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Citation
Barkeshli, Maissam and Wen, Xiao-Gang. “Anyon Condensation and Continuous Topological Phase Transitions in Non-Abelian Fractional Quantum Hall States.” Phys. Rev. Lett. 105.21 (2010) © 2010 The American Physical Society.

As Published
http://dx.doi.org/10.1103/PhysRevLett.105.216804

Publisher
American Physical Society

Version
Final published version

Citable link
http://hdl.handle.net/1721.1/62172

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Anany Condensation and Continuous Topological Phase Transitions in Non-Abelian Fractional Quantum Hall States

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(Received 30 May 2010; published 19 November 2010)

We find a series of possible continuous quantum phase transitions between fractional quantum Hall states at the same filling fraction in two-component quantum Hall systems. These can be driven by tuning the interlayer tunneling and/or interlayer repulsion. One side of the transition is the Halperin \((p, p, p - 3)\) Abelian two-component state, while the other side is the non-Abelian \(Z_4\) parafermion (Read-Rezayi) state. We predict that the transition is a continuous transition in the 3D Ising class. The critical point is described by a \(Z_2\) gauged Ginzburg-Landau theory. These results have implications for experiments on two-component systems at \(\nu = 2/3\) and single-component systems at \(\nu = 8/3\).

DOI: 10.1103/PhysRevLett.105.216804

PACS numbers: 73.43.-f, 11.15.-q, 11.25.Hf

One of the most challenging problems in the study of quantum many-body systems is to understand transitions between topologically ordered states [1]. Since topological states cannot be characterized by broken symmetry and local order parameters, we cannot use the conventional Ginzburg-Landau theory. When non-Abelian topological states are involved, the transitions that are currently understood are essentially all equivalent to the transition between weak and strong-paired BCS states [2,3]. Over the last ten years, while there has been much work on the subject, there has not been another quantum phase transition in a physically realizable system, involving a non-Abelian phase, for which we can answer the most basic questions of whether the transition can be continuous and what the critical theory is. Here we present an additional example in the context of fractional quantum Hall (FQH) systems.

The quasiparticle excitations in FQH states carry fractional statistics and fractional charge [1]. In particular, in a \((ppq)\) bilayer FQH state [1,4], there is a type of excitation, called a fractional exciton (f exciton), which is a bound state of a quasiparticle in one layer and an oppositely charged quasihole in the other layer. It carries fractional statistics. As we increase the repulsion between the electrons in the two layers, the energy gap of the f exciton will be reduced; when it is reduced to zero, the f exciton will condense and drive a phase transition. When the anyon number has only a mod \(n\) conservation, this can even lead to a non-Abelian FQH state [2], yet little is known about “anany condensation” [5–8]. A better understanding of these phase transitions may aid the quest for experimental detection of non-Abelian FQH states, because one side of the transition—in our case the \((330)\) state at \(\nu = 2/3\)—can be accessed experimentally [9,10]. The results of this paper suggest a new way of experimentally tuning to a non-Abelian state in bilayer FQH states, similar to the transition from the \((331)\) state to the Moore-Read Pfaffian at \(\nu = 1/2\) [2,3,11].

In the \((ppq)\) state, when the energy gap of the f exciton at \(k \equiv 0\) is reduced to zero, the f exciton will condense [2]. The transition can be described by the \(\phi = 0 \rightarrow \phi \neq 0\) transition in a Ginzburg-Landau theory with a Chern-Simons (CS) term: 

\[ \mathcal{L} = |(\partial_0 + i a_0)\phi|^2 - v^2 |(\partial_1 + i a_1)\phi|^2 - f |\phi|^2 - g |\phi|^4 - \frac{1}{4\pi} a_\mu \partial_\mu a_\lambda \epsilon_{\mu\lambda} \] 

where \(\theta\) is the statistical angle of the f exciton. Such a transition changes the Abelian \((ppq)\) FQH state to another Abelian charge-\(2e\) FQH state [2,3].

In the presence of interlayer electron tunneling, the number of f excitons is conserved only mod \(p - q\). A new term \(\delta \mathcal{L} = t (\phi \tilde{M})^\mu_\nu + \text{H.c.} \) must be included, where \(\tilde{M}\) is an operator that creates \(2\pi\) flux of the \(U(1)\) gauge field \(a_\mu\). With this new term, what is the fate of the \(\phi = 0 \rightarrow \phi \neq 0\) transition?

When \(p - q = 2\), the f excitons happen to be fermions (i.e., \(\theta = \pi\)), so we can map the \(\mathcal{L} + \delta \mathcal{L}\) theory to a free fermion theory and solve the problem [2]. The problem is closely related to the transition from weak to strong pairing of a \(p_s + ip_s\) paired BCS superconductor [3]. The interlayer electron tunneling splits the single continuous transition between the \((p, p, p - 2)\) and the charge-\(2e\) FQH states into two continuous transitions. The new phase between the two new transitions is the non-Abelian Pfaffian state [12]. This is the only class of phase transitions involving a non-Abelian FQH state for which anything is known.

When \(p - q \neq 2\), the f excitons are anyons. The problem becomes so hard that we do not even know where to start. But we may guess that even when \(p - q \neq 2\), an interlayer electron tunneling may still split the transition between the \((ppq)\) and charge-\(2e\) FQH states. The new phase between the two new transitions may be a non-Abelian FQH state [2]. When \(p - q = 3\), it was suggested that the new phase is a \(Z_4\) parafermion (Read-Rezayi) [13] FQH state [14]. This is because antisymmetrizing the \((330)\) wave function between the coordinates of the two layers yields the \(Z_4\) parafermion wave function, in direct analogy to the known continuous transition from \((331)\) to Pfaffian.
where antisymmetrizing the (331) wave function yields the single-layer Pfaffian wave function.

In this Letter, we show that the Abelian \((p, p, p - 3)\) state can change into the \(Z_4\) parafermion state through a continuous quantum phase transition. The transition is in the 3D Ising class. The critical point is described by a \(Z_2\) gauged Ginzburg-Landau theory.

These results are experimentally relevant in the case \(p = 3\). The (330) state has been experimentally realized in double layer and wide quantum wells [10]. The existence of a neighboring single-layer non-Abelian state in the phase diagram, which can be realized by tuning the interlayer tunneling or repulsion, suggests that experiments have a chance of realizing this transition. Furthermore, recent detailed experimental studies of the energy gaps of the \(\nu = 8/3\) FQH state in single-layer systems indicate that it might be an exotic state, as opposed to a conventional Laughlin or particle that carries a state in the phase diagram; similarly, the results here [17], while it is also known that the Pfaffian state is near the \(\nu = 2\) plateau, numerical studies [16]—in explaining the plateau at \(\nu = 8/3\). In the case of the 5/2 plateau, numerical studies have recently suggested that the finite layer thickness of the \(Z_4\) parafermion state ought to be considered as a candidate—in addition to other proposed possibilities (e.g. [16])—in explaining the plateau at \(\nu = 8/3\). In the case of the 5/2 plateau, numerical studies have recently suggested that the finite layer thickness of the quantum well may stabilize the non-Abelian Pfaffian state [17], while it is also known that the Pfaffian state is near the (331) state in the phase diagram; similarly, the results here suggest that the finite layer thickness may also help stabilize the \(Z_4\) parafermion state at \(\nu = 8/3\) because of its proximity in the phase diagram to the (330) state.

The conceptual breakthrough in our understanding is a recently discovered low energy effective theory for the \(Z_4\) parafermion state, which was found to be a \(U(1) \times U(1)\) class of CS theory [a \(U(1) \times U(1)\) CS theory coupled with a \(Z_2\) gauge symmetry] [18]. This is closely related to the effective theory for the \((p, p, p - 3)\) state, which is a \(U(1) \times U(1)\) CS theory. So, effective theories for the \(Z_4\) parafermion state and the \((p, p, p - 3)\) states only differ by a \(Z_2\) gauge symmetry. Thus the transition between the \((p, p, p - 3)\) and the \(Z_4\) states may just be a \(Z_2\) “gauge symmetry-breaking” transition induced by the condensation of a \(Z_2\) charged field. We find that the \(U(1) \times U(1)\) class of CS theory contains a certain electrically neutral, bosonic quasiparticle that carries a \(Z_2\) gauge charge. We argue that this bosonic quasiparticle becomes gapless at the transition, and its condensation breaks the \(Z_2\) gauge symmetry and yields the \((p, p, p - 3)\) state.

To obtain the above results, without losing generality, let us choose \(p = 3\), and consider the (330) state and the corresponding filling fraction \(\nu = 2/3\) \(Z_4\) parafermion state. The same results would also apply to filling fractions \(\nu = 2n + 2/3\), where \(n\) is an integer. We begin by explaining the quasiparticle content of the \(Z_4\) states; then we show that there exists an electrically neutral bosonic quasiparticle in the \(Z_4\) state whose condensation yields the (330) state and that carries a \(Z_2\) gauge charge in the low energy effective theory. Finally, we discuss some consequences for physically measurable quantities.

One way to understand the topologically inequivalent excitations is through ideal wave functions, which admit a great variety of powerful tools for analysis of their physical properties [12,13,19–25]. In the ideal wave function approach, the ground state and quasiparticle wave functions of a \(\nu = 8\) FQH state are taken to be correlation functions of a 2D conformal field theory (CFT): \(\Phi_\nu(z_j) \sim \langle V_\nu(0) \prod V_{\gamma_i}(z_j) \rangle\), where \(\Phi\) is a wave function with a single quasiparticle of type \(\gamma\) located at the origin and \(z_j = x_j + iy_j\) is the coordinate of the \(j\)th electron. \(V_\nu\) is a quasiparticle operator in the CFT and \(V_\nu\) are electron operators. The electron operator, through its operator product expansions (OPE), forms the chiral algebra of the CFT. Quasiparticles correspond to representations of the chiral algebra. Two operators \(V_\gamma\) and \(V_\nu\) correspond to topologically equivalent quasiparticles if they differ by electron operators.

The \(Z_4\) parafermion states, which exist at \(\nu = 2/(2M + 1)\), have \(5(2M + 1)\) topologically distinct quasiparticles. These can be organized into three representations of a magnetic translation algebra [21], which each contain \(2(2M + 1), 2(2M + 1),\) and \(2M + 1\) quasiparticles—see Table I, where we also list a representative operator in the corresponding CFT description of these states. The CFT description of these states is formulated in terms of the \(Z_4\) parafermion CFT [26] and a free boson CFT. The \(Z_4\) parafermion CFT can be formulated in terms of an

**TABLE I. Quasiparticles in the \(Z_4\) parafermion FQH state at \(\nu = 2/3\) \((M = 1)\).**

| \(\Phi\) | \(Z_2\) orbifold label | \(\{n\}\) | \(h_{pf} + h_{gs}\) |
|---|---|---|---|
| 1 | \(\Phi_0 e^{i\varphi_0}\) | 0110011 | 0 + 0 |
| 1 | \(\Phi_1 e^{i\varphi_1}\) | 111101 | 0 + 0 |
| 1 | \(\Phi_2 e^{i\varphi_2}\) | 111101 | 0 + 0 |
| 1 | \(\Phi_3 e^{i\varphi_3}\) | 001111 | 0 + 0 |
| 1 | \(\Phi_4 e^{i\varphi_4}\) | 100111 | 0 + 0 |
| 1 | \(\Phi_5 e^{i\varphi_5}\) | 110101 | 0 + 0 |
| 1 | \(\Phi_6 e^{i\varphi_6}\) | 111001 | 0 + 0 |
| 1 | \(\Phi_7 e^{i\varphi_7}\) | 011011 | 0 + 0 |
| 1 | \(\Phi_8 e^{i\varphi_8}\) | 110111 | 0 + 0 |
| 1 | \(\Phi_9 e^{i\varphi_9}\) | 011011 | 0 + 0 |
| 1 | \(\Phi_{10} e^{i\varphi_{10}}\) | 110111 | 0 + 0 |
| 1 | \(\Phi_{11} e^{i\varphi_{11}}\) | 011011 | 0 + 0 |
| 1 | \(\Phi_{12} e^{i\varphi_{12}}\) | 110111 | 0 + 0 |
| 1 | \(\Phi_{13} e^{i\varphi_{13}}\) | 011011 | 0 + 0 |
| 1 | \(\Phi_{14} e^{i\varphi_{14}}\) | 110111 | 0 + 0 |
SU(2)\textsubscript{d}/U(1) coset CFT \cite{27} or, equivalently, as the theory of a scalar boson $\varphi_r$, compactified at a special radius $R^2 = 6$ so that $\varphi_r \sim \varphi_r + 2\pi R$, and that is gauged by a $Z_2$ action: $\varphi_r \sim -\varphi_r$. \cite{28}. Such a CFT is called the $U(1)/Z_2$ orbifold CFT. In Table I, we have included labels of the operators in the CFT using both of these formulations.

The fusion rules of these quasiparticles can be obtained from the fusion rules of the $Z_2$ parafermion CFT: $\Phi^0_m \times \Phi^0_l = \Phi^0_{m+l}$, $\Phi^1_m \times \Phi^1_l = \Phi^0_{m+l}$, and $\Phi^2_m \times \Phi^2_l = (l + m) \Phi^0_m + \Phi^0_m$. $\Phi^0_m$ exists for $l + m$ even, $0 \leq l \leq n$, and is subject to the following equivalences: $\Phi^0_n \sim \Phi^0_{n+2m} \sim \Phi^0_{n+2m-2}$, where $n = 4$ for the $Z_4$ parafermion CFT.

Another useful way to understand the topological order of the $Z_4$ FQH state is through its bulk effective field theory, for which there are several different formulations \cite{18,29,30}. Here we use the $U(1) \times U(1) \times Z_2$ CS theory \cite{18}. This is the theory of two $U(1)$ gauge fields, $\varphi$ and $\varphi_r$, with an additional $Z_2$ gauge symmetry that corresponds to interchanging $\varphi$ and $\varphi_r$ at any given point in space. The Lagrangian is

$$\mathcal{L} = \frac{1}{4} F^\mu_\nu F^\mu_\nu + \frac{1}{4} \mathcal{F}^\mu_\nu \mathcal{F}^\mu_\nu$$

where $a \partial a$ is shorthand for $\epsilon^\mu\nu\rho a_\mu \partial_\nu a_\rho$. For $p - q = 3$, it was found that the ground state degeneracy on genus $g$ surfaces of this theory agrees with that of the $Z_4$ parafermion FQH states at $\nu = 2/(2p - 3)$. In addition, it was found that the $Z_2 \times Z_2$ vortices, which correspond to defects around which $\varphi$ and $\varphi_r$ are interchanged, correspond to the non-Abelian $\Phi^0_4$ quasiparticles (6–11 in Table I) \cite{18}. It was also found that quasiparticle 3, which corresponds to the operator $\Phi^0_4$ (see Table I), is charged under the $Z_2$ gauge symmetry. This latter result is suggested by the orbifold formulation of the $Z_4$ parafermion CFT \cite{28}, where $\Phi^0_4$ corresponds to a $U(1)$ current $j_r \sim \partial \varphi_r$. In the $Z_2$ orbifold, the scalar boson $\varphi_r$ is gauged by the $Z_2$ action $\varphi_r \sim -\varphi_r$. The $Z_2$ gauge symmetry in the bulk CS theory is the $Z_2$ gauging in the orbifold CFT, which suggests that the quasiparticle $\Phi^0_4$ would carry a $Z_2$ gauge charge.

From Table I, we see that the $Z_4$ states contain a special quasiparticle, 3 ($\Phi^0_4$), which is electrically neutral, fuses with itself to the identity, and has Abelian fusion rules with all other quasiparticles. $\Phi^0_4$ has scaling dimension 1 and is a bosonic operator. In the following we show that the condensation of this neutral bosonic quasiparticle yields the topological order of the (330) phase. Before condensation, two excitations are topologically equivalent if they differ by an electron, which is a local excitation. After condensation, all allowed quasiparticles must be local with respect to both the electron and $\Phi^0_4$, and two quasiparticles will be topologically equivalent if they differ either by an electron or by $\Phi^0_4$. In the CFT language, this means that $\Phi^0_4$ has been added to the chiral algebra and will appear in the Hamiltonian. Such a situation was analyzed in a general mathematical setting for topological phases in Ref. [5].

From the OPE of $\Phi^0_4$ and the other quasiparticle operators in the CFT description, we find that $\Phi^0_4$ is locally mutual with respect to the quasiparticles in the first and third representations of the magnetic algebra, which consist of the quasiparticles made of $\Phi^0_m$ and $\Phi^0_0$ (see Table I).

However, its mutual locality exponent with the $\Phi^0_m$ quasiparticles is half-integer, which means that $\Phi^0_4$ is nonlocal with respect to those quasiparticles. This is expected, because the $\Phi^0_4$ quasiparticles were found to correspond to $Z_2$ vortices in the $U(1) \times U(1) \times Z_2$ CS theory while $\Phi^0_4$ was found to carry $Z_2$ charge. Thus we would expect that $\Phi^0_4$ would be nonlocal with respect to the $\Phi^0_4$ quasiparticles, with a half-integer mutual locality exponent. As a result, quasiparticles 6–11 are no longer valid (particlike) topological excitations after condensation.

Since quasiparticles that differ by $\Phi^0_4$ are regarded as topologically equivalent after condensation, quasiparticles 0, 1, and 2 become topologically equivalent to quasiparticles 3, 4, and 5 (see Table I), leaving three topologically distinct quasiparticles from this representation. Furthermore, the three quasiparticles in the third representation split into six topologically distinct quasiparticles. The reason for this was discussed in Ref. [5]. Consider the fusion of a $\Phi^0_4$ quasiparticle, which we will label as $\gamma$, and its conjugate: $\gamma \times \bar{\gamma} = 0 + 3 + 13$. After condensation, we identify 3 ($\Phi^0_3$) with the vacuum sector, so if $\gamma$ does not split into at least two different quasiparticles, then there would be two different ways for it to annihilate into the vacuum with its conjugate. A basic property of topological phases is that particles annihilate into the vacuum in a unique way, so $\gamma$ must split into at least two different quasiparticles. Since the quantum dimension of $\gamma$ is 2, it must split into exactly two quasiparticles, each with quantum dimension 1: $\gamma \Rightarrow \gamma_1 + \gamma_2$. $\gamma_1$ and $\gamma_2$ are now Abelian quasiparticles because they have unit quantum dimension. Therefore, we see that the 15 quasiparticles in the $Z_4$ parafermion state become, after condensation of $\Phi^0_4$, the nine Abelian quasiparticles of the (330) state.

As the energy gap to quasiparticle 3 ($\Phi^0_3$) is reduced, the low energy effective theory will simply be the theory of this bosonic field coupled to a $Z_2$ gauge field. The phase transition is a Higgs transition of this $Z_2$ charged boson (at least in the case where we explicitly break the global $Z_2$ symmetry of layer exchange). Note that there is no $U(1)$ symmetry that conserves the density of this kind of excitation because $\Phi^0_4$ can annihilate with itself into the vacuum. Such a theory of a real scalar coupled to a $Z_2$ gauge field was studied in Ref. [31], and it was found that the transition is continuous and in the 3D Ising universality class. As we mentioned before, the same transition, when viewed as a transition from the (330) state to the $Z_4$ state, is induced by an anyon condensation with a mod 3 conservation. This suggests that the anyon condensation can, surprisingly, be described by a $Z_2$-charged boson condensation.

Given this result, a natural question is why fluctuations of $\Phi^0_4$ should physically be related to interlayer density fluctuations, which can be tuned by the interlayer tunneling and interlayer repulsion. One answer to this question comes from the analysis of the wave functions. The (330)
wave function in real space is $\Phi_{(330)}(\{z_i, w_i\}) = \langle 0 | \prod_{i=1}^n \psi_{e_1}(z_i) \psi_{e_2}(w_i) | 330 \rangle$, where $\psi_{e_i}$ is the electron annihilation operator in the $i$th layer. When interlayer tunneling is increased, the gap between the single-particle symmetric and antisymmetric states is increased, until eventually all electrons occupy the symmetric set of orbitals, created by $\psi_{e+}$, where $\psi_{e\pm} \equiv \psi_{e_1}^\dagger \pm \psi_{e_2}^\dagger$. The natural wave function to guess in the limit of infinite interlayer tunneling is thus the projection onto the symmetric states: $\Phi(|z_i\rangle) = \langle 0 | \prod_{i=1}^n \psi_{e_1}(z_i) + \psi_{e_2}(z_i) | 330 \rangle$, which in this case is the $Z_4$ parafermion wave function [14]. Thus starting from the (330) state, the wave function analysis suggests that interlayer tunneling will yield the $Z_4$ parafermion state. Since the (330) state can be obtained from the $Z_4$ parafermion state by condensing $\Phi_{d}^0$, we are led to take this wave function analysis seriously and to conclude that the gap to $\Phi_{d}^0$ can be controlled by interlayer tunneling. Note that tuning interlayer tunneling will tune fluctuations of the operator $\psi_{e+}^\dagger \psi_{e-}^\dagger$, which is related to interlayer density fluctuations (since $\psi_{e+}^\dagger \psi_{e-} + \text{H.c.} = \psi_{e_1}^\dagger \psi_{e_1} - \psi_{e_2}^\dagger \psi_{e_2}$). Since the dimensionless parameters that we are concerned with are $t/V_{\text{inter}}$ and $V_{\text{inter}}/V_{\text{intra}}$ we see that properly tuning the interlayer or intralayer Coulomb repulsions (given by $V_{\text{inter/intra}}$) should also drive the interlayer density fluctuations and the corresponding transition. For yet a different perspective, we refer to [32].

Since the transition between these two FQH states is driven by the condensation of a neutral quasiparticle, it will be difficult to observe in experiments. Experiments have observed a phase transition in bilayer systems at $\nu = 2/3$ by measuring the kink in the charged excitation gap in charge transport measurements, though that may be an indication of a different transition [10]. One can also observe the (330) to $Z_4$ parafermion transition directly by probing the gapless neutral mode. The analysis here predicts that the bulk of the sample should remain an electrical insulator but become a thermal conductor at the transition. Furthermore, as the transition is approached, the fluctuations in the operator $\psi_{e+}^\dagger \psi_{e-}^\dagger$ should correspond to fluctuating electric dipole moments between the two layers, which can be probed through surface acoustic waves [33]. Alternatively, experiments measuring minimal quasiparticle charge would be able to detect this transition, because the $Z_4$ state at $\nu = 2/3$ has a minimal quasiparticle charge of $e/6$, while the (330) state has a minimal quasiparticle charge of $e/3$.

This research is supported by NSF Grant No. DMR-0706078.

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[1] X.-G. Wen, *Quantum Field Theory of Many-Body Systems—From the Origin of Sound to an Origin of Light and Electrons* (Oxford University Press, Oxford, 2004).

[2] X.-G. Wen, Phys. Rev. Lett. **84**, 3950 (2000).

[3] N. Read and D. Green, Phys. Rev. B **61**, 10267 (2000).

[4] R. Prange and S. Girvin, *The Quantum Hall Effect* (Springer, New York, 1987).

[5] F. A. Bais and J. K. Slingerland, Phys. Rev. B **79**, 045316 (2009).

[6] C. Gils, E. Ardonne, S. Trebst, A. W. W. Ludwig, M. Troyer, and Z. Wang, Phys. Rev. Lett. **103**, 070401 (2009).

[7] F. A. Bais, J. K. Slingerland, and S. M. Haaker, Phys. Rev. Lett. **102**, 220403 (2009).

[8] E. Grosfeld and K. Schoutens, Phys. Rev. Lett. **103**, 076803 (2009).

[9] Y. W. Suen, H. C. Manoharan, X. Ying, M. B. Santos, and M. Shayegan, Phys. Rev. Lett. **72**, 3405 (1994).

[10] T. S. Lay, T. Jungwirth, L. Smrcka, and M. Shayegan, Phys. Rev. B **56**, R7092 (1997).

[11] M. R. Peterson and S. Das Sarma, Phys. Rev. B **81**, 165304 (2010).

[12] G. Moore and N. Read, Nucl. Phys. **B360**, 362 (1991).

[13] N. Read and E. Rezayi, Phys. Rev. B **59**, 8084 (1999).

[14] E. Rezayi, X.-G. Wen, and N. Read, arXiv:1007.2022.

[15] H. C. Choi, W. Kang, S. Das Sarma, L. N. Pfeiffer, and K. W. West, Phys. Rev. B **77**, 081301(R) (2008).

[16] P. Bonderson and J. K. Slingerland, Phys. Rev. B **78**, 125323 (2008).

[17] M. R. Peterson, T. Jolicoeur, and S. Das Sarma, Phys. Rev. Lett. **101**, 016807 (2008).

[18] M. Barkeshli and X.-G. Wen, Phys. Rev. B **81**, 045323 (2010).

[19] X.-G. Wen and Z. Wang, Phys. Rev. B **77**, 235108 (2008).

[20] X.-G. Wen and Z. Wang, Phys. Rev. B **78**, 155109 (2008).

[21] M. Barkeshli and X.-G. Wen, Phys. Rev. B **79**, 195132 (2009).

[22] M. Barkeshli and X.-G. Wen, arXiv:0906.0337.

[23] B. A. Bernevig and F. D. M. Haldane, Phys. Rev. Lett. **100**, 246802 (2008).

[24] E. J. Bergholtz and A. Karlhede, J. Stat. Mech. (2006) L04001.

[25] A. Seidel and D.-H. Lee, Phys. Rev. Lett. **97**, 056804 (2006).

[26] A. Zamolodchikov and V. Fateev, Sov. Phys. JETP **62**, 215 (1985).

[27] D. Gepner and Z. Qiu, Nucl. Phys. **B285**, 423 (1987).

[28] R. Dijkgraaf, C. Vafa, E. Verlinde, and H. Verlinde, Commun. Math. Phys. **123**, 485 (1989).

[29] D. Cabra, E. Fradkin, G. Rossini, and F. Schaposnik, Int. J. Mod. Phys. A **15**, 4857 (2000).

[30] M. Barkeshli and X.-G. Wen, Phys. Rev. B **81**, 155302 (2010).

[31] E. Fradkin and S. H. Shenker, Phys. Rev. D **19**, 3682 (1979).

[32] M. Barkeshli and X.-G. Wen, arXiv:1010.4270.

[33] R. L. Willett, R. R. Ruel, M. A. Paalanen, K. W. West, and L. N. Pfeiffer, Phys. Rev. B **47**, 7344 (1993).