Optimal dynamic measurements in presence of the random interference

A L Shestakov¹, M A Sagadeeva², N A Manakova², A V Keller², S A Zagrebina², A A Zamyshlyaeva² and G A Sviridyuk²
¹Department of Information-Measuring Technique, South-Ural State University, 76 Lenin ave, Chelyabinsk 454080, Russian Federation
²Institute of Natural and Exact Sciences, South-Ural State University, 76 Lenin ave, Chelyabinsk 454080, Russian Federation

E-mail: zamyshliaevaaa@susu.ru

Abstract. The stochastic problem of optimal dynamic measurements in spaces of differentiable “noises” is considered in the article. To solve this problem, the theory of optimal dynamic measurements, which has actively been developing recently, is applied. The main purpose of this article is to reconstruct a dynamically distorted input signal from a given observation. This theory is at the intersection of several scientific areas: the theory of dynamic measurements, the theory of optimal control for Leontief type systems and the theory of Sobolev type equations. Based on the results obtained earlier by the authors, an interval estimation of the optimal dynamic measurement with known characteristics of random interference at the input is constructed.

1. Introduction
At present, the theory of dynamic measurements develops in different directions. One of them is the theory of optimal dynamic measurements [1], based on the theory of optimal control [1, 2]. Here the model of the measuring transducer (MT) is described by the Leontief type system

\[
\begin{aligned}
L\dot{x}(t) &= Mx(t) + u(t), \\
y(t) &= Cx(t),
\end{aligned}
\]

(1)

where \(L\) and \(M\) are square matrices of order \(n\), characterizing the construction of the MT, \(x = (x_1, x_2, \ldots, x_n)\) and \(\dot{x} = (\dot{x}_1, \dot{x}_2, \ldots, \dot{x}_n)\) are the vector-functions of a state and a rate of state change for the MT respectively; \(y = (y_1, y_2, \ldots, y_m)\) is a vector-function of observations; \(C\) is a rectangular matrix characterizing the relationship between the system state and observation; \(u = (u_1, u_2, \ldots, u_n)\) is a vector function of measurements. The main goal of the theory of optimal dynamic measurement is the reconstruction of a dynamically distorted input signal \(u(t)\) according to a given observation \(y_0(t)\). When using this approach the key concept is the optimal dynamic measurement \(v(t)\), which is constructed as a minimum of the functional

\[J(v) = \min_{u \in U_\phi} J(x(u), u)\]

on a set of admissible measurements \(U_\phi\), where the pair \((x(u), u)\) satisfies system (1), and \(U_\phi\) contains a priori information about measurements. The functional \(J(x(u), u)\) reflects the evaluation of
the proximity of actual observation $y_0(t)$ and virtual observation $y(t)$, described by the system (1). At present, within the framework of the theory of optimal dynamic measurements, the deterministic case is well studied [2], algorithms for solving such problems are constructed [3]. However, the deterministic problem does not take into account the effects of random interferences, which are always present in real processes, so it was suggested to use the stochastic model of MT [1]. In contrast to previous results, in this study, based on a priori information about the inference, an interval estimation of the optimal dynamic measurement is constructed using the Bayesian approach.

Note that the developed methods are consistent with the philosophical concepts of the evolution of scientific theories. For example, E. Mach in his article “Erkenntnis und Irrtum” (1905), presented the development of science as a translational, cumulative and progressive process in the course of which erroneous theories are rejected, and the true theories have become increasingly important. Criticizing this thesis, T. Kuhn in the monograph “The Structure of Scientific Revolution” (1962) showed that the development of science on Mach is possible only in the interval between scientific revolutions, during which there is a replacement of one paradigm for another. An unusual position in this dispute was taken by P. Feyerabend. In the treatise “Against the Method. Outline of an Anarchistic Theory of Knowledge” (1975) he reasonably proved the possibility of coexistence in science, even contradictory paradigms. For example, in mathematics, these are the geometries of Euclid, Riemann and Lobachevsky. And the system of C. Ptolemaeus is not all discarded by modern science. It is actively used in topography and geodesy.

In our studies, we adhere to the thesis of P. Feyerabend on coexistence, possibly conflicting paradigms in the field of metrology as dynamic measurements [4–6]. Moreover, under the paradigm we understand the scientific theory that satisfies as well as the Au. Comte principle of verification so K. Popper principle of falsification. Moreover, according to I. Kant’s thesis “in any science exactly as much science as mathematics in it” we demand that each theory has a very strong mathematical foundation. Examples of such theories can be [4] and [5]. In [4] the whole theory of dynamic measurements is based on the theory of inverse problems. In [5] such an important in dynamic dimensions concept as “uncertainty” is analyzed from the point of view of fuzzy sets theory. The mathematical foundation of the theory of optimal dynamic measurements is the theory of groups of operators [7] and the theory of optimal control over solutions of Leontief type equation systems [3].

2. Stochastic Leontief Type System in Spaces of “Noises”

Let $\Omega = (\Omega, \mathcal{A}, P)$ be a completely probability space, $\mathbb{R}$ be a set of real numbers, endowed with Borel $\sigma$-algebra. The measurable mapping $\xi : \Omega \to \mathbb{R}$ is called a random variable. The set of random variables such that $E\xi = 0$ forms a Hilbert space $L_2$ with inner product $(\xi_1, \xi_2) = E\xi_1\xi_2$. Let $\mathbb{R} \in \mathbb{R}$ be some interval. The mapping $\eta : \mathbb{R} \times \Omega \to \mathbb{R}$ of the form $\eta = \eta(t, \omega)$ is called a (one-dimensional) stochastic process, thus for every fixed $t \in \mathbb{R}$ value of mapping $\eta = \eta(t, \cdot)$ is a random variable, i.e. $\eta = \eta(t, \cdot) \in L_2$ and for every fixed $\omega \in \Omega$ the value of a stochastic process $\eta = \eta(\cdot, \omega)$ is called the (sample) trajectory. The random process $\eta$ is called continuous, if almost surely (a.s.) all its trajectories are continuous. Let us denote by $CL_2$ the space of the continuous random processes. Continuous random process, which independent random variables are Gaussian, is called Gaussian.

By the $\xi^{(t)}$ let us denote the $l$-th Nelson-Gliklikh derivative of the stochastic process $\eta$ [8]. The set of continuous stochastic processes having continuous Nelson – Gliklikh derivatives until the order $k \in \mathbb{N}$ at each point of the set $\mathbb{R}$ forms a space, which we denote by $C^kL_2$. The Wiener process $\beta(t) = \sum_{k=0}^{\infty} \xi_k \sin(\pi(2k+1)t)$, where $\xi_k \in L_2$ are independent Gaussian random variables, $E\xi_k = 0$ and $D\xi_k = \left[\sqrt{\frac{\pi}{2}}(2k+1)^2\right]^2$, $k \in \mathbb{N}$ is an example of a Gaussian stochastic process, which is continuous on $\mathbb{R}$.

Consider the stochastic Leontief type system
where $\xi_0 = \sum_{k=0}^{n} \xi_{0,k} e_k$, and $\xi_{0,k}$ are the pairwise independent Gaussian random variables and $\{e_k\}_{k=1}^{n}$ is an orthonormal basis in $\mathbb{R}^n$. Previously authors showed, that for all $\xi_0 \in L_2(\mathbb{R}^n)$ and $\varphi \in C^0 L_2(\mathcal{S}, \mathbb{R}^n)$ independent for every time $t \in \mathcal{S}$ there exists a unique solution $\xi \in C^0 L_2(\mathcal{S}, \mathbb{R}^n)$ of the problem (2), (4), which has the form
\begin{equation}
\xi(t) = \mathcal{Y} \xi_0 + \int_0^t \mathcal{Y}^{-1} Q \varphi(s) ds + \sum_{q=0}^{n} \left(M^{-1} (I_n - Q) L \right)^q M^{-1} (Q - I_n) ^q \varphi^{(q)}(t),
\end{equation}
where $\mathcal{Y} = \lim_{t \to \infty} \left((L - \frac{1}{2} M)^{-1} L \right)^r$, $Q = \lim_{t \to \infty} \left(I^r_n(M) \right)^p$, $I^r_n(M) = L \left(L - \frac{1}{2} M \right)^{-1}$, and $I_n$ is a unit matrix of order $n$.

3. Interval estimate of the optimal dynamic measurement
The stochastic process $\varphi(t)$ in equation (2) can be represented as the sum of the measured deterministic signal $u(t)$ and random interference $\theta(t)$. We will assume, that the random interference $\theta$ is a Gaussian stochastic process with mathematical expectation $E\theta(t) = 0$ and variance $D\theta(t) = \sigma^2(t) < +\infty$ for all $t \in [0, \tau]$ for some fixed $\tau > 0$, that is the density of the process distribution $\theta$ has the form
\begin{equation}
p_{\theta}(z,t) = \frac{1}{(2\pi)^{n/2} \sigma(t)} \exp \left(-\frac{1}{2\sigma^2(t)} \sum_{k=1}^{n} z_k^2 \right), \quad z \in \mathbb{R}^n.
\end{equation}
Analogously to the deterministic case when investigating the problem of reconstructing a dynamically distorted signal by random interference at the input, we consider the control problem
\begin{equation}
J(\nu) = \min_{u \in U_\nu} J(\xi(u), u)
\end{equation}
with a functional of the form
\begin{equation}
J(\nu) = \delta \sum_{k=0}^{l} \mathbb{E} \left[ \left( \tilde{\eta}^{(k)}(t) - \tilde{\eta}^{(0)}_0(t) \right)^2 \right] dt + (1 - \delta) \sum_{k=0}^{l} \mathbb{E} \left[ \left( \tilde{\eta}^{(k)}(t) + \tilde{\eta}^{(0)}_0(t) \right)^2 \right] dt, \quad \delta > 0,
\end{equation}
where the pair $(\xi(u),u)$ satisfies (2) – (4). Then we will call $\nu \in U_\nu$ the optimal dynamic measurement.

The actual observation $\tilde{\eta}_0(t)$, obtained in the course of the experiment, is a deterministic function corresponding to specific realizations $\tilde{\theta}(t)$ of random interference $\theta(t)$ and $\tilde{\xi}_0$ of random variable $\xi_0$ for fixed value $\tilde{\omega} \in \Omega$. Therefore, we can consider the realization of the problem (2)–(4), (6) for $\tilde{\omega} \in \Omega$:
\begin{equation}
\begin{aligned}
L \frac{d \tilde{\xi}}{dt} &= M \tilde{\xi} + u + \hat{\theta}, \\
\hat{\eta} &= C \tilde{\xi}, \\
\left(\alpha L - M \right)^{-1} L \right)^{p+1} (\tilde{\xi}(0) - \tilde{\xi}_0) &= 0
\end{aligned}
\end{equation}
with a functional of the form
\[ J(\hat{\phi}) = \delta \sum_{k=0}^{r} \left\| \hat{\eta}^{(k)}(t) - \eta_0^{(k)}(t) \right\|^2 dt + (1 - \delta) \sum_{k=0}^{K} \left\| \phi(t) \right\|^2 dt , \]  

where \( \hat{\phi}(t) = u(t) + \hat{\theta}(t) \), \( \hat{\xi}(t) = \xi(t, \hat{\omega}) \) and \( \hat{\eta}(t) = \eta(t, \hat{\omega}) \).

Let us consider the space of states of MT \( \Xi = \{ \hat{\xi} \in L_2((0, \tau); \mathbb{R}^n) : \frac{d}{dt} \hat{\xi} \in L_2((0, \tau); \mathbb{R}^n) \} \) and the space of measurements \( \Phi = \{ \hat{\phi} \in L_2((0, \tau); \mathbb{R}^n) : \frac{d}{dt} \hat{\phi} \in L_2((0, \tau); \mathbb{R}^n) \} \). The function \( \hat{\xi} \in \Xi \) is called a strong solution of the system (8), if for every \( \hat{\phi} \) the function satisfies the system almost surely on \( (0, \tau) \). To solve problem (6), (8) we apply the methods of the theory of optimal dynamic measurements developed for the deterministic case [2]. For a close convex set \( U_\delta \) there exists a unique solution of the problem (6), (8) with the functional (9). As a result of the solution, we find the minimum point \( \psi(t) \) of the functional (9). Thus, according to \( \eta_0(t) \) we can restore \( \psi(t) \), and for the measured deterministic signal \( u(t) \) with probability 0.997 for all \( t \in [0, \tau] \) the estimate

\[ \left\| \psi(t) - u(t) \right\| < 3\sigma(t) \]

holds.

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