Interval Estimation of a $P(X_1 < X_2)$ Model for Variables having General Inverse Exponential Form Distributions with Unknown Parameters

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Abstract: In this article the interval estimation of a $P(X_1 < X_2)$ model is discussed when $X_1$ and $X_2$ are non-negative independent random variables, having general inverse exponential form distributions with different unknown parameters. Different interval estimators are derived, by applying different approaches. A simulation study is performed to compare the estimators obtained. The comparison is carried out on basis of average length, average coverage, and tail errors. The results are illustrated, using inverse Weibull distribution as an example of the general inverse exponential form distribution.

Keywords: General Inverse Exponential Form Distribution, Coverage Probability, Bootstrap Confidence Interval, Generalized Pivotal Quantity

1. Introduction

There has been continuous interest in the problem of estimating the stress–strength reliability, $R = P(X_1 < X_2)$, where $X_1$ and $X_2$ are independent random variables. The parameter $R$ is referred to as the reliability parameter. This problem arises in the classical stress–strength reliability, where the random strength $X_2$ of a component exceeds the random stress $X_1$ to which the component is subjected. If $X_1 \geq X_2$, then the component fails. Kotz et al. [1] have surveyed most of the theoretical and the practical results on the theory and applications of the stress-strength reliability problem up to year 2003. However, after year 2003 several authors have considered the stress-strength reliability problem by different approaches for example, Al-Mutairi et al. [2], Amiri et al. [3], Rezaei et al. [4], among others.

Recently Mokhlis et al. [5] have obtained the point and the interval estimation of $R = P(X_1 < X_2)$ by different methods, under the assumption that $X_1$ and $X_2$ are non-negative independent and continuous random variables, having the general inverse exponential form with the cumulative distribution functions (CDFs) and probability density functions (PDFs) given respectively by

$$F_{X_i}(x; \eta_i) = \exp(-\eta_i g(x;c)) \quad \text{and} \quad f_{X_i}(x; \eta_i) = -\eta_i g'(x;c) \exp(-\eta_i g(x;c)); i = 1, 2.$$  \hspace{1cm} (1)

where $c$ is a common known parameter, $\eta_i \in \zeta_i$; $i = 1, 2$, are unknown parameters, $\zeta_i$ is the parametric space, $g(x;c)$ is a continuous, monotone decreasing, differentiable function, such that, $g(x;c) \to \infty$ as $x \to 0$ and $g(x;c) \to 0$ as $x \to \infty$. They proved that the reliability function, $R$, is given by

$$R = \frac{n_{2} \eta_{2}}{\eta_{1} + n_{2}}.$$  \hspace{1cm} (2)

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The present paper, presents estimation of $R$, when $c$ is unknown, and $\eta_i(b_1,c)$ is a function of the unknown parameters $b_1$ and $c; i = 1, 2$. The reliability, $R$, takes the form

$$R = \frac{n_{2}(b_2,c)}{n_{1}(b_2,c) + n_{2}(b_2,c)}.$$  \hspace{1cm} (2)

The present paper, presents estimation of $R$, when $X_1$ and $X_2$ follow the general inverse exponential form distributions with CDFs and PDFs given respectively by
where \( \eta_i = \eta_i(b_i, c_i) \) is a differentiable function in two unknown parameters \( b_i \) and \( c_i \), where \( c_i \in C_i \), \( b_i \in B_i \), and \( C_i \) and \( B_i \) are the parametric spaces of \( c_i \) and \( b_i \), respectively. The function \( g(x; c_i) \) is a continuous, monotone decreasing, differentiable function, such that, \( g(x; c_i) \rightarrow \infty \) as \( x \rightarrow 0 \) and \( g(x; c_i) \rightarrow 0 \) as \( x \rightarrow \infty \). \( g'(x; c_i) \) is the first derivative of \( g(x; c_i) \) w.r.t \( x \). Using (3), the reliability is given by

\[
R = \int_{0}^{\infty} -\eta_2(b_2, c_2) g'(x; c_2) \exp[-\eta_1(b_1, c_1)g(x; c_1)] - \eta_2(b_2, c_2)g(z; c_2) \, dz
\]

The above integral can be evaluated numerically. If \( c_1 = c_2 = c \), then the reliability can be expressed as in (2) as Mokhlis et al. [6].

Different interval estimators are constructed by applying different approaches. (i) An approximate confidence interval for \( R \) is constructed using the maximum likelihood estimator (MLE) of \( R \). (ii) A generalized confidence interval is obtained, using the generalized variable (GV) approach. (iii) Two bootstrap confidence intervals (percentile and t) are also presented. (iv) Two Bayesian credible intervals of \( R \) are obtained, using Markov chain Monte Carlo (MCMC) method, with different priors. The different interval estimators obtained are illustrated using inverse Weibull distributions as examples of the underlying distributions. A comparison is performed, by means of simulation among the estimators obtained on the basis of average length, average coverage, and tail errors.

This paper is organized as follows: the approximate confidence interval (ACI) of \( R \) is obtained, in Section 2. In Section 3, using generalized variable approach, the generalized confidence interval of \( R \) is derived. The bootstrap intervals are obtained, using percentile and t-bootstrap methods, in Section 4. In Section 5, using MCMC method, two Bayesian credible intervals of \( R \) are presented by applying different sets of priors. In Section 6, taking the inverse Weibull distribution as an example of the underlying distributions, the results obtained are illustrated and a numerical comparison of the interval estimators is performed.

### 2. Approximate Confidence Interval (ACI) of \( R \)

Let \( X_i = \{X_{i1}, X_{i2}, ..., X_{in_i}\}; i = 1, 2 \), be two independent random samples from populations with general inverse exponential form distributions given by (3). The likelihood function is

\[
L(x_1, x_2|\eta_1, \eta_2, c_1, c_2) = \exp \left[ \sum_{i=1}^{2} n_i \ln \eta_i (b_i, c_i) + \sum_{i=1}^{n} \ln \left( -g'(x_{ij}; c_i) \right) - \sum_{i=1}^{2} n_i (b_i, c_i) \sum_{j=1}^{n_i} g(x_{ij}; c_i) \right],
\]

where \( x_{ij} \) is the jth observation in the sample \( X_{i} \); \( j = 1, ..., n_i \). For simplicity write \( \eta_i = \eta_i(b_i, c_i) ; i = 1,2 \), the log-likelihood function is

\[
\ln L = \ln L(x_1, x_2|\eta_1, \eta_2, c_1, c_2) = \sum_{i=1}^{2} n_i \ln \eta_i + \sum_{i=1}^{n} \ln \left( -g'(x_{ij}; c_i) \right) - \sum_{i=1}^{2} n_i \sum_{j=1}^{n_i} g(x_{ij}; c_i).
\]

To derive the MLEs \( \hat{\eta}_i \), \( \hat{c}_i \), and \( \hat{b}_i \) of \( \eta_i, c_i \), and \( b_i \); \( i = 1, 2 \), respectively, solve the following system of equations for \( \hat{\eta}_i, \hat{c}_i \), and \( \hat{b}_i \):

\[
\frac{\partial \ln L}{\partial b_i} \bigg|_{\eta=\hat{\eta}_i, c=\hat{c}_i, b=\hat{b}_i} = \frac{n_i}{\eta_i} - \hat{\eta}_i g(x_{ij}; c_i) \left( \frac{\partial \eta_i}{\partial b_i} \right)_{\eta=\hat{\eta}_i, c=\hat{c}_i, b=\hat{b}_i} = 0, \quad \text{and}
\]

\[
\frac{\partial \ln L}{\partial c_i} \bigg|_{\eta=\hat{\eta}_i, c=\hat{c}_i, b=\hat{b}_i} = \frac{n_i}{\eta_i} + \hat{\eta}_i \frac{\partial \ln \left( -g'(x_{ij}; c_i) \right)}{\partial c_i} - \hat{\eta}_i \frac{\partial g(x_{ij}; c_i)}{\partial c_i} \left( \frac{\partial \eta_i}{\partial c_i} \right)_{\eta=\hat{\eta}_i, c=\hat{c}_i, b=\hat{b}_i} = 0.
\]

As known \( \frac{\partial \eta_i}{\partial b_i} \neq 0 \), from (7) the MLE, \( \hat{\eta}_i \) of \( \eta_i \); \( i = 1, 2 \), is expressed as

\[
\hat{\eta}_i = -\frac{n_i}{\sum_{j=1}^{n_i} g(x_{ij}; c_i)}; \quad i = 1, 2.
\]

The MLE, \( \hat{c}_i \) of \( c_i \); \( i = 1, 2 \), can be obtained by substituting (9) in (8), and then solving (8) numerically. Finally, the generalization of variable approach, the generalized confidence interval of \( R \) is derived. The bootstrap intervals are obtained, using percentile and t-bootstrap methods, in Section 4. In Section 5, using MCMC method, two Bayesian credible intervals of \( R \) are presented by applying different sets of priors. In Section 6, taking the inverse Weibull distribution as an example of the underlying distributions, the results obtained are illustrated and a numerical comparison of the interval estimators is performed.
information matrix, \( T \), of \( \theta = (\theta_1, \theta_2, \theta_3, \theta_4) = (c_1, c_2, b_1, b_2) \), \( S^2 \) is the transpose of matrix \( S \), where \( S = \left[ \frac{\partial R}{\partial \theta} \right] \), and \( T = -E \left[ \frac{\partial^2 \ln L}{\partial \theta \partial \theta^T} \right] \), with

\[
\frac{\partial^2 \ln L}{\partial c_i^2} = -n \left( \frac{\partial \ln g(x_i; c_i)}{\partial c_i} \right)^2 + \frac{n}{2} \frac{\partial^2}{\partial c_i^2} \ln \left[ \frac{g(x_i; c_i)}{g(x_i; c_i)} \right] - \frac{n}{2} \frac{\partial^2}{\partial c_i^2} g(x_i; c_i),
\]

Clearly, the explicit expression of \( \sigma^2_R \) depends on \( \eta_i \), \( g(x_i; c_i) \), and \( g(x_i; c_i) \); \( i = 1, ..., n_i, i = 1, 2 \). The \((1 - \alpha)100\%\) ACI for \( R \) is \((\bar{R} \pm z_{(1-\alpha/2)} \sigma_{\bar{R}})\), where \( \sigma^2_R = \sum_{i=1}^{n} \left( \frac{\partial \eta_i}{\partial c_i} \right)^2 \); \( i = 1, 2 \).

3. Generalized Confidence Interval (GCI) of \( R \)

Let \( \bar{X}_i = (X_{i1}, X_{i2}, ..., X_{iN_i}) \); \( i = 1, 2 \), be two independent random samples from populations with distributions given in (3), having unknown parameters \( c_i \) and \( b_i; i = 1, 2 \). It is known that, a GPQ of any parameter is a function of observed statistics and random variables whose distribution is free of unknown parameters. For constructing the GCI, applying a useful feature of the GV approach, which states that if \( G_{b_i} \) and \( G_{c_i} \) are GPQs of \( b_i \) and \( c_i \), then \( \eta_i(G_{b_i}, G_{c_i}) \) is a GPQ of \( \eta_i; i = 1, 2 \). This feature enables us obtaining a GPQ for \( R \), given as \( G_R = R(G_{b_1}, G_{b_2}, G_{c_1}, G_{c_2})\) by replacing the parameters in (4) with their GPQs, then using \( G_R \) in constructing confidence interval for \( R \). The \((1 - \alpha)100\%\) GCI for \( R \) is obtained as \( G_{R(a/2)} \), \( G_{R(1-a/2)} \), \( G_{R(a/2)} \) and \( G_{R(1-a/2)} \) are the \((a/2)\)th and \((1 - a/2)\)th quantiles of \( R \). 

4. Bootstrap Confidence Interval (boot) of \( R \)

There are several ways to construct bootstrap confidence intervals. Clearly, the percentile and t-bootstrap confidence intervals are commonly used for the reliability, (see, Efron [7]). Algorithm 1 is applied for constructing the bootstrap confidence intervals.

Algorithm 1.
1. From the original data \( x_{i\mu} \), compute the MLEs, \( \hat{c}_i \) and \( \hat{\eta}_i \) of \( c_i \) and \( \eta_i; i = 1, 2 \), respectively, and the MLE, \( \bar{R} \) of \( R \).
2. Resample two independent random samples \( X_{i\mu} \); \( i = 1, 2 \), from \( x_{i\mu} \); \( i = 1, 2 \), respectively; compute the MLEs, \( \hat{c}_{i*} \), \( \hat{\eta}_{i*} \), and \( \hat{R}^* \) of \( c_i, \eta_i, \) and \( R \), respectively.
3. Repeat step 2, \( N \) times to derive \( \{R_{j}; j = 1, ..., N\} \), and order \( R_{j}; j = 1, ..., N \), like that \( R_{(1)} \leq ... \leq R_{(N)} \).
4. Construct the percentile and t-bootstrap confidence intervals of \( R \), as follows:
   a. Percentile bootstrap (P-boot)
      The \((1 - \alpha)100\%\) P-boot for \( R \) is given by \((\bar{R}_{(1/a)}; \bar{R}_{(1-a/2)})\), where \( \bar{R}_{(a/2)} \) and \( \bar{R}_{(1-a/2)} \) are the \((a/2)\)th and \((1 - a/2)\)th quantiles of \( R \), respectively.
   b. T-bootstrap (T-boot)
      The \((1 - \alpha)100\%\) T-boot for \( R \) is expressed as \((\bar{R} - t_{(1-a/2)}S^*; \bar{R} - t_{(a/2)}S^*)\), where \( S^* \) is the sample standard deviation of \( \{R_{j}; j = 1, ..., N\} \) and \( t_{(a)} \) is the \((a)\)th quantile of \( \{\bar{R} - \bar{R}; j = 1, ..., N\} \).

5. Bayesian Credible Interval (BCI) of \( R \)

a. Gamma priors (G-BCI)
   Let \( X_{i\mu}; i = 1, 2 \) be two independent random samples from general inverse exponential form distributions in (3) with unknown parameters \( b_i \) and \( c_i \). Consider, \( \eta_i = \eta_i(b_i, c_i) \), as a single parameter, assume that the prior distributions of \( \eta_i; i = 1, 2 \) are independent with pdfs
   \[
   \pi(\eta_i) = \frac{\eta_i^{-1} \cdot e^{-\eta_i}}{\Gamma(\alpha_i)} \cdot \eta_i^{-\alpha_i} \cdot e^{-\beta_i} \cdot i = 1, 2, \eta_i, \alpha_i, \beta_i > 0. \tag{11}
   \]

   Moreover assume that, \( c_i; i = 1, 2 \) have independent gamma priors with pdfs
   \[
   \pi(c_i) = \frac{\lambda_i^\mu \cdot e^{-\lambda_i c_i}}{\Gamma(\mu)} \cdot \lambda_i^{\mu} \cdot c_i^{-\mu-1} \cdot e^{-\lambda_i c_i} \cdot i = 1, 2, c_i, \mu, \lambda_i > 0. \tag{12}
   \]
   The joint posterior distribution of \( \eta_i \) and \( c_i; i = 1, 2 \) is as follows

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Asgharzadeh et al. [8].

It is observed that (13) cannot be obtained in a closed form. The BCI can be computed by a combination of Metropolis-Hastings and Gibbs sampling. Moreover the marginal posterior distribution of \( \eta_i \) is

\[
\pi_\eta (c_i | x_i, x_j) = \frac{L(x_1, x_2 | \eta_1, \eta_2, c_1, c_2) \pi_\eta (\eta_i) \pi_\eta (c_i) \pi_{\eta_2} (c_2)}{\int \int \int \int L(x_1, x_2 | \eta_1, \eta_2, c_1, c_2) \pi_\eta (\eta_i) \pi_\eta (c_i) \pi_{\eta_2} (c_2) d\eta_1 d\eta_2 d\eta_3 d\eta_4}
\]

(13)

which is Gamma\((n_i + \alpha_i), (\beta_1 + \sum_{j=1}^{n_i} g(x_j; c_i))\); \(i = 1, 2\). The marginal posterior distribution of \( c_i \) is

\[
\pi_{c_i} (c_i | x_1, x_2) = \frac{A_i^{-1} \exp \left[ (\mu_i - 1) \ln c_i - c_i \lambda_i + \sum_{j=1}^{n_i} \ln (-g'(x_j; c_i)) - (n_i + \alpha_i) \ln (\beta_i + \sum_{j=1}^{n_i} g(x_j; c_i)) \right] d\eta_i; i = 1, 2.
\]

(14)

where \( A_i = \exp \left[ (\mu_i - 1) \ln c_i - c_i \lambda_i + \sum_{j=1}^{n_i} \ln (-g'(x_j; c_i)) - (n_i + \alpha_i) \ln (\beta_i + \sum_{j=1}^{n_i} g(x_j; c_i)) \right] \).

Notice that the marginal posterior distribution of \( c_i; i = 1, 2 \), is not known and so the Metropolis-Hastings with Gibbs sampling algorithm can be used to solve it as follows (see, Asgharzadeh et al. [8]).

Algorithm 2.
1. Choose a starting values \( c_i^{(0)}; i = 1, 2 \).
2. For \( j = 1 \) to \( N \) times.
3. Generate \( \eta_i^{(j)} \) from Gamma\((n_i + \alpha_i), (\beta_1 + \sum_{j=1}^{n_i} g(x_j; c_i))\); \(i = 1, 2\), respectively.
4. Generate \( c_i^{(j)} \) from (15) using the Metropolis-Hastings algorithm with the normal proposal distribution \( \rho_j \sim N(c_i^{(j-1)}, 1) \); \(i = 1, 2\).
   a. Define \( Q_i = \min \left\{ 1, \frac{n_i f_i(k_2, x_j)}{n_i f_i(k_2, x_j) + \rho_j(k_2)} \right\}; i = 1, 2\).
   b. Define \( u_i \) from Uniform \((0, 1)\). Take \( c_i^{(j)} = c_i^{(j-1)} \); otherwise.
5. Calculate the \( R^j \) using
\[
R^j = \int_0^\infty \eta_1^{(j)} g(z; c_2^{(j)}) \exp \left[ -\eta_1^{(j)} g(z; c_2^{(j)}) \right] d\eta_1^{(j)}
\]

\[
-\pi_{c_i} (c_i | x_1, x_2) = B_i^{-1} \exp \left[ \sum_{j=1}^{n_i} \ln (-g'(x_j; c_i)) - (n_i + \alpha_i) \ln (\beta_i + \sum_{j=1}^{n_i} g(x_j; c_i)) \right] d\eta_i; i = 1, 2.
\]

(16)

The joint posterior distribution of \( \eta_i \) and \( c_i; i = 1, 2 \), cannot be obtained in a closed form. The marginal posterior distribution of \( c_i \) is given by

6. Numerical Illustration

This section, presents a numerical illustration of the results obtained. The confidence intervals; ACI, GCI, P-boot, T-boot, G-BCI, and M-BCI for R with some general inverse exponential form distributions are compared. 1000 samples of sample sizes \((n_1, n_2) = (10, 10) \) and \((30, 30) \) from the underlying distributions of \( X_1 \) and \( X_2 \), with unknown
parameters are generated. Taking $\alpha = 0.05$, average length, average coverage probability, left and right tail errors of the $(1 - \alpha)100\%$ confidence intervals are calculated. The parameter values that produce the values of $R$ to be approximately $0.6, 0.7, 0.8, 0.9, 0.95, 0.97,$ and $0.99$ are selected.

The inverse Weibull distribution is chosen as an example of the general inverse exponential form. The inverse Weibull distribution is flexible and includes a variety of distributions. For the inverse Weibull distributions, the CDFs with $\eta (b_i, c_i) = \frac{1}{b_i}$, $g(x; c_i) = \frac{1}{x^{c_i}}$ and $g'(x; c_i) = -\frac{c_i}{x^{c_i+1}}$; $i = 1, 2$, are

$$F_x(x; b_i, c_i) = \exp \left[ -\left( \frac{1}{b_i} \right)^{c_i} \right].$$

(17)

and the reliability is obtained by substituting these values in (4) as

$$R = \int_0^{\infty} c_i b_i^{2c_i} x^{-c_i-1} \exp \left[ -x^{-c_i} - b_i^{2c_i} x^{-c_i} \right] dz.$$  

(18)

As noticed from (9), $\hat{\eta}_i = \frac{1}{b_i} = \frac{n_i}{\sum x_i^{c_i+1}}$ and hence $\hat{b}_i = \left( \frac{\sum x_i^{c_i+1}}{n_i} \right)^{1/c_i}$. Using the Newton–Raphson iterative method $\hat{c}_i$ is obtained from (8) after applying the suitable substitutions.

(i) The $(1 - \alpha)100\%$ CI of $R$ can be obtained as $\hat{R} \pm z_{(1-\alpha/2)} \hat{\sigma}_R$, where $\hat{R}$ is obtained from (10), and $\hat{\sigma}_R^2 = S^{c_i-1} i |b_i|=b_i,c_i|\xi_i|=\eta_i$ is obtained from the following equations.

$$\frac{\partial^2 \ln L}{\partial c_i^2} = \frac{n_i}{c_i} - \frac{1}{b_i^{c_i+1}} \sum_{j=1}^{n_i} \ln \left( b_i x_j \right)^{c_i},$$

$$\frac{\partial^2 \ln L}{\partial b_i \partial c_i} = \frac{n_i}{b_i} - \frac{1}{b_i^{c_i+2}} \sum_{j=1}^{n_i} \frac{c_i}{x_j^{c_i+1}} - \frac{c_i}{b_i^{c_i+1}} \sum_{j=1}^{n_i} \ln \left( b_i x_j \right)$$

$$\frac{\partial^2 \ln L}{\partial b_i^2} = \frac{\partial^2 \ln L}{\partial c_i^2} - \frac{\partial^2 \ln L}{\partial b_i^2} = \frac{n_i}{b_i} - \frac{1}{b_i^{c_i+2}} \sum_{j=1}^{n_i} \frac{c_i}{x_j^{c_i+1}} - \frac{c_i}{b_i^{c_i+1}} \sum_{j=1}^{n_i} \ln \left( b_i x_j \right).$$

(ii) As mentioned in Section 3, taking $G_{c_i} = \left( \hat{c}_i \right)$, $\hat{c}_i = \hat{c}_i$, and $G_{b_i} = \left( \hat{b}_i \right)^{c_i} \hat{b}_i = \left( \frac{1}{b_i} \right)^{c_i} \hat{b}_i$; $i = 1, 2$, and $\hat{c}_i$ and $\hat{b}_i$ are the observed values of $\hat{c}_i$ and $\hat{b}_i$. The distributions of $\hat{c}_i^*$ and $\hat{b}_i^*$ are $\Gamma(c_i, b_i)$ and $\Gamma(b_i, c_i)$ respectively, based on two independent random samples from standard inverse exponential distributions (see, Krishnamoorthy et al. [10]). Then consequently the GPQs of $\eta_i$ are $G_{\eta_i} = \left( \frac{1}{\hat{b}_i} \right)^{\hat{c}_i}$. The GCI is obtained by computing the $G_R = \Gamma(G_{c_1}, G_{c_2}, G_{c_1}, G_{c_2})$. The next Algorithm 3 is used to estimate the GCI of $R$ (see, Krishnamoorthy and Lin [11]).

**Algorithm 3.**

1. Let $X_i$: $i = 1, 2, 2$ be two independent random samples from (17), respectively. Compute the $\hat{c}_i$ and $\hat{b}_i$ of $c_1$ and $b_1$; $i = 1, 2$, respectively.

2. Generate two independent random samples $X_i^*$; $i = 1, 2$, from standard inverse exponential distributions.

3. Compute the $G_{c_1}, G_{b_1}, G_{\eta_i},$ and hence $G_R$; $i = 1, 2$.

4. Repeat the steps 2-3, $N$ times to obtain a set of values of $G_R$, say $\{G_{R_j}: j = 1, ..., N\}$.

5. Order $G_{R_j}; j = 1, ..., N$, ascending to obtain $G_{R(j)}^{(1)} \leq \cdots \leq G_{R(j)}^{(N)}$.

6. Construct the $(1 - \alpha)100\%$ P-boot and T-boot of $R$ as

$$(G_{R(i/2)^{P}} G_{R(1-\alpha/2)^{T}}).$$

(iii) The $(1 - \alpha)100\%$ P-boot and T-boot of $R$ are obtained, using Algorithm 1.

| $R$ | Average length | boot | T | Gamma | Mixed |
|-----|----------------|------|---|-------|-------|
| $n=10$ | $n=30$ | $n=10$ | $n=30$ |
| ACI | GCI | P | T | Gamma | Mixed | ACI | GCI | P | T | Gamma | Mixed |
| $c_1 = 2, c_2 = 5, b_1 = 1.2, b_2 = 0.9$ | 0.6207 | 0.4605 | 0.4512 | 0.4812 | 0.4812 | 0.3909 | 0.3947 | 0.2759 | 0.2915 | 0.2814 | 0.2814 | 0.2517 | 0.2530 |
| $c_1 = 3, c_2 = 4, b_1 = 1.2, b_2 = 0.9$ | 0.7050 | 0.4161 | 0.4230 | 0.4180 | 0.4180 | 0.3931 | 0.3875 | 0.2525 | 0.2915 | 0.2535 | 0.2535 | 0.2439 | 0.2408 |
| $c_1 = 2, c_2 = 3, b_1 = 2, b_2 = 1$ | 0.8068 | 0.3429 | 0.3748 | 0.3042 | 0.3042 | 0.3741 | 0.3591 | 0.2106 | 0.2594 | 0.2066 | 0.2066 | 0.2206 | 0.2131 |
| $c_1 = 2, c_2 = 3, b_1 = 2.8, b_2 = 0.9$ | 0.9127 | 0.2223 | 0.2822 | 0.1767 | 0.1767 | 0.3199 | 0.2803 | 0.1341 | 0.1666 | 0.1280 | 0.1280 | 0.1633 | 0.1475 |
| $c_1 = 2, c_2 = 3, b_1 = 4.8, b_2 = 0.9$ | 0.9557 | 0.1455 | 0.2128 | 0.1118 | 0.1118 | 0.2857 | 0.2366 | 0.0865 | 0.1170 | 0.0828 | 0.0828 | 0.1264 | 0.1105 |
| $c_1 = 2, c_2 = 3, b_1 = 6.6, b_2 = 0.9$ | 0.9799 | 0.0855 | 0.1511 | 0.0628 | 0.0628 | 0.2489 | 0.1992 | 0.0498 | 0.0745 | 0.0482 | 0.0482 | 0.0919 | 0.0764 |
| $c_1 = 2, c_2 = 3, b_1 = 9.2, b_2 = 0.9$ | 0.9910 | 0.0513 | 0.1088 | 0.0422 | 0.0422 | 0.2175 | 0.1699 | 0.0274 | 0.0451 | 0.0277 | 0.0277 | 0.0663 | 0.0529 |

Table 1. Average length of the confidence intervals of $R$ for inverse Weibull distributions, $\eta_i = \frac{1}{b_i}$; $i = 1, 2$. 

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| R                          | Average coverage probability |
|---------------------------|-----------------------------|
| n=10                      |                             |
| ACI | GCI | boot | BCI | ACI | GCI | boot | BCI | ACI | GCI | boot | BCI |                |
| P   | T   | Gamma | Mixed | P   | T   | Gamma | Mixed | P   | T   | Gamma | Mixed |                |
| 0.6207 | 0.887 | 0.951 | 0.884 | 0.838 | 0.985 | 0.973 | 0.907 | 0.944 | 0.904 | 0.866 | 0.956 | 0.945 |
| 0.7050 | 0.865 | 0.942 | 0.867 | 0.811 | 0.954 | 0.959 | 0.893 | 0.941 | 0.879 | 0.853 | 0.924 | 0.933 |
| 0.8068 | 0.835 | 0.94 | 0.793 | 0.746 | 0.874 | 0.963 | 0.874 | 0.944 | 0.864 | 0.82 | 0.914 | 0.947 |
| 0.9127 | 0.794 | 0.923 | 0.732 | 0.666 | 0.741 | 0.915 | 0.871 | 0.953 | 0.846 | 0.801 | 0.865 | 0.937 |

| n=30       |                             |
| ACI | GCI | boot | BCI | ACI | GCI | boot | BCI | ACI | GCI | boot | BCI |                |
| P   | T   | Gamma | Mixed | P   | T   | Gamma | Mixed | P   | T   | Gamma | Mixed |                |
| 0.6207 | 0.887 | 0.951 | 0.884 | 0.838 | 0.985 | 0.973 | 0.907 | 0.944 | 0.904 | 0.866 | 0.956 | 0.945 |
| 0.7050 | 0.865 | 0.942 | 0.867 | 0.811 | 0.954 | 0.959 | 0.893 | 0.941 | 0.879 | 0.853 | 0.924 | 0.933 |
| 0.8068 | 0.835 | 0.94 | 0.793 | 0.746 | 0.874 | 0.963 | 0.874 | 0.944 | 0.864 | 0.82 | 0.914 | 0.947 |
| 0.9127 | 0.794 | 0.923 | 0.732 | 0.666 | 0.741 | 0.915 | 0.871 | 0.953 | 0.846 | 0.801 | 0.865 | 0.937 |

**Table 3.** Left tail error of the confidence intervals of R for inverse Weibull distributions, $\eta = \frac{1}{n_i}; i = 1, 2.$

| R                          | Left tail error |
|---------------------------|----------------|
| n=10                      |                |
| ACI | GCI | boot | BCI | ACI | GCI | boot | BCI | ACI | GCI | boot | BCI |                |
| P   | T   | Gamma | Mixed | P   | T   | Gamma | Mixed | P   | T   | Gamma | Mixed |                |
| 0.6207 | 0.2219 | 0.2540 | 0.2435 | 0.2210 | 0.2373 | 0.1956 | 0.1318 | 0.1536 | 0.1270 | 0.1421 | 0.1416 | 0.1222 |
| 0.7050 | 0.1961 | 0.2555 | 0.2145 | 0.1796 | 0.2930 | 0.2282 | 0.1238 | 0.1550 | 0.1212 | 0.1274 | 0.1670 | 0.1331 |
| 0.8068 | 0.1658 | 0.2562 | 0.1552 | 0.1376 | 0.3083 | 0.2530 | 0.1035 | 0.1475 | 0.0999 | 0.1031 | 0.1631 | 0.1359 |
| 0.9127 | 0.1158 | 0.2240 | 0.1013 | 0.0846 | 0.3016 | 0.2342 | 0.0677 | 0.1152 | 0.0689 | 0.0604 | 0.1384 | 0.1101 |
| 0.9557 | 0.0783 | 0.1816 | 0.0712 | 0.0517 | 0.2821 | 0.2156 | 0.0459 | 0.0883 | 0.0495 | 0.0384 | 0.1156 | 0.0912 |
| 0.9799 | 0.0475 | 0.1363 | 0.0440 | 0.0284 | 0.2516 | 0.1916 | 0.0273 | 0.0604 | 0.0322 | 0.0209 | 0.0885 | 0.0685 |
| 0.9910 | 0.0297 | 0.1019 | 0.0336 | 0.0168 | 0.2224 | 0.1685 | 0.0154 | 0.0384 | 0.0200 | 0.0110 | 0.0656 | 0.0496 |

| n=30       |                |
| ACI | GCI | boot | BCI | ACI | GCI | boot | BCI | ACI | GCI | boot | BCI |                |
| P   | T   | Gamma | Mixed | P   | T   | Gamma | Mixed | P   | T   | Gamma | Mixed |                |
| 0.6207 | 0.2385 | 0.1972 | 0.2376 | 0.2601 | 0.1535 | 0.1991 | 0.1441 | 0.1379 | 0.1544 | 0.1393 | 0.1100 | 0.1307 |
| 0.7050 | 0.2200 | 0.1674 | 0.2035 | 0.2383 | 0.1000 | 0.1592 | 0.1286 | 0.1164 | 0.1323 | 0.1261 | 0.0769 | 0.1076 |
| 0.8068 | 0.1771 | 0.1186 | 0.1489 | 0.1665 | 0.0657 | 0.1061 | 0.1071 | 0.0918 | 0.1066 | 0.1035 | 0.0575 | 0.0771 |
| 0.9127 | 0.1005 | 0.0582 | 0.0753 | 0.0920 | 0.0183 | 0.0460 | 0.0664 | 0.0513 | 0.0591 | 0.0676 | 0.0248 | 0.0374 |
| 0.9557 | 0.0672 | 0.0312 | 0.0406 | 0.0601 | 0.0036 | 0.0210 | 0.0406 | 0.0287 | 0.0332 | 0.0443 | 0.0108 | 0.0192 |
| 0.9799 | 0.0379 | 0.0147 | 0.0188 | 0.0344 | 0.0026 | 0.0076 | 0.0224 | 0.0141 | 0.0160 | 0.0273 | 0.0034 | 0.0079 |
| 0.9910 | 0.0215 | 0.0068 | 0.0086 | 0.0253 | 0.0048 | 0.0013 | 0.0120 | 0.0067 | 0.0076 | 0.0166 | 0.0006 | 0.0032 |

**Table 4.** Right tail error of the confidence intervals of R for inverse Weibull distributions, $\eta = \frac{1}{n_i}; i = 1, 2.$
(iv) Finally, the $(1 - \alpha)100\%$ G-BCI and $(1 - \alpha)100\%$ M-BCI of $R$, are obtained using Algorithm 2. The hyper-parameters of priors for G-BCI and M-BCI are chosen on basis of the same mean but different variances as the follow:

G-BCI: let $(\alpha_1, \beta_1) = (2, 4), (\alpha_2, \beta_2) = (1, 2), (\mu_1, \lambda_1) = \left(\frac{3}{2}, 3\right)$, and $(\mu_2, \lambda_2) = \left(\frac{1}{2}, 1\right)$.

M-BCI: let $(\alpha_1, \beta_1) = (2, 4)$, and $(\alpha_2, \beta_2) = (1, 2)$.

The comparisons on the basis of average length, average coverage, and left and right tail errors are introduced in Tables (1-4), respectively. From Tables 1-4, the following results are observed:

1. As expected, the average length of all confidence intervals decreases when $n$ and $R$ increase.
2. When $R = 0.8068-0.9910$, the average length of boot and the left tail error of T-boot are smallest, and the left tail error of all confidence intervals decrease when $n$ and $R$ increase.
3. The average coverage of GCI is approximately around the $(1 - \alpha)100\%$.
4. The average coverage of ACI, boot, and G-BCI decrease when $R$ increases. But the average coverage of ACI and boot increase when $n$ increases.
5. For all the confidence intervals, the right tail error is decreasing when $R$ is increasing. Moreover the right tail error of all confidence intervals except BCI are decreasing when $n$ is increasing.
6. The right tail error of BCI is the smallest.

7. Conclusion

In this article, several approaches for estimating the confidence intervals of Stress-Strength Reliability $P(X_1 < X_2)$ are provided when random variables having general inverse exponential form distributions with different unknown parameters. A simulation study is performed to compare its performance using the inverse Weibull distribution as an example of the general inverse exponential form distribution. The comparison is carried out on basis of average length, average coverage, and tail errors. The results obtained were very close to each other.

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