A Note on UV/IR for Noncommutative Complex Scalar Field

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Abstract

Noncommutative quantum field theory of a complex scalar field is considered. There is a two-coupling noncommutative analogue of $U(1)$-invariant quartic interaction $(\phi^* \phi)^2$, namely $A\phi^* \phi \star \phi^* \phi + B\phi^* \phi \star \phi \star \phi$. For arbitrary values of $A$ and $B$ the model is nonrenormalizable. However, it is one-loop renormalizable in two special cases: $B = 0$ and $A = B$. Furthermore, in the case $B = 0$ the model does not suffer from IR divergencies at least at one-loop insertions level.
1 Introduction

Recently, there is a renovation of the interest in noncommutative quantum field theories (or field theories on noncommutative space-time [1, 2]). As emphasized in [3], the important question is whether or not the noncommutative quantum field theory is well-defined. Note that one of earlier motivations to consider noncommutative field theories is a hope that it would be possible to avoid quantum field theory divergencies [4, 5, 2, 6, 7]. Now a commonly accepted belief is that a theory on a noncommutative space is renormalizable iff the corresponding commutative theory is renormalizable. Results on one-loop renormalizability of noncommutative gauge theory [8] and two-loop renormalizability of noncommutative scalar $\phi^4$ theory [9] as well as general considerations [10, 11] support this belief. In this paper we show that for more complicated models this is not true.

Note that renormalizability does not guarantee that the theory is well-defined. There is a mixing of the UV and the IR divergencies [12]. In particular, multi one-loop insertions in $\phi^3$ theory [12] and multi tadpole insertions in $\phi^4$ theory [9] produce infrared divergencies. UV/IR mixing depends on the model. The $U(1)$ noncommutative gauge theory does not exhibit a mixing of the UV and the IR dynamics [13]. For further discussions see [14]-[18].

The IR behaviour of noncommutative theories is closely related with an existence of a commutative limit of a noncommutative quantum theory under consideration. In particular, the IR behaviour of noncommutative $\phi^4$ theory makes an existence of the commutative limit impossible.

In this paper we consider noncommutative quantum field theories of complex scalar field [19] whose commutative analogue $(\phi^*\phi)^2$ is renormalizable in four-dimensional case. There is a two-coupling noncommutative analogue of $U(1)$-invariant quartic interaction $(\phi^*\phi)^2$, namely $A\phi^*\phi^*\phi^*\phi + B\phi^*\phi^*\phi^*\phi$. For arbitrary values of $A$ and $B$ the model is nonrenormalizable. However it is one-loop renormalizable in two special cases: $B = 0$ and $A = B$. Moreover, in the case $B = 0$ the model does not suffer from IR divergencies at least at one-loop insertions level.

2 The model

Consider complex scalar field. There are only two noncommutative structures that generalize a commutative quartic interaction $(\phi^*\phi)^2$:

(a) $\text{Tr} \phi^* \phi^* \phi^* \phi$,

(b) $\text{Tr} \phi^* \phi^* \phi^* \phi$,

where $\ast$ is the Moyal product $(f \ast g)(x) = e^{i\xi \theta_{\mu\nu}\partial_{\mu} \otimes \partial_{\nu}} f(x) \otimes g(x)$, $\xi$ is a deformation parameter, $\theta_{\mu\nu}$ is a nondegenerate skew-symmetric real constant matrix. In the commutative case the quartic interaction $(\phi^*\phi)^2$ is invariant under local $U(1)$-transformations. In the noncommutative theory we can consider a ”deformed” $U(1)$-symmetry ($U \ast U^* = 1$). One sees that only the structure (a) is invariant under these transformations. Using (a) and (b) we can construct an interaction $V[\phi^*, \phi] = A \text{Tr} \phi^* \phi^* \phi^* \phi + B \text{Tr} \phi^* \phi^* \phi^* \phi = (A - B) \text{Tr} \phi^* \phi^* \phi^* \phi + \frac{B}{2} \text{Tr}([\phi^*, \phi]_{AM} \ast [\phi^*, \phi]_{AM})$, (1)

where $[,]_{AM}$ is the Moyal antibracket $[f, g]_{AM} = f \ast g + g \ast f$. The action of the theory is $S = \int d^4x \left[ \partial_{\mu} \phi^* \partial_{\mu} \phi + m^2 \phi^* \phi \right] + V[\phi^*, \phi]$. (2)
Let us rewrite the interaction term in the Fourier components and symmetrize it, i.e.

\[
V[\phi^*, \phi] = \frac{1}{(2\pi)^4} \int dp_1 \ldots dp_4 \delta(\sum p_i) \times \\
\Phi \left[ A \cos(p_1 \wedge p_2 + p_3 \wedge p_4) + B \cos(p_1 \wedge p_3) \cos(p_2 \wedge p_4) \right] \\
\times \left[ \phi^*(p_1)\phi(p_2)\phi^*(p_3)\phi(p_4) \right].
\]  

(3)

3 One Loop

In this section we analyze counterterms to one loop Feynman graphs in the theory (2) and find conditions when this theory is renormalizable. All one-loop graphs are presented on Fig. 1:b,c,d. "In" arrows are the fields "\(\phi\)" and "out" arrows are the fields "\(\phi^*\)."

The following analytic expression corresponds to the graph on Fig. 1:b

\[
\Gamma_{1b} = \frac{N_b}{(2\pi)^d} \int d^4k \frac{\mathcal{P}_{1b}(p,k)}{(k^2 + m^2)((k + P)^2 + m^2)},
\]

(4)

where \(N_b\) is a number of graphs (\(N_b = 8\)), \(P = p_2 + p_4 = -p_1 - p_3\) and \(\mathcal{P}_{1b}(p,k)\) is the trigonometric polynomial

\[
\mathcal{P}_{1b}(p,k) = [A \cos(k \wedge p_2 + (-k - p_2) \wedge p_4) + B \cos(p_2 \wedge p_4) \cos(k \wedge P)] \\
\times [A \cos(p_1 \wedge (-k) + p_3 \wedge (k - p_1)) + B \cos(p_1 \wedge p_3) \cos(k \wedge P)].
\]

(5)

The terms containing \(\exp[(\ldots) \wedge k]\) give a finite contribution to (4). Divergencies come from the terms \(\Delta \mathcal{P}_{1b}\) of the polynomial \(\mathcal{P}_{1b}\),

\[
\Delta \mathcal{P}_{1b} = \frac{B^2}{2} \cos(p_1 \wedge p_3) \cos(p_2 \wedge p_4).
\]

(6)

The graphs Fig. 1:c and 1:d mutually differ by permutation of momenta \(1 \leftrightarrow 3\) only and the analytic expressions for these graphs coincide. For the graph Fig. 1:c we have

\[
\Gamma_{1c} = \frac{N_c}{(2\pi)^d} \int d^4k \frac{\mathcal{P}_{1c}(p,k)}{(k^2 + m^2)((k + P)^2 + m^2)},
\]

(7)

where \(N_c\) is a number of graphs (\(N_c = 16\)), \(P = p_1 + p_2 = -p_3 - p_4\) and \(\mathcal{P}_{1c}(p,k)\) is the trigonometric polynomial

\[
\mathcal{P}_{1c}(p,k) = [A \cos(p_1 \wedge p_2 + (-k - P) \wedge k) + B \cos(p_1 \wedge (k + P)) \cos(p_2 \wedge k)] \\
\times [A \cos(p_3 \wedge p_4 + (-k) \wedge (k + P)) + B \cos(p_3 \wedge k) \cos(4 \wedge (k + P))].
\]

(8)
The polynomial $\Delta P_1c$ that gives contribution to a divergent part of this graph is equal (after symmetrization $p_2 \leftrightarrow p_4$) to

$$\Delta P_1c = \cos(p_1 \wedge p_2 + p_3 \wedge p_4) \left[ \frac{A^2}{2} + \frac{B^2}{8} \right] + \frac{AB}{2} \cos(p_1 \wedge p_3) \cos(p_2 \wedge p_4). \quad (9)$$

We obtain the same answer for the graph on Fig.1d, i.e.

$$N_d = N_c, \quad \Delta P_1d = \Delta P_1c.$$ 

It is easy to see that the following condition is equal to one-loop renormalizability of the theory

$$N_b \Delta P_1b + 2N_c \Delta P_1c = C \left[ A \cos(p_1 \wedge p_2 + p_3 \wedge p_4) + B \cos(p_1 \wedge p_3) \cos(p_2 \wedge p_4) \right], \quad (10)$$

where $C$ is a constant. The condition (10) yields two algebraic equations:

$$N_c \left[ A^2 + \frac{B^2}{4} \right] = AC \quad (11)$$

$$N_b \frac{B^2}{2} + N_c AB = BC \quad (12)$$

This system is self consistent if

$$B(BN_c - 2AN_b) = 0.$$ 

The last equation has two solutions: $B = 0$ and $A = B$. Therefore, one-loop renormalizability takes place only in two cases

$$B = 0 \quad \text{and} \quad V[\phi^*, \phi] = A \text{Tr}(\phi^* \phi)^2, \quad (13)$$

$$A = B \quad \text{and} \quad V[\phi^*, \phi] = B \frac{1}{2} \text{Tr}([\phi^*, \phi]_{AM})^2. \quad (14)$$

Theories with a real scalar field have problems with infrared behaviour [12, 9] originated in multi one-loop insertions.

Considering a tadpole Fig.1e in our case of complex scalar field we have

$$\Gamma(p) = \frac{1}{(2\pi)^d} \int d^dk \frac{A + B \cos^2(k \wedge p)}{k^2 + m^2} = \frac{A + \frac{B}{2}}{(2\pi)^d} \int d^dk \frac{1}{k^2 + m^2} + \frac{B}{2(2\pi)^d} \int d^dk \frac{\xi^{2k \wedge p}}{k^2 + m^2}. \quad (15)$$

Integrating this expression over momentum $k$ we obtain

$$\Gamma(p) = \frac{m^{d-2}}{(4\pi)^{d/2}} (A + \frac{B}{2}) \Gamma(1 - d/2) + \frac{B}{(4\pi)^{d/2}} \left[ \frac{m}{\xi |\theta p|} \right]^{d/2-1} K_{d/2-1}(2m|\theta p|). \quad (16)$$

If $d = 4$ the second term is singular when $p \to 0$. But in the case $B = 0$ this term disappears and hence there is no IR problem at all.

In conclusion, we have considered two-coupling noncommutative analogue of $U(1)$-invariant quartic interaction $(\phi^* \phi)^2$ of the complex scalar field and shown that renormalizability takes place only in two special cases, in one of this cases the theory is free of infrared divergencies.

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