OPERADIC SEMI-INFINITE HOMOLOGY

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Abstract. We propose the notion of semi-infinite homology for algebras over operads using the relative homology theory for operadic algebras.

Contents

0. Introduction 1
Notations 4
1. Operads in relative setting 5
1.1. Operads 5
1.2. Algebras over operads 9
1.3. Modules of algebras over operads 10
1.4. Enveloping Operads 11
1.5. Derivations 12
1.6. Cooperads 13
2. Operadic homology algebra 14
2.1. Homology of algebras over operads 16
2.2. Operadic twisting morphism 18
2.3. Operadic bar and cobar construction 20
2.4. Operadic (co)chain complexes 21
3. Relative homology of operads 22
3.1. Cotriple homology 22
3.2. Relative homology of algebras over operads 23
4. Operadic semi-infinite cohomology 24
4.1. Semi-infinite structure 24
4.2. Standard semi-injective resolution 25
References 26

0. Introduction

The motivation of this note is to understand semi-infinite (co)homology for Lie algebras and associative algebras with the viewpoint of operads. Although there are already much literature on semi-infinite homology algebra, we believe this note sheds new light on the topic.

Let us cite a concise explanation of semi-infinite homology from [P10 Introduction]: Roughly speaking, the semi-infinite cohomology is defined for a Lie or associative algebra-like object which is split in two halves; the semi-infinite cohomology has the features of a homology theory (left derived functor) along one half of the variables and a cohomology theory (right derived functor) along the other half.

Semi-infinite homology for Lie algebras was first introduced by B. Feigin in [F84] in the 1980s. Voronov proposed in [V93] a homology algebraic treatment (see also
For associative algebras, Arkhipov [A97a, A97b] first built the theory of semi-infinite homology. Further study is done by Sevostyanov [Se01]. Positselski constructed in [P10] a huge theory using semi-infinite version of comodule-contramodule correspondence. See also [P11] for the theory of derived categories in his framework.

In this note we develop a relative homology theory for algebras over operads, and use it to define a standard semi-injective resolution. The theory of operadic relative (co)homology is built with the help of cotriple homology, as explained in § 3. The construction of standard semi-injective resolution is explained in § 4.

Our construction is pretty simple, and may give a transparent understanding of various theories of semi-infinite (co)homology. We plan to apply our method to the algebras in the category of Tate vector spaces in future.

Notations. $\mathbb{N} = \{0, 1, 2, \ldots \}$ denotes the set of non-negative integers. For $n \in \mathbb{Z}_{>0}$, we denote by $\Sigma_n$ the $n$-th symmetric group, and we set $\Sigma_0 := \Sigma_1 = \{e\}$, the trivial group.

For a category $A$, we write $X \in A$ in the meaning of $X$ being an object of $A$. The set of morphisms $X \to Y$ for $X, Y \in A$ is denoted by $A(X, Y)$, or sometimes by $\text{Hom}_A(X, Y)$. Set denotes the category of (U-small) sets (if we fix a universe $U$).

We use the word limit in the meaning of inverse limit, and colimit in the meaning of direct limit. The word dg means differential $\mathbb{Z}$-graded.

Calligraphy symbols like $A, B, \ldots$ denote categories. Sans-Serif symbols like $P, Q, \ldots$ denote operads.

1. Operads in relative setting

Following [Fr09, Chapters 1–3], we give a preliminary on operads in relative setting. Namely we work over a symmetric monoidal category $\mathcal{C}$, which will be called the base category. Operads $P$ we will consider live in this base category $\mathcal{C}$, and algebras over $P$ will be defined in another category $\mathcal{E}$ which is “over $\mathcal{C}$”.

Following [Fr09] §0.1, we put the following conditions on categories and functors.

Assumption 1.0.1. On categories and functors we assume the following conditions.

- Every category $A$ considered in this note contains a small subcategory $A_f$ such that every object $X \in A$ is the filtered colimit of a diagram of $A$.
- Every functor $F : A \to B$ preserves filtered colimits.

The assumption on the existence of $A_f$ implies that a functor $\varphi : A \to B$ admits a right adjoint $B \to A$ if and only if $\varphi$ preserves colimits. This observation will be used at Definition 1.2.2 of the endomorphism operad.

1.1. Operads. We fix a symmetric monoidal category $\mathcal{C} = (\mathcal{C}, \otimes, 1)$ satisfying the following conditions.

C1 All small colimits and small limits exist.
C2 $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ commutes with all small colimits in each variable.

This assumption implies that $\mathcal{C}$ has the initial object, which will be denoted by $0$. We call $\mathcal{C}$ the base category.

Definition. A $\Sigma_*$-object in $\mathcal{C}$ consists of a sequence $M = \{M(n)\}_{n \in \mathbb{Z}}$ of objects $M(n) \in \mathcal{C}$ with a right $\Sigma_n$-action. A morphism $M \to N$ between $\Sigma_*$-objects consists of a sequence $\{M(n) \to N(n)\}_{n \in \mathbb{Z}}$ of morphisms in $\mathcal{C}$ which commute with the $\Sigma_n$-actions. We denote by $\mathcal{C}^{\Sigma_*}$ the category of $\Sigma_*$-objects in $\mathcal{C}$.
The identity $\Sigma_n$-object $1 \in \mathcal{C}^\Sigma$ is defined by
\[
1(n) = \begin{cases} 1 & (n = 1) \\ 0 & (n \neq 1) \end{cases}.
\]

The category $\mathcal{C}^\Sigma$ has a natural monoidal structure induced by the one in $\mathcal{C}$. Before explaining that, let us define the bifunctor $\otimes : \mathcal{C} \times \mathcal{Set} \to \mathcal{C}$ by
\[
C \otimes K := \otimes_{k \in K} C.
\]

Here $\mathcal{Set}$ denotes the category of sets. Since $\mathcal{C}$ has a colimit by the condition $C_2$, this definition makes sense. Note also that it makes sense for any category $\mathcal{C}$ with all small colimits. Next let $G$ be a group with $m : G \otimes G \to G$ the multiplication map, $M$ be a left $G$-module with $\lambda : G \otimes M \to M$ the action map, and assume that $C \in \mathcal{C}$ has a right $G$-action $\rho$. Then the tensor product $C \otimes G \otimes M$ makes sense. Now we define $C \otimes_G M \in \mathcal{C}$ to be the following coequalizer.

\[
\begin{array}{ccc}
C \otimes G \otimes M & \xrightarrow{\rho \otimes id} & C \otimes M \\
\otimes id & \xrightarrow{id \otimes \lambda} & C \otimes G \otimes M.
\end{array}
\]

We can now explain the monoidal structure on $\mathcal{C}^\Sigma$. For $M, N \in \mathcal{C}^\Sigma$ we set
\[
(M \otimes N)(n) := \bigoplus_{p+q=n} (M(p) \otimes M(q)) \otimes_{\Sigma_p \times \Sigma_q} \Sigma_n.
\]

On the right hand side we used the coequalizer definition of $\otimes_{\Sigma_p \times \Sigma_q} \Sigma_n$. The left action of $\Sigma_p \times \Sigma_q$ on $\Sigma_n$ is defined as follows. We consider $\Sigma_p \times \Sigma_q$ as a subgroup of $\Sigma_n$ by identifying $\sigma \in \Sigma_p$ with a permutation of $\{1, \ldots, p\}$ of $\Sigma_p$, and by identifying $\tau \in \Sigma_q$ with a permutation of $\{p+1, \ldots, n\}$ of $\Sigma_q$. Then $\Sigma_p \times \Sigma_q$ acts on $\Sigma_n$ by translations on the right, which is the desired action. Now the right action of $\Sigma_n$ on itself makes $(M \otimes N)(n)$ a right $\Sigma_n$-module. Thus we have a $\Sigma_n$-object $M \otimes N = \{(M \otimes N)(n)\}_{n \in \mathbb{N}}$. One can check that the triple
\[(\mathcal{C}^\Sigma, \otimes, 1)\]
is a symmetric monoidal category.

The category $\mathcal{C}^\Sigma$ is equipped with another monoidal structure $(\mathcal{C}^\Sigma, \circ, 1)$, where the monoidal operation $\circ$ is defined by the following.

**Definition 1.1.1.** For $M, N \in \mathcal{C}^\Sigma$, define $M \circ N \in \mathcal{C}^\Sigma$ by
\[
(M \circ N)(r) := \bigoplus_{k \geq 0} (M(k) \otimes N^{\otimes k}(r)) \Sigma_k.
\]

In the right hand side the group $\Sigma_k$ acts diagonally on $M(k) \otimes N^{\otimes k}(r)$, where the action on $N^{\otimes k}(r)$ is the permutation of $k$ factors. The symbol $X_G$ denotes the coinvariants in $X \in \mathcal{C}$ with respect to the action of a group $G$.

Since we assume that $\mathcal{C}$ has colimits, and that $\otimes$ commutes with colimits, this definition makes sense and $M \circ N$ is actually a $\Sigma_n$-object. More explicitly, the $r$-th component $(M \circ N)(r)$ is given by
\[
(M \circ N)(r) = \bigoplus_{k \geq 0} M(k) \otimes_{\Sigma_k} \left( \bigotimes_{\Sigma_{i_1} \times \cdots \times \Sigma_{i_k}} N(i_1) \otimes \cdots \otimes N(i_k) \right).
\]

Here the second direct sum runs over $(i_1, \ldots, i_k) \in \mathbb{N}^k$ such that $i_1 + \cdots + i_k = n$. The right $\Sigma_k$-action on the right-hand side factor is the permutation of the $k$-tuple $(i_1, \ldots, i_k)$. This presentation makes the natural $\Sigma_r$-action on $(M \circ N)(r)$ explicit. See [LV12] [5.1.4] for a detailed explanation.

In a set-theoretic context, we denote an element of
\[
M(k) \otimes_{\Sigma_k} \bigotimes_{\Sigma_{i_1} \times \cdots \times \Sigma_{i_k}} N(i_1) \otimes \cdots \otimes N(i_k) \subset (M \circ N)(r)
\]
by the following notation \[ \text{LV12} \ [5.1.7]. \]

\[(m; n_1, \ldots, n_k; \sigma), \quad m \in M(k), \quad n_j \in N(i_j), \quad \sigma \in Sh(i_1, \ldots, i_k). \quad (1.1)\]

Here \(Sh(i_1, \ldots, i_k)\) is the set of \((i_1, \ldots, i_k)\)-shuffles. In other words, \(\sigma \in \Sigma_r\) is an \((i_1, \ldots, i_k)\)-shuffle if it satisfies

\[
\sigma(1) < \sigma(2) < \cdots < \sigma(i_1), \quad \sigma(i_1 + 1) < \cdots < \sigma(i_1 + i_2), \quad \ldots, \\
\ldots, \quad \sigma(i_1 + \cdots + i_{k-1} + 1) < \cdots < \sigma(i_1 + \cdots + i_k) = \sigma(r).
\]

The set \(Sh(i_1, \ldots, i_k)\) is bijective to the quotient \((\Sigma_{i_1} \times \cdots \times \Sigma_{i_k})/\Sigma_r\).

**Definition.** An operad in \(\mathcal{C}\) is a triple

\[(P, \mu, \eta)\]

consisting of a \(\Sigma_r\)-object \(P\) in \(\mathcal{C}\) and two morphisms of \(\Sigma_r\)-objects \(\mu : P \circ P \rightarrow P\) and \(\eta : 1 \rightarrow P\) making the following diagrams commutative.

\[
\begin{array}{ccc}
\text{id} & \downarrow & \mu \\
\text{id} & \downarrow & \mu \\
P \circ P & \mu & \text{id}
\end{array}
\quad \quad
\begin{array}{ccc}
\text{id} & \downarrow & \mu \\
\text{id} & \downarrow & \mu \\
P \circ l & \mu & \text{id}
\end{array}
\]

The morphism \(\mu\) is called the composition map, and \(\eta\) is called the unit map.

One can see that the identity \(\Sigma_r\)-object \(I\) has an obvious operad structure. This operad is called the identity operad and denoted by the same symbol \(I\).

The composition map \(\mu\) is encoded by the partial compositions

\[
o_i : P(r) \otimes P(s) \rightarrow P(r + s - 1)
\]

for \(1 \leq i \leq r\). These are defined to be the composites

\[
P(r) \otimes P(s) \xrightarrow{i_{th}} P(r) \otimes (1 \otimes \ldots \otimes P(s) \otimes \ldots \otimes 1) \\
\rightarrow P(r) \otimes (P(1) \otimes \ldots \otimes P(s) \otimes \ldots \otimes P(1)) \xrightarrow{\mu} P(r + s - 1),
\]

where the second morphism is \text{id} \otimes (\eta \otimes \ldots \otimes \text{id} \otimes \ldots \otimes \eta). See \(\text{LV12} \ [5.3.4]\) for the equivalence of the definitions of operads in terms of the composition map \(\mu\) and the partial composition maps \(o_i\).

**Definition.** A morphism \(\varphi : P \rightarrow Q\) between operads in \(\mathcal{C}\) is defined to be a morphism of \(\Sigma_r\)-objects that preserves operad structures. Such \(\varphi\) will be called operad morphism. We denote by \(\text{Op} \mathcal{C}\) the category of operads in \(\mathcal{C}\).

Let us also recall the notion of free operads. The forgetful functor \(\text{Op} \mathcal{C} \rightarrow \mathcal{C}^{\Sigma}\) from operads to \(\Sigma_r\)-objects has a left adjoint

\[
F : \mathcal{C}^{\Sigma} \rightarrow \text{Op} \mathcal{C}
\]

making each object \(M\) to an operad \(F(M)\) in \(\mathcal{C}\).

Explicitly, the underlying \(\Sigma_r\)-object of \(F(M)\) is given by

\[
F(M) := \varinjlim F_n(M),
\]

where \(F_n(M)\) is determined inductively by

\[
F_0(M) := I, \quad F_n(M) := M \circ F_{n-1}(M),
\]

and the inclusion map \(i_n : F_n(M) \hookrightarrow F_{n+1}(M)\) is given by \(i_1 : 1 \hookrightarrow I \oplus M\), inclusion in the first factor, and \(i_n := \text{id} \oplus (\text{id} \circ i_{n-1}).\) Any induced inclusion map \(F_m(M) \hookrightarrow F_n(M)\) is denoted by \(i\). We also have the inclusion map \(M \circ F_{n-1}(M) \hookrightarrow F_n(M)\) into second factor. The induced map \(M \hookrightarrow F(M)\) is denoted by \(j\).
The composition map $\mu$ on $F(M)$ is determined by $\mu_{m,n} : F_m(M) \circ F_n(M) \to F_{m+n}(M)$, and we construct $\mu_{m,n}$ inductively. First we set $\mu_{0,n} := \text{id}$ on $F_0(M) \circ F_n(M) \to F_n$. Then we define $\mu_{m,n}$ to be the following composite.

$$
F_m(M) \circ F_n(M) \xrightarrow{\sim} F_n(M) \oplus (M \circ (F_{n-1}(M) \circ F_m(M)))
\xrightarrow{(\text{id}, \text{id} \circ \mu_{m-1,n})} F_n(M) \oplus (M \circ F_{m-1}(M)) \circ F_n(M) \xrightarrow{\iota + j} F_{m+n}(M).
$$

We can also define the unit map $\eta : I \to F(M)$ to be the composite $I \xrightarrow{\sim} F_0(M) \hookrightarrow F(M)$. The triple $(F(M), \mu, \eta)$ obtained in this way is indeed an operad. See [LV12 §5.5.1] for the proof.

**Definition 1.1.2.** The operad $F(M)$ is called the free operad associated to $M$.

The natural projection $\varepsilon : F(M) \to I$ is an augmentation map of the free operad $F(M)$, i.e., $\varepsilon$ is a morphism of operads to the identity operad satisfying the relation $\varepsilon \eta = \text{id}$. Thus $F(M)$ is an augmented operad in the following sense.

**Definition 1.1.3.** An operad $P$ is called augmented if it has the augmentation map $\varepsilon : P \to I$. The kernel $\text{Ker}(\varepsilon)$ is called the augmentation ideal and denoted by $P_{\text{aug}}$.

Concrete examples of operads are constructed via generators and relations. Let $M, F \in \mathcal{C}$ be $\Sigma$-objects. Then we can define an operad $P$ in $\mathcal{C}$ to be a coequalizer of the form

$$
F(R) \xrightarrow{d_0} F(M) \xrightarrow{d_1} P.
$$

In the classical setting like $\mathcal{C} = k\text{-Mod}$, the category of $k$-modules over a commutative ring $k$, one usually take $R \subset F(M)$ and consider $P := F(M)/(R)$. Here $(R)$ denotes the ideal generated by $R$ in the operadic sense, and it is spanned by composites which include a factor of the form $w\rho$ with $w \in \Sigma_n$ and $\rho \in R(n)$. In terms of the coequalizer definition, one takes $d_1 = 0$ to get this operad $P$. See [LV12 Chapter 5] for the detail.

Closing this subsection, let us recall the classical examples of operads, namely the associative, commutative and Lie operads.

**Example 1.1.4.** Let $\mathcal{C} = (k\text{-Mod}, \otimes, k)$ be the category of $k$-modules over a commutative ring $k$ with $\otimes = \otimes_k$ the ordinary tensor product. Then

1. The commutative operad $\text{Com}$ is

$$
\text{Com} = F(k\mu)/(\tau \mu - \mu \circ_1 \mu - \mu \circ_2 \mu).
$$

Thus the generating $\Sigma$-object $M = k\mu$ is spanned by a binary operation $\mu = \mu(x_1, x_2)$, and as a $k$-module $F(k\mu)$ is isomorphic to the tensor algebra $\oplus_{n \geq 0}(k\mu)^{\otimes n}$. The defining relations are given by two elements $\tau \mu - \mu \in F(k\mu)/(2)$ and $\mu \circ_1 \mu - \mu \circ_2 \mu \in F(k\mu)/(3)$. Here $\tau = (1, 2)$ denotes the permutation. Thus the relation means the commutativity $\mu(x_1, x_2) = \mu(x_2, x_1)$ and the associativity $\mu(\mu(x_1, x_2), x_3) = \mu(x_1, \mu(x_2, x_3))$ of the multiplication $\mu$.

2. The associative operad $\text{Asc}$ is

$$
\text{Asc} = F(k\mu \otimes k\tau \mu)/(\mu \circ_1 \mu - \mu \circ_2 \mu).
$$

3. The Lie operad $\text{Lie}$ is

$$
\text{Lie} = F(k\mu)/(\mu + \tau \mu, (1 + c + c^2) \mu \circ_1 \mu).
$$

Here we denoted $c = (1, 2, 3) \in \Sigma_3$. The first relation means the anti-commutativity of the operation $\mu = \mu(x_1, x_2)$, and the second means the Jacobi identity.
We also have a sequence of morphisms of operads
\[ \text{Lie} \longrightarrow \text{Asc} \longrightarrow \text{Com.} \] (1.3)

The first morphism is induced by the natural embedding of $\Sigma_n$-modules $\text{Lie}(n)$ into the regular $\Sigma_n$-modules $\text{Asc}(n)$. The second one is induced by the natural quotient.

1.2. Algebras over operads. Let us fix a symmetric monoidal category $\mathcal{C}$ with the same assumptions as in the previous subsection. Following [Fr09, Chapter 1] we introduce the notion of symmetric monoidal category over $\mathcal{C}$.

**Definition 1.2.1.** A symmetric monoidal category $\mathcal{E}$ over $\mathcal{C}$ is $\mathcal{E} = (\mathcal{E}, \otimes, 1)$ together with a bifunctor $\otimes: \mathcal{C} \times \mathcal{E} \to \mathcal{E}$ such that

- **E1** $1_{\mathcal{C}} \otimes X \simeq X$ for all $X \in \mathcal{E}$,
- **E2** $(C \otimes D) \otimes X \simeq C \otimes (D \otimes X)$ for all $C, D \in \mathcal{C}$ and $X \in \mathcal{E}$,
- **E3** $C \otimes (X \otimes Y) \simeq (C \otimes X) \otimes Y \simeq X \otimes (C \otimes Y)$ for all $C \in \mathcal{C}$ and $X, Y \in \mathcal{E}$.

At the first condition we wrote $1_{\mathcal{C}}$ the unit object of the monoidal category $\mathcal{C} = (\mathcal{C}, \otimes, 1_{\mathcal{C}})$. This bifunctor $\otimes: \mathcal{C} \times \mathcal{E} \to \mathcal{E}$ is called the external tensor product, and $\otimes: \mathcal{C} \times \mathcal{E} \to \mathcal{E}$ is called the internal tensor product. We also assume the following conditions.

- **E4** All small colimits and small limits exist in $\mathcal{E}$.
- **E5** The internal tensor product $\otimes: \mathcal{C} \times \mathcal{E} \to \mathcal{E}$ commutes with small colimits in each variable.
- **E6** The external tensor product $\otimes: \mathcal{C} \times \mathcal{E} \to \mathcal{E}$ commutes with small colimits in each variable.

A symmetric monoidal category $\mathcal{E}$ will be the place where an algebra over an operad lives. For the definition of algebras over operads, we need one more preparation.

**Definition.** For a $\Sigma_n$-object $P$ in $\mathcal{C}$ and an object $E \in \mathcal{E}$, we define

\[ S(P, E) := \bigoplus_{n \geq 0} (P(n) \otimes E^\otimes n)_{\Sigma_n} \in \mathcal{E}. \]

Here we used the internal tensor product to form $E^\otimes n$, and used the external tensor product to form $P(n) \otimes E^\otimes n$. The existence of colimits assures that the coinvariants $(P(n) \otimes X^\otimes n)_{\Sigma_n}$ is well-defined. We see from the definition that if $\mathcal{E} = \mathcal{C}$ and the external and internal tensor products coincide, then $S(P, E) = P \otimes E$.

The operation $P \to S(P, -)$ has the following nice property. Let $\text{Func}(\mathcal{C}, \mathcal{E})$ be the category of functors on $\mathcal{E}$.

**Fact.** The operation $P \to S(P, -)$ defines a functor

\[ (\mathcal{C}^{\Sigma_n}, \circ, 1) \longrightarrow (\text{Func}(\mathcal{C}, \mathcal{E}), \circ, \text{id}) \]

of monoidal categories. Here the $\circ$ in the target denotes the composition of functors.

**Definition.** Let $P$ be an operad in $\mathcal{C}$. A $P$-algebra in $\mathcal{E}$ is a pair $(A, \mu_A)$ consisting of an object $A \in \mathcal{E}$ and a morphism $\mu_A: S(P, A) \to A$ in $\mathcal{E}$ making the following diagrams commutative.
The morphism $\mu_A$ is called the *evaluation map*. We denote such $(A, \mu_A)$ simply by $A$.

More explicitly, $\mu_A$ consists of a collection of morphisms $P(n) \otimes A^{\otimes n} \to A$ which are equivariant with respect to the $\Sigma_n$-actions, and which also satisfy the unit and associative conditions in terms of operad actions.

One can restate $P$-algebra structures on $A \in \mathcal{E}$ in terms of endomorphism operads. See [LV12, §5.2.11] for an explanation in the non-relative setting. To explain it in the relative setting, let us introduce the endomorphism operad $\text{End}_A$ for $A \in \mathcal{E}$ following [Fr09, §3.4].

Recall the assumption that the external tensor product $\otimes: \mathcal{C} \times \mathcal{E} \to \mathcal{E}$ preserves colimits. Then the argument after Assumption 1.0.1 says that there is a bifunctor $\text{Hom}_{\mathcal{E}}: \mathcal{E} \to \text{End}_A$ such that for any $C \in \mathcal{C}$ and $X, Y \in \mathcal{E}$ we have

$$E(C \otimes X, Y) = E(C, \text{Hom}_{\mathcal{E}}(X, Y)). \quad (1.4)$$

**Definition 1.2.2.** Let $X \in \mathcal{E}$. The *endomorphism operad* $\text{End}_X$ of $X$ is the $\Sigma_\ast$-object in $\mathcal{C}$ consisting of

$$\text{End}_X(r) := \text{Hom}_{\mathcal{E}}(X^{\otimes r}, X),$$

where $\Sigma_\ast$-acts by permuting the tensor components of $X^{\otimes r}$, together with the composition map $\mu$ given by the natural composition of endomorphisms.

Using the set-theoretic symbol (1.1), we can write the action of $\mu$ as

$$\mu(f; f_1, \ldots, f_k; 1) = f(f_1 \otimes \cdots \otimes f_k)$$

for $f \in \text{Hom}_{\mathcal{E}}(X^{\otimes k}, X)$ and $f_j \in \text{Hom}_{\mathcal{E}}(X^{\otimes i_j}, X)$. In the case $\sigma \neq \text{id}$, the right $\sigma_r$-action and the symmetric monoidal structure of $\mathcal{E}$ over $\mathcal{C}$ determines $\mu(f; f_1, \ldots, f_k; \sigma)$.

Now we have the well-known equivalence between $P$-algebra structures on a given object $A$ and operad morphisms between $P$ and $\text{End}_A$.

**Fact 1.2.3** ([Fr09 Proposition 3.4.3]). For any $A \in \mathcal{E}$, there is a one-to-one correspondence between operad morphisms $P \to \text{End}_A$ and $P$-algebra structures on $A$.

**Proof.** By (1.3) we have $\mathcal{C}(P(n), \text{End}_A(n)) = E(P(n) \otimes A^{\otimes n}, A)$. Thus we have

$$\mathcal{C}(P, \text{End}_A) = E(\otimes_{n \geq 0}(P(n) \otimes A^{\otimes n})_{\Sigma_n}, A) = E(S(P, A), A),$$

which yields the statement. \hfill $\square$

**Definition 1.2.4.** A *morphism* $f: A \to B$ between $P$-algebras $A$ and $B$ in $\mathcal{E}$ is a morphism in $\mathcal{E}$ commuting with the evaluation maps $\mu_A$ and $\mu_B$. In other words, the following diagram commutes.

$$\begin{array}{ccc}
S(P, A) & \xrightarrow{\mu_A} & A \\
S(id, f) \downarrow & & \downarrow f \\
S(P, B) & \xrightarrow{\mu_B} & B
\end{array} \quad (1.5)$$

The category of $P$-algebras in $\mathcal{E}$ is denoted by $P\mathcal{E}$.

**Example 1.2.5.** Algebras over the classical operads in Example 1.1.4 are commutative, associative and Lie algebras in the ordinary meaning. Precisely speaking, setting $\mathcal{E} = \mathcal{C}$ to be the category of $k$-modules over a commutative ring $k$, we have the followings.
The morphism \( \lambda \) action maps free of diagrams commutative.

Let \( \text{Definition 1.3.1.} \)

\( E \) consisting of \( P \) since \( M \) on \( A \) naturally gives an associative algebra structure, and an associative algebra structure on \( A \) gives rise to a Lie algebra structure by setting the Lie bracket to be the commutator.

1.3. Modules of algebras over operads. Following [Fr09] §4.2, we introduce modules over a \( P \)-algebra in the relative setting. Note that in [Fr09] they are called representations of a \( P \)-algebra. See [LV12] §12.3.1 for the non-relative setting.

Fix \( \mathcal{C} \) and \( \mathcal{E} \) as in the previous subsections. Recall that \( \mathcal{E}^{\Sigma} \) denotes the category of \( \Sigma \)-objects in \( \mathcal{E} \). For \( P \in \mathcal{E}^{\Sigma} \) and \( M, N \in \mathcal{E} \), define \( S(P, M; N) \in \mathcal{E} \) to be

\[
S(P, M; N) := \bigoplus_{k \geq 0} \left( P(k) \otimes \left( \bigoplus_{1 \leq i \leq k} M \otimes \cdots \otimes N \otimes \cdots \otimes M \right) \right)_{\Sigma_k}
\]

where in the right hand side \( M \) appears \( k - 1 \) times. The symbol \((-)_\Sigma\) denotes the coinvariants as before. Thus we have \( S(P, M; M) = S(P, M) \). Note that for \( P, Q \in \mathcal{E}^{\Sigma} \) and \( M, N \in \mathcal{E} \) we have a natural isomorphism

\[
S(P, S(Q, M); S(Q, M; N)) \simeq S(P \circ Q, M; N).
\]

We also have an isomorphism

\[
S(I, M; N) = 1 \otimes N \simeq N
\]

since \( l(1) = 1 \) and \( l(n) = 0 \) for \( n \neq 1 \).

**Definition 1.3.1.** Let \( P = (P, \mu, \eta) \) be an operad in \( \mathcal{C} \), and \( A = (A, \mu_A) \) be a \( P \)-algebra in \( \mathcal{E} \). An \( A \)-module (over \( P \) in \( \mathcal{E} \)) is a pair

\((E, \lambda_E)\)

consisting of \( E \in \mathcal{E} \) and a morphism \( \lambda_E : S(P, A; E) \to E \) in \( \mathcal{E} \) making the following diagrams commutative.

\[
\begin{array}{ccc}
S(P \circ P, A; E) & \xrightarrow{S(\mu, \text{id}, \text{id})} & S(P, S(P(A; A), S(P, A; E)) \xrightarrow{S(\mu, \mu_A, \lambda_E)} S(P, A; E) \\
S(\mu, \text{id}, \text{id}) & & \lambda_E \\
S(P, A; E) & \xrightarrow{\lambda_E} & E
\end{array}
\]

\[
\begin{array}{ccc}
S(I, A; E) & \xrightarrow{S(\eta, \text{id}, \text{id})} & S(P, A; E) \\
\lambda_E & & \lambda_E \\
S(1, A; E) & \xrightarrow{\lambda_E} & E
\end{array}
\]

The morphism \( \lambda_E \) is called the action map. We denote such \( (E, \lambda_E) \) simply by \( E \).

A morphism \( f : E \to E' \) of \( A \)-modules is a morphism in \( \mathcal{E} \) commuting with the action maps \( \lambda_E \) and \( \lambda_{E'} \).

We denote by \( \text{Mod}_A^{\mathcal{E}} \) the category of \( A \)-modules in \( \mathcal{E} \).

Let us explain a universal construction of \( A \)-modules, namely the construction of free \( A \)-module generated by an object in \( \mathcal{E} \).

Given \( M \in \mathcal{E} \) we define \( A \otimes^P M \in \mathcal{E} \) to be the coequalizer

\[
S(P, S(P, A); M) \xrightarrow{\bar{\mu}_A} S(P, A; M) \longrightarrow A \otimes^P M,
\]
where \( \tilde{\mu} \) and \( \tilde{\mu}_A \) are given by
\[
\tilde{\mu} : S(P, S(P, A); M) \xrightarrow{\sim} S(P \circ P, A; M) \xrightarrow{S(\mu, \text{id}; \text{id})} S(P, A; M)
\]
and
\[
\tilde{\mu}_A : S(P, S(P, A); M) \xrightarrow{S(\text{id}, \mu_A; \text{id})} S(P, A; M).
\]

A \( \otimes^P \) M is actually an \( A \)-module. The \( A \)-action on \( A \otimes^P M \) is given by the following fact: the composite
\[
S(P, A; S(P, A; M)) \rightarrow S(P \circ P, A; M) \xrightarrow{S(\mu, \text{id}; \text{id})} S(P, A; M)
\]
factors through the quotient \( S(P, A; M) \rightarrow A \otimes^P M \). See [LY12, Lemma 12.3.3] for the proof.

**Definition 1.3.2.** Let \( A \in \mathcal{P} \mathcal{E} \) and \( M \in \mathcal{E} \). The \( A \)-module \( A \otimes^P M \) is called the *free \( A \)-module* generated by \( M \).

One can also check that
\[
A \otimes^P - : \mathcal{E} \rightarrow \text{Mod}^P_A
\]
is a functor, and moreover it is a left adjoint of the forgetful functor \( \text{fog} : \text{Mod}^P_A \rightarrow \mathcal{E} \).

In other words, we have

**Fact** ([LV12 Theorem 12.3.4]). For any \( A \in \mathcal{P} \mathcal{E} \), the two functors
\[
A \otimes^P - : \mathcal{E} \xrightarrow{\sim} \text{Mod}^P_A : \text{fog}
\]
form an adjoint pair. Thus for any \( M \in \text{Mod}^P_A \) and \( N \in \mathcal{E} \) we have
\[
\text{Mod}^P_A(A \otimes^P M, N) = \mathcal{E}(M, \text{fog}(N)).
\]

Another example of \( A \)-modules is given by

**Lemma 1.3.3.** Let \( f : A \rightarrow B \) be a morphism in \( \mathcal{P} \mathcal{E} \). Then \( B \) has a natural structure of \( A \)-module.

**Proof.** Define \( \lambda_B : S(P, A; B) \rightarrow B \) to be the composition
\[
\lambda_B := (S(P, A; B) \xrightarrow{S(\text{id}, f; \text{id})} S(P, B; B) = S(P, B) \xrightarrow{\mu_B} B).
\]

By the commutativity of [L3], we have the following commutative diagram.
\[
\begin{array}{ccc}
S(P, A; A) & \xrightarrow{\text{id}} & S(P, A) \\
S(\text{id}, \text{id}, f) \downarrow & & \downarrow \mu_A \\
S(P, A; B) & \xrightarrow{\lambda_B} & B
\end{array}
\]

Using this diagram repeatedly, one can check that the two diagrams in Definition 1.3.1 commute.

In particular, \( A \) is itself an \( A \)-module whose action map is given by \( \lambda_A = \mu_A \).

1.4. **Enveloping Operads.** Following [LY09] §4.1 we recall the enveloping operads for an operad. Let \( \mathcal{C} \) and \( \mathcal{E} \) be symmetric monoidal categories as in the previous subsections. Recall that \( \mathcal{O} \mathcal{P}_\mathcal{E} \) denotes the category of operads in \( \mathcal{E} \).

We have a symmetric monoidal functor \( \eta : (\mathcal{C}, \otimes, 1) \rightarrow (\mathcal{E}, \otimes, 1) \) given by \( \eta(C) = C \otimes 1 \), where \( C \otimes 1 \) denotes the external tensor product. Using \( \eta \), we can map an operad in \( \mathcal{C} \) to an operad in \( \mathcal{E} \).

Let \( P \) be an operad in \( \mathcal{C} \). Now take \( \mathcal{E} \) to be a symmetric monoidal category over \( \mathcal{C} \), and consider the comma category \( \mathcal{P}/\mathcal{O} \mathcal{P}_\mathcal{C} \) of objects under \( P \). More explicitly, we have
**Definition.** Consider the category whose objects are operad morphisms \( \phi : P \to Q \) in \( E \) with \( P \) regarded as an operad in \( E \) via the functor \( \eta \), and whose morphisms are commutative diagrams

\[
\begin{array}{c}
\phi \\
\downarrow \phi' \\
Q \\
\downarrow \phi' \\
Q'
\end{array}
\]

We denote it by \( P\mathcal{O}p_{E} \).

Using this category \( P\mathcal{O}p_{E} \) we can give a universal definition of enveloping operad.

**Definition.** The *enveloping operad* of a \( P \)-algebra \( A \) is an operad \( U_{P} \in P\mathcal{O}p_{E} \) defined by the following adjunction.

\[
(P\mathcal{O}p_{E})(U_{P}(A), Q) = \rho E(A, Q(0)).
\]

The right hand side makes sense since \( Q(0) \) is the initial object in the category \( QE \) and the structure map \( P \to Q \) makes \( Q(0) \) a \( P \)-algebra. The existence of \( U_{P}(A) \) follows from the fact that the functor \( Q \to Q(0) \) preserves limits.

In a set-theoretic context, we have the following description of \( U_{P}(A)(m) \). It is spanned by formal elements

\[
u(x_{1}, \ldots, x_{m}) = p(x_{1}, \ldots, x_{m}, a_{1}, \ldots, a_{r}),
\]

where \( p \in P(m + r) \) with arbitrary \( r \in \mathbb{N} \), \( x_{i} \) are formal variables and \( a_{i} \in A \). These elements satisfy the relations of the form

\[
p(x_{1}, \ldots, x_{m}, a_{1}, \ldots, a_{r}) = p \circ_{m+r} q(x_{1}, \ldots, x_{r}, a_{1}, \ldots, a_{r}).
\]

By taking the unary part of the enveloping operad, we get an algebra object in the monoidal category \( E \). This is the operadic enveloping algebra.

**Definition 1.4.1.** The *enveloping algebra* \( U_{P}(A) \) of a \( P \)-algebra \( A \) is a unital associative ring object \( U_{P}(A)(1) \) in \( E \) whose multiplication \( U_{P}(A) \otimes U_{P}(A) \to U_{P}(A) \) is induced by the composition map \( \mu_{A} \).

Using the set-theoretical description of \( U_{P}(A) \), we can consider the universal algebra \( U_{P}(A) \) as the ring with the generators

\[
u = p(x, a_{1}, \ldots, a_{n}), \quad p \in P(1 + n), \quad a_{i} \in A
\]

and the defining relations

\[
p(x, a_{1}, \ldots, a_{r}, q(a_{e}, \ldots, a_{e+n-1}), a_{e+n}, \ldots, a_{m+n-1}) = p \circ_{1+n} q(x, a_{1}, \ldots, a_{m+n-1}).
\]

We have a more explicit description using the free \( A \)-module in Definition 1.3.2.

**Fact** ([LV12, §12.3.4]). For \( P = (P, \mu, \eta) \in \mathcal{O}p_{E} \) and \( A \in \rho E \), the unital associative ring \( U_{P}(A) \) is isomorphic to

\[
(A \otimes^{P} 1_{E}, m, u),
\]

where the multiplication \( m \) is given by

\[
m : (A \otimes^{P} 1) \otimes (A \otimes^{P} 1) \longrightarrow A \otimes^{P} (A \otimes^{P} 1) \longrightarrow A \otimes^{P} 1
\]

with the second morphism induced by the composition map \( \mu \), and where the unit \( u : 1 \to A \otimes^{P} 1 \) is induced by the unit map \( \eta \).

As in the case of the ordinary Lie algebras, we have
Fact 1.4.2 ([Fr09, 4.3.2 Proposition]). Let \( A \) be a \( P \)-algebra in \( \mathcal{E} \). The category of \( A \)-modules in \( \mathcal{E} \) is equivalent to the category of left \( U_P(A) \)-modules.

Example 1.4.3. For the classical operads in Example 1.1.4 and Example 1.2.5, we have the followings.

1. For the commutative operad \( P = \text{Com} \), the enveloping algebra \( U_{\text{Com}}(A) \) is the unitary commutative algebra \( A_+ = 1 \otimes A \).
2. For the associative operad \( P = \text{Asc} \), the enveloping algebra \( U_{\text{Asc}}(A) \) is the classical enveloping algebra \( A_e = A + A \otimes A^{op} \).
3. For the Lie operad \( P = \text{Lie} \), the enveloping algebra \( U_{\text{Lie}}(g) \) is the classical enveloping algebra \( U(g) \) if we take the base ring \( k \) to be a field.

Since the enveloping operad is defined by adjunction, the correspondence \( A \mapsto U_P(A) \) enjoys a functoriality.

Fact 1.4.4 ([LV12, Proposition 12.3.9]). For any operad \( P \) in \( \mathcal{C} \), the enveloping algebra construction gives a functor

\[
U_P : \text{PAlg}(\mathcal{C}) \rightarrow \text{Mod}(\mathcal{C})
\]

We close this subsection by the recollection on the relative free module ([LV12, §12.3.5]). First we note that any morphism \( f : B \rightarrow A \) of \( P \)-algebras induces a functor

\[
f^* : \text{Mod}_B \rightarrow \text{Mod}_A
\]

Indeed, by Fact 1.4.4 we have a morphism \( U_P(f) : U_P(B) \rightarrow U_P(A) \) of associative algebras, and it gives the functor \( U_P(f)^* \) from the category of left \( U_P(A) \)-modules to the category of left \( U_P(B) \)-modules by the restriction of scalar. Then by Fact 1.4.2 we have the desired functor \( f^* \).

Definition. We call \( f^* \) the restriction functor.

Remark 1.4.5. Let us note that there is a natural \( A \)-module structure on \( A \) itself by Lemma 1.3.3 applied to \( id_A : A \rightarrow A \), and that there is also a \( B \)-module structure on \( A \) by the same lemma applied to \( f : B \rightarrow A \). Let us denote by \( A_A \) the former \( A \)-module and by \( B_A \) the latter \( B \)-module. Then we have \( f^* B_A \simeq A_A \) as \( A \)-modules.

Next we consider another functor

\[
f_1 : \text{Mod}_B \rightarrow \text{Mod}_A
\]

given by the following coequalizer.

\[
\begin{array}{c}
U_P(A) \otimes U_P(B) \otimes M \\
\downarrow_{U_P(f) \otimes \text{id}} \\
U_P(A) \otimes M \\
\downarrow_{\text{id} \otimes \lambda} \\
f(M)
\end{array}
\]

Here \( \lambda : U_P(B) \otimes M \rightarrow M \) denotes the left \( U_P(B) \)-module structure on \( M \). The functor \( f_1 \) is in fact a left adjoint of the restriction functor \( f^* \).

Fact 1.4.6 ([LV12, Proposition 12.3.10]). For any operad \( P \) in \( \mathcal{C} \) and any morphism \( f : B \rightarrow A \) of \( P \)-algebras in \( \mathcal{E} \), we have an adjoint pair of functors

\[
f_1 : \text{Mod}_B \rightleftarrows \text{Mod}_A : f^*.
\]

In particular we have a natural isomorphism

\[
\text{Mod}_A(f_1(M), N) \simeq \text{Mod}_B(M, f^*(N))
\]

for each \( M \in \text{Mod}_B \) and \( N \in \text{Mod}_A \).
1.5. Derivations. Now we recall derivations of algebras over operads following [Fr09] §4.4 and [LV12] §12.3.7.

**Definition 1.5.1.** Let \( f : B \to A \) be a morphism in the category \( p \mathcal{E} \) of \( P \)-algebras in \( \mathcal{E} \). Let \( E = (E, \lambda_E) \) be an \( A \)-module in \( \mathcal{E} \). A morphism \( \theta \in \mathcal{E}(B, E) \) is called an \( A \)-derivation if it makes the following diagram commutative.

\[
\begin{array}{c}
\begin{array}{ccc}
S(P, B) & \xrightarrow{S(\text{id}, \theta)} & S(P, B; B) \\
\downarrow{\mu_P} & & \downarrow{S(\text{id}, \theta)} \\
B & \xrightarrow{\theta} & E
\end{array}
\end{array}
\]

We denote by \( \text{Der}_A^P(B, E) \) the set of all such derivations.

Using the partial composition \( \phi_i \) in \([1.2]\), the condition for \( \theta \) is restated as \( \theta \circ p = \sum_{i=1}^n p \phi_i \theta : B^{\otimes n} \to E \) for all \( n \in \mathbb{N} \) and \( p \in P(n) \). Using this presentation one can recover the classical notion of derivations in the case \( P = \text{Com}, \text{As}, \text{Lie} \).

Next we will discuss the functor given by the operation \( B \mapsto \text{Der}_A^P(B, E) \). The source category of this functor is set to be the comma category \( p \mathcal{E}/A \) of objects over \( A \) in \( p \mathcal{E} \). A direct definition is

**Definition.** Let \( A \in p \mathcal{E} \). Consider the category whose objects are morphisms \( f : B \to A \) in \( p \mathcal{E} \), and whose morphisms are the commutative diagrams

\[
\begin{array}{c}
\begin{array}{ccc}
B & \xrightarrow{f} & B' \\
\downarrow{f} & & \downarrow{f'} \\
A
\end{array}
\end{array}
\]

We call it the category of \( P \)-algebras over \( A \), and denote it by \( p \mathcal{E}/A \).

The functor

\[
\text{Der}_A^P(-, E) : p \mathcal{E}/A \to \text{Set}
\]

is representable by the module of Kähler differentials. Its definition is as follows. Recall the free \( A \)-module functor \( A \otimes^P \cdot \) in Definition \([1.3.2]\). Let \( A \) be a \( P \)-algebra in \( \mathcal{E} \). We define \( \Omega_P A \in \mathcal{E} \) to be the following coequalizer.

\[
\begin{array}{c}
\begin{array}{ccc}
A \otimes^P S(P, A) & \xrightarrow{\tilde{\mu}_{(1)}} & A \otimes^P A \\
\downarrow{\tilde{\mu}_A} & & \downarrow{\Omega_P A} \\
A \otimes^P A
\end{array}
\end{array}
\]

where the map \( \tilde{\mu}_{(1)} \) comes from

\[
S(P, A; S(P, A)) = S(P, A; S(P, A; A)) \to S(P, A; A; A) \xrightarrow{S(\mu, \text{id}, \text{id})} S(P, A; A)
\]

and the map \( \tilde{\mu}_A \) is comes from

\[
S(P, A; S(P, A)) \xrightarrow{S(\text{id}, \text{id}, \text{id}, \mu_A)} S(P, A; A).
\]

The \( A \)-module structure on \( A \otimes^P A \) passes to the quotient \( \Omega_P A \).

**Definition 1.5.2.** The \( A \)-module \( \Omega_P A \) in \( \mathcal{E} \) is called the module of Kähler differentials.

Now let \( f : B \to A \) be a morphism in \( p \mathcal{E} \). Recall the functor \( f_! : \text{Mod}_B^P \to \text{Mod}_A^P \) (see Fact \([1.4.0]\)).

**Fact** ([Fr09] §4.4, [LV12] §12.3). For any \( E \in \text{Mod}_A^P \) we have

\[
\text{Mod}_A^P(\Omega_P B, f_! E) = \text{Der}_A^P(B, E).
\]
1.6. Cooperads. Cooperad is a natural dual notion of operad. See [LV12, §5.8] for an explanation in the non-relative setting. Here we give definitions in the relative setting. Let $\mathcal{C}$ and $\mathcal{E}$ be as in the previous subsections.

First we introduce a new monoidal operation on the category $\mathcal{C}_\Sigma^*$ of $\Sigma_*$-objects, which can be considered as a natural dual operation of $\circ$ in Definition 1.1.1.

**Definition.** For $M, N \in \mathcal{C}_\Sigma^*$, define $M \circ N \in \mathcal{C}_\Sigma^*$ by

$$M \circ N := \prod_{k \geq 0} (M(k) \otimes N^{\otimes k})^{\Sigma_k}.$$

Here $\Sigma_k$ acts diagonally on $M(k) \otimes N^{\otimes k}$ where the action on $N^{\otimes k}$ is the permutation.

One can check that $(\mathcal{C}_\Sigma^*, \circ, I)$ is a monoidal category. Using $\circ$ instead of $\circ$, one can define cooperads just dually as operads.

**Definition.** A cooperad in $\mathcal{C}$ is a $\Sigma_*$-object $C$ in $\mathcal{C}$ together with two morphisms of $\Sigma_*$-objects $\Delta : C \to C \circ C$ and $\varepsilon : C \to I$ making the following diagrams commutative.

The morphism $\Delta$ is called the *decomposition map*, and $\varepsilon$ is called the *counit map*.

A morphism between cooperads is defined naturally. The category of cooperads in $\mathcal{C}$ is denoted by $\mathcal{C}oop_{\mathcal{C}}$.

As in the case of the identity operad, the identity $\Sigma_*$-object $I$ has a cooperad structure. The corresponding cooperad is called the *identity cooperad* and denoted by the same symbol $I$.

One can also define coalgebras over cooperads in a dual manner of algebras over operads For $C \in \mathcal{C}_\Sigma^*$ and $X \in \mathcal{E}$, set

$$S(C, X) := \prod_{k \geq 0} (C(k) \otimes X^{\otimes k})^{\Sigma_k},$$

where $\Sigma_k$ acts diagonally on $C(k) \otimes X^{\otimes k}$.

**Definition.** Let $C$ be a cooperad in $\mathcal{C}$. A $C$-coalgebra $C$ in $\mathcal{E}$ is an object $C \in \mathcal{E}$ equipped with a morphism $\Delta_C : C \to S(C, C)$ such that the following diagrams commute.

The structure of a cooperad is encoded by the *partial decompositions*

$$\Delta_{(i)} : C(r + s - 1) \to C(r) \otimes C(s)$$

for $1 \leq i \leq r + s - 1$ as in the case of operads and partial compositions.
Thus, we consider the standard symmetric monoidal structure on dg/CZ of the forgetful functor.

In the adjoint language, the cofree cooperad is defined as the image of a given object \( M \in \mathcal{C} \) by the left adjoint \( F : \mathcal{C} \rightarrow \text{coop}_\mathcal{C} \) of the forgetful functor \( \text{coop}_\mathcal{C} \rightarrow \mathcal{C} \).

Explicitly, the underlying \( \Sigma \)-object is given by the same one as the free operad \( F(M) \). In other words, we set
\[
F(M) := \lim\limits_{\rightarrow} F_n(M),
\]
where \( F_n(M) \) is defined by
\[
F_0(M) := 1, \quad F_n(M) := 1 \oplus (M \triangleright F_{n-1}(M)).
\]

The decomposition map is inductively defined in the following way. First we set \( \Delta(\text{id}) := \text{id} \triangleright \text{id} \), \( \Delta(m) := \text{id} \triangleright m + m \triangleright \text{id} \) (\( m \in M(n) \)). Thus \( \Delta \) on \( F_1(M) \) is defined. Suppose \( \Delta \) is defined on \( F_{n-1}(M) \). Then for \( p \in M \triangleright F_{n-1}(M) \subset F_n(M) \) we set
\[
\Delta(p) := \text{id} \triangleright p + \Delta^+(p),
\]
where \( \Delta^+ \) is the following composition.

\[
M \triangleright F_{n-1}M \xrightarrow{id \triangleright \Delta} M \triangleright (F_{n-1}M \triangleright F_{n-1}M) \xrightarrow{\sim} (M \triangleright F_{n-1}M) \triangleright F_{n-1}M \xrightarrow{j_n \partial n} F_nM \triangleright F_nM.
\]

Here \( i_n : F_{n-1}(M) \hookrightarrow F_{n-1}(M) \) and \( j_n : M \triangleright F_{n-1}(M) \hookrightarrow F_n(M) \) are natural inclusions.

Define \( \varepsilon : F(M) \rightarrow M \) by \( F_1(M) = 1 \oplus M \rightarrow M \). Then \( (F(M), \Delta, \varepsilon) \) is a cooperad in \( \mathcal{C} \).

**Definition 1.6.1.** The cooperad \( F(M) \) is called the **cofree cooperad** associated to \( M \).

Moreover, the inclusion map \( \eta : 1 \rightarrow F(M) \) induced by by the natural inclusions \( j_n \) is a **coaugmentation** map of the cooperad \( F(M) \). In other words, \( \eta \) is a cooperad morphism such that \( \varepsilon \eta = \text{id} \).

**Definition 1.6.2.** A cooperad equipped with a coaugmentation map is called a **coaugmented cooperad**. For a coaugmented cooperad \( \mathcal{C} \) with the coaugmentation map \( \eta : 1 \rightarrow \mathcal{C} \), the cokernel \( \text{Coker}(\eta) \) is called the **coaugmentation coideal**, denoted by \( \mathcal{C}_{\text{coaug}} \).

## 2. Operadic homology algebra

In this section we review the homology theory for algebras over operads following \[Fr09\] Chap. 13 and \[LV12\] Chap. 12. Unless otherwise stated, we take \( \mathcal{C} = \mathcal{E} = \text{dg k-Mod} \), the category of complexes of \( k \)-modules, or \( \text{dg k-modules} \), over a fixed commutative ring \( k \). An object of \( \text{dg k-Mod} \) will be denoted like \( C = (\oplus_{n \in \mathbb{Z}} C_n, d) = (C_\bullet, d) \) with \( d : C_n \rightarrow C_{n+1} \). For a homogeneous element \( x \in C_p \), we set \( |x| := p \).

We consider the standard symmetric monoidal structure on \( \text{dg k-Mod} \). Explicitly, for \((C_\bullet, d_C), (D_\bullet, d_D) \in \text{dg k-Mod} \) we define \( C \otimes D = ((C \otimes D)_\bullet, d) \) by
\[
(C \otimes D)_n := \bigoplus_{p+q=n} C_p \otimes D_q,
\]
and
\[
d(x \otimes y) := d_C(x) \otimes y + (-1)^p x \otimes d_D(y)
\]
2.1. Homology of algebras over operads. Following [Fr09 Chapter 13] we recall the homology theory of algebras over operads. Since we take \( \mathcal{C} = \text{dg } \text{k-Mod} \), it has a standard model structure, making \( \mathcal{C} \) to be a cofibrantly generated symmetric monoidal category. Let \( \mathcal{P} \) be a \( \Sigma \)-cofibrant operad in \( \mathcal{C} \). Then by the argument on semi-model structure on operads [Fr09 Chap. 11] the category \( p \mathcal{E} \) of \( \mathcal{P} \)-algebras in \( \mathcal{E} = \text{dg } \text{k-Mod} \) has a semi-model structure. In particular for any \( A \in p \mathcal{E} \) one can take a cofibrant replacement \( Q_A \sim A \) in \( p \mathcal{E} \).

For the definition of (co)homology of algebras over operads, it is convenient to introduce the universal coefficient. Recall the set \( \Omega_p(\cdot) \) of Kähler differentials in Definition [1.5.2]. Since we take \( \mathcal{C} = \mathcal{E} = \text{dg } \text{k-Mod} \), \( \Omega_p(M) \) is actually a dg \( \text{k} \)-module for any \( M \in \text{Mod}_p^\mathcal{P} \). In particular, taking a cofibrant replacement \( Q_A \sim A \), one has the dg \( \text{k} \)-module \( \Omega_p(Q_A) \in \text{Mod}_p^\mathcal{P} \). Note that by Fact [1.4.2] \( \Omega_p(Q_A) \) is a left \( U_p(Q_A) \)-module, where \( U_p(Q_A) \) denotes the enveloping algebra of \( Q_A \). (see Definition [1.4.1]). Note also that by the morphism \( Q_A \rightarrow A \) the enveloping algebra \( U_p(Q_A) \) acts on \( U_p(A) \) from right.

**Definition.** For \( A \in p \mathcal{E} \) we define the left \( U_p(A) \)-module \( T_p(Q_A) \) to be

\[
T_p(Q_A) := U_p(A) \otimes_{U_p(Q_A)} \Omega_p(Q_A).
\]

Let us cite the following important result.

**Fact 2.1.1 ([Fr09 13.1.12 Lemma]).** The morphism \( T_p(f) : T_p(Q_A) \rightarrow T_p(Q_B) \) induced by a weak equivalence \( f : Q_A \sim Q_B \) of \( \mathcal{P} \)-algebras over \( A \) in \( \mathcal{E} \) is a weak equivalence of left \( U_p(A) \)-modules if \( Q_A \) and \( Q_B \) are cofibrant.

Now we can define the (co)homology of \( \mathcal{P} \)-algebras with coefficients using \( \otimes \) and \( \text{Hom} \) over the modules of the associative algebra \( U_p(Q_A) \).

**Definition 2.1.2.** Let \( A \in p \mathcal{E} \) and \( Q_A \) be a cofibrant replacement of \( A \) in \( \mathcal{E} \).

1. For a right \( U_p(A) \)-module \( M \), the homology of \( A \) with coefficient in \( M \) is a dg \( \text{k} \)-module defined by

\[
H^\bullet_p(A, M) := H^\bullet(M \otimes_{U_p(A)} T_p(Q_A)).
\]

2. For a left \( U_p(A) \)-module \( N \), the cohomology of \( A \) with coefficient in \( N \) is a graded \( \text{k} \)-module defined by

\[
H^\bullet_p(A, N) := H^\bullet \text{Hom}_{U_p(A)}(T_p(Q_A), N).
\]

By Fact 2.1.1 these definitions are independent of the choice of cofibrant replacement \( Q_A \) [Fr09 13.1.2, 13.1.4 Proposition].

Let us close this subsection by explaining the origin of the definition of homology. Recall the set \( \text{Der}^p_p(B, E) \) of derivations in Definition [1.5.1]. In the present setting \( \mathcal{C} = \mathcal{E} = \text{dg } \text{k-Mod} \), this set is a dg \( \text{k} \)-module.

**Fact ([Fr09 Chap. 13]).** For \( A \in p \mathcal{E} \) and \( N \in \text{Mod}_p^\mathcal{P} \), we have the following dg \( \text{k} \)-module isomorphism.

\[
H^\bullet_p(A, N) \simeq H^\bullet \text{Der}^p_p(Q_A, N).
\]

In [LV12 Chap. 12] the right hand side is taken to be the definition of the homology of \( A \) with coefficient. It is reminiscent of the André-Quillen homology of commutative algebras.
2.2. Operadic twisting morphism. The remaining subsections in the present section are devoted to the explanation of operadic cochain complexes for Koszul operads. The main subject will be given in §6.4.

We give a relative version of operadic twisting morphism explained in [LV12 §6.4]. In this subsection $\mathcal{C}$ and $\mathcal{E}$ are taken generally as in [1]. For an operad $(\mathcal{P}, \mu_\mathcal{P}, \eta)$ and a cooperad $(\mathcal{C}, \Delta_\mathcal{C}, \varepsilon)$ in the base category $\mathcal{C}$, consider

$$\mathcal{C}(\mathcal{C}, \mathcal{P}) := \{\mathcal{C}(n), \mathcal{P}(n)\}_{n \geq 0},$$

which is a $\Sigma_\ast$-object under the action

$$(f \circ \sigma)(x) := f(x \circ \sigma^{-1}) \circ \sigma$$

for $f \in \mathcal{D}(\mathcal{C}(n), \mathcal{P}(n)), x \in \mathcal{C}(n)$ and $\sigma \in \Sigma_n$.

Now we define $\mu(f; g_1, \ldots, g_k; \sigma) \in \mathcal{C}(n)$ for $\sigma \in \Sigma_n, f \in \mathcal{C}(\mathcal{P}(k))$ and $\chi_j \in \mathcal{C}(\mathcal{P}(i_j), \mathcal{P}(i_j), 1 \leq j \leq k$ with $i_1 + \cdots + i_k = n$ to be the following composition of morphisms.

$$(f \otimes g_1 \otimes \cdots \otimes g_k \otimes \sigma) : \mathcal{C}(n) \xrightarrow{\Delta_{\mathcal{C}}} (\mathcal{C} \otimes \mathcal{C})(n) \xrightarrow{\mu} \mathcal{C}(\mathcal{P}(i_1) \otimes \cdots \otimes \mathcal{P}(i_k) \otimes \Sigma_n) \rightarrow \mathcal{P}(k) \otimes \mathcal{P}(i_1) \otimes \cdots \otimes \mathcal{P}(i_k) \otimes \Sigma_n \rightarrow (\mathcal{P} \circ \mathcal{P})(n) \xrightarrow{\mu_\mathcal{P}} \mathcal{P}(n)$$

Then we have

**Fact** ([LV12 Proposition 6.4.1]). $\mathcal{C}(\mathcal{C}, \mathcal{P})$ with $\mu$ the composition map is an operad in $\mathcal{C}$.

Following [LV12 §6.4] we use

**Definition.** The operad $\mathcal{C}(\mathcal{C}, \mathcal{P})$ is called the *convolution operad*.

Now we change the base category to the dg category $\mathcal{D}$ explained in the beginning of this section. Each object $X$ in $\mathcal{D}$ has a $\mathbb{Z}$-grading $X_\bullet$ and a differential $d : X_\bullet \rightarrow X_{\bullet-1}$. A $\Sigma_\ast$-object $M = \{M(n)\}_{n \geq 0}$ in $\mathcal{D}$ consists of objects $M(n)$ in $\mathcal{D}$ with $\Sigma_n$-action.

**Definition.** For a homogeneous morphism $f : M \rightarrow N$ in $\mathcal{D}^\Sigma$ of degree $|f|$, define its derivative $\partial(f)$ to be

$$(f) := d_N \circ f - (-1)^{|f|} f \circ d_M.$$

The importance of the convolution operad lies in

**Fact 2.2.1** ([LV12 Proposition 6.4.5]). The convolution operad $(\mathcal{D}(\mathcal{C}, \mathcal{P}), \partial)$ is an operad in the dg category $\mathcal{D}$. It is also equipped with a pre-Lie product

$$f \ast g : C \xrightarrow{\Delta_1} C \otimes (1) C \xrightarrow{f \circ (1) g} P \circ (1) P \xrightarrow{\mu_1} P$$

for $f, g \in \text{Hom}(\mathcal{C}, \mathcal{P})$, so that $(\mathcal{D}(\mathcal{C}, \mathcal{P}), [, ], \partial)$ with $[f, g] := f \ast g - g \ast f$ is a dg Lie algebra.

For the definition of pre-Lie algebra, see [LV12 §1.4].

A solution $\alpha \in \mathcal{D}(\mathcal{C}, \mathcal{P})$ of the Maurer-Cartan equation

$$\partial(\alpha) + \alpha \ast \alpha = 0$$

of degree $-1$ is called an *operadic twisting morphism*. For such $\alpha$, one can define a complex

$$\mathcal{P} \circ_\alpha \mathcal{C} = (\mathcal{P} \circ \mathcal{C}, d_{\mathcal{P} \circ \mathcal{C}} + d_\alpha)$$

called the *left twisting composite product*, and another one

$$\mathcal{C} \circ_\alpha \mathcal{P} = (\mathcal{C} \circ \mathcal{P}, d_{\mathcal{C} \circ \mathcal{P}} + d_\alpha)$$

(2.1) called the *right twisting composite product*. See [LV12 §6.4.5] for the detail.
2.3. Operadic bar and cobar construction. We explain a relative version of the operadic bar and cobar constructions given in [LV12, §6.5].

Let $\mathcal{D}$ be a symmetric monoidal dg category linear over a field $\mathbb{k}$. The operadic bar construction means the following functor

$$\text{Bar} : \{\text{augmented operads in } \mathcal{D}\} \longrightarrow \{\text{coaugmented cooperads in } \mathcal{D}\}.$$ 

Let us explain the definition of $\text{Bar}$. Let $ks$ be the one-dimensional graded vector space spanned by $s$ with grading $|s| = 1$. The suspension of $V \in \mathcal{D}$ is defined to be $sV := ks \otimes V$.

In particular we have $(SV)_i \simeq V_{i-1}$.

For an augmented operad $P = (P, \mu, \eta, \varepsilon)$ in $\mathcal{D}$, we set

$$\text{Bar}(P) := (F(sP_{\text{aug}}), d_1 + d_2).$$

Here $F$ denotes the cofree cooperad functor explained around Definition [1.6.1] and we applied it to the suspended $\Sigma_*$-object $sP_{\text{aug}}$, where $P_{\text{aug}}$ is the augmentation ideal (see Definition [1.4.3]).

The differentials $d_1$ and $d_2$ are defined as follows. $d_1$ is the map on $F(sP_{\text{aug}})$ from the differential $d_P$ on $P$. Thus we have $d_1^2 = 0$. $d_2$ is given by the following composite.

$$\begin{align*}
F(sP_{\text{aug}}) &\longrightarrow sP_{\text{aug}} \circ (1) \ sP_{\text{aug}} \xrightarrow{\sim} (ks \otimes P_{\text{aug}}) \otimes (ks \otimes P_{\text{aug}}) \\
&\xrightarrow{id \otimes \tau \otimes id} (ks \otimes ks) \otimes (P_{\text{aug}} \otimes P_{\text{aug}}) \xrightarrow{\mu \otimes \mu_{\text{aug}}} ks \otimes P_{\text{aug}}.
\end{align*}$$

Here $\tau$ is the permutation map, and $\mu : ks \otimes ks \rightarrow ks$ is the map of degree $-1$ defined by $\mu(s \otimes s) = s$. By [LV12, Proposition 6.5.1] we know $d_2^2 = 0$. We also have $d_1 d_2 + d_2 d_1 = 0$, so that $\text{Bar}(P)$ is a cooperad in $\mathcal{D}$.

The cooperad $\text{Bar}(P)$ obtained is coaugmented by the coaugmentation on the cofree cooperad. In summary, the functor $\text{Bar}$ is well-defined.

Dually one can construct a functor

$$\text{Cobar} : \{\text{coaugmented cooperads in } \mathcal{D}\} \longrightarrow \{\text{augmented operads in } \mathcal{D}\}.$$ 

For an coaugmented cooperad $C = (C, \Delta, \varepsilon, \eta)$ in $\mathcal{D}$, we set

$$\text{Cobar}(C) := (F(s^{-1}C_{\text{coaug}}), d_1 + d_2).$$

Here $F$ denotes the free operad functor, and $s^{-1}$ denotes the inverse suspension, i.e., the graded vector space $ks^{-1}$ generated by $s^{-1}$ with $|s^{-1}| = -1$. $C_{\text{coaug}}$ denotes the coaugmentation coideal of $C$, see Definition [1.6.2].

The differential $d_1$ is induced by the one $d_C$ on the cooperad $C$. The differential $d_2$ is defined by

$$\begin{align*}
ks^{-1} \otimes C_{\text{coaug}} \xrightarrow{\Delta \otimes \Delta} (ks^{-1} \otimes ks^{-1}) \otimes (C_{\text{coaug}} \circ (1) C_{\text{coaug}}) \xrightarrow{id \otimes \tau \otimes id} (s^{-1} \otimes C_{\text{coaug}}) \circ (1) (ks^{-1} \otimes C_{\text{coaug}}) \xrightarrow{\sim} s^{-1} C_{\text{coaug}} \circ (1) s^{-1} C_{\text{coaug}} \xrightarrow{\sim} F(s^{-1} C_{\text{coaug}}).
\end{align*}$$

Now we can state the fundamental result on the operadic bar and cobar constructions.

Fact ([LV12 Theorem 6.5.7]). The functors $\text{Bar}$ and $\text{Cobar}$ form an adjoint pair

$$\text{Cobar} : \{\text{coaugmented cooperads in } \mathcal{D}\} \xrightarrow{\sim} \{\text{augmented operads in } \mathcal{D}\} \xleftarrow{\text{Bar}}.$$

More precisely, for an augmented operad $P$ and a coaugmented cooperad $C$ in $\mathcal{D}$, there exists a natural isomorphism

$$\mathcal{O}p_{\mathcal{D}}(\text{Cobar}(C), P) \simeq \mathcal{C}oo\mathcal{P}_{\mathcal{D}}(C, \text{Bar}(P)).$$
Fact ([LV12, Theorem 6.6.3]). The unit \( \eta : \text{Bar Cobar } \mathcal{C} \xrightarrow{\sim} \mathcal{C} \) is a quasi-isomorphism of cooperads in \( \mathcal{D} \). Dually, the counit \( \epsilon : \text{Cobar Bar } \mathcal{P} \xrightarrow{\sim} \mathcal{P} \) is a quasi-isomorphism of operads in \( \mathcal{D} \).

2.4. Operadic (co)chain complexes. Following [LV12, §12.1], we recall a general construction of complexes computing the (co)homology of algebras over quadratic operads. We take \( \mathcal{C} = \mathcal{C} = \text{dg k-Mod} \) as in the beginning of this section.

For a graded \( \Sigma_* \)-module \( M \) in \( \mathcal{C} \), consider the free operad \( \mathcal{F}(M) \) in \( \mathcal{C} \). It has a weight grading given by
\[
\begin{align*}
    w(\text{id}) &:= 0, \\
    w(\xi) &:= 1 \ (\xi \in M(n)), \\
    w(\xi; \eta_1, \ldots, \eta_k; \sigma) &:= w(\xi) + w(\eta_1) + \cdots + w(\eta_k).
\end{align*}
\]

Here we used the set-theoretic notation (1.1) and \( \text{id} \in 1 = k \subset \mathcal{F}(M)(1) \). We denote by \( \mathcal{F}(M)^{(m)} \) the submodule of weight \( m \).

Now an operadic quadratic data is a pair \((E, R)\) of a graded \( \Sigma_* \)-module \( E \) and a graded sub-\( \Sigma_* \)-module \( R \subset \mathcal{F}(E)^{(2)} \).

Associated to such \((E, R)\), we define the quadratic operad to be
\[
\mathcal{P}(E, R) := \mathcal{F}(E)/(R),
\]
the quotient operad of \( \mathcal{F}(E) \) by the operadic ideal generated by \( R \). Thus it is a universal operad among the quotient operads \( \mathcal{F}(E) \twoheadrightarrow \mathcal{P} \) such that the composition \( (R) \hookrightarrow \mathcal{F}(E) \twoheadrightarrow \mathcal{P} \) is trivial. An operad which is isomorphic to such \( \mathcal{P}(E, R) \) is called quadratic. The classical operads in Example 1.1.4 are thus quadratic.

Dually we define the quadratic cooperad \( \mathcal{C}(E, R) \) to be the cooperad which is universal among sub-cooperads \( \mathcal{C} \hookrightarrow \mathcal{Cobar}(E) \) such that the composition \( \mathcal{C} \hookrightarrow \mathcal{Cobar}(E) \twoheadrightarrow \mathcal{Cobar}(E)^{(2)} \) is trivial.

Definition. The Koszul dual cooperad of the quadratic operad \( \mathcal{P}(E, R) \) is defined to be the quadratic cooperad
\[
\mathcal{C}(sE, s^2 R).
\]

Here \( s \) denotes the degree shifting in \( \mathcal{C} = \text{dg k-Mod} \).

Note that \( \mathcal{P}(E, R) \) and \( \mathcal{C}(sE, s^2 R) \) are both \( \Sigma_* \)-modules in \( \mathcal{C} = \text{dg k-Mod} \) with trivial differentials. We now have

Fact ([LV12, Lemma 7.4.1]). Define a morphism \( \kappa \) of \( \Sigma_* \)-modules in \( \mathcal{C} \) to be
\[
\kappa : \mathcal{C}(sE, s^2 R) \longrightarrow E \xrightarrow{s^{-1}} E \hookrightarrow \mathcal{P}(E, R).
\]

Then \( \kappa \circ \kappa = 0 \), where \( \circ \) is the pre-Lie product in Fact 2.2.1.

Thus \( \kappa \) is an operadic twisting morphism. Now recall the construction of the left twisting composite product [2.1].

Definition. Let \( \mathcal{P} = \mathcal{P}(E, R) \) be a quadratic operad and \( \mathcal{C} = \mathcal{C}(sE, s^2 R) \) be the associated Koszul dual cooperad.

(1) The complex \( \mathcal{C} \circ_\kappa \mathcal{P} \) is called the Koszul complex of the operad \( \mathcal{P} \).

(2) \( \mathcal{P} \) is called a Koszul operad if the Koszul complex \( \mathcal{C} \circ_\kappa \mathcal{P} \) is acyclic.

The bar-cobar construction explained in the previous subsection implies

Fact ([LV12, Theorem 7.4.2]). For a quadratic cooperad \( \mathcal{P} = \mathcal{P}(E, R) \), the followings are equivalent.

(1) \( \mathcal{P} \) is Koszul.

(2) The right twisting composite product \( \mathcal{P} \circ_\kappa \mathcal{C} \) with \( \mathcal{C} = \mathcal{C}(sE, s^2 R) \) is acyclic.

(3) The canonical inclusion \( \mathcal{C} \hookrightarrow \text{Bar}(\mathcal{C}) \) induced by the adjunction \((\text{Cobar}, \text{Bar})\) is a quasi-isomorphism.
(4) The canonical projection \( C \mapsto \text{Cobar}(C) \) is a quasi-isomorphism.

Now we can given the definition of operadic chain complex.

**Definition.** Let \( P = P(E, R) \) be a quadratic operad and \( C = C(sE, s^2R) \) be its Koszul dual cooperad. For a \( P \)-algebra \( A \), we set

\[ C^P_\bullet(A) := (C \circ P \circ \text{op} A). \]

Here \( \text{op} \) denotes the relative composite product, which is defined to be the coequalizer

\[
\begin{array}{ccc}
M \circ P \circ N & \xrightarrow{\text{id} \circ \lambda} & M \circ N \\
\rho \circ \text{id} & \downarrow & \downarrow
\end{array}
\]

for a right \( P \)-module \( M \) with the action map \( \rho : M \circ P \to M \) and a left \( P \)-module \( N \) with the action map \( \lambda : P \circ N \to N \). Thus as a \( \Sigma_* \)-module we have \( C^P_\bullet(A) \simeq C \circ A = \overline{S}(C, A) \), and we can present

\[ C^P_\bullet(A) = (\overline{S}(C, A), d). \]

For a binary quadratic operad \( P \), we have the following explicit description of this complex.

**Fact 2.4.1** ([LV12 Proposition 12.1.1]). Let \( P = P(E, R) \) be a binary quadratic operad, i.e., \( E = (0, 0, E, 0, \ldots) \) as a \( \Sigma_* \)-module. Then for a \( P \)-algebra \( A \), we have

\[ C^P_\bullet(A) = C^{(n+1)} \simeq \Sigma(n+1) A^{\otimes (n+1)} \]

with the differential

\[ d(\zeta \otimes (a_1, \ldots, a_{n+1})) = \sum \xi \otimes (a_{\sigma^{-1}(i)}, \ldots, a_{\sigma^{-1}(i-1)}, \eta(a_{\sigma^{-1}(i)}, a_{\sigma^{-1}(i+1)}), a_{\sigma^{-1}(i+2)}, \ldots, a_{\sigma^{-1}(n+1)}) \]

for \( \Delta(1) \zeta = \sum (\xi; \text{id}, \ldots, \text{id}, \eta; \text{id}, \ldots, \text{id}; \sigma), \xi \in C^{(n+1)}, \eta \in C(2) = E \) and \( \sigma \in \Sigma_{n+1} \).

Similarly, the operadic cochain complex is defined as follows.

**Definition.** Let \( P = P(E, R) \) be a quadratic operad and \( C = C(sE, s^2R) \) be its Koszul dual cooperad. For a \( P \)-algebra \( A \) and an \( A \)-module \( M \), we set

\[ C^P_\bullet(A, M) := (\text{Hom}(C \circ A, M), \partial). \]

where the differential is given by

\[ \partial_n(g) := \partial(g) - (-1)^{|g|} d. \]

Here \( d \) is the differential of \( C^P_\bullet(A) \), and \( \partial \) is given by

\[ \overline{S}(C, A) \xrightarrow{\Delta} \overline{S}(C \circ C, A) \xrightarrow{S(\varepsilon, \text{id})} \overline{S}(C, A), M) \xrightarrow{\lambda_{\sigma}} M. \]

The map \( \tilde{\varepsilon} = \overline{S}(\varepsilon, \text{id}) : \overline{S}(C, A) \to 1 \circ A \simeq A \) is induced by the counit \( \varepsilon : C \to 1 \).

The comparison to Definition 2.1.2 is given by

**Fact** ([LV12 Theorem 12.4.3]). Let \( P = P(E, R) \) be a quadratic operad in \( \mathcal{C}, A \) be a \( P \)-algebra and \( M \) be an \( A \)-module in \( \mathcal{E} \). If the operad \( P \) is Koszul and the complexes \( A, M \) are bounded below, then the cohomology \( H^P_\bullet(A, M) \) in Definition 2.1.2 is calculated by the complex \( C^P_\bullet(A, M) \).

**Example 2.4.2.** Using Fact 2.4.1 one can check that for algebras over the classical operads in Example 1.1.4 the operadic (co)chain complex \( C^P_\bullet(A) \) (or \( C^P_\bullet(A, M) \)) is essentially the classical complex calculating the (co)homology. In other words they are

1. the Harrison (co)chain complex for \( P = \text{Com} \),
(2) the Hochschild (co)chain complex for $P = \text{Asc}$.
(3) the Chevalley-Eilenberg (co)chain complex for $P = \text{Lie}$.

3. Relative homology of operads

3.1. Cotriple homology. Let us briefly recall the theory of cotriple homology following [W94, Chapter 8]. We begin with the introduction of simplicial objects and the associated complexes.

We denote by $\Delta$ the category of finite ordered sets $[n] := \{0 < 1 < \cdots < n\}$ ($n \in \mathbb{N}$) and non-decreasing monotone maps.

**Definition.** For a category $\mathcal{C}$, a simplicial object in $\mathcal{C}$ means a functor $C : \Delta^{\text{op}} \to \mathcal{C}$.

Let $C$ be a simplicial object in $\mathcal{C}$. We set $C_n := C([n])$. Then the structure of simplicial object is uniquely determined by $\{C_n\}_{n \in \mathbb{N}}$ together with face operations $\partial_i : C_n \to C_{n-1}$ and degeneracy operations $\sigma_i : C_n \to C_{n+1}$ for $i = 0, 1, \ldots, n$ satisfying the simplicial identities (see [W94, Proposition 8.1.3]).

Let $A$ be an abelian category. For a simplicial object $A$ in $\mathcal{A}$, we set

$$C_n(A) := \begin{cases} A_n & (n \geq 0) \\ 0 & (n < 0) \end{cases}, \quad d := \sum_{i=0}^{n} (-1)^i \partial_i : C_n(A) \to C_{n-1}(A),$$

where $\partial_i : A_n \to A_{n-1}$ is the face operation of $A$. Then we have $d^2 = 0$ so that $C_\bullet(A) = (\{C_n(A)\}_{n \in \mathbb{Z}}, d)$ is a chain complex in $\mathcal{A}$.

**Definition.** The chain complex $C_\bullet(A)$ is called the unnormalized chain complex of $A$.

Dually we define a cosimplicial object in $\mathcal{C}$ to be a functor $\Delta \to \mathcal{C}$. For a cosimplicial object $A$ in an abelian category, setting $A^n := A([n])$, we have a cochain complex $C^\bullet(A) = (\{C^n(A)\}_{n \in \mathbb{Z}}, d)$ with $C^n(A) = A^n$ and $d : A^n \to A^{n+1}$.

**Definition.** The cochain complex $C^\bullet(A)$ is called the unnormalized cochain complex of $A$.

Next we turn to the notion of cotriple and a construction of simplicial object from a given cotriple.

**Definition.** A cotriple $(T, \varepsilon, \delta)$ in a category $\mathcal{C}$ consists of a functor $T : \mathcal{C} \to \mathcal{C}$, natural transformations $\varepsilon : T \to \text{id}_\mathcal{C}$ and $\delta : T \to TT$ such that the following diagrams commute.

\[
\begin{array}{ccc}
TC & \xrightarrow{\delta_C} & TTC \\
\downarrow{\delta} & & \downarrow{\delta_{TC}} \\
TTC & \xrightarrow{T \delta} & TTTC
\end{array}
\quad
\begin{array}{ccc}
TC & \xrightarrow{id} & TTC \\
\downarrow{\delta} & & \downarrow{\delta_{TC}} \\
TC & \xrightarrow{T \delta} & TTC
\end{array}
\]

**Fact 3.1.1** ([W94, 8.6.4]). Let $(T, \varepsilon, \delta)$ be a cotriple in a category $\mathcal{C}$. For an object $C \in \mathcal{C}$, we have a simplicial object $T_\Delta C$ in $\mathcal{C}$ determined by

$$(T_\Delta C)_n := T^{n+1}C, \quad \partial_i := T^i \varepsilon T^{n-i}, \quad \sigma_i := T^i \delta T^{n-i}.$$  

Let $(T, \varepsilon, \delta)$ be a cotriple in a category $\mathcal{C}$, and $E : \mathcal{C} \to \mathcal{M}$ be a functor to an abelian category $\mathcal{M}$. Take an object $A \in \mathcal{C}$. Then the image $E(T_\Delta A)$ is a simplicial object in $\mathcal{M}$, since a simplicial object is defined to be a contravariant functor. Thus we have the unnormalized chain complex $C_\bullet(E(T_\Delta A))$.

**Definition 3.1.2.** The cotriple homology $H_n(A, E)$ of $A \in \mathcal{C}$ with coefficients in a functor $E : \mathcal{C} \to \mathcal{M}$ is defined to be the homology of $C_\bullet(E(T_\Delta A))$:

$$H_n(A, E) := H_n C_\bullet(E(T_\Delta A)).$$
Dually, given a functor \( F : \mathcal{C}^{op} \to \mathcal{M} \) to an abelian category \( \mathcal{M} \), we have a cosimplicial object \( F(T\Delta A) \) in \( \mathcal{M} \), so that we have the unnormalized cochain complex \( C^\bullet(F(T\Delta A)) \). It is natural to introduce

**Definition 3.1.1.** The cotriple cohomology \( H^n(A, F) \) of \( A \in \mathcal{C} \) with coefficients in a functor \( F : \mathcal{C}^{op} \to \mathcal{M} \) is defined to be the homology of \( C^\bullet(F(T\Delta A)) \):

\[
H^n(A, F) := H^nC^\bullet(F(T\Delta A)).
\]

Finally we recall a construct of cotriple from a given adjoint pair of functors, following [W94, 8.6.2]. Let \( \mathcal{B} : \mathcal{C} \rightleftarrows \mathcal{U} : \mathcal{D} \) be an adjoint pair of functors. Set

\[
\mathcal{M} \quad \text{and} \quad \mathcal{C} \quad \text{of functors}, \quad \text{where we omitted the restriction functor}
\]

\[\mathcal{M} \quad \text{of monoidal categories as in} \ \mathcal{P} \]

\[\mathcal{C} \quad \text{category of} \ \mathcal{P} \]

\[\mathcal{M} \quad \text{applying Fact 3.1.1.} \]

\[\mathcal{C} \quad \text{of functors, where} \ f \]

\[\mathcal{M} \quad \text{a simplicial object} \ \text{Bar} \]

\[\mathcal{C} \quad \text{is the restriction functor and} \ f \]

\[\mathcal{M} \quad \text{with} \ \mathcal{C} \]

\[\mathcal{M} \quad \text{the unnormalized (co)chain complex. It gives the notion of relative}
\]

\[\mathcal{M} \quad \text{(co)homology for algebras over operads.}
\]

\[\mathcal{M} \quad \text{Let} \ f : \mathcal{B} \rightleftarrows \mathcal{C} : \mathcal{U} \]

\[\mathcal{M} \quad \text{be a adjoint pair of functors. Set}
\]

\[\mathcal{M} \quad \text{The adjunction gives natural transformations}
\]

\[\mathcal{M} \quad \text{where} \ \varepsilon : T = \mathcal{F} \mathcal{U} \to \text{id}_\mathcal{C}, \ \eta : \text{id}_\mathcal{B} \to \mathcal{U} \mathcal{F}.
\]

\[\mathcal{M} \quad \text{Also we define a natural transformation} \ \delta : T \to TT \text{ to be}
\]

\[\mathcal{M} \quad \text{for} \ C \in \mathcal{C}.
\]

**Fact 3.1.4.** The obtained data \((\mathcal{F}\mathcal{U}, \varepsilon, \delta)\) is a cotriple in \( \mathcal{C} \).

### 3.2. Relative homology of algebras over operads.

Let \( \mathcal{C} \) and \( \mathcal{E} \) be symmetric monoidal categories as in \( \mathcal{P} \) and \( \mathcal{P} \) be an operad in \( \mathcal{C} \). Recall that \( \mathcal{P}\mathcal{E} \) denotes the category of \( \mathcal{P} \)-algebras in \( \mathcal{E} \) (see Definition 1.2.4).

Let us fix \( f : \mathcal{B} \to \mathcal{A} \) in \( \mathcal{P}\mathcal{E} \) for a while. Recall that by Fact 1.4.6 we have an adjoint pair

\[
f_1 : \text{Mod}_\mathcal{A}^\mathcal{P} \rightleftarrows \text{Mod}_\mathcal{B}^\mathcal{P} : f^*
\]

of functors, where \( f^* \) is the restriction functor and \( f_1 := \mathcal{U}_\mathcal{P}(A) \otimes_{\mathcal{U}_\mathcal{P}(\mathcal{B})} - \). Now we apply Fact 3.1.4 to this adjoint pair, and get a cotriple \((f_1 f^*, \varepsilon, \delta)\) in \( \text{Mod}_\mathcal{A}^\mathcal{P} \). Then applying Fact 3.1.4 to this cotriple we have a simplicial object \( \text{Bar}_\Delta^\mathcal{P}(A, B, M) \) in \( \text{Mod}_\mathcal{A}^\mathcal{P} \) for \( M \in \text{Mod}_\mathcal{A}^\mathcal{P} \) with

\[
\text{Bar}_\Delta^\mathcal{P}(A, B, M)_n = (f_1 f^*)^{n+1} M.
\]

As a left \( \mathcal{U}_\mathcal{P}(A) \)-module we have

\[
\text{Bar}_\Delta^\mathcal{P}(A, B, M)_n = (\mathcal{U}_\mathcal{P}(A) \otimes_{\mathcal{U}_\mathcal{P}(\mathcal{B})})^{n+1} M,
\]

where we omitted the restriction functor \( f^* \). In particular, taking \( M = A \), we have a simplicial left \( \mathcal{U}_\mathcal{P}(A) \)-module \( \text{Bar}_\Delta^\mathcal{P}(A, B, A) \).

We now assume \( \mathcal{E} = \text{dg} \mathcal{E} \)-Mod and consider the functors

\[
\mathcal{M} \otimes_{\mathcal{U}_\mathcal{P}(A)} (-) : \text{Mod}_\mathcal{A}^\mathcal{P} \to \text{Mod}_\mathcal{A}^\mathcal{P}
\]

and

\[
\text{Hom}_{\mathcal{U}_\mathcal{P}(A)} (-, N) : (\text{Mod}_\mathcal{A}^\mathcal{P})^{op} \to \text{Mod}_\mathcal{A}^\mathcal{P},
\]

where \( \mathcal{M} \) (resp. \( \mathcal{N} \)) is a right (resp. left) \( \mathcal{U}_\mathcal{P}(A) \)-module in \( \mathcal{E} \). Then applying Definition 3.1.2 to these functors and the simplicial object \( \text{Bar}_\Delta^\mathcal{P}(A, B, A) \) in \( \text{Mod}_\mathcal{A}^\mathcal{P} \), we obtain the unnormalized (co)chain complex. It gives the notion of relative (co)homology for algebras over operads.

**Definition 3.2.1.** Let \( f : \mathcal{B} \to \mathcal{A} \) be a morphism in \( \mathcal{P}\mathcal{E} \) with \( \mathcal{E} = \text{dg} \mathcal{E} \)-Mod.

1. For a right \( \mathcal{U}_\mathcal{P}(A) \)-module \( \mathcal{M} \) in \( \mathcal{E} \), we define the relative homology of \( A \) over \( B \) with coefficient in \( M \) to be

\[
H_n(A, B, M) := H_nC^\bullet(\mathcal{M} \otimes_{\mathcal{U}_\mathcal{P}(A)} \text{Bar}_\Delta^\mathcal{P}(A, B, A)).
\]
Operadic Semi-Infinite Homology 22

(2) For a left $U_p(A)$-module (or equivalently, an $A$-module) $N$ in $\mathcal{E}$, we define the relative cohomology of $A$ over $B$ with coefficient in $N$ to be

$$H^n(A, B, N) := H^n C^* \operatorname{Hom}_{U_p(A)}(\operatorname{Bar}_A^P(A, B, A), N).$$

The following statement is a relative version of [109, 13.3.4 Theorem], which claims that the cotriple homology coincides with the homology of algebras over operads in Definition 2.1.2.

**Theorem 3.2.2.** Let $P$ be a $\Sigma_+$-cofibrant operad in $\mathcal{E} = \text{dg } \mathbb{k} \text{-Mod}$, and $A$ be a $P$-algebra in $\mathcal{E} = \text{dg } \mathbb{k} \text{-Mod}$. Then the relative (co)homology in Definition 3.2.1 with $B = k$ the trivial $P$-algebra is isomorphic to the (co)homology in Definition 2.1.2. In other words, we have

$$H_n(A, M) = H_n C^* (M \otimes_{U_p(A)} \operatorname{Bar}_A^P(A, k, A))$$

and

$$H^n(A, N) = H^n C^* \operatorname{Hom}_{U_p(A)}(\operatorname{Bar}_A^P(A, k, A), N).$$

4. Operadic Semi-Infinite Cohomology

In this section we introduce the main ingredient of this note. Let $\mathbb{k}$ be a field. We set $\mathcal{E} = \mathcal{E} = \text{dg } \mathbb{k} \text{-Mod}$. The monoidal structure will be denote by $\otimes_k$ or by $\otimes$ for simplicity. We fix an operad $P$ in $\mathcal{E}$.

4.1. Semi-infinite structure. Let $A$ be a $P$-algebra in $\mathcal{E}$. Recall that $U_p(A)$ denotes the enveloping algebra of $A$, which is a dg algebra over $k$ in the present situation. We denote by $U_p(A) = \oplus_{n \in \mathbb{Z}} U_p(A)_n$ the grading structure. Mimicking the notion of semi-infinite structure for associative algebras in [97b], we introduce

**Definition 4.1.1.** The semi-infinite structure on $A$ is the monomorphism

$$f : B \hookrightarrow A$$

in $\mathcal{E}$ such that the following conditions hold. We set $N := \operatorname{Coker}(f)$.

1. The enveloping algebra $U_p(A)$ contains $U_p(N)$ and $U_p(B)$ as graded sub-algebras.
2. $U_p(N)$ is non-negatively graded, $U_p(N)_0 = k$ and $\dim_k U_p(N)_n < \infty$.
3. $U_p(B)$ is non-positively graded.
4. The multiplication in $U_p(A)$ gives isomorphisms of graded vector spaces $U_p(B) \otimes_k U_p(N) \xrightarrow{\sim} U_p(A)$ and $U_p(N) \otimes_k U_p(B) \xrightarrow{\sim} U_p(A)$.
5. The two linear isomorphisms in the previous item are continuous in the following sense. Let $\varphi : U_p(B) \otimes U_p(N) \xrightarrow{\sim} U_p(B) \otimes U_p(B)$ be the composition of the isomorphisms. Then that for every $m, n \in \mathbb{Z}$ there exist $k_+, k_- \in \mathbb{N}$ such that $\varphi(U_p(B)_m \otimes U_p(N)_n) \subset \oplus_{k_- \leq k \leq k_+} U_p(N)_{n-k} \otimes U_p(B)_{m+k}$. We demand the same condition for the other composition $U_p(N) \otimes U_p(B) \xrightarrow{\sim} U_p(B) \otimes U_p(N)$.

**Example.** Recall that enveloping algebras for $P = \text{Asc}, \text{Lie}$ are nothing but the classical enveloping algebras by Example 1.4.3. Then one can immediately check that

1. The semi-infinite structure of an associative algebra given in [97b] gives a semi-infinite structure in our sense for $P = \text{Asc}$.
2. The semi-infinite structure of a Lie algebra given in [93] gives a semi-infinite structure for $P = \text{Lie}$. 
4.2. Standard semi-injective resolution. Let $A$ be an $P$-algebra in $\mathcal{E}$ with a semi-infinite structure $B \hookrightarrow A$. We will use the symbol $N := \text{Coker}(B \hookrightarrow A)$.

Definition. For an $A$-module (or a left $U_P(A)$-module) $M$ in $\mathcal{E}$, we set

$$\text{Bar}_P^{\infty}\Delta^\bullet(A, M) := \text{Hom}_{\text{Mod}_P}(\text{Bar}_P^{\Delta}(A, B, A), M) \otimes_{U_P(A)} \text{Bar}_P^{\Delta}(A, N, A).$$

The operadic semi-infinite homology of $A$ with coefficient in $M$ is defined to be the homology of the complex $\text{Bar}_P^{\infty}\Delta^\bullet(A, M)$.

By the argument for associative algebras in [Se01, Proposition 2.6.3], we have

Theorem 4.2.1. $\text{Bar}_P^{\infty}\Delta^\bullet(A, M)$ is a semi-injective resolution of $M$, In other words, it is a complex of $A$-modules satisfying

1. $K$-injective as a complex of $N$-modules,
2. $K$-projective relative to $N$.

By construction, setting $P = \text{Asc}$ or $P = \text{Lie}$, we recover the complexes appearing in the literature [Fe84, V93, A97b, Se01]. In particular we have

Theorem 4.2.2. For the classical operad $\text{op} = \text{Asc}$ or $P = \text{Lie}$, the homology of the complex $\text{Bar}_P^{\infty}\Delta^\bullet(A, M)$ gives the semi-infinite cohomology of $A$ with coefficient in $M$ in the literature.

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