C-oscillators and new outlook on cluster dynamics

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Abstract. The elementary physical theory of cluster dynamics is proposed. It is shown that cluster formation of dynamical processes in latticing structures lies within the framework of classic representations on a synchronization of dynamical systems. It is shown that any cluster structure is just a simple synchronization of certain number so-called cluster oscillators. The full set of types of cluster oscillators in one-dimensional homogeneous chain and two-dimensional homogeneous lattice of elementary oscillators are established. The principles of coupling of C-oscillators in cluster structures and principles of their possible transformation are explained.

Investigation of regular structures caused by processes of synchronization in coupled discrete lattices of dynamical systems is a one of the most important problems in modern nonlinear dynamics [1-8]. Importance of this problem is caused by the multiplicity of problems concerned with the structuring of dynamical processes that appear in different fields of science: in radio-physics, electronics, hydrodynamics, chemistry, biology etc. and also in technical applications [9,10,11,12].

By definition, cluster is a group of synchronized oscillators of a lattice. The theory of cluster structures is based on the existence of integral manifolds of lattices of dynamical systems: an integral manifold of a coupled system corresponds to each cluster structure of a lattice [13–17]. In this case, the problem of studying the existence of different cluster structures is reduced to finding a set of integral manifolds in a coupled system. Accordingly, the problem of the stability of a structure is reduced to the investigation of the stability of corresponding integral manifold. However, it is not always so. In some cases, parameters of the individual oscillator together with the parameter of coupling could have such magnitudes that can provide stability of “cluster” manifold, but cluster structure corresponding to this manifold may not exist. The formalism of integral manifolds does not account for a physical background of cluster dynamics in an explicit form.

In figures outlining results of numerical studies each cluster normally is painted by some colour. Accordingly, cluster structure of a lattice is represented as some mosaic figure. In the case of two- and three-dimensional lattices of large dimensions, a “spectrum” of such figures is so wide and figures themselves are so intricate [17] that dynamics of a whole system could be supposed to be mysterious and physically incomprehensible.

In this paper, the physical simplicity of the effect of cluster formation of dynamical processes in lattices based on the general definitions of the oscillation’s theory related to the synchronization of oscillators using simple theses and methods of elementary physics is explained. It is shown that this effect does not lie outside classic representations on a synchronization of dynamical systems and, essentially, is not a new physical effect. And at the same time the language of integral manifolds is not used at all.
1. Cluster oscillators and structures in a homogeneous chain of coupled systems

The simplest example of a coupled system is a chain of diffusively-coupled elementary oscillators with Neiman boundary conditions of the form

\[ \dot{X}_i = F(X_i) + \varepsilon C(X_{i+1} - 2X_i + X_{i-1}), \quad i = 1, N, \quad X_i \in \mathbb{R}^n, \quad F(X_i) : \mathbb{R}^n \to \mathbb{R}^n, \]

\[ \varepsilon \geq 0, \quad C = \text{diag}(c_1, c_2, \ldots, c_m), \quad c_i \geq 0. \]

\[ X_0 = X_1, \quad X_N = X_{N+1}. \]

Here matrix \( C \) defines the group of variables of oscillators, through which the coupling is done. Elements of \( c_i \) are equal either 0 or 1, \( \varepsilon \) is a scalar parameter of coupling. Let us suppose that each elementary oscillator is a chaotic oscillator (not a necessary condition) that is described by the following vector equation

\[ \dot{X} = F(X) \]

Let us consider a classic Chua’s oscillator:

\[ X = (x, y, z)^T, \]

\[ \dot{x} = \alpha(y - h(x)), \]

\[ \dot{y} = x - y + z, \]

\[ \dot{z} = -\beta y - \gamma z, \quad h(x) = m_0 x + \frac{1}{2}(m_0 - m_1)(|x+1| - |x-1|). \]

Hereinafter, the following set of parameters is used

\[ \{\alpha; \beta; \gamma; m_0; m_1; \varepsilon\} = \left\{9.5, 14.0, 0.1, -\frac{1}{7}, \frac{2}{7}, \varepsilon\right\} \]

And value of the parameter of coupling \( \varepsilon \) will be indicated at the left upper corner of each figure showing a phase portrait.

We assume that the dynamical properties of an elementary oscillator are completely known. In particular, we suppose that maximal Lyapunov index \( \lambda(1) \) of attractor \( A(1) \) (see figure 1) is known.

![Figure 1. Chua’s attractor \( A(1) \).](image)
2. Symmetrical cluster oscillators
Let us start with the simplest examples. Namely, we will consider the system of two
dissipative- and symmetrically-coupled oscillators of the form

\[
\begin{align*}
\dot{X}_1 &= F(X_1) + \varepsilon C(X_2 - X_1) \\
\dot{X}_2 &= F(X_2) + \varepsilon C(X_1 - X_2)
\end{align*}
\]  

(2)

The question is: what do we know about possible stationary regimes of two coupled
oscillators despite the number of their degrees of freedom and other individual dynamical
properties? Firstly, for some magnitude of the parameter of coupling \( \varepsilon \), which depends on
maximal Lyapunov index \( \lambda(1) \) of the attractor \( A(1) \) for certain initial conditions a regime
of mutual chaotic synchronization of self-oscillations will be observed [1,2]: \( X_1 = X_2 \) for
\( t \to \infty \) (see figure 2).

![Figure 2](image)

**Figure 2.** Simple chaotic synchronization of two elementary Chua’s attractors.

In the course of time, oscillations of oscillators become identical. A schematic
representation of this system in a regime of synchronization is depicted in figure 3

![Figure 3](image)

**Figure 3.** Schematic representation of the system of two coupled Chua’s attractors in a
regime of synchronization

Note that voltages at the oscillators (at the points of connection) are equal to each other at
any moment of time. Points of connections 1 and 2 are equipotential, and if we “cut” them,
we would not damage the regime of synchronization. In the case of radio-engineering
oscillators such cut is literal (the importance of the possibility of “cutting” will be clear later).
Secondly, in general case, besides the regime of synchronization, a regime of stationary
regular or stationary chaotic beatings can exist. Figure 4 shows an example of such a regime.
Both regimes are realized depending on the initial conditions.
In a regime of stationary beatings, values of all same-titled variables of oscillators are different at any moment of time and this system can be represented schematically in a following form (see figure 5):

![Figure 5. Schematic representation of the system of two coupled Chua’s attractors in a regime of stationary beatings.](image)

During the attempt to “cut” the connections between such elements one will get a shortcut. In a regime of stationary beatings system (2) is inseparable into elementary oscillators and represents single object. If there exists attractor \( A_s(2) \), then we will call system (2) as cluster oscillator (C-oscillator) of the form \( O_s(2) \). Let us rewrite system (2) in a double vector form:

\[
\begin{align*}
\dot{X} &= F(X), \\
X &= (X_1, X_2)^T, \\
F(X) &= (F(X_1), F(X_2))^T + \varepsilon B \otimes CX,
\end{align*}
\]

(3)

with \( \otimes \) the symbol of direct multiplication of matrices. Index \( s \) in denotation for C-oscillator means symmetry of the cluster matrix \( B \).

Now we will couple two oscillators of the type \( O_s(2) \) and consider the following system

\[
\begin{align*}
\dot{X} &= F(X) + \varepsilon C_s (Y - X) \\
\dot{Y} &= F(Y) + \varepsilon C_s (X - Y)
\end{align*}
\]

(4)

with \( C_s = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \otimes C \) (compare to system (2)).

System (4) represents a pair of interacting oscillators. In contrast to the previous case, these oscillators just have doubled number of degrees of freedom. The last circumstance does not add any novelty in interpretation of stationary dynamics of the system: starting from some value of the parameter of coupling \( \varepsilon^* \) that depends on the maximal Lyapunov index \( \lambda(2) \) of
the attractor $A_1(2)$, for certain initial conditions a simple synchronization of oscillators on this attractor will occur: $X = Y$ for $t \to \infty$. Schematic representation of this system in a regime of cluster synchronization is depicted in figure 6.

![Figure 6](image)

**Figure 6.** Schematic representation of the system of four coupled oscillators in a regime of cluster synchronization.

On the other hand, if $\varepsilon^* = \varepsilon$, and $Y = (X_4, X_1)^T$ then system (4) becomes system (1) for $N = 4$. Thus, in the regime of simple synchronization in a chain of C-oscillators a so-called “central” cluster structure $S_{c}^{1}(2) \colon X_1 = X_4, \ X_2 = X_1$ [14]. Note that chain can be cut into a pair of equal C-oscillators without making any harm to the regime of synchronization. Let us connect one more C-oscillator to the pair of C-oscillators:

$$
\begin{align*}
\dot{X} &= F(X) + \varepsilon^* C_s (Y - X) \\
\dot{Y} &= F(Y) + \varepsilon^* C_s (X - Y) + \varepsilon^* C^*(Z - Y) \\
\dot{Z} &= F(Z) + \varepsilon^* C^*(Y - Z)
\end{align*}
$$

with $C^* = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \otimes C$.

In the regime of simple synchronization $X = Y = Z$. Corresponding schematic representation is depicted in figure 7.

![Figure 7](image)

**Figure 7.** Schematic representation of the system (5) in the regime of simple synchronization.

On the other hand, if one will put $\varepsilon^* = \varepsilon$, and $Z = (X_5, X_6)^T$ in the system (5) then for $N = 6$ this system will become system (1). Thus, simple synchronization of three C-oscillators of the type $O_3(2)$ produces in a so-called “alternative” cluster structure $S_{a}^{11}(2) \colon X_1 = X_4, \ X_2 = X_3 = X_6$ [14]. As an example, the alternative cluster structure in a chain of $N = 6$ elementary Chua’s oscillators is depicted in figure 8. One can compare the cluster attractor $A_1(2)$ depicted in figure 4 and attractor of the structure – they are the same. Again, we draw the attention to the property of splitting of a chain into equal C-oscillators in the regime of synchronization.
$\varepsilon = 0.476$

![Figure 8](image)

**Figure 8.** Alternative cluster structure produced by the simple synchronization of three C-oscillators $O_s(2)$ in a chain of $N = 6$ elementary Chua’s oscillators.

The chain of cluster oscillators can be extended further by adding and coupling C-oscillators.

We will call the system of three symmetrically-coupled elementary oscillators of the form

$$
\dot{X}_i = F(X_i) + \varepsilon(-X_i + X_2) \\
\dot{X}_2 = F(X_2) + \varepsilon(X_1 - 2X_2 + X_3) \\
\dot{X}_3 = F(X_3) + \varepsilon(X_2 - X_3)
$$

as cluster oscillator $O_s(3)$ if there exists cluster attractor $A_s(3)$ corresponding to the regime of stationary beatings of all three elementary oscillators. An example of such cluster attractor $A_s(3)$ is depicted in figure 9. Schematic representation of C-oscillator $O_s(3)$ is depicted in figure 10.

$\varepsilon = 0.43$

![Figure 9](image)

**Figure 9.** Attractor $A_s(3)$ of C-oscillator $O_s(3)$ in projections on coordinate planes.
Now we rewrite system (5) as one equation
\[ X = F(X), \]
\[ X = (X_1, X_2, X_3)^T, F(X) = (F(X_1), F(X_2), F(X_3))^T + \varepsilon B \otimes CX \]
\[ B = \begin{pmatrix} -1 & 1 & 0 \\ 1 & -2 & 1 \\ 0 & 1 & -1 \end{pmatrix} \]

Then we will couple a pair of such C-oscillators and consider the following system
\[ \dot{X} = F(X) + \varepsilon^* C_s (-X + Y) \]
\[ \dot{Y} = F(Y) + \varepsilon^* C_s (X - Y), \]
with \[ C_s = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \otimes C. \]

Again we have got a system of two interacting oscillators. The interpretation of the stationary dynamics remains the same. In the regime of simple synchronization \( X = Y \). If \( \varepsilon^* = \varepsilon \), and \( Y = (X_6, X_5, X_4)^T \) then for \( N = 6 \) this system is identical to system (1). In other words, a central cluster structure occurs in the chain \( S^*_1(3) : X_1 = X_6, \ X_2 = X_5, \ X_3 = X_4 \). As an example, a central cluster structure in a chain of \( N = 6 \) elementary Chua’s oscillators is depicted in figure 11. A schematic representation of this structure is depicted in figure 12.

\[ \varepsilon = 0.43 \]
Attaching the third C-oscillator to the pair of C-oscillators of the type $O_3(3)$ in the same way, we obtain the following system

\[
\begin{align*}
\dot{X} &= F(X) + \epsilon^* C_s(-X + Y) \\
\dot{Y} &= F(Y) + \epsilon^* C_s(X - Y) + \epsilon^* C_s^*(-Y + Z) \\
\dot{Z} &= F(Z) + \epsilon^* C_s^*(Y - Z)
\end{align*}
\]

with $C^* = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, $Z = (X_7, X_8, X_9)$

In the regime of simple synchronization $X = Y = Z$ that corresponds to the alternative cluster structure $S_{alt}^3(3)$: $X_1 = X_6 = X_7$, $X_2 = X_8 = X_9$, $X_3 = X_4 = X_9$ in a chain of $N = 9$ elementary oscillators. Fig. 13 shows the schematic representation of this structure.

By adding new C-oscillators on can proceed with further up-building of the chain. For even numbers of C-oscillators one will get central structures and for the odd numbers of C-oscillators one will get alternative three-cluster structures.

**Definition 1**: we will call a system of $n$ elementary oscillators of the system (1) with the boundary condition $X_n = X_{n+1}$ if there exists attractor $A_{s}^n$ corresponding to the regime of stationary beatings of all these elementary oscillators as symmetric cluster oscillator of the type $O_s^n$.

For some conditions all the chain can also be considered as a symmetric cluster oscillator. Symmetric cluster oscillators can be coupled in a chain according to the principle, depicted in figure 14.

**Figure 12.** Schematic representation of central cluster structure in a chain of $N = 6$ Chua’s oscillators.

**Figure 13.** Schematic representation of the alternative cluster structure in a chain of $N = 9$ elementary oscillators.

**Figure 14.** Schematic representation of the coupling in a chain of cluster oscillators.
The simple synchronization of \( m \) coupled C-oscillators of the type \( O_e(n) \) for even \( m \) defines central \( n \)-cluster structure \( S^n_e(n) \) in a chain of \( N = mn \) elementary oscillators and defines the alternative cluster structure \( S^n_a(n) \) for the odd \( m \).

3. Asymmetric cluster oscillators

Consider the system of two asymmetrically-coupled elementary oscillators of the form

\[
\begin{align*}
\dot{X}_1 &= F(X_1) + \epsilon(-X_1 + X_2) \\
\dot{X}_2 &= F(X_2) + 2\epsilon(X_1 - X_2)
\end{align*}
\]

(7)

In this system, besides the regime of synchronization, a regime of stationary beatings (whose image is attractor \( A_a(2) \)) does also exist. The example of attractor \( A_a(2) \) for two asymmetrically-coupled Chua’s oscillators is depicted in figure.15.

\[ \epsilon = 0.215 \]

Figure 15. Attractor \( A_a(2) \) corresponding to the regime of stationary beatings of two asymmetrically-coupled Chua’s oscillators in projections on coordinate planes.

If there exists a regime of stationary beatings, then we will call the system (7) as C-oscillator of the type \( O_e(2) \). This system is defined by the following equation

\[
\begin{align*}
\dot{X} &= F(X) \\
X &= (X_1, X_2)^T, F(X) = (F(X_1), F(X_2))^T + \epsilon B \otimes CX, \\
B &= \begin{pmatrix} -1 & 1 \\ 2 & -2 \end{pmatrix}
\end{align*}
\]

Consider a system of two coupled C-oscillators of the type \( O_a(2) \)

\[
\begin{align*}
\dot{X} &= F(X) + \epsilon^* D(-X + Y) \\
\dot{Y} &= F(Y) + \epsilon^* D(X - Y) \\
D &= \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \otimes C
\end{align*}
\]

(8)
with \( X = (X_1, X_2)^T, Y = (Y_1, Y_2)^T, X_2 = Y_2 \) the additional internal coupling. This system can be physically interpreted by the circuit that is depicted in figure 16.

![Figure 16](image)

**Figure 16.** Physical interpretation of the system of two inertly-coupled C-oscillators of the type \( O_n(2) \).

In this circuit, elementary oscillators are denoted by encircled crosses. Two oscillators in the middle are coupled at the same points and thereby, they are always synchronized. Being coupled in a chain, asymmetric C-oscillators always coexist by pairs.

In the regime of simple synchronization of these C-oscillators \( X = Y \). On the other hand, for \( \varepsilon' = \varepsilon, Y = (X_3, X_2)^T \) and for \( N = 3 \) this system becomes system (1). A central cluster structure \( S^c_n(2) \) will be realized in the chain. Figure 17 shows schematic representation of this structure.

![Figure 17](image)

**Figure 17.** Schematic representation of a central cluster structure in the regime of simple synchronization in a system of two inertly-coupled C-oscillators of the type \( O_n(2) \) with additional internal coupling.

Consider the system of two coupled pairs of C-oscillators:

\[
\begin{align*}
\dot{X} &= F(X) + \varepsilon' D(-X + Y) \\
\dot{Y} &= F(Y) + \varepsilon' D(X - Y) + \varepsilon' C'(-Y + Z) \\
\dot{Z} &= F(Z) + \varepsilon' D(-Z + W) + \varepsilon' C'(Y - Z) \\
\dot{W} &= F(W) + \varepsilon' D(Z - W)
\end{align*}
\]

with additional internal couplings \( X_2 = Y_2, Z_2 = W_2 \).

In the regime of simple synchronization \( X = Y = Z = W \). On the other hand, if \( \varepsilon' = \varepsilon \), \( X = (X_1, X_2)^T, Y = (X_3, X_2)^T, Z = (X_4, X_5)^T, W = (X_6, X_5)^T \) then this system transfers to system (1) for \( N = 6 \). A central cluster structure \( S^c_n(2) \) is realized in the chain, which schematic representation is depicted in figure 18.
Figure 18. Central cluster structure $S^c_a(2)$ in a chain of two dissipative-coupled pairs of C-oscillators.

Figure 19 shows central cluster structure in a chain of $N = 6$ elementary Chua’s oscillators. One can compare this structure to attractor $A_a(2)$ of $O_a(2)$, which is depicted in figure. 15. It is seen that they are the same.

$$\varepsilon = 0.215$$

Figure 19. Central cluster structure based on the C-oscillator of the type $O_a(2)$ in a chain of $N = 6$ elementary Chua’s oscillators.

**Definition 2**: we will call a subsystem of $n$ elementary oscillators of the system (1) with the boundary condition $X_{n+1} = X_{n+1}$ if there exists attractor $A_a(n)$ corresponding to the regime of stationary beatings of all these elementary oscillators as *asymmetric cluster oscillator of the type* $O_a(n)$.

Asymmetric C-oscillators are always co-exist by pairs and can be coupled in a chain in the manner, depicted in figure. 20.

Figure 20. Schematic representation of a chain of asymmetric C-oscillators.

Simple synchronization of $m$ pairs of cluster oscillators of the type $O_a(n)$ defines central cluster structure $S^c_a(n)$ in a chain of $N = (2n - 1)m$ elementary oscillators.
Remark: Indexes \( s \) and \( a \) in notations for cluster oscillators denote symmetry or asymmetry of the cluster matrix \( B \). Hereinafter, these indexes will not be concerned with symmetry or asymmetry of cluster structures, produced by the synchronization of C-oscillators.

Theorem: Cluster oscillators \( O_s(n) \) and \( O_a(n) \) represent the full set of types of cluster oscillators in a homogeneous chain of elementary oscillators with Neiman boundary conditions.

The proof: Suppose that some cluster structure is realized in a chain and there is a segment of this chain consisting of first \( n \) not synchronized oscillators (cluster oscillator). This means that oscillator with number \((n+1)\) is synchronized with some of the elementary oscillators of the considered segment. Suppose, this is oscillator with number \( n \). In this case, values of all variables of these oscillators (currents, voltages) are equal to each other at any moment of the time. Points at the inputs of oscillators are equipotential so there is no current through the resistances connecting them (see circuit, depicted in figure 21).

Currents at inputs of synchronized oscillators fulfill the following equations

\[
V_{n-1} - V = IR, \quad V_{n+1} - V = IR \quad \Rightarrow \quad V_{n-1} = V_{n+1}
\]

It means that oscillators with numbers \((n-1)\) and \((n+2)\) are also synchronized by pairs.

Writing down equations for the currents of these oscillators, we obtain

\[
V_{n-2} + V - 2V_{n} = I_{n-1}R, \quad V_{n+3} + V - 2V_{n+2} = I_{n+2}R \quad \Rightarrow \quad V_{n-2} = V_{n+3}
\]

Continuing this we finally obtain that the condition \( X_{n} = X_{n+1} \) corresponds to the coupling of C-oscillators of the type \( O_s(n) \). Thus, formation of cluster structure is based on this C-oscillator. Suppose now that \((n+1)\)-th oscillator is synchronized with \((n-1)\)-th one. Considering equations for currents of synchronized oscillators, we obtain

\[
V_{n-2} - 2V_{n-1} + V_{n} = I_{n-1}R, \quad V_{n+2} - 2V_{n+1} + V_{n} = I_{n+1}R \quad \Rightarrow \quad V_{n-2} = V_{n+2}
\]

Continuing doing this we obtain that condition \( X_{n} = X_{n+1} \) corresponds to the coupling of C-oscillators of the type \( O_a(n) \). Finally we suppose that oscillator with number \((n+1)\) is synchronized with some other \( k \)-th elementary oscillator of the cluster oscillator \((k \leq n-2)\). By coupling the appropriate equipotential points and by making further transformations of the circuit (which do not harm the dynamical regime of the chain), we obtain a sequence of equivalent circuits represented in the figure 22.
Note that in the first and the third circuits there is no current through the connector. We obtain

\[ V_n - 2V_{n+1} + V_{n+2} = IR, V_{k+1} - 2V_k + V_{n+2} = IR \Rightarrow V_n = V_{n+1} \]

Here comes the contradiction: the system of first \( n \) elementary oscillators contains the synchronized oscillators and therefore is not a cluster oscillator. Thus all C-oscillators of the types \( O_s(n) \) and \( O_a(n) \) constitute full set of types of cluster oscillators in a chain. Now we conclude the following:

The effect of cluster synchronization consists in a simple synchronization of cluster oscillators ("synchronization of regimes of beatings").

Each cluster structure is a result of simple synchronization of certain number of equal cluster oscillators of one of these two types: \( O_s(n) \) or \( O_a(n) \).

Each cluster structure can be “cut” through the connections of equipotential points into a certain number of equal C-oscillators of the type \( O_s(n) \) or into a certain number of equal pairs of C-oscillators of the type \( O_a(n) \) with their further reduction into one cluster oscillator by means of connection of the equipotential points.

Each structure that can not be “cut” and thereby simplified is not physically feasible (a necessary condition).

The problem of existence of different types of cluster structures for assigned “length” of a chain \( N \) can be reduced to the elementary problem of covering a segment of elementary oscillators by equal cluster oscillators with different number of clusters \( n \).

Remark 1 all known theorems concerning existence of alternative invariant manifolds in the system (1) [14,15] are now just a simple consequence of the aforementioned theorem. Moreover, it follows from this theorem that there are no other types of integral manifolds in the system (1).

Example: figure 23 shows the complex of cluster structures in a chain of \( N = 12 \) elementary oscillators. According to the aforementioned theorem, there are no other cluster structures in a chain of \( N = 12 \) elementary oscillators.
4. Cluster oscillators and structures in two-dimensional lattices of coupled oscillators

Consider a homogeneous rectangular lattice of coupled oscillators with Neiman boundary conditions

\[
\dot{X}_{ij} = F(X_{ij}) + \varepsilon C(X_{i+1,j} + X_{i-1,j} + X_{ij+1} + X_{ij-1} - 4X_{ij})
\]

\[i = 1,N_1, \quad j = 1,N_2.\]  

\[X_{0j} \equiv X_{ij}, \quad X_{N_1+1,j} \equiv X_{N_1,j}, \quad X_{10} \equiv X_{i1}, \quad X_{iN_2+1} \equiv X_{iN_2} \]

Again we will solve the problem of existence of different cluster structures by finding a set of basic types of cluster oscillators and by following the elementary idea of covering of a lattice by equal cluster oscillators and by blocks of cluster oscillators.

**Definition 3** We will call lattice of the size \( p \times q \), whose nodes are filled up by the elements of one cluster oscillator or by the elements of a block of equal cluster oscillators, which can not be cut as simple cell.

In the case of a chain, such cells are segments, filled up by elements of C-oscillator of type \( O_s(n) \) or by elements of pair of C-oscillators of type \( O_a(n) \). In a two-dimensional case, such cells will be filled up by elements of one C-oscillator or by elements of a pair, four, eight but not more C-oscillators. Note that the entire lattice \( N_1 \times N_2 \) under some conditions can be considered as a simple cell. This simple fact allows us to suppose that simple cells have “rectangular” form. We will use different types of the symmetry of plane figures, understanding the symmetry more in physical than in geometrical sense: as a equality of potentials of certain points of simple cell with respect to its axes and centres of symmetry.

5. List of cluster oscillators and simple cells

1, 2, 3, 4. The simplest type of C-oscillators are the one-dimensional C-oscillators of the type \( O_s(1 \times n), O_s(n \times 1), O_a(1 \times n), O_a(n \times 1) \) that are heritable by two-dimensional lattice from a chain. Simple synchronization of such C-oscillators realizes “band-like” cluster structures. An example of such a structure is depicted in figure 24.
Figure 24. Band-like cluster structures based on C-oscillators of types $O_j (1 \times 3), O_a (1 \times 2), O_j (3 \times 1), O_a (2 \times 1)$, respectively.

C-oscillators of the type $O_j (.)$ and blocks of C-oscillators of the type $O_a (.)$ (simple cells) can be coupled. “Two-dimensional” C-oscillators in a regime of simple synchronization define mosaic cluster structures.

5. C-oscillator of the type $O_{as} (m \times n)$ is a subsystem of $mn$ elementary oscillators of the system (9) with boundary conditions $X_{mn} = X_{m+1n}, X_{in} = X_{in+1}$ if there exists attractor $A_{as} (m \times n)$. C-oscillators of this type always co-exist by pairs. A building of a simple cell and of a posterior cluster structure into a “large” lattice can be done in the way that is depicted in figure 26. The size of a simple cell of this type is $(m \times 2n - 1)$.

Figure 25. Example of coupling of C-oscillators of the type $O_{as} (2 \times 3)$.

Figure 26. Example building of a simple cell and posterior cluster structure based on C-oscillator $O_{as} (2 \times 2)$.
b) Bending through the horizontal axis.

7. C-oscillator $O_{mn} (m \times n)$ is a subsystem of $mn$ elementary oscillators of system (9) with boundary conditions $X_{m,j} \equiv X_{m+1,j}, X_{m,j} \equiv X_{m,j+1}$ if there exists attractor $A_{mn} (m \times n)$. C-oscillators of this type also always co-exist by pairs. Coupling of C-oscillators in a cell and coupling of cells in a lattice can be done in the way that is depicted in figure 27. The size of a simple cell of this type is $(2m-1 \times n)$.

![Figure 27](image)

Figure 27. Example of building of cell and lattice based on C-oscillator $O_{2 \times 2}$.

c) Suppose that simple cell represents a block of four C-oscillators. In this case, reduction of the cell in one C-oscillator causes by double sequential bending through both axes.

8. C-oscillator $O_{mn} (m \times n)$ is a subsystem of $mn$ elementary oscillators of system (9) with boundary conditions $X_{m-1,j} = X_{m+1,j}, X_{m,j-1} = X_{m,j+1}$ under condition of existence of attractor $A_{mn} (m \times n)$ corresponding to the regime of stationary beatings of all these elementary oscillators. C-oscillators of this type also always co-exist by fours and can be coupled in a simple cell as depicted in figure 28. The size of a simple cell of this type is $(2m-1 \times 2n-1)$.

![Figure 28](image)

Figure 28. An example building of cell and lattice based on C-oscillator $O_{2 \times 2}$.

There are no other bends of a simple cell allowed. This fact can be easily proven by means of Kirchoff laws. Simple cell can not contain more than for C-oscillators of such configuration.

d) Oscillators, concerned with diagonal symmetry of a square. Suppose that simple cell is square-like and has a size $(n \times n)$. Let us make transforms of such a cell concerned with connection of possible equipotential points. We combine a lattice by bending it through one of diagonals and by bending all consequent figures through their axes of symmetry. Making these transforms we get sequences of figures shown in figure 29.
At the second iteration of the lattice transformation with even number of elements we cut the figure through its axis of symmetry. All further transformations can be done in the same way as in the case of lattice with odd number of elements. C-oscillators are the systems of not synchronized oscillators placed at the joints of latticing figures. Unfolding C-oscillators into a corresponding square (simple cell), the order of their coupling goes in the reverse sequence. During the inverse transformations, the following conditions have to be fulfilled: (a) the number of coupled cluster oscillators forming a simple cell should not exceed eight, b) elementary oscillators from one cluster can not be both on a boundary of a square and inside it. These conditions can be elementary proven within the framework of the Kirchhoff laws as it has been done for a chain. Examples of construction of simple cells and corresponding cluster structures are shown in figure 30. Further cells can be coupled in a “large” lattice. It is easy to see that there are five qualitatively different types of cluster oscillators concerned with diagonal symmetry of a square. We will skip determining the set of equations for C-oscillators, but we will make their “geometrical” description.

**Figure 29.** Sequences of figures concerned with connection of possible equipotential points.

**Figure 30.** Examples of construction of simple cells and corresponding cluster structures.
Figure 30. (continuation) Examples of construction of simple cells and corresponding cluster structures.

9,10. “Flags–2” are the C-oscillators of the type $O_{2a} \left( n(n+1)/2 \right)$ and $O_{2a}^{1} \left( n(n+1)/2 \right)$ unfolding to $n$–square in one iteration as shown in figure. 30 (b) and (e). $O_{2a}^{1} \left( . \right)$ results in rotating $O_{2a} \left( . \right)$ into $90^\circ$. In a simple cell, these oscillators define structures with diagonal symmetry.

11. “Pyramid” is a cluster oscillator $O_{4a} \left( \left( n+1 \right)^2/4 \right)$ unfolding to $n=2k+1$ –square in two iterations as depicted in figure. 30 (d) and (g). This C-oscillator and all the further ones define structures with central symmetry.

12. “Mausoleum” is a cluster oscillator $O_{4a} \left( \left( n^2+2n \right)/4 \right)$ unfolding to $n=2k$ –square (simple cell) in two iterations as shown in Fig. 30 (a) and (h).

13. “Flag–8” is a cluster oscillator $O_{8a} \left( \left( n+1 \right)\left( n+3 \right)/8 \right)$ unfolding to $n=2k+1$ –square in three iterations as shown in figure. 30 (c) and (f).

C-oscillators and simple cells, concerned with central symmetry of rectangles (squares).

14. “π–rectangle”, a cluster oscillator $O_{2\pi} \left( mn/2 \right)$ is a rectangular lattice of the size $(m/2 \times n)$ for $m=2k$ or of the size $(m \times n/2)$ for $n=2k$. Simple cell of the size $(m \times n)$ can be derived by coupling C-oscillator and its image. The latter results in rotation of C-oscillator into $180^\circ$. Such C-oscillators co-exist by pairs. The rule of formation of a simple cell and further cluster structures is shown in figure 31.
Figure 31. Example of the rule of formation of a simple cell and further cluster structures based on $O_{(6)}^{5}(6)$.

15. “π–flag” is a C-oscillator of the type $O_{(6)}^{5}\left((mn+1)/2\right)$. Simple cell of the size $(m \times n)$, $m = 2k + 1$, $n = 2p + 1$ that can be derived by connecting C-oscillator and its image. The latter results in rotation of C-oscillator into $180^\circ$. Such C-oscillators co-exist by pairs. The rule of formation of a simple cell and further cluster structures is shown in figure 31.

Figure 32. Example of the rule of formation of a simple cell and further cluster structures based on $O_{(5)}^{5}(5)$.

Statement: all aforementioned 15 types of C-oscillators constitute the full set of types of cluster oscillator in a two-dimensional lattice (9).

Remark: the problem of existence of different types of integral manifolds in system (9) can be solved based on the set of types of C-oscillators and simple cells in combination of properties of numbers $N_1$ and $N_2$ (sizes of lattice).

Example: figure 33 shows the full set of cluster oscillators and structures in the lattice $(3 \times 6)$.

Let us discuss briefly some consequences of the proposed theory.

6. On cluster oscillators and structures in three-dimensional lattice of coupled oscillators

The list of C-oscillators and cluster structures, generated by them in a spatial lattice is too large to be covered by this paper. Due to this, we will limit ourselves to short remarks.

1. The most simple are the cluster oscillators of the type $O(n \times 1 \times 1)$, $O_a(n \times 1 \times 1)$, $O(1 \times n \times 1)$, $O_a(1 \times n \times 1)$..., that are inherited from two-dimensional lattice. Simple synchronization of these oscillators defines layered cluster structures: band-like structure that is made by elementary oscillators of certain layer, which can be coupled to each other. As a result one has layered spatial cluster structure. There are only 12 such C-oscillators.
2. All two-dimensional mosaic cluster oscillators also “pass in to the space” together with their equations. In particular, C-oscillators concerned with axial symmetry of rectangles pass into C-oscillators of the type $O_{ss} (m \times n \times 1)$, $O_{ss} (m \times 1 \times n)$, $O_{ss} (1 \times m \times n)$, $O_{ss} (m \times n \times 1)$... Altogether, there are 33 such C-oscillators. Simple synchronization of two-dimensional mosaic C-oscillators in a three-dimensional lattice in colour representation gives parallel multicoloured “threads” – mosaic layer couples dissipative with the identical one (colour to colour). In result one has aforementioned cluster structure.

Figure 33. Cluster oscillators and structures in homogeneous lattice $(3 \times 6)$ of coupled oscillators.
3. Spatial mosaic cluster structures are defined by simple synchronization, in particular by synchronization of C-oscillators of types $O_{ssv}(m \times n \times p)$, $O_{ssv}(m \times n \times p)$, $O_{ssv}(m \times n \times p)$... Altogether, there are 27 such C-oscillators.

4. All other spatial mosaic C-oscillators are concerned with different “equipotential” transforms of parallelepiped and cube that concerned with different types of symmetry of these figures.

7. Cluster trees of coupled oscillators
Suppose we have enough number of equal one-dimensional C-oscillators of a certain type. Let us compose a chain from these C-oscillators and provide the regime of cluster synchronization. In the course time, motions of corresponding elementary oscillators composing C-oscillators will be almost identical (we suppose that they are absolutely identical). We will cut the chain through equipotential points into initial number of C-oscillators. At the same time, every one of these will remain in initial dynamical regime. The difference of phases of corresponding elementary oscillators entering C-oscillators will be equal to zero. Now, “working under the voltage” following the rule “colour to colour”, we can couple C-oscillators not only in a chain but also in various figures of most miraculous forms, for instance “trees”. Such trees instead of “branches” can also contain rings – branches that are grown together (See figure 36). Dynamical processes in elementary oscillators, composing the figure will be the same as in Neiman chain. A symbolic representation of cluster tree is depicted in figure 34.

![Figure 34](image1.png)

Figure 34. Symbolic representation of cluster tree.

In this paper, we will not pay attention to the stability of cluster structures in such systems and to physical interpretation or any other interpretation of cluster trees. We will just pay attention to the properties of differential equations that govern their dynamics based on an example of a simplest tree (see figure 35). These equations have the form

\[
\begin{align*}
\dot{X}_1 &= F(X_1) + \varepsilon C(X_2 - X_1) \\
\dot{X}_2 &= F(X_2) + \varepsilon C(X_1 - 2X_2 + X_3) \\
\dot{X}_3 &= F(X_3) + \varepsilon C(X_2 + X_4 + X_5 - 3X_3) \\
\dot{X}_4 &= F(X_4) + \varepsilon C(X_3 - X_4) \\
\dot{X}_5 &= F(X_5) + \varepsilon C(X_3 + X_6 - 2X_5) \\
\dot{X}_6 &= F(X_6) + \varepsilon C(X_5 - X_6)
\end{align*}
\]  

(10)

This system has the following properties. First, system (10), as it has been done above, can be rewritten as system of three interacting dissipative-coupled symmetric C-oscillators of the type $O_s(2)$. Simple synchronization of these C-oscillators causes cluster structure that is
depicted in figure 34. Second, it is easy to establish that system (10) has integral manifold \( M = \{ X_2 = X_3 = X_5, X_4 = X_6 \} \). Thus, existence of cluster trees, built from cluster oscillators is justified as from physical as from formal standpoint.

![Figure 36](image) An example of cluster tree built based on C-oscillator \( O_x(2) \).

8. On a cluster formation of dynamical processes in irregular homogeneous lattices
Suppose we have homogeneous lattice that is large enough in all directions and a cluster structure based on certain type of cluster oscillator is realized in this lattice. By cutting the equipotential points we can obtain any number of C-oscillators from this chain. Such a cutting can be done as for boundary C-oscillators as for the internal ones. After this procedure, the border of the lattice becomes irregular and, if necessary, multivariable. At the same time, its physical properties as a “formative” basic cluster oscillator remain the same as for the initial “regular” lattice. If initial lattice “physically” is rather small-sized and cluster oscillator contains small number of elements than the boundary of “new” lattice can be formed to the shape, defined beforehand as close as one wants. The border could be adjusted as almost smooth and lattice itself could have a shape of a disc or a ring.

9. On a cluster formation of dynamical processes in ordered inhomogeneous coupled systems
Always, during the definition of cluster in homogeneous systems as a group of synchronized oscillators, the obvious fact is not discussed. This fact is that synchronization is not “identical”: \( X_i = X_j \) for \( t \to \infty \) with \( X_i \) and \( X_j \) the variables of elementary oscillators from one cluster or, as it was mentioned above, the variables of the self-named elementary oscillators entering different cluster oscillators of the same type (the core of cluster synchronization). It is possible, being in the frames of this definition including identity of synchronization to violate the condition of homogeneity of a “medium”. Namely, if a lattice is built from a specific cluster oscillator (!) then there is no necessity for elementary oscillators forming C-oscillator to be identical (in particular, they can be of different types). Moreover, in case of not identity of elements a cluster oscillator, the main requirement consisting in the absence of synchronization between its elements will be fulfilled even better if one would account for influence of parameters of coupling on the “duration of life” of cluster attractors of C-oscillators. By coupling C-oscillators with non-identical elements according to aforementioned principles we will get a “large” ordered-inhomogeneous lattice, which dynamics will be clustered as in a homogeneous medium.

In conclusion we will make the remark concerning investigation of cluster dynamics of systems. In systems (1) and (9), the intervals of existence of cluster attractors for typical elementary oscillators (Chua’s oscillator, Lorenz oscillator etc.) are the finite intervals
depending on the parameter of coupling. There are no cluster attractors outside these intervals and corresponding C-oscillators cease to be cluster oscillators. It is natural that the appropriate cluster structures will not be observed since they even do not exist independently on the stability of corresponding integral manifold. Thus, studying the conditions of existence of cluster attractors is the main problem of cluster dynamics.

10. Conclusions
In this paper, we have proposed the elementary physical theory of cluster dynamics. It is shown that cluster formation of dynamical processes in lattices lies within the framework of classic representations on a synchronization of dynamical systems. According to this theory, any cluster structure is just a simple synchronization of certain number so-called cluster oscillators.

The types of cluster oscillators in one-dimensional chain of elementary oscillators are established. Those are symmetric and asymmetric C-oscillators. It is proven that only these two types of C-oscillators can exist in a homogeneous chain.

For a two-dimensional lattice of coupled oscillators, 15 types of C-oscillators representing the full set of types of cluster oscillators in such a lattice are defined.

The principles of coupling of C-oscillators in cluster structures and principles of transformation of these structures providing their essential simplification and thereby better understanding of dynamics processes in complicated cluster structures are explained.

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