Remarks on non-perturbative three–body dynamics and its application to the $K K \bar{K}$ system

Xu Zhang$^{1,2,3,a}$, Christoph Hanhart$^{1,b}$, Ulf-G. Meißner$^{4,1,5,c}$, Ju-Jun Xie$^{2,3,6,d}$

$^1$ Institute for Advanced Simulation, Institut für Kernphysik and Jülich Center for Hadron Physics, Forschungszentrum Jülich, 52425 Jülich, Germany
$^2$ Institute of Modern Physics, Chinese Academy of Sciences, Lanzhou 730000, China
$^3$ School of Nuclear Science and Technology, University of Chinese Academy of Sciences, Beijing 101408, China
$^4$ Helmholtz-Institut für Strahlen- und Kernphysik and Bethe Center for Theoretical Physics, Universität Bonn, 53115 Bonn, Germany
$^5$ Tbilisi State University, 0186 Tbilisi, Georgia
$^6$ School of Physics and Microelectronics, Zhengzhou University, Zhengzhou, Henan 450001, China

Received: 10 August 2021 / Accepted: 30 December 2021 / Published online: 6 February 2022
© The Author(s), under exclusive licence to Società Italiana di Fisica and Springer-Verlag GmbH Germany, part of Springer Nature 2022
Communicated by E. Oset

Abstract A formalism is discussed that allows for a straightforward treatment of the relativistic three-body problem while keeping the correct analytic structure. In particular it is demonstrated that sacrificing covariance for analyticity can be justified by the hierarchy of different contributions in the spirit of an effective field theory. For definiteness the formalism is applied to the $K K \bar{K}$ system allowing for the emergence of the $a_0(980)$ and the $f_0(980)$ as hadronic molecules. For simplicity all inelastic channels are switched off.

1 Introduction

Over the past decade, a large number of so-called exotic states have been observed experimentally, especially in the heavy quark sector [1], which can not be easily explained either as $q\bar{q}$ or $qqq$ states. The interpretation of these states has motivated many theoretical studies [2–6]. Understanding their structure will clearly extend our knowledge of the strong interaction dynamics. Many of these states, including the lightest scalar mesons, can be described as emerging from hadron-hadron dynamics and therefore qualify as hadronic molecules, structures analogous to the deuteron in nuclear physics. Although this interpretation is not yet fully accepted in the literature, it is intriguing to ask, if also three- or even more-body bound states can emerge as well. This question is addressed, e.g., in Ref. [7] and in a series of follow-up works.

Here we put the focus on the implications of using relativistic kinematics in the scattering equations. For studies investigating this issue for different systems, however, employing non-relativistic kinematics see, e.g., Refs. [8–12].

If a two-body system gets embedded into a three-body system, the intrinsic variables need to be handled with care. In particular, the self-energy of a two-body subsystem needs to be evaluated at the subenergy available given the presence of the third particle. While in a non-relativistic system this is all well established and straightforward [13], in relativistic systems usually certain approximations are applied to deal with the kinematic dependence of the subamplitudes. In this work we present the three-body scattering equations in a form close to what is known for two-body scattering that at the same time allow one to properly treat this kinematic dependence even when relativistic variables are employed. The focus of this work is to demonstrate that certain approximations to the choice of kinematic variables can lead to wrong conclusions regarding the emergence of three-body bound states.

The necessity to properly treat subsystem self-energies was stressed already, e.g., in Ref. [14] and the advantages of using relativistic kinematics in three particle systems are presented in Ref. [15]. In particular, the emergence of Efimov states [16,17] is in this way avoided. Here we extend the discussion by studying the significance of the violation of covariance that comes with the equations. In particular we evaluate all the diagrams necessary to restore covariance at the one-loop level. For simplicity the two-body system is treated in the isobar approximation. In particular this avoids the emergence of left-hand cuts in the two-body subamplitudes. It should be understood, however, that also those singularities
can be handled properly in a straightforward extension of the formalism presented here.

For definiteness we focus on the $K K \bar{K}$ system in the absence of the $\pi \pi K$ and $\pi \eta K$ inelastic channels, while allowing for $f_0(980)$ and $a_0(980)$ intermediate states, which are included as bound states. While this clearly does not fully represent reality, it still allows us to address the issues mentioned above quantitatively and to investigate the implications of certain choices of kinematics on the appearance of three-body states. As both the $f_0(980)$ and $a_0(980)$ are located close to the $K K \bar{K}$ threshold, and couple strongly to this channel, they are widely regarded as bound states with dominant $K K \bar{K}$ component in their wave functions [18–28].

Direct experimental support for this view comes from the observation of a very prominent isospin-violating signal in between the charged and neutral kaon thresholds [29] that was predicted to emerge due to $a_0(980)$-$f_0(980)$ mixing [30–32]. A natural candidate for a resulting $K K \bar{K}$ bound state is then the $K(1460)$ with $I(J^P) = \frac{1}{2}(0^-)$ [1], although it is not yet clearly established experimentally. First indications for this state were seen at SLAC in the $K \pi \pi$ channel in the reaction $K^\pm p \rightarrow K^\pm \pi^+ \pi^- p$ [33]. The $J^P = 0^-$ partial-wave analysis yields a mass around 1400 MeV and width around 250 MeV. Later, this state was reported at about 1460 MeV by the ACCMOR Collaboration in the diffractive process $K^- p \rightarrow K^- \pi^+ \pi^- p$ [34]. Recently, the LHCb Collaboration showed further evidence for the $K(1460)$ in the $\bar{K}^*(892)^0 \pi^-$ and $[\pi^+ \pi^-]^L=0 K^-$ channels with mass $M_{K(1460)} = 1482.40 \pm 3.58 \pm 15.22$ MeV and width $\Gamma_{K(1460)} = 335.60 \pm 6.20 \pm 8.65$ MeV [35]. The $K(1460)$ is a good candidate for a three kaon bound state, since its quantum numbers are consistent with all kaons in a relative $S$-wave and its mass is only a few MeV below the three-kaon threshold.

The idea of the $K(1460)$ as a three-kaon bound state was investigated, e.g., in Ref. [36], where a certain triangle diagram was employed as the driving potential. In Ref. [37], a study was carried out by solving the Faddeev equations for the $K K \bar{K}$, $K \pi \pi$ and $K \pi \eta$ channels using the two-body inputs provided by unitarized chiral perturbation theory (in the on-shell approximation). A three-body $K K \bar{K}$ quasi-bound state with $I(J^P) = \frac{1}{2}(0^-)$ was found with a mass around 1420 MeV which was identified with the $K(1460)$. In a more recent work [38], by solving the Faddeev equations in configuration space within the Gaussian expansion method, the $K(1460)$ was identified with a three-body $K K \bar{K}$ bound state with a mass of 1460 MeV. We note that in Ref. [39], within the isobar assumption, even the $K a_0(980)$ interaction in $I(J^P) = \frac{3}{2}(0^-)$ channel generated a resonance above the $K a_0(980)$ threshold with a mass around 1500 MeV. Note, however, that the $K(1460)$ was explained in a relativistic quark model as the $2^1S_0$ excitation of the kaon [40].

In the present work as well as most of those mentioned above, the $K K \bar{K}$ three-body system in $I(J^P) = \frac{1}{2}(0^-)$ channel is studied using the isobar approach, where the two-body $K \bar{K}$ interaction is parametrised via the $f_0(980)$ and $a_0(980)$ poles. Such a formalism satisfies two-body and three-body unitarity [41,42]. The latter plays an important role as shown in Refs. [43,44] for the $\rho \pi$ scattering. A very general formalism for $3 \rightarrow 3$ scattering in the isobar approach was presented in Ref. [45]; decay amplitudes with three particles in the final-state may be calculated employing Khuri-Treiman equations [46] (those were used more recently in Refs. [47–53]). It turns out that the resulting equations are quite involved and demanding to solve. The goal we aim at here is in contrast to this to present an easy to handle formalism that keeps relativistic kinematics, but sacrifices covariance for the sake of simplicity. Our formalism is derived employing time ordered perturbation theory (TOPT) rigorously. In particular, our amplitudes are constructed to keep track of the leading singularities of the amplitudes. An alternative formulation, that leads to very similar expressions is presented in Ref. [42]. The formal covariance of this treatment is achieved by modifications of some the contributions to the potential. As stressed in Ref. [54] this introduces unphysical singularities. We demonstrate that avoiding those modifications removes the unphysical singularities and at the same time only introduces a very mild violation of covariance that moreover can be removed systematically order by order.

The paper is organized as follows. In Sect. 2, the derivation of the interaction potential for the concrete example employed here for illustration is given. In Sect. 3, we present the Lippmann-Schwinger-type equation which fulfills two-body and three-body unitarity. The numerical results are discussed in Sect. 4. In this section we also study the impact of different approximations on the numerical results.

### 2 Effective Potentials

#### 2.1 The Lagrangian and coupling constant

For the $f_0(980)\bar{K}K$ and $a_0(980)\bar{K}K$ vertices, we use a scalar coupling [55] (note that in a more sophisticated calculation, the Goldstone boson nature of the kaon should be accounted for)

$$\mathcal{L} = \frac{f_1}{\sqrt{2}} f_0 \bar{K}K + \frac{f_2}{\sqrt{2}} \bar{K} (a_0 \cdot \tau) K$$  \hspace{1cm} (1)

with

$$\bar{K} \equiv \left( \frac{K^+}{K_0} \right), \quad \bar{K} \equiv \left( K^-, \bar{K}^0 \right) \hspace{1cm} (2)$$
Here, $f_0$ and $a_0$ denote the fields of the scalar-isoscalar $f_0(980)$ and the scalar-isovector $a_0(980)$, respectively, where the three components of the latter refer to the different charge states. The coupling constants $f_S$ ($f_1$ and $f_2$) can be determined under the assumption that the $f_0(980)$ and $a_0(980)$ are pure bound states of $K\bar{K}$ system [23,26,28].

$$\frac{f_{S}^2}{4\pi} = 32m_K \sqrt{m_K \varepsilon_S}, \quad (3)$$

with $m_K$ the mass of $K$ meson, and $\varepsilon_S$ is the binding energy of the scalar bound state $S$. It should be stressed in this context that Eq. (3) provides an upper bound for the coupling fixed by the normalisation of a bound state — if a state contains a non-molecular component, the coupling would be lower [4,26] (for an extension of the concept to virtual states see Ref. [27]).

Here we chose the binding energy equal for the two scalar states and we use the following masses

$$m_K = 495 \text{ MeV}, \quad m_{f_0(980)} = m_{a_0(980)} = 980 \text{ MeV}. \quad (4)$$

This leads to

$$f_1 = f_2 = f_S = 3.74 \text{ GeV}, \quad (5)$$

which agrees with the experimental result of Ref. [25]. Clearly, for a realistic calculation both the $\pi\pi K$ and the $\pi\eta K$ system need to be taken into account as well, however, since our focus lies on the formalism, what is introduced here is sufficient. For the same reason we also do not try to better constrain the input data for the $f_0(980)$ and the $a_0(980)$. All this will be improved in a subsequent study.

2.2 Coupled-channel matrix elements

The formalism for the three-body scattering used here employs the scattering of two quasi-particles by means of a Lippmann-Schwinger (LS) type equation. This implies that the scattering potential needs to contain the exchanges of the constituents of the quasi-particles. Accordingly we may decompose the $S$-wave $Kf_0(980)-Ka_0(980)$ interaction to second order in the coupling $f_S$ as

$$V(E, p', p) = V_t(E, p', p) + V_s(E, p', p), \quad (6)$$

where $V_t(E, p', p)$ and $V_s(E, p', p)$ are the $t$- and $s$-channel one-kaon exchange, respectively.

The interaction potential $V_t(E, p', p)$ reads in channel space

$$V_t(E, p', p) = \begin{pmatrix} V_{11}^{tS}(E, p', p) & V_{12}^{tS}(E, p', p) \\ V_{21}^{tS}(E, p', p) & V_{22}^{tS}(E, p', p) \end{pmatrix}. \quad (7)$$

The same structure holds for $V_s(E, p', p)$. In each matrix element $V_{ij}^{tS}(E, p', p)$ and $V_{ij}^{sS}(E, p', p)$, the index $\lambda(\lambda') = 1, 2$ denotes the particle channel ($Kf_0(980) = 1, Ka_0(980) = 2$) and the $S$ denotes the $S$-wave projection of the potential [55] (see also Ref. [56])

$$V_{ij}^{tS}(E, p', p) = \frac{1}{2} \int_{-1}^{1} V_{ij}^{S}(E, p', p) \, d\cos\theta. \quad (8)$$

In the expressions above $E$ denotes the total energy of the system and $p$ and $p'$ are the incoming and the outgoing momenta.

Since we focus on a possible bound state with $I(J^P) = \frac{1}{2} (0^-)$, the flavor wave functions of the $Kf_0(980)$ and $Ka_0(980)$ systems can be written as

$$|\frac{1}{2}, \frac{1}{2} \rangle = -|K^+ f_0(980)\rangle,$$

$$|\frac{1}{2}, -\frac{1}{2} \rangle = -\sqrt{\frac{\gamma}{3}} |K^0 a_0(980)\rangle - \sqrt{\frac{1}{3}} |K^+ a_0(980)\rangle. \quad (9)$$

using the convention that $|K^+\rangle = -|\frac{1}{2}, -\frac{1}{2}\rangle$.

In TOPT, the $t$-channel one-kaon exchange potential acquires two contributions depicted in Fig. 1a and b. In the center-of-mass frame, for the scattering process $12 \rightarrow 1'2'$ with $q = -p - p'$, the potential can be written as

$$V_{ij}^{tS}(E, p', p) = f_{S}^{2} T N \frac{1}{2\omega_{K}(\mathbf{q})} \times \left[ \frac{1}{E - \omega_{1}(p') - \omega_{K}(\mathbf{q}) - \omega_{1}(p)} + \frac{1}{E - \omega_{2}(p') - \omega_{K}(\mathbf{q}) - \omega_{2}(p)} \right], \quad (10)$$

with

$$\omega_{i}(p) = \sqrt{m_{i}^{2} + p^{2}}, \quad i = 1(\sigma), 2(\text{p}),$$

$$\omega_{K}(\mathbf{q}) = \sqrt{m_{K}^{2} + p^{2} + p'^{2} + 2pp'\cos\theta}, \quad (11)$$

where $\theta$ is the angle between $p$ and $p'$. The isospin factors $T$ are listed in Table 1.

Since we work in TOPT, all the potentials contain the normalization factor

$$N = \frac{1}{\sqrt{16\omega_{1}(p)\omega_{2}(p')\omega_{1}(p')\omega_{2}(p')}}. \quad (12)$$

Analogously, the $s$-channel one-kaon exchange potential acquires the two contributions depicted in Fig. 2a and b. The potential can be written as
Fig. 1 Diagrams for the $t$-channel kaon exchange contribution. The left and right diagrams correspond to $t(a)$ and $t(b)$, respectively. The double-solid and dashed line represent the $f_0(980)$ or $a_0(980)$ and the $K$ or $\bar{K}$ meson, respectively.

Table 1 Isospin factors for one-kaon exchange potentials and box diagram contributions

| Potential / Box Type | $Kf_0(980) \rightarrow Kf_0(980)$ | $Kf_0(980) \rightarrow Ka_0(980)$ | $Ka_0(980) \rightarrow K\bar{a}_0(980)$ |
|----------------------|----------------------------------|----------------------------------|-------------------------------------|
| $t$-channel          | $\frac{1}{2}$                    | $\frac{\sqrt{3}}{4}$           | $\frac{-1}{7}$                      |
| $s$-channel          | $\frac{1}{2}$                    | $\frac{\sqrt{3}}{4}$           | $\frac{3}{2}$                      |
| Single-channel:      |                                   |                                  |                                     |
| stretched boxes      | $\frac{1}{4}$                    | $\frac{1}{3}$                   | $1$                                 |
| coupled-channel:     |                                   |                                  |                                     |
| stretched boxes      | $\frac{1}{2}$                    | $\frac{1}{3}$                   | $1$                                 |
| coupled-channel:     |                                   |                                  |                                     |
| crossed boxes        | $\frac{1}{2}$                    | $\frac{1}{3}$                   | $1$                                 |
| coupled-channel:     |                                   |                                  |                                     |
| crossed boxes        | $\frac{1}{2}$                    | $\frac{1}{3}$                   | $1$                                 |

Fig. 2 Diagrams for the $s$-channel kaon exchange contribution. The left and right diagrams correspond to $s(a)$ and $s(b)$, respectively. The double-solid and dashed line represent the $f_0(980)$ or $a_0(980)$ and the $K$ or $\bar{K}$ meson, respectively.

\[ V_{s}\lambda^{\lambda'}(E, p', p) = \mathcal{I} N \frac{1}{2m_K} \times \left[ \frac{f_S^2}{E - \omega_1(p) - \omega_1(p') - m_K - \omega_2(p) - \omega_2(p')} + \frac{f_{\lambda'}^{(0)} f_{\lambda}^{(0)}}{E - m_K^{(0)}} \right] , \]

where again the isospin factors are listed in Table 1. In the expression above it is already used that the scattering equation is solved in the overall center-of-mass frame. Moreover, the notation distinguishes explicitly between the physical parameters $f_S$ and $m_K$, and the bare parameters $f_{\lambda}^{(0)}$ and $m_K^{(0)}$ that get renormalized by the scattering equation. How the latter parameters are determined is described in the next section. We can see that $V_{s}\lambda^{\lambda'}(E, p', p)$ is independent of the scattering angle $\theta$ and thus does not need to be partial-wave projected.

3 Lippmann-Schwinger-type equation

The partial wave decomposed LS equation can be written as

\[ T(E, p', p) = V(E, p', p) \]

\[ + \int_0^\Lambda \frac{4\pi k^2 dk}{(2\pi)^3} V(E, p', k) G(E, k) T(E, k, p) , \]
with the definitions
\[
G(E, k) = \begin{pmatrix} G_r^1(E, k) & 0 \\ 0 & G_r^2(E, k) \end{pmatrix},
\]
where the renormalized isobar-K propagators are
\[
G_r^2(E, k) = \left[ Z(E - \omega_r(k) - \omega_K(k)) - \alpha f_S^2 \Sigma_R^{(\lambda)}(E, k) \right]^{-1},
\]
as we show in Appendix A.

The full potential contains all contributions free of \(f_0(980)/a_0(980)K\) cuts. The part of it to second order in the coupling was defined in the previous section. In the formalism employed here the matrix \(G\) describes the propagation of an \(a_0(980)/f_0(980)K\) intermediate state. Accordingly, \(\omega^{(1)}(k) (\omega^{(2)}(k))\) is the energy of an on mass-shell \(f_0(980)\) \((a_0(980))\) with momentum \(k\). The self-energy \(f_S^2 \Sigma_R^{(\lambda)}(E, k)\) captures the effect of the two-meson loops on the resonance propagators. The renormalization factor \(Z\) is defined as \(Z = f_S^2 / f_S^0\), which is introduced in Eq. (16) to satisfy the condition that the residue of the renormalized propagator \(G_r(E, k)\) is one at \(E = \omega_r(k_{\text{on}}) + \omega_K(k_{\text{on}})\). (We will come back to this issue in following.) A key study of this work is to investigate the impact of the self-energy, and in particular certain approximations thereof, on the potentially emerging three-body bound states. For that study we introduced the parameter \(\alpha\) that will eventually be varied \(0 \leq \alpha \leq 1\), with \(\alpha = 1\) representing the fully unitary treatment, while \(\alpha = 0\) leads above the three-kaon threshold to a violation of unitarity and accordingly below this threshold to a violation of analyticity. The reason for this is that above the three-K threshold the self-energy generates an imaginary part that is necessary for the equations to be unitary. Below threshold this term needs to be continued analytically as otherwise the amplitude suffers from unphysical non-analyticities.

In Fig. 3, all relevant momenta are shown explicitly for the self-energy correction of \(f_0(980)\) or \(a_0(980)\) meson in TOPT.

The expression corresponding to Fig. 3 may be written in the following form
\[
f_S^2 \Sigma^{(\lambda)}(E, k) = \frac{f_S^2}{2\omega^{(\lambda)}(k)} \int \frac{d^3s}{(2\pi)^3} \frac{1}{4\omega_K(s+)\omega_K(s-) 1} \times \frac{1}{E - \omega_K(k) - \omega_K(s+) - \omega_K(s-) + i\epsilon},
\]
where \(s_\pm = s \pm k/2\). Since in this exploratory study the inelasticities of the \(a_0(980)\) and \(f_0(980)\) are neglected, both states appear as stable bound states. To ensure that the amplitudes generate the \(a_0(980)/f_0(980)K\) branch cuts correctly, instead of Eq. (17) in the three-body equations we need to employ a renormalized self-energy as shown in [57] and Appendix A.

\[
f_S^2 \Sigma^{(\lambda)}(E, k) = f_S^2 \Sigma^{(\lambda)}(E, k) - \text{Re} \left( f_S^2 \Sigma^{(\lambda)}(E, k_{\text{on}}) \right),
\]
where
\[
k_{\text{on}} = \sqrt{[E^2 - (m^{(\lambda)}_+ + m_K^2)] [E^2 - (m^{(\lambda)}_+ - m_K^2)]} / 2E.
\]
Note that this procedure needs to be generalized when inelastic channels [for our example \(\pi\pi\) for the \(f_0(980)\) and \(\pi\eta\) for the \(a_0(980)\)] are switched on, since the mentioned cut does not disappear, but moves into the complex plane of the unphysical sheet [58], with the location of the corresponding branch point being related to the complex pole position of the resonance involved.

As the inelastic sub-channels \(\pi\pi\) for \(f_0(980)\) and \(\pi\eta\) for \(a_0(980)\) are switched on as additional channels in the isobar self-energies, circular cuts will arise for the unequal mass of the constituent of the isobar. Furthermore, in a more refined model the isobars may be replaced by states emerging from finite range meson exchanges, giving rise to left-hand cuts of the two-body interactions. While those improvements call for more sophisticated methods for the numerical evaluation of
the pertinent integrals, the formalism ensures that even then only physical singularities need to be handled. Moreover, a more complete study calls for the inclusion of three-body forces. Those will change the pole positions compared to those generated by the interaction of the current model. These issues will be subject to future studies.

The expression of the self-energy as defined in Eq. (17) contains both \( k \) and \( E \) in a non-trivial way. In case of non-relativistic kinematics, one finds

\[
\omega_K(s_+) + \omega_K(s_-) = 2\omega_K(s) + k^2/(4m_K)
\]

(20)

where the \( k^2 \) term captures the kinetic energy of the two-kaon system in the overall center-of-mass frame. With this, the \( k \) and \( E \) dependence of the self-energy can be absorbed into an effective energy

\[
E_{\text{eff}}(E, k) = E - \omega_K(k) - k^2/(4m_K)
\]

and the self-energy depends on this single variable only. However, for relativistic kinematics this appears to be not possible and both the \( k \) and the \( E \) dependence need to be kept explicitly. While \( k \ll m_K \) and \( s \ll m_K \) might be a good approximation for the system studied here in the absence of inelastic channels, it is certainly invalid as soon as the \( \pi \pi \) and \( \pi \eta \) channels are included. Because of this and the reason presented in the introduction, we proceed using relativistic kinematics. Note that, to speed up the numerical solution of the integral equation, the self-energy can be calculated for the energies of interest outside the routine that fills the potential integral in the LS equation.

To render the LS equation in Eq. (14) well defined, it is regularized by a finite momentum cutoff \( \Lambda \). For consistency the same cutoff is employed for the self-energy defined in Eq. (18), although the regularized expression is formally convergent. In our calculation, we vary the value of \( \Lambda \) in the range from 0.5 to 2.0 GeV. To illustrate the resulting cutoff dependence of \( f_s^2 \Sigma_R(E, k) \), the numerical results corresponding to \( E = 1.474 \) GeV are shown in Fig. 5.

From the condition that the residue of the renormalized propagator \( G_r(E, k) \) in Eq. (16) is one at

\[
E = E_{\text{on}} = \omega^{(k)}(k_{\text{on}}) + \omega_K(k_{\text{on}})
\]

(21)

we get

\[
Z = 1 + \frac{d}{dE} f_s^2 \Sigma^{(k)}(E, k_{\text{on}}) \bigg|_{E=E_{\text{on}}}
\]

(22)

The derivation of this expression is presented in Appendix A. The physical value of \( f_s \) is fixed to be 3.74 GeV. The values of \( Z \) corresponding calculated in this way are quoted in Table 2. In addition to showing the values of \( Z \) for fixed energies, we show its energy dependence for \( \Lambda = 1 \) GeV and 2 GeV in Fig. 6. Thus we find a negligible energy dependence of the \( Z \) factor as it should be in general, however, this feature could have been distorted here by the non-covariance of the formalism.

Most of the integrals entering the LS equation are formally convergent. Only those that contain the \( s \)-channel diagrams lead to a divergence and correspondingly may lead to a sizeable regulator dependence. However, at least the divergence in the one particle reducible diagrams introduced via the kaon pole diagram can be absorbed into mass and wave function regularization. In this procedure the bare parameters \( f_1^{(0)}, f_2^{(0)}, f_3^{(0)} \) and \( m_K^{(0)} \), introduced in Eq. (13), are determined from

\[
\begin{aligned}
(f_1^{(0)})^2 &= \frac{1}{2} f_s^2 \left[ \frac{1}{2} (\Gamma_{11} + R \Gamma_{12}) \left( \Gamma_{11}^T + R \Gamma_{21}^T \right) + \frac{1}{2} f_s^2 \\ &\times \left( \Sigma_{11}^{(3)} + 2 R \Sigma_{12}^{(3)} + R^2 \Sigma_{22}^{(3)} \right) \right], \\
(f_2^{(0)})^2 &= f_1^{(0)} R, \\
m_K^{(0)} &= m_K - \left( f_1^{(0)} \Sigma_{11}^{(3)} + f_2^{(0)} \Sigma_{12}^{(3)} + f_3^{(0)} \Sigma_{22}^{(3)} \right),
\end{aligned}
\]

(23, 24, 25)
Table 2 The renormalization factor $Z$. In all cases $f_S = 3.74\text{ GeV}$ was employed. The $t$-symbol is added to the $\alpha$-value when the subleading contribution to the self energy, Fig. 4, was also included

| $E \text{ [GeV]}$ | $\alpha$ | $\Lambda \text{ [GeV]}$ | $Z$ |
|-------------------|----------|-----------------|-----|
| 0.495             | 1        | 0.5             | 0.223 |
| 0.495             | 1        | 1               | 0.168 |
| 0.495             | 1        | 1.5             | 0.154 |
| 0.495             | 1        | 2               | 0.148 |
| 0.495             | $1^t$    | 0.5             | 0.221 |
| 0.495             | $1^t$    | 1               | 0.162 |
| 0.495             | $1^t$    | 1.5             | 0.145 |
| 0.495             | $1^t$    | 2               | 0.137 |
| 1.474             | 1        | 0.5             | 0.224 |
| 1.474             | 1        | 1               | 0.168 |
| 1.474             | 1        | 1.5             | 0.154 |
| 1.474             | 1        | 2               | 0.148 |
| 1.474             | $1^t$    | 0.5             | 0.222 |
| 1.474             | $1^t$    | 1               | 0.162 |
| 1.474             | $1^t$    | 1.5             | 0.145 |
| 1.474             | $1^t$    | 2               | 0.137 |

Fig. 6 The $E$ dependence of the renormalization constant $Z$. The red (green) dashed and blue (brown) dashed lines correspond to $\Lambda = 1.0$ and 2.0 GeV, respectively. For the green dashed and brown dashed lines, the subleading contribution to the self energy, Fig. 4, was also included

The solution of the LS equation is found by straightforward numerical matrix inversion. For this we use the method given in Ref. [59]. In our calculation, a 40-point Gaussian quadrature yields stable results. Note that since we only study energies below the $a_0/f_0K$ threshold, no three-body singularities need to be dealt with numerically.

The requirement that Eq. (14) has a pole at some energy $E$ is equivalent to the condition

$$\det [I - V(E)G(E)] = 0.$$  \hfill (28)

For a given pole the binding energy is $E_B = m^{(1)} + m_K - E$, since we measure the energy relative to the $f_0(980)K$ threshold (which for the parameters employed here equals to the $a_0(980)K$ threshold).

4 Numerical results for the potential quadratic in $f_S^2$

For our study all parameters are fixed as discussed above, however, we still quote the bare, calculated parameters for the single and the coupled channel calculation in Table 3 to illustrate that the renormalization effects can in fact be quite sizeable. Moreover, note that for the coupled channel case, although the dressed couplings of $a_0(980)$ and $f_0(980)$ to kaon-antikaon are equal, the corresponding bare couplings are different due to the different isospin factors in the different channels.

We start the discussion by omitting the effect of the self-energy $\Sigma^{(k)}(E, k)$ in Eq. (16) by setting $\alpha = 0$ and $Z = 1$. In this case the three-body scattering generates a bound state pole very close to the threshold on the physical sheet, when the coupled-channel $Kf_0(980)K$ formalism is employed — the corresponding binding energies that arise when more and more terms are added to the potential are shown by the columns $f(a) - s(b)$ in the lines marked by $\alpha = 0$ in Table 4 and as the first four green bars in Fig. 7, where the relative importance of the different contributions calculated for $\Lambda = 1$ GeV is illustrated. For the single-channel $Kf_0(980)$ formalism, the three-body scattering does not generate a bound state, reflecting that the coupled-channel effect has a strong influence on the scattering process. Here, we employed $f_S = 3.74\text{ GeV}$, fixed via Eq. (3) by the masses of $f_0(980)$ and $a_0(980)$.

The dependence of the resulting binding energy on the four different cut-offs for the coupled channel case is also shown in Table 4. Clearly for the contributions discussed so far the cut-off dependence is rather weak. Although not directly reflected in the numbers reported in the table, it turns out that the most dominant contribution to the emergence of the bound state comes from the first diagram of the $t$-channel kaon exchange labeled as $t(a)$: When using only individual contributions in solving the LS equation, only this part of
the potential generates binding. This is expected, since this contribution contains the leading three-body singularity. The second \( t \)-channel contribution, although not binding by itself, is still important quantitatively as it increases the binding energy by about 30%. Also the two \( s \)-channel contributions are large individually, however, there are quite effective cancellations among \( t(b), s(a) \) and \( s(b) \) such that the binding energy deduced from the full potential to order \( f_2^3 \) for all cut-offs is within 2% of the one calculated from \( t(a) \) only (cf. the sixth column of Table 4).

Since both \( f_0(980) \) and \( a_0(980) \) couple strongly to the \( K\bar{K} \) system, the implications of three-body unitarity above the three kaon-threshold and its impact on analyticity should play an important role in the three-body dynamics. We thus repeat the calculation with \( \alpha = 1 \) and the values of \( Z \) given in Fig. 6. We can see that the binding energies calculated for the four different cut-offs become larger by more than a factor of 3, see the lines marked with \( \alpha = 1 \) in Table 4. Moreover, also the relative contributions from the individual other diagrams get enhanced (e.g. when diagram \( t(b) \) is added the binding energy get enhanced by almost a factor of 2; the contributions from the \( s \)-channel diagrams are even larger). Also the cut-off dependence is now larger. However, the binding energies deduced from the sum of the full potential to order \( f_2^3 \) remains very close to that calculated from \( t(a) \) only as shown by the numbers in bold face in the sixth column of Table 4 as well as the fourth bar in Fig. 7.

Thus, our numerical results suggest that a \( I(J^P) = \frac{1}{2} (0^+) \) bound state can be generated from coupled-channel \( \bar{K}f_0(980)-K a_0(980) \) interactions, at least as long as inelastic channels are omitted. The binding energies deduced from the interactions discussed so far are below 2 MeV. We also show that for reliable quantitative results the full potential to order \( f_2^3 \) needs to be included, since the individual pieces of the potential undergo significant cancellations.

### 5 Study of the violation of covariance

The potentials introduced in Sect. 2 have only physical singularities, but are not invariant under Lorentz transformations. In this section we investigate how much the results change when also the contributions of the stretched boxes are included in the potential that restore covariance at the one-loop level. We demonstrate below that their effect on the results is rather small — in any case of the same order a other contributions to the potential higher order in the couplings \( f_S \).

To show how the violation of covariance emerges in time-ordered perturbation theory, we start from the expression for the \( t \)-channel potential given in Eq. (10). The two terms can be combined to

\[
V_{i\lambda}^{i\lambda}(E, p', p) = \frac{f_2^2 T N}{\omega_{K}(q)} \left( \frac{\omega_{K}(q) - E_{K}}{\Delta E^2 - (\omega_{K}(q) - E_{K})^2} \right),
\]

where

\[
E_{\text{off}} = E - (\omega_1(p') + \omega_2(p') + \omega_1(p) + \omega_2(p))/2
\]

is the average off-shellness of the initial and final state for any give pair of momenta \( p \) and \( p' \) and

\[
\Delta E = (\omega_1(p') - \omega_2(p') + \omega_1(p) - \omega_2(p))/2
\]

denotes the energy transfer for initial and final particles on their mass shell. For both particles entering and leaving the potential being on the energy shell, \( E_{\text{off}} \) vanishes and the potential reduces to the well known, covariant Feynman amplitude.
Table 4 The numerical results for the binding energies for the coupled-channel $K_{f_0}(980)-\Lambda a_0(980)$ formalism for the individual contributions of the potential added in one-by-one with $f_S = 3.74$ GeV. The labels for the first 4 contributions refer to those of Figs. 1 and 2. The last two columns are labeled collectively via the types of the diagrams. The numbers quoted in the columns 3 and up are the binding energy in MeV as well as in brackets those binding energies in units of the leading contribution, $t(a)$, for the given calculation. The $\dagger$-symbol is added to the $\alpha$-value when the subleading contribution to the self energy, Fig. 4, was also included. The numbers in bold face denote the full result of the leading calculation and the calculation with higher order interactions and those that correct for Lorentz symmetry violation are included.

| $\alpha$ | $\Lambda$ [GeV] | $t(a)$ | $+t(b)$ | $+s(a)$ | $+s(b)$ | +stretched boxes | +crossed boxes |
|----------|-----------------|--------|----------|---------|---------|--------------------|----------------|
| 0        | 0.5             | 0.51 (1) | 0.66 (1.29) | 0.83 (1.63) | 0.52 (1.02) | 0.58 (1.14) | 0.68 (1.33) |
| 1        | 0.5             | 1.63 (1) | 2.51 (1.54) | 3.93 (2.41) | 1.63 (1) | 1.86 (1.14) | 2.21 (1.36) |
| 1$\dagger$ | 0.5            | 1.58 (1) | 2.44 (1.54) | 3.81 (2.41) | 1.59 (1.01) | 1.81 (1.15) | 2.16 (1.37) |
| 0        | 1               | 0.51 (1) | 0.68 (1.33) | 0.90 (1.76) | 0.52 (1.02) | 0.59 (1.16) | 0.70 (1.37) |
| 1        | 1               | 1.72 (1) | 3.03 (1.76) | 7.29 (4.24) | 1.67 (0.97) | 1.97 (1.15) | 2.45 (1.42) |
| 1$\dagger$ | 1              | 1.59 (1) | 2.79 (1.75) | 6.57 (4.13) | 1.60 (1.01) | 1.90 (1.19) | 2.36 (1.48) |
| 0        | 1.5             | 0.51 (1) | 0.68 (1.33) | 0.91 (1.78) | 0.51 (1) | 0.59 (1.16) | 0.70 (1.37) |
| 1        | 1.5             | 1.67 (1) | 3.02 (1.81) | 9.05 (5.42) | 1.68 (1.01) | 1.99 (1.19) | 2.49 (1.49) |
| 1$\dagger$ | 1.5           | 1.49 (1) | 2.66 (1.79) | 7.65 (5.13) | 1.62 (1.09) | 1.94 (1.30) | 2.43 (1.63) |
| 0        | 2               | 0.51 (1) | 0.68 (1.33) | 0.92 (1.80) | 0.51 (1) | 0.59 (1.16) | 0.70 (1.37) |
| 1        | 2               | 1.63 (1) | 2.95 (1.81) | 9.85 (6.04) | 1.70 (1.04) | 2.02 (1.24) | 2.54 (1.56) |
| 1$\dagger$ | 2              | 1.41 (1) | 2.51 (1.78) | 7.92 (5.62) | 1.66 (1.18) | 1.99 (1.41) | 2.51 (1.78) |

Fig. 7 The numerical results for the binding energies for the coupled-channel $K_{f_0}(980)-\Lambda a_0(980)$ formalism for the individual contributions of the potential added in one-by-one with $f_S = 3.74$ GeV and $\Lambda = 1$ GeV. The green bars correspond to $\alpha = 0$, the blue bars correspond to $\alpha = 1$ and the red bars correspond to $\alpha = 1$ where the subleading contribution to the self energy, Fig. 4, was also included. The labels for the first 4 contributions refer to those of Figs. 1 and 2. The last two contributions are labeled collectively via the types of the diagrams.

\[ V^{\lambda, \bar{\omega}}(E, p', p) = f_S^2 ITN \left( \frac{1}{t - m_K^2} \right), \]  

where $t$ is the four-momentum transfer squared, as it should be. However, when being put into the LS equation, both $p$ and $p'$ are integration variables and thus the potential is typically evaluated off-shell. Then clearly there is a difference between the two potentials. To keep covariance also then, the formalism of Ref. [42] calls for putting Eq. (30) into an LS type equation. This keeps formal covariance, however, it introduces unphysical singularities [54]. We propose to use the full potential of Eq. (29) or equivalently Eq. (10) instead. This avoids unphysical singularities, but violates covariance, since the potential depends on the particle energies which are not invariant under Lorentz transformations.

To quantify the amount of violation of Lorentz invariance, we calculate explicitly the contributions that restore it at the one loop level. This is achieved by the inclusion of the so-called stretched boxes. The corresponding diagrams are shown in Figs. 8, 9, 10 and 11, the related amplitudes are given in Eqs. (C.19)–(C.38) in the appendix. The effect of the inclusion of these diagrams into the potential on the resulting binding energies for the different calculations is illustrated by the second to last column in Table 4. The stretched boxes change the binding energies in all calculations by about 20% and we may regard this as a subleading contribution. It should be noted that to restore covariance also at the two-loop level higher stretched boxes (contributing at order $f_S^4$) would need to be included. Those are even more suppressed kinematically than the ones at order $f_S^2$, since more parti-
Fig. 8 The stretched boxes contribution

Fig. 9 The stretched boxes arise from $f_0(980)$ or $a_0(980)$ running backward in time

Fig. 10 The stretched boxes arise from kaons running backward in time

Fig. 11 The stretched boxes arise from both $f_0(980)$ or $a_0(980)$ and kaons running backward in time
icles are included in the equal time slices. We therefore conclude that the violation of Lorentz invariance for the equations employed here is in the energy range studied indeed mild and can be restored in a controlled way. In contrast to this introducing the mentioned unphysical singularities generates an error in the calculation that cannot be controlled quantitatively.

Moreover, there are additional diagrams at order $f_4^S$ that turn out to be of the same size as the stretched boxes, but are of a different topology. Those are the crossed boxes as shown in Figs. 12, 13, 14 and 15. The related amplitudes are given in Eqs. (C.39)–(C.62) in the appendix.

The effect of the mentioned contributions can again be read off from the last column of Table 4. It appears therefore
not appropriate to employ a formalism where the covariance of the ladder type diagrams is enforced but diagrams of the crossed box type are ignored.

The final contribution necessary to restore covariance at the one-loop level is an additional one loop-correction to the self-energy of the scalar mesons. The corresponding diagram is shown in Fig. 4. Since this contribution is not an addition to the scattering potential, but modifies the $f_0(980)/a_0(980)K$ propagator in the LS equation, it also modifies the binding energies calculated from the scattering potential to order $f_2^S$. Therefore we report its effect by additional lines in Table 4. Those are labeled with $\alpha = 1$.† The effect of this contribution is a moderate reduction of the calculated binding energies, however, the effect is no larger than the one of the crossed boxes. This can be traced to the fact that the energy dependence of the resulting self-energy-contribution is quite weak and thus the threshold subtraction to renormalize the self energy largely removes the contribution of the second time ordering.

It should be noted that the final results now including the restoration of the covariance to one loop shows a larger dependence of the regulator than the one without those corrections. The reason for this is most probably that the added terms contain in the time slices more of the heavier scalar fields that introduce larger momentum scales into the integrals. This observation indicates that a proper renormalization to this order might require a three-kaon counter term. However, addressing this issue, which calls for a proper power counting of the system, goes beyond the scope of this paper.

6 Dependence of the binding energies on the coupling constants

Finally, we study the dependence of the binding energies on the coupling constants. To be concrete, we consider the following two cases: In case A1, we increase the coupling constants $f_1$ and $f_2$ by a factor of $\sqrt{2}$, still demanding the masses to satisfy the Weinberg condition of Eq. (3). In case A2, the coupling $f_1$ and the masses are same as that in case A1, but the value of $f_2$ used is same as the original value (this means we include some elementary component in the $a_0$ wave function). Now, we discuss the two cases explicitly.

In case A1 the coupling constants and masses we employ are

$$f_1 = f_2 = f_S = 3.74 \times \sqrt{2} = 5.29 \text{ GeV},$$

and

$$m_{f_0(980)} = m_{a_0(980)} = 950 \text{ MeV},$$

which follow from Eq. (3). The resulting values for $Z$ are shown in Table 5 and the deduced bare parameters for the single-channel and coupled-channel calculation are shown in Table 6.

Again we start the discussion by omitting the effect of the self-energy by setting $\alpha = 0$ and $Z = 1$. For the coupled-channel $Kf_0(980) - Ka_0(980)$ formalism, the corresponding binding energies are shown in Table 7. As before the single-channel $Kf_0(980)$ formalism for the three-body scattering does not generate a bound state.

Comparing Tables 4 and 7, we can see as $f_2^S$ is increased by a factor of 2, the binding energy from $t(a)$ get enhanced by almost a factor of 3; the cut-off dependence remains weak. Also there are quite effective cancellations among $t(b)$, $s(a)$ and $s(b)$ such that the binding energy deduced from the full potential to order $f_2^S$ for all cut-offs is within 15% of the one calculated from $t(a)$ only.

As $f_2^S$ becomes large by a factor of 2, $f_4^S$ becomes larger by a factor of 4 and the contributions from stretched and crossed boxes get enhanced accordingly. At $\Lambda = 1 \text{ GeV}$, as illustrated in Table 7, the stretched boxes and crossed boxes change the
Table 5 The renormalization factor $Z$. In all cases $f_S = 5.29$ GeV was employed. The labels are same with that in Table 2

| $E$ [GeV] | $\alpha$ | $\Lambda$ [GeV] | $Z$ |
|-----------|----------|-----------------|-----|
| 0.495     | 1        | 0.5             | 0.393 |
| 0.495     | 1        | 1               | 0.288 |
| 0.495     | 1        | 1.5             | 0.260 |
| 0.495     | 1        | 2               | 0.247 |
| 0.495     | $1^\dagger$ | 0.5             | 0.388 |
| 0.495     | $1^\dagger$ | 1               | 0.274 |
| 0.495     | $1^\dagger$ | 1.5             | 0.239 |
| 0.495     | $1^\dagger$ | 2               | 0.222 |
| 1.444     | 1        | 0.5             | 0.395 |
| 1.444     | 1        | 1               | 0.290 |
| 1.444     | 1        | 1.5             | 0.262 |
| 1.444     | 1        | 2               | 0.250 |
| 1.444     | $1^\dagger$ | 0.5             | 0.390 |
| 1.444     | $1^\dagger$ | 1               | 0.277 |
| 1.444     | $1^\dagger$ | 1.5             | 0.243 |
| 1.444     | $1^\dagger$ | 2               | 0.227 |

Table 6 The calculated bare parameters for the single-channel (s) and coupled-channel (cc) calculation. The coupling constant and mass are employed in case A1. The labels are same with that in Table 3

| type | $\alpha$ | $\Lambda$ [GeV] | $f_1^{(0)}$ [GeV] | $f_2^{(0)}$ [GeV] | $m_k^{(0)}$ [GeV] |
|------|----------|-----------------|------------------|------------------|------------------|
| s    | 0        | 0.5             | 5.24             | –                | 0.505            |
| s    | 0        | 1               | 5.16             | –                | 0.532            |
| s    | 0        | 1.5             | 5.12             | –                | 0.558            |
| s    | 0        | 2               | 5.10             | –                | 0.580            |
| s    | 1        | 0.5             | 5.16             | –                | 0.517            |
| s    | 1        | 1               | 4.88             | –                | 0.585            |
| s    | 1        | 1.5             | 4.68             | –                | 0.650            |
| s    | 1        | 2               | 4.55             | –                | 0.703            |
| s    | $1^\dagger$ | 0.5             | 5.16             | –                | 0.517            |
| s    | $1^\dagger$ | 1               | 4.84             | –                | 0.584            |
| s    | $1^\dagger$ | 1.5             | 4.61             | –                | 0.646            |
| s    | $1^\dagger$ | 2               | 4.45             | –                | 0.695            |
| cc   | 0        | 0.5             | 5.08             | 5.32             | 0.537            |
| cc   | 0        | 1               | 4.78             | 5.35             | 0.652            |
| cc   | 0        | 1.5             | 4.61             | 5.35             | 0.766            |
| cc   | 0        | 2               | 4.52             | 5.34             | 0.860            |
| cc   | 1        | 0.5             | 4.77             | 5.28             | 0.583            |
| cc   | 1        | 1               | 3.59             | 5.05             | 0.856            |
| cc   | 1        | 1.5             | 2.79             | 4.74             | 1.087            |
| cc   | 1        | 2               | 2.31             | 4.47             | 1.242            |
| cc   | $1^\dagger$ | 0.5             | 4.76             | 5.27             | 0.583            |
| cc   | $1^\dagger$ | 1               | 3.49             | 4.92             | 0.837            |
| cc   | $1^\dagger$ | 1.5             | 2.62             | 4.46             | 1.019            |
| cc   | $1^\dagger$ | 2               | 2.12             | 4.09             | 1.122            |

binding energies in all calculations by about 45% and 75%, respectively. As before their contributions are individually similar in size.

We repeat the calculation $\alpha = 1$ and the values of $Z$ given in Table 5. Comparing Tables 4 and 7, we can see as $f_S^2$ becomes large by a factor of 2, the binding energy from $t(\alpha)$ get enhanced by almost a factor of 2. The binding energies deduced from the sum of the full potential to order $f_S^2$ remain very close to those calculated from $t(\alpha)$ only as shown by the numbers in bold face in the sixth column of Table 7. Furthermore, at $\Lambda = 1$ GeV, the streched boxes and crossed boxes change the binding energies in all calculations by about 50% and 80%, respectively. The cut-off dependence of the streched boxes and crossed boxes contribution is very strong. And with the cut-off increasing, the streched and crossed boxes contribution become even larger, indicating that for this subleading order a counter term needs to be included to arrive at a consistent effective field theory.

In case A2, we use different values of $f_1$ and $f_2$. The coupling constants we employed are

$$f_1 = 3.74 \times \sqrt{2} = 5.29 \text{ GeV},$$

$$f_2 = 3.74 \text{ GeV},$$

and the masses used are same with that in case A1 in Eq. (32). In case A2, the values of $Z$ corresponding to renormalization of $f_0(980)$ is same with that in case A1, the values of $Z$ corresponding to renormalization of $a_0(980)$ are shown in Table 8.

For unequal values of $f_1$ and $f_2$, in order to calculate the bare coupling constants $f_1^{(0)}$ and $f_2^{(0)}$, $R$ in Eq. (26) should be replaced by

$$R = \frac{\sqrt{3}c \Gamma_{11} - \Gamma_{21}}{-\sqrt{3}c \Gamma_{12} + \Gamma_{22}},$$

with $c = f_2/f_1 = 1/\sqrt{2}$. Then we repeat the procedure with the same coupling constant to get the values of $f_1^{(0)}$ and $f_2^{(0)}$ as shown in Table 9.

We start the discussion by omitting the effect of the self-energy by setting $\alpha = 0$ and $Z = 1$. For the coupled-channel $K f_0(980)-K a_0(980)$ formalism, the corresponding binding energies are shown in Table 10. Comparing with case A1, we can see as $f_2$ is reduced by a factor $\sqrt{2}$, the binding energy from $t(\alpha)$ get reduced by almost a factor 4 but again, the binding energies deduced from the sum of the full potential to order $f_S^2$ remains very close to that calculated from $t(\alpha)$ only as shown by the numbers in the sixth column of Table 10.

In order to include the streched and crossed boxes with different values of $f_1$ and $f_2$, we need to the isospin factors for $a_0(980)$ and $f_0(980)$ in the loops individually. Those are given in Tables 11 and 12, respectively.
with case A1, we note that with different values of energies by about 70% and 140%, respectively. Compared as can be seen from Table 1. However, with different values of the potential added in one-by-one. The coupling constant and mass are employed in case A1. The labels are same as in Table 4.

| $\alpha$ | $\Lambda$ [GeV] | $t(a)$ | $+t(b)$ | $+s(a)$ | $+s(b)$ | $+\text{stretched boxes}$ | $+\text{crossed boxes}$ |
|---------|----------------|--------|---------|---------|---------|---------------------------|---------------------------|
| 0       | 0.5            | 1.32 (1) | 2.58 (1.95) | 4.36 (3.30) | 1.32 (1) | 1.81 (1.37) | 2.62 (1.98) |
| 1       | 0.5            | 2.54 (1) | 5.29 (2.08) | 9.81 (3.86) | 2.46 (0.97) | 3.35 (1.32) | 4.80 (1.89) |
| 1$^+$   | 0.5            | 2.4 (1)  | 5.05 (2.10) | 9.41 (3.92) | 2.35 (0.98) | 3.22 (1.34) | 4.62 (1.93) |
| 0       | 1              | 1.44 (1) | 3.0 (2.08) | 5.99 (4.16) | 1.28 (0.89) | 1.93 (1.34) | 3.03 (2.10) |
| 1       | 1              | 2.87 (1) | 7.30 (2.54) | 21.70 (7.56) | 2.59 (0.90) | 3.94 (1.37) | 6.26 (2.18) |
| 1$^+$   | 1              | 2.47 (1) | 6.40 (2.59) | 19.17 (7.76) | 2.49 (1.01) | 3.82 (1.55) | 6.10 (2.47) |
| 0       | 1.5            | 1.45 (1) | 3.04 (2.10) | 6.41 (4.42) | 1.24 (0.86) | 1.89 (1.30) | 3.0 (2.07) |
| 1       | 1.5            | 2.74 (1) | 7.29 (2.66) | 28.03 (10.23) | 2.71 (0.99) | 4.19 (1.53) | 6.72 (2.45) |
| 1$^+$   | 1.5            | 2.18 (1) | 5.98 (2.74) | 23.18 (10.63) | 2.74 (1.26) | 4.27 (1.96) | 6.89 (3.16) |
| 0       | 2              | 1.45 (1) | 3.04 (2.10) | 6.57 (4.53) | 1.22 (0.84) | 1.87 (1.29) | 2.97 (2.05) |
| 1       | 2              | 2.60 (1) | 7.03 (2.70) | 30.95 (11.90) | 2.89 (1.11) | 4.46 (1.72) | 7.15 (2.75) |
| 1$^+$   | 2              | 1.94 (1) | 5.47 (2.82) | 24.30 (12.53) | 3.02 (1.56) | 4.70 (2.42) | 7.58 (3.91) |

### Table 7
The numerical results for the binding energies for the coupled-channel $K f_0(980) - K a_0(980)$ formalism for the individual contributions of the potential added in one-by-one. The coupling constant and mass are employed in case A1. The labels are same as in Table 4.

### Table 8
The renormalization factor $Z$. In all cases $f_2 = 3.74$ GeV and $m_{a_0}(980) = 950$ MeV were employed. The labels are same as in Table 2.

### Table 9
The calculated bare parameters for the coupled-channel (cc) calculation. The coupling constant and mass are employed in case A2. The labels are same as in Table 3.

| $E$ [GeV] | $\alpha$ | $\Lambda$ [GeV] | $Z$ |
|----------|----------|----------------|-----|
| 0.495    | 1        | 0.5            | 0.696 |
| 0.495    | 1        | 1              | 0.644 |
| 0.495    | 1        | 1.5            | 0.630 |
| 0.495    | 1        | 2              | 0.624 |
| 0.495    | 1$^+$    | 0.5            | 0.694 |
| 0.495    | 1$^+$    | 1              | 0.637 |
| 0.495    | 1$^+$    | 1.5            | 0.620 |
| 0.495    | 1$^+$    | 2              | 0.611 |
| 1.444    | 1        | 0.5            | 0.697 |
| 1.444    | 1        | 1              | 0.645 |
| 1.444    | 1        | 1.5            | 0.631 |
| 1.444    | 1        | 2              | 0.625 |
| 1.444    | 1$^+$    | 0.5            | 0.695 |
| 1.444    | 1$^+$    | 1              | 0.639 |
| 1.444    | 1$^+$    | 1.5            | 0.622 |
| 1.444    | 1$^+$    | 2              | 0.614 |

The box contributions with different values of $f_1$ and $f_2$ are shown in Table 10. We can see that at $\Lambda = 1$ GeV, the stretched boxes and crossed boxes change the binding energies by about 70% and 140%, respectively. Compared with case A1, we note that with different values of $f_1$ and $f_2$, the stretched and crossed boxes contributions become even larger. The reason of this is that with same values of $f_1$ and $f_2$, the off-diagonal element $K f_0(980) \rightarrow K a_0(980)$ is zero as can be seen from Table 1. However, with different values of $f_1$ and $f_2$, the off-diagonal element $K f_0(980) \rightarrow K a_0(980)$ starts to contribute, as can be seen from Tables 11 and 12.

We repeat the calculation $\alpha = 1$ and the values of $Z$ given in Table 8. The corresponding binding energies are shown in Table 10. Also for this parameter set some quite effective cancellations exist among $t(b)$, $s(a)$ and $s(b)$ (see Table 10).

The box contributions with different coupling constant of $f_1$ and $f_2$ can be read from Table 10. We can see at $\Lambda = 1$ GeV, the stretched and crossed boxes change the binding energies by about 70% and 130%, respectively.

To summarize this chapter, while varying the parameters changes the results quantitatively, the patterns reported in the previous chapters persist: There is still some significant
The numerical results for the binding energies for the coupled-channel $Kf_0(980)-K\alpha_0(980)$ formalism for the individual contributions of the potential added in one-by-one. The coupling constant and mass are employed in case A2. The labels are same with that in Table 4.

| $\alpha$ | $\Lambda$ [GeV] | $t(a)$ | $+t(b)$ | $+s(a)$ | $+s(b)$ | $+$stretched boxes | $+$crossed boxes |
|---------|-----------------|--------|---------|---------|---------|-------------------|-----------------|
| 0       | 0.5             | 0.31 (1) | 0.88 (2.84) | 1.60 (5.16) | 0.38 (1.23) | 0.57 (1.84) | 0.91 (2.94) |
| 1       | 0.5             | 0.67 (1) | 1.85 (2.76) | 3.43 (5.12) | 0.81 (1.21) | 1.17 (1.75) | 1.78 (2.66) |
| $1^+$   | 0.5             | 0.62 (1) | 1.74 (2.81) | 3.26 (5.26) | 0.75 (1.21) | 1.10 (1.77) | 1.69 (2.73) |
| 0       | 1               | 0.35 (1) | 1.04 (1.98) | 2.17 (6.20) | 0.38 (1.09) | 0.64 (1.83) | 1.12 (3.20) |
| 1       | 1               | 0.74 (1) | 2.43 (3.27) | 6.06 (8.19) | 0.82 (1.11) | 1.35 (1.82) | 2.29 (3.09) |
| $1^+$   | 1               | 0.59 (1) | 2.08 (3.53) | 5.30 (8.98) | 0.69 (1.17) | 1.17 (1.98) | 2.04 (3.46) |
| 0       | 1.5             | 0.35 (1) | 1.06 (3.03) | 2.29 (6.54) | 0.37 (1.06) | 0.63 (1.80) | 1.12 (3.20) |
| 1       | 1.5             | 0.68 (1) | 2.37 (3.49) | 6.74 (9.91) | 0.76 (1.12) | 1.28 (1.88) | 2.24 (3.29) |
| $1^+$   | 1.5             | 0.48 (1) | 1.87 (3.90) | 5.51 (11.48) | 0.60 (1.25) | 1.06 (2.21) | 1.91 (3.98) |
| 0       | 2               | 0.35 (1) | 1.06 (3.03) | 2.33 (6.66) | 0.36 (1.03) | 0.63 (1.80) | 1.11 (3.17) |
| 1       | 2               | 0.63 (1) | 2.26 (3.59) | 6.83 (10.84) | 0.72 (1.14) | 1.24 (1.97) | 2.17 (3.44) |
| $1^+$   | 2               | 0.40 (1) | 1.66 (4.15) | 5.29 (13.23) | 0.54 (1.35) | 0.98 (2.45) | 1.81 (4.53) |

Table 11: Isospin factors for box diagram contributions with $Kf_0(980)$ intermediate state

| $Kf_0(980) \rightarrow Kf_0(980)$ | $Kf_0(980) \rightarrow K\alpha_0(980)$ | $K\alpha_0(980) \rightarrow K\alpha_0(980)$ |
|----------------------------------|--------------------------------------|----------------------------------|
| Coupled-channel: boxes in Figs. 8, 11, 13 and 14 | $\frac{1}{4}$ | $\sqrt{2}$ | $\frac{3}{4}$ |
| Coupled-channel: boxes in Figs. 9, 10, 12 and 15 | $\frac{1}{4}$ | $\sqrt{2}$ | $-\frac{1}{4}$ |

Table 12: Isospin factors for box diagram contributions with $K\alpha_0(980)$ intermediate state

| $Kf_0(980) \rightarrow Kf_0(980)$ | $Kf_0(980) \rightarrow K\alpha_0(980)$ | $K\alpha_0(980) \rightarrow K\alpha_0(980)$ |
|----------------------------------|--------------------------------------|----------------------------------|
| Coupled-channel: boxes in Figs. 8, 11, 13 and 14 | $\frac{3}{4}$ | $-\sqrt{2}$ | $\frac{1}{4}$ |
| Coupled-channel: boxes in Figs. 9, 10, 12 and 15 | $\frac{3}{4}$ | $-\sqrt{2}$ | $\frac{5}{4}$ |

cancellation between $t(b)$, $s(a)$ and $s(b)$ and the crossed and stretched boxes contribute with similar strengths in all cases.

7 Summary and outlook

In this work we provide additional support for the proposal put forward in Ref. [15] to employ time ordered perturbation theory with relativistic particle energies in calculations of three-body scattering. To be concrete, we demonstrate on the example of isospin 1/2 three-body $KK\bar{K}$ scattering in the isobar formalism, where both $\alpha_0(980)$ and $f_0(980)$ are assumed to be shallow near threshold $KK$ bound states, that the effect of the violation of covariance on the binding energies for three-body bound states is rather mild and can even be removed in a systematic way by inclusion of the stretched boxes — although a restoration of covariance at the two-loop level in practice would be a formidable task given the large number of time orderings that would need to be included. Moreover, we demonstrate in addition that the so-called crossed box contributions are of similar importance than the stretched boxes. This shows that a systematic calculation of three-body bound states using time ordered perturbation theory could be set up by including at leading order just the contributions to the potential at order $f_0^2$ as well as the leading contribution to the self-energies (omitting the latter leads to binding energies of the three-body system about a factor of three too small), at next-to-leading order the two-particle reducible contributions (here two-particle refers for the concrete example studied here to an intermediate state of an isobar and a kaon) at order $f_0^3$ and so on, where the latter group should not only contain the new topologies that appear at this order but also the diagrams that restore covariance at the one loop level. All two-particle reducible contributions are enhanced kinematically and are automatically generated via the LS equation. Our results suggest that the binding energies of the three kaon system can be well estimated by including the contribution from $t(a)$ only in the potential, although this violates covariance.
This kind of procedure avoids the need to use covariant potentials in a three dimensional formalism that can introduce unphysical singularities [42,54] or to solve the very complicated four-dimensional scattering equations [61,62]. Clearly, a further improved calculation needs to consider the contribution from inelastic channels as well as the Goldstone boson nature of the participating particles.

Furthermore, we studied the dependence of the results on the coupling constants. We find that if we keep the couplings equal but increase their values, the cancellations among $t(b)$, $s(a)$ and $s(b)$ remain effective and our leading result remains close to that from diagram $t(a)$ alone. Naturally enhancing the couplings leads to an enhancement of the contributions from the stretched and crossed boxes. Thus with the coupling constants increasing, the effect of the violation of covariance of the leading order TOPT calculation of the binding energies of the three-body system increases. However, the diagram $t(a)$ still gives the dominant contribution to the binding and crossed and stretched boxes are of similar size such that the main conclusions given above are still in place. Also for unequal coupling constants, the effective cancellations among $t(b)$, $s(a)$ and $s(b)$ remains large, but now the effect of the violation of covariance on the binding energies for three-body bound states gets enhanced even more.

A systematic, quantitative understanding of the relative importance of the different contributions still needs to be found. On the path to a proper effective field theory employing relativistic three-body equations the crucial findings of this work are that the leading three-body interaction is provided by diagram $t(a)$. This contribution is crucial for the emergence of the three-body bound state and shows only mild regulator dependence. While this contribution by itself violates relativistic covariance, we demonstrated that the inclusion of the diagrams necessary to restore covariance at the one-loop level provides only subleading contributions in the sense that the inclusion of only those in the potential does not lead to bound states, although for some parameter sets they can shift the pole location significantly. Moreover, those contributions turn out to be of similar size than the so-called crossed boxes and in addition show a significant regulator dependence, indicating that a properly renormalised effective field theory would need a counter term at this order.

Acknowledgements This work is supported in part by the National Natural Science Foundation of China (NSFC) and the Deutsche Forschungsgemeinschaft (DFG) through the funds provided to the Sinogerman Collaborative Research Center TRR110 “Symmetries and the Emergence of Structure in QCD” (NSFC Grant no. 12070131001, DFG Project-ID 196253076), by the Chinese Academy of Sciences (CAS) through a President’s International Fellowship Initiative (PIFI) (Grant no. 2018DM0034), by the VolkswagenStiftung (Grant no. 93562), and by the EU Horizon 2020 research and innovation programme, STRONG-2020 project under grant agreement no. 824093. It is also supported by the National Natural Science Foundation of China under Grant nos. 12075288, 11735003, and 1196114012. Furthermore, Xu Zhang acknowledges financial support from the China Scholarship Council.

Data Availability Statement This manuscript has no associated data or the data will not be deposited. [Authors’ comment: This is a theoretical paper and there are no data associated with the work.]

Appendix A: The renormalized propagator

The unrenormalized or bare propagator in channel $\lambda$ reads

$$G^{(\lambda)}_b(E, k) = \frac{1}{E - \sqrt{m_0^{(\lambda)^2} + k^2 - \omega_K(k)} - f_S^0 \Sigma^{(\lambda)}(E, k)},$$

(A.1)

where $\sqrt{m_0^{(\lambda)^2} + k^2}$ is the bare free energy of the isobar, $f_S^0$ is the bare coupling for the isobar to $K \bar{K}$, and the unrenormalized self-energy $f_S^0 \Sigma^{(\lambda)}(E, k)$ is given Eq. (17).

By requiring that $G^{(\lambda)}_b(E, k)$ has pole at the $E_{on}$ defined in Eq. (21), we may write

$$G^{(\lambda)}_b(E, k) = \left( E - \omega^{(\lambda)}(k) - \omega_K(k) - f_S^0 \Sigma^{(\lambda)}(E, k) + f_S^0 \Sigma^{(\lambda)}(E, k_{on}) \right)^{-1},$$

(A.2)

where we have fixed the bare energy via

$$\sqrt{m_0^{(\lambda)^2} + k^2} = \omega^{(\lambda)}(k) - f_S^0 \Sigma^{(\lambda)}(E, k_{on}).$$

(A.3)

Then we may write

$$G^{(\lambda)}_b(E, k) = \left( E - \omega^{(\lambda)}(k) - \omega_K(k) - f_S^0 \Sigma^{(\lambda)}_R(E, k) \right)^{-1},$$

(A.4)

where

$$\Sigma^{(\lambda)}_R(E, k) = \Sigma^{(\lambda)}(E, k) - \Sigma^{(\lambda)}(E, k_{on}).$$

(A.5)

Note that for $E < \sqrt{m_0^{(\lambda)^2} - m_K^2}$, the continuation $\omega_K(k_{on}) \rightarrow -\omega_K(k_{on})$ needs to be employed.

The renormalized propagator, $G^{(\lambda)}_b(E, k)$ is related to the bare propagator, $G^{(\lambda)}_b(E, k)$, by $G^{(\lambda)}_b(E, k) = G^{(\lambda)}_b(E, k)Z^{-1}$. Its residue at the pole is

$$\lim_{E \rightarrow E_{on}} \frac{G^{(\lambda)}_b(E - \omega^{(\lambda)}(k) - \omega_K(k))Z^{-1}G^{(\lambda)}_b(E, k)}{E - E_{on}} = Z^{-1} \left( \frac{1}{1 - \frac{d}{dE} f_S^0 \Sigma^{(\lambda)}_R(E, k_{on})} \right).$$

(A.6)
By definition this residue needs to be one. Using the definition of the renormalized coupling $f_S^0 = f_S^0 Z$, we get

$$\frac{f_S^0}{f_S^0} = 1 - \frac{d}{dE} f_S^0 \sum^{(3)}(E, k_{on})|_{E=E_{on}}.$$ (A.7)

Hence, we have

$$f_S^0 = 1 + \frac{d}{dE} f_S^0 \sum^{(3)}(E, k_{on})|_{E=E_{on}},$$ (A.8)

and

$$Z = 1 + \frac{d}{dE} f_S^0 \sum^{(3)}(E, k_{on})|_{E=E_{on}}.$$ (A.9)

Accordingly the expression for the renormalized propagator reads

$$G^\gamma_r(E, k) = \left[Z\left(E - \sqrt{m(k)^2 + k^2 - \omega_k(k)}\right) - \alpha f_S^0 \sum^{(3)}(E, k)\right]^{-1}.$$ (A.10)

**Appendix B: Renormalization of the kaon pole contribution**

In this appendix we outline the renormalization procedure for the kaon pole contribution following the procedure used in Ref. [60] to renormalize the nucleon pole in $\pi N$ scattering. Clearly, when the kaon $s$-channel pole is included into the potential of the LS-equation both its coupling to the scalar fields and a kaon as well as its mass get renormalized. Thus, to have in the final amplitude kaon pole and residue correct we need to employ a potential that is formulated in terms of bare parameters (c.f. Eq. (13)). In general one has

$$\frac{1}{2} f_1 f_1 = \left(\Gamma_{11} f_1^{(0)} + \Gamma_{12} f_2^{(0)}\right)\left(1 - f_1^{(0)} \sum^{(3)} \right) - \frac{1}{2} f_2^{(0)} \sum^{(3)} (E, k_{on}) = \left(\Gamma_{21} f_1^{(0)} + \Gamma_{22} f_2^{(0)}\right),$$ (B.11)

$$\sqrt{3} \frac{1}{2} f_1 f_2 = \left(\Gamma_{11} f_1^{(0)} + \Gamma_{12} f_2^{(0)}\right)\left(1 - f_1^{(0)} \sum^{(3)} \right) - \frac{1}{2} f_2^{(0)} \sum^{(3)} (E, k_{on}) = \left(\Gamma_{21} f_1^{(0)} + \Gamma_{22} f_2^{(0)}\right),$$ (B.12)

$$\frac{3}{2} f_2 f_2 = \left(\Gamma_{21} f_1^{(0)} + \Gamma_{22} f_2^{(0)}\right)\left(1 - f_1^{(0)} \sum^{(3)} \right) - \frac{1}{2} f_2^{(0)} \sum^{(3)} (E, k_{on}) = \left(\Gamma_{11} f_1^{(0)} + \Gamma_{12} f_2^{(0)}\right).$$ (B.13)

In these expressions the kaon self energy $f_1^{(0)} f_2^{(0)} \sum^{(3)}(E)$ and dressed vertex function $\Gamma_{1j}^{T}(E, k)$ get generated by the LS-equation. Explicitly the $\sum^{(3)}(E)$ can be written as

$$\sum^{(3)}(E) = \sum_{\lambda, \lambda'} \left(\mathcal{I}^3 \mathcal{I}^{\lambda'}\right) \frac{1}{2 m_K} \times \left(\delta_{\lambda, \lambda'} \int (2\pi)^3 2\omega_{(1)}(q) 2\omega_K(q) G_E(E, q)\right)$$

$$\times \left\{\frac{d^3q d^3q'}{2 m_K} \left[(2\pi)^3 2\omega_{(1)}(q) 2\omega_K(q) 2\omega_{(1)}(q') 2\omega_K(q')\right] \times G_E(E, q') \sqrt{2\omega_{(1)}(q) 2\omega_K(q)} \sqrt{2\omega_{(1)}(q') 2\omega_K(q')} \times \tilde{T}^{\lambda, \lambda'}(E, q', q) G_E(E, q),\right\}.$$ (B.14)

where $\tilde{T}^{\lambda, \lambda'}(E, q', q)$ is the solution of the LS-equation employing the non-pole part of the potential, namely the $t$-channel contribution in Fig. 1a and b and $s$-channel contribution in Fig. 2a. Isospin factors $\mathcal{I}^3 \mathcal{I}^{\lambda'}$ are $\mathcal{I}^1 \mathcal{I}^1 = \frac{1}{2}$, $\mathcal{I}^1 \mathcal{I}^2 = \sqrt{\frac{3}{2}}$ and $\mathcal{I}^2 \mathcal{I}^2 = \frac{3}{2}$. Clearly the procedure can be straightforwardly generalised to include also higher orders in the potential.

The vertex functions $\Gamma_{\lambda, \lambda'}(E, k)$ can be written as

$$\Gamma_{\lambda, \lambda'}(E, k) = \mathcal{I}^{\lambda'} \frac{d^3q}{(2\pi)^3 2\omega_{(1)}(q) 2\omega_K(q)} \frac{\sqrt{2\omega_{(1)}(k) 2\omega_K(k)}}{\sqrt{2\omega_{(1)}(q) 2\omega_K(q)} \sqrt{2\omega_{(1)}(q') 2\omega_K(q')} G_E(E, q)}.$$ (B.15)

where the isospin factors are $\mathcal{I}^1 = \sqrt{\frac{1}{2}}$ and $\mathcal{I}^2 = \sqrt{\frac{3}{2}}$.

As explained in the main text, based on general considerations, the physical coupling $f_S$ is known when we assume that both $a_0(980)$ and $f_0(980)$ are $\tilde{K} K$ bound states. Moreover, the physical kaon mass is known, while the bare parameters are unknown. Thus we need to express the latter in terms of the former. Solving Eqs. (B.12) and (B.13) for $f_1^{(0)}$ and $f_2^{(0)}$ and taking $f_1 = f_2 = f_S$ gives

$$f_1^{(0)} = \frac{1}{2} f_S^2 \left[ \left(\Gamma_{11} + R \Gamma_{12}\right) \left(\Gamma_{11} + R \Gamma_{12}\right) + \frac{1}{2} f_S^2 \right]$$

$$f_2^{(0)} = f_1^{(0)} R,$$ (B.16)

$$f_2^{(0)} = f_1^{(0)} R.$$ (B.17)
which agrees to Eqs. (23) and (24). With the bare coupling determined we can now also calculate the bare mass from
\[ m_K^{(0)} = m_K - \left( f_1^{(0)} \Sigma_{11}^{(3)} + 2 f_2^{(0)} \Sigma_{12}^{(3)} + f_2^{(0)} \Sigma_{22}^{(3)} \right). \]  
(B.18)

**Appendix C: Contributions proportional to \( f_S^4 \)**

In the framework of TOPT, the expressions corresponding to the diagrams of Fig. 8 may be written in the following form
\[ V_{\text{strut-\theta}}(E, p', p) = f_S^4 \int_0^\Lambda d^3 q \frac{1}{(2\pi)^3} \]
\[ \times \frac{1}{16\omega_K(p+q)\omega_K(q)\omega_K(p'+q)} \]
\[ \times \frac{1}{E - \omega_1(p) - \omega_K(p + q) - \omega_K(q) + i\varepsilon} \]
\[ \times \frac{1}{E - \omega_1(p) - \omega_K(p + q) - \omega_2(p' + q) + i\varepsilon} \]
\[ \times \frac{1}{E - \omega_2(p) - \omega_K(p + q) - \omega_K(p' + q) + \omega_1(p) + i\varepsilon} \]
\[ \times \frac{1}{E - \omega_1(p') - \omega_K(p' + q) - \omega_K(q) + i\varepsilon} \]
(C.19)

In the framework of TOPT, the expressions corresponding to the diagrams of Fig. 9 may be written in the following form
\[ V_{\text{strut-\theta}}(E, p', p) = f_S^4 \int_0^\Lambda d^3 q \frac{1}{(2\pi)^3} \]
\[ \times \frac{1}{16\omega_K(p+q)\omega_K(q)\omega_K(p'+q)} \]
\[ \times \frac{1}{E - \omega_1(p) - \omega_K(p + q) - \omega_K(q) + i\varepsilon} \]
\[ \times \frac{1}{E - \omega_1(p) - \omega_K(p + q) - \omega_1(p') + i\varepsilon} \]
\[ \times \frac{1}{E - \omega_1(p') - \omega_K(p' + q) - \omega_K(q) + i\varepsilon} \]
\[ \times \frac{1}{E - \omega_1(p') - \omega_K(p' + q) - \omega_1(p') + i\varepsilon} \]
(C.20)

In the framework of TOPT, the expressions corresponding to the diagrams of Fig. 10 may be written in the following form
\[ V_{\text{strut-\theta}}(E, p', p) = f_S^4 \int_0^\Lambda d^3 q \frac{1}{(2\pi)^3} \]
\[ \times \frac{1}{16\omega_K(p+q)\omega_K(q)\omega_K(p'+q)} \]
\[ \times \frac{1}{E - \omega_1(p) - \omega_K(p + q) - \omega_K(q) + i\varepsilon} \]
\[ \times \frac{1}{E - \omega_1(p) - \omega_K(p + q) - \omega_1(p') + i\varepsilon} \]
\[ \times \frac{1}{E - \omega_1(p') - \omega_K(p' + q) - \omega_K(q) + i\varepsilon} \]
\[ \times \frac{1}{E - \omega_1(p') - \omega_K(p' + q) - \omega_1(p') + i\varepsilon} \]
(C.21)
\begin{align}
V_{\text{stret-negk--a}}^\lambda(E, p', p) &= f_{3}^{\lambda} N \int_{0}^{\Lambda} \frac{d^{3}q}{(2\pi)^{3}} \\
& \times \frac{1}{16\omega_{K}(p' - q)\omega_{K}(q)\omega^{(1)}(p + p' - q)\omega_{K}(p - q)} \\
& \times \frac{1}{1/[E - \omega_{1}(p) - \omega_{2}(p) - \omega_{2}^{\prime}(p') - \omega_{K}(p' - q)} \\
& - \omega_{K}(q) + i\epsilon \big] \\
& \times E - \omega_{1}(p) - \omega_{2}(p') - \omega_{K}(p - q) - \omega_{K}(p' - q) + i\epsilon \\
& \times E - \omega_{2}(p') - \omega_{K}(p - q) - \omega^{(1)}(p + p' - q) + i\epsilon,
\end{align}

\begin{align}
V_{\text{stret-negk--b}}^\lambda(E, p', p) &= f_{3}^{\lambda} N \int_{0}^{\Lambda} \frac{d^{3}q}{(2\pi)^{3}} \\
& \times \frac{1}{16\omega_{K}(p' - q)\omega_{K}(q)\omega^{(1)}(p + p' - q)\omega_{K}(p - q)} \\
& \times \frac{1}{1/[E - \omega_{2}(p) - \omega_{K}(p' - q) - \omega^{(1)}(p + p' - q)} \\
& - \omega_{K}(q) + i\epsilon \big] \\
& \times E - \omega_{2}(p) - \omega_{1}(p') - \omega_{K}(p - q) - \omega_{K}(p' - q) + i\epsilon \\
& \times E - \omega_{2}(p') - \omega_{K}(p - q) - \omega^{(1)}(p + p' - q) + i\epsilon.
\end{align}

\begin{align}
V_{\text{stret-negk--c}}^\lambda(E, p', p) &= f_{3}^{\lambda} N \int_{0}^{\Lambda} \frac{d^{3}q}{(2\pi)^{3}} \\
& \times \frac{1}{16\omega_{K}(p' - q)\omega_{K}(q)\omega^{(1)}(p + p' - q)\omega_{K}(p - q)} \\
& \times \frac{1}{1/[E - \omega_{1}(p) - \omega_{2}(p) - \omega_{2}(p') - \omega_{K}(p' - q)} \\
& - \omega_{K}(q) + i\epsilon \big] \\
& \times E - \omega_{2}(p) - \omega_{2}(p') - \omega_{K}(q) - \omega^{(1)}(p + p' - q) + i\epsilon \\
& \times E - \omega_{2}(p') - \omega_{K}(p - q) - \omega^{(1)}(p + p' - q) + i\epsilon.
\end{align}

\begin{align}
V_{\text{stret-negk--d}}^\lambda(E, p', p) &= f_{3}^{\lambda} N \int_{0}^{\Lambda} \frac{d^{3}q}{(2\pi)^{3}} \\
& \times \frac{1}{16\omega_{K}(p' - q)\omega_{K}(q)\omega^{(1)}(p + p' - q)\omega_{K}(p - q)} \\
& \times \frac{1}{1/[E - \omega_{2}(p) - \omega_{K}(p' - q) - \omega^{(1)}(p + p' - q)} \\
& - \omega_{K}(q) + i\epsilon \big] \\
& \times E - \omega_{2}(p) - \omega_{2}(p') - \omega_{K}(q) - \omega^{(1)}(p + p' - q) + i\epsilon \\
& \times E - \omega_{2}(p') - \omega_{K}(p - q) - \omega^{(1)}(p + p' - q) + i\epsilon.
\end{align}

\begin{align}
V_{\text{stret-negk--e}}^\lambda(E, p', p) &= f_{3}^{\lambda} N \int_{0}^{\Lambda} \frac{d^{3}q}{(2\pi)^{3}} \\
& \times \frac{1}{16\omega_{K}(p' - q)\omega_{K}(q)\omega^{(1)}(p + p' - q)\omega_{K}(p - q)} \\
& \times \frac{1}{1/[E - \omega_{1}(p) - \omega_{2}(p) - \omega_{2}(p') - \omega_{K}(p' - q)} \\
& - \omega_{K}(q) + i\epsilon \big] \\
& \times E - \omega_{2}(p) - \omega_{2}(p') - \omega_{K}(q) - \omega^{(1)}(p + p' - q) + i\epsilon \\
& \times E - \omega_{2}(p') - \omega_{K}(p - q) - \omega^{(1)}(p + p' - q) + i\epsilon.
\end{align}

\begin{align}
V_{\text{stret-negk--f}}^\lambda(E, p', p) &= f_{3}^{\lambda} N \int_{0}^{\Lambda} \frac{d^{3}q}{(2\pi)^{3}} \\
& \times \frac{1}{16\omega_{K}(p' - q)\omega_{K}(q)\omega^{(1)}(p + p' - q)\omega_{K}(p - q)} \\
& \times \frac{1}{1/[E - \omega_{2}(p) - \omega_{K}(p' - q) - \omega^{(1)}(p + p' - q)} \\
& - \omega_{K}(q) + i\epsilon \big] \\
& \times E - \omega_{2}(p) - \omega_{2}(p') - \omega_{K}(q) - \omega^{(1)}(p + p' - q) + i\epsilon \\
& \times E - \omega_{2}(p') - \omega_{K}(p - q) - \omega^{(1)}(p + p' - q) + i\epsilon.
\end{align}

In the framework of TOPT, the expression corresponding to Fig. 11 may be written in the following form

\begin{align}
V_{\text{stret-negks--a}}^\lambda(E, p', p) &= f_{3}^{\lambda} N \int_{0}^{\Lambda} \frac{d^{3}q}{(2\pi)^{3}} \\
& \times \frac{1}{16\omega_{K}(p' - q)\omega_{K}(q)\omega^{(1)}(p + p' - q)\omega_{K}(p - q)} \\
& \times \frac{1}{1/[E - \omega_{1}(p) - \omega_{2}(p) - \omega_{2}(p') - \omega_{K}(p' - q)} \\
& - \omega_{K}(q) + i\epsilon \big] \\
& \times E - \omega_{2}(p) - \omega_{1}(p') - \omega_{2}(p') - \omega_{K}(p' - q) - \omega_{K}(q) + i\epsilon \\
& \times E - \omega_{2}(p') - \omega_{K}(p - q) - \omega^{(1)}(p + p' - q) + i\epsilon.
\end{align}

\begin{align}
V_{\text{stret-negks--b}}^\lambda(E, p', p) &= f_{3}^{\lambda} N \int_{0}^{\Lambda} \frac{d^{3}q}{(2\pi)^{3}} \\
& \times \frac{1}{16\omega_{K}(p' - q)\omega_{K}(q)\omega^{(1)}(p + p' - q)\omega_{K}(p - q)} \\
& \times \frac{1}{1/[E - \omega_{2}(p) - \omega_{K}(p' - q) - \omega^{(1)}(p + p' - q)} \\
& - \omega_{K}(q) + i\epsilon \big] \\
& \times E - \omega_{2}(p) - \omega_{1}(p') - \omega_{2}(p') - \omega_{K}(p' - q) - \omega_{K}(q) + i\epsilon \\
& \times E - \omega_{2}(p') - \omega_{K}(p - q) - \omega^{(1)}(p + p' - q) + i\epsilon.
\end{align}

\begin{align}
V_{\text{stret-negks--c}}^\lambda(E, p', p) &= f_{3}^{\lambda} N \int_{0}^{\Lambda} \frac{d^{3}q}{(2\pi)^{3}} \\
& \times \frac{1}{16\omega_{K}(p' - q)\omega_{K}(q)\omega^{(1)}(p + p' - q)\omega_{K}(p - q)} \\
& \times \frac{1}{1/[E - \omega_{1}(p) - \omega_{2}(p) - \omega_{2}(p') - \omega_{K}(p' - q)} \\
& - \omega_{K}(q) + i\epsilon \big] \\
& \times E - \omega_{2}(p) - \omega_{1}(p') - \omega_{2}(p') - \omega_{K}(p' - q) - \omega_{K}(q) + i\epsilon \\
& \times E - \omega_{2}(p') - \omega_{K}(p - q) - \omega^{(1)}(p + p' - q) + i\epsilon.
\end{align}

\begin{align}
V_{\text{stret-negks--d}}^\lambda(E, p', p) &= f_{3}^{\lambda} N \int_{0}^{\Lambda} \frac{d^{3}q}{(2\pi)^{3}} \\
& \times \frac{1}{16\omega_{K}(p' - q)\omega_{K}(q)\omega^{(1)}(p + p' - q)\omega_{K}(p - q)} \\
& \times \frac{1}{1/[E - \omega_{2}(p) - \omega_{K}(p' - q) - \omega^{(1)}(p + p' - q)} \\
& - \omega_{K}(q) + i\epsilon \big] \\
& \times E - \omega_{2}(p) - \omega_{1}(p') - \omega_{2}(p') - \omega_{K}(p' - q) - \omega_{K}(q) + i\epsilon \\
& \times E - \omega_{2}(p') - \omega_{K}(p - q) - \omega^{(1)}(p + p' - q) + i\epsilon.
\end{align}
\[
\begin{align*}
&\times \frac{1}{16\omega_K(p'-q)\omega_K(q)\omega^{(1)}(q)\omega_K(p-q)} \\
&\times \frac{1}{1/[E - \omega_1(p) - \omega_2(p') - \omega_1'(p') - \omega_K(p' - q) - \omega^{(1)}(q) + i\varepsilon]} \\
&\times \frac{1}{1/[E - \omega_1(p) - \omega_2(p) - \omega_1'(p') - \omega_2(p') - \omega_K(q) + i\varepsilon]} \\
&\times \frac{1}{1/[E - \omega_2(p) - \omega_1'(p') - \omega_2(p') - \omega_K(p) - \omega_K(q) + i\varepsilon]}, \\
&V_{\text{strct} - \text{negkt} - \varepsilon}(E, p', p) = f_s^2 \int_0^\Lambda \frac{d^3q}{(2\pi)^3} \\
&\times \frac{1}{16\omega_K(p'-q)\omega_K(q)\omega^{(1)}(p)\omega_K(p-q)} \\
&\times \frac{1}{1/[E - \omega_1(p) - \omega_2(p) - \omega_1'(p') - \omega_2(p') - \omega_K(p' - q) - \omega^{(1)}(q) + i\varepsilon]} \\
&\times \frac{1}{1/[E - \omega_1(p) - \omega_2(p) - \omega_1'(p') - \omega_2(p') - \omega_K(q) + i\varepsilon]} \\
&\times \frac{1}{1/[E - \omega_2(p) - \omega_1'(p') - \omega_2(p') - \omega_K(p) - \omega_K(q) + i\varepsilon]}, \\
&V_{\text{strct} - \text{negkt} - f}(E, p', p) = f_s^2 \int_0^\Lambda \frac{d^3q}{(2\pi)^3} \\
&\times \frac{1}{16\omega_K(p'-q)\omega_K(q)\omega^{(1)}(p)\omega_K(p-q)} \\
&\times \frac{1}{1/[E - \omega_1(p) - \omega_2(p) - \omega_1'(p') - \omega_2(p') - \omega_K(p' - q) - \omega^{(1)}(q) + i\varepsilon]} \\
&\times \frac{1}{1/[E - \omega_2(p) - \omega_1'(p') - \omega_2(p') - \omega_K(p) - \omega_K(q) + i\varepsilon]}, \\
&\times \frac{1}{16\omega_K(p+q)\omega_K(q)\omega^{(1)}(p+p')\omega_K(p'+q)} \\
&\times \frac{1}{1/[E - \omega_2(p) - \omega_K(p+q) - \omega_1'(p') + i\varepsilon]} \\
&\times \frac{1}{1/[E - \omega_1(p) - \omega_K(p+q) - \omega_2(p') + i\varepsilon]} \\
&\times \frac{1}{1/[E - \omega_2(p) - \omega_K(p+q) - \omega^{(1)}(p+p') + i\varepsilon]}, \\
&V_{\text{cro} - \delta}(E, p', p) = f_s^2 \int_0^\Lambda \frac{d^3q}{(2\pi)^3} \\
&\times \frac{1}{16\omega_K(p+q)\omega_K(q)\omega^{(1)}(p+p')\omega_K(p'+q)} \\
&\times \frac{1}{1/[E - \omega_2(p) - \omega_K(p+q) - \omega^{(1)}(p+p') + i\varepsilon]} \\
&\times \frac{1}{1/[E - \omega_1(p) - \omega_K(p+q) - \omega_2(p') + i\varepsilon]} \\
&\times \frac{1}{1/[E - \omega_2(p) - \omega_K(p+q) - \omega^{(1)}(p+p') + i\varepsilon]}, \\
&\times \frac{1}{16\omega_K(p+q)\omega_K(q)\omega^{(1)}(p+p')\omega_K(p'+q)} \\
&\times \frac{1}{1/[E - \omega_2(p) - \omega_K(p+q) - \omega^{(1)}(p+p') + i\varepsilon]} \\
&\times \frac{1}{1/[E - \omega_1(p) - \omega_K(p+q) - \omega_2(p') + i\varepsilon]} \\
&\times \frac{1}{1/[E - \omega_2(p) - \omega_K(p+q) - \omega^{(1)}(p+p') + i\varepsilon]}, \\
&V_{\text{cro} - \gamma}(E, p', p) = f_s^2 \int_0^\Lambda \frac{d^3q}{(2\pi)^3} \\
&\times \frac{1}{16\omega_K(p+q)\omega_K(q)\omega^{(1)}(p+p')\omega_K(p'+q)} \\
&\times \frac{1}{1/[E - \omega_2(p) - \omega_K(p+q) - \omega^{(1)}(p+p') + i\varepsilon]} \\
&\times \frac{1}{1/[E - \omega_1(p) - \omega_K(p+q) - \omega_2(p') + i\varepsilon]} \\
&\times \frac{1}{1/[E - \omega_2(p) - \omega_K(p+q) - \omega^{(1)}(p+p') + i\varepsilon]}, \quad (C.39)
\end{align*}
\]
In the framework of TOPT, the expressions corresponding to the diagrams of Fig. 13 may be written in the following form

\[
\mathcal{V}_{\text{cross-\text{\_}negs-a}}^{\lambda, \lambda} (E, \mathbf{p}' , \mathbf{p}) = f_S^\lambda \mathcal{I} \int_0^\Lambda \frac{d^3q}{(2\pi)^3} \frac{1}{16\omega_K(p+q)\omega_K(q)\omega(1)(q)\omega_K(p'+q)} \\
\times \frac{1}{E - \omega_1(p) - \omega_K(p + q) - \omega_K(q) + i\varepsilon} \\
\times \frac{1}{E - \omega_2(p) - \omega_2'(p') - \omega_K(p + q) - \omega_K(q) + i\varepsilon} \\
\times \frac{1}{E - \omega_1(p) - \omega_1'(p') - \omega_K(p + q) - \omega_K(q) + i\varepsilon} \\
\times \frac{1}{E - \omega_1'(p') - \omega_K(p + q) - \omega_K(q) + i\varepsilon}.
\]

\[
\mathcal{V}_{\text{cross-\text{\_}negs-b}}^{\lambda, \lambda} (E, \mathbf{p}' , \mathbf{p}) = f_S^\lambda \mathcal{I} \int_0^\Lambda \frac{d^3q}{(2\pi)^3} \frac{1}{16\omega_K(p + q)\omega_K(q)\omega(1)(q)\omega_K(p'+q)} \\
\times \frac{1}{E - \omega_1(p) - \omega_1'(p') - \omega_K(p' + q) + i\varepsilon} \\
\times \frac{1}{E - \omega_2(p) - \omega_2'(p') - \omega_K(p' + q) + i\varepsilon} \\
\times \frac{1}{E - \omega_1(p) - \omega_1'(p') - \omega_K(p' + q) + i\varepsilon} \\
\times \frac{1}{E - \omega_1'(p') - \omega_K(p' + q) - \omega_K(q) + i\varepsilon}.
\]

\[
\mathcal{V}_{\text{cross-\text{\_}negs-c}}^{\lambda, \lambda} (E, \mathbf{p}' , \mathbf{p}) = f_S^\lambda \mathcal{I} \int_0^\Lambda \frac{d^3q}{(2\pi)^3} \frac{1}{16\omega_K(p + q)\omega_K(q)\omega(1)(q)\omega_K(p'+q)} \\
\times \frac{1}{E - \omega_1(p) - \omega_2(p) - \omega_2'(p') - \omega_K(p'+q) - \omega_K(q) + i\varepsilon} \\
\times \frac{1}{E - \omega_1(p) - \omega_1'(p') - \omega_K(p + q) - \omega_K(q) + i\varepsilon} \\
\times \frac{1}{E - \omega_1(p) - \omega_1'(p') - \omega_K(p + q) - \omega_K(q) + i\varepsilon} \\
\times \frac{1}{E - \omega_1'(p') - \omega_K(p + q) - \omega_K(q) + i\varepsilon}.
\]

\[
\mathcal{V}_{\text{cross-\text{\_}negs-d}}^{\lambda, \lambda} (E, \mathbf{p}' , \mathbf{p}) = f_S^\lambda \mathcal{I} \int_0^\Lambda \frac{d^3q}{(2\pi)^3} \frac{1}{16\omega_K(p + q)\omega_K(q)\omega(1)(q)\omega_K(p'+q)} \\
\times \frac{1}{E - \omega_2(p) - \omega_2'(p') - \omega_K(p' + q) + i\varepsilon} \\
\times \frac{1}{E - \omega_2(p) - \omega_2'(p') - \omega_K(p' + q) + i\varepsilon} \\
\times \frac{1}{E - \omega_2(p) - \omega_2'(p') - \omega_K(p' + q) + i\varepsilon} \\
\times \frac{1}{E - \omega_2(p) - \omega_2'(p') - \omega_K(p' + q) + i\varepsilon}.
\]

\[
\mathcal{V}_{\text{cross-\text{\_}negs-e}}^{\lambda, \lambda} (E, \mathbf{p}' , \mathbf{p}) = f_S^\lambda \mathcal{I} \int_0^\Lambda \frac{d^3q}{(2\pi)^3} \frac{1}{16\omega_K(p + q)\omega_K(q)\omega(1)(q)\omega_K(p'+q)} \\
\times \frac{1}{E - \omega_1(p) - \omega_1'(p') - \omega_K(p' + q) + i\varepsilon} \\
\times \frac{1}{E - \omega_1(p) - \omega_1'(p') - \omega_K(p' + q) + i\varepsilon} \\
\times \frac{1}{E - \omega_1(p) - \omega_1'(p') - \omega_K(p' + q) + i\varepsilon} \\
\times \frac{1}{E - \omega_1(p) - \omega_1'(p') - \omega_K(p' + q) + i\varepsilon}.
\]

\[
\mathcal{V}_{\text{cross-\text{\_}negs-f}}^{\lambda, \lambda} (E, \mathbf{p}' , \mathbf{p}) = f_S^\lambda \mathcal{I} \int_0^\Lambda \frac{d^3q}{(2\pi)^3} \frac{1}{16\omega_K(p + q)\omega_K(q)\omega(1)(q)\omega_K(p'+q)} \\
\times \frac{1}{E - \omega_1(p) - \omega_1'(p') - \omega_K(p' + q) + i\varepsilon} \\
\times \frac{1}{E - \omega_1(p) - \omega_1'(p') - \omega_K(p' + q) + i\varepsilon} \\
\times \frac{1}{E - \omega_1(p) - \omega_1'(p') - \omega_K(p' + q) + i\varepsilon} \\
\times \frac{1}{E - \omega_1(p) - \omega_1'(p') - \omega_K(p' + q) + i\varepsilon}.
\]

In the framework of TOPT, the expressions corresponding to the diagrams of Fig. 14 may be written in the following form

\[
\mathcal{V}_{\text{cross-\text{\_}negk-a}}^{\lambda, \lambda} (E, \mathbf{p}' , \mathbf{p}) = f_S^\lambda \mathcal{I} \int_0^\Lambda \frac{d^3q}{(2\pi)^3} \frac{1}{16\omega_K(p + q)\omega_K(q)\omega(1)(q)\omega_K(p'-q)} \\
\times \frac{1}{E - \omega_1(p) - \omega_2(p) - \omega_2'(p') - \omega_K(p'-q) - \omega_K(q) + i\varepsilon} \\
\times \frac{1}{E - \omega_1(p) - \omega_2(p) - \omega_2'(p') - \omega_K(p'-q) - \omega_K(q) + i\varepsilon} \\
\times \frac{1}{E - \omega_1(p) - \omega_2(p) - \omega_2'(p') - \omega_K(p'-q) - \omega_K(q) + i\varepsilon} \\
\times \frac{1}{E - \omega_1(p) - \omega_2(p) - \omega_2'(p') - \omega_K(p'-q) - \omega_K(q) + i\varepsilon}.
\]

\[
\mathcal{V}_{\text{cross-\text{\_}negk-b}}^{\lambda, \lambda} (E, \mathbf{p}' , \mathbf{p}) = f_S^\lambda \mathcal{I} \int_0^\Lambda \frac{d^3q}{(2\pi)^3} \frac{1}{16\omega_K(p' + q)\omega_K(q)\omega(1)(q)\omega_K(p-q)} \\
\times \frac{1}{E - \omega_2(p) - \omega_2'(p') - \omega_K(p'-q) - \omega_K(q) + i\varepsilon} \\
\times \frac{1}{E - \omega_2(p) - \omega_2'(p') - \omega_K(p'-q) - \omega_K(q) + i\varepsilon} \\
\times \frac{1}{E - \omega_2(p) - \omega_2'(p') - \omega_K(p'-q) - \omega_K(q) + i\varepsilon} \\
\times \frac{1}{E - \omega_2(p) - \omega_2'(p') - \omega_K(p'-q) - \omega_K(q) + i\varepsilon}.
\]
In the framework of TOPT, the expressions corresponding to the diagrams of Fig. 15 may be written in the following form

$$V_{\text{cros-negk-c}}^{\lambda}(E, \mathbf{p}', \mathbf{p}) = f_{S}^{4} \lambda \varpi^{2} \int_{0}^{\Lambda} \frac{d^{3}q}{(2\pi)^{3}}$$

$$\times \frac{1}{16\omega_{K}(p'-q)\omega_{K}(q)\omega^{(1)}(q)\omega_{K}(p-q)}$$

$$\times \frac{1}{E - \omega_{2}(p) - \omega_{2}'(p') - \omega_{K}(p' - q) - \omega_{K}(q) - \omega^{(1)}(q) + i\varepsilon}$$

$$\times \frac{1}{E - \omega_{2}(p) - \omega_{2}'(p') - \omega_{K}(p - q) - \omega_{K}(p' - q) - \omega_{K}(q) - \omega^{(1)}(q) + i\varepsilon}$$

$$\times \frac{1}{E - \omega_{2}(p) - \omega_{2}G(p') - \omega_{K}(p' - q) - \omega_{K}(q) - \omega^{(1)}(q) + i\varepsilon}, \quad \text{(C.53)}$$

$$V_{\text{cros-negk-d}}^{\lambda}(E, \mathbf{p}', \mathbf{p}) = f_{S}^{4} \lambda \varpi^{2} \int_{0}^{\Lambda} \frac{d^{3}q}{(2\pi)^{3}}$$

$$\times \frac{1}{E - \omega_{2}(p) - \omega_{2}'(p') - \omega_{K}(p - q)}$$

$$\times \frac{1}{E - \omega_{2}(p) - \omega_{2}'(p') - \omega_{K}(p - q) - \omega_{K}(p' - q) - \omega_{K}(q) - \omega^{(1)}(q) + i\varepsilon}$$

$$\times \frac{1}{E - \omega_{2}(p) - \omega_{2}'(p') - \omega_{K}(p - q) - \omega_{K}(p' - q) - \omega_{K}(q) - \omega^{(1)}(q) + i\varepsilon \varepsilon}, \quad \text{(C.54)}$$

$$V_{\text{cros-negk-e}}^{\lambda}(E, \mathbf{p}', \mathbf{p}) = f_{S}^{4} \lambda \varpi^{2} \int_{0}^{\Lambda} \frac{d^{3}q}{(2\pi)^{3}}$$

$$\times \frac{1}{E - \omega_{2}(p) - \omega_{2}'(p') - \omega_{K}(p - q)}$$

$$\times \frac{1}{E - \omega_{2}(p) - \omega_{2}'(p') - \omega_{K}(p - q) - \omega_{K}(p' - q) - \omega_{K}(q) - \omega^{(1)}(q) + i\varepsilon}$$

$$\times \frac{1}{E - \omega_{2}(p) - \omega_{2}'(p') - \omega_{K}(p - q) - \omega_{K}(p' - q) - \omega_{K}(q) - \omega^{(1)}(q) + i\varepsilon \varepsilon}, \quad \text{(C.55)}$$

$$V_{\text{cros-negk-f}}^{\lambda}(E, \mathbf{p}', \mathbf{p}) = f_{S}^{4} \lambda \varpi^{2} \int_{0}^{\Lambda} \frac{d^{3}q}{(2\pi)^{3}}$$

$$\times \frac{1}{16\omega_{K}(p'-q)\omega_{K}(q)\omega^{(1)}(q)\omega_{K}(p-q)}$$

$$\times \frac{1}{E - \omega_{2}(p) - \omega_{2}'(p') - \omega_{K}(p' - q) - \omega_{K}(q) - \omega^{(1)}(q) + i\varepsilon}$$

$$\times \frac{1}{E - \omega_{2}(p) - \omega_{2}'(p') - \omega_{K}(p - q) - \omega_{K}(p' - q) - \omega_{K}(q) - \omega^{(1)}(q) + i\varepsilon}$$

$$\times \frac{1}{E - \omega_{2}(p) - \omega_{2}'(p') - \omega_{K}(p' - q) - \omega_{K}(q) - \omega^{(1)}(q) + i\varepsilon \varepsilon}, \quad \text{(C.56)}$$
\[
V_{\text{cross} - n\text{egkt} - f}(E, p', p) = f_{2} A IN \int_{0}^{\Lambda} \frac{d^{3}q}{(2\pi)^{3}} \times \\
\frac{1}{16a(K'(p'-q)\omega(q)\omega(p-q)\omega(p')\omega(p-q)} \\
\times 1/[E - \omega_{1}(p' - q) - \omega_{2}(p' - q) - \omega_{K}(p - q)] \\
- \omega_{K}(q) + i\varepsilon], \quad (C.60)
\]

\[
\frac{1}{16a(K'(p'-q)\omega(q)\omega(p-q)\omega(p')\omega(p-q)} \\
\times 1/[E - \omega_{1}(p) - \omega_{2}(p) - \omega_{2}(p') - \omega_{K}(p - q)] \\
- \omega_{K}(p') - \omega_{K}(q) + i\varepsilon] \\
\times 1/[E - \omega_{1}(p) - \omega_{1}(p') - \omega_{2}(p') - \omega_{K}(p - q)] \\
- \omega_{K}(p' - q) - \omega_{K}(q) + i\varepsilon] \\
\times 1/[E - \omega_{1}(p) - \omega_{1}(p') - \omega_{2}(p') - \omega_{K}(p - q)] \\
- \omega_{K}(q) + i\varepsilon], \quad (C.61)
\]

\[
\frac{1}{16a(K'(p'-q)\omega(q)\omega(p-q)\omega(p')\omega(p-q)} \\
\times 1/[E - \omega_{1}(p) - \omega_{2}(p) - \omega_{1}(p') - \omega_{K}(p - q)] \\
- \omega_{1}(p' - q) + i\varepsilon] \\
\times 1/[E - \omega_{1}(p) - \omega_{1}(p') - \omega_{2}(p') - \omega_{K}(p - q)] \\
- \omega_{K}(p') - \omega_{K}(q) + i\varepsilon] \\
\times 1/[E - \omega_{1}(p) - \omega_{1}(p') - \omega_{2}(p') - \omega_{K}(p - q)] \\
- \omega_{K}(p' - q) + i\varepsilon]. \quad (C.62)
\]

References

1. P.A. Zyla et al., [Particle Data Group], PTEP 2020(8), 083C01 (2020).
2. F.K. Guo, X.H. Liu, S. Sakai, Prog. Part. Nucl. Phys. 112, 103757 (2020). arXiv:1912.07030 [hep-ph].
3. Y.R. Liu et al., Prog. Part. Nucl. Phys. 107, 237 (2019). arXiv:1903.11976 [hep-ph].
4. F.K. Guo et al., Rev. Mod. Phys. 90(1), 015004 (2018). arXiv:1705.00141 [hep-ph].
5. H.X. Chen et al., Rept. Prog. Phys. 80(7), 076201 (2017). arXiv:1609.08928 [hep-ph].
6. N. Brambilla et al., Phys. Rept. 873, 1–154 (2020). arXiv:1907.05783 [hep-ex].
7. K.P. Khemchandani, A. Martinez-Torres, E. Oset, Eur. Phys. J. A 37, 233 (2008). arXiv:0804.4670 [nucl-th].
8. D.L. Canham, H.W. Hammer, R.P. Springer, Phys. Rev. D 80, 014009 (2009). arXiv:0906.1263 [hep-ph].
9. V. Baru, A.A. Filin, C. Hanhart, Y.S. Kalashnikova, A.E. Kudryavtsev, A.V. Nefediev, Phys. Rev. D 84, 074029 (2011). arXiv:1108.5644 [hep-ph].
10. L. Ma, Q, Wang, U.-G. Mei, Phys. Rev. D 98, 7 (2018). arXiv:1805.07453 [hep-ph].
11. Q. Wang, V. Baru, A.A. Filin, C. Hanhart, A.V. Nefediev, J.L. Wynen, Phys. Rev. D 89, 7 (2016). arXiv:1805.07453 [hep-ph].
