Approximate Canonical Quantization for Cosmological Models

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Abstract

In cosmology minisuperspace models are described by nonlinear time-reparametrization invariant systems with a finite number of degrees of freedom. Often these models are not explicitly integrable and cannot be quantized exactly. Having this in mind, we present a scheme for the (approximate) quantization of perturbed, non-integrable, time-reparametrization invariant systems that uses (approximate) gauge invariant quantities. We apply the scheme to a couple of simple quantum cosmological models.

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1 Introduction

Historically, the first attempt in quantizing gravity moved from the canonical formalism. The origin of quantum gravity dates back to the pioneeristic works of Bergmann and Goldberg [1] and Dirac [2] who first investigated in the late 50’s the canonical analysis of classical general relativity and applied to the gravitational field the theory of quantization of constrained systems [3]. This line of research led to the Hamiltonian formulation of the Einstein theory.
by Arnowitt, Deser, and Misner \cite{ADM} and to the application of Dirac quantization to the ADM formalism by Wheeler \cite{Wheeler} and DeWitt \cite{DeWitt} (Wheeler-DeWitt equation).

In spite of an effort lasting more than 40 years, the conceptual problems arising in the quantum canonical formulation of gravity are hale and hearty and a large amount of time is presently devoted to the discussion of “structural issues in quantum gravity” \cite{4}. Let us just mention the so-called “problem of time” \cite{5, 6}: since the first-class Hamiltonian for general relativity is identically zero, due to general covariance, the evolution of the system is “hidden” in the constraints.

Minisuperspace models are the natural arena to investigate these structural issues. By retaining only a finite number of degrees of freedom quantum field theory reduces to quantum mechanics and typical problems due to the field nature of the system (for instance, anomalies – see Ref. \cite{10}) disappear. However, other conceptual problems, as the mentioned “problem of time”, survive to the reduction to a finite number of degrees of freedom \cite{11, 12}. The standard techniques of quantization of constrained gauge systems cannot be naively applied, for instance, even to the simple case of a Friedmann-Robertson-Walker (FRW) universe minimally coupled to a massive scalar field: for nonintegrable systems the absence of a global time parameter forbids the removal of the gauge degree of freedom and a sensible quantum theory cannot be obtained. So some approximation must be introduced – at some stage – in the quantization procedure. A possible approach is given by the semiclassical approximation. In this framework a (approximate) Schrödinger equation for the system can be derived \cite{13}. However, the definition of time in the Schrödinger equation – and the Schrödinger equation itself – depend on the particular wave function considered. As a consequence, it is not obvious how to construct a Hilbert space.

In this paper we present a method to obtain an approximate global time definition and approximate gauge invariant quantities for finite-dimensional, nonintegrable, perturbed systems, i.e. systems whose nonintegrable sector can be written as a small perturbation. We develop a scheme that allows the explicit construction of these quantities to any order in the perturbation parameter $\lambda$. Then the approximate system to the order $O(\lambda^n)$ can be quantized using the standard operator approach. Finally, we illustrate the method discussing a couple of quantum cosmological models.

2 Quantization of Integrable Finite-Dimensional Time-Reparametrization Invariant Systems

As a warming up exercise let us discuss finite-dimensional systems that are explicitly integrable. Due to time-reparametrization invariance their action reads

$$S = \int_{t_1}^{t_2} dt \left[ \dot{q}_\mu p^\mu - u(t) H(q, p) \right], \quad \tag{2.1}$$

2
where $\mu = 0, 1, \ldots, N - 1$ and $u(t)$ is a nondynamical variable that imposes the constraint $H = 0$. It is straightforward to see that the action (2.1) remains invariant under the time redefinition $t \rightarrow \tilde{t} = f(t)$ [14]. This property implies that the system is invariant under the (gauge) transformation generated by $H$

$$
\delta q_\mu = \epsilon \frac{\partial H}{\partial p_\mu} = \epsilon [q_\mu, H]_P,
$$

$$
\delta p_\mu = -\epsilon \frac{\partial H}{\partial q_\mu} = \epsilon [p_\mu, H]_P,
$$

$$
\delta u = \frac{d\epsilon}{dt},
$$

where $\epsilon(t_1) = \epsilon(t_2) = 0$. Note that the gauge transformation (2.2) has the same content of the equations of motion. Since the system is integrable Eqs. (2.2) can be explicitly solved.

The standard operator approaches to the quantization of the system (2.1) are the Dirac method and the reduced method. In both approaches the redundant gauge degree of freedom is eliminated via the imposition of an extra condition $F(q, p; t) = 0$ (gauge identity) [15]: in the Dirac method the constraint $H = 0$ is promoted to a quantum operator and then the gauge degree of freedom is eliminated by the gauge fixing; conversely, the reduced method leads first to a classical reduced gauge-fixed phase space where quantization can be carried out as usual (Schrödinger equation) and wave functions have the customary interpretation.

Since the gauge fixing condition is given in the form of a local relation, it may happen that the gauge fixing condition does not globally hold in the entire phase space (Gribov obstruction, see for instance Ref. [14]). So the quantization cannot be carried out. (Geometrically, the existence of a global time parameter implies that gauge conditions intersect the gauge orbits on the constraint surface once and only once.) However, when the system is integrable it is certainly possible to choose a global gauge fixing. Indeed, we can construct a canonical transformation $\{q_\mu, p_\mu\} \rightarrow \{Q_\mu, P_\mu\}$ where one of the new canonical coordinates, say $P_0$, coincides with the constraint, its conjugate $Q_0$ transforms linearly under the gauge transformations, and the remaining canonical coordinates are gauge invariant (Shanmugadasan canonical variables) [16]. The conjugate variable to the Hamiltonian, $Q_0$, defines a global time parameter and there is no Gribov obstruction. Let us see this in detail.

Consider Eq. (2.1) and neglect the constraint $H = 0$. Since the system is integrable we can write and solve the equation for the Hamilton characteristic function

$$
H \left( q_\mu, \frac{\partial W}{\partial q_\mu} \right) = \alpha_0, \ \ \rightarrow \ \ W \equiv W(q_\mu, \alpha_\mu).
$$

(2.3)

where $\alpha_\mu$ are $N$ constants (of motion). The Hamilton characteristic function generates a canonical transformation to cyclic coordinates $\{Q_\mu, P_\mu\}$

$$
Q_0 = \frac{\partial W}{\partial \alpha_0} = \tau + \beta_0,
$$

(2.4)
\[ Q_i = \frac{\partial W}{\partial \alpha_i} = \beta_i, \quad i = 1, 2, \ldots \]  
\[ P_\mu = \alpha_\mu, \]  
\[ \beta_\mu = \frac{\partial W}{\partial q_\mu}, \quad p_\mu = \frac{\partial W}{\partial q_\mu}. \]  
\[ \tau(t) = \int_{t_0}^{t} u(t') dt'. \]  
(2.8)

The quantities \{\(Q_\mu, P_\mu\}\} form a set of Shanmugadhasan variables of the system. Indeed, since \(\beta_\mu\) are constants of motion, \(Q_i\) and \(P_i\) are \(2(N-1)\) gauge-invariant quantities, \(P_0\) is the Hamiltonian and \(Q_0\) transforms linearly for the gauge transformation generated by \(H\). \(\alpha_\mu\) and \(\beta_\mu\) can be written as functions of \(q\) and \(p\) evaluating Eqs. (2.4-2.6) at \(\tau = 0\) and inverting the relations
\[ Q_\mu = \beta_\mu(q, p), \quad P_\mu = \alpha_\mu(q, p). \]  
(2.10)

The quantities \{\(Q_i, P_\mu\}\} form a complete set of gauge-invariant quantities (observables) that are in a one-to-one correspondence with the initial conditions. Note that the definition of Shanmugadhasan variables given by Eq. (2.10) is not unique. For instance, the generator of the canonical transformation is defined up to an additive generic function of the new momenta \(f(\alpha_\mu)\)
\[ P_\mu \rightarrow P'_\mu = P_\mu, \quad Q_\mu \rightarrow Q'_\mu = Q_\mu + \frac{\partial f}{\partial P_\mu}. \]  
(2.11)

Clearly, \{\(Q'_\mu, P'_\mu\}\} form a different set of Shanmugadhasan variables.

In the Shanmugadhasan representation (2.10) the action reads
\[ S = \int dt (\dot{Q}_0 P_0 + \dot{Q}_i P_i - u P_0). \]  
(2.12)

The quantity \(Q_0\) can be used to fix the gauge because its transformation properties under the gauge transformation imply that time defined by this variable covers once and only once the symplectic manifold, i.e. time defined by \(Q_0\) is a global time (see e.g. Ref. [17]). Using the Shanmugadhasan variables the quantization procedure becomes trivial and both Dirac and reduced approaches lead to the same Hilbert space.

It is well-known that classical canonical transformations and canonical quantization gen-
erally do not commute. Graphically (operators are marked with the hat symbol)

\[ H(q, p) \text{ Classical Theory} \quad \xrightarrow{\text{Quantum Theory}} \quad \hat{H}(\hat{q}, \hat{p}) \text{ Quantum Theory} \]

\[ H(Q, P) \text{ Classical Theory} \quad \xrightarrow{\text{Quantum Theory}} \quad \hat{H}(\hat{Q}, \hat{P}) \text{ Quantum Theory} \]

(2.13)

So the quantization of a classical system described by different – but classically equivalent – sets of canonical coordinates may lead to different quantum theories. The Shanmugadhasan variables are not immune from this disease: quantization in the \{Q, P\} representation may be inequivalent to quantization in the \{q, p\} representation. However, the Shanmugadhasan set of canonical variables seems to represent a preferred choice of canonical coordinates to be used in the quantization procedure. Indeed, using the Shanmugadhasan variables both Dirac and reduced approaches lead to the same Hilbert space. This property is not true in a generic representation and gives the Shanmugadhasan set of coordinates a preferred status.

In the Dirac approach we start setting the commutation relations

\[ [\hat{Q}_\mu, \hat{P}_\nu] = i\delta_{\mu\nu}. \quad (2.14) \]

The quantization of the system can be achieved imposing that the constraint becomes an operator on wave functions. This enforces the gauge invariance of the theory. Once one has imposed the quantum relation \( \hat{H}\Psi = 0 \), the gauge must be fixed. This can be done using the Faddeev-Popov procedure to define the inner product. We have

\[ (\Psi_2, \Psi_1) = \int d[Q_\mu] \Psi_2^*(Q_\mu)\delta(F)\Delta_{FP}\Psi_1(Q_\mu), \quad (2.15) \]

where \( d[Q_\mu] \) is defined in the unconstrained phase space and represents the measure invariant under the symmetry transformations of the system (rigid and gauge transformations). \( \Delta_{FP} \) is the Faddeev-Popov determinant (see e.g. Ref. [14]).

Using the measure \( d[Q_\mu] = \prod dQ_\mu \) (we neglect possible issues related to the self-adjointness nature of the operators) and the operator representation \( \hat{Q}_\mu \to Q_\mu, \hat{P}_\mu \to -i\partial_\mu \) the constraint equation reads \( -i\partial_0\Psi(Q_i) = 0 \). So the physical wave functions do not depend on \( Q_0 \) and the zeroth degree of freedom is pure gauge. A suitable gauge fixing condition is \( F \equiv Q_0 - k = 0 \), where \( k \) is a parameter. With this choice the Faddeev-Popov determinant is \( \Delta_{FP} = 1 \) and the inner product (2.15) reads

\[ (\Psi_2, \Psi_1) = \int \prod_i dQ_i \Psi_2^*(Q_j)\Psi_1(Q_j). \quad (2.16) \]
This completes the quantization of the system. The Hilbert space can also be obtained reducing first the system to the physical degrees of freedom by the gauge fixing and then applying the quantization algorithm. In this case we impose the gauge fixing condition \( F \equiv Q_0 - t = 0 \). This determines the Lagrange multiplier as \( u = 1 \) because of the gauge equations and of the definition of \( Q_0 \). Now the gauge fixing can be implemented by the (time dependent) canonical transformation

\[
Q'_0 = Q_0 - t, \quad P'_0 = P_0,
\]

imposing \( Q'_0 = 0 \) (gauge fixing) and \( P'_0 = 0 \) (constraint). The effective Hamiltonian on the gauge shell becomes \( H_{\text{eff}} = P_0 = 0 \) and the Schrödinger equation reads

\[
i \frac{\partial}{\partial t} \psi(t, Q_i) = 0.
\]

Wave functions do not depend on \( t = Q_0 \) and we recover the same Hilbert space that we have previously found following the Dirac approach.

We have seen that for integrable systems a maximal set of gauge invariant canonical variables can be constructed. Then the quantization is carried out using the new variables. Now let us discuss a concrete example. Consider the Hamiltonian constraint

\[
H = -\frac{1}{2}(p_0^2 + q_0^2) + \frac{1}{2}(p_1^2 + q_1^2) = 0.
\]

(Equation (2.19) describes, for instance, a closed FRW universe coupled to a homogeneous conformal scalar field. In this case \( q_0 \) is the scale factor of the metric \( a \) and \( \chi = q_1/a \) is the conformal scalar field \([11]\).)

The Hamilton characteristic function can be chosen (we consider without loss of generality \( p_0 \) and \( p_1 \) positive defined)

\[
W = \int dq_0 \sqrt{2(\alpha_1 - \alpha_0) - q_0^2} + \int dq_1 \sqrt{2\alpha_1 - q_1^2}.
\]

Using Eqs. (2.9) it is straightforward to obtain the Shanmugadhasan variables

\[
Q_0 = \arccos \frac{q_0}{\sqrt{p_0^2 + q_0^2}},
\]

\[
Q_1 = -\arccos \frac{q_0}{\sqrt{p_0^2 + q_0^2}} - \arccos \frac{q_1}{\sqrt{p_1^2 + q_1^2}},
\]

\[
P_0 = H(q_i, p_i),
\]

\[
P_1 = \frac{1}{2}(p_1^2 + q_1^2).
\]

The system can be quantized using \( Q_0 \) as time parameter. In the original variables the time parameter reads

\[
t = \arctan \frac{p_0}{q_0}.
\]
In this case the system is separable and time is defined only by one of the two degrees of freedom. So we can reduce the system to the Shanmugadhasan form only in the sector defined by \((q_0, p_0)\). In this representation the “hybrid” canonical variables are \((Q'_0, P'_0, q_1, p_1)\), where

\[
Q'_0 = \arccos \frac{q_0}{\sqrt{p_0^2 + q_0^2}}, \tag{2.26}
\]

\[
P'_0 = -\frac{1}{2}(p_0^2 + q_0^2). \tag{2.27}
\]

In the hybrid representation the Schrödinger equation reads

\[
i\frac{\partial}{\partial t} \psi(q_1; t) = \frac{1}{2} \left( -\frac{\partial^2}{\partial q_1^2} + q_1^2 \right) \psi(q_1; t), \tag{2.28}
\]

where we have used the usual representation for the operators \(\hat{q}_1\) and \(\hat{p}_1\). The eigenfunctions of the Hamiltonian are

\[
\psi_n(q_1) = \frac{1}{\sqrt{\pi^{1/2} 2^n n!}} H_n(q_1) e^{-q_1^2/2}, \tag{2.29}
\]

where \(H_n\) are the Hermite polynomial of order \(n\). The eigenfunctions \(2.29\) form a orthonormal basis in the Hilbert space.

### 3 Quantization of Perturbed Nonintegrable Systems

Any integrable time-reparametrization invariant system with a finite number of degrees of freedom can be exactly quantized in the Shanmugadhasan (or hybrid) form. The canonical coordinate conjugate to the Hamiltonian can be used to define the time parameter for the system and a Schrödinger equation can be written. However, the quantization procedure illustrated in the previous section relies deeply on integrability and on the existence of a global time. If the system under consideration does not admit a global time exact quantization is not possible. So an approximate method of quantization for these systems is worth exploring.

In this paper we are interested in systems that are described by a Hamiltonian constraint of the form

\[
H \equiv H_{\text{int}}(q_\mu, p_\mu) + \lambda H_{\text{pert}}(q_\mu, p_\mu), \tag{3.1}
\]

where \(\lambda\) is a small dimensionless parameter, \(H_{\text{int}}\) identifies the integrable sector of the theory, and \(H_{\text{pert}}\) represents the nonintegrable perturbation.

We have mentioned in the introduction the semiclassical approximation \([13]\). In this paper we follow a different approach. Our aim is to obtain approximate gauge invariant quantities for the system, i.e. approximate Shanmugadhasan variables up to a certain order \(n\) in the parameter \(\lambda\). The action is reduced to the form \((2.12)\) plus a term of the order
$O(\lambda^{n+1})$ and the (approximate) system can be quantized along the lines of the previous section. Let us see the procedure in detail.

Since the first term in the Hamiltonian is completely integrable, let us suppose without loss of generality that the Hamiltonian (3.1) is

$$H \equiv p_0 + \lambda H_1(q_\mu, p_\mu).$$

(Is it always possible to reduce Eq. (3.1) in the form (3.2) using the canonical transformation described in the previous section.) The reduction to the Shanmugadhasan form is obtained by implementing the canonical transformation generated by

$$W(q_\mu, P_\mu) = q_\mu P_\mu + \lambda W_1(q_\mu, P_\mu) + \lambda^2 W_2(q_\mu, P_\mu) + \ldots,$$

where $W_1(q_\mu, P_\mu), W_2(q_\mu, P_\mu), \ldots$ are determined by the request that the transformed Hamiltonian has the form

$$H = P_0 + O(\lambda^{n+1}).$$

Calculating $Q_\mu$ and $p_\mu$ from Eq. (3.3), expanding in powers of $\lambda$, and equating terms of the same order we find

$$\frac{\partial W_1}{\partial q_0} + H_1(q_\mu, p_\mu)|_{p_\mu = P_\mu} = 0,$$

$$\frac{\partial W_2}{\partial q_0} + \frac{\partial W_1}{\partial q_\mu} \frac{\partial H_1}{\partial p_\mu}|_{p_\mu = P_\mu} = 0,$$

$$\ldots$$

From Eqs. (3.5) it is straightforward to obtain the expressions for $W_n$. For instance $W_1$ reads

$$W_1(q_\mu, P_\mu) = -\int d q' \{ H_1(q'_0, q_i, P_\mu)|_{p_\mu = P_\mu} + Z(q_i, P_\mu),$n

where $Z$ is an arbitrary function that we set for simplicity equal to zero. (Note that all $W_n$ are determined up to additive arbitrary functions of the new momenta: $W_n \rightarrow W_n + Z_n(P_\mu)$ since Eqs. (3.3) involve only derivatives of $W_n$ w.r.t. $q$. ) The first-order canonical transformation is generated by

$$W(q_\mu, P_\mu) = q_\mu P_\mu - \lambda \int d q' H_1(q'_0, q_i, P_\mu)|_{p_\mu = P_\mu} + O(\lambda^2).$$

The second term reads

$$W_2(q_\mu, P_\mu) = \int d q' \left\{ H_1 \frac{\partial H_1}{\partial p_0} + \int d q'' \frac{\partial H_1}{\partial q_i} \frac{\partial H_1}{\partial p_i} \right\}|_{p_\mu = P_\mu},$$

and so on. Implementing the canonical transformation generated by Eq. (3.3) to the order $n$ one finds approximate gauge invariant variables to the order $O(\lambda^n)$. The Poisson brackets read

$$[Q_\mu, P_\nu]_P = \delta_{\mu\nu} + O(\lambda^{n+1}).$$
Finally, the canonical variable conjugate to $P_0$

$$Q_0 = q_0 + \lambda \frac{\partial W_1}{\partial P_0} + \lambda^2 \frac{\partial^2 W_2}{\partial P_0^2} + \ldots$$  \hspace{1cm} (3.10)

truncated to the order $O(\lambda^n)$ is the approximate time parameter of the system.

We do not expect in general that the series (3.10) are converging in the entire phase space because a nonintegrable system does not admit a global time (see e.g. Ref. [17]). Nevertheless $Q_0$ defined in Eq. (3.10) is a good time parameter in those regions of the phase space where the Poisson bracket of $Q_0$ with the Hamiltonian is positive defined, i.e. when the quantity $O(\lambda^{n+1})$ in Eq. (3.9) is less than 1. This happens usually when the perturbation $H_{\text{pert}}$ is small compared to the integrable part. Conversely, when $H_{\text{pert}} \gg 1$ nonperturbative effects are present and the series (3.10) will in general not converge. Clearly, for $\lambda = 0$ the condition is satisfied and the zero-order term of $Q_0$ in Eq. (3.10) is a global time, as expected because now the system is integrable.

At this point one can quantize the system using the (approximate) Shanmugadhasan variables along the lines of Sect. 4. The quantization procedure can be completed exactly and formally does not depend on the expansion scheme. Indeed, the expansion is chosen such that the new Hamiltonian is $P_0$ and the wave functions are diagonalized by suitable operators defined by the gauge invariant variables $Q_i$ and $P_i$. The approximation scheme affects only the definition of the new canonical quantities as functions of the original variables of the system. In conclusion, the procedure simply allows to calculate the finite (approximate) canonical transformation to a set of Shanmugadhasan variables. For nonintegrable, time-reparametrization invariant, perturbed systems time and gauge invariant quantities are approximate concepts defined only in suitable regions of the phase space.

In the next section we shall apply the method illustrated above to a simple minisuperspace model.

## 4 A Flat FRW Universe Coupled to a Scalar Field

Let us consider a flat FRW universe coupled to a scalar field $\varphi$. In Planck units ($l_{\text{Planck}} = \sqrt{4\pi G/3}$) the action density for this model reads

$$S = \int dt \left[ -\frac{1}{2} \frac{\dot{a}^2}{N} + \frac{1}{2} \frac{\dot{\varphi}^2 a^3}{N} - N a^3 V(\varphi) \right],$$  \hspace{1cm} (4.1)

where $a \geq 0$ is the scale factor of the FRW universe, $N$ is the lapse function, $V(\varphi)$ is the potential of the scalar field, and dots represent derivatives w.r.t. the time $t$. Equation (4.1) can be cast in the Hamiltonian form. Defining $u = N/a^3$ we have

$$S = \int dt \left\{ \dot{a} p_a + \dot{\varphi} p_\varphi - u \left[ \frac{1}{2} (p_\varphi^2 - a^2 p_a^2) + a^6 V(\varphi) \right] \right\},$$  \hspace{1cm} (4.2)
Clearly, the action (4.2) has the form (2.1); \( u \) is the nondynamical variable. When the potential is constant the system is separable and integrable; conversely, when the potential for the scalar field is not constant the system is not integrable. However, if the conditions

\[
a^6V(\varphi) \ll a^2p_\varphi^2 \approx p_\varphi^2
\]

are satisfied, the last term in Eq. (4.2) can be considered as a small perturbation and we can apply the techniques developed in the previous section. (For instance, a massive minimally coupled scalar field with potential \( V(\varphi) = m^2\varphi^2/2 \) can be described by a perturbed model when \( m \ll 1 \).) In this case Eq. (4.3) defines the region of the minisuperspace where the perturbative approximation is well-defined.

Using the canonical transformation

\[
\tilde{q}_0 = \ln a, \quad \tilde{p}_0 = ap_a,
\]

the Hamiltonian can be cast in a useful form. We have

\[
H = \frac{1}{2}(\tilde{p}_1^2 - \tilde{p}_0^2) + \lambda e^{6\tilde{q}_0} \tilde{V}(\tilde{q}_1),
\]

where \( \tilde{q}_1 = \varphi, \tilde{p}_1 = p_\varphi, \) and \( \tilde{V}(\tilde{q}_1) = V(\tilde{q}_1)/\lambda \). So the system in the “barred” variables is equivalent to a two-dimensional Klein-Gordon particle with a time-dependent potential. In these variables the conditions (4.3) read

\[
|\lambda|e^{6\tilde{q}_0} \tilde{V}(\tilde{q}_1) \ll \tilde{p}_0^2 \approx \tilde{p}_1^2.
\]

We shall first discuss the system in the case of constant potential. In this case the system is integrable and the exact reduction to the Shanmugadhasan form can be found. Then we shall move to the case of a potential of the form \( \tilde{V}(\tilde{q}_1) = (\tilde{q}_1^n)/n! \) and apply the perturbative techniques. We shall see that for \( n = 0 \) the perturbative result coincides with the exact one.

### 4.1 Constant Potential: \( \tilde{V}(\tilde{q}_1) = 1 \)

Let us reduce the system to the Shanmugadhasan form using the method of Sect. 2. The generator of the canonical transformation is

\[
W(\tilde{q}_\mu, P_\mu) = \tilde{q}_1\sqrt{P_1 + P_0} + \frac{1}{3}\sqrt{P_1 - P_0}\left(\sqrt{1 + \frac{2\lambda}{P_1 - P_0}e^{6\tilde{q}_0}} - \text{arctanh} \sqrt{1 + \frac{2\lambda}{P_1 - P_0}e^{6\tilde{q}_0}}\right).
\]

Using Eq. (4.7) we find

\[
Q_\mu = \frac{1}{2}\tilde{p}_1 \pm \frac{1}{2\tilde{p}_1}\text{arcsinh} \sqrt{\frac{\tilde{p}_0^2}{2\lambda}} e^{-6\tilde{q}_0} - 1, \quad (4.8)
\]

\[
P_\mu = \frac{1}{2}(\tilde{p}_1^2 + \tilde{p}_0^2) \pm \lambda e^{6\tilde{q}_0}, \quad (4.9)
\]
where different signs refer to “0” and “1” variables respectively. In the next section we shall investigate the case of non-constant potential in the limit of small $\lambda$ and compare the perturbative result to the constant potential case. So it is convenient to study the behavior of $Q_0$ in the limit of small $\lambda$. It is easy to see that the Shanmugadhasan variables (4.8) are ill defined for $\lambda \to 0$. This means that this set of variables cannot be used to describe the evolution of the system when the potential is close to zero. Since the Shanmugadhasan variables are not unique we can redefine them and eliminate the singularity at $\lambda = 0$.

Recalling that the generator of the canonical transformation to Shanmugadhasan variables is determined up to a generic function of the new momenta we define the new generator

$$W'(\bar{q}_\mu, P_\mu) = W(\bar{q}_\mu, P_\mu) - \int dP_0 \frac{1}{12 \sqrt{P_1 - P_0}} \ln \left[ \frac{2}{\lambda} (P_1 - P_0) \right].$$

With this definition the singularity at $\lambda = 0$ in the Shanmugadhasan variables is removed. $P_\mu$ are clearly unaffected. The conjugate variables read now

$$Q_\mu = \frac{1}{2} \frac{\bar{q}_1}{\bar{p}_1} \pm \frac{1}{6 \bar{p}_0} \sqrt{1 - \frac{2}{\bar{p}_0^2} e^{-\bar{q}_0}} \left[ -3 \bar{q}_0 + \ln \left( \frac{1 + \sqrt{1 - \frac{2}{\bar{p}_0^2} e^{6\bar{q}_0}}}{2 \sqrt{1 - \frac{2}{\bar{p}_0^2} e^{6\bar{q}_0}}} \right) \right].$$

In the limit of small $\lambda$ we have

$$Q_\mu = \frac{1}{2} \left( \frac{\bar{q}_1}{\bar{p}_1} \mp \frac{\bar{q}_0}{\bar{p}_0} \right) \mp \lambda \frac{2 \bar{p}_0^3}{\bar{p}_0^4} e^{6\bar{q}_0} \left( \bar{q}_0 - \frac{1}{6} \right) + O(\lambda^2).$$

The quantity $Q_0$ in Eqs. (4.11) can be used as time parameter for the system because $[Q_0, H]_P = 1$. The Poisson bracket of the first order approximation (4.12) with the Hamiltonian is

$$[Q_\mu, H]_P = \delta_{\mu 0} \mp 9 \lambda^2 e^{12\bar{q}_0} \frac{1}{\bar{p}_0^3}.$$  

The region of the phase space where the first order approximation for the time parameter holds is then defined by the relation

$$1 - 9 \lambda^2 e^{12\bar{q}_0} \frac{1}{\bar{p}_0^3} > 0, \quad \Rightarrow \quad |\lambda| e^{6\bar{q}_0} < \frac{\bar{p}_0^2}{3},$$

or, using the original variables,

$$3 |\lambda| a^4 < p_a^2.$$  

The region of validity of the first order approximation is consistent with Eqs. (4.3, 4.4) therefore the perturbation techniques can be successfully implemented. In the next subsection we discuss the perturbative technique in the case of a nonconstant potential $\bar{V}(\bar{q}_1) = \bar{q}_1^n / n!$ and recover the perturbative result of this section in the case $n = 0$. 

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4.2 Nonconstant Potential: $\tilde{V}(\bar{q}_1) = \bar{q}_1^n/n!$

Let us apply the formalism of Sect. 3 to the case $\tilde{V}(\bar{q}_1) = \bar{q}_1^n/n!$. Our starting point is the reduction of the Hamiltonian (4.13) to the form (3.2). We define the canonical transformation to the new variables $(q_\mu, p_\mu)$

$$\bar{q}_\mu = (q_1 \mp q_0)\sqrt{p_1 \mp p_0} ,$$
$$\bar{p}_\mu = \sqrt{p_1 \mp p_0} .$$

Using Eqs. (4.16,4.17) the Hamiltonian is cast into the form

$$H = p_0 + \lambda e^{6(q_1-q_0)\sqrt{p_1-p_0}}\tilde{V} \left[ (q_1 + q_0)\sqrt{p_1 + p_0} \right] .$$

The first order Hamilton characteristic function can be easily calculated from Eq. (3.7). The result is

$$W(q_\mu, p_\mu) = q_\mu p_\mu - \lambda \frac{n}{n!} \frac{\gamma^n}{(-6)^{n+1}} e^{12\beta q_1} \frac{\partial^n}{\partial \beta^n} \left( \frac{e^{-6\beta\xi}}{\beta} \right) + O(\lambda^2) ,$$

where $\beta = \sqrt{p_1 - p_0}$, $\gamma = \sqrt{p_1 + p_0}$, and $\xi = q_1 + q_0$. Starting from Eq. (4.19) the first order invariant quantities and the first order time parameter can be calculated. We have

$$Q_0 = q_0 + \frac{\lambda}{2(n-1)!} \frac{\gamma^{n-2}}{(-6)^n} e^{12\beta q_1} \left[ \left( \frac{12q_1 \gamma^2}{n} - 1 \right) \frac{\partial^n}{\partial \beta^n} + \frac{\gamma^2}{n^2} \frac{\partial^{n+1}}{\partial \beta^{n+1}} \right] \left( \frac{e^{-6\beta\xi}}{\beta} \right) + O(\lambda^2) ,$$

$$Q_1 = q_1 - \frac{\lambda}{2(n-1)!} \frac{\gamma^{n-2}}{(-6)^n} e^{12\beta q_1} \left[ \left( \frac{12q_1 \gamma^2}{n} + 1 \right) \frac{\partial^n}{\partial \beta^n} + \frac{\gamma^2}{n^2} \frac{\partial^{n+1}}{\partial \beta^{n+1}} \right] \left( \frac{e^{-6\beta\xi}}{\beta} \right) + O(\lambda^2) ,$$

$$P_0 = p_0 + \frac{\lambda}{n!} (\gamma\xi)^n e^{6\beta(q_1-q_0)} + O(\lambda^2) ,$$

$$P_1 = p_1 + \frac{\lambda}{n!} (\gamma\xi)^n e^{6\beta(q_1-q_0)} - \frac{2\lambda}{n!} \frac{\beta\gamma^n}{(-6)^n} e^{12\beta q_1} \frac{\partial^n}{\partial \beta^n} \left( \frac{e^{-6\beta\xi}}{\beta} \right) + O(\lambda^2) ,$$

where now $\beta$ and $\gamma$ are evaluated to the order $O(\lambda)$, i.e. $\beta = \sqrt{p_1 - p_0}$, $\gamma = \sqrt{p_1 + p_0}$. In the barred variables the above canonical quantities reads

$$Q_0 = \frac{1}{2} \left( \frac{\bar{q}_1}{\bar{p}_1} - \frac{\bar{q}_0}{\bar{p}_0} \right) + \frac{\lambda}{2(n-1)!} \frac{\gamma^{n-2}}{(-6)^n} e^{6\phi_0} \left( \frac{\bar{q}_1}{\bar{p}_1} + \frac{\bar{q}_0}{\bar{p}_0} \right) A_n^- + O(\lambda^2) ,$$

$$Q_1 = \frac{1}{2} \left( \frac{\bar{q}_1}{\bar{p}_1} + \frac{\bar{q}_0}{\bar{p}_0} \right) - \frac{\lambda}{2(n-1)!} \frac{\gamma^{n-2}}{(-6)^n} e^{6\phi_0} \left( \frac{\bar{q}_1}{\bar{p}_1} + \frac{\bar{q}_0}{\bar{p}_0} \right) A_n^+ + O(\lambda^2) ,$$

$$P_0 = \frac{1}{2} (\bar{p}_1^2 - \bar{p}_0^2) + \frac{\lambda}{n!} \bar{q}_1^2 e^{6\phi_0} + O(\lambda^2) ,$$

$$P_1 = \frac{1}{2} (\bar{p}_1^2 + \bar{p}_0^2) + \frac{\lambda}{n!} \bar{q}_1^2 e^{6\phi_0} - \frac{2\lambda}{n!} \frac{\beta\gamma^n}{(-6)^n} e^{12\beta q_1} \bar{p}_1 e^{6\phi_0} \left( \frac{\bar{q}_1}{\bar{p}_1} + \frac{\bar{q}_0}{\bar{p}_0} \right) B_n + O(\lambda^2) ,$$

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where
\[
A_n^{(\pm)} = \left\{ \left[ \frac{6\bar{p}_1^2}{np_0} \left( \frac{\bar{q}_1}{\bar{p}_1} + \frac{\bar{q}_0}{\bar{p}_0} \right) \pm 1 \right] \frac{\partial^n}{\partial \bar{p}_1^n} + \frac{\bar{p}_1^2}{np_0} \frac{\partial^{n+1}}{\partial \bar{p}_0^{n+1}} \right\} \left( \frac{e^{-6\bar{p}_0\bar{q}_1/\bar{p}_1}}{\bar{p}_0} \right), \tag{4.28} \right.
\]
\[
B_n = \frac{\partial^n}{\partial \bar{p}_0^n} \left( e^{-6\bar{p}_0\bar{q}_1/\bar{p}_1} \right). \tag{4.29} \]

After some algebra and using the Poisson brackets
\[
\left[ B_n, \bar{p}_0^2 \right]_P = 0, \tag{4.30} \]
\[
\left[ B_n, \bar{p}_1^2 \right]_P = -12 \left( \frac{-6\bar{q}_1}{\bar{p}_1} \right)^n e^{-6\bar{p}_0\bar{q}_1/\bar{p}_1}, \tag{4.31} \]
\[
\left[ B_n, \bar{q}_0 \right]_P = -\frac{1}{\bar{p}_0} B_{n+1}, \tag{4.32} \]
\[
\left[ B_n, \bar{q}_1 \right]_P = 0, \tag{4.33} \]

it is easy to verify that \([Q_\mu, P_\nu]_P = \delta_{\mu\nu} + O(\lambda^2)\). Hence, \(Q_0\) defined in Eq. (4.24) is the first order approximation of the global time parameter of the system and the quantities defined in Eqs. (4.25-4.27) form a complete set of approximate gauge invariant quantities. Setting \(n = 0\) it is straightforward to check that Eqs. (4.24-4.27) reduce to Eqs. (4.12) and Eqs. (4.9). Therefore the first-order expansions (4.24-4.27) approximate the exact expression for a constant potential. When the potential is not constant the result (4.24-4.27) is of particular relevance. In this case the system is not integrable and a global time and gauge invariant quantities cannot be found in an analytic form. However, finite expressions like (4.24-4.27) can be found at any order in the \(\lambda\) expansion.

The procedure illustrated above can be easily generalized to the case of a closed FRW universe. In this case the Hamiltonian is defined as in Eq. (4.2) with an extra term \(-a^4/2\). Using the canonical transformation
\[
\bar{q}_0 = \frac{1}{2} \ln \left( \frac{a}{p_a + \sqrt{p_a^2 + a^2}} \right), \tag{4.34} \]
\[
\bar{p}_0 = a \sqrt{p_a^2 + a^2}, \tag{4.35} \]

the kinetic part becomes equal to the kinetic part of Eq. (1.5) and the (perturbed) potential part reads
\[
H_1 = \left( \frac{2\bar{p}_0}{1 + e^{4\bar{q}_0}} \right)^3 e^{6\bar{q}_0} \tilde{V}(\bar{q}_1). \tag{4.36} \]

The discussion of this case proceeds along the lines of the flat FRW case, the only difference being the form of the perturbed Hamiltonian in the barred variables.
5 Conclusions

In this paper we have illustrated a procedure that allows the approximate quantization of finite-dimensional, time-reparametrization invariant, non integrable systems. The method is based on the construction of a set of Shanmugadhasan variables approximated to the order $O(\lambda^n)$ in the perturbation parameter $\lambda$ which identifies the nonintegrable sector of the theory. Using the new canonical variables the system becomes trivial and quantization can be carried out by standard techniques.

The quantization scheme illustrated above has some similarities with the method developed by Barvinsky [18] and Barvinsky and Krykhtin [19]. In Ref. [19] Barvinsky and Krykhtin prove the equivalence of BFV, Dirac, and reduced quantization at the one-loop (semiclassical) approximation by the equivalence of the physical inner product in the two approaches. In the present paper we have proved the equivalence of Dirac and reduced methods by the same observation (see below Eq. (2.18)). However, in our case the result is obtained by considering not a semiclassical expansion, but a perturbative expansion around the integrable sector of the theory. Further, the equivalence of Dirac and reduced approaches is proved to an arbitrary order in the perturbation parameter. A deeper discussion on the relation between our method and the approach illustrated in Ref. [19] is certainly worth being investigated.

What are advantages and disadvantages of the quantization scheme presented in the previous sections? We have seen that Dirac and reduced quantization coincide – to the order $O(\lambda^n)$ – when (approximate) Shanmugadhasan variables are used. Further, the observation that Shanmugadhasan variables represent a preferred choice of coordinates in the phase space may solve, at least partially, the ambiguity in the choice of gauge fixing and the problem of inequivalence of quantum theories obtained starting from different classical sets of canonical coordinates. In a sense, Shanmugadhasan variables play the role of the “Cartesian coordinates” in elementary quantum mechanics: quantization prescriptions must be given in the Shanmugadhasan representation. One of the main limitations of the method is that the relation between Shanmugadhasan and original canonical variables is generally nonlinear. As a consequence, it is very difficult, if not impossible, to write down quantum operators corresponding to the original fields. However, let us stress that one should be very careful in assigning a privileged meaning to the original fields. In general relativity no one can identify a priori which quantities determine the physical properties of the system: it is the nature of the system itself that selects “the physical” fields. In the canonical description of the Schwarzschild black hole, for instance, the relevant field is not the metric tensor but the mass of the black hole (see Ref. [20]), i.e a gauge invariant, nonlinear, function of the “original field” (the metric) that appears in the Einstein-Hilbert action.
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References

[1] P.G. Bergmann and I. Goldberg, *Phys. Rev.* **98**, 531 (1955).

[2] P.A.M. Dirac, *Proc. Royal Soc. of London* A**246**, 333 (1958).

[3] P.A.M. Dirac, “Lectures on Quantum Mechanics”, *Lectures Given at Yeshiva University* (Belfer Graduate School of Science, Yeshiva University, New York, 1964).

[4] R. Arnowitt, S. Deser, and C.W. Misner, “The Dynamics of General Relativity” in *Gravitation: An Introduction to Current Research*, ed. L. Witten (J. Wiley and Sons, New York, 1962).

[5] J.A. Wheeler, “Superspace and the Nature of Quantum Geometrodynamics”, in *Batelle Rencontres: 1967 Lectures in Mathematics and Physics*, ed. C. DeWitt and J.A. Wheeler (W.A. Benjamin, New York, 1968) pp. 242-307.

[6] B.S. DeWitt, *Phys. Rev.* **160**, 1113 (1967).

[7] C.J. Isham, “Structural Issues in Quantum Gravity”, Plenary session lecture given at the GR14 conference, Florence, August 1995; Report No. Imperial-TP-95-96-07, e-Print Archive: gr-qc/9510063.

[8] K.V. Kuchař, “Time and Interpretations of Quantum Gravity”, in *Proceedings of the 4th Canadian Conf. on General Relativity and Relativistic Astrophysics* (World Scientific, Singapore) pp. 211-314.

[9] C.J. Isham, “Canonical Quantum Gravity and the Problem of Time”, Lectures Presented at the Nato Advanced Study Institute *Recent problems in Mathematical Physics*, Salamanca, June 15-27, 1992; Report No. Imperial-TP-91-92-25, e-Print Archive: gr-qc/9210011.

[10] See for example E. Benedict, R. Jackiw and H.-J. Lee, *Phys. Rev.* D**54**, 6213 (1996) and references therein.
[11] M. Cavaglià, V. de Alfaro, and A.T. Filippov, “A Schrödinger Equation for Mini Universes”, *Int. J. Mod. Phys.* **A10**, 611 (1995).

[12] M. Cavaglià and V. de Alfaro, *Gen. Rel. Grav.* **29**, 773 (1997).

[13] A. Vilenkin, *Phys. Rev.* **D39**, 116 (1989).

[14] M. Henneaux and C. Teitelboim, “Quantization of Gauge Systems” (Princeton Univ. Press, New Jersey, 1992).

[15] There are alternative approaches to the quantization of constrained systems that do not use extra conditions to eliminate the gauge degrees of freedom. (We are very grateful to Carlo Rovelli for this remark.) We will not discuss these techniques here. Let us only stress that the system must be purged of the redundant gauge degrees of freedom, whatever approach is being used.

[16] S. Shanmugadhasan, *J. Math. Phys.* **14**, 677 (1973).

[17] P. Hájíček, *Phys. Rev.* **D34**, 1040 (1986).

[18] A.O. Barvinsky, *Class. Quantum Grav.* **10**, 1985 (1993).

[19] A.O. Barvinsky and V. Krykhtin, *Class. Quantum Grav.* **10**, 1957 (1993).

[20] M. Cavaglià, V. de Alfaro, and A.T. Filippov, *Phys. Lett.* **B424**, 265 (1998), an extended version can be found in the e-Print Archive: [hep-th/9802158](http://arxiv.org/abs/hep-th/9802158).