Anomalies, Chern-Simons Terms
and Chiral Delocalization in Extra Dimensions

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(Dated: January 18, 2006)

Abstract

Gauge invariant topological interactions, such as the $D = 5$ Chern-Simons term, are required in models in extra dimensions that split anomaly free representations. The Chern-Simons term is necessary to maintain the overall anomaly cancellations of the theory, but it can have significant, observable, physical effects. The CS-term locks the KK-mode parity to the parity of space-time, leaving a single parity symmetry. It leads to new processes amongst KK-modes e.g., the decay of a KK-mode to a 2-body final state of KK-modes. A formalism for the effective interaction amongst KK-modes is constructed, and the decay of a KK-mode to KK-mode plus zero mode is analyzed as an example. We elaborate the general KK-mode current and anomaly structure of these theories. This includes a detailed study of the triangle diagrams and the associated “consistent anomalies” for Weyl spinors on the boundary branes. We also develop the non-abelian formalism. We illustrate this by showing in a simple way how a $D = 5$ Yang-Mills “quark flavor” symmetry leads to the $D = 4$ chiral lagrangian of mesons and the quantized Wess-Zumino-Witten term.

PACS numbers: 11.10.-z, 11.10.Kk, 11.15.-q, 11.25.Mj, 11.25.Uv, 11.30.Rd, 11.40.-q

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I. INTRODUCTION

We explore the physics of a Chern-Simons term in a model of a gauge theory in a compactified extra dimension. A Chern-Simons term leads to new physical processes, i.e., new interactions amongst the KK-modes of gauge fields, characterized by a quantized coefficient. The Chern-Simons term locks KK-mode parity to the parity of space-time via the 5-\(\epsilon\) symbol permitting processes in which KK-parity is violated, while KK-parity combined with \(D = 4\) space-time parity is conserved. That is, KK-mode parity is a spurious symmetry and is independent of space-time parity, until a Chern-Simons term appears, which unites these into a single parity symmetry.

The effects of Chern-Simons terms are intertwined with the anomalies of matter fields. In the present discussion matter fields will be localized on boundary branes. Since the Chern-Simons term is a bulk interaction, it probes physics away from the boundary (branes). Taken together, however, the anomalies and Chern-Simons terms produce gauge invariant amplitudes containing new physics. Perhaps the most novel new physics is the decay of a single KK-mode into a two KK-mode final state. We study the latter process in this paper as an explicit example of the formalism.

It should not be surprising that there is nontrivial physics associated with the CS term. For example, in the case of \(D = 3\) QED, the CS term gives the photon a mass \([1]\). Moreover, the \(D = 5\) Yang-Mills CS term for \(SU(N)\) can be deformed into a Wess-Zumino-Witten (WZW) term \([2, 3]\) of an \(SU(N)_L \times SU(N)_R\) chiral lagrangian of mesons, under certain compactification schemes \([4]\) (we give a simple derivation of this in Section V of the present paper \([5]\)). Many essential physical processes in QCD, such as \(\pi \to 2\gamma, \phi \to K\overline{K}\), and \(\phi \to 3\pi\), etc., are controlled by the WZW term. In particular, the pion parity in a chiral lagrangian is defined by the WZW term via the 4-\(\epsilon\) symbol and would be a spurious symmetry without it. This is the analogue of the fate of KK-parity in the presence of the CS term in the \(D = 5\) bulk.

Split fermion theories \([6]\) are compelling in that anomalous representations of chiral fermions occur in a delocalized way in extra dimensions. Given that the standard model involves flavor and chirality in a nontrivial (non-vectorlike) way, it would seem plausible that chiral delocalization would occur if extra dimensions should exist. Chirality is then, in a sense, emergent from the spatial localization in extra dimensions. A key discriminant
for such a theory is, indeed, likely to be the KK-mode parity violation through CS term interactions.

We study in detail a $U(1)$ theory, but we also indicate how things work in non-abelian theories as well. Our goal is to develop a solid formalism for correctly obtaining gauge invariant amplitudes. In the non-abelian case we use this formalism to demonstrate how the Wess-Zumino-Witten term, with its quantized coefficient, arises by morphing a $D = 5$ Yang-Mills theory of quark flavor into a compactified $D = 4$ theory, where it becomes a chiral lagrangian of mesons, the “mesons” being the $A^5$ gauge fields.

In $D = 5$, for a $U(1)$ gauge theory, there exists a CS term of the form $\epsilon_{ABCDE} A^A \partial^B A^C \partial^D A^E$, and we’ll consider this operator to be part of a lagrangian of a $D = 5$ theory compactified to $D = 4$ with boundary branes. Under a gauge transformation this term produces anomalies on the boundary branes, which we’ll refer to as the “CS anomalies.” Anomalies represent nonconserved currents, i.e., $\partial_A j^A = (\text{anomaly})$. Any gauge theory would be a priori inconsistent with such anomalies since the equation of motion, $\partial_A F^{AB} = j^B$, implies $\partial_A j^A = 0$, by the antisymmetry of $F^{AB}$. We thus require some mechanism to cancel anomalies, and this typically implies something like chiral matter fields attached to different boundary branes that produce their own “matter anomalies” [7, 8, 9, 10]. The CS anomalies must then cancel against the matter anomalies [11].

The relevant fermionic anomalies on the boundaries are the “consistent anomalies,” i.e., the form that arises directly from Feynman diagrams. We explicitly verify in detail the results of Bardeen [9] for the consistent anomalies of massless Weyl spinors, (and the massive “left-right symmetric” case) arising from the effective action operators in Appendix C. A key point, often confused in the literature, is that the Weyl spinor anomaly (a consistent anomaly) is not “half the Dirac spinor axial current anomaly of QED,” [8] (the “covariant anomaly”). The consistent and covariant forms of the anomalies differ by the addition of a counterterm to the action that makes the vector (electromagnetic) current conserved, even in the presence of a background axial vector photon. Remarkably (though perhaps not surprisingly), we find that this counterterm is precisely “boundary term” that arises when we integrate out the fermions on the boundary branes in the large fermion mass limit. In combination with the CS anomaly, which has the form of the consistent anomaly, this maintains the zero-mode gauge invariance, and it shifts the lowest axial vector current anomaly into the covariant form. We obtain the resulting tower of KK-mode covariant
current anomalies in Appendix B.

Our formal problem, in part, is to bring the full effective action into a manifestly gauge invariant form. Indeed, one of the beautiful aspects of gauge field theories under compactification is the manner in which the KK-mode mass terms are automatically generated in a manifestly gauge invariant way. The modes $A^5_n$ become packaged together with their longitudinal components $\partial_\mu A^\mu_n$, as they acquire mass – they become gauge invariant “Stueckelberg fields.” Once we are assured that the action can be brought into Stueckelberg combinations of massive fields, then we are assured of unambiguous Feynman rules that respect gauge invariance for the massive fields. We find that a Wilson line gauge transformation that brings $A^5 = 0$ in the bulk simultaneously packages all of the massive KK-modes in the CS-term into Stueckelberg form. This leads to a remarkably compact formula for the CS-term part of the effective action. In the case of chiral fermions on the boundaries, the triangle diagrams maintain the zero-mode gauge invariance. A physical amplitude is thus a sum of the CS term contribution in the bulk plus a boundary term that comes from the chiral fermions on the branes. In the case of delocalized chiral fermions with a Wilson line mass term, and a $U(1)$ gauge group, the boundary term arises from triangle diagrams involving the fermions on the branes.

We consider two distinct ways to implement matter fields. The conceptually simpler method employs axions, $\phi_L$ and $\phi_R$, on the $x^5 = 0$ ($L$) and $x^5 = R$ ($R$) coupled to $F\tilde{F}(0)$ and $F\tilde{F}(R)$ on their respective boundary branes. This construction involves no triangle diagrams, yet it demonstrates, as a matter of principle that the CS term bulk interactions are physical. However, this theory is somewhat less interesting because here the zero mode photon itself becomes massive, by eating one linear combination, $\phi_L + \phi_R$, of the axions. A massive Stueckelberg photon field now appears explicitly in the CS term. A second, orthogonal, combination $\phi_L - \phi_R$ remains as a physical uneaten axion. We discuss this in Appendix A.

We are mainly interested in QED in $D = 5$ with “split chiral electrons” on the boundary branes. Though QED is a vectorlike theory, if we want to promote it to $D = 5$, yet maintain a naturally small electron mass (compared to $1/R$, the compactification scale), then we require chiral delocalization e.g., split chiral electrons on the boundary branes. We thus place $\psi_L$ on the left-brane $x^5 = 0$ and $\psi_R$ on the right-brane $x^5 = R$. The electron mass term is then a bilocal operator involving a Wilson line, $m\bar{\psi}_L \exp(\int i dx^5 A_5) \psi_R + h.c.$ where
the integral runs from $x^5 = 0$ to $x^5 = R$. This is the technically natural setting for QED in $D = 5$, and it necessarily contains the CS-term due to the split fermion representation. The anomalies on the branes cancel the CS term anomalies for a particular choice of the coefficient of the CS term. Thus the CS term is “quantized.” In this model the photon stays massless, and the full effective interaction consisting of the CS term, together with the fermion triangle loop contributions is now explicitly gauge invariant for all modes \[15\].

The non-abelian theory can be developed along parallel lines. It can also contain a $\text{tr}(AdAdA) + ...$ CS-term, and boundary brane chiral fermions. We develop the non-abelian case to the point of establishing the quantized coefficient of the CS-term. We then revisit the problem of morphing a $D = 5$ Yang-Mills theory of quark flavor into the low energy Wess-Zumino-Witten term by showing how the $\text{Tr}(\pi d\pi d\pi d\pi d\pi)$ term arises with the Witten quantization in a theory with flipped orbifold boundary conditions \[4\].

Most technical details have been relegated to a series of Appendices. The main text attempts to give the sequential arguments in a more conceptual form. We have included relevant background issues concerning anomalies to enhance the reader’s familiarity with some of the subtleties.

II. THE $U(1)$ CASE: GENERAL ARGUMENT

Consider a $U(1)$ gauge theory in $D = 5$, with the covariant derivative, field strength and kinetic term lagrangian density:

\[
D_A = \partial_A - iA_A, \quad F_{AB} = i[D_A, D_B], \quad L_0 = -\frac{1}{4\tilde{e}^2} F_{AB}F^{AB}.
\]  

We define the gauge fields to have canonical dimension for $D = 4$, i.e., $A \sim M^1$, where then $1/\tilde{e}^2$ has dimension of $M^1$ (the e.t.c. is then $[A_i(\vec{x}), \tilde{A}_j(\vec{y})] = i\tilde{e}^2\delta^4(\vec{x} - \vec{y})$).

The theory admits a Chern-Simons term, defined by the local lagrangian density:

\[
L_{CS} = c \epsilon^{ABCDE} A_A \partial_B A_C \partial_D A_E = \frac{c}{4} \epsilon^{ABCDE} A_A F_{BC} F_{DE}
\]  

We can then define the non-compactified $D = 5$ theory by the action $S = S_0 + S_{CS}$ where:

\[
S_0 = \int d^5x \ L_0, \quad S_{CS} = \int d^5x \ L_{CS}.
\]  

The variation of the action with respect to $A_A$ generates the equation of motion:

\[
\tilde{e}^2 \frac{\delta S}{\delta A_A} = \partial_B F^{BA} - J^A = 0.
\]
We see that a conserved “Chern-Simons current” appears as the source term in the theory,

\[ J^A = \frac{3c}{4} \epsilon^{ABCDEF} F_{BC} F_{DE}, \quad \partial^A J_A = 0. \quad (5) \]

The non-compactified action is readily seen to be gauge invariant, provided we forbid surface terms, \textit{i.e.}, we view all fields as approaching zero sufficiently rapidly at infinity. Under a gauge transformation:

\[ A_A \rightarrow A_A + \partial_A \theta \quad (6) \]

we see that \( L_0 \) is strictly invariant and we also have upon integrating by parts:

\[ S_{CS} \rightarrow S_{CS} + \frac{c}{4} \int d^5 x \epsilon^{ABCDE} \partial_A \theta F_{BC} F_{DE} \]

\[ = S_{CS} - \frac{c}{4} \int d^5 x \epsilon^{ABCDE} \theta \partial_A F_{BC} F_{DE} = S_{CS}. \quad (7) \]

The situation changes, however, when we compactify the theory and must accommodate surface terms. Let us compactify the fifth dimension, \( 0 \leq x^5 \leq R \). We thus imagine that, located at \( x^5 = 0 \) and \( x^5 = R \), are surfaces (branes) denoted respectively as \( I \) and \( II \), upon which we may choose to apply various boundary conditions.

For example, if we apply the condition, \( F_{\mu 5}|_I = F_{\mu 5}|_{II} = 0 \), then we have a “magnetic superconducting parallel plate capacitor” \cite{16}. This corresponds to an orbifold since \( F_{\mu 5} = 0 \) can be satisfied with the gauge choice, \( \partial_\mu A_5 = 0 \) and \( \partial_5 A_\mu = 0 \). This, in turn, implies that \( A_5 \) is an odd function on the extended interval, \( 0 \leq x^5 \leq 2R \), while \( A_\mu \) is even, corresponding to the normal orbifold configurations for a theory compactified on \( S_1/Z_2 \). The advantage of stating boundary conditions in the “capacitor” language is that they are then manifestly gauge invariant, \textit{i.e.}, orbifolding does not break gauge invariance but, rather, corresponds to a particular gauge choice for a physical capacitor system.

If we now repeat the check of gauge invariance, we see that there is a surface term generated on each of the branes that follows from performing the integration by parts:

\[ S_{CS} \rightarrow S_{CS} + \frac{c}{4} \int_{II} d^4 x \theta(R) \epsilon^{\mu \rho \sigma} F_{\mu \nu} F_{\rho \sigma}(R) - \frac{c}{4} \int_I d^4 x \theta(0) \epsilon^{\mu \rho \sigma} F_{\mu \nu} F_{\rho \sigma}(0). \quad (8) \]

The theory is no longer gauge invariant since the action has shifted by the two surface terms that take the form of anomalies. We refer to these terms as the “CS anomalies.” If, however, the branes contain additional matter fields with anomalous gauge currents, then they can cancel the above CS anomalies on the branes, and the overall theory, bulk plus branes, becomes gauge invariant.
Let us presently assume that we have arranged a generic mechanism of cancelling the CS anomaly. Since we are compactifying the theory on $0 \leq x^5 \leq R$, it is useful to put the CS term into a form that consists of two terms, one that isolates $A_5$ and another that isolates $\partial_5$. It is readily seen that $L_{CS}$ can be written as:

$$L_{CS} = \frac{3c}{4} \epsilon^{\mu\nu\rho\sigma} A_5 F_{\mu\nu} F_{\rho\sigma} + c \epsilon^{\mu\nu\rho\sigma} (\partial_5 A_\mu) A_\nu F_{\rho\sigma}. \quad (9)$$

To write $L_{CS}$ in this form we have discarded total divergences in the $D = 4$ theory that do not affect the physics. In the compactified case our problem is to bring the CS term into a manifestly gauge invariant form. We will see that this is possible for all massive modes, but not for the zero mode.

Consider a Wilson line that emanates from, e.g., brane I, $x^5 = 0$, toward an arbitrary point in the bulk, $x^5 = y$:

$$U(y) = \exp \left( i \int_0^y dx^5 A_5(x^5) \right) \quad (10)$$

and we have:

$$\partial_y U = i A_5(y) U \quad (11)$$

Using the Wilson line as a gauge transformation, we have:

$$A_A \rightarrow A_A + i U^\dagger \partial_A U \quad (12)$$

and we thus see that:

$$A_5 \rightarrow A_5(y) + i U^\dagger \partial_y U = A_5(y) - \partial_y \int_0^y dx^5 A_5(x^5) = 0 \quad (13)$$

Assuming the cancellation of CS anomalies occurs with some generic matter fields, this gauge transformation thus annihilates the first term of eq.$(9)$, and takes the CS term into the form $[17]$:

$$L_{CS} = c \epsilon^{\mu\nu\rho\sigma} (\partial_5 B_\mu) B_\nu F_{B\rho\sigma} \quad (14)$$

where we define the gauge transformed $A_\mu$’s as $B_\mu$’s,

$$B_\mu = A_\mu - \partial_\mu \int_0^y A_5 dx^5; \quad F_{B\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu. \quad (15)$$

Thus, the resulting Chern-Simons action takes the form:

$$S_{CS} = c \epsilon^{\mu\nu\rho\sigma} \int d^4x \int_0^R dy (\partial_y B_\mu) B_\nu F_{B\rho\sigma}. \quad (16)$$
Eq. (16) is the form we desire. As we’ll see below, the $B_\mu$ field will lead to gauge invariant ("Stueckelberg") combinations for each massive KK-mode in the compactified theory when we do the mode expansion. The massless zero mode (photon) gauge field, however, will appear explicitly in the result, and the full zero mode gauge invariance requires the addition of the matter effects (e.g., the triangle diagrams). Note the $\partial_y$ in the integrand which leads to the breaking of KK-parity. This is associated with $\epsilon_{\mu\nu\rho\sigma}$ so that, overall, only the product of KK-parity and space-time parity is conserved.

We now turn to QED with chiral electrons as a means of providing the anomaly cancellation on the branes. An alternative theory with axions on the branes is developed in Appendix A.

III. $D = 5$ QED, ORBITFOLD COMPACTIFICATION

A. Chiral Fermions

Consider QED in $D = 5$ on an orbifold with periodic domain $0 \leq x^5 \leq R$. We place electrons on the boundary branes located at $x^5 = 0$ and $x^5 = R$. The electrons are chiral, with $\psi_L (\psi_R)$ on the left-brane, I at $x^5 = 0$ (right-brane, II, at $x^5 = R$). These fermions have anomalies on their respective branes. These matter anomalies will cancel the CS anomalies provided the coefficient $c$ takes on a special value dictated by the fermionic anomalies. We first establish this special value of $c$.

This model has the same divergence structure for the fermion loops as does ordinary QED in $D = 4$. The only nontrivial new bulk interaction is the topological CS term. It is not hard to verify that the topological CS term is not subject to renormalization at the one loop level, from diagrams involving internal gauge fields in the continuum $D = 5$ theory. This non-renormalization happens because the CS-term is a topological invariant. This probably holds to all orders in the $D = 5$ perturbation theory, and is related to the $D = 4$ Adler-Bardeen non-renormalization theorem for the anomaly. The CS term does renormalize the bulk kinetic term with a quadratic divergence, and it may induce additional non-topological counterterms with factors of $e^2$. This is the usual problem of a $D = 5$ theory, so we imagine some kind of UV cut-off, such as an overarching string theory as the UV completion.
The full action of the theory is defined as,

$$ S = S_0 + S_{CS} + S_{branes} $$  \hspace{1cm} (17)$$

where the bulk kinetic term action for the theory is:

$$ S_0 = -\frac{1}{4e^2} \int_0^R dy \int d^4x F_{\mu\nu}F^{\mu\nu} - \frac{1}{2e^2} \int_0^R dy \int d^4x F_{\mu 5}F^{\mu 5}. $$  \hspace{1cm} (18)$$

The fermionic matter action on the branes is:

$$ S_{branes} = \int_I d^4x \overline{\psi}_L i\slashed{D}_L \psi_L + \int_{II} d^4x \overline{\psi}_R i\slashed{D}_R \psi_R $$  \hspace{1cm} (19)$$

where:

$$ D_{L\mu} = \partial_{\mu} - iA_{\mu}(x_{\mu}, 0), \quad D_{R\mu} = \partial_{\mu} - iA_{\mu}(x_{\mu}, R) $$  \hspace{1cm} (20)$$

and

$$ \psi_L = \frac{1 - \gamma^5}{2} \psi, \quad \psi_R = \frac{1 + \gamma^5}{2} \psi $$  \hspace{1cm} (21)$$

We have thus “split” QED into two chiral theories living on distinct branes, I and II. The $\psi_L$ and $\psi_R$ chiral projections are key ingredients of the theory. This structure can, of course,
come about if there is a thin domain wall (kink) at \( x^5 = 0 \) and an anti-domain wall (anti-kink) at \( x^5 = R \), where \( \psi_L \) and \( \psi_R \) are then the fermionic zero modes. We presently have no bulk propagation of the fermions, and this can be engineered if the fermions have a very large Dirac mass in the bulk away from the domain walls.

The chiral electrons have anomalies on their respective branes. Consider a gauge transformation in the bulk:

\[
A_A(x_\mu, y) \rightarrow A_A(x_\mu, y) + \partial_\mu \theta(x_\mu, y) \tag{22}
\]

and we therefore have:

\[
S_{\text{branes}} \rightarrow S_{\text{branes}} + \int_I d^4x \overline{\psi}_L \gamma_\mu \partial_\mu \psi_L(x_\mu, 0) + \int_{II} d^4x \overline{\psi}_R \gamma_\mu \partial_\mu \psi_R(x_\mu, R)
\]

\[
\rightarrow S_{\text{branes}} - \int_I d^4x \theta(x_\mu, 0) \partial_\mu J^\mu_L - \int_{II} d^4x \theta(x_\mu, R) \partial_\mu J^\mu_R \tag{23}
\]

where \( J^\mu_{L,R} = \overline{\psi} \gamma_\mu \psi_{L,R} \). Note that this can be induced by a gauge transformation of the electrons on the branes,

\[
\psi_L \rightarrow \exp(i\theta(x_\mu, 0)) \psi_L, \quad \psi_R \rightarrow \exp(i\theta(x_\mu, R)) \psi_R. \tag{24}
\]

At the quantum loop level this transformation is anomalous, the currents are not conserved. We can view it as generating the Noether terms on the branes of eq.(23) through the fermionic functional measure of the path integral \[12\]. To proceed, we must determine the forms of the anomalies on the branes, and this requires care.

**B. Consistent Anomalies on the Branes**

Consider the theory of a single Weyl spinor in \( D = 4 \):

\[
S = \int d^4x \overline{\psi}_L(i\slashed{\partial} + A_L) \psi_L \tag{25}
\]

The theory is anomalous and the anomaly is unambiguously determined as:

\[
\partial_\mu J^\mu_L = -\frac{1}{48\pi^2} F^\mu_{A,L} \tilde{F}_{A,L\mu} , \quad J^\mu_L = \overline{\psi} \gamma^\mu \psi_L , \quad \text{and} \quad \tilde{F}_{\mu\nu} = \frac{1}{2} \epsilon_{\mu\nu\rho\sigma} F^{\rho\sigma}. \tag{26}
\]

Moreover, if we have a pair of Weyl spinors in \( D = 4 \):

\[
S = \int d^4x \left[ \overline{\psi}_L(i\slashed{\partial} + A_L) \psi_L + \overline{\psi}_R(i\slashed{\partial} + A_R) \psi_R \right] \tag{27}
\]
then we can treat these fields symmetrically, each as in eq.(25). The anomalies are given implicitly in Bardeen’s paper for the massive case, [9], and we confirm these results in Appendix C for the massless Weyl spinor theory of eq.(25):

\[ \partial_\mu J_\mu^L = -\frac{1}{48\pi^2} F_{\mu\nu}^L F^A_{\Lambda\mu\nu}, \quad \partial_\mu J_\mu^R = \frac{1}{48\pi^2} F_{\mu\nu}^R \tilde{F}^A_{\Lambda\mu\nu} \]  

(28)

where:

\[ J_\mu^L = \bar{\psi} \gamma^\mu \psi^L, \quad J_\mu^R = \bar{\psi} \gamma^\mu \psi^R. \]  

(29)

If we construct the vector and axial vector currents, \( J = J_L + J_R \), and \( J^5 = J_R - J_L \), and we define define \( A_R = V + A \) and \( A_L = V - A \), we have by simple algebra:

\[ \partial_\mu J_\mu = \frac{1}{12\pi^2} F_{\mu\nu}^V \tilde{F}^A_{\mu\nu}, \quad \partial_\mu J_5^\mu = \frac{1}{24\pi^2} \left( F_{\mu\nu}^V \tilde{F}^V_{\mu\nu} + F_{\mu\nu}^A \tilde{F}^A_{\mu\nu} \right) \]  

(30)

This latter result is the form quoted in eq.(44) of [9]. One should pay particular attention to the coefficients in eq.(26), eq.(28) and eq.(30).

This form of the vector and axial-vector current anomalies is known as the “consistent anomaly,” as it is consistent with the direct calculation of the triangle diagrams of the Weyl spinors. As stated above, the form of the anomaly for a theory containing only a single pure left-handed Weyl fermion is unambiguous, given by the first expression in eq.(28). There is no ambiguity when there is only one gauge field \( A_L \) and there is no nonvanishing counterterm (a term of the form \( \epsilon_{\mu\nu\rho\sigma} A_{\mu A_L}^\rho A_{\sigma A_L}^\nu \) is zero) that can be added to the theory to redefine the anomaly.

The form in eq.(30) is just the sum of left-handed and right-handed Weyl fermion consistent anomalies, and is referred to as the “left-right symmetric anomaly.” However, taken together with both the left-handed and right-handed Weyl spinors and two gauge fields, \( A_L \) and \( A_R \), there is now an ambiguity in the form of the anomaly. We now have the freedom to introduce counterterms such as \( \epsilon_{\mu\nu\rho\sigma} A_{\mu A_L}^\rho A_{\sigma A_L}^\nu \), etc., and these can modify the form of the anomaly. We can now force the vector current to be conserved by adding to the lagrangian a particular counterterm.

To see this, consider a term in the action of the form:

\[ S' = \frac{1}{6\pi^2} \int d^4x \epsilon_{\mu\nu\rho\sigma} A^\mu V^\nu \partial^\rho V^\sigma \]  

(31)

This term is unique, having even parity and nonvanishing. Upon variation wrt \( V \) or \( A \) it
adds corrections to the vector and axial currents:

\[
\frac{\delta S'}{\delta V_\mu} = \delta J^\mu = -\frac{1}{3\pi^2} \epsilon_{\mu\nu\rho\sigma} A^\nu \partial^\rho V^\sigma + \frac{1}{6\pi^2} \epsilon_{\mu\nu\rho\sigma} V^\nu \partial^\rho V^\sigma
\]

\[
\frac{\delta S'}{\delta A_\mu} = \delta J^5_\mu = \frac{1}{6\pi^2} \epsilon_{\mu\nu\rho\sigma} V^\nu \partial^\rho V^\sigma
\]

(32)

The full currents, \( \tilde{J} = J + \delta J \), now satisfy:

\[
\partial \mu \tilde{J}^\mu = 0, \quad \partial \mu \tilde{J}^5_\mu = \frac{1}{8\pi^2} \left( F_{V \mu \nu} \tilde{F}_{V \mu \nu} + \frac{1}{3} F_{A \mu \nu} \tilde{F}_{A \mu \nu} \right).
\]

(33)

This is called the "covariant" form of the anomaly. The theory is now invariant, and operators transform covariantly with respect to the vector gauge symmetry. We thus see that the coefficient of the \( F_{V \mu \nu} \tilde{F}_{V \mu \nu} \) in the divergence of the axial current now corresponds to Adler’s result in QED \[8\]. The vector current is conserved even though there is an axial vector background field. Thus, Adler’s coefficient of the anomaly \[8\] arises as a mixture of the Weyl fermion (consistent) anomaly and the counterterm. In fact, for the \( F_{V \mu \nu} \tilde{F}_{V \mu \nu} \) part, the Adler coefficient is \( 1/3 \times \) (triangle diagrams) plus \( 2/3 \times \) (counterterm). These coefficients are often confused in the literature when authors incorrectly assume that the left-handed Weyl current has an anomaly coefficient that, in magnitude, is half that of Adler’s result.

One might wonder what kind of UV completion theory leads to this counterterm in the effective action. In fact, as we see below (and in Appendix B), this counterterm is just the Chern-Simons term expressed as \( D = 4 \) effective interaction, arising from our \( D = 5 \) theory, when we truncate on the zero-mode and first KK-mode.

In our present situation, therefore, a theory with the spatial delocalization of the chiral fermions where anomalous representations are placed on distinct branes in \( D = 5 \), dictates the use of the consistent anomaly on each brane. We have the correspondence \( B_\mu(0) = A_L = V_\mu - A_\mu \) and \( B_\mu(R) = A_R = V_\mu + A_\mu \), hence the anomalies take the form on the branes:

\[
\partial \mu J^\mu_L = -\frac{1}{48\pi^2} F_{\mu \nu}(0) \tilde{F}_{\mu \nu}(0) \quad (\text{brane I})
\]

\[
\partial \mu J^\mu_R = \frac{1}{48\pi^2} F_{\mu \nu}(R) \tilde{F}_{\mu \nu}(R) \quad (\text{brane II})
\]

(34)

From eq.(23), eq.34, the shift in the action under the gauge transformation,

\[
\psi_L(x_\mu) \rightarrow \exp(i\theta(0, x_\mu))\psi_L(x_\mu)
\]

\[
\psi_R(x_\mu) \rightarrow \exp(i\theta(R, x_\mu))\psi_R(x_\mu)
\]

(35)
thus takes the form:

\[
S_{\text{branes}} \to S_{\text{branes}} + \frac{1}{48\pi^2} \int_I d^4x \theta(x_\mu, 0) F^{\mu\nu} \tilde{F}_{\mu\nu}(0) - \frac{1}{48\pi^2} \int_{II} d^4x \theta(x_\mu, R) F^{\mu\nu} \tilde{F}_{\mu\nu}(R)
\]  

(36)

On the other hand, the CS term produced, under the gauge transformation, the result of eq.(36) on the boundaries:

\[
S_{\text{CS}} \to S_{\text{CS}} - \frac{c}{2} \int_I d^4x \theta(x_\mu, 0) F^{\mu\nu} \tilde{F}_{\mu\nu} + \frac{c}{2} \int_{II} d^4x \theta(x_\mu, R) F^{\mu\nu} \tilde{F}_{\mu\nu}
\]

(37)

(note the factor of 1/2 from the introduction of \( \tilde{F} \)). The cancellation of the anomalies therefore requires:

\[
c = \frac{1}{24\pi^2}
\]

(38)

Our discussion may have been somewhat tedious, but this result is essential to the correct implementation of the CS-term in a higher dimensional theory. This is one of several ways to obtain the quantization of the coefficient of the CS-term in \( D = 5 \). With multiple copies of boundary fermions we can have integer multiples of \( c \). We caution the reader that different configurations of boundary branes, or chiral fermions in the bulk, can lead to differing results for \( c \) in different domains of the extra dimension (e.g., see the discussion in the non-abelian case below eq.(38)). For the configuration of a physical domain \([0, R]\) with a pair of boundary branes \( I \) and \( II \) the result of eq.(38) is the correct coefficient of the Chern-Simons term in the \( U(1) \) and non-abelian cases.

C. Mode Expansion

Let us now consider the compactification in flat space-time. For the orbifold (magnetic superconducting branes) \( A_\mu \) is defined as an even function in the doubled interval \([0, 2R]\) and \( A^5 \) is odd. The physical extra dimension spans the interval \([0, R]\), which dictates the normalization of the fields. We perform a conventional mode expansion for the KK-mode...
tower of gauge fields:

\[ A_0^0(x, y) = \sqrt{\frac{1}{R}} \tilde{e} A_\mu^0(x) \]

\[ A_\mu(x, y) = \sum_{n=1}^{\infty} (-1)^n \sqrt{\frac{2}{R}} \tilde{e} \cos(n \pi y/R) A_n^\mu(x) \]

\[ A_5(x, y) = \sum_{n=1}^{\infty} (-1)^{n+1} \sqrt{\frac{2}{R}} \tilde{e} \sin(n \pi y/R) A_n^5(x) \]  

(39)

The sign conventions, \((-1)^n\), are designed so that the \( A_n^\mu, B_n^\mu \) with \( n \) odd couple with a positive sign to the axial current, \( \bar{\psi} \gamma^5 \psi \).

The kinetic terms contained in \( S_0 \equiv S_1 + S_2 \) become:

\[ S_1 = -\frac{1}{4\tilde{e}^2} \int_0^R dy \int d^4x \ F_{\mu\nu} F^{\mu\nu} = -\frac{1}{4} \sum_n \int d^4x \ F_n^{\mu\nu} F_n^{\mu\nu} \]

\[ S_2 = \frac{1}{2\tilde{e}^2} \int_0^R dy \int d^4x \ F_{\mu5} F^{\mu5} = \frac{1}{2} \sum_n M_n^2 \int d^4x \ B_n^\mu B_n^\mu \]

(40)

(41)

where the \( B_n^\mu(x) ( A_n^\mu(x)) \) without the argument \( y \) are \( D = 4 \) fields:

\[ M_n = n\pi/R; \quad B_n^\mu = A_n^\mu + \frac{1}{M_n} \partial_\mu A_n^5; \quad F_n^{\mu\nu} = \partial_\mu B_n^\nu - \partial_\nu B_n^\mu. \]

(42)

We thus observe that the gauge field mass term of eq.(41) is manifestly gauge invariant. That is, it is automatically expressed in terms of the \( B_n^\mu \), which are “Stueckelberg fields.” The Stueckelberg fields are combinations of transverse and longitudinal gauge fields that are manifestly gauge invariant, \( i.e. \), if we shift \( A_n^\mu \to A_n^\mu + \partial_\mu \theta_n \) then we can also shift \( A_5^n \to A_5^n - M_n^0 \theta_n \) and we see that \( B_n^\mu \) is invariant. The \( B_n^\mu \) fields have the same mode expansion as the \( A_n^\mu \) in eq.(39).

The physical value of the electric charge follows by considering the zero mode component of a coupling \( A_\mu J^\mu \), where \( J^\mu \) is the vector current on the branes (sum of left current on \( L \) and right current on \( R \)), and we see that:

\[ e = \tilde{e}/\sqrt{R} \equiv e_0 \]

(43)

Note that \( e \) is dimensionless, since \( \tilde{e} \) has dimensions of \( M^{-1/2} \). Likewise, if we consider a transverse KK-mode coupling to a current on the brane, \( A_n^\mu J^\mu \) we see that the coupling differs by a normalization factor of \( \sqrt{2} \), and we thus define:

\[ e' = \sqrt{2} \tilde{e}/\sqrt{R} = \sqrt{2} e \equiv e_n \quad (n \neq 0) \]  

(44)
The $B_n^\mu$ couple as:

$$\bar{\psi}\gamma_\mu \psi_L \sum_n (-1)^n e_n B_n^{\mu} + \bar{\psi}\gamma_\mu \psi_R \sum_n e_n B_n^{\mu}$$

$$= \bar{\psi}\gamma_\mu \psi \left( \frac{1}{2} \sum_n (1 + (-1)^n) B_n^{\mu} \right) + \bar{\psi}\gamma_\mu \gamma^5 \psi \left( \frac{1}{2} \sum_n (1 - (-1)^n) B_n^{\mu} \right) \quad (45)$$

We now turn to the Chern-Simons term, eq.(16) with the quantized coefficient of eq.(38). We substitute the mode expansion:

$$S_{CS} = \frac{1}{24\pi^2} \int_0^R dy \int d^4 x \epsilon^{\mu\nu\rho\sigma} (\partial_y B_\mu) B_\nu F_{\rho\sigma}$$

$$\equiv \frac{1}{12\pi^2} \sum_{nmk} \int d^4 x (e_n e_m e_k) c_{nmk} (B_n^{\mu} B_m^{\mu} \tilde{F}^{k\mu\nu}) \quad (46)$$

The structure constants, $c_{nmk}$, are determined by performing the wave-function overlap integrals in the bulk, which is straightforward:

$$c_{nmk} = (-1)^{(k+n+m)} \int_0^1 dz \partial_z \left[ \cos(n\pi z) \cos(m\pi z) \cos(k\pi z) \right]$$

$$= \frac{n^2(k^2 + m^2 - n^2) \left[ (-1)^{(k+n+m)} - 1 \right]}{(n + m + k)(n + m - k)(n - k - m)(n - m + k)} \quad (47)$$

Note that in particular cases of interest to us these reduce to:

$$c_{nm0} = c_{n0m} = -\frac{n^2}{n^2 - m^2} \left[ (-1)^{n+m} - 1 \right]$$

$$c_{0nm} = c_{000} = 0$$

$$c_{n00} = [1 - (-1)^n]. \quad (48)$$

The selection rules for KK-mode production and decay can almost be inferred from these results, but the effects of the matter fields must also be incorporated. For example, while the CS-term appears to allow a KK-mode decay to two zero modes, since $c_{n00}$ is nonzero, we actually find in Section IV that this is completely cancelled by the triangle diagrams in the $m^2 > M_n^2$ limit, (while it remains allowed in the case of axions on the branes). The decay of an odd (even) KK-parity mode to an even (odd) KK parity mode plus a zero mode, through the nonzero $c_{n00}$ is, however, allowed when the triangle diagram effects are included.

The effective action for the full theory can now be written. It is again convenient to reabsorb the coupling constants into the gauge fields, and write the effective action in the
following compact form as an effective $D = 4$ theory:

$$S_{\text{full}} = \int d^4x \left[ \bar{\psi}(i\gamma^\mu + V + A_5)\gamma^5 - m\right]\psi + \frac{1}{12\pi^2} \sum_{nmk} c_{nmk} B^m_\mu B^m_\nu \tilde{F}^{\mu\nu}$$

$$- \frac{1}{4e^2} F^0_{\mu\nu} F^{0\mu\nu} - \frac{1}{4e^2} \sum_{n\geq 1} F^n_{\mu\nu} F^{n\mu\nu} + \sum_{n\geq 1} \frac{1}{2e^2} M^2_n B^n_\mu B^n_\mu \right] \quad (49)$$

where:

$$V_\mu = \sum_{n \text{ even}} B^n_\mu, \quad A_\mu = \sum_{n \text{ odd}} B^n_\mu \quad (50)$$

and the photon zero mode is defined as: $A_\mu \equiv B^0_\mu$. Sums over even $n$ now include $n = 0$, unless otherwise indicated. We caution that $S_{\text{tree}}$ must still be supplemented by the boundary brane matter effects, i.e., the triangle diagram loops of Appendix C.

In eq.(49) we have supplemented the action with an electron mass term. The electron mass has to be viewed as arising from a bilocal bulk term in the parent theory of the form:

$$m \bar{\psi}L(x_\mu,0) W \psi_R(x_\mu,R) + \text{h.c.}, \quad W = \exp(i \int_0^R A_5(x_\mu,x_5) dx^5) \quad (51)$$

This is gauge invariant in the full $D = 5$ theory owing to the Wilson line, $W$. When we perform the gauge transformation of eq.(12) the Wilson line becomes $W = U(0)WU(R)^\dagger = 1$ and the mass term goes into the Dirac form, $-\bar{\psi}\psi$, as displayed in eq.(49). The full set of gauge transformations on the branes thus form a $U(1)_L \times U(1)_R$ gauged chiral lagrangian. Since the $B^n_\mu$ axial fields are in the Stueckelberg form, there is no conflict with the gauge invariance of the electron mass term. Moreover, the vectorial gauge transformations, i.e., those with $\theta(0) = \theta(R)$ commute with the electron mass term.

The KK-modes with even (odd) $n$ must now be interpreted as vectors, with $J^P = 1^-$ (axial-vectors, with $J^P = 1^+$). The $\epsilon$ symbol has locked the internal KK-mode parity of the $x_5$ wave-functions in the bulk to the parity of space-time. Put another way, the CS-term is explicitly violating the notion of independent symmetries of KK-mode parity and space-time parity. All of the massive $B^n_\mu$ are seen to be gauge invariant in the sense of Stueckelberg fields, i.e., they appear in the gauge invariant combinations as written in eq.(42).

Note that if we truncate the theory on the zero mode $B^0$ and first KK-mode, $B^1$, the CS term goes into the form:

$$\frac{1}{12\pi^2} c_{100} B^1_\mu B^0_\nu \tilde{F}^{0\mu\nu} = \frac{1}{6\pi^2} \epsilon^{\mu\nu\rho\sigma} A_\mu V_\nu \partial_\rho V_\sigma \quad (52)$$
where the $c_{111}$ term vanishes by antisymmetry. This is precisely the same form as the counterterm of eq. (31), and correspondingly the full vector and axial vector currents have covariant anomalies as in eq. (33). Using a remarkable identity amongst the $c_{nmk}$ structure constants, obtained in eq. (B6), the full current and anomaly structure for the tower of KK-modes is derived in Appendix B. This yields the covariant form of the anomaly for the tower of KK modes, and the vector zero-mode current is indeed conserved.

D. The Full Action Including Matter Effects

We now supplement the full tree action of eq. (49) with the contributions of the matter fields. We do this by integrating out the matter fields. The gauge field part of the full action now takes the form:

$$S_{\text{tree}} = \int d^4x \left[ \frac{1}{12\pi^2} \sum_{nmk} c_{nmk} B_{\mu}^n B_{\nu}^m \tilde{F}^{k\mu\nu} - \frac{1}{4e^2} F_{\mu\nu}^0 F_{\mu\nu}^0 - \frac{1}{4e'^2} \sum_{n \geq 1} F_{\mu\nu}^n F_{\mu\nu}^n + \sum_{n = 0} 2e_n^2 M_n^2 B_{\mu}^n B_{\mu}^n \right]$$

(53)

where the $c_{nmk}$ have been replaced by the $\tau_{nmk}$. In Appendix A we show that with axions as matter fields the CS term is unmodified:

$$\tau_{nmk} = c_{nmk} \quad \text{(axions)}$$

(54)

In this case the photon has acquired a mass, and we include a nonzero $M_0^2$ term. There is also a residual physical axion, $\phi^{(-)}$ and terms containing it must be included into eq. (53) from eq. (A10). The full mass matrix of KK-modes further involves small off-diagonal mixings of the massive photon with the $n$ even KK-modes, as displayed in eq. (A10). These off-diagonal mixings are negligible, but the diagonal photon mass in a unitary gauge makes the photon itself a Stueckelberg field and gauge invariance is protected by shifts in the $\phi^{(+)} = (\phi_L - \phi_R)/2$ combination of the brane axions.

The more interesting case of chiral fermions is studied in Appendix C. The simplest case is that of the decoupled fermion, in which $m$ is large compared to $M^a$, $M^b$ and $M^c$, and $a \rightarrow b + c$ is the exclusive tree body decay mode mediated by the CS term and anomaly. We obtain for the boundary term an effective operator, $O_3$, describing the 3-gauge boson amplitude of the triangle diagrams from Appendix C, in eq. (C45) (with coupling constants
restored):
\[
O_3 = -\frac{1}{12\pi^2} \epsilon^{\mu\nu\rho\sigma} \sum_{nmk} (e_n e_m e_k) a_{nmk} B^a_\mu B^m_\nu \partial_\rho B^k_\sigma
\]
(55)

where:
\[
a_{nmk} = \frac{1}{2}(1 - (-1)^{n+m+k})(-1)^{m+k}
\]
(56)

Note that this has the form of the counterterm that mediates consistent and covariant anomalies when truncated on the lowest modes. In this case, we therefore have:
\[
\bar{c}_{nmk} = c_{nmk} - a_{nmk} \quad \text{(massive spinors)}
\]
(57)

hence we can write:
\[
\bar{c}_{nmk} = \left[(-1)^{(k+n+m)} - 1\right] \left(\frac{n^2(k^2 + m^2 - n^2)}{(n + m + k)(n + m - k)(n - k - m)(n - m + k)} + \frac{1}{2}(-1)^{m+k}\right)
\]
(58)

By adding these terms we are decoupling the fermions and the effective action becomes purely bosonic.

**IV. NEW PHYSICS FROM THE CHERN-SIMONS TERM**

From the CS term in eq.(53) we deduce the Feynman rule for a vertex as shown in Fig. 2 for the process \(B^a \rightarrow B^b + B^c\):
\[
T_{CS} = -\frac{e e'^2}{12\pi^2} \left[ (-\bar{\epsilon}_{abc} + \bar{\epsilon}_{bac} - \bar{\epsilon}_{cba})[B] + (\bar{\epsilon}_{acb} - \bar{\epsilon}_{cab} + \bar{\epsilon}_{bca} - \bar{\epsilon}_{cba})[A] \right]
\]
(59)

where \([A]\) and \([B]\) are (as in Appendix C):
\[
[A] = \epsilon^{\mu\nu\rho\sigma} \epsilon^a_\mu \epsilon^b_\nu \epsilon^c_\rho \gamma^\sigma \quad [B] = \epsilon^{\mu\nu\rho\sigma} \epsilon^a_\mu \epsilon^b_\nu \epsilon^c_\rho \gamma^\sigma
\]
(60)

Here we have used momentum conservation \(p = k + q\). and we have rescaled coupling constants back into the interaction of eq.(53). As written, \(q\) and \(k\) are outgoing momenta, and give a factor of \(+i\); we also have \(+i\) from \(e^{iS}\) and \(-1\) from the CS-term coefficient.

**A. Decay of KK-mode to KK-mode plus \(\gamma\)**

As an example application of the formalism, let us now compute the tree approximation decay width of the \(a\)th KK-mode into the \(b\)th KK-mode plus a massless zero-mode (photon).
FIG. 2: Chern-Simons term plus triangle diagrams yield a three body vertex describing the decay of KK-modes $a \rightarrow b + c$. In the text we study the special case where $c = 0$ and $B^c$ is the photon $\gamma$.

For the $a$th KK-mode of 4-momentum $p_\mu$, polarization $\epsilon^a_\mu$, decaying to the $b$th mode of momentum $k_\mu$, polarization $\epsilon^b_\mu$ and the $\gamma$ of momentum $q_\mu$, polarization $\epsilon_\mu^\gamma$, eq.(53) leads to the Feynman rule for the 3-body vertex of Fig.(2):

$$T_{CS} = \frac{ee'}{12\pi^2} \left[ \left( -\epsilon^a_{ab0} + \epsilon^a_{b0a} - \epsilon^a_{0ba} \right) [B] + \left( \epsilon^a_{a0b} + \epsilon^a_{b0a} - \epsilon^a_{0ba} \right) [A] \right]$$  \hspace{1cm} (61)

The pure CS term using the coefficients $c_{nmk}$ yields

$$T_{CS} = \frac{ee'}{6\pi^2} \left[ \left( a^2 + 2b^2 \right) [B] - [A] \right]$$  \hspace{1cm} (62)

Here we have used the pure $c_{nmk}$ structure constants of eqs.(47,48) and we have assumed that $a + b$ is odd, as required for a nonzero result. Note that the KK-mode masses are $M_a = \pi a / R$, so the vertex rule can be written as:

$$T_{CS} = \frac{ee'}{6\pi^2} \left[ \left( 1 + \frac{3M_b^2}{M_a^2 - M_b^2} \right) [B] - [A] \right]$$  \hspace{1cm} (63)

As we have stressed, it is the combination of the matter fields and the CS-term that is gauge invariant.

For the massive fields, the condition on the outgoing polarization $p^\mu \epsilon^a_\mu$ is fixed (essentially by our gauge choice, $A^5 = 0$). This arises from the free field equation of motion of the Stueckelberg fields, $\partial_\mu F^{a\mu\nu} + M_a^2 B_\nu = 0$ whence $\partial_\mu B^\mu = 0$. However, for the massless zero mode (photon) there is no such restriction on the polarization. Correspondingly under the zero mode gauge transformation:

$$\epsilon^\gamma_\mu \rightarrow \epsilon^\gamma_\mu + \kappa q_\mu$$  \hspace{1cm} (64)
we see that the term \([B]\) is gauge invariant, while \([A]\) is \textit{not} invariant, undergoing a shift with eq.(64).

We now include the fermion triangle loops. The triangle diagrams in the large \(m\) limit produce a simple expression for the amplitude that has been computed in eq.(55), or eq.(C45) and thus contribute the \(a_{nmk}\) coefficients of eq.(56) to the \(c_{nmk} = c_{nmk} - a_{nmk}\) combination. For the process of interest, \(B^a \rightarrow B^b + \gamma\) we see that the eq.(55) contribution takes the form:

\[
T_L + T_R = \frac{ee'^2}{6\pi^2} \left[ [A] - \frac{1}{2} (3(-1)^b - 1)[B] \right] + \mathcal{O}(1/m^2) \tag{65}
\]

We can check this result directly from the vertices in eqs.(C39,C41) where we take \(c = 0\) and \(a + b\) is then odd. Hence, \(f_{ab0} = 2\) and \(g_{ab0} = -(3(-1)^b - 1)\) and eq.(C39) yields the above result.

Thus, the resulting full Feynman vertex rule is the sum of eq.(63) and eq.(65), and takes the form:

\[
\overline{T} \equiv T_{CS} + T_L + T_R = \frac{ee'^2}{2\pi^2} \left( \frac{M^2_a}{M^2_a - M^2_b} - \frac{1}{2}((-1)^b - 1) \right) [B]. \tag{66}
\]

We see that the \([A]\) term, which violated the electromagnetic gauge invariance, has miraculously cancelled, as indeed it must! We now have a result that is fully gauge invariant: the longitudinal component of the zero-mode, \(\epsilon^\gamma \propto q\), decouples from the full amplitude, since \(B \rightarrow 0\) when \(\epsilon^\mu \rightarrow q\).

The physical transition amplitudes for abnormal parity, \(1^+ [a \text{ odd}, b \text{ even}]\), and normal parity, \(1^- [a \text{ even}, b \text{ odd}]\), \(B^a\) decay thus take the compact forms:

\[
\overline{T}^+ = \frac{ee'^2}{2\pi^2} \left( \frac{M^2_a}{M^2_a - M^2_b} \right) [B] \quad 1^- \rightarrow 1^+ + \gamma
\]

\[
\overline{T}^- = \frac{ee'^2}{2\pi^2} \left( \frac{M^2_b}{M^2_a - M^2_b} \right) [B] \quad 1^+ \rightarrow 1^- + \gamma \tag{67}
\]

Several comments on the physical structure of the decay amplitude are in order. We see that if the massive gauge fields are nearby in mass, \(M^2_a \approx M^2_b\), then the amplitude is mainly dominated by the CS term, eq.(33). On the other hand, in the limit \(M^2_b \ll M^2_a\) the decay amplitude is dominated by a coherent superposition of the Chern-Simons term and the matter anomalies (triangle diagrams) of eq.(65). This does not mean that the partial width is predominantly governed by the \(M^2_a \approx M^2_b\) limit, since here the phase space is becoming small (in fact, we’ll see that it is dominated by the decays to the lightest KK-mode longitudinal components, since those have the smallest “decay constants,” \textit{i.e.}, the
final state couples as \( \sim k^\mu /M_b \) giving an \((M_a/M_b)^2\) enhancement. The smallest allowed difference \( M_a^2 - M_b^2 \) for large \( a \), is given by \( b = a - 1 \), whence the mass \( M_a^2 = a^2 \pi^2/R \) and mass difference is \( M_a^2 - M_{a-1}^2 \approx 2a\pi^2/R^2 \). In the limit of large \( a \) and \( b = a - 1 \) we then have:

\[
\mathcal{T}^+ \approx \mathcal{T}^+ \approx \frac{ee^2a}{4\pi^2}[B] \tag{68}
\]

This growth of the amplitudes with large \( a \approx b \) can be viewed as a short distance limit. The factor \( a \) is yielding the power-law running of the coupling constant, \( e^2a \equiv e^2(M_a) \). In the limit \( M_b^2 \ll M_a^2 \) the decay amplitudes are suppressed, and by the anomaly (triangle diagrams) CS-term interference amplitude:

\[
\mathcal{T} \rightarrow \frac{ee^2}{4\pi^2} \left((-1)^b + 1\right) [B] \tag{69}
\]

The odd-even effect here, is a consequence of the \( D = 5 \) overlapping wavefunctions. It is reminiscent of a chiral flip suppression as in, \( e.g., \pi^\pm \) decay to \( e\nu \), but it involves the particular matching of the CS term with the boundary anomaly in a way that makes the result somewhat opaque. This can presumably vary if a different theory is taken on the branes. For example, in the case of the brane axions as in Appendix A there is no brane triangle diagram contribution and this odd-even effect is washed out.

Turning to the partial width calculation, the zero-mode gauge invariance implies, as usual, that in summing over final state \( \gamma \) polarizations we can use the familiar:

\[
\sum_\lambda \epsilon^a_\mu(\lambda)\epsilon^a_\nu(\lambda) = -g_{\mu\nu} \tag{70}
\]

and conveniently drop the (singular) \( q_\mu q_\nu /q^2 \) terms.

For the heavy vector meson polarizations the source-free equations of motion, eq.(B11), imply that the \( B^n_\mu \) (Stueckelberg) fields must obey the Lorentz gauge condition, \( \partial^\mu B^n_\mu = 0 \) (as usual, the outgoing and incoming fields are treated as freely propagating fields, so we ignore their anomalous source terms, which are part of the interaction generating the transition). Thus, the polarization sums for massive fields are:

\[
\sum_\lambda \epsilon^a_\mu(\lambda)\epsilon^a_\nu(\lambda) = -\left(g_{\mu\nu} - \frac{p_\mu p_\nu}{M_a^2}\right)
\]

\[
\sum_\lambda \epsilon^b_\mu(\lambda)\epsilon^b_\nu(\lambda) = -\left(g_{\mu\nu} - \frac{k_\mu k_\nu}{M_b^2}\right) \tag{71}
\]
Note that if we had axions as matter fields there would be no loop correction and the original vertex, $T_{CS}$, of eq.(63) would be the full result. However, then the zero mode photon would be a massive Stueckelberg field, and we would use the polarization sums:

$$\sum_{\lambda} \epsilon^\gamma_{\mu}(\lambda) \epsilon^\gamma_{\nu}(\lambda) = - \left( g_{\mu\nu} - \frac{p_\mu p_\nu}{m_\gamma^2} \right)$$  \hspace{1cm} (72)

Squaring the amplitude and summing over the $b$ and $\gamma$ polarizations and averaging over the $a$ polarization yields:

$$\langle T^2 \rangle = \frac{1}{3} \left( \frac{e'}{\pi^2} \right)^2 \left( \frac{M_a^2}{M_b^2} \right) (M_a^2 + M_b^2) \hspace{1cm} 1^- \rightarrow 1^+ + \gamma$$

$$\langle \bar{T}^2 \rangle = \frac{1}{3} \left( \frac{e'}{\pi^2} \right)^2 \left( \frac{M_b^2}{M_a^2} \right) (M_a^2 + M_b^2) \hspace{1cm} 1^+ \rightarrow 1^- + \gamma$$  \hspace{1cm} (73)

where we have set $e'^2 = 2e^2$. The former case is quite singular in the $M_b << M_a$ limit. This arises from the decay of the transverse $1^-$ particle into the longitudinal $1^+$. The $M_b^2/M_a^2$ suppression in the $1^+ \rightarrow 1^- + \gamma$ is the analogue of a chiral suppression, such as in $\pi \rightarrow e\nu$.

Thus, putting in phase space, the partial decay width for $a \rightarrow b + \gamma$ in the $M_a^2 >> M_b^2$ limit, is:

$$\Gamma_{1^-\rightarrow1^+\gamma} = \frac{2\alpha^3}{3\pi^3} \left( \frac{M_a^3}{M_b^2} \right) \hspace{1cm} \Gamma_{1^+\rightarrow1^-\gamma} = \frac{2\alpha^3}{3\pi^3} M_b.$$  \hspace{1cm} (74)

In the limit $\Delta M = M_a - M_b << M_a$ we have:

$$\Gamma_{1^\pm\rightarrow1^{\mp}\gamma} = \frac{2\alpha^3}{3\pi^3} \Delta M.$$  \hspace{1cm} (75)

The most conspicuous effect is the $M_a^2/M_b^2$ enhancement. This could be absorbed into $\alpha'^2$ factors ($\alpha' = 2\alpha$) and viewed as a power-law running of the coupling constants:

$$\Gamma_{1^-\rightarrow1^+\gamma} = \frac{\alpha(0)\alpha'^2(M_a)}{6\pi^3} \left( \frac{M_a}{R^2 M_b^2} \right)$$

$$\Gamma_{1^+\rightarrow1^-\gamma} = \frac{\alpha(0)\alpha'^2(M_a)}{6\pi^3} \left( \frac{M_b}{R^2 M_a^2} \right)$$  \hspace{1cm} (76)

The point of writing this latter result is that there is nothing particularly pathological about the enhanced decay rates of superheavy KK modes through the CS term, compared to the usual pathologies of extra dimensions. The expected order of magnitude for such decays would have been $\sim \alpha(0)[\alpha'(M_a)]^2 M_a$ and we see that typically $R^2 M_b^2 = \pi^2 b^2 >> 1$. Thus, these decay widths are consistent with naive expectations.
The quadratic growth of the width $\Gamma_{1^{-}\to 1^+}\gamma \sim M_a^2$ is presumably related to the quadratic divergence of the two point function in the continuum $D = 5$ theory where the Chern-Simons term plays the role of the interaction as in Fig.(2). While we have not performed the detailed analysis, this loop presumably satisfies a $D = 5$ dispersion relation (a sum-rule in the $D = 4$ effective theory), and yields the radiative corrections to the power-law growth of the coupling constant.

B. Zero Mode + Zero Mode $\to$ KK-Mode Vanishes

We note that the CS-term contains the vertex describing $a \to 0 + 0$:

$$ T_{CS} = -\frac{ee'^2}{12\pi^2} \left[ (-\bar{c}_{a00} + \bar{c}_{0a0} + \bar{c}_{00a} - \bar{c}_{00a})[B] + (\bar{c}_{a00} - \bar{c}_{0a0} + \bar{c}_{00a} - \bar{c}_{00a})[A] \right] $$

(77)

Consider the case in which $\bar{c}_{nmk} = c_{nmk}$ and using $c_{a00} = (1 - (-1)^a)$, $c_{0mk} = 0$ we have the amplitude:

$$ T_{CS} = -(1 - (-1)^a) \frac{ee'^2}{12\pi^2} ([A] - [B]) $$

(78)

This is the result for the axionic case.

For fermionic matter fields we see from eq.(56) that the $a_{a00} = (1 - (-1)^a)/2$, $a_{0a0} = a_{00a} = (-1)^a(1 - (-1)^a)/2$, thus:

$$ T_L + T_R = (1 - (-1)^a) \frac{ee'^2}{12\pi^2} ([A] - [B]), $$

(79)

and, the combined amplitude is:

$$ T_{CS} + T_L + T_R = 0. $$

(80)

There are no couplings of a single KK-mode to two zero-modes in the $m^2 > M_a^2$ limit. This is a consequence of gauge invariance of the zero mode.

In the case of axions on the branes we would have a zero-mode + zero mode $\to$ KK-mode vertex, but then the zero-mode is not massless. The cancellation of eq.(80) may be a general result for a massless zero-mode. We caution that we have proved it in the large $m$ case and the effects of massless fermions, or an off-shell zero-mode, may be non-zero. It would be of interest to explore the standard model with split fermions, e.g., in which the $U(1)_Y$ anomalies are delocalized, to examine if processes like $Z + \gamma \to B^a$ can occur via mixing, or the nonzero $Z$ mass.
V. THE NON-ABELIAN CASE

We now consider an $SU(N)$ gauge theory in $D = 5$, with the covariant derivative:

$$D_A = \partial_A - iA_A \equiv A_A^a \frac{\lambda^a}{2}$$

(81)

The field strength and the kinetic term lagrangian density are:

$$G_{AB} = i[D_A, D_B] = \partial_A A_B - \partial_B A_A - i[A_A, A_B]; \quad L_0 = -\frac{1}{2g^2} \text{Tr}(G_{AB}G^{AB}),$$

(82)

and $1/\tilde{g}^2$ again has dimension of $M^1$. A gauge transformation involves a local gauge rotation, $U = \exp(i\theta^a(x^A)\lambda^a/2)$ and the gauge field and covariant field strength transform as:

$$A_A \rightarrow U^\dagger iD_A U + A_A - \partial_A \theta^a \frac{\lambda^a}{2} - i\theta^a A^b \left[ \frac{\lambda^a}{2}, \frac{\lambda^b}{2} \right] + ...$$

(83)

The Yang-Mills Chern-Simons term, also known as the “second Chern character,” takes the form:

$$L_{CS} = c \epsilon^{ABCDE} \text{Tr} \left( A_A \partial_B A_C \partial_D A_E - \frac{3i}{2} A_A A_B A_C \partial_D A_E - \frac{3}{5} A_A A_B A_C A_D A_E \right),$$

(84)

This can be rewritten in a convenient form involving gauge covariant field strengths,

$$L_{CS} = \frac{c}{4} \epsilon^{ABCDE} \text{Tr} \left( A_A G_{BC} G_{DE} + iA_A A_B A_C G_{DE} - \frac{2}{3} A_A A_B A_C A_D A_E \right),$$

(85)

It should be noted that these expressions vanish for gauge groups having no $d$ symbols. (See [4] for further discussion and references).

We now include the Chern character into the action of a $D = 5$ theory as in the the QED case:

$$S_0 = \int d^5x \ L_0; \quad S_{CS} = \int d^5x \ L_{CS}.$$  

(86)

The variation of the action with respect to $A_A$ again generates the equation of motion:

$$\tilde{e}^2 \frac{\delta S}{\delta A^a_A} = [D_B, G^{BA}]^a - J^{aA} = 0.$$  

(87)

and there is again a conserved Chern-Simons current appearing as the source term in the theory,

$$J^a_A = \frac{3c}{2} \epsilon^{ABCDE} \text{Tr} \left( \frac{\lambda^a}{2} \{G^{BC}, G^{DE}\} \right).$$

(88)
The current explicitly requires that $SU(N)$ possess a $d$-symbol, hence $N \geq 3$; and it is covariantly conserved, $[D^A, J^a_A \lambda^a/2] = 0$.

The CS-anomaly arising on boundaries under a gauge transformation is a more complicated expression. In anomaly matching to boundary matter fields it suffices to keep track only of the $\text{Tr}(dAdA)$ terms. If we compactify the theory as we did in the QED case with boundary branes $I$ and $II$, we see that, under an infinitesimal gauge transformation, there are surface terms:

$$S_{CS} \rightarrow S_{CS} + c \int d^4x \, \theta^a(R) \, \epsilon^{\mu\nu\rho\sigma} \text{Tr}(\frac{\lambda^a}{2} \partial_\mu A_\nu \partial_\rho A_\sigma) + \ldots$$

$$-c \int d^4x \, \theta^a(0) \, \epsilon^{\mu\nu\rho\sigma} \text{Tr}(\frac{\lambda^a}{2} \partial_\mu A_\nu \partial_\rho A_\sigma) + \ldots$$

(89)

The action has shifted by the two CS anomalies.

We again consider the orbifold compactification with branes $I$ and $II$ now containing, respectively, quarks $q_L$ and $q_R$ transforming as $N$ under the $SU(N)$ gauge group. For our envisioned application we view $SU(N)$ to be the flavor symmetry, so we’ll simply assume the quarks also carry an (ungauged) color index $N_c$.

The anomalies on the branes of the chiral quarks are again required to be the consistent Yang-Mills anomalies. These are given in full form in Bardeen’s paper (and we can infer the $\text{Tr} dAdA$ terms from Appendix C). Written in terms of the chiral quarks on their respective boundary branes we have:

$$\partial_\mu J^a_\mu = -\frac{N_c}{24\pi^2} \epsilon^{\mu\nu\rho\sigma} \text{tr} \left( \frac{\lambda^a}{2} \partial_\mu A_\nu \partial_\rho A_\sigma \right) + \ldots$$

$$\partial_\mu J^a_\mu = \frac{N_c}{24\pi^2} \epsilon^{\mu\nu\rho\sigma} \text{tr} \left( \frac{\lambda^a}{2} \partial_\mu A_R \partial_\rho A_R \right) + \ldots$$

(90)

where:

$$J^a_\mu = \bar{q} \gamma^\mu \frac{\lambda^a}{2} q_L, \quad J^a_\mu = \bar{q} \gamma^\mu \frac{\lambda^a}{2} q_R, \quad \text{and} \quad \tilde{G}_{\mu\nu} = \frac{1}{2} \epsilon_{\mu\nu\rho\sigma} G^{\rho\sigma}.$$  

(91)

Under the $SU(N)$ flavor gauge transformation the quarks undergo the transformation:

$$q_L \rightarrow \exp \left( i \theta^a(0) \frac{\lambda^a}{2} \right) q_L \quad q_R \rightarrow \exp \left( i \theta^a(R) \frac{\lambda^a}{2} \right) q_L$$

(92)

and the consistent anomalies produce the shifts on the branes:

$$S_{\text{quark}} \rightarrow S_{\text{quark}} - \frac{N_c}{24\pi^2} \int_{II} d^4x \, \theta^a(R) \, \epsilon^{\mu\nu\rho\sigma} \text{Tr}(\frac{\lambda^a}{2} \partial_\mu A_\nu \partial_\rho A_\sigma)$$

$$+ \frac{N_c}{24\pi^2} \int_I d^4x \, \theta^a(0) \, \epsilon^{\mu\nu\rho\sigma} \text{Tr}(\frac{\lambda^a}{2} \partial_\mu A_\nu \partial_\rho A_\sigma) + \ldots$$

(93)
Thus, we conclude:

\[ c = \frac{N_c}{24\pi^2} \]  \hspace{1cm} (94)

This coefficient applies in the case we have constructed of two branes \( I \) and \( II \) bounding the physical interval \([0, R]\). One can consider a different construction in which the physical interval is extended to \([0, 2R]\), but the branes \( I \) and \( II \) remain located at \( x^5 = 0 \) and \( x^5 = R \), and we use periodic boundary conditions on all fields. This physically corresponds to a kink + anti-kink soliton with fermionic zero-modes on \( S_1 \). If we impose a parity symmetry on \([0, R] \leftrightarrow [R, 2R]\) then we would have a CS term in the domain \([0, R]\) with coefficient \( c_1 = -1/48\pi^2 \) and an anti-CS term in the domain \([R, 2R]\) with \( c_2 = 1/48\pi^2 \). In fact, without the parity symmetry, the domains can have CS terms with arbitrary coefficients, \( c_1 \) and \( c_2 \). The anomaly matching simply requires \( c_1 - c_2 = -1/24\pi^2 \). For concreteness we use the orbifold with physical domain \([0, R]\) and \( c_1 = -1/24\pi^2 \) in the application below.

Note that we can also introduce a Wilson line mass term for our separated quarks of the form:

\[
\int d^4x \, m \, \overline{q}_L(0) P \exp \left( i \int_0^R A_5 \, dx^5 \right) q_R(R) + h.c. \]  \hspace{1cm} (95)

This will play the role of a constituent quark mass below when we derive the constituent chiral quark model, and \( P \exp(-i \int_0^R A_5 dx^5) = \exp(2i\tilde{\pi}/f_\pi) \) plays the role of the chiral field of mesons when we truncate on the zero-mode of the theory. In this way, the Yang-Mills theory of flavor can be morphed into an \( SU(N)_L \times SU(N)_R \) chiral lagrangian of mesons, and the CS term becomes the Wess-Zumino-Witten term, as developed using deconstruction in ref.\([4]\).

The Chern-Simons term may again be written in a form, as in the \( U(1) \) case, that separates the \( A_5 \) and \( \partial_5 \) terms,

\[
L_1 = \frac{c}{2} \text{Tr}((\partial_5 A_\mu) K^\mu) + \frac{3c}{4} \epsilon^{\mu\nu\rho\sigma} \text{Tr}(A_5 G_{\mu\nu} G_{\rho\sigma}), \]  \hspace{1cm} (96)

where we find \([4]\):

\[
K^\mu \equiv \epsilon^{\mu\nu\rho\sigma} (i A_\nu A_\rho A_\sigma + G_{\nu\rho} A_\sigma + A_\nu G_{\rho\sigma}). \]  \hspace{1cm} (97)

In deriving this result, some total divergences in the \( D = 4 \) subspace have been discarded, which play no role in the physics or in the anomaly matching, as in the \( U(1) \) case.
Again we can perform the gauge transformation that sets $A^5 = 0$ using the Wilson line $U(y)$ and defining the “Stueckelberg fields” $B_\mu$ as:

$$U(y) = P \exp(i \int_0^y A_5(x^5) dx^5) \quad B_\mu = U^\dagger [i D_\mu, U].$$  \hspace{1cm} (98)

The Chern-Simons action thus becomes:

$$L_1 = \frac{c}{2} \epsilon^{\mu \nu \rho \sigma} \int d^4 x \int_0^R dy \, \Tr \left[ \partial_y B_\mu (i B_\nu B_\rho B_\sigma + G_{\nu \rho} B_\sigma + B_\nu G_{\rho \sigma}) \right]$$  \hspace{1cm} (99)

and the quark mass term goes into the Dirac form $m q_L q_R + h.c. = m \bar{q} q$.

As we did in the $U(1)$ case, we can pass to a KK-mode expansion and calculate the effective interaction amongst the KK-modes. Again, the KK-mode parity is now locked to space-time parity and the CS term yields new interactions, including 4-body amplitudes.

As an example application of this formalism we turn presently to something slightly different and give a simple derivation of the chiral constituent quark model and the Wess-Zumino-Witten term [2, 3] from a Yang-Mills theory in $D = 5$.

A. Application: The Wess-Zumino-Witten Term

Consider the theory truncated on the zero-mode $A^5$ field. All Stueckelberg gauge fields $B_\mu$ now become pure gauge fields:

$$B_\mu(x^\mu, y) = i U^\dagger (y) \partial_\mu U(y)$$  \hspace{1cm} (100)

and the field strength, $G_{\mu \nu}$, is now zero.

The Wilson line, $U$, now contains only the zero mode $A^5$ and the path-ordering is no longer necessary because the $x^5$ wavefunction is factorized from the flavor orientation:

$$U(y) = \exp \left( i \int_0^y A_5 dx^5 \right)$$  \hspace{1cm} (101)

We identify the Wilson line extending between the two branes with a chiral field of mesons:

$$W = \exp \left( i \int_0^R A_5 dx^5 \right) = \exp(2i\pi / f_\pi), \quad \pi = \pi^a \lambda^a / 2$$  \hspace{1cm} (102)

For the pseudoscalar octet, $f_\pi = 93$ MeV with this normalization. We remark that this completely fixes the normalization of our $\pi$ field. That is, the kinetic term, $\Tr(G_{\mu 5} G^{\mu 5}) \sim \Tr(\partial^2 \pi)^2$ normalization determines $f_\pi$ in terms of $R$ and $\tilde{g}$, but the Wilson line completely
specifies the definition of $f_\pi$ in terms of the $A^5$ line integral which is all we need presently (i.e., we can just assume the kinetic terms are correctly normalized by the relationship between $\tilde{e} R$ and $f_\pi$).

The $A^5$ wavefunction on the orbifold in the $x^5$ coordinate is $\propto \sin(\pi x^5/R)$, and using eq.(102) we can simply write:

$$U(y) = \exp(-2i h(y) \tilde{\pi}/f_\pi); \quad h(y) = \frac{1}{2}(1 - \cos(\pi y/R))$$

(103)

We can actually use any monotonic wave-function $h(y)$ satisfying $h(0) = 0$ and $h(R) = 1$. Thus, expanding the pure gauge vector potential, we see:

$$B_\mu(x_\mu, y) = -2 \times \left( -\frac{1}{f_\pi} h(y) \partial_\mu \tilde{\pi} + i \frac{1}{f_\pi} h(y)^2 [\tilde{\pi}, \partial_\mu \tilde{\pi}] + O([\tilde{\pi}, [\tilde{\pi}, \partial \tilde{\pi}]] \right)$$

(104)

Since we now have pure gauge configurations, the only surviving term in the CS-term of eq.(99) is the $\text{Tr}((\partial_y B_\mu) B_\nu B_\rho B_\sigma) \epsilon^{\mu\nu\rho\sigma}$ term. We note that $\text{Tr}(B_\mu B_\nu B_\rho B_\sigma) \epsilon^{\mu\nu\rho\sigma} = 0$ by the cyclicity of the trace, which makes eq.(99) easy to evaluate when we substitute the expression of eq.(104), and it takes the form:

$$\mathcal{L}_1 = \frac{8i c}{f_\pi^5} \epsilon^{\mu\nu\rho\sigma} \int d^4 x \int_0^R dy (\partial_y h(y)) h(y)^4 \text{Tr} [[\tilde{\pi}, \partial_\mu \tilde{\pi}] [\partial_\nu \tilde{\pi}, \partial_\rho \tilde{\pi}, \partial_\sigma \tilde{\pi}]] + ...$$

(105)

where the ellipsis refers to higher powers in the $\tilde{\pi}$ field.

Since $h(0) = 0$ and $h(y) = 1$ we see that the integral over $y$ is trivial – the integrand is an exact differential and the result doesn’t depend upon the particular shape of the wavefunction $h(y)$! For example, the zero-mode $A^5$ could have been a constant $x^5$ whence, $h(y) = y/R$, and we would obtain the same result. The resulting expression is:

$$\mathcal{L}_1 = \frac{2 N_c i}{15 f_\pi^5} \epsilon^{\mu\nu\rho\sigma} \int d^4 x \text{Tr} [\tilde{\pi} \partial_\mu \tilde{\pi} \partial_\nu \tilde{\pi} \partial_\rho \tilde{\pi} \partial_\sigma \tilde{\pi} ]$$

(106)

Where we have removed the commutator in eq.(105) using cyclicity of the trace, and we have substituted the coefficient, $c = 1/24\pi^2$. This agrees with Witten’s famous result \[3\].

In ref.[5] we develop the fully gauged WZW term from the present formalism.

We can develop a matching to the QCD spectrum from a compactified $D = 5$ Yang-Mills theory by considering the boundary conditions dual to an orbifold, $\epsilon^{ABCD5} F_{AB}|_I = \epsilon^{ABCD5} F_{AB}|_{II} = 0$, equivalently, $F_{\mu\nu}|_I = F_{\mu\nu}|_{II} = 0$. This implies a “flipped” or an “anti-orbifold” in which $A^5$ now has a zero-mode with a flat wave-function in $x^5$ (and even basis functions on the $[0, 2R]$ interval), while $A_\mu$ is now odd, beginning with a $J^P = 1^-$ massive
mode, then a $J^P = 1^+$ recurrence, etc. For an $SU(3)$ gauge group of flavor this, remarkably, has the spectrum corresponding to the QCD mesons. The Wilson line, $\int dx^5 A_5$, over the zero-modes forms the octet of pseudoscalar mesons (containing $\pi, K, \eta$), while the first $A_5$ KK-modes correspond to the $0^+$ octet ($a_0$). The $A_\mu$ KK-mode tower begins with the $(\rho)$ vector meson octet, then the $(A_1)$ axial-vector meson octet, etc. We expect the quantized coefficient of the CS term remains the same in this model, hence the WZW term remains the same. Here the CS term clearly exclusively becomes the WZW term, because the boundary conditions $F_{\mu\nu}|_I = F_{\mu\nu}|_{II} = 0$ prohibit any Stueckelberg field anomalies that might contribute to the WZW term.

VI. CONCLUSIONS

This paper has investigated the physics of the Chern-Simons term in gauge theories in compactified extra dimensions. The main thrust is that Chern-Simons terms must occur in association with “chiral delocalization,” whereby anomalous chiral fermions are placed in different locations in a $D = 5$ bulk (i.e., “split” anomalies). We view chiral delocalization as a compelling attribute of extra dimensional theories, providing a rationale for the existence of flavor-chirality (non-vectorlike representations) as is seen in the standard model. We observe that even a vectorlike theory, such as QED, must become chirally delocalized when it is imbedded into $D = 5$, to naturally protect the small electron mass. The electron mass becomes a Wilson line connected the chiral partners. In these cases the Chern-Simons term is inevitable. We study two models, (1) chiral fermions on opposing branes and (2) axions on branes.

For bulk propagating gauge fields, The Chern-Simons term locks KK-mode parity to the parity of space-time. Indeed, KK-mode parity, if it is present, is a spurious symmetry, independent of space-time parity. The Chern-Simons terms blends these symmetries into a single surviving parity. This is the exact analogue of the pion parity for the Wess-Zumino-Witten term.

Let us summarize how the analysis procedes in general. We begin in a $D = 5$ gauge theory, compactified in $0 \leq x^5 \leq R$, with chirally delocalized fermions on the boundaries (branes). The theory contains a bulk-filling Chern-Simons term. The chiral fermions have a gauge invariant mass term that is bilocal, $\sim \bar{\psi}_L(x,0)W\psi_R(x,R) + h.c.$, and involves the
Wilson line, $W = P \exp \left( i \int_0^R A_5 dx^5 \right)$ that spans the bulk. A general gauge transformation in the bulk produces anomalies on the boundaries coming from the Chern-Simons term. Likewise, this gauge transformation produces anomalies, coming from the fermions on the boundaries. These anomalies take the consistent form, i.e., they are the direct result of the Feynman triangle loops for the fermions, and have the identical same form as the anomalies from the CS term (see Appendix). We demand that these anomalies cancel, and this fixes the coefficient of the CS term, generally to $c = 1/24\pi^2$.

We now rewrite the CS term into a form that displays separately $A_5$ and $\partial_5$. We then perform a master gauge transformation that converts $A_5 \rightarrow B_5 = 0$, and $A_\mu \rightarrow B_\mu$. This also sets the Wilson line spanning the bulk between the branes to unity. The massive components of the $B_\mu$ are now gauge invariant Stueckelberg fields, having “eaten” their longitudinal degrees of freedom contained in the non-zero modes of $A_5$.

Finally, we integrate out the fermions in the large $m$ limit. This produces effective interactions (the log of the Dirac determinant) on the boundaries. The form of this effective interaction is just Bardeen’s counterterm that maps consistent anomalies into covariant ones. We thus have an expression for total action, $S_{full}$, the sum of $S_{CS}$, the Chern-Simons term, and the boundary terms from the fermionic Dirac determinant, summarized by the matrix elements of $\mathcal{O}_3$. These are functionals of the Stueckelberg fields contained in the mode expansion of $B_\mu$. The new interactions involve the “structure constants,” $c_{nmk}$, for the different KK-modes, $(n, m, k)$. We demonstrate that $S_{full}$ leads to new physical processes involving 3-body decay amplitudes amongst KK-modes. These processes violate naive KK-mode parity, but conserve the combined space-time and KK-mode parity. As an example of the formalism, we explicitly compute the decay widths for massive KK-modes into lighter KK-modes, plus the zero mode.

The Chern-Simons term coefficient was determined at the outset by cancelling the anomalies of matter fields that are localized on branes. We are lead to a study of the massless and massive consistent anomalies of Weyl fermions. The relationship between the “consistent” and “covariant” anomalies involves a counterterm, (which is shown to be our boundary term, term truncated on the zero-mode and first KK-mode of a $D = 5$ theory). The tower of KK-mode currents and their anomalies are determined by the “structure constants,” $c_{nmk}$, that appear in the Chern-Simons term in a mode expansion. The form of the covariant anomalies involves a miraculous identity amongst the $c_{nmk}$. 

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We finally develop the non-abelian formalism, to the point of computing the coefficient of the Chern-Simons term. This is similar to the $U(1)$ case, though we postpone the construction of the KK-mode effective lagrangian and derivation of the structure constants. As an example, however, we show how an $SU(N)$ Yang-Mills theory of quark flavor can be compactified into a low energy $SU(N)_L \times SU(N)_R$ chiral langrangian, where the $A^{5n}$ gauge fields have become “mesons.” In this case the Chern-Simons term built of KK-mode of gauge fields, becomes the Wess-Zumino-Witten term. We immediately obtain Witten’s coefficient for this WZW term, and the formalism will be developed in a subsequent paper, yielding the fully gauged WZW term $[3]$.

There is large number of theories to which these considerations apply. These include various incarnations of Randall-Sundrum models, Little Higgs theories, and models of (anomaly) split fermion representations in extra dimensions. We further envision applications to string theory, and AdS-CFT QCD as well. The WZW term of gravitation in a split anomaly mode, e.g., in $D = 6$ and $D = 7$, would also be an intriguing application. An interesting candidate for further study is the standard model with left-handed fermions on brane $I$ and right-handed fermions on brane $II$. An $SU(2) \times U(1)$ theory is a subgroup of $SU(3)$ and does contain $d$-symbols, and a non-abelian CS term is present to cancel the delocalized consistent matter anomalies. In this theory the Higgs mechanism has to yield the Wilson line connecting the two branes, so the Higgs field is spread out in the bulk. This is reminiscent of many “Higgsless” theories. It is of interest to explore the induced physics via the CS term in these models. It would likewise be interesting to explore an axi-gluonic QCD that can be constructed by splitting left-color onto $I$ and right-color onto $II$. Again, a quantized CS term will occur.

The background geometry has been taken flat in the present discussion. The results may have limited sensitivity to the introduction of curvature. It would be interesting to reformulate the $U(1)$ model in a Randall-Sundrum geometry to test the ideas. Moreover, gravitational CS terms could be developed in a parallel manner.

Originally, in beginning this analysis, we had hoped that the production of single KK modes through the CS term would be an available channel. For example, gluon fusion into a colored axi-gluon would be a spectacular LHC signature, or $e^+e^- \to \gamma + Z'$ might occur at a sufficiently energetic linear collider. The $U(1)$ case in the large $m_{\text{electron}}$ limit disallows such processes by the cancellation of the triangle diagrams with the CS-term for
one KK plus two zero-mode vertices, as in the coefficients $\tau_{mnk}$. However, the axion model shows that such processes can occur (though in association with photon mass). We have not explicitly checked, however, that this process does not occur for massless zero-modes in Yang-Mills, or considered processes such as “associated KK mode production” e.g., zero + zero $\rightarrow$ zero + KK. Thus, the collider physics implications of the Chern-Simons term remain to be investigated.

The $U(1)$ $D = 5$ continuum bulk theory consisting only of gauge kinetic terms and the Chern-Simons term, with matter restricted to the branes, is in itself an interesting system. The bulk topological theory is subject to some nonrenormalization constraints and it would be interesting to study its loop structure in $D = 5$ further.

**APPENDIX A: AXIONS ON THE BRANES**

The simplest anomaly free model with the bulk filling CS term involves the incorporation of “left” ($\phi_L$) and “right” ($\phi_R$) axion fields confined to the respective boundary branes. The axions are coupled to the left- and right- anomalies, and they can freely shift to absorb the induced boundary anomalies from the CS term under a gauge transformation.

The kinetic terms of the axions must be invariant under this shift, and this requires that they couple longitudinally to the left- and right-gauge fields on the respective branes. This further locks the axion shift to the gauge transformation. In the $D = 4$ effective theory, we then find that one linear combination of the axions, $\phi^{(+)}$, is eaten to become a longitudinal massive photon, where:

$$
\phi^{(+)} = \frac{1}{2}(\phi_L + \phi_R) \quad \phi^{(-)} = \frac{1}{2}(\phi_L - \phi_R) \quad (A1)
$$

$\phi^{(-)}$ remains as a massless state in the spectrum, coupled longitudinally to the pseudovector KK-modes. This is the physical axion. The model exhibits the fact that the bulk CS term interactions are physical, and yields $\tau_{nmk} = c_{nmk}$.

Consider QED in $D = 5$ on an orbifold with periodic domain $0 \leq x^5 \leq R$. The full action of the theory is,

$$
S = S_0 + S_{CS} + S_{branes} \quad (A2)
$$
where the gauge field kinetic term action for the theory is:

\[
S_0 = -\frac{1}{4e^2} \int_0^R dy \int d^4x \, F_{\mu\nu}F^{\mu\nu} - \frac{1}{2e^2} \int_0^R dy \int d^4x \, F_{\mu5}F^{\mu5}
\]  
(A3)

The action \( S_{CS} \) is defined in eq.(3). On brane I at \( y = 0 \) (II at \( y = R \)), we place an axion field \( \phi_L(x^\mu) \) (\( \phi_R(x^\mu) \)), and we define the action:

\[
S_{\text{branes}} = \frac{1}{2} \int_1 d^4x \, m^2(A_\mu(0,x_\mu) - \frac{1}{m} \partial_\mu \phi_L)^2 + \frac{1}{2} \int_{II} d^4x \, m^2(A_\mu(R,x_\mu) - \frac{1}{m} \partial_\mu \phi_R)^2 \\
+ \frac{c}{4m} \int d^4x \, \phi_L \epsilon^{\mu\nu\rho\sigma} F_{\mu\nu}F_{\rho\sigma}|_I - \frac{c}{4m} \int d^4x \, \phi_R \epsilon^{\mu\nu\rho\sigma} F_{\mu\nu}F_{\rho\sigma}|_{II}
\]  
(A4)

This construction converts our \( U(1) \) gauge theory into an effective chiral theory, i.e., with distinct axion “chiralities” living on the two distinct branes. The \( m^2 \) terms contain the axion kinetic terms, as well as longitudinal couplings to gauge fields on the respective branes, locking the \( \phi \)'s to the gauge fields under gauge transformations. Thus a gauge transformation in the bulk:

\[
A_A(x_\mu, y) \rightarrow A_A(x_\mu, y) + \partial_A \theta(x_\mu, y),
\]  
(A5)

implies, through the \( m^2 \) terms, a transformation of the axions on the branes:

\[
\phi_L \rightarrow \phi_L + m\theta(x_\mu, 0); \quad \phi_R \rightarrow \phi_R + m\theta(x_\mu, R).
\]  
(A6)

The CS term under this transformation generates anomalous terms on the branes, as in eq.(8). These anomalies are now cancelled by the matching shifts in the axion fields coupled to \( F\tilde{F}_I \) and \( F\tilde{F}_{II} \) as in eq.(A4), and the theory is gauge invariant.

We can thus perform the Wilson line gauge transformation of eq.(12) with impunity. The brane action then becomes a functional of the Stueckelberg fields \( B_\mu \):

\[
S_{\text{branes}} = \frac{1}{2} \int_1 d^4x \, m^2(B_\mu(0,x_\mu) - \frac{1}{m} \partial_\mu \phi_L)^2 + \frac{1}{2} \int_{II} d^4x \, m^2(B_\mu(R,x_\mu) - \frac{1}{m} \partial_\mu \phi_R)^2 \\
+ \frac{c}{4m} \int d^4x \, \phi_L \epsilon^{\mu\nu\rho\sigma} F_{B\mu\nu}F_{B\rho\sigma}|_I - \frac{c}{4m} \int d^4x \, \phi_R \epsilon^{\mu\nu\rho\sigma} F_{B\mu\nu}F_{B\rho\sigma}|_{II}
\]  
(A7)

where all gauge fields are now of the form eq.(15).

It is useful to write \( B_\mu(x^\mu, y) \) in terms of a zero mode, \( A^0_\mu(x^\mu) \) which is independent of \( y \), and the non-zero mode \( x^5 \) parity components, \( B^+_\mu(x^\mu, y) \) and \( B^-_\mu(x^\mu, y) \), which have non-zero KK-mode momentum. We have the \( x^5 \) parity assignments:

\[
A^0_\mu(x^\mu, y) = A^0_\mu(x^\mu, R - y), \quad B^+_\mu(x^\mu, y) = B^+_\mu(x^\mu, R - y), \\
B^-_\mu(x^\mu, y) = -B^-_\mu(x^\mu, R - y).
\]  
(A8)
Thus, on the branes we define:

$$B^+(x^\mu) \equiv B^+(x^\mu, 0) = B^+(x^\mu, R), \quad B^-(x^\mu) \equiv B^-(x^\mu, 0) = -B^-(x^\mu, R).$$  \hspace{1cm} (A9)$$

The decomposition thus takes the form:

$$B_\mu(x^\mu, y) = A^0_\mu(x^\mu) + B^\mu_+(x^\mu, y) + B^\mu_-(x^\mu, y)$$  \hspace{1cm} (A10)$$

We presently require only that $A^0_\mu(x^\mu), B^\mu_+(x^\mu, y)$ and $B^\mu_-(x^\mu, y)$ are orthogonal fields upon integration over $y$ from 0 to $R$.

After performing the gauge transformation of eq. (A5), we have brought the field $A_5(x^\mu, y) = 0$ everywhere throughout the bulk. There remains a residual gauge transformation that we can do which maintains this gauge condition, which redefines the zero mode (photon) field:

$$B^\mu_0(x^\mu) = A^0_\mu(x^\mu) + \partial_\mu \theta(x^\mu)$$  \hspace{1cm} (A11)$$

where:

$$\theta(x) = \frac{1}{2m}(\phi_L(x^\mu) + \phi_R(x^\mu)) \equiv \frac{1}{m} \phi^{(+)}$$  \hspace{1cm} (A12)$$

We do not shift the $\phi^{(+)}$ and $\phi^{(-)}$ fields under this transformation. The bulk CS term, however, generates the surface anomalies under this gauge transformation, which cancel the $\phi^{(+)}$ anomalous coupling on the branes (this can be explicitly checked by substituting eq. (A11) into the CS term and noting the integrand is an exact differential in $y$). This brings the photon field into the form of a massive Stueckelberg field and $\phi^{(+)}$, thus disappears from the action. The brane action now takes the form:

$$S_{\text{branes}} = \int d^4x \left[ \frac{1}{2} (\partial_\mu \phi^{(-)})^2 + m^2 (B^0_\mu(x^\mu) + B^\mu_+(x^\mu))^2 + m^2 (B^-_\mu(x^\mu))^2 - mB^-_\mu(x^\mu)\partial^\mu \phi^{(-)} \right]$$

$$+ \frac{c}{m} \int d^4x \phi^{(-)} [F_{B^\mu_0\nu}\tilde{F}^{\mu\nu}_{B^0} + 2F_{B^\nu_\mu\nu}\tilde{F}^{\mu\nu}_{B^\nu} + F_{B^+\mu\nu}\tilde{F}^{\mu\nu}_{B^+} + F_{B^-\mu\nu}\tilde{F}^{\mu\nu}_{B^-}].$$  \hspace{1cm} (A13)$$

This shows that the theory contains a residual massless axion, $\phi^{(-)}$ coupled to the anomalies, and longitudinally to the pseudovector KK-modes. This is reminiscent of the $\pi^0$. It is not eaten since the KK-modes, $B^+$ and $B^-$, acquire (large) masses from the bulk kinetic terms. There will also be a mass term induced for the photon of leading order order: $\approx m^2 |B^0(x^\mu)|^2$ with corrections due to mixing with the KK-modes of order $m^4/M_{KK}^2 + ...$
Let us now consider the orbifold compactification in flat space-time. The mode expansion follows the identical form as in the text, eqs. (39). The axion coupling to the KK-modes is obtained from substituting the mode expansions into eq. (A13). We define:

\[ \tilde{m}^2 = \frac{2m^2 e^2}{R} \quad (A14) \]

and we have the Stueckelberg field for the massive photon with our zero mode normalization:

\[ B_\mu^0 = A_\mu^0 - \frac{1}{\sqrt{2} \tilde{m}} \partial_\mu \phi^{(+)} \quad (A15) \]

The full \(D = 4\) effective action, including the brane axion component of eq. (A13), together with the bulk gauge field kinetic terms and CS term, becomes:

\[
S_{\text{axion}} = \int d^4x \left[ \frac{1}{2} \left( \partial_\mu \phi^{(-)} \right)^2 + \tilde{m}^2 \left( \frac{1}{\sqrt{2}} B^0_\mu + \sum_{n \text{ even}} B^n_\mu \right)^2 + \tilde{m}^2 \left( \sum_{n \text{ odd}} B^n_\mu \right)^2 - \tilde{m} B_\mu^0 \partial_\mu \phi^{(-)} \right] \\
+ \frac{c}{2\tilde{m}} \int d^4x \phi^{(-)} \epsilon^{\mu\nu\rho\sigma} \times \\
\left[ e^2 e' F^{B^0}_\mu F^{B^0}_\nu + 2ee'^2 F^{B^0}_\mu F^{B^0}_\nu \sum_{n \text{ even}} F^n_{\rho\sigma} + e'^3 \sum_{n,m \text{ even}} F^n_{\mu\nu} F^m_{\rho\sigma} + e'^3 \sum_{n,m \text{ odd}} F^n_{\mu\nu} F^m_{\rho\sigma} \right] \\
+ \int d^4x \left[ + \frac{1}{12\pi^2} \epsilon_{nmc} c_{nmc} B^m_\mu B^n_\nu \tilde{F}^{k\mu\nu} + \sum_n \left( -\frac{1}{4} F^{n\rho\sigma}_{\mu\nu} F^{n\mu\nu} + \frac{1}{2} M^n_{\mu\nu} B^n_{\mu\nu} \right) \right]. \\
(A16)
\]

This effective action is characterized by the presence of the physical axion, \(\pi^-\), the photon mass term, \(\tilde{m}^2 B^0_\mu B^0_\nu\) and the pure CS term structure constants, \(\tilde{c}_{nmk} = c_{nmk}\).

**APPENDIX B: KK-MODE CURRENTS AND COVARIANT ANOMALIES**

We derive the currents of the theory of eq. (49) by variation of the full action wrt \(B^n_\mu\). The spinor currents are supplemented by current contributions from the Chern-Simons term:

\[ \tilde{J}^n_\mu = \frac{\delta S}{\delta B^n_\mu} = \bar{\psi} \gamma_\mu \psi|_{n \text{ even}} + \bar{\psi} \gamma_\mu \gamma^5 \psi|_{n \text{ odd}} + J^{n \text{CS}}_\mu \quad (B1) \]

where \(J^{n \text{CS}}_\mu\) is the Chern-Simons current:

\[ J^{n \text{CS}}_{\mu\nu} = \frac{c_{\mu\nu\rho\sigma}}{12\pi^2} \sum_{nmk} \left[ (c_{nmk} - c_{mnk} + c_{kmn} - c_{mkn}) B^{mn\rho\sigma} \phi \right] \quad (B2) \]
In what follows we use the fermionic current consistent anomalies as computed in [9] (Appendix C.4).

We can now compute the current divergences. The consistent anomalies can be written in one compact formula for both axial and vector currents as (from eq.(30)):

$$\partial^\mu J^n_\mu = \frac{1}{48\pi^2} \sum_{mk} (1 - (-1)^{n+m+k}) F^m_{\mu\nu} \tilde{F}^{k\mu\nu}$$

(B3)

The $J^n_\mu$ are the axial currents for $n$ odd ($J^{(n \text{ odd})} = \bar{\psi}\gamma_\mu \gamma^5 \psi$) and vector currents for $n$ even ($J^{(n \text{ even})} = \bar{\psi}\gamma_\mu \psi$).

Moreover, the divergence of the Chern-Simons current takes the form:

$$\partial^\mu J^n_{\mu}^{\text{CS}} = \frac{1}{48\pi^2} \sum_{m,k} (c_{nmk} - c_{mnk} + c_{nkm} - c_{knm}) F^m_{\mu\nu} \tilde{F}^{k\mu\nu}$$

(B4)

where we have exploited the even symmetry under interchange of indices of $k \leftrightarrow m$ to resymmetrize the summand. Thus, the full current divergence takes the form

$$\partial^\mu \tilde{J}^n_\mu = \frac{1}{24\pi^2} \sum_{m,k} d_{nmk} F^m_{\mu\nu} \tilde{F}^{k\mu\nu}$$

(B5)

where we have a remarkable identity:

$$d_{nmk} = \frac{1}{2} \left[ (1 - (-1)^{n+m+k}) + (c_{nmk} - c_{mnk} + c_{nkm} - c_{knm}) \right]$$

$$= \frac{3}{2} \left[ (-1)^{n+m+k} - 1 \right] \frac{n^2(k^2 + m^2 - n^2)}{(k + m - n)(k + m + n)(k + n - m)(k - m - n)}$$

$$= \frac{3}{4} c_{nmk}$$

(B6)

where the $c_{nmk}$ are defined in eq.(47). Hence, we have the final result for the entire KK-tower of currents:

$$\partial^\mu \tilde{J}^n_\mu = \frac{1}{16\pi^2} \sum_{m,k} c_{nmk} F^m_{\mu\nu} \tilde{F}^{k\mu\nu}$$

(B7)

where $n$ even (odd) is a vector (axial vector) current. The conservation of the electromagnetic current is now manifest, since:

$$c_{0mk} = 0$$

(B8)

while the remaining $n \geq 0$ currents have anomalies.
In particular, as a check of this result, if the theory contains only the $n = 0$ photon, $\gamma$, and the $n = 1$ axial vector meson, $B$, we then have a conserved vector electromagnetic current and we find the divergence of the full axial vector current:

$$\partial^\mu J^5_\mu = \frac{1}{16\pi^2} \left( c_{100} F_{\gamma \mu \nu} \tilde{F}_{\gamma}^{k\mu\nu} + c_{111} F_B \mu \nu \tilde{F}_B^{k\mu\nu} \right)$$

$$= \frac{1}{8\pi^2} F_{\gamma \mu \nu} \tilde{F}_{\gamma}^{\mu\nu} + \frac{1}{24\pi^2} F_B \mu \nu \tilde{F}_B^{\mu\nu}$$

(B9)

The latter expression is Bardeen’s result for the axial vector anomaly when he enforces a conserved vector current. The CS term is playing the role of Bardeen’s counterterm, which brings the current divergences into the present form.

The general issue of anomaly consistency is that the full currents are sources of the gauge field equations of motion:

$$\partial^\mu F^0_{\mu \nu} = e \tilde{J}^0_\nu$$

$$\partial^\mu F^n_{\mu \nu} + M_n^2 B^n_\nu = e' \tilde{J}^n_\nu \quad (a \neq 0)$$

(B10)

Since $\partial^\mu \partial^\nu F^0_{\mu \nu} = 0$ by antisymmetry of $F_{\mu \nu}$ we see that the full $\tilde{J}^0_\mu$ must be conserved, as indeed it is. Since the axial vector mesons have masses (and are Stueckelberg fields with imbedded longitudinal components) there is not an inconsistency here either, as the double divergence implies:

$$M_n^2 \partial^\nu B^n_\nu = \partial^\nu \tilde{J}^n_\nu = (\text{anomaly})_n.$$  

(B11)

This is just the equation of motion of the longitudinal modes, the $A_5^a$ which are eaten by the $B^n_\mu$ fields.

**APPENDIX C: TRIANGLE DIAGRAMS**

The present Appendix is schematic and we only quote the main results. A detailed description of these calculations is available elsewhere.

The massless calculation yields a result equivalent to Bardeen’s result for the consistent anomalies. Bardeen performs, however, a massive spinor loop calculation and quotes the anomalies in $V \pm A$ form. All anomalies we obtain presently fully confirm Bardeen’s result in both the massless and massive cases. We do, however, see a slight subtlety in the form of the pure current divergences expressed in the $V_L$ and $V_R$ forms obtained in the massive
case (the current divergence are not the full anomaly in that case, since the \(im\bar{\psi}\gamma^5\psi\) term must be subtracted. We also obtain the effective operator description of the these currents used in the text.

1. **Single Massless Weyl Spinor**

Consider the action:

\[
S_L = \int d^4x \bar{\psi}_L(i\partial + V_L)\psi_L
\]  
(C1)

where the gauge fields:

\[
V_{L\mu} = B^a_{L\mu} + B^b_{L\mu} + B^c_{L\mu}
\]  
(C2)

couple to the current:

\[
J_{L\mu} = \bar{\psi}_L\gamma_\mu\psi_L
\]  
(C3)

and the components of \(V_L\) have the respective masses \(M^a, M^b, M^c\). We are compute the triangle loop with three distinct external fields, \(B^a, B^b\) and \(B^c\), these can be alternatively viewed as distinct momentum components of the single field \(V\). If all three fields were identical (exact Bose invariance) the amplitude would vanish, since it would involve an operator \(VVdV\) which is zero. It is the external momentum differences or flavor indices that distinguish these fields and allow non-zero operators such as \([B] \sim B^aB^b dB\) and \([B] \sim B^aB^c dB\), etc.. In the massless Weyl fermion case of interest presently, we compute in a limit \(M^a >> M^b \sim M^a \sim 0\). We can view this as an operator product expansion of the triangle diagrams in which the internal lines carrying \(p^2 = M_a^2\) are treated as a short-distance expansion.

With the particular choice of momentum routing in the figure, we have the following expression for the sum of the triangle diagram and its Bose symmetric counterpart, which have a common denominator:

\[
T = (-1)(i)^3(i)^3 \int \frac{d^4\ell}{(2\pi)^4} \frac{N_1 + N_2}{D}
\]

\[
N_1 = \text{Tr}[f_aL(\ell - \hat{q})f_cL(\ell + \hat{k})f_bL(\ell + \hat{k})]
\]

\[
N_2 = -\text{Tr}[f_aL(\ell + \hat{k})f_bL(\ell)\xi_cL(\ell - \hat{q})]
\]

\[
D = (\ell + k)^2(\ell^2)(\ell - q)^2
\]  
(C4)
FIG. 3: Bose symmetric triangle diagrams for $B^a(p) \rightarrow B^b(k) + B^c(q)$ The external lines are on mass-shell, $p^2 = M_a^2$, $k^2 = M_b^2$ and $q^2 = M_c^2$. The respective polarizations are $\epsilon^a_\mu$, $\epsilon^b_\mu$ and $\epsilon^c_\mu$. The internal momentum routing and integration momenta are chosen so that both diagrams have a common denominator.

where,

$$p = k + q, \quad L = \frac{1}{2}(1 - \gamma^5), \quad R = \frac{1}{2}(1 + \gamma^5).$$

The overall sign contains: $\times (i)^3$ (vertices; note that our vector potentials have the opposite sign to the conventions of Bjorken and Drell, hence flipping the vertex rule from $-i\gamma_\mu \rightarrow +i\gamma_\mu$), $\times (i)^3$ (propagators), $\times (-1)$ (Fermi statistics). In $N_2$ we’ve factored out an overall minus sign. Note that one must use extreme care to write the given correct cyclic ordering of the factors that make up the numerator, relative to the momentum routing signs $^{[13]}$. This affects the overall sign of the triangle loop with three gauge vertices (but is has no effect upon the $i\overline{\psi}\gamma^5\psi$ loop computed in the massive case).

We unify the denominator using:

$$\frac{1}{ABC} = 2\int_0^1 dy \int_0^y dz \frac{1}{(Az + B(y - z) + C(1 - y))^3}$$

The unified denominator becomes:

$$\frac{1}{D} = 2\int_0^1 dy \int_0^y dz \frac{1}{(\ell^2 + 2\ell \cdot (zk - (1 - y)q) + zk^2 + (1 - y)q^2)^3}$$
Shifting the loop momentum to a symmetric integration momenta, $\ell$:

$$\ell = \ell - zk + (1 - y)q$$  \hspace{1cm} (C8)

the unified denominator becomes:

$$\left(\ell^2 + z(1 - z)k^2 + y(1 - y)q^2 + 2k \cdot qz(1 - y)\right)$$  \hspace{1cm} (C9)

(see the comment on shifting momenta below).

We define the following vertex tensors:

\[ A = \epsilon_{\mu\nu\rho\sigma}^a \epsilon_{b\nu}^\mu \epsilon_{c\rho}^\sigma k^\sigma \quad \longleftrightarrow \quad -i \langle b, k; c, q | \epsilon_{\mu\nu\rho\sigma} B^{a\mu} B^{c\nu} \partial^\rho B^{b\sigma} | a, p \rangle \]

\[ B = \epsilon_{\mu\nu\rho\sigma}^a \epsilon_{b\nu}^\mu \epsilon_{c^\mu}^\sigma q^\sigma \quad \longleftrightarrow \quad i \langle b, k; c, q | \epsilon_{\mu\nu\rho\sigma} B^{a\mu} B^{c\nu} \partial^\rho B^{b\sigma} | a, p \rangle \]

\[ C = \epsilon_{\mu\nu\rho\sigma}^a \epsilon_{b\nu}^\mu \epsilon_{c\rho}^\sigma q^\sigma \quad \longleftrightarrow \quad \frac{1}{2} \langle b, k | F_{\mu\nu}^{a} \tilde{F}^{b \mu\nu} | a, p \rangle \]

\[ D = \epsilon_{\mu\nu\rho\sigma}^a \epsilon_{b\nu}^\mu \epsilon_{c^\mu}^\sigma q^\sigma \quad \longleftrightarrow \quad -\frac{1}{2} \langle c, q | F_{\mu\nu}^{a} \tilde{F}^{c \mu\nu} | a, p \rangle \]

\[ E = \epsilon_{\mu\nu\rho\sigma}^a \epsilon_{b\nu}^\mu \epsilon_{c^\mu}^\sigma q^\sigma \quad \longleftrightarrow \quad \frac{1}{2} \langle b, k; c, q | F_{\mu\nu}^{b} \tilde{F}^{c \mu\nu} | 0 \rangle \]  \hspace{1cm} (C10)

where we have indicated the corresponding operator matrix elements, $\langle \text{out} | \mathcal{O} | \text{in} \rangle$ and note:

$$\tilde{F}_{\mu\nu} = \frac{1}{2} \epsilon_{\mu\nu\rho\sigma} F^{\rho\sigma}$$  \hspace{1cm} (C11)

is the standard definition of the dual field strength.

It is a well known ambiguity of the triangle loops in momentum space that shifting loop momenta can lead to residual terms, owing to the superficial linear divergence. Such terms can only be of the form $\propto [A] - [B]$, and they would yield an anomaly that does not respect Bose symmetry in the three $a$, $b$ and $c$ channels. For example, setting $\epsilon_a \rightarrow p_\mu$ we obtain $\rightarrow -2[E] \sim -2F_L \tilde{F}_L$, while setting $\epsilon_c \rightarrow -q$ we obtain $\rightarrow +[C] \sim +F_L \tilde{F}_L$. Imposing this symmetry on the triangle loops is a luxury we have only in the massless Weyl case, since we do not have to subtract $\langle b, c | \imath m \bar{\psi}_L \gamma^5 \psi_R + h.c. | 0 \rangle$ to obtain the anomaly. Imposing Bose symmetry on the anomaly as constraint on the calculation removes the surface term ambiguity. It turns out, however, for the particular momentum routing we have chosen the result for the anomaly is fully and nontrivially Bose symmetric, thus there is no surface term. Moreover, even the superficial log divergence (also $\propto [A] - [B]$) is cancelled, as we see below, and the triangle loops are UV finite.
We now compute the triangle loops. Since we are mainly interested in a heavy KK mode decaying to low mass KK-modes, kinematically we have:

\[ p = k + q \quad M_b^2 = k^2 \approx 0 \quad M_c^2 = q^2 \approx 0 \quad M_a^2 \approx 2k \cdot q \]  

(C12)

Hence the large \( M_a^2 \) limit corresponds to a symmetrical expansion in \( k^2/2k \cdot q \) and \( q^2/2k \cdot q \).

For the large electron mass limit, \( M_a^2 \ll m^2 \), we define the loop integrals with the usual Wick rotation on the loop energy \( \ell_0 \) and a Euclidean momentum space cut-off \( \Lambda^2 \):

\[
\int \frac{d^4\ell}{(2\pi)^4} \frac{(1, \ell^2)}{(\ell^2 - m^2 + i\epsilon)^3} = \left[ \frac{-i}{16\pi^2} \left( \frac{1}{2m^2} \right), \frac{i}{16\pi^2} \left[ \ln \left( \frac{\Lambda^2}{m^2} \right) - \frac{3}{2} \right] \right] \\
\int \frac{d^4\ell}{(2\pi)^4} \frac{(1, \ell^2)}{(\ell^2 - m^2 + i\epsilon)^4} = \left[ \frac{i}{16\pi^2} \left( \frac{1}{6(m^2)^2} \right), \frac{-i}{16\pi^2} \left( \frac{1}{3(m^2)^2} \right) \right] 
\]

(C13)

The familiar Wick rotation is a counterclockwise rotation of the contour of the \( \ell_0 \) integral in the complex plane. The rotation is clockwise to avoid the poles at \( \pm \sqrt{\ell^2 + m^2} \mp i\epsilon \), in the resulting Euclidean integral. For us, \( m^2 \) is actually \( m^2 - z(1 - y)M_a^2 \) from the unified denominator. In the case of \( M_a^2 >> m^2 \) the \( \ell^2 \) integrals develop a cut structure. Nonetheless, the results are analytic functions of large \( m^2 \), and in the limit \( M_a^2 >> m^2 \) we can simply replace \( m^2 \rightarrow -z(1 - y)M_a^2 \). In the massless spinor case the \( zy \) integrals now acquire infrared singularities, but massive case and the the anomaly are infrared finite.

The diagrams yield a superficial log divergence. With \( k^2 = q^2 = 0 \) (the \( \alpha_0 \) contribution) and upon doing the denominator unification integrals, this yields a finite result:

\[
T_{\text{in}(\Lambda^2)} = \frac{2i}{16\pi^2} \int_0^1 dy \int_0^y dz (2i)[(1 - 3z)A + (2 - 3y)B] \left[ \ln \left( \frac{\Lambda^2}{-z(1 - y)M_a^2} \right) - \frac{3}{2} \right] \\
= -\frac{1}{24\pi^2}[A] + \frac{1}{24\pi^2}[B] 
\]

(C14)

Note that \( T_{\text{in}(\Lambda^2)} \) is Bose symmetric under interchange of the photon and \( b \)-KK mode (let \( q \leftrightarrow k \), hence \( M_b^2 \rightarrow 0 \), \( \epsilon_a \leftrightarrow \epsilon_b \), and note \( A \leftrightarrow B \)). However, as mentioned above, this term cannot be the full result, because its anomaly is not Bose symmetric, and we indeed must keep residual finite terms.

Note, as a check on the large \( m^2 \) case, that if the argument of the log, \(-\Lambda^2/\pm(1 - y)M_a^2\) is replaced by \( \Lambda^2/m^2 \) then \( T_{\text{in}(\Lambda^2)} = 0 \). The result is finite because:

\[
\int_0^1 dy \int_0^y dz \left[ (1 - 3z)A + (2 - 3y)B \right] = 0 
\]

(C15)
This furthermore implies that the imaginary part of the expression is vanishing.

Combining all of terms of the Feynman diagrams yields the following full result for the triangle diagrams (see \cite{13} for the detailed calculation):

\[
T = -\frac{1}{12\pi^2} [A] + \frac{1}{12\pi^2} [B] \\
+ \frac{1}{4\pi^2} [Ck \cdot \epsilon_c + Ak \cdot q] \frac{I_b}{M_a^2} \\
- \frac{1}{4\pi^2} [Cq \cdot \epsilon_c + Aq^2] \frac{I_c}{M_a^2} \\
+ \frac{1}{4\pi^2} [D\epsilon_b \cdot k + Bk^2] \frac{I_b}{M_a^2} \\
- \frac{1}{4\pi^2} [D\epsilon_b \cdot q + Bk \cdot q] \frac{I_c}{M_a^2} \\
- \frac{1}{4\pi^2} M_a^2 [E] (\epsilon_a \cdot k I_b + \epsilon_a \cdot q I_c') + \mathcal{O}(q^2, k^2).
\] (C16)

The integrals \(I_i\) and \(I_i'\) are infrared divergent in our expansion. The result is manifestly Bose symmetric only if we perform the unification integrals, \(I_i\) and \(I_i'\) with a Bose symmetric IR cut-off. For a particular choice of small IR cut-offs \(x_i\), the leading log divergent terms are:

\[
I_b = \int_0^{1-x_b} \int_0^y dz \, dy \frac{z(y - y)}{z(1-y)} = \frac{1}{2} \ln(x_b) + k_b \\
I_c = \int_0^1 \int_x^y dz \, dy \frac{(1-y)(z-y)}{z(1-y)} = \frac{1}{2} \ln(x_c) + k_c \\
I'_b = \int_0^{1-x} \int_0^y dz \, dy \frac{2z - zy - z^2}{z(1-y)} = -\frac{1}{2} \ln(x_b) + k'_b \\
I'_c = \int_0^1 \int_x^y dz \, dy \frac{z + y - zy - y^2}{z(1-y)} = -\frac{1}{2} \ln(x_c) + k'_c
\] (C17)

Note that \(I_b = I_c\) and \(I'_b = I'_c\) if \(x_b = x_c\) and Bose symmetry is maintained. The physical cutoffs are of order \(x_b \sim M_b^2/M_a^2\) and \(x_c \sim M_c^2/M_a^2\), and \(k\) and \(k'\) are indeterminate. This can be replaced with a more physical procedure by resumming \(k^2\) and \(q^2\) into the denominators.

The logarithmic IR singularities in, e.g., the \(q^2 = 0\) limit are presumably cancelled by collinear \(\overline{\psi}\psi\) propagation in the \(B^c \rightarrow B^b + \overline{\psi} + \psi\) process, where \(\overline{\psi} + \psi\) rescatter into a photon.

Note, however, a final lemma that is relevant to the anomaly:

\[
I_a + I_b + I'_b + I'_c = 2
\] (C18)

which is an infra-red “safe” quantity.
2. Massless Weyl Spinor Anomaly

The amplitude we have just computed is:

\[ T = \langle b, c | T \ldots i \int d^4x \exp(-ip \cdot x)\epsilon^a_\mu \bar{\psi}\gamma^\mu \psi_L \ldots |0 \rangle \]  \hspace{1cm} (C19)

On the other hand, the amplitude we want is the matrix element of the current divergence:

\[ W = \langle b, c | T \ldots \int d^4x \exp(-ip \cdot x)\partial_\mu \bar{\psi}\gamma^\mu \psi_L \ldots |0 \rangle \]
\[ = \langle b, c | T \ldots \int d^4x (-\partial_\mu \exp(-ip \cdot x))\bar{\psi}\gamma^\mu \psi_L \ldots |0 \rangle \]
\[ = \langle b, c | T \ldots \int d^4x \exp(-ip \cdot x)ip_\mu \bar{\psi}\gamma^\mu \psi_L \ldots |0 \rangle \]  \hspace{1cm} (C20)

We thus obtain \( W \) from \( T \) by the replacement:

\[ W = T(\epsilon^a_\mu \rightarrow p_\mu) \]  \hspace{1cm} (C21)

Under this substitution we have: \( [A] \rightarrow -[E], [B] \rightarrow [E], [C] \rightarrow 0, [D] \rightarrow 0 \), and \( \epsilon_a \cdot k \rightarrow k \cdot q, \epsilon_a \cdot q \rightarrow k \cdot q. \)

We thus obtain:

\[ T \rightarrow W = \frac{1}{12\pi^2}[E] + \frac{1}{12\pi^2}[E] - \frac{1}{8\pi^2}(I_a + I_b + I'_b + I'_c)[E] \]
\[ = -\frac{1}{12\pi^2}[E] \]  \hspace{1cm} (C22)

where we use eq.(C18). The result is infra red non singular. Note that we have the operator correspondence \( [E] \rightarrow (1/4)F\bar{F} \) from eq.(C10). Our result for the anomaly thus corresponds to the operator equation:

\[ \partial^\mu \bar{\psi}\gamma_\mu \psi_L = -\frac{1}{48\pi^2}F_{\mu\nu}\bar{F}^{\mu\nu}_L \]  \hspace{1cm} (C23)

It is trivial to infer the right-handed current anomaly by flipping signs:

\[ \partial^\mu \bar{\psi}\gamma_\mu \psi_R = \frac{1}{48\pi^2}F_{\mu\nu}\bar{F}^{\mu\nu}_R \]  \hspace{1cm} (C24)

Defining \( L = V - A \) and \( R = V + A \) we can write this in the form:

\[ \partial^\mu \bar{\psi}\gamma_\mu \psi = \frac{1}{12\pi^2}F_{\mu\nu}\bar{F}^{\mu\nu}_A \]
\[ \partial^\mu \bar{\psi}\gamma_\mu \gamma^5 \psi = \frac{1}{24\pi^2}(F_{\mu\nu}\bar{F}^{\mu\nu}_V + F_{\mu\nu}\bar{F}^{\mu\nu}_A) \]  \hspace{1cm} (C25)
which agrees with Bardeen’s result for the left-right symmetric anomaly \[9\].

As a further check on the calculation, we can also examine the anomaly in the \(B^c\) current, by letting \(\epsilon_c \to -q\) (the minus sign occurs since \(B^c\) is outgoing), and we take the \(c\) field to be on-shell and massless, \(i.e.,\) set \(q^2 = 0\). Whence \([A] = [C]\) and \([B] = [D] = [E] = 0:\)

\[
T \to - \frac{1}{12\pi^2}[C] \tag{C26}
\]

Using eq.(C10) this corresponds to eq.(C23), consistent with the \(a\) channel result. Likewise, we can check the \(B^b\) channel, and verify the same result. We can, furthermore, check the off-shell gauge invariance for \(c\) identified with a photon, and \(M^2_c = 0\), setting \(\epsilon_c \to -q\) and examining the \(\mathcal{O}(q^2)\) terms. These are found to cancel \[13\]. This implies that the only non-gauge invariant part of the amplitude is the anomaly.

### 3. Finite Electron Mass

We now turn to the case of a finite, and large electron mass, where “large” means in comparison to external momenta and masses. We carry out the analysis of the loops in the presence of the full electron mass term, with the couplings

\[
\int d^4x \left( \overline{\psi}_L(i\partial + V_L)\psi_L + \overline{\psi}_R(i\partial + V_R)\psi_R - m(\overline{\psi}_L\psi_R + h.c.) \right) \tag{C27}
\]

where we take separate \(L\) and \(R\) fields, \(B^a_{\mu} L,R:\)

\[
V_{L\mu} = B^a_{\mu} L + B^b_{\mu} L + B^c_{\mu} L \quad V_{R\mu} = B^a_{\mu} R + B^b_{\mu} R + B^c_{\mu} R \tag{C28}
\]

Note that in comparison to the KK-mode normalizations used in the text we have:

\[
B^n_{L\mu} = (-1)^n B^n_\mu \quad B^n_{R\mu} = B^n_\mu \tag{C29}
\]

We will implement this relationship subsequently, but presently we work in the independent and generic \(V_L, V_R\) basis.

We presently adopt an obvious generalized notation for vertices, \(e.g.,\)

\[
A^{LRL} = \epsilon_{\mu\nu\rho\sigma} \epsilon_a^{L\mu} \epsilon_b^{R\nu} \epsilon_c^{L\rho} k^\sigma, \quad A^{LRR} = \epsilon_{\mu\nu\rho\sigma} \epsilon_a^{L\mu} \epsilon_b^{R\nu} \epsilon_c^{R\rho} k^\sigma \quad \ldots \\
C^{LR} = \epsilon_{\mu\nu\rho\sigma} \epsilon_a^{L\mu} \epsilon_b^{R\nu} k^\rho q^\sigma \quad \ldots \tag{C30}
\]

and so forth.

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We have just computed the LLL (RRR) loops arising from the pure massless $\psi_L$ ($\psi_R$). In the case of a massive electron the LLL (RRR) loops have the same numerator structure, but the denominator now contains electron mass terms:

$$D = [(\ell + k) - m^2][(\ell^2) - m^2][(\ell - q)^2 - m^2]$$

(C31)

This causes all of the previously computed LLL (RRR) terms to become suppressed in the large $m^2$ limit. For example, the $\alpha_0$ term previously computed for $m^2 = 0$ now becomes:

$$T_{\alpha_0} = -\frac{1}{4\pi^2} \int_0^1 dy \int_0^y dz \left[(1 - 3z)A + (2 - 3y)B\right] \ln \left(\frac{\Lambda^2}{m^2 - z(1 - y)M_a^2}\right) - \frac{3}{2}$$

$$\rightarrow -\frac{M_a^2}{480\pi^2 m^2} ([A] - [B]),$$

(C32)

and now vanishes in the large $m^2$ limit. All of the new terms of interest in the massive electron case arise from the numerator terms containing mass insertions. This represents mixing from $\psi_L$ to the $\psi_R$, and thus generates new vertices, such as $[A]^{LRL}$, etc..

We compute the triangle loops with a single pure left-handed $\ell^a_L \gamma^\mu L$ vertex, carrying in momentum $p$, and again noting the the cyclic order in which numerator terms are written, we have:

$$T_L = (-1)(i)^3(i)^3 \int \frac{d^4\ell}{(2\pi)^4} \frac{N_1 + N_2}{D}$$

$$N_1 = \text{Tr}[\ell^a_L (\ell - \hat{q} + m)(\ell^c_L + \ell^c_R)(\ell + m)(\ell^b_L + \ell^b_R)(\ell + \hat{q} + m)]$$

$$N_2 = -\text{Tr}[\ell^a_L (\ell + \hat{q} - m)(\ell^b_L + \ell^b_R)(\ell - m)(\ell^c_L + \ell^c_R)(\ell + \hat{q} - m)]$$

$$D = [(\ell + k)^2 - m^2][\ell^2 - m^2][(\ell - q)^2 - m^2]$$

(C33)

Note the sign flips in the momentum and $m$ terms in $N_1$ and momenta in $N_2$, a consequence of having factored out an overall minus sign.

We obtain the result (full details are available in [13]):

$$T_L = (-4im^2) \times 2 \int_0^1 dy \int_0^y dz \int \frac{d^7\ell}{(2\pi)^4}$$

$$\left(-z[A]^{LRL} - y[B]^{LRL} + (1 - z)[A]^{LRR} + (1 - y)[B]^{LRR} + z[A]^{LRR} - (1 - y)[B]^{LRR}\right)$$

$$\left(\ell^2 + z(1 - z)k^2 + y(1 - y)q^2 + 2k \cdot qz(1 - y) - m^2\right)^3$$

(C34)

This result is negligible in the limit $k^2$, $2k \cdot q$, $q^2 >> m^2$. However, in the limit of large $m^2$ the result reduces to:

$$T_L = \frac{1}{24\pi^2}([A]^{LRL} + 2[B]^{LRL} - 2[A]^{LRR} - [B]^{LRR} - [A]^{LRR} + [B]^{LRR})$$

(C35)
From this we can easily infer the result for a computation of the triangle loops with a single pure $\ell^{aR}R$ (right-handed) vertex:

$$T_R = -\frac{1}{24\pi^2}(A^{RRL} + 2[B]^{RLR} - 2[A]^{RRL} - [B]^{RRL} - [A]^{RLL} + [B]^{RLL}) \quad (C36)$$

Combining these we have:

$$T_L + T_R = \frac{1}{24\pi^2}(A^{LRL} + 2[B]^{RLR} - 2[A]^{LLR} - [B]^{LRR} - [A]^{LRR} + [B]^{LRR}) - [A]^{RLR} - 2[B]^{RLR} + 2[A]^{RRL} + [B]^{RRL} + [A]^{RLL} - [B]^{RLL}) \quad (C37)$$

For KK-mode $B^n_\mu$ we have an $x^5$ wave-function parity of $(-1)^n$, and $B^n_{\mu L} = (-1)^n B^n_{\mu R} = B^n_\mu$. The KK-modes are normalized so that an axial vector (odd $n$) couples to $\bar{\psi}\gamma_\mu \gamma^5\psi$ with positive sign. Thus, we can write:

$$T_L + T_R = \frac{1}{12\pi^2}(f_{abc}A + g_{abc}[B]) \quad (C39)$$

where:

$$f_{abc} = \frac{1}{2}((-1)^{a+c} - 2(-1)^{a+b} - (-1)^a - (-1)^b + 2(-1)^c + (-1)^{b+c})$$

$$g_{abc} = \frac{1}{2}(2(-1)^{a+c} - (-1)^{a+b} + (-1)^a - 2(-1)^b + (-1)^c - (-1)^{b+c}). \quad (C40)$$

Note that if $a + b + c$ is even, then $f = g = 0$, which is the condition that a transition cannot occur! But, of course, the condition that a transition can occur is $a + b + c$ odd. When $a + b + c$ is odd, we can therefore write:

$$f_{abc} = -(-1)^a - (-1)^b + 2(-1)^c$$

$$g_{abc} = (-1)^a - 2(-1)^b + (-1)^c \quad (C41)$$

Under $b \leftrightarrow c$ we have $A \leftrightarrow -B$ and thus $g_{abc} \leftrightarrow -f_{acb}$, which confirms Bose symmetry.

Under the exchange $a \leftrightarrow b$ we have $B \rightarrow -B$ and $A \rightarrow A + B$ (since the $k$ in the $A$ vertex now becomes $-k - q$ with the sign flip, since $a$ is incoming momentum $k + q$). Thus the vertex becomes:

$$T_L + T_R \rightarrow \frac{1}{12\pi^2}(f_{bac}[A] + (f_{bac} - g_{bac})[B]) \quad (C42)$$

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and we immediately verify that \( f_{bac} = f_{abc} \) and \( f_{bac} - g_{bac} = g_{abc} \). The amplitude is seen to be fully Bose symmetric (we leave the verification of \( a \leftrightarrow c \) Bose symmetry to the reader).

The vertex calculation can be represented by an operator of the form:

\[
\mathcal{O}_3 = -\frac{1}{12\pi^2} \varepsilon^{\mu\nu\rho\sigma} \sum_{nmk} a_{nmk} B_\mu^m B_\nu^n \partial_\rho B_\sigma^k
\]

where:

\[
a_{nmk} = \frac{1}{2} [1 - (-1)^{n+m+k}] (-1)^{m+k}
\]

For the process \( a \to b + c \) the matrix element of \( \mathcal{O} \) takes the form (we’ve multiplied by \( +i \) from \( e^{iS} \)):

\[
i \langle a | \mathcal{O} | b, c \rangle = \frac{1}{12\pi^2} \left[ (-a_{abc} + a_{bac} + a_{bca} - a_{cba})[B] + (a_{acb} - a_{cab} + a_{bca} - a_{cba})[A] \right]
\]

and we see that (for \( a + b + c \) odd):

\[
- a_{abc} + a_{bac} + a_{bca} - a_{cba} = g_{abc}
\]

\[
a_{acb} - a_{cab} + a_{bca} - a_{cba} = f_{abc}
\]

4. Massive Left-Right Symmetric Anomaly

The current divergence, \( \partial_\mu \bar{\psi} \gamma_\mu \psi_L \), is obtained by the replacement \( \epsilon_\mu \to p_\mu \) in \( T_L \). We thus have that \( A \to -E \) and \( B \to E \) and we make use of the operator correspondence eq.(C10):

\[
\partial_\mu \bar{\psi} \gamma_\mu \psi_L = \frac{1}{48\pi^2} (F_{\mu \nu} \tilde{F}_{\mu \nu}^L + F_{\mu \nu}^R \tilde{F}_{\mu \nu}^R)
\]

\[
\partial_\mu \bar{\psi} \gamma_\mu \psi_R = \frac{1}{48\pi^2} (F_{\mu \nu} \tilde{F}_{\mu \nu}^R + F_{\mu \nu}^L \tilde{F}_{\mu \nu}^L)
\]

We emphasize that this result is not the anomaly. To extract the anomaly, we note that the equations of motion yield the divergences of the spinor currents:

\[
\partial^\mu \bar{\psi} \gamma_\mu \psi_L = -im(\bar{\psi}_L \psi_R - \bar{\psi}_R \psi_L) + \text{anomaly}
\]

\[
\partial^\mu \bar{\psi} \gamma_\mu \psi_R = -im(\bar{\psi}_R \psi_L - \bar{\psi}_L \psi_R) + \text{anomaly}
\]
We thus need to subtract the vacuum to 2-gauge field matrix element of the mass term, which is the operator $-i m \bar{\psi} \gamma^5 \psi$, to obtain the anomaly. The mass term yields a similar triangle diagram structure, and we define:

$$M^5 = (-1)(i)^2(i)^3 \int \int \frac{d^4 \ell}{(2\pi)^4} \frac{N_1 + N_2}{D}$$

$$N_1 = (-i)(-im) \text{Tr}[\gamma^5(\ell - \gamma + m)(\ell^L c + \ell^R c)(\ell + m)(\ell^L b + \ell^R b)(\ell + \gamma + m)]$$

$$N_2 = (+i)(-im) \text{Tr}[\gamma^5(\ell + \gamma - m)(\ell^L b + \ell^R b)(\ell - m)(\ell^L c + \ell^R c)(\ell - \gamma - m)]$$

$$D = (\ell^2 + 2\ell \cdot (zk - (1 - y)q) + zk^2 + (1 - y)q^2 - m^2)^3$$

(C51)

The result is (see [13] for details):

$$M^5 = \langle 0 | -i m \bar{\psi} \gamma^5 \psi | b, c \rangle = \frac{1}{24\pi^2} [2(E^{LL} + E^{RR}) + (E^{LR} + E^{RL})] , \quad (C52)$$

or, the operator correspondence:

$$i m \bar{\psi} \gamma^5 \psi \rightarrow -\frac{1}{48\pi^2} [F_L \tilde{F}_L + F_R \tilde{F}_R + F_L \tilde{F}_R] . \quad (C53)$$

Forming the difference of the current divergence with $-i m \bar{\psi} \gamma^5 \psi$ we have:

$$\partial_\mu \bar{\psi} \gamma_\mu \psi_L + i m(\bar{\psi}_L \psi_R - \bar{\psi}_R \psi_L) = \frac{1}{48\pi^2} [F_L \tilde{F}_R + F_R \tilde{F}_L] - \frac{1}{48\pi^2} [F_L \tilde{F}_L + F_R \tilde{F}_R + F_L \tilde{F}_R]
= -\frac{1}{48\pi^2} F_L \tilde{F}_L . \quad (C54)$$

Likewise:

$$\partial_\mu \bar{\psi} \gamma_\mu \psi_R + i m(\bar{\psi}_R \psi_L - \bar{\psi}_L \psi_R) = \frac{1}{48\pi^2} [F_R \tilde{F}_L + F_L \tilde{F}_R] + \frac{1}{48\pi^2} [F_L \tilde{F}_L + F_R \tilde{F}_R + F_L \tilde{F}_R]
= \frac{1}{48\pi^2} F_R \tilde{F}_R . \quad (C55)$$
5. Summary

Pseudoscalar Mass Term:

\[ im \bar{\psi} \gamma^5 \psi \rightarrow - \frac{1}{48 \pi^2} \left[ F_{L\mu\nu} \tilde{F}^{\mu\nu}_L + F_{R\mu\nu} \tilde{F}^{\mu\nu}_R + F_{L\mu\nu} \tilde{F}^{\mu\nu}_L + F_{R\mu\nu} \tilde{F}^{\mu\nu}_R \right] \] (C56)

Consistent Anomalies:

(1) Pure Massless Weyl Spinors \((p_i \cdot p_j >> m^2)\):

\[ \partial^\mu \bar{\psi} \gamma^\mu \psi_L = \frac{1}{48 \pi^2} F_{L\mu\nu} \tilde{F}^{\mu\nu}_L \]
\[ \partial^\mu \bar{\psi} \gamma^\mu \psi_R = \frac{1}{48 \pi^2} F_{R\mu\nu} \tilde{F}^{\mu\nu}_R \] (C57)

(2) Heavy Massive Weyl Spinors \((p_i \cdot p_j << m^2)\):

\[ \partial^\mu \bar{\psi} \gamma^\mu \psi_L + im(\bar{\psi}_L \gamma^\mu \psi_R - \bar{\psi}_R \gamma^\mu \psi_L) = - \frac{1}{48 \pi^2} F_{L\mu\nu} \tilde{F}^{\mu\nu}_L \]
\[ \partial^\mu \bar{\psi} \gamma^\mu \psi_R + im(\bar{\psi}_R \gamma^\mu \psi_L - \bar{\psi}_L \gamma^\mu \psi_R) = \frac{1}{48 \pi^2} F_{R\mu\nu} \tilde{F}^{\mu\nu}_R \] (C58)

(3) Heavy Massive Weyl Spinors \((p_i \cdot p_j << m^2)\):

\[ \partial^\mu \bar{\psi} \gamma^\mu \gamma^5 \psi_L = \frac{1}{48 \pi^2} \left( F_{V\mu\nu} \tilde{F}^{\mu\nu}_V + F_{A\mu\nu} \tilde{F}^{\mu\nu}_A \right) \]
\[ \partial^\mu \bar{\psi} \gamma^\mu \gamma^5 \psi_R = - \frac{1}{48 \pi^2} \left( F_{L\mu\nu} \tilde{F}^{\mu\nu}_L + F_{R\mu\nu} \tilde{F}^{\mu\nu}_R \right) \] (C59)

Consistent \(L = V - A\) and \(R = V + A\) Forms:

(1) Pure Massless Weyl Spinors \((p_i \cdot p_j >> m^2)\):

\[ \partial^\mu \bar{\psi} \gamma^\mu \psi = \frac{1}{12 \pi^2} F_{V\mu\nu} \tilde{F}^{\mu\nu}_A \]
\[ \partial^\mu \bar{\psi} \gamma^5 \psi = \frac{1}{24 \pi^2} \left( F_{V\mu\nu} \tilde{F}^{\mu\nu}_V + F_{A\mu\nu} \tilde{F}^{\mu\nu}_A \right) \] (C60)

(2) Heavy Massive Weyl Spinors \((p_i \cdot p_j << m^2)\):

\[ \partial^\mu \bar{\psi} \gamma^\mu \psi = \frac{1}{12 \pi^2} F_{V\mu\nu} \tilde{F}^{\mu\nu}_A \]
\[ \partial^\mu \bar{\psi} \gamma^\mu \gamma^5 \psi - 2im \bar{\psi} \gamma^5 \psi = \frac{1}{24 \pi^2} \left( F_{V\mu\nu} \tilde{F}^{\mu\nu}_V + F_{A\mu\nu} \tilde{F}^{\mu\nu}_A \right) \] (C61)
(3) Heavy Massive Weyl Spinors \((p_i \cdot p_j << m^2)\):
\[
\partial^\mu \bar{\psi} \gamma_\mu \psi = \frac{1}{12\pi^2} F_{V\mu\nu} \tilde{F}^{\mu\nu}
\]
\[
\partial^\mu \bar{\psi} \gamma_\mu \gamma^5 \psi = -\frac{1}{12\pi^2} (F_{V\mu\nu} \tilde{F}^{\mu\nu})
\]
(C62)

**Covariant Forms:**

Add a term to the lagrangian of the form \((1/6\pi^2)\epsilon_{\mu\nu\rho\sigma} A^\mu V^\nu \partial^\rho V^\sigma\). The currents are now modified to \(\bar{J} = J + \delta J\) and \(\bar{J}^5 = J^5 + \delta J^5\) as described in the text.

(1) Pure Massless Weyl Spinors \((p_i \cdot p_j >> m^2)\):
\[
\partial^\mu \bar{J}_\mu = 0 \\
\partial^\mu \bar{J}^5 \mu = \frac{1}{8\pi^2} (F_{V\mu\nu} \tilde{F}^{\mu\nu} + \frac{1}{3} F_{A\mu\nu} \tilde{F}^{\mu\nu})
\]
(C63)

(2) Heavy Massive Weyl Spinors \((p_i \cdot p_j << m^2)\):
\[
\partial^\mu \bar{J}_\mu = 0 \\
\partial^\mu \bar{J}^5 \mu - 2im\bar{\psi} \gamma^5 \psi = \frac{1}{8\pi^2} (F_{V\mu\nu} \tilde{F}^{\mu\nu} + \frac{1}{3} F_{A\mu\nu} \tilde{F}^{\mu\nu})
\]
(C64)

(3) Heavy Massive Weyl Spinors \((p_i \cdot p_j << m^2)\):
\[
\partial^\mu \bar{J}_\mu = 0 \\
\partial^\mu \bar{J}^5 \mu = 0
\]
(C65)

The latter case is completely summarized by the fact that, for KK-modes, the three-gauge boson amplitude is described by the operator:
\[
\mathcal{O}_3 = -\frac{1}{12\pi^2} \epsilon^{\mu\nu\rho\sigma} \sum_{nmk} a_{nmk} B^m_\mu B^n_\nu \partial^\rho B^k_\sigma
\]
(C66)

where:
\[
a_{nmk} = \frac{1}{2}[1 - (-1)^{n+m+k}(-1)^{m+k}]
\]
(C67)

This operator is equivalent to \((-1/6\pi^2)\epsilon_{\mu\nu\rho\sigma} A^\mu V^\nu \partial^\rho V^\sigma\) when we truncate on the first two KK-modes, and identify \(B^0 = V\) and \(B^1 = A\). Adding the \((1/6\pi^2)\epsilon_{\mu\nu\rho\sigma} A^\mu V^\nu \partial^\rho V^\sigma\) term cancels this quantity, completely cancels the triangle diagrams, and the resulting currents then have vanishing divergences.
Acknowledgments

We especially thank Bill Bardeen and Bogdan Dobrescu for numerous helpful discussions. We also wish to thank Tao Han and Sherwin Love for helpful discussions in the early phase of this work. This work is supported in part by the US Department of Energy, High Energy Physics Division, Contract W-31-109-ENG-38, and grant DE-AC02-76CH03000.

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[15] Note, of course, that the branes can be explicitly constructed by coupling a $D = 5$ bulk electron to a kink plus antikink soliton. This would yield the brane chiral electrons as domain wall fermions. Our branes are presently imposed by hand, yet the present model can be viewed as a description of domain wall fermions in the limiting case of a thin domain wall, and a large fermionic Dirac mass in the bulk.

[16] This is dual to a the conventional electric superconducting parallel plate capacitor; a magnetic superconductor is a confining phase of the theory since it admits electric flux tubes that would confine electric charges.

[17] The “annihilation” of the first term of eq. (9) can be seen to occur in detail upon performing the gauge transformation for any particular matter anomaly cancellation with the CS anomaly. It involves, however, both the first ($A^5$), as well as the second ($\partial_5$), terms of eq. (9). The second term yields $\partial_5 \partial_\mu A^5$ terms that must be integrated by parts, and the cancellation is then just the full matter anomaly and CS term anomaly cancellation.