Moduli Kähler Potential for M-theory on a $G_2$ Manifold

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Abstract

We compute the moduli Kähler potential for M-theory on a compact manifold of $G_2$ holonomy in a large radius approximation. Our method relies on an explicit $G_2$ structure with small torsion, its periods and the calculation of the approximate volume of the manifold. As a verification of our result, some of the components of the Kähler metric are computed directly by integration over harmonic forms. We also discuss the modification of our result in the presence of co-dimension four singularities and derive the gauge-kinetic functions for the massless gauge fields that arise in this case.

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# 1 Introduction

Seven-dimensional spaces with holonomy $G_2$ provide the general setting for relating M-theory to four-dimensional theories with $N = 1$ supersymmetry. It has been known for some time [1] that 11-dimensional supergravity on smooth seven-dimensional manifolds reduces to a non-chiral theory in four dimensions. More specifically, for smooth manifolds of holonomy $G_2$ the four-dimensional spectrum consists of Abelian gauge multiplets, which descend from the three-form of 11-dimensional supergravity, and uncharged chiral multiplets that contain the metric moduli of the $G_2$ manifold and associated axions [2].

The situation changes if the $G_2$ space acquires singularities. Specifically, singularities of co-dimension four lead to non-Abelian gauge multiplets and co-dimension seven singularities to chiral matter (possibly charged under these gauge multiplets) [3]–[5]. These features make M-theory on singular $G_2$ spaces an interesting framework for M-theory “particle phenomenology” and have triggered much activity in the subject recently [6]–[18].

A more detailed analysis of the phenomenology of such models requires explicit knowledge of the four-dimensional effective theory. It is the main purpose of this paper to work out some of its features. Concretely, we will compute the four-dimensional moduli Kähler potential and the gauge kinetic functions obtained from M-theory on a $G_2$ manifold. This requires working with compact $G_2$ manifolds rather than with the non-compact examples [19, 20] that have been widely used in recent work. The moduli Kähler potential is obviously relevant to a number of problems in this context, for example to the study of supersymmetry breaking and the cosmological dynamics of moduli, to name only two.

For reasons of simplicity, we will initially consider a smooth $G_2$ manifold $X$ and later allow for co-dimension four singularities leading to non-Abelian gauge fields. In this paper, we will not attempt to include co-dimension seven singularities. For concreteness, we will focus on the specific compact $G_2$ manifold constructed by Joyce in Ref. [21]. However, our method applies to a large class of compact $G_2$ manifolds constructed in a similar fashion [22, 23].

For some classes of internal manifolds a general and explicit formula for the moduli Kähler potential (at least at tree level) can be given in terms of certain topological data. For example, in the case of the Kähler moduli space of Calabi-Yau three folds the moduli Kähler potential is determined by a cubic polynomial with coefficients given by the triple intersection numbers of the Calabi-Yau space in question [24, 25]. At the heart of this result is a quasi topological formula for the Hodge dual of two-forms on the Calabi-Yau space. Unfortunately, an analogous formula for three-forms on $G_2$ manifolds does not seem to exist. Therefore, while abstract formulae for the moduli Kähler metric in terms of harmonic three-forms on the $G_2$ manifold, and for the Kähler potential in terms of the volume of the $G_2$ manifold, are known [12], these expressions cannot be evaluated generically for all $G_2$ manifolds in the way they can for the Calabi-Yau Kähler moduli space. Our approach will, therefore, be to focus on the particular $G_2$ manifold of Ref. [21] and explicitly construct all the objects required. Concretely, on a specific $G_2$ manifold $X$, we will construct a family of $G_2$ structures $\varphi$ with
small torsion, following Ref. [21], and determine the associated family of “almost Ricci-flat” metrics $g$. By computing the periods of $\varphi$ and the volume as measured by the metrics $g$ we are able to compute the moduli Kähler potential in a controlled approximation.

Let us now summarise the main result of this paper. We have computed the four-dimensional moduli Kähler potential for M-theory on the compact $G_2$ manifold $X$ constructed in Ref. [21]. The main features of this manifold are as follows. The starting point of the construction is the seven-torus $T^7 = \mathbb{R}^7/\mathbb{Z}^7$ with coordinates $x^A$, where $A, B, \cdots = 1, \ldots, 7$. This torus is divided by three $\mathbb{Z}_2$ symmetries, generated by $\alpha, \beta$ and $\gamma$, whose precise action on the coordinates $x^A$ is given in Eqs. (3.1), (3.2) and (3.3). The resulting orbifold has 12 co-dimension four fixed points, four associated with each of the three $\mathbb{Z}_2$ symmetries. We will label these fixed points by a pair $(\tau, n)$, where $\tau = \alpha, \beta, \gamma$ indicates the type of the fixed point (that is, under which of the $\mathbb{Z}_2$ symmetries it remains inert) and $n = 1, 2, 3, 4$ labels the fixed points of the same type. The manifold $X$ is then obtained by blowing up each of these 12 points using Eguchi-Hanson spaces. We will refer to these regions of $X$ as “blow-ups” and to the surrounding torus-like region as “bulk”.

The metric moduli of this space can be organised into bulk moduli and moduli associated with the blow-ups. It turns out that the only bulk parameters that survive the orbifolding are the seven radii of the torus, producing seven corresponding moduli $a^A$. The precise relation between $a^A$ and the geometrical radii of the torus is given in Eq. (5.3). Further, each Eguchi-Hanson blow-up comes with three additional moduli, one being the radius of the blow-up, the other two describing the orientation of the Eguchi-Hanson space relative to the bulk. We denote these moduli by $a^{(\tau, n, a)}$, where $a, b, \cdots = 1, 2, 3$, and their relation to the underlying geometrical parameters is given in Eq. (5.4). In total, we therefore have $7 + 3 \cdot 12 = 43$ metric moduli. They pair up with 43 axions, which descend from the three-form of 11-dimensional supergravity, to form $N = 1$ chiral superfields. We denote these chiral superfields by $T^A$ and $U^{(\tau, n, a)}$, where Re$(T^A) = a^A$ and Re$(U^{(\tau, n, a)}) = a^{(\tau, n, a)}$. For the Kähler potential $K$ of these fields we find

$$K = - \sum_{A=1}^{7} \ln(T^A + \bar{T}^A) - 3 \ln \left[ 1 - \frac{8}{3} \sum_{\tau, n, a} \frac{(U^{(\tau, n, a)} + \bar{U}^{(\tau, n, a)})^2}{(T^A)^2 + T^A(T^{B^\tau} + T^{B^\bar{\tau}})} \right] + c \quad (1.1)$$

$$\simeq - \sum_{A=1}^{7} \ln(T^A + \bar{T}^A) + 8 \sum_{\tau, n, a} \frac{(U^{(\tau, n, a)} + \bar{U}^{(\tau, n, a)})^2}{(T^A)^2 + T^A(T^{B^\tau} + T^{B^\bar{\tau}})} + c \quad (1.2)$$

Here $A(\tau, a)$ and $B(\tau, a)$ are two specific indices of $1, \ldots, 7$ that determine the two bulk moduli by which the blow-up moduli $U^{(\tau, n, a)}$ associated with the fixed point $(\tau, n)$ are divided. The specific values of these index functions, which are directly linked to the structure of the orbifolding, are given in Table 1. The above Kähler potential constitutes an approximate results for two reasons. Firstly, as usual one has to require all moduli to be sufficiently large for the supergravity approximation to be valid. Secondly, the blow-up moduli $U^{(\tau, n, a)}$ have to be small compared to the bulk moduli $T^A$, so that the expected corrections of quartic and higher order in $U/T$ can be neglected. We have expanded the logarithm in the second line (1.2) to indicate that our result is accurate to leading, quadratic order in $U/T$. The constant $c$ is irrelevant for the moduli kinetic terms associated with $K$ but it does play a role in the presence of a non-trivial superpotential, such as that
based on flux proposed in [26, 6, 12]. The value of $c$ for our normalisation of the fields is $6 \ln(8\pi) + \ln(2)$ (see Eq. (5.11)). Note that one cannot consistently work with a universal bulk modulus while varying the blow-up moduli independently for different values of $(\tau, a)$. From Eq. (1.1), each pair $(\tau, a)$ is sensitive to two particular bulk moduli so that such a non-universal evolution of the blow-up moduli almost inevitably leads to anisotropy in the bulk evolution.

In addition to the moduli chiral superfields, the four-dimensional effective theory also contains Abelian vector multiplets. For our specific $G_2$ manifold we have 12 such multiplets, one for each blow-up. The gauge-kinetic functions for these multiplets depend on the type $\tau$ of the blow-up only and are given by

$$f_{(\tau)} = \begin{cases} 
T^7 & \text{for } \tau = \alpha \\
T^6 & \text{for } \tau = \beta \\
T^5 & \text{for } \tau = \gamma
\end{cases} \quad (1.3)$$

What happens if any of the blow-ups collapse to an orbifold singularity? One expects the moduli Kähler potential to still be of the form (1.1) but with all terms corresponding to singularities dropped. This amounts to blowing down all singularities $(\tau, a)$ via $U^{(\tau, n, a)} \to 0$. For each such singularity the gauge group enhances from $U(1)$ to $SU(2)$. The gauge kinetic functions $f$ of the associated $SU(2)$ gauge multiplets are still given by the same expressions (1.3) as in the Abelian case.

The plan of the paper is as follows. In the next section, we will review some necessary facts about $M$-theory on $G_2$ manifolds and outline our strategy to compute the Kähler potential (1.1). In Section 3, we construct the manifold $X$ and a basis of its third homology, following Ref. [21]. A family of $G_2$ structures $\varphi$ and the associated family of metrics $g$ on $X$ is presented in Section 4. Using these ingredients, in Section 5, we compute the periods of $\varphi$ and the volume of $X$ in a controlled approximation. Combining these results, we finally obtain the moduli Kähler potential. In Section 6, as a check for our result, we verify that some of the components of the associated Kähler metric can be reproduced by a direct calculation using some of the harmonic forms on $X$. Finally, in Section 7, we discuss gauge-kinetic functions and the effect of co-dimension four singularities.

To keep the main text more readable we have collected many of the technical details in three Appendices; on Eguchi-Hanson spaces, smoothed Eguchi-Hanson spaces and $G_2$ structures.

Let us summarise the main conventions we will use in this paper. We denote seven-dimensional coordinates by $x = (x^A)$ with associated indices $A, B, \cdots = 1, \ldots, 7$. We will frequently need to split these seven coordinates into a group of four coordinates, denoted by $\zeta = (\zeta^\mu)$, with indices $\mu, \nu, \cdots = 0, 1, 2, 3$ and a complementary group of three coordinates denoted by $\xi = (\xi^a)$ with indices $a, b, \cdots = 1, 2, 3$. We will use four-dimensional coordinates $z = (x, y, z, t) = (z^\mu)$ on the Eguchi-Hanson spaces. Tangent space indices are denoted by an underlined version of their curved counterparts.

As mentioned above, the various Eguchi-Hanson blow-ups are labelled by a pair $(\tau, n)$, where $\tau = \alpha, \beta, \gamma$
indicates the type and \( n = 1, 2, 3, 4 \) numbers the four blow-ups of each type. In dealing with the specific \( G_2 \) manifold we consider, we will find it practical to work in the “upstairs” picture for the orbifold, that is, we will consider the full torus \( T^7 \) rather than a fundamental domain. In this picture, there exist four equivalent copies of each blow-up. We will label these degenerate blow-ups by adding an additional index \( d = 1, 2, 3, 4 \), that is, we use a triple \((\tau, n, d)\) of indices. For notational simplicity, we will frequently abbreviate this triple by \((i) = (\tau, n, d)\). Generic moduli of the \( G_2 \) manifold will be indexed by \( I, J, \ldots \).

2 M-theory on \( G_2 \) manifolds

Let us start by considering a seven-dimensional real, compact manifold \( X \) with \( G_2 \) holonomy and coordinates \( x = (x^A) \). Such a manifold is equipped with a \( G_2 \) structure \( \varphi \), that is, a smooth three-form which is isomorphic to the “flat” \( G_2 \) structure \( \bar{\varphi} = dx^1 \wedge dx^2 \wedge dx^7 + dx^1 \wedge dx^3 \wedge dx^6 + dx^1 \wedge dx^4 \wedge dx^5 
+ dx^2 \wedge dx^3 \wedge dx^5 + dx^4 \wedge dx^2 \wedge dx^6 + dx^3 \wedge dx^4 \wedge dx^7 + dx^5 \wedge dx^6 \wedge dx^7 \). (2.1)

By way of this isomorphism, a \( G_2 \) structure induces a Riemannian metric \( g \) on \( X \) that can be explicitly computed from \( \varphi \) using Eqs. (C.5) and (C.6). We call \( g \) the metric associated with the \( G_2 \) structure \( \varphi \). The associated metric can be used to define the map \( \Theta \) on \( G_2 \) structures by \( \Theta(\varphi) = \ast \varphi \), where the Hodge-star is with respect to the associated metric of the argument \( \varphi \) of \( \Theta \). The additional \( \varphi \) dependence hidden in the Hodge star makes this map highly non-linear. On a manifold with holonomy \( G_2 \) there exists a torsion-free \( G_2 \) structure, that is, a \( G_2 \) structure satisfying \( d\varphi = d\Theta(\varphi) = 0 \), or, equivalently, for compact \( X \), a \( G_2 \) structure which is harmonic with respect to its associated metric. We will denote such a torsion-free \( G_2 \) structure \( \tilde{\varphi} \). Its associated metric \( \tilde{g} \) is a Ricci-flat metric on \( X \).

The Ricci-flat deformations of the metric \( \tilde{g} \) can be described by the torsion-free deformations of the \( G_2 \) structure \( \tilde{\varphi} \) and, hence, by the third cohomology \( H^3(X, \mathbb{R}) \). Consequently, the number of independent metric moduli is given by the third Betti number \( b_3(X) \). To define these moduli more explicitly, we first introduce an integral basis \( \{ C^I \} \) of three-cycles, where \( I, J, \cdots = 1, \ldots, b_3(X) \), and an associated dual basis \( \{ \Phi_I \} \) of harmonic three-forms. Duality implies that

\[
\int_{C^I} \Phi_J = \delta^I_J .
\] (2.2)

A torsion-free \( G_2 \) structure \( \tilde{\varphi} \) can then be expanded as

\[
\tilde{\varphi} = \sum_I a^I \Phi_I ,
\] (2.3)

where the expansion coefficients \( a^I \) represent the metric moduli in question. From Eq. (2.2), these moduli \( a^I \) can be computed in terms of certain underlying geometrical parameters (on which a family of \( G_2 \) structures \( \tilde{\varphi} \) depends) by performing the period integrals

\[
a^I = \int_{C^I} \tilde{\varphi} .
\] (2.4)
Let us also introduce an integral basis \( \{ D^I \} \) of two-cycles, where \( I, J, \cdots = 1, \ldots, \beta^2(X) \) and a dual basis \( \{ \omega_I \} \) of two-forms satisfying
\[
\int_{D^I} \omega_J = \delta^I_J. \tag{2.5}
\]
Then, the three-form field \( C \) of 11-dimensional supergravity can be expanded in terms of the basis \( \{ \Phi_I \} \) and \( \{ \omega_I \} \) as
\[
C = \nu^I \Phi_I + A^I \wedge \omega_I. \tag{2.6}
\]
The expansion coefficients \( \nu^I \) represent \( \beta^3(X) \) axionic fields in the four-dimensional effective theory, while the Abelian gauge fields \( A^I \), with field strengths \( F^I \), are part of \( \beta^2(X) \) Abelian vector multiplets. The \( \nu^I \) pair up with the metric moduli \( a^I \) to form the bosonic parts of \( \beta^3(X) \) four-dimensional chiral superfields
\[
T^I = a^I + i \nu^I. \tag{2.7}
\]
It is the Kähler potential for these fields \( T^I \) we wish to compute explicitly. A general formula for the associated Kähler metric \( K_{IJ} \) is given by [12]
\[
K_{IJ} = \frac{1}{4V} \int_X \Phi_I \wedge \star \Phi_J, \tag{2.8}
\]
where the Hodge star is defined with respect to the Ricci-flat metric \( \tilde{g} \) and \( V \) is the volume
\[
V = \int_X \sqrt{\det(\tilde{g})} \tag{2.9}
\]
of \( X \). This formula is most easily proven by reducing the kinetic term for \( C \) in the action of 11-dimensional supergravity by inserting the expansion (2.6). With some more effort, it can also be derived by reducing the 11-dimensional Einstein-Hilbert term [27].

Using general properties of \( G_2 \) manifolds, it was shown in Ref. [12] that the Kähler metric (2.8) descends from the Kähler potential
\[
K = -3 \ln \left( \frac{V}{2\pi^2} \right) \tag{2.10}
\]
with the volume \( V \) defined in (2.9). This means that the associated Kähler metric
\[
K_{IJ} = \frac{\partial^2 K}{\partial T^I \partial T^J} \tag{2.11}
\]
must coincide with the one obtained directly from Eq. (2.8). It can also be shown that the first derivatives of the Kähler potential
\[
K_I = \partial K/\partial T^I \tag{2.12}
\]
can be directly computed from the harmonic forms \( \Phi_I \) using
\[
K_I = \frac{1}{2V} \int_X \Phi_I \wedge \Theta(\tilde{\varphi}) . \tag{2.13}
\]
These facts will later provide us with a useful check of our explicit result for the Kähler potential.

It is clear from Eq. (2.10) that the Kähler potential (and the Kähler metric) only depends on the metric moduli \( a^I \) but not on the axions \( \nu^I \). In terms of superfields, this means that \( K \) is a function of the real parts \( T^I + \bar{T}^I \) only.
Reduction of the Chern-Simons term of 11-dimensional supergravity by inserting the gauge field part of (2.6) leads to the four-dimensional term \[ \int_{M_4} c_{IJK} \nu^I F^J \wedge F^K, \] (2.14)
where the coefficients \( c_{IJK} \) are given by
\[ c_{IJK} \propto \int_X \Phi_I \wedge \omega_J \wedge \omega_K. \] (2.15)
This implies that the gauge-kinetic function \( f_{JK} \), which couples \( F^J \) and \( F^K \), is of the form
\[ f_{JK} \propto \sum_I T^I c_{IJK}. \] (2.16)

Let us summarise our strategy to compute \( K \). First, we will explicitly construct a specific manifold \( X \), a basis \( \{ C^I \} \) of its third homology and a family of \( G_2 \) structures on \( X \) that depend on a number of geometrical parameters, such as radii. We denote these geometrical parameters collectively by \( R^I \). This part of the construction closely follows Ref. [21]. Using the relations, (C.5) and (C.6), between \( G_2 \) structures and metrics, we then compute the family of associated metrics. These metrics can be used, via Eqs. (2.9) and (2.10), to compute the volume and the Kähler potential as a function of \( R^I \). Next, we evaluate the periods to obtain the moduli \( a^I \) as a function of \( R^I \), which, in turn, allows us to rewrite the Kähler potential in terms of the proper superfields \( T^I \).

Finally, by computing the associated Kähler metric from Eq. (2.11) we obtain predictions for the integrals (2.8). As a check of our result, we will determine some of the harmonic forms \( \Phi_I \) on \( X \) and verify these predictions by computing some of the integrals (2.8) directly.

Ideally, one would like to carry out this program using a family of torsion-free \( G_2 \) structures \( \tilde{\varphi} \). However, such torsion-free structures are not explicitly known on compact \( G_2 \) manifolds. Rather, the construction of Ref. [21] involves explicitly writing down \( G_2 \) structures \( \varphi \) with small torsion and proving the existence of “nearby” torsion-free \( G_2 \) structures. We will perform our computation using these small torsion \( G_2 \) structures and show that this allows one to compute the Kähler potential in a controlled approximation.

3 Construction of the manifold

We will now review the construction of the manifold \( X \). The starting point is the seven-dimensional standard torus \( T^7 = \mathbb{R}^7/\mathbb{Z}^7 \) with coordinates \( x = (x^A) \) and the associated orbifold \( O = T^7/\mathbb{Z}_2^3 \). Here, the three \( \mathbb{Z}_2 \) symmetries are generated by \( \alpha, \beta \) and \( \gamma \) acting on the torus coordinates as
\[ \alpha((x^1, \ldots, x^7)) = (-x^1, -x^2, -x^3, -x^4, x^5, x^6, x^7) \] (3.1)
\[ \beta((x^1, \ldots, x^7)) = \left( -x^1, \frac{1}{2} - x^2, x^3, x^4, -x^5, -x^6, x^7 \right) \] (3.2)
\[ \gamma((x^1, \ldots, x^7)) = \left( \frac{1}{2} - x^1, x^2, \frac{1}{2} - x^3, x^4, -x^5, x^6, -x^7 \right). \] (3.3)
Let us discuss the fixed loci of this orbifold, which are of co-dimension four. Inspection of the above generators shows that each of the three $\mathbb{Z}_2$ symmetries leaves 16 three-tori, $T^3 = \mathbb{R}^3/\mathbb{Z}^3$, invariant. Some of these are mapped into each other by the remaining symmetries, leaving four inequivalent fixed tori for each $\mathbb{Z}_2$ and twelve in total. However, since we are going to use the “upstairs” picture, that is consider the full torus $T^7$ rather than a fundamental domain of the orbifold, we will effectively work with 48 fixed tori, which have to be identified in groups of four. These 48 three-tori are denoted by $T(i)$ where the label $(i)$ is split as $(i) = (\tau, n, d)$ whenever necessary. Here, the type $\tau = \alpha, \beta, \gamma$ indicates under which of the three $\mathbb{Z}_2$ symmetries the three-torus remains invariant, $n = 1, 2, 3, 4$ labels the four inequivalent tori for a given type and $d = 1, 2, 3, 4$ counts the four-fold degeneracy due to the upstairs picture.

To describe these fixed tori in more detail, it is useful to split, for each symmetry type $\tau = \alpha, \beta, \gamma$, the coordinates $x = (x^4)$ into a group $\zeta_\tau = (\zeta^\mu_\tau)$ of four coordinates which transforms non-trivially under the $\mathbb{Z}_2$ and a complementary group $\xi_\tau = (\xi^\mu_\tau)$ of three coordinates which transform trivially. For the three symmetries we have

- $\mathbb{Z}_2$ generated by $\alpha$

$$\zeta^\alpha = (x^1, x^2, x^3, x^4), \quad \xi^\alpha = (x^7, x^6, x^5) \quad (3.4)$$

$$T_{(\alpha, n, d)} = \left\{ \left( \zeta^\mu_{(\alpha)} \right) \in \left\{ \frac{1}{2} \right\} \right\} \quad (3.5)$$

- $\mathbb{Z}_2$ generated by $\beta$

$$\zeta^\beta = (x^5, x^6, x^1, x^2), \quad \xi^\beta = (x^7, x^4, x^3) \quad (3.6)$$

$$T_{(\beta, n, d)} = \left\{ \left( \zeta^0_{(\beta)}, \zeta^1_{(\beta)}, \zeta^2_{(\beta)} \right) \in \left\{ \frac{1}{2} \right\} \right\} \quad (3.7)$$

- $\mathbb{Z}_2$ generated by $\gamma$

$$\zeta^\gamma = (x^7, x^1, x^3, x^3), \quad \xi^\gamma = (x^2, x^4, x^6) \quad (3.8)$$

$$T_{(\gamma, n, d)} = \left\{ \left( \zeta^0_{(\gamma)}, \zeta^3_{(\gamma)} \right) \in \left\{ \frac{1}{2} \right\} \right\} \quad (3.9)$$

The next step in the construction of $X$ is to remove, for each fixed point $(i)$, a four-dimensional ball centred around this fixed point times the associated fixed three-torus $T(i)$. We will refer to the remaining parts of the torus $T^7$ as the bulk $B$. The holes in $B$ are then replaced by $F(i) = U \times T(i)$, where $U$ is the blow-up of $\mathbb{C}^2/\mathbb{Z}_2$ as discussed in Appendix A. Hence, the manifold $X$ consists of a bulk chart $B$ and charts $F(i)$ for each of the fixed points of the underlying orbifold. We denote the coordinates in $F(i)$ by $z(i) = (z^\mu_{(i)})$ and allow for a general linear transformation

$$\zeta_{(i)} = G_{(i)} z_{(i)} \quad (3.10)$$

where $G_{(i)} \in \text{Gl}(4)$, as a transition function in the four-dimensional part of the overlap between $B$ and $F(i)$.

Let us now present a basis of three-cycles. First of all, we have seven bulk three-cycles $C^A \subset B$ which correspond to the seven terms in the flat $G_2$ structure (2.1). They can be defined by setting four of the
coordinates $x^A$ to constants (chosen so there is no intersection with the blow-ups $F_{(i)}$) and adding on the seven images identified under $Z_2^7$. For concreteness, we define the cycle $C^A$ by setting the four coordinates on which the $A^{th}$ term in (2.1) does not depend to constants, that is, for example

$$C^1 = \{x^3, x^4, x^5, x^6 = \text{const}\} \cup \{\text{seven copies under } Z_2^7\}$$

(3.11)

and similarly for the other cycles. We move on to the three-cycles localised in the blow-up $F_{(i)} = U \times T_{(i)}$. Let us first denote the exceptional divisor (see Appendix A for a definition) in $U$ by $E_{(i)}$ and the one-cycle along the coordinate direction $\xi_i$ in $T_{(i)}$ by $L^a_i$. Then, for each fixed point $(\tau, n)$, we can define three-three-cycles

$$C^{(\tau,n,a)} = \bigcup_{d=1}^4 E_{(\tau,n,d)} \times L^a_{(\tau,n,d)} \subset \bigcup_{d=1}^4 F_{(\tau,n,d)}$$

(3.12)

The union over $d$ is a result of us working in the upstairs picture and it accounts for the images identified by the orbifolding. The collection $\{C^A, C^{(\tau,n,a)}\}$ of cycles then provides a basis for $H_3(X, Z)$. In total there are $7 + 12 \cdot 3 = 43$ cycles and we have $b^3(X) = 43$.

Two-cycles can, in general originate from the bulk and the blow-ups. Inspection of the orbifold action (3.1)–(3.3) shows that there are no $Z_2^7$ invariant forms $dx^A \wedge dx^B$ and, hence, no bulk two-cycles. Each blow-up comes with a single two-cycle given by the exceptional divisor. We, therefore, have 12 two-cycles which we denote by $\{D^i\} = \{D^{(\tau,n)}\}$. Explicitly, they are given by

$$D^{(\tau,n)} = \bigcup_{d=1}^4 E_{(\tau,n,d)}.$$  

(3.13)

As a result we have $b^2(X) = 12$.

4 $G_2$ structures and associated metrics

Let us now explain how to construct a family of $G_2$ structures $\varphi$ on the manifold $X$ of the previous section. In doing so, we need to include the full dependence on all 43 moduli.

We begin with the $G_2$ structures on the bulk $B$. It is easy to see that, for a constant metric on the torus $T^7$, only the diagonal components survive the orbifolding by (3.1), (3.2) and (3.3). These diagonal components are the seven radii $R^A$ of the torus. We easily obtain the bulk $G_2$ structure from the flat $G_2$ structure (2.1) by rescaling $x^A \to R^A x^A$, leading to

$$\varphi = R_1 R_2 R_7 dx^1 \wedge dx^2 \wedge dx^7 + R_1 R_3 R_6 dx^1 \wedge dx^3 \wedge dx^6 + R_1 R_4 R_5 dx^1 \wedge dx^4 \wedge dx^5$$

$$+ R_2 R_3 R_5 dx^2 \wedge dx^3 \wedge dx^5 + R_4 R_2 R_6 dx^4 \wedge dx^2 \wedge dx^6 + R_3 R_4 R_7 dx^3 \wedge dx^4 \wedge dx^7$$

$$+ R_5 R_6 R_7 dx^5 \wedge dx^6 \wedge dx^7.$$  

(4.1)

We will also sometimes find it convenient to work with the rescaled coordinates $\tilde{x}^A = R^A x^A$. Obviously, this part of the $G_2$ structure is torsion-free.

To deal with the blow-ups we split the seven radii $R^A$ into a group $R^a_{(\tau)}$ of four and a complementary group $R^a_{(\tau)}$ of three depending on the type $(\tau)$ of the blow-up and in analogy with the split (3.4), (3.6) and (3.8) of...
the coordinates. For a given blow-up \( F_{(i)} = U \times T_{(i)} \), we then introduce rescaled coordinates \( \bar{z}_{(i)}^\mu = R_{(\tau)}^\mu \zeta_{(i)}^\mu \) on \( U \) and \( \bar{\xi}_{(\tau)}^a = R_{(\tau)}^a \xi_{(\tau)}^a \) on \( T_{(i)} \). With these definitions, the \( G_2 \) structure on \( F_{(i)} \) can be written as

\[
\varphi = \sum_{a,b} w^a(z_{(i)}, \rho_{(i)}) \wedge d\bar{\xi}_{(\tau)}^b \mathcal{O}_{(i)ab} - d\bar{\xi}_{(\tau)}^1 \wedge d\bar{\xi}_{(\tau)}^2 \wedge d\bar{\xi}_{(\tau)}^3 \tag{4.2}
\]

where

\[
w_1(z, \rho) = \frac{\mathcal{F}(u, \rho)}{2} du \wedge \sigma_1 + \mathcal{F}(u, \rho) \sigma_2 \wedge \sigma_3 \tag{4.3}
\]

\[
w_2(z) = u du \wedge \sigma_2 + u^2 \sigma_3 \wedge \sigma_1 \tag{4.4}
\]

\[
w_3(z) = u du \wedge \sigma_3 + u^2 \sigma_1 \wedge \sigma_2 \tag{4.5}
\]

and \( u = |z| \). Further, the function \( \mathcal{F} \) has been computed to order \( \rho^6 \), in Appendix B (see Eq. (B.18)) and is given by

\[
\mathcal{F}(u, \rho) = u^2 + \frac{1}{2u^2} (\epsilon^2 - u \epsilon') \rho^4 + \mathcal{O}(\rho^8), \tag{4.6}
\]

Here \( \epsilon \) is an interpolating function with \( \epsilon(u) = 1 \) for \( u \leq u_0 \) and \( \epsilon(u) = 0 \) for \( u \geq u_1 \); the two fixed radii \( u_0, u_1 \) satisfying \( \rho \ll u_0 < u_1 \). The one-forms \( \sigma^a \) are the Maurer-Cartan forms on \( S^3 \simeq SU(2) \) defined in Appendix A. Finally, \( \mathcal{O}_{(i)} \in SO(3) \).

The technicalities associated with the \( G_2 \) structure (4.2) are explained in detail in the Appendices, particularly in Appendix C. Here, we will discuss its interpretation. The two-form \( w_1 \) is the Kähler form associated with a “smoothed” Eguchi-Hanson space on the blow-up \( U \) and interpolates, via the function \( \epsilon \), between Eguchi-Hanson space for \( u < u_0 \) and flat space for \( u > u_1 \). The parameter \( \rho_{(i)} \) can be interpreted as the radius of the blow-up \( (i) \). On the Eguchi-Hanson space \( (u < u_0) \), and flat space \( (u > u_1) \), the two-forms \( w_2 \) and \( w_3 \) constitute the two other Kähler forms that are expected on these hyperkähler spaces. Since the forms \( w^a \) are closed everywhere, the same is true for the \( G_2 \) structure (4.2). Due to the hyperkähler structures at \( u < u_0 \) and \( u > u_1 \) it is even torsion-free in these regions. It departs from non-zero torsion in the “collar” region \( u \in [u_0, u_1] \) where it interpolates between a non-trivial torsion-free \( G_2 \) structure at small radius and the flat \( G_2 \) structure at large radius. However, it can be shown that the torsion \( d\Theta(\varphi) \) in these collar regions is proportional to \( \rho_{(i)} \) and derivatives of \( \epsilon \). Hence, for sufficiently small blow-ups, \( \rho_{(i)} \ll 1 \), and a “smooth” interpolation, the deviation from a torsion-free \( G_2 \) structure is small. This fact will be used to show that the Kähler potential can be reliably computed, to leading order in \( \rho_{(i)} \), using the above \( G_2 \) structure.

Let us explicitly verify that the \( G_2 \) structure (4.2) indeed exactly matches the bulk \( G_2 \) structure (4.1) for \( u > u_1 \). We know from Appendix B that, in this region, the forms \( \bar{w}^a \) coincide with the three Kähler forms \( \bar{\omega}^a \) on flat space, defined in Eq. (A.40). Hence, for \( u > u_1 \) we have

\[
\varphi = \sum_{a,b} \bar{w}^a(z_{(i)}) \wedge d\bar{\xi}_{(\tau)}^b \mathcal{O}_{(i)ab} - d\bar{\xi}_{(\tau)}^1 \wedge d\bar{\xi}_{(\tau)}^2 \wedge d\bar{\xi}_{(\tau)}^3 \tag{4.7}
\]

Let us now relate the coordinates \( \bar{z}_{(i)} \) on \( F_{(i)} \) and the bulk coordinates \( \bar{\xi}_{(\tau)} \) by an \( SO(4) \) rotation \( \Lambda_{(i)} \), that is \( \bar{z}_{(i)} = \Lambda_{(i)} \bar{z}_{(i)} \), by choosing the transition function in Eq. (3.10) appropriately. Further, we require \( \Lambda_{(i)} \) to be such that the \( SO(3) \) rotation \( \mathcal{O}_{(i)} \) represents the right-handed vector representation of \( SO(4) \), that is
\( \mathcal{O}_i = O_+ (\Lambda_i) \). The left- and right-handed vector representations of SO(4) have been explicitly defined in Eq. (A.11). With this choice, \( \Lambda_i \) and \( \mathcal{O}_i \) together define an embedding of SO(4) into \( G_2 \) as explained in Appendix C. This means that both rotations drop out of \( \varphi \) and at \( u > u_1 \) we obtain

\[
\varphi = \sum_a w^a (\bar{\zeta}_\tau^a) \wedge \frac{d}{d\bar{\zeta}_\tau^a} - \frac{d}{d\bar{\zeta}_\tau^a} \wedge \frac{d}{d\bar{\zeta}_\tau^a} + d \bar{\xi}_\tau^a \wedge d \bar{\xi}_\tau^a \quad (4.8)
\]

Using Eq. (C.3), and after replacing the bared coordinates with their rescaled counterparts, this indeed matches the bulk \( G_2 \) structure (4.1) for the three coordinate identifications (3.4), (3.6), (3.8).

Let us summarise by listing all parameters on which the above \( G_2 \) structure depends. First, we have the seven radii \( R^A \) of the torus. Note that due to its dependence on the bared, rescaled coordinates the \( G_2 \) structure (4.2) on the blow-ups is also a function of these radii. In addition, for each fixed point \( (\tau, n) \), we have a radius \( \rho_{(\tau, n)} \), which measures the size of the blow-up, and a SO(3) rotation \( \mathcal{O}_{(\tau, n)} \). As we will see, the \( G_2 \) structure only depends on the normal vector \( n^a_{(\tau, n)} = \mathcal{O}_{(\tau, n)1a} \quad (4.9) \)
and, hence, only on two of the three angles of this rotation. This normal vector parametrises the orientation of the blow-up with respect to the bulk. The \( G_2 \) structure, therefore, depends on

\[
R^A, \quad \rho_{(\tau, n)}, \quad n^a_{(\tau, n)} \quad (4.10)
\]

where \( |n_{(\tau, n)}| = 1 \), which makes a total of 43 independent parameters, as expected.

We will now compute the family of metrics \( g \), or rather vielbeins, associated with the above \( G_2 \) structures using the relations (C.5) and (C.6).

For the bulk \( B \), the vielbein is computed from the bulk \( G_2 \) structure (4.1) and is, of course, given by

\[
e_A^A = R^A \delta_A^A \quad (4.11)
\]

Likewise, the vielbein on the blow-up \( F_{(i)} \) is obtained from the \( G_2 \) structure (4.2). The essence of this calculation has been described in the part of Appendix C leading up to the metric (C.15). Here, we need to slightly generalise this calculation to include the radii \( R^A \) and the rotation of the blow-up. To deal with this we introduce the matrices

\[
A_{(i)} = \text{diag}(R^a_{(\tau)}), \quad B_{(i)} = \text{diag}(R^a_{(\tau)}) \quad (4.12)
\]

The vielbein is then given by

\[
\begin{pmatrix}
e_A^a \\ e_A^a
\end{pmatrix} = D(u, \rho_{(i)}) P(\bar{z}_{(i)}) A_{(i)}, \quad \begin{pmatrix}
e_A^a \\ e_A^a
\end{pmatrix} = E(u, \rho_{(i)}) \mathcal{O}_i B_{(i)} \quad (4.13)
\]

where

\[
D(u, \rho_{(i)}) = G(u, \rho_{(i)})^{-1/6} \text{diag} \left( \frac{F(u, \rho_{(i)})}{2u}, \frac{\sqrt{F(u, \rho_{(i)}^1)}}{2u}, \frac{\sqrt{F(u, \rho_{(i)})}}{u} \right) \quad (4.14)
\]

\[
E(u, \rho_{(i)}) = \text{diag} \left( G(u, \rho_{(i)})^{1/3}, G(u, \rho_{(i)})^{-1/6}, G(u, \rho_{(i)})^{-1/6} \right) \quad (4.15)
\]
Here, $F$ is the function defined in Eq. (4.6), $G$ is given by
\[
G(u, \rho) \equiv \frac{F(u, \rho)F'(u, \rho)}{2u^3} = 1 + \frac{\rho^4}{4u^3} \left( \epsilon^2 - u \epsilon \epsilon' \right) + O(\rho^8)
\]
and $u = |\bar{z}|$. Note, that the matrix $P$, defined in Eq. (A.16) is an element of $SO(4)$.

From these results, one can easily obtain the measure $\sqrt{g}$ of the associated metric. One finds that
\[
\sqrt{g} = \prod_A R^A
\]
for the bulk $B$ and
\[
\sqrt{g} = G(u, \rho^{(i)})^{1/3} \prod_A R^A
\]
for the blow-up $F^{(i)}$.

5 Periods, volume and Kähler potential

An important ingredient in the construction of compact $G_2$ manifolds given in Ref. [21] is the proof that, for sufficiently small blow-up radii $\rho^{(i)}$, the $G_2$ structure $\varphi$ given in the previous section differs from its torsion-free counterpart $\tilde{\varphi}$ by an exact form $d\eta$, that is,
\[
\tilde{\varphi} = \varphi + d\eta.
\]
This implies that the periods (2.4) can, in fact, be computed exactly from
\[
a^I = \frac{1}{8} \int_{C^I} \varphi
\]
using the $G_2$ structure $\varphi$ with small torsion. Let us now carry this out starting with the bulk cycles $C^A$ defined in Eq. (3.11) and the associated periods, which we denote by $a^A$. Since these cycles are entirely contained within the bulk, we can use the bulk expression (4.1) for $\varphi$. One easily finds
\[
a^1 = R_1 R_2 R_7, \quad a^2 = R_1 R_3 R_6, \quad a^3 = R_1 R_4 R_5, \quad a^4 = R_2 R_3 R_5,
\]
\[
a^5 = R_2 R_4 R_6, \quad a^6 = R_3 R_4 R_7, \quad a^7 = R_5 R_6 R_7.
\]
Let us now move to the cycles related to the blow-ups. Recall, that for each of the 12 fixed points $(\tau, a)$ we have three three-cycles $C^{(\tau, n, a)}$, defined in Eq. (3.12), which are localised in the four equivalent patches $F^{(\tau, n, d)}$, where $d = 1, 2, 3, 4$. The associated periods are denoted by $a^{(\tau, n, a)}$. With the expression (4.2) for the $G_2$ structure on the blow-ups we find
\[
a^{(\tau, n, a)} = -\frac{\pi}{2} R^a_{(\tau)} R^2_{(\tau, n)} n^a_{(\tau, n)},
\]
where we have used the definitions (4.3)–(4.5) of the two-forms $w^a$ and the explicit representations of the Maurer-Cartan forms $\sigma^a$ in terms of angular coordinates given in Appendix A. Also note that the cycles $C^{(\tau, n, a)}$ are located at $u = 0$, that is, in a region where the four-dimensional part of the space is identical

\[\text{1Here, we include a factor of 1/8 relative to Eq. (2.4) to take care of the eight-fold over-counting in the upstairs picture.}\]
to an Eguchi-Hanson space. Hence, we can use the exact form (A.28) of the function $F$ associated with the Eguchi-Hanson metric. In particular, we have $F(u = 0, \rho) = \rho^2$ and $F'(u = 0, \rho) = 0$. Also, recall the definition (4.9) of the normal vector $n(\tau,n) = (n^a(\tau,n))$ that parametrises the orientation of the blow-up relative to the bulk.

Our next task is to compute the volume (2.9) of the manifold $X$ with respect to the Ricci-flat metric $\tilde{g}$ associated with the torsion-free $G_2$ structure $\tilde{\phi}$. Since we do not know either $\tilde{g}$ or $\tilde{\phi}$ explicitly this cannot be done exactly. However, as we will see, the volume can be computed in a controlled approximation using the $G_2$ structure $\phi$ with small torsion and its associated metric $g$ instead. From the definition of $V$, Eq. (2.9) and the relation (5.1) we have

$$V = \frac{1}{7} \int_X \tilde{\phi} \wedge \Theta(\tilde{\phi}) = \frac{1}{7} \int_X \phi \wedge \Theta(\tilde{\phi}),$$

(5.5)

where the second equality follows after partial integration. Further, one can expand the map $\Theta(\tilde{\phi}) = \Theta(\phi + d\eta)$ around $\phi$ to linear order in $d\eta$. This leads to [21]

$$\Theta(\phi + d\eta) = \Theta(\phi) + \frac{4}{3} \ast \pi_1(d\eta) + \ast \pi_7(d\eta) - \ast \pi_{27}(d\eta) + O((d\eta)^2)$$

(5.6)

where $\pi_1$, $\pi_7$ and $\pi_{27}$ denote the projections of three-forms onto their components transforming as one of the irreducible $G_2$ representations in $(7 \times 7 \times 7)_{\text{anti-symmetric}} = 1 + 7 + 27$. The fact [21] that both $d\Theta(\phi)$ and the $L_2$ norm of $d\eta$ are of order $\rho^8$ implies that the terms in Eq. (5.6) linear in $d\eta$ only contribute to the volume at order $\rho^8$. The same is obviously the case for quadratic and higher terms in (5.6). Hence, we learn that the volume computed with the $G_2$ structure $\phi$, or, equivalently, with its associated metric $g$ approximates the exact volume up to terms of order $\rho^8$, that is, we have

$$V = \frac{1}{8} \int_X \sqrt{g} + O(\rho^8) = \frac{1}{8} \sqrt{g} + O(\rho^8).$$

(5.7)

Applying this formula to the measure (4.17), (4.18) computed in the previous section we find

$$V = \frac{1}{8} \left[ \prod_A R_A^4 - \frac{2\pi^2}{3} \sum_{\tau,n} \rho^4(\tau,n) \prod_a R_a^{(\tau)} \right].$$

(5.8)

We remind the reader that $R_a^{(\tau)}$ denote three of the seven radii $R^A$, depending on the type $\tau$ of the blow-up $(\tau,n)$, in analogy with the definition of the coordinates $\xi(\tau)$ in Eqs. (3.4), (3.6) and (3.8).

We are now ready to compute the Kähler potential. Using the results (5.3) and (5.4) for the periods, we can rewrite the volume (5.8) in terms of $a^A$ and $a^{(\tau,n,a)}$, which constitute the real, bosonic parts of superfields. We denote these superfields by $T^A$ and $U^{(\tau,n,a)}$ such that

$$\text{Re}(T^A) = a^A, \quad \text{Re}(U^{(\tau,n,a)}) = a^{(\tau,n,a)}.$$

(5.9)

Again, we have included a factor of 1/8 to account for the over-counting in the upstairs picture.
Table 1: Values of the index functions $(A(\tau, a), B(\tau, a))$ specifying the bulk moduli $T^A$ by which the blow-up moduli $U^{(\tau,n,a)}$ are divided in the Kähler potential.

| $(\tau, a)$ | 1   | 2   | 3   |
|------------|-----|-----|-----|
| $\alpha$  | (1,6)| (2,5)| (3,4)|
| $\beta$   | (1,7)| (3,5)| (2,4)|
| $\gamma$  | (1,4)| (3,6)| (2,7)|

From Eq. (2.10) we then find for the Kähler potential

$$K = -\sum_{A=1}^{7} \ln(T^A + \bar{T}^A) - 3 \ln \left[ 1 - \frac{8}{3} \sum_{\tau,n,a} \frac{(U^{(\tau,n,a)} + \bar{U}^{(\tau,n,a)})^2}{(T^A(\tau,a) + \bar{T}^A(\tau,a))(T^B(\tau,a) + \bar{T}^B(\tau,a))} \right] + c,$$

(5.10)

where the constant $c$ is given by

$$c = 6 \ln(8\pi) + \ln(2).$$

(5.11)

The index functions $A(\tau, a), B(\tau, a) \in \{1, \ldots, 7\}$ indicate by which two of the seven bulk moduli $T^A$ the blow-up moduli $U^{(\tau,n,a)}$ are divided in the Kähler potential (5.10). Their values depend on the type $\tau$ and the orientation index $a$ but not on $n$. The nine possible values of these index functions are summarised in Table 1. With the blow-up moduli $U^{(\tau,n,a)}$ being of order $\rho^2_{(\tau,n)}$ (see Eq. (5.4)), and possible corrections to the volume of order $\rho^8_{(\tau,n)}$, we conclude that the leading corrections to the Kähler potential (5.10) arise at order $(U/T)^4$.

Hence, Eq. (5.10) represents a viable approximation to the Kähler potential if, firstly, all moduli are larger than one (in units where the Planck length is set to one) so that the supergravity approximation is valid and, secondly, if all blow-up moduli $U^{(\tau,n,a)}$ are small compared to $T^A$ so that corrections of order $(U/T)^4$ can be neglected.

6 Kähler metric and harmonic forms

In this section we shall verify our result for the Kähler potential (5.10) using Eqs. (2.8) and (2.13). These equations relate derivatives of the Kähler potential to certain integrals involving the harmonic forms $\{\Phi_I\}$. Specifically, we will focus on the first derivative

$$K_{(\tau,n,a)} = 16 \frac{U^{(\tau,n,a)} + \bar{U}^{(\tau,n,a)}}{(T^A(\tau,a) + \bar{T}^A(\tau,a))(T^B(\tau,a) + \bar{T}^B(\tau,a))}$$

(6.1)

and the component

$$K_{(\tau,n,a)(\tau,n,a)} = \frac{16}{(T^A(\tau,a) + \bar{T}^A(\tau,a))(T^B(\tau,a) + \bar{T}^B(\tau,a))}$$

(6.2)

of the Kähler metric, here produced by differentiating the Kähler potential (5.10). We would now like to construct some of the harmonic forms, compute the integrals in Eqs. (2.8) and (2.13) and verify that the results indeed coincide with the derivatives of the Kähler potential given above.
Recall, that the integral basis of three-cycles we have used consists of \( \{ C' \} = \{ C^A, C^{(\tau, n, a)} \} \), where \( C^A \) are the seven bulk cycles and \( C^{(\tau, n, a)} \) represent three three-cycles for each of the 12 blow-ups \( (\tau, n) \). We denote the dual basis of harmonic three-forms, satisfying Eq. (2.2), by \( \{ \Phi_I \} = \{ \Phi_A, \Phi^{(\tau, n, a)} \} \). Of course, we will not be able to determine these harmonic forms exactly because we do not know the exact Ricci-flat metric on \( X \).

However, we will find suitable approximations for the localised forms \( \Phi^{(\tau, n, a)} \) that allow us to compute some of the relevant integrals to the accuracy required.

From Appendix (B) we know that smoothed Eguchi-Hanson spaces have an anti-selfdual two-form

\[
\nu(\bar{z}(i), \rho(i)) = \frac{2}{F^2} du \wedge \sigma_1 + \frac{1}{F} \sigma_2 \wedge \sigma_3 ,
\]

with the function \( F \) as in Eq. (4.6). For \( u < u_0 \), the smoothing function \( \epsilon \) is one and this two-form is identical to the known localised harmonic two-form on the Eguchi-Hanson space [32]. In this range the three-forms

\[
\Phi^{(\tau, n, a)} \simeq -\frac{2\rho^2}{\pi} \nu(\bar{z}(\tau, n, d), \rho(i)) \wedge d\xi^a
\]

on \( F(\tau, n, d) \) are harmonic with respect to the metric \( g \) associated to the small-torsion \( G_2 \) structure \( \varphi \). The prefactor is chosen so that

\[
\frac{1}{8} \int_{C^{(\tau, n, a)}} \Phi^{(\tau, n, a)} = 1 ,
\]

as required for the dual basis. The forms (6.4) fall off as \( 1/u^4 \) for large \( u \) but do not vanish exactly in the bulk. Consequently, the above form is not valid in the bulk as this would lead to non-vanishing integrals over the bulk cycles, contradicting Eq. (2.2). We do not know how to smoothly interpolate between the above expressions for the harmonic forms on the blow-ups and bulk expressions that lead to vanishing integrals over the bulk cycles. However, for some of the integrals we need to perform, the behaviour at large \( u \) will be irrelevant and it is for these that (6.4) is useful. On the other hand, we can find three-forms

\[
\varphi^{(\tau, n, a)} = \frac{2}{\pi} \tilde{\nu}(\bar{z}(\tau, n, d)) \wedge d\xi^a
\]

on \( F(\tau, n, d) \), where

\[
\tilde{\nu}(\bar{z}) = \frac{\nu'}{2} du \wedge \sigma_1 + \epsilon \sigma_2 \wedge \sigma_3 ,
\]

that vanish in the bulk and are closed but not harmonic. They are correctly normalised, that is

\[
\frac{1}{8} \int_{C^{(\tau, n, a)}} \varphi^{(\tau, n, a)} = 1
\]

and indeed satisfy

\[
\int_{C^A} \varphi^{(\tau, n, a)} = 0 ,
\]

as they vanish identically in the bulk. As a result, they are non-harmonic representatives of the cohomology classes specified by \( \Phi^{(\tau, n, a)} \).

Let us now compute some of the integrals in Eq. (2.8) and (2.13). We start with the one we expect to reproduce the component (6.2) of the Kähler metric. Using

\[
* \Phi^{(\tau, n, a)} \simeq \frac{\rho^2}{\pi A(\tau, a) B(\tau, a)} \epsilon_{abc} \nu(\bar{z}(\tau, n, d)) \wedge d\xi^b \wedge d\xi^c
\]

(6.10)
on $F_{(\tau,n,d)}$ for $u < u_0$ we obtain

$$\frac{1}{4V} \int_X \Phi_{(\tau,n,a)} \wedge \ast \Phi_{(\tau,n,a)} = \frac{16}{(T^A(\tau,a) + T^A(\tau,a))(T^B(\tau,a) + T^B(\tau,a))}. \quad (6.11)$$

This indeed matches the component (6.2) of the Kähler metric that we have obtained from the Kähler potential (5.10) exactly. Note that the main contribution to the above integral comes from small values of $u$. The contribution at large $u = u_0$ behaves like $\rho^{4\alpha}/u_0^4$ and so can be neglected to the order in $\rho$ we are working.

The range of small $u$ values is precisely the one where we can trust the expression for the three-form (6.4), which is why its use in the present context is justified.

Next, we would like to consider the integrals

$$\frac{1}{2V} \int_X \Phi_{(\tau,n,a)} \wedge \ast \varphi, \quad (6.12)$$

which, from Eq. (2.13), should reproduce the first derivatives $K_{(\tau,n,a)}$ of the Kähler potential. Since the $G_2$ structure $\varphi$ is non-vanishing in the bulk we cannot use the expression (6.4) for $\Phi_{(\tau,n,a)}$, which is valid for small $u$ only. However, since $d \ast \varphi \simeq 0$ to a good approximation, we can evaluate the above integral with the non-harmonic representatives $\varphi_{(\tau,n,a)}$, Eq. (6.6), which are exactly zero in the bulk. Using the expression (4.2) for $\varphi$, some of the properties of the Maurer-Cartan forms $\sigma^\alpha$ listed in Appendix A and the results (5.3) and (5.4) for the periods, one finds

$$\frac{1}{2V} \int_X \varphi_{(\tau,n,a)} \wedge \ast \varphi = 16 \cdot \frac{U_{(\tau,n,a)} + \bar{U}_{(\tau,n,a)}}{(T^A(\tau,a) + T^A(\tau,a))(T^B(\tau,a) + T^B(\tau,a))}, \quad (6.13)$$

which exactly reproduces the first derivative (6.1) of the Kähler potential. Hence, we have checked our main result (5.10) for the Kähler potential by reproducing some of its derivatives directly from integrals over harmonic forms.

### 7 Gauge-kinetic functions and singularities

The manifold we are considering has $b^2(X) = 12$ and, hence, there exist 12 Abelian gauge multiplets $A^{(\tau,n)}$, one for each blow-up $(\tau,n)$. We would now like to compute the gauge-kinetic functions (2.16) for these vector multiplets. First we need explicit expressions for the basis $\{\omega_{(\tau,n)}\}$ of two-forms dual to the basis of two-cycles $\{D^{(\tau,n)}\}$ defined in Eq. (3.13). Since the integrals (2.15) are topological we can use any representatives for these dual two-forms, including non-harmonic ones. A convenient choice is provided by

$$\omega_{(\tau,n)} = \frac{2}{\pi} \bar{\nu}(\bar{\varphi}_{(\tau,n,d)}), \quad (7.1)$$

on $F_{(\tau,n,d)}$, where $\bar{\nu}$ is defined in Eq. (6.7). It is immediately clear from this form that the gauge kinetic function $f_{(\tau,n)(\tau',n')}^\prime$ coupling the field strengths $F^{(\tau,n)}$ and $F^{(\tau',n')}$ is non-zero only for gauge fields of the same type, that is when $\tau = \tau'$ and $n = n'$. Moreover, their value only depends on the type $\tau$ but not on $n$. 

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15
so that we have three types of gauge couplings \( f(\tau) \), where \( \tau = \alpha, \beta, \gamma \). A short calculation using Eqs. (2.15) and (2.16) shows that

\[
    f(\tau) = \begin{cases} 
        T^7 & \text{for } \tau = \alpha \\
        T^6 & \text{for } \tau = \beta \\
        T^5 & \text{for } \tau = \gamma 
    \end{cases} \tag{7.2}
\]

Note that \( c_{IJK} \) is only non-zero when the modulus \( T^A \) corresponds to a bulk harmonic form \( \Phi_A \) living on the three torus parallel to the blow-up under consideration. That is, \( A = 7 \) for \( \tau = \alpha \); \( A = 6 \) for \( \tau = \beta \) and \( A = 5 \) for \( \tau = \gamma \), which is precisely the pattern emerging in (7.2).

We now move on to discuss the four-dimensional effective theory when some of the blow-ups in our \( G_2 \) manifold \( X \) collapse back to an orbifold singularity.

Generally, a co-dimensional four singularity that is locally of the form \( \mathbb{C}^2 / \mathbb{Z}_n \times Y \) with a three-cycle \( Y \subset X \), that is, a singularity of type \( A_{n-1} \), leads to additional massless gauge fields with gauge group \( \text{SU}(n) \). These gauge fields are localised on the seven-dimensional manifold \( Y \times M_4 \), where \( M_4 \) is four-dimensional space-time. They can be incorporated by adding to 11-dimensional supergravity on \( X \times M_4 \) the action

\[
    S_{\text{YM}} = -\frac{1}{4\lambda^2} \int_{Y \times M_4} \left[ d^7x \sqrt{-g^7} \text{tr} F^2 + C \wedge \text{tr}(F \wedge F) \right] \tag{7.3}
\]

for the \( \text{SU}(n) \) gauge field \( A \) with associated field strength \( F \). Here \( g_7 \) is the induced metric on \( Y \times M_4 \). In Ref. [17], the gauge coupling \( \lambda \) has been determined as

\[
    \lambda = (16\pi^2)^{2/3} , \quad \tag{7.4}
\]

in units where the 11-dimensional Newton constant \( \kappa \) is set to one. The second term in this action has been inferred from anomaly cancellation arguments in Ref. [8]. This term allows one to explicitly determine the four-dimensional gauge-kinetic function \( f \). Let us expand the three-cycle \( Y \) in terms of our basis \( \{C^I\} \) of three-cycles as

\[
    Y = \sum_I m_I C^I \tag{7.5}
\]

with integer expansion coefficients \( m_I \). Then reducing the action (7.3) by integrating over \( Y \) and using the expansion (2.6) for \( C \) one finds

\[
    \text{Im}(f) = \sum_I m_I \nu^I , \quad \text{Re}(f) = \text{vol}(Y) . \tag{7.6}
\]

We recall that the axions \( \nu^I \) are the imaginary parts of the chiral superfields \( T^I \). Holomorphicity of the gauge-kinetic function then tells us that

\[
    f = \sum_I m_I T^I . \tag{7.7}
\]

By combining the expression for \( \text{Re}(f) \) with the one deduced from holomorphicity, Eq. (7.7), we learn that the volume of \( Y \) can be written as

\[
    \text{vol}(Y) = \sum_I m_I a^I \tag{7.8}
\]
in terms of the metric moduli $a'$. 

We would now like to apply these general remarks to the specific $G_2$ manifold $X$ we have focused on in this paper. Each of the 12 blow-ups of this manifold originates from an $A_1$ singularity, so a collapse of each of these blow-ups will lead to an SU(2) gauge field in the four-dimensional effective theory. What does the moduli Kähler potential look like when some of the blow-ups $(\tau,a)$ have collapsed? The natural conjecture is that the Kähler potential is still of the form (5.10) but with the terms corresponding to collapsed blow-ups set to zero. Formally, this can be achieved by the blow-down operation $\text{Re}(U^{\tau,a}) \to 0$ for all singular $(\tau,a)$ in Eq. (5.10).

Let us next determine the gauge-kinetic functions for these SU(2) gauge fields. Their precise form depends on the type $\tau = \alpha, \beta, \gamma$ of the orbifold singularity. Comparing the definition of the orbifold action in Eqs. (3.1), (3.2) and (3.3) with the definition (3.11) of the basis $\{C^I\}$ of three cycles one finds that

$$Y = \begin{cases} 
  C^7 & \text{for } \tau = \alpha \\
  C^6 & \text{for } \tau = \beta \\
  C^5 & \text{for } \tau = \gamma 
\end{cases} \quad (7.9)$$

The gauge-kinetic function is then obtained from Eqs. (7.5) and (7.7) and it coincides with the result (7.2) that we found for the Abelian gauge multiplets in the smooth case. This is not unexpected given that the Abelian vector multiplet present in the smooth case corresponds to the U(1) vector multiplet contained within the SU(2) that forms in the singular limit.

Acknowledgements
We would like to thank Philip Candelas, Jan Louis, Paul Saffin and Kelly Stelle for helpful discussion. A. L. is supported by a PPARC Advanced Fellowship, and S. M. by a PPARC Postgraduate Studentship.

Appendix

A Blow-up and Eguchi-Hanson metric

In this Appendix, we collect some standard material on the blow-up of $\mathbb{C}^2/\mathbb{Z}_2$, and the associated Eguchi-Hanson metric, that will be used in our calculations. We mainly follow Refs. [28]-[31].

Let us first recall how to construct the blow-up of the origin in $\mathbb{C}^2/\mathbb{Z}_2$, where the $\mathbb{Z}_2$ action on the complex coordinates $Z = (z_1,z_2)$ is defined by $(z_1,z_2) \to (-z_1,-z_2)$. More precisely, we focus on a four-dimensional ball $B_\sigma$, with radius $\sigma$, centred around the origin of $\mathbb{C}^2$. Introducing homogeneous coordinates $L = [l_1,l_2]$ on
the blow-up $U \subset B^4_{\sigma} \times \mathbb{CP}^1$ can be defined as
\[ U = \{(Z, L) \in B_{\sigma} \times \mathbb{CP}^1 | z_1 l_2 = z_2 l_1 \} . \tag{A.1} \]
If this definition is extended to all of $\mathbb{C}^2$, the resulting space can be identified with $T^*\mathbb{CP}^1$, the cotangent bundle over $\mathbb{CP}^1$. The blow-down projection $\pi$ is defined by $\pi(Z, l) = Z$. For $z_1 \neq 0$ or $z_2 \neq 0$, $\pi^{-1}(z)$ consists of a single point, as the condition in (A.1) shows. Hence, away from the origin the blow-up looks like $\mathbb{C}^2$ locally. At the origin, $z_1 = z_2 = 0$, on the other hand, we have
\[ E \equiv \pi^{-1}(0) \simeq \mathbb{CP}^1 . \tag{A.2} \]
The cycle $E$ is called the exceptional divisor of the blow-up. The manifold $U$ can be covered by two coordinate charts, $U_1 = \{l_1 \neq 0\}$ and $U_2 = \{l_2 \neq 0\}$, with associated coordinates
\[ \frac{l_2}{l_1} = \frac{z_2}{z_1} , \quad z_1 \tag{A.3} \]
for $U_1$ and
\[ \frac{l_1}{l_2} = \frac{z_1}{z_2} , \quad z_2 \tag{A.4} \]
for $U_2$.

Subsequently, we will need both real and polar coordinates on $\mathbb{C}^2$. We introduce real coordinates $z = (z^\mu) = (x, y, z, t)$, where $\mu, \nu, \cdots = 0, 1, 2, 3$, by
\[ z_1 = x + iy , \quad z_2 = z + it \tag{A.5} \]
and polar coordinates $(u, \theta, \phi, \psi)$ by
\[ z_1 = u \cos \frac{\theta}{2} \exp \left( \frac{i}{2} (\psi + \phi) \right) , \quad z_2 = u \sin \frac{\theta}{2} \exp \left( \frac{i}{2} (\psi - \phi) \right) . \tag{A.6} \]
Here, the three angular coordinates vary within the ranges
\[ \theta \in [0, \pi] , \quad \phi \in [0, 2\pi] , \quad \psi \in [0, 4\pi] . \tag{A.7} \]
The action of $\mathbb{Z}_2$ in polar coordinates is given by $\psi \to \psi + 2\pi$ with $u, \theta$ and $\phi$ unchanged. In a “downstairs” picture, where one works with the fundamental domain only, the range of $\psi$ should be restricted to $\psi \in [0, 2\pi]$. Here, we will usually use the “upstairs” picture and, consequently, work with the full range as in Eq. (A.7). Note that, from Eqs. (A.3) and (A.4), the angles $\theta$ and $\phi$ can be considered good coordinates on the exceptional divisor $E$ so that we can write, in polar coordinates
\[ E = \{u = 0, \theta \in [0, \pi], \phi \in [0, 2\pi]\} . \tag{A.8} \]
This interpretation of the exceptional divisor will be particularly useful for explicit calculations.

To proceed further, we need to recall some $\text{SO}(4)$ group properties. We first introduce a basis of the $\text{SO}(4)$ Lie algebra consisting of left- and right-handed generators $T^a_{\pm}$. They take the explicit form
\[ (T^a_{\pm})_{00} = \delta^a_0 , \quad (T^a_{\pm})_{bc} = -(T^a_{\mp})_{bc} = \epsilon_{abc} , \tag{A.9} \]
where $\mu, \nu, \cdots = 1, 2, 3$, and they satisfy the standard commutation relations $[T^a_+, T^b_+] = 2\epsilon^{ab}_c T^c_+$ and $[T^a_+, T^b_-] = 0$. It is useful to add the unit matrix as $T^0_+$ to obtain a “covariant” version

\[(T^\mu_+) = (1_4, T^\mu_+)\]

where $\mu, \nu, \cdots = 0, 1, 2, 3$. We also note that the left- and right-handed vector representations $R_\pm$ of SO(4) are obtained from the relations

\[\Lambda^T T^\mu_\pm \Lambda = R_\pm(\Lambda) \mu, \nu, T^\nu_\pm\]

(A.11)

with $O_\pm(\Lambda) \in SO(3)$.

We can now introduce the Maurer-Cartan one-forms $\sigma^\mu$ on $S^3 \simeq SU(2)$ satisfying

\[d\sigma^\mu = \epsilon^\mu_{\nu\rho} \sigma^\nu \wedge \sigma^\rho.\]

(A.13)

In terms of the one-forms $(\epsilon^\mu) = (du, u\sigma^2)$ on $\mathbb{C}^2$, they can be explicitly defined as

\[\epsilon^\mu = \frac{1}{|z|} z^T T^\mu_\pm dz,\]

(A.14)

Alternatively, in terms of the left-handed generators $T^a_-$ we can write

\[\epsilon^\mu = P(z) \mu, \nu, dz^\nu,\]

(A.15)

where the matrix $P(z)$ takes the form

\[P(z) = \frac{1}{|z|} \left( \begin{array}{cccc} x & y & z & t \\ -y & x & -t & z \\ -z & t & x & -y \\ t & z & -y & x \end{array} \right) = \frac{1}{|z|} \sum_\mu z^\mu T^\mu_-.\]

(A.16)

We remark that $P(z) \in SO(4)$. One can verify by straightforward calculation that the so-defined forms $\sigma^\mu$ indeed satisfy the SO(3) Maurer-Cartan equation (A.13). Further, it is easy to see from Eqs. (A.14) and (A.11) that they transform in the right-handed vector representation of SO(4), that is,

\[\sigma^\mu(\Lambda z) = O_+(\Lambda) \mu, \nu, \sigma^\nu(z).\]

(A.17)

In polar coordinates, the forms $\sigma^\mu$ can be written as

\[\sigma_1 = \frac{1}{2} (\cos \theta d\phi + d\psi)
\]

\[\sigma_2 = \frac{1}{2} (\cos \psi d\theta + \sin \theta \sin \psi d\phi)
\]

\[\sigma_3 = \frac{1}{2} (\sin \psi d\theta - \sin \theta \cos \psi d\phi).\]

(A.18)

We will write explicit indices $\mu$ on $\sigma^\mu$ as lower indices for notational convenience.
We also collect the wedge products

\[ \sigma_2 \wedge \sigma_3 = -\frac{1}{4} \sin \theta d\theta \wedge d\phi \]
\[ \sigma_3 \wedge \sigma_1 = \frac{1}{4} (\cos \theta \sin \psi d\theta \wedge d\phi + \sin \psi d\theta \wedge d\psi - \sin \theta \cos \psi d\phi \wedge d\psi) \]
\[ \sigma_1 \wedge \sigma_2 = \frac{1}{4} (\cos \theta \sin \psi d\theta \wedge d\phi + \cos \psi d\theta \wedge d\psi + \sin \theta \sin \psi d\phi \wedge d\psi) \]  
(A.19)

Note that the first of these forms, \( \sigma_2 \wedge \sigma_3 \) is well-defined on the exceptional divisor \( E \) because it depends on \( \theta \) and \( \phi \) only. Therefore it can be extended to the blow-up \( U \). It is, in fact, proportional to the volume form on \( E \cong S^2 \). The other two wedge products, and indeed \( \sigma_2 \) and \( \sigma_3 \) themselves, do depend on \( \psi \) and are, therefore, not well-defined at the origin. Nevertheless, as we will see below, they can appear in certain forms that are well-defined on the blow-up \( U \) as long as they are multiplied with a function of the radial coordinate \( u \) that vanishes at the origin \( u = 0 \). Further, we have

\[ \sigma_1 \wedge \sigma_2 \wedge \sigma_3 = -\frac{1}{8} \sin \theta d\theta \wedge d\phi \wedge d\psi \]  
(A.20)

and

\[ \sigma_2^2 + \sigma_3^2 = \frac{1}{4} (d\theta^2 + \sin^2 \theta d\phi^2) \]  
(A.21)

The flat volume form on \( \mathbb{C}^2 \) can then be written as

\[ dx \wedge dy \wedge dz \wedge dt = -\frac{u^3}{8} \sin \theta du \wedge d\theta \wedge d\phi \wedge d\psi = u^3 du \wedge \sigma_1 \wedge \sigma_2 \wedge \sigma_3. \]  
(A.22)

Let us briefly recall the definition of a hyperkähler space before proceeding with our example. A hyperkähler space is a 4m–dimensional Riemannian manifold with metric \( g \), called the hyperkähler metric, and a triplet \( J^\mu \) of covariantly constant complex structures satisfying the algebra

\[ J^\mu J^\nu = -\delta^{\mu\nu} + \epsilon^\mu_{\rho\nu} J^\rho. \]  
(A.23)

As a consequence, the Kähler forms \( w^\mu \) associated with \( J^\mu \) via

\[ w^\mu_{\rho\nu} = (J^\mu)_{\rho\nu} g_{\rho\nu} \]  
(A.24)

are also covariantly constant and, hence, closed and co-closed, that is \( dw^\mu = 0 \) and \( d \ast w^\mu = 0 \).

The Eguchi-Hanson hyperkähler metric on the blow-up \( U \) can be obtained from the Kähler potential

\[ \mathcal{K} = \sqrt{u^4 + \rho^4} + 2\rho^2 \ln u - \rho^2 \ln \left( \sqrt{u^4 + \rho^4} + \rho^2 \right), \]  
(A.25)

where

\[ u^2 = |z_1|^2 + |z_2|^2 \]  
(A.26)
and $\rho$ is a real parameter that measures the radius of the exceptional divisor $E$. For such $U(2)$ invariant Kähler potentials, which depend on the complex coordinates through the radial coordinate only, it is useful to introduce two auxiliary functions

$$ F = \frac{uK'}{2}, \quad G = \frac{FF'}{2u^3}, $$

where the prime denotes the derivative with respect to $u$. For the concrete example (A.25) we find for these functions

$$ F = \sqrt{u^4 + \rho^4}, \quad G = 1. $$

By straightforward calculation, the metric associated with the Kähler potential (A.25) can be obtained as

$$ ds_{\text{EH}}^2 = \frac{F'}{2u} du^2 + \frac{uF'}{2} \sigma_1^2 + F(\sigma_2^2 + \sigma_3^2), $$

where $F$ is given in Eq. (A.28). This metric is Ricci-flat and constitutes a hyperkähler metric on the blow-up $U$, that is, it has associated with it a triplet of three integrable complex structures and three covariantly constant Kähler potentials. We will present these Kähler forms explicitly below.

The metric (A.29) takes a more familiar form when written in the radial coordinate $r$ defined by

$$ r^2 = F = \sqrt{u^4 + \rho^4}. $$

It then turns into

$$ ds_{\text{EH}}^2 = \frac{dr^2}{1 - \frac{\rho^4}{r^4}} + r^2 \left[ 1 - \frac{\rho^4}{r^4} \right] \sigma_1^2 \sigma_2^2 + \sigma_3^2. $$

The coordinate $r$ is restricted to $r \in [\rho, \infty]$ and, hence, its range depends on the parameter $\rho$. Throughout this paper, we will work with the coordinate $u$ whose range $u \in [0, \infty]$ is independent of $\rho$.

On the exceptional divisor, at $u = 0$, the metric (A.29) degenerates into

$$ ds_{\text{EH}}^2(u = 0) = \rho^2(\sigma_2^2 + \sigma_3^2) = \frac{\rho^2}{4} (d\theta^2 + \sin^2 \theta d\phi^2), $$

where the second equality follows from Eq. (A.21). This is indeed the standard metric on a two-sphere $E \simeq S^2$ with radius $\rho/2$.

The vierbein one-forms $e^\mu$ associated with the metric (A.29) are given by

$$ (e^\mu) = \left( \sqrt{\frac{F'}{2u}} du, \sqrt{\frac{uF'}{2}} \sigma_1, \sqrt{F} \sigma_2, \sqrt{F} \sigma_3 \right). $$

The Kähler form $w_1$, associated with the Kähler potential (A.25), takes the form

$$ w_1 = \frac{F'}{2} du \wedge \sigma_1 + F \sigma_2 \wedge \sigma_3, $$

where $F$ is as given in Eq. (A.28). As discussed above, $\sigma_2 \wedge \sigma_3$ can be understood as a form on $U$, while $F(u = 0, \rho) = 0$. Therefore, $w_1$ is indeed a well-defined form on the blow-up $U$. The two other Kähler forms, which must exist on a hyperkähler space, are given by

$$ w_2 = \text{Re}(dz_1 \wedge dz_2), \quad w_3 = \text{Im}(dz_1 \wedge dz_2). $$
and can be expressed in polar coordinates as
\[ w_2 = u \, du \wedge \sigma_2 + u^2 \sigma_3 \wedge \sigma_1 \]  
\[ w_3 = u \, du \wedge \sigma_3 + u^2 \sigma_1 \wedge \sigma_2 . \]  
(A.36)  
(A.37)

Note that these forms vanish on the exceptional divisor at \( u = 0 \) and are, therefore, well-defined on the full blow-up \( U \), although this is not the case for some of their constituent forms, such as \( \sigma_2, \sigma_3, \sigma_3 \wedge \sigma_1 \) and \( \sigma_1 \wedge \sigma_2 \). From the relation (A.13) all three forms \( w_a \) are closed, as they should be. Moreover, in terms of the vierbein (A.33), they can be written as
\[ w_a = e^0 \wedge e_a + \frac{1}{2} e^a_{\mu} e^\mu \wedge e_a , \]  
(A.38)

which implies their self-duality, \( *w_a = w_a \), by virtue of the identity
\[ * (e^\mu \wedge e^\nu) = \frac{1}{2} e^{\mu\nu} \, e^\mu \wedge e^\nu . \]  
(A.39)

Hence, the forms \( w_a \) are closed and co-closed as expected on a hyperkähler manifold. One can also explicitly verify that the complex structures \( J_a \) associated with \( w_a \) do indeed satisfy the algebra (A.23).

Let us discuss the asymptotic form of these Kähler forms for \( u \gg \rho \). As the vierbein approaches the flat space expression \( e^\mu \to dz^\mu \) in this limit, we have from Eq. (A.38)
\[ w^a \to \bar{w}^a \equiv dz^0 \wedge dz^a + \frac{1}{2} e^a_{\mu} dz^b dz^c = \frac{1}{2} (T_+^a)_{\mu\nu} dz^\mu dz^\nu . \]  
(A.40)

These are precisely the three constant Kähler forms associated with flat space (regarded as a hyperkähler space).

There also exists a closed anti-selfdual two-form [32] \( \nu \) on \( U \) which can be written as
\[ \nu = \frac{1}{\mathcal{F}} \left( e^0 \wedge e^1 - e^2 \wedge e^3 \right) = \frac{\mathcal{F}}{2 \mathcal{F}_2} \, du \wedge \sigma_1 + \frac{1}{\mathcal{F}} \sigma_2 \wedge \sigma_3 , \]  
(A.41)

with \( \mathcal{F} \) as in Eq. (A.28). Unlike the above Kähler forms this form is “localised”, that is, it falls off as \( 1/u^4 \) for \( u \gg \rho \).

B Smoothed Eguchi-Hanson spaces

The Eguchi-Hanson space discussed in the previous Appendix approaches flat space asymptotically. However, what is really needed for the construction of \( G_2 \) manifolds are smoothed versions of this space which become exactly flat for sufficiently large radius. In this Appendix, we will discuss such smoothed Eguchi-Hanson spaces, which interpolate between Eguchi-Hanson space at small radius and flat space at large radius, following Ref. [21].

In this context, it is useful to start by analysing general Kähler potentials
\[ \mathcal{K} = \mathcal{K}(u) , \quad u^2 = |z_1|^2 + |z_2|^2 \]  
(B.1)
on a four-dimensional ball $B_\sigma$ with radius $\sigma$ around the origin of $\mathbb{C}^2/\mathbb{Z}_2$ (or on the blow-up $U$ of this space, depending on the properties of $K$). As we will see, many of the properties and relations we require can be obtained within this general framework. Two examples for such Kähler potentials are provided by the Kähler potential (A.25) associated with the Eguchi-Hanson space and that for flat space

$$K = u^2.$$  

(B.2)

More complicated examples will be given below but for now we keep $K = K(u)$ arbitrary.

As before, it is helpful to introducing the auxiliary functions $F$ and $G$ by

$$F = \frac{uK'}{2}, \quad G = \frac{FF'}{2u^3},$$  

(B.3)

where the prime denotes the derivative with respect to $u$. The metric associated with (B.1) is then given by

$$ds^2_{EH} = g_{\mu\nu}dz^\mu dz^\nu = \frac{F'}{2u}du^2 + \frac{uF'}{2}\sigma_1^2 + F(\sigma_2^2 + \sigma_3^2),$$  

(B.4)

and the vierbein reads

$$(e^\mu) = \left(\sqrt{\frac{F'}{2u}}du, \sqrt{\frac{uF'}{2}}\sigma_1, \sqrt{F}\sigma_2, \sqrt{F}\sigma_3\right).$$  

(B.5)

It is interesting to note that the measure derived from this metric, given by

$$\sqrt{\det(g)} = G = \frac{1}{4u^3} \frac{d}{du} (F^2),$$  

(B.6)

only depends on the derivative of $F^2$. With the factor $u^3$ from $d^4z$ cancelling that in $G$, the volume $\text{vol}(u_0, u_1)$ of the part of the space defined by $u \in [u_0, u_1]$ takes the form

$$\text{vol}(u_0, u_1) = \int_{u \in [u_0, u_1]} \sqrt{g}d^4z = \frac{\pi^2}{2} (F(u_1)^2 - F(u_0)^2),$$  

(B.7)

where we have used Eq. (A.22). Hence, remarkably, this volume can be calculated from the first derivative of the Kähler potential directly without the need for explicit integration.

The Kähler form $w_1$ derived from (B.1) is

$$w_1 = \frac{F'}{2}du \wedge \sigma_1 + F\sigma_2 \wedge \sigma_3 = e^0 \wedge e^1 + e^2 \wedge e^3,$$  

(B.8)

which, from Eq. (A.13) is closed, as it should be, and selfdual. This form is well-defined on $\mathbb{C}^2/\mathbb{Z}_2$, and well-defined on the blow-up $U$ as long as $F'(u = 0, \rho) = 0$. We can still introduce a triplet $w_\alpha$ of closed two-forms by defining $w_2$ and $w_3$ as in Eqs. (A.35), (A.36) and (A.37). However, the so-defined forms are in general no longer selfdual and, hence, no longer co-closed. This reflects the fact that we are merely working with Kähler spaces and only certain Kähler potentials of the form (B.1) correspond to hyperkähler spaces.

Remarkably, the generalisation of the anti-selfdual closed form (A.41) can be found for the general Kähler potential (B.1). It is given by

$$\nu = \frac{1}{F^2} (e^0 \wedge e^1 - e^2 \wedge e^3) = \frac{F'}{2F^2}du \wedge \sigma_1 + \frac{1}{F}\sigma_2 \wedge \sigma_3,$$  

(B.9)
and is well-defined on $\mathbb{C}^2/\mathbb{Z}^2$ as long as the function $F$ is everywhere different from zero. If in addition $F'(u = 0, \rho) = 0$ it is also well-defined on the blow-up $U$.

As we have seen, all the relevant objects on a Kähler space defined by a Kähler potential of the form (B.1) are determined in terms of the two functions $F$ and $G$, defined in Eq. (B.3). Let us now apply the above formalism to a number of examples by computing these functions in each case.

We start with the trivial case of flat space

$$\mathcal{K} = u^2$$  \hspace{1cm} (B.10)

which, of course, constitutes a hyperkähler space. One finds that

$$F = u^2, \quad G = 1.$$  \hspace{1cm} (B.11)

From Eq. (B.7) we find for the total volume of the space

$$\text{vol}(0, \sigma) = \frac{\pi^2}{2} \sigma^4.$$  \hspace{1cm} (B.12)

For the Eguchi-Hanson hyperkähler space the Kähler potential reads

$$\mathcal{K} = \sqrt{u^4 + \rho^4} + 2\rho^2 \ln u - \rho^2 \ln \left(\sqrt{u^4 + \rho^4} + \rho^2\right),$$  \hspace{1cm} (B.13)

which leads to

$$F = \sqrt{u^4 + \rho^4}, \quad G = 1.$$  \hspace{1cm} (B.14)

The total volume is identical to the one for flat space, that is

$$\text{vol}(0, \sigma) = \frac{\pi^2}{2} \sigma^4.$$  \hspace{1cm} (B.15)

The third example [21] is defined by the Kähler potential

$$\mathcal{K} = \sqrt{u^4 + \epsilon(u)^2 \rho^4} + 2\epsilon(u)\rho^2 \ln u - \epsilon(u)\rho^2 \ln \left(\sqrt{u^4 + \epsilon(u)^2 \rho^4} + \epsilon(u)\rho^2\right),$$  \hspace{1cm} (B.16)

where $\epsilon(u)$ is a function with

$$\epsilon(u) = \begin{cases} 1 & \text{if } u \leq u_0 \\ 0 & \text{if } u \geq u_1 \end{cases},$$  \hspace{1cm} (B.17)

where $u_0$ and $u_1$ are two characteristic radii satisfying $\rho \ll u_0 < u_1 < \sigma$. Hence, the space described by the Kähler potential (B.16) is identical to the Eguchi-Hanson space for radii $u \leq u_0$ and identical to flat space for $u \geq u_1$, that is, it interpolates between the Eguchi-Hanson space and flat space. Although this space interpolates between two hyperkähler spaces it is not a hyperkähler space by itself. Accordingly the forms $w_2$ and $w_3$ are no longer co-closed in the “collar” region $u \in [u_0, u_1]$. For the functions $F$ and $G$ we find, to order $\rho^6$ in the blow-up radius

$$F = u^2 + \frac{1}{2u^2} (\epsilon^2 - u\epsilon') \rho^4 + O(\rho^8)$$  \hspace{1cm} (B.18)

$$G = 1 + \frac{\rho^4}{4u^3} \frac{d}{du} (\epsilon^2 - u\epsilon') + O(\rho^8).$$  \hspace{1cm} (B.19)
These functions interpolate between their counterparts for Eguchi-Hanson and flat space, as they should. Moreover, the correction terms arising in the collar \( u \in [u_0, u_1] \) are at least of order \( \rho^4 \) and proportional to derivatives of the interpolating function \( \epsilon \). Hence, this space can be thought of as being close to hyperkähler as long as the blow-up radius \( \rho \) is sufficiently small compared to one and the function \( \epsilon \) is slowly varying. The function \( \mathcal{G} \) will eventually be the crucial ingredient in computing the volume of the \( G_2 \) space. Note that the \( \mathcal{O}(\rho^4) \) correction to \( \mathcal{G} \) can be written as a total derivative. From Eq. (B.7), the total volume is given by

\[
\text{vol}(0, \sigma) = \frac{\pi^2}{2}(\sigma^4 - \rho^4)
\]
and so is independent of the precise form of the smoothing function \( \epsilon \). This property, which we will recover in slightly different circumstances when we compute the volume of the \( G_2 \) manifold, is crucial for our calculation. The second term in (B.20) represents the amount subtracted from the volume due to the presence of the blow-up and it equals the volume of a four-dimensional ball with radius \( \rho \).

## C  \( G_2 \) structures

This Appendix collects some useful information on the group \( G_2 \) and \( G_2 \) structures on seven-dimensional manifolds. We also describe the specific example of a \( G_2 \) structure on \( U \times T^3 \) where \( U \) is the blow-up of \( \mathbb{C}^2 / \mathbb{Z}_2 \) described in Appendix A. In parts, we follow Ref. [21, 23].

We start by defining the flat \( G_2 \) structure

\[
\varphi = dx^1 \wedge dx^2 \wedge dx^7 + dx^1 \wedge dx^3 \wedge dx^6 + dx^1 \wedge dx^4 \wedge dx^5 + dx^2 \wedge dx^3 \wedge dx^5 + dx^4 \wedge dx^2 \wedge dx^6 + dx^3 \wedge dx^4 \wedge dx^7 + dx^5 \wedge dx^6 \wedge dx^7
\]
(C.1)
on \( \mathbb{R}^7 \) with coordinates \( x = (x^A) \), where \( A, B, \cdots = 1, \ldots, 7 \). Consider elements of the seven-dimensional special orthogonal group, \( g \in \text{SO}(7) \) acting linearly on \( x \). The group \( G_2 \) can be defined as the subgroup consisting of elements \( g \in \text{SO}(7) \) that leave the three-form \( \varphi \) invariant under the linear action on \( x \).

For our purpose, it is useful to split up the seven-dimensional coordinates \( x \) into a four-dimensional part with coordinates \( \zeta = \zeta^\mu \), where \( \mu, \nu, \cdots = 0, 1, 2, 3 \), and a complementary three-dimensional part with coordinates \( \xi = \xi^a \), where \( a, b, \cdots = 1, 2, 3 \). Let us also introduce the two-forms

\[
\bar{w}^a = \frac{1}{2} (T_+^a)_{\mu\nu} d\zeta^\mu d\zeta^\nu
\]
(C.2)
on the four-dimensional part of the space. The matrices \( T_+^a \) have been defined in Eq. (A.9). We have already encountered these two-forms in Eq. (A.40) as the triplet of Kähler forms on \( \mathbb{R}^4 \). For appropriate identifications of the seven-dimensional coordinates \( x \) with \( \zeta \) and \( \xi \) the three-form \( \varphi \) can be written as

\[
\varphi = \sum_{a=1}^{3} \bar{w}^a \wedge d\xi^a - d\xi^1 \wedge d\xi^2 \wedge d\xi^3.
\]
(C.3)
A particular example of an identification for which the above relation holds is provided by
\[ \zeta = (x^1, x^2, x^3, x^4) \]
and
\[ \xi = (x^7, x^6, x^5). \]
However, there are other possibilities, obtained by suitable permutations of the coordinates, that we will encounter in the construction of the \( G_2 \) manifold. Each such coordinate identification, associated with a relation (C.3), leads to an embedding \( \text{SO}(4) \subset G_2 \). Indeed, define the action \( g_\Lambda \) of any \( \Lambda \in \text{SO}(4) \) on \( x \) as
\[
g_\Lambda(x) = \begin{pmatrix} \Lambda \zeta \\ O_+(\Lambda) \xi \end{pmatrix}. \tag{C.4}\]
It is then clear from the definition (A.11) of the right-handed vector representation \( O_+ \), and Eqs. (C.2) and (C.3), that the three-form \( \tilde{\varphi} \) is invariant under \( g_\Lambda \) and, hence, \( g_\Lambda \in G_2 \).

Let \( X \) be a seven-dimensional oriented manifold. A \( G_2 \) structure on \( X \) is defined by a smooth three-form \( \varphi \) which is isomorphic to the “flat” \( G_2 \) structure \( \bar{\varphi} \) given in Eq. (C.1). The isomorphism induces a metric \( g \) on \( X \) that is referred to as the metric associated with \( \varphi \). Given such a \( G_2 \) structure \( \varphi \) the associated metric can be explicitly computed. Defining
\[
\gamma_{AB} = \frac{1}{144} \varphi_{ACD} \varphi_{BEF} \varphi_{GHI} \hat{\epsilon}^{CDEFGH} \tag{C.5}
\]
with the “pure-number” Levi-Civita pseudo-tensor \( \hat{\epsilon} \), the associated metric \( g \) is given by
\[
g_{AB} = \det(\gamma)^{-1/9} \gamma_{AB}, \quad \sqrt{\det(g)} = \det(\gamma)^{1/9}. \tag{C.6}
\]
A number of useful properties of \( \varphi \) can be directly deduced from its flat counterpart \( \bar{\varphi} \). For example, one has
\[
\varphi^{ABC} \varphi^{ABC} = 42 \tag{C.7}
\]
where the indices have been raised with the associated metric \( g \). The volume of the manifold \( X \) measured with the metric \( g \) can then be written as
\[
\text{vol}(X) = \int_X \sqrt{\det(g)} d^7 x = \frac{1}{7} \int_X \varphi \wedge \Theta(\varphi). \tag{C.8}
\]
Here, the map \( \Theta \) is defined as
\[
\Theta(\varphi) = \star \varphi, \tag{C.9}
\]
where the Hodge star is taken with respect to the metric \( g \) associated with \( \varphi \). By virtue of (C.5) and (C.6), \( \Theta \) is a highly non-linear map acting on \( G_2 \) structures \( \varphi \).

A \( G_2 \) structure \( \varphi \) is said to have vanishing torsion if it is covariantly constant with respect to the Levi-Civita connection \( \nabla \) induced by the associated metric \( g \). This condition is equivalent to \( d\varphi = d\Theta(\varphi) = 0 \), or to \( \varphi \) being harmonic with respect to the metric \( g \). It can be shown that the holonomy group of \( X \) with respect to \( \nabla \) is a subgroup of \( G_2 \) if the \( G_2 \) structure is torsion-free. Then, the associated metric \( g \) is Ricci-flat. If, in addition, the first fundamental group \( \pi_1(X) \) is finite, the holonomy group is precisely \( G_2 \).

In practice, torsion-free \( G_2 \) structures have not been constructed explicitly on compact manifolds. Instead, the construction of compact manifolds with \( G_2 \) holonomy in Ref. [21, 22, 23] relies on explicit \( G_2 \) structures.
with small torsion. It is then shown that torsion-free $G_2$ structures exist “nearby”. An essential ingredient in this construction are small-torsion $G_2$ structures on $F \equiv U \times T^3$, where $U$ is the blow-up of $\mathbb{C}^2/\mathbb{Z}^2$ as defined in Appendix A and $T^3 = \mathbb{R}^3/\mathbb{Z}^3$ is the standard three-torus. Let us now review these specific $G_2$ structures and derive some of their properties.

Following Ref. [21], the $G_2$ structure on $F$ is taken to be

$$\varphi = \sum_a w^a \wedge d\xi^a - d\xi^1 \wedge d\xi^2 \wedge d\xi^3 . \quad (C.10)$$

Here the two-form $w_1$ has the structure (B.8) and the functions $F$ and $G$ are taken to be the ones associated with the smoothed Eguchi-Hanson Kähler potential (B.16), and are explicitly given in Eqs. (B.18) and (B.19).

The two other forms $w_2$ and $w_3$ are defined in Eqs. (A.36) and (A.37), respectively. Further, $\xi^a$, where $a,b,\cdots = 1,2,3$, are the coordinates of the three-torus $T^3$. Since the two-forms $w^a$ are closed the same is true for $\varphi$, that is, we have

$$d\varphi = 0 . \quad (C.11)$$

For the radial coordinate $u$ in the range $u \in [0,u_0]$ the forms $w^a$ coincide with the three Kähler forms of the Eguchi-Hanson space. On the other hand, in the range, $u \in [u_1,\sigma]$ the $w^a$ are identical to the three Kähler forms (C.2) of flat space and, hence, from Eq. (C.3), $\varphi$ is given by the flat $G_2$ structure (C.1) for a suitable identification of the coordinates $z^\mu$ and $\xi^a$ with $x^A$. Then in both those regions $\varphi$ is actually torsion-free, that is we have $d\Theta(\varphi) = 0$ for $u \leq u_0$ or $u \geq u_1$. In the collar region $u \in [u_0,u_1]$, however, $d\Theta(\varphi)$ is nonzero and can be shown to be of order $\rho^4$ and proportional to derivatives of the interpolating function $\epsilon$. Hence, for small blow-up radius $\rho$ and a smooth interpolation with a slowly-varying function $\epsilon$, the $G_2$ structure $\varphi$ has small torsion.

The previous statements can be explicitly verified using the metric $g$ associated with $\varphi$. We will now compute this metric using Eqs. (C.5) and (C.6). From Eq. (C.10) and the definition of the two-forms $w^a$, Eqs. (B.8), (A.36), (A.37), (B.18), along with the expressions for the Maurer-Cartan forms $\sigma^a$ given in Appendix A, we find the only non-vanishing components of $\varphi$ are given by

$$\varphi_{\mu\nu a} = 2m_a P^0_{[\mu} P^a_{\nu]} + n_a \epsilon^a_{bc} P^b_{\mu} P^c_{\nu} \quad (C.12)$$

$$\varphi_{abc} = -\epsilon_{abc} \quad (C.13)$$

where

$$(m_a) = \left( \frac{F'}{2u}, 1, 1 \right) , \quad (n_a) = \left( \frac{F}{u^2}, 1, 1 \right) . \quad (C.14)$$

We recall that the matrix $P$, defined in Eq. (A.16), is an element of $\text{SO}(4)$, a fact which considerably simplifies the subsequent calculation. Inserting the above components of $\varphi$ into Eqs. (C.5) and (C.6) leads, after some algebra, to the associated metric

$$ds^2 = g_{AB} dx^a dx^b = G^{-1/3} \left[ \frac{F'}{2u} du^2 + \frac{u F'}{2} \sigma_1^2 + F (\sigma_2^2 + \sigma_3^2) \right] + G^{2/3} d\xi_1^2 + G^{-1/3} (d\xi_2^2 + d\xi_3^2) . \quad (C.15)$$

27
We note, that the four-dimensional part of this metric differs from the smoothed Eguchi-Hanson metric (B.4) by the conformal factor $G^{-1/3}$. This difference is due to the complicated relation (C.5), (C.6) between $\varphi$ and $g$ and it matters precisely in the collar region where, from Eq. (B.19), $G$ is different from one. For the measure associated with the above metric we find

$$\sqrt{\det(g)} = G^{1/3}. \quad (C.16)$$

Note the power $1/3$ by which this result differs from the measure (B.6) for the smoothed Eguchi-Hanson metric. Using the explicit expression for $G$ in Eq. (B.19) we can now compute the volume to order $\rho^6$. One finds

$$\text{vol}(0, \sigma) = \frac{\pi^2}{2} \left( \sigma^4 - \frac{1}{3} \rho^4 \right) + O(\rho^8). \quad (C.17)$$

The second term in this expression can be interpreted as an effect of the blow-up. Due to the non-trivial exponent in (C.16) this term is only $1/3$ of the corresponding term in Eq. (B.20) obtained from the smoothed Eguchi-Hanson metric. This fact can be directly traced to the cubic nature of the relation (C.5) between $\varphi$ and $g$.

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