PROPAGATION OF STOCHASTIC TRAVELLING WAVES OF COOPERATIVE SYSTEMS WITH NOISE

HAO WEN AND JIANHUA HUANG∗
College of Liberal Arts and Science
National University of Defense Technology
Changsha 410073, China

YUHONG LI
School of Civil and Hydraulic Engineering
Huazhong University of Science and Technology
Wuhan 430074, China

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ABSTRACT. We consider the cooperative system driven by a multiplicative Itō type white noise. The existence and their approximations of the travelling wave solutions are proven. With a moderately strong noise, the travelling wave solutions are constricted by choosing a suitable marker of wavefront. Moreover, the stochastic Feynman-Kac formula, sup-solution, sub-solution and equilibrium points of the dynamical system corresponding to the stochastic cooperative system are utilized to estimate the asymptotic wave speed, which is closely related to the white noise.

1. Introduction. In the study of reaction-diffusion equation, there exists a special kind of invariant solution, called travelling wave solution (TWS), which attracts a lot of attentions in various fields [4]. It is generally believed that the traveling wave of reaction-diffusion equation was originally studied in [7, 11], which proposed the Fisher-KPP equation

\[ u_t = Du_{xx} + ku(1-u). \]  

(1)

Kolmogorov et al. [11] proved the traveling wave solutions properties for equation (1) with initial data \( u(t, x) \) as a Heaviside function, e.g. for \( x \geq 0 \), \( u(0, x) = 0 \); and for \( x < 0 \), \( u(0, x) = 1 \), that is as time \( t \) tends to infinity, the solution to (1) has the following form

\[ u(t, x) = U(x - c(t)) = U(z), \quad z = x - c(t), \]

(2)

where \( \lim_{t \to \infty} \frac{c(t)}{t} = 2\sqrt{kD} \) is the critical wave speed. In addition, the wavefront is decreasing and satisfies \( U(-\infty) = 1 \), \( U(+\infty) = 0 \). If wave speed is not smaller than

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∗ Corresponding author: Jianhua Huang.

1
the critical speed and \( c \geq 2\sqrt{kD} \), the traveling wave solutions exist and propagate along the \( x_+ \)-axis.

However, the natural systems are often interfered by external factors, which leads to certain deviations and limitations in the description of natural phenomena and objective reality in the deterministic model. Therefore, it is an inevitable trend to introduce random factors into the model. In 1994 Shiga [22] commanded that the most important property of stochastic travelling wave solution, which called supported compact property (SCP), and it perfectly illustrates the property of wavefront marker of the special solution. Hence, we know that in the one-stable system, the SCP of the trajectory between the two status (stable and unstable) will imply the existence of travelling wave solutions.

Based on this, Tribe [24] studied such following Fisher-KPP equation driven by Brownian motion

\[
\begin{align*}
    \frac{du}{dt} &= \frac{du}{dx} + \theta u - u^2 + \frac{|u|^{1/2}}{\sqrt{u(1-u)}} \dot{W}, \\
    u(0,x) &= u_0(x).
\end{align*}
\] (3)

Denote by \( R_0(t) = \sup\{x \in \mathbb{R} : u(t,x) > 0\} \) is the wavefront marker, he proposed two sufficient conditions to obtain the existence of travelling wave solutions:

- the wavefront marker \( R_0(t) \) is bounded for all \( t > 0 \);
- the solution \( u(t, \cdot + R_0(t)) \) is stationary process in time.

In 1995, M"uller [16] proved the existence of travelling solutions to Fisher-KPP equation driven by Brownian motion in another way, but different from Tribe, the strength of multiplicative noise is replaced by \( \sqrt{u(t,x)(1-u(t,x))} \)

\[
\begin{align*}
    \left\{ \begin{array}{l}
    \frac{du}{dt} = \frac{du}{dx} + \theta u - u^2 + u^{1/2}dW_t, \\
    u(0,x) = u_0(x).
\end{array} \right. 
\] (4)

In that paper, M"uller used a large deviation principle and the property of super-process to prove the length of the trajectory is bounded and the solution is stationary, which verifies the idea of Shiga. Moreover, both of Tribe and M"uller started their work with the Heaviside function as the initial data.

In 2017, Sandra [10] studied the model proposed by Tribe with non-negative and supported compactly initial data in \( C_c^+ \)

\[
\begin{align*}
    \left\{ \begin{array}{l}
    \frac{du}{dt} = \frac{du}{dx} + \theta u - u^2 + u^{1/2}dW_t, \\
    u(0,x) = u_0(x) \in C_c^+.
\end{array} \right. 
\] (5)

Under the condition that \( \theta > \theta_c \), Sandra constructed the travelling wave solution by the wavefront marker directly instead of seeking for help of new wavefront marker \( R_1(t) \) presented by Tribe. Moreover, he proved that the support of the process \( u(t) \) is recurrent.

Besides the existence of travelling waves, the propagation of travelling waves is worthy more attentions. By the stationary of travelling waves and ergodicity of system, Tribe [24] obtained the upper bound of wave speed

\[
\lim_{t \to \infty} \frac{R_0(t)}{t} = A \in [-\infty, 2\sqrt{\theta}] \text{ a.s.} 
\] (6)

In [6, 19, 20], Zhao et al. consider the following model

\[
\left\{ \begin{array}{l}
    \frac{du}{dt} = \left( \frac{1}{2} u_{xx}(t,x) + c(u(t,x))u(t,x) \right)dt + k(t)u(t,x)dW_t, \\
    u(0,x) = \chi_{(-\infty,t]}(x),
\end{array} \right. 
\] (7)
where \( c : \mathbb{R}^+ \to \mathbb{R} \) is strictly decreasing. If \( t, k \) are both deterministic, the existence of travelling waves is proven by Hamilton-Jacobi theory. Furthermore, the asymptotic behavior of the solution depends on the strength of noise. If \( \liminf_{t \to \infty} \frac{1}{t} \int_0^t k^2(s) ds > c_0 = c(0) \), the noise is called strong noise and the solution almost surely tends to 0. If the noise is weak, that is \( \int_0^t k^2(s) ds < \infty \), then the solution to (7) converges to the same travelling wave as the solution of the corresponding unperturbed deterministic KPP equation. If the noise is neither strong nor weak, we call it moderately strong noise, the solution may tend to travelling wave solution or may be vanished. Suppose that the limit \( k_\infty = \lim_{t \to \infty} \frac{1}{t} \int_0^t k^2(s) ds \) exists, thus the wavefront for large time is known as \( x = \sqrt{D(2c_0 - 2k_\infty)}t \). For any \( h > 0 \) and for \( t \) large enough, there are positive constants \( c_1, c_2 \) and \( c_3 \), if \( x > (\sqrt{D(2c_0 - 2k_\infty)} + h)t \), then

\[
\begin{align*}
  u(t, x) &< e^{-c_1t} \text{ a.s.,} \\
\end{align*}
\]

and if \( x < (\sqrt{D(2c_0 - 2k_\infty)} + h)t \), then

\[
\begin{align*}
  e^{-c_3\sqrt{2t \ln t}} &< u(t, x) < e^{c_2\sqrt{2t \ln t}} \text{ a.s.}
\end{align*}
\]

where \( c(u) \) is decreasing and satisfies \( c_0 = c(0) > 0 \). The existence of travelling waves is proven following the Feynman-Kac formula in [24] and the ergodicity of stochastic Fisher-KPP equation given by [20], and the wave speed is obtained

\[
c = \sqrt{4c_0 - 4k_\infty} \text{ a.s.,}
\]

As for equation (4), Müeller et al. [14] presented the Brunet-Derrida conjecture for its wave speed. The sup-solution and sub-solution were constructed through the corresponding deterministic equation to equation (4). Then the upper bound of the wave speed of sup-solution and the lower bound of the wave speed of sub-solution were given respectively for \( \epsilon \) small enough by the phase plane analysis approach, and

\[
\frac{\pi^2}{(\log \epsilon^2 - \log \alpha (\log \epsilon^2)^{-3} - 4)^2} \leq v_{\text{com}} \leq \frac{\pi^2}{(\log \epsilon^2 + 2)^2},
\]

where \( \alpha = \alpha (\log \epsilon^2)^{-3} \) is introduced as constructing the sub-solution. Next, the influence of probability distribution term on the wave speed tends to zero is analyzed through the relationship between the deterministic and the random items with \( \epsilon \) small enough. Furthermore, Müeller obtained the estimation of wave speed by using stopping times and iteration method

\[
\begin{align*}
  \bar{v}_\epsilon &\geq v_0 - \frac{\pi^2}{|\log \epsilon^2|} - \frac{2\pi^2 |9 \log |\log \epsilon|-\log \alpha (|\log \epsilon|^{-3})|}{|\log \epsilon^2|}, \\
  \underline{v}_\epsilon &\leq v_0 - \frac{8\pi^2 |3 \log |\log \epsilon|-\log \alpha (|\log \epsilon|^{-3})|}{|\log \epsilon^2|} + \frac{8\pi^2 |9 \log |\log \epsilon|-\log \alpha (|\log \epsilon|^{-3})|}{|\log \epsilon^2|},
\end{align*}
\]

where \( v_0 \) is the critical wave speed of deterministic problem.
Recently, Müller et al. [15] considered such a stochastic reaction-diffusion equations with strong noise
\[
\begin{align*}
\partial_t u &= \frac{1}{2} \partial_x^2 u + f(u) + \sqrt{u(1-u)}W(t,x), & t \geq 0, & x \in \mathbb{R}, \\
\ u(0,x) &= u_0(x),
\end{align*}
\] (14)
arising in population genetics. Under assumptions where \( |f(u)| \leq K \sqrt{|u(1-u)|} \), \( f \) is continuous and \( u_0(x) \) is a Heaviside function, the authors proved that there always exists a travelling wave solution when \( \sigma > 0 \). Cao and Gao [3] investigated a KPP-type lattice equation in random media
\[
\dot{u}_i(t) = u_{i+1}(t) - 2u_i(t) + u_{i-1}(t) + a(\theta(t)\omega)u_i(t)(1-u_i(t)), \quad i \in \mathbb{Z},
\] (15)
where \( \theta + t \) is an ergodic metric dynamical system on \( \Omega \), \( \omega \in \Omega \), \( (\Omega, \mathcal{F}, \mathbb{P}) \) is a probability space, \( a : \Omega \rightarrow (0,\infty) \) is measurable, and \( a(\theta(t)\omega) \) is locally Hölder continuous in \( t \in \mathbb{R} \) for every \( \omega \in \Omega \). By constructing super-solutions and sub-solutions, the authors proved the existence of random transition fronts \( u_i(t;\omega) = \Phi(i - \int_0^t c(s;\omega)ds;\theta(t)\omega) \), and showed that \( c_0 := \inf_{\mu > 0} \frac{e^{\mu t} + e^{-\mu t} - 2 + \lim_{\tau \to \infty} \inf_{\mu > 0} \int_0^\tau a(\theta(\tau+\omega))d\tau}{\mu} \) as the least mean speed.

The above mentioned papers focus on the scale equation. However in nature, we have to consider the interaction between different species, including interspecific competition, interspecific cooperation and predator-prey relationship. Li and Zhang [13] studied Lotka-Volterra type competitive system with time delay, and then the transformed cooperative system are studied. The necessary and sufficient conditions for the existence of traveling wave solutions are proved, and the estimation of wave speed is given. Hou and Li [8] studied the travelling wave solutions to a three species competition cooperation system, which is derived from a spatially averaged and temporally delayed Lotka-Volterra system, and they proved the existence of travelling solutions by the constructions of sup-solution and sub-solution, then they also estimated the wave speed. Bao and Shen [1] studied the spatial spreading speeds and travelling wave solutions of cooperative systems in space-time periodic habitats with nonlocal dispersal. These works can well depict the biological relationship in nature, but for the natural environment of the organisms, it is not invariable. The environment carrying capacity of the biosphere in which various biological populations live must be affected by various factors, such as geological disasters and human factors. Hence, we shall make the model more realistic by introducing environmental noise, and analyze the properties of stochastic cooperative system and its travelling wave solutions. Firstly we propose our model
\[
\begin{align*}
\dot{u}_t &= u_{xx} + u(1 - a_1 u + b_1 v) + \epsilon udW_t, \\
\dot{v}_t &= v_{xx} + v(1 - a_2 v + b_2 u) + \epsilon vdW_t, \\
\ u(0) = u_0, & v(0) = v_0,
\end{align*}
\] (16)
where \( W(t) \) is a white noise, \( u_0, v_0 \) are both Heaviside functions, and \( a_i, b_i \) are positive constants satisfying \( \min \{a_i\} > \max \{b_i\} \). The element “1” of \( 1 - a_1 u + b_1 v \) and \( 1 - a_2 v + b_2 u \) in equation (16) is the formal environment carrying capacity. However, the environment carrying capacity will be affected by various factors, thus we assume the carrying capacity is stochastic and given by \( 1 + \epsilon W(t) \), substituting stochastic carrying capacity into cooperative system gives the stochastic partial differential equation (16). In this paper, we always assume that \( \epsilon \) satisfies \( \epsilon^2 < 2 \),
otherwise the solution would tend to 0. The dynamic system corresponding to equation (16) is
\[
\begin{align*}
    u_t &= u(1 - a_1u + b_1v), \\
    v_t &= v(1 - a_2v + b_2u), \\
    u(0) &= u_0, v(0) &= v_0.
\end{align*}
\]

Obviously, there are four equilibrium points for equation (17), in which (0, 0) is unstable, \((\frac{1}{a_1}, 0)\) and \((0, \frac{1}{a_2})\) are saddle points, \((p_1, p_2) := \left(\frac{a_2 + b_1}{a_1a_2 - b_1b_2}, \frac{a_1 + b_2}{a_1a_2 - b_1b_2}\right)\) is the only stable point which means coexistence. Actually, the travelling wave solution to equation (16) is the trajectory connecting the two points \((0, 0)\) and \((p_1, p_2)\).

The main difficulties in studying cooperative system are dealing with coupling term \(uv\), and depicting the wavefront marker, which includes definition and wave speed estimation. Since the cooperative system is monotonic dynamic system and the solution is also monotonic, then we can use comparison method to construct sup-solution and sub-solution after proving the boundedness of solution by stochastic Feynman-Kac formula. Hence, the key to proving the existence of travelling wave solution is to verify the trajectory connecting the two states poses SCP property. Similarly, we define the wavefront marker for cooperative system as \(R_0(t) = \max\{R_0(u(t)), R_0(v(t))\}\), where \(R_0(u(t)) = \sup\{x \in R : u(t, x) > 0\}\), \(R_0(v(t)) = \sup\{x \in R : v(t, x) > 0\}\). Then by this definition, both agents of cooperative system can reach the same status at the point \(R_0(t)\). Similar to the work of Tribe, an appropriate wavefront marker \(R_0(t)\) is defined to approximate the \(R_0(t)\), thus the existence of travelling wave solution is easy to deal with.

**Theorem 1.1.** If \(u_0, v_0 \in C^+_{tem} \setminus \{0\}\) for a.e. \(\omega \in \Omega\), there exists a travelling wave solution to equation (16).

When estimating the asymptotic wave speed, we are supposed to estimate the wave speed of sup-solution equation and sub-solution equation respectively. Thus we can obtain the upper and lower bound of the wave speed of the travelling wave solution via monotonicity. Since the strength of noise is moderate, we construct sup-solution and sub-solution in the form similar to equation (7). Furthermore, with the definition of asymptotic wave speed and comparison method, we can obtain the asymptotic wave speed of travelling wave solution to (16) by the estimation of wave speed of sup-solution and sub-solution.

**Theorem 1.2.** If \(u_0, v_0 \in C^+_{tem} \setminus \{0\}\), denote by \(c^*\) the asymptotic wave speed of travelling wave solution to equation (16), then
\[
\sqrt{4 - 2\epsilon^2} \leq c^* \leq \sqrt{4p - 2\epsilon^2} \quad \text{a.s.,}
\]
where \(p = \max\{a_i\} \times \max\{\sqrt{|u_0|^2 + |v_0|^2 + \frac{\epsilon^2}{12\epsilon}} + 9 + 1, C(\epsilon, t) = \sup_{0 \leq r \leq t} \int_0^r \epsilon dW_s\}\).

2. Preliminaries and notation. Throughout this paper, we set \(\Omega\) be the space of temper distributions, \(\mathcal{F}\) be the \(\sigma\)-algebra on \(\Omega\), and \((\Omega, \mathcal{F}, \mathbb{P})\) be the white noise probability space. Denote by \(\mathbb{E}\) the expectation with respect to \(\mathbb{P}\). Denote by \(\phi_\lambda(x) = \exp(-\lambda|x|)\), here are some notations:

- \(C^+ = \{f | f : R \to [0, \infty) \text{ and } f \text{ is continuous}\}\);
- \(||f||_\lambda = \sup_{x \in R} (f(x)\phi_\lambda(x))\);
Lemma 2.1. \cite{24} A set $K \subset C_\lambda^+$ is called relatively compact if and only if
(a) $K$ is equicontinuous on a compact set;
(b) $\lim_{K \to \infty} \sup_{f \in K} \sup_{|x| \geq R} |f(x)e^{-\lambda|x|}| = 0$.

Lemma 2.2. \cite{24} $K \subset C_{tem}^+$ is (relatively) compact if and only if it is (relatively)
compact in $C_\lambda^+$ for all $\lambda > 0$.

Lemma 2.3. \cite{24}(Kolmogorov tightness criterion) For $C < \infty$, $\delta > 0$, $\mu <
\lambda, \gamma > 0$, define
\[ K(C, \delta, \gamma, \mu) = \{ f : |f(x) - f(x')| \leq C|x - x'|^\gamma e^{\mu|x|} \text{ for all } |x - x'| \leq \delta \}, \]
then with above conditions we know that $K(C, \delta, \gamma, \mu) \cap \{ f : \int_R f(x)\phi_1dx \leq a \}$ is
compact in $C_\lambda^+$, where $a$ is a constant.

(1) If $\{X_n(\cdot)\}$ are $C_\lambda$-valued processes, with $\{ \int_R X_n\phi_1dx \}$ tight and there are
$C_0 < \infty, p > 0, \gamma > 1, \mu < \lambda$ such that for all $n \geq 1, |x - y| \leq 1$,
\[ \mathbb{E}(|X_n(x) - X_n(y)|^p) \leq C_0|x - y|^\gamma e^{\mu|x|}, \]
then $\{X_n\}$ are tight.

(2) Similarly, if $\{X_n\}$ are $C([0, T], C_\lambda^+)$-valued processes, with $\{ \int_0^T X_n(0)\phi_1dx \}$
tight, and there are $C_0 < \infty, p > 0, \gamma > 2, \mu < \lambda$ such that for all $n \geq 1,|x - y| \leq 1,|t - t'| \leq 1, t, t' \in [0, T]$,
\[ \mathbb{E}(|X_n(x, t) - X_n(y, t')|^p) \leq C_0(|x - y|^\gamma + |t - t'|^\gamma)e^{\mu|x|}, \]
then $\{X_n\}$ are tight.

3. Existence of travelling wave solution. For convenience, let $Y = (u, v)^T$, $F(Y) = u(1 - a_1u + b_1v), v(1 - a_2v + b_2u))^T$, $H(Y) = (u, v)^T$, $F_1(Y) = u(1 - a_1u + b_1v), F_2(Y) = v(1 - a_2v + b_2u)$, $H_1(Y) = u$, $H_2(Y) = v$, then the stochastic
cooperative system \eqref{16} can be rewritten as
\begin{align}
\begin{cases}
Y_t = Y_{xx} + F(Y) + \epsilon H(Y) dW_t, \\
Y(0, x) = Y_0 = (u_0, v_0)^T,
\end{cases}
\end{align}
(19)
where $u_0 = p_1\chi_{(-\infty, 0]}, v_0 = p_2\chi_{(-\infty, 0]}$.

For any matrix $M = (m_{ij})_{n \times m}$, define the norm $| \cdot |$ as $|M| = \Sigma_{i, j=1} |m_{ij}|$, and
the vector norm is $||A||_\infty = \max_i (A_i)$.

Lemma 3.1. For $u_0, v_0 \in C_{tem}^+$, and a.e. $\omega \in \Omega$, there exists a unique solution
to \eqref{19} in law with the form
\begin{align}
Y(t, x) = \int_R G(t, x, y)Y_0dy \\
+ \int_0^t \int_R G(t - s, x, y)F(Y)dsdy + \epsilon \int_0^t \int_R G(t - s, x, y)H(Y)dW_sdy,
\end{align}
(20)
where $G(t, x, y)$ is Green function.

Proof. Firstly, we do some brief truncations

\[ F_n(Y^n) = Y^n - (a_1((u^n)^2 \land n), a_2((v^n)^2 \land n))^T + (b_1(u^n \land \sqrt{n})(v^n \land \sqrt{n}), b_2(u^n \land \sqrt{n})(v^n \land \sqrt{n}))^T, \]

and it is not difficult to verify that $F_n$ is Lipschitz continuous and linear growth. We will check one case and others can be obtained similarly.

If $\sqrt{u_1} < nu_1$, $\sqrt{u_2} < nu_2$, $\sqrt{v_1} < nv_1$, $\sqrt{v_2} < nv_2$, then

\[ |F_n(Y_1) - F_n(Y_2)| \leq |Y_1 - Y_2| + a_1(|u_1|^2 \land n - (u_2)^2 \land n) + a_2(|v_1|^2 \land n - (v_2)^2 \land n) \]

\[ + b_1(|u_1 \land \sqrt{n})(v_1 \land \sqrt{n}) - (u_2 \land \sqrt{n})(v_2 \land \sqrt{n})| \]

\[ + b_2(|u_1 \land \sqrt{n})(v_1 \land \sqrt{n}) - (u_2 \land \sqrt{n})(v_2 \land \sqrt{n}). \]

Consequently, we do some discussion about the relationship between $u_i$, $v_i$ and $n$ respectively, then our conclusion can be obtained via estimations of $I_1, I_2, I_3$ and $I_4$.

- If $u_i, v_i < \sqrt{n}$
  \[ I_1 = a_1|u_1^2 - u_2^2| \leq 2a_1\sqrt{n}|u_1 - u_2|, \]
  \[ I_2 \leq 2a_2\sqrt{n}|v_1 - v_2|, \]
  \[ I_3 = b_1|u_1v_1 - u_2v_2 + u_1v_2 - u_2v_2| \leq b_1\sqrt{n}(|u_1 - u_2| + |v_1 - v_2|), \]
  \[ I_4 \leq b_2\sqrt{n}(|u_1 - u_2| + |v_1 - v_2|), \]
  then we have
  \[ |F_n(Y_1) - F_n(Y_2)| \leq C(a_1, a_2, b_1, b_2, n)|Y_1 - Y_2|. \]

- If $u_i, v_i > \sqrt{n}$, it is obvious that
  \[ |F_n(Y_1) - F_n(Y_2)| = |Y_1 - Y_2|. \]

- If $u_1 < \sqrt{n} < u_2$, we obtain
  \[ I_1 = a_1|u_1^2 - n| \leq 2a_1\sqrt{n}|u_1 - u_2|. \]

Hence our analysis can be divided into four parts. If $v_1, v_2 < \sqrt{n}$, we have that

\[ I_3 = b_1|u_1v_1 - \sqrt{n}v_2| \leq b_1\sqrt{n}|v_1 - v_2| + b_1\sqrt{n}|u_1 - u_2| = b_1\sqrt{n}|Y_1 - Y_2|. \]

If $v_1, v_2 > \sqrt{n}$, then

\[ I_3 = b_1|\sqrt{n}u_1 - n| \leq b_1\sqrt{n}|u_1 - u_2|. \]

If $v_1 < \sqrt{n} < v_2$, then

\[ I_3 = b_1|u_1v_1 - n| \leq b_1\sqrt{n}|v_1 - v_2| + b_1\sqrt{n}|u_1 - u_2| = b_1\sqrt{n}|Y_1 - Y_2|. \]

Moreover, if $v_2 < \sqrt{n} < v_1$, then

\[ I_3 = b_1|\sqrt{n}u_1 - n + v_2| \leq b_1\sqrt{n}|Y_1 - Y_2|. \]
In summary, we find that
\[|F_n(Y_1) - F_n(Y_2)| \leq C(a_1, a_2, b_1, b_2, n)|Y_1 - Y_2|.
\]
As for the linear growth property, it is easy to verify by letting \(Y_2 = 0\).

Here, the truncated equation
\[
\begin{aligned}
Y^n_t &= Y^n_{xx} + F_n(Y^n) + \epsilon H(Y^n) dW_t, \\
Y^n(0) &= Y_0,
\end{aligned}
\]
has a unique solution \(\{Y^n(t)\}_{n \in \mathbb{N}}\) in law. Furthermore, refer to Theorem 2.6 in [22], we know that \(Y^n(t) \in C^+_{tem}, \forall n \in \mathbb{N}\).

Next refer to Theorem 2.2 in [24], one can show that for a.e. \(\omega \in \Omega\) there is a unique solution \(Y(t)\) to equation (16) such that \(Y^n(t)\) converges to \(Y(t)\) as \(n \to \infty\), where
\[
Y(t, x) = \int_R G(t, x, y) Y_0 dy + \int_0^t \int_R G(t - s, x, y) F(Y) ds dy + \epsilon \int_0^t \int_R G(t - s, x, y) H(Y) dW_s dy,
\]
and \(Y(t) \in C^+_{tem} \).

Furthermore, along the idea of Tribe [24], we have the following conclusion,

**Lemma 3.2** [24]. All solutions to (16) started at \(Y_0\) have the same law which we denote by \(Q^{Y_0, a_1, a_2, b_1, b_2}\), and the map \((Y_0, a_1, a_2, b_1, b_2) \to Q^{Y_0, a_1, a_2, b_1, b_2}\) is continuous. The law \(Q^{Y_0, a_1, a_2, b_1, b_2}\) for \(Y_0 \in C^+_{tem}\) form a strong Markov family.

We have obtained the solution to equation (16) with some properties, now we will do some analysis about the solution \(Y(t)\).

Firstly, refer to [24, 12, 17], the comparison methods for stochastic reaction-diffusion equations is given as following:

**Lemma 3.3.** There is a coupling solution \(Y(t, x)\) to (19) started at \(Y_0 \in C^+_{tem}\) with \(\Theta(t, x)\) a solution to
\[
\begin{aligned}
\Theta_t &= \Theta_{xx} + P(\Theta) + \epsilon H(\Theta) dW_t, \\
\Theta_0 &= Y_0,
\end{aligned}
\]
if \(P(Y) \geq F(Y)\) and \(P(Y)\) is Lipschitz continuous, then for any \(Y_0, \Theta_0 \in C^+_{tem}\).

(1) Fix \(\Theta_0^{(1)}, \Theta_0^{(2)} \in C^+_{tem}\) with \(\Theta_0^{(1)} \leq \Theta_0^{(2)}\), then for every \(t > 0, x \in R, and for a.e. \(\omega \in \Omega\)
\[\Theta^{(1)}(t, x) \leq \Theta^{(2)}(t, x)\].

(2) For every \(t > 0, x \in R, and for a.e. \(\omega \in \Omega\)
\[Y(t, x) \leq \Theta(t, x)\].

Next, we do some estimation about \(Y(t, x)\), which is of great importance for our further research.

**Theorem 3.4.** For any \(u_0, v_0 \in C^+_{tem} \setminus \{0\}\), and \(t > 0\) fixed, a.e. \(\omega \in \Omega\), it permits that
\[
\mathbb{E}[u(t, x) + v(t, x)] \leq C(\epsilon, t)(u_0 + v_0 + \frac{2}{c} - \frac{\epsilon^2}{c}), \forall x \in R,
\]
where \(C(\epsilon, t)\) is a constant, \(c = \min\{a_i\} - \max\{b_i\}\).
Proof. Back to equation (16)

\[
\begin{aligned}
&u_t = u_{xx} + u(1 - a_1 u + b_1 v) + \epsilon u dW_t, \\
&v_t = v_{xx} + v(1 - a_2 v + b_2 u) + \epsilon v dW_t, \\
&w(0) = u_0, v(0) = v_0,
\end{aligned}
\]

(25)
denote by \(\phi(t, x) = u(t, x) + v(t, x)\), we have

\[
\begin{aligned}
&\phi_t = \phi_{xx} + u(1 - a_1 u + b_1 v) + v(1 - a_2 v + b_2 u) + \epsilon \phi dW_t, \\
&\phi(0, x) = \phi_0 = u_0 + v_0,
\end{aligned}
\]

(26)
since \(\min\{a_i\} > \max\{b_i\}\), then

\[
\begin{aligned}
&u(1 - a_1 u + b_1 v) + v(1 - a_2 v + b_2 u) \\
&\leq u + v - \frac{c}{2}(u + v)^2 \\
&= (u + v)(1 - \frac{c}{2}(u + v)).
\end{aligned}
\]

Let \(\psi\) be the solution of the following equation

\[
\begin{aligned}
&\psi_t = \psi_{xx} + \psi(1 - \frac{c}{2}\psi) + \epsilon \psi dW_t, \\
&\psi_0 = u_0 + v_0,
\end{aligned}
\]

(27)
then, \(u(t, x) \leq \psi(t, x)\) a.s. and \(v(t, x) \leq \psi(t, x)\) a.s.

Let \(\zeta\) be a solution to the following equation

\[
\begin{aligned}
&\zeta_t = \zeta_{xx} + \zeta(1 - \frac{c}{2}\zeta) - \frac{\epsilon^2}{2} \zeta, \\
&\zeta_0 = \psi_0,
\end{aligned}
\]

(28)
thus we assert that for every \((t, x) \in [0, \infty) \times R\), we have

\[
\exp(\inf_{0 \leq r \leq t} \int_r^t \epsilon dW_s) \zeta(t, x) \leq \psi(t, x) \leq \exp(\sup_{0 \leq r \leq t} \int_r^t \epsilon dW_s) \zeta(t, x)\text{ a.s.}
\]

(29)
To obtain a contradiction, it can be supposed that there is \((t_0, x_0) \in [0, \infty) \times R\), such that

\[
\psi(t_0, x_0) > \exp(\sup_{0 \leq r \leq t_0} \int_r^{t_0} \epsilon dW_s) \zeta(t_0, x_0),
\]

(30)
then

\[
\psi(t_0, x_0) > \zeta(t_0, x_0).
\]

Construct a new probability space \((\hat{\Omega}, \hat{\mathcal{F}}, \hat{\mathbb{P}})\), let \(\hat{W} = (\hat{W}(t) : t \geq 0)\) be a Brownian motion over the new probability space. Let \(X_{s_0, x_0}^{t_0} = (t_0 - s, x_0 + \sqrt{2}\hat{W}(s)), s > 0\), and define a stoping time

\[
\tau = \inf\{s > 0 : \zeta(X_{s_0, x_0}^{t_0}) = \psi(X_{s_0, x_0}^{t_0})\},
\]
for each $\omega \in \hat{\Omega}$. Using stochastic Feynman-Kac formula and by the strong Markov property, we have

$$
\psi(t_0, x_0) = \hat{E}[\psi(X_{t_0}^{t_0, x_0})] \leq \hat{E}[\zeta(X_{t_0}^{t_0, x_0})] = \exp(\sup_{0 \leq r \leq t_0} \int_{t_0-r}^{t_0} \epsilon dW_s) \zeta(t_0, x_0) a.s.,
$$

which contradicts (30) and the upper bound is proved.

Similarly we have

$$
\psi(t_0, x_0) \geq \exp(\inf_{0 \leq r \leq t_0} \int_{t_0-r}^{t_0} \epsilon dW_s) \zeta(t_0, x_0) a.s.
$$

For arbitrary $t > 0$ fixed, for any $\sigma > 0$, multiplying $G(t - s + \sigma, x - y)$ in (28) and integrating over $R$, we obtain

$$
\frac{\partial}{\partial s} \int_R \zeta(s, y)G(t - s + \sigma, x - y)dy = (1 - \frac{\epsilon^2}{2}) \int_R \zeta(s, y)G(t - s + \sigma, x - y)dy - \frac{c}{2} \int_R \zeta^2(s, y)G(t - s + \sigma, x - y)dy \\
\leq (1 - \frac{\epsilon^2}{2}) \int_R \zeta(s, y)G(t - s + \sigma, x - y)dy - \frac{c}{2} (\int_R \zeta(s, y)G(t - s + \sigma, x - y)dy)^2.
$$

Let $\varphi(s) = \int_R \zeta(s, y)G(t - s + \sigma, x - y)dy$, thus we get

$$
\begin{cases}
\frac{d\varphi(s)}{ds} \leq (1 - \frac{\epsilon^2}{2}) \varphi(s) - \frac{c}{2} \varphi^2(s), \\
\varphi_0 = \int_R \zeta_0 G(t + \sigma, x - y)dy.
\end{cases}
$$

In general, we have

$$
\varphi(s) \leq \varphi_0 + \frac{2}{c} - \frac{\epsilon^2}{c},
$$

which implies

$$
\int_R \zeta(t, y)G(\sigma, x - y)dy \leq \int_R \zeta_0 G(t + \sigma, x - y)dy + \frac{2}{c} - \frac{\epsilon^2}{c}.
$$

Let $\sigma \to 0$, then

$$
\zeta(t, x) \leq \int_R \zeta_0 G(t, x - y)dy + \frac{2}{c} - \frac{\epsilon^2}{c} a.s.
$$

Hence, combining with (29), we obtain

$$
u(t, x) + v(t, x) \leq \psi(t, x) \leq \exp(\sup_{0 \leq r \leq t} \int_r^t \epsilon dW_s) \\
\times (\int_R \zeta_0 G(t, x - y)dy + \frac{2}{c} - \frac{\epsilon^2}{c}) a.s.
$$
With the initial data \( u_0 = p_1\chi(-\infty, 0); \ v_0 = p_2\chi(-\infty, 0) \), taking expectation, then we get
\[
E[u(t, x) + v(t, x)] \leq C(\epsilon, t)(u_0 + v_0 + \frac{2}{c} - \frac{\epsilon^2}{c}), \tag{36}
\]
where \( C(\epsilon, t) = E[\exp(\sup_{0 \leq r \leq t} \int_{r}^{t} \epsilon dW_s)] \).

**Theorem 3.5.** For any \( u_0, v_0 \in C^\infty_{tem} \setminus \{0\} \), a.e. \( \omega \in \Omega \) and any \( t > 0 \), then \( Y(t) \) satisfies
\[
E[|u(t)|^2 + |v(t)|^2] \leq E[|u_0|^2 + |v_0|^2]e^{-t} + K(1 - e^{-t}), \tag{37}
\]
where \( K > 0 \) is a constant.

**Proof.** For convenience, let \( V(t) := |u(t)|^2 + |v(t)|^2 \). Applying Itô formula and we obtain
\[
dV(t) = 2\langle u, u_{xx} \rangle dt + 2\langle v, v_{xx} \rangle dt + 2\langle u, u - a_1u^2 + b_1uv \rangle dt \\
+ 2\langle v, v - a_2v^2 + b_2uv \rangle dt + \epsilon^2 |u^2 + v^2| dt + 2\epsilon[u^2 + v^2] dW_t.
\]
Integrating both sides in \([0, t]\) and taking expectation, then
\[
E[V(t)] = E[|u_0|^2 + |v_0|^2] + 2E \int_0^t \langle u, u_{xx} \rangle ds + 2E \int_0^t \langle v, v_{xx} \rangle ds + \epsilon^2 E \int_0^t (u^2 + v^2) ds \\
+ 2E \int_0^t \langle u, u - a_1u^2 + b_1uv \rangle ds + 2E \int_0^t \langle v, v - a_2v^2 + b_2uv \rangle ds \\
\leq E[|u_0|^2 + |v_0|^2] - 2E \int_0^t |\nabla u|^2 ds - 2E \int_0^t |\nabla v|^2 ds + \epsilon^2 E \int_0^t (u^2 + v^2) ds \\
+ 2E \int_0^t (u^2 + v^2) ds - 2\epsilon E \int_0^t (u^3 + v^3) ds \\
\leq E[|u_0|^2 + |v_0|^2] - 2\epsilon E \int_0^t (u^3 + v^3) ds + 2E \int_0^t (u^2 + v^2) ds \\
+ \epsilon^2 E \int_0^t (u^2 + v^2) ds + E \int_0^t (u^2 + v^2) ds - E \int_0^t (u^2 + v^2) ds.
\]
Thus with Young inequality, it can be obtained that
\[
3E \int_0^t (u^2 + v^2) ds \leq E \int_0^t \frac{3}{2} \times \frac{2c}{3} u^3 + \frac{3}{2} \times \frac{2c}{3} v^3 ds + 9t
\]
\[
= \epsilon E \int_0^t (u^3 + v^3) ds + 9t, \tag{38}
\]
\[
\epsilon^2 E \int_0^t (u + v) ds \leq E \int_0^t \frac{\epsilon^2}{2} \frac{2c}{c^2} u^3 + \frac{\epsilon^2}{2} \frac{2c}{c^2} v^3 ds + \frac{\epsilon^6}{12c^2} t
\]
\[
= \epsilon E \int_0^t (u^3 + v^3) ds + \frac{\epsilon^6}{12c^2} t. \tag{39}
\]
Combining (38) with (39), we get
\[
E[|u(t)|^2 + |v(t)|^2] \leq E[|u_0|^2 + |v_0|^2] + \left( \frac{\epsilon^6}{12c^2} + 9 \right) t - E \int_0^t (u^2 + v^2) ds.
\]
Furthermore, we have
\[ E[|u(t)|^2 + |v(t)|^2] \leq E[|u_0|^2 + |v_0|^2]e^{-t} + \left( \frac{e^6}{12c^2} + 9 \right)(1 - e^{-t}). \]  
(40)

Studying the boundedness of \( Y(t,x) \) is crucial for proving the boundedness of wavefront marker, and estimating the wave speed through sup-solution and sub-solution. Based on this, by improving Lemma 2.1 in [24], we can estimate how fast the support of \( Y(t,x) \) can spread.

**Lemma 3.6.** Let \( Y(t,x) \) be a solution to (19) started at \( Y_0 \), suppose for some \( R > 0 \) that \( Y_0 \) is supported outside \((-R - 2, R + 2)\), then for any \( t \geq 1 \),
\[ \mathbb{P} \left( \int_0^t \int_{-R}^R |Y(s,x)|_\infty dsdx > 0 \right) \leq C e^t \int \frac{\sqrt{t}}{|x| - (R + 1)} \exp \left( -\frac{(|x| - (R + 1))^2}{2t} \right) ||Y_0||_\infty dx. \]  
(41)

**Proof.** From Theorem 3.4 and Theorem 3.5 we know the solution \( Y(t,x) \) is uniformly bounded, thus the sup-solution solves
\[
\begin{align*}
    u_t^* &= u_{xx}^* + u^*(k - a_1 u^*) + \epsilon u^* dW_t, \\
    v_t^* &= v_{xx}^* + v^*(k - a_2 v^*) + \epsilon v^* dW_t, \\
    u(0) &= u_0, v(0) &= v_0,
\end{align*}
\]  
(42)

where \( k > 0 \) is a constant satisfying \( F_1(Y) \leq u(k - a_1 u) \) and \( F_2(Y) \leq v(k - a_2 v) \). Refer to [24, 5], the proof can be completed.

**Remark 1.** When \( R_0(t) \) is defined as a wavefront marker, the SCP of \( Y(t,x) \) can not hold. And we can not ensure the translational invariance of the solution \( Y(t,x) \) with respect to \( R_0(t) \). However thanks to Lemma 3.6, we can choose a suitable wavefront marker to ensure the SCP of \( Y(t) \) holds.

We will verify that \( Y(t,x) \) satisfy Kolmogorov tightness criterion, and \( Y(t,x) \in K(C, \delta, \mu, \gamma) \), which helps constructing a probability measure sequence, which is convergent.

**Lemma 3.7.** For any \( u_0,v_0 \in C_{tem}^+ \setminus \{0\} \), \( t > 0 \), fixed \( p \geq 2 \) and a.e. \( \omega \in \Omega \), if \( |x - x'| \leq 1 \), there exists positive constant \( C(t) \), such that
\[ Q^{1/6}(|Y(t,x) - Y(t,x')|^p) \leq C(t)|x - x'|^{p/2 - 1}. \]

**Proof.** Since the solution \( Y(t,x) \) can be expressed as
\[
Y(t,x) = \int_R G(t,x - y)Y_0 dy \\
+ \int_0^t \int_R G(t-s,x,y)F(Y)dsdy + \epsilon \int_0^t \int_R G(t-s,x,y)H(Y)dW_s dy.
\]
Through directly calculating, we get
\[
|Y(t, x) - Y(t, x')|^p 
\leq 3^{p-1}| \int_R (G(t, x - y) - G(t, x' - y))u_0dy|^p 
+ 3^{p-1}| \int_R (G(t, x - y) - G(t, x' - y))v_0dy|^p 
+ 3^{p-1}| \int_R \int_0^t (G(t-s, x - y) - G(t-s, x' - y))F_1(Y)dsdy|^p 
+ 3^{p-1}| \int_R \int_0^t (G(t-s, x - y) - G(t-s, x' - y))F_2(Y)dsdy|^p 
+ 3^{p-1}| \int_R \int_0^t (G(t-s, x - y) - G(t-s, x' - y))H_1(Y)dW_sdy|^p 
+ 3^{p-1}| \int_R \int_0^t (G(t-s, x - y) - G(t-s, x' - y))H_2(Y)dW_sdy|^p.
\]

Refering to Shiga [22] Lemma 6.2, as
\[
\int_0^t \int_R (G(t-s, x - y) - G(t-s, x' - y))^2dsdy \leq C(t)|x - x'|,
\]
for III, with Theorem 3.5, we obtain
\[
E[III] \leq C(p)e^pE(\int_R \int_0^t (G(t-s, x - y) - G(t-s, x' - y))^2dsdy)^{p/2-1} 
\times (\int_0^t \int_R (G(t-s, x - y) - G(t-s, x' - y))^2u^pdsdy) 
\leq C_1(p, t)|x - x'|^{p/2-1}.
\]
Similarly, for IV we have
\[
E[IV] \leq C_2(p, t)|x - x'|^{p/2-1}.
\]

For I, with Hölder inequality we get
\[
E[I] = 3^{p-1}E| \int_0^t \int_R (G(t-s, x - y) - G(t-s, x' - y))(u - a_1u^2 + b_1uv)dsdy|^p 
\leq 3^{p-1}(| \int_0^t \int_R (G(t-s, x - y) - G(t-s, x' - y))^2dsdy)^{p/2-1} 
\times (\int_0^t \int_R (G(t-s, x - y) - G(t-s, x' - y))^2dsdy)^{p/2-1} 
\times (\int_0^t \int_R (G(t-s, x - y) - G(t-s, x' - y))^2dsdy) 
\leq C_3(p, t)|x - x'|^{p/2-1}.
\]
Similarly
\[
E[II] \leq C_4(p, t)|x - x'|^{p/2-1}.
\]
For the rest terms, we have
\[
\mathbb{E}\left| \int_R (G(t, x - y) - G(t, x' - y)) u_0 dy \right|^p
\]
\[
= \mathbb{E}\left| \int_R \int_{x'}^x \frac{(y - r)}{2t\sqrt{4\pi y}} \exp\left(-\frac{(y - r)^2}{4t}\right) u_0 dr dy \right|^p
\]
\[
\leq K(t) (\int_R \int_{x'}^x \frac{1}{\sqrt{t}} \exp\left(-\frac{(y - r)^2}{5t}\right) u_0 dr dy)^p
\]
\[
\leq K(t) |x - x'|^p (\int_R \frac{1}{\sqrt{t}} \exp\left(-\frac{(y - x)^2}{5t}\right) u_0 dy)
\]
\[
\leq C_5 (p, t) |x - x'|^{p/2 - 1}, \quad (\text{since } |x - x'| \leq 1),
\]
and
\[
\mathbb{E}\left| \int_R (G(t, x - y) - G(t, x' - y)) v_0 dy \right|^2 \leq C_6 (p, t) |x - x'|^{p/2 - 1}.
\]

In summary, we complete the proof with above inequalities, that is
\[
\mathbb{E}[|Y(t, x) - Y(t, x')|^p] \leq C(p, t) |x - x'|^{p/2 - 1}. \quad (43)
\]

**Remark 2.** Lemma 3.7 verifies that \(Y(t, x) \in K(C, \delta, \mu, \gamma)\), and \(Y(t, x)\) satisfies Kolmogrov tightness criterion. Thus, we can start to construct a travelling wave solution.

Define \(Q_Y^{\nu}\) as the law of the unique solution to equation (19) with initial data \(Y(0) = Y_0\). For a probability measure \(\nu\) on \(C_{tem}^+\), we define
\[
Q^\nu(A) = \int_{C_{tem}^+} Q_Y^{\nu}(A)d\nu(Y_0).
\]

In order to construct a travelling wave solution to equation (16), we must ensure that the translation of solution with respect to a wavefront marker is stationary and the solution poses SCP property. However, \(R_0(Y(t))\) does not meet this demand. So we have to choose a new suitable wavefront marker. As the solution to (19) with Heaviside initial condition is exponentially small almost surly as \(x \to \infty\), with the stochastic Feynmac-Kac formula we may turn to \(R_1(t) : C_{tem}^+ \to [-\infty, \infty]\) defined as
\[
R_1(f) = \ln \int_R e^x f dx,
\]
\[
R_1(u(t)) = \ln \int_R e^x u(t) dx,
\]
and
\[
R_1(t) := R_1(Y(t)) = \max\{R_1(u(t)), R_1(v(t))\}.
\]

The marker \(R_1(t)\) is an approximation to \(R_0(Y(t)) = \max\{R_0(u(t)), R_0(v(t))\}\).

Let \(Z(t) = Y(t, \cdot + R_1(t)) = (Z_1(t), Z_2(t))^T, Z_0(t) = Y(t, \cdot + R_0(Y(t))),\) and define
\[
Z(t) = \begin{cases} (0, 0)^T, & R_1(t) = -\infty, \\ (u(t, \cdot + R_1(t)), v(t, \cdot + R_1(t)))^T, & -\infty < R_1(t) < \infty, \\ \left(\frac{a_2 + b_1}{a_1 a_2 - b_1 b_2}, \frac{a_1 + b_2}{a_1 a_2 - b_1 b_2}\right)^T, & R_1(t) = \infty. \end{cases}
\]
Hence $Z(t)$ is the wave shifted so that the wavefront marker $R_1(t)$ lies at the origin. Note that whenever $R_0(Y_0) < \infty$, the compact support property in Lemma 3.6 implies that $R_0(t) < \infty$, $\forall t > 0$, $Q^{Y_0}$-a.s.

**Remark 3.** Here we define $R_1(t)$ in the maximum form, not only since it simplifies the discussion about boundedness, but also because the asymptotic wave speed is the minimum wave speed which keeps the travelling wave solution monotonic. As mentioned before, we calculate asymptotic wave speed via $c = \lim_{t \to \infty} \frac{R_1(t)}{t}$.

Therefore, the wavefront marker $R_1(t)$ defined in such form can ensure the traveling wave solutions of the two subsystems monotonic.

Next, define

$$\nu_T = \text{the law of } \frac{1}{T} \int_0^T Z(s) ds \text{ under } Q^{Y_0}.$$  

Now we summarise the method for constructing travelling wave solution. With the initial data $(u_0 = p_1 \chi_{(-\infty,0]}, v_0 = p_2 \chi_{(-\infty,0])} \in C_{tem}^+$ as Heaviside function, we shall show that the sequence $\{\nu_T\}_{T \in \mathbb{N}}$ is tight (see Lemma 3.9) and any limit point is nontrivial (see Theorem 3.10). Hence for any limit point $\nu$ (the limit is not unique), $Q^\nu$ is the law of a travelling wave solution. Two parts constituting the proof of tightness are Kolmogorov tightness criterion for the unshifted waves (see Lemma 3.7) and the control on the movement of the wavefront marker $R_1(t)$ ensuring the shifting will not destroy the tightness (see Lemma 3.8).

Firstly, we complete preparations to prove the tightness of the sequence $\{\nu_T\}_{T \in \mathbb{N}}$.

**Lemma 3.8.** For any $u_0, v_0 \in C_{tem}^+ \setminus \{0\}$, $t \geq 0$, $d > 0$, $T \geq 1$, and a.e. $\omega \in \Omega$ there exists a positive constant $C(t) < \infty$, such that

$$\mathbb{P}(|R_1(t)| > d) \leq \frac{C(t)}{d}. \quad (44)$$

**Proof.** By the comparison method, we can construct a sup-solution satisfying

$$\begin{align*}
\tilde{u}_t &= \tilde{u}_{xx} + k_0 \tilde{u} + c\tilde{u} dW_t, \\
\tilde{v}_t &= \tilde{v}_{xx} + k_0 \tilde{v} + c\tilde{v} dW_t, \\
\tilde{u}_0 &= u_0, \tilde{v}_0 = v_0,
\end{align*} \quad (45)$$

where $k_0 > 0$ is a constant which can be obtained by Theorem 3.4 and Theorem 3.5 such that $F_1(Y) < k_0 u$, $F_2(Y) < k_0 v$. Therefore, we know that $u(t) \leq \tilde{u}(t)$ and $v(t) \leq \tilde{v}(t)$ hold on $[0, T]$ uniformly, and for a.e. $\omega \in \Omega$ the solution $\tilde{Y}(t, x)$ to equation (45) is

$$\tilde{Y}(t, x) = \int_R e^{k_0 t} G(t, x-y) Y_0(y) dy + \epsilon \int_R \int_0^t G(t-s, x-y) H(\tilde{Y}) dW_s dy. \quad (46)$$

Applying the comparison method, we obtain

$$Q^{u_0}(\int_R u(t, x) e^x dx) \leq \mathbb{E}\left[ \int_R \tilde{u}(t, x) e^\tilde{x} dx \right]$$

$$= \mathbb{E}\left[ \int_R \int_0^t e^{k_0} G(t, x-y) u_0(y) dy e^{y} dx \right]$$

$$= e^{k_0 t+t} \int_R u_0(x) e^x dx.$$
Similarly we have
\[ Q^{u_0}(\int_R v(t,x)e^{\xi}dx) \leq e^{k_0 t + t} \int_R v_0(x)e^{\xi}dx. \]

Without generality, we assume that \( R_1(t) = R_1(u(t)) \), then it permits that
\[ \int_R u(t,x + R_1(t))e^{\xi}dx = e^{-R_1(t)} \int_R u(t,x)e^{\xi}dx = 1. \]

On the other hand, we have
\[ \int_R v(t,x + R_1(t))e^{\xi}dx \leq 1. \]

Associating with inequalities above, we have
\[
Q^{u_0}(R_1(t) \geq d) = \frac{1}{T} \int_0^T Q^{u_0}(Q^{u(s)}(R_1(t) \geq d))ds \\
= \frac{1}{T} \int_0^T Q^{u_0}(Q^{u(s)}(e^{-d} \int_R u(t,x)e^{\xi}dx \geq 1))ds \\
\leq e^{-d} \frac{1}{T} \int_0^T Q^{u_0}(Q^{u(s)}(\int_R u(t,x)e^{\xi}dx))ds \\
\leq e^{-d} e^{k_0 t + t} \frac{1}{T} \int_0^T \int_R u(s,x + R_1(s))e^{\xi}dxds \\
= e^{-d} e^{k_0 t + t}.
\]

Next, Jensen’s inequality gives
\[
Q^{u_0}(R_1(t)) \leq \ln(e^{k_0 t + t} \int_R u_0(x)e^{\xi}dx) \\
\leq k_0 t + t + R_1(u_0),
\]
in addition, we have such estimation
\[
\frac{1}{T} Q^{u_0}(\int_0^{T+t} R_1(s)ds - \int_0^T R_1(s)ds) \\
= \frac{1}{T} Q^{u_0}(\int_0^T R_1(t + s) - R_1(s)ds) \\
= \frac{1}{T} \int_0^T \int_{\{R_1(t+s) - R_1(s) > -d\}} (R_1(t+s) - R_1(s))Q^{u_0}(du)ds \\
+ \frac{1}{T} \int_0^T \int_{\{R_1(t+s) - R_1(s) \leq -d\}} (R_1(t+s) - R_1(s))Q^{u_0}(du)ds \\
\leq \frac{1}{T} \int_0^T \int_{\{R_1(t+s) - R_1(s) > 0\}} (R_1(t+s) - R_1(s))Q^{u_0}(du)ds \\
- \frac{d}{T} \int_0^T Q^{u_0}(R_1(t + s) - R_1(s) \leq -d)ds \\
\leq \frac{1}{T} \int_0^T \int_0^\infty Q^{u_0}(R_1(t + s) - R_1(s) \geq y)dyds.
\]
Remark 4. Every subsequential limit \( \nu \) of the tight sequence \( \{ \nu_T : T \in \mathbb{N} \} \) yields the law \( \mathbb{P}_\nu \) of the travelling wave solution, but the lack of uniqueness of \( \nu \) leads to the lack of uniqueness of travelling wave solution.

\[
- \frac{d}{T} \int_0^T Q^{u_0}(R_1(t+s) - R_1(s)) \leq -d ds
= \int_0^\infty Q^{\nu_T}(R_1(t) \geq y) dy - dQ^{\nu_T}(R_1(t) \leq -d).
\]

Rearranging the inequalities gives
\[
Q^{\nu_T}(R_1(t) \leq -d) \leq \frac{1}{d} \int_0^\infty Q^{\nu_T}(R_1(t) \geq y) dy + \frac{1}{dT} \int_0^T Q^{\nu_T}(R_1(s)) ds
- \frac{1}{d} \int_y^{T+t} Q^{u_0}(R_1(s)) ds
\leq \frac{1}{d} \int_0^\infty e^{-y+k_0 t} dy + \frac{1}{dT} \int_0^t k_0 s + R_1(u_0) ds
\leq \frac{C(t)}{d}.
\]

Now we can say the marker \( R_1(t) \) is bounded, which helps proving that the sequence \( \{ \nu_T : T \in \mathbb{N} \} \) is tight and wavefront marker \( R_0(t) \) is bounded. Then, we will show the tightness of \( \{ \nu_T : T \in \mathbb{N} \} \) with \( Y(t, x) \in K(C, \delta, \mu, \gamma) \).

Lemma 3.9. For any \( u_0, v_0 \in C^+_{tem} \setminus \{0\} \), and a.e. \( \omega \in \Omega \), the sequence \( \{ \nu_T : T \in \mathbb{N} \} \) is tight.

Proof. Similar to Theorem 3.8, we discuss with \( u(t, x) \). According to Lemma 3.7, \( Y(t, x) \in K(C, \delta, \mu, \gamma) \) gives \( u(t, x) \in K(C, \delta, \mu, \gamma) \), then it can be obtained
\[
\nu_T(K(C, \delta, \mu, \gamma)) = \frac{1}{T} \int_0^T Q^{u_0}(u(t, \cdot + R_1(t)) \in K(C, \delta, \mu, \gamma)) ds
\geq \frac{1}{T} \int_0^T Q^{u_0}((u(t, \cdot + R_1(t-1)) \in K(Ce^{-\mu d}, \delta, \gamma, \mu))
\times |R_1(t) - R_1(t-1)| \leq d) ds
\geq \frac{1}{T} \int_1^T Q^{u_0}(Q^{Z_1(t-1)}(u(1) \in K(Ce^{-\mu d}, \delta, \gamma, \mu))) dt
- \frac{1}{T} \int_1^T Q^{u_0}(|R_1(t) - R_1(t-1)| \geq d) dt
:= I - II.
\]

With Lemma 3.8, \( II \to 0 \) as \( d \to \infty \). Via Kolmogorov tightness and Lemma 3.7, for given \( d, \mu > 0 \), one can choose \( C, \delta, \gamma \) to make \( I \) as close to \( \frac{T-1}{T} \) as desired. In addition, we have
\[
\nu_T\{u_0 : \int_R u_0(x)e^{-|x|} dx \leq \int_R u_0(x)e^x dx = 1\} = 1,
\]
by the definition of tightness, for given \( \mu > 0 \), one can choose \( C, \delta, \gamma \) such that \( \nu_T(K(C, \delta, \mu, \gamma) \cap \{u_0 : \int_R u_0(x)e^{-|x|} dx\}) \) as close to 1 as desired for \( T \) and \( d \) sufficient large, which implies that the sequence \( \{ \nu_T : T \in \mathbb{N} \} \) is tight.

\[\square\]
Theorem 3.10. For any \( u_0, v_0 \in C^+_\text{tem} \setminus \{0\} \), and a.e. \( \omega \in \Omega \), there is a travelling wave solution to equation (16), and \( Q^p(x) \) is the law of travelling wave solution.

Proof. Denote by \( (f, g) = \int_R fg dx \). Firstly, taking a subsequence \( \{\nu_n\} \) converging to \( \nu \), then we choose \( g(x) \in C^+_\text{tem} \) satisfying \( \int_R g(x)e^xdx = p_1 \). Choose \( g_1(x), g_2(x) \in C^+_\text{tem} \) with \( g = g_1 + g_2, (g_1, I_{d/3, \infty}) = 0 \) and \( (g_2, I_{-\infty, 2d/3}) = 0 \). Take \( g_1, g_2 \) independent solutions to (45) with respect to \( \tilde{u} \) started at \( g_1, g_2 \), then with the comparison method shows that \( g \leq g_1 + g_2 \) is a solution to (45) with respect to \( \tilde{u} \) started at \( g \). Applying Lemma 3.6 and taking large \( d \), we have

\[
Q^p((u(t,x), I_{d,\infty}) > 0) \leq \mathbb{P}((g_1(t,x), I_{d,\infty}) > 0) + \mathbb{P}((g_2(t,x), 1) > 0) \\
\leq C(k_0, t)e^{-d/3}.
\]

Taking \( h(x) \) with \( \int_R h(x)e^xdx = p_2 \), we also have

\[
Q^p((v(t,x), I_{d,\infty}) > 0) \leq C(k_0, t)e^{-d/3}.
\]

So

\[
\nu_T(u_0 : (u_0, I_{2d,\infty}) = 0) = \frac{1}{T} \int_0^T Q^{\nu_0}((Z_1(t), I_{2d,\infty}) = 0)\,dt \\
\geq \frac{1}{T} \int_0^T Q^{\nu_0}((u(t), I_{d+R_1(t-1),\infty}) = 0, \\
- |R_1(t) - R_1(t-1)| \leq d)\,dt \\
\geq \frac{1}{T} \int_1^T Q^{\nu_0}(Q^\nu_0(t = 1)((u(1), I_{d,\infty}) = 0)\,dt \\
- Q^{\nu_T}(|R_1(1) \geq d) \\
\geq \frac{T - 1}{T} - \frac{C(1)}{d}.
\]

By Lemma 3.8 we have

\[
\lim_{T \to \infty, d \to \infty} \nu_T(u_0 : (u_0, I_{2d,\infty}) = 0) = 1. \tag{47}
\]

Similarly, we have

\[
\lim_{T \to \infty, d \to \infty} \nu_T(v_0 : (v_0, I_{2d,\infty}) = 0) = 1. \tag{48}
\]

In order to prove the boundedness of \( R_0(t) \), from \( \nu_{T_n}(u_0 : (u_0, e^x) = p_1) = 1 \), we get

\[
\nu(u_0 : (u_0, e^x) \leq p_1 = 1.
\]

Taking \( e_1^d(x) = \exp(d - |x - d|) \), then

\[
\nu(u_0 : (u_0, e^x) \geq p_1) \geq \nu(u_0 : (u_0, e^d_1) \geq p_1) \\
\geq \lim_{n \to \infty} \sup_{\nu_{T_n}}(u_0 : (u_0, e^d_1) = p_1) \\
= \lim_{n \to \infty} \sup_{\nu_{T_n}}(u_0 : (u_0, I_{d,\infty}) = 0) \to 1, \text{ as } d \to \infty.
\]

Since \( \nu(u_0 : (u_0, e^x) \equiv p_1) = 1 \), we obtain \( \nu(u_0 : R_0(u_0) > -\infty) = 1 \).

Similarly, we have \( \nu(v_0 : R_0(v_0) > -\infty) = 1 \).
Now, we move to prove the boundedness of wavefront marker $R_0(t)$. Taking $\psi_d \in \Phi$ with $(\psi_d > 0) = (d, \infty)$, then

$$\nu(u_0 : R_0(u_0) \leq d) = \nu(u_0 : (u_0, \psi_d) = 0)$$

$$\geq \limsup_{n \to \infty} \nu_{\mathcal{T}_n}(u_0 : (u_0, \psi_d) = 0)$$

$$= \limsup_{n \to \infty} \nu_{\mathcal{T}_n}(u_0 : (u_0, I_{(d, \infty)}) = 0) \to 1 \text{ as } d \to \infty.$$ 

So we have $\nu(Y_0 : -\infty < R_0(Y_0) < \infty) = 1$ and complete the proof of boundedness of wavefront marker $R_0(t)$.

To verify that the solution $Y(t)$ is nontrivial, taking $R_1^d(t) = \ln \int ||Y(t)||_\infty e^d_tdx$, we have

$$Q^\nu(\exists s \leq t, |Y(s)| = 0) \leq Q^\nu(R_1^d(t) < -d)$$

$$\leq \limsup_{n \to \infty} Q^{\nu_{\mathcal{T}_n}}(R_1^d(t) < -d)$$

$$\leq \limsup_{n \to \infty} (Q^{\nu_{\mathcal{T}_n}}(R_1(t) < -d) + Q^{\nu_{\mathcal{T}_n}}((u(t), I_{(d, \infty)}) > 0))$$

$$\leq C(T) \to 0 \text{ as } d \to \infty.$$

We now show that $Z(t)$ is a stationary process and $Q^\nu$ is the law of a travelling wave solution to (16). Let $F : C_{tem}^+ \to R$ be bounded and continuous, and take $u(t, x)$ for example, for any $t > 0$ fixed

$$|Q^{\nu_{\mathcal{T}_n}}(F(Z_1(t))) - Q^\nu(F(Z_1(t)))|$$

$$\leq |Q^{\nu_{\mathcal{T}_n}}(F(u(t, \cdot + R_1^d(t)))) - Q^\nu(F(u(t, \cdot + R_1^d(t))))|$$

$$+ \sup_{x \in R} |F(u_0)||Q^{\nu_{\mathcal{T}_n}}(R_1(t) \neq R_1^d(t)) + Q^\nu(R_1(t) \neq R_1^d(t))|.$$ 

Since $\nu_{\mathcal{T}_n}(u_0 : (u_0, e^x) = p_1) = 1$, we have

$$Q^{\nu_{\mathcal{T}_n}}(R_1(t) \neq R_1^d(t)) \leq Q^{\nu_{\mathcal{T}_n}}((u(t), I_{(d, \infty)}) > 0) \leq C(k_0, t)/d. \quad (49)$$

With $\nu(u_0 : (u_0, e^x) = p_1) = 1$, we have

$$Q^\nu(R_1(t) \neq R_1^d(t)) \leq Q^\nu((u(t), I_{(d, \infty)}) > 0) \leq C(k_0, t)/d. \quad (50)$$

By the continuity of $u_0 \to Q^{\nu_{\mathcal{T}_n}}$ one has $Q^{\nu_{\mathcal{T}_n}} \to Q^\nu$. Since $F$ is bounded and continuous, then

$$|Q^{\nu_{\mathcal{T}_n}}(F(u(t, \cdot + R_1^d(t)))) - Q^\nu(F(u(t, \cdot + R_1^d(t))))| \to 0, \text{ as } n \to \infty.$$

Therefore we have

$$Q^\nu(F(Z_1(t))) = \lim_{n \to \infty} Q^{\nu_{\mathcal{T}_n}}(F(Z_1(t)))$$

$$= \lim_{n \to \infty} \frac{1}{T_n} \int_0^{T_n} Q^{u_0}(F(Z_1(s + t)))ds$$

$$= \lim_{n \to \infty} \frac{1}{T_n} \int_0^{T_n} Q^{u_0}(F(Z_1(s)))ds$$

$$= \nu(F).$$

Similarly, we have $Q^\nu(F(Z_2(t))) = \nu(F)$. It is straightforward to check that $\{Z(t) : t \geq 0\}$ is Markov, hence $\{Z(t) : t \geq 0\}$ is stationary. Since the map $Y_0 \to Y_0(-R_0(Y_0))$ is measurable on $C_{tem}^+$, the process $\{Z_0(t) : t \geq 0\}$ is also stationary, which implies that $Q^\nu$ is the law of travelling wave solution to equation (16). \qed
4. **Asymptotic wave speed.** In this section, we investigate the asymptotic property of the travelling wave solution. By constructing sup-solution and sub-solution, we obtain the asymptotic wave speed for the two travelling wave solutions respectively. Then we have the estimation of the wave speed of travelling wave solutions to (16). Since the asymptotic wave speed $c$ of the travelling wave solution defined as

$$c = \lim_{t \to \infty} \frac{R_0(t)}{t} \text{ a.s.,}$$

we denote by $R_0(u(t)) = \sup\{x \in \mathbb{R} : u(t,x) > 0\}$ and $R_0(v(t)) = \sup\{x \in \mathbb{R} : v(t,x) > 0\}$ for the sub-systems of the cooperative system. Since the wave-front marker $R_0(t)$ of the cooperative system is $R_0(t) = \max\{R_0(u(t)), R_0(v(t))\},$ and the asymptotic wave speed is the maximum value between $\lim_{t \to \infty} \frac{R_0(u(t))}{t}$ and $\lim_{t \to \infty} \frac{R_0(v(t))}{t},$ we can define the wave speed $c^\ast$ as

$$c^\ast = \lim_{t \to \infty} \frac{R_0(Y(t))}{t} \text{ a.s.}$$

We now construct a sup-solution. Let $Y(t,x) = (\bar{u}(t,x), \bar{v}(t,x))^T$ satisfy

$$
\begin{align*}
\bar{u}_t &= \bar{u}_{xx} + \bar{u}(p - a_1 \bar{u}) + \epsilon \bar{u}dW_t, \\
\bar{v}_t &= \bar{v}_{xx} + \bar{v}(p - a_2 \bar{v}) + \epsilon \bar{v}dW_t, \\
\bar{u}_0 &= u_0, \bar{v}_0 = v_0,
\end{align*}
$$

(51)

where $F_1(Y) \leq u(p - a_1 u),$ $F_2(Y) \leq v(p - a_2 v)$ and

$$p = \max\{b_1\} \times \max\left\{\sqrt{|u_0|^2 + |v_0|^2 + \frac{\epsilon^6}{12c^2}} + 9 + 1, \right.$$

$$C(\epsilon, t)(u_0 + v_0 + \frac{2}{c} - \frac{\epsilon^2}{c}, p_1, p_2) + 1.\left.\right\}
$$

Then we construct a sub-solution and let $Y(t,x) = (\underline{u}(t,x), \underline{v}(t,x))^T$ satisfy

$$
\begin{align*}
\underline{u}_t &= \underline{u}_{xx} + \underline{u}(1 - a_1 \underline{u}) + \epsilon \underline{u}dW_t, \\
\underline{v}_t &= \underline{v}_{xx} + \underline{v}(1 - a_2 \underline{v}) + \epsilon \underline{v}dW_t, \\
\underline{u}_0 &= u_0, \underline{v}_0 = v_0.
\end{align*}
$$

(52)

Obviously, $F_1(Y) \geq u(1 - a_1 u)$ and $F_2(Y) \geq v(1 - a_2 v).$ With equation (51) and equation (52), we have such following conclusion:

**Theorem 4.1.** For any $u_0, v_0 \in C_{\text{term}}^\ast \setminus \{0\},$ let $c^\ast$ be the asymptotic wave speed of equation (16), then

$$\sqrt{4 - 2c^2} \leq c^\ast \leq \sqrt{4p - 2c^2} \text{ a.s.}$$

(53)

In order to prove Theorem 4.1, we need the following lemmas. We first introduce the comparison method for the asymptotic wave speed.

**Lemma 4.2.** Let $Y(t,x)$ and $\bar{Y}(t,x)$ be solutions to (52) and (51) respectively, if $\zeta$ is the asymptotic wave speed of $Y(t, \cdot + R_0(Y(t)))$ and $\bar{c}$ is the asymptotic wave speed of $\bar{Y}(t, \cdot + R_0(\bar{Y}(t))),$ then

$$\zeta \leq c^\ast \leq \bar{c} \text{ a.s.}$$
Proof. The comparison method for the stochastic diffusion equation (see Lemma 3.3) gives that
\[ Y(t,x) \leq \bar{Y}(t,x) \leq \underline{Y}(t,x), \]
which implies \( u(t,x) \leq \bar{u}(t,x) \) a.s. and \( v(t,x) \leq \bar{v}(t,x) \) a.s. Denote the wavefront markers by \( R_1(Y(t)) \), \( R_1(\bar{Y}(t)) \) and \( R_1(\underline{Y}(t)) \), with the definition of asymptotic wave speed
\[ c = \lim_{t \to \infty} \frac{R_1(t)}{t} \text{ a.s.}, \]
and the definition of wavefront marker
\[ R_1(Y(t)) = \max\{ \ln \int_R u(t,x)e^xdx, \ln \int_R v(t,x)e^xdx \}, \]
thus it gives
\[ \lim_{t \to \infty} \frac{R_1(Y(t))}{t} \leq \lim_{t \to \infty} \frac{R_1(\bar{Y}(t))}{t} \leq \lim_{t \to \infty} \frac{R_1(\underline{Y}(t))}{t} \text{ a.s.}, \tag{54} \]
which implies
\[ \epsilon \leq c^* \leq \bar{c} \text{ a.s.}. \tag{55} \]

4.1. **Asymptotic wave speed of sub-solution.** Now we show the asymptotic property of wavefront marker of the sub-solution. Consider equation (52)
\[
\begin{cases}
  \dot{u}_t = u_{xx} + u(1 - a_1 u) + \epsilon u dW_t, \\
  \dot{v}_t = v_{xx} + v(1 - a_2 v) + \epsilon v dW_t, \\
  u_0 = u_0, v_0 = v_0,
\end{cases} \tag{56}
\]
obviously \( u \) and \( v \) are independent, thus we can divide (52) into two equation to study. For each equation one can have the asymptotic wave speed \( c(u) \) and \( c(v) \) respectively, so the asymptotic wave speed of (52) is \( c(Y) = \max\{c(u), c(v)\} \).

**Theorem 4.3.** For any \( u_0, v_0 \in C^+_\text{tem} \setminus \{0\}, Y(t,x) \) is solution to (52), then the asymptotic wave speed \( c(Y) \) satisfies
\[ c(Y) = \sqrt{4 - 2\epsilon^2} \text{ a.s.}. \tag{57} \]

Proof. For any \( h > 0 \), take \( \kappa \in (0, \frac{h^2}{4} + \sqrt{1 - \frac{\epsilon^2}{2}} h) \). Define
\[ \eta_t(\omega) = \exp\int_0^t dW_s - \frac{1}{2} \int_0^t e^sds, \quad 0 \leq t \leq \infty, \]
construct new probability space \((\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}}), \tilde{W} = (\tilde{W}(t) : t \geq 0)\) is Brownian motion. Then there exists \( T_1 > 0 \), such that for \( t \geq T_1 \) and a.e. \( \omega \in \Omega \)
\[ \exp(-\frac{\epsilon^2}{2} t + \kappa t) \leq \eta_t(\omega) \leq \exp(-\frac{\epsilon^2}{2} t + \kappa t), \]
thus stochastic Feynman-Kac formula gives
\begin{align*}
  u(t,x) & \leq \exp(t - \frac{1}{2} \epsilon^2 t + \kappa t) \tilde{\mathbb{P}}(\tilde{W}(t) \leq -\frac{x}{\sqrt{2}}) \\
  & \leq \exp(t - \frac{1}{2} \epsilon^2 t + \kappa t - \frac{x^2}{4t}) \text{ a.s.,}
\end{align*}
for \( t \geq T_1 \). Set \( x \geq (k+h)t \), where \( k \) is a constant. Multiple \( e^x \) with both sides and integrate in \([(k+h)t, \infty)\), then we have

\[
\int_{(k+h)t}^{\infty} u(t,x)e^x \, dx \leq \int_{(k+h)t}^{\infty} \exp(\frac{1}{2} \epsilon^2 t + \kappa t - \frac{x^2}{4t} + x) \, dx
\]

\[
= 2\sqrt{t} \exp\left(t - \frac{1}{2} \epsilon^2 t + \kappa t + t\right) \int_{(k+h)t-2t}^{\infty} e^{-\sqrt{t}^2} \, dx
\]

\[
\leq \sqrt{t} \exp\left(1 + \kappa - \frac{k^2}{4} - \frac{k h}{2} - \frac{h^2}{4} - k - h - \frac{\epsilon^2}{2} t\right) \text{ a.s.},
\]

for \( t \geq T_1 \). Let \( k = \sqrt{4 - 2\epsilon^2} + 2 - 4 \). Then we obtain

\[
\lim_{t \to \infty} \int_{(k+h)t}^{\infty} u(t,x)e^x \, dx = 0 \text{ a.s.} \quad (58)
\]

Similarly, integrating \( u(t,x)e^x \) in \([(\sqrt{4 - 2\epsilon^2} + h)(k-h)t), \text{we have}

\[
\int_{(k-h)t}^{(k-h)t} u(t,x)e^x \, dx
\]

\[
\leq \int_{(k-h)t}^{(k-h)t} \exp(\frac{1}{2} \epsilon^2 t + \kappa t - \frac{x^2}{4t} + x) \, dx
\]

\[
= 2\sqrt{t} \exp\left(t - \frac{1}{2} \epsilon^2 t + \kappa t + t\right) \int_{(k-h)t-2t}^{\infty} e^{-\sqrt{t}^2} \, dx
\]

\[
\leq \sqrt{t} \exp\left(t - \frac{\epsilon^2}{2} t + \kappa t - \frac{4 - 2\epsilon^2}{4} t - \frac{(\sqrt{4 - 2\epsilon^2}h) t}{2} - \frac{h^2}{4} t + \sqrt{4 - 2\epsilon^2} t + ht\right)
\]

\[
- \sqrt{t} \exp\left(t - \frac{\epsilon^2}{2} t + \kappa t - \frac{k^2}{4} t + \frac{k h}{2} t - \frac{h^2}{4} t + kt - ht\right)
\]

\[
\leq \sqrt{t} \exp(\kappa t + \sqrt{4 - 2\epsilon^2} t - \frac{(\sqrt{4 - 2\epsilon^2}h) t}{2} - \frac{h^2}{4} t + ht)
\]

\[
- \sqrt{t} \exp(\kappa t + \frac{k h}{2} t - \frac{h^2}{4} t - ht) \text{ a.s.},
\]

for \( t \geq T_1 \). Analogously, we have

\[
\int_{(\sqrt{4 - 2\epsilon^2} + h)t}^{(k-h)t} u(t,x)e^x \, dx
\]

\[
\leq \sqrt{t} \exp(\kappa t + \sqrt{4 - 2\epsilon^2} t + \frac{(\sqrt{4 - 2\epsilon^2}h) t}{2} - \frac{h^2}{4} t + ht)
\]

\[
- \sqrt{t} \exp(\kappa t + \frac{k h}{2} t - \frac{h^2}{4} t - ht) \text{ a.s.},
\]

and

\[
\int_{(k-h)t}^{(k+h)t} u(t,x)e^x \, dx \leq \sqrt{t} \exp(\kappa t + \frac{k h}{2} t - \frac{h^2}{4} t + ht)
\]

\[
- \sqrt{t} \exp(\kappa t - \frac{k h}{2} t - \frac{h^2}{4} t + ht) \text{ a.s.},
\]
for \( t \geq T_1 \). Referring to [20], there exists \( T_2 > 0 \), such that for all \( t \geq T_2 \) and 
\[ x < (\sqrt{4 - 2\epsilon^2} - h) t, \]
there exist \( \rho_1, \rho_2 > 0 \) satisfying
\[ \exp(-\rho_1 \sqrt{2t \ln \ln t}) \leq u(t, x) \leq \exp(\rho_2 \sqrt{2t \ln \ln t}) \quad \text{a.s.,} \] (59)
which goes into
\[ \int_{-\infty}^{(\sqrt{4 - 2\epsilon^2} - h)t} u(t, x)e^x \, dx \leq \exp(\rho_2 \sqrt{2t \ln \ln t} + (\sqrt{4 - 2\epsilon^2} - h)t) \quad \text{a.s.} \] (60)

Since \( \int_{(k + h)t}^{(k + h)t} u(t, x)e^x \, dx \leq 1 \), then we have
\[ \int_{R} u(t, x)e^x \, dx \leq \exp(\rho_2 \sqrt{2t \ln \ln t} + (\sqrt{4 - 2\epsilon^2} - h)t)(2 + H(t) + G(t)) \quad \text{a.s.,} \] (61)
where
\[ H(t) = \sqrt{t} \exp\left(\frac{1}{2} \epsilon^2 - \frac{\epsilon^2}{2} t + \kappa t + \frac{\kappa h}{2} - h^2 - \frac{\epsilon^2}{2} t - \rho_2 \sqrt{2t \ln \ln t} - \sqrt{4 - 2\epsilon^2} t\right), \]
and
\[ G(t) = \sqrt{t} \exp\left(\frac{1}{2} \epsilon^2 - \frac{\epsilon^2}{2} t + \kappa t - \frac{\epsilon^2}{2} h t - \rho_2 \sqrt{2t \ln \ln t} - h^2 \right) + 2h - \frac{1}{t} \rho_2 \sqrt{2t \ln \ln t} \quad \text{a.s.} \]

With the arbitrariness of \( h \) and \( \kappa \) we know that \( H(t) \leq 1 \) a.s. for large \( t \). Simple calculation shows that
\[ \frac{1}{t} \ln G(t) = \frac{1}{2t} \ln 4t - \frac{1}{t} (\ln 2 - \frac{\epsilon^2}{2} + \frac{\epsilon^2}{2} t) \]
\[ + \kappa - \frac{4 - 2\epsilon^2}{4} h - h^2 + 2h - \frac{1}{t} \rho_2 \sqrt{2t \ln \ln t} \quad \text{a.s.} \]

Again with the arbitrariness of \( h \) and \( \kappa \)
\[ \lim_{t \to \infty} \frac{1}{t} \ln G(t) = 0 \quad \text{a.s.} \] (62)

In summary, we have the upper bound of the asymptotic wave speed of travelling wave solution to (52)
\[ \frac{R_1(t)}{t} \leq \frac{1}{t} \rho_2 \sqrt{2t \ln \ln t} + \sqrt{4 - 2\epsilon^2} - h + \frac{1}{t} \ln 2 + \frac{1}{t} \ln G(t) \quad \text{a.s.} \] (63)

Furthermore, we have
\[ \limsup_{t \to \infty} \frac{R_1(t)}{t} \leq \sqrt{4 - 2\epsilon^2} \quad \text{a.s.} \] (64)

In addition, we have
\[ \frac{R_1(t)}{t} \geq -\frac{1}{t} \rho_1 \sqrt{2 \ln \ln t} + \sqrt{4 - 2\epsilon^2} - h \quad \text{a.s.} \] (65)

Thus the lower bounded can be obtained
\[ \liminf_{t \to \infty} \frac{R_1(t)}{t} \geq \sqrt{4 - 2\epsilon^2} \quad \text{a.s.} \] (66)

Combining (64) with (66) gives
\[ \lim_{t \to \infty} \frac{R_1(t)}{t} = \sqrt{4 - 2\epsilon^2} \quad \text{a.s.} \] (67)
Remark 5. In the study of \( u(t) \) we obtain the asymptotic wave speed as \( c(u) = \sqrt{4 - 2\epsilon^2} \). Analogously to \( v(t) \) we have \( c(v) = \sqrt{4 - 2\epsilon^2} \), thus \( c^* \geq \sqrt{4 - 2\epsilon^2} \) a.s.

4.2. Asymptotic wave speed of sup-solution. By the method used in Theorem 4.3, we consider the sup-solution \( \bar{Y}(t, x) \) satisfying the following equation

\[
\begin{align*}
\bar{u}_t &= \bar{u}_{xx} + \bar{u}(p - a_1 \bar{u}) + \epsilon \bar{u}dW_t, \\
\bar{v}_t &= \bar{v}_{xx} + \bar{v}(p - a_2 \bar{v}) + \epsilon \bar{v}dW_t, \\
\bar{u}_0 &= u_0, \\
\bar{v}_0 &= v_0.
\end{align*}
\]  

(68)

Similarly, we construct a new probability space \( (\bar{\Omega}, \bar{\mathcal{F}}, \bar{\mathbb{P}}) \), and \( \bar{W} = (\bar{W}(t) : t \geq 0) \) is a Brownian motion defined on \( (\bar{\Omega}, \bar{\mathcal{F}}, \bar{\mathbb{P}}) \). For any \( h > 0 \), choosing \( 0 < \tau < \frac{\kappa^2}{4} + \sqrt{1 - \epsilon^2}h \), defining

\[
\eta(t) = \exp\left(\int_0^t \epsilon dW_s - \frac{1}{2} \int_0^t \epsilon^2 ds\right), \quad 0 \leq t \leq \infty,
\]

there exists \( T_1 > 0 \) such that

\[
\bar{u}(t, x) \leq \exp\left(p t - \frac{1}{2} \epsilon^2 t + \tau t - \frac{x^2}{4t}\right) \text{ a.s.},
\]

for \( t \geq T_1 \). Thus similar to Theorem 4.3, we have the conclusion:

Theorem 4.4. For any \( u_0, v_0 \in C^+ \setminus \{0\} \), \( \bar{Y}(t, x) \) is a solution to (51), then the asymptotic wave speed \( c(\bar{Y}) \) satisfies

\[
c(\bar{Y}) = \sqrt{4p - 2\epsilon^2} \quad \text{a.s.}
\]

(69)

Proof of Theorem 4.1. Associating Theorem 4.3 and Theorem 4.4, with Lemma 4.2 we can achieve the conclusion:

\[
\sqrt{4 - 2\epsilon^2} \leq c^* \leq \sqrt{4p - 2\epsilon^2} \quad \text{a.s.}
\]

(70)

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E-mail address: wenhao12@nudt.edu.cn
E-mail address: jhuang32@nudt.edu.cn
E-mail address: liyuhong@hust.edu.cn