Transport in ultradilute solutions of $^3$He in superfluid $^4$He

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We calculate the effect of a heat current on transporting $^3$He dissolved in superfluid $^4$He at ultralow concentration, as will be utilized in a proposed experimental search for the electric dipole moment of the neutron (nEDM). In this experiment, a phonon wind will generate to drive (partly depolarized) $^3$He down a long pipe. In the regime of $^3$He concentrations $\lesssim 10^{-9}$ and temperatures $\sim 0.5$ K, the phonons comprising the heat current are kept in a flowing local equilibrium by small angle phonon-phonon scattering, while they transfer momentum to the walls via the $^3$He first viscosity. On the other hand, the phonon wind drives the $^3$He out of local equilibrium via phonon-$^3$He scattering. For temperatures below 0.5 K, both the phonon and $^3$He mean free paths can reach the centimeter scale, and we calculate the effects on the transport coefficients. We derive the relevant transport coefficients, the phonon thermal conductivity and the $^3$He diffusion constants from the Boltzmann equation. We calculate the effect of scattering from the walls of the pipe and show that it may be characterized by the average distance from points inside the pipe to the walls. The temporal evolution of the spatial distribution of the $^3$He atoms is determined by the time dependent $^3$He diffusion equation, which describes the competition between advection by the phonon wind and $^3$He diffusion. As a consequence of the thermal diffusivity being small compared with the $^3$He diffusivity, the scale height of the final $^3$He distribution is much smaller than that of the temperature gradient. We present exact solutions of the time dependent temperature and $^3$He distributions in terms of a complete set of normal modes.

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I. INTRODUCTION

The physics underlying the transport properties of mixtures of $^3$He and superfluid $^4$He changes markedly as the concentration of $^3$He varies. We determine here the transport properties of these mixtures at very low concentrations, $x_3 = n_3/(n_3 + n_4) \lesssim 10^{-9}$, where $n_3$ and $n_4$ are the $^3$He and $^4$He densities, and low temperatures, $T \lesssim 0.6$ K, where phonons are the dominant superfluid excitation. In this case, the phonons are in local thermal equilibrium; their interactions with the $^3$He distort the $^3$He distribution and dominate the $^3$He diffusion. For concentrations $x_3 \lesssim 10^{-4}$, the reverse situation holds: the $^3$He are in local equilibrium due to rapid $^3$He-$^3$He scattering and the phonon distribution is distorted due to phonon-$^3$He interactions [1]. In the intermediate concentration regime, the phonon and $^3$He distributions are both distorted and must be determined by solving the coupled evolution, or Boltzmann, equations [2]. At the highest concentrations, $x_3 \sim 1\%$, Fermi-Dirac statistics for the $^3$He become important [3]. These transport properties are of interest as an example of a two-component fluid with excitations of comparable energy but very different momenta, and where the excitations of the two species obey different statistics.

The transport properties of $^3$He in superfluid $^4$He at low concentrations are also important for the proposed experiment [7] to measure the neutron electric dipole moment (nEDM) at the Oak Ridge National Laboratory Spallation Neutron Source. There, the neutron precession frequency will be determined using the absorption of polarized ultracold neutrons on polarized $^3$He atoms in solution in superfluid $^4$He via the reaction

$$n + ^3\text{He} \rightarrow p + t + 764 \text{ keV},$$

which has a strong spin dependence, since capture proceeds primarily through the spin-singlet channel. Two key considerations accrue from this choice of detection technique. In order to maximize the precision with which the precession frequency can be measured, the optimal $^3$He concentration, $x_3 \sim 10^{-10}$, corresponds to a capture rate comparable to the decay rate of the neutrons. However, primarily due to wall collisions, the $^3$He will gradually become depolarized. In order to reduce the background from neutron capture on unpolarized $^3$He, it is crucial to be able to periodically sweep out the $^3$He by means of a heat current [7]. In this paper, we calculate both the heat and $^3$He particle currents based on well-established microscopic theory of phonon-phonon [8] and phonon-$^3$He scatterings [9], as well as the evolution of the temperature and $^3$He concentration.

At the concentrations and temperatures of interest in the experiment, in addition to phonon-phonon, phonon-$^3$He and $^3$He-$^3$He scattering, the scattering of both phonons and $^3$He from the walls of the containers can also be important. Here we extend the solution of the Boltzmann equations in Ref. [2] to include these effects, in addition to providing some examples for $x_3 \lesssim 10^{-9}$. For
illustration, we consider the effect of a heat current in an essentially one-dimensional geometry, with the $^3$He superfluid $^4$He mixture in a long pipe with a diameter of a few cm. The phonon-wall interactions affect the thermal conductivity as well as the phonon velocity distribution within the pipe; the $^3$He–wall interactions affect the transport of the $^3$He in the presence of a heat current.

This paper is arranged as follows: Section II describes the basic scattering mechanisms and calculates the transport coefficients from the Boltzmann equation given in Sec. III. Subsequently, we calculate the temporal and spatial evolution of the temperature (Sec. IV) and the $^3$He density (Sec. V). We summarize results in Sec. VI. In Appendix A we analyze the transport when scattering of phonons is predominantly from the walls of the pipe, and in Appendix B we solve analytically the equation for the temporal evolution of the $^3$He concentration.

II. PHONON AND $^3$HE RELAXATION

We begin by considering the relevant microscopic relaxation mechanisms (detailed in Ref. [2]), first for the phonons. The momentum-dependent mean free path of phonons is predominantly from the walls of the pipe; the $^3$He–wall interactions affect the thermal equilibrium, where the mean phonon drift velocity, $v_{ph}$, is

$$T S_{ph} v_{ph} = -K_{ph} \nabla T.$$  

(6)

The thermal conductivity of the phonons, $K_{ph}$, can, at low concentrations, be written as $[2, 10],

$$K_{ph} = \frac{5}{8} \frac{s S_{ph} R^2}{\ell_{visc}}, \quad (7)$$

in a pipe of radius $R$ and where $\ell_{visc}$ is the effective mean free path

$$\frac{1}{\ell_{visc}} = \frac{1}{\ell_{visc}} + \frac{16}{5 R}. \quad (8)$$

The second term represents scattering of the phonons on the walls; the thermal conductivity assumes the Casimir form in this limit.

The $^3$He contribution to the overall heat flux is negligible at low $x_3 [1, 2]$. The mean free path of a $^3$He scattering on unpolarized $^3$He is $[2]

$$\ell_{33} = \frac{1}{(n_3/2) \sigma_{33}} = \frac{8.66 \times 10^{-8}}{x_3} \text{ cm}, \quad (9)$$

where $\sigma_{33}$ is the corresponding cross section. Thus for $x_3 < 10^{-3}$, one has $\ell_{33} \sim 1 \text{ m}$; we therefore neglect $^3$He–$^3$He scattering. On the other hand, the mean free path for $^3$He scattering on phonons $[2],

$$\ell_{3ph} = \frac{\sqrt{3}}{2 J} \left( \frac{n_4}{S_{ph}} \right)^2 \frac{m_{\ast}^{1/2} s^2}{T^{3/2}} \quad (10)$$

$$= 0.077 \left( \frac{0.45 \text{ K}}{T} \right)^{15/2} \text{ cm}, \quad (11)$$

where $m_{\ast}$ is the $^3$He effective mass in superfluid $^4$He, is small compared to the pipe diameter for $T \gtrsim 0.3 \text{ K}$. Thus the dominant process for bringing the $^3$He toward equilibrium for temperatures of interest in the experiment is scattering against phonons. In the next section, we outline the calculation of the $^3$He transport coefficients; more details can be found in Ref. [2].
III. $^3$He BOLTZMANN EQUATION AND TRANSPORT COEFFICIENTS

The $^3$He Boltzmann equation has the general form
\[
\frac{\partial f_p}{\partial t} + \vec{p} \cdot \mathbf{\nabla} f_p = \sum_{p',q'q} \mathcal{T} \left[ f_{p'q'}^{\text{le}}(\vec{r}) (1 + n_{q'}^{\text{le}}(\vec{r})) - f_{p'q'}^{\text{le}}(\vec{r}) (1 + n_{q'}^{\text{le}}(\vec{r})) \right] - \frac{\delta f_p}{\tau_3} - \frac{\delta f_p}{\tau_{3 wd}} - \frac{\delta f_p}{\tau_{3 w}}. \tag{13}
\]

Here $f_p$ is the $^3$He distribution function,
\[
f_p^0 = e^{-\beta(p^2/2m^* - \mu_3)} \tag{14}
\]
is the equilibrium distribution function, and we write the deviations from local equilibrium as
\[
\delta f_p = f_p - f_p^0, \tag{15}
\]
where
\[
f_p^{le0} = e^{-\beta(p^2/2m^* - \mu_3(\vec{r})/T(\vec{r}))} \tag{16}
\]
is the local equilibrium distribution function, i.e., the distribution towards which collisions with phonons drive the $^3$He. The first term on the right represents the scattering on the phonons, the second term the scattering from other $^3$He (numerically insignificant for the concentrations of interest) and the last two terms isotropic diffuse (d) and specular (s) scattering of the $^3$He from the walls. In the term describing collisions with phonons, $\vec{p}$ and $\vec{p}'$ are the initial and final $^3$He momenta, respectively, and $\vec{q}$ and $\vec{q}'$ are the corresponding phonon momenta. The phonon–$^3$He scattering kernel is $\mathcal{T} = \{|p'q'|T|pq|\}^2\pi\delta(p^2/2m^* + sq - p'^2/2m^* - sq')$, and momentum conservation, $\vec{p} + \vec{q} = \vec{p}' + \vec{q}'$, is understood in the collision term.

In order to calculate the effects of a phonon wind on the $^3$He we solve Eq. (13) for $\delta f_p$. On the left side of the Boltzmann equation, we approximate the distribution by its local equilibrium form, $f_p^{le0}$. We neglect the contribution from $\partial n_{ph}/\partial z$ because of the relatively small temperature gradient (see Eq. (35b) below), while the gradient of $\beta$ in this term gives a second order contribution, which we neglect. The left side of the Boltzmann equation is then
\[
\frac{\partial f_p^{le0}}{\partial z} = \left[ \frac{\vec{p}^2}{2m^*} - \frac{3}{2T} \right] f_p^{le0} \frac{\partial T}{\partial z} + f_p^{le0} \frac{1}{n_3} \frac{\partial n_3}{\partial z}. \tag{17}
\]

On the right side we write
\[
\delta f_p = \beta f_p^0 p_z w_p, \tag{18}
\]
as shown in Ref. [2] the $^3$He-phonon collision term is diagonalized by expanding $w_p$ in Sonine polynomials [2].

To solve for $w_p$ we multiply the Boltzmann equation by $p_z$ and integrate over all $\vec{p}$. Noting that the distortion $\delta f_p$ is proportional to $p_z$, we see first that on the right side of the Boltzmann equation the two term $\propto \delta f_p - \beta f_p^0 v_3$ do not contribute, since both the $^3$He–$^3$He scattering and the specular $^3$He scattering from the walls conserve momentum in the $z$ direction. Following Eq. (81) of Ref. [2] for the phonon–$^3$He scattering, we see that the remaining terms on the right side comprise
\[
- \frac{\beta T}{3m^*} \left[ \delta f_p - \beta f_p^0 p_z v_{ph} \right] - \delta f_p, \tag{19}
\]

With the inclusion of the recoil effect in the phonon–$^3$He scattering to lowest order (see the Appendix of Ref. [1] and Sec. 6 of Ref. [2]), the solution of the Boltzmann equation is
\[
\delta f_p = \frac{3}{m^*} \left[ \frac{\beta}{3} \frac{\Gamma_{rec}}{\Gamma_{3 wd}} \frac{p_z v_{ph}}{3} - \frac{\partial f_p^{le0}}{\partial z} \right], \tag{20}
\]

where
\[
\frac{1}{\tau_3} = \frac{\beta \Gamma_{rec}}{m^*} + \frac{1}{\tau_{3 wd}}, \tag{21}
\]
is the effective $^3$He scattering rate, including both scattering from phonons, encoded in $\Gamma_{rec}$, and diffuse scattering from the walls of the pipe.

Integrating the Boltzmann equation, Eq. (13), over $\vec{p}$ we recover the continuity equation
\[
\frac{\partial n_3}{\partial t} + \mathbf{\nabla} \cdot \vec{j}_3 = 0, \tag{22}
\]

with the $^3$He particle current given by
\[
\vec{j}_3 = \nu \int \frac{d^3p}{(2\pi)^3 m^*} \vec{p} \delta f_p, \tag{23}
\]

where $\nu$ is the number of spin degrees of freedom of the $^3$He: 1 for a fully polarized sample and 2 in the unpolarized case. It is straightforward to evaluate the current (including the effect of recoil in the phonon–$^3$He scattering)
\[
\vec{j}_3 = n_3 v_{ph} - D_{rec} \frac{\partial n_3}{\partial z} - D_{T,rec} \frac{\partial T}{\partial z}, \tag{24}
\]

where the $^3$He diffusion constant, including recoil corrections, is [2]
\[
D_{rec} = 3\xi(R, \zeta) T^2 / \Gamma_{rec}, \tag{25}
\]

and the “thermoelectric” coefficient is
\[
D_{T,rec} = 3\xi(R, \zeta) T n_3 / \Gamma_{rec}. \tag{26}
\]

In these expressions the basic forms of $D$ and $D_T$ are modified by the wall scattering factor
\[
\xi(R, \zeta) = \left( 1 + \frac{3m^*}{\beta \Gamma_{rec} \tau_{3 wd} (R)} \right)^{-1}, \tag{27}
\]
of the nEDM experiment, which in the vicinity of the operating temperature regime becomes more important for lower temperatures for example, is shown in Fig. 1 for $\zeta = 1$. We assume here that the $^3$He wall scattering is diffuse, $\zeta = 1$.

$$\zeta = \frac{\tau_{3\text{wd}}^{-1}}{\tau_{3\text{wd}}^{-1} + \tau_{3\text{ses}}}$$  \hspace{1cm} (28)

is the fraction of the $^3$He wall scattering rate that is diffuse.

To see the effect of scattering of $^3$He from the walls, we take the total wall scattering rate to be simply

$$\frac{1}{\tau_{3\text{wd}}(R)} = \frac{\bar{v}_3}{D_{\text{eff}}},$$  \hspace{1cm} (29)

where $\bar{v}_3 = \sqrt{3T/m^*}$ is the mean $^3$He thermal velocity, and $D_{\text{eff}} = 2R/3$ [see Eq. (A8)] is the effective average distance from an interior point to the wall of an infinitely long pipe of radius $R$ entering the transport $^3$. The effect of wall scattering on the diffusion constant, $D_{\text{rec}}$, for example, is shown in Fig. 1 for $\zeta = 1$. As we see, the effect becomes more important for lower temperatures because of the decrease in the phonon density. As $R \to \infty$, $D_{\text{rec}}$ approaches the result without wall scattering, which in the vicinity of the operating temperature regime of the nEDM experiment, $T \sim 0.45$ K, is

$$D_{\text{rec}} \approx \frac{0.88}{T^2} \text{cm}^2/\text{s}.$$  \hspace{1cm} (30)

We also show, in Fig. 2 the corresponding effect of phonon–wall scattering on the phonon thermal conductivity, Eq. (7).

At temperatures of approximately 0.3K and below, the mean free path of a $^3$He quasiparticle in the bulk medium is greater than 1 cm, which is comparable to the radii of the pipes considered for the nEDM experiment. In this regime, the $^3$He quasiparticles can lose momentum only not by collisions with phonons but also directly to the walls of the pipe. This situation is analogous to the (Knudsen) flow of a low-density gas in response to a pressure gradient, except that in the dilute helium solutions the force driving the flow of $^3$He has two components, one due to the $^3$He pressure gradient and another due to the collisions with phonons. When the $^3$He distribution function is stationary, the two contributions to the force are equal and opposite. The $^3$He distribution function in principle depends not only on direction in momentum space but also on the radial coordinate in the pipe. However, the time scales for smoothing out the radial dependence via diffusion or ballistic transport are of order several milliseconds, and therefore short compared with the overall evolution time scales of the system. A detailed study of this regime lies outside the scope of the present article.

### IV. TIME EVOLUTION OF THE TEMPERATURE

We now estimate, using the heat diffusion equation, the timescale for heating the fluid. We assume, as above, that the heat is carried by the phonons (see also Eq. (89) of Ref. 2 and discussion following) and that the relative temperature variation and, hence, the variation of $K_{\text{ph}}$ is small, so that

$$\frac{\partial \epsilon}{\partial t} = 3 S_{\text{ph}} \frac{\partial T}{\partial t} = -\nabla \cdot \bar{Q} = K_{\text{ph}} \nabla^2 T,$$  \hspace{1cm} (31)

where $\epsilon$ is phonon energy density. Within a few scattering times after the application of heat at one end of the pipe $(z = 0)$, the temperature there is approximately fixed at $T_0 + \Delta T$. We assume that the temperature at $z = L$, the other end of the pipe, is kept constant at temperature $T_0$ by a refrigerator.

To solve the heat diffusion equation it is sufficient to consider the average of the temperature, $T(z, t)$, over the
cross-section of the pipe, thus avoiding having to take into account details of the counterflows within the pipe. The solution is given in terms of the modes in the pipe that vanish at \( z = 0 \) and \( L, \sin k_\nu z, \) where \( k_\nu = \nu \pi / L, \) with \( \nu \) here a positive integer:

\[
T(z,t) = T_0 + \Delta T(1 - z/L) + \sum_{\nu \neq 0} c_\nu e^{-D_{th} k_\nu^2 t} \sin k_\nu z.
\]

(32)

We denote the thermal diffusivity by

\[
D_{th} = K_{ph} / 3 S_{ph},
\]

(33)

and recognize \( 3 S_{ph} = (2\pi^2 / 15)(T/s)^3 \) as the \(^4\text{He} \) specific heat. The condition that \( T(z,t = 0) = T_0 \) except immediately at \( z = 0, \) implies that the mode weights are given by \( c_\nu = -2\Delta T / \nu \pi. \) The characteristic time, \( \tau_{th} \) to set up a steady state phonon wind is essentially that of the \( \nu = 1 \) mode,

\[
\tau_{th} = \frac{1}{D_{th} k_1^2} = \frac{L^2 K_{ph}}{3 \pi^2 S_{ph}}.
\]

(34)

For typical conditions in the experiment, 5 mW of heat in a 3 cm diameter, 100 cm long pipe at \( T = 0.45 \) K, the phonon thermal conductivity is \( 2.4 \times 10^8 \) erg/s/cm-K, \( \tau_{th} \approx 11 \) ms, and \( \Delta T = 3 \) mK.

V. TIME EVOLUTION OF THE \(^3\text{He} \) CONCENTRATION

To begin examining the \(^3\text{He} \) concentration, we consider its steady-state distribution in the presence of a heat current or phonon wind. Because, as we shall see below, the term involving \( D_T \) is relatively small for low concentrations, the condition that the \(^3\text{He} \) particle current, Eq. (24), vanishes, is simply

\[
D_{rec} \frac{\partial n_3}{\partial z} = v_{ph} n_3,
\]

(35)

which has the solution

\[
n_3(z) = \tilde{n}_3 e^{z/h} \equiv n_{3,\infty}(z),
\]

(36)

where we define the scale height, \( h = D_{rec} / v_{ph} \) (for the example parameters above, \( D_{rec} = 225 \) cm/s, \( v_{ph} = 17 \) cm/s and \( h = 13 \) cm), and

\[
\tilde{n}_3 = \frac{n_0}{e^{L/h} - 1} h,
\]

(37)

with \( n_0 \) the initial uniform \(^3\text{He} \) density. We note that the relative size of the term involving \( D_{r,rec} \) is simply the ratio of the scale heights of the concentration and the temperature,

\[
\frac{D_{T,rec} \partial T / \partial z}{D_{rec} \partial n_3 / \partial z} = \frac{|\partial \ln T / \partial z|}{|\partial \ln n_3 / \partial z|} = \frac{D_{rec} S_{ph}}{K_{ph}}.
\]

(38)

about 1/1000 for the example parameters given above.

In the nEDM experiment, the \(^3\text{He} \) in the system depolarizes in time, primarily due to interactions with walls. The depolarized \(^3\text{He} \) will be removed by a phonon wind before the system is recharged with more highly polarized \(^3\text{He}. \) As above, we consider the simple situation of a long pipe with a heater at \( z = 0 \) and closed ends. The evolution of the \(^3\text{He} \) is governed by a competition between two processes: the phonon wind, which were to act alone would push all the \(^3\text{He} \) to the downstream (large \( z \)) end of the pipe, and diffusion of the \(^3\text{He} \), limited by scattering with the phonons, which allows the \(^3\text{He} \) to drift back towards smaller \( z \).

This evolution of the \(^3\text{He} \) concentration in the presence of a phonon wind is described by the diffusion equation resulting from Eq. (22),

\[
\frac{\partial n_3(z,t)}{\partial t} + v_{ph} \frac{\partial n_3}{\partial z} - D_{rec} \frac{\partial^2 n_3}{\partial z^2} = 0,
\]

(39)

where we have dropped the \( D_T \) term in Eq. (24). Once a steady phonon wind, with a small temperature gradient, is established, we may neglect the temperature dependence of \( D_{rec} \) and take \( v_{ph} \) and \( D_{rec} \) to be constant. For a pipe with a large length to diameter ratio, we may treat the problem as one dimensional, averaging over its cross section as we did above for the heat flow. The boundary conditions are that the \(^3\text{He} \) current, \( j_3, \) Eq. (24), vanishes at the two ends of the pipe,

\[
\frac{\partial n_3}{\partial z} = \frac{n_3}{h} \quad (z = 0, L).
\]

(40)

To solve Eq. (39) with constant \( v_{ph} \) and \( D \), we write the \(^3\text{He} \) density as \( e^{z/2h} \tilde{n}(z,t) \) and decompose \( \tilde{n}(z,t) \) as a sum of time dependent modes \( \tilde{n}_\nu(z,t) \) periodic in \( 2L \) (see Appendix B):

\[
n_3(z,t) = e^{z/2h} \sum_{\nu = 0}^{\infty} \tilde{n}_\nu(z,t),
\]

(41)

where \( \tilde{n} \) satisfies the boundary condition

\[
\frac{\partial \tilde{n}}{\partial z} = \frac{\tilde{n}}{2h} \quad (z = 0, L).
\]

(42)

The spatial parts of the mode functions \( \tilde{n}_\nu(z,t) \) are the complete orthonormal set

\[
\phi_\nu(z) = \begin{cases} \frac{e^{z/2h}}{h (e^{L/h} - 1)^{1/2}} & \nu = 0 \\ \alpha_\nu \left( \cos k_\nu z + \frac{1}{2k_\nu} \sin k_\nu z \right) & \nu \geq 1 \end{cases}
\]

(43)

with \( k_\nu = \pi \nu / L \) and \( \alpha_\nu = ([L/2 (1 + 1/(2h k_\nu)^2)]^{-1/2}. \) The time dependence of the modes is \( e^{-t/\tau_\nu} \) where

\[
\frac{1}{\tau_\nu} = \begin{cases} 0, & \nu = 0 \\ k_\nu^2 D + v_{ph}^2 / 4D = (k_\nu^2 + 4h^2) D, & \nu \geq 1 \end{cases}
\]

(44)
The solution of Eq. \((39)\) for an initially uniform density \(n_{3,0}\) is then

\[
n_3(z, t) = n_{3,0} e^{-z^2/2h} \sum_{\nu=0}^{\infty} c_\nu \phi_\nu(z) e^{-t/\tau_\nu}, \tag{45}
\]

where

\[
\begin{align*}
c_\nu &= \int_0^L \left( \frac{n(z, 0)}{n_{3,0}} \right) \phi_\nu(z) dz \\
&= \begin{cases} \\
\frac{L}{h (e^{L/h} - 1)^{1/2}}, & \nu = 0 \\
\frac{8 \nu \omega_\nu}{\Gamma (2 \nu + 1)} \left( 1 + (-1)^{\nu+1} e^{-L/2h} \right), & \nu \geq 1
\end{cases}
\end{align*} \tag{46}
\]

As we show in Appendix B, the general solution may be written in compact form in terms of a Green’s function

\[
\hat{n}(z, t) = \int_0^L G(z, z', t) \hat{n}(z', 0) dz', \tag{47}
\]

where

\[
G(z, z', t) = \sum_{\nu=0}^{\infty} \phi_\nu(z) \phi_\nu(z') e^{-t/\tau_\nu} \theta(t). \tag{48}
\]

The \(z\) and \(t\) dependences of \(x_3\) are shown in Figs. 3 and 4, respectively, for the case of the uniform initial distribution and typical experimental values (5 mW heat into a 3 cm diameter, 100 cm long pipe at \(T = 0.45\) K). As the figures illustrate, the concentration scale height, \(h\), is substantially smaller than the pipe length. Figure 5 plots the difference between \(x_3\) and its steady state value for several points along the pipe, showing that, after a few seconds, the lowest mode, with \(\tau_1 = 1.8\) s, dominates the time evolution throughout the pipe.

The results for the evolution of the \(^3\)He concentration presented here are equally applicable to a heat flush experiment being carried out at Harvard at natural \(^3\)He concentration \([14]\). There one must use the more general phonon thermal conductivity as derived in \([2]\); phonon-wall scattering in this regime plays a negligible role.

\[\text{VI. SUMMARY}\]

We have calculated the transport properties of dilute mixtures, \(x_3 \lesssim 10^{-9}\), of \(^3\)He in superfluid \(^4\)He at temperatures around 0.5 K where phonons are the dominant excitations of the superfluid. In this regime, we considered a simple one dimensional geometry (a pipe), a heat current generates a phonon wind with phonons in local equilibrium corresponding to the temperature at that point in the pipe. On the other hand, phonon scattering distorts the \(^3\)He distribution from equilibrium. Starting from the known phonon-phonon and phonon-\(^3\)He scattering, we calculate the transport coefficients from the Boltzmann equation. We show that, in the presence of a heat current which generates a temperature scale height much larger than the length of the pipe (i.e., a small relative temperature gradient), the scale height for the \(^3\)He concentration can be much less than the pipe length. This leads to...
FIG. 5: (color online) The absolute value of the $^3$He concentration difference, $|x_z - x_{3\infty}|$, from the solution of Eq. (11) as a function of time, $t$, for various positions along the pipe: the curves correspond to $z = 0$ (dotted), 20 (dash double dot), 40 (dash dot), 60 (short dash), 80 (long dash) and 100 cm (solid). We note that the lowest mode, corresponding to $z = 0$ (dotted), 20 (dash double dot), 40 (dash dot), 60 (short dash), 80 (long dash) and 100 cm (solid). We note that the lowest mode, corresponding to the time constant $\tau_1$, dominates the time evolution after a few seconds. The result is shown for typical parameters in the nEDM experiment: $x_{3,0} = 10^{-10}$, and 5 mW of heat into a 3 cm diameter, 100 cm long pipe at a nominal temperature of 0.45 K, for which $\tau_1 = 1.8$ s.

A large decrease in concentration at the hot boundary and a corresponding increase at the cold end. For temperatures below 0.5 K the mean free paths of both the phonons and the $^3$He can reach the centimeter scale; in these cases, scattering from the walls of the container becomes important. Finally, we calculate the timescales associated with the evolution of both the temperature and concentration distributions: because of the large superfluid thermal conductivity, the thermal timescales are on the ms scale, whereas the corresponding scale for evolution of the concentration is on the scale of seconds.

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Appendix A: Phonon drift velocity in the ballistic limit

Here we derive the spatial dependence of the phonon drift velocity when the phonon-phonon scattering mean free path for large angle scattering, Eq. (14), is much larger than the pipe radius, $R$. For the pipe geometry considered in the text, this condition holds at a temperature $T \lesssim 0.3$ K. We assume that the pipe axis is in the $z$ direction, and that the transverse coordinates are $x$ and $y$. We also assume that a phonon striking the cylinder wall is diffusively reflected, with a distribution of final momenta given by the local temperature, $T(z) = T_0 + T'z$, where $T' < 0$ is the temperature gradient. Then $n_q(\vec{r})$, the number of phonons of momentum $\vec{q}$ at point $\vec{r}$, is given by the equilibrium distribution $n_q^0(\vec{q})$ at the point $\vec{r}^\prime = (x', y', z')$ on the pipe wall where the phonon at $\vec{r}$ originated:

$$n_q(\vec{r}) = \frac{1}{e^{\frac{\epsilon_q}{T(z(\vec{q}))}} - 1} \approx n_q^0 - \vec{z}' \cdot T' \frac{\partial n_q^0}{\partial q},$$

(A1)

to lowest order in $T'$, where $n_q^0$ is the equilibrium distribution.

The point of origin is determined by simple geometry, namely, $\vec{r} - \vec{r}' = \vec{q} \mathcal{D}$, where $\mathcal{D} = |\vec{r} - \vec{r}'|$. We measure $\vec{q}$ in polar coordinates $\vec{q} = (\theta_q, \phi_q)$. Then $z' = z - D \cos \theta_q$, $x' = x - D \sin \theta_q \cos \phi_q$, and $y' = y - D \sin \theta_q \sin \phi_q$. Using $x'^2 + y'^2 = R^2$ on the cylinder wall, we have then

$$\sin^2 \theta_q D^2 - 2\rho D \sin \theta_q \cos(\phi_q - \phi_r) - R^2 + \rho^2 = R^2,$$

(A2)

where $\phi_r$ is the azimuthal angle of $\vec{r}$ and $\rho = \sqrt{x'^2 + y'^2}$. The solution is

$$\mathcal{D} = \frac{1}{\sin \theta_q} \left[ \rho \cos(\phi_q - \phi_r) + \sqrt{R^2 - \rho^2 \sin^2(\phi_q - \phi_r)} \right];$$

(A3)

without loss of generality we take $\phi_r = 0$.

The local phonon flow velocity $v_{ph,z}(\rho)$ is given by the local total momentum flux density in the $z$ direction divided by the normal mass density of the phonons, $\rho_{ph}$.

$$v_{ph,z}(\rho) \equiv \frac{1}{\rho_{ph}} \int \frac{d^3q}{(2\pi)^3} q \cos \theta_q n_q(\vec{r}).$$

(A4)

To first order in $T'$, only the $\mathcal{D}$ term in the distribution function survives the angular average in the numerator, so that

$$v_{ph,z}(\rho) = \frac{T'}{T_0} \int \frac{d^3q \cos^2 \theta_q D \partial n_q^0}{\partial q} \frac{d^3q \partial n_q^0}{\partial q} / \frac{d^3q n_q^0}{\partial q},$$

(A5)

independent of $z$. Since $\mathcal{D}$ is independent of $q$, the integrals over $q$ in numerator and denominator cancel, and

$$v_{ph,z}(\rho) = \frac{1}{\rho_{ph}} \frac{T'}{T_0} \langle D \cos^2 \theta_q \rangle,$$

(A6)

where the angular brackets denote the average over angles of $\vec{q}$.

The flow velocity averaged over the cross section of the pipe (denoted by an overline) is simply

$$\overline{v_{ph,z}} = \frac{1}{\pi R^2} \int_0^R 2\pi \rho d\rho v_{ph,z}(\rho) = -3s \frac{T'}{T_0} \mathcal{D}_{\text{eff}},$$

(A7)
where
\[ D_{\text{eff}} = \frac{1}{\pi R^2} \int_0^R 2\pi \rho \, d\rho \, \langle D \cos^2 \theta_q \rangle. \] (A8)

In evaluating \( D_{\text{eff}} \), the integral over \( \theta_q \) decouples from those over \( \rho \) and \( \phi_q \), and the latter are easily performed if one integrates over \( \rho \) before integrating over \( \phi_q \). One finds
\[ D_{\text{eff}} = \frac{2R}{3} = \frac{1}{2} \langle D \rangle, \] (A9)

which expresses the fact that the length important for thermal conduction is one half of \( \langle D \rangle \), the average distance to the wall of the pipe, averaged over the cross section of the pipe. Since \( \Pi_{ph,z} = -3(K_{ph}/TC_{ph})T' \), where \( C_{ph} \) is the phonon heat capacity, we find the phonon thermal conductivity,
\[ K_{ph} = sC_{ph}D_{\text{eff}} = \frac{2}{3} sC_{ph}R, \] (A10)

which is the Casimir result, Ref. \[11\].

Locally,
\[ \int \frac{d\Omega_3}{4\pi} D \cos^2 \theta_q = \frac{\pi R}{4} I(\rho^2/R^2) \] (A11)

where the elliptic integral,
\[ I(t^2) = \frac{\pi}{2} \int_0^{\pi/2} d\phi \sqrt{1 - t^2 \sin^2 \phi}, \] (A12)

must be done numerically. As we see in Fig. 6, the velocity profile
\[ v_{ph,z}(\rho) = -\frac{3\pi s R T'}{4 T_0} I(\rho^2/R^2), \] (A13)

is independent of \( z \) and nearly quadratic almost to the edge of the pipe, where it falls more rapidly, but unlike when viscosity dominates, it does not fall to zero at the pipe wall.

**Appendix B: Exact solution of the time dependent diffusion equation with advection**

Here we construct the general solutions of the time-dependent diffusion equation \[43\] by first transforming the equation into self-adjoint form by writing \( \hat{n}_3(z,t) = e^{z^2/2h} \tilde{n}(z,t) \). As a result, \( \tilde{n}(z,t) \) obeys
\[ \left( \frac{\partial}{\partial t} + \hat{D}_{ph} \frac{\partial^2}{4D} - D \frac{\partial^2}{\partial z^2} \right) \tilde{n}(z,t) = 0, \] (B1)

in the interval \( 0 \leq z \leq L \) (we simply write a generic diffusion constant, \( D \), to simplify the notation). The boundary condition of vanishing current at the two ends of the pipe then becomes
\[ \frac{\partial \tilde{n}}{\partial z} = \frac{\tilde{n}}{2h} \quad (z = 0, L), \] (B2)

where \( h = D/v_{ph} \). To realize the boundary conditions we expand \( \tilde{n} \) in a complete set of normalized solutions of Eq. (B1) that are periodic in the interval 0 to 2L,
\[ \phi_\nu(z) = \alpha_\nu \left( \cos k z + \frac{1}{2h} \sin k z \right), \] (B3)

with \( k = \pi \nu / L \ (\nu \geq 1) \), and
\[ \alpha_\nu = [(L/2) (1 + 1/(2hk_\nu)^2)]^{-1/2}, \] (B4)

together with the stationary solution
\[ \phi_0(z) = \frac{e^{z^2/2h}}{[h (e^{L/h} - 1)]^{1/2}}. \] (B5)

The modes \( \phi_\nu(z) \) form an orthonormal set obeying
\[ \int_0^L \phi_\nu(z) \phi_{\nu'}(z) dz = \delta_{\nu,\nu'}, \] (B6)

as well as the completeness relation in the interval 0 < \( z, z' < L \),
\[ \sum_{\nu=0}^{\infty} \phi_\nu(z) \phi_\nu(z') = \delta(z - z'). \] (B7)

The time dependence of the modes \( \phi_\nu(z,t) \) is \( e^{-t/\tau_\nu} \), with
\[ \frac{1}{\tau_\nu} = \left( \frac{1}{4h^2} + k_\nu^2 \right) D, \quad \nu \geq 1, \] (B8)

and \( 1/\tau_0 = 0 \). Then
\[ \hat{n}(z,t) = \sum_{\nu=0}^{\infty} b_\nu \phi_\nu(z) e^{-t/\tau_\nu}, \] (B9)

with
\[ b_\nu = \int_0^L \hat{n}(z,0) \phi_\nu(z) dz. \] (B10)
We can thus write the solution \( \hat{n}(z, t) \) in terms of the initial density distribution as

\[
\hat{n}(z, t) = \int_0^L G(z, z', t) \hat{n}(z', 0) dz',
\]

(B11)

where

\[
G(z, z', t) = \sum_{\nu=0}^{\infty} \phi_\nu(z) \phi_\nu(z') e^{-t/\tau_\nu} \theta(t)
\]

(B12)
is the Green’s function for the diffusion equation in the form (B1), with \( \theta \) the Helmholtz unit step function. At \( t = 0 \), the completeness relation implies \( G(z, z', t) = \delta(z - z') \), and thus

\[
\left( \frac{\partial}{\partial t} + \frac{v_{ph}^2}{4D} \frac{\partial^2}{\partial z^2} \right) G(z, z', t) = \delta(z - z') \delta(t),
\]

(B13)
in the interval \( 0 \leq z \leq L \).

We now convert back to the original form of the diffusion equation, (39), and have

\[
n_3(z, t) = \int_0^L G(z, z', t) n_3(z', 0) dz',
\]

(B14)

where

\[
G(z, z', t) = \delta(z + z')/2h \delta(t)
\]

(B15)
is the Green’s function for the original diffusion equation (39), i.e.,

\[
\left( \frac{\partial}{\partial t} + v_{ph} \frac{\partial}{\partial z} - D \frac{\partial^2}{\partial z^2} \right) G(z, z', t)
\]

\[
= \delta(z + z')/2h \delta(z - z') \delta(t).
\]

(B16)

[1] G. Baym, D. H. Beck, and C. J. Pethick, Phys. Rev. B 88, 014512 (2013), arXiv:1212.2946.
[2] G. Baym, D. H. Beck, and C. J. Pethick, J. Low Temp. Phys., 178, 200 (2015); arXiv:1408.1619.
[3] G. Baym, J. Bardeen and D. Pines, Phys. Rev. 156, 207 (1967); G. Baym and C. J. Pethick, Landau Fermi liquid theory: concepts and applications, (J. Wiley and Sons, New York, 1991).
[4] R. Golub and S. K. Lamoreaux, Phys. Rep. 237, 1 (1994).
[5] M. Hayden, S. K. Lamoreaux, and R. Golub, AIP Conf. Proc. 850, 147 (2006).
[6] H. J. Maris, Rev. Mod. Phys. 49, 341 (1977).
[7] G. Baym and C. Ebner, Phys. Rev. 164, 235 (1967).
[8] The measurement is optimized when the neutron capture rate is approximately the same as the neutron decay rate—this occurs for concentrations \( x_3 \sim 3 \times 10^{-10} \).
[9] D. Benin and H. J. Maris, Phys. Rev. B 18, 3112 (1978).
[10] D. Greywall, Phys. Rev. B 23, 2152 (1981).
[11] H. B. G. Casimir, Physica 5, 495 (1938). A basic assumption here is that phonons leaving the walls of the tube are in thermal equilibrium at the local temperature of the wall. If there is significant specular reflection at the wall, the result is modified.
[12] S. K. Lamoreaux, G. Archibald, P. D. Barnes, W. T. Butler, D. J. Clark, M. D. Cooper, M. Espy, G. L. Greene, R. Golub, M. E. Hayden, C. Lei, L. J. Marek, J.-C. Peng, and S. Penttila, Europhys. Lett. 58, 718 (2002).
[13] For wall scattering alone, the diffusion constant would be simply \( D = \bar{v}_3 \lambda/3 \), where \( \lambda \) is the average mean free path, here simply \( 2R \), the pipe diameter.
[14] D. H. Beck et al. (to be published).