - \frac{\sin m_\xi}{m_\xi} \) and \( H(\xi) = (1 - m_\xi) \cos m_\xi + (1 + m_\xi) \sin m_\xi \), with the properties \( \lim_{\xi \to 0} G(\xi) = -1 \) and \( \lim_{\xi \to 0} H(\xi) = 1 \). Since we have an oscillating Green function (it is not asymptotically zero) the "gravitational potentials" (36) at infinity are zero unless a constant value (\( \lim_{a \to \infty} 2 r_g m_\xi G(\xi)(\sin ma - \cos ma) \)).

The spatial behavior of Ricci scalar (36) and metric component (37) are shown in FIGs. 6 and 7. The considerations of preceding subsection hold also for the solutions (36) - (37). The only difference is that now we have oscillating behaviors instead of exponential behaviors. The correction term to the Newtonian potential in the external solution can be interpreted as the Fourier transform of the matter density \( \rho(\mathbf{x}) \). In fact, we have:

\[
\lim_{\xi \to 0} \int d^3 \mathbf{x}' \rho(\mathbf{x}') e^{-i \mathbf{k} \cdot \mathbf{x}'} = - \lim_{\xi \to 0} M G(|\mathbf{k}|) = M
\]

(38)

Also in this case we conclude the section showing in FIG. 8 the comparison between gravitational forces induced in GR and in \( f(R) \)-theory in the Newtonian limit. Obviously also in this last case we obtained a force stronger than in GR.
FIG. 7: Plot of metric potential $\zeta m^{-1}\Phi$ vs distance from central mass with the choice $\zeta = m\xi = .5$ in the oscillating case. The dashed line is the GR behavior.

FIG. 8: Comparison between gravitational forces induced by GR and $f(R)$-theory with $\zeta = m\xi = .5$ in the oscillating case. The dashed line is the GR behavior.

C. Solutions at $O(4)$-order

The metric potentials and the function $\Xi(x)$, respectively defined in (30) and (31), satisfy the following properties with respect to derivative of coordinate $l$-th\(^4\) in the matter.

\[ |x|_l = |x|^{-1} x_l \]

\(^4\) we remember that $|x|_l = |x|^{-1} x_l$
\[
\Xi,_{,l}(x) = \frac{m^2(1+m\xi)}{\xi^2} e^{-m\xi} F(x) x_l = \Xi_0(x) x_l
\]
\[
\Phi,_{,l}(x) = \left[ \frac{1}{\xi^2} + \frac{(1+m\xi)}{\xi^2} e^{-m\xi} F(x) \right] x_l = \Phi_0(x) x_l
\]
\[
\Psi,_{,l}(x) = \left[ \frac{1}{\xi^2} - \frac{(1+m\xi)}{\xi^2} e^{-m\xi} \frac{(m^2|x|^2+6) \sinh m|x|+m|x|}{m^2|x|^3} \right] x_l = \Psi_0(x) x_l
\]
\[
\Lambda,_{,l}(x) = \frac{2(1+m\xi)}{\xi^2} e^{-m\xi} \frac{(4m^2|x|^2+9) \sinh m|x|}{m^2|x|^3} x_l = \Lambda_0(x) x_l
\]
\[
\Xi,_{,ln}(x) = \Xi_0(x) \delta_{ln} + \frac{3(1+m\xi)}{\xi^2} e^{-m\xi} \frac{(m^2|x|^2+3) \sinh m|x|-3m|x| \cosh m|x|}{m^2|x|^3} x_l x_n = \Xi_0(x) \delta_{ln} + \Xi_1(x) x_l x_n
\]
\[
\Phi,_{,ln}(x) = \Phi_0(x) \delta_{ln} + \frac{(1+m\xi)}{\xi^2} e^{-m\xi} \frac{(m^2|x|^2+3) \sinh m|x|-3m|x| \cosh m|x|}{m^2|x|^3} x_l x_n = \Phi_0(x) \delta_{ln} + \Phi_1(x) x_l x_n
\]
and in the vacuum
\[
\Xi,_{,l}(x) = \frac{m^2(m|x|+1)}{|x|^3} F(\xi) e^{-m|x|} x_l = \Xi_0(x) x_l
\]
\[
\Phi,_{,l}(x) = \left[ \frac{1}{|x|^2} + \frac{m|x|+1}{|x|^2} \frac{F(\xi)e^{-m|x|}}{3} \right] x_l = \Phi_0(x) x_l
\]
\[
\Psi,_{,l}(x) = \left[ \frac{m^2|x|^2-2}{|x|^3} - \frac{m^2|x|^3-2m|x|^2-6m|x|-6 F(\xi)e^{-m|x|}}{m^2|x|^3} \right] x_l = \Psi_0(x) x_l
\]
\[
\Lambda,_{,l}(x) = \left[ \frac{6}{m^2|x|^3} - \frac{m^3|x|^2+4m^2|x|^2+9m|x|+9 \frac{2F(\xi)e^{-m|x|}}{3}}{m^2|x|^3} \right] x_l = \Lambda_0(x) x_l
\]
\[
\Xi,_{,ln}(x) = \Xi_0(x) \delta_{ln} - \frac{m^2(m^2|x|^2+3m|x|+3)}{|x|^3} F(\xi) e^{-m|x|} x_l x_n = \Xi_0(x) \delta_{ln} + \Xi_1(x) x_l x_n
\]
\[
\Phi,_{,ln}(x) = \Phi_0(x) \delta_{ln} - \left[ \frac{1}{|x|^2} + \frac{m^2|x|^2+3m|x|+3 \frac{F(\xi)e^{-m|x|}}{3}}{|x|^3} \right] x_l x_n = \Phi_0(x) \delta_{ln} + \Phi_1(x) x_l x_n
\]

Obviously when we consider the physics in the matter or in the vacuum we choose the "right" quantities \(\Xi_0(x), \Xi_1(x), \Phi_0(x), \Phi_1(x), \Psi_0(x), \Lambda_0(x)\).

The expression of Ricci scalar at fourth order, (18), is

\[
R^{(4)}(t, x) = \frac{r_g^2}{2m|x|} \int_0^\infty \int d|x'||d|x| \left\{ e^{-m|x|-|x'||} - e^{-m(|x|+|x'|)} \right\} \left\{ \frac{m^4}{\mu^4} \Xi(x')^2 + \frac{|x'|^2}{m^2} \Xi_0(x')^2 \right\}
\]
\[
+ \left( 3 \Lambda(x') + \Phi_0(x') - \Psi_0(x') + \Lambda_0(x') \right) |x'|^2 \Xi_0(x') + m^2 \Psi(x') \Xi(x') + \Xi_1(x') \Lambda(x') |x'|^2 \right\}
\]

(41)

from the which we note two contributes. The first one still depends on the quadratic term \((\propto R^2)\) in the action (11), while the second one is related to cubic term \((\propto R^3)\). By introducing two functions \(\Xi_I(x)\) and \(\Xi_{II}(x)\), the (11) is re writable as follows

\[
R^{(4)}(t, x) = r_g^2 \left[ \Xi_I(x) + \frac{m^4}{\mu^4} \Xi_{II}(x) \right]
\]

(42)

An analogous situation is found for the \(tt\)-component of metric tensor at fourth order. In fact the (11) becomes
\[ g^{(4)}_{tt}(t, \mathbf{x}) = \frac{r_g \mathcal{X}}{\lvert \mathbf{x} \rvert} \int_0^\infty \int d||\mathbf{x}'||d\mathbf{x}' \left\{ ||\mathbf{x}|| - ||\mathbf{x}'|| - ||\mathbf{x}|| - ||\mathbf{x}'||\right\} \rho (\mathbf{x}') \Phi (\mathbf{x}'), \]

\[ + \frac{r_g^2}{\lvert \mathbf{x} \rvert} \int_0^\infty d||\mathbf{x}'||d\mathbf{x}' \left\{ ||\mathbf{x}|| - ||\mathbf{x}'|| - ||\mathbf{x}|| - ||\mathbf{x}'||\right\} \left\{ \frac{1}{2} \left[ \Xi_I(\mathbf{x}') + \Phi(\mathbf{x}') \Xi(\mathbf{x}') + \Phi_0(\mathbf{x}')^2 ||\mathbf{x}'||^2 \right] \right\} \]

\[ - \frac{r_g^2}{3r^2 \lvert \mathbf{x} \rvert} \int_0^\infty d||\mathbf{x}'||d\mathbf{x}' \left\{ ||\mathbf{x}|| - ||\mathbf{x}'|| - ||\mathbf{x}|| - ||\mathbf{x}'||\right\} \left\{ \frac{\Xi(\mathbf{x}')^2}{4} + \frac{\Xi_0(\mathbf{x}')}{2} \left[ 6\Delta(\mathbf{x}') + \left( 5\Phi_0(\mathbf{x}') + 3\Lambda_0(\mathbf{x}') \right) ||\mathbf{x}'||^2 \right] + \frac{\Xi(\mathbf{x}')}{2} \left[ 2m^2 \left( \Phi(\mathbf{x}') + \Psi(\mathbf{x}') \right) - 3\Phi_0(\mathbf{x}') - \Phi_1 ||\mathbf{x}'||^2 \right] \right\} \]

\[ + \frac{m^4 r_g^2}{\mu^4 \lvert \mathbf{x} \rvert} \int_0^\infty d||\mathbf{x}'||d\mathbf{x}' \left\{ ||\mathbf{x}|| - ||\mathbf{x}'|| - ||\mathbf{x}|| - ||\mathbf{x}'||\right\} \left\{ \frac{\Xi_I(\mathbf{x}')}{2} - \frac{m^2 \Xi(\mathbf{x}')^2}{3m^4} + \frac{||\mathbf{x}'||^2 \Xi_0(\mathbf{x}')^2}{3m^4} - \frac{\Delta(\mathbf{x}) \Xi_I(\mathbf{x}')}{3m^2} \right\} \] (43)

and by introducing other new functions \( \Phi_I(x), \Phi_{II}(x) \) we have

\[ g^{(4)}_{tt}(t, \mathbf{x}) = r_g^2 \left[ \Phi_I(x) + \frac{m^4}{\mu^4} \Phi_{II}(x) \right] \] (44)

It is useful to note that we have generally four contributions to \( g^{(4)}_{tt} \) in (43). The first one is induced by the non-linearity of the metric tensor even in our static spherically symmetric case. The product \( \rho(\mathbf{x})\Phi(\mathbf{x}) \) is not zero only in the matter but contributes to determination of \( tt \)-component in any point of space. The second one holds account of the induced contribution, by solution of previous order, to the determination of the \( tt \)-component of the Ricci tensor at fourth order. These first two terms are present also in GR. While the second two ones are derived from the modification of theory. In fact the third contribution depends on the addition of the quadratic term \((\propto R^2)\) in the action and finally the fourth one from the addition of the cubic term \((\propto R^3)\).

The choice of free parameter \( \mu \) (which is linked to third derivative of \( f(R) \)) is a crucial point in both the expressions (42) and (44) to obtain the right behavior. From mathematical interpretation of Newtonian limit one has \( f'''(0) < 0 \) and if \( \mu^4 > 0 \) \((i.e. \ sign[f''(0)] = - \ sign[f'''(0)]\), otherwise \( \mu^4 \) is not a length) we have \( m^4/\mu^4 = |f'''(0)|/3f''(0)^2 \), so we find the constraint \( 0 < m^4/\mu^4 < 1 \). In FIG. (9) we report the spatial behavior of (42) in the matter and in the vacuum, \((0.5 \leq m^4/\mu^4 \leq 0.9)\), showing that far from source we obtain a spacetime with a vanishing scalar curvature. At Newtonian level the Ricci scalar \( R^{(2)} \) is negative defined (20) while at Post-Newtonian limit is positive defined.

In FIG. (10) we report the time-time component of metric tensor, \( g^{(4)}_{tt} \), on the same interval of values of \( m^4/\mu^4 \), although the behavior is quite insensitive to changes induced by the contributions of the cubic term in the Lagrangian. Besides we can observe an important analogy with respect the results of GR. In both cases we have a potential barrier, but for \( f(R) \)-gravity it is higher (as in the Newtonian limit we found a deeper potential well).

**D. Solutions from isotropic coordinates to standard coordinates**

The found solutions of metric are expressed in isotropic coordinates and often for spherically symmetric problems is conveniently rewritten in standard coordinates (the usual form in which we write the Schwarzschild solution). Here the relativistic invariant of metric (41) is

\[ ds^2 = \left[ 1 + r_g \Phi(x) + r_g^2 \left( \Phi_I(x) + \frac{m^4}{\mu^4} \Phi_{II}(x) \right) \right] dt^2 - \left[ 1 - r_g \Psi(x) \right] |dx|^2 + r_g \Lambda(x) \frac{(x \cdot dx)^2}{|x|^2} \] (45)
FIG. 9: Plot of dimensionless function $\zeta^4 m^{-4}(\Xi_I + z \Xi_{II})$, representing the Ricci scalar at fourth order, where $z = m^4/\mu^4$ and $\zeta = m\xi = .5$. The spatial behavior is shown for $.3 \leq z \leq .9$ (solid lines) while the dotted line corresponds to $R - \frac{1}{6m^2}R^2$-theory.

FIG. 10: Plot of dimensionless function $\zeta^2 m^{-2}(\Phi_I + z \Phi_{II})$ (solid lines) where $z = m^4/\mu^4$ and of function $1/2|x|^2$ (dashed line). For $z = 0$ (dotted line) we have the behavior of $R - \frac{1}{6m^2}R^2$-theory. The solid lines are obtained for $.3 \leq z \leq .9$ and for $\zeta = m\xi = .5$.

From spherically symmetric form of (45) it is convenient to replace the position $\mathbf{x}$ with spherical polar coordinates $r, \theta, \phi$ defined as usual by

$$x^1 = r \sin \theta \cos \phi, \quad x^2 = r \sin \theta \sin \phi, \quad x^3 = r \cos \phi.$$  (46)

The proper time interval (45) then becomes

$$ds^2 = \left[1 + r_g \Phi(r) + r_g^2 \left(\Phi_I(r) + \frac{m^4}{\mu^4} \Phi_{II}(r)\right)\right]dt^2 - \left[1 - r_g \left(\Psi(r) + \Lambda(r)\right)\right]dr^2 - \left[1 - r_g \Psi(r)\right]r^2 d\Omega$$  (47)

where $d\Omega = d\theta^2 + \sin^2 \theta d\phi^2$ is the solid angle. To get the metric in the standard form it needs to impose a radial coordinate transformation

$$\left[1 - r_g \Psi(r)\right]r^2 = \tilde{r}^2$$  (48)

and we have a new set of coordinates $\tilde{r}, \theta, \phi$. The metric (47) becomes

$$ds^2 = \left[1 + r_g \tilde{\Phi}(\tilde{r}) + r_g^2 \left(\tilde{\Phi}_I(\tilde{r}) + \frac{m^4}{\mu^4} \tilde{\Phi}_{II}(\tilde{r})\right)\right]d\tilde{r}^2 - \left[1 - r_g \left(\tilde{\Psi}(\tilde{r}) + \tilde{\Lambda}(\tilde{r})\right)\right] \left(\frac{dr}{d\tilde{r}}\right)^2 d\tilde{r}^2 - \tilde{r}^2 d\Omega$$  (49)
the expression previously desired.

The explicit expression of \((49)\) is not displayed because the equation \((48)\) can not be solved algebraically. However, this technical problem is overcome with the help of numerical methods when we are interesting to test experimentally the theory. In a further work that is taking place, we are considering the main experiments tested for GR and we want retest them with respect to the results obtained.

IV. CONCLUSIONS

In this paper, we have generally reformulated the Newtonian limit of \(f(R)\)-gravity by applying the Green function method. Moreover the post-Newtonian limit has been studied in the harmonic gauge condition and the relative spatial behaviors (Yukawa-like and oscillating) of gravitational potential in the matter and in the vacuum have been shown. The Taylor expansion of a generic \(f\) has been considered obtaining general solutions in term of the derivatives up to third degree when an uniform mass ball-like source is considered. All metric potentials, however, depend strictly on the coupling parameters appearing indirectly in the Lagrangian of the theory (the derivatives of \(f\) in \(R = 0\) are arbitrary constants).

A detailed discussion has been developed for systems presenting spherical symmetry. In this case, the role of corrections to the Newtonian potential is clearly evident. This means that one of the effects introducing a generic function of scalar curvature is to select a characteristic scale length which could have physical interests.

Furthermore, it has been shown that the Birkhoff theorem is not a general result for \(f(R)\)-gravity. This is a fundamental difference between GR and fourth order gravity. While in GR a spherically symmetric solution is, in any case, stationary and static, here time-dependent evolution can be achieved depending on the order of perturbations.

Hypothesizing a nonlinear Lagrangian we obtained a gravitational attraction stronger than in GR. The hypothesis of dark matter is needed in GR to have more gravitational attraction. Here without hypothesis of alternative matter and without modifying the gravitational constant we have qualitatively the same outcome. This occurrence could be particularly useful to solve the problem of missing matter in large astrophysical systems like galaxies and clusters of galaxies. In fact dark matter could be nothing else but the effects that GR, experimentally tested only up to Solar System scales, does not work at extragalactic scales and then it has to be corrected.

Then it is worth pointing out that \(f(R)\)-gravity seem good candidate to solve several shortcomings of Modern Astrophysics and Cosmology. Taking into account the results presented, it is clear that only GR presents directly the Newtonian potential in the weak field limit while corrections appear as soon as the theory is non-linear in the Ricci scalar.

In forthcoming researches, there is the intention to confront such solutions with experimental data in order to see if large self-gravitating systems could be modelled by them and if the experimental test of GR in the Solar System are compatible with \(f(R)\)-gravity.

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