On the classification of solutions to a weighted elliptic system involving the Grushin operator.

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Abstract

We investigate here the following weighted degenerate elliptic system

\[-\Delta_s u = \left(1 + \|x\|^{2(s+1)}\right)^{\frac{2}{2s+1}} u^p, \quad -\Delta_s v = \left(1 + \|x\|^{2(s+1)}\right)^{\frac{2}{2s+1}} u^\theta, \quad u, v > 0 \quad \text{in} \quad \mathbb{R}^N := \mathbb{R}^{N_1} \times \mathbb{R}^{N_2}.\]

where \(\Delta_s = \Delta_x + |x|^{2s} \Delta_y\), is the Grushin operator, \(s \geq 0\), \(\alpha \geq 0\) and \(1 < p \leq \theta\). Here

\[\|x\| = \left(|x|^{2(s+1)} + |y|^2\right)^{\frac{1}{2s+1}}, \quad \text{and} \quad x := (x, y) \in \mathbb{R}^N := \mathbb{R}^{N_1} \times \mathbb{R}^{N_2}.\]

In particular, we establish some new Liouville-type theorems for stable solutions of the system, which recover and considerably improve upon the known results \[2, 10, 12, 7, 5\]. As a consequence, we obtain a nonexistence result for the weighted Grushin equation

\[-\Delta_s u = \left(1 + \|x\|^{2(s+1)}\right)^{\frac{2}{2s+1}} u^p, \quad u > 0 \quad \text{in} \quad \mathbb{R}^N.\]

Keywords: Stable solutions, Liouville-type theorem, Weighted Grushin equation, Weighted elliptic system.

1. Introduction

We start by noting that throughout this article, \(N_s := N_1 + (1 + s)N_2\) is called the homogeneous dimension associated to the Grushin operator:

\[\Delta_s = \Delta_x + |x|^{2s} \Delta_y,\]

where \(s \geq 0\), and

\[\Delta_x := \sum_{i=1}^{N_1} \frac{\partial^2}{\partial x_i^2} \quad \text{and} \quad \Delta_y := \sum_{j=1}^{N_2} \frac{\partial^2}{\partial y_j^2},\]

are Laplace operators with respect to \(x \in \mathbb{R}^{N_1}, y \in \mathbb{R}^{N_2}\), and \(|x|^{2s} = \left(\sum_{i=1}^{N_1} x_i^2\right)^s\).

We devote this paper to the study of stable solutions to the following weighted degenerate elliptic system

\[-\Delta_s u = \left(1 + \|x\|^{2(s+1)}\right)^{\frac{2}{2s+1}} u^p, \quad -\Delta_s v = \left(1 + \|x\|^{2(s+1)}\right)^{\frac{2}{2s+1}} u^\theta, \quad u, v > 0 \quad \text{in} \quad \mathbb{R}^N.\]
where $1 < p \leq \theta$, $\alpha \geq 0$, and the weighted Grushin equation

$$-\Delta_s u = \left(1 + \|x\|^{2(s+1)}\right)^{\frac{1}{2(s+1)}} u^p, \quad u > 0 \quad \text{in} \quad \mathbb{R}^N,$$

(1.2)

where $p > 1$, $\alpha \geq 0$, and

$$\|x\| = \left(|x|^{2(s+1)} + |y|^2\right)^{\frac{1}{2(s+1)}}, \quad s \geq 0, \quad \text{and} \quad x := (x, y) \in \mathbb{R}^N = \mathbb{R}^{N_1} \times \mathbb{R}^{N_2},$$

is the norm corresponding to the Grushin distance, where $|x|$ and $|y|$ are the usual Euclidean norms in $\mathbb{R}^{N_1}$ and $\mathbb{R}^{N_2}$, respectively. Then, we can verify that the $\|x\|$-norm is 1-homogeneous for the group of anisotropic dilations attached to $\Delta_s$. It is defined by

$$\delta_{\lambda}(x) = (\lambda x, \lambda^{1+s} y), \quad \lambda > 0 \quad \text{and} \quad x := (x, y) \in \mathbb{R}^N = \mathbb{R}^{N_1} \times \mathbb{R}^{N_2}.$$

It is not hard to see that $d\delta_{\lambda}(x) = \lambda^{N_1+(1+s)N_2} dxdy = \lambda^N dx$, where $dxdy$ denotes the Lebesgue measure in $\mathbb{R}^N$.

Our aim in this paper is to classify stable solutions of systems (1.1), which can be regarded as a natural generalization of the weighted Grushin equation (1.2). In order to state our results more accurately we defined the notion of stability, we consider a general system given by

$$-\Delta_s u = f(x, v), \quad -\Delta_s v = g(x, u), \quad \text{in} \quad \mathbb{R}^N,$$

(1.3)

with $f, g \in C^1(\mathbb{R}^{N+1}, \mathbb{R})$ satisfying $f_t := \frac{\partial f(x, t)}{\partial t}, g_t := \frac{\partial f(x, t)}{\partial t} \geq 0$ in $\mathbb{R}$. A smooth solution $(u, v)$ of (1.3) is said stable if there exist positive smooth functions $\zeta, \xi$ verifying

$$-\Delta_s \zeta = f_v(x, v)\zeta, \quad -\Delta_s \zeta = g_u(x, u)\zeta, \quad \text{in} \quad \mathbb{R}^N.$$

This definition is motivated by [14, 12, 2]. Let us review some results related to our problem.

Firstly, we recall the case $s = \alpha = 0$, the Lane–Emden equation and system have been extensively studied by many experts.

A celebrated result of Farina [6], gives a complete classification up to the so called Joseph-Lundgren exponent (see [9, 6]). In [2], using Farina’s approach, Fazly established Liouville type theorem of the weighted Lane–Emden equation (1.2) for $s = 0$, $N$ satisfying

$$N < N_\star = 2 + \frac{2(2 + \alpha)}{p - 1} \left(p + \sqrt{p^2 - p}\right),$$

and $p \geq 2$. A large amount of works have been done generalizing this result in various directions. To cite a few we refer to [4, 2, 22, 12, 10].

We now turn to the celebrated Lane–Emden system where $\alpha = s = 0$, (1.1) reduces to

$$-\Delta u = u^p, \quad -\Delta v = u^\theta, \quad u, v > 0 \quad \text{in} \quad \mathbb{R}^N, \quad \text{where} \quad \theta \geq p > 1.$$

(1.4)

There is a famous conjecture who states that: Let $p, \theta > 0$. If the pair $(p, \theta)$ is subcritical, i.e. if

$$\frac{1}{p+1} + \frac{1}{\theta+1} > \frac{N-2}{N},$$

(1.5)

then there is no smooth solution to (1.4). This conjecture is proved to be true for radial functions by Mitidieri [13], (see also Serrin-Zou [20]), and for the nonradial solutions in dimensions $N = 3, 4$, by Souplet and his collaborators, see [24, 22]. Other significant work in these topic can be found in [23, 8, 1, 21] and reference therein.

In [12], the authors have obtained the nonexistence of stable at infinity solutions of (1.1) for any $p, \theta > 0$, satisfying (1.5). We should also mention that the classification for stable solutions of (1.4), was obtained by Cowan [2]. Indeed, Using the stability inequality of system (1.1) and an interesting iteration
argument, it was proved that there is no smooth stable solution to (1.4), if \( \max(1, 2t_0^-) \leq \theta \), and \( N \) satisfies \( N < 2 + \frac{4(\theta + 1)}{p\theta - 1}t_0^+ \), where

\[
t_0^\pm = \sqrt{\frac{p\theta(\theta + 1)}{\theta + 1} \pm \sqrt{\frac{p\theta(\theta + 1)}{\theta + 1} - \sqrt{\frac{p\theta(p + 1)}{\theta + 1}}}.
\]

In particular, if \( N < 10 \), (1.3) has no stable solution for any \( 2 < p \leq \theta \). This result was extended in a work due to Hu [12], for the case of weighted system (1.4) with \( s = 0 \), namely

\[
-\Delta u = \rho(x) v^p, \quad -\Delta v = \rho(x) u^\theta, \quad u, v > 0 \quad \text{in} \quad \mathbb{R}^N, \quad \text{where} \quad \theta > p > 1,
\]

and \( \rho \equiv (1 + |x|^2)^{\alpha} \), \( \alpha > 0 \). It was shown that, if \( 2t_0^- < p \leq \theta \) and \( N \) satisfies

\[
N < 2 + \frac{2(2 + \alpha)(\theta + 1)}{p\theta - 1}t_0^+.
\]

Furthermore, in [1], the authors established a Liouville type theorem for the Lane-Emden system with general weights (1.6), where \( \rho(x) \) is a radial function satisfying \( \rho(x) \geq A(1 + |x|^2)^{\alpha} \) at infinity. This result was then improved the previous works [2, 7, 12], and mainly obtained a new inverse comparison principle which is the key to deal with the case \( 1 < p \leq \frac{4}{3} \). In particular, the range of nonexistence result in [10] is larger than that in [2, 12].

For the general equation or system with \( s \neq 0 \), the Liouville property is less understood and is more complicated to deal with than \( s = 0 \). In the special case \( \alpha = 0 \), the system (1.4) and Eq (1.2), become

\[
-\Delta_s u = u^p, \quad -\Delta_s v = u^\theta, \quad u, v > 0 \quad \text{in} \quad \mathbb{R}^N,
\]

where \( 1 < p \leq \theta \), and Grushin equation

\[
-\Delta_s u = u^p, \quad p > 1, \quad u > 0 \quad \text{in} \quad \mathbb{R}^N.
\]

Let us first recall some facts about the problem involving the Grushin operator. The Liouville type theorem for solutions of (1.8), has been established in [26, 17] for the case \( 1 < p < \frac{N+2}{N-2} \). The main tool in [26, 17], is the Kelvin transform combined with technique of moving planes.

Very recently, in [18] the author extended some of Farina’s results [1] in order to prove the nonexistence for nontrivial stable solution of the weighted equation \(-\Delta_s u = |x|^\alpha |u|^{p-1}u \) in \( \mathbb{R}^N \), if

\[
\begin{cases}
1 < p < \infty & \text{si} \quad N_s \leq 10 + 4\alpha \\
1 < p < p_{\text{JL}}(N_s, \alpha) & \text{si} \quad N_s > 10 + 4\alpha,
\end{cases}
\]

with

\[
p_{\text{JL}}(N_s, \alpha) = \frac{(N_s - 2)^2 - 2(\alpha + 2)(N_s) + 2\sqrt{(\alpha + 2)^3(\alpha + 2N_s - 2)}}{(N_s - 2)(N_s - 10)}
\]

It should be notice that this condition \( 1 < p < p_{\text{JL}}(N_s, \alpha) \) is equivalent to \( N_s < N_\star \), where \( N_\star \) is given explicitly in the above.

Concerning the nonexistence of classical stable solutions of (1.7) for \( \theta \geq p > 1 \), new results were shown in [2], where the authors used the technics developed in [2, 10] in order to obtain a direct extension of Theorem 1 in [2] for \( s = 0 \):

**Theorem A.** 1. Suppose that \( \frac{4}{3} < p \leq \theta \) and

\[
N_s < 2 + \frac{4(\theta + 1)}{p\theta - 1}t_0^+,
\]

Then there is no stable positive solution of (1.7). In particular, the assertion is true if \( N_s \leq 10 \).
2. Suppose that $1 < p \leq \min\left(\frac{4}{3}, \theta\right)$, and

$$Ns < 2 + \left[2 + \frac{2(p + 1)}{p\theta - 1} + \frac{4(2 - p)}{\theta + p - 2}\right] t_0^\ast.$$  

Then there is no bounded stable positive solution of (1.7).

From now on, we assume that $\alpha > 0$. Our main objective is to prove the following Liouville-type theorem for stable solutions of (1.1) or (1.2) in $\mathbb{R}^N$.

**Theorem 1.1.** Let $x_0$ be the largest root of the polynomial

$$H(x) = x^4 - \frac{16p\theta(p + 1)(\theta + 1)}{(p\theta - 1)^2} x^2 + \frac{16p\theta(p + 1)(\theta + 1)(p + \theta + 2)}{(p\theta - 1)^3} x - \frac{16p\theta(p + 1)^2(\theta + 1)^2}{(p\theta - 1)^4}. \quad (1.9)$$

1. If $\frac{4}{3} < p \leq \theta$ then (1.1) has no stable solution if $Ns < 2 + (2 + \alpha)x_0$. In particular, if $Ns \leq 10 + 4\alpha$, then (1.1) has no stable solution for all $\frac{4}{3} < p \leq \theta$.
2. If $1 < p \leq \min\left(\frac{4}{3}, \theta\right)$, then (1.1) has no bounded stable solution, if

$$Ns < 2 + \left[\frac{p + 2(2 - p)(p\theta - 1)}{(p + \theta - 2)(\theta + 1)}\right] (\alpha + 2)x_0.$$  

Therefore, if $Ns \leq 6 + 2\alpha$, the system (1.1) has no bounded stable solution for all $\theta \geq p > 1$.

Thus, the following classification result for stable solution of (1.2), is a direct consequence of Theorem 1.1.

**Corollary 1.1.**

1. If $\frac{4}{3} < p$ then (1.2) has no stable solution if

$$Ns < 2 + \frac{2(2 + \alpha)}{p - 1} \left(p + \sqrt{p^2 - p}\right). \quad (1.10)$$

In particular, if $Ns \leq 10 + 4\alpha$, then (1.2) has no stable solution for all $\frac{4}{3} < p$.
2. If $1 < p \leq \frac{4}{3}$, (1.2) has no bounded stable solution for $Ns$ verifying (1.10).

Therefore, there is no bounded stable solution of (1.2) for all $p > 1$ if $Ns \leq 10 + 4\alpha$.

**Remark 1.1.**

- If $s = 0$, then the results in Theorem 1.1 and Corollary 1.1 coincide with that in [10].
- Using Remark 3.1 below, we see that $2t_0^{\ast} \frac{d - 1}{p\theta - 1} < x_0$ for any $1 < p \leq \theta$, where $x_0$ is the largest root of the polynomial $H$ given by (1.9). So, Theorem 1.1 improves the bound given in Theorem A with $\alpha = 0$.

Establishing a Liouville type result for stable solution of (1.1) is delicate, even we can borrow some ideas from [2, 10]. More precisely, the proof is based on nonlinear integral estimates and the comparison property between $u$ and $v$. Nevertheless, unlike the system (1.3), the comparison property has not been obtained for the case of weighted system (1.1), since the operator $\Delta_s$ no longer has symmetry and it degenerates on the manifold $\{0\} \times \mathbb{R}^N_2$ and this introduces some essential difficulties in the proof of Theorem 1.1. Furthermore, to derive the comparison property for the weighted Grushin operators, we follow the general lines of the methods and techniques developed in [3]. Another difficulty, the $L^1$-estimates to the bootstrap argument of Cowan [2], does not work in the case of weighted Grushin operator. In order to overcome the difficulties, we instead pass to the $L^2$-estimates in the bootstrap argument, which plays an essential role in iteration process.

Our paper is organized as follows. In section 2, we prove some preliminaries results. The proofs of Theorem 1.1 and Corollary 1.1 are given in section 3.
2. Preliminary technical lemmas.

In order to prove our results, we need some technical lemmas. In the following, \( C \) denotes always a generic positive constant independent on \((u, v)\), which could be changed from one line to another. The ball of center 0 and radius \( r > 0 \) will be denoted by \( B_r \).

2.1. Stability inequality

We can proceed similarly as the proof of Lemma 2.1 in [5], we establish the following inequality.

**Lemma 2.1.** If \((u, v)\) is a nonnegative stable solution of (1.1), then

\[
\sqrt{p\theta} \int_{\mathbb{R}^N} \left(1 + \|x\|^{2(s+1)}\right) \frac{u^{p-1}v^{p-1}}{\phi^2} dxdy \leq \int_{\mathbb{R}^N} |\nabla_s \phi|^2 dxdy, \quad \forall \phi \in C^1_c(\mathbb{R}^N). \tag{2.1}
\]

**Proof.** Let \((u, v)\) denote a stable solution of (1.1). By the definition of stability, there exist positive smooth functions \( \varphi, \psi \) verifying

\[
-\Delta_s \varphi = p\left(1 + \|x\|^{2(s+1)}\right) \frac{u^{p-1}v^{p-1}}{\varphi^2}, \quad -\Delta_s \psi = \theta\left(1 + \|x\|^{2(s+1)}\right) \frac{u^{\theta-1}v^{\theta}}{\psi} \quad \text{in } \mathbb{R}^N.
\]

Let \( \gamma, \chi \in C^1_c(\mathbb{R}^N) \) and multiply the first equation by \( \gamma^2 \), and the second by \( \chi^2 \) and integrate over \( \mathbb{R}^N \), to arrive at

\[
p \int_{\mathbb{R}^N} \left(1 + \|x\|^{2(s+1)}\right) \frac{u^{p-1}v^{p-1}}{\varphi^2} \gamma^2 dxdy \leq -\int_{\mathbb{R}^N} \frac{\Delta_s \varphi}{\varphi} \gamma^2 dxdy,
\]

and

\[
\theta \int_{\mathbb{R}^N} \left(1 + \|x\|^{2(s+1)}\right) \frac{u^{\theta-1}v^{\theta}}{\psi} \chi^2 dxdy \leq -\int_{\mathbb{R}^N} \frac{\Delta_s \psi}{\psi} \chi^2 dxdy.
\]

The simple calculation implies that

\[
\int_{\mathbb{R}^N} \left(-\frac{\Delta_s \varphi}{\varphi} \gamma^2 - |\nabla_s \gamma|^2\right) dxdy \leq \int_{\mathbb{R}^N} \left(\nabla_s \varphi \cdot \nabla_s (\gamma^2 \varphi^{-1}) - |\nabla_s \gamma|^2\right) dxdy
\]

\[
\leq \int_{\mathbb{R}^N} \left(-\varphi^{-2} |\nabla_s \varphi|^2 \gamma^2 + 2\varphi^{-1} \gamma \nabla_s \varphi \cdot \nabla_s \gamma - |\nabla_s \gamma|^2\right) dxdy
\]

\[
\leq \int_{\mathbb{R}^N} \left(-\varphi^{-1} \gamma \nabla_s \varphi - \nabla_s \gamma\right)^2 dxdy \leq 0
\]

Proceeding as above, we can easily show that

\[
-\int_{\mathbb{R}^N} \frac{\Delta_s \psi}{\psi} \chi^2 dxdy \leq \int_{\mathbb{R}^N} |\nabla_s \chi|^2 dxdy.
\]

Using the inequality \(2ab \leq a^2 + b^2\), we deduce then

\[
2 \left(1 + \|x\|^{2(s+1)}\right) \frac{\sqrt{p\theta}u^{p-1}v^{p-1}}{\phi^2} \leq \left(1 + \|x\|^{2(s+1)}\right) \frac{2\varphi^{-1}\gamma^2}{\phi^2 + \theta u^{\theta-1}v^{\theta}}.
\]

Taking \( \varphi = \chi \), and combining all these inequalities, we get readily the estimate (2.1). \( \Box \)

Inspired by the previous works [13, 7, 14], we obtain the following integral estimates for all solutions of the system of (1.1).
Lemma 2.2. Suppose that \((u, v)\) is a smooth solution of \((1.1)\), with \(\theta \geq p > 1\). Then
\[
\int_{B_R \times B_{R^{1+s}}} \left(1 + \|x\|^{2(s+1)}\right) \frac{\alpha}{\alpha + 1} v^p \, dx \, dy \leq CR^{N_s - \frac{2(p+1)\theta - (p+1)\alpha}{p\theta - \alpha}}. \tag{2.2}
\]
\[
\int_{B_R \times B_{R^{1+s}}} \left(1 + \|x\|^{2(s+1)}\right) \frac{\alpha}{\alpha + 1} u^\theta \, dx \, dy \leq CR^{N_s - \frac{2(p+1)\theta - (p+1)\alpha}{p\theta - \alpha}}. \tag{2.3}
\]

Proof. Let \(\chi_j \in C_c^\infty(\mathbb{R}, [0, 1]), j = 1, 2\) be a cut-off function verifying \(0 \leq \chi_j \leq 1\),
\[
\chi_j = 1 \quad \text{on} [-1, 1], \quad \text{and} \quad \chi_j = 0 \quad \text{outside} \quad [-2^{1+(j-1)s}, 2^{1+(j-1)s}].
\]
For \(R \geq 1\), put \(\psi_R(x, y) = \chi_1\left(\frac{x}{R}\right)\chi_2\left(\frac{y}{R}\right)\), it is easy to verify that there exists \(C > 0\) independent of \(R\) such that
\[
|\nabla_x \psi_R| \leq \frac{C}{R} \quad \text{and} \quad |\nabla_y \psi_R| \leq \frac{C}{R^{1+s}}.
\]
\[
|\Delta_x \psi_R| \leq \frac{C}{R^2} \quad \text{and} \quad |\Delta_y \psi_R| \leq \frac{C}{R^{2(1+s)}}.
\]
Multiplying the equation \(-\Delta_x u = \left(1 + \|x\|^2\right)^\frac{\alpha}{\alpha + 1} v^p\) by \(\psi^m\), and integrating by parts to yield
\[
\int_{B_R \times B_{(2R)^{1+s}}} \left(1 + \|x\|^{2(s+1)}\right) \frac{\alpha}{\alpha + 1} \psi^m \, dx \, dy
\]
\[
= - \int_{B_R \times B_{(2R)^{1+s}}} u \Delta_x (\psi^m_R) \, dx \, dy \leq \frac{C}{R^{2s}} \int_{B_R \times B_{(2R)^{1+s}}} u \psi^m_R - 2 \, dx \, dy.
\]
Let \(\frac{1}{\theta} + \frac{1}{p} = 1\). Apply Hölder’s inequality, we obtain
\[
\int_{B_R \times B_{(2R)^{1+s}}} \left(1 + \|x\|^{2(s+1)}\right) \frac{\alpha}{\alpha + 1} \psi^m_R \, dx \, dy
\]
\[
\leq \frac{C}{R^{2s}} \left(\int_{B_R \times B_{(2R)^{1+s}}} \left(1 + \|x\|^{2(s+1)}\right) \frac{\alpha}{\alpha + 1} \psi^m_R \, dx \, dy\right)^{\frac{1}{p}}
\]
\[
\times \left(\int_{B_R \times B_{(2R)^{1+s}}} \left(1 + \|x\|^{2(s+1)}\right) \frac{\alpha}{\alpha + 1} u^\theta \psi^{(m-2)\theta}_R \, dx \, dy\right)^{\frac{1}{\theta}}
\]
\[
\leq CR^{\frac{N_s - \frac{\alpha}{\alpha + 1} - \frac{\alpha}{\alpha + 1} - 2}{\theta}} \left(\int_{B_R \times B_{(2R)^{1+s}}} \left(1 + \|x\|^{2(s+1)}\right) \frac{\alpha}{\alpha + 1} u^\theta \psi^{(m-2)\theta}_R \, dx \, dy\right)^{\frac{1}{\theta}}.
\]
Adopting the similar argument as above where we use the second equation in \((1.1)\), we deduce then for \(k \geq 2\)
\[
\int_{B_R \times B_{(2R)^{1+s}}} \left(1 + \|x\|^{2(s+1)}\right) \frac{\alpha}{\alpha + 1} u^\theta \psi^k_R \, dx \, dy
\]
\[
\leq CR^{\frac{N_s - \frac{\alpha}{\alpha + 1} - \frac{\alpha}{\alpha + 1} - 2}{\theta}} \left(\int_{B_R \times B_{(2R)^{1+s}}} \left(1 + \|x\|^{2(s+1)}\right) \frac{\alpha}{\alpha + 1} u^p \psi^{(k-2)p}_R \, dx \, dy\right)^{\frac{1}{p}},
\]
where \(\frac{1}{p} + \frac{1}{\theta} = 1\). Take now \(k\) and \(m\) large verifying \(m \leq (k-2)p\) and \(k \leq (m-2)\theta\). Combining the two
above inequalities, one concludes
\[ \int_{B_{2R} \times B(2R)^{1+s}} \left( 1 + \|x\|^{2(s+1)} \right)^{\frac{\alpha}{2(s+1)}} \nu^p \psi_R^m \, dx \, dy \]
\leq CR^N \frac{\|x\|^{2(s+1)}}{\|x\|^{2(s+1)}} \left( \int_{B_{2R} \times B(2R)^{1+s}} \left( 1 + \|x\|^{2(s+1)} \right)^{\frac{\alpha}{2(s+1)}} \nu^p \psi_R^{m-k\nu} \, dx \, dy \right)^{\frac{1}{\nu^p}}
\leq CR^N \frac{\|x\|^{2(s+1)}}{\|x\|^{2(s+1)}} \left( \int_{B_{2R} \times B(2R)^{1+s}} \left( 1 + \|x\|^{2(s+1)} \right)^{\frac{\alpha}{2(s+1)}} \nu^p \psi_R^m \, dx \, dy \right)^{\frac{1}{\nu^p}}.
So, we get
\[ \int_{B_{R} \times B(1)^{1+s}} \left( 1 + \|x\|^{2(s+1)} \right)^{\frac{\alpha}{2(s+1)}} \nu^p \, dx \, dy \]
\leq \int_{B_{2R} \times B(2R)^{1+s}} \left( 1 + \|x\|^{2(s+1)} \right)^{\frac{\alpha}{2(s+1)}} \nu^p \psi_R^m \, dx \, dy \leq CR^N \frac{\|x\|^{2(s+1)}}{\|x\|^{2(s+1)}} \left( \int_{B_{2R} \times B(2R)^{1+s}} \left( 1 + \|x\|^{2(s+1)} \right)^{\frac{\alpha}{2(s+1)}} \nu^p \psi_R^{m-k\nu} \, dx \, dy \right)^{\frac{1}{\nu^p}}.

Finally, the estimate (2.3) follows from using the same argument as above. \(\square\)

2.2. Comparison principle

Inspired by the previous works in [5, 10], we can find the point-wise estimate of solution \((u, v)\).

**Lemma 2.3.** (Comparaison property.) Let \(\theta \geq p > 1\), \(\alpha \geq 0\), and \((u, v)\) be positive solution of (1.1). Then there holds
\[ u^{\theta+1} \leq \frac{\theta + 1}{p + 1} v^{\theta+1} \text{ in } \mathbb{R}^N. \] (2.4)

**Proof.** Let \(\sigma = \frac{p+1}{\theta+1} \in (0, 1]\), \(\lambda = \sigma^{-1}\), and \(w = u - \lambda v^\sigma\). A straightforward computation implies
\[ \Delta_s w = \Delta_s u - \lambda \sigma v^{\sigma-1} \Delta_s v - \lambda \sigma (\sigma - 1) |\nabla_s v|^{2} v^{\sigma-2} \geq \left( 1 + \|x\|^{2(s+1)} \right)^{\frac{\alpha}{2(s+1)}} \left[ -u^p + \lambda \sigma v^{\sigma-1} u^\theta \right] 
= \left( 1 + \|x\|^{2(s+1)} \right)^{\frac{\alpha}{2(s+1)}} \left[ -v^{p-\sigma+1} + \lambda \sigma u^\theta \right] 
= \left( 1 + \|x\|^{2(s+1)} \right)^{\frac{\alpha}{2(s+1)}} \left[ \lambda^{-\theta} u^\theta - v^{\theta \sigma} \right]. \]
Therefore, for any \(\sigma \in (0, 1]\), there exists \(C > 0\) such that
\[ C v^{\sigma-1} \left[ u^\theta - \left( \lambda v^\sigma \right)^\theta \right] \leq \left( 1 + \|x\|^{2(s+1)} \right)^{\frac{\alpha}{2(s+1)}} \left[ u^\theta - \left( \lambda v^\sigma \right)^\theta \right] \leq \Delta_s w. \] (2.5)

We need to prove that
\[ u \leq \lambda v^\sigma. \]

We shall show that \(w \leq 0\), by a contradiction argument. Suppose that
\[ \sup_{\mathbb{R}^N} w > 0. \] (2.6)

Next, we split the proof into two cases.

**Case 1:** we consider the case where the supremum of \(w\) is attained at infinity.
Choose now $\phi_R(x,y) = \psi(m(x,y), R, R_+\sigma)$, where $m > 0$, and $\psi$ a cut-off function in $C^\infty_c(\mathbb{R}^N, [0, 1])$, such that
\[ \psi = 1 \quad \text{on} \quad B_1 \times B_1, \quad \text{and} \quad \psi = 0 \quad \text{outside} \quad B_2 \times B_{2^1+\cdot}. \]

A simple calculation, implies that
\[ \frac{|\nabla \phi_R|^2}{\phi_R} \leq \frac{C}{R^2} \phi_R^{-\frac{m-2}{2}} \quad \text{and} \quad |\Delta \phi_R| \leq \frac{C}{R^2} \phi_R^{-\frac{m-2}{2}}. \]

Set
\[ w_R = \phi_R w, \]
which is a compactly supported function. Then there exists $(x_R, y_R) \in B_{2R} \times B_{(2R)^{1+\cdot}}$, such that
\[ w_R(x_R, y_R) = \max_{\mathbb{R}^N} w_R(x,y) \to \sup w(x,y) \quad \text{as} \quad R \to \infty, \]
which implies
\[ \nabla_s w_R(x_R, y_R) = 0 \quad \text{and} \quad \Delta_s w_R(x_R, y_R) \leq 0, \]
which means that at $(x_R, y_R)$,
\[ \nabla_s w = -\phi_R^{-1} \nabla_s \phi_R w \quad \text{and} \quad \phi_R \Delta_s w \leq 2w \phi_R^{-1} |\nabla_s \phi_R|^2 - w \Delta_s \phi_R. \tag{2.7} \]

From (2.7), and using the properties of $\phi_R$, we can conclude then
\[ \Delta_s w \leq \frac{C}{R^2} \phi_R^{-\frac{m-2}{2}} w. \tag{2.8} \]

Furthermore, for $w = u - \lambda v^\sigma \geq 0$, we observe that
\[ \frac{u^\theta}{w^\sigma} - \frac{(\lambda v^\sigma)^\theta}{w^\sigma} \geq 1, \quad \text{or equivalently} \quad \lambda^{-\theta} u^\theta - v^\theta \sigma \geq \lambda^{-\theta} w^\theta. \tag{2.9} \]

Multiplying (2.5) by $\phi_R$, combining it with (2.9) and (2.8), one obtains
\[ w^{\sigma-1} \phi_R^m \phi_R^\frac{m+2}{2} \leq \frac{C}{R^2} w_R \phi_R. \]

As $\sigma \leq 1$. If the sequence $v(x_R, y_R)$ is bounded, and we choose
\[ \theta = \frac{m + 2}{m} \quad \text{so that} \quad m = \frac{2}{\theta - 1}, \]
there holds then
\[ w_R^{\theta-1} \leq \frac{C}{R^2}. \]
Taking the limit $R \to \infty$, we have $\sup_{\mathbb{R}^N} w = 0$, which contradicts (2.6), the claim follows.

**Case 2**: If there exists $(x_0, y_0)$, such that $\sup_{\mathbb{R}^N} w(x_0, y_0) = w(x_0, y_0) - \lambda v^\sigma(x_0, y_0) > 0$, then $\frac{\partial w}{\partial x}(x_0, y_0) = 0$, or $\frac{\partial w}{\partial y}(x_0, y_0) = 0$, and $\frac{\partial^2 w}{\partial x^2}(x_0, y_0) \leq 0$, or $\frac{\partial^2 w}{\partial y^2}(x_0, y_0) \leq 0$. This contradicts the fact that $\Delta_s w(x_0, y_0) \geq 0$, the proof is completed. \qed

Exploiting the technique introduced in (10), we establish the following comparison property between $u$ and $v$, which is somehow an inverse version of the estimate (2.4).

**Lemma 2.4.** Suppose that $(u,v)$ be a solution of (1.1), with $\theta \geq p > 1$ and assume that $u$ is bounded,
then
\[ v \leq \|u\|_{\infty}^{\frac{\theta}{\theta-p}} u. \] (2.10)

**Proof.** Let \( w = v - lu \), where \( l = \|u\|_{\infty}^{\frac{\theta}{\theta-p}} \). We have, as \( \theta \geq p \),
\[
\Delta w = \left(1 + \|x\|^{2(s+1)}\right) \frac{\theta}{\theta-p} \left( lv^p - u^\theta \right) = \left(1 + \|x\|^2\right) \frac{\theta}{\theta-p} \left[ lv^p - \left( \frac{u}{\|u\|_\infty} \right)^p \|u\|_\infty^\theta \right] \\
\geq \left(1 + \|x\|^{2(s+1)}\right) \frac{\theta}{\theta-p} \left[ lv^p - \left( \frac{u}{\|u\|_\infty} \right)^p \|u\|_\infty^\theta \right] \\
= \left(1 + \|x\|^{2(s+1)}\right) \frac{\theta}{\theta-p} \|u\|_\infty^{\theta-p} \left( lv^p - u^p \right) \\
= \left(1 + \|x\|^{2(s+1)}\right) \frac{\theta}{\theta-p} \|u\|_\infty^{\theta-p} \left( l^{-p}v^p - u^\theta \right).
\] (2.11)

For the rest of the proof, can be proceeded as in Lemma 2.3 where we replace just (2.5) by (2.11), so we omit the details. \( \square \)

A crucial ingredient in our proof of Theorem 1.1 for the case \( 1 < p \leq \frac{4}{3} \), is given by the following Lemma is a consequence of the stability inequality (2.1), and the inverse comparison principle (2.10).

**Lemma 2.5.** Let \((u, v)\) be a stable solution to (1.1) with \( 1 < p \leq \min\left(\frac{4}{3}, \theta\right) \). Assume that \( u \) is bounded, then holds
\[
\int_{B_R \times B_{R^{1+s}}} \left(1 + \|x\|^{2(s+1)}\right)^{\frac{\theta}{\theta-p}} v^2 dxdy \leq CR^{N \cdot \frac{2(s+1)p-2(\theta+1)n}{p \theta - \theta p - 2}}. \] (2.12)

**Proof.** Let \((u, v)\) be a stable solution of (1.1), where \( u \) is bounded. Take \( \chi \in C^\infty_c(\mathbb{R}^N, [0, 1]) \), be a cut-off function satisfying \( \chi = 1 \) on \( B_1 \times B_1 \), and \( \chi = 0 \) outside \( B_2 \times B_{2^{s+1}} \). Put \( \eta_R(x, y) = \chi(x \cdot R, y \cdot \frac{R}{R^{1+s}}) \).

Thanks to the approximation argument, the stability property (2.1) holds true with \( \phi = v\eta_R \), we deduce then
\[
\sqrt{p \theta} \int_{\mathbb{R}^N} \left(1 + \|x\|^{2(s+1)}\right)^{\frac{\theta}{\theta-p}} u^\frac{\theta}{\theta-p} v^2 \eta_R^2 dxdy \\
\leq \int_{\mathbb{R}^N} \ |
\nabla_x v|^2 \eta_R^2 dxdy + \int_{\mathbb{R}^N} v^2 \left| \nabla_x \eta_R \right|^2 dxdy - \frac{1}{2} \int_{\mathbb{R}^N} v^2 \Delta_x (\eta_R^2) dxdy.
\] (2.13)

Multiplying \(-\Delta v = \left(1 + \|x\|^{2(s+1)}\right)^{\frac{\theta}{\theta-p}} u^\theta \) by \( v\eta_R^2 \) and integrating by parts, one gets:
\[
\int_{\mathbb{R}^N} \ |
\nabla_x v|^2 \eta_R^2 dxdy = \int_{\mathbb{R}^N} \left(1 + \|x\|^{2(s+1)}\right)^{\frac{\theta}{\theta-p}} u^\theta v\eta_R^2 dxdy + \frac{1}{2} \int_{\mathbb{R}^N} v^2 \Delta_x (\eta_R^2) dxdy.
\]

From (2.14), one has
\[
\int_{\mathbb{R}^N} \ |
\nabla_x v|^2 \eta_R^2 dxdy \leq \sqrt{\frac{\theta + 1}{p + 1}} \int_{\mathbb{R}^N} \left(1 + \|x\|^{2(s+1)}\right)^{\frac{\theta}{\theta-p}} u^\frac{\theta}{\theta-p} v^\frac{\theta}{\theta-p} + \frac{1}{2} \int_{\mathbb{R}^N} v^2 \Delta_x (\eta_R^2) dxdy.
\]

Substituting this in (2.13), we obtain readily
\[
\left(\sqrt{p \theta} - \sqrt{\frac{\theta + 1}{p + 1}}\right) \int_{\mathbb{R}^N} \left(1 + \|x\|^{2(s+1)}\right)^{\frac{\theta}{\theta-p}} u^\frac{\theta}{\theta-p} v^\frac{\theta}{\theta-p} \eta_R^2 dxdy \leq \int_{\mathbb{R}^N} v^2 \left| \nabla_x \eta_R \right|^2 dxdy,
\]

Take \( \eta_R = \varphi_R^m \) with \( m > 2 \). Using Lemma 2.4 there exists a positive constant \( C \) such that
\[
\int_{\mathbb{R}^N} \left( 1 + \|x\|^{2(s+1)} \right)^{\frac{\alpha}{p(s+1)}} v^{\frac{\theta + p - 2}{2}} \varphi_R^{2m} \, dx \, dy \leq \frac{C}{R^{2 + \alpha}} \int_{\mathbb{R}^N} \left( 1 + \|x\|^{2(s+1)} \right)^{\frac{\alpha}{p(s+1)}} v^2 \varphi_R^{2m-2} \, dx \, dy. \quad (2.14)
\]

Since \(1 < p \leq \min\left(\frac{4}{3}, \theta\right)\), a direct calculation shows that
\[
2 = \frac{p + 2}{2} (1 - \lambda) \quad \text{with } \lambda = \frac{\theta + p - 2}{\theta + 2 - p} \in (0, 1).
\]

By Hölder’s inequality, the integral in the right hand side of (2.14), can be estimated as
\[
\int_{\mathbb{R}^N} \left( 1 + \|x\|^{2(s+1)} \right)^{\frac{\alpha}{p(s+1)}} v^2 \varphi_R^{2m-2} \, dx \, dy
\]
\[
\leq \left( \int_{\mathbb{R}^N} \left( 1 + \|x\|^{2(s+1)} \right)^{\frac{\alpha}{p(s+1)}} v^{\frac{\theta + p - 2}{2}} \varphi_R^{2m} \, dx \, dy \right)^{1 - \lambda} \left( \int_{\mathbb{R}^N} \left( 1 + \|x\|^{2(s+1)} \right)^{\frac{\alpha}{p(s+1)}} v^p \varphi_R^{2m-2} \, dx \, dy \right)\lambda.
\]
\[
\leq \left( \int_{\mathbb{R}^N} \left( 1 + \|x\|^{2(s+1)} \right)^{\frac{\alpha}{p(s+1)}} v^{\frac{\theta + p - 2}{2}} \varphi_R^{2m} \, dx \, dy \right)^{1 - \lambda} \left( \int_{B_{2R} \times B_{(2R)^1 + s}} \left( 1 + \|x\|^{2(s+1)} \right)^{\frac{\alpha}{p(s+1)}} v^p \, dx \, dy \right)\lambda. \quad (3.15)
\]

For the last inequality, we used \(0 \leq \varphi_R \leq 1\) and chosen \(m\) large such that \(m\lambda > 1\). Combining (2.12) and (2.13)-(2.15), there holds
\[
\int_{\mathbb{R}^N} \left( 1 + \|x\|^{2(s+1)} \right)^{\frac{\alpha}{p(s+1)}} v^2 \varphi_R^{2m-2} \, dx \, dy
\]
\[
\leq R^\lambda (N - \frac{2(\theta + p - 2)}{p - 1} \frac{(p + 1) \alpha}{p - 1} - (2 + \alpha)(1 - \lambda)) \left( \int_{\mathbb{R}^N} \left( 1 + \|x\|^{2(s+1)} \right)^{\frac{\alpha}{p(s+1)}} v^2 \varphi_R^{2m-2} \, dx \, dy \right)^{1 - \lambda}.
\]

Therefore
\[
\int_{B_R \times B_{R1 + s}} \left( 1 + \|x\|^{2(s+1)} \right)^{\frac{\alpha}{p(s+1)}} v^2 \, dx \, dy \leq CR^\lambda \left( N - \frac{2(\theta + p - 2)}{p - 1} \frac{(p + 1) \alpha}{p - 1} - \frac{2(2 + \alpha)(2 - p)}{p - 2} \right),
\]
so we are done. \(\square\)

3. Proofs of Theorem 3.1 and Corollary 3.1

Adopting the similar approach as in Lemma 3.1 in [11], we establish the following lemma which plays an important role in proving Theorems 3.1.

Lemma 3.1. Let \(\alpha \geq 0\). Assume that \((u, v)\) is a stable solution of (1.1) such that \(u\) is bounded. Then for any \(z > \frac{p + 1}{2}\) verifying \(L(z) < 0\), there exists \(C < \infty\) such that
\[
\int_{B_R \times B_{R1 + s}} \left( 1 + \|x\|^{2(s+1)} \right)^{\frac{\alpha}{p(s+1)}} u^\theta v^{z-1} \, dx \, dy \leq \frac{C}{R^2} \int_{B_{2R} \times B_{(2R)^1 + s}} v^2 \, dx \, dy. \quad (3.1)
\]

where
\[
L(z) := z^4 - \frac{16p\theta(p + 1)}{\theta + 1} z^2 + \frac{16p\theta(p + 1)(p + \theta + 2)}{(\theta + 1)^2} z - \frac{16p\theta(p + 1)^2}{(\theta + 1)^2}. \quad (3.2)
\]

Proof. Let \((u, v)\) be a stable solution of (1.1). Let \(\phi \in C_c^\infty \left( \mathbb{R}^N = \mathbb{R}^{N_1} \times \mathbb{R}^{N_2}, [0, 1] \right)\), and \(\varphi = u^{\frac{\theta + 1}{2}} \phi\) with \(q > 0\). Integrating by parts, we get
\[ \int_{\mathbb{R}^N} |\nabla \phi|^2 \, dx dy = \frac{(q+1)^2}{4q} \int_{\mathbb{R}^N} \left( 1 + \|x\|^{2(1+\theta)} \right)^{\frac{q+1}{2(1+\theta)}} u^q v^p \phi^2 \, dx dy + \int_{\mathbb{R}^N} u^{q+1} |\nabla \phi|^2 \, dx dy \]

(3.3)

Take \( \phi \) into the stability inequality and using \( (3.3) \), we obtain

\[ \sqrt{p \theta} \int_{\mathbb{R}^N} \left( 1 + \|x\|^{2(1+\theta)} \right)^{\frac{q+1}{2(1+\theta)}} u^{\frac{q-1}{2}} v^{\frac{p-1}{2}} u^{q+1} \phi^2 \, dx dy \]

\[ \leq \int_{\mathbb{R}^N} |\nabla \phi|^2 \, dx dy \leq \frac{(q+1)^2}{4q} \int_{\mathbb{R}^N} \left( 1 + \|x\|^{2(1+\theta)} \right)^{\frac{q+1}{2(1+\theta)}} u^q v^p \phi^2 \, dx dy + C \int_{\mathbb{R}^N} u^{q+1} \left[ |\nabla \phi|^2 + \Delta_s(\phi^2) \right] \, dx dy, \]

so we get

\[ a_1 \int_{\mathbb{R}^N} \left( 1 + \|x\|^{2(1+\theta)} \right)^{\frac{q+1}{2(1+\theta)}} u^{\frac{q-1}{2}} v^{\frac{p-1}{2}} u^{q+1} \phi^2 \, dx dy \]

\[ \leq \int_{\mathbb{R}^N} \left( 1 + \|x\|^{2(1+\theta)} \right)^{\frac{q+1}{2(1+\theta)}} u^q v^p \phi^2 \, dx dy + C \int_{\mathbb{R}^N} u^{q+1} \left[ |\nabla \phi|^2 + \Delta_s(\phi^2) \right] \, dx dy, \]

where \( a_1 = \frac{4\sqrt{p \theta}}{(q+1)^2} \). Choose now \( \phi(x, y) = \psi(\frac{x}{R}, \frac{y}{R^{1+\theta}}) \), where \( \psi \) a cut-off function in \( C_{\infty} (\mathbb{R}^N = \mathbb{R}^{N_1} \times \mathbb{R}^{N_2}, [0, 1]) \), such that

\[ \psi = 1 \quad \text{on} \quad B_1 \times B_1, \quad \text{and} \quad \psi = 0 \quad \text{outside} \quad B_2 \times B_{2^{1+\theta}}. \]

A simple calculation, implies that

\[ |\nabla \phi| \leq \frac{C}{R} \quad \text{and} \quad |\Delta_s(\phi^2)| \leq \frac{C}{R^2}. \]

Hence,

\[ I_1 := \int_{\mathbb{R}^N} \left( 1 + \|x\|^{2(1+\theta)} \right)^{\frac{q+1}{2(1+\theta)}} u^{\frac{q-1}{2}} v^{\frac{p-1}{2}} u^{q+1} \phi^2 \, dx dy \]

\[ \leq \frac{1}{a_1} \int_{\mathbb{R}^N} \left( 1 + \|x\|^{2(1+\theta)} \right)^{\frac{q+1}{2(1+\theta)}} u^q v^p \phi^2 \, dx dy + \frac{C}{R^2} \int_{B_{2R} \times B_{(2R)^{1+\theta}}} u^{q+1} \, dx dy. \]

Furthermore, using \( v^{\frac{p-1}{2}} \phi, \ r > 0 \) as test function in \( (2.1) \). As above, we get readily

\[ I_2 := \int_{\mathbb{R}^N} \left( 1 + \|x\|^{2(1+\theta)} \right)^{\frac{q+1}{2(1+\theta)}} u^{\frac{q-1}{2}} v^{\frac{p-1}{2}} u^{r+1} \phi^2 \, dx dy \]

\[ \leq \frac{1}{a_2} \int_{\mathbb{R}^N} \left( 1 + \|x\|^{2(1+\theta)} \right)^{\frac{q+1}{2(1+\theta)}} u^q v^p \phi^2 \, dx dy + \frac{C}{R^2} \int_{B_{2R} \times B_{(2R)^{1+\theta}}} v^{r+1} \, dx dy. \]

with \( a_2 = \frac{4\sqrt{p \theta}}{(r+1)^2} \). Combining the two last inequalities, we have then

\[ I_1 + a_2 \frac{2^{r+1}}{r+1} I_2 \]

\[ \leq \frac{1}{a_1} \int_{\mathbb{R}^N} \left( 1 + \|x\|^{2(1+\theta)} \right)^{\frac{q+1}{2(1+\theta)}} u^q v^p \phi^2 \, dx dy + a_2 \frac{2^{r+1} - r}{r+1} \int_{\mathbb{R}^N} \left( 1 + \|x\|^{2(1+\theta)} \right)^{\frac{q+1}{2(1+\theta)}} u^q v^p \phi^2 \, dx dy + \frac{C}{R^2} \int_{B_{2R} \times B_{(2R)^{1+\theta}}} \left( u^{q+1} + v^{r+1} \right) \, dx \]

(3.4)

Fix

\[ q = \frac{(\theta + 1)r}{p+1} + \frac{\theta - p}{p+1}, \quad \text{or equivalently} \quad q + 1 = \frac{(\theta + 1)(r+1)}{p+1}. \]

(3.5)
Let $r > \frac{p-1}{2}$. Applying Young’s inequality and using (3.3), the first term on the right hand side of (3.4), can be estimated as

\[
\frac{1}{a_1} \int_{\mathbb{R}^N} \left( 1 + \|x\|^{2(s+1)} \right)^{\frac{2}{2s+1}} u^q v^r \phi^2 \, dx dy
\]

\[
= \frac{1}{a_1} \int_{\mathbb{R}^N} \left( 1 + \|x\|^{2(s+1)} \right)^{\frac{2}{2s+1}} u^{\frac{p}{2} + 1} + v^{\frac{p}{2} + 1} \, dx dy
\]

\[
= \frac{1}{a_1} \int_{\mathbb{R}^N} \left( 1 + \|x\|^{2(s+1)} \right)^{\frac{2}{2s+1}} u^{\frac{p}{2}} v^{\frac{p}{2} + 1} \, dx dy
\]

\[
\leq \frac{2r + 1 - p}{2(r + 1)} \int_{\mathbb{R}^N} \left( 1 + \|x\|^{2(s+1)} \right)^{\frac{2}{2s+1}} u^{\frac{p}{2}} v^{\frac{p}{2} + 1} \, dx dy
\]

\[
+ \frac{p + 1}{2(r + 1)} a_1 \int_{\mathbb{R}^N} \left( 1 + \|x\|^{2(s+1)} \right)^{\frac{2}{2s+1}} u^{\frac{p}{2}} v^{\frac{p}{2} + 1} \, dx dy
\]

\[
= \frac{2r + 1 - p}{2(r + 1)} I_1 + \frac{p + 1}{2(r + 1)} a_1 \frac{2(r+1)}{p+1} I_2,
\]

and similarly

\[
a_2 \frac{2(r+1)}{p+1} I_2 \leq \left[ \frac{2r + 1 - p}{2(r + 1)} a_2 \frac{2(r+1)}{p+1} + \frac{p + 1}{2(r + 1)} a_1 \frac{2(r+1)}{p+1} \right] I_2 + \frac{C}{R^2} \int_{B_{2R} \times B_{(2R)^{1+s}}} (u^{q+1} + v^{r+1}) \, dx dy.
\]

Inserting the two above estimates in (3.3), we arrive at

\[
a_2 \frac{2(r+1)}{p+1} I_2 \leq \left[ \frac{2r + 1 - p}{2(r + 1)} a_2 \frac{2(r+1)}{p+1} + \frac{p + 1}{2(r + 1)} a_1 \frac{2(r+1)}{p+1} \right] I_2 + \frac{C}{R^2} \int_{B_{2R} \times B_{(2R)^{1+s}}} (u^{q+1} + v^{r+1}) \, dx dy.
\]

Combining (3.5) and (3.4), one obtains

\[
u^{q+1} \leq C' v^{r+1} \quad \text{and} \quad u^{\frac{p}{2}} v^{\frac{p}{2} + 1} \geq u^{\theta} v^r.
\]

We get then

\[
\frac{p + 1}{2(r + 1)} \left[ (a_1 a_2) \frac{2(r+1)}{p+1} - 1 \right] \int_{\mathbb{R}^N} u^{\theta} v^r \phi^2 \, dx dy \leq \frac{C R^{-2} a_1 \frac{2(r+1)}{p+1}}{a_2 \frac{2(r+1)}{p+1}} \int_{B_{2R} \times B_{(2R)^{1+s}}} v^{r+1} \, dx dy.
\]

Thus, if $a_1 a_2 > 1$, there holds

\[
\int_{B_{R} \times B_{(R)^{1+s}}} \left( 1 + \|x\|^{2(s+1)} \right)^{\frac{2}{2s+1}} u^{\theta} v^r \, dx dy \leq \int_{\mathbb{R}^N} u^{\theta} v^r \phi^2 \, dx dy \leq \frac{C}{R^2} \int_{B_{2R} \times B_{(2R)^{1+s}}} v^{r+1} \, dx dy.
\]

Denote $z = r + 1$, we conclude that if $a_1 a_2 > 1$ and $z > \frac{p+1}{2}$,

\[
\int_{B_{R} \times B_{(R)^{1+s}}} \left( 1 + \|x\|^{2(s+1)} \right)^{\frac{2}{2s+1}} u^{\theta} v^{z-1} \, dx dy \leq \frac{C}{R^2} \int_{B_{2R} \times B_{(2R)^{1+s}}} v^{z} \, dx dy.
\]

Furthermore, we can check that $a_1 a_2 > 1$ is equivalent to $L(z) < 0$, the proof is completed. □

Performing the change of variables $x = \frac{\theta + 1}{p - \theta} \tilde{x}$ in (1.9), a direct computation shows that

\[
H(x) = \left( \frac{\theta + 1}{p - \theta} \right)^4 (z),
\]

where $L$ is given by (3.22). Hence $H(x) < 0$ if and only if $L(z) < 0$. In addition, using Lemma 6 in [10], we have
Remark 3.1.  
• Let $1 < p \leq \theta$, then $L(2) < 0$ and $L$ has a unique root $z_0$ in $(2, \infty)$ and $2t_0^+ < z_0$.
• If $p > \frac{4}{3}$, then $L(p) < 0$ and $z_0$ is the unique root of $L$ in $(p, \infty)$, hence $x_0 = \frac{p+1}{p\theta-1}z_0$.
• Therefore, from Remark 3 in [13], we find that
  \[ x_0 > 2t_0^+ \frac{\theta + 1}{p\theta - 1} > 4, \quad \forall \theta \geq p > 1. \]

We need to recall the following properties of $t_0^+$ and $t_0^-$ before we complete the proof of Theorem 1.1.

Remark 3.2.  
• It is known that for $1 < p \leq \theta$, there hold $t_0^- < 1 < t_0^+$, $t_0^-$ is decreasing and $t_0^+$ is increasing in $\varpi := \frac{p\theta(p+1)}{\theta+1}$. Moreover, $\lim_{\varpi \to \infty} t_0^- = \frac{1}{2}$ and $\lim_{\varpi \to \infty} t_0^+ = 1$.
• Obviously $2t_0^- < p$ if $p > \frac{4}{3}$. Indeed, if $p > \frac{4}{3}$ then $\theta \geq p > \frac{4}{3}$ and $\varpi > \frac{16}{3}$. Since $f(\varpi) := \sqrt{\varpi} - \sqrt{\varpi} - \sqrt{\varpi}$ is decreasing in $\varpi$, there holds $2t_0^- = 2f(\varpi) < 2f(\frac{16}{3}) = \frac{4}{3} < p$.

3.1. End of the proof of Theorem 1.1

In this subsection, we use $L^2$-estimates for Grushin operator, and we apply the bootstrap iteration as in [3, 10, 2]. For the completeness, we present the details.

Let $\eta \in C_c^\infty (\mathbb{R}^N = \mathbb{R}^{N_1} \times \mathbb{R}^{N_2}, [0, 1])$, be a cut-off function such that

\[ \eta = 1 \text{ on } B_1 \times B_1, \quad \text{and } \eta = 0 \text{ outside } B_2 \times B_2. \tag{3.6} \]

We divide the proof in three parts.

Step 1. Denote by $\lambda_s = \frac{N_1}{2\alpha}$. We claim that for any smooth non-negative function $w$, there exists a positive constant $C > 0$ such that

\[
\left( \int_{B_R \times B_R} w^{2\lambda_s} \, dx \, dy \right)^{\frac{1}{2\lambda_s}} \leq CR^{N_s} \left( \frac{1}{\lambda_s} - 1 \right)^2 \int_{B_{2R} \times B_{(2R)^1+s}} |\nabla_s w|^2 \, dx \, dy + CR^{N_s} \left( \frac{1}{\lambda_s} - 1 \right) \int_{B_{2R} \times B_{(2R)^1+s}} w^2 \, dx \, dy. \tag{3.7}
\]

In fact, by using Sobolev inequality [26] and integration by parts, imply that

\[
\left( \int_{B_1 \times B_1} w^{2\lambda_s} \, dx \, dy \right)^{\frac{1}{2\lambda_s}} \leq \left( \int_{B_2 \times B_2} (w\eta)^{2\lambda_s} \, dx \, dy \right)^{\frac{1}{2\lambda_s}} \leq C \left( \int_{B_2 \times B_2} |\nabla_s (w\eta)|^2 \, dx \, dy \right)^{\frac{1}{2}} \leq C \left[ \int_{B_2 \times B_2} \left( |\nabla_s w|^2 \eta^2 + w^2 |\nabla_s \eta|^2 - \frac{w^2}{2} \Delta_s (\eta) \right) \, dx \, dy \right]^{\frac{1}{2}},
\]

So, we get

\[
\left( \int_{B_1 \times B_1} w^{2\lambda_s} \, dx \, dy \right)^{\frac{1}{2\lambda_s}} \leq C \int_{B_2 \times B_2} (|\nabla_s w|^2 + w^2) \, dx \, dy.
\]

By scaling argument, we obtain readily the estimate (3.7).
Step 2. Let \((u, v)\) be a stable solution of (1.1), with \(1 < p \leq \theta\). Then for any \(\lambda_s = \frac{N}{N_s - 2}\) and \(2t_0^- < z_0\), we claim that there exists a positive constant \(C > 0\) such that

\[
\left(\int_{B_{2R} \times B_{(2R)^{1+s}}} v^{z_0 \lambda_s} \, dx dy\right)^{\frac{1}{\lambda_s}} \leq CR^{N_s \left(\frac{1}{\lambda_s} - 1\right)} \int_{B_{2R} \times B_{(2R)^{1+s}}} v^{z_0} \, dx dy. \tag{3.8}
\]

To prove this, for \(2t_0^- < z_0\), in what follows, we choose

\[w = v^{\frac{t_0^-}{2}}.\]

Let us put \(\eta_R(x, y) = \eta(\frac{x}{R}, \frac{y}{R^{1+s}})\), where \(\eta\) is given in (3.6). A simple calculation, we obtain readily

\[
\int_{B_{2R} \times B_{(2R)^{1+s}}} |\nabla_s w|^2 \, dx dy \leq C \int_{B_{2R} \times B_{(2R)^{1+s}}} v^{z_0 - 2} |\nabla_s v|^2 \eta^2_R \, dx dy. \tag{3.9}
\]

Multiplying \(-\Delta v = \left(1 + ||x||^2\right)^{\frac{a}{2}} u^\theta\) by \(v^{z_0 - 2} \eta^2_R\) and integrating by parts, we derive

\[
(z_0 - 1) \int_{B_{2R} \times B_{(2R)^{1+s}}} v^{z_0 - 2} |\nabla_s v|^2 \eta^2_R \, dx dy = \int_{B_{2R} \times B_{(2R)^{1+s}}} \left(1 + ||x||^{2(s+1)}\right)^{\frac{a}{2}} v^{z_0 - 2} u^\theta \eta^2_R \, dx dy
\]

\[
- 2 \int_{B_{2R} \times B_{(2R)^{1+s}}} \eta_R v^{z_0 - 2} \nabla_s v \cdot \nabla_s \eta_R \, dx dy. \tag{3.10}
\]

By Young’s inequality,

\[
2 \int_{B_{2R} \times B_{(2R)^{1+s}}} v^{z_0 - 2} |\nabla_s v|^2 \eta^2_R \, dx dy \leq \frac{z_0 - 1}{2} \int_{B_{2R} \times B_{(2R)^{1+s}}} v^{z_0 - 2} |\nabla_s v|^2 \eta^2_R \, dx dy + C \int_{B_{2R} \times B_{(2R)^{1+s}}} v^{z_0} |\nabla_s \eta_R|^2 \, dx dy.
\]

Substituting this in (3.10), and by virtue of estimate (3.9), we arrive at

\[
\int_{B_{2R} \times B_{(2R)^{1+s}}} |\nabla_s w|^2 \, dx dy
\]

\[
\leq \int_{B_{2R} \times B_{(2R)^{1+s}}} v^{z_0 - 2} |\nabla_s v|^2 \eta^2_R \, dx dy
\]

\[
\leq C \int_{B_{2R} \times B_{(2R)^{1+s}}} \left(1 + ||x||^{2(s+1)}\right)^{\frac{a}{2}} v^{z_0 - 2} u^\theta \eta^2_R \, dx dy + \frac{C}{R^2} \int_{B_{2R} \times B_{(2R)^{1+s}}} v^{z_0} \, dx dy.
\]

In view of estimate (3.7), and using Lemmas 3.1, we obtain easily the estimate (3.8).

Step 3. Let \((u, v)\) be a stable solution of (1.1), with \(1 < p \leq \theta\) and \(q \in (2t_0^-, z_0)\). Then for any \(\lambda_s = \frac{N}{N_s - 2}, \ q < z_m \lambda_s\), we claim that there exists a positive constant \(C > 0\) such that

\[
\left(\int_{B_{2R} \times B_{(2R)^{1+s}}} v^{z_0 \lambda_s} \, dx dy\right)^{\frac{1}{\lambda_s}} \leq CR^{N_s \left(\frac{1}{\lambda_s} - 1\right)} \left(\int_{B_{R_m} \times B_{(R_m)^{1+s}}} v^q \, dx dy\right)^{\frac{1}{q}}. \tag{3.11}
\]

To prove this, by Remark 3.2, we known that \(2t_0^- < p\). Fix a real positive number \(q\) satisfying

\[2t_0^- < q < p.\]
Let $m$ be the nonnegative integer such that $q \lambda_s^{m-1} < z_0 < q \lambda_s^m$. Put

$$z_1 = qk, \quad z_2 = qk \lambda_s, \ldots, \quad z_m = qk \lambda_s^{m-1},$$

verifying

$$2z_0 < z_1 < z_2 < \ldots < z_m < z_0.$$  

Here the constant $k \in [1, \lambda_s]$, is chosen such that $z_m$ is arbitrarily close to $z_0$. Set $R_n = 2^n R$. by (3.8), and an induction argument, we deduce then

$$(\int_{B_R \times B_{R^{1+s}}} v^{z_m} dx dy)^{\frac{1}{z_m}} \leq CR^{N_s}(\int_{B_{R^2} \times B_{(1+s)}} v^{z_m} dx dy)^{\frac{1}{z_m}},$$

$$= CR^{N_s}(\int_{B_{R^2} \times B_{(1+s)}} v^{z_m} dx dy)^{\frac{1}{z_m}},$$

$$\leq CR^{N_s}(\int_{B_{Rm} \times B_{(Rm)^{1+s}}} v^{z_m} dx dy)^{\frac{1}{z_m}},$$

$$\leq CR^{N_s}(\int_{B_{Rm} \times B_{(Rm)^{1+s}}} v^{z_s} dx dy)^{\frac{1}{z_s}},$$

Furthermore, by Hölder’s inequality, there holds

$$(\int_{B_{Rm} \times B_{(Rm)^{1+s}}} v^{qk} dx dy)^{\frac{1}{qk}} \leq \left[ \left( \int_{B_{Rm} \times B_{(Rm)^{1+s}}} v^{qk} dx dy \right)^{\frac{1}{qk}} \left( \int_{B_{Rm} \times B_{(Rm)^{1+s}}} dx dy \right)^{1-\frac{1}{qk}} \right]^{\frac{1}{qk}},$$

$$\leq C \left[ \left( \int_{B_{Rm} \times B_{(Rm)^{1+s}}} v^{qk} dx dy \right)^{\frac{1}{qk}} CR^{N_s}(1-\frac{1}{qk}) \right]^{\frac{1}{qk}},$$

$$\leq CR^{N_s}(\int_{B_{Rm} \times B_{(Rm)^{1+s}}} v^{qk} dx dy)^{\frac{1}{qk}} R^{N_s}(\frac{1}{qk} - \frac{1}{q}) \left( \int_{B_{Rm} \times B_{(Rm)^{1+s}}} v^{qk} dx dy \right)^{\frac{1}{qk}}.$$

Therefore, the claim follows by combining the last two inequalities.

**Proof of Theorem 1.1 completed.** We are now in position to conclude. Let $\alpha \geq 0$, and $(u, v)$ be a stable solution of (1.1) with $\theta \geq p > 1$. We split the proof into two cases: $p > \frac{4}{3}$ and $1 < p \leq \min(\frac{4}{3}, \theta)$.

**Case 1:** $p > \frac{4}{3}$. Let $p > q > 0$. From (2.2), we use Hölder’s inequality, to obtain

$$\int_{B_R \times B_{R^{1+s}}} v^q dx dy \leq \left( \int_{B_R \times B_{R^{1+s}}} \left( 1 + \|x\|^{2(s+1)} \right)^{\frac{2(s+1)}{q} - 1} v^p dx dy \right)^{\frac{q}{p}}$$

$$\times \left( \int_{B_R \times B_{R^{1+s}}} \left( 1 + \|x\|^{2(s+1)} \right)^{-\frac{2(s+1)}{q} - 1} dx dy \right)^{\frac{q}{2(s+1)}}$$

$$\leq CR^{N_s}(\frac{2(s+1)}{q} - 1) \left( 1 + \frac{\alpha}{p} \right)^{\frac{2(s+1)}{q} - 1} = CR^{N_s}(\frac{2(s+1)}{q} - 1)^\theta.$$
Combining (3.14) and (3.11), we deduce that

$$\left( \int_{B_R \times B_R^{1+s}} v^{z_m}\lambda_s \, dx dy \right)^{\frac{1}{z_m \lambda_s}} \leq CR_{\alpha}^{\frac{N_s}{N_s - 2}} (2 + (\alpha)(\theta + 1))^{\frac{N_s}{p^\theta - 1}}.$$

(3.15)

Recall that \(\lambda_s = \frac{N_s}{N_s - 2}\). Since

\[
N < 2 + \left( \frac{2 + (\alpha)(\theta + 1)}{p^\theta - 1} \right) z_0,
\]

we chose \(k \in [1, \lambda_s]\), such that \(z_m\) is close to \(z_0\). Then, it implies from (3.15) that \(\|v\|_{L^{z_m} \lambda_s(\mathbb{R}^N)} = 0\) as \(R \to \infty\), i.e., \(v \equiv 0\) in \(\mathbb{R}^N\). This is a contraction. Therefore, we get the desired result: the equation (1.1) has no stable solution if \(N < 2 + (2 + \alpha)x_0\) where \(x_0 = \frac{\theta + 1}{p^\theta - 1}z_0\).

Finally, it follows from Remark 3.1 that if \(N \leq 10 + 4\alpha\), (1.1) has no stable solution for any \(\theta \geq p > \frac{4}{3}\).

Case 2: \(1 < p \leq \frac{4}{3}\) and \(u\) is bounded. Let now \(2 > q > 0\), using Lemma 2.5 we obtain

\[
\int_{B_R \times B_R^{1+s}} v^q \, dx dy \leq CR_{\alpha}^{\left\lceil \frac{N_s}{p^\theta - 1} \right\rceil \left( \frac{2 + (\alpha)(\theta + 1)}{p^\theta - 1} \right)^{\frac{N_s - 2}{p^\theta - 1}} \left( \frac{2 + (\alpha)(\theta + 1)}{p^\theta - 1} \right)^{\frac{N_s - 2}{p^\theta - 1}}. 
\]

Substituting this in (3.11),

\[
\left( \int_{B_R \times B_R^{1+s}} v^{z_m}\lambda_s \, dx dy \right)^{\frac{1}{z_m \lambda_s}} \leq CR_{\alpha}^{\frac{N_s}{N_s - 2}} \left( \frac{2 + (\alpha)(\theta + 1)}{p^\theta - 1} \right)^{\frac{N_s - 2}{p^\theta - 1}}. 
\]

Arguing as Case 1, we get the desired result.

\[\square\]

3.2. End of the proof of Corollary 1.1

If \(p = \theta > 1\), then, from Lemma 2.3, we get that \(v = u\) and the weighted Lane-Emden system (1.1) is reduced to the weighted Lane-Emden equation (1.2). From Remark 3.1 and arguing as above, we get

\[
L(z) = z^4 - 16p^2z^2 + 32p^2z - 16p^2 = (z^2 + 4p(z - 1))(z - 2t_0^+)(z - 2t_0^-) \quad \text{with} \quad t_0^+ = p \pm \sqrt{p^2 - p}.
\]

Adopting the similar argument as in the proof of Corollary 1.2, we obtain \(2t_0^+\) is the largest root of \(L\) as \(t_0^+ > p > 1\). Therefore

\[
x_0 = \frac{2p + 2\sqrt{p^2 - p}}{p - 1} > 4 \quad \text{for all} \ p > 1.
\]

The result follows immediately by applying Theorem 1.1.

\[\square\]

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