Bounds on Capacity Region of Optical Intensity Multiple Access Channel

Jing Zhou, Member, IEEE, and Wenyi Zhang, Senior Member, IEEE

Abstract—This paper provides new inner and outer bounds on the capacity region of optical intensity multiple access channel (OIMAC) with per-user average or peak power constraint. For the average power constrained OIMAC, our bounds at high power are asymptotically tight, thereby characterizing the asymptotic capacity region. The bounds are extended to $K$-user OIMAC with average power constraint without loss of asymptotic optimality. For the peak power constrained OIMAC, at high power, we bound the asymptotic capacity region to within 0.09 bits, and determine the asymptotic capacity region in the symmetric case. At moderate power, for both types of constraints, the capacity regions are bounded to within fairly small gaps.

Index Terms—Channel capacity, intensity modulation, multiple access channel, optical wireless communications.

I. INTRODUCTION

Intensity modulation and direct detection (IM/DD) based optical wireless communications (OWC), such as visible light communications (VLC), has received increasing attention in recent years [1], [2]. In IM/DD systems, information is transmitted by varying the intensity of emitted light, i.e., the optical power transmitted per unit area. A widely accepted channel model for IM/DD based indoor OWC is the Gaussian optical intensity channel [3], [4], which captures key properties including nonnegativity of optical intensity, input-independent additive Gaussian noise [4] and practical constraints such as limited average and/or peak optical power. The Gaussian optical intensity channel has been used in studies on coding and modulation design [5]–[7] as well as channel capacity [8]–[12].

In many indoor OWC applications there are multiple non-cooperative users (or devices) transmitting data simultaneously [2]. To explore fundamental limits of multiuser indoor OWC, capacities of several multiuser optical intensity channels including parallel channel [13], multiple access channel [14], broadcast channel [15], etc., have been studied. These channels are building blocks of more complex systems of multiuser indoor OWC.

We consider discrete-time optical intensity multiple access channel (OIMAC) with Gaussian noise. In [14], several bounds on the capacity region of OIMAC have been established where the input of each user is constrained in both its average and its peak power. Specifically, the inner bounds were obtained by using truncated Gaussian input and uniformly-spaced discrete input for each user, respectively; the outer bounds were obtained by known results for single-user optical intensity channels. By optimizing both types of input distributions numerically with respect to signal-to-noise ratio (SNR), in [14], the low SNR capacity region of OIMAC is determined. However, at moderate to high SNR, the gaps left in [14] are still evident, and the high-SNR capacity region of OIMAC is still unknown.

In this paper, by introducing new input distributions and bounding techniques, we provide new inner and outer bounds on the capacity region of OIMAC with per-user average power constraint or peak power constraint. For the average power constrained OIMAC, we derive asymptotically tight inner and outer bounds at high SNR, thereby determining the high-SNR capacity region. At moderate SNR the bounds are also fairly tight. Moreover, we extend the bounds to the $K$-user case without loss of asymptotic optimality, and provide some discussions related to system design. For the peak power constrained case, at high peak-to-noise ratio (PNR), the asymptotic capacity region of OIMAC is bounded to within 0.09 bits, and this gap vanishes in the symmetric case; at moderate PNR, by combining our outer bound and the inner bound based on discrete input in [14], the capacity region is bounded to within a small gap. A key step to our main results is utilizing capacity results of two additive noise channels where the noises obey certain maxentropic distributions, namely, exponential distribution for the average power constrained case and uniform distribution for the peak power constrained case.

The remaining part of this paper is organized as follows. In Sec. II we introduce the OIMAC with power constrains and some useful notations. Our results for average power constrained and peak power constrained OIMAC are provided in Sec. III and Sec. IV, respectively. Finally, the paper is concluded in Sec. V.

Notation: Throughout this paper, $C$ stands for capacity, $H(\cdot)$ and $h(\cdot)$ stand for entropy and differential entropy, respectively, $I(\cdot;\cdot)$ stands for mutual information, and $I(SNR)$ and $I(PNR)$ stand for the mutual information between $X$ and $X+Z$, $Z \sim \mathcal{N}(0,\sigma^2)$, with respect to SNR and PNR, respectively. We use $\sigma$ and $a$ to denote upper and lower bounds on a quantity $a$, respectively. For $i \in \{1,2\}$, we use $i$ to denote the other element in $\{1,2\}$. The convex closure (or convex hull) of a set of points $P$ is denoted by $\text{Conv}(P)$. The
asymptotic expression
\[ \lim_{t \to \infty} [A(t) - B(t)] = 0 \]  
(1)
is denoted as \( A(t) \equiv B(t) \).

II. OPTICAL INTENSITY MULTIPLE ACCESS CHANNEL

A discrete-time single-user optical intensity channel with Gaussian noise is given by [9], [10]
\[ Y = X + Z, \]  
(2)
where \( X \geq 0 \) and \( Z \sim \mathcal{N}(0, \sigma^2) \). A discrete-time two-user OIMAC has two transmitters and one receiver, and the received signal is the linear superposition of the inputs and a noise [3], [11]:
\[ Y = X_1 + X_2 + Z, \]  
(3)
where \( X_i \geq 0, i = 1, 2, \) and \( Z \sim \mathcal{N}(0, \sigma^2) \). This paper considers two types of input power constraints, namely, the per-user average power constraint as
\[ \mathbb{E}[X_i] \leq \mathcal{E}_1, \mathbb{E}[X_2] \leq \mathcal{E}_2, \]  
(4)
and the per-user peak power constraint as
\[ 0 \leq X_1 \leq \mathcal{A}_1, 0 \leq X_2 \leq \mathcal{A}_2. \]  
(5)
We define the optical SNR and PNR as \( \text{SNR} \triangleq \frac{\mathbb{E}}{\sigma^2} \) and \( \text{PNR} \triangleq \frac{\mathbb{E}}{\sigma^2} \), respectively, and denote the SNR and PNR of user \( i \) as \( \text{SNR}_i \) and \( \text{PNR}_i \), respectively. These notations can be extended directly to a \( K \)-user OIMAC
\[ Y = \sum_{i=1}^{K} X_i + Z. \]  
(6)
Further consideration on a combined constraint of average and peak power is left to future study.

Throughout the paper, in high-power analysis, we let all \( \text{SNR}_i \) or \( \text{PNR}_i \) increase simultaneously, i.e., we keep the ratio \( \frac{\text{SNR}_1}{\text{SNR}_i} \) or \( \frac{\text{PNR}_1}{\text{PNR}_i} \) fixed as input power increases.

The following single-letter characterization of the capacity region of OIMAC readily follows from the capacity region of discrete memoryless multiple access channel and the discretization procedure [16, Sec. 3.4] (cf. [14]).

Lemma 1 (Capacity region of OIMAC): The capacity region of the OIMAC [2] is the convex closure of
\[ \bigcup_{p(x_1, x_2)} \mathcal{R}(X_1, X_2), \]  
where \( \mathcal{R}(X_1, X_2) \) is the set of rate pairs \((R_1, R_2)\) satisfying
\[ R_1 \leq I(X_1; Y | X_2), \]  
(7)
\[ R_2 \leq I(X_2; Y | X_1), \]  
(8)
\[ R_1 + R_2 \leq I(X_1, X_2; Y), \]  
(9)
for a fixed product distribution \( p_X(x_1)p_X(x_2) \) satisfying given input constraint.

However, evaluating this capacity region is difficult since the inputs have continuous amplitude. Even for the single-user optical intensity channel, no analytic expression for the capacity is known. To characterize the capacity region, we will provide outer and inner bounds.

For an additive multiple access channel as [3], two simple but useful facts are given as follows. The first is
\[ I(X_i; Y | X_j) = I(X_i; X_i + Z), \]  
(10)
due to
\[ I(X_i; Y | X_j) = h(Y | X_i) - h(Y | X_1, X_2) \]
\[ = h(X_i + Z) - h(Z) \]
\[ = I(X_i; X_i + Z). \]  
(11)
The second is
\[ I(X_1, X_2; Y) = I(X_1 + X_2; Y), \]  
(12)
due to
\[ I(X_1, X_2; Y) = h(Y) - h(Y | X_1, X_2) \]
\[ = h(Y) - h(Y | X_1 + X_2) \]
\[ = I(X_1 + X_2; Y). \]  
(13)
These facts help us utilize single-user capacity results in our study on the capacity region of OIMAC.

III. AVERAGE POWER CONSTRAINED OIMAC

A. Known Single-User Capacity Results

For an OIMAC with average power constraint as [4], we utilize results of single-user optical intensity channel and additive exponential noise channel [17], [18] to derive capacity bounds. We introduce these results as follows.

Lemma 2 [8]–[10]: The capacity of the Gaussian optical intensity channel [2] with an average power constraint as \( \mathbb{E}[X] \leq \mathcal{E} \) is upper bounded by
\[ C_{\text{AP-OIC}} \leq \frac{1}{2} \log \left( \frac{e}{2\pi} (\text{SNR} + 2)^2 \right), \]  
(14)
where \( \text{SNR} \triangleq \frac{\mathbb{E}}{\sigma^2} \). The capacity is lower bounded by
\[ C_{\text{AP-OIC}} \geq \frac{E(\text{SNR})}{2} \geq I(X^E; X^E + Z) \]  
(15)
\[ \geq \frac{1}{2} \log \left( 1 + \frac{e}{2\pi} \text{SNR}^2 \right), \]  
(16)
where \( X^E \) is an exponential random variable with mean \( \mathcal{E} \). The capacity is also lower bounded by
\[ C_{\text{AP-OIC}} \geq I^G(\text{SNR}) \geq \max_{\ell > 0} I(X^G; X^G + Z), \]  
(17)
where \( X^G \) is a geometric random variable with mean \( \mathcal{E} \) and probability mass function (PMF)
\[ p_X(x, \ell) = \sum_{m=0}^{\infty} \frac{\ell}{\ell + \mathcal{E}} \left( \frac{\mathcal{E}}{\ell + \mathcal{E}} \right)^m \delta(x - m\ell), \]  
(18)
At high SNR,
\[ C_{\text{AP-OIC}} \geq \frac{1}{2} \log \left( \frac{e}{2\pi} \text{SNR}^2 \right). \]  
(19)

Lemma 3 [17]: The capacity of an additive exponential noise (AEN) channel
\[ Y = X + Z, X \geq 0, \mathbb{E}[X] \leq \mathcal{E}, \]  
(20)
where $Z$ is an exponential random variable with mean $\xi_n$, is given by
\[
C_{AEN} = \log \left(1 + \frac{\xi}{\xi_n}\right).
\] (21)

The probability density function (PDF) of the capacity-achieving input distribution is
\[
p_X(x) = \frac{\xi}{\xi + \xi_n} \delta(x) + \frac{\xi_n}{(\xi + \xi_n)^2} \exp \left(-\frac{x}{\xi + \xi_n}\right), \quad x \geq 0,
\] (22)
and the corresponding output distribution is an exponential distribution with mean $\xi + \xi_n$.

B. Bounds on Capacity Region of Average Power Constrained OIMAC

Our main results for the average power constrained OIMAC are given in the following two propositions.

**Proposition 1 (Outer bound):** The capacity region of the OIMAC with a per-user average power constraint is outer bounded by
\[
\bar{C}_i = \frac{1}{2} \log \left(\frac{e}{2\pi} (\text{SNR}_i + 2)^2\right), \quad i = 1, 2;
\] (23)
\[
\bar{C}_{\text{sum}} = \frac{1}{2} \log \left(\frac{e}{2\pi} (\text{SNR}_1 + \text{SNR}_2 + 2)^2\right).
\] (24)

**Proof:** From (7), the rate of user $i$ must satisfy $R_i \leq \max_{p_{X_i}(x_1)p_{X_2}(x_2)} I(X_i; Y|X_i)$. Combining this with the fact (10) and the single-user capacity upper bound (14), we obtain (23). From (9), the sum rate must satisfy $R_1 + R_2 \leq \max_{p_{X_1}(x_1)p_{X_2}(x_2)} I(X_1, X_2; Y)$. Combining this with (12), by noting that $X_1 + X_2$ must satisfy an average power constraint $E[X_1 + X_2] \leq \xi_1 + \xi_2$, and applying the single-user upper bound (14), we obtain (24).

**Proposition 2 (Inner bound):** The capacity region of the OIMAC with a per-user average power constraint is inner bounded by a polytope with the following five rate pairs as corner points:
\[
(R_1, R_2) = \{(0, 0), (I^E(\text{SNR}_1), 0), (I^E(\text{SNR}_1) + I^E(\text{SNR}_2), I^E(\text{SNR}_1)), (I^E(\text{SNR}_1) + I^E(\text{SNR}_2) - I^E(\text{SNR}_1), I^E(\text{SNR}_2)), (0, I^E(\text{SNR}_2))\}.
\] (25)

A closed-form inner bound weaker than (25) is given by
\[
\bar{C}_i = \frac{1}{2} \log \left(1 + \frac{e}{2\pi} \text{SNR}_i^2\right), \quad i = 1, 2;
\] (26)
\[
\bar{C}_{\text{sum}} = \frac{1}{2} \log \left(1 + \frac{e}{2\pi} (\text{SNR}_1 + \text{SNR}_2)^2\right).
\] (27)

**Proof:** The achievability of the second and last rate pairs in (25) follows directly from Lemma 2. To prove the achievability of the third and fourth rate pairs, we employ an input distribution $p_{X_i}(x_i)p_{X_2}(x_2)$. Let $p_{X_i}(x_i)$ be an exponential distribution with mean $\xi_i$, and let $p_{X_2}(x_2)$ be as (22) in which we set $\xi_2 = \xi_i$ and $\xi_n = \xi_i$. According to Lemma 3, the sum random variable $X_1 + X_2$ is exponentially distributed with mean $\xi_i + \xi_2$. By combining (7), (8) with (10) we obtain that a rate $R_i = I^E(\text{SNR}_i)$ is achievable for user $i$, and by combining (9) with (12) we obtain that a sum rate $R_1 + R_2 = I^E(\text{SNR}_1 + \text{SNR}_2)$ is achievable. So user $i$ can achieve $R_i = I^E(\text{SNR}_i)$, and the third and fourth rate pairs in (25) are both achievable. All other rate pairs in the inner bound can be achieved by time sharing (16).

According to (15) and (16), the region determined by (26) and (27) is a subset of the convex closure of (25) (noting that $I^E(\text{SNR}) \geq I^E(\text{SNR})$). Therefore it is also an inner bound.

Combining Proposition 1 and the inner bound (26) (27) in Proposition 2, we immediately obtain the following corollary.

**Corollary 1:** Let $\bar{R}$ be the achievable rate region given by
\[
R_i \leq \bar{C}_i = \frac{1}{2} \log \left(\frac{e}{2\pi} \text{SNR}_i^2\right), \quad i = 1, 2;
\] (28)
\[
R_1 + R_2 \leq \bar{C}_{\text{sum}} = \frac{1}{2} \log \left(\frac{e}{2\pi} (\text{SNR}_1 + \text{SNR}_2)^2\right).
\] (29)

Then $\bar{R}$ approximates the capacity region of the average power constrained OIMAC to within a vanishing gap as SNR grows without bound.

The rate region $\bar{R}$ is a pentagon as shown in Fig. 1. It is determined by $\bar{C}_1$ and $\bar{C}_{\text{sum}}$, which are high-SNR asymptotic expressions of the single-user capacity and the sum capacity, respectively. We call $\bar{R}$ the asymptotic capacity region of the average power constrained OIMAC.

**Remark 1 (Rate of the second user):** Combining (28) and (29), we note that when user $i$ asymptotically achieves $C_i$, the rate of the second user satisfies
\[
R_i = \bar{R}_i = \frac{1}{2} \log \left(1 + \frac{\text{SNR}_i^2}{\text{SNR}_i}\right).
\] (30)

This can be interpreted as follows. From Proposition 2, when user $i$ employs an exponential input distribution and achieves the rate $\bar{R}_i = I^E(\text{SNR}_i)$, which is lower bounded by the RHS of (26), the other user $i$, employing an input distribution like...
can achieve $R_i = I^E(SNR_1 + SNR_2) - I^E(SNR_i)$, which is lower bounded by

$$R_i = \frac{1}{2} \log \left( 1 + \frac{e}{2\pi} (SNR_1 + SNR_2)^2 \right). \tag{31}$$

This lower bound can be obtained by 1) combining \[14\], \[15\], and \[16\] to obtain upper and lower bounds of $I^E(SNR)$, and 2) applying both bounds as

$$R_i \geq I^E(SNR_1 + SNR_2) - I^E(SNR_i). \tag{32}$$

Similarly, an upper bound on $R_i$ can be obtained as

$$\overline{R}_i = I^E(SNR_1 + SNR_2) - I^E(SNR_i) \leq \frac{1}{2} \log \frac{e}{2\pi} (SNR_1 + SNR_2 + 2)^2 \left( 1 + \frac{e}{2\pi} SNR_i^2 \right). \tag{33}$$

Comparing $31$ and $33$ we obtain $30$. Therefore, the corner points of $\mathcal{R}$ are given by

$$(\overline{C}_i, \overline{R}_i) = \left( \frac{1}{2} \log \left( \frac{e}{2\pi} SNR_i^2 \right), \log \left( 1 + \frac{SNR_i}{SNR_i} \right) \right), i = 1, 2. \tag{34}$$

Fig. 2 shows our capacity bounds for average power constrained OIMAC by two examples. At high SNR, the closed-form inner bound in Proposition 2 is very tight. At moderate SNR, the closed-form inner bound \[26\], \[27\] becomes looser, but the inner bound given by \[22\], which can be evaluated numerically, is still fairly tight.

C. Extension to K-User OIMAC

Consider a $K$-user OIMAC as \[6\]. Let $\mathcal{K} = \{1, 2, \ldots, K\}$, $\mathcal{J} \subseteq \mathcal{K}$, $|\mathcal{J}| = J \geq 0$ (i.e., $\mathcal{J} = \emptyset$ is allowed), $X_{\mathcal{J}} = \{X_k : k \in \mathcal{J}\}$, and $R_{\mathcal{J}} = \sum_{k \in \mathcal{J}} R_k$. Let $\mathcal{J}^c$ denote the complement of $\mathcal{J}$. By directly extending Lemma 1, we obtain that the capacity region of the $K$-user OIMAC is the convex closure of the rate tuples $(R_1, R_2, \ldots, R_K)$ satisfying

$$R_{\mathcal{J}} \leq I(X_{\mathcal{J}}; Y|X_{\mathcal{J}}), \text{ for all } \mathcal{J} \subseteq \mathcal{K}, \tag{35}$$

for some product distribution $\prod_{k \in \mathcal{J}} p_{X_k}(x_k)$ satisfying given input power constraint.

Denote the maximum achievable $R_{\mathcal{J}}$ for all feasible input distributions as $C_{\mathcal{J}}$. The following results on the capacity region of the $K$-user OIMAC can be obtained following the same approach in our study on the two-user case. For brevity we only give outlines of proofs.

Proposition 3 (Outer bound): The capacity region of the $K$-user OIMAC \[6\] with a per-user average power constraint as $E[X_i] \leq \mathcal{E}_k, k \in \mathcal{K}$ is outer bounded by

$$C_{\mathcal{J}} = \frac{1}{2} \log \left( \frac{e}{2\pi} \left( \sum_{k \in \mathcal{J}} SNR_k + 2 \right)^2 \right), \forall \mathcal{J} \subseteq \mathcal{K}. \tag{36}$$

Outline of Proof: The bound can be derived from \[35\] by noting that

$$E \left[ \sum_{k \in \mathcal{J}} X_k \right] \leq \sum_{k \in \mathcal{J}} \mathcal{E}_k \tag{37}$$

must be satisfied $\forall \mathcal{J} \subseteq \mathcal{K}$. \[\Box\]

Proposition 4 (Inner bound): Let $\tau$ be a permutation on $\mathcal{K}$ and $\tau(k)$ be the order of $k$ in $\tau$. For a given $\mathcal{J} \subseteq \mathcal{K}$, let $\mathcal{V}_{\mathcal{J}}$ be the set of rate tuples $(R_1, \ldots, R_K)$ satisfying \[35\].

\[4\] When $\ell'$ does not exist (i.e., when $\tau(k) = 1$), we let $SNR_{\ell'} = 0$.\footnote{Note that although the noise is Gaussian, to approach this rate pair, user $i$ may use a maximum energy decoder (see \[18\]) other than a nearest neighbor decoder (a commonly adopted decoder in channels with Gaussian noise). This is because for an AEN channel, a maximum energy decoder (which is a maximum likelihood decoder in the AEN channel) achieves capacity. This result is proved in \[18\] by using the generalized mutual information (GMI).}
The capacity region of the $K$-user OIMAC \cite{6} with a per-user average power constraint as $E[X_i] \leq \mathcal{E}_k, k \in K$ is inner bounded by

$$R^K = \text{Conv} \left( \bigcup_{\mathcal{J} \subseteq K} \mathcal{V}_J \right). \quad (39)$$

A closed-form inner bound slightly weaker than \cite{38} is given by

$$C_J = \frac{1}{2} \log \left( 1 + \frac{e}{2\pi} \left( \sum_{k \in J} \text{SNR}_k \right)^2 \right), \forall \mathcal{J} \subseteq K. \quad (40)$$

**Proof:** An outline of the proof is given in Appendix A.

The following asymptotic behavior of the capacity region can be obtained by noting that the gap between the upper and lower bounds on $C_J$ vanishes in the high-SNR limit.

**Corollary:** As SNR grows without bound, the capacity region of the $K$-user OIMAC \cite{6} with a per-user average power constraint as $E[X_i] \leq \mathcal{E}_k, k \in K$ can be approximated within a vanishing gap by the achievable rate region determined by

$$R_J \leq \hat{C}_J = \frac{1}{2} \log \left( \frac{e}{2\pi} \left( \sum_{k \in J} \text{SNR}_k \right)^2 \right), \forall \mathcal{J} \subseteq K. \quad (41)$$

Note that each of \cite{40} and \cite{41} includes $2^K - 1$ equations. The rate regions determined by \cite{38, 40} and the high SNR rate region determined by \cite{41} are all convex polytopes in the $K$-dimensional space.

**Remark 2:** According to Proposition 4, a sum rate

$$\max_{(R_1, \ldots, R_K) \in \mathbb{R}^K} \sum_{k \in K} R_k = I^E \left( \sum_{k \in K} \text{SNR}_k \right) \quad (42)$$

is achievable. We can also provide an inner bound on the capacity region by

$$R_J \leq I^E \left( \sum_{k \in J} \text{SNR}_k \right), \forall \mathcal{J} \subseteq K, \quad (43)$$

which is equivalent to \cite{41}. At first glance, the inner bound \cite{43} is equivalent to the inner bound determined by \cite{38} except when $|J| = 1$. However, this is true only at high SNR. When SNR is sufficiently low, the rate $I^E(SNR_1 + SNR_2) - I^E(SNR_2)$ may exceed $I^E(SNR_1)$. In this case the inner bound \cite{43} is strictly smaller than that determined by \cite{38}, even if we replace $I^G(SNR_k)$ in \cite{38} by $I^E(SNR_k)$.

**D. Discussions on K-User OIMAC**

In contrast to the Gaussian MAC, to achieve the capacity region of average power constrained OIMAC, different users need to employ different types of input distribution. Consider the asymptotic capacity region determined by \cite{41}. At high SNR, the boundary of this region includes a face on which the sum rate is maximized (max-sum-rate face). A corner point of this face is given by

$$R_1 = \frac{1}{2} \log \left( \frac{e}{2\pi} \text{SNR}_1 \right), \quad (45)$$

$$R_k = \log \left( 1 + \frac{\text{SNR}_k}{\sum_{j<k} \text{SNR}_j} \right), \quad k > 1, k \in K, \quad (46)$$

which is achieved by employing the input distribution described in the proof of Proposition 4. That input distribution and the achieved rate for different users, however, is highly
asymmetric. Take, for example, an OIMAC with a symmetric per-user average power constraint (i.e., \( \forall k, \mathcal{E}_k = \mathcal{E} \)). In this case, as SNR increases, the rate \( R_1 \) in (45) grows without bound, and the corresponding input distribution is an exponential distribution with mean \( \mathcal{E} \). But according to (46),

\[
R_k = \log \left( 1 + \frac{1}{k - 1} \right), \quad k > 1,
\]

and the corresponding input distribution of the \( k \)th user has a singleton at zero satisfying \( \Pr(X_k = 0) = \frac{k - 1}{k} \). If our target is maximizing the sum rate with equal rates for all users (called symmetric capacity in [14], [16]), then time sharing or rate splitting must be used, while in Gaussian MAC a single Gaussian input distribution suffices [16]. The symmetric capacity can also be achieved using time-division multiple access (TDMA) with power (intensity) control [16], which has lower detection complexity than transmitting simultaneously. However, the optimality of TDMA in terms of sum capacity does not hold if there exists a per-user peak power constraint (some examples on this fact can be found in [14]).

According to our results, to achieve the sum capacity at high SNR, the input distribution for each user must be carefully chosen based on the input power constraints of all users. A natural question is that if the users still follow single-user transmission strategy (i.e., employing some near-optimal input distributions for the single-user OIMAC), then how large is the loss on the sum rate? We give an example to shed some insight on this. Consider the sum rates achieved by two types of input distributions as follows.

- Type I: users follow the asymptotically optimal input distributions given in the proof of Proposition 4.
- Type II: the input of each user obeys an exponential distribution with maximum allowed average intensity (asymptotically optimal at high SNR in the single-user case).

For simplicity let us focus on the symmetric case with \( \mathcal{E}_k = \mathcal{E}, \forall k \in K \). For Type I, the sum of the inputs (we denote it by \( S^I \)) is exponentially distributed with mean \( K \mathcal{E} \), while for Type II, the sum of the inputs \( S^II = \sum_{k=1}^K X_k \) obeys an Erlang distribution with PDF [19]

\[
p_S(s) = \frac{s^{K-1}e^{-s/(K-1)}}{(K-1)!}, \quad s \geq 0.
\]

Using the fact

\[
\begin{align*}
I(S; \text{SNR} \cdot S + Z) &= h(\text{SNR} \cdot S + Z) - h(\text{SNR} \cdot S + Z | S) \\
&= h(\text{SNR} \cdot S + Z) - h(Z) \\
&\doteq h(\text{SNR} \cdot S),
\end{align*}
\]

the high-SNR gap between the sum capacity (asymptotically achieved by Type I) and the sum rate achieved by Type II can be evaluated by the gap between the differential entropies of \( S^I \) and \( S^II \). The differential entropy of \( S^I \) is log \( K e \mathcal{E} \), and the differential entropy of \( S^II \) is [20]

\[
h(S^II) = \log \left( e^{K(1-1)!\mathcal{E}} \right) + (1 - K) \psi(K),
\]

where \( \psi(\cdot) \) is the digamma function

\[
\psi(K) = \sum_{k=1}^K k^{-1} - \gamma,
\]

and \( \gamma \) is Euler’s constant:

\[
\gamma = \lim_{n \to \infty} \left[ \sum_{k=1}^n k^{-1} - \ln n \right] \approx 0.5772.
\]

Then we obtain

\[
\Delta(\text{SNR}) \triangleq [C_{\text{sum}}(\text{SNR}) - I(S^II; S^II + Z)] \\
\doteq (K - 1)\psi(K) - \log (e^{K-1}(K-1)!K^{-1}).
\]

In Fig. 4, the values of (53), denoted as \( \hat{\Delta}(K) \), are plotted. It is shown that the gap increases linearly as \( K \) increases exponentially (in fact \( \hat{\Delta}(K) = \Theta(\log K) \)). Therefore, at high SNR, the performance loss of Type II input is more important for relatively small number of users. In Fig. [5] by numerically evaluating the input-output mutual information, the sum rates achieved by Type I and Type II inputs are plotted for different numbers of users and finite SNR values. It is shown that the performance loss of Type II input is more severe when SNR is lower.

IV. PEAK POWER CONSTRAINED OIMAC

A. Known Single-User Capacity Results

For an OIMAC with peak power constraint as (5), we utilize results of single-user optical intensity channels and certain kinds of peak power constrained channels to derive capacity bounds. Here we review these results as follows.

The following lemma comes from upper bounds on the capacity of peak power constrained optical noise (AWGN) channels given in [21], [22]. Here we have translated the result to optical intensity channels by noting that an optical intensity channel with peak power constraint \( A \) is equivalent to an AWGN channel with peak power constraint \( |X| \leq \sqrt{P} \) when \( A = 2\sqrt{P} \).

**Lemma 4:** The capacity of the Gaussian optical intensity channel [24] with a peak power constraint as \( 0 \leq X \leq A \) is upper bounded by the McKellips bound [24] as

\[
C_{\text{PP-OIC}} \leq C_{\text{TPB}}(\text{PNR}) \\
\triangleq \min \left\{ \log \left( 1 + \sqrt{\frac{1}{2\pi e} \text{PNR}} \right), \frac{1}{2} \log \left( 1 + \frac{1}{4} \text{PNR}^2 \right) \right\}.
\]

When PNR satisfies \( \frac{1}{2} - \mathcal{Q}(\text{PNR}) \geq \frac{\text{PNR}}{\text{PNR} + \sqrt{2\pi e}} \), the capacity is also upper bounded by [22]

\[
C_{\text{PP-OIC}} \leq C_{\text{TPB}}(\text{PNR}) \\
= H_2 \left( \frac{1}{2} - \mathcal{Q}(\text{PNR}) \right) + \left( \frac{1}{2} - \mathcal{Q}(\text{PNR}) \right) \log \sqrt{\frac{1}{2\pi e}} \text{PNR},
\]

where \( \mathcal{Q}(\cdot) \) is the complementary cumulative distribution function of the standard normal distribution.
where $H_2(p) = p \log \frac{1}{p} + (1 - p) \log \frac{1}{1 - p}$ is the binary entropy function, $Q(x) = \frac{1}{\sqrt{2\pi}} \int_x^\infty \exp \left( -\frac{u^2}{2} \right) du$ is the Q function. The capacity is lower bounded by \[ C_{PP-OIC} \geq I^U(X; Y) \geq \frac{1}{2} \log \left( 1 + \frac{1}{2\pi e} \text{PNR}^2 \right), \] where $X^U$ is a uniformly distributed random variable with support $[0, A]$. At high PNR, \[ C_{PP-OIC} \approx \frac{1}{2} \log \left( \frac{1}{2\pi e} \text{PNR}^2 \right). \] \[ \text{Lemma 5 [23 Problem 7.5, pp. 556].}^5 \text{ Consider an additive noise channel } Y = X + Z \text{ with input constraint } |X| \leq a, \text{ and } Z \text{ uniformly distributed over } [-1, 1]. \text{ The capacity of this channel is} \]
\[ C(a) = \log(n + 1) - (n - a) \log \frac{n + 1}{n}, \] where $n = \lceil a \rceil$, and the capacity-achieving input distribution is
\[ f_X(x) = \sum_{m=0}^{n-1} \frac{n - m}{n(n + 1)} \left( \delta(a - 2m) + \delta(-a + 2m) \right). \] When $a$ is an integer, we have $C(a) = \log(n + 1)$ and $f_X(x) = 1/(n+1) \sum \delta(n - 2m)$, which is a discrete uniform distribution.

**B. Bounds on Capacity Region of Peak Power Constrained OIMAC**

Our main results for the peak power constrained OIMAC are given in the following two propositions.

**Proposition 5 (Outer bound):** The capacity region of the OIMAC \( \mathcal{C} \) with a per-user peak power constraint as \( \mathcal{A} \) is outer bounded by
\[ \mathcal{C}_i = \mathcal{C}(\text{PNR}_i), \quad i = 1, 2, \] \[ \mathcal{C}_{\text{sum}} = \mathcal{C}(\text{PNR}_1 + \text{PNR}_2). \]

where \[ \mathcal{C}(\text{PNR}) \triangleq \begin{cases} \min \{ \mathcal{C}_{\text{M}}(\text{PNR}), \mathcal{C}_{\text{TKB}}(\text{PNR}) \}, & \text{PNR} \leq \text{PNR}^* \\ \mathcal{C}_{\text{M}}(\text{PNR}), & \text{PNR} > \text{PNR}^*, \end{cases} \]

where $\mathcal{C}_{\text{M}}$ and $\mathcal{C}_{\text{TKB}}$ are given in (54) and (55), respectively, and $\text{PNR}^*$ is the unique solution of
\[ \frac{1}{2} - Q(\text{PNR}) = \frac{\text{PNR}}{\text{PNR} + \sqrt{2\pi e}}. \]

**Proof:** This outer bound can be obtained following the same approach of the proof of Proposition 1. Specifically, the bound $\mathcal{C}_i$ is obtained by applying the single-user capacity upper bound in Lemma 4 directly; The bound $\mathcal{C}_{\text{sum}}$ is obtained by $R_{\text{sum}} \leq \max_{X_1 \in \mathcal{A}_1, X_2 \in \mathcal{A}_2} I(X_1 + X_2; Y)$, noting that $X_1 + X_2 \leq A_1 + A_2$, and applying the single-user capacity upper bound in Lemma 4.

Using techniques pioneered in [23], the capacity of the Gaussian optical intensity channel \( \mathcal{C} \) with a peak power constraint can be numerically evaluated accurately with respect to PNR. So the outer bound in Proposition 5 can be refined by replacing $\mathcal{C}(\text{PNR})$ in (60) and (61) by the numerical result of $C_{PP-OIC}(\text{PNR})$. The gain of this refinement is limited, however, since the upper bound in Lemma 4 is already very tight for moderate to high PNR. But the numerical result of $C_{PP-OIC}(\text{PNR})$ is indeed helpful for tightening the inner bound.

**Proposition 6 (Inner bound):** The capacity of the OIMAC \( \mathcal{C} \) with a per-user peak power constraint as \( \mathcal{A} \) is inner bounded by a polytope with the following five rate pairs as corner points:
\[ (R_1, R_2) = \left\{ (0, 0), (C_{PP-OIC}(\text{PNR}_1), 0), \right. \]
\[ \left. (I^U(\text{PNR}_1), I^{U+}(a_2) - I^U(\text{PNR}_1)), \right. \]
\[ \left. (I^{U+}(a_1) - I^U(\text{PNR}_2), I^U(\text{PNR}_2)), \right. \]
\[ \left. (0, C_{PP-OIC}(\text{PNR}_2)) \right\}. \]

\[ \text{The solution is approximately } 4.1324. \]
where $C_{OIC}$ is the capacity of a single-user OIC with peak power constraint; for $i = 1, 2$, $a_i = \frac{A_i}{A}$,

$$I^{U^+}(a_i) = I(X_i^U + X_i^D; X_i^U + X_i^D + Z),$$

where $Z \sim N(0, \sigma^2)$, $0 \leq X_i^U \leq A_i$, and $X_i^D$ has a PDF as

$$f_X(x) = \sum_{m=0}^{n_i-1} \frac{n_i - m}{n_i(n_i + 1)} (\delta(A_i - mA_i) + \delta(mA_i)),$$

where $n_i = \lfloor a_i \rfloor$. A closed-form inner bound weaker than (62) is given by

$$R_i \leq C_i = \frac{1}{2} \log \left( 1 + \frac{1}{2\pi e} \frac{P_{NR}^2}{2} \right), \quad i = 1, 2,$$

(67)

$$(C_i + C_2 - C_{12})R_1 + (C_1 + C_2 - C_{12})R_2 \leq C_{1}C_{11} + C_{2}C_{12} - C_{12}C_{11},$$

(68)

where for $i = 1, 2$,

$$C_{1i} = \frac{1}{2} \log \left( 1 + \left( \frac{n_i}{n_i + 1} \right)^2 \frac{2}{2\pi e} \frac{P_{NR}^2}{2} \right).$$

(69)

Proof: The achievability of the second and last rate pairs in (64) follows directly from Lemma 4. To prove the achievability of the third and fourth rate pairs, we employ an input distribution $p_{X_i}(x_i)p_{X_j}(x_j)$, where $p_{X_i}(x_i)$ is the PDF of a uniform input distribution over $[0, A_i]$, and $p_{X_j}(x_j)$ is as (66). By combining (7), (8) with (10), we obtain that a rate $R_i = I^U(P_{NR})$ for user $i$ is achievable; simultaneously, by combining (9) with (12), we obtain that a sum rate $R_1 + R_2 = I^U(X_1^U + X_1^D, X_2^U + X_2^D; Z)$ is achievable, which is exactly $I^{U^+}(a_i)$. So user $i$ can achieve $I^{U^+}(a_i) - I^U(P_{NR})$, and therefore the third and fourth rate pairs in (64) are both achievable. All other rate pairs in the convex closure of (64) can be achieved using time sharing (10).

To establish the closed-form lower bound (67) (68), we first prove that the rate pair $(R_1, R_2) = (C_{1}, C_{12} - C_1)$ is achievable. We employ the input distributions $X_i^D$ and $X_i^D$ (where we set $i = 1$ and $i = 2$) for user 1 and 2, respectively. Then the achieved sum rate and single-user rate are exactly $I^{U^+}(a_2)$ and $I^U(P_{NR})$. The achievability of $R_1 = C_1$, can be obtained directly by the single-user capacity lower bound (56). Thus, to show the achievability of $R_2 = C_{12} - C_1$, we only need a proof of

$$I^{U^+}(a_2) \geq C_{12}.$$  

(70)

First, the LHS of (70) can be lower bounded as

$$I^{U^+}(a_2) = I(X_1^U + X_2^D; X_1^U + X_2^D + Z) = h(X_1^U + X_2^D + Z) - h(X_1^U + X_2^D + Z|X_1^U + X_2^D) \geq \frac{1}{2} \log \left( \exp \left( 2h(X_1^U + X_2^D) \right) + \exp \left( 2h(Z) \right) \right) - h(Z),$$

(71)

where the inequality follows from the entropy power inequality (EPI) (26). To evaluate $h(X_1^U + X_2^D)$, we note that

$$I(X_2^D; X_1^U + X_2^D) = C(a_2),$$

(72)

where the LHS equals to

$$I(X_2^D; X_1^U + X_2^D) = h(X_1^U + X_2^D) - h(X_1^U + X_2^D|X_2^D) = h(X_1^U + X_2^D) - h(X_1^U) = h(X_1^U + X_2^D) - \log A_1.$$  

(73)

Combining (72), (73), and (58) yields

$$h(X_1^U + X_2^D) = C(a_2) + \log A_1 = \log (n_2 + 1) - (n_2 - a_2) \log \frac{n_2 + 1}{n_2} + \log A_1.$$  

(74)

Substituting (74) into (71) yields

$$I^{U^+}(a_2) \geq \frac{1}{2} \log \left( 1 + \left( \frac{n_2}{n_2 + 1} \right)^2 \frac{2}{2\pi e} \frac{P_{NR}^2}{2} \right) = C_{12},$$

(75)

and (70) is obtained. So $(R_1, R_2) = (C_1, C_{12} - C_1)$ is achievable. By symmetry $(C_2, C_{12} - C_2)$ is also achievable. Using time sharing we can further achieve the rate pair

$$(R_1(\eta), R_2(\eta)) = (\eta C_1 + (1-\eta)(C_{12} - C_2), \eta(C_{12} - C_1) + (1-\eta)C_2),$$

(76)

where $0 \leq \eta \leq 1$. Note that $(R_1(\eta), R_2(\eta))$ satisfies the linear inequality (68) with equality, and it also satisfies (76).

Therefore, the set of rate pairs satisfying both $R_1 \leq R_1(\eta)$ and $R_2 \leq R_2(\eta)$ for all $0 \leq \eta \leq 1$, which is an achievable rate region, is equivalent to the set of rate pairs satisfying both (67) and (68). This establishes the closed-form inner bound (67), (68) in Proposition 6.

C. Asymptotic Analysis and Numerical Results

In this subsection the tightness of the closed-form inner bound discribed by (67) and (68) at high PNR is evaluated. When $PNR > PNR^*$, the outer bound in Proposition 5 can be simplified as

$$C_i = \log \left( 1 + \sqrt{\frac{1}{2\pi e} \frac{P_{NR}}{A_i}} \right), \quad i = 1, 2,$$

(77)

$$C_{sum} = \log \left( 1 + \sqrt{\frac{1}{2\pi e} \frac{P_{NR}1 + P_{NR2}}{A_i}} \right).$$

(78)

Thus the asymptotic tightness of (67), and also the asymptotic result

$$C_i = \log \sqrt{\frac{1}{2\pi e} \frac{P_{NR}}{A_i}},$$

(79)

can be obtained directly. Note that $C_i$ is a lower bound on the sum capacity (it corresponds to the achievable rate pair $(R_1, R_i) = (C_i, C_i - C_i)$); see the proof of (67) and (68). From

$$C_{sum} = \log \left( \sqrt{\frac{1}{2\pi e} \frac{(a_i + 1)A_i}{A_i}} \right) = \log \frac{(n_i + 1)A_i}{\sqrt{2\pi e} A_i} + \log \frac{a_i + 1}{n_i + 1},$$

(80)
and

\[
C_{ii} = \log \left( \frac{n_i}{n_i+1} \right) (n_i-a_i) \frac{(n_i+1)A_i}{\sqrt{2\pi e \sigma}} = \log \frac{n_i+1}{\sqrt{2\pi e \sigma}} + (n_i-a_i) \log \frac{n_i}{n_i+1},
\]

(81)

the asymptotic gap between our upper bound (61) and lower bound (69) on the sum capacity can be obtained as (without causing ambiguity, indices are omitted hereinafter)

\[
\Delta(n, \lambda) = C_{\text{sum}} - C_{ii} = \log \frac{a+1}{n+1} - (n-a) \log \frac{n}{n+1}
\]

\[
= \log \left( \frac{a+1}{n+1} \left( 1 + \frac{1}{n} \right)^{n-a} \right)
\]

\[
= \log \left( \left( 1 - \frac{\lambda}{n+1} \right) \left( 1 + \frac{1}{n} \right)^{\lambda} \right),
\]

(82)

where \( \lambda \triangleq n - a \), which satisfies \( 0 \leq \lambda < 1 \). Note that

\[
e^{\Delta(n, \lambda)} = \frac{\left( 1 - \frac{\lambda}{n+1} \right) \left( 1 + \frac{1}{n} \right)^{\lambda}}{\left( 1 - \frac{\lambda}{n+2} \right) \left( 1 + \frac{1}{n+1} \right)^{\lambda}}
\]

\[
= \frac{n+1 - \lambda}{n+2 - \lambda} \left( 1 + \frac{1}{n+1} \right)^{1-\lambda} \left( 1 + \frac{1}{n} \right)^{\lambda}
\]

\[
\geq \frac{n+1 - \lambda}{n+2 - \lambda} \left( 1 + \frac{1}{n+1} \right)^{1-\lambda} \left( 1 + \frac{1}{n} \right)^{\lambda}
\]

\[
= \frac{1 + \lambda(1-\lambda)}{n(n+1)}
\]

\[
\geq 1,
\]

(83)

where the first inequality follows from the fact

\[
\left( 1 + \frac{1}{n} \right)^{\lambda} \geq 1 + \frac{\lambda}{n}.
\]

(84)

Therefore, for a given \( \lambda \), the gap \( \Delta \) is nonincreasing with \( n \). When \( n = 1 \),

\[
\Delta(1, \lambda) = \log \frac{2-\lambda}{2-1-\lambda},
\]

(85)

which is continuous, nonnegative, and approaching zero as \( \lambda \) tends to zero or one. Letting \( \Delta_{\lambda} \rightarrow 0 \), we obtain a unique solution \( \lambda^* = 2 - \log_2 e \approx 0.5573 \). Therefore, when \( \lambda = \lambda^* \), the maximum asymptotic gap of our bounds on sum capacity is achieved, which is

\[
\Delta(1, 2 - \log_2 e) = \log_2 \frac{2}{e \ln 2} \approx 0.0861 \text{ bits}.
\]

(86)

This maximum value is achieved when \( a_i = \frac{A_i}{A_{\text{max}}} = \log_2 e - 1 \approx 0.4427 \), at the rate pair \( (R_i, \tilde{R}_i) = \left( C_{ii}, C_{\text{sum}} - C_{ii} \right) \). Since the asymptotic single-user capacity has been found in (77), by noting that the sum rate of all rate pairs determined by (76) (also 68) is lower bounded by \( \min \{C_{12}, C_{21} \} \), we can infer that the maximum asymptotic gap (86) is also the maximum asymptotic gap between our outer bound (60), 61 and inner bound (67), (68) on the capacity region.

Specifically, for the symmetric case where \( A_1 = A_2 = A \), we have \( \lambda = 0 \), and the asymptotic gap is \( \Delta(1, 0) = 0 \). So the asymptotic capacity region is determined \( \tilde{C_i} \) as

\[
R_i \leq \tilde{C_i} = \log \left( \sqrt{\frac{2}{\pi e}} \right), \quad i = 1, 2
\]

(87)

\[
R_1 + R_2 \leq \tilde{C}_{\text{sum}} = \log \left( \sqrt{\frac{2}{\pi e}} \right),
\]

(88)

where \( \text{PNR} = \frac{A}{\sigma} \). Note that in this case

\[
\tilde{C}_{\text{sum}} - \tilde{C_i} = 1 \text{ bit}.
\]

(89)

In summary, in the worst case, we bound the asymptotic capacity region of peak power constrained OIMAC to within 0.0861 bits; in the symmetric case, we obtain the asymptotic capacity region.

Fig. 6 and Fig. 7 show our capacity bounds for peak power constrained OIMAC by two examples. In Fig. 6 at high PNR, our outer and inner bounds almost coincide, and both bounds perform better than the bounds given in [14]. The inner bound of [14], derived by employing truncated Gaussian input distribution for each user, is fairly close to the outer bounds, but it cannot approach the sum capacity. This is because the sum of two truncated Gaussian inputs cannot approach a uniform distribution, which is the asymptotically optimal distribution at high PNR.

In Fig. 7 at moderate PNR, our inner bound (67), (68) is not as tight as in the high-PNR regime. The inner bound (63) is tighter, but the inner bound given in [14] Fig. 5-a], which is obtained numerically by optimizing a uniformly-spaced discrete distribution, performs the best. Our outer bound is close to this inner bound, thereby bounding the capacity region to within a small gap. The outer bounds given in [14], however, cannot achieve satisfactory tightness.

Remark 3: Like the translation in Lemma 4, all our results in this section can be directly translated to a Gaussian multiple access channel (GMAC) with per-user peak power constraint as new results. The peak power constrained GMAC has been studied in, e.g., [27]-[29]. In [27], it has been shown that if the PNRs are sufficiently small, then the capacity region is achieved by employing equiprobable antipodal signaling with maximum allowable amplitude for each user. Using this result, numerical examples of the low power capacity region of the peak power constrained GMAC have been provided in [28]. In [28] and [29], it has been proved that the boundary of the capacity region of the peak power constrained GMAC is

7When the input constraint satisfies \( \frac{A_i}{A_{\text{max}}} = n_i \), we have \( \Delta(n_i, \lambda) = 0 \), which corresponds to the rate pair \( (R_i, \tilde{R}_i) = (C_{ii}, C_{\text{sum}} - C_{ii}) \). However, the gap \( \Delta(n_i, \lambda) \) corresponds to the rate pair \( (R_i, \tilde{R}_i) = (C_{ii}, C_{\text{sum}} - C_{ii}) \) is not zero since \( \frac{A_i}{A_{\text{max}}} = \frac{n_i}{n} \) is not an integer except when \( n = 1 \). In the symmetric case, both gaps vanishes at high PNR and the asymptotic capacity can be determined.

8In [14], for each user, both an average power constraint \( E_i \), and a peak power constraint \( A_i \) are considered. When the ratio \( \frac{E_i}{A_i} \geq \frac{1}{2} i = 1, 2 \), the average power constraint is inactive and the corresponding results of [14] can be compared with our results.
achieved by discrete input distributions with a finite number of mass points; however, no explicit outer or inner bounds on the capacity region were given therein.

V. CONCLUSION

In this paper, new outer and inner bounds on the capacity region of OIMAC are established under two types of input power constraints, namely, an average power constraint, and a peak power constraint. We determine the asymptotic capacity region of the average power constrained OIMAC. For the peak power constrained OIMAC, we determine the asymptotic capacity region for the symmetric case, and we bound the asymptotic capacity region to within 0.09 bits in general. For both cases, at moderate power, several non-asymptotic bounds we determined are also fairly tight.

APPENDIX A
OUTLINE OF PROOF OF PROPOSITION 4

The proof follows the same pattern as the proof of Proposition 2. First we establish the achievability of $\mathcal{V}_k$, which is the set of all $K!$ corner points of the max-sum-rate face of $R^K$. Based on the first step, we then establish the achievability of
\( \bigcup_{\mathcal{J} \subseteq \mathcal{K}} \mathcal{V}_J \), which includes all corner points of \( \mathcal{R}^K \). Finally, all rate tuples in \( \mathcal{R}^K \) can be achieved using time sharing. Some details are given as follows.

Consider a permutation \( \tau \) on \( \mathcal{K} \). We omit the indices of users, and alternatively index a user by its order: \( X_k \) is denoted as \( X_{(m)} \) if \( \tau(k) = m \). We employ an input distribution \( \prod_{m \in \mathcal{K}} p_{X_{(m)}}(x_{(m)}) \), and let \( 1 \) \( p_{X_{(1)}}(x_{(1)}) = 1 \) be an exponential \( m \in \mathcal{K} \) distribution with mean \( \mathcal{E}_{(1)} \); 2) for every \( 1 < m \leq K \), \( p_{X_{(m)}}(x_{(m)}) \) be as (22) in which

\[
\mathcal{E}_x = \mathcal{E}_{(m)}, \quad \mathcal{E}_n = \sum_{\ell \in L_{\ell}} \mathcal{E}_{(\ell)},
\]

(90)

where \( L_{\ell} = \{ \ell \} \ : \ell \in \mathcal{K}, 1 \leq \ell < m \}. \) In this setting, for every \( \mathcal{E}_{(m)} \subseteq \mathcal{E}_{(m)} \), \( X_{(m)} + \sum_{\ell \in L_{\ell}} X_{(\ell)} \) and \( \mathcal{E}_k + \sum_{\ell \in S_{\ell}} \mathcal{E}_{(\ell)} \), respectively. Following the same approach as that in the proof of Proposition 2, we obtain an achievable rate tuple \( (\mathcal{R}_{(1)}, \ldots, \mathcal{R}_{(K)}) \) which is determined by

\[
\mathcal{R}_{(m)} = I^E \left( \sum_{m \in \mathcal{L} \setminus (m+1)} \text{SNR}_m, m \in \mathcal{K} \right). \quad (91)
\]

Note that when \( \mathcal{J} = \mathcal{K} \), (101) is equivalent to (38). Therefore, by considering all \( \mathcal{K}! \) different permutations on \( \mathcal{K} \) we can establish the achievability of \( \mathcal{V}_\mathcal{K} \), which further guarantees the achievability of \( \mathcal{V}_\mathcal{J} \) except when \( |\mathcal{J}| = 1 \). The achievability of \( \mathcal{V}_\mathcal{J} \) follows directly from Lemma 2. The rate region \( \mathcal{R}_K \) is achieved using time sharing between rate tuples in \( \bigcup_{\mathcal{J} \subseteq \mathcal{K}} \mathcal{V}_\mathcal{J} \).

According to (15) and (19), the rate region determined by (40) is a subset of \( \mathcal{R}_K \) and it is thus achievable.

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