NOTE ON WEIGHTED BOHR’S INEQUALITY

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ABSTRACT. In this paper, first we give a new generalization of the Bohr’s inequality for the class of bounded analytic functions \( B' \) and for the class of sense-preserving \( K \)-quasiconformal harmonic mappings of the form \( f = h + g \), where \( h \in B' \). Finally we give a new generalization of the Bohr’s inequality for the class of analytic functions subordinate to univalent functions and for the class of sense-preserving \( K \)-quasiconformal harmonic mappings of the form \( f = h + g \), where \( h \) is subordinated to some analytic function.

1. Introduction and Preliminaries

Throughout we let \( B \) denote the class of all analytic functions \( \omega \) in the open unit disk \( D = \{ z \in \mathbb{C} : |z| < 1 \} \) such that \( |\omega(z)| \leq 1 \) for all \( z \in D \). Bohr’s inequality says that if \( f \in B \) and \( f(z) = \sum_{n=0}^{\infty} a_n z^n \), then we have

\[
\sum_{n=0}^{\infty} |a_n| r^n \leq 1
\]

for all \( z \in D \) with \( |z| = r \leq \frac{1}{3} \). This inequality was discovered by Bohr in 1914 [8]. Bohr actually obtained the inequality for \( |z| \leq \frac{1}{6} \). Later M. Riesz, I. Schur and F. W. Wiener independently, established the inequality for \( |z| \leq \frac{1}{3} \) and showed that \( \frac{1}{3} \) is sharp. The number \( \frac{1}{3} \) is called Bohr radius for the family \( B \). A space of analytic or harmonic functions \( f \) in \( D \) is said to have Bohr’s phenomenon if an inequality of this type holds in some disk of radius \( \rho > 0 \) and for all such functions in unit ball of the space. In [6], it is shown that not every space of functions has Bohr’s phenomenon. On the other hand, Abu-Muhanna [1] proved the existence of Bohr phenomenon in the case of subordination and bounded harmonic classes. Many mathematicians have contributed towards the understanding of this problem in several settings [9][10]. Extensions of Bohr’s inequality to more general domains or higher dimensional spaces were investigated by many. See for instance, [7][11][15]. We refer to the recent survey on this topic by Abu-Muhanna et al. [2] and Garcia et al. [13], for the importance and the several other results. For certain recent results, see [5][17][19].

More generally, a harmonic version of Bohr’s inequality was discussed by Kayumov et al. [20]. For certain other results on harmonic Bohr’s inequality, we refer to [13][20]. Recently, a new generalization of Bohr’s ideas was introduced and investigated by Kayumov et
Let \( F \) denote the set of all sequences \( \{ \varphi_n(r) \}_{n=0}^{\infty} \) of nonnegative continuous functions in \([0, 1]\) such that the series \( \sum_{n=0}^{\infty} \varphi_n(r) \) converges locally uniformly with respect to \( r \in [0, 1] \). Let \( F_{\text{dec}} \subset F \) consist of decreasing sequences of functions from \( F \), and for convenience, we let \( \Phi_1(r) = \sum_{n=1}^{\infty} \varphi_n(r) \) so that \( \Phi_1'(r) = \sum_{n=1}^{\infty} \varphi_n'(r) \) whenever each \( \varphi_n \) \((n \geq 1)\) is differentiable on \([0, 1]\).

**Theorem A.** ([16]) Let \( f \in B, f(z) = \sum_{k=0}^{\infty} a_k z^k \), and \( p \in (0, 2) \). If \( \varphi_0(r) > (2/p) \Phi_1(r) \), then the following sharp inequality holds:

\[
B_f(\varphi, p, r) := |a_0|^p \varphi_0(r) + \sum_{k=1}^{\infty} |a_k| \varphi_k(r) \leq \varphi_0(r) \text{ for all } r \leq R,
\]

where \( R \) is the minimal positive root of the equation \( \varphi_0(x) = (2/p) \Phi_1(x) \). In the case when \( \varphi_0(x) < (2/p) \Phi_1(x) \) in some interval \((R, R+\varepsilon)\), the number \( R \) cannot be improved. If the functions \( \varphi_k(x) \) \((k \geq 0)\) are smooth functions, then the last condition is equivalent to the inequality \( \varphi_0(R) < (2/p) \Phi_1'(R) \).

Further investigation and refinements of several earlier known results on Bohr-type inequality, we refer to [22].

For two analytic functions \( f \) and \( g \) in \( \mathbb{D} \), we say that \( g \) is subordinate to \( f \) (denoted simply by \( g \sim f \)) if there exists a function \( \omega \), analytic in \( \mathbb{D} \) with \( \omega(0) = 0 \) and \( |\omega(z)| < 1 \), satisfying \( g = f \circ \omega \). We denote the class of all analytic functions \( g \) in \( \mathbb{D} \) that are subordinate to a fixed function \( f \) by \( S(f) \), and \( f(\mathbb{D}) = \Omega \). We say that \( S(f) \) has Bohr’s phenomenon if for any \( g(z) = \sum_{n=0}^{\infty} b_n z^n \in S(f) \) and \( f(z) = \sum_{n=0}^{\infty} a_n z^n \), there is a \( \rho_0 \), \( 0 < \rho_0 \leq 1 \), so that

\[
\sum_{n=1}^{\infty} |b_n z^n| \leq \text{dist}(f(0), \partial \Omega),
\]

for \( |z| < \rho_0 \). We remark that the class \( S(f) \) has Bohr’s phenomenon when \( f \) is univalent (see [11, Theorem 1]). For each \( f(z) = \sum_{k=0}^{\infty} a_k z^k \) belonging to \( B \), it is well-known that \( |a_n| \leq 1 - |a_0|^2 \) for all \( n \geq 1 \). Besides the fact that \( 1 - |a_0| \leq 1 - |a_0|^2 \) for \( |a_0| \leq 1 \), as demonstrated by Aizenberg and Vïdras (see [3, p. 736]), there exists a nice subclass of functions \( f \in B \) for which \( |a_n| \leq 1 - |a_0| \) all \( n \geq 1 \). We now recall this result.

**Theorem B.** ([3]) Let \( f \in B \), such that the Taylor coefficients \( a_{mn} = 0 \) for a given \( m > 1 \) and all \( n \geq 1 \). Then \( |a_n| \leq 1 - |a_0| \) for all \( n \geq 1 \).

Thus, it is natural to consider

\[
B' = \left\{ f(z) = \sum_{k=0}^{\infty} a_k z^k \in B : |a_n| \leq 1 - |a_0| \text{ for all } n \geq 1 \right\}.
\]

In [4, Theorem 1], it was shown that the Bohr radius for functions in \( B' \) is \( \frac{1}{2} \), and the constant \( 1/2 \) cannot be improved.

In this article, we first investigate the Bohr radius for the family \( B' \) in a general setting which is indeed an analog of Theorem A for the family \( B' \) (See Theorem [14]). Our second
result (Theorem 3) extends Theorem A to the case of sense-preserving $K$-quasiconformal harmonic mappings of the form $f = h + \overline{g}$, where $h \in \mathcal{B}'$. In Section 4, we establish that the family $\mathcal{S}(f)$ has Bohr’s phenomenon in our new setting (see Theorems 4 and 5), especially when $f$ is either univalent or convex (univalent) in $D$. Finally, we extend this result (Theorem 5) for sense-preserving $K$-quasiconformal harmonic mappings.

2. Bohr radius for a special family of analytic functions

The following theorem displays the sharp Bohr radius for $\mathcal{B}'$.

**Theorem 1.** Let $f \in \mathcal{B}'$, $f(z) = \sum_{n=0}^{\infty} a_n z^n$, and $p \in (0, 1]$. If $\{\varphi_n(r)\}_{n=0}^{\infty} \in \mathcal{F}$ such that $\Phi_1(r) = \sum_{n=1}^{\infty} \varphi_n(r)$, and satisfies the inequality

\[
\varphi_0(r) \geq \frac{1}{p} \Phi_1(r).
\]

Then the following sharp inequality holds:

\[
B_f(\varphi, p, r) := |a_0|^p \varphi_0(r) + \sum_{n=1}^{\infty} |a_n| \varphi_n(r) \leq \varphi_0(r) \text{ for all } r \leq R,
\]

where $R$ is the minimal positive root of the equation

\[
\varphi_0(x) = \frac{1}{p} \Phi_1(x).
\]

In the case when $\varphi_0(x) < \frac{1}{p} \Phi_1(x)$ in some interval $(R, R + \epsilon)$, the number $R$ cannot be improved.

**Proof.** Let $f \in \mathcal{B}'$. Then $|a_n| \leq 1 - |a_0|$ for all $n \geq 1$ and thus, we get that

\[
|a_0|^p \varphi_0(r) + \sum_{n=1}^{\infty} |a_n| \varphi_n(r) \leq |a_0|^p \varphi_0(r) + (1 - |a_0|) \Phi_1(r)
\]

\[
= \varphi_0(r) + (1 - |a_0|) \left[ \Phi_1(r) - \left( \frac{1 - |a_0|^p}{1 - |a_0|} \right) \varphi_0(r) \right]
\]

\[
\leq \varphi_0(r) + (1 - |a_0|) \left[ \Phi_1(r) - p \varphi_0(r) \right]
\]

\[
\leq \varphi_0(r), \text{ by Eqn. (1)},
\]

for all $r \leq R$, by the definition of $R$. In the third inequality above, we have used the fact that the function

\[
B(x) = \frac{1 - x^p}{1 - x}, \quad x \in [0, 1),
\]

is decreasing on $[0, 1)$ for $0 < p \leq 1$ so that

\[
B(x) \geq \lim_{x \to 1^{-}} \frac{1 - x^p}{1 - x} = p.
\]

This proves the desired inequality (2). Now let us prove that $R$ is an optimal number. For $a \in [0, 1)$, we consider the function

\[
f(z) = \frac{a - (1 - a + a^2) z}{1 - az} = a - (1 - a) \sum_{n=1}^{\infty} a^{n-1} z^n, \quad z \in \mathbb{D}.
\]
A simple exercise shows that \( f \in \mathcal{B}' \). For this function, we have
\[
|a_0|^p \varphi_0(r) + \sum_{n=1}^{\infty} |a_n| \varphi_n(r) = a^p \varphi_0(r) + (1 - a) \sum_{n=1}^{\infty} a^{n-1} \varphi_n(r)
= \varphi_0(r) + p(1 - a) \left[ \frac{1}{p} \sum_{n=1}^{\infty} a^{n-1} \varphi_n(r) - \varphi_0(r) \right]
+ (1 - a) \left[ (p - \frac{1 - a^p}{1 - a}) \varphi_0(r) \right].
\]
Now it is easy to see that number is \( > \varphi_0(r) \) when \( a \) is close to 1. The proof of the theorem is complete.

**Remark 1.** Note that the function \( B(x) \) in the above proof is increasing on \([0, 1]\) for \( p \geq 1 \) so that \( B(x) \geq B(0) = 1 \). This means that the inequality (2) holds for \( r \leq \frac{1}{2} \) in the case when \( \varphi_n(r) = r^n (n \geq 1) \).

**Corollary 1.** Suppose that \( f \in \mathcal{B}' \), \( f(z) = \sum_{n=0}^{\infty} a_n z^n \), and \( p \in (0, 1] \). Then
\[
|a_0|^p + \sum_{n=1}^{\infty} |a_n| r^n \leq 1 \text{ for } r \leq R(p) = \frac{p}{1 + p},
\]
and the constant \( R(p) \) cannot be improved.

The case \( p = 1 \) of Corollary \( \Box \) is the Bohr inequality for special family of bounded analytic functions \( \mathcal{B}' \), obtained in \([3, \text{ Theorem 1}]\).

3. **Bohr radius for harmonic mappings as an extension of Theorem \( \Box \)**

We recall that a sense-preserving harmonic mappings \( f \) of the form \( f = h + \overline{g} \), is said to be \( K \)-quasiconformal if \( |g'(z)| \leq k |h'(z)| \) in the unit disk, for \( k = \frac{K - 1}{K + 1} \in [0, 1] \). See \([20] \) for discussion on Bohr radius for quasiconformal mappings.

**Lemma C.** \([23] \) Let \( \{\psi_n(r)\}_{n=1}^{\infty} \) be a decreasing sequence of nonnegative functions in \([0, r_\psi]\), and \( g, h \) be analytic functions in the unit disk \( \mathbb{D} \) such that \( |g'(z)| \leq k |h'(z)| \) in \( \mathbb{D} \) and for some \( k \in [0, 1] \), where \( h(z) = \sum_{n=0}^{\infty} a_n z^n \) and \( g(z) = \sum_{n=0}^{\infty} b_n z^n \). Then
\[
\sum_{n=1}^{\infty} |b_n|^2 \psi_n(r) \leq k^2 \sum_{n=1}^{\infty} |a_n|^2 \psi_n(r) \quad \text{for } r \in [0, r_\psi].
\]

Next, we find Bohr radius for the family of sense-preserving \( K \)-quasiconformal harmonic mappings of the form \( f = h + \overline{g} \), where \( h \in \mathcal{B}' \) and show the sharpness of it.

**Theorem 2.** Suppose that \( f(z) = h(z) + \overline{g(z)} = \sum_{n=0}^{\infty} a_n z^n + \sum_{n=1}^{\infty} b_n z^n \) is harmonic mapping of the disk \( \mathbb{D} \) such that \( |g'(z)| \leq k |h'(z)| \) in \( \mathbb{D} \) and for some \( k \in [0, 1] \), where \( h \in \mathcal{B}' \). Assume that \( \varphi_0(r) = 1 \) and \( \{\varphi_n(r)\}_{n=0}^{\infty} \) belongs to \( \mathcal{F}_{\text{dec}} \) with \( \Phi_1(r) = \sum_{n=1}^{\infty} \varphi_n(r) \), and \( p \in (0, 1] \). If
\[
p \geq (1 + k) \Phi_1(r),
\]
then the following sharp inequality holds:

\[
|a_0|^p + \sum_{n=1}^{\infty} |a_n| \varphi_n(r) + \sum_{n=1}^{\infty} |b_n| \varphi_n(r) \leq \|h\|_\infty \text{ for all } r \leq R,
\]

where \( R \) is the minimal positive root of the equation

\[
p = (1 + k) \Phi_1(x).
\]

In the case when \( p < (1 + k) \Phi_1(x) \) in some interval \((R, R + \epsilon)\), the number \( R \) cannot be improved.

**Proof.** For simplicity, we suppose that \( \|h\|_\infty = 1 \). For \( h \in \mathcal{B}' \), gives the inequality \( |a_n| \leq 1 - |a_0| \) for all \( n \geq 1 \). By assumption \( |g'(z)| \leq k|h'(z)| \) in \( \mathbb{D} \), where \( k \in [0, 1] \) and so, by Lemma C it follows that

\[
\sum_{n=1}^{\infty} |b_n|^2 \varphi_n(r) \leq k^2 \sum_{n=1}^{\infty} |a_n|^2 \varphi_n(r) \leq k^2 (1 - |a_0|)^2 \sum_{n=1}^{\infty} \varphi_n(r) = k^2 (1 - |a_0|)^2 \Phi_1(r).
\]

Consequently, it follows from the classical Schwarz inequality that

\[
\sum_{n=1}^{\infty} |b_n| \varphi_n(r) \leq \sqrt{\sum_{n=1}^{\infty} |b_n|^2 \varphi_n(r)} \sqrt{\sum_{n=1}^{\infty} \varphi_n(r)} \leq k(1 - |a_0|) \Phi_1(r)
\]

and thus, as in the proof of Theorem 1 [1] we get that

\[
|a_0|^p + \sum_{n=1}^{\infty} |a_n| \varphi_n(r) + \sum_{n=1}^{\infty} |b_n| \varphi_n(r) \leq |a_0|^p + (1 - |a_0|)(1 + k) \Phi_1(r)
\]

\[
= 1 + (1 - |a_0|) \left( (1 + k) \Phi_1(r) - \left( \frac{1 - |a_0|^p}{1 - |a_0|} \right) \right)
\]

\[
\leq 1 + (1 - |a_0|) \left[ (1 + k) \Phi_1(r) - p \right]
\]

\[
\leq 1, \text{ by Eqn. 3},
\]

for all \( r \leq R \), by the definition of \( R \). This proves the desired inequality [3]. Now let us prove that \( R \) is an optimal number. We consider the function

\[
h(z) = \frac{a - (1 - a + a^2)z}{1 - az} = a - (1 - a) \sum_{n=1}^{\infty} a^{n-1} z^n, a \in [0, 1), z \in \mathbb{D}
\]

and \( g(z) = \lambda kh(z) \), where \( |\lambda| = 1 \). Then it is a simple exercise to see that

\[
|a_0|^p + \sum_{n=1}^{\infty} |a_n| \varphi_n(r) + \sum_{n=1}^{\infty} |b_n| \varphi_n(r)
\]

\[
= a^p + (1 - a) \sum_{n=1}^{\infty} a^{n-1} \varphi_n(r) + k(1 - a) \sum_{n=1}^{\infty} a^{n-1} \varphi_n(r)
\]

\[
= 1 + p(1 - a) \left[ \frac{1}{p} (1 + k) \sum_{n=1}^{\infty} a^{n-1} \varphi_n(r) - 1 \right] + (1 - a) \left( p - \frac{1 - a^p}{1 - a} \right).
\]
Now it is easy to see that number is > 1 when a is close to 1. The proof of the theorem is complete.  

**Corollary 2.** Suppose that \( f(z) = h(z) + g(z) = \sum_{n=0}^{\infty} a_n z^n + \sum_{n=1}^{\infty} b_n z^n \) is a sense-preserving \( K \)-quasiconformal harmonic mapping of the disk \( \mathbb{D} \), i.e., \( |g'(z)| \leq k|h'(z)| \) in \( \mathbb{D} \) for some \( k = \frac{K-1}{K+1} \in [0,1] \), where \( h \in \mathcal{B}' \). Then we have the sharp inequality

\[
|a_0|^p + \sum_{n=1}^{\infty} |a_n|r^n + \sum_{n=1}^{\infty} |b_n|r^n \leq 1 \text{ for } r \leq R_k(p)
\]

where \( p \in (0,1] \), and

\[
R_k(p) = \frac{p}{k+1+p} = \frac{p(K+1)}{(p+2)K+p}
\]

and the constant \( R_k(p) \) cannot be improved.

In particular, the case \( p = 1 \) in (5) yields the recently obtained result [4, Theorem 2].

### 4. Bohr Phenomenon in Subordination

The following lemma will be used to prove that the family \( \mathcal{S}(f) \) has Bohr’s phenomenon in our new setting (see Theorem 3).

**Lemma D.** [12, p. 195-196] Let \( f \) be an analytic univalent map from \( \mathbb{D} \) onto a simply connected domain \( \Omega := f(\mathbb{D}) \) and \( g(z) = \sum_{n=0}^{\infty} b_n z^n \prec f(z) \). Then

\[
\frac{1}{4} |f'(0)| \leq \text{dist}(f(0), \partial \Omega) \leq |f'(0)|, \text{ and } |b_n| \leq n |f'(0)| \leq 4n \text{ dist}(f(0), \partial \Omega).
\]

**Theorem 3.** Suppose that \( g(z) = \sum_{n=0}^{\infty} b_n z^n \in \mathcal{S}(f) \) and \( f(z) = \sum_{n=0}^{\infty} a_n z^n \) is univalent in \( \mathbb{D} \). If \( \{\varphi_n(r)\}_{n=1}^{\infty} \in \mathcal{F} \) satisfies the inequality

\[
1 \geq 4\Psi_1(r),
\]

where \( \Psi_1(r) = \sum_{n=1}^{\infty} n \varphi_n(r) \), then the following sharp inequality holds:

\[
\sum_{n=1}^{\infty} |b_n| \varphi_n(r) \leq \text{dist}(f(0), \partial \Omega) \text{ for all } r \leq R,
\]

where \( R \) is the minimal positive root of the equation \( 1 = 4\Psi_1(x) \). In the case when \( 1 < 4\Psi_1(x) \) in some interval \( (R, R + \epsilon) \), the number \( R \) cannot be improved.

**Proof.** By assumption \( g \prec f \) and \( f \) is univalent in \( \mathbb{D} \). Then, by Lemma D, we have

\[
|b_n| \leq 4n \text{ dist}(f(0), \partial \Omega).
\]

Thus, we have

\[
\sum_{n=1}^{\infty} |b_n| \varphi_n(r) \leq 4\text{dist}(f(0), \partial \Omega) \sum_{n=1}^{\infty} n \varphi_n(r) = 4\text{dist}(f(0), \partial \Omega) \Psi_1(r)
\]

\[
\leq \text{dist}(f(0), \partial \Omega), \text{ by Eqn.(6)},
\]
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for all $r \leq R$, by the definition of $R$. This proves the desired inequality (7). Now let us prove that $R$ is an optimal number. We consider the function

$$g(z) = f(z) = \frac{z}{(1-z)^2} = \sum_{n=1}^{\infty} nz^n, \quad z \in \mathbb{D}.$$ 

Then it is easy to show that

$$\text{dist}(f(0), \partial \Omega) = \frac{1}{4} \quad \text{and} \quad \sum_{n=1}^{\infty} |b_n| \varphi_n(r) = \sum_{n=1}^{\infty} n \varphi_n(r).$$

Now it is easy to see that number is $> \frac{1}{4}$ when $r > R$. The proof of the theorem is complete. \qed

**Remark 2.** It is a simple exercise to see that if $\varphi_n(r) = r^n (n \geq 1)$, then Theorem 3 yields the result of Abu-Muhanna [1, Theorem 1] with $R = 3 - \sqrt{8}$.

The next lemma will be used to prove Theorems 4 and 5.

**Lemma E.** [12, p. 195-196] Let $\psi$ be an analytic univalent map from $\mathbb{D}$ onto a convex domain $\Omega := \psi(\mathbb{D})$ and $g(z) = \sum_{n=0}^{\infty} b_n z^n \prec \psi(z)$. Then

$$\frac{1}{2} |\psi'(0)| \leq \text{dist}(\psi(0), \partial \Omega) \leq |\psi'(0)|, \quad \text{and} \quad |b_n| \leq |\psi'(0)| \leq 2 \text{ dist}(\psi(0), \partial \Omega).$$

**Theorem 4.** Suppose that $g(z) = \sum_{n=0}^{\infty} b_n z^n \in \mathcal{S}(f)$ and $f(z) = \sum_{n=0}^{\infty} a_n z^n$ is univalent and convex in $\mathbb{D}$. If $\{\varphi_n(r)\}_{n=0}^{\infty} \in F$ satisfies the inequality

$$1 \geq 2 \Phi_1(r),$$

where $\Phi_1(r) = \sum_{n=1}^{\infty} \varphi_n(r)$, then the following sharp inequality holds:

$$\sum_{n=1}^{\infty} |b_n| \varphi_n(r) \leq \text{dist}(f(0), \partial \Omega) \quad \text{for all} \quad r \leq R,$$

where $R$ is the minimal positive root of the equation $1 = 2 \Phi_1(x)$. In the case when $1 < 2 \Phi_1(x)$ in some interval $(R, R + \epsilon)$, the number $R$ cannot be improved.

**Proof.** The proof follows if we use the method of proof of Theorem 3 and use Lemma E in place of by Lemma D. Sharpness follows by considering the following function

$$g(z) = f(z) = \frac{1}{1-z} = \sum_{n=0}^{\infty} z^n \text{ for } z \in \mathbb{D},$$

so that

$$\text{dist}(f(0), \partial \Omega) = \frac{1}{2} \quad \text{and} \quad \sum_{n=1}^{\infty} |b_n| \varphi_n(r) = \sum_{n=1}^{\infty} \varphi_n(r).$$

Now it is easy to see that number is $> \frac{1}{2}$ when $r > R$. The proof of the theorem is complete. \qed
Remark 3. It is a simple exercise to see that if $\varphi_n(r) = r^n \ (n \geq 1)$, then Theorem 4 yields the remark of Abu-Muhanna [1, Remark 1] with $R = 1/3$.

Theorem 5. Suppose that $f(z) = h(z) + g(z) = \sum_{n=0}^{\infty} a_n z^n + \sum_{n=1}^{\infty} b_n z^n$ is harmonic mapping of the disk $\mathbb{D}$ such that $|g'(z)| \leq k|h'(z)|$ in $\mathbb{D}$ and for some $k \in [0, 1]$ and $h \prec \psi$, where $\psi$ is univalent and convex in $\mathbb{D}$. Assume that $\{\varphi_n(r)\}_{n=0}^{\infty}$ belongs to $\mathcal{F}_{\text{dec}}$ and $\Phi_1(r) = \sum_{n=1}^{\infty} \varphi_n(r)$. If

$$1 > 2(1 + k)\Phi_1(r),$$

then the following sharp inequality holds:

$$\sum_{n=1}^{\infty} |a_n| \varphi_n(r) + \sum_{n=1}^{\infty} |b_n| \varphi_n(r) \leq \text{dist}(\psi(0), \partial \psi(\mathbb{D})) \Phi_1(r) \leq \text{dist}(\psi(0), \partial \psi(\mathbb{D})), \text{ by Eqn. (8)},$$

where $R$ is the minimal positive root of the equation $1 = 2(1 + k)\Phi_1(x)$. In the case when $1 < 2(1 + k)\Phi_1(x)$ in some interval $(R, R + \epsilon)$, the number $R$ cannot be improved.

Proof. By assumption $h \prec \psi$ and $\psi(\mathbb{D})$ is a convex domain. Then, by Lemma E, we have $|a_n| \leq 2 \text{dist}(\psi(0), \partial \psi(\mathbb{D}))$.

Consequently,

$$\sum_{n=1}^{\infty} |a_n| \varphi_n(r) \leq 2 \text{dist}(\psi(0), \partial \psi(\mathbb{D})) \Phi_1(r).$$

By assumption $|g'(z)| \leq k|h'(z)|$ in $\mathbb{D}$, where $k \in [0, 1]$ and so, by Lemma C and the classical Schwarz inequality, it follows that

$$\sum_{n=1}^{\infty} |b_n| \varphi_n(r) \leq k \left( \sum_{n=1}^{\infty} |a_n|^2 \varphi_n(r) \right)^{1/2} \left( \sum_{n=1}^{\infty} \varphi_n(r) \right)^{1/2} \leq 2k \text{dist}(\psi(0), \partial \psi(\mathbb{D})) \Phi_1(r).$$

Thus, we have

$$\sum_{n=1}^{\infty} |a_n| \varphi_n(r) + \sum_{n=1}^{\infty} |b_n| \varphi_n(r) \leq 2(1 + k)\text{dist}(\psi(0), \partial \psi(\mathbb{D})) \Phi_1(r) \leq \text{dist}(\psi(0), \partial \psi(\mathbb{D})), \text{ by Eqn. (8)},$$

for all $r \leq R$, by the definition of $R$. This proves the desired inequality (9). Now let us prove that $R$ is an optimal number. We consider the function

$$\psi(z) = h(z) = \frac{1}{1 - z} = \sum_{n=0}^{\infty} z^n, \quad z \in \mathbb{D}$$
and \(g'(z) = \lambda kh'(z)\), where \(|\lambda| = 1\). Then it is easy to see that

\[
\text{dist}(\psi(0), \partial \psi(D)) = \frac{1}{2} \quad \text{and} \quad g(z) = k\lambda \sum_{n=1}^{\infty} z^n,
\]

so that

\[
\sum_{n=1}^{\infty} |a_n|\varphi_n(r) + \sum_{n=1}^{\infty} |b_n|\varphi_n(r) = (1 + k) \sum_{n=1}^{\infty} \varphi_n(r).
\]

Now it is easy to see that this number is \(> \frac{1}{2}\) when \(r > R\). The proof of the theorem is complete.

Example 1. Theorem 5 for the case of \(\varphi_n(r) = r^n (n \geq 1)\), gives the following result which was originally obtained at first in [21, Theorem 1]:

\[
\sum_{n=1}^{\infty} |a_n|r^n + \sum_{n=1}^{\infty} |b_n|r^n \leq \text{dist}(\psi(0), \partial \psi(D)) \quad \text{for} \quad r \leq \frac{1}{3 + 2k}.
\]

The constant \(\frac{1}{3+2k}\) is sharp.

Acknowledgment. I would like to thank my supervisor Prof. S. Ponnusamy for his support during the course of this work, fruitful discussions and valuable comments on this manuscript.

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