On Poincaré extensions of rational maps.

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Abstract

There is a classical extension, of Möbius automorphisms of the Riemann sphere into isometries of the hyperbolic space $\mathbb{H}^3$, which is called the Poincaré extension. In this paper, we construct extensions of rational maps on the Riemann sphere over endomorphisms of $\mathbb{H}^3$ exploiting the fact that any holomorphic covering between Riemann surfaces is Möbius for a suitable choice of coordinates. We show that these extensions define conformally natural homomorphisms on suitable subsemigroups of the semigroup of Blaschke maps. We extend the complex multiplication to a product in $\mathbb{H}^3$ that allows to construct a visual extension of any given rational map.

1 Introduction

In the literature there are some constructions of extensions of rational dynamics from $\mathbb{C}$ to $\mathbb{H}^3$, see for example [9], [12] and [14]. The constructions in [12] and [14] are based on Choquet’s barycentric construction introduced and studied by A. Douady and C. Earle in their paper [5]. Other important contributions on the barycentric constructions appear in [1] and [4].

As it was mention in the abstract, the basic idea of this paper is the following fact: “Any holomorphic covering between Riemann surfaces is a Möbius map on suitable coordinates.” Then this covering can be extended to suitable Möbius manifolds. Let us discuss this idea in details.

First, remind that a Möbius $n$-orbifold is a $n$-orbifold endowed with an atlas such that the transition maps are Möbius transformations.

Given a discrete subgroup $\Gamma$ of Möbius transformations of the $n$-sphere $S^n$ acting properly discontinuous and freely on a domain $\Omega \subset S^n$, the quotient manifold $\Omega/\Gamma$ admits a Möbius structure. In the case when $n = 3$, any manifold modeled on one of the following spaces $\mathbb{R}^3$, $S^3$, the unit ball $B^3$ in $\mathbb{R}^3$, $S^2 \times \mathbb{R}$ or $B^2 \times \mathbb{R}$ admits a Möbius structure, see [16].

Let $S_1$ and $S_2$ be two Möbius 2-orbifolds and let $R : S_1 \to S_2$ be a finite degree covering which is Möbius on the respective Möbius structures. Assume that there exist two Kleinian groups $\Gamma_1$ and $\Gamma_2$ and two components $W_1$ and $W_2$ of the discontinuity sets $\Omega(\Gamma_1)$ and $\Omega(\Gamma_2)$ respectively, such that

$$S_i = W_i/\text{Stab}_{W_i}(\Gamma_i)$$

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for \( i = 1, 2 \). Now assume that there exist a Möbius map \( \alpha(R) : W_1 \rightarrow W_2 \) making the following diagram commutative

\[
\begin{array}{ccc}
W_1 & \xrightarrow{\alpha(R)} & W_2 \\
\pi_1 \downarrow & & \downarrow \pi_2 \\
S_1 & \xrightarrow{R} & S_2
\end{array}
\]

so that \( \alpha(R) \) induces a homomorphism from \( \Gamma_1 \) to \( \Gamma_2 \). If

\[
M_i = (\mathbb{H}^3 \cup W_i)/\Gamma_i,
\]

then \( \alpha(R) \) induces a unique Möbius morphism

\[
\tilde{R} : M_1 \rightarrow M_2
\]

which is an extension of \( R : S_1 \rightarrow S_2 \). We call the map \( \tilde{R} : M_1 \rightarrow M_2 \) a Poincaré extension of \( R \). The map \( \tilde{R} \) depends on the uniformizing groups \( \Gamma_1 \) and \( \Gamma_2 \). Hence, in general, for a given covering map \( R \) there are many possibilities to construct a Poincaré extension. Note that the degree \( \text{deg}(\tilde{R}) \) is equal to the index \( [\Gamma_2 : \alpha(R) \circ \Gamma_1 \circ \alpha(R)^{-1}] \). Hence \( \text{deg}(\tilde{R}) \leq \text{deg}(R) \) with equality when

\[
\text{Stab}_{W_i}(\Gamma_i) = \Gamma_i.
\]

Given a Riemann surface \( S \) with a fixed Möbius structure, in [6, sect.8] R. Kulkarni and U. Pinkal constructed a Möbius 3-manifold \( M \) such that the surface \( S \) is canonically contained in the boundary of \( M \). If the structure of \( S \) is uniformizable by a non trivial Kleinian group then, this construction is given by the Classical Poincaré extension of the uniformizing group and produces a complete hyperbolic manifold \( M \). The construction is based on the following idea: Let \( D \) be a round disk on \( S \) with respect to the Möbius structure; that is, there exist a coordinate under which \( D \) is a round disk in the plane. Using this coordinate, we attach a round half-ball in \( \mathbb{H}^3 \) to \( D \). Then the 3-manifold \( M \) is the union of all the half open balls over all round disks in \( S \).

On the Riemann sphere \( \mathbb{C} \) there is a unique complete Möbius structure \( \sigma_0 \), this is the standard Möbius structure on \( \mathbb{C} \). The construction of Kulkarni and Pinkal is clearer when \( S \) is a planar surface with the standard Möbius structure. Let \( R \) be a branched self-covering of \( \mathbb{C} \). If \( \text{deg}(R) > 1 \), then \( R \) is not a Möbius covering with respect to the standard structure on any domain in \( \mathbb{C} \).

When \( S = \mathbb{C}^* \) with the standard Möbius structure, Kulkarni-Pinkal construction gives a canonical non-complete Möbius 3-manifold which is Möbius equivalent to the 3-dimensional ball with the vertical diameter removed and endowed with the standard conformal structure on \( B^3 \). Now, consider a complete Möbius structure on \( \mathbb{C}^* \). In this case, Kulkarni-Pinkal construction gives a complete hyperbolic 3-manifold with the same underlying space as before. More generally, if \( S = \mathbb{C} \setminus F \), where \( F \) is a closed set, then the Kulkarni-Pinkal’s extension \( M \) is homeomorphic to \( \mathbb{H}^3 \setminus \text{convhull}(F) \), where \( \text{convhull}(F) \) is the hyperbolic convex hull in \( \mathbb{H}^3 \) of all points in \( F \). The standard Möbius structure on \( M \) is the extension of the standard Möbius structure on \( S \). The construction of Kulkarni and Pinkal motivates the idea of a model manifold for the Poincaré extension of a rational map.
We will restrict our attention to the case when $R$ is a rational map and $S_1$ and $S_2$ are two Riemannian orbifolds with underlying spaces contained in $\mathbb{C}$, and such that $R : S_1 \to S_2$ is a holomorphic covering. Let $\sigma_2$ be an uniformizable Möbius structure on $S_2$ and suppose that the pullback $\sigma_1 := R_*(\sigma_2)$ is also a uniformizable Möbius structure on $S_1$. If $\Gamma_1$ and $\Gamma_2$ are the uniformizing groups. Let $\hat{R}$ be a Poincaré extension of $R$, such that $\Omega(\Gamma_1)$ are connected. Let $\phi_i : \partial M_i \to S_i$ be some identification maps and assume that there are homeomorphic extensions $\Phi_i : M_i \to \tilde{\mathbb{H}}^3$ for each $\phi_i$. Then the map $\Phi_2 \circ \hat{R} \circ \Phi_1^{-1}$ is called geometric extension if and only if satisfies the following conditions.

1. The sets $\Phi_i(M_i \cup \partial M_i)$ are of the form $\tilde{\mathbb{H}}^3 \setminus \bigcup \gamma_j$ where each $\gamma_j$ is either a quasi-geodesic or a family of finitely many quasi-geodesic rays with common starting point. There are no more than countably many curves $\gamma_j$. Here by quasi-geodesic we mean the image of a hyperbolic geodesic by a quasiconformal automorphism.

2. There exist a continuous extension, on all $\mathbb{H}^3$, which maps complementary quasi-geodesics to complementary quasi-geodesics.

Hence, a geometric extension is an endomorphism of $\mathbb{H}^3$ such that its restriction to $\Phi_1(M_1)$ is a Poincaré extension.

Let $\text{Rat}_d(\mathbb{C})$ denote the set of rational maps $R$ of degree $d$. Let $A \subset \text{Rat}_d(\mathbb{C})$. Assume that there exist a map

$$\text{Ext} : A \to \text{End}(\mathbb{H}^3)$$

such that $\text{Ext}(R)$ is an extension of $R$ for every $R$ in $A$. Then for every pair of maps $h, g$ in the Möbius group $\text{Mob}$ we define

$$\hat{\text{Ext}}(g \circ R \circ h) = \hat{g} \circ \text{Ext}(R) \circ \hat{h}$$

where $\hat{g}$ and $\hat{h}$ are the classical Poincaré extensions of the maps $g$ and $h$ in the hyperbolic space, respectively.

If $\hat{\text{Ext}}$ is a well defined map from the Möbius bi-orbit of $A$ to $\text{End}(\mathbb{H}^3)$, then we call $\text{Ext}$ a conformally natural extension of $A$.

In particular, when the bi-action of $\text{PSL}(2, \mathbb{C})$ on $A$ has no fixed points on $A$, then any map $\text{Ext} : A \to \text{End}(\mathbb{H}^3)$ defines a map $\hat{\text{Ext}}$ on the Möbius bi-orbit which is a conformal natural extension. If the action has fixed points then, in order to obtain a conformally natural extension the map $\text{Ext}$ has to be consistent with the Möbius action. The situation is tricky, even in the case when $A$ consists of a single point $R$.

Let $\hat{R}$ be a geometric extension of a rational map $R$, then any rational map on the Möbius bi-orbit of $R$ has a geometric extension. Namely, if $h$ and $g$ are elements in $\text{PSL}(2, \mathbb{C})$, then $Q = g \circ R \circ h$ has a geometric extension with the same uniformizing groups $\Gamma_1$ and $\Gamma_2$, the projections $p_1 = \pi_1 \circ \hat{h}$ and $p_2 = \pi_2 \circ \hat{g}^{-1}$ and the associated manifolds are $N_1 = \hat{h}^{-1}(M_1)$ and $N_2 = \hat{g}(M_2)$. We define an extension of $Q$ by the formula $Q = \hat{g} \circ \hat{R} \circ \hat{h}$.

However, one should be careful in the situation when, for a given $R$, there are other elements $g'$ and $h'$ in $\text{PSL}(2, \mathbb{C})$ such that $Q = g' \circ R \circ h'$. This situation happens when there are elements $h_1$ and $h_2$ in $\text{PSL}(2, \mathbb{C})$ such that

$$R \circ h_2 = h_1 \circ R. \quad (*)$$
If there are no such $h_1$ and $h_2$ then, any geometric extension of $R$ is conformally natural. However, if such elements do exist but the Poincaré extensions of $h_i$ are Möbius automorphisms of $M_i$ with respect to their Möbius structures, then the extension $R \mapsto \hat{R}$ is conformally natural.

In this article, we will investigate the existence of extensions of $R$, defined in the hyperbolic space $\mathbb{H}^3$, satisfying as many as possible of the following desirable conditions.

1. **Geometric.** As defined above.

2. **Same degree.** The index $[\Gamma_2 : \Gamma_1]$ is equal to the degree of the map $R$.

3. **Dynamical.** Let $\hat{R}$ be a Poincaré extension such that $M_1 = M_2$, then $\hat{R}$ is *dynamical*. In particular, these are extensions $\text{Ext}$ such that $\text{Ext}(R^n) = \text{Ext}(R^n)$ for $n = 1, 2, \ldots$.

4. **Semigroup Homomorphisms.** A stronger version of the previous property is to find semigroups $S$, of rational maps, for which there is an extension $\text{Ext}$ defined in all $S$ such that $\text{Ext}(R \circ Q) = \text{Ext}(R) \circ \text{Ext}(Q)$.

5. **Equivariance under Möbius actions.** We look for conformally natural extensions of subsets $A$ of $\text{Rat}_d(\mathbb{C})$.

Assume that $R$ has a geometric extension, and let $\Gamma_2$ be the group that uniformizes the Möbius structure on $S_2$. Then the discontinuity set $\Omega(\Gamma_2)$ consists of the orbit of a unique component $C$. The stabilizer of $C$ uniformizes the surface $S_2$. In this case the group $\Gamma_2$ is either totally degenerated, of Schottky type or of Web group type. When $\Gamma_2$ is a web group, the orbit of the component $C$ is infinite. Thus in general it is possible that the condition (2) may not be satisfied.

On the other hand, if the component $C$ uniformizing the surface $S_2$ is invariant under $\Gamma_2$, then condition (2) is satisfied. For this reason, we restrict our discussion to this case. A group having an invariant component is called a function group. By Maskit’s theorem any function group can be represented as a Klein-Maskit combination of the following groups:

- Totally degenerated groups.
- Schottky type.

According to this list of groups we call a geometric extension *totally degenerated*, or of *Schottky type*, whenever the uniformizing group has the corresponding property. Totally degenerated groups appear as geometric limits of quasifuchsian groups. In fact, totally degenerated groups belong to the boundary of a Bers’ slice.

This paper is organized as follows.

In Section 2 we will discuss Hurwitz spaces and quasifuchsian extensions. We will construct an extension, the radial extension, that satisfies (2), (3) and (4) in the previous list. We will give some conditions for which the radial extension is geometric.

Section 3 is devoted to Schottky type extensions.

In Section 4, we discuss a visual extension of all rational maps which is connected to a product structure defined on the hyperbolic space.
Finally, in Section 5, we discuss examples of extensions and surgeries of Maskit type.

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2 Fuchsian structures, degenerated and radial extensions

Given a rational map $R$, let $CV(R)$ denote the critical values of $R$ and take $S_2 = \mathbb{C} \setminus CV(R)$ and $S_1 = R^{-1}(S_2)$ then $R : S_1 \to S_2$ is a covering. We assume that the set of critical values of $R$ contains at least three points, say $b_1, b_2$ and $b_3$, so that $S_2$ is a hyperbolic Riemann surface. In order to get normalized maps we pick three points $a_1, a_2$ and $a_3$ in $S_1$ such that $R(a_i) = b_i$ for $i = 1, 2, 3$.

We say that two branched coverings $R$ and $Q$, of the Riemann sphere onto itself, are Hurwitz equivalent if there are quasiconformal homeomorphisms $\phi$ and $\psi$, making the following diagram commutative

$$
\begin{array}{ccc}
\hat{\mathbb{C}} & \xrightarrow{\psi} & \hat{\mathbb{C}} \\
\xline{R} & & \xline{Q} \\
\hat{\mathbb{C}} & \xrightarrow{\phi} & \hat{\mathbb{C}}
\end{array}
$$

Given a rational map $R$, the Hurwitz space $H(R)$ is the set of all rational maps $Q$ that are Hurwitz equivalent to $R$. The topology we are considering on $H(R)$ is the compact-open topology.

Let $f : S_2 \to S'_2$ be a representative of a point in $T(S_2)$ with Beltrami coefficient $\mu$ and fixing the points $b_i$. Let $R_*(\mu)$ be the pull-back of $\mu$ under $R$, and $h_f$ be the solution, defined on the Riemann sphere, of the Beltrami equation for the coefficient $R_*(\mu)$, and take $S'_1 = h_f(S_1)$. Let us define the map $\tau : T(S_2) \to H(R)$ so that $\tau(f)$ is the rational map making the following diagram commutative

$$
\begin{array}{ccc}
S_1 & \xrightarrow{h_f} & S'_1 \\
\xline{R} & & \xline{\tau(f)} \\
S_2 & \xrightarrow{f} & S'_2
\end{array}
$$

The map $\tau$ is well defined and continuous, since the solution of the Beltrami equation depends analytically on $\mu$. We call the space $H_*(R) = \tau(T(S_2))$, the reduced Hurwitz space of $R$. The closure of the bi-orbit by the Möbius group of $H_*(R)$ is the whole $H(R)$. The group $PSL(2, \mathbb{C})$ also acts by conjugation on $H(R)$. The space of orbits by conjugation fibres over the reduced Hurwitz space $H_*(R)$.

Let $f$ be an element of the Mapping Class Group $MCG(S_2)$ such that $f(b_i) = b_i$. We say that $f$ is liftable with respect to $R$ if there exist a map $g : S_1 \to S_1$, such that $g(a_i) = a_i$ and makes the following diagram commutative:
In this case, we say that $g$ is the lifted map of $f$ with respect to $R$. Let $G$ be the subgroup of the mapping class group $\text{MCG}(S_2)$ which consists of all liftable elements with respect to $R$. We identify $H_\tau(R)$ with the space $T(S_2)/G$ as the following theorem suggests.

**Theorem 1.** The space $H_\tau(R)$ is continuously bijective to $T(S_2)/G$.

**Proof.** We will show that $\tau(\phi_1) = \tau(\phi_2)$ if and only if $\phi_2^{-1} \circ \phi_1 \in G$.

Let $f = \phi_2^{-1} \circ \phi_1$, and assume that $f$ belongs to $G$ and let $g$ be the lifted map of $f$ with respect to $R$, so if $h_{\phi_1}$ and $h_{\phi_2}$ are the maps associated with $\tau(\phi_1)$ and $\tau(\phi_2)$ respectively, we have $h_{\phi_2} \circ g = h_{\phi_1}$ by the normalization of $g$. Hence

$$\phi_2 \circ f \circ R = \phi_1 \circ R = \tau(\phi_1) \circ h_{\phi_1},$$

on the other hand,

$$\phi_2 \circ f \circ R = \phi_2 \circ R \circ g = \tau(\phi_2) \circ h_{\phi_2} \circ g$$

so we get

$$\tau(\phi_1) = \tau(\phi_2).$$

Reciprocally, if $\tau(\phi_1) = \tau(\phi_2)$, then $f = \phi_2^{-1} \circ \phi_1$ fixes the points $b_i$. Since $h_{\phi_2}^{-1} \circ h_{\phi_1}$ is a lift of $f$ with respect to $R$, the map $f$ belongs to $G$. \hfill $\Box$

The following theorem gives a description of compact subsets of $H(R)$. For a quasiconformal map $f$, let $K_f(z)$ be the distortion of $f$ at the point $z$.

**Theorem 2.** If $\{f_i\}$ is a family of quasiconformal maps on the sphere $\hat{C}$ fixing the points $b_1, b_2$ and $b_3$. Let

$$A_n = \{z : K_{f_i}(z) \geq n \text{ for } i \text{ big enough.}\}$$

Assume that $A_\infty = \bigcap A_n$ is a compact subset of $S_2$. So we have:

- If $A_\infty$ does not separate the critical values of $R$, then the family $\{[f_i]\}$ is bounded in $T(S_2)$.

- If there exist a domain $D_0$ contained in $S_2 \setminus A_\infty$ such that $D_0$ contains at least two of the points $b_i$ and $W_0 = R^{-1}(D_0)$ is connected then the respective classes $\{\tau([f_i])\}$ are bounded in $\text{Rat}_d(\hat{C})$.

**Proof.** For the first item, let $U$ be a neighborhood of $A_\infty$ such that is compactly contained in $S_2$, does not separate $S_2$, and such that every component of $U$ is simply connected with analytic boundary.

The restrictions of $f_i$ on $\partial U$ are quasisymmetric maps with uniform bound of distortion. Using Douady-Earle extension, the maps $f_i|_{\partial U}$ extend to maps $\tilde{f}_i$, defined on the interior of $U$, with uniformly bounded distortion. Since $f_i$ and $\tilde{f}_i$...
have the same values on the boundary, the maps \( f_i \) and \( \tilde{f}_i \) are homotopic, and define the same points in \( T(S_2) \). Hence the family \( f_i \) have uniformly bounded Beltrami coefficients, so defines a bounded set in \( T(S_2) \).

Since \( \text{Rat}_q(\mathbb{C}) \) can be identified with an open and dense subset of the projective space, it is enough to prove that all the limit maps of \( \{\tau(f_i)\} \) are rational maps of the same degree. Let \( g_i \) be quasiconformal automorphisms of \( \mathbb{C} \) fixing the points \( a_i \) and such that \( f_i \circ R = \tau(f_i) \circ g_i \). Under the conditions of the second item, the accumulation functions of the families \( \{f_i|D\} \) and \( \{g_i|W_0\} \) are non constant quasiconformal functions. Let \( R_\infty \) be a rational map which is accumulation point of the maps \( \tau(f_i) \), then there exist two non constant quasiconformal functions \( f_\infty \) and \( g_\infty \) and domains \( O = h_\infty(D) \) and \( X = g_\infty^{-1}(W_0) \) on which \( f_\infty \circ R = R_\infty \circ g_\infty \). Then \( \deg(R_\infty) = \deg(R|W_0) = \deg(R) \), as we wanted to prove.

2.1 Bers slices

Let \( \Delta = \{ z : |z| < 1 \} \) and \( \Delta^* = \{ z : |z| > 1 \} \). We denote the boundary of \( \Delta \) by \( S^1 \). Now let us consider again the rational map \( R \) and surfaces \( S_i \). Let \( \Gamma_i \) be Fuchsian groups that uniformizes the surfaces \( S_i \). By the Monodromy Theorem there exist a Möbius map \( \alpha \) such that \( \Gamma = \alpha \Gamma_1 \alpha^{-1} \) is a subgroup of \( \Gamma_2 \). If \( p : \Delta \to S_1 \cong \Delta/\Gamma \) is the orbit projection. Then defining \( \pi_1 = p \circ \alpha^{-1} \) gives the following diagram

\[
\begin{array}{ccc}
\Delta & \xrightarrow{\text{Id}} & \Delta \\
\downarrow{\pi_1} & & \downarrow{\pi_2} \\
S_1 \cong \Delta/\Gamma_1 & \xrightarrow{R} & S_2 \cong \Delta/\Gamma_2.
\end{array}
\]

Moreover, \([\Gamma_2 : \Gamma_1] = \deg(R)\) since \( R \) is a covering. We call the pair \((\Gamma_1, \Gamma_2)\) a uniformization of \( R \). Simultaneously we have uniformization of the surfaces \( S_i^* = \Delta^*/\Gamma_i \) and the map \( Q : S_1^* \to S_2^* \) given by \( Q(z) := \overline{R(z)} \). Now these groups are acting on the complement of the unit disk \( \Delta^* \).

Let \( D(\Gamma_2) \) be the space of groups \( \Gamma \) such that there exist a quasiconformal map \( f \) such that \( \Gamma = f \circ \Gamma_2 \circ f^{-1} \) and such that the Beltrami differential \( \mu_f = 0 \) in \( \Delta^* \). Now put

\[
\text{Def}(\Gamma_2) = D(\Gamma_2)/\text{PSL}(2, \mathbb{C}.
\]

Analogously, we define \( \text{Def}^*(\Gamma_2) \) as the space of deformations of \( \Gamma \) on \( \Delta^* \). By Bers Theorem, both spaces \( \text{Def}(\Gamma_2) \) and \( \text{Def}^*(\Gamma_2) \) have compact closure on the space of classes of faithful and discrete representations

\[
\Gamma_2 \mapsto \text{PSL}(2, \mathbb{C}.
\]

These closures of \( \text{Def}(\Gamma_2) \) and \( \text{Def}^*(\Gamma_2) \) are called Bers slices of the Teichmüller space. In these cases, the Bers slices consist of function groups with a simply connected invariant component. Geometrically finite groups contained in the Bers slice are either quasi-fuchsian or cusps. Definitions and properties can be found in the papers by Bers [3], by Maskit [10] and by McMullen [11]. If a group \( G \) in the boundary of the Bers slice has a connected region of discontinuity, then \( G \) is totally degenerated. By theorems of Bers, Maskit and McMullen (see [3], [10], [11]).
and totally degenerated groups and cusps are both dense on the boundary of the Bers slice.

Any group $G$ in the Bers slice defines a 3-hyperbolic manifold $M(G)$ with boundary. Given a uniformization $(\Gamma_1, \Gamma_2)$ of $R$, let us consider $M_1 := M(\Gamma_1)$ and $M_2 := M(\Gamma_2)$ the associated 3-hyperbolic manifolds with boundary. Then the inclusion $\Gamma_1$ in $\Gamma_2$ defines a Möbius map

$$F : M_1 \to M_2.$$ 

The restriction of $F$ to the boundary components of $M_1$ define maps which are rational in coordinates. Let $\Sigma_i \subset \partial M_i$ be the respective invariant components of $\Gamma_i$. Then the map $F : \Sigma_1 \to \Sigma_2$ belongs to either the Hurwitz space $H(R)$ or $H(Q)$.

If a group $G$ in the Bers slice $Def(\Gamma_2)$ is geometrically finite, then $G$ is a cusp or quasifuchsian. In those situations there is a boundary component $\hat{S}$ in $M(G)$ conformally equivalent to $\Delta^*/\Gamma_2$. In the case that there are cusps on the group, we regard the set of all components as a connected surface with nodes $\hat{S}$.

**Question.** Assume that $G_i$ converges, in Bers slice, to a totally degenerated group $G$. Is it true that the accumulation set of the associated rational maps may contain constants maps?

Now we are ready to proof the main result of this section. Let us begin with the following definition. Let $G$ be a totally degenerated group. Then we call the group $G$ acceptable for the rational map $R$ if and only if the following conditions hold:

- There are two uniformizable Möbius orbifolds $S_i$ supported on the Riemann sphere, such that $R : S_1 \to S_2$ is a holomorphic covering.
- If $\Gamma_2$ is Fuchsian group uniformizing $S_2$, then $G$ belongs to $Def^*(\Gamma_2)$.
- The manifold $M(G)$ is homeomorphic to $\partial M(G) \times \mathbb{R}_+$, here $\mathbb{R}_+$ denotes the set of non-negative real numbers.

Let $\pi : \mathbb{H}^3 \cup \Omega(G) \to M(G)$ be the orbit projection. Under the homeomorphism of the last item, let us define, for every $t$ in $\mathbb{R}_+$ the set $\Omega(G)_t = \pi^{-1}(\partial M \times t)$. Hence, the space $\mathbb{H}^3 \cup \Omega(G)$ is foliated by the sets $\Omega(G)_t$ and there exists a continuous family of homeomorphisms $f_t : \Omega(G) \to \Omega(G)_t$ which commutes with $G$ and $f_0 = Id$.

Let $\mathbb{B}^3$ denote the unit ball model for the hyperbolic space. Given a rational map $R$, we define the radial extension $\hat{R}$ as follows. For every $\lambda \in [0, 1]$ and $(x, y, z) \in \mathbb{R}^3$, let $H_\lambda(x, y, z) = (\lambda x, \lambda y, \lambda z)$. Then we have

$$\hat{\mathbb{B}}^3 = \bigcup_{\lambda \in [0, 1]} H_\lambda(\partial \mathbb{B}^3).$$

Now define $\hat{R}(0, 0, 0) = (0, 0, 0)$ and for $v \in \hat{\mathbb{B}}^3$, different from $0$, define $\hat{R}(v) = H_{||v||} \circ R \circ H_{||v||}^{-1}$ where $||v||$ denotes the euclidean distance to $v$ from the origin.

**Theorem 3.** If there exist an acceptable group for $R$ then the radial extension of $R$ is geometric.
Proof. Let $G_2$ be an acceptable group for $R$, then there exist $\phi : \Omega(G_2) \to \Delta$ such that induces an isomorphism $\phi_* : G_2 \to \Gamma_2$, then $G_2$ has a finite index subgroup $G_1 = \phi^{-1}(\Gamma_1)$, such that map $\alpha(R) : M(G_1) \to M(G_2)$ is Möbius. Moreover, the manifold $M(G_1)$ is a manifold homeomorphic to $\partial M(G_1) \times \mathbb{R}_+$. We have to show that the radial extension is equivalent to $\alpha(R)$, so it is geometric. Again, each $M(G_i)$ is homeomorphic to $S_i \times \mathbb{R}_+$, and the horizontal foliation in $M_1$ is the pull back by $F : M_1 \to M_2$ of the horizontal foliation in $M_2$. Hence there exist a covering $\phi : S_1 \times \mathbb{R}_+ \to S_2 \times \mathbb{R}_+$ and homeomorphisms $h_1, h_2$ such that the following diagram commutes

\[ \begin{array}{ccc}
M_1 & \xrightarrow{\phi} & M_2 \\
\downarrow{\ h_1} & & \downarrow{\ h_2} \\
S_1 \times \mathbb{R}_+ & \xrightarrow{\phi} & S_2 \times \mathbb{R}_+
\end{array} \]

such that $h_1(x) = x$ and $h_2(y) = y$ for all $x$ in $S_1$ and $y$ in $S_2$. Hence $\phi(x) = F(x) = R(x)$ for $x$ in $S_1$. There are two families of homeomorphisms $\psi_t$ and $\chi_t$ such that

\[ \begin{array}{ccc}
S_1 \times \{t\} & \xrightarrow{\phi} & S_2 \times \{t\} \\
\downarrow{\psi_t} & & \downarrow{\chi_t} \\
S_1 \times \{0\} & \xrightarrow{\phi} & S_2 \times \{0\}
\end{array} \]

So that $\phi$ preserves the parameter $t$. Now consider a homeomorphism $k : [0, \infty) \to [0, 1]$ such that $k(0) = 1$ and $k(\infty) = 0$. For $i = 0, 1$, let us identify the sets $S_i \times 0$ with the corresponding $S_i$ on the Riemann sphere. Hence we define two homeomorphisms $\psi$ and $\chi$ such that $\psi(x, t) = H_{k(t)}(\psi_t^{-1}(x))$ and $\chi(x, t) = H_{k(t)}(\chi_t^{-1}(x))$, where $H_t(x, y, z) = (tx, ty, tz)$. Since $\psi$ and $\chi$ are the identity on the boundary, these homeomorphisms uniformize the extension of $\phi$ over $S_1$ and $S_2$.

We have not found a reference that shows that the manifold of a totally degenerated group in the Bers slice of a finitely generated group is always a product. However Michael Kapovich kindly gave us arguments to show this happens. The arguments are based upon a work of Waldhausen and the solution of the Tame Conjecture.

### 3 Schottky type extensions of rational maps

In this section, we prove that any map in the Hurwitz space of a Blaschke map has an extension of Schottky type that satisfies the properties (1), (2) and (3) in the introduction. We also construct an extension of Blaschke maps that satisfy almost all five conditions: this extension does not satisfies condition (4). However, if we take the group of Möbius transformations preserving the unit circle instead of $PSL(2, \mathbb{C})$, then the modified condition (4) holds.

A Blaschke map $B : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ is a rational map that leaves the unit disk $\Delta$ invariant. If $d$ is the degree of $B$, then there exist $\theta \in [0, 2\pi]$ and $d$ points
The class of Blaschke maps allow us to build a specific topological construction based upon Schottky coverings of Riemann surfaces. What is special about Blaschke maps is that every Blaschke map commutes with the involution based upon Schottky coverings of Riemann surfaces.  

\[ \{a_1, ..., a_d\} \text{ in } \Delta \text{ such that} \]

\[ B(z) = e^{i\theta} \left( \frac{z - a_1}{1 - \bar{a}_1z} \right) \cdots \left( \frac{z - a_d}{1 - \bar{a}_dz} \right). \]

Let us denote by \( B_1 = B|_\Delta \) and \( B_2 = B|_{\Delta^*} \), then \( B_2(z) = \frac{1}{z} \circ B_1(z) \circ \frac{1}{z} \).

Let \( CV(B) \) be the set of critical values of \( B \) then define \( \tilde{S}_2 = \mathbb{C} \setminus CV(B) \) and \( S_1 = B^{-1}(\tilde{S}_2) \). Thus \( B : S_1 \to S_2 \) is a holomorphic covering and the surfaces \( S_i \) are symmetric with respect to \( S^1 \).

The class of Blaschke maps allow us to build a specific topological construction based upon Schottky coverings of Riemann surfaces. What is special about Blaschke maps is that every Blaschke map commutes with the involution \( \tau(z) = \frac{1}{z} \). Recall that Fuchsian groups of second type are Schottky type Fuchsian groups.

**Theorem 4.** Given a Blaschke map \( B \), such that \( B : S_1 \to S_2 \) is a covering and the surfaces \( S_i \) are symmetric surfaces with respect to \( S^1 \). There are two Fuchsian groups of second type \( \Gamma_1 \) and \( \Gamma_2 \) such that \( \Omega(\Gamma_i)/\Gamma_i = S_i \), where \( \Omega(\Gamma_i) \) is the discontinuity set of \( \Gamma_i \) for \( i = 1, 2 \). Furthermore, there exist an Möbius map \( \alpha : \Omega(\Gamma_1) \to \Omega(\Gamma_2) \) making following diagram commutative

\[
\begin{array}{ccc}
\Omega(\Gamma_1) & \xrightarrow{\alpha} & \Omega(\Gamma_2) \\
\pi_1 & & \pi_2 \\
S_1 & \xrightarrow{B} & S_2.
\end{array}
\]

Also \( [\Gamma_2 : \alpha \Gamma_1 \alpha^{-1}] = \deg(B) \).

**Proof.** For \( i = 1, 2 \) let \( \Delta_i = \Delta \cap S_i \) and \( \Delta_i^* = \Delta^* \cap S_i \).

The Simultaneous Uniformization Theorem of Bers [2], ensures that there exist a Fuchsian group \( \Gamma_2 \) acting in \( \hat{\mathbb{C}} \), with \( \Delta/\Gamma_2 = \Delta_2 \) and \( \Delta^*/\Gamma_2 = \Delta_2^* \). The limit set \( \Lambda(\Gamma_2) \) is contained in \( S^1 \).

Similarly, there is a Fuchsian group \( \Gamma_1 \) such that \( \Delta^*/\Gamma_1 = \Delta_1^* \) and \( \Delta/\Gamma_1 = \Delta_1 \) and \( \Lambda(\Gamma_1) \subset S^1 \).

The map \( B_i \) lifts to Möbius maps \( \alpha_1 : \Delta \to \Delta \) and \( \alpha_2 : \Delta^* \to \Delta^* \). Moreover, since \( \Omega_2 \) is a Riemann surface anti-conformally equivalent to \( \Omega_1 \), we can choose \( \alpha_2 \) such that \( \alpha_2(z) = \frac{1}{z} \circ \alpha_1 \circ \frac{1}{z} \) and these maps agree at \( S^1 \setminus \Lambda(G) \). Being \( \Lambda(G) \) a Cantor set, then the map \( \alpha : \hat{\mathbb{C}} \setminus \Lambda(G) \to \hat{\mathbb{C}} \setminus \Lambda(G) \) defined as \( \alpha|_{\Delta_i} = \alpha_i \) extends to a Möbius map \( \alpha \) defined on the Riemann sphere.

By Theorem 4 a Blaschke map admits a Poincaré extension which follows from the diagram below:

\[
\begin{array}{ccc}
\mathbb{B}^3 & \xrightarrow{\alpha} & \mathbb{B}^3 \\
\downarrow & & \downarrow \\
\mathbb{B}^3/\Gamma_1 & \xrightarrow{B} & \mathbb{B}^3/\Gamma_2.
\end{array}
\]

Let us observe that \( \Gamma_1 \) and \( \Gamma_2 \) are Schottky type groups with parabolic generators. Hence \( \mathbb{B}^3/\Gamma_1 \) and \( \mathbb{B}^3/\Gamma_2 \) are homeomorphic to the complement in
B³ of a finite number of geodesics connecting symmetric perforations of the surfaces Sᵢ.

The arguments in Theorem 3 work in a more general situation. The key facts are Bers Simultaneous Uniformization Theorem and the symmetry of respective orbifolds. So we have

**Corollary 5.** Let W₁ and W₂ be any two given connected symmetric orbifolds supported on the Riemann sphere. If \( B : W₁ \to W₂ \) is a covering symmetric with respect to \( S¹ \), then the conclusion of Theorem 4 still holds for the covering \( B \).

Let \( \text{GO}(P(B)) \) be the grand orbit of the postcritical set \( P(B) \) and take \( S = \overline{\mathbb{C}} \setminus \text{GO}(P(B)) \), then \( S \) is an open disconnected surface consisting of two components \( D = S \cap \Delta \) and \( D^* = S \cap \Delta^* \) and \( B : S \to S \) is a holomorphic self-covering.

**Corollary 6.** There exist a Fuchsian group \( \Gamma \) and \( \alpha \) in \( \text{PSL}(2, \mathbb{R}) \) such that \( \alpha_\ast(\Gamma) = \alpha \Gamma \alpha^{-1} \) is a subgroup of \( \Gamma \). Moreover, we have that \( \Omega(\Gamma) = \Delta \cup \Delta^* \) and \( \Delta/\Gamma = D \). Finally, for every \( n \) the following diagram is commutative

\[
\begin{array}{ccc}
\Omega(\Gamma) & \xrightarrow{\alpha^n} & \Omega(\Gamma) \\
\downarrow{\pi₁} & & \downarrow{\pi₁} \\
S & \xrightarrow{B^n} & S.
\end{array}
\]

**Proof.** The proof is essentially the same as in Theorem 4 using the symmetry of the surfaces plus the fact that \( B \) defines a self-covering of the surface \( S \). ☐

As noted after Theorem 4 we have that the Poincaré extension of \( B \) is an endomorphism of \( \mathbb{B}³/\Gamma \), so this Poincaré extension is dynamical.

Again, the argument in the corollary above can be generalized as in the following corollary:

**Corollary 7.** Given a Blaschke map \( B \), let \( A \) be a completely invariant symmetric closed subset of \( \overline{\mathbb{C}} \) which contains all critical points of \( B \). Assume that \( \overline{\mathbb{C}} \setminus A = W \) consists of exactly two components \( U = W \cap \Delta^* \) and \( U = W \cap \Delta^* \). Then \( \{B^n : U \to U^*\} \) and \( \{B^n : U \to U\} \) are semigroups of coverings and the Poincaré extension in this situation is dynamical.

Now let us consider the case of decomposable Blaschke maps. Assume that \( B = B₁ \circ B₂ \) is a decomposable Blaschke map where \( B₁ \) and \( B₂ \) are Blaschke maps. Then if \( Q = B₂ \circ B₁ \) there are semiconjugacies.

\[
\begin{array}{ccc}
\mathbb{C} & \xrightarrow{B = B₁ \circ B₂} & \mathbb{C} \\
\downarrow{B₂} & & \downarrow{B₂} \\
\mathbb{C} & \xrightarrow{Q = B₂ \circ B₁} & \mathbb{C} \\
\downarrow{B₁} & & \downarrow{B₁} \\
\mathbb{C} & \xrightarrow{B} & \mathbb{C}.
\end{array}
\]
Corollary 8. If $B = B_1 \circ B_2$ and $Q = B_2 \circ B_1$ there exist two Fuchsian groups $\Gamma(B)$ and $\Gamma(Q)$ satisfying the conditions of Corollary and $\alpha_B$ and $\alpha_Q$ in $PSL(2, \mathbb{R})$. So that there are two elements $\beta_1$ and $\beta_2$ in $PSL(2, \mathbb{R})$ with $\alpha(B) = \beta_1 \circ \beta_2$ and $\alpha(Q) = \beta_2 \circ \beta_1$. Such that the following diagrams are commutative.

**Proof.** Note that $B_2 : S_B \to S_Q$ and $B_1 : S_Q \to S_B$ define coverings, hence there are Möbius maps $\beta_1 : \Omega(\Gamma_Q) \to \Omega(\Gamma_B)$ and $\beta_2 : \Omega(\Gamma_B) \to \Omega(\Gamma_Q)$ which make the diagrams commutative.

As a consequence of Corollary 8 we have the following conclusion.

**Proposition 9.** Let $B = B_1 \circ B_2$, let $\hat{B}$ be a dynamical extension, then there are Poincaré extensions $\hat{B}_1$ and $\hat{B}_2$ of $B_1$ and $B_2$ respectively, such that

$$\hat{B} = \hat{B}_1 \circ \hat{B}_2.$$  

Moreover, there exist $\hat{Q}$ a dynamical extension of $Q := B_2 \circ B_1$ such that

$$\hat{Q} = \hat{B}_2 \circ \hat{B}_1.$$  

The following theorem summarize the results above and show that the corresponding extensions are geometric.

**Theorem 10.** Let $B$ be a Blaschke map, then:
i) The extension constructed in Theorem 4 is geometric.

ii) The dynamical extension constructed in Corollary 6 is geometric.

iii) The extensions in Corollary 3 are all geometric.

Each of the extensions in items (i)-(iii) is conformally natural with respect to the group of Möbius transformations that leaves the unit circle invariant.

Proof. We will show item (i), the proof the other items apply similar arguments to the extensions constructed in Corollary 6 and Proposition 9. Again, the important feature is that the corresponding surfaces are symmetric with respect to the unit circle. According to Theorem 4 there are manifolds $M_1$ and $M_2$, Möbius projections $p_1$ and $p_2$, and a Poincaré extension $\hat{B}$ of $B$ such that the following diagram is commutative

$$
\begin{array}{ccc}
\mathbb{H}^3 \cup \Omega(\Gamma_1) & \xrightarrow{Id} & \mathbb{H}^3 \cup \Omega(\Gamma_2) \\
p_1 & \downarrow & p_2 \\
M_1 & \xrightarrow{\hat{B}} & M_2.
\end{array}
$$

such that $p_i|_{\Omega(\Gamma_i)} = \pi_i$. Now, we construct universal coverings $q_i$ which maps $\mathbb{H}^3 \cup \Omega(\Gamma_i)$ in $\bar{B}_3$ and $q_i(x) = q_i(y)$ if, and only if, there exist a $\gamma_i$ in $\Gamma_i$ with $\gamma_i(x) = y$, so that

$q_i|_{\Omega(\Gamma_i)} = p_i|_{\Omega(\Gamma_i)} = \pi_i$.

By Theorem 4 the group $\Gamma_2$ acts on the unit disk which belongs to the boundary of $\mathbb{B}_3 \cap \mathbb{R}^2$ so that $\pi_2(\Delta) = S_2 \cap \Delta$ also belongs to the boundary of $\mathbb{B}_3 \cap \mathbb{R}^2$. Let $\tau_\phi$ be the Möbius rotation, in $\mathbb{R}^3$, with respect to $\partial \Delta$ of angle $\phi$. Then

$$\mathbb{B}_3 = \bigcup_{0 \leq \phi \leq \pi} \tau_\phi(\Delta)$$

and $\tau_\phi : \Delta \to \Delta^*$ is the map $z \mapsto 1/\bar{z}$ in the holomorphic coordinate of $\Delta$. Define $q_i(z, \phi) = (\tau_\phi \circ \pi_i(z))$ such that $\tau_\phi$ commutes with $\text{Aut}(\Delta) \simeq \text{PSL}(2, \mathbb{R})$. Furthermore, $\tau_\phi$ commutes with any Möbius map that leaves the unit circle invariant (for instance $z \mapsto 1/z$).

Then $M_i = q_i(\mathbb{H}^3 \cup \Omega(\Gamma))$ are subsets of $\mathbb{B}_3$ and the respective Poincaré extension is conformally natural with respect to the group of Möbius map that leaves the unit circle invariant.

Corollary 11. If in Proposition 6 $B_1 = B_2$ then $\hat{B}_1 = \hat{B}_2$. Using induction If $B = B^n_1$ then for every dynamical extension $\hat{B}$ of $B$, there exist a dynamical extension $\hat{B}_1$ of $B_1$ such that $\hat{B} = \hat{B}_1^n$.

In the case (i) of Theorem 10 let $Q_i : \mathbb{H}^3 \to \mathbb{B}_3$ be other extensions of the projections $\pi_i$, then there are continuous maps $h_i : M_i \to \mathbb{B}_3$ such that $Q_i = h_i \circ q_i$. Where $q_i$ are the extensions constructed on the proof of the Theorem 10.

Let us put $K = Q_2 \circ Q_1^{-1}$ where the composition is defined. If $\hat{B}$ is the geometric extension of $B$ from Theorem 10 then $K \circ h_2 = h_1 \circ \hat{B}$.

In the case (ii) of Theorem 10 assume $Q_i$ is another extension of $\pi_i$. Again, put $K = Q \circ \alpha_B \circ Q^{-1}$. If $K$ is a map, then $K$ is semiconjugated to $\hat{B}$. 


Theorem 12. Let $S$ be a semigroup of Blaschke maps, then the extension constructed in Theorem 14 defines in $S$ a geometric homomorphic conformally natural extension preserving degree if, and only if, $S$ does not intersect the bi-orbit of $f(z) = z^n$ with respect to $\text{Aut}(\Delta)$.

Proof. If the extension is not conformally natural with respect to $\text{PSL}(2, \mathbb{C})$ then there are two elements $B_1$ and $B_2$ in $S$ with two maps $g_1$ and $g_2$ in $\text{PSL}(2, \mathbb{C})$ such that $B_1 \circ g_1 = g_2 \circ B_2$. Then by Theorem 10 the maps $g_1$ and $g_2$ cannot leave the unit circle invariant. Then there two circles $C_1 = g_1(S^1)$ and $C_2 = g_2(S^1)$ with $B_1^{-1}(C_2) = C_1$. Let us show that the circles $C_i$ do not intersect $S^1$. Assume that there is an element $x$ in $C_1 \cap S^1$ then $C_2$ intersects $S^1$ in all the preimages of $x$ with respect to $B_1$. But this is possible only if the preimage of $x$ under $B_1$ is a single critical point, but a Blaschke map cannot have critical points on $S^1$. Therefore, $C_1$ and $C_2$ cannot intersect $S^1$. Then $C_1$ and $S^1$ bound an annulus. By the reflection principle, we have that $B_1$ has in the unit disk a critical point of multiplicity $d - 1$. Hence either $B_1$ or $1/B_1$ belongs to the bi-orbit of $z^n$ with respect of $\text{Aut}(\Delta)$, but the extension in Theorem 14 is compatible with the map $z \mapsto 1/z$. Hence the conclusion of the theorem holds. Reciprocally, the extension of Theorem 14 is not conformally natural on the map $z^n$.

Corollary 13. Assume $B$ is a Blaschke map of the form $B = g_1 \circ z^d \circ g_2$ such that $g_2(z) \neq e^{i\alpha} \circ g_1^{-1}(z)$, then the extensions constructed on Theorem 14 of $(B^n)$ are conformally natural whenever $n \geq 2$.

Proof. If for $n \geq 2$, $B^n$ does not satisfies the conditions of Theorem 12 then $B^n$ should be in the bi-orbit of $z^{dn}$. Hence $B^n$ has one critical point and one critical value, this implies that the critical point $x$ of $B$ is fixed. Hence $g_1 \circ z^d \circ g_2(x) = x$ then $g_1(0) = g_2^{-1}(0)$, which implies $g_2 \circ g_1(0) = 0$ and $g_2 = e^{i\alpha} \circ g_1^{-1}(z)$.

3.1 Geometric extensions in Hurwitz spaces

Let us recall that a branched covering $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ of degree $d$ is in general position, if the number of critical points is the same than the number of critical values and equal to $2d - 2$. According to [17] a theorem of Luroth and Clebsch states that:

Lemma 14. Any two branched coverings $f_i : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ of the same degree in general position are Hurwitz equivalent.

The existence of a Schottky type extension is a property of the whole Hurwitz space as we show in the following.

Lemma 15. Let $B$ be a map with a Schottky type geometric extension and $R$ a rational map in $H(B)$, then $R$ has also an extension of Schottky type.

Proof. Let $\Gamma_1$ and $\Gamma_2$ be the uniformizing groups for $B$. Let $\alpha : \mathbb{C} \to \mathbb{C}$ be the Möbius map extending $B$. By definition of $H(B)$, there are two quasiconformal maps $f, g$ on the Riemann sphere such that $f \circ B = R \circ g$. Solving the Beltrami equation, we get quasiconformal extensions $\tilde{f} : \mathbb{C} \to \mathbb{C}$ and $\tilde{g} : \mathbb{C} \to \mathbb{C}$ of $f$ and $g$ respectively, such that $\beta = \tilde{g}^{-1} \circ \alpha \circ \tilde{f}$ is a Möbius map extending $R$. Let us
assume first that \( f \) and \( g \) have small distortion, then by a theorem in [5, Th. 5] each map, \( f \) and \( g \), admits a homeomorphic extension, say \( \hat{f} \) and \( \hat{g} \), compatible with the groups \( \Gamma_1 \) and \( \Gamma_2 \). Hence we obtain two Kleinian groups \( \hat{\Gamma}_1 = \hat{f} \circ \Gamma_1 \circ \hat{f}^{-1} \) and \( \hat{\Gamma}_2 = \hat{g} \circ \Gamma_2 \circ \hat{g}^{-1} \) with manifolds \( M(\hat{\Gamma}_1) \) and \( M(\hat{\Gamma}_2) \) that extend the map \( R \). To see that this extension is geometric we have to embed each manifold \( M(\hat{\Gamma}_1) \) and \( M(\hat{\Gamma}_2) \) into \( B^3 \), such that the image of these embeddings are the complement of a finite set of quasigeodesics.

To do so, we use the geometric extension of \( B \). We know that \( M(\Gamma_1) \) and \( M(\Gamma_2) \) are already realized as submanifolds of \( B^3 \), hence by conjugating \( M(\Gamma_1) \) by \( \hat{f} \) and \( M(\Gamma_2) \) by \( \hat{g} \), we obtain the desired embeddings of \( M(\hat{\Gamma}_1) \) and \( M(\hat{\Gamma}_2) \) into \( B^3 \).

To complete the proof we note that \( H(B) \) is connected, so for maps \( f \) and \( g \) with big distortion, we can use a path on \( H(B) \) and extend on small distortion changes.

Now let us show that there are Schottky type extensions for a large set of rational maps.

**Theorem 16.** There is an open and everywhere dense subset in \( \text{Rat}_d(\mathbb{C}) \) which has a geometric extension of Schottky type, of the same degree.

**Proof.** By Lemma [14] the union of all the Hurwitz spaces of all Blaschke maps of fixed degree is open and everywhere dense in \( \text{Rat}_d(\mathbb{C}) \). Hence Theorem [4] and Lemma [15] imply this theorem.

It follows that structurally stable rational maps have a Schottky type extension. We believe that any rational map has a geometric extension such that the respective manifold belongs to the closure of the Schottky space. Since Hurwitz space of any branched covering of finite degree of the sphere contains a rational map, we conjecture that the closure of the Schottky space of given degree \( d \), contains all realizable Hurwitz combinatorics.

### 3.2 Extensions of exceptional maps

Now we discuss the situation when, for a given rational map \( R : S_1 \rightarrow S_2 \), the orbifolds \( S_1 \) and \( S_2 \) are equal, so the map \( R \) is an orbifold endomorphism. The class of maps \( R \) are called exceptional, the reader will find a more detailed discussion of these maps in [13]. In particular, the Euler characteristic \( \chi(S_1) \) is zero. Hence \( S_1 \) is a parabolic orbifold, this only occurs when the map \( R \) is conjugate to either a Tchebichev map, a Lattés map or \( z \mapsto z^n \).

**Theorem 17.** Let \( G \) be a semigroup of rational maps which are self-coverings of a parabolic orbifold \( S \) supported on the Riemann sphere. Then there exist a geometric extension satisfying the following conditions

- For every \( g \in G \), the extension \( \hat{g} \) has the same degree as \( g \).
- Each extension is geometric.
- The set of extensions \( \hat{G} \) is a semigroup, and the extension map is a homomorphism from \( G \) to \( \hat{G} \).
Proof. The proof exploits the fact that the elements in \( G \) have known uniformizations. Consider lattice 
\[ L_\tau := \langle z \mapsto z + 1, z \mapsto z + \tau : \Im \tau > 0 \rangle \]
in the Lattés case, and the lattice 
\[ L_0 := \langle z \mapsto z + 1, z \mapsto z + \tau : \Im \tau > 0 \rangle \]
in the case of \( z^n \) and Tchebichev. Let \( \sigma \) be the involution \( z \mapsto -z \). If \( \Gamma_\tau \) is the group generated by \( L_\tau \) and the involution, then \( S \) is equivalent to \( \mathbb{C}/\Gamma_\tau \) or \( \mathbb{C}/L_0 \). Now, we have three groups \( \Gamma_\tau, \Gamma_0 \) and the group \( L_0 \). In terms of the lattice \( L_\tau \), \( G \) is a semigroup of affine endomorphisms of \( L_\tau \). In each case, \( G \) has a simultaneous Poincaré extensions on the orbifolds \( \mathbb{H}^3/\Gamma \) where \( \Gamma \) is one of the three groups mentioned. These Poincaré extensions satisfy the properties of the Theorem.

As an example we give a detailed description of a three dimensional orbifold supporting the Poincaré extension of integral Lattés maps.

In other words, this is the case when \( \bar{R} \) is a holomorphic endormorphism of the orbifold of type \((\bar{\mathbb{C}}, 2, 2, 2, 2)\).

Let us consider the filled torus \( T \) in \( \mathbb{C}^2 \), given in coordinates \((z_1, z_2)\) by \(|z_1| \leq 1 \) and \(|z_2| = 1 \). This space is uniformized by the Poincaré extension of a lattice \( L_\tau \) in \( \mathbb{H}^3 \) parallel to the plane, for a suitable choice of \( \tau \).

Let \( I \) be the involution map 
\[ (z_1, z_2) \mapsto (\bar{z}_1, \bar{z}_2). \]

The map \( I \) acts on the filled torus \( T \) as an involution. The quotient \( T/I \) gives an orbifold \( O \) supported in \( \mathbb{H}^3 \) with two ramification lines. Let \( \pi : T \to T/I \) be the orbit projection. The core of \( T \) is the unit circle \( C = \{(z_1, z_2) : z_1 = 0 \} \).

Let us consider the family of endomorphisms \( \Psi(l, m, k) \) of \( T \) given by the formulae
\[ (z_1, z_2) \mapsto (z_1^{l/k}, z_2^n). \]
Where \( l, m, k \) are integers. In other words, \( \Psi(l, m, k) \) contains all the Poincaré extensions of the semigroup of integer multiplications on \( T \setminus C \). Then this family commutes with the involution \( I \) and generates a semigroup \( J \) of endomorphisms of the orbifold \( O \).

**Corollary 18.** Let \( \bar{R} \) be an element of \( J \), then \( \pi(C) \) is an interval which is invariant with respect to \( \bar{R} \) and the restriction of \( \bar{R} \) on \( \pi(C) \) is topologically conjugate to a Tchebichev polynomial of degree \( k \). Moreover, there exist a continuous projection \( h : \partial O \to \pi(C) \) so that \( h \circ \bar{R} = \bar{R} \circ h \).

**Proof.** Note that the projection \( P : T \to C \) given by \((z_1, z_2) \mapsto (0, z_2) \) commutes with the maps \( \Psi(l, m, k) \) and the involution \( I \). Moreover, the restriction of the map \( \Psi(l, m, k) \) on \( C \) is the power map \( \bar{z}_2 \mapsto \bar{z}_2^n \). The action of \( \bar{R} \) restricted on \( \pi(C) \) is topologically conjugate the Julia set of a Tchebichev map of degree \( m \), hence \( \pi(C) \) is an interval. Since the projection \( P \) commutes with the action of \( I \), descends to a projection \( h \) as desired. \( \square \)
One would expect that Tchebichev polynomials are obtained by pinching $T$ onto $C$. The projection $P$ defines a foliation $\partial T$ by circles. Hence, the projection $h$ defines a foliation $F$ on $\partial O$, all leaves in $F$ are topological circles with the exception of two leaves homeomorphic to intervals. In other words, one would think that the foliation $F$ shrinks to a model of the Tchebichev polynomial. So there would be a deformation of the foliation in the boundary $\partial O$ that produces a Tchebichev map. The corollary above suggests an argument to construct such deformation. So, it is natural to ask: is it true that the closure of the space of quasiconformal deformations of flexible Lattés maps in $\bigcup_{d<\deg(R)} \text{Rat}_d(\mathbb{C})$ contains Tchebichev maps? Is it true that any point in the boundary of flexible Lattés map is rigid? Finally, is it true that any point in the boundary has degree strictly smaller than degree of the given Lattés map?

3.3 Non-Galois affine extensions

The main idea of this paper is to transform a rational map to a Möbius morphism. On the sections above, we discussed Galois coverings, which is the uniformizable situation. In this subsection, let us consider non Galois coverings which also transform rational dynamics into Möbius dynamics.

Simple examples of non Galois coverings are given by Poincaré functions associated to repelling cycles of $R$. These are functions $f$ satisfying the functional equation $f(\lambda z) = R^n \circ f$ for some $n$ and some $\lambda$.

Let us suppose that there exist an extension $\hat{f}$ of $f$ in $\mathbb{H}^3$, then there exist a multivalued map defined

$$K_f = \hat{f} \circ \lambda z \circ \hat{f}^{-1}.$$ 

When $K_f$ is an ordinary map, we have a dynamical extension. If $f$ is a Galois covering then we are in the parabolic situation described in the previous section. In general, is not clear when $K_f$ is a map even in the case $\hat{f}$ is a visual extension of a Poincaré function $f$. In the case $K_f$ is a map, we call $\hat{K}_f$ a non-Galois extension of $R$.

Let $B$ be a Blaschke map, and $R$ a quasiconformal deformation of $B$. Now we show that for any Poincaré function $f$ of $R$ there exist a non Galois extension.

**Theorem 19.** Let $R$ be quasiconformally conjugate to a Blaschke map. For every Poincaré function of $R$ there exist a non Galois extension of $R$.

**Proof.** Let us first consider a Blaschke map $B$. Any Poincaré function of $B$ satisfies

$$f(\bar{z}) = \frac{1}{f(z)},$$

hence $f$ maps the lower half plane $\mathbb{H}^2$ to the unit disk. Now we are in position to use a similar argument of the proof of Theorem 10 to define an extension $\hat{f} : \mathbb{H}^3 \to B^3$ as follows, first identify $\mathbb{H}^3$ with the “open book” coordinates $(z, \phi)$ where $z \in \mathbb{H}^2$ and $\phi$ in the interval $(0, \pi)$ and put

$$\hat{f}(z, \phi) = \tau_\phi(f(z))$$

where $\tau_\phi$ is the Möbius rotation of angle $\phi$ in $\mathbb{R}^3$ with respect to the unit circle. In this case, from the equation satisfied by a Poincaré function, we have that
$K_f = \hat{B}^n$ for some iterate of $\hat{B}$, where $\hat{B}$ is the dynamical extension constructed in Theorem 10. Let $R = \phi \circ B \circ \phi^{-1}$ then any Poincaré function for $R$ belongs to the Hurwitz space of a suitable Poincaré function of $B$. Now we can apply the arguments in Lemma 15 to finish the proof.

Let us recall that, for every complex affine line $L$ there is a process of hyperbolization $T : L \to H(L)$ which associates a hyperbolic manifold $H(L)$ to $L$. This hyperbolization process is used in the construction of 3-hyperbolic Lyubich-Minsky laminations $S$. By this process there is an identification $H(L) \cong \mathbb{C} \times \mathbb{R}_+$. Given an affine line $L$, let us assume that we have fixed any such identification.

Let $h_i : L \to L \times t \subset H(L)$ be the horospherical inclusion, then we have

$$h_i^{-1}(\lambda z) = h^{-1}(\lambda z).$$

Now let $L_1$ and $L_2$ be complex affine lines. Let $F : L_1 \to L_2$ be any map, then for any $\lambda > 0$ there is a family of extensions $\hat{F}_\lambda : H(L_1) \to H(L_2)$ given in coordinates by

$$\hat{F}_\lambda(x,t) = (F(x), \lambda t).$$

Note that if $F$ is affine then there exists a unique $\lambda_0$ such that $F_{\lambda_0}$ is the Poincaré extension of $F$ in $H(L)$. Let $q_i : L_i \to \mathbb{C}$ be maps for $i = 1, 2$. Assume we have a polynomial $P$ and an affine map $\gamma : L_1 \to L_2$ satisfying

$$P \circ q_1 = q_2 \circ \gamma.$$

Then for all $\lambda, \omega$ and $\rho$ positive real numbers, we have

$$(q_2)_\omega \circ \gamma_\lambda(x,t) = (q_2)_\omega(\gamma(x), \lambda t) = (q_2(\gamma(x)), \omega \lambda t) = (P \circ \pi_1(x), \omega \lambda t) = P_\omega(\gamma_\lambda(x), \omega \lambda t).$$

Now let $\lambda_0$ be the number such that $\gamma_{\lambda_0}$ is the Poincaré extension of $\gamma$ in $\mathbb{H}^3$ and put $\rho = \lambda_0$ in the formula above, then we have

$$(q_2)_\omega \circ \gamma_{\lambda_0} = P_{\lambda_0} \circ (q_1)_\omega.$$

Assume that $q_1 = q_2 = f$ where $f$ is the Poincaré function of a fixed point with multiplier $\lambda_0$, then for every $\omega$ the map $P_{\lambda_0}$ is a geometric dynamical extension with the same degree. In this extension, the orbit of every point in $\mathbb{H}^3$ converges to infinity. In other words, the Julia set of the extension belongs to $\hat{\mathbb{C}}$. To show that $P_{\lambda_0}$ is geometric let $M_1$ be the complement in $\mathbb{H}^3$ of all vertical lines based on the $P$-preimages of the postcritical set, then $M_2$ is the complement in $\mathbb{H}^3$ of all vertical lines based on the postcritical set and $P_{\lambda_0} : M_1 \to M_2$. Then $f_\omega$ endows $M_1$ with incomplete Möbius structures, making $P_{\lambda_0}$ geometric.

Another situation is when $\rho = 1$, again let $q_1 = q_2 = f$ as above. Then $P_1$ is a Poincaré extension, with the manifolds $M_1$. However, the Möbius structures on $M_1$ are different, on $M_1$ is given with $f_\omega$ and on $M_2$ is given by $f_{\lambda_0 \omega}$. Note that $\rho = 1$ gives a homomorphic extension defined on the semigroup of polynomials.

For the reader familiar with the construction of Lyubich-Minsky $S$, we note that natural extension of either $P_{\lambda_0}$ or $P_1$ is equivalent to the 3-hyperbolic Lyubich-Minsky lamination.
4 Product extension

At least for us, it is very surprising that there is a product structure on $\mathbb{H}^3$ which, in a sense, is a “conformal natural” extension of the complex product on $\mathbb{C}$. To construct this product, first let us extend the exponential map $\operatorname{Exp}(z) = e^z$.

We consider the coordinates $(z, t)$ in $\mathbb{H}^3$. Let $h_\alpha : \mathbb{H}^3 \to \mathbb{H}^3$ be the translation given by

$$(x, y, t) \mapsto (x, y - \alpha, t),$$

and let $p : \triangle \to \mathbb{H}^3$ be the stereographic projection that maps the unit disk $\triangle$ in the unit semisphere in $\mathbb{H}^3$.

Put $\Phi(x, 0, t) = p \circ \operatorname{Exp}(x + it)$ and let $V$ be the vertical semiplane over the imaginary line. Then $\Phi$ maps $V$ onto the unit semisphere in $\mathbb{H}^3$. Finally, for $w = (x, y, t)$ let

$$\hat{\operatorname{Exp}}(w) = H_{e^{-2\pi y}} \circ \Phi \circ h_y^{-1}(w).$$

By construction $\hat{\operatorname{Exp}}$ maps $\mathbb{H}^3$ onto $M$, the complement of the $t$-axis in $\mathbb{H}^3$, and is a covering. When $t = 0$ the map $\hat{\operatorname{Exp}}$ coincides with the $\operatorname{Exp}$. Also $\hat{\operatorname{Exp}}$ defines a complete Möbius structure $\delta$ on $M$. Any Möbius map that leaves $M$ invariant is Möbius in $\delta$.

Since $\hat{\operatorname{Exp}}$ defines an homomorphism of the additive structure on $\mathbb{C}$ onto the multiplicative structure of $\mathbb{C}^*$. Then $\hat{\operatorname{Exp}}$ gives a multiplication “$*$” in $M$, which is the push-forward of the additive structure on $\mathbb{H}^3$. Let $a$ and $b$ elements in $M$, and let $a_1$ and $b_1$ be elements such hat $a = \hat{\operatorname{Exp}}(a_1)$ and $b = \hat{\operatorname{Exp}}(b_1)$. Then

$$a \ast b = \hat{\operatorname{Exp}}(a_1 + b_1).$$

Lemma 20. The multiplicative structure in $M$ extends to a multiplicative structure on $\mathbb{H}^3$.

Proof. Let $\|\|_3$ the standard norm in the euclidean space $\mathbb{R}^3$. Then for every $x$ and $y$ in $\mathbb{H}^3$. We have

$$\|x \ast y\| = \|x\| \|y\|.$$  

Now, let $x = (0, 0, t)$. For any $y \in \mathbb{H}^3$ define $y \tau = (0, 0, t\|y\|)$. The restriction of this multiplication to points in the axis $t$ in $\mathbb{H}^3$ coincides with the multiplication on $\mathbb{R}_+$. This definition continuously extends the multiplication $\ast$ to $\mathbb{H}^3$.

The multiplication in $\mathbb{H}^3$ is commutative and associative and has the following properties:

- The multiplication on the boundary is the usual multiplication in $\mathbb{C}$.
- Let $\lambda$ be a non-zero complex number then, for any $x \in \mathbb{H}^3$, we have $\lambda \ast x = H_\lambda(x)$. Where $H_\lambda(x)$ is the Poincaré extension of the map $z \mapsto \lambda z$.  

19
• The unique unit element is \((1, 0, 0)\). For any \(x \neq 0, \infty\) in \(\mathbb{H}^3\) there exist \(y\) such that \(x * y = y * x = (1, 0, 0)\) and \(y = H(x)\) where \(H\) is the Poincaré extension of the map \(z \mapsto 1/z\).

Now we can define an extension of rational maps. Let \(R\) be a rational map and can be represented as a product of Möbius maps:

\[
R(z) = \prod \gamma_i(z)
\]

where \(\gamma_i \in PSL(2, C)\). Hence for \(x \in \mathbb{H}^3\) we have a extension with respect to the maps \(\gamma_i\)

\[
\hat{R}(x) = \prod \gamma_i P(\gamma_i)(x)
\]

where \(P(\gamma_i)\) is the Poincaré extension of \(\gamma_i\) in \(\mathbb{H}^3\). Since the multiplication is commutative then the definition does not depend on the order of the factors \(\gamma_i\).

The following proposition follows from the definition of the product extension.

**Proposition 21.** The product extension has the following properties:

1. If \(\sigma_i\) is the geodesic that connects the pole with the zero of \(\gamma_i\). Then \(\hat{R}(\sigma_i)\) is the \(t\)-axis.
2. The extension \(R \mapsto \hat{R}\) is visual.
3. Any rational map \(R\) has a finite number of decompositions in Möbius factors. Hence there are only finitely many product extensions for each rational map \(R\).

Another extension from \(\mathbb{C}\) to \(\mathbb{H}^3\) is induced with the product, is given by a monomorphism \(\Phi\) from the ring of formal series over the usual multiplication on the complex plane to the ring of formal series with the \(*\) multiplication. The map \(\Phi\) is continuous on the subring of polynomials. However, it is not clear whether it is still continuous on the subring of absolutely convergent series. Note that this extension is not conformally natural, is not even visual. The map \(\Phi\) is not a homomorphism with respect to composition. The biggest semigroup \(S\), where \(\Phi\) defines a homomorphism with respect to composition, is the generated by \(\lambda z^n\) for any complex \(\lambda\) and \(n\) a natural number. Moreover, \(\Phi\) on \(S\) is conformally natural, geometric and the same degree. In general is not clear when product and ring extensions are geometric, numerical calculations of the ring extension of \(z^2 + c\), with \(c\) real, suggests that this extension is geometric.

### 4.1 Some examples of Poincaré extensions of quadratic polynomials

Here we compute some Poincaré extensions. These computations are based in the following formula for the exponential map defined on the previous section.

\[
\hat{\text{Exp}}(x, y, t) = \left( \frac{2e^t \cos(y)}{1 + e^{2t}}, \frac{2e^t \sin(y)}{1 + e^{2t}}, \frac{e^{2t} - 1}{1 + e^{2t}} e^x \right).
\]

We have the following facts:

• The map \(\hat{\text{Exp}}\) is a Poincaré extension of the map \(e^z\).
• Let $T$ be group generated by the translation $z \mapsto z + 2\pi i$, then the action of $T$ in $\mathbb{C}$ extends an action in $\mathbb{H}^3$ generated by the map $(z, t) \mapsto (z + 2\pi i, t)$.

• The orbit space $\mathbb{H}^3/T$ is homeomorphic to $B_L := \mathbb{H}^3 \setminus L$ where $L$ is the $t$-axis.

• There exist a complete hyperbolic Möbius structure on $B_L$ so that $\tilde{\text{Ext}} : \mathbb{H}^3 \to B_L$ defines a Möbius universal covering map.

• The extension from $\text{Exp}$ to $\tilde{\text{Exp}}$ is conformally natural.

Let $H_2$ be the Poincaré extension of the Möbius map $z \mapsto 2z$, hence the map $\tilde{Q} = \tilde{\text{Exp}} \circ H_2 \circ \tilde{\text{Exp}}^{-1} : B_L \to B_L$ is a Poincaré extension of the map $Q(z) = z^2$.

For a circle $S$ in $\partial \mathbb{H}^3$, let us define the dome over $S$ as the 2-sphere with equator $S$ intersected with $\mathbb{H}^3$ and will be denoted by $\text{Dome}(S)$.

Using the equations above, we obtain the equation

$$\tilde{Q}(\lambda(X, Y, T)) = \lambda^2 \left( \frac{X^2 - Y^2}{1 + T^2} - \frac{2XY}{1 + T^2}, \frac{2T}{1 + T^2} \right)$$

with $\lambda \in \mathbb{R}$ and $(X, Y, T) \in \text{Dome}(S^1)$.

If $x = \lambda X$, $y = \lambda Y$ and $t = \lambda T$, then $||p||^2 = x^2 + y^2 + t^2 = \lambda^2$. We have that

$$\tilde{Q}(x, y, t) = (||p||^2 \frac{x^2 - y^2}{||p||^2 + t^2}, ||p||^2 \frac{2xy}{||p||^2 + t^2}, ||p||^2 \frac{2t||p||}{||p||^2 + t^2})$$

In this case, by the formula above, we have that $\tilde{Q}$ extends to the whole $\mathbb{H}^3$ and $\tilde{Q}(0, 0, t) = (0, 0, t^2)$. Moreover, for every $w$ in $\mathbb{H}^3$, we have $\tilde{Q}(w) = w \ast w$ where $\ast$ is the product defined above. The map $\tilde{Q}$ commutes with the reflection with respect to the dome.

We have the following invariant foliations for the action of $\tilde{Q}$:

1. **Semi-spheres centered at the origin.** Observe that parallel planes of the form $(x_0, y, t)$: under $H_2$ are mapped to themselves $(2x_0, 2y, 2t)$. The map $\tilde{\text{Exp}}$ send this foliation to a foliation of domes over circles centered at the origin. Hence this domes is a foliation invariant under $\tilde{Q}$.

2. **Cones centered at the origin:** Horizontal planes (horocyclic foliation of $\mathbb{B}^3$) of the form $(x, y_0, z)$ are invariant under $H_2$, hence their image under $\tilde{\text{Exp}}$ also. These are cones centered at $0$.

3. **Onion like foliation:** Planes of the form $(x, y, kx)$, are invariant under $H_2$. Its image is the onion-like foliation surrounding the vertical axis $(0, 0, t)$. This is a book decomposition, hence the bind of the book is the unit circle in the boundary plane.

Let $T_c(x, y, t) = (x + Re(c), y + Im(c), t)$ be the Poincaré extension of the map $z \mapsto z + c$ and $L_c$ be the vertical line over $c$ in $\mathbb{H}^3$ and let $B_{L_c} = \mathbb{H}^3 \setminus L_c$. If $Q_c(z) = z^2 + c$, then we have an extension $\tilde{Q}$ of $Q_c$ depicted in the following commutative diagram.
The diagram implies that $\hat{Q}_c = T_c \circ \hat{Q}$. In this case, the line $L$ is also the critical line but has complicated dynamics. Let us define $K(\hat{Q}_c):= \text{spatial filled Julia set}$ of $Q_c$, the set of $(x, y, t)$ such that $\hat{Q}_c^n(x, y, t)$ does not tends to $\infty$ as $n \to \infty$.

Using similar arguments as in the one dimensional case, one can show that $K(\hat{Q}_c)$ is always bounded in $\mathbb{H}^3$. Also, for parameters $c$ with $|c|$ large enough, the critical line converge to infinity. If $V_0$ denotes the semiplane $\{(x, y, t) \in \mathbb{H}^3 : y = 0\}$. The section $K(\hat{Q}_c) \cap V_0$ is a bi-dimensional set that we have illustrated in Figure 4.1 for different values of $c$.

![Figure 1: The sets $K(\hat{Q}_c) \cap V_0$ for $c = 0.25, -0.75, -0.77, -1, -1.28$ from left to right and top to bottom.](image)

5 Remarks and conclusions

As it was mentioned in the introduction, there are some constructions in the literature of extensions of rational maps into endomorphisms of $\mathbb{H}^3$. Most of these extensions are based on the barycentric construction suggested by Choquet’s theorem. Let us briefly describe the barycentric extension. Let $\mathbb{B}^3$ be the closed unit ball in $\mathbb{R}^3$. Let $\mathcal{M}$ be the space of probability measures on $\partial \mathbb{B}^3$. Then, for every $\mu$ in $\mathcal{M}$ the barycenter of $\mu$ is the unique point $x$ such that for
every functional $L$ on $\mathbb{R}^3$, the following equation holds

$$L(x) = \int_{\mathbb{R}^3} L(y) d\mu(y).$$

We define $\text{Bar}(\mu) = x$, by Choquet’s theorem (See [15], page 48) the map $\text{Bar}$ sends $\mathcal{M}$ onto $\mathbb{R}^3$. The semigroup $\text{Rat}(\mathbb{C})$ acts in $\mathcal{M}$ by push-forward, for every $R$ in $\text{Rat}(\mathbb{C})$ we denote by $R\mu = R_{\ast}(\mu)$ the push-forward of $\mu$ by $R$. For every point $x \in \mathbb{R}^3$, let $\nu_x$ be the visual measure based on $x$ we define the barycentric extension of $R$ by

$$\hat{R}(x) = \text{Bar}(R\nu_x).$$

Then the barycentric extension is visual as is proved in [12]. If instead we use the conformal barycenter we obtain a conformally natural extension as is proved in [14]. It is very difficult to get any geometric information of these extensions. For instance, it is not clear that these extensions defines a branched covering of the same degree from $\mathbb{H}^3$ to $\mathbb{H}^3$.

The following proposition was already mentioned in [7]. We include the proof for completeness.

**Proposition 22.** Let $R$ be a rational map, then all conformally natural extensions of $R$ are homotopic, with a homotopy that consist of conformally natural extensions of $R$.

**Proof.** Let $\hat{Q}$ and $\hat{S}$ be extensions of a rational map $R$, and let $x \in \mathbb{H}^3$. For $\lambda$ in $[0,1]$, let $E_\lambda(x)$ be the point along the geodesic from $\hat{Q}(x)$ to $\hat{S}(x)$, which is at distance $\lambda d(\hat{Q}(x), \hat{S}(x))$. Since for every $x \in \partial\mathbb{H}^3$, $\hat{Q}(x) = \hat{S}(x) = R(x)$. The map $E_\lambda(x)$ extends to $\partial\mathbb{H}^3$ as an extension of $R$. If the maps $\hat{Q}$ and $\hat{S}$ are either visual, conformally natural or Poincaré, the map $E_\lambda(x)$ also holds the same property.

It follows that if for a given rational map $R$ there are two visual (or conformally natural) extensions, then there are uncountably many visual (or conformally natural) extensions. This situation contrasts with the product extensions which are only finitely many.

The extension discussed in [9] is uniformly quasiregular dynamical, has the same degree as the starting map $R$. Moreover, it can be shown that is geometric. However, for most rational maps these extensions do not exist [9].

Another aspect of our geometric construction is about Maskit surgery on the respective Möbius manifolds. A rational map $R : S_1 \to S_2$ is modeled with two groups $\Gamma_1 \leq \Gamma_2$. Let us assume that $\Gamma_2 = \langle \gamma_1, ..., \gamma_n \rangle$. For $1 < k < n$ $G = \langle \gamma_1, ..., \gamma_k \rangle$ and $H = \langle \gamma_{k+1}, ..., \gamma_n \rangle$, and consider the intersections $G_i = \Gamma_i \cap G$ and $H_i = \Gamma_i \cap H$. Then $G_1$ and $H_1$ are subgroups of finite index in $G_2$ and $H_2$ respectively. This construction defines two rational maps $R_G$ and $R_H$ associated to groups $G_1$ and $H_1$ respectively. Is possible to define analogously amalgamated products and HNN-extension.

Now that we have associated a Schottky group to each Blaschke maps, we can use Maskit combination theorems in order to construct rational maps out of a pair of Blaschke maps. A theorem of B. Maskit shows that every Schottky group is product of cyclic groups. Let us consider the following example, let $B$ and $B'$ be two Blaschke products of degree 2, let $O_1$ and $O_2$ be uniformizations
associated to $B$, and $O_1'$ and $O_2'$ be the uniformization of Schottky type associated to $B'$ such that the automorphism group of each $O_i$ is Schottky. Let $\Gamma_1$ and $\Gamma_2$, $\Gamma_1'$ and $\Gamma_2'$ be the associated Schottky groups, then we construct the product $\tilde{\Gamma}_1 = \Gamma_1 \ast \Gamma_1'$ and $\tilde{\Gamma}_2 = \Gamma_2 \ast \Gamma_2'$. The diagonal action of the inclusions gives an inclusion $\tilde{\alpha} : \tilde{\Gamma}_1 \to \tilde{\Gamma}_2$. Defined on the connected sum of $O_1$ with $O_2$. After taking quotients, we obtain a rational map $R$ which depends only on the combinatorial data of $B$ and $B'$. We believe that classical combination constructions in holomorphic dynamics, such as mating, tuning and surgery are special cases of the combinations just described.

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