Magnetic Black Holes in the Vector-Tensor Horndeski Theory

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Abstract

We construct novel exact solutions of magnetically charged Black Holes in the vector-tensor Horndeski gravity and discuss their main features. Unlike the analogous electric case, the field equations are linear in a simple (quite standard) parametrization of the metric tensor and they can be solved analytically. The solutions are presented in terms of hypergeometric functions which makes the analysis of the black hole properties relatively straightforward. Some of the aspects of these black holes are quite ordinary like the existence of extremal configurations with maximal magnetic charge for a given mass, or the existence of a mass with maximal temperature for a given charge, but others are somewhat unexpected, like the existence of black holes with a repulsive gravitational field. We perform our analysis for both signs of the non-minimal coupling constant and find black hole solutions in both cases but with significant differences between them. The most prominent difference is the fact that the black holes for the negative coupling constant have a spherical surface of curvature singularity rather than a single point. On the other hand, the gravitational field produced around this kind of black holes is always attractive. Also, for small enough magnetic charge and negative coupling constant, extremal black holes do not exist and all magnetic black holes have a single horizon.

1 Introduction

In the effort aimed to understand the dark matter and dark energy problems of the Universe, numerous extensions of General Relativity (GR) have been studied in the last few decades. Among these generalized gravities the extensions of the minimal Einstein-Hilbert Lagrangian by scalar fields play an important role. With the main objective of maintaining field equations of the second order in field derivatives, thus avoiding the Ostrogradsky instability, the works of G. Horndeski [1] provoked a considerable revival of interest in the last years with a vast range of applications from cosmology to black hole (BH) physics.

A family of vector-tensor theories was also found by Horndeski [2] as an answer to the analogous question: what is the most general extension of the Einstein-Hilbert Lagrangian by a vector field which analogously keeps the field equation of second order. Unlike the scalar-tensor Horndeski theory which has a huge freedom, the vector-tensor theory is essentially unique. It is characterized by a single interaction term between the geometry and the vector field $\mathcal{L}(g, A)$, which we call Horndeski term, with a single coupling constant. The action considered is of the form

$$S = \int d^4x \sqrt{-g} \left[ \frac{1}{2\kappa} R - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{\gamma}{4} (F_{\mu\nu} F^{\kappa\lambda} R_{\kappa\lambda}^{\mu\nu} - 4 F_{\mu\kappa} F_{\nu\kappa} R_{\mu\nu} + F_{\mu\nu} F^{\mu\nu} R) \right]$$

(1.1)

where $F_{\mu\nu}$ is the electromagnetic field strength and $R_{\kappa\lambda}^{\mu\nu}$ is the Riemann tensor with $R_{\mu\nu}^{\kappa\lambda} = R_{\lambda}^{\kappa\nu}$ and $R_{\mu\nu} = R$. We use $\kappa = 8\pi G$ which we later take to be 1 by rescaling. $\gamma$ is a dimensionless
parameter which fixes the strength of the Horndeski non-minimal coupling. The last term $I(g, A)$ is the non-minimal coupling term of the vector field to the geometry introduced by Horndeski \cite{2} as the only possible interaction term which still keeps the field equations of second order:

$$I(g, A) = \frac{1}{4} \star \star R^\mu_\nu \kappa \lambda F^\kappa \lambda F^\mu \nu = -\frac{1}{4} (F_\mu \nu F^\kappa \lambda R^\mu \nu \kappa \lambda - 4F_\mu \kappa F^\mu \nu R^\nu \kappa \lambda + F^\mu \nu F^\nu \mu R)$$

where $\star \star R^\mu_\nu \kappa \lambda$ is the doubly dual Riemann tensor. Similarly, we use the dual field strength $\star F^\mu \nu$.

Both are defined by applying appropriately the Levi-Civita tensor $\sqrt{-g} \epsilon^\kappa \lambda \mu \nu$.

There are various ways to write down the field equations derived from (1.1). We use the following form:

$$\nabla_\mu \left( F^\mu \nu - \gamma \kappa \star \star R^\mu_\nu \kappa \lambda F^\kappa \lambda \right) = 0$$

$$G_\mu \nu + \kappa T^{(Max)}_\mu \nu - \gamma \kappa^2 H_\mu \nu = 0$$

where $T^{(Max)}_\mu \nu$ is the Maxwell standard contribution of the energy-momentum tensor and the contribution from the Horndeski term is:

$$H_\mu \nu = \frac{1}{4} \sqrt{-g} \star \star R^\mu_\nu \kappa \lambda \rho \sigma F^\kappa \lambda F^\rho \sigma - \frac{1}{2} \left( \star \star R^\mu_\lambda \nu \rho \sigma F^\kappa \lambda F^\rho \sigma + \star \star R^\mu_\lambda \nu \rho \sigma F^\rho \sigma \right) F^\kappa \lambda F^\mu \nu$$

$$+ \frac{1}{4} \left( \nabla_\kappa \star F^\mu \nu (\nabla_\kappa \star F^\mu \nu) - \nabla_\mu \star F^\nu \kappa (\nabla_\nu \star F^\kappa \nu) \right)$$

In a sharp distinction with respect to the scalar-tensor Horndeski theory, relatively little effort was invested in its vector-tensor relative. The first studies of this theory were naturally done in the spherically-symmetric static case, first by Horndeski himself \cite{3} and then in more detail by Müller-Hoissen and Sippel \cite{4} who found that the electric solutions contain deformations of the Reissner-Nordstrom (RN) solutions, which may be described as Horndeski-Reissner-Nordstrom (HRN) electrically charged black holes. Some further work clarifying open points followed several years later \cite{5}. Further work was done recently in the context of a scalarized version of these solutions \cite{6}.

In a parallel path, the cosmological aspects of this vector-tensor theory were studied by several authors \cite{7, 8, 9} and others considered also the non-Abelian version \cite{10, 11}.

In this paper we return to the localized spherically-symmetric static solutions, but now magnetically charged. Only very little exists in the literature about the magnetic counterpart of the electrostatic non-minimal BHs mentioned above. Perhaps the reason is the experience from the pure Einstein-Maxwell system where the magnetic black hole is essentially identical to the Reissner-Nordstrom solution, although it is obvious that this should not be the case since the duality symmetry is broken by the Horndeski term $I(g, A)$.

The first work about magnetically charged black holes with the Horndeski non-minimal coupling was a short study in a quite unknown paper by Horndeski \cite{12} that has accumulated 11 citation to date. In addition there are several more recent works \cite{13, 14} which concentrate on the non-Abelian generalization of the Horndesky vector-tensor theory usually containing a larger family of non-minimal coupling terms which yield field equations of order higher than 2. These papers present self-gravitating magnetic monopoles of the Wu-Yang type and magnetic BHs with further extensions like adding a cosmological constant. Some exact solutions have been found too \cite{15}, but they are solutions to some special cases which do not include the Horndeski coupling.

In this paper we revisit the Abelian theory and show that it deserves further study. Most of the study here is based on the finding that the field equations for the magnetically charged Horndeski BHs can be casted as two decoupled linear differential equations that can be solved analytically.
2 The model: Magnetic Spherically-Symmetric Solutions

We are interested in magnetic spherically-symmetric solutions for the Einstein-Maxwell-Horndeski field equations (1.3)-(1.4) with (1.5).

2.1 Ansatz and Field Equations

In order to obtain static spherically symmetric solutions we will adopt a very popular parametrization of the metric

\[ ds^2 = f(r)a^2(r)dr^2 - \frac{1}{f(r)}dr^2 - r^2d\Omega_2^2 \]  

(2.1)

completed by a spherically-symmetric magnetic field derived from the vector potential \( A_\mu dx^\mu = P(1-\cos \theta)d\phi \). The magnetic component \( P \) must be constant since we insist on spherical symmetry. In that case \( P \) is just the magnetic charge. Without loss of generality we assume \( P > 0 \).

Eq. (1.3) is thus satisfied trivially, but from Eq (1.4), or (what is easier), using directly the Lagrangian \( \sqrt{-g}\mathcal{L} \), one finds after some elementary manipulations the following 2 decoupled linear equations:

\[ (r^4 + \gamma \kappa^2 P^2)\frac{da'}{a} + 3\gamma \kappa^2 P^2 = 0 \]  

(2.2)

\[ (r^4 + \gamma \kappa^2 P^2)rf' + (r^4 - 6\gamma \kappa^2 P^2)f + \frac{\kappa P^2}{2}r^2 - r^4 = 0 \]  

(2.3)

There exists a third (second order) equation which is not independent and we do not present here.

2.2 Solutions of the Field Equations

The equation for the function \( a(r) \) is easily solved by:

\[ a(r) = \left| 1 + \frac{\gamma \kappa^2 P^2}{r^4} \right|^{3/4} \]  

(2.4)

where the integration constant is taken such that \( a(r) \to 1 \) asymptotically. The absolute value is added in order to take care of the case \( \gamma < 0 \). The second equation is less trivial to solve, and it is simpler to distinguish between two cases: \( \gamma > 0 \) and \( \gamma < 0 \).

2.2.1 \( f(r) \) for \( \gamma > 0 \)

For \( \gamma > 0 \) we change variables such that \( z = \gamma \kappa^2 P^2/r^4 \) and get for \( f(z) \) the following linear and quite simple equation:

\[ 4(z + 1)zf' + (6z - 1)f - p z^{1/2} + 1 = 0 \]  

(2.5)

where \( p = P/2\gamma^{1/2} \). Notice that this equation contains a single free parameter, \( p \). A second one will be an integration constant which will determine the mass of the BH.

The solution of this equation can be written explicitly and analytically in terms of the Gauss hypergeometric functions \( F(a,b,c,z) \) as (see Appendix):

\[ f(z) = \frac{1}{(1+z)^{7/4}} \left[ -\mu z^{1/4} + pz^{1/2} F \left( \frac{-3}{4}, \frac{1}{4}, \frac{5}{4}, -z \right) + F \left( \frac{-3}{4}, -\frac{1}{4}, \frac{3}{4}, -z \right) \right] \]  

(2.6)
The solution is generally singular at \( z = r/\gamma^{1/4}\sqrt{\kappa P} = z^{-1/4} \) as we see shortly. The dependence on the non-minimal coupling constant \( \gamma \) is actually absorbed in the dimensionless parameters \( \mu \) and \( p \). In terms of the dimensionless radial coordinate \( x = r/\gamma^{1/4}\sqrt{\kappa P} \), the solution reads:

\[
f(x) = \frac{1}{(1 + 1/x^4)^{7/4}} \left[ -\frac{\mu}{x} + \frac{p}{x^2} F \left( -\frac{3}{4}, -\frac{1}{4}, \frac{5}{4}, \frac{1}{x^4} \right) + F \left( -\frac{3}{4}, -\frac{1}{4}, -\frac{3}{4}, -\frac{1}{x^4} \right) \right] \tag{2.7}
\]

and the mass of the black hole will be obtained from the asymptotic behavior of \( f(r) \):

\[
f(r) = 1 - \frac{2M}{r} + \frac{\kappa P^2}{2r^2} - \frac{2\gamma \kappa^2 P^2}{r^4} \left( \frac{27\gamma^2 P^2 M}{2r^5} - \frac{4\gamma \kappa P^4}{5r^6} \right) + ... \tag{2.8}
\]

that is, the coefficient of the \( 1/r \) term is related to the integration constant such that \( 2M = \mu \gamma^{1/4}\sqrt{\kappa P} \). The explicit form of \( f(r) \) is obtained trivially from (2.7):

\[
f(r) = \left( 1 + \frac{\gamma \kappa^2 P^2}{r^4} \right)^{-7/4} \left[ F \left( -\frac{3}{4}, -\frac{1}{4}, \frac{3}{4}, \frac{1}{r^4} \right) - \frac{2M}{r} + \frac{\kappa P^2}{2r^2} F \left( \frac{3}{4}, -\frac{1}{4}, -\frac{5}{4}, -\frac{\gamma \kappa^2 P^2}{r^4} \right) \right] \tag{2.9}
\]

As in the electric case, it depends on the two “hairs”, mass and magnetic charge. Note also that taking \( \gamma = 0 \) in the solution, it goes over to the magnetic Reissner-Nordstrom solution, i.e. the first 3 terms in the asymptotic expansion above. Actually, the solution (2.9) is just a modification of the RN solution by the overall prefactor \( (1 + \gamma \kappa^2 P^2/r^4)^{-7/4} \) and the hypergeometric functions which multiply the RN terms. However, the magnetic RN solution is not a solution of the full vector-tensor system. On the other hand, the Schwarzschild (S) solution is a solution (if \( \gamma \neq 0 \)) if there is no magnetic charge.

### 2.2.2 \( f(r) \) for \( \gamma < 0 \)

For \( \gamma < 0 \) one defines \( z = |\gamma| \kappa^2 P^2/r^4 \) and \( p = P/2|\gamma|^{1/2} \) and get for \( f(z) \):

\[
4(1-z)zf' - (1+6z)f - p z^{1/2} + 1 = 0 \tag{2.10}
\]

The solution is generally singular at \( z = 1 \) and the branch for \( 0 < z < 1 \) (which includes the asymptotic region since \( r^4 \sim 1/z \)) is obtained similarly to the case \( \gamma > 0 \) above to be:

\[
f(z) = \frac{1}{(1-z)^{7/4}} \left[ -\frac{\mu z^{1/4}}{1} + \frac{p z^{1/2}}{1} F \left( -\frac{3}{4}, -\frac{1}{4}, -\frac{5}{4}, z \right) + F \left( -\frac{3}{4}, -\frac{1}{4}, -\frac{3}{4}, z \right) \right] \tag{2.11}
\]

and \( f(x) \) (now \( x = r/(\gamma)^{1/4}\sqrt{\kappa P} = z^{-1/4} \)) is:

\[
f(x) = \frac{1}{(1-1/x^4)^{7/4}} \left[ -\frac{\mu}{x} + \frac{p}{x^2} F \left( -\frac{3}{4}, -\frac{1}{4}, \frac{5}{4}, \frac{1}{x^4} \right) + F \left( -\frac{3}{4}, -\frac{1}{4}, -\frac{3}{4}, \frac{1}{x^4} \right) \right], \quad x > 1 \tag{2.12}
\]

The asymptotic behavior is still given by Eq. (2.8) with \( \gamma \to -|\gamma| \) and the same goes for the explicit form of \( f(r) \) which is still given by Eq. (2.9). However, there is an important difference with respect to the \( \gamma > 0 \) solutions, which is the singularity of Eq. (2.10) and consequently of the generic solutions at \( z_s = x_s = 1 \) or \( r_s = |\gamma|^{1/4}\sqrt{\kappa P} \). This singularity has a significant effect on the solutions since it is not a coordinate singularity, but a “real” curvature singularity at which the Ricci scalar, and the two quadratic invariants \( R_{\mu \nu} R^{\mu \nu} \) and \( R_{\kappa \lambda \mu \nu} R^{\kappa \lambda \mu \nu} \) all diverge. This means that the \( \gamma < 0 \) BH solutions are defined outside a spherical region of a circumferential radius of \( r_s \).
3 Main Characteristics of the Solutions: $\gamma > 0$

The first thing to be studied is the horizon structure of the solutions. So we start by presenting some profiles of the metric components of the solutions. Figure 1 shows the metric components for the typical value $\gamma = 1$ of the coupling constant. The upper two panels of the figure show the common pattern that is found throughout most of parameter space where $\gamma > 0$. In these profile plots we rescale the radial coordinate by the length parameter $r_s$, so we actually use the dimensionless coordinate $x = r/r_s$. In the next plots we will rescale the radial coordinate and the mass by $\sqrt{\kappa}$ only in order to identify more clearly the role of each of the mass, magnetic charge and horizon separately. Unless ambiguity may occur, we will use the same symbols $r$ and $M$ for the rescaled quantities.

First we note that the function $f(r)$ vanishes at the origin, unlike the RN case. This is a direct result from the modification of the RN solution by the hypergeometric factors. It is obvious that away from the origin and asymptotically, the behavior is similar to RN, but near the origin the behavior is modified drastically.

In addition, $f(r)$ may have two nodes for $r > 0$ or no nodes, and in between, there is the

![Figure 1: Upper part: Profiles of $g_{00}(r)$ and of $f(r) = -1/g_{rr}(r)$ for $\gamma = 1$ and $P = 8$ for several values of the dimensionless mass parameter: $2M/r_s = \mu = 2, 3, 3.988$ (for the extremal solution), 4.5, 5.5, 6.5. The mass increases in a “spectral order” from red to blue, or lower curves correspond to larger mass. Lower part: Profiles of negative mass BHs which appear for small $P$ (see text). The $\mu$ values are: $\mu = -1.390$ (extremal), $-1.2, -1.0, -0.75, -0.5, -0.25, 0$. Note especially the decreasing $g_{00}(r)$ beyond the maximum.](image-url)
special (“extremal”) case where the two nodes degenerate to one. Figure 2 summarizes the general dependence of the points where \( f(r_0) = 0 \) on the mass and magnetic charge. Note that the function is doubly-valued. The larger of the two zeroes (when they exist) is of course the event horizon of the black hole which we will denote by \( r_H \). Otherwise, we have naked singularities. The special value of \( r_0 \) where the two zeroes merge corresponds to the extremal BH. From the fact that for an extremal BH both \( f(r) \) and \( f'(r) \) vanish at \( r_0 \), it can be deduced that the extremal BH radius is determined by the magnetic charge as \( r_{ext} = \sqrt{\kappa P^2/2} \) as in the RN case. Unlike the RN case, the extremal mass is not equal to \( \sqrt{\kappa P^2/2} \). It is smaller as seen e.g. from the extremal mass value cited in Fig. which is smaller than \( \sqrt{\kappa P^2/2} \).

The simplest method to obtain all those results is to get from the equation \( f(r_0) = 0 \) an explicit expression for \( M(\gamma, P, r_0) \):

\[
M(\gamma, P, r_0) = \frac{r_0}{2} F\left(\frac{3}{4}, \frac{1}{4}, \frac{3}{4}, -\frac{\gamma \kappa^2 P^2}{r_0^4}\right) + \frac{\kappa P^2}{4r_0} F\left(-\frac{3}{4}, \frac{1}{4}, \frac{5}{4}, -\frac{\gamma \kappa^2 P^2}{r_0^4}\right)
\]

(3.1)

Differentiating \( M(\gamma, P, r_0) \) with respect to \( r_0 \) (using (6.5)) in order to find the extremal point gives directly the linear relation \( r_{ext} = \sqrt{\kappa P^2/2} \). Substituting this relation back in Eq. yield the equation for the extremal mass curve in the \( P-M \) plane:

\[
M_{ext}(\gamma, P) = \sqrt{\frac{\kappa P^2}{8}} \left[ F\left(-\frac{3}{4}, -\frac{1}{4}, \frac{3}{4}, -\frac{4\gamma}{P^2}\right) + F\left(-\frac{3}{4}, -\frac{1}{4}, \frac{5}{4}, -\frac{4\gamma}{P^2}\right)\right]
\]

(3.2)

Figure 2: Left: Dependence of \( r_0 \) on the mass and magnetic charge for \( \gamma = 1 \). Right: Sections of the surface at several values of \( M \) in even intervals from \( M = 0 \) to \( M = 8 \). The mass values increase in a “spectral order” from red to violet.
It is quite easy to see from this expression that $M_{\text{ext}}(\gamma, P)$ is always (for $\gamma > 0$) smaller than the corresponding RN value and that the difference decreases with increasing magnetic charge.

So we find it more illuminating to add the other two sections of the three-dimensional surface of Fig. 2 which are presented below in Fig. 3.

A new feature of this Einstein-Maxwell-Horndeski system is the existence of BH-like solutions with negative mass as is obvious from the lower two panels of Fig. 1 and from both parts of Fig. 3. Of course, $M = 0$ RN solutions exist too, but they have naked singularities. As opposed to RN, in the present circumstances if the magnetic charge is small enough, the singularity is hidden by the two horizons as seen also at the lower left corner of Fig. 2 – where it is obvious that a whole line in the $P-r_0$ plane with $M = 0$ exists. Moreover, $M = 0$ is not a limiting case, but can be crossed and a new type of solutions appears that have the same horizon structure (inner and outer) but presenting a negative mass. This is clearly reflected in the $g_{00}(r)$ metric component of Fig. 1 which asymptotically decreases with $r$. So these solutions have a repulsive gravitational field. This of course has no analogue in the RN solution where the horizon structure is determined by the mass and charge such that $M$ should be not only positive, but also larger than $\sqrt{\kappa P^2/2}$. In the present case there is an additional Horndeski contribution to the extremal mass and it is easy to see that it is negative and its absolute value increases with $\gamma$.

The domain where negative BH masses are possible can be obtained from Eq. (3.2) for the extremal mass, by the condition $M_{\text{ext}} < 0$. This translates to a transcendental algebraic equation in the sum of the two hypergeometric functions which gives the maximal $\gamma$-dependent value of $P$ where negative mass BHs are possible, to satisfy $4\gamma/P^2 = 25.61190$. This means that negative mass BHs can be found for magnetic charges of $0 < P < 0.395193^{1/2}$. Note that for any finite $P$

![Figure 3: The two other sections of the surface in the $M-P-r_0$ space of Fig. 2. Left: $r_0-M$ section for several values of $P$. Right: $P-M$ section for several values of $r_0$. Note the negative mass region near the origin in both plots.](image)
Figure 4: The BH temperature as a function of the BH mass for several values of the magnetic charge. All curves start at the extremal BHs where $T = 0$. The line in the RHS figure corresponds to Eq. (3.2).

in this interval the extremal BHs may be negative, but they are still bounded from below.

Next, we turn to the temperature of these black holes, for which we adopt the conventional definition \cite{17} in terms of the surface gravity $\mathcal{K}$: $T = \mathcal{K}/2\pi$. In the static spherically-symmetric case with our parametrization, the temperature is given by $T(\gamma, P, r_H) = a(r_H)f'(r_H)/(4\pi)$ and by using the field equations we find the explicit result

$$T(\gamma, P, r_H) = \frac{r_H^2 - \kappa P^2/2}{4\pi r_H^2(r_H^4 + \gamma\kappa^2 P^2)^{1/4}}. \quad (3.3)$$

First we note that the extremal BHs have zero temperature as usual. We also can check that when the BH is not magnetically charged we get back the Bekenstein-Hawking temperature $T = 1/4\pi r_H$. More generally, we can learn how the BH temperature depends on the mass and magnetic charge from Fig. 4. The function $T(\gamma, P, M)$ cannot be written down explicitly, so the best we can do is to use the parametric representation $(M(\gamma, P, r_H), T(\gamma, P, r_H))$ with $r_H$ as a parameter. The most prominent feature of all curves is the existence of a maximal temperature, very much like the behavior of the RN BH. However, while the maximal RN temperature is inversely proportional to the magnetic charge, in the presence of the Horndeski term it is not exactly $\sim 1/P$ for all $P$, although the difference is quite small and the behavior becomes $1/P$ asymptotically.

This brings us to the final point which is the effect of varying the non-minimal coupling parameter $\gamma$. It has of course a decisive effect, but technically it is quite simple to understand since most of it is realized through a scaling behavior which originates from the fact that there are actually two independent free parameters which determine the solutions: $\mu$ and $p$. So the effect of $\gamma$ is done only through these two parameters. One example of the scaling behavior can be seen in Eq. (3.2) for the extremal mass where $M_{\text{ext}}(\gamma, P)$ depends on $\gamma$ through the ratio $\gamma/P^2$. This scaling behavior is not valid for vanishing magnetic charge, but we may exclude this case in the present discussion since anyhow we know already that $P = 0$ gives the Schwarzschild (S) solution for vanishing as well as for non-vanishing $\gamma$.

The same can be done for the temperature if we use the radial coordinate $\xi = r/r_{\text{ext}} = r\sqrt{2/\kappa P^2}$. Then we can write the temperature as
Figure 5: Profiles of $g_{00}(r)$ and of $f(r) = -1/g_{rr}(r)$ for $\gamma = -1$ and $P = 8$ and several values of the mass parameter: $\mu = 2.5$, 4.013 (the extremal solution), 4.5, 4.603 (the intermediate solution), 5, 6.

\[ T(\gamma, P, \xi_H) = \frac{\sqrt{2}}{4\pi\sqrt{\kappa P^2}} \frac{\xi_H^2 - 1}{\xi_H^2 (\xi_H^4 + 4\gamma/P^2)^{1/4}}. \] (3.4)

So here too, most of the effect of increasing $\gamma$ can be done also by decreasing the magnetic field and keeping $\gamma$ fixed – of course, as long as it stays finite.

4 Main Characteristics of the Solutions: $\gamma < 0$

The nature of the solutions for $\gamma < 0$ is quite different from that of $\gamma > 0$, although some similarities do exist. One similarity is the fact that the mass function is still given by Eq. (3.1) and $r_{\text{ext}} = \sqrt{\kappa P^2/2}$ still plays the role of the horizon of the Extremal BHs, although the opposite sign changes the behavior noticeably. As for other differences, first we mention again the fact that the $\gamma < 0$ solutions are well-defined only outside the singular sphere at $r = r_s$. A direct consequence from this condition is that the horizon size must always be larger than $r_s$, so it seems that $\gamma < 0$ RN-like BH solutions (see below) are more constrained due to the additional condition $r_{\text{ext}} \geq r_s$ which in terms of $P$ reads $P/|\gamma|^{1/2} = p \geq 1$.

Next, we present in Fig. 5 typical profiles for the same $P$ and $|\gamma|$ as in Fig. 4 and notice immediately that whereas the $\gamma > 0$ BHs all have a RN-like behavior ($g_{00}(r) \to \infty$ as $r \to 0$), the $\gamma < 0$ ones are split between RN-like behavior for small masses and S-like behavior for large masses. In between there exists a solution which seems totally finite and regular for all $r > 0$. However, this solution too suffers from the same curvature singularity at $r = r_s$ as the calculation of the curvature invariants shows immediately. The “culprit” is of course the function $a^2(r)$ which for $\gamma < 0$ is differentiable only once at $r = r_s$. We also note that the extremal solution has its double zero at the same point as for $\gamma > 0$ but the mass parameter is higher. Indeed in these two figures the mass parameter is close to $\mu = 4$, but it is a result of the large $P$ behavior of the function $M_{\text{ext}}(\gamma, P)$ in Eq. (3.2). The general pattern is that for $\gamma < 0$ the extremal masses are also larger than the RN bound of $\sqrt{\kappa P^2/2}$. This is in accord with the previous observation that increasing $\gamma$ tends to decrease the BH mass.

\[ \text{Recall however, that for } \gamma < 0, g_{00}(r) \text{ diverges at } r_s. \]
Figure 6: BH mass and magnetic charge vs. \( r_0 \) for \( \gamma = -1 \) and several values of \( P \) and \( M \) respectively. The line at the left in each panel represents the curvature singularity which is located at \( x = r/(\gamma^{1/4}\sqrt{\kappa P}) = 1 \).

The same function of (3.2) can give an additional view on the fact that the RN-like solutions exist in a limited domain in parameter space: First of all we recall that \( M_{\text{ext}}(\gamma, P) \) is defined for RN-like solutions only. It says nothing about S-like ones. Second, \( M_{\text{ext}}(\gamma, P) \) is defined (actually, is real) only for \(-4\gamma/P^2 < 1\) which for \( \gamma < 0 \) is the same as \( p > 1 \). So extremal RN BHs can exist for \( p > 1 \) only. The condition \( p < 1 \) is realized by the S-like BHs. In other words, if \( p < 1 \), all BH solutions are S-like. This points up to another special feature of these S-like BHs: since the horizon size cannot be arbitrarily small, there is a lower bound on the mass of the S-like BHs with a given \( P \) provided also that \( P < 2|\gamma|^{1/2} \):

\[
M_{\text{min}}^{(S)}(\gamma, P) = \Gamma\left(\frac{7}{4}\right) |\gamma|^{1/4}\sqrt{\kappa P} \left(\frac{\Gamma(3/4)}{\sqrt{\pi}} + \frac{\Gamma(5/4)P}{4|\gamma|^{1/2}}\right), \quad P < 2|\gamma|^{1/2}
\]

This is the analogue of the S-like BHs to the extremal mass given by Eq. (3.2).

To return to \( p > 1 \), this is not a sufficient condition to have RN-like solutions. The reason is more obvious from Fig. 6 which presents the dependence of the BH mass and magnetic charge on the zeroes \( r_0 \) of the metric components. Notice the “forbidden zone” bounded by the curvature singularity. Inspecting the left-hand-side panel of Fig. 6 we can see that above the \( p = 1 \) line, the curves of \( M(\gamma, P, r_0) \) develop a minimum (at a certain \( r_0 \)), but unlike the RN case where for each \( M \) there are two values of \( r_0 \), in the present case there is an additional upper bound, say \( M_{\text{inter}}(\gamma, P) \) above which all BH solutions will have a single zero. This \( M_{\text{inter}}(\gamma, P) \) is the mass where the curve \( M(\gamma, P, r_0) \) crosses the line of curvature singularity. For the case depicted in Fig. 5 with \( p = 4 \), the corresponding mass parameter is \( \mu_{\text{inter}} = 4.603 \) which determines the intermediate solution shown in the figure. So, RN-like solutions exist only for masses \( M_{\text{ext}}(\gamma, P) < M < M_{\text{inter}}(\gamma, P) \). If \( M > M_{\text{inter}}(\gamma, P) \) there exist only S-like BHs, as is seen clearly in Fig. 5.
Notice also in Fig. 6 the region of small $M$ and $P$ where only one zero exists for each $M$ and $P$. This is the region of $p < 1$ where all BHs are S-like. We did not include a figure containing profiles corresponding to this case. They are similar to those of the lower curves of Fig. 5.

Also obvious from these plots (and from direct study of (3.1) for $\gamma < 0$) is that no negative mass BHs exist for $\gamma < 0$.

Next we move to the temperature which is obtained similarly to (3.3) as

$$T(\gamma < 0, P, r_H) = \frac{\gamma_H^2 - \kappa P^2/2}{4\pi r_H^2 (r_H^4 - |\gamma| \kappa^2 P^2)^{1/4}}.$$ \hspace{1cm} (4.2)

Of course, in order to stay away from the singularity at $r = r_s$ we have to assume $r_H > r_s$.

We expect to find for the temperature two different types of behavior corresponding to the two types of $\gamma < 0$ BH solutions, and indeed this is the case as demonstrated by Fig. 7 which presents the mass dependence of the temperature for several values of $P$. For large enough magnetic charge the temperature behaves similarly to the $\gamma > 0$ RN-like case: it starts with $T(r_{ext}) = 0$ rises to a maximum and then decreases monotonically, asymptotically to zero. For the smaller values of $P$, no extremal BHs exist since the BH solutions are S-like. Thus the temperature is always strictly positive and generally decreases with $M$. The decrease is monotonic for small enough $P$, while some “bumps” develop for higher values of $P$ which are still too small to allow for extremal BHs, i.e. still $p < 1$.

By comparison with Fig. 4 we find that for the same values of mass and charge, the temperatures of the $\gamma < 0$ RN-like BHs are very close to the $\gamma > 0$ ones, but not identical.

Figure 7: The BH temperature as a function of the BH mass for several values of the magnetic charge. Notice the two different types of behavior: The RN-like curves start at the extremal BHs where $T = 0$, attain a maximum and decrease. The others correspond to the small $P$ S-like BHs. The “spikes” are not singularities, but look like that because of resolution. The dashed line corresponds to the S BH.
5 Conclusion

In this paper we constructed the magnetically charged Black Hole solution of the Einstein-Maxwell system with an additional non-minimal coupling between the vector field and the gravitational field of the form $F^{\kappa\lambda} F_{\mu\nu} R_{\kappa\lambda\mu\nu}$. Unlike the analogous electric case, the field equations become linear and they are solved analytically in terms of hypergeometric functions and other elementary functions. These solutions describe new kinds of black holes. A large portion of the solutions are similar to the ordinary RN solutions, but others may have some exotic properties. Some may have repulsive gravitational field around them, while others have a spherical curvature singularity rather than a point-like. We have calculated their mass-charge-horizon relations, their temperature behavior and other characteristics. We found a significant difference between positive and negative $\gamma$. For $\gamma > 0$ all BHs are RN-like, i.e. $g_{00}(r) \to \infty$ as $r \to 0$. On the other hand, for $\gamma < 0$ all BH solutions have a spherical surface of curvature singularity but they are divided into two subclasses: RN-like and S-like. This difference is reflected by their temperature dependence on their masses: The temperature of an RN-like BH starts at zero in an extremal solution, increases with $M$ until it reaches its maximum, and then decreases. The S-like BHs have a temperature with similar decreasing behavior for large $M$, but the temperature is not bounded from above but diverges for a minimal (non-zero) value of the mass given by Eq. (4.1).

Still, further analysis of these systems is in order. Some of the immediate directions are: studying more systematically the negative mass BHs for $\gamma > 0$ and the properties of the spherical singularity of the BH solutions for for $\gamma < 0$, finding the trajectories of point particles (neutral or charged) around the BHs, studying their entropy, generalizing to the non-Abelian case and coupling scalar fields and discussing scalarization. Another direct ion of study is the question whether the fact that the field equations become linear enables to superpose these elementary solutions to construct richer structures.

6 Appendix: Solving Equation (2.5)

The equation (2.5) is:

$$4(z + 1)zf' + (6z - 1)f - p z^{1/2} + 1 = 0 \quad (6.1)$$

This is a linear inhomogeneous ODE. The standard procedure to solve this equation is to obtain the general solution of the homogeneous equation. This is a trivial matter of a direct integration:

$$f_0(z) = c_1 \frac{z^{1/4}}{(1 + z)^{7/4}} \quad (6.2)$$

Next we replace the integration constant $c_1$ by a function $u(z)$, so the solution is the product $f(z) = f_0(z)u(z)$. Consequently, the function $u(z)$ must solve

$$4(z + 1)z f_0(z)u' - p z^{1/2} + 1 = 0 \quad (6.3)$$

The solution for $u(z)$ is again obtained by direct integration which is expressed in terms of two hypergeometric functions:

$$u(z) = pz^{1/4}F\left(-\frac{3}{4},\frac{1}{4},\frac{5}{4},-z\right) + \frac{1}{z^{1/4}}F\left(-\frac{3}{4},\frac{1}{4},\frac{3}{4},-z\right) - \mu \quad (6.4)$$
where $\mu$ is the integration constant. The integration is performed by the following identity for hypergeometric functions:

$$
\int (1 + z)^\alpha z^\beta dz = \frac{z^{\beta+1}}{\beta + 1} F (-\alpha, \beta + 1, \beta + 2, -z)
$$

which is obtained in a straightforward way from the standard properties of the hypergeometric functions (see e.g. Abramovitz and Stegun [16]). The function $f(z)$ which solves Eq. (2.5) is just a product of the two functions in (6.2) and (6.4):

$$
f(z) = \frac{z^{1/4}}{(1 + z)^{7/4}} \left[ p z^{1/4} F \left( -\frac{3}{4}, \frac{1}{4}, \frac{5}{4}, -z \right) + \frac{1}{z^{1/4}} F \left( -\frac{3}{4}, -\frac{1}{4}, \frac{3}{4}, -z \right) - \mu \right]
$$

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