ANALYTIC SOLUTION TO AN INTERFACIAL FLOW WITH KINETIC UNDERCOOLING IN A TIME-DEPENDENT GAP HELE-SHAW CELL

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(Communicated by Thomas P. Witelski)

ABSTRACT. Hele-Shaw cells where the top plate is lifted uniformly at a prescribed speed and the bottom plate is fixed have been used to study interface related problems. This paper focuses on an interfacial flow with kinetic undercooling regularization in a radial Hele-Shaw cell with a time dependent gap. We obtain the local existence of analytic solution of the moving boundary problem when the initial data is analytic. The methodology is to use complex analysis and reduce the free boundary problem to a Riemann-Hilbert problem and an abstract Cauchy-Kovalevskaya evolution problem.

1. Introduction. The problem of a less viscous fluid displacing a more viscous fluid in a Hele-Shaw cell has been the subject of numerous studies since Saffman and Taylor’s seminal papers [31, 38] in the 1950s. Ignoring the surface tension, Saffman and Taylor [31] found an one-parameter family of exact steady solutions, parameterized by width $\lambda$. This theoretical shape (usually referred to in the literature as the Saffman-Taylor finger) agreed well with experiments for relatively large displacement rates, or equivalently for small surface tension. It was found [31] experimentally that an unstable planar interface evolves through finger competition to a steady translating finger, with relative finger width $\lambda$ close to one half. However, in the zero-surface-tension steady-state theory, $\lambda$ remained undetermined in the $(0, 1)$ interval. The selection of $\lambda$ remained unresolved until the mid 1980s. Numerical calculations [22], [42], [19], supported by formal asymptotic calculations in the steady finger [8, 9], [34], [18], [36], [5] and the closely-related steady Hele-Shaw bubble problem [10, 35], suggest that a discrete family of solutions exist for which the limiting shape, as surface tension tends to zero, approaches the Saffman-Taylor finger with $\lambda = \frac{1}{2}$. Rigorous results were later obtained in [43, 37].

The Hele-Shaw problem is similar to the Stefan problem in the context of melting or freezing. Besides surface tension, the physical effect most commonly incorporated in regularizing the ill-posed Stefan problem is kinetic undercooling [15, 20, 2], where the temperature on the moving interface is proportional to the normal velocity of the interface. For the Hele-Shaw problem, kinetic undercooling regularization...
first appeared in [29, 30]. Local existence of analytic solution was obtained in [26, 28] for the time dependent Hele-Shaw problem with kinetic undercooling. Using exponential asymptotics, Chapman and King [6] analyzed the selection problem of determining the discrete set of widths of a traveling finger for varying kinetic undercooling. Some numerical studies have been attempted in [11, 16]. A continuum of corner-free traveling fingers were obtained numerically in [11] for any finger width above a critical value, while a discrete set of analytic fingers, as predicted in [6], were computed in [16]. The physical connection between the kinetic undercooling effect and the action of the dynamic contact angle was established in [1]. The selection problem with undercooling regularization was studied rigorously in [44].

Besides the classical Hele-Shaw setup, there are several variants related to the viscous fingering problem [17, 4, 12, 13]. One of the variants is interfacial flows in a Hele-Shaw cell where the top plate is lifted uniformly at a prescribed speed and the bottom plate is fixed (lifting plate problems) [32, 39, 40, 7, 33, 14, 23, 45]. In the lifting plate problem, the gap \( b(t) \) between the two plates is increasing in time but uniform in space. As the plate is pulled, an inner viscous fluid shrinks in the center plane between the two plates and increases in the \( z \)-direction to preserve volume. An outer less viscous fluid invades the cell and generates fingering patterns. The patterns are visually similar to those in the classical radial Hele-Shaw problem, but the driving physics is different. In [32], the authors derived the governing equations for the lifting problem and they established the existence, uniqueness and regularity of solutions for analytic data when the surface tension is zero. Some exact solutions were also constructed, both with or without surface tension. Analytic results were also be generalized to higher dimensions in [41]. Numerical simulation and the pattern formation of the interface were presented in [32, 45].

In this paper, we study the lifting problem where the kinetic undercooling regularization is used instead of the surface tension. A radially symmetric solution is found and its linear stability is studied. We also obtain the local existence of analytic solution of the moving boundary problem when the initial data is analytic. The methodology is to use complex analysis and reduce the free boundary problem to a Riemann-Hilbert problem and an abstract Cauchy-Kovalevskaya evolution problem. Although the methodology is similar to that in [28], due to the effect of the time dependent gap on the governing equations, the analysis of the Riemann-Hilbert problem and the construction of the Cauchy-Kovalevskaya evolution problem in the lifting problem exhibit a number of subtleties (see the detailed discussion in section 4).

2. Mathematical formulation. Our studies center on the free interface problem in a Hele-Shaw cell with a time dependent gap \( b(t) \); see Figure 1. The upper plate is lifted perpendicular to the cell, while the lower plate stays fixed. We assume that

\[
b(t) \in C^1([0, \infty)), \ b(t) \geq b_0 \text{ for some positive constants } b_0.
\]

We consider the displacement of a viscous fluid by another fluid of negligible viscosity. Let \( \Omega(t) \) in the \((x, y)\) plane be the more viscous fluid domain with free boundary \( \partial \Omega \). Following M. Shelley, F. Tian, and K. Wlodarski [32], we have the following governing equations: The fluid velocity is

\[
u = -\frac{b^2(t)}{12 \mu} \nabla p(x, y, t), \tag{1}
\]

where \( p \) is the pressure and \( \mu \) is the viscosity of the fluid. Conservation of mass equation is
The kinematic boundary condition is
\[- \frac{b^2(t)}{12\mu} \frac{\partial p}{\partial n} = V_n \text{ on } \partial \Omega(t), \tag{3}\]
where \( \frac{\partial}{\partial n} \) denotes the derivative in the direction of the outward normal \( n \) to \( \partial \Omega \), and \( V_n \) is the velocity of the \( \partial \Omega(t) \) in the direction of outward normal vector \( n \); and the dynamic boundary condition is
\[ p = \tau V_n \text{ on } \partial \Omega(t), \tag{4}\]
where \( \tau \) is a kinetic undercooling coefficient.

Non-dimensionalizing the length and time and the pressure, the nondimensional version of (1), (2), (3) and (4) is
\[ u = -b^2(t) \nabla p(x, y, t) \text{ in } \Omega(t), \tag{5}\]
\[ \nabla \cdot u = -\frac{\dot{b}(t)}{b(t)} \text{ in } \Omega(t), \tag{6}\]
\[ -b^2(t) \frac{\partial p}{\partial n} = V_n \text{ on } \partial \Omega(t), \tag{7}\]
\[ p = cV_n \text{ on } \partial \Omega(t); \tag{8}\]
where \( c \) is the nondimensional kinetic undercooling coefficient.

Plugging (5) into (6), we obtain
\[ \nabla^2 p = \frac{\dot{b}(t)}{b^3(t)} \text{ in } \Omega(t). \tag{9}\]

Hence the lifting He-Shaw problem consists of the equations (7), (8) and (9).

2.1. **Exact solution and Linear stability analysis.** It can be seen that (9), (7) and (8) have a radially symmetric solution \( \Omega(t) = \{ (x, y) : r = \sqrt{x^2 + y^2} < R(t) \} \), where
\[ \frac{R(t)}{R(0)} = \frac{\dot{b}(t)}{2b(t)}, \quad R(t) = \frac{R(0)b(0)}{\sqrt{b(t)}}; \tag{10}\]
and
\[ p(t, r) = -\frac{cR(0)\sqrt{b(0)}\dot{b}(t)}{2b(t)\sqrt{b(t)}} - \frac{R^2(0)b(0)\dot{b}(t)}{4b^4(t)} + \frac{\dot{b}(t)}{4b^3(t)} r^2. \tag{11}\]
We consider the interface to be perturbed, \( r(t, \alpha) = R(t) + \epsilon \delta(t) \cos(k \alpha) \), where \( \epsilon \) is small, \( k \geq 2 \) is the perturbation mode, \( \alpha \in [0, 2\pi] \) is the polar angle and \( \delta(t) \) is the amplitude of the perturbation. From (9), (7) and (8), following standard perturbation analysis, we find that

\[
\frac{\dot{\delta}(t)}{\delta(t)} = \frac{(k - 1) \sqrt{R(0)b(0)\dot{b}(t)}}{2b(t)(\sqrt{R(0)b(0)} + k\epsilon^5/2(t))},
\]

or alternatively

\[
\frac{\dot{\delta}(t)}{\delta(t)} - \frac{\dot{R}(t)}{R(t)} = \left( \frac{\delta(t)}{R(t)} \right)^{-1} \frac{d}{dt} \left( \frac{\delta(t)}{R(t)} \right) = \frac{k\dot{b}(t)}{2b(t)} \left[ \frac{\sqrt{R(0)b(0)} + k\epsilon^5/2(t)}{\sqrt{R(0)b(0)} + k\epsilon^5/2(t)} \right] \cdot \delta(t).
\]

We can see that the perturbation grows when \( \dot{b}(t) \) is positive while the perturbation decays when \( \dot{b}(t) \) is negative. From the linear stability analysis, one expects that the global smooth solution to the lifting problem may not exist. Nevertheless, we are going to prove the existence of local smooth solution to the lifting problem.

2.2. Reformulation of the problem. Let \( \tilde{p} = p - \frac{\dot{b}(t)}{4b^2(t)}(x^2 + y^2) \), then \( \tilde{p} \) satisfies

\[
\nabla^2 \tilde{p} = 0, \text{ in } \Omega(t)
\]

\[
\tilde{p} = cV_n - \frac{\dot{b}(t)}{4b^2(t)}(x^2 + y^2), \text{ on } \partial\Omega(t),
\]

\[
V_n = -b^2(t) \frac{\partial \tilde{p}}{\partial n} - \frac{\dot{b}(t)}{2b(t)}(x, y) \cdot n \text{ on } \partial\Omega(t),
\]

Let \( z = f(t, \xi) \) be the conformal mapping that maps \( \Omega(t) \) onto the unit disk \( |\xi| < 1 \). Without loss of generality, we assume that \( \Omega(t) \) contains \( z = 0 \) for all \( t \geq 0 \) and \( f(t, 0) = 0 \) and \( f'(t, 0) \) is real. The unit outward normal vector \( n \) is given by \( n = \frac{\xi f'}{|f'|} \), and the normal velocity is given by

\[
V_n = \text{Re} \left[ \frac{f \xi f'}{|f'|} \right]
\]

Let \( W(t, \xi) = \tilde{p} + i\tilde{q} \), where \( \tilde{q} \) is a harmonic conjugate of \( \tilde{p} \), then \( W(t, \xi) \) is analytic in \( |\xi| < 1 \); (15) becomes

\[
\text{Re} \left( \frac{\partial f}{\partial t} \left( \xi \frac{\partial f}{\partial \xi} \right) \right) = -b^2(t) \text{Re} \left[ \xi W_\xi \right] - \frac{\dot{b}(t)}{2b(t)} \text{Re} \left( f \frac{\partial f}{\partial \xi} \right), \text{ on } |\xi| = 1.
\]

(14) becomes

\[
\text{Re} W = c \frac{\text{Re} \left( \frac{\partial t}{\partial \xi} \left( \xi \frac{\partial f}{\partial \xi} \right) \right)}{|f'|} - \frac{\dot{b}(t)}{4b^2(t)} |f|^2;
\]

where \( \xi = e^{i\theta} \) is on the unit circle \( |\xi| = 1 \).

Let

\[
\phi(t, \xi) = \left( \frac{\partial f}{\partial \xi}(t, \xi) \right)^{-1}, \text{ } w(t, \xi) = \xi W_\xi(t, \xi)
\]
Now we turn to the equation for \(w(t, \xi)\). First \(w(t, \xi)\) must be analytic in \(|\xi| < 1\). We note that on the unit circle \(\xi = e^{i\theta}, \partial_{\theta} = i\xi \partial_{\xi}\). In (18), taking derivative with respect to \(\theta\), and using (17) and (18) we obtain on the unit circle \(|\xi| = 1\),

\[
Im \, |w| = \chi b^2(t)\partial_{\theta} \left( |\phi| Re \, |w| + e^{\frac{\bar{b}(t)}{2b(t)}} \partial_{\theta} \left( |\phi|^{-1} Re \, |\xi^{-1} f \phi| \right) \right) + \frac{\bar{b}(t)}{4b^2(t)} \partial_{\theta} \left( |f|^2 \right) .
\]

Hence, \(w(t, \xi)\) is the solution of the following Riemann-Hilbert problem:

**Riemann-Hilbert problem:** Given \(\phi(t, \xi)\) and \(f(t, \xi)\), find \(w(t, \xi)\) so that \(w(t, \xi)\) is analytic in \(|\xi| < 1\) and satisfies (20) on \(|\xi| = 1\).

The equation (17) can be written as

\[
Re \left( \frac{f_t}{\xi f_\xi} \right) = -b^2(t) |f_\xi|^{-2} Re \, |w| - \frac{\bar{b}(t)}{2b(t)} Re \left( \frac{f}{\xi f_\xi} \right), \text{ on } |\xi| = 1.
\]

Since \(f(t, 0) = 0\) implies that \(\frac{f_t}{\xi f_\xi}\) and \(\frac{f}{\xi f_\xi}\) are analytic in \(|\xi| < 1\); the Poisson formula gives

\[
\frac{f_t}{\xi f_\xi} = -b^2(t) T(\phi, f) - \frac{\bar{b}(t)}{2b(t)} \left( \frac{f}{\xi f_\xi} \right) + i\alpha(t), \text{ in } |\xi| < 1.
\]

where \(\alpha(t)\) is real and the operator \(T(\phi, f)\) is defined as

\[
T(\phi, f)(t, \xi) = \frac{1}{2\pi i} \int_{|\eta| = 1} [\phi(t, \eta)]^2 Re[w(t, \eta)] \frac{(\eta + \xi) d\eta}{\eta(\eta - \xi)},
\]

where \(w\) is the solution of the Riemann-Hilbert problem (20).

Using the fact that \(f_\xi(t, 0)\) is real, we have \(\alpha(t) = 0\). So (22) becomes

\[
f_t = -b^2(t) \xi f_\xi T(\phi, f) - \frac{\bar{b}(t)}{2b(t)} f, \text{ in } |\xi| < 1,
\]

Taking derivative with respect to \(\xi\) in the above equation and using (19), we have

\[
\frac{\partial \phi}{\partial t} = -b^2(t) \xi \left( \frac{\partial \phi}{\partial \xi} \right) T(\phi, f) + b^2(t) \phi \frac{\partial}{\partial \xi} (\xi T(\phi, f)) + \frac{\bar{b}(t)}{2b(t)} \phi, \text{ in } |\xi| < 1.
\]

We impose initial condition on \(f(t, \xi)\)

\[
f(0, \xi) = f_0(\xi)
\]

where \(f_0(0) = 0, \frac{\partial f_0}{\partial \xi} \neq 0\) in a neighborhood of the unit disk \(|\xi| < 1\). From (19), the initial condition for \(\phi\) is

\[
\phi(0, \xi) = \frac{1}{\frac{\partial f_0}{\partial \xi}(\xi)}.
\]

So we obtain the following problem:

**Problem one:** find functions \(f(t, \xi), \phi(t, \xi)\) and \(w(t, \xi)\) such that they are analytic in a neighborhood of the unit disk \(|\xi| < 1\) as functions of \(\xi\) and continuously differentiable with respect to \(t\), and they satisfy (20) and (24)-(27).

3. **Existence of analytic solution.** In this section, we are going to prove the existence of analytic solution to the problem one consisting of (24)-(27). We first introduce some function spaces which are Banach spaces.

Let \(r_1\) and \(r\) be two fixed numbers such that \(1 < r_1 < r\). Let \(\mathcal{R}_s\) be the disk in complex \(\xi\) plane with radius \(s\), i.e. \(\mathcal{R}_s = \{\xi, |\xi| < s\}\); we define function space \(\mathcal{B}_s\) so that
\( \mathbf{B}_s = \{ f(\xi) : f(\xi) \text{ is analytic in } \mathcal{R}_s, \partial_\xi f \text{ is continuous on } \overline{\mathcal{R}}_s, \| f \|_s < \infty \} \)

with norm \( \| f \|_s = \| f \| := \sup_{\xi \in \mathcal{R}} | f(\xi) | + \sup_{\xi \in \mathcal{R}} | \partial_\xi f(\xi) | \).

**Lemma 3.1.** The function space \( \mathbf{B}_s \) is a Banach space and an algebra with \( \| fg \|_s \leq \| f \|_s \| g \|_s \).

We define the Holder space \( C^{k,\alpha}(\overline{\mathcal{R}}_1) \) as the set of functions \( f \) on \( \overline{\mathcal{R}}_1 \) such that:

1. The derivatives \( \partial_{\xi_1}^k \partial_{\xi_2}^k f \) are continuously extended to \( \overline{\mathcal{R}}_1 \), where \( \xi_1 \) and \( \xi_2 \) are the real and imaginary part of \( \xi \) respectively, i.e. \( \xi = \xi_1 + i\xi_2; k_1 \) and \( k_2 \) are non-negative integers such that \( k_1 + k_2 \leq k \).

2. The norm \( | f |_{k,\alpha} \) is finite, here \( | f |_{k,\alpha} \) is defined as

\[
| f |_{k,\alpha} = \sum_{k_1 + k_2 \leq k} \sup_{\xi \in \mathcal{R}_1} | \partial_{\xi_1}^{k_1} \partial_{\xi_2}^{k_2} f(\xi) | + \sum_{k_1 + k_2 = k} \sup_{\xi, \eta \in \mathcal{R}_1} \frac{| \partial_{\xi_1}^{k_1} \partial_{\xi_2}^{k_2} f(\xi) - \partial_{\xi_1}^{k_1} \partial_{\xi_2}^{k_2} f(\eta) |}{| \xi - \eta |^\alpha}.
\]

**Lemma 3.2.** Let \( f \in \mathbf{B}_s, r > 1 \), then \( f \in C^{k,\alpha}(\overline{\mathcal{R}}_1) \) for any positive integer \( k \) and \( \alpha \in (0,1) \), and \( | f |_{k,\alpha} \leq C \| f \|_r \), where \( C \) is a positive constant depending only on \( k, r \) and \( \alpha \).

**Proof.** For \( f \in \mathbf{B}_s, r > 1 \), \( f \) is analytic in \( \mathcal{R}_r \). Fix \( r_1 \) so that \( 1 < r_1 < r \), using Cauchy integral formula, we obtain

\[
\partial_\xi^k f(\xi) = \frac{k!}{2\pi i} \int_{|\eta| = r_1} \frac{f(\eta)}{(\eta - \xi)^{k+1}} d\eta \quad \text{for } \xi \in \overline{\mathcal{R}}_1.
\] (28)

From (28), all derivatives \( \partial_\xi^k f \) can be bounded by \( C \sup_{|\eta| = r_1} | f(\eta) | \) since \( | \xi - \eta | \geq r_1 - 1 \) for \( \xi \in \overline{\mathcal{R}}_1 \) and \( |\eta| = r_1 \). Hence the lemma follows. \( \square \)

### 3.1. Riemann-Hilbert problem and analytic continuation to \( \mathcal{R}_s \). We first establish an existence result for the Riemann-Hilbert problem (20) on the region \( \overline{\mathcal{R}}_1 \), then the solution is extended analytically to the region \( \overline{\mathcal{R}}_s, s > 1 \). In the Riemann-Hilbert problem, the variable \( t \) is only a parameter. For brevity, we omit the variable \( t \) in this section and use \( \phi(\xi), f(\xi), w(\xi) \), etc.

**Lemma 3.3.** Let \( f, \phi \in \mathbf{B}_s, \phi \neq 0 \) in \( \mathcal{R}_s \), then there exists a unique function \( w \in C^{2,\alpha}(\overline{\mathcal{R}}_1) \) which is analytic in \( \mathcal{R}_s \) and satisfies (20), \( w(0) = 0 \) and

\[
| w |_{2,\alpha} \leq C(|\phi^{-1}|_{2,\alpha} + | f |_{2,\alpha} + | \phi^{-1} |_{2,\alpha} | \phi |_{2,\alpha} | f |_{2,\alpha}).
\] (29)

**Proof.** We denote the Hilbert transform on the unit circle by \( \mathcal{H}(f) \):

\[
\mathcal{H}(f)(\theta) = \frac{1}{2\pi} \int_0^{2\pi} f(t) \cot \left( \frac{\theta - t}{2} \right) dt.
\] (30)

If \( w \) is analytic in \( \mathcal{R}_1 \), then

\[
\text{Im} | w |(\theta) = \mathcal{H}(\text{Re} | w |)(\theta), \quad \text{Im} | w |(0) = 0.
\] (31)

Taking the Hilbert transform of (20) to obtain

\[
- \text{Re} | w | = \epsilon b^2 \partial_\theta \mathcal{H} (|\phi| \text{Re} | w |) + c \frac{b(t)}{2b(t)} \partial_\theta \mathcal{H} (|\phi|^{-1} \text{Re} | x^{-1} f \phi |) + \frac{b(t)}{4b(t)} \partial_\theta \mathcal{H} (| f |^2).
\] (32)
If $U(\xi, \bar{\xi})$ is the harmonic function in $\mathcal{R}_1$ and satisfies $U(\xi, \bar{\xi}) = |\phi(\xi)| Re \{w(\xi)| \}$ for $\xi \in \partial \mathcal{R}_1$, then from Cauchy Riemann equations $\frac{\partial U}{\partial n} = \partial_{\bar{\xi}} \mathcal{H}(|\phi(\xi)| Re \{w(\xi)\})$. So (31) can be written as

$$c b^2 \frac{\partial U}{\partial n} + \frac{U}{|\phi|} = -c \frac{\dot{b}(t)}{2b(t)} \partial_{\bar{\xi}} \mathcal{H} \{ |\phi|^{-1} Re \{\xi^{-1} f(\phi)\} \} - \frac{b(t)}{4b^3(t)} \partial_{\bar{\xi}} \mathcal{H} \{ |f|^2 \}. \quad (33)$$

From Theorem 3.1 and Theorem 3.2 in [21], there exists a unique solution defined on the Riemann- Hilbert problem (20) as in Lemma 3.3, then $h$ has a unique solution in $\mathcal{R}_1$.

We next extend the solution $w$ to the Riemann-Hilbert problem (20) as in Lemma 3.3, then

$$w_1 - w_2|_{\mathcal{R}_1} \leq C(|\phi_1 - \phi_2|_{\mathcal{R}_1} + |\phi_1^{-1} - \phi_2^{-1}|_{\mathcal{R}_1} + |f_1 - f_2|_{\mathcal{R}_1}), \quad (35)$$

where the constant $C = C(|f_j|_{\mathcal{R}_1}, |\phi_j|_{\mathcal{R}_1}, |\phi_j^{-1}|_{\mathcal{R}_1})$ such that if $|f_j|_{\mathcal{R}_1} \leq M, |\phi_j|_{\mathcal{R}_1} \leq M, |\phi_j^{-1}|_{\mathcal{R}_1} \leq M$, then $C \leq C(M)$.

**Proof.** Since $w_j, j = 1, 2$ satisfies (20) corresponding to $\phi_j$ and $f_j$, the difference $w = w_1 - w_2$ satisfies

$$Im \{w\} = c b^2 \partial_{\bar{\xi}} \{(|\phi_1| Re \{w\}) + c \frac{\dot{b}(t)}{2b(t)} \partial_{\bar{\xi}} \{(|\phi_1|^{-1} - |\phi_2|^{-1}|) Re \{\xi^{-1} f(\phi)\} \}$$

$$+ c \frac{\dot{b}(t)}{2b(t)} \partial_{\bar{\xi}} \{(|\phi_2|^{-1} Re \{\xi^{-1} (f_1 - f_2) \phi_1\})$$

$$+ c \frac{\dot{b}(t)}{2b(t)} \partial_{\bar{\xi}} \{(|\phi_2|^{-1} Re \{\xi^{-1} (\phi_1 - \phi_2) f_2\})$$

$$+ c b^2 \partial_{\bar{\xi}} \{(|\phi_1| - |\phi_2|) Re \{w_2\}) + \frac{\dot{b}(t)}{4b^3(t)} \partial_{\bar{\xi}} \{(|f_1| - |f_2|)(|f_1| + |f_2|)\} \}. \quad (36)$$

Applying the same steps as in the proof of Lemma 3.3 to $w$ and (36), we obtain the lemma. 

We next extend the solution $w$ of the Riemann- Hilbert problem in Lemma 3.3 defined on $\overline{\mathcal{R}_1}$ analytically to the region $\overline{\mathcal{R}_s}, s > 1$. We are going to use differential equation method to do the extension. In particular, we first derive the differential equation that $w$ satisfies on $\overline{\mathcal{R}_1}$. Since all coefficients and the nonhomogeneous term of the derived first order differential equation are analytic in $\overline{\mathcal{R}_s}, s > 1$, the equation has a unique solution in $\overline{\mathcal{R}_s}, s > 1$, which is the analytic extension of $w$. We first introduce some functions and notations.

**Definition 3.5.** Let $\phi \in \mathcal{B}_s$, $\phi \neq 0$, we define the function $h^+(\xi) = (\phi(\xi))^2$ for $\xi \in \mathcal{R}_s$, where we choose any branch of the square root.

We define $h^- (\xi) = \left(h^+(\xi)\right)^{\frac{1}{2}}$ for $\xi \in \{\xi : |\xi| > \frac{1}{2}\}$. $m(\xi)$ is defined by $m(\xi) := h^+(\xi) h^- (\xi)$. $m(\xi)$ is analytic in the annulus region $A := \{\xi : \frac{1}{2} < |\xi| < s\}$ and $m(\xi) = |\phi(\xi)|$ on $\partial \mathcal{R}_1$. 

In the same fashion, for \( f \in B_s \), we define \( f^+(\xi) = f(\xi) \) for \( \xi \in \mathcal{R}_s \), and \( f^-(\xi) = \frac{f^+(\xi)}{1 + \xi} \) for \( \xi \in \{ \xi : |\xi| > \frac{1}{s} \} \). \( n(\xi) \) is defined by \( n(\xi) := f^+(\xi)f^-(\xi) \). \( n(\xi) \) is analytic in the annulus region \( A = \{ \xi : \frac{1}{s} < |\xi| < 1 \} \) and \( n(\xi) = |f(\xi)|^2 \) on \( \partial \mathcal{R}_1 \).

Let \( w \) be the solution of the Riemann-Hilbert problem as in Lemma 3.3 defined on \( \overline{\mathcal{R}}_1 \). We define \( w^+(\xi) := w(\xi) \) for \( \xi \in \mathcal{R}_1 \), and \( w^-(\xi) := \frac{w^+(\xi)}{1 + \xi} \) for \( \xi \in \{ \xi : |\xi| > 1 \} \). \( w^- \) is analytic in \( \xi \in \{ \xi : |\xi| > 1 \} \) and \( w^-(\xi) = w(\xi) \) on \( \partial \mathcal{R}_1 \).

Using the above notations and \( i \partial_v = -\xi \partial_\xi \), we can write the boundary condition (20) on \( \partial \mathcal{R}_1 \) as the following:

\[
\begin{align*}
&cb^2 \partial_\xi \left( m(\xi)w^+(\xi) \right) + w^+(\xi) \\
&+ c \frac{b(t)}{2b(t)} \partial_\xi \left( m(\xi)^{-1}\xi^{-1}f^+\phi^+ \right) + \frac{b(t)}{4b^3(t)} \xi \partial_\xi n(\xi) \\
&- c \frac{b(t)}{2b(t)} \partial_\xi \left( m(\xi)w^-(\xi) \right) + w^- (\xi) \\
&- c \frac{b(t)}{4b^3(t)} \xi \partial_\xi n(\xi).
\end{align*}
\]

(37)

We note that all terms on the left side of equation is analytic in the annulus region \( A_i := \{ \xi : \frac{1}{s} < |\xi| < 1 \} \), while all terms on the right side of equation is analytic in the annulus region \( A := \{ \xi : 1 < |\xi| < s \} \). If we define \( F(\xi) \) as follows:

\[
F(\xi) \begin{cases}
= cb^2 \partial_\xi \left( m(\xi)w^+(\xi) \right) + w^+(\xi) + c \frac{b(t)}{2b(t)} \partial_\xi \left( m(\xi)^{-1}\xi^{-1}f^+\phi^+ \right) \\
+ \frac{b(t)}{4b^3(t)} \xi \partial_\xi n(\xi) \quad \text{for } \xi \in A_i \\
= -cb^2 \partial_\xi \left( m(\xi)w^-(\xi) \right) + w^- (\xi) - c \frac{b(t)}{2b(t)} \partial_\xi \left( m(\xi)^{-1}\xi^{-1}f^-\phi^- \right) \\
- \frac{b(t)}{4b^3(t)} \xi \partial_\xi n(\xi) \quad \text{for } \xi \in A_c;
\end{cases}
\]

(38)

then \( F(\xi) \) is analytic in the annulus region \( A = \{ \xi : \frac{1}{s} < |\xi| < s \} \). Consequently, we can use the differential equation

\[
\begin{align*}
&cb^2 \partial_\xi \left( m(\xi)w^+(\xi) \right) + w^+(\xi) \\
&= -c \frac{b(t)}{2b(t)} \partial_\xi \left( m(\xi)^{-1}\xi^{-1}f^+\phi^- \right) - \frac{b(t)}{4b^3(t)} \xi \partial_\xi n(\xi) + F(\xi)
\end{align*}
\]

(39)

to analytically continue \( w^+ \) from \( A_i \) to \( A_c \). So we have obtained the following lemma:

**Lemma 3.6.** Let \( f, \phi \in B_s \), and \( w \) is the solution of the Riemann-Hilbert problem (20) as in the above lemma 3.3, then \( w \) can be analytically extended to the domain \( \mathcal{R}_s \).

We suppose the initial data \( \phi_0(\xi) \neq 0 \) in \( \overline{\mathcal{R}}_r \) with \( r > 1 \). Let

\[
\rho_0 = \inf_{\xi \in \overline{\mathcal{R}}_r} |\phi_0(\xi)| = \inf_{\xi \in \overline{\mathcal{R}}_r} \left| \frac{1}{(f_0)(\xi)} \right| > 0.
\]

(40)

**Lemma 3.7.** If \( \phi \in B_s, 1 < s \leq r \), and \( |\phi - \phi_0|_s < \rho \) with \( \rho < \rho_0 \), then \( |\phi^-|_s \leq C|\phi|_s, |\phi^-|_{2, \alpha} \leq C|\phi|_{2, \alpha} \).
Proof. For \( \xi \in \mathcal{R}_s \), we have
\[
|\phi(\xi)| \geq |\phi_0(\xi)| - |\phi_0(\xi) - \phi(\xi)| > \rho_0 - \rho > 0,
\]
which implies
\[
|\phi^{-1}(\xi)| \leq \frac{1}{\rho_0 - \rho}. \tag{41}
\]
Also
\[
\left| \frac{\partial(\phi^{-1})}{\partial \xi}(\xi) \right| = \left| \frac{\partial \xi \phi(\xi)}{\phi^2(\xi)} \right| \leq \frac{|\phi_s|}{(\rho_0 - \rho)^2}. \tag{42}
\]
So we have \( |\phi^{-1}|_s \leq C|\phi|_s \), the other part can be proved similarly. \( \square \)

Lemma 3.8. If \( f, \phi \in \mathcal{B}_s, 1 < s \leq r \), and \( |\phi - \phi_0|_s < \rho \) with \( \rho < \rho_0 \), \( |f|_s \leq M \), \( M \) is a positive constant. Let \( w \) be as in Lemma 3.6, then \( |w|_s \leq C \), where \( C = C(\rho, \rho_0, M, r) \) is a positive constant depending only on \( \rho, \rho_0, M, r \).

Proof. From Lemma 3.2 and Lemma 3.3, we have \( |w(\xi)| \leq C, |\partial \xi w(\xi)| \leq C \) in \( \xi \in \mathcal{R}_1 \), so it suffices to prove that \( |w(\xi)| \leq C, |\partial \xi w(\xi)| \leq C \) for \( \xi \in \mathcal{A}_c \).

From (39), \( w \) satisfies the following equation in \( \mathcal{A}_c \):
\[
\partial \xi w(\xi) + p(\xi) w(\xi) = G(\xi); \tag{43}
\]
where
\[
p(\xi) = \frac{c}{c^2 + m(\xi)}; \tag{44}
\]
\[
G(\xi) = \frac{1}{c b^2 m(\xi)} \left( F(\xi) - c \frac{\dot{b}(t)}{2 b(t)} \partial \xi \left( m(\xi)^{-1} \xi^{-1} f(\phi) - \frac{\dot{b}(t)}{4 b^3(t)} \xi \partial \xi n(\xi) \right) \right). \tag{45}
\]
Let \( \xi_k = e^{i \frac{2 \pi k}{n}}, k = 0, 1, \ldots, n-1, \) \( A_k = \{ \xi \in A_c, 2\pi k \leq \arg \xi \leq \frac{2\pi (k+1)}{n} \} \), \( n \) is a sufficiently large positive integer. Solving (43) for \( \xi \in A_k \), we obtain
\[
w(\xi) = w(\xi_k) q(\xi) + q(\xi) \int_{\xi_k}^{\xi} \frac{G(\eta)}{q(\eta)} d\eta; \tag{46}
\]
where
\[
q(\xi) = e^{- \int_{\xi_k}^{\xi} p(\eta) d\eta}. \tag{47}
\]
From Definition 3.5 and Lemma 3.7, and (38), we have
\[
|m(\xi)|^{-1} \leq \frac{1}{\rho_0 - \rho}, \quad |\partial \xi m(\xi)| \leq C(\rho, \rho_0), \quad |\partial \xi n(\xi)| \leq C(M), \quad |F(\xi)| \leq C(\rho, \rho_0, r, M), \quad |w(\xi_k)| \leq C(\rho, \rho_0, r, M). \tag{48}
\]
It follows from (44)-48) that \( |w(\xi)| \leq C(\rho, \rho_0, r, M) \), which implies also \( |w_{\xi}(\xi)| \leq C(\rho, \rho_0, r, M) \) in view of (43). \( \square \)

Lemma 3.9. If \( f_j, \phi_j \in \mathcal{B}_s, 1 < s \leq r \), and \( |\phi_j - \phi_0|_s < \rho \) with \( \rho < \rho_0 \), \( |f_j|_s \leq M \), \( M \) is a positive constant. Let \( w_j \) be as in Lemma 3.6 corresponding to \( f_j \) and \( \phi_j \), then \( |w_1 - w_2|_s \leq C(|\phi_1 - \phi_2|_s + |f_1 - f_2|_s) \), where \( C = C(\rho, \rho_0, M, r) \) is a positive constant depending only on \( \rho, \rho_0, M, r \).

Proof. Let \( w = w_1 - w_2 \). Since \( w_j \) satisfies \( \partial \xi w_j(\xi) + p_j(\xi) w_j(\xi) = G_j(\xi), j = 1,2 \), \( w \) satisfies the following equation:
\[
\partial \xi w(\xi) + p_1(\xi) w(\xi) = (G_1(\xi) - G_2(\xi)) + (p_2(\xi) - p_1(\xi)). \tag{49}
\]
Using \((44)\) we can write
\[
P_2(\xi) - P_1(\xi) = \frac{-cb^2(m_2(\xi) - m_1(\xi))}{cb^2m_1(\xi)m_2(\xi)} + \frac{cb^2[(m_2(\xi) - m_1(\xi))\partial_\xi m_1(\xi) + m_1(\xi)(\partial_\xi m_1(\xi) - \partial_\xi m_2(\xi))]}{cb^2m_1(\xi)m_2(\xi)}.
\]

Using \((38)\), we can write
\[
F_1(\xi) - F_2(\xi) = -cb^2\partial_\xi ((m_1(\xi) - m_2(\xi))w^-(\xi) + m_2(\xi)(w_1^-(\xi) - w_2^- (\xi)))
+ (w_1^-(\xi) - w_2^- (\xi)) - \frac{b(t)}{4b^3(t)}\xi(\partial_\xi m_n(\xi) - \partial_\xi m_{n-1}(\xi))
- \frac{b(t)}{2b^3(t)}\partial_\xi [((m_1(\xi)^{-1} - m_2(\xi)^{-1})\xi^{-1}f^-_1 \phi^-_1(\xi)]
+ m_2(\xi)^{-1}\xi^{-1}(f^-_1(\xi) - f^-_2(\xi))\phi^-_1(\xi)
+ m_2(\xi)^{-1}\xi^{-1}f^-_2(\xi)(\phi^-_1(\xi) - \phi^-_2(\xi)).
\]

Using \((45)\), we have
\[
G_1(\xi) - G_2(\xi)
= \left(\frac{1}{cb^2m_1(\xi)} - \frac{1}{cb^2m_2(\xi)}\right)\left(F_1(\xi) - \frac{b(t)}{2b^3(t)}\partial_\xi (m_1(\xi)^{-1}f_1(\xi)\phi(\xi))\right)
- \left(\frac{1}{cb^2m_1(\xi)} - \frac{1}{cb^2m_2(\xi)}\right)\left(\frac{b(t)}{4b^3(t)}\xi\partial_\xi m_n(\xi)\right)
+ \frac{1}{cb^2m_2(\xi)}\left((F_1(\xi) - F_2(\xi)) - \frac{b(t)}{4b^3(t)}\xi\partial_\xi (m_1(\xi) - m_2(\xi))\right)
- \frac{b(t)}{2b^3(t)}\partial_\xi [(m_1(\xi)^{-1} - m_2(\xi)^{-1})\xi^{-1}f_1(\xi)\phi_1(\xi)]
+ m_2(\xi)^{-1}\xi^{-1}(f_1(\xi) - f_2(\xi))\phi_1(\xi) + m_2(\xi)^{-1}\xi^{-1}f_2(\xi)(\phi_1(\xi) - \phi_2(\xi)).
\]

Applying the same steps as in the proof of Lemma 3.8 to \((49)\), and using \((50)-(52)\), we obtain the lemma. 

We now turn our attention to \((24)\) and \((25)\). The operator \(T(\phi, f)(\xi)\) is defined for \(\xi \in \mathcal{R}_1\) by \((23)\). In order to analytically extend the equations \((24)\) and \((25)\) from \(\xi \in \mathcal{R}_1\) to \(A_{c} \), we need analytically extend the operator \(T(\phi, f)(\xi)\) from \(\mathcal{R}_1\) to \(A_{c}\).

**Lemma 3.10.** If \(f, \phi \in B_u, 1 < r_1 < s \leq r, \) and \(|\phi - \phi_0|_s < \rho\) with \(\rho < \rho_0, |f|_s \leq M, M\) is a positive constant. Let \(w\) be as in Lemma 3.8. Let
\[
T^-(\phi, f)(\xi) := \frac{1}{4\pi i} \int_{|\eta| = 1} \phi(\eta)\phi^-(\eta)(w(\eta) + w^-(\eta))\frac{(\eta + \xi)d\eta}{\eta(\eta - \xi)}
+ \phi(\xi)\phi^-(\xi)(w(\xi) + w^-(\xi)) \text{ for } \xi \in A_{c}
\]

Then \(T^-(\phi, f)\) is the analytic continuation of \(T(\phi, f)\) into \(A_{c} = \mathcal{R}_s \cap \{ |\xi| > 1 \}\) and
\[
|T(\phi, f)|_s \leq C|\phi|^2|_s w|_s.
\]

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**Xuming Xie**

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Proof. Let \( \eta_0 \in \partial R_1 \), from (23) and Plemelj formula [3] we have

\[
\lim_{\xi \to \eta \to \eta_0} T'(\phi, f)(\xi) = \frac{1}{2\pi i} (PV) \int_{|\eta|=1} |\phi|^2 \Re[w] \frac{(\eta + \eta_0) d\eta}{\eta(\eta - \eta_0)} + \Re[w](\eta_0)|\phi(\eta_0)|^2.
\]  

(55)

From (53) and Plemelj formula we have

\[
\lim_{\xi \to \eta \to \eta_0} T^-(\phi, f)(\xi) = \frac{1}{2\pi i} (PV) \int_{|\eta|=1} \phi(\eta) \phi^-(\eta)(\eta + \eta_0) d\eta \frac{(\eta + \eta_0) d\eta}{\eta(\eta - \eta_0)} - \frac{1}{2} \phi(\eta_0) \phi^-(\eta_0)(\eta + \eta_0) \eta(\eta - \eta_0) \eta.
\]

(56)

From Definition 3.5, it follows that \( \phi(\eta_0) \phi^-(\eta_0) = |\phi_0(\eta_0)|^2 \), \( (\eta + \eta_0)(\eta + \eta_0) = 2 \Re[w](\eta_0) \). Hence \( T^-(\phi, f)(\eta_0) = T(\phi, f)(\eta_0) \), which implies that \( T^-(\phi, f) \) is the analytic continuation of \( T(\phi, f) \) into \( A_c \). To prove (54), we note that \( \phi^- \) and \( w^- \) is analytic in \( A = \{ \xi : \frac{1}{r} < |\xi| < s \} \) and \( \frac{1}{r} < \frac{1}{r_1} < 1 \), hence from Cauchy’s theorem it follows

\[
\int_{|\eta|=1} \phi(\eta) \phi^-(\eta)(\eta + \eta_0)(\eta + \eta_0) d\eta \frac{(\eta + \eta_0) d\eta}{\eta(\eta - \eta_0)} = \int_{|\eta| = \frac{1}{r}} \phi(\eta) \phi^-(\eta)(\eta + \eta_0)(\eta + \eta_0) d\eta \frac{(\eta + \eta_0) d\eta}{\eta(\eta - \eta_0)}.
\]

(57)

The integral on the right side can be bounded by \( C|\phi|^{2}|w|_{s} \) since \( |\xi - \eta| > 1 - \frac{1}{r_1} \) for \( |\xi| > 1, |\eta| = \frac{1}{r} \). The same bound can be obtained similarly for the derivative of the integral, so the lemma follows from (53).

\[\square\]

Lemma 3.11. Assume \( f_j, \phi_j \in B_\infty, 1 < r_1 < s < r \), and \( |\phi_j - \phi_0|_s < \rho \) with \( \rho < \rho_0 \), \( |f_j|_s \leq M \), \( M \) is a positive constant. Let \( \omega_j \) be as in Lemma 3.6, then \( |T(\phi_1, f_1) - T(\phi_2, f_2)|_s \leq C(|\phi_1 - \phi_2|_s + |f_1 - f_2|_s) \), where \( C = C(\rho, \rho_0, M, r_1, r) \) is a positive constant depending only on \( \rho, \rho_0, M, r_1, r \).

Proof. We can write

\[
\phi_1(\xi) \phi_1^-(\xi)(\omega_1(\xi) + \omega_1^-)(\xi) - \phi_2(\xi) \phi_2^-(\xi)(\omega_2(\xi) + \omega_2^-)(\xi) = (\phi_1(\xi) - \phi_2(\xi)) \phi_1^+(\xi)(\omega_1(\xi) + \omega_1^-)(\xi) + \phi_2(\xi)(\phi_1^+(\xi) - \phi_2^-(\xi))(\omega_1(\xi) + \omega_1^-)(\xi) + \phi_2(\xi) \phi_2^+(\xi)(\omega_1(\xi) - \omega_2^-)(\xi) + (\omega_1^-)(\xi) - \omega_2^-)(\xi),
\]

the lemma can be obtained in the same manner as (54).

\[\square\]

3.2. An abstract Cauchy-Kowalewskaya problem and main result. The previous sections have enable us to prove the main result of this paper, i.e. the existence and uniqueness of the solution to the problem one consisting of (25) - (27). The problem can be rewritten as the following abstract Cauchy-Kowalewskaya Problem:

\[
\frac{dU}{dt} = L(t, U), U|_{t=0} = 0;
\]

(58)

where \( U = [f(t, \xi) - f_0(\xi), \phi(t, \xi) - \phi_0] \) and \( L(t, U) = [L_1(t, U), L_2(t, U)] \);

\[
L_1(t, U) = -b^2(t) \xi f, L_2(t, U) = \frac{\dot{b}(t)}{2b(t)} f.
\]

(59)
$\mathcal{L}_2(t, \mathcal{U}) = -\nu^2(t)\xi \left( \frac{\partial \phi}{\partial \xi} \right) T(\phi, f) + \nu^2(t)\phi \frac{\partial}{\partial \xi} (\xi T(\phi, f)) + \frac{\dot{b}(t)}{2b(t)} \phi$.  \hfill (60)\\

In order to verify the conditions of Nishida-Nirenberg Theorem \cite{24, 25}, we first establish the following key lemma.

**Lemma 3.12.** If $f \in \mathcal{B}_s$, $r_1 < s' < s \leq r$; then $f_\xi \in \mathcal{B}_{s'}$ and

$$\|f_\xi\|_{s'} \leq \frac{K_1}{s-s'} \|f\|_s,$$  \hfill (61)

where $K_1 > 0$ is independent of $s, s'$ and $f$.

**Proof.** Since $\text{dist}(\partial \mathcal{R}_{s'}, \partial \mathcal{R}_s) = s - s'$, for $\xi \in \mathcal{B}_{s'}$, we are able to find a disk $D(\xi)$ centered at $\xi$ with radius $s - s'$ such that $D(\xi)$ is contained in $\mathcal{R}_s$. Using Cauchy integral formula, we have

$$f_\xi(\xi) = \frac{1}{2\pi i} \int_{|t-\xi|=s-s'} \frac{f(t)}{(t-\xi)^2} dt, \quad f_{\xi\xi}(\xi) = \frac{1}{2\pi i} \int_{|t-\xi|=s-s'} \frac{f_\xi(t)}{(t-\xi)^2} dt$$

so

$$|f_\xi(\xi)| \leq \frac{1}{2\pi} \int_{|t-\xi|=s-s'} \frac{|f(t)|}{|t-\xi|^2} |dt|$$

$$\leq \frac{1}{2\pi} \int_0^{2\pi} \frac{|f(\xi + |s-s'|e^{i\theta})|}{s-s'} |d\theta| \leq K_1 \sup_{\xi \in \mathcal{R}_s} |f(\xi)| \leq \frac{K_1}{s-s'} \|f\|_s,$$

$$|f_{\xi\xi}(\xi)| \leq \frac{1}{2\pi} \int_{|t-\xi|=s-s'} \frac{|f_\xi(t)|}{|t-\xi|^2} |dt|$$

$$\leq \frac{1}{2\pi} \int_0^{2\pi} \frac{|f_\xi(\xi + |s-s'|e^{i\theta})|}{s-s'} |d\theta| \leq K_1 \sup_{\xi \in \mathcal{R}_s} |f_\xi(\xi)| \leq \frac{K_1}{s-s'} \|f\|_s,$$

which gives the lemma. \hfill $\Box$

We assume that the initial data $f_0(\xi)$ is analytic in $\mathcal{R}_r$, $r > 1$ and $\frac{\partial f_0}{\partial \xi}(\xi) \neq 0$ for $\xi \in \mathcal{R}_r$. Let $M = |f_0|_r$ and $\rho_0$ be given in (40), we define the following spaces:

$$\mathcal{B}_s(M) = \{ f(\xi) : f \in \mathcal{B}_s, \quad \| f \|_s \leq M \},$$

$$\mathcal{B}_s(\rho_0) = \{ \phi(\xi) : f \in \mathcal{B}_s, \quad \| \phi \|_s \leq \frac{\rho_0}{2} \}.$$ 

For $u = [u_1, u_2] \in \mathcal{B}_s(M) \times \mathcal{B}_s(\rho_0)$, we define $\|u\|_s = |u_1|_s + |u_2|_s$. The operator $\mathcal{L}$ has the following properties:

**Property 1:** For some constant $\delta > 0$ and every pair of numbers $s, s'$ such that $r_1 < s' < s < r$, $(u, t) \rightarrow \mathcal{L}(t, u)$ is a continuous mapping of

$$\{ t : |t| < \delta \} \times \{ u \in \mathcal{B}_s(M) \times \mathcal{B}_s(\rho_0) \} \rightarrow \mathcal{B}_{s'} \times \mathcal{B}_{s'}.$$  \hfill (62)

**Proof.** The lemma follows from Lemma 3.10, Lemma 3.11, Lemma 3.12 and (59). \hfill $\Box$
Lemma 3.14. Let $f$ one solution. If
\[
\text{Property 2: } \text{for every } t, |t| < \delta, \mathcal{L} \text{ satisfies}
\]
\[
\|\mathcal{L}(t, u) - \mathcal{L}(t, v)\|_{s'} \leq C\|u - v\|_{s}
\]
where $C$ is some positive constant independent of $t, u, v, s, s'$.

Proof. Let $u = [u_1, u_2], v = [v_1, v_2]$, using (59) and (60) we can write
\[
\mathcal{L}_1(t, u) - \mathcal{L}_1(t, v) = -b^2(t)\xi\partial_\xi(u_1 - v_1)T(u_2 + \phi_0, u_1 + f_0) - \frac{\dot{b}(t)}{2b(t)}(u_1 - v_1)
\]
\[
- b^2(t)\xi\partial_\xi(v_1 + f_0)[T(u_2 + \phi_0, u_1 + f_0) - T(v_2 + \phi_0, v_1 + f_0)],
\]
(64)
\[
\mathcal{L}_2(t, u) - \mathcal{L}_2(t, v) = -b^2(t)\xi\partial_\xi(u_2 - v_2)T(u_2 + \phi_0, u_1 + f_0) + \frac{\dot{b}(t)}{2b(t)}(u_2 - v_2)
\]
\[
- b^2(t)\xi\partial_\xi(v_2 + \phi_0)[T(u_2 + \phi_0, u_1 + f_0) - T(v_2 + \phi_0, v_1 + f_0)]
\]
\[
+ b^2(t)(u_2 - v_2)\partial_\xi(\xi T(u_2 + \phi_0, u_1 + f_0))
\]
\[
+ b^2(t)(v_2 + \phi_0)\partial_\xi(\xi T(u_2 + \phi_0, u_1 + f_0) - T(v_2 + \phi_0, v_1 + f_0))).
\]
(65)
Applying Lemma 3.10, Lemma 3.11, Lemma 3.12 to (64) and (65), we obtain (63).

Property 3: $\mathcal{L}(t, 0)$ is a continuous function of $t, |t| < \delta$ with values in $\mathcal{B}_s \times \mathcal{B}_s$ for every $r_1 < s < r$ and satisfies, with some positive constant $K$,
\[
\|\mathcal{L}(t, 0)\|_s \leq \frac{K}{(r-s)}
\]
(66)

Proof. From (59) and (60), it follows
\[
\mathcal{L}_1(t, 0) = -b^2(t)\xi f_\xi T(\phi_0, f_0) - \frac{\dot{b}(t)}{2b(t)} f_0,
\]
(67)
\[
\mathcal{L}_2(t, 0) = -b^2(t)\xi\left(\frac{\partial \phi_0}{\partial \xi}\right) T(\phi_0, f_0) + b^2(t)\phi_0 \frac{\partial}{\partial \xi}(\xi T(\phi_0, f_0)) + \frac{\dot{b}(t)}{2b(t)} \phi_0.
\]
(68)
Applying Lemma 3.10, Lemma 3.12 to (67) and (68), we obtain (66).

Applying the Nishida-Nirenberg Theorem, we obtain the following main result:

Theorem 3.13. If $f_0 \in \mathcal{B}_r$ and $\partial_\xi f_0(\xi) \neq 0$ in $\mathcal{R}_r$, then there exists one and only one solution $f \in C^1([0, T], \mathcal{B}_s(M)), \phi \in C^1([0, T], \mathcal{B}_s(\rho_0)), 1 < r_1 < s < r$, to the problem one consisting of (24) (27), where $T = a_0(r-s), a_0$ is a suitable positive constant independent of $s$.

Let $[\phi(t, \xi), f(t, \xi)]$ be the solution obtained in Theorem 3.13, from Lemma 3.3, we can solve the Riemann-Hilbert problem (20) to obtained $w(t, \xi)$ with $w(t, 0) = 0$, hence we can obtain $W(t, \xi)$ from (19). For $[f(t, \xi), W(t, \xi)]$ to be the solution to the problem (17)-(18), it has to be true that $\phi(t, \xi) = \frac{1}{f_\xi(t, \xi)}$ which is proved in the following lemma.

Lemma 3.14. Let $[\phi(t, \xi), f(t, \xi)]$ be the solution obtained in Theorem 3.13, then $\phi(t, \xi) = \frac{1}{f_\xi(t, \xi)}$, for $\xi \in \mathcal{R}_s, t \in [0, T]$. 
Integrating (75), we obtain the lemma.

\[ \xi \]

Theorem 3.16. Given an initial simply connected domain \( \Omega_0 \) with analytic boundary \( \partial \Omega_0 \), the initial value problem for the lifting Hele-Shaw problem (9)-(8) has one and only one analytic solution \( \{ \Omega(t) \} \) for sufficiently small time \( t \).

Proof. Let \( g(t, \xi) = \phi(t, \xi) - \frac{1}{\int_{t(t, \xi)}^{1}} \). Taking derivative with respect to \( \xi \) in (25) and using (20), we obtain that \( g(\xi) \) satisfies

\[
\frac{\partial g}{\partial t}(t, \xi) + b^2(t) \xi T(\phi, f)(t, \xi) \frac{\partial g}{\partial \xi}(t, \xi) = D(t, \xi)g(t, \xi),
\]

\[ g|_{t=0} = 0; \]

where \( D(t, \xi) \) is defined by

\[
D(t, \xi) = b^2(t) \frac{\partial}{\partial \xi} (\xi T(\phi, f)(t, \xi)) + \frac{\dot{b}(t)}{2b(t)}.
\]

For fixed \( \eta \in \mathcal{R}_+ \) and \( T_1 \in (0, T] \), we have the characteristics \( \xi = \xi(t, \eta, T_1) \) of (69), where \( \xi(t, \eta, T_1) \) satisfies the following ODE:

\[
\frac{d\xi}{dt} = b^2(t) \xi T(\phi, f)(t, \xi), \text{ for } t \in [0, T_1)
\]

\[ \xi|_{t=T_1} = \eta. \]

The ODE (71) is equivalent to the following integral equation:

\[
\xi(t, \eta, T_1) = \eta \exp \left( - \int_{0}^{T_1} b^2(\tau) T(\phi, f)(\tau, \xi(\tau, \eta, T_1)) d\tau \right).
\]

It follows from (72) that \( |\xi(t, \eta, T_1)| \leq 2|\eta| < 1 \) for all \( t \in [0, T_1] \) and sufficiently small \( T_1 \) since \( |T(\phi, f)(\tau, \xi)| \) is bounded due to (54).

Now along the characteristics \( \xi = \xi(t, \eta, T_1) \), we can integrate (69) to obtain:

\[
g(t, \xi(t, \eta, T_1)) = \int_{0}^{T} D(\tau, \xi(\tau, \eta, T_1))g(\tau, \xi(\tau, \eta, T_1)) d\tau, \text{ for } t \in [0, T_1].
\]

It follows from (70) and Lemma 3.10 that \( |D(t, \xi)| \leq C \) for \( |\xi| < 1 \), \( t \in [0, T_1] \), hence, using (73), we have

\[
\sup_{t \in [0, T_1]} |g(t, \xi(t, \eta, T_1))| \leq CT_1 \sup_{t \in [0, T_1]} |g(t, \xi(t, \eta, T_1))|
\]

which implies that \( \sup_{t \in [0, T_1]} |g(t, \xi(t, \eta, T_1))| = 0 \) for sufficiently small \( T_1 \). So we have proven that \( g(T_1, \eta) = 0 \) for any \( |\eta| < \frac{1}{2}, T_1 \in (0, T] \). The analyticity of \( g(T_1, \xi) \) in \( \xi \) then implies \( g(T_1, \eta) = 0 \) for any \( \eta \in \mathcal{R}_+, T_1 \in (0, T] \). \( \square \)

Lemma 3.15. Let \( [\phi, f] \) be the solution obtained in Theorem 3.13, and \( f_0(0) = 0 \), then \( f(t, 0) = 0 \) for all \( t \in [0, T] \).

Proof. Setting \( \xi = 0 \) in equation (25), we obtain

\[
\frac{\partial f}{\partial t}(t, 0) = -\frac{\dot{b}(t)}{2b(t)} f(t, 0), \text{ } f(0, 0) = 0
\]

Integrating (75), we obtain the lemma. \( \square \)

Theorem 3.16. Given an initial simply connected domain \( \Omega_0 \) with analytic boundary \( \partial \Omega_0 \), the initial value problem for the lifting Hele-Shaw problem (9)-(8) has one and only one analytic solution \( \{ \Omega(t) \} \) for sufficiently small time \( t \).
\textbf{Proof.} Without loss of generality we assume that \( \Omega_t \) contains the origin \( z = 0 \). From the analyticity of \( \partial \Omega_0 \) and Riemann Mapping Theorem, there exists unique \( z = f_0(\xi) \) with \( f_0(0) = 0, f_0'(0) > 0 \) so that \( f_0(\xi) \) is conformal in \( \mathcal{R}_r \) for some \( r > 1 \) and \( f_0(\xi) \) maps the unit disk \( \mathcal{R}_1 \) onto \( \Omega_0 \). Combining Theorem 3.13, Lemma 3.14-Lemma 3.15, \( [\phi(t, \xi) = \frac{1}{i[t_1, \xi]}, f] \) is the unique solution obtained in Theorem 3.13, from Lemma 3.3, we can solve the Riemann-Hilbert problem (20) to obtained \( w(t, \xi) \) with \( w(t, 0) = 0 \), hence we can obtain \( W(t, \xi) \) from (21). \( [f(t, \xi), W(t, \xi)] \) is the solution to the problem (17)-(18), so it follows that \( \Omega(t) = \{ z : z = f(t, \xi) : |\xi| < 1, t \in (0, T]\} \) is the unique solution of the lifting Hele-Shaw problem (7)-(9). \( \square \)

4. Conclusion and discussion. In this paper, we are concerned with the existence of analytic solutions to an interfacial problem with kinetic undercooling regularization in a Hele-Shaw cell with time-dependent gap \( b(t) \). The problem has radially symmetric exact solutions which are linearly stable for \( \dot{b}(t) < 0 \) and unstable for \( \dot{b}(t) > 0 \). The linear stability analysis shows that global solutions seem impossible for general \( b(t) \), this paper shows that analytic solutions to the problem do exist locally in time if the initial data are analytic. The problem can be reformulated as a set of PDEs for functions related to the conformal map function from the unit disk to the interface profile and the complex velocity potential. Analyticity is shown by means of solving a Riemann-Hilbert problem for the complex velocity potential and solving a system of abstract evolution equations with a Cauchy-Kowalevskaya theorem.

When \( \dot{b}(t) = 0 \), i.e. the gap is a constant \( b \) which will be assumed to be one, and the flow is driven by a source at the origin with injection (or suction) rate \( Q(t) \), then the problem (9) - (8) becomes the following problem:

\[
\nabla^2 p = Q(t)\delta(x, y) \text{ in } \Omega(t),
\]

where \( \delta(x, y) \) is the Dirac delta function.

\[
-\frac{\partial p}{\partial n} = V_n \text{ on } \partial \Omega(t),
\]

\[
p = cV_n \text{ on } \partial \Omega(t).
\]

Let \( z = f(t, \xi) \) be the conformal mapping that maps \( \Omega(t) \) onto the unit disk \( |\xi| < 1 \), and \( W(t, \xi) = Q(t) \log |\xi| + W(t, \xi) \) be the complex velocity potential where the real part of \( W \) is \( p \) and \( W(t, \xi) \) is analytic in \( |\xi| < 1 \), then (77) becomes

\[
\text{Re} \left( \frac{\partial f}{\partial t} \left( \frac{\xi}{\partial f} \right) \right) = -\text{Re} \left[ \xi W_\xi \right] - Q(t)
\]

and (78) becomes

\[
\text{Re} \ W = c\frac{\text{Re} \left( \frac{\partial f}{\partial t} \left( \frac{\xi}{\partial f} \right) \right)}{|f_\xi|};
\]

where \( \xi = e^{i\theta} \) is on the unit circle \( |\xi| = 1 \).

Let

\[
\phi(t, \xi) = \left( \frac{\partial f}{\partial \xi} (t, \xi) \right)^{-1}, \ w(t, \xi) = \xi W_\xi (t, \xi).
\]

Taking derivative with respect to \( \theta \), and using (79) and (80) we obtain that on the unit circle \( |\xi| = 1 \), \( w(t, \xi) \) satisfies

\[
\text{Im} \ |w| = c\partial_\theta \left( |\phi| \text{Re} \ |w| \right) + cQ(t)\partial_\theta \left( |\phi|^{-1} \right).
\]
Using the Poisson formula in (79) we obtain
\[ f_t = -\xi f_\xi T_1(\phi) \text{ in } |\xi| < 1, \] (83)
where the operator \( T_1(\phi) \) is defined as
\[ T_1(\phi)(t, \xi) = \frac{1}{2\pi i} \int_{|\eta|=1} |\phi(t, \eta)|^2 \left( Re[w(t, \eta)] + Q(t) \right) \frac{(\eta + \xi) d\eta}{\eta(\eta - \xi)}. \] (84)
Taking derivative with respect to \( \xi \) in (83) and using (81), we have
\[ \frac{\partial \phi}{\partial t} = -\xi \left( \frac{\partial \phi}{\partial \xi} \right) T_1(\phi) + \phi \frac{\partial}{\partial \xi} (\xi T_1(\phi)) \text{ in } |\xi| < 1. \] (85)
Hence solving the problem (76)-(78) is equivalent to solving (82) and (85) for \( \phi \).

The problem (76)-(78) was studied in [28] and [11]. Using a similar approach, the authors of [28] obtained an analytic solution \( \phi \) to (82) and (85). Let us compare the problem (82) - (85) solved in [28] with the problem (20), (24) and (25) in this paper. The terms involving \( \dot{b}(t) \) on the right side of (20) depend on both \( f \) and \( \phi \) while (82) depends on \( \phi \) but is independent of \( f \); consequently this causes some complications and difficulties. First it is more difficult to solve the Riemann-Hilbert problem (20) in this work than solving (82) in [28]; especially the analytic continuation of the Riemann-Hilbert problem beyond the unit disk needs more delicate analysis. We overcame these difficulties by working with different function spaces in section 3.1. Secondly we need to construct a system of abstract equations for \( f \) and \( \phi \) in this work while only one abstract equation was needed for \( \phi \) in [28]. The proof of the equivalency of the system of abstract evolution equations to the original lifting problem is not trivial, and Lemma 3.14 plays an important role in the proof.

There are a few unanswered questions about the problem discussed in this work. For the problem (76)-(78), Dallaston and McCue [11] numerically demonstrated corner formation for sufficiently high kinetic undercooling in finite time. Although preliminary computation showed similar corner formation in the lifting problem, more numerical and analytical work on singularity formation needs to be done. As demonstrated in section 2.1, when \( \dot{b}(t) < 0 \), the exact circular solutions to the problem are linearly stable. That means that it is possible to obtain global solutions to the problem when the initial shape is close to a circle, we are going to address this in a forthcoming paper.

Acknowledgments. The author would like to thank the Isaac Newton Institute for Mathematical Sciences for support and hospitality during the workshop “Complex analysis in mathematical physics and applications” held from October 28 to November 1, 2019 when work on this paper was presented. The author would like to thank Dr. Scott McCue and Dr. Koya SaKakibara for helpful discussions.

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Received May 2020; 1st revision August 2020; 2nd revision August 2020.

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