Recursive-Biased-Generalized-Inverse Method for Thrust Allocation for Dynamic Positioning Vessels with Non-Rotatable Thrusters and Force Constraints

Bo Zhao, Guoqing Xia
College of Automation, Harbin Engineering University, Harbin, China

bo_zhao@hrbeu.edu.cn, xiaguoqing@hrbeu.edu.cn

Abstract. This paper presents a novel method for thrust allocation for DP vessels with non-rotatable thrusters and force constraints. It uses a modified cost function of the thrust allocation problem for non-rotatable thruster and without force constraints, such that the force constraint can be handled. The proposed method runs in a recursive way to minimize the allocation error. The proposed method is different from the known methods in the literature. The completeness and continuity of the proposed algorithm are studied to verify the effectiveness of the proposed method.

1. Introduction

Thrust allocation (TA), which is control allocation (CA) on thruster systems of marine crafts, is a difficult problem. CA is a frequently encountered problem in motion control system with redundant actuators. If a system is equipped with redundant actuators, since a control command can be achieved in various ways by different actuators. The CA problem is to calculate and select the best control among all feasible solutions. [1] has systematically reviewed the up-to-date technologies in control allocation problem.

DP vessels are the vessels, which can keep their position and heading by only using their thrusters in the presence of wind, wave, and ocean current ([2], [3]). They have been widely used in ocean oil and gas exploitation, offshore engineering, oceanic research, and other marine operations. Due to the possible catastrophic consequence of failures of DP vessels, DP control system is required to be redundant by international regulations. Therefore, it is necessary to have more than three thrusters installed on a DP vessel. However, it is common for a DP vessel to have even more thrusters. As the DP vessel is to be regulated at a certain position and heading, which is in 3 degree-of-freedom, the thruster system is over-actuated. This means the thrust allocation problem have to be solved in DP system design. [2] details the thrust allocation problem.

This paper considers the constrained linear control allocation problem in [1], which equals to the problem of thrust allocation for dynamic positioning (DP) vessels with non-rotatable thrusters and force constraints in [2]. This paper proposed recursive-biased-generalized-inverse method for this problem. To the knowledge of the authors, the proposed method is different from any existing method. Comparing with redistributed pseudo-inverse method, daisy chaining method and direct allocation method, the proposed method is complete. Comparing with methods based on mathematical programming, this method is more efficient.
This paper introduces the problem of TA in Section 2 and explains the proposed method in Section 3. The properties of the proposed method are analyzed in Section 4. Section 5 gives the conclusion.

2. Thrust allocation problem
Tunnel thrusters, azimuth thrusters, and main propellers are the common thrusters used for DP vessels. Defining the body-frame of a vessel as Figure 3, the general force and moment produced by a thruster can be written as
\[
\tau = t(\alpha)f
\]
\[
= \begin{bmatrix}
\cos(\alpha) \\
\sin(\alpha) \\
l_x \sin(\alpha) - l_y \cos(\alpha)
\end{bmatrix} f,
\]
where \(\alpha\) is the angle of the thrust force, which is measured from x-axis to the force vector, \(f\) is the magnitude of the thrust, and \(l_x, l_y\) are the location of the thruster. The resulting force and moment of all the thrusters of a DP vessel is a collection of (1), such as
\[
\tau = T(\alpha)f,
\]
where \(T = [t(\alpha_1) \ldots t(\alpha_n)]\) is named thrust configuration matrix following [2], \(\alpha = [\alpha_1 \ldots \alpha_n], f = [f_1 \ldots f_n], \alpha_1, \ldots, \alpha_n\) and \(f_1, \ldots, f_n\) are the angles and magnitudes of the forces produced by thrusters, respectively.
Usually, the control command of a DP controller can be expressed as
\[
\tau_c = [X \quad Y \quad N]^T,
\]
where \(X, Y, N\) are the force command in surge, sway direction and moment command in yaw, respectively. In TA problem, the thrusters are to be commanded by the TA algorithm to produce a resulting force which follows the generalized force command \(\tau_c\), such as \(\tau = \tau_c\). A TA algorithm can be written as a mapping
\[
[\alpha \quad f]^T = U(\tau_c), \text{ subject to } \tau - T(\alpha)f = 0,
\]
where \(U(\cdot)\) is the TA mapping, and \(\tau_c\) is the generalized force to be allocated.
Since the force produced by thrusters is limited, a virtual control \(\tau_c\) may not be able to be allocated to a given group of thrusters. This is usually referred as force constraint and can be expressed as \(f_{\min} < f < f_{\max}\).
Azimuth thrusters and rudders are subject to angular constraints. However, we only consider tunnel thruster and azimuth thrusters with a fixed angle in this paper.

3. Recursive-Biased-Generalized-Inverse Method

3.1. Biased-generalized-inverse (BGI) method
The original idea of thrust allocation using generalized inverse is described in [2]. The problem can be defined as
\[
J = \min_f \left\{ f^T W f \right\}
\]
subject to: \(\tau - Tf = 0\)
where \(W\) is a weighting matrix. This formula is adopted from [2]. But it is only suitable for solving control allocation problem for non-rotatable thrusters without force constraint, which is an ideal case in TA.
Consider the following modification to the cost function.
\[
J = \min_f \left\{ (f - f_0)^T W_1 (f - f_0) + f^T W_2 f \right\}
\]
subject to: \(\tau - Tf = 0\)
where $W_1$ and $W_2$ are positive definite diagonal weighting matrices, $f_0$ is a parameter which will be discussed in the following. The second term in the cost function is the same as the one in (6), which represent the cost of the usage of thrusters. The first term represents the deviation of the control force to the parameter $f_0$.

The first term in the cost function (6) can serve different practical purposes.

- Consider a discrete time DP control system. If $f_0$ is the control force of the thrusters of a DP vessel at the last time step, and $f$ is the control force at current step. Minimizing $\|f - f_0\|$ is to reduce the change of the thrust force. Because the change rate of the thrust force is limited due to physical constraint, it is meaningful to have this term in the cost function.

- The term can be used to avoid allocated control forces exceeding the limits. For example, assuming $\tau_{e,0}$ is allocated as $f_0$ by the BGI algorithm using cost function (6). But assume that some of the elements of $f_0$ exceed the corresponding bounds. For instance, assume $f_k > f_{k,\text{max}}$ and other control forces are within their corresponding bound. Then by choosing $f_0 = [0, \ldots, 0, f_{k,\text{max}} - f_k, 0, \ldots, 0]^T$, where $f_k - f_{k,\text{max}}$ is the $k$th element. In this case, because of the first term in the cost function, the $k$th element in the control force allocated using cost function (6) will be moved to the direction which reduces the size of the excess.

The optimization problem (6) can be solved by Lagrange Multipliers. Assuming $W_1$ and $W_2$ are symmetric. Consider the Lagrangian

$$L(f, \lambda) = (f - f_0)^T W_1 (f - f_0) + f^T W_2 f + \lambda^T (\tau - T^T f)$$

where $\lambda \in \mathbb{R}^3$ is a vector of Lagrange multipliers. Consequently, differentiating the Lagrangian $L$ with respect to $f$ yields

$$\frac{\partial L}{\partial f} = 2 (W_1 + W_2) f - 2 W_1 f_0 - T^T \lambda$$

Let $\frac{\partial L}{\partial f} = 0$, yields

$$f = \frac{1}{2} (W_1 + W_2)^{-1} \left(T^T \lambda + 2 W_1 f_0\right)$$

Next

$$\tau = T^T f = \frac{1}{2} T (W_1 + W_2)^{-1} T^T \lambda + T (W_1 + W_2)^{-1} W_1 f_0$$

Then

$$\lambda = 2 \left(T (W_1 + W_2)^{-1} T^T\right)^{-1} \left(\tau - T (W_1 + W_2)^{-1} W_1 f_0\right)$$

Then

$$f = W_{12}^{-1} T^T \left(T W_{12}^{-1} T^T\right)^{-1} \left(\tau - T W_{12}^{-1} W_1 f_0\right) + 2 W_{12}^{-1} W_1 f_0$$

$$= T^T \tau + \left(2 I - T^T\right) W_{12}^{-1} W_1 f_0,$$

where $W_{12} = W_1 + W_2$ and $T^T = W_{12}^{-1} T^T \left(T W_{12}^{-1} T^T\right)^{-1}$.

3.2. Recursive-biased-generalized-inverse (Rec-BGI) method

In this section, the BGI method is modified using cost function (6) and run in a recursive way to handle the constrained control allocation problem for non-rotatable actuators. The algorithm is as following.
4. Analysis and Verification of Rec-BGI

4.1. Completeness

In this section, the completeness of Rec-BGI is studied with examples. If a TA algorithm is complete, it means that the algorithm is capable to allocate any attainable generalized control command to the feasible region of the thrusters. Furthermore, the completeness of a TA algorithm means the degree of the TA algorithm being complete, i.e. in which subset of the attainable set, the TA algorithm is complete.

The completeness of the proposed algorithm is studied associated with a give thrust configuration. Using the example ship in Appendix, considering T4 and T5 output force in $-\pi/4$ rad and $\pi/4$ rad, respectively, the study of the completeness gives the results in Figure 1.

**Require**: $\tau_c$ - the generalized force to be allocated.

**Require**: $T$ - the thrust configuration matrix.

**Require**: $f_0$ - the initial point.

**Require**: $W_1, W_2$ - weighting matrices.

**Require**: $b_l = [b_{l,1}, \ldots b_{l,p}]^T, b_u = [b_{u,1}, \ldots b_{u,p}]^T$ - lower and upper boundary of the control force.

**Require**: $i_{max}$ - the maximum times of iteration.

**Ensure**: $f$ - allocated control force.

```plaintext
function Rec-BGI($\tau_c, T, f_0, W_1, W_2, b_l, b_u, i_{max}$)  
i = 0  
while $i \leq i_{max}$ do  
    $i += 1$
    $f = BGI(\tau, f_0)$
    if $b_l \leq f$ AND $f \leq b_u$ then break the While loop
    else
        for $i_f = 1 : p$ do
            $f_{ex,i_f} = 0$
            if $b_{i,i_f} > f_{i_f}$ then
                $f_{ex,i_f} = f_{i_f} - b_{i,i_f}$
            end if
            if $f_{i_f} > b_{u,i_f}$ then
                $f_{ex,i_f} = f_{i_f} - b_{u,i_f}$
            end if
        end for
        $f = f - f_{ex}$
        $f_0 = f_0 - f_{ex}$
    end if
end while
end function
```
In Figure 1, the grey surface is the attainable set, which is the set that can be researched by the thrusters. Red scattered spots are the generalized control commands $\tau^e$ which cannot be allocated by the proposed method. The maximum number of iterations is 100 in (a) and 1000 in (b).

From Figure 1, it can be concluded that the Rec-BGI can allocate almost any generalized force in the attainable set except for the generalized force close to the boundary. Results also show that the completeness increases with the number of iterations. From the above results, it can be concluded that the completeness of the Rec-BGI algorithm is high.

4.2. Continuity

The continuity of the solution is an important aspect of TA algorithms. The continuity of the solution means that given generalized force $\tau_0$ and corresponding TA solution $f_0$, such that $f_0 = TA(\tau_0)$, the TA shall satisfy that

$$\lim_{\tau \to \tau_0} TA(\tau) = f_0$$  

(13)

The BGI algorithm is continuous, if the constraints are not considered. To show this, take the derivative of the TA solution (12), such as

$$\frac{\partial f}{\partial \tau} = W_{12}^{-1} T^\top T (W_{12})^{-1} T^\top$$  

(14)

which is bounded. This means that if the BGI can find a solution in the feasible region, the solution is continuous.

The continuity of Rec-BGI is to be discussed in different cases.

**CASE 1**: Both $f_1$ and $f_2$ are in the attainable set. In this case, Rec-BGI leads to the solution being exactly the same as BGI, and then is continuous.

**CASE 2**: Consider the situation that $f_1$ and $f_2$, which are the allocation of $\tau_1$ and $\tau_2$ from Rec-BGI. Assuming that $\tau_1$ and $\tau_2$ are close enough such that they can be allocated with the same bias term $f_0$. Analyzing the difference, resulting

$$\Delta f = f_1 - f_2 = W_{12}^{-1} T^\top T (W_{12})^{-1} T^\top$$  

(15)

which means that the Rec-BGI method is continuous.

5. Conclusion

This paper presented a novel method for thrust allocation for DP vessels with non-rotatable thrusters and force constraints. The main difference comparing with the literature is that the force constraints is handled with an additional bias term in the cost function. The detailed Rec-BGI algorithm is proposed.
The completeness and continuity study of the proposed algorithm shows that the proposed algorithm can allocate almost any generalized control and keeps the change of the allocated force small if the change of the input is small. This makes the proposed algorithm very practical and robust in practice.

6. Appendix - Example vessel

Refer to Figure 2, the vessel in this example is adopted from a real rescue vessel with DP system. The vessel is equipped with three tunnel thrusters to the bow, and two azimuth thrusters to the aft, as shown in Figure 2. The setup of the thrusters is listed in Table 1.

**Figure 2.** The configuration of the thrusters

**Table 1.** Thruster setup.

| Thruster | Location [m]   | Thrust Limit [kN] |
|----------|----------------|-------------------|
|          | x  | y   |                   |
| T1       | 50 | 0   | 150               |
| T2       | 40 | 0   | 150               |
| T3       | -40| 0   | 150               |
| T4       | -65| -5  | 300               |
| T5       | -65| 5   | 300               |

**Acknowledgments**

This work is supported by the 7th Generation Ultra Deep Water Drilling Platform (Ship) Innovation Project, the Fundamental Research Funds for the Central Universities (HEUCFJ180404), and the National Natural Science Foundation of China (Grant No.51879049) research on green and safe mooring-assisted dynamic positioning technology for all-weather deepwater platform.

**References**

[1] Tor A. Johansen and Thor I. Fossen. Control allocation|A survey. Automatica, 49(5): 1087-1103, may 2013.

[2] Thor I. Fossen. Handbook of Marine Craft Hydrodynamics and Motion Control. John Wiley & Sons, Ltd, 1 edition edition, 2011.

[3] Asgeir J. Sørensen. A survey of dynamic positioning control systems. Annual Reviews in Control, 35(1):123-136, apr 2011.