ON OZAWA KERNELS

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Abstract. We write explicitly Ozawa kernels for group extensions, for discrete metric spaces of finite asymptotic dimension, of large enough Hilbert space compression, and for suitable actions of countable groups on metric spaces. We also obtain an alternative proof of stability results concerning Yu’s property A.

1. Introduction

Property A is a weak form of amenability which was first introduced by G. Yu in [Yu00]. He builds on ideas developed by M.B. Bekka, P-A. Cherix and A. Valette in [BCV95] and shows that a finitely generated group with property A is uniformly embeddable in some Hilbert space, the main result of [Yu00] being that such a group satisfies the coarse Baum-Connes Conjecture and the Novikov Higher Signature Conjecture.

The class of finitely generated groups satisfying property A contains, for instance, amenable groups, hyperbolic groups (and groups which are hyperbolic relatively to a finite family of subgroups with property A), one relator groups, Coxeter groups (moreover any group of finite asymptotic dimension), any discrete subgroup of a connected Lie group, groups acting properly on finite dimensional CAT(0) cube complexes, or more generally every group acting by isometries on a metric space (with bounded geometry) having property A with at least one point stabilizer having property A. Furthermore, Yu’s property A is known to be closed under taking subgroups, extensions, direct limits, amalgamated free products and HNN extensions (see for instance [Tu01], [CN04], [DG03], [Gu01], [GHW05] and [HR00]). Actually, the only known examples of groups which do not satisfy property A are due to M. Gromov (see [Gr03]).

Property A admits several equivalent definitions. Here we focus on a formulation of this property in terms of the existence of an approximation of the unity by positive definite kernels of finite width (called Ozawa kernels, see definitions below). One aim of this paper is to study the behaviors of these kernels and to find explicit formulas. We write explicitly these kernels for group extensions, for discrete metric spaces of finite asymptotic dimension, for discrete metric spaces of large enough Hilbert space compression and moreover for groups acting in a suitable way on metric spaces with property A. In the last section we apply formulas obtained to particular examples like hyperbolic groups, CAT(0) cubical groups and Baumslag-Solitar groups.

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2. Property A and Ozawa kernels

First of all we recall the definition of Yu’s property A:

2.1. Definition. A discrete metric space $(X, d)$ is said to have property A if for every $R > 0$ and every $\varepsilon > 0$ there exists a family of finite sets $\{A_x\}_{x \in X}$ in $X \times \mathbb{N}$ satisfying:

(1) $\exists S > 0$ such that $d(x, y) \leq S$ whenever $(x, m) \in A_y$;

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(2) \( \forall x, y \in X \) such that \( d(x, y) \leq R \), we have \( |A_x \Delta A_y| < \varepsilon|A_x \cap A_y| \) (\(|A|\) denoting the cardinality of \( A \)).

If \( X = \Gamma \) is a countable group, up to coarse equivalence, there is a unique way to endow \( \Gamma \) with a left invariant metric (induced naturally by a proper length function \( l \), i.e., \( d(x, y) := l(x^{-1}y) \)) for which the resulting metric space has bounded geometry (see [Tu01] Lemma 2.1 and Lemma 4.1). In the sequel, all groups will be considered endowed with such a metric. For finitely generated groups, the metric will always be considered to be induced by a length function associated to a fixed finite generating set. This metric will be denoted by “\( d_1 \)”, and \( B_T(\gamma, S) \) will denote the closed ball centered at \( \gamma \) of radius \( S \) in \( \Gamma \) with respect to that metric.

2.2. Theorem. Let \( \Gamma \) be a countable group, then the following assertions are equivalent:

(i) \( \Gamma \) has property A;
(ii) The action of \( \Gamma \) on its Stone-Čech compactification by left translations is topologically amenable;
(iii) The Roe \( C^* \)-algebra of \( \Gamma \) is nuclear;
(iv) The reduced \( C^* \)-algebra of \( \Gamma \) is exact.

The equivalence “\( (i) \iff (ii) \)” is due to N. Higson and J. Roe [HR00], the equivalence “\( (ii) \iff (iii) \)” can be found in [AD00] and the equivalence “\( (iii) \iff (iv) \)” is due to N. Ozawa [Oz00]. In his proof, N. Ozawa introduces positive definite kernels to emphasize the links between the geometric properties of a group and properties of its reduced \( C^* \)-algebra.

2.3. Definition. Let \( X \) be a set. A function \( \psi : X \times X \to \mathbb{R} \) is said to be a positive definite kernel if \( \psi(x, y) = \psi(y, x) \) for all \( x, y \in X \), and if for every integer \( n \geq 1 \), for every \( x_1, \ldots, x_n \in X \) and for every \( \lambda_1, \ldots, \lambda_n \in \mathbb{R} \), the following inequality holds:

\[
\sum_{1 \leq i, j \leq n} \lambda_i \lambda_j \psi(x_i, x_j) \geq 0.
\]

2.4. Definition. A discrete metric space \( X \) satisfies Ozawa’s property if for every \( R > 0 \) and every \( \varepsilon > 0 \) there exist a positive definite kernel \( \psi : X \times X \to \mathbb{R} \) and a constant \( S \geq R \) such that \( \text{supp}(\psi) \subseteq \{(x, y) \in X \times X \mid d(x, y) \leq S\} \) and \( |1 - \psi(x, y)| < \varepsilon \) for every \( x, y \in X \) such that \( d(x, y) \leq R \). Such kernels \( \psi \) are called Ozawa kernels (or, more precisely, \((R, \varepsilon)\)-Ozawa kernels).

In the case of countable groups, the following result is a consequence of Theorem 2.2 but a direct proof (without any reference to \( C^* \)-algebras) is the subject of Proposition 3.2 in [Tu01] together with Lemma 3.5 of [HR00].

2.5. Theorem. Let \( X \) be a discrete metric space of bounded geometry. The following assertions are equivalent:

(i) \( X \) has property A;
(ii) \( X \) has Ozawa’s property.

2.6. Remarks. If one wants to compare property A and amenability, Ozawa’s property plays the role of Hulanicki’s property (see for instance appendix G in [BH05]). The proof of J-L. Tu in [Tu01] allows us to systematically deduce Ozawa kernels from the sets in Definition 2.1.

2.7. Proposition. Let \( X \) be a discrete metric space. Let \( \varepsilon > 0, R > 0, S > 0, \) and \( \{A_x\}_{x \in X} \) as in Definition 2.1. Then

\[
\psi_X(x, y) := \sum_{z \in X} \left( \frac{\sum_{n \in \mathbb{N}} \chi_{A_x}(z, n)}{|A_x|} \right)^{1/2} \left( \frac{\sum_{n \in \mathbb{N}} \chi_{A_y}(z, n)}{|A_y|} \right)^{1/2}
\]

is an \((R, 2\varepsilon)\)-Ozawa kernel, and its support is contained in \( \{(x, y) \in X \times X \mid d(x, y) \leq 2S\} \).
Proof of Proposition 2.7. One has $\frac{|A \cap A_\gamma|}{|A_\gamma|} < \varepsilon$, whenever $d(x, y) \leq R$. This is equivalent to $\frac{\|X_{A_\gamma} - X_A\|}{\|X_{A_\gamma}\|} < \varepsilon^{1/2}$, whenever $d(x, y) \leq R$ (where $\| \cdot \| := \| \cdot \|_{L^2(X \times N)}$). Then,
\[
\frac{\|X_{A_\gamma}\|}{\|X_A\|} \leq \frac{\|X_{A_\gamma} - X_A\|}{\|X_{A_\gamma}\|} \leq 2 \frac{\|X_{A_\gamma} - X_A\|}{\|X_{A_\gamma}\|} < 2\varepsilon^{1/2},
\]
whenever $d(x, y) \leq R$. For every $x \in X$, let us consider the function $\eta_x : X \to \mathbb{R}, z \mapsto \frac{\|X_{(z, \eta)(x)}\|}{\|X_{\eta(z)}\|} = \left(\sum_{n \in \mathbb{N}} \frac{\lambda_{A_\gamma}(z, n)}{|A_x|}\right)^{1/2}$. We have $\|\eta_x - \eta_y\|_{L^2(X)} \leq \frac{\|X_{A_\gamma}\|}{\|X_{A_\gamma}\|} \leq \frac{\|X_{A_\gamma} - X_A\|}{\|X_{A_\gamma}\|} < 2\varepsilon^{1/2}$ whenever $d(x, y) \leq R$.

Finally, put
\[
\psi(x, y) := \langle \eta_x, \eta_y \rangle = \sum_{z \in X} \left(\frac{\sum_{n \in \mathbb{N}} \lambda_{A_\gamma}(z, n)}{|A_x|}\right)^{1/2} \left(\frac{\sum_{n \in \mathbb{N}} \lambda_{A_\gamma}(z, n)}{|A_y|}\right)^{1/2}.
\]
This is a positive definite kernel satisfying $\text{supp}(\psi_X) \subset \{(x, y) \in X \times X \mid d(x, y) \leq 2S\}$ and $1 - \psi(x, y) = \frac{1}{2} \|\eta_x - \eta_y\|_{L^2(X)} \leq 2\varepsilon$ for all $x, y \in X$ with $d(x, y) \leq R$.

2.8. Remark. If there is only one “level” in the definition of property $A$, i.e., if $X \times \mathbb{N}$ may be replaced by $X$ in Definition 2.1 then Ozawa kernels can be taken of the form $(x, y) \mapsto \frac{|A \cap A_\gamma|}{\sqrt{|A| |A_\gamma|}} = \frac{1}{\sqrt{|A| |A_\gamma|}} \sum_{z \in X} \lambda_{A_\gamma}(z, 1)$. This is actually the case for all known examples of groups with property $A$. In particular, using examples 2.3 and 2.4 in [Yu09], we easily recover Ozawa kernels obtained by S. Campbell in [C08] (see the following examples).

Suppose $X = \Gamma$ is a group, and the Ozawa kernels are $\Gamma$-invariant. Set $\phi(\gamma) := \psi(e, \gamma)$. Then we take an approximation of the unity in $\Gamma$ by positive definite functions with finite support. Actually this characterizes amenability: if a group $\Gamma$ has property $A$ and admits $\Gamma$-invariant Ozawa kernels, then $\Gamma$ is amenable. The converse is also true, see the first example below.

2.9. Examples. Let $\varepsilon, R > 0$.

(i) Let $\Gamma$ be a countable amenable group. The length function on $\Gamma$ being assumed to be proper, $F := B_T(e, R)$ is a finite set. Using Folner’s definition of amenability, there exists a finite subset $A \subset \Gamma$ such that $\frac{|A \Delta A_\gamma|}{|A_\gamma|} < \frac{2\varepsilon}{2\varepsilon + 1}$ for every $\gamma \in F$. Then we define $A_\gamma := (\gamma A) \setminus \{1\}$ (note that $|A_\gamma| = |A|$ for every $\gamma$). If $\gamma, \gamma' \in \Gamma$ are such that $d_T(\gamma, \gamma') \leq R$, we have $\gamma^{-1}\gamma' \in F$ and $|A_\gamma \Delta A_{\gamma'}| = |(\gamma^{-1}\gamma')A_\gamma A_{\gamma'}| < \frac{2\varepsilon}{2\varepsilon + 1}|A_\gamma|$. We also have
\[
|A_\gamma \cap A_{\gamma'}| = \frac{1}{2}||A_\gamma| + |A_{\gamma'}| - |A_\gamma \Delta A_{\gamma'}|| > \frac{4}{4 + \varepsilon}|A_\gamma|
\]
Therefore $|A_\gamma \Delta A_{\gamma'}| < \frac{2}{3}|A_\gamma \cap A_{\gamma'}|$. Moreover, if $S := \max_{a \in A} d_T(e, a)$, $(\gamma, 1) \in A_{\gamma'}$, implies $d_T(\gamma, \gamma') \leq S$.

Hence, the family $\{A_\gamma\}_{\gamma \in \Gamma}$ satisfies Definition 2.1 and by Proposition 2.7 we have shown that
\[
\psi(\gamma, \gamma') \mapsto \frac{|(\gamma A) \cap (\gamma' A)|}{|A|}
\]
is an $(R, \varepsilon)$-Ozawa kernel for $K$ (and this kernel is clearly $\Gamma$-invariant).

(ii) Let $\Gamma$ be a finitely generated free group, and $T$ be the standard Cayley graph (which is a tree) of $\Gamma$. We fix a geodesic ray $r_0$ in $T$, and for any $\gamma \in \Gamma$, we denote by $r(\gamma)$ the unique geodesic ray starting from $\gamma$ such that $r(\gamma) \cap r_0$ is a non-empty geodesic ray. Let $S > \frac{2R}{2S + 1} = \frac{2}{3}$, and for every $\gamma \in \Gamma$, let us define $A_\gamma := [r(\gamma) \cap B_T(\gamma, S)] \times \{1\} \subset \Gamma \times \mathbb{N}$ (note that $|A_\gamma| = S + 1$ for every $\gamma$). Now, let $\gamma, \gamma' \in \Gamma$ such that $d_T(\gamma, \gamma') \leq R$. As $S > \frac{2R}{2S + 1}$, $A_\gamma \cap A_{\gamma'} \neq \emptyset$, and $|A_\gamma \Delta A_{\gamma'}|$ is maximal (being equal to $d_T(\gamma, \gamma')$) exactly when $A_{\gamma'}$ and $A_{\gamma'}$ intersect in just one point, (i.e. when $A_{\gamma'}$ and $A_{\gamma'}$ realize the unique geodesic ray between $\gamma$ and $\gamma'$ in $T$), then for any $\gamma, \gamma' \in \Gamma$ such that $d_T(\gamma, \gamma') \leq R$, one has $|A_\gamma \Delta A_{\gamma'}| \leq R$. We also have
\[
|A_\gamma \cap A_{\gamma'}| = \frac{1}{2}||A_\gamma| + |A_{\gamma'}| - |A_\gamma \Delta A_{\gamma'}|| \geq \frac{2(S + 1) - R}{2}.
\]
Therefore, by our choice of $S$, we obtain
\[ |A_\gamma \triangle A_{\gamma'}| \leq \frac{2R}{2(S + 1) - R} |A_\gamma \cap A_{\gamma'}| < \frac{\epsilon}{2} |A_\gamma \cap A_{\gamma'}|. \]
Moreover, by construction, $(\gamma, 1) \in A_{\gamma'}$ implies $d_\Gamma(\gamma, \gamma') \leq S$. Hence, we have shown that
\[ (\gamma, \gamma') \mapsto \frac{|A_\gamma \cap A_{\gamma'}|}{S + 1} \]
is an $(R, \epsilon)$-Ozawa kernel for $\Gamma$.

2.10. Remark. Using, the so-called GNS construction (see for instance Proposition 3 of Chapter 5 in [HV]), one knows that any Ozawa kernel on a discrete metric space $X$,

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Furthermore, by construction, $(\gamma, 1) \in A_{\gamma'}$ implies $d_\Gamma(\gamma, \gamma') \leq S$. Hence, we have shown that
\[ (\gamma, \gamma') \mapsto \frac{|A_\gamma \cap A_{\gamma'}|}{S + 1} \]
is an $(R, \epsilon)$-Ozawa kernel for $\Gamma$.

2.11. Proposition. Let $X$ be a discrete metric space with bounded geometry, if $X$ has property A, then an $(R, \epsilon)$-Ozawa kernel for $X$ can always be taken of the form $\psi : (x, y) \mapsto \langle \lambda(x), \lambda(y) \rangle$, for some $\lambda : X \to l^2(X)_1 := \{ \xi \in l^2(X) \mid \|\xi\|_{l^2(X)} = 1 \}$. Unfortunately, in general, we have no information about the support of the functions in the range of $\lambda$. However, if $X$ has bounded geometry, as a corollary of Theorem 2.10, we have the following:

3. Group extensions

The exactness of $C^*$-algebras is stable under taking extensions. This is a result due to E. Kirchberg and S. Wassermann (see [KW95]), but the proof is quite technical and done in the context of $C^*$-algebras. G. Yu [Yu00] indicates how to prove the stability of property A under taking semi-direct products leaving details to the reader. Here we give a detailed simple proof of the stability under general extensions by expliciting Ozawa kernels. It is inspired by [DG03] (see also [ADR00]).

3.1. Theorem. Let $1 \to H \xrightarrow{i} \Gamma \xrightarrow{\pi} G \to 1$ be a short exact sequence of countable groups. If $H$ and $G$ have property A then $\Gamma$ has property A.

As an easy consequence, we obtain the following:

3.2. Corollary. Let $\Gamma$ be a countable group, and let $H$ be a finite index subgroup. Then property A for $H$ implies property A for $\Gamma$.

Proof of Corollary 3.2. Let $\{\gamma_1, \ldots, \gamma_n\}$ be a set of representatives for the left cosets of $H$ in $\Gamma$. Then $N := \bigcap_{i=1}^n \gamma_i H \gamma_i^{-1}$ is also a finite index subgroup of $\Gamma$, and moreover (by construction) $N$ is normal in $\Gamma$. Being a subgroup of $H$, $N$ also has property A. Now we can conclude applying theorem 3.1 to the extension $1 \to N \to \Gamma \to \Gamma/N \to 1$ using the fact that a finite group obviously has property A.

3.3. Remark. Actually the preceding result is also true if we replace the finite index hypothesis by the property A of $G/H$ viewed as an abstract metric space endowed with the quotient metric (see Corollary 3.7).

3.4. Notations. For the proof of theorem 3.1, given a group extension $1 \to H \xrightarrow{i} \Gamma \xrightarrow{\pi} G \to 1$, we deduce Ozawa kernels for $\Gamma$ from Ozawa kernels for $H$ and $G$. Let $l_\Gamma$ be an arbitrary proper length function on $\Gamma$. In the sequel, we identify $H$ with $i(H) \subset \Gamma$, and then we consider the restriction $l_H := (l_\Gamma)_{|H}$ of $l_\Gamma$ on $H$. On $G$, we will consider the length function $l_G$ defined by
\[ l_G(g) := \min\{l_\Gamma(\gamma) \mid \gamma \in \Gamma, \pi(\gamma) = g\}. \]
One can easily check that $l_G$ is a proper length function and that the minimum is realized. We endow $G$, $H$, and $\Gamma$ with the left invariant metrics denoted $d_G$, $d_H$ and $d_\Gamma$ associated respectively to $l_G$, $l_H$ and $l_\Gamma$. Note that $l_G$ and $l_H$ being proper, the associated metrics on $G$ and $H$ are coarsely equivalent to the original metrics on this groups.

Let us fix a set theoretical section $\sigma$ of $\pi$ in such a way that $l_\Gamma(\sigma(g)) = l_G(g)$ for every $g \in G$. The idea of the proof is then to decompose elements of $\Gamma$, using $\sigma$, into two parts, one in $H$ and one in $G$ as in
the case of a split sequence. Hence we will need the following cocycle:

\[ c : \Gamma \times G \to H, \ (\gamma, g) \mapsto \sigma(g)^{-1} \gamma \sigma(\pi(\gamma)^{-1} g). \]

For any short exact sequence, the map \( \gamma \mapsto (\pi(\gamma), c(\gamma, \pi(\gamma))) \) is a bijection between \( \Gamma \) and \( G \times H \).

The following inequalities will be useful.

### 3.5. Lemma

For any \( \gamma_1, \gamma_2 \in \Gamma \), and any \( g \in G \), one has:

(i) \( d_G(\pi(\gamma_1), \pi(\gamma_2)) \leq d_H(\gamma_1, \gamma_2) \);

(ii) \( d_H(c(\gamma_1, g), c(\gamma_2, g)) + d_G(g, \pi(\gamma_2)) \)

(iii) \( d_H(c(\gamma_1, g), c(\gamma_2, g)) \leq d_G(g, \pi(\gamma_1)) + d_H(\gamma_1, \gamma_2) + d_G(g, \pi(\gamma_2)) \).

**Proof of Lemma 3.5.** Inequality (i) comes from the definition of \( d_G \). To see inequalities (ii) and (iii), it suffices to use sub-additivity of length functions and to write that:

\[ |\gamma_1^{-1} \gamma_2 = \sigma(\pi(\gamma_1)^{-1} g) c(\gamma_1, g)^{-1} c(\gamma_2, g)(\sigma(\pi(\gamma_2)^{-1} g))^{-1}. \]

By Theorem 2.6, Theorem 3.1 is a consequence of the following statement:

### 3.6. Theorem

Let \( 1 \to H \to \Gamma \xrightarrow{\pi} G \to 1 \) be a short exact sequence of countable groups, \( H \) and \( G \) having property \( A \). Let \( \psi_G : (g_1, g_2) \mapsto (\lambda(g_1), \lambda(g_2)) \) and \( \psi_H : (h_1, h_2) \mapsto (\mu(h_1), \mu(h_2)) \) be Ouzawa kernels for \( G \) and \( H \) given by Proposition 2.11 (with \( \lambda : G \to \ell^2(G) \) and \( \mu : H \to \ell^2(H) \)). Then maps of the form

\[ \psi_T : (\gamma_1, \gamma_2) \mapsto \sum_{g \in G} \lambda(\pi(\gamma_1))(g) \cdot \lambda(\pi(\gamma_2))(g) \cdot \psi_H(c(\gamma_1, g), c(\gamma_2, g)) \]

are Ouzawa kernels for \( \Gamma \) (where \( c \) denotes the cocycle defined above).

**Proof of Theorem 3.6.** First, note that, because \( \lambda(\pi(\gamma)) \) has finite support, the sum defining \( \psi_T(\gamma_1, \gamma_2) \) is a finite sum. To check that \( \psi_T \) is indeed a positive definite kernel, it suffices to note that for every \( (\gamma_1, \gamma_2) \in \Gamma \times \Gamma \), \( \psi_T(\gamma_1, \gamma_2) = \langle \nu(\gamma_1), \nu(\gamma_2) \rangle \), where

\[ \nu : \Gamma \to \ell^2(G), \gamma \mapsto \nu(\gamma) : g \mapsto \lambda(\pi(\gamma))(g) \cdot \mu(c(\gamma, g)). \]

Now, let \( \varepsilon > 0 \) and \( R > 0 \) be fixed. By hypothesis, there exist positive constants \( S_1 \) and \( S_2 \) such that:

(a) \( d_G(g_1, g_2) \leq R \Rightarrow |1 - \psi_G(g_1, g_2)| < \frac{\varepsilon}{2} \);

(b) \( d_H(h_1, h_2) \leq R + 2S_1 \Rightarrow |1 - \psi_H(h_1, h_2)| < \frac{\varepsilon}{2} \);

(c) \( \text{supp}(\lambda(g)) \subseteq B_G(g, S_1), \forall g \in G \);

(d) \( d_H(h_1, h_2) > S_2 \Rightarrow \psi_H(h_1, h_2) = 0 \).

For any \( \gamma_1, \gamma_2 \in \Gamma \), we have:

\[ |1 - \psi_T(\gamma_1, \gamma_2)| \leq \sum_{g \in G} |1 - \psi_H(c(\gamma_1, g), c(\gamma_2, g))| \cdot \lambda(\pi(\gamma_1))(g) \cdot \lambda(\pi(\gamma_2))(g) + |1 - \psi_G(\pi(\gamma_1), \pi(\gamma_2))|. \quad (*) \]

Then, if \( d_T(\gamma_1, \gamma_2) \leq R \), by Lemma 3.5 (i) and by (a), one has \( |1 - \psi_G(\pi(\gamma_1), \pi(\gamma_2))| < \frac{\varepsilon}{2} \).

On the other hand, by (c), the sum in the first term of \( (*) \) is just over the set of \( g \in G \) such that \( d_G(g, \pi(\gamma_1)) \leq S_1 \) and \( d_G(g, \pi(\gamma_2)) \leq S_1 \). But for such \( g \in G \), by (iii) of Lemma 3.5, one has \( d_H(c(\gamma_1, g), c(\gamma_2, g)) \leq R + 2S_1 \). Therefore, by (b) and by the Cauchy-Schwarz inequality (as \( \sum_{g \in G} \lambda(x)(g)^2 = 1 \) for all \( x \in G \), one obtains:

\[ \sum_{g \in G} (1 - \psi_H(c(\gamma_1, g), c(\gamma_2, g))) \cdot \lambda(\pi(\gamma_1))(g) \cdot \lambda(\pi(\gamma_2))(g) \leq \frac{\varepsilon}{2}. \]

Hence, \( |1 - \psi_T(\gamma_1, \gamma_2)| < \varepsilon \) for all \( \gamma_1, \gamma_2 \in \Gamma \) such that \( d_T(\gamma_1, \gamma_2) \leq R \).

Moreover, again by (c) and by the Cauchy-Schwarz inequality, we have:

\[ |\psi_T(\gamma_1, \gamma_2)| \leq \max\{|\psi_H(c(\gamma_1, g), c(\gamma_2, g))| : g \in G, d_G(g, \pi(\gamma_k) \leq S_1, k = 1, 2\}. \quad (***) \]
If $\gamma_1, \gamma_2 \in \Gamma$ are that $d_{l}(\gamma_1, \gamma_2) > 2S_1 + S_2$, by (ii) of Lemma 3.5, one has:

$$d_{l}(c(\gamma_1, g), c(\gamma_2, g)) \geq d_{l}(\gamma_1, \gamma_2) - d_{c}(g, \pi(\gamma_1)) - d_{c}(g, \pi(\gamma_2)) > S_2.$$  

Thus, by (d), we deduce from (****) that $\text{supp}(\psi_{l}) \subset \{(\gamma_1, \gamma_2) \in \Gamma \times \Gamma \mid d_{l}(\gamma_1, \gamma_2) \leq 2S_1 + S_2 \}$. This concludes the proof.

3.7. Remark. Suppose that short exact sequence splits, i.e., when $\Gamma \cong H \rtimes G$. Then viewing $G$ as a subgroup of $\Gamma$, the metric corresponding to the restriction to $G$ of $l_{\Gamma}$ is coarsely equivalent to the metric $d_{c}$ defined above. Hence, in this case, up to coarse equivalence, $c(\gamma, g)$ may be replaced by the action of $g^{-1}\pi(\gamma)$ on $\pi(\gamma)^{-1}\Gamma$ (the component of $\gamma$ in $H$).

4. Metric spaces with finite asymptotic dimension

Among finitely generated groups, there are important examples of groups of finite asymptotic dimension, for instance, Coxeter groups [109], one relator groups [106] and hyperbolic groups [105]. It is well known that for a discrete metric space $X$ of bounded geometry, finite asymptotic dimension implies property A (see [HR00] and [DG05]). We follow the proof of this result but make it more precise as far as Ozawa kernels are concerned. First, recall:

4.1. Definition. Let $(X, d)$ be a metric space and $k \in \mathbb{N}$. The space $X$ is said to have asymptotic dimension less than $k$, denoted by $\text{asdim} X \leq k$, if for any $L > 0$, there is an open cover $\mathcal{U} := \{U_i\}_{i \in I}$ of $X$ satisfying:

(i) $\sup_{i \in I} \text{diam}(U_i) < \infty$ ($\mathcal{U}$ is said to be uniformly bounded);

(ii) The Lebesgue number of $\mathcal{U}$, $L(\mathcal{U}) := \inf \{\max \{d(x, X \setminus U_i) \mid x \in X\} \mid x \in X\}$, satisfies $L(\mathcal{U}) \geq L$;

(iii) For every $x \in X$, $\{|\{i \in I \mid x \in U_i\}| \leq k + 1$ ($\mathcal{U}$ is said to have multiplicity $\leq k + 1$).

4.2. Lemma. Let $(X, d)$ be a discrete metric space admitting a uniformly bounded cover $\mathcal{U} := \{U_i\}_{i \in I}$ with multiplicity $\leq k + 1$. Then the kernel $\psi_{l}$ defined by

$$\psi_{l} : X \times X \to \mathbb{R}, \ (x, y) \mapsto \sum_{i \in I} \left( \frac{d(x, X \setminus U_i)}{\sum_{j \in I} d(x, X \setminus U_j)} \right)^{1/2} \left( \frac{d(y, X \setminus U_i)}{\sum_{j \in I} d(y, X \setminus U_j)} \right)^{1/2}$$

is positive definite and satisfies $|1 - \psi_{l}(x, y)| \leq \frac{(k+1)(2k+3)}{L(\mathcal{U})} d(x, y)$ for every $x, y \in X$.

Proof of Lemma 4.2. The kernel $\psi_{l}$ is a positive definite kernel as $\psi_{l}(x, y) = \langle \lambda(x), \lambda(y) \rangle$, where $\lambda : X \to l^2(I), \ x \mapsto \left( \frac{d(x, X \setminus U_i)}{\sum_{j \in I} d(x, X \setminus U_j)} \right)_{i \in I}$. Moreover, we have

$$1 - \psi_{l}(x, y) = \frac{1}{2} \|\lambda(x) - \lambda(y)\|_{l^2(I)}^2 = \frac{1}{2} \sum_{i \in I} \left( \frac{d(x, X \setminus U_i)}{\sum_{j \in I} d(x, X \setminus U_j)} \right)^{1/2} \left( \frac{d(y, X \setminus U_i)}{\sum_{j \in I} d(y, X \setminus U_j)} \right)^{1/2}$$

$$\leq \frac{1}{2} \sum_{i \in I} \left| \frac{d(x, X \setminus U_i)}{\sum_{j \in I} d(x, X \setminus U_j)} - \frac{d(y, X \setminus U_i)}{\sum_{j \in I} d(y, X \setminus U_j)} \right|$$

$$\leq (k + 1) \max_{i \in I} \left| \frac{d(x, X \setminus U_i)}{\sum_{j \in I} d(x, X \setminus U_j)} - \frac{d(y, X \setminus U_i)}{\sum_{j \in I} d(y, X \setminus U_j)} \right|$$

the last inequality coming from the fact that the multiplicity of $\mathcal{U}$ is less than $k + 1$, i.e.,

$$\{|\{i \in I \mid d(x, X \setminus U_i) \text{ or } d(y, X \setminus U_i) \neq 0\}| \leq 2k + 2$$

However for each $i \in I$

$$\left| \frac{d(x, X \setminus U_i)}{\sum_{j \in I} d(x, X \setminus U_j)} - \frac{d(y, X \setminus U_i)}{\sum_{j \in I} d(y, X \setminus U_j)} \right|$$

$$\leq \left| \frac{d(x, X \setminus U_i)}{\sum_{j \in I} d(x, X \setminus U_j)} - \frac{d(y, X \setminus U_i)}{\sum_{j \in I} d(y, X \setminus U_j)} \right| + \left| \frac{d(y, X \setminus U_i)}{\sum_{j \in I} d(y, X \setminus U_j)} - \frac{d(y, X \setminus U_i)}{\sum_{j \in I} d(y, X \setminus U_j)} \right|$$

(*)
Then, as \( \sum_{j \in I} d(x, X \setminus U_j) \geq L(\mathcal{U}) \) and \( \frac{d(y, X \setminus U_j)}{\sum_{j \in I} d(y, X \setminus U_j)} \leq 1 \), we obtain
\[
\left| \frac{d(x, X \setminus U_i)}{\sum_{j \in I} d(x, X \setminus U_j)} - \frac{d(y, X \setminus U_i)}{\sum_{j \in I} d(y, X \setminus U_j)} \right| \leq \frac{1}{L(\mathcal{U})} \left[ |d(x, X \setminus U_i) - d(y, X \setminus U_i)| + \sum_{j \in I} |d(x, X \setminus U_j) - d(y, X \setminus U_j)| \right]
\]
therefore, by (+) and by the inequality \( |d(x, X \setminus U_j) - d(y, X \setminus U_j)| \leq d(x, y) \), we deduce
\[
\left| \frac{d(x, X \setminus U_i)}{\sum_{j \in I} d(x, X \setminus U_j)} - \frac{d(y, X \setminus U_i)}{\sum_{j \in I} d(y, X \setminus U_j)} \right| \leq \frac{(2k + 1)R}{L(\mathcal{U})}
\]
\[\square\]

4.3. Theorem. Let \((X, d)\) be a discrete metric space with \( \text{asdim} X \leq k \), and let \( \epsilon, R > 0 \). We fix a uniformly bounded cover of \( X \) with multiplicity \( \leq k + 1 \), \( \mathcal{U} := \{U_i\}_{i \in I} \), such that \( L(\mathcal{U}) > \frac{(k+1)(2k+3)R}{\epsilon} \).

Then the kernel \( \psi_{\mathcal{U}} \) defined in Lemma 4.2 is an \((R, \epsilon)\)-Ozawa kernel for \( X \). In particular, \( X \) has property A.

Proof of Theorem 4.3. If \( S := \sup_{i \in I} \text{diam}(U_i) \), then for every \( x, y \in X \) such that \( d(x, y) > S \) and for any \( i \in I \), we have \( d(x, X \setminus U_i) = 0 \) or \( d(y, X \setminus U_i) = 0 \), thus \( \psi_{\mathcal{U}}(x, y) = 0 \). Moreover, for every \( x, y \in X \) such that \( d(x, y) \leq R \), by Lemma 4.2 and by our choice for \( L(\mathcal{U}) \), we have
\[
|1 - \psi_{\mathcal{U}}(x, y)| \leq \frac{(k + 1)(2k + 3)R}{L(\mathcal{U})} \leq \epsilon.
\]
\[\square\]

5. Metric spaces with Hilbert space compression \( > 1/2 \)

It is well known that any discrete metric space with bounded geometry having property A is uniformly embeddable in a Hilbert space. The converse is not known. A result of E. Guentner and J. Kaminker gives a partial converse: a finitely generated group with Hilbert space compression \( > 1/2 \) has property A (see Theorem 3.2 in [GK04]). In this section, we give a proof (strongly inspired by [GK04]) of a slightly more general result. First of all, let us recall the definition of the Hilbert space compression.

5.1. Definition. Let \((X, d)\) be a metric space and \( \mathcal{H} \) be an Hilbert space. A map \( f : X \to \mathcal{H} \) is said to be a uniform embedding if there exist non-decreasing functions \( \rho_{\pm}(f) : \mathbb{R} \to \mathbb{R} \) such that:
\[
(i) \quad \rho_{-}(f)(d(x, y)) \leq ||f(x) - f(y)||_{\mathcal{H}} \leq \rho_{+}(f)(d(x, y)), \text{ for all } x, y \in X;
(ii) \quad \lim_{r \to +\infty} \rho_{\pm}(f)(r) = +\infty.
\]
Then the Hilbert space compression of the metric space \( X \), denoted by \( R(X) \), is defined as the sup of all \( \beta \geq 0 \) for which there exists a uniform embedding into a Hilbert space \( f \) with \( \rho_{+}(f) \) affine and \( \rho_{-}(f)(r) = r^\beta \) for \( r \) large enough.

5.2. Definition. A metric space \((X, d)\) is said to be quasi-geodesic if there exist \( \delta > 0 \) and \( \lambda \geq 1 \) such that for all \( x, y \in X \) there exists a sequence \( x_0 = x, x_1, \ldots, x_n = y \) of elements of \( X \) satisfying
\[
d(x_{i-1}, x_i) \leq \delta \text{ for every } i = 1, \ldots, n, \text{ and }
\sum_{i=1}^{n} d(x_{i-1}, x_i) \leq \lambda d(x, y).
\]
Such a sequence \( x_0 = x, x_1, \ldots, x_n = y \) will be called a \((\lambda, \delta)\)-chain of length \( n \) from \( x \) to \( y \).

We will need the following two lemmas.

5.3. Lemma. Let \((X, d)\) be a quasi-geodesic metric space. There exists an integer-valued metric \( \tilde{d} \) on \( X \) such that \((X, \tilde{d})\) is quasi-geodesic and quasi-isometric to \((X, d)\).

Proof of Lemma 5.3. Let \( \delta > 0 \) and \( \lambda \geq 1 \) as in Definition 5.2. We define a metric \( \tilde{d} \) on \( X \) by setting
\[
\tilde{d}(x, y) := \min \{ n \in \mathbb{N} \mid \text{there exists a } (\lambda, \delta)\text{-chain of length } n \text{ from } x \text{ to } y \}.
\]
This integer-valued metric is clearly quasi-geodesic (with $\lambda = \delta = 1$). It remains to show that $(X, \tilde{d})$ is quasi-isometric to $(X, d)$. Let us fix $x, y \in X$ and set $\tilde{d}(x, y) := n$. On the one hand, by definition of $n$, one can find a $(\lambda, \delta)$-chain of length $n$ from $x$ to $y$. In particular we have

$$d(x, y) \leq \sum_{i=1}^{n} d(x_{i-1}, x_{i}) \leq \delta n = \delta \tilde{d}(x, y).$$

On the other hand, again by definition of $n$, we must have $d(x_{i-1}, x_{i}) + d(x_{i}, x_{i+1}) > \delta$ for every $i = 1, \ldots, n - 1$ (otherwise, we could forget one of the $x_{i}$'s and this would contradict the minimality of $n$). Hence, in the chain, there is at least $\lfloor \frac{n}{2} \rfloor \geq \frac{n}{2}$ successive distances which are greater than $\frac{\delta}{2}$. Then, we obtain that

$$\left( \frac{n - 1}{2} \right) \frac{\delta}{2} \leq \sum_{i=1}^{n} d(x_{i-1}, x_{i}) \leq \lambda d(x, y)$$

and we deduce that

$$\tilde{d}(x, y) \leq 4\delta d(x, y) + 1.$$

\[\square\]

5.4. Lemma. Let $(X, d)$ be a discrete quasi-geodesic metric space of bounded geometry. Then the growth of $X$ is at most exponential, i.e., there exist constants $B, L > 0$ such that $|B(x, R)| \leq BL^R$ for every $R > 0$ and for every $x \in X$ (where $B(x, R) := \{ y \in X \mid d(x, y) \leq R \}$ denotes the closed ball of radius $R$ centered at $x$).

Proof of Lemma 5.4. Let $\delta > 0$ and $\lambda \geq 1$ as in Definition 5.2. By the bounded geometry assumption on $X$, $B_{\delta} := \max_{x \in X} |B(x, \delta)| < \infty$. Let us fix $R > 0$ and $x \in X$. In order to estimate the number of elements in $B(x, R)$ it suffices to estimate the number of minimal $(\lambda, \delta)$-chains starting at $x$ with end-point in $B(x, R)$. On the one hand, if $(x_0 = x, x_1, \ldots, x_n = y)$ is a $(\lambda, \delta)$-chain of length $n = \tilde{d}(x, y)$ with $y \in B(x, R)$, by the proof of Lemma 5.3 we have $n \leq \frac{4\lambda R}{\delta} + 1$. On the other hand, by the definition of $B_{\delta}$, for a fixed $n$, the number of $(\lambda, \delta)$-chains of length $n$ starting at $x$ with end-point in $B(x, R)$ is at most $B_{\delta}^n$. Therefore, we deduce that

$$|B(x, R)| \leq \sum_{n=0}^{\frac{4\lambda R}{\delta} + 1} B_{\delta}^n \leq BL^R$$

with $L := B_{\frac{4\lambda}{\delta}}$ and $B := B_{\delta}^2$. \[\square\]

5.5. Theorem. Let $(X, d)$ be a discrete quasi-geodesic metric space of bounded geometry such that $R(X) > 1/2$, then $X$ has property $A$.

Proof of Theorem 5.5. By Lemma 5.3, we can suppose that the metric $d$ on $X$ takes integer values. For every $n \in \mathbb{N}$ and $x \in X$, we will denote $S(x, n) := \{ y \in X \mid d(x, y) = n \}$.

Let $R > 0$ and $0 < \epsilon < 1$ fixed. By hypothesis, there exist a Hilbert space $\mathcal{H}$, a map $f : X \to \mathcal{H}$, and positive constants $\alpha, C, D$ such that $\| f(x) - f(y) \|_{\mathcal{H}} \leq \alpha + C d(x, y) + D$ for every $x, y \in X$ and

$$\| f(x) - f(y) \|_{\mathcal{H}} \geq n^{\frac{1+\alpha}{2}}$$

for every $n, x, y \in X$ such that $d(x, y) \geq n \geq n_0$.

Then, for a fixed $k > \frac{CR + D}{\ln(1 - \frac{\epsilon}{2})}$, we consider

$$\phi : X \times X \to \mathbb{R}, \ (x, y) \mapsto \exp \left( - \frac{\| f(x) - f(y) \|_{\mathcal{H}}^2}{k} \right)$$

By the Schoenberg’s theorem (see for instance [HV]), $\phi$ is a positive definite kernel on $X$. Moreover, by our choice of $k$, for every $x, y \in X$ such that $d(x, y) \leq R$ we have

$$|1 - \phi(x, y)| < \frac{\epsilon}{2}.$$

Unfortunately, $\phi$ does not have finite width and therefore is not an Ozawa kernel on $X$. Actually, the remainder of the proof will be devoted to approximate $\phi$ in a suitable way using the hypothesis on the
Hilbert space compression of $X$. Let us formally consider the kernel operator associated to $\phi$ defined on $\xi \in l^2(X)$ by

$$U_\phi(\xi)(x) := \sum_{y \in X} \phi(x, y)\xi(y).$$

The key point of the proof is that $U_\phi$ defines a bounded positive operator from $l^2(X)$ into itself. Indeed, for every $\xi \in l^2(X)$, on the one hand, by Cauchy-Schwarz’s inequality, we have

$$\sum_{x \in X} \left( \sum_{y \in X} \phi(x, y)\xi(y) \right)^2 \leq \sum_{x \in X} \left( \sum_{y \in X} \phi(x, y)^{1/2} \phi(x, y)^{1/2}\xi(y) \right)^2 \leq \left( \sup_{x \in X} \sum_{y \in X} \phi(x, y) \right)^2 \|\xi\|_{l^2(X)}^2.$$  \quad (**)  

On the other hand, by Lemma 5.4 one can find $B, L > 0$ such that $|S(x, n)| \leq BL^n$ for every $n$ and every $x$. Then, if we fix $N \geq n_0$ satisfying $Le^{-N^n/k} < 1$, for any $x \in X$ we obtain that

$$\sum_{y \in X} \phi(x, y) = \sum_{n \geq 0} \sum_{y \in S(x,n)} \phi(x, y) = \sum_{0 \leq n \leq N} \sum_{y \in S(x,n)} \phi(x, y) + \sum_{n > N} \sum_{y \in S(x,n)} \phi(x, y)$$

As $\phi(x, y) \leq 1$, the first term is at most $B \sum_{0 \leq n \leq N} L^n$. Concerning the second term, by our choice of $N$, we have $\phi(x, y) \leq e^{-\frac{n+1}{k}}$ for every $n > N$ and $y \in S(x, n)$. Therefore,

$$\sum_{n > N} \sum_{y \in S(x,n)} \phi(x, y) \leq B \sum_{n > N} L^n e^{-\frac{n+1}{k}} \leq B \sum_{n > N} \left( Le^{-N^n/k} \right)^n$$

which is the remainder of a convergent geometric series and does not depend on $x$.

Hence $U_\phi : l^2(X) \to l^2(X)$ is a bounded operator with $1 \leq \|U_\phi\|$, as $U_\phi(\delta_x)(x) = 1$ for every $x \in X$. Moreover, $U_\phi$ is self-adjoint and positive as $\phi$ is symmetric and positive definite. In particular, one can consider the positive square root of $U_\phi$ which can be represented as (see for instance Theorem VI.9 in [RS72])

$$\mathcal{V}_\phi := \|U_\phi\|^{1/2} \sum_{n \geq 0} a_n \left( I - \frac{U_\phi}{\|U_\phi\|} \right)^n$$

where $I := \text{Id}_{l^2(X)}$ and $\sqrt{1-z} = \sum_{n \geq 0} a_n z^n$ converges absolutely for every $z$ such that $|z| \leq 1$.

Now fix $M_0$ large enough such that

$$\left\| \mathcal{V}_\phi - \|U_\phi\|^{1/2} \sum_{0 \leq n \leq M_0} a_n \left( I - \frac{U_\phi}{\|U_\phi\|} \right)^n \right\| < \frac{\epsilon}{4(4\|U_\phi\|^{1/2} + 1)}.$$  \quad (***)  

We fix also $M \geq R$ such that

$$\sum_{n > M} \left( Le^{-N^n/k} \right)^n < \frac{\|U_\phi\|^{1/2}\epsilon}{2^{M_0+3}(4\|U_\phi\|^{1/2} + 1)B}$$

Then we denote

$$\phi_M : X \times X \to \mathbb{R}, \; (x, y) \mapsto \begin{cases} \phi(x, y) & \text{if } d(x, y) \leq M \\
0 & \text{otherwise} \end{cases}$$

and we define $U_{\phi_M} : l^2(X) \to l^2(X)$ by $U_{\phi_M}(\xi)(x) := \sum_{y \in X} \phi_M(x, y)\xi(y)$ for every $\xi \in l^2(X)$.

If $\phi_M$ was positive definite the proof would be finished but a priori there is no reason for that.

By (**), (***) and our choice of $M$, we have

$$\|U_\phi - U_{\phi_M}\| \leq \frac{\|U_\phi\|^{1/2}\epsilon}{2^{M_0+3}(4\|U_\phi\|^{1/2} + 1)} \quad (***)$$
Finally, by setting
\[ W := \|\mathcal{U}_0\|^{1/2} \sum_{0 \leq n \leq M_0} a_n \left( I - \frac{\mathcal{U}_0}{\|\mathcal{U}_0\|} \right)^n \]
we can consider the positive definite kernel
\[ \psi : X \times X \rightarrow \mathbb{R}, \ (x, y) \mapsto \langle W(\delta_x), W(\delta_y) \rangle. \]
To prove that \(|1 - \psi(x, y)| < \epsilon\) whenever \(d(x, y) \leq R\), it suffices to show that \(|\phi(x, y) - \psi(x, y)| < \epsilon/2\) for every \(x, y \in X\). We have
\[ |\phi(x, y) - \psi(x, y)| = |(\mathcal{U}_0(\delta_x), \delta_y) - (W^*W(\delta_x), \delta_y)| \]
\[ = |\langle (V^0\phi^0 - W^*W)(\delta_x), \delta_y \rangle| \]
\[ \leq \|V^0\phi^0 - W^*W\| \]
\[ \leq (\|\phi\| + \|W\|) \|\phi - W\| \]
\[ \leq 2\|\phi - W\| \|\phi - W\|^2 \]
\[ \leq (2\|\phi\| + 1) \|\phi - W\| \]
\[ \leq 4(4\|\mathcal{U}_0\|^{1/2} + 1) \|\phi - W\| \]
But we have
\[ \|\phi - W\| \leq \left\| V - \mathcal{U}_0 \right\|^{1/2} \sum_{0 \leq n \leq M_0} a_n \left( I - \frac{\mathcal{U}_0}{\|\mathcal{U}_0\|} \right)^n \]
\[ \leq \left( \frac{\epsilon}{4(4\|\mathcal{U}_0\|^{1/2} + 1)} \right)^{1/2} \sum_{0 \leq n \leq M_0} a_n \left( I - \frac{\mathcal{U}_0}{\|\mathcal{U}_0\|} \right)^n - W \]
and moreover, as \(|a_n| \leq 1\) for every \(n\), using the inequality \(\|A^n - B^n\| \leq 2^n \|A - B\|\) (when \(\|A\| \leq 1\) and \(\|B\| \leq 2\)), we obtain
\[ \left\| \sum_{0 \leq n \leq M_0} a_n \left( I - \frac{\mathcal{U}_0}{\|\mathcal{U}_0\|} \right)^n - W \right\| \leq 2^{M_0 + 1} \frac{\|\mathcal{U}_0 - \mathcal{U}_0\|}{\|\mathcal{U}_0\|^{1/2}} \]
\[ < \frac{\epsilon}{4(4\|\mathcal{U}_0\|^{1/2} + 1)} \]
the last inequality coming from (**). Hence we deduce that
\[ \|\phi - W\| < \frac{\epsilon}{2(4\|\mathcal{U}_0\|^{1/2} + 1)} \]
and therefore \(|\phi(x, y) - \psi(x, y)| < \epsilon/2\) for every \(x, y \in X\).

It remains to show that \(\psi\) has finite width (more precisely, it has width at most \(2M_0M\)). Indeed, if \(x, y \in X\) are such that
\[ \psi(x, y) = \sum_{z \in X} \mathcal{W}(\delta_x)(z)\mathcal{W}(\delta_y)(z) \neq 0, \]
then there exists at least one \(z \in X\) such that \(\mathcal{W}(\delta_x)(z) \neq 0\) and \(\mathcal{W}(\delta_y)(z) \neq 0\). But for every \(t \in X\),
\[ \mathcal{W}(\delta_t)(z) = \|\mathcal{U}_0\|^{1/2} \sum_{s \in X} \sum_{0 \leq n \leq M_0} a_n \left( \delta - \frac{\phi_M}{\|\mathcal{U}_0\|} \right)^n(z, s) \delta_t(s) = \|\mathcal{U}_0\|^{1/2} \sum_{0 \leq n \leq M_0} a_n \left( \delta - \frac{\phi_M}{\|\mathcal{U}_0\|} \right)^n(z, t), \]
where \(\delta(z, z) = 1, \delta(z, t) = 0\) if \(z \neq t\), and where \(\lambda * \mu\) denotes the convolution product, i.e. \(\lambda * \mu(z, t) = \sum_{s \in X} \lambda(z, s)\mu(s, t)\) (\(\nu^*\) being the \(n\)-fold convolution product of \(\nu\) with itself). Hence, with these notations, for some \(p, q \leq M_0\) we have
\[ \left( \delta - \frac{\phi_M}{\|\mathcal{U}_0\|} \right)^p(z, x) \neq 0 \quad \text{and} \quad \left( \delta - \frac{\phi_M}{\|\mathcal{U}_0\|} \right)^q(z, y) \neq 0 \]
6. Groups acting on metric spaces

If $X$ is a discrete metric space, by the Švarc-Milnor Lemma, any countable group $G$ acting properly and co-compactly by isometries on $X$ is quasi-isometric (thus coarsely equivalent) to $X$. Therefore, property A of $G$ is equivalent to property A of $X$. In this setting, it is easy to make explicit Ozawa kernels:

6.1. Proposition. Let $G$ be a countable group acting properly by isometries on a discrete metric space $X$. Suppose that there exists $x_0 \in X$ such that the orbit $G \cdot x_0$ has property A. Let $R, \epsilon > 0$, let $K(R) := \max_{g \in B_k(g,R)} d_X(x_0,gx_0)$ and let $\phi$ be a $(K(R),\epsilon)$-Ozawa kernel on $G \cdot x_0$. Then

$$\tilde{\phi} : G \times G \to \mathbb{R}, \ (g,g') \mapsto \phi(gx_0,g'x_0)$$

defines an $(R,\epsilon)$-Ozawa kernel on $G$.

Proof of Proposition 6.1. By definition, $\phi$ is a positive definite kernel on $G$ satisfying $|1 - \tilde{\phi}(g,g')| < \epsilon$, whenever $d_G(g,g') \leq R$. Moreover, there exists $S > 0$ such that $\supp(\phi) \subset \{(gx_0,g'x_0) \in G \cdot x_0 \times G \cdot x_0 \mid d(gx_0,g'x_0) \leq S\}$. Hence, for every $g, g' \in G$ such that $\phi(g,g') \neq 0$, we have $g^{-1}g'x_0 \in B_X(x_0,S) \cap G \cdot x_0$ which is finite (by properness). Then, if $B_X(x_0,S) \cap G \cdot x_0 = \{g_1 \cdot x_0, \ldots, g_N \cdot x_0\}$, we obtain that $\supp(\tilde{\phi}) \subset \bigcup_{k=1}^N g_k G_0$, where $G_0$ is the (finite) stabilizer of $x_0$. This last set is finite, and if $\tilde{S}$ denotes the diameter of this set, we have $\supp(\tilde{\phi}) \subset \{(g,g') \in G \times G \mid d(g,g') \leq \tilde{S}\}$.

It is well-known that the 0-skeleton (endowed with the induced metric) of a finite dimensional CAT(0) cube complex with bounded geometry has property A (see [CN04]), then Proposition 6.1 gives a partial generalization of Theorem B in [CN04]:

6.2. Corollary. Any countable group acting properly by isometries on a finite dimensional CAT(0) cube complex with bounded geometry has property A.

6.3. Remark. Note that according to a recent result of Brodzki, Campell, Guentner, Niblo and Wright, the previous corollary remains true without the assumption of bounded geometry.

It was pointed out to me by E. Guentner that a stronger result actually holds. The following theorem is a direct consequence of ideas developed in [DG05] but does not appear explicitly. Here we give a self-contained proof, expliciting Ozawa kernels, in the setting of countable groups.

6.4. Theorem. Let $G$ be a countable group acting by isometries on a discrete metric space with bounded geometry $X$. Assume that there exists $x_0 \in X$ such that both the orbit $G \cdot x_0$ and the stabilizer $G_0$ of $x_0$ have property A. Then $G$ has property A.

Proof of Theorem 6.4. Let $R, \epsilon > 0$. We begin by proceeding as in the proof of Proposition 6.1. Denote by $K(R) := \max_{g \in B_G(g,R)} d_X(x_0,gx_0)$, and let $\phi_0$ be a $(K(R),\epsilon^2)$-Ozawa kernel for $G \cdot x_0$ given by Proposition 2.11. That is, $\phi_0 : (gx_0,g'x_0) \mapsto \langle \lambda(gx_0), \lambda(g'x_0) \rangle$ for some $\lambda : G \to l^2(G \cdot x_0)$ with $\supp(\lambda(gx_0)) \subset B_X(gx_0,S_0)$ for every $x$ and $\lambda(gx_0)(g'x_0) \geq 0$ for every $g$. For every $g \in G$, we define

$$U_g := \{g' \in G \mid gx_0 \in \supp(\lambda(g'x_0))\}$$

and $\alpha_g : G \to \mathbb{R}, \ g' \mapsto \lambda(g'x_0)(gx_0)$. Hence we obtain a cover $U_G := \{U_g\}_{g \in G}$ of $G$ and a partition of unity $\alpha := \{\alpha_g\}_{g \in G}$ subordinated to it (i.e., $\alpha_g(g') = 0$ if $g' \notin U_g$ and $\sum_{g \in G} \alpha_g(g') = 1$ for every $g' \in G$).
Moreover, if \( d_G(g_1, g_2) \leq R \), by Cauchy-Schwarz’s inequality,
\[
\sum_{g \in G} |\alpha_g(g_1) - \alpha_g(g_2)| = \|\lambda(g_1 x_0)^2 - \lambda(g_2 x_0)^2\|_{\ell^1(G \cdot x_0)} \\
\leq 2 \|\lambda(g_1 x_0) - \lambda(g_2 x_0)\|_{\ell^1(G \cdot x_0)} \\
= 2\sqrt{2} - 2\phi_0(g_1 x_0, g_2 x_0) \leq \epsilon.
\]

For any subset \( A \) of \( X_g \) and for any \( L > 0 \), we denote by \( A(L) := \{ x \in X_g \mid d_G(x, A) \leq L \} \) the \( L \)-neighborhood of \( A \) in \( X_g \).

Now the main part of the proof is to construct an Ozawa kernel on each \( X_g \) (then on each \( U_g \)) using Ozawa’s property on \( G \).

Let \( R_1 := 3R, \epsilon_1 := \frac{\epsilon^2}{4} \), and let \( L \geq \frac{4N(2N+1)R_1}{\epsilon^2} \). Then \( X_g^j(L) \) is a finite cover of \( X_g \) of multiplicity \( \leq N \) and with Lebesgue number \( \geq L \) (see Definition 4.14). Then defining for \( x \in X_g \),
\[
\delta_g^j(x) := \frac{d_G(x, X_g \setminus X_g^j(L))}{\sum_{j=1}^N d_G(x, X_g \setminus X_g^j(L))},
\]
we obtain a partition of unity subordinated to the \( \{ X_g^j(L) \}_{j=1}^N \) satisfying, by Lemma 4.2 and by our choice for \( L \),
\[
\sum_{i=1}^N |\delta_g^i(x) - \delta_g^j(y)| \leq \frac{\epsilon_1}{2},
\]
whenever \( d_G(x, y) \leq R_1 \).

Now, we fix a \( (2(L + R_1), \frac{\epsilon_2}{12}) \)-Ozawa kernel on \( G_0, \psi_0 : (z, z') \mapsto (\mu(z), \mu(z')) \), for some \( \mu : G_0 \to l^2(G_0) \) with \( \text{supp}(\mu(z)) \subset B_G(z, S_1) \) for every \( z \). Hence we deduce a \( (2(L + R_1), \frac{\epsilon_2}{12}) \)-Ozawa kernel (with the same support) on each \( X_g^j \) by setting
\[
\psi_1(gg_i z, gg_j z') := \psi_0(z, z')
\]
for every \( z, z' \in G_0 \). In order to obtain an appropriate Ozawa kernel on \( X_g \), for any \( z \in X_g \), we fix \( p_i(z) \in X_g^j \) such that \( d_G(z, X_g^j) = d_G(z, p_i(z)) \). Then we put \( \sigma_g : X_g \to l^2(X_g) \), \( z \mapsto \sigma_g(z) \), where
\[
\sigma_g(z)(i, x) := \delta_g^j(z) \mu((gg_i)^{-1} p_i(z))(gg_j)^{-1} x.
\]

Observe that if \( \sigma_g(z)(i, x) \neq 0 \) for some \( z \in X_g^j \), then \( \mu((gg_i)^{-1} p_i(z))(gg_j)^{-1} x \neq 0 \), \( z \in X_g^j(L) \) and we have \( d_G((gg_i)^{-1} p_i(z), (gg_j)^{-1} x) \leq S_1 \). Therefore,
\[
d_G(x, z) \leq d_G(x, p_i(z)) + d_G(p_i(z), z) \leq S_1 + L. \quad (*)
\]

Moreover, for \( z_1, z_2 \in X_g \) such that \( d_G(z_1, z_2) \leq R_1 \), one has
Minkowski's inequality, in particular, as it follows that have $\tau \in (X_g', L')$, then $d_G(p_1(z_1), p_2(z_2)) \leq 2(L + R_I)$. By Cauchy-Schwarz's inequality,
\[
\sum_{x \in G_0} |\mu((gg_i)^{-1} p_1(z_1))^2(x) - \mu((gg_i)^{-1} p_1(z_2))^2(x)| = \|\mu((gg_i)^{-1} p_1(z_1))^2 - \mu((gg_i)^{-1} p_1(z_2))^2\|_{\ell_1(G_0)}^2 \\
\leq 2\|\mu((gg_i)^{-1} p_1(z_1)) - \mu((gg_i)^{-1} p_1(z_2))\|_{\ell_1(G_0)}^2 \\
= 2\sqrt{2\sqrt{1 - \psi_0((gg_i)^{-1} p_1(z_1), (gg_i)^{-1} p_1(z_2))}} \\
\leq \frac{\epsilon_1}{2}.
\]
We conclude that (I) $\leq \frac{\epsilon_1}{2}$. On the other hand,
\[
\sum_{x \in G_0} \mu((gg_i)^{-1} p_2(z_2))^2(x) = \|\mu((gg_i)^{-1} p_2(z_2))\|_{\ell_1(G_0)}^2 = 1.
\]
Hence, part (II) is less or equal to
\[
\sum_{i=1}^{N} \left| \delta_g^i(z_1) - \delta_g^i(z_2) \right| \leq \frac{\epsilon_1}{2}.
\]
It follows that $\|\sigma_g(z_1) - \sigma_g(z_2)\|_{\ell_1(X_g)}^2 \leq \epsilon_1$ whenever $d_G(z_1, z_2) \leq R_I$.

Now let us deduce an Ozawa kernel on $U_g$. For every $x \in X_g$ let us fix $q_g(x) \in U_g$ such that $d_G(x, q_g(x)) = d_G(x, U_g)$ and define $\tau_g : U_g \to l^2(U_g)$, $u \mapsto \tau_g(u)$, where
\[
\tau_g(u) : w \mapsto \left( \sum_{(i, x) \in X_g : q_g(x) = w} \sigma_g(u)^2(i, x) \right)^{1/2}.
\]
In particular, as $\bigcup_{w \in U_g} \{(i, x) \in X_g : q_g(x) = w\} = X_g$, we have $\|\tau_g(u)\|_{l^2(U_g)} = \|\sigma_g(u)\|_{l^2(X_g)} = 1$. By Minkowski's inequality, $d_G(u, u') \leq R_I$ implies
\[
\|\tau_g(u) - \tau_g(u')\|_{l^2(U_g)}^2 \leq \|\sigma_g(u) - \sigma_g(u')\|_{l^2(X_g)}^2 \leq \epsilon_1
\]
Moreover, if $\tau_g(u)(w) \neq 0$, there exists $(i, x) \in X_g$ such that $q_g(x) = w$ with $\sigma_g(u)(i, x) \neq 0$. Thus, by $(*)$, $d_G(u, x) \leq S_1 + L$ and
\[
d_G(u, w) \leq d_G(u, x) + d_G(x, q_g(x)) \leq 2d_G(u, x) \leq 2(S_1 + L).
\]
Then supp$(\tau_g(u)) \subset B_G(u, 2(S_1 + L))$ for every $u$.

Now, for every $g \in G$ and for every $x \in U_g(R)$, we fix $r_g(x) \in U_g$ such that $d_G(x, r_g(x)) = d_G(x, U_g) \leq R$. Then we set
\[
\nu_g : U_g(R) \to l^2(U_g), \quad x \mapsto \nu_g(x) := \tau_g(r_g(x)).
\]
By definition, if \( d_G(x, z) \leq R \) we have \( d_G(r_g(x), r_g(z)) \leq 3R = R_1 \), hence by (**)
\[
\|\nu_g(x) - \nu_g(z)\|^2_{\mathcal{P}(U_g)} = \|\tau_g(r_g(x)) - \tau_g(r_g(z))\|^2_{\mathcal{P}(U_g)} \leq \epsilon_1
\]
and if \( \nu_g(x) := \tau_g(r_g(x))(u) \neq 0 \), we have \( d_G(u, r_g(x)) \leq 2(S_1 + L) \) and \( d_G(u, x) \leq R + 2(S_1 + L) \), i.e., \( \text{supp}(\nu_g(x)) \subseteq \mathcal{B}_G(x, 2(S_1 + L) + R) \) for every \( x \).

Finally, we extend each \( \nu_g \) on \( G \) by 0 outside \( U_g(R) \), we put \( \mathcal{U}_G := \bigsqcup_{g \in G} \{g\} \times U_g \) and we consider the positive definite kernel
\[
\psi : G \times G \rightarrow \mathbb{R}, \ (g_1, g_2) \mapsto (\kappa(g_1), \kappa(g_2)) = \sum_{g \in G} \sum_{x \in U_g} \kappa(g_1)(g, x) \kappa(g_2)(g, x)
\]
where \( \kappa : G \rightarrow l^2(\mathcal{U}_G) \) is defined by
\[
\kappa(g')(g, x) := \alpha_g^{1/2}(g')\nu_g(g')(x).
\]

On the one hand, if \( \psi(g_1, g_2) \neq 0 \) there exists \( (g, x) \in \mathcal{U}_G \) such that \( \kappa(g_1)(g, x) := \alpha_g^{1/2}(g_1)\nu_g(g_1)(x) \neq 0 \) and \( \kappa(g_2)(g, x) := \alpha_g^{1/2}(g_2)\nu_g(g_2)(x) \neq 0 \), i.e. \( \nu_g(g_1)(x) \neq 0, \nu_g(g_2)(x) \neq 0 \) and \( g_1, g_2 \in U_g \). Then \( d_G(x, g_1) \leq 2(S_1 + L) + R \) and \( d_G(x, g_2) \leq 2(S_1 + L) + R \). Hence \( d_G(g_1, g_2) \leq 4(S_1 + L) + 2R \). Therefore,
\[
\text{supp}(\psi) \subseteq \{(g_1, g_2) \in G \times G \mid d_G(g_1, g_2) \leq 4(S_1 + L) + 2R\}.
\]

On the other hand,
\[
\|\kappa(g_1) - \kappa(g_2)\|^2_{\mathcal{P}(\mathcal{U}_G)} = \sum_{g \in G} \sum_{x \in U_g} |\kappa(g_1)(g, x) - \kappa(g_2)(g, x)|^2
\]
\[
= \sum_{g \in G} \sum_{x \in U_g} |\alpha_g^{1/2}(g_1)\nu_g(g_1)(x) - \alpha_g^{1/2}(g_2)\nu_g(g_2)(x)|^2
\]
\[
\leq \sum_{g \in G} \sum_{x \in U_g} |\nu_g(g_1)\nu_g(g_1)(x) - \nu_g(g_2)\nu_g(g_2)(x)|^2
\]
\[
\leq \left( \sum_{g \in G} \alpha_g(g_1) \right) \left( \sum_{x \in U_g} |\nu_g(g_1)(x) - \nu_g(g_2)(x)|^2 \right) \quad (A)
\]
\[
+ \left( \sum_{x \in U_g} \nu_g(g_2)^2(x) \right) \left( \sum_{g \in G} |\alpha_g(g_1) - \alpha_g(g_2)| \right) \quad (B)
\]

If \( d_G(g_1, g_2) \leq R \), \( \sum_{g \in G} \alpha_g(g_1) = 1 \) and there exists at least one \( g \in G \) such that \( g_1 \in U_g \), then \( g_2 \in U_g(R) \). Thus, by Cauchy-Schwarz’s inequality
\[
\sum_{x \in U_g} |\nu_g(g_1)^2(x) - \nu_g(g_2)^2(x)| \leq 2\|\nu_g(g_1)^2 - \nu_g(g_2)^2\|_{\mathcal{P}(U_g)} \leq 2\sqrt{\epsilon} = \epsilon
\]
i.e. \( (A) \leq \epsilon \). For the term \( (B) \), one has \( \sum_{x \in U_g} \nu_g(g_2)^2(x) = 0 \) if \( g_2 \notin U_g(R) \) and \( \sum_{x \in U_g} \nu_g(g_2)^2(x) = \|\nu_g(g_2)^2\|_{\mathcal{P}(U_g)}^2 = 1 \) if \( g_2 \in U_g(R) \). Therefore in all cases
\[
(B) \leq \sum_{g \in G} |\alpha_g(g_1) - \alpha_g(g_2)| \leq \epsilon
\]
hence \( 1 - \psi(g_1, g_2) = \frac{1}{2} \|\kappa(g_1) - \kappa(g_2)\|^2_{\mathcal{P}(\mathcal{U}_G)} \leq \epsilon \). This concludes the proof.

What precedes can be summarized as follows:
6.6. Proposition. Let $G$ be a countable group acting by isometries on a discrete metric space with bounded geometry $X$. Assume that there exists $x_0 \in X$ such that both the orbit $G \cdot x_0$ and the stabilizer $G_0$ of $x_0$ have property A. Let $R, \epsilon > 0$. Let $K(R) := \max_{g \in B_G(R)} d_X(x_0, g x_0)$ and fix an $(K(R), \epsilon')$-Ozawa kernel for $G \cdot x_0$, $\phi_0 : (g x_0, g' x_0) \mapsto \langle \lambda(x_0), \lambda(g' x_0) \rangle$, for some $\lambda : G \cdot x_0 \to l^2(G \cdot x_0)$ with $\text{supp}(\lambda(x_0)) \subset B_X(g x_0, S_0)$ for every $x$ and $\lambda(x_0)(g' x_0) \geq 0$ for every $g$. For every $g \in G$ we define $U_g := \{g' \in G \mid g x_0 \in \text{supp}(\lambda(g' x_0))\}$.

If $B_X(x_0, S_0) \cap G \cdot x_0 = \{g_1 \cdot x_0, \ldots, g_N \cdot x_0\}$, for each $g \in G$ we have $U_g \subset \bigcup_{i=1}^{N} g_i G_0 := X_g$. Let $L \geq \frac{48N(2N+1)R}{\epsilon'}$ fix a $(2L + 3R, \epsilon')$-Ozawa kernel on $G_0$, $\psi_0 : (z, z') \mapsto \langle \mu(z), \mu(z') \rangle$, for some $\mu : G_0 \to l^2(G_0)$ with $\text{supp}(\mu(z)) \subset B_{G_0}(z, S_0)$ for every $z$. Finally set for every $x \in X_g$

$$\delta_g^j(x) := \frac{d_G(x, X_g \smallsetminus (g g_i G_0)(L))}{\sum_{j=1}^{N} d_G(x, X_g \smallsetminus (g g_i G_0)(L))} 1/2$$

defines an Ozawa kernel on $U_g$. Here $X_g := \prod_{i=1}^{N} \{0\} \times gg_i G_0$ and $p_i$, $q_g$ satisfy $p_i(z) \in X_g$, $q_g(z) \in U_g$, $d_G(z, p_i(z)) = d_G(z, gg_i G_0)$ and $d_G(z, q_g(z)) = d_G(z, U_g)$ for any $z \in X_g$.

Therefore

$$\psi(g_1, g_2) := \sum_{g \in G} \chi_{U_g}(g_1) \chi_{U_g}(g_2) \lambda(g_1 x_0)(g x_0) \lambda(g_2 x_0)(g x_0) \zeta_\theta(r_g(g_1), r_g(g_2))$$

is an $(R, \epsilon)$-Ozawa kernel on $G$ with $\text{supp}(\psi) \subset \{g_1, g_2 \in G \times G \mid d_G(g_1, g_2) \leq 4(S_1 + L) + 2R\}$, where $r_g(x) \in U_g$ and $d_G(x, r_g(x)) = d_G(x, U_g)$ for every $x \in U_g(R)$.

6.7. Corollary. Let $G$ be a countable group and $H$ a subgroup with property A. If the set of left cosets $G/H$ (endowed with the quotient metric) has property A, then $G$ has property A.

Note that we recover (in a rather complicated way) Theorem 3.1 and, moreover, we obtain an alternative proof of the stability of property A under taking certain amalgamated free products and HNN-extensions (see [DG03] for the general case):

6.8. Corollary. Let $G$ and $H$ be two countable groups.

(i) If $G$ and $H$ have property A, and if $K$ is a common subgroup of finite index both in $G$ and $H$, then $G \ast_K H$ has property A.

(ii) Assume that $G$ has property A, and that $H$ is a finite index subgroup of $G$. Let $\theta : H \to G$ be a monomorphism such that $\theta(H)$ also have finite index in $G$, then the HNN-extension $\text{HNN}(G, H, \theta)$ has property A.

Proof of Corollary 6.8. The finite index hypothesis ensures bounded geometry of the corresponding Bass-Serre trees (endowed with the simplicial metric) on which the groups act by isometries. The stabilizers of the vertex are all isometric to $G$ (in the case of HNN-extensions), or, to $G$ or $H$ (in the case of amalgamated free products). Hence, it suffices to use property A for trees (which can be shown exactly in the same way as for free groups) and to apply Theorem 6.4.

7. Applications

7.1. Hyperbolic groups. It is a result due to J. Roe (see [Roe05]) that finitely generated hyperbolic groups have finite asymptotic dimension. Explicit covers given in [Roe05] allow us to exhibit explicit Ozawa kernels. Let $\Gamma = (S)$ be a finitely generated $\delta$-hyperbolic group (endowed with the length function $\cdot \mid _S$ associated to $S$), and let $(\cdot \mid \cdot)$ denote the Gromov product on $\Gamma$. Let $N_5$ denote the number such
that each ball of radius $R + 6\delta$ can be covered by at most $N_\delta$ balls of radius $R$. Let $\epsilon, R > 0$, and $L \geq \frac{2(2N_\delta + 1)(4N_\delta + 3)R}{\epsilon}$. For each $k \geq 1$, we fix a maximal subset $\{\gamma_{ik}\}_{i=1}^{n_k}$ in the sphere of radius $kL$ such that $dR(\gamma_{ik}, \gamma_{jk}) > L$ for every $i \neq j$. Then we define

$$U_{ik} := \{\gamma \in \Gamma \mid kL \leq |\gamma|_S \leq (k + 1)L, \ (\gamma, \gamma_{ik}) \geq (k - \frac{1}{2})L - \delta\}$$

This is shown in [C06] that for every $k$, $\{U_{ik}\}_{i=1}^{n_k}$ is a uniformly bounded cover of the annulus $\{\gamma \in \Gamma \mid kL \leq |\gamma|_S \leq (k + 1)L\}$ such that every ball of radius less than $L$ meets at most $N_\delta$ elements of this cover. Hence, defining $U := \bigcup_{k \in \mathbb{N}} U_{ik}(L)$ (where $U_{ik}(L)$ denote the $L$-neighborhood of $U_{ik}$), it is easy to see that $U$ is a uniformly bounded cover of $\Gamma$ with multiplicity less than $2N_\delta$ and such that $L(U) \geq L$. Therefore asdim $\Gamma \leq 2N_\delta - 1$, hence, by Theorem [15][3] the kernel

$$\psi : \Gamma \times \Gamma \to \mathbb{R}, \ (\gamma, \gamma') \mapsto \sum_{i, k} \left(\frac{d(\gamma, \gamma_{ik} \cup \gamma_{ik})}{\sum_{j, l} d(\gamma_{jk}, \gamma_{jk})}\right)^{1/2} \left(\frac{d(\gamma', \gamma_{ik} \cup \gamma_{ik})}{\sum_{j, l} d(\gamma_{jk}, \gamma_{jk})}\right)^{1/2}$$

is an $(R, \epsilon)$-Ozawa kernel for $\Gamma$.

7.2. CAT(0) cubical groups. Let $X$ denote a CAT(0) cube complex with bounded geometry and let $X^{(0)}$ denote its 0-skeleton endowed with the path metric on the 1-skeleton of $X$ (when $X$ is finite dimensional this metric is equivalent to the metric induced by $X$). Using Theorem [5.5] and ideas developed in [CN04], we deduce Ozawa kernels on $X^{(0)}$. Fix a basepoint $v \in X^{(0)}$ and denote by $H$ the set of “hyperplanes” in $X$. For every $s \leq X^{(0)}$ there is a unique “normal cube path” $\{C_i, \ldots, C_n\}$. Let us define $w_s : H \to \mathbb{N}$ by $w_s(h) = i + 1$ if $h$ intersects the cube $C_i$ and $w_s(h) = 0$ otherwise. Then for every $0 < \alpha < 1/2$, $f_\alpha : X^{(0)} \to l^2(H)$, $s \mapsto \sum_{h \in H} w_s(h)^\alpha \delta_h$ is a uniform embedding with $\alpha + (f_\alpha)$ linear and $\rho(f_\alpha) = e^{1/2 + \alpha}$ for $v$ large enough. Hence, the proof of Theorem [5.5] provides Ozawa kernels on $X^{(0)}$. More precisely, for a fixed $\alpha > 0$ and for $M$ large enough, setting

$$\phi : X^{(0)} \times X^{(0)} \to \mathbb{R}, \ (x, y) \mapsto \exp \left(-\frac{\|f_\alpha(x) - f_\alpha(y)\|^2}{M}\right)$$

with the notations in the proof of Theorem [5.5] we obtain that

$$\psi : (x, y) \mapsto \|U_0\| \sum_{x \in X^{(0)}} \sum_{0 \leq m, n \leq M} a_n a_m (\delta - \phi_M \|U_0\|)^{-m} \left(\delta - \phi_M \|U_0\|\right)^{-n} (z, y)$$

is an Ozawa kernel on $X^{(0)}$. Now, let us fix a vertex $v_0 \in X^{(0)}$. By Proposition [6.1] for every group $G$ acting properly (and co-compactly) by isometries on $X$, the kernel $\psi_G : G \times G \to \mathbb{R}, \ (g, h) \mapsto \psi(gv_0, hv_0)$ defines an Ozawa kernel on $G$.

7.3. Baumslag-Solitar groups. Let $p, q \geq 1$, and let BS$(p, q) := \langle a, b \mid ab^p a^{-1} = b^q \rangle$ be a standard presentation of Baumslag-Solitar group. It is the HNN-extension $\text{HNN}(G, H, \theta)$, where $G = \langle b \rangle \leq Z$, $H = \langle b^p \rangle \leq Z$ and $\theta : H \to G$, $b^q \mapsto b^p$. The group BS$(p, q)$ acts transitively by isometries on its Bass-Serre tree $T_{p, q}$ (which is $(p + q)$-regular) and all the stabilizers of the vertex are isometric to $Z$. Hence Ozawa kernels on BS$(p, q)$ are given by formula (2) in Proposition [6.6] with (for $k, l$ large enough)

$$\lambda : T_{p, q} \to l^2(T_{p, q}), \ v \mapsto \frac{X_{A_v}}{k+1}, \ \mu : G \to l^2(G), \ x \mapsto \frac{X_{A_v} B_v(x, l)}{2l + 1},$$

where $A_v$ denotes the intersection of $B_{T_{p, q}}(v, k)$ with the unique geodesic ray starting form $v$ and intersecting a fixed geodesic ray in $T_{p, q}$ as a geodesic ray.

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