Research on Stress-Strain State in the Angular Domain Cut-Out Zone Using Photoelastic Method

Ludmila Frishter¹

¹National Research Moscow State University of Civil Engineering, Yaroslavskoye shosse, 26, Moscow, 129337, Russian Federation
lfishter@mail.ru

Abstract. Current challenges in civil engineering practice call for creating and substantiating structural theory research approaches that correlate with the objects, processes and phenomena under study (physical models) to the fullest possible extent, and that provide reliable results for tackling engineering challenges. The topicality of this paper is associated with the choice of research method, i.e. the experimental photoelastic method. This method allows to obtain the local stress-strain state with the highest reliability when solving problems with specified rupturing forced deformations in the domain of significant structural non-homogeneity, i.e. geometrical stress concentrators. Boundary shape and finite discontinuity of specified forced deformations translate into emerging singularity of stress-strain state. The research aim consists in theoretical and experimental analysis of the stress-strain state in neighborhood of irregular point on the boundary of plane domain which the finite discontinuity (jump) of forced deformations emerges into. The singularities of strain-stress state of structures and constructions that are characterized by “structural non-homogeneity” and rupturing forced deformations are determined using polymer models of the photoelastic and deformation freezing method as stress concentrators whose domains are the focus of this article. Research results demonstrate that a zone of stress-strain state similarity exists in the domain of elasticity problem singular solution where isochromatics on the photelastic method polymer model are unreadable or “poorly” readable.

1. Introduction
The stress-strain state of structures and constructions in areas with significant structural non-homogeneity (re-entrant corners, special lines, and points) is characterized by appearance of stress concentration. Acting forced deformations that do not comply with conditions of compatibility have their finite discontinuity on the line (surface) of domain contact emerging into an irregular boundary point. These deformations result in appearance of stresses. Boundary shape and finite discontinuity of specified forced deformations translate into emerging singularity of stress-strain state of structures.

The research objective in the paper consists in theoretical and experimental analysis of the stress-strain state in neighborhood of irregular point on the boundary of plane domain which the finite discontinuity (jump) of forced deformations emerges into.

The singularities of strain-stress state of structures and constructions that are characterized by “structural non-homogeneity” and rupturing forced deformations are determined using polymer models of the photoelastic and deformation freezing method as stress concentrators whose domains
are the focus of this article. The novelty of research approach is ensured by the choice of research method, i.e. the experimental photoelastic method [1–5]. This method allows to obtain the local stress-strain state with the highest reliability when solving problems with specified rupturing forced deformations in the domain of significant structural non-homogeneity, i.e. geometrical stress concentrators.

The model of the photoelasticity method is made of mesh polymeric material by curing epoxy resin with anhydrite. The method of defrosting forced deformations [1, 3–5], using the procedure of preliminary freezing of model elements with subsequent defrosting of the entire model, is an effective method for simulation of stresses as a result of specified forced deformations. In the study of composite structures by the method of photoelasticity and defrosting "defrosting" of forced deformations, a model composed of elements with previously created forced deformations is created. The basics of the photoelasticity method are given in [1–5].

Theoretical and experimental research of stress concentration associated with the boundary shape is described in the papers by Neyber, R. Peterson, N. G. Savin and V. I. Tulchiya, B. N. Ushakov, I. P. Fomin, N. A. Makhutov, V. V. Vasilyev and many others. A unified theoretical, computational or experimental approach to studying local stress concentrations does not exist.

The photoelastic method helps obtain a continual solution in the domain with angular boundary cut-out [1–5]. Experimental solution on a polymer model in neighborhood of geometrical stress concentrator (apex of angular boundary cut-out) is “unreadable” or poorly “readable” under any magnification of the neighborhood fragment. At a certain distance from the stress concentration source, confident experimental data are available that are changing continuously and monotonously while approaching the irregular boundary point. To extrapolate confident experimental data into the domain where band pattern is unreadable, a comprehensive approach to obtaining and analyzing stressed state in neighborhood of an irregular point on the plane domain boundary is proposed.

2. Materials and Methods

2.1. Problem statement and solution

A plane elasticity problem in neighborhood of irregular boundary point [6–8], which the line of domain contact with the jump (discontinuity) of forced deformations emerges into, is considered.

A homogeneous or piecewise homogeneous body in plane stress state has an angular point in the boundary domain. At the contact boundary $\Gamma = \Gamma_1 \cup \Gamma_2$ of domains $\Omega_1$ and $\Omega_2$, that make up the elastic body, forced deformations, volume forces have a jump (finite discontinuity) of the form:

$$
\Delta \varepsilon_{ij} = \varepsilon_{ij}^\Gamma - \varepsilon_{ij}^\Omega_1; \quad \Delta F_i = F_i^\Gamma - F_i^\Omega_1, \quad i, j = x, y.
$$

Elasticity moduli, Poisson's ratios, linear expansion coefficients of domains $\Omega_1$ and $\Omega_2$ are constant and different: $E_i, \nu_i, \alpha_i, i, j \in \Omega_1$ and $E_2, \nu_2, \alpha_2, i, j \in \Omega_2$, correspondingly. Boundary conditions in neighborhood of irregular point $O(0,0)$ at the boundary $L_0$ are homogeneous.

Let us consider small neighborhood $O_{\delta}(0)$ of irregular point $O(0,0)$ of elastic body fragment $B: x^2 + y^2 < \varepsilon_0^2; \quad z < \varepsilon_0, \quad \varepsilon_i, \varepsilon_0$ are positive small numbers.

We apply similarity group [8]:

$$
x_1 = tx; \quad y_1 = ty; \quad z_1 = z; \quad \sigma_{ij} = t \sigma_{ij}; \quad \varepsilon_{ij} = t \varepsilon_{ij}; \quad U_i = U_i.
$$

Resolving system of equations for the plane elasticity problem with new variables of $O_{\delta}(0)$ taking into account (2) will be rewritten as follows:
\[
\sum_{j1} \frac{\partial \bar{\sigma}_{ij}}{\partial y_j} + \frac{1}{t^2} F_j = 0, \quad \bar{\varepsilon}_{ij} = \frac{\partial \bar{U}_i}{\partial y_j} + \frac{\partial \bar{U}_j}{\partial y_i}, \quad \mathcal{V}_1 \cap \mathcal{V}_2, i, j \in \Omega_1, \Omega_2.
\]

\[
\sum_{j1} \bar{\sigma}_{ij} n_j |_{\Gamma_0} = 0; \quad \sum_{j1} \bar{\sigma}_{ij} n_j |_{\Gamma_B} = \frac{1}{t} \sigma_b^b |_{\Gamma_B},
\]

\[
\bar{\sigma}_{in} |_{\Gamma_1} = \bar{\sigma}_{mn} |_{\Gamma_2}; \quad \bar{U}_i |_{\Gamma_1} = \bar{U}_i |_{\Gamma_2},
\]

\[
\bar{\varepsilon}_{ij} = \frac{1}{E_k} \left[(1+\nu_k) \bar{\sigma}_{ij} - \nu_k \bar{S} \delta_{ij} \right] + \frac{1}{t} \varepsilon_{ij}^b + \frac{1}{t} \alpha_k T \delta_{ij},
\]

Where \( k = 1, (x_1, y_1) \in \Omega_1; \quad k = 2, (x_1, y_1) \in \Omega_2, \quad i, j = x_1, y_1, \quad n_j = n_j \) is the normal to domain contact line \( \Gamma = \Gamma_1 \cup \Gamma_2, \) GB is the boundary of domain that includes the neighborhood of boundary irregular point.

Depending on withdrawal from or approaching to the irregular boundary point, which is defined by modifying geometrical parameter \( t, \) the form of resolving equation system (2) in \( \Omega_b(0) \) will change as follows:

a) In case of infinitely increasing geometrical parameter \( t = \frac{\varepsilon_2}{\varepsilon_1} \to \infty, \quad \varepsilon_1 \ll \varepsilon_2, \quad \varepsilon_1 \to 0, \quad \varepsilon_2 \to 0, \)

the plane problem of piecewise homogeneous body elasticity (\( E_k, \nu_k, \alpha_k, k = 1, 2 \)) with specified loads (forced deformations, temperature deformations, and volume forces), in neighborhood of irregular boundary point \( \Omega(0,0) \) is reduced to a homogeneous boundary value problem \( \bar{\xi}^O = (\bar{\sigma}_{ij}^O, \bar{\varepsilon}_{ij}^O, \bar{U}_i^O) \) for a piecewise homogeneous body with homogeneous boundary conditions (canonical singular value form).

Solution of obtained homogeneous boundary value problem with homogeneous boundary conditions characterizes the singularity of stress-strain state in irregular point \( \Omega(0,0) \) and its neighborhood. It depends on the specified boundary shape, type of homogeneous boundary conditions and values of material mechanical characteristics (\( E_k, \nu_k, k = 1, 2 \)). We shall define the nontrivial solution of obtained homogeneous problem as eigensolution. The order of stress state power singularity is determined by solving the characteristic equation of homogeneous boundary value problem [6–11].

b) When \( t \to 1, \) equation system (3) is identical to the initial system with acting specified loads. When \( t \to 1, \) the stress-strain state is defined by specified loads and boundary conditions: \( \xi^l = (\sigma_{ij}^l, \varepsilon_{ij}^l, U_i^l). \)

c) In the intermediate variation range of \( t \in (1 + \alpha, N) \) parameter values, where \( N > 0 \) is fairly large, \( \alpha > 0 \) is fairly small, both the intrinsic stress-strain state \( \bar{\xi}^O \) and the stress-strain state produced by specified loads are effective. \( \xi^l. \)

According to the considered cases of parameter \( t \) variation, the solution of the piecewise homogeneous plane problem in neighborhood of irregular boundary point may be expressed as:

\[
\bar{\sigma}_{ij} = \bar{\sigma}_{ij}^O + \bar{\sigma}_{ij}^l; \quad \bar{\varepsilon}_{ij} = \bar{\varepsilon}_{ij}^O + \bar{\varepsilon}_{ij}^l; \quad U_i = \bar{U}_i = \bar{U}_i^O + \bar{U}_i^l
\]

or

\[
\xi = \bar{\xi}^O + \xi^l,
\]

where \( \bar{\xi}^O = (\bar{\sigma}_{ij}^O, \bar{\varepsilon}_{ij}^O, \bar{U}_i^O) \) is the singular (eigen) solution of homogeneous boundary value problem that characterizes the singularity of stress-strain state in neighborhood of the irregular boundary point; \( \xi^l = (\sigma_{ij}^l, \varepsilon_{ij}^l, U_i^l) \) is the solution of system (3), defined by the influence of action of
specified loads having the following form:

\[ F_i^\Delta = \frac{1}{l^2} F_i; \quad \sigma_m^\Delta \big|_{\Gamma B} = \frac{1}{l} \sigma_m^d \big|_{\Gamma B}; \quad \varepsilon_{ij}^\Delta = \frac{1}{t} \varepsilon_{ij}^d + \frac{1}{t} \alpha_i T \delta_{ij}, \]

as well as by the influence of action of forced deformations and volume forces jump along the contact line of \( \Gamma = \Gamma_1 \cup \Gamma_2 \) of domains \( \Omega_1 \) and \( \Omega_2 \):

\[
\begin{align*}
\Delta F_i^\Delta &= \frac{1}{t^2} \left( F_i \big|_{\Gamma_2} - F_i \big|_{\Gamma_1} \right) = \frac{1}{t^2} \Delta F_i, \\
\Delta \varepsilon_{ij}^\Delta &= \frac{1}{t} \left( \varepsilon_{ij}^d \big|_{\Gamma_2} - \varepsilon_{ij}^d \big|_{\Gamma_1} \right) = \frac{1}{t} \Delta \varepsilon_{ij}^d, \\
\Delta \varepsilon_{ij}^{\Delta l} &= \frac{1}{t} \left( \alpha_2 T \big|_{\Gamma_2} - \alpha_1 T \big|_{\Gamma_1} \right) \delta_{ij} = \frac{1}{t} \Delta \alpha T \delta_{ij},
\end{align*}
\]

When we consider the ratios between summands in stress-strain state representation: \( \xi = \xi^O + \xi^I \) in neighborhood of the irregular boundary point, one can distinguish the following characteristic domains of stress-strain state action.

a) A neighborhood of plane domain irregular boundary point exists where the singular solution of homogeneous boundary value problem is valid, that is characterized by \( \sigma_{ij} \rightarrow \sigma_{ij}^O, \sigma_{ij}^I \rightarrow 0 \). The singularity of eigen stresses \( \sigma_{ij}^O \) (deformations \( \varepsilon_{ij}^O \)) has a power form \( r^{\Re \lambda - 1}, \lambda \in [0,0.5], \Re \lambda \) - the minimum value of the real part of the complex root of the characteristic equation of the homogeneous boundary value problem for the model wedge. Orders of bands in the stress concentrator domain (singular solution domain) are unreadable on the model under any magnification of irregular point neighborhood.

b) A neighborhood of domain boundary irregular point exists where \( \sigma_{ij} = \sigma_{ij}^O, \sigma_{ij}^I = 0 \) and the nonsingular homogeneous elasticity problem is valid with the same “eigen” value \( \min \Re \lambda \) as in the singular problem. The nonsingular solution domain does not contain the singular solution neighborhood and the irregular point itself, but is adjacent to it. As they approach the boundary of stress singular solution domain from the outside, deformations change continuously, and their values are great but finite. Orders of bands on the model that correspond to the nonsingular solution domain, are readable with a few exceptions.

c) At a sufficient distance from the irregular boundary point, there is a neighborhood where \( \sigma_{ij} = \sigma_{ij}^O, \sigma_{ij}^I = 0 \) and stresses are defined by the specified loads (common stress field).

In the nonsingular solution domain of homogeneous plane elasticity problem, evaluations may be given \([8-9, 12]\), using which one can extrapolate data to cross-sections positioned close to the boundary irregular point, taking into account experimental data and practical accuracy of data measurement using the photoelastic method.

By analyzing the stress state in the apex of right-angled wedge, using the example of a well-known \([1]\) experimental solution by M. Frocht (figure 1), choosing the nonsingular solution domain \( (m=7) \), adjacent to the singular domain, we can restore the load value. Based on that, we can conclude that it is possible to restore the order of bands in the small neighborhood of wedge apex, on the assumption that the experimental solution is practically possible and that the photoelastic method is accurate in the framework of linearly elastic statement.
2.2. Experimental solution of the problem, data correlation

Let us consider an experimental solution of a thermoelastic problem for a beam, where free temperature deformations $\alpha T \delta_{ij}$ are induced in one domain, while the other domain is free of loads. Forced deformations jump $\Delta \varepsilon_{ij} = \alpha T \delta_{ij}$ along the contact line of domains comprising the model emerges into the irregular point $O(0,0)$ of beam straight end boundary. Band pattern obtained using the “defreezing” method [1–5] for one of the beam domains is given in figure 1b.

Inherent stresses in neighborhood of irregular boundary point $O(0,0)$ of beam straight end have the form of:

$$\sigma_r = \frac{\alpha ET}{r} (c_1 \cos \theta + c_2 \sin \theta); \quad \sigma_\theta = \tau_\theta = 0.$$  

(8)

The singularity of inherent radial stresses in the beam straight end domain (figure 1b) is the same as the singularity of radial stresses for right-angled wedge (figure 1a) under the action of force in the problem by M. Frocht: when $r \to 0 \quad \sigma_r \to \frac{1}{r} \to \infty$. Therefore, the pattern of isochromatics in figure 1 corresponding to radial stresses in the experimental solution by M. Frocht is the pattern of inherent radial stresses in the form of (4) in neighborhood of the beam straight end irregular boundary (figure 1b).

In the beam straight end domain, diagrams of band orders (isochromatics) have been built for several radial cross-sections, given in figure 3. Similarity of band order diagrams has been established [11]. Based on experimental data, d-d design section in the non-singular solution domain of homogeneous problem was selected. Taking into account the continuity and variation similarity of band orders, diagrams of isochromatic orders have been built in cross-section e-e (figures 3, 4), and, with reference to them, diagrams of inherent radial stresses (4) shown in figure 4 (solid line). Cross-section e-e is located in the singular solution domain where the band pattern is unreadable or “poorly” readable.

**Figure 1.** a) Theoretical and experimental band patterns correlation as given in the paper by M. Frocht b) Theoretical and experimental band patterns correlation for beam straight end domain
Based on the theoretical and experimental analysis of stressed state in neighborhood of irregular point on the boundary of plane domain which the jump of forced deformations emerges into, the following formula is proposed for experimental data extrapolation:

\[ m_{i+1} = \left( \frac{r_i}{r_{i+1}} \right)^{1-\lambda_0} m_i, \]  

(9)

where \( m_i \) are band orders based on experimental data in the design cross-section \( r_i \) in neighborhood of nonsingular solution of homogeneous boundary value problem, \( m_{i+1} \) are band orders in cross-section of a smaller radius \( r_{i+1} < r_i \), located in the domain with unreadable or “poorly” readable model isochromatics pattern, and \( \lambda_0 = \min \Re \lambda \) is the minimum value of the real part of characteristic equation complex root of the homogeneous boundary value problem for the model wedge.

3. Results and discussions

Using the “microscopic” principle, the influence of geometrical parameter \( t \), characterizing the proximity to boundary irregular point, on changes in the form of boundary value problem in the domain boundary cut-out zone, has been studied. Specific domains of stress-strain state action in the domain boundary cut-out zone are distinguished: domain of homogeneous boundary value problem singular solution with power singularity, domain adjacent to the singular domain, and domain of common stress field under the action of specified loads. Based on the theoretical and experimental analysis of stressed state in neighborhood of irregular point on the boundary of plane domain, which the jump of forced deformations emerges into, formula (5) is proposed for experimental data extrapolation. Theoretical and experimental analysis of stressed state in the zone of domain boundary irregular point zone is performed in the framework of linearized elasticity problem statement and based on the accuracy of obtaining experimental data using photoelastic method.

4. Conclusions

Theoretical and experimental analysis of stressed state in neighborhood of irregular point on the boundary of plane domain, which the finite discontinuity (jump) of forced deformations emerges into,
and the experimental data extrapolation formula (5) proposed based on this analysis make it possible to restore band orders in the domain of elasticity problem singular solution where isochromatics on the model are unreadable or “poorly” readable. Cross-section proximity to the boundary irregular point is associated with the linearly elastic problem statement, experimental data measurement accuracy and the practical accuracy of photoelastic method.

References
[1] M. M. Frocht, Photoelasticity: Translated from English/edited by N. I. Prigorovsky, M.-L.: GITTL, vol. 1, p. 432; vol. 2, p. 488, 1950
[2] M. L. Williams, J. Appl. Mech, vol. 19, 4, p. 526, 1952
[3] G. L. Hesin and others, The photoelasticity method, vol. 3, p. 311, 1975
[4] A. Ya. Aleksandrov, M. Kh. Akhmetzyanov, Polarized light methods of deformable solid mechanics, M.: Nauka, p. 576, 1974
[5] I. A. Razumovsky, Interference-optical Methods of Solid Mechanics, Moscow: Bauman MSTU, p. 240, 2007
[6] S. P. Timoshenko, J. N. Goodier, Theory of elasticity, p. 576, 1975
[7] V. Z. Parton, P. I. Perlin, Methods of the mathematical theory of elasticity, p. 688, 1981
[8] G. P. Cherepanov, Mechanics of brittle fracture, M.: Nauka, p. 640, 1974
[9] G. S. Vardanyan, L. Yu. Frishter, International journal for computational civil and structural engineering, 3, Issue 2, pp. 75–81, 2007
[10] L. Frishter, Stress-deform state in the plane domain boundary angle cutout zone, XXVII R-S-P Seminar 2018, Theoretical Foundation of Civil Engineering, matec web of conferences 196, 01036, 2018, https://doi.org/10.105/matecconf/201819601036
[11] L. Frishter, The stress intensity factors in the corner cut-out area of the plane domain boundary, IO P Publishing FORM 2018, IOP Con. Ser.: Mat. Sci. Eng. 365(2018) 042020, https://doi.org/10.1088/1757-899X/365/4/042020
[12] L. Frishter, Evaluations of the solution to the homogeneous plane problem of the theory of elasticity in the neighborhood of an irregular boundary point, XXVI R-S-P Seminar 2017, v. 117, 2017