The theorems of Caratheodory and Gluskin for $0 < p < 1$

by

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1. Introduction and notation.

Throughout the paper $X$ will denote a real vector space and $p$ will be a real number, $0 < p < 1$. A set $A \subseteq X$ is called $p$-convex if $\lambda x + \mu y \in A$, whenever $x, y \in A$, and $\lambda, \mu \geq 0$, with $\lambda^p + \mu^p = 1$. Given $A \subseteq X$, the $p$-convex hull of $A$ is defined as the intersection of all $p$-convex sets that contain $A$. Such set is denoted by $p\text{-conv}(A)$.

A $p$-norm on $X$ is a map $\| \cdot \| : X \to \mathbb{R}$ verifying:

(i) $\| x \| \geq 0, \forall x \in X$ and $\| x \| = 0 \iff x = 0$.

(ii) $\| ax \| = |a| \| x \|, \forall a \in \mathbb{R}, x \in X$.

(iii) $\| x + y \|^p \leq \| x \|^p + \| y \|^p, \forall x, y \in X$.

We will say that $(X, \| \cdot \|)$ is a $p$-normed space. The unit ball of a $p$-normed space is a $p$-convex set and will be denoted by $B_X$.

We denote by $\mathcal{M}_n^p$ the class of all $n$-dimensional $p$-normed spaces. If $X, Y \in \mathcal{M}_n^p$ the Banach-Mazur distance $d(X, Y)$ is the infimum of the products $\| T \| \cdot \| T^{-1} \|$, where the infimum is taken over all the isomorphisms $T$ from $X$ onto $Y$.

We shall use the notation and terminology commonly used in Banach space theory as it appears in [Tmcz].

In this note we investigate some aspects of the local structure of finite dimensional $p$-Banach spaces. The well known theorem of Gluskin gives a sharp lower bound of the diameter of the Minkowski compactum. In [Gl] it is proved that $\text{diam}(\mathcal{M}_n^p) \geq cn$ for some absolute constant $c$. Our purpose is to study this problem in the $p$-convex setting. In [Pe], T. Peck gave an upper bound of the diameter of $\mathcal{M}_n^p$ namely, $\text{diam}(\mathcal{M}_n^p) \leq n^{2/p-1}$. We will show that such bound is optimum.

The method used by Gluskin to prove his result can be directly applied, with some minor variations, to our case. At some point of the proof it is necessary to find some volumetric estimates for convex envelopes. In particular if $\{P_i\}_{i=1}^m$ are $m$ points in the euclidean sphere in $\mathbb{R}^n$ we need to estimate from above $\left( \frac{|\text{conv} \{\pm P_i\}|}{|B_{\ell^p_2}|} \right)^{1/n}$.

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Szarek, [Sz], and other authors gave the estimate \( \left( \frac{\text{conv} \left\{ \pm P_i \right\}}{|B_{\ell^2}|} \right)^{1/n} \leq Cmn^{-3/2} \) (\( C \) is an absolute constant). Caratheodory’s convexity theorem turned out to be an important ingredient of the proof. For \( p < 1 \) we will proceed in this fashion and so we will need to have the corresponding version of Caratheodory convexity theorem.

The main results of the paper are the following:

**Theorem 1.** Let \( A \subseteq \mathbb{R}^n \) and \( 0 < p < 1 \). For every \( x \in p\text{-conv} \,(A), x \neq 0 \) there exist linearly independent vectors \( \{P_1 \ldots P_k\} \subseteq A \) with \( k \leq n \), such that \( x \in p\text{-conv} \{P_1 \ldots P_k\} \). Moreover, if \( 0 \in p\text{-conv} \,(A) \), there exits \( \{P_1 \ldots P_k\} \subseteq A \) with \( k \leq n + 1 \) such that \( 0 \in p\text{-conv} \{P_1 \ldots P_k\} \).

**Theorem 2.** Let \( 0 < p < 1 \). There exits a constant \( C_p > 0 \) such that for every \( n \in \mathbb{N} \)
\[
C_p n^{2/p-1} \leq \text{diam} (\mathcal{M}_n^p) \leq n^{2/p-1}.
\]

The first result can be viewed as the \( p \)-convex analogue of Caratheodory’s theorem. Apparently, the result for \( p < 1 \) is stronger than the Caratheodory’s one in the sense that we get \( k \leq n \) and only \( k \leq n + 1 \) can be assured for \( p = 1 \) (see [Eg], pg 35). Proposition 3 ii) will show that this is not such since vector 0 plays a particularly special role.

The second result is the analogue of Gluskin’s theorem in the \( p \)-convex setting, that is the diameter of the Minkowski compactum grows asimptotically like \( n^{2/p-1} \).

2. **Caratheodory’s theorem for \( p \)-convex hulls.**

In this section we want to prove Theorem 1. We begin by recalling the first properties of \( p \)-convex hulls. They are probably known but since we have not found them in any reference we sketch their proofs.

**Proposition 3.** For every \( \emptyset \neq A \subseteq X \).

i) \( p\text{-conv} \,(A) = \left\{ \sum_{i=1}^{n} \lambda_i x_i \mid \lambda_i \geq 0, \sum_{i=1}^{n} \lambda_i^p = 1, x_i \in A, n \in \mathbb{N} \right\} \).

ii) \( p\text{-conv} \,(A \cup \{0\}) = \{0\} \cup p\text{-conv} \,(A) = \left\{ \sum_{i=1}^{n} \lambda_i x_i \mid \lambda_i \geq 0, \sum_{i=1}^{n} \lambda_i^p \leq 1, x_i \in A, n \in \mathbb{N} \right\} \).

iii) \( p\text{-conv} \,(T(A)) = T(p\text{-conv} \,(A)) \), for any linear map \( T \).

**Proof:** i) and iii) are straightforward.

ii) We only have to prove that \( p\text{-conv} \,(A \cup \{0\}) \subseteq \{0\} \cup p\text{-conv} \,(A) \). It is enough to show that every non zero element \( x \) of the form \( x = \sum_{i=1}^{n} \lambda_i x_i, x_i \in A, \sum_{i=1}^{n} \lambda_i^p < 1 \) can be written as \( x = \sum_{i=1}^{m} \mu_i y_i, y_i \in A, \sum_{i=1}^{m} \mu_i^p = 1 \).
Suppose $\lambda_1 \neq 0$. Write $\lambda_1 = \sum_{i=1}^{k} \beta_i$, with $\beta_i \geq 0$. We have

$$\sum_{i=1}^{n} \lambda_i^p \leq \sum_{i=1}^{k} \beta_i^p + \sum_{i=2}^{n} \lambda_i^p \leq k^{1-p} \lambda_1^p + \sum_{i=2}^{n} \lambda_i^p.$$ 

It is now clear, by a continuity argument, that we can find $k$ and $\beta_i \geq 0$, $1 \leq i \leq k$, such that $\lambda_1 = \sum_{i=1}^{k} \beta_i$ and $\sum_{i=1}^{k} \beta_i^p + \sum_{i=2}^{n} \lambda_i^p = 1$. Finally, the representation $x = \sum_{i=1}^{k} \beta_i x_i + \sum_{i=2}^{n} \lambda_i x_i$ does the job. ///

Remark. In particular ii) says that for every $0 \neq x \in X$, $p$-conv $\{x\} = (0, x] = \{\lambda x; 0 < \lambda \leq 1\}$. This situation is rather different from the case when $p = 1$.

Another useful particular case of ii) is the following: If $A = \{P_1, \ldots, P_n\} \subset X, P_i \neq 0, P_i \neq P_j, \forall 1 \leq i \neq j \leq n$, then $0 \neq x \in p$-conv $(A)$ $\Rightarrow x = \sum_{i=1}^{n} \lambda_i P_i$, with $\sum_{i=1}^{n} \lambda_i^p \leq 1, \lambda_i \geq 0$. Observe that we allow no more than $n$ non-zero summands while in i) and ii) there is no restriction.

Next, we are going to prove a particular case of Theorem 1, which will help us in the general case.

Lemma 4. Let $\{P_1 \ldots P_n, Q\} \subseteq \mathbb{R}^n$ with $\{P_i\}_{i=1}^{n}$ linealy independent. If $M \in p$-conv $\{P_1 \ldots P_n, Q, 0\}$ then there exist $P_1 \ldots P_n \in \{P_1 \ldots P_n, Q\}$ such that $M \in p$-conv $\{P_1 \ldots P_n, 0\}$.

Proof: By Proposition 3 iii), it’s enough to consider the case $\{e_1 \ldots e_n, Q, 0\}$ where $\{e_i\}_{i=1}^{n}$ is the canonical basis in $\mathbb{R}^n$ and $Q = (a_1 \ldots a_n) \neq 0$. Denote by $\mathcal{P}$ the subset of $p$-conv $\{e_1 \ldots e_n, Q, 0\}$ for which the thesis of the Lemma holds.

Let

$$K = \{ (\lambda_1 \ldots \lambda_n) \in \mathbb{R}^n \mid \sum_{i=1}^{n} \lambda_i^p \leq 1, \lambda_i \geq 0, 1 \leq i \leq n \}$$

Write $\mu = \mu(\lambda_1 \ldots \lambda_n) = (1 - \sum_{i=1}^{n} \lambda_i^p)^{1/p}$, and consider the map $\varphi: K \rightarrow \mathbb{R}^n$ defined as $\lambda = (\lambda_1 \ldots \lambda_n) \rightarrow \varphi(\lambda) = \sum_{i=1}^{n} \lambda_i e_i + \mu Q$. Denote by $J\varphi(\lambda)$ the Jacobian of the function $\varphi$ at a point $\lambda$.

The proof of the Lemma rests on the following:

Claim. For every $\lambda \in \text{Int}(K)$ such that $J\varphi(\lambda) = 0$ we have $\varphi(\lambda) \in \mathcal{P}$.

Assume the claim is true and continue with the proof of the Lemma.
Let $M \in p\text{-conv}\{e_1 \ldots e_n, Q, 0\}$ i.e. $M = \sum_{i=1}^n \lambda_i e_i + \nu Q$, with $\lambda_i, \nu \geq 0$, $1 \leq i \leq n$, $\sum_{i=1}^n \lambda_i^p + \nu^p \leq 1$.

Suppose first that $\sum_{i=1}^n \lambda_i^p + \nu^p = 1$, that is $M = \varphi(\lambda), \lambda \in K$. If $\lambda \in \partial(K)$ then, either $\lambda_i$ or $\nu$ are equal to 0 and clearly $\varphi(\lambda) \in \mathcal{P}$. If $\lambda \in \text{Int}(K)$, we also have two possibilities: a) $J \varphi(\lambda) = 0$ and the claim says that $M \in \mathcal{P}$ or b) $J \varphi(\lambda) \neq 0$. By the inverse function theorem we necessarily have that $M \in \text{Int} \mathcal{P}$. Since $\varphi(K)$ is compact there exists $t > 1$ such that $tM \in \partial \varphi(K)$, and therefore $tM = \varphi(\lambda')$ with either $\lambda' \in \partial(K)$ or $\lambda' \in \text{Int}(K)$ and $J \varphi(\lambda') = 0$. In any case we deduce that $tM$ belongs to $\mathcal{P}$ and so does $M$.

If, on other hand, $\sum_{i=1}^n \lambda_i^p + \nu^p = s^p < 1$ the results easily follows by considering $\frac{M}{s}$ and applying the preceding case. ///

**Proof of the Claim:** It is an easy exercise to show that the Jacobian of $\varphi$ is

$$J \varphi(\lambda) = 1 - \sum_{i=1}^n a_i \left(\frac{\mu}{\lambda_i}\right)^{1-p}.$$

For every $\lambda \in \text{Int}(K)$ we write $\lambda = tv, v = (v_1 \ldots v_n), v_i > 0, 1 \leq i \leq n, 0 < t < 1, \sum_{i=1}^n v_i^p = 1$. We have $J \varphi(\lambda) = 0$ if and only if

$$\left(\frac{\mu}{\lambda_i}\right)^{1-p} \sum_{i=1}^n a_i \frac{1}{v_i^{1-p}} = 1 \quad \text{and} \quad \mu^p = 1 - t^p.$$

Write $R = \left(\sum_{i=1}^n \frac{a_i}{v_i^{1-p}}\right)^{1-p} > 0$. It is easy to see that for every $v$, there is a unique $t \in (0, 1)$ such that $J \varphi(tv) = J \varphi(\lambda) = 0$. Explicitly $\lambda = \frac{R}{(1 + R^p)^{1/p}} v$. Therefore with this new notation, the points $\lambda$ with $J \varphi(\lambda) = 0$ are such that $M = \sum_{i=1}^n \frac{Rv_i + a_i}{(1 + R^p)^{1/p}} e_i$

where $v_i > 0, 1 \leq i \leq n, \sum_{i=1}^n v_i^p = 1, R^{1-p} = \sum_{i=1}^n \frac{a_i}{v_i^{1-p}} > 0$.

**Case 1.** If $Rv_i + a_i \geq 0$ for all $i$, then $M \in p\text{-conv}\{e_1 \ldots e_n, 0\}$. Indeed, let’s show that

$$\sum_{i=1}^n (Rv_i + a_i)^p < 1 + R^p$$

This is equivalent to

$$\sum_{i=1}^n \left(v_i + \frac{a_i}{R}\right)^p - \sum_{i=1}^n v_i^p - \frac{1}{R} \sum_{i=1}^n \frac{a_i}{v_i^{1-p}} < 0$$
and to
\[ \sum_{i=1}^{n} v_i^p \left( 1 + \frac{a_i}{Rv_i} \right)^p - \sum_{i=1}^{n} v_i^p \left( 1 + \frac{a_i}{Rv_i} \right) < 0. \]

But this is obvious by the elementary inequality:
\[ (1 + x)^p \leq 1 + px, \quad x \geq -1 \tag{\ast} \]

**Case 2.** If there is some \( i, 1 \leq i \leq n \) such that \( Rv_i + a_i < 0 \), then \( a_i < 0 \). We suppose without loss of generality that \( \min \{ a_i/v_i \mid 1 \leq i \leq n \} \) is achieved at \( i = 1 \).

We shall prove \( M \in p\text{-conv} \{ e_2 \ldots e_n, Q, 0 \} \). Recall that \( M = \sum_{i=1}^{n} \frac{Rv_i + a_i}{(1 + R^p)^{1/p} e_i} \). We will show that \( M \) can be represented as \( M = \sum_{i=2}^{n} \alpha_i e_i + \beta Q \) with \( \sum_{i=2}^{n} \alpha_i^p + \beta^p < 1 \).

Comparing the two representations, it is easy to see that
\[ \alpha_i = \left( v_i - \frac{v_1 a_i}{a_1} \right) \frac{R}{(1 + R^p)^{1/p}}, \quad 2 \leq i \leq n \]
\[ \beta = \left( 1 + \frac{Rv_1}{a_1} \right) \frac{1}{(1 + R^p)^{1/p}}. \]

By hypothesis we have \( \beta, \alpha_i \geq 0, \quad 2 \leq i \leq n \). It remains to show that \( \sum_{i=2}^{n} \alpha_i^p + \beta^p < 1 \), which is the same as
\[ \sum_{i=2}^{n} \left( v_i - \frac{v_1 a_i}{a_1} \right)^p + \left( 1 + \frac{Rv_1}{a_1} \right)^p \frac{1}{R^p} < 1 + \frac{1}{R} \sum_{i=1}^{n} \frac{a_i}{v_i^{1-p}} \]
or
\[ \left( \sum_{i=1}^{n} \frac{a_i}{Rv_i^{1-p}} \right) \left( 1 + \frac{Rv_1}{a_1} \right)^p + \sum_{i=2}^{n} v_i^p \left( 1 - \frac{v_1 a_i}{a_1 v_i} \right)^p - \sum_{i=1}^{n} v_i^p \left( 1 + \frac{a_i}{Rv_i} \right) < 0 \]

Again (\ast) establishes \( \left( 1 + \frac{Rv_1}{a_1} \right)^p \leq 1 + p \frac{Rv_1}{a_1} \) and \( \left( 1 - \frac{v_1 a_i}{a_1 v_i} \right)^p \leq 1 - p \frac{v_1 a_i}{a_1 v_i} \)
and the result easily follows.

We are now ready to state and prove the main theorem of the section.

**Theorem 1.** Let \( A \subseteq \mathbb{R}^n \) and \( 0 < p < 1 \). For every \( x \in p\text{-conv} \ (A), x \neq 0 \) there exist linearly independent vectors \( \{ P_1 \ldots P_k \} \subseteq A \) with \( k \leq n \), such that \( x \in p\text{-conv} \ {P_1 \ldots P_k} \). Moreover, if \( 0 \in p\text{-conv} \ (A) \), there exits \( \{ P_1 \ldots P_k \} \subseteq A \) with \( k \leq n + 1 \) such that \( 0 \in p\text{-conv} \ {P_1 \ldots P_k} \).

**Proof:** Let \( x \in p\text{-conv} \ (A), x \neq 0 \), then \( x = \sum_{i=1}^{N} \lambda_i P_i \) with \( P_i \in A, P_i \neq 0, \sum_{i=1}^{N} \lambda_i^p \leq 1, \lambda_i > 0 \) and \( 1 \leq i \leq N \). Let \( \dim(\text{span} \{ P_i \}) = m \leq n \). By
Proposition 3 iii) and without loss of generality, we can suppose that we are in $\mathbb{R}^m$ and that $x = \sum_{i=1}^{N} \lambda_i P_i$ with $P_1 \ldots P_m$ linearly independent.

Write $s^p = \sum_{i=1}^{m+1} \lambda_i^p$ and $\tilde{x} = \sum_{i=1}^{m+1} \frac{\lambda_i}{s} P_i$. Clearly $\tilde{x} \in p$-conv $\{P_1 \ldots P_{m+1}\}$ and therefore, by Lemma 4 there exists $\{P_{k_1} \ldots P_{k_m}\} \subset \{P_1 \ldots P_{m+1}\}$ such that $\tilde{x} = \sum_{i=1}^{m} \beta_i P_{k_i}, \sum_{i=1}^{m} \beta_i^p \leq 1$. Hence

$$x = s\tilde{x} + \sum_{i=m+2}^{N} \lambda_i P_i = \sum_{i=m+2}^{m} s\beta_i P_{k_i} + \sum_{i=m+2}^{N} \lambda_i P_i$$

with $\sum_{i=1}^{m} \beta_i^p s^p + \sum_{i=m+2}^{N} \lambda_i^p \leq s^p + \sum_{i=m+2}^{N} \lambda_i^p \leq 1$.

We have represented $x$ as a combination of points of $A$ of length $N - 1$. Consider now, span$\{P_{k_1} \ldots P_{k_m}, P_{m+2} \ldots P_N\}$ and repeat the argument until reaching a representation of length $\leq n$.

If $0 \in p$-conv $(A)$ then $0 = \sum_{i=1}^{N} \lambda_i P_i, P_i \in A, \lambda_i > 0, 1 \leq i \leq N$ and $\sum_{i=1}^{N} \lambda_i^p = 1$. As before, we can suppose $P_1 \ldots P_m$ linearly independent with $m \leq n$. We consider $\sum_{i=1}^{m+1} \lambda_i P_i = - \sum_{i=m+2}^{N} \lambda_i P_i$. If we apply Lemma 4 to $\tilde{x} = \sum_{i=1}^{m+1} \frac{\lambda_i}{s} P_i, s^p = \sum_{i=1}^{m+1} \lambda_i^p$ we obtain

$$\sum_{i=1}^{m} \beta_i P_{k_i} = - \sum_{i=m+2}^{N} \lambda_i P_i$$

with $\sum_{i=1}^{m} \beta_i^p \leq 1$. Hence $0 \in p$-convex envelope of $N - 1$ points. Repeat the argument until reaching a representation of length $\leq n + 1$. ///

3. Gluskin’s theorem for $0 < p < 1$.

In this section we are going to prove Theorem 2. As quoted above, Peck showed that diam $(\mathcal{M}_n^p) \leq n^{2/p-1}$. Given an $n$-dimensional $p$-normed space $X$, he considered its Banach envelope $X^b$ (the normed space whose unit ball is the convex envelope of the unit ball of $X$) and proved $d(X, X^b) \leq n^{1/p-1}$ (see [Pe] or [G-K]). By using John’s theorem he obtained the estimate. We want to prove that this result is sharp. More precisely what we are going to show is

**Theorem 2.** Let $0 < p < 1$. There exits a constant $C_p > 0$ such that for every $n \in \mathbb{N}$

$$C_p n^{2/p-1} \leq \text{diam}(\mathcal{M}_n^p) \leq n^{2/p-1}.$$
The proof of Theorem 2 follows Gluskin’s original ideas. We first introduce some notation. $S^{n-1}$ will denote the euclidean sphere in $\mathbb{R}^n$ with its normalized Haar measure $\mu_{n-1}$ and $\Omega$ will be the product space $S^{n-1} \times \ldots \times S^{n-1}$ endowed with the product probability $\mathcal{P}$. If $K \subseteq \mathbb{R}^n$, $|K|$ is the Lebesgue measure of $K$. If $A = (P_1, \ldots, P_n) \in \Omega$, we write $Q_p(A) = p$-conv $\{ \pm e_i, \pm P_i | 1 \leq i \leq n \}$, being $\{e_i\}_{i=1}^n$ the canonical basis of $\mathbb{R}^n$. We denote by $\|: \|Q_p(A)$ the $p$-norm in $\mathbb{R}^n$ whose unit ball is $Q_p(A)$.

We only need to prove that for some absolute constant $C_p > 0$, there exist $A, A' \in \Omega$ such that simultaneously

$$\|T\|_{Q_p(A) \rightarrow Q_p(A')} \geq C_p n^{1/p-1/2} \quad \text{and} \quad \|T^{-1}\|_{Q_p(A') \rightarrow Q_p(A)} \geq C_p n^{1/p-1/2}$$

hold for any $T \in \text{SL}(n)$ (that is, any linear isomorphism in $\mathbb{R}^n$ with $\det T = 1$). Straightforward argument shows that it is enough to see that for any fixed $A' \in \Omega$ we have,

$$\mathbb{P}\{ A \in \Omega \mid \|T\|_{Q_p(A) \rightarrow Q_p(A')} < C_p n^{1/p-1/2} \text{ for some } T \in \text{SL}(n) \} < \frac{1}{2}$$

Fix $A \in \Omega$ and $t > 0$. Consider the set

$$\Omega(A', t) = \{ A \in \Omega \mid \|T\|_{Q_p(A) \rightarrow Q_p(A')} < t \text{ for some } T \in \text{SL}(n) \}$$

The proof of the following three lemmas are analogous to the ones in the case $p = 1$ (see [Tmze], §38).

**Lemma 5.** Let $A' \in \Omega$ and $t > 0$. There exists a $t$-net $N(A', t)$ in $\{ T \in \text{SL}(n) \mid \|T\|_{Q_p(A')} \leq t \}$ with respect to the metric induced by $\|: \|_{\ell_2^p \rightarrow Q_p(A')}$ of cardinality

$$|N(A', t)| \leq (3n^{1/p-1/2})^{n^2} \frac{|Q_p(A')|^n}{|\{ T \in \text{SL}(n) \mid \|T\|_{\ell_2^p \rightarrow \ell_2^p} \leq 1 \}|}$$

**Lemma 6.** For every $A' \in \Omega$ and $t > 0$ we have,

$$\Omega(A', t) \subseteq \bigcup_{T \in N(A', t)} \{ A \in \Omega \mid \|T(P_i)\|_{Q_p(A')} \leq 2^1/pt, \forall 1 \leq i \leq n \}$$

**Lemma 7.** Given $T \in \text{SL}(n), A' \in \Omega$ and $t > 0$,

$$\mathbb{P}\{ A \in \Omega \mid \|T(P_i)\|_{Q_p(A')} \leq 2^1/pt, \forall 1 \leq i \leq n \} \leq (2^{1/pt})^{n^2} \left( \frac{|Q_p(A')|}{|B_{\ell_2^p}|} \right)^n$$

**Proof of Theorem 2:** Numerical constants are always denoted by the same letters $C$ (or $C_p$, if it depends only on $p$) although they may have different value
from line to line. Using consecutively the three preceding lemmas we have for every 
\( A' \in \Omega \) and \( t > 0 \),
\[
\mathbb{P}(\Omega(A', t)) \leq (C_p t n^{1/p - 1/2})^{n^2} \frac{|Q_p(A')|^{2n}}{|B_{e_2}^n| \cdot |\{T \in \text{SL}(n) \mid \|T\|_{\ell_2^n} \rightarrow \ell_2^n \leq 1\}|}
\]

It is well known that for some absolute constant \( C > 0 \), (see \([\text{Tmcz}]\)),
\[
|\{T \in \text{SL}(n) \mid \|T\|_{\ell_2^n} \rightarrow \ell_2^n \leq 1\}| \geq C n^2 |B_{e_2}^n|
\]

Now using Theorem 1 it is clear that if \( A' = \{P_1, \ldots, P_n\} \), then \( Q_p(A') \subseteq \bigcup_{p-\text{conv}} \{P_{k_1}, \ldots, P_{k_n}\} \) where the union runs over the \((4n)\) choices of \( \{P_{k_i}\}_{i=1}^n \subseteq \{\pm e_i, \pm P_i, 1 \leq i \leq n\} \). Since \( \|P_i\|_2 = 1 \) and
\[
|p-\text{conv} \{P_{k_1}, \ldots, P_{k_n}\}| = |\det \{P_{k_1}, \ldots, P_{k_n}\}| \cdot |p-\text{conv} \{e_1, \ldots, e_n\}|
\]

we get
\[
|Q_p(A')| \leq \left(\frac{4n}{n}\right)^\frac{|B_{e_2}^n|}{2^n} \leq C_p n^{-n/p} 2^{-n}
\]

for some constant \( C_p \) (see \([\text{Pi}], \text{pg} \ 11\) ). Hence,
\[
\mathbb{P}(\Omega(A', t)) \leq (C_p t n^{1/2 - 1/p})^{n^2}
\]

If we take a suitable \( t > 0 \), we can assure \( \mathbb{P}(\Omega(A', t)) < \frac{1}{2} \) and the result follows.

\[\[\]

**Remark.** In the same way as quoted above, given a \( p \)-normed space \( X \) and \( p < q \leq 1 \), we can define the \( q \)-Banach envelope of \( X \) as the \( q \)-normed space, \( X^q \) whose unit ball is the \( q \)-convex envelope of the unit ball of \( X \). It is easy to see that \( d(X, X^q) \leq d(X, Y) \) for any \( n \)-dimensional \( q \)-normed space \( Y \). Theorem 1 shows that \( d(X, X^q) \leq n^{1/p - 1/q} \). Indeed, for every \( x \in B_{X^q} = q-\text{conv} (B_X) \) and \( \|x\|_{X^q} = 1 \) there exist \( P_1, \ldots, P_n \in B_X \) such that \( x = \sum_{i=1}^n \lambda_i P_i \) with \( \lambda_i \geq 0, 1 \leq i \leq n, \sum_{i=1}^n \lambda_i^q \leq 1 \) and
\[
1 \leq \|x\|_X \leq \sum_{i=1}^n \lambda_i^q \|P_i\|_X \leq \sum_{i=1}^n \lambda_i^q \leq n^{1/p - 1/q}
\]

by homogeneity we achieve the result. Now it is easy to see that if \( X, Y \) are the spaces appearing in Theorem 2, then \( d(X, X^q) \geq C_p n^{1/p - 1/q}, d(Y, Y^q) \geq C_p n^{1/p - 1/q} \) and \( d(X^q, Y^q) \geq C_p n^{2/q - 1} \). In particular, for \( q = 1 \), \( d(X, X^b) \geq C_p n^{1/p - 1}, d(Y, Y^b) \geq C_p n^{1/p - 1} \) and \( d(X^b, Y^b) \geq C_p n \).

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