Riesz potential and its commutators on Orlicz spaces
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Abstract
In the present paper, we shall give necessary and sufficient conditions for the strong and weak boundedness of the Riesz potential operator $I_\alpha$ on Orlicz spaces. Cianchi (J. Lond. Math. Soc. 60(1):247-286, 2011) found necessary and sufficient conditions on general Young functions $\Phi$ and $\Psi$ ensuring that this operator is of weak or strong type from $L^{\Phi}$ into $L^{\Psi}$. Our characterizations for the boundedness of the above-mentioned operator are different from the ones in (Cianchi in J. Lond. Math. Soc. 60(1):247-286, 2011). As an application of these results, we consider the boundedness of the commutators of Riesz potential operator $[b, I_\alpha]$ on Orlicz spaces when $b$ belongs to the BMO and Lipschitz spaces, respectively.

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1 Introduction
Norm inequalities for several classical operators of harmonic analysis have been widely studied in the context of Orlicz spaces. It is well known that many of such operators fail to have continuity properties when they act between certain Lebesgue spaces and, in some situations, the Orlicz spaces appear as adequate substitutes.

The Hardy-Littlewood maximal operator $M$ and the Riesz potential operator $I_\alpha$ ($0 < \alpha < n$) are defined by

$$Mf(x) = \sup_{t>0} \left| B(x, t) \right|^{-1} \int_{B(x, t)} |f(y)| \, dy, \quad I_\alpha f(x) = \int_{\mathbb{R}^n} \frac{|f(y)|}{|x-y|^{n-\alpha}} \, dy.$$

Here and everywhere in the sequel $B(x, r)$ is the ball in $\mathbb{R}^n$ of radius $r$ centered at $x$ and $|B(x, r)| = v_n r^n$ is its Lebesgue measure, where $v_n$ is the volume of the unit ball in $\mathbb{R}^n$.

The commutators generated by a suitable function $b$ and the operator $I_\alpha$ are formally defined by

$$[b, I_\alpha]f = I_\alpha(bf) - bI_\alpha(f),$$

respectively.

Given a measurable function $b$ the operators $M_b$ and $|b, I_\alpha|$ are defined by

$$M_b(f)(x) = \sup_{t>0} \left| B(x, t) \right|^{-1} \int_{B(x,t)} |b(x) - b(y)| |f(y)| \, dy$$

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and

\[ |b, I_\alpha|f(x) = \int_{\mathbb{R}^n} \frac{|b(x) - b(y)|}{|x - y|^{n-\alpha}} f(y) \, dy. \]

In [1], Cianchi found necessary and sufficient conditions on general Young functions \( \Phi \) and \( \Psi \) ensuring that the operator \( I_\alpha \) is of weak or strong type from \( L^\Phi \) into \( L^\Psi \). Another boundedness statement with only sufficient conditions for the operator \( I_\alpha \) on Orlicz spaces was given by Nakai [2]. Note that in [2] a more general case of generalized fractional integrals was studied. Commutators of classical operators of harmonic analysis play an important role in various topics of analysis and PDE; see for instance [3, 4], and the references therein. In [5], Fu et al. gave the sufficient conditions for the boundedness of the commutator \([b, I_\alpha]\) on Orlicz spaces.

The main purpose of this paper is to give characterizations for the strong and weak boundedness of the Riesz potential on Orlicz spaces. Our characterizations for the boundedness of the operator \( I_\alpha \) are different from the ones in [1]. As an application of these results, we consider the boundedness of the commutators of Riesz potential operator on Orlicz spaces when \( b \) belongs to the BMO and Lipschitz spaces, respectively.

We use the notation \( A \lesssim B \), which means that \( A \leq CB \) with some positive constant \( C \) independent of appropriate quantities. If \( A \lesssim B \) and \( B \lesssim A \), we write \( A \approx B \) and say that \( A \) and \( B \) are equivalent.

2 Preliminaries; on Young functions and Orlicz spaces

We recall the definition of Young functions.

**Definition 2.1** A function \( \Phi : [0, \infty) \to [0, \infty] \) is called a Young function if \( \Phi \) is convex and left-continuous, \( \lim_{r \to 0^+} \Phi(r) = \Phi(0) = 0 \) and \( \lim_{r \to \infty} \Phi(r) = \infty. \)

From the convexity and \( \Phi(0) = 0 \) it follows that any Young function is increasing.

The set of Young functions such that

\[ 0 < \Phi(r) < \infty \quad \text{for} \quad 0 < r < \infty \]

is denoted by \( \mathcal{Y} \). If \( \Phi \in \mathcal{Y} \), then \( \Phi \) is absolutely continuous on every closed interval in \( [0, \infty) \) and bijective from \( [0, \infty) \) to itself.

For a Young function \( \Phi \) and \( 0 \leq s \leq \infty \), let

\[ \Phi^{-1}(s) = \inf \{ r \geq 0 : \Phi(r) > s \}. \]

If \( \Phi \in \mathcal{Y} \), then \( \Phi^{-1} \) is the usual inverse function of \( \Phi \).

It is well known that

\[ r \leq \Phi^{-1}(r) \Phi^{-1}(r) \leq 2r \quad \text{for} \quad r \geq 0, \tag{2.1} \]

where \( \Phi(r) \) is defined by

\[ \Phi(r) = \begin{cases} \sup \{ rs - \Phi(s) : s \in [0, \infty) \}, & r \in [0, \infty), \\ \infty, & r = \infty. \end{cases} \]
A Young function $\Phi$ is said to satisfy the $\Delta_2$-condition, denoted also as $\Phi \in \Delta_2$, if

$$\Phi(2r) \leq C \Phi(r), \quad r > 0,$$

for some $C > 1$. If $\Phi \in \Delta_2$, then $\Phi \in \nabla^2$. A Young function $\Phi$ is said to satisfy the $\nabla_2$-condition, denoted also by $\Phi \in \nabla_2$, if

$$\Phi(r) \leq \frac{1}{2C} \Phi(Cr), \quad r \geq 0,$$

for some $C > 1$.

**Definition 2.2** (Orlicz space) For a Young function $\Phi$, the set

$$L^\Phi(\mathbb{R}^n) = \left\{ f \in L^1_{\text{loc}}(\mathbb{R}^n) : \int_{\mathbb{R}^n} \Phi(k|f(x)|) \, dx < \infty \text{ for some } k > 0 \right\}$$

is called Orlicz space. If $\Phi(r) = rp, 1 \leq p < \infty$, then $L^\Phi(\mathbb{R}^n) = L^p(\mathbb{R}^n)$. If $\Phi(r) = 0, (0 \leq r \leq 1)$ and $\Phi(r) = \infty, (r > 1)$, then $L^\Phi(\mathbb{R}^n) = L^\infty(\mathbb{R}^n)$. The space $L^\Phi_{\text{loc}}(\mathbb{R}^n)$ is defined as the set of all functions $f$ such that $f \chi_B \in L^\Phi(\mathbb{R}^n)$ for all balls $B \subset \mathbb{R}^n$.

$L^\Phi(\mathbb{R}^n)$ is a Banach space with respect to the norm

$$\|f\|_{L^\Phi} = \inf \left\{ \lambda > 0 : \int_{\mathbb{R}^n} \Phi \left( \frac{|f(x)|}{\lambda} \right) \, dx \leq 1 \right\}.$$

For a measurable set $\Omega \subset \mathbb{R}^n$, a measurable function $f$ and $t > 0$, let $m(\Omega, f, t) = |\{ x \in \Omega : |f(x)| > t \}|$. In the case $\Omega = \mathbb{R}^n$, we for brevity denote it by $m(f, t)$.

**Definition 2.3** The weak Orlicz space

$$WL^\Phi(\mathbb{R}^n) = \left\{ f \in L^1_{\text{loc}}(\mathbb{R}^n) : \|f\|_{WL^\Phi} < \infty \right\}$$

is defined by the norm

$$\|f\|_{WL^\Phi} = \inf \left\{ \lambda > 0 : \sup_{t > 0} \Phi(t)m \left( \frac{f}{\lambda}, t \right) \leq 1 \right\}.$$

We note that $\|f\|_{WL^\Phi} \leq \|f\|_{L^\Phi}$,

$$\sup_{t > 0} \Phi(t)m(\Omega, f, t) = \sup_{t > 0} tm(\Omega, f, \Phi^{-1}(t)) = \sup_{t > 0} tm(\Omega, \Phi(|f|), t)$$

and

$$\int_{\Omega} \Phi \left( \frac{|f(x)|}{\|f\|_{L^\Phi(\Omega)}} \right) \, dx \leq 1, \quad \sup_{t > 0} \Phi(t)m \left( \Omega, \frac{f}{\|f\|_{WL^\Phi(\Omega)}}, t \right) \leq 1,$$

(2.2)

where $\|f\|_{L^\Phi(\Omega)} = \|f \chi_\Omega\|_{L^\Phi}$ and $\|f\|_{WL^\Phi(\Omega)} = \|f \chi_\Omega\|_{WL^\Phi}$.

The following analogue of the Hölder inequality is well known (see, for example, [6]).
Theorem 2.4 Let $\Omega \subset \mathbb{R}^n$ be a measurable set and functions $f$ and $g$ measurable on $\Omega$. For a Young function $\Phi$ and its complementary function $\tilde{\Phi}$, the following inequality is valid:

$$\int_{\Omega} |f(x)g(x)| \, dx \leq 2\|f\|_{L^\Phi(\Omega)} \|g\|_{L^{\tilde{\Phi}}(\Omega)}.$$ 

By elementary calculations we have the following property.

Lemma 2.5 Let $\Phi$ be a Young function and $B$ be a set in $\mathbb{R}^n$ with finite Lebesgue measure. Then

$$\|\chi_B\|_{L^\Phi} = \|\chi_B\|_{WL^\Phi} = \frac{1}{\Phi^{-1}(|B|^{-1})}.$$ 

By Theorem 2.4, Lemma 2.5 and (2.1) we get the following estimate.

Lemma 2.6 For a Young function $\Phi$ and $B = B(x, r)$, the following inequality is valid:

$$\int_B |f(y)| \, dy \leq 2|B|\Phi^{-1}(|B|^{-1})\|f\|_{L^\Phi(B)}.$$ 

In the next section, where we prove our main estimates, we use the following theorem.

Theorem 2.7 ([7]) Let $\Phi$ be a Young function.

(i) The operator $M$ is bounded from $L^\Phi(\mathbb{R}^n)$ to $WL^\Phi(\mathbb{R}^n)$, and the inequality

$$\|Mf\|_{W^\Phi} \leq C_0\|f\|_{L^\Phi}$$

holds with constant $C_0$ independent of $f$.

(ii) The operator $M$ is bounded on $L^\Phi(\mathbb{R}^n)$, and the inequality

$$\|Mf\|_{L^\Phi} \leq C_0\|f\|_{L^\Phi}$$

holds with constant $C_0$ independent of $f$ if and only if $\Phi \in \nabla_2$.

3 Riesz potential in Orlicz spaces

In this section we find necessary and sufficient conditions for the strong/weak boundedness of the Riesz potential operator on Orlicz spaces.

We recall that, for functions $\Phi$ and $\Psi$ from $[0, \infty)$ into $[0, \infty]$, the function $\Psi$ is said to dominate $\Phi$ globally if there exists a positive constant $c$ such that $\Phi(s) \leq \Psi(cs)$ for all $s \geq 0$.

In the theorem below we also use the notation

$$\Psi_p(s) = \int_0^s r^{p'-1}(\psi(r^p))^{p'} \, dr,$$

where $1 < p \leq \infty$ and $\Psi_p(s)$ is the Young conjugate function to $\psi_p(s)$, and

$$\Phi_p(s) = \int_0^s r^{p'-1}(A_p^{-1}(r^p))^{p'} \, dr,$$

where $A_p(r) = \int_0^r t^{p'-1} \, dt$ for $1 < p \leq \infty$ and $A_1(r) = \int_0^r \frac{1}{t} \, dt$ for $p = 1$. 


where $B_p^{-1}(s)$ and $A_p^{-1}(s)$ are inverses to

$$B_p(s) = \int_0^1 \frac{\Psi(t)}{t^{1+sp}} \, dt \quad \text{and} \quad A_p(s) = \int_0^1 \frac{\Phi(t)}{t^{1+sp}} \, dt,$$

respectively. These functions $\Psi_p(s)$ and $\Phi_p(s)$ are used below with $P = \frac{2}{\alpha}$.

In [1], Cianchi found the necessary and sufficient conditions for the boundedness of $I_a$ on Orlicz spaces.

**Theorem 3.1** ([1]) Let $0 < \alpha < n$. Let $\Phi$ and $\Psi$ Young functions and let $\Phi_{nil}$ and $\Psi_{nil}$ be the Young functions defined as in (3.2) and (3.1), respectively. Then

(i) $I_a$ is bounded from $L^\Phi(\mathbb{R}^n)$ to $WL^\Psi(\mathbb{R}^n)$ if and only if

$$\int_0^1 \frac{\Phi(t)}{t^{1+\alpha(n-1)}} \, dt < \infty \quad \text{and} \quad \Phi_{nil} \text{ dominates } \Psi \text{ globally.} \quad (3.3)$$

(ii) $I_a$ is bounded from $L^\Phi(\mathbb{R}^n)$ to $L^\Psi(\mathbb{R}^n)$ if and only if

$$\int_0^1 \frac{\Phi(t)}{t^{1+\alpha(n-1)}} \, dt < \infty, \quad \int_0^1 \frac{\Phi(t)}{t^{1+\alpha(n-1)}} \, dt < \infty, \quad (3.4)$$

$\Phi$ dominates $\Psi_{nil}$ globally and $\Phi_{nil}$ dominates $\Psi$ globally.

For proving our main results, we need the following estimate.

**Lemma 3.2** If $B_0 := B(x_0, r_0)$, then $r_0^\alpha \leq Cl_a x_{B_0}(x)$ for every $x \in B_0$.

**Proof** If $x, y \in B_0$, then $|x - y| \leq |x - x_0| + |y - x_0| < 2r_0$. Since $0 < \alpha < n$, we get $r_0^{\alpha-n} \leq C|x - y|^{\alpha-n}$. Therefore

$$I_a x_{B_0}(x) = \int_{\mathbb{R}^n} x_{B_0}(y) |x - y|^{\alpha-n} \, dy = \int_{B_0} |x - y|^{\alpha-n} \, dy \geq Cr_0^{\alpha-n}|B_0| = Cr_0^\alpha. \quad \square$$

The following theorem gives necessary and sufficient conditions for the boundedness of the operator $I_a$ from $L^\Phi(\mathbb{R}^n)$ to $WL^\Psi(\mathbb{R}^n)$ and from $L^\Phi(\mathbb{R}^n)$ to $L^\Psi(\mathbb{R}^n)$.

**Theorem 3.3** Let $0 < \alpha < n$ and $\Phi, \Psi \in \mathcal{Y}$.

1. The condition

$$r^\alpha \Phi^{-1}(r^n) + \int_r^\infty \Phi^{-1}(t^n) t^\alpha \frac{dt}{t} \leq C\Psi^{-1}(r^n) \quad (3.5)$$

for all $r > 0$, where $C > 0$ does not depend on $r$, is sufficient for the boundedness of $I_a$ from $L^\Phi(\mathbb{R}^n)$ to $WL^\Psi(\mathbb{R}^n)$. Moreover, if $\Phi \in \mathcal{Y}_2$, the condition (3.5) is sufficient for the boundedness of $I_a$ from $L^\Phi(\mathbb{R}^n)$ to $L^\Psi(\mathbb{R}^n)$.

2. The condition

$$r^\alpha \Phi^{-1}(r^n) \leq C\Psi^{-1}(r^n) \quad (3.6)$$

for all $r > 0$, where $C > 0$ does not depend on $r$, is necessary for the boundedness of $I_a$ from $L^\Phi(\mathbb{R}^n)$ to $WL^\Psi(\mathbb{R}^n)$ and from $L^\Phi(\mathbb{R}^n)$ to $L^\Psi(\mathbb{R}^n)$.
(3) If the regularity condition

\[
\int_{r}^{\infty} \Phi^{-1}(t^{-n}) \frac{dt}{t} \leq C r^{\alpha} \Phi^{-1}(r^{-n})
\]  

(3.7)

holds for all \( r > 0 \), where \( C > 0 \) does not depend on \( r \), then the condition (3.6) is necessary and sufficient for the boundedness of \( I_{\alpha} \) from \( L^{\Phi}(\mathbb{R}^n) \) to \( WL^{\Psi}(\mathbb{R}^n) \).

Moreover, if \( \Phi \in \nabla_2 \), the condition (3.6) is necessary and sufficient for the boundedness of \( I_{\alpha} \) from \( L^{\Phi}(\mathbb{R}^n) \) to \( L^{\Psi}(\mathbb{R}^n) \).

Proof (1) For an arbitrary ball \( B = B(x, r) \) we represent \( f \) as

\[
f = f_1 + f_2, \quad f_1(y) = f(y) \chi_B(y), \quad f_2(y) = f(y) \chi_{\mathbb{R}^n \setminus B}(y),
\]

and have

\[
I_{\alpha} f(x) = I_{\alpha} f_1(x) + I_{\alpha} f_2(x).
\]

For \( I_{\alpha} f_1(x) \), following Hedberg's trick, see [8], we obtain

\[
|I_{\alpha} f_1(x)| \lesssim r^\alpha Mf(x).
\]

For \( I_{\alpha} f_2(x) \) by Lemma 2.6 we have

\[
\int_{B} \frac{|f(y)|}{|x - y|^{n-\alpha}} \, dy \lesssim \int_{B} \frac{|f(y)|}{|x - y|^{n+1-\alpha}} \, dy \lesssim \int_{\mathbb{R}^n} |f(y)| \int_{r \leq |x - y|} \frac{dt}{t^{n+1-\alpha}} \lesssim \int_{r}^{\infty} \Phi^{-1}(\|B(x, t)\|^{-1}) r^{\alpha-1} \|f\|_{L^\infty(B(x, t))} \, dt.
\]

Consequently we have

\[
|I_{\alpha} f(x)| \lesssim r^\alpha Mf(x) + \|f\|_{L^\infty} \int_{r}^{\infty} t^{\alpha-1} \Phi^{-1}(t^{-n}) \frac{dt}{t}.
\]

Thus, by (3.5) we obtain

\[
|I_{\alpha} f(x)| \lesssim Mf(x) \Phi^{-1}(r^{-n}) + \|f\|_{L^\infty} \Psi^{-1}(r^{-n}).
\]

Choose \( r > 0 \) so that \( \Phi^{-1}(r^{-n}) = \frac{Mf(x)}{C_0 \|f\|_{L^\Phi}} \). Then

\[
\frac{\Psi^{-1}(r^{-n})}{\Phi^{-1}(r^{-n})} = \frac{\Psi^{-1}(r^{-n})}{\Phi^{-1}(r^{-n})} = \frac{Mf(x)}{C_0 \|f\|_{L^\Phi}}.
\]

Therefore, we get

\[
|I_{\alpha} f(x)| \leq C_1 \|f\|_{L^\infty} (\Psi^{-1} \circ \Phi) \left( \frac{Mf(x)}{C_0 \|f\|_{L^\Phi}} \right).
\]

(3.8)
Let $C_0$ be as in (2.3). Then by Theorem 2.7,
\[
\sup_{r>0} r m \left( \frac{|I\alpha f(x)|}{C_1 \|f\|_{L^\Phi}}, r \right) = \sup_{r>0} r m \left( \frac{|I\alpha f(x)|}{C_1 \|f\|_{L^\Phi}}, r \right) \leq \sup_{r>0} r \Phi \left( \frac{Mf(x)}{C_0 \|f\|_{L^\Phi}}, r \right) \leq \sup \Phi(r) m \left( \frac{Mf(x)}{\|Mf\|_{WL^\Phi}}, r \right) \leq 1,
\]
i.e.
\[
\|I\alpha f\|_{WL^\Psi} \lesssim \|f\|_{L^\Phi}.
\]

Let $C_0$ be as in (2.4). Since $\Phi \in \mathcal{V}_2$, by Theorem 2.7, we have
\[
\int_{\mathbb{R}^n} \Psi \left( \frac{|I\alpha f(x)|}{C_1 \|f\|_{L^\Phi}} \right) dx \leq \int_{\mathbb{R}^n} \Phi \left( \frac{Mf(x)}{C_0 \|f\|_{L^\Phi}} \right) dx \leq \int_{\mathbb{R}^n} \Phi \left( \frac{Mf(x)}{\|Mf\|_{L^\Phi}} \right) dx \leq 1,
\]
i.e.
\[
\|I\alpha f\|_{L^\Phi} \lesssim \|f\|_{L^\Phi}.
\]

(2) We shall now prove the second part. Let $B_0 = B(x_0, r_0)$ and $x \in B_0$. By Lemma 3.2, we have $r_0^\alpha \leq C_\alpha \chi_{B_0}(x)$. Therefore, by Lemma 2.5, we have
\[
r_0^\alpha \lesssim \Psi^{-1} \left( |B_0|^{-1} \right) \|I\alpha \chi_{B_0} \|_{WL^\Psi} \lesssim \Psi^{-1} \left( |B_0|^{-1} \right) \|I\alpha \chi_{B_0} \|_{L^\Psi} \lesssim \frac{\Psi^{-1}(r_0^{\alpha n})}{\Phi^{-1}(r_0^{\alpha n})}
\]
and
\[
r_0^\alpha \lesssim \Psi^{-1} \left( |B_0|^{-1} \right) \|I\alpha \chi_{B_0} \|_{L^\Psi} \lesssim \Psi^{-1} \left( |B_0|^{-1} \right) \|I\alpha \chi_{B_0} \|_{L^\Psi} \lesssim \frac{\Psi^{-1}(r_0^{\alpha n})}{\Phi^{-1}(r_0^{\alpha n})}.
\]

Since this is true for every $r_0 > 0$, we are done.

(3) The third statement of the theorem follows from the first and second parts of the theorem. \qed

From Theorems 3.1 and 3.3 we have the following corollary.

Corollary 3.4 Let $0 < \alpha < n$, $\Phi, \Psi \in \mathcal{V}$ and the regularity condition (3.7) holds, then:

(1) Condition (3.3) holds if and only if condition (3.6) holds.

(2) Moreover, if $\Phi \in \mathcal{V}_2$, then condition (3.4) holds if and only if (3.6) holds.

The following result is due to Nakai [2].
Theorem 3.5 ([2]) Let \( 0 < \alpha < n \) and \( \Phi, \Psi \in \mathcal{Y} \). Assume that the conditions (3.6) and (3.7) hold. Then the operator \( I_\alpha \) is bounded from \( L^\Phi(\mathbb{R}^n) \) to \( WL^\Psi(\mathbb{R}^n) \). Moreover, if \( \Phi \in \nabla_2 \), then \( I_\alpha \) is bounded from \( L^\Phi(\mathbb{R}^n) \) to \( L^\Psi(\mathbb{R}^n) \).

Remark 3.6 Note that in Theorem 3.5 Nakai found the sufficient conditions which ensures the boundedness of the operator \( I_\alpha \) from \( L^\Phi(\mathbb{R}^n) \) to \( L^\Psi(\mathbb{R}^n) \), including weak version. Theorem 3.3 improves Theorem 3.5 by adding the necessity. Theorems 3.1 and 3.3 are different characterizations for the boundedness of the operator \( I_\alpha \) from \( L^\Phi(\mathbb{R}^n) \) to \( L^\Psi(\mathbb{R}^n) \), including a weak version.

4 Maximal commutator in Orlicz spaces
In this section we investigate the boundedness of the maximal commutator \( M_b \) in Orlicz spaces.

We recall the definition of the space of BMO(\( \mathbb{R}^n \)).

Definition 4.1 Suppose that \( f \in L^1_{\text{loc}}(\mathbb{R}^n) \), let

\[
\|f\|_* = \sup_{x \in \mathbb{R}^n, r > 0} \frac{1}{|B(x, r)|} \int_{B(x, r)} |f(y) - f_{B(x, r)}| dy,
\]

where

\[
f_{B(x, r)} = \frac{1}{|B(x, r)|} \int_{B(x, r)} f(y) dy.
\]

Define

\[
\text{BMO}(\mathbb{R}^n) = \{ f \in L^1_{\text{loc}}(\mathbb{R}^n) : \|f\|_* < \infty \}.
\]

Modulo constants, the space BMO(\( \mathbb{R}^n \)) is a Banach space with respect to the norm \( \| \cdot \|_* \).

Before proving our theorems, we need the following lemmas and theorem.

Lemma 4.2 ([9]) Let \( b \in \text{BMO}(\mathbb{R}^n) \). Then there is a constant \( C > 0 \) such that

\[
|b_{B(x, r)} - b_{B(x, t)}| \leq C \|b\|_* \ln \frac{t}{r} \quad \text{for } 0 < 2r < t,
\]

where \( C \) is independent of \( b, x, r, \) and \( t \).

Lemma 4.3 ([10]) Let \( f \in \text{BMO}(\mathbb{R}^n) \) and \( \Phi \) be a Young function with \( \Phi \in \Delta_2 \), then

\[
\|f\|_* \approx \sup_{x \in \mathbb{R}^n, r > 0} \Phi^{-1}(\|B(x, r)\|^{-1}) \|f(\cdot) - f_{B(x, r)}\|_{L^\Phi(B(x, r))}.
\]

Theorem 4.4 ([11]) Let \( b \in \text{BMO}(\mathbb{R}^n) \) and \( \Phi \in \nabla_2 \cap \mathcal{Y} \).

Then the operator \( M_b \) is bounded on \( L^\Phi(\mathbb{R}^n) \), and the inequality

\[
\|M_bf\|_{L^\Phi} \leq C_0 \|b\|_* \|f\|_{L^\Phi}
\]

holds with constant \( C_0 \) independent of \( f \).
The following theorem is valid.

**Theorem 4.5** Let \( b \in \text{BMO}(\mathbb{R}^n) \) and \( \Phi \) be a Young function. Then the condition \( \Phi \in \mathcal{V}_2 \) is necessary for the boundedness of \( M_b \) on \( L^\Phi(\mathbb{R}^n) \).

**Proof** Assume that (4.3) holds. For the particular symbol \( b(x) = \log |x| \in \text{BMO}(\mathbb{R}^n) \) and \( f(x) = \chi_{B_r}(x) \), (4.3) becomes

\[
\|M_b \chi_{B_r}\|_{L^\Phi} \leq C \|\chi_{B_r}\|_{L^\Phi},
\]

(4.4)

where \( r = (a_1uv)^{-1/n} \), \( B_r = B(0, r) \), \( a_r = |B_r|, u > 0 \) and \( v > 1 \). By Lemma 2.5 and (2.1), we have

\[
\|\chi_{B_r}\|_{L^\Phi} = \frac{1}{\Phi^{-1}(1/|B_r|)} = \frac{1}{\Phi^{-1}(1/(r^n|B_1|))} = \frac{1}{\Phi^{-1}(uv)} \leq \frac{1}{uv} \Phi^{-1}(uv).
\]

On the other hand, if \( x \notin B_r \) then \( B_r \subset B(x, 2|x|) \) since for \( y \in B_r \) we have

\[
|x - y| \leq |x| + |y| \leq |x| + r \leq 2|x|.
\]

Also for each \( y \in B_r \), we have

\[
b(x) - b(y) \geq \log \left( \frac{|x|}{r} \right).
\]

Therefore

\[
M_b \chi_{B_r}(x) \geq \frac{1}{|B(x, 2|x|)|} \int_{B(x, 2|x|) \cap B_r} |b(x) - b(y)| \, dy \geq \left( \frac{r}{2|x|} \right)^n \log \left( \frac{|x|}{r} \right).
\]

Following the ideas of [12], for \( g = \Phi^{-1}(u) \chi_{B_1} \) with \( s = (a_1u)^{-1/n} \) we obtain

\[
\int_{\mathbb{R}^n} \Phi(|g(x)|) \, dx \leq u|B_1| = us^n|B_1| = 1.
\]

Since the Luxemburg-Nakano norm is equivalent to the Orlicz norm

\[
\|f\|_2^* := \sup \left\{ \int_{\mathbb{R}^n} |f(x)g(x)| \, dx : \|g\|_{L^\Phi} \leq 1 \right\}
\]

(more precisely, \( \|f\|_{L^2} \leq \|f\|_2^* \leq 2\|f\|_{L^\Phi} \)), it follows that

\[
\|M_b \chi_{B_1}\|_{L^\Phi} \geq \Phi^{-1}(u) \int_{B_1} M_b \chi_{B_1}(x) \, dx \geq \Phi^{-1}(u) \int_{B_1 \setminus B_0} \left( \frac{r}{2|x|} \right)^n \log \left( \frac{|x|}{r} \right) \, dx
\]

\[
= \Phi^{-1}(u) \int_{|x| < r} \frac{1}{|x|^n} \log \left( \frac{|x|}{r} \right) \, dx
\]

\[
= \Phi^{-1}(u) \frac{na_1}{2^{n+1}a_1uv} \left( \log \frac{s}{r} \right)^2 = \Phi^{-1}(u) \left( \frac{na_1}{2^{n+1}uv} \right) (\log v)^2.
\]
Hence (4.4) implies that

\[ \frac{\Phi^{-1}(u)}{2^{n+1} nuv} (\log v)^2 \leq 2C_1 \frac{1}{uv} \Phi^{-1}(uv) \]

for \( u > 0 \) and \( v > 1 \). Thus, taking \( v = \exp(\sqrt{nC_1} \cdot 2^{n+1}) \) we obtain \( 2\Phi^{-1}(u) \leq \Phi^{-1}(u \exp(\sqrt{nC_1} \cdot 2^{n+1})) \) for \( u > 0 \) or \( \Phi(2t) \leq \exp(\sqrt{nC_1} \cdot 2^{n+1}) \Phi(t) \) for every \( t > 0 \), and so \( \Phi \) satisfies the \( \Delta_2 \) condition. \[ \square \]

By Theorems 4.4 and 4.5 we have the following result.

**Corollary 4.6** Let \( b \in \text{BMO}(\mathbb{R}^n) \) and \( \Phi \in \mathcal{Y} \). Then the condition \( \Phi \in \mathcal{V}_2 \) is necessary and sufficient for the boundedness of \( M_b \) on \( L^\Phi(\mathbb{R}^n) \).

**Theorem 4.7** \( b \in L^1_{\text{loc}}(\mathbb{R}^n) \) and \( \Phi \) be a Young function. The condition \( b \in \text{BMO}(\mathbb{R}^n) \) is necessary for the boundedness of \( M_b \) on \( L^\Phi(\mathbb{R}^n) \).

**Proof** Suppose that \( M_b \) is bounded from \( L^\Phi(\mathbb{R}^n) \) to \( L^\Phi(\mathbb{R}^n) \). Choose any ball \( B = B(x, r) \) in \( \mathbb{R}^n \), by (2.1)

\[
\frac{1}{|B|} \int_B |b(y) - b_B| dy \leq \frac{1}{|B|} \int_B \frac{1}{|B|} \int_B |b(y) - b(z)| \chi_B(z) dz dy \\
\leq \frac{1}{|B|} \int_B M_b(\chi_B)(y) dy \\
\leq \frac{2}{|B|} \|M_b(\chi_B)\|_{L^\Phi(B)} \|1\|_{L^\Phi(B)} \\
\leq \frac{2}{|B|} \|\chi_B\|_{L^\Phi} \|\chi_B\|_{L^\Phi} \leq C.
\]

Thus \( b \in \text{BMO}(\mathbb{R}^n) \). \[ \square \]

By Theorems 4.4 and 4.7 we have the following result.

**Corollary 4.8** Let \( \Phi \) be a Young function with \( \Phi \in \mathcal{V}_2 \). Then the condition \( b \in \text{BMO}(\mathbb{R}^n) \) is necessary and sufficient for the boundedness of \( M_b \) on \( L^\Phi(\mathbb{R}^n) \).

## 5 Commutators of Riesz potential in Orlicz spaces

In this section we find necessary and sufficient conditions for the boundedness of the commutators of Riesz potential on Orlicz spaces with the help of the previous section.

In [5], Fu et al. found the sufficient conditions for the boundedness of the commutator \([b, I_\alpha] \) on Orlicz spaces as follows.

**Theorem 5.1** ([5]) Let \( 0 < \alpha < n \) and \( b \in \text{BMO}(\mathbb{R}^n) \). Let \( \Phi \) be a Young function and \( \Psi \) defined, via its inverse, by setting, for all \( t \in (0, \infty) \), \( \Psi^{-1}(t) := \Phi^{-1}(t)^{\frac{1}{\alpha}} \). If \( \Phi, \Psi \in \Delta_2 \cap \mathcal{V}_2 \), then \([b, I_\alpha] \) is bounded from \( L^\Phi(\mathbb{R}^n) \) to \( L^\Psi(\mathbb{R}^n) \).

The following lemma is the analogue of the Hedberg trick for \([b, I_\alpha] \).
Lemma 5.2 If $0 < \alpha < n$ and $f, b \in L^1_{\text{loc}}(\mathbb{R}^n)$, then for all $x \in \mathbb{R}^n$ and $r > 0$ we get
\[
\int_{B(x,r)} \frac{|f(y)|}{|x-y|^{n-\alpha}} |b(x) - b(y)| \, dy \lesssim r^\alpha M_b f(x).
\]
Proof We have
\[
\int_{B(x,r)} \frac{|f(y)|}{|x-y|^{n-\alpha}} |b(x) - b(y)| \, dy \\
= \sum_{j=0}^{\infty} \int_{2^{-j-1}r \leq |x-y| < 2^{-j}r} \frac{|f(y)|}{|x-y|^{n-\alpha}} |b(x) - b(y)| \, dy \\
\lesssim \sum_{j=0}^{\infty} (2^j r)^\alpha (2^j r)^{-\alpha} \int_{|x-y| < 2^{-j}r} |f(y)||b(x) - b(y)| \, dy \lesssim r^\alpha M_b f(x).
\]
\[\square\]

Lemma 5.3 If $b \in L^1_{\text{loc}}(\mathbb{R}^n)$ and $B_0 := B(x_0, r_0)$, then
\[
r_0^\alpha |b(x) - b_{B_0}| \leq C |b|_{L^\infty} |\chi_{B_0}(x)
\]
for every $x \in B_0$, where $b_{B_0} = \frac{1}{|B_0|} \int_{B_0} b(y) \, dy$.
Proof If $x, y \in B_0$, then $|x-y| \leq |x-x_0| + |y-x_0| < 2r_0$. Since $0 < \alpha < n$, we get $r_0^\alpha \lesssim C|x-y|^{\alpha-n}$. Therefore
\[
|b, I_\alpha|_{\chi_{B_0}(x)} = \int_{B_0} |b(x) - b(y)||x-y|^{\alpha-n} \, dy \geq C r_0^\alpha \int_{B_0} |b(x) - b(y)| \, dy \\
\geq C r_0^\alpha \int_{B_0} (b(x) - b(y)) \, dy = C r_0^\alpha |b(x) - b_{B_0}|.
\]
\[\square\]

The following theorem gives necessary and sufficient conditions for the boundedness of the operator $|b, I_\alpha|$ from $L^\Phi(\mathbb{R}^n)$ to $L^\Psi(\mathbb{R}^n)$.

Theorem 5.4 Let $0 < \alpha < n$, $b \in \text{BMO}(\mathbb{R}^n)$ and $\Phi, \Psi \in \mathcal{V}$.
\begin{enumerate}
\item If $\Phi \in \nabla_2$ and $\Psi \in \Delta_2$, then the condition
\[
r^\alpha \Phi^{-1}(r^{-\alpha}) + \int_r^\infty \left( 1 + \ln \frac{t}{r} \right) \Phi^{-1}(t^{-\alpha}) t^{\alpha} \frac{dt}{t} \leq C \Psi^{-1}(r^{-\alpha})
\]  
(5.1)
for all $r > 0$, where $C > 0$ does not depend on $r$, is sufficient for the boundedness of $|b, I_\alpha|$ from $L^\Phi(\mathbb{R}^n)$ to $L^\Psi(\mathbb{R}^n)$.
\item If $\Psi \in \Delta_2$, then the condition (3.6) is necessary for the boundedness of $|b, I_\alpha|$ from $L^\Phi(\mathbb{R}^n)$ to $L^\Psi(\mathbb{R}^n)$.
\item Let $\Phi \in \nabla_2$ and $\Psi \in \Delta_2$. If the condition
\[
\int_r^\infty \left( 1 + \ln \frac{t}{r} \right) \Phi^{-1}(t^{-\alpha}) t^{\alpha} \frac{dt}{t} \leq C r^\alpha \Phi^{-1}(r^{-\alpha})
\]  
(5.2)
holds for all $r > 0$, where $C > 0$ does not depend on $r$, then the condition (3.6) is necessary and sufficient for the boundedness of $|b, I_\alpha|$ from $L^\Phi(\mathbb{R}^n)$ to $L^\Psi(\mathbb{R}^n)$.
\end{enumerate}
Proof. (1) For arbitrary $x_0 \in \mathbb{R}^n$, set $B = B(x_0, r)$ for the ball centered at $x_0$ and of radius $r$.

Write $f = f_1 + f_2$ with $f_1 = f\chi_{2B}$ and $f_2 = f\chi_{(2B)^c}$.

For $x \in B$ we have

$$
|b, L_a[f_2](x)| \lesssim \int_{\mathbb{R}^n} |b(y) - b(x)| \frac{|f_2(y)|}{|x - y|^{n-\alpha}} \, dy \approx \int_{\mathbb{R}^n} \frac{|b(y) - b(x)|}{|x - y|^{n-\alpha}} |f(y)| \, dy
$$

$$
\lesssim \int_{\mathbb{R}^n} \frac{|b(y) - b_B|}{|x - y|^{n-\alpha}} |f(y)| \, dy + \int_{\mathbb{R}^n} \frac{|b(x) - b_B|}{|x - y|^{n-\alpha}} |f(y)| \, dy = J_1 + J_2(x),
$$

since $x \in B$ and $y \in \mathbb{R}^n \setminus (2B)$ implies $|x - y| \approx |x_0 - y|$. Let us estimate $J_1$.

$$
J_1 = \int_{(2B)^c} \frac{|b(y) - b_B|}{|x - y|^{n-\alpha}} |f(y)| \, dy \approx \int_{(2B)^c} |b(y) - b_B| |f(y)| \int_{\{x_0 - y\}} dt \frac{dt}{t^{n+1-\alpha}}
$$

$$
\lesssim \int_{2r} \int_{2r} |b(r) - b_{B(x_0, r)}| |f(y)| \, dy \frac{dt}{t^{n+1-\alpha}}
$$

Applying Hölder’s inequality, by (2.1), (4.1), (4.2) and Lemma 2.6 we get

$$
J_1 \lesssim \int_{2r} \int_{B(x_0, t)} |b(y) - b_{B(x_0, t)}| |f(y)| \, dy \frac{dt}{t^{n+1-\alpha}}
$$

$$
\lesssim \int_{2r} \left(1 + \ln \frac{t}{r}\right) \|f\|_{L^p(B(x_0, t))} \Phi^{-1}\left(|B(x_0, t)|^{-1}\right) \frac{dt}{t^{1-\alpha}}
$$

$$
\lesssim \|b\| \|f\|_{L^p} \int_{2r} \left(1 + \ln \frac{t}{r}\right) \Phi^{-1}\left(r^n\right) r^n \frac{dt}{t}.
$$

A geometric observation shows $2B \subset B(x, 3r)$ for all $x \in B$. Using Lemma 5.2, we get

$$
J_0(x) := \left|[b, L_a]f_1(x)\right| \lesssim \int_{2B} \frac{|b(y) - b(x)|}{|x - y|^{n-\alpha}} |f(y)| \, dy
$$

$$
\lesssim \int_{B(x, 3r)} \frac{|b(y) - b(x)|}{|x - y|^{n-\alpha}} |f(y)| \, dy \lesssim r^n M_{bf}(x).
$$

Consequently, we have

$$
J_0(x) + J_1 \lesssim \|b\|_{L^p} M_{bf}(x) + \|b\|_{L^p} \|f\|_{L^p} \int_{2r} \left(1 + \ln \frac{t}{r}\right) \Phi^{-1}(r^n) \frac{dt}{t}.
$$

Thus, by (5.1) we obtain

$$
J_0(x) + J_1 \lesssim \|b\|_{L^p} \left( M_{bf}(x) \psi^{-1}(r^n) + \psi^{-1}(r^n) \|f\|_{L^p} \right).
$$
Choose \( r > 0 \) so that \( \Phi^{-1}(r^{-\alpha}) = \frac{M_b f(x)}{C_0 \|f\|_{L^\infty}} \). Then

\[
\Phi^{-1}(r^{-\alpha}) = \frac{(\Psi^{-1} \circ \Phi)(\frac{M_b f(x)}{C_0 \|f\|_{L^\infty}})}{\frac{M_b f(x)}{C_0 \|f\|_{L^\infty}}}.
\]

Therefore, we get

\[
J_0(x) + J_1 \leq C_1 \|b\|_+ \|f\|_{L^\infty} (\Psi^{-1} \circ \Phi)(\frac{M_b f(x)}{C_0 \|f\|_{L^\infty}}).
\]

Let \( C_0 \) be as in (4.3). Consequently, by Theorem 4.4 we have

\[
\int_B \Phi \left( \frac{J_0(x) + J_1}{C_1 \|b\|_+ \|f\|_{L^\infty}} \right) dx \leq \int_B \Phi \left( \frac{M_b f(x)}{C_0 \|f\|_{L^\infty}} \right) dx \leq \int_{\mathbb{R}^n} \Phi \left( \frac{M_b f(x)}{\|M_b f\|_{L^\infty}} \right) dx \leq 1,
\]

i.e.

\[
\|J_0(\cdot) + J_1\|_{L^\infty(B)} \lesssim \|b\|_+ \|f\|_{L^\infty}.
\] (5.3)

In order to estimate \( J_2 \), by (4.2), Lemma 2.6 and condition (5.1), we also get

\[
\|J_2\|_{L^\infty(B)} = \left\| \int_{\mathbb{R}^n} \frac{|b(y) - b|}{|x_0 - y|^{n-\alpha}} \|f(y)\|_{L^\infty(B)} dy \right\|_{L^\infty(B)}
\approx \|b(y) - b\|_{L^\infty(B)} \int_{\mathbb{R}^n} \frac{|f(y)|}{|x_0 - y|^{n-\alpha}} dy
\lesssim \|b\|_+ \left( \frac{1}{\Psi^{-1}(|B|^{-1})} \right) \int_{\mathbb{R}^n} \|f(y)\|_{L^\infty(B)} \int_{|x_0 - y|^{n-\alpha}} dt \frac{dt}{t_{n+1-\alpha}}
\approx \|b\|_+ \left( \frac{1}{\Psi^{-1}(|B|^{-1})} \right) \int_{|x_0 - y|^{n-\alpha}} \|f(y)\|_{L^\infty(B)} \int_{2r}^{\infty} \frac{dt}{t_{n+1-\alpha}}
\lesssim \|b\|_+ \left( \frac{1}{\Psi^{-1}(|B|^{-1})} \right) \int_{2r}^{\infty} \Phi^{-1}(\|f\|_{L^\infty(B(x_0,t))}) \int_{|x_0 - y|^{n-\alpha}} \frac{dt}{t_{n+1-\alpha}}
\lesssim \|b\|_+ \left( \frac{1}{\Psi^{-1}(|B|^{-1})} \right) \|f\|_{L^\infty} \int_{2r}^{\infty} \frac{dt}{t_{n+1-\alpha}}
\lesssim \|b\|_+ \|f\|_{L^\infty}.
\]

Consequently, we have

\[
\|J_2\|_{L^\infty(B)} \lesssim \|b\|_+ \|f\|_{L^\infty}.
\] (5.4)

Combining (5.3) and (5.4), we get

\[
\| [b, L_a] f \|_{L^\infty(B)} \lesssim \|b\|_+ \|f\|_{L^\infty}.
\] (5.5)
By taking the supremum over $B$ in (5.5), we get
\[ \| [b, I_n] f \|_{L^\infty} \lesssim \| f \|_{L^\Phi}, \]

since the constants in (5.5) do not depend on $x_0$ and $r$.

(2) We shall now prove the second part. Let $B_0 = B(x_0, r_0)$ and $x \in B_0$. By Lemma 5.3, we have $r_0 |b(x) - b_{B_0}| \leq C |b, I_n | \chi_{B_0}(x)$. Therefore, by Lemmas 4.3 and 2.5
\[ r_0 \lesssim \left\| \left[ b, I_n \right] \chi_{B_0} \right\|_{L^\Phi(B_0)} \lesssim \Psi^{-1}(|B_0|^{-1}) \left\| \left[ b, I_n \right] \chi_{B_0} \right\|_{L^\Phi} \lesssim \frac{\Psi^{-1}(r_0^{\alpha})}{\Phi^{-1}(r_0^{\alpha})}. \]

Since this is true for every $r_0 > 0$, we are done.

(3) The third statement of the theorem follows from the first and second parts of the theorem. \qed

**Remark 5.5** Theorems 5.1 and 5.4 give different sufficient conditions for the boundedness of the operator $[b, I_n]$ from $L^p(R^n)$ to $L^q(R^n)$. But in Theorem 5.4 we also have necessary conditions for the boundedness of the operator $[b, I_n]$ from $L^p(R^n)$ to $L^q(R^n)$.

The following theorem is valid.

**Theorem 5.6** Let $0 < \alpha < n$, $b \in L^1_{\text{loc}}(R^n)$ and $\Phi, \Psi \in \mathcal{Y}$.

1. If $\Phi \in \mathcal{V}_2$, $\Psi \in \Delta_2$ and the condition (5.1) holds, then the condition $b \in \text{BMO}(R^n)$ is sufficient for the boundedness of $[b, I_n]$ from $L^p(R^n)$ to $L^q(R^n)$.

2. If $\Psi^{-1}(t) \lesssim \Phi^{-1}(t) t^{-\alpha/n}$, then the condition $b \in \text{BMO}(R^n)$ is necessary for the boundedness of $[b, I_n]$ from $L^p(R^n)$ to $L^q(R^n)$.

3. If $\Phi \in \mathcal{V}_2$, $\Psi \in \Delta_2$, $\Psi^{-1}(t) \approx \Phi^{-1}(t) t^{-\alpha/n}$ and the condition (5.2) holds, then the condition $b \in \text{BMO}(R^n)$ is necessary and sufficient for the boundedness of $[b, I_n]$ from $L^p(R^n)$ to $L^q(R^n)$.

**Proof** (1) The first statement of the theorem follows from the first part of Theorem 5.4.

(2) We shall now prove the second part. Choose any ball $B = B(x, r)$ in $R^n$, by Lemmas 2.5 and 2.6
\[ \frac{1}{|B|} \int_B |b(y) - b_B| \, dy = \frac{1}{|B|} \int_B \frac{1}{|B|} \int_B (b(y) - b(z)) \, dz \, dy \leq \frac{1}{|B|^{1+\frac{\alpha}{n}}} \int_B \frac{1}{|B|^{1+\frac{\alpha}{n}}} \int_B (b(y) - b(z)) \chi_B(z) \, dz \, dy \leq \frac{C}{|B|^{1+\frac{\alpha}{n}}} \int_B \frac{|b(y) - b(z)|}{|y - z|^{n+\alpha}} \chi_B(z) \, dz \, dy \leq \frac{C}{|B|^{1+\frac{\alpha}{n}}} \int_B |b, I_n| (\chi_B)(y) \, dy \leq \frac{C}{|B|^{1+\frac{\alpha}{n}}} \Psi^{-1}(|B|^{-1}) \lesssim \frac{1}{|B|^{1+\frac{\alpha}{n}}} \Phi^{-1}(|B|^{-1}) \lesssim C. \]

Thus $b \in \text{BMO}(R^n)$. 
6 Characterization of Lipschitz spaces via commutators

In this section, as an application of Theorem 3.3 we consider the boundedness of \([b, I_\alpha]\) on Orlicz spaces when \(b\) belongs to the Lipschitz space, by which some new characterizations of the Lipschitz spaces are given. Such a characterization was given in [13] as an application of the boundedness of \(M_b\) on Lebesgue spaces.

**Definition 6.1** Let \(0 < \beta < 1\), we say a function \(b\) belongs to the Lipschitz space \(\dot{\Lambda}_\beta(\mathbb{R}^n)\) if there exists a constant \(C\) such that, for all \(x, y \in \mathbb{R}^n\),

\[
\left| b(x) - b(y) \right| \leq C|x - y|^{\beta}.
\]

The smallest such constant \(C\) is called the \(\dot{\Lambda}_\beta(\mathbb{R}^n)\) norm of \(b\) and is denoted by \(\|b\|_{\dot{\Lambda}_\beta(\mathbb{R}^n)}\).

To prove the theorems, we need auxiliary results. The first one is the following characterization of Lipschitz space, which is due to DeVore and Sharply [14].

**Lemma 6.2** Let \(0 < \beta < 1\), we have

\[
\|f\|_{\dot{\Lambda}_\beta(\mathbb{R}^n)} \approx \sup_B \frac{1}{|B|^{1+\beta/n}} \int_B |f(x) - f_B| \, dx,
\]

where \(f_B = \frac{1}{|B|} \int_B f(y) \, dy\).

**Lemma 6.3** Let \(0 < \beta < 1\), \(0 < \alpha < n\), \(0 < \alpha + \beta < n\) and \(b \in \dot{\Lambda}_\beta(\mathbb{R}^n)\), then the following pointwise estimate holds:

\[
|b, I_\alpha|(|f|)(x) \lesssim \|b\|_{\dot{\Lambda}_\beta(\mathbb{R}^n)} I_{\alpha+\beta}(|f|)(x).
\]

**Proof** If \(b \in \dot{\Lambda}_\beta(\mathbb{R}^n)\), then

\[
|b, I_\alpha|(|f|)(x) = \int_{\mathbb{R}^n} \frac{|b(x) - b(y)|}{|x - y|^{n-\alpha}} |f(y)| \, dy 
\lesssim \|b\|_{\dot{\Lambda}_\beta(\mathbb{R}^n)} I_{\alpha+\beta}(|f|)(x).\]

The following theorem is valid.

**Theorem 6.4** Let \(0 < \beta < 1\), \(0 < \alpha < n\), \(0 < \alpha + \beta < n\), \(b \in \dot{\Lambda}_\beta(\mathbb{R}^n)\), \(\Phi, \Psi \in \mathcal{Y}\).

1. If \(\Phi \in \mathcal{V}_2\) and the conditions

\[
\int_t^\infty \Phi^{-1}(t^{-n}) r^{\alpha+\beta} \frac{dr}{r} \leq C t^{\alpha+\beta} \Phi^{-1}(t^{-n}),
\]

\[
t^{-\alpha+\beta} \Phi^{-1}(t) \leq C \Psi^{-1}(t),
\]

hold for all \(t > 0\), where \(C > 0\) does not depend on \(t\), then the condition \(b \in \dot{\Lambda}_\beta(\mathbb{R}^n)\) is sufficient for the boundedness of \([b, I_\alpha]\) from \(L^\Phi(\mathbb{R}^n)\) to \(L^\Psi(\mathbb{R}^n)\).
(2) If the condition
\[ \Psi^{-1}(t) \leq C \Phi^{-1}(t) t^{-\frac{\alpha + \beta}{\alpha}} \tag{6.3} \]
holds for all \( t > 0 \), where \( C > 0 \) does not depend on \( t \), then the condition \( b \in \dot{A}_\beta(\mathbb{R}^n) \) is necessary for the boundedness of \( |b, I_\alpha| \) from \( L^\Phi(\mathbb{R}^n) \) to \( L^\Psi(\mathbb{R}^n) \).

3. If \( \Phi \in \mathcal{V}_2 \), condition (6.1) holds and \( \Psi^{-1}(t) \approx \Phi^{-1}(t) t^{-\frac{\alpha + \beta}{\alpha}} \), then the condition \( b \in \dot{A}_\beta(\mathbb{R}^n) \) is necessary and sufficient for the boundedness of \( |b, I_\alpha| \) from \( L^\Phi(\mathbb{R}^n) \) to \( L^\Psi(\mathbb{R}^n) \).

Proof (1) The first statement of the theorem follows from Theorem 3.3 and Lemma 6.3.

(2) We shall now prove the second part. Suppose that \( \Psi^{-1}(t) \lesssim \Phi^{-1}(t) t^{-\frac{\alpha + \beta}{\alpha}} \) and \( |b, I_\alpha| \) is bounded from \( L^\Phi(\mathbb{R}^n) \) to \( L^\Psi(\mathbb{R}^n) \). Choose any ball \( B \) in \( \mathbb{R}^n \), by Lemmas 2.5 and 2.6

\[
\frac{1}{|B|^{\frac{n}{2n}}} \int_B |b(y) - b_B| \, dy = \frac{1}{|B|^{\frac{n}{2n}}} \int_B \left| \int_B (b(y) - b(z)) \, dz \right| \, dy
\leq \frac{C}{|B|^{\frac{n}{2n}}} \int_B \int_B |b(y) - b(z)| |y - z|^{\alpha - \alpha} \chi_B(z) \, dz \, dy
\leq \frac{C}{|B|^{\frac{n}{2n}}} \int_B |b, I_\alpha|(\chi_B)(y) \, dy
\leq \frac{C \Psi^{-1}(|B|^{-1})}{|B|^{\frac{n}{2n}}} \|b, I_\alpha|\chi_B\|_{L^\Psi(B)}
\leq \frac{C \Psi^{-1}(|B|^{-1})}{|B|^{\frac{n}{2n}}} \Phi^{-1}(|B|^{-1}) \leq C.
\]

Thus by Lemma 6.2 we get \( b \in \dot{A}_\beta(\mathbb{R}^n) \).

(3) The third statement of the theorem follows from the first and second parts of the theorem. \( \square \)

The following theorem is valid.

Theorem 6.5 Let \( 0 < \beta < 1 \), \( 0 < \alpha < n \), \( 0 < \alpha + \beta < n \), \( b \in L^1_{\text{loc}}(\mathbb{R}^n), \Phi, \Psi \in \mathcal{V} \).

(1) If the conditions (6.1) and (6.2) are satisfied, then the condition \( b \in \dot{A}_\beta(\mathbb{R}^n) \) is sufficient for the boundedness of \( |b, I_\alpha| \) from \( L^\Phi(\mathbb{R}^n) \) to \( WL^\Psi(\mathbb{R}^n) \).

(2) If the condition (6.3) holds and \( \frac{1}{|B|^{n/2n}} \) is almost decreasing for some \( \varepsilon > 0 \), then the condition \( b \in \dot{A}_\beta(\mathbb{R}^n) \) is necessary for the boundedness of \( |b, I_\alpha| \) from \( L^\Phi(\mathbb{R}^n) \) to \( WL^\Psi(\mathbb{R}^n) \).

(3) If \( \Psi^{-1}(t) \approx \Phi^{-1}(t) t^{-\frac{\alpha + \beta}{\alpha}} \), condition (6.1) holds and \( \frac{1}{|B|^{n/2n}} \) is almost decreasing for some \( \varepsilon > 0 \), then the condition \( b \in \dot{A}_\beta(\mathbb{R}^n) \) is necessary and sufficient for the boundedness of \( |b, I_\alpha| \) from \( L^\Phi(\mathbb{R}^n) \) to \( WL^\Psi(\mathbb{R}^n) \).

Proof (1) The first statement of the theorem follows from Theorem 3.3 and Lemma 6.3.

(2) For any fixed ball \( B_0 \) such that \( x \in B_0 \) by Lemma 5.3 we have \( |B_0|^{\alpha/n}|b(x) - b_{B_0}| \lesssim |b, I_\alpha|\chi_{B_0}(x) \). Thus, together with the boundedness of \( |b, I_\alpha| \) from \( L^\Phi(\mathbb{R}^n) \) to \( WL^\Psi(\mathbb{R}^n) \) and
Lemma 2.5,

\[
\left| \left\{ x \in B_0 : |b(x) - b_{B_0}| > \lambda \right\} \right| \leq \frac{1}{\Psi \left( \frac{\lambda}{C \| \chi_{B_0} \|_{L^1}} \right)} = \frac{1}{\Psi \left( \frac{\Phi^{-1}(\lambda)^{-1}}{C} \right)}.
\]

Let \( t > 0 \) be a constant to be determined later, then

\[
\int_{B_0} |b(x) - b_{B_0}| \, dx = |B_0|^{-\alpha/n} \int_0^\infty \left| \left\{ x \in B_0 : |b(x) - b_{B_0}| > |B_0|^{-\alpha/n} \lambda \right\} \right| \, d\lambda.
\]

\[
= |B_0|^{-\alpha/n} \int_0^t \left| \left\{ x \in B_0 : |b(x) - b_{B_0}| > |B_0|^{-\alpha/n} \lambda \right\} \right| \, d\lambda
\]

\[
+ |B_0|^{-\alpha/n} \int_t^\infty \left| \left\{ x \in B_0 : |b(x) - b_{B_0}| > |B_0|^{-\alpha/n} \lambda \right\} \right| \, d\lambda
\]

\[
\leq t |B_0|^{1-\alpha/n} + |B_0|^{-\alpha/n} \int_t^\infty \frac{1}{\Psi \left( \frac{\Phi^{-1}(\lambda)^{-1}}{C} \right)} \, d\lambda
\]

\[
\lesssim t |B_0|^{1-\alpha/n} + \frac{|B_0|^{-\alpha/n} t}{\Psi \left( \frac{\Phi^{-1}(\lambda)^{-1}}{C} \right)},
\]

where we use \( \Psi(t) \) being almost decreasing in the last step.

Set \( t = C |B_0|^{\frac{\alpha}{n}} \) in the above estimate, we have

\[
\int_{B_0} |b(x) - b_{B_0}| \, dx \lesssim |B_0|^{1+\beta/n}.
\]

Thus by Lemma 6.2 we get \( b \in \dot{A}_\mu(R^n) \) since \( B_0 \) is an arbitrary ball in \( R^n \).

(3) The third statement of the theorem follows from the first and second parts of the theorem. \( \square \)

7 Conclusions

We have obtained necessary and sufficient conditions for the boundedness of the Riesz potential and its commutators on Orlicz spaces. We have also compared our results with the existing results. Lastly, we conclude this paper by remarking that some new characterizations of the Lipschitz spaces have been given as an application of the above-mentioned results.

Competing interests

The authors declare that they have no competing interests.

Authors’ contributions

This work was carried out in collaboration between all authors. VSG raised these interesting problems in the research. VSG, FD and SGH proved the theorems, interpreted the results and wrote the article. All authors defined the research theme, read and approved the manuscript.

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