A NEW PROOF OF A BISMUT-ZHANG FORMULA FOR SOME CLASS OF REPRESENTATIONS

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Abstract. Bismut and Zhang computed the ratio of the Ray-Singer and the combinatorial torsions corresponding to non-unitary representations of the fundamental group. In this note we show that for representations which belong to a connected component containing a unitary representation the Bismut-Zhang formula follows rather easily from the Cheeger-Müller theorem, i.e. from the equality of the two torsions on the set of unitary representations. The proof uses the fact that the refined analytic torsion is a holomorphic function on the space of representations.

1. Introduction

Let $M$ be a closed oriented odd-dimensional manifold and let $\text{Rep}(\pi_1(M), \mathbb{C}^n)$ denote the space of representations of the fundamental group $\pi_1(M)$ of $M$. For each $\alpha \in \text{Rep}(\pi_1(M), \mathbb{C}^n)$, let $(E_\alpha, \nabla_\alpha)$ be a flat vector bundle over $M$, whose monodromy representation is equal to $\alpha$. We denote by $H^\bullet(M, E_\alpha)$ the cohomology of $M$ with coefficients in $E_\alpha$. Let $\text{Det}(H^\bullet(M, E_\alpha))$ denote the determinant line of $H^\bullet(M, E_\alpha)$.

Reidemeister [21] and Franz [10] used a cell decomposition of $M$ to construct a combinatorial invariant of the representation $\alpha \in \text{Rep}(\pi_1(M), \mathbb{C}^n)$, called the Reidemeister torsion. In modern language it is a metric on the determinant line $\text{Det}(H^\bullet(M, E_\alpha))$, cf. [19, 2]. If $\alpha$ is unitary, then this metric is independent of the cell decomposition and other choices. In general to define the Reidemeister metric one needs to make some choices. One of such choices is a Morse function $F : M \to \mathbb{R}$. Bismut and Zhang [2] call the metric obtain using the Morse function $F$ the Milnor metric and denote it by $\| \cdot \|_M$.

Ray and Singer [20] used the de Rham complex to give a different construction of a metric on $\text{Det}(H^\bullet(M, E_\alpha))$. This metric is called the Ray-Singer metric and is denoted by $\| \cdot \|^\text{RS}$. Ray and Singer conjectured that the Ray-Singer and the Milnor metrics coincide for unitary representation of the fundamental group. This conjecture was proven by Cheeger [8] and Müller [16] and extended by Müller [17] to unimodular representations. For non-unitary representations the two metrics are not equal in general. In the seminal paper [2] Bismut and Zhang computed the ratio of the two metrics using very non-trivial analytic arguments.

In this note we show that for a large class of representations the Bismut-Zhang formula follows quite easily from the original Ray-Singer conjecture. More precisely, let $\alpha_0 \in \text{Rep}(\pi_1(M), \mathbb{C}^n)$ be a unitary representation which is a regular point of the complex

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analytic set $\text{Rep}(\pi_1(M), \mathbb{C}^n)$ and let $C \subset \text{Rep}(\pi_1(M), \mathbb{C}^n)$ denote the connected component of $\text{Rep}(\pi_1(M), \mathbb{C}^n)$ which contains $\alpha_0$. We derive the Bismut-Zhang formula for all representations in $C$ from the Cheeger-Müller theorem. In other words, we show that knowing that the Milnor and the Ray-Singer metrics coincide on unitary representations one can derive the formula for the ratio of those metrics for all representations in the connected component $C$.

The proof uses the properties of the refined analytic torsion $\rho_{\text{an}}(\alpha)$ introduced in [3, 6, 5] and of the refined combinatorial torsion $\rho_{\epsilon, o}(\alpha)$ introduced in [27, 9]. Both refined torsions are non-vanishing elements of the determinant line $\text{Det}(H^\bullet(M, E_\alpha))$ which depend holomorphically on $\alpha \in \text{Rep}(\pi_1(M), \mathbb{C}^n)$. The ratio of these sections is a holomorphic function

$$\alpha \mapsto \frac{\rho_{\text{an}}(\alpha)}{\rho_{\epsilon, o}(\alpha)}$$

on $\text{Rep}(\pi_1(M), \mathbb{C}^n)$. We first use the Cheeger-Müller theorem to compute this function for unitary $\alpha$. Let now $C \subset \text{Rep}(\pi_1(M), \mathbb{C}^n)$ be a connected component and suppose that a unitary representation $\alpha_0$ is a regular point of $C$. The set of unitary representations can be viewed as the real locus of the connected complex analytic set $C$. As we know $\frac{\rho_{\text{an}}(\alpha)}{\rho_{\epsilon, o}(\alpha)}$ for all points of the real locus, we can compute it for all $\alpha \in C$ by analytic continuation. Since the Ray-Singer norm of $\rho_{\epsilon, o}$ and the Milnor norm of $\rho_{\text{an}}$ are easy to compute, we obtain the Bismut-Zhang formula for all $\alpha \in C$.

The paper is organized as follows. In Section 2, we briefly outline the main steps of the proof. In Subsection 3.5 and Section 3 we recall the construction and some properties of the Milnor metric and of the Farber-Turaev torsion. In Section 4 we recall some properties of the refined analytic torsion. In Section 5 we recall the construction of the holomorphic structure on the determinant line bundle and show that the ratio of the refined analytic and the Farber-Turaev torsions is a holomorphic function on $\text{Rep}(\pi_1(M), \mathbb{C}^n)$. Finally, in Section 6 we present our new proof of the Bismut-Zhang theorem for representations in the connected component $C$.

2. The idea of the proof

Our proof of the Bismut-Zhang theorem for representations in the connected component $C$ consists of several steps. In this section we briefly outline these steps.

Step 1. In [25, 26], Turaev constructed a refined version of the combinatorial torsion associated to an acyclic representation $\alpha$. Turaev’s construction depends on additional combinatorial data, denoted by $\epsilon$ and called the Euler structure, as well as on the cohomological orientation of $M$, i.e., on the orientation $o$ of the determinant line of the cohomology $H^\bullet(M, \mathbb{R})$ of $M$. In [9], Farber and Turaev extended the definition of the Turaev torsion to non-acyclic representations. The Farber-Turaev torsion associated to a representation $\alpha$, an Euler structure $\epsilon$, and a cohomological orientation $o$ is a non-zero element $\rho_{\epsilon, o}(\alpha)$ of the determinant line $\text{Det}(H^\bullet(M, E_\alpha))$. 
Let us fix a Hermitian metric $h^{E_{\alpha}}$ on $E_{\alpha}$. This scalar product induces a norm $\| \cdot \|_{RS}$ on $\text{Det}(H^\bullet(M, E_{\alpha}))$, called the Ray-Singer metric. In Subsection 3.5 we use the Cheeger-Müller theorem to show that for unitary $\alpha$

$$\| \rho_{\varepsilon, \sigma}(\alpha) \|_{RS} = 1. \quad (2.1)$$

**Remark 2.1.** Theorem 10.2 of [9] computes the Ray-Singer norm of $\rho_{\varepsilon, \sigma}$ for arbitrary representation $\alpha \in \text{Rep}(\pi_1(M), \mathbb{C}^n)$, however the proof uses the result of Bismut and Zhang, which we want to prove here for $\alpha \in \mathcal{C}$.

Theorem 1.9 of [5] computes the Ray-Singer metric of $\rho_{\text{an}}(\alpha)$. Combining this result with (2.1) we conclude, cf. Subsection 5.7, that if $\alpha$ is a unitary representation, then

$$\left| \frac{\rho_{\text{an}}(\alpha)}{\rho_{\varepsilon, \sigma}(\alpha)} \right| = \frac{\| \rho_{\text{an}}(\alpha) \|_{RS}}{\| \rho_{\varepsilon, \sigma}(\alpha) \|_{RS}} = 1. \quad (2.2)$$

**Step 2.** The Farber-Turaev torsion $\rho_{\varepsilon, \sigma}(\alpha)$ is a holomorphic section of the determinant line bundle

$$\mathcal{D}et := \bigcup_{\alpha \in \text{Rep}(\pi_1(M), \mathbb{C}^n)} \text{Det}(H^\bullet(M, E_{\alpha}))$$

over $\text{Rep}(\pi_1(M), \mathbb{C}^n)$. We denote by $\rho_{\text{an}}(\alpha)/\rho_{\varepsilon, \sigma}(\alpha)$ the unique complex number such that

$$\rho_{\text{an}}(\alpha) = \frac{\rho_{\text{an}}(\alpha)}{\rho_{\varepsilon, \sigma}(\alpha)} \cdot \rho_{\varepsilon, \sigma}(\alpha) \in \text{Det}(H^\bullet(M, E_{\alpha})).$$

Since both $\rho_{\varepsilon, \sigma}$ and $\rho_{\text{an}}$ are holomorphic sections of $\mathcal{D}et$,

$$\alpha \mapsto \frac{\rho_{\text{an}}(\alpha)}{\rho_{\varepsilon, \sigma}(\alpha)}$$

is a holomorphic function on $\text{Rep}(\pi_1(M), \mathbb{C}^n)$.

**Step 3.** Let $\alpha'$ denote the representation dual to $\alpha$ with respect to a Hermitian scalar product on $\mathbb{C}^n$. Then the Poincaré duality induces, cf. [9, §2.5] and [5, §10.1], an anti-linear isomorphism

$$D : \text{Det}(H^\bullet(M, E_{\alpha})) \longrightarrow \text{Det}(H^\bullet(M, E_{\alpha}')).$$

In particular, when $\alpha$ is a unitary representation, $D$ is an anti-linear automorphism of $\text{Det}(H^\bullet(M, E_{\alpha}))$. Hence,

$$\frac{D(\rho_{\text{an}}(\alpha))}{D(\rho_{\varepsilon, \sigma}(\alpha))} = \frac{\rho_{\text{an}}(\alpha)}{\rho_{\varepsilon, \sigma}(\alpha)} \quad (2.3)$$

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$^1$There is a sign difference in the definition of the duality operator in [9] and [5], which is not essential for the discussion in this paper.
Using Theorem 7.2 and formula (9.4) of [9] we compute the ratio $D(\rho_{\varepsilon, \sigma}(\alpha))/\rho_{\varepsilon, \sigma}(\alpha)$, cf. (6.6) (here $\alpha$ is a unitary representation). On the analytic side Theorem 10.3 of [5] computes the ratio $D(\rho_{\text{an}}(\alpha))/\rho_{\text{an}}(\alpha)$. Combining these two results we get

$$\frac{\rho_{\text{an}}(\alpha)}{\rho_{\varepsilon, \sigma}(\alpha)} = f_2(\alpha) \cdot \frac{\rho_{\text{an}}(\alpha)}{\rho_{\varepsilon, \sigma}(\alpha)},$$

(2.4)

where $f_2$ is a function on $\text{Rep}(\pi_1(M), \mathbb{C}^n)$ computed explicitly in (6.7).

From (2.3) and (2.4) we conclude that

$$\left(\frac{\rho_{\text{an}}(\alpha)}{\rho_{\varepsilon, \sigma}(\alpha)}\right)^2 = f_1(\alpha)^2 \cdot f_2(\alpha)$$

(2.5)

for any unitary representation $\alpha$, cf. (6.9), where $f_1(\alpha) = \rho_{\text{an}}(\alpha)/\rho_{\varepsilon, \sigma}(\alpha)$.

**Step 4.** The right hand side of (2.5) is an explicit function of a unitary representation $\alpha$. It turns out that it is a restriction of a holomorphic function $f(\alpha)$ on $\text{Rep}(\pi_1(M), \mathbb{C}^n)$ to the set of unitary representations. Recall that the connected component $C$ contains a regular point which is a unitary representation. The set of unitary representations can be viewed as the real locus of the complex analytic set $C$. Hence any two holomorphic functions which coincide on the set of unitary representations, coincide on $C$. We conclude now from (2.5) that

$$\left(\frac{\rho_{\text{an}}(\alpha)}{\rho_{\varepsilon, \sigma}(\alpha)}\right)^2 = f(\alpha), \quad \text{for all } \alpha \in C.$$

(2.6)

**Step 5.** Recall that we denote by $\| \cdot \|_M^F$ the Milnor metric associated to the Morse function $F$. In Section 3 we compute the Milnor metric

$$\|\rho_{\varepsilon, \sigma}(\alpha)\|_F^M = h_1(\alpha),$$

(2.7)

where $h_1(\alpha)$ is a real valued function on $\text{Rep}(\pi_1(M), \mathbb{C}^n)$ given explicitly by the right hand side of (3.14).

Theorem 1.9 of [5] computes the Ray-Singer norm of the refined analytic torsion:

$$\|\rho_{\text{an}}(\alpha)\|_{\text{RS}} = h_2(\alpha),$$

(2.8)

where $h_2(\alpha)$ is a real valued function on $\text{Rep}(\pi_1(M), \mathbb{C}^n)$ given explicitly by the right hand side of (4.5). Combining (2.6) with (2.8), we get

$$\frac{\| \cdot \|_{\text{RS}}}{\| \cdot \|_M^F} = \frac{\|\rho_{\text{an}}(\alpha)\|_{\text{RS}}}{\|\rho_{\varepsilon, \sigma}(\alpha)\|_F^M} \cdot \frac{\rho_{\varepsilon, \sigma}(\alpha)}{\rho_{\text{an}}(\alpha)} = \frac{h_2(\alpha)}{h_1(\alpha) \cdot |f(\alpha)|}. $$

(2.9)

This is exactly the Bizmut-Zhang formula [2, Theorem 0.2].

The rest of the paper is occupied with the details of the proof outlined above.
3. THE MILNOR METRIC AND THE FARBER-TURAEV TORSION

In this section we briefly recall the definitions and the main properties of the Milnor metric and the Farber-Turaev refined combinatorial torsion. We also compute the Milnor norm of the Farber-Turaev torsion.

3.1. The Thom-Smale complex. Set

\[ C^k(K, E_\alpha) = \bigoplus_{x \in \text{Cr}(F) \atop \text{ind}_F(x) = k} E_{\alpha, x}, \quad k = 1, \ldots, n, \]

where \( E_{\alpha, x} \) denotes the fiber of \( E_\alpha \) over \( x \) and the direct sum is over the critical points \( x \in \text{Cr}(F) \) of the Morse function \( F \) with Morse-index \( \text{ind}_F(x) = k \). If the Morse function is \( F \) generic, then using the gradient flow of \( F \) one can define the Thom-Smale complex \( (C^\bullet(K, E_\alpha), \partial) \) whose cohomology is canonically isomorphic to \( H^\bullet(M, E_\alpha) \), cf. for example [2, §I c].

3.2. The Euler structure. The Euler structure \( \varepsilon \) on \( M \) can be described as (an equivalence class of) a pair \((F, c)\) where \( c \) is a 1-chain in \( M \) such that

\[ \partial c = \sum_{x \in \text{Cr}(F)} (-1)^{\text{ind}_F(x)} \cdot x, \quad (3.1) \]

cf. [7, §3.1]. We denote the set of Euler structures on \( M \) by \( \text{Eul}(M) \).

Remark 3.3. The Euler structure was introduced by Turaev [26]. Turaev presented several equivalent definitions and the equivalence of these definitions is a nontrivial result. Burghelea and Haller [7] found a very nice way to unify these definitions. They suggested a new definition which is obviously equivalent to the two definitions of Turaev. In this paper we use the definition introduced by Burghelea and Haller.

3.4. The Kamber-Tondeur form. To define the Milnor and the Ray-Singer metrics on \( \text{Det}(H^\bullet(M, E_\alpha)) \) we fix a Hermitian metric \( h_{E_\alpha} \) on \( E_\alpha \). This metric is not necessary flat and the measure of non-flatness is given by taking the trace of \( (h_{E_\alpha})^{-1} \nabla_\alpha h_{E_\alpha} \in \Omega^1(M, \text{End}E_\alpha) \) which defines the Kamber-Tondeur form

\[ \theta(h_{E_\alpha}) := \text{Tr} \left[(h_{E_\alpha})^{-1} \nabla_\alpha h_{E_\alpha}\right] \in \Omega^1(M), \quad (3.2) \]

cf. [14] (see also [2, Ch. IV]).

Let \( \text{Det}(E_\alpha) \to M \) denote the determinant line bundle of \( E_\alpha \), i.e. the line bundle whose fiber over \( x \in M \) is equal to the determinant line \( \text{Det}(E_{\alpha, x}) \) of the fiber \( E_{\alpha, x} \) of \( E_\alpha \). The connection \( \nabla_\alpha \) and the metric \( h_{E_\alpha} \) induce a flat connection \( \nabla_{\alpha}^{\text{Det}} \) and a metric \( h_{\text{Det}(E_\alpha)} \) on \( \text{Det}(E_\alpha) \). Then

\[ \theta(h_{\text{Det}(E_\alpha)}) = \theta(h_{E_\alpha}). \quad (3.3) \]

For a curve \( \gamma : [a, b] \to M \) let

\[ \alpha(\gamma) : E_{\alpha, \gamma(a)} \to E_{\alpha, \gamma(b)}; \quad \alpha^{\text{Det}}(\gamma) : \text{Det}(E_{\alpha, \gamma(a)}) \to \text{Det}(E_{\alpha, \gamma(b)}) \quad (3.4) \]
denote the parallel transports along $\gamma$. Then

$$\det(\alpha(\gamma)) = \alpha^\det(\gamma).$$

(3.5)

Let $\tilde{\gamma}(t) \in \det(E_{a,\gamma(t)})$ denote the horizontal lift of the curve $\gamma$. By the definition of the Kamber-Tondeur form we have

$$\log \frac{h_{\det(E_a)}(\tilde{\gamma}(b), \tilde{\gamma}(b))}{h_{\det(E_a)}(\tilde{\gamma}(a), \tilde{\gamma}(a))} = \int_\gamma \theta(h_{\det(E_a)}) = \int_\gamma \theta(h_{E_a}),$$

(3.6)

where in the last equality we used (3.3).

If $\gamma$ is a closed curve, $\gamma(a) = \gamma(b)$, we obtain

$$\frac{h_{\det(E_a)}(\tilde{\gamma}(b), \tilde{\gamma}(b))}{h_{\det(E_a)}(\tilde{\gamma}(a), \tilde{\gamma}(a))} = |\alpha^\det(\gamma)|^2 = |\det(\alpha(\gamma))|^2.$$

Hence from (3.6) we obtain

$$|\det(\alpha(\gamma))| = e^{\frac{1}{2} \int_\gamma \theta(h_{E_a})}.$$  

(3.7)

3.5. The Milnor metric. The Hermitian metric $h_{E_a}$ on $E_a$ defines a scalar product on the spaces $C^*(K, E_a)$ and, hence, a metric $\| \cdot \|_{\det(C^*(K, E_a))}$ on the determinant line of $C^*(K, E_a)$. Using the isomorphism

$$\phi : \det(C^*(K, E_a)) \to \det(H^*(M, E_a))$$

(3.8)

cf. formula (2.13) of [5], we thus obtain a metric on $\det(H^*(M, E_a))$, called the Milnor metric associated with the Morse function $F$ and denoted by $\| \cdot \|^M_F$.

3.6. The Farber-Turaev torsion. Turaev [26] showed that if an Euler structure is fixed, then the scalar product on the spaces $C^k(K, E_a)$ allows one to construct not only a metric on the determinant line $\det(C^*(K, E_a))$ but also an element of this line, defined modulo sign.

We recall briefly Turaev’s construction. Fix a base point $x_* \in M$. Then every Euler structure $\varepsilon$ can be represented by a pair $(F, c)$ such that

$$c = \sum_{x \in C_r(F)} (-1)^{\text{ind}_F(x)} \gamma_x,$$

with $\gamma_x : [0, 1] \to M$ being a smooth curve such that $\gamma_x(0) = x_*$ and $\gamma_x(1) = x$. The chain $c$ is often referred to as a Turaev spider.

We need to construct an element of the determinant line $\det(C^*(K, E_a))$ of the cochain complex $C^*(K, E_a)$. It is easier to start with constructing an element in the determinant line of the chain complex. Since the cochain complex is dual to the chain complex of the bundle $E_{a'}$, where $a'$ denote the representation dual to $a$, we construct an element in the determinant line $\det(C_a(K, E_{a'}))$. This is done as follows:

Fix an element $v_* \in \det(E_{a',x_*})$ whose norm with respect to the Hermitian metric $h_{\det(E_{a'})}$ is equal to 1 and set

$$v_x := \alpha^\det(\gamma_x)(v_*) \in \det(E_{a',x}),$$
where \( \alpha'^{\text{Det}} \) is the monodromy of the induced connection on the determinant line bundle \( \text{Det}(E_{\alpha'}) \), cf. (3.4). Let

\[
|v|^{\text{Det}(E_{\alpha'})} := \sqrt{h^{\text{Det}(E_{\alpha'})}(v, v)}
\]

denote the norm induced on \( \text{Det}(E_{\alpha'}) \) by the Hermitian metric \( h^{\text{Det}(E_{\alpha'})} \). Then from (3.6) we obtain

\[
|v_x|^{\text{Det}(E_{\alpha'})} = |v_x|^{\text{Det}(E_{\alpha'})} e^{\frac{1}{2} \int_{\gamma_x} \theta(h^{\text{Det}(E_{\alpha'})})} = e^{-\frac{1}{2} \int_{\gamma_x} \theta(h^{\text{Det}(E_{\alpha'})})}.
\] (3.9)

Let

\[
v = \prod_{x \in \mathcal{C}(F)} v_x^{(-1)\text{ind}_F(x)} \in \text{Det} \left( \mathcal{C}^*(K, E_{\alpha'}) \right)/\pm.
\]

(The sign indeterminacy comes from the choice of the order of the critical points of \( F \).) From (3.9) we conclude that

\[
\|v\|^{\text{Det}(\mathcal{C}^*(K, E_{\alpha}))} = e^{-\frac{1}{2} \int_c \theta(h^{\text{Det}(E_{\alpha})})}.
\] (3.10)

Let \( \langle \cdot, \cdot \rangle \) denote the natural pairing

\[
\text{Det} \left( \mathcal{C}^*(K, E_{\alpha}) \right) \times \text{Det} \left( \mathcal{C}^*(K, E_{\alpha'}) \right) \to \mathbb{C}
\]

and let \( \nu \in \text{Det} \left( \mathcal{C}^*(K, E_{\alpha}) \right)/\pm \) be the unique element such that \( \langle \nu, v \rangle = 1 \). From (3.10) we now obtain

\[
\|\nu\|^{\text{Det}(\mathcal{C}^*(K, E_{\alpha}))} = e^{\frac{1}{2} \int_c \theta(h^{\text{Det}(E_{\alpha})})}.
\] (3.11)

Using the isomorphism (3.8) we obtain an element

\[
\phi(\nu) \in \text{Det} \left( H^\bullet(M, E_{\alpha}) \right)/\pm.
\] (3.12)

To fix the sign one can choose a cohomological orientation \( \sigma \), i.e. an orientation of the determinant line \( \text{Det}(H^\bullet(M, \mathbb{R})) \). Thus, given the Euler structure \( \varepsilon \) and the cohomological orientation \( \sigma \) we obtain a sign refined version of \( \phi(\nu) \) which we call the Farber-Turaev torsion and denote by

\[
\rho_{\varepsilon, \sigma}(\alpha) \in \text{Det} \left( H^\bullet(M, E_{\alpha}) \right).
\] (3.13)

3.7. The Milnor norm of the Farber-Turaev torsion. From (3.11) we immediately get

\[
\|\rho_{\varepsilon, \sigma}(\alpha)\|_{F}^{M} = e^{\frac{1}{2} \int_c \theta(h^{E_{\alpha}})}.
\] (3.14)

In particular, if \( \alpha \) is a unitary representation, then \( h^{E_{\alpha}} \) is a flat Hermitian metric and \( \theta(h^{E_{\alpha}}) = 0 \). Hence, if \( \alpha \) is unitary, then

\[
\|\rho_{\varepsilon, \sigma}(\alpha)\|_{F}^{M} = 1.
\] (3.15)

We now use the Cheeger-Müller theorem to conclude that

\[
\|\rho_{\varepsilon, \sigma}(\alpha)\|_{RS}^{M} = 1, \quad \text{if } \alpha \text{ is unitary.}
\] (3.16)
3.8. Dependence of the Farber-Turaev torsion on the Euler structure. For a homology class $h \in H_1(M, \mathbb{Z})$ and an Euler structure $\varepsilon = (F, c) \in \text{Eul}(M)$ we set

$$h\varepsilon := (F, c + h) \in \text{Eul}(M).$$

(3.17)

This defines a free and transitive action of $H_1(M, \mathbb{Z})$ on $\text{Eul}(M)$, cf. [9, §5] or [7, §3.1].

One easily checks, cf. [9, page 211], that

$$\rho_{h\varepsilon, o}(\alpha) = \det(\alpha(h)) \cdot \rho_{\varepsilon, o}(\alpha).$$

(3.18)

From (3.7) and (3.14) we now obtain

$$\|\rho_{h\varepsilon, o}(\alpha)\|_F^M = e^{-\frac{1}{2} \int_{c+h} \theta(h\varepsilon)}. $$

(3.19)

4. The Ray-Singer norm of the Refined Analytic Torsion

In [5] Braverman and Kappeler defined an element of $\text{Det}(H^\bullet(M, E_\alpha))$ called the refined analytic torsion and denoted by $\rho_{\text{an}}(\alpha)$. They also computed the Ray-Singer norm $\|\rho_{\text{an}}(\alpha)\|_{\text{RS}}$ of the refined analytic torsion. In this section we recall the result of this computation.

4.1. The odd signature operator. Fix a Riemannian metric $g^M$ on $M$ and let $\ast : \Omega^\bullet(M, E_\alpha) \to \Omega^{m-\bullet}(M, E_\alpha)$ denote the Hodge $\ast$-operator, where $m = \dim M$. Define the chirality operator

$$\Gamma = \Gamma(g^M) : \Omega^\bullet(M, E_\alpha) \to \Omega^\bullet(M, E_\alpha)$$

by the formula

$$\Gamma \omega := i^r (-1)^{\frac{k(k+1)}{2}} \ast \omega, \quad \omega \in \Omega^k(M, E),$$

(4.1)

where $r = \frac{m+1}{2}$. The numerical factor in (4.1) has been chosen so that $\Gamma^2 = 1$, cf. Proposition 3.58 of [1].

The odd signature operator is the operator

$$\mathcal{B} = \mathcal{B}(\nabla_\alpha, g^M) := \Gamma \nabla_\alpha + \nabla_\alpha \Gamma : \Omega^\bullet(M, E_\alpha) \to \Omega^\bullet(M, E_\alpha).$$

(4.2)

4.2. The eta invariant. We recall from [5, §3] the definition of the sign-refined $\eta$-invariant $\eta(\nabla_\alpha, g^M)$ of the (not necessarily unitary) connection $\nabla_\alpha$.

Let $\Pi_> \ (\text{resp. } \Pi_<)$ be the projection whose image contains the span of all generalized eigenvectors of $\mathcal{B}$ corresponding to eigenvalues $\lambda$ with $\text{Re} \lambda > 0 \ (\text{resp. with } \text{Re} \lambda < 0)$ and whose kernel contains the span of all generalized eigenvectors of $\mathcal{B}$ corresponding to eigenvalues $\lambda$ with $\text{Re} \lambda \leq 0 \ (\text{resp. with } \text{Re} \lambda \geq 0)$, cf. [18, Appendix B]. We define the $\eta$-function of $\mathcal{B}$ by the formula

$$\eta_\theta(s, \mathcal{B}) = \text{Tr} \left[ \Pi_> \mathcal{B}_\theta^s \right] - \text{Tr} \left[ \Pi_< (\mathcal{B}_\theta)^*_s \right],$$

(4.3)

where $\theta$ is an Agmon angle for both operators $\mathcal{B}$ and $-\mathcal{B}$ and $\mathcal{B}_\theta^s$ denotes the complex power of $\mathcal{B}$ defined relative to the spectral cut along the ray $\{re^{i\theta} : r > 0\}$, cf. [22, 24]. It was shown by Gilkey, [11], that $\eta_\theta(s, \mathcal{B})$ has a meromorphic extension to the whole complex plane $\mathbb{C}$ with isolated simple poles, and that it is regular at $s = 0$. Moreover, the number $\eta_\theta(0, \mathcal{B})$ is independent of the Agmon angle $\theta$. 

Let \( m_+(B) \) (resp., \( m_-(B) \)) denote the number of eigenvalues of \( B \), counted with their algebraic multiplicities, on the positive (resp., negative) part of the imaginary axis. Let \( m_0(B) \) denote algebraic multiplicity of 0 as an eigenvalue of \( B \).

**Definition 4.3.** The \( \eta \)-invariant \( \eta(\nabla_\alpha, g^M) \) of the pair \( (\nabla_\alpha, g^M) \) is defined by the formula
\[
\eta(\nabla_\alpha, g^M) = \frac{\eta_0(0, B) + m_+(B) - m_-(B) + m_0(B)}{2}.
\]

If the representation \( \alpha \) is unitary, then the operator \( B \) is self-adjoint and \( \eta(\nabla_\alpha, g^M) \) is real. If \( \alpha \) is not unitary then, in general, \( \eta(\nabla_\alpha, g^M) \) is a complex number. Notice, however, that while the real part of \( \eta(\nabla_\alpha, g^M) \) is a non-local spectral invariant, the imaginary part \( \text{Im} \eta(\nabla_\alpha, g^M) \) of \( \eta(\nabla_\alpha, g^M) \) is local and relatively easy to compute, cf. \[11, 15\].

We also note that the imaginary part of the \( \eta \)-invariant is independent of the Riemannian metric \( g^M \).

**4.4. The Ray-Singer norm of the refine analytic torsion.** Let \( \eta(\nabla_\alpha, g^M) \) denote the \( \eta \)-invariant of the odd signature operator corresponding to the connection \( \nabla_\alpha \). By Theorem 1.9 of \[5\]
\[
\|\rho_{\text{an}}(\alpha)\|_{\text{RS}} = e^{\pi \text{Im} \eta(\nabla_\alpha, g^M)}.
\]
In particular, when \( \alpha \) is a unitary representation, \( \eta(\nabla_\alpha, g^M) \) is real and we get
\[
\|\rho_{\text{an}}(\alpha)\|_{\text{RS}} = 1.
\]

**5. The Determinant Line Bundle over the Space of Representations**

The space \( \text{Rep}(\pi_1(M), \mathbb{C}^n) \) of complex \( n \)-dimensional representations of \( \pi_1(M) \) has a natural structure of a complex analytic space, cf., for example, \[6, \S 13.6\]. The disjoint union
\[
\text{Det} := \bigsqcup_{\alpha \in \text{Rep}(\pi_1(M), \mathbb{C}^n)} \text{Det} \left( H^*(M, E) \right)
\]
(5.1)
is a line bundle over \( \text{Rep}(\pi_1(M), \mathbb{C}^n) \), called the determinant line bundle. In \[4, \S 3\], Braverman and Kappeler constructed a natural holomorphic structure on \( \text{Det} \), with respect to which both the refined analytic torsion \( \rho_{\text{an}}(\alpha) \) and the Farber-Tureav torsion \( \rho_{\varepsilon, o}(\alpha) \) are holomorphic sections. In this section we first recall this construction and then consider the ratio \( \rho_{\text{an}}/\rho_{\varepsilon, o} \) of these two sections as a holomorphic function on \( \text{Rep}(\pi_1(M), \mathbb{C}^n) \).

**5.1. The flat vector bundle induced by a representation.** Denote by \( \pi : \tilde{M} \to M \) the universal cover of \( M \). For \( \alpha \in \text{Rep}(\pi_1(M), \mathbb{C}^n) \), we denote by
\[
E_\alpha := \tilde{M} \times_\alpha \mathbb{C}^n \to M
\]
(5.2)
the flat vector bundle induced by \( \alpha \). Let \( \nabla_\alpha \) be the flat connection on \( E_\alpha \) induced from the trivial connection on \( \tilde{M} \times \mathbb{C}^n \).

For each connected component (in classical topology) \( \mathcal{C} \) of \( \text{Rep}(\pi_1(M), \mathbb{C}^n) \), all the bundles \( E_\alpha, \alpha \in \mathcal{C} \), are isomorphic, see e.g. \[12\].
5.2. **The combinatorial cochain complex.** Fix a CW-decomposition \( K = \{e_1, \ldots, e_N\} \) of \( M \). For each \( j = 1, \ldots, N \), fix a lift \( \tilde{e}_j \), i.e., a cell of the CW-decomposition of \( \tilde{M} \), such that \( \pi(\tilde{e}_j) = e_j \). By (5.2), the pull-back of the bundle \( E_\alpha \) to \( \tilde{M} \times \mathbb{C}^n \to \tilde{M} \). Hence, the choice of the cells \( \tilde{e}_1, \ldots, \tilde{e}_N \) identifies the cochain complex \( C^\bullet(K, \alpha) \) of the CW-complex \( K \) with coefficients in \( E_\alpha \) with the complex

\[
0 \to \mathbb{C}^{n-k_0} \overset{\partial_0(\alpha)}{\to} \mathbb{C}^{n-k_1} \overset{\partial_1(\alpha)}{\to} \cdots \overset{\partial_{m-1}(\alpha)}{\to} \mathbb{C}^{n-k_m} \to 0, (5.3)
\]

where \( k_j \in \mathbb{Z}_{\geq 0} \) (\( j = 0, \ldots, m \)) is equal to the number of \( j \)-dimensional cells of \( K \) and the differentials \( \partial_j(\alpha) \) are \((nk_j \times nk_{j-1})\)-matrices depending analytically on \( \alpha \in \text{Rep}(\pi_1(M), \mathbb{C}^n) \).

The cohomology of the complex (5.3) is canonically isomorphic to \( H^\bullet(M, E_\alpha) \). Let

\[
\phi_{C^\bullet(K, \alpha)} : \text{Det} \left( C^\bullet(K, \alpha) \right) \longrightarrow \text{Det} \left( H^\bullet(M, E_\alpha) \right) (5.4)
\]
denote the natural isomorphism between the determinant line of the complex and the determinant line of its cohomology, cf. [5, §2.4]

5.3. **The holomorphic structure on \( \text{Det} \).** The standard bases of \( \mathbb{C}^{n-k_j} \) (\( j = 0, \ldots, m \)) define an element \( c \in \text{Det} \left( C^\bullet(K, \alpha) \right) \), and, hence, an isomorphism

\[
\psi_{\alpha} : \mathbb{C} \longrightarrow \text{Det} \left( C^\bullet(K, \alpha) \right), \quad z \mapsto z \cdot c.
\]

Then the map

\[
\sigma : \alpha \mapsto \phi_{C^\bullet(K, \alpha)}(\psi_{\alpha}(1)) \in \text{Det} \left( H^\bullet(M, E_\alpha) \right), (5.5)
\]

where \( \alpha \in \text{Rep}(\pi_1(M), \mathbb{C}^n) \) is a nowhere vanishing section of the determinant line bundle \( \text{Det} \) over \( \text{Rep}(\pi_1(M), \mathbb{C}^n) \).

**Definition 5.4.** We say that a section \( s(\alpha) \) of \( \text{Det} \) is holomorphic if there exists a holomorphic function \( f(\alpha) \) on \( \text{Rep}(\pi_1(M), \mathbb{C}^n) \), such that \( s(\alpha) = f(\alpha) \cdot \sigma(\alpha) \).

This defines a holomorphic structure on \( \text{Det} \), which is independent of the choice of the lifts \( \tilde{e}_1, \ldots, \tilde{e}_N \) of \( e_1, \ldots, e_N \), since for a different choice of lifts the section \( \sigma(\alpha) \) will be multiplied by a constant. It is shown in [4, §3.5] that this holomorphic structure is also independent of the CW-decomposition \( K \) of \( M \).

**Theorem 5.5.** Both the refined analytic torsion \( \rho_{\text{an}}(\alpha) \) and the Farber-Turaev torsion \( \rho_{e,\phi}(\alpha) \) are holomorphic sections of \( \text{Det} \) with respect to the holomorphic structure described above.

**Proof.** The fact that the Farber-Turaev torsion is holomorphic is established in Proposition 3.7 of [4]. The fact that the refined analytic torsion is holomorphic is proven in Theorem 4.1 of [4]. \( \square \)
5.6. **The ratio of the torsions as a holomorphic function.** Since both $\rho_{\varepsilon,\sigma}$ and $\rho_{\text{an}}$ are holomorphic nowhere vanishing sections of the same line bundle there exists a holomorphic function

$$\kappa : \text{Rep}(\pi_1(M), \mathbb{C}^n) \rightarrow \mathbb{C}\setminus\{0\}$$

such that

$$\rho_{\text{an}}(\alpha) = \kappa(\alpha) \cdot \rho_{\varepsilon,\sigma}(\alpha).$$

We shall denote this function by

$$\kappa(\alpha) = \frac{\rho_{\text{an}}(\alpha)}{\rho_{\varepsilon,\sigma}(\alpha)}.$$  \hfill (5.6)

5.7. **The absolute value of $\frac{\rho_{\text{an}}(\alpha)}{\rho_{\varepsilon,\sigma}(\alpha)}$ for unitary representations.** Combining (4.6) with (3.16) we obtain

$$\left|\frac{\rho_{\text{an}}(\alpha)}{\rho_{\varepsilon,\sigma}(\alpha)}\right| = \frac{\|\rho_{\text{an}}(\alpha)\|_{\text{RS}}}{\|\rho_{\varepsilon,\sigma}(\alpha)\|_{\text{RS}}} = 1, \quad \text{if } \alpha \text{ is unitary.} \hfill (5.7)$$

6. **The Bismut-Zhang theorem for some non-unitary representations**

We now present our proof of the Bismut-Zhang theorem [2, Theorem 0.2] for representations in the connected component $\mathcal{C}$.

6.1. **The duality operator.** Let $\alpha'$ denotes the representation dual to $\alpha$. The Poincaré duality defines a non-degenerate pairing

$$\text{Det} \left( H^k(M, E_\alpha) \times \text{Det} \left( H^{m-k}(M, E_{\alpha'}) \right) \right) \rightarrow \mathbb{C}, \quad k = 0, \ldots, m,$$

and, hence, an anti-linear map

$$D : \text{Det} \left( H^\bullet(M, E_\alpha) \right) \rightarrow \text{Det} \left( H^\bullet(M, E_{\alpha'}) \right)$$ \hfill (6.1)

see [9, §2.5] and [5, §10.1] for details.

By Theorem 10.3 of [5] we have

$$D \rho_{\text{an}}(\alpha) = \rho_{\text{an}}(\alpha') \cdot e^{2\pi i \left( \eta(\nabla_\alpha, g^M) - (\text{rank } E) \eta_{\text{trivial}}(g^M) \right)},$$ \hfill (6.2)

where $\eta(\nabla_\alpha, g^M)$ is defined in Definition 4.3 and $\eta_{\text{trivial}}$ is the $\eta$-invariant corresponding to the standard connection on the trivial line bundle $M \times \mathbb{C} \rightarrow M$.

6.2. **The dual of the Farber-Turaev torsion.** By Theorem 7.2 of [9]

$$D \rho_{\varepsilon,\sigma}(\alpha) = \pm \rho_{\varepsilon^*,\sigma}(\alpha'),$$ \hfill (6.3)

where $\varepsilon^* := (-F, -c)$ is the dual Euler structure on $M$.

We shall use formula (3.18) in the following situation: if $\varepsilon = (F, c) \in \text{Eul}(M)$ then the Euler structure $\varepsilon^* := (-F, -c)$ is called dual to $\varepsilon$. Since $H_1(M, \mathbb{Z})$ acts freely and transitively on $\text{Eul}(M)$ there exists $c_\varepsilon \in H_1(M, \mathbb{Z})$ such that

$$\varepsilon = c_\varepsilon \varepsilon^*.$$ \hfill (6.4)
The homology class \( c_\varepsilon \) was introduced by Turaev [26] and is called the characteristic class of the Euler structure. From (3.18) and (6.3) we now conclude that
\[
D \rho_{\varepsilon, \sigma}(\alpha) = \pm \rho_{\varepsilon, \sigma}(\alpha') = \pm \Det \left( \alpha'(c_\varepsilon) \right) \cdot \rho_{\varepsilon, \sigma}(\alpha').
\] (6.5)

If \( \alpha \) is a unitary representation, then \( \alpha = \alpha' \). Hence, it follows from (6.5) that
\[
\rho_{\varepsilon, \sigma}(\alpha') = \pm \left( \Det \left( \alpha(c_\varepsilon) \right) \right)^{-1} \cdot D \rho_{\varepsilon, \sigma}(\alpha).
\] (6.6)

6.3. The ratio of torsions for unitary representations. Combining (6.2) and (6.6) we conclude that for unitary \( \alpha \)
\[
\frac{D \rho_{\text{an}}(\alpha)}{D \rho_{\varepsilon, \sigma}(\alpha)} = \pm \Det \left( \alpha(c_\varepsilon) \right) \cdot \rho(\alpha) \cdot \rho\left( \alpha \right)\frac{\rho_{\text{triv}}(\alpha)}{\rho_{\varepsilon, \sigma}(\alpha)}.
\] (6.7)

Since \( D \) is an anti-linear involution we have
\[
\left( \frac{\rho_{\text{an}}(\alpha)}{\rho_{\varepsilon, \sigma}(\alpha)} \right)^2 = \pm \Det \left( \alpha(c_\varepsilon) \right) \cdot e^{2i\pi \left( \eta(\nabla_{\alpha, g^M} - (\text{rank } E) \eta_{\text{triv}}(g^M)) \right)} \cdot \left| \frac{\rho_{\text{an}}(\alpha)}{\rho_{\varepsilon, \sigma}(\alpha)} \right|^2.
\] (6.8)

Combining this equality with (5.7) we obtain for unitary \( \alpha \)
\[
\left( \frac{\rho_{\text{an}}(\alpha)}{\rho_{\varepsilon, \sigma}(\alpha)} \right)^2 = \pm \Det \left( \alpha(c_\varepsilon) \right) \cdot e^{-2i\pi \left( \eta(\nabla_{\alpha, g^M} - (\text{rank } E) \eta_{\text{triv}}(g^M)) \right)} \cdot \left| \frac{\rho_{\text{an}}(\alpha)}{\rho_{\varepsilon, \sigma}(\alpha)} \right|^2.
\] (6.9)

6.4. The ratio of torsions for non-unitary representations. Suppose now that \( \mathcal{C} \subset \text{Rep}(\pi_1(M), \mathbb{C}^n) \) is a connected component and \( \alpha_0 \subset \mathcal{C} \) is a unitary representation which is a regular point of the complex analytic set \( \mathcal{C} \). The set of unitary representations is the fixed point set of the anti-holomorphic involution
\[
\tau : \text{Rep}(\pi_1(M), \mathbb{C}^n) \to \text{Rep}(\pi_1(M), \mathbb{C}^n), \quad \tau : \alpha \mapsto \alpha'.
\]
Hence it is a totally real submanifold of \( \text{Rep}(\pi_1(M), \mathbb{C}^n) \) whose real dimension is equal to \( \dim_{\mathbb{C}} \mathcal{C} \), see for example [13, Proposition 3]. In particular there is a holomorphic coordinates system \( (z_1, \ldots, z_r) \) near \( \alpha_0 \) such that the unitary representations form a real neighborhood of \( \alpha_0 \), i.e. the set \( \text{Im } z_1 = \ldots = \text{Im } z_r = 0 \). Therefore, cf. [23, p. 21], if two holomorphic functions coincide on the set of unitary representations they also coincide on \( \mathcal{C} \). We conclude that the equation (6.9) holds for all representations \( \alpha \in \mathcal{C} \). Hence, using (4.5) and (3.14) we obtain for every \( \alpha \in \mathcal{C} \)
\[
\frac{\| \cdot \|_{RS}^{1/2}}{\| \cdot \|_{M_F}^{1/2}} = \frac{\left| \rho_{\text{an}}(\alpha) \right|^2}{\left| \rho_{\varepsilon, \sigma}(\alpha) \right|^2} \cdot \left| \frac{\rho_{\varepsilon, \sigma}(\alpha)}{\rho_{\varepsilon, \sigma}(\alpha)} \right|^2 \cdot \left| \Det \left( \alpha(c_\varepsilon) \right) \right|^{-1/2} \cdot e^{-\frac{1}{2} \int_{\theta'} f_{\theta_{\text{triv}}}}.
\] (6.10)
6.5. **The absolute value of the determinant of** $\alpha(c_\varepsilon)$. Let

$$PD : H_1(M, \mathbb{R}) \to H^{n-1}(M, \mathbb{R})$$

denote the Poincaré isomorphism. By Proposition 3.9 of [7] there exists a map

$$P : \text{Eul}(M) \to \Omega^{n-1}(M, \mathbb{R})$$

such that

$$P(h\varepsilon) = P(\varepsilon) + PD(h),$$

$$P(\varepsilon^*) = -P(\varepsilon),$$

(6.11)

and if $\varepsilon = (X, c)$ then for every $\omega \in \Omega^1(M, \mathbb{R})$

$$\int_c \omega = \int_M \omega \wedge X^*\Psi(g) - \int_M \omega \wedge P(\varepsilon).$$

(6.12)

Here $\Psi(g)$ is the Mathai-Quillen current on $TM$, cf. [2, §III c] and $X^*\Psi(g)$ denotes the pull-back of this current by $X : M \to TM$.

Combining (6.4) with (6.11) we obtain

$$P(\varepsilon) = P(\varepsilon^*) + PD(c_\varepsilon) = -P(\varepsilon) + PD(c_\varepsilon).$$

Thus

$$PD(c_\varepsilon) = 2P(\varepsilon).$$

(6.13)

Combining this equality with (6.12) we get

$$\int_c \omega = \int_M \omega \wedge X^*\Psi(g) - \frac{1}{2} \int_M \omega \wedge PD(c_\varepsilon).$$

(6.14)

Notice now that

$$\int_M \omega \wedge PD(c_\varepsilon) = \int_{c_\varepsilon} \omega.$$ 

Hence, from (6.14) we obtain

$$\int_{c_\varepsilon} \omega = -2 \int_c \omega + 2 \int_M \omega \wedge X^*\Psi(g).$$

(6.15)

In particular, setting $\omega = \theta(h^{E_\alpha})$ and using (3.7) we obtain

$$\left| \text{Det} \left( \alpha(c_\varepsilon) \right) \right| = e^{-\int_c \theta(h^{E_\alpha}) + \int_M \theta(h^{E_\alpha}) \wedge X^*\Psi(g)}. $$

(6.16)

Combining this equality with (6.10)

$$\| \cdot \|_{M}^{\text{RS}} \| \cdot \|_{F}^{\text{M}} = e^{-\frac{1}{2} \int_M \theta(h^{E_\alpha}) \wedge X^*\Psi(g)},$$

(6.17)

which is exactly the Bizmut-Zhang formula [2, Theorem 0.2].
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