Calculation of Heavy Meson Potential Coefficients by Solving Lippman-Schwinger Equation

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Abstract

In the present work, we study meson systems consisting of quark-antiquark. We solve Lippman-Schwinger equation numerically for heavy meson systems. We attempt to find a non-relativistic potential model through which we can solve the quark-antiquark bound state problem. The coefficients so obtained are in agreement with Martin potential coefficients. Via this method we also determine the strong coupling constant for the mesons \(t\bar{t}, t\bar{c}\) and \(t\bar{b}\) which is a coefficient of Cornel potential.

Keywords: binding energy, lippman-Schwinger, Martin potential, strong coupling constant

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1 Introduction

The heavy quark potential is an important quantity associated with confinement. This quantity can be almost accurately calculated via lattice simulations [1]. This potential is specified through a non-relativistic effective theory, yet it also deserves serious investigation in perturbative QCD. Lattice QCD simulations are used for analyzing at long distances, but for short distances, perturbation theory works better.

There exists a variety of experimental data on light and heavy mesons, and it is still increasing [2]. When the heavy quark mass ($m_Q$) remarkably exceeds $\Lambda_{QCD}$ (the QCD scale), the running coupling $\alpha_s(m_Q)$ is small. This means that at this scale of the order of the Compton wavelength $\lambda \sim 1/m_Q$, perturbative QCD can serve to describe hadrons. In the non-relativistic limit, it has been shown that the interaction between the two $\bar{Q}$ and $Q$ states can be described by a local potential $V(r)$, where $r$ is the relative coordinate between $Q$ and $\bar{Q}$ (spin is ignored for the moment) [1].

We exploit Lippman-Schwinger equation to analyze the quark-antiquark bound state. This equation has recently been used in some studies of non-relativistic bound state [3, 4, 5]. We have investigated some potential models for heavy mesons. We have also identified the stability intervals of some suitable potentials for these systems and obtained the mass spectrum of these mesons. In our recent work [6], we have studied the tetraquark and solved the two-body problem for tetraquark systems of diquark-antidiquark.

Having studied some types of appropriate potentials such as Martin [7] and Cornell [8, 9], we found Martin potential as the best one for heavy mesons. This is because of the potential’s larger stability interval [3]. In the present work, we calculate the coefficients of Cornell and Martin potentials (which are suitable for describing heavy mesons) through numerical solution of Lippman-Schwinger equation and obtain the physical eigenvalue. It must be notified that we have used input parameters different from those of ref [3] (quark mass, binding energy, and potential). In ref [3], we utilized a revised potential, but we have made use of Cornell potential in the present paper.

It is worth mentioning that in our previous work [3], we detected appropriate potentials through this method and obtained $\lambda = 1$ within an extensive interval of $r$; therefore, we
have been certain about the appropriateness of Cornell and Martin potentials for heavy mesons in our method. That is, we took the binding energy as already identified and solved Schrodinger equation to find two potential coefficients for each meson.

The procedure of the study is given in sec 1. We explain how to find Martin potential coefficients in sec 2. In sec 3, we have also used Coulomb potential confinement to predict the strong coupling constant for $t\bar{t}$, $t\bar{c}$ and $t\bar{b}$. The results and discussion appear in sec. 4 and finally, sec. 5 gives a concise conclusion of the study.

## 2 Method

In this section, we obtain the coefficients of a suitable potential for heavy meson systems by using heavy mesons’ binding energies and by solving Lippman-Schwinger equation numerically.

The two-body bound state of Schrodinger equation with potential $V$ is:

$$| \psi_b > = G_0 V | \psi_b >$$ (1)

where $G_0$ is the propagator of a free particle. In configuration space, it is shown as:

$$\psi_b(r) = -m \frac{1}{4\pi} \int_0^\infty dr' r'^2 \int_{-1}^1 dx' \int_0^{2\pi} d\phi' \frac{exp(-\sqrt{m|E_b|} \frac{r - r'}{r - r'})}{|r - r'|} V(r') \psi_b(r')$$ (2)

$$\psi_b(r) = \int_0^\infty dr' \int_{-1}^1 dx' M(r, r', x') \psi_b(r')$$ (3)

where:

$$M(r, r', x') = -\frac{m}{2} \frac{exp((\sqrt{m|E_b|}) \sqrt{r'^2 + r^2 - 2rr'x})}{\sqrt{r'^2 + r^2 - 2rr'x}} V(r'^2)$$ (4)

$E_b$ stands for the binding energy of two-body bound system. The eigenvalue form of equation (3) is:

$$K(E_b) | \psi_b > = \lambda(E_b) | \psi_b >$$ (5)

To solve this equation, we use a Fortran code. It is used to calculate the eigenvalues for a real non-symmetric square matrix. The magnitude of eigenvalue $\lambda$ depends on energy. The energy that results in $\lambda = 1$ is the binding energy. $\lambda = 1$ is the highest positive eigenvalues. The eigenvalue equation can be solved by direct iteration method. To
discretize it, the integral Gauss-legendre method \cite{11} is used. It is worth mentioning that
to get better results out of the calculations, we consider Gaussian quadrature points for
\( r, r', \) and \( x' \) as 100 (The more the points, the more accurate the results, although this
lowers the running speed of the program).
The required input for the program includes system mass, \( E_b \), potential coefficients, and r-
cutoff. R-cutoff, which is of the same order of magnitude as the meson radius, is supposed
to be the point where the potential tends to zero. \( E_b \) is obtained via solving an eigenvalue
equation by using quark-antiquark interaction (spin-spin interaction in the potential and
spin splitting are ignored). The masses of constituent quarks (MeV) are fitted as:

\[
m_c = 1800, m_b = 5174, m_t = 174000
\]

Ref \cite{12} also makes use of the same inputs for Martin potential. Table 1 shows heavy
meson masses and heavy meson binding energies. Binding energy is dened as the energy
used when breaking a meson into its components, i.e. quark and antiquark, so it is negative
\(^1\). As we know, meson mass is calculated through equation (6)

\[
M(meson) = m_{quark} + m_{antiquark} + E_b \tag{6}
\]

| Spin | Meson  | Meson Mass (PDG\cite{13}) | \( E_b \)  |
|------|--------|--------------------------|-----------|
| S=1  | \( c\bar{c}(J/\psi) \) | 3096.9                   | -503.1    |
| S=0  | \( c\bar{c}(\eta_c) \)  | 2979.9                   | -620.1    |
| S=1  | \( b\bar{b}(\Upsilon) \) | 9460                     | -888      |
| S=0  | \( b\bar{b}(\eta_b) \)  | 9398                     | -850      |

\( E_b \) in table (1) is obtained from equation (6). Based on the information presented in
table 1, we introduce the reduced mass of the meson’s constituent quarks and also the
energies into the program \cite{3}. First, we place the potential in the program:

\[
V(r) = (ra_n/\hbar c)^2 \tag{7}
\]

where \( a_n = 1000 MeV \) is an input. We have introduced the coefficient \( a_n \) to correct the
dimension so that the dimension of each term of the potential is given in MeV. The param-

\(^1\)In order to separate the quark from the antiquark, huge energy of the order of several MeV is required,
i.e. the rest mass energy of the quarks in free state exceeds their total energy while they are inside the
meson. The energy required to separate the quarks is the binding energy of the meson.
eter $hc$ is included in the equation for dimension coordination. $V$ is expressed in terms of $MeV$ in the program and $r$ in terms of $fm$. In this part, we assigned $z$ an arbitrary quantity. The purpose was to obtain $\lambda = 1$, so $z$ was assigned lower or higher values. We carried out this process of iteration until $\lambda = 1$ was obtained as the program output. There is a direct relationship between the coefficient $z$ and the program output, i.e. the $\lambda$ spectrum. The higher the coefficient, the higher the values obtained in the $\lambda$ spectrum. The size and direction of each iteration depends on what $\lambda$ spectrum is obtained following each iteration. If the values in the spectrum are much larger/smaller than 1, the next assigned $z$ should naturally be much larger/smaller than the previous $z$. But when we obtain a $\lambda$ spectrum very near to 1, the next assigned $z$ will be very near to the previous one. Of course, the rate of accuracy set in this stage of the study is 0.01, then $0.99 \leq \lambda \leq 1.01$ is acceptable. Thus, the quantities that yielded eigenvalues either greater than 1.01 or smaller than 0.99 were ignored. The results of running the program show that at $z = 0.1$, the desired output is obtained.

In the next stage, we introduce the coefficient $k$ and take the potential as $k(r^{0.1})$. We then follow the above procedure, i.e. we obtain $\lambda = 1$ by assigning different values to $k$. The rate of accuracy; however, is 0.001. Therefore, any $0.999 \leq \lambda \leq 1.001$ is acceptable. Finally, we introduce the constant $d$ and increase the accuracy up to 0.0001. In fact, we don’t introduce the potential at once but rather stage by stage. Initially, the coefficient $z$ is introduced and the predetermined acceptable error is 0.01. Later as we introduce the other coefficients, we approach the full form of the potential and the tolerance is enhanced to 0.01 for $k$ and finally to 0.0001 for $d$.

$$V(r) = k(a_n r)^z + d$$  \hspace{1cm} (8)

We are using tables 1 and 2 as input. Table 2 illustrates the results. The cells marked with an asterisk (*) contain results that comply with the coefficients of Martin potential. In this potential, $r$ is defined in terms of $GeV^{-1}$. The cells containing a dash (−) indicate potential coefficients which didn’t enjoy the desired accuracy. Through the procedure already described, we have managed to obtain values of $z$, $k$, and $d$ which are significant for different potentials. Our results are spin-dependent because the potential models we are considering are spin-dependent.
Table 2: Potential exponent and coefficients obtained from the program

| Meson   | z     | k(MeV) | d(MeV) |
|---------|-------|--------|--------|
|         | 0.0931| 6886   | -      |
|         | 0.0972*| 6899*  | 8071*  |
|         |       | 6910   | -      |
|         | 0.0981*| 6990*  | -      |
|         |       | 6930*  | 8980*  |
|         |       |        | 8591   |
|         | 0.1*  | 6849   | 6849   |
|         |       | 6862   | -      |
|         | 0.0983*| 6883*  | 8669*  |
|         |       |        | 9465   |

Martin potential coefficients [7] 0.1 6898 8093

3 Calculating Heavy Meson Strong Coupling Constant

The running coupling in QCD is small at high energies, while it increases to order 1 at low energies. This feature is asymptotic freedom of QCD. Mesons containing top quark are unstable because of their short lifetime. However, within this short lifetime, the strong coupling constant can be anticipated (we are aware that this application may be purely academic because of the non-existence of the corresponding bound states).

In this section, our anticipated strong coupling constant for $t\bar{t}$, $t\bar{c}$ and $t\bar{b}$ are presented, thus here we use Cornell potential. Cornell potential is composed of two terms. One is Coulomb potential used for interaction in small distances i.e. $V \propto -\alpha_s/r$ at $r \to 0$. The other is a linear potential capable of describing coennement. It is used for large distances, i.e. $V \propto r$ at large $r$. This potential has absorbed much attention in hadron physics and
is definitely of strong interaction type. This potential is of the following form [8, 9]:

$$V(r) = fr - \frac{4\alpha_s\hbar c}{3} r$$

(9)

where $f = 150000\text{MeV}^2$[15] is the string tension. Here, $\hbar c$ is also used for dimension coordination. $\alpha_s$ is the coefficient to be worked out. Now, we introduce the energy (table [3]) and potential (eq. 9) into the program. We assign an arbitrary initial value to $\alpha_s$ (we assigned 0.01) and then search through the immediate neighbourhood iteratively with a step-size of 0.0001 until $\lambda = 1 \pm 0.01$ is encountered. The results are presented in table (5). Varying $\alpha_s$ leads to different $\lambda$ spectra. However, we seek to arrive at results (i.e. spectrum) which are closest to 1. In other words, accuracy depends on the minimum allowed error of calculation outcome. It is worth mentioning that strong coupling constant for $c\bar{c}$ and $b\bar{b}$ mesons can also be calculated by this method.

We are using table 3 as input. Our results are available in tables 4 and 5. According to [14], coupling constant is inversely correlated with mass, i.e.

$$m_{c\bar{c}} \leq m_{b\bar{b}} \leq m_{t\bar{t}} \Rightarrow \alpha_{c\bar{c}} \leq \alpha_{b\bar{b}} \leq \alpha_{t\bar{t}}$$

Our results in table 4 also indicate this inverse correlation. We used this method to identify the coupling constant of $c\bar{c}$ and $b\bar{b}$ mesons. The $\alpha_s$ we obtained is similar to the $\alpha_s$ in [16]. The results appear in table 5.

As stated in the introduction, we have earlier investigated certain potential models appropriate for heavy mesons to calculate the binding energy of these systems. Since this method is potential-independent, the obtained binding energy applies to other potentials too [17]. In [3], we arrived at the binding energy of $t\bar{t}$ and $t\bar{c}$ systems through a revised potential (we have also calculated the binding energy of $t\bar{b}$ by means of this potential). In this paper, we used that binding energy as the input of the program and obtained potential coefficients different from the previous work. As two examples, we obtained Martin potential coefficients for $b\bar{b}$ and $c\bar{c}$ systems and obtained the coupling constant in Cornell potential for $t\bar{b}$, $t\bar{c}$, $t\bar{t}$, $b\bar{b}$ and $c\bar{c}$ systems. However, we could have alternatively calculated Martin potential coefficients for t-quark mesons.
Table 3: Introduced energies for heavy mesons

| Meson | S | E(MeV)  |
|-------|---|--------|
| $\bar{t}t$ | 1 | -11.296 |
| $\bar{t}\bar{t}$ | 0 | -36.948 |
| $\bar{t}\bar{b}$ | 0 | -195.0870 |
| $\bar{t}\bar{c}$ | 0 | -550 |
| $\bar{t}\bar{b}$ | 1 | -641.7242 |

Table 4: Strong coupling constants obtained for $t\bar{t}$, $t\bar{b}$, and $t\bar{c}$ mesons

| Meson | $\bar{t}t(S = 0)$ | $\bar{t}t(S = 1)$ | $\bar{t}\bar{b}(S = 0)$ | $\bar{t}\bar{b}(S = 1)$ | $\bar{t}\bar{c}(S = 0)$ |
|-------|-------------------|-------------------|-------------------|-------------------|-------------------|
| $\alpha_s$ | 0.2340 | 0.1998 | 0.3240 | 0.3060 | 0.4260 |

4 Results and Discussion

We made use of mass, r-cutoff, and binding energy of each meson to solve Schrodinger equation and diagonalize the kernel. We searched iteratively through parameter space to obtain coefficients for proposed potential model until desired eigenvalues were met within acceptable tolerance. We found exclusive potential coefficients for each meson. The procedure led to the identification of two appropriate potential coefficients for heavy mesons.

It must be notified that the two potentials used in this paper are spin-independent. The coefficient differences observed across $S = 0$ and $S = 1$ states result from the different binding energies that we used for these two states (tables 1 and 3). Thus, spin indirectly affects the mass and coefficients of the potential.

Table 5: Strong coupling constants obtained for $c\bar{c}$ and $b\bar{b}$ mesons

| Meson | $\alpha_s$ in ref[16] | Calculated $\alpha_s$ |
|-------|---------------------|---------------------|
| $c\bar{c}$ | 0.51 | 0.48 |
| $b\bar{b}$ | 0.33 | 0.31 |
5 Conclusions

In this study, we managed to calculate the coefficients of one potential through solving Lippman-Schwinger equation numerically and obtain the physical eigenvalue. This potential's coefficients are very close to Martin potential coefficients. Additionally, since we arrived at acceptable results in the program, we managed to obtain $\alpha_s$, considered as an unidentified coefficient in the Cornell potential for $t\bar{t}$, $t\bar{b}$, $t\bar{c}$, $b\bar{b}$ and $c\bar{c}$ mesons. The method practiced, therefore, is a good one to identify the potential coefficients used for heavy mesons.

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