MY ENCOUNTERS — AS A PHYSICIST — WITH
MATHEMATICS

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It is a pleasure for me — as a physicist — to speak at the twenty-fifth anniversary of your mathematical institute, when also a new building is being dedicated. Such an occasion calls for accolades of appreciation, both to the profession and to its practitioners, and over the years many bouquets have been offered to mathematics.

For example, Gauss was quite pleased with his line of work, stating that

“Mathematics is the queen of the sciences.”

(Gauss)

Even an outsider, the psychologist Havelock Ellis, evaluated mathematicians as having

“. . . reached the highest rung on the ladder of human thought”

(Ellis)

To be sure, there are also dissenters, for example Plato:

“I have hardly ever known a mathematician who was capable of reasoning.”

(Plato)

Physicists’ opinion lies between these extremes. Well known is Wigner’s appreciation of the

“unreasonable effectiveness of mathematics”

(Wigner)
A forceful position in favor of mathematics in physics was stated by Dirac:

“The most powerful method of advance [in physics] . . . is to employ all the resources of pure mathematics in attempts to perfect and generalize the mathematical formalism that forms the existing basis of theoretical physics, and . . . to try to interpret the new mathematical features in terms of physical entities.”

(Dirac)

The list of apt quotations can be extended to great length, and I shall not attempt adding my own words. Let me merely repeat that mathematics is indeed good for us physicists, but also we are good for mathematics by providing new ideas for mathematical research and by finding fresh applications of old ideas.

These days there is intense cross fertilization between mathematics and physics, specifically between geometry and field theory. The contact, first established through Einstein’s general relativity, surged again about two decades ago. Some of my own research took place at this new beginning, so I thought I would present here a reminiscence, thereby providing a case history of a physics-mathematics encounter.

By the early 1970’s, quantum field theory was very much in favor with theoretical physicists, but the quantized equations resisted solution. It then occurred to many people that it would be worthwhile to ignore the quantal nature of the fields, and to solve the equations as if they describe non-linear, classical dynamical systems. Interesting, localized and non-dissipative solutions were found very quickly. These were the kinks in
one dimension — relevant to physics on a line, vortices in two dimensions — in planar physics, Skyrmions and magnetic monopoles in the three dimensions of our physical world; collectively such solutions were called “solitons,” the name being taken from applied mathematics. Another class of solutions comprised the “instantons” in four-dimensional space-time.

With colleagues at M.I.T., I addressed the question of how to extract from these classical results information on the quantum theory — i.e. we wanted to determine the quantum meaning of classical fields. Progress was made on this problem, and at a certain stage Claudio Rebbi and I decided that we needed to study both the linear small fluctuations about the non-linear soliton and instanton field profiles, and also the coupling of other linear systems, like fermions, to solitons and instantons. Thus we were led to linear eigenvalue equations and we realized that the “zero-modes,” corresponding to vanishing eigenvalues, contain especially important information about the quantum physics. The zero-modes in the small fluctuation equations measure allowed deformations of the soliton or instanton, while the number of these modes gives the dimension of the moduli space for the solution of the non-linear equation. In the fermionic Dirac equation, the eigenvalues measure energy, and positive-energy modes describe quantum particles, negative-energy modes correspond to anti-particles, while zero-modes — when they exist — signal a degeneracy that gives rise to unexpected quantum numbers, e.g. fractional fermion number.

Rebbi and I were delighted to find precisely such zero-modes, and to
establish their physical consequences. But we were surprised that the existence of these special solutions did not depend on the details of the localized profiles in the background solitons and instantons; rather only their large-distance behavior mattered. The long-range features of course characterize the topological properties of solitons and instantons, so we began to suspect that the occurrence of zero-modes was not an accident of our analysis, but a consequence of having non-trivial topological backgrounds.

We wanted to find out what mathematicians knew about this. At M.I.T. all buildings are connected and the math department is in the same structure as my work space. However, locked doors as well as the chemistry department intervene, so communication is obstructed. Nevertheless, we walked the corridors of the mathematics offices, but could not immediately find anyone who wanted to spend time understanding our questions, and answering them in a way comprehensible to non-specialists, to us physicists. Shortly later we met Barry Simon, who did not have specific information on our problem, but suggested that work of Atiyah and Singer might be relevant.

Singer had temporarily moved from M.I.T. to Berkeley, but as it happened my colleague and collaborator Goldstone knew Atiyah from student days in Cambridge, England, and had information that he was coming to visit his mathematics friends in Cambridge, Massachusetts.

So we arranged a meeting in my office. We invited physicists who were working on soliton-instanton questions, and we listened to Atiyah explain how his index theorem with Singer counts instanton zero-modes, and how
their spectral flow theorem with Patodi is relevant to fractional charge. Learning that the four-dimensional index is given by an integral over the curvature-form $F$, specifically by $\int F \wedge F$, was especially thrilling to us since the integrand, $F \wedge F$, had also arisen in the physics literature as the anomalous divergence of the chiral fermion current, thereby controlling neutral pion decay. Evidently the chiral anomaly and the index theorem are related; they had been elaborated in the late 1960’s at different ends of the same M.I.T. corridor, by people working in ignorance of each other!

We appreciated very much Atiyah’s efforts to make his presentation understandable to us; still exchanging information was not easy. One young member of the audience impressed Atiyah, who encouraged the fellow to speak because he seemed to understand, better than anyone else, what Atiyah was saying. That person was Witten; as all of you know, he has continued to impress Atiyah and other mathematicians.

Soon thereafter, I was asked to review these exciting new results about quantum field theory at a meeting of the American Physical Society. Since Singer was present, I yielded some of my time to him, with the suggestion that he describe the mathematical connection. But a detailed presentation could not be fit in, so he merely eulogized collaboration between mathematics and physics with the following ode.

“In this day and age
The physicist sage
Writes page after page
On the current rage
The gauge

Mathematicians so blind
Follow slowly behind
With their clever minds
A theorem they’ll find
Duly written and signed

But gauges have flaws
God hems and haws
As the curtain He draws
O’er His physical laws
It may be a lost cause”

(Singer)

Index theory also received a contribution from physics. The Atiyah-Singer theorem applies to even-dimensional spaces on which a connection is defined. However, physicists are also interested in odd-dimensional spaces — where one-dimensional kinks or three-dimensional Skyrmions and monopoles reside. These configurations can lead to zero modes, even in the absence of a gauge connection. So we asked the mathematicians about odd dimensions; apparently nothing was known. At that time I had a mathematically-minded student, Costas Callias, and I asked him
to prove an odd-dimensional index theorem. He succeeded and this further prompted Bott and Seeley to publish a mathematical exegesis of the result, immediately following Callias’ paper in *Communications in Mathematical Physics*. Since then I have been happy to see the “Callias index theorem” used and cited.

Figure 1: Front pieces of papers by Callias, Bott and Seeley.
The two approaches to solving problems — the explicit, goal-oriented methods of the physicists and the general theorems of the mathematicians — are well illustrated by the determination of the dimensionality for instanton moduli space: the n–instanton SU(2) solution depends on $8n-3$ parameters. This result appears in the same issue of *Physics Letters*, once by Schwarz, who used the Atiyah-Singer theorem, and a few pages later by Rebbi and me, who solved differential equations to find explicitly $8n-3$ zero-modes.

Figure 2: Front pieces of papers by Schwarz, Jackiw and Rebbi.
Gauge theories in general and instantons in particular continued to interest mathematicians. They established the topological properties of the instanton moduli space and produced a construction — but not an explicit formula — for the general solution. The most general explicit expression, which does not exhaust all the parameters, was given by physicists.

Further developments on four-dimensional gauge fields led mathematicians to Donaldson theory. In three dimensions, the Chern-Simons term — another gauge structure first emphasized in the physics literature — has been related by mathematicians to knot theory on manifolds with various topologies, while physicists applied this term to experimental phenomena on the plane, like the Hall effect.

These parallel investigations by physicists and mathematicians also highlight our differences: physicists use mathematics as a language for recording observations about physical systems, and this limits our interest in the full range of mathematical possibility, which fascinates mathematicians. For example, the general instanton solution does not appear to be physically relevant; only the original one-instanton and the explicit, but limited, multi-instanton solutions have illuminated physical theory. Even the physicists’ language need not always be mathematical. The fractional charge phenomenon, which can be inferred from zero-modes or from spectral flow, was also independently established by Su, Schrieffer and Heeger, who found a physical realization in linear polyacetylene chains. One of their derivations uses the pictorial language of chemical bonds, where the only mathematics consists of counting!
An analogy comes here to mind: the English language contains over 200,000 words and all of them interest the lexicographer; for Shakespeare 20,000 words sufficed to express his ideas in plays and sonnets; while Churchill used less than 2000 words in his historically decisive speeches. Physicists, like Churchill, achieve their goals by effective use of a limited vocabulary.

The process of gaining knowledge goes through the same steps in physics and mathematics: first there is the intuition/guess, then follows the proposal/conjecture and finally comes the verification. For the mathematician verification consists of constructing a proof, establishing a theorem according to rules whose legitimacy evolves slowly under the direction of the entire mathematics community. But the physicist verifies his ideas by finding a physical correlative: neutral pion decay validates chiral anomalies, properties of solitons in polyacetylene establish fractional charge. The rules for giving a proof are constantly and rapidly changing — presuppositions can become modified, experimental facts can evolve.

Because “proof” and “theorem” carry intellectual prestige and pleasure, occasionally there are attempts to employ them in physics. To my mind this is mostly futile and sterile. For example, physicists wanted very much to combine internal and space-time symmetries in a non-trivial fashion and were not daunted by a proven “impossibility theorem.” Rather the “theorem” was circumvented by the simple device of replacing commutators with anti-commutators, by grading the algebra, and supersymmetry was born, which now is also influencing mathematics. Similarly, when field-theory “constructivists” proved the existence of quantum $\lambda\phi^4$
theory in \((1 + 1)\) dimensions, they were correct, but missed the entire quantum soliton phenomenon, which is the only physically interesting feature of that model.

A statement by Yang accurately describes physicists’ historical use of mathematics.

“...physics is not mathematics, just as mathematics is not physics. Somehow nature chooses only a subset of the very beautiful and complex and intricate mathematics that mathematicians develop, and that precise subset is what the theoretical physicist is trying to look for.”

(Yang)

This conservative view of mathematics differs from Dirac’s radical advice, cited earlier, that physicists should

“...try to interpret ... mathematical features in terms of physical entities.”

(Dirac)

However, today faced with the absence of new experimental data about fundamental phenomena, particle physics theory, as realized in the string program, is driven by mathematics in the manner advocated by Dirac. This was not the way things worked in the past, not even for Dirac: when first confronting his negative energy solutions, he identified them with the proton — the only then-known positively charged particle — as a
physicist he did not at first trust his mathematics enough to postulate the existence of the positron!

I am immensely curious about the ultimate fate of the new physics, built entirely on mathematics, indeed on new mathematics that it helps to create. I trust that in the next quarter century the Centre de Recherches Mathématiques will play a role in settling this question.