Multi-Granulation Modified Rough Bipolar Soft Sets and their Applications in Decision-Making

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ABSTRACT The classical theory of rough sets (RSs) established by Pawlak, mainly focused on the approximation of sets characterized by a single equivalence relation (ER) over the universe. However, most of the current single granulation structure models cannot meet the user demand or the target of solving problems. Multi-granulation rough sets (MGRS) can better deal with the problems where data might be spread over various locations. In this article, based on modified rough bipolar soft sets (MRBSs), the concept of multi-granulation MRBSs (MGMRBSs) is introduced. A finite collection of bipolar soft sets (BSs) has been used for this purpose. Several important structural properties and results of the suggested model are carefully analyzed. Meanwhile, to measure the uncertainty of MGMRBSs, some important measures associated with MGMRBSs are presented in MGMRBS-approximation space, and some of their interesting properties are examined. In the framework of multi-granulation, we developed optimistic MGMRBSs (OMGMRBSs) and pessimistic MGMRBSs (PMGMRBSs). The relationships among the MGMRBSs, OMGMRBSs, and PMGMRBSs are also established. After that, a novel multi-criteria group decision-making (MCGDM) approach based on OMGMRBSs is developed to solve some problems in decision-making (DM). The basic principles and the detailed steps of the DM model are presented in detail. To demonstrate the applicability and potentiality of the developed model, we give a practical example of a medical diagnosis. Finally, we conduct a comparative study of the proposed MCGDM approach with some existing techniques to endorse the advantages of the proposed model.

INDEX TERMS MGRS, MRBSs, MGMRBS-approximations, MCGDM.

I. INTRODUCTION

There are various problems in our everyday lives that include uncertainty in their data. Such problems are prominent in economics, engineering, environmental sciences, medical sciences, and a variety of other disciplines dependent on modeling uncertainties that traditional mathematical techniques cannot tackle.

In recent times, several researchers have endeavored to establish suitable techniques and mathematical theories to cope with these uncertainties. These include fuzzy set (FS) theory, RS theory, DM theory, etc., which have been formulated to tackle these problems but have only been found partly effective. These theories bridge the gap between conventional mathematical models and ambiguous real-world data. These theories have some issues because of the insufficiency of the parameterization tools mentioned in [37].

Pawlak proposed RS theory [41] to handle uncertainty and imprecision in the data. Uncertainty is presented in this theory through the boundary region of a set. To date, many extended RS models have been developed, such as rough fuzzy sets and fuzzy RS [11], decision-theoretic RS [9], and probabilistic RS [28], etc. A single ER typically characterizes RS models. However, in practical life, we often need to describe the concept via multiple relations over the universe based on user requirement or the target of tackling the problem. Hence, Qian et al. [42] extended the single granulation RS model to the MGRS model, which has recently emerged as a prominent topic in artificial intelligence, attracting a wide range of research from both theoretical and application perspectives.
Molodtsov [37] presented a soft set (SS) in 1999, which can be viewed as an entirely new mathematical technique for modeling uncertainty, where a SS is connected with a set of attributes and therefore free from the difficulties mentioned above. Unlike classical mathematics, which demands an exact solution to a mathematical model, SS theory embraces an approximate description of an item as its initial point. Appropriate parameterization tools, like functions, linguistic phrases, numbers, etc., makes SS theory highly practical and simple to implement in reality.

Recently, numerous scholars have been intrigued by the properties and applications of SS. In 2003, Maji et al. [33] provided several new algebraic operations on SS and focused on a detailed theoretical study of SS. In 2001, Maji et al. [34] suggested the fuzzy SS and investigated its structure. Ali and Shabir [6] proposed De Morgan’s Laws in fuzzy SS by rectifying some pointed errors in [34].

Feng et al. [12], [13] pioneered a fusion between SS and RS and invented the soft rough set (SRS), which may yield better approximations than RS. As a result, some unusual situations have occurred, like:

- The upper approximation of the non-empty set may be empty.
- An upper approximation of a subset \( M \) of the universe \( \mathcal{O} \) may not contain the set \( M \).

These situations do not happen in RS theory. Shabir et al. [48] created a variant of SRS termed “modified soft rough sets (MSRSs)” to address these deficiencies. Shaheen et al. [53] proposed the dominance-based SRS with applications in DM.

In data analysis, bipolarity is a vital factor in constructing mathematical formulations for some problems. Bipolarity Discusses the positive and negative sides of the data. Positive data shows what is expected to be true, while negative data shows what is unlikely, prohibited, or certainly wrong. The idea underlying the existence of bipolarity is that a vast range of human DM is dependent on bipolar perception. For example, the sweetness and sourness of food, cooperation and competition, effects and side effects of medicines are the two sides of data in DM and coordination. The coexistence, equilibrium, and harmony of these two sides are considered core features for the stability of social systems. The SS, the FS, and the RS are inadequate techniques for dealing with bipolarity. For instance, a dress that is not beautiful, may not be necessarily ugly. Subsequently, Zhang [74] proposed bipolar FSs (BFSs) as an extension of FSs. In BFSs, the membership degree is enlarged from the interval \([0, 1]\) to the interval \([-1, 1]\). In a BFS, the membership degree 0 of an element implies that the element is irrelevant to the related property, the membership degree in \((0, 1]\) of an element shows that the element somewhat satisfies the property and the membership degree in \([-1, 0)\) of an element demonstrates that the element somewhat meets the implicit counter-property. Dubois and Prade [81] studied three main classifications of bipolarity.

Due to the importance of bipolarity, Shabir and Naz [49] laid the foundation of the bipolar soft sets (BSs) and proposed their set-theoretic operations with applications to DM. Following this study, BSs have grown in popularity among researchers. Karaaslan and Karataş [26] redefined BSs using a new approximation, allowing researchers to examine the topological structures of BSs. Shabir and Bakhtawar [50] explored the bipolar soft connected, bipolar soft disconnected, and bipolar soft compact spaces for the first time in 2017. Karaaslan et al. [24] developed bipolar soft groups. Kmaci and Petchimuthu [22] pioneered bipolar N-SS with applications in DM problems. Naz and Shabir [39] initiated the fuzzy BSs and examined their algebraic structures. Recently, Dalkılıç [10] proposed a DM approach to reduce the margin of error of decision-makers for BSs theory.

Karaaslan and Çağman [25] were the first to establish the bipolar soft rough sets (BSRSs). Shabir and Gul [51] point out some basic problems in the definition of BSRSs and redefine a new model of BSRSs known as MRBSs. They also discussed an MCGDM approach based on MRBSs. Gul and Shabir [17] proposed \((\alpha, \beta)\)-bipolar fuzzified RS. Akram and Ali [1] suggested a technique for DM based on rough Pythagorean fuzzy bipolar soft information. Gul et al. [19] developed a novel approach towards the roughness of BSs and applied this approach in MCGDM. Malik and Shabir [36] use rough bipolar fuzzy approximations based consensus model. Shabir et al. [52] established rough approximations of BSs based on soft relations with applications in DM.

A. LITERATURE REVIEW

MGRS has emerged as a popular topic in artificial intelligence, attracting a wide range of research in both theoretical and applied perspectives. Many researchers have worked on the MGRS theory in the literature. In 2013, Xu et al. [63] developed two novel kinds of MGRS. After that, Xu et al. [61] proposed the MGRS model in a fuzzy environment, the MGRS based on tolerance relations (Xu et al. [62]), the MGRS model in ordered information systems (Xu et al. [59]), and the MGRS in a fuzzy tolerance approximation space (Xu et al. [64]). Zhan and Xu [75] suggested two coverings-based MGRS in the FS environment with applications to DM. Yang et al. [68] established the hierarchical structure properties of the MGRS. Yang et al. [66] demonstrated a test cost-sensitive MGRS model. Lin et al. [32] developed a neighborhood-based MGRS. Xu et al. [60] initiated the generalized MGRS. Sun et al. [57] proposed multi-granulation vague RS over two universes with applications to DM. She et al. [54] studied the topological structures of MGRS. Qian et al. [46] pioneered three models of multi-granulation decision-theoretic RS. Feng and Mi [14] investigated variable precision multi-granulation fuzzy decision-theoretic RS in an information system. Li et al. [27] pioneered a double-quantitative multi-granulation decision-theoretic rough FS. Lin et al. [31] proposed a two-grade fusion approach involved in the evidence theory.
and MGRS using a well-defined distance function among granulation structures and developed three sorts of covering-based MGRS. Pan et al. [40] studied MGRS for the ordinal system using preference relations. Zhang et al. [78] studied MGRS in the context of hesitant FS with applications in DM. Huang et al. [20] established an intuitionistic hesitant MGRS (IFMGRS). Liang et al. [29] pioneered an efficient algorithm for rough feature selection for large-scale data using MGRS. Ali et al. [3] suggested dominance-based MGRS (DB-MGRS) with applications in conflict analysis. Rehman et al. [47] established the soft DB-MGRS with applications in conflict analysis. You et al. [70] pioneered MGRS (DB-MGRS) with applications in conflict analysis. You et al. [70] came up with the idea of $(\alpha, \beta)$-multi-granulation bipolar fuzzy RS with applications to MCGDM. Li et al. [79] initiated multi-granulation double-quantitative decision-theoretic RSs based on logical operations. Zue et al. [80] proposed three-way decision models based on multi-granulation rough intuitionistic hesitant FSs.

B. MOTIVATION

The motivation of this article is described by the following arguments:

- As mentioned earlier, there are certain shortcomings in BSRSs. To overcome these shortcomings, Shabir and Gul [51] proposed MRBSs with applications in DM. Also, many studies for MGRS have been developed in recent years, but they remain theoretical and qualitative, and less effort has been focused on the applications of MGRS. Therefore, motivated by the aforementioned earlier studies and the efficiency of the MRBSs in solving DM problems, in this work, the concept of MGRSs is developed in which the set approximations are defined by using a finite collection of BSs over the universe.

- Also, we explore a new approach to DM under uncertainty using the theory of MGRBSs. We consider a DM problem of medical diagnosis. Medical diagnosis is the process of identifying a disease from its symptoms. But sometimes, symptoms are not obvious and show bipolarity in behavior. In this situation, MRBSs can be of great interest because they provide information about the positive and negative parameters of certain objects. As we know, uncertainty is an inherent component of medical diagnosis since a symptom is an uncertainty index about whether or not a disease is occurring. Symptoms and diseases belong to different universes, although they are interrelated. Thus, uncertainty arises when describing the interrelations between symptoms and diseases in clinical settings. A particular patient may exhibit various symptoms, just as each disease may exhibit a variety of symptoms. Hence, it is difficult for a doctor to decide which disease the patient has. So, different but highly interrelated universes describe this DM problem. Further, considering the complexity of the disease for a given patient, the doctor may invite different experts such as physicians, surgeons, and orthopedists to provide consultation to make an exact diagnosis. Each expert will give an opinion in the form of BS based on the patient’s symptoms according to the individual expert’s experience and professional knowledge. Finally, the doctor incorporates all the opinions provided by different experts and concludes the diagnosis.

- From the perspective of RS theory, this process of medical diagnosis can be described as a multi-granulation setting in the approximation space. To improve DM in practical situations such as this, research on MGRMRBSs is necessary.

C. AIM OF THE PROPOSED STUDY

The primary goal of this study is to extend the idea of MRBSs to MGRMRBSs.

The following innovative research highlights the paper:

- A novel idea of MGRMRBSs has been developed.
- Essential structural properties of MGRMRBSs are thoroughly explored.
- Several significant measures related to MGRMRBSs have been proposed along with their properties.
- Furthermore, two types of MGRMRBSs models, called the OGMGRBSs, and the PMGRMRBSs have been developed. Also, a detailed discussion of some fundamental structural properties of the OGMGRBSs and PMGRMRBSs models is given.
- The relationship among MGRMRBSs, OGMGRBSs, and PMGRMRBSs is established.
- A comprehensive MCGDM method under the OGMGRBSs environment has been established. The DM methodology, as well as the algorithm of the proposed technique, are given. A practical example demonstrates the applicability of this approach by considering a medical diagnosis problem.
- A detailed comparison with some other MCGDM approaches is made to demonstrate the merits of the proposed set and methodology.

D. ORGANIZATION OF THE PAPER

The rest of this paper is structured as follows: Section 2 provides a quick survey of some foundational notions necessary for understanding our study. In section 3, we define the idea of MGRMRBSs and then prove some of their interesting structural properties. It is shown that MRBSs are a special case of those of MGRMRBSs. Section 4 discusses various essential measures associated with MGRMRBSs. In section 5, we propose the notion of OGMGRBSs and their important structural properties. In section 6, we offer the concept of PMGRMRBSs and investigate some of their important structural properties. Section 7 establishes the relationship amongst the MGRMRBSs, OGMGRBSs, and PMGRMRBSs. Section 8 offers a DM methodology based on OGMGRBSs and verifies the main steps of the decision technique with a
Definition 4: [41] An object of the form \( \Delta = (O, \pi) \) is termed as an approximation space \((A, s)\), where \(O\) is a finite non-empty universe and \(\pi\) is an ER over \(O\).

If \(\emptyset \neq M \subseteq O\), then the lower and upper approximations of \(M\) w.r.t \(\Delta\) are respectively defined as:

\[
\tilde{\text{apr}}_\pi(M) = \{x \in O: [x]_\pi \subseteq M\},
\]

(1)

\[
\text{apr}_\pi(M) = \{x \in O: [x]_\pi \cap M \neq \emptyset\},
\]

(2)

where,

\([x]_\pi = \{y \in O \mid (x, y) \in \pi\} \).

Moreover, the boundary region of RS is defined as:

\[ \text{Bnd}_\pi(M) = \tilde{\text{apr}}_\pi(M) - \text{apr}_\pi(M). \]

Thus the set \(M\) is called definable when \(\text{apr}_\pi(M) = \tilde{\text{apr}}_\pi(M)\); otherwise, it is called RS.

Pawlak’s RS theory uses a single ER. Qian et al. [42] proposed MGRS by using more than one ER, as stated below.

Definition 2: [42] Let \(\pi_1, \pi_2, \ldots, \pi_n\) be \(n\) ERs over \(O\) and \(M \subseteq O\). The lower and upper MGRS-approximations of \(M\) are respectively defined as follows:

\[
\sum_{i=1}^{n} \pi_i(M) = \{x \in O: [x]_{\pi_i} \subseteq M \text{ for some } i, 1 \leq i \leq n\},
\]

(3)

\[
\sum_{i=1}^{n} \pi_i(M) = \{x \in O: [x]_{\pi_i} \cap M \neq \emptyset \text{ for all } i, 1 \leq i \leq n\}.
\]

(4)

The boundary region of \(M\) under MGRS-approximations is given as:

\[
\text{Bnd}_\pi(M) = \sum_{i=1}^{n} \pi_i(M) - \sum_{i=1}^{n} \pi_i(M).
\]

(5)

Definition 3: [37] A SS over \(O\) is an object of the form \((\xi, A)\), where \(\xi: A \rightarrow \mathcal{P}(O)\).

In other words, a SS over \(O\) gives a parameterized collection of subsets of \(O\). For \(\rho \in A\), \(\xi(\rho)\) may be viewed as the collection of \(\rho\)-approximate objects of \(O\) by the SS \((\xi, A)\). A SS over \(O\) may also be denoted as:

\[
(\xi, A) = \{(\rho, \xi(\rho)) : \rho \in A, \xi(\rho) \subseteq \xi(\rho) \subseteq \mathcal{P}(O)\}.
\]

(6)

Definition 4: [33] Let \(A\) be a set of parameters. Then, NOT set of parameters of \(A\) is defined as:

\[
\neg A = \{\neg \rho : \rho \in A\}, \text{ where } \neg \rho = \text{ not } \rho \text{ for } \rho \in A.
\]

Definition 5: [54] A BS over \(O\) is an object of the form \((\xi, \eta: A)\), where \(\xi: A \rightarrow \mathcal{P}(O)\) and \(\eta: \neg A \rightarrow \mathcal{P}(O)\) such that for all \(\rho \in A\), \(\xi(\rho) \cap \eta(\neg \rho) = \emptyset\).

\[
\text{(7)}
\]

Henceforth, \(\text{BSS}_A(O)\) represents the set of all BSs over \(O\) with parameters set \(A\).

Karaaslan and Çağman [25] introduced the notion of BSRs, which is a hybridization of Rs and BSs, which is stated below.

Definition 6: [25] Let \((\xi, \eta: A) \in \text{BSS}_A(O)\). Then a bipolar soft approximation space \((\text{BSRA}_O)\) is an object of the form \(\Omega = (O, (\xi, \eta: A))\). Based on \(\Omega\), we can define the following operators for any \(M \subseteq O\):

\[
\text{S}_{\xi}(M) = \{x \in O: \exists \rho \in A, x \in \xi(\rho) \subseteq M\},
\]

\[
\text{S}_{\eta}(M) = \{x \in O: \exists \neg \rho \in \neg A, x \in \eta(\neg \rho) \subseteq M\},
\]

\[
\text{S}_{\xi \eta}(M) = \{x \in O: \exists \rho \in A, x \in \xi(\rho) \cap \eta(\neg \rho) \subseteq M\},
\]

\[
\text{S}_{\xi \neg \eta}(M) = \{x \in O: \exists \neg \rho \in \neg A, x \in \eta(\neg \rho) \cap \xi(\rho) \subseteq M\}.
\]

(8)

which are said to be soft \(\Omega\)-lower positive, soft \(\Omega\)-lower negative, soft \(\Omega\)-upper positive and soft \(\Omega\)-upper negative approximations of \(M\), respectively. Here \(M^c = O - M\). Generally, the ordered pairs given by:

\[
\text{BPS}_{\Omega}(M) = \left\{(\xi, \eta: A) \in \text{BSS}_A(O) : \text{S}_{\xi}(M) \subseteq \text{S}_{\eta}(M) \right\},
\]

\[
\text{BPS}_{\Omega}(M) = \left\{(\xi, \eta: A) \in \text{BSS}_A(O) : \text{S}_{\xi}(M) \subseteq \text{S}_{\eta}(M) \right\},
\]

\[
\text{BPS}_{\Omega}(M) = \left\{(\xi, \eta: A) \in \text{BSS}_A(O) : \text{S}_{\xi}(M) \subseteq \text{S}_{\eta}(M) \right\},
\]

\[
\text{BPS}_{\Omega}(M) = \left\{(\xi, \eta: A) \in \text{BSS}_A(O) : \text{S}_{\xi}(M) \subseteq \text{S}_{\eta}(M) \right\},
\]

(9)

are regarded as bipolar soft rough approximations (BSR-approximations) of \(M\) w.r.t \(\text{BSA}_A\). When \(\text{BPS}_{\Omega}(M) \neq \emptyset\), then \(M\) is referred to as BSRs; otherwise, it is called bipolar soft definable. The boundary region under BSR-approximations is defined as:

\[
\text{Bnd}_{\Omega}(M) = \left(\text{S}_{\xi}(M) - \text{S}_{\eta}(M), \text{S}_{\eta}(M) - \text{S}_{\xi}(M)\right).
\]

(10)

BSRs model was firstly introduced by Karaaslan and Çağman [25] to handle roughness through BSs which was later modified and enhanced by Shabir and Gul [51] by introducing the notion of MRBSs, which is given as follows:

Definition 7: [51] Let \((\xi, \eta: A) \in \text{BSS}_A(O)\), where \(\xi: A \rightarrow \mathcal{P}(O)\) and \(\eta: \neg A \rightarrow \mathcal{P}(O)\). Construct two different mappings as follows:

\[
\Phi: O \rightarrow \mathcal{P}(O),
\]

\[
\Phi(x) = \{\rho : x \in \xi(\rho)\}.
\]
are named as $\Phi$-lower positive, $\Phi$-upper positive, $\Phi$-lower negative and $\Phi$-upper negative MRBS-approximations of $M$. When $\overline{MBS}\_T(M) \neq \overline{MBS}_T(M)$, then $M$ is called MRBS-definable. The boundary region of the MRBSs is described as:

$$
\text{Bnd}_T(M) = \overline{M}_{\Phi^+} \backslash \overline{M}_{\Phi^+} \cup \overline{M}_{\Phi^-} \backslash \overline{M}_{\Phi^-}.
$$

III. MULTI-GRANULATION MODIFIED ROUGH BIPOLAR SOFT SETS (MGRBSS)

In this section, we extend the notion of the MRBSs to MGRBSSs. Firstly, we discuss the approximations of a set by using two BSs over the universe, i.e., the target concept is described by two granulation spaces. After that, we extend the MRBSs model to MGRBSSs, where the set approximations are characterized by a finite collection of BSs over the universe. Also, we discuss several essential structural properties of MGRBSSs in-depth with some concrete examples.

Definition 8: Let $(\xi_1, \eta_1 : A), (\xi_2, \eta_2 : A) \in \text{BSS}\_A(O)$, where $\xi_1, \xi_2 : A \rightarrow \mathcal{P}(A)$ and $\eta_1, \eta_2 : A \rightarrow \mathcal{P}(A)$. Now construct four different mappings $\Phi_1, \Phi_2, \Psi_1$ and $\Psi_2$ as:

$$
\Phi_1 : O \rightarrow \mathcal{P}(A)
$$

$$
\Phi_1(x) = \{\xi : x \in \xi_1(\xi)\},
$$

$$
\Phi_2 : O \rightarrow \mathcal{P}(A)
$$

$$
\Phi_2(x) = \{\xi : x \in \xi_2(\xi)\},
$$

$$
\Psi_1 : O \rightarrow \mathcal{P}(\neg A)
$$

$$
\Psi_1(x) = \{\neg \xi : x \in \eta_1(\eta)\},
$$

and

$$
\Psi_2 : O \rightarrow \mathcal{P}(\neg A)
$$

$$
\Psi_2(x) = \{\neg \xi : x \in \eta_2(\eta)\} \text{ for all } x \in O.
$$

Then $P = \langle O, (\Phi_1, \Phi_2), (\Psi_1, \Psi_2) \rangle$ is called TGRMBS-$A_\_s$ (two-granulation MRBS-$A_\_s$). For $M \subseteq O$, the lower and the upper two-granulation modified bipolar pairs w.r.t TGRMBS-$A_\_s$ are respectively, defined in the following manner:

$$
\text{TGRMB}_P(M) = \left(\overline{M}(\Phi_1^+, \Phi_2^+), \overline{M}(\Psi_1^+, \Psi_2^+)\right),
$$

$$
\text{TGRMB}_P(M) = \left(\overline{M}(\Phi_1^+, \Phi_2^+), \overline{M}(\Psi_1^+, \Psi_2^+)\right),
$$

where

$$
\overline{M}(\Phi_1^+, \Phi_2^+) = \left\{x \in M : \Phi_1(x) \neq \Phi_1(y) \text{ for all } y \in M^c\right\}
$$

$$
\text{or } \Phi_2(x) \neq \Phi_2(z) \text{ for all } z \in M^c,
$$

$$
\overline{M}(\Phi_1^+, \Phi_2^+) = \left\{x \in O : \Phi_1(x) = \Phi_1(y) \text{ for some } y \in M\right\}
$$

$$
\text{and } \Phi_2(x) = \Phi_2(z) \text{ for some } z \in M,
$$

$$
\overline{M}(\Phi_1^+, \Phi_2^+) = \left\{x \in O : \Psi_1(x) = \Psi_1(y) \text{ for some } y \in M\right\}
$$

$$
\text{and } \Psi_2(x) = \Psi_2(z) \text{ for some } z \in M,
$$

$$
\overline{M}(\Phi_1^+, \Phi_2^+) = \left\{x \in M : \Psi_1(x) \neq \psi_1(y) \text{ for all } y \in M^c\right\}
$$

$$
\text{or } \psi_2(x) \neq \psi_2(z) \text{ for all } z \in M^c.
$$

will be referred to as $\Phi$-lower positive, $\Phi$-upper positive, $\Phi$-lower negative, and $\Phi$-upper negative MRBS-approximations of $M$, respectively.

Definition 9: Let $\text{TGRMB}_P(M)$ and $\text{TGRMB}_P(M)$ be lower and upper two-granulation modified bipolar pairs of $M \subseteq O$ w.r.t $P$, respectively. Then $M$ is said to be TGRMBS w.r.t $P$, if $\text{TGRMB}_P(M) \neq \text{TGRMB}_P(M);\text{ otherwise } M$ is said to be TGRMBS-definable.

The corresponding positive, boundary and negative regions w.r.t TGRMBS-approximations are listed as follows:

- $\text{POS}_{(\Phi, \Psi)}(M) = \left(\overline{M}(\Phi^+, \Phi^+) \cap \overline{M}(\Psi^+, \Psi^+)\right)$

- $\text{BND}_{(\Phi, \Psi)}(M) = \left(\overline{M}(\Phi^+, \Phi^+) \cap \overline{M}(\Psi^+, \Psi^+)\right)$

- $\text{NEG}_{(\Phi, \Psi)}(M) = \left(\overline{M}(\Phi^+, \Phi^+) \cap \overline{M}(\Psi^+, \Psi^+)\right)$

Remark 1: We can see from Definition 9 that $M \subseteq O$ is TGRMBS-definable w.r.t $P$ if and only if $\text{BND}_{(\Phi, \Psi)}(M) = \emptyset$.

Remark 2: It is important to note that, if $(\xi_1, \eta_1 : A) = (\xi_2, \eta_2 : A)$, then (20) degenerates into $\overline{M}_{\Phi^+}$, (21) degenerates into $\overline{M}_{\Phi^+}$, (22) degenerates into $\overline{M}_{\Phi^-}$, and (23) degenerates into $\overline{M}_{\Phi^-}$. They are identical as the $\Phi$-lower positive, $\Phi$-upper positive, $\Psi$-lower negative, and $\Psi$-upper
negative MRBS-approximations of \( M \subseteq O \), respectively as given in [51]. So, the TGMRBSs are a natural generalization of the MRBSs.

**Example 1:** As an illustration, let \((\xi_1, \eta_1 : A), (\xi_2, \eta_2 : A) \in BSS_A(O)\), where \( O = \{u_1, u_2, u_3, u_4, u_5, u_6\} \) and \( A = \{\rho_1, \rho_2, \rho_3, \rho_4\} \). The mappings \( \xi_1, \xi_2, \eta_1 \) and \( \eta_2 \) are described as follows:

\[
\begin{align*}
\xi_1 : A &\rightarrow \mathcal{P}(O), \\
\rho &\mapsto \begin{cases} 
\{u_1, u_6\} & \text{if } \rho = \rho_1, \\
\{u_3\} & \text{if } \rho = \rho_2, \\
\{u_1, u_2, u_5\} & \text{if } \rho = \rho_3, \\
\{u_1, u_5, u_6\} & \text{if } \rho = \rho_4, 
\end{cases} \\
\xi_2 : A &\rightarrow \mathcal{P}(O), \\
\rho &\mapsto \begin{cases} 
\{u_2\} & \text{if } \rho = \rho_1, \\
\{u_3, u_4\} & \text{if } \rho = \rho_2, \\
\{u_1, u_5, u_6\} & \text{if } \rho = \rho_3, \\
\{u_1, u_5, u_6\} & \text{if } \rho = \rho_4, 
\end{cases} \\
\eta_1 : \neg A &\rightarrow \mathcal{P}(O), \\
\neg \rho &\mapsto \begin{cases} 
\{u_3\} & \text{if } \neg \rho = \neg \rho_1, \\
\{u_5\} & \text{if } \neg \rho = \neg \rho_2, \\
\{u_1, u_5, u_6\} & \text{if } \neg \rho = \neg \rho_3, \\
\{u_1, u_2, u_3\} & \text{if } \neg \rho = \neg \rho_4, 
\end{cases} \\
\eta_2 : \neg A &\rightarrow \mathcal{P}(O), \\
\neg \rho &\mapsto \begin{cases} 
\{u_1, u_4\} & \text{if } \neg \rho = \neg \rho_1, \\
\{u_5\} & \text{if } \neg \rho = \neg \rho_2, \\
\{u_1, u_5, u_6\} & \text{if } \neg \rho = \neg \rho_3, \\
\{u_2, u_3\} & \text{if } \neg \rho = \neg \rho_4. 
\end{cases}
\end{align*}
\]

From above mappings, we can get the tabular representation (TABLES 1 and 2) of \((\xi_1, \eta_1 : A)\) and \((\xi_2, \eta_2 : A)\) as follows:

\[
\begin{array}{|c|c|c|c|c|c|c|}
\hline
\xi_1 & u_1 & u_2 & u_3 & u_4 & u_5 & u_6 \\
\hline
\rho_1 & 1 & 0 & 0 & 0 & 0 & 1 \\
\rho_2 & 0 & 0 & 1 & 0 & 0 & 0 \\
\rho_3 & 0 & 0 & 0 & 0 & 0 & 0 \\
\rho_4 & 1 & 1 & 0 & 0 & 1 & 0 \\
\hline
\end{array}
\]

\[
\begin{array}{|c|c|c|c|c|c|c|}
\hline
\eta_1 & u_1 & u_2 & u_3 & u_4 & u_5 & u_6 \\
\hline
\neg \rho_1 & 0 & 0 & 1 & 0 & 0 & 0 \\
\neg \rho_2 & 0 & 0 & 0 & 0 & 1 & 0 \\
\neg \rho_3 & 0 & 1 & 1 & 0 & 0 & 1 \\
\neg \rho_4 & 0 & 0 & 1 & 0 & 0 & 0 \\
\hline
\end{array}
\]

**TABLE 1: Tabular representation of \((\xi_1, \eta_1 : A)\)**

Now, let \( P = \langle O, (\Phi_1, \Phi_2), (\Psi_1, \Psi_2) \rangle \) be the \( TGMRBS \)-approximations...
of $\mathcal{M} = \{u_3, u_4, u_5\} \subseteq \mathcal{M}$ w.r.t $TGMRSBS-A_s$ as:
\[\overline{M}(\phi_i^+, \phi_j^+) = \{u_3, u_4\},\]
\[\overline{M}(\psi_i^+, \psi_j^+) = \{u_3, u_4, u_5\},\]
\[\overline{M}(\phi_i^-, \psi_j^-) = \{u_3, u_4, u_5\}.
Thus, \[TGMBS(M) = \{\{u_3, u_4, u_5\}, \{u_3, u_4, u_5\}, \{u_3, u_4, u_5\}\}.\]

Consequently, $\mathcal{M}$ is $TGMRSBS$, since $\overline{TGMBS}(\mathcal{M}) \neq TGMBS(M)$. Moreover, by direct computations we have:
\[\overline{POS}(\phi_i, \phi_j)(\mathcal{M}) = \{(u_3, u_4), \{u_3, u_4, u_5\}\},\]
\[\overline{BND}(\phi_i, \phi_j)(\mathcal{M}) = \{(u_5), \}\}
\[\overline{NEG}(\phi_i, \phi_j)(\mathcal{M}) = \{(u_1, u_2, u_6), \{u_1, u_2, u_6\}\}.

Remark 3: The relationship between $TGMBS(M)$ and $\overline{TGMBS(M)}$ is $M(\phi_i^+, \phi_j^+) \subseteq \overline{M}(\phi_i^+, \phi_j^+)$ and $\overline{M}(\phi_i^+, \phi_j^+) \supseteq \overline{M}(\phi_i^+, \phi_j^+)$. Proposition 1: Let $(\xi_1, \eta_1 : A), (\xi_2, \eta_2 : A) \in BSSD(A)$, such that $P = \langle \{1, \{\phi_1, \phi_2\}, \{\psi_1, \psi_2\}\rangle$ be a $TGMRSBS-A_s$ and $\mathcal{M} \subseteq \mathcal{O}$. Then,
\[1. \overline{M}(\phi_1^+, \phi_2^+) \subseteq \overline{M}(\phi_1^+, \phi_2^+), \quad \overline{M}(\phi_1^+, \phi_2^+) \subseteq \overline{M}(\phi_1^+, \phi_2^+);
\[2. \overline{M}(\phi_1^+, \phi_2^+) \supseteq \overline{M}(\phi_1^+, \phi_2^+), \quad \overline{M}(\phi_1^+, \phi_2^+) \supseteq \overline{M}(\phi_1^+, \phi_2^+);
\[3. \overline{M}(\phi_1^+, \phi_2^+) \supseteq \overline{M}(\phi_1^+, \phi_2^+), \quad \overline{M}(\phi_1^+, \phi_2^+) \supseteq \overline{M}(\phi_1^+, \phi_2^+);
\[4. \overline{M}(\phi_i^+, \phi_j^+) = \overline{M}(\phi_i^+, \phi_j^+) \cup \overline{M}(\phi_i^+, \phi_j^+) = \overline{M}(\phi_i^+, \phi_j^+) \cap \overline{M}(\phi_i^+, \phi_j^+);
\[5. \overline{M}(\psi_i^+, \psi_j^-) = \overline{M}(\psi_i^+, \psi_j^-) \cup \overline{M}(\psi_i^+, \psi_j^-) = \overline{M}(\psi_i^+, \psi_j^-) \cap \overline{M}(\psi_i^+, \psi_j^-).
Proof 1: Straightforward.
Definition 10: Let $(\xi_1, \eta_1 : A), (\xi_2, \eta_2 : A) \in BSSD(A)$, such that $P = \langle \{1, \{\phi_1, \phi_2\}, \{\psi_1, \psi_2\}\rangle$ be a $TGMRSBS-A_s$ and $\mathcal{M}, \mathcal{N} \subseteq \mathcal{O}$. Then,
\[1. \overline{M}(\phi_1^+, \phi_2^+) \subseteq \overline{M}(\phi_1^+, \phi_2^+) \subseteq \overline{M}(\phi_1^+, \phi_2^+), \quad \overline{M}(\phi_1^+, \phi_2^+) \subseteq \overline{M}(\phi_1^+, \phi_2^+);
\[2. \overline{M}(\phi_1^+, \phi_2^+) \subseteq \overline{M}(\phi_1^+, \phi_2^+) \subseteq \overline{M}(\phi_1^+, \phi_2^+), \quad \overline{M}(\phi_1^+, \phi_2^+) \subseteq \overline{M}(\phi_1^+, \phi_2^+);\]

Proposition 2: Let $(\xi_1, \eta_1 : A), (\xi_2, \eta_2 : A) \in BSSD(A)$, such that $P = \langle \{1, \{\phi_1, \phi_2\}, \{\psi_1, \psi_2\}\rangle$ be a $TGMRSBS-A_s$. Then for any $\mathcal{M}, \mathcal{N} \subseteq \mathcal{O}$, we have
\[1. \mathcal{M} \subseteq \mathcal{N} \implies \overline{M}(\phi_1^+, \phi_2^+) \subseteq \overline{M}(\phi_1^+, \phi_2^+);\]
\[2. \mathcal{M} \subseteq \mathcal{N} \implies \overline{M}(\phi_1^+, \phi_2^+) \subseteq \overline{M}(\phi_1^+, \phi_2^+);\]
\[3. \mathcal{M} \subseteq \mathcal{N} \implies \overline{M}(\phi_1^+, \phi_2^+) \subseteq \overline{M}(\phi_1^+, \phi_2^+);\]
\[4. \mathcal{M} \subseteq \mathcal{N} \implies \overline{M}(\phi_1^+, \phi_2^+) \subseteq \overline{M}(\phi_1^+, \phi_2^+);\]
\[\overline{POS}(\psi_i, \psi_j)(\mathcal{M}) = \{(u_3, u_4), \{u_3, u_4, u_5\}\},\]
\[\overline{NEG}(\psi_i, \psi_j)(\mathcal{M}) = \{(u_1, u_2, u_6), \{u_1, u_2, u_6\}\}.
Here we elaborate an example to demonstrate the preceding proposition.
Example 2: Consider $(\xi_1, \eta_1 : A), (\xi_2, \eta_2 : A) \in BSSD(A)$ as given in Example 1, where $\mathcal{O} = \{u_1, u_2, u_3, u_4, u_5, u_6\}$ and $\mathcal{A} = \{\rho_1, \rho_2, \rho_3, \rho_4\}$. Let $\mathcal{M}, \mathcal{N} \subseteq \mathcal{O}$ be such that $\mathcal{M} = \{u_4, u_5\}$ and $\mathcal{N} = \{u_3, u_4, u_5\}$. Then by direct computation we get:
\[\overline{M}(\phi_1^+, \phi_2^+) = \{u_4\},\]
\[\overline{M}(\phi_1^+, \phi_2^+) = \{u_4, u_5\},\]
\[\overline{M}(\phi_1^+, \phi_2^+) = \{u_4, u_5\}.\]

Similarly,
\[\overline{N}(\phi_1^+, \phi_2^+) = \{u_3, u_4\},\]
\[\overline{N}(\phi_1^+, \phi_2^+) = \{u_3, u_4, u_5\},\]
\[\overline{N}(\phi_1^+, \phi_2^+) = \{u_3, u_4, u_5\}.
Since $\mathcal{M} \subseteq \mathcal{N}$, so one can easily see that $\overline{M}(\phi_1^+, \phi_2^+) \subseteq \overline{N}(\phi_1^+, \phi_2^+), \overline{M}(\phi_1^+, \phi_2^+) \subseteq \overline{N}(\phi_1^+, \phi_2^+)$.\]
Theorem 1: Let $(\xi_1, \eta_1 : A), (\xi_2, \eta_2 : A) \in \text{BSS}_A(O)$, such that $P = \langle \Omega, (\Phi_1, \Phi_2), (\Psi_1, \Psi_2) \rangle$ be a TGMRBS-A and $M \subseteq O$. Then the following properties hold:

1. $\mathcal{M}(\Phi_1^+, \Phi_2^+) \subseteq M \subseteq \overline{\mathcal{M}}(\Phi_1^+, \Phi_2^+);$
2. $\overline{\mathcal{O}}(\Phi_1^+, \Phi_2^+) = \emptyset = \overline{\mathcal{O}}(\Phi_1^+, \Phi_2^+);$
3. $\overline{\mathcal{O}}(\Phi_1^+, \Phi_2^+) = \mathcal{O}(\Phi_1^+, \Phi_2^+);$
4. $\overline{\mathcal{M}}(\Phi_1^+, \Phi_2^+) = \left( (\mathcal{M}^c)(\Phi_1^+, \Phi_2^+) \right)^c;$
5. $\mathcal{M}(\Phi_1^+, \Phi_2^+) = \left( (\mathcal{M}^c)(\Phi_1^+, \Phi_2^+) \right)^c.$

Proof: If $(\xi_1, \eta_1 : A) = (\xi_2, \eta_2 : A)$, then (1) to (5) hold obviously. If $(\xi_1, \eta_1 : A) \neq (\xi_2, \eta_2 : A)$, we prove them as follows:

1. By definition, $\mathcal{M}(\Phi_1^+, \Phi_2^+) \subseteq M$. To prove the other inclusion, let $x \in M \subseteq O$. Then $\Phi_1(x) = \Phi_1(x)$ and $\Phi_2(x) = \Phi_2(x); x \in M$. Thus, $x \in \mathcal{M}(\Phi_1^+, \Phi_2^+);$
2. Obvious from the definition of $\Phi$-lower and upper positive TGMRBS-approximations.

(3)

Since,

$$\overline{\mathcal{O}}(\Phi_1^+, \Phi_2^+) = \left\{ x \in O : \Phi_1(x) \neq \Phi_1(y) \text{ for all } y \in O^c = \emptyset \right\},$$

Thus, $\overline{\mathcal{O}}(\Phi_1^+, \Phi_2^+) = \emptyset = \mathcal{O}(\Phi_1^+, \Phi_2^+);$

(4) Let $x \in \left( (\mathcal{M}^c)(\Phi_1^+, \Phi_2^+) \right)^c = \mathcal{O} = \left( (\mathcal{M}^c)(\Phi_1^+, \Phi_2^+) \right)^c.$

So $x \in O$ and $x \notin \left( (\mathcal{M}^c)(\Phi_1^+, \Phi_2^+) \right)^c$. This gives that $\Phi_1(x) = \Phi_1(y)$ for some $y \in (\mathcal{M}^c)^c = M$ and $\Phi_2(z) = \Phi_2(z)$ for some $z \in (\mathcal{M}^c)^c = M$. Therefore, we have $x \in O : \Phi_1(x) = \Phi_1(y)$ for some $y \in M$ and $\Phi_2(z) = \Phi_2(z)$ for some $z \in M$. So, $x \in \mathcal{M}(\Phi_1^+, \Phi_2^+).$

Consequently, $(\mathcal{M}^c)(\Phi_1^+, \Phi_2^+) \subseteq (\mathcal{M}^c)(\Phi_1^+, \Phi_2^+)$. 

(5) Taking $\mathcal{M}$ in place of $M$ in part (4) we obtain $\mathcal{M}(\Phi_1^+, \Phi_2^+) = \mathcal{M}(\Phi_1^+, \Phi_2^+).$ This implies that $\left( (\mathcal{M}^c)(\Phi_1^+, \Phi_2^+) \right)^c = \mathcal{M}(\Phi_1^+, \Phi_2^+).$

Hence, $\mathcal{M}(\Phi_1^+, \Phi_2^+) = \left( (\mathcal{M}^c)(\Phi_1^+, \Phi_2^+) \right)^c.$

This completes the proof.

Theorem 2: Let $(\xi_1, \eta_1 : A), (\xi_2, \eta_2 : A) \in \text{BSS}_A(O)$, such that $P = \langle \Omega, (\Phi_1, \Phi_2), (\Psi_1, \Psi_2) \rangle$ be a TGMRBS-A and $M \subseteq O$. Then the following properties hold:

1. $\overline{\mathcal{M}}(\Psi_1^+, \Psi_2^+) \subseteq M \subseteq \overline{\mathcal{M}}(\Psi_1^+, \Psi_2^+);$
2. $\overline{\mathcal{O}}(\Psi_1^+, \Psi_2^+) = \emptyset = \overline{\mathcal{O}}(\Psi_1^+, \Psi_2^+);$
3. $\overline{\mathcal{O}}(\Psi_1^+, \Psi_2^+) = \mathcal{O}(\Psi_1^+, \Psi_2^+);$
4. $\overline{\mathcal{M}}(\Psi_1^+, \Psi_2^+) = \left( (\mathcal{M}^c)(\Psi_1^+, \Psi_2^+) \right)^c;$
5. $\mathcal{M}(\Psi_1^+, \Psi_2^+) = \left( (\mathcal{M}^c)(\Psi_1^+, \Psi_2^+) \right)^c.$

Proof: If $(\xi_1, \eta_1 : A) = (\xi_2, \eta_2 : A)$, then (1) to (5) hold obviously. If $(\xi_1, \eta_1 : A) \neq (\xi_2, \eta_2 : A)$, we prove them as follows:

1. By definition, $\overline{\mathcal{M}}(\Psi_1^+, \Psi_2^+) \subseteq M$. To prove the next inclusion, let $x \in M \subseteq O$. Then $\Psi_1(x) = \Psi_1(x)$ and $\Psi_2(z) = \Psi_2(z); x \in M$. Thus, $x \in \overline{\mathcal{M}}(\Psi_1^+, \Psi_2^+).$

(2) Obvious from the definition of $\Psi$-lower and upper negative TGMRBS-approximations.

(3) Since,

$$\overline{\mathcal{O}}(\Psi_1^+, \Psi_2^+) = \left\{ x \in O : \Psi_1(x) \neq \Psi_1(y) \text{ for some } y \in O^c \right\}$$

Thus, $\overline{\mathcal{O}}(\Psi_1^+, \Psi_2^+) = \emptyset$. 

And,

$$\overline{\mathcal{O}}(\Psi_1^+, \Psi_2^+) = \left\{ x \in O : \Psi_1(x) \neq \Psi_1(y) \text{ for all } y \in O^c \right\}$$

Thus, $\overline{\mathcal{O}}(\Psi_1^+, \Psi_2^+) = \emptyset.$

(4) Let $x \in \left( (\mathcal{M}^c)(\Psi_1^+, \Psi_2^+) \right)^c = \mathcal{O} = \left( (\mathcal{M}^c)(\Psi_1^+, \Psi_2^+) \right)^c.$

So $x \in O$ and $x \notin \left( (\mathcal{M}^c)(\Psi_1^+, \Psi_2^+) \right)^c$. This gives that $\Psi_1(x) = \Psi_1(y)$ for all $y \in M^c$ and $\Psi_2(z) = \Psi_2(z)$ for all $z \in M^c$. Thus, $x \in \overline{\mathcal{M}}(\Psi_1^+, \Psi_2^+).$

Consequently, $(\mathcal{M}^c)(\Psi_1^+, \Psi_2^+) \subseteq (\mathcal{M}^c)(\Psi_1^+, \Psi_2^+)$. 

(5) Taking $\mathcal{M}$ in place of $M$ in part (4) we obtain $\mathcal{M}(\Psi_1^+, \Psi_2^+) = \mathcal{M}(\Psi_1^+, \Psi_2^+).$ This implies that $\left( (\mathcal{M}^c)(\Psi_1^+, \Psi_2^+) \right)^c = \mathcal{M}(\Psi_1^+, \Psi_2^+).$

Hence, $\mathcal{M}(\Psi_1^+, \Psi_2^+) = \left( (\mathcal{M}^c)(\Psi_1^+, \Psi_2^+) \right)^c.$

This completes the proof.
Hence, we get $\overline{M}(\psi_1^+, \psi_2^+)^c_c = (M^c)^c_c(\psi_1^+, \psi_2^+)^c_c$.

(5) Using $M^c$ in place of $M$ in part (4), we have $\overline{(M^c)}(\psi_1^+, \psi_2^+)^c_c = (M(\psi_1^+, \psi_2^+))^c_c$. This implies that $\overline{(M^c)}(\psi_1^+, \psi_2^+)^c_c = M(\psi_1^+, \psi_2^+)^c_c$.

Hence, $\overline{M}(\psi_1^+, \psi_2^+)^c_c = (M^c)^c_c(\psi_1^+, \psi_2^+)^c_c$.

This completes the proof.

**Theorem 3:** Let $(\xi_1, \eta_1 : A), (\xi_2, \eta_2 : A) \in BSS_A(O)$, such that $P = \langle O, (\Phi_1, \Psi_1, \Phi_2, \Psi_2) \rangle$ be a TGRMRBS-A$_x$. Then for any $M \subseteq O$, the following properties hold:

1. $\overline{M}(\phi_1^+, \phi_2^+) = M(\phi_1^+, \phi_2^+)$
2. $\overline{M}(\phi_1^+, \phi_2^+) = M(\phi_1^+, \phi_2^+)$
3. $\overline{M}(\phi_1^+, \phi_2^+) = M(\phi_1^+, \phi_2^+)$
4. $\overline{M}(\phi_1^+, \phi_2^+) \subseteq M(\phi_1^+, \phi_2^+)$

**Proof 5:** If $(\xi_1, \eta_1 : A) = (\xi_2, \eta_2 : A)$, then (1) to (4) hold obviously. If $(\xi_1, \eta_1 : A) \neq (\xi_2, \eta_2 : A)$, we prove them as follows:

(1) According to part (1) of Theorem 1, $M^c(\phi_1^+, \phi_2^+) \subseteq M(\phi_1^+, \phi_2^+)$.

(2) For all $x \in M(\phi_1^+, \phi_2^+)$, we have $x \in M^c(\phi_1^+, \phi_2^+)$.

(3) $\overline{M}(\phi_1^+, \phi_2^+) \subseteq M(\phi_1^+, \phi_2^+)$.

(4) Clear from part (1) of Theorem 1.

This completes the proof.

The next example demonstrates that the containment in part (4) of preceding theorem might be strict.

**Example 3:** If we take $M = \{u_3, u_4, u_5\}$ in Example 1, then $\overline{M}(\phi_1^+, \phi_2^+) = \{u_3, u_4, u_5\}$.

Thus, $\overline{M}(\phi_1^+, \phi_2^+) \subset M(\phi_1^+, \phi_2^+)$, which demonstrates that the containment in part (4) of Theorem 3 might be strict.

**Theorem 4:** Let $(\xi_1, \eta_1 : A), (\xi_2, \eta_2 : A) \in BSS_A(O)$, such that $P = \langle O, (\Phi_1, \Psi_1, \Phi_2, \Psi_2) \rangle$ be a TGRMRBS-A$_x$ and $M \subseteq \Phi$. Then the following properties hold:

1. $\overline{M}(\psi_1^+, \psi_2^+) = M(\psi_1^+, \psi_2^+)$.
2. $\overline{M}(\psi_1^+, \psi_2^+) = M(\psi_1^+, \psi_2^+)$.
3. $\overline{M}(\psi_1^+, \psi_2^+) = M(\psi_1^+, \psi_2^+)$.
4. $\overline{M}(\psi_1^+, \psi_2^+) \subseteq M(\psi_1^+, \psi_2^+)$.

**Proof 6:** If $(\xi_1, \eta_1 : A) = (\xi_2, \eta_2 : A)$, then (1) to (4) hold obviously. If $(\xi_1, \eta_1 : A) \neq (\xi_2, \eta_2 : A)$, we prove them as follows:
According to part (1) of Theorem 2, \( \overline{M(\Psi_1^-, \Psi_2^-)} \subseteq \left( \overline{M(\Psi_1^-, \Psi_2^-)} \right)_{(\Psi_1^-, \Psi_2^-)} \).

Conversely, let \( x \in \left( \overline{M(\Psi_1^- \Psi_2^-)} \right)_{(\Psi_1^-, \Psi_2^-)} \). Then according to Definition 8, \( x \in O : \Psi_1(x) = \Psi_1(y) \) for some \( y \in \overline{M(\Psi_1^-, \Psi_2^-)} \) and \( \Psi_2(x) = \Psi_2(z) \) for some \( z \in \overline{M(\Psi_1^-, \Psi_2^-)} \). According to definition of \( \Psi \)-upper negative TGMRBSS-approximation of \( M \), \( y \in M : \Psi_1(y) \neq \Psi_1(x) \) for all \( n \in M^c \) or \( z \in M : \Psi_2(z) \neq \Psi_2(x) \) for all \( n \in M^c \). As \( \Psi_1(x) = \Psi_1(y) \) and \( \Psi_2(x) = \Psi_2(z) \), we have \( \Psi_1(x) \neq \Psi_1(y) \) for all \( n \in M^c \) or \( \Psi_2(x) \neq \Psi_2(z) \) for all \( n \in M^c \). This indicates that \( x \notin M^c \). Therefore, \( x \in M \) and so, \( x \in \overline{M(\Psi_1^-, \Psi_2^-)} \).

Hence, we get \( \left( \overline{M(\Psi_1^-, \Psi_2^-)} \right)_{(\Psi_1^-, \Psi_2^-)} = \overline{M(\Psi_1^-, \Psi_2^-)} \).

(4) It trivially follows from part (1) of Theorem 2. This completes the proof.

The next example indicates that the containment in part (4) of preceding theorem might be strict.

Example 4: If we take \( M = \{u_3, u_4, u_5\} \) in Example 1, then

\[ \overline{M(\Psi_1^-, \Psi_2^-)} = \{u_3, u_4, u_5\}, \]

Thus, \( \left( \overline{M(\Psi_1^-, \Psi_2^-)} \right)_{(\Psi_1^-, \Psi_2^-)} \subseteq \overline{M(\Psi_1^-, \Psi_2^-)} \), which demonstrates that the containment in part (4) of Theorem 4 may hold strictly.

Remark 4: From parts (1) and (2) of Theorem 3 and Theorem 4, we observed that \( M(\Psi_1^+, \Phi_2^+) \) and \( M(\Psi_1^-, \Phi_2^-) \) are definable in \( P = \langle O, \Phi_1, \Phi_2, (\Psi_1, \Psi_2) \rangle \). Furthermore, \( \Phi \)-lower and upper positive TGMRBSS-approximation of \( M(\Phi_1, \Phi_2) \) and \( \Psi \)-lower and upper negative TGMRBSS-approximations \( M(\Psi_1, \Psi_2) \) w.r.t \( P \) are invariant.

Theorem 5: Let \( (\xi_1, \eta_1 : A), (\xi_2, \eta_2 : A) \) be BSS \( A(O) \), such that \( P = \langle O, (\Phi_1, \Phi_2), (\Psi_1, \Psi_2) \rangle \) be a TGMRBSS-\( A \), and \( M, N \subseteq O \). Then, the following properties hold:

(1) \( \left( M \cap N \right)_{(\Phi_1^+, \Phi_2^+)} \subseteq \left( M_{(\Phi_1^+, \Phi_2^+)} \right) \cap \left( N_{(\Phi_1^+, \Phi_2^+)} \right) \).

(2) \( \left( M \cap N \right)_{(\Phi_1^+, \Phi_2^-)} \subseteq \left( M_{(\Phi_1^+, \Phi_2^-)} \right) \cap \left( N_{(\Phi_1^+, \Phi_2^-)} \right) \).

(3) \( \left( M \cup N \right)_{(\Phi_1^+, \Phi_2^+)} \subseteq \left( M_{(\Phi_1^+, \Phi_2^+)} \right) \cup \left( N_{(\Phi_1^+, \Phi_2^+)} \right) \).

(4) \( \left( M \cup N \right)_{(\Phi_1^+, \Phi_2^-)} \subseteq \left( M_{(\Phi_1^+, \Phi_2^-)} \right) \cup \left( N_{(\Phi_1^+, \Phi_2^-)} \right) \).

Proof: If \( (\xi_1, \eta_1 : A) = (\xi_2, \eta_2 : A) \), then (1) to (4) hold obviously. If \( (\xi_1, \eta_1 : A) \neq (\xi_2, \eta_2 : A) \), we prove them as follows:

(1) Since \( M \cap N \subseteq M \) and \( M \cap N \subseteq N \), so according to part (1) of Proposition 2, \( \left( M \cap N \right)_{(\Phi_1^+, \Phi_2^+)} \subseteq \left( M_{(\Phi_1^+, \Phi_2^+)} \right) \) and \( \left( M \cap N \right)_{(\Phi_1^+, \Phi_2^-)} \subseteq \left( M_{(\Phi_1^+, \Phi_2^-)} \right) \). Thus, \( \left( M \cap N \right)_{(\Phi_1^+, \Phi_2^+)} \subseteq \left( M_{(\Phi_1^+, \Phi_2^+)} \right) \) and \( \left( M \cap N \right)_{(\Phi_1^+, \Phi_2^-)} \subseteq \left( M_{(\Phi_1^+, \Phi_2^-)} \right) \).

(2) As \( M \cap N \subseteq M \) and \( M \cap N \subseteq N \), therefore, according to part (2) of Proposition 2, \( \left( M \cap N \right)_{(\Phi_1^+, \Phi_2^+)} \subseteq \left( M_{(\Phi_1^+, \Phi_2^+)} \right) \) and \( \left( M \cap N \right)_{(\Phi_1^+, \Phi_2^-)} \subseteq \left( M_{(\Phi_1^+, \Phi_2^-)} \right) \). Therefore, \( \left( M \cap N \right)_{(\Phi_1^+, \Phi_2^+)} \subseteq \left( M \cap N \right)_{(\Phi_1^+, \Phi_2^+)} \).

(3) Since \( M \subseteq M \cup N \) and \( N \subseteq M \cup N \), so by part (1) of Proposition 2, \( M_{(\Phi_1^+, \Phi_2^+)} \subseteq \left( M \cup N \right)_{(\Phi_1^+, \Phi_2^+)} \) and \( N_{(\Phi_1^+, \Phi_2^-)} \subseteq \left( M \cup N \right)_{(\Phi_1^+, \Phi_2^-)} \). Hence, \( M_{(\Phi_1^+, \Phi_2^+)} \subseteq \left( M \cup N \right)_{(\Phi_1^+, \Phi_2^+)} \) and \( N_{(\Phi_1^+, \Phi_2^-)} \subseteq \left( M \cup N \right)_{(\Phi_1^+, \Phi_2^-)} \).

(4) Let \( x \in \left( \overline{M(\Psi_1^+, \Phi_2^+)} \right) \). Then according to Definition 8, \( x \in O : \Phi_1(x) = \Phi_1(y) \) for some \( y \in \left( \overline{M(\Psi_1^+, \Phi_2^+)} \right) \) and \( \Phi_2(x) = \Phi_2(z) \) for some \( z \in \left( \overline{M(\Psi_1^+, \Phi_2^+)} \right) \). This gives \( x \in \left( \overline{M(\Psi_1^+, \Phi_2^+)} \right) \).
\[ \mathcal{M}(\Phi^+, \Phi^+) \cup \mathcal{N}(\Phi^+, \Phi^+) \subset \mathcal{M}(\Phi^+, \Phi^+) \cup \mathcal{N}(\Phi^+, \Phi^+) \]

Conversely, as \( M \subseteq \mathcal{M} \cup \mathcal{N} \) and \( N \subseteq \mathcal{M} \cup \mathcal{N} \). Therefore according to part (2) of Proposition 2, \( \mathcal{M}(\Phi^+, \Phi^+) \subseteq (\mathcal{M} \cup \mathcal{N})(\Phi^+, \Phi^+) \) and \( \mathcal{N}(\Phi^+, \Phi^+) \subseteq (\mathcal{M} \cup \mathcal{N})(\Phi^+, \Phi^+) \).

Thus, \( \mathcal{M}(\Phi^+, \Phi^+) \cup \mathcal{N}(\Phi^+, \Phi^+) \subseteq (\mathcal{M} \cup \mathcal{N})(\Phi^+, \Phi^+) \).

Hence, \( (\mathcal{M} \cup \mathcal{N})(\Phi^+, \Phi^+) = \mathcal{M}(\Phi^+ \cup \Phi^+) \cup \mathcal{N}(\Phi^+, \Phi^+) \). This completes the proof.

The containment in parts (1), (2) and (3) of above theorem may be proper. This can be seen in the next example.

**Example 5:** Let \((\xi_1, \eta_1 : A), (\xi_2, \eta_2 : A) \in BSS_A(O)\), where \( O = \{u_1, u_2, u_3, u_4, u_5, u_6, u_7\} \) and \( A = \{\rho_1, \rho_2, \rho_3\} \). The mappings \( \xi_1, \eta_1, \xi_2, \eta_2 \) are given as follows:

\[
\begin{align*}
\xi_1 : A &\rightarrow \mathcal{P}(O), \\
\rho &\mapsto \begin{cases} 
\{u_1, u_4, u_6, u_7\} & \text{if } \rho = \rho_1, \\
\{u_1, u_2, u_3\} & \text{if } \rho = \rho_2, \\
\{u_5, u_6\} & \text{if } \rho = \rho_3.
\end{cases} \\
\eta_1 : \neg A &\rightarrow \mathcal{P}(O), \\
\neg \rho &\mapsto \begin{cases} 
\{u_2, u_3\} & \text{if } \neg \rho = \neg \rho_1, \\
\{u_4, u_6, u_7\} & \text{if } \neg \rho = \neg \rho_2, \\
\{u_1, u_3\} & \text{if } \neg \rho = \neg \rho_3.
\end{cases}
\end{align*}
\]

\[
\begin{align*}
\xi_2 : A &\rightarrow \mathcal{P}(O), \\
\rho &\mapsto \begin{cases} 
\{u_1, u_2, u_3, u_4, u_5, u_7\} & \text{if } \rho = \rho_1, \\
\{u_1, u_4, u_7\} & \text{if } \rho = \rho_2, \\
\{u_2, u_5, u_6\} & \text{if } \rho = \rho_3, \\
\end{cases} \\
\eta_2 : \neg A &\rightarrow \mathcal{P}(O), \\
\neg \rho &\mapsto \begin{cases} 
\{\} & \text{if } \neg \rho = \neg \rho_1, \\
\{u_3, u_5, u_6\} & \text{if } \neg \rho = \neg \rho_2, \\
\{u_1, u_7\} & \text{if } \neg \rho = \neg \rho_3.
\end{cases}
\end{align*}
\]

Tabular representation (TABLES 3 and 4) of \((\xi_1, \eta_1 : A)\) and \((\xi_2, \eta_2 : A)\) is given as:

**TABLE 3: Tabular representation of \((\xi_1, \eta_1 : A)\)**

| \(\xi_1\) | \(u_1\) | \(u_2\) | \(u_3\) | \(u_4\) | \(u_5\) | \(u_6\) | \(u_7\) |
|---|---|---|---|---|---|---|---|
| \(\rho_1\) | 1 | 1 | 0 | 0 | 1 | 1 | 1 |
| \(\rho_2\) | 1 | 1 | 1 | 0 | 0 | 0 | 0 |
| \(\rho_3\) | 0 | 0 | 0 | 1 | 0 | 0 | 1 |

**TABLE 4: Tabular representation of \((\xi_2, \eta_2 : A)\)**

Now, assume that \(P = \langle O, (\Phi_1, \Phi_2), (\Psi_1, \Psi_2) \rangle\) be a TGMRBS-\(A_s\). Then by above tables, we can construct \(\Phi_1, \Phi_2, \Psi_1 \) and \(\Psi_2\) as:

\[
\begin{align*}
\Phi_1 : O &\rightarrow \mathcal{P}(A), \\
\rho &\mapsto \begin{cases} 
\{\rho_1, \rho_2\} & \text{if } u = u_1, \\
\{\rho_2\} & \text{if } u = u_2, \\
\{\rho_2\} & \text{if } u = u_4, \\
\{\rho_1\} & \text{if } u = u_3, \\
\{\rho_3\} & \text{if } u = u_5, \\
\{\rho_1, \rho_3\} & \text{if } u = u_6, \\
\{\rho_1\} & \text{if } u = u_7.
\end{cases}
\end{align*}
\]

\[
\begin{align*}
\Psi_1 : O &\rightarrow \mathcal{P}(\neg A), \\
\rho &\mapsto \begin{cases} 
\{\neg \rho_3\} & \text{if } u = u_1, \\
\{\neg \rho_1\} & \text{if } u = u_2, \\
\{\neg \rho_1, \neg \rho_3\} & \text{if } u = u_3, \\
\{\neg \rho_2\} & \text{if } u = u_4, \\
\{\neg \rho_1\} & \text{if } u = u_5, \\
\neg \rho_1 & \text{if } u = u_6, \\
\{\neg \rho_1\} & \text{if } u = u_7.
\end{cases}
\end{align*}
\]

\[
\begin{align*}
\Phi_2 : O &\rightarrow \mathcal{P}(A), \\
\rho &\mapsto \begin{cases} 
\{\rho_1, \rho_2\} & \text{if } u = u_1, \\
\{\rho_1\} & \text{if } u = u_2, \\
\{\rho_1, \rho_3\} & \text{if } u = u_3, \\
\{\rho_1\} & \text{if } u = u_4, \\
\{\rho_1, \rho_3\} & \text{if } u = u_5, \\
\{\rho_3\} & \text{if } u = u_6, \\
\{\rho_1, \rho_2\} & \text{if } u = u_7.
\end{cases}
\end{align*}
\]

\[
\begin{align*}
\Psi_2 : O &\rightarrow \mathcal{P}(\neg A), \\
\rho &\mapsto \begin{cases} 
\{\neg \rho_3\} & \text{if } u = u_1, \\
\{\neg \rho_2\} & \text{if } u = u_2, \\
\{\neg \rho_2\} & \text{if } u = u_4, \\
\neg \rho_2 & \text{if } u = u_5, \\
\{\neg \rho_2\} & \text{if } u = u_6, \\
\{\neg \rho_3\} & \text{if } u = u_7.
\end{cases}
\end{align*}
\]
Let $\mathcal{M}, \mathcal{N} \subseteq \mathcal{O}$ be such that $\mathcal{M} = \{u_1, u_2, u_3, u_4, u_7\}$ and $\mathcal{N} = \{u_1, u_2, u_5\}$ then $\mathcal{M} \cap \mathcal{N} = \{u_1, u_2\}$. Then,

$$\mathcal{M} \cap \mathcal{N} = \{u_1, u_2, u_3, u_4, u_7\},$$

$$\mathcal{N} = \{u_1, u_2, u_5\},$$

$$\big(\mathcal{M} \cap \mathcal{N}\big)_{(\xi^+, \xi^-)} = \{u_1\}.$$ 

Clearly, $\big(\mathcal{M} \cap \mathcal{N}\big)_{(\xi^+, \xi^-)} = \{u_1\} \subseteq \{u_1, u_2\} = \mathcal{M} \cap \mathcal{N}$, which shows that the containment in part (1) of Theorem 5 may hold strictly.

Now, assuming $\mathcal{M}_1 = \{u_1, u_2, u_3, u_7\}$ and $\mathcal{N}_1 = \{u_3, u_4\}$, then $\mathcal{M}_1 \cap \mathcal{N}_1 = \{u_3\}$. Then,

$$\mathcal{M}_1 \cap \mathcal{N}_1 = \{u_1, u_2, u_3, u_4, u_7\},$$

$$\mathcal{N}_1 = \{u_3, u_4\},$$

$$\big(\mathcal{M}_1 \cap \mathcal{N}_1\big)_{(\xi^+, \xi^-)} = \{u_3\}.$$ 

Clearly, $\big(\mathcal{M}_1 \cap \mathcal{N}_1\big)_{(\xi^+, \xi^-)} = \{u_3\} \subset \{u_3, u_4\} = \mathcal{M}_1 \cap \mathcal{N}_1$, which demonstrates that the containment in part (2) of Theorem 5 may hold strictly.

Now, if we take $\mathcal{M}_2 = \{u_1, u_2, u_4\}$ and $\mathcal{N}_2 = \{u_3, u_6\}$, then $\mathcal{M}_2 \cup \mathcal{N}_2 = \{u_1, u_2, u_3, u_4, u_6\}$. Then,

$$\mathcal{M}_2 \cup \mathcal{N}_2 = \{u_1, u_2, u_3, u_4, u_6\},$$

$$\mathcal{N}_2 = \{u_3, u_6\},$$

$$\big(\mathcal{M}_2 \cup \mathcal{N}_2\big)_{(\xi^+, \xi^-)} = \{u_1, u_2, u_3, u_4, u_6\}.$$ 

Clearly, $\big(\mathcal{M}_2 \cup \mathcal{N}_2\big)_{(\xi^+, \xi^-)} = \{u_1, u_2, u_3, u_4, u_6\} \supset\{u_1, u_2, u_3, u_4, u_6\} = \mathcal{M}_2 \cup \mathcal{N}_2$, which indicates that the containment in part (3) of Theorem 5 might be strict.

Theorem 6: Let $(\xi_1, \eta_1 : A), (\xi_2, \eta_2 : A) \in \text{BSS}_{\lambda}(\mathcal{O})$, such that $\mathcal{P} = \langle \mathcal{O}, (\Phi_1, \Phi_2), (\Psi_1, \Psi_2) \rangle$ be a TGMRBS-$\lambda$ and $\mathcal{M}, \mathcal{N} \subseteq \mathcal{O}$. Then, the following properties hold:

1. $\mathcal{M} \cap \mathcal{N} \subseteq \mathcal{M}$ and $\mathcal{M} \cap \mathcal{N} \subseteq \mathcal{N}$, so by part (3) of Proposition 2, $\mathcal{M} \cap \mathcal{N} \subseteq \mathcal{M} \cap \mathcal{N}$.

2. As $\mathcal{M} \cap \mathcal{N} \subseteq \mathcal{M}$ and $\mathcal{M} \cap \mathcal{N} \subseteq \mathcal{N}$, therefore according to part (4) of Proposition 2, $\mathcal{M} \cap \mathcal{N} \subseteq \mathcal{M} \cap \mathcal{N}$.

3. Let $x \in \mathcal{M} \cap \mathcal{N}$, then $x \in \mathcal{O}$ such that $\Psi_1(x) = \Psi_1(y)$ for some $y \in \mathcal{M}$ and $\Psi_2(x) = \Psi_2(z)$ for some $z \in \mathcal{N}$. This gives $\{x \in \mathcal{O} : \Psi_1(x) = \Psi_1(y) \text{ for some } y \in \mathcal{M} \text{ and } \Psi_2(x) = \Psi_2(z) \text{ for some } z \in \mathcal{N}\} = \{x \in \mathcal{O} : x \in \mathcal{M} \cap \mathcal{N}\}$.

The Next Example demonstrates that the containment in parts (1), (2), and (4) of the preceding Theorem may hold strictly.

Example 6: Let $(\xi_1, \eta_1 : A), (\xi_2, \eta_2 : A) \in \text{BSS}_{\lambda}(\mathcal{O})$, where $\mathcal{O} = \{u_1, u_2, u_3, u_4, u_5, u_6, u_7\}$ and $A = \{\rho_1, \rho_2, \rho_3\}$. The mappings $\xi_1, \xi_2, \eta_1$ and $\eta_2$ are given as follows:

$$\xi_1 : A \rightarrow \mathcal{P}(\mathcal{O}),$$

$$\rho_1 \mapsto \{u_2, u_3\} \text{ if } \rho_1 = \rho_1,$$

$$\{u_4, u_5, u_7\} \text{ if } \rho_1 = \rho_2,$$

$$\{u_1, u_3\} \text{ if } \rho_1 = \rho_3,$$

$$\eta_1 : \mathcal{P}(\mathcal{O}) \rightarrow \mathcal{O},$$

$$\bar{\rho}_1 \mapsto \{u_1, u_4, u_6, u_7\} \text{ if } \bar{\rho}_1 = \bar{\rho}_1,$$

$$\{u_1, u_2, u_3\} \text{ if } \bar{\rho}_1 = \bar{\rho}_2,$$

$$\{u_5, u_6\} \text{ if } \bar{\rho}_1 = \bar{\rho}_3.$$
TABLE 5: Tabular representation of ($\xi_1, \eta_1 : A$)

(a)

| $\xi_1$ | $u_1$ | $u_2$ | $u_3$ | $u_4$ | $u_5$ | $u_6$ | $u_7$ |
|---------|-------|-------|-------|-------|-------|-------|-------|
| $\rho_1$ | 0     | 1     | 1     | 0     | 0     | 0     | 0     |
| $\rho_2$ | 0     | 0     | 0     | 1     | 0     | 1     | 1     |
| $\rho_3$ | 1     | 0     | 1     | 0     | 0     | 0     | 0     |

(b)

| $\eta_1$ | $u_1$ | $u_2$ | $u_3$ | $u_4$ | $u_5$ | $u_6$ | $u_7$ |
|----------|-------|-------|-------|-------|-------|-------|-------|
| $\neg \rho_1$ | 1     | 0     | 0     | 1     | 0     | 1     | 1     |
| $\neg \rho_2$ | 1     | 1     | 1     | 0     | 0     | 0     | 0     |
| $\neg \rho_3$ | 0     | 0     | 0     | 0     | 1     | 1     | 0     |

TABLE 6: Tabular representation of ($\xi_2, \eta_2 : A$)

(a)

| $\xi_2$ | $u_1$ | $u_2$ | $u_3$ | $u_4$ | $u_5$ | $u_6$ | $u_7$ |
|---------|-------|-------|-------|-------|-------|-------|-------|
| $\rho_1$ | 0     | 0     | 0     | 0     | 0     | 0     | 0     |
| $\rho_2$ | 0     | 0     | 1     | 0     | 1     | 1     | 0     |
| $\rho_3$ | 1     | 0     | 0     | 0     | 0     | 0     | 1     |

(b)

| $\eta_2$ | $u_1$ | $u_2$ | $u_3$ | $u_4$ | $u_5$ | $u_6$ | $u_7$ |
|----------|-------|-------|-------|-------|-------|-------|-------|
| $\neg \rho_1$ | 1     | 1     | 1     | 1     | 1     | 0     | 1     |
| $\neg \rho_2$ | 1     | 0     | 0     | 1     | 0     | 0     | 1     |
| $\neg \rho_3$ | 0     | 1     | 0     | 0     | 1     | 1     | 0     |

If we consider $M, N \subseteq O$ such that $M = \{u_1, u_2, u_3, u_7\}$ and $N = \{u_3, u_4\}$ then $M \cap N = \{u_3\}$. Then,

$$\Phi_1 : O \rightarrow \mathcal{P}(A),$$

$$u \mapsto \begin{cases} 
\{\rho_3\} & \text{if } u = u_1, \\
\{\rho_1\} & \text{if } u = u_2, \\
\{\rho_1, \rho_3\} & \text{if } u = u_3, \\
\{\rho_2\} & \text{if } u = u_4, \\
\{\rho_1\} & \text{if } u = u_5, \\
\{\rho_1\} & \text{if } u = u_6, \\
\{\rho_1\} & \text{if } u = u_7.
\end{cases}$$

$$\Phi_2 : O \rightarrow \mathcal{P}(\neg A),$$

$$u \mapsto \begin{cases} 
\{\neg \rho_1, \neg \rho_2\} & \text{if } u = u_1, \\
\{\neg \rho_2\} & \text{if } u = u_2, \\
\{\neg \rho_2\} & \text{if } u = u_3, \\
\{\neg \rho_1\} & \text{if } u = u_4, \\
\{\neg \rho_3\} & \text{if } u = u_5, \\
\{\neg \rho_1, \neg \rho_3\} & \text{if } u = u_6, \\
\{\neg \rho_1\} & \text{if } u = u_7.
\end{cases}$$

Remark 5: It is worth noticing that, in MRBSs [51]:

1. When $\Phi(u) = \Phi(v)$ for some $u, v \in O$, then for any $M \subseteq O$, either $u, v \in \mathcal{M}_{\Phi^+}$ or $u \notin \mathcal{M}_{\Phi^+}$.
2. When $\Psi(u) = \Psi(v)$ for some $u \in O$, then for any $M \subseteq O$, either $u, v \in \mathcal{M}_{\Psi^+}$ or $u \notin \mathcal{M}_{\Psi^-}$.

But in MGRBSs, the above two conditions are not true. For instance, in Example 1 we can see that for $M =$
Clearly, in case of MRBSs $\Phi_1(u_2) = \Phi_1(u_5)$ implies $u_2, u_5 \in \mathcal{M}_{\Phi_1}$. Similarly, $\Phi_2(u_1) = \Phi_2(u_5)$ implies $u_1, u_5, u_6 \in \mathcal{M}_{\Phi_2}$. But in case of MGMRBSs, only $u_5 \in \mathcal{M}_{(\Phi_1, \Phi_2)}$. Also,

$$\mathcal{M}_{\Phi_1} = \{u_1, u_3, u_4, u_5\},$$

$$\mathcal{M}_{\Phi_2} = \{u_2, u_3, u_4, u_5\},$$

$$\mathcal{M}_{(\Phi_1, \Phi_2)} = \{u_3, u_4, u_5\}.$$

Clearly, in case of MRBSs $\Psi_1(u_1) = \Psi_1(u_4)$ implies $u_1, u_4 \in \mathcal{M}_{\Psi_1}$. Similarly, $\Psi_2(u_2) = \Psi_2(u_3)$ implies $u_2, u_3 \in \mathcal{M}_{\Psi_2}$. But in case of MGMRBSs, only $u_3, u_4 \in \mathcal{M}_{(\Psi_1, \Psi_2)}$.

Based on the above discussions, now we extend the MRBSs model to a MGMRBSs, where the set approximations are defined by a finite collection of BSs over the universe $\mathcal{O}$.

**Definition 11**: Let $(\xi_i, \eta_i : A)_{i \in I}$ be a finite collection of BSs over $\mathcal{O}$, where $\xi_i : A \rightarrow \mathcal{P}(\mathcal{O})$ and $\eta_i : A \rightarrow \mathcal{P}(\mathcal{O})$. Construct the mappings $\Phi_i$ and $\Psi_i$ as follows:

$$\Phi_i : \mathcal{O} \rightarrow \mathcal{P}(A)$$

$$\Phi_i(x) = \{\rho : x \in \xi_i(\rho)\} \text{ for all } x \in \mathcal{O}, i \in I,$$

and

$$\Psi_i : \mathcal{O} \rightarrow \mathcal{P}(\neg A)$$

$$\Psi_i(x) = \{\neg \rho : x \in \eta_i(\neg \rho)\} \text{ for all } x \in \mathcal{O}, i \in I.$$

Then $\mathcal{P} = \langle \mathcal{O}, (\Phi_i, \Psi_i) \rangle_{i \in I}$ is called MGMRBS-$A$, (multi-granulation MRBSA$_s$). For any $\mathcal{M} \subseteq \mathcal{O}$, the lower and upper multi-granulation modified bipolar pairs are respectively defined as follows:

$$\text{MGMB}_B^-(\mathcal{M}) = \left(\sum_{i \in I} \mathcal{M}_{\Phi_i^+}, \left(\sum_{i \in I} \mathcal{M}_{\Phi_i^-}\right)\right),$$

$$\text{MGMB}_B^+(\mathcal{M}) = \left(\sum_{i \in I} \mathcal{M}_{\Phi_i^+}, \left(\sum_{i \in I} \mathcal{M}_{\Phi_i^-}\right)\right),$$

where

$$\left(\sum_{i \in I} \mathcal{M}_{\Phi_i^+}\right) = \bigcup_{i \in I}\left\{x \in \mathcal{M} : \Phi_i(x) \neq \Phi_i(y) \text{ for all } y \in \mathcal{M}^c\right\},$$

$$\left(\sum_{i \in I} \mathcal{M}_{\Phi_i^-}\right) = \bigcap_{i \in I}\left\{x \in \mathcal{O} : \Phi_i(x) = \Phi_i(y) \text{ for some } y \in \mathcal{M}\right\}.$$
Theorem 8: Let \((\xi_i, \eta_i : A_i)_{i \in I}\) be a finite collection of BSs over \(O\), such that \(\hat{P} = \langle O, (\Phi_i, \Psi_i) \rangle \) be a MGRMRS-As and \(M, N \subseteq O\). Then, 
1. \((\sum_{i \in I} M) \psi^- \subseteq M \leq (\sum_{i \in I} M) \psi^- ; \\
2. (\sum_{i \in I} \emptyset) \psi^- = \emptyset = (\sum_{i \in I} \emptyset) \psi^- ; \\
3. (\sum_{i \in I} \emptyset) \psi^- = O = (\sum_{i \in I} \emptyset) \psi^- ; \\
4. (\sum_{i \in I} M) \psi^- = (\sum_{i \in I} (M'))^c \psi^- ; \\
5. (\sum_{i \in I} M) \psi^- = (\sum_{i \in I} (M'))^c \psi^- ; \\
6. \(M \subseteq N \implies (\sum_{i \in I} M) \psi^- \leq (\sum_{i \in I} N) \psi^- \) and 
   \( (\sum_{i \in I} M) \psi^- \leq (\sum_{i \in I} N) \psi^- ; \\
7. \(\sum_{i \in I} M \psi^- = (\sum_{i \in I} M) \psi^- ; \\
8. (\sum_{i \in I} M) \psi^- = (\sum_{i \in I} M) \psi^- ; \\
9. (\sum_{i \in I} M) \psi^- = (\sum_{i \in I} M) \psi^- ; \\
10. (\sum_{i \in I} M \psi^- \subseteq (\sum_{i \in I} M) \psi^- ; \\
11. (\sum_{i \in I} (M \cap N)) \psi^- \leq (\sum_{i \in I} M) \psi^- \cap (\sum_{i \in I} N) \psi^- ; \\
12. (\sum_{i \in I} (M \cap N)) \psi^- \leq (\sum_{i \in I} M) \psi^- \cap (\sum_{i \in I} N) \psi^- ; \\
13. (\sum_{i \in I} (M \cup N)) \psi^- = (\sum_{i \in I} M) \psi^- \cup (\sum_{i \in I} N) \psi^- ; \\
14. (\sum_{i \in I} (M \cup N)) \psi^- \geq (\sum_{i \in I} M) \psi^- \cup (\sum_{i \in I} N) \psi^- .

Proof 10: It can be directly obtained from Definition 11.

IV. SEVERAL MEASURES IN MGRMRRS

Pawlak proposed two statistical measures for quantifying the imprecision of RS approximations in [41], which might help us obtain an idea of how precise information relating to some equivalence relation for a particular classification is. Generally, the existence of a boundary area causes a set’s uncertainty. The greater the set’s boundary region, the lesser the set’s accuracy.

According to Pawlak [41], the accuracy measure of \(M \subseteq O\) w.r.t \(A_s\) is defined as:

\[
A_s(M) = \frac{\text{apr}_s(M)}{\text{apr}_s(M)} ,
\]

which is designed to convey the completeness of the information regarding set \(M\). Here \(|\bullet|\) denotes the set’s cardinality. Similarly, the roughness measure is defined as:

\[
R_s(M) = 1 - A_s(M) ,
\]

which is viewed as the incompleteness of information regarding the set \(M\).

In the present section, we introduce some measures for MGRMRRSs.

**Definition 12:** Let \((\xi_i, \eta_i : A_i)_{i \in I}\) be a finite collection of BSs over \(O\) and \(\hat{P} = \langle O, (\Phi_i, \Psi_i) \rangle \) be a MGRMRS-As. Then the measure of accuracy for \(\emptyset \neq M \subseteq O\) under MGRMRRSs environment w.r.t \(\hat{P}\) is defined as:

\[
MA(M) = \left( M^\Phi, M^\Psi \right) ,
\]

where

\[
M^\Phi_i = (\sum_{i \in I} M) \psi^+_i
\]

and

\[
M^\Psi_i = (\sum_{i \in I} M) \psi^-_i .
\]

The measure of roughness for \(\emptyset \neq M \subseteq O\) under MGRMRRSs environment w.r.t \(\hat{P}\) is defined as follows:

\[
MR(M) = (1, 1) - \left( M^\Phi, M^\Psi \right) = \left( 1 - M^\Phi, 1 - M^\Psi \right).
\]

Obviously, \(0 \leq M^\Phi \leq 1\) and \(0 \leq M^\Psi \leq 1\) for any \(M \subseteq O\).

From Definition 12, one can derive the following results.

**Proposition 3:** For \(M, N \subseteq O\), the measure of accuracy \(MA(M) = \left( M^\Phi, M^\Psi \right)\) owning the following properties:

1. \( MA(M) = (0, 0) \iff \left( \sum_{i \in I} M \right) \psi^+_i = 0 = \left( \sum_{i \in I} M \right) \psi^-_i ; \\
2. \( MA(M) = (1, 1) \iff M = O \) or \( \left( \sum_{i \in I} M \right) \psi^+_i = \left( \sum_{i \in I} M \right) \psi^-_i ; \\
3. \( MA(M \cup N) \geq MA(M) \) and \( MA(M \cup N) \geq MA(N) ; \\
4. \( MA(M \cap N) \leq MA(M) \) and \( MA(M \cap N) \leq MA(N) ; \\
5. \( M \subseteq N \implies MA(M) \leq MA(N) \).

**Proof 11:** Straightforward.

**Remark 6:** If \(M \subseteq O\) and \(MA(M) = \left( M^\Phi, M^\Psi \right)\) the accuracy measure of \(M\). Then,

(i) If \(M \subseteq O\) is MGRMRS-definable w.r.t \(\hat{P} = \langle O, (\Phi_i, \Psi_i) \rangle\) if and only if \(MR(M) = (0, 0)\).

(ii) If \(M^\Phi < 1\) and \(M^\Psi < 1\) the set \(M\) has some non-empty boundary region and consequently is MGRMRS.

Gediga and Düntsch [16] proposed measure of precision, which is not affected by the approximation \(M^c\), is a simple statistic given by the formula:

\[
P_\pi(M) = \frac{\text{apr}_\pi(M)}{|M|} .
\]

This is just a relative number of objects of \(M\) which can be approximated by \(ER\) \(\pi\). It is important to note that \(P_\pi(M)\) requires complete information of \(M\), whereas \(A_s(M)\) does not, since the latter uses only RS approximations \(\text{apr}_s(M), \text{apr}_s^{-}(M)\).

In MGRMRRSs context, it can be characterized as follows:
Definition 13: Let \((\xi_i, \eta_i : A_i)_{i \in I}\) be a finite collection of BSs over \(O\) and \(\mathbf{P} = \langle O, (\Phi_i, \Psi_i)_{i \in I} \rangle\) a MGRBS-\(A_s\). Then the measure of precision for \(O \neq M \subseteq O\) under MGRBSs environment w.r.t \(\mathbf{P}\) is defined by an ordered pair:

\[
\mathbf{MP}(M) = (\mathcal{M}_\pi^{\Phi}, \mathcal{M}_\pi^{\Psi}),
\]

where

\[
\mathcal{M}_\pi^{\Phi} = \left(\frac{\sum_{i \in I} M_i^{\Phi}}{\left|\mathcal{M}_\pi\right|}\right),
\]

and

\[
\mathcal{M}_\pi^{\Psi} = \left(\frac{\sum_{i \in I} M_i^{\Psi}}{\left|\mathcal{M}_\pi\right|}\right).
\]

Obviously, \(0 \leq \mathcal{M}_\pi^{\Phi} \leq 1\) and \(0 \leq \mathcal{M}_\pi^{\Psi} \leq 1\) for any \(O \neq M \subseteq O\).

From the above definition, one can prove the following properties:

Proposition 4: For \(M, N \subseteq O\), the measure of precision \(\mathbf{MP}(M) = (\mathcal{M}_\pi^{\Phi}, \mathcal{M}_\pi^{\Psi})\) own the following properties:

1. \(\mathbf{MP}(M) = (0, 0) \iff \left(\sum_{i \in I} M_i^{\Phi}\right) = 0 = \left(\sum_{i \in I} M_i^{\Psi}\right);\)
2. \(\mathbf{MP}(M) = (1, 1) \iff \mathcal{M} = \emptyset\) or \(\left(\sum_{i \in I} M_i^{\Phi}\right) = \mathcal{M} = \left(\sum_{i \in I} M_i^{\Psi}\right);\)
3. \(\mathbf{MP}(M \cup N) \geq \mathbf{MP}(M)\) and \(\mathbf{MP}(M \cup N) \geq \mathbf{MP}(N);\)
4. \(\mathbf{MP}(M \cap N) \leq \mathbf{MP}(M)\) and \(\mathbf{MP}(M \cap N) \leq \mathbf{MP}(N);\)
5. \(M \subseteq N \implies \mathbf{MP}(M) \leq \mathbf{MP}(N).\)

Proof 12: Straightforward.

Proposition 5: For any \(M \subseteq O\), \(\mathbf{MP}(M) \geq \mathbf{MA}(M).\)

That is, \(\mathcal{M}_\pi^{\Phi} \geq \mathcal{M}_\pi^{\Phi}\) and \(\mathcal{M}_\pi^{\Psi} \geq \mathcal{M}_\pi^{\Psi}\).

Proof 13: Since,

\[
\mathcal{M}_\pi^{\Phi} = \left(\frac{\sum_{i \in I} M_i^{\Phi}}{\left|\mathcal{M}_\pi\right|}\right)
\]

Thus, \(\mathcal{M}_\pi^{\Phi} \geq \mathcal{M}_\pi^{\Phi}\).

Also,

\[
\mathcal{M}_\pi^{\Psi} = \left(\frac{\sum_{i \in I} M_i^{\Psi}}{\left|\mathcal{M}_\pi\right|}\right)
\]

Therefore, \(\mathcal{M}_\pi^{\Phi} \geq \mathcal{M}_\pi^{\Psi}\).

Consequently, \(\left(\mathcal{M}_\pi^{\Phi}, \mathcal{M}_\pi^{\Psi}\right) \geq \left(\mathcal{M}_\pi^{\Phi}, \mathcal{M}_\pi^{\Psi}\right).\)

Hence, \(\mathbf{MP}(M) \geq \mathbf{MA}(M).\)

Yao [69] developed another measure, named as the measure of completeness of knowledge, which is given as:

\[
\mathbf{C}_k(M) = \left(\frac{\left|\text{apr}_\pi(M)\right|}{\left|O\right|}\right) + \left(\frac{\left|\text{apr}_\pi(M^c)\right|}{\left|O\right|}\right).
\]

In the framework of MGRBSs, it can be characterized as:

Definition 14: Let \((\xi_i, \eta_i : A_i)_{i \in I}\) be a finite collection of BSs over \(O\) and \(\mathbf{P} = \langle O, (\Phi_i, \Psi_i)_{i \in I} \rangle\) a MGRBS-\(A_s\). Then the measure of completeness of knowledge for \(O \neq M \subseteq O\) under MGRBSs environment w.r.t \(\mathbf{P}\) is defined as:

\[
\mathbf{C}_k(M) = \left(\mathcal{M}_\pi^{\Phi}, \mathcal{M}_\pi^{\Psi}\right),
\]

where

\[
\mathcal{M}_\gamma^{\Phi} = \left(\frac{\sum_{i \in I} M_i^{\Phi}}{\left|\mathcal{M}_\gamma\right|}\right)
\]

and

\[
\mathcal{M}_\gamma^{\Psi} = \left(\frac{\sum_{i \in I} M_i^{\Psi}}{\left|\mathcal{M}_\gamma\right|}\right).
\]

Obviously, \(0 < \mathcal{M}_\pi^{\Phi} \leq 1\) and \(0 < \mathcal{M}_\pi^{\Psi} \leq 1\) for \(O \neq M \subseteq O\). That is, \(\mathbf{C}_k(M)\) cannot be zero for any \(M \subseteq O\).

Just from the above definition, one can get the following result.

Proposition 6: \(\mathbf{C}_k(M) = (1, 1)\) if \(M = \emptyset\) or \(M = O\).

Proof 14:

Case 1: When \(M = \emptyset\), then

\[
\mathcal{M}_\gamma^{\Phi} = \left(\frac{\sum_{i \in I} (\emptyset)^{\phi_i}}{\left|\mathcal{M}_\gamma\right|}\right) = \emptyset + \left|\emptyset\right| = 0 + 1 = 1
\]

Similarly,

\[
\mathcal{M}_\gamma^{\Psi} = \left(\frac{\sum_{i \in I} (\emptyset)^{\psi_i}}{\left|\mathcal{M}_\gamma\right|}\right) = \emptyset + \left|\emptyset\right| = 0 + 1 = 1
\]

Thus, it follows that \(\mathbf{C}_k(M) = \left(\mathcal{M}_\pi^{\Phi}, \mathcal{M}_\pi^{\Psi}\right) = (1, 1).\)
Case 2: When \( \mathcal{M} = \mathcal{O} \), then
\[
\mathcal{M}^\Phi = \left( \sum_{i \in I} \mathcal{O} \right) \Phi^+_i + \left( \sum_{i \in I} \mathcal{O}^c \right) \Phi^-_i
\]
\[
\mathcal{O} = \left| \mathcal{O} \right| + \left| \mathcal{O}^c \right| = \left| \mathcal{O} \right| + 0 = 1
\]
Also,
\[
\mathcal{M}^\Psi = \left( \sum_{i \in I} \mathcal{O} \right) \Psi^-_i + \left( \sum_{i \in I} \mathcal{O}^c \right) \Psi^-_i
\]
\[
\mathcal{O} = \left| \mathcal{O} \right| + \left| \mathcal{O}^c \right| = \left| \mathcal{O} \right| + 0 = 1
\]
Consequently, \( C_k(\mathcal{M}) = (\mathcal{M}^\Phi, \mathcal{M}^\Psi) = (1, 1) \).
Hence, in both cases we obtain \( C_k(\mathcal{M}) = (1, 1) \).

Here, we elaborate on the following example to explain the concept of the aforementioned measures.

Example 7: (Continued from Example 6) If we consider \( \mathcal{M} = \{u_1, u_2, u_3, u_7\} \subseteq \mathcal{O} \), Then, \( TGMRBS \)-approximations of \( \mathcal{M} \) are given as follows:
\[
\mathcal{M}^\Phi = \{u_1, u_3, u_7\},
\]
\[
\mathcal{M}^\Psi = \{u_1, u_2, u_3, u_6, u_7\},
\]
Also, we can calculate \( \Phi \)-lower positive and \( \Psi \)-upper negative \( TGMRBS \)-approximations of \( \mathcal{M} \) as follows:
\[
\mathcal{M}^c = \Phi^+_i \cup \Psi^-_i = \{u_4, u_5\},
\]
Therefore,
\[
\mathbf{MA}(\mathcal{M}) = \left( \mathcal{M}^\Phi, \mathcal{M}^\Psi \right) = \left( \frac{3}{5}, \frac{3}{5} \right) = (0.600, 0.600),
\]
\[
\mathbf{MP}(\mathcal{M}) = \left( \mathcal{M}_{\Phi^+_i}, \mathcal{M}_{\Psi^-_i} \right) = \left( \frac{3}{4}, \frac{3}{4} \right) = (0.750, 0.750),
\]
\[
\mathbf{BND}(\mathcal{M}) = \left( \mathcal{M}^\Phi, \mathcal{M}^\Psi \right) = \left( \frac{3+2}{7}, \frac{3+2}{7} \right) = (0.714, 0.714).
\]

V. OPTIMISTIC MULTI-GRAUANULATION MODIFIED ROUGH BIPOLAR SOFT SETS (OMGMRBS)
Qian et al. [43], [44] provided two different MGRS models, including optimistic and pessimistic ones. In the current section, we propose the notion of OMGMRBSs and study their basic properties.

Definition 15: Let \( (\xi_i, \eta_i : \mathcal{A})_{i \in I} \) be a finite collection of BSs over \( \mathcal{O} \) and \( \mathcal{P} = \langle \mathcal{O}, (\Phi_i, \Psi_i) \rangle_{i \in I} \) be a \( MGMBRS \)-\( A \), where \( \xi_i : \mathcal{A} \rightarrow \mathcal{P}(\mathcal{O}), \eta_i : \mathcal{A} \rightarrow \mathcal{P}(\mathcal{O}) \), and the maps \( \Phi_i \) and \( \Psi_i \) are given as follows:
\[
\Phi_i : \mathcal{O} \rightarrow \mathcal{P}(\mathcal{A})
\]
\[
\Phi_i(x) = \{ \rho : x \in \xi_i(\rho) \} \text{ for all } x \in \mathcal{O}, i \in I,
\]
and
\[
\Psi_i : \mathcal{O} \rightarrow \mathcal{P}(\mathcal{A})
\]
\[
\Psi_i(x) = \{ \neg \rho : x \in \eta_i(\neg \rho) \} \text{ for all } x \in \mathcal{O}, i \in I.
\]
The lower and upper \( OMGMBRS \)-approximations for any \( \mathcal{M} \subseteq \mathcal{O} \) are respectively defined:
\[
\text{OMGMBRS}_P^\Phi(\mathcal{M}) = \left( \sum_{i \in I} \mathcal{M} \right) \Phi^+_i \cup \left( \sum_{i \in I} \mathcal{M} \right) \Psi^-_i
\]
\[
\text{OMGMBRS}_P^\Psi(\mathcal{M}) = \left( \sum_{i \in I} \mathcal{M} \right) \Phi^+_i \cup \left( \sum_{i \in I} \mathcal{M} \right) \Psi^-_i
\]
where
\[
\left( \sum_{i \in I} \mathcal{M} \right) \Phi^+_i = \bigcup_{i \in I} \{ x \in \mathcal{M} : \Phi_i(x) \neq \Phi_i(y) \} \text{ for all } y \in \mathcal{M}^c
\]
\[
\left( \sum_{i \in I} \mathcal{M} \right) \Psi^-_i = \bigcup_{i \in I} \{ x \in \mathcal{M} : \Psi_i(x) = \Psi_i(y) \} \text{ for some } y \in \mathcal{M}
\]
\[
\left( \sum_{i \in I} \mathcal{M} \right) \Psi^-_i = \bigcap_{i \in I} \{ x \in \mathcal{M} : \Psi_i(x) \neq \Psi_i(y) \} \text{ for } y \in \mathcal{M}^c
\]
Moreover, if \( \text{OMGMBRS}_P^\Phi(\mathcal{M}) \neq \text{OMGMBRS}_P^\Psi(\mathcal{M}) \), then \( \mathcal{M} \) is named as \( OMGMBRS \); otherwise it is called \( OMGMBRS \)-definable. The \( boundary \) region of \( \mathcal{M} \subseteq \mathcal{O} \) under \( OMGMBRS \) environment is defined as follows:
\[
\text{BND}^\mathcal{M} = \left( \sum_{i \in I} \mathcal{M} \right) \Phi^+_i \cup \left( \sum_{i \in I} \mathcal{M} \right) \Psi^-_i
\]

Remark 7: From Definition 15, we observed that \( \mathcal{M} \subseteq \mathcal{O} \) is \( OMGMBRS \)-definable w.r.t \( \mathcal{P} \) if and only if \( \text{BND}^\mathcal{M} = (\mathcal{O}, \mathcal{O}) \).

Next, we elaborate the following example to explain the notion of the lower and upper \( OMGMBRS \)-approximations of \( \mathcal{M} \subseteq \mathcal{O} \).

Example 8: (Continued from Example 1) We can calculate
the lower and upper OMGMRBS-approximations of \( M = \{u_3, u_4, u_5\} \subseteq \mathcal{O} \) as follows:

\[
M^o_{\phi_+} = \{u_3, u_4\},
\]

\[
M^o_{\phi_+} = \{u_1, u_2, u_3, u_4, u_5\},
\]

\[
M^o_{\phi_-} = \{u_3, u_4, u_5\},
\]

Thus,

\[
M_{\mathcal{O}}(M) = \{u_3, u_4, u_5\},
\]

\[
M_{\mathcal{O}}(M) = \{u_1, u_2, u_3, u_4, u_5, u_6\}, \text{ for } A \subseteq \mathcal{O}.
\]

Consequently, \( X \) is OMGMRBS, since \( M_{\mathcal{O}}(M) \neq M_{\mathcal{O}}(M) \). Moreover,

\[
\sum_{i \in I} M = \{\{u_1, u_2, u_5, u_6\}, \{u_3, u_4\}\}.
\]

The following properties identify the relationship between the lower and upper OMGMRBS-approximations of a single set and the lower and upper OMGMRBS-approximations of two sets described over \( \mathcal{O} \).

**Theorem 9:** Let \( (\xi_i, \eta_i : A)_{i \in I} \) be a finite collection of BSs over \( \mathcal{O} \) and \( \mathcal{P} = \{(\phi_i, \psi_i)\}_{i \in I} \) be a MGRBS-A. Then for any \( A, N \subseteq \mathcal{O} \), the lower and upper OMGMRBS-approximations own the following properties:

1. \( \bigcup_{M \subseteq N} \bigcup_{i \in I} M^o_{\phi_+} \subseteq M \subseteq \bigcup_{i \in I} M^o_{\phi_+} \)
2. \( \bigcup_{M \subseteq N} \bigcup_{i \in I} M^o_{\phi_-} \subseteq M \subseteq \bigcup_{i \in I} M^o_{\phi_-} \)
3. \( \bigcup_{M \subseteq N} \bigcup_{i \in I} M^o_{\phi_+} = \bigcup_{i \in I} M^o_{\phi_+} \)
4. \( \bigcup_{M \subseteq N} \bigcup_{i \in I} M^o_{\phi_-} = \bigcup_{i \in I} M^o_{\phi_-} \)

**Example 9:** (Continued from Example 8) We can calculate \( \text{MA}^o(M) \) and \( \text{MR}^o(M) \) for \( M = \{u_3, u_4, u_5\} \) as follows:

\[
\text{MA}^o(M) = \left( \frac{2}{6} \right) = 0.33, 0.33,
\]

\[
\text{MR}^o(M) = (1, 1) - (0.33, 0.33) = (0.67, 0.67).
\]

**VI. PESSIONISTIC MULTI-GRANULATION MODIFIED ROUGH BIPOLAR SOFT SETS (PMGMRBSS)**

In the current section, we propose the notion of PMGMRBSs and study their essential structural properties.
Definition 17: Let $(ξ_i, η_i : A)_{i ∈ I}$ be a finite collection of BSs over $O$ and $P = \langle O, (Φ_i, Ψ_i) \rangle_{i ∈ I}$ be a $MGMRBS_A$-s, where $ξ_i : A → \mathcal{P}(O)$, $η_i : \neg A → \mathcal{P}(O)$, and the maps $Φ_i$ and $Ψ_i$ are given as follows:

$$Φ_i : O → \mathcal{P}(A)$$

$$Φ_i(x) = \{ ρ : x ∈ ξ_i(ρ) \} \text{ for all } x ∈ O, i ∈ I,$$

and

$$Ψ_i : O → \mathcal{P}(\neg A)$$

$$Ψ_i(x) = \{ \neg ρ : x ∈ η_i(\neg ρ) \} \text{ for all } x ∈ O, i ∈ I.$$

For any $M ⊆ O$, the lower and upper PMGMRBS-approximations are defined as:

$$MGMB_P^\nu(M) = \left\{ \left( \sum_{i ∈ I} \mathcal{M}_i^\nu, \sum_{i ∈ I} \mathcal{M}_i^\nu \right) : \right\}$$

$$\sum_{i ∈ I} \mathcal{M}_i^\nu = \left\{ x ∈ M : Φ_i(x) = Φ_i(y) \right\}, \text{ for all } y ∈ M^c,$$

$$\sum_{i ∈ I} \mathcal{M}_i^\nu = \left\{ x ∈ O : Ψ_i(x) = Ψ_i(y) \right\}, \text{ for some } y ∈ M^c,$$

where

$$\sum_{i ∈ I} \mathcal{M}_i^\nu = \bigcup_{i ∈ I} \left\{ x ∈ M : Φ_i(x) = Φ_i(y) \right\}, \text{ for all } y ∈ M^c$$

Moreover, if $MGMB_P^\nu(M) ≠ MGMB_P^\nu(M)$, then $M$ is titled as PMGMRBS; otherwise it is called PMGMRBS-definable. The boundary region of $M ⊆ O$ under PMGMRBS environment is defined as follows:

$$BND^\nu_P = \left( \sum_{i ∈ I} \mathcal{M}_i^\nu \right)^\nu - \left( \sum_{i ∈ I} \mathcal{M}_i^\nu \right)^\nu,$$

Remark 8: From the above Definition, we observed that $M ⊆ U$ is PMGMRBS-definable w.r.t $P$ if and only if $\sum_{i ∈ I} \mathcal{M}_i^\nu = (O, O)$.

To illustrate the notion of the lower and upper PMGMRBS-approximations of $M ⊆ O$, here we employ the following example.

Example 10: (Continued from Example 1) We can calculate the lower and upper PMGMRBS-approximations of $M = \{ u_3, u_4, u_5 \} \subseteq O = \{ u_1, u_2, u_3, u_4, u_5 \}$ as follows:

$$\sum_{i ∈ I} \mathcal{M}_i^\nu = \{ u_3, u_4 \},$$

$$\sum_{i ∈ I} \mathcal{M}_i^\nu = \{ u_3, u_4, u_5 \},$$

$$\sum_{i ∈ I} \mathcal{M}_i^\nu = \{ u_1, u_2, u_3, u_4, u_5 \},$$

Thus,

$$\sum_{i ∈ I} \mathcal{M}_i^\nu = \{ u_3, u_4, u_1, u_2, u_3, u_4, u_5 \},$$

$$\sum_{i ∈ I} \mathcal{M}_i^\nu = \{ u_3, u_4, u_5 \},$$

Consequently, $M$ is PMGMRBS, since $\sum_{i ∈ I} \mathcal{M}_i^\nu ≠ \sum_{i ∈ I} \mathcal{M}_i^\nu (M)$. Moreover,

$$BND^\nu = \{ u_5 \}, \{ u_1, u_2 \}.$$
Example 11: Let \( (\xi, \eta, A)_{i \in I} \) be a finite collection of BSs over \( O, \hat{P} = \{O, (\Phi_i, \Psi_i)\}_{i \in I} \) be a MGMRBS-\( A \)-s and \( M \subseteq O \). Then, the following properties hold:

1. \( (\sum_{i \in I} M)_{\Phi_i}^o \subseteq (\sum_{i \in I} M)_{\Phi_i}^o + \)
2. \( (\sum_{i \in I} M)_{\Psi_i}^o \subseteq (\sum_{i \in I} M)_{\Psi_i}^o + \)
3. \( (\sum_{i \in I} M)_{\Phi_i}^s \subseteq (\sum_{i \in I} M)_{\Phi_i}^s + \)
4. \( (\sum_{i \in I} M)_{\Psi_i}^s \subseteq (\sum_{i \in I} M)_{\Psi_i}^s + \)

Proof 17: Clear from Definition 11, Definition 15, and Definition 17.

Proposition 8: Let \( (\xi, \eta, A)_{i \in I} \) be a finite collection of BSs over \( O, \hat{P} = \{O, (\Phi_i, \Psi_i)\}_{i \in I} \) be a MGMRBS-\( A \)-s and \( M \subseteq O \). Then, the following properties hold:

1. \( (\sum_{i \in I} M)_{\Phi_i}^o \subseteq (\sum_{i \in I} M)_{\Phi_i}^o + \)
2. \( (\sum_{i \in I} M)_{\Psi_i}^o \subseteq (\sum_{i \in I} M)_{\Psi_i}^o + \)
3. \( (\sum_{i \in I} M)_{\Phi_i}^s \subseteq (\sum_{i \in I} M)_{\Phi_i}^s + \)
4. \( (\sum_{i \in I} M)_{\Psi_i}^s \subseteq (\sum_{i \in I} M)_{\Psi_i}^s + \)

Proof 18: Straightforward.

The preceding proposition demonstrates the relationship of containment among MGMRBS-approximations, OMGMRBS-approximations, and PMGMRBS-approximations. The above properties reveal that:

- \( \Phi \)-upper positive OMGMRBS-approximations of \( M \) is finer than the \( \Phi \)-upper positive MGMRBS-approximations of \( M \). Similarly, \( \Psi \)-upper negative OMGMRBS-approximations of \( M \) is coarser than the \( \Psi \)-upper negative MGMRBS-approximations of \( M \).
- Also, \( \Phi \)-lower positive MGMRBS-approximations of \( M \) is finer than the \( \Phi \)-lower positive PMGMRBS-approximations of \( M \). Moreover, \( \Psi \)-lower negative MGMRBS-approximations of \( M \) is coarser than the \( \Psi \)-lower negative PMGMRBS-approximations of \( M \).

The following result depicts the relationship of containment between the OMGMRBS-approximations and the PMGMRBS-approximations of \( M \subseteq O \).

Proposition 9: Let \( (\xi, \eta, A)_{i \in I} \) be a finite collection of BSs over \( O, \hat{P} = \{O, (\Phi_i, \Psi_i)\}_{i \in I} \) be a MGMRBS-\( A \)-s and \( M \subseteq O \). Then,

1. \( (\sum_{i \in I} M)_{\Phi_i}^o \subseteq (\sum_{i \in I} M)_{\Phi_i}^o + \)
2. \( (\sum_{i \in I} M)_{\Psi_i}^o \subseteq (\sum_{i \in I} M)_{\Psi_i}^o + \)
3. \( (\sum_{i \in I} M)_{\Phi_i}^s \subseteq (\sum_{i \in I} M)_{\Phi_i}^s + \)
4. \( (\sum_{i \in I} M)_{\Psi_i}^s \subseteq (\sum_{i \in I} M)_{\Psi_i}^s + \)

Proof 19: Straightforward.

VII. RELATIONSHIPS AMONG THE MGMRBS, OMGMRBS AND PMGMRBS

In this section, the relationships among the MGMRBS, OMGMRBSs, and PMGMRBSs will be explored further.
The following result demonstrates the relationship of the measure of the accuracy of MGMRBSs, OMGMRBSs, and PMGMRBSs.

**Proposition 10:** Let \( \text{MA}(\mathcal{M}) = (\mathcal{M}^{\Phi_i}, \mathcal{M}^{\Psi_i}) \), \( \text{MA}^o(\mathcal{M}) = (\mathcal{M}^{\Phi_i}, \mathcal{M}^{\Psi_i}) \), and \( \text{MA}^{o_o}(\mathcal{M}) = (\mathcal{M}^{\Phi_i}, \mathcal{M}^{\Psi_i}) \) be the measure accuracy of MGMRBSs, OMGMRBSs, and PMGMRBSs, respectively. Then,

1. \( \mathcal{M}^{\Phi_i} \leq \mathcal{M}^{\Phi_i} \) and \( \mathcal{M}^{\Psi_i} \leq \mathcal{M}^{\Psi_i} \);
2. \( \mathcal{M}^{\Phi_i} \leq \mathcal{M}^{\Phi_i} \) and \( \mathcal{M}^{\Psi_i} \leq \mathcal{M}^{\Psi_i} \).

**Proof 20:** Obvious.

**VIII. MCGDM UNDER THE FRAMEWORK OF OMGMRBS**

MCGDM is one of the most active research fields in decision analysis, which has attracted much attention in the past decades. In GDM problems, a group of decision-makers (DMs) selects the most appropriate one among a set of alternatives. The MCGDM is devoted to choosing the most suitable alternative under the DMs’ evaluation values on a set of parameters concerning a set of criteria. MCGDM has been successfully applied to many real-world problems in different domains, like engineering project management, green supplier selection, portfolio allocation, etc.

The following section explores a comprehensive MCGDM technique under the proposed MGMRBSs. In MCGDM with MGMRBSs, a group of DMs evaluates a set of alternatives w.r.t. a set of parameters. Every expert provides their evaluation information for all \( u \in \Omega \) according to their experience and professional knowledge. Generally, the evaluations of DMs are in the form of BSs. According to the evaluation information, the most appropriate alternative is selected. First of all, we provide a brief description of the MCGDM problem based on MGMRBSs and then give a DM procedure for MCGDM by using the theory of MGMRBSs.

**A. PROBLEM DESCRIPTION**

Here, we give the basic statement of the considered MCGDM problem as follows.

Let \( \Omega = \{ o_1, o_2, \ldots, o_n \} \) be the universe of \( n \) objects under consideration, \( \mathcal{A} = \{ a_1, a_2, \ldots, a_m \} \) be the collection of parameters related to the objects of \( \Omega \) and \( \Omega_i = (\xi_i, \eta_i : \mathcal{A}) \in \mathcal{BSS}_A(\Omega) \) be a finite collection of BSs over \( \Omega \). Suppose that \( \mathcal{D} = \{ D_1, D_2, \ldots, D_k \} \) is a panel of \( k \) invited experts/DMs, \( M_1, M_2, \ldots, M_k \) are non-empty subsets of \( \Omega \), represent results of primary assessments of experts \( D_1, D_2, \ldots, D_k \), respectively and \( \Omega_1, \Omega_2, \ldots, \Omega_k \in \mathcal{BSS}_A(\Omega) \) are the actual results that previously obtained for same or similar problems in different times or different places. Then the DM for this MCGDM problem is: “how to obtain the evaluation of these particular experts so that the selected object is optimal for all criteria”.

All three models of MGMRBSs, OMGMRBSs, and PMGMRBSs can be used to discuss this DM problem. In this paper, we use the OMGMRBSs model to present the decision methodology for the MCGDM problem.

**B. METHODOLOGY OF DM**

Here we give the mathematical formulation of the MCGDM technique using OMGMRBSs.

**Definition 19:** Let \( \Omega_i = (\xi_i, \eta_i : \mathcal{A})_{i \in I} \) be a finite collection of BSs over \( \mathcal{O} \). Let \( M^\phi_{\Omega_i}(M_j) = \left( \sum_{i \in I} M^\phi_i \right)^{\phi_i}, \sum_{i \in I} M^\psi_i \right)^{\psi_i} \) and \( M^\psi_{\Omega_i}(M_j) = \left( \sum_{i \in I} M^\phi_i \right)^{\phi_i}, \sum_{i \in I} M^\psi_i \right)^{\psi_i} \) be the lower and upper OMGMRBS-approximations of \( M_j \), \( j = 1, 2, \ldots, k \) related to \( \Omega_q \); \( q = 1, 2, \ldots, r \). Then,

\[
[M^\phi](\phi^+, \psi^-) = \left[ \left( M^\phi_{\Omega_i}, M^\psi_{\Omega_i} \right) \right]_{1 \times k}, \quad (63)
\]

\[
[M^\psi](\phi^+, \psi^-) = \left[ \left( M^\phi_{\Omega_i}, M^\psi_{\Omega_i} \right) \right]_{1 \times k}, \quad (64)
\]

are called the lower and upper OMGMRBS-approximations matrices, respectively. Here

\[
M^\phi \left( \phi^+, \psi^- \right) = \left( \sum_{i \in I} \frac{\Omega_i}{ \Omega_i \in I } \right)^{\phi_i} \quad \text{and} \quad M^\psi \left( \phi^+, \psi^- \right) = \left( \sum_{i \in I} \frac{\Omega_i}{ \Omega_i \in I } \right)^{\psi_i} \quad (65)
\]

\[
M^\psi \left( \phi^+, \psi^- \right) = \left( \sum_{i \in I} \frac{\Omega_i}{ \Omega_i \in I } \right)^{\psi_i} \quad (66)
\]

\[
M^\phi \left( \phi^+, \psi^- \right) = \left( \sum_{i \in I} \frac{\Omega_i}{ \Omega_i \in I } \right)^{\phi_i} \quad (67)
\]

\[
M^\psi \left( \phi^+, \psi^- \right) = \left( \sum_{i \in I} \frac{\Omega_i}{ \Omega_i \in I } \right)^{\psi_i} \quad (68)
\]

Where,

\[
u_i^{\phi \phi} = \begin{cases} 1 & \text{if } u_i \in \left( \sum_{i \in I} \frac{\Omega_i}{ \Omega_i \in I } \right)^{\phi_i} \\ 0 & \text{if } u_i \notin \left( \sum_{i \in I} \frac{\Omega_i}{ \Omega_i \in I } \right)^{\phi_i} \end{cases} \quad (69)
\]

\[
u_i^{\psi \psi} = \begin{cases} 1 & \text{if } u_i \in \left( \sum_{i \in I} \frac{\Omega_i}{ \Omega_i \in I } \right)^{\psi_i} \\ 0 & \text{if } u_i \notin \left( \sum_{i \in I} \frac{\Omega_i}{ \Omega_i \in I } \right)^{\psi_i} \end{cases} \quad (70)
\]

\[
u_i^{\phi \phi} = \begin{cases} 1 & \text{if } u_i \in \left( \sum_{i \in I} \frac{\Omega_i}{ \Omega_i \in I } \right)^{\phi_i} \\ 0 & \text{if } u_i \notin \left( \sum_{i \in I} \frac{\Omega_i}{ \Omega_i \in I } \right)^{\phi_i} \end{cases} \quad (71)
\]

\[
u_i^{\psi \psi} = \begin{cases} 1 & \text{if } u_i \in \left( \sum_{i \in I} \frac{\Omega_i}{ \Omega_i \in I } \right)^{\psi_i} \\ 0 & \text{if } u_i \notin \left( \sum_{i \in I} \frac{\Omega_i}{ \Omega_i \in I } \right)^{\psi_i} \end{cases} \quad (72)
\]

**Definition 20:** Let \( [M^\phi](\phi^+, \psi^-) \) and \( [M^\psi](\phi^+, \psi^-) \) be the lower and upper OMGMRBS-approximations matrices, re-
respectively. Then the expressions:

\[
(V^+)^o = \sum_{j=1}^{k} \sum_{q=1}^{r} (M_{j\Phi_q^+}^o + M_{j\Phi_q^+}^o), \tag{73}
\]

\[
(V^-)^o = \sum_{j=1}^{k} \sum_{q=1}^{r} (M_{j\Phi_q^-}^o + M_{j\Phi_q^-}^o), \tag{74}
\]

are called the positive and negative OMGMRBS-approximations vectors, respectively. The symbols \(\sum\) and \(\oplus\) denote the vector summation.

**Definition 21:** Let \((V^+)\) and \((V^-)\) be the positive and negative OMGMRBS-approximations vectors, respectively. Then the expressions:

\[
V_\delta = (V^+)^o \oplus (V^-)^o = (\mu_1, \mu_2, \ldots, \mu_n), \tag{75}
\]

is termed as decision vector.

**Definition 22:** Let \(V_\delta = (\mu_1, \mu_2, \ldots, \mu_n)\) be the decision vector. Then each \(\mu_i\) is termed as the score value (SV) of \(u_i \in \mathcal{O}\).

(i) An alternative \(u_i \in \mathcal{O}\) is named as an optimal alternative if its SV is a maximum of \(\mu_i\); \(i = 1, 2, \ldots, n\).

(ii) An alternative \(u_i \in \mathcal{O}\) is named as the worst alternative if its SV is a minimum of \(\mu_i\); \(i = 1, 2, \ldots, n\).

**C. ALGORITHM FOR MCGDM PROBLEM UNDER OMGMRBS**

Now, we provide a DM algorithm for the established MCGDM problem discussed in the previous subsection. The corresponding steps are listed as:

**Step 1:** Take primary assessments \(\mathcal{M}_1, \mathcal{M}_2, \ldots, \mathcal{M}_k\) of experts \(\mathcal{D}_1, \mathcal{D}_2, \ldots, \mathcal{D}_k\).

**Step 2:** Evaluate \(\Omega_1, \Omega_2, \ldots, \Omega_r \in \text{BSS}_A(\mathcal{D})\) using the actual results.

**Step 3:** Compute \(\bar{M}_{\Omega_j}^0(M_j)\) and \(\bar{M}_{\Omega_q}^0(M_j)\) for all \(j = 1, 2, \ldots, k\) and \(q = 1, 2, \ldots, r\).

**Step 4:** Evaluate \(\bar{M}_{\Omega_j}^0(\Phi^+),\Phi^-\) and \(\bar{M}_{\Omega_q}^0(\Phi^+),\Phi^-\) by using Definition 19.

**Step 5:** Calculate \((V^+)\) and \((V^-)\) according to Definition 20.

**Step 6:** Compute \(V_\delta\) according Definition 21.

**Step 7:** Find \(\max_{1 \leq i \leq n} \mu_i\). An alternative of largest SV should be chosen for the final selection.

Figure 1 depicts a flow chart representation of the aforementioned algorithm.

**D. A DESCRIPTIVE EXAMPLE (APPLICATION IN MEDICAL DIAGNOSIS)**

In this section, we will use a medical diagnosis problem to illustrate the principal methodology step-by-step for the decision technique presented in Section VIII.

**Example 12:** Suppose a panel of doctors (DMs) wants to diagnose the correct disease in a patient with multiple symptoms. According to the patient’s symptoms, the panel makes a pre-diagnosis (primary evaluations). They consider five possible diseases, including, viral fever, malaria, pneumonia, typhoid, and dengue. They want to make a common decision with a correct diagnosis based on the above-mentioned disease symptoms.

Let \(\mathcal{O} = \{d_1, d_2, d_3, d_4, d_5\}\) is the universe of possible diseases, where \(d_1 = \text{Viral Fever}, d_2 = \text{Malaria}, d_3 = \text{Pneumonia}, d_4 = \text{Typhoid} \) and \(d_5 = \text{Dengue} \), respectively. Suppose that \(\mathcal{A} = \{s_1, s_2, s_3\}\) is the set of parameters (symptoms of diseases), where \(s_1 = \text{throat pain}, s_2 = \text{headache and s}_3 = \text{fatigue} \).

**Step 1:** Suppose that \(\mathcal{D} = \{\mathcal{D}_1, \mathcal{D}_2, \mathcal{D}_3\}\) is a panel of doctors who give their pre-diagnosis (Primary evaluations) for the same patients as:

\(\mathcal{M}_1 = \{d_1, d_2, d_3\}, \mathcal{M}_2 = \{d_1, d_3, d_5\}, \mathcal{M}_3 = \{d_2, d_4, d_5\}\).

**Step 2:** Actual results in three different meetings and clinics for the same patient are represented in the form of BS-sets as:

\(\Omega_1 = (\xi_1, \eta_1 : \mathcal{A}), \Omega_2 = (\xi_2, \eta_2 : \mathcal{A})\) and \(\Omega_3 = (\xi_3, \eta_3 : \mathcal{A})\) as follows, where positive membership map of BSs denote severity of symptoms and negative membership denote mildness of symptoms:

\[\xi_1 : \mathcal{A} \rightarrow \mathcal{P}(\mathcal{O}),\]

\[s \mapsto \{d_1\} \quad \text{if} \quad s = s_1, \]

\[\{d_1, d_2\} \quad \text{if} \quad s = s_2, \]

\[\{d_4, d_5\} \quad \text{if} \quad s = s_3, \]

\[\eta_1 : \mathcal{A} \rightarrow \mathcal{P}(\mathcal{O}),\]

\[s \mapsto \{d_3, d_5\} \quad \text{if} \quad s = \neg s_1, \]

\[\{d_3, d_4\} \quad \text{if} \quad s = \neg s_2, \]

\[\{d_1, d_3\} \quad \text{if} \quad s = \neg s_3. \]

\[\xi_2 : \mathcal{A} \rightarrow \mathcal{P}(\mathcal{O}),\]

\[s \mapsto \{d_2\} \quad \text{if} \quad s = s_1, \]

\[\{d_2, d_4\} \quad \text{if} \quad s = s_2, \]

\[\{d_3, d_4\} \quad \text{if} \quad s = s_3, \]

\[\eta_2 : \mathcal{A} \rightarrow \mathcal{P}(\mathcal{O}),\]

\[s \mapsto \{d_1, d_4\} \quad \text{if} \quad s = \neg s_1, \]

\[\{d_1\} \quad \text{if} \quad s = \neg s_2, \]

\[\{d_1, d_5\} \quad \text{if} \quad s = \neg s_3. \]

\[\xi_3 : \mathcal{A} \rightarrow \mathcal{P}(\mathcal{O}),\]

\[s \mapsto \{d_3, d_5\} \quad \text{if} \quad s = s_1, \]

\[\{d_3, d_5\} \quad \text{if} \quad s = s_2, \]

\[\{d_2, d_5\} \quad \text{if} \quad s = s_3, \]

\[\eta_3 : \mathcal{A} \rightarrow \mathcal{P}(\mathcal{O}),\]

\[s \mapsto \{d_1, d_2\} \quad \text{if} \quad s = \neg s_1, \]

\[\{d_4\} \quad \text{if} \quad s = \neg s_2, \]

\[\{d_1, d_3\} \quad \text{if} \quad s = \neg s_3. \]

**Step 3:** Compute \(\bar{M}_{\Omega_1}^0(M_j)\) and \(\bar{M}_{\Omega_3}^0(M_j)\) for \(j = 1, 2, 3\) and
Step 4: By using Definition 19, we get:

\[
M_{\Omega}^0(M_1) = \left\{ \{d_1, d_2, d_5\}, \{d_1, d_2, d_5\}\right\},
\]

\[
M_{\Omega}^0(M_2) = \left\{ \{d_1, d_3, d_5\}, \{d_1, d_3, d_5\}\right\},
\]

\[
M_{\Omega}^0(M_3) = \left\{ \{d_2, d_4, d_5\}, \{d_2, d_4, d_5\}\right\}.
\]

Step 5: By using Definition 20, \((\mathbf{V}^+)^o\) and \((\mathbf{V}^-)^o\) can be calculated as follows:

\[
(\mathbf{V}^+)^o = (5, 5, 4, 4, 6),
\]

\[
(\mathbf{V}^-)^o = (4, 2, 1, 2, 6).
\]

Step 6: By using Definition 21, we get:

\[
\mathbf{V}_5 = (9, 7, 5, 6, 12).
\]

Step 7: As \(\max_{1 \leq i \leq 5} d_i = d_5 = 12\). So, the patient is suffering from disease \(d_5(Dengue)\). Finally, we can obtain the ranking order among the diseases as follows:

\[
d_5 \geq d_4 \geq d_2 \geq d_4 \geq d_3.
\]

The pictorial portrayal for the ranking of the companies is shown in Figure 2.

**IX. COMPARATIVE ANALYSIS AND DISCUSSION**

This part highlights the advantages of the suggested MCGDM approach and conducts a comparative analysis with some other methods.

**A. ADVANTAGES OF THE PROPOSED TECHNIQUE**

In the realistic DM process, the results were provided by multiple experts are more accurate than those based on
Gul et al. [19]

Karaaslan and Çağman [25]

Shabir and Gul [51]

Our proposed method

d_5 \geq d_1 \geq d_2 \geq d_3 \geq d_4

d_5 \geq d_2 \geq d_4 \geq d_3 \geq d_1

d_5 \geq d_4 \geq d_1 = d_2 \geq d_3

d_5 \geq d_1 \geq d_2 \geq d_4 \geq d_3

TABLE 7: The results of different methods for Example 12

FIGURE 2: (Ranking of diseases)

one single expert. Therefore, the MGRS approach is more accurate in DM problems. Real-world MCGDM problems occur in complex environments with uncertain and imprecise data, which is difficult to tackle. The proposed technique is particularly well suited to the scenario in which the data is complicated and uncertain, particularly when the existing data is dependent on bipolar information and is located across multiple locations. In the proposed approach, DMs express their opinions as BSs. Afterward, all opinions given by DMs are aggregated using the OMGMRBS-approximations, and finally, an optimal compromise solution is obtained. So, the MGRBS model to MCGDM employs a novel strategy to aggregate the preferences of DMs. Therefore, the proposed approach is valid and offers a novel perspective for investigating MCGDM problems in real life. The key advantages of the proposed method over the existing studies are summarized below:

(i) In many real-life problems, we have to deal with the situations where bipolarity of the alternatives (positive and negative parameters) are required but with the help of existing models we cannot handle these situations. MGRBSs can better deal with these types of situations.

(ii) One of the important problems in medical diagnosis is how to judge whether a patient has a disease or not according to his symptoms. However, the information about the various symptoms used for diagnosis is not always clear and sometimes shows bipolarity in behavior. Doctors with different expertise and clinical experience also have different diagnoses for the same patient. Therefore, in order to improve the accuracy of medical diagnosis results, it is necessary to require multiple doctors to diagnose multiple symptoms, and then fuse this diagnostic information to obtain the optimal diagnosis. The MGRBS model can deal with this problem well.

(iii) The proposed model is more effective because it is capable of dealing with situations involving multiple experts as well bipolar parameters in a multi-granulation environment.

(iv) The suggested technique considers positive and negative aspects of each alternative in the form of BS. This hybrid model is more generalized and suitable for dealing with aggressive DM.

(v) Using the OMGMRBS-approximations, this approach gives another way to obtain the group preference evaluation based on the individual preference evaluation for a considered MCGDM problem.

(vi) This technique is also ideal because the DMs are liberated from any external restrictions and requirements in this approach.

(vii) Our proposed technique effectively solves MCGDM problems when the weight information for the criteria is entirely unknown.

(viii) The suggested approach considers not only the opinions of DMs but also past experiences (primary assessments) by OMGMRBS-approximations in actual scenarios. Hence, it is a more comprehensive approach for a better interpretation of available information and thus making decisions using artificial intelligence.

(ix) The proposed MCGDM approach is easy to understand and can be applied to real-life DM problems.

B. COMPARISON WITH OTHER APPROACHES

In the literature, there are various MCGDM methods. Each of these methods has advantages and disadvantages. The capability of any method is determined by the problem under consideration. Here, we compare the proposed technique to several existing methods in the fuzzy and bipolar fuzzy contexts and examine the significance of the proposed MCGDM
method.

(i) If we compare our proposed technique with approaches presented in [3], [32], [47], [57], [59], [60], [61], [62], [78], and [82], we have seen that these methods are incapable of detecting bipolarity in the DM process, which is a key element of human thinking and behavior. Therefore, the proposed DM technique in this study has wider practicability and stronger effectiveness.

(ii) In the fuzzy and bipolar fuzzy MCGDM frameworks, there is always a need for a membership function to fuzzify the data. In the proposed technique, we have used the notion of the OMGMRBS-approximations to derive the uncertainty from the original set of data without additional adjustment and membership functions. In hybrid fuzzy and bipolar fuzzy techniques, the membership function depends on the choice and thinking of the DMS which makes the results more biased.

(iii) When we apply the current MCGDM techniques presented in [19], [25], and [51] to our Example 12, we obtain the following ranking among the diseases (shown in TABLE 7), and the corresponding graphical portrayal is displayed in Figure 3. From TABLE 7, we observe that the optimal solution via all four methods is the same, making our technique feasible and effective.

(iv) A characteristic comparison of different methods with our proposed method is also recapitulated in TABLE 8.

X. CONCLUSION AND FUTURE DIRECTIONS
In this section, we first provide the main conclusions obtained in the article. Then, we consider further prospects for future research studies.

A. MAIN CONCLUSIONS
One of the desired directions in RS theory is MGRS, which approximates lower and upper approximations via granular structures obtained by multiple binary relations. Based on RS, it offers a novel approach for decision analysis. In this article, we have proposed a novel model of MGMRBSs by using a finite collection of BSs to construct the approximation operators. Furthermore, the structural properties and results of the proposed MGMRBS-approximation operators have also been carefully analyzed. Some measures associated with MGMRBSs are also offered. Meanwhile, we provide two MGMRBSs models: the OMGMRBSs and the PMGMRBSs. Moreover, the interrelationship among MGMRBSs, OMGMRBSs, and PMGMRBSs is established. Furthermore, we construct a general framework for dealing with a type of MCGDM under uncertainty by using the theory of MGMRBSs and demonstrating the basic steps of the decision methodology in detail. Finally, we demonstrated the method’s applicability in medical diagnosis to demonstrate that it can be fruitfully implemented to solve real-world problems, including uncertainty. The results are compared to existing approaches for MCGDM, and the superiority of our proposed method is highlighted.

B. FUTURE RESEARCH WORK
The notions of MGMRBSs are rudimentary, but they may lead to more adequate mathematical tools to approximate reasoning in soft computing. The results of this study enrich decision analysis and provide novel ideas for tackling complicated DM problems. Bearing in mind those as mentioned earlier, future studies will concentrate on:

- The practical applications of the suggested technique in solving a wider variety of selection problems,
like TOSIS, VIKOR, ELECTRE, AHP, COPRAS, PROMETHEE, etc.

- Researchers may study the algebraic structures of MGRBSSs.
- The various types of correlation coefficients can also be explored within the context of MGRBSSs.
- The notion of MGRBSSs can also be extended to covering-based MGRBSSs.
- The attribute reduction of MGRBSSs should be analyzed, and comprehensive experimental investigations, and comparisons with existing methodologies should also be justified and explored.
- Strategies for decision support in real-time and dynamic DM tasks are also our next target.
- The MGRBSSs can be extended in a fuzzy environment, and effective DM techniques might be developed.
- Further study can be done to establish fruitful algorithms for different kinds of DM problems.
- Another direction is to investigate the topological properties and similarity measures of MGRBSSs to create a concrete foundation for future studies.
- We will also investigate the possibility of hybridizing the suggested approach to improve precision in results and apply these strategies to real-world problems with large data sets. In this manner, we can acquire and demonstrate the use of our suggested framework.
- Although this article concentrates on the basic theory of the MCGDM principle with the MGRBSSs, it is encouraged that a real-life large data set can be used to validate the approach presented in this paper.

**TABLE 8: Characteristic comparison of different methods with proposed method**

| Methods                | Handle Bipolarity | Computational Complexity | Competent to manage parametrization | Ranking  |
|------------------------|-------------------|--------------------------|-------------------------------------|----------|
| Gul et al. [19]        | Yes               | High                     | Yes                                 | Yes      |
| Feng [82]              | Failed to handle  | Less                     | Yes                                 | Yes      |
| Shabir and Gul [51]    | Yes               | High                     | Yes                                 | Yes      |
| Li et al. [30]         | Failed to handle  | High                     | No                                  | No       |
| Mandal and Ranadive [83]| Failed to handle | High                     | No                                  | No       |
| Sun et al. [58]        | Failed to handle  | Less                     | No                                  | Yes      |
| Xu et al. [60]         | Failed to handle  | High                     | No                                  | No       |
| Proposed Method        | Yes               | Less                     | Yes                                 | Yes      |

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