A ROBUST REDUCED-ORDER OBSERVERS DESIGN APPROACH FOR LINEAR DISCRETE PERIODIC SYSTEMS

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Abstract. This paper investigates the problem of designing reduced-order observers for linear discrete-time periodic (LDP) systems. In case that the linear discrete-time periodic system is observable, an algebraic equivalent system is obtained by non-singular linear transformation, and the partial states to be observed are separated simultaneously. Then the considered problem is transformed into the problem of solving a class of periodic Sylvester matrix equation and an iterative algorithm for periodic reduced-order state observers design is derived. In addition, robust consideration based on periodic reduced-order state observers for LDP systems is also conducted. At last, one numerical example is worked out to illustrate the effectiveness of the proposed approaches.

1. Introduction. Linear discrete-time periodic systems are the simplest and most important time-varying systems. Many time-varying systems in practical engineering can be approximated as a simple linear discrete period periodic system. With the development of aerospace, networks and some high technology in recent years, linear discrete periodic systems have been paid renewed attentions in the control theory community, and have achieved many important results (see [13]-[2] and references therein). In [18], a novel lifting method was proposed that converted the linear periodic system to an augmented Linear Time-Invariant (LTI) system, and the design problem of spacecraft attitude control using magnetic torques was solved. [20] studied the local control of discrete-time periodic linear systems subject to input saturation by using the multi-step periodic invariant set approach. In addition, stabilization of periodic systems with input and output delays was investigated in [21].

State observer is an important research topic in control theory and control engineering, the purpose of observer is to estimate the state of another dynamic system. The state observer’s theory stems from the Kalman filter in [4] and the Luenberger
observer (see [8]-[15] and references therein) designed for linear systems in the 1960s and 1970s. The predecessors of observers design have made certain achievements. For examples, in [16], full-order state observers were used to generate the residual for the purpose of robust fault detection in linear systems with unknown disturbances. In [22], observer based output feedback control of linear systems with both input and output delays was concerned, and truncated predictor feedback approach for state feedback stabilization of time-delay systems was generalized to the design of observers.

As we all know, a reduced-order state observer is structurally simpler than a full-dimensional state observer of a linear system, since reducing the dimension means that the observer only needs to be composed of fewer integrators. By virtue of this point, it has a wide range of applications in control theory and engineering practice (see [14]-[23] and references therein). In [24], observer-based actuator fault detection and reconstruction problems for uncertain nonlinear systems were considered, a kind of reduced-order observer was developed by choosing a special observer gain matrix. Based on the reduced-order observer, a kind of unknown information estimating method was provided. Output feedback consensus of both continuous-time and discrete-time multi-agent systems were concerned in literature [23], and distributed reduced-order observer based protocols were established by only using the relative outputs and inputs of neighboring agents. At present, the research results of the state observer of the linear time-invariant system and the nonlinear system are relatively mature (see [12]-[17] and references therein), but the state observer of the linear discrete-time periodic system is involved in less.

Robust design is an ongoing research focus in the control community, which is aimed to design a controller, a filter, an observer, a strategy, etc., for the presence of uncertainties or disturbances. In this respect, great progress has been made in recent years and many important achievements have been made. For examples, a robust minimum variance beamformer design was proposed without full knowledge of the channel mismatch in [5]. The two-stage distributionally robust optimization problem that depends on a random vector and the distributionally robust two-stage stochastic convex programming problem were solved in [6] and [7], respectively. The problem of probabilistic robust stabilization for uncertain stochastic systems by using scenario optimization approach was addressed in [9], where the uncertainties are not assumed to be norm-bounded and state feedback controllers were designed via linear matrix inequality (LMI) technique. The problem of probabilistic robust stabilization for uncertain systems subject to input saturation was studied in [19], where a probabilistic solution framework for robust control analysis and synthesis problems was addressed by a scenario optimization approach.

In this paper, the problem of reduced-order state observers design for LDP systems is studied. In case that the linear discrete-time periodic system can be observed, an algebraic equivalent system is obtained by non-singular linear transformation, and the partial states to be observed are separated simultaneously. By strict mathematics deduction, periodic gain matrices of the reduced-state observer for linear discrete-time system can be determined by using the solutions to a class of periodic Sylvester matrix equation. On this basis, in case that uncertain disturbances exist in the system parameters, robust reduced-order observers design algorithm for LDP systems is presented.

The paper is organized as follows. In Section 2, preliminaries and problem formulation are provided. In Section 3, the algorithms and their theoretical analysis
are formulated. Section 4 offers an illustrative numerical example to verify the effectiveness of the proposed method. We come to conclusion in Section 5.

**Notation** The superscripts “T” and “−1” stand for matrix transposition and matrix inverse, respectively; $\mathbb{R}^n$ denotes the n-dimensional Euclidean space; $i,j$ represents the integer set $\{i, i+1, \ldots, j-1, j\}$; $\text{tr}(A)$ means the trace of matrix $A$ and $\Psi_A$ denotes the monodromy matrix $A_{T-1}A_{T-2}\cdots A_0$ with period $T$.

2. **Preliminaries and problem formulation.** Consider LDP systems with the following state space representation

$$\begin{align*}
x_{t+1} &= A_t x_t + B_t u_t \\
y_t &= C_t x_t
\end{align*}$$

(1)

where $t \in \mathbb{Z}$, the set of integers, $x_t \in \mathbb{R}^n$, $u_t \in \mathbb{R}^r$ and $y_t \in \mathbb{R}^m$ are respectively the state vector, the input vector and the output vector, $A_t$, $B_t$, $C_t$ are matrices of compatible dimensions satisfying

$$A_{t+T} = A_t, \quad B_{t+T} = B_t, \quad C_{t+T} = C_t.$$ 

To recall standard concept and results for periodic systems, for $t \in \mathbb{Z}$ denote by

$$\Phi_A(t+T,t) := A(t+T-1)A(t+T-2)\cdots A(t) \in \mathbb{R}^{n\times n}$$

(2)

$$A_t := \text{diag}\{A(t), A(t+1), \ldots, A(t+T-1)\} \in \mathbb{R}^{nT\times nT}$$

(3)

$$B_t := \text{diag}\{B(t), B(t+1), \ldots, B(t+T-1)\} \in \mathbb{R}^{nT\times rT}$$

(4)

$$C_t := \text{diag}\{C(t), C(t+1), \ldots, C(t+T-1)\} \in \mathbb{R}^{mT\times nT}$$

(5)

$$\mathcal{R}_t(\lambda) := \begin{bmatrix} 0_{n(T-1),n} & I_{n(T-1)} \\ \lambda I_n & 0_{n,n(T-1)} \end{bmatrix} \in \mathbb{R}^{nT\times nT}$$

(6)

where $0_{m,n}$ denotes the zero element of $\mathbb{R}^{m\times n}$, and $I_n$ denotes the identity matrix of dimension $n$.

The notions and results concerning the reachability and observability of the periodic system (1) have been introduced and analyzed in [3]. We provide them in the following.

**Lemma 2.1.** [3] The periodic system (1) is reachable at time $t$ if and only if the following matrix

$$\begin{bmatrix} A_t - \mathcal{R}_t(\lambda) & B_t \end{bmatrix}$$

has full row rank for all $\lambda \in \mathbb{C}$, or equivalently, for all the eigenvalues of $\Phi_A(t+T,t)$.

**Lemma 2.2.** [3] The periodic system (1) is observable at time $t$ if and only if the following matrix

$$\begin{bmatrix} A_t - \mathcal{R}_t(\lambda) \\ C_t \end{bmatrix}$$

has full column rank for all $\lambda \in \mathbb{C}$, or equivalently, for all the eigenvalues of $\Phi_A(t+T,t)$.

When there exists some restrictions in practice, the state of system (1) can not be fully gotten by hardware, but the input $u(t)$ and the output $y(t)$ can be measured. Considering that the output $y_t$ of the system already contains part of information of the system state $x_t$, the direct use of this part of information can construct a state observer with dimension lower than the estimated system.

Here, we conclude the problem of periodic reduced-order state observer for linear discrete-time periodic (LDP) systems as follows:
Problem. Given a completely reachable and completely observable LDP system (1), find a state observer with dimension \( n-m \), whose state variable is denoted by \( z_t \), such that the original states \( x_t \) can be reconstituted by the combination of system output \( y_t \) and the observer state \( z_t \).

Based on the above idea, we would like to present an non-singular linear transformation firstly, by which the partial states to be observed can be separated.

Lemma 2.3. Given estimated system (1), selecting a group of \((n-m) \times n\) dimension matrix \( R_t, t \in [0,T-1] \), arbitrarily such that the following \( n \times n \) matrix \( P_t, t \in [0,T-1] \), are non-singular.

\[
P_t = \begin{bmatrix} C_t \\ R_t \end{bmatrix}
\]

Let

\[
\bar{A}_t = P_t A_t P_t^{-1} = \begin{bmatrix} \bar{A}_{11t} & \bar{A}_{12t} \\ \bar{A}_{21t} & \bar{A}_{22t} \end{bmatrix}
\]

\[
\bar{B}_t = P_t B_t = \begin{bmatrix} \bar{B}_{1t} \\ \bar{B}_{2t} \end{bmatrix}
\]

\[
\bar{C}_t = C_t P_t^{-1} = \begin{bmatrix} I_m & 0 \end{bmatrix}
\]

where \( \bar{A}_{11t} \in \mathbb{R}^{m \times m}, \bar{A}_{12t} \in \mathbb{R}^{m \times (n-m)}, \bar{A}_{21t} \in \mathbb{R}^{(n-m) \times m}, \bar{A}_{22t} \in \mathbb{R}^{(n-m) \times (n-m)} \),

\( \bar{B}_{1t} \in \mathbb{R}^{m \times r}, \bar{B}_{2t} \in \mathbb{R}^{(n-m) \times r} \), then the matrix pair \((\bar{A}_{22t}, \bar{A}_{12t})\) is observable at time \( t \), if and only if the matrix pair \((A_t, C_t)\) is observable at time \( t \).

Proof. Let

\[
Q_t = P_t^{-1} = \begin{bmatrix} Q_{1t} & Q_{2t} \end{bmatrix}
\]

where \( Q_{1t} \in \mathbb{R}^{n \times m}, Q_{2t} \in \mathbb{R}^{n \times (n-m)} \). Then the following formula can be established

\[
I_n = P_t Q_t = \begin{bmatrix} C_t \\ R_t \end{bmatrix} \begin{bmatrix} Q_{1t} & Q_{2t} \end{bmatrix} = \begin{bmatrix} C_t Q_{1t} & C_t Q_{2t} \\ R_t Q_{1t} & R_t Q_{2t} \end{bmatrix} = \begin{bmatrix} I_m & 0 & 0 \\ 0 & I_{n-m} \end{bmatrix}
\]

Then we have

\[
C_t Q_{1t} = I_m, \quad C_t Q_{2t} = 0
\]

Therefore,

\[
\bar{C}_t = C_t P_t^{-1} = \begin{bmatrix} I_m \\ C_t Q_{1t} \end{bmatrix} = \begin{bmatrix} I_m & 0 \end{bmatrix}.
\]

Since algebraic equivalent systems have the same controllability and observability, \((A_t, C_t)\) is observable if and only if \((\bar{A}_t, \bar{C}_t)\) is observable. According to Lemma 2.2, by denoting the column rank as \( \text{rank}_c \), the observability of matrix pair \((\bar{A}_t, \bar{C}_t)\) at time \( t \) means

\[
nT = \text{rank}_c \left[ \frac{\bar{A}_t - R_t(\lambda)}{\bar{C}_t} \right]
\]

\[
= \text{rank}_c \left[ \begin{array}{cccccc} \bar{A}_t & -I_n & 0 & \cdots & 0 \\ 0 & \bar{A}_{t+1} & -I_n & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ -\lambda I_m & 0 & 0 & \cdots & \bar{A}_{t+T-1} \\ I_m & 0 & 0 & \cdots & 0 \\ 0 & [ I_m ] & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & [ I_m ] \end{array} \right].
\]
According to (11), the above equation can be rewritten as

\[
\begin{bmatrix}
\bar{A}_{11t} & \bar{A}_{12t} & -I_m & 0 & \cdots & 0 \\
\bar{A}_{21t} & \bar{A}_{22t} & 0 & -I_{n-m} & \cdots & 0 \\
0 & 0 & \bar{A}_{11(t+1)} & \bar{A}_{12(t+1)} & \cdots & 0 \\
0 & 0 & \bar{A}_{21(t+1)} & \bar{A}_{22(t+1)} & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & \lambda I_m & 0 & \cdots & \bar{A}_{11(t+T-1)} \\
0 & 0 & 0 & \cdots & \bar{A}_{12(t+T-1)} \\
0 & 0 & \lambda I_{n-m} & 0 & \cdots & \bar{A}_{21(t+T-1)} \\
0 & 0 & 0 & \cdots & \bar{A}_{22(t+T-1)} \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & \lambda I_m \\
0 & 0 & 0 & \cdots & I_m \\
\end{bmatrix}
= nT.
\]

Then, we can get

\[
\begin{bmatrix}
\bar{A}_{12t} & 0 & 0 & \cdots & 0 \\
\bar{A}_{22t} & -I_{n-m} & 0 & \cdots & 0 \\
0 & \bar{A}_{12(t+1)} & 0 & \cdots & 0 \\
0 & \bar{A}_{22(t+1)} & -I_{n-m} & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots \\
0 & 0 & 0 & \cdots & \bar{A}_{12(t+T-1)} \\
\end{bmatrix}
= (n - m)T.
\]

Rearranging the rows of this matrix gives

\[
\begin{bmatrix}
\bar{A}_{22t} & -I_{n-m} & 0 & \cdots & 0 \\
0 & \bar{A}_{22(t+1)} & -I_{n-m} & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots \\
\vdots & \vdots & \vdots & \vdots & \ddots \\
0 & 0 & 0 & \cdots & \bar{A}_{22(t+T-2)} \\
0 & 0 & 0 & \cdots & \bar{A}_{12(t+T-1)} \\
\end{bmatrix}
= (n - m)T.
\]

That is to say

\[
\text{rank}\left[ \frac{\bar{A}_{22t} - \mathcal{R}_t(\lambda)}{\bar{A}_{12t}} \right] = (n - m)T,
\]

where \(\mathcal{A}\) and \(\mathcal{R}\) are give by (3) and (6) with compatible matrix dimensions. Again, according to Lemma 2.2, the above equation denote that matrix pair \((\bar{A}_{22t}, \bar{A}_{12t})\) is observable at time \(t\). Thus the proof is accomplished.

Utilizing the linear non-singular transformation \(\bar{x}_t = P_t x_t\) provided in Lemma 2.3, we can see that the estimated system (1) is algebraically equivalent to the
following system

\[
\begin{aligned}
\begin{bmatrix}
\mathbf{x}_1(t+1) \\
\mathbf{x}_2(t+1)
\end{bmatrix} &=
\begin{bmatrix}
A_{11t} & A_{12t} \\
A_{21t} & A_{22t}
\end{bmatrix}
\begin{bmatrix}
\mathbf{x}_1(t) \\
\mathbf{x}_2(t)
\end{bmatrix} +
\begin{bmatrix}
B_{1t} \\
B_{2t}
\end{bmatrix} u_t \\
y_t &=
\begin{bmatrix}
I_m & 0
\end{bmatrix}
\begin{bmatrix}
\mathbf{x}_1(t) \\
\mathbf{x}_2(t)
\end{bmatrix}
= \mathbf{x}_1(t)
\end{aligned}
\]  

(16)

where \( \mathbf{x}_1 \in \mathbb{R}^m, \mathbf{x}_2 \in \mathbb{R}^{n-m} \) are the state vectors.

To design a periodic observer for system (1), a natural thought is to design a periodic observer for system (16) firstly, then by linear inverse transformation of state \( \mathbf{x}_t \), to obtain the estimation of original state \( \mathbf{x}(t) \). From (16), we can see that for the transformed state \( \mathbf{x}_t \), its substate \( \mathbf{x}_1 \) is the output \( y_t \) of the original system (1). So it can be used directly without reconstruction. What need to be reconstructed here is the \( n-m \) dimension substate vector \( \mathbf{x}_2 \) of \( \mathbf{x}_t \), so lower order periodic state observer with dimension \( n-m \) can meet requirements. Thus, problem 2 can be degenerated into the following problem.

**Problem.** For completely reachable and completely observable LDP system (16), find periodic matrix \( \mathcal{L}_t \in \mathbb{R}^{(n-m) \times m}, t \in 0, T-1 \), such that the state observer of the substate vector \( \mathbf{x}_2 \) can be given as follows

\[
\begin{aligned}
z_{t+1} &= (\mathcal{A}_{22t} - \mathcal{L}_t \mathcal{A}_{12t}) z_t + [(\mathcal{A}_{22t} - \mathcal{L}_t \mathcal{A}_{12t}) \mathcal{L}_t + (\mathcal{A}_{21t} - \mathcal{L}_t \mathcal{A}_{11t})] y_t + (\mathcal{B}_{2t} + \mathcal{L}_t \mathcal{B}_{1t}) u_t \\
\hat{\mathbf{x}}_{2t} &= z_t + \mathcal{L}_t y_t
\end{aligned}
\]  

(17)

And for any \( \mathbf{x}_0, z_0 \) and \( u_t \), there holds

\[
\lim_{t \to \infty} \left[ \mathbf{x}_2(t) - \hat{\mathbf{x}}_{2t} \right] = 0
\]

**Remark 1.** The observer (17) is an \( n-m \) order dynamic system with the input vectors \( u_t \) and \( y_t \).

3. Main results.

3.1. Periodic reduced-order observer design. The first thing to consider is the existence condition for periodic reduced-order state observer. To do this, we would like to give the following theorem.

**Theorem 3.1.** For a given LDP system (16), assume that periodic matrix pair \( (\mathcal{A}_{22t}, \mathcal{A}_{12t}) \) is observable and \( \mathcal{L}_t, t \in 0, T-1 \), are a group of matrices that make periodic matrix pair \( (\mathcal{A}_{22t} - \mathcal{L}_t \mathcal{A}_{12t}) \) stable, then system (17) constitutes an observer of the state component \( \mathbf{x}_2 \) of the system (16).

**Proof.** By formula (16) we can obtain

\[
\begin{aligned}
\mathbf{x}_{2(t+1)} &= \mathcal{A}_{22t} \mathbf{x}_{2t} + (\mathcal{A}_{21t} y_t + \mathcal{B}_{2t} u_t) \\
y_{t+1} &= \mathcal{A}_{11t} y_t - \mathcal{B}_{1t} u_t = \mathcal{A}_{12t} \mathbf{x}_{2t}
\end{aligned}
\]  

(18)

Define

\[
\begin{aligned}
\bar{u}_t &= (\mathcal{A}_{21t} y_t + \mathcal{B}_{2t} u_t), \\
w_t &= y_{t+1} - \mathcal{A}_{11t} y_t - \mathcal{B}_{1t} u_t
\end{aligned}
\]  

(19)

Formula (18) can be expressed as canonical as follows:

\[
\begin{aligned}
\mathbf{x}_{2(t+1)} &= \mathcal{A}_{22t} \mathbf{x}_{2t} + \bar{u}_t \\
w_t &= \mathcal{A}_{12t} \mathbf{x}_{2t}
\end{aligned}
\]  

(20)
Since \((\bar{A}_{22t} - \bar{L}_t\bar{A}_{12t})\) is observable, the system (20) is the \((n - m)\) dimensional subsystem of system (16), the full-dimensional state observer of system (20) exists and has the following form

\[
\hat{x}_{2(t+1)} = (\bar{A}_{22t} - \bar{L}_t\bar{A}_{12t})\hat{x}_{2t} + \bar{L}_tw_t + u_t \tag{21}
\]

and one can configure all the eigenvalues of \((\bar{A}_{22t} - \bar{L}_t\bar{A}_{12t})\) by selecting \(\bar{L}_t\).

Then we substitute equation (19) into system (21),

\[
\hat{x}_{2(t+1)} = (\bar{A}_{22t} - \bar{L}_t\bar{A}_{12t})\hat{x}_{2t} + \bar{L}_t(\dot{y}_{t+1} - \bar{A}_{11t}y_t - \bar{B}_{11t}u_t) + (\bar{A}_{21t}y_t + \bar{B}_{21t}u_t) \tag{22}
\]

Let

\[
z_t = \hat{x}_{2t} - \bar{L}_ty_t \tag{23}
\]

From system (22) and formula (23), we can obtain

\[
z_{t+1} = \hat{x}_{2(t+1)} - \bar{L}_t\dot{y}_{t+1}
= (\bar{A}_{22t} - \bar{L}_t\bar{A}_{12t})z_t + [(\bar{A}_{21t} - \bar{L}_t\bar{A}_{11t}) + (\bar{A}_{22t} - \bar{L}_t\bar{A}_{12t})\bar{L}_t]\dot{y}_t
+ (\bar{B}_{2t} - \bar{L}_t\bar{B}_{11t})u_t \tag{25}
\]

Observer (17) can be obtained from equation (23) and equation (25). Thus the proof is accomplished.

According to the above theorem, the problem of reduced-order observer design can be converted into the stabilization of periodic matrix pair \((\bar{A}_{22t}, \bar{A}_{12t})\). In the following, we would like to adopt poles assignment technique to achieve this purpose.

In the following, we will firstly present an iterative algorithm to compute periodic reduced order observer gains \(\bar{L}(t), t \in 0, T - 1\), which can solve Problem 2.

**Algorithm 1.** (An iterative algorithm for computing the periodic reduced-order observer gains)

1. Construct matrix \(P_t\) by equations (9), and calculate matrices \(Q_t, \bar{A}_t, \bar{B}_t, \bar{C}_t\) according to equations (12) and (11).
2. Let \(F_t \in \mathbb{R}^{(n-m) \times (n-m)}, t \in 0, T - 1\), be a group of real matrices, satisfying that all the poles of \(\Lambda(\Psi_{F_t}) = \Gamma\) lie in the unit circle and \(\Lambda(\Psi_{F_t}) \cap \Lambda(\Psi_{A_{22t}}) = 0\).

Further, let \(D_t \in \mathbb{R}^{m \times (n-m)}, t \in 0, T - 1\), are real parametric matrix such that periodic matrix pair \((F_t, D_t)\) is completely observable.
3. Set tolerance error \(\varepsilon\) and choose arbitrary initial periodic matrix \(X_t(0) \in \mathbb{R}^{(n-m) \times (n-m)}, t \in 0, T - 1\); Further, calculate as follows:

\[
W_t(0) = A_{12t}^T D_t - A_{22t}^T X_t(0) - X_{t+1}(0) F_t^L;
N_t(0) = A_{22t} W_t(0) + W_{t-1}(0) F_{t-1}^L;
H_t(0) = -N_t(0);
k := 0;
\]

4. If \(\sum_{t=0}^{T-1} \|N_t(k)\| \leq \varepsilon\), stop; else, go to next step;
By this equation, we have:

\[ \alpha_t(k) = \frac{\sum_{t=0}^{T-1} \| H_{t}^T(k)N_t(k) \|}{\sum_{t=0}^{T-1} \| A_{22}^T H_{t}(k) + H_{t+1}(k) A_{22}^T \|}; \]

\[ X_t(k+1) = X_t(k) + \alpha_t(k) H_t(k); \]

\[ W_t(k+1) = A_{22}^T D_t - A_{22}^T X_t(k+1) - X_{t+1}(k+1) F_t; \]

\[ N_t(k+1) = A_{22} W_t(k+1) + W_{t-1}(k+1) F_{t-1}^T; \]

\[ H_t(k+1) = -N_t(k+1) + \frac{\sum_{t=0}^{T-1} \| N_t(k+1) \|^2}{\sum_{t=0}^{T-1} \| N_t(k) \|^2} H_t(k); \]

\[ k = k + 1; \]

6. Let \( X_t = X_t(k). \) The real periodic matrix \( \overline{L}_t \) can be obtained as

\[ \overline{L}_t = (D_t X_t^{-1})^T, \ t \in 0, T - 1 \tag{26} \]

**Remark 2.** The main part of the algorithm does not contain nested loops, so the computational complexity of the algorithm is \( O(n - m). \)

On the convergence and correctness of the Algorithm 1, we give the following lemma, which is similar with the corresponding lemma of literature [10] and its proof is omitted here.

**Lemma 3.2.** The sequences \( \{N_t(k)\}, t \in 0, T - 1, \) generated by Algorithm 1 satisfy

\[ \lim_{k \to \infty} \| N_t(k) \| = 0. \]

Next, we will give the main result of this section.

**Theorem 3.3.** Consider the completely observable periodic discrete-time linear system (1). If matrices \( Q_t, \overline{A}_t, \overline{B}_t, \overline{C}_t \) are determined by equations (11)-(12), and matrices \( \overline{L}_t \) are generated from Algorithm 1, then a periodic reduced-order observer for system (1) with dimension \( n - m \) can be given by

\[
\begin{align*}
    z_{t+1} &= (\overline{A}_{22} - \overline{L}_t \overline{A}_{12t}) z_t + [(\overline{A}_{21t} - \overline{L}_t \overline{A}_{11t}) + (\overline{A}_{22t} - \overline{L}_t \overline{A}_{12t}) \overline{L}_t] y_t \\
    &\quad + (\overline{B}_{2t} - \overline{L}_t \overline{B}_{1t}) u_t \\
    \hat{x}_t &= Q_{2t} z_t + (Q_{1t} + Q_{2t} \overline{L}_t) y_t
\end{align*}
\]

That is to say, for any \( x_0, z_0, \) and \( u_t \) the composite system composed of system (1) and system (27) satisfies the following relationship

\[ \lim_{t \to \infty} |x_t - \hat{x}_t| = 0 \]

**Proof.** According to Lemma 3.2, \( N_t(k), t \in 0, T - 1, \) are convergent sequences. Denote \( \lim_{k \to \infty} N_t(k) = N_t, t \in 0, T - 1. \) Then \( N_t = 0, t \in 0, T - 1. \) Construct the following index function:

\[ J = \sum_{t=0}^{T-1} \frac{1}{2} \| A_{12t}^T D_t - A_{22t}^T X_t - X_{t+1} F_t \|^2. \tag{28} \]

By this equation, we have:

\[ \frac{\partial J}{\partial X_t} = A_{22t} (A_{12t}^T D_t - A_{22t}^T X_t - X_{t+1} F_t) + (A_{12t}^T D_t - A_{22t}^T X_t - X_{t+1} F_t) F_{t-1}^T = N_t \]
Then the least squares solution \((X_0^*, X_1^*, \ldots, X_{T-1}^*)\) satisfies

\[
\frac{\partial J}{\partial X_t} \bigg|_{X_t = X_t^*} = 0,
\]

for \(t = 0, 1, \ldots, T-1\).

Since \(N_t = 0\), we can conclude \(J = 0\), which leads to

\[
\mathcal{A}_t^\top X_t - \mathcal{A}_t^\top D_t = -X_{t+1}F_t, \tag{29}
\]

By equation (26), it can be rewritten as

\[
\mathcal{A}_t^\top X_t - \mathcal{A}_t^\top L_t^\top X_t = -X_{t+1}F_t, \tag{30}
\]

premultiplying the both sides of equation (30) with \(X_t^{-1}\) gives

\[
X_t^{-1}(\mathcal{A}_t^\top - \mathcal{A}_t^\top L_t^\top)X_t = -F_t. \tag{31}
\]

Continuous multiplication of the above equation for \(t \in \mathbb{T}^{-1,0}\), gives

\[
\prod_{T=1}^{0} X_t^{-1}(\mathcal{A}_t^\top - \mathcal{A}_t^\top L_t^\top)X_t = \prod_{T=1}^{0} (-F_t). \tag{32}
\]

This means

\[
X_0^{-1}\Psi_{\mathcal{A}_t^\top - \mathcal{A}_t^\top L_t^\top}X_0 = -\Psi_F. \tag{33}
\]

when \(T\) is an odd number, or

\[
X_0^{-1}\Psi_{\mathcal{A}_t^\top - \mathcal{A}_t^\top L_t^\top}X_0 = \Psi_F. \tag{34}
\]

when \(T\) is an even number.

According to Algorithm 1, all the poles of \(\Psi_F\) lie in the unit circle. By the above relation, matrices \(L_t\) can stabilize periodic matrix pair \((\mathcal{A}_{2t}, \mathcal{A}_{12t})\). This is to say, the periodic observer gain matrices \(L_t\) generated from Algorithm 1 is a solution to Problem 2.

Once problem 2 is solved, the reconstruction of the transformed state \(\hat{x}_t\) can be attained. Considering \(x_t = P_t^{-1}x_t = Q_t\hat{x}_t\), there is also \(\hat{x}_t = Q_t\hat{x}_t\). Then we can get the reconstruction state \(\hat{x}_t\) of the system state \(x_t\) as

\[
\hat{x}_t = P_t^{-1}\hat{x}_t = \begin{bmatrix} Q_{1t} & Q_{2t} \end{bmatrix} \begin{bmatrix} y_t \\ z_t + L_t y_t \end{bmatrix} = Q_{1t}y_t + Q_{2t}(z_t + L_t y_t) \tag{35}
\]

To summarize, combining equations (35) and (17), we can obtain a \((n-m)\) dimensional state observer for system (1) with the following expression

\[
\begin{cases}
  z_{t+1} = (\mathcal{A}_{2t} - \mathcal{A}_t \mathcal{A}_{12t})z_t + [(\mathcal{A}_{21t} - \mathcal{L}_t \mathcal{A}_{12t}) + (\mathcal{A}_{22t} - \mathcal{L}_t \mathcal{A}_{12t}) \mathcal{L}_t ]y_t + (\mathcal{B}_{2t} - \mathcal{L}_t \mathcal{B}_{12t})u_t \\
  \hat{x}_t = Q_{1t}y_t + Q_{2t}(z_t + L_t y_t) = Q_{2t}z_t + (Q_{1t} + Q_{2t}L_t)y_t
\end{cases} \tag{36}
\]
3.2. Robust consideration. In practice, it is inevitable that there exist some uncertain disturbances in system running, which can lead to the deviation of the real system data from the nominal system data. However, the periodic observers are designed according to the nominal system. This will result in inaccuracy even instability on the observation error. Therefore, it is indispensable to design a robust observer which is insensitive to the small perturbation on system data. In this section, based on a perturbation analysis result on periodic data, we would like to provide a periodic robust observer design program.

For normalization, the problem of robust observer design for linear discrete-time periodic system (1) can be described as follows.

**Problem.** Consider the completely observable linear discrete-time periodic system (1), seek the periodic matrix \( L_t \in \mathbb{R}^{(n-m) \times m} \), \( t \in 0, T-1 \), such that the following conditions are met:

1. Observer system (36) gives an asymptotic estimation of state \( x_t \).
2. The reduced-order observer gains are as insensitive as possible to small perturbations on system matrices.

Here, we review a previous research result on perturbation analysis, which can be found in reference [11].

**Lemma 3.4.** Let \( \Psi = A(T-1)A(T-2) \cdots A(0) \in \mathbb{R}^{(n-m) \times (n-m)} \) be diagonalizable and \( Q \in \mathbb{C}^{(n-m) \times (n-m)} \) be a nonsingular matrix such that \( \Psi = Q^{-1} \Lambda Q \in \mathbb{R}^{(n-m) \times (n-m)} \), where \( \Lambda = \text{diag}\{\lambda_1, \lambda_2, \cdots, \lambda_n\} \) is the Jordan canonical form of matrix \( \Psi \). For a real scalar \( \varepsilon > 0 \), \( \Delta_i(\varepsilon) \in \mathbb{R}^{n \times n}, i \in 0, T-1 \), are matrix functions of \( \varepsilon \) satisfying

\[
\lim_{\varepsilon \to 0^+} \frac{\Delta_i(\varepsilon)}{\varepsilon} = \Delta_i,
\]

where \( \Delta_i \in \mathbb{R}^{n \times n}, i \in 0, T-1 \), are constant matrices. Then for any eigenvalue \( \lambda \) of matrix

\[
\Psi(\varepsilon) = (A(T-1) + \Delta_{T-1}(\varepsilon))(A(T-2) + \Delta_{T-2}(\varepsilon)) \cdots (A(0) + \Delta_0(\varepsilon)),
\]

the following relation holds:

\[
\min_i \{|\lambda_i - \lambda|\} \leq \varepsilon \kappa_F(Q) \left( \sum_{i=0}^{T-1} \|A(i)\|_F^{T-1} \right) \max_i \{\|\Delta_i\|_F\} + O(\varepsilon^2).
\]

(37)

According to Lemma 3.4, one could take the robust performance index of problem 3.2 as

\[
J_1(D_t) = \kappa_F(X(0)) \sum_{t=0}^{T-1} \|A_{22t} - T_t A_{12t}\|_F^{T-1}.
\]

(38)

On the other hand, since small gains also mean robustness in some sense, one can adopt another robustness index as

\[
J_2(D_t) = \sum_{t=0}^{T-1} \|T_t\|_F^2.
\]

(39)

Taking a tradeoff between the two index gives

\[
J(D_t) = \alpha J_1(D_t) + (1 - \alpha) J_2(D_t),
\]

(40)

where \( 0 \leq \alpha \leq 1 \) is a weighting factor. Combined with the iterative algorithm for reduced-order observer design, the problem of robust periodic reduced-order
observer design can be converted into an optimization problem. The corresponding algorithm can be summarized in the following.

**Algorithm 2.** (An iterative algorithm for computing periodic robust reduced-order observer gains)

1. Perform the operations of steps 1-5 of Algorithm 1.
2. Based on gradient-based search methods and the index (40), solve the optimization problem
   \[
   \text{Minimize } J(D_t),
   \]
   and denote the optimal decision matrix by \(D_t^{opt}, t \in 0, T-1\).
3. Substituting \(D_t^{opt}\) into steps 3-5 of algorithm 1 gives optimization matrices \(X_t^{opt}\).
4. The robust periodic reduced-order observer gain matrices can be obtained as
   \[
   L_t^{opt} = (D_t^{opt}(X_t^{opt})^{-1})^T, t \in 0, T-1.
   \]

4. **A numerical example.** Consider LDP system (1) with periodic \(T = 3\). Its parameters are given as follows:
   \[
   A_0 = \begin{bmatrix} 1 & 3 & 2 \\ -2 & 1 & 3 \\ 2 & 2 & -1 \end{bmatrix}, \quad A_1 = \begin{bmatrix} 1 & 2 & 2 \\ -2 & -1 & -2 \\ 1 & 3 & 3 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & 1 \\ 2 & 1 & 3 \end{bmatrix}
   \]
   \[
   B_0 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \quad B_1 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}
   \]
   \[
   C_0 = [ 1 \ 1 \ 1 ], \quad C_1 = [ 1 \ 1 \ 1 ], \quad C_2 = [ 1 \ 1 \ 1 ]
   \]

It is known by simple verification that this system is completely reachable and completely observable. The state dimension is 3 and the output dimension is 1. Therefore, a reduce-order observer with dimension 2 can be expected.

According to lemma 2.3, we can obtain the non-singular matrix \(P_t, t \in 0, 1, 2\), as shown below
\[
P_t = \begin{bmatrix} C_t \\ R_t \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, t \in 0, 1, 2.
\]

Let the parameter matrix \(D_t, t \in 0, 1, 2\), be in the following form:
\[
D_t = \begin{bmatrix} -1 \ 1 \end{bmatrix}, t \in 0, 1, 2.
\]

According to Algorithm 1, by choosing parameter matrices \(D_t\), we can obtain a group of gains for reduced-order observer as
\[
\left\{
\begin{array}{l}
L_0^{rand} = \begin{bmatrix} -0.9856 \\ 0.3230 \end{bmatrix}^T \\
L_1^{rand} = \begin{bmatrix} -0.6952 \\ -0.3519 \end{bmatrix}^T \\
L_2^{rand} = \begin{bmatrix} -2.5528 \\ -0.8837 \end{bmatrix}^T
\end{array}
\right.
\]

Furthermore, to realize robustness purpose, applying Algorithm 2 gives a group of robust periodic reduced-order state observer gains as follows
\[
\left\{
\begin{array}{l}
L_0^{robu} = \begin{bmatrix} -0.4752 \\ -0.1353 \end{bmatrix}^T \\
L_1^{robu} = \begin{bmatrix} -0.6280 \\ -0.1224 \end{bmatrix}^T \\
L_2^{robu} = \begin{bmatrix} 2.0459 \\ -1.2898 \end{bmatrix}^T
\end{array}
\right.
\]
To check the effectiveness of the designed observers, the observed errors by the two observers are taken into consideration. Let reference input be \( v(t) = 0.3[\sin(\frac{\pi}{2} + t) + \cos(\frac{\pi}{2} + t)] \) and assume that initial states of the system (1) and the observer (36) are respectively taken as 

\[
\begin{bmatrix}
0.5 \\
-5 \\
1
\end{bmatrix},
\begin{bmatrix}
0 \\
0 \\
0
\end{bmatrix}
\]

The response history of the observed errors by \( L^{\text{rand}} \) and \( L^{\text{robust}} \) in step 40 are shown in Figure 1. From the simulation results, we can see that both the two observers can track the system states very well, even the reduced order observer gains \( L^{\text{rand}} \) are generated randomly by Algorithm 1. Moreover, the error trajectories by \( L^{\text{robust}} \) have smaller overshoot and faster convergence speed, which can lead better transient and steady performance. Therefore, we can conclude that the robust reduced-order observer design algorithm proposed in the paper is very effective.

5. Conclusion. This paper mainly introduces the design method of periodic reduced-order state observer for a class of LDP systems. Following the design ideas of reduce-order observer for LTI systems, by non-singular linear transformation, a part of state information can be completely represented by system output and the other state information can be reconstructed via the periodic poles assignment technique. In addition, robust periodic reduced-order observer is also considered in this paper. Correspondingly, two detailed design algorithms are provided and their effectiveness are illustrated by a simulation example.

Interesting future research topics include: (i) to study distributionally robust design for LDP systems under the uncertainties with unknown distributions; (ii) to consider probabilistic control of Markov jump linear discrete periodic systems.
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