EXTENSIONS OF THE CLASSICAL TRANSFORMATIONS OF THE HYPERGEOMETRIC FUNCTION $3F_2$

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Abstract. It is shown that the classical quadratic and cubic transformations of the generalized hypergeometric function $3F_2$ have extensions involving hypergeometric functions of higher order, the additional parameter pairs of which have integral differences. The added parameters are nonlinearly constrained: they are the negated roots of certain dual Hahn and Racah polynomials. Applications of the new function transformations include the extending of Whipple’s formula relating very well poised $7F_6(1)$ series and balanced $4F_3(1)$ series, to versions that have additional, nonlinearly constrained parameters.

1. Introduction

The Gauss hypergeometric function $2F_1$ and its non-confluent generalizations $3F_2, 4F_3$, etc., are parametrized higher transcendental functions of continuing importance. They satisfy many identities of the form $F(\phi(x)) = A(x)\tilde{F}(x)$, where $\phi$ is a rational function satisfying $\phi(0) = 0$, $A$ is a product of zero or more powers of rational functions, and the parameters of the left-hand hypergeometric function $F$ and its lifted version $\tilde{F}$ are constrained and related. The best known identities of this type are Euler’s and Pfaff’s transformations of $2F_1$, for which $\phi$ is of degree 1, and the many quadratic and cubic transformations of $2F_1$. The transformations of $2F_1$ with at least one free parameter were determined by Goursat [9].

Only a few of the transformations of $2F_1$ to itself extend to ones of $3F_2$ to itself [3]. On the $3F_2$ level, the classical identities include Whipple’s quadratic transformation [2, (3.1.15)] and Bailey’s two cubic ones [2, Ch. 3, Ex. 3.8]. In each, the left-hand $3F_2$ has parametric excess equal to $\frac{1}{2}$. (The parametric excess or Saalschützian index is the sum of the lower parameters, less the sum of the upper ones; throughout this paper, it will be denoted by $S$.) Each of these three has a ‘companion’ in which the left-hand function has $S = -\frac{1}{2}$ and the right-hand one is not a $3F_2$ but a $4F_3$. (See [3, p. 97, Example 6] and [8, (4.1), (5.4), (5.7)].)

If the hypergeometric functions $F, \tilde{F}$ are of the same order, a transformation of the form $F(\phi(x)) = A(x)\tilde{F}(x)$ may be attributable to the differential equation satisfied by $F$ being lifted (i.e., pulled back) by the map $x \mapsto \phi(x)$ to the equation satisfied by $\tilde{F}$. (For the case of $2F_1$, see [2, §3.9] and [19].) Recently, Kato determined all transformations of $3F_2$ to $3F_2$ which are of this sort [11]. They include Whipple’s quadricic, Bailey’s two cubics, and several more obscure ones.

It is shown here that each of these three classical transformations of a $3F_2$ (with $S = \frac{1}{2}$) to another $3F_2$ can be extended to one of a $3F_2$ (with $S = \frac{1}{2} + k$,
k = 0, 1, 2, \ldots \) to a \( 3_{+2k} F_{2+2k} \). The parameters of the latter function, \( \tilde{F} \), are non-linearly constrained: they arise from the (negated) roots of a certain polynomial. An example is the extension of Whipple’s quadratic, which is

\[
(1.1) \quad 3 F_2 \left[ \begin{array}{c} \frac{a}{2}, \frac{1+a}{2}, 1-k+a-b-c \\ 1+a-b, 1+a-c \end{array} \middle| \ -\frac{4x}{(1-x)^2} \right] = (1-x)^\alpha \left[ \begin{array}{c} a, b, c, 1+\xi_1, \ldots, 1+\xi_{2k} \\ 1+a-b, 1+a-c, \xi_1, \ldots, \xi_{2k} \end{array} \middle| x \right],
\]

with \( 2k \) unit-difference parameter pairs; and \( \xi_1, \ldots, \xi_{2k} \) are the negated roots of

\[
(1.2) \quad Q_k^{(2)}(n; a, b, c) = 3 F_2 \left[ \begin{array}{c} -n, n+a, -k \\ b, c \end{array} \middle| 1 \right],
\]

which is a polynomial of degree \( 2k \) in \( n \).

This result touches the theory of orthogonal polynomials of a discrete variable, because \( Q_k^{(2)}(n; a, b, c) \) is essentially a dual Hahn polynomial [12, §9.6]; it is invariant under \( n \rightarrow -n-a \) and can be written as \( R_k(\lambda(n); a, b, c) \), where \( R_k(\lambda; a, b, c) \) is of degree \( k \) in \( \lambda(n) = n(n+a) \), the coordinate of a quadratic lattice. The case \( k = 0 \) of \( Q_k^{(2)} \) is the classical one; the case \( k = 1 \) was proved more recently [14], as was its \( q \)-analogue [1]. It should be noted that for all \( k \geq 0 \), the \( 3_{+2k} F_{2+2k} \), having \( 2k \) unit-difference parameter pairs, can be written as a combination of \( 3 F_2 \)'s [19].

The two cubic transformations of Bailey can be extended to \( k \geq 0 \) in the same way, though the corresponding degree-\( 2k \) polynomials \( Q_k^{(3)}, Q_k^{(3)} \) are asymmetric and may lack an interpretation as orthogonal polynomials. One of the resulting identities is the curious specialization

\[
(1.3) \quad 3 F_2 \left[ \begin{array}{c} -\frac{\theta}{2} + \frac{\sqrt{3}}{2} \sin \theta, \frac{\theta}{2} + \frac{\sqrt{3}}{2} \sin \theta, \frac{\theta}{2} + \frac{\sqrt{3}}{2} \sin \theta \\ 1 + \sin(\theta + \frac{\pi}{6}), 1 + \sin(\theta - \frac{\pi}{6}) \end{array} \middle| -\frac{27x}{(1-4x)^3} \right] = (1-4x)^{-\frac{\theta}{2} + \sqrt{3}\sin \theta} \times 4 F_3 \left[ \begin{array}{c} -\frac{\theta}{2} + \sqrt{3}\sin \theta, -\frac{\theta}{2} + \sqrt{3}\sin \theta, \frac{\theta}{2} + \sqrt{3}\sin \theta, \frac{\theta}{2} + \sqrt{3}\sin \theta \\ 1 + \sin(\theta + \frac{\pi}{6}), 1 + \sin(\theta - \frac{\pi}{6}), 1 + \sin(\theta - \frac{\pi}{6}) \end{array} \middle| x \right].
\]

The left-hand \( 3 F_2 \) has \( S = \frac{3}{2} = \frac{1}{2} + k \) with \( k = 1 \). One would expect the right-hand function to be \( 3_{+2k} F_{2+2k} = 5 F_3 \), but the left-hand parameters are chosen here in such a way that the right-hand pairs \( \left( \frac{1+\xi_1}{\xi_1}, \frac{1+\xi_2}{\xi_2} \right) \) that come from the negated roots \( \xi_1, \xi_2 \) of \( Q_1^{(3)} \) satisfy \( \xi_2 = 1 + \xi_1 \). This makes possible their merging into the single final pair seen in [13], which is of the form \( \left( \frac{2+\xi_1}{\xi_1} \right) \).

The identities extending Whipple’s quadratic transformation and Bailey’s cubic ones are shown to have generalizations to \( 4 F_3 \). In each identity a new parameter-pair \( \langle k+d, d \rangle \), with \( d \) supplying a degree of freedom, can be added to the parameter array of the left-hand \( 3 F_2 \), converting it to a \( 4 F_3 \). The resulting generalized polynomials \( Q_k^{(2)}, Q_k^{(3)}, Q_k^{(3)} \) on the right-hand side depend on \( d \) and have representations in terms of \( 4 F_3 \), and the latter two are now of degree \( 3k \) in \( n \). The generalized \( Q_k^{(2)} \) is essentially a Racah polynomial [12, §9.2]. The results of this paper on quadratic \( 3 F_2 \) and \( 4 F_3 \) transformations make contact with work of Miller and Paris [13] and Rathie, Rakha et al. [15–16, 20], who have considered the effects of adding some number \( r \geq 1 \) of parameter-pairs with integral differences, such as \( \left( \frac{m_1+d_1}{d_1}, \ldots, \frac{m_r+d_r}{d_r} \right) \), to the left-hand functions in quadratic transformations of \( 2 F_1 \).
On the $3F_2$ level, it is shown that each of the ‘companion’ transformations of a
parametrized $3F_2$ with $S = -\frac{1}{2}$ to a $4F_3$ (i.e., the companions of Whipple’s qua-
dratic and Bailey’s cubics) can also be extended. Each extends to a transformation of
a parametrized $3F_2$ with $S = -\frac{1}{2} - k, k = 0, 1, 2, \ldots,$ to a $4+4kF_{3+4k}$. The

parameter arrays of the latter function $\tilde{F}$ include $1 + 4k$ parameter-pairs with unit
differences, of the form \((1 + \xi_1, \ldots, 1 + \xi_{1+4k})\). Here $\xi_1, \ldots, \xi_{1+4k}$ are the negated roots
of a new polynomial $Q_k^{(2)}$, resp. $Q_k^{(3)}$, resp. $Q_k^{(3')}$. These $k$-indexed polynomials have
no obvious hypergeometric representation or interpretation involving orthogonality,
but recurrences for them are supplied. Interestingly, the new family $Q_k^{(2)}$, like the
dual Hahn and Racah ones denoted by $Q_k^{(2)}$, is defined on a quadratic lattice.

Gessel and Stanton [8] showed that by pairing $3F_2$ transformations with their
companions, one can derive many hypergeometric evaluation formulas, including
Whipple’s identity relating $7F_5(1)$ and $4F_3(1)$, and ‘strange’ evaluations discov-
ered by Gosper. Applying the same technique to the extensions of this paper yields
extended versions of several of the Gessel–Stanton formulas, which incorporate non-
linear parametric constraints. These new formulas, in particular two extensions of
Whipple’s identity with extension parameter $k = 0, 1, 2, \ldots,$ overlap those recently
found by Srivastava, Vyas and Fatawat [18].

Finally, another technique (multiplying both sides of a hypergeometric transfor-
mation by a power of \((1 - x)\) and equating the coefficients of $x^m$ on the two sides)
is shown to yield extensions of certain known summation formulas [5 § 4.5(1,2)].

The main extension theorems are stated in §3 and most are proved in §4. The
recurrences satisfied by the $Q_k$ and $Q_k$, which resemble and include those satisfied
by the dual Hahn and Racah polynomials, are derived in §5. The summation
identities mentioned in the two preceding paragraphs are derived in §§6 and 7.

2. Preliminaries

The generalized hypergeometric function $F = \frac{r+1}{r+1}F_r$, with \((a) = a_0, a_1, \ldots, a_r\)
and \((b) = b_1, \ldots, b_r\) as its arrays of $C$-valued parameters, is defined by

\[
(2.1) \quad F \left[ \begin{array}{c} a_0, a_1, \ldots, a_r \\ b_1, \ldots, b_r \end{array} \right] (x) = \sum_{n=0}^{\infty} \frac{(a)_n(a_1)_n \ldots (a_r)_n}{(1)_n(b_1)_n \ldots (b_r)_n} x^n,
\]

where \((a)_0 = 1, (a)_n = (a)(a+1)\ldots(a+n-1)\). It is assumed that no lower parameter
is a nonpositive integer, to avoid division by zero; and if an upper one is a
nonpositive integer, the series will terminate. The series converges on \(|x| < 1\), and
at $x = 1$ if Re $S > 0$; if $x = 1$, the argument is usually omitted. Hypergeometric
identities of the form $F(\varphi(x)) = A(x)\tilde{F}(x)$ with $\varphi(0) = 0$ are taken to hold on the
largest neighborhood of $x = 0$ to which both sides can be continued.

Any $\frac{r+1}{r+1}F_r$ with excess $S$ is said to be $S$-balanced. It is called well-poised if
$a_0 + 1 = a_1 + b_1 = \cdots = a_r + b_r$, and nearly poised if only one of these $r + 1$
parameter-pair sums differs from the others. It is called $(M, N)$-poised if $Ma_0 + N =
Ma_1 + Nb_1 = \cdots = Ma_r + Nb_r$, where $M, N$ are positive integers. It is called very
well poised if it is well-poised and a parameter-pair, e.g., \((a_1)_{b_1}\), equals \((1+a_0/2)_{a_0/2}\).

It is convenient to extend the definition (2.1) to

\[
(2.2) \quad F \left[ \begin{array}{c} (a) \\ (b) \end{array} \right] (Q(n)) (x) = \sum_{n=0}^{\infty} \frac{(a)_n(b)_n}{(1)_n(\beta)_n} Q(n) x^n,
\]
where \((\alpha), (\beta)\) are arrays of parameters and \(Q: \mathbb{N} \to \mathbb{C}\) is any function of growth no more rapid than exponential. If \(Q(n)\) is a polynomial of degree \(\ell\) satisfying \(Q(0) = 1\), with \((\alpha) = a_0, a_1, \ldots, a_r\) and \((\beta) = b_1, \ldots, b_r\), then by examination

\[
F \left[ \begin{array}{c} a_0, a_1, \ldots, a_r \\ b_1, \ldots, b_r \\ \end{array} | Q(n) \right] x
\]

(2.3)

\[
= r+\ell+1 F_{r+\ell} \left[ \begin{array}{c} a_0, a_1, \ldots, a_r, 1 + \xi_1, \ldots, 1 + \xi_\ell \\ b_1, \ldots, b_r, \xi_1, \ldots, \xi_\ell \\ \end{array} | x \right],
\]

where \(\xi_1, \ldots, \xi_\ell\) are the negated roots of \(Q(n)\), counted with multiplicity. The right-hand side of (2.3) is a hypergeometric function with \(\ell\) unit-difference parameter pairs, which can optionally be written as a combination of \(r+1\)’s [10]. In the formulas that employ this notation, the normalization \(Q(0) = 1\) will hold.

The key lemma is the following (cf. [7, (5.7)]). Here, \(\Delta(\cdot)\) is a polynomial of degree \(\ell\) unit-difference parameter pairs, which can optionally be written as a combination of \(r+1\)’s [10]. In the formulas that employ this notation, the normalization \(Q(0) = 1\) will hold.

Lemma 2.1. For \(l, m \geq 1\), arbitrary parameter arrays \((\alpha), (\beta)\) of lengths \(A, B\), and arbitrary \(a\) and \(x_0 \neq 0\), one has the identity

\[
l+m+A \ B F \left[ \begin{array}{c} \Delta(l+m; a) \\ \Delta(l+m; 1) \\ \end{array} \right] F \left[ \begin{array}{c} \Delta(l; -n) \\ \Delta(m; n+a) \\ \end{array} \right] = (1 - x/x_0)^a F \left[ \begin{array}{c} a \\ R(n) \end{array} \right] \]

(2.4)

where

\[
R(n) = l+m+A \ B F \left[ \begin{array}{c} \Delta(l; -n) \\ \Delta(m; n+a) \\ \end{array} \right] F \left[ \begin{array}{c} a \\ 1 \end{array} \right],
\]

assuming the convergence of the series for the latter \(l+m+A \ B F (1)\).

Only the case \(l + m + A = B + 1\) will be needed. This is an identity of the double-summation type: to prove it, one expands the hypergeometric argument \(\varphi(x)\) of the left-hand \(l+m+A \ B F\) in a geometric series, and converts the left side (multiplied by \(1 - x/x_0)^a\) to the right one (multiplied by same) by interchanging the order of the two summations. It could be called classical; it was stated by Bailey [4, §4], and the \(l = m = 1\) case was rediscovered by Chaundy. A generalization was proved in [7]. Special cases are scattered in the literature; for details, see [17, §2.6].

3. Main Theorems

The theorems are arranged in a 3 \(\times\) 3 array. Sections 3.1, 3.2, and 3.3 contain the extensions of the classical transformations of \(3F_2\) to itself, their generalizations to \(4F_3\), and the extensions of the companion transformations of \(3F_2\) to \(4F_3\). Each of these sections contains three transformations: one quadratic and two cubic.

3.1. Extended transformations of \(3F_2\). The following theorems, indexed by \(k \geq 0\), reduce to Whipple’s quadratic transformation and Bailey’s two cubic ones when \(k = 0\). In each, the left-hand \(3F_2\) has \(S = \frac{1}{2} + k\).

Theorem 3.1. For all \(k \geq 0\), one has the quadratic transformation

\[
3F_2 \left[ \begin{array}{c} \frac{3}{2}, \frac{1}{2} + \frac{r}{2}, 1 - k + a - b - c \\ 1 + a - b, 1 + a - c \\ \end{array} \right] \frac{4x}{(1-x)^2}
\]

\[
= (1 - x)^a 3_{2k+2k}F_{2+2k} \left[ \begin{array}{c} a, b, c \\ 1 + a - b, 1 + a - c \end{array} | Q^{(2)}_k(n) \right] x,
\]
where \( Q_k^{(2)}(n) = Q_k^{(2)}(n; a, b, c) \) is a degree-2k polynomial in \( n \) or a degree-k one in \( \lambda = \lambda(n; a) = n(n + a) \), defined by

\[
Q_k^{(2)}(n; a, b, c) = {_3F_2}\left[ \begin{array}{c} -n, n + a, -k \\ b, c \end{array} \right].
\]

Here the right-hand \( 3+2kF_{2+2k} \) is well-poised for all \( k \geq 0 \). Owing to the \( n \mapsto -n - a \) invariance, the negated roots \( \xi_1, \ldots, \xi_{2k} \) of \( Q_k^{(2)} \) are symmetric about \( \xi = \frac{a}{2} \), and the lower parameters \( \xi_1, \ldots, \xi_{2k} \) of the \( 3+2kF_{2+2k} \) that are implicit in this formula (recall (2.3)) can be permuted so that each parameter-pair sums to \( 1 + a \).

The \( k = 1 \) case of this quadratic \( 3F_2 \) transformation, the first to exhibit nonlinear parametric constraints, was discovered by Niblett [14] (22). One finds

\[
Q_1^{(2)}(n; a, b, c) = 1 + \frac{\lambda}{bc} = \frac{n^2 + an + bc}{bc},
\]

suggesting a subcase of interest: if \( a = -b - c \), then \( Q_k^{(2)}(n) = (n - b)(n - c)/bc \) and the negated roots \( \{ \xi_1, \xi_2 \} \) are \( \{-b, -c\} \). The resulting specialization is

\[
3F_2\left[ \begin{array}{c} -\frac{1}{2}, -\frac{1}{2}, \frac{1}{2} - \frac{2}{n} - b - 2c, 1 - b + 2c \end{array} \right] - \frac{4x}{(1 - x)^2}
\]

\[
= (1 - x)^{-b - c} {_5F_4}\left[ \begin{array}{c} -b - c, b, 1 - 2b - c, 1 - b - 2c, 1 - c \end{array} \right],
\]

in which the hypergeometric parameters are constrained linearly. (Compare [14] (16).) Another notable \( k = 1 \) subcase occurs when \( a^2 = 1 + 4bc \). Then the negated roots \( \xi_1, \xi_2 \) of \( n^2 + an + bc \) differ by unity, and the parameter-pairs \( \left( \frac{1 + \xi_1}{\xi_1} \right), \left( \frac{1 + \xi_2}{\xi_2} \right) \) can be merged into \( \left( \frac{2 + \xi_1}{\xi_1} \right) \), reducing the right-hand \( 5F_4 \) to a \( 4F_3 \).

Other specializations of interest include the case \( c = b = \frac{a}{2} \), when the identity reduces to a transformation of a \( 2F_1 \) with \( S = \frac{1}{2} + k \) to a well-poised \( 2+2kF_{1+2k} \). The \( k = 0 \) subcase of this is classical, but the \( k > 0 \) subcases are new.

**Theorem 3.2.** For all \( k \geq 0 \), one has the first cubic transformation

\[
3F_2\left[ \begin{array}{c} \frac{a}{2}, \frac{a}{2}, \frac{a}{2}, \frac{a}{2}, \frac{a}{2}, \frac{a}{2} - b \end{array} \right] - \frac{27x}{(1 - 4x)^3}
\]

\[
= (1 - 4x)^a 3+2kF_{2+2k}\left[ a, \frac{3}{4} + \frac{k}{2} + b, \frac{3}{4} + \frac{k}{2} + b, \frac{3}{4} + \frac{k}{2} + b, \frac{3}{4} + \frac{k}{2} + b \right] Q_k^{(3)}(n) \quad \frac{x}{x},
\]

where \( Q_k^{(3)}(n) = Q_k^{(3)}(n; a, b) \) is a degree-2k polynomial in \( n \), equal to

\[
\frac{4^k(\frac{1}{2} - \frac{b}{2} + \frac{a}{2} - \frac{3}{2})k(\frac{1}{2} - \frac{b}{2} - \frac{3}{2})}{(\frac{1}{2} + b)(\frac{1}{2} - b)} 3F_2\left[ \begin{array}{c} -n, \frac{a}{2}, \frac{a}{2}, -k \end{array} \right] - \frac{n^2 + 4(1 + 2a)n + (1 - 4b^2)}{1 - 4b^2}.
\]

Here the right-hand \( 3+2kF_{2+2k} \) is \( (1, 2) \)-poised if \( k = 0 \) (the classical case), but not otherwise. This is illustrated by the \( k = 1 \) case. One finds

\[
Q_1^{(3)}(n; a, b) = \frac{12n^2 + 4(1 + 2a)n + (1 - 4b^2)}{1 - 4b^2}
\]

(the denominator being required by the normalization \( Q_k^{(3)}(n = 0) = 1 \); the subcase \( b = \pm \frac{1}{2} \) is singular). From this, the negated roots \( \xi_1, \xi_2 \) needed for the \( k = 1 \) case can be computed. The resulting upper parameters \( 1 + \xi_1, 1 + \xi_2 \) and lower parameters...
Theorem 3.3. For all \( k \geq 0 \), one has the second cubic transformation

\[
3F_2 \left[ \frac{3}{4} + \frac{3}{4}, \frac{1}{2} + \frac{3}{4}, \frac{3}{4} + \frac{3}{4} \right] \begin{pmatrix} \frac{27x^2}{(4-x)^3} \\ \end{pmatrix} = (1 - \frac{x}{2})^3 3F_2 \left[ \frac{3}{4} + \frac{3}{4}, \frac{1}{2} + \frac{3}{4}, \frac{3}{4} + \frac{3}{4} \right] \begin{pmatrix} a, b, c \end{pmatrix} = 12n^2 - 4(1 - 4n) + (1 - 2a - b)(1 - 2a + b) \]

where \( Q_k^{(3)}(n) = Q_k^{(3)}(n; a; b) \) is a degree-2k polynomial in \( n \), equal to \( (1 - \frac{x}{2})^3 3F_2 \left[ \frac{3}{4} + \frac{3}{4} - \frac{2}{2} + \frac{2}{2} - n \right] \begin{pmatrix} a, b, c \end{pmatrix} = 12n^2 - 4(1 - 4n) + (1 - 2a - b)(1 - 2a + b) \)

Here the right-hand \( 3F_2 \) is \( (2,1) \)-poised if \( k = 0 \) (the classical case), but not otherwise. The polynomials \( Q_k^{(3)} \) differ from the \( Q_k^{(3)} \); for instance,

\[
3F_2 \left[ \frac{3}{4} + \frac{3}{4}, \frac{1}{2} + \frac{3}{4}, \frac{3}{4} + \frac{3}{4} \right] \begin{pmatrix} 1/2, 1/2, 1 \end{pmatrix} = \frac{12n^2 - 4(1 - 4n) + (1 - 2a - b)(1 - 2a + b)}{(1 - 2a - b)(1 - 2a + b)}.
\]

As with Theorem 3.2 there are interesting specializations.

3.2. Generalizations to \( 4F_3 \). Each left-hand \( 4F_3 \) in the following theorems has \( S = \frac{1}{2} \) and contains a parameter-pair \( (k + \frac{d}{a}) \), where \( d \) is an additional free parameter. These identities reduce to Whipple’s and Bailey’s classical transformations when \( k = 0 \), and to the extensions of \( 3F_2 \) when \( d \to \infty \). It should be noted that any \( 4F_3 \) with a parameter-pair \( (k + \frac{d}{a}) \) can be written as a combination of \( 1 + k 3F_2 \)’s.

Theorem 3.4. For all \( k \geq 0 \), one has the quadratic transformation

\[
4F_3 \left[ \frac{3}{4}, \frac{1}{2} + \frac{3}{4}, 1 - k + a - b - c, \frac{k + d}{1 + a - b, 1 + a - c} - \frac{4x}{(1 - x)^2} \right] = (1 - x)^n 3F_2 \left[ \frac{a, b}{1 + a - b, 1 + a - c} \right] \begin{pmatrix} Q_k^{(2)}(n) \end{pmatrix} \]
where \( Q_k^{(2)}(n) = Q_k^{(2)}(n; a; b, c, d) \) is a degree-2k polynomial in \( n \) or a degree-k one in \( \lambda = \lambda(n; a) = n(n + a) \), defined by

\[
Q_k^{(2)}(n; a; b, c, d) = 4F_3 \left[ \begin{array}{c} -n, n + a, -k, \frac{k - 1}{2} - a + b + c + d \\ b, c, d \end{array} \right].
\]

Here the right-hand \( _3+2kF_{2+2k} \) is well-poised for all \( k \geq 0 \), as in Theorem 3.4

The four-parameter \( Q_k^{(2)}(n) \) is essentially a Racah polynomial [12 § 9.2], just as the three-parameter one in Theorem 3.1 was a dual Hahn polynomial. For instance,

\[
Q_k^{(2)}(n; a; b, c, d) = 1 + \frac{(b + c + d - a)\lambda}{bcd} = (b + c + d - a)n^2 + a(b + c + d - a)n + bcd,
\]

the \( d \to \infty \) limit of which is the \( Q_k^{(2)}(n; a, b, c) \) of (3.1). Owing to the \( n \mapsto -n - a \) invariance, the negated roots \( \xi_1, \ldots, \xi_{2k} \) of \( Q_k^{(2)} \) are symmetric about \( \xi = \frac{a}{2} \).

Specializations of interest include the case \( c = \frac{1}{2} + \frac{a}{2} \), which leads to

\[
\begin{align*}
3F_2 & \left[ \frac{a}{2}, \frac{b}{2} - k + \frac{a}{2} - b, \frac{k + d}{2} \right] \quad \left[ \begin{array}{c} 1 + a - b, -k \\ d \end{array} \right] \quad \frac{-4x}{(1 - x)^2} \\
= & (1 - x)^{a \cdot 2 + 2k} \frac{F_1 + 2k}{F_3} \left[ a, b, \frac{1}{1 + a - b}, \frac{k + d}{2} \right] \quad \left[ \begin{array}{c} 1 - a - b, x \end{array} \right],
\end{align*}
\]

where \( Q_k^{(2)}(n) := Q_k^{(2)}(n; a; b, \frac{1}{2} + \frac{a}{2}; d) \). The \( k = 1 \) cases of (3.6) and (3.8) are known (see [20 Thm. 1], resp. [15 (3.1)]). It must be mentioned that other transformations of a \( _3F_2 \) with a parameter-pair \( (1 + d) \) to a \( _4F_3 \) have been found (see [13 § 6] and [16]). The others have lifting functions \( \varphi(x) \) equal to \( \frac{x^2}{(2-x)^2}, \frac{4x}{(1-x)^2}, 4x(1-x) \).

**Theorem 3.5.** For all \( k \geq 0 \), one has the first cubic transformation

\[
\begin{align*}
4F_3 & \left[ \frac{a}{3}, \frac{b}{3} + \frac{a}{3}, \frac{c}{3} + \frac{a}{3}, \frac{d}{3} + \frac{a}{3} \right] \quad \left[ \begin{array}{c} \frac{k}{3} + d \\ d \end{array} \right] \quad \frac{-27x}{(1 - x)^3} \\
= & (1 - 4x)^{a \cdot 3 + 3k} \frac{F_1 + 3k}{F_3} \left[ a, \frac{b}{3} - k - b, \frac{c}{3} - k + b \right] \quad \left[ \begin{array}{c} 1 - a - b, \frac{k + d}{3} \end{array} \right],
\end{align*}
\]

where \( Q_k^{(3)}(n) = Q_k^{(3)}(n; a; b; d) \) is a degree-3k polynomial in \( n \), defined as in Theorem 3.2 but with the \( _3F_2(1) \) in the definition extended to

\[
4F_3 \left[ \frac{a}{3}, \frac{b}{3} + \frac{a}{3}, \frac{c}{3} + \frac{a}{3}, \frac{d}{3} + \frac{a}{3} \right] \quad \left[ \begin{array}{c} \frac{k}{3} + d \\ d \end{array} \right] \quad \frac{-27x}{(1 - x)^3}.
\]

**Theorem 3.6.** For all \( k \geq 0 \), one has the second cubic transformation

\[
\begin{align*}
4F_3 & \left[ \frac{a}{3}, \frac{b}{3} + \frac{a}{3}, \frac{c}{3} + \frac{a}{3}, \frac{d}{3} + \frac{a}{3} \right] \quad \left[ \begin{array}{c} \frac{k}{3} + d \\ d \end{array} \right] \quad \frac{27x^2}{(1 - x)^3} \\
= & (1 - x)^{a \cdot 3 + 3k} \frac{F_1 + 3k}{F_3} \left[ a, \frac{b}{3} - k - b, \frac{c}{3} - k + b \right] \quad \left[ \begin{array}{c} 1 - a - b, \frac{k + d}{3} \end{array} \right],
\end{align*}
\]

where \( Q_k^{(3)}(n) = Q_k^{(3)}(n; a; b; d) \) is a degree-3k polynomial in \( n \), defined as in Theorem 3.2 but with the \( _3F_2(1) \) in the definition extended to

\[
4F_3 \left[ \frac{a}{3}, \frac{b}{3} + \frac{a}{3}, \frac{c}{3} + \frac{a}{3}, \frac{d}{3} + \frac{a}{3} \right] \quad \left[ \begin{array}{c} \frac{k}{3} + d \\ d \end{array} \right] \quad \frac{27x^2}{(1 - x)^3}.
\]
3.3. Extended companion transformations of $3F_2$. The following theorems, indexed by $k \geq 0$, reduce to the companions of Whipple’s quadratic transformation and Bailey’s two cubic ones when $k = 0$. In each, the left-hand $3F_2$ has $S = -\frac{1}{2} - k$.

**Theorem 3.7.** For all $k \geq 0$, one has the quadratic transformation

$$3F_2 \left[ \frac{1}{2} + k + \frac{a}{2}, 1 + k + \frac{b}{2}, 1 - k - a - b - c \right] \left[ \frac{a}{2}, \frac{b}{2}, \frac{c}{2} \right] = (1 + x)^{1-2k} \times (1 - x)^{1+2k+a} \times 3F_3 \left[ \frac{a}{3}, \frac{b}{3}, \frac{c}{3} \right] \left[ \frac{1}{2} + k + \frac{a}{2}, 1 + k + \frac{b}{2}, 1 + k + \frac{c}{2} \right],$$

where $Q^{(2)}_k(n) = Q^{(2)}_k(n; a; b, c)$ is a degree-$(1+4k)$ polynomial in $n$, equal to $1 + \frac{2n}{a}$ times $Q^{(2)}_k(n) = Q^{(2)}_k(n; a; b, c)$, which is a degree-4k polynomial in $n$ or a degree-2k one in $\lambda = \lambda(n; a) = n(n + a)$, determined by $Q^{(2)}_0 \equiv 1$ and the $k$-raising relation

$$(k + \frac{a}{2})(1 + a)bc(n + \frac{a}{2})Q^{(2)}_k(n) = (n + k + \frac{a}{2})(n + a)(n + b)(n + c)(n + \frac{1}{2} + \frac{a}{2})Q^{(2)}_{k-1,+}(n) + (n - k + \frac{a}{2})n(n + a - b)(n + a - c)(n - \frac{1}{2} + \frac{a}{2})Q^{(2)}_{k-1,+}(n - 1),$$

with $Q^{(2)}_{k-1,+}(n) := Q^{(2)}_{k-1}(n; a + 1, b + 1, c + 1)$.

Here the right-hand $4+4kF_{3+4k}$ is very well poised for all $k \geq 0$, because one negated root is $\xi_1 = \frac{a}{2}$, coming from the factor $1 + \frac{2n}{a}$, and the remaining ones $\xi_2, \ldots, \xi_{1+4k}$ are symmetric about $\xi = \frac{a}{2}$, as the recurrence for $Q^{(2)}_k$ is invariant under $n \mapsto -n - a$. An example of $Q^{(2)}_k$ being of degree 2k in $\lambda = n(n + a)$ is

$$Q^{(2)}_1(n; a; b, c) = 1 + \frac{\lambda(4\lambda + (a - 1)(a - 2) + (2b + 3)(2c + 3) - 9)}{(a + 1)(a + 2)bc}. \quad \text{(3.9)}$$

Specializations of interest include the case $c = \frac{1}{2} + \frac{a}{2}$, when the right-hand $4+4kF_{3+4k}$ reduces to a $3+4kF_{2+4k}$, and $c = \frac{1}{2} - k + \frac{a}{2}$ and $c = -k + \frac{a}{2}$, when the left-hand $3F_2$ reduces to a $2F_1$. One can show from the raising relation that, e.g.,

$$Q^{(2)}_k(n; a; b, \frac{1}{2} - k + \frac{a}{2}) = \left( \frac{\frac{1}{2} + k + \frac{a}{2}}{\frac{1}{2} - k + \frac{a}{2}} \right)_n 3F_2 \left[ \frac{-n}{2}, \frac{n + a}{2}, -k \right] \left[ \frac{1}{2}, 1 + \frac{a}{2} \right], \quad \text{(3.10)}$$

2k of the 4k negated roots of which are $\frac{1}{2} - k + \frac{a}{2}, \ldots, -\frac{1}{2} + k + \frac{a}{2}$. But for general parameter choices, a hypergeometric representation of $Q^{(2)}_k(n; a; b, c)$ is lacking.

**Theorem 3.8.** For all $k \geq 0$, one has the first cubic transformation

$$3F_2 \left[ \frac{1}{3} + \frac{2k}{3} + \frac{a}{3}, \frac{2}{3} + \frac{2k}{3} + \frac{b}{3}, \frac{1}{3} + \frac{2k}{3} + \frac{c}{3} \right] \left[ \frac{a}{3}, \frac{b}{3}, \frac{c}{3} \right] = (1 + 8x)^{1-2k} \times (1 - 4x)^{1+2k+a} \times 3F_3 \left[ \frac{a}{4}, \frac{b}{4}, \frac{c}{4} \right] \left[ \frac{1}{2} + \frac{a}{4}, \frac{1}{2} + \frac{b}{4}, \frac{1}{2} + \frac{c}{4} \right].$$
where \( Q^{(3)}_k(n) = Q^{(3)}_k(n; a; b) \) is a degree-\((1 + 4k)\) polynomial in \( n \), determined by \( Q^{(3)}_0(n) = 1 + \frac{3n}{a} \) and the \( k \)-raising relation
\[
a \left( \frac{1}{2} - k - b \right) \left( \frac{1}{2} - k + b \right) Q^{(3)}_k(n) = \left( \frac{3n + 2k + a}{2k + a} \right) (n + a)(n + \frac{1}{2} - k - b)(n + \frac{1}{2} - k + b) Q^{(3)}_{k-1, +}(n) + 8 \left( \frac{3n - 4k + a}{2k + a} \right) n(n - \frac{1}{2} + \frac{k}{2} + \frac{a}{2} + \frac{b}{2})(n - \frac{1}{2} + \frac{k}{2} + \frac{a}{2} - \frac{b}{2}) Q^{(3)}_{k-1, +}(n - 1),
\]
with \( Q^{(3)}_{k-1, +}(n) := Q^{(3)}_k(n; a + 1, b) \).

**Theorem 3.9.** For all \( k \geq 0 \), one has the second cubic transformation
\[
3F_2 \left[ \frac{1}{3} + \frac{2k}{3} + \frac{a}{3}, \frac{2}{3} + \frac{2k}{3} + \frac{a}{3}, 1 + \frac{2k}{3} + \frac{a}{3} \middle| \frac{27x^2}{(4 - x)^3} \right] = \left( 1 + \frac{x}{3} \right)^{-1 - 2k} \times (1 - x)^{1 + 2k + a} 4_{4}F_{3+4k} \left[ \frac{a}{3}, \frac{1}{2} - k + \frac{k}{3} + \frac{a}{3} - \frac{b}{3}, \frac{1}{2} - k + \frac{k}{3} + \frac{a}{3} + \frac{b}{3}, \frac{1}{2} + k + a - b; \ 1 + a \right] Q^{(3)'}_{k}(n),
\]
where \( Q^{(3)'}_{k}(n) = Q^{(3)'}_{k}(n; a; b) \) is a degree-\((1 + 4k)\) polynomial in \( n \), determined by \( Q^{(3)'}_0(n) = 1 + \frac{3n}{a} \) and the \( k \)-raising relation
\[
2a \left( \frac{1}{4} - k + \frac{a}{4} - \frac{b}{4} \right) \left( \frac{1}{4} - k + \frac{a}{4} + \frac{b}{4} \right) Q^{(3)'}_{k}(n) = \left( \frac{3n + 4k + 2a}{2k + a} \right) (n + a)(n + \frac{1}{4} - k + \frac{a}{4} - \frac{b}{4})(n + \frac{1}{4} - k + \frac{a}{4} + \frac{b}{4}) Q^{(3)'}_{k-1, +}(n) + \frac{1}{8} \left( \frac{3n - 4k + 2a}{2k + a} \right) n(n - \frac{1}{4} + \frac{k}{4} + \frac{a}{4} + \frac{b}{4})(n - \frac{1}{4} + \frac{k}{4} + \frac{a}{4} - \frac{b}{4}) Q^{(3)'}_{k-1, +}(n - 1),
\]
with \( Q^{(3)'}_{k-1, +}(n) := Q^{(3)'}_k(n; a + 1, b) \).

### 4. Proofs

The following are the proofs of the first six theorems of \( \text{§3} \) those of the final three being deferred to the next section. The proofs employ the Sheppard–Andersen transformation of terminating \( 3F_2(1) \)'s, which is [2] Cor. 3.3.4
\[(4.1) \quad 3F_2 \left[ \frac{a}{n}, \frac{A}{D}, \frac{B}{E} \right] = \left[ \frac{D - A}{D}, \frac{E - A}{E} \right] \quad 3F_2 \left[ \frac{-n}{1 + A - D - n}, \frac{1 + A - E - n}{1 + A - E - n} \right],
\]
where \( S = n - A - B + D + E \) is the parametric excess of the left-hand \( 3F_2(1) \).
(The notation \( \left[ \begin{array}{c} \alpha \\ \beta \end{array} \right]_{n} \) signifies \( \prod_{i=0}^{n-1} \frac{(\alpha_i)_n}{(\beta_i)_n} \). This extends to Whipple’s transformation of 1-balanced terminating \( 4F_3(1) \)'s, which is [2] Thm. 3.3.3
\[(4.2) \quad 4F_3 \left[ \frac{-n}{1 + A - D - n}, \frac{1 + A - E - n}{1 + A - E - n} \right] = 4F_3 \left[ \frac{-n}{1 + A - D - n}, \frac{1 + A - E - n}{1 + A - E - n} \right].
\]

It is assumed in \( (4.2) \) that the parametric excess of the left-hand \( 4F_3(1) \), which is \( n - A - B - C + D + E + F \), equals unity. Equation \( (4.2) \) can be deduced from Euler’s transformation of \( 2F_1 \), and \( (4.1) \) comes by taking \( C, F \to \infty \) with \( F - C = \text{const} \).

The quadratic identity of Theorem \( 3.1 \) and the cubic ones of Theorems \( 3.2 \) and \( 3.3 \) follow respectively from the \((l, m) = (1, 1), (1, 2), (2, 1)\) cases of Lemma \( 2.1 \) provided that the parameter arrays \((\alpha), (\beta)\) are taken to be \((1 - k + a - b - c), (1 + a - b, 1 + a - c)\) in the \((1, 1)\) case, and \((-), \left( \frac{a}{3} + \frac{b}{3} + \frac{a}{3} + \frac{b}{3} + \frac{a}{3} + \frac{b}{3} + \frac{a}{3} - \frac{b}{3} \right) \)
in the two others, with $x_0$ respectively equal to $1, \frac{1}{2}, 4$. The $(l, m) = (1, 1), (1, 2)$ cases can then be written as

$$3F_2 \left[ \begin{array}{c} \frac{a}{2}, \frac{1}{2} + \frac{a}{2}, 1 - k + a - b - c \\ 1 + a - b, 1 + a - c \end{array} \right] - \frac{4x}{(1-x)^2}$$

(4.3a)

$$= (1 - x)^a 3F_2 \left[ \begin{array}{c} a \\ R^{(2)}(n) \end{array} \right] | x \right],$$

$$3F_2 \left[ \begin{array}{c} \frac{a}{2}, \frac{1}{2} + \frac{a}{2}, \frac{3}{2} + \frac{a}{2} \\ \frac{1}{2} + \frac{k}{2} + \frac{a}{2}, \frac{1}{2} + \frac{k}{2} + \frac{a}{2} - b \end{array} \right] - \frac{27x}{(1-4x)^3}$$

(4.3b)

$$= (1 - 4x)^a 3F_2 \left[ \begin{array}{c} a \\ R^{(3)}(n) \end{array} \right] | 4x \right],$$

where

(4.4a)

$$R^{(2)}(n) = 3F_2 \left[ \begin{array}{c} -n, n + a, 1 - k + a - b - c \\ 1 + a - b, 1 + a - c \end{array} \right]$$

$$= \left[ \begin{array}{c} b, c \\ 1 + a - b, 1 + a - c \end{array} \right] n 3F_2 \left[ \begin{array}{c} -n, n + a, -k \\ b, c \end{array} \right],$$

(4.4b)

$$R^{(3)}(n) = 3F_2 \left[ \begin{array}{c} \frac{a}{2}, \frac{1}{2} + \frac{a}{2}, \frac{3}{2} + \frac{a}{2} + \frac{k}{2}, \frac{1}{2} + \frac{k}{2} + \frac{a}{2} - b \end{array} \right]$$

$$= \left[ \begin{array}{c} \frac{3}{2} + \frac{k}{2} + \frac{a}{2} - b, \frac{3}{2} + \frac{k}{2} + \frac{a}{2} - c \\ \frac{1}{2} - \frac{k}{2} + \frac{a}{2} - b, \frac{1}{2} - \frac{k}{2} + \frac{a}{2} - c \end{array} \right] n 3F_2 \left[ \begin{array}{c} -n, n + a, -k \\ \frac{1}{2} - \frac{k}{2} + \frac{a}{2} - b, \frac{1}{2} - \frac{k}{2} + \frac{a}{2} - c \end{array} \right].$$

The second expressions for $R^{(2)}(n)$, $R^{(3)}(n)$ are obtained by applying the transformation (4.1). The prefactor on the right-hand side in (4.4b) equals (4.5)

$$4^{-n} \left[ \begin{array}{c} \frac{1}{2} - k - b, \frac{1}{2} - k + b \\ \frac{3}{2} + \frac{k}{2} + \frac{a}{2} - b, \frac{3}{2} + \frac{k}{2} + \frac{a}{2} - c \end{array} \right] n \frac{4^k (\frac{1}{2} - \frac{k}{2} + \frac{a}{2} - b)_k (\frac{1}{2} - \frac{k}{2} - \frac{a}{2} - c)_k}{(\frac{1}{2} + b)_k (\frac{1}{2} - b)_k}$$

by elementary manipulations.

Substituting (4.4a), (4.4b) [with (4.5)] into (4.3a), (4.3b) immediately yields the identities of Theorems 3.1 and 3.2. The derivation of the cubic transformation in Theorem 3.3 from the $(l, m) = (2, 1)$ case of the lemma proceeds similarly, with a minor difference: its even-$n$ and odd-$n$ subcases must be treated separately.

The proofs of the $\Phi_3$ transformations in Theorems 3.4, 3.5, and 3.6 are identical to the preceding three, except that Whipple’s transformation (4.2) is used instead of the Sheppard–Andersen transformation (4.1). The added parameter-pairs $(C_F), (C - C)$ on the left and right of (4.2) are taken to equal $(k + d), (-k)$.

It is worth mentioning that the Sheppard–Andersen and Whipple transformations do not have close analogues on higher levels. Transformations of terminating $9F_8(1)$ series are known, but the series must satisfy restrictive conditions (e.g., that they be very well poised as well as 2-balanced).

5. The Polynomials $Q_k$ and $Q_k$: Raising Relations

Each of the polynomials $Q_k^{(2)}$, $Q_k^{(3)}$, $Q_k^{(4)}$ and $Q_k^{(2)}$, $Q_k^{(3)}$, $Q_k^{(4)}$ in the transformations of §3 satisfies a recurrence on $k$, a so-called $k$-raising relation. In §§3.1 and 3.2 the $Q_k$ have hypergeometric representations from which recurrences can be derived. But in all cases it is easier to go directly from a transformation $F(t) = A(x) \tilde{F}(x)$, based on a lifting function $t = \varphi(x)$, to the corresponding recurrence.
Suppose \( t = \varphi(x) \) and that \( F(t), \tilde{F}(x) \) are hypergeometric functions. Define \( \vartheta = t \frac{1}{m} \) and \( \tilde{\vartheta} = x \frac{1}{m} \), so that \( \vartheta = \chi(x) \tilde{\vartheta} \) with \( \chi(x) = \varphi(x)/x \frac{1}{m} (x) \). (Compare the manipulations of Burchnall [6].) Then, the differential operators \( T[e] := 1 + \vartheta^{-1} \vartheta \) and \( \tilde{T}[\tilde{e}] := 1 + \tilde{\vartheta}^{-1} \tilde{\vartheta} \) will increment the upper parameters of the hypergeometric series in \( t \) and \( x \) which define \( F \) and \( \tilde{F} \). That is, if one of the upper parameters of \( F \) is \( e \), in \( T[e]F \) it will be replaced by \( 1 + e \), and if none of the upper parameter is \( e \), \( T[e]F \) will have an extra parameter-pair \( (1 + e) \). The action of \( T[e] \) on \( F \) is similar.

The \( _3F_2 \) transformations in \( 3.4 \) are treated as follows. Their common form is

\[
(5.1) \quad F \left[ \frac{\Delta(l + m; a), (\alpha)}{\beta} \Bigg| t \right] = \left(1 - \frac{x}{x_0}\right)^a \tilde{F} \left[ a, (\gamma) \Bigg| Q_k(n) \Bigg| x \right],
\]

where the lifting function \( t = \varphi(x) \) comes from Lemma 2.1, i.e.,

\[
(5.2) \quad \varphi(x) = \varphi_{l,m;x_0}(x) := \frac{(l + m)^{l+m}}{l! \, m^m} \frac{(-x/x_0)^l}{(1 - x/x_0)^{l+m}}.
\]

(Recall that \((l, m; x_0)\) is \((1, 1; 1)\), \((1, 2; 1)\), and \((2, 1; 4)\) for the quadratic, first cubic, and second cubic identities.) It follows readily that \( \chi(x) = \frac{x - a}{x_0 - m x} \) and that \( T, \tilde{T} \) are related by, e.g.,

\[
(5.3) \quad T \left[ \frac{a}{l+m} \right] = \left(1 + \frac{m}{l} \frac{x}{x_0}\right)^{-1} \left(1 - \frac{x}{x_0}\right)^{1+a} \tilde{T} \left[ \frac{la}{l+m} \right] \left(1 - \frac{x}{x_0}\right)^{-a}.
\]

Equation (5.3) is a symbolic restatement of a well-known result of Gessel and Stanton [8, Prop. 2]. They applied what was essentially the operator \( T \left[ \frac{a}{l+m} \right] \) to the classical \((k = 0)\) cases of Theorems 3.1, 3.2, and 3.3 to obtain their companions: the classical \((k = 0)\) cases of Theorems 3.7, 3.8, and 3.9.

Now, consider the following two alternative actions on any of the three transformation identities in \( 3.4 \) of the form (5.1), when \( k \geq 1 \):

(I) act with \( T \left[ \frac{a}{l+m} \right] \) on it, rewriting the right side with the aid of (5.3); or,

(II) increment \( a \) (and also \( b, c \) in the quadratic case), and decrement \( k \).

It is easy to see that the left-hand sides resulting from actions (I)(II) are the same, thus the resulting right sides must also be equal. This implies that

\[
(5.4) \quad \tilde{F} \left[ a, (\gamma) \Bigg| Q_k(n) \Bigg| x \right] = \left(1 + \frac{m}{l} \frac{x}{x_0}\right)^{1+a} \left[ \frac{l}{x} \right] \tilde{F} \left[ 1 + a, (\gamma+) \Bigg| Q_{k-1,+}(n) \Bigg| x \right],
\]

where the subscript \(+\) indicates the incrementing of \( a \) (and \( b, c \) in the quadratic case); and for the arrays \((\gamma)\) and \((\tilde{\delta})\), the decrementing of \( k \) as well. (One sees at a glance that in all three transformations, \((\gamma+) = 1 + (\gamma)\) and \((\tilde{\delta}+) = (\tilde{\delta})\).) It follows by equating the coefficients of \( x^n \) on the two sides of (5.4) that

\[
(5.5) \quad K \cdot \left\{ \prod (a, (\gamma)) \right\} Q_k(n) = A_0 \cdot \left\{ \prod \left[ n + (a, (\gamma)) \right] \right\} Q_{k-1,+}(n) + A_1 \cdot \left\{ \prod \left[ (n-1) + (1, (\tilde{\delta}) \right]\right\} Q_{k-1,+}(n-1),
\]

with the coefficients

\[
(5.6) \quad K = \frac{n + [l a/(l + m)]}{[l a/(l + m)]}, \quad A_0 = 1, \quad A_1 = \frac{m}{l} \frac{1}{x_0}.
\]
Equation (5.5), with (5.1), is a master \( k \)-raising relation for \( Q_k \), standing for each of the polynomials \( Q_k^{(2)} \), \( Q_k^{(3)} \), and \( Q_k^{(3')} \) of \( §3.3 \). It is based on a backward difference operator on \( n \). By specializing \( (l, m; x_0) = (1, 1, 1) \), one obtains an explicit \( k \)-raising relation for each. For example, setting \( (l, m; x_0) = (1, 1, 1) \) yields

\[
abc \frac{n + (a/2)}{(a/2)} Q_k^{(2)}(n) = (n + a)(n + b)(n + c) Q_{k-1,+}^{(2)}(n)
\]

\[
+ n(n + a - b)(n + a - c) Q_{k-1,+}^{(2)}(n - 1)
\]

as the recurrence satisfied by \( Q_k^{(2)} = Q_k^{(2)}(n; a; b, c) \). This is essentially the degree-raising relation for the dual Hahn polynomials \([12, (9.6.8)]\).

The \( d \)-dependent \( _4F_3 \) transformations in \( §3.2 \) can be treated similarly if the action (II) is altered to include an application of \( T[d] \). In the resulting master \( k \)-raising relation, the coefficients (5.6) are replaced by

\[
K = \frac{n + [la/(l + m)]}{[la/(l + m)]}, \quad A_0 = \frac{n + ld}{ld}, \quad A_1 = -\frac{n + (a - md)}{lx_0d},
\]

which tend to the previous values as \( d \to \infty \). Setting \( (l, m; x_0) = (1, 1, 1) \) yields

\[
abcd \frac{n + (a/2)}{(a/2)} Q_k^{(2)}(n) = (n + a)(n + b)(n + c)(n + d) Q_{k-1,+}^{(2)}(n)
\]

\[
- n(n + a - b)(n + a - c)(n + a - d) Q_{k-1,+}^{(2)}(n - 1)
\]

as the recurrence satisfied by the four-parameter \( Q_k^{(2)} = Q_k^{(2)}(n; a; b, c, d) \). This is essentially the degree-raising relation for the Racah polynomials (see \([12, (9.2.8)]; \) cf. \([2, (3.7.6)]\)). However, the recurrences for \( Q_k^{(3)}, Q_k^{(3')} \) are less familiar.

The companion transformations of \( §3.3 \) can be treated in much the same way, mutatis mutandis. Their common form is

\[
F \left[ \begin{array}{c} \Delta(l + m; 1 + 2k + a), \\
(\alpha) \\
(\beta) \\
(\gamma) \\
(\delta) 
\end{array} \right] t
\]

\[
= \left(1 + \frac{m}{l} \frac{x}{x_0} \right)^{-1-2k} \left(1 - \frac{x}{x_0} \right)^{1+2k+a} F \left[ \begin{array}{c} a, \\
(\gamma) \\
(\delta) 
\end{array} \right] Q_k(n) \left| x \right]
\]

To treat this form, \( T \left[ \frac{\varphi}{\varphi + m} \right] \) must be replaced in (I) by \( T \left[ \frac{1+2k+a}{l+m} \right] \), the effect of which on each right-hand side can be worked out by expressing it in terms not of \( \vartheta \) but of \( \delta \). Also, (II) must be replaced by its inverse, which decrements \( a \), etc., and increments \( k \). By equating the coefficients of \( x^a \) in the right-hand sides coming from (I) and (II), one finds after much algebraic labor an identity resembling (5.5), but with \( Q_k \) replaced by \( Q_k \) and with the new coefficient values

\[
K = l, \quad A_0 = \frac{(l + m)n + 2lk + la}{2k + a}, \quad A_1 = \left( \frac{m}{lx_0} \right) \frac{(l + m)n - 2mk + la}{2k + a}.
\]

It is the master \( k \)-raising relation for the polynomials \( Q_k^{(2)}, Q_k^{(3)}, \) and \( Q_k^{(3')} \) of \( §3.3 \).

By setting \( (l, m; x_0) \) equal to \((1, 1, 1), (1, 2; \frac{1}{3}), \) and \((2, 1; 4)\), one obtains the relations in the statements of Theorems \(5.7, 3.8, \) and \(5.9 \). (The relation in the first is phrased in terms of \( Q_k^{(2)} \) rather than \( Q_k^{(2)} \).) This completes their common proof.
6. Summation Identities (I)

Besides being of intrinsic interest, the extended function transformations of §3 yield new summation formulas: evaluations of hypergeometric functions with nonlinearly constrained parameters at fixed values of their argument, such as \( x = 1 \). These can be constructed by a technique of Gessel and Stanton, which pairs transformations and their companions. The following is a restatement of their result [8 Thm. 2], which is a version of the residue composition theorem. It is adapted to the lifting function \( t = \varphi(x) = \varphi_{1,m;x_0}(x) \) of [5, 2]. In it, \( C_{l,m} \) denotes the prefactor \((l+m)^{l+m}/l!m^m\), and \([x^n] \) indicates the extraction of the coefficient of \( x^n \). Only the \( l = 1 \) case is stated here.

**Lemma 6.1.** Suppose one has a pair of hypergeometric function transformations

\[
G \left[ \begin{array}{c} (A) \\ (B) \end{array} \right] \varphi_{1,m;x_0}(x) = \left( 1 - \frac{x}{x_0} \right)^{a} \tilde{G} \left[ \begin{array}{c} (\tilde{A}) \\ (\tilde{B}) \end{array} \right] \]

\[
G_c \left[ \begin{array}{c} (A_c) \\ (B_c) \end{array} \right] \varphi_{1,m;x_0}(x) = \left( 1 + \frac{m}{x_0} \right)^{-a} \left( 1 - \frac{x}{x_0} \right)^{1+a_c} \tilde{G}_c \left[ \begin{array}{c} (\tilde{A}_c) \\ (\tilde{B}_c) \end{array} \right] \]

in which \( a,a_c \) appear as parameters in the arrays \((\tilde{A}), (\tilde{A}_c)\), respectively, and that \( N = (1-a-a_c)/(1+m) \) is a nonnegative integer. Then, \([x^N] \{G(x)\tilde{G}_c(x)\} \) equals \((-C_{1,m;x_0}/x_0)^N \times \{G(t)\tilde{G}_c(t)\} \). Equivalently,

\[
\left[ \frac{(\tilde{A})}{(B)} \right] \left[ \begin{array}{c} (A) \\ (B) \end{array} \right] -N \left[ \begin{array}{c} (A_c) \\ (B_c) \end{array} \right] = \left[ \frac{C_{1,m;x_0}}{x_0} \right]^N \left[ \begin{array}{c} (A) \\ (B) \end{array} \right] \left[ \begin{array}{c} (A_c) \\ (B_c) \end{array} \right].
\]

In [8], this is applied to the pair consisting of Whipple’s quadratic transformation of \( 3F_2 \) and its companion (the \( k = 0 \) cases of Theorems 3.1 and 3.7), and yields Whipple’s formula relating any very well poised \( \tau F_6(1) \) to a \( 1 \)-balanced \( 4F_3(1) \). (See [8, (5.2)].) An extension is possible. The lemma can be applied to the unstructured case \((k \geq 0) \) of Theorem 3.4, paired with the \( k = 0 \) case of Theorem 3.7. Moreover, as an alternative to Theorem 3.1 \((k \geq 0) \), the \( d \)-dependent \( 4F_3 \) transformation of Theorem 3.4 \((k \geq 0) \) can be used. These two alternatives yield:

**Theorem 6.2.** For all \( k \geq 0 \) and \( N \geq 0 \), the finite \( \tau_{+2k} F_{0+2k} \)

\[
\left[ \begin{array}{c} a, 1 + \frac{a}{2}, b, c, d, e, -N \\ \frac{a}{2}, 1 + a - b, 1 + a - c, 1 + a - d, 1 + a - e, 1 + a + N \end{array} \right] R_k(n),
\]

where \( R_k(n) \) denotes \( Q_k^{(2)}(N-n)/Q_k^{(2)}(N) \), equals

\[
\left[ Q_k^{(2)}(N) \right]^{-1} \left[ \begin{array}{c} 1 + a, 1 - k + a - d - e \\ 1 + a - d, 1 + a - e \end{array} \right]_N \times 4F_3 \left[ \begin{array}{c} 1 + a - b - c, d, e, -N \\ 1 + a - b, 1 + a - c, k - a + d + e - N \end{array} \right]
\]

if \( Q_k^{(2)}(n) := Q_k^{(2)}(n; -a - 2N; d - a - N, e - a - N) \), and equals

\[
\left[ Q_k^{(2)}(N) \right]^{-1} \left[ \begin{array}{c} 1 + a, 1 - k + a - d - e, 1 - k + a - f \\ 1 + a - d, 1 + a - e, 1 + a - f \end{array} \right]_N \times 5F_4 \left[ \begin{array}{c} 1 + a - b - c, d, e, 1 + a - f, -N \\ 1 + a - b, 1 + a - c, k - a + d + e - N, 1 - k + a - f \end{array} \right]
\]
if \( Q_k^{(2)} := Q_k^{(2)}(a; -a - 2N; d - a - N, e - a - N, f - a - N). \)

These identities reduce to Whipple’s formula when \( k = 0 \), and the second reduces to the first when \( f \to \infty \). The left sides are very well poised and the right sides have \( S = 1 + k \), resp. \( S = 1 \). The \( k = 1 \) case of the second can be shown to agree with a result of Srivastava, Vyas and Fatawat [18, Thm. 3.2] by using the formula (3.7) for the four-parameter quadratic polynomial \( Q_4^{(2)} \). Like Whipple’s formula (cf. [2 §§3.4, 3.5]), they have interesting specializations and limits; e.g., if the \( \tau + 2kF_6 + 2k \) has \( S = 2 \), the right-hand parameters \( 1 + a - b - c, k - a + d + e - N \) will cancel. The identities then become extensions of Dougall’s theorem [2 Thm. 3.5.1].

One can also apply Lemma 6.1 to the pair consisting of the unrestricted Theorem 3.2 (the first cubic transformation of \( _3F_2 \)), resp. Theorem 3.5 (the first cubic transformation of \( _4F_3 \)), and the \( k = 0 \) case of its companion, Theorem 3.8. The two summation formulas that result are extensions to \( k \geq 0 \) of the first cubic summation formula of Gessel and Stanton [8 (1.7)]. Details are left to the reader.

7. Summation Identities (II)

One can obtain a parametrized finite summation formula from any of the extended function transformations of \( _4F_3 \) by a classical technique: multiplying both sides by a power of \( 1 - x \) and equating the coefficients of \( x^m \) on the two sides. This technique was applied by Bailey to many hypergeometric transformations, including Whipple’s quadratic transformation of \( _3F_2 \) and its companion (the \( k = 0 \) cases of Theorems 3.1 and 3.7); see [2, p. 97, Examples 5, 6]. Applying it to the unrestricted versions of Theorems 3.1, 3.4, and 3.7 is straightforward and yields:

**Theorem 7.1.** For all \( k \geq 0 \) and \( m \geq 0 \), one has (i) the finite summation formula

\[
5F_4 \left[ \begin{array}{c} \frac{a}{2}, \frac{1 + a}{2}, 1 - k + a - b - c, 1 + a - w, 1 \end{array} \right]_m 1 + a - b, 1 + a - c, \frac{1}{2} + \frac{1}{2}(a - w - m), 1 + \frac{1}{2}(a - w - m) \\
= \left[ \begin{array}{c} w \\
-w-a \end{array} \right]_{m+2k} F_{3+2k} \left[ \begin{array}{c} a, b, c, -m \\
1 + a - b, 1 + a - c, w \end{array} \right] Q_k^{(2)}(n),
\]

where \( Q_k^{(2)}(n) := Q_k^{(2)}(a; b, c) \), and (ii) a similar formula in which a parameter-pair \((k, d)\) is added to the left-hand side, and \( Q_k^{(2)}(n) := Q_k^{(2)}(a; b, c, d) \).

**Theorem 7.2.** For all \( k \geq 0 \) and \( m \geq 0 \), one has the finite summation formula

\[
5_{k+1}F_{4+k} \left[ \begin{array}{c} \frac{1}{2} + k + \frac{a}{2}, 1 + k + \frac{a}{2}, 1 - k + a - b - c, 1 + a - w, 1 - a - c, \end{array} \right]_m \frac{1}{2} + \frac{1}{2}(a - w - 2k - m), \frac{1}{2} + \frac{1}{2}(a - w - 2k - m) \\
= \left[ \begin{array}{c} 1 + a - w, 1 + a - b, 1 + a - c, \end{array} \right]_{m+4k} F_{4+k} \left[ \begin{array}{c} a, b, c, -m \\
1 + a - b, 1 + a - c, w \end{array} \right] P_k(n),
\]

where \( Q_k^{(2)}(n) := Q_k^{(2)}(a; b, c) \), which is of degree \( 1 + 4k \) in \( n \), and \( P_k(n) := P_k(n; 1 + a - w, -m) \) is a polynomial of degree \( k \) in \( n \) defined by

\[
P_k(n; A, B) := (n + A)_{2k+1} 2F_1 \left[ \begin{array}{c} -1 - 2k, n + B \\
-n - A - 2k \end{array} \right] -1,
\]

which (by series reversal) is odd under the interchange of \( A, B \).
In the left-hand $5_{-k}F_{4+k}(1)$ of Theorem 7.2 the convention introduced in §2 is not adhered to, for simplicity of expression: the weighting function, here $P_k(n) = P_k(n; A, B)$, does not equal unity at $n = 0$. For instance, $P_0(n)$ equals $A - B$.

The summation formulas in Theorems 7.1 and 7.2 reduce when $k = 0$ to those given by Bailey [3, §4.5(1,2)]. In each, the left-hand series ($5F_4(1)$ or $6F_5(1)$, resp. $5_{-k}F_{4+k}(1)$) is either $(1 + k)$-balanced or 1-balanced, and the right-hand one ($4_{-2k}F_{3+2k}(1)$, resp. $5_{-k}F_{4+4k}(1)$) is nearly poised. The $k = 1$ case of Theorem 7.1(ii) was recently proved by Wang and Rathie [20, Cor. 4].

Bailey noted that there is an equivalence between Whipple’s quadratic transformation of $3F_2$ and his formula relating a 1-balanced $5F_4(1)$ to a nearly poised $4F_3(1)$, i.e., the $k = 0$ case of Theorem 7.1(i): one implies the other. (For a $q$-analogue, see [11].) One sees that this equivalence holds in greater generality.

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