Detection of Total Rotations on 2D-Vector Fields with Geometric Correlation

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Abstract. Correlation is a common technique for the detection of shifts. Its generalization to the multidimensional geometric correlation in Clifford algebras additionally contains information with respect to rotational misalignment. It has been proven a useful tool for the registration of vector fields that differ by an outer rotation.

In this paper we prove that applying the geometric correlation iteratively has the potential to detect the total rotational misalignment for linear two-dimensional vector fields. We further analyze its effect on general analytic vector fields and show how the rotation can be calculated from their power series expansions.

Keywords: geometric algebra, Clifford algebra, registration, total rotation, correlation, iteration.

1. INTRODUCTION

[1] In signal processing correlation is one of the elementary techniques to measure the similarity of two input signals. It can be imagined like sliding one signal across the other and multiplying both at every shifted location. The point of registration is the very position, where the normalized cross correlation function takes its maximum, because intuitively explained there the integral is built over squared and therefore purely positive values. For a detailed proof compare [2]. Correlation is widely used for signal analysis, image registration, pattern recognition, and feature extraction [3, 4].

For quite some time the generalization of this method to multivariate data has only been parallel processing of the single channel technique. Multivectors, the elements of geometric or Clifford algebras $\mathbb{C}_{p,q}$ [5, 6] have a natural geometric interpretation. So the analysis of multidimensional signals expressed as multivector valued functions is a very reasonable approach.

Scheuermann made use of Clifford algebras for vector field analysis in [7]. Together with Ebling [8, 9] they applied geometric convolution and correlation to develop a pattern matching algorithm. They were able to accelerate it by means of a Clifford Fourier transform and the respective convolution theorem.

At about the same time Moxey, Ell, and Sangwine [10, 11] used the geometric properties of quaternions to represent color images, interpreted as vector fields. They introduced a generalized hypercomplex correlation for quaternion valued functions. Moxey et. al. state in [11], that the hypercomplex correlation of translated and outer rotated images will have its maximum peak at the position of the shift and that the correlation at this point also contains information about the outer rotation. From this they were able to approximately correct rotational distortions in color space.

In [12] we extended their work and ideas analyzing vector fields with values in the Clifford algebra $\mathbb{C}_{3,0}$ and their copies produced from outer rotations. We proved that iterative application of the rotation encoded in the cross correlation at the point of registration completely eliminates the outer misalignment of the vector fields.

In this paper we go one step further and analyze if iteration can not only lead to the detection of outer rotations but also to the detection of total rotations of vector fields.

The term rotational misalignment with respect to multivector fields is ambiguous. We distinguish three cases, visualized for a simple example in Figure 1. Let $R_\alpha$ be an operator, that describes a mathematically positive rotation by the angle $\alpha$.

Two multivector fields $A(x), B(x) : \mathbb{R}^m \rightarrow \mathbb{C}_{p,q}$ differ by an inner rotation if they suffice

$$A(x) = R_{-\alpha}(R_{\alpha}(B(x))).$$ \hspace{1cm} (1.1)

It can be interpreted like the starting position of every vector is rotated by $\alpha$. Then the old vector is reattached at the new position, but it still points into the old direction. The inner rotation is suitable to describe the rotation of a color image. The color is represented as a vector and does not change when the picture is turned.

Another kind of misalignment we want to mention is the outer rotation

$$A(x) = R_{\alpha}(B(x)).$$ \hspace{1cm} (1.2)

Here every vector on the vector field $A$ is the rotated copy of every vector in the vector field $B$. The vectors are rotated independently from their positions. This kind of rotation appears for example in color images, when
The fundamental idea for this paper stems from the correlation of a two-dimensional vector field and its copy from outer rotation

\[ (R_\alpha(v) \ast v)(0) = \int_{\mathbb{R}^2} R_\alpha(v(x))v(x) \, d^2x \]
\[ = \int_{\mathbb{R}^2} e^{-\alpha_12}v(x)v(x) \, d^2x \]
\[ = ||v(x)||_L^2 e^{-\alpha_12}. \]

Since \( ||v(x)||_L^2 \in \mathbb{R} \) the alignment can be restored by rotating back \( R_\alpha(v) \) by the angle encoded in the argument.

We want to develop this idea further to analyze total rotations. In \( Cl_{2,0} \) they take the shape

\[ u(x) = R_\alpha(v(R_{-\alpha}(x))) = e^{-\alpha_12}v(e^{\alpha_12}x), \]

so it is not possible to predict the rotation that is encoded in the geometric correlation without knowing the shape of \( v \).

Vector fields that depend only on the magnitude of \( x \) are invariant with respect to inner rotations. It is easy to see that in this case the correlation takes the same shape as in (2.1) and that the misalignment can be corrected applying a rotation by the angle in the argument, too. But in general the vector fields and the rotor can not be separated from the integral of the correlation

\[ (u \ast v)(0) = \int_{\mathbb{R}^m} R_\alpha(v(x))v(x) \, dm \]
\[ = e^{-\alpha_12} \int_{\mathbb{R}^m} v(e^{\alpha_12}x)v(x) \, dm. \]

We dealt with a similar problem in [12] when we treated the three-dimensional outer rotation. For this case we could prove that the encoded rotation is at least a fair approximation to the one sought after and that iterative application leads to the detection of the misalignment sought after.

Trying to adapt this idea to total rotations we discovered that this result does not apply to all two-dimensional vector fields, compare the following counterexample.
imagine starting the iterative algorithm from (12) with
misalignment was, we always detect its negative. So
correct direction and double the misalignment
with its inverse like in (2.1) we would rotate in the
If we want to correct the misalignment by rotating back
inside the unit circle and the correlation of the two is
Assume a linear vector field in two dimensions
let \( v : B_1(0) \to \mathbb{R}^2 \) be the vector field from
Figure 2 vanishing outside the unit circle take the shape
\[
v(r, \varphi) = e_1 e^{2 \varphi e_{12}}
\]
expressed in polar coordinates. Then its rotated copy
suffices
\[
u(r, \varphi) = e_1 e^{(2\varphi - \alpha) e_{12}}
\]
inside the unit circle and the correlation of the two is
\[
(u(x) \cdot v(x))(0) = \int_{B_1(0)} e_1 e^{(2\varphi - \alpha) e_{12}} e_1 e^{2\varphi e_{12} r} dr d\varphi
\]
\[
= \int_{B_1(0)} e_1 e^{-(2\varphi - \alpha) e_{12}} e_1 e^{2\varphi e_{12} r} dr d\varphi
\]
\[
= \int_{B_1(0)} e^{\alpha e_{12} r} dr d\varphi
\]
\[
= \pi e^{\alpha e_{12}}.
\] (2.6)

If we want to correct the misalignment by rotating back
with its inverse like in (2.1) we would rotate in the
completely wrong direction and double the misalignment
with each step, because no matter how the rotational
misalignment was, we always detect its negative. So
imagine starting the iterative algorithm from [12] with
\( \alpha = \frac{2\pi}{3} \). It would become periodic \( \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{8\pi}{3} = \frac{2\pi}{3} \)
and not converge at all.

But the idea applies to all linear fields. We will show
in the next sections, that iteratively rotating back with
the inverse of the normalized geometric correlation will
detect the correct misalignment of any two-dimensional
linear vector field and its copy from total rotation.

3. EASY LINEAR EXAMPLES

Assume a linear vector field in two dimensions
\[
v(x) = (a_{11}x_1 + a_{12}x_2)e_1 + (a_{21}x_1 + a_{22}x_2)e_2
\] (3.1)

with real coefficients. Before analyzing the general linear
case, let us look the examples in Figure 3, the saddles
\[
a(x) = x_1 e_1 - x_2 e_2,
\]
\[
b(x) = x_2 e_1 + x_1 e_2,
\] (3.2)

the source
\[
c(x) = x_1 e_1 + x_2 e_2,
\] (3.3)

and the vortex
\[
d(x) = x_2 e_1 - x_1 e_2.
\] (3.4)

Remark 3.1. Instead of using the coefficients the vector
fields from Figure 3 can analogously be expressed by
basic transformations
\[
c(x) = x,
\]
\[
a(x) = -e_2 x e_2,
\]
\[
d(x) = e_1 x,
\]
\[
b(x) = -e_2 d(x)e_2.
\]

The first three of these are the identity, a reflection at the
hyperplane perpendicular to \( e_2 \) and a rotation about \(-\frac{\pi}{2}\).
The last one \( b(x) = -e_2 d(x) e_2 \) can be seen as a rotation about \(-\frac{\pi}{2}\)
followed by a reflection at the hyperplane perpendicular to \( e_2 \) or as \( b(x) = -e_2 a(x) \) a reflection at the
hyperplane perpendicular to \( e_2 \) followed by a rotation about \(\frac{\pi}{2}\) or alternatively as \( b(x) = -1/2(e_1 - e_2)x(e_1 - e_2) \) just a reflection at the hyperplane perpendicular to
\( e_1 - e_2 \). From this description we immediately get
\[
a(x) \perp b(x),
\]
\[
c(x) \perp d(x),
\] (3.6)

and
\[
a(x)^2 = b(x)^2 = c(x)^2 = d(x)^2 = x^2.
\] (3.7)

The geometric products of them and their rotated
copies at any position take very simple forms.

Example. For a saddle \( a(x) = x_1 e_1 - x_2 e_2 = -e_2 x e_2 \) we get
\[
R_\alpha(a(R_{-\alpha}(x))) = e^{-\alpha e_{12}} (-e_2 e^{\alpha e_{12} x} e_2)
\]
\[
= e^{-2\alpha e_{12}} (-e_2 x e_2)
\] (3.8)

and therefore the product
\[
R_\alpha(a(R_{-\alpha}(x))) a(x) = a(x)^2 e^{-2\alpha e_{12}}
\] (3.9)

reveals twice the angle we looked for.
Remark 3.2. The argument of $a(x)^2 e^{-2\alpha e_{12}}$ is only $-2\alpha$ for $\alpha \in [-\pi/2, \pi/2]$. Keeping track of the case differentiation is a hassle, so we restrict ourselves to this interval during the whole paper. For two-dimensional linear vector fields this is no restriction, because the by $\pi$ totally rotated copy of $v(x)$ suffices

$$R_\pi(v(R_\pi(x))) = e^{-\pi e_{12}}(v(e^{\pi e_{12}}(x))) = -v(-x) \quad (3.10)$$

That means an angle $\alpha$ in $[-\pi/2, \pi/2]$ can always be found to describe the rotational difference of $v(x)$ and its rotated copy.

Example. For a vortex $d(x) = x_2 e_1 - x_1 e_2 = e_{12} x$ we get

$$R_\alpha(d(R_{-\alpha}(x))) = e^{-\alpha e_{12}} e_{12} e^{\alpha e_{12}} x$$

$$= e^{-\alpha e_{12}} e^{\alpha e_{12}} x \quad (3.11)$$

and therefore the product

$$R_\alpha(d(R_{-\alpha}(x))) d(x) = d(x)^2 \quad (3.12)$$

is real valued, what is to be interpreted as an angle of zero. That is not a surprise, because a vortex is invariant with respect to total rotations in the plane. Therefore it is not disturbing that we do not get the rotational information either.

Example. A saddle with the shape $b(x) = x_2 e_1 + x_1 e_2 = e_1 x e_2$ suffices

$$R_\alpha(b(R_{-\alpha}(x))) = e^{-2\alpha e_{12}} e_1 e^{\alpha e_{12}} x e_2$$

$$= e^{-2\alpha e_{12}} e_1 x e_2 \quad (3.13)$$

So the product of the saddles always leads to $-2\alpha$, too.

Example. A source $c(x) = x_1 e_1 + x_2 e_2 = x$ leads to

$$R_\alpha(c(R_{-\alpha}(x))) = e^{-\alpha e_{12}} e^{\alpha e_{12}} x$$

$$= x \quad (3.14)$$

So the product of sources also leads to zero angle.

4. COMPOSITION OF LINEAR FIELDS FROM THE EXAMPLES

Lemma 4.1. Any linear vector field can be expressed as a linear combination of the four examples from the preceding section. That means for

$$a(x) = x_1 e_1 - x_2 e_2,$$

$$b(x) = x_2 e_1 + x_1 e_2,$$

$$c(x) = x_1 e_1 + x_2 e_2,$$

$$d(x) = x_2 e_1 - x_1 e_2$$

there are $a, b, c, d \in \mathbb{R}$, such that

$$v(x) = a(a(x)) + b(b(x)) + c(c(x)) + d(d(x)). \quad (4.2)$$

Proof. For the proof set

$$a = \frac{1}{2}(a_{11} - a_{22}),$$

$$b = \frac{1}{2}(a_{12} + a_{21}),$$

$$c = \frac{1}{2}(a_{11} + a_{22}),$$

$$d = \frac{1}{2}(-a_{12} + a_{21}),$$

that leads to

$$a_{11} = a + c,$$

$$a_{12} = b - d,$$

$$a_{21} = b + d,$$

$$a_{22} = -a + c,$$

and therefore

$$v(x) = a_{11} x_1 e_1 + a_{12} x_1 e_2 + a_{21} x_2 e_1 + a_{22} x_2 e_2$$

$$= (a + c) x_1 e_1 + (b - d) x_1 e_2$$

$$+ (b + d) x_2 e_1 + (-a + c) x_2 e_2$$

$$= a(x_1 e_1 - x_2 e_2) + b(x_2 e_1 + x_1 e_2)$$

$$+ c(x_1 e_1 + x_2 e_2) + d(x_2 e_1 - x_1 e_2)$$

$$= aa(x) + bb(x) + cc(x) + dd(x). \quad (4.5)$$

□
Remark 4.2. The decomposition from Lemma 4.1 is analogous to the description of two-dimensional linear vector fields by Scheuermann in [7]. He stresses the isomorphism of the vectors and the rotors in $\mathcal{O}_{2,0}$, denoting

$$v(x) = \tilde{E}(r)$$

$$= E(z, \bar{z})e_1$$

$$= \left((\alpha + e_{12}\beta)z + (\gamma + e_{12}\delta)\bar{z}\right)e_1 \quad (4.6)$$

with $z = x_1 + x_2e_{12}, \bar{z} = x_1 - x_2e_{12}$. It leads to

$$v(x) = (\alpha x_1 e_1 - x_2 e_2) + (\gamma x_1 e_1 + x_2 e_2)$$

$$= \alpha (x_1 e_1 - x_2 e_2) + \beta \left(\bar{z} + x_1 e_1 - x_2 e_2\right) \quad (4.7)$$

which is the same decomposition as above if we identify $\alpha = a, -\beta = b, \gamma = c$ and $\delta = d$.

5. INFLUENCE OF SUPERPOSITION ON THE PRODUCT

Lemma 5.1. Let the part in Lemma 4.1 of the two-dimensional linear vector field $v(x)$ consisting of the two saddles be denoted by

$$v_1(x) = a a(x) + b b(x) = (a - be_{12})(-e_{2}xe_{2}) \quad (5.1)$$

and the part consisting of the vortex and the source by

$$v_2(x) = c c(x) + d d(x) = (c + de_{12})x \quad (5.2)$$

Then their totally rotated copies take the shapes

$$R_{\alpha}(v_1(R_{-\alpha}(x))) = e^{-2ae_{12}}v_1(x),$$

$$R_{\alpha}(v_2(R_{-\alpha}(x))) = v_2(x). \quad (5.3)$$

Proof. The assertion follows from the linearity of the rotation and the calculations from (3.9) and (3.12).}

Remark 5.2. By means of the description by Scheuermann in [7] this result takes the following shape: The product of a two-dimensional linear vector field and its copy from total rotation by $\alpha$ yields an argument of $-2\alpha$ iff the vector field $E(z, \bar{z})$, compare (4.6), only depends on $z$ and an argument of zero, iff it only depends on $\bar{z}$.

Lemma 5.3. Let $v_1(x) = (a - be_{12})(-e_{2}xe_{2}), v_2(x) = (c + de_{12})x$ be the fields from Lemma 5.1. The product of any two-dimensional linear vector field $v(x)$ and its totally rotated copy $u(x) = R_{\alpha}(v(R_{-\alpha}(x)))$ takes the shape

$$u(x)v(x) = e^{-2ae_{12}}v_1(x)^2 + e^{-2ae_{12}}v_1(x)v_2(x)$$

$$+ v_2(x)v_1(x) + v_2(x)^2 \quad (5.4)$$

with

$$v_1(x)^2 = (a^2 + b^2)(x_1^2 + x_2^2),$$

$$v_1(x)v_2(x) = (a - be_{12})(c - de_{12})(x_1^2 - x_2^2 + 2x_1x_2e_{12}),$$

$$v_2(x)v_1(x) = (a + be_{12})(c + de_{12})(x_1^2 - x_2^2 + 2x_1x_2e_{12}),$$

$$v_2(x)^2 = (c^2 + d^2)(x_1^2 + x_2^2). \quad (5.5)$$

Proof. We have seen in Lemma 4.1 that the vector field can be split into $v(x) = v_1(x) + v_2(x)$, with $v_1(x) = aa(x) + bb(x)$ and $v_2(x) = cc(x) + dd(x)$. Applying Lemma 5.1 leads to

$$R_{\alpha}(v(R_{-\alpha}(x))) = R_{\alpha}((v_1 + v_2)(R_{-\alpha}(x)))$$

$$= R_{\alpha}(v_1(R_{-\alpha}(x))) + R_{\alpha}(v_2(R_{-\alpha}(x)))$$

$$= e^{-2ae_{12}}v_1(x) + v_2(x) \quad (5.6)$$

and therefore the product suffices (5.4). The assertions about the exact shape of the summands follow from straightforward calculation, we only give the derivation of one of the mixed parts representatively

$$v_1(x)v_2(x) = (a - be_{12})(c + de_{12})c(x_1 e_1 - x_2 e_2)x$$

$$= (a - be_{12})(c - de_{12})(x_1^2 - x_2^2 + 2x_1x_2e_{12}). \quad (5.7)$$

We want to determine what angle is encoded in the expression (5.4) and look at some examples.

Example. For the sum of a saddle and a vortex $v(x) = a(x) + d(x)$ the totally rotated copy suffices

$$R_{\alpha}(v(R_{-\alpha}(x))) = R_{\alpha}(a(R_{-\alpha}(x))) + R_{\alpha}(d(R_{-\alpha}(x)))$$

$$= e^{-2ae_{12}}a(x) + d(x) \quad (5.8)$$

so their product takes the shape

$$R_{\alpha}(v(R_{-\alpha}(x)))v(x)$$

$$= e^{-2ae_{12}}x - e^{-2ae_{12}}(-2x_1x_2 + x_1^2 - x_2^2)e_{12}$$

$$= (e^{-ae_{12}} + e^{ae_{12}})(x^2 + 2x_1x_2)$$

$$= (2\cos(\alpha)(x_1 + x_2)^2 + 2\sin(\alpha)(x_1^2 - x_2^2))e^{-ae_{12}}. \quad (5.9)$$

Its inverse reveals the correct misalignment by $\alpha$.

All the examples we had might lead to the assumption, that the argument $\theta$ of the polar form of the geometric product always takes a value in $[0, -2\alpha]$. But the following counterexample shows that this is not true.

Example. The product of the linear vector field $v(x) = a(x) + 2c(x) = 3x_1e_1 + x_2e_2$, which is a superposition of
a saddle and a source, and its copy rotated by $\alpha = \frac{\pi}{4}$ has the value
\[
R_\alpha (v(R_\alpha (x))) = ((2x_1 + x_2)e_1 + (x_1 + 2x_2)e_2)(3x_1e_1 + x_2e_2) = (2(3x_1^2 + 2x_1x_2 + x_2^2)) - (3x_1^2 + 4x_1x_2 - x_2^2)e_12
\]
If we evaluate it at the position $x = -e_1 + e_2$ we get
\[
R_\alpha (v(R_\alpha (-e_1 + e_2))) = 4 + 2e_{12}, \quad (5.11)
\]
the argument of which is positive.

In the coming sections we will show that in contrast to the product the assumption will hold for the geometric correlation over a symmetric area.

### 6. INFLUENCE OF SUPERPOSITION ON THE CORRELATION

Heuristic shows, that a case like the last example appears relatively sparsely. Using the average over a larger area could erase such appearances. We will show that the integral is equivalent to the correlation at the origin, if we assume the vector fields to vanish outside this area.

**Theorem 6.1.** Let the two-dimensional vector field $v(x)$ be linear within and zero outside of an area $A$ symmetric with respect to both coordinate axes. The correlation at the origin with its totally rotated copy $u(x) = R_\alpha(v(R_\alpha(x)))$ satisfies

\[
(u \cdot v)(0) = e^{-2ae_12}\|v_1(x)\|^2_{L^2(A)} + \|v_2(x)\|^2_{L^2(A)} \quad (6.1)
\]

with $v_1(x) = (a - be_12)(-e_2e_2), v_2(x) = (c + de_12)x$ from Lemma 5.1.

**Proof.** We already know from Lemma 5.3 that the product of the vector field and its rotated copy takes the form

\[
u(x)v(x) = e^{-2ae_12}v_1(x)^2 + e^{-2ae_12}v_1(x)v_2(x) + v_2(x)v_1(x) + v_2(x)^2.
\]

Taking into account (5.5) and the fact, that the integral over the symmetric domain $A$ over $x_1^2 - x_2^2$ is zero as well as the integral over $x_1x_2$, we get

\[
\int_A v_1(x)v_2(x)^2 \text{d}^2x = \int_{-l_{\perp}^2} (a - be_12)(c - de_12)
\]

\[
\cdot (x_1^2 - x_2^2 + 2x_1x_2e_{12}) \text{d}^2x \quad (6.3)
\]

and $\int_A v_2(x)v_1(x) \text{d}^2x = 0$, too. That is why the integral over the product reduces to

\[
\int_A u(x)v(x) \text{d}^2x = \int_A e^{-2ae_12}v_1(x)^2 + v_2(x)v_1(x)
\]

\[
+ e^{-2ae_12}v_1(x)v_2(x) + v_2(x)^2 \text{d}^2x = e^{-2ae_12}\|v_1(x)\|^2_{L^2(A)} + \|v_2(x)\|^2_{L^2(A)}, \quad (6.4)
\]

**Remark 6.2.** Please note that an integral over an unsymmetric area does in general not lead to a result without the mixed terms $v_1(x)v_2(x)$.

### 7. DETECTION OF THE ANGLE

Now we want to use Theorem 6.1 to evaluate the angle $\alpha$ by which our pattern and our vector field differ. First assume we have the analytical description of the pattern.

**Corollary 7.1.** If we know the shape of $v_1(x)$ and $v_2(x)$ from Lemma 5.1, we can determine the angle $\alpha$ of the rotational misalignment of the reference pattern and the vector field from

\[
\alpha = -\frac{1}{2} \arg((u \cdot v_1)(0)). \quad (7.1)
\]

Please note, that knowledge about the other vector field, the counterpart to the pattern, is not necessary and that we usually have analytic information about the pattern, because we generally know what we are looking for.

If we do not have the analytic description of the pattern, we can still use the correlation, because the argument is a more or less good approximation to the true rotational difference $\alpha$.

**Lemma 7.2.** Let the two-dimensional vector field $v(x)$ be linear within and zero outside of an area $A$ symmetric with respect to both coordinate axes. The angle $\phi$ which is the argument of the correlation at the origin with its totally rotated copy $u(x) = R_\alpha(v(R_\alpha(x)))$ satisfies

\[
0 \geq \phi \geq -2\alpha, \quad \text{for} \ \alpha \geq 0,
\]

\[
0 \leq \phi \leq -2\alpha, \quad \text{else}. \quad (7.2)
\]

The proof of Lemma 7.2 is very technical. Figure 4 provides a more fundamental insight of its assertion by exploiting the homomorphism of the rotors in $Cl_{2,0}$ and the complex numbers.
**Proof.** The argument satisfies

\[
\varphi = \arg \left( \int_{\Delta \mathcal{R}_{\alpha}} R_{\alpha} (v(R_{-\alpha}(x))) v(x) \, d^2 x \right) \\
= \arg (e^{-2\alpha e_{12}} \|v_1(x)\|^2 + \|v_2(x)\|^2) \\
= \arctan (\frac{-\sin(2\alpha) \|v_1(x)\|^2}{\cos(2\alpha) \|v_1(x)\|^2 + \|v_2(x)\|^2}),
\]

where \( \|v_1(x)\|^2 = 0 \) and \( \|v_2(x)\|^2 = 0 \) the statement is trivially true, because then \( \varphi = 0 \) or \( \varphi = -2\alpha \). So let \( \|v_1(x)\|^2, \|v_2(x)\|^2 > 0 \). Now we have to make a case differentiation.

1. The assumptions \( \cos(2\alpha) \|v_1(x)\|^2 + \|v_2(x)\|^2 > 0 \) and \( -\sin(2\alpha) \|v_1(x)\|^2 > 0 \) lead to

\[
\varphi = \arctan (\frac{-\sin(2\alpha) \|v_1(x)\|^2}{\cos(2\alpha) \|v_1(x)\|^2 + \|v_2(x)\|^2}) = 2\alpha,
\]

so \( \varphi \) is positive. If we leave out \( \|v_2(x)\|^2 \) the denominator gets smaller. If the denominator \( \cos(2\alpha) \|v_1(x)\|^2 > 0 \) remains positive the positive fraction gets larger and we have

\[
\varphi \leq \arctan (\frac{-\sin(2\alpha) \|v_1(x)\|^2}{\cos(2\alpha) \|v_1(x)\|^2}) = -2\alpha,
\]

with positive \(-2\alpha \) and therefore negative \( \alpha \). If the denominator \( \cos(2\alpha) \|v_1(x)\|^2 \leq 0 \) becomes negative we have \( -2\alpha \in [\frac{\pi}{2}, \pi] \), because of \( -\sin(2\alpha) \|v_1(x)\|^2 > 0 \), so \( \varphi \in (-\frac{\pi}{2}, \frac{\pi}{2}) \) the magnitude of the denominator gets larger so the magnitude of the fraction gets smaller. Since the fraction is negative and the arctangent is monotonic increasing a lower magnitude increases the whole right side and we have

\[
\varphi \geq \arctan (\frac{-\sin(2\alpha) \|v_1(x)\|^2}{\cos(2\alpha) \|v_1(x)\|^2}) = -2\alpha.
\]

2. The assumptions \( \cos(2\alpha) \|v_1(x)\|^2 + \|v_2(x)\|^2 > 0 \) and \( -\sin(2\alpha) \|v_1(x)\|^2 < 0 \) lead to

\[
\varphi = \arctan (\frac{-\sin(2\alpha) \|v_1(x)\|^2}{\cos(2\alpha) \|v_1(x)\|^2 + \|v_2(x)\|^2})
\]

so \( \varphi \) is negative. If we leave out \( \|v_2(x)\|^2 \) the denominator gets smaller. If the denominator \( \cos(2\alpha) \|v_1(x)\|^2 > 0 \) remains positive the negative fraction gets smaller and we have

\[
\varphi \geq \arctan (\frac{-\sin(2\alpha) \|v_1(x)\|^2}{\cos(2\alpha) \|v_1(x)\|^2}) = -2\alpha.
\]

3. The assumptions \( \cos(2\alpha) \|v_1(x)\|^2 + \|v_2(x)\|^2 < 0 \) and \( -\sin(2\alpha) \|v_1(x)\|^2 < 0 \) lead to

\[
\varphi = \arctan (\frac{-\sin(2\alpha) \|v_1(x)\|^2}{\cos(2\alpha) \|v_1(x)\|^2 + \|v_2(x)\|^2} + \pi)
\]

so \( \varphi \) is positive. If we leave out \( \|v_2(x)\|^2 \) the magnitude of the denominator gets larger so the magnitude of the fraction gets smaller. Since the fraction is negative and the arctangent is monotonic increasing a lower magnitude increases the whole right side and we have

\[
\varphi \leq \arctan (\frac{-\sin(2\alpha) \|v_1(x)\|^2}{\cos(2\alpha) \|v_1(x)\|^2}) - \pi = -2\alpha.
\]

Because the numerator is positive and the denominator is negative this equals \(-2\alpha \), which is positive and therefore \( \alpha \) is negative.

4. The assumptions \( \cos(2\alpha) \|v_1(x)\|^2 + \|v_2(x)\|^2 < 0 \) and \( -\sin(2\alpha) \|v_1(x)\|^2 < 0 \) lead to

\[
\varphi = \arctan (\frac{-\sin(2\alpha) \|v_1(x)\|^2}{\cos(2\alpha) \|v_1(x)\|^2 + \|v_2(x)\|^2} - \pi)
\]

so \( \varphi \) is negative. If we leave out \( \|v_2(x)\|^2 \) the magnitude of the denominator gets larger so the magnitude of the fraction decreases. It is positive so the fraction gets smaller, so does the arctangent and the whole right side and we have

\[
\varphi \geq \arctan (\frac{-\sin(2\alpha) \|v_1(x)\|^2}{\cos(2\alpha) \|v_1(x)\|^2}) - \pi = -2\alpha.
\]

Because the numerator and the denominator are negative this equals \(-2\alpha \), which is negative and therefore \( \alpha \) is positive.
Since we covered all possible configurations, we see that \(\alpha\) and \(\phi\) always have different signs. The right estimation for positive \(\alpha\) is a result of the even cases and for negative \(\alpha\) of the odd ones.

**Theorem 7.3.** Let the two-dimensional vector field \(v(x)\) be linear within and zero outside of an area \(A\) symmetric with respect to both coordinate axes and \(\phi : (-\frac{\pi}{2}, \frac{\pi}{2}) \rightarrow (-\frac{\pi}{2}, \frac{\pi}{2})\) be the function defined by the rule
\[
\phi(\alpha) = \arctan((R_{\alpha}(v(R_{-\alpha}))) \times v)(0).
\]

Then the series \(a_0 = \alpha, a_{n+1} = \alpha_n + \phi(\alpha_n)\) converges to zero for all \(\alpha \in (-\frac{\pi}{2}, \frac{\pi}{2})\), if \(||v_1(x)||^2 \neq 0 ||v_2(x)||^2||\).

**Proof.** Lemma 7.2 shows that the series \(a_0 = \alpha, a_{n+1} = \alpha_n + \phi(\alpha_n)\) decreases with respect to its magnitude, because for \(\alpha_n \in (-\frac{\pi}{2}, 0)\) we have \(0 \leq \phi(\alpha_n) \leq -2\alpha_n\) and therefore
\[
a_n = a_n + 0 \leq \alpha_n + \phi(\alpha_n) = a_{n+1},
a_{n+1} = a_n + \phi(\alpha_n) \leq -2\alpha_n = -a_n
\]
and for \(a_n \in (0, \frac{\pi}{2})\) we have \(0 \geq \phi(\alpha_n) \geq -2\alpha_n\) and therefore
\[
a_n = a_n + 0 \geq \alpha_n + \phi(\alpha_n) = a_{n+1},
a_{n+1} = a_n + \phi(\alpha_n) \geq -2\alpha_n = -a_n.
\]

Since the series of magnitudes is monotonically decreasing and bounded from below by zero it is convergent.

Let the limit of the sequence of magnitudes be \(a = \lim_{n \to \infty} |a_n|\) then using the definition of the series and applying the limit leads to
\[
\lim_{n \to \infty} (|a_{n+1}|) = \lim_{n \to \infty} (|a_n + \phi(\alpha_n)|).
\]

The modulus function and \(\phi(\alpha_n)\) are continuous in \(\alpha_n \in (-\frac{\pi}{2}, \frac{\pi}{2})\). That allows us to swap the limit and the functions and write
\[
a \equiv \lim_{n \to \infty} (|a_n|) + \lim_{n \to \infty} (|\phi(\alpha_n)|)
\]
\[
= |a + \phi(a)|.
\]

We apply a case differentiation to the previous equation.

1. For \(a + \phi(a) \geq 0\) it is equivalent to
\[
a = a + \phi(a) \Leftrightarrow \phi(a) = 0.
\]

Since
\[
\phi(\alpha) = \arctan2([-\sin(2\alpha)]|v_1(x)|^2, 
\cos(2\alpha)|v_1(x)|^2 + |v_2(x)|^2)
\]

the claim \(\phi(a) = 0\) is true for \(\cos(2a)|v_1(x)|^2 + |v_2(x)|^2 > 0, -\sin(2a)|v_1(x)|^2 = 0\) which is fulfilled either for \(|v_1(x)|^2 = 0\) and arbitrary \(a\) or for \(|v_1(x)|^2 > 0\) and \(a = 0\).

2. \(a + \phi(a) < 0\) leads to
\[
a = a - \phi(a) \Leftrightarrow \phi(a) = -2a,
\]
which is only fulfilled for \(||v_2(x)||^2 = 0\) and arbitrary \(a\).

Combination of the two cases leads to the proposition \(a = 0\) if \(||v_1(x)||^2 \neq 0 \neq ||v_2(x)||^2||\). Since the sequence of the magnitudes converges to zero the sequence itself converges to zero as well.

\[\square\]

## 8. ALGORITHM AND EXPERIMENTS

The claim \(||v_1(x)||^2 = 0\) means \(v_1(x) = 0\) almost everywhere. For a linear vector field this is equivalent to \(v_1(x) = 0\), analogously \(||v_2(x)||^2 = 0 \Leftrightarrow v_2(x) = 0\). In the case \(v_1(x) = 0\) Lemma 5.1 shows that \(\forall \alpha \in (-\frac{\pi}{2}, \frac{\pi}{2}) : \phi(\alpha) = 0\). An iterative algorithm would stop after one step and return the correct result, because these vector fields are rotational invariant anyway.

In the case \(v_2(x) = 0\) Lemma 5.1 shows that \(\forall \alpha \in (-\frac{\pi}{2}, \frac{\pi}{2}) : \phi(\alpha) = -2\alpha\). The algorithm would alternate between \(-2\alpha\) and zero. That means if the algorithm takes the value zero in the \(\alpha\) variable after its first iteration the underlying vector field must be a saddle \(v(x)\) and the correct misalignment is half the calculated \(\phi\). This exception is handled in Line 11 in Algorithm 1.

**Algorithm 1 Detection of total misalignment of vector fields**

**Input:** vector field: \(v(x)\), rotated pattern: \(u(x)\), desired accuracy: \(\epsilon > 0\),
1. \(\phi = \pi, \alpha = 0, \text{iter} = 0, \text{exception} = \text{false},\)
2. while \(\phi > \epsilon\) do
3. \(\text{iter} += 1,\)
4. \(\text{Cor} = (u(x) \star v(x))(0),\)
5. \(\phi = \arctan(\text{Cor}),\)
6. \(\alpha = \phi \pm \phi,\)
7. if \(\text{iter} = 1\) and \(\alpha = 0\) then
8. \(\alpha = \phi = \pi/4,\)
9. exception = true,
10. end if
11. if \(\text{iter} = 2\) and not \text{exception} and \(\alpha = 0\) then
12. \(\alpha = \phi = \pi/2,\)
13. end if
14. if \(\text{iter} = 2\) and \text{exception} and \(\phi = -\pi/2\) then
15. \(\alpha = \pi/2, \phi = \pi/4,\)
16. end if
17. \(u(x) = e^{-\phi x^2}u(e^{\phi x^2} x),\)
18. end while

**Output:** misalignment: \(\alpha\), corrected pattern: \(u(x)\), iterations needed: \(\text{iter}\).

In the case of \(\alpha = \pm \frac{\pi}{2}\) the correlation will be real valued, compare Theorem 6.1. This case can only appear in
TABLE 1. Results of Algorithm 1 depending on the required accuracy.

| determined accuracy $\epsilon_p$ | 0.1  | 0.01 | 0.001 | 0.0001 | 0.00001 |
|----------------------------------|------|------|-------|--------|--------|
| average error                   | 0.098| 0.015| 0.002 | 0.0002 | 0.00002|
| maximal error                   | 1.927| 0.903| 0.312 | 0.089  | 0.028  |
| average number of iterations    | 4.07 | 16.95| 45.72 | 91.69  | 129.05 |

the first step of the algorithm. It would return the angle zero like in the case where in deed no rotation is necessary. Therefore we need to include another exception handling. We suggest to apply a total rotation by $\frac{\pi}{4}$ to the pattern, if the first step returns $\alpha = 0$, compare Line 7 in Algorithm 1. The disadvantage of this treatment is that it might disturb the alignment in the nice case, when vector field and pattern incidentally match at the beginning, but will guarantee the convergence.

The last exception to be treated appears when both $\alpha \in \{-\frac{\pi}{4}, 0, \frac{\pi}{4}\}$ and $v(x) = v_1(x)$. In this case $\alpha$ gets the value $\pm \frac{\pi}{4}$ from the first exception handling and will alternate between $\pm \frac{\pi}{4}$ for the rest of the algorithm. We fixed this problem in Line 14 in Algorithm 1.

Together with Remark 3.2 this leads to the Corollary.

**Corollary 8.1.** Algorithm 1 returns the correct rotational misalignment for any two-dimensional linear vector field and its totally rotated copy by arbitrary angle.

We practically tested Algorithm 1 applying it to continuous, linear vector fields $\mathbb{R}^2 \to Cl_{2,0}$, that vanish outside the unit square. The angle $\alpha \in (-\pi, \pi]$ and the four coefficients with magnitude not bigger than one describing the vector fields were determined randomly. The results for one million applications can be found in Table 1. The error was measured from the square root of the sum of the squared differences of the determined and the given coefficients. The experiments showed that high numbers of necessary iterations occur when the magnitudes of $v_1$ and $v_2$ differ gravely. Table 1 also implies that the error decreases linearly with the required accuracy while the number of iterations increases sublinearly. But most importantly we could see Algorithm 1 converges in all cases, just as the theory suggested.

**9. CONCLUSIONS AND OUTLOOK**

The geometric cross correlation of two vector fields is scalar and bivector valued. We proved in Lemma 7.2 that for all linear two-dimensional vector fields this rotor has an argument with opposite sign and magnitude less or equal to twice the angle of the misalignment. Therefore application of the encoded rotation to the outer rotated copy of the vector field does not increase the misalignment to its original. In Theorem 7.3 we showed that iterative application completely erases the misalignment of the rotationally misaligned vector fields, if $||v_1(x)|| \neq 0 \neq ||v_2(x)||$. These exceptions could also be treated in Algorithm 1. We implemented it and experimentally confirmed the theoretic results.

Currently we analyze the application of this approach to total rotations of three-dimensional vector fields. For praxis it will be interesting how the algorithm is able to treat vector fields, that are discrete, disturbed, or also dissimilar with respect to translation. The algorithm converges quite fast, if $||v_1(x)||$ and $||v_2(x)||$ do not differ from each other too much. In case one of them dominates it leads to oscillation or deceleration. We think about developing an adaptive algorithm, that estimates the ratio of $||v_1(x)||$ and $||v_2(x)||$ and weights the result respectively. Another way of improving the speed of the algorithm is to use a fast Fourier transform and a geometric convolution theorem [14]. We plan on using the algorithm for vector field registration of real world data and compare it to established algorithms with respect to reliability and runtime.

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