Adaptive Huber Regression on Markov-dependent Data

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Abstract

High-dimensional linear regression has been intensively studied in the community of statistics in the last two decades. For convenience of theoretical analyses, classical methods usually assume independent observations and subGaussian-tailed errors. However, neither of them hold in many real high-dimensional time-series data. Recently [Sun, Zhou, Fan, 2019, J. Amer. Stat. Assoc., in press] proposed Adaptive Huber Regression (AHR) to address the issue of heavy-tailed errors. They discover that the robustification parameter of the Huber loss should adapt to the sample size, the dimensionality and $(1 + \delta)$-moments of the heavy-tailed errors. We progress in a vertical direction and justify AHR on dependent observations. Specially, we consider an important dependence structure — Markov dependence. Our results show that the Markov dependence impacts on the adaption of the robustification parameter and the estimation of regression coefficients in the way that the sample size should be discounted by a factor depending on the spectral gap of the underlying Markov chain.

Keywords: Adaptive Huber Regression, dependent observations, Markov chain, high-dimensional regression, heavy-tailed errors.

1 Introduction

In the Big Data era, massive and high-dimensional data characterize many modern statistical problems, arising from biomedical sciences, econometrics, finance, engineering and social sciences. Examples include the gene expression data, the functional magnetic resonance imaging (fMRI) data, the macroeconomic data, the high-frequency financial data, the high-resolution image data, the e-commerce data, among others. The data abundance, the high dimensionality and other complex structures have given rise to a few statistical and computational challenges (Fan, Han and Liu, 2014).

An important and fundamental problem is the high-dimensional linear regression, in which the dimensionality (the number of covariates) is much larger than the sample size so that the ordinary least squares estimation of the regression coefficients is highly unstable. To solve this problem, statisticians have made extensive progress in the development of high-dimensional statistical inference (Tibshirani, 1996; Fan and Li, 2001; Candès and...
Tao, 2007; Bickel, Ritov and Tsybakov, 2009; Su and Candès, 2016). Under the sparsity assumption that only a small number of covariates contribute to the response variable, they enforce the sparsity of regression coefficients by regularization techniques. For a comprehensive and systematic overview of this (sparse) high-dimensional linear regression, we refer to Bühlmann and van de Geer (2011) and Tibshirani, Wainwright and Hastie (2015).

For convenience of theoretical analyses, many high-dimensional regression methods assume Gaussian or subGaussian tails of the errors in the model. However, this assumption is unsatisfied in a broad range of real datasets. See Cont (2001) for asset return data, Stock and Watson (2002); McCracken and Ng (2016) for macroeconomic data, Gupta et al. (2014) for RNA-seq gene expression data, Wang, Peng and Li (2015) for microarray gene expression data, Eklund, Nichols and Knutsson (2016) for fMRI data, to name a few. Fan, Wang and Zhu (2016) argue that heavy tails of the errors are stylized features of high-dimensional data. As the ordinary least squares loss function is non-robust to outliers, many high-dimensional regression estimators may be inconsistent in the presence of heavy-tailed errors.

Robust loss functions, which are less sensitive to outliers, have been considered to address the issue of heavy-tailed errors. Huber’s seminar work (Huber, 1964, 1973) introduced robust M-estimators (“M” for “maximum likelihood-type”) and provided the initial theory on robust regression methods. Asymptotic properties of robust M-estimators in the low-dimensional setting have been well studied by Yohai and Maronna (1979); Portnoy (1985); Mammen (1989); He and Shao (1996); Bai and Wu (1997); He and Shao (2000). Recently Loh (2017) investigated theoretical properties of a general class of regularized robust M-estimators for the high-dimensional linear model with heavy-tailed errors.

More recently, Sun, Zhou and Fan (2017) studied a specific regularized robust M-estimator, which minimizes the $\ell_1$-regularized Huber loss, for the high-dimensional linear model under the assumption that errors have heavy tails but finite $(1+\delta)$-moments for some $\delta > 0$. Remarkably, Sun, Zhou and Fan (2017) observe that the robustification parameter of the Huber loss should adapt to the sample size, the dimensionality and $(1+\delta)$-moments of heavy-tailed errors for an optimal tradeoff between bias and robustness. The adaption of the robustification parameter exhibits a smooth phase transition between $0 < \delta < 1$ and $\delta > 1$. Thereafter, they name the method Adaptive Huber Regression (AHR), to highlight its difference from others’ Huber regression methods with fixed robustification parameter.

Apart from heavily-tailed errors, another common feature of high-dimensional data, especially those collected in a temporal order, is dependent observations. For example, fMRI time series data are usually collected from a few different regions over a time period (Friston et al., 1995; Worsley and Friston, 1995); the monthly data of macroeconomic variables spanning the time period of decades are now benchmark datasets used by many econometric studies (Stock and Watson, 2002; Ludvigson and Ng, 2009; McCracken and Ng, 2016).

High-dimensional regression has been applied to fMRI time-series data (Smith, 2012; Ryali et al., 2012; Tang et al., 2012) and economic time-series data (Ludvigson and Ng,
2009; Fan et al., 2011; Belloni et al., 2012), although there is little consideration of the dependence structure of observations in their theoretical framework. For fMRI time-series data, the lasso or elastic-net regression method is one of the main tools to find a small number of functionally-connected regions of a specific region in human brains from temporally-ordered observations of all the regions (Smith, 2012). On a macroeconomic dataset, (Ludvigson and Ng, 2009) regressed the U.S. bond premia on macroeconomic variables over the time period spanning from January, 1964 to December, 2003. Nevertheless, the applicability of high-dimensional regression methods to these time-series data is not fully understood, because they assume either unconditional independence of samples directly, as in the random design setup, or conditional independence of samples given covariates satisfying some conditions, as in the fixed design setup (Bühlmann and van de Geer, 2011; Tibshirani, Wainwright and Hastie, 2015). In the latter setup, high probabilities of these conditions are verified with independent observations of covariates.

This paper aims to close the gap between theories of high-dimensional regression methods and practical needs in addressing both heavy-tailed errors and dependent observations of real data. Inspired by the optimality of AHR dealing with heavy-tailed errors in the independent setup, we consider extending AHR to cope with the dependence structure of observations. Albeit of the theoretical results on AHR by Sun, Zhou and Fan (2017), whether AHR work for the dependent samples is still unclear. This unclearness puts the applicability of AHR to many real high-dimensional datasets in doubt. Even if AHR is justifiable under some type of dependence structure, we are still curious about the degree to which AHR maintain the error rate of regression coefficient estimation against sample dependence. As an initial step towards full answers for these questions, we narrow down to the Markov dependence, an important and widely-used dependence structure, and analyze AHR on Markov-dependent samples. Specially, we assume that covariates are functions of an underlying Markov chain and heteroskedastic heavy-tailed errors are dependent on the Markov chain.

In this Markov-dependent setup, we show under moderate conditions that AHR exhibits a similar phase transition of the adaptation of the robustification parameter and the estimation of regression coefficients between $0 < \delta < 1$ and $\delta > 1$, compared to that in the usual independent setup. The only difference is that the sample size should be discounted by a factor depending on the spectral gap of the underlying Markov chain.

The core of the proof is to bound in $\ell_\infty$-norm the gradient of the Huber loss at the true sparse vector of coefficients, denoted by $\beta^*$, and to establish the restricted eigenvalue condition of the Hessian of the Huber loss over a neighborhood of $\beta^*$. In the usual independent setting, these tasks are accomplished by applying Bernstein’s inequality for independent random variables. The Bernstein-type mixture of the subGaussian and subexponential tails is the key to derive the trade-off of the bias and robustness in AHR. However, the analogous tasks in the Markov-dependent setup are non-trivial, due to the lack of Bernstein’s inequalities for (possibly non-identical) functions of Markov chains. A very recent work of ours
(Jiang, Sun and Fan, 2018) establishes the exact counterpart of Bernstein’s inequality for bounded, non-identical functions of Markov chains. But it still does not fully meet the requirements of the theoretical analyses in this paper, because covariates and errors involved in the theoretical analyses are unbounded. We develop a truncation argument for the extension of the Berstein’s inequality in (Jiang, Sun and Fan, 2018) to unbounded functions.

The rest of this paper is organized as follows. Section 2 introduces the high-dimensional linear model and the methodology of AHR in the Markov-dependent setup. Section 3 presents the assumptions and the main theorem of this paper. Section 4 sketches the proof of the main theorem. Other technical proofs are collected in Section 5. Section 6 concludes the paper with a brief discussion.

2 Model and Methodology

The goal is to estimate the high-dimensional linear model with Markov-dependent covariates and heavy-tailed errors. Start with the linear model as follows.

\[ y_i = x_i^T \beta + \varepsilon_i, \quad i = 1, \ldots, n, \]

where \( y_i \) is the response, \( x_i \in \mathbb{R}^d \) is the vector of \( d \) covariates, \( \varepsilon_i \) is the error, \( \beta \in \mathbb{R}^d \) is the vector of \( d \) regression coefficients. In the high-dimensional regime, \( d \) is much larger than the sample size \( n \). To make the model identifiable, assume only a small number \( s \) of covariates contribute to the response, i.e., the vector of true regression coefficients \( \beta^* \) contains at most \( s \) non-zero elements.

Suppose covariates \( \{x_i\}_{i=1}^n \) are functions of a stationary Markov chain \( \{Z_i\}_{i=1}^n \) on a general state space, i.e., for a collection of \( d \)-dimensional vectorial functions \( \{f_i\}_{i=1}^n \), where \( f_i = (f_{i1}, \ldots, f_{id})^T \),

\[ x_{ij} = f_{ij}(Z_i). \]

The errors are conditionally independent given the underlying Markov chain and possibly heteroskedastic. There exists a conditional distribution \( g(\cdot | z) \) such that

\[ \varepsilon_i | Z_i \sim g(\cdot | Z_i). \]

If \( \{Z_i\}_{i=1}^n \) are i.i.d. then this regression setup reduces to the usual one in which \( \{(x_i, \varepsilon_i)\}_{i=1}^n \) are independent.

For the task of estimating the true \( s \)-sparse coefficients \( \beta^* \), consider the \( \ell_1 \)-regularized robust M-estimator with the Huber loss as follows.

\[ \hat{\beta}_{r, \lambda} = \arg\min_{\beta} H_r(\beta) + \lambda \| \beta \|_1, \quad H_r(\beta) = \frac{1}{n} \sum_{i=1}^n h_r(y_i - x_i^T \beta), \] (1)

where \( h_r \) is the Huber loss (Huber, 1964)

\[ h_r(w) = \begin{cases} w^2/2 & \text{if } |w| \leq \tau \\ \tau|w| - \tau^2/2 & \text{if } |w| > \tau, \end{cases} \] (2)
with the so-called robustification parameter $\tau$ (Fan, Li and Wang, 2017), and $\lambda$ is the regularization parameter encouraging the sparsity of $\beta$ (Tibshirani, 1996).

Heuristically, a larger robustification parameter $\tau$ reduces the bias of $\hat{\beta}_{\tau,\lambda}$ at the cost of less robustness. The extreme case of $\tau = \infty$ corresponds to the ordinary least squares estimation. We find that $\tau$ should adapt to the sample size $n$, the dimensionality $d$, the heavity of the tails of the errors and the dependence of the Markov chain. Suppose the heavy-tailed errors have finite $(1+\delta)$-moments (conditionally on $Z_i$’s) for some $\delta > 0$. A large $\delta$ indicates light tails of errors. The dependence of the Markov chain is measured by $\gamma \in [0, 1]$, denoting the norm of the Markov operator (induced by transition kernel) acting on the Hilbert space of all squared-integrable and mean-zero functions with respect to the invariant distribution. A small $\gamma$ indicates a fast speed of the Markov chain converging towards its stationarity from non-stationarity (Rudolf, 2012). Our analyses show under moderate conditions that the choice of

$$\tau \asymp \left( \frac{1 - \gamma}{1 + \gamma} \cdot \frac{n}{\log d} \right)^{1/(1+\min\{\delta,1\})},$$

achieves the optimal trade-off between bias and robustness in the Markov dependent setup.

3 Assumptions and Theorems

This section presents the main results of this paper under four assumptions. The first assumption is on the converge speed of the underlying Markov chain.

Assumption 1 (Markov Chain with Non-zero Spectral Gap). The Markov chain $\{Z_i\}_{i=1}^n$ is stationary with its unique invariant measure $\pi$ and admits a non-zero spectral gap $1 - \gamma$.

Recall that the quantity $\gamma$ is defined as the norm of the Markov operator (induced by transition kernel) acting on the Hilbert space of all $\pi$-squared-integrable and $\pi$-mean-zero functions. $1 - \gamma$ is called spectral gap of the Markov chain. It has been involved as constants in mean squared error bound for Markov chain Monte Carlo (Rudolf, 2012), Hoeffding-type and Bernstein-type inequalities for Markov chains (Lezaud, 1998; León and Perron, 2004; Chung, Lam, Liu and Mitzenmacher, 2012; Miasojedow, 2014; Paulin, 2015; Fan, Jiang and Sun, 2018a; Jiang, Sun and Fan, 2018). A non-zero spectral gap is closely related to other convergence criteria of Markov chains (Roberts and Rosenthal, 1997; Roberts and Tweedie, 2001; Kontoyiannis and Meyn, 2012).

Next two assumptions allow both covariates and heavy-tailed errors to be heteroskedastic, but impose on them some moment conditions. For the covariates, the boundedness of the fourth moment of an envelop function is required. For the heavy-tailed errors, the boundedness of the conditional $(1+\delta)$-moment is required.

Assumption 2 (Random Design with Bounded Fourth Moments). There exists an envelop function $M : z \mapsto \mathbb{R}$ for the family of functions $f_{ij}$’s, i.e., $M(z) \geq \max_{1 \leq i \leq n, 1 \leq j \leq d} |f_{ij}(z)|$ for $\pi$-almost every $z$. And, $\sigma^2 := \int M^4(z)\pi(dz) < \infty$. 

Assumption 3 (Heavy-tailed Errors with Bounded \((1 + \delta)\)-Moments). \(E[\varepsilon_i | Z_i] = 0\) and \(E[|\varepsilon_i|^{1 + \delta}| Z_i] < v_\delta\) almost surely, i.e., \(\int \varepsilon g(\varepsilon | z) d\varepsilon = 0\) and \(\int |\varepsilon|^{1 + \delta} g(\varepsilon | z) d\varepsilon < v_\delta\) for \(\pi\)-almost every \(z\).

The last assumption is on the restricted eigenvalue of the (aggregated) covariance matrix of covariates. It is a unified condition in the literature of high-dimensional statistics, see e.g., Bickel, Ritov and Tsybakov (2009) and Fan et al. (2018b). Let \(S = \{ j : \beta^*_j \neq 0 \}\) be the index set of active covariates. Define the \(\ell_1\)-cone

\[
C := \{ u \in \mathbb{R}^d : \| u_{S^c} \|_1 \leq 3 \| u_S \|_1 \},
\]

where \(u_{S^c}\) is the subvector assembling all \(u_j, j \in S^c\) and \(u_S\) is the subvector assembling all \(u_j, j \in S\). The constant 3 in the definition of \(C\) has no specific meaning. It can be replaced by other constant larger than 1. Write the (aggregated) covariance matrix of covariates as

\[
\Sigma_n := \frac{1}{n} \sum_{i=1}^{n} E[x_i x_i^T].
\]

Assumption 4 (Restricted Eigenvalue of Covariance Matrix). There exists constant \(\kappa > 0\) such that, for sufficiently large \(n\),

\[
\inf \{ u^T \Sigma_n u : \| u \|_2 = 1, u \in C \} \geq 2\kappa.
\]

It is not hard to see that this condition holds if the smallest eigenvalue of \(\Sigma_n\) is strictly bounded away from 0 for sufficiently large \(n\). Furthermore, if \(f_{ij}\)'s do not vary with \(i\) for each given \(i\), \(\{ x_i \}_{i=1}^n\) are stationary time series as a time-independent function of the underlying Markov chain. In this case, \(\Sigma_n = \Sigma = E[x_1 x_1^T]\), and Assumption 4 holds if the smallest eigenvalue of \(\Sigma\) is strictly bounded away from 0.

Now we are ready to present the main result of this paper. Note that “w.h.p.” stands for “with high probability \(1 - o(1)\)

Theorem 1. Suppose Assumptions 1-4 hold and

\[
s \left( \frac{1 + \gamma}{1 - \gamma} \cdot \frac{\log d}{n} \right)^{(1+\min\{\delta,1\})} = o(1).
\]

Then the AHR estimator \(\hat{\beta}_{r,\lambda}\) in (1) with robustification and regularization parameters

\[
\tau \asymp \left( \frac{1 - \gamma}{1 + \gamma} \cdot \frac{n}{\log d} \right)^{1/(1+\min\{\delta,1\})}, \quad \lambda \asymp \left( \frac{1 + \gamma}{1 - \gamma} \cdot \frac{\log d}{n} \right)^{\min\{\delta,1\}/(1+\min\{\delta,1\})}
\]

achieves estimation errors

\[
\| \hat{\beta}_{r,\lambda} - \beta^* \|_1 \lesssim s\lambda, \quad \| \hat{\beta}_{r,\lambda} - \beta^* \|_2 \lesssim \sqrt{s}\lambda \quad \text{w.h.p.}
\]
Similar to the discovery of Sun, Zhou and Fan (2017) in the independent setup, there is a smooth phase transition of \(\tau\)-adaption and \(\beta^*\)-estimation. A small \(0 < \delta < 1\) suffices for AHR to consistently estimate \(\beta^*\), albeit the rates \(s(\log d/n)^{\delta/(1+\delta)}\) or \(s^{1/2}(\log d/n)^{\delta/(1+\delta)}\) (given fixed \(\gamma < 1\)) are slower than those for the case with \(\delta = 1\). The latter are \(s(\log d/n)^{1/2}\) for \(\ell_1\)-error and \(s^{1/2}(\log d/n)^{1/2}\) for \(\ell_2\)-error, which match those achieved by classical high-dimensional regression methods, but only require bounded second moments of \(\varepsilon_i\). A larger \(\delta > 1\) gains no more estimation accuracy than \(\delta = 1\), though the choices of \(\tau\) is less sensitive. For example, if \(\delta = 2\), we have more flexible choice of \(\tau\) than that for \(\delta = 1.5\), say.

The cases of subGaussian errors belong to the regime of \(\delta > 1\) for any \(\delta\), which has much wider choice of \(\tau\).

The Markov dependence impacts on the adaption of the robustification parameter and the estimation of regression coefficients in the way that the sample size \(n\) is discounted by a factor \((1 - \gamma)/(1 + \gamma) < 1\). In other words, to achieve comparable \(\tau\)-adaption and \(\beta^*\)-estimation, the required sample size increases by \((1 + \gamma)/(1 - \gamma)\) when moving from the independent setup to the Markov-dependent setup. Furthermore, this theorem allows \(\gamma\) to approach to 1 as \(n\) increases, so long as \(s\sqrt{(1 + \gamma)/(1 - \gamma)} \cdot \log d/n \to 0\).

4 Proof of Theorem 1

We break the proof of Theorem 1 into three propositions. Proposition 1 bounds the \(\ell_1\)- and \(\ell_2\)-error of a generic \(\ell_1\)-regularized M-estimator \(\hat{\beta}\) minimizing \(L(\beta) + \lambda ||\beta||_1\). Roughly speaking, given a localized restricted eigenvalue (LRE) condition, it establishes an \(\ell_2\)-error bound

\[
\|\hat{\beta}_{r,\lambda} - \beta^*\|_2 \lesssim \sqrt{s} \|
abla H_{r}(\beta^*)\|_{\infty},
\]

and a similar \(\ell_1\) error bound with an additional factor \(\sqrt{s}\), by applying Proposition 1 to \((L, \hat{\beta}) = (H_r, \hat{\beta}_{r,\lambda})\). This LRE condition requires strictly positive restricted eigenvalues over a local \(\ell_1\)-neighborhood. It is a simplified version of (Sun, Zhou and Fan, 2017, Definition 2). Another LRE condition in \(\ell_2\)-neighborhood has found applications in (Fan, Liu, Sun and Zhang, 2018b, Definition 4.1).

Proposition 1. Consider a \(\ell_1\)-regularized minimizer \(\hat{\beta} = \arg\min L(\beta) + \lambda ||\beta||_1\) of a convex, twice differentiable function \(L : \mathbb{R}^d \mapsto \mathbb{R}\). Suppose the following two conditions hold:

(a) A \(s\)-sparse \(\beta^*\) satisfies \(\|\nabla L(\beta^*)\|_{\infty} \leq \lambda/2\).

(b) (localized restricted eigenvalue) Recall that \(S\) is the support of \(\beta^*\) and that \(C\) is defined as \((3)\). There exists a constant \(\kappa > 0\) such that

\[
\inf \{ u^T \nabla^2 L(\beta) u : u \in C, \|u\|_2 = 1, \|\beta - \beta^*\|_1 \leq 48s\lambda/\kappa \} \geq \kappa.
\]

Then

\[
\|\hat{\beta} - \beta^*\|_1 \leq 48s\lambda/\kappa, \quad \|\hat{\beta} - \beta^*\|_2 \leq 12\sqrt{s}\lambda/\kappa.
\]
To facilitate the proof of Theorem 1 as an application of Proposition 1 to \((L, \hat{\beta}) = (H_\tau, \hat{\beta}_{\tau,\lambda})\), we establish the LRE condition for the Hessian of the Huber loss \(\nabla^2 H_\tau(\beta)\) in Proposition 2 and bound its gradient \(\nabla H_\tau(\beta^*)\) in Proposition 3.

**Proposition 2.** Under Assumptions 1-4, we have

\[
\inf \{ u^T \nabla^2 H_\tau(\beta) u : \|u\|_2 = 1, u \in C, \|\beta - \beta^*\|_1 \leq r \} = 2\kappa - sO_p \left( \frac{1 + \gamma}{1 - \gamma} \cdot \log \frac{d}{n} + \frac{1}{\tau^{1+\delta}} + \frac{r^2}{\tau^2} \right).
\]

**Proposition 3.** Under Assumptions 1-3, for some constant \(C > 0\)

\[
\|\nabla H_\tau(\beta^*)\|_\infty \leq \frac{1 + \gamma}{1 - \gamma} \cdot \frac{2\sigma^2 v_{\min}(\delta, 1)^{1-\min\{\delta, 1\}} \log d}{n} + \frac{1 + \gamma}{1 - \gamma} \cdot \frac{20\tau \log d}{n} + C\tau^{-\min\{\delta, 1\}}, \quad \text{w.h.p.}
\]

Since \(\|\hat{\beta}_{\tau,\lambda} - \beta^*\|_2 \lesssim \sqrt{s}\|\nabla H_\tau(\beta^*)\|_\infty\), optimizing the bound of \(\|\nabla H_\tau(\beta^*)\|_\infty\) in Proposition 3 over \(\tau\) get the optimal \(\ell_2\)-error bound for \(\hat{\beta}_{\tau,\lambda}\). Collecting these pieces together proves Theorem 1.

**Proof of Theorem 1.** Setting

\[
\tau = \left( \frac{1 - \gamma}{1 + \gamma} \cdot \frac{n}{\log d} \right)^{1/(1+\min\{\delta, 1\})},
\]
\[
\lambda = 2\left( \sqrt{2\sigma^2 v_{\min}(\delta, 1) + 20 + C}\right)\tau^{-\min\{\delta, 1\}},
\]
\[
r = 48s\lambda/\kappa
\]

in Propositions 2-3 yields

\[
\inf \{ u^T \nabla^2 H_\tau(\beta) u : \|u\|_2 = 1, u \in C, \|\beta - \beta^*\|_1 \leq r \} \geq \kappa, \quad \text{w.h.p.}
\]
\[
\|\nabla H_\tau(\beta^*)\|_\infty \leq \lambda/2, \quad \text{w.h.p.}
\]

Applying Proposition 1 to \((L, \hat{\beta}) = (H_\tau, \hat{\beta}_{\tau,\lambda})\) completes the proof.

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**5 Other Technical Proofs**

**5.1 Proof of Proposition 1**

The proof of Proposition 1 consists of three steps.

(a) \(\hat{\beta} - \beta^* \in C\), where \(C\) is defined in (3).

(b) \(\|\hat{\beta} - \beta^*\|_1 \leq 4\sqrt{s}\|\hat{\beta} - \beta^*\|_2 \leq r := 48s\lambda/\kappa\).
Proof of Proposition 1(a). By the optimality of $\hat{\beta}$,
\[
\mathcal{L}(\hat{\beta}) - \mathcal{L}(\beta^*) \leq \lambda(\|\beta^*\|_1 - \|\hat{\beta}\|_1).
\]
By the convexity of $\mathcal{L}$, Hölder’s inequality, and the condition that $\|\nabla \mathcal{L}(\beta^*)\|_\infty \leq \lambda/2$,
\[
\mathcal{L}(\hat{\beta}) - \mathcal{L}(\beta^*) \geq \langle \nabla \mathcal{L}(\beta^*), \hat{\beta} - \beta^* \rangle \geq -\|\nabla \mathcal{L}(\beta^*)\|_\infty \|\hat{\beta} - \beta^*\|_1 \geq -\lambda \|\hat{\beta} - \beta^*\|_1/2.
\]
It follows that
\[
2(\|\hat{\beta}\|_1 - \|\beta^*\|_1) \leq \|\hat{\beta} - \beta^*\|_1.
\]
On the left-hand side, using the fact that $\beta_{S^C}^* = 0$,
\[
\|\hat{\beta}\|_1 - \|\beta^*\|_1 = \|\hat{\beta}_{S^C}\|_1 + \|\hat{\beta}_S\|_1 - \|\beta_{S^C}^*\|_1
= \|\hat{\beta}_S - \beta^*\|_1 + \|\beta_{S^C}^* + (\hat{\beta} - \beta^*)_S\|_1 - \|\beta_{S^C}^*\|_1
\geq \|\hat{\beta}_S - \beta^*\|_1 - \|\hat{\beta}_S - \beta^*\|_1.
\]
On the right-hand side,
\[
\|\hat{\beta} - \beta^*\|_1 = \|(\hat{\beta}_S - \beta^*)_S\|_1 + \|(\hat{\beta}_S - \beta^*)_S\|_1.
\]
Putting the last three displays together and rearranging terms completes the proof. \qed

Proof of Proposition 1(b). First, it follows from step (a) that $\|\hat{\beta} - \beta^*\|_1 \leq 4\sqrt{\lambda}\|\hat{\beta} - \beta^*\|_2$.
It is left to show $4\sqrt{\lambda}\|\hat{\beta} - \beta^*\|_2 \leq r$. Suppose for the sake of contradiction that $4\sqrt{\lambda}\|\hat{\beta} - \beta^*\|_2 > r$. By the optimality of $\hat{\beta}$ and the integral form of the Taylor expansion,
\[
0 \geq [\mathcal{L}(\hat{\beta}) + \lambda(\|\hat{\beta}\|_1)] - [\mathcal{L}(\beta^*) + \lambda(\|\beta^*\|_1)]
= \lambda(\|\hat{\beta}\|_1 - \|\beta^*\|_1) + \langle \nabla \mathcal{L}(\beta^*), \hat{\beta} - \beta^* \rangle
+ \int_0^1 (1-t)(\hat{\beta} - \beta^*)^T \nabla^2 \mathcal{L}(\beta^* + t(\hat{\beta} - \beta^*))(\hat{\beta} - \beta^*) dt.
\]
For the first term,
\[
\lambda(\|\hat{\beta}\|_1 - \|\beta^*\|_1) \geq -\lambda \|\hat{\beta} - \beta^*\|_1 \geq -4\sqrt{\lambda}\|\hat{\beta} - \beta^*\|_2.
\]
For the second term, from the condition that $\|\nabla \mathcal{L}(\beta^*)\|_\infty \leq \lambda/2$, it follows that
\[
\langle \nabla \mathcal{L}(\beta^*), \hat{\beta} - \beta^* \rangle \leq \|\nabla \mathcal{L}(\beta^*)\|_s \|\hat{\beta} - \beta^*\|_s + \|\nabla \mathcal{L}(\beta^*)\|_s^\infty \|\hat{\beta} - \beta^*\|_s^\infty \|\hat{\beta} - \beta^*\|_s
\leq \sqrt{s}\|\nabla \mathcal{L}(\beta^*)\|_s \|\hat{\beta} - \beta^*\|_s + \|\nabla \mathcal{L}(\beta^*)\|_s^\infty \|\hat{\beta} - \beta^*\|_s^\infty \|\hat{\beta} - \beta^*\|_s
\leq \frac{\sqrt{s}\lambda}{2} \|\hat{\beta} - \beta^*\|_s^2 + \frac{3\lambda}{2} \|\hat{\beta} - \beta^*\|_s^2 \leq 2\sqrt{s}\lambda \|\hat{\beta} - \beta^*\|_2.
\]
Proceed to lower bound the third term. To this end, note that $r/4\sqrt{s}\|\hat{\beta} - \beta^*\|_2 < 1$ by the initial assumption. For any $0 \leq t \leq r/4\sqrt{s}\|\hat{\beta} - \beta^*\|_2$,
\[
\|\hat{\beta} + t(\hat{\beta} - \beta^*) - \beta^*\|_1 \leq t\|\hat{\beta} - \beta^*\|_1 \leq t \times 4\sqrt{s}\|\hat{\beta} - \beta^*\|_2 \leq r.
\]
Combining it with step (a) that $\hat{\beta} - \beta^* \in C$ and the LRE condition yield a lower bound for the third term

$$\int_0^1 (1-t)(\hat{\beta} - \beta^*)^\top \nabla^2 \mathcal{L}(\beta^* + t(\hat{\beta} - \beta^*))(\hat{\beta} - \beta^*)dt$$

$$\geq \int_0^{r/4\sqrt{s}||\hat{\beta} - \beta^*||_2} (1-t)(\hat{\beta} - \beta^*)^\top \nabla^2 \mathcal{L}(\beta^* + t(\hat{\beta} - \beta^*))(\hat{\beta} - \beta^*)dt$$

$$\geq \int_0^{r/4\sqrt{s}||\hat{\beta} - \beta^*||_2} (1-t)\kappa ||\hat{\beta} - \beta^*||_2^2 dt = \frac{\kappa r}{4\sqrt{s}}||\hat{\beta} - \beta^*||_2 - \frac{\kappa r^2}{32s}.$$ 

Putting the lower bounds of three terms together with the scaling of $r = 48s\lambda/\kappa$ yields

$$0 \geq \left(\frac{\kappa r}{4\sqrt{s}} - 6\sqrt{s}\lambda\right)||\hat{\beta} - \beta^*||_2 - \frac{\kappa r^2}{32s} = 6\sqrt{s}\lambda||\hat{\beta} - \beta^*||_2 - \frac{3\lambda r}{2},$$

which contradicts the initial assumption that $4\sqrt{s}||\hat{\beta} - \beta^*||_2 > r$. 

\[\Box\]

### 5.2 Proof of Proposition 2

We first present two lemmas, which are useful in the proof of Proposition 2. Proofs of these two lemmas are put at the end of this subsection.

**Lemma 1.** Under Assumptions 1-2,

$$\max_{1 \leq j,k \leq d} \left| \frac{1}{n} \sum_{i=1}^n x_{ij}x_{ik} - \frac{1}{n} \sum_{i=1}^n E[x_{ij}x_{ik}] \right| = O_p\left(\sqrt{\frac{1 + \gamma}{1 - \gamma} \cdot \frac{\log d}{n}}\right).$$

**Lemma 2.** Under Assumptions 1-3,

$$\frac{1}{n} \sum_{i=1}^n M^2(Z_i)1\{|\varepsilon_i| > \tau/2\} \leq \sigma^2 \left(\frac{2}{\tau}\right)^{1+\delta} \nu_\delta + O_p\left(\sqrt{\frac{1 + \gamma}{1 - \gamma} \cdot \frac{\log d}{n}}\right).$$

**Proof of Proposition 2.** For any $\beta$ such that $||\beta - \beta^*||_1 \leq r$,

$$1\{|y_i - x_i^\top \beta| > \tau\} \leq 1\{|\varepsilon_i| > \tau/2\} + 1\{|x_i^\top (\beta - \beta^*)| > \tau/2\}$$

$$\leq 1\{|\varepsilon_i| > \tau/2\} + 1\{|x_i|_\infty > \tau/2r\}$$

$$\leq 1\{|\varepsilon_i| > \tau/2\} + 1\{|M(Z_i)| > \tau/2r\},$$

where $M$ is the envelop function introduced by Assumption 2. Let

$$\epsilon_{1n} = \max_{1 \leq j,k \leq d} \left| \frac{1}{n} \sum_{i=1}^n x_{ij}x_{ik} \right| = \max_{1 \leq j,k \leq d} \left| \frac{1}{n} \sum_{i=1}^n x_{ij}x_{ik} - \frac{1}{n} \sum_{i=1}^n E[x_{ij}x_{ik}] \right|$$

$$\epsilon_{2n} = \max_{1 \leq j,k \leq d} \left| \frac{1}{n} \sum_{i=1}^n x_i x_i^\top 1\{|\varepsilon_i| \geq \tau/2\} \right| \leq \frac{1}{n} \sum_{i=1}^n M^2(Z_i)1\{|\varepsilon_i| \geq \tau/2\}$$

$$\epsilon_{3n} = \max_{1 \leq j,k \leq d} \left| \frac{1}{n} \sum_{i=1}^n x_i x_i^\top 1\{|M(Z_i)| \geq \tau/2r\} \right| \leq (2r/\tau)^2 \times \frac{1}{n} \sum_{i=1}^n M^4(Z_i).$$
For any \( u \in C \) such that \( \|u\|_2 = 1 \), we have \( \|u\|_1 \leq 4\sqrt{s} \). Thus,

\[
\begin{align*}
\mathbf{u}^T \nabla^2 H_{\tau}(\beta) \mathbf{u} &= \mathbf{u}^T \left[ \frac{1}{n} \sum_{i=1}^{n} \mathbf{x}_i \mathbf{x}_i^T \mathbf{1}\{|y_i - \mathbf{x}_i^T \beta| \leq \tau\} \right] \mathbf{u}, \\
&\geq \mathbf{u}^T \Sigma_n \mathbf{u} - \mathbf{u}^T \left[ \Sigma_n - \frac{1}{n} \sum_{i=1}^{n} \mathbf{x}_i \mathbf{x}_i^T \right] \mathbf{u} \\
&\quad - \mathbf{u}^T \left[ \frac{1}{n} \sum_{i=1}^{n} \mathbf{x}_i \mathbf{x}_i^T \mathbf{1}\{|\varepsilon_i| > \tau/2\} \right] \mathbf{u} \\
&\quad - \mathbf{u}^T \left[ \frac{1}{n} \sum_{i=1}^{n} \mathbf{x}_i \mathbf{x}_i^T \mathbf{1}\{M(Z_i) > \tau/2r\} \right] \mathbf{u}.
\end{align*}
\]

Further bounding \( \epsilon_{1n} \) by Lemma 1, \( \epsilon_{2n} \) by Lemma 2 and \( \epsilon_{3n} \) by the law of large number for Markov chains (Meyn and Tweedie, 2012, Theorem 17.1.2) completes the proof.

Proof of Lemma 1. Define a truncation operator

\[
T_t(w) = \begin{cases} 
-t & \text{if } w < -t \\
w & \text{if } |w| \leq t \\
+ t & \text{if } w > +t.
\end{cases}
\]

For each \( 1 \leq j \leq d \) and each \( 1 \leq k \leq d \),

\[
\left| \frac{1}{n} \sum_{i=1}^{n} x_{ij} x_{ik} - \frac{1}{n} \sum_{i=1}^{n} \mathbb{E}[x_{ij} x_{jk}] \right| \leq D_{1jk} + D_{2jk} + D_{3jk},
\]

where

\[
\begin{align*}
D_{1jk} &= \left| \frac{1}{n} \sum_{i=1}^{n} \mathbb{E}[T_t(x_{ij} x_{ik})] - \frac{1}{n} \sum_{i=1}^{n} \mathbb{E}[x_{ij} x_{ik}] \right|, \\
D_{2jk} &= \left| \frac{1}{n} \sum_{i=1}^{n} T_t(x_{ij} x_{ik}) - \frac{1}{n} \sum_{i=1}^{n} x_{ij} x_{ik} \right|, \\
D_{3jk} &= \left| \frac{1}{n} \sum_{i=1}^{n} T_t(x_{ij} x_{ik}) - \frac{1}{n} \sum_{i=1}^{n} \mathbb{E}[T_t(x_{ij} x_{ik})] \right|.
\end{align*}
\]

Using the fact that \( |T_t(w) - w| \leq |w|1\{|w| > t\} \leq |w|^2/t \), Cauchy-Schwarz inequality
and the bounds on fourth moment of the envelop function $M : z \mapsto \mathbb{R}$ in Assumption 2,

$$
\max_{j,k} D_{1jk} \leq \max_{j,k} \frac{1}{tn} \sum_{i=1}^{n} \mathbb{E}[|x_{ij}x_{ik}|^2] \leq \max_{j,k} \frac{1}{tn} \sum_{i=1}^{n} \mathbb{E}[x_{ij}^4] \times \frac{1}{tn} \sum_{i=1}^{n} \mathbb{E}[x_{ik}^4] \\
\leq \frac{1}{tn} \sum_{i=1}^{n} \mathbb{E}[M^4(Z_i)] \leq \frac{\sigma^4}{t}.
$$

By a similar argument and the law of large number for Markov chains (Meyn and Tweedie, 2012, Theorem 17.1.2)

$$
\max_{j,k} D_{2jk} \leq \frac{1}{tn} \sum_{i=1}^{n} M^4(Z_i) = \frac{\sigma^4 + o_p(1)}{t}. 
$$

Noting that $|T_w - \mathbb{E}[T_w]| \leq 2t$ almost surely, we apply the Bernstein’s inequality for Markov chains in (Jiang, Sun and Fan, 2018, Theorem 1.1) and yields

$$
P(D_{3jk} > \epsilon) \leq 2 \exp \left( -\frac{n\epsilon^2}{1+\gamma \cdot V_{n,t} + 10t\epsilon} \right) \leq 2 \exp \left( -\frac{n\epsilon^2}{1+\gamma \cdot \sigma^4 + 10t\epsilon} \right),$$

where

$$V_{n,t} = \frac{n}{n} \sum_{i=1}^{n} \text{Var}(T_{w} x_{ij} x_{ik}) \leq \frac{1}{n} \sum_{i=1}^{n} \mathbb{E}[x_{ij}^2 x_{ik}^2] \leq \sigma^4.$$

A union bound delivers that

$$P \left( \max_{j,k} D_{3jk} > \epsilon \right) \leq 2p^2 \exp \left( -\frac{n\epsilon^2}{1+\gamma \cdot \sigma^4 + 10t\epsilon} \right).$$

Let $t = \frac{1}{10c} \sqrt{\frac{1-\gamma}{1+\gamma} \cdot \frac{n}{\log d}}$, and $\epsilon = c \sqrt{\frac{1+\gamma}{1-\gamma} \cdot \frac{\log d}{n}}$ for some $c > \sqrt{2(\sigma^4 + 1)}$. Then

$$\max_{j,k} D_{3jk} \geq c \sqrt{\frac{1+\gamma}{1-\gamma} \cdot \frac{\log d}{n}}$$

with probability at most

$$2 \exp \left( -\left( \frac{c^2}{\sigma^4 + \frac{1-\gamma}{1+\gamma} - 2} \log d \right) \right) \leq 2 \exp \left( -\left( \frac{c^2}{\sigma^4 + 1} - 2 \log d \right) \right).$$

Collecting these pieces together completes the proof. \qed

**Proof of Lemma 2.** Note that $\{(Z_i, \varepsilon_i)\}_{i=1}^{n}$ are identically distributed. Write

$$
\mathbb{E} \left[ M^2(Z_1) 1\{|\varepsilon_1| > \tau/2\} \right] = \mathbb{E} \left[ M^2(Z_1) \mathbb{P}(\{|\varepsilon_1| > \tau/2\}|Z_1) \right] \\
\leq \mathbb{E} \left[ M^2(Z_1) \left( \frac{2}{\tau} \right)^{1+\delta} \mathbb{E}[|\varepsilon_1|^{1+\delta}|Z_1] \right] \\
\leq \mathbb{E}[M^2(Z_1)] \left( \frac{2}{\tau} \right)^{1+\delta} \sigma^2 \left( \frac{2}{\tau} \right)^{1+\delta} \varepsilon_\delta.
$$
It is left to show
\[
\left| \frac{1}{n} \sum_{i=1}^{n} \sum_{i=1}^{n} M^2(Z_i) 1\{| \varepsilon_i | > \tau/2 \} - E \left[ M^2(Z_1) 1\{| \varepsilon_1 | > \tau/2 \} \right] \right| = O_p \left( \sqrt{\frac{1+\gamma}{1-\gamma} \cdot \log d} \right).
\]

Recall that \( \mathcal{T}_t \) is the truncation operator with threshold \( t \) defined in (5). Break down the quantity on the left-hand side of the last display into three terms.

\[ D_1 + D_2 + D_3, \]

where
\[
D_1 = \left| \mathbb{E} \mathcal{T}_t [M^2(Z_1) 1\{| \varepsilon_1 | > \tau/2 \}] - \mathbb{E} [M^2(Z_1) 1\{| \varepsilon_1 | > \tau/2 \}] \right|,
\]
\[
D_2 = \frac{1}{n} \sum_{i=1}^{n} \mathcal{T}_t [M^2(Z_i) 1\{| \varepsilon_i | > \tau/2 \}] - \frac{1}{n} \sum_{i=1}^{n} M^2(Z_i) 1\{| \varepsilon_i | > \tau/2 \},
\]
\[
D_3 = \frac{1}{n} \sum_{i=1}^{n} \mathcal{T}_t [M^2(Z_i) 1\{| \varepsilon_i | > \tau/2 \}] - \mathbb{E} \mathcal{T}_t [M^2(Z_1) 1\{| \varepsilon_1 | > \tau/2 \}].
\]

Using the fact that \( | \mathcal{T}_t (w) - w | \leq |w| 1\{|w| > t\} \leq |w|^2 / t \), Cauchy-Schwarz inequality and the bounds on fourth moment of \( M : z \mapsto \mathbb{R} \) in Assumption 2,

\[ D_1 \leq \frac{\mathbb{E} [M^4(Z_1)]}{t} = \frac{\sigma^4}{t}. \]

By a similar argument and the law of large number for Markov chains (Meyn and Tweedie, 2012, Theorem 17.1.2)

\[ D_2 \leq \frac{1}{tn} \sum_{i=1}^{n} M^4(Z_i) = \frac{\sigma^4 + o_p(1)}{t}. \]

Noting that \( | \mathcal{T}_t (w) - \mathbb{E} \mathcal{T}_t (w) | \leq 2t \) almost surely, we apply the Bernstein’s inequality for geometrically ergodic Markov chains in (Jiang, Sun and Fan, 2018, Theorem 1.1) and yields

\[ P (D_3 > \epsilon) \leq 2 \exp \left( -\frac{nc^2}{1+\gamma} \cdot V_t + 10t\epsilon \right) \leq 2 \exp \left( -\frac{nc^2}{1+\gamma} \cdot \sigma^4 + 10t\epsilon \right), \]

where
\[ V_t = \text{Var} \{ \mathcal{T}_t [M^2(Z_1) 1\{| \varepsilon_2 | > \tau/2 \}] \} \leq \mathbb{E} [M^4(Z_1)] = \sigma^4. \]

Let \( t = \frac{1}{10c} \sqrt{\frac{1+\gamma}{1-\gamma} \cdot \frac{n}{\log d}} \) and \( \epsilon = c \sqrt{\frac{1+\gamma}{1-\gamma} \cdot \frac{\log d}{n}} \) for some \( c > 0 \) then

\[ P \left( D_3 \geq c \sqrt{\frac{1+\gamma}{1-\gamma} \cdot \frac{\log d}{n}} \right) \leq 2 \exp \left( -\frac{c^2 \log d}{\sigma^4 + 1+\gamma} \right). \]

Collecting these pieces together completes the proof. \( \square \)
5.3 Proof of Proposition 3

Proof of Proposition 3. We merely consider the case of $0 < \delta \leq 1$. The proof for the case of $\delta > 1$ is the same with that for $\delta = 1$. Note that $\mathcal{T}_\tau(w)$, defined by (5), is exactly the first-order derivative of the huber loss function $h_\tau$ in (2). For each $1 \leq j \leq d$,

$$-\nabla_j h_\tau(\beta^*) = -\frac{\partial h_\tau(\beta^*)}{\partial \beta_j} = \frac{1}{n} \sum_{i=1}^{n} \mathcal{T}_\tau(\varepsilon_i)x_{ij},$$

Note that $\mathbb{E}[\varepsilon_i | Z_i] = 0$ almost surely in Assumption 3. Write

$$|\mathbb{E}[\nabla_j h_\tau(\beta^*)]| \leq \frac{1}{n} \sum_{i=1}^{n} \mathbb{E}[\mathcal{T}_\tau(\varepsilon_i)|x_{ij}|] = \frac{1}{n} \sum_{i=1}^{n} \mathbb{E}[\mathbb{E}[\mathcal{T}_\tau(\varepsilon_i)|x_i]|x_{ij}|]$$

$$= \frac{1}{n} \sum_{i=1}^{n} \mathbb{E}[\{\mathbb{E}[\varepsilon_i | x_i] - \mathbb{E}[\varepsilon_i | Z_i]\}|x_{ij}]$$

$$\leq \frac{1}{n} \sum_{i=1}^{n} \mathbb{E}[\mathbb{E}[|\varepsilon_i|1\{|\varepsilon_i| > \tau\}|Z_i]|x_{ij}]$$

$$\leq \frac{1}{n} \sum_{i=1}^{n} \mathbb{E}[\mathbb{E}[|\varepsilon_i|^{1+\delta}\tau^{-\delta}|Z_i]|x_{ij}] \leq \sigma \nu_\delta \tau^{-\delta}$$

Next, we use a similar argument to that in Lemmas 1-2 to bound the deviations of $\nabla_j h_\tau(\beta^*)$ from their expectations. Break down the deviation into three terms as follows.

$$|\nabla_j h_\tau(\beta^*) - \mathbb{E}[\nabla_j h_\tau(\beta^*)]| \leq D_{1j} + D_{2j} + D_{3j}$$

where

$$D_{1j} = \left| \frac{1}{n} \sum_{i=1}^{n} \mathbb{E}[\mathcal{T}_\tau(\varepsilon_i)x_{ij}] - \frac{1}{n} \sum_{i=1}^{n} \mathbb{E}[\mathcal{T}_\tau(\varepsilon_i)x_{ij}] \right|, $$

$$D_{2j} = \left| \frac{1}{n} \sum_{i=1}^{n} \mathbb{T}_i[\mathcal{T}_\tau(\varepsilon_i)x_{ij}] - \frac{1}{n} \sum_{i=1}^{n} \mathcal{T}_\tau(\varepsilon_i)x_{ij} \right|, $$

$$D_{3j} = \left| \frac{1}{n} \sum_{i=1}^{n} \mathcal{T}_i[\mathcal{T}_\tau(\varepsilon_i)x_{ij}] - \frac{1}{n} \sum_{i=1}^{n} \mathbb{E}[\mathcal{T}_\tau(\varepsilon_i)x_{ij}] \right|. $$

Using the fact that $|\mathcal{T}_\tau(w) - w| \leq |w| 1\{|w| > t\} \leq |w|^2 / t$, Cauchy-Schwarz inequality, the bounds on moments in Assumptions 2-3,

$$\max_j D_{1j} \leq \frac{1}{tn} \sum_{i=1}^{n} \mathbb{E}[|\mathcal{T}_\tau(\varepsilon_i)|^{2}M^2(Z_i)] \leq \frac{\mathbb{E}[|\varepsilon_1|^{1+\delta}M^2(Z_1)]}{t} \leq \frac{\sigma^2 \nu_\delta \tau^{1-\delta}}{t}.$$ 

Note that the augmented sequence $\{(Z_i, \varepsilon_i)\}_{i=1}^{n}$ is still a stationary Markov chain. By a similar argument and the law of large number for Markov chains (Meyn and Tweedie, 2012, Theorem 17.1.2)

$$\max_j D_{2j} \leq \frac{1}{tn} \sum_{i=1}^{n} \tau^{1-\delta}|\varepsilon_i|^{1+\delta}M^2(Z_i)$$

$$= \frac{\tau^{1-\delta}(\mathbb{E}[|\varepsilon_1|^{1+\delta}M^2(Z_1)] + o_p(1))}{t} \leq \frac{\tau^{1-\delta}(\sigma^2 \nu_\delta + o_p(1))}{t}.$$
Noting that $|T_i[W] - \mathbb{E} T_i[W]| \leq 2t$ almost surely, we apply the Bernstein’s inequality for Markov chains in (Jiang, Sun and Fan, 2018, Theorem 1.1) and yields

$$\mathbb{P}(D_{3j} > \epsilon) \leq 2 \exp\left(-\frac{n\epsilon^2}{\frac{2\gamma}{1-\gamma} \cdot V_{n,t} + 10t\epsilon}\right),$$

where

$$V_{n,t} = \frac{1}{n} \sum_{i=1}^{n} \text{Var}\{T_i[T_T(\epsilon_i)x_{ij}]\} \leq \frac{1}{n} \sum_{i=1}^{n} \mathbb{E}[|T_T(\epsilon_i)|^2x_{ij}^2] \leq \sigma^2v_\delta \tau^{1-\delta}.$$

A union bound delivers that

$$\mathbb{P}\left(\max_j D_{3j} > \epsilon\right) \leq 2p \exp\left(-\frac{n\epsilon^2}{\frac{2\gamma}{1-\gamma} \cdot \sigma^2v_\delta \tau^{1-\delta} + 10t\epsilon}\right).$$

Let $t = \frac{1+\gamma}{1-\gamma} \cdot \tau$ then

$$\max_j D_{3j} \leq \sqrt{\frac{1+\gamma}{1-\gamma} \cdot \frac{2\sigma^2v_\delta \tau^{1-\delta} \log d}{n} + \frac{1+\gamma}{1-\gamma} \cdot 20\tau \log d \cdot \frac{d}{n}}$$

with probability at least $1 - 2/p$. Collecting these pieces together completes the proof.

6 Discussion

Heavy-tailed errors and dependent observations are two stylized features of many real high-dimensional data. However, the current framework of high-dimensional regression assumes subGaussian tails of errors and independent observations for convenience of theoretical analyses. Our long-term goal is to generalize the current framework to cover data with complex structures such as heavy-tailed errors and dependent observations. While (Sun, Zhou and Fan, 2017) proposed AHR for robust estimation against heavy-tailed errors with bounded $(1 + \delta)$-moments, this paper makes progresses in the vertical direction and deals with data under an important dependence structure — Markov dependence.

We find that the $\tau$-adaption and $\beta$-estimation of AHR in the Markov-dependent setup exhibits a similar phase transition to that in the independent setup. The only difference is the sample size $n$ is discounted by a factor $(1 - \gamma)/(1 + \gamma)$, where $\gamma$ is the norm of Markov operator acting on the Hilbert space $L_2(\pi)$ and measures the Markov dependence. The conclusion is still valid if $\gamma \to 1$ as $n \to \infty$, so long as

$$s\sqrt{\frac{1+\gamma}{1-\gamma} \cdot \frac{\log d}{n}} \to 0.$$  

The core technique to prove the main result is the Bernstein’s inequality for possible non-identical functions of Markov chains in our very recent work (Jiang, Sun and Fan, 2018). The mixture of subGaussian and subexponential tails in the Berstein’s inequality
quantitatively characterizes the tradeoff between bias and robustness in the Huber regression. Nevertheless, the Berstein’s inequality in Jiang, Sun and Fan (2018) does not fully meet the need of theoretical analyses in this paper, because the Berstein’s inequality works for bounded functions of Markov chains only in contrast that the covariates are errors are modeled as unbounded functions of the underlying Markov chain. We develop a truncation argument to circumvent this difficulty. This truncation argument is potential useful for other high-dimensional statistical problems in the Markov-dependent setup, e.g., the high-dimensional covariance matrix estimation on Markov-dependent samples.

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