Research Article

Unsteady Unidirectional Flow of Second-Grade Fluid through a Microtube with Wall Slip and Different Given Volume Flow Rate

Cha’o-Kuang Chen, Hsin-Yi Lai, and Wei-Fan Chen

Department of Mechanical Engineering, National Cheng Kung University, No.1, University Road, Tainan 70101, Taiwan

Correspondence should be addressed to Cha’o-Kuang Chen, ckchen@mail.ncku.edu.tw

Received 9 March 2010; Accepted 3 August 2010

Academic Editor: Saad A. Ragab

Copyright © 2010 Cha’o-Kuang Chen et al. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

The second-grade flows through a microtube with wall slip are solved by Laplace transform technique. The effects of rarefaction and elastic coefficient are considered with an unsteady flow through a microtube for a given but arbitrary inlet volume flow rate with time. Five cases of inlet volume flow rate are as follows: (1) trapezoidal piston motion, (2) constant acceleration, (3) impulsively started flow, (4) impulsively blocked fully developed flow, and (5) oscillatory flow. The results obtained are compared to those solutions under no-slip and slip condition.

1. Introduction

During the past decades, a great deal of literatures used the Navier-Stokes equation to describe the Newtonian fluid. However, the Newtonian fluid is the simplest to be solved and its application is very limited. In practice, many complex fluids such as blood, soaps, clay coatings, certain oils and greases, elastomers, suspensions, and many emulsions have been treated as non-Newtonian fluids. From literatures, the non-Newtonian fluids are mainly classified into three categories on the basis of their behavior in shear. (a) The shear stress of the fluids depends only on the rate of shear. (b) A fluid with a relationship between the shear stress and shear rate. (c) The fluids possess both elastic and viscous properties. One of the most popular models for non-Newtonian fluids is called the second-grade fluid. Erdoğan and İmрак [1] used the second-order Rivlin-Ericksen fluid, or the so-called second-grade fluid to model the non-Newtonian fluid. They solved some unsteady flows, such as unsteady flow over a plane wall, unsteady Couette flow, and unsteady Poiseuille flow. Hayat et al. [2]
analysed the influence of variable viscosity and viscous dissipation on the non-Newtonian flow. Bandelli and Rajagopal [3] solved various startup flows of second-grade fluids in domains with one finite dimension by integral transform method. Some unsteady flows of the fluids of second grade have been investigated by many authors [4–7].

Microfluidics is the significant technologies developed in the engineering field. As the microflow is considered, the no-slip condition is not sufficient for a fluid of second grade. Rarefaction phenomena should be considered when fluid flows in a microtube. The typically flow field can be divided into the four regimes by Knudsen number [8]: $K_n < 10^{-3}$, continuum flow; $10^{-3} \leq K_n < 10^{-1}$, slipflow; $10^{-1} \leq K_n < 10$, transition flow; and $10 \leq K_n$ free molecular flow. Much research in the literature does not consider the effect of rarefaction in the second-grade fluid. The study examined the effects of rarefaction of an unsteady flow through a microtube by Chen et al. [9].

In practice application, generally the inlet volume flow rate is a given condition instead of pressure gradient. Das and Arakeri [10] solved the unsteady laminar duct flow with a given volume flow rate variation. They discussed the problem with various types of given inlet piston motion in the channel and duct. Also Das and Arakeri [11] verified their earlier experimental work. Chen et al. [12–14] extended Das and Arakeri’s work by considering various non-Newtonian fluids. Several studies [15, 16] have suggested the no-slip condition that is deduced as the limiting cases when the slip parameter is equal to zero. Hayat et al. [17] considered the unsteady flow of an incompressible second-grade fluid in a circular duct with a given volume flow rate variation. The effects of Hall current are taken into account. For the above reason, this study considers the wall slip condition and the second-grade fluid with different given volume flow rate. For $\alpha_1 \to 0$, they reduce to the similar solutions for Newtonian fluids. The results show that the analytical solutions of velocity profile and pressure gradient are affected by the slip conditions and the viscoelastic parameter.

2. Constitutive Equations

The constitutive equation of second-grade Rivlin-Ericksen fluid is in the following form:

$$\vec{T} = -p \vec{I} + \mu \vec{A}_1 + \alpha_1 \vec{A}_2 + \alpha_2 \vec{A}_1^2,$$  \hspace{1cm} (2.1)

where $\vec{T}$ is the stress tensor, $\vec{I}$ is identity tensor, $p$ is the static fluid pressure, $\mu$ is the dynamic viscosity coefficient, $\alpha_1$ is the elastic coefficient and $\alpha_2$ is the transverse viscosity coefficient, and $\vec{A}_1$ and $\vec{A}_2$ represent the Rivlin-Ericksen tensors. Here, $\mu$, $\alpha_1$ and $\alpha_2$ are material modules which are assumed constant. The kinematic tensors $\vec{A}_1$ and $\vec{A}_2$ are defined as

$$\vec{A}_1 = \left( \text{grad} \vec{V} \right) + \left( \text{grad} \vec{V} \right)^T,$$

$$\vec{A}_2 = \frac{d \vec{A}_1}{dt} + \vec{A}_1 \left( \text{grad} \vec{V} \right) + \left( \text{grad} \vec{V} \right)^T \vec{A}_1.$$  \hspace{1cm} (2.2)
In the above equation, \( \vec{V} \) is the velocity and \( d/dt \) denotes the material time derivative. Since the fluid is incompressible, it can undergo only isochoric motion, and hence

\[
\text{div} \vec{V} = 0, \tag{2.3}
\]

and substituting constitutive equation (2.1) into the balance of linear momentum

\[
\text{div} \vec{T} + \rho \vec{b} = \rho \frac{d \vec{V}}{dt}, \tag{2.4}
\]

where \( \rho \) is the density of the fluid and \( \vec{b} \) is the body force. In the sense of the Clausius-Duhem inequality and the condition that the Helmholtz free energy is a minimum when the fluid is at rest, then the material modules must be satisfied [7] as follows:

\[
\mu \geq 0, \quad \alpha_1 \geq 0, \quad \alpha_1 + \alpha_2 = 0. \tag{2.5}
\]

In our study, we use the cylindrical polar coordinates \((r, \theta, x)\), where \( r \) is radial distance from the center of the pipe, \( \theta \) is the angular direction, and \( x \) is the axial direction. Velocity in the \( x, r, \) and \( \theta \)-direction are \( u, u_r, \) and \( u_\theta \), respectively. We also investigate the fluid rarefaction effect in a microtube, the Knudsen number is an important nondimensional parameter

\[
\text{Kn} \equiv \frac{\lambda}{L}, \tag{2.6}
\]

where \( \lambda \) is the molecular mean free path, which is defined as the mean secondary collision distance of a gas molecule, and \( L \) is the characteristic length.

In order to find the fluid of second grade for unsteady unidirectional flows, we seek a velocity field of the form \( u = u(r, t), u_r = 0, u_\theta = 0 \). The governing equations can be derived from (2.4), which gives

\[
\left( \mu + \alpha_1 \frac{\partial}{\partial t} \right) \left( \frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \frac{\partial u}{\partial r} \right) - \rho \frac{\partial u}{\partial t} = \frac{\partial p}{\partial x},
\]

\[
\frac{\partial p}{\partial r} = \frac{\partial p}{\partial \theta} = 0, \tag{2.7}
\]

where \( \nu \) is the kinematic viscosity. This implies that the pressure gradient is a function of time only.

3. Methodology of Solution

The problem can be solved if the governing equation, boundary condition, and initial condition are known. This third-order nonhomogeneous partial differential equation is not convenient to use the method of separation of variable to solve. In this paper, we give the
Laplace transform method reducing the two variables into single variable. In other words, we transform PDE into ODE that will effectively reduce the original difficult equation.

The governing equation of motion in $x$-direction is

$$\frac{\partial u}{\partial t} = -\frac{1}{\rho} \frac{\partial p}{\partial x} + \left( \nu + \theta \frac{\partial}{\partial t} \right) \left( \frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \frac{\partial u}{\partial r} \right),$$

where $\nu = \mu/\rho$ and $\theta = \alpha_1/\rho$.

With $R$ is the radius of duct, the boundary conditions are

$$u(R, t) = -\beta \nu \lambda \frac{\partial u(R, t)}{\partial r}, \quad \frac{\partial u(0, t)}{\partial r} = 0,$$

where $\beta \nu \lambda$ is the velocity slip coefficient and is defined as

$$\beta \nu = \frac{2 - F_v}{F_v},$$

and $F_v$ is the tangential momentum accommodation coefficient that describes the interaction between fluid and wall and is related to constituents of fluid, temperature, velocity, wall temperature, roughness, and chemical status. $F_v$ is defined as

$$F_v \equiv \frac{u_i - u_{re}}{u_i - U_w},$$

where $u_i$, $u_{re}$, and $U_w$ are tangential momentum of incoming molecules, reflected molecules, and re-emitted molecules, respectively.

We need an initial condition for the velocity, $u(r, 0)$, and the problem can be solved if the pressure gradient function is known. In our case, we determined the pressure gradient indirectly by the volume flow rate, which is given. The velocity is related to the inlet volume flow rate by

$$\int_0^R 2\pi ru(r, t) dr = u_p(t) \pi R^2 = Q(t),$$

where $u_p$ is the average inlet velocity.
Using the Laplace transform technique of (3.1), (3.2), and (3.5) yields the following equations:

\[
\frac{d^2 u(r, s)}{dr^2} + \frac{1}{r} \frac{du(r, s)}{dr} - \frac{s}{\nu + s\theta} u(r, s) = \frac{1}{\rho(v + s\theta)} \frac{dP(x, s)}{dx}, \tag{3.6}
\]

\[
u \frac{du(0, s)}{dr} = 0, \tag{3.8}
\]

\[
Q(s) = \int_0^S 2\pi ru(r, s)dr = u_p(s)\pi R^2. \tag{3.9}
\]

Equation (3.6) is a second-order inhomogeneous ordinary differential equation. The homogeneous part is the modified Bessel’s equation of zeroth order and assuming the particular integral as \(\Psi_p\), the general solution is

\[
u(r, s) = C_1 I_0(mr) + C_2 K_0(mr) + \Psi_p, \tag{3.10}
\]

where \(m = \sqrt{s/(\nu + s\theta)}\).

Using the boundary conditions (3.7) and (3.8) into (3.6) to solve these two unknown coefficients \(C_1\) and \(C_2\), substituting \(C_1\) and \(C_2\) into (3.6) give

\[
u(r, s) = \Psi_p \left( 1 - \frac{I_0(mr)}{I_0(mR) + \beta_\nu \lambda m I_1(mR)} \right), \tag{3.11}
\]

where \(I_1\) is the modified Bessel’s equation of the first order.

To solve for the unknown \(\Psi_p\), we substitute (3.11) into (3.9) and \(\Psi_p\) is obtained as

\[
\Psi_p = \frac{u_p(s)}{(1 - (2 I_1(mR)/mR[I_0(mR) + \beta_\nu \lambda m I_1(mR)]))^\nu}, \tag{3.12}
\]

Substituting \(\Psi_p\) into (3.11), we get

\[
u(r, s) = \frac{u_p(s)[I_0(mR) + am RI_1(mR)] - I_0(mr)]}{[[I_0(mR) + am RI_1(mR)] - 2 I_1(mR)/mR]} \tag{3.13}
\]

or

\[
u(r, s) = u_p(s) \cdot G(r, s), \tag{3.14}
\]
where

\[
G(r, s) = \frac{[I_0(mR) + amR I_1(mR)] - I_0(mr)}{[I_0(mR) + amR I_1(mR)] - 2I_1(mR)/mR}
\]

and \( \alpha = \beta_v \lambda / R = \beta_v Kn \approx Kn \).

Taking the inverse Laplace transform, the velocity profile is

\[
u(r, t) = \frac{1}{2\pi i} \int_{r-i\infty}^{r+i\infty} u_p(s)G(r, s)e^{st}ds.
\]

Furthermore, the pressure gradient is found by substituting (3.11) into (3.6) to obtain

\[
dp(x,s) dx = -u_p(s) \frac{[I_0(mR) - amR I_1(mR)]}{[I_0(mR) - amR I_1(mR)] - 2I_1(mR)/mR},
\]

We obtain the expressions for the variation of nondimensional pressure gradient with time by taking the inverse transform formula.

### 4. Illustration of Examples

We consider some examples proposed by Das and Arakeri [10] with the second-grade fluid and the effect of wall-slip conditions on the unsteady flow patterns in a microtube.

For the following case, the velocity moves with a constant acceleration of the piston starting from rest, and the other one, the piston suddenly starts from rest and then keeping this velocity. These two solutions we apply to the trapezoidal motion, that is, the piston has three stages: constant acceleration of the piston starting from rest, a period of constant velocity, and a constant deceleration of the piston to a stop.

#### 4.1. Trapezoidal Piston Motion

We get the solution for the three stages piston velocities which vary with time as follows:

\[
u_p(t) = \begin{cases}
U_p, & \text{for } 0 \leq t \leq t_0, \\
\frac{t_0}{t_0}, & \text{for } t_0 \leq t \leq t_1, \\
U_p \frac{(t_2 - t)}{(t_2 - t_1)}, & \text{for } t_1 \leq t \leq t_2, \\
0, & \text{for } t_2 \leq t \leq \infty,
\end{cases}
\]
where $U_p$ is the constant velocity after acceleration, and $t_0$, $t_1$, and $t_2$ are the time periods for changing piston velocity. We use the Heaviside unit step function to describe the piston motion as follows:

$$u_p(t) = \frac{U_p}{t_0} t H(t) - \frac{U_p}{t_0} t H(t - t_0) + U_p H(t - t_0) - U_p H(t - t_1) + U_p \frac{t_2 - t}{t_2 - t_1} H(t - t_1) - U_p \frac{t_2 - t}{t_2 - t_1} H(t - t_2).$$

For the constant acceleration period $(0 \leq t \leq t_0)$, taking the Laplace transform of $u_p(t) = \frac{U_p}{t_0} t H(t)$, we get

$$u_p(s) = \frac{U_p}{t_0 s^2}.$$  \hfill (4.3)

From (3.16) expression, the integration is determined using complex variable theory, as discussed by Arparci [18]. We obtain the velocity distribution

$$u(r, t) = \frac{1}{2\pi i} \left[ \frac{2\pi i}{\sum_{j=1}^{R_j}} \right],$$

where $R_j$ is the residual of poles of $U_p e^{st} G(r, s) / t_0 s^2$.

It can be easily observed that $s = 0$ is a pole of order 2. Therefore, the residue at $s = 0$ is

$$\text{Res}(0) = \frac{U_p}{t_0} \left\{ \frac{2f \left[ 1 - (r/R)^2 + 2\alpha \right]}{(1 + 4\alpha)} + \frac{R^2 \left[ 1 - (r/R)^4 + 4\alpha \right]}{8\nu(1 + 4\alpha)} - \frac{R^2 (1 + 6\alpha) \left[ 1 - (r/R)^2 + 2\alpha \right]}{6\nu(1 + 4\alpha)^2} \right\}.$$ \hfill (4.5)

The other singular points are the zeroes of

$$I_0(mR) + amR I_1(mR) - \frac{2I_1(mR)}{mR} = 0.$$ \hfill (4.6)

Setting $mR = i\lambda$, we find that

$$\alpha \lambda J_1(\lambda) + J_2(\lambda) = 0.$$ \hfill (4.7)

If $\lambda_n$, $n = 1, 2, 3, \ldots, \infty$ are zeros of (4.7), then $s_n = -\lambda_n^2 \nu / (R^2 + \theta \lambda_n^2)$, $n = 1, 2, 3, \ldots, \infty$ are the simple poles. Since all $\lambda_n$ are symmetrically placed about zero on the real axis, all the poles
Adding $\text{Res}(0)$ and $\text{Res}(s_n)$, the dimensionless velocity distribution is obtained as

$$u^*(c, t^*) = \frac{1}{2t_0} \left\{ \frac{2t^*(1 - c^2 + 2\alpha)}{(1 + 4\alpha)} \right\} + \frac{2}{t_0} \sum_{n=1}^{\infty} \left\{ e^{-\frac{1}{2} t^* / (1 + \beta t_n^2)} J_0(\lambda_n) - J_0(c \lambda_n) - \alpha \lambda_n J_1(\lambda_n) \right\} \frac{1}{(1 + 2\alpha) \lambda_n^3 f_1(\lambda_n) + \alpha \lambda_n^4 f_0(\lambda_n)} \right\},$$

where $u^* = u_p / U_p$, $c = r / R$, $\alpha = \beta \nu\kappa$, $t^* = tv / R^2$, $t_0^* = t_0v / R^2$, $\beta = \theta / R^2$.

By the same method, the dimensionless velocity profile during the constant piston velocity ($t_0 \leq t \leq t_1$) is obtained as

$$u_p(t) = \frac{U_p}{t_0} t H(t) - \frac{U_p}{t_0} t H(t - t_0) + \frac{U_p}{t_0} H(t - t_0),$$

$$u^*(c, t^*) = 2 \left\{ \frac{(1 - c^2) + 2\alpha}{1 + 4\alpha} \right\} + 2 \sum_{n=1}^{\infty} \left\{ e^{-\frac{1}{2} t^* / (1 + \beta t_n^2)} - e^{-\frac{1}{2} t^* / (1 + \beta t_n^2)} \right\} \frac{J_0(\lambda_n) - J_0(c \lambda_n) - \alpha \lambda_n J_1(\lambda_n)}{(1 + 2\alpha) \lambda_n^3 f_1(\lambda_n) + \alpha \lambda_n^4 f_0(\lambda_n)} \right\},$$

during the constant deceleration of the piston motion ($t_1 \leq t \leq t_2$).

$$u_p(t) = \frac{U_p}{t_0} t H(t) - \frac{U_p}{t_0} t H(t - t_0) + \frac{U_p}{t_0} H(t - t_0),$$

$$u^*(c, t^*) = \frac{1}{t_2^* - t_1^*} \left\{ \frac{2(t_2^* - t_1^*)^2 + 2\alpha}{1 + 4\alpha} \right\} + \frac{(1/6)(1 + 6\alpha)(1 - c^2 + 2\alpha)}{(1 + 4\alpha)^2} + \sum_{n=1}^{\infty} \left\{ e^{-\frac{1}{2} t^* / (1 + \beta t_n^2)} - e^{-\frac{1}{2} t^* / (1 + \beta t_n^2)} \right\} \frac{J_0(\lambda_n) - J_0(c \lambda_n) - \alpha \lambda_n J_1(\lambda_n)}{(1 + 2\alpha) \lambda_n^3 f_1(\lambda_n) + \alpha \lambda_n^4 f_0(\lambda_n)} \right\},$$

where $t_1^* = t_1v / R^2$, $t_2^* = t_2v / R^2$. 
And after the piston has stopped ($t_2 \leq t \leq \infty$),

$$u_p(t) = \frac{U_p}{t_0} t H(t) - \frac{U_p}{t_0} t H(t-t_0) + U_p H(t-t_0) - U_p H(t-t_1)$$

$$+ U_p \frac{t_2-t}{t_2-t_1} H(t-t_1) - U_p \frac{t_2-t}{t_2-t_1} H(t-t_2),$$

$$u^* (c, t^*) = 2 \sum_{n=1}^{\infty} \left\{ \frac{e^{-\lambda_n^2 t^*}/(1+\beta_n^2)}{t^*_0} - \frac{e^{-\lambda_n^2 (t^*-t^*_0)/(1+\beta_n^2)}}{t^*_2-t^*_1} - \frac{e^{-\lambda_n^2 (t^*-t^*/(1+\beta_n^2))}}{t^*_2-t^*_1} \right\}$$

$$\times \frac{f_0(\lambda_n) - f_0(c \lambda_n) - \alpha \lambda_n f_1(\lambda_n)}{(1+2\alpha) \lambda_n f_1(\lambda_n) + \alpha \lambda_n^2 f_0(\lambda_n)} \right\}.$$

(4.12)

We also obtain the expressions of the dimensionless pressure gradient during these four different stages. During the constant acceleration period ($0 \leq t \leq t_0$),

$$\frac{dp^*}{dx^*} = -\frac{1}{t^*_0} \left[ t^* + (\alpha/2) + (1/4) \right] \frac{(1/12) + (\alpha/2)}{(1+4\alpha)^2}$$

$$+ \frac{1}{4t^*_0} \sum_{n=1}^{\infty} \left\{ e^{-\lambda_n^2 t^*/(1+\beta_n^2)} \frac{f_0(\lambda_n) - \alpha \lambda_n f_1(\lambda_n)}{(1+2\alpha) \lambda_n f_1(\lambda_n) + \alpha \lambda_n^2 f_0(\lambda_n)} \right\},$$

(4.13)

where $P^* = P/(8\mu U_p/R)$, $x^* = x/R$.

During the constant velocity period ($t_0 \leq t \leq t_1$),

$$\frac{dp^*}{dx^*} = -\frac{1}{1+4\alpha} + \frac{1}{4t^*_0} \sum_{n=1}^{\infty} \left\{ e^{-\lambda_n^2 t^*/(1+\beta_n^2)} - e^{-\lambda_n^2 (t^*-t^*_0)/(1+\beta_n^2)} \right\} \frac{f_0(\lambda_n) - \alpha \lambda_n f_1(\lambda_n)}{(1+2\alpha) \lambda_n f_1(\lambda_n) + \alpha \lambda_n^2 f_0(\lambda_n)} \right\},$$

(4.14)

during the constant deceleration period ($t_1 \leq t \leq t_2$),

$$\frac{dp^*}{dx^*} = \frac{1}{t^*_2-t^*_1} \left[ \frac{(t^*_2-t^*_1) - ((\alpha/2) + (1/4))}{(1+4\alpha)^2} \right]$$

$$+ \sum_{n=1}^{\infty} \left\{ e^{-\lambda_n^2 t^*/(1+\beta_n^2)} - e^{-\lambda_n^2 (t^*-t^*_0)/(1+\beta_n^2)} \right\} \frac{f_0(\lambda_n) - \alpha \lambda_n f_1(\lambda_n)}{(1+2\alpha) \lambda_n f_1(\lambda_n) + \alpha \lambda_n^2 f_0(\lambda_n)} \right\},$$

(4.15)
and after the piston has stopped ($t_2 \leq t \leq \infty$),

$$
\frac{dp^*}{dx^*} = \sum_{n=1}^{\infty} \left[ \frac{e^{-\lambda_n^2 (t^* - t_1^*)} - e^{-\lambda_n^2 (t^* - t_0^*)}}{4t_0^*} - \frac{e^{-\lambda_n^2 (t^* - t_2^*)} - e^{-\lambda_n^2 (t^* - t_1^*)}}{4(t_2^* - t_1^*)} \right]
\times \frac{J_0(\lambda_n) - \alpha \lambda_n J_1(\lambda_n)}{[(1 + 2\alpha)\lambda_n J_1(\lambda_n) + \alpha \lambda_n^2 J_0(\lambda_n)]}.
$$

(4.16)

Above these infinite series, equations are convergent and we set the $n = 50$ as enough for the cases. For trapezoidal piston motion with different nondimensional times ($t^* = t\nu/R$) are $t_0\nu/R^2$, $t_1\nu/R^2$ and $t_2\nu/R^2 = 0.0012$, 0.0305, and 0.0366, respectively. The velocity profiles at Kn = 0.1 and $\beta = 0.05$ are illustrated in Figure 1.
These values are chosen for the purpose of comparing the results obtained by Das and Arakeri [10] and Chen et al. [9]. When \( a \approx Kn = 0 \) (no-slip condition) and \( \beta = 0 \) (no-elastic effect), the velocity profiles in (4.9) to (4.12) are exactly like Das and Chen’s results. Figure 1 shows the second-grade flow with slippage on the microtube wall during four different time periods. The development of velocity is similar to that in Das et al. and Chen et al.’s works. However, the elastic coefficient retarded the change of velocity in the microtube. Because the effect of slippage, the shift of velocity from the wall to centerline is smoother than that in Das et al. and Chen et al.’s works. During the time period when the piston decelerates and stops at time \( t^*_p \), it is observed that the flow reverses its direction near the wall (see Figure 1(c)). After the piston motion ceases, the velocity profile (see Figure 1(d)) continues to have reverse flow near the wall to satisfy the zero mass flow condition. Figure 2 shows the variation of nondimensional pressure gradients with time at \( Kn = 0.1 \) and \( \beta = 0.05 \). During the acceleration and deceleration stages, the pressure gradients are large mainly because of fluid inertia. Finally, when the piston stops, the pressure gradient slowly decays to zero. Figure 3 shows the effect of different \( \beta \) values (\( \beta = 0, 0.05, 0.1 \)) on the velocity profiles at \( Kn = 0.1 \). During the four stages of piston motion, the larger \( \beta \) values the smoother the velocity profile.

The degree of smoothness is proportional to the \( \beta \) value. In the special case, it is worth mentioning that when \( \beta \to 0 \) (means that \( \alpha_1 \to 0 \)), corresponding to Newtonian fluids, all solutions that have been obtained are going to be those for Newtonian fluids performing the same motions. Figure 4 shows the effect of \( Kn \) various values (\( Kn = 0, 0.05, 0.1 \)) on the velocity profiles at \( \beta = 0.05 \). The analytical result demonstrates that a larger Kn value will flatten the velocity profile. It is observed that the slip condition occurs near the wall.

### 4.2. Constant Acceleration Case

For a piston with constant acceleration can be described by the following equation:

\[
 u_p(t) = a_p t = \left( \frac{U_p}{t_0} \right) t,
\]

where \( a_p \) is the constant acceleration, \( U_p \) is the final velocity after acceleration, and \( t_0 \) is the time period of acceleration.

The velocity profile can be obtained by putting \( t = t_0 \) of (4.9) as follows:

\[
 u^*(c, t^*) = \frac{1}{t^*} \left\{ \frac{2t^* (1 - c^2 + 2\alpha) + (1/8)(1 - c^4 + 4\alpha)}{(1 + 4\alpha)^2} - \frac{(1/6)(1 + 6\alpha)(1 - c^2 + 2\alpha)}{(1 + 4\alpha)^2} \right\}
 + \frac{2}{t^*} \sum_{n=1}^{\infty} \left\{ e^{-\frac{\lambda_n t^*}{1 + \beta \lambda_n^2}} \frac{I_0(\lambda_n) - J_0(c\lambda_n) - \alpha \lambda_n J_1(\lambda_n)}{\left[ (1 + 2\alpha)\lambda_n J_1(\lambda_n) + \alpha \lambda_n^2 I_0(\lambda_n) \right]} \right\}
\]

(4.18)
and when the pressure gradient, as time approaches infinity, is

$$\frac{dp^*}{dx^*} = \frac{1}{(1 + 4\alpha)} \left[ -\frac{(\alpha/2 + (1/4))a_pR^2}{(1 + 4\alpha)\nu_p} + \frac{((1/12) + (\alpha/2))a_pR^2}{(1 + 4\alpha)^2\nu_p} \right]$$

$$- \frac{1}{t^*} \left[ \frac{t^* + (\alpha/2 + (1/4))}{(1 + 4\alpha)} - \frac{(1/12) + (\alpha/2)}{(1 + 4\alpha)^2} \right].$$

(4.19)

### 4.3. Suddenly Started Flow

The solution to the suddenly started flow in a circular duct is as follows:

$$u_p(t) = \begin{cases} 
0, & \text{for } t \leq 0, \\
U_p, & \text{for } t > 0,
\end{cases}$$

(4.20)

where $U_p$ is the constant velocity

$$u^*(c, t^*) = 2 \left[ \frac{(1 - c^2) + 2\alpha}{1 + 4\alpha} \right] - 2 \sum_{n=1}^{\infty} \left\{ e^{-\lambda_n^2 t^*/(1 + \beta \nu_p)} \frac{\lambda_n}{\lambda_n^2 f_1(\lambda_n) + 2\alpha \lambda_n f_1(\lambda_n) + \alpha \lambda_n^2 f_0(\lambda_n)} \right\}.$$
and the pressure gradient is

\[
dp^*/dx^* = -\frac{1}{1 + 4\alpha} \left\{ \sum_{n=1}^{\infty} e^{-\lambda_n^2 r^2/(1 + \beta \lambda_n^2)} \frac{\lambda_n [J_0(\lambda_n) - \alpha \lambda_n J_1(\lambda_n)]}{[(1 + 2\alpha) J_1(\lambda_n) + \alpha \lambda_n J_0(\lambda_n)]} \right\}.
\]

(4.22)

4.4. Suddenly Blocked Fully Developed Flow

The exact solution of this problem with no-slip wall condition was considered by Weinbaum and Parker [19]. The initial condition for this problem is \( u(r, 0) = 1 - c^2 \), and the mass flow condition is \( \int_0^R 2 \pi ru \, dr = 0 \). The resulting velocity profile is

\[
u^*(c, t^*) = -2 \sum_{n=1}^{\infty} e^{-\lambda_n^2 r^2/(1 + \beta \lambda_n^2)} \frac{J_0(\lambda_n) - f_0(c \lambda_n)}{\lambda_n J_1(\lambda_n)}.
\]

(4.23)
When the wall slip is considered, the corresponding velocity profile and pressure gradient are
\[ u^*(c, t^*) = -2 \sum_{n=1}^{\infty} \left\{ e^{-\lambda_n^2 t^* / (1 + \beta \lambda_n^2)} \frac{J_0(\lambda_n) - J_0(c \lambda_n) - \alpha \lambda_n f_1(\lambda_n)}{[\lambda_n f_1(\lambda_n) + 2 \alpha \lambda_n f_1(\lambda_n) + \alpha \lambda_n^2 J_0(\lambda_n)]} \right\}, \]
\[ \frac{dp^*}{dx^*} = \frac{1}{4t^*} \sum_{n=1}^{\infty} \left\{ e^{-\lambda_n^2 t^* / (1 + \beta \lambda_n^2)} \frac{J_0(\lambda_n) - \alpha \lambda_n f_1(\lambda_n)}{[(1 + 2\alpha) \lambda_n f_1(\lambda_n) + \alpha \lambda_n^2 J_0(\lambda_n)]} \right\}. \] (4.24)

5. Oscillatory Flow

Here, the oscillating piston motion starting from rest is considered. The piston motion is defined as
\[ u_p(t) = \begin{cases} 0, & \text{for } t \leq 0, \\ U_p \sin(\omega t), & \text{for } t > 0. \end{cases} \] (5.1)
Taking the Laplace transform of (5.1), we have

$$u_p(s) = \frac{U_p \omega}{s^2 + \omega^2}. \tag{5.2}$$

Substitute (5.2) into (3.16) to find the velocity profile. The poles are simple poles at $s = \pm i \omega$ and the roots of $\alpha \lambda f_1(\lambda) + f_2(\lambda) = 0$. The solution is

$$u^*(c, t^*) = \frac{i}{2} \left[ e^{-i \omega t} G(r, -i \omega) - e^{i \omega t} G(r, i \omega) \right]$$

$$+ \sum_{n=1}^{\infty} \left\{ \frac{e^{-\lambda_n \omega / (1 + \beta \lambda_n^2)} 2 R^3 \nu \lambda_n^3 [J_0(\lambda_n) - J_0(c \lambda_n) - \alpha \lambda_n f_1(\lambda_n)]}{\lambda_n^4 \nu^2 + \omega^2 (R^2 + \theta \lambda_n^2)^2 [(1 + 2 \alpha) \lambda_n^2 f_1(\lambda_n) + \alpha \lambda_n f_0(\lambda_n)]} \right\}, \tag{5.3}$$

where $G(r, s)$ is defined by (3.15).

And the pressure gradient is obtained as

$$\frac{dp^*}{dx^*} = -\frac{R^2 \omega}{16 \nu} \left[ e^{i \omega t} \Gamma(i \omega) + e^{-i \omega t} \Gamma(-i \omega) \right]$$

$$+ \frac{1}{4} \sum_{n=1}^{\infty} \left\{ \frac{e^{-\lambda_n \omega / (1 + \beta \lambda_n^2)} R^4 \omega \lambda_n^4 [J_0(\lambda_n) - \alpha \lambda_n f_1(\lambda_n)]}{\rho [\lambda_n^4 \nu^2 + \omega^2 (R^2 + \theta \lambda_n^2)^2] [(1 + 2 \alpha) \lambda_n^2 f_1(\lambda_n) + \alpha \lambda_n f_0(\lambda_n)]} \right\}, \tag{5.4}$$

where $\Gamma(r, s) = [I_0(mR) - am R I_1(mR)] / ([I_0(mR) - am R I_1(mR)] - 2 I_1(mR) / mR)$ and $m = \sqrt{s / (\nu + s \theta)}$.

### 6. Conclusion

In this paper, the second-grade flows through a microtube with wall slip are solved by Laplace transform technique. The analytical solutions of velocity profiles and pressure gradients are compared to those obtained by Das and Arakeri’s work [10] for no-slip flow and Chen et al.’s work [9] for wall slip condition. We found that the Kn number represents the degree of rarefaction and the $\alpha_1$ is characterized as the elastic coefficient. Those two variables play a significant role in influencing the velocity profile.

From the equation we deduced that the larger Kn value decreases the change of velocity distribution. We could find that the wall slip condition moves more fluid at one cross section than at a cross section without slip. Also, the role of elastic coefficient is to retard the development of flow in the microtube.
Nomenclature

\(a_p\): Constant acceleration
\(A_1\): Rivlin-Ericksen tensor of first order
\(A_2\): Rivlin-Ericksen tensor of second order
\(b\): Body force field
\(c\): \(r/R\)
\(C_1, C_2\): Arbitrary coefficients
\(F_v\): Tangential momentum accommodation coefficient
\(H(t)\): Heaviside unit step function
\(I\): Identity tensor
\(I_0, I_1\): Modified Bessel's function of the first kind of zeroth and first order
\(J_0, J_1\): Bessel's function of zeroth and first order
\(Kn\): Knudsen number \((Kn \equiv \lambda/L)\)
\(K_0, K_1\): Modified Bessel's function of the second kind of zeroth and first order
\(L\): Characteristic length of the microtube
\(m\): \(\sqrt{s/(\nu + s\theta)}\)
\(P\): Static pressure
\(P^*\): Nondimensional pressure \((P^* = P/(8\mu U_p/R))\)
\(Q\): Inlet volume flow rate
\(R\): Radius of microtube
\(r, \theta, x\): Cylindrical coordinates
\(s\): Parameter of the Laplace transform
\(t\): Time
\(t_0, t_1, t_2\): Time period of acceleration, constant velocity, and deceleration, respectively
\(T\): Stress tensor
\(u_r, u_\theta, u\): Velocity components in the \(r, \theta,\) and \(x\)-directions, respectively
\(u_r^e\): Tangential momentum of reflected molecules
\(u_i\): Tangential momentum of incoming molecules
\(u_p\): Average inlet velocity
\(u^*\): Nondimensional average velocity over cross section
\(U_p\): Constant inlet piston velocity
\(U_w\): Tangential momentum of re-emitted molecules
\(V\): Velocity vector.

Greek Symbols

\(\alpha\): Nondimensional velocity slip coefficient
\(\alpha_1\): Elastic coefficient
\(\alpha_2\): Transverse viscosity coefficient
\(\beta_v\): Velocity slip parameter
\(\lambda\): Molecular mean free path
\(\Psi_p\): Assumed particular solution
\(\rho\): Fluid density
\(\nu\): Kinematic viscosity
\(\mu\): Dynamic viscosity.
Acknowledgment

This work was partly sponsored by the National Science Council of Taiwan under Contract no. NSC93-2212-E-006-062.

References

[1] M. Emin Erdön and C. Erdem İmra, “On some unsteady flows of a non-Newtonian fluid,” *Applied Mathematical Modelling*, vol. 31, no. 2, pp. 170–180, 2007.
[2] T. Hayat, R. Ellahi, and S. Asghar, “The influence of variable viscosity and viscous dissipation on the non-Newtonian flow: an analytical solution,” *Communications in Nonlinear Science and Numerical Simulation*, vol. 12, no. 3, pp. 300–313, 2007.
[3] R. Bandelli and K. R. Rajagopal, “Start-up flows of second grade fluids in domains with one finite dimension,” *International Journal of Non-Linear Mechanics*, vol. 30, no. 6, pp. 817–839, 1995.
[4] T. Hayat, S. Asghar, and A. M. Siddiqui, “Some unsteady unidirectional flows of a non-Newtonian fluid,” *International Journal of Engineering Science*, vol. 38, no. 3, pp. 337–346, 2000.
[5] C. Fetecau and J. Zierep, “On a class of exact solutions of the equations of motion of a second grade fluid,” *Acta Mechanica*, vol. 150, no. 1-2, pp. 135–138, 2001.
[6] A. M. Benharbit and A. M. Siddiqui, “Certain solutions of the equations of the planar motion of a second grade fluid for steady and unsteady cases,” *Acta Mechanica*, vol. 94, no. 1-2, pp. 85–96, 1992.
[7] J. E. Dunn and K. R. Rajagopal, “Fluids of differential type: critical review and thermodynamic analysis,” *International Journal of Engineering Science*, vol. 33, no. 5, pp. 689–729, 1995.
[8] A. Beskok and G. E. Karniadakis, “Simulation of slip-flows in complex micro-geometries,” in *Proceedings of the Annual Meeting of the American Society of Mechanical Engineers*, vol. 40, pp. 355–370, 1992.
[9] C.-I. Chen, C.-K. Chen, and H.-J. Lin, “Analysis of unsteady flow through a microtube with wall slip and given inlet volume flow rate variations,” *Journal of Applied Mechanics*, vol. 75, no. 1, Article ID 014506, 7 pages, 2008.
[10] D. Das and J. H. Arakeri, “Unsteady laminar duct flow with a given volume flow rate variation,” *Journal of Applied Mechanics*, vol. 67, no. 2, pp. 274–281, 2000.
[11] D. Das and J. H. Arakeri, “Transition of unsteady velocity profiles with reverse flow,” *Journal of Fluid Mechanics*, vol. 374, pp. 251–283, 1998.
[12] C.-I. Chen, C.-K. Chen, and Y.-T. Yang, “Unsteady unidirectional flow of Bingham fluid between parallel plates with different given volume flow rate conditions,” *Applied Mathematical Modelling*, vol. 28, no. 8, pp. 697–709, 2004.
[13] C. Chen, Y.-T. Yang, and C. Chen, “Unsteady unidirectional flow of a Voigt fluid between the parallel surfaces with different given volume flow rate conditions,” *Applied Mathematics and Computation*, vol. 144, no. 2-3, pp. 249–260, 2003.
[14] C. I. Chen, C. K. Chen, and Y. T. Yang, “Unsteady unidirectional flow of an Oldroyd-B fluid in a circular duct with different given volume flow rate conditions,” *Heat Mass Transfer*, vol. 40, pp. 203–209, 2004.
[15] C. Fetecau, T. Hayat, C. Fetecau, and N. Ali, “Unsteady flow of a second grade fluid between two side walls perpendicular to a plate,” *Nonlinear Analysis: Real World Applications*, vol. 9, no. 3, pp. 1236–1252, 2008.
[16] R. Ellahi, T. Hayat, F. M. Mahomed, and S. Asghar, “Effects of slip on the non-linear flows of a third grade fluid,” *Nonlinear Analysis: Real World Applications*, vol. 11, no. 1, pp. 139–146, 2010.
[17] T. Hayat, S. Nadeem, R. Ellahi, and S. Asghar, “The influence of Hall current in a circular duct,” *Nonlinear Analysis: Real World Applications*, vol. 11, no. 1, pp. 184–189, 2010.
[18] V. S. Arparci, *Conduction Heat Transfer*, Addison-Wesley, Redwood City, Calif, USA, 1996.
[19] S. Weinbaum and K. H. Parker, “The laminar decay of suddenly blocked channel and pipe flows,” *Journal of Fluid Mechanics*, vol. 69, pp. 729–752, 1975.
Submit your manuscripts at
http://www.hindawi.com