Abstract. In this paper, claims by Lemmens and Seidel in 1973 about equiangular sets of lines with angle 1/5 are proved by carefully analyzing pillar decompositions, with the aid of the uniqueness of two-graphs on 276 vertices. The Neumann Theorem is generalized in the sense that if there are more than 2r – 2 equiangular lines in \( \mathbb{R}^r \), then the angle is quite restricted. Together with techniques on finding saturated equiangular sets, we determine the maximum size of equiangular sets "exactly" in an \( r \)-dimensional Euclidean space for \( r = 8, 9, \) and 10.

1. Introduction

A set of lines in a Euclidean space is called equiangular if any pair of lines forms the same angle. For examples, the four diagonal lines of a cube are equiangular in \( \mathbb{R}^3 \) with the angle \( \arccos(1/3) \), and the six diagonal lines of an icosahedron form 6 equiangular lines with angle \( \arccos(1/\sqrt{5}) \). The structure of methan CH\(_4\) also contains equiangular lines: carbon-hydrogen chemical bounds form the same angle (about 109.5 degrees). Equiangular lines in real and complex spaces are related to many beautiful mathematic topics and even quantum physics, such as SIC-POVM [RBKSC04, SG10, Sco06, Zau]. First, equiangular lines in real spaces are equivalent to the notion of two-graphs which caught much attention in algebra [GR13]. A classical way to construct equiangular lines comes from combinatorics designs. For instance, the 90 equiangular lines in \( \mathbb{R}^{20} \) and 72 equiangular lines in \( \mathbb{R}^{19} \) can be obtained from the Witt design. The details can be found in Taylor’s thesis in 1971 [Tay71]. The spherical embedding of certain strongly regular graphs can also give rise to equiangular lines [Cam04]: the maximum size of equiangular lines in \( \mathbb{R}^{23} \) is 276 which can be constructed from the strongly regular graphs with parameters (276, 135, 78, 54). Such configuration is the solution to the energy minimizing problems [SK97], also known as the Thomson Problem. The Thomson problem is to determine the minimum electrostatic potential energy configuration of \( N \) electrons constrained to the surface of a unit sphere that repel each other with a force given by Coulomb’s law. The physicist J. J. Thomson posed the problem in 1904 [Tho04]. The configuration of several maximum equiangular lines would give rise to the minimizers of a large class of energy minimizing problems called the universal optimal codes [CK07]. Furthermore, if we have \( \frac{r(r+1)}{2} \) equiangular lines in \( \mathbb{R}^r \) (which is known as the Gerzon bounds [LS73]), then they will offer the construction of tight spherical 5-designs [Del77] which are also universal optimal codes. So far, only when \( r = 2, 3, 7, \) and 23 can the Gerzon bounds be achieved. The special sets of equiangular lines, called equiangular tight frames (ETFs) refer to the optimal line packing problems [MS18]. ETFs achieve the classical Welch bounds [Wel74] which are the lower bounds for maximum absolute value of inner product values between distinct points on unit sphere, i.e. if we have \( M \) points \( \{x_i\}_{i=1}^M \) on the unit sphere in \( \mathbb{R}^r \), then

\[
\max_{i \neq j} |\langle x_i, x_j \rangle| \geq \sqrt{\frac{M-r}{r(M-1)}} \]

The study of ETFs has numerous references [FJMP18, SH03, FMJ16, JMF14, BGOY15, Wall09, SH03].

From another point of view, a set of equiangular lines can be regarded as the collection of points on the unit sphere such that distinct points in the set have mutual inner products either \( \alpha \) or \( -\alpha \) for some \( \alpha \in [0, 1) \). Below we formally state its definition.

Definition 1.1. We say that a finite set of unit vector \( X = \{x_1, \ldots, x_s\} \) in \( \mathbb{R}^r \) is an equiangular set if for some \( \alpha \in [0, 1) \),

\[
\langle x_i, x_j \rangle \in \{-\alpha, \alpha\} \quad \text{whenever } i \neq j.
\]
By abuse of language, we will say that a set of vectors which satisfy the condition (1) are equiangular with angle \( \alpha \), although the actual angle of intersection is \( \arccos \alpha \). A natural question in this context is: what is the maximum size of equiangular sets in \( \mathbb{R}^r \)? We denote by \( M(r) \) for this quantity. The values of \( M(r) \) were extensively studied over the last 70 years. It is easy to see that \( M(2) = 3 \) and the maximum construction is realized by the three diagonal lines of a regular hexagon. In 1948, Haantjes [Haa48] showed that \( M(3) = M(4) = 6 \). In 1966, van Lint and Seidel [vLS66] showed that \( M(5) = 10, M(6) = 16, \) and \( M(7) \geq 28 \). Currently, there are only 35 known values for \( M(r) \) and all of them have that \( r \leq 43 \). To the best of our knowledge, the ranges of \( M(r) \) for \( 2 \leq r \leq 43 \) are listed in Table 1 (see [AM16, BY14, GKMS16, Gre18, Gre18, Szö17, Yu15]).

**Table 1. Maximum cardinalities of equiangular lines for small dimensions**

| \( r \) | 2  | 3–4 | 5  | 6  | 7–13 | 14 | 15 | 16 | 17 |
|---|---|---|---|---|---|---|---|---|---|
| \( M(r) \) | 2  | 3  | 6  | 10 | 16  | 28 | 28–29 | 36 | 40–41 | 48–49 |
| \( r \) | 18 | 19 | 20 | 21 | 22  | 23–41 | 42 | 43 |
| \( M(r) \) | 54–60 | 72–75 | 90–95 | 126 | 176  | 276 | 276–288 | 344 |

Note that for the dimensions \( r = 14, 16, 17, 18, 19, 20 \), determining the exact values of \( M(r) \) are still open problems; though we know that the current well-known maximum constructions of equiangular lines are saturated [LY18], i.e. the current maximum constructions of equiangular lines cannot be added any more line while keeping equiangular. The estimation of upper bounds for equiangular lines can be considered from several different methods. The bounds could be achieved by semidefinite programming method [BY14, OY16, Gre18], the analysis of eigenvalues of the Seidel matrices [GKMS16, Gre18, Gre18], polynomial methods [Gre18], Ramsey Theorem for asymptotic bounds [BDKS18], forbidden subgraphs for graphs of bounded spectral radius [JP17], and algebraic graphs theory [GR13, Szö17].

The motivation for the study of equiangular lines can also be various. For instance, Bannai, Okuda and Tagami [BOT15] considered the tight harmonic index 4-designs problems and proved that the existence of tight harmonic index 4-designs is equivalent to the existence of \( \frac{(r+1)(r+2)}{6} \) equiangular lines with angle \( \sqrt{\frac{2}{r+1}} \) in \( \mathbb{R}^r \). Later, Okuda-Yu [OY16] proved such equiangular lines do not exist for all \( r > 2 \). For more informations about harmonic index \( t \)-designs, please see the references [BOT15, ZBB+17, BZZ+18, BBX+18].

The main contribution for this paper is that we proved the result which Lemmens-Seidel claimed true in 1973. In [LS73], Lemmens and Seidel claimed that the following conjecture holds when the base size \( K = 2, 3, 5 \) (for the definition of base size, see Definition 2.5):

**Conjecture 1.2** ([LS73], Conjecture 5.8). The maximum size of equiangular sets in \( \mathbb{R}^r \) for angle \( \frac{\alpha}{2} \) is 276 for \( 23 \leq r \leq 185 \), and \( \frac{1}{2}(r - 5) + r + 1 \) for \( r \geq 185 \).

Although the conjecture was prominent in the study of equiangular lines, no proof was found in the literature for the cases \( K = 3, 5 \). Following the discussion of pillar methods, we use techniques from linear algebra, linear programming, and the uniqueness of the two-graphs with 276 vertices to prove the \( K = 3, 5 \) cases, and offer a partial solution for \( K = 4 \). We also offer better upper bounds for the equiangular sets for some special setting on pillar conditions.

There is another interesting phenomenon that receives our attention. It is well known that \( M(8) = 28 \) (see Table 1), but those 28 lines always live in a 7-dimensional subspace of \( \mathbb{R}^8 \) ([GY18], Theorem 4). Glazyrin and Yu [GY18] asks the maximum size of equiangular sets of general ranks. The following theorem essentially states that the angle is restricted when the size of equiangular set is large enough.

**Theorem 1.3** (Neumann, cf. [LS73]). Let \( X \) be an equiangular set with angle \( \alpha \) in \( \mathbb{R}^r \). If \( |X| > 2r \), then \( \frac{1}{\alpha} \) is an odd integer.

We first give a generalization of the Neumann theorem (see Theorem 5.3), then we employ the techniques about saturated equiangular sets in [LY18] to determine the maximum size of equiangular sets of ranks \( 8, 9, \) and \( 10 \).

The organization of the paper is as follows. In Section 2 we review the basic notations in the study of equiangular sets and recall the pillar decompositions introduced by Lemmens and Seidel [LS73]. In Section 3 we determine the maximum size of a pillar with orthogonal vectors only. In Section 4 we provide a proof for the Lemmens-Seidel conjecture when the base size \( K = 3 \).
or 5, and also give a new upper bound for $K = 4$. In Section 5 we discuss the maximum size of equiangular sets of prescribed rank. We close this paper with some discussions and proposing two conjectures based on our computations.

2. Prerequisites

Throughout this paper, $\hat{e}$ denotes the unit vector in the same direction as a non-zero vector $x$ in an Euclidean space. We start with some basic definitions for equiangular sets. Let $X$ be an equiangular set with angle $\alpha$ in $\mathbb{R}^r$. There are a few mathematical objects that could be associated to $X$.

**Definition 2.1.** Let $X = \{x_1, \ldots, x_s\} \subset \mathbb{R}^r$ be a finite set of vectors. The Gram matrix of $X$, denoted by $G(X)$ or $G(x_1, \ldots, x_s)$, is the matrix of mutual inner products of $x_1, \ldots, x_s$; that is,

$$G(X) = X^T X = [(x_i, x_j)]_{i,j=1}^s$$

When $X$ is equiangular with angle $\alpha$, then its Gram matrix $G(X)$ is symmetric and positive semidefinite, with entries $1$ along its diagonal and $\pm \alpha$ elsewhere. The rank of $G(X)$ is the dimension of the span of vectors in $X$; $X$ is linearly independent if and only if $G(X)$ is of full rank (or equivalently, positive definite).

**Definition 2.2.** For an equiangular set $X = \{x_1, \ldots, x_s\}$ with angle $\alpha$, the Seidel graph of $X$ is a simple graph $S(X)$ whose vertex set is $X$, and two vertices $x_i$ and $x_j$ of $S(X)$ are adjacent if and only if $\langle x_i, x_j \rangle = -\alpha$.

Since we are interested in equiangular lines in $\mathbb{R}^r$, choices need to be made between two unit vectors that span the same line. However, the choices could affect the signs of their mutual inner products. If two sets of vectors represent the same set of lines, they are called in the same switching class. This terminology comes from the graph theory: if we switch a vertex $v$ in a simple graph, the resulting graph is obtained by removing all edges that are incident to $v$ but adding edges connecting $v$ to all vertices that were not adjacent to $v$. We also have the freedom to relabel the vertices of the graph. All these actions lead to the following proposition about the switching equivalence for two Gram matrices.

**Proposition 2.3** ([KT16], Definition 4). Two sets of unit vectors $X, Y$ in $\mathbb{R}^r$ are in the same switching class if and only if there are a diagonal $(1, -1)$-matrix $B$ and a permutation matrix $C$ such that

$$(CB)^T \cdot G(X) \cdot (CB) = G(Y).$$

We would also say that $G(X)$ is switching equivalent to $G(Y)$, and write $G(X) \simeq G(Y)$.

As usual, let $I_s$ (resp. $J_s$) denote the identity matrix (resp. all-one matrix) of size $s \times s$; the subscript $s$ will sometimes be dropped when the size is clear from the context.

**Proposition 2.4** ([LS73], Section 4). If there are $k \geq 2$ equiangular vectors $p_1, \ldots, p_k$ such that

$$G(p_1, \ldots, p_k) \simeq (1 + \alpha)I - \alpha J, \quad \alpha > 0,$$

then $k \leq \frac{1}{\alpha} + 1$. Furthermore, if $k < \frac{1}{\alpha} + 1$, then the vectors $p_1, \ldots, p_k$ are linearly independent; but if $k = \frac{1}{\alpha} + 1$, then the vectors $p_1, \ldots, p_k$ are linearly dependent. In fact, if $k = \frac{1}{\alpha} + 1$ and $G(p_1, \ldots, p_k) = (1 + \alpha)I - \alpha J$, the vectors $p_1, \ldots, p_k$ form a $k$-simplex in $\mathbb{R}^{k-1}$.

Under a suitable choice of signs, the vectors $\pm p_1, \ldots, \pm p_k$ from an equiangular set $X$ will form a $k$-clique in its Seidel graph. Following [LS73], we will define two important notions that are associated to an equiangular set $X$ (Definitions 2.5 and 2.7).

**Definition 2.5.** Let $X$ be an equiangular set in $\mathbb{R}^r$ with angle $\alpha$. The base size of $X$, denoted by $K(X)$, is defined as

$$K(X) := \max \{k \in \mathbb{N} : \text{there exist } p_1, \ldots, p_k \text{ in } X \text{ such that } G(p_1, \ldots, p_k) \simeq (1 + \alpha)I - \alpha J\}.$$ 

In other words, $K(X)$ is the maximum of the clique numbers of Seidel graphs that are switching equivalent to that of $X$.

Note that the clique numbers of Seidel graphs in the switching class of $X$ are not constant, therefore we need to take their maximum. Nevertheless $K(X)$ is always bounded by $\frac{1}{\alpha} + 1$ by Proposition 2.4. Since we are interested in large equiangular sets, we will assume that $\frac{1}{\alpha}$ is an odd integer, thanks to Theorem 1.3. The following proposition states that the only meaningful range of base size is $2, 3, \ldots, \frac{1}{\alpha} + 1$. 
Proposition 2.6 ([KT16], Proposition 3). Let $X$ be an equiangular set in $\mathbb{R}^r$. If $|X| \geq 2$, then $K(X) \geq 2$.

Proof. If two vertices in the Seidel graph $S(X)$ are independent, then we switch one of the them to form a 2-clique. □

Definition 2.7. Let $X$ be an equiangular set with angle $\alpha$ and base size $K$. A set of $K$ vectors $p_1, \ldots, p_K$ is called a $K$-base of $X$ if $p_1, \ldots, p_k$ belong to some set which is switching equivalent to $X$, and $G(p_1, \ldots, p_K) = (1 + \alpha)I - \alpha J$.

Let $K$ be the base size of an equiangular set $X$. We will fix a $K$-base $P = \{p_1, \ldots, p_K\}$ that forms a $K$-clique in the Seidel graph of $X$. Now we introduce the pillar decomposition of $X$ with respect to $P$, following [LS73]. (More details can also be found in [KT16].)

For each vector $x \in X \setminus P$, there is a $(1, -1)$-vector $\varepsilon(x) \in \mathbb{R}^K$ such that $(x, p_1), \ldots, (x, p_K) = \alpha \cdot \varepsilon(x)$.

A vector $x$ in $X$ will be replaced by $-x$ if $\varepsilon(x)$ has more positive entries than $\varepsilon(-x)$, or $\varepsilon(x)$ has the same number of positive entries as negative entries, thus only $(x, p_1), \ldots, (x, p_K)$ are $\alpha$; otherwise the vector $x$ stays put.

Let $\Sigma(\varepsilon(x))$ denote the number of positive entries in $\varepsilon(x)$. A pillar (with respect to a $K$-base $P$) containing a vector $x \in X \setminus P$, denoted by $\bar{x}$, is the subset of vectors $x' \in X \setminus P$ such that $\varepsilon(x') = \varepsilon(x)$; $\bar{x}$ is called a $(K, n)$ pillar when $\Sigma(\varepsilon(x)) = n$. Thus the vectors in $X \setminus P$ are partitioned into several $(K, n)$ pillars for $1 \leq n \leq \lfloor K/2 \rfloor$. The number of different $(K, n)$ pillars is at most $\binom{K}{n}$ when $1 \leq n < \frac{K}{2}$, but it is at most $\frac{K}{2} \binom{K/2}{n}$ when $n = \frac{K}{2}$. However, if $K = \frac{n}{2} + 1$, then $p_1, \ldots, p_K$ form a $K$-simplex and $\sum_{i=1}^K p_i = 0$. Therefore $\varepsilon(x)$ has the same number of positive entries as negative entries, and there is only $(K, \frac{K}{2})$ pillars can exist. The collection of all $(K, n)$ pillars in an equiangular set $X$ will be denoted by $\hat{X}(K, n)$.

The following fact will be used in many occasions.

Proposition 2.8. Let $X$ be an equiangular set with angle $\alpha$ and base size $K$, and $P = \{p_1, \ldots, p_K\}$ be a $K$-base. If two vectors $x, y$ belong to the same $(K, 1)$ pillar with respect to $P$, then $(x, y) = \alpha$.

Proof. By definition of $x$ and $y$ being in the same $(K, 1)$ pillar, there are $K - 1$ vectors in $P$ to which both $x$ and $y$ are adjacent in the Seidel graph $S(X)$ of $X$. If $x$ and $y$ are also adjacent to each other in $S(X)$, $x$ and $y$ together with those $K - 1$ vectors that they are connected to form a $(K + 1)$-clique in $S(X)$, which contradicts to the definition of the base size $K = K(X)$. Hence there is no edge connecting $x$ and $y$ in $S(X)$, which is equivalent of saying that $(x, y) = \alpha > 0$. □

3. Schur decomposition for symmetric positive semidefinite matrices

In checking a matrix being positive (semi-)definite, we use the Schur decomposition.

Theorem 3.1 (Schur decomposition [BV04]). Let $M$ be a symmetric real matrix, given by blocks

$$M = \begin{bmatrix} A & B' \\ B & C \end{bmatrix}$$

Suppose that $A$ is positive definite. Then $M$ is positive (semi-)definite if and only if $C - B^T A^{-1} B$ is positive (semi-)definite.

Let $X$ be an equiangular set with angle $\alpha = \frac{1}{2(n + 1)}$ and base size $K = K(X) = \frac{n + 2}{2n}$ in $\mathbb{R}^r$. The reason for this particular combination of $\alpha$ and $K$ will be clear soon. Let $P = \{p_1, \ldots, p_K\}$ be a $K$-base of $X$, $\Gamma$ be the subspace spanned by $P$, and $\Gamma^\perp$ be the orthogonal complement of $\Gamma$ in $\mathbb{R}^r$. For the vectors $x_1, x_2 \in X \setminus P$ belonging to the same $(K, 1)$ pillar, let $x_1 = h + c_1, x_2 = h + c_2$ be their pillar decompositions, that is, $h \in \Gamma$, and $c_1, c_2 \in \Gamma^\perp$. As $h$ is a linear combination of $p_1, \ldots, p_K$, we can write $h = \sum_{i=1}^K c_i p_i$ for some unknown coefficients $c_1, \ldots, c_K$. Since $x_1$ belongs to a $(K, 1)$ pillar, there is an index $k_0 \in \{1, \ldots, K\}$ such that

$$\langle x, p_k \rangle = \langle h, p_k \rangle = \begin{cases} \alpha, & \text{if } k = k_0; \\ -\alpha, & \text{if } k \neq k_0. \end{cases}$$

Rewriting (2) as a matrix equation, we see that

$$G \cdot \begin{bmatrix} c_1 \\ \vdots \\ c_K \end{bmatrix} = \alpha \cdot (2c_{k_0} - \sum_{i=1}^K c_i),$$

where $G = \begin{bmatrix} 2 & \cdots & 2 \\ \vdots & \ddots & \vdots \\ 2 & \cdots & 2 \end{bmatrix}$.
Proof. Let us look at the situation where two vectors come from different pillars. Suppose that

\[ G = G(P) = (1 + \alpha)I - \alpha J \]

is the Gram matrix for \( P \), and \( \{e_1, \ldots, e_K\} \) is the standard orthonormal basis for \( \mathbb{R}^K \). Since \( G \) is positive and invertible, we compute

\[
G^{-1} = \frac{1}{1 + \alpha}I + \frac{\alpha}{(1 + \alpha)(1 + \alpha - K\alpha)}J.
\]

Hence by (3) we obtain that

\[
c_k = \begin{cases} 0, & \text{if } k = k_0, \\ -(K - 1)^{-1}, & \text{if } k \neq k_0; \end{cases}
\]

that is,

\[
h = \frac{-1}{K - 1} \left( \sum_{i=1}^K p_i - p_{k_0} \right).
\]

From this expression we conclude that \( (h, h) = \alpha \). Since \( \langle x_1, x_2 \rangle = \alpha \) by Proposition 2.8, we conclude that \( \langle \hat{c}_1, \hat{c}_2 \rangle = 0 \), that is, the \( c \)-vectors within a single \((K, 1)\) pillar are orthogonal. (The orthogonality condition among the \( c \)-vectors does not hold for any other combinations of \( \alpha \) and \( K \).)

**Theorem 3.2.** Let \( n \) be a positive integer with \( n \geq 2 \), and \( \alpha = \frac{1}{2(n+1)} \). Let \( X \) be an equiangular set with angle \( \alpha \) and base size \( K = n + 2 \) in \( \mathbb{R}^r \), and we fix a base \( P = \{p_1, \ldots, p_K\} \) for \( X \). If there is a \((K, 1)\) pillar with at least two vectors, then for any other \((K, 1)\) pillar \( \bar{x} \),

\[
|\bar{x}| \leq \begin{cases} 2n^2(n+1), & \text{if } n \leq 3; \\ \frac{1}{2}n^2(n+1)^2, & \text{if } n \geq 3. \end{cases}
\]

**Proof.** Let us look at the situation where two vectors come from different pillars. Suppose that \( x = h_1 + c_1 \) and \( u = h_2 + c_2 \) in \( X \) belong to distinct \((K, 1)\) pillars. Because the Hamming distance of \( \varepsilon(x) \) and \( \varepsilon(u) \) is 2, we have

\[
(h_1, h_2) = \frac{n - 1}{(n + 1)(2n + 1)}.
\]

Therefore

\[
\langle \hat{c}_1, \hat{c}_2 \rangle = \frac{(x, u) - (h_1, h_2)}{\|c\|^2} = \frac{\pm \frac{1}{2n+1} - \frac{n-1}{n(n+1)(2n+1)}}{1 - \frac{1}{2n+1}} = \frac{1}{n(n+1)} - \frac{1}{n+1}.
\]

Now suppose that the pillar \( \bar{u} \) contains two vectors \( u_1, u_2 \), and \( \bar{x} \) contains \( N \) vectors \( x_1, \ldots, x_N \). Let \( x_i = h_1 + c_i \) and \( u_i = h_2 + d_i \) be their pillar decompositions. Then the Gram matrix of \( \{\hat{c}_1, \ldots, \hat{c}_N, \hat{d}_1, \hat{d}_2\} \) has the following form:

\[
G = G(\hat{c}_1, \ldots, \hat{c}_N, \hat{d}_1, \hat{d}_2) = \begin{bmatrix}
I_N & v_1 & v_2 \\
v_1^T & 1 & 0 \\
v_2^T & 0 & 1
\end{bmatrix},
\]

where \( v_1 \) and \( v_2 \) are vectors in \( \mathbb{R}^N \) with entries in \( \{\pm 1, \pm \frac{1}{n+1}\} \). Let us assume that in \( \bar{x} \),

- there are \( \ell_{11} \) vectors \( x \) such that \( \langle x, u_1 \rangle = \alpha \), \( \langle x, u_2 \rangle = \alpha \);
- there are \( \ell_{12} \) vectors \( x \) such that \( \langle x, u_1 \rangle = \alpha \), \( \langle x, u_2 \rangle = -\alpha \);
- there are \( \ell_{21} \) vectors \( x \) such that \( \langle x, u_1 \rangle = -\alpha \), \( \langle x, u_2 \rangle = \alpha \);
- there are \( \ell_{22} \) vectors \( x \) such that \( \langle x, u_1 \rangle = -\alpha \), \( \langle x, u_2 \rangle = -\alpha \).

Certainly \( \ell_{11} + \ell_{12} + \ell_{21} + \ell_{22} = N \). It follows that

\[
\langle v_1, v_1 \rangle = \frac{\ell_{11} + \ell_{12}}{n^2(n+1)^2} + \frac{\ell_{21} + \ell_{22}}{n(n+1)^2};
\]

\[
\langle v_2, v_1 \rangle = \frac{\ell_{11} + \ell_{21}}{n^2(n+1)^2} + \frac{\ell_{12} + \ell_{22}}{n(n+1)^2};
\]

\[
\langle v_1, v_2 \rangle = \frac{\ell_{11}}{n(n+1)^2} - \frac{\ell_{12} + \ell_{21}}{n(n+1)^2} + \frac{\ell_{22}}{(n+1)^2}.
\]

Since the Gram matrix \( G \) is positive semidefinite, the following \( 2 \times 2 \) matrix is also positive semidefiniteness by Theorem 3.1:

\[
M := \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - v_1^T I_N^{-1} v_1 v_2 = \begin{bmatrix} 1 - \langle v_1, v_1 \rangle & -\langle v_1, v_2 \rangle \\ -\langle v_2, v_1 \rangle & 1 - \langle v_2, v_2 \rangle \end{bmatrix} \succeq 0.
\]
Therefore the problem becomes

\begin{align*}
\text{to maximize} & \quad N = s + 2t \\
\text{subject to} & \quad s, t \in \mathbb{Z}, \quad s, t \geq 0, \\
& \quad n^2(n+1)^2 - s - (n^2+1)t \geq 0, \\
& \quad (n^2-t)(n^2(n+1)^2 - 2s - (n-1)^2t) \geq 0.
\end{align*}

Because $M$ is symmetric, $M$ is positive semidefinite if and only if $\text{tr} M \geq 0$ and $\det M \geq 0$. We compute

\begin{align*}
(4) \quad \frac{n^2(n+1)^2}{2} \text{tr} M &= n^2(n+1)^2 - (\ell_{11} + \frac{n^2+1}{2}(\ell_{12} + \ell_{21}) + n^2\ell_{22}); \\
\quad n^4(n+1)^4 \det M &= \det(n^2(n+1)^2 M) \\
(5) \quad &= n^4(n+1)^4 - n^2(n+1)^2(2(\ell_{11} + n^2\ell_{22}) + (n^2+1)(\ell_{12} + \ell_{21})) \\
&\quad + (\ell_{11} + n^2\ell_{22} + \ell_{12} + n^2\ell_{21})(\ell_{11} + n^2\ell_{22} + \ell_{21} + n^2\ell_{12}) \\
&\quad - (\ell_{11} + n^2\ell_{22} - n(\ell_{12} + \ell_{21}))^2.
\end{align*}

Keep in mind that we want to maximize $N = \ell_{11} + \ell_{12} + \ell_{21} + \ell_{22}$ subject to $\text{tr} M \geq 0$, $\det M \geq 0$, and the variables $\ell_{ij}$ are all non-negative integers. If we look closely to (4) and (5), the terms $\ell_{11} + n^2\ell_{22}$ always appear as a pair, and there is no other separate term for $\ell_{11}$ and $\ell_{22}$; as a result, the sum $\ell_{11} + \ell_{22}$ is maximized when $\ell_{22} = 0$. Henceforth we let $\ell_{22} = 0$ and continue the computation from (5):

\begin{align*}
(6) \quad n^4(n+1)^4 \det M &= n^4(n+1)^4 - n^2(n+1)^2(2\ell_{11} + (n^2+1)(\ell_{12} + \ell_{21})) \\
&\quad + (\ell_{11} + \ell_{12} + n^2\ell_{21})(\ell_{11} + \ell_{21} + n^2\ell_{12}) - (\ell_{11} - n(\ell_{12} + \ell_{21}))^2 \\
&\quad = n^4(n+1)^4 - n^2(n+1)^2(2\ell_{11} + (n^2+1)(\ell_{12} + \ell_{21})) \\
&\quad + (n+1)^2\ell_{11}(\ell_{12} + \ell_{21}) + (n^2-1)^2\ell_{12}\ell_{21}.
\end{align*}

The expressions and (4) and (6) are symmetric with respect to $\ell_{12}$ and $\ell_{21}$, and if the sum $\ell_{12} + \ell_{21}$ is fixed, (6) is maximized when $\ell_{12} = \ell_{21}$ by the A.M.-G.M. inequality. So we set $s = \ell_{11}$ and $t = \ell_{12} = \ell_{21}$ and continue the computation:

\begin{align*}
&\quad n^4(n+1)^4 \det M = n^4(n+1)^4 - 2n^2(n+1)^2(s + (n^2+1)t) + 2(n+1)^2st + (n^2-1)^2t^2 \\
&\quad = (n+1)^2(n^2 - t)(n^2(n+1)^2 - 2s - (n-1)^2t).
\end{align*}

Therefore the problem becomes

\begin{align*}
\text{to maximize} & \quad N = s + 2t \\
\text{subject to} & \quad \begin{cases} 
\quad s, t \in \mathbb{Z}, \quad s, t \geq 0, \\
\quad n^2(n+1)^2 - s - (n^2+1)t \geq 0, \\
\quad (n^2-t)(n^2(n+1)^2 - 2s - (n-1)^2t) \geq 0.
\end{cases}
\end{align*}
This is a standard problem in linear programming, whose feasible domain is shaded in Figure 1. We solve the problem and write the maximum $N_0$ of $N$ as

$$N_0 = \begin{cases} 
2n^2(n+1), & \text{achieved at } (s,t) = (2n^2, n^2), \text{ when } n \leq 3; \\
\frac{1}{7}n^2(n+1)^2, & \text{achieved at } (s,t) = \left(\frac{1}{7}n^2(n+1)^2, 0\right), \text{ when } n \geq 3.
\end{cases}$$

The proof is now completed. \hfill \Box

**Example.** For $n = 3$, we are looking at the angle $\alpha = \frac{1}{3}$ and the base size $K = 5$. By Theorem 3.2, if there is a $(5,1)$ pillar with two or more vectors, then the size of another $(5,1)$ pillar is bounded by 72. This maximum is achieved in two ways: the quadruple $(\ell_{11}, \ell_{12}, \ell_{21}, \ell_{22})$ defined in the proof of the theorem can be $(72, 0, 0, 0)$ or $(54, 9, 9, 0)$.

**Remark.** Following the proof of their Lemma 16, King and Tang [KT16] proved that $|\hat{x}| \leq n^2(n+1)^2$ for a $(K,1)$ pillar $\hat{x}$ if there is another nonempty $(K,1)$ pillar. Theorem 3.2 cuts their bound by half.

### 4. The Lemmens-Seidel Conjecture

Throughout this section we assume that the common angle is $\alpha = \frac{1}{5}$. Let us first recall a theorem in [LS73].

**Theorem 4.1** ([LS73], Theorem 5.7). Any set of unit vectors with inner product $\pm \frac{1}{5}$ in $\mathbb{R}^r$, which contains 6 unit vectors with inner product $-\frac{1}{5}$, has maximum cardinality 276 for $23 \leq r \leq 185$, $\lfloor \frac{r}{5} \rfloor + r + 1$ for $r \geq 185$.

This theorem corresponds to the case where the common angle $\alpha = \frac{1}{5}$ and base size $K = 6$. Lemmens and Seidel concluded Section 5 of [LS73] with the following remark, which we quote here:

*It would be interesting to know whether Theorem 5.7 holds true without the requirement of the existence of 6 unit vectors with inner product $-\frac{1}{5}$. . . . The authors have obtained only partial results in this direction. In fact, the cases where [the base size $K_1 = 2,3,5$ have been proved, but the case [K = 4] remains unsettled. Yet, there is enough evidence to support the following conjecture. . . .*

So they raised their conjecture (Conjecture 1.2), but the proofs, even for the cases $K = 3,5$, have been elusive. Sections 3 and 4 of [KT16] provided some upper bounds for $\alpha = \frac{1}{5}$. It is well known that $|X| \leq r$ if $X \subset \mathbb{R}^r$ and $K = 2$ (cf. [KT16], Corollary 2). In this section we are going to sharpen their results and prove the conjecture when $K = 3,5$.

#### 4.1. $K = 3$.

Let $X \subset \mathbb{R}^r$ be an equiangular set with angle $\frac{1}{5}$ in $\mathbb{R}^r$, with the base size $K = K(X) = 3$. Let $P = \{p_1, p_2, p_3\}$ be a 3-base in $X$, and the rest of the vectors in $X \setminus P$ are partitioned into three $(3,1)$ pillars. By symmetry, for an unit vector $x \in X \setminus P$ that satisfies

$$(\langle x, p_1 \rangle, \langle x, p_2 \rangle, \langle x, p_3 \rangle) = \frac{1}{5}(1, -1, -1),$$

we can decompose $x$ into $x = h + c$, where $h \in \Gamma$ and $c \in \Gamma^\perp$. A little computation shows that

$$h = \frac{1}{9}(p_1 - 2p_2 - 2p_3).$$

So $\|h\|^2 = \frac{1}{9}$ and $\|c\|^2 = \frac{8}{9}$. If $x_1 = h + c_1$ and $x_2 = h + c_2$ come from the same $(3,1)$ pillar, then $\langle x_1, x_2 \rangle = \frac{1}{5}$ by Proposition 2.8, henceforth $\langle c_1, c_2 \rangle = \frac{1}{45}$. If $x = h_1 + c_1$ and $y = h_2 + c_2$ come from different $(3,1)$ pillars, then (by symmetry again)

$$\langle h_1, h_2 \rangle = \langle \frac{1}{9}(p_1 - 2p_2 - 2p_3), \frac{1}{9}(-2p_1 + p_2 - 2p_3) \rangle = -\frac{1}{45}.$$  

Since

$$\pm \frac{1}{5} = \langle x, y \rangle = \langle h_1, h_2 \rangle + \langle c_1, c_2 \rangle,$$

hence $\langle c_1, c_2 \rangle \in \left\{\frac{1}{5}, -\frac{1}{5}\right\}$.

**Lemma 4.2.** Suppose that there are two nonempty $(3,1)$ pillars. If one of them has 4 vectors, then the other has at most 54 vectors.
Proof. Let
\[ \bar{x} = \{ h_1 + c_i : h_1 \in \Gamma, c_i \in \Gamma^1, i = 1, \ldots, n \}, \]
\[ \bar{u} = \{ h_2 + d_i : h_2 \in \Gamma, d_i \in \Gamma^1, i = 1, 2, 3, 4 \}, \]
be two nonempty \((3, 1)\) pillars. Then the Gram matrix of \(\hat{c}_i\) and \(\hat{d}_i\) has the following form:
\[
G = G(\hat{c}_1, \ldots, \hat{c}_n, \hat{d}_1, \ldots, \hat{d}_4) = \begin{bmatrix}
\frac{n}{10}I_n + \frac{1}{10}J_n & v_1 & v_2 & v_3 & v_4 \\
v_1^T & \frac{n}{10}I_4 + \frac{1}{10}J_4 \\
v_2^T & \\
v_3^T & \\
v_4^T & 
\end{bmatrix},
\]
where \(v_1, \ldots, v_4\) are column vectors whose entries are \(\frac{1}{2}\) or \(-\frac{1}{2}\). Since \(G\) needs to be positive semidefinite, by Theorem 3.1 we see that
\[
M := \left( \frac{n}{10}I_4 + \frac{1}{10}J_4 \right) - V^T \left( \frac{n}{10}I_n + \frac{1}{10}J_n \right)^{-1} V \succeq 0, \quad \text{where } V := \begin{bmatrix} v_1 & v_2 & v_3 & v_4 \end{bmatrix}.
\]

The following setup is used to facilitate the computation. Consider the Seidel graph \(S'\) generated by the vectors in \(\bar{x} \cup \bar{u}\). By Proposition 2.8, \(S'\) is a bipartite graph because every edge must connect a vertex in \(\bar{x}\) to a vertex in \(\bar{u}\). Let us classify the vectors in \(\bar{x}\) by how they are connected to the vectors \(u_1, \ldots, u_4\) in \(\bar{u}\). Let \(B_1, B_2, B_3, B_4\) denote the subset of \(B_4\) consisting of those binary strings \(b_1b_2b_3b_4\) such that \(\sum_j b_j = i\) for \(i = 0, 1, 2, 3, 4\). For \(B = b_1b_2b_3b_4 \in B_4\), let \(t_B\) denote the number of vectors \(h_1 + c\) in the pillar \(\bar{x}\) such that
\[
\langle \hat{c}_i, \hat{d}_i \rangle = \begin{cases} 
\frac{1}{4}, & \text{if } b_i = 1, \\
-\frac{1}{4}, & \text{if } b_i = 1, \quad i = 1, 2, 3, 4.
\end{cases}
\]

In total there are \(2^4 = 16\) variables \(t_B, B \in B_4\), of non-negative integral values. Obviously \(n = \sum_{B \in B_4} t_B\), which is the total number of vectors in \(\bar{x}\), and \(\sum_{B \in B_k} t_B\) is the number of vertices of degree \(i\) in \(\bar{x}\), for \(i = 0, 1, 2, 3, 4\).

The vectors \(v_1, v_2, v_3, v_4\) in the Gram matrix \(G\) in (8) have the following mutual inner products:
\[
\langle v_i, v_j \rangle = \frac{1}{16} \sum_{B \in B_{k,j}^{b_i, b_j}} t_B - \frac{1}{20} \sum_{B \in B_{1,j}^{b_i, b_j}} t_B + \frac{1}{25} \sum_{B \in B_{2,j}^{b_i, b_j}} t_B, \quad i, j \in \{1, 2, 3, 4\},
\]
where \(B_{k,j}^{b_i, b_j}\) is the subset of \(B_4\) consisting of \(B = b_1b_2b_3b_4\) such that \(\{b_i, b_j\} = \{k, \ell\}\), for \(k, \ell \in \{0, 1\}\).
For instance,
\[
\langle v_1, v_2 \rangle = \frac{1}{16} (t_{0000} + t_{0001} + t_{0010} + t_{0011}) - \frac{1}{20} (t_{0100} + t_{0101} + t_{0110} + t_{0111}) + \frac{1}{25} (t_{1100} + t_{1101} + t_{1110} + t_{1111}).
\]

We also need
\[
w_i := \frac{1}{4} \sum_{B = b_1b_2b_3b_4 \in B_4, b_i = 0} t_B - \frac{1}{5} \sum_{B = b_1b_2b_3b_4 \in B_4, b_i = 1} t_B, \quad i \in \{1, 2, 3, 4\}.
\]

For example,
\[
w_1 = \frac{1}{4} (t_{0000} + t_{0001} + t_{0010} + t_{0011} + t_{0100} + t_{0101} + t_{0110} + t_{0111}) - \frac{1}{5} (t_{1000} + t_{1001} + t_{1010} + t_{1011} + t_{1100} + t_{1101} + t_{1110} + t_{1111}).
\]

Since
\[
\left( \frac{n}{10}I_n + \frac{1}{10}J_n \right)^{-1} = \frac{10}{9} \left( I_n - \frac{1}{9+n} J_n \right),
\]
\[
V^T I_n V = [(v_i, v_j)]_{i,j=1}^4, \quad V^T J_n V = [(w_iw_j)]_{i,j=1}^4.
\]
we use these informations to expand the left-hand side of (9) as
\[
M = \frac{9}{10}I_4 + \frac{1}{10}J_4 - \frac{10}{9}V^T I_n V + \frac{10}{9(9+n)}V^T J_n V = [m_{ij}]_{i,j=1}^{4},
\]
where the entries \(m_{ij}\) are
\[
m_{ij} = \begin{cases} 
1 - \frac{10}{9}v_i v_i^T + \frac{10}{9(9+n)}w_i^2, & \text{if } i = j, \\
- \frac{10}{9}v_i v_j + \frac{10}{9(9+n)}w_i w_j, & \text{if } i \neq j,
\end{cases} \quad i, j \in \{1, 2, 3, 4\}.
\]
Remind that we want to maximize the sum \(n = \sum_{B \in B_4} t_B\) subject to the conditions \(t_B \in \mathbb{Z}, t_B > 0\) for all \(B \in B_4\), and \(M \geq 0\). Notice that when we set some of the variables \(t_B\) to be zero, we are focusing on a particular subset of vectors in the pillar \(\mathcal{I}\). We argue that each of the variables \(t_B\) has an upper bound as follows:

- Set \(t_{0000} = n\) and \(t_B = 0\) for all \(B \neq 0000\). Then
  \[
  M = \frac{9}{10}I_4 + \left(\frac{1}{10} - \frac{5n}{8(9+n)}\right)J_4.
  \]
  By considering its eigenvalues, we see that \(M\) is positive semidefinite if and only if \(9/10 + 4 \cdot \left(\frac{1}{10} - \frac{5n}{8(9+n)}\right) \geq 0\).
  Solving this inequality for \(n\), we get \(-9 \leq n \leq \frac{39}{4}\). Since \(n\) only assumes a non-negative integral values, we see that \(0 \leq n \leq 9\); this is the range for \(t_{0000}\).
- Set \(t_{1000} = n\) and \(t_B = 0\) for all \(B \neq 1000\). Then
  \[
  M = \begin{bmatrix} 
  1 - \frac{5n}{8(9+n)} & 1 - \frac{5n}{8(9+n)} & 1 - \frac{5n}{8(9+n)} & \frac{n}{8(9+n)} \\
  - \frac{5n}{8(9+n)} & 1 - \frac{5n}{8(9+n)} & 1 - \frac{5n}{8(9+n)} & \frac{n}{8(9+n)} \\
  - \frac{5n}{8(9+n)} & 1 - \frac{5n}{8(9+n)} & 1 - \frac{5n}{8(9+n)} & \frac{n}{8(9+n)} \\
  \frac{n}{8(9+n)} & \frac{n}{8(9+n)} & \frac{n}{8(9+n)} & \frac{n}{8(9+n)}
  \end{bmatrix}
  \]
  By considering non-negative values for \(n\) only, our computation shows that \(M\) is positive semidefinite if and only if \(0 \leq n \leq 7\). By symmetry, we conclude that \(0 \leq t_B \leq 7\) for each \(B \in B_{1,1}\).
- Set \(t_{1100} = n\) and \(t_B = 0\) for all \(B \neq 1100\). Then
  \[
  M = \begin{bmatrix} 
  1 - \frac{5n}{8(9+n)} & 1 - \frac{5n}{8(9+n)} & 1 - \frac{5n}{8(9+n)} & \frac{n}{8(9+n)} \\
  - \frac{5n}{8(9+n)} & 1 - \frac{5n}{8(9+n)} & 1 - \frac{5n}{8(9+n)} & \frac{n}{8(9+n)} \\
  - \frac{5n}{8(9+n)} & 1 - \frac{5n}{8(9+n)} & 1 - \frac{5n}{8(9+n)} & \frac{n}{8(9+n)} \\
  \frac{n}{8(9+n)} & \frac{n}{8(9+n)} & \frac{n}{8(9+n)} & \frac{n}{8(9+n)}
  \end{bmatrix}
  \]
  By considering non-negative values for \(n\) only, our computation shows that \(M\) is positive semidefinite if and only if \(0 \leq n \leq 7\). By symmetry, we conclude that \(0 \leq t_B \leq 7\) for each \(B \in B_{1,2}\).
- Set \(t_{1110} = n\) and \(t_B = 0\) for all \(B \neq 1110\). Then
  \[
  M = \begin{bmatrix} 
  1 - \frac{5n}{8(9+n)} & 1 - \frac{5n}{8(9+n)} & 1 - \frac{5n}{8(9+n)} & \frac{n}{8(9+n)} \\
  - \frac{5n}{8(9+n)} & 1 - \frac{5n}{8(9+n)} & 1 - \frac{5n}{8(9+n)} & \frac{n}{8(9+n)} \\
  - \frac{5n}{8(9+n)} & 1 - \frac{5n}{8(9+n)} & 1 - \frac{5n}{8(9+n)} & \frac{n}{8(9+n)} \\
  \frac{n}{8(9+n)} & \frac{n}{8(9+n)} & \frac{n}{8(9+n)} & \frac{n}{8(9+n)}
  \end{bmatrix}
  \]
  By considering non-negative values for \(n\) only, our computation shows that \(M\) is positive semidefinite if and only if \(0 \leq n \leq 9\). By symmetry, we conclude that \(0 \leq t_B \leq 9\) for each \(B \in B_{1,3}\).
- Set \(t_{1111} = n\) and \(t_B = 0\) for all \(B \neq 1111\). Then
  \[
  M = \frac{9}{10}I_4 + \left(\frac{1}{10} - \frac{2n}{5(9+n)}\right)J_4.
  \]
  Hence \(M\) is positive semidefinite if and only if \(0 \leq n \leq 39\); this is the range for \(t_{1111}\).

Up to this point, we find that there are only a finite number of combinations of 16-tuples \((t_B : B \in B_4)\) that will make the matrix \(M\) positive semidefinite; so far there are \(10^8 \cdot 4^3 \cdot 10^4 \cdot 0.4 \approx 2.8 \times 10^{13}\) cases to check. To further reduce the computations, we have observed the following:\(^1\)

\(^1\)The SAGE script for this part of computations can be downloaded at http://math.ntnu.edu.tw/~yclin/two-31-pillars.sage.
Table 2. Upper bounds for $t_B$ for specified values of $t_{1111}$

| $t_{1111}$ | Upper bounds for $t_B$ | $M_2$ |
|------------|------------------------|-------|
| 0          | 9                      | 54    |
| 1          | 5                      | 51    |
| 2          | 3                      | 50    |
| 3          | 2                      | 46    |
| 4          | 2                      | 47    |
| 5          | 1                      | 48    |
| 6          | 1                      | 44    |
| 7          | 1                      | 45    |
| 8          | 1                      | 42    |
| 9          | 0                      | 42    |
| 10, 11     | 0                      | 42, 43|
| 12, 13     | 0                      | 44, 45|
| 14         | 0                      | 42    |
| 15         | 0                      | 37    |
| 16–19      | 0                      | 38–41 |
| 20–22      | 0                      | 42–44 |
| 23         | 0                      | 39    |
| 24–27      | 0                      | 36–39 |
| 28–31      | 0                      | 36–39 |
| 32–35      | 0                      | 36–39 |
| 36–39      | 0                      | 36–39 |

(i) Let us consider the upper bounds on the number of vertices in $\bar{x}$ of each of the degrees in the Seidel graph $S'$ (generated by $\bar{x} \cup \bar{u}$), that is, upper bounds for $\sum_{B \in B_i} t_B$, $i = 0, 1, 2, 3, 4$. For example, when we only look for vertices of degree 1, we set $t_B = 0$ whenever $B \in B \setminus B_{1,1}$. Since $0 \leq t_B \leq 7$ for $B \in B_{1,1}$, we only need to pick out those quadruples $(t_{0001}, t_{0010}, t_{0100}, t_{1000}) \in \{0, 1, \ldots, 7\}^4$ such that the resulting matrix $M$ in (10) is positive semidefinite (there are only $(7 + 1)^4 = 4096$ cases to check). Among those quadruples which survive the test, the maximum for the sum $\sum_{B \in B_{1,1}} t_B$ is 16, which occurs at $t_B = 4$ for each $B \in B_{1,1}$.

The computations for other degrees are similar and we find that

$$\sum_{B \in B_{1,1}} t_B \leq 16, \quad \sum_{B \in B_{1,2}} t_B \leq 13, \quad \text{and} \quad \sum_{B \in B_{1,3}} t_B \leq 16.$$ 

This is not good enough to beat the Lemmens-Seidel bound\(^2\), so we proceed further.

(ii) We fix the value of the variable $t_{1111}$ in the range $0 \leq t_{1111} \leq 39$, and consider the maximum possible value for another variable $t_B$ for $B \in B \setminus B_{1,4}$ subject to that the matrix $M$ in (10) is positive semidefinite. To do this, we set $t_{B'} = 0$ whenever $B' \neq 1111, B' \neq B$. Table 2 lists the upper bounds for $t_B$, $B \in B_{1,4}, i = 0, 1, 2, 3$, when the value of $t_{1111}$ is specified.

Denote the upper bound for $t_B$ for $B \in B_{1,4}$ found in Table 2 by $m_i$, $i = 0, 1, 2, 3$. Since $|B_{1,0}| = 1$, $|B_{1,1}| = 4$, $|B_{1,2}| = 6$, and $|B_{1,3}| = 4$, an upper bound for the size of the pillar $\bar{x}$ is given by

$$M_2 = m_0 + \min\{4m_1, 16\} + \min\{6m_2, 13\} + \min\{4m_3, 16\} + t_{1111}.$$ 

The values for $M_2$ are also listed in Table 2. From here we conclude that the size of a $(3, 1)$ pillar cannot exceed 54 when another $(3, 1)$ pillar with 4 or more vectors is present. \(\Box\)

**Remark.** We note here that when a $(3, 1)$ pillar $\bar{u}$ has 3 vectors only, it is possible to have another $(3, 1)$ pillar $\bar{x}$ with as many vectors as possible. This occurs when the inner product between any one vector in $\bar{x}$ and any one vector in $\bar{u}$ is $-\frac{1}{2}$. Assume that $|\bar{x}| = n$. Then the Gram matrix

---

\(^2\)When there are two $(3, 1)$ pillars with 4 or more vectors, our computations shows that the size of whole equiangular set is bounded by $3 + 93 \cdot 3 = 282$ (see also the comparison done in Theorem 4.3). But this is not enough to beat the Lemmens-Seidel’s bound of 276.
\[ G = G(\bar{c}_1, \ldots, \bar{c}_n, \bar{d}_1, \bar{d}_2, \bar{d}_3) \text{ is} \]
\[
G = \left[ \begin{array}{ccc}
\frac{9}{10} I_n + \frac{1}{10} J_n & v & v \\
v^T & \frac{9}{10} I_3 + \frac{1}{10} J_3 \\
v^T & \frac{9}{10} I_3 + \frac{1}{10} J_3
\end{array} \right],
\]
where \( v \) is the vector \((-\frac{1}{5}, -\frac{1}{5}, \ldots, -\frac{1}{5})\) in \( \mathbb{R}^n \), and \( G \) has the Schur decomposition:
\[
\frac{9}{10} I_3 + \frac{1}{10} I_3 - \left[ \begin{array}{ccc}
v^T
\end{array} \right] \left( \frac{9}{10} I_n + \frac{1}{10} J_n \right)^{-1} \left[ \begin{array}{ccc}
v & v
\end{array} \right] = \frac{9}{10} I_3 + \left( \frac{1}{10} - \frac{2n}{5(9 + n)} \right) J_3,
\]
which is always positive definite for any \( n \in \mathbb{N} \).

**Theorem 4.3.** Let \( X \) be an equiangular set with angle \( \frac{1}{5} \) and base size \( K(X) = 3 \) in \( \mathbb{R}^r \). Then
\[ |X| \leq \max\{165, r + 6\}. \]

**Proof.** The equiangular set \( X \) is decomposed as a disjoint union of \( P = \{p_1, p_2, p_3\} \) and three \((3, 1)\) pillars. If there are two \((3, 1)\) pillars with four or more vectors, then by Lemma 4.2 we have
\[ |X| = |P| + |X(3, 1)| \leq 3 + 54 \cdot 3 = 165. \]

Otherwise there is only one big \((3, 1)\) pillar and the other two pillars can have at most 3 vectors each. Since vectors in a single \((3, 1)\) pillar is linearly independent of rank \( r - 3 \), we see that in this case
\[ |X| = |P| + |X(3, 1)| \leq 3 + (r - 3) + 3 + 3 = r + 6. \]
These inequalities finish the proof of the theorem. \(\Box\)

Note that \( \max\{165, r + 6\} \) is certainly less than the bound \( \max\{276, r + 1 + \left\lceil \frac{\sqrt{n}}{2} \right\rceil \} \) given in the Lemmens-Seidel conjecture, hence we have finished the proof when the base size \( K(X) = 3 \).

4.2. \( K = 4 \). King and Tang ([KT16], Lemma 16) showed that \( |\bar{x}| \leq 36 \) for a \((4, 1)\) pillar \( \bar{x} \) if there is another nonempty \((4, 1)\) pillar \( \bar{x} \). We get a better upper bound for \( |\bar{x}| \) for \( \bar{x} \in X(4, 1) \) if there is another nonempty \((4, 1)\) pillar \( \bar{u} \) with two or more vectors by applying Theorem 3.2. In the situation \( n = 2 \), the maximum of \( |\bar{x}| \) is \( 2n^2(n + 1) = 24 \) if there is another \((4, 1)\) pillar with two or more vectors. Hence we have the following result.

**Proposition 4.4.** In an equiangular set \( X \) with angle \( \frac{1}{5} \) and the base size \( K(X) = 4 \) in \( \mathbb{R}^r \), the maximum number of vectors that are contained in the four \((4, 1)\) pillars is \( \max\{96, r - 1\} \).

**Proof.** If there are two \((4, 1)\) pillars with two or more vectors, there are at most \( 24 \times 4 = 96 \) vectors in those pillars. Otherwise, there can be one large pillar \( \bar{x} \) together with three other pillars each of which contains at most one vector. In the case, since the vectors in \( \bar{x} \) are linearly independent in the \((r - 4)\)-dimensional subspace \( \Gamma^\perp \), the number of vectors in these \((4, 1)\) pillars is at most \((r - 4) + 3 = r - 1. \)

**Remark.** Under computations similar to Theorem 3.2, we find that if there are two nonempty \((4, 1)\) pillars, then another \((4, 1)\) pillar can hold at most 25 vectors. Hence in the case where there is only one large pillar of size \( r - 4 \) in Proposition 4.4, there can only be one other nonempty \((4, 1)\) pillar consisting of one vector when \( r - 4 > 25 \), i.e., \( r \geq 30 \).

For each of the three \((4, 2)\) pillars, the best known bound of its cardinality is \( s(r - 4, \frac{1}{13}, -\frac{5}{13}) \), obtained in [KT16], which denotes the number of vectors in a 2-distance set in \( \mathbb{R}^{r-4} \) with angles \( \frac{1}{13} \) and \( -\frac{5}{13} \). With a little improvement under Proposition 4.4, we state the result for \( K = 4 \).

**Proposition 4.5.** Let \( X \) be an equiangular set with the angle \( \frac{1}{5} \) and base size 4 in \( \mathbb{R}^r \). Then
\[ |X| \leq 100 + 3 \cdot s(r - 4, \frac{1}{13}, -\frac{5}{13}). \]

**Proof.** The equiangular set \( X \) can be partitioned into the following pairwise disjoint subsets: the 4-base \( P \), four \((4, 1)\) pillars, and three \((4, 2)\) pillars. By Lemma 16 of [KT16], any \((4, 1)\) pillar \( \bar{x} \)
will satisfy $|\bar{x}| \leq 39$ if there is a nonempty $(4,2)$ pillar. Since $s(r - 4, \frac{1}{13}, -\frac{5}{13}) \geq r - 4$ (which can be realized if all vectors within a $(4,2)$ pillar are linearly independent), we see that

$$|X| \leq |P| + |X(4,1)| + |X(4,2)| \leq 4 + 4 \cdot 24 + 3 \cdot s(r - 4, \frac{1}{13}, -\frac{5}{13}) = 100 + 3 \cdot s(r - 4, \frac{1}{13}, -\frac{5}{13}).$$

Notice that the right-hand side of (11) will never beat the Lemmens-Seidel bound. Details will be elaborated in Section 6.

4.3. $K = 5$. Let $X \subseteq \mathbb{R}^r$ be an equiangular set with angle $\frac{1}{4}$ in $\mathbb{R}^r$, with the base size $K = K(X) = 5$. Let $P = \{p_1, p_2, p_3, p_4, p_5\}$ be a 5-base in $X$. With respect to $P$, $X \setminus P$ can be partitioned into 5 possible $(5,1)$ pillars and 10 possible $(5,2)$ pillars. By carefully analyzing those pillars, we answer affirmatively to the Lemmens-Seidel conjecture for the case $K = 5$.

**Theorem 4.6.** Let $X$ be an equiangular set with angle $\frac{1}{4}$ and base size $K(X) = 5$ in $\mathbb{R}^r$.

1. If there are two or more nonempty $(5,2)$ pillars, then $|X| \leq 272$.

2. If there is at most one nonempty $(5,2)$ pillar, then $|X| \leq r + 15$.

**Proof.** By Lemma 18 of [KT16], we know that $|X(5,1)| \leq 15$. Let us now consider the rest of the vectors $P \cup X(5,2)$. Note that $Y := P \cup \{p_6\} \cup X(5,2)$ is still an equiangular set with $K(Y) = 6$ in $\mathbb{R}^r$, where $p_6 = -\sum_{i=1}^{5} p_i$. Those $(5,2)$ pillars in $X$ will become $(6,3)$ pillars in $Y$, and their classifications have been discussed thoroughly by Lemmens and Seidel [LS73]. Let us recall a key fact found in the proof of Theorem 5.7 of [LS73].

**Lemma 4.7 ([LS73]).** Let $Y$ be an equiangular set with angle $\frac{1}{8}$ and base size $K(Y) = 6$. Let $P_Y$ be a 6-base in $Y$ and $Y$ be decomposed into $P_Y$ and various $(6,3)$ pillars. Suppose there are at least two nonempty pillars in $Y$.

(i) If there are two distinct pillars each of which contains a pair of adjacent vertices, then $|Y| \leq 276$.

(ii) If there is only one pillar containing a pair of adjacent vertices and all other pillars contain independent vertices only, then $|Y| \leq 222$.

(iii) If each of these nonempty pillars contains independent vertices only, then $|Y| \leq 258$.

If there are two or more nonempty $(6,3)$ pillars and $|Y| > 258$, then $Y$ must be a subset of the equiangular set $Z$ with 276 lines in $\mathbb{R}^{23}$ with a 6-base $P \cup \{p_6\}$ by Lemma 4.7. Goethals and Seidel [GS75] proved that the structure of these 276 equiangular lines is unique, i.e., there is only one such switching class. Here we need an explicit description of these lines. The following detailed information can be found in [Neu84]. Let $\mathfrak{W}$ be the collection of 759 8-subsets of $\{1, 2, \ldots, 24\}$ that comes from the Steiner triple system $S(5,8,24)$ (or the Witt design $\text{Wit37}$), and $\mathfrak{W}_1$ be the subcollection of $\mathfrak{W}$ consisting of those 253 8-subsets that contains $\{1\}$. For any $\sigma \in \mathfrak{W}_1$, define $w_\sigma$ be the vector in $\mathbb{R}^{24}$:

$$w_\sigma := 4 \sum_{i \in \sigma} e_i - 4e_1 - \sum_{j=1}^{24} e_j.$$

For each $k \in \{2, 3, \ldots, 24\}$, let $v_k := 4e_1 + 8e_k - \sum_{j=1}^{24} e_j$ (with $e_1, \ldots, e_{24}$ being the standard basis for $\mathbb{R}^{24}$). Thus

$$Z_0 := \{w_\sigma : \sigma \in \mathfrak{W}_1\} \cup \{v_k : k = 2, 3, \ldots, 24\}$$

gives rise to the 276 equiangular set with angle $\frac{1}{8}$. Note that all these 276 vectors lie in the hyperplane $5x_1 + \sum_{j=2}^{24} x_j = 0$, and it is easy to see that $v_2, \ldots, v_{24}$ are linearly independent, so the span of $Z_0$ is of dimension 23. Consider the following 6 elements from $\mathfrak{W}_1$:

$$\sigma_1 = \{1, 2, 5, 8, 13, 15, 18, 20\}, \quad \sigma_4 = \{1, 2, 5, 8, 9, 11, 22, 24\},$$
$$\sigma_2 = \{1, 2, 3, 4, 9, 10, 11, 12\}, \quad \sigma_5 = \{1, 2, 3, 4, 17, 18, 19, 20\},$$
$$\sigma_3 = \{1, 3, 5, 7, 17, 19, 22, 24\}, \quad \sigma_6 = \{1, 3, 5, 7, 10, 12, 13, 15\}.$$

and define

$$p_i = \begin{cases} 
\bar{w}_{\sigma_i}, & \text{if } i = 1, 2, 3, \\
-\bar{w}_{\sigma_i}, & \text{if } i = 4, 5, 6.
\end{cases}$$

³ The complete list of these 253 8-subsets of $[24]$ can be found at http://math.ntnu.edu.tw/~yclin/253-8.txt.
Then the unit vectors \( p_1, \ldots, p_6 \) have mutual inner products \(-\frac{1}{5}\). For the remaining 270 vectors from \( \mathbb{Z}_0 \setminus \{ \pm p_i : i = 1, \ldots, 6 \} \), we normalize them and pick a suitable direction for each vector so that the resulting unit vectors all have inner products \( \frac{1}{5} \) with \( p_6 \). Then these vectors have a pillar decomposition

\[
Z = \{ p_1, \ldots, p_6 \} \cup \bigcup_{i=1}^{10} \bar{z}_i,
\]

where each \( \bar{z}_i \) is a \((6, 3)\) pillar consisting of 27 unit vectors, whose Seidel graph is a disjoint union of nine 3-cliques. So there are 90 3-cliques upstairs in the pillars. By uniqueness, we can assume that the set \( Y \) above is a subset of \( Z \) which contains the base set \( p_1, \ldots, p_6 \). If \( |Y| > 258 \), then \( Y \) misses at most 17 vectors in the pillars upstairs, therefore \( Y \) must contain at least one of those 90 3-cliques.

Now let us assume that there is exactly one nonempty \((5, 2)\) pillar \( \bar{x} \). We now make the following observation.

**Lemma 4.8.** Let \( \bar{x} \) be a \((5, 2)\) pillar. If \( |\bar{x}| > \text{rank}(\bar{x}) \), then the Seidel graph of \( \bar{x} \) contains a 3-clique.

**Proof of Lemma 4.8.** Again we consider \( Y = P \cup \{p_6\} \cup \bar{x} \). Now \( \bar{x} \) becomes a \((6, 3)\) pillar in \( Y \). By Theorem 5.1 of [LS73], any connected component of the Seidel graph of \( \bar{x} \) is a subgraph of one of the graphs in Figure 2, which are those connected graphs with maximum eigenvalue 2.

Except for \( C_3 = K_3 \), we check each case to ensure that \( |\bar{x}| = \text{rank}(\bar{x}) \):

- Type I, \( C_n \) with \( n \geq 4 \). This follows from the property of circulant matrices (for example, see [Mey00]).
- Type II: Let \( G \) be the Gram matrix for a Type II graph of \( n \) vertices, and \( x = (x_1, x_2, \ldots, x_n) \) be any vector in \( \mathbb{R}^n \). Then

\[
x^\top G x = \frac{1}{5} \left( \sum_{k=1}^{n} x_k^2 + (2x_1 - x_3)^2 + (2x_2 - x_3)^2 + 2 \sum_{k=3}^{n-3} (x_k - x_{k+1})^2 + (2x_n - x_{n-2})^2 + (2x_{n-1} - x_{n-2})^2 \right) \geq 0,
\]

and the equality holds if and only if \( x = 0 \). Therefore \( G \) is positive definite and of full rank.
- Types III, IV, V: Direct checks.

And the lemma is now proved. \( \square \)
Back to the proof of the main theorem. If $\bar{x}$ is a $(5, 2)$ pillar and $|\bar{x}| > \text{rank}(\bar{x})$, then $\bar{x}$ must contain a 3-clique by Lemma 4.8. Together with $P$, $X$ must contain a 6-clique in its switching class, which is a contradiction to the assumption that $K(X) = 5$. Therefore $|\bar{x}| = \text{rank}(\bar{x}) = r - 5$. Hence $|X| = |P| + |X(5, 1)| + |X(5, 2)| \leq 5 + 15 + (r - 5) = r + 15$, and the proof is now completed. □

5. Maximum equiangular sets of certain ranks

Besides the maximum cardinality of equiangular sets in $\mathbb{R}^r$, Glazyrin and Yu considered a similar question in [GY18].

**Definition 5.1.** Let $r$ be a positive integer. We define the number $M^*(r)$ to be the maximum cardinality of equiangular lines of rank $r$.

For example, we know that maximum size of equiangular line in $\mathbb{R}^8$ is 28. However, such 28 equiangular lines in $\mathbb{R}^8$ actually live in a 7-dimensional subspace by the Theorem 4 in [GY18], yet $M^*(8)$ is unknown. It is well known that $M^*(7) = 28$ and $M^*(23) = 276$. It seems that $M^*(r)$ is an increasing function on $n$, but Glazyrin and Yu [GY18] refuted this by showing $M^*(24) < 276 = M^*(23)$. Moreover, not every value of $M^*(r)$ is known even for small $r$ in the literature, for instance $M^*(8)$.

We first deal with $M^*(8)$ and start with the following result. The main technique of identifying saturated equiangular sets can be found in the authors’ previous work [LY18].

**Proposition 5.2.** There are at most 14 equiangular lines of angle $\frac{1}{3}$ of rank 8.

**Proof.** We first construct $8 \times 8$ symmetric matrices whose diagonals are 1, and $\pm \frac{1}{2}$ elsewhere. By considering their switching classes, we may assume that the entries in the first column and the first rows are all $\frac{1}{2}$, except that the top-left corner being 1. Since these matrices are Gram matrices for some bases for $\mathbb{R}^8$, they are required to be positive definite. The associated graph of such a matrix is a disjoint union of a graph of 7 vertices and one isolated vertex. By checking all 1044 such graphs (see [FS09], Example II.5), we find that there are only 3 graphs that satisfy all conditions listed above. For each of those 3 graphs, we collect all the unit vectors whose mutual inner products with each vector represented by the graph are $\pm \frac{1}{2}$, and transform these vectors as vertices of a new graph in which two vectors are adjacent if and only if their mutual inner products are $\pm \frac{1}{2}$. The clique number of the new graph plus 8 will be the size of a saturated equiangular set, and we identify the maximum in these clique numbers. Saturated equiangular sets containing these three sets of 8 basis vectors consist of 8, 14, and 14 lines respectively, from which we conclude that $M^*_4(8) = 14$. □

**Remark.** Lemmens and Seidel showed that $M^*_4(r) = 2r - 2$ for $r \geq 8$ (cf. [LS73], Theorem 4.5). The same technique as in the proof of Proposition 5.2 is applied to produce Table 3.

We indicate that the technique in [LY18] is more powerful than semidefinite programming method in [BY14]. For instance, the semidefinite programming bound on equiangular sets with angle $\frac{1}{3}$ in $\mathbb{R}^8$ is 11.2 and the technique in [LY18] obtains the bound 10.

Before we proceed further, we find the following generation of the Neumann theorem (Theorem 1.3) is necessary.

**Theorem 5.3** (Generalization of Neumann Theorem). Let $r > 3$ be a positive integer. If there are more than $2r - 2$ equiangular lines with angle $\alpha$ in $\mathbb{R}^r$, then:

- When $r$ is odd, $\frac{1}{\alpha}$ is either an odd integer, or $\sqrt{2r - 1}$.
- When $r$ is even, $\frac{1}{\alpha}$ is either an odd integer, $\sqrt{2r - 1}$, or $\frac{\sqrt{6r - 3} + 1}{2}$. 

---

Table 3. Maximum sizes of equiangular lines with specified angles for small ranks

| angle $\alpha$ | $\frac{1}{7}$ | $\frac{1}{8}$ | $\frac{1}{9}$ | $\frac{1}{10}$ | $\frac{1}{11}$ | $\frac{1}{12}$ | $\frac{1}{13}$ | $\frac{1}{14}$ | $\frac{1}{15}$ | $\frac{1}{16}$ | $\frac{1}{17}$ | $\frac{1}{18}$ |
|---------------|--------------|--------------|--------------|--------------|--------------|--------------|--------------|--------------|--------------|--------------|--------------|--------------|
| $M^*_\alpha(8)$ | 14           | 10           | 8            |              |              |              |              |              |              |              |              |              |
| $M^*_\alpha(9)$ | 16           | 12           | 10           | 18           |              |              |              |              |              |              |              |              |
| $M^*_\alpha(10)$ | 18           | 16           |              |              |              |              |              |              |              |              |              |              |
Proof. Let $X$ be an equiangular set with angle $\alpha$ in $\mathbb{R}^r$ with $|X| = 2r - 1$. Consider the matrix $A = \frac{1}{\alpha}(G(X) - I)$, which is a $(2r - 1) \times (2r - 1)$ symmetric matrix with integer coefficients and diagonal entries are all zeros, but non-diagonal entries are either 1 or $-1$. The matrix $A$ will have an eigenvalue $a = -\frac{1}{\alpha}$ with multiplicity at least $r - 1$, since $G$ has an eigenvalue zero with multiplicity at least $r - 1$. If $a$ is rational, then $a$ must be an odd integer using the same argument as in the proof of the original Neumann theorem (cf. [LS73], Theorem 3.4). Otherwise $a$ is irrational, and by degree count $a$ must a zero of an irreducible quadratic polynomial over $\mathbb{R}$; let $a^*$ be its conjugate, which must also be an eigenvalue of $A$.

The characteristic polynomial of $A$ assumes the following form: $\text{char}(A) = (x^2 - c_1 x + c_2)^\nu - 1(x - c_3)$, with $c_i$ being all integers. Comparing the coefficients, we get:

(12) \[ c_3 + (r - 1)c_1 = \text{tr} A = 0, \quad \text{i.e.,} \quad c_3 = -(r - 1)c_1. \]

Next, we see that

(13) \[ (r - 1)(a^2 + a^{*2}) + c_3^2 = \text{tr} A^2 = (2r - 1)(2r - 2). \]

By Vieta’s formula, $a^2 + a^{*2} = (a + a^*)^2 - 2aa^* = c_1^2 - 2c_2$. Plug in this relation and (12) back to (13), we see that

\[ rc_1^2 - 2c_2 = 4r - 2, \quad \text{i.e.,} \quad c_2 = \frac{r}{2}(c_1^2 - 4) + 1. \]

Because $a$ and $a^*$ are distinct real roots of the quadratic equation $x^2 - c_1 x + c_2 = 0$, its discriminant must be positive, that is,

\[ 0 < \Delta = c_1^2 - 4c_2 = (1 - 2r)(c_1^2 - 4). \]

Since $r \in \mathbb{N}$, that $\Delta > 0$ implies that $c_1 \in \{0, 1, -1\}$. We look into these three cases separately:

• $c_1 = 0$. Then $(c_2, c_3) = (-2r + 1, 0)$, and the angle is $\alpha = \frac{1}{\sqrt{2r - 1}}$.

• $c_1 = 1$. Then $(c_2, c_3) = (-3r^2 + 1, -(r + 1))$. But this case is not allowed, because the matrix $G$, being a Gram matrix, must be positive semidefinite, hence 0 is the smallest eigenvalue of $G$, and this implies that $a = -\frac{1}{\alpha}$ is the smallest eigenvalue of $A$. However, $c_3 = -(r + 1)$, which is also an eigenvalue of $A$ by assumption, is always smaller than $a$, contradiction.

• $c_1 = -1$. Then $(c_2, c_3) = (-3r^2 + 1, r + 1)$. Because $c_2 \in \mathbb{Z}$, we see that $r$ has to be an even integer. A straightforward computation shows that the angle is $\alpha = \frac{2}{\sqrt{6r - 3} + 1}$. \qed

We also need the inequality (14), which is the so-called relative bound for equiangular lines.

Theorem 5.4 ([vLS66], p.342). Let $X$ be an equiangular set with angle $\alpha$ in $\mathbb{R}^r$. If $r < \frac{1}{\alpha^2}$, then

(14) \[ |X| \leq \frac{r(1 - \alpha^2)}{1 - r\alpha^2}. \]

Together with Theorems 5.3 and 5.4, we realize that for each positive integer $r$ there are only a finite number of angles to be checked to determine $M^*(r)$.

Theorem 5.5. We have

\[ M^*(8) = 14, \quad M^*(9) = 18, \quad \text{and} \quad M^*(10) = 18. \]

Proof. For the case $r = 8$, we first note that from the proof of Theorem 5.3, we have $c_2 = 4c_1^2 - 15$ and $c_3 = -7c_1$. Therefore $(c_1, c_2, c_3) \in \{(1, -11, 7), (0, -15, 0)\}$. The zeros of char$(A)$ will be the only possible values for $-\frac{1}{\alpha}$. Hence we only need to check the $\alpha$ values for $\{\frac{1}{5}, \frac{1}{15}, \frac{1}{45}, \frac{2}{45+1}\}$. Consulting with Table 3 and checking directly that $M_\alpha(8) = 8$ where $\alpha \in \{\frac{1}{5}, \frac{1}{15}, \frac{2}{45+1}\}$, we conclude that $M^*(8) = 14$.

For $n = 9$, we read from Table 3 that $M_\frac{1}{5}(9) = 16$ and $M_\frac{1}{15}(9) = 12$. By Theorem 5.4 we obtain that $M_\frac{1}{5}(9) \leq 10$ for positive integers $n \geq 3$. Finally we find that $M_{\frac{2}{45+1}}(9) = 18$, which can be constructed by the Paley graph with cardinality 17 (see [Wal99]). Hence $M^*(9) = 18$.

Table 3 shows that $M_\frac{1}{5}(10) = 18$, and $M_\frac{2}{45+1}(10) \leq 16$ for every positive integer $n \geq 2$ by Theorem 5.4. According to Theorem 5.3, the remaining significant cases are $M_\frac{1}{5}(10) = 10$, and $M_{\frac{2}{45+1}}(10) = 11$. 


Hence we prove that $M^*(10) = 18$. 

Notice that Theorem 5.3 is universal for every dimension $r$. We may solve for more exact values of $M^*(r)$ if we spend more time on computer calculation. However the work will be repetitious so we stop here.

6. Closing remarks

We note that the results of Theorem 3.2 and Lemma 4.2 are not optimal in the sense that the upper bound on the cardinality of a pillar can be lowered if more vectors are presents in another pillar. For instance, with the angle $\frac{\pi}{4}$ and base size 4, it should not be possible to have four $(4,1)$ pillars with 24 vectors each (this produces the number 96 in Proposition 4.4). Nevertheless our bounds are sufficient to beat Lemmens-Seidel’s conjecture, so we did not pursue further. On the other hand, these bounds are valid regardless of the dimensions or ranks where the equiangular set lives.

Based on our experimentations, we believe that there can only be a large pillar; by this we form the following conjecture.

Conjecture 6.1. There is a constant $C$ that depends on the angle $\alpha$ and the base size $K$, but not to the dimension or rank, of any equiangular set, such that there could not be two pillars of size at least $C$.

This conjecture is coherent to Sudakov’s result that when the angle is fixed except for $\frac{\pi}{4}$, the upper bound for equiangular sets in $\mathbb{R}^r$ is at most $1.92r$ asymptotically (see [BDKS18]). Sudakov had a construction of equiangular sets with angle $\alpha = \frac{\pi}{2r+1}$ and rank $r$ which concentrates in one pillar whose cardinality is asymptotic to $\frac{(n+1)r}{n}$ for every positive integer $n$ (see [BDKS18], Conjecture 6.1).

The only unsolved case towards the Lemmens-Seidel conjecture is the $(4,2)$ pillars. King and Tang [KT16] showed that the unit vectors within one $(4,2)$ pillar form a 2-distance set of angles $\frac{\pi}{13}$ and $-\frac{5\pi}{13}$. But the semidefinite linear programming bound $s(r, \frac{\pi}{13}, -\frac{5\pi}{13})$ cannot be small. Consider the $3\ell \times 3\ell$ matrix of the following block form:

$$
\begin{bmatrix}
B & \frac{1}{\sqrt{3}} J_3 & \cdots & \frac{1}{\sqrt{3}} J_3 \\
\frac{1}{\sqrt{3}} J_3 & B & \cdots & \frac{1}{\sqrt{3}} J_3 \\
\vdots & \vdots & \ddots & \vdots \\
\frac{1}{\sqrt{3}} J_3 & \frac{1}{\sqrt{3}} J_3 & \cdots & B
\end{bmatrix}
$$

where $B = \begin{bmatrix} 1 & -\frac{5}{\sqrt{13}} & -\frac{5}{\sqrt{13}} \\ -\frac{5}{\sqrt{13}} & 1 & -\frac{5}{\sqrt{13}} \\ -\frac{5}{\sqrt{13}} & -\frac{5}{\sqrt{13}} & 1 \end{bmatrix}_{3 \times 3}$

This matrix has rank $2\ell+1$ and positive semidefinite, so it is the Gram matrix of $3\ell$ vectors of rank $2\ell+1$. On the other hand, the base size of the equiangular set generated from this matrix is 6, for in such a pillar there are many independent 3-cliques, and two independent 3-cliques are switching equivalent to a 6-clique (by switching all three vertices in one of the 3-cliques). So we raise another conjecture which is related to Theorem 5.1 of [LS73].

Conjecture 6.2. In the case where $\alpha = \frac{\pi}{4}$ and base size $K = 4$, there are only a finite number of families of connected graphs $S_i$’s such that the connected components of the Seidel graph of any $(4,2)$ pillar in an equiangular set is either a graph or a subgraph of a graph in $S_i$.

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