Time evolution of an unstable soliton solution to dust acoustic plasma with trapped electrons

Yuttakarn Rattanachai\textsuperscript{1} and Sarun Phibanchon\textsuperscript{2}

\textsuperscript{1} Faculty of Sciences and Liberal Arts, Rajamangala University of Technology Isan
744 Saranarai Road, Muang, Nakorn Ratchasima, 30000, Thailand
\textsuperscript{2} Faculty of Education, Burapha University
169 Longhardbangsaen road, Sansuk, Muang, Chonburi, 20131, Thailand
E-mail: sarunp@go.buu.ac.th

Abstract. The positive dust charged grains is considered as the immobile particles compared to ions and electrons. Using the assumption that some electrons can be trapped in the dust charge potential while others move freely in the plasma with external magnetic field. The reductive perturbation method shows that the weakly nonlinear waves equation gives a soliton solution. The spectral method is applied for the time evolution of the perturbed soliton solution to the sinusoidal function with the long-wavelength. This shows that the unstable plane soliton will be transformed into the higher dimensions.

1. Introduction
Plasma is the fourth state of the matter which provides the variety of the oscillations. It is also believed that our universe consists more than 60\% of this state.\cite{1}. However, in general, there are various of sizes and charges in plasmas as well as a neutral particles. These neutral particle can also be charged as both the negative and positive charges\cite{2} called the dusty plasmas. There are many examples of dusty plasmas which found in planetary rings, cometary comae and tails as well as in asteroid zones and mesosphere and magnetosphere\cite{2, 3}. As like the plasmas, dusty plasmas contains many modes of oscillations such as dust acoustic (DA) waves\cite{4}, where dust mass gives the inertia and the thermal pressures to the electrons and dust ion acoustic waves (DIA)\cite{5}. The dusty plasma is also found in man-made laboratories such as a dust in fusion devices or during the processes of microelectronic fabrication. We have found this phenomenon in various physical systems, this is the main reason why some recently works are dedicated to the dusty plasmas. One remarkable phenomena that can be found in the weakly nonlinear ion-acoustic waves is the soliton\cite{6}. There are many scientific works on this phenomenon\cite{7} as well as DA and DIA\cite{2, 3}. To obtain the soliton equation, the electron distribution function can be described as the Maxwellian function\cite{6, 8} or non-thermal case\cite{9, 10, 11, 12}.

Soliton is not only found in plasmas but also in others continuous media such as fiber optics\cite{13, 14}, surface and internal water waves\cite{15} or superfluid $^4$He\cite{16}. The main characters of soliton are traveling without change its structure and particle-like collisions.

The environment of the dusty plasma mainly contains the electrons, positive ions and obviously some dust particles. The interaction between charged particles and dust particles gives a charging of the dust grains. The size of dust particles is ranged from ten nanometers
to hundreds of microns. They are quite big compared to the electrons and ions. To observe the dusty acoustic waves, we will only consider the dust charged grains.

In this work we propose that some positive charged dust grains [17] with some trapped electrons in their electric potentials. This situation can be provided the soliton equation for the weakly nonlinear wave equation. The transverse perturbation with a long-wavelength to the soliton solution gives the transformation of the perturbed plane soliton to the higher dimensional soliton.

2. Dusty plasma with trapped electrons

To obtain the weakly nonlinear equation for the dusty plasma with some trapped electrons [9, 18], we will apply the reductive perturbation method on the governed equations for the dust particles. These equations are mass continuity, momentum and Poisson equations in which are written as follows

\[
\frac{\partial n}{\partial t} + \nabla \cdot (n v) = 0, \quad (1)
\]

\[
\frac{\partial v}{\partial t} + (v \cdot \nabla) v = -\nabla \phi + v \times \Omega \hat{x}, \quad (2)
\]

and

\[
\nabla^2 \phi = \mu_e n_e - \mu_i n_i - n_d, \quad (3)
\]

respectively, where \( n_i, n_e \) and \( n_d \) are ion, electron and dust particle density, respectively, \( \Omega = B \sqrt{\epsilon_0 / n_{d0} m_d} \), \( n_{d0} \) is an unperturbed dust particle density and \( m_d \) is a dust particle mass. The unperturbed number of electrons residing on the electrons and positive ions are \( \mu_e \) and \( \mu_i \), respectively. All these quantities are dimensionless, they are normalized by the following characteristic quantities: \( n_d \) by the \( n_{d0} \), \( v \) by the ion sound velocity \( (k_B T_e / m_i)^{1/2} \), \( \phi \) by \( (k_B T_e / e) \), \( t \) by the plasma frequency, \( \omega_{pi} = (4\pi e^2 n_{e0} / m_i)^{1/2} \) and \( x \) by the electron Debye length, \( \lambda_d = (k_B T_e / 4\pi e^2 n_{e0})^{1/2} \), where \( k_B \) denotes the Boltzmann’s constant. On the assumption that some electrons can be trapped into the dust charged potential, the electron distribution function can be considered as followed Schamel’s works [9, 18],

\[
n_e = e^\phi \text{erfc}(\sqrt{\phi}) + \frac{1}{\sqrt{\beta}} \left\{ e^{\beta \phi} \text{erf}(\sqrt{\beta \phi}) ; \beta \geq 0 \right\} \frac{1}{\sqrt{\pi}} W(\sqrt{-\beta \phi}) ; \beta < 0
\]

where \( W(x) \) is a Dawson function,

\[
W(x) = e^{x^2} \int_0^x e^{-t^2} dt,
\]

and \( \beta = T_{ef} / T_{et} \) denotes the ratio of temperatures between free and trapped electrons. The stretch variables is introduced as follows,

\[
\tau = \epsilon^{3/4} t, \quad \xi = \epsilon^{1/2}(x - t), \quad \chi = \epsilon^{1/2} y, \quad \text{and} \quad \mu = \epsilon^{1/2} z,
\]

where \( \epsilon \) is a small parameter to measure the weakness of the wave amplitude, \( 0 < \epsilon < 1 \). The dependent variables \( (n, v, \phi, v_x, v_y, v_z) \) are expanded as

\[
\begin{align*}
n &= 1 + \epsilon n_1 + \epsilon^2 n_2 + \ldots \\
\phi &= \epsilon \phi_1 + \epsilon^2 \phi_2 + \ldots \\
v_x &= \epsilon v_{x1} + \epsilon^2 v_{x2} + \ldots \\
v_y &= \epsilon^{3/2} v_{y1} + \epsilon^2 v_{y2} + \ldots \\
v_z &= \epsilon^{3/2} v_{z1} + \epsilon^2 v_{z2} + \ldots
\end{align*}
\]

(6)
and consider (4) and $n_i$ as

$$n_i(\phi) = 1 + \phi + \frac{\phi^2}{2} + O(\phi^3),$$

$$n_2(\phi) = 1 + \phi - \frac{4}{3} b \phi^{3/2} + \frac{\phi^2}{2} + O(\phi^{5/2}),$$

where $b = (1 - \beta) / \sqrt{\pi}$. We substitute (5)-(7) into (1)-(3), the first order, $\epsilon_1$, can be expressed as

$$-\frac{\partial n_1}{\partial \tau} + \frac{\partial v_{x1}}{\partial x} = 0, \quad \frac{\partial v_{x1}}{\partial \tau} = \frac{\partial \phi_1}{\partial x}.$$

and three-half order, $\epsilon_3/2$, can be shown as

$$-\frac{\partial v_{y1}}{\partial \chi} = -\frac{\partial v_{z1}}{\partial \mu}, \quad \frac{\partial v_{y1}}{\partial \xi} = -\frac{\partial \phi_2}{\partial \tau}, \quad \frac{\partial v_{z1}}{\partial \xi} = \Omega v_{z1}, \quad \phi_1 = n_1.$$

We next consider the second order, $\epsilon_2$, including $b = \epsilon_1^{1/2} \hat{b}$,

$$\frac{\partial n_1}{\partial \tau} - \frac{\partial n_2}{\partial \xi} + \frac{\partial}{\partial \xi} [v_{x2} + n_1 v_{x1}] + \frac{\partial v_{y2}}{\partial \chi} + \frac{\partial v_{z2}}{\partial \mu} = 0,$$

$$\frac{\partial v_{x1}}{\partial \tau} - \frac{\partial v_{x2}}{\partial \xi} + v_{x1} \frac{\partial v_{x1}}{\partial \xi} = -\frac{\partial \phi_2}{\partial \tau}.$$

We can now rearrange all these equations and express in terms of the first order variables,

$$2 \frac{\partial \phi_1}{\partial \tau} + 2 \phi_1 \frac{\partial \phi_1}{\partial \xi} + 2 b \mu_e \phi_1^{1/2} \frac{\partial \phi_1}{\partial \xi} + \frac{\partial^3 \phi_1}{\partial \xi^3} + \left(1 + \frac{1}{\Omega^2}\right) \frac{\partial}{\partial \xi} \left[ \frac{\partial^2 \phi_1}{\partial \xi^2} + \frac{\partial^2 \phi_1}{\partial \mu^2} \right] = 0.$$

This is a weakly nonlinear equation for the dusty plasma with trapped electrons. For more convenience, we introduce some new variables to eliminate all coefficients,

$$\phi_1 = \Gamma^2 \phi, \quad \tau = \frac{t}{\Gamma^3 \sqrt{2}}, \quad \xi = \frac{x}{\Gamma \sqrt{2}}, \quad \chi = \frac{\sqrt{\sigma y}}{\Gamma}, \quad \mu = \frac{\sqrt{\sigma z}}{\Gamma},$$

where $\sigma = (1 + 1/\Omega^2) / 2$ and $\Gamma = b \mu_e$, the dimensionless form can be written as

$$\frac{\partial \phi}{\partial \tau} + \phi \frac{\partial \phi}{\partial x} + \phi^{1/2} \frac{\partial \phi}{\partial x} + \frac{\partial^3 \phi}{\partial \xi^3} + \frac{\partial}{\partial \xi} \left[ \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2} \right] = 0.$$

We will refer this equation as Schamel-Korteweg-de Vries-Zakharov-Kutznetsov (SKdVZK) equation, this equation also admits the soliton solution [19] for 1D case, namely,

$$\phi = \left( \frac{60 \eta^2}{1 + \sqrt{1 + 75 \eta^2 \cosh 2 \eta (x - x_0)}} \right)^2.$$
3. Time evolution of an unstable soliton

To study the unstable soliton, we will apply the long-wavelength function to the soliton solution in the perpendicular direction to the soliton direction [20, 21, 19]. This ansatz can be used as the initial condition for the time evolution,

\[ \phi = \phi_0 + \epsilon u(x)e^{iky + \gamma t}, \]

where \( \gamma \) determines the growth rate of soliton, \( \epsilon \) denotes a small number, \( \epsilon << k \), \( u(x) \) can be any real functions and \( \phi_0 \) is the unperturbed soliton, (9). After substitute this ansatz into (8) and collect to the 1-st order of \( \epsilon \), we then have

\[ \frac{d}{dx} (Lu(x)) = -\gamma u(x), \] (10)

where

\[ L = \frac{d^2}{dx^2} + \phi_0 + \phi_0^{1/2} - (c-k^2). \]

c denotes the wave speed and \( \phi_0 \) is the equation in (9). Moreover, there is another method called the variational method [22] for calculating the growth rate. This method is mostly used for the cubic nonlinear Schrödinger equation and its modification [23, 24]. The Lagrangian density will be constructed as

\[ \mathcal{L} = \frac{\psi_x \psi_t}{2} - \frac{c \psi_x^2}{2} + \frac{4}{15} \psi_5/2 + \frac{\psi_x^3}{6} + \frac{\psi_{xx}^2}{2} + \frac{\psi_{xy}^2}{2}, \] (11)

where \( \psi_x = \phi(x) \). To calculate the growth rate, we need to guess the trial function for (11). For these methods, the growth rate can be found all the possible values of \( k \) to make soliton unstable. To evaluate this result, we will apply the spectral method to SKdVZK equation. The basic idea is to apply a Fourier transformation, for the 2D case,

\[ \Phi_{\xi,\chi} = F(\phi_{l,m}) = \sum_{l=0}^{N_x-1} \sum_{m=0}^{N_y-1} \phi_{l,m} e^{il\xi_x + i\chi y_m}, \]

where \( N_x \) and \( N_y \) are the number of mesh points in the \( x \) and \( y \) direction, respectively.

For the temporal space, the Runge-Kutta method [25] is applied to the time derivative and the initial condition is set as

\[ \dot{\phi}_0 = \left( \frac{60\eta^2}{1 + \sqrt{1 + 75\eta^2 \cosh 2\eta(x-x_0)}} \right)^2 \left( 1 + \epsilon \cos \left( \frac{2\pi y}{\lambda} \right) \right), \]

where \( x_0 \) denotes a starting point and the sinusoidal function is used for the long wave-length function. All parameters are set for time evolution is \( L_x \in [-25, 25] \) and \( L_y \in [-25, 25] \), \( x_0 = 0 \), \( \eta = 0.2 \), \( dt = 0.0005 \), \( N_x = 128 \) and \( N_y = 128 \). The periodic boundary conditions in \( x \)-direction has also been applied. The initial perturbed plane soliton is shown in Figure 1(1). Figure 1(2)-(4) show the evolution of the unstable soliton, and become the higher solitons as shown in Figure 1(5)-(6), this 2D soliton moves faster than the unperturbed plane soliton. There are also others two solitons found because there are possible space that more solitons can be existed.
Figure 1. Time evolution of an unstable soliton with $\eta = 0.2$ with (1) $t=0$, (2) $t=85$, (3) $t=130$, (4) $t=150$, (5) $t=180$ and (6) $t=200$.

To obtain more 2D solitons, we double the period of the sinusoidal function and the transformation of unstable soliton is shown in Figure 2. For this case, 3 solitons emerge simultaneously.

4. Conclusion
The reductive perturbation method can be used to obtain the soliton equation for the weakly nonlinear wave equation. The long wavelength perturbation method is used to determine the unstable growth rate of the soliton. The unstable dust ion-acoustic soliton will be transformed into the higher dimensional solitons. The number of higher solitons depends on the length of perpendicular direction. The higher soliton dimension might be useful for study some behaviors such as their collisions [26].

5. Acknowledgments
SP wishes to acknowledge support from Faculty of Education, Burapha University. YR would like to thank the Higher Education Commission for the support.

6. References
[1] Chen F 2016 Introduction to Plasma Physics and Controlled Fusion, 3rd Ed. (Springer)
Figure 2. Time evolution of an unstable soliton with $\eta = 0.2$ with (1) $t=0$, (2) $t= 15$, (3) $t= 55$, (4) $t= 88$, (5) $t = 120$ and (6) $t = 140$.

[2] Shukla P and Mamun A 2002 Introduction to dusty plasma physics (IOP)
[3] Verheest F 2000 Waves in Dusty Space Plasmas (Kluwer Academics)
[4] Rao N, Shukla P and Yu M 1990 Planet. Space Sci 38 543
[5] Barkan A, D’Angelo N and Merlino R 1996 Planetary and Space Science 44 239–242
[6] Washimi H and Taniuti T 1966 Phys. Rev. Lett. 17 996–8
[7] Infield E and Rowlands G 2000 Nonlinear Waves, Solitons and Chaos 2nd ed (Cambridge: Cambridge University Press)
[8] El-Wakil S A, Zahran M A, El-Shewy E K and Mowafy A E 2006 Physica Scripta 74 503–509
[9] Schamel H 1972 Plasma Phys. 14 905–24
[10] Cairns R A, Mamun A A, Bingham R and Shukla P K 1996 Phys. Scr. T63 80–6
[11] Anowar M and Mamun A 2009 IEEE Transactions on Plasma Science 37 1638–1645
[12] Misra A and Wang Y 2015 Communications in Nonlinear Science and Numerical Simulation 22 1360–1369
[13] Agrawal G 2001 Nonlinear fiber optics 3rd ed (Academic Press)
[14] Mollenauer L F and Gordon J P 2006 Solitons in Optical Fibers: Fundamentals and Applications (ElsevierAcademic Press)
[15] Khursudinov K R, Stepanyants Y A and Tranter M R 2018 Physics of Fluids 30 022104
[16] Ancilotto F, Levy D, Pimentel J and Eloranta J 2018 Physical Review Letters 120
[17] Baluku T K, Hellberg M A and Mace R L 2008 Physics of Plasmas 15 033701
[18] Schamel H 1973 J. Plasma Phys. 9 377–87
[19] Allen M A, Phibanchon S and Rowlands G 2007 JPP 73 215–29
[20] Rowlands G 1969 J. Plasma Phys. 3 567–76
[21] Allen M A and Rowlands G 1993 J. Plasma Phys. 50 413–24
[22] Bettinson D C and Rowlands G 1998 J. Plasma Phys. 59 543–54
[23] Anderson D, Lisak M and Berntson A 2001 Pramana J. Phys. 57 917–36
[24] Phibanchon S and Rattanachai Y 2013 The soliton instability to the quintic nonlinear Schrödinger equation
Burapha University International Conference pp 621–627

[25] Press W H, Teukolsky S A, Vetterling W T and Flannery B P 1992 Numerical Recipes in C 2nd ed
(Cambridge: Cambridge University Press)

[26] Phibanchon S and Allen M A 2007 Time evolution of perturbed solitons of modified Kadomtsev-Petviashvili
equations The 2007 International Conference on Computational Science and its Applications (ICCSA 2007) (Los Alamitos, CA, USA: IEEE Computer Society) pp 20–3