Entanglement and Berry Phase in a $9 \times 9$ Yang-Baxter system

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A M-matrix which satisfies the Hecke algebraic relations is presented. Via the Yang-Baxterization approach, we obtain a unitary solution $\hat{R}(\theta, \varphi_1, \varphi_2)$ of Yang-Baxter Equation. It is shown that any pure two-qutrit entangled states can be generated via the universal $\hat{R}$-matrix assisted by local unitary transformations. A Hamiltonian is constructed from the $\hat{R}$-matrix, and Berry phase of the Yang-Baxter system is investigated. Specifically, for $\varphi_1 = \varphi_2$, the Hamiltonian can be represented based on three sets of SU(2) operators, and three oscillator Hamiltonians can be obtained. Under this framework, the Berry phase can be interpreted.

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I. INTRODUCTION

In 1984, Berry [1] predicted that a quantum system acquires a geometrical phase, in addition to a dynamical phase, if the environment or the Hamiltonian returns to its initial state adiabatically. Simon [2] was the first to recast the mathematical formalism of berry phase within the elegant language of differential geometry and fibre bundles. Berry’s theory has been generalized by extending it to non-adiabatic evolution, non-cyclic and non-unitary evolution, mixed states or non-abelian geometric phases. A vast number of experiments have verified its characteristics in various systems, including NMR, the closely related technique of NQR, optical systems, and so on. Recently more and more works have been attributed to it because of its possible applications to quantum computation. Such interest is motivated by the belief that geometric quantum gates should exhibit an intrinsic fault tolerance in the presence of some kind of external noise due to the geometric nature of the Berry phase. Quantum gate operations can be implemented through the geometric effects on the wave function of the systems. On the other hand, entanglement is a bizarre feature of quantum theory, has been recognized as an important resource for applications in quantum information and computation processing.

Very recently, braiding operators and Yang-Baxter equation (YBE) have been introduced to the field of quantum information and quantum computation. Ref. [26] investigated quantum computation by anyons based on quantum braids. Ref. [27] has explored the role of unitary braiding operators in quantum computation. It is shown that the braid matrix can be identified as the universal quantum gate. This motivates a novel way to study quantum entanglement and the Berry phase based on the theory of braiding operators, as well as YBE. The first step along this direction is initiated by Zhang, Kauffman and Ge. In Ref. [29], the Bell matrix generating two-qubit entangled states has been recognized to be a unitary braid transformation. Later on, an approach to describe Greenberger-Horne-Zeilinger (GHZ) states, or N-qubit entangled states based on the theory of unitary braid representations has been presented in Ref. [30]. Chen and his co-workers utilized unitary braiding operators to realize entanglement swapping and generate the GHZ states, as well as the linear cluster states. With the unitary $\hat{R}(\theta, \phi)$ matrix, the authors constructed a Hamiltonian. The Berry phase and quantum criticality of the Yang-Baxter system have been explored accordingly. In a very recent work, it is found that any pure two-qudit entangled state can be achieved by a universal Yang-Baxter Matrix assisted by local unitary transformations. However, The unitary solution $\hat{R}(x)$ of YBE, which is presented in Ref. [33], only depends on a parameter $\theta$ which is time-independent. We can’t explore the evolution of the Yang-Baxter system. Motivated by this, in this paper we present a unitary solution of YBE, $\hat{R}(\theta, \varphi_1, \varphi_2)$, where $\varphi_1$ and $\varphi_2$ are time-dependent, while $\theta$ is time-independent. Thus, we can explore the evolution of the Yang-Baxter system by constructing a Hamiltonian from the matrix $\hat{R}(\theta, \varphi_1, \varphi_2)$. This in turn allows us to investigate the Berry phase in the entanglement space.

The paper is organized as follows: In Sec [31] we present a $9 \times 9$ Yang-Baxter matrix $\hat{R}(\theta, \varphi_1, \varphi_2)$. In the following, we investigate the entanglement properties of it. It is shown that the arbitrary degree of entanglement for two-qutrit entangled states can be generated via the unitary $\hat{R}$-matrix acting on the standard basis. In fact, we can prove that this unitary matrix $\hat{R}$ is local equivalent to the solution in Ref. [33]. So we can say that all pure two-qutrit entangled...
states can be generated via the universal $\tilde{R}$-matrix assisted by local unitary transformations. In Section III, we construct a Hamiltonian from the unitary $\tilde{R}$-matrix. The Berry phase of this system is investigated. Specifically, for $\varphi_1 = \varphi_2$, the three nonzero Hamiltonian subsystems all are shown to be equivalent to oscillator systems of two fermions. We end with a summary.

II. UNITARY SOLUTION OF YANG-BAXTER EQUATION AND ENTANGLEMENT

As is known, The YBE is given by

$$\tilde{R}_i(x)\tilde{R}_{i+1}(xy)\tilde{R}_i(y) = \tilde{R}_{i+1}(y)\tilde{R}_i(xy)\tilde{R}_{i+1}(x),$$

where $x$ and $y$ are spectrum parameters. The notation $\tilde{R}_i(x) \equiv \tilde{R}_{i,i+1}(x)$ is used, $\tilde{R}_{i,i+1}(x)$ implies $1_1 \otimes 1_2 \otimes 1_3 \cdots \otimes 1_n$, $1_j$ represents the unit matrix of the $j$-th particle, and $x = e^{i\theta}$ is a parameter related to the degree of entanglement. Let the unitary Yang-Baxter $\tilde{R}$-matrix for two qutrits be the form

$$\tilde{R}_i(x) = \rho(x)[1_i + G(x)M_i],$$

where $\rho(x)$ and $G(x)$ are some functions needed to determine later on, $1_i = 1_i \otimes 1_{i+1}$, and the Hermitian matrices $M_i$‘s (i.e., $M_i = M_i^\dagger$) satisfy the Hecke algebraic relations: $M_i M_{i+1} M_i + g M_i + M_{i+1} M_i M_{i+1} + g M_{i+1} M_i = \alpha M_i + \beta 1_i$, with $\beta - g = 2a^2$ while $a \neq 0$. Substituting Eq. (2) into Eq. (1), one has $G(x) + G(x) + \alpha G(x)G(y) = [1 + gG(x)G(y)]G(xy)$. The initial condition $\tilde{R}_i(1) = I_i$ leads to $G(1) = 0, \rho(1) = 1$. In addition, the unitary condition $\tilde{R}_i^+(x) = \tilde{R}_i^{-1}(x) = \tilde{R}_i^+(x^{-1})$ yields $G(x) + G(x^{-1}) + \alpha G(x)G(x^{-1}) = 0, \rho(x)\rho(x^{-1})[1 + \beta G(x)G(x^{-1})] = 1$. For convenience, we restrict ourselves on $a = 1$, and $\beta = g = 2$. As a result, one has $G(x) = -\frac{x-x^{-1}}{2x + x^{-1}}, \rho(x) = \frac{2x + x^{-1}}{3}$.

In this work, we choose basis $\{|00\rangle, |01\rangle, |02\rangle, |10\rangle, |11\rangle, |12\rangle, |20\rangle, |21\rangle, |22\rangle\}$ as the standard basis. The $9 \times 9$ matrix $M$ is realized as,

$$M = \begin{pmatrix}
0 & 0 & 0 & 0 & q_2 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & q_2 & q_1 & 0 & 0 & 0 \\
0 & 0 & q_1 & 0 & 0 & 0 & q_1 & 0 & 0 \\
0 & q_1 & 0 & 0 & q_1 & 0 & 0 & 0 & q_2 \\
0 & 0 & q_1 & 0 & 0 & 0 & q_1 & 0 & 0 \\
0 & q_2 & q_1 & 0 & 0 & 0 & q_1 & 0 & 0 \\
0 & q_2 & q_2 & 1 & 0 & 0 & 0 & 0 & 0 \\
\frac{1}{q_2} & 0 & q_2 & q_2 & 1 & 0 & 0 & 0 & 0 \\
\frac{1}{q_2} & 0 & q_2 & q_2 & 1 & 0 & 0 & 0 & 0 \\
\end{pmatrix},$$

where $q_1 = e^{i\varphi_1}$ and $q_2 = e^{i\varphi_2}$, with the parameters $\varphi_1$ and $\varphi_2$ both are real.

The Gell-mann matrices $\lambda_{\mu}$ satisfy $[I_{\lambda}, I_{\mu}] = i\lambda_{\mu\nu}I_{\nu}(\lambda, \mu, \nu = 1, \cdots, 8)$, where $I_{\lambda} = \frac{1}{2}\lambda_{\mu}$. To the later convenience, we denote $I_{\lambda}$ by, $I_{\pm} = I_1 \pm i I_2, V_{\pm} = I_4 \mp i I_5, U_{\pm} = I_6 \pm i I_7, Y = \frac{2}{\sqrt{3}}I_8$. Introducing three sets of $SU(3)$ realizations

$$I^{(i)} = I^{(i)}_1 I^{(i)}_2, \quad U^{(i)} = U^{(i)}_1 U^{(i)}_2, \quad V^{(i)}_\pm = V^{(i)}_1 \mp V^{(i)}_2,$$

(i) : 

$$I^{(i)}_1 = \frac{1}{3}(I^3_1 + I^3_2) + \frac{1}{2}(I^3_1 Y_2 + Y_1 I^3_2),$$

$$Y^{(i)} = \frac{1}{3}(Y_1 + Y_2) + \frac{2}{3}I^3_1 Y_2 - \frac{1}{2}Y_1 Y_2;$$

(ii) :

$$I^{(ii)}_1 = U^{(ii)}_1 V^{(ii)}_2, \quad U^{(ii)}_\mp = U^{(ii)}_1 \mp U^{(ii)}_2, \quad V^{(ii)}_\pm = I^{(ii)}_1 \mp I^{(ii)}_2,$$

$$I^{(ii)}_3 = \frac{1}{3}(I^3_1 - I^3_2) - \frac{1}{6}(Y_1 + Y_2) + \frac{1}{3}I^3_1 Y_2 - \frac{1}{2}Y_1 Y_2;$$

$$Y^{(ii)} = -\frac{1}{3}(I^3_1 - I^3_2) - \frac{1}{6}(Y_1 + Y_2) + \frac{1}{3}I^3_1 Y_2 - \frac{1}{2}Y_1 Y_2;$$
one is assisted by local unitary transformations. We can say the unitary \( \hat{R}(\theta) \) represents of SU(3) algebra. In addition, each block of \( \hat{R} \)-matrix can be represented by fundamental representations of SU(3) algebra.

We eventually arrive at the unitary Yang-Baxter matrix for two qutrits as,

\[
M = \begin{cases} 
\frac{q_2}{q_1} I_1^{(1)} + \frac{q_1}{q_2} I_1^{(1)} + \frac{1}{q_1^2} V_1^{(1)} + \frac{2}{q_2^2} V_2^{(1)} + \frac{q_1^2 U_1^{(1)}}{q_1} + \frac{1}{q_1} U_1^{(2)} \\
\frac{q_1}{q_2} I_1^{(2)} + \frac{q_2}{q_1} I_1^{(2)} + \frac{1}{q_2^2} V_2^{(2)} + \frac{2}{q_1^2} V_1^{(2)} + \frac{1}{q_2} U_2^{(2)} + \frac{1}{q_1} U_2^{(1)} \\
\frac{q_1}{q_2} I_1^{(3)} + \frac{q_2}{q_1} I_1^{(3)} + \frac{1}{q_2^2} V_2^{(3)} + \frac{2}{q_1^2} V_1^{(3)} + \frac{1}{q_2} U_2^{(3)} + \frac{1}{q_1} U_2^{(2)} 
\end{cases}
\]

We can study these entangled states. The negativity for two qutrits is given by,

\[
\mathcal{N}(\rho) = \frac{\|\rho^{TA}\| - 1}{2},
\]

where \( \|\rho^{TA}\| \) denotes the trace norm of \( \rho^{TA} \), which denotes the partial transpose of the bipartite state \( \rho \). In fact, \( \mathcal{N}(\rho) \) corresponds to the absolute value of the sum of negative eigenvalues of \( \rho^{TA} \), and negativity vanishes for unentangled states. By calculation, we can obtain the negativity of the state \( |\psi\rangle_{YB} \) as

\[
\mathcal{N}(\theta) = \frac{4}{9} (\sin^2 \theta + |\sin \theta| \sqrt{1 + 8 \cos^2 \theta}).
\]

When \( x = e^{i \pi/3} \), the state \( |\psi\rangle_{YB} \) becomes the maximally entangled state of two qutrits as \( |\psi\rangle_{YB} = \frac{1}{\sqrt{3}} (e^{i \pi/3} |00\rangle + e^{i \pi/3} |11\rangle + e^{i \pi/3} |22\rangle) \). In general, if one acts the unitary \( \hat{R}(\theta) \) on the basis \( |00\rangle, |01\rangle, |02\rangle, |10\rangle, |11\rangle, |12\rangle, |20\rangle, |21\rangle, |22\rangle \), he will obtain the same negativity as Eq. (8). It is easy to check that the negativity ranges from 0 to 1 when the parameter \( \theta \) runs from 0 to \( \pi \). But for \( \theta \in [0, \pi] \), the entanglement is not a monotonic function of \( \theta \). And when \( x = e^{i \pi/3} \), one will generate nine complete and orthogonal maximally entangled states of two qutrits. It is worth mention in that the entanglement doesn’t depend on the parameters \( \varphi_1 \) and \( \varphi_2 \). So one can verify that the parameters \( \varphi_1 \) and \( \varphi_2 \) may be absorbed into a local operation. Actually, we can introduce a local unitary operation \( P \) whose form is \( P = \text{diag} \left\{ \frac{q_1}{q_2}, 1, q_1 \right\} \). By means of this local transformation \( (P \otimes P) \hat{R}(\theta, \varphi_1, \varphi_2) (P^{-1} \otimes P^{-1}) = \hat{R}(\theta) \), we can say the unitary \( \hat{R}(\theta, \varphi_1, \varphi_2) \)-matrix is local equivalent to the universal \( \hat{R}(\theta) \)-matrix, which is the solution of \( n = 3 \) in Ref. [33], where the proof of universality for a \( n^2 \times n^2 \) Yang-Baxter matrix has been presented. So the same as the property of \( \hat{R}(\theta) \)-matrix in Ref. [33] we can also say all pure entangled states of two 3-dimensional quantum systems (i.e., two qutrits) can be generated from an initial separable state via the universal \( \hat{R}(\theta, \varphi_1, \varphi_2) \)-matrix if one is assisted by local unitary transformations.
III. HAMILTONIAN AND BERRY PHASE

A Hamiltonian of the Yang-Baxter system can be constructed from the $R(\theta, \varphi_1, \varphi_2)$-matrix. As is shown in Ref. \[31\], the Hamiltonian is obtained through the Schrödinger evolution of the entangled states. Let the parameters $\theta$ be time-independent and $\varphi_i = \omega_i t$ be time-dependent, where $\omega_i = n_i \omega (i=1,2; n_i$ is a fraction in lowest terms), the Hamiltonian reads,

$$\dot{H} = \frac{i \hbar}{\partial t} \hat{R}(\theta, \varphi_1, \varphi_2) \hat{R}(\theta, \varphi_1, \varphi_2) = \sum_{k=1}^{3} H^{(k)},$$

(12)

where the superscript $k$ denotes the $k$-th subsystem. The $k$-th subsystem's Hamiltonian $H^{(k)}$ can be expressed as follows,

$$H^{(1)} = -\frac{8 \sqrt{2} \hbar \omega \sin \theta}{3} \left[ \frac{\sqrt{3}}{12} (ib^n - ibn_2) e^{-2i(\varphi_1 - \varphi_2)I^{(1)}_+} + \frac{\sqrt{3}}{12} (-ibn_1 + ib^n_2) e^{2i(\varphi_1 - \varphi_2)I^{(1)}_-} \right.$$

$$+ \frac{\sqrt{3}}{12} (ib^n_2 - 2n_1 \sin \theta) e^{-2i\varphi_2 V^{(1)}_+} - \frac{\sqrt{3}}{12} (ibn_2 + 2n_1 \sin \theta) e^{2i\varphi_2 V^{(1)}_-}$$

$$- \frac{\sqrt{3}}{12} (ibn_1 + 2n_2 \sin \theta) e^{2i\varphi_1 U^{(1)}_+} + \frac{\sqrt{3}}{12} (ib^n_1 - 2n_2 \sin \theta) e^{-2i\varphi_1 U^{(1)}_-}$$

$$- \frac{\sqrt{3}}{2} (n_1 - n_2) \sin \theta I^{(1)}_3 - \frac{\sqrt{3}}{4} (n_1 + n_2) \sin \theta Y^{(1)} \right]$$

(13)

Hereafter the superscript $i(i=2,3)$ denotes the second or the third subsystem.

Based on the operators $I^{(k)}_\lambda (\lambda = 1, 2, \cdots, 8; k = 1, 2, 3)$, the $k$-th subsystem's Hamiltonian can be rewritten as follows,

$$H^{(k)} = C(k) \sum_{\lambda=1}^{8} B^{(k)}_\lambda I^{(k)}_\lambda,$$

(15)

where $C(1) = -\frac{8 \sqrt{2} \hbar \omega \sin \theta}{3}$ and $C(i) = -\frac{4 \sqrt{2} \hbar \omega \sin \theta}{3} (i = 2, 3)$. By comparing Eq.\[13\], Eq.\[14\] with Eq.\[15\], one can obtain $B^{(k)}_\lambda$ as follows,

$$B^{(1)}_1 = \frac{\sqrt{2}}{2} (n_1 - n_2) \cos \theta \sin 2(\varphi_1 - \varphi_2) + \frac{\sqrt{2}}{6} (n_1 + n_2) \sin \theta \cos 2(\varphi_1 - \varphi_2),$$

$$B^{(1)}_2 = -\frac{\sqrt{2}}{2} (n_1 - n_2) \cos \theta \cos 2(\varphi_1 - \varphi_2) + \frac{\sqrt{2}}{6} (n_1 + n_2) \sin \theta \sin 2(\varphi_1 - \varphi_2),$$

$$B^{(1)}_3 = -\frac{\sqrt{2}}{2} (n_1 - n_2) \sin \theta,$$

$$B^{(1)}_4 = \frac{\sqrt{2}}{6} n_2 \sin \theta \cos 2(\varphi_2) + \frac{\sqrt{2}}{2} n_2 \cos \theta \sin 2(\varphi_2) - \frac{\sqrt{2}}{3} n_1 \sin \theta \cos 2(\varphi_2),$$

$$B^{(1)}_5 = -\frac{\sqrt{2}}{6} n_2 \sin \theta \sin 2(\varphi_2) + \frac{\sqrt{2}}{2} n_2 \cos \theta \cos 2(\varphi_2) + \frac{\sqrt{2}}{3} n_1 \sin \theta \sin 2(\varphi_2),$$

$$B^{(1)}_6 = \frac{\sqrt{2}}{6} n_1 \sin \theta \cos 2(\varphi_1) + \frac{\sqrt{2}}{2} n_1 \cos \theta \sin 2(\varphi_1) - \frac{\sqrt{2}}{3} n_2 \sin \theta \cos 2(\varphi_1),$$

$$B^{(1)}_7 = -\frac{\sqrt{2}}{6} n_1 \sin \theta \sin 2(\varphi_1) + \frac{\sqrt{2}}{2} n_1 \cos \theta \cos 2(\varphi_1) + \frac{\sqrt{2}}{3} n_2 \sin \theta \sin 2(\varphi_1),$$

$$B^{(1)}_8 = \frac{\sqrt{2}}{6} (n_1 + n_2) \sin \theta,$$

(16)
the components of vector $B$ Hamiltonian. After a periods $2\pi$ the components
\begin{align}
B_1^{(i)} &= \frac{\sqrt{2}}{2} (n_1 - n_2) \sin(\varphi_1 - \varphi_2) \cos \theta - \frac{\sqrt{2}}{6} (n_1 + n_2) \cos(\varphi_1 - \varphi_2) \sin \theta, \\
B_2^{(i)} &= \frac{\sqrt{2}}{2} (n_1 - n_2) \cos(\varphi_1 - \varphi_2) \cos \theta + \frac{\sqrt{2}}{6} (n_1 + n_2) \sin(\varphi_1 - \varphi_2) \sin \theta, \\
B_3^{(i)} &= \frac{\sqrt{2}}{6} (n_1 - n_2) \sin \theta, \\
B_4^{(i)} &= -\frac{\sqrt{2}}{6} n_2 \cos \varphi_2 \sin \theta + \frac{\sqrt{2}}{2} n_2 \sin \varphi_2 \cos \theta + \frac{\sqrt{2}}{6} n_1 \cos \varphi_2 \sin \theta, \\
B_5^{(i)} &= -\frac{\sqrt{2}}{6} n_2 \sin \varphi_2 \sin \theta - \frac{\sqrt{2}}{2} n_2 \cos \varphi_2 \cos \theta + \frac{\sqrt{2}}{6} n_1 \sin \varphi_2 \sin \theta, \\
B_6^{(i)} &= -\frac{\sqrt{2}}{6} n_1 \cos \varphi_1 \sin \theta + \frac{\sqrt{2}}{2} n_1 \sin \varphi_1 \cos \theta + \frac{\sqrt{2}}{6} n_2 \cos \varphi_1 \sin \theta, \\
B_7^{(i)} &= -\frac{\sqrt{2}}{6} n_1 \sin \varphi_1 \sin \theta - \frac{\sqrt{2}}{2} n_1 \cos \varphi_1 \cos \theta + \frac{\sqrt{2}}{6} n_2 \sin \varphi_1 \sin \theta, \\
B_8^{(i)} &= -\frac{\sqrt{2}}{6} (n_1 + n_2) \sin \theta.
\end{align}

The Hamiltonian of the $k$-th subsystem, $H(B(t)^{(k)})^{(k)}$, depends on the parameters $B_{\lambda}^{(k)} (\lambda = 1, 2, \cdots, 8)$, which are the components of vector $B^{(k)}$. Namely, $B^{(k)}$ are a series of time-varying parameters controlling the $k$-th subsystem’s Hamiltonian. After a periods $T^{(k)}$, Hamiltonian returns to its original form, i.e. $H(B(0))^{(k)} = H(B(T^{(k)}))^{(k)}$. According to this, one can easily verify the periods of the subsystems are, $T^{(1)} = \frac{\pi}{\omega}$ and $T^{(i)} = \frac{2\pi}{\omega}$, when $\frac{m}{n_2} = m$ (i.e. $n_1 = m, n_2 = 1$, where $m$ is integer and $m \neq 0$).

To the later convenience, we denote $\sqrt{n_1^2 - n_1 n_2 + n_2^2} = n, 3n_1 + 2\sqrt{2}n \sin \theta = \alpha_+; 3n_1 + \sqrt{3}ib^*n = \beta_+; 3n_1 - 2\sqrt{2}n \sin \theta = \alpha_-; 3n_1 - \sqrt{3}ibn = \beta_-; \alpha_+; 3n_2 + 2\sqrt{2}n \sin \theta = \delta_+; 3n_2 + \sqrt{3}ibn = \delta_-; n_2 - \sqrt{2}bn = \eta_-$, $N^{(k)}_0(\alpha) = \{\alpha = +, 0, -\}$ are normalization coefficients. $\mathcal{N}_0^{(1)} = 12n^{2}[6n^2 + 2\sqrt{2}(n_1 - n_2)n \sin \theta - 3n_1^2]; \mathcal{N}_0^{(1)} = 12n^{2}[3n^2 + 2\sqrt{2}(n_1 + n_2)n \sin \theta + 3n_1 n_2]; \mathcal{N}_0^{(i)} = 2n^2, \mathcal{N}_0^{(i)} = 12n^{2}[3n^2 + 2\sqrt{2}(n_1 + n_2)n \sin \theta + 3n_1 n_2]$.

The eigenstates of the first subsystem are found to be
\begin{align}
|E_+^{(1)}\rangle &= \frac{1}{\sqrt{\mathcal{N}_+^{(1)}}}((n_1 - 2n_2)\alpha_+ + 6n_2^2)e^{2i\varphi_2}|00\rangle + (n_2\beta_- + \sqrt{2}ibn_1)e^{2i\varphi_1}|11\rangle + (n_1\alpha_+ - n_2\beta_+)|22\rangle, \\
|E_0^{(1)}\rangle &= 1\sqrt{\mathcal{N}_0^{(1)}}(-n_1 e^{2i\varphi_2}|00\rangle + n_2 e^{2i\varphi_1}|11\rangle + (n_1 - n_2)|22\rangle), \\
|E_-^{(1)}\rangle &= 1\sqrt{\mathcal{N}_-^{(1)}}((n_1 - 2n_2)\alpha_- + 6n_2^2)e^{2i\varphi_2}|00\rangle + (n_2\beta_- - \sqrt{2}ibn_1)e^{2i\varphi_1}|11\rangle + (n_1\alpha_- - n_2\beta_-)|22\rangle,
\end{align}

with the corresponding eigenvalues $E_+^{(1)} = \frac{4\sqrt{2}}{n_2}h\omega \sin \theta$, $E_0^{(1)} = 0$ and $E_-^{(1)} = -\frac{4\sqrt{2}}{n_2}h\omega \sin \theta$. For the second and the third subsystems, the eigenstates are found to be,
\begin{align}
|E_+^{(2)}\rangle &= \frac{1}{\sqrt{\mathcal{N}_+^{(2)}}}((n_1\alpha_- + n_2\delta_-)e^{i\varphi_2}|01\rangle + (n_1\alpha_- - n_2\beta_-^*)|12\rangle + (n_2\delta_- - n_1\eta_+)e^{-i(\varphi_1 - \varphi_2)}|20\rangle), \\
|E_0^{(2)}\rangle &= \frac{1}{\sqrt{\mathcal{N}_0^{(2)}}}((-n_1 + n_2)e^{i\varphi_2}|01\rangle + n_1|12\rangle - n_2 e^{-i(\varphi_1 - \varphi_2)}|20\rangle), \\
|E_-^{(2)}\rangle &= \frac{1}{\sqrt{\mathcal{N}_-^{(2)}}}((n_1\alpha_+ + n_2\delta_+)e^{i\varphi_2}|01\rangle + (n_1\alpha_- - n_2\beta_+^*)|12\rangle + (n_2\delta_+ - n_1\eta_-)e^{-i(\varphi_1 - \varphi_2)}|20\rangle),
\end{align}
\begin{align}
|E_+^{(3)}\rangle &= \frac{1}{\sqrt{\mathcal{N}_+^{(3)}}}((n_2\delta_- - n_1\eta_+)e^{-i(\varphi_1 - \varphi_2)}|02\rangle + (n_1\alpha_- + n_2\delta_-)e^{i\varphi_2}|10\rangle + (n_1\alpha_- - n_2\beta_+^*)|21\rangle), \\
|E_0^{(3)}\rangle &= \frac{1}{\sqrt{\mathcal{N}_0^{(3)}}}(-n_2 e^{-i(\varphi_1 - \varphi_2)}|02\rangle + (-n_1 + n_2)e^{i\varphi_2}|10\rangle + n_1|21\rangle), \\
|E_-^{(3)}\rangle &= \frac{1}{\sqrt{\mathcal{N}_-^{(3)}}}((n_2\delta_+ - n_1\eta_-)e^{-i(\varphi_1 - \varphi_2)}|02\rangle + (n_1\alpha_+ + n_2\delta_+)e^{i\varphi_2}|10\rangle + (n_1\alpha_+ - n_2\beta_+^*)|21\rangle),
\end{align}
with the corresponding eigenvalues $E^{(i)}_\alpha = \frac{2\sqrt{2}}{3} \hbar \omega \sin \theta$, $E^{(i)}_0 = 0$ and $E^{(i)}_\alpha = -\frac{2\sqrt{2}}{3} \hbar \omega \sin \theta$ ($i = 2, 3$).

According to the definition of Berry phase [1], when the parameter $B^{(k)}$ is slowly changed around a circuit, then at the end of circuit, the eigenstates $|E^{(k)}_\alpha \rangle (\alpha = +, 0, -)$ evolves adiabatically from 0 to $T^{(k)}$, the Berry phases accumulated by the states $|E^{(k)}_\alpha \rangle$ are,

$$\gamma^{(k)}_\alpha = i \int_0^{T^{(k)}} (E^{(k)}_\alpha | \frac{\partial}{\partial t} E^{(k)}_\alpha \rangle) dt. \quad (21)$$

By substituting (18) (19) (20) into (21), one can obtain the Berry phases for these eigenstates,

Subsystem 1:

$$\gamma^{(1)}_+ = \frac{K_1 + K_2}{N^{(1)}_+} 2\omega T^{(1)}$$

$$\gamma^{(1)}_0 = -\frac{n_1 n_2 (n_1 + n_2)}{2\pi} 2\omega T^{(1)}$$

$$\gamma^{(1)}_- = \frac{K_1 - K_2}{N^{(1)}_-} 2\omega T^{(1)}$$

Subsystem 2 or 3:

$$\gamma^{(i)}_+ = \frac{K_1 + K_2}{N^{(i)}_+} \omega T^{(i)}$$

$$\gamma^{(i)}_0 = -\frac{n_2 (n_1 - n_2)(n_1 - 2n_2)}{2n_2^2} \omega T^{(i)}$$

$$\gamma^{(i)}_- = \frac{K_1 - K_2}{N^{(i)}_-} \omega T^{(i)}$$

where

$$K_1 = -10n_1^5 + 13n_1^4n_2 + 11n_1^3n_2^2 - 82n_1^2n_2^3 + 94n_1n_2^4 - 52n_2^5 - 8\cos(2\theta)(n_1 - 2n_2)n_4^4,$$

$$K_2 = -6\sqrt{2} \sin \theta mn_2(n_1^3 - 9n_1^2n_2 + 12n_1n_2^2 - 8n_2^2),$$

$$K_3 = 2n_2^2((9n_1^2(n_1 - 2n_2) - 8\sin^2 \theta (n_1^3 + n_2^3)) - 9n_2(n_1^4 - n_1^3n_2 + 5n_1^2n_2 - 3n_1n_2^3 + 2n_4^4),$$

$$K_4 = 6\sqrt{2} \sin \theta mn_2(n_1^3 + 6n_1n_2^2 - 3n_1n_2^2 + 4n_2^3).$$

The above Berry phases are the general result for the whole system. Despite the complexity in the expressions for the Berry phase in its full generality, we can use the general formulæ to discuss some special cases for convenience to understand.

**Example 1:** For $\varphi_1 = \varphi_2$, i.e. $n_1 = n_2 = 1$. In this case, $T^{(1)} = \frac{\pi}{2\omega}$ and $T^{(i)} = \frac{2\pi}{2\omega}$. When one substitutes the conditions into Eq. (22), he then gets the following explicit expressions of the Berry phase(all phases are defined modulo $2\pi$ throughout this paper),

$$\gamma^{(k)}_+ = -\gamma^{(k)}_- = -\pi(1 - \frac{2\sqrt{2}}{3} \sin \theta),$$

$$\gamma^{(k)}_0 = 0,$$

where $k = 1, 2, 3$. In fact, for $\varphi_1 = \varphi_2$, the Hamiltonian can be represented based on three sets of $SU(2)$ operators. Under this framework, the Berry phase can be interpreted. In the following, we have a thorough discussing for the case.

Introducing three sets of $SU(2)$ realizations in terms of three sets of operators [1] (3) (10),

$$S^{(k)}_+ = \frac{1}{\sqrt{2}} (V^{(k)}_- + U^{(k)}_+),$$

$$S^{(k)}_- = \frac{1}{\sqrt{2}} (V^{(k)}_+ + U^{(k)}_-),$$

$$S^{(k)}_0 = \frac{3}{4} Y^{(k)} + \frac{1}{4} (I^{(k)}_+ + I^{(k)}_-).$$

They satisfy the algebra relations of $SU(2)$ group: $[S^{(i)}_+, S^{(j)}_-] = 2\delta_{ij} S^{(i)}_0$, $[S^{(i)}_0, S^{(j)}_-] = \pm \delta_{ij} S^{(i)}_\pm$, $(S^{(i)}_\pm)^2 = 0 (i, j = 1, 2, 3)$, with $S^{(k)}_\pm = S^{(k)}_1 \pm i S^{(k)}_2 (k = 1, 2, 3)$. By the way, their second-order Casimir operators are $J^{(k)} = \frac{1}{2}(S^{(k)}_+ S^{(k)}_- + S^{(k)}_- S^{(k)}_+) + (S^{(k)}_0)^2$. One can verify that the eigenvalues of $J^{(k)}$ are $\frac{1}{2}(1 + 1) = \frac{3}{2}$ and $0 (0 + 1) = 0$ which correspond to spin-$\frac{1}{2}$ system and spin-0 system. In terms of the operators (24), the Hamiltonian for subsystems Eq. (13) and Eq. (14) can be rewritten as follows:

$$H^{(1)} = C(1)(-\frac{1}{6} ib^* e^{2i\varphi} S^{(1)}_+ + \frac{1}{6} ib e^{-2i\varphi} S^{(1)}_- + \frac{2\sqrt{2}}{3} \sin \theta S^{(1)}_0),$$

$$H^{(2)} = C(2)(-\frac{1}{6} ib^* e^{2i\varphi} S^{(2)}_+ + \frac{1}{6} ib e^{-2i\varphi} S^{(2)}_- + \frac{2\sqrt{2}}{3} \sin \theta S^{(2)}_0),$$

$$H^{(3)} = C(3)(-\frac{1}{6} ib^* e^{2i\varphi} S^{(3)}_+ + \frac{1}{6} ib e^{-2i\varphi} S^{(3)}_- + \frac{2\sqrt{2}}{3} \sin \theta S^{(3)}_0).$$
\[ H^{(2)} = C(2)(-\frac{1}{6} \hat{b} e^{i\varphi} S_{+}^{(2)} + \frac{1}{6} \hat{b}^* e^{-i\varphi} S_{-}^{(2)} + \frac{2\sqrt{2}}{3} \sin \theta S_{3}^{(2)}) \] (26)

\[ H^{(3)} = C(3)(-\frac{1}{6} \hat{b} e^{i\varphi} S_{+}^{(3)} + \frac{1}{6} \hat{b}^* e^{-i\varphi} S_{-}^{(3)} + \frac{2\sqrt{2}}{3} \sin \theta S_{3}^{(3)}) \] (27)

\( C(1), C(2) \) and \( C(3) \) are defined in Eq. (15). So we can say the whole system is equivalent to three spin-\( \frac{1}{2} \) subsystems and three spin-0 subsystems. Actually, we can introduce a 9 × 9 orthogonal matrix \( P \) which is time-independent (see Appendix A). By means of \( P \), the whole system’s Hamiltonian \( \hat{H} \) and Casimir operators \( \mathcal{J}^{(k)} \) are transformed into block-diagonal matrices. Namely, \( \hat{H} = P \mathcal{H} P^T \) and \( \mathcal{J}^{(k)} = P \mathcal{J}^{(k)} P^T \) are block-diagonal matrices, where \( P^T \) denotes the transpose of matrix \( P \).

For the subsystem 1, from Eq. (A2) we can obtain its Hamiltonian \( \hat{H}^{(1)} = H_{\frac{1}{2}}^{(1)} \oplus H_0^{(1)} \). For \( H_0^{(1)} \), the eigenvalue of Casimir operator \( \mathcal{J}^{(1)} \) is 0, and the Berry Phase is 0. So we can say the subsystem Hamiltonian \( H_0^{(1)} \) is equivalent to a spin-0 subsystem. For \( H_{\frac{1}{2}}^{(1)} \), we introduce the transformation, \( \cos \alpha = \frac{\sqrt{3}}{2} \sin \theta \) and \( \cos \beta = -\frac{\sin \theta \cos 2\varphi + 3 \cos \theta \sin 2\varphi}{\sqrt{9 - 8 \sin^2 \theta}} \), where \( \alpha \in (\arccos \frac{2\sqrt{3}}{3}, \arccos -\frac{2\sqrt{3}}{3}) \) and \( \beta \in [0, 2\pi] \). \( \alpha \) is time-independent, and \( \beta \) is time-dependent. Then the Berry phase Eq (29) can be recast as,

\[ \gamma_{\pm}^{(1)} = \mp \pi (1 - \cos \alpha) = \mp \frac{\Omega(C)}{2} \] (28)

where \( \Omega(C) = 2\pi (1 - \cos \alpha) \) is the familiar solid angle enclosed by the loop on the Bloch sphere, and the parameter \( \alpha \) comes from \( \theta \) which comes from the Yang-Baxterization of the Hermitian matrix \( M \). So the Berry phase depends on the spectral parameter. By means of \( P \), the eigenstates \( |E_{\pm}^{(1)} \rangle \) can be recast as follows (we neglected the global phase factor),

\[ |E_{+}^{(1)} \rangle = -e^{i\beta} \sin \frac{\alpha}{2} |00 \rangle + \cos \frac{\alpha}{2} |01 \rangle, \] (29)

\[ |E_{-}^{(1)} \rangle = \cos \frac{\alpha}{2} |00 \rangle + e^{-i\beta} \sin \frac{\alpha}{2} |01 \rangle. \] (30)

The Hamiltonian \( H^{(1)}_{\frac{1}{2}} \) in Eq (26) can be recast based on the operators Eq (A3) to the form

\[ H^{(1)}_{\frac{1}{2}} = -2\hbar \cos \alpha (2 \cos \alpha S_3^{(1)} + \sin \alpha e^{i\beta} \tilde{S}_+^{(1)} + \sin \alpha e^{-i\beta} \tilde{S}_-^{(1)}) \]

\[ = -2\hbar \cos \alpha \tilde{H}_0^{(1)}, \] (31)

where \( \tilde{H}_0^{(1)} \) is of the following form,

\[ \tilde{H}_0^{(1)} = 2 \cos \alpha \tilde{S}_3^{(1)} + \sin \alpha e^{i\beta} \tilde{S}_+^{(1)} + \sin \alpha e^{-i\beta} \tilde{S}_-^{(1)}. \]

Thus, the Hamiltonian \( H_{\frac{1}{2}}^{(1)} \) has the same physical meaning as that given in (32). That is, \( H_{\frac{1}{2}}^{(1)} \) is an oscillator Hamiltonian formed by two fermions with frequency \( 2\omega \cos \alpha \) where \( \alpha \in (\arccos \frac{2\sqrt{3}}{3}, \arccos -\frac{2\sqrt{3}}{3}) \). For the Yang-Baxter subsystem, \( \alpha = 0 \) is a critical point. However, because of \( \alpha \neq 0 \), the subsystem Hamiltonian Eq (31) can’t reduce to the standard oscillator. In other words, \( \sin \alpha e^{i\beta} \) plays a role of the “energy gap” and the wave function of the subsystem takes the form of spin-coherent states [38]. The quantum criticality can’t occur in the Hamiltonian subsystem \( H_{\frac{1}{2}}^{(1)} \) and the Berry phases can’t vanish since \( \alpha \neq 0 \).

Via the same method, the Berry phases for subsystems 2 and 3 may be obtained, \( \gamma_{\pm}^{(2)} = \mp \pi (1 - \cos \alpha) = \mp \frac{\Omega(C)}{2} \) and \( \gamma_{0}^{(i)} = 0 (i = 2, 3) \). The Hamiltonian \( H^{(i)}_{\frac{1}{2}} \) in Eq (26) and Eq (27) can be recast to the form

\[ H^{(2)}_{\frac{1}{2}} = H^{(3)}_{\frac{1}{2}} = -\hbar \omega \cos \alpha \tilde{H}_0^{(i)}, \] (32)

where \( \tilde{H}_0^{(i)} = 2 \cos \alpha \tilde{S}_3^{(i)} + \sin \alpha e^{i\beta} \tilde{S}_+^{(i)} + \sin \alpha e^{-i\beta} \tilde{S}_-^{(i)} (i = 2, 3) \). Thus, the Hamiltonian \( H^{(2)}_{\frac{1}{2}} \) and \( H^{(3)}_{\frac{1}{2}} \) both have the same physical meaning as that given in (32). That is, \( H^{(2)}_{\frac{1}{2}} \) and \( H^{(3)}_{\frac{1}{2}} \) both are oscillator Hamiltonians formed by two
entangled states can be generated via the universal $\gamma$-matrix. In this case, one can find that the arbitrary degree of entanglement for two-qutrit entangled states can be generated via the unitary $\gamma$-matrix assisted by local unitary transformations. Specifically, the arbitrary degree of entanglement for two-qutrit entangled states can be generated via the unitary $\gamma$ matrix acting on the standard basis. Then the evolution of the Yang-Baxter system is explored by constructing a Hamiltonian from the unitary $\gamma$-matrix. In addition, the Berry phase of the system is investigated, and general expressions of Berry phase are figured out. Finally, we use the general expressions to discuss some special cases. For $\varphi_1 = \varphi_2$, based on three sets of SU(2) operators, the Hamiltonian has been represented and the three nonzero Hamiltonian subsystems all are shown to be equivalent to oscillator systems of two fermions. Under this framework, the Berry phase can be interpreted.

Example 2: For $\varphi_1 = -\varphi_2$, i.e. $n_1 = -n_2 = -1$. In this case $T^{(1)} = \frac{i\pi}{2}$ and $T^{(i)} = \frac{2\pi}{\omega}$. One can get the following explicit expressions of the Berry phase,

$$
\begin{align*}
\text{Subsystem 1:} & \quad \gamma_+^{(1)} = \frac{\sqrt{6} \sin \theta}{3} 2\pi \\
& \quad \gamma_0^{(1)} = 0 \\
& \quad \gamma_-^{(1)} = -\frac{\sqrt{6} \sin \theta}{3} 2\pi \\
\text{Subsystem 2 or 3:} & \quad \gamma_+^{(i)} = \frac{\sqrt{6} \sin \theta}{3} 2\pi \\
& \quad \gamma_0^{(i)} = 0 \\
& \quad \gamma_-^{(i)} = -\frac{\sqrt{6} \sin \theta}{3} 2\pi
\end{align*}
$$

(33)

In this case, one can find that $\gamma_+^{(1)} = -\gamma_-^{(i)}$, $\gamma_0^{(1)} = -\gamma_0^{(i)}$ and $\gamma_-^{(1)} = -\gamma_+^{(i)}$.

Example 3: For $\varphi_1 = 2\varphi_2$, i.e. $n_1 = 2n_2 = 2$. In this case $T^{(1)} = \frac{\pi}{\omega}$ and $T^{(i)} = \frac{2\pi}{\omega}$. We then get the following explicit expressions of the Berry phase,

$$
\begin{align*}
\text{Subsystem 1:} & \quad \gamma_+^{(1)} = \frac{\sqrt{6} \sin \theta}{3} 2\pi \\
& \quad \gamma_0^{(1)} = 0 \\
& \quad \gamma_-^{(1)} = -\frac{\sqrt{6} \sin \theta}{3} 2\pi \\
\text{Subsystem 2 or 3:} & \quad \gamma_+^{(i)} = \frac{\sqrt{6} \sin \theta}{3} 2\pi \\
& \quad \gamma_0^{(i)} = 0 \\
& \quad \gamma_-^{(i)} = -\frac{\sqrt{6} \sin \theta}{3} 2\pi
\end{align*}
$$

(34)

In this case, one can find that $\gamma_+^{(1)} = -\gamma_-^{(i)}$, $\gamma_0^{(1)} = -\gamma_0^{(i)}$ and $\gamma_-^{(1)} = -\gamma_+^{(i)}$.

Example 4: For $\varphi_1 = -2\varphi_2$, i.e. $n_1 = -2n_2 = -2$. In this case $T^{(1)} = \frac{\pi}{\omega}$ and $T^{(i)} = \frac{2\pi}{\omega}$. We then get the following explicit expressions of the Berry phase,

$$
\begin{align*}
\text{Subsystem 1:} & \quad \gamma_+^{(1)} = (\frac{4}{\sqrt{7}} + \frac{\sqrt{7}}{3} \sin \theta) 2\pi \\
& \quad \gamma_0^{(1)} = -\frac{2\pi}{\sqrt{7}} \\
& \quad \gamma_-^{(1)} = (\frac{4}{\sqrt{7}} - \frac{\sqrt{7}}{3} \sin \theta) 2\pi \\
\text{Subsystem 2 or 3:} & \quad \gamma_+^{(i)} = -(\frac{4}{\sqrt{7}} - \frac{\sqrt{7}}{3} \sin \theta) 2\pi \\
& \quad \gamma_0^{(i)} = \frac{2\pi}{\sqrt{7}} \\
& \quad \gamma_-^{(i)} = -(\frac{4}{\sqrt{7}} + \frac{\sqrt{7}}{3} \sin \theta) 2\pi
\end{align*}
$$

(35)

In this case, one can find that $\gamma_+^{(1)} = -\gamma_-^{(i)}$, $\gamma_0^{(1)} = -\gamma_0^{(i)}$ and $\gamma_-^{(1)} = -\gamma_+^{(i)}$.

IV. SUMMARY

In this paper, we have presented a $9 \times 9$ $M$-matrix which satisfies the Hecke algebraic relations and derived a unitary $\hat{R}(\theta, \varphi_1, \varphi_2)$-matrix via Yang-Baxterization of the $M$-matrix. In the following, we can say that all pure two-qutrit entangled states can be generated via the universal $\hat{R}$-matrix assisted by local unitary transformations. Specifically, the arbitrary degree of entanglement for two-qutrit entangled states can be generated via the unitary $\hat{R}$ matrix acting on the standard basis. Then the evolution of the Yang-Baxter system is explored by constructing a Hamiltonian from the unitary $\gamma$-matrix. In addition, the Berry phase of the system is investigated, and general expressions of Berry phase are figured out. Finally, we use the general expressions to discuss some special cases. For $\varphi_1 = \varphi_2$, based on three sets of SU(2) operators, the Hamiltonian has been represented and the three nonzero Hamiltonian subsystems all are shown to be equivalent to oscillator systems of two fermions. Under this framework, the Berry phase can be interpreted.

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APPENDIX A: BLOCK-DIAGONALIZE $\hat{H}$ AND $J^{(k)}$

The $9 \times 9$ orthogonal matrix $P$ which is time-independent reads,

$$
P = \begin{pmatrix}
\frac{1}{\sqrt{2}} & 0 & 0 & 0 & \frac{1}{\sqrt{2}} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
\frac{1}{\sqrt{2}} & 0 & 0 & 0 & -\frac{1}{\sqrt{2}} & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\
0 & 0 & 0 & 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 & 0 & 0 & 0 & \frac{1}{\sqrt{2}} \\
0 & 0 & 0 & 0 & \frac{1}{\sqrt{2}} & 0 & 0 & 0 & -\frac{1}{\sqrt{2}} \\
\end{pmatrix}.
$$

(A1)

The orthogonal matrix $P$ satisfies the relation $PP^T = P^TP = I_{9 \times 9}$, where $P^T$ denotes the transpose of matrix $P$.

By means of $P$, the Hamiltonian $\hat{H}$ for $\varphi_1 = \varphi_2$ can be recast as follows,

$$
\hat{\tilde{H}} = P\hat{H}P^T = \text{diag}\{H^{(1)}_{1/2}, H^{(1)}_0, H^{(2)}_0, H^{(2)}_{1/2}, H^{(3)}_0, H^{(3)}_{1/2}\}
$$

(A2)

where $\hat{\tilde{H}}^{(k)} = H^{(k)}_{1/2} \oplus H^{(k)}_0$, $H^{(k)}_{1/2}$'s are $2 \times 2$ matrices, and $H^{(k)}_0$'s are $1 \times 1$ matrices with $H^{(k)}_0 = 0$. The $(3 \times 3)$-dimensional interactional Hamiltonian system is decomposed into six subsystems. Three sets of $SU(2)$ realizations can be recast as,

$$
\tilde{S}^{(1)}_{+} = |00\rangle\langle 01|, \quad \tilde{S}^{(1)}_{-} = |01\rangle\langle 00|, \quad \tilde{S}^{(1)}_{3} = \frac{1}{2}(|00\rangle\langle 00| - |01\rangle\langle 01|),
$$

(A3)

$$
\tilde{S}^{(2)}_{+} = |11\rangle\langle 20|, \quad \tilde{S}^{(2)}_{-} = |20\rangle\langle 11|, \quad \tilde{S}^{(2)}_{3} = \frac{1}{2}(|11\rangle\langle 11| - |20\rangle\langle 20|),
$$

(A4)

$$
\tilde{S}^{(3)}_{+} = |21\rangle\langle 22|, \quad \tilde{S}^{(3)}_{-} = |22\rangle\langle 21|, \quad \tilde{S}^{(3)}_{3} = \frac{1}{2}(|21\rangle\langle 21| - |22\rangle\langle 22|),
$$

(A5)

and the seconde-order Casimir operators are $\tilde{J}^{(1)} = \frac{3}{4}(|00\rangle\langle 00| + |01\rangle\langle 01|)$, $\tilde{J}^{(2)} = \frac{3}{4}(|11\rangle\langle 11| + |20\rangle\langle 20|)$, $\tilde{J}^{(3)} = \frac{3}{2}(|21\rangle\langle 21| + |22\rangle\langle 22|)$.

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