A PARAMETRIZATION OF THE THETA DIVISOR OF THE QUARTIC DOUBLE SOLID

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Abstract. Let $M_X(2; 0, 3)$ be the moduli space of rank-2 stable vector bundles with Chern classes $c_1 = 0, c_2 = 3$ on the Fano threefold $X$, the double solid $\mathbb{P}^3$ of index two. We prove that the vector bundles obtained by Serre’s construction from smooth elliptic quintic curves on $X$ form an open part of an irreducible component $M$ of $M_X(2; 0, 3)$ and that the Abel-Jacobi map $\phi : M \rightarrow J(X)$ into the intermediate Jacobian $J(X)$ of $X$ defined by the second Chern class is generically finite of degree 84 onto a translate $\Theta + \text{const}$ of the theta-divisor. We also prove that the family of elliptic quintics on a general $X$ is irreducible and of dimension 10.

0. Introduction

After the famous paper of Clemens and Griffiths [CG], in which they determined the intermediate Jacobian and its theta divisor for the cubic threefold, the next Fano threefold to study was naturally the quartic double solid, that is the double cover $\pi : X_2 \xrightarrow{2:1} \mathbb{P}^3$ ramified in a quartic surface $W \subset \mathbb{P}^3$. In 1981, Welters [Ve] found a family of curves parametrizing the intermediate Jacobian $J(X_2)$, the septs of genus 4, and this is the simplest known family of generically irreducible curves with this property. He developed also techniques that allow one to decide whether the Abel-Jacobi images of certain families of curves lie in a translate $\Theta + \text{const}$ of the theta divisor, but failed to find one parametrizing the whole of $\Theta + \text{const}$. Such a family was found in 1986 (with a use of the techniques of Welters) by Tikhomirov [T-2]: these are the sextics of genus 3, or the so called Reye sextics. In 1991, Clemens [C-2] reproved Tikhomirov’s result by degenerating the double solid into a pair of $\mathbb{P}^3$’s meeting each other transversely along a quadric, a technique he developed earlier in [C-1]. Several other papers have contributed, since 1981, to the study of quartic double solids: [T-1], [SV], [De], [B], the Fano threefolds $X_{g-2}$ of the main series of genus 7 [IM-3] and 9 [IR], and the double solid $X_2$, treated in the present paper and in [T-3]. It is noteworthy that in most cases the

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interesting components of moduli are those parametrizing the vector bundles obtained by Serre’s construction from elliptic quintics. For the cubic \( X_3 \), the relevant moduli space is \( M_{X_3}(2; 0, 2) \), and the Abel-Jacobi map identifies it with an open set in the intermediate Jacobian \( J(X_3) \). For the double solid, the corresponding moduli space \( M_X(2; 0, 3) \) is reducible: it has at least two components, one of which, say \( M \), is associated to elliptic quintics, and the other to disjoint unions of two elliptic curves in \( X \) whose images in \( \mathbb{P}^3 \) are a plane cubic and a line. Leaving apart the question whether there are any other components, we restrict ourselves to the study of \( M \).

The main result of this paper states that the Abel-Jacobi map \( \Phi \) sends the family of elliptic quintics \( C_3^1(X) \) onto an open subset of \( \Theta + \text{const} \), and that the Serre construction defines a factorization \( C_3^1(X) \xrightarrow{\text{Serre}} M \xrightarrow{g} \Theta + \text{const} \) with \( g \) generically finite of degree 84 (Theorems 2.7 and 5.1). We also prove the irreducibility of \( C_3^1(X) \) (Theorem 1.4), so that \( M \) is really an irreducible component of the moduli space, rather than a union of components. The facts that the image of \( \Phi \) is dense in \( \Theta + \text{const} \) and that \( g \) is generically finite were stated in [1-3], so the main goal of our paper is to determine the degree of \( g \). Nonetheless, we provide a complete proof of these facts too, for their proof was only sketched in loc. cit.

We use Welters’ criterion (Proposition 2.4) which gives sufficient conditions for a certain subfamily \( \{ C_t \}_{t \in T} \) of a given family \( \{ C_s \}_{s \in S} \) of curves of degree \( k(k+1) - 3 \) to be mapped by the Abel-Jacobi map into a translate of the theta divisor. Here \( l \) is a line in \( X \) and

\[
T_l = \{ t \in T \mid C_t \cap l = \emptyset \text{ and } C_t \cup l \text{ lies on a surface from } |\mathcal{O}_X(k)|, \]

where \( \mathcal{O}_X(k) := \pi^*\mathcal{O}_{\mathbb{P}^3}(k) \). We construct such a family \( T \) with \( k = 4 \) as a partial compactification of the family of reducible curves \( C = C_1 \cup C_2 \), where \( C_1 \) is a genus-4 septic, \( C_2 \) the pullback of a genus 2 quintic \( C_2 \subset \mathbb{P}^3 \), and \( C_1, C_2 \) meet each other quasi-transversely in 13 points. We show that \( T_l \) contains a subfamily of curves of the form \( C' \cup C'' \cup C_2 \) such that \( C' \in C_3^1(X) \) and \( C'' = \pi^{-1}(m) \), where \( m \subset \mathbb{P}^3 \) is a trisecant of \( \pi(C') \), whose Abel-Jacobi images differ from those of \( C' \) by a constant translation. Hence \( \Phi_{C_3^1(X)}(C_3^1(X)) \subset \Phi_{T_l}(T_l) + \text{const} \subset \Theta + \text{const} \). The opposite inclusion follows from the computation of the differential of the Abel-Jacobi map. The generic fiber of \( \Phi_{C_3^1(X)} \) is a union of \( \nu \) copies of \( \mathbb{P}^1 \) which can be interpreted either as pencils of elliptic quintics \( \{ C_\lambda \}_{\lambda \in \mathbb{P}^1} \) in the K3 surfaces \( S(C_\lambda) \) from the linear system \( |\mathcal{O}_X(2)| \), or as pencils of the zero loci of sections \( s \in H^0(X, \mathcal{E}) \), where \( \mathcal{E} \) is the vector bundle obtained by Serre’s construction from \( C_\lambda \).

To compute the number \( \nu \) of copies of \( \mathbb{P}^1 \) in the generic fiber of \( \Phi_{C_3^1(X)} \), we use the degeneration \( \{ X_t \}_{t \in \Delta} \) of \( X \) into the union \( X_0 = \mathbb{P}^3 \cup \mathbb{P}^3' \) of two copies of \( \mathbb{P}^3 \) meeting each other in a smooth quadric \( Q = \{ G = 0 \} \), where \( \Delta \) is a disc in \( C \). The fiber \( X_t \) is defined as the double cover \( \pi_t : X_t \longrightarrow \mathbb{P}^3 \) ramified in \( W_t = \{ tF + G^2 = 0 \} \), where \( W = \{ F = 0 \} \) is the branch locus of \( X \longrightarrow \mathbb{P}^3 \). The Hodge theory and the Neron model of the relative intermediate Jacobian of such a degeneration were studied by Clemens in [4-5]. He made also the following important observation: let \( C_0 \subset X_0 \) be an irreducible curve of degree \( d \) which is the limit of a family of curves \( C_s \subset X_t \), \( t = s^e \). Assume that \( C_s \) for \( s \neq 0 \) is projected birationally onto its image \( C_s \subset \mathbb{P}^3 \); this means that \( C_s \) is tangent to \( W_t \) in 2d points. Then \( C_0 \) is a 2d-secant to the octic curve \( B = W \cap Q \). Thus the limits of elliptic quintics from \( C_3^1(X_t) \) are elliptic quintics in \( \mathbb{P}^3 \) or \( \mathbb{P}^3' \) which are 10-secant to \( B \). The work of Clemens allows us to extend the Abel-Jacobi map to such curves in \( X_0 \) and the question on the number of copies of \( \mathbb{P}^1 \) in the generic fiber of the Abel-Jacobi map \( \Phi_0 \) at \( t = 0 \) is reduced to that on the number of elliptic quintics in \( \mathbb{P}^3 \) which are 10-secant to a given octic \( B \).
By a result of Getzler [Ge], there are 42 elliptic quintics passing through 10 generic points in \( \mathbb{P}^3 \). We show that the 10-uples of points lying in the smooth complete intersections of a quartic and a quadric in \( \mathbb{P}^3 \) form a divisor in the main component \( H \) of \( \text{Hilb}^{10}(\mathbb{P}^3) \) and that this divisor is not contained in the branch locus of the 42-sheeted covering defined by the elliptic quintics, hence there are 2 \( \cdot \) 42 = 84 copies of \( \mathbb{P}^1 \) in the generic fiber of \( \Phi_0 \) (42 copies in each one of the two \( \mathbb{P}^3 \)'s constituting \( X_0 \)). We show also that the elliptic quintics, 10-secant to \( B \), are acquired with multiplicity 1 in the relative Hilbert scheme of the family \( \{ X_t \}_{t \in \Delta} \), hence the number of copies of \( \mathbb{P}^1 \) in the fiber of \( \Phi_t \) is invariant under the deformation and \( \nu = 84 \).

To this end, we show that the family of 10-secant elliptic quintics to a given generic \( B \) is irreducible and construct a family \( \{ C_t \}_{t \in \Delta} \) of cycles of 5 lines, or pentagons in \( X_t \), such that \( C_0 \) is 10-secant to \( B \) and is strongly smoothable into an elliptic quintic in \( X_0 \), 10-secant to \( B \). This implies that a generic 10-secant elliptic quintic in \( X_0 \) admits a local cross-section passing through it in the relative Hilbert scheme of \( \{ X_t \}_{t \in \Delta} \), hence the fiber over \( t = 0 \) is acquired with multiplicity 1.

We now describe the contents of the paper by sections.

The main result of Section 1 is the proof of the irreducibility of the family of elliptic quintics \( C_5^1(X) \). We start by proving the irreducibility of the families \( C_2^0(X) \) of conics and \( C_3^0(X) \) of twisted cubics. The proofs are based on the splitting of the curves under consideration into two components of smaller degrees. As concerns elliptic quintics, we split them into a twisted cubic and an ”elliptic conic”, that is, \( \pi^{-1}(m) \) for a line \( m \subset \mathbb{P}^3 \). As soon as the irreducibility of the family of split curves is established, we get a distinguished component of \( C_5^1(X) \), namely the one containing the smoothings of the split curves. Then we use the irreducibility of the monodromy action on the set of all the components of \( C_5^1(X) \) to deduce that the number of components is 1.

In Section 2 we prove the irreducibility of the family \( C_2^4(X) \) of genus-4 septics in \( X \), which dominate \( J(X) \) according to Welters, and construct the above mentioned family \( T \) of curves of degree \( k(k + 1) - 3 = 17 \) for \( k = 4 \), to which the criterion of Welters is applied. We conclude by the proof of Theorem [2.7] stating that the Abel-Jacobi image of the elliptic quintics is a translate of \( \Theta \).

In Section 3 we explain the Clemens’ degeneration technique and describe the limiting Abel-Jacobi map \( \Phi_0 \) on the degenerate double solid, which consists of two copies of \( \mathbb{P}^3 \). We also show that the central fiber in the domain of the Abel-Jacobi map is reduced by studying the deformations of 10-secant pentagons. This implies that the number of connected components in the generic fiber of \( \Phi_t \) is the same as for \( \Phi_0 \).

In Section 4 we prove that the number of elliptic quintics in \( \mathbb{P}^3 \) passing through 10 generic points remains unchanged if we specialize the 10-uple of points to a 10-uple lying on a generic complete intersection of a quadric and a quartic. This implies that the number of connected components in the generic fiber of \( \Phi_0 \) is 84.

In Section 5 we introduce the Serre construction and reformulate the main theorem in terms of moduli of vector bundles.

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1. Conics, twisted cubics and elliptic quintics on \( X \)

1.1. Notation.

- \( X \), a general quartic double solid, \( X \subset Y = \text{Proj}(\mathcal{O}_{\mathbb{P}^3} \oplus \mathcal{O}_{\mathbb{P}^3}(2)) \), where \( \mathcal{L} = \mathcal{O}_{\mathbb{P}^3}(1) \) (see [T-1] or [We]).
- \( \rho : Y \longrightarrow \mathbb{P}^3 \), \( \pi = \rho|_X : X \longrightarrow \mathbb{P}^3 \), the natural maps.
the ample sheaf $O(X)$ (Theorem 1.4). First we need to prove that the families $X$

1.2. Conics. First recall the well known facts about lines on a general $X$ (see, e.g., [T-1], [T-2], [We]).

Lemma 1.1. Let $X$ be a general quartic double solid. Then the following assertions hold:

(i) The Fano surface $F = F(X)$ of lines on $X$ is an irreducible surface having at most isolated quadratic points as singularities, and the incidence divisor $D_F = \{(l_1, l_2) \in F \times F | l_1 \cap l_2 \neq \emptyset \}$ on $F \times F$ is irreducible.

(ii) For a general line $l \in F(X)$, the normal bundle $N_{l/X}$ is isomorphic to $2O_{\mathbb{P}^3}$.

Next we will prove the following statement:

Lemma 1.2. Let $X$ be a general quartic double solid. Then:

(i) $C_2^0(X)$ is irreducible of dimension 4.

(ii) For a general conic $C \in C_2^0(X)$, the normal bundle $N_{C/X}$ is isomorphic either to $2O_{\mathbb{P}^1}$ or to $O_{\mathbb{P}^1} \oplus O_{\mathbb{P}^1}(2)$.

Proof. (i) Let $W = C_2^0(X) \subset \mathbb{P}^3$ be the dual surface of the quartic $W$ and consider the morphism $v : C_2^0(X) \to \mathbb{P}^3 : C \mapsto \text{Span}(\pi(C))$. Then $v^{-1}(Y)$ for a plane $Y \in \mathbb{P}^3$ consists of all conics on the surface $S_Y = \pi^{-1}(Y)$. If $Y \in U := \mathbb{P}^3 \setminus W$, then $S_Y$ is a smooth del Pezzo surface and for any conic $C \in C_2^0(X)$ lying on $S_Y$ one easily checks that the family of conics in $S_Y$ containing $C$ is a pencil containing 6 reducible conics. These reducible conics, considered as pairs of lines, when $C$ runs through the open subset $C_2^{0,\ast}(X) := v^{-1}(U)$ of $C_2^0(X)$, constitute the subset $D_2^* - D_2$ of dimension 3, which is a dense open subset of $D_2$ since $D_2$ is irreducible of dimension 3 by Lemma 1.1. Thus we have a map $\psi : D_2^* \to C_2^{0,\ast}(X)$ which is 2:1 onto $D_1 = \psi(D_2^*)$. Moreover, the map $v$ factors as $v : C_2^{0,\ast}(X) \xrightarrow{\text{can}} C \xrightarrow{\pi} U$ such that the maps $v_2$ and $v_1|D_1$ are quasifinite and surjective, $\text{deg}(v_1|D_1) = 6$ and $v_1^{-1}(c) \simeq \mathbb{P}^1$ for $c \in C$. Thus since $D_2^*$ is irreducible, it follows that $C_2^{0,\ast}(X)$ is irreducible of dimension 4.

Now take $Y \in W$, so that $Y$ is a tangent projective plane to $W$ at some point $x : Y = PT_x W$. Since $W$ is a general quartic in $\mathbb{P}^3$ for a general $X$, the set $T_Y$ of points $x$ such that $Y = PT_x W$ is finite (generically it is a unique point). Assume that $\dim v^{-1}(Y) \geq 2$. Since $\pi : S_Y \to Y$ is a finite morphism, the map $\bar{v} : v^{-1}(Y) \to |O_Y(2)| : C \mapsto \pi(C)$ is quasifinite, hence $\dim V(Y) \geq 2$, where $V(Y) := \bar{v}(v^{-1}(Y))$. Let $V_1(Y) = \{C \in V(Y)|C \ni x$ for some $x \in T_Y\}$. If $\dim V_1(Y) \geq 2$, then some point $x \in T_Y$ belongs to at least a 1-dimensional family of reducible conics from $V_1(Y)$, i.e. there is a 1-dimensional family of lines through $x$ in $Y$, each of which is a double tangent to the quartic curve $W \cap Y$, which is impossible when $W$ is general. Hence $\dim V_1(Y) \leq 1$, so that $\dim v^{-1}(Y)' \geq 2$, where
Now for any conic $C \in v^{-1}(Y)'$ the surface $S_Y = \pi^{-1}(Y)$ is smooth along $C$, so that, as above, the conic $C$ varies in a pencil of conics on $S_Y$, contrary to the above inequality \( \dim_C v^{-1}(Y)' \geq 2 \). Hence \( \dim v^{-1}(Y) \leq 1 \) for $Y \in \mathcal{W}$, and so \( \dim v^{-1}(\mathcal{W}) \leq 3 \). Now, by Riemann-Roch \( \chi(N_{C/X}) = 4 \) for any conic $C \in C_2^0(X)$, hence by deformation theory \( \dim_C C_2^0(X) \) and the inequality \( \dim v^{-1}(\mathcal{W}) \leq 3 \) implies that \( v^{-1}(\mathcal{W}) \) is not a component of $C_2^0(X)$. Since by construction $C_2^0(X) = \mathcal{C}_2(X) \cup v^{-1}(\mathcal{W})$, the irreducibility of $C_2^0(X)$ follows.

(ii) Let \((l_1, l_2) \in D_F\) be generic. Then $l_1$ meets $l_2$ quasi-transversely (that is, with different tangents) at some point $x \in X$ and, by Lemma 1.1, \( N_{l_i/X} \simeq 2\mathcal{O}_\mathbb{P}_1 \). Let $C = l_1 \cup l_2$. We have the exact triple $0 \to N_{l_i/X} \to N_{C/X} \big| l_i \xrightarrow{\pi} C_x \to 0$, \( i = 1, 2 \), so that $N_{C/X} \big| l_i \simeq \mathcal{O}_{\mathbb{P}_1} \oplus \mathcal{O}_{\mathbb{P}_1}(1)$, \( i = 1, 2 \), and the exact triple $0 \to N_{C/X} \to N_{C/X} \big| l_1 \oplus N_{C/X} \big| l_2 \to N_{C/X} \to 0$, \( N_{C/X} \big| x \simeq C_2^0 \), implies that \( h^1(N_{C/X}) = 0 \). Also, the equality $N_{C/X} \big| l_1 = \mathcal{O}_{\mathbb{P}_1} \oplus \mathcal{O}_{\mathbb{P}_1}(1)$, and the exact triple $0 \to N_{C/X} \big| l_2(-x) \to N_{C/X} \big| l_1 \to 0$, in which $R$ is the usual restriction map, imply the surjectivity of the map $H^0 R : H^0(N_{C/X}) \to H^0(N_{C/X} \big| l_1)$.

Hence the composition $\tau_x : H^0(N_{C/X}) \xrightarrow{H^0R} H^0(N_{C/X} \big| l_1) \xrightarrow{H^0} H^0(C_x)$ is surjective. Hence the equality $h^1(N_{C/X}) = 0$ and [Hir] Prop. 1.1 imply that the conic $C = l_1 \cup l_2$ is strongly smoothable in $X$, that is there exists an analytic family $\pi : S \xrightarrow{i} X \times \Delta \xrightarrow{\pi_1} \Delta$ over the disk $\Delta$ such that $S$ is a smooth complex surface, $\pi$ is a flat morphism, $\pi^{-1}(0) = C$ and $C_t = i(\pi^{-1}(t)) \simeq \pi^{-1}(t)$ is a smooth conic in $X$ for any $t \in \Delta \setminus \{0\}$. Hence we can take a section $s : \Delta \xrightarrow{i} S$, $t \mapsto y_t$, of $\pi$ such that $y_0 \neq x, y_0 \in l_1$. It follows now from the above description of $N_{C/X} \big| l_1$ and the exact triple $0 \to N_{C/X} \big| l_2(-x) \to N_{C/X} \big| l_1(-y_0) \to 0$ that $h^1(N_{C/X} \big| l_1(-y_0) = 0$, so that by semicontinuity (possibly after shrinking $\Delta$) we obtain $h^1(N_{C/X} \big| l_t = 0$ for $t \in \Delta \setminus \{0\}$. Combining this with the observation that $\det(N_{C/X}) \simeq \mathcal{O}_{\mathbb{P}_1}(2)$ for $t \in \Delta \setminus \{0\}$, we see that (ii) holds for a general conic.

\[\Box\]

1.3. Twisted cubics.

Lemma 1.3. Let $X$ be a general quartic double solid. Then the following assertions hold:

(i) $C_2^0(X)$ is irreducible of dimension 6.

(ii) For a general smooth cubic $C \in C_3^0(X)$, the normal bundle $N_{C/X}$ is isomorphic to $\mathcal{O}_{\mathbb{P}_1}(k) \oplus \mathcal{O}_{\mathbb{P}_1}(4 - k)$, \( 0 \leq k \leq 4 \).

Proof. The proof is similar to that of Lemma 1.2. Let us start by proving (ii). Using Lemmas 1.1 and 1.2, pick a general line $l \in \mathcal{F}$ and a conic $C' \in C_2^0(X)$ meeting each other quasi-transversely in a unique point $x$ and such that $N_{l/X} \simeq 2\mathcal{O}_{\mathbb{P}_1}$, $N_{C'/X} \simeq \mathcal{O}_{\mathbb{P}_1}(k) \oplus \mathcal{O}_{\mathbb{P}_1}(2 - k)$, \( 0 \leq k \leq 1 \). Let $C = l \cup C'$. Then we have the exact sequences $0 \to N_{l/X} \to N_{C'/X} \big| l \xrightarrow{\pi} C_x \to 0$, $0 \to N_{C'/X} \to N_{C/X} \big| C' \to C_x \to 0$. Hence $N_{C/X} \big| C' \simeq \mathcal{O}_{\mathbb{P}_1}(a) \oplus \mathcal{O}_{\mathbb{P}_1}(b)$, \( 0 \leq a \leq b \), \( a + b = 3 \), and the exact triple $0 \to N_{C/X} \to N_{C/X} \big| C' \to N_{C/X} \big| x \to 0$, $N_{C/X} \big| x \simeq C_2^0$, implies that $h^1(N_{C/X}) = 0$. As in the proof of Lemma 1.2 (ii) with $(l, C')$ in place of $(l_1, l_2)$, we obtain the wanted assertion for a generic cubic from any irreducible component of $C_3^0(X)$ which contains smoothings of a generic decomposable curve. Let us show that there is only one such component.

By [We1], Corollary (5.4), the curve $D_l \subset \mathcal{F}$ of lines in $X$ meeting a given generic line $l$ is irreducible and is a 6-sheeted covering of $l$, because there are 6 lines passing through the generic point of $X$. So, if we take a generic reducible conic $C = l_1 + l_2 \subset X$, then the curve $D_C$ of lines meeting $C$ is a 6-sheeted covering of $C$, which is irreducible as a covering, that is there is no subcurve $D' \subset D_C$ which is a $d$-sheeted covering of $C$ for some $d < 6$. The irreducibility of a covering is stable under small deformations, so we can conclude that $D_C$ stays irreducible when we deform $C$ into a smooth conic. The family of conics in $X$ being irreducible, the incidence variety $\mathcal{D}$ of pairs $(l, C') \in \mathcal{F} \times C_3^0$ such that
l ∩ C′ ≠ ∅ is irreducible. By the above, l + C′ is a smooth point of Hilb(X) for generic 
(l, C′) ∈ D, so there is only one component C^{0*, 1}(X) of C^{0*}(X) containing the smoothings 
of the curves l + C′ for generic (l, C′) ∈ D.

Now we can prove the assertion (i) which implies, in particular, that C^{0*}(X) is the 
whole of C^{0*}(X) and hence (ii) is verified for a generic cubic C from C^{0*}(X).

We have dim C^{0*, 1}(X) = 6, and any irreducible component C^{0*}(X) of C^{0*}(X) has dimension 
at least χ(N_{C/X}) = 6.

Let X ↦ Y × P^{45} be the total space of the linear system L^2 on Y and Y ↾ X ↦ P^{45} 
the natural projections. Let U = {t ∈ P^{45} | X_t = p(q^{-1}(t)) is smooth}. Any C ∈ C^{0*}(X_t) 
for t ∈ U is projected by ρ to a twisted cubic C_0 ⊂ P^3. Clearly, (O_{P^3} ⊕ O_{P^3}(2))|_{C_0} ≃ 
O_{P^3} ⊕ O_{P^3}(6) for any C_0 ∈ C^{0*}(P^3). Hence there exists a rank-8 vector bundle E over 
C^{0*}(P^3) with fiber E_{(C_0)} = H^0(O_{P^3} ⊕ O_{P^3}(6)) over C_0 ∈ C^{0*}(P^3), and C^{0*}(Y) is a dense open 
subset of Proj(E) such that the natural projection C^{0*}(Y) → C^{0*}(P^3), C ↦ ρ(C), coincides 
with the composition C^{0*}(Y) ↦ Proj(E) ↾ P^3 → C^{0*}(P^3), where pr is the structure morphism. 
C^{0*}(P^3) being irreducible, C^{0*}(Y) is irreducible too and, since dim C^{0*}(P^3) = 12, we obtain 
dim C^{0*}(Y) = 19.

Consider the incidence variety Γ = {(t, C) ∈ U × C^{0*}(Y) | C ⊂ X_t}, U ↾ P^3 ↦ C^{0*}(Y).

We have p^{1-1}(C) = U ∩ |I_{C/Y} ⊕ L^{0*}| ↾ P^3 → |I_{C/Y} ⊕ L^{0*}|. Any C_0 ∈ C^{0*}(P^3) has the following 
resolution of its structure sheaf: 0 → 2O_{P^3}(-3) → 3O_{P^3}(-2) → O_{P^3} → O_{C_0} → 0.

Hence the twisted ideal sheaf I_{C/Y} ⊕ L^{0*} of any C ∈ C^{0*}(Y) has a resolution of the form 
0 → 2L(-3h) → 3L(-2h) → L ⊕ 2L^{0*}(-2h) → I_{C/Y} ⊕ L^{0*} → 0. An easy computation gives |
I_{C/Y} ⊕ L^{0*}| = P^{32}. Hence Γ is an irreducible variety of dimension 51. As p_1 : Γ → U is dominant and dim U = 45, all the irreducible components of 
p^{1-1}(t) = C^{0*}(X_t), t ∈ U, are generically of dimension 6. There is a distinguished one, 
C^{0*}(X_t) ∩ C^{0*}(X_t), so by the irreducibility of Γ and U, C^{0*}(X_t) = C^{0*}(X_t).

1.4. Elliptic quintics. The following theorem is the main result of Section I.

Theorem 1.4. Let X be general. Then C^{1*}(X) is irreducible and of dimension 10.

Proof. The proof is similar to that of Lemma [L3].

Pick a general cubic C_1 ∈ C^{0*} and a smooth elliptic conic C_2 ∈ C^{1*}(X) meeting each other 
transversely at a unique point x and such that N_{C_1/X} ≃ O_{P^3}(k) ⊕ O_{P^3}(4 - k), 0 ≤ 
k ≤ 2, N_{C_2/X} ≃ 2O_{C_2}(x + y), \{x, y\} = π^{-1}(π(x)). Let C = C_1 ∪ C_2. Then we have the 
exact sequences 0 → N_{C_1/X} → N_{C_2/X}|_{C_1} → C_x → 0, i = 1, 2. Repeating the arguments from 
the proof of Lemma [L3] we see that there exists a unique irreducible component 
C^{1*}_X(X) of Hilb(X) which contains the generic reducible curve C = C_1 ∪ C_2 as above. From 
the values of h^1(N_{C/X}), we see that dim C^{1*}_X(X) = 10 and dim C^{1*}_X(X) ≥ 10 for any other 
irreducible component C^{1*}_X(X) of C^{1*}_X(X).

For t ∈ U any quintic C ∈ C^{1*}(X_t) is projected by ρ to an elliptic quintic C_0 ⊂ P^3. It is 
well known that C^{1*}(P^3) is irreducible of dimension 20 [H1]. Arguing as in the proof of 
Lemma [L3] we obtain that there exists a rank-11 vector bundle E over C^{1*}(P^3) with fiber 
E_{(C_0)} = H^0((O_{P^3} ⊕ O_{P^3}(2))|_{C_0}) over C_0 ∈ C^{1*}(P^3), and C^{1*}(Y) is a dense open 
subset of Proj(E) such that the natural projection C^{1*}(Y) → C^{1*}(P^3), C ↦ ρ(C), coincides with the 
composition C^{1*}(Y) ↦ Proj(E) ↾ P^3 → C^{1*}(P^3), where pr is the structure morphism. Hence 
C^{1*}(Y) is irreducible and dim C^{1*}(Y) = 30.

Consider the incidence variety Γ = {(t, C) ∈ U × C^{1*}(Y) | C ⊂ X_t}, U ↾ P^3 ↦ C^{1*}(Y).

We have p^{1-1}(C) = U ∩ |I_{C/Y} ⊕ L^{0*}| ↾ P^3 → |I_{C/Y} ⊕ L^{0*}|. Now take any quintic 
C ∈ C^{1*}(Y). Let ρ_C : C → C_0 be the natural isomorphic projection of C onto a 
quintic C_0 ∈ C^{1*}(P^3). We have pr_*(L^{0*} ⊕ O_C) ≃ O_{P^3}(4)|_{C_0}. Consider the standard 
exact triple 0 → I_{C/Y} ⊕ L^{0*} → L^{0*} → L^{0*} ⊗ O_C → 0. Applying ρ_*, we obtain the
Lemma 1.5. Let $X$ be general so that $C^4_5(X)$ is irreducible and let $\overline{C^4_5(X)}$ be the closure of $C^4_5(X)$ in $\text{Hilb}(X)$. Then there exists a strongly smoothable reduced nodal curve $C_0 \in \overline{C^4_5(X)}$ of the form $C_0 = l_1 \cup ... \cup l_5$, $l_i \in \mathcal{F}$, $i = 1, ..., 5$, such that its only singularities are the 5 distinct points $a_1 = l_1 \cap l_2$, $a_2 = l_2 \cap l_3$, $a_3 = l_3 \cap l_4$, $a_4 = l_4 \cap l_5$, $a_5 = l_1 \cap l_5$, and $N_{i/X} = 2O_{l_i}$, $i = 1, ..., 5$, $N_{C_0/X}|l_i \simeq O_{l_i}(1) \oplus O_{l_i}(2)$, $N_{C_0/X}|l_i \simeq 2O_{l_i}(1)$, $2 \leq i \leq 4$, $N_{C_0/X}|l_5 \simeq \mathcal{O}_{l_5} \oplus \mathcal{O}_{l_5}(1)$.

In Section 2 we will use the following result which can be easily proved by the techniques of [HHi].

**Lemma 1.5.** Let $X$ be general so that $C^4_5(X)$ is irreducible and let $\overline{C^4_5(X)}$ be the closure of $C^4_5(X)$ in $\text{Hilb}(X)$. Then there exists a strongly smoothable reduced nodal curve $C_0 \in \overline{C^4_5(X)}$ of the form $C_0 = l_1 \cup ... \cup l_5$, $l_i \in \mathcal{F}$, $i = 1, ..., 5$, such that its only singularities are the 5 distinct points $a_1 = l_1 \cap l_2$, $a_2 = l_2 \cap l_3$, $a_3 = l_3 \cap l_4$, $a_4 = l_4 \cap l_5$, $a_5 = l_1 \cap l_5$, and $N_{i/X} = 2O_{l_i}$, $i = 1, ..., 5$, $N_{C_0/X}|l_i \simeq O_{l_i}(1) \oplus O_{l_i}(2)$, $N_{C_0/X}|l_i \simeq 2O_{l_i}(1)$, $2 \leq i \leq 4$, $N_{C_0/X}|l_5 \simeq \mathcal{O}_{l_5} \oplus \mathcal{O}_{l_5}(1)$.

2. THE ABEL-JACOBI MAP OF THE FAMILY OF ELLIPTIC QUINTICS

In this section we will study the Abel-Jacobi maps of three families of curves on a general double solid $X$: genus-4 septic $C^4_5(X)$, elliptic quintics $C^5_5(X)$ and a certain family $T$ of reducible curves of degree 17 and arithmetic genus 29 which will set a connection between the two other families.

2.1. Notation. We will use the following symbols, in addition to those introduced in [HHi]

- $C^4_5(\mathbb{P}^3) \subset \text{Hilb}(\mathbb{P}^3)$, the base of the family of smooth irreducible septic curves of genus 4 in $\mathbb{P}^3$.
- $C^4_5(X)$ (resp. $C^4_5(Y)$), the base of the family of septic curves of genus 4 in $X$ (resp. $Y$) which are mapped by $\rho$ isomorphically onto a curve from $C^4_5(\mathbb{P}^3)$.
- $\overline{C^4_5(X)}$, the closure of $C^4_5(X)$ in $\text{Hilb}(X)$.
- $\overline{C^4_5(X)}^*: = \{ C \in \overline{C^4_5(X)} | C \text{ is a reduced curve with at worst ordinary double points} \}$, a dense open subset of $\overline{C^4_5(X)}$ containing $C^4_5(X)$.
- $C^2_5(\mathbb{P}^3) := \{ C \in \text{Hilb}(\mathbb{P}^3) | C \text{ is a reduced quintic curve of arithmetic genus 2 and bidegree (3,2) on a smooth quadric surface } Q \subset \mathbb{P}^3 \}$.
- $\pi^* C^2_5(\mathbb{P}^3) := \{ C \in \text{Hilb}(X) | C = \pi^{-1}(C), C \in C^2_5(\mathbb{P}^3) \}$.
- $C_{3,3}(\mathbb{P}^3) := \{ C \in \text{Hilb}(\mathbb{P}^3) | C \text{ is a curve of bidegree (3,3) on a smooth quadric surface } Q \subset \mathbb{P}^3 \text{ that is, } C \text{ has arithmetic genus 4 and is a complete intersection of } Q \text{ and a cubic surface} \}$.
- $\pi^* C_{3,3}(\mathbb{P}^3) := \{ C \in \text{Hilb}(X) | C = \pi^{-1}(C), C \in C_{3,3}(\mathbb{P}^3) \}$.
- $\overline{C^4_5(X)}_5$, the closure of $C^4_5(X)$ in $\text{Hilb}(X)$.
- $\overline{C^4_5(X)}_5^* = \{ C \in \overline{C^4_5(X)} | C \text{ is a reduced curve with at most ordinary double points} \}$, an open subset of $\overline{C^4_5(X)}_5$.
- $C_{5+2} := \{ C \in \text{Hilb}(X) | C \text{ is a reduced curve of the form } C = C' \cup C'' \text{, where } C' \in \overline{C^4_5(X)}_5 \text{ and } C'' \in \overline{C^4_5(X)}_5 \text{ and } C' \text{ meets } C'' \text{ quasi-transversely in } 3 \text{ points} \}$.
- $\Phi_B: B \to J(X)$, the Abel-Jacobi map of a given subvariety $B$ of the Hilbert scheme of curves on $X$ (defined uniquely up to a choice of a reference point in $B$).
2.2. Septics of genus 4. We are interested in the family of septics $C_7^4(X)$ because of the following result of Welters [We 6.18]:

**Proposition 2.1.** For $X$ general, the Abel-Jacobi map $\Phi_{C_7^4(X)} : C_7^4(X) \to J(X)$ is dominant.

We begin with the following lemma.

**Lemma 2.2.** Let $X$ be a general quartic double solid. Then:

(i) $\mathcal{C}_{5+2}$ is irreducible of dimension 21 and the natural projection $p'' : \mathcal{C}_{5+2} \to C_5^4(X)$, $C' \cup C'' \mapsto C'$, is surjective.

(ii) Let $\mathcal{C}_{5+2}$ be the closure of $\mathcal{C}_{5+2}$ in $C_{5+2}$. There exists a strongly smoothable curve $C_0' \cup C_0'' \in C_{5+2} \setminus \mathcal{C}_{5+2}$ such that $C_0 := l_1 \cup \ldots \cup l_5 \in C_5^4(X) \setminus C_7^4(X)$ is the curve from Lemma [1.5] and $C_0' \subset C_7^4(X)$ is a smooth elliptic conic such that $Z_6 := C_0' \cap C_0'' = \{b_1, b_2, b_3\}$, $b_1 \in l_2$, $b_2 \in l_4$, $b_3 \in l_5$, $Z_6 \cap Z_a = \emptyset$, where $Z_a = \text{Sing} C_0'$.

(iii) The set $C_{5+2} := \{C \in \mathcal{C}_{5+2} \mid C$ is strongly smoothable$\} \subset C_{5+2} \cap C_7^4(X)$ is an irreducible dense open subset of $C_{5+2}$ and the projection $p'' : C^+_{5+2} \to C_5^4(X)$, $C' \cup C'' \mapsto C'$, is dominant.

(iv) Let $C_7^4(X)$ be an irreducible component of $C_7^4(X)$ containing $C^*_7$. Then $\dim C_7^4(X) = 14$ and $C_7^4(X)/\langle \pi \rangle = \{C \in C_7^4(X) \mid C$ is smooth and $\pi|C : C \to C\}$ is an isomorphism is a nonempty open subset of $C_7^4(X)$.

**Proof.** (i) $B = \pi(C')$ is an elliptic quintic in $\mathbb{P}^3$. For any $x \in B$, there are at most two trisecant lines of $B$ passing through $x$, and the family $Z$ of trisecant lines of $B$ is an irreducible curve. To prove this, consider the projection $\pi_x$ with center $x$ from $B$ to $\mathbb{P}^2$. The image of $B$ is a plane quartic of geometric genus 1, so we have the following two possibilities for its singular locus: 1) two singular points, each one of which is either a node, or a cusp, 2) one tacnode. The inverse image in $B$ of a singular point of $\pi_x(B)$ is a pair of points, say $z_1, z_2$, and the line $\overline{z_1z_2}$ is a trisecant of $B$, which meets $B$ in three distinct points $x, z_1, z_2$. Hence the family $Z$ of trisecant lines of $B$ is a curve. To prove its irreducibility, consider the incidence curve $D = \{(z, l) \in B \times Z \mid z \in l\}$ in $B \times Z$ with its natural projections $B \xrightarrow{\pi} D \xrightarrow{\pi} Z$. The above argument shows that $p$ is of degree 2. To see that both $D$ and $Z$ are irreducible, it suffices to verify that $p$ has at least one simple ramification point. The latter is easily done in the manner of [SR], Chapter XII, by interpreting such ramification points as fixed points of an appropriate correspondence $G \subset B \times B$ and applying to $G$ the Chasles-Cayley-Brill formula ([SR], Chapter XII, Corollary 3.52); we leave the details to the reader. Now $p''(C')$ is identified with a nonempty open subset of $Z$, so it is 1-dimensional and irreducible, which implies (i).

(ii) Remark that, in Lemma [1.5], one can choose the curve $C_0' = l_1 \cup \ldots \cup l_5$ in such a way that the lines $m_i = \pi(l_i)$, $i = 2, 4, 5$ in $\mathbb{P}^3$ are disjoint. Then it is obvious that a small (in the classical topology over $\mathbb{C}$) smooth deformation $C'_0$ of $C_0'$ can be lifted to a small deformation $C'_1 \cup C''_1 \in \mathcal{C}_{5+2}$ of $C'_0 \cup C''_0$. By the strong smoothability of $C_0'$ (Lemma [1.5]), $C := C'_0 \cup C''_0 \in \mathcal{C}_{5+2}$. To see that $C$ is strongly smoothable, we consider the natural exact triples: (a) $0 \to N_a \to N_{C/X} \to \mathcal{O}_{Z_a} \to 0$, $0 \to N'_a \to N_a \to N_{C_1/X} \to 0$, and (b) $0 \to N_b \to N_{C_0/X} \to \mathcal{O}_{Z_b} \to 0$, $0 \to N'_b \to N_b \to N_{C''_0/X} \to 0$, where $C_1 := l_1 \cup l_2 \cup l_4 \cup C''_0$, $N'_a := (N_{C/X}|l_3 \cup l_5)(-Z_a)$, $N'_b := (N_{C_1/X}|C''_0)(-Z_b)$. Then the strong smoothability of $C$ will follow from the equalities $h^1(N_a) = 0$, $h^1(N_b) = 0$. To prove the first of them, one deduces from Lemma [1.5] that $N'_a = 2\mathcal{O}_{l_3}(-1) \oplus 2\mathcal{O}_{l_5}(-1)$. Then the exact triples $0 \to 2\mathcal{O}_{l_3}(-1) \oplus 2\mathcal{O}_{l_5}(-1) \to N_{C_1/X} \to N_{C_1/X}|C''_0 \oplus 2\mathcal{O}_{l_1}(1) \to 0$, $0 \to N_{C''_0/X} \to N_{C_1/X}|C''_0 \to C_{b_1} \oplus C_{b_2} \mapsto 0$ together with the second triple in (b) give $h^1(N_a) = 0$.
To prove that \( h^1(N_b) = 0 \), we include \( N'_b \) into the series of exact triples following from the construction of \( C \): 
\[
0 \rightarrow N_1 \rightarrow N'_b \rightarrow \mathcal{O}_{l_5} \oplus \mathcal{O}_{l_5}(-1) \rightarrow 0, \quad 0 \rightarrow 2\mathcal{O}_{l_5}(-1) \rightarrow N_1 \rightarrow N_2 \rightarrow 0, \quad 0 \rightarrow N_3 \rightarrow N_2 \rightarrow \mathcal{O}_{l_2} \oplus \mathcal{O}_{l_2}(1) \rightarrow 0, \quad 0 \rightarrow \mathcal{O}_{l_4} \oplus \mathcal{O}_{l_4}(-1) \rightarrow N_3 \rightarrow \mathcal{O}_{l_4} \oplus \mathcal{O}_{l_4}(-1) \rightarrow 0.
\]
As \( C'' \equiv \pi^{-1}(m) \), where \( m \) denotes a line in \( \mathbb{P}^3 \), one immediately verifies that \( h^1(N_{C''/X}) = 0 \). This together with the second triple in (b) gives \( h^1(N_b) = 0 \).

(iii) follows from (i) and (ii) since the strong smoothability is an open condition in \( \tilde{C}_{5+2} \).

(iv) Let \( C = C'_0 \cup C'' \in \tilde{C}_{5+2} \cup \tilde{C}_{5+2} \) from (ii) above. From the above exact sequences, one immediately obtains \( h^1(N_{C/X}) = 0 \), \( h^0(N_{C/X}) = 14 \), hence \( \dim C''(X) = 14 \). Next, consider \( C \) as the union \( C = C_3 \cup C_4 \), where \( C_3 := l_5 \cup C''_0 \) and \( C_4 := l_1 \cup l_2 \cup l_3 \cup l_4 \).

One easily verifies that \( C_3 \) is strongly smoothable and \( \dim_{\{C_3\}} \text{Hilb}(X) = 6 \). Thus there exists a unique 6-dimensional irreducible component \( C''(X) \) of \( \text{Hilb}(X) \) containing \( C_3 \).


Lemma 2.3. For \( X \) general, \( C''(X) \) is irreducible of dimension 14.

Proof. This is proved in the same way as Lemma 1.3 and Theorem 1.4. First, remark that \( C''(\mathbb{P}^3) \) is an irreducible variety of dimension 28 (see, e.g., [H, Ch.4]), and for any \( C_0 \in C''(\mathbb{P}^3) \) we have \( (\mathcal{O}_{p_3} \oplus \mathcal{O}_{p_3}(2))(C_0) \simeq \mathcal{O}_{C_0} \oplus \mathcal{O}_{C_0}(D) \), \( \deg D = 14 \), so that \( h^0((\mathcal{O}_{P_3} \oplus \mathcal{O}_{P_3}(2))(C_0) = 12 \). This implies that there exists a rank-12 vector bundle \( E \) over \( C''(\mathbb{P}^3) \) with fiber \( E_{|C_0} = H^0((\mathcal{O}_{p_3} \oplus \mathcal{O}_{p_3}(2))(C_0) \) over \( C_0 \in C''(\mathbb{P}^3) \). Thus \( C''(Y) \) is a dense open subset of \( \text{Proj}(E) \) such that the natural projection \( C''(Y) \rightarrow C''(\mathbb{P}^3) \), \( C \mapsto \rho(C) \), coincides with the composition \( C''(Y) \hookrightarrow \text{Proj}(E) \xrightarrow{p_r} C''(\mathbb{P}^3) \), where \( pr \) is the structure morphism. Hence \( C''(Y) \) is irreducible of dimension 39.

Now take any \( C \in C''(Y) \) and compute \( \dim |\mathcal{I}_{C/Y} \otimes \mathcal{O}_{\mathbb{P}^2}|. \) As in the proof of Theorem 1.4, it suffices to show that \( h^1(\mathcal{I}_{C/Y} \otimes \mathcal{O}_{\mathbb{P}^2}) = 0 \). To prove this, take two general cubic surfaces \( S_1, S_2 \) through \( C_0 \) (such cubics exist since, by Riemann-Roch, \( h^0(\mathcal{O}_{C_0}(3)) = 18 \).

There are two possibilities for \( S_1 \) and \( S_2 \): 1) both \( S_1 \) and \( S_2 \) are irreducible, and 2) both \( S_1 \) and \( S_2 \) are reducible. We will consider the case 1), the case 2) being treated similarly. In this case \( S_1 \cap S_2 = C_0 \cap C', \) where \( C' \) is a curve of degree 2. We have the following two subcases: 1.a) \( C' \) a plane conic, or 1.b) \( C' \) a pair of two skew lines, possibly coincident, that is a double line on a smooth quadric. Consider the scheme \( Z = C_0 \cap C' \). One easily verifies that \( \chi(\mathcal{O}_Z) = 7 \) in case 1.a) and \( \chi(\mathcal{O}_Z) = 8 \) in case 1.b), and that \( \chi(\mathcal{O}_{Z'}) = 4 \) for any line \( l \subset C' \). Thus, in both cases one obtains \( h^1(\mathcal{I}_{Z,C'}(4)) = 0 \). Then the exact triples \( 0 \rightarrow \mathcal{I}_{C_0 \cup C'/\mathbb{P}^3}(4) \rightarrow \mathcal{I}_{C_0 / \mathbb{P}^3}(4) \rightarrow \mathcal{I}_{Z,C'/\mathbb{P}^3}(4) \rightarrow 0 \) and \( 0 \rightarrow \mathcal{O}_{p_3}(-2) \rightarrow \mathcal{O}_{p_3}(1) \rightarrow \mathcal{I}_{C_0 \cup C'/\mathbb{P}^3}(4) \rightarrow 0 \) imply that \( h^1(\mathcal{I}_{C_0 / \mathbb{P}^3}(4)) = 0 \).

Hence \( h^0(\mathcal{I}_{C/Y} \otimes \mathcal{O}_{\mathbb{P}^2}) = \chi(\mathcal{I}_{C/Y} \otimes \mathcal{O}_{\mathbb{P}^2}) = 21 \) and \( \dim |\mathcal{I}_{C/Y} \otimes \mathcal{O}_{\mathbb{P}^2}| = \mathbb{P}^2 \) for \( C \in C''(Y) \). Thus, considering the incidence variety \( \Gamma = \{(t, C) \in U \times C''(Y) \mid C \subset X_t\} \) with its natural projections \( U \xrightarrow{\mathbb{P}^2} C''(Y) \), we see that \( p_{21}^{-1}(C) \), is open in \( |\mathcal{I}_{C/Y} \otimes \mathcal{O}_{\mathbb{P}^2}| \), so that \( \Gamma \) is irreducible of dimension 59. The remaining part of the proof goes along the same lines as the end of the proofs of Lemma 1.3 and Theorem 1.4. Namely, by Lemma 2.2 (iv), for a general double solid \( X \in U \), there is a distinguished component \( C''(X) \) of \( p_1^{-1}(X) = C''(X) \) which plays the role of \( C_3^{10}(X), C_5^{18}(X) \) in the proofs of the above mentioned lemma and
Curves of degree 17 and arithmetic genus 29. In this subsection we construct a family $\Gamma_0$ of reducible curves $C = C_1 \cup C_2$ of degree 17 and arithmetic genus 29 on $X$, where $C_1$ is a general septic from $C^*_1(X)$ and $C_2 \in \pi^*C^*_2(\mathbb{P}^3)$ intersects $C_1$ in 13 points, and a subfamily $\Sigma_0 \subset \tilde{\Gamma}_0$ (see (13)) of curves of the form $C = C' \cup C'' \cup C_2$, where $C'$ is a general quintic from $C^*_1(X)$ and $C'' = \pi^{-1}(m)$, where $m$ is a line in $\mathbb{P}^3$ trisecant to $\pi(C')$. These two families will be used later, see (21) and Proposition (25).

Let

$$M = \left\{ (Q, l, Z, Z_1) \mid Q \text{ is a smooth quadric in } \mathbb{P}^3, l \text{ a line on } Q, Z = \{y_1, \ldots, y_{10}\} \text{ a 7-uple of distinct points on } Q \smallsetminus l, \right. \left. Z_1 = \{y_1, y_2, y_3\} \text{ a triple of distinct points on } l \right\}$$

and $M^* = \{(Q, l, Z, Z_1) \in M \mid \dim \{J_{Z \cup Z_1}(Q, 3, 2)\} = 1\}$, where we assume that $l$ is of bidegree $(0, 1)$ on $Q$. Clearly, $M$ is irreducible and $M^*$ is nonempty, hence dense in $M$.

Next, let

$$\Pi = \{(C, (Q, l, Z, Z_1)) \in C^*_1(X) \times M \mid C \cap Q = Z \cup Z_1, \text{ where } C = \pi(C)\}.$$ 

It has two natural projections: $\Pi \xrightarrow{p_\Pi} C^*_1(X) \xrightarrow{\pi} M$. It is easy to see that $\Pi$ is nonempty. Indeed, for any $C \in C^*_1(X)$, consider the smooth quintic $C' = \pi(C)$, take any trisecant line $l$ of $C$ and any smooth quadric $Q$ through $l$ and let $Z = C \cap (Q \smallsetminus l)$, $Z_1 = C \cap l$; then, by construction, $(C, (Q, l, Z, Z_1)) \in \Pi$. We can factor $p_\Pi$ as

\begin{align*}
(1) \quad p_\Pi : \Pi \xrightarrow{p'} C^*_1(X) \\
(2) \quad p' : (C', (Q, l, Z, Z_1)) \mapsto C' \cup \pi^{-1}(l), \quad p'' : C' \cup C'' \mapsto C',
\end{align*}

and our construction shows that $p'$ is surjective with an irreducible and rational fiber of dimension 7. Besides, for $C' \in C^*_1(X)$, $p''^{-1}(C')$ is isomorphic to a dense open subset of the 1-dimensional family of trisecant lines of the smooth quintic $\pi(C')$ in $\mathbb{P}^3$. This description of the fibres of $p'$ and $p''$ together with (1) implies, in particular, that $\Pi$ is irreducible for general $X$.

Now show that

$$\Pi^* := p'^{-1}(C^*_1(X)) \cap q_\Pi^{-1}(M^*) \neq \emptyset.$$ 

Choose any $C \in C^*_1(X)$ and a trisecant line $l$ of $C$ such that $C \cup \pi^{-1}(l) \in C^*_5$, where $C = \pi(C)$. Then $Z_l = C \cap l = \{x_1^0, x_2^0, x_3^0\}$ is a triple of distinct points. Through each $x_i^0$ there is, in addition to $l$, one more trisecant $l'_i$ of $C$, and we denote $\{x_{2i+2}^0, x_{2i+3}^0\} = l'_i \cap C \setminus \{x_i^0\}$, $i = 1, 2, 3$. Since $C$ has degree 5, it follows immediately that $l'_1, l'_2, l'_3$ are disjoint, hence there exists a unique smooth quadric $Q_0$ containing $l'_1, l'_2$ and $l'_3$. Let $x_{10}^0$ be the residual point of intersection of $C$ and $Q_0$: $C \cap Q_0 = \{x_1^0, \ldots, x_{10}^0\}$. One can easily verify that for sufficiently general $l$, the 10 points $x_1^0, \ldots, x_{10}^0$ are distinct. Denote $Z^0 = \{x_1^0, \ldots, x_{10}^0\}$ and let $l_i$ be the line on $Q^0$ through $x_{10}^0$ in the same ruling as $l$, say $|\mathcal{O}_{Q_0}(0, 1)|$. Then $|J_{Z \cup Z_1}(Q^0, 3, 2)| = \bigcup_{i=1}^4 l_i^0 + |\mathcal{O}_{Q_0}(0, 1)| \approx \mathbb{P}^4$. Now consider the triple $x_0 = (x_1^0, x_2^0, x_3^0)$ as a point of $\text{Sym}^3(l) \simeq \mathbb{P}^3$ and take a sufficiently small neighborhood $U$ of $x_0$ in the classical topology of $\text{Sym}^3(l)$ so that for any $x = (x_1, x_2, x_3) \in U$ the following properties are satisfied: (i) for each $i = 1, 2, 3$, there is a unique trisecant line $l_i$ of $C$ passing through $x_i$ and close to $l_i^0$ in the sense that $\{x_{2i+2}, x_{2i+3}\} = l_i \cap C \setminus \{x_i^0\}$ is close to the pair $\{x_{2i+2}^0, x_{2i+3}^0\}$ in $\text{Sym}^3(C)$; (ii) there exists a unique quadric $Q$ containing $l_1, l_2, l_3$ and close to $Q_0$ in the space of quadrics; for this quadric, the residual point $x_{10} = C \cap Q \smallsetminus \{x_1, \ldots, x_9\}$ is close to $x_{10}^0$. 

Denote \( Z_l = \{x_1, x_2, x_3\} \), \( Z = \{x_4, ..., x_{10}\} \) and let \( l_q \) be the line on \( Q \) through \( x_{10} \) belonging to the same ruling as \( l \). These data define a point \( m = (Q, l, Z, Z_l) \in M \). Now we will determine the exact sense of the condition that \( U \) is sufficiently small:

\[ |\mathcal{J}_{Z \cup Z_l/Q}(3, 2)| \simeq \mathbb{P}^1. \]

Remark that, by construction, \( \bigcup_{i=1}^4 l_i + |\mathcal{O}_Q(0, 1)| \subset |\mathcal{J}_{Z \cup Z_l/Q}(3, 2)| \), so that \( |\mathcal{J}_{Z \cup Z_l/Q}(3, 2)| = \bigcup_{i=1}^4 l_i + |\mathcal{O}_Q(0, 1)| \). The condition (3) is open and nonempty. So, for any \( x_0 \in \text{Sym}^3(l) \), we obtain a point \( m = m(x_0) = (Q, l, Z, Z_l) \in M^* \). Then \((C, m) \in \Pi^*\), and (2) follows. As \( \Pi \) is irreducible and \( M^* \) is open in \( M \), this implies that \( \Pi^* \) is dense and open in \( \Pi \) and, consequently,

\[ p': \Pi^* \to C^*_{3+2} \text{ is dominant.} \]

Hence (1) and Lemma 2.2 (i) show that

\[ p_{\Pi}: \Pi^* \to C^*_1(X), (C', (Q, l, Z, Z_l)) \mapsto C', \text{ is dominant.} \]

Remark that, for any \( C_2 \in \pi^*|\mathcal{J}_{Z \cup Z_l/Q}(3, 2)| \), we have:

\[ \text{Define, for any given } z = (C', (Q, l, Z, Z_l)) \in \Pi^*, \]

\[ C_1 = p'(z) = C' \cup C'', \quad C'' = \pi^{-1}(l), \]

\[ f_z = \{ C_2 \in \pi^*C^2_5(\mathbb{P}^3) \mid C_2 = \pi^{-1}(C_2), C_2 \in |\mathcal{J}_{Z \cup Z_l/Q}(3, 2)|\}, \quad f_z \simeq \mathbb{P}^1, \]

\[ f_z^* = \{ C_2 \in f_z \mid C_2 \text{ is a reduced curve}\}. \]

Clearly, \( f_z^* \) is a nonempty open subset of \( f_z \). By construction, for any \( C_2 \in f_z^* \), the curves \( C_1 \) and \( C_2 \) are reduced and meet each other quasi-transversely in 13 points:

\[ C_1 \cap C_2 = \{y_1, y_2, y_3, y_1', y_2', y_3', y_4, ..., y_{10}\}, \quad \tilde{Z}_l = \pi^{-1}(Z_l) = \{y_4, ..., y_{10}\} \]

\[ = \{y_1, y_2, y_3, y_1', y_2', y_3'\} = C'' \cap C_2, \quad \tilde{Z} = (\pi|C|)^{-1}(Z) = \{y_4, ..., y_{10}\} = C' \cap C_2. \]

Hence

\[ C_0 = C' \cup C'' \cup C_2 \]

is a curve on \( X \) of degree 17 and arithmetic genus 29. Remark also that \( \pi^*C^2_5(\mathbb{P}^3) \simeq C^2_5(\mathbb{P}^3) \) is a rational irreducible variety of dimension 20.

Now consider the incidence variety

\[ \Gamma = \left\{ (C_1, C_2) \in \overline{C^2_1(X)} \times \pi^*C^2_5(\mathbb{P}^3) \mid C_1 \text{ and } C_2 \text{ are reduced and meet each other quasi-transversely in 13 points} \right\} \]

with projections \( \overline{C^2_1(X)} \xrightarrow{p_{\Pi}} \Gamma \xrightarrow{p_{\Sigma^*}} \pi^*C^2_5(\mathbb{P}^3) \) and define its locally closed subset \( \Sigma^* = p_{\Pi}^{-1}(p'(\Pi^*)) \). From (1), (4) and (5) it follows immediately that the natural projection

\[ p_{\Sigma^*} := p'' \circ p_1: \Sigma^* \to C^*_1(X), (C' \cup C'', C_2) \mapsto C', \text{ is dominant.} \]

Moreover, for \( z \in \Pi^* \), by (7) and the description of \( p' \) in (1), we have for \( C_1 = p'(z) \):

\[ p_{\Pi}^{-1}(C_1) \text{ is a rational } 7\text{-dimensional variety.} \]

Hence, for general \( X \), \( \Sigma^* \) is irreducible of dimension 17, and there is an isomorphism

\[ \psi: \Sigma^* \xrightarrow{\sim} \Sigma', (C' \cup C'', C_2) \mapsto C' \cup C'' \cup C_2, \]

where \( \Sigma' = \{C_0 \in \text{Hilb}(X) \mid C_0 \text{ is defined by (8)-9 for some } z \in \Pi^* \} \). In view of (12), we rewrite (10) as follows:

\[ p_{\Sigma'}: \Sigma' \to C^*_1(X), C_0 = (C' \cup C'' \cup C_2) \mapsto C', \text{ is dominant.} \]

Also, by (6), we have \( \text{Im}(q_{\Sigma'}) \subset \pi^*C_{3,3}(\mathbb{P}^3) \), where \( q_{\Sigma'}: C_0 = (C' \cup C'' \cup C_2) \mapsto C'' \cup C_2 \).
Next, looking at the local equations of \( \Gamma \) in \( W = \overline{\mathcal{C}_4(X)} \times \pi^* \mathcal{C}_2^g(\mathbb{P}^3) \) at a point \( w = (C_1, C_2) \in \Sigma^* \), we see that the transversal intersection of \( C_1 \) and \( C_2 \) in each one of the 13 points \( \mathbf{8} \) gives one local equation for \( \Gamma \) in \( Y \); hence, since \( \dim \pi^* \mathcal{C}_2^g(\mathbb{P}^3) = 20 \), \( \dim p_1^{-1}p_1(w) \geq 20 - 13 = 7 \). Since by \( |\mathbf{11}| \) this dimension is precisely 7, it follows that there exists an irreducible component of \( \Gamma \), say \( \Gamma_0 \), which contains \( w \) and dominates \( \overline{\mathcal{C}_4(X)} \) via \( p_1 \):

\[
\overline{p_1(\Gamma_0)} = \overline{\mathcal{C}_4(X)},
\]

where the closure is taken in \( \overline{\mathcal{C}_4(X)} \) and we use the irreducibility of \( \overline{\mathcal{C}_4(X)} \) by Lemma \( \mathbf{2.3} \). Hence, taking into account the irreducibility of \( \Sigma^* \), \( |\mathbf{11}| \) and \( |\mathbf{12}| \), we see that \( \Sigma_0 = \psi(\Gamma_0 \cap \Sigma^*) \) is a dense open subset of \( \Sigma \); also, by \( |\mathbf{10}| \), the natural projection

\[
p_{\Sigma_0} : \Sigma_0 \to \mathcal{C}_g^1(X), \quad C_0 = (C' \cup C'' \cup C_2) \mapsto C', \quad \text{is dominant}.
\]

Now consider in \( \text{Hilb}(X) \) the subset \( \tilde{\Gamma}_0 = \{ C \in \text{Hilb}(X) | C \) is a reducible curve of the form \( C = C_1 \cup C_2 \), where \( (C_1, C_2) \in \Gamma_0 \} \). There is an isomorphism \( \psi : \Gamma_0 \xrightarrow{\sim} \tilde{\Gamma}_0, \ (C_1, C_2) \mapsto C_1 \cup C_2 \), similar to \( |\mathbf{12}| \), and by construction,

\[
\Sigma_0 \subset \tilde{\Gamma}_0.
\]

2.4. Criterion of Welters.

Now we invoke the following important technical result of G.Welters [We, Prop. 6.17], giving a sufficient condition for a family of curves to parametrize, via the Abel-Jacobi map, a translate of the theta divisor of \( J(X) \).

**Proposition 2.4.** Let \( T \) be a smooth irreducible (not necessarily complete) variety parametrizing a family \( \{ C_t | t \in T \} \) of reduced curves of degree \( d = k(k + 1) - 3 \) on \( X \) with the following properties: (i) The Abel-Jacobi map \( \Phi_T : T \to J(X) \) is dominant; (ii) for all \( t \in T \), the linear system of surfaces \( |J_{C_t/X}(k)| \) has dimension \( k \) and the surfaces of this linear system do not contain all the lines of \( X \); (iii) the incidence divisor in \( T \times \mathcal{F} \) is reduced. Then for any line \( l \in \mathcal{F} \), the subvariety of \( T \) defined by

\[
T_l = \{ t \in T | C_t \cap l = \emptyset \text{ and } C_t \cup l \text{ lies on a surface from } |O_X(k)| \}
\]

is sent by the Abel-Jacobi map into a translate of the theta divisor \( \Theta \subset J(X) \).

Let us construct such a \( T \) from the curves introduced in the previous subsection. Consider the Abel-Jacobi map \( \Phi_{\tilde{\Gamma}_0} : \tilde{\Gamma}_0 \to J(X) \) and the induced map

\[
\Phi_0 : \Gamma_0 \xrightarrow{\psi} \tilde{\Gamma}_0 \xrightarrow{\Phi_{\tilde{\Gamma}_0}} J(X), \quad (C_1, C_2) \mapsto \Phi_{\tilde{\Gamma}_0}(C_1 \cup C_2),
\]

where

\[
\Phi_{\tilde{\Gamma}_0}(C_1 \cup C_2) = \Phi_{\overline{\mathcal{C}_4(X)}}(C_1) + \Phi_{\pi^* \mathcal{C}_2^g(\mathbb{P}^3)}(C_2), \quad C_1 \in \overline{\mathcal{C}_4(X)}, \quad C_2 \in \pi^* \mathcal{C}_2^g(\mathbb{P}^3).
\]

As \( \pi^* \mathcal{C}_2^g(\mathbb{P}^3) \cong \mathcal{C}_5^2(\mathbb{P}^3) \) is rational,

\[
\Phi_{\pi^* \mathcal{C}_2^g(\mathbb{P}^3)} = \text{const}, \quad \text{Im}(\Phi_{\pi^* \mathcal{C}_2^g(\mathbb{P}^3)}) = \{ c \}.
\]

Hence, taking the closure of the image of the Abel-Jacobi map in \( J(X) \) and using \( |\mathbf{13}| \), \( |\mathbf{16}| - |\mathbf{18}| \) and Proposition \( \mathbf{2.11} \), we obtain:

\[
\text{Im}(\Phi_{\tilde{\Gamma}_0}) = \text{Im}(\Phi_0) = c + \text{Im}(\Phi_{\overline{\mathcal{C}_4(X)}}(p_1(\Gamma_0))) = c + \text{Im}(\Phi_{\overline{\mathcal{C}_4(X)}}) = c + \text{Im}(\Phi_{\mathcal{C}_4^g(X)}) = c + J(X) = J(X),
\]
that is the Abel-Jacobi map $\Phi_{\tilde{\Gamma}_0} : \tilde{\Gamma}_0 \to J(X)$ is dominant. Now, consider the Abel-Jacobi map $\Phi_{\Sigma_0} : \Sigma_0 \to J(X)$. As $\pi^*C_{(3,3)}(\mathbb{P}^3)$ is rational, $\text{Im}(\Phi_{\pi^*C_{(3,3)}(\mathbb{P}^3)}) = \{c'\}$ is a point, and we obtain from (14):

$$\text{Im}(\Phi_{\Sigma_0}) = \text{Im}(\Phi_{C_{(3,3)}^{(1)}(X)}) + c'.$$

Now let $\sigma : T \to \tilde{\Gamma}_0$ be any desingularization of $\tilde{\Gamma}_0$. Using (15), define

$$\Sigma \subset \sigma^{-1}(\Sigma_0),$$

an irreducible component of $\sigma^{-1}(\Sigma_0)$ such that $\sigma : \Sigma \to \Sigma_0$.

Consider the Abel-Jacobi maps $\Phi_T = \Phi_{\tilde{\Gamma}_0} \circ \sigma : T \to J(X)$ and $\Phi_\Sigma = \Phi_{\Sigma_0} \circ \sigma : \Sigma \to J(X)$. Then (20) gives

$$\text{Im}(\Phi_\Sigma) = \text{Im}(\Phi_{C_{(3,3)}^{(1)}(X)}) + c'.$$

We obtain the following proposition:

**Proposition 2.5.** For $X$ general, there exists a smooth irreducible variety $T$ parametrizing some family of curves of degree 17 and arithmetic genus 29 such that: (i) the Abel-Jacobi map $\Phi_T : T \to J(X)$ is dominant; (ii) $T$ contains the subset $\Sigma$, defined by (21) and satisfying (22), such that the curves $C_t$ for $t \in \Sigma$ split into three components as in (3)-(2); (iii) the incidence divisor $D = \{(C,l) \in T \times F | C \cap l \neq \emptyset\}$, endowed with its natural scheme-theoretic structure, is reduced.

**Proof.** (i) and (ii) follow from (19) and (22). Let us prove (iii). There are 6 distinct lines, counted with multiplicity 1 in $F$, which pass through a general point of $X$. So, it is sufficient to show that there exists a curve $C$ from $T$ which passes through a general point of $X$: in this case, $D_C$ is a reduced 6-sheeted covering of $C$, and hence $D$ is reduced. But this is clear, since by the construction of $\Sigma$, the components of the reduced curves $C_0 \in \Sigma$ sweep all of $X$ as $C_0$ runs through $\Sigma$. \qed

We proceed now to the proof of the remaining hypotheses of the Welters criterion for our family $T$.

**Lemma 2.6.** Let $X$ be a general quartic double solid. Then the following assertions are verified: (i) For any $C \in \Sigma$, $h^0(J_{C/X}(4)) = 5$; (ii) for a general (hence any) line $l \in F$ such that $C \cap l = \emptyset$, there exists a surface $S \in |O_X(4)|$ containing $C \cup l$; (iii) for general $C \in \Sigma$, the surfaces of the linear system $|J_{C/X}(4)|$ do not contain all the lines of $X$.

**Proof.** (i) Let $C \in \Sigma$. By the definition of $\Sigma$ we have $C = C_1 \cup C_2$, where $C_1 \in C_{(3,3)}^{(1)}(X)$, $\pi : C_1 \to C_1 := \pi(C_1)$ is an isomorphism, $C_2 = \pi^{-1}(C_2)$, where $C_2 \in C_{(3,3)}^{(2)}(\mathbb{P}^3)$, and $Z := C_1 \cap C_2 = \{y_1, \ldots, y_{29}\}$ is a subset of the 13-uple of points (3). This means, in particular, that $C_1$ lies in a uniquely defined quadric, say $Q$. Let $Z_0 := \pi(Z)$, $\deg(Z_0) = \deg(Z) = 10$. Since by construction $C_1 \cap Q \supseteq Z_0$ (as schemes) and $\deg(C_1 \cap Q) = 10 = \deg(Z_0)$, it follows that, scheme-theoretically, $Z_0 = C_1 \cap Q$. Using the fact that $O_Q(C_2) = O_Q(3)$, we get the following exact triple:

$$0 \to J_{Q,C_1^{(1)}(\mathbb{P}^3)}(4) \to J_{C_1^{(1)}(\mathbb{P}^3)}(4) \to O_Q(1) \to 0.$$  

Let $S \subset \mathbb{P}^3$ be any smooth cubic surface passing through $C_1$. We have the exact triple $0 \to J_{Q,S(\mathbb{P}^3)} \to J_{Q,C_1^{(1)}(\mathbb{P}^3)} \to J_{S(\mathbb{P}^3),C_1^{(1)}(\mathbb{P}^3)} \to 0$. Here $J_{S(\mathbb{P}^3),C_1^{(1)}(\mathbb{P}^3)} \simeq O_S(-2)(-C_1)$, because $S$ is smooth, hence twisting by $O_{\mathbb{P}^3}(4)$, we obtain: $0 \to O_{\mathbb{P}^3}(-1) \to J_{Q,C_1^{(1)}(\mathbb{P}^3)} \to O_S(2)(-C_1) \to 0$. One easily verifies that the natural map $H^0(O_S(2)) \to H^0(O_{C_1^{(1)}(\mathbb{P}^3)})$ is surjective, so that $h^1(O_S(2)(-C_1)) = 0$. Besides, $h^1(O_{\mathbb{P}^3}(-1)) = 0$. Hence the last triple implies $h^1(J_{Q,C_1^{(1)}(\mathbb{P}^3)}(4)) = 0$. This together with (23) and the equality $h^1(O_Q(1)) = 0$.
gives $h^1(J_{C_1 \cup C_2}/P^3(4)) = 0$. Hence the exact triple $0 \to J_{Z_0/\Sigma}(4) \to J_{C_1/\Sigma}(4) \to 0$ yields the surjectivity
\begin{equation}
(24) \quad H^0(J_{C_2/\Sigma}(4)) \twoheadrightarrow H^0(J_{Z_0/\Sigma}(4)).
\end{equation}

Now, since $\pi : C_1 \to C_1$ is an isomorphism, we have $\pi_*J_{Z/C_1}(4) \simeq J_{Z_0/C_1}(4)$, which provides the exact triple $0 \to J_{Z_0/C_1}(4) \to \pi_*\mathcal{O}_C(4) \to \pi_*\mathcal{O}_{C_1}(4) \to 0$. Hence $h^0(\pi_*\mathcal{O}_C(4)) = \chi(\mathcal{O}_C(4)) = 40$. Next, consider the natural restriction map $g : \pi_*\mathcal{O}_X(4) \rightarrow \pi_*\mathcal{O}_{C_1}(4)$. Here $\ker(g) = \pi_*J_{C_2/X}(4)$, so passing to the map on global sections $f : H^0(\pi_*\mathcal{O}_X(4)) \rightarrow H^0(\pi_*\mathcal{O}_C(4))$, we obtain: $\text{coker}(f) = \text{coker}(H^0(\pi_*J_{C_2/X}(4)) \twoheadrightarrow H^0(J_{Z_0/\Sigma}(4)))$. By the projection formula for $\pi$, we have $\pi_*J_{C_2/X}(4) = \pi_*\pi^*J_{C_2/\Sigma}(4) = J_{C_3/p_3}(4) \oplus J_{C_4/p_3}(2)$, and the composition $H^0(J_{C_3/p_3}(4)) \hookrightarrow H^0(\pi_*J_{C_2/X}(4)) \twoheadrightarrow H^0(J_{Z_0/\Sigma}(4))$ coincides with the surjective map $(24)$, so $\text{coker}(f) = 0$. Hence the equalities $h^0(\pi_*\mathcal{O}_C(4)) = 40$ and $h^0(\mathcal{O}_X(4)) = h^0(\mathcal{O}_{p_3}(4)) + h^0(\mathcal{O}_{p_3}(2)) = 45$ imply that $H^0(J_{C_3/X}(4)) = H^0(\pi_*J_{C_2/X}(4)) = \ker(f) = C^2$.

(ii) Take a cubic surface $F$ containing $\mathcal{C}_1$ and write $\mathcal{C}_2 = F \cap Q$, where $Q$ is a general quadric. Then $C_1 \cup C_2 \subset \pi^{-1}(F)$. Take a general plane $\mathbb{P}^2$ in $\mathbb{P}^3$ and a line $l$ in the del Pezzo surface $\pi^{-1}(\mathbb{P}^2)$ which does not meet $C_1 \cup C_2$. Clearly such $l$ exists for a general choice of $\mathbb{P}^2$. Then $S = \pi^{-1}(F \cup \mathbb{P}^2) \in |\mathcal{O}_X(4)|$ is the desired surface.

(iii) This is a standard dimension count. One embeds $C$ into $Y$ and considers the linear system $L = |J_{C_1/Y} \otimes \mathcal{O}_Y|p_3(2)|$. From (i), we deduce that $\dim L = 5$. Take now a line $l \subset Y$, mapped by $\rho$ isomorphically onto a line in $\mathbb{P}^3$. If it is sufficiently general, then there exists a unique smooth $X \subset Y$ containing $l$; this $X$ is the wanted double solid, and the lemma is proved. \hfill $\square$

We are ready now to prove the main result of this section:

**Theorem 2.7.** $\text{Im}(\Phi_{C_1/(X)}) = \Theta + \text{const}$.

**Proof.** The family $T$ constructed in Proposition 2.5 satisfies the hypotheses of Proposition 2.4 with $k = 4$. According to Lemma 2.6, for general $l \in \mathcal{F}$, we have $T_l \supset \Sigma$, where we replace $\Sigma$ by a nonempty open subset, if necessary. Hence Proposition 2.4 and (22) imply:
\begin{equation}
(25) \quad \text{Im}(\Phi_{C_1/(X)}) \subset \Theta + \text{const}.
\end{equation}

To prove the opposite inclusion, let us recall a well known description of the image of the differential of the Abel-Jacobi map (see, e.g., [We, (2.11.0) and (6.19)]), in applying it specifically to the map $\Phi := \Phi_{C_1/(X)} : C_1^1(X) \rightarrow J(X)$. Take any unobstructed quintic $C \subset C_5^5(X)$, so that $T_C C_5^1(X) = H^0(N_{C_5/X}) = C_{10}$. Assume, moreover, that $C \not\subseteq W$ and denote $C_0 = \pi(C), \ Z_0 = \pi(C \cap W)$, where $W$ is the branch quartic of $X$. Consider the composition $\phi : C_{10} = T_C C_5^1(X) \xrightarrow{df} T_{\Phi_1}(J(X) \xrightarrow{\sim} T_0J(X) \simeq (H^0(\mathcal{O}_{p_3}(2)))^\vee$. Then the kernel of the dual map $\phi^\vee$ is described as follows: $\text{Ker}(\phi^\vee) = \text{Ker}(C_{10} = H^0(\mathcal{O}_{p_3}(2)) \xrightarrow{H^0(r_{C_0})} H^0(\mathcal{O}_{C_0}(2)) \xrightarrow{H^0(r_{Z_0})} H^0(\mathcal{O}_{Z_0}(2)))$, where $r_{C_0} : \mathcal{O}_{p_3}(2) \rightarrow \mathcal{O}_{C_0}(2), \ r_{Z_0} : \mathcal{O}_{C_0}(2) \rightarrow \mathcal{O}_{Z_0}(2)$ are the restriction maps, and $\text{Im} \phi$ is the orthogonal complement of $\text{Ker}(\phi^\vee)$. By definition, $C_0$ is an elliptic quintic in $\mathbb{P}^3$, hence it does not lie on a quadric, so $H^0(r_{C_0})$ is an isomorphism, and $\text{Ker}(H_0^0(r_{Z_0})) \simeq H^0(\mathcal{O}_{C_0}) = \mathbb{C}$, as follows from the exact triple $0 \rightarrow \mathcal{O}_{C_0} \xrightarrow{r_{Z_0}} \mathcal{O}_{C_0}(2) \xrightarrow{r_{Z_0}} \mathcal{O}_{Z_0}(2) \rightarrow 0$. Hence $\dim \text{Ker}(\phi^\vee) = 1$ and $\dim \text{Im} \Phi = 9$. This together with (25) and the irreducibility of $\Theta$ (see [We]) implies the wanted assertion. \hfill $\square$

3. Pentagons on a degenerate double solid

3.1. Degeneration into a pair of $P^3$'s.
In this section, we will use the technique of degeneration of the double solid into a reducible variety, developed by Clemens in [C-1], [C-2]. Let \( \pi : X \to \Delta \) be a family of double covers \( \pi_t : X_t \to \mathbb{P}^3 \) branched in the quartics \( W_t = \{ tF + G^2 = 0 \} \subset \mathbb{P}^3 \) \( (t \in \Delta) \), where \( \Delta \) is an open disc in \( \mathbb{C} \), the equation \( F = 0 \) defines a smooth quartic \( W \) and \( G = 0 \) a smooth quadric \( Q \) such that \( W \cap Q = B \) is a smooth octic curve. For \( t \in \Delta^* = \Delta \setminus \{ 0 \} \), the fiber \( X_t \) is a smooth double solid, and for \( t = 0 \) a union of two copies of \( \mathbb{P}^3 \), say \( \mathbb{P}^3, \mathbb{P}^3' \), meeting each other transversely along \( Q \).

One can associate to \( \pi \) the compactified family \( \mathcal{F} \to \Delta \) of Fano surfaces, whose fiber \( \mathcal{F}_t \) over \( t \neq 0 \) parametrizes lines in \( X_t \), and the Neron model \( J \to \Delta \) of the family of the intermediate Jacobians \( J_t \) of the threefolds \( X_t \). The compactifications of both families are described by Clemens. Namely, \( \mathcal{F}_0 \) is the union of two components \( \mathcal{F}' \cup \mathcal{F}'' \), where \( \mathcal{F}' \) (resp. \( \mathcal{F}'' \)) parametrizes the bisection lines to \( C \) in \( \mathbb{P}^3 \) (resp. \( \mathbb{P}^3' \)). As follows from a remark of Clemens on p. 211 in [C-1], the fiber \( \mathcal{F}_0 \) is acquired in the family \( \mathcal{F}/\Delta \) with multiplicity 1. The lines parametrized by \( \mathcal{F}_0 \), that is, the bisections to \( C \) in either one of the two copies of \( \mathbb{P}^3 \), will be called lines in \( X_0 \). The Neron model \( J \) is a family of abelian Lie groups such that the fiber \( J_0 \) over \( t = 0 \) is the union of two components \( J^+, J^- \), both isomorphic to a \( \mathbb{C}^* \)-extension of the Jacobian \( J(B) \) of \( B \). The Neron model has a natural compactification \( \tilde{J} \), obtained by pasting in two copies of \( J(B) \), so that the central fiber \( \tilde{J}_0 \) is the union of two components \( \tilde{J}^\pm \) which are both \( \mathbb{P}^1 \)-bundles over \( J(B) \) and which meet each other transversely along two disjoint sections of the \( \mathbb{P}^1 \)-bundles.

In [C-2], Clemens studies the family of sextics of genus 3 in the varieties \( X_t \). We will treat in the same way the one of elliptic quintics.

Let \( H \to \Delta \) be the subscheme of the relative Hilbert scheme \( \text{Hilb}(X/\Delta) \), such that the fiber \( H_t \) over \( t \neq 0 \) is the component of \( \text{Hilb}(X_t) \) whose generic point represents an elliptic quintic in \( X_t \), and the points of \( H_0 \), the fiber of \( H \) over \( t = 0 \), represent the curves which are limits of elliptic quintics in \( X_t \) as \( t \to 0 \), that is, \( H \) is the closure of \( \bigcup_{t \in \Delta^*} H_t \) in \( \text{Hilb}(X/\Delta) \). Let \( \tilde{\Delta} \to \Delta, s \mapsto t = s^e \), be some base change and \( C_s \in H_s \) a family of curves in \( X_s \) which are smooth elliptic quintics for \( s \neq 0 \). Their images \( C_s = \pi_t(C_s) \subset \mathbb{P}^3 \) are elliptic quintics which are totally tangent to \( W_t \). Each \( C_s \) for \( s \neq 0 \) defines a unique K3 surface \( S(C_s) \in |O_{X_t}(2)| \) such that \( C_s \subset S(C_s) \), where we denote by \( O_{X_t}(k) \) the sheaf \( \pi_t^*(\mathcal{O}_{\mathbb{P}^3}(k)) \). The image \( \pi_t(S(C_s)) \) is a quartic \( S(C_s) \in \mathbb{P}^3 \), containing \( C_s \), and tangent to \( W_t \), along the curve \( \overline{S(C_s) \cap W_t} \), so that \( \pi_t^{-1}(\overline{S(C_s)}) \) is the union of two components: \( S(C_s) \) and \( \iota S(C_s) \), where \( \iota \) denotes the Galois involution of \( \pi_t \). For \( s = 0 \), there is a unique limit quartic in \( \mathbb{P}^3 \), possibly singular, which we denote by \( \overline{S(C_0)} \). So the family \( S(C_s) \) extends over \( s = 0 \) with fiber \( S(C_0) \cong S(C_0) \) lying in one of the two components \( \mathbb{P}^3, \mathbb{P}^3' \) of \( X_0 \), and the limiting curve \( C_0 \) lies entirely in \( \mathbb{P}^3 \) or in \( \mathbb{P}^3' \). As remarks Clemens, a component of \( C_0 \) of degree \( d \) which does not lie in \( Q = \mathbb{P}^3 \cap \mathbb{P}^3' \) has to meet \( B = W \cap Q \) in \( 2d \) points counting multiplicities, because otherwise the deformed curve \( C_s \) for small \( s \) will not be totally tangent to \( W_t \), which is absurd. We will call such components 2d-sectants to \( B \).

Let \( H' \), resp. \( H'' \) be the subset of \( H_0 \) consisting of the curves \( C \subset X_0 \) such that: (1) \( C \) is smoothable in \( X_0 \), (2) \( C_{\text{red}} \) has no components contained in \( Q \), and (3) all the components of \( C \) lie in \( \mathbb{P}^3 \), resp. in \( \mathbb{P}^3' \). Denote \( H^0 = H' \cup H'' \).

As \( \mathcal{F}_0 \) is a reduced fiber of \( \mathcal{F} \), there exists a local analytic family of lines \( l_t \in \mathcal{F}_t \) near \( t = 0 \), and we can use \( 5l_t \) as the section of reference curves for the definition of a relative Abel-Jacobi map \( \alpha = \alpha_H : H \to J \). According to [C-2], Sect. 8, \( \alpha \) extends as a regular map to all of \( H^0 \subset H_0 \), and the image of \( H' \) (resp. \( H'' \)) lies in one component \( J' \) (resp. \( J'' \)) of the smooth locus \( J_0 = J^+ \cup J^- \) of \( J_0 \) (thus, \( J' = J^+ \) or \( J'' \) and \( J'' \) may a priori coincide either with \( J' \), or with the other component of \( J_0 \)). If we consider the composition \( \phi \) of \( \alpha|_{H^0} \) with the natural projection \( \eta : J_0 \to J(B) \), we obtain the following description modulo a constant translation by a fixed divisor on \( B \) depending on the choice of the
reference point: \( \phi(C) \) is the class of \( B \cdot C \) considered as a degree-10 divisor on \( B \). In particular, \( H' \) dominates \( J(B) \) via \( \phi \) if and only if \( H'' \) does.

We will use these considerations to prove the following proposition.

**Proposition 3.1.** Let \( \pi : X \to \Delta \) be a family of degenerating quartic double solids as above which is generic, that is, \( F' \) and \( G \) are generic. Then there exists a morphism

\[
\alpha = \alpha_H : H \setminus A \to J
\]

over a possibly shrunk disc \( \Delta \), where \( A = H_0 \setminus H^0 \) is a proper closed subset of \( H_0 \), such that the following properties are verified:

(i) The restriction \( \alpha_t : H_t \to J_t \) of \( \alpha \) to any fiber \( H_t \) with \( t \in \Delta^* \) is, modulo a constant translation in \( J_t \), the Abel–Jacobi map of \( H_t \).

(ii) The generic fiber of \( \alpha_t \) for any \( t \in \Delta \) is the union of finitely many copies of \( \mathbb{P}^1 \) corresponding to elliptic pencils in the \( K3 \) surfaces in \( [C-2] \), (4.2.1)-(4.2.2). It states that: 1) only one of the irreducible components of \( \lims \) is generic, that is, \( \lims \) is a section of the natural projection \( \to \) over a possibly shrunk disc \( \Delta \).

(iii) The closure \( D \) of \( \alpha(H \setminus A) \) is a relative divisor in \( J \) over \( \Delta \), that is \( D_t = D \cap J_t \) is a divisor in \( J_t \) for every \( t \in \Delta \), and \( D_t \) is a translate of the theta divisor of \( J_t \) if \( t \in \Delta^* \).

(iv) The smooth locus \( J_0 \) of the central fiber \( J_0 \) of \( J \) consists of two components \( J^\pm \), and only one of them, say \( J' \), contains \( \alpha_0(H^0) \). Moreover, \( \alpha_0(H^0) = \alpha_0(H') = \alpha_0(H'') \subset J' \).

(v) \( \alpha_0(H^0) \) is a section of the natural projection \( J' \to J(B) \) over an open subset of \( J(B) \).

**Proof.** (i) and (ii) follow from the above discussion, (iii) is a consequence of Theorem [C-2].

To prove (iv) and (v), we will use the characterization of the divisors in \( J_0 \) which can be limits of a family of principal polarizations in the neighboring fibers \( J_t \), given by Clemens in \([C-2]\), (4.2.1)-(4.2.2). It states that: 1) only one of the irreducible components of \( D_0 \), say \( D' \), dominates \( J(B) \), and 2) if \( J' (\epsilon = + \text{ or } -) \) is the component of \( J_0 \) containing \( D' \), then \( D' \) is a section of the natural projection \( J' \to J(B) \) over an open subset of \( J(B) \).

As we have remarked, \( H' \) dominates \( J(B) \) via \( \phi \) if and only if \( H'' \) does. Hence, if we assume that \( H' \) dominates \( J(B) \), then \( \alpha_0(H') \) and \( \alpha_0(H'') \) both dominate \( J(B) \), hence \( J' = J'' \) and \( \alpha_0(H') = \alpha_0(H'') \) is one and the same rational section of \( J' \to J(B) \).

It remains to see that \( H' \) dominates \( J(B) \). This follows from the fact that for generic \( W, Q \) and for a generic 10-uple of points \( Z \) in \( B = W \cap Q \), there is at least one elliptic quintic in \( \mathbb{P}^3 \) passing through \( Z \) (see Corollary [1.3]). Alternatively, one can prove, by an argument similar to that of Clemens on p. 97 of \([C-2]\), that the adherence values of \( \alpha \) along \( A \) project to a proper closed subset of \( J(B) \), and thus at least one of the components of \( H_0 \setminus A = H' \cup H'' \) has to project to a rational section of \( J \to J(B) \).

**\[ \square \]**

### 3.2. Deforming pentagons from \( X_0 \) to \( X_t \).

We want now to prove that the fiber \( H_0 \) is reduced at a generic point of \( H' \). To this end, we will construct a local cross-section of the projection \( H \to \Delta \) in the neighborhood of 0 taking its values in the subschemes of \( H_t \) parametrizing cycles of five lines, or pentagons in \( X_t \).

**Definition 3.2.** Let \( X \) be a quartic double solid. A pentagon in \( X \) is a reducible curve \( \Gamma = \bigcup_{i=1}^5 \ell_i \), where \( \ell_i \) are lines in \( X \) such that \( \ell_i \cap \ell_{i+1} \neq \emptyset \) (addition of subscripts modulo 5). If \( \ell_i \) meets \( \ell_{i+1} \) quasi-transversely at one point and these are the only intersection points of \( \ell_i, \ell_j \) (\( i \neq j \)), we will call \( \Gamma \) a good pentagon.

Consider the 5-uple fibered product of \( \mathcal{F} \) with itself over \( \Delta \):

\[
Y = \mathcal{F}_\Delta^5 = \mathcal{F} \times_\Delta \mathcal{F} \times_\Delta \mathcal{F} \times_\Delta \mathcal{F} \times_\Delta \mathcal{F}
\]

Let \( Y_t \) denote the fiber of \( Y \) over \( t \in \Delta \) and \( D_t \) the subvariety of \( Y \) parametrizing the 5-uples \((\ell_1, \ldots, \ell_5)\) of lines in the \( X_t \) defined by the incidence relation \( \ell_i \cap \ell_{i+1} \neq \emptyset \) \((i = 1, \ldots, 5, 5 + 1 = 1)\). Then the variety parametrizing all the pentagons in \( X_t, t \in \Delta \) can be represented as the intersection \( \Sigma := \bigcap_{i=1}^5 D_i \).
Proposition 3.3. Let the quartic \( W \) and the quadric \( Q \) defining the family \( X \rightarrow \Delta \) be generic. Let \( \Sigma_0 \) be the fiber of \( \Sigma \) over \( t = 0 \). Then there exists a component of \( \Sigma_0 \), such that its generic point \( y \) represents a good pentagon in \( X_0 \) whose five vertices are not in \( Q \), the 6 subvarieties \( Y_0, D_1, \ldots, D_6 \) are nonsingular at \( y \) and their intersection is transversal at \( y \).

Proof. It suffices to prove the assertion for a special \( y_0 \in Y_0 \) and a special degenerate quartic \( W \). Choose for \( y_0 \) the pentagon \( \bigcup_{i=1}^{5} \ell_i^0 \), where \( \ell_i^0 = P_iP_{i+1} \) (addition of subscripts modulo 5), \( P_i = (1 : 0 : 0 : 0) \), \( P_2 = (0 : 1 : 0 : 0) \), \( P_3 = (0 : 0 : 1 : 0) \), \( P_4 = (0 : 0 : 0 : 1) \), \( P_5 = (1 : 1 : 1 : 1) \). Let \( W \) be the union of 4 planes \( \Pi_k, k = 1, \ldots, 4 \), where \( \Pi_1 = \{x_3-\lambda x_4 = 0\}, \Pi_2 = \{x_3-\mu x_4 = 0\}, \Pi_3 = \{x_3-\nu x_2 = 0\}, \Pi_4 = \{x_3-\eta(x_2-x_3) = 0\} \), and \( Q \) a generic quadric passing through the two points \( A_0^1 = \Pi_2 \cap \ell_0^0 \) and \( A_2^0 = \Pi_3 \cap \ell_0^0 \), so that \( C = W \cap Q \) is the union of 4 conics \( C_k = \Pi_k \cap Q \), \( k = 1, \ldots, 4 \). An affine neighborhood of \( y_0 \) in \( Y \) can be parametrized by 11 rational functions \( t, u_1, \ldots, u_5, v_5 \), where \( (u_i, v_i) \) for \( i = 1, \ldots, 4 \) are affine coordinates in the plane \( \mathbb{P}^2 \), parametrizing lines in \( \Pi_i \), and \( u_5, v_5 \) are the parameters representing \( A_1^0 \subset C_2 \) and \( A_2^0 \subset C_3 \) respectively. Writing out the linearized equations of \( D_i \) at \( y_0 \) (depending on \( \lambda, \mu, \nu, \eta \) and two more constants defining tangent directions of \( C_2, C_3 \) at \( A_1^0, A_2^0 \), one immediately verifies that they do not depend on \( t \) and are linearly independent as soon as \( \mu \neq 0, \nu - \mu \neq 0, \nu - 1 \neq 0 \), which ends the proof. \( \square \)

3.3. Smoothability of good pentagons.

Now we will check that the good pentagons can be deformed into elliptic quintics.

Proposition 3.4. Under the hypotheses and in the notation of Proposition 3.3, there exists a local cross-section \( \Delta \rightarrow \mathbb{H}, t \mapsto y(t) \), over a possibly shrunk disc \( \Delta \), such that \( y_0 \in H^t \) and the corresponding curves \( \Gamma_{y(t)} \subset X(t) \) are good pentagons, strongly smoothable into elliptic quintics in \( X_t \) for all \( t \in \Delta \).

Recall that \( \Gamma_0 = \Gamma \) is called smoothable if there exists a flat family of curves \( \Gamma_s \) such that its generic member is smooth. If there exists such a deformation whose base and total space are both smooth, then \( \Gamma_0 \) is strongly smoothable.

Proof of Proposition 3.4. As above, we specialize our family \( X \rightarrow \Delta \) to the case when \( C \) is the union of four conics \( C_i \). Let \( \Gamma = \bigcup_{i=1}^{5} \ell_i^0 \) be a good pentagon as constructed in the proof of Proposition 3.3.

The following lemma is proved by a standard application of the techniques of [HH].

Lemma 3.5. The Hilbert scheme of \( X_0 = \mathbb{P}^r \cup \mathbb{P}^r \) is smooth of local dimension 20 at \( \Gamma \).

Now we will impose on the curves parametrized by \( \text{Hilb}(X_0) \) the incidence conditions with the four conics \( C_i \).

Lemma 3.6. In the situation of Lemma 3.5, let \( U_C \) be the locally closed subset of \( \text{Hilb}(X_0) \) whose points represent the curves meeting transversely \( C \) at 10 points. Then the point \( y_0 \) representing \( \Gamma \) is a smooth point of \( U_C \), \( \dim_{y_0} U_C = 10 \), and the generic point of the component of \( U_C \) containing \( y_0 \) represents a nonsingular curve in \( \mathbb{P}^r \).

Proof. By Lemma 3.5, \( y_0 \) is a smooth point of \( \text{Hilb}(X_0) \). It has a neighborhood \( U \) consisting only of smooth points and contained entirely in \( \text{Hilb}(\mathbb{P}^r) \). Let \( F \subset \mathbb{P}^r \) be any reduced curve meeting \( \Gamma \) transversely at \( r \) points \( x_1, \ldots, x_r \) which are nonsingular both on \( F \) and on \( \Gamma \), and let \( \xi_i \) for \( i = 1, \ldots, r \) denote the point of \( \mathbb{P}(\mathcal{N}_F) \) corresponding to the direction of the branch of \( F \) passing through \( x_i \). Denote by \( \sum_{i=1}^{r} \xi_i \), \( \mathcal{N}_F \) the negative elementary transformation of the normal bundle \( \mathcal{N}_F \) w.r.t. the \( \xi_i \) (see [HH]). Let \( U_F \subset U \) be the locus of curves meeting \( F \) transversely at \( r \) distinct points and represented by elements
of $U$. Then $U_F$ is smooth at $y_0$ if $h^1(\Gamma, \text{elm}_{\xi_1, \ldots, \xi_6}(\mathcal{N}_T)) = 0$ and in this case we have for the tangent space $T_{y_0} U_F = H^0(\Gamma, \text{elm}_{\xi_1, \ldots, \xi_6}(\mathcal{N}_F))$ (see [Ran], Remark 5.1).

Let us apply this observation to $F = C = \bigcup_{i=1}^4 C_i$, $r = 10$, $C_i \cap \ell_i = \{x_{2i-1}, x_{2i}\}$, $i = 1, \ldots, 5$. Let $\mathcal{N} = \text{elm}_{\xi_1, \ldots, \xi_6}(\mathcal{N}_F)$. We have $\mathcal{N}_F|_{\ell_i} \cong 2\mathcal{O}(2)$ for any $i$. As the branches of $C$ passing through $x_{2i-1}, x_{2i}$ lie in one plane for each $i = 1, \ldots, 4$ and they do not for $i = 5$, we have $\mathcal{N}|_{\ell_i} \cong \mathcal{O} \oplus \mathcal{O}(2)$ for $i = 1, \ldots, 4$ and $\mathcal{N}|_{\ell_5} \cong 2\mathcal{O}(1)$. This easily implies that $h^0(\mathcal{N}) = 10$, $h^1(\mathcal{N}) = 0$. It remains to prove that $\Gamma$ is smooth out by a generic deformation inside $U_C$. It suffices to show that the natural map $T_{y_0} U_C \rightarrow T_{y_0}^1$ to the Schlesinger’s space $T^1$ is surjective for any $i = 1, \ldots, 5$, where $p_i = \ell_i \cap \ell_{i+1}$ (the addition of subscripts modulo 5). This can be done in the same way as in the proof of Theorem 4.1 in [HH].

Now we go back to the proof of Proposition 3.4. By Proposition 3.3, we can find a local cross-section of $\Sigma$ representing a flat family of good pentagons in the varieties $X_t$, whose fiber over 0 is a good pentagon in $\mathbb{P}^4$. By the universal property of the Hilbert scheme, this family is induced via a cross-section $\Delta \rightarrow \text{Hilb}(X/\Delta)$ over $\Delta$, which we will denote by $t \mapsto y(t)$. It suffices to show that $y_0 \in H'$. We have verified in Lemma 3.6 that a good pentagon $\Gamma$ in $\mathbb{P}^4$ becomes strongly smoothable when $C$ is specialized into a union of four conics. By the semi-continuity of $h^i(\mathcal{N})$, and because the surjectivity of the maps $H^0(\mathcal{N}) \rightarrow T_{y_0}^1$ is an open condition, a generic good pentagon in some component of good pentagons in $\mathbb{P}^4$ is strongly smoothable for generic $X \rightarrow \Delta$. In fact, we do not check that the family of good pentagons in $\mathbb{P}^4$ is irreducible, so we are speaking here about one of its components.

As $\Gamma$ is a connected curve of degree 5 and arithmetic genus 1, so is its smoothing, hence it is an elliptic quintic in $\mathbb{P}^4$. By construction, it meets $C$ transversely at 10 points and is an element of $H'$. This ends the proof of Proposition 3.4. □

This immediately implies

**Corollary 3.7.** With the same assumptions as above, there is a component of $H'$ which is acquired with multiplicity 1 in the fiber $H_0$ of $H \rightarrow \Delta$.

We will prove in Section 3 that $H'$ is irreducible, hence $H'$ and $H''$ are irreducible components of $H_0$ acquired with multiplicity 1.

### 4. Elliptic Quintics through Ten Points

#### 4.1. Elliptic Quintics, 10-Secant to $B = W \cap Q$.

We are going to prove the irreducibility of the subscheme $H'$ of $\text{Hilb}(X_0)$ introduced in Sect. 3.11

**Lemma 4.1.** Let $Q$ (resp. $S$) be a generic quadric (resp. quartic) in $\mathbb{P}^3$, and $B = Q \cap S$. Let $\mathcal{C}_5^1[B]_{10} \subset \mathcal{C}_5^1(\mathbb{P}^3)$ be the base of the family of elliptic quintics meeting $B$ in a subscheme of length 10. Then $\mathcal{C}_5^1[B]_{10}$ is irreducible.

**Proof.** Let $\mathbb{P}^{34}$ be the space of quartic surfaces in $\mathbb{P}^3$, $\Gamma = \{(S, C) \in \mathcal{C}_5^1(\mathbb{P}^3) \times \mathbb{P}^{34} \mid C \subset S\}$ and $\mathbb{P}^{33} \xrightarrow{\mathcal{F}_1} \Gamma \xrightarrow{p_2} \mathcal{C}_5^1(\mathbb{P}^3)$ the natural projections. Any $C \in \mathcal{C}_5^1(\mathbb{P}^3)$ does not lie on a quadric, hence the restriction map $H^0(\mathcal{O}_{\mathbb{P}^3}(2)) \rightarrow H^0(\mathcal{O}_C(2))$ is bijective. Then $h^1(\mathcal{I}_{C/\mathbb{P}^3}(2)) = 0$ and a fortiori $h^1(\mathcal{I}_{C/\mathbb{P}^3}(4)) = 0$, which implies $h^0(\mathcal{I}_{C/\mathbb{P}^3}(4)) = 15$. Hence $p_2^{-1}(C) \simeq \mathbb{P}^{14}$, and as $\mathcal{C}_5^1(\mathbb{P}^3)$ is irreducible of dimension 20, $\Gamma$ is irreducible as well and $\dim \Gamma = 34$. Moreover, denoting $\mathcal{C}_5^1(\mathbb{P}^3)^* := \mathcal{C}_5^1(\mathbb{P}^3) \setminus \text{Sing}(\mathcal{C}_5^1(\mathbb{P}^3))$, $\Gamma^* = p_2^{-1}(\mathcal{C}_5^1(\mathbb{P}^3)^*)$, we see that $\Gamma^*$ is smooth and is a projective bundle over $\mathcal{C}_5^1(\mathbb{P}^3)^*$.

Next, a general quartic $S$ in $p_2^{-1}(C)$ is smooth for general $C \in \mathcal{C}_5^1(\mathbb{P}^3)$. Hence $D = p_1(\Gamma)$ is irreducible. Besides, for any smooth quartic $S = p_1(S, C) \in D$ the fibre $p_1^{-1}(S)$ is a
dense open subset in $|O_S(C)| \simeq \mathbb{P}^1$; hence dim $D = 33$, i.e. $D$ is an irreducible divisor in $\mathbb{P}^{34}$. Moreover, $p_1 : \Gamma \rightarrow D$ is a smooth morphism of relative dimension 1.

Let $\overline{D}$ be the closure of $D$ in $\mathbb{P}^{34}$. We will prove that codim$_{\overline{D}}$ Sing $\overline{D} \geq 2$. In fact, if codim$_{\overline{D}}$ Sing $\overline{D} = 1$, i.e. dim Sing $\overline{D} = 32$, then $p_1(p_2^{-1}(C)) \cap$ Sing $\overline{D} = \mathbb{P}^{14} \cap$ Sing $\overline{D} \neq \emptyset$ for any $C \in C_5^1(\mathbb{P}^3)^*$, hence, by the smoothness of $p_1|\Gamma^*$, $p_2^{-1}(C) \cap$ Sing $\Gamma^* \subset p_2^{-1}(C) \cap p_1^{-1}(\text{Sing} \overline{D}) \neq \emptyset$, which contradicts the smoothness of $\Gamma^*$.

Now let $C_{2.4}(\mathbb{P}^3) := \{C \in \text{Hilb}(X) \mid C \text{ a smooth complete intersection curve of a smooth quartic surface and a smooth quadric surface in } \mathbb{P}^3\}$. Clearly, $C_{2.4}(\mathbb{P}^3)$ is a smooth irreducible variety of dimension 33. Let $U$ be the open subset of smooth quartics in $\mathbb{P}^{34}$, $\Pi = \{(C, F) \in C_{2.4}(\mathbb{P}^3) \times \mathbb{P}^{34} \mid C \subset F\}$ and $C_{2.4}(\mathbb{P}^3)$ be the projections. One immediately verifies, using the resolution $0 \rightarrow O_{\mathbb{P}^3}(-2) \rightarrow O_{\mathbb{P}^3}(2) \oplus O_{\mathbb{P}^3} \rightarrow \mathcal{I}_C/\mathbb{P}^3(4) \rightarrow 0$ for $C \in C_{2.4}(\mathbb{P}^3)$, $h^0(\mathcal{I}_C/\mathbb{P}^3(4)) = 11$. By the definition of $C_{2.4}(\mathbb{P}^3)$, a general quartic in $q_1^{-1}(C)$ is smooth, hence $q_1^{-1}(C)$ is an open subset in $\mathbb{P}^{10} = P(H^0(\mathcal{I}_C/\mathbb{P}^3(4)))$. Thus, $\Pi$ is irreducible of dimension 43. Moreover, for any $(C, S) \in \Pi$, we have $(C, S) \in q_2^{-1}(S')$, with $S' \in U$, and $q_2^{-1}(S')$ is an open subset in $|O_{S'}(C)| = |\text{Sing}(\mathbb{P}^9)| \simeq \mathbb{P}^9$ consisting of smooth curves. Clearly, $D' := U \cap D \neq \emptyset$ and $D'$ is open in $D$, hence $D_{\Pi} := q_2^{-1}(D')$ is an irreducible divisor in $\Pi$.

Let us look now at the general fiber of $q_D := q_1|D_{\Pi} : D_{\Pi} \rightarrow C_{2.4}(\mathbb{P}^3)$. Let $\overline{\Pi}$ be the closure of $\Pi$ in $\overline{C} \times \mathbb{P}^{34}$ with natural projections $\overline{C} \xrightarrow{\overline{\pi}_1} \overline{\Pi} \xrightarrow{\overline{\pi}_2} \mathbb{P}^{34}$, where $\overline{C}$ is the closure of $C_{2.4}(\mathbb{P}^3)$ in $\text{Hilb}(\mathbb{P}^3)$, and $\overline{D}_{\Pi} = \overline{q_2}^{-1}(\overline{D})$. Let $Z \subset \overline{D}$ be the subset of $\overline{D}$ which consists of reducible quartics, codim$_{\overline{D}}Z \geq 11$, $D^* = \overline{D} \setminus Z$, $D^*_{\Pi} := \overline{q_2}^{-1}(D^*)$. For any $S \in D^*$, $|O_{S}(2)| \simeq \mathbb{P}^9$ lies in $\overline{C}$. Hence $\overline{q_2}_*|D^*_{\Pi} : D^*_{\Pi} \rightarrow D^*$ is a projective bundle with fiber $\mathbb{P}^9$, in particular, it is a smooth morphism of relative dimension 9. By construction, the fiber of $\overline{q}_D = \overline{q_1}|D : D_{\Pi} \rightarrow \overline{C}$ over any $C \in C_{2.4}(\mathbb{P}^3)$ is a divisor in $\overline{q}_1^{-1}(C) = \mathbb{P}^{10}$ which contains $q_D^{-1}(C)$ as an open subset. Assume that $q_D^{-1}(C)$ is reducible. Then its closure $\overline{q_D}^{-1}(C)$ is also a reducible divisor in $\mathbb{P}^{10}$, hence it has singularities in codimension 1. Since $C \in C_{2.4}(\mathbb{P}^3)$ is smooth, codim$_{\overline{D}_{\Pi}}\text{Sing} \overline{D}_{\Pi} = 1$. As codim$_{\overline{D}}Z \geq 11$, codim$_{D^*_{\Pi}}\text{Sing} D^*_{\Pi} = 1$. Hence, since $\overline{q_2}_*|D^*_{\Pi} : D^*_{\Pi} \rightarrow D^*$ is a smooth morphism, codim$_{D^*}\text{Sing} D^* = 1$ and a fortiori codim$_{\overline{D}}\text{Sing} \overline{D} \geq 2$, which is absurd.

Thus, for general $C \in C_{2.4}(\mathbb{P}^3)$, both $q_D^{-1}(C)$ and $\overline{q_D}^{-1}(C)$ are irreducible. The generic quartic $S$ from $q_D^{-1}(C)$ is smooth and contains a unique pencil $|E|$ of elliptic curves, and these curves belong to $C_5^1[B]_{10}$, where $B = Q \cap S$, $Q$ being the unique quadric containing $C$. Conversely, taking $B$ as in the hypothesis of the lemma, any smooth elliptic quintic $E$ meeting $B$ in 10 points lies in a generically smooth quartic surface $S$ through $B$, i.e. such that $(B, S) \in q_D^{-1}(B)$, and varies on $S$ in a pencil. Hence we have a fibration $H' \rightarrow \overline{q}_D^{-1}(C)$ with an open subset of $\mathbb{P}^1$ as a fiber. This implies the irreducibility of $C_5^1[B]_{10}$. \hfill $\Box$

**Corollary 4.2.** Let $Q \subset \mathbb{P}^3$ be a smooth quadric surface, $W \subset \mathbb{P}^3$ a generic quartic, $B = Q \cap W$. Then the subscheme $H'$ (resp. $H''$) of $\text{Hilb}(X_0)$ introduced in Sect. 3.1 and $C_5^1[B]_{10}$ contain isomorphic dense open subsets, hence $H'$ (resp. $H''$) is irreducible. This implies, by Corollary 3.7, that the fiber $H_0$ of $H$ over $t = 0$ is reduced at the generic points of $H'$ and $H''$.

4.2. 42 elliptic quintics through 10 points

Let $U$ be the open subset of the smooth locus of $\text{Hilb} \mathbb{P}^3$ parametrizing elliptic quintics. There is a universal family $\mathcal{C} \subset U \times \mathbb{P}^3$ over $U$ with projection $p_C : \mathcal{C} \rightarrow U$. Consider the 10-th relative symmetric power $\mathcal{S} := \text{Sym}^{10}(\mathcal{C}/U)$ of $\mathcal{C}$ over $U$ with projection $p : \mathcal{S} \rightarrow U$ and fiber $p^{-1}(\{C\}) = \text{Sym}^{10}C$. Thus $\mathcal{S}$ is a smooth irreducible variety of dimension 30. We can embed it into $U \times \text{Hilb}^{10} \mathbb{P}^3$:

$$\mathcal{S} = \{(\{C\}, \{Z\}) \mid \{C\} \in U, \{Z\} \in \text{Hilb}^{10} \mathbb{P}^3 \text{ and } Z \subset C \text{ as a scheme}\},$$
and we have a natural projection $\mu : S \to \text{Hilb}^{10} \mathbb{P}^3$, $(\{C\}, \{Z\}) \mapsto \{Z\}$. There exists at least one quintic from the closure of $C^1_3(\mathbb{P}^3)$ passing through 10 distinct points in $\mathbb{P}^3$, and for a general 10-uple of points, the number of such quintics is finite and they are smooth. In fact, we have:

**Lemma 4.3** (E. Getzler, [Ge], Table 2). For a general 10-uple $Z$ of points in $\mathbb{P}^3$, there are 42 elliptic quintics passing through $Z$.

Denote by $H^{10}$ the component of $\text{Hilb}^{10} \mathbb{P}^3$ containing a 10-uple of distinct points. It follows that $H := \mu(S)$ is an open subset of $H^{10}$ and that $\mu : S \to H$ is generically finite of degree 42. We need to replace in the above lemma “a general 10-uple $Z$ of points in $\mathbb{P}^3$” by “a general 10-uple $Z$ of points in a smooth quadric $Q \subset \mathbb{P}^3$”, that is, we should prove that the 10-uples of points lying in smooth quadrics form a divisor in $H$ which is not contained in the branch locus of $\mu$.

Let $D = \{(\{C\}, \{Z\}) \in S \mid Z = C \cap Q$ for some quadric $Q$ in $\mathbb{P}^3\}$. Clearly, $D$ is a divisor in $S$, and for any $\{C\} \in U$,

$$D \cap p^{-1}(\{C\}) = \{\{Z\} \in \text{Sym}^{10} C \mid \mathcal{O}_C(Z) = \mathcal{O}_{\mathbb{P}^3}(2)|C\} = \{\mathcal{O}_{\mathbb{P}^3}(2)|C\} \simeq \mathbb{P}^9.$$

As $U$ is irreducible, $D$ is irreducible by (26), hence so is $B := \mu(D)$. It is easy to see that $B$ is a divisor. Indeed, for any smooth elliptic quintic $C$ in $\mathbb{P}^3$ and for any $Z \in |\mathcal{O}_C(2)|$, there is a unique quadric $Q$ passing through $Z$, so dim $B = 29$.

**Proposition 4.4.** Let $R \subset S$ be the ramification divisor of $\mu$. Then we have: (i) $D$ is not contained in $R$, hence $\mu : S \to H$ is étale in a general point of $D$.

(ii) $\mu$ is an étale covering over an open subset $V$ of $H$ such that $B \cap V \neq \emptyset$.

**Proof.** First we show that (i) implies (ii). For a general 10-uple of points $Z$ on a general smooth quadric $Q$, there exists a quartic surface passing through $Z$ and intersecting $Q$ in a smooth curve, say $C_0$. Further, by Corollary 4.2, the set of all the elliptic quintics meeting $C_0$ in 10-uples of points is irreducible. Hence $(\mu^{-1}(\text{Sym}^{10}(C_0)))_{\text{red}}$ is irreducible. Hence $\mu$ has the same ramification indices in all the points of $\mu^{-1}(Z)$, where $\{Z\} \in B$ is the above general point of $B$. But according to (i), $\mu$ is unramified in at least one point of $\mu^{-1}(Z)$. Hence $\mu$ is unramified, that is, étale, in all the points of $\mu^{-1}(Z)$.

Let us prove (i). A pair $(Z, C)$ is a ramification point of $\mu$ if and only if $C$ has infinitesimal deformations which fix $Z$, that is, $h^0(\mathcal{N}_{C/\mathbb{P}^3} \otimes \mathcal{I}_Z) \neq 0$. But if $Z$ lies on a quadric, $\mathcal{N}_{C/\mathbb{P}^3} \otimes \mathcal{I}_Z \simeq \mathcal{N}_{C/\mathbb{P}^3}(-2)$, and $h^0(\mathcal{N}_{C/\mathbb{P}^3}(-2)) = 0$ by [EL] (see also Theorem (VIII.2.7) in [Hu]).

The following statements are obvious corollaries of what we have proved:

**Corollary 4.5.** Let $Q \subset \mathbb{P}^3$ be a smooth quadric surface. Then there are 42 elliptic quintics in $\mathbb{P}^3$ passing through 10 generic points of $Q$. There are also 42 elliptic quintics in $\mathbb{P}^3$ passing through 10 generic points on the curve $B = Q \cap W$ for a generic quartic surface $W \subset \mathbb{P}^3$.

**Corollary 4.6.** Let $\alpha_0 : H' \cup H'' \to J^+$ be the map introduced in Proposition 3.1, (ii). Then the number of copies of $\mathbb{P}^1$ in the generic fiber of $\alpha_0$ is equal to 84.

As $H' \cup H''$ contains all the elliptic quintics which are the limits of elliptic quintics from the neighboring fibers, we deduce from Corollaries 4.6 and 4.2 that the number of $\mathbb{P}^1$’s in the generic fiber of $\alpha_t$ is 84 for all sufficiently small $t \in \Delta$. This implies the following result.

**Theorem 4.7.** Let $X$ be a general quartic double solid. Then the generic fiber of the Abel Jacobi map $\Phi : C^1_3(\mathbb{P}^3) \to \Theta + \text{const}$ is a dense open subset in the union of 84 copies of $\mathbb{P}^1$. 
5. Vector bundles

Let $X$ be general, so that $C^1_5(X)$ is irreducible. Let $C \in C^1_5(X)$. The Serre construction allows us to define an $O_X$-sheaf $E_C$ as an extension of $O_X$-sheaves:

$$(27) \quad 0 \rightarrow O_X(-1) \rightarrow E_C \rightarrow J_{C/X}(1) \rightarrow 0.$$  

Any such extension is determined uniquely up to an isomorphism by a 1-dimensional subspace of the group $\text{Ext}^1(O_{C/X}(1), O_X(-1)) \simeq H^0(\text{Ext}^1(J_{C/X}(1), O_X(-1))) \simeq H^0(\text{Ext}^2(O_C(1), O_X(-1))) \simeq H^0(\text{Ext}^2(O_C, \omega_X)) \simeq H^0(\omega_C) \simeq \mathbb{C}$ (we are using $h^iO_X(-2) = 0$, $i < 3$, and $\omega_C \simeq O_C$). Thus $E_C$ is determined uniquely up to an isomorphism by $C$. Besides, by a well known property of the Serre construction, the local triviality of $\text{Ext}^1(J_{C/X}(1), O_X(-1)) \simeq \omega_C$ implies that $E_C$ is locally free.

From $(27)$, one easily computes the Chern classes:

$$c_1(E_C) = 0, \quad c_2(E_C) = 3$$

(here we understand $c_2$ as an element of the group $B_1$ of classes of algebraic equivalence of 1-dimensional cycles on $X$, generated by the class of a line on $X$). As $\pi : C \rightarrow \pi(C)$ is an isomorphism onto a quintic $\pi(C)$ that does not lie in a plane and $H^0(O_X(1)) \simeq \pi^*H^0(O_{\mathbb{P}^3}(1))$, it follows that $H^0(J_{C/X}(1)) \simeq H^0(J_{\pi(C)/\mathbb{P}^3}(1)) = 0$. Hence the triple $(27)$ gives $h^0(E_C) = 0$. This means that $E_C$ is Gieseker stable (see [MT-1 Prop. 2.6]). Thus, denoting by $M_X(2; 0, 3)$ the Gieseker-Maruyama moduli scheme of stable rank-2 vector bundles on $X$ with Chern classes $c_1 = 0$, $c_2 = 3$, we have $[E_C] \in M_X(2; 0, 3)$, where $[E]$ stands for the isomorphism class of $E$. Remark that the equality $h^0(J_{C/X}(2)) = 1$ immediately implies $h^i(J_{C/X}(2)) = 0$, $i > 0$; hence by $(27)$ also

$$h^i(E_C) = 0, \quad i > 0.$$  

Let $M = \{[E] \in M_X(2; 0, 3) | h^i(E) = 0, \ i > 0\}$. By semicontinuity, this is an open subset of $M_X(2; 0, 3)$ containing $[E_C]$. Let $M$ be the irreducible component of $M$ containing $[E_C]$. We have a natural map

$$f : C^1_5(X) \rightarrow M, \ C' \mapsto [E_{C'}].$$

It is a standard matter to show that $f$ is a dominant morphism (see [MT-1 Lemmas 5.2 and 5.3]), whose fibre $f^{-1}(f(C'))$ is open in $P(H^0(E_{C'}(1)) \simeq \mathbb{P}^1$:

$$f^{-1}(f(C')) = \mathbb{P}^1, \ C' \in C^1_5(X).$$

The closure is taken in $C^1_5(X)$, and we keep in mind that, by Riemann-Roch, $h^0(E_{C'}(1)) = 2$, since by the definition of $M$, $h^i(E_{C'}(1)) = 0, \ i > 0$. The equality $h^0(J_{C'/X}(2)) = 1$ means that the linear series $|J_{C'/X}(2)|$ consists of a unique K3 surface $S(C')$, and by Serre construction, the fibre $f^{-1}(f(C')) = \mathbb{P}^1$ is just the pencil of elliptic curves on $S(C')$, which are nothing else than the zero loci of sections of $E_{C'}(1)$:

$$(28) \quad \overline{f^{-1}(f(C'))} = |O_{S(C')}(C')| = \mathbb{P}^1.$$  

Now consider the Stein factorization of the Abel-Jacobi map $\Phi = \Phi_{C^1_5(X)} : C^1_5(X) \rightarrow \Theta + \text{const}$. Using the above description $(28)$ of fibres of $f$ and the well-known interpretation of the differential of the Abel-Jacobi map in terms of the branch quartic $W$ of $\pi : X \rightarrow \mathbb{P}^3$ (see the end of Section 2 or [T-1 Corollary 1], or [We Prop.2.13]), we obtain the following

**Theorem 5.1.** Let $X$ be a general quartic double solid. Then there exists a quasi-finite dominant morphism $g : M_0 \rightarrow \Theta + \text{const}$ of degree 84 from an open part $M_0$ of the above component $M$ of the moduli space $M_X(2; 0, 3)$ to a translate of the theta divisor of $J(X)$.
which gives the Stein factorization of the Abel-Jacobi map $\Phi$ of $C^1_5(X)$:

$$
\begin{array}{ccc}
C^1_5(X) & \xrightarrow{\Phi} & \Theta + \text{const} \\
\downarrow f & & \downarrow g \\
M_0 & \xrightarrow{\sim} & M_0
\end{array}
$$

The fibers of $f : C^1_5(X) \to M_0$ are open subsets of $\mathbb{P}^1$.

The proof is similar to that of Theorem 5.6 in [MT-1].

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