Optimal building block of multipartite quantum battery

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To take quantum advantage of collective effects in many-body system, we design an elementary block for building multipartite quantum battery, which enables charging an atomic ensemble with optimal numbers in a common thermal bath. One achieves maximum free energy as the stored energy in the steady state, which is prior to each atom parallel charging independently. It ascribes to quantum collective effects in the ensemble of atoms induced by the competition between the coherent driving and decoherent dissipation. The corresponding thermodynamic efficiency of the energy storage is analyzed. The existence of the optimal elementary units of multipartite quantum battery provide a guideline for designing a realizable charging scheme.

I. INTRODUCTION

Recently, diligent efforts are devoted to explore the possibility of taking advantage of quantum resources to achieve superior performances in the energy conversion and storage with the control achievement on multipartite quantum system.¹³ Quantum battery (QB) is a quantum system for storing energy supplied by an external source. The battery exploits quantum effects for efficient charging in comparison to its classical counterpart.¹⁴ ¹⁷ A renewed effort is devoted to enhance charging of multipartite batteries in a closed system as a consequence of quantum correlations in many-body systems, which is known as collective effects.¹⁸ ¹⁹

When a multipartite battery is subjected to a common thermal bath, it gives rise to interesting phenomena such as an increasing entropy, which establishes a link to quantum thermodynamics.¹² ¹³ An emergence of collective effects in quantum thermodynamics like the free energy as the stored energy and work extraction are attractive quantum phenomenon in open systems, while the influence of thermodynamics on such quantum effects are overlooked. Many efforts have been devoted to investigate the QB for the energy storage in the thermal environment using different charging protocols.¹⁶ ¹⁸ More recently, a dissipative charging process of a battery was suggested,¹⁰ in which an efficient thermodynamic equilibrium state is approached by work extraction under cyclic unitary operations with help of auxiliary systems. This engineering needs to find a unitary evolution for the battery and the auxiliary system with a globally conserved quantity, which is controlled by a post selected driving agent.

A harmonic driving as an external source has been proposed as an alternative powerful charging field due to the tunable driving frequency for maximal stored energy.² Inspired by the advantage of the collective effects and the harmonic driving, a multipartite QB in a common thermal bath, collectively coupled to a harmonic driving field, is an attractive battery model for optimal energy storage. In such many-body system, the interplay between the coherent driving and decoherent charging induced by the thermal dissipation plays an essential role in the cooperative many-body effects for the charging. The question is whether such collective effects in many-body system be harnessed to improve thermodynamically meaningful features in the driven-dissipative charging protocols.

In this paper, we present the irreversible thermodynamic charging of an ensemble of N two-level atoms charged by a harmonic driving field in a common thermal bath. It is different from previous entropy-preserving or energy-conserving charging process under unitary operations. With the increment of atoms, the quantum collective effects lead to a non-monotonic behavior of free energy and a decreasing entropy per atom, which are induced by the competition between the coherent driving force and the decoherent dissipation of the common thermal bath. We find the optimal number of atoms as an elementary unit of the QB interacting with a common thermal bath, which results in maximizing the stored energy per atom. Engineering such optimal-atom battery as a building block one achieves more free energy by compared to parallel charging for N independent atoms. Meanwhile, thermodynamic efficiency of the energy storage in terms of the work done by the external charging field is analyzed.

II. A PROTOCOL FOR A MULTIPARTITE QB CHARGING

We consider an open charging system of a multipartite battery, which consists of two-level atoms coupled to an external driving as a charger to transfer energy. Fig. 1 (a) shows normal parallel charging strategy with independent thermal bath. Our charging protocol focus on the elementary building block (blue box), illustrated in Fig. 1 (b), with finite number of atoms with shared thermal bath. Atoms in each units are collectively charged by
Dicke states induce the state transitions among energy levels of the J for later discussion. The first term on the right side is the temperature number of bath mode with the frequency where N is two-level atoms in parallel. The importance is to define the usable stored energy. The Boltzmann’s constant is set by kB = 1 for later discussion. The first term on the right side describes the normal-parallel charging for N independent atoms. The basis set for representing the QB system of N atoms is the Dicke states |J, l − N/2⟩ (l = 0, 1, ..., N), which are eigenstates of J^2 and J_z with the total pseudospin J = N/2. Both external driving and the dissipation induce the state transitions among energy levels of the Dicke states |N/2, l − N/2⟩ with l = 0, 1, ..., N.

III. AN OPTIMAL ELEMENTARY UNIT FOR CHARGING

The importance is to define the usable stored energy. In the normal charging without thermal environment, the internal energy change of QB can be utilized in the later retraction. Yet, only part of the internal energy can be extracted in the charging process within the thermal environment. For such evaluation, we have considered the entropy S(ρ_s) = −Tr(ρ_s ln ρ_s) of the QB system. The useful energy stored in the QB is measured by the free energy

\[ F(\rho_s) = E(\rho_s) - k_B T S(\rho_s), \]

where E(ρ_s) = Tr(H_sρ_s) is the internal energy of the QB system. The difference in free energy ΔF = ΔE − k_BTΔS measures the useful energy stored in the QB. At zero temperature T = 0, the free energy change is equivalent to the mean energy, ΔF(ρ_s) = ΔE(ρ_s). Here we only consider the situation with one thermal bath during the charging and later retraction process with the same temperature. The similar definition of the useful work is well discussed in the quantum thermodynamic resource discussions [15].

Initially, N atoms decouple with the charging field, and the QB system is prepared in the Gibbs thermal state of N atoms, ρ_0 = e^{−H_s/(k_BT)}/Z with the partition function Z = Tr[e^{−H_s/(k_BT)}]. For the single-atom QB, the temporal mean energy E(ρ_s) = ω_0⟨σ_z⟩/2 is obtained analytically as (see the appendix)

\[ E(\rho_s)/\omega_0 = \left. \frac{-γ^2χ}{\gamma^2χ^2 + A^2/2} \left[ 1 - \frac{e^{-3γχ}}{2γ^2χ^2} \{ (2γ^2χ^2 + 1 + αχ + α^2)\cos(Ωt) \} \right] \right| \]

where the oscillation Rabi frequency is Ω = \sqrt{A^2 − γ^2χ^2}/4 with χ = [1 + 2n(T)]. The mean energy E of the QB becomes larger as the driving amplitude A increases or the dissipation rate γ decreases. At the zero temperature with n(T) = 0, when the driving strength A becomes larger than the dissipative rate γ, E tends to be zero in the steady state and the corresponding stored energy ΔE equals to ω_0/2. (See Appendix A)

For the large value of N, we use the Holstein-Primakoff transformation in terms of auxiliary bosonic operators b^† and b: J_z = b^†b − N/2, J_+ = b^† √N and J_− = b √N. It gives dq(t)/dt = −i[ω_0(b^†b − N/2) + A√N cos(ωt)(b^† + b), ρ] + γN|n(T) + 1|2(b^†bρ − b^†bρ) + γN|n(T) + 1|2(b^†bρ − b^†bρ). The driving strength is proportional to A√N, while the decay rate is proportional to γN. Due to the competition between the external driving force and the thermal dissipation, one would expect the existence of the efficient elementary unit of the QB with optimal number of atoms for maximizing the stored energy.

Due to the difficulty of finding analytical results for the multipartite batteries, we present dynamical charging process via the free energy and the entropy by numerical calculations. Fig. 2(a) shows oscillations of the scaled free energy change ΔF/N, which exhibits a stable value at the steady state. The charging period for

![FIG. 1: (a) Charging protocol of N two-level atoms in parallel. Each atom is charged by a harmonic driving and couples to a thermal environment. (b) The elementary building block (blue box) consists of a few atoms with a common thermal bath. During the charging time, the QB interacts with a harmonic driving field A cos(ωt).](image-url)
the steady state becomes shorter as the atom number $N$ increases, because the system relaxes to the steady state rapidly due to the dominated dissipation. Interesting, the stored free energy in the steady state increases firstly and then decreases as the atom number $N$ increases, because the system relaxes to the steady state as a function of $N$ for different dissipation rate $\gamma = 0, 0.03\omega, 0.05\omega$ and $0.1\omega$ with $A = 0.5\omega$.

One expects a steady state with larger population in higher-energy states with the increasing of dissipation $\gamma$, yet the occupation of the lower-energy states would result in a non-monotonic dependence of the free energy $F$ on the number of atoms $N$ in the steady state, exhibiting a maximum value at an optimal number $N_{op}$ of atoms. Especially, in the absence of the dissipation $\gamma = 0$, the free energy is proportional to $N$, $\Delta F \propto N$, which is consistent with previous results [9,10]. For different dissipation rate $\gamma$ and the driving strength $A$, the efficient elementary unit of the QB consists of the optimal number $N_{op}$ of atoms in a common thermal bath in Fig. 1(b), which has maximum stored energy by comparing to results of atoms charging independently in parallel.

**IV. CHARGING EFFICIENCY**

In the charging process, the stored free energy is an important quantity to evaluate the performance. However, we have shown the energy from the agent dissipates into the thermal bath. The extent, to which the work done by the agent is stored, is also important and typically evaluated as the efficiency. We quantify the efficiency of the charging process by the ratio

$$\eta = \frac{\Delta F}{W},$$

where the amount of the work done on the QB by the external driving field is given by

$$W = \text{Tr} \left( \int_0^\tau \frac{dH}{dt} \rho dt \right) = -A\omega \int_0^\tau \sin(\omega t) \text{Tr}(J_x \rho) dt.$$

Due to the thermal dissipation of the irreversible process, some amount of work $W$ is transferred into heat $Q$ flowing into the thermal bath, which is measured by $Q = \text{Tr} \left( \int_0^\tau \rho Hdtdt \right)$. The intrinsic energy of the charging system including the interactions driving part $H_I$ is measured as $\langle H \rangle = \text{Tr} [H(\rho(t))] = E + \langle H_I \rangle$, which is different from the mean energy $E$ of the QB system.
Fig. 3 shows behaviors of the work $W$ done by the external driving force, the heat flowing to the common bath $Q$. For a closed system with $\gamma = 0$ in Fig. 3(a), the work $W$ is fully transferred into the variance of the intrinsic energy $\Delta(H)$ periodically without heat dissipation. One interesting finding is that the variance of the intrinsic energy $\Delta(H)$ sometimes is lower than the variance of free energy $\Delta F = \Delta E$ due to the negative value of the interacting energy $\Delta(H’)$, resulting in the efficiency $\eta > 1$. Such efficiency is caused by the energy exchange between the coherent driving field and the QB. For the dissipation case with $\gamma = 0.03\omega$ in Fig. 3(b), the laws of thermodynamics, $W = Q + \Delta H$, holds in the open charging system as well as the closed system. The corresponding efficiency $\eta$ decreases as the number $N$ of atoms for an elementary unit of the QB increases in Fig. 3(c). The efficiency for the efficient elementary unit with the optimal number of atoms $N_{op} = 17$ is worse than that of single-battery case, which ascribes to heat dissipation.

V. CONCLUSION

We have shown the driven-dissipative charging of the multipartite battery comprising an ensemble of two-level atoms, which interact with an external harmonic driving and a common thermal bath. The stored energy in the battery is measured by the free energy due to increasing of the entropy in the open charging system. Due to the competition between the coherent driving and the decoherent dissipation in atoms system, the free energy per atom in the steady state behaves non-monotonically dependent on the number of the atoms. We find the optimal elementary unit of the QB with finite atoms in a common thermal bath, which provides maximal average stored energy. To characterize the thermodynamic efficiency of the charging process, we elucidate the work done by the external field as well as the intrinsic energy of the QB system. We prove the first law of thermodynamics holds in the open charging system. The quantum thermodynamics in the open charging system involves quantum collective effects induced by quantum coherence in many-body system in comparison to classical thermodynamics, resulting in optimal building block of multipartite quantum battery. Our results is a fundamental attributions to the powerful energy storage with optimal atoms for physically realizable charging schemes.

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[1] P. Strasberg, G. Schaller, T. Brandes, and M. Esposito, Phys. Rev. X 7, 021003 (2017).
[2] G. Watanabe, B. P. Venkatesh, P. Talkner, and A. del Campo, Phys. Rev. Lett. 118, 050601 (2017).
[3] M. Campisi, P. Hänggi, and P. Talkner, Rev. Mod. Phys. 83, 1653 (2011).
[4] K. V. Hovhannisyan, M. Perarnau-Llobet, and A. Acín, Phys. Rev. Lett. 111, 240401 (2013).
[5] G. M. Andolina, M. Kecl, A. Mari, M. Campisi, V. Giovannetti, and M. Polini, Phys. Rev. Lett. 122, 047702 (2019).
[6] N. Friis, M. Huber, and M. Perarnau-Llobet, Phys. Rev. E 93, 042135 (2016).
[7] N. Friis, and M. Huber, Quantum 2, 61 (2018).
[8] F. Campaioli, F. A. Pollock, F. C. Binder, L. Cleri, J. Goold, S. Vinjanampathy, and K. Modi, Phys. Rev. Lett. 118, 150601 (2017).
[9] Y. Y. Zhang, T. R. Yang, L. B. Fu, and X. G. Wang, Phys. Rev. E 99, 052106 (2019).
[10] D. Ferraro, M. Campisi, G. M. Andolina, V. Pellegrini, and M. Polini, Phys. Rev. Lett. 120, 117702 (2018).
[11] F. Binder, S. Vinjanampathy, K. Modi, and J. Goold, New. J. Phys. 17, 075015 (2015).
[12] P. Faist, F. Dupuis, J. Oppenheim, and R. Renner, Nat. Commun. 6, 7669 (2015).
[13] P. Skrzypack, A. J. Short, and S. Popescu, Nat. Commun. 5, 4185 (2014).
[14] M. N. Bera, A. Riera, M. Lewenstein, and A. Winter, Nat. Commun. 8, 2180 (2017).
[15] Y. H. Ma, R. X. Zhai, C. P. Sun, and H. Dong, arXiv: 1910.13434.
[16] F. Barra, Phys. Rev. Lett. 122, 210601 (2019).
[17] D. Farina, G. M. Andolina, A. Mari, M. Polini, and V. Giovannetti, Phys. Rev. B 99, 035421 (2019).
[18] F. Pirmoradian, and K. Mølmer, Phys. Rev. A 100, 043833 (2019).
[19] R. Alicki, J. Phys. A Math. Gen. 12, L103-L107 (1979).
oscillating terms. The Bloch equations are derived as
\[
\langle \sigma_z \rangle_\tau = \frac{A}{2} \langle (\sigma_- - \langle \sigma_+ \rangle_\tau) - 2\gamma \langle (\sigma_z \rangle_\tau (2n(T) + 1) + 1, 
\]
\[
\langle \sigma_+ \rangle_\tau = i\frac{A}{4} \langle (\sigma_z \rangle_\tau - \gamma \langle (\sigma_-) \rangle_\tau (2n(T) + 1), 
\]
\[
\langle \sigma_- \rangle_\tau = -i\frac{A}{4} \langle (\sigma_z) \rangle_\tau - \gamma \langle (\sigma_+ \rangle_\tau (2n(T) + 1). 
\]

For the initial Gibbs state with \( \alpha = T \rho \sigma_z[0] \), it gives analytically
\[
\langle \sigma_z \rangle_\tau = -\frac{2\gamma^2 \eta}{2\gamma^2 \eta^2 + A^2} \left[ 1 - e^{-3\gamma \gamma/2} + \frac{1}{2}\gamma \right] 
\]
\[
\left[ (2\gamma^2 \eta^2 (1 + \eta \gamma) + \alpha A^2) \cos(\Omega \tau) \right.
\]
\[
-\frac{\gamma}{2\gamma} \left[ 2\gamma^2 \eta^2 (1 + \eta \gamma) + A^2(4 + \eta \gamma) \right] \sin(\Omega \tau)
\]
\[
\]
where \( \eta = [1 + 2n(T)] \), and the oscillation Rabi frequency is
\[
\Omega = \sqrt{A^2 - \gamma^2/4}. 
\]
Especially, at zero temperature with \( n(T) = 0 \) for \( \langle \sigma_z \rangle = -1, \langle \sigma_- \rangle = 0, \langle \sigma_+ \rangle = 0 \), the corresponding temporal energy \( E(\rho_s) = \omega_0 \langle \sigma_z \rangle \tau / 2 \) is obtained
\[
E(\rho_s) = \frac{-4\omega_0 \gamma^2}{8\gamma^2 + A^2} \left[ 1 + \frac{A^2}{8\gamma^2} e^{-3\gamma \gamma/2} (\cos(\Omega \tau) + \frac{3\gamma}{2\Omega} \sin(\Omega \tau)) \right],
\]
with \( \Omega = \sqrt{A^2 - \gamma^2/4} \). When the driving strength \( A \) becomes larger than the dissipative rate \( \gamma \), \( \langle \sigma_z \rangle_{\tau} \) tends to be zero and the stored energy \( \Delta E \) equals to \( \omega_0 / 2 \).

We calculate \( E(\rho_s) \) according to Eq. (A1) numerically without the RWA for the dissipative charging process in Fig. A.1. The analytical solutions in Eq. (A8) agree well with numerical ones for \( A / \omega_0 = 0.05 \) and 0.5. And the oscillation frequency of the mean energy are correctly captured by the analytical one \( \Omega \) dependent on the driving strength \( A \) and dissipative rate \( \gamma \). As \( A \) increases to be much larger than \( \gamma \), \( A = 0.5 \omega_0 \), \( E(\rho_s) \) increases to become zero in the steady state in Fig. A.1(b), which is consistent with the analytical ones from Eq. (A8).

For a thermal bath with \( n(T) = 1 \), excellent agreement between the analytical results and the numerical ones is observed in Fig. A.1(c) and (d). As the temperature increases, the oscillation Rabi frequency \( \Omega \) becomes smaller in Eq. (A7) due to the stronger thermal dissipation. It demonstrates that the charging time to reach the steady state becomes shorter at finite temperature, and the oscillation Rabi frequency of the QB energy \( E \) can be modulated by the driving strength \( A \), the loss rate \( \gamma \) and the temperature \( T \).

Fig. A.2 shows the thermodynamics charging controlled by the temperature. Since the decay rate of the

**Appendix A: Analytical solutions for single-atom battery**

We analyze the thermodynamics of a single-atom battery charging process with the master equation. The two-level system evolution involves energy transfer from the external driving field and the dissipation of the energy into the thermal environment.

The master equation of the single-atom battery system becomes
\[
d\rho_s/dt = -i\omega_0 \rho_s - iA/4 (e^{i\omega t} + e^{-i\omega t}) \rho_s 
\]
\[
+ \gamma n(T) + 1 \{2 \rho_s \sigma_+ - \rho_s \sigma_- \} 
\]
\[
\rho_s \sigma_+ \rho_s - \{ \sigma_- \sigma_+ \}, \quad \text{(A1)}
\]

We perform a rotating-frame transformation using \( U = \exp(i\omega t \sigma_z / 2) \) to give
\[
d\rho'_s/dt = -i\omega_0 \rho'_s - iA/2 \rho'_s 
\]
\[
- iA/4 (e^{i\omega t} + e^{-i\omega t}) [e^{i\omega t} \sigma_+ + e^{-i\omega t} \sigma_- \rho'_s] 
\]
\[
+ \gamma n(T) + 1 \{2 \rho'_s \sigma_+ - \rho'_s \sigma_- \} 
\]
\[
\rho'_s \sigma_+ \rho'_s - \{ \sigma_- \sigma_+ \}, \quad \text{(A2)}
\]
where \( \rho'_s = U \rho_s U^\dagger \). When the driving strength \( A \) is much smaller than the two-level energy \( \omega_0 \) on resonance with the QB system \( \omega_0 = \omega \), it is reasonable to making a rotating-wave approximation (RWA) by ignoring fast

**FIG. A.1:** Scaled mean energy \( E/\omega_0 \) obtained by the analytical solution (dashed black line) as a function of \( \tau \) for one-atom quantum battery with different driving strength (a)(c) \( A/\omega_0 = 0.05 \), and (b)(d) \( A/\omega_0 = 0.5 \) at temperature \( n(T) = 0 \) and 1. The driving frequency is chosen as \( \omega = \omega_0 = 2 \), and the dissipation rate is \( \gamma = 0.01 \omega_0 \). The numerical results are listed for comparison (solid red line).
two-level atom is proportional to the rate $\gamma [n(T) + 1]$ and $\gamma n(T)$ in Eq. (A1). As $n(T)$ increases, the mean energy of the QB system and the free energy decay rapidly as in Fig. A.2(a). The corresponding variance of the entropy $\Delta S$ decreases in Fig. A.2(b). The increasing of the temperature induces more energy flowing to the environment, while it reduces the information flow. In the entropy non-preserving process, the heat flow to the thermal bath decreases of the energy of the QB system due to the entropy production. In the absence of the dissipation $\gamma = 0$, it exhibits periodic oscillation of the free energy $\Delta F$ and the entropy is conserved in Fig. A.2(c) (d).

FIG. A.2: (a) Scaled free energy $\Delta F/\omega_0$ and (b) the Von Neumann entropy $\Delta S$ of one-atom quantum battery for different temperature $n(T) = 0.1, 1$ with dissipation rate $\gamma = 0.05\omega$. (c) Scaled free energy $\Delta F/\omega_0$ and (d) the Von Neumann entropy for different dissipative rate $\gamma = 0$ and $0.05\omega$ with $n(T) = 0.2$. The driving frequency is chosen as $\omega = \omega_0 = 2$, and the driving strength $A/\omega_0 = 0.5$. 