The Lambek-Grishin calculus is NP-complete

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Abstract. The Lambek-Grishin calculus $LG$ is the symmetric extension of the non-associative Lambek calculus $NL$. In this paper we prove that the derivability problem for $LG$ is NP-complete.

1 Introduction

In his 1958 and 1961 papers, Lambek formulated two versions of the Syntactic Calculus: in (Lambek, 1958), types are assigned to strings, which are then combined by an associative operation; in (Lambek, 1961), types are assigned to phrases (bracketed strings), and the composition operation is non-associative. We refer to these two versions as $L$ and $NL$ respectively.

As for generative power, Kandulski (1988) proved that $NL$ defines exactly the context-free languages. Pentus (1993) showed that this also holds for associative $L$. As for the complexity of the derivability problem, de Groote (1999) showed that for $NL$ this belongs to $PTIME$, for $L$, Pentus (2003) proves that the problem is NP-complete and Savateev (2009) shows that NP-completeness also holds for the product-free fragment of $L$.

It is well known that some natural language phenomena require generative capacity beyond context-free. Several extensions of the Syntactic Calculus have been proposed to deal with such phenomena. In this paper we look at the Lambek-Grishin calculus $LG$ (Moortgat, 2007, 2009). $LG$ is a symmetric extension of the nonassociative Lambek calculus $NL$. In addition to $\otimes, \setminus, /$ (product, left and right division), $LG$ has dual operations $\oplus, \circ, \odot$ (coproduct, left and right difference). These two families are related by linear distributivity principles. Melissen (2009) shows that all languages which are the intersection of a context-free language and the permutation closure of a context-free language are recognizable in $LG$. This places the lower bound for $LG$ recognition beyond LTAG. The upper bound is still open.

The key result of the present paper is a proof that the derivability problem for $LG$ is NP-complete. This will be shown by means of a reduction from SAT.\footnote{This paper has been written as a result of my Master thesis supervised by Michael Moortgat. I would like to thank him, Rosalie Iemhoff and Arno Bastenhof for comments and I acknowledge that any errors are my own.}

2 Lambek-Grishin calculus

We define the formula language of $LG$ as follows.
Let \( \text{Var} \) be a set of primitive types, we use lowercase letters to refer to an element of \( \text{Var} \). Let formulas be constructed using primitive types and the binary connectives \( \otimes, /, \backslash, \oplus, \odot \) and \( \odot \) as follows:

\[
A, B ::= p | A \otimes B | A/B | B\backslash A | A \oplus B | A \odot B | B \odot A
\]

The sets of input and output structures are constructed using formulas and the binary structural connectives \( \cdot \otimes \cdot, \cdot / \cdot, \cdot \backslash \cdot, \cdot \oplus \cdot, \cdot \odot \cdot \) and \( \cdot \odot \cdot \) as follows:

\[
\text{(input)} \quad X, Y ::= A | X \otimes Y | X \odot P | P \odot X
\]

\[
\text{(output)} \quad P, Q ::= A | P \oplus Q | P / X | X \backslash P
\]

The sequents of the calculus are of the form \( X \rightarrow P \), and as usual we write \( \vdash_{\text{LG}} X \rightarrow P \) to indicate that the sequent \( X \rightarrow P \) is derivable in LG. The axioms and inference rules are presented in Figure 1, where we use the display logic from (Goré, 1998), but with different symbols for the structural connectives.

It has been proven by Moortgat (2007) that we have Cut admissibility for LG. This means that for every derivation using the Cut-rule, there exists a corresponding derivation that is Cut-free. Therefore we will assume that the Cut-rule is not needed anywhere in a derivation.

3 Preliminaries

3.1 Derivation length

We will first show that for every derivable sequent there exists a Cut-free derivation that is polynomial in the length of the sequent. The length of a sequent \( \varphi \), denoted as \( |\varphi| \), is defined as the number of (formula and structural) connectives used to construct this sequent. A subscript will be used to indicate that we count only certain connectives, for example \( |\varphi|_\otimes \).

**Lemma 1.** If \( \vdash_{\text{LG}} \varphi \) there exists a derivation with exactly \( |\varphi| \) logical rules.

**Proof.** If \( \vdash_{\text{LG}} \varphi \) then there exists a Cut-free derivation for \( \varphi \). Because every logical rule removes one logical connective and there are no rules that introduce logical connectives, this derivation contains \( |\varphi| \) logical rules. \( \square \)

**Lemma 2.** If \( \vdash_{\text{LG}} \varphi \) there exists a derivation with at most \( \frac{1}{4}|\varphi|^2 \) Grishin interactions.

**Proof.** Let us take a closer look at the Grishin interaction principles. First of all, it is not hard to see that the interactions are irreversible. Also note that the interactions happen between the families of input connectives \( \{\otimes, /, \backslash\} \) and output connectives \( \{\oplus, \odot, \odot\} \) and that the Grishin interaction principles are the only rules of inference that apply on both families. So, on any pair of one input and one output connective, at most one Grishin interaction principle can be applied.
\[
\begin{align*}
\frac{p \rightarrow p}{Ax} \\
\frac{X \rightarrow A}{A \rightarrow P} \quad \text{Cut} \\
\frac{Y \rightarrow X \cdot \backslash \cdot P}{X \cdot \otimes \cdot Y \rightarrow P} \quad r \\
\frac{X \cdot \otimes \cdot Y \rightarrow P \cdot / \cdot Y}{X \rightarrow P \cdot / \cdot Y} \quad r \\
\frac{X \cdot \otimes \cdot Q \rightarrow P}{P \cdot \otimes \cdot X \rightarrow Q} \quad dr \\
\frac{X \cdot \otimes \cdot Q \rightarrow P}{X \rightarrow P \cdot \otimes \cdot Q} \quad dr \\
\end{align*}
\]

(a) Display rules

\[
\begin{align*}
\frac{X \cdot \otimes \cdot Y \rightarrow P \cdot \oplus \cdot Q}{X \cdot \otimes \cdot Q \rightarrow P \cdot / \cdot Y} \quad d \otimes / \\
\frac{X \cdot \otimes \cdot Y \rightarrow P \cdot \oplus \cdot Q}{P \cdot \otimes \cdot X \rightarrow Q \cdot / \cdot Y} \quad d \otimes / \\
\end{align*}
\]

(b) Distributivity rules (Grishin interaction principles)

\[
\begin{align*}
\frac{A \cdot \otimes \cdot B \rightarrow P}{A \otimes B \rightarrow P} \quad \otimes L \\
\frac{X \rightarrow B \cdot \oplus \cdot A}{X \rightarrow B \oplus A} \quad \oplus R \\
\frac{X \rightarrow A \cdot / \cdot B}{X \rightarrow A / B} \quad / R \\
\frac{B \cdot \otimes \cdot A \rightarrow P}{B \otimes A \rightarrow P} \quad \otimes L \\
\frac{X \rightarrow B \cdot \backslash \cdot A}{X \rightarrow B \backslash A} \quad \backslash R \\
\frac{A \cdot \otimes \cdot B \rightarrow P}{A \otimes B \rightarrow P} \quad \otimes L \\
\frac{X \rightarrow A \cdot Y \rightarrow B}{X \cdot \otimes \cdot Y \rightarrow A \otimes B} \quad \otimes R \\
\frac{B \rightarrow P}{B \oplus A \rightarrow P} \quad \oplus L \\
\frac{X \rightarrow B \cdot A \rightarrow P}{P \cdot \otimes \cdot X \rightarrow A \otimes B} \quad \otimes R \\
\end{align*}
\]

(c) Logical rules

Fig. 1: The Lambek-Grishin calculus inference rules
If $\vdash_{LG} \varphi$ there exists a Cut-free derivation of $\varphi$. The maximum number of possible Grishin interactions in 1 Cut-free derivation is reached when a Grishin interaction is applied on every pair of one input and one output connective. Thus, the maximum number of Grishin interactions in one Cut-free derivation is $|\varphi|_{\{\otimes, \land\}} \cdot |\varphi|_{\{\otimes, \land\}}$.

By definition, $|\varphi|_{\{\otimes, \land\}} + |\varphi|_{\{\otimes, \land\}} = |\varphi|$, so the maximum value of $|\varphi|_{\{\otimes, \land\}} \cdot \frac{1}{2}$ is reached when $|\varphi|_{\{\otimes, \land\}} = \frac{1}{4} |\varphi|$. Then the total number of Grishin interactions in 1 derivation is $\frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4} |\varphi|^2$, so any Cut-free derivation of $\varphi$ will contain at most $\frac{1}{4} |\varphi|^2$ Grishin interactions. \hfill \Box

**Lemma 3.** In a derivation of sequent $\varphi$ at most $2|\varphi|$ display rules are needed to display any of the structural parts.

**Proof.** A structural part in sequent $\varphi$ is nested under at most $|\varphi|$ structural connectives. For each of these connectives, one or two $r$ or $dr$ rules can display the desired part, after which the next connective is visible. Thus, at most $2|\varphi|$ display rules are needed to display any of the structural parts.

**Lemma 4.** If $\vdash_{LG} \varphi$ there exists a Cut-free derivation of length $O(|\varphi|^3)$.

**Proof.** From Lemma 3 and Lemma 2 we know that there exists a derivation with at most $|\varphi|$ logical rules and $\frac{1}{2} |\varphi|^2$ Grishin interactions. Thus, the derivation consists of $|\varphi| + \frac{1}{4} |\varphi|^2$ rules, with between each pair of consecutive rules the display rules. From Lemma 3 we know that at most $2|\varphi|$ display rules are needed to display any of the structural parts. So, at most $2|\varphi| \cdot (|\varphi| + \frac{1}{4} |\varphi|^2) = 2|\varphi|^2 + \frac{1}{2} |\varphi|^3$ derivation steps are needed in the shortest possible Cut-free derivation for this sequent, and this is in $O(|\varphi|^3)$. \hfill \Box

### 3.2 Additional notations

Let us first introduce some additional notations to make the proofs shorter and easier readable.

Let us call an input structure $X$ which does not contain any structural operators except for $\cdot \otimes$ a $\otimes$-structure. A $\otimes$-structure can be seen as a binary tree with $\cdot \otimes$ in the internal nodes and formulas in the leaves. Formally we define $\otimes$-structures $U$ and $V$ as:

$$U, V ::= A \mid U \cdot \otimes V$$

We define $X[]$ and $P[]$ as the input and output structures $X$ and $P$ with a hole in one of their leaves. Formally:

$$X[] ::= [] \mid X[] \cdot \otimes Y \mid Y \cdot \otimes X[] \mid X[] \cdot \otimes Q \mid Y \cdot \otimes P[] \mid Q \cdot \otimes X[] \mid P[] \cdot \otimes Y$$

$$P[] ::= [] \mid P[] \cdot \otimes Q \mid Q \cdot \otimes P[] \mid P[] \cdot / \cdot Y \mid Q \cdot / \cdot X[] \mid Y \cdot \backslash \cdot P[] \mid X[] \cdot \backslash \cdot Q$$

This notation is similar to the one of de Groot [1999] but with structures. If $X[]$ is a structure with a hole, we write $X[Y]$ for $X[]$ with its hole filled with structure $Y$. We will write $X^{\circ}[]$ for a $\otimes$-structure with a hole.
Furthermore, we extend the definition of hole to formulas, and define $A[\ ]$ as a formula $A$ with a hole in it, in a similar manner as for structures. Hence, by $A[B]$ we mean the formula $A[\ ]$ with its hole filled by formula $B$.

In order to distinguish between input and output polarity formulas, we write $A^\bullet$ for a formula with input polarity and $A^\circ$ for a formula with output polarity. Note that for structures this is already defined by using $X$ and $Y$ for input polarity and $P$ and $Q$ for output polarity. This can be extended to formulas in a similar way, and we will use this notation only in cases where the polarity is not clear from the context.

### 3.3 Derived rules of inference

Now we will show and prove some derived rules of inference of $L_G$.

**Lemma 5.** If $\vdash_{L_G} A \rightarrow B$ and we want to derive $X^\otimes[A] \rightarrow P$, we can replace $A$ by $B$ in $X^\otimes[\ ]$. We have the inference rule below:

\[
\begin{array}{c}
A \rightarrow B \quad X^\otimes[B] \rightarrow P \\
\hline
X^\otimes[A] \rightarrow P
\end{array}
\]

Proof. We consider three cases:

1. If $X^\otimes[A] = A$, it is simply the cut-rule:

\[
A \rightarrow B \quad B \rightarrow P \quad \text{Cut}
\]

2. If $X^\otimes[A] = Y^\otimes[A] \cdot \otimes \cdot V$, we can move $V$ to the righthand-side and use induction to prove the sequent:

\[
\begin{array}{c}
A \rightarrow B \quad Y^\otimes[B] \cdot \otimes \cdot V \rightarrow P \\
\hline
Y^\otimes[A] \rightarrow P \cdot / \cdot V
\end{array}
\]

then use induction to prove the sequent:

\[
\begin{array}{c}
Y^\otimes[A] \cdot \otimes \cdot V \rightarrow P \\
\hline
\text{Repl}
\end{array}
\]

3. If $X^\otimes[A] = U \cdot \otimes \cdot Y^\otimes[A]$, we can move $U$ to the righthand-side and use induction to prove the sequent:

\[
\begin{array}{c}
A \rightarrow B \quad U \cdot \otimes \cdot Y^\otimes[B] \rightarrow P \\
\hline
U \cdot \otimes \cdot Y^\otimes[A] \rightarrow U \cdot / \cdot P
\end{array}
\]

then use induction to prove the sequent:

\[
\begin{array}{c}
U \cdot \otimes \cdot Y^\otimes[A] \rightarrow U \cdot \backslash \cdot P \\
\hline
Y^\otimes[A] \rightarrow U \cdot \backslash \cdot P
\end{array}
\]

\[
\text{Repl}
\]

\[
\begin{array}{c}
Y^\otimes[A] \rightarrow U \cdot \backslash \cdot P \\
\hline
U \cdot \otimes \cdot Y^\otimes[A] \rightarrow P
\end{array}
\]

\[
\text{Repl}
\]

\[
\text{Move}
\]

- □

**Lemma 6.** If we want to derive $X^\otimes[A \otimes B] \rightarrow P$, then we can move the expression $\otimes B$ out of the $\otimes$-structure. We have the inference rule below:

\[
\begin{array}{c}
X^\otimes[A] \cdot \otimes \cdot B \rightarrow P \\
\hline
X^\otimes[A \otimes B] \rightarrow P
\end{array}
\]

**Move**
Proof. We consider three cases:

1. If $X \otimes [A \otimes B] = A \otimes B$, then this is simply the $\otimes L$-rule:

\[
\begin{align*}
A \otimes B & \rightarrow Y \\
A \otimes B & \rightarrow Y \quad \otimes L
\end{align*}
\]

2. If $X \otimes [A \otimes B] = Y \otimes [A \otimes B] \cdot \otimes V$, we can move $V$ to the righthand-side and use induction together with the Grishin interaction principles to prove the sequent:

\[
\begin{align*}
(Y \otimes [A] \cdot \otimes V) \cdot \otimes B & \rightarrow P \\
(Y \otimes [A] \cdot \otimes B) & \rightarrow P \cdot / V \\
Y \otimes [A \otimes B] & \rightarrow P \cdot / V \\
Y \otimes [A \otimes B] \cdot \otimes V & \rightarrow P \\
U \otimes [A \otimes B] & \rightarrow P
\end{align*}
\]

3. If $X \otimes [A \otimes B] = U \cdot \otimes Y \otimes [A \otimes B]$, we can move $U$ to the righthand-side and use induction together with the Grishin interaction principles to prove the sequent:

\[
\begin{align*}
(U \cdot \otimes Y \otimes [A]) \cdot \otimes B & \rightarrow P \\
U \cdot \otimes Y \otimes [A] & \rightarrow P \cdot / B \\
Y \otimes [A] \cdot \otimes B & \rightarrow U \cdot / P \\
Y \otimes [A \otimes B] & \rightarrow U \cdot / P \\
U \cdot \otimes Y \otimes [A \otimes B] & \rightarrow P
\end{align*}
\]

$\square$

Lemma 7. $\vdash_{LG} A_1 \otimes (A_2 \otimes \ldots (A_{n-1} \otimes A_n)) \rightarrow P$ iff $\vdash_{LG} A_1 \cdot \otimes (A_2 \cdot \otimes \ldots (A_{n-1} \cdot \otimes A_n)) \rightarrow P$

Proof. The if-part can be derived by the application of $n - 1$ times the $\otimes L$ rule together with the $r$ rule:

\[
\begin{align*}
A_1 \cdot \otimes (A_2 \cdot \otimes \ldots (A_{n-1} \cdot \otimes A_n)) & \rightarrow P \\
A_{n-1} \otimes A_n \rightarrow \ldots \cdot \otimes (A_2 \cdot \otimes (A_1 \cdot \otimes P)) & \rightarrow P \\
A_{n-1} \otimes A_n \rightarrow \ldots \cdot \otimes (A_2 \cdot \otimes (A_1 \cdot \otimes P)) & \rightarrow P \\
A_1 \otimes (A_2 \otimes \ldots (A_{n-1} \otimes A_n)) & \rightarrow P
\end{align*}
\]

The only-if-part can be derived by application of $n - 1$ times the $\otimes R$ rule followed by a $Cut$:
A_1 \rightarrow A_1 \rightarrow A_n \rightarrow A_n \otimes R
\frac{A_2 \rightarrow A_2 \quad \ldots \quad (A_{n-1} \cdot \otimes A_n) \rightarrow \ldots (A_{n-1} \otimes A_n)}{A_1 \rightarrow A_1 \rightarrow A_2 \otimes \ldots \otimes (A_{n-1} \cdot A_n) \otimes R}
\frac{A_1 \cdot \otimes (A_2 \cdot \otimes \ldots \otimes (A_{n-1} \cdot \otimes A_n))}{A_1 \cdot \otimes (A_2 \cdot \otimes \ldots \otimes (A_{n-1} \otimes A_n)) \rightarrow P}

\text{Cut}

Note that because of the Cut elimination theorem, there exists a cut-free derivation for this sequent.

\[ \square \]

### 3.4 Type similarity

The type similarity relation \( \sim \), introduced by Lambeck (1958), is the reflexive transitive symmetric closure of the derivability relation. Formally we define this as:

**Definition 1.** \( A \sim B \) iff there exists a sequence \( C_1 \ldots C_n (1 \leq i \leq n) \) such that \( C_1 = A, C_n = B \) and \( C_i \rightarrow C_{i+1} \) or \( C_{i+1} \rightarrow C_i \) for all \( 1 \leq i < n \).

It was proved by Lambeck that \( A \sim B \) iff one of the following equivalent statements holds (the so-called diamond property):

- \( \exists C \) such that \( A \rightarrow C \) and \( B \rightarrow C \) (join)
- \( \exists D \) such that \( D \rightarrow A \) and \( D \rightarrow B \) (meet)

This diamond property will be used in the reduction from SAT to create a choice for a truthvalue of a variable.

**Definition 2.** If \( A \sim B \) and \( C \) is the join type of \( A \) and \( B \) so that \( A \rightarrow C \) and \( B \rightarrow C \), we define \( A C \cap B = (A/(C/C) \cap C)) \otimes ((C/C) \cap B) \) as the meet type of \( A \) and \( B \).

This is also the solution given by Lambeck (1958) for the associative system \( L \), but in fact this is the shortest solution for the non-associative system \( NL \) (Foret, 2003).

**Lemma 8.** If \( A \sim B \) with join-type \( C \) and \( \vdash_{LG} A \rightarrow P \) or \( \vdash_{LG} B \rightarrow P \), then we also have \( \vdash_{LG} A C \cap B \rightarrow P \). We can write this as a derived rule of inference:

\[
\frac{A \rightarrow P \quad \text{or} \quad B \rightarrow P}{A \cap B \rightarrow P}
\]

**Meet**

**Proof.**
1. If $A \rightarrow P$:

$$
\begin{array}{c}
C \rightarrow C \cdot C \rightarrow C \\
\frac{C/C \rightarrow C \cdot C}{B \rightarrow C} [L] \\
\frac{(C/C) \\setminus B \rightarrow (C/C) \cdot C}{A \rightarrow P} [L] \\
\frac{A/(((C/C)\setminus C)) \rightarrow P}{(A/(((C/C)\setminus C)) \cdot \setminus ((C/C)\setminus B)) \rightarrow P} \otimes [L]
\end{array}
$$

2. If $B \rightarrow P$:

$$
\begin{array}{c}
C \rightarrow C \cdot C \rightarrow C \\
\frac{C/C \rightarrow C \cdot C}{A \rightarrow C} [L] \\
\frac{(C/C) \cdot \setminus C \rightarrow C}{A \rightarrow (C/C) \cdot C} [L] \\
\frac{A/(((C/C)\setminus C)) \rightarrow C}{A/(((C/C)\setminus C)) \rightarrow C \cdot C} [L] \\
\frac{A/(((C/C)\setminus C)) \cdot \setminus (C/C)\setminus B}{B \rightarrow P} [L] \\
\frac{(A/(((C/C)\setminus C)) \cdot \setminus ((C/C)\setminus B)) \rightarrow P}{(A/(((C/C)\setminus C)) \cdot ((C/C)\setminus B) \rightarrow P} \otimes [L]
\end{array}
$$

The following lemma is the key lemma of this paper, and its use will become clear to the reader in the construction of Section 4.

**Lemma 9.** If $\vdash LG A \cap B \rightarrow P$ then $\vdash LG A \rightarrow P$ or $\vdash LG B \rightarrow P$, if it is not the case that:

- $P = P'[A'[A_1 \otimes A_2]^*]]$
- $\vdash LG A/((C/C)\setminus C) \rightarrow A_1$
- $\vdash LG (C/C)\setminus B \rightarrow A_2$

**Proof.** We have that $\vdash LG (A/((C/C)\setminus C)) \otimes ((C/C)\setminus B) \rightarrow P$, so from Lemma 7 we know that $\vdash LG (A/((C/C)\setminus C)) \cdot \setminus ((C/C)\setminus B) \rightarrow P$. Remark that this also means that there exists a cut-free derivation for this sequent. By induction on the length of the derivation we will show that if $\vdash LG (A/((C/C)\setminus C)) \cdot \setminus ((C/C)\setminus B) \rightarrow P$, then $\vdash LG A \rightarrow P$ or $\vdash LG B \rightarrow P$, under the assumption that $P$ is not of the form that is explicitly excluded in this lemma. We will look at the derivations in a top-down way.

The induction base is the case where a logical rule is applied on the lefthand-side of the sequent. At a certain point in the derivation, possibly when $P$ is an atom, one of the following three rules must be applied:
1. The $\otimes R$ rule, but then $P = A_1 \otimes A_2$ and in order to come to a derivation it must be the case that $\vdash_{LG} A/(((C/C)\backslash C) \rightarrow A_1$ and $\vdash_{LG} (C/C)\backslash B \rightarrow A_2$. However, this is explicitly excluded in this lemma so this can never be the case.

2. The $/L$ rule, in this case first the $r$ rule is applied so that we have $\vdash_{LG} A/(((C/C)\backslash C) \rightarrow P \cdot / \cdot ((C/C)\backslash B)$. Now if the $/L$ rule is applied, we must have that $\vdash_{LG} A \rightarrow P$.

3. The $\backslash L$ rule, in this case first the $r$ rule is applied so that we have $\vdash_{LG} (C/C)\backslash B \rightarrow (A/(((C/C)\backslash C)) \cdot \backslash \cdot P$. Now if the $\backslash L$ rule is applied, we must have that $\vdash_{LG} B \rightarrow P$.

The induction step is the case where a logical rule is applied on the righthand-side of the sequent. Let $\delta = \{ r, dr, d \odot /, d \odot \backslash, d \odot /, d \odot \} \} \) and let $\delta^*$ indicate a (possibly empty) sequence of structural residuation steps and Grishin interactions. For example for the $\otimes R$ rule there are two possibilities:

- The lefthand-side ends up in the first premiss of the $\otimes R$ rule:

  \[
  (A/(((C/C)\backslash C)) \cdot \odot \cdot ((C/C)\backslash B) \rightarrow P'[A']
  \]

  \[
  P'[A/(((C/C)\backslash C)) \cdot \odot \cdot ((C/C)\backslash B)] \rightarrow A' \delta^* B' \rightarrow Q \otimes R
  \]

  \[
  P'[A/(((C/C)\backslash C)) \cdot \odot \cdot ((C/C)\backslash B)] \cdot \odot \cdot Q \rightarrow A' \odot B' \delta^*
  \]

  \[
  (A/(((C/C)\backslash C)) \cdot \odot \cdot ((C/C)\backslash B) \rightarrow P'[A' \odot B']
  \]

In order to be able to apply the $\otimes R$ rule, we need to have a formula of the form $A' \odot B'$ on the righthand-side. In the first step all structural rules are applied to display this formula in the righthand-side, and we assume that in the lefthand-side the meet-type ends up in the first structural part (inside a structure with the remaining parts from $P$ that we call $P'$). After the $\otimes R$ rule has been applied, we can again display our meet-type in the lefthand-side of the formula by moving all other structural parts from $P'$ back to the righthand-side ($P''$).

In this case it must be that $\vdash_{LG} (A/(((C/C)\backslash C)) \cdot \odot \cdot ((C/C)\backslash B) \rightarrow P'[A']$, and by induction we know that in this case also $\vdash_{LG} A \rightarrow P'[A']$ or $\vdash_{LG} B \rightarrow P'[A']$. In the case that $\vdash_{LG} A \rightarrow P''[A']$, we can show that $\vdash_{LG} A \rightarrow P'[A' \odot B']$ as follows:

\[
A \rightarrow P''[A']
\]

\[
P'[A] \rightarrow A' \delta^* B' \rightarrow Q \otimes R
\]

\[
P'[A] \cdot \odot \cdot Q \rightarrow A' \odot B' \delta^*
\]

\[
A \rightarrow P'[A' \odot B']
\]

The case for $B$ is similar.

- The lefthand-side ends up in the second premiss of the $\otimes R$ rule:

  \[
  Q \rightarrow A'
  \]

  \[
  B' \rightarrow P'[A/(((C/C)\backslash C)) \cdot \odot \cdot ((C/C)\backslash B)] \delta^* \otimes R
  \]

  \[
  Q \cdot \odot \cdot P'[A/(((C/C)\backslash C)) \cdot \odot \cdot ((C/C)\backslash B)] \rightarrow A' \odot B' \delta^*
  \]

  \[
  (A/(((C/C)\backslash C)) \cdot \odot \cdot ((C/C)\backslash B) \rightarrow P'[A' \odot B']
  \]

  \[
  \]
This case is similar to the other case, except that the meet-type ends up in the other premisse. Note that, although in this case it is temporarily moved to the righthand-side, the meet-type will still be in an input polarity position and can therefore be displayed in the lefthand-side again.

In this case it must be that $\vdash_{LG} (A/(C/C)\backslash C) \otimes ((C/C)\backslash B) \rightarrow P^m[B']$, and by induction we know that in this case also $\vdash_{LG} A \rightarrow P^m[B']$ or $\vdash_{LG} B \rightarrow P^m[B']$. In the case that $\vdash_{LG} A \rightarrow P^m[B']$, we can show that $\vdash_{LG} A \rightarrow P[A' \otimes B']$ as follows:

$$
\begin{array}{c}
A \rightarrow P^m[B'] \\
Q \rightarrow A' \\
B' \rightarrow P'[A] \\
\delta^* \\
\hline
Q \otimes P'[A] \rightarrow A' \otimes B' \\
\odot R \\
A \rightarrow P[A' \otimes B'] \\
\delta^*
\end{array}
$$

The case for $B$ is similar.

The cases for the other logical rules are similar.

\[\square\]

4 Reduction from SAT to LG

In this section we will show that we can reduce a Boolean formula in conjunctive normal form to a sequent of the Lambek-Grishin calculus, so that the corresponding LG sequent is provable if and only if the CNF formula is satisfiable. This has already been done for the associative system $L$ by Pentus (2003) with a similar construction.

Let $\varphi = c_1 \wedge \ldots \wedge c_n$ be a Boolean formula in conjunctive normal form with clauses $c_1 \ldots c_n$ and variables $x_1 \ldots x_m$. For all $1 \leq j \leq m$ let $\lnot x_j$ and $\lnot c_j$ stand for the literal $\lnot x_j$ and $\lnot c_j$, respectively. Now $(t_1, \ldots, t_m) \in \{0, 1\}^m$ is a satisfying assignment for $\varphi$ if and only if for every $1 \leq i \leq n$ there exists a $1 \leq j \leq m$ such that the literal $\lnot c_j$ appears in clause $c_i$.

Let $p_i$ (for $1 \leq i \leq n$) be distinct primitive types from $\text{Var}$. We now define the following families of types:

- $E_j^1(t) = \left\{ \begin{array}{ll} p_i \otimes (p_i \otimes p_i) & \text{if } \lnot x_j \text{ appears in clause } c_i \\
p_i & \text{otherwise} \end{array} \right.$ if $1 \leq i \leq n$, $1 \leq j \leq m$
- $E_j(t) = E_j^1(t) \otimes (E_j^2(t) \otimes (\ldots (E_j^n(t) \otimes E_j^1(t))))$ if $1 \leq j \leq m$ and $t \in \{0, 1\}$
- $H_j = p_1 \otimes (p_2 \otimes (\ldots (p_{n-1} \otimes p_n)))$ if $1 \leq j \leq m$
- $F_j = E_j(1) \uparrow E_j(0)$ if $1 \leq j \leq m$
- $G_0 = H_1 \otimes (H_2 \otimes (\ldots (H_{m-1} \otimes H_m)))$
- $G_i = G_{i-1} \otimes (p_i \otimes p_i)$ if $1 \leq i \leq n$

Let $\bar{\varphi} = F_1 \otimes (F_2 \otimes (\ldots (F_{m-1} \otimes F_m))) \rightarrow G_n$ be the LG sequent corresponding to the Boolean formula $\varphi$. We now claim that the $\vdash \varphi$ if and only if $\vdash_{LG} \bar{\varphi}$. 
4.1 Example

Let us take the Boolean formula \((x_1 \lor \neg x_2) \land (\neg x_1 \lor \neg x_2)\) as an example. We have the primitive types \(\{p_1, p_2\}\) and the types as shown in Figure 2. The formula is satisfiable (for example with the assignment \((1, 0)\)), thus \(\vdash_{LG} F_1 \otimes F_2 \rightarrow G_2\). A sketch of the derivation is given in Figure 2; some parts are proved in lemma’s later on.

4.2 Intuition

Let us give some intuitions for the different parts of the construction, and a brief idea of why this would work. The basic idea is that on the lefthand-side we create a type for each literal \((F_j\) is the formula for literal \(j\)), which will in the end result in the base type \(H_j\), so \(F_1 \otimes (F_2 \otimes (\ldots (F_{m-1} \otimes F_m))\)) will result in \(G_0\). However, on the righthand-side we have an occurrence of the expression \(\otimes (p_i \otimes p_j)\) for each clause \(i\), so in order to come to a derivation, we need to apply the \(\otimes R\) rule for every clause \(i\).

Each literal on the lefthand-side will result in either \(E_j(1)\) \((x_j \text{ is } true)\) or \(E_j(0)\) \((x_j \text{ is } false)\). This choice is created using a join type \(H_j\) such that \(\vdash_{LG} E_j(1) \rightarrow H_j\) and \(\vdash_{LG} E_j(0) \rightarrow H_j\), which we use to construct the meet type \(F_j\). It can be shown that in this case \(\vdash_{LG} F_j \rightarrow E_j(1)\) and \(\vdash_{LG} F_j \rightarrow E_j(0)\), i.e. in the original formula we can replace \(F_j\) by either \(E_j(1)\) or \(E_j(0)\), giving us a choice for the truthvalue of \(x_j\).

Let us assume that we need \(x_1 = true\) to satisfy the formula, so on the lefthand-side we need to replace \(F_j\) by \(E_1(1)\). \(E_1(1)\) will be the product of exactly \(n\) parts, one for each clause \((E_1^1(1) \ldots E_1^n(1))\). Here \(E_1^1(1) = p_i \otimes (p_i \otimes p_i) \text{ iff } x_1\) does appear in clause \(i\), and \(p_i\) otherwise. The first thing that should be noticed is that \(\vdash_{LG} p_i \otimes (p_i \otimes p_i) \rightarrow p_i\), so we can rewrite all \(p_i \otimes (p_i \otimes p_i)\) into \(p_i\) so that \(\vdash_{LG} E_1(1) \rightarrow H_1\).

However, we can also use the type \(p_i \otimes (p_i \otimes p_i)\) to facilitate the application of the \(\otimes R\) rule on the occurrence of the expression \(\otimes (p_i \otimes p_i)\) in the righthand-side. From Lemma 6 we know that \(\vdash_{LG} X \otimes [p_i \otimes (p_i \otimes p_i)] \rightarrow G_i\) if \(\vdash_{LG} X \otimes [p_i] \cdot \otimes (p_i \otimes p_i) \rightarrow G_i\), so if the expression \(Y\) occurs somewhere in a \(\otimes\) structure we can move it to the outside. Hence, from the occurrence of \(p_i \otimes (p_i \otimes p_i)\) on the lefthand-side we can move \(\otimes (p_i \otimes p_i)\) to the outside of the \(\otimes\) structure and \(p_i\) will be left behind within the original structure (just as if we rewrote it to \(p_i\)). However, the sequent is now of the form \(X \otimes [p_i] \cdot \otimes (p_i \otimes p_i) \rightarrow G_{i-1} \otimes (p_i \otimes p_i)\), so after applying the \(\otimes R\) rule we have \(X \otimes [p_i] \rightarrow G_{i-1}\).

Now if the original CNF formula is satisfiable, we can use the meet types on the lefthand-side to derive the correct value of \(E_j(1)\) or \(E_j(0)\) for all \(j\). If this assignment indeed satisfies the formula, then for each \(i\) the formula \(p_i \otimes (p_i \otimes p_i)\) will appear at least once. Hence, for all occurrences of the expression \(\otimes (p_i \otimes p_i)\) on the righthand-side we can apply the \(\otimes R\) rule, after which the rest of the \(p_i \otimes (p_i \otimes p_i)\) can be rewritten to \(p_i\) in order to derive the base type.

If the formula is not satisfiable, then there will be no way to have the \(p_i \otimes (p_i \otimes p_i)\) types on the lefthand-side for all \(i\), so there will be at least one occurrence
Fig. 2: Sketch proof for LG sequent corresponding to \((x_1 \lor \neg x_2) \land (\neg x_1 \lor x_2)\)
of \( \odot(p_i \odot p_i) \) on the righthand-side where we cannot apply the \( \odot R \) rule. Because the \( \odot \) will be the main connective we cannot apply any other rule, and we will never come to a valid derivation.

Note that the meet type \( F_j \) provides an explicit switch, so we first have to replace it by \( E_j(1) \) or \( E_j(0) \) before we can do anything else with it. This guarantees that if \( \vdash_{LG} \varphi \), there also must be some assignment \( \langle t_1, \ldots, t_m \rangle \in \{0, 1\}^m \) such that \( \vdash_{LG} E_1(t_1) \otimes (E_2(t_2) \otimes (\ldots (E_{m-1}(t_{m-1}) \otimes E_m(t_m)))) \rightarrow G_n \), which means that \( \langle t_1, \ldots, t_m \rangle \) is a satisfying assignment for \( \varphi \).

5 Proof

We will now prove the main claim that \( \models \varphi \) if and only if \( \vdash_{LG} \varphi \). First we will prove that if \( \models \varphi \), then \( \vdash_{LG} \varphi \).

5.1 If-part

Let us assume that \( \models \varphi \), so there is an assignment \( \langle t_1, \ldots, t_m \rangle \in \{0, 1\}^m \) that satisfies \( \varphi \).

Lemma 10. If \( 1 \leq i \leq n \), \( 1 \leq j \leq m \) and \( t \in \{0, 1\} \) then \( \vdash_{LG} E_j^i(t) \rightarrow p_i \).

Proof. We consider two cases:

1. If \( E_j^i(t) = p_i \) this is simply the axiom rule.
2. If \( E_j^i(t) = p_i \odot (p_i \odot p_i) \) we can prove it as follows:

\[
\begin{align*}
& p_i \rightarrow p_i \odot p_i \rightarrow p_i \odot p_i \odot R \\
& p_i \rightarrow p_i \odot p_i \otimes (p_i \odot p_i) dr \\
& p_i \odot (p_i \odot p_i) \rightarrow p_i dr \\
& p_i \odot (p_i \odot p_i) \rightarrow p_i \odot L
\end{align*}
\]

Lemma 11. If \( 1 \leq j \leq m \) and \( t \in \{0, 1\} \), then \( \vdash_{LG} E_j(t) \rightarrow H_j \).

Proof. From Lemma 10 we know that \( \vdash_{LG} E_j^i(t) \rightarrow p_i \), so using Lemma 8 we can replace all \( E_j^i(t) \) by \( p_i \) in \( E_j(t) \) after which we can apply the \( \odot R \) rule \( n - 1 \) times to prove the lemma.

Lemma 12. If \( 1 \leq j \leq m \), then \( \vdash_{LG} F_j \rightarrow E_j(t) \).

Proof. From Lemma 11 we know that \( \vdash_{LG} E_j(1) \rightarrow H_j \) and \( \vdash_{LG} E_j(0) \rightarrow H_j \), so \( E_j(1) \sim E_j(0) \) with join-type \( H_j \). Now from Lemma 8 we know that \( \vdash_{LG} E_j(1) \vdash_{H_j} E_j(0) \rightarrow E_j(1) \) and \( \vdash_{LG} E_j(1) \vdash_{H_j} E_j(0) \rightarrow E_j(0) \).
Lemma 13. We can replace each $F_i$ in $\varnothing$ by $E_j(t_j)$, so:

$$E_1(t_1) \odot (E_2(t_2) \odot \cdots (E_m(t_m-1) \odot (E_m(t_m)))) \rightarrow G_n$$

$$F_1 \odot (F_2 \odot (\cdots (F_m-1 \odot F_m))) \rightarrow G_n$$

Proof. This can be proven by using Lemma 7 to turn it into a $\odot$-structure, and then apply Lemma 12 in combination with Lemma 5 $m$ times. 

Lemma 14. In $E_1(t_1) \odot (E_2(t_2) \odot \cdots (E_m(t_m-1) \odot E_m(t_m))) \rightarrow G_n$, there is at least one occurrence of $p_i \odot (p_i \odot p_i)$ in the lefthand-side for every $1 \leq i \leq n$.

Proof. This sequence of $E_1(t_1), \ldots, E_m(t_m)$ represents the truthvalue of all variables, and because this is a satisfying assignment, for all $i$ there is at least one index $k$ such that $\neg x_k$ appears in clause $i$. By definition we have that $E_k(t_k) = p_i \odot (p_i \odot p_i)$. 

Definition 3. $Y_i^j := E_j(t_j)$ with every occurrence of $p_k \odot (p_k \odot p_k)$ replaced by $p_k$ for all $i < k \leq n$.

Lemma 15. $\vdash_{LG} Y_1^0 \odot (Y_2^0 \odot \cdots (Y_m^0 \odot Y_m^0)) \rightarrow G_0$

Proof. Because $Y_j^0 = H_j$ by definition for all $1 \leq j \leq m$ and $G_0 = H_1 \odot (H_2 \odot (\cdots (H_{m-1} \odot H_m)))$, this can be proven by applying the $\odot$-rule $m-1$ times. 

Lemma 16. If $\vdash_{LG} Y_1^{i-1} \odot (Y_2^{i-1} \odot \cdots (Y_m^{i-1} \odot Y_m^{i-1})) \rightarrow G_{i-1}$, then \[ \vdash_{LG} Y_1^i \odot (Y_2^i \odot \cdots (Y_m^i \odot Y_m^i)) \rightarrow G_i \]

Proof. From Lemma 14 we know that $p_i \odot (p_i \odot p_i)$ occurs in $Y_1^i \odot \cdots (Y_2^i \odot \cdots (Y_m^i \odot Y_m^i))$ (because the $Y_i^j$ parts are $E_j(t_j)$ but with $p_k \odot (p_k \odot p_k)$ replaced by $p_k$ only for $k > i$). Using Lemma 4 we can move the expression $\odot (p_i \odot p_i)$ to the outside of the lefthand-side of the sequent, after which we can apply the $\odot R$-rule. After this we can replace all other occurrences of $p_i \odot (p_i \odot p_i)$ by $p_i$ using Lemma 10 and Lemma 5. This process can be summarized as:

$$\vdash_{LG} Y_1^i \odot (Y_2^i \odot \cdots (Y_m^i \odot Y_m^i)) \rightarrow G_i$$

Lemma 17. $\vdash_{LG} Y_1^n \odot (Y_2^n \odot \cdots (Y_m^n \odot Y_m^n)) \rightarrow G_n$

Proof. We can prove this using induction with Lemma 15 as base and Lemma 16 as induction step. 

Lemma 18. If $\vdash \varphi$, then $\vdash_{LG} \varphi$. 


Proof. From Lemma 17 we know that \( \vdash_{LG} Y^n_j \cdot \otimes (Y^n_{m-1-j} \cdot \otimes \cdots \cdot Y^n_m) \rightarrow G_n \), and because by definition \( Y^n_j = E_j(t_j) \), we also have that \( \vdash_{LG} E_1(t_1) \cdot \otimes \cdots \cdot (E_{m-1}(t_{m-1}) \cdot \otimes E_m(t_m)) \rightarrow G_n \). Finally combining this with Lemma 13 we have that \( \vdash_{LG} \varphi = F_1 \otimes (F_2 \otimes \cdots \cdot (F_{m-1} \otimes F_m)) \rightarrow G_n \), using the assumption that \( \vdash \varphi \).

\[ \square \]

5.2 Only-if part

For the only if part we will need to prove that if \( \vdash_{LG} \varphi \), then \( \vdash \varphi \). Let us now assume that \( \vdash_{LG} \varphi \).

Lemma 19. If \( \vdash_{LG} X \rightarrow P'((P \otimes Y)^\circ) \), then there exist a \( Q \) such that \( Q \) is part of \( X \) or \( P' \) (possibly inside a formula in \( X \) or \( P' \)) and \( \vdash_{LG} Y \rightarrow Q \).

Proof. The only rule that matches a \( \otimes \) in the righthand-side is the \( \otimes R \) rule, so somewhere in the derivation this rule must be applied on the occurrence of \( P \otimes Y \). Because this rule needs a \( \cdot \otimes \cdot \) connective in the lefthand-side, we know that if \( \vdash_{LG} X \rightarrow P'((P \otimes Y)^\circ) \) it must be the case that we can turn this into \( X' \cdot \otimes \cdot Q \rightarrow P \otimes Y \) such that \( \vdash_{LG} Y \rightarrow Q \).

\[ \square \]

Lemma 20. If \( \vdash_{LG} E_1(t_1) \cdot \otimes \cdot (E_{m-1}(t_{m-1}) \cdot \otimes E_m(t_m)) \rightarrow G_n \), then there is an occurrence \( p_i \otimes (p_i \otimes p_i) \) on the lefthand-side at least once for all \( 1 \leq i \leq n \).

Proof. \( G_n \) by definition contains an occurrence of the expression \( \otimes (p_i \otimes p_i) \) for all \( 1 \leq i \leq n \). From Lemma 19 we know that somewhere in the sequent we need an occurrence of a structure \( Q \) such that \( \vdash_{LG} p_i \otimes p_i \rightarrow Q \). From the construction it is obvious that the only possible type for \( Q \) is in this case \( p_i \otimes p_i \), and it came from the occurrence of \( p_i \otimes (p_i \otimes p_i) \) on the lefthand-side.

\[ \square \]

Lemma 21. If \( \vdash_{LG} E_1(t_1) \cdot \otimes \cdot (E_{m-1}(t_{m-1}) \cdot \otimes E_m(t_m)) \rightarrow G_n \), then \( \langle t_1, t_2, \ldots, t_{m-1}, t_m \rangle \) is a satisfying assignment for the CNF formula.

Proof. From Lemma 20 we know that there is a \( p_i \otimes (p_i \otimes p_i) \) in the lefthand-side of the formula for all \( 1 \leq i \leq n \). From the definition we know that for each \( i \) there is an index \( j \) such that \( E_j(t_j) = p_i \otimes (p_i \otimes p_i) \), and this means that \( \neg t_j, x_j \) appears in clause \( i \), so all clauses are satisfied. Hence, this choice of \( t_1 \ldots t_m \) is a satisfying assignment.

\[ \square \]

Lemma 22. If \( 1 \leq j \leq m \) and \( \vdash_{LG} X^{\otimes}[F_j] \rightarrow G_n \), then \( \vdash_{LG} X^{\otimes}[E_j(0)] \rightarrow G_n \) or \( \vdash_{LG} X^{\otimes}[E_j(1)] \rightarrow G_n \).

Proof. We know that \( X^{\otimes}[F_j] \) is a \( \otimes \)-structure, so we can apply the \( r \) rule several times to move all but the \( F_j \)-part to the righthand-side. We then have that \( \vdash_{LG} F_j \rightarrow \ldots \cdot \otimes \cdot G_n \cdot / \ldots \). From Lemma 9 we know that we now have that \( \vdash_{LG} E_j(0) \rightarrow \ldots \cdot \otimes \cdot G_n \cdot / \ldots \) or \( \vdash_{LG} E_j(1) \rightarrow \ldots \cdot \otimes \cdot G_n \cdot / \ldots \). Finally we can apply the \( r \) rule again to move all parts back to the lefthand-side, to show that \( \vdash_{LG} X^{\otimes}[E_j(0)] \rightarrow G_n \) or \( \vdash_{LG} X^{\otimes}[E_j(1)] \rightarrow G_n \).

\[ \square \]
Note that, in order for Lemma 9 to apply, we have to show that this sequent satisfies the constraints. $G_n$ does contain $A_1 \otimes A_2$ with output polarity, however the only connectives in $A_1$ and $A_2$ are $\otimes$. Because no rules apply on $A/((C/C)\setminus C) \rightarrow A_1' \otimes A_2''$, we have that $\not \vdash_{LG} A/((C/C)\setminus C) \rightarrow A_1$. In $X^\otimes[]$, the only $\otimes$ connectives are within other $F_k$, however these have an input polarity and do not break the constraints either.

So, in all cases $F_j$ provides an explicit switch, which means that the truthvalue of a variable can only be changed in all clauses simultaneously.

Lemma 23. If $\vdash_{LG} \bar{\varphi}$, then $\models \varphi$.

Proof. From Lemma 22 we know that all derivations will first need to replace each $F_j$ by either $E_j(1)$ or $E_j(0)$. This means that if $\vdash_{LG} F_1 \otimes (F_2 \otimes (\ldots(F_{m-1} \otimes F_m))) \rightarrow G_n$, then also $\vdash_{LG} E_1(t_1) \otimes \ldots \otimes (E_{m-1}(t_{m-1}) \otimes \ldots \otimes E_m(t_m)) \rightarrow G_n$ for some $\langle t_1, t_2, \ldots, t_{m-1}, t_m \rangle \in \{0, 1\}^m$. From Lemma 21 we know that this is a satisfying assignment for $\varphi$, so if we assume that $\vdash_{LG} \bar{\varphi}$, then $\models \varphi$.

5.3 Conclusion

Theorem 1. $LG$ is NP-complete.

Proof. From Lemma 2 we know that for every derivable sequent there exists a proof that is of polynomial length, so the derivability problem for $LG$ is in NP. From Lemma 18 and Lemma 23 we can conclude that we can reduce SAT to $LG$. Because SAT is a known NP-hard problem (Garey and Johnson, 1979), and our reduction is polynomial, we can conclude that derivability for $LG$ is also NP-hard.

Combining these two facts we conclude that the derivability problem for $LG$ is NP-complete.
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