A single gradient step finds adversarial examples on random two-layers neural networks

Sébastien Bubeck
Microsoft Research

Yeshwanth Cherapanamjeri
UC Berkeley

Gauthier Gidel*
Mila, Université de Montréal

Rémi Tachet des Combes
Microsoft Research

Abstract

Daniely and Schacham recently showed that gradient descent finds adversarial examples on random undercomplete two-layers ReLU neural networks. The term “undercomplete” refers to the fact that their proof only holds when the number of neurons is a vanishing fraction of the ambient dimension. We extend their result to the overcomplete case, where the number of neurons is larger than the dimension (yet also subexponential in the dimension). In fact we prove that a single step of gradient descent suffices. We also show this result for any subexponential width random neural network with smooth activation function.

1 Introduction

We study the following random two-layers neural network model: let $f : \mathbb{R}^d \to \mathbb{R}$ be a random function defined by

$$f(x) = \frac{1}{\sqrt{k}} \sum_{\ell=1}^k a_{\ell} \psi(w_{\ell} \cdot x),$$

where $\psi : \mathbb{R} \to \mathbb{R}$ is a fixed non-linearity, the weight vectors $w_{\ell} \in \mathbb{R}^d$ are i.i.d. from a Gaussian distribution $\mathcal{N}(0, \frac{1}{d}I_d)$ (so that they are roughly unit norm vectors), and the coefficients $a_{\ell} \in \mathbb{R}$ are independent from the weight vectors and i.i.d. uniformly distributed in $\{-1, +1\}$. With this parametrization, the central limit theorem says that, for $x \in \sqrt{d} \cdot S^{d-1}$ (so that $w_{\ell} \cdot x \sim \mathcal{N}(0, 1)$) and large width $k$, the distribution of $f(x)$ is approximately a centered Gaussian with variance $\mathbb{E}_{X \sim \mathcal{N}(0, 1)}[\psi(X)^2]$.

*Canada CIFAR AI Chair
Our goal is to study the concept of adversarial examples in this random model. We say that $\delta \in \mathbb{R}^d$ is an adversarial perturbation at $x \in \mathbb{R}^d$ if $\|\delta\| \ll \|x\|$ and $\text{sign}(f(x)) \neq \text{sign}(f(x + \delta))$, and in this case we call $x + \delta$ an adversarial example. Our main result is that, while $|f(x)| \approx 1$ with high probability, a single gradient step on $f$ (i.e., a perturbation of the form $\delta = \eta \nabla f(x)$ for some $\eta \in \mathbb{R}$) suffices to find such adversarial examples, with roughly $\|\delta\| \approx \frac{\|x\|}{\sqrt{d}} = 1$. We prove this statement for a network with smooth non-linearity and subexponential width (e.g., $k \ll \exp(o(d))$), as well as for the Rectified Linear Unit (ReLU) $\psi(t) = \max(0,t)$ in the overcomplete and subexponential regime (e.g. $d \ll k \ll \exp(d^c)$ for some constant $c > 0$).

**Theorem 1** Let $\gamma \in (0,1)$ and $\psi$ be non-constant, Lipschitz and with Lipschtiz derivative. There exists constants $C_1,C_2,C_3,C_4$ depending on $\psi$ such that the following holds true. Assume $k \geq C_1 \log^3(1/\gamma)$ and $d \geq C_2 \log(k/\gamma) \log(1/\gamma)$, and let $\eta \in \mathbb{R}$ such that $|\eta| = C_3 \sqrt{\log(1/\gamma)}$ and $\text{sign}(\eta) = -\text{sign}(f(x))$. Then with probability at least $1 - \gamma$ one has:

$$
\text{sign}(f(x)) \neq \text{sign}(f(x + \eta \nabla f(x))).
$$

Moreover we have $\|\eta \nabla f(x)\| \leq C_4 \sqrt{\log(1/\gamma)}$.

Note that our proof of Theorem 1 in Section 2 easily gives explicit values for $C_1,C_2,C_3,C_4$. Also note that the subexponential width condition in the above Theorem is of the form $k \ll \exp(o(d))$.

**Theorem 2** Let $\gamma \in (0,1)$ and $\psi(t) = \max(0,t)$. There exists constants $C_1,C_2,C_3,C_4$ such that the following holds true. Assume $k \geq C_1 d \log^2(d)$ and $\frac{d}{\log(d)} \geq C_2 \log^3(k) \log(1/\gamma)$, and let $\eta \in \mathbb{R}$ such that $|\eta| = C_3 \sqrt{\log(1/\gamma)}$ and $\text{sign}(\eta) = -\text{sign}(f(x))$. Then with probability at least $1 - \gamma$ one has:

$$
\text{sign}(f(x)) \neq \text{sign}(f(x + \eta \nabla f(x))).
$$

Moreover we have $\|\eta \nabla f(x)\| \leq C_4 \sqrt{\log(1/\gamma)}$.

Note that the subexponential condition on the width in the above Theorem is of the form $k \ll \exp(d^{0.24})$. In fact by modifying a bit the proof we can get a condition of the form $k \ll \exp(d^{\rho})$ for any $\rho < 1/2$, but for the sake of clarity we only prove the weaker version stated above.

### 1.1 Related works

The existence of adversarial examples in neural network architectures was first evidenced in the seminal paper of Szegedy et al. [2014], where the authors found adversarial examples by using the L-BFGS optimization procedure. Shortly after this work, it was hypothesized in Goodfellow et al. [2015] that the existence of adversarial examples stems from an excessive “linearity” of neural network models. This hypothesis was experimentally confirmed by showing that a single step of gradient descent suffices to find adversarial perturbations (the so-called fast gradient sign method -FGSM-). Our theorems can be thought of as a theoretical confirmation of the hypothesis in Goodfellow et al. [2015]. In fact, as we explain
in Section 1.2 below, our proofs proceed exactly by showing that “most” two-layers neural networks behave “mostly” linearly over “vast” regions of input space.

We note that not all networks are susceptible to one-step gradient attacks to find adversarial examples. Indeed, in Goodfellow et al. [2015], it was shown that adversarial training can be used to build networks that are somewhat robust to one-step gradient attacks. Interestingly in Madry et al. [2018] it was then shown that such models remain in fact susceptible to multi-steps gradient attacks, and empirically they demonstrated that better robustness can be achieved with adversarial training using multi-steps gradient attacks. Understanding this phenomenon theoretically remains a challenge, see for example Allen-Zhu and Li [2020] for one proposed approach, and Moosavi-Dezfooli et al. [2019], Qin et al. [2019] for discussion/algorithmic consequences of the relation with the phenomenon of gradient obfuscation (see Athalye et al. [2018], Papernot et al. [2017]).

Our work is a direct follow-up of Daniely and Schacham [2020] (which itself is a follow-up on Shamir et al. [2019]). Daniely and Schacham prove that multi-steps gradient descent finds adversarial examples for ReLU random networks of the form (1), as long as the number of neurons is much smaller than the dimension (i.e., \( k = o(d) \)). They explicitly conjecture that this condition is not necessary, and indeed we exponentially improve their condition to requiring \( k = \exp(o(d)) \) in Theorem 2 (see below for a discussion of the case where \( k \) is exponential in the dimension). We note that there remains a small window of widths around \( k \approx d \) where the conjecture of Daniely and Schacham remains open, as we require \( k \geq d \log^2(d) \) in Theorem 2. Moreover Daniely and Schacham went beyond two-layers neural networks, and they conjecture (and prove for shrinking layers) that gradient descent finds adversarial examples on random multi-layers neural networks. We give some experimental confirmation of this multi-layer conjecture in Section 4.

The ultra-wide case \( k = \exp(\Omega(d)) \) remains open. This exponential size case seems of a different nature than the polynomial size we tackle here, at least for the ReLU activation function. In particular it is likely that the behavior with exponential width would be closely tied to the actual limit case \( k = +\infty \), where the random model (1) yields a Gaussian process. Namely for \( k = +\infty \) one has that \( f \) is a Gaussian process indexed by the sphere (say if we restrict to inputs \( x \in \sqrt{d} \cdot S^{d-1} \)), with \( f(x) \sim \mathcal{N}(0, \mathbb{E}_{X \sim \mathcal{N}(0,1)}[\psi(X)]) \) and \( \mathbb{E}[f(x)f(y)] = \mathbb{E}_{X,Y \sim \mathcal{N}(0,1), |X| = |Y|}[\psi(X)\psi(Y)] \). For example if the activation function is a Hermite polynomial of degree \( p \), then \( f \) would be a spherical \( p \)-spin glass model. This polynomial case is particularly well-understood, and in fact the landscape we describe below in Section 1.2 was already described in this case in Ben Arous et al. [2020] (see in particular Corollary 59). It would be interesting to see if the \( p \)-spin glass landscape literature can be extended to non-polynomial activation functions, and to a finite (but possibly exponential in \( d \)) \( k \). A step in this latter direction was recently taken in Eldan et al. [2021], where convergence rates to the Gaussian process limit where given both for polynomial activations and for the ReLU. Finally we note that for a smooth activation it might be that there is a more direct argument to remove the subexponential width condition in Theorem 1 (technically in the proof of Lemma 7 there might a better argument than using the naive upper bound on \( \text{Lip(\Phi)} \)).
Finally, we note that, in practice, it has been found that there exists “universal” adversarial perturbations that generalize across both inputs and neural networks, Moosavi-Dezfooli et al. [2017]. For the case of ReLU activation (Theorem 2), we could in fact prove our result by replacing the gradient step with a step in the direction \( \sum_{\ell=1}^{k} a_{\ell} w_{\ell} \), which is indeed a direction \textit{independent} of the input \( x \), thus proving the existence of “universal” perturbation (generalizing across inputs) for our model.

### 1.2 The landscape of random two-layers neural networks

For a smooth \( \psi \), we have

\[
\nabla f(x) = \frac{1}{\sqrt{k}} \sum_{\ell=1}^{k} a_{\ell} w_{\ell} \psi'(w_{\ell} \cdot x),
\]

and

\[
\nabla^2 f(x) = \frac{1}{\sqrt{k}} \sum_{\ell=1}^{k} a_{\ell} w_{\ell} w_{\ell}^\top \psi''(w_{\ell} \cdot x).
\]

We already claimed in the introduction that, with high probability,

\[
|f(x)| = O(1).
\]  

(2)

We alluded to the CLT for this claim, but it is also easy to guess it intuitively by noting that (since \( \mathbb{E}[a_{\ell} a_{\ell'}] = 1 \{ \ell = \ell' \} \)):

\[
\mathbb{E}[f(x)^2] = \mathbb{E} \left[ \frac{1}{k} \sum_{\ell, \ell'=1}^{k} a_{\ell} a_{\ell'} \psi(w_{\ell} \cdot x) \psi(w_{\ell'} \cdot x) \right] = \mathbb{E}_{X \sim \mathcal{N}(0,1)}[\psi(X)^2].
\]

The formal proof of (2) (and all other claims we make here) will eventually be a simple application of the classical Bernstein concentration inequality. Similarly, it is easy to see that (note for example that \( \mathbb{E}[\|\nabla f(x)\|^2] = \mathbb{E}_{X \sim \mathcal{N}(0,1)}[\psi'(X)^2] \)), with high probability,

\[
\|\nabla f(x)\| = \Theta(1).
\]  

(3)

A slightly more difficult calculation, although classical too, is that

\[
\|\nabla^2 f(x)\|_{\text{op}} = \tilde{O} \left( \frac{1}{\sqrt{d}} \right).
\]  

(4)

Indeed one can simply note that, for any \( u \in \mathbb{S}^{d-1} \), \( u^\top \nabla^2 f(x) u = \frac{1}{\sqrt{k}} \sum_{\ell=1}^{k} a_{\ell} (w_{\ell} \cdot u)^2 \psi''(w_{\ell} \cdot x) \) is approximately distributed as a centered Gaussian with variance

\[
\mathbb{E}_{X,Y \sim \mathcal{N}(0,1):\mathbb{E}[XY]=x \cdot u} \left[ \left( \frac{X}{\sqrt{d}} \right)^4 \psi'' \left( \frac{Y}{\sqrt{d}} \right)^2 \right],
\]

so that with probability at least \( 1 - \gamma \) one can expect \( u^\top \nabla^2 f(x) u \) to be of order \( \sqrt{\log(1/\gamma)} \), and thus by taking a union bound over a discretization of the sphere \( \mathbb{S}^{d-1} \), one expects the
inequality (4). In fact, interestingly, one can even hope that (4) holds true for an entire ball around $x$: with appropriate smoothness over $\psi$, this could be obtained by doing another union bound over a second discretization of a $d$-dimensional ball. In other words, we can expect that with high probability:

$$\forall x \in \mathbb{R}^d : ||x|| = \text{poly}(d), \text{ one has } ||\nabla^2 f(x)||_{\text{op}} = O \left( \frac{1}{\sqrt{d}} \right).$$  \hfill (5)

The equations (2), (3), and (5) paint a rather clear geometric picture. There are essentially two scales around a fixed $x \in \sqrt{d} \cdot S^{d-1}$: The macroscopic scale, where one considers a perturbation $x + \delta$ with $||\delta|| = \Omega(\sqrt{d})$, and the mesoscopic scale where $||\delta|| = o(\sqrt{d})$ (we use this term because for the ReLU network there will also be a microscopic scale, with $||\delta|| = o(1)$). At the macroscopic scale the landscape of $f$ might be very complicated, but our crucial observation is that the picture at the mesososcopic scale is dramatically simpler. Namely at the mesoscopic scale the function $f$ is essentially linear, since one has (thanks to (3) and (5))

$$||\nabla f(x) - \nabla f(x + \delta)|| = o(||\nabla f(x)||), \forall \delta : ||\delta|| = o(\sqrt{d}).$$  \hfill (6)

Moreover, since the height of the function is constant (by (2)) and the norm of the gradient is constant, it suffices to step at a constant distance in the direction of the gradient (or negative gradient) to change the sign of $f$. In other words, this already proves our main point: a single step of gradient descent (or ascent) suffices to find an adversarial example, and moreover the adversarial perturbation $\delta$ satisfies $||\delta|| = O(1) = O(||x||/\sqrt{d})$. Formally one easily concludes from (2), (3), and (6) by using the following simple lemma for gradient descent:

**Lemma 1** For any continuous and almost everywhere differentiable function $f$, and any $x \in \mathbb{R}^d$ and $\eta \in \mathbb{R}$, one has:

$$\left| f \left( x + \frac{\eta}{||\nabla f(x)||^2} \nabla f(x) \right) - (f(x) + \eta) \right| \leq \sup_{\delta \in \mathbb{R}^d : ||\delta|| \leq \frac{\eta}{||\nabla f(x)||}} ||\nabla f(x) - \nabla f(x + \delta)|| \cdot \frac{||\nabla f(x)||}{||\nabla f(x)||}. $$

**Proof.** Let $g(t) = f \left( x + t \frac{\eta}{||\nabla f(x)||^2} \nabla f(x) \right)$ so that

$$g'(t) = \frac{\eta}{||\nabla f(x)||^2} \nabla f(x) \cdot \nabla f \left( x + t \frac{\eta}{||\nabla f(x)||^2} \nabla f(x) \right)$$

$$= \eta + \frac{\eta}{||\nabla f(x)||} \cdot \frac{\nabla f \left( x + t \frac{\eta}{||\nabla f(x)||^2} \nabla f(x) \right) - \nabla f(x)}{||\nabla f(x)||}. $$

Thus we have:

$$|g(1) - g(0) - \eta| \leq \int_0^1 |g'(t) - \eta| dt \leq |\eta| \int_0^1 \left| \nabla f \left( x + t \frac{\eta}{||\nabla f(x)||^2} \nabla f(x) \right) - \nabla f(x) \right| \frac{dt}{||\nabla f(x)||},$$

which concludes the proof. \qed
1.3 Proof strategy

The starting point of the proof for both the smooth and ReLU case is to show (2) and (3), which we essentially do below in Section 1.4. In the smooth case, one could then prove formally (4) and conclude as indicated in the last paragraph of Section 1.2. Of course, (4) is simply ill-defined for the ReLU case, so one has to take a different route there. Instead we propose to directly prove (6), that is we study the difference of gradients at the mesoscopic scale. Using that \( \|h\| = \sup_{v \in \mathbb{S}^{d-1}} v \cdot h \), we thus need to control (for some \( R = o(\sqrt{d}) \)):

\[
\sup_{\delta \in \mathbb{R}^d : \|\delta\| \leq R} \|\nabla f(x) - \nabla f(x + \delta)\| = \sup_{\mathbb{S}^{d-1}, \delta \in \mathbb{R}^d : \|\delta\| \leq R} \frac{1}{k} \sum_{\ell=1}^{k} a_\ell (w_\ell \cdot v)(\psi'(w_\ell \cdot x) - \psi'(w_\ell \cdot (x + \delta))).
\] (7)

We execute this strategy first for the smooth case in Section 2. We then prove the ReLU case in Section 3, where we face an extraneous difficulty since the gradient is not Lipschitz at very small scale, which introduces a third scale (the microscopic scale) that has to be dealt with differently. Technically, this issue appears when we try to move from the discretization over \( v \) and \( \delta \) in (7) to the whole space (a so-called \( \varepsilon \)-net argument).

1.4 Scaling of value and gradient

Here we show how to prove (2) and (3) (in fact, for our purpose, we only need the one-sided inequality \( \|\nabla f(x)\| = \Omega(1) \)) under very mild conditions on \( \psi \) which will be satisfied for both ReLU and smooth activations. We will repeatedly use Bernstein’s inequality which we restate here for convenience (see e.g., Theorem 2.10 in Boucheron et al. [2013]):

**Theorem 3 (Bernstein’s inequality)** Let \( (X_\ell) \) be i.i.d. centered random variables such that there exists \( \sigma, c > 0 \) such that for all integers \( q \geq 2 \),

\[
\mathbb{E}[|X_\ell|^q] \leq \frac{q!}{2} \sigma^2 c^{q-2}.
\]

Then with probability at least \( 1 - \gamma \) one has:

\[
\sum_{\ell=1}^{k} X_\ell \leq 2\sqrt{\sigma^2 k \log(1/\gamma)} + c \log(1/\gamma).
\]

We will also use repeatedly that \( \mathbb{E}_{X \sim \mathcal{N}(0,1)}[|X|^q] \leq (q-1)!! \leq \frac{q!}{2} \), as well as the following concentration of \( \chi^2 \) random variables (see e.g., (2.19) in Wainwright [2019]): let \( X_1, \ldots, X_k \) be i.i.d. standard Gaussians, then with probability at least \( 1 - \gamma \), one has:

\[
\left| \sum_{\ell=1}^{k} X_\ell^2 - k \right| \leq 4 \sqrt{k \log(2/\gamma)}.
\] (8)

We can now proceed to our various results.
Lemma 2 Assume that there exists $\sigma, c > 0$ such that for all integers $q \geq 2$,
\[ \mathbb{E}_{X \sim \mathcal{N}(0, 1)}[|\psi(X)|^q] \leq \frac{q!}{2} \sigma^2 e^{q-2}. \] (9)

Then with probability at least $1 - \gamma$ one has
\[ |f(x)| \leq \sqrt{2 \sigma^2 \log(1/\gamma)} + \frac{c \log(1/\gamma)}{\sqrt{k}}. \]

Proof. Let $X_\ell = a_\ell \psi(w_\ell \cdot x)$. Then $\mathbb{E}[X_\ell] = 0$ and
\[ \mathbb{E}[|X_\ell|^q] \leq \frac{q!}{2} \sigma^2 e^{q-2}, \quad \text{for all integers } q \geq 2. \]

Thus Bernstein’s inequality states that with probability at least $1 - \gamma$ one has
\[ \sqrt{k} f(x) = k \sum_{\ell=1}^k X_\ell \leq \sqrt{2 \sigma^2 k \log(1/\gamma)} + c \log(1/\gamma). \]

Lemma 3 Let $\psi$ be differentiable almost everywhere. Then with probability at least $1 - \gamma$ for $0 < \gamma < 2/e$ one has:
\[ \|\nabla f(x)\| \geq \left(1 - 5 \sqrt{\frac{\log(2/\gamma)}{d}}\right) \sqrt{\frac{1}{k} \sum_{\ell=1}^k \psi'(w_\ell \cdot x)^2}. \]

Proof. Let $P = I_d - \frac{xx^\top}{d}$ be the projection on the orthogonal complement of the span of $x$. We have $\|\nabla f(x)\| \geq \|P \nabla f(x)\|$. Moreover $a_\ell P w_\ell$ is independent of $w_\ell \cdot x$, and thus conditioning on the values $(w_\ell \cdot x)_{\ell \in [k]}$ we obtain (using that $a_\ell P w_\ell$ is distributed as $\mathcal{N}(0, \frac{1}{d} I_{d-1})$):
\[ P \nabla f(x) = \frac{1}{\sqrt{k}} \sum_{\ell=1}^k a_\ell P w_\ell \psi'(w_\ell \cdot x) \overset{(d)}{=} \left(\sqrt{\frac{1}{kd} \sum_{\ell=1}^k \psi'(w_\ell \cdot x)^2}\right) Y \text{ where } Y \sim \mathcal{N}(0, I_{d-1}). \]

Using (8) we have that with probability at least $1 - \gamma$:
\[ \|Y\|^2 \geq d - 4 \sqrt{d \log(2/\gamma)} \geq d - 5 \sqrt{d \log(2/\gamma)}. \]

where we used that $d \geq 1$ and $\gamma < 2/e$. The two above displays easily conclude the proof. \(\square\)
Lemma 4 Let \( \psi \) be differentiable almost everywhere, and assume that there exists \( \sigma' , c' > 0 \) such that for all integers \( q \geq 2 \),
\[
\mathbb{E}_{X \sim \mathcal{N}(0,1)}[|\psi'(X)|^{2q}] \leq \frac{q!}{2}\sigma'^2 c'^{q-2}.
\]
Then with probability at least \( 1 - \gamma \),
\[
\|\nabla f(x)\| \geq \left( \mathbb{E}_{X \sim \mathcal{N}(0,1)}[|\psi'(X)|^2] - \left( \sqrt{\frac{2\sigma'^2 \log(2/\gamma)}{k}} + \frac{c' \log(2/\gamma)}{k} \right) \right)^{1/2} \left( 1 - 5\sqrt{\frac{\log(4/\gamma)}{d}} \right).
\]

Proof. Straightforward application of Bernstein’s inequality yields with probability at least \( 1 - \gamma \) one has:
\[
\frac{1}{k} \sum_{\ell=1}^{k} \psi'(w_\ell \cdot x)^2 \geq \mathbb{E}_{X \sim \mathcal{N}(0,1)}[|\psi'(X)|^2] - \left( \sqrt{\frac{2\sigma'^2 \log(1/\gamma)}{k}} + \frac{c' \log(1/\gamma)}{k} \right).
\]
It suffices to combine this inequality with Lemma 3 and apply a direct union bound. \( \square \)

2 Proof of Theorem 1

In this section, we consider a 1-Lipschitz and \( L \)-smooth activation function, that is for all \( s, t \in \mathbb{R} \),
\[
|\psi(s) - \psi(t)| \leq |s - t| \quad \text{and} \quad |\psi'(s) - \psi'(t)| \leq L|s - t|.
\]
We also assume \( \psi(0) = 0 \) and denote \( c^2_\psi = \mathbb{E}_{X \sim \mathcal{N}(0,1)}[|\psi'(X)|^2] \) which we assume to be non-zero (that is \( \psi \) is not a constant function).

Lemma 5 Under the above assumptions, one has with probability at least \( 1 - \gamma \),
\[
|f(x)| \leq \sqrt{2\log(1/\gamma)} \left( 1 + \sqrt{\frac{\log(2/\gamma)}{k}} \right),
\]
and
\[
\|\nabla f(x)\| \geq \left( c^2_\psi - \sqrt{\frac{2\log(4/\gamma)}{k}} \left( 1 + \sqrt{\frac{\log(4/\gamma)}{k}} \right) \right)^{1/2} \left( 1 - 5\sqrt{\frac{\log(8/\gamma)}{d}} \right)
\]

Proof. With the assumptions we have \( |\psi(X)| \leq |X| \) and thus in Lemma 2 we can take \( \sigma = c = 1 \) which yields the first claimed equation. For the second equation we use that \( |\psi'(X)| \leq 1 \) (since \( \psi \) is 1-Lipschitz) and thus, in Lemma 4, we can also take \( \sigma' = c' = 1 \) which yields the second claimed equation. \( \square \)

Next we need to control (7) where we use crucially the smoothness of the activation function.
Lemma 6 Fix $\delta \in \mathbb{R}^d$ such that $\|\delta\| \leq R$ and $v \in \mathbb{S}^{d-1}$. Then with probability at least $1 - \gamma$ one has:

$$\frac{1}{\sqrt{k}} \sum_{\ell=1}^{k} a_\ell (w_\ell \cdot v)(\psi'(w_\ell \cdot x) - \psi'(w_\ell \cdot (x + \delta))) \leq \frac{4RL}{d} \sqrt{\log(1/\gamma)} \left( 1 + \frac{\sqrt{\log(1/\gamma)}}{k} \right).$$

Proof. We apply Bernstein’s inequality with $X_\ell = \frac{a_\ell}{2} (w_\ell \cdot v)(\psi'(w_\ell \cdot x) - \psi'(w_\ell \cdot (x + \delta)))$. We have $\mathbb{E}[X_\ell] = 0$ and (by smoothness of $\psi$)

$$\mathbb{E}[|X_\ell|^q] \leq \mathbb{E}[|w_\ell \cdot v|^q |w_\ell \cdot \delta|^q] \leq \sqrt{\mathbb{E}[|w_\ell \cdot v|^{2q}] \mathbb{E}[|w_\ell \cdot \delta|^{2q}]} = \frac{\|\delta\|^q}{d^q} \mathbb{E}_{X \sim \mathcal{N}(0,1)}[|X|^{2q}] \leq (2q - 1)!! \left( \frac{R}{d} \right)^q \leq \frac{q!}{2} \left( \frac{2R}{d} \right)^q.$$

Thus we can apply Bernstein with $\sigma = c = \frac{2R}{d}$ which yields the claimed bound. \hfill $\square$

Lemma 7 Let $R \geq 1$. With probability at least $1 - \gamma$ one has

$$\sup_{v \in \mathbb{S}^{d-1}, \delta \in \mathbb{R}^d, \|\delta\| \leq R} \frac{1}{\sqrt{k}} \sum_{\ell=1}^{k} a_\ell (w_\ell \cdot v)(\psi'(w_\ell \cdot x) - \psi'(w_\ell \cdot (x + \delta))) \leq 20RL \left( \sqrt{\frac{\log(k/\gamma)}{d}} + \frac{\log(1/\gamma)}{\sqrt{k}} \right).$$

Proof. Denote $\Phi(v, \delta) = \frac{1}{\sqrt{k}} \sum_{\ell=1}^{k} a_\ell (w_\ell \cdot v)(\psi'(w_\ell \cdot x) - \psi'(w_\ell \cdot (x + \delta)))$. In Lemma 6, we controlled $\Phi(v, \delta)$ for a fixed $v$ and $\delta$. We now want to control it uniformly over $\Omega = \{(v, \delta) : \|v\| = 1, \|\delta\| \leq R\}$. To do so, we apply an union bound over an $\varepsilon$-net for $\Omega$, denote it $N_\varepsilon$, whose size is then at most $(10R/\varepsilon)^{2d}$. In particular; we obtain with probability at least $1 - \gamma$:

$$\sup_{(v, \delta) \in \Omega} |\Phi(v, \delta) - \Phi(v', \delta')| \leq 4RL \frac{\sqrt{2d \log(10/\varepsilon) + \log(1/\gamma)}}{d} \left( 1 + \frac{\sqrt{2d \log(10/\varepsilon) + \log(1/\gamma)}}{k} \right) + \varepsilon \times \text{Lip}(\Phi). \quad (11)$$

Thus, it only remains to estimate the Lipschitz constant of the mapping $\Phi$. To do so, note that for any $\delta, \delta'$,

$$|\Phi(\delta, v) - \Phi(\delta', v)| \leq \frac{L\|\delta - \delta'\|}{\sqrt{k}} \sum_{\ell=1}^{k} \|w_\ell\|^2,$$

and similarly for any $v, v'$,

$$|\Phi(\delta, v) - \Phi(\delta, v')| \leq \frac{RL\|v - v'\|}{\sqrt{k}} \sum_{\ell=1}^{k} \|w_\ell\|^2.$$

Using (8), we have with probability at least $1 - \gamma$ that

$$\sum_{\ell=1}^{k} \|w_\ell\|^2 \leq k + 4\sqrt{\frac{k \log(1/\gamma)}{d}}. \quad (12)$$
Thus with see that with probability at least $1 - \gamma$,  
\[ \text{Lip}(\Phi) \leq RL \left( \sqrt{k} + 4\sqrt{\frac{\log(1/\gamma)}{d}} \right). \]
Combining this with (11) concludes the proof (by taking $\varepsilon = 1/k$ and with straightforward algebraic manipulations).

Finally we can turn to the proof of Theorem 1:

\textbf{Proof.} [of Theorem 1] We make the following claims which hold with probability at least $1 - \gamma$. With the assumptions on $k$, $d$ and $\eta$, Lemma 5 shows that $|f(x)| \leq 0.1|\eta|$ and \( \|\nabla f(x)\| \geq c \) for some small constant $c > 0$. Moreover Lemma 7 shows that for all $\delta$ such that $\|\delta\| \leq \frac{|\eta|}{\|\nabla f(x)\|}$ we have $\|\nabla f(x) - \nabla f(x + \delta)\| \leq c/10$. Thus Lemma 1 easily allows us to conclude (using in particular that $f(x)(f(x) + \eta) < 0$).

\[ \square \]

3 Proof of Theorem 2

In this section, we consider $\psi(t) = \max(0, t)$.

\textbf{Lemma 8} With probability at least $1 - \gamma$,  
\[ |f(x)| \leq \sqrt{2\log(2/\gamma)} \left( 1 + \sqrt{\frac{\log(2/\gamma)}{k}} \right), \]
and
\[ \|\nabla f(x)\| \geq \left( \frac{1}{2} - \sqrt{\frac{2\log(4/\gamma)}{k}} \left( 1 + \sqrt{\frac{\log(1/\gamma)}{k}} \right) \right)^{1/2} \left( 1 - 5\sqrt{\frac{\log(4/\gamma)}{d}} \right). \]

\textbf{Proof.} In Lemma 2 and Lemma 4, we can take $\sigma = c = \sigma' = \sigma' = 1$ (since $|\psi(X)| \leq |X|$ and $|\psi'(X)| \leq 1$), which concludes the proof.

We now turn to the control of (7). In the smooth case we did so via Lemma 6 and Lemma 7, which both used crucially the smoothness of the activation function. Here, instead of smoothness, we will use that only few activations can change when you make microscopic move (i.e., between $x + \delta$ and $x + \delta'$ with $\|\delta - \delta'\| = o(1)$). The key observation is the following lemma:

\textbf{Lemma 9} For any $\delta$ such that $\|\delta\| \leq R$,

\[ \mathbb{P}(\text{sign}(w_\ell \cdot x) \neq \text{sign}(w_\ell \cdot (x + \delta))) \leq R \sqrt{\frac{2\log(d)}{d}} + \frac{1}{d}. \]

Moreover, for any $\delta$ with $\|\delta\| \leq \sqrt{d}/2$, we have

\[ \mathbb{P}(\exists \delta': \|\delta - \delta'\| \leq \varepsilon \text{ and } \text{sign}(w_\ell \cdot (x + \delta)) \neq \text{sign}(w_\ell \cdot (x + \delta'))) \leq 2\varepsilon \left( 1 + 2\sqrt{\frac{\log(2/\varepsilon)}{d}} \right). \]
Thus we can apply Bernstein with \( \sigma \) where the last inequality holds for any threshold \( t \in \mathbb{R} \). Now, note that \( w_\ell \cdot \delta \sim \mathcal{N}(0, \frac{||\delta||^2}{d}) \) and \( w_\ell \cdot x \sim \mathcal{N}(0, 1) \). Thus picking \( t = R \sqrt{\frac{2 \log(d)}{d}} \) shows that

\[
\mathbb{P}(\text{sign}(w_\ell \cdot x) \neq \text{sign}(w_\ell \cdot (x + \delta))) \leq R \sqrt{\frac{2 \log(d)}{d}} + \frac{1}{d},
\]

which concludes the proof of (13).

For (14) we have:

\[
\mathbb{P}(\exists \delta' : ||\delta - \delta'|| \leq \varepsilon \text{ and } \text{sign}(w_\ell \cdot (x + \delta)) \neq \text{sign}(w_\ell \cdot (x + \delta'))) \\
\leq \mathbb{P}(\exists \delta' : ||\delta - \delta'|| \leq \varepsilon \text{ and } |w_\ell \cdot (\delta' - \delta)| \geq t) \text{ + } \mathbb{P}(|w_\ell \cdot (x + \delta)| \leq t) \\
\leq \mathbb{P}(||w_\ell|| \geq t/\varepsilon) \text{ + } \mathbb{P}(|w_\ell \cdot (x + \delta)| \leq t).
\]

where \( w_\ell \cdot (x + \delta) \sim \mathcal{N}(0, \sigma^2) \) with \( \sigma^2 \geq \frac{1}{2} \) since \( ||\delta|| \leq \sqrt{d}/2 \). Thus picking \( t = \varepsilon \sqrt{1 + 4 \sqrt{\frac{\log(2/\varepsilon)}{d}}} \) and applying (8) concludes the proof.

We now give the equivalent of Lemma 6:

**Lemma 10** Fix \( \delta \in \mathbb{R}^d \) such that \( ||\delta|| \leq R \) (with \( R \geq 1 \)) and \( v \in \mathbb{S}^{d-1} \). Then with probability at least \( 1 - \gamma \) one has:

\[
\frac{1}{\sqrt{k}} \sum_{\ell=1}^{k} a_\ell (w_\ell \cdot v)(\psi'(w_\ell \cdot x) - \psi'(w_\ell \cdot (x + \delta))) \leq 2 \sqrt{\frac{\log(1/\gamma)}{d}} \left( 2R \sqrt{\frac{\log(d)}{d}} \right)^{1/4} + \sqrt{\frac{\log(1/\gamma)}{k}}.
\]

**Proof.** We apply Bernstein’s inequality with \( X_\ell = a_\ell (w_\ell \cdot v)(\psi'(w_\ell \cdot x) - \psi'(w_\ell \cdot (x + \delta))) \). We have \( \mathbb{E}[X_\ell] = 0 \) and (using (13) in Lemma 9)

\[
\mathbb{E}[|X_\ell|^q] = \mathbb{E}[|w_\ell \cdot v||\psi'(w_\ell \cdot x) - \psi'(w_\ell \cdot (x + \delta))|^q] \\
\leq \sqrt{\mathbb{E}[|w_\ell \cdot v|^{2q}]} \times \mathbb{P}(\text{sign}(w_\ell \cdot x) \neq \text{sign}(w_\ell \cdot (x + \delta))) \\
\leq \sqrt{(2q)! \over 2d^q} \times \sqrt{2R \sqrt{\frac{\log(d)}{d}}} \\
\leq {q \over 2} \left( {2 \over \sqrt{d}} \right)^q \times \sqrt{2R \sqrt{\frac{\log(d)}{d}}}.
\]

Thus we can apply Bernstein with \( \sigma = {2 \over \sqrt{d}} \times \left( 2R \sqrt{\frac{\log(d)}{d}} \right)^{1/4} \) and \( c = 2R \) which yields the claimed bound.

Finally, we give the equivalent of Lemma 7:
Lemma 11  Let $1 \leq R \leq \sqrt{d}/2$, $\sqrt{k} \geq 52$ and $d \geq \log(1/\gamma)$. Then, with probability at least $1 - \gamma$, one has

$$
\sup_{v \in \mathbb{S}^{d-1}, \delta \in \mathbb{R}^d; \delta \leq R} \frac{1}{\sqrt{k}} \sum_{\ell=1}^k a_\ell (w_\ell \cdot v)(\psi'(w_\ell \cdot x) - \psi'(w_\ell \cdot (x + \delta))) \\
\leq 20 \left( R \log^2(Rk) \sqrt{\frac{\log d}{d}} \right)^{1/4} + 40 \sqrt{\frac{d}{k}} \log(Rk).
$$

**Proof.** Similarly to the proof of Lemma 7, we define $\Phi(v, \delta) = \frac{1}{\sqrt{k}} \sum_{\ell=1}^k a_\ell (w_\ell \cdot v)(\psi'(w_\ell \cdot x) - \psi'(w_\ell \cdot (x + \delta)))$, and $N_\varepsilon$ an $\varepsilon$-net for $\Omega = \{(v, \delta), \|v\| = 1, \|\delta\| \leq R\}$ (recall that $|N_\varepsilon| \leq (10R/\varepsilon)^{2d}$). Using Lemma 10, we obtain with probability at least $1 - \gamma$:

$$
\sup_{(v, \delta) \in \Omega} \Phi(v, \delta) \leq \sup_{(v, \delta) \in N} \Phi(v, \delta) + \sup_{(v, \delta), (v', \delta') \in \Omega: \|v-v'\|+\|\delta-\delta'\| \leq \varepsilon} |\Phi(v, \delta) - \Phi(v', \delta')| \\
\leq 2 \sqrt{\frac{2d \log(10R/\varepsilon) + \log(1/\gamma)}{d}} \left( \frac{2R \sqrt{\log d}}{\sqrt{d}} \right)^{1/4} + \sqrt{\frac{2d \log(10R/\varepsilon) + \log(1/\gamma)}{k}} \\
+ \sup_{(v, \delta), (v', \delta') \in \Omega: \|v-v'\|+\|\delta-\delta'\| \leq \varepsilon} |\Phi(v, \delta) - \Phi(v', \delta')|.
$$

Thus, it remains again to estimate the “Lipschitz constant” of the mapping $\Phi$ but crucially only at scale $\varepsilon$ (the crucial point is that we don’t need to argue about infinitesimal scale, where a ReLU network is not smooth). For $v, v'$, one has

$$
|\Phi(\delta, v) - \Phi(\delta, v')| \leq \frac{\|v-v'\|}{\sqrt{k}} \sum_{\ell=1}^k \|w_\ell\|.
$$

Using (8), we see that with probability at least $1 - \gamma$, one has for all $\ell \in [k]$,

$$
\|w_\ell\|^2 \leq 1 + 4 \sqrt{\frac{\log(k/\gamma)}{d}},
$$

so that in this event we have:

$$
|\Phi(\delta, v) - \Phi(\delta, v')| \leq \|v-v'\| \sqrt{k + 4k \sqrt{\frac{\log(k/\gamma)}{d}}}. \tag{16}
$$
On the other hand, for \( \delta, \delta' \) we write:

\[
|\Phi(\delta, v) - \Phi(\delta', v)| \leq \frac{1}{\sqrt{k}} \left| \sum_{\ell=1}^{k} \mathbb{1} \{ \text{sign}(w_{\ell} \cdot (x + \delta)) > \text{sign}(w_{\ell} \cdot (x + \delta')) \} a_{\ell} w_{\ell} \cdot v \right| + \frac{1}{\sqrt{k}} \left| \sum_{\ell=1}^{k} \mathbb{1} \{ \text{sign}(w_{\ell} \cdot (x + \delta)) < \text{sign}(w_{\ell} \cdot (x + \delta')) \} a_{\ell} w_{\ell} \cdot v \right|
\]

\[
\leq \frac{1}{\sqrt{k}} \left| \sum_{\ell=1}^{k} \mathbb{1} \{ \text{sign}(w_{\ell} \cdot (x + \delta)) > \text{sign}(w_{\ell} \cdot (x + \delta')) \} a_{\ell} w_{\ell} \right| + \frac{1}{\sqrt{k}} \left| \sum_{\ell=1}^{k} \mathbb{1} \{ \text{sign}(w_{\ell} \cdot (x + \delta)) < \text{sign}(w_{\ell} \cdot (x + \delta')) \} a_{\ell} w_{\ell} \right|,
\]

(17)

Letting \( X_{\ell}(\delta) = \mathbb{1} \{ \exists \delta' : \|\delta - \delta'\| \leq \varepsilon \text{ and } \text{sign}(w_{\ell} \cdot (x + \delta)) \neq \text{sign}(w_{\ell} \cdot (x + \delta')) \} \), we now control with exponentially high probability \( \sum_{\ell=1}^{k} X_{\ell}(\delta) \). By (14) in Lemma 9, we know that \( X_{\ell}(\delta) \) is a Bernoulli of parameter at most \( 2\varepsilon \left( 1 + 2\sqrt{\frac{\log(2/\varepsilon)}{d}} \right) \). So we have:

\[
\mathbb{P}\left( \sum_{\ell=1}^{k} X_{\ell}(\delta) \geq s \right) \leq \left( 2k\varepsilon \left( 1 + 2\sqrt{\frac{\log(2/\varepsilon)}{d}} \right) \right)^{s}.
\]

And thus, thanks to an union bound, we obtain:

\[
\mathbb{P}\left( \exists (v, \delta) \in N_{\varepsilon} : \sum_{\ell=1}^{k} X_{\ell}(\delta) \geq s \right) \leq \left( \frac{10R}{\varepsilon} \right)^{2d} \left( 2k\varepsilon \left( 1 + 2\sqrt{\frac{\log(2/\varepsilon)}{d}} \right) \right)^{s}.
\]

(18)

With \( s = 4d \) the latter is upper bounded by \( (26k\sqrt{R}\varepsilon^{3d/8})^{4d} \) (using the fact that \( \sqrt{\varepsilon}(1 + 2\sqrt{\log(2/\varepsilon)}) \leq 4\varepsilon^{5/8}, \forall 1 \geq \varepsilon > 0 \)). Taking \( \varepsilon = R^{-4/3}k^{-4} \) we get that this probability is less than \( (26k/\sqrt{k})^{4d} \leq \gamma \) for \( \sqrt{k} \geq 52 \) and \( d \geq \log(1/\gamma) \).

Furthermore, we have by another union bound and the concentration of Lipschitz functions of Gaussians [Boucheron et al., 2013, Theorem 5.5] (\( \| \cdot \| \) is a 1-Lipschitz function):

\[
\mathbb{P}\left( \exists S \subset [k], \| S \| \leq 4d : \left\| \frac{1}{\sqrt{k}} \sum_{i \in S} a_{i} w_{i} \right\| \geq \sqrt{\frac{|S|}{k}}(1 + t) \right) \leq k^{4d} e^{-\frac{4t^{2}}{2}}
\]

By setting \( t = 2\sqrt{\log 4k + \frac{\log 8/\gamma}{d}} \), we get that with probability at least \( 1 - \gamma/8 \):

\[
\forall S \subset [k], \| S \| \leq 4d : \left\| \frac{1}{\sqrt{k}} \sum_{i \in S} a_{i} w_{i} \right\| \leq 9\sqrt{\frac{d}{k}} \sqrt{\log 4k + \frac{\log 8/\gamma}{d}}
\]

(19)

Finally, noting that for all \((v, \delta) \in N, \| \delta' - \delta \| \leq \varepsilon\):

\[
\mathbb{1} \{ \text{sign}(w_{\ell} \cdot (x + \delta)) < \text{sign}(w_{\ell} \cdot (x + \delta')) \} \leq X_{\ell}(\delta)
\]

\[
\mathbb{1} \{ \text{sign}(w_{\ell} \cdot (x + \delta)) > \text{sign}(w_{\ell} \cdot (x + \delta')) \} \leq X_{\ell}(\delta),
\]
we may combine (16), (17), (18) and (19) to obtain that with probability at least $1 - \gamma$, we have for all $\delta, v, \delta', v'$ with $\|\delta - \delta'\| \leq \frac{1}{R^{4/3}k^4}$ and $\|v - v'\| \leq \frac{1}{R^{4/3}k^4}$,

$$|\Phi(\delta, v) - \Phi(\delta, v')| \leq \frac{1}{R^{4/3}k^4} \sqrt{k + 4k \sqrt{\frac{\log(4k/\gamma)}{d}}}.$$ 

and

$$|\Phi(\delta, v) - \Phi(\delta', v)| \leq 18 \sqrt{\frac{d}{k}} \sqrt{\frac{\log 4k + \log 8/\gamma}{d}}.$$

Combining this with (15) we obtain with probability at least $1 - \gamma$:

$$\sup_{(v, \delta) \in \Omega} \Phi(v, \delta) \leq 2 \sqrt{\frac{10d \log(Rk) + \log(2/\gamma)}{d}} \left( \frac{2R \sqrt{\frac{\log(d)}{d}}}{d} \right)^{1/4} + \sqrt{\frac{10d \log(Rk) + \log(2/\gamma)}{k}}$$

$$+ 20 \sqrt{\frac{d}{k}} \sqrt{\frac{\log 4k + \frac{\log 8/\gamma}{d}}{d}}$$

$$\leq 3 \sqrt{\frac{10d \log(Rk) + \log(2/\gamma)}{d}} \left( \frac{2R \sqrt{\frac{\log(d)}{d}}}{d} \right)^{1/4} + \sqrt{\frac{10d \log(Rk) + \log(2/\gamma)}{k}}$$,

which concludes the proof up to straightforward algebraic manipulations. □

Proof. [of Theorem 2] The proof is the same as for Theorem 1 with Lemma 8 instead of Lemma 5 and Lemma 11 instead of Lemma 7. □

4 Experiments

Setting In order to verify our theoretical findings, we ran some experiments to measure empirically the values of $\|\nabla f(x)\|$ and the probability of finding an adversarial example in that direction. More precisely, we take a random point $x$ of norm $\sqrt{d}$ and initialize a network using the procedure described in Section 1. We then find the smallest $\eta$ such that a gradient step $\eta \nabla f(x)$ changes the sign of the function. $\eta$ is of the opposite sign of $f(x)$ and we limit our search to $|\eta| < 20$. We explore various values of $d$ and $k$. We also consider deeper networks with $L = 1$ through $L = 6$ hidden layers. All the hidden layers are of width $k$.

Results Figure 1a shows the average of the smallest $\eta$ required to switch the sign of the function. We note that the average only includes cases where an $\eta$ was indeed found. Figure 1b shows the gradient norm in $x$ (all cases included). As we see, both the smallest $\eta$ and the gradient norm are approximately constant both in $d$ and in $k$. This finding also holds for deeper networks (see Appendix A). In Figure 2, we show the fraction of examples (out of 10,000 samples) whose sign is switched for $|\eta| < 20$. We see that with $L = 1$ and
Figure 1: Smallest step-size $\eta$ switching the prediction (left) and average gradient norm $\|\nabla f(x)\|$ (right) for $L = 1$. Averages over 100 network initializations and 100 values of $x$ per initialization. The colored area represents one standard deviation.

values of $d$ and $k$ larger than 50, 100% of samples are switched. This confirms our theoretical results. Additionally, we also observe that for deeper networks, the same statement holds. The values of $d$ and $k$ at which 100% switching is reached appears to grow with $L^1$.

Acknowledgment

We thank Mark Sellke for pointing out to us the reference Ben Arous et al. [2020], and Peter Bartlett for several discussions on this problem.

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Due to GPU memory limitations, $k$ could not reach 1,000,000 for deeper networks.
Figure 2: Fraction of inputs with an adversarial example found with $\eta < 20$.

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A Appendix

For the sake of completeness, we report in Fig. 3-7 the smallest $\eta$ to switch the sign of the prediction and the gradient norm at $x$ for depths $L \in \{2, \ldots, 6\}$. In all our plots, the results are Averaged over 100 network initializations and 100 values of $x$ per initialization and the colored area represents one standard deviation.

**Figure 3:** Smallest $\eta$ switching the prediction and average gradient norm $\|\nabla f(x)\|$ for $L = 2$.

**Figure 4:** Smallest $\eta$ switching the prediction and average gradient norm $\|\nabla f(x)\|$ for $L = 3$. 
Figure 5: Smallest $\eta$ switching the prediction and average gradient norm $\|\nabla f(x)\|$ for $L=4$.

Figure 6: Smallest $\eta$ switching the prediction and average gradient norm $\|\nabla f(x)\|$ for $L=5$.

Figure 7: Smallest $\eta$ switching the prediction and average gradient norm $\|\nabla f(x)\|$ for $L=6$. 