Spatially covariant gravity with a dynamic lapse function

Jiong Lin,¹,* Yungui Gong,¹,† Yizhou Lu,¹,‡ and Fengge Zhang¹,§

¹School of Physics, Huazhong University of Science and Technology, Wuhan, Hubei 430074, China

Abstract

In the framework of spatially covariant gravity (SCG), it is natural to extend a gravitational theory by putting the lapse function $N$ and the spatial metric $h_{ij}$ on an equal footing. We find two sufficient and necessary conditions for ensuring two physical degrees of freedoms (DoFs) for the theory with the lapse function being dynamical by Hamiltonian analysis. A class of quadratic actions with only two DoFs is constructed. In the case that the coupling functions depend on $N$ only, we find that the spatial curvature term cannot enter the Lagrangian and thus this theory possesses no wave solution and cannot recover general relativity (GR). In the case that the coupling functions depend on the spatial derivatives of $N$, we perform a spatially conformal transformation on a class of quadratic actions with non-dynamical lapse function to obtain a class of quadratic actions with $\dot{N}$. We confirm this theory has two DoFs by checking the two conditions. Besides, we find that a class of quadratic actions with two DoFs can be transformed from GR by disformal transformation.

* jionglin@hust.edu.cn
† Corresponding author. yggong@hust.edu.cn
‡ louischou@hust.edu.cn
§ fenggezhang@hust.edu.cn
I. INTRODUCTION

To explain the early and late-time accelerated expansion of the universe [1–5], many extended theories of gravity have been proposed. A natural and simple extension to general relativity (GR) is the inclusion of an extra scalar physical degree of freedom (DoFs) [6]. Scalar-tensor theories, especially Horndeski theory [7–12] have played an important role in building models of inflation and dark energy [13–20]. Horndeski theory which contains Brans-Dicke theory [21, 22], is the most general scalar-tensor theory involving up to second-order derivatives in the Lagrangian while retaining second-order field equations in four dimensions. By performing disformal transformation [23]

\[
\tilde{g}_{\mu\nu} = C(\phi, X)g_{\mu\nu} + D(\phi, X)\phi_\mu\phi_\nu
\]  

(1)

on Horndeski theory, a class of degenerate higher-order scalar-tensor theories (DHOST) [24–34] can be obtained. DHOST possess higher-order derivatives in the Lagrangian while the degeneracy of the kinetic matrix indicates the existence of hidden constraints [35, 36] and consequently the possible absence of Ostrogradsky ghost [37, 38].

A timelike scalar field offers a natural spacetime foliation. After fixing the unitary gauge, i.e. \( \phi = t \), scalar-tensor theories can be regarded as spatially covariant gravity (SCG) theories [39–49]. Although the scalar field vanishes in the unitary gauge, the scalar-type physical degree of freedom remains due to the breaking of spacetime diffeomorphism. Thus to some extent, introducing the extra physical degree of freedom beyond GR is equivalent to constructing a class of SCG. A general Lagrangian of SCG, which depends on the lapse function \( N \), spatial metric \( h_{ij} \), extrinsic curvature \( K_{ij} \equiv \mathcal{L}_{\vec{n}} h_{ij}/2 \) and their spatial derivative \( D_i \), has been investigated in Refs. [41, 42]. This kind of SCG includes the Horndeski theory, beyond Horndeski theory, effective field theory (EFT) of inflation [50–52] and the Hořava gravity [53–55], and there are three propagating DoFs. A more general class of SCG has been studied by including another important building block \( \mathcal{L}_{\vec{n}} N \) to the Lagrangian [43, 45]. In GR, \( N \) acts as an auxiliary field to ensure the spacetime diffeomorphism. While in SCG, there is no need to keep \( N \) as an auxiliary field and it is natural to extend a gravitational theory by putting the lapse function and the spatial metric on an equal footing. On the other hand, from the viewpoint of effective field theory (EFT), the operator \( \mathcal{L}_{\vec{n}} N \) should be included in the effective Lagrangian. Besides, under disformal transformation or minmetric transformation [56, 57], the transformed theories may acquire \( \dot{N} \). Generally, if the lapse
function is dynamic, an extra scalar mode arises and the theory has 4 DoFs. In Ref. [43] the author obtained two sufficient and necessary conditions to get rid of one scalar mode through detailed Hamiltonian analysis. An alternative derivation of the two conditions in the Lagrangian level has been studied by performing the perturbation analysis in Ref. [45].

Recently, a more aggressive attempt to seek a class of modified theories of gravity with two tensorial DoFs has been made within the framework of SCG with a non-dynamic lapse function [46]. This kind of SCG naturally contains cuscuton theory [58–61] where the scalar field is non-dynamic and the theory only propagates two tensorial DoFs in the unitary gauge. However, a further question arises: what if the lapse function is treated on an equal footing with the spatial metric. On the other hand, motivated by field transformation, one may wonder whether there is a relation between GR and SCG with two DoFs. In this paper, as a first step, by performing detailed Hamiltonian analysis, we derive two sufficient and necessary conditions for the SCG theory with a dynamic lapse function to propagate two DoFs. The correspondence between GR and SCG with two DoFs is also discussed.

The paper is organized as follows. In Sec.II, we derive the two sufficient and necessary conditions for ensuring two DoFs for the theory. Then a class of quadratic actions is constructed and the correspondence between SCG and GR is discussed in Sec.III. We conclude our paper in Sec.IV.

II. SPATIALLY COVARIANT GRAVITY WITH TWO PHYSICAL DEGREES OF FREEDOM

We start with the Arnowitt-Deser-Misner 3+1 decomposition of the four dimensional spacetime [62]

$$ds^2 = -N dt^2 + h_{ij} (dx^i + N^i dt)(dx^j + N^j dt),$$

(2)

where $N$ is the lapse function, $N^i$ is the shift function and $h_{ij}$ is the induced metric on the spatial hypersurfaces. In general, the Lagrangian involves both the kinetic term of the induced spatial metric, i.e. the extrinsic curvature $K_{ij} = \mathcal{L}_{\tilde{n}} h_{ij} / 2$, and the kinetic term of the lapse function $F = \mathcal{L}_{\tilde{n}} N$, where $\mathcal{L}_{\tilde{n}}$ is the Lie derivative with respect to the timelike vector $\tilde{n}$ normal to the spatial hypersurfaces. Note that to keep the spatial diffeomorphism, $N^i$ can only enter the Lagrangian through $K_{ij}$ and $F$. In addition, terms involving the
spatial derivative such as $D_m F, D_m K_{ij}$ are also allowed. The most general action of SCG is

$$S = \int dt d^3x \sqrt{h} N L(N, h_{ij}, K_{ij}, F, D_i).$$

(3)

In this section, we will derive sufficient and necessary conditions for ensuring two DoFs for the theory by Hamiltonian analysis. For the sake of Hamiltonian analysis, we introduce two auxiliary fields $A$ and $B_{ij}$ to rewrite the action (3) as

$$S = S_B + \int d^4x \left[ \frac{\delta S_B}{\delta A} (F - A) + \frac{\delta S_B}{\delta B_{ij}} (K_{ij} - B_{ij}) \right],$$

(4)

where $S_B$ is obtained from the action (3) by replacing $K_{ij}$ with $B_{ij}$ and $F$ with $A$. In the case that $S_B$ doesn’t involve the spatial derivative of $A, B_{ij}$, varying the action with respect to $A$ and $B_{ij}$, we get

$$\frac{\delta^2 S_B}{\delta A^2} (F - A) = 0,$$

(5)

$$\frac{\delta^2 S_B}{\delta B_{ij} \delta B_{kl}} (K_{kl} - B_{kl}) = 0.$$  

(6)

To ensure $F = A$ and $K_{ij} = B_{ij}$, we require

$$\frac{\delta^2 S_B}{\delta A^2} \neq 0, \quad \frac{\delta^2 S_B}{\delta B_{ij} \delta B_{kl}} \neq 0.$$  

(7)

In the SCG, the 17 canonical variables are

$$\{\Phi_I\} = \{N^i, A, B_{ij}, N, h_{ij}\},$$

(8)

and the corresponding conjugate momenta are

$$\{\Pi^I\} = \{\pi_i, p, p^{ij}, \pi, \pi^{ij}\}$$

(9)

with

$$\pi_i = 0, \quad p = 0, \quad p^{ij} = 0, \quad \pi = \frac{1}{N} \frac{\delta S_B}{\delta A}, \quad \pi^{ij} = \frac{1}{2N} \frac{\delta S_B}{\delta B_{ij}}.$$  

(10)

From Eq. (10), we get the primary constraints

$$\{\varphi^I\} = \{\pi_i, p, p^{ij}, \pi, \pi^{ij}\} \approx 0,$$

(11)

where

$$\tilde{\pi} = \pi - \frac{1}{N} \frac{\delta S_B}{\delta A}, \quad \tilde{\pi}^{ij} = \pi^{ij} - \frac{1}{2N} \frac{\delta S_B}{\delta B_{ij}}.$$  

(12)
and “≈” denotes weak equality which is valid on the subspace of phase space determined by the primary constraints. The corresponding Lagrange multipliers are

$$\{\lambda_I\} = \{\lambda^i, \Lambda, \Lambda_{ij}, \lambda, \lambda_{ij}\}. \quad (13)$$

The canonical Hamiltonian is obtained from Lagrangian by Legendre transformation

$$H_C = \int d^3x (\Pi^I \dot{\Phi}_I - N \sqrt{h} \mathcal{L}_B)$$
$$= \int d^3x (NC + \pi \mathcal{L}_N N + \pi^i \mathcal{L}_N h_{ij}), \quad (14)$$

where

$$C = \pi A + 2\pi^i B_{ij} - \sqrt{h} \mathcal{L}_B,$$ \quad (15)

and $\mathcal{L}_B$ is obtained from $\mathcal{L}$ in Eq. (3) by replacing $K_{ij}$ with $B_{ij}$ and $F$ with $A$. Using the constraints $p^{ij} \approx 0$, $p \approx 0$, $\pi_i \approx 0$, integrating by parts and choosing Minkowski spacetime as boundary condition at infinity, the canonical Hamiltonian can be recasted to the form

$$H_C \approx \int d^3x (NC + N^i C_i), \quad (16)$$

where

$$C_i = \pi D_i N - 2\sqrt{h} D_j \left( \frac{\pi^j}{\sqrt{h}} \right) + p D_i A + p^{kl} D_i B_{kl} - 2\sqrt{h} D_j \left( \frac{p_{ik}}{\sqrt{h}} B_{ik} \right) + \pi_k D_i N^k$$
$$+ \sqrt{h} D_k \left( \frac{\pi^k N^j}{\sqrt{h}} \right). \quad (17)$$

Defining the Poisson bracket

$$[F, G] = \int d^3z \left( \frac{\delta F}{\delta \Phi_I} \frac{\delta G}{\delta \Pi^I} - \frac{\delta F}{\delta \Pi^I} \frac{\delta G}{\delta \Phi_I} \right), \quad (18)$$

and using the Hamiltonian (16) and the Poisson bracket (18), we get the consistency conditions

$$\dot{\varphi}^I (\vec{x}) = \int d^3y D^{IJ} \lambda_J + [\varphi^I (\vec{x}), H_C] \approx 0, \quad (19)$$

where the values of the Dirac matrix $D^{IJ} = [\varphi^I (\vec{x}), \varphi^J (\vec{y})]$ are presented in Appendix A. The consistency condition of the constraint $\pi_i \approx 0$ leads to three secondary constraints

$$\dot{\pi}_i (\vec{x}) = -C_i (\vec{x}) \approx 0. \quad (20)$$

Note that it was proved in Ref. [43] that the Poisson brackets between $C_i \approx 0$ and any tensor field $T \approx 0$ vanish. Thus $C_i \approx 0$ are the first-class constraints.
If the Lagrangian contains no $\dot{N}$, generally the theory has three DoFs due to the breaking of spacetime diffeomorphism [41]. When $\dot{N}$ enters the Lagrangian, the lapse function $N$ contributes a scalar mode and generally the theory has four DoFs. In Ref. [43], the author has derived two sufficient and necessary conditions to eliminate a scalar mode. The two conditions are

$$D(\vec{x}, \vec{y}) = \frac{\delta^2 S_B}{\delta A(\vec{x}) \delta A(\vec{y})} - \int d^3 x' d^3 y' \frac{\delta^2 S_B}{\delta A(\vec{x}) \delta B_{ij}(\vec{x})} G_{ijkl}(\vec{x}', \vec{y}') \frac{\delta^2 S_B}{\delta B_{kl}(\vec{y}') \delta A(\vec{y})} = 0$$

(21)

and

$$F(\vec{x}, \vec{y}) = [\bar{\pi}(\vec{x}), \bar{\pi}(\vec{y})] + \int d^3 z ([\bar{\pi}(\vec{x}), \bar{\pi}^{ij}(\vec{z})] V_{ij}(\vec{z}, \vec{y}) - [\bar{\pi}(\vec{y}), \bar{\pi}^{ij}(\vec{z})] V_{ij}(\vec{z}, \vec{x}))$$

$$+ \int d^3 x' d^3 y' V_{ij}(\vec{x}', \vec{x}) [\bar{\pi}^{ij}(\vec{x}'), \bar{\pi}^{kl}(\vec{y}')] V_{kl}(\vec{y}', \vec{y}) = 0.$$  

(22)

where $G_{ijkl}(\vec{z}, \vec{x})$ is the inverse of $\delta^2 S_B / \delta B_{ij}(\vec{x}) \delta B_{kl}(\vec{y})$, i.e.

$$\int d^3 x G_{ijmn}(\vec{z}, \vec{x}) \frac{\delta^2 S_B}{\delta B_{mn}(\vec{x}) \delta B_{kl}(\vec{y})} = \delta^k_i \delta^l_j \delta^3 (\vec{z} - \vec{y})$$

(23)

and

$$V_{ij}(\vec{x}, \vec{y}) = -2N(\vec{x}) \int d^3 y' G_{ijkl}(\vec{x}, \vec{y}') [p^{kl}(\vec{y}'), \bar{\pi}(\vec{y})].$$

(24)

After imposing the two conditions (21) (22), another secondary constraint

$$C'(\vec{x}) = [\bar{\pi}(\vec{x}), H_c] + \int d^3 y [\bar{\pi}^{ij}(\vec{y}), H_c] V_{ij}(\vec{y}, \vec{x}) \approx 0$$

(25)

aries. For the sake of counting the number of the physical DoFs, we show the Dirac matrix of the constraints in Tab.I, where the non-zero elements are indicated by $\ast$ and

$$\bar{p}(\vec{x}) = p(\vec{x}) + \int d^3 y p^{ij}(\vec{y}) U_{ij}(\vec{y}, \vec{x}),$$

(26)

$$\bar{\pi}(\vec{x}) = \bar{\pi}(\vec{x}) + \int d^3 y p^{ij}(\vec{y}) U_{ij}(\vec{y}, \vec{x}) + \int d^3 y \bar{\pi}^{ij}(\vec{y}) V_{ij}(\vec{y}, \vec{x}),$$

(27)

$$C'(\vec{x}) = C'(\vec{x}) + \int d^3 z [S_{ij}(\vec{x}, \vec{z}) p^{ij}(\vec{z}) + T_{ij}(\vec{x}, \vec{z}) \bar{\pi}^{ij}(\vec{z})],$$

(28)

$$\{\varphi_1^f\} = \{\pi_i, \bar{p}, p^{ij}, \bar{\pi}, \bar{\pi}^{ij}, C_i, C'\}$$

(29)
with

$$U_{ij}(\vec{x}, \vec{y}) = \int d^3x' G_{ijkl}(\vec{x}, \vec{x}') 2N(\vec{x}') [\bar{\pi}^{kl}(\vec{x}'), p(\vec{y})],$$

$$X_{ij}(\vec{x}, \vec{y}) = \int d^3x' G_{ijkl}(\vec{x}, \vec{x}') 2N(\vec{x}') \left( [\bar{\pi}^{kl}(\vec{x}'), \bar{\pi}(\vec{y})] + \int d^3y' [\bar{\pi}^{kl}(\vec{x}'), \bar{\pi}^{mn}(\vec{y}')] \gamma_{mn}(\vec{y}', \vec{y}) \right),$$

$$T_{ij}(\vec{x}, \vec{w}) = \int d^3y' 2N(\vec{w}) G_{ijkl}(\vec{w}, \vec{y}') [\mathcal{C}'(\vec{x}), p^{kl}(\vec{y}')],$$

$$S_{ij}(\vec{x}, \vec{w}) = -\int d^3y' 2N(\vec{y}') G_{ijkl}(\vec{w}, \vec{y}') [\mathcal{C}'(\vec{x}), \bar{\pi}^{kl}(\vec{y}')]$$
$$- \int d^3y' \int d^3z' 2N(\vec{y}') G_{ijkl}(\vec{w}, \vec{y}') T_{ij'}(\vec{x}, \vec{z}') [\bar{\pi}^{i'j'}(\vec{z}'), \bar{\pi}^{kl}(\vec{y}')] \approx 0,$$
and
\[
\mathcal{J}(\vec{x}, \vec{y}) = \int d^3y' \int d^3y'' N(\vec{y}') \mathcal{G}_{ijkl}(\vec{y}', \vec{y}'') \frac{\delta C'(\vec{x})}{\delta h_{kl}(\vec{y}'')} \frac{\delta C'(\vec{y})}{\delta B_{ij}(\vec{y}'')} \\
+ \int d^3y' \int d^3y'' \int d^3z' \mathcal{G}_{ijkl}(\vec{y}, \vec{y}'') N(\vec{z}) \mathcal{G}_{i'j'k'l'}(\vec{z}', \vec{y}'') \frac{\delta C'(\vec{x})}{\delta B_{k'l'}(\vec{y}'')} \frac{\delta C'(\vec{y})}{\delta h_{ij}(\vec{y}'')} (\vec{x} \leftrightarrow \vec{y}) \approx 0,
\]
where
\[
C'(\vec{x}) = \frac{\delta S_B}{\delta N} - \frac{1}{N(\vec{x})} \frac{\delta S_B}{\delta B_{ij}(\vec{x})} B_{ij}(\vec{x}).
\]
In our case that \( N \) becomes dynamical, the two TT conditions (35) and (36) need to be generalized. To eliminate another scalar mode, the Dirac matrix in Table I must be degenerate. Deleting the first two columns and rows of the Dirac matrix whose elements are zero, the sub-matrix is
\[
D(\vec{x}, \vec{y}) = \begin{pmatrix}
0 & 0 & [p^{ij}(\vec{x}), \tilde{\pi}^{kl}(\vec{y})] & 0 \\
0 & 0 & 0 & [\tilde{\pi}(\vec{x}), C'(\vec{y})] \\
[\tilde{\pi}^{ij}(\vec{x}), p^{kl}(\vec{y})] & 0 & [\tilde{\pi}^{ij}(\vec{x}), \tilde{\pi}^{kl}(\vec{y})] & 0 \\
0 & [C'(\vec{x}), \tilde{\pi}(\vec{y})] & 0 & [C'(\vec{x}), C'(\vec{y})]
\end{pmatrix}
\]
(38)

\[
D_1(\vec{x}, \vec{y}) = \begin{pmatrix}
0 & D_1(\vec{x}, \vec{y}) \\
-D_1^T(\vec{y}, \vec{x}) & D_2(\vec{x}, \vec{y})
\end{pmatrix},
\]

where
\[
D_1(\vec{x}, \vec{y}) = \begin{pmatrix}
[p^{ij}(\vec{x}), \tilde{\pi}^{kl}(\vec{y})] & 0 \\
0 & [\tilde{\pi}(\vec{x}), C'(\vec{y})]
\end{pmatrix},
\]
(39)
and
\[
D_2(\vec{x}, \vec{y}) = \begin{pmatrix}
[\tilde{\pi}^{ij}(\vec{x}), \tilde{\pi}^{kl}(\vec{y})] & 0 \\
0 & [C'(\vec{x}), C'(\vec{y})]
\end{pmatrix}.
\]
(40)
Note that the degeneracy of \( D \) is equivalent to degeneracy of \( D_1 \). To keep the two tensorial degrees, the degeneracy condition of \( D_1 \) is
\[
S_1(\vec{x}, \vec{y}) \equiv [\tilde{\pi}(\vec{x}), C'(\vec{y})] \approx 0.
\]
(41)
Using the explicit expressions of the Poisson brackets presented in Appendix A, we get
\[
S_1(\vec{x}, \vec{y}) \approx -\frac{\delta C'(\vec{y})}{\delta N(\vec{x})} - \int d^3y' X_{ij}(\vec{y}', \vec{x}) \frac{\delta C'(\vec{y})}{\delta B_{ij}(\vec{y})} - \int d^3y' \mathcal{V}_{ij}(\vec{y}', \vec{x}) \frac{\delta C'(\vec{y})}{\delta h_{ij}(\vec{y}')}. 
\]
(42)
Our condition (41) can be regarded as the extension of the first TT condition (35). In fact, when the Lagrangian doesn’t contain \( \bar{N} \), \( C'(\bar{x}) \) reduces to \( C'(\bar{x}) \) and

\[
\mathcal{V}_{ij} = 0, \quad \mathcal{X}_{ij} = \int d^3x' G_{ijkl}(\bar{x}, \bar{x}')[2N(\bar{x}')[\pi^{kl}(\bar{x}'), \bar{\pi}(\bar{y})] \tag{43}
\]

with

\[
[\pi^{kl}(\bar{y}), \bar{\pi}(\bar{x})] = \frac{1}{2} \delta^3(\bar{x} - \bar{y}) \left[ \frac{1}{2 N^2(\bar{y})} \delta S - \frac{1}{2 N(\bar{y})} \delta N(\bar{x}) \delta B_{kl}(\bar{y}) \right]. \tag{44}
\]

Substituting Eqs. (37) and (43) to Eq. (42), we find our condition (41) reduces to the first TT condition (35).

If \([\bar{C}'(\bar{x}), \bar{C}'(\bar{y})] \neq 0\), then the condition (41) makes \( \bar{\pi} \approx 0 \) a first class constraint and no secondary constraint appears. In this case, we have 8 first class constraints \( \pi_i, C_i, \bar{p} \) and \( \bar{\pi} \), and 13 second class constraints \( p^{ij}, \pi^{ij} \) and \( \bar{C}' \). The dimension of phase space is

\[
2 \times 17 - 2 \times 8 - 13 = 5,
\]

which is odd and the theory is not self consistent. Thus another condition

\[
\mathcal{J}_1(\bar{x}, \bar{y}) \equiv [\bar{C}'(\bar{x}), \bar{C}'(\bar{y})] \approx 0 \tag{45}
\]

should be imposed. Using the explicit expressions of the Poisson brackets in Appendix A, we get

\[
\mathcal{J}_1(\bar{x}, \bar{y}) \approx [\bar{C}'(\bar{x}), \bar{C}'(\bar{y})] = -\int d^3y'y' S_{ij}((\bar{x}, \bar{y}'), \delta C'(\bar{y})) \delta B_{ij}(\bar{y}) - \int d^3y'y' T_{ij}((\bar{x}, \bar{y}'), \delta C'(\bar{y})) \delta h_{ij}(\bar{y})
\]

\[
= \int d^3y' \int d^3y'' 2N(\bar{y}'') G_{ijkl}(\bar{y}, \bar{y}'') \delta C'(\bar{x}) \delta C'(\bar{y}) \delta B_{ij}(\bar{y}) \delta h_{kl}(\bar{y}'')
\]

\[
+ \int d^3y' \int d^3y'' \int d^3z' G_{ijkl}(\bar{y}, \bar{y}'') 2N(\bar{z}') G_{k'l'k''l''}(\bar{z}, \bar{y}'') \delta C'(\bar{x}) \delta C'(\bar{y}) \delta B_{ij}(\bar{y}) \delta h_{kl}(\bar{y}'')
\]

\[
\times \delta B_{k'l'}(\bar{y}'') \delta B_{ij}(\bar{y}) - (\bar{x} \leftrightarrow \bar{y}). \tag{46}
\]

Our condition (45) can be regarded as the extension of the second TT condition (36). In fact, when the Lagrangian doesn’t contain \( \bar{N} \), \( C'(\bar{x}) \) reduces to \( C'(\bar{x}) \) and our condition (45) becomes the second TT condition (36).

In the case that \( C'(\bar{x}) \) only depends on \( K_{ij}, R_{ij}, N \) and \( h_{ij} \), i.e. \( C' = C'(K_{ij}, R_{ij}, N, h_{ij}) \), the second condition (45) can be simplified. After introducing test functions \( \alpha(\bar{x}) \) and \( \beta(\bar{y}) \)
with $\alpha, \beta \rightarrow \infty$ 0, the second condition (45) becomes

$$\int d^3x \alpha(\bar{x}) \beta(\bar{y}) J_1(\bar{x}, \bar{y})$$

$$= \int d^3y' d^3y'' 2N(\bar{y}') G_{ijkl}(\bar{y}', \bar{y}'') \left[ \frac{\delta C'[\alpha]}{\delta h_{kl}(\bar{y}'')} \frac{\delta C'[\beta]}{\delta B_{ij}(\bar{y}'')} - (\alpha \leftrightarrow \beta) \right]$$

$$= \int d^3x 2G_{ijkl}(\bar{x}) \left[ \beta(\bar{x}) \frac{\partial C'}{\partial B_{ij}} D_n D_m \left( \frac{\alpha}{\sqrt h} \frac{\partial C'}{\partial R_{ij}} T_{ijkl}^{mnkl} \right) - (\alpha \leftrightarrow \beta) \right]$$

$$(47)$$

$$(\alpha D_m \beta - \beta D_m \alpha) (T_{ijkl}^{mnkl} \Omega_{ij} \Theta_{kl} - \Theta_{kl} D_n (T_{ijkl}^{mnkl} \Omega_{ij})) \approx 0,$$

where

$$C'[\alpha] = \int d^3x C'(\bar{x}) \alpha(\bar{x}),$$

$$\bar{G}_{ijkl} \delta^3(\bar{x} - \bar{y}) = N \sqrt h G_{ijkl}(\bar{x}, \bar{y}),$$

$$T_{ijkl}^{mnkl} = h^m(l_i h_{k,l}^n) - \frac{1}{2} h^{mn} h_{(k}^i h_{l)}^n - \frac{1}{2} h^{kl} h^m_i h^n_j,$$

$$\Theta_{kl} = \bar{G}_{ijkl} \frac{\partial C'}{\partial B_{ij}}, \quad \Omega_{ij} = \frac{\partial C'}{\partial R_{ij}}.$$

In the case that $C'(\bar{x})$ depends on the Ricci tensor $R_{ij}$ only through the Ricci scalar $R$, we have

$$T_{ijkl}^{mnkl} \Omega_{ij} \propto h^{n(k} h^m_l) - h^{mn} h_{kl},$$

so $T_{ijkl}^{mnkl} \Omega_{ij}$ possesses $m \leftrightarrow n$ exchange symmetry. The second condition (47) can be further simplified as

$$\int d^3x \frac{2}{\sqrt h} [(\alpha D_m \beta - \beta D_m \alpha) (T_{ijkl}^{mnkl} \Omega_{ij} D_n D_n (T_{ijkl}^{mnkl} \Omega_{ij}))] \approx 0. \quad (53)$$

After the two conditions (41) and (45) are imposed, a tertiary constraint

$$\Phi(\bar{x}) = [\bar{C}'(\bar{x}), H_c]$$

arises. In general, we have

$$[\Phi, \bar{p}] \neq 0, \quad [\Phi, \bar{n}] \neq 0, \quad [\Phi, \bar{C}'] \neq 0. \quad (55)$$

But we can introduce the combinations

$$\bar{p}(\bar{y}) = \int d^3z [R_1(\bar{y}, \bar{z}) \bar{p}(\bar{z}) + R_2(\bar{y}, \bar{z}) \bar{n}(\bar{z})], \quad (56)$$
and

\[ \tilde{C}'(\vec{y}) = \int d^3 z [R_3(\vec{y}, \vec{z})\tilde{C}'(\vec{z}) + R_4(\vec{y}, \vec{z})\bar{\pi}(\vec{z})], \tag{57} \]

such that

\[ [\bar{\pi}, \Phi] = 0, \quad [\tilde{C}', \Phi] = 0. \tag{58} \]

Now we have 8 first class constraints \(\{\pi_i, \bar{p}, C_i, \tilde{C}'\}\) and 14 second class constraints \(\{p_{ij}, \bar{\pi}, \tilde{\pi}_{ij}, \Phi\}\), thus the theory has DoFs=(2\times17-2\times8-14)/2 = 2. We show the Dirac matrix of these constraints in Table II, where \(\{\varphi_2^I\} = \{\pi_i, \bar{p}, p_{ij}, \bar{\pi}, \tilde{\pi}_{ij}, C_i, \tilde{C}', \Phi\}\).

### TABLE II. The Dirac matrix of the constraints. The non-zero elements are indicated by *.

| \(\pi_i(\vec{y})\) | \(\bar{p}(\vec{y})\) | \(p_{kl}(\vec{y})\) | \(\bar{\pi}(\vec{y})\) | \(\tilde{\pi}_{kl}(\vec{y})\) | \(C_i(\vec{y})\) | \(\tilde{C}'(\vec{y})\) | \(\Phi(\vec{y})\) | \(\varphi_2^I(\vec{x}), H_c\) |
|------------------|------------------|------------------|------------------|------------------|------------------|------------------|------------------|------------------|
| \(\pi_i(\vec{x})\) | \(\bar{p}(\vec{x})\) | \(0\) | \(0\) | \(0\) | \(0\) | \(0\) | \(0\) | \(0\) |
| \(p_{ij}(\vec{x})\) | \(*\) | \(*\) | \(*\) | \(*\) | \(*\) | \(*\) | \(*\) | \(*\) |
| \(\bar{\pi}(\vec{x})\) | \(0\) | \(0\) | \(0\) | \(0\) | \(0\) | \(0\) | \(0\) | \(0\) |
| \(\tilde{\pi}_{kl}(\vec{x})\) | \(*\) | \(*\) | \(*\) | \(*\) | \(*\) | \(*\) | \(*\) | \(*\) |
| \(C_i(\vec{x})\) | \(0\) | \(0\) | \(0\) | \(0\) | \(0\) | \(0\) | \(0\) | \(0\) |
| \(\tilde{C}'(\vec{x})\) | \(0\) | \(0\) | \(0\) | \(0\) | \(0\) | \(0\) | \(0\) | \(0\) |
| \(\Phi(\vec{x})\) | \(*\) | \(*\) | \(*\) | \(*\) | \(*\) | \(*\) | \(*\) | \(*\) |

### III. QUADRATIC ACTION AND FIELD TRANSFORMATION

In this section, we construct a class of quadratic action with two DoFs in the case that the lapse function is dynamic. Let us start from the action

\[ S = \int d^4x N\sqrt{\bar{h}}[aK_{ij}K^{ij} + \beta K^2 + c_1KF + c_2F^2 + \gamma (^{(3)}R + f)], \tag{59} \]

where \(^{(3)}\)R is the spatial curvature. As a first step, we assume that the coupling functions \(a, \beta, c_1, c_2, \gamma \) and \(f\) only depend on \(N\). Generally, this theory has 4 DoFs. Applying the conditions (21) and (22) to eliminate a scalar mode, we get

\[ c_2 = \frac{3}{4}a + \frac{c_1^2}{3\beta}. \tag{60} \]
Note that there is no constraint on $\gamma$ and $f$. Substituting this condition (60) into the action (59), we get

$$S = \int d^4x N \sqrt{\hat{h}} [a \hat{K}_{ij} \hat{K}^{ij} + b(K + cF)^2 + \gamma (3)R + f],$$

(61)

where the traceless part of the extrinsic curvature

$$\hat{K}_{ij} = K_{ij} - \frac{1}{3} Kh_{ij},$$

(62)

and

$$b = \frac{1}{3}(a + 2\beta), \quad c = \frac{c_1}{2b}.$$

(63)

Substituting the equivalent action

$$S = S_B + \int dtd^3x \left[ \frac{\delta S_B}{\delta B_{ij}} (K_{ij} - B_{ij}) + \frac{\delta S_B}{\delta A} (F - A) \right],$$

(64)

into Eq. (25), we get

$$\mathcal{C}'(\vec{x}) = \sqrt{h}[\bar{a}\hat{B}_{ij} \hat{B}^{ij} + \bar{b}(B + cA)^2 + \bar{\gamma} (3)R + \bar{f}_1 + \bar{f}_2],$$

(65)

where

$$\hat{B}_{ij} = B_{ij} - \frac{1}{3} Bh_{ij}$$

(66)

is the traceless part of $B_{ij}$,

$$\bar{a} = Na' - a - Nac, \quad \bar{b} = Nb' - b - Nbc,$$

$$\bar{\gamma} = (N\gamma)' - c\gamma N/3, \quad \bar{f}_1 = f + Nf' - Ncf, \quad \bar{f}_2 = \frac{4c}{3}D_kD^k(\gamma N),$$

(67)

and the prime denotes the derivative with respect to $N$.

Plugging Eq. (65) into the first condition (41), we get

$$- \frac{1}{\sqrt{h}}S_1(\vec{x}, \vec{y}) \approx \delta^3(\vec{x} - \vec{y}) \left[ \left( a' - \frac{2a\bar{a}'}{a} + c\bar{a} \right) \hat{B}_{ij} \hat{B}^{ij} + (\bar{b} - 2\bar{b}'\bar{b}/b + c\bar{b})(B + cA)^2 + (\bar{\gamma}' - \bar{\gamma}c/3) (3)R + \frac{4}{3}(c' + \frac{2}{3}c^2)D_kD^k(\gamma N) + \bar{f}'_1 - c\bar{f}_1 - c\bar{f}_2 \right]$$

(68)

Using the constraint $\mathcal{C}'(\vec{x}) \approx 0$, we get

$$- \frac{1}{\sqrt{h}}S_1(\vec{x}, \vec{y}) \approx \delta^3(\vec{x} - \vec{y}) \left[ (a' - 2a\bar{a}'/a) \hat{B}_{ij} \hat{B}^{ij} + (\bar{b} - 2\bar{b}'\bar{b}/b)(B + cA)^2 + (\bar{\gamma}' - 4\bar{\gamma}c/3) (3)R + \frac{4}{3}(c' - \frac{4}{3}c^2)D_kD^k(\gamma N) + \bar{f}'_1 - 2c\bar{f}_1 \right]$$

(69)

$$+ \frac{4}{3}(\gamma N)'D_kD^k[c\delta^3(\vec{x} - \vec{y})] + \frac{4c}{3}D_kD^k[\gamma \delta^3(\vec{x} - \vec{y})] \approx 0.$$
In order to avoid dealing with the derivatives of the delta function, we multiply Eq. (69) by the test function \( \alpha(\vec{y}) \) and perform the spatial integral \( \int d^3y \), then Eq. (69) becomes

\[
Q_1 D^k D_k \alpha + Q_2^k D_k \alpha + Q_3 \alpha \approx 0, \tag{70}
\]

where

\[
Q_1 = \frac{4}{3}((\gamma N)'c + c\gamma), \\
Q_2^k = \frac{8}{3}((\gamma N)'c' + c\gamma')D^k N, \\
Q_3 = \frac{4}{3}[P_1 D_k D^k N + P_2 D^k N D_k N + P_3]
\tag{71}
\]

with

\[
P_1 = (c' - 4c^2/3)(\gamma N)' + c\gamma' + (\gamma N)'c', \\
P_2 = (c' - 4c^2/3)(\gamma N)'' + c\gamma'' + (\gamma N)'c'', \\
P_3 = (\alpha' - 2\alpha'/a)\dot{B}_{ij}\dot{B}^{ij} + (\ddot{b}' - 2\ddot{b}/b)(B + cA)^2 + (\gamma' - 4\gamma c/3) (3) R + \tilde{f}' - 2c\tilde{f}.
\tag{72}
\]

Thus the solution to the first condition (41) is

\[
Q_1 = 0, \quad Q_2^k = 0, \quad P_1 = 0, \quad P_2 = 0, \quad P_3 = 0. \tag{73}
\]

In the case that \( N \) is dynamic, i.e. \( c \neq 0 \), solving Eq. (73), we get

\[
a = \frac{\alpha_1(t)N}{\alpha_2(t) + N}, \quad b = \frac{\alpha_3(t)N}{\alpha_4(t) + N}, \quad c = -\frac{3}{4N + \alpha_5(t)}, \quad \gamma = 0, \tag{74}
\]

where \( \alpha_i(t) \) are general functions of time. The solution \( \gamma = 0 \) indicates that the spatial curvature term \((3) R\) does not enter the Lagrangian and this theory possesses no wave solution and cannot recover GR, so we are not interested in this case.

In the case that \( N \) is non-dynamic, i.e. \( c = 0 \), the quadratic action was constructed in Ref. [46] as

\[
S = \int d^4x N\sqrt{h} \left[ \frac{N}{\beta_2 + N} \dot{K}_{ij}\dot{K}^{ij} - \frac{2N}{3(\beta_4 + N)}K^2 + \left( \beta_5 + \frac{\beta_6}{N} \right) (3) R + \beta_7 + \frac{\beta_8}{N} \right], \tag{75}
\]

where \( \beta_i(t) \) are arbitrary functions of time.

Note that throughout the above analysis, we assume that the coupling functions only depend on \( N \). One may ask what if the coupling functions also depend on the spatial derivative of \( N \). However, in this case, the two conditions (41) and (45) become very
complicated to solve. Fortunately, it was proved that an invertible field transformation does not change the number of physical DoFs [27, 63, 64]. Thus, we take advantage of this result and perform a conformal transformation on the quadratic action (75) to obtain a new action that contains functions of \( N, \dot{N} \) and \( D_i N \). For this purpose, we choose the following conformal transformation

\[ h_{ij} \rightarrow e^{2w(N)} h_{ij}, \quad N \rightarrow N, \quad N^i \rightarrow N^i, \quad (76) \]

where \( w(N) \) is an arbitrary function of \( N \). Under the conformal transformation (76), the spatial curvature becomes

\[ (3)R \rightarrow e^{-2w}[\dot{(3)R} - 2D_k w D^k w - 4D^k D_k w]. \quad (77) \]

The traceless part and the trace of the extrinsic curvature become

\[ \dot{K}_{ij} \rightarrow e^{2w} \dot{K}_{ij}, \quad K \rightarrow K + 3w' \mathcal{L}_{\dot{n}} N. \quad (78) \]

In the case that \( w' \neq 0 \), the lapse function becomes dynamic after the conformal transformation and the corresponding quadratic action is

\[
S = \int dt d^4x \sqrt{h} \left[ e^{3w N} \hat{K}_{ij} \hat{K}^{ij} - \frac{2e^{3w N}}{3(\beta_1 + N)}(K + 3w' F)^2 + e^{w} \left( \beta_5 + \frac{2\beta_6}{N} \right) \right] (79)
\]

\[
\equiv \int dt d^4x \sqrt{h} \left[ \hat{a} \hat{K}_{ij} \hat{K}^{ij} + \hat{b}(K + cF)^2 + \hat{\beta} (3)R + \hat{f_1} D_k N D^k N + \hat{f_2} D_k D^k N + \hat{g} \right],
\]

where

\[
\hat{a} = \frac{e^{3w N}}{\beta_2 + N}, \quad \hat{b} = -\frac{2e^{3w N}}{3(\beta_1 + N)}, \quad \hat{c} = 3w', \quad \hat{\gamma} = e^{w} \left( \beta_5 + \frac{2\beta_6}{N} \right),
\]

\[
\hat{f_1} = -\hat{\gamma}(2w'^2 + 4w''), \quad \hat{f_2} = -4w' \hat{\gamma}, \quad \hat{g} = e^{3w} \left( \beta_7 + \frac{\beta_8}{N} \right). \quad (80)
\]

To confirm this theory really propagates two DoFs, we check whether the two conditions (41) and (45) are satisfied. After tedious calculations, the first condition (41) becomes

\[
- \int d^3x \alpha(\hat{x}) S_1(\hat{x}, \hat{y})/\sqrt{h} = (\hat{a}' - 2\hat{a}\hat{a}'/\hat{a} + c\hat{a}) \hat{B}_{ij} \hat{B}^{ij} + (\hat{b}' - 2\hat{b}\hat{b}'/\hat{b} + c\hat{b})(B + \hat{c}A)^2 + (\hat{\beta}' - \hat{\gamma}\hat{c}/3) (3)R
\]

\[
+ \hat{g}' - \hat{\gamma} \hat{g} + (\hat{f_1}' - \hat{c}\hat{f_1}/3) D_k N D^k N + (\hat{f_2}' - \hat{c}\hat{f_2}/3) D_k D^k N
\]

\[
- 2D_k (\hat{f_1} \alpha D^k N) + D_k D^k (\hat{f_2} \alpha) + \frac{4\hat{c}}{3} D_k D^k (\hat{\gamma} \alpha) + \hat{\gamma} D^m (\hat{f_2} \alpha D_m N)
\]

\[
\equiv Q_1 \alpha + Q_2^k D_k \alpha + Q_3 D^k D_k \alpha,
\]

\[ (81) \]
where $\alpha(\vec{x})$ is a test function and

\[
\begin{align*}
\ddot{a} &= N\dot{a}' - \ddot{a} - N\ddot{c}, \quad \ddot{b} = N\dot{b}' - \ddot{b} - N\ddot{c}, \quad \ddot{\gamma} = (\ddot{\gamma}N)' - \frac{1}{3}\ddot{c}\dot{\gamma}N, \\
\ddot{f}_1 &= -(\ddot{f}_1) + (f_2)'' - \frac{1}{3}N\ddot{c}\ddot{f}_1 + \frac{4\ddot{c}}{3}(\ddot{\gamma}N)' + \frac{\ddot{c}}{3}(N\ddot{f}_2)', \\
\ddot{f}_2 &= 2(\ddot{f}_2) - 2\dot{f}_1 - \frac{1}{3}N\ddot{c}\ddot{f}_2 + \frac{4\ddot{c}}{3}(\ddot{\gamma}N)' + \frac{\ddot{c}}{3}N\ddot{f}_2, \\
\ddot{g} &= (N\ddot{g})' - N\ddot{c}\ddot{g}, \\
Q_1 &= \ddot{f}_2 + \frac{4\ddot{c}}{3}\ddot{\gamma}, \\
Q_2 &= (-\ddot{f}_1 + \ddot{f}_2 + \frac{4\ddot{c}}{3}\ddot{\gamma} + \frac{\ddot{c}}{6}(\dot{f}_2))D^kN, \\
Q_3 &= P_1 + P_2D^kD_kN + P_3D_kND^kN
\end{align*}
\]

with

\[
\begin{align*}
P_1 &= (\dddot{a}' - 2\dddot{a}'/\dot{a} + c\dddot{a})\dot{B}_{ij}\dot{B}^{ij} + (\dddot{b}' - 2\dddot{b}'/\dot{b} + \dddot{c})\dot{B}^{ij}(B + \dddot{c}A)^2 \\
&\quad + (\dddot{\gamma}' - \dddot{\gamma}/3)R + \dddot{g}' - \dddot{c}\ddot{g}, \\
P_2 &= \ddot{f}_2 - \ddot{c}\ddot{f}_2/3 - 2\ddot{f}_1 + \ddot{f}_2 + \frac{4\ddot{c}}{3}\ddot{\gamma} + \frac{\ddot{c}}{3}\dddot{f}_2, \\
P_3 &= \dddot{f}_1 - \dddot{c}\ddot{f}_1/3 - 2\ddot{f}_1 + \dddot{f}_2 + \frac{4\ddot{c}}{3}\dddot{\gamma} + \frac{\ddot{c}}{3}\dddot{f}_2.
\end{align*}
\]

Using the explicit expression of coupling functions $\dddot{a}, \dddot{b}, \dddot{c}, \dddot{\gamma}, \dddot{f}_1, \dddot{f}_2$ and $\dddot{g}$ from Eq. (80), we find that

\[
Q_1 = 0, \quad Q_2 = 0, \quad Q_3 = 0,
\]

thus the first condition $S_1(\vec{x}, \vec{y}) \approx 0$ is satisfied. The second condition (45) is

\[
\begin{align*}
\frac{1}{\sqrt{\hbar}}&(T^{mnkl}_{ij}\Omega^{ij}D_n\Theta_{kl} - \Theta_{kl}D_nT^{mnkl}_{ij}\Omega^{ij}) \\
= &\sqrt{\hbar}\left(\dddot{\gamma}D_n\left[\frac{\dddot{a}}{\dddot{a}}B^{mn} + \left(-\frac{\dddot{a}}{3\dddot{a}} - \frac{2\dddot{b}}{3\dddot{b}}\right)Bh^{mn} - \frac{2\dddot{b}}{3\dddot{b}}\dddot{c}Ah^{mn}\right]ight) \\
&\quad - \left[\frac{\dddot{a}}{\dddot{a}}B^{mn} + \left(-\frac{\dddot{a}}{3\dddot{a}} - \frac{2\dddot{b}}{3\dddot{b}}\right)Bh^{mn} - \frac{2\dddot{b}}{3\dddot{b}}\dddot{c}Ah^{mn}\right]D_n\dddot{\gamma}.
\end{align*}
\]

Using the explicit expressions of the coupling functions $\dddot{a}, \dddot{b}$ and $\dddot{c}$ from Eq. (80), we find the following relations

\[
\frac{\dddot{a}}{\dddot{a}} = -e^{-3w}\dddot{a}, \quad -\frac{\dddot{a}}{3\dddot{a}} - \frac{2\dddot{b}}{3\dddot{b}} = -e^{-3w}(\dddot{b} - \dddot{a}/3), \quad -\frac{2\dddot{b}}{3\dddot{b}}\dddot{c} = -e^{-3w}\dddot{b}\dddot{c}.
\]
Substituting Eq. (86) into Eq. (85), we get

\[
\frac{1}{\sqrt{h}}(T_{ij}^{mnkl}\Omega^{ij}D_n\Theta_{kl} - \Theta_{kl}D_nT_{ij}^{mnkl}\Omega^{ij})
\]

\[
= -\sqrt{h}\left(\bar{\xi}D_n\left[e^{-3w}\left[\tilde{a}B^{mn} + (\tilde{b} - \tilde{a}/3)Bh^{mn} + \tilde{b}\tilde{c}Ah^{mn}\right]\right] - e^{-3w}\left[\tilde{a}B^{mn} + (\tilde{b} - \tilde{a}/3)Bh^{mn} + \tilde{b}\tilde{c}Ah^{mn}\right]D_n\bar{\xi}\right).
\]

Applying the constraint

\[
C^i(\vec{x}) \approx D_j\left(\tilde{a}B^{ij} + (\tilde{b} - \tilde{a}/3)Bh^{ij} + \tilde{b}\tilde{c}Ah^{ij}\right) \approx 0,
\]

and choosing the Minkowski spacetime as the boundary condition at infinity, we have

\[
\tilde{a}B^{ij} + (\tilde{b} - \tilde{a}/3)Bh^{ij} + \tilde{b}\tilde{c}Ah^{ij} \approx 0.
\]

Therefore, the second condition (45) is also satisfied and we confirm that the theory (79) has only two propagating DoFs.

Motivated by the field transformation, one may wonder whether there exists the correspondence between GR and SCG with two DoFs. Under the disformal transformation

\[
h_{ij} \to e^{2w(N)}h_{ij}, \quad N \to e^{\lambda(N)}N, \quad N^i \to N^i,
\]

the action for GR

\[
S_{GR} = \int d^4xN\sqrt{h}\left[\hat{K}_{ij}\hat{K}^{ij} - \frac{2}{3}K^2 + (3)R\right]
\]

becomes

\[
S = \int d^4xN\sqrt{h}\left[e^{3w-\lambda}\hat{K}_{ij}\hat{K}^{ij} - \frac{2}{3}e^{3w-\lambda}(K + 3w')^2\right.\]

\[
+ e^{w+\lambda}\left[(3)R - (2w'^2 + 4w'')D_kND^N - 4w'D_kD^kN\right],
\]

where \(w(N)\) and \(\lambda(N)\) are arbitrary functions of \(N\). Choosing

\[
\lambda(N) = \ln\left(\frac{\gamma_1 + \gamma_2 N}{N}\right),
\]

we get

\[
S = \int d^4xN\sqrt{h}\left[\frac{e^{3w}N}{\gamma_1 + \gamma_2 N}\hat{K}_{ij}\hat{K}^{ij} - \frac{2e^{3w}N}{3(\gamma_1 + \gamma_2 N)}(K + 3w')^2\right.\]

\[
+ e^w\left(\frac{\gamma_1}{N} + \gamma_2\right)\left[(3)R - (2w'^2 + 4w'')D_kND^N - 4w'D_kD^kN\right].
\]
It is obvious that the action (94) is a subclass of the quadratic action (79), so GR can be transformed to SCG with dynamic $N$ by the disformal transformation. In the case $w = 0$, we get

$$S = \int d^4x N \sqrt{h} \left[ \frac{N}{\gamma_1 + \gamma_2 N} \dot{K}^{ij} \dot{K}^{ij} - \frac{2N}{3(\gamma_1 + \gamma_2 N)} K^2 + \left( \frac{\gamma_1}{N} + \gamma_2 \right)^{(3)} R \right] ,$$  

which belongs to a subclass of SCG quadratic action with non-dynamic $N$ constructed in Ref. [46].

IV. CONCLUSION

In this paper, within the framework of spatially covariant gravity with a dynamic lapse function $N$, we investigate the sufficient and necessary conditions for a theory to have two physical degrees of freedom by performing the Hamiltonian analysis. Generally, the dynamic lapse function contributes a scalar mode and the theory has 4 DoFs. In Ref. [43] two conditions have been obtained to eliminate a scalar mode. We further obtain the sufficient and necessary conditions (41) and (45) to eliminate another one scalar mode through the detailed Hamiltonian analysis and they are

$$[\bar{\pi}(\vec{x}), \bar{C}'(\vec{y})] \approx 0, \ [\bar{C}'(\vec{x}), \bar{C}'(\vec{y})] \approx 0.$$

The first condition (41) ensures the Dirac Matrix is degenerate and turns a second-class constraint to a first-class constraint. If only the first condition is imposed, the dimension of phase space at each point of spacetime becomes odd and the theory is not self consistent. Thus another condition should be imposed. The second condition (45) ensures that a tertiary constraint arises. The dimension of the phase space then becomes even and the theory has two DoFs.

In the case that coupling functions only depend on $N$, we find that the spatial curvature term cannot enter the Lagrangian, and the theory possesses no wave solution and cannot recover general relativity. To construct SCG with coupling functions dependent on the spatial derivative of $N$, we perform a spatially conformal transformation, i.e. $h_{ij} \rightarrow e^{2w} h_{ij}$, on a class of quadratic actions with a non-dynamic lapse function and two DoFs. Due to the $N$-dependence of spatial conformal factor $w$, lapse function becomes dynamic after the conformal transformation, and a class of quadratic action with $\dot{N}$ and two DoFs is obtained.
We confirm that this theory propagates two DoFs by checking the two conditions (41) and (45). Besides, we also investigate the correspondence between SCG with two DoFs and GR. We find that a subclass of the quadratic action (79) with dynamic $N$ can be related to GR by performing the spatially conformal transformation $h_{ij} \rightarrow e^{2\lambda} h_{ij}$ and rescaling the lapse function $N \rightarrow e^{2\lambda} N$, with $\lambda = \ln [(\gamma_1 + \gamma_2 N)/N]$.

**Appendix A: The Poisson bracket**

The Poisson brackets among primary constraints are

\[
[p(\bar{x}), \pi^i(\bar{y})] = 0, \quad [p(\bar{x}), p(\bar{y})] = 0, \quad [p(\bar{x}), p^{ij}(\bar{y})] = 0, \quad [p^{ij}(\bar{x}), p^{kl}(\bar{y})] = 0, \quad (A1)
\]

\[
[p(\bar{x}), \tilde{\pi}(\bar{y})] = \frac{1}{N(\bar{y})} \frac{\delta^2 S_B}{\delta A(\bar{x}) \delta A(\bar{y})}, \quad [p(\bar{x}), \tilde{\pi}^{kl}(\bar{y})] = \frac{1}{2N(\bar{y})} \frac{\delta^2 S_B}{\delta B_{ij}(\bar{x}) \delta B_{kl}(\bar{y})}, \quad (A2)
\]

\[
[p^{ij}(\bar{x}), \tilde{\pi}(\bar{y})] = \frac{1}{N(\bar{y})} \frac{\delta^2 S_B}{\delta B_{ij}(\bar{x}) \delta A(\bar{y})}, \quad [p^{ij}(\bar{x}), \tilde{\pi}^{kl}(\bar{y})] = \frac{1}{2N(\bar{y})} \frac{\delta^2 S_B}{\delta B_{ij}(\bar{x}) \delta B_{kl}(\bar{y})}, \quad (A3)
\]

\[
[\tilde{\pi}(\bar{x}), \pi(\bar{y})] = \frac{1}{N(\bar{y})} \frac{\delta^2 S_B}{\delta N(\bar{x}) \delta A(\bar{y})} - \frac{1}{N(\bar{x})} \frac{\delta^2 S_B}{\delta A(\bar{x}) \delta N(\bar{y})}, \quad (A4)
\]

\[
[\tilde{\pi}(\bar{x}), \tilde{\pi}^{ij}(\bar{y})] = -\frac{1}{2} \delta^3(\bar{x} - \bar{y}) \frac{1}{N^2(\bar{y})} \frac{\delta S_B}{\delta B_{ij}(\bar{y})} + \frac{1}{2N(\bar{y})} \frac{\delta^2 S_B}{\delta N(\bar{x}) \delta B_{ij}(\bar{y})} - \frac{1}{N(\bar{x})} \frac{\delta^2 S_B}{\delta A(\bar{x}) \delta h_{ij}(\bar{y})}, \quad (A5)
\]

\[
[\tilde{\pi}^{ij}(\bar{x}), \tilde{\pi}^{kl}(\bar{y})] = \frac{1}{2N(\bar{y})} \frac{\delta^2 S_B}{\delta h_{ij}(\bar{x}) \delta B_{kl}(\bar{y})} - \frac{1}{N(\bar{x})} \frac{\delta^2 S_B}{\delta B_{ij}(\bar{x}) \delta h_{kl}(\bar{y})}. \quad (A6)
\]

The Poisson brackets between primary constraints and the canonical Hamiltonian are

\[
[p(\bar{x}), H_C] \approx 0, \quad [p^{ij}(\bar{x}), H_C] \approx 0, \quad [\pi(\bar{x}), H_C] = -C_i(\bar{x}), \quad
\]

\[
[\tilde{\pi}(\bar{x}), H_C] \approx \frac{\delta S_B}{\delta N(\bar{x})} - \frac{1}{N(\bar{x})} \frac{\delta S_B}{\delta B_{ij}(\bar{x})} B_{ij}(\bar{x})
\]

\[
- \frac{1}{N(\bar{x})} \int d^3 y N(\bar{y}) \left( \frac{\delta^2 S_B}{\delta A(\bar{x}) \delta N(\bar{y})} + \frac{\delta^2 S_B}{\delta A(\bar{x}) \delta h_{ij}(\bar{y})} 2B_{ij}(\bar{y}) \right), \quad (A7)
\]

\[
[\tilde{\pi}^{ij}(\bar{x}), H_C] = \frac{\delta S_B}{\delta N} + \frac{A(\bar{x})}{2N(\bar{x})} \frac{\delta S_B}{\delta B_{ij}(\bar{x})}
\]

\[
- \frac{1}{2N(\bar{x})} \int d^3 y N(\bar{y}) \left( \frac{\delta^2 S_B}{\delta B_{ij}(\bar{x}) \delta N(\bar{y})} A(\bar{y}) + \frac{\delta^2 S_B}{\delta B_{ij}(\bar{x}) \delta h_{kl}(\bar{y})} 2B_{kl}(\bar{y}) \right).\]
ACKNOWLEDGMENTS

J. Lin thanks Prof. Xian Gao for useful discussions. This research was supported in part by the National Natural Science Foundation of China under Grant No. 11875136 and the Major Program of the National Natural Science Foundation of China under Grant No. 11690021.

[1] A. A. Starobinsky, JETP Lett. 30, 682 (1979).
[2] A. H. Guth, Adv. Ser. Astrophys. Cosmol. 3, 139 (1987).
[3] K. Sato, Mon. Not. Roy. Astron. Soc. 195, 467 (1981).
[4] A. G. Riess et al. (Supernova Search Team), Astron. J. 116, 1009 (1998), arXiv:astro-ph/9805201.
[5] S. Perlmutter et al. (Supernova Cosmology Project), Astrophys. J. 517, 565 (1999), arXiv:astro-ph/9812133.
[6] T. Clifton, P. G. Ferreira, A. Padilla, and C. Skordis, Phys. Rept. 513, 1 (2012), arXiv:1106.2476 [astro-ph.CO].
[7] G. W. Horndeski, Int. J. Theor. Phys. 10, 363 (1974).
[8] C. Deffayet, G. Esposito-Farese, and A. Vikman, Phys. Rev. D 79, 084003 (2009), arXiv:0901.1314 [hep-th].
[9] C. Deffayet, S. Deser, and G. Esposito-Farese, Phys. Rev. D 80, 064015 (2009), arXiv:0906.1967 [gr-qc].
[10] C. Deffayet, X. Gao, D. Steer, and G. Zahariade, Phys. Rev. D 84, 064039 (2011), arXiv:1103.3260 [hep-th].
[11] C. Deffayet and D. A. Steer, Class. Quant. Grav. 30, 214006 (2013), arXiv:1307.2450 [hep-th].
[12] T. Kobayashi, Rept. Prog. Phys. 82, 086901 (2019), arXiv:1901.07183 [gr-qc].
[13] C. Armendariz-Picon, T. Damour, and V. F. Mukhanov, Phys. Lett. B 458, 209 (1999), arXiv:hep-th/9904075.
[14] J. Garriga and V. F. Mukhanov, Phys. Lett. B 458, 219 (1999), arXiv:hep-th/9904176.
[15] T. Kobayashi, M. Yamaguchi, and J. Yokoyama, Phys. Rev. Lett. 105, 231302 (2010), arXiv:1008.0603 [hep-th].
[16] M. Crisostomi and K. Koyama, Phys. Rev. D 97, 084004 (2018), arXiv:1712.06556 [astro-ph.CO].

[17] M. Crisostomi, K. Koyama, D. Langlois, K. Noui, and D. Steer, JCAP 01, 030, arXiv:1810.12070 [hep-th].

[18] N. Frusciante, R. Kase, K. Koyama, S. Tsujikawa, and D. Vernieri, Phys. Lett. B 790, 167 (2019), arXiv:1812.05204 [gr-qc].

[19] J. Lin, Q. Gao, Y. Gong, Y. Lu, C. Zhang, and F. Zhang, Phys. Rev. D 101, 103515 (2020), arXiv:2001.05909 [gr-qc].

[20] Z. Yi, Y. Gong, B. Wang, and Z.-h. Zhu, (2020), arXiv:2007.09957 [gr-qc].

[21] C. Brans and R. Dicke, Phys. Rev. 124, 925 (1961).

[22] R. Dicke, Phys. Rev. 125, 2163 (1962).

[23] J. D. Bekenstein, Phys. Rev. D 48, 3641 (1993), arXiv:gr-qc/9211017.

[24] M. Zumalacárrregui and J. García-Bellido, Phys. Rev. D 89, 064046 (2014), arXiv:1308.4685 [gr-qc].

[25] D. Bettoni and S. Liberati, Phys. Rev. D 88, 084020 (2013), arXiv:1306.6724 [gr-qc].

[26] D. Langlois and K. Noui, JCAP 02, 034, arXiv:1510.06930 [gr-qc].

[27] F. Arroja, N. Bartolo, P. Karmakar, and S. Matarrese, JCAP 09, 051, arXiv:1506.08575 [gr-qc].

[28] D. Langlois and K. Noui, JCAP 07, 016, arXiv:1512.06820 [gr-qc].

[29] M. Crisostomi, K. Koyama, and G. Tasinato, JCAP 04, 044, arXiv:1602.03119 [hep-th].

[30] J. Ben Achour, D. Langlois, and K. Noui, Phys. Rev. D 93, 124005 (2016), arXiv:1602.08398 [gr-qc].

[31] M. Crisostomi, M. Hull, K. Koyama, and G. Tasinato, JCAP 03, 038, arXiv:1601.04658 [hep-th].

[32] J. Ben Achour, M. Crisostomi, K. Koyama, D. Langlois, K. Noui, and G. Tasinato, JHEP 12, 100, arXiv:1608.08135 [hep-th].

[33] D. Langlois, M. Mancarella, K. Noui, and F. Vernizzi, JCAP 05, 033, arXiv:1703.03797 [hep-th].

[34] K. Takahashi and T. Kobayashi, JCAP 11, 038, arXiv:1708.02951 [gr-qc].

[35] H. Motohashi, K. Noui, T. Suyama, M. Yamaguchi, and D. Langlois, JCAP 07, 033, arXiv:1603.09355 [hep-th].
[36] R. Klein and D. Roest, JHEP 07, 130, arXiv:1604.01719 [hep-th].

[37] M. Ostrogradsky, Mem. Acad. St. Petersbourg 6, 385 (1850).

[38] R. P. Woodard, Scholarpedia 10, 32243 (2015), arXiv:1506.02210 [hep-th].

[39] J. Khoury, G. E. Miller, and A. J. Tolley, Phys. Rev. D 85, 084002 (2012),
    arXiv:1108.1397 [hep-th].

[40] T. Fujita, X. Gao, and J. Yokoyama, JCAP 02, 014, arXiv:1511.04324 [gr-qc].

[41] X. Gao, Phys. Rev. D 90, 081501 (2014), arXiv:1406.0822 [gr-qc].

[42] X. Gao, Phys. Rev. D 90, 104033 (2014), arXiv:1409.6708 [gr-qc].

[43] X. Gao and Z.-B. Yao, JCAP 05, 024, arXiv:1806.02811 [gr-qc].

[44] X. Gao, M. Yamaguchi, and D. Yoshida, JCAP 03, 006, arXiv:1810.07434 [hep-th].

[45] X. Gao, C. Kang, and Z.-B. Yao, Phys. Rev. D 99, 104015 (2019), arXiv:1902.07702 [gr-qc].

[46] X. Gao and Z.-B. Yao, Phys. Rev. D 101, 064018 (2020), arXiv:1910.13995 [gr-qc].

[47] X. Gao, (2020), arXiv:2003.11978 [gr-qc].

[48] X. Gao and Y.-M. Hu, Phys. Rev. D 102, 084006 (2020), arXiv:2004.07752 [gr-qc].

[49] X. Gao, (2020), arXiv:2006.15633 [gr-qc].

[50] N. Arkani-Hamed, H.-C. Cheng, M. A. Luty, and S. Mukohyama, JHEP 05, 074,
    arXiv:hep-th/0312099.

[51] P. Creminelli, M. A. Luty, A. Nicolis, and L. Senatore, JHEP 12, 080, arXiv:hep-th/0606090.

[52] C. Cheung, P. Creminelli, A. Fitzpatrick, J. Kaplan, and L. Senatore, JHEP 03, 014,
    arXiv:0709.0293 [hep-th].

[53] P. Horava, Phys. Rev. D 79, 084008 (2009), arXiv:0901.3775 [hep-th].

[54] P. Horava, Phys. Rev. Lett. 102, 161301 (2009), arXiv:0902.3657 [hep-th].

[55] M. Visser, Phys. Rev. D 80, 025011 (2009), arXiv:0902.0590 [hep-th].

[56] A. H. Chamseddine and V. Mukhanov, JHEP 11, 135, arXiv:1308.5410 [astro-ph.CO].

[57] L. Sebastiani, S. Vagnozzi, and R. Myrzakulov, Adv. High Energy Phys. 2017, 3156915 (2017),
    arXiv:1612.08661 [gr-qc].

[58] N. Afshordi, D. J. Chung, and G. Geshnizjani, Phys. Rev. D 75, 083513 (2007),
    arXiv:hep-th/0609150.

[59] N. Afshordi, D. J. Chung, M. Doran, and G. Geshnizjani, Phys. Rev. D 75, 123509 (2007),
    arXiv:astro-ph/0702002.

[60] H. Gomes and D. C. Guariento, Phys. Rev. D 95, 104049 (2017), arXiv:1703.08226 [gr-qc].
[61] A. Iyonaga, K. Takahashi, and T. Kobayashi, JCAP 12, 002, arXiv:1809.10935 [gr-qc].

[62] R. L. Arnowitt, S. Deser, and C. W. Misner, Phys. Rev. 116, 1322 (1959).

[63] G. Domènech, S. Mukohyama, R. Namba, A. Naruko, R. Saitou, and Y. Watanabe, Phys. Rev. D 92, 084027 (2015), arXiv:1507.05390 [hep-th].

[64] K. Takahashi, H. Motohashi, T. Suyama, and T. Kobayashi, Phys. Rev. D 95, 084053 (2017), arXiv:1702.01849 [gr-qc].