Qualms regarding “Dynamical Foundations of Nonextensive Statistical Mechanics” by C. Beck (cond-mat/0105374)

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Abstract

The derivation of Student’s pdf from superstatistics is a mere coincidence due to the choice of the \( \chi^2 \) distribution for the inverse temperature which is actually the Maxwell distribution for the speed. The difference between the estimator and the variance introduces a fluctuating temperature that is generally different than the temperature of the variance. Fluctuations in the energy of a composite system are transferred to fluctuations in the temperature via Lévy’s transform. The randomization of the operational parameter leading to a subordinated process requires the same number of degrees of freedom in the directing as in the original process. In place of Student’s pdf we find the subordinated process to Maxwell’s speed distribution has a Pareto distribution which is interpreted as the density of the length of the random velocity vector.

There is always a certain delight to see old ideas from a new perspective, but there is no delight to see professed ‘new’ ideas that are not new. The notion of ‘superstatistics’¹ grew out of an idea that if one began with a stationary Maxwell distribution and introduced a randomized operational temperature, then a new process could be derived which has something to do with Tsallis’ distribution², which he obtained as a stationary condition from maximizing a nonadditive entropy of degree-\( q \) with respect to the energy constraint³. The generalization to ‘superstatistics’ then consisted in replacing the exponential in the Maxwell distribution by \( e^{-\beta H} \), where \( H \) is any generic hamiltonian, and using an arbitrary probability distribution for the conjugate \( \beta \) one could obtain a new probability distribution for \( H \) by integrating the product of the two probability distributions over all values of \( \beta \). We have questioned the physical justification of this procedure elsewhere⁴.

The idea of randomizing temperature is not new, as Beck would have us believe; it can be found in ref.⁵. It is based on subordination, which can
be found in Feller’s book [6]. If $X(t)$ represents a brownian motion then by randomizing the operational time $t$, a variety of new processes can be derived. That is, if the displacements of brownian motion are governed by the probability density function (pdf),

$$f_{t_0}(x) = \frac{1}{\sqrt{2\pi t_0}} e^{-x^2/2t_0},$$  

where $x$ is the random variable and $t_0$ the parameter, then exchanging their roles leads to a Lévy pdf,

$$f_{x_0}(t) = \frac{x_0}{\sqrt{2\pi t^3}} e^{-x_0^2/2t},$$  

for the randomized time, $t$, where $x_0$ is the parameter. The two pdf are related by the transform $x_0^2/t_0 = n^2/x_0^2$.

The two processes are evaluated at the same time. The Maxwell speed, $U(T)$, in a single dimension, and the temperature, $T$, stand in the same relation as the displacement of a brownian motion process, $X(t)$, and the time, $t$ [5]. The Cauchy distribution for the speed is therefore subordinated to the Maxwell distribution in one dimension [5].

Therefore, Beck [2] does not have to suppose that the fluctuations in inverse temperature, $\beta$, follow a $\chi^2$ distribution given by his equation (2),

$$f_{\beta_0}(\beta) = \frac{1}{\Gamma(n/2)} \left( \frac{n}{2\beta_0} \right)^{n/2} \beta^{n/2-1} e^{-n\beta/2\beta_0},$$  

which we would prefer to write as

$$f_{u_0}(\beta) = \frac{1}{\Gamma(n/2)} \left( \frac{u_0^2}{2} \right)^{n/2} \beta^{n/2-1} e^{-1/2^2 \beta u_0^2},$$  

on the strength of equipartition, $u_0^2\beta_0 = n$. By what we shall show is an incorrect association of Tsallis’ distribution with the subordinated process, Beck finds $n = 2/(q-1)$, where $q$ is the degree of the nonadditive entropy, or, what is commonly known as the Tsallis exponent. A single mode of [3] exists at $\beta = (1 - 2/n)\beta_0$. The single mode therefore vanishes for all $q \geq 2$. We should expect a qualitative change in the behavior of the process for $q < 2$ and $q \geq 2$. That no behavioral change has been predicted, already casts doubts on the association of half the number of the degrees of freedom with the inverse of $(q - 1)$.

It follows from the Lévy transform

$$\beta u_0^2 = \beta_0 u^2$$  

(5)
that the \( n \)-dimensional Maxwell speed distribution

\[
f_{\beta_0}(u) = \frac{2}{\Gamma(n/2)} \left( \frac{\beta_0}{2} \right)^{n/2} u^{n-1} e^{-\frac{1}{2} \beta_0 u^2},
\]

(6)
corresponds to the \( \chi^2 \) distribution for inverse temperature, \( \xi \), and not Beck’s one dimensional Maxwell pdf (6)[eqn. 14] below. What has happened is that, in a space of \( n \) dimensions, we began with a Markov process, \( U(\beta) \), whose stationary transition probability is given by \( \xi \). A whole host of new processes can be derived by randomizing the temperature, or its inverse, \( \beta > 0 \), corresponding to a new random variable \( \beta(\beta) \) whose pdf is \( 4 \). The process \( \beta(\beta) \) is called the ‘directing’ process [6, p. 347][5, p. 56]. The pdf of the new process \( U(\beta(\beta)) \) will be given by

\[
f_{\beta_0}(u) = \int_0^\infty f_\beta(u) f_{\beta_0}(\beta) d\beta = \frac{2}{B(n/2, n/2)} \frac{(u/u_0)^{n-1}}{(1 + (u/u_0)^2)^n u_0},
\]

(7)
where we have set \( \beta = \beta_0 \) so as to ensure that the two processes are in thermal equilibrium, and \( B(\cdot, \cdot) \) is the beta function. Contrary to what has been claimed [2], (7) is not Tsallis’ canonical probability distribution [2, eqn (9)]

\[
f_{\beta_0}(u) = \frac{\Gamma \left( \frac{n+1}{2} \right)}{\sqrt{n/2} \Gamma(n/2)} \frac{\beta_0}{2\pi} \left( 1 + \frac{\beta_0 u^2}{n} \right)^{-(n+1)/2}.
\]

(8)
From symmetry considerations, all odd order moments of \( \xi \) are zero. In particular, the second moment is

\[
\overline{u^2} = \frac{n\beta_0^{-1}}{n-2},
\]

(9)
which does not reflect equipartition that follows from \( \xi \) at all [cf., eqn 18] below.

For a generic value of \( n \), \( 4 \) is a special case of the inverse beta pdf

\[
f(x) = \frac{1}{B(n/2, n/2)} x^{n/2-1},
\]

(10)
where \( x = (u/u_0)^2 \). In economics, \( 10 \) is known as the Pareto pdf, since “it was thought (rather naively from a modern statistical standpoint) that income distributions should have a tail with a density \( \sim Ax^{-\alpha} \) as \( x \to \infty \), and \( 10 \) fulfills this requirement” [6, p. 50].

In \( n = 1 \) dimension, \( 4 \) is the half-Cauchy pdf for a positive variate. Just as the Cauchy process for the displacement is subordinated to brownian motion in one dimension when time is randomized, the Cauchy process for the kinetic energy is subordinated to the Maxwellian in a single dimension when the temperature, or its inverse, is randomized [5].
In $n = 3$ dimensions, (7) is [5, p. 95]

$$f_{n=3}(v) = \frac{8}{\pi} \frac{v^2}{(1 + v^2)^3}. \quad (11)$$

This is the density of the length of a random velocity vector $v = \sqrt{v_x^2 + v_y^2 + v_z^2}$ in $\mathbb{R}^3$ dimensions. It is related to the density

$$f_{n=1}(v) = \frac{1}{B\left(\frac{1}{2}; m - \frac{1}{2}\right)} \frac{1}{(1 + v^2)^m}. \quad (12)$$

in a fixed direction, by [3, p. 32]

$$f_{n=3}(v) = -vf'_{n=1}(v), \quad (13)$$

where the prime denotes differentiation with respect to the argument. The pdf (12) has $2m - 1$ degrees of freedom. For $m = 1$, (12) reduces to the Cauchy pdf, while for $m = 2$, implying that the distribution has 3 degrees of freedom, we obtain (11) from (13). Hence, (11) is the density of the length $v$ of a random velocity vector in $n = 3$ dimensions [5, p. 95].

Although (6) is valid for any $n > 0$, Beck’s ‘conditional’ pdf for the speed, (6), is our (9) for $n = 1$. For any $n \neq 1$ there is an incompatibility in the dimensions of the two distributions. This is responsible for the seemingly close appearance of the pdf of the subordinated process with the Tsallis distribution. In other words, if Beck’s (6) is meant to be the speed distribution, it does not reflect the dimensionality of the $\chi^2$ pdf for $\beta$, (9). Rather, Beck finds that the subordinated process is governed by Student’s $t$ distribution, (8). The first factor tends to unity as $n \to \infty$, and for every fixed $u$ [8]

$$-\frac{n + 1}{2} \log \left(1 + \frac{\beta_0 u^2}{n}\right) \to -\frac{1}{2} \beta_0 u^2$$

so that in the limit as $n \to \infty$ we have

$$f_{\beta_0}(u) = \sqrt{\frac{\beta_0}{2\pi}} e^{-\frac{1}{2} \beta_0 u^2}. \quad (14)$$

This is the $n = 1$ Maxwellian of (9) which is the conditional pdf (6) that Beck started out with!

What has happened is that the derivation of the Student pdf (8) formally resembles subordination, but has really nothing to do with it. Suppose that $n + 1$ random variables $u$ and $u_1, u_2, \ldots, u_n$ are independent and identically distributed according to the normal distribution with zero mean and standard deviation $1/\sqrt{n \beta_0}$. The distribution of $u$ is given by (14), while the distribution of the square root of the average of the sum of squares, $v = \sqrt{\frac{1}{n} \sum_{i=1}^{n} u_i^2}$ is

$$f_{\beta_0}(v) = \frac{2}{\Gamma(n/2)} \left(\frac{n \beta_0}{2}\right)^{n/2} v^{n-1} e^{-\frac{1}{2} n \beta_0 v}. \quad (15)$$
On account of the independence of the random variables \( u \) and \( v \), their joint pdf will be given by

\[
f_{\beta_0}(u)f_{\beta_0}(v) = \frac{2}{\Gamma(n/2)} \sqrt{\frac{2}{\pi}} \left( \frac{n}{2} \right)^{n/2} \beta_0^{(n+1)/2} e^{-\frac{1}{2} \beta_0 (u^2 + n v^2)}. \tag{16}
\]

The probability that \( u/v \leq t \) is the integral of (16) over the region \( v > 0 \) and \( u \leq tv \). Introducing the transformation \( u = xy \) and \( v = y \), whose Jacobian is \( y \), gives the distribution

\[
F(t) = \frac{2}{\Gamma(n/2)} \sqrt{\frac{2}{\pi}} \left( \frac{n}{2} \right)^{n/2} \beta_0^{(n+1)/2} \int_{-\infty}^{t} dx \int_{0}^{\infty} dy y^{n} e^{-\frac{1}{2} \beta_0 (1 + x^2/n) y^2}.
\]

Its derivative yields Student’s pdf

\[
f(t) = \frac{1}{\sqrt{n\pi}} \frac{\Gamma \left( \frac{n+1}{2} \right)}{\Gamma(n/2)} \left( 1 + \frac{t^2}{n} \right)^{-(n+1)/2}.	ag{17}
\]

It is just a mere coincidence that the \( \chi^2 \) pdf which Beck chose to represent fluctuations in the inverse temperature, (3), has the same form as Maxwell’s density for the speed \( \sqrt{\frac{1}{n} \sum_{i=1}^{n} u_i^2} \). Quite surprisingly, the Student distribution (17) is independent of the variance, \( 1/\beta_0 \). Although this is important insofar as testing hypotheses of the mean of a population, because it is independent of the variance, it nevertheless implies that information has been lost [cf., (21) and (23) below].

The reason why Beck has \( \beta_0 \) as a parameter in his pdf (8) is that he obtained

\[
\int_{0}^{\infty} \beta_{(n-1)/2} e^{\frac{1}{2} \beta (u^2 + n/\beta_0)} d\beta = \frac{1}{B \left( \frac{1}{2}, n/2 \right)} \sqrt{\frac{\beta_0}{n}} \left( 1 + \frac{\beta_0 u^2}{n} \right)^{-(n+1)/2},
\]

using (3) instead of using (4). Had he done so, he would have obtained

\[
f_{u_0}(u) = \frac{1}{B \left( \frac{1}{2}, n/2 \right)} u_0^{-1} \left( 1 + \frac{u^2}{u_0^2} \right)^{-(n+1)/2}.
\]

The two are the same if equipartition holds,

\[
u_0 = \sqrt{\frac{\sum_{i=1}^{n} u_i^2}{\beta_0}} = \sqrt{\frac{n}{\beta_0}}. \tag{18}
\]

In general, the estimator of the variance, \( \frac{1}{n} \sum_{i=1}^{n} u_i^2 \), will be different from the variance \( \beta_0^{-1} \). The estimator is \( \beta^{-1} \), and this temperature will usually be different from the temperature of the reservoir, \( \beta_0^{-1} \). In other words, the statistic for the Student distribution, \( t = u/\sqrt{\sum_{i=1}^{n} u_i^2} \), is not distributed as a normal...
random variable, as it would be had the statistic been given by 
$t = u/\sqrt{n/\beta_0}$. The actual variance can be completely unknown, and it suffices the sample variance alone in order to make statistical predictions. Nevertheless, we expect, that as the number of degrees of freedom increases the distribution of $t$ will be very similar to that of a standard normal variable \(i.e.,\) it is a consistent estimator. Accordingly, we can identify the $X_2^2$ in Beck’s eqn (3), which are distributed as $\chi^2$, with the kinetic energies of the individual particles, and the left-hand side should be $n/\beta$, and not $\beta$ itself.

The idea of ‘superstatistics’ is that the exponential pdf \[17\]
\[
f_{\beta_0}(E) = \frac{e^{-\beta_0 E} \rho(E)}{Z(\beta_0)} \quad (19)
\]
is actually a conditional pdf, and specifying a normalized pdf for what was previously a mere parameter, say, by the $\chi^2$ pdf
\[
f_{E_0}(\beta) = \frac{1}{\Gamma(n/2)} E_0^{n/2} \beta^{n/2-1} e^{-\beta E_0}. \quad (20)
\]
leads to a new distribution of $E$ which is precisely the Tsallis pdf. However, (20) is no arbitrarily chosen pdf for the inverse temperature, but, rather the directing process which results when the parameter of the original distribution is randomized.

Consider the usual case of a power law for the structure function \[9\],
\[
\rho(E) = \frac{E^{m-1}}{\Gamma(m)},
\]
which leads to equipartition because the partition function is $Z(\beta) = \beta^{-m}$. Then the process which is subordinated to the exponential pdf \[19\] has a pdf
\[
f_{E_0}(E) = \int_0^\infty f_\beta(E) f_{E_0}(\beta) d\beta = \frac{1}{B(m, n/2)} \frac{(E/E_0)^{m-1}}{(1 + E/E_0)^{m+n/2}} \frac{1}{E_0} \quad (21)
\]
which is still the inverted beta pdf \[cf., eqn \[7\]]. This is because the Lévy transform,
\[
\beta_0 E = \beta E_0, \quad (22)
\]
in \[19\] produces a randomized inverse temperature which is distributed according to \[20\] with $n = 2m$ degrees of freedom. Consequently, the inverse beta pdf \[21\] is the Fisher-Snedecor pdf with equal degrees of freedom, $m = n/2$, or what is commonly known as the Pareto distribution. The transformation to a new process by randomizing the operational temperature cannot change the number of degrees of freedom of the system.

‘F-superstatistics’ \[1, 7\] is not a directing process, but, rather, a subordinated process that is obtained from \[4\] and \[6\] by averaging the product of these densities over a common value of the kinetic energy, viz. \[5\]
\[
f_{\beta_0}(\beta) = \int_0^\infty f_\beta(u) f_u(\beta) du = \frac{1}{B(n/2, n/2)} \frac{(\beta/\beta_0)^{n/2-1}}{(1 + \beta/\beta_0)^{n/\beta_0}} \quad (23)
\]
On account of the necessity that both the original and directing processes have the same number of degrees of freedom, (23) is a symmetrical inverse beta distribution, which necessarily has the same number of degrees of freedom. Comparing (23) with (21), it is easily seen that they are interchangeable on the strength of the Lévy transformation (22). Reference to a Tsallis distribution in β space is completely inappropriate.

The symmetry of the inverse beta pdf (23) means that we are dealing with a composite system comprised of two subsystems with the same number of degrees of freedom. The total energy $E_t = E_0 + E$, formed from two subsystems with energies $E$ and $E_0$, is fixed. The inverse beta pdf (21) is thus transformed into the beta pdf, which can be written as the composition law [5, p. 80]

$$\rho(E_t) = \int_0^{E_t} \rho(E_t - E)\rho(E) dE = \frac{E_t^{2m-1}}{\Gamma(2m)}$$

for the structure function.

Thus, we conclude that the statement “Of course, other distribution functions $f(\beta)$ can also be considered which may lead to other generalized statistics” [10] is devoid of meaning when the original process is distributed according to (19). More importantly, as we have concluded elsewhere, “subordination can be considered as the probabilistic origin of power laws in physics” [5, p. 58].

References

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