THE ORLICZ MINKOWSKI PROBLEM INVOLVING $0 < p < 1$: FROM ONE CONSTANT TO AN INFINITE INTERVAL

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Abstract. In this paper we study the existence of convex bodies for the Orlicz Minkowski problem
$$c \varphi(h_K) dS(K, \cdot) = d\mu \quad \text{on } S^{n-1}$$
where $\mu$ is the given Borel measure on $S^{n-1}$, $h_K$ is the support function of $K$, $S_K$ is the surface area measure of $K$, and $c$ is a real parameter. We prove that, under assumptions on $\varphi$ at infinity, there exists $c_*>0$ such that, if $c \in [c_*, +\infty)$ this problem always has a solution $K_c$.

1. Introduction. The classical Minkowski problem was proposed by Minkowski in 1897, which is about a geometric measure generated by convex bodies. From the classical one to the $L_p$ Minkowski problem and now to the Orlicz Minkowski problem, the Brunn-Minkowski theory has developed rapidly during the past two decades and many challenges remain for $p < 1$. For the Brunn-Minkowski theory, see the book by Schneider [44] for a comprehensive introduction.

The classical Minkowski problem asks the following: let $\mu$ be a Borel measure on $S^{n-1}$. Under what conditions does there exist a convex body $K$ such that $\mu = S(K, \cdot)$? Furthermore, if such a $K$ exists, is it unique? The surface area measure $S(K, \cdot)$ of a convex body $K$ is a Borel measure on $S^{n-1}$, defined for each Borel set $\omega \subset S^{n-1}$ by
$$S_K(\omega) = \int_{x \in \gamma_K^{-1}(\omega)} d\mathcal{H}^{n-1}(x),$$
where $\gamma_K : \partial' K \to S^{n-1}$ is the Gauss map of $K$, defined on $\partial' K$, the set of points of the boundary of $K$ that have a unique outer unit normal, and $\mathcal{H}^{n-1}$ is $(n-1)$-dimensional Hausdorff measure. Note here that $\gamma_K : \partial' K \to S^{n-1}$ is surjective since $K$ is a nonempty compact convex set and that $\mathcal{H}^{n-1}$-measure of singular boundary points (i.e., boundary point has more than one unit normal vector) of a convex

2020 Mathematics Subject Classification. Primary: 52A40.
Key words and phrases. Orlicz Minkowski problem, variational method.
The authors are supported by NSFC grants 11971027 and 11771468.
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body equals to 0. If the given measure $\mu$ has a smooth density $f$ with respect to the Lebesgue measure of $S^{n-1}$, the Minkowski problem is equivalent to the equation:

$$det(\partial_{x,ij}u) = f, \quad u > 0 \quad \text{on} \ S^{n-1}$$

where $(u_{i,j})$ is the Hessian matrix of $u$ with respect to an orthonormal frame on $S^{n-1}$ and $\delta_{i,j}$ is the Kronecker delta. Minkowski [41] solved the discrete measure (i.e., polytopal version) and proved the uniqueness of the solution. For general measure this problem was solved by Alexandrov [2] and independently by Fenchel and Jessen [16]. The existence uses the Alexandrov variational formula and the uniqueness employs the Minkowski inequality for mixed volumes. The regularity are due to Lewy [30], Nirenberg [42], Calabi [10], Pogorelov [43], Cheng and Yau [13], Caffarelli, Nirenberg, and Spruck [9], and others.

In the early 1960s, Firey showed that the classical Minkowski addition of convex sets is the special case $p = 1$ of a general $L_p$-addition which is defined indirectly in terms of the $p$-means of the support functions of convex sets being added (now known as Minkowski-Firey $L_p$-addition). Around 1993, the $L_p$-surface area measure of a convex body $K$ containing the origin in its interior (i.e., $K \in \mathcal{K}_0^n$) was introduced by Lutwak [33, 34], which is a core notation in the modern convex geometric analysis: $S_p(K, \cdot)$ is a Borel measure on $S^{n-1}$, defined for each Borel $\omega \subset S^{n-1}$ by

$$S_p(K, \omega) = \int_{x \in \gamma_K^{-1}(\omega)} (x \cdot \gamma_K(x))^{1-p} d\mathcal{H}^{n-1}(x)$$

for all $p \in \mathbb{R}$. $S_1(K, \cdot) = S(K, \cdot)$ (also denoted by $S_K$) is the classical surface area measure of $K$, and $1/nS_0(K, \cdot)$ is the cone-volume measure of $K$. Therefore, similar questions have been posed for the $L_p$-surface area measure of convex bodies in $\mathcal{K}_0^n$, i.e., is there a convex body $K \in \mathcal{K}_0^n$ such that

$$h_K^{1-p}dS_K = d\mu?$$

Here, $h_K$ is the support function of $K$ and $S_K$ is the surface area measure of $K$. If the given measure $\mu$ has a smooth density $f$ w.r.t. the Lebesgue measure of $S^{n-1}$, the $L_p$ Minkowski problem is equivalent to the Monge-Ampère equation:

$$u^{1-p}det(\partial_{x,ij}u) = f, \quad u > 0 \quad \text{on} \ S^{n-1}.$$ 

During the past two decades, the combination of volume and $L_p$-addition has led to a rapidly evolving $L_p$ Brunn-Minkowski theory (see, e.g., [1-4, 6-22, 24-26, 28-30, 33-37, 40-43, 45-47, 49-52, 55-58]). Now, the $L_p$ Minkowski problem has not been solved completely and many challenges remain for $p < 1$.

The even $L_p$ Minkowski problem for $p > 1$ but $p \neq n$ was solved by Lutwak [33]. An equivalent volume-normalized version of the $L_p$ Minkowski problem was proposed by Lutwak, Yang, and Zhang [36] and the even case was solved for $p = n$. The regularity was established by Lutwak and Oliker [35]. The noneven $L_p$ Minkowski problem, for all $p \geq n$, was solved by Chou and Wang [14], in which they also solved the relevant Monge-Ampère equation for all $p > 1$, while an alternate approach to this problem was presented by Hug et al. [25].

For $p < 1$, the situation is intricate and some special cases were discussed. When $p = 0, n = 3$ and $f$ is a constant, the constant function is the unique solution and Firey’s conjecture that the ultimate shape of centrally symmetric convex stones undergoing an idealized wearing process is a sphere, were completely proved by Andrews [3]. Tzitzeica [52] studied $L_{-n}$ Minkowski problem when $f \equiv 1$ and announced that all solutions are ellipsoids centered at the origin. Recently, Böröczky
et al. [7] announced that the subspace concentration condition is the necessary and sufficient condition for the existence of a solution to the even $L_0$ Minkowski problem. Without the even assumption, the existence of a solution of the PDE with $p \in (-n, 0)$ was proved by Chou and Wang [14]. Moreover, there are examples of data (smooth and positive) for which the PDE with $p \in (-n, 0)$ possesses multiple solutions by Jian, Lu, and Wang [28]. Zhu [55-57] established the existence of a solution to the $L_p$ Minkowski problem for $p = 0, p = -n, 0 < p < 1$ when $\mu$ is discrete. More recent developments in the non-even direction are in Chen, Li, and Zhu [12], where the existence of a convex body was proved for $0 < p < 1$ when the measure $\mu$ is not concentrated on a great subsphere, and Zhu [58] for all $p < 0$ when $\mu$ is a discrete measure.

Very little is known about the uniqueness of $L_p$ Minkowski problem when $p < 1$. For the $L_0$ Minkowski problem, the uniqueness was established when the data is an even, smooth and strictly positive function by Gage and Li [18, 19] in the plane, when $\mu$ is a discrete measure by Stancu [47] in the plane, and when the data is a constant by Firey [17] and Andrews [3] in 3-dimension. Around 2012, Böröczky, Lutwak, Yang, and Zhang [6] gave the uniqueness to the planar $L_p$ Minkowski problem for $0 \leq p < 1$ when the given measure is even. Recently, in 3-dimension, Huang, Liu, and Xu [24] proved the uniqueness of some type convex bodies for $0 < p < 1$ when the data is a constant.

In the plane ($n = 2$), related works have been carried out for all $p \in \mathbb{R}$. The $L_p$ Minkowski problem was treated by Andrews [4] for all real $p$ when the data is a constant, Jiang [29] when $2 > p > 0$ and the data is $T$-periodic continuous function positive somewhere with $T \leq \pi$, Böröczky and Trinh [8] when $0 < p < 1$ and the data is nonnegative $L^1(\mathbb{S})$-function, Gage and Li [18, 19] when $p = 0$ with an arbitrary strictly positive smooth data, Stancu [46] when $p = 0$ and the data is a discrete measure, Umanskiy [53] when $p \not\in \{0, 2\}$ and the data is continuous $T$-periodic with $0 < T < 1$, Ivaki [26] when $p = -2$ and the data is even, Chen [11] when $-2 \leq p \leq 0$ and the data is continuous and not necessarily positive, Dou and Zhu [15] when $p \leq -2$ and the data is continuous, positive at, at least, one point and $\pi/k$-periodic function with integer $k > 1$, and the second author and Long [50] for all $p < 0$ when the data is nonnegative, nontrivial $\pi/k$-periodic $L^1(\mathbb{S})$-function with integer $k > 1$.

Recently, the next step in the evolution of the Brunn-Minkowski theory toward an Orlicz Brunn-Minkowski theory has been made. The Orlicz function $\varphi$ constitutes far reaching generalizations of the $L_p$ Minkowski problem within this new Orlicz Brunn-Minkowski theory. The geometric insights are in Lutwak, Yang, and Zhang [38, 39] for Orlicz projection bodies and Orlicz centroid bodies, Haberl, Schuster, and Xiao [21, 22] for the asymmetric $L_p$ Brunn-Minkowski theory, Ludwig and Reitzner [31, 32] for the discovery of Orlicz type affine surface areas. A major difficulty of the Orlicz Minkowski problem is that it is no longer homogeneous. In [20], Haberl, Lutwak, Yang, and Zhang solved the Orlicz Minkowski problem

$$\varphi(u)dS(K, \cdot) = d\mu$$

for $\varphi : (0, +\infty) \to (0, +\infty)$ which is continuous, $\phi(t) = \int_0^t \frac{1}{\varphi(s)} ds$ exists for every positive $t$ and unbounded as $t \to +\infty$, provided $\mu$ is an even finite Borel measure on $\mathbb{S}^{n-1}$ which is not concentrated on a great subsphere of $\mathbb{S}^{n-1}$. They proved that there exists an origin symmetric convex body $K \subset \mathbb{R}^n$ and a constant $c > 0$ such that $c\varphi(u)dS(K, \cdot) = d\mu$. They also gave an explicit value of $c = V(K)^{\frac{\alpha}{n-1}}, \alpha \in$
(0,1). More recent research on the symmetry, stating that one can get rid of the even assumption on $\mu$, appeared then in Huang and He [23] provided additionally $\varphi(s) \to +\infty$ as $s \to 0^+$, in Jian and Lu [27] provided additionally $\varphi(s) \to 0$ as $s \to 0^+$ and $\varphi$ is non-decreasing. It is natural to ask whether or not the constant $c$ can be chosen as independent of the convex body $K$ or whether or not the existence result holds also if $c \neq V(K)\hat{\beta}^{-1}$. In this paper, we are interested in understanding the constant $c$. Our theorem is as follows:

**Theorem 1.1.** Let $S^{n-1}$ be the unit sphere in $\mathbb{R}^n$, $\mu$ be an even finite Borel measure on $S^{n-1}$ whose support is not concentrated on a great subsphere of $S^{n-1}$, and we also let $\varphi : (0, +\infty) \to (0, +\infty)$ be a continuous function such that $F(s) = \int_0^s \frac{1}{\varphi(t)} \, dt$ exists for every positive $s$. If $\varphi$ satisfies additionally that for some $R > 0$, and $\beta > n$,

$$s \frac{1}{\varphi(s)} \geq \beta F(s) := \beta \int_0^s \frac{1}{\varphi(t)} \, dt, \forall s \geq R \quad (*)$$

then there exists $c_\ast > 0$ such that for each $c \in [c_\ast, +\infty)$ there exists a convex body $K_c \in \mathcal{K}_e$ such that

$$c \varphi(h_{K_c})dS_{K_c} = d\mu.$$

**Remark 1.** The condition (*) is a general growth condition at infinity in nonlinear functional analysis, see, e.g., the books by Badiale and Serra [5, p.75 (f3)], Struwe [48, p.110 Theorem 6.2].

**Remark 2.** Improvements in the case $n = 2$ are in the second author [49] (Sun, 2018) where $0 < p < 1$ is involved, and the second author and Zhang [51] (Sun, 2019) where $p = 0$ is involved.

To clarify the content of Theorem 1.1, let us describe its consequence in case $1/\varphi(s) = s^{p-1} + s^{l-1}, \forall s > 0$, where $0 < p < 1$ and $l > n$.

**Example 1.** Let $S^{n-1}$ be the unit sphere in $\mathbb{R}^n$, $0 < p < 1$ and $l > n$. Let $\mu$ be an even finite Borel measure on $S^{n-1}$ whose support is not concentrated on a great subsphere of $S^{n-1}$. Then there exists $c_\ast > 0$ such that for each $c \in [c_\ast, +\infty)$ there corresponds a convex body $K_c \in \mathcal{K}_e$ such that

$$c \varphi(h_{K_c})dS_{K_c} = d\mu,$$

with $1/\varphi(s) = s^{p-1} + s^{l-1}$.

Since $0 < p < 1, l > n$, we can choose $\beta \in (n, l)$, and $R = \left[\frac{l(l-\beta)}{p(l-\beta)}\right]^{1/(l-p)}$ such that

$$s \frac{1}{\varphi(s)} = s^p + s^l \geq \beta s^p + \frac{\beta}{l} s^l = \beta \int_0^s \frac{1}{\varphi(t)} \, dt, \forall s \geq R.$$

That is, the Orlicz function $\varphi(s) = [s^{p-1} + s^{l-1}]^{-1}$ with $0 < p < 1$ and $l > n$ satisfies (*) in Theorem 1.1, which guarantees the existence of a convex body $K$ satisfying $c \varphi(h_K)dS_K = d\mu$ for each $c$ in some infinite interval.

2. **Preliminaries.** For quick later read we introduce some preliminary terminology and notation about convex bodies.

The standard inner product of the vectors $x, y \in \mathbb{R}^n$ is denoted by $\langle x, y \rangle$. We write $|x|^2 = \langle x, x \rangle$, and $\mathbb{S}^{n-1} = \{ x \in \mathbb{R}^n : |x| = 1 \}$ for the boundary of the Euclidean unit ball $B$ in $\mathbb{R}^n$. 


The set of continuous functions on the sphere $\mathbb{S}^{n-1}$ will be denoted by $C(\mathbb{S}^{n-1})$ and will always be viewed as equipped with the max-norm:
$$
\|u\|_\infty = \max_{x \in \mathbb{S}^{n-1}} |u(x)|
$$

for $u \in C(\mathbb{S}^{n-1})$. The set of strictly positive continuous functions will be denoted by $C^+(\mathbb{S}^{n-1})$ and $C_c^+(\mathbb{S}^{n-1})$ will denote the subspace of $C^+(\mathbb{S}^{n-1})$ consisting of only the even functions.

For $k$-dimensional Hausdorff measure, we write $\mathcal{H}^k$. The letter $\mu$ will be used exclusively to denote a finite Borel measure on $\mathbb{S}^{n-1}$. For such a measure $\mu$, we denote by $|\mu|$ its total mass, i.e., $|\mu| = \mu(\mathbb{S}^{n-1})$.

A convex body is a compact convex subset of $\mathbb{R}^n$ with non-empty interior. The set of convex bodies in $\mathbb{R}^n$ containing the origin in their interiors is denoted by $\mathcal{K}_n^0$. The set of convex bodies in $\mathbb{R}^n$ that are symmetric about the origin will be denoted by $\mathcal{K}_n^s$.

The support function $h_K : \mathbb{R}^n \to \mathbb{R}$ of a compact convex set $K \subset \mathbb{R}^n$ is defined, for $x \in \mathbb{R}^n$, by
$$
h_K(x) = \max \{ \langle x, y \rangle : y \in K \}.
$$

For example, the support function of the line segment $[v]$ joining the points $\pm v \in \mathbb{R}^n$ is given by $h_v(x) = |\langle x, v \rangle|$, $\forall x \in \mathbb{R}^n$. The support function of $K \in \mathcal{K}_n^0$ is strictly positive and continuous on the unit sphere $\mathbb{S}^{n-1}$.

A function $u \in C^+(\mathbb{S}^{n-1})$ gives rise to the convex body
$$
K = \bigcap_{y \in \mathbb{S}^{n-1}} \{ x \in \mathbb{R}^n : x \cdot y \leq u(y) \}.
$$

The body $K$ is called the Aleksandrov body (also known as the Wulff shape) associated with $u$. Note here that since $u$ is strictly positive and continuous its Aleksandrov body $K$ must be an element of $\mathcal{K}_n^0$. Obviously, for the Aleksandrov body $K$ associated with the function $u$, we have
$$
h_K(x) \leq u(x), \forall x \in \mathbb{S}^{n-1}.
$$

If $u$ is the support function of a convex body $K \in \mathcal{K}_n^0$, then $K$ itself is the Aleksandrov body associated with $u$. If $u$ is an even function, then the Aleksandrov body associated with $u$ is origin-symmetric.

The volume $V(u)$ of a function $u \in C^+(\mathbb{S}^{n-1})$ is defined as the volume of the Aleksandrov body associated with $u$. Since the Aleksandrov body associated with the support function $h_K$ of a convex body $K \in \mathcal{K}_n^0$ is the body $K$ itself, we have $V(h_K) = V(K)$. Moreover, $V : C^+(\mathbb{S}^{n-1}) \to \mathbb{R}$ is continuous.

3. Proof of Theorem 1.1.

3.1. Uniform estimates. The following lemma provides the uniform estimates:

**Lemma 3.1.** Let $\beta > 0$. If $\mu$ is an even finite Borel measure on $\mathbb{S}^{n-1}$ whose support is not concentrated on a great subsphere of $\mathbb{S}^{n-1}$, then for all $a \in \mathbb{S}^{n-1}$ there exists $c_0 > 0$ independent of $a \in \mathbb{S}^{n-1}$ such that
$$
\int_{x \in \mathbb{S}^{n-1}} |\langle x, a \rangle|^\beta \, d\mu \geq c_0.
$$
Proof. Fixed $a \in S^{n-1}$. Note that since the support of $\mu$ is not concentrated on a great subsphere of $S^{n-1}$, we have that

$$\int_{S^{n-1}} |\langle x, a \rangle|^\beta d\mu > 0. \quad (1)$$

It remains to prove that uniform bound from below by some positive number can be established for all $a \in S^{n-1}$. Assume that, on the contrary, there is a sequence $\{a_j\} \subset S^{n-1}$ such that $\int_{S^{n-1}} |\langle x, a_j \rangle|^\beta d\mu \to 0$ as $j \to \infty$. Then, after passing to a subsequence (still denoted by $a_j$), we have that $a_j \to a_0$ with $a_0 \in S^{n-1}$. This leads to

$$|\langle x, a_j \rangle|^\beta \to |\langle x, a_0 \rangle|^\beta,$$

$$|\langle x, a_j \rangle|^\beta \leq 1, \ \forall x \in S^{n-1}.$$

Thanks to the Dominated convergence theorem, we have that

$$\int_{S^{n-1}} |\langle x, a_j \rangle|^\beta d\mu \to \int_{S^{n-1}} |\langle x, a_0 \rangle|^\beta d\mu,$$

so that $\int_{S^{n-1}} |\langle x, a_0 \rangle|^\beta d\mu = 0$ which is in contradiction with (1). This ends the proof of Lemma 3.1.

The next lemma provides the lower estimates on the primitive of $1/\varphi$, which is standard in nonlinear functional analysis.

Lemma 3.2. Let $\varphi : (0, +\infty) \to (0, +\infty)$ be a continuous function such that $F(s) = \int_0^s \frac{1}{\varphi(t)} dt$ exists for every positive $s$. If $\varphi$ satisfies additionally that for some $R > 0$, and $\beta > 0$,

$$s \frac{1}{\varphi(s)} \geq \beta F(s) := \beta \int_0^s \frac{1}{\varphi(t)} dt, \forall s \geq R,$$

then there exist two constants $c_i > 0, i = 1, 2$ such that

$$F(s) \geq c_1 s^{-\beta} - c_2, \forall s \geq 0.$$

Proof. For this purpose, we first write that

$$\frac{d}{ds} (s^{-\beta} F(s)) = s^{-\beta - 1} \left[ s \frac{1}{\varphi(s)} - \beta F(s) \right] \geq 0, \forall s \geq R$$

which leads after the integration on $[R, s]$ to

$$F(s) \geq R^{-\beta} F(R) s^\beta, \forall s \geq R. \quad (2)$$

We thus obtain, thanks to the fact that $F(s) = \int_0^s \frac{1}{\varphi(t)} dt > 0, \forall s > 0$ and to (2), that

$$F(s) \geq R^{-\beta} F(R) s^\beta - F(R), \forall s \geq 0.$$

This ends the proof of Lemma 3.2.

3.2. The proof of Theorem 1.1. Let $\mu$ be the given finite Borel measure on $S^{n-1}$ with total mass $|\mu| > 0$. We let $c_0 > 0$ and $\lambda_0 > 0$ be chosen so that

$$c_0 V(B) - \lambda_0 F(c_0)|\mu| > 0 \quad (3)$$

thanks to the assumption that $F(s) = \int_0^s \frac{1}{\varphi(t)} dt$ exists for every positive $s$. Then for each $\lambda \in (0, \lambda_0]$, we let $\Phi_\lambda$ be the functional defined by

$$\Phi_\lambda(u) = V(u) - \lambda \int_{x \in S^{n-1}} F(u(x)) d\mu, \forall u \in C_c^+ (S^{n-1})$$
where \( V(u) \) is the volume of the Aleksandrov body associated with \( u \), and that
\[
F(u(x)) = \int_0^{u(x)} \frac{1}{\varphi(t)} \, dt, \forall x \in S^{n-1}.
\]

Now, we consider \( \sup_{u \in C^+_f(S^{n-1})} \Phi_\lambda(u) \) although we have no idea whether or not it is a finite value at the moment. Let \( \{u_i\} \) be a sequence of continuous positive even functions on \( S^{n-1} \) such that
\[
\Phi_\lambda(u_i) \to \sup_{u \in C^+_f(S^{n-1})} \Phi_\lambda(u).
\]

We let \( K_i \) be the corresponding Aleksandrov body associated with \( u_i \), and \( h_i \) be the support function of \( K_i \). Note that for \( u_i \in C^+_f(S^{n-1}) \), \( K_i \in \mathcal{K}_n \) and that, as a consequence, \( h_i \in C^+_f(S^{n-1}) \). Standard convex bodies theory (see, e.g., [44]) tells us that \( V(u_i) = V(h_i) = V(K_i) \) and \( u_i(x) \geq h_i(x), \forall x \in S^{n-1} \). This leads to
\[
F(u_i(x)) = \int_0^{u_i(x)} \frac{1}{\varphi(t)} \, dt \geq \int_0^{h_i(x)} \frac{1}{\varphi(t)} \, dt = F(h_i(x)), \forall x \in S^{n-1}
\]
thanks to the assumption that \( \varphi : (0, +\infty) \to (0, +\infty) \), so that, since \( \lambda > 0 \), we have that
\[
\Phi_\lambda(u_i) = V(u_i) - \lambda \int_{S^{n-1}} F(u_i(x)) \, d\mu \leq V(h_i) - \lambda \int_{S^{n-1}} F(h_i(x)) \, d\mu = \Phi_\lambda(h_i).
\]

Then we can write, with these relations between \( u_i \) and \( h_i \), that
\[
\Phi_\lambda(h_i) \to \sup_{u \in C^+_f(S^{n-1})} \Phi_\lambda(u). \tag{4}
\]

For each \( K \in \mathcal{K}_n \), we write \( R_K := \max_{x \in \partial K} |x| \), so that \( \pm R_K x_K \in \partial K \) for some \( x_K \in S^{n-1} \), \( K \subset R_K B \), and \( K \supset [-R_K x_K, R_K x_K] \) the line segment joining the two points \( \pm R_K x_K \). Then it follows from the assumption (*) and from Lemma 3.2 that
\[
\int_{S^{n-1}} F(h_K) \, d\mu \geq \int_{S^{n-1}} F(R_K |\langle x, x_K \rangle|) \, d\mu \\
\geq \int_{S^{n-1}} c_1 R_K \langle x, x_K \rangle^\beta - c_2 d\mu \\
= c_1 R_K^\beta \int_{S^{n-1}} |\langle x, x_K \rangle|^\beta \, d\mu - c_2 |\mu|, \tag{5}
\]
where we have used the fact that
\[
h_K(x) \geq h_{[-R_K x_K, R_K x_K]}(x) = |\langle x, R_K x_K \rangle|, \forall x \in \mathbb{R}^n.
\]
As a consequence, we can write, thanks to Lemma 3.1, Lemma 3.2, and to (5), that for each \( K \in \mathcal{K}_n \),
\[
\Phi_\lambda(h_K) = V(h_K) - \lambda \int_{S^{n-1}} F(h_K(x)) \, d\mu \\
= V(K) - \lambda \int_{S^{n-1}} F(h_K(x)) \, d\mu \\
\leq V(R_K B) - \lambda \left[ c_1 R_K^\beta \int_{S^{n-1}} |\langle x, x_K \rangle|^\beta \, d\mu - c_2 |\mu| \right]
\]
Thus, since $\beta > n$, we have that
\[ \Phi_\lambda(h_K) \to -\infty, \quad \text{as } R_K \to +\infty. \]

In particular, we obtain the following result: there exists $R_0 > 0$ such that for all $K \in K_c$ with $R_K \geq R_0$ there holds
\[ \Phi_\lambda(h_K) \leq -1. \tag{6} \]

Thanks to (3), we have that
\[
\Phi_\lambda(h_{co,B}) = V(h_{co,B}) - \lambda \int_{S^{n-1}} F(h_{co,B})d\mu \\
= c_0^n V(B) - \lambda F(c_0)|\mu| \geq c_0^n V(B) - \lambda_0 F(c_0)|\mu| > 0,
\]
since $h_B(x) \equiv 1, \forall x \in S^{n-1}$. Thus we obtain that
\[ \sup_{u \in C^+_e(S^{n-1})} \Phi_\lambda(u) \geq \Phi_\lambda(h_{co,B}) > 0, \tag{7} \]
and that for $i$ large,
\[ \Phi_\lambda(h_i) > 0 \]

thanks to (4). It comes from (6) that $R_{K_i} < R_i$ for all $i$ large, so that $\{ K_i \} \subset R_0 B$, i.e., $K_i$ is a bounded sequence of convex bodies. Blaschke’s selection theorem (see, e.g., [44, Theorem 1.8.7]) then gives the existence of some compact convex subset $K_0 \subset \mathbb{R}^n$ such that, after passing to a subsequence, $K_i \to K_0$ in Hausdorff metric.

By the definition of Hausdorff metric, we thus have that $h_i \to h_0$ uniformly on $S^{n-1}$ (see, e.g., [44, Lemma 1.8.14]), with $h_0$ the support function of $K_0$. Since $\lim_{i \to \infty} K_i = K_0$ in the Hausdorff metric is equivalent to the two conditions taken together: (a) each point in $K_0$ is the limit of a sequence $(x_i)$ with $x_i \in K_i$; (b) the limit of any convergent sequence $(x_i)$ with $x_i \in K_i$ belongs to $K_0$, and $K_i \in K_c$, we have that $0 \in K_0$, and compact convex subset $K_0$ is symmetric about the origin. Since volume functional $V$ is continuous on all nonempty compact subsets of $\mathbb{R}^n$, with the Hausdorff metric, we have that
\[
V(K_0) = \lim_{i \to \infty} V(K_i) \\
= \lim_{i \to \infty} \left[ \Phi_\lambda(h_i) + \lambda \int_{S^{n-1}} F(h_i)d\mu \right] \\
\geq \lim_{i \to \infty} \Phi_\lambda(h_i) > 0.
\]

thanks to (4) and (7). This implies $K_0$ has nonempty interior. In turn, this clearly implies that $0 \notin \partial K_0$ as $K_0$ is symmetric about the origin so that $K_0$ contains the origin in its interior. Thus, $K_0 \in K_c^+$, and $h_0 \in C^+_e(S^{n-1})$. Thanks to the uniform convergence of $h_i$, we can choose $C > 0$ large enough such that
\[ \frac{1}{C} < \min_{S^{n-1}} h_0 \leq \max_{S^{n-1}} h_0 < C, \quad \frac{1}{C} \leq h_i(x) \leq C, \forall x \in S^{n-1}, \tag{8} \]
so that
\[
\left| \int_{\mathbb{S}^{n-1}} F(h_i) d\mu - \int_{\mathbb{S}^{n-1}} F(h_0) d\mu \right|
\]
\[
= \left| \int_{\mathbb{S}^{n-1}} \left[ \int_0^{h_i(x)} \frac{1}{\varphi(t)} dt \right] d\mu - \int_{\mathbb{S}^{n-1}} \left[ \int_0^{h_0(x)} \frac{1}{\varphi(t)} dt \right] d\mu \right|
\]
\[
= \left| \int_{\mathbb{S}^{n-1}} \left[ \int_0^{h_i(x)} \frac{1}{\varphi(t)} dt \right] d\mu \leq \int_{\mathbb{S}^{n-1}} \left[ \int_0^{h_i(x)} \frac{1}{\varphi(t)} dt \right] d\mu \right|
\]
\[
\leq \max_{1/C \leq t \leq C} \frac{1}{\varphi(t)} \int_{\mathbb{S}^{n-1}} |h_i(x) - h_0(x)| d\mu \Rightarrow 0 \text{ as } i \to \infty.
\]
where we have used the fact that \( \varphi : (0, +\infty) \to (0, +\infty) \) is a continuous function. Thus we have proved that
\[
\sup_{u \in C^+_{\varphi}(\mathbb{S}^{n-1})} \Phi_\lambda(u) = \lim_{i \to \infty} \Phi_\lambda(h_i)
\]
\[
= \lim_{i \to \infty} \left[ V(h_i) - \lambda \int_{\mathbb{S}^{n-1}} F(h_i) d\mu \right]
\]
\[
= V(h_0) - \lambda \int_{\mathbb{S}^{n-1}} F(h_0) d\mu = \Phi_\lambda(h_0),
\]
so that, the support function \( h_0 \) of the convex body \( K_0 \in K^n_c \) verifies
\[
\frac{d}{dt} \Phi_\lambda(h_0 + tf) = 0
\]
for any even continuous function \( f \) on \( \mathbb{S}^{n-1} \) since \( h_0 + tf \in C^+_{\varphi}(\mathbb{S}^{n-1}) \) for \( |t| \) small. Note here that, applying the dominated convergence theorem, we get, thanks to the continuity of \( \varphi \) on \( (0, +\infty) \), and to (8), that
\[
\int_{\mathbb{S}^{n-1}} F(h_0 + tf) d\mu - \int_{\mathbb{S}^{n-1}} F(h_0) d\mu
\]
\[
= \int_{\mathbb{S}^{n-1}} \left[ \int_0^{h_0(x) + tf(x)} \frac{1}{\varphi(t)} dt \right] d\mu - \int_{\mathbb{S}^{n-1}} \left[ \int_0^{h_0(x)} \frac{1}{\varphi(t)} dt \right] d\mu
\]
\[
= \int_{\mathbb{S}^{n-1}} \left[ \int_0^{h_0(x)} \frac{1}{\varphi(t)} dt \right] d\mu \to \int_{\mathbb{S}^{n-1}} \frac{1}{\varphi(h_0(x))} f(x) d\mu \quad \text{(10)}
\]
as \( t \to 0 \), where \( \square \) is between \( h_0(x) \) and \( h_0(x) + tf(x) \), since
\[
\frac{1}{\varphi(\square)} f(x) \to \frac{1}{\varphi(h_0(x))} f(x), \forall x \in \mathbb{S}^{n-1}
\]
as \( t \to 0 \), and for \( |t| \) small,
\[
\left| \frac{1}{\varphi(\square)} f(x) \right| \leq \max_{1/C \leq t \leq C} \frac{1}{\varphi(t)} \cdot |f(x)|.
\]
Coming back to (9) with (10) and the variation formula for volume (see, e.g., [20, Corollary 1]), we have that, for any \( \lambda \in (0, \lambda_0) \),
\[
\int_{\mathbb{S}^{n-1}} f dS_{K_0} - \lambda \int_{\mathbb{S}^{n-1}} \frac{1}{\varphi(h_0)} f d\mu = 0.
\]
or say,
\[ \frac{1}{\lambda} \varphi(h_0(x)) dS_{K_0} = d\mu. \]

Thus we have obtained the following result: there exists \( c_* = 1/\lambda_0 > 0 \) such that for any \( c \in [c_*, +\infty) \) there exists a convex body \( K_c \in K^n_0 \) such that
\[ c \varphi(h_{K_c}) dS_{K_c} = d\mu. \]

This ends the proof of Theorem 1.1.

Acknowledgments. The authors thank the two referees for carefully reading this paper and suggesting useful comments. This work was supported by the National Science Foundation of China (Grants 11971027 and 11771468).

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Received October 2020; revised February 2021.

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