Excitation Spectrum at the Yang-Lee Edge Singularity of the 2D Ising model

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Abstract

This paper studies the Yang-Lee edge singularity of 2-dimensional (2D) Ising model through a quantum spin chain. In particular, finite-size scaling measurements on the quantum spin chain are used to determine the low-lying excitation spectrum and central charge at the Yang-Lee edge singularity. The measured values are consistent with predictions for the $(A_4, A_1)$ minimal conformal field theory.

In 1978, Fisher[1] argued forcefully that Yang-Lee edge singularities[2, 3] are similar to ordinary critical points. Later, Cardy[4] provided another strong argument that the Yang-Lee edge singularity of the 2D Ising model is identified with the $(A_4, A_1)$ minimal conformal field theory (CFT)[5, 6] of the ADE classification.[7] Cardy’s identification enables a number of CFT predictions about the Yang-Lee edge singularity of the 2D Ising model.

Some of these CFT predictions have been confirmed. Using transfer matrix methods, Itzykson et al measured the central charge and exponent $\nu$ at the Yang-Lee edge singularity of the 2D Ising model.[8] Using short quantum spin chains,[9] Uzelac et al also measured the exponent $\nu$ at the Yang-Lee edge singularity of the 2D Ising model.[10] Both measurements are in agreement with predictions derived from Cardy’s identification.

This article provides other measurements that can be predicted based on Cardy’s identification. The measurements provide the low-lying excitation spectrum at this Yang-Lee edge singularity of the 2D Ising model. This article also
compares the measured low-lying excitation spectrum with predictions based on Cardy’s identification of the \((A_4, A_1)\) minimal CFT with this Yang-Lee edge singularity of the 2D Ising model.\(^4\)\(^8\)\(^10\)

In this article, measurements\(^11\) are based on finite-size scaling in quantum spin chains. For the 2D Ising model in an imaginary external magnetic field, the associated \(N\)-site quantum spin chain has a Hamiltonian, \(H_{\text{Ising}}\), given by:\(^12\)

\[
H_{\text{Ising}} = - \sum_{n=1}^{N} \{ t \sigma_z(n) \sigma_z(n+1) + iB \sigma_z(n) + \sigma_x(n) \}. \tag{1}
\]

In Eq. (1), \(\sigma_z(n)\) and \(\sigma_z(n)\) are 2x2 Pauli spin matrices at site \(n\), the parameter \(^"t"\) is a positive coupling for ferromagnetic spin-spin interactions, and \(iB\) is a purely imaginary external magnetic field. In Eq. (1), the last term results from single inter-row spin flips in the associated 2D transfer matrix.\(^13\)

In this article, the phenomenological renormalization group (PRG) is used to determine critical values of the purely imaginary magnetic field, \(iB_{\text{YL}}(N)\), for various chain lengths, \(N\). For such purely imaginary magnetic fields, the PRG equation becomes:\(^14\)\(^8\)

\[
\left[ N - 1 \right] m(B_{\text{YL}}(N), N - 1) = \left[ N \right] m(B_{\text{YL}}(N), N). \tag{2}
\]

In Eq. (2), \(m(B, N) = [E_1(B, N) - E_0(B, N)]\) where \(E_0(B, N)\) and \(E_1(B, N)\) are the energies of the ground state \(^"0"\) and first excited state \(^"1"\) of the quantum spin chain having length \(N\). At the \(B_{\text{YL}}(N)\)’s, the Ising quantum spin chain exhibits finite-size scaling behavior associated with the Yang-Lee edge singularity of the 2D Ising spin model. In particular, if the \(B_{\text{YL}}(N)\)’s converge to a nonzero value as \(N \to \infty\), that value corresponds to the critical point for the Yang-Lee edge singularity of the 2D Ising spin model.

At the \(B_{\text{YL}}(N)\)’s, excitation energies and other physical quantities should scale. CFT predicts the scaling behavior of such physical quantities at the \(B_{\text{YL}}(N)\)’s.

First, CFT predicts how excitation energies will scale with the length, \(N\), of the quantum spin chain. For an excited energy eigenstate \(^"i"\) of the quantum spin chain, the CFT prediction is that the excitation energy, \(E_i(N) - E_0(N)\), will scale as:\(^15\)

\[
E_i(N) - E_0(N) = \zeta 2\pi \frac{\Delta_i + \bar{\Delta}_i - (\Delta + \bar{\Delta})}{N}. \tag{3}
\]

In Eq. (3), \(\Delta_i\) and \(\bar{\Delta}_i\) are left and right conformal dimensions of the field \(^"i"\) in the associated CFT, and \(\Delta\) and \(\bar{\Delta}\) are the conformal dimensions of the primary field of lowest \(^"negative"\) scaling dimension in the CFT. Such negative dimension fields occur in various non-unitary CFTs. In Eq. (3), the constant \(\zeta\) is non-universal and depends, e.g., on the normalization of the Hamiltonian of the quantum spin chain.

Second, CFT predicts how the ground state energy, \(E_0(N)\), will scale with the length, \(N\), of the quantum spin chain. In particular, the ground state energy
scales as:[17, 16]  
\[
E_0(N) = AN + B - \zeta \pi C_{eff}/(6N) + \ldots \tag{4}
\]

While the constants A and B are non-universal, $C_{eff}$ and the exponents of the finite-size corrections in $1/N$ of the last two terms are universal numbers predicted by CFT. In particular, $C_{eff}$, is the effective central charge of the CFT. $C_{eff}$ is defined by $C_{eff} = C - (\Delta + \bar{\Delta})$ where "C" is the normal central charge of the CFT.[8]

For the minimal CFTs, the ADE classification provides modular invariant partition functions[7, 18] from which the low-lying excitation spectrum and central charge are easily extracted. For the $(A_4, A_1)$ minimal CFT, Table 1 shows the low-lying excitation energies and associated degeneracies of the spectrum obtained from the associated modular invariant partition function of the ADE classification. Rather than absolute excitation energies, Table 1 lists normal-

| CFT       | $(A_4, A_1)$ |
|-----------|--------------|
| Normalized Energies | 0 1 2.5 5.0 6.0 7.5 |
| Degeneracy     | 1 1 2 3 2 4 |

Table 1: Low-lying excitation spectrum of $(A_4, A_1)$ minimal CFT

For a state "i", the associated normalized excitation energy is defined as the ratio is the actual excitation energy of the state "i" over the actual excitation energy of the lowest excited state "1". Here, absolute excited energies are measured with respect to the ground state energy. The ratios of Table 1 have the advantage of not depending on non-universal constants such as $\zeta$. For the $(A_4, A_1)$ minimal CFT, the form of the modular invariant partition function also implies that the effective central charge, $C_{eff}$, is equal to $2/5$. [8] Below, finite-size scaling measurements on the Ising quantum spin chain are used to determine the low-lying excitation spectrum and effective central charge at the Yang-Lee edge singularity.

The measurements of the critical magnetic fields, $B_{YL}(N)$, were obtained by numerically solving PRG eq. (2) for Ising quantum chains of different lengths. For the numerical solutions, the state energies were obtained by applying the Lanczos algorithm to $H_{Ising}$ of eq. (1). Table 2 shows the values for the critical fields, i.e., $B_{YL}(N)$'s, ground state energies, and lowest excitation energies, i.e., Gap(N)'s, as obtained via the Lanczos algorithm. Table 2 lists measurements of these physical quantities for Ising quantum spin chains in which the coupling, t, had the value of 0.1. The measurements of Table 2 also show that $N \times$ Gap(N) scales to a constant as $N \to \infty$ as expected for the PRG.

Based on the finite-size scaling measurements of Table 2, we first evaluated the effective central charge, $C_{eff}$, at the Yang-Lee edge singularity of the 2D Ising model to test the numerical methods. In particular, ground state energies for adjacent triplets of chain lengths $[(N-1), N, (N+1)]$ and an estimate, $\zeta(N)$, for the non-universal constant, $\zeta$, were used to solve Eq. (4) for estimates, i.e., $C_{eff}(N)$'s, to the effective central charge, $C_{eff}$. In each evaluation, the Gap(N)
was used to estimate the non-universal constant $\zeta(N)$ of Eq. (3) at chain length $"N"$. The resulting estimates, i.e., the $C_{eff}(N)$’s, are plotted in Figure 1. The $C_{eff}(N)$’s have a ”$1/N$” dependence due to the higher order ”$1/N$” corrections to eqs. (3) and (4).

From a visual inspection of Figure 1, one sees that the estimates for the effective central charge have a crossover behavior in $1/N$. After the crossover, the $C_{eff}(N)$’s converge smoothly towards a limit, i.e., $C_{eff}$, as $1/N \rightarrow 0$. The effective central charge may be obtained by extrapolating the estimates towards $1/N = 0$. A visual inspection of Figure 1 also shows that the $C_{eff}(N)$’s behave as $C_{eff} + \alpha/N$ as $1/N \rightarrow 0$. Based on this form, one easily extracts from the measured values that $C_{eff}$ is approximately equal to 0.41. This measured value for $C_{eff}$ agrees well with the value of 2/5 that the ($A_4, A_1$) minimal CFT predicts for $C_{eff}$.

The critical magnetic field values of Table 2, i.e., the $B_{YL}$’s, were also used to find the low-lying excitation spectra of Ising quantum spin chains of various lengths. While the Lanczos algorithm can provide the low-lying excitation spectra, it is inconvenient for determining the complete low-lying excitation spectrum. In particular, the Lanczos algorithm requires one to select a starting vector and then, produce a Krylov space over the starting vector. For simple choices of the starting vector, we found that the Lanczos algorithm spreads the low-lying excitations over several Krylov spaces. Thus, an evaluation of the complete low-lying excitation spectrum via the Lanczos algorithm would require finding a first Krylov space and then, finding other Krylov spaces orthogonal to the first space. To avoid such numerical complications, we used the commercial linear algebra package of MapleTM 9.5 to generate complete excitation spectra.

The MapleTM package provided excitation spectra for Ising quantum spin chains with 6 - 12 sites. The measured low-lying parts of said spectra including

| Number of Sites | $B_{YL}(N)$ | Energy of ground state | Gap(N) | $N \times \text{Gap}(N)$ |
|-----------------|-------------|------------------------|--------|------------------------|
| 3               | .24591801   | -2.8811043             | .8103423 | 2.4310                |
| 4               | .23841271   | -3.8028211             | .6629112 | 2.6516                |
| 5               | .23523391   | -4.7341982             | .5613016 | 2.8065                |
| 6               | .23376371   | -5.6688215             | .4858628 | 2.9152                |
| 7               | .2302791    | -6.6048003             | .4275400 | 2.9928                |
| 8               | .2263471    | -7.5414746             | .3811698 | 3.0494                |
| 9               | .2241181    | -8.4785910             | .345105  | 3.0916                |
| 10              | .2227931    | -9.4160213             | .3123765 | 3.1237                |
| 11              | .2219721    | -10.353696             | .286250  | 3.1488                |
| 12              | .2214421    | -11.291568             | .264041  | 3.1685                |
| ...             | ...         | ...                    | ...     | ...                   |
| $\infty$       | .23193i     | $-\infty$              | 0.0     | 3.2840                |

Table 2: PRG results for $B_{YL}(N)$, Ground state energy, Gap, and $N \times \text{Gap}$ as a function of the number of sites, $N$.
both energies and degeneracies are provided in Table 3. There, the excitation energies are again normalized by division by the lowest excitation energy of the Ising quantum spin chain. As already described, such a normalization removes any dependence on the non-universal constant $\zeta$.

From the measured spectra of Table 3, one can determine the form of the low-lying excitation spectrum in the limit where $1/N \to 0$. The determination requires first, identifying classes of eigenstates that correspond for different chain lengths and then, determining the scaling behavior for each of the identified classes as, $1/N \to 0$. The identification of corresponding eigenstates for different values of $N$ is achieved with the aid of some simple rules. The first rule is that the energies of corresponding states vary monotonically and smoothly with $N$ provided that the initial value of $N$ is sufficiently large. The second rule is that the correspondences between states must account for degeneracies. In particular, while new eigenstates appear as $N$ increases, old eigenstates do not disappear.\(^1\) Thus, in each class, degeneracies of eigenstates either stay constant or increase with $N$.

\(^1\)A few "special" low-lying excited eigenstates have excitation energies whose magnitudes grow rapidly with $N$. The "special" eigenstates are not part of the low-energy excitation spectrum as $N \to \infty$. In fact, the number eigenstates with energies of smaller magnitude than those of the "special" eigenstates grows with $N$. Such "special" states were also seen at the Yang-Lee edge singularity of the 3-state Potts quantum spin chain where they were not part of the spectrum in the thermodynamic limit.\(^2\) The "special" excited eigenstates are not discussed further.
Table 3: The table shows normalized low-lying excitation energies of states A - F for Ising quantum spin chains with 6 to 12 sites. Degeneracies of the states are as listed in the left column.

| State | Degeneracy | 6    | 7    | 8    | 9    | 10   | 11   | 12   |
|-------|------------|------|------|------|------|------|------|------|
| A     | 2          | 2.68432 | 2.64386 | 2.61415 | 2.59207 | 2.57540 | 2.56260 | 2.55253 |
| B     | 1          | 4.18193 | 4.27896 | 4.36713 | 4.44474 | 4.51197 | 4.56977 | 4.61912 |
| C     | 2          | 4.51738 | 4.63236 | 4.70368 | 4.75182 | 4.78652 | 4.81281 | 4.83329 |
| D     | 2          | 5.85889 | 5.89208 | 5.91240 | 5.92644 | 5.93703 | 5.94544 | 5.95210 |
| E     | 2          | –     | 5.68559 | 6.03104 | 6.27270 | 6.45018 | 6.58573 | 6.69223 |
| F     | 2          | –     | 6.24252 | 6.35798 | 6.46344 | 6.55966 | 6.64694 | 6.72535 |

Application of the above rules leads the classification of low-lying excited eigenstates of Table 3. The classification includes classes A - F of state types. Figures 2 - 5 plot the normalized excitation energies for the states of classes A - F as a function of the inverse of the length of the Ising quantum spin chain.

Figure 2: The measured energies of type A states are plotted in 1/N.

A visual inspection of Figures 2 - 5 readily shows that the excited states A, B, C, D, E, and F combine into the four distinct sets A, B & C, D, and E & F. Within each distinct set, the excited eigenstates have energies that approach the same value as 1/N → 0.

2Here, the lowest excited state has been ignored. The lowest excited eigenstate was however, observed to be non-degenerate as predicted for the (A_4, A_1) minimal CFT. The lowest lying excited eigenstate automatically has an energy of 1 in the convention of Table 1.
Figure 3: The measured energies of type B states (squares) and of type C states (circles) are plotted in $1/N$.

Figure 4: The measured energies of type D states are plotted in $1/N$. 
Figure 5: The measured energies of type E states (squares) and of type F states (circles) are plotted in $1/N$.

A visual inspection of Figures 2 - 5 shows that the normalized energies of the states of the sets A, B & C, D, and E & F approach about 2.45, 5.0, 6.03, and 7.6, respectively, as $1/N \to 0$. A BST analysis[20] was performed to obtain more reliable values for the energies of these states in the limit where $1/N \to 0$. The BST analysis indicated that the normalized energies of the states of type A, B, C, D, E, and F scale to 2.4995(5), 5.005(1), 5.003(3), 5.99(1), 7.54(8), and 7.60(7), respectively, in this limit. These finite-size scaling measurements of the low-lying excitation energies agree well with those of the $(A_4, A_1)$ CFT as shown in Table 1.

Also, an inspection of Table 3 shows that the distinct state sets A, B & C, D, and E & F, of different limiting excitation energies have 2, 3, 2, and 4 states, respectively. Thus, the PRG measurements of the normalized excitation spectra also provide values for the degeneracies that agree well with those of the $(A_4, A_1)$ CFT of Table 1.

In conclusion, our finite-size scaling measurements on the Ising quantum spin chain at the Yang-Lee edge singularity have produced a low-lying excitation spectrum that is in very good agreement with that predicted from the $(A_4, A_1)$ minimal CFT. These results further confirm Cardy’s identification of the Yang-Lee edge singularity of the 2D Ising model with the $(A_4, A_1)$ minimal CFT.

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$^3$The BST analysis also provided evidence that finite-size scaling corrections to the normalized energies of these sets of states scale as $1/N^b$ where the b’s are in $[1, 2]$ as $1/N \to 0$. 

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