CHARACTERISATIONS OF TRIVIAL EXTENSIONS

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Abstract. In this paper we give a characterisation of trivial extension algebras in terms of quivers with relations. This result is based on an explicit description of the ideal of relations of the trivial extension of an algebra, given by the first author in the appendix. We also give a new proof of Wakamatsu’s theorem in terms of their quiver and relations, which determines when two given algebras have isomorphic trivial extensions.

1. Introduction

Split extensions of a ring by a bimodule are classical constructions that appear in many different contexts. For example, Hochschild [14] showed that the trivial extension of a ring $R$ by an $R$-$R$-bimodule $M$ corresponds to the zero element in the second cohomology group $H^2(R, M)$. In commutative algebra, Nagata [18] used split-extensions to show that any module over a commutative ring can be thought of as an ideal. More recently, split extensions have played a central role in the study of cluster-tilted algebras [1, 5], in the connections between gentle algebras and symmetric special biserial algebras [12, 19], in higher homological algebra [10] and the Hochschild (co)homology of split-extensions has been studied, for example, in [3, 4, 6].

Although the theory of trivial extensions has been greatly developed, in general, given an explicit algebra in terms of quiver and relations, there has not been an explicit construction of the structure of the trivial extension as an algebra, except for certain cases [7, 8, 13]. Note that by the trivial extension of an algebra $A$, we mean the algebra $T(A) = A \rtimes D(A)$ where $D(A)$ is $A$-$A$-bimodule given by the dual of $A$. In the appendix of this paper, an explicit construction of the trivial extension is developed by the first author (see Theorem 1.1). Namely given an algebra $A = KQA/I_A$ the ideal of relations of the trivial extension $T(A)$ is explicitly constructed, thus completing the description of $T(A)$ since in [7] the quiver of $T(A)$ was already constructed.

Furthermore, it has been an open question to know which algebras are isomorphic to the trivial extension of a finite dimensional algebra. Since trivial extensions are symmetric, the question reduces to asking when is a symmetric algebra isomorphic to a trivial extension. We address this question for all finite dimensional symmetric $K$-algebras, see Theorem 1.2. Finally, we turn to the question of when two algebras have isomorphic trivial extensions, a question already considered in the abstract setting by Wakamatsu in [20]. Lastly, in Theorem 1.3, we give a precise characterisation of the algebras that have isomorphic trivial extensions in the language of admissible cuts which were introduced in [9].
We now state the main results of this paper. For this we recall that in [7] the quiver \( Q_{T(A)} \) of the trivial extension \( T(A) \) of a finite dimensional algebra \( A = KQ/I \) has been described. Namely, the set of vertices \((Q_A)_0\) and \((Q_{T(A)})_0\) coincide. Then given a \( K \)-basis \( \mathbb{M} = \{ p_1, \ldots, p_n \} \) of \( \text{soc} A \cdot A \), the set of arrows \((Q_{T(A)})_1\) is the disjoint union of \((Q_A)_1\) and \( \{ \beta_{p_1}, \ldots, \beta_{p_n} \} \) where \( s(\beta_{p_i}) = t(p_i) \) and \( t(\beta_{p_i}) = s(p_i) \) for every \( p_i \in \mathbb{M} \). Given a path \( p \), we call a path \( q \) such that \( pq \) is a cycle a supplement of \( p \) (see Definition 2.2). For the notion of elementary cycle, we refer to Definition 2.1. In the appendix of this paper the following is shown.

**Theorem 1.1 (Theorem A.1).** Let \( A = KQ/A \) be a finite-dimensional algebra and let \( T(A) = KQ_{T(A)}/I_{T(A)} \) be its trivial extension. Then the quiver \( Q_{T(A)} \) is as above and the ideal \( I_{T(A)} \) is generated by the union of the following sets.

1. A generating set of the ideal of relations \( I_A \) of \( A \).
2. The paths that are not contained in an elementary cycle.
3. For any vertices \( x \) and \( y \) in \( Q_{T(A)} \), the linear combinations of paths \( \rho \in e_x KQ_{T(A)} e_y \) such that \( q \rho \in I'_x \) or \( \rho q \in I'_y \) for any supplement path \( q \) in an elementary cycle \( C \).

Having this complete description of the ideal of relations of a trivial extension, we are able to give a complete characterisation of when a symmetric algebra is isomorphic to a trivial extension of some finite-dimensional \( K \)-algebra.

**Theorem 1.2 (Theorem 3.6).** Let \( A = KQ/I \) be an algebra. Then \( A \) is isomorphic to the trivial extension of some finite-dimensional \( K \)-algebra if and only if:

(a) There is a presentation of \( A \) for which there exists a set \( E \) of distinguished cycles in \( Q_A \) with weight function \( \omega : E \to K \) and

(b) There is an allowable cut \( D = \{ \gamma_1, \cdots, \gamma_l \} \) of \( A \) such that the quotient \( B = A/\langle D \rangle \) verifies the following:

(i) \( A \) is a split-by-nilpotent extension of \( B \).

(ii) The supplements of the cut arrows in the cycles in \( E \) are in one-to-one correspondence with the elements of a basis of \( \text{soc} B \cdot B \).

In this case the two-sided ideal \( \langle D \rangle \) is isomorphic to \( DB \) as a \( B \)-\( B \)-bimodule and \( A \) is isomorphic to \( T(B) \).

We note that the previous result reduces the problem of identifying if an algebra is a trivial extension to the problem of determining whether an algebra is a split-by-nilpotent extension, a problem that has been solved in [2]. Thus our result gives a complete answer which in combination with [2] should be implementable in terms of a computer algorithm.

In the 70’s, Müller [17], Green and Reiten [11], Iwanaga and Wakamatsu [16], and Hughes and Waschbüsch [15] studied the relationship between the representation type of the algebra \( T(A) \) and the algebra \( A \). In particular, in [15], they gave a complete description of the representation-finite trivial extension of algebras which relies on whether two trivial extensions are isomorphic or not. This result motivated Wakamatsu to study when two trivial extensions are isomorphic, see [20]. He gave, for two Artin algebras, necessary and sufficient conditions for having isomorphic trivial extensions. In more precise terms, he shows that two algebras \( A \) and \( A' \) have isomorphic trivial extensions if and only if
they are split-by-nilpotent extensions of a common subalgebra $S$ by a $S$-$S$-bimodule $M$ and its dual $D(M)$, respectively.

The original motivation for this paper was to give an explicit description of the relationship of two algebras $A$ and $A'$ that have isomorphic trivial extensions. This is achieved in our next result, which gives an explicit description of the algebra $S$ and the extending $S$-$S$-bimodules in the case where $A$ and $A'$ are given by quiver and relations. We also give an explicit characterisation of when the trivial extensions of two algebras are isomorphic. Our result is shown by a proof which is independent of Wakamatsu’s proof.

**Theorem 1.3** (Theorem 4.1). Let $A = KQ_A/I_A$ be a finite-dimensional algebra with trivial extension $T(A)$ and set $(Q_A)_1 = \{\alpha_1, \ldots, \alpha_n\}$ and $\{\beta_1, \ldots, \beta_l\} = (Q_{T(A)})_1 \setminus (Q_A)_1$. The following are equivalent

(a) $T(A) \cong T(A')$.

(b) There exists an admissible cut $\mathcal{D}$ of the form

$$\mathcal{D} = \{\alpha_1, \ldots, \alpha_r, \beta_1, \ldots, \beta_s : \alpha_i \in (Q_A)_1 \text{ and } \beta_j \notin (Q_A)_1\}$$

with $A' \cong T(A)/(\mathcal{D})$, $T(A)$ is a split-by-nilpotent extension of $A'$ and the supplements in the elementary cycles of the cut arrows are in one-to-one correspondence with the elements of $soc_A A'$.

(c) $A \cong S \ltimes M$ and $A' = S \ltimes D(M)$, where

(i) $S$ is the subalgebra of $T(A)$ generated by $\sum_{x \in (Q_T(A))_0} e_x$ the identity of $T(A)$ and $\alpha_{r+1}, \ldots, \alpha_n$,

(ii) $M = S (\alpha_1, \ldots, \alpha_r) S$,

(iii) and $D(M) = S (\beta_{s+1}, \ldots, \beta_l) S$.

2. Background

In this paper, by an algebra we mean a basic connected finite-dimensional algebra over an algebraically closed field $K$. It is well-known that every such algebra $A$ is Morita equivalent to the path algebra of a (finite) quiver $Q$ modulo an admissible ideal $I$ of $KQ$. For the purposes of this paper, we always identify $A$ with $KQ_A/I_A$. If $A = KQ_A/I_A$, we say that $Q_A$ is the quiver of $A$. In addition, every element of a set of generators of $I_A$ is called a relation of $A$ and we refer to $I_A$ as the ideal of relations of $A$.

Given a quiver $Q$, we denote by $Q_0$ the set of vertices of $Q$ and $Q_1$ the set of arrows of $Q$. For every arrow $\alpha \in Q_1$, $s(\alpha)$ and $t(\alpha)$ denote the source and the target of $\alpha$, respectively. A path $p$ of length $t$ from the vertex $x$ to the vertex $y$ in $Q$ is an ordered set of arrows $\{\alpha_1, \ldots, \alpha_t\}$ such that $s(\alpha_1) = x$, $t(\alpha_t) = y$ and $t(\alpha_i) = s(\alpha_{i+1})$ for all $1 \leq i < t-1$. By abuse of notation we write $p = \alpha_1 \alpha_2 \ldots \alpha_t$ and we define $s(p) := s(\alpha_1)$ and $t(p) := t(\alpha_t)$. Also, for every vertex $x \in (Q_A)_0$ we denote by $e_x$ the stationary path at vertex $x$, that is the unique path of length 0 from $x$ to $x$. We say that a path $p$ is a cycle if $p$ is of strictly positive length and $s(p) = t(p)$. A cycle $p$ is said to be non-zero if $p \notin I_A$.

Let $A$ be an algebra and $M$ be a $A$-$A$-bimodule equipped with a multiplication map $\mu : M \otimes_A M \to M$. Then the $K$-vector space $B = A \oplus M$ has the structure of an algebra
with the following multiplication

\[(a_1, m_1)(a_2, m_2) = (a_1a_2, a_1m_2 + m_1a_2 + \mu(m_1 \otimes m_2)),\]

where \(a_1, a_2 \in A\) and \(m_1, m_2 \in M\). If the map \(\sigma : A \to B\) defined by \(\sigma(a) = (a, 0)\) is a morphism of algebras we say that \(B\) is a split extension of \(A\) by \(M\). Moreover, if \(M\) is nilpotent for \(\mu\) we say that \(B\) is a split-by-nilpotent extension of \(A\) by \(M\) and we denote it as \(B = A \ltimes M\). If \(\mu(m_1, m_2) = 0\) for all \(m_1, m_2 \in M\) we say that \(B\) is the trivial extension of \(A\) by \(M\) and we denote it by \(A \rtimes M\). Finally, if \(M = D(A)\) with its natural bimodule structure, where \(D = \text{Hom}_A(-, K)\), we say that \(B = A \ltimes D(A)\) is simply the trivial extension of \(A\) and we denote it by \(T(A)\).

We recall from [7] an explicit description of the ordinary quiver of \(T(A)\) based on the quiver of \(A\). This description depends on a \(K\)-basis \(\mathbb{M} = \{p_1, \ldots, p_n\}\) of \(\text{soc}_A A\). The quiver \(Q_{T(A)}\) is then defined as follows. The set of vertices \((Q_A)_0\) and \((Q_{T(A)})_0\) coincide. The set of arrows \((Q_{T(A)})_1\) is the disjoint union of \((Q_A)_1\) and \(\{\beta_{p_1}, \ldots, \beta_{p_n}\}\) where \(s(\beta_{p_i}) = t(p_i)\) and \(t(\beta_{p_i}) = s(p_i)\) for every \(p_i \in \mathbb{M}\). A set of generators of the ideal of relations \(I_{T(A)}\) of \(T(A)\) is given in Theorem A.1.

Finally, we recall the notion of admissible cut and elementary cycles, introduced in [7] to study when two algebras of a certain family have isomorphic trivial extension.

**Definition 2.1.** Let \(\mathbb{M} = \{p_1, \ldots, p_n\}\) be a basis of \(\text{soc}_A A\). An oriented cycle \(C = \alpha_1 \ldots \alpha_1 \beta_{p_j} \ldots \beta_{p_i} \) in \(KQ_{T(A)}\) is said to be elementary if \(\alpha_1 \ldots \alpha_n\) is a path in \(KQA\), \(p_j \in \mathbb{M}\) and \(p_j^*(\alpha_1 \ldots \alpha_n) \neq 0\), where \(p_j^*\) is the dual of \(p_j\). In this case, we say that \(\omega(C) = p_j^*(\alpha_1 \ldots \alpha_n) \in K\) is the weight of \(C\).

**Definition 2.2.** Let \(Q\) be a quiver, \(C\) a cycle of \(Q\), and \(q\) a path contained in \(C\). The supplement of \(q\) in \(C\) is the path \(p\) such that \(C = qp\) up to rotation. Note that if \(C = q\) then \(p\) is the stationary path \(e_{(s(q))}\).

**Definition 2.3.** Let \(A = KQ/I\) be a finite-dimensional algebra. Then an admissible cut \(D\) of \(T(A)\) is a subset of arrows of \(Q_{T(A)}\) containing exactly one arrow in each elementary cycle of \(T(A)\) in such a way that if an arrow in \(D\) appears in an elementary cycle \(C\) then it appears only once in \(C\).

**Remark 2.4.** The set of new arrows \(D = \{\beta_{p_1}, \ldots, \beta_{p_j}\}\) added in the construction of the quiver \(Q_{T(A)}\) form an admissible cut of \(T(A)\). Moreover, it is easy to see that \(A\) is isomorphic to the quotient \(T(A)/\langle D \rangle\) where \(\langle D \rangle\) is the two-sided of \(T(A)\) ideal generated by \(D\).

We now give three examples. In the first example, we show that the weights of the elementary cycles are not always equal to one, in the second we show that not every arrow in an elementary cycle is included in an admissible cut. In the third example, we show that not every admissible cut \(D\) of \(T(A)\) gives rise to an algebra \(B = T(A)/\langle D \rangle\) such that \(T(A)\) is a split-by-nilpotent extension of \(B\).
Example 2.5. Consider the algebra \( A = KQ_A/I_A \) with quiver

\[
Q_A := \begin{array}{c}
1 \\
\alpha_1 \\
\gamma_1 \\
\epsilon_1 \\
4
\end{array} \quad \begin{array}{c}
2 \\
\alpha_2 \\
\gamma_2 \\
\epsilon_2 \\
5
\end{array}
\]

quotiented by the ideal \( I_A = \langle \lambda_1 \alpha_1 \alpha_2 + \lambda_2 \gamma_1 \gamma_2 + \lambda_3 \epsilon_1 \epsilon_2 \rangle \). Then \( \text{soc}_{A'} A \) is two dimensional with basis \( \{ p_1 = \gamma_1 \gamma_2, p_2 = \epsilon_1 \epsilon_2 \} \). Then \( T(A) = KQ_{T(A)}/I_{T(A)} \) where

\[
Q_{T(A)} := \begin{array}{c}
1 \\
\alpha_1 \\
\beta_{p_1} \\
\gamma_1 \\
\epsilon_1 \\
4
\end{array} \quad \begin{array}{c}
2 \\
\alpha_2 \\
\beta_{p_2} \\
\gamma_2 \\
\epsilon_2 \\
5
\end{array}
\]

In this case we have four different elementary cycles, \( C_1 = \beta_{p_1} \gamma_1 \gamma_2, C_2 = \beta_{p_2} \epsilon_1 \epsilon_2, C_3 = \beta_{p_1} \alpha_1 \alpha_2 \) and \( C_4 = \beta_{p_2} \alpha_1 \alpha_2 \). We calculate

\[
\begin{align*}
w(C_1) &= p_1^* (\gamma_1 \gamma_2) = 1 \\
w(C_2) &= p_2^* (\epsilon_1 \epsilon_2) = 1 \\
w(C_3) &= p_1^* (\alpha_1 \alpha_2) = p_1^* (-\lambda_2 \gamma_1 \gamma_2 - \lambda_3 \epsilon_1 \epsilon_2) = \frac{-\lambda_2}{\lambda_1} \\
w(C_4) &= p_2^* (\alpha_1 \alpha_2) = p_2^* (-\lambda_2 \gamma_1 \gamma_2 - \lambda_3 \epsilon_1 \epsilon_2) = \frac{\lambda_3}{\lambda_1}
\end{align*}
\]

Example 2.6. Let \( Q \) be the quiver

\[
\begin{array}{c}
1 \\
\alpha \\
2
\end{array} \quad \begin{array}{c}
2 \\
\beta \\
1
\end{array}
\]

and \( I \) the ideal generated by \( \alpha^3 \). Denote by \( A \) the algebra \( KQ/I \). Then the set \( M = \{ \alpha^2 \beta \} \) is a \( K \)-basis for \( \text{soc}_{A'} A \) and \( Q_{T(A)} \) is the following quiver

\[
\begin{array}{c}
1 \\
\alpha \\
2
\end{array} \quad \begin{array}{c}
2 \\
\beta \\
\beta_1
\end{array}
\]

Then \( C = \beta_1 \alpha^2 \beta \) is the only elementary cycle in \( KQ_{T(A)} \), up to cyclic permutations. It follows from Definition 2.3 that the only admissible cuts are \( \{ \beta \} \) and \( \{ \beta_1 \} \).
Observe that if we consider the set \( \{ \alpha \} \) as a cut, then \( A/\langle \alpha \rangle \) is isomorphic to the algebra \( B = KQ'/I' \) where \( Q' \) is the quiver

\[
\begin{array}{c}
1 \\
\downarrow \beta \\
2 \\
\end{array}
\]

and \( I' = (\beta_1 \beta) \), which is an algebra of dimension 6 and therefore \( T(B) \) is an algebra of dimension 12, while \( T(A) \) is a 14-dimensional algebra. Hence \( T(A) \) is not isomorphic to \( T(B) \).

We note that in this example there are nonzero cycles in \( T(A) \) which are not elementary cycles, namely \( \alpha, \alpha^2 \) and \( \beta_1 \).

**Example 2.7.** Let \( A \) be an algebra given by \( Q \)

\[
\begin{array}{c}
1 \\
\downarrow \epsilon_1 \\
2 \\
\downarrow \epsilon_2 \\
3 \\
\downarrow \epsilon_3 \\
4 \\
\end{array}
\begin{array}{c}
5 \\
\downarrow \beta_1 \\
6 \\
\end{array}
\begin{array}{c}
7 \\
\downarrow \beta_2 \\
8 \\
\end{array}
\begin{array}{c}
9 \\
\downarrow \eta_1 \\
10 \\
\end{array}
\]

and the relations

\[
\alpha_1 \alpha_2 + \beta_1 \beta_2 + \gamma_1 \gamma_2 + \eta_1 \eta_2 = 0
\]

\[
\epsilon_1 \gamma_1 \gamma_2 = 0, \quad \epsilon_1 \eta_1 \eta_2 = 0, \quad \epsilon_2 \gamma_1 \gamma_2 = 0, \quad \epsilon_2 \eta_1 \eta_2 = 0,
\]

\[
\epsilon_3 \alpha_1 \alpha_2 = 0, \quad \epsilon_3 \beta_1 \beta_2 = 0, \quad \epsilon_4 \alpha_1 \alpha_2 = 0, \quad \epsilon_4 \beta_1 \beta_2 = 0.
\]

Then the following elements are a \( K \)-basis for \( \text{soc}_A A \).

\[
\begin{align*}
\epsilon_1 \alpha_1 \alpha_2 &= -\epsilon_1 \beta_1 \beta_2, \\
\epsilon_2 \alpha_1 \alpha_2 &= -\epsilon_2 \beta_1 \beta_2, \\
\epsilon_3 \gamma_1 \gamma_2 &= -\epsilon_3 \eta_1 \eta_2, \\
\alpha_1 \alpha_2 &= \beta_1 \beta_2, \\
\gamma_1 \gamma_2 &= -\eta_1 \eta_2, \\
\epsilon_1 \gamma_1, \epsilon_1 \eta_1, \epsilon_2 \gamma_1, \epsilon_2 \eta_1, \epsilon_3 \alpha_1, \epsilon_3 \beta_1, \epsilon_4 \alpha_1, \text{ and } \epsilon_4 \beta_1
\end{align*}
\]

The quiver \( Q_{T(A)} \) is the following.

The quiver \( Q_{T(A)} \) is the following.
Where the vertices labelled by 6, 7, 8, 9 should be identified.

There are 18 elementary cycles, which are listed below.

- $C_1 = \alpha_1 \alpha_2 \delta_1$
- $C_2 = \alpha_1 \beta_1 \beta_2 \delta_1$
- $C_3 = \beta_1 \beta_2 \delta_2$
- $C_4 = \alpha_1 \alpha_2 \delta_2$
- $C_5 = \gamma_1 \gamma_2 \delta_3$
- $C_6 = \epsilon_1 \eta_1 \eta_2 \delta_3$
- $C_7 = \epsilon_2 \gamma_1 \gamma_2 \delta_4$
- $C_8 = \epsilon_4 \eta_1 \eta_2 \delta_4$
- $C_9 = \epsilon_3 \alpha_1 \delta_5$
- $C_{10} = \gamma_1 \gamma_2 \delta_6$
- $C_{11} = \epsilon_1 \gamma_1 \delta_7$
- $C_{12} = \epsilon_2 \gamma_1 \delta_8$
- $C_{13} = \epsilon_1 \eta_1 \delta_9$
- $C_{14} = \epsilon_2 \eta_1 \delta_{10}$
- $C_{15} = \epsilon_3 \alpha_1 \delta_{11}$
- $C_{16} = \epsilon_3 \beta_1 \delta_{12}$
- $C_{17} = \epsilon_4 \alpha_1 \delta_{13}$
- $C_{18} = \epsilon_4 \beta_1 \delta_{14}$

Observe that the set $D = \{\alpha_1, \beta_1, \delta_3, \delta_4, \delta_6, \delta_8, \delta_9, \delta_{10}\}$ is an admissible cut of $T(A)$, however $B = T(A)/\langle D \rangle$ is not a subalgebra of $T(A)$ since $\gamma_1 \gamma_2 + \eta_1 \eta_2 = 0$ is the induced relation from the minimal relation $\alpha_1 \alpha_2 + \beta_1 \beta_2 + \gamma_1 \gamma_2 + \eta_1 \eta_2 = 0$ in $A$ and therefore $T(A)$ is not a split-by-nilpotent extension of $B$ by \cite{2} Theorem 2.5.

3. Characterisation of trivial extensions

In this section we give a complete characterisation of the algebras that are isomorphic to the trivial extension of some finite dimensional $K$-algebra.

Before proving the main result of this section, Theorem 3.6, we introduce the necessary terminology and prove some preliminary results.

**Definition 3.1.** Let $A = KQ/I$ be an algebra and let $\mathcal{E} = \{C_1, \ldots, C_n\}$ be a set of non-zero cycles of length at least two. Then an **allowable cut** $D$ of $A$ with respect to $\mathcal{E}$ is a subset of arrows of $QA$ containing exactly one arrow in each cycle of $\mathcal{E}$ in such a way that if an arrow in $D$ appears in a cycle $C$ of $\mathcal{E}$ then it appears only once in $C$.

**Remark 3.2.** If $A$ is a trivial extension of an algebra $A'$, then any admissible cut $D$ of $A$ is an allowable cut with respect the set of elementary cycles.

In what follows, whenever the set $\mathcal{E}$ of cycles is clear from the context, we will simply say that $D$ is an allowable cut of $A$, omitting the reference to $\mathcal{E}$.
Lemma 3.5. Let $A = KQ_A/I_A$ be an algebra and consider a set $\mathcal{E}$ of non-zero cycles of length at least two and a weight function $\omega: \mathcal{E} \to K$. Then for every vertex $x \in (Q_A)_0$ we define $I_x^\mathcal{E}$ to be the two-sided ideal in $KQ$ generated by

(i) Oriented cycles starting and ending at $x$ that do not belong to $\mathcal{E}$.

(ii) The elements of the form $\omega(C')C - \omega(C)C'$, where $C, C' \in \mathcal{E}$.

Definition 3.4. Let $A = KQ/I$ be an algebra. A finite set $\mathcal{E}$ of non-zero cycles of length at least two is said to be a set of distinguished cycles if it admits an allowable cut $\mathcal{D} = \{\alpha_1, \cdots, \alpha_r\}$ of $A$ and a weight function $\omega: \mathcal{E} \to K$ such that the following holds.

(i) Every path not contained in a cycle $C$ of $\mathcal{E}$ is an element of $I$.

(ii) If $\rho$ is a linear combination of paths in $e_x KQ_B e_y$ such that $\rho q \in I$ (or $\rho q \in I$) for every supplement path $q$ in a cycle $C \in \mathcal{E}$, then $\rho$ is an element of $I$.

(iii) If $\rho$ is a linear combination of paths in $e_x KQ_A e_y$ such that each summand of $\rho$ is an element of $\langle \mathcal{D} \rangle$ and $\rho q \in I_x^\mathcal{E}$ or $\rho q \in I_{x'}^\mathcal{E}$ for every supplement $q$ in a cycle $C \in \mathcal{E}$, then $\rho$ is an element of $I$.

Lemma 3.5. Let $A$ and $A'$ be Artin algebras such that $T(A) \cong T(A')$. Then $T(A)$ is a split-by-nilpotent extension of $A'$.

Proof. Since $T(A')$ is a split-by-nilpotent extension of $A'$, there exists a short exact sequence of $A'$-$A'$-bimodules

$$0 \longrightarrow DA' \xrightarrow{i} T(A') \xrightarrow{\pi} A' \longrightarrow 0$$

such that $\pi \circ i = \text{Id}_{A'}$.

Now, let $f: T(A') \to T(A)$ be an isomorphism of $K$-algebras. Consider $\pi \circ f^{-1}: T(A) \to A'$ and $f \circ i: A' \to T(A)$. Then we have a short exact sequence

$$0 \longrightarrow \ker(\pi \circ f^{-1}) \longrightarrow T(A) \xrightarrow{f \circ i} A' \longrightarrow 0$$

Moreover, $(\pi \circ f^{-1}) \circ (f \circ i) = \text{Id}_{A'}$, then $T(A)$ is a split-by-nilpotent extension of $A'$. \hfill \square

Theorem 3.6. Let $A = KQ/I$ be an algebra. Then $A$ is isomorphic to the trivial extension of some finite-dimensional $K$-algebra if and only if:

(a) There is a presentation of $A$ for which there exists a set $\mathcal{E}$ of distinguished cycles in $Q_A$ with weight function $\omega: \mathcal{E} \to K$ and

(b) there is an allowable cut $\mathcal{D} = \{\gamma_1, \cdots, \gamma_\ell\}$ of $A$ such that the quotient $B = A/\langle \mathcal{D} \rangle$ verifies the following:

(i) $A$ is a split-by-nilpotent extension of $B$.

(ii) The supplements of the cut arrows in the cycles in $\mathcal{E}$ are in one-to-one correspondence with the elements of a basis of soc$_B B$.

In this case $A$ is isomorphic to $T(B)$ and the two-sided ideal $\langle \mathcal{D} \rangle$ is isomorphic to $DB$ as a $B$-$B$-bimodule.

Proof. Suppose first that $A$ is isomorphic to $T(B)$. Then it follows from Theorem A.1 that for any $K$-basis $\mathbb{M}$ for soc$_B B$, the set of elementary cycles $\mathcal{E}$ induced by $\mathbb{M}$ is a set of distinguished cycles and the set $\{\beta_p: p \in \mathbb{M}\}$ is an allowable cut of $A$ with respect to $\mathcal{E}$. Moreover, (b)(i) and (b)(ii) hold by construction.
To prove the converse, we show that \langle D \rangle \cong DB as B-B-bimodules and A \cong T(B). By Lemma \ref{lem:auto} we can assume that the presentation of A is as in the statement.

Let us see first that \langle D \rangle \subset DB. Then by (b)(i) A is a split-by-nilpotent extension of B by \langle D \rangle and thus A = B \ltimes \langle D \rangle \subset B \ltimes DB = T(B). Now in order to show \langle D \rangle \subset DB, consider a distinguished cycle \( C \in \mathcal{E} \). Up to cyclic permutation, we can write \( C = \gamma q \) where \( \gamma \in D \) is an arrow from \( x \) to \( y \) and \( q \) is a path from \( y \) to \( x \). By hypothesis, the path \( q \) is in one-to-one correspondence with an element \( p \) of soc\(_B^*\)B from \( y \) to \( x \). Then, by (b)(ii), \( \gamma = p^* \in DB \) and \langle D \rangle \subset DB.

To finish the proof it is enough to show that DB \subset \langle D \rangle. Let \( p \) be an element in the basis of soc\(_B^*\)B. Then \( p \) induces at least one elementary cycle \( C \) of \( T(B) \). Let \( q_p \) be a path from \( x \) to \( y \) such that \( \beta_p q_p = C \), thus \( q_p \) is a path in \( B \). Since \( q_p \) is a subpath of an elementary cycle in \( T(B) \), \( q_p \) is non-zero in \( T(B) \) and thus non-zero in \( B \). Hence, \( q_p \) is non-zero in \( A \) and there exists a path \( q' \) such that \( q_p q' = C' \) is a distinguished cycle of \( A \).

We claim that \( q' \) is in fact an arrow. From the fact that \( q_p \) belongs to \( B \), we have that \( q' \) contains a unique arrow \( \gamma \in D \) which appears only once in \( q' \), i.e., \( q' = u\gamma v \) where \( u \) and \( v \) are paths in \( B \). If \( q_p \) is a maximal non-zero path in \( B \) then \( q_p \) is an element of the soc\(_B^*\)B, and we get that \( u \) and \( v \) are idempotents, therefore \( q' = \gamma \). It follows that \( q_p \) is a supplement of \( \gamma \) in \( C' \) and then we can identify \( p^* \) with \( \gamma \).

Otherwise, \( q_p \) is in one-to-one correspondence with an element \( p \) in the soc\(_B^*\)B of the form \( p = a_p q_p + \sum_{i=1}^t a_i q_i \) in soc\(_B^*\)B. Suppose that there are two supplements \( \alpha \) and \( r \) of \( q_p \), where \( \alpha \) is an arrow. Then, by Definition \ref{def:supplementary} \( \alpha - r \) is in \( I_A \), which contradicts the assumption that \( I_A \) is an admissible ideal. Hence, if there is a supplement \( \alpha \) of \( q_p \) which is an arrow, then it is unique and it can be identified with \( p^* \).

Now suppose that any supplement \( r \) of \( q_p \) in \( A \) is of the form \( r = uav \) where \( \alpha \in D \) and at least one of \( u \) or \( v \) is a path of positive length. Moreover, at least one of the arrows of every \( r \) is an arrow of \( B \). Since \( A \) is a split-by-nilpotent extension of \( B \), we have in particular that \( B \) is a subalgebra of \( A \). Moreover \( p \) is in soc\(_B^*\)B, if the length of \( u \) is greater than zero, \( p(u\gamma v) = (pu)(\gamma v) = 0(\gamma v) = 0 \) or, similarly, if the length of \( v \) is greater than zero \( rp = 0 \). Hence \( rp = 0 \) or \( pr = 0 \) for every supplement \( r \) of \( q_p \). By Definition \ref{def:supplementary} we get that \( p \in I_A \). This, together with the fact that \( p \) is a linear combination of paths in \( B \) implies that \( p \in I_B \). But this is a contradiction of our hypothesis that \( p \) is an element of a basis of soc\(_B^*\)B. This contradiction comes from the supposition that \( r \) is not an arrow. Thus we can conclude that \( DB \) is contained in \( \langle D \rangle \).

From the above we conclude that \( T(B) \) and \( A \) have the same quiver and that the elementary cycles of \( T(B) \) coincide with the distinguished cycles of \( A \) with the same weight function. Thus, it follows from Definition \ref{def:iso} and Theorem \ref{thm:chara} that \( A \) is isomorphic to \( T(B) \) and \langle D \rangle is isomorphic to \( DB \) as B-B-bimodules. In particular \( A \) is a symmetric algebra.

In Theorem \ref{thm:chara} we have characterised algebras that are isomorphic to a trivial extension in terms of allowable cuts. We now use this result to study when two algebras have isomorphic trivial extensions in terms of admissible cuts.

**Corollary 3.7.** Let \( A = KQ_A/I_A \) be a finite-dimensional algebra with trivial extension \( T(A) \). The following are equivalent
(a) $T(A) \cong T(A')$

(b) There exists a presentation of $T(A)$, and an admissible cut $\mathcal{D}$ of that presentation such that

(i) $A' \cong T(A)/\langle \mathcal{D} \rangle$,

(ii) $T(A)$ is a split-by-nilpotent extension of $A'$,

(iii) the supplements of each arrow of $\mathcal{D}$ in the elementary cycles of $T(A)$ are in one-to-one correspondence with the elements of $\text{soc}_{A'} A'$.

Proof. Suppose that $T(A) \cong T(A')$. Then the quiver of $T(A)$ is equal to the quiver of $T(A')$. Hence $T(A')$ is a different presentation of $T(A)$. Then the result follows from Theorem A.1. For the converse it suffices to apply Theorem 3.6 to the algebra $T(A)$. □

In the following example, we illustrate an algorithm to decide whether an algebra is a trivial extension.

Example 3.8. Consider the algebra $A$ given by the quiver $Q_A$

\[ \begin{array}{c}
\alpha_1 \\
\alpha_2 \\
\alpha_3 \\
\alpha_4 \\
\alpha_5 \\
\end{array} \quad \begin{array}{c}
1 \\
2 \\
3 \\
4 \\
5 \\
\end{array} \]

with ideal of relations

$I_A = \langle \alpha_1^2, \alpha_4 \alpha_2, \alpha_5 \alpha_3, \alpha_5 \alpha_2, \alpha_4 \alpha_3, \alpha_4 \alpha_1 \alpha_3, \alpha_5 \alpha_1 \alpha_2, \alpha_2 \alpha_4 - \alpha_3 \alpha_5 \rangle$.

We claim that the cycles $C_1 = \alpha_1 \alpha_2 \alpha_4$ and $C_2 = \alpha_1 \alpha_3 \alpha_5$ and all its permutations are the distinguished cycles of $Q_A$ with weights $w(C_1) = w(C_2) = 1$. In order to show that the claim holds we need to verify that $I_A$ is determined by $C_1$, $C_2$ and its cyclic permutations, as indicated in Definition 3.4. Indeed, all the monomial relations correspond to paths that are not subpaths of a distinguished cycle and the relation $\alpha_2 \alpha_4 - \alpha_3 \alpha_5$ is obtained from Definition 3.4(iii). Moreover the set $\mathcal{D} = \{ \alpha_4, \alpha_5 \}$ form an allowable cut of $Q_A$ with respect to this set of cycles. Finally, the algebra $B = A/\langle \alpha_4, \alpha_5 \rangle$ is isomorphic to the path algebra of the quiver $Q_B$

\[ \begin{array}{c}
\alpha_1 \\
\alpha_2 \\
\alpha_3 \\
\alpha_4 \\
\alpha_5 \\
\end{array} \quad \begin{array}{c}
1 \\
2 \\
3 \\
4 \\
5 \\
\end{array} \]

modulo $I_B = \langle \alpha_1^2 \rangle$. A basis of the two-sided socle $\text{soc} B_{B^e}$ of $B$ is given by the paths $\alpha_1 \alpha_2$ and $\alpha_1 \alpha_3$. Hence there is a correspondence between a basis of $\text{soc} B_{B^e}$ and the set $\mathcal{D}$. Finally, it follows from [2] Theorem 2.5 that $A$ is a split-by-nilpotent extension of $B$. Then, we can apply Theorem 3.6 to conclude that $A$ is the trivial extension of $B$. 
4. Wakamatsu’s Theorem for Bound Path Algebras

One of the motivating questions for this paper was to determine the explicit relationship between two finite dimensional algebras that have isomorphic trivial extensions. This has been abstractly described by Wakamatsu in [20], where he also gives necessary and sufficient conditions to decide when two trivial extensions of Artin algebras are isomorphic. In this section, we give an explicit description of Wakamatsu’s result in terms of quivers and relations by providing an independent proof.

**Theorem 4.1.** Let $A = KQ_A/I_A$ be a finite-dimensional algebra with trivial extension $T(A)$ and set $(Q_A)_1 = \{\alpha_1, \ldots, \alpha_n\}$ and $\{\beta_1, \ldots, \beta_l\} = (Q_{T(A)})_1 \setminus (Q_A)_1$. The following are equivalent

(a) $T(A) \cong T(A')$.

(b) There exists an admissible cut $D$ of the form

$$D = \{\alpha_1, \ldots, \alpha_r, \beta_1, \ldots, \beta_s : \alpha_i \in (Q_A)_1 \text{ and } \beta_j \notin (Q_A)_1\}$$

with $A' \cong T(A)/\langle D \rangle$, $T(A)$ is a split-by-nilpotent extension of $A'$ and the supplements in the elementary cycles of the cut arrows are in one-to-one correspondence with the elements of $soc_A A'$.

(c) $A \cong S \ltimes M$ and $A' = S \ltimes D(M)$, where

(i) $S$ is the subalgebra of $T(A)$ generated by $\sum_{x \in (Q_{T(A)})_0} e_x$ the identity of $T(A)$ and $\alpha_{r+1}, \ldots, \alpha_n$,

(ii) $M = S \langle \alpha_1, \ldots, \alpha_r \rangle S$,

(iii) and $D(M) = S \langle \beta_{s+1}, \ldots, \beta_l \rangle S$.

**Proof.** By Corollary 3.7 (a) and (b) are equivalent.

We now show that (b) implies (c). Suppose that $A$ and $A'$ are as in (b) and let $S$ be the subalgebra of $T(A)$, and $M$ and $D(M)$ be the $S$-$S$-bimodules as defined in (c).

The first step is to show that $A \cong S \ltimes M$. Let $N \subset T(A)$ be the $S$-$S$-bimodule generated by $\{\beta_{s+1}, \ldots, \beta_l\}$, hence $N^2 = 0$. By construction $M \subset \langle D \rangle$ and therefore we also have that $M^2 = 0$. We claim that $S \cap (M + N) = 0$ in $T(A)$. Suppose that we have $0 \neq s \in S \cap (M + N)$. Then $s = \sum a_i u_i = \sum b_j v_j$ where each $u_i$ is a path from $x$ to $y$ in $S$ and each $v_j$ is a path from $x$ to $y$ containing at least an arrow in $\{\alpha_1, \ldots, \alpha_r\}$ or in $\{\beta_{s+1}, \ldots, \beta_l\}$. If $s$ is a non-zero element of $T(A)$, at least one of the $u_i$ is non-zero, let’s call it $u_k$. By Theorem [A1] there exists a path $q_k \in T(A)$ from $y$ to $x$ such that $q_k u_k$ is an elementary cycle in $T(A)$, that is, $q_k$ is a supplement of $u_k$. Recall that each elementary cycle of $T(A)$ contains exactly one arrow of each admissible cut. Hence the elementary cycle $q_k u_k$ contains exactly one arrow in $D$ and one arrow in $\{\beta_1, \ldots, \beta_l\}$. Since $u_k$ is in $S$, it contains no arrow in $D \cup \{\beta_1, \ldots, \beta_l\}$. Thus $q_k$ contains exactly one arrow in $D$ and one arrow in $\{\beta_1, \ldots, \beta_l\}$. Furthermore $q_k s$ is non-nilpotent and thus there exists $j$ such that $q_k v_j$ is non-zero. By definition $v_j$ has at least one arrow of $D$ or $\{\beta_1, \ldots, \beta_l\}$. This implies that $q_k v_j$ has two arrows in the same admissible cut. This implies that $q_k v_j$ is not a subpath of an elementary cycle and thus $q_k v_j = 0$ by Theorem [A1] and we arrive at a contradiction. Thus $S \cap (M + N) = 0$. In particular, we have that $S \cap M = 0$ and $S \cap N = 0$.

Recall that $S$ is a subalgebra of $T(A)$ generated by $e_x$ and $\alpha_{r+1}, \ldots, \alpha_n$. By Theorem [A1] the relations of $T(A)$ only depend on the relations of $A$ and the elementary cycles
in $T(A)$. Thus $S$ can be seen as a subalgebra of $A$ generated by $e_x$ and $\alpha_{x+1}, \ldots, \alpha_n$. Now let $a \in A$, then $a = u + v$ where $u \in S$ and $v \in M$. Then for $m \in M$, $am = um + vm = um$ since $vm \in M^2 = 0$. By symmetry $M$ is a two-sided ideal of $A$. Since $S \cap M = 0$, the map $h : S \ltimes M \to A$ defined by $h(s, m) = s + m$ is an isomorphism and $S \ltimes M \cong A$.

Now, we prove that $A' = S \ltimes DM$. Since $N$ is a $S$-$S$-bimodule such that $N^2 = 0$ we get that $\mathcal{A}' := S + N$ is a subalgebra of $T(A)$. We have that $\mathcal{A}'$ is isomorphic to $S \ltimes N$ using a similar argument that we used to show that $A \cong S \ltimes M$.

We now show that $\mathcal{A}' \cong A'$. Consider the canonical map $\pi : T(A) \to A'$ restricted to $\mathcal{A}'$. Since $\pi|_{\mathcal{A}'}$ surjects the generators of $\mathcal{A}'$ onto the generators of $A'$ and the relations of $A'$ are induced by the relations in $T(A)$, we have that $\pi|_{\mathcal{A}'}$ is a surjective algebra morphism.

To show that $\pi|_{\mathcal{A}'}$ is an isomorphism, we need to show that the kernel $\ker(\pi|_{\mathcal{A}'}) = (\mathcal{D}) \cap \mathcal{A}' = (\mathcal{D}) \cap (S + N) = 0$. Let $0 \neq \rho \in (\mathcal{D}) \cap (S + N)$. Then $\rho$ is in $(\mathcal{D})$ and there are $s \in S$ and $n \in N$ such that $\rho = s + n$, where $s = \sum a_k u_k$ and $n = \sum b_k v_k$. Assume $\rho$ is a linear combination of paths from $x$ to $y$. Note that $(\mathcal{D}) \cap N = 0$ since $(\mathcal{D})$ is generated by $\{\beta_1, \ldots, \beta_s\}$, $N'$ is the ideal generated by $\{\beta_{s+1}, \ldots, \beta_t\}$ and $\{\beta_1, \ldots, \beta_t\}$ is an admissible cut of $T(A)$. If $s$ is non-zero, there exists a $k$ such that $u_k$ is a non-zero path in $(T(A))$. Then there exists a path $q_k$ from $y$ to $x$ such that $q_k u_k$ is an elementary cycle. Thus $q_k u_k$ contains exactly one arrow of the admissible cut $\mathcal{D}$ and one arrow of $\{\beta_1, \ldots, \beta_t\}$. Since $u_k$ is a path in $S$ we have that $q_k$ must contain one arrow of the admissible cut $\mathcal{D}$ and one arrow of $\{\beta_1, \ldots, \beta_t\}$. This implies that $q_k v_k = 0$ for all $k$ because $N$ is generated by $\{\beta_{s+1}, \ldots, \beta_t\}$. Thus that $q_k \rho = q_k s \neq 0$. On the other hand, $\rho \in (\mathcal{D}) \cap (S + N)$. Then we can write $\rho = \sum c_j w_j \in (\mathcal{D}) = \langle \alpha_1, \ldots, \alpha_r, \beta_1, \ldots, \beta_t \rangle$. So $w_j$ contains an arrow of $\mathcal{D}$, for all $j$. Since we have seen that $q_k$ also contains an arrow of $\mathcal{D}$ we have that $q_k w_j = 0$, for all $j$. Hence $q_k \rho = 0$. A contradiction that arises from the assumption that $\rho \neq 0$. Then $(S + N) \cap (\mathcal{D}) = 0$, as claimed.

To finish, we need to show that $D(M) \cong N$. We recall that $N = \langle \beta_{s+1}, \ldots, \beta_t \rangle$ where $\beta_k = (0, p_k^{12})$ and $p_k^{12} : A \to K$ is an element of the basis of $D(soc_{A'}(A))$ which we extend to a basis of $D(A)$. Thus $N \subset D(A)$. We define $f : N \to D(M)$ where $f(w) : M \to K$ is given by $f(w)(m) = w(m)$ for all $w \in N$ and $m \in M$.

We show that $f : N \to D(M)$ is injective. Let $0 \neq w \in N$ such that $f(w)(m) = 0$ for all $m \in M$. Without loss of generality, $w = \sum b_k v_k$, where $v_k$ is a path from the vertex $x$ to $y$. If $w$ is not zero, there exists a $k$ such that $v_k$ is not zero in $T(A)$. Then there exists a path $q_k$ from $y$ to $x$ such that $q_k v_k$ is an elementary cycle. By construction, $v_k$ contains exactly one arrow $\{\beta_{s+1}, \ldots, \beta_t\}$. Hence $q_k$ does not contain any arrow in $\{\beta_1, \ldots, \beta_t\}$, and $q_k$ is an element of $A$. On the other hand, the elementary cycle $q_k v_k$ contains exactly one arrow $\alpha$ in $\mathcal{D}$ and $v_k$ does not, thus $q_k$ contains exactly one arrow $\alpha$ of $\mathcal{D}$. This implies that $q_k \in M$. Then $f(w)(q_k) = w(q_k) \neq 0$, which is a contradiction. So $f$ is injective.

Since (b) implies (a) by Corollary 3.7, $T(A) \cong T(A')$. Then $\dim_k T(A) = \dim_k T(A')$ and this implies that $\dim_k T(A) = 2 \dim_k A$. In particular $\dim_k A = \dim_k A'$. From $A = S \ltimes M$ and $A' = S \ltimes N'$ it follows that $\dim_k DM = \dim_k M = \dim_k N$. Then $f$ is surjective and $N$ is isomorphic to $DM$. We conclude that $A' \cong S \ltimes DM$. 
Now we prove that (c) implies (a). Since \( A = S \ltimes M \) and \( A' = S \ltimes DM \) we have that \( T(A) = A \ltimes DA = S \ltimes M \ltimes (DS \ltimes DM) \) and \( T(A) = A' \ltimes DA' = S \ltimes DM \ltimes (DS \ltimes M) \). It is not difficult to show that \( T(A) \cong T(A') \). \( \square \)

### Appendix A. The ideal of relations of the trivial extension of an algebra by Elsa Fernandez

In this appendix we give an explicit description of the ideal of relations \( I_{T(A)} \) of the trivial extension \( T(A) = A \ltimes D(A) \) for any finite-dimensional algebra \( A = KQ/I \).

For completeness, we first restate the explicit description of the ordinary quiver of \( T(A) \) based on the quiver of \( A \), as given in Section 2. This description depends on a \( K \)-basis \( M = \{ p_1, \ldots, p_n \} \) of \( \text{soc}_A A \). The set of vertices \((Q_A)_0\) and \((Q_{T(A)})_0\) coincide. The set of arrows \((Q_{T(A)})_1\) is the disjoint union of \((Q_A)_1\) and \( \{ \beta_{p_1}, \ldots, \beta_{p_n} \} \) where \( s(\beta_{p_i}) = t(p_i) \) and \( t(\beta_{p_i}) = s(p_i) \) for every \( p_i \in M \).

Before giving an explicit description of the ideal of relations \( I_{T(A)} \) of \( T(A) \), we recall the following definition from \( \mathbb{2} \). For each vertex \( x \in Q_{T(A)} \) we define the two-sided ideal \( I'_x \) in \( KQ_{T(A)} \) generated by

1. oriented cycles from \( x \) to \( x \) which are not contained in an elementary cycle;
2. the elements of the form \( \omega(C')C - \omega(C)C' \), where \( C \) and \( C' \) are elementary cycles starting and ending at \( x \).

#### Theorem A.1

Let \( A = KQ_A/I_A \) be a finite-dimensional algebra and let \( T(A) = KQ_{T(A)}/I_{T(A)} \) be its trivial extension. Then the quiver \( Q_{T(A)} \) is as above and the ideal \( I_{T(A)} \) is generated by the union of the following sets.

1. A generating set of the ideal of relations \( I_A \) of \( A \).
2. The paths that are not contained in an elementary cycle.
3. For any vertices \( x \) and \( y \) in \( Q_{T(A)} \), the linear combinations of paths \( \rho \in e_x KQ_{T(A)} e_y \) such that \( q \rho \in I'_x \) or \( q \rho \in I'_y \) for any supplement path \( q \) in an elementary cycle \( C \).

In order to prove our result we need to recall and show some preliminary results. We start with the following remark.

#### Remark A.2.

(i) Note that it follows directly from Definition \( \mathbb{2.1} \) that each non-zero path \( q \) in \( KQ_A \) can be completed to an elementary cycle in \( KQ_{T(A)} \). In other words, each non-zero path in \( KQ_A \) is contained in an elementary cycle \( C \).

(ii) In Item 3. of Theorem \( \mathbb{A.1} \) if we set \( x = y \), then \( \rho \) is a relation generated by the second condition in the definition of \( I'_x \) above. Moreover, every relation of the form \( \omega(C')C - \omega(C)C' \) arises in this way by taking \( q = e_x \) the trivial path at \( x \).

(iii) Also note that it follows from Item 2. of Theorem \( \mathbb{A.1} \) that if \( C = \alpha_1 \ldots \alpha_m \) is an elementary cycle in \( T(A) \), then the paths \( C \alpha_1 \) and \( \alpha_n C \) are zero in \( T(A) \).

Consider the morphism of \( K \)-algebras \( \Phi : KQ_{T(A)} \to T(A) \) defined on the stationary paths and arrows as follows.

\[
\Phi(e_x) = (e_x, 0), \quad \text{for all } x \in (Q_A)_0 \\
\Phi(\alpha) = (\alpha, 0), \quad \text{for all } \alpha \in (Q_A)_1, \\
\Phi(\beta_{p_i}) = (0, p^*), \quad \text{and } p \in M.
\]
It follows immediately from the definition of $\Phi$ that it is a surjective morphism of algebras and $\ker \Phi \cong I_{T(A)}$. Associated with $\Phi$ we consider two morphisms

$$\phi_1 = \pi_1 \Phi : KQ_{T(A)} \to A \quad \text{and} \quad \phi_2 = \pi_2 \Phi : KQ_{T(A)} \to D(A)$$

where $\pi_1$ and $\pi_2$ are the natural projections induced by the decomposition of $T(A) = A \oplus D(A)$ as $K$-vector spaces. We denote by $I_{M} = (\beta_p)_{p \in \mathbb{M}}$ the ideal of $T(A)$ generated by the elements $\beta_p$ in $T(A)$. The following lemma was shown in \cite[Lemma 3.5]{7} provided that each cycle in $A$ is zero, but the same proof holds for any algebra $A = KQ/1$.

**Lemma A.3.** Let $A = KQA/I_A$ be an algebra and let $q,u$ be paths in $KQ_{T(A)}$.

(a) If $v \in KQ_{T(A)}$ is such that $v = v_1 + v_2$ with $v_1 \in KQA$ and $v_2 \in I_M$, then $\Phi(v) = (\phi_1(v_1), \phi_2(v_2))$.

(b) $\phi_2(q) \neq 0$ implies $q \in I_M$.

(c) $\phi_2(q) = 0$ if $q$ contains at least two arrows $\beta_p$, $p \in \mathbb{M}$.

(d) $\phi_2(q)(u) \neq 0$ implies that $u$ is a supplement of $q$.

(e) $\phi_2(v)(u) = \phi_2(vu)(e_2) = \phi(uv)(e_2)$ if $u$ is a path from $x$ to $y$ in $Q_A$.

(f) If $v = \sum a_i q_i$, with $q_i$ different paths and $\phi_2(v) \neq 0$, then there exists a supplement $u$ of the $q_i$ such that $\phi_2(uv) \neq 0$ and $\phi_2(vu) \neq 0$.

(g) Let $C$ be an elementary cycle with origin $e$. Then $\phi_2(C)(e) = \omega(C)$ and $\phi_2(C)(u) = 0$ for any path $u \neq e$.

(h) If $q$ has a supplement, then $\Phi(q) \neq 0$.

As a consequence of Lemma A.3(g) we have that any elementary cycle is nonzero in $T(A)$. It was proven in \cite[Corollary 2.8]{7} that if any oriented cycle in $Q_A$ is zero in $A$, then any nonzero cycle in $T(A)$ is an elementary cycle. We illustrate with the following example that the previous statement is not true in general, see Example 2.6

The following result was shown in \cite[Proposition 3.6]{7} under the assumption that $A$ is monomial. However, the proof directly generalises to every finite-dimensional algebra $A$.

**Proposition A.4.** Let $A$ be a finite-dimensional algebra and $\Phi$ the $K$-algebra morphism defined as above. Then $I'_x$ is a subset of $\ker \Phi \cap (e_x K_{T(A)} e_x)$ for any $x \in (Q_A)_0$.

**Remark A.5.** As a direct consequence of the preceding proposition we have that the classes of elementary cycles starting at a vertex $x$ of $KQ_{T(A)}$ generate a one dimensional subspace of $KQ_{T(A)}/I'_x$.

We now show Theorem A.7.

**Proof of Theorem A.7.** Let $I'$ be the ideal generated by 1.-3. in the statement. It is sufficient to prove that $I' = \ker \Phi$.

First we show that $I' \subset \ker \Phi$. It is clear that $I_A \subset \ker \Phi$. Now, Remark A.2(i) and Lemma A.3(d) implies that $\Phi(u) = 0$ for every path $u$ which is not contained in an elementary cycle, since if $u$ is contained in an elementary cycle it is non-zero on its supplement.

Assume now that $w = \sum_{k=1}^{l} a_k v_k \notin \ker \Phi$, where $v_k$ are different paths in $I_M$ from $x$ to $y$. Then $\Phi(w) = (0, \phi_2(w))$. We know that $I'_x \subset \ker \Phi$ by Proposition A.4 for all $x \in (Q_{T(A)})_0$. Then Lemma A.3(f) states that if $\phi_2(w) \neq 0$ then there are no supplements $\rho_k$ of $v_k$ such that $\sum_{k=1}^{l} a_k \rho_k v_k \in I'_x$. Then $w$ is not in $I'$. This shows that $I' \subset \ker \Phi$. 
Since \( \Phi : KQ_{T(A)} \to T(A) \) is surjective and \( I' \subset \ker \Phi \), it is enough to show that \( \dim_K(KQ_{T(A)}/I') = \dim_K T(A) = 2 \dim_K A \). Note that the inclusion morphism \( \sigma : A \to T(A) \) factors through \( KQ_{T(A)}/I' \) because \( I_A \subset I' \), which implies that \( \iota : A \to KQ_{T(A)}/I' \) is an algebra monomorphism.

We have that \( KQ_{T(A)} = KQ_A + I_M \). Then \( e_x KQ_{T(A)} e_y = e_x KQ_A e_y + e_x I_M e_y \) for each \( x, y \in (Q(T_A))_0 \). Let \( \pi : KQ_{T(A)} \to KQ_{T(A)}/I' \) be the canonical epimorphism. We define the subspaces \( \mathcal{P}_{xy} = \iota (e_x A e_y) \) and \( \mathcal{F}_{xy} = \pi (e_x I_M e_y) \). Note that \( \sum_{xy} \dim_K \mathcal{P}_{xy} = \dim_K A \).

We start by showing that \( \mathcal{P}_{xy} \neq 0 \) if and only if \( \mathcal{F}_{yx} \neq 0 \). Indeed, if \( \mathcal{F}_{yx} \neq 0 \) then there exists a non-zero path \( q \in \mathcal{F}_{yx} \) which admits a supplement \( r \) in an elementary cycle \( C \) by Remark [A.2](i). Moreover, \( r \in KQ_A \) since \( q \) contains an arrow \( \beta \in I_M \) and \( C \) is an elementary cycle. Hence \( 0 \neq r \in \mathcal{P}_{xy} \) and \( \mathcal{P}_{xy} \neq 0 \).

Conversely, if \( \mathcal{P}_{xy} \neq 0 \) there exists a non-zero path \( q \in \mathcal{P}_{xy} \) which is contained in an elementary cycle \( C \) with supplement \( r \). Moreover, we know that there exists a unique arrow \( \beta \in I_M \) which is contained in the cycle \( C \). Since \( C =rq \) and \( q \in \mathcal{P}_{xy} \), we have that \( r \) contains \( \beta \). Thus \( r \in \mathcal{F}_{yx} \) and \( \mathcal{F}_{yx} \) is non-zero.

We now prove that \( \dim_K \mathcal{P}_{xy} \geq \dim_K \mathcal{F}_{yx} \). Suppose to the contrary that \( n := \dim_K \mathcal{P}_{xy} < \dim_K \mathcal{F}_{yx} \). Then there is a set of linearly independent paths \( \{ \mu_1, \ldots, \mu_{n+1} \} \) in \( \mathcal{F}_{yx} \). Furthermore, \( \mu_t \) does not belong to \( I' \) for all \( 1 \leq t \leq n + 1 \). This implies that \( \mu_t \) is included in an elementary cycle \( C_t \) and admits a supplement \( \gamma_t \) for all \( 1 \leq t \leq n + 1 \).

Note that, \( 0 \neq \gamma_t \in \mathcal{P}_{xy} \) for all \( t \). Since \( n = \dim_K \mathcal{P}_{xy} \), there exists \( 1 \leq s \leq n + 1 \) such that \( \gamma_s = \sum_{i=1}^{s-1} a_i \gamma_i \), where \( \{ \gamma_1, \ldots, \gamma_{n-1} \} \) is a linearly independent set in \( \mathcal{P}_{xy} \). Then there exists \( 1 \leq r \leq s - 1 \) such that \( \mu_s \gamma_r \neq 0 \) since \( 0 \neq a_s \mu_s \gamma_s = \sum_{i=1}^{s-1} a_i \mu_s \gamma_i \).

Now, since \( \mu_s \gamma_r \) is a non-zero path going from the vertex \( x \) to itself we have the existence of a cycle \( \rho \) from \( x \) to \( x \) in \( KQ_A \) such that \( \rho \mu_s \gamma_r \) is an elementary cycle in \( KQ_A \). By hypothesis \( A \) is a finite-dimensional algebra, implying the existence of \( m \in \mathbb{N} \) such that \( \rho^{m-1} \neq 0 \) and \( \rho^m = 0 \). Moreover we have that \( \{ \rho^{m-1} \mu_1, \ldots, \rho^{m-1} \mu_{n+1} \} \) is a linearly independent set in \( \mathcal{F}_{yx} \). We know from Remark [A.3](i) that the set of elementary cycles from \( x \) to \( x \) generate a subspace of dimension one in \( KQ_{T(A)}/I' \). Thus, there exists a non-zero \( b \in K \) such that \( \rho \mu_s \gamma_r = b \mu_r \gamma_r \) and then the element \( \rho \mu_s - b \mu_r \) belongs to class \( (3) \) of \( I' \). Therefore, \( \rho \mu_s - b \mu_r = 0 \). As a consequence \( \mu_r = b^{-1} \rho \mu_s \). Then \( 0 \neq \rho^{m-1} \mu_r = b^{-1} \rho^m \mu_s = 0 \), a contradiction. So \( \dim_K \mathcal{P}_{xy} \geq \dim_K \mathcal{F}_{yx} \) as claimed.

From this we can conclude that \( \dim_K T(A) \geq \dim_K KQ_{T(A)}/I' \), since
\[
\dim_K T(A) = 2 \dim_K A = \sum_{x,y \in Q_0} \dim_K \mathcal{P}_{xy} + \dim_K \mathcal{F}_{yx} \\
\geq \sum_{x,y \in Q_0} \dim_K \mathcal{P}_{xy} + \dim_K \mathcal{F}_{yx} \geq \dim(KQ_{T(A)}/I').
\]

Now, using that \( KQ_{T(A)}/I' \) maps onto \( T(A) \), we obtain that \( \dim_K(KQ_{T(A)}/I') \geq \dim_K T(A) \). This completes the proof of the theorem. \( \square \)

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