CC-distance and metric normal of smooth hypersurfaces in sub-Riemannian Carnot groups

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Abstract

In this paper we study the main geometric properties of the Carnot-Carathéodory (abbreviated CC) distance $d_H$ in the setting of $k$-step sub-Riemannian Carnot groups from many different points of view; see Section 2.1 and Section 2.2 An extensive study of the so-called normal CC-geodesics is given. We state and prove some related variational formulae and we find suitable Jacobi-type equations for normal CC-geodesics; see Section 2.2 One of our main results is a sub-Riemannian version of the Gauss Lemma; see Section 2.3 We show the existence of the metric normal for smooth non-characteristic hypersurfaces; see Corollary 2.31 In Section 2.3 we compute the sub-Riemannian exponential map $\exp_{SR}$ for the case of 2-step Carnot groups. Other features of normal CC-geodesics are then studied. In Section 2.6 we show how the system of normal CC-geodesic equations can be integrated step by step. Finally, in Section 3 we show a regularity property of the CC-distance function $\delta_H$ from a $C^k$-smooth hypersurface $S$; see Theorem 3.5.

Key words and phrases: Carnot groups; Sub-Riemannian geometry; CC-metrics; distance from hypersurfaces; metric normal; normal geodesics.

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1 Introduction

In the last thirty years, many progresses have been done in developing Analysis through very general geometries and metric spaces. This trend in mathematical research, already initiated by Federer’s treatise [21], has been pursued by many authors, with different point of views: Ambrosio [3], Ambrosio and Kirchheim [4, 5], Capogna, Danielli and Garofalo [10], Cheeger [12], Cheeger and Kleiner [13], David and Semmes [15], De Giorgi [19], Gromov [35, 36], Franchi, Gallot and Wheeden [25], Franchi and Lanconelli [26], Franchi, Serapioni and Serra Cassano [27, 30], Garofalo and Nhieu [31], Heinonen and Koskela [37], Jerison [40], Korany and Riemann [42], Pansu [56] but, of course, this list is not complete.

Sub-Riemannian or Carnot-Carathéodory geometries have become a research field of great interest, also because of their wide connections with different fields in Mathematics and Physics, such as PDE’s, Control Theory, Mechanics, Theoretical Computer Science.

For references, comments and some different perspectives on sub-Riemannian geometry we refer the reader to Agrachev and Ghautier [1], Agrachev and Sarychev [2], Gromov [36], Montgomery [54], Pansu [56, 57], and Strichartz [61], Vershik and Gershkovich [62].

Very recently, the so-called Visual Geometry has also received new impulses from this field; see [14], [15] and references therein.

In this article we begin a study of fine properties of the function “sub-Riemannian distance from a hypersurface” in Carnot groups. Before giving a formal outline of the context and content of the paper, we would like to briefly and colloquially discuss what we think are the main features of this research.

The elementary analysis of the function “distance from a closed set” $E$ with smooth boundary $\partial E$ in the Euclidean space $\mathbb{R}^n$ is based on the intuitive idea of “moving” a sufficiently small closed Euclidean ball $B$ in $\mathbb{R}^n \setminus E$ until it touches $\partial E$ at some point $P$. Let $B_0$ be the ball $B$ in its final position and let $O$ be its center. The Euclidean segment $OP$ is (part of) the segment perpendicular to $\partial E$ at $P$. For points $Q$ in $OP$, the distance from $Q$ to $E$ is realized by the segment $OQ$. The idea of a ”sufficiently small moving ball”, however intuitive, is not easy to deal with in calculations and it is not really necessary. In fact, it is easier to work with the final configuration: given a point $P$ on $\partial E$ and the direction $\nu$ normal to $\partial E$ at $P$, we can recover $B_0$ as any of the sufficiently small balls centered at a point $O$ on the segment leaving $P$ in the direction $\nu$, having radius $OP$. The normal direction $\nu$ is a first order differential object; while the fact that the ball $B_0$ touches $E$ at $P$ only, generally depends on the “extrinsic” curvatures of $\partial E$ at $P$, which is a second order differential object.
Exactly the same line of reasoning works if $E$ lives in a Riemannian manifold. The direction $\nu$, normal to $\partial E$ at $P$, determines the geodesic $\gamma$ having speed $\nu$ at $P$, the geodesic normal to $\partial E$ at $P$. The ball $B_0$ can then be recovered as a ball having center on $\gamma$, sufficiently close to $P$.

The sub-Riemannian case presents new difficulties. Consider, for expository reasons, the case of $\mathbb{R}^n$ endowed with a sub-Riemannian structure of codimension $v \geq 1$. In particular, to each point $x \in \mathbb{R}^n$ is associated a linear space $H_x \subset T_x\mathbb{R}^n$ of admissible horizontal directions, $\dim(H_x) := h = n - v$, and a metric $g_{H_x}$ to measure lengths of vectors in $H_x$. We stress that the metric $g_{H_x}$ can also be defined (non-canonically) as the restriction of a Riemannian metric $g$ defined on the whole tangent bundle $T\mathbb{R}^n$.

We can use this structure to define lengths of admissible curves (those whose velocity belong to $H$) and, under suitable hypotheses, this leads to a new distance $d_{H_x}$ in $\mathbb{R}^n$, called Carnot-Carathéodory (abbreviated CC) distance. This distance $d_{H_x}$ is a length-metric, realized by geodesics. In this introduction, we assume they are smooth, although the regularity of the so-called CC-geodesics is, in general, an open problem; see Section 2.1 and Section 2.3.

We can mentally repeat the previous procedure, moving a small metric ball $B$ until it touches $\partial E$ at $x$. We assume that $x$ is non-characteristic, i.e., roughly speaking, the linear space $H_x$ is not totally contained in $T_x\partial E$, the tangent space to $\partial E$ at $x$. In particular, we can determine a direction $\nu_{H_x}(x)$ in $H_x$ (the unit horizontal normal to $\partial E$ at $x$) which is orthogonal with respect to $g_{H_x}$ to all directions in $H_x \cap T_x\partial E$. The novelty is that there are infinitely many CC-geodesics $\gamma$ passing through $x$ and having tangent vector $\nu_{H_x}(x)$ there. In fact, there is a $\nu$-parameter family of them, where $v = \text{codim} H$. Hence, we can not identify the final ball $B_0$ based on the information contained in $\nu_{H_x}$ alone.

However, in several cases there exists a unique metric normal $\gamma_{\mathcal{N}}$ to $\partial E$ at $x$. By this we mean the (maximal) CC-geodesic curve $\gamma_{\mathcal{N}}$ starting from $x$, with the property that, for each $y$ in $\gamma_{\mathcal{N}} \setminus E$, $d_{H_x}(x, y)$ realizes the distance between $y$ and $E$. i.e., the metric ball centered at $y$ and having radius $d_{H_x}(x, y)$ touches $\partial E$ at $x$. The notion of metric normal was introduced by the first two authors in the context of the (lowest dimensional) Heisenberg group $\mathbb{H}^1$; see [6].

In order to specify the metric normal among the CC-geodesics tangent to $\nu_{H_x}$ at $x \in \partial E$ we need $i$ real parameters. In the Heisenberg case, $i = 1$, the extra parameter could be read off from the parametrization of $\partial E$. This fact is crucial, because from the equation for $\partial E$ one can explicitly write down the “exponential map” associated with $\partial E$, and, for instance, deduce regularity properties of the function “distance from $E$”.

In [7] it was observed that, in the Heisenberg case, the extra parameter could be read as a “imaginary curvature” for $\partial E$. At the same time and independently, the third author [31], continuing his work [30] on variational formulas in sub-Riemannian Carnot groups, associated to each non-characteristic point $x \in \partial E$ a “vertical vector” $\varpi$ of parameters, which reduces to the imaginary curvature in the Heisenberg case. These parameters, which can be interpreted as “curvatures” (or Lagrangian multipliers; see Section 2.2), have no immediate counterpart in Riemannian geometry and play an important role in the geometric analysis of $\partial E$ in this sub-Riemannian setting. An interesting feature of these “curvatures”, which appears in seminal form in [6] and in much greater generality in [50], [51], is that they can best be computed in terms of the Riemannian gradient (with
respect to the fixed Riemannian metric $g$ on the base manifold) of the smooth function $f$ defining $\partial E$ locally (i.e., $\partial E = \{ f = 0 \}$, locally). Roughly speaking $\varpi$ can be regarded as the “non-horizontal” (more precisely, vertical; see Section 1.1) part of the Riemannian gradient of $f$; see Definition 1.30.

Researchers working with “first order” geometric measure theory of hypersurfaces in Carnot group (isoperimetric inequalities, Lipschitz submanifolds, functions of $H$-bounded variation, etc.) are generally concerned with the horizontal gradient of the given defining function only.

It is in fact an accepted rule-of-thumb that horizontal vectors contain all the first order geometric information of the group, and that this extends to some second order objects (e.g., $H$-mean curvature and minimal surfaces). When dealing with such “second order information” as that encoded by the distance function, the “forgotten component” of the Riemannian gradient has to be taken into account. In this paper, we bring together these two different perspectives, relating these “curvatures” $\varpi$ to the metric normal $\gamma_N$, hence to the distance function from a smooth hypersurface.

This way, among others things, we shall extend to Carnot groups many results of [6] and [7], but with a somewhat different approach.

We now give a less informal view of the paper and of its context.

Definition 1.1 (Metric Normal). Let $(X,d)$ be a metric space and let $E$ be closed in $X$. The metric normal to $E$ at $x \in E$ is the set $\gamma_N := \{ y \in X : d(x,y) = d(y,E) \}$, where $d(y,E) := \inf_{z \in E} d(z,y)$.

In this paper, the metric space $(X,d)$ is a Carnot group $\mathbb{G}$ endowed with its “natural” CC-distance $d_H$.

A $k$-step Carnot group $\mathbb{G}$ is a Lie group, whose Lie algebra $\mathfrak{g}$ admits a stratification $\mathfrak{g} = H \oplus V$, where $V := \bigoplus_{j=2}^k H_k$, $H_1 := H$, $H_{j+1} = [H_j, H_1]$ and $H_{k+1} = \{0\}$. In $H$ we find the so-called horizontal vectors, while in $V$ we find the vertical vectors.

Notation 1.2 (Projections). Throughout this paper, the mappings $P_H : T\mathbb{G} \longrightarrow H_i$, $P_V : T\mathbb{G} \longrightarrow V$, will denote the projection operator onto the subbundles $H_i$ ($i = 1,\ldots,k$) and $V$, respectively.

Remind that, as for any Lie group, an internal group of translations, called “left-translations”, is given on $\mathbb{G}$. In addition, any Carnot group $\mathbb{G}$ has intrinsic dilations making it a homogeneous group; see Section 1.1. In general, we shall say that a property in $\mathbb{G}$ is intrinsic whenever it is invariant with respect to these transformations.

The horizontal space $H$ is endowed with a Riemannian metric $g_H$ which is used in the obvious way to define lengths of horizontal curves $\gamma$ in $\mathbb{G}$ (i.e. $\gamma : \mathbb{R} \longrightarrow \mathbb{G}$ is absolutely continuous and $\dot{\gamma}(t) \in H$ for a.e. $t \in \mathbb{R}$). The distance between points in $\mathbb{G}$ is the infimum of the lengths of curves joining them, and it can be proved that such distance is realized by the length of (not necessarily unique) CC-geodesic curves; see Section 2.1 and Section 2.2 for a precise definition of CC-geodesic.

We stress that is a difficult open problem showing, or disproving, that CC-geodesics are smooth for all Carnot groups; see [54, 55].
In the Euclidean case, by assuming reasonable hypotheses on \( \partial E \), the metric normal at a point \( x \in \partial E \) can be identified with the normal direction associated with the surface at \( x \in \partial E \), because it is a segment; see \[52\] for further details). As a consequence, whenever we recall the normal direction, we emphasize the linear aspect of the metric normal, forgetting that it is, first of all, a path. In the Riemannian setting it is known that, if we stay near to a smooth hypersurface \( S \) there exists one, and only one, unit vector belonging to the tangent space to the base Riemannian manifold that select the metric normal to the hypersurface \( S \) at \( x \). As already said, in the sub-Riemannian setting this problem it is not so straightforward. Indeed, just to fix the ideas in the simplest Heisenberg group \( \mathbb{H}^1 \) (see \[52\] for further details), even when we fix a non-trivial horizontal vector \( \nu_y(x) \) at a point \( (x,y,r) \), where \( H = \text{span}_R \{X, Y\} \), infinite different CC-geodesics starting from \( x \in \mathbb{H}^1 \) exist with the same initial velocity \( \nu_y(x) \). In \[6\] this problem was solved by remarking that a path that locally parameterizes the metric normal of a smooth surface \( S \) at \( x \in S \) (whenever \( X \) is non-characteristic) can be selected just by considering two intrinsic object associated with the surface \( S \) at \( x \), i.e. the intrinsic unit normal vector \( \nu_y(x) \) and the so-called “imaginary curvature” of \( S \) at \( x \); see \[6\].

Roughly saying, if \( S = \{ y \in \mathbb{H}^1 : f(y) = 0 \} \), with \( f \in C^2(\mathbb{H}^1) \) and \( x \equiv [x_1, y_1, t_1] \in S \) is non-characteristic, i.e. \( Xf(x) \equiv \partial_{x_1}f(x) + 2y_1\partial_{t_1}f(x) \neq 0 \) or \( Yf(x) \equiv \partial_{y_1}f(x) - 2x_1\partial_{t_1}f(x) \neq 0 \), where \( x \equiv [x_1, y_1, t_1] \), then

\[
\nu_y(x) = \frac{Xf(x)X(x) + Yf(x)Y(x)}{\sqrt{(Xf(x))^2 + (Yf(x))^2}}, \quad \varpi(x) = \frac{|X,Y|f(x)}{\sqrt{(Xf(x))^2 + (Yf(x))^2}}
\]

(which are, respectively, the intrinsic unit normal along \( S \) at \( x \) and the imaginary curvature of \( S \) at \( x \)) select an horizontal path that is a subset of the metric normal \( \gamma_N \) to \( S \) at \( x \).

The problem we trait in this paper concerns the generalization of this program to any Carnot group. Later on we shall describe our approach.

Assume that \( S \) is the boundary of an open bounded subset \( \Omega \) of a Carnot group \( G \). Now let us assume that the set \( \Omega \) satisfies the so-called property of the internal ball. Namely, any point \( x \in \Omega \) can be touched by a closed metric ball \( B(y,r) \subset \Omega \) and \( \partial B(y,r) \cap S = \{ x \} \). In this case \( \partial B(y,r) \) and \( S \) have the same tangent space at \( x \). The objects we are searching for, whenever they exist, are somehow “hidden” in the invariants associated with this tangent space. On the other hand the CC-ball \( B(y,r) \) is the union of all the horizontal paths of length less than \( r \), starting from \( y \). This flux of CC-geodesics is usually determined by the solutions of a suitable Hamiltonian system; see Section 2.1.

At this point we have to select the path \( \gamma : [0,r] \longrightarrow \mathbb{G}; \gamma(0) = y, \gamma(r) = x \), connecting \( y \) to \( x \) and satisfying the property \( d_{\gamma}(x,\gamma(t)) = \inf_{z \in \partial B(y,r)} d_{\gamma}(\gamma(t),z) \).

We will need two key facts: (i) the CC-distance from a point satisfies the eikonal equation (see \[53\]), i.e. \( |\text{grad}_u d_{\gamma} | = 1 \), at each regular point of \( d_{\gamma} \); (ii) even in this sub-Riemannian structure, a Gauss-type lemma holds (see Section 2.4), namely if \( \gamma \) is a normal CC-geodesic leaving the center of the ball, then \( \gamma(t) = \text{grad}_u d_{\gamma}(\gamma(t)) \). We stress that the last identity is a straightforward consequence of the eikonal equation.

The remaining part of the proof can be described as follows.

---

\(^1\)According to the notation used in \[6, 17\], in this introduction we adopt the following convention: every point \( x \in \mathbb{H}^1 \) is given, using exponential coordinates, by the triple \( x \equiv [x_1, y_1, t_1] \). Moreover, we use, as a vector basis \( \{X, Y\} \) for the horizontal space \( H \), the following vectors field: \( X(x) := \partial_{x_1} + 2y_1\partial_{t_1} \), \( Y(x) := \partial_{y_1} - 2x_1\partial_{t_1} \). This convention for \( \mathbb{H}^1 \) will be slightly changed in the sequel.
Let us introduce a left-invariant frame $\underline{X} := \{X_1, \ldots, X_h, X_{h+1}, \ldots, X_n\}$ for $T\mathbb{G}$ where $H = \text{span}_\mathbb{R}\{X_1, \ldots, X_h\}$ and $V = \text{span}_\mathbb{R}\{X_{h+1}, \ldots, X_n\}$. Moreover we fix a Riemannian metric $g(\cdot, \cdot) := \langle \cdot, \cdot \rangle$ on $T\mathbb{G}$ that makes orthonormal the frame $\underline{X}$. We shall assume that the restriction of $g$ to $H$ equals the sub-Riemannian metric $g_H$, i.e. $g|_H = g_H$.

Set $P_l(t) := (X_l d_H)(\gamma(t))$ ($l = 1, \ldots, n$), where $d_H(\cdot) := d_H(\cdot, y)$.

In particular, one has

$$\dot{P}_l(t) = \langle \text{grad} (X_l d_H)(\gamma(t)), \dot{\gamma}(t) \rangle \quad (l = 1, \ldots, n).$$

By the Gauss’ lemma (i.e. $\dot{\gamma}(t) = \text{grad}_H d_H(\gamma(t))$), we get

$$\dot{P}_l(t) = \sum_{j=1}^{h} (X_j X_l) (d_H(\gamma(t))) P_j(t).$$

On the other hand $X_j X_l = [X_j, X_l] + X_l X_j$, thus

$$\dot{P}_l(t) = \sum_{j=1}^{h} (X_j X_l) (d_H(\gamma(t))) P_j(t) + \sum_{j=1}^{h} [X_j, X_l] (d_H(\gamma(t))) P_j(t).$$

Here plays a role the eikonal equation, namely $\sum_{j=1}^{h} (X_j d_H)^2(\gamma(t)) = 1$, because

$$\dot{P}_l(t) = \frac{1}{2} X_l \left( \sum_{j=1}^{h} (X_j d_H)^2(\gamma(t)) P_j(t) \right) + \sum_{j=1}^{h} [X_j, X_l] (d_H(\gamma(t))) P_j(t)$$

$$= \sum_{j=1}^{h} [X_j, X_l] (d_H(\gamma(t))) P_j(t).$$

It turns out that $[X_j, X_l] (d_H(\gamma(t))) = \sum_{\alpha=h+1}^{n} C^\alpha_{jl} X_\alpha (d_H(\gamma(t)))$, where the coefficients $C^\alpha_{jl}$ are the \textit{structural constants} of the Lie algebra $\mathfrak{g}$; see Section 1.1.

This way, we obtain the following system of O.D.E.’s:

$$\dot{P}_l(t) = \sum_{j=1}^{h} \sum_{\alpha=h+1}^{n} C^\alpha_{jl} P_j(t) P_\alpha(t) \quad (l = 1, \ldots, n). \quad (1)$$

By using the properties of the Carnot structural constants (in particular, see \ref{6}), this system can be solved step by step. A similar algorithm will be discussed in the Appendix of Section 2.

The previous discussion, for which we refer the reader to Section 2.4, yields the next:

\textbf{Lemma 1.3.} Let $\gamma : [0, r] \rightarrow \mathbb{G}$ ($r > 0$) be any normal CC-geodesic\footnote{See Section 2.1 for a precise definition of normal CC-geodesic.} of unit-speed and parameterized by arc-length. Let $\gamma(0) = y$, $P(0) = P_0$ be its initial data and set $d_H(x) = d_H(x, y)$ ($x \in \mathbb{G}$). Then we have

(i) $\text{grad}_H d_H(\gamma(t)) = P_\alpha(t)$ for every $t \in [0, r]$;

(ii) $\text{grad}_H d_H(\gamma(t)) = P_\alpha(t)$ for every $t \in [0, r]$;
(ii) \[ \text{grad}_v d_H(\gamma(t)) = P_v(t) \] for every \( t \in [0, r] \);

(iii) \[ \dot{P}_l(t) = \sum_{j=1}^{h} \sum_{\alpha=h+1}^{n} \alpha C_{jl}^\alpha P_j(t)P_\alpha(t) \] for every \( t \in [0, r] \) \((l = 1, ..., n)\).

This lemma solves the problem of selecting, for any regular point \( x \) belonging to the CC-sphere \( S^n_{\text{SR}}(y, r) := \partial B(y, r) \), the unique normal CC-geodesic having velocity vector equal to the horizontal normal direction at that point (i.e. \( P_v(0) = \nu_v(x) \)) and connecting this point to the center \( y \) of the CC-sphere. In fact, by uniqueness of solutions of O.D.E.’s systems, one gets that the desired curve must be the normal CC-geodesic defined by

\[ \gamma(t) := \exp_{\text{SR}}(y, -N(x))(t) \quad t \in [0, r]. \]

Here we have set

\[ N := \frac{\nu}{|P_H\nu|} = (\nu_H, \varpi), \]

where \( \nu \) denotes the Riemannian unit normal along the CC-sphere \( S^n_{\text{SR}}(y, r) \), thus \( \nu_v \) is just the horizontal unit normal along \( S^n_{\text{SR}}(y, r) \) and \( \varpi = \frac{P_v\nu}{|P_H\nu|} \). Moreover, \( \exp_{\text{SR}} \) denote the sub-Riemannian exponential map; see Section 2.3.

An immediate “geometric” consequence can be given:

**Corollary 1.4.** Let \( S = \{ x \in G : f(x) = 0 \} \), where \( f \) is a \( C^2 \) function. Assume that there exists a CC-ball \( B(y, r) \) with center \( y \) and radius \( r \), such that \( B(y, r) \subset \{ f(x) < 0 \} \), or \( B(y, r) \subset \{ f(x) > 0 \} \), and \( B(y, r) \cap S = \{ x \} \), where \( x \in S \) is non-characteristic. Then there exists the metric normal \( \gamma_N \) to \( S \) at \( x \) and for every \( t \in [0, r] \), \( \gamma(t) \in \gamma_N \), where

\[ \gamma(t) := \exp_{\text{SR}}(y, -N(x))(t) \quad t \in [0, r]. \]

Till now we have described some of the main results of this paper, but there are many others aspects and related questions. Here we briefly give an account of the rest of the paper.

In Section 1.1 we shall introduce the main ingredients to deal with Carnot groups and their sub-Riemannian structures: Lie algebraic preliminaries, Carnot dilations, CC-distance, properties of the Carnot structural constants and the notion of curvature of a distribution. Moreover we shall discuss some other tools as, for instance, the Levi-Civita connection, the so-called \( H \)-connection (or horizontal connection and, the related notion of \( E \)-connection) and covariant differentiation along curves. We also give some basic examples.

In Section 1.2 we will just recall notation and some basic definitions to work with hypersurfaces of Carnot groups, such as the notion of characteristic point and that of horizontal perimeter.

In Section 2 we begin our study of Carnot-Carathéodory metrics and their geodesics.

In Section 2.1 we will introduce CC-geodesics from the so-called Hamiltonian point of view, well-known in literature; see [8], [54], [34]. We will give the equations for normal and abnormal curves (and minimizers) and discuss some examples. Here we just remind the so-called normal CC-geodesic equations:

\begin{align*}
\text{(Normal Equations)} \quad \begin{cases}
\dot{x} = P_H \\
\dot{P} = -C(P_v)P_H,
\end{cases}
\end{align*}
where $P = (P_H, P_V)$ is the $n$-vector of the momentum functions associated with a fixed orthonormal (left-invariant) moving frame $\mathbf{X} = \{X_1, ..., X_n\}$ for $T\mathbb{G}$. Furthermore, $C(P_V) \in \mathcal{M}_n$ is a linear combination of $n \times n$ constant matrices which only depends on the Carnot structural constants. More precisely, one has

$$C(P_V) := \sum_{\alpha = h+1}^{n} P_\alpha C^\alpha,$$

where $C^\alpha := [C^\alpha_{ij}]_{i,j=1,...,n}$ ($\alpha = h + 1, ..., n$) and, by definition, $C^\alpha_{ij} := \langle [X_i, X_j], X_\alpha \rangle$.

In Section 2.2 we shall discuss another “natural” point of view on this subject: the Lagrangian one. This is because it seems the more natural way to obtain some additional information about normal CC-geodesics and their minimizing features. In particular, we will derive both the first and the second variation of the natural sub-Riemannian Lagrangian. For precise statements see Proposition 2.13 and Theorem 2.14. Then, after introducing the notion of CC-geodesic variation, we shall prove the validity of another interesting second variation formula; see Corollary 2.17. Starting from these formulae, we will deduce the natural Jacobi equations for normal CC-geodesics; see Definition 2.18.

Furthermore, we will discuss another Jacobi-type system of O.D.E.’s, which is obtained under suitable assumptions on the variations. These “restricted” Jacobi-type equations read as follows:

$$\nabla^2_t Y + R(P_H, Y)P_H + C(P_V)\nabla_t Y = 0,$$

where $\nabla_t$ denotes “covariant differentiation” and $R$ denotes the Riemannian curvature tensor. This material can be useful in the study of the conjugate and cut loci of a point, in this context.

In Section 2.3 we shall define the sub-Riemannian exponential map $\exp_{sr}$ and we will show some of their basic features.

In Section 2.4 we will prove a sub-Riemannian version of the Gauss Lemma; see Proposition 2.29 and Corollary 2.31.

In Section 2.5 we will compute the sub-Riemannian exponential map $\exp_{sr}$ for the special case of 2-step Carnot groups. Remind that the sub-Riemannian exponential map based at the point $x_0 \in \mathbb{G}$ is a mapping

$$\exp_{sr} (x_0, \cdot)(\cdot) : UH \times H_2 \times \mathbb{R} \longrightarrow \mathbb{G},$$

where $UH = \{X \in H : |X| = 1\}$ denotes the bundle of all horizontal unit vectors. We will show that

$$\exp_{sr} (x_0, P_0)(t) := x_0 + \int_0^t e^{-C_H(P_{H_2})s} P_0(0) ds - \frac{1}{2} \sum_{\alpha = h+1}^{n} \left( \int_0^t (C^\alpha_{H_1} x_{H_1}, \dot{x}_{H_1}) ds \right) e_\alpha,$$

where $P_0 = (P_H(0), P_{H_2}) \in UH \times H_2$, $C^\alpha_H := [C^\alpha_{ij}]_{i,j=1,...,h} \in \mathcal{M}_h$ ($\alpha = h + 1, ..., n$) and $C_H(P_{H_2}) := \sum_{\alpha = h+1}^{n} P_\alpha C^\alpha_H$. Moreover

$$x_{H}(t) := x_{H}(0) + \int_0^t e^{-C_H(P_{H_2})s} P_H(0) ds.$$
In the previous formula we have used the notation for the exponential of a linear operator. More precisely, \( e^{-CH(P_{h^2})s} \) denotes the exponential of a square-matrix, i.e. the \( h \times h \)-matrix defined by:

\[
e^{-CH(P_{h^2})s} := \text{Id}_h - CH(P_{h^2})s + \frac{[CH(P_{h^2})s]^2}{2!} - \frac{[CH(P_{h^2})s]^3}{3!} + \ldots.
\]

We shall analyze the existence of \( T \)-periodic solutions for the “auxiliary” horizontal path \( x_H(t) = (x_1(t), \ldots, x_h(t)) \) previously defined. Indeed, the \( T \)-periodic of \( x_H(t) \) is somehow connected with the study of the conjugate and cut loci of a point; see Remark 2.41.

In Section 2.6 we will show how, at least in principle, the system of normal CC-geodesic equations can be integrated step by step.

In Section 3 we will apply some of the tools previously developed toward the study of the CC-distance function \( \delta_H \) from a \( C^k \)-smooth (\( k \geq 2 \)) hypersurface \( S \), i.e.

\[
\delta_H(x) := \inf_{y \in S} \text{d}_H(x,y).
\]

This will be done only for 2-step Carnot groups, by using the explicit structure of \( \exp_{sr} \) in this case. More precisely, we shall define a mapping \( \Phi : S \times ]\epsilon,\epsilon[ \rightarrow \mathbb{G} (\epsilon > 0) \) by

\[
\Phi(y,t) := \exp_{sr}(y, N(y))(t),
\]

and then we will compute its Jacobian; see Lemma 3.2. To be more precise, let us fix Riemannian normal coordinates \( (u_1, \ldots, u_{n-1}) \) around \( y_0 \in S \). Then \( V(y) := \frac{\partial y}{\partial u_1} \wedge \ldots \wedge \frac{\partial y}{\partial u_{n-1}} \) is a normal (non-unit) vector along \( S \), in a neighborhood of \( y_0 \in S \) and it turns out that

\[
| \det [J_{(y,0)}\Phi] | = |P_H V|,
\]

where \( J_{(y,0)}\Phi \) denotes the Jacobian matrix operator at \( (y,0) \in S \times ]-\epsilon,\epsilon[ \). Therefore, out of the characteristic set \( C_S \), the map \( \Phi \) turns out to be invertible.

Finally, we shall prove the following (see Theorem 3.5):

**Theorem 1.5.** Let \( \mathbb{G} \) be a 2-step Carnot group. Let \( S \subset \mathbb{G} \) be a \( C^k \)-smooth hypersurface with \( k \geq 2 \) and let \( \delta_H \) denote the CC-distance function for \( S \). Set \( S_0 := S \setminus C_S \), where \( C_S \) denote the characteristic set of \( S \). Then, for every open set \( U_0 \) compactly contained in \( S_0 \), there exists a neighborhood \( U \subset \mathbb{G} \) of \( U_0 \) having the unique nearest point property, with respect to the CC-distance. Furthermore, the CC-distance function from \( U_0 \cap S \) is \( \delta_H|_{U \cap U_0} \) is a \( C^k \)-smooth function.

The proof of this result is based on explicit computations and on the sub-Riemannian Gauss Lemma (see Proposition 2.29).

### 1.1 Sub-Riemannian Geometry of Carnot groups

In this section, we will introduce the definitions and the main features concerning the sub-Riemannian geometry of Carnot groups. References for this subject are, for instance, [10], [32], [31], [35, 36], [45, 46], [49], [51], [56, 57], [61]. First, let us consider a \( C^\infty \)-smooth
connected \( n \)-dimensional manifold \( N \) and let \( H \subset TN \) be a \( h_1 \)-dimensional smooth subbundle of \( TN \). For any \( p \in N \), let \( T_p^k \) denote the vector subspace of \( T_pN \) spanned by a local basis of smooth vector fields \( X_1(p), \ldots, X_{h_1}(p) \) for \( H \) around \( p \), together with all commutators of these vector fields of order \( \leq k \). The subbundle \( H \) is called generic if for all \( p \in N \) \( \dim T_p^k \) is independent of the point \( p \) and horizontal if \( T_p^k = TN \) for some \( k \in \mathbb{N} \). The pair \((N, H)\) is a \( k \)-step CC-space if it is generic and horizontal and if \( k := \inf \{ r : T_p^r = TN \} \). In this case, we have that

\[
0 = T^0 \subset H = T^1 \subset T^2 \subset \ldots \subset T^k = TN
\]

is a strictly increasing filtration of smooth subbundles of constant dimensions \( n_i := \dim T^i \) (\( i = 1, \ldots, k \)). Setting \((H_i)_p := T^i_p \setminus T^{i-1}_p\), then \( \mathfrak{gr}(T_pN) := \oplus_{i=1}^k (H_k)_p \) is the associated graded Lie algebra, at the point \( p \in N \), with Lie product induced by \([\cdot, \cdot]\). Moreover, we shall set \( h_i := \dim H_i = n_i - n_{i-1} (n_0 = h_0 = 0) \). The \( k \)-vector \((h_1, \ldots, h_k)\) is the growth vector of \( H \). Notice that every \( H_i \) is a smooth subbundle of the tangent bundle \( \pi : TN \to N \), i.e. \( \pi_{H_i} : H_i \to N \), where \( \pi_{H_i} = \pi | H_i (i = 1, \ldots, k) \).

**Definition 1.6.** We will call graded frame \( \mathcal{X} = \{X_1, \ldots, X_n\} \) for \( N \), any frame for \( N \) such that, for any \( p \in N \) we have that \( \{X_i(p) : n_{j-1} < i_j \leq n_j\} \) is a basis for \( H_{j-1}{p} := H_j(p) (j = 1, \ldots, k) \).

**Definition 1.7.** A sub-Riemannian metric \( g_\mu = \langle \cdot, \cdot \rangle_\mu \) on \( N \) is a symmetric positive bilinear form on \( H \). If \((N, H)\) is a CC-space, then the CC-distance \( d_\mu(p, q) \) between \( p, q \in N \) is

\[
d_\mu(p, q) := \inf \int \sqrt{\langle \dot{\gamma}, \dot{\gamma} \rangle_\mu} dt,
\]

where the infimum is taken over all piecewise-smooth horizontal paths \( \gamma \) joining \( p \) to \( q \).

In fact, Chow’s Theorem (see [35], [54]) implies that \( d_\mu \) is actually a metric on \( N \), since any two points can be joined with (at least one) horizontal path; moreover the topology induced by the CC-metric turns out to be compatible with the given topology of \( N \).

The general setting introduced above is the starting point of sub-Riemannian geometry. A nice and very large class of examples of these geometries is represented by Carnot groups which for many reasons play, in sub-Riemannian geometry, an analogous role to that of Euclidean spaces in Riemannian geometry. Below we will introduce their main features. For an introduction to the following topics, we suggest Helgason’s book, [38], and the survey paper by Milnor, [47], regarding the geometry of Lie groups, and Gromov, [35], Pansu, [50, 57], and Montgomery, [54], specifically for sub-Riemannian geometry.

By definition a \( k \)-step Carnot group \((G, \bullet)\) is a \( n \)-dimensional, connected, simply connected, nilpotent and stratified Lie group (with respect to the group law \( \bullet \)). This means that its Lie algebra \( \mathfrak{g} \cong \mathbb{R}^n \) satisfies:

\[
\mathfrak{g} = H_1 \oplus \ldots \oplus H_k, \quad [H_1, H_{i-1}] = H_i, \quad (i = 2, \ldots, k), \quad H_{k+1} = \{0\},
\]

We shall denote by \( 0 \) the identity on \( G \). Any \( x \in G \) defines smooth maps \( L_x, R_x : G \to G \), called left-translation and right-translation, respectively, by \( L_x(y) := x \bullet y, R_x(y) := y \bullet x \), for any \( y \in G \). Remind that the Lie algebra \( \mathfrak{g} \) is naturally isomorphic to \( T_0G \) by identifying
any left-invariant vector field $X$ with its value at 0. The isomorphism is explicitly given by $L_{x*} : T_0\mathbb{G} \rightarrow T_x\mathbb{G}$. The smooth subbundle $H_1$ of the tangent bundle $TG$ is said horizontal and henceforth denoted by $H$. We will set $V := H_2 \oplus \ldots \oplus H_k$ and call $V$ the vertical subbundle of $TG$. We shall set $v := \text{dim} \, V$. One has

$$v := \text{codim}H, \quad n = h + v.$$  

As before, we will assume that $\dim H_i = h_i$ ($i = 1, \ldots, k$) and that $H$ is generated by some basis of left-invariant horizontal vector fields $X_\mu := \{X_1, \ldots, X_{h_1}\}$. This one can be completed to a global basis (frame) of left-invariant sections of $TG$, $\underline{X} := \{X_1, \ldots, X_n\}$, which is graded or adapted to the stratification. This can be done by re-labelling the canonical basis $\{e_i : i = 1, \ldots, n\}$ of $\mathfrak{g} \cong \mathbb{R}^n$ in a way that it turns out to be adapted to the stratification and then by setting

$$X_i(x) := L_{x*}e_i = \left. \frac{\partial x \cdot y}{\partial y} \right|_{y=0} e_i \quad (i = 1, \ldots, n).$$

We shall set $n_i := h_1 + \ldots + h_l$ ($n_0 = h_0 := 0$, $n_k = n$) and $H_l = \text{span}_\mathbb{R}\{X_i : n_{i-1} < i \leq n_l\}$ ($l = 1, \ldots, k$).

**Notation 1.8.** We shall set $I_\mu := \{1, \ldots, h_1\}$, $I_{\mu_2} := \{h_1 + 1, \ldots, n_2(= h_1 + h_2)\}, \ldots$, $I_{\mu_k} := \{n_{k-1}+1, \ldots, n_k(= n)\}$, and $I_\nu := \{h_1+1, \ldots, n\}$. Moreover, we will use Latin letters $i, j, k, \ldots$, for indices belonging to $I_\mu$ and Greek letters $\alpha, \beta, \gamma, \ldots$, for indices belonging to $I_\nu$. Unless otherwise specified, capital Latin letters $I, J, K, \ldots$, may denote any generic index. We also define the function $\text{ord} : \{1, \ldots, n\} \rightarrow \{1, \ldots, k\}$ by $\text{ord}(I) := i$ if, and only if, $n_{i-1} < I \leq n_i$ ($i = 1, \ldots, k$).

If $p \in \mathbb{G}$ and $X \in \mathfrak{g}$ we set $\vartheta_{(X,p)}(t) := \exp_{\mathfrak{g}}[tX](p) \in \mathbb{R}$, i.e. $\vartheta_{(X,p)}$ denotes the integral curve of $X$ starting from $p$ and it is a 1-parameter sub-group of $\mathbb{G}$. The Lie group exponential map is then defined by

$$\exp : \mathfrak{g} \rightarrow \mathbb{G}, \quad \exp_{\mathfrak{g}}(X) := \exp_{\mathfrak{g}}[X](1).$$

It turns out that $\exp_{\mathfrak{g}}$ is an analytic diffeomorphism between $\mathfrak{g}$ and $\mathbb{G}$ whose inverse will be denoted by $\log_{\mathfrak{g}}$. Moreover we have

$$\vartheta_{(X,p)}(t) = p \cdot \exp_{\mathfrak{g}}(tX) \quad \forall \, t \in \mathbb{R}.$$

From now on we shall fix on $\mathbb{G}$ the so-called exponential coordinates of 1st kind, i.e. the coordinates associated to the map $\log_{\mathfrak{g}}$.

As for any nilpotent Lie group, the Baker-Campbell-Hausdorff formula (see [16]) uniquely determines the group multiplication $\cdot$ of $\mathbb{G}$, from the “structure” of its own Lie algebra $\mathfrak{g}$. In fact, one has

$$\exp_{\mathfrak{g}}(X) \cdot \exp_{\mathfrak{g}}(Y) = \exp_{\mathfrak{g}}(X \star Y) \quad (X, Y \in \mathfrak{g}),$$

where $\star : \mathfrak{g} \times \mathfrak{g} \rightarrow \mathfrak{g}$ is the Baker-Campbell-Hausdorff product defined by

$$X \star Y = X + Y + \frac{1}{2}[X, Y] + \frac{1}{12}[X, [X, Y]] - \frac{1}{12}[Y, [X, Y]] + \text{brackets of length } \geq 3. \quad (4)$$

Using exponential coordinates, (4) implies that the group multiplication $\cdot$ of $\mathbb{G}$ is polynomial and explicitly computable (see [16]). Moreover, $0 = \exp_{\mathfrak{g}}(0, \ldots, 0)$ and the inverse of $x \in \mathbb{G}$ ($x = \exp_{\mathfrak{g}}(x_1, \ldots, x_n)$) is $x^{-1} = \exp_{\mathfrak{g}}(-x_1, \ldots, -x_n)$. 

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Notation 1.9. Using exponential coordinates for \( G \), every point \( x = \exp_g(\sum_i x_i e_i) \in G \) can be regarded as \( n \)-tuple \( x = (x_1, \ldots, x_h, x_{h+1}, \ldots, x_n) \in \mathbb{R}^n \). We shall set
\[
x_h := (x_1, \ldots, x_h) \in \mathbb{R}^h, \quad x_v := (x_{h+1}, \ldots, x_n) \in \mathbb{R}^v \quad (n = h + v).
\]
Hence \( x = \exp_g(x_h, x_v) \equiv (x_h, x_v) \).

When we endow the horizontal subbundle with a metric \( g_H = \langle \cdot, \cdot \rangle_H \), we say that \( G \) has a sub-Riemannian structure. It is important to note that it is always possible to define a left-invariant Riemannian metric \( g = \langle \cdot, \cdot \rangle \) in such a way that the frame \( X_1, X_2, \ldots, X_h \) turns out to be orthonormal and such that \( g|_H = g_H \). For this, it is enough to choose a Euclidean metric on \( g = T_0G \) which can be left-translated to the whole tangent bundle and, by this way, the direct sum becomes an orthogonal direct sum.

Since for Carnot groups the hypotheses of Chow’s Theorem trivially apply, the Carnot-Carathéodory distance \( d_h \) associated with \( g_H \) can be defined as before, and \( d_h \) makes \( G \) a complete metric space in which every couple of points can be joined by (at least) one \( d_h \)-geodesic; see [3], [54].

We remind that Carnot groups are homogeneous groups (see [60]), i.e., they have a 1-parameter group of automorphisms \( \delta_t : G \to G \) \((t > 0)\). Using exponential coordinates, we have \( \delta_t x = \exp_g(\sum_i t^i x_i e_i) \) for all \( x = \exp_g(\sum_i x_i e_i) \in G \). The homogeneous dimension of \( G \) is the integer \( Q := \sum_{i=1}^k i h_i \), coinciding with the Hausdorff dimension of \((G, d_h)\) as a metric space (see [49], [54], [35]).

We remind that the structural constants of the Lie algebra \( g \) associated with the (left-invariant) frame \( X \) are defined by
\[
C^R_{IJ} := \langle [X_I, X_J], X_R \rangle \quad (I, J, R = 1, \ldots, n).
\]
They satisfy the customary properties:

- \( C^R_{IJ} + C^R_{JI} = 0 \) (skew-symmetry)
- \( \sum_{J=1}^n C^I_{JL} C^J_{RM} + C^I_{JM} C^J_{LR} + C^I_{JR} C^J_{ML} = 0 \) (Jacobi’s identity).

The stratification hypothesis on the Lie algebra implies the following further property:
\[
X_i \in H_l, \quad X_j \in H_r \implies [X_i, X_j] \in H_{l+r}, \quad (5)
\]
Therefore, if \( i \in I_h \) and \( j \in I_r \), one has
\[
C^m_{ij} \neq 0 \implies m \in I_h + r. \quad (6)
\]

Definition 1.10. Throughout this paper we will use the following notation:

(i) \( C^\alpha_{ij} := [C^\alpha_i]_{j \in I_H} \in M_{h_1 \times h_1}(\mathbb{R}) \quad (\alpha \in I_H) \);

(ii) \( C^\alpha := [C^\alpha_{IJ}]_{I, J = 1, \ldots, n} \in M_{n \times n}(\mathbb{R}) \quad (\alpha \in I_V) \).

Furthermore, for any \( Z = \sum_{\alpha \in I_V} z^\alpha X_\alpha \in V \), we will set

---

3Here, \( j \in \{1, \ldots, k\} \) and \( i_j \in I_{u_j} = \{n_{j-1} + 1, \ldots, n_j\} \).
(iii) \( C_H(Z) := \sum_{\alpha \in I_H} z_{\alpha} C_H^\alpha; \)

(iv) \( C(Z) := \sum_{\alpha \in I_Y} z_{\alpha} C^\alpha. \)

**Definition 1.11.** Let \( i \in \{1, ..., k-1\}. \) Then the \( i \)-th curvature of \( H \) is the skew-symmetric, bilinear map

\[
\Omega_{H_i} : H \otimes H_i \rightarrow H_{i+1}, \quad \Omega_{H_i}(X \otimes Y) := [X, Y] \mod T^i
\]

whenever \( X \in H \) and \( Y \in H_i. \) By definition of \( k \)-step Carnot group, one has \( \Omega_{H_i}(\cdot, \cdot) = 0. \)

Since the bracket map \([\cdot, \cdot] : H \otimes H_i \rightarrow H_{i+1} (i = 1, ..., k-1)\) is surjective, the definition is well posed. We stress that the 1st curvature \( \Omega_H(\cdot, \cdot) := \Omega_{H_1}(\cdot, \cdot) \) of \( H \) is the “curvature of a distribution”; see \([32], [36], [54].\)

If \( Y \in TG, \) let \( Y = (Y_1, ..., Y_k) \) be the canonical decomposition of \( Y \) with respect to the Carnot grading, i.e. \( Y = \sum_{i=1}^k \mathcal{P}_{H_i}(Y), \) where \( \mathcal{P}_{H_i} \) is the orthogonal projection onto \( H_i. \) Set

\[
\Omega(X, Y) := \sum_{i=1}^{k-1} \Omega_{H_i}(X, Y_i),
\]

for \( X \in H \) and \( Y \in TG. \) Then we have

**Lemma 1.12.** Let \( X \in H \) and \( Y, Z \in TG. \) Then we have

(i) \( \langle \Omega_{H_i}(X, Y), Z \rangle = \langle C_H(Z)Y, X \rangle; \)

(ii) \( \langle \Omega(X, Y), Z \rangle = \langle C(Z)Y, X \rangle. \)

**Proof.** The proof is an immediate consequence of Definition \([1.11] \) and Definition \([1.10]. \)

\[\]

In the sequel, we will give a definition of connection which recovers the usual definitions of Riemannian, partial and non-holonomic connections. Classical notions of connection (linear, affine or Riemannian) and related topics can be found in \([38], [39].\)

Partial connections was defined by Z. Ge in \([32]; \) see also \([36] \) and \([41].\) Non-holonomic connections were used by É. Cartan in his studies on non-holonomic mechanics and by the Russian school; see the survey by Vershik and Gershkovich, \([62].\)

**Definition 1.13.** Let \( N \) be a \( C^\infty \) smooth manifold and let \( \pi_E : E \rightarrow N, \pi_F : F \rightarrow N \) be smooth subbundles of \( TN. \) An \( E \)-connection \( \nabla^{(E,F)} \) on \( F \) is a rule which assigns to each vector field \( X \in C^\infty(N, E) \) an \( \mathbb{R} \)-linear transformation \( \nabla_X^{(E,F)} : C^\infty(N, F) \rightarrow C^\infty(N, F) \) such that

(i) \( \nabla_{fX+gY}^{(E,F)}Z = f\nabla_X^{(E,F)}Z + g\nabla_Y^{(E,F)}Z \quad \forall X, Y \in C^\infty(N, E) \forall Z \in C^\infty(N, F) \forall f, g \in C^\infty(N); \)

(ii) \( \nabla_X^{(E,F)}fY = f\nabla_Y^{(E,F)}X + (Xf)Y \quad \forall X, Y \in C^\infty(N, E) \forall f \in C^\infty(N). \)

If \( E = F \) we shall set \( \nabla^E := \nabla^{(E,E)} \) and call \( \nabla^E \) an \( E \)-connection. Any such connection will be called a **partial connection** of \( TN. \) If \( E = TN, \) then \( \nabla^{(TN,F)} \) is called a
non-holonomic $F$-connection\footnote{This definition recovers the usual one of “vector bundle connection” (see \cite{HS}) where instead of a generic vector bundle $\pi : F \to N$ we make use of a subbundle of the tangent bundle.}. If $E$ has a positive definite inner product $g_E$, then an $E$-connection $\nabla^E$ is said metric preserving if

$$(iii) \quad Zg_E(X,Y) = g_E(\nabla^E_Z X, Y) + g_E(X, \nabla^E_Z Y) \quad \forall X, Y, Z \in C^\infty(N, E).$$

The torsion $T_E$ associated to the $E$-connection $\nabla^E$ is defined by

$$T_E(X,Y) := \nabla^E_X Y - \nabla^E_Y X - P_E [X,Y] \quad \forall X,Y \in C^\infty(N,E),$$

where $P_E : TN \to E$ denotes the orthogonal projection onto $E$. An $E$-connection is torsion free if $T_E(X,Y) = 0$ for every $X,Y \in C^\infty(N,E)$. We shall say that $\nabla^E$ is the Levi-Civita $E$-connection on $E$ if it is metric preserving and torsion-free. Note that if $E = TN$, terminology and definitions adopted here are the customary ones. In this case, we will denote by $\nabla$ the Levi-Civita connection on $TN$.

We stress that the difference between the definitions of partial and non-holonomic connection is that the latter allows us to covariantly differentiate along any curve of $N$ whereas using the first one only curves that are tangent to the subbundle $E$ can be considered.

**Definition 1.14.** In the sequel, $\nabla$ will denote the unique left-invariant Levi-Civita connection on $G$ associated with the fixed left invariant metric $g$. Moreover, for any $X, Z \in \mathfrak{x}(H) := C^\infty(G,H)$, we shall set $\nabla^H Y := P_h(\nabla_X Y)$. We note that $\nabla^H$ is a partial connection, also called horizontal connection or H-connection. For notational convenience, in the sequel we will denote by the same symbol the non-holonomic connection on $G$, i.e. $\nabla^H = \nabla^{(TG,H)}$.

**Definition 1.15.** We define the horizontal curvature $R^H$ of the $H$-connection $\nabla^H$ to be the trilinear map $R^H : H \times H \times H \to H$ given by

$$R^H(X,Y)Z := \nabla^H_X \nabla^H Y Z - \nabla^H_Y \nabla^H X Z - \nabla^H_{P^H[Y,X]} Z,$$

where $X, Y, Z \in \mathfrak{x}(H)$. In the sequel, $R$ will denote the Riemannian curvature tensor, defined by

$$R(X,Y)Z := \nabla_Y \nabla_X Z - \nabla_X \nabla_Y Z - \nabla_{[Y,X]} Z \quad (X,Y,Z \in \mathfrak{x}(G)).$$

**Remark 1.16.** From Definition [1.14], using the properties of the structural constants of any Levi-Civita connection, we get that the horizontal connection $\nabla^H$ is flat, i.e.

$$\nabla^H_{X_i} X_j = 0 \quad (i, j \in I_H).$$

Note that $\nabla^H$ turns out to be compatible with the sub-Riemannian metric $g_H$, i.e.

$$X\langle Y, Z \rangle_H = \langle \nabla^H_X Y, Z \rangle_H + \langle Y, \nabla^H_X Z \rangle_H \quad \forall X,Y,Z \in \mathfrak{x}(H).$$

This follows immediately from the very definition of $\nabla^H$ using the compatibility property of the left-invariant Levi-Civita connection $\nabla$ with respect to the Riemannian metric $g$. Furthermore, $\nabla^H$ is torsion-free, i.e.

$$\nabla^H_X Y - \nabla^H_Y X - \nabla^H_{P^H}[X,Y] = 0 \quad \forall X,Y \in \mathfrak{x}(H).$$

In particular, it turns out that the horizontal curvature $R^H$ is identically zero.
Definition 1.17. If \( \psi \in C^\infty(\mathbb{G}) \) we define the horizontal gradient of \( \psi \), \( \text{grad}_H \psi \), as the unique horizontal vector field such that

\[
\langle \text{grad}_H \psi, X \rangle_H = d\psi(X) = X \psi \quad \forall X \in \mathfrak{X}(H).
\]

Later on, we will denote by \( \mathcal{J}_H \) the Jacobian matrix of a vector-valued function, computed with respect to any given orthonormal frame \( \underline{\tau}_H = \{ \tau_1, \ldots, \tau_{\text{h}_1} \} \) for \( H \).

We shall now define the left-invariant co-frame \( \varpi := \{ \omega_I : I = 1, \ldots, n \} \) dual to \( X \) (with respect to the left invariant metric \( g \)). The left-invariant 1-forms \( \omega_I \) are uniquely determined by the condition:

\[
\omega_I(X_J) = \langle X_I, X_J \rangle = \delta^I_J \quad \text{(Kroneker)} \quad (I,J = 1, \ldots, n).
\]

The Cartan’s structure equations for the left-invariant co-frame \( \varpi \) are given by:

\[
\begin{align*}
(I) \quad d\omega_I &= \sum_{J=1}^n \omega_{IJ} \wedge \omega_J, \\
(II) \quad d\omega_{JK} &= \sum_{L=1}^n \omega_{JL} \wedge \omega_{LK} - \Omega_{JK} \quad (I,J,K = 1, \ldots, n),
\end{align*}
\]

where \( \omega_{IJ}(X) = \langle \nabla_X X_I, X_J \rangle \) are the connection 1-forms for \( \varpi \) while \( \Omega_{JK} \) are the curvature 2-forms, defined by

\[
\Omega_{JK}(X,Y) = \omega_K(R(X,Y)X_J) \quad (X,Y \in \mathfrak{X}(\mathbb{G})).
\]

For what concerns the theory of connections on Lie group and left-invariant differential forms, see [38]. Moreover, for many topics about the geometry of nilpotent Lie groups equipped with a left-invariant connection, see [17]; for the Carnot case see [50, 51].

Remark 1.18. We have

\[
\nabla_{X_I} X_J = \frac{1}{2} \sum_{R=1}^n (C^R_{IJ} - C^R_{JI} + C^R_{JI}) X_R \quad (I,J = 1, \ldots, n). \tag{8}
\]

This formula and condition (5) are needed to make explicit computations in terms of the structural constants. For instance, from (8) it follows that the 1st structure equation for the coframe \( \varpi \) becomes

\[
d\omega_R = -\frac{1}{2} \sum_{1 \leq I,J \leq n_i-1} C^R_{IJ} \omega_I \wedge \omega_J, \tag{9}
\]

where \( R \in I_{n_i} = \{ j : n_{i-1} < j \leq n_i \} \) and \( i = 1, \ldots, k \).

In the sequel we will need the notion of covariant derivative along a path; we refer the reader to [11] for a detailed introduction.

\footnotetext[5]{That is, \( L^*_x \omega_I = \omega_I \) for every \( x \in \mathbb{G} \).}
Definition 1.19. Let $\gamma : [a, b] \subset \mathbb{R} \rightarrow G$ be a $C^1$ path and let $X : [a, b] \subset \mathbb{R} \rightarrow T \mathbb{G}$, $X = \sum_t \xi_t(t)X_t(\gamma)$, be a vector field along $\gamma$. Then the covariant derivative of $X$ along $\gamma$, denoted by $\nabla_t$, is defined as

$$\nabla_t X := \nabla_\gamma X = \sum_{I=1}^n \left\{ \xi_I + \sum_{J,K=1}^n \Gamma^I_{JK}(\gamma) \xi_J \xi_K \right\} X_L(\gamma),$$

where $\nabla$ denotes the Levi-Civita connection on $G$ and

$$\Gamma^I_{JK} := (\nabla_{X_K} X_J, X_I) = \omega_{IJ}(X_K) \quad (I, J, K = 1, \ldots, n)$$

are the Christoffel symbols of $\nabla$, with respect to the left invariant frame $X = \{X_1, \ldots, X_n\}$ on $G$.

We end this section with some important examples.

Example 1.20 (Heisenberg group $\mathbb{H}^1$). Let $\mathfrak{h}_1 := T_0 \mathbb{H}^1 = \mathbb{R}^3$ denote the Lie algebra of the Heisenberg group $\mathbb{H}^1$, that is the most simple example of 2-step Carnot group. Its Lie algebra $\mathfrak{h}_1$ satisfies:

$$[e_1, e_1] = e_3$$

and all other commutators vanish. We have $\mathfrak{h}_1 = H \oplus \mathbb{R}^3$ where $H = \text{span}_\mathbb{R}\{e_1, e_2\}$. In particular, the 2nd layer of the grading $\mathbb{R}^3$ is the center of the Lie algebra $\mathfrak{h}_n$. These conditions determine the group law $\bullet$ via the Baker-Campbell-Hausdorff formula. Indeed, if $x = \exp_\theta(\sum_{i=1}^3 x_i e_i)$, $y = \exp_\theta(\sum_{i=1}^3 y_i e_i) \in \mathbb{H}^1$, one has

$$x \bullet y = \exp_\theta \left( x_1 + y_1, x_2 + y_2, x_3 + y_3 + \frac{1}{2}(x_1 y_2 - x_2 y_1) \right).$$

The standard frame of orthonormal left invariant vector fields for $\mathbb{H}^1$ is given, using exponential coordinates, by

$$X_1(x) := L_x e_1 = e_1 - \frac{x_2}{2} e_3;$$

$$X_2(x) := L_x e_2 = e_2 + \frac{x_1}{2} e_3;$$

$$X_3(x) := L_x e_3 = e_3.$$

Example 1.21 (Heisenberg group $\mathbb{H}^n$). Let $\mathfrak{h}_n := T_0 \mathbb{H}^n = \mathbb{R}^{2n+1}$ denote the Lie algebra of the Heisenberg group $\mathbb{H}^n$, that is the most important example of 2-step Carnot group. Its Lie algebra $\mathfrak{h}_n$ is defined by the following rules:

$$[e_i, e_{i+1}] = e_{2n+1} \quad \text{for every } i = 2k - 1, \ k = 1, \ldots, n = \frac{h}{2}$$

and all other commutators vanish. One has $\mathfrak{h}_n = H \oplus \mathbb{R}^{2n+1}$, where

$$H = \text{span}_\mathbb{R}\{e_i : i = 1, \ldots, 2n\}.$$

The 2nd layer of the grading $\mathbb{R}^{2n+1}$ is the center of $\mathfrak{h}_n$. The above conditions uniquely determine the group law $\bullet$ via the Baker-Campbell-Hausdorff formula. More precisely, if $x = \exp_\theta(\sum_{i=1}^{2n+1} x_i e_i)$, $y = \exp_\theta(\sum_{i=1}^{2n+1} y_i e_i) \in \mathbb{H}^n$, then

$$x \bullet y = \exp_\theta \left( x_1 + y_1, \ldots, x_{2n} + y_{2n}, x_{2n+1} + y_{2n+1} + \frac{1}{2} \sum_{k=1}^n (x_{2k-1} y_{2k} - x_{2k} y_{2k-1}) \right).$$
If \( i \in \{1, ..., n\} \), the standard frame of orthonormal left invariant vector fields for \( H^n \) is given, using exponential coordinates, by

\[
X_{2i-1}(x) := L_{x*}e_{2i-1} = e_{2i-1} - \frac{x_{2i}}{2} e_{2n+1};
\]

\[
X_{2i}(x) := L_{x*}e_{2i} = e_{2i} + \frac{x_{2i-1}}{2} e_{2n+1};
\]

\[
X_{2n+1}(x) := L_{x*}e_{2n+1} = e_{2n+1}.
\]

**Example 1.22** (2-step Carnot groups). We have \( g = H \oplus H_2 \), where \( H_2 \) is the center of \( g \). Furthermore, we have the following rules:

\[
[e_i, e_j] = \sum_{\alpha \in I_{H_2}} C^\alpha_{ij} e_\alpha \quad \text{for every } i, j \in I_H = \{1, ..., h_1\}
\]

and all other commutators vanish. Let \( x = \exp_g(\sum_{i=1}^n x_i e_i) \), \( y = \exp_g(\sum_{i=1}^n y_i e_i) \in G \). Then

\[
x \cdot y = \exp_g\left( x + y - \frac{1}{2} \sum_{\alpha \in I_{H_2}} (C^\alpha_{ij} x, y) e_\alpha \right).
\]

The standard frame of orthonormal left invariant vector fields for \( G \) is given by

\[
X_i(x) = L_{x*}e_i = e_i - \frac{1}{2} \sum_{\alpha \in I_{H_2}} (C^\alpha_{ij} x, e_i) e_\alpha \quad (i \in I_H)
\]

\[
X_\alpha(x) = L_{x*}e_\alpha = e_\alpha \quad (\alpha \in I_{H_2}).
\]

Remind that \( C^\alpha_{ij} = [C^\alpha_{ij}]_{i,j \in I_H} \); see Definition 1.10.

**Example 1.23** (Engel group \( E^1 \)). The Engel group is the simpler example of a 3-step Carnot group. Its Lie algebra \( \mathfrak{e} \) is 4-dimensional and is defined by the following rules:

\[
[e_1, e_2] = e_3, \quad [e_1, e_3] = e_4
\]

and all other commutators vanish. We have \( \mathfrak{e} = H \oplus \mathbb{R}e_3 \oplus \mathbb{R}e_4 \), where \( H = \text{span}_\mathbb{R}\{e_1, e_2\} \) and the center of the Lie algebra \( \mathfrak{e} \) is \( \mathbb{R}e_4 \). Therefore \( C^3_H = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \) and

\[
C^4 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.
\]

The group law \( \cdot \) is given, for \( x = \exp_g(\sum_{i=1}^4 x_i e_i) \), \( y = \exp_g(\sum_{i=1}^4 y_i e_i) \in E^1 \), by

\[
x \cdot y = \exp_g\left( x + y - \frac{1}{2} (C^3_H x, y) e_3 - \left( \frac{1}{12} (C^4 x, y) + \frac{1}{12} (C^3_H y, y) (C^4 e_3, (x - y)) \right) e_4 \right).
\]
The standard frame of orthonormal left invariant vector fields for $\mathbb{E}^1$ is given by

$$X_i(x) := L_x e_i = e_i - \frac{1}{2} \langle C^3_i x, e_i \rangle e_3 - \left( \frac{1}{2} \langle C^4 x, e_i \rangle + \frac{1}{12} \langle C^3_i x, e_i \rangle \langle C^4 e_3, x \rangle \right) e_4 \quad (i = 1, 2)$$

$$X_3(x) := L_x e_3 = e_3 - \frac{1}{2} \langle C^4 x, e_3 \rangle e_4$$

$$X_4(x) := L_x e_4 = e_4.$$

**Example 1.24** (3-step Carnot groups). We have that $\mathfrak{g} = H \oplus H_2 \oplus H_3$ where $H_3$ is the center of $\mathfrak{g}$. In order to describe $\mathfrak{g}$, we make use of the matrices of the structure constants of $\mathfrak{g}$, i.e.

$$C^\alpha_H = [C^\alpha_H]_{i,j} \in H \quad (\alpha \in I_{H_2})$$

$$C^\alpha = [C^\alpha_{IJ}]_{I,J=1,...,n} \quad (\alpha \in I_{H_3}).$$

The group law $\bullet$ is given, for $x = \exp_\mathfrak{g}(\sum_{i=1}^n x_i e_i)$, $y = \exp_\mathfrak{g}(\sum_{i=1}^n y_i e_i) \in \mathfrak{g}$, by

$$x \bullet y = \exp_\mathfrak{g} \left( x + y - \frac{1}{2} \sum_{\alpha \in I_{H_2}} \langle C^\alpha_H x, y \rangle e_\alpha - \sum_{\beta \in I_{H_3}} \left[ \frac{1}{2} \langle C^\beta x, y \rangle - \frac{1}{12} \sum_{\alpha \in I_{H_2}} \langle C^\beta (x-y), e_\alpha \rangle \langle C^\alpha_H x, y \rangle \right] e_\beta \right).$$

The standard frame of orthonormal left invariant vector fields for $\mathbb{G}$ is given by

$$X_i(x) = L_x e_i$$

$$= e_i - \frac{1}{2} \sum_{\alpha \in I_{H_2}} \langle C^\alpha_H x, e_i \rangle e_\alpha - \sum_{\beta \in I_{H_3}} \left[ \frac{1}{2} \langle C^\beta x, e_i \rangle - \frac{1}{12} \sum_{\alpha \in I_{H_2}} \langle C^\beta (x-y), e_\alpha \rangle \langle C^\alpha_H x, e_i \rangle \right] e_\beta$$

$$(i \in I_H);$$

$$X_\gamma(x) = L_x e_\gamma = e_\gamma - \frac{1}{2} \sum_{\beta \in I_{H_3}} \langle C^\beta x, e_\gamma \rangle e_\beta \quad (\gamma \in I_{H_2});$$

$$X_\beta(x) = L_x e_\alpha = e_\beta \quad (\beta \in I_{H_3}).$$

We remind that, for any $z = \exp_\mathfrak{g}(\sum_I z_I e_I) \in \mathbb{G}$ we usually set $z_H := (z_1, ..., z_h) \in \mathbb{R}^h$; see Notation [1.9]

### 1.2 Hypersurfaces: some basic facts

We now remind some basic facts about hypersurfaces which will be needed in the sequel. This material can be found in [50, 51].

Later on, $\mathcal{H}_cc^n_{\mathbb{G}}$ and $\mathcal{S}_cc^n_{\mathbb{G}}$ will denote, respectively, the usual and the spherical Hausdorff measures associated with the CC-distance. The (left-invariant) **Riemannian volume form** on $\mathbb{G}$ is defined as $\sigma^n := \Lambda^n \omega = \Lambda^n(T\mathbb{G}).$

**Remark 1.25.** By integrating $\sigma^n$, we obtain a measure $\text{vol}^n_R$, which is the Haar measure of $\mathbb{G}$. Since the determinant of $L_x$ is equal to 1, this measure equals the measure induced on...
by the push-forward of the n-dimensional Lebesgue measure \( \mathcal{L}^n \) on \( \mathbb{R}^n \cong \mathfrak{g} \). Moreover, up to a constant multiple, \( \text{vol}^n_x \) equals the \( Q \)-dimensional Hausdorff measure \( \mathcal{H}^Q_{\text{cc}} \) on \( \mathfrak{g} \). This follows because they are both Haar measures for the group and therefore equal up to a constant; see \cite{54}. Here we assume this constant equal to 1.

In the study of hypersurfaces of Carnot groups we have to introduce the notion of characteristic point.

**Definition 1.26.** If \( S \subset \mathfrak{g} \) is a \( C^r \)-smooth \((r = 1, \ldots, \infty)\) hypersurface, we say that \( S \) is characteristic at \( x \in S \) if \( \dim H_x = \dim (H_x \cap T_x S) \) or, equivalently, if \( H_x \subset T_x S \). The characteristic set of \( S \) is denoted by \( C_S \), i.e.

\[
C_S := \{ x \in S : \dim H_x = \dim (H_x \cap T_x S) \}.
\]

A hypersurface \( S \subset \mathfrak{g} \), oriented by its unit normal vector \( \nu \), is non-characteristic if, and only if, the horizontal subbundle \( H \) is transversal to \( S \) \((H \pitchfork TS)\). We have then

\[
H_x \cap T_x S \iff \mathcal{P}_H \nu(x) \neq 0 \iff \exists X \in \mathfrak{X}(H) : \langle X(x), \nu(x) \rangle \neq 0 \quad (x \in S),
\]

where \( \mathcal{P}_H : T\mathfrak{g} \longrightarrow H \) denotes the orthogonal projection onto \( H \).

**Remark 1.27 (Hausdorff measure of \( C_S \); see \cite{25}).** If \( S \subset \mathfrak{g} \) is a \( C^1 \)-smooth hypersurface, \( r \geq 1 \) then the \( Q-1 \)-dimensional Hausdorff measure associated with \( d_H \) of \( C_S \) is zero, i.e.

\[
\mathcal{H}^{Q-1}_{\text{cc}}(C_S) = 0.
\]

**Remark 1.28 (Riemannian measure on hypersurfaces).** Let \( S \subset \mathfrak{g} \) be a \( C^r \)-smooth hypersurface and let \( \nu \) denote the unit normal vector along \( S \). By definition, the \( n-1 \)-dimensional Riemannian measure along \( S \) is given by

\[
\sigma^{n-1}_{\mathbb{R}} \mid S := (\nu \downarrow \sigma^{n}_{\mathbb{R}})|_S,
\]

where \( \downarrow \) denotes the “contraction”, or interior product, of a differential form.

Since we shall study smooth hypersurfaces, instead of the usual weak definition of \( H \)-perimeter measure (see \cite{3, 10, 10, 17, 27, 28, 29, 31}) we now introduce a \((n-1)\)-differential form which, by integration, coincides with the \( H \)-perimeter measure.

**Definition 1.29 (\( \sigma^{n-1}_{\mu} \)-measure on hypersurfaces).** Let \( S \subset \mathfrak{g} \) be a \( C^r \)-smooth non-characteristic hypersurface and let us denote by \( \nu \) its unit normal vector. We will call \( H \)-normal along \( S \), the normalized projection onto \( H \) of \( \nu \), i.e.

\[
\nu_{\mu} := \frac{\mathcal{P}_H \nu}{|\mathcal{P}_H \nu|}.
\]

We then define the \((n-1)\)-dimensional measure \( \sigma^{n-1}_{\mu} \) along \( S \) to be the measure associated with the \((n-1)\)-differential form \( \sigma^{n-1}_{\mu} \in \Lambda^{n-1}(TS) \) given by the contraction of the volume form \( \sigma^{n}_{\mathbb{R}} \) of \( \mathfrak{g} \) with the horizontal unit normal \( \nu_{\mu} \), i.e.

\[
\sigma^{n-1}_{\mu} \mid S := (\nu_{\mu} \downarrow \sigma^{n}_{\mathbb{R}})|_S.
\]

**If we allow \( S \) to have characteristic points, we may trivially extend the definition of \( \sigma^{n-1}_{\mu} \) by setting \( \sigma^{n-1}_{\mu} \mid C_S = 0 \). We stress that \( \sigma^{n-1}_{\mu} \mid S = |\mathcal{P}_H \nu| \cdot \sigma^{n-1}_{\mathbb{R}} \mid S \).**

\( \downarrow \)The linear map \( \downarrow : \Lambda^k(T\mathfrak{g}) \rightarrow \Lambda^{k-1}(T\mathfrak{g}) \) is defined, for \( X \in T\mathfrak{g} \) and \( \omega^k \in \Lambda^k(T\mathfrak{g}) \), by \( (X \downarrow \omega^k)(Y_1, \ldots, Y_{k-1}) := \omega^k(X, Y_1, \ldots, Y_{k-1}) \); see \cite{38, 21}. 

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Definition 1.30. If $\nu_H$ is the horizontal unit normal along $S$, at each regular point $x \in S \setminus C_S$ one has that $H_x = (\nu_H)_x \oplus H_x S$, where we have set $H_x S := H_x \cap T_x S$.

We call $H_x S$ the **horizontal tangent space** at $x$ along $S$. We define in the obvious way the associated subbundles $HS(\subset TS)$ and $\nu_H S$, called, respectively, **horizontal tangent bundle** and **horizontal normal bundle** of $S$. Moreover we shall set:

1. $N := \nu_{| \nu_H} = (\nu_H, \bar{\omega})$, where $\bar{\omega} := \frac{\nu}{|\nu|}$;
2. $\omega_\alpha := \frac{\nu_\alpha}{|\nu_H|}$ ($\alpha \in I_V$);
3. $C_H(\bar{\omega}) := \sum_{\alpha \in I_H} \omega_\alpha C_\alpha$;
4. $C(\bar{\omega}) := \sum_{\alpha \in I_V} \omega_\alpha C_\alpha$.

2 The CC-distance in Carnot groups

2.1 On normal and abnormal CC-geodesics in Carnot groups

Below we shall introduce the main notions about normal and abnormal geodesics in the setting of Carnot groups and we shall explicitly write down the associated equations. Our approach is the Hamiltonian one and we follow that given in [34] (see also [55]). However, at the end of this section, we shall also briefly recall and discuss the Lagrangian point of view, which will turn out to be useful in the sequel.

From now on we shall set $h := h_1 = \dim H$.

As already recalled in the previous section, the CC-metric $d_H$ measures the distance between two given points $p$ and $q$ by minimizing the length of all (absolutely continuous) horizontal curves (i.e. tangent to the horizontal subbundle $H \subset TG$) joining $p$ and $q$. Thus we have to study minimizers for this metric. A **minimizer** is any absolutely continuous horizontal curve $\gamma : I \subset \mathbb{R} \to G$ which is such that for every $t \in I$ there exists $\epsilon > 0$ such that $\gamma$ minimizes the length between $\gamma(t_0)$ and $\gamma(t_1)$ whenever $t_0, t_1$ belong to $(t - \epsilon, t + \epsilon) \subset I$. A specific feature of sub-Riemannian geometry is that minimizers can be of two different types. However they are not necessarily mutually exclusive. Roughly speaking, the minimizers of the first type, called **normal**, are projections of solutions of a Hamiltonian system and, in a sense, they generalize the Riemannian situation since, in particular, they are differentiable. Minimizers of the second type are called **abnormal** or **singular**. Although their existence was originally deduced from the Pontryagin Maximum Principle (see [54] and discussion therein), they can also be defined as projection onto $G$ of characteristic curves (in the symplectic sense) of the annihilator of $H$ in the cotangent bundle $T^*G$, as we shall see below.
For sake of completeness we recall the main definitions of the hamiltonian formalism in our particular setting. A Hamiltonian is a function $\mathcal{H} : T^*G \rightarrow \mathbb{R}$ where the cotangent bundle $T^*G$ is the phase space. The exponential coordinates $x = \exp_g(x_1, ..., x_n)$, which have been fixed on the whole $G$, induce fiber coordinates on $T^*G$ by expanding any arbitrary covector $p \in T^*_xG$ in terms of the coordinate covector fields $\{dx_1, ..., dx_n\}$, i.e. $p = \sum_{I=1}^n p_I dx_I$. The $2n$ functions $(x, p) = (x_1, ..., x_n, p_1, ..., p_n)$ are said canonical coordinates on the phase space and the 1-form $\Theta = \sum_{I=1}^n p_I dx_I$ is the tautological one form on $T^*_xG$ (which is actually independent of the choice of coordinates on $G$). The canonical symplectic form on $T^*G$ is, by definition, the non-degenerate 2-form $\omega = -d\Theta$. Now, given a function $H$, $\omega$ uniquely determines a vector field $X_H$ satisfying $dH = \omega(X_H, \cdot)$, which is called the Hamiltonian vector field for $H$ (or symplectic gradient of $H$). The Hamilton’s equations for a smooth Hamiltonian $H$ are the O.D.E.’s for the integral curves of $X_H$. In canonical coordinates, they are given by

$$\dot{x}_I = \frac{\partial H}{\partial p_I}, \quad \dot{p}_I = -\frac{\partial H}{\partial x_I} \quad (I = 1, ..., n).$$ (12)

We remind that the momentum function $P_Y : T^*G \rightarrow \mathbb{R}$ is defined by

$$P_Y(x, p) := p(Y(x)).$$

In the sequel $p_I$ will denote the momentum function associated to the $I$-th coordinate vector field $\partial/\partial x_I$. Hence, in canonical coordinates one has $P_Y(x, p) = \sum_{I=1}^n p_I Y_I(x)$, where $Y = \sum_{I=1}^n Y_I \partial/\partial x_I$.

We now start with the derivation of the CC-geodesic equations by defining the sub-Riemannian Hamiltonian (or kinetic energy) $\mathcal{H}_{SR}$. To this aim let us set

$$P_i := P_{X_i} \quad \text{for every } i \in I_H = \{1, ..., h\},$$

to denote the momentum functions associated with a orthonormal (left-invariant) moving frame $X_H = \{X_1, ..., X_h\}$ for $H$ and note that, if $X_i(x) = \sum_{I=1}^n (X_i(x))_I e_I$, one has

$$P_i(x, p) = P_{X_i}(x, p) = \sum_{I=1}^n (X_i(x))_I p_I \quad \text{for every } i \in I_H.$$

Notation 2.1. We shall set

- $P_H := \sum_{i \in I_H} P_i X_i$,
- $P_V := \sum_{\alpha \in I_V} P_{\alpha} X_{\alpha}$,
- $P := \sum_I P_I X_I$.

7 We are using the notation:

$$\partial/\partial x_I \equiv e_I = (0, ..., 1 \underset{I-\text{th place}}{\cdots}, 0) \quad (I = 1, ..., n).$$
Definition 2.2. The sub-Riemannian Hamiltonian is defined by
\[ H_{sr}(x, p) := \frac{1}{2} \sum_{i \in I_H} P_i^2(x, p). \]

The Hamiltonian equations (12) associated with the sub-Riemannian Hamiltonian \( H_{sr} \) are called the normal geodesics equations. A normal curve is the projection onto \( G \) of a solution of the normal geodesics equations.

The following result is well-known and a proof can be found in [54].

Theorem 2.3. Every sufficiently short arc of a normal curve \( x \subset G \) is a minimizing CC-geodesic. Moreover \( x \) is the unique minimizing CC-geodesic connecting its endpoints.

The normal CC-geodesic equations are given by the following system:
\[
\begin{aligned}
\dot{x} &= P_H, \\
\dot{P} &= -\sum_{\alpha \in I_V} P_\alpha C^\alpha P_H, \\
\end{aligned}
\]
see Definition 1.10 for the notation \( C^\alpha \). Note that the first equation express the fact that the normal curve is horizontal. The second equation can be deduced by noting that, for any function \( f : T^*G \to \mathbb{R} \) and for any solution \( x : I \subset \mathbb{R} \to G \) of the Hamiltonian equations, one has
\[
\frac{d}{dt} f(\gamma(t)) = \{ f, H_{sr} \},
\]
where \( \{\cdot, \cdot\} \) denotes Poisson bracket. In particular \( \dot{P}_I = \{P_I, H_{sr}\} \) for every \( I = 1, \ldots, n \).

Therefore, it follows that
\[
\dot{P}_I = \{P_I, \frac{1}{2} \sum_{i \in I_H} P_i^2(x, p)\} = \frac{1}{2} \sum_{i \in I_H} \{P_I, P_i\} P_i = -\sum_{i \in I_H} \sum_{\alpha \in I_V} C^\alpha P_\alpha P_i.
\]
The last computation yields (13).

Below we shall discuss some other features of the normal geodesic equations but let us first introduce the following (see [34], [54]):

Definition 2.4. An abnormal curve is a horizontal curve which is the projection onto \( G \) of an absolutely continuous curve in the annihilator \( H^\perp \subset T^*G \) of \( H \), with square integrable derivative, which does not intersect the zero section of \( H^\perp \) and whose derivative, whenever it exists, is in the kernel of the canonical symplectic form restricted to \( H^\perp \). An abnormal minimizer is an abnormal curve which is a minimizer. A strictly abnormal curve -resp. minimizer- is an abnormal curve -resp. minimizer- which is not normal.

For a deep study on abnormal extremals and related problems in sub-Riemannian geometry, see [1], [2] and references therein. The equations for abnormal curves in our Carnot setting can explicitly be derived along the lines of [34] (see also [53, 54]). More precisely, they are given by
\[
\begin{aligned}
\sum_{\alpha \in I_V} P_\alpha C^\alpha x_H &= 0, \\
\dot{P}_V &= -\sum_{\alpha \in I_V} P_\alpha C^\alpha x, \\
\dot{x}_V &= 0, \\
\dot{P}_H &= 0.
\end{aligned}
\]
Unlike the system (13), the equations of the system (16) are mixed algebraic-differential equations and they cannot be expressed as O.D.E.’s. Note also that abnormal curves only depends on $H$ and not on the metric.

We remind that, for any $x = \exp_g(\sum_I x_I e_I) \in G$ we set $x_H := (x_1, ..., x_h) \in \mathbb{R}^h \cong H$ and $x_V := (x_{h+1}, ..., x_n) \in \mathbb{R}^v \cong V$; see Notation 1.9.

**Definition 2.5.** According to Definition 1.10, we shall set:

(i) $C_H(P_V) := \sum_{\alpha \in I_V} P_\alpha C_H^\alpha$

(ii) $C(P_V) := \sum_{\alpha \in I_V} P_\alpha C^\alpha$

Using this notation, (13) and (16) can be rewritten more compactly as follows:

(Normal Equations) \[
\begin{aligned}
\dot{x} &= P_H \\
\dot{P}_H &= -C(P_V)P_H;
\end{aligned}
\] (15)

(Abnormal Equations) \[
\begin{aligned}
C_H(P_V)x_H &= 0 \\
\dot{P}_V &= -C(P_V)x \\
\dot{x}_V &= 0 \\
P_H &= 0.
\end{aligned}
\] (16)

In the next examples we shall write down the normal CC-geodesic equations. Some of them will be studied in greater detail in the sequel.

**Example 2.6** (Heisenberg group $\mathbb{H}^1$). We get

\[
\begin{aligned}
\dot{x} &= P_H \\
\dot{P}_H &= -C_H(P_3)P_H \\
\dot{P}_3 &= 0.
\end{aligned}
\] (17)

More explicitly, one has \[\dot{P}_H = -P_3 \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} P_H, \text{ where } P_H = \begin{pmatrix} P_1 \\ P_2 \end{pmatrix}.\]

**Example 2.7** (Heisenberg group $\mathbb{H}^n$). We get

\[
\begin{aligned}
\dot{x} &= P_H \\
\dot{P}_H &= -C_H(P_{2n+1})P_H \\
\dot{P}_{2n+1} &= 0.
\end{aligned}
\] (18)

We have

\[
\dot{P}_H = -P_{2n+1} \begin{pmatrix} 0 & 1 & 0 & 0 & \cdots & 0 \\ -1 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 1 & \cdots & 0 \\ 0 & 0 & -1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 0 & \cdots & -1 \end{pmatrix} P_H, \quad P_H = \begin{pmatrix} P_1 \\ P_2 \\ P_3 \\ P_4 \\ \vdots \\ P_{2n-1} \\ P_{2n} \end{pmatrix}.\]
Example 2.8 (2-step Carnot groups). We get

\[
\begin{cases}
\dot{x} = P_H \\
\dot{P}_H = -C_H(P_{H_2})P_H \\
\dot{P}_{H_2} = 0,
\end{cases}
\]

(19)

where \(C_H(P_{H_2}) = \sum_{\alpha \in I_{H_2}} P_\alpha C_H^\alpha\).

Example 2.9 (Engel group \(E^1\)). We have

\[
\begin{cases}
\dot{x} = P_H \\
\dot{P}_H = -(P_3 C_H^3 + P_4 C^4)P_H \\
\dot{P}_3 = -P_4 C^4 P_H \\
\dot{P}_4 = 0.
\end{cases}
\]

(20)

Example 2.10 (3-step Carnot groups). For the general 3-step case from (15) we get

\[
\begin{cases}
\dot{x} = P_H \\
\dot{P}_H = -C_H(P_{H_2})P_H \\
\dot{P}_{H_2} = -C(P_{H_3})P_H \\
\dot{P}_{H_3} = 0,
\end{cases}
\]

(21)

where \(P_{H_2} := \sum_{\alpha \in I_{H_2}} P_\alpha X_\alpha, \ P_{H_3} := \sum_{\alpha \in I_{H_3}} P_\alpha X_\alpha\). We also remind that

\[C_H(P_{H_2}) = \sum_{\alpha \in I_{H_2}} P_\alpha C_H^\alpha, \quad C(P_{H_3}) = \sum_{\alpha \in I_{H_3}} P_\alpha C^\alpha.\]

2.2 CC-normal geodesics, variational formulae and Jacobi fields

In this section we shall again discuss normal CC-geodesics and their minimizing properties. We first recall some results which can be found in \[62\].

If we look for minimizing geodesics joining two (fixed) points \(x, y \in \mathbb{G}\), we are solving the minimization problem (with fixed endpoints):

\[\min \int_a^b |\dot{x}_H| dt, \quad \dot{x}(t) \in H_{x(t)} \ \forall t \in [a,b], \quad x = x(a), \ y = x(b).\]

Notation 2.11. If \(Z \in \mathfrak{X}(\mathbb{G})\), then \(Z_H, Z_V\) denote the orthogonal projection of \(Z\) onto \(H\) and \(V\), respectively.

The Euler-Lagrange equations for this problem, written with respect to the global left invariant frame \(\mathfrak{X} = \{X_1, \ldots, X_n\}\), are\(^8\)

\[
\frac{d}{dt} L_x - L_{\dot{x}} = \sum_{\alpha \in I_V} \left(\dot{P}_\alpha \omega_\alpha - P_\alpha (\dot{x}_H \mathcal{J} d\omega_\alpha)\right)^\sharp
\]

\(^8\)Hereafter the symbols \(\flat\) and \(\sharp\) will be used to denote the so-called musical isomorphisms between vectors and co-vectors, defined in an obvious way with the help of the metric; see \[44\]. For the definition of the “contraction operator” \(\mathcal{J}\), see footnote \(\natural\).
where $L(t, x, \dot{x}) = |\dot{x}_H|$ and $P_\alpha (\alpha \in I_V)$ denote the $\alpha$-th Lagrange multiplier; see [62].

The horizontality for the curve $x$ is expressed by the equations $\omega_\alpha (\dot{x}) = 0 (\alpha \in I_V)$. These equations can be rewritten in invariant form by replacing the Lagrangian $L$ of the unconditional problem by $L_{SR} := L + \sum_{\alpha \in I_V} P_\alpha \omega_\alpha (\dot{x})$. Using the Levi-Civita connection related to the fixed (left invariant) Riemannian metric on $G$, the normal CC-geodesic equations are given (see [62]) by

$$\begin{cases}
\nabla_t \dot{x}_H + \sum_{\alpha \in I_V} \left( \dot{P}_\alpha \omega_\alpha - P_\alpha (\dot{x}_H \lrcorner \, d\omega_\alpha) \right) = 0 \\
\omega_\alpha (\dot{x}) = 0 \quad (\alpha \in I_V).
\end{cases}$$

(22)

Note that the second equation is again the horizontality condition which can equivalently be expressed by $\dot{x}_V = 0$. Setting $P_\alpha := \sum_{\alpha \in I_V} P_\alpha X_\alpha$ and $C (P_\alpha) := \sum_{\alpha \in I_V} P_\alpha C^\alpha$, the system (37) can be rewritten as

$$\begin{cases}
\nabla_t \dot{x}_H + \dot{P}_\alpha + C (P_\alpha) \dot{x}_H = 0 \\
\dot{x}_V = 0.
\end{cases}$$

(23)

To see this it is enough to express the right-hand side of the first equation in (37) using formula (9) for the exterior derivative $d\omega_\alpha (\alpha \in I_V)$. Note that the Lagrangian multiplier $P_\alpha$ can be regarded as a curve in the vertical subbundle $V$. Obviously, (15) and (23) coincide. This immediately follows by explicitly calculating the first equation in (23). More precisely, as in the Riemannian case, it turns out that

$$\nabla_t \dot{x}_H = \sum_{L=1}^{n} \left( \frac{d^2 x_L}{dt^2} + \sum_{i,j \in I_H} \Gamma^L_{ij} \dot{x}_i \dot{x}_j \right) X_L,$$

where $\Gamma^L_{ij} := \langle \nabla X_i, X_j, X_L \rangle$ denote Christoffel Symbols. Using formula (8) together with condition (6) on the Carnot structure constants, we get that $\Gamma^L_{ij} = \frac{1}{2} C^L_{ij} = 0$ whenever $L \in I_H (i, j \in I_H)$. Moreover, by skew-symmetry of the structure constants $C^L_{ij}$, for every $L \in I_V$ we get that $\sum_{i,j \in I_H} \Gamma^L_{ij} \dot{x}_i \dot{x}_j = 0$. Therefore $\dot{x}_H = \nabla_t \dot{x}_H$. Setting $P_H := \dot{x}_H$ and projecting (23) onto $H$ and $V$, respectively, one gets

$$\begin{cases}
\dot{x} = \dot{x}_H = P_H \\
\dot{x}_V = 0 \\
\dot{P}_H = \dot{x}_H = \nabla_t \dot{x}_H = -C_H (P_V) P_H \\
\dot{P}_V = -C (P_V) P_H
\end{cases}$$

(24)

which is the projected form of (15), as we wished to prove.

As already said in the previous discussion, the normal CC-geodesic equations can be directly deduced by minimizing the constrained Lagrangian

$$L_{SR} (t, x, \dot{x}) = |\dot{x}_H| + \langle P_V, \dot{x} \rangle = |\dot{x}_H| + \sum_{\alpha \in I_V} P_\alpha \omega_\alpha (\dot{x}).$$

---

9See [11], [38], [39] for a classical setting, or [41] for a discussion of geodesics equations for Nonholonomic geometries.

10Actually, by using again [6], $C^L_{ij}$ can be different from 0 only if $L$ belong to $I_{H^2}$. 

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This is equivalent to compute the so-called 1st variation of the functional

\[ I_{SR}(x) := \int_{0}^{b} L_{SR}(t, x, \dot{x}) dt. \]

For this reason and in order to explicitly computing the 2nd variation of \( I_{SR}(x) \), we need preliminarily the next:

**Definition 2.12.** If \( x : [a, b] \rightarrow \mathbb{G} \) is a smooth path, we call variation of \( x \) any smooth mapping \( \vartheta : [a, b] \times ]-\epsilon_{0}, \epsilon_{0}[ \rightarrow \mathbb{G} \), \( \epsilon_{0} > 0 \), for which \( x(t) = \vartheta(t, 0) \) for all \( t \in [a, b] \).

We say that \( \vartheta \) fixes endpoints if \( x(a) = \vartheta(a, s), \ x(b) = \vartheta(b, s) \) for all \( s \in ]-\epsilon_{0}, \epsilon_{0}[ \). In this case we say that \( \vartheta \) is a homotopy of \( x \). Moreover, if for every \( s \in ]-\epsilon_{0}, \epsilon_{0}[ \) the path \( \vartheta_{s} : [a, b] \rightarrow \mathbb{G}, \ \vartheta_{s}(t) = \vartheta(t, s) \), is a normal CC-geodesic, we say that \( \vartheta \) is a sub-Riemannian or CC-geodesic variation of \( x \).

In the sequel we shall write \( \partial_{t}, \partial_{s} \) for \( \vartheta_{s}(\partial_{t}), \vartheta_{t}(\partial_{s}) \) respectively, and denote covariant differentiation of vector fields along \( \vartheta \) with respect to \( \partial_{t}, \partial_{s} \) by \( \nabla_{t}, \nabla_{s} \) respectively. Furthermore, let us recall the following general identities:

\[
\begin{align*}
\nabla_{s} \partial_{t} \vartheta &= \nabla_{t} \partial_{s} \vartheta \tag{25} \\
\nabla_{s} \nabla_{t} - \nabla_{t} \nabla_{s} &= R(\partial_{t} \vartheta, \partial_{s} \vartheta), \tag{26}
\end{align*}
\]

where \( R \) is the Riemannian curvature tensor of \( \mathbb{G} \).

**Proposition 2.13** (1st variation of \( I_{SR}(x) \)). Let us assume that \( x : [a, b] \rightarrow \mathbb{G} \) is a differentiable path and let \( \vartheta : [a, b] \times ]-\epsilon_{0}, \epsilon_{0}[ \rightarrow \mathbb{G} \) be a differentiable variation of \( x \). Then

\[
\delta^{1} I_{SR} := \left. \frac{d}{ds} I_{SR}(\vartheta_{s}) \right|_{s=0} = \left\langle \partial_{s} \vartheta|_{s=0}, \left( \frac{\partial x}{\partial t|x}, P_{v} \right) \right\rangle \bigg|_{a}^{b} - \int_{a}^{b} \left\{ \left\langle \partial_{s} \vartheta|_{s=0}, \left[ \nabla_{t} \frac{\partial x}{\partial t|x} + \partial_{t} P_{v} + C(P_{v}) \partial_{t} x \right] \right\rangle \right\} dt. \tag{27}
\]

Let us assume that \( |\partial_{t} x| = 1 \) and let \( \vartheta \) be a homotopy of \( x \). Setting

\[
Y(t) := \partial_{s} \vartheta(t, 0), \quad Q_{v}(t) := \partial_{s} P_{v}(t, 0)
\]

we get that

\[
\delta^{1} I_{SR} = \int_{a}^{b} \left\{ \left\langle Q_{v}, \partial_{t} x \right\rangle - \left\langle Y, \left[ \nabla_{t} \partial_{s} x + \partial_{t} P_{v} + C(P_{v}) \partial_{t} x \right] \right\rangle \right\} dt. \tag{28}
\]

Note that (23) immediately follows from the previous result once we require that the path \( x \) is an extremal of the functional \( I_{SR}(x) \). Indeed, in such a case, the fist term under integral sign in (28) must vanish for every \( Y \), which gives the first equation in (23). The same procedure implies that the second term under integral sign vanishes for every \( Q_{v} \), which in turn implies the horizontality condition for \( x \).
Proof of Proposition 2.13. First we have to compute
\[
\partial_s \int_a^b \left\{ |\partial_t \vartheta| + \langle P_V, \partial_t \vartheta \rangle \right\} dt = \partial_s \int_a^b |\partial_t \vartheta| dt + \partial_s \int_a^b \langle P_V, \partial_t \vartheta \rangle dt.
\]
Actually, the first term \(A_1\) can be directly deduced from the Riemannian computation (see, for instance, [11], Theorem 2.3) and, with our notation, we get that
\[
A_1 = \left. \left\{ \langle \partial_s \vartheta, \frac{\partial_x}{|\partial_s x|} \rangle \right\} \right|_a^b - \int_a^b \left\langle \partial_s \vartheta, \nabla \frac{\partial_x}{|\partial_s x|} \right\rangle dt.
\]
So we have to compute the other term. We have
\[
A_2 = \partial_s \int_a^b \langle P_V, \partial_t \vartheta \rangle dt = \int_a^b \partial_s \langle P_V, \partial_t \vartheta \rangle dt
\]
\[
= \int_a^b \left\{ \langle \partial_s P_V + \nabla_{\partial_t \vartheta} P_V, \partial_t \vartheta \rangle + \langle P_V, \nabla_t \partial_t \vartheta \rangle \right\} dt
\]
\[
= \int_a^b \left\{ \langle \partial_s P_V, \partial_t \vartheta \rangle + \langle \nabla_{\partial_t \vartheta} P_V, \partial_t \vartheta \rangle + \partial_t \langle P_V, \partial_t \vartheta \rangle - \langle \nabla_t P_V, \partial_t \vartheta \rangle \right\} dt
\]
\[
= \int_a^b \left\{ \langle \partial_s P_V, \partial_t \vartheta \rangle + \langle \nabla_{\partial_t \vartheta} P_V, \partial_t \vartheta \rangle + \partial_t \langle P_V, \partial_t \vartheta \rangle - \langle \partial_t P_V + \nabla_{\partial_t \vartheta} P_V, \partial_t \vartheta \rangle \right\} dt
\]
\[
= \langle \partial_t \vartheta, P_V \rangle \big|_a^b + \int_a^b \left\{ \langle \partial_s P_V, \partial_t \vartheta \rangle - \langle \partial_t P_V, \partial_t \vartheta \rangle + \langle \nabla_{\partial_t \vartheta} P_V, \partial_t \vartheta \rangle - \langle \nabla_t P_V, \partial_t \vartheta \rangle \right\} dt.
\]
We claim that \(B = \langle [\partial_s \vartheta, \partial_t \vartheta], P_V \rangle\). To prove this claim we may proceed by using a well-known formula for the Riemannian connection which can be found in [11] (see formula 1.29, p.15). This way we get
\[
B = \langle \nabla_{\partial_t \vartheta} P_V, \partial_t \vartheta \rangle - \langle \nabla_t P_V, \partial_t \vartheta \rangle
\]
\[
= \frac{1}{2} \left\{ \langle [\partial_s \vartheta, P_V], \partial_t \vartheta \rangle \right\} - \langle [P_V, \partial_t \vartheta], \partial_t \vartheta \rangle + \langle [\partial_s \vartheta, \partial_t \vartheta], P_V \rangle + \langle [\partial_s \vartheta, \partial_t \vartheta], P_V \rangle
\]
\[
= \langle [\partial_s \vartheta, \partial_t \vartheta], P_V \rangle.
\]
Moreover, it is easy to show that \(\langle [\partial_s \vartheta, \partial_t \vartheta], P_V \rangle = -\langle C(P_V) \partial_x, \partial_s \vartheta \rangle\). Indeed one has
\[
\langle [\partial_s \vartheta, \partial_t \vartheta], P_V \rangle = \sum_{I,J=1}^n \sum_{\alpha \in I_V} (\partial_s \vartheta)_I (\partial_t \vartheta)_J p_\alpha \langle [X_I, X_J], X_\alpha \rangle = \sum_{I,J=1}^n \sum_{\alpha \in I_V} C^\alpha_{IJ} (\partial_s \vartheta)_I (\partial_t \vartheta)_J p_\alpha
\]
and the claim immediately follows by using (ii) of Definition 2.5. So we have shown that
\[
A_2 = \langle \partial_t \vartheta, P_V \rangle \big|_a^b + \int_a^b \left\{ \langle \partial_s P_V, \partial_t \vartheta \rangle - \langle \partial_t P_V + C(P_V) \partial_t \vartheta, \partial_s \vartheta \rangle \right\} dt.
\]
By adding \(A_1\) and \(A_2\) (27) claim easily follows. Finally, (28) follows from (27).
**Theorem 2.14** (2nd Variation of $I_{sr}(x)$). Under the notation of Proposition 2.13 let $x$ be a normal CC-geodesic satisfying $|\partial_t x| = 1$, and assume that $\vartheta$ is a homotopy of $x$. Moreover, set $Y(t) := \partial_s \vartheta(t, 0)$ and $Q_v(t) := \partial_s P_v(t, 0)$. Then we have

$$\delta^2 I_{sr} = \int_a^b \left\{ 2 \left\langle Q_v, \left[ \nabla_t Y - \frac{3}{4} [Y, \partial_t x] - \frac{1}{4} C(Y) \partial_t x \right] \right\rangle ight. - \left\langle Y, \left( \nabla_t^2 Y + C(P_v)(\nabla_t Y + [Y, \partial_t x]) + [Y, \partial_t x] + R(\partial_t x, Y) \partial_t x \right) \right\rangle \right\} dt \quad (30)$$

**Proof.** Since, by hypothesis, $x$ is a normal CC-geodesic and $\vartheta$ is a homotopy of $x$, we may start by considering the following identity:

$$\frac{d}{ds} I_{sr}(\vartheta_s) = - \int_a^b \left\{ \left\langle \partial_s \vartheta, \left[ \nabla_t \partial_t \vartheta_h + \partial_t P_v + C(P_v) \partial_t \vartheta_h \right] \right\rangle - \left\langle \partial_s P_v, \partial_t \vartheta \right\rangle \right\} dt. \quad (31)$$

This identity can easily be deduced by the proof of Proposition 2.13. So we have

$$\frac{d^2}{ds^2} I_{sr}(\vartheta_s) = \left. -\partial_s \int_a^b \left\{ \left\langle \partial_s \vartheta, \left[ \nabla_t \partial_t \vartheta_h + \partial_t P_v + C(P_v) \partial_t \vartheta_h \right] \right\rangle - \left\langle \partial_s P_v, \partial_t \vartheta \right\rangle \right\} dt \right. \quad (32)$$

$$= - \int_a^b \left\{ \partial_s \left\langle \partial_s \vartheta, \left[ \nabla_t \partial_t \vartheta_h + \partial_t P_v + C(P_v) \partial_t \vartheta_h \right] \right\rangle - \partial_s \left\langle \partial_s P_v, \partial_t \vartheta \right\rangle \right\} dt \quad (33)$$

$$= - \int_a^b \left\{ \left\langle \nabla_s \partial_s \vartheta, \left[ \nabla_t \partial_t \vartheta_h + \partial_t P_v + C(P_v) \partial_t \vartheta_h \right] \right\rangle \right. \quad (34)$$

$$+ \left. \left\langle \partial_s \vartheta, \nabla_s \left[ \nabla_t \partial_t \vartheta_h + \partial_t P_v + C(P_v) \partial_t \vartheta_h \right] \right\rangle - \partial_s \left\langle \partial_s P_v, \partial_t \vartheta \right\rangle \right\} dt. \quad (35)$$

Now it is obvious that, at $s = 0$, one has $A_1 = 0$ because $x$ is assumed to be a normal CC-geodesic.

**Step 1.** (Computation of $A_2$) We have

$$\nabla_s \left( \nabla_t \partial_t \vartheta_h + \partial_t P_v + C(P_v) \partial_t \vartheta_h \right) \bigg|_{s=0} = \nabla_t^2 Y + [Y, \partial_t x] + R(\partial_t x, Y) \partial_t x + \nabla_t Q_v + C(Q_v) \partial_t x + C(P_v) \left( \nabla_t Y + [Y, \partial_t x] \right).$$

**Proof.** This computation generalizes the classical deduction of Jacobi’s equation. We have

$$B := \nabla_s \left( \nabla_t \partial_t \vartheta_h + \partial_t P_v + C(P_v) \partial_t \vartheta_h \right) = \nabla_s \nabla_t \partial_t \vartheta_h \quad (36)$$

$$\quad + \nabla_s \partial_t P_v + \nabla_s \left( C(P_v) \partial_t \vartheta_h \right) \quad (37)$$

The first term $B_1$ can be computed in analogy with the Riemannian case; see [20], p.111. More precisely, by (26) we get

$$B_1 = \nabla_s \nabla_t (\partial_t \vartheta)_h - \nabla_t \nabla_s (\partial_t \vartheta)_h - R(\partial_s \vartheta, \partial_t \vartheta) (\partial_t \vartheta)_h.$$

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At this point we have to compute the term $\nabla_s(\partial_t \vartheta)_u$. Note that this must be done in a different way with respect to the Riemannian case because the term $\partial_t \vartheta$ is not horizontal, a priori, and so it may be $\partial_t \vartheta \neq \partial_t \vartheta_u$. We claim that

$$\nabla_s(\partial_t \vartheta)_u = \nabla_t(\partial_s \vartheta)_u + [\partial_s \vartheta, (\partial_t \vartheta)_u]. \tag{32}$$

Indeed we have

$$\nabla_s(\partial_t \vartheta)_u = \nabla_s(\mathcal{P}_u(\partial_t \vartheta)) = \partial_s(\mathcal{P}_u(\partial_t \vartheta)) + \nabla_{\partial_t \vartheta} \mathcal{P}_u(\partial_t \vartheta)$$

$$= \partial_t(\mathcal{P}_u(\partial_s \vartheta)) + \sum_{j \in I_u} \sum_{I=1}^n (\partial_t \vartheta)_j(\partial_s \vartheta)_I(\nabla_{X_I} X_J)|_{\vartheta}$$

$$= \partial_t(\mathcal{P}_u(\partial_s \vartheta)) + \sum_{\alpha \in I_V} \left((C^\alpha \mathcal{P}_u(\partial_t \vartheta), \partial_s \vartheta)_\alpha \right)_{\vartheta}$$

$$= \nabla_t(\partial_s \vartheta)_u + \sum_{\alpha \in I_V} (C^\alpha(\partial_t \vartheta)_u, \partial_s \vartheta)_\alpha$$

and (32) follows. Therefore we obtain

$$B_1 = \nabla_s \nabla_t(\partial_t \vartheta)_u = \nabla_t \nabla_t(\partial_s \vartheta)_u + [\partial_s \vartheta, (\partial_t \vartheta)_u] + R(\partial_t \vartheta, \partial_s \vartheta)(\partial_t \vartheta)_u. \tag{33}$$

**Remark 2.15.** In what above, we have used the identity

$$\partial_s(\mathcal{P}_u(\partial_t \vartheta)) = \partial_t(\mathcal{P}_u(\partial_s \vartheta)).$$

In order to prove this identity, we may argue componentwise. More precisely, for $i \in I_u$ we compute

$$\partial_s(\partial_t \vartheta, X_i) = \langle \partial_s \partial_t \vartheta, X_i \rangle + \langle \partial_t \vartheta, \partial_s X_i \rangle = \langle \partial_s \partial_t \vartheta, X_i \rangle + \langle \partial_t \vartheta, J X_i \partial_s \vartheta \rangle$$

$$\partial_t(\partial_s \vartheta, X_i) = \langle \partial_t \partial_s \vartheta, X_i \rangle + \langle \partial_s \vartheta, \partial_t X_i \rangle = \langle \partial_t \partial_s \vartheta, X_i \rangle + \langle \partial_s \vartheta, J X_i \partial_t \vartheta \rangle.$$

By subtracting the second identity from the first one, we easily get that

$$A_i := \partial_s(\partial_t \vartheta, X_i) - \partial_t(\partial_s \vartheta, X_i) = \langle \partial_t \vartheta, [J X_i - (J X_i)^T] \partial_s \vartheta \rangle.$$

From the last relation we infer that

$$A_i = \sum_{I, J=1}^n \langle [J X_i - (J X_i)^T] X_I, X_J \rangle (\partial_s \vartheta)_I(\partial_t \vartheta)_J.$$

Now since

$$\langle J X_i X_I, X_J \rangle = -\langle \nabla_{X_J} X_I, X_i \rangle,$$

$$\langle (J X_i)^T X_I, X_J \rangle = \langle (J X_i)^T X_I, X_J \rangle = -\langle \nabla_{X_J} X_I, X_i \rangle,$$
we get that
\[ A_i = \sum_{I,J=1}^{n} \langle \nabla X_I, X_J \rangle (\partial_s \vartheta)_I (\partial_t \vartheta)_J = \sum_{I,J=1}^{n} C^i_{IJ} (\partial_s \vartheta)_I (\partial_t \vartheta)_J = 0 \]
for every \( i \in I_n \), which implies the claim.

**Remark 2.16.** If the variation \( \vartheta \) is a CC-geodesic variation of \( x \), the term \( B_1 \) can be computed as in the Riemannian case; see [20], p.111. Indeed, in such a case we have \( \partial_t \vartheta = \mathcal{P}_u (\partial_t \vartheta) \) for every \( s \in ]-\epsilon, \epsilon[ \). Using (25) and (26) yields
\[ B_1 = \nabla_s \nabla_t (\partial_t \vartheta)_u = \nabla_s \nabla_t (\partial_t \vartheta) = \nabla_t \nabla_s (\partial_t \vartheta) - R(\partial_s \vartheta, \partial_t \vartheta) \partial_t \vartheta \]
\[ = \nabla_t \nabla_s (\partial_t \vartheta) + R(\partial_t \vartheta, \partial_s \vartheta) \partial_t \vartheta. \]
This can also be deduced from (33), by noting that
\[ [\partial_s \vartheta, (\partial_t \vartheta)_u] = [\partial_s \vartheta, \partial_t \vartheta] = \partial_s [\partial_s, \partial_t] = 0, \]
because \( [\partial_s, \partial_t] = 0 \).

The term \( B_2 \) can be computed by means of (25) and we immediately get that
\[ B_2 = \nabla_s \partial_t P_V = \nabla_t \partial_s P_V. \quad (35) \]

Analogously, the term \( B_3 \) can be computed by means of (25), (26) and (32), as follows:
\[ B_3 = \nabla_s (\mathcal{P}_V (\partial_t \vartheta)_u) = \nabla_s \left( \sum_{\alpha \in I_V} P_\alpha C^\alpha \partial_t \vartheta_u \right) = \sum_{\alpha \in I_V} (\partial_s P_\alpha C^\alpha \partial_t \vartheta_u + P_\alpha C^\alpha \nabla_s \partial_t \vartheta_u) \]
\[ = C(\partial_s P_\alpha) \partial_t \vartheta_u + C(\mathcal{P}_V) \nabla_s \partial_t \vartheta_u = C(\partial_s P_\alpha) \partial_t \vartheta_u + C(\mathcal{P}_V) (\nabla_t (\partial_s \vartheta)_u + [\partial_s \vartheta, (\partial_t \vartheta)_u]). \]

Finally, by adding the terms \( B_1, B_2, B_3 \), we get
\[ B = \nabla_t \nabla_s (\partial_t \vartheta)_u + [\partial_s \vartheta, (\partial_t \vartheta)_u] + R(\partial_t \vartheta, \partial_s \vartheta) (\partial_t \vartheta)_u + \nabla_t \partial_s P_V \]
\[ + C(\partial_s P_\alpha) \partial_t \vartheta_u + C(\mathcal{P}_V) (\nabla_t (\partial_s \vartheta)_u + [\partial_s \vartheta, (\partial_t \vartheta)_u]). \]

The thesis follows by substituting \( Y(t) = \partial_s \vartheta(t, 0) \) and \( Q_V(t) = \partial_s P_V (t, 0) \) into the last expression.

**Step 2.** We have \( \left\{ \int_a^b A_3 dt \right\} |_{s=0} = - \int_a^b \langle Y, [\partial_t Q_V + C(Q_V) \partial_t x] \rangle dt \).

**Proof.** By arguing as in Proposition 2.13, we get that
\[ \int_a^b A_3 dt = \langle \partial_s P_V, \partial_s \vartheta \rangle |_a^b + \int_a^b \left\{ \langle \frac{\partial^2 P_V}{\partial s^2}, \partial_t \vartheta \rangle - \langle \partial_s \vartheta, [C(\partial_s P_V) \partial_t x + \partial_t \partial_s P_V] \rangle \right\} dt. \]

Now we consider this expression at \( s = 0 \) and we make the substitutions: \( \partial_t \vartheta|_{s=0} = \partial_t x, \partial_s \vartheta|_{s=0} = Y \) and \( \partial_s P_V |_{s=0} = Q_V \). By hypothesis, \( \vartheta \) is a homotopy. Hence
\[ \langle Q_V, Y \rangle |_a^b = \langle \partial_s P_V |_{s=0}, \partial_s \vartheta |_{s=0} \rangle |_a^b = 0. \]

Set \( s = 0 \). Then \( \left\langle \frac{\partial^2 P_V}{\partial s^2}, \partial_t \vartheta \right\rangle |_{s=0} = \left\langle \frac{\partial^2 P_V}{\partial s^2} |_{s=0}, \partial_t x \right\rangle = 0 \) and the thesis follows. \( \square \)
Using Step 1 and Step 2, we get that
\[
\delta^2 I_{\text{sr}} = \frac{d^2}{ds^2} I_{\text{sr}}(\partial_s) \bigg|_{s=0} = -\int_a^b \left\{ \left\langle Y, \left( \nabla_t^{(2)} Y_x + [Y, \partial_t x] + R(\partial_t x, Y) \partial_t x + \nabla_t Q_v + C(Q_v) \partial_t x + C(P_v) \left( \nabla_t Y_x + [Y, \partial_t x] \right) \right) \right\rangle \right\} dt.
\]
Since \( \nabla_t Q_v = \partial_t Q_v + \nabla_{\partial_t x} Q_v \), it follows that
\[
\delta^2 I_{\text{sr}} = -\int_a^b \left\{ \left\langle Y, \left[ \nabla_t^{(2)} Y_x + [Y, \partial_t x] + R(\partial_t x, Y) \partial_t x + 2 \left( \nabla_t Q_v + C(Q_v) \partial_t x - \frac{1}{2} \nabla_{\partial_t x} Q_v \right) + C(P_v) \left( \nabla_t Y_x + [Y, \partial_t x] \right) \right] \right\rangle \right\} dt.
\]
By reordering a bit the last expression we easily get that
\[
\delta^2 I_{\text{sr}} = -\int_a^b \left\{ \left\langle Y, \left[ \nabla_t^{(2)} Y_x + C(P_v) \left( \nabla_t Y_x + [Y, \partial_t x] \right) + [Y, \partial_t x] + R(\partial_t x, Y) \partial_t x \right] \right\rangle \right\} dt. \tag{36}
\]

**Step 3.** The following identities hold true:

(i) \( \langle \nabla_{\partial_t x} Q_v, Y \rangle = \frac{1}{2} \left( \langle C(Y_v) Q_v, \partial_t x \rangle + \langle C(Q_v) \partial_t x, Y \rangle \right) \);

(ii) \( \langle C(Q_v) \partial_t x, Y \rangle = -\langle [\partial_t x, Y], Q_v \rangle \).

**Proof.** To prove the first identity we compute
\[
\langle \nabla_{\partial_t x} Q_v, Y \rangle = \sum_{i \in I_H} \sum_{\alpha \in I_V} \sum_{J=1}^n (\partial_t x)_i Q_{i\alpha} Y_J \langle \nabla x, X_{\alpha}, X_J \rangle
\]
\[
= \frac{1}{2} \sum_{i \in I_H} \sum_{\alpha \in I_V} \sum_{J=1}^n (\partial_t x)_i Q_{i\alpha} Y_J \left( C_{i\alpha}^J - \underbrace{C_{i\alpha}^J}_{=0 \text{ by } \Box} + C_{\alpha J}^i \right)
\]
\[
= \frac{1}{2} \left( \langle C(Y_v) Q_v, \partial_t x \rangle + \langle C(Q_v) \partial_t x, Y \rangle \right),
\]
which proves (i), while (ii) follows from Definition [2.5] and Lemma [1.12].

Using Step 3 and integrating by parts, yields
\[
\int_a^b \left\langle Y, \left[ \nabla_t Q_v + C(Q_v) \partial_t x - \frac{1}{2} \nabla_{\partial_t x} Q_v \right] \right\rangle dt
\]
\[
= \langle Y, Q_v \rangle \bigg|^b_a - \langle \nabla_t Y, Q_v \rangle - \langle [\partial_t x, Y], Q_v \rangle - \frac{1}{4} \left( \langle C(Y_v) Q_v, \partial_t x \rangle + \langle C(Q_v) \partial_t x, Y \rangle \right) dt
\]
Finally, by using the last expression and (36), we get that

\[ \delta^2 I_{SR} = - \int_a^b \left\{ Q_v \left[ \nabla_t Y + \frac{3}{4} [\partial_t x, Y] - \frac{1}{4} C(Y_v) \partial_t x \right] \right\} dt, \]

which is equivalent to the thesis.

Accordingly with the Hamiltonian theory already discussed in Section 2.1, we set

\[ P_H := \partial_t x. \]

Starting from the second variation formula (30), it is natural to consider the following system of O.D.E.’s:

\[
\begin{aligned}
\nabla_t^{(2)} Y_H + C(P_v) (\nabla_t Y_H + [Y, \partial_t x]) + [Y, P_H] + R(\partial_t x, Y) P_H = 0 \\
\mathcal{P}_v (\nabla_t Y - \frac{3}{4} [Y, \partial_t x] - \frac{1}{4} C(Y_v) \partial_t x) &= 0.
\end{aligned}
\]

(37)

**Corollary 2.17** (2nd derivative of \( I_{SR}(x) \) through CC-geodesic variations). **Under the notation of Proposition 2.13** let \( x \) be a normal CC-geodesic satisfying \( |\partial_t x| = 1 \), and let us assume that:

(i) \( \vartheta \) is a homotopy of \( x \);

(ii) \( \vartheta \) is a CC-geodesic variation of \( x \).

Moreover, set \( Y(t) := \partial_s \vartheta(t, 0) \) and \( Q_v(t) := \partial_s P_v(t, 0) \). Then

\[ \delta^2 I_{SR} = \int_a^b \left\{ \left\langle Q_v , \left( \nabla_t Y + [\partial_t x, Y] \right) \right\rangle - \left\langle Y , \left( \nabla_t^{(2)} Y + R(\partial_t x, Y) \partial_t x + C(P_v) \nabla_t Y \right) \right\rangle \right\} dt. \]

(38)

**Proof.** The proof follows the same lines of the general case. Since, by hypothesis, \( x \) is a normal CC-geodesic and the CC-geodesic variation \( \vartheta \) is also a homotopy of \( x \), we may start by considering the identity

\[ \frac{d}{ds} I_{SR}(\partial_s) = - \int_a^b \left\{ \partial_s \partial_t \left[ \nabla_t \partial_t \vartheta + \partial_t P_v + C(P_v) \partial_t \vartheta \right] \right\} dt. \]

(39)
Indeed note that \( \partial_t \vartheta = \partial_t \vartheta_t \) for every \((t, s) \in [a, b] \times [-\epsilon_0, \epsilon_0] \). This identity can easily be deduced, as in the proof of Proposition 2.13, by using the hypotheses on \( x \) and \( \vartheta \). Hence

\[
\frac{d^2}{ds^2} I_{sr} (\vartheta_s) = - \partial_s \int_a^b \left\{ \left( \partial_t \vartheta_s, \left[ \nabla_t \partial_t \vartheta_s + \partial_t P_{\vartheta} + C(P_{\vartheta}) \partial_t \vartheta_s \right] \right) \right\} dt
\]

\[
= - \int_a^b \left\{ \partial_s \left( \partial_t \vartheta_s, \left[ \nabla_t \partial_t \vartheta_s + \partial_t P_{\vartheta} + C(P_{\vartheta}) \partial_t \vartheta_s \right] \right) \right\} dt
\]

\[
= - \int_a^b \left\{ \left( \nabla_s \partial_s \vartheta_s, \left[ \nabla_t \partial_t \vartheta_s + \partial_t P_{\vartheta} + C(P_{\vartheta}) \partial_t \vartheta_s \right] \right) \right\} dt.
\]

Now it is obvious that, at \( s = 0 \), one has \( A_1 = 0 \), because \( x \) is a normal CC-geodesic. So we need to prove the following fact:

**Step 1.** (Computation of \( A_2 \)) One has

\[
\nabla_s \left( \nabla_t \partial_t \vartheta_s + \partial_t P_{\vartheta} + C(P_{\vartheta}) \partial_t \vartheta_s \right) \big|_{s=0} = \nabla_t(2) Y + R(\partial_t x, Y) \partial_t x + \nabla_t Q_{\vartheta} + C(Q_{\vartheta}) \partial_t x + C(P_{\vartheta}) \nabla_t Y.
\]

**Proof.** The proof mimics that of Theorem 2.14. We have

\[
B := \nabla_s \left( \nabla_t \partial_t \vartheta_s + \partial_t P_{\vartheta} + C(P_{\vartheta}) \partial_t \vartheta_s \right) = \nabla_s \nabla_t \partial_t \vartheta_s + \nabla_s \partial_t P_{\vartheta} + \nabla_s \left( C(P_{\vartheta}) \partial_t \vartheta_s \right).
\]

The first term \( B_1 \) can be computed exactly as in the Riemannian case (see [20], p.111 or the previous Remark 2.16). More precisely, by using (25) and (26), we get

\[
B_1 = \nabla_s \nabla_t \partial_t \vartheta_s = \nabla_t \nabla_s \partial_t \vartheta_s - R(\partial_s \vartheta, \partial_t \vartheta)(\partial_t \vartheta).
\]

Also \( B_2 \) can be computed exactly as in the previous proof of Theorem 2.14 (see Step 1) and we get that

\[
B_2 = \nabla_s \partial_t P_{\vartheta} = \nabla_t \partial_s P_{\vartheta}.
\]

Analogously, for the term \( B_3 \) we get

\[
B_3 = \nabla_s \left( C(P_{\vartheta}) \partial_t \vartheta_s \right) = \nabla_s \left( \sum_{\alpha \in I_{\vartheta}} P_{\alpha} C^\alpha \partial_t \vartheta_s \right) = \sum_{\alpha \in I_{\vartheta}} \left( \partial_s P_{\alpha} C^\alpha \partial_t \vartheta_s + \partial_s P_{\alpha} C^\alpha \nabla_s \partial_t \vartheta_s \right)
\]

\[
= C(\partial_s P_{\vartheta}) \partial_t \vartheta_s + C(P_{\vartheta}) \nabla_s \partial_t \vartheta_s = C(\partial_s P_{\vartheta}) \partial_t \vartheta_s + C(P_{\vartheta}) \nabla_t (\partial_s \vartheta).
\]

Finally, by adding the terms \( B_1, B_2 \) and \( B_3 \) we have

\[
B = \nabla_t \nabla_t (\partial_s \vartheta) + R(\partial_t \vartheta, \partial_s \vartheta)(\partial_t \vartheta) + \nabla_t \partial_s P_{\vartheta} + C(\partial_s P_{\vartheta}) \partial_t \vartheta_s + C(P_{\vartheta}) \nabla_t (\partial_s \vartheta).
\]

The thesis follows by substituting \( Y(t) = \partial_t \vartheta(t, 0) \) and \( Q_{\vartheta}(t) = \partial_s P_{\vartheta}(t, 0) \) into the last expression. □
By what proved in Step 1 we get that
\[ \delta^2 I_{SR} = \left. \frac{d^2}{ds^2} I_{SR} (\vartheta_s) \right|_{s=0} = - \int_a^b \left\{ \left\langle Y, \left( \nabla_t^2 Y + R(\partial_t x, Y) \partial_t x + \nabla_t Q_v + C(Q_v) \partial_t x + C(P_v) \nabla_t Y \right) \right\rangle \right\} \, dt. \]

Now, by arguing again as in proof of Theorem 2.14, we obtain
\[ \int_a^b \left\langle Y, \left( \nabla_t Q_v + C(Q_v) \partial_t x \right) \right\rangle \, dt = \left. \left\langle Y, Q_v \right\rangle \right|_a^b - \int_a^b \left\langle \left( \nabla_t Y + [\partial_t x, Y] \right), Q_v \right\rangle \, dt \]
and since \( \vartheta \) is a homotopy of \( x \) we have \( \left. \left\langle Y, Q_v \right\rangle \right|_a^b = 0 \). Putting all together, (38) follows.

As before (see (37)), by setting \( P_H := \partial_t x \) and using (38), we deduce the following O.D.E.’s system:
\[
\begin{align*}
\nabla_t^2 Y + R(P_H, Y) P_H + C(P_v) \nabla_t Y &= 0 \\
P_v (\nabla_t Y - [Y, P_H]) &= 0.
\end{align*}
\]

**Definition 2.18** (Jacobi equations for normal CC-geodesics). We say that (40) represents the system of Jacobi equations for normal CC-geodesics. Let \( x : [0, a] \rightarrow G \) be a normal CC-geodesic satisfying \( |\partial_t x| = 1 \). Then we say that a vector field \( J \in \mathfrak{X}(G) \) is a Jacobi field along the normal CC-geodesic \( x \) if and only if \( J \) satisfies (40) for all \( t \in [0, a] \).

**Remark 2.19.** An interesting corollary of the second variation formula can be formulated by considering only some particular CC-geodesic variations. Indeed, we could consider variations of a given normal CC-geodesic \( x \), or, more precisely \( (x, P_v) \), through normal CC-geodesics \( \vartheta_s \) such that, for every \( s \in ]-\epsilon_0, \epsilon_0[ \) one has \( Q_v = 0 \). Roughly speaking, the variation \( \vartheta \) is chosen in such a way that \( \vartheta_s \) satisfies (23) for each \( s \in ]-\epsilon_0, \epsilon_0[ \), with the same \( P_v \). By making this substitution into (38) we obtain the following formula:
\[ \delta^2 I_{SR} = - \int_a^b \left\langle Y, \left( \nabla_t^2 Y + R(\partial_t x, Y) \partial_t x + C(P_v) \nabla_t Y \right) \right\rangle \, dt. \]

We therefore obtain the following 2nd order linear system of O.D.E.’s:
\[
\nabla_t^2 Y + R(P_H, Y) P_H + C(P_v) \nabla_t Y = 0.
\]

**Definition 2.20** (Jacobi equations with constant Lagrangian multiplier \( P_v \)). We say that (41) represent the system of Jacobi equations for normal CC-geodesics having the same Lagrangian multiplier \( P_v \). Furthermore, let \( x : [0, a] \rightarrow G \) be a normal CC-geodesic satisfying \( |\partial_t x| = 1 \) and having Lagrangian multiplier \( P_v \). Then we say that a vector field \( J \in \mathfrak{X}(G) \) is a Jacobi field along the normal CC-geodesic \( (x, P_v) \) if, and only if, \( J \) satisfies (41) for all \( t \in [0, a] \).
As in the classical setting, a Jacobi field is uniquely determined by its initial conditions: \( J(0), \nabla_t J(0) \). To see this, it is sufficient to develop either (40) or (41) along the left invariant frame \( \mathcal{X} = \{ X_1, \ldots, X_n \} \) by using the very definition of \( \nabla_t \) and the linearity of the curvature tensor \( R(\dot{x}, \cdot)\dot{x} \) (remind that \( \dot{x} = \dot{x}_u = P_u(x) \)). Indeed, one easily deduce that (41) is a linear system of O.D.E.’s of the 2nd order. Hence, for given initial conditions \( J(0), \nabla_t J(0) \), there exists a \( C^\infty \) solution of the system defined on \( [0, a] \). Thus there exist \( 2n \) linearly independent Jacobi fields along \( (x, P_V) \).

**Remark 2.21.** As in Riemannian Geometry, Jacobi equations for normal CC-geodesics and Jacobi equations with constant Lagrangian multiplier \( P_V \) -together with the related notions of Jacobi fields- are important tools and they can be used to analyze the sub-Riemannian exponential map and to perform a precise study of the conjugate and cut loci of a point. Nevertheless, we will not pursue this task here.

### 2.3 Sub-Riemannian exponential map and CC-spheres

The main references for this section are [58], [59], [62].

Starting from the system (15) for normal CC-geodesics, from the standard O.D.E.’s theory we easily obtain existence, uniqueness, regularity and smooth dependence on the initial data for small times. So let \( x_0 \in G \) be a fixed point. Then we shall denote by

\[
\exp_{\text{SR}}(x_0, P_0) : [0, r] \subset \mathbb{R} \to G
\]

the (unique) normal CC-geodesic starting from \( x_0 \) with initial condition \( P(0) = P_0 \) for some fixed vector \( P_0 = P_u(0) + P_V(0) \in H_{x_0} \oplus V_{x_0} \). Here \( r < r_{\max}(P_0) \) where \( r_{\max}(P_0) \) is the maximal time of existence and uniqueness of the solution \( x(t) = \exp_{\text{SR}}(x_0, P_0)(t) \) of (15). Actually, by standard “extendibility” results for O.D.E.’s, it can be shown that each solution of system (15) is globally defined\(^{11}\) on \( G \), i.e. \( r_{\max}(P_0) = +\infty \).

In order to define a sub-Riemannian equivalent to the ordinary exponential map we preliminarily state the following:

**Lemma 2.23** (Homogeneity). If the normal CC-geodesic \( x(t) = \exp_{\text{SR}}(x_0, P_0)(t) \) is defined on the interval \( [0, r] \), then the normal CC-geodesic \( \exp_{\text{SR}}(x_0, aP_0)(t), a > 0, \) is defined on the interval \( [0, \frac{r}{a}] \) and it turns out that

\[
\exp_{\text{SR}}(x_0, aP_0)(t) = \exp_{\text{SR}}(x_0, P_0)(at) \quad \text{for every } t \in \left[ 0, \frac{r}{a} \right].
\]

\(^{11}\)More generally, in the setting of sub-Riemannian manifolds, it can be shown that each solution of the system of normal CC-geodesic can be continued as long as \( x(t) \) remains in the base manifold, so blow-up in the dual variable \( P \) never occurs. This claim can easily be proved using the following:

**Lemma 2.22** ([61]). Let \( x(t) \) be any normal CC-geodesic defined for \( t \in [0, r] \) and assume that \( x(t) \) remains inside a compact subset of the base manifold. Then \( x(t) \) can be extended beyond \( t = r \).

**Proof.** The proof can be found in [61], Lemma 4.1. \( \square \)

This result can be used to canonically define a sub-Riemannian exponential map; see [61].
Proof. Let \((y, Q) : [0, \frac{r}{a}] \rightarrow T\mathbb{G}\) be the curve given by \((y(t), Q(t)) = (x(at), aP(at))\). Now we claim that \((y, Q)\) satisfies \(\text{[15]}\) with \(y(0) = x_0\) and \(Q_0 = Q(0) = aP_0\). Indeed, by the very definition of \(x(t)\), we get that \(\dot{y}(t) = a\dot{x}(at) = aP_\mu(at)\) and that
\[
\dot{Q}(t) = a^2 \dot{P}(at) = -a^2 C(P_\nu(at))P_\mu(at) = -C(Q_\nu(t))Q_\mu(t)
\]
for \(t \in [0, \frac{r}{a}]\) and the claim follows. So by uniqueness we get, in particular, that
\[
y(t) = \exp_{\text{SR}}(x_0, aP_0)(t) = x(at) = \exp_{\text{SR}}(x_0, P_0)(at)\quad \text{for every } t \in \left[0, \frac{r}{a}\right],\]
and the thesis follows. \(\square\)

Definition 2.24 (Sub-Riemannian exponential map). From now on we shall set
\[
\exp_{\text{SR}}(x_0, \cdot) : T_{x_0}\mathbb{G} \rightarrow \mathbb{G}, \quad \exp_{\text{SR}}(x_0, P_0) = \exp_{\text{SR}}(x_0, P_0)(1).
\]
The map \(\exp_{\text{SR}}(x_0, \cdot)\) is called the sub-Riemannian exponential map at \(x_0 \in \mathbb{G}\).

The sub-Riemannian exponential map parameterizes normal CC-geodesics. Note that every minimizing curve connecting \(x_0\) to a point of \(\mathbb{G} \setminus \exp(x_0, T_{x_0}\mathbb{G})\) is necessarily strictly abnormal.

The map \(\exp_{\text{SR}}(x_0, \cdot)\) plays in sub-Riemannian geometry a similar role with respect to the ordinary exponential map in Riemannian geometry. Nevertheless, there are many differences and its structure is much more complicated. An important difference is that \(\exp_{\text{SR}}(x_0, \cdot)\) is not a diffeomorphism on any neighborhood of the origin in \(H_{x_0} \oplus V_{x_0}\). To see this it is enough to choose \(P_0 \in V_{x_0}\) (i.e. \(P_\mu(0) = 0\)); in this case \(\exp_{\text{SR}}(x_0, P_0) = x_0\). Furthermore, in any arbitrarily small neighborhood of the origin in \(H_{x_0} \oplus V_{x_0}\) there are points \(P_0 = P_\mu(0) + P_\nu(0)\) with \(P_\mu(0) \neq 0\) at which the rank of \(d\exp_{\text{SR}}(x_0, P_0)\) is not maximal. We shall discuss some of these facts later on.

We begin by stating the notion of sub-Riemannian wave front.

Let \(UH\) denote the set of all unit horizontal vector of \(H\), i.e. \(UH \subset H \cong \mathbb{S}^{h-1}\) and define the map
\[
\tilde{\gamma} : H \setminus \{0\} \oplus V \rightarrow UH \oplus V, \quad \tilde{\gamma} = \frac{P}{|P_\mu|}.
\]
The sub-Riemannian wave front \(W_{\text{SR}}(x_0, r)\) centered at \(x_0\) and with radius \(r > 0\) is defined as the set of points \(x(r) = \exp_{\text{SR}}(x_0, r\tilde{\gamma}) = \exp_{\text{SR}}(x_0, \tilde{\gamma})\) for \(P_0 \in H_{x_0} \oplus V_{x_0}\). In other words
\[
W_{\text{SR}}(x_0, r) = \exp_{\text{SR}}(x_0, r(UH_{x_0} \oplus V_{x_0})).
\]
Note that in the Riemannian setting the wave front simply coincides with the \(r\)-sphere. In our case only the inclusion \(\mathbb{S}^n_{\text{SR}}(x_0, r) \subset W_{\text{SR}}(x_0, r)\) holds true. Actually the structure of \(W_{\text{SR}}(x_0, r)\) is very complicated and, in general, the sub-Riemannian wave fronts are not manifolds.

Definition 2.25 (Conjugate locus). A point \(x \in \exp_{\text{SR}}(x_0, T_{x_0}\mathbb{G})\) is said conjugate to \(x_0\) if and only if it is a critical value of \(\exp_{\text{SR}}(x_0, \cdot)\), i.e. there exists \(P_0 \in H_{x_0}\) such that \(x = \exp_{\text{SR}}(x_0, P_0)\) and \(d\exp_{\text{SR}}(x_0, P_0)\) is not onto. The conjugate locus \(\text{Conj}(x_0)\) of \(x_0\) is then the set of all points conjugate to \(x_0\).
Notice that, from what we have said above, it turns out that \( x_0 \in \text{Conj}(x_0) \). Moreover, by Sard Theorem applied to \( \exp_{SR}(x_0, \cdot) \) one gets that \( \text{Conj}(x_0) \) is a set of (Lebesgue) measure zero in \( G \).

**Definition 2.26 (Cut locus).** Let us fix \( x_0 \in G \) and let us choose a normal CC-geodesic \( x(t) = \exp_{SR}(x_0, P_0)(t) \). If \( t > 0 \) is sufficiently small, \( d_{\mathcal{H}}(x(0), x(t)) = t \), i.e. \( x([0, t]) \) is a minimizing normal CC-geodesic. Moreover, if for some \( t_1 \) \( x([0, t_1]) \) is not minimizing the same is true for all \( t > t_1 \). By continuity, the set of numbers \( t > 0 \) such that \( d_{\mathcal{H}}(x(0), x(t)) = t \) is of the form \([0, t_0]\) or \([0, +\infty[\). In the first case we say that \( x(t_0) \) is the cut point of \( x_0 \) along \( x(t) \) while, in the second case, we say that the cut point does not exist. The set \( \text{Cut}(x_0) \) defined as the union of the cut points of \( x_0 \) along all the normal CC-geodesics starting from \( x_0 \) is called the cut locus of \( x_0 \).

### 2.4 A sub-Riemannian version of the Gauss Lemma

**Remark 2.27 (Working hypothesis).** Let \( G \) be a k-step Carnot group. Throughout this section we shall assume that there are not strictly abnormal minimizers in \( G \); see Definition 2.4.

Here below we shall perform an explicit and general computation that will be an important tool for the rest of this paper. It is somehow based on the validity of the eikonal equation, first proven in [53].

We may start by the O.D.E.’s system [15]. We assume the solution is parameterized by arc-length. Furthermore, let us fix the initial conditions: \( x(0) = x_0, P(0) = P_0 \)\(^{12}\) and set \( d_{\mathcal{H}}(x) := d_{\mathcal{H}}(x, x_0) \). We also assume the solution to be with unit speed (i.e. \( |P_{\mathcal{H}}| = 1 \)). All together, these assumptions uniquely determine a normal unit-speed CC-geodesic. In particular, we get

\[
d_{\mathcal{H}}(x(t)) = t
\]

for every \( t \in [0, \epsilon] \) (\( \epsilon > 0 \) small enough). By differentiating this identity, we obtain

\[
\frac{d}{dt} d_{\mathcal{H}}(x(t)) = \langle \operatorname{grad} d_{\mathcal{H}}(x(t)), \dot{x}(t) \rangle = \langle \operatorname{grad}_{\mathcal{H}} d_{\mathcal{H}}(x(t)), \dot{x}(t) \rangle = 1.
\]

Since \( \dot{x} = P_{\mathcal{H}} \) and \( |P_{\mathcal{H}}| = 1 \), the eikonal equation implies that \( \nabla\langle \operatorname{grad}_{\mathcal{H}} d_{\mathcal{H}}(x(t)), \dot{x}(t) \rangle = 0 \), or equivalently, that

\[
\operatorname{grad}_{\mathcal{H}} d_{\mathcal{H}}(x(t)) = \dot{x}(t)
\]

for every small enough \( t \geq 0 \). This can be written more explicitly as follows:

\[
X_i d_{\mathcal{H}}(x(t)) = P_i \quad \text{for } i \in I_{\mathcal{H}} = \{1, \ldots, \mathcal{H}\}.
\]

\( \text{(42)} \)

Let now \( \alpha \in I_{\mathcal{V}} = \{h + 1, \ldots, n\} \) and let us differentiate the quantity \( X_\alpha d_{\mathcal{H}} \) along the normal CC-geodesic \( x(t) \) defined by the previous assumptions. We have\(^{13}\)

\[
\frac{d}{dt} X_\alpha d_{\mathcal{H}}(x) = \langle \operatorname{grad}(X_\alpha d_{\mathcal{H}})(x), \dot{x} \rangle = \langle \operatorname{grad}_{\mathcal{H}} (X_\alpha d_{\mathcal{H}})(x), \dot{x} \rangle
\]

\[
= \langle \operatorname{grad}_{\mathcal{H}} (X_\alpha d_{\mathcal{H}})(x), P_{\mathcal{H}} \rangle = \sum_{i \in I_{\mathcal{H}}} P_i X_i (X_\alpha d_{\mathcal{H}})(x).
\]

\(^{12}\) \( P(0) = P_0 = P_0(0) + P_\epsilon(0) \).

\(^{13}\) For sake of simplicity, in these computations we shall drop the dependence on the variable \( t \).
Since \( X_i X_\alpha = X_\alpha X_i + [X_i, X_\alpha] = X_\alpha X_i + \sum_{\beta \in \ell_{\text{ord}(\alpha)}+1} C^\beta_{i\alpha} X_\beta \), we get

\[
\frac{d}{dt} X_\alpha d_H(x) = \sum_{i \in I_H} P_i X_\alpha (X_i d_H)(x) + \sum_{\beta \in \ell_{\text{ord}(\alpha)}+1} C^\beta_{i\alpha} X_\beta d_H(x)
\]

\[
= \sum_{i \in I_H} P_i (X_\alpha (P_i)(x) + \sum_{\beta \in \ell_{\text{ord}(\alpha)}+1} C^\beta_{i\alpha} X_\beta d_H(x))
\]

\[
= \frac{1}{2} X_\alpha (|P_d|^2)(x) + \sum_{i \in I_H} \sum_{\beta \in \ell_{\text{ord}(\alpha)}+1} P_i C^\beta_{i\alpha} X_\beta d_H(x)
\]

The first term in the sum is zero because \(|P_d| = 1\) (unit-speed). Using Definition 1.10 and the skew-symmetry of \( C^\beta \), we finally obtain

\[
\frac{d}{dt} X_\alpha d_H(x) = -\sum_{\beta \in \ell_{\text{ord}(\alpha)}+1} X_\beta d_H(x) (C^\beta P_d, X_\alpha) \quad (43)
\]

We summarize the previous discussion in the following:

**Lemma 2.28.** Let \( x : [0, r] \to \mathbb{G} \) (\( r > 0 \)) be any normal CC-geodesic of unit-speed and parameterized by arc-length. Let \( x(0) = x_0 \), \( P(0) = P_0 \) be its initial data and set \( d_H(x) = d_H(x_0, x) \) \((x \in \mathbb{G})\). Then we have

(i) \( \text{grad}_H d_H(x(t)) = P_d(t) \) for every \( t \in [0, r] \);

(ii) \( \frac{d}{dt} \text{grad}_H d_H(x(t)) = -\sum_{\beta \in \ell_{\text{ord}(\alpha)}+1} X_\beta d_H(x(t)) C^\beta P_d(t) \) for every \( t \in [0, r] \).

**Proof.** The first claim is (12), while the second one is (13), both rewritten using vector notation. \( \square \)

We reformulate Lemma 2.28 by using the notation given in Definition 2.5. One has

\[
\frac{d}{dt} \text{grad}_H d_H(x) = -C(\text{grad}_H d_H(x)) P_d.
\]

At this point Lemma 2.28 can be restated, in geometric terms, as follows:

**Proposition 2.29** (Sub-Riemannian Gauss’ Lemma). Let \( S_{\text{SR}}^n(x_0, t) \) the CC-sphere centered at \( x_0 \) of radius \( t \in [0, r] \) and set \( \nu_H = \nu_{H_{S_{\text{SR}}^n(x_0, t)}} \) and \( \omega = \omega_{S_{\text{SR}}^n(x_0, t)} \) \((t \in [0, r])\). Let \( x : [0, r] \to \mathbb{G} \) (\( r > 0 \)) be any normal CC-geodesic of unit-speed, parameterized by arc-length, with initial data \( x(0) = x_0 \), \( P(0) = P_0 \), i.e. \( x(t) = \exp_{\text{SR}}(x_0, P_0)(t) \) \((t \in [0, r])\). Then, for every \( t \in [0, r] \) the following O.D.E.’s system holds:

\[
\begin{cases}
\frac{dx}{dt} = \nu_H \\
\frac{d\nu_H}{dt} = -C_H(\omega)\nu_H \\
\frac{d\omega}{dt} = -C(\omega)\nu_H.
\end{cases} \quad (44)
\]
Notice that the first equation in (44) says that each normal CC-geodesic starting from $x_0$ intersects $S^n_{SR}(x_0, t)$ orthogonally (in the horizontal sense), i.e. at the intersection point $x(t)$ the velocity vector of the normal CC-geodesic coincides with the horizontal unit normal. Furthermore, the second and third equations in (44) express how change $\nu_t = \nu_t|_{SR(x_0, t)}$ and $\varpi = \varpi|_{SR(x_0, t)}$ along $x(t)$.

**Proof of Proposition 2.29.** We stress that our hypothesis about the absence of abnormal minimizers in $G$ implies the smoothness of $d_H$. Since every CC-sphere $S^n_{SR}(x_0, t)$ ($t \in [0, r]$) turns out to be defined as $S^n_{SR}(x_0, t) = \{ x : d_H(x) = t \}$, the Riemannian unit normal vector $\nu$ along $S^n_{SR}(x_0, t)$ (at each regular non-characteristic point) may be written just by normalizing the following (non unit) normal vector along $S^n_{SR}(x_0, t)$

$$\mathcal{N} := \text{grad} d_H = (\text{grad} d_H, \text{grad} d_H).$$

Clearly, $\nu_t$ is uniquely determined by $\mathcal{N}$. The eikonal equation implies that

$$\nu_t = \text{grad} d_H, \quad \varpi = \text{grad} d_H \quad \forall t \in [0, r].$$

Therefore, by the first equation of system (15) and by (i) of Lemma 2.28 we immediately get that $\dot{x}(t) = P_h(t) = \nu_t(x(t))$. This identity together with (ii) of Lemma 2.28 implies the third equation in (44). Finally, the second equation in (44) immediately follows by using (i) of 2.28 together with the third equation of (44).

**Remark 2.30.** We stress that Lemma 2.29 solves the problem of selecting, for any regular point $x_1$ belonging to the CC-sphere $S^n_{SR}(x_0, r)$, the unique normal CC-geodesic having velocity vector equals to the horizontal normal direction at that point (i.e. $P_h(0) = \nu_t(x_1)$) and connecting this point to the center $x_0$ of the CC-sphere. Actually, Lemma 2.29 and the uniqueness of solutions of O.D.E.’s imply that the desired curve must be the normal CC-geodesic defined by

$$\bar{x}(t) := \exp(x_1, -\mathcal{N}(x_1))(t) \quad t \in [0, r]$$

where $\mathcal{N} = (\nu_t, \varpi)$.

**Corollary 2.31.** Let $S = \{ x \in G : f(x) = 0 \}$, where $f$ is a $C^2$ function. Assume that there exists a CC-ball $B(y, r)$ with center $y$ and radius $r$, such that $B(y, r) \subset \{ f(x) < 0 \}$, or $B(y, r) \subset \{ f(x) > 0 \}$, and $B(y, r) \cap S = \{ x \}$, where $x \in S$ is non-characteristic. Then there exists the metric normal $\gamma_N$ to $S$ at $x$ and for every $t \in [0, r]$, $\gamma(t) \in \gamma_N$, where

$$\gamma(t) := \exp_{SR}(y, -\mathcal{N}(x))(t) \quad t \in [0, r].$$

### 2.5 2-step case: explicit integration and other features

In this section we shall explicitly analyze the case of 2-step Carnot groups. Remind that in the 2-step setting there exist no abnormal minimizers and our working hypothesis is satisfied. In this case, it is well-known that CC-geodesics are smooth; see [34].

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14Remark that, by definition, $\nu_t := \frac{P_h \nu}{|P_h \nu|}$, and $\varpi := \frac{P_h \nu}{|P_h \nu|}$. 

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In order to describe the sub-Riemannian exponential map, we note that the system for normal CC-geodesics can explicitly be integrated. Using (19), i.e.

\[
\begin{align*}
\dot{x} &= P_H \\
\dot{P}_H &= -C_H(P_{H_2})P_H \\
P_{H_2} &= 0,
\end{align*}
\]

we get that \( P_{H_2} \in \mathbb{R}^{h_2} \cong H_2 \) is a constant vector. By standard results about O.D.E.’s, we therefore get that

\[ P_H(t) = e^{-C_H(P_{H_2})t}P_H(0). \]

To obtain the solution in exponential coordinates first note that the equation \( \dot{x} = P_H \) is equivalent\(^\text{15}\) to

\[
\begin{align*}
\dot{x}_i &= P_i \\
\dot{x}_\alpha &= -\frac{1}{2}\langle C_{H}^\alpha x_H, P_H \rangle (\alpha \in I_{H_2}).
\end{align*}
\]

Therefore, setting \( x_H(t) := (x_1(t), ..., x_h(t)) \in \mathbb{R}^h \) to denote the projection of the solution \( x(t) \) of (19) onto the firsts \( h \) variables\(^\text{16}\), one gets

\[ x_H(t) = x_H(0) + \int_0^t e^{-C_H(P_{H_2})s}P_H(0) \, ds \quad (45) \]

and

\[ x_\alpha(t) = x_\alpha(0) - \frac{1}{2} \int_0^t \langle C_{H}^\alpha x_H, \dot{x}_H \rangle \, ds. \quad (46) \]

These equations describe the sub-Riemannian exponential map in the 2-step case. More precisely, fixing a base point \( x_0 \), we have that

\[ \exp_{SR}(x_0, \cdot)(\cdot) : UH \times H_2 \times \mathbb{R} \longrightarrow \mathbb{G} \]

is given by

\[ \exp_{SR}(x_0, P_0)(t) := x_0 + \int_0^t e^{-C_H(P_{H_2})s}P_H(0) \, ds - \frac{1}{2} \sum_{\alpha \in I_{H_2}} \left\{ \int_0^t \langle C_{H}^\alpha x_H, \dot{x}_H \rangle \, ds \right\} e_{\alpha}. \quad (47) \]

Here above \( UH \) denotes the bundle of all unit horizontal vectors, i.e. \( UH \subset H \cong S^{h-1}. \)

\(^\text{15}\)Explicitly, we have

\[ \dot{x}_\alpha = \langle P_H, e_\alpha \rangle = \sum_{i \in I_H} P_i \langle X_i, e_\alpha \rangle = \sum_{i \in I_H} P_i \left( -\frac{1}{2} \langle C_{H}^\alpha x, e_i \rangle \right) = -\frac{1}{2} \langle C_{H}^\alpha x, P_H \rangle (\alpha \in I_{H_2}). \]

\(^\text{16}\)Using exponential coordinates for \( \mathbb{G} \), every point \( x \in \mathbb{G} \) is \( n \)-tuple \( x = (x_1, ..., x_h, x_{h+1}, ..., x_n) \). So it seems natural to “divide” the variables as follows:

\[ x_H := (x_1, ..., x_h) \in \mathbb{R}^h, \quad x_{H_2} := (x_{h+1}, ..., x_n) \in \mathbb{R}^{h_2} \quad (n = h + h_2) \]

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Remark 2.32. Setting $x(t) := \exp_{SR}(x_0, P_0)(t)$ and

$$\mathcal{E}(t) := \int_0^t e^{-C_H(P_{h_2})s} \, ds \in \mathcal{M}_h,$$

we get that

$$x(t) = x_0 + \left( \langle \mathcal{E}(t)P_H(0), e_1 \rangle, \ldots, \langle \mathcal{E}(t)P_H(0), e_h \rangle, \ldots \right) \left( \frac{1}{2} \int_0^t \langle C_H^\alpha \dot{x}_H, x_H \rangle \, ds \right) \ldots .$$

Notation 2.33. In the sequel, we shall denote by $dx_H$ the following vector valued 1-form:

$$dx_H := (dx_1, \ldots, dx_h)^\top \in \bigotimes_{h \times \alpha} H^* \times \cdots \times H^*.$$

Notation 2.34. Let $x : [0, T] \rightarrow \mathbb{G}$ be a CC-normal geodesic such that $x(0) = x_0$ and $x(T) = x_1$. Later on we shall set $[x_0, x_1] := x([0, T]) = \{ x(t) \in \mathbb{G} : t \in [0, T] \}$.

Remark 2.35. For any $\alpha \in I_n$, the integral

$$T^\alpha_{\pi_H}(t) := \int_0^t \langle C^v_H x_H, \dot{x}_H \rangle \, ds = \int_{[x_H(0), x_H(t)]} \langle C^v_H x_H, dx_H \rangle$$

can explicitly be evaluated, by means of standard linear algebra arguments, by noting that, as every skew-symmetric linear operator, $C^v_H$ can be written, after an orthogonal change of basis, in a “canonical” form. More precisely, there exists $O^\alpha \in O_h(\mathbb{R})^{17}$ such that

$$(O^\alpha)^{-1} C^v_H O^\alpha = \begin{pmatrix}
0 & \lambda_1^\alpha & 0 & 0 & \cdots \\
-\lambda_1^\alpha & 0 & 0 & 0 & \cdots \\
0 & 0 & 0 & \lambda_2^\alpha & \cdots \\
0 & 0 & -\lambda_2^\alpha & 0 & \cdots \\
\vdots & \vdots & \vdots & \vdots & \ddots \\
0 & \lambda_R^\alpha & -\lambda_R^\alpha & 0 & \cdots \\
& & & & 0_N
\end{pmatrix}$$

where $\pm i \lambda_j^\alpha$ ($j = 1, \ldots, R$) are the eigenvalues - purely imaginary - of $C^v_H$ computed with their multiplicity, and $0_N$ denotes the zero $N \times N$-matrix where $N$ is the nullity of $C^v_H$.

Here $R = \frac{1}{2} \text{rank} C^v_H$ and $h = 2R + N$. Setting

$$y_H(t) = (O^\alpha)^{-1} x_H(t), \quad y_H(0) = (O^\alpha)^{-1} x_H(0),$$

we get that

$$T^\alpha_{\pi_H}(t) = \int_{[x_H(0), x_H(t)]} \langle C^v_H x_H, dx_H \rangle = \sum_{j=1}^R \lambda_j^\alpha \int_{[y_H(0), y_H(t)]} y_{j+1}dy_j - y_jdy_{j+1}.$$

We stress that similar integrals was previously introduced and studied by Pansu in his deep study of differentiability in CC-spaces; see [56], Definition 4.6, p.15.

\textsuperscript{17}i.e. the orthogonal group on $\mathbb{R}^h(\cong H)$.
Using a standard result about periodic solutions of linear O.D.E.'s\textsuperscript{18}, we may study the existence of $T$-periodic solutions, $T > 0$, of the $h$-th order linear O.D.E.'s system

$$\dot{P}_h = -C_h(P_{u_2})P_h,$$  \hspace{1cm}  \text{(49)}$$

where $P_{u_2} \in H_2$ is constant. More precisely, we get that there exists a $k$-dimensional sub-space $S_k$ of the solution space $S_h$ of the equation (49) such that each member of $S_k$ is $T$-periodic, and no member of $S_h \setminus S_k$ is $T$-periodic, if and only if the following holds:

$$\text{rank}(e^{-C_h(P_{u_2})T} - \text{Id}_h) = h - k.$$  \hspace{1cm}  \text{(50)}$$

Hereafter, we shall analyze this condition. First, note that $P_{u_2} = 0$ implies that $P_h$ is constant and so we may assume $P_{u_2} \neq 0$. As in Remark 2.35, we make use of a standard Linear Algebra argument. More precisely, as every skew-symmetric linear operator, $C_h(P_{u_2})$ can be written, after an orthogonal change of basis, in canonical form. In particular, there exists $O \in \mathbf{O}_h(\mathbb{R})$ such that

$$\tilde{C}_h(P_{u_2}) := O^{-1}C_h(P_{u_2})O = \begin{pmatrix}
0 & \lambda_1 & 0 & 0 \\
-\lambda_1 & 0 & 0 & 0 \\
0 & 0 & 0 & \lambda_2 \\
0 & 0 & -\lambda_2 & 0 \\
& \cdots & \\
& & 0 & \lambda_R \\
& & -\lambda_R & 0 \\
& & & 0_N
\end{pmatrix},$$

where $\pm i \lambda_j$ ($j = 1, \ldots, R$) are the - purely imaginary - eigenvalues of $C_h(P_{u_2})$, computed with their multiplicity, and $0_N$ denotes the zero $N \times N$-matrix, where $N$ is the nullity of $C_h(P_{u_2})$. Here $R = \frac{1}{2}\text{rank}C_h(P_{u_2})$ and $h = 2R + N$. Notice that the eigenvalues are functions of $P_{u_2}$, i.e. $\lambda_j = \lambda_j(P_{u_2})$ ($j = 1, \ldots, R$). Setting

$$\tilde{P}_h(t) = O^{-1}P_h(t), \quad \tilde{P}_h(0) = O^{-1}P_h(0),$$

we obtain the following equivalent equation:

$$\tilde{\dot{P}}_h = -\tilde{C}_h(P_{u_2})\tilde{P}_h.$$  \hspace{1cm}  \text{(50)}$$

\textbf{Theorem 2.36.} Let

$$\dot{y} := A(t)y$$  \hspace{1cm}  \text{(48)}$$

be a $n$-th order system of linear homogeneous differential equations, where $A(t) = [a_{ij}(t)]_{i,j=1,\ldots,n}$ is a $n \times n$ matrix and each element $a_{ij}(t)$ is a real-valued function continuous on the real line $\mathbb{R}$. Let $W$ denote a fundamental matrix of (48). Let $k$ be a non-negative integer, $0 \leq k \leq n$. Then there exists a $k$-dimensional sub-space $S_k$ of the solution space $S_n$ of (48) such that each member of $S_k$ is periodic of period $T$ and no member of $S_n \setminus S_k$ is periodic of period $T$ if and only if the rank of the matrix $W(T) - W(0)$ is $n - k$.  \hspace{1cm}  \text{(42)}$$

\textbf{\textsuperscript{18}The following result holds true (see \textsuperscript{9}):}
By applying the previous argument, we infer that there exists a $k$-dimensional sub-space $S_k$ of the solution space $S_h$ of the equation (50) such that each member of $S_k$ is $T$-periodic, and no member of $S_h \backslash S_k$ is $T$-periodic, if and only if:

$$\text{rank}(e^{-CH(P_{h_2})T} - \text{Id}_h) = h - k.$$ 

Therefore, it remains us to analyze the matrix $e^{-CH(P_{h_2})T}$ given by

$$
\begin{pmatrix}
\cos(\lambda_1(P_{h_2})T) & \sin(\lambda_1(P_{h_2})T) & 0 & \ldots \\
-\sin(\lambda_1(P_{h_2})T) & \cos(\lambda_1(P_{h_2})T) & 0 & \ldots \\
0 & 0 & \ddots & \\
\vdots & \vdots & \ddots & \cos(\lambda_R(P_{h_2})T) & \sin(\lambda_R(P_{h_2})T) \\
& & & -\sin(\lambda_R(P_{h_2})T) & \cos(\lambda_R(P_{h_2})T) \\
& & & & \text{Id}_N
\end{pmatrix}
$$

We claim that for every given $P_{h_2} \neq 0$, there exist $T$-periodic solutions of (49), for some positive $T = T(P_{h_2})$. Indeed, if $N \neq 0$, i.e. the nullity $N$ of $C_H(P_{h_2})$ is non-zero, the claim follows by the above discussion since $\text{rank}(e^{-CH(P_{h_2})T} - \text{Id}_h) \geq h - N$ and $k \geq N > 0$. Furthermore, if $N = 0$, the claim follows by choosing $T = 2\pi/\lambda_j(P_{h_2})$ for some $j = 1, \ldots, R$. Indeed in such a case one sees that $k \geq 2$. With an analogous argument, one can easily show that the dimension of the space $S_k$ of $T$-periodic solutions of (49) satisfies:

$$N + 2 \leq k \leq h.$$

Note that these solutions can also be constant functions. In similar way, we see that for every $P_{h_2} \neq 0$ there exists $T = T(P_{h_2}) > 0$ and there exist -at least- two non-constant, linearly independent, $T$-periodic solutions of (49).

We summarize the above discussion in the next:

**Proposition 2.37.** For every $P_{h_2} \in H_2$, $P_{h_2} \neq 0$, there exists $T = T(P_{h_2}) > 0$ and there exists a $k$-dimensional sub-space $S_k$ of the solution space $S_h$ of (50) such that each member of $S_k$ is $T$-periodic and no member of $S_h \backslash S_k$ is $T$-periodic. Furthermore it turns out that $N + 2 \leq k \leq h$, where $N$ is the nullity of $C_H(P_{h_2})$. Finally, there exists a positive integer $r \geq 2$ and there exists a $r$-dimensional sub-space $S_r$ of $S_k$ such that each member of $S_r$ is $T$-periodic and non-constant.

**Remark 2.38.** In the previous Proposition 2.37 the numbers $k$ and $r$ can be characterized in a slightly different way. Indeed it turns out that $k$ is the multiplicity of the eigenvalue $\lambda = 1$ of the scalar matrix $e^{-CH(P_{h_2})T}$. Moreover $r = k - N$. To prove this claim one can use another characterization of $T$-periodic solutions of linear homogeneous systems of O.D.E.’s which can be found in [2].

We may use the above Proposition 2.37 to study the $T$-periodicity of $x_H$; see [15].
Remark 2.39. If \( x_u \) is \( T \)-periodic then \( \int_0^T P_u(s) \, ds = 0 \). This follows by hypothesis, using the first equation of (19). Note also that if \( P_u \) is \( T \)-periodic, then \( x_u \) is \( T \)-periodic if and only if \( \int_0^T P_u(s) \, ds = 0 \).

Example 2.40. Let \( P_{n_2} \neq 0 \) and let us assume that \( C_H(P_{n_2}) \) is invertible. For instance, this is the case of the Heisenberg group \( H^n \). In such case, by a direct computation based on the very definition of \( e^{-C_H(P_{n_2})T} \), it turns out that

\[
 x_u(t) = x_u(0) + (C_H(P_{n_2}))^{-1}(\mathrm{Id}_n - e^{-C_H(P_{n_2})T})P_u(0).
\]

Remark 2.41. Let us consider the system of normal CC-geodesics \( (19) \). We would like to remark that the \( T \)-periodicity of \( x_u \) is related with some other things. To this aim, let us assume that \( x_u \) be \( T \)-periodic. More precisely, we assume that there exists a minimal \( T = T(P_{n_2}) > 0 \) such that \( x_u(t) \) is \( T \)-periodic, for any given \( P_{n_2} \neq 0 \). Furthermore, let

\[
 x_1 := \exp_{SR}(x_0, P_0)(T), \quad \text{where} \quad P_0 = (P_u(0), P_{n_2}) \in UH \times H_2. \quad \text{Then it can be shown that:}
\]

(i) The point \( x_1 \) is conjugate to \( x_0 \) along the normal CC-geodesic \( x(t) = \exp_{SR}(x_0, P_0)(t), t \in [0, T] \);

(ii) The “segment” \( [x_0, x_1] := x([0, T]) = \{ x(t) \in G : t \in [0, T] \} \) is a minimizing normal CC-geodesic and \( x_1 \) is the cut-point of \( x_0 \) along the normal CC-geodesic \( x(t) \).

In this case, we will say that \( [x_0, x_1] \) is a minimizing CC-geodesic segment. Note that the CC-length of segment \( [x_0, x_1] \) is simply \( T \). The proofs of these claims can be done by following a classical pattern, for which we refer the reader to [20]. However, this is beyond the scope of this paper.

Remark 2.42. For 2-step Carnot groups, similar arguments can be used to show that the CC-sphere \( S_{SR}^n(x, r) = \{ y \in G : d_H(x, y) = r \} \) is \( C^\infty \)-smooth out of the set

\[
 H_2(x) \cap S_{SR}^n(x, r),
\]

where

\[
 H_2(x) := \{ z = \exp_g(z_H, z_{n_2}) \in G : z_{n_2} = x_{n_2} \}.
\]

More precisely, each point \( y \in S_{SR}^n(x, r) \setminus H_2(x) \) can be joined to the center \( x \) of \( S_{SR}^n(x, r) \) by a unique minimizing normal CC-geodesic. In particular, one can show that for every \( y \in S_{SR}^n(x, r) \setminus H_2(x) \) there exists a unique \( P_0 \in UH \times H_2 \) such that \( y = \exp_{SR}(x, P_0)(r) \) and \( \exp_{SR}(x, P_0) \) has maximal rank. On the other hand, if \( y \in S_{SR}^n(x, r) \cap H_2(x) \), then it turns out that \( y \in \mathrm{Conj}(x) \).

2.6 Appendix: iterative integration for normal CC-geodesics

In this section we will how, at least in principle, the system of normal CC-geodesics can be integrated, step by step. In what follows, we shall use the notation \( P_{n_2} = P_{n_2}(P) \).

Remind that \( (15) \) is given by

\[
 \begin{cases}
 \dot{x} = P_u, \\
 \dot{P} = -C(P_V)P_u;
\end{cases}
\]

\[
(\iff) \quad \dot{x}_{n_2} = P_u, \quad \dot{x}_V = 0
\]

\[\text{Remark 2.39. If } f : \mathbb{R} \to \mathbb{R} \text{ is a continuous } T \text{-periodic function, then } \int_0^T f(s) \, ds \text{ is } T \text{-periodic if and only if } \int_0^T f(s) \, ds = 0.\]
By orthogonal projection onto $H_k$, we get that $\dot{P}_{u_k} = 0$. Hence $P_{u_k}$ is constant. Then the $k - 1$-th vector equation becomes $\dot{P}_{u_{k-1}} = -C(P_{u_k}) \dot{x}$ and so

$$P_{u_{k-1}}(t) = P_{u_{k-1}}(0) - C(P_{u_k})(x(t) - x_0).$$

By iterating the same procedure one gets $\dot{P}_{u_{k-2}} = -C(P_{u_{k-1}}) \dot{x}_H$. Hence

$$P_{u_{k-2}}(t) = P_{u_{k-2}}(0) - \int_0^t C(P_{u_{k-1}}(s)) \dot{x}_H \, ds$$

$$\ldots$$

$$P_{u_1}(t) = P_{u_1}(0) - \int_0^t C(P_{u_2}(s)) \dot{x}_H \, ds$$

$$\ldots$$

$$P_{u_2}(t) = P_{u_2}(0) - \int_0^t C(P_{u_3}(s)) \dot{x}_H \, ds$$

$$P_{u}(t) = P_{u}(0) - \int_0^t C_H(P_{u_2}(s)) \dot{x}_H \, ds.$$

Finally we get that

$$x_H(t) = x_H(0) + \int_0^t P_u(s) \, ds$$

$$= x_H(0) + \int_0^t \left( P_{u_1}(0) - \int_0^{s_1} C_H(P_{u_2}(s_2)) \dot{x}_H \, ds_2 \right) \, ds_1$$

$$= x_H(0) + P_{u_1}(0) t - \int_0^t \left( \int_0^{s_1} C_H(P_{u_2}(s_2)) \dot{x}_H \, ds_2 \right) \, ds_1,$$

or equivalently, that

$$\ddot{x}_H(t) = C_H(P_{u_2}(t)) \dot{x}_H.$$

In the sequel we shall apply this procedure to the case of general 3-step Carnot groups. We stress that the first vector equation in [21], i.e. $\dot{x} = P_{u}$, in exponential coordinates, is equivalent to the following system:

$$\begin{cases} 
\dot{x}_i = P_i & (i \in I_u) \\
\dot{x}_\alpha = -\frac{1}{2} \langle C^\alpha x_H, P_u \rangle & (\alpha \in I_{u_2}) \\
\dot{x}_\beta = -\left( \frac{1}{2} \langle C^\beta x, P_u \rangle - \frac{1}{12} \sum_{\alpha \in I_{u_2}} \langle C^{\beta} x, e_\alpha \rangle \langle C^\alpha x_H, P_u \rangle \right) & (\beta \in I_{u_3}).
\end{cases}$$

\text{By} the formula for left-invariant vector field stated in Example[1, 24], we get that

$$\dot{x}_\alpha = \langle P_u, e_\alpha \rangle = \sum_{i \in I_u} P_i (X_i, e_\alpha) = \sum_{i \in I_u} P_i \left( -\frac{1}{2} \langle C^\alpha x_H, e_i \rangle \right) = -\frac{1}{2} \langle C^\alpha x_H, P_u \rangle$$  \hspace{1cm} (\alpha \in I_{u_2})$$

$$\dot{x}_\beta = \langle P_u, e_\beta \rangle = \sum_{i \in I_u} P_i (X_i, e_\beta) = -\left( \frac{1}{2} \langle C^\beta x, P_u \rangle - \frac{1}{12} \sum_{\alpha \in I_{u_2}} \langle C^{\beta} x, e_\alpha \rangle \langle C^\alpha x_H, P_u \rangle \right)$$  \hspace{1cm} (\beta \in I_{u_3}).$$

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Therefore, for any $\alpha \in I_{\mu_2}$, we have
\[
x_\alpha(t) = x_\alpha(0) - \frac{1}{2} \int_{[x_H(0),x_H(t)]} \langle C_\alpha^\mu, x_H, dx_H \rangle
\] (52)
and, for any $\beta \in I_{\mu_3}$, we have
\[
x_\beta(t) = x_\beta(0) - \frac{1}{2} \int_0^t \left( \frac{1}{2} \langle C_\beta^\mu, x, \dot{x}_H \rangle - \frac{1}{12} \sum_{\alpha \in I_{\mu_2}} \langle C_\alpha^\mu, x_H, \dot{x}_H \rangle \langle C_\beta^\mu, x, e_\alpha \rangle \right) ds
\] (53)
Now we proceed with the integration of (21) as we have explained in general, at the beginning of this section. As before, we have that $\dot{P}_{\mu_3} = 0$ and so $P_{\mu_3}$ is constant. Since $\dot{P}_{\mu_2} = -C(P_{\mu_3}) \dot{x}_H$ we get
\[
P_{\mu_2}(t) = P_{\mu_2}(0) - C(P_{\mu_3})(x_H(t) - x_H(0)).
\]
Moreover $\dot{P}_H = -C(P_{\mu_2}) \dot{x}_H$, and so
\[
P_H(t) = P_H(0) - \int_0^t C_\mu(P_{\mu_2}(s)) \dot{x}_H(s) ds
\] (52)
Finally, we have
\[
x_H(t) = x_H(0) + \int_0^t P_H(s) ds
\]
\[
x_H(t) = x_H(0) + \int_0^t \left( P_H(0) - \int_0^s C_\mu(P_{\mu_2}(s)) \dot{x}_H(s) ds \right) ds
\]
\[
x_H(t) = x_H(0) + \int_0^t \left( P_H(0) - \int_0^s C_\mu \left( P_{\mu_2}(0) - C(P_{\mu_3})[x_H(s) - x_H(0)] \right) \dot{x}_H(s) ds \right) ds
\]
or, equivalently,
\[
\dot{x}_H = -C_\mu \left( P_{\mu_2}(0) - C(P_{\mu_3})[x_H(t) - x_H(0)] \right) \dot{x}_H.
\] (54)
### 3 CC-distance from hypersurfaces in 2-step Carnot groups

Throughout this section we shall assume that $\mathbb{G}$ be a 2-step Carnot group. Let $S \subset \mathbb{G}$ be a smooth hypersurface (i.e. closed $(n-1)$-dimensional submanifold of $\mathbb{G}$) and let

$$\delta_H: \mathbb{G} \to \mathbb{R}^+ \cup \{0\}$$

denote the CC-distance function for $S$, i.e. $\delta_H(x) = \inf_{y \in S} d_H(x,y)$.

In the sequel we shall use the notation

$$\mathcal{N} := \nu_{\mathbb{H}}(\nu) = (\nu_H, \nu),$$

where $\nu$ is the Riemannian unit normal along $S$. We stress that the normal (non-unit) vector field $\mathcal{N}$ is defined at each non-characteristic point $x \in S \setminus C_S$.

**Remark 3.1.** We would like to remind some classical results about the regularity of the distance function to smooth hypersurfaces (or, more generally, submanifolds) which hold in the Euclidean -and Riemannian- setting. If $N \subset \mathbb{R}^n$ is a $C^k$-smooth manifold with $k \geq 2$, then one easily sees that, near $N$, the distance function $\delta_N$ is $C^{k-1}$-smooth. A first, somewhat surprising result, was proved by Gilbarg Trudinger in the Appendix of their celebrated book [33]. Indeed they prove the existence of a neighborhood $U$ of $N$ such that the distance function $\delta_N$ is of class $C^k$ on $U \setminus N$. A similar problem was also considered by Federer in its theory of Sets of Positive Reach; see [22]. The complete solution of this problem, in the $C^1$ case, can be found in a paper by Krantz and Parks; see [43]. We stress that, in such a case, a further hypothesis is needed. More precisely, if we assume that $N$ is just $C^1$-smooth and that there exists a neighborhood $U$ of $N$ having the so-called “unique nearest point property”, then $\delta_N$ is of class $C^1$ on $U \setminus N$. Later on, a simplified proof of this result was proved by Foote; see [24]. We would like to remind that, some previous results of this type, in a sub-Riemannian setting, was proved by Arcozzi and Ferrari for the first Heisenberg group $\mathbb{H}^1$; see[6].

So let us begin by assuming that $S$ is $C^k$-smooth with $k \geq 2$. As a first thing, let us define the mapping $\Phi: S \times [-\epsilon, \epsilon] \to \mathbb{G} (\epsilon > 0)$,

$$\Phi(y, t) := \exp_{SR}(y, \mathcal{N}(y))(t).$$

By construction, it turns out that $\Phi \in C^{k-1}(S \setminus C_S \times ]-\epsilon, \epsilon[)$ for any - small enough - $\epsilon > 0$. The “size” of $\epsilon$ depends on the local geometry of $S$.

Next we shall apply the Inverse Mapping Theorem to this map. For this reason, we have to compute the differential of $\Phi$ at $t = 0$. To this aim, for any fixed point $y_0 \in S$, let us introduce a system of Riemannian normal coordinates in a neighborhood of $y_0$;

21 More precisely, $\epsilon$ represents the CC-length of a minimizing CC-geodesic segment having “lagrangian multiplier” $P_{\alpha} = \omega(y)$; see Remark 2.41.

22 Let $(\tau_1, ..., \tau_{n-1})$ be an orthonormal basis of $T_{y_0}S$. For $1 \leq j \leq n-1$ define a real-valued function $u_j$ on a neighborhood of $y_0$ by

$$u_j \left( \exp_{\mathbb{H}}(y_0) \left( \sum_{j=1}^{n-1} t_i \tau_i \right) \right) = t_j,$$

where $\exp_{\mathbb{H}}$ denotes the Riemannian exponential map. Then, by definition, $(u_1, ..., u_{n-1})$ is a system of normal coordinates corresponding to the orthonormal basis $(\tau_1, ..., \tau_{n-1})$.  

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see, for instance, [11] or [39]. Hence \( y \equiv y(u_1, \ldots, u_{n-1}) \) and
\[
\begin{align*}
\frac{\partial}{\partial u_1}, \ldots, \frac{\partial}{\partial u_{n-1}}
\end{align*}
are the coordinate vector fields associated with the normal coordinate system \((u_1, \ldots, u_{n-1})\). Using these coordinates, it turns out that \( V(y) := \frac{\partial y}{\partial u_1} \wedge \ldots \wedge \frac{\partial y}{\partial u_{n-1}} \) is a normal (non-unit) vector along \( S \), in a neighborhood of \( y_0 \in S \). By definition, the Riemannian unit normal \( \nu \) along \( S \) (in this neighborhood of \( y_0 \)) is given, up to the sign, by \( \nu = \frac{V}{|V|} \), while the horizontal unit normal \( \nu_H \) can be written out as \( \nu_H = \frac{P_H V}{|P_H V|} \); see Section 1.2.

**Lemma 3.2.** Let \( S \subset G \) be \( C^k \)-smooth with \( k \geq 2 \). Then
\[
| \det [J(y,0)\Phi] | = |P_H V|,
\]
where \( J(y,0)\Phi \) denotes the Jacobian matrix operator at \((y,0) \in S \times ]-\epsilon,\epsilon[\). Therefore \( | \det [J(y,0)\Phi] | \neq 0 \) at each non-characteristic point of \( y \in S \setminus C_S \).

**Proof.** Throughout this proof we will need some results of Section 2.5. By (47) we get
\[
\Phi(y,t) = y + \int_0^t e^{-C_H (\omega(y))s}v_H(y) \, ds - \frac{1}{2} \sum_{\alpha \in I_{H2}} \left( \int_0^t \langle C_H^\alpha x_H(s), \dot{x}_H(s) \rangle \, ds \right) e_\alpha,
\]
where
\[
x_H(t) := y_H + \int_0^t e^{-C_H (\omega(y))s}v_H(y) \, ds
\]
\[= \left( \ldots, y_i + \left( \int_0^t e^{-C_H (\omega(y))s}v_H(y) \, ds \right) e_i, \ldots \right) \in \mathbb{R}^h.
\]
To sake of simplicity, we also set
\[
\Phi(y,t) := y + A(y,t).
\]
By using the explicit expression of \( \Phi \) and the Fundamental Theorem of Calculus we get\[23\]
\[
\begin{align*}
\frac{\partial \Phi_H}{\partial t} &= e^{-C_H (\omega)(t)}v_H, \\
\frac{\partial \Phi_{Hu}}{\partial t} &= \frac{1}{2} \sum_{\alpha \in I_{H2}} \langle C_H^\alpha \dot{x}_H, x_H \rangle e_\alpha.
\end{align*}
\]
\[\text{Since } x(t) := \Phi(y,t) \in G, \text{ in exponential coordinates we have } \Phi = \exp_g((\Phi_H, \Phi_{Hu})) \equiv (\Phi_H, \Phi_{Hu}), \text{ where } x_H = \Phi_H \in \mathbb{R}^h \text{ and } x_{Hu} = \Phi_{Hu} \in \mathbb{R}^v; \text{ see Notation 1.13.}\]
In the last formula we have also used the skew-symmetry of the structure constants operators $C^\alpha_H (\alpha \in I_{H_2})$. Note that from the previous computations we obtain:

$$\left( \frac{\partial \Phi}{\partial t} \bigg|_{t=0} \right)^T = \sum_{i \in I_H} \nu^i(y) e_i + \frac{1}{2} \sum_{\alpha \in I_{H_2}} \langle C^\alpha_H \nu_H(y), y_H \rangle e_\alpha$$

$$= \sum_{i \in I_H} \nu^i(y) X_i(y)$$

$$= \nu_H(y).$$

Furthermore, using the system of normal coordinates $u = (u_1, ..., u_{n-1})$, we get that

$$\frac{\partial \Phi}{\partial u_i} = \frac{\partial y}{\partial u_i} + \frac{\partial A}{\partial u_i} \quad (i = 1, ..., n-1).$$

As usual we will set $A = \exp_y((A_H, A_{H_2})) \equiv (A_H, A_{H_2})$. We compute

$$\frac{\partial A_H}{\partial u_i} = \int_0^t \left( \mathcal{J}_y \left( e^{-C_H(\varpi(y))s} \nu_H(y) \right) \frac{\partial y}{\partial u_i} \right) ds \in \mathbb{R}^h,$n

$$\frac{\partial A_{H_2}}{\partial u_i} = -\frac{1}{2} \sum_{\alpha \in I_{H_2}} \left( \int_0^t \langle \text{grad}_y((C^\alpha_H x_H, \dot{x}_H)), \frac{\partial y}{\partial u_i} \rangle ds \right) e_\alpha \in \mathbb{R}^v \quad (i = 1, ..., n-1).$$

Therefore, choosing $t = 0$, one gets

$$\mathcal{J}_{(y,0)} \Phi = \text{col} \left[ \frac{\partial y}{\partial u_1}, ..., \frac{\partial y}{\partial u_{n-1}}, \frac{\partial \Phi}{\partial t} \bigg|_{t=0} \right]$$

$$= \text{col} \left[ \frac{\partial y}{\partial u_1}, ..., \frac{\partial y}{\partial u_{n-1}}, \nu_H(y) \right].$$

Finally, we may compute the Jacobian determinant of $\mathcal{J}_{(y,0)} \Phi$. By using standard Linear Algebra arguments, one gets

$$\left| \det \left[ \mathcal{J}_{(y,0)} \Phi \right] \right| = \left| \det \left( \text{col} \left[ \frac{\partial y}{\partial u_1}, ..., \frac{\partial y}{\partial u_{n-1}}, \nu_H(y) \right] \right) \right|$$

$$= \left| \langle \left( \frac{\partial y}{\partial u_1} \wedge ... \wedge \frac{\partial y}{\partial u_{n-1}} \right), \nu_H(y) \rangle \right|$$

$$= \left| \frac{\partial y}{\partial u_1} \wedge ... \wedge \frac{\partial y}{\partial u_{n-1}} \right| \left| \langle \nu(y), \nu_H(y) \rangle \right|$$

$$= \left| \frac{\partial y}{\partial u_1} \wedge ... \wedge \frac{\partial y}{\partial u_{n-1}} \right| |\mathcal{P}_H \nu(y)|$$

$$= |\mathcal{P}_H \mathcal{V}(y)|,$n

which achieves the proof. \hfill \Box

24Remind that we are working in exponential coordinates. So we have

$$\nu_H(y) = \sum_{i \in I_H} \nu^i_H(y) X_i(y) = \sum_{i \in I_H} \nu^i_H(y) \left( e_i - \frac{1}{2} \sum_{\alpha \in I_{H_2}} \langle C^\alpha_H y, e_i \rangle \right).$$
Remark 3.3 (Invertibility at the non-characteristic set). Let us set $S_0 := S \setminus C_S$. It turns out that $S_0$ is an open subset of $S$, in the relative topology. Moreover, since we are assuming that $S$ is $C^k$-smooth with $k \geq 2$, one gets that $\dim C_S \leq (n - 2)$; see [43, 46]. Now let $U_0 \Subset S_0$ be an open set compactly contained in $S_0$. By Lemma 3.2 we know that the Jacobian of the mapping $\Phi : U_0 \times ] - \epsilon, \epsilon[ \longrightarrow \mathbb{G}$ is non-zero along $U_0 \times \{0\}$. The Inverse Mapping Theorem implies that there exists $\epsilon_0 \in ]0, \epsilon]$ such that

$$\Phi : U_0 \times ] - \epsilon_0, \epsilon_0[ \longrightarrow \Phi(U_0 \times ] - \epsilon_0, \epsilon_0[)$$

is a $C^{k-1}$-diffeomorphism.

Notation 3.4 (Projection mapping). The previous Remark 3.3 enables us to define the following mapping:

$$\Psi := \Phi^{-1} : \Phi(U_0 \times ] - \epsilon_0, \epsilon_0[) \longrightarrow U_0 \times ] - \epsilon_0, \epsilon_0[.$$

By construction, $\Psi$ is $C^{k-1}$-smooth. In the sequel, we shall denote by $\Psi_S$ the projection of the map $\Psi$ onto its 1st factor, i.e. $\Psi(x) = (\Psi_S(x), t(x))$.

Let us set $U := \Phi(U_0 \times ] - \epsilon_0, \epsilon_0[) \subset \mathbb{G}$ and let $x \in U$. The previous discussion can be summarized by saying that every open set $U_0$, which is compactly contained in $S_0$, has a neighborhood $U \subset \mathbb{G}$ satisfying the unique nearest point property [25], i.e. for every $x \in U$ there exists a unique point $y \in U_0 \subset S_0$ such that $\delta_\mu(x) = d_\mu(x, y)$. By using the previous notation, one has $\Psi(x) = (y, t)$, where $y = \Psi_S(x)$ and $t(x) = d_\mu(x, y) = \delta_\mu(x)$.

Theorem 3.5. Let $\mathbb{G}$ be a 2-step Carnot group. Let $S \subset \mathbb{G}$ be a $C^k$-smooth hypersurface with $k \geq 2$ and let $\delta_\mu$ denote the CC-distance function for $S$, i.e. $\delta_\mu(x) = \inf_{y \in S} d_\mu(x, y)$. Set $S_0 := S \setminus C_S$, where $C_S$ denote the characteristic set of $S$. Then, for every open set $U_0$ compactly contained in $S_0$, there exists a neighborhood $U \subset \mathbb{G}$ of $U_0$ having the unique nearest point property with respect to the CC-distance $d_\mu$. Finally, the CC-distance function from $U_0 \cap S$ is $\delta_\mu|_{U \cap V_0}$ is a $C^k$-smooth function.

Proof. We just have to prove the last claim. To this aim, let $X \in \mathfrak{X}(\mathbb{G})$ and set $y := \Psi_S(x)$, where $\Psi_S(x)$ denotes the projection along $U_0 \subset S_0$ of the point $x \in U := \Phi(U_0 \times ] - \epsilon_0, \epsilon_0[)$. Moreover set $t := d_\mu(x, y)$. We have

$$\langle \text{grad} \delta_\mu(x), X \rangle = \langle \text{grad} d_\mu(x, \Psi_S(x)), X \rangle = \left\langle \left( \text{grad}_x d_\mu(x, y) \right) \bigg|_{y = \Psi_S(x)}, X \right\rangle + \left\langle \left[ \mathcal{J}_x \Psi_S(x) \right] X, \left( \text{grad}_y d_\mu(x, y) \right) \bigg|_{y = \Psi_S(x)} \right\rangle. \quad (57)$$

Now let us introduce the following further notation.

Remark 3.6. Let $x \in \mathbb{G}$, $y \in \mathbb{G} \setminus H_{2}(x) = \{ z = \exp_{g}(z_{\mu}, z_{\nu}) \in \mathbb{G} : z_{\mu} = x_{\mu} \}$ and set $t := d_\mu(x, y)$. Moreover, let

$$\widetilde{\nu}(y) := \nu_{S_{SR}(x, t)}(y)$$

25With respect to the CC-distance $d_\mu$. [40, 41]
denote the Riemannian unit normal along the CC-sphere $S_{SR}(x,t)$ at any regular point $y \in S_{SR}(x,t)$. We stress that, under our hypotheses, the point $y$ turns out to be a regular point of $S_{SR}(x,t)$; see, for more details, Remark 2.42. Below we shall set
\[ N_{x,t}(y) := \frac{\bar{\nu}(y)}{|P_H(\bar{\nu}(y))|}. \]
According to the results of Section 2.4, we immediately get that:
\[ \text{grad}_x d_H(x,y) = N_{y,t}(x), \]
\[ \text{grad}_y d_H(x,y) = N_{x,t}(y). \]
By applying the results of Section 2.4, together with the explicit form of CC-geodesic for 2-step Carnot groups (see Section 2.5), we get that
\[ N_{y,t}(x) = \left( e^{-C_H(\varpi(y))} \nu_H(y), \varpi(y) \right). \]
Notice that $N_{y,t}$ is $C^{k-1}$-smooth as well as $N = (\nu_H, \varpi)$. Furthermore, it is easy to see that $N_{x,t}(y) = \pm N(y)$, where the sign only depends on the given orientation of $S$.

By the previous discussion and (57) we get that
\[ \langle \text{grad} \delta_H(x), X \rangle = \langle N_{\Psi_S(x),t}(x), X \rangle + \langle \left[ J_x \Psi_S(x) \right] X, N_{x,t}(\Psi_S(x)) \rangle. \]
Since $\left[ J_x \Psi_S(x) \right] X \in T_{\Psi_S(x)}S$, by using the fact that $N_{x,t}(\Psi_S(x))$ is normal to $S$ at $\Psi_S(x)$ one gets
\[ \langle \left[ J_x \Psi_S(x) \right] X, N_{x,t}(\Psi_S(x)) \rangle = 0. \]
Therefore $\langle \text{grad} \delta_H(x), X \rangle = \langle N_{\Psi_S(x),t}(x), X \rangle$. By the arbitrariness of $X \in \mathcal{X}(G)$, it follows that $\text{grad} \delta_H$ is of class $C^{k-1}$ on $U \setminus U_0$. Hence $\delta_H$ is of class $C^k$ on $U \setminus U_0$. This achieves the proof. \[ \square \]

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