MULTIDIMENSIONAL QUANTUM COSMOLOGY: QUANTUM WORMHOLES, THIRD QUANTIZATION, INFLATION FROM "NOTHING", etc

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Abstract

A multidimensional cosmological model with space-time consisting of \( n \) \( (n \geq 2) \) Einstein spaces \( M_i \) is investigated in the presence of a cosmological constant \( \Lambda \) and \( m \) \( (m \geq 1) \) homogeneous minimally coupled scalar fields as a matter source. Classes of the models integrable at classical as well as quantum levels are found. These classes are equivalent to each other. Quantum wormhole solutions are obtained for them and the procedure of the third quantization is performed. An inflationary universe arising from classically forbidden Euclidean region is investigated for a model with a cosmological constant.

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1 Introduction

We believe that for description of quantum gravitational processes at high energies the multidimensional approach is more adequate. Modern theories of unified physical interactions use ideas of hidden (or extra) dimensions. In order to study different phenomena at early stage of the universe one should use these theories or at any rate models keeping their main ideas. In any case, multidimensional models have to explain the observed four dimensionality of space-time at present time. This is realized in models where one or a number of internal spaces are compact and contracted to Planckian scales during the evolution of the universe (dynamical compactification) or the symmetry between the external and internal dimensions is broken from the very beginning and the internal spaces are static and compactified at Planck’s length (spontaneous compactification). A further possibility is given in the quantum theory where only the external spaces can be created by tunneling while the internal dimensions may be hidden because they stay behind a potential barrier.

Of special interest are exact solutions because they can be used for a detailed study of the evolution of our space, of the compactification of the internal spaces and of the behavior of matter fields.

One of the most natural multidimensional cosmological models (MCM) generalizing the Friedmann-Robertson-Walker (FRW) universe is given by a toy model with the topology $R \times M_1 \times \ldots \times M_n$ where $M_i \ (i = 1, \ldots, n)$ denotes Einstein space. One of these spaces, say $M_1$, describes the external space but all others are internal spaces. The gauge covariant form of the Wheeler-De Witt (WDW) equation \[1, 2\] for a model with this topology was proposed in \[3\] and some integrable models were investigated in \[4]-\[6\].

In the recent paper we consider a general model with a cosmological constant $\Lambda$ and $m \ (m \geq 1)$ homogeneous minimally coupled scalar fields $\varphi^{(a)} \ (a = 1, \ldots, m)$ with potentials $U^{(a)}(\varphi^{(a)})$. We show that some integrable models considered before \[4]-\[6\] and some new ones are equivalent to each other. Among solutions of the WDW equations for these models there are ones which describe tunneling universes, in particular, birth of universes from classically forbidden Euclidean region (birth from ”nothing” \[7\]). Quantum wormholes \[8\] representing a special class of solutions of the WDW equation are constructed for considered models also. Full sets of the orthonormal solutions in these models give possibility to perform third quantization \[9\] and to get a spectrum of created universes. An inflationary universe arising due to quantum tunneling is investigated for a model with a cosmological constant. Parameters of the model which ensure inflation of the ex-
ternal space and dynamical compactification of internal spaces are found. In particular, the dimension of internal spaces should be $d > 40$. It is shown that the tunneling from “nothing” of this universe is strongly suppressed because of very large spatial volume of the arisen universe.

2 General description of the model

The metric of the model

$$g = -\exp [2\gamma(\tau)]d\tau \otimes d\tau + \sum_{i=1}^{n} \exp [2\beta^i(\tau)]g^{(i)}$$

(2.1)

is defined on the manifold

$$M = R \times M_1 \times \ldots \times M_n,$$

where the manifold $M_i$ with the metric $g^{(i)}$ is an Einstein space of dimension $d_i$, i.e.

$$R_{m_i n_i}[g^{(i)}] = \lambda^i g^{(i)m_in_i}$$

(2.2)

$$i = 1, \ldots, n; \ n \geq 2.$$

The total dimension of the space-time $M$ is $D = 1 + \sum_{i=1}^{n} d_i$. This describes the case where the topology of a factorized space-time manifold is assumed from the very beginning and compactification of internal spaces is described as shrinking to or freezing on the Planck scale.

Here we investigate the general model with cosmological constant $\Lambda$ and $m$ ($m \geq 1$) non-interacting homogeneous minimally coupled scalar fields $\varphi^{(a)}$ ($a = 1, \ldots, m$) with potentials $U^{(a)}(\varphi^{(a)})$. The action of the model is adopted in the following form:

$$S = \frac{1}{2\kappa^2} \int d^D x \sqrt{|g|} \left[ R[g] - 2\Lambda \right] + S_\varphi + S_{GH},$$

(2.3)

where $R[g]$ is the scalar curvature of the metric (2.1) and $\kappa^2$ is $D$-dimensional gravitational constant. $S_{GH}$ is the standard Gibbons-Hawking boundary term [10]. $S_\varphi = \sum_{a=1}^{m} S_\varphi^{(a)}$ is the action of $m$ non-interacting minimally coupled homogeneous scalar fields

$$S_\varphi^{(a)} = \int d^D x \sqrt{|g|} \left[ -\frac{1}{2} g^{MN} \partial_M \varphi^{(a)} \partial_N \varphi^{(a)} - U^{(a)}(\varphi^{(a)}) \right].$$

(2.4)

For the metric (2.1) the action (2.3) reads

$$S = \mu \int d\tau L$$

(2.5)

with the Lagrangian $L$ being

$$L = \frac{1}{2} e^{-\gamma + \gamma_0} \left( G_{ij} \dot{\beta}^i \dot{\beta}^j + \kappa^2 \sum_{a=1}^{m} \left( \dot{\varphi}^{(a)} \right)^2 \right) - V.$$

(2.6)
Here $\gamma_0 = \sum_{i=1}^n d_i \beta^i$ and the overdot denotes differentiation with respect to the time $\tau$. The components of the minisuperspace read

$$G_{ij} = d_i \delta_{ij} - d_i d_j,$$  \hspace{1cm} (2.7)

and the potential is given by

$$V = e^{\gamma + \gamma_0} \left( -\frac{1}{2} \sum_{i=1}^n \theta_i e^{-2\beta^i} + \kappa^2 \sum_{a=1}^m U^{(a)}(\varphi^{(a)}) + \Lambda \right), \hspace{1cm} (2.8)$$

where $\theta_i = \lambda^i d_i$. If the $M_i$ are spaces of constant curvature, then $\theta_i$ may be normalized in such a way that $\theta_i = k_i d_i (d_i - 1)$, $k_i = \pm 1, 0$. The parameter $\mu = \prod_{i=1}^n V_i / \kappa^2$ where $V_i$ is the volume of $M_i$ and we may put $\mu = 1$ [11].

The constraint equation reads

$$-\frac{\partial L}{\partial \gamma} = \frac{1}{2} e^{-\gamma + \gamma_0} \left( G_{ij} \dot{\beta}^i \dot{\beta}^j + \kappa^2 \sum_{a=1}^m (\dot{\varphi}^{(a)})^2 \right) + V = 0. \hspace{1cm} (2.9)$$

The minisuperspace metric $G = G_{ij} d\beta^i \otimes d\beta^j$ may be diagonalized in different coordinate systems. In the present paper we use two of them. In the first one the minisuperspace metric reads [4, 5, 12]

$$G = -dv^0 \otimes dv^0 + \sum_{i=1}^{n-1} dv^i \otimes dv^i, \hspace{1cm} (2.10)$$

where

$$q_1 v^0 = (d_1 - 1) \beta^1 + \sum_{i=2}^n d_i \beta^i,$$

$$q_1 v^1 = \left[ (D - 2) / (d_1 \Sigma_2) \right]^{1/2} \sum_{i=2}^n d_i \beta^i,$$

$$q_1 v^i = \left[ (d_1 - 1) d_i / d_1 \Sigma_1 \Sigma_{i+1} \right]^{1/2} \sum_{j=i+1}^n d_j (\beta^j - \beta^i), \ i = 2, \ldots, n - 1. \hspace{1cm} (2.11)$$

Here we used the notations $q_1 = [(d_1 - 1) / d_1]^{1/2}$ and $\Sigma_i = \sum_{j=i}^n d_j$. In the second coordinate system

$$G = -dz^0 \otimes dz^0 + \sum_{i=1}^{n-1} dz^i \otimes dz^i, \hspace{1cm} (2.12)$$

where [3, 4, 13]

$$z^0 = q_2^{-1} \sum_{i=1}^n d_i \beta^i,$$

$$z^i = [d_i / \Sigma_j \Sigma_{j+1}]^{1/2} \sum_{j=i+1}^n d_j (\beta^j - \beta^i), \ i = 1, \ldots, n - 1. \hspace{1cm} (2.13)$$
Here $q_2 = [(D - 1)/(D - 2)]^{1/2}$. The spatial volume of the universe is proportional to $v = \prod_{i=1}^{n} a_i^{d_i}$ where scale factors $a_i = \exp \beta^i$. In the coordinates (2.13) it takes the following form

$$v = \prod_{i=1}^{n} a_i^{d_i} = \exp (q_2 z^0). \quad (2.14)$$

3 Wheeler-De Witt equations. Integrable cosmologies

At the quantum level the constraint (2.9) is modified into the WDW equation [1, 2]. Now, we consider classes of cosmological models which are integrable at classical as well as quantum levels, and we show the equivalence of these models.

3.1 Models with one non-Ricci-flat factor space

In the present chapter we consider the integrable case of a MCM where only one of the factor space $M_i$, say $M_1$, is not Ricci-flat: $\theta_1 \neq 0$, $\theta_i = 0$, $i = 2, \ldots, n$. Using the coordinates (2.11) we set for free scalar fields (it is clear that in the case of free scalar fields it is sufficient to take $m = 1$) the following form of the constraint (2.9)

$$-\left(\dot{\psi}^0\right)^2 + \sum_{i=1}^{n-1} \left(\dot{\psi}^i\right)^2 + \dot{\varphi}^2 - \theta_1 e^{2q_1 v^0} = 0,$$  

(3.1)

where we used the harmonic time gauge $\gamma = \gamma_0$ and the scalar field $\varphi^{(1)} \equiv \varphi$ is redefined: $\kappa \varphi \rightarrow \varphi$. The WDW equation in this case reads [4, 5, 12, 14]

$$\left(-\frac{\partial}{\partial v^0} \frac{\partial}{\partial v^0} + \sum_{i=1}^{n-1} \frac{\partial}{\partial v^i} \frac{\partial}{\partial v^i} + \frac{\partial^2}{\partial \varphi^2} + \theta_1 e^{2q_1 v^0}\right) \Psi = 0.$$  

(3.2)

It is easy to obtain solutions of this equation by separation of variables

$$\Psi(v) = e^{ipv} \Phi(v^0),$$  

(3.3)

where $p = (p^1, \ldots, p^n)$ is a constant vector, $v = (v^1, \ldots, v^{n-1}, \varphi)$, $pv = \sum_{i=1}^{n} p_i v^i$ and $p_i = p^i$. The substitution of (3.3) into (3.2) gives

$$\left[-\frac{1}{2} \left(\frac{d}{dv^0}\right)^2 + \frac{1}{2} \theta_1 e^{2q_1 v^0}\right] \Phi = \varepsilon \Phi,$$  

(3.4)

where

$$\varepsilon = \frac{1}{2} \sum_{i=1}^{n} (p^i)^2.$$  

(3.5)
3.2 Models with cosmological constant

Here we consider the integrable case of a MCM with all Ricci-flat factor-spaces: \( \theta_i = 0 \) \((i = 1, \ldots, n)\) in the presence of the cosmological constant \( \Lambda \) and free scalar field as a matter source. Using the coordinates (2.13) and the harmonic time gauge we get for (2.9)

\[
- (z^0)^2 + \sum_{i=1}^{n-1} (z^i)^2 + \varphi^2 + 2\Lambda e^{2q_2z^0} = 0 ,
\]

which is modified into the WDW equation [3, 6, 13]

\[
\left( -\frac{\partial}{\partial z^0} \frac{\partial}{\partial z^0} + \sum_{i=1}^{n-1} \frac{\partial}{\partial z^i} \frac{\partial}{\partial z^i} + \frac{\partial^2}{\partial \varphi^2} - 2\Lambda e^{2q_2z^0} \right) \Psi = 0 .
\]

We are seeking the solution of (3.7) in the form

\[
\Psi(z) = e^{ipz}\Phi(z^0) .
\]

where \( p = (p^1, \ldots, p^n) \) and \( z = (z^0, \ldots, z^{n-1}, z^n = \varphi) \). \( \Phi(z^0) \) satisfies the equation

\[
\left[ -\frac{1}{2} \left( \frac{d}{dz^0} \right)^2 - \Lambda e^{2q_2z^0} \right] \Phi = \varepsilon \Phi
\]

with \( \varepsilon \) defined by the relation (3.5).

It is easy to see that the models of the subsections 3.1 and 3.2 are equivalent to each other with an accuracy to the evident substitutions:

\[
v^i \leftrightarrow z^i, \; i = 0, \ldots, n - 1,
\]

\[
\frac{1}{2} \theta_1 \leftrightarrow -\Lambda,
\]

\[
q_1 \leftrightarrow q_2 .
\]

Thus, to investigate the quantum behavior of these models it is sufficient to consider one of them, e.g. that of subsection 3.2 only.

3.3 Exact scalar field cosmologies

Here we consider a special class of the integrable MCM with \( m \) \((m \geq 1)\) scalar fields. The action of these models is given by the relations (2.3), (2.4) where \( \Lambda = 0 \). The Lagrangian (2.9) for these models reads
The energy-momentum tensor of a matter with the action
\[ S = \int d^Dx \sqrt{|g|} L \] is defined by
\[ T_{ik} = -2 \frac{\partial L}{\partial g^ik} + g_{ik} L. \] (3.12)

Using this formula we get the non-zero components of the scalar field \( \varphi^{(a)} \) energy-momentum tensor:
\[
T^{(a)}_0^0 = -\frac{1}{2} e^{2\gamma} \left( \dot{\varphi}^{(a)} \right)^2 - U^{(a)}(\varphi^{(a)}) \equiv -\rho^{(a)}, \]
(3.13)
\[
T_{im}^{(a)m_i} = \frac{1}{2} e^{2\gamma} \left( \dot{\varphi}^{(a)} \right)^2 - U^{(a)}(\varphi^{(a)}) \equiv P^{(a)}, \quad i = 1, \ldots, n, \]
(3.14)
where we introduced the energy density \( \rho^{(a)} \) and the pressure \( P^{(a)} \) which correspond to the scalar field \( \varphi^{(a)} \). Now we suppose that these quantities are connected by the equation of state
\[
P^{(a)} = \left( \alpha^{(a)} - 1 \right) \rho^{(a)}, \]
(3.15)
where \( \alpha^{(a)} = \text{const}, \quad a = 1, \ldots, m. \)

It is not difficult to prove that these models are equivalent to the cosmological models in the presence of \( m \)-component perfect fluid with the energy-momentum tensor
\[ T^M_N = \sum^m_{a=1} T^{(a)M}_N, \]
(3.16)
\[
T^{(a)M}_N = \text{diag} \left( -\rho^{(a)}(\tau), P^{(a)}(\tau)\delta^{m_i}_{k_1}, \ldots, P^{(a)}(\tau)\delta^{m_n}_{k_n} \right), \]
(3.17)
where for any \( a \)-th component of perfect fluid the pressure and the energy density are connected by the equation of state (3.13) and conservation equations are imposed on each component separately:
\[ T^{(a)M}_{N;M} = 0. \]
(3.18)
The non-trivial conservation equations (3.18) are
\[
\dot{\rho}^{(a)} + \dot{\gamma}_0 \left( \rho^{(a)} + P^{(a)} \right) = 0, \quad a = 1, \ldots, m, \]
(3.19)
which results in
\[
\rho^{(a)} = A^{(a)} \exp \left( -\alpha^{(a)} \gamma_0 \right) = A^{(a)} v^{-\alpha^{(a)}}, \quad a = 1, \ldots, m, \]
(3.20)
where $A^{(a)} = \text{const}$ and the spatial volume $v$ is defined by (2.14).

To prove the proposition about equivalence between these models we note first that the Klein-Gordon equations

$$\frac{\partial}{\partial \tau} \left( e^{-\gamma_0} \frac{\partial \varphi^{(a)}}{\partial \tau} \right) + e^{\gamma_0} \frac{\partial U^{(a)}}{\partial \varphi^{(a)}} = 0, \; a = 1, \ldots, m,$$

(3.21)

following from the Lagrangian (3.11) are equivalent to the conservation equations (3.19) if the relations (3.13) and (3.14) have place. Second, using the relations (3.13) – (3.15) and (3.20) we obtain

$$\frac{\partial L_s}{\partial \dot{\gamma}} = \frac{\partial L_\rho}{\partial \gamma},$$

(3.22)

and

$$d \frac{\partial L_s}{d \tau \partial \dot{\beta}_i} - \frac{\partial L_s}{\partial \beta_i} = d \frac{\partial L_\rho}{d \tau \partial \dot{\beta}_i} - \frac{\partial L_\rho}{\partial \beta_i}, \; i = 1, \ldots, n,$$

(3.23)

where the lagrangian $L_s$ is given by (3.11) and the effective Lagrangian $L_\rho$ corresponding to the models with the energy-momentum tensor (3.16) – (3.18) is [15, 16, 17]

$$L_\rho = \frac{1}{2} e^{-\gamma_0} G_{ij} \dot{\beta}_i \dot{\beta}_j + e^{\gamma_0} \left( \frac{1}{2} \sum_{i=1}^n \lambda_i d_i e^{-2\beta_i} - \kappa^2 \sum_{a=1}^m \rho^{(a)}(v) \right),$$

(3.24)

with $\rho^{(a)}$ defined by (3.20).

To get integrable models we consider the Ricci-flat case ($\lambda^i = 0, i = 1, \ldots, n$). Then, in the $z$-coordinates (2.13) the Lagrangian and the constraint read respectively

$$L_\rho = -\frac{1}{2} (\dot{z}^0)^2 + \frac{1}{2} (\dot{z}^i)^2 - \kappa^2 \sum_{a=1}^m A^{(a)} \exp \left[ (2 - \alpha^{(a)}) q_2 z^0 \right]$$

(3.25)

and

$$-\frac{1}{2} (\dot{z}^0)^2 + \frac{1}{2} (\dot{z}^i)^2 + \kappa^2 \sum_{a=1}^m A^{(a)} \exp \left[ (2 - \alpha^{(a)}) q_2 z^0 \right],$$

(3.26)

where we use the harmonic time gauge $\gamma = \gamma_0$. The equations of the motion

$$\ddot{z}^i = 0, \; i = 1, \ldots, n-1$$

(3.27)

have the first integrals

$$\dot{z}^i = p^i.$$

(3.28)

The constraint (3.26) can be rewritten as follows

$$\dot{v} = \pm \sqrt{2} q_2 v \left[ \varepsilon + \kappa^2 v^2 \sum_{a=1}^m \rho^{(a)}(v) \right],$$

(3.29)
where the parameter $\varepsilon$ is defined by the relation similar to (3.8). Using the relation (3.28) and (3.13) it is not difficult to get $\phi^{(a)}$ as a function of the spatial volume (18)

$$\phi^{(a)} = \pm \sqrt[\alpha(a)/2]{q_2} \int \sqrt[\rho(a)(v)]{[\varepsilon + \kappa^2 v^2 \sum_{a=1}^m \rho(a)(v)]^{1/2}} \phi^{(a)}_0. \quad (3.30)$$

Inverting this expression we can find the spatial volume as a function of the scalar field $\phi^{(a)}: v = v(\phi^{(a)})$ and consequently a dependence of the energy density $\rho^{(a)}$ on the scalar field $\phi^{(a)}: \rho^{(a)} = \rho^{(a)}(\phi^{(a)})$. To reconstruct the potentials $U^{(a)}$ we can write them with the help of the relations (3.13) – (3.15) in the form

$$U^{(a)}(\phi^{(a)}) = \frac{1}{2}(2 - \alpha^{(a)})\rho^{(a)}(\phi^{(a)}), \quad a = 1, \ldots, m, \quad (3.31)$$

where

$$\rho^{(a)}(\phi^{(a)}) = A^{(a)}[v(\phi^{(a)})]^{-\alpha^{(a)}}. \quad (3.32)$$

If $\alpha^{(a)} = 0$, then $\phi^{(a)}, U^{(a)}$ and $\rho^{(a)}$ are constant. This scalar field is equivalent to the cosmological constant $\Lambda \equiv \kappa^2 U^{(a)}$ with the equation of state $P^{(a)} = -\rho^{(a)}$. For $\alpha^{(a)} = 2$ we have $U^{(a)} \equiv 0$. The scalar field $\phi^{(a)}$ in this case is equivalent to the perfect fluid with the ultra-stiff equation of state $P^{(a)} = \rho^{(a)}$.

It is clear that it is hardly possible to integrate models with an arbitrary set of exponents in the relations (3.23) and (3.26). Therefore, we consider now the integrable case of the two-component ($m = 2$) scalar field with an arbitrary $\alpha^{(1)} \equiv \alpha \neq 0$, 2 and $\alpha^{(2)} = 2$ which corresponds to the the two-component perfect fluid with the energy density

$$\rho = \rho^{(1)} + \rho^{(2)} = A^{(1)} v^{-\alpha} + A^{(2)} v^{-2}. \quad (3.33)$$

The potentials $U^{(a)}$ in this case read (18) $U^{(2)} \equiv 0$ and

$$U^{(1)}(\phi^{(1)}) = \frac{(2 - \alpha)\Lambda}{2\kappa^2} \exp \left[ \pm \sqrt{2\alpha} \kappa q_2 (\phi^{(1)} - \phi^{(1)}_0) \right], \quad E = 0, \quad \Lambda > 0,$$

$$= U^{(1)}_0 \left[ \sinh \left( \frac{\kappa q_2(2 - \alpha)}{\sqrt{2\alpha}} (\phi^{(1)} - \phi^{(1)}_0) \right) \right]^{-2\alpha/(2-\alpha)}, \quad E > 0, \quad \Lambda > 0,$$

$$= -U^{(1)}_0 \left[ \sin \left( \frac{\kappa q_2(2 - \alpha)}{\sqrt{2\alpha}} (\phi^{(1)} - \phi^{(1)}_0) \right) \right]^{-2\alpha/(2-\alpha)}, \quad E > 0, \quad \Lambda < 0,$$

$$= U^{(1)}_0 \left[ \cosh \left( \frac{\kappa q_2(2 - \alpha)}{\sqrt{2\alpha}} (\phi^{(1)} - \phi^{(1)}_0) \right) \right]^{-2\alpha/(2-\alpha)}, \quad E < 0, \quad \Lambda < 0, \quad (3.34)$$

where $\Lambda = \kappa^2 A^{(1)}$ and

$$E = \varepsilon + \kappa^2 A^{(2)}. \quad (3.35)$$
As usual, at quantum level the constraints $\partial L_s/\partial \gamma = 0$ and $\partial L_\rho/\partial \gamma = 0$ are modified into the WDW equations. It is more simple to investigate a quantum behavior of the geometry in these models studying the WDW equation which was obtained from the effective Lagrangian (3.23). For the integrable two-component model (3.33) we get

$$\left( -\frac{\partial}{\partial z_0} \frac{\partial}{\partial z_0} + \sum_{i=1}^{n-1} \frac{\partial}{\partial z_i} \frac{\partial}{\partial z_i} - 2\kappa^2 A^{(2)} - 2\kappa^2 A^{(1)} e^{(2-\alpha)q_2z_0} \right) \Psi = 0. \quad (3.36)$$

Separating the variables in the form

$$\Psi(z) = e^{ipz} \Phi(z^0), \quad (3.37)$$

where $p = (p^1, \ldots, p^{n-1})$ is a constant vector and $z = (z^1, \ldots, z^{n-1})$, we get for $\Phi(z^0)$ the equation

$$\left[ -\frac{1}{2} \left( \frac{d}{dz_0} \right)^2 - \kappa^2 A^{(1)} e^{(2-\alpha)q_2z_0} \right] \Phi = E \Phi \quad (3.38)$$

with $E$ defined by (3.35).

It is easy to see again that the models of the subsections 3.2 and 3.3 are equivalent to each other also with an accuracy to the substitutions:

$$\varepsilon \leftrightarrow E,$$

$$\Lambda \leftrightarrow \kappa^2 A^{(1)},$$

$$2q_2 \leftrightarrow (2-\alpha)q_2. \quad (3.39)$$

4 Quantum wormholes

Using the equivalency between these three models we can consider one of them only to investigate quantum behavior of the universe. Let us consider the second of them. Simple analysis of the equation (3.9) shows [6] that quantum behavior depends strongly on the signs of $\Lambda$ and $\varepsilon$. For example, if $\Lambda > 0$ then for $\varepsilon \geq 0$ the Lorentzian regions exist only. For $\varepsilon < 0$ both regions, the Lorentzian as well as Euclidean one exist. In this case quantum transitions with topology changes take place (tunneling universes or birth from "nothing"). If $\Lambda < 0$ then for $\varepsilon \leq 0$ the Euclidean region exists only. For $\varepsilon > 0$ both regions, the Lorentzian as well as Euclidean one exist. In this case quantum transitions with topology changes take place (quantum wormholes).

Let us consider the latter case in more detail. Solving (3.9), we get

$$\Phi(z^0) = B_{\sqrt{2\varepsilon}/q_2} \left( \sqrt{-2M_2^{-1}} \exp (q_2z_0^0) \right), \quad (4.1)$$

10
where $\sqrt{2\varepsilon/q} = |p|/q_2$ and $B = I, K$ are modified Bessel functions. The general solution of the equation (3.7) has the following form:

$$
\Psi(z) = \sum_{B=I,K} d^n p \ C_B(p) \exp(ipz B_{i|p|/q_2} \left( \sqrt{-2\Lambda q_2^{-1}} \exp(q_2 z^0) \right),
$$

(4.2)

where functions $C_B (B = I, K)$ belong to an appropriate class.

Quantum wormholes represent a special class of solutions of the WDW equation with the following boundary conditions [8]:

(i) the wave function is exponentially damped for large spatial geometry,
(ii) the wave function is regular when the spatial geometry degenerates.

We restrict our consideration to real values of $p_i, (i = 1, \ldots, n)$. This corresponds to real geometries in Lorentzian region. In this case we have $\varepsilon \geq 0$.

If $\Lambda > 0$ the wave function (3.8) $\Psi = \exp(ipz) \, \Phi(z^0)$ with $\Phi(z^0)$ defined by (4.1) is not exponentially damped when the spatial volume $v \to \infty$, i.e. the condition (i) for quantum wormholes is not satisfied. It oscillates and may be interpreted as corresponding to the classical Lorentzian solutions.

For $\Lambda < 0$ the wave function (3.8) is exponentially damped for large $v$ only when $B = K$ in (4.1). But in this case the function $\Phi$ oscillates an infinite number of times when $v \to 0$. Thus, the condition (ii) is not satisfied. The wave function describes the transition between Lorentzian and Euclidean regions.

The function

$$
\Psi_p(z) = \exp(ipz) \, K_{i|p|/q_2} \left( \sqrt{-2\Lambda q_2^{-1}} \exp(q_2 z^0) \right)
$$

(4.3)

may be used for constructing quantum wormhole solutions. We consider the superposition of singular solutions

$$
\Psi_{\lambda,n}(z) = \frac{1}{\pi} \int_{-\infty}^{+\infty} dk \Psi_{q_2k\lambda}(z) \exp(-ik\lambda),
$$

(4.4)

where $\lambda \in R$, $n$ is a unit vector ($n^2 = 1$) and the quantum number $k$ is connected with the quantum number $\varepsilon = \frac{1}{2}|p|^2$ by the formula $2\varepsilon = q_2^2 k^2$. The calculation gives [13]

$$
\Psi_{\lambda,n}(z) = \exp \left( -\frac{\sqrt{-2\Lambda}}{q_2} \exp(q_2 z^0) \cosh(\lambda - q_2zn) \right).
$$

(4.5)

It is not difficult to verify that the formula (4.5) leads to solutions of the WDW equation (3.7), satisfying the quantum wormholes boundary conditions. Similar quantum wormholes for the model of subsection 3.1 were obtained in [5, 12].

These results are the straightforward generalization of the discussion in [19] – [21] to the multidimensional case. Therefore, the set of wave functions $\Psi_p$ and $\Psi_{\lambda,n}$ are spanning
the same space of physical states and are both bases of the Hilbert space of the model in the corresponding representation. The connection between these bases $\Psi_p$ and $\Psi_{\lambda,n}$ is given by the equation (4.4).

The function

$$\Psi_{m,n} = H_m(x^0)H_m(x^1) \exp \left\{ -\frac{1}{2} \left[ (x^0)^2 + (x^1)^2 \right] \right\},$$

where $H_m$ are the Hermite polynomials and

$$x^0 = (2/q_2)^{1/2}(-2\Lambda)^{1/4} \exp(q_2z^0/2) \cosh \left( \frac{1}{2}q_2zn \right), \quad (4.7)$$

$$x^1 = (2/q_1)^{1/2}(-2\Lambda)^{1/4} \exp(q_2z^0/2) \sinh \left( \frac{1}{2}q_2zn \right), \quad (4.8)$$

$m = 0, 1, \ldots$ are also solutions of the WDW equation with the quantum wormholes boundary conditions. Solutions of such type are called discrete spectrum quantum wormholes [5, 8, 12, 13], [19] – [21] and form a discrete basis for the Hilbert space of the system.

5 Third quantization

The WDW equations (3.2), (3.7) and (3.36) are similar to the scalar field equation in the curved space-time. These equations are gauge covariant (conformal covariant) [4]. It is not difficult to show [4, 22] that the minisuperspace metric $G$ (2.7) is conformally equivalent to the Milne metric and for a special gauge the WDW equations (3.2), (3.7) and (3.36) coincide with a field equation for a scalar field conformally coupled to a Milne space-time.

By analogy with the quantum field theory it might be worth-while to perform the second quantization of the universe wave function $\Psi$ expanding it on the creation and annihilation operators. The WDW equation itself is a result of the quantization of a geometry and matter. Thus, the procedure of the wave function $\Psi$ quantization is called the third quantization [9] . Similar to the quantum scalar field theory in the curved space-time we can expect that the vacuum state in a third quantized theory is unstable and creation of particles (in our case, universes) from the initial vacuum state takes place.

To perform the procedure of scalar field quantization on the time-dependent gravitational field background we should determine the vacuum state. This is quite a problem. Since there is no global timelike Killing vector and, hence, there is no global vacuum state, it is only possible to define different vacuum states which, in general, are not equivalent to each other and have different physical nature. By analogy with this, in the third quantization procedure we have a similar situation. Different Fock spaces constructed from the exact
solutions of the WDW equations are not equivalent to each other. It is natural to define the initial vacuum state with respect to the orthonormal set of mode solutions which are positive frequency modes in the limit of vanishing spatial volume: \( v \to 0 \). As a result, the birth of particles (universes) from "nothing" may have place where "nothing" is defined above initial vacuum state.

Let us consider now the model of subsection 3.2 with \( \Lambda > 0 \), which corresponds to a scalar field with the positive square of mass. It is not difficult to find two complete set of modes

\[
\Psi_p = \frac{1}{(2\pi)^{n/2}} \frac{\sqrt{\pi}}{2q^2 \sinh (\pi \sqrt{2\varepsilon}/q^2)} \frac{1}{2^{1/2}} e^{ipz} J_{-i\sqrt{2\varepsilon}/q^2} \left( \frac{\sqrt{2\Lambda}}{q^2} e^{i\varepsilon z^0} \right) \tag{5.1}
\]

and

\[
\hat{\Psi}_p = \frac{1}{(2\pi)^{n/2}} \frac{\sqrt{\pi}}{2q^2} \exp (\pi \sqrt{\varepsilon}/q^2) e^{ipz} H^{(2)}_{i\sqrt{2\varepsilon}/q^2} \left( \frac{\sqrt{2\Lambda}}{q^2} e^{i\varepsilon z^0} \right), \tag{5.2}
\]

which are orthonormal:

\[
\left( \Psi_p, \hat{\Psi}_{p'} \right) = -i \int_{z^0=\text{const}} \Psi_p \hat{\partial}_z \hat{\Psi}_{p'}^* dz = \delta(p - p'). \tag{5.3}
\]

The modes (5.1) are excited states above the Hartle-Hawking vacuum state [23] with \( \varepsilon = 0 \). As both sets, (5.1) and (5.2) are complete, they are related to each other by the Bogolubov transformation

\[
\Psi_p = \alpha_\varepsilon \hat{\Psi}_p + \beta_\varepsilon \hat{\Psi}_p^*, \tag{5.4}
\]

where the Bogolubov coefficients are

\[
\alpha_\varepsilon = \left[ \frac{\exp (\pi \sqrt{2\varepsilon}/q^2)}{2 \sinh (\pi \sqrt{2\varepsilon}/q^2)} \right]^{1/2} \tag{5.5}
\]

and

\[
\beta_\varepsilon = \left[ \frac{\exp (-\pi \sqrt{2\varepsilon}/q^2)}{2 \sinh (\pi \sqrt{2\varepsilon}/q^2)} \right]^{1/2}. \tag{5.6}
\]

The coefficients \( \beta_\varepsilon \) are not equal to zero. Thus, two Fock spaces constructed with the help of the modes \( \Psi_p \) and \( \hat{\Psi}_p \) are not equivalent and we have two different third quantized vacuum states (voids): \( |0> \) and \( |\hat{0}> \). The modes (5.1) have the asymptote

\[
\Psi_p \sim \exp \left[ i(pz - \sqrt{2\varepsilon} z^0) \right], \ v \to 0, \tag{5.7}
\]

which are positive frequency modes with respect to the conformal "time" \( z^0 \). Thus, the vacuum \( |0> \) which is defined with respect to these modes is connected to the minisuperspace conformal Killing vector \( \partial z^0 \). The modes (5.1) are no longer positive frequency ones
under $v \to \infty$. In this limit the modes (5.2) have the asymptote

$$\hat{\Psi}_p \sim \exp \left[ i(py - \sqrt{2\Lambda}y^0) \right], \quad v \to \infty,$$

(5.8)

where

$$y^0 = q_2^{-1} \exp(q_2 z^0); \quad y^i = z^i, \quad i = 1, \ldots, n.$$

(5.9)

The modes (5.2) in this limit are positive frequency ones with respect to the "time" $y^0$.

Since the vacuum states $|0\rangle$ and $|\hat{0}\rangle$ are not equivalent, the birth of the universes from "nothing" may have place, where "nothing" is the vacuum state $|0\rangle$. If $|0\rangle$ is the initial state when $v \to 0$, then an observer defined with respect to the vacuum state $|\hat{0}\rangle$ will detect in the limit $v \to \infty$

$$n_\varepsilon = |\beta_\varepsilon|^2 = \left[ \exp \left( 2\pi \sqrt{2\varepsilon / q_2} \right) - 1 \right]^{-1}$$

(5.10)

universes in mode $p$ (we remind that $2\varepsilon = p^2$). This is precisely Planck spectrum for radiation at temperature $T = q_2 / 2\pi$.

Now we consider the model with $\Lambda < 0$. It is not difficult to see that we can get the WDW equation (3.7) from the action

$$S = \frac{1}{2} \int d^{n+1}z \hat{H} \hat{\Psi},$$

(5.11)

which coincides with the action for a scalar field in the Minkowski space-time with the potential

$$V(\Psi) = \frac{M^2}{2} \Psi^2,$$

(5.12)

where

$$M^2(z) = 2\Lambda \exp(2q_2 z^0).$$

(5.13)

If $\Lambda < 0$, then $M^2 < 0$ and this model has an unstable vacuum state. The spectrum of energy is unbounded from below. The theory is well defined if we add the self-interaction term. Then

$$V(\Psi) = -\frac{\nu^2}{2} \Psi^2 + \frac{\lambda}{4} \Psi^4,$$

(5.14)

where we define

$$M^2 \equiv -\nu^2 = -2|\Lambda| \exp(2q_2 z^0).$$

(5.15)

The minimum of the potential (5.14) has place at

$$\Psi_0 = \pm \frac{\nu}{\sqrt{\lambda}} = \pm \sqrt{\frac{2|\Lambda|}{\lambda}} \exp(q_2 z^0).$$

(5.16)
It follows from this expression that symmetry breaking takes place dynamically, because

$$\Psi_0 \to 0, \text{ if } v \to 0. \quad (5.17)$$

The depth of wells at minima is:

$$V(\Psi = \Psi_0) = -\nu^4/(4\lambda) = -(\Lambda^2/\lambda) \exp(4q_2z^0). \quad (5.17)$$

The square of mass of the field $\Psi$ excitations after symmetry breaking becomes positive:

$$m^2(\Psi = \Psi_0) = \frac{d^2V}{d\Psi^2}{|}_{\Psi=\Psi_0} = 2\nu^2. \quad (5.18)$$

Let us consider now the field $\bar{\Psi} = \Psi - \Psi_0$ which describes oscillations near minima of the potential $V(\Psi)$. This field satisfies the equation

$$\left(-\frac{\partial}{\partial z^0} \frac{\partial}{\partial z^0} + \sum_{i=1}^{n} \frac{\partial}{\partial z^i} \frac{\partial}{\partial z^i} - 2\nu^2\right)\bar{\Psi} = j(z^0) + 3\lambda \Psi_0 \bar{\Psi}^2 + \lambda \bar{\Psi}^3, \quad (5.19)$$

where the source $j$ is

$$j(z^0) = \frac{\partial}{\partial z^0} \frac{\partial}{\partial z^0} \Psi_0 = \pm q_2^2 \sqrt{\frac{2|\Lambda|}{\lambda}} \exp(q_2z^0). \quad (5.20)$$

As it follows from the relation (5.18), the field $\bar{\Psi}$ has positive square of mass which depends on the "time" $z^0$. Thus, in linear approximation and without the source term the birth from "nothing" takes place as for the case $\Lambda > 0$. We should make in the formulas (5.1) and (5.2) the only replacement: $\Lambda \to 2|\Lambda|$. Presence of the source term in the equation (5.19) leads to an additional universes production. The source term has its origin in the dependence of the classical minimum $\Psi_0$ on "time" (see Eq. (5.20)).

Presence of the interaction terms $\sim \bar{\Psi}^2$ and $\bar{\Psi}^3$ in the relation (5.19) (respectively, $\sim \bar{\Psi}^3$ and $\bar{\Psi}^4$ in the potential $V(\bar{\Psi})$) gives the possibility to consider the processes with the topology alteration. For example, the cubic term in the potential is analogous to the interaction term which arises naturally in the string theory. This term describes the fission of the universe into two or the fusion of two universes into one.

It is important to note that the third quantization may have an influence on choice of the topology of models. If we demand the renormalizability of third quantized theory, then, by analogy with the scalar field theory with self-interaction, it follows that its dimension should be equal or less four [25]. In our case it means that we should take models with $n \leq 3$, i.e. in the models without scalar field $\varphi$ we can take at most four factor-spaces $M_i$ and in the presence of scalar field we can consider at most three factor-spaces.

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6 Inflation from "nothing"

Inflationary models are very popular now in cosmology because they explain why our universe is homogeneous, isotropic and almost spatially flat \[26\]. So, it might be worthwhile to get inflationary models in multidimensional cosmology also. However, contrary to usual 4-dimensional space-time cosmologies, in the multidimensional case we should solve two problems simultaneously. Namely, it is necessary to get inflation of our external space and compactification of internal dimensions near Planck length \(L_P \sim 10^{-33}\) cm to make them unobservable at present time.

Another interesting hypothesis consists in the proposal that inflationary universe arose by quantum tunneling from classically forbidden Euclidean region. This process is called the birth from "nothing" \[7\] as in the previous chapter 5, but its nature is quite different to former one.

In present chapter we investigate the multidimensional inflationary universe which arose from "nothing" by quantum tunneling process. It is clear that in the harmonic time gauge the solutions of the constraints (3.1), (3.6) and (3.26) and the equation of the form (3.27) are equivalent to each other, but inverting them we get quite different behavior of the scale factors \(a_i = \exp \beta^i\) in these models. Thus, the analysis of classical behavior of the universe for each model should be performed separately.

Here we investigate the model of the subsection 3.2 with the cosmological constant \(\Lambda > 0\). As it follows from the chapter 4, in this case the quantum tunneling takes place if \(\varepsilon < 0\). As we demand the reality of metric in the Lorentzian region, the condition \(\varepsilon < 0\) takes place for imaginary scalar field \(\varphi\) only \[3\].

In the harmonic time gauge the solution of the constraint (3.6) reads

\[ v = \exp (q_2 z^0) = \frac{\sqrt{|\varepsilon|/\Lambda}}{\cos (q_2 \sqrt{2|\varepsilon|} \tau)} \quad , \quad |\tau| \leq \frac{\pi/2}{q_2 \sqrt{2|\varepsilon|}} , \]

with the turning point at minimum of the spatial volume

\[ v_{min} = \sqrt{|\varepsilon|/\Lambda} \equiv v_t . \]

The analytic continuation \(\tau_L \to -i\tau_E\) gives the solution in the Euclidean region

\[ v = \frac{\sqrt{|\varepsilon|/\Lambda}}{\cosh (q_2 \sqrt{2|\varepsilon|} \tau)} \quad , \quad -\infty < \tau < +\infty \]

with the turning point at maximum: \(v_{max} = v_t\).
In synchronous system \((\gamma = 0)\) the scale factors read

\[ a_i = A_i \left[ \cosh \left( \frac{t}{T} \right) \right]^\sigma \left[ f \left( \frac{t}{2T} \right) \right]^{\sigma_i}, \quad i = 1, \ldots, n, \tag{6.4} \]

where \(t\) is the proper time, \(\sigma = 1/(D - 1)\) and \(T = [(D - 2)/2\Lambda(D - 1)]^{1/2}\). Here

\[ f(x) = \exp \left[ -2 \arctan e^{-2x} \right] \tag{6.5} \]

is smooth monotonically increasing function with the asymptotes: \(f(x) \to \exp (-\pi)\) as \(x \to -\infty\), \(f(x) \to 1\) as \(x \to +\infty\), and at zero: \(f(0) = \exp (-\pi/2)\). The parameters \(\sigma_i\) satisfy the relation

\[ \sum_{i=1}^{n} d_i \sigma^i = 0 \tag{6.6} \]

and

\[ \sum_{i=1}^{n} d_i \sigma_i^2 + \sigma_{n+1}^2 = \frac{D - 2}{D - 1}. \tag{6.7} \]

The spatial volume reads

\[ v = \left( \prod_{i=1}^{n} A_i^{d_i} \right) \cosh \frac{t}{T} \tag{6.8} \]

and has its minimum at \(t = 0\). It is not difficult to verify that

\[ \prod_{i=1}^{n} A_i^{d_i} = \sqrt{|\varepsilon|/\Lambda}. \tag{6.9} \]

The scale factors \(a_i\) have their minima at

\[ \frac{t^{(0)i}}{T} = \operatorname{arsinh} \frac{\sigma_i}{\sigma} = -\ln \left[ \frac{\sigma_i}{\sigma} + \sqrt{\left( \frac{\sigma_i}{\sigma} \right)^2 + 1} \right], \quad i = 1, \ldots, n, \tag{6.10} \]

from which it follows that \(\operatorname{sign} t_{(0)i} = -\operatorname{sign} \sigma_i\).

We suppose now that the universe arose tunneling from the Euclidean region and from the turning point \(t = 0\) (see Eq. (6.8)) its behavior can be described by classical equations. For simplicity, we consider the model with two factor spaces \((n = 2)\) where one of them (say \(M_1\)) is our external space. The generalization to the case \(n > 2\) is straightforward. We suppose also that after birth the external space \(M_1\) monotonically expands. Thus, it follows from the equations (6.6) and (6.10) that \(t_{(0)1} < 0 (\sigma_1 > 0)\) and \(t_{(0)2} > 0 (\sigma_2 < 0)\). Let all dimensions at the moment of the universe creation from ”nothing” have equal rights:

\[ a_1(t = 0) = a_2(t = 0) = 10^x L_{Pl}, \tag{6.11} \]

where \(2 \leq x \leq 3\). If we take \(x > 3\) then the probability of the birth becomes too small because of too large spatial volume. If \(x < 2\) then the scale factor \(a_2\) goes to \(L_{Pl}\) too fast.
and it is not sufficient time for inflation of the scale factor $a_1$. From the relations (6.4) and (6.11) we get
\[ A_i = 10^x \exp \frac{\pi}{2} \sigma_i, \quad i = 1, 2. \]

Using these relations we find for the scale factor $a_1$ that [27]
\[ a_1 \approx 10^x \exp \frac{\pi}{2} \sigma_1, \]
if
\[ 4 \lesssim t/T \ll D - 1. \]

To solve the flatness and horizon problems the scale factor $a_1$ during inflation should expand in $10^{30}$ times [28]. Thus,
\[ \frac{\pi}{2} \sigma_1 \gtrsim 70. \]

It gives the lower boundary for the parameter $\sigma_1$. If the size of $M_1$ to the end of inflation is approximately equal to the size of observable at the present time universe, i.e. $\sim 10^{28}$ cm, then
\[ \frac{\pi}{2} \sigma_1 \approx 140. \]

For the parameter $\sigma_2$ we get
\[ \sigma_2 \approx - \frac{d_1}{d_2} \frac{140}{\pi}. \]

It can be easily seen that within the limits $140/\pi \leq \sigma_1 \leq 280/\pi$ we have for the position of the minima of $a_2$
\[ t_{0(2)}/T \approx 6, \]
if $d_2 \gg d_1 = 3$. Here we consider the model where the space $M_2$ shrinks at the end of the inflation to its minimum size near Planck length, i.e.
\[ t_{0(2)} = t^* \]
and
\[ a_2(t_{0(2)}) \approx L_{Pl}. \]

Thus, for the scale factor $a_2$ we get
\[ a_2(t_{0(2)}) \approx L_{Pl} \approx 10^x \exp \left( \frac{\pi}{2} \sigma_2 \right) \lesssim 10^x \exp \left( - \frac{70d_1}{d_2} \right), \]
which gives an estimate
\[ \frac{70d_1}{d_2} \lesssim x \ln 10. \]
For example, if $d_1 = 3$, $\frac{3}{2}\sigma_1 = 70$ and $2 \leq x \leq 3$ we get respectively

$$45 \geq d_2 \geq 30.$$  \hspace{1cm} (6.23)

In general, we find that to ensure the inflation of the external space the dimension of the internal space should be $d > 40$ in accordance with the paper [29].

After inflation the external space $M_1$ should have a power-law expansion and the internal space $M_2$ should remain frozen near Planck scale. The transition to such stage can be performed if the cosmological constant $\Lambda$ goes very fast to zero. As a result we have the Kasner-like solution [30]:

$$a_i = a_{(0)}t^{\alpha_i}, \varphi = \ln t^{\alpha_{n+1}} + \text{const},$$

where $\sum_{i=1}^{n} d_i \alpha_i = 1$ and $\sum_{i=1}^{n} d_i \alpha_i^2 = 1 - \alpha_{n+1}^2$. In particular, the solution with the freezed internal spaces exists when $\alpha_i = 0$ ($i = 2, \ldots, n$). In this case for the external space we get $\alpha_1 = 1/d_1$.

Thus, the factor-space $M_1$ expands as a FRW universe filled with ultra-stiff matter (for $d_1 = 3$).

Now we consider the probability of the birth from ”nothing” for the inflationary universe. The amplitude of transition between the states with zero spatial volume $v = 0$ at the moment $\tau_i$ and some value of $v$ at $\tau_f$ is given by the path integral

$$\langle v, \tau_f \mid \emptyset, \tau_i \rangle = \int [dg][d\varphi]e^{iS_L},$$ \hspace{1cm} (6.24)

where $S_L$ is a Lorentzian action and the path integral is taken over all trajectories between points $v = 0$ and $v$. In our case the action $S_L$ is given by the relation (2.5) (for all Ricci-flat factor spaces and one-component free scalar field ($m = 1$)) and we consider the transition between ”points” with $v = 0$ and the classical turning point $v_t = \sqrt{|\epsilon|/\Lambda}$. To make the oscillating integral (6.24) convergent it is necessary to perform the Vick rotation to Euclidean time: $\tau_L \rightarrow -i\tau_E$. The probability between points $v = 0$ and $v_t$ is proportional to square of modulus of amplitude:

$$P \sim |\langle v_t, \tau_f \mid \emptyset, \tau_i \rangle|^2.$$ \hspace{1cm} (6.25)

In semiclassical limit

$$\langle v_t, \tau_f \mid \emptyset, \tau_i \rangle \sim e^{-S_E},$$ \hspace{1cm} (6.26)

where the Euclidean action for our model

$$S_E = \frac{1}{2\kappa^2} \int_{\tau_i}^{\tau_f} d\tau \left[ -(z^0)^2 + 2|\epsilon| + 2\Lambda e^{2qz^0} \right] - \frac{1}{2\kappa^2} \frac{\dot{v}}{v}|_{\tau_i} = \left. \frac{\dot{v}}{v} \right|_{\tau_i} = 2\frac{\Lambda}{\kappa^2} \int_{\tau_i}^{\tau_f} d\tau e^{2qz^0} - \frac{1}{2\kappa^2} \frac{\dot{v}}{v}|_{\tau_i}$$ \hspace{1cm} (6.27)

is calculated on classical solutions of the Euclidean field equations (instantons) interpolating between the vanishing geometry $v = 0$ and the turning point $v_t$. 

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Classical solutions of the Euclidean field equations in our model are given by the relation (6.3) which shows that \( \tau_i = -\infty \) (\( v = 0 \)) and \( \tau_f = 0 \) (\( v = v_t \)). Substituting this relation into (6.27) we get

\[
S_E = C \left[ 8\pi \prod_{i=2}^{n} a_{(c)i}^{d_i} \right]^{-1} \sqrt{|\varepsilon|},
\]

where

\[
C = \left( \frac{\sqrt{2}}{q_2} - \frac{q_2}{\sqrt{2}} \right)
\]

and we took into account that \( D \)-dimensional gravitation constant \( \kappa^2 \) is connected with the Newton constant \( G_N \) by the relation

\[
\kappa^2 = 8\pi G_N \prod_{i=2}^{n} a_{(c)i}^{d_i},
\]

where \( a_{(c)i} \) are the scale factors of freezed internal spaces (in the formula (6.28) we put \( G_N = 1 \)). The parameter \( C > 0 \) for \( D > 3 \) and the presence of the boundary term in the action (6.27) does not change the sign of \( S_E \). Let us estimate \( S_E \) for the two-component \((n = 2)\) inflationary model. It follows from the relations (6.6), (6.9) and (6.12) that

\[
v_t = \sqrt{|\varepsilon|/\Lambda} = A_1^{d_1} A_2^{d_2} = 10^{x(d_1+d_2)}.
\]

Then, with the help of the estimate (6.22) we get

\[
\sqrt{|\varepsilon|} = \Lambda^{1/2} 10^{x(d_1+d_2)} \gtrsim \Lambda^{1/2} 10^{d_1(x+70/\ln 10)}.
\]

Thus,

\[
S_E = C \left[ 8\pi a_{(c)2}^{d_2} \right]^{-1} \sqrt{|\varepsilon|} \gtrsim C \frac{\Lambda^{1/2} 10^{d_1(x+70/\ln 10)}}{8\pi},
\]

where \( a_{(c)2} \approx L_{Pl} \) (see Eq. (5.20)). The quantum birth of universe will be not suppressed if \( S_E \lesssim 1 \). It follows from the formula (6.33) that it takes place for \( \Lambda \lesssim 10^{-124} \text{cm}^{-2} \). This quantity is much less than possible value of the cosmological constant in the observable universe \( \Lambda_0 \lesssim 10^{-57} \text{cm}^{-2} \). Thus, for realistic theories \( S_E \gg 1 \) and, consequently, the quantum birth of this system is strongly suppressed. The reason of it consists in the large number of the internal dimensions, as it follows from the relation (6.31).

In some of papers (see, e.g. [28]) it was suggested that, instead of the standard Euclidean rotation \( \tau_L \to -i\tau_E \), the action (6.27) should be obtained by rotation in the opposite sense, \( \tau_L \to +i\tau_E \). However, we get in this case non-physical result that the probability of the birth of the universe is proportional to the volume of the arisen universe.
References

[1] J.A.Wheeler, Geometrodynamics (Academic, New York, 1962).
[2] B.C.De Witt, Phys.Rev. 160 (1967) 1113.
[3] V.D.Ivashchuk, V.N.Melnikov and A.I.Zhuk, Nuovo Cimento B140 (1989) 575.
[4] A.Zhuk, Class.Quant.Grav. 9 (1992) 2029.
[5] A.Zhuk, Phys.Rev. D45 (1992) 1192.
[6] U.Bleyer, V.D.Ivashchuk, V.N.Melnikov and A.Zhuk, Nucl.Phys. B429 (1994) 177.
[7] P.I.Fomin, Dokl.Akad.Nauk Ukr.SSR 9A (1975) 831;
   A.Vilenkin, Phys.Rev. D27 (1983) 2848, D50 (1994) 2581;
   V.A. Rubakov, Phys.Lett. B148 (1984) 280;
   A.Linde, Lett.Nuovo Cimento 39 (1984) 401;
   Ya.B.Zeldovich and A.A.Starobinsky, Sov.Astron.Lett. 10 (1984) 135;
   A.O.Barvinsky and A.Yu.Kamenshchik, Phys.Rev. D50 (1994) 5093.
[8] S.W.Hawking and D.N.Page, Phys.Rev. D42 (1990) 2665.
[9] V.A.Rubakov, Phys.Lett. B214 (1988) 503;
   S.Giddings and A.Strominger, Nucl.Phys. B321 (1989) 481;
   A.A.Kirillov, Pis’ma Zh.Eksp.Teor.Fiz. 55 (1992) 541.
[10] G.W.Gibbons and S.W.Hawking, Phys.Rev. D15 (1977) 2752.
[11] U.Bleyer and A.Zhuk, Gravitation and Cosmology 1 (1995) 37.
[12] A.Zhuk, Sov.J.Nucl.Phys. 55 (1992) 149.
[13] V.D.Ivashchuk and V.N.Melnikov, Teor.Mat.Fiz. 98 (1994) 312.
[14] U.Bleyer and A.Zhuk, Gravitation and Cosmology 1 (1995) 106.
[15] U.Bleyer, D.-E.Liebscher, H.-J.Schmidt and A.Zhuk, Wissenschaftliche Zeitschrift (Jena) 39 (1990) 22.
[16] V.D.Ivashchuk and V.N.Melnikov, Int.J.Mod.Phys. D3 (1994) N4;
   V.R.Gavrilo, V.D.Ivashchuk and V.N.Melnikov, Multidimensional Cosmology with multicomponent perfect fluid and Toda lattices, Preprint gr-qc/9407013.
[17] V.D.Ivashchuk and V.N.Melnikov, Billiard representation for multidimensional cosmology with multicomponent perfect fluid near the singularity, Preprint gr-qc/9407028.

[18] A.Zhuk, Integrable scalar field multidimensional cosmologies, (submitted to Class.Quant.Grav., 1995).

[19] L.M.Campbell and L.J.Garay, Phys.Lett. B254 (1991) 49.

[20] L.J.Garay, Phys.Rev. D48 (1993) 1710.

[21] G.A.Mena Marugan, Phys.Rev. D50 (1994) 3923.

[22] A.Zhuk, Sov.J.Nucl.Phys. 58 (1995) N11.

[23] Y.Peleg, Class.Quant.Grav. 8 (1991) 827.

[24] J.B.Hartle and S.W.Hawking, Phys.Rev. D28 (1983) 2960.

[25] J.Zinn-Justin, Quantum Field Theory and Critical Phenomena (Claredon Press, Oxford, 1989).

[26] A.D.Linde, Phys.Rev. D45 (1994) 748;
A.Linde, D.Linde and A.Mezhlumian, Phys.Rev. D49 (1994) 1783.

[27] A.Zhuk, Inflation from ”nothing” in multidimensional cosmology (to appear in Sov.J.Nucl.Phys., 59 (1996)).

[28] A.Linde, Rep.Prog.Phys. 47 (1984) 925.

[29] R.Abbot, S.Barr and S.Ellis, Phys.Rev. D30 (1984) 720.

[30] U.Bleyer and A.Zhuk, Kasner-like, inflationary and steady state solutions in multidimensional cosmology, Preprint of Astrophysical Institute of Potsdam, AIP 95-03.