The column and row immanants of matrices over a split quaternion algebra.

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Abstract

The theory of the column-row determinants has been considered for matrices over a non-split quaternion algebra. In this paper the concepts of column-row determinants are extending to a split quaternion algebra. New definitions of the column and row immanants (permanents) for matrices over a non-split quaternion algebra are introduced, and their basic properties are investigated. The key theorem about the column and row immanants of a Hermitian matrix over a split quaternion algebra is proved. Based on this theorem an immanant of a Hermitian matrix over a split quaternion algebra is introduced.

Keywords: quaternion algebra; split quaternion; noncommutative determinant; immanant

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1 Introduction

The immanent of a matrix is a generalization of the concepts of determinant and permanent. The immanent of a complex matrix was defined by Dudley E. Littlewood and Archibald Read Richardson in [1] as follows.

Definition 1.1 Let $\sigma \in S_n$ denote the symmetric group on $n$ elements. Let $\chi : S_n \rightarrow \mathbb{C}$ be a complex character. For any $n \times n$ matrix $A = (a_{ij}) \in \mathbb{C}^{n \times n}$

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define the immanent of $A$ as

$$\text{Imm}_\chi(A) = \sum_{\sigma \in S_n} \chi(\sigma) \prod_{i=1}^n a_{i\sigma(i)}$$

Special cases of immanants are determinants and permanents. In the case where $\chi$ is the constant character ($\chi(x) = 1$ for all $x \in S_n$), $\text{Imm}_\chi(A)$ is the permanent of $A$. In the case where $\chi$ is the sign of the permutation (which is the character of the permutation group associated to the (non-trivial) one-dimensional representation), $\text{Imm}_\chi(A)$ is the determinant of $A$. The main goal of this paper is the extending the concept of immanent to a split quaternion algebra using methods of the theory of the row and column determinants. The theory of the row and column determinants was introduced in [2, 3] for matrices over the quaternion non-split algebra. This theory over the quaternion skew field is being actively developed as by the author [4]-[6], and others (see, for ex. [7]-[9]).

The paper is organized as follows. In Section 2 we consider briefly the main provisions of the quaternion algebra. In Section 3 definitions of the row and column immanents (consequently, determinants and permanents) are given. Their properties of an arbitrary quadratic matrix over the quaternion algebra are described in Section 4. In Section 5 the key theorem about the column and row immanants of a Hermitian matrix over a split quaternion algebra is proved and based on it we introduce the immanant (determinant, permanent) of a Hermitian matrix.

2 Quaternion algebra

A quaternion algebra $\mathbb{H}(a, b)$ over a field $\mathbb{F}$ is a central simple algebra over $\mathbb{F}$ that is a four-dimensional vector space over $\mathbb{F}$ with basis $\{1, i, j, k\}$ and the following multiplication rules:

$$i^2 = a, \quad j^2 = b, \quad ij = k, \quad ji = -k.$$  

A quaternion algebra $\mathbb{H}(a, b)$ over $\mathbb{F}$ is denoted $(\frac{a \beta}{\mathbb{F}})$ as well. To every quaternion algebra $\mathbb{H}(a, b)$, one can associate a quadratic form $n$ (called the norm form) on $\mathbb{H}$ such that $n(xy) = n(x)n(y)$ for all $x$ and $y$ in $\mathbb{H}$. A linear mapping $x \rightarrow \bar{x} = t(x) - x$ is also defined on $\mathbb{H}$. It is an involution, i.e. $\bar{\bar{x}} = x$, $x + y = \bar{x} + \bar{y}$ and $x \cdot y = \bar{y} \cdot \bar{x}$. An element $\bar{x}$ is called the conjugate of $x \in \mathbb{H}$. 


t(x) and n(x) are called the trace and the norm of x respectively, at that \( \{n(x), t(x)\} \subset F \) for all x in H. They also satisfy the following conditions: \( n(x) = n(\overline{x}) \), \( t(x) = t(\overline{x}) \) and \( t(p \cdot q) = t(p) \cdot t(q) \). The last property is the rearrangement property of the trace.

Depending on the choice of F, a and b we have only two possibilities ([10]):
1. \((\frac{a}{b}, F)\) is a division algebra,
2. \((\frac{a}{b}, F)\) is isomorphic to the algebra of all \(2 \times 2\) matrices with entries from F.

If an F-algebra is isomorphic to a full matrix algebra over F we say that the algebra is split, so (2) is the split case.

The most famous example of a non-split quaternion algebra is Hamilton’s quaternions \(\mathbb{H} = (\frac{-1-1}{\mathbb{R}})\).

An example of a split quaternion algebra is split quaternions of James Cockle \(\mathbb{H}_S(\frac{-1}{\mathbb{R}})\). Recently there was conducted a number of studies in split quaternion matrices (see, for ex. [11]-[14]). The matrix representation for the complex quaternions, which is also a split quaternion algebra, has been introduced in [15].

3 Definitions of the column and row immanants

Denote by \(H^{n \times m}\) a set of \(n \times m\) matrices with entries in H. For \(A = (a_{ij}) \in H^{n \times n}\) we define n row immanents as follows.

**Definition 3.1** The ith row immanent of \(A = (a_{ij}) \in H^{n \times n}\) is defined by putting

\[
r\text{Imm}_i A = \sum_{\sigma \in S_n} \chi(\sigma)a_{i_1i_{k_1}}a_{i_{k_1}i_{k_1+1}} \ldots a_{i_{k_1+l_1}i_{i_{k_1+l_1}}} \ldots a_{i_{k_r}i_{k_r+1}} \ldots a_{i_{k_r+l_r}i_{k_r}},
\]

where left-ordered cycle notation of the permutation \(\sigma\) is written as follows

\[
\sigma = (i_{k_1}i_{1+k_1} \ldots i_{k_1+l_1}) (i_{k_2}i_{k_2+1} \ldots i_{k_2+l_2}) \ldots (i_{k_r}i_{k_r+1} \ldots i_{k_r+l_r}).
\]  

(1)

Here the index i starts the first cycle from the left and other cycles satisfy the following conditions

\[
i_{k_2} < i_{k_3} < \ldots < i_{k_r}, \quad i_{k_t} < i_{k_{t+s}}.
\]  

(2)

for all \(t = 2, r\) and \(s = 1, l_t\).
Consequently we have the following definitions.

**Definition 3.2** The $i$th row permanent of $A = (a_{ij}) \in \mathbb{H}^{n \times n}$ is defined as

\[
\text{rper}_i A = \sum_{\sigma \in S_n} a_{i_k_1} a_{i_k_1 i_{k_1 + 1}} \cdots a_{i_{k_r i_{k_r + 1}}} \cdots a_{i_{k_r + l_r i_{k_r}}},
\]

where left-ordered cycle notation of the permutation $\sigma$ satisfies the conditions (1) and (2). (Here $\text{sign}(\sigma) = (-1)^{n-\text{r}}$).

**Definition 3.3** The $i$th row determinant of $A = (a_{ij}) \in \mathbb{H}^{n \times n}$ is defined as

\[
\text{rdet}_i A = \sum_{\sigma \in S_n} (-1)^{n-\text{r}} a_{i_k_1} a_{i_k_1 i_{k_1 + 1}} \cdots a_{i_{k_r i_{k_r + 1}}} \cdots a_{i_{k_r + l_r i_{k_r}}},
\]

where left-ordered cycle notation of the permutation $\sigma$ satisfies the conditions (1) and (2), (since $\text{sign}(\sigma) = (-1)^{n-\text{r}}$).

For $A = (a_{ij}) \in \mathbb{H}^{n \times n}$ we define $n$ column immanents as well.

**Definition 3.4** The $j$th column immanent of $A = (a_{ij}) \in \mathbb{H}^{n \times n}$ is defined as

\[
\text{cImm}_j A = \sum_{\tau \in S_n} \chi(\tau) a_{j_k_1} a_{j_k_1 j_{k_1 + 1}} \cdots a_{j_{k_r + l_r} j_{k_r}} \cdots a_{j_{k_1 + 1 j_1}} \cdots a_{j_{k_3 + l_3} j_{k_3}} a_{j_{k_1 j_1}},
\]

where right-ordered cycle notation of the permutation $\tau \in S_n$ is written as follows

\[
\tau = (j_{k_1 + l_1} \cdots j_{k_3 + l_3} j_{k_3}) \cdots (j_{k_2 + l_2} \cdots j_{k_3 + l_3} j_{k_2}) \cdots (j_{k_1 + l_1} \cdots j_{k_r} j_{k_r}).
\]

Here the first cycle from the right begins with the index $j$ and other cycles satisfy the following conditions

\[
j_{k_2} < j_{k_3} < \ldots < j_{k_r}, \quad j_{k_t} < j_{k_{t+1}},
\]

for all $t = 2, r$ and $s = 1, l_t$.

Consequently we have the following definitions as well.
Definition 3.5  The $j$th column permanent of $A = (a_{ij}) \in \mathbb{H}^{n \times n}$ is defined as
\[
\text{rper}_j A = \sum_{\tau \in S_n} a_{j_1} \cdots a_{j_{k_1} + 1} \cdots a_{j_{k_r} + 1} \cdots a_{j_{k_r + 1} \cdots a_{j_1}},
\]
where right-ordered cycle notation of the permutation $\sigma$ satisfies the conditions (3) and (4).

Definition 3.6  The $j$th column determinant of $A = (a_{ij}) \in \mathbb{H}^{n \times n}$ is defined as
\[
\text{rdet}_j A = \sum_{\tau \in S_n} (-1)^{n-r} a_{j_1} \cdots a_{j_{k_1} + 1} \cdots a_{j_{k_r} + 1} \cdots a_{j_{k_r + 1} \cdots a_{j_1}},
\]
where right-ordered cycle notation of the permutation $\sigma$ satisfies the conditions (3) and (4).

4 Basic properties of the column and row immanants

Consider the basic properties of the column and row immanants of a square matrix over $\mathbb{H}$.

Proposition 4.1  (The first theorem about zero of an immanant) If one of the rows (columns) of $A = (a_{ij}) \in \mathbb{H}^{n \times n}$ consists of zeros only, then $r\text{Imm}_i A = 0$ and $c\text{Imm}_i A = 0$ for all $i = 1, n$.

Proof. The proof immediately follows from the definitions. \[\square\]

Denote by $Ha$ and $aH$ left and right principal ideals of $\mathbb{H}$, respectively.

Proposition 4.2  (The second theorem about zero of an row immanant) Let $A = (a_{ij}) \in \mathbb{H}^{n \times n}$ and $a_{ki} \in Ha_i$ and $a_{ij} \in a_iH$, where $n(a_i) = 0$ for $k, j = 1, n$ and for all $i \neq k$. Let $a_{11} \in Ha_1$ and $a_{22} \in a_1H$ if $k = 1$, and $a_{kk} \in Ha_k$ and $a_{11} \in a_kH$ if $k = i > 1$, where $n(a_k) = 0$. Then $r\text{Imm}_k A = 0$.

Proof. Let $i \neq k$. Consider an arbitrary monomial of $r\text{Imm}_k A$, if $i \neq k$,
\[
d = \chi(\sigma)a_{ki}a_{ij} \cdots a_{lm}
\]
where $\{l, m\} \subset \{1, \ldots, n\}$. Since there exists $a_i \in \mathbb{H}$ such that $n(a_i) = 0$, and $a_{ki} \in Ha_i$, $a_{ij} \in a_iH$, than $a_{ki}a_{ij} = 0$ and $d = 0$. 

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Let \( i = k = 1 \). Then an arbitrary monomial of \( r\text{Imm}_1 \mathbf{A}, \)
\[
d = \chi(\sigma)a_{11}a_{22} \ldots a_{lm}.
\]
Since there exists \( a_1 \in \mathbf{H} \) such that \( n(a_1) = 0 \), and \( a_{11} \in \mathbf{H}a_1, a_{22} \in \mathbf{a}_1\mathbf{H}, \)
than \( a_{11}a_{22} = 0 \) and \( d = 0 \). If \( k = i > 1 \), then an arbitrary monomial of \( r\text{Imm}_k \mathbf{A}, \)
\[
d = \chi(\sigma)a_{kk}a_{11} \ldots a_{lm}.
\]
Since there exists \( a_k \in \mathbf{H} \) such that \( n(a_k) = 0 \), and \( a_{kk} \in \mathbf{H}a_k, a_{11} \in \mathbf{a}_k\mathbf{H}, \)
than \( a_{kk}a_{11} = 0 \) and \( d = 0 \). ■

**Proposition 4.3** (The second theorem about zero of a column immanant)
Let \( \mathbf{A} = (a_{ij}) \in \mathbf{H}^{n \times n} \) and \( a_{ik} \in a_i \mathbf{H} \) and \( a_{ji} \in \mathbf{H}a_i \), where \( n(a_i) = 0 \) for \( k, j = \frac{1}{n} \) and for all \( i \neq k \). Let \( a_{11} \in a_1 \mathbf{H} \) and \( a_{22} \in \mathbf{H}a_1 \) if \( k = 1 \), and \( a_{kk}a_{11} \in \mathbf{H} \) and \( a_{11} \in \mathbf{H}a_k \) if \( k = i > 1 \), where \( n(a_k) = 0 \). Then \( c\text{Imm}_k \mathbf{A} = 0 \).

**Proof.** The proof is similar to the proof of Proposition 4.2. ■

The proofs of the next theorems immediately follow from the definitions.

**Proposition 4.4** If the \( i \)th row of \( \mathbf{A} = (a_{ij}) \in \mathbf{H}^{n \times n} \) is left-multiplied by \( b \in \mathbf{H} \), then \( r\text{Imm}_i \mathbf{A}_i (b \cdot a_{i \cdot}) = b \cdot r\text{Imm}_i \mathbf{A} \) for all \( i = \frac{1}{n} \).

**Proposition 4.5** If the \( j \)th column of \( \mathbf{A} = (a_{ij}) \in \mathbf{H}^{n \times n} \) is right-multiplied by \( b \in \mathbf{H} \), then \( c\text{Imm}_j \mathbf{A}_j (a_{j \cdot} \cdot b) = c\text{Imm}_j \mathbf{A} \cdot b \) for all \( j = \frac{1}{n} \).

**Proposition 4.6** If for \( \mathbf{A} = (a_{ij}) \in \mathbf{H}^{n \times n} \) there exists \( t \in \{1, \ldots, n\} \) such that \( a_{ij} = b_j + c_j \) for all \( j = \frac{1}{n} \), then for all \( i = \frac{1}{n} \)
\[
r\text{Imm}_i \mathbf{A} = r\text{Imm}_i \mathbf{A}_t (b) + r\text{Imm}_i \mathbf{A}_t (c),
\]
\[
c\text{Imm}_j \mathbf{A} = c\text{Imm}_j \mathbf{A}_t (b) + c\text{Imm}_j \mathbf{A}_t (c),
\]
where \( b = (b_1, \ldots, b_n) \), \( c = (c_1, \ldots, c_n) \).

**Proposition 4.7** If for \( \mathbf{A} = (a_{ij}) \in \mathbf{H}^{n \times n} \) there exists \( t \in \{1, \ldots, n\} \) such that \( a_{it} = b_i + c_i \) for all \( i = \frac{1}{n} \), then for all \( j = \frac{1}{n} \)
\[
r\text{Imm}_j \mathbf{A} = r\text{Imm}_j \mathbf{A}_i (b) + r\text{Imm}_j \mathbf{A}_i (c),
\]
\[
c\text{Imm}_j \mathbf{A} = c\text{Imm}_j \mathbf{A}_i (b) + c\text{Imm}_j \mathbf{A}_i (c),
\]
where \( b = (b_1, \ldots, b_n)^T \), \( c = (c_1, \ldots, c_n)^T \).

**Proposition 4.8** If \( \mathbf{A}^* \) is the Hermitian adjoint matrix (the conjugate transpose) of \( \mathbf{A} = (a_{ij}) \in \mathbf{H}^{n \times n} \), then \( r\text{Imm}_i \mathbf{A}^* = c\text{Imm}_i \mathbf{A} \) for all \( i = \frac{1}{n} \).

Particular cases of these properties for the row-column determinants and permanents are evident.
5 An immanent of a Hermitian matrix

If $A^* = A$ then $A \in H^{n \times n}$ is called a Hermitian matrix. In this section we consider the key theorem about row-column immanants of a Hermitian matrix.

The following lemma is needed for the sequel.

**Lemma 5.1** [2] Let $T_n$ be the sum of all possible products of the $n$ factors, each of which are either $h_i \in H$ or $\overline{h_i}$ for all $i = 1, n$, by specifying the ordering in the terms, $T_n = h_1 \cdot h_2 \cdot \ldots \cdot h_n + \overline{h_1} \cdot h_2 \cdot \ldots \cdot h_n + \ldots + \overline{h_1} \cdot \overline{h_2} \cdot \ldots \cdot \overline{h_n}$. Then $T_n$ consists of the $2^n$ terms and $T_n = t(h_1) \cdot t(h_2) \ldots \cdot t(h_n)$.

**Theorem 5.1** If $A \in H^{n \times n}$ is a Hermitian matrix, then $r\text{Imm}_1 A = \ldots = r\text{Imm}_n A = c\text{Imm}_1 A = \ldots = c\text{Imm}_n A \in F$.

**Proof.** At first we note that if $A = (a_{ij}) \in H^{n \times n}$ is Hermitian, then we have $a_{ii} \in F$ and $a_{ij} = \overline{a_{ji}}$ for all $i, j = 1, n$.

We divide the set of monomials of $r\text{Imm}_i A$ for some $i \in \{1, \ldots, n\}$ into two subsets. If indices of coefficients of monomials form permutations as products of disjoint cycles of length 1 and 2, then we include these monomials to the first subset. Other monomials belong to the second subset. If indices of coefficients form a disjoint cycle of length 1, then these coefficients are $a_{jj}$ for $j \in \{1, \ldots, n\}$ and $a_{jj} \in F$.

If indices of coefficients form a disjoint cycle of length 2, then these entries are conjugated, $a_{ik_{k+1}} = \overline{a_{ik_{k+1}i}}$, and

$$a_{ik_{k+1}} \cdot a_{ik_{k+1}i} = \overline{a_{ik_{k+1}i}} \cdot a_{ik_{k+1}i} = n(a_{ik_{k+1}i}) \in F.$$ 

So, all monomials of the first subset take on values in $F$.

Now we consider some monomial $d$ of the second subset. Assume that its index permutation $\sigma$ forms a direct product of $r$ disjoint cycles. Denote $i_{k_1} := i$, then

$$d = \chi(\sigma) a_{i_{k_1}i_{k_1+1}} \ldots a_{i_{k_1+1}i_{k_1}} a_{i_{k_2+i_{k_2+1}}} \ldots a_{i_{k_2+1}i_{k_2}} \ldots a_{i_{km+i_{km+1}}} \ldots \chi(\sigma) h_1 h_2 \ldots h_m \ldots h_r,$$

where $h_s = a_{i_{ks}i_{ks+1}} \ldots a_{i_{ks+s}i_{ks}}$ for all $s = 1, r$, and $m \in \{1, \ldots, r\}$. If $l_s = 1$, then $h_s = a_{i_{ks}i_{ks+1}} a_{i_{ks+1}i_{ks}} = n(a_{i_{ks}i_{ks+1}}) \in F$. If $l_s = 0$, then $h_s = a_{i_{ks}i_{ks}} \in F$.
If \( l_s = 0 \) or \( l_s = 1 \) for all \( s = 1, r \) in (3), then \( d \) belongs to the first subset. Let there exists \( s \in I_n \) such that \( l_s \geq 2 \). Then

\[
\bar{h}_s = a_{i_{k_s}i_{k_s+1}} \cdots a_{i_{k_s+l_s}i_{k_s}} = a_{i_{k_s+l_s}i_{k_s}} \cdots a_{i_{k_s}i_{k_s+1}} = a_{i_{k_s}i_{k_s+l_s}} \cdots a_{i_{k_s+l_s}i_{k_s}}.
\]

Denote by \( \sigma_s(i_{k_s}) = (i_{k_s}i_{k_s+1} \cdots i_{k_s+l_s}) \) a disjoint cycle of indices of \( d \) for some \( s \in \{1, \ldots, r\} \), then \( \sigma = \sigma_1(i_{k_1}) \sigma_2(i_{k_2}) \cdots \sigma_r(i_{k_r}) \). The disjoint cycle \( \sigma_s(i_{k_s}) \) corresponds to the factor \( h_s \). Then \( \sigma_s^{-1}(i_{k_s}) = (i_{k_s+l_s}i_{k_s+1} \cdots i_{k_s+1}) \) is the inverse disjoint cycle and \( \sigma_s^{-1}(i_{k_s}) \) corresponds to the factor \( \bar{h}_s \). By Lemma 5.1 there exist another \( 2^p - 1 \) monomials for \( d \), (where \( p = r - \rho \) and \( \rho \) is the number of disjoint cycles of length 1 and 2), such that their index permutations form the direct products of \( r \) disjoint cycles either \( \sigma_s(i_{k_s}) \) or \( \sigma_s^{-1}(i_{k_s}) \) by specifying their ordering by \( s \) from 1 to \( r \). Their cycle notations are left-ordered according to Definition 3.1. These permutations are unique decomposition of the permutation \( \sigma \) including their ordering by \( s \) from 1 to \( r \). Suppose \( C_1 \) is the sum of these \( 2^p - 1 \) monomials and \( d \), then by Lemma 5.1 we obtain

\[
C_1 = \chi(\sigma) \alpha t(h_{\nu_1}) \cdots t(h_{\nu_p}) \in F.
\]

Here \( \alpha \in F \) is the product of coefficients whose indices form disjoint cycles of length 1 and 2, \( \nu_k \in \{1, \ldots, r\} \) for all \( k = 1, p \).

Thus for an arbitrary monomial of the second subset of row immanants of \( A \), we can find the \( 2^p \) monomials such that their sum takes on a value in \( F \). Therefore, \( r\text{Imm}_j A \in F \).

Now we prove the equality of all row immanents of \( A \). Consider an arbitrary row immanant of \( j \neq i \) for all \( j = 1, n \). We divide the set of monomials of \( r\text{Imm}_j A \) into two subsets using the same rule as for \( r\text{Imm}_i A \). Monomials of the first subset are products of entries of the principal diagonal or norms of entries of \( A \). Therefore they take on a value in \( F \) and each monomial of the first subset of \( r\text{Imm}_j A \) is equal to a corresponding monomial of the first subset of \( r\text{Imm}_j A \).

Now consider the monomial \( d_1 \) of the second subset of monomials of \( r\text{Imm}_j A \) consisting of coefficients that are equal to the coefficients of \( d \) but they are in another order. Consider all possibilities of the arrangement of coefficients in \( d_1 \).

(i) Suppose that the index permutation \( \sigma' \) of its coefficients form a direct product of \( r \) disjoint cycles and these cycles coincide with the \( r \) disjoint cycles of \( d \) but differ by their ordering. Then \( \sigma' = \sigma \) and we have

\[
d_1 = \chi(\sigma) \alpha h_\mu \cdots h_\lambda,
\]
where \{\mu, \ldots, \lambda\} = \{\nu_1, \ldots, \nu_p\}. By Lemma 5.1 there exist \(2^p - 1\) monomials of the second subset of \(rImm_j A\) such that each of them is equal to a product of \(p\) factors either \(h_s\) or \(\overline{h}_s\) for all \(s \in \{\mu, \ldots, \lambda\}\). Hence by Lemma 5.1 we obtain

\[C_2 = \chi(\sigma) \alpha \ t(h_\mu) \cdots t(h_\lambda) = \chi(\sigma) \alpha \ t(h_{\nu_1}) \cdots t(h_{\nu_p}) = C_1.\]

(ii) Now suppose that in addition to the case (i) the index \(j\) is placed inside some disjoint cycle of the index permutation \(\sigma\) of \(d\), e.g. \(j \in \{i_{k_{m+1}}, \ldots, i_{k_{m+l}}\}\). Denote \(j = i_{k_{m+q}}\). Considering the above said and \(\sigma_{k_{m+1}}(i_{k_{m+1}}) = \sigma_{k_{m+q}}(i_{k_{m+q}})\), we have \(\sigma' = \sigma\). Then \(d_1\) is represented as follows:

\[d_1 = \chi(\sigma)a_{i_{k_{m+q}}i_{k_{m+q}+1}} \cdots a_{i_{k_{m+l}i_{k_{m+l}}}}a_{i_{k_{m+l}}i_{k_{m+1}}} \cdots \times \]
\[\times a_{i_{k_{m+q-1}i_{k_{m+q}}}a_{i_{k_{mu}i_{k_{mu}+1}}a_{i_{k_{mu+1}i_{k_{mu}}}}} \cdots a_{i_{k_{m+1}i_{k_{m+1}}}a_{i_{k_{m+1}}i_{k_{m+1}}}}} = (6)\]

where \{\(m, \mu, \ldots, \lambda\} = \{\nu_1, \ldots, \nu_p\}\). Except for \(\tilde{h}_m\), each factor of \(d_1\) in (6) corresponds to the equal factor of \(d\) in (5). By the rearrangement property of the trace, we have \(t(\tilde{h}_m) = t(h_m)\). Hence by Lemma 5.1 and by analogy to the previous case, we obtain,

\[C_2 = \chi(\sigma) \alpha \ t(\tilde{h}_m) t(h_\mu) \cdots t(h_\lambda) = \chi(\sigma) \alpha \ t(h_{\nu_1}) \cdots t(h_m) \cdots t(h_{\nu_p}) = C_1.\]

(iii) If in addition to the case (i) the index \(i\) is placed inside some disjoint cycles of the index permutation of \(d_1\), then we apply the rearrangement property of the trace to this cycle. As in the previous cases we find \(2^p\) monomials of the second subset of \(rImm_j A\) such that by Lemma 5.1 their sum is equal to the sum of the corresponding \(2^p\) monomials of \(rImm_i A\). Clearly, we obtain the same conclusion at association of all previous cases, then we apply twice the rearrangement property of the trace.

Thus, in any case each sum of \(2^p\) corresponding monomials of the second subset of \(rImm_j A\) is equal to the sum of \(2^p\) monomials of \(rImm_i A\). Here \(p\) is the number of disjoint cycles of length more than 2. Therefore, for all \(i, j = 1, n\) we have

\[rImm_i A = rImm_j A \in F.\]

The equality \(cImm_j A = rImm_i A\) for all \(i = 1, n\) is proved similarly. □

Since Theorem 5.1 we have the following definition.
Definition 5.1 Since all column and row immanents of a Hermitian matrix over $\mathbb{H}$ are equal, we can define the immanant (permanent, determinant) of a Hermitian matrix $A \in \mathbb{H}^{n \times n}$. By definition, we put for all $i = 1, \ldots, n$

$$\text{Imm } A := r_{\text{Imm}} A = c_{\text{Imm}} A,$$
$$\text{per } A := r_{\text{per}} A = c_{\text{per}} A,$$
$$\det A := r_{\text{det}} A = c_{\text{det}} A.$$ 

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