On the Effect of Data Dimensionality on Eigenvector Centrality

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January 31, 2022

Abstract

Graphs (i.e., networks) have become an integral tool for the representation and analysis of relational data. Advances in data gathering have lead to multi-relational data sets which exhibit greater depth and scope. In certain cases, this data can be modeled using a hypergraph. However, in practice analysts typically reduce the dimensionality of the data (whether consciously or otherwise) to accommodate a traditional graph model. In recent years spectral hypergraph theory has emerged to study the eigenpairs of the adjacency hypermatrix of a uniform hypergraph. We show how analyzing multi-relational data, via a hypermatrix associated to the aforementioned hypergraph, can lead to conclusions different from those when the data is projected down to its co-occurrence matrix. In particular, we provide an example of a uniform hypergraph where the most central vertex (à la eigencentrality) changes depending on the order of the associated matrix. To the best of our knowledge this is the first known hypergraph to exhibit this property.

1 Introduction

There is a class of problems which seeks to quantify the importance of vertices (i.e., nodes) in a graph (i.e., network) according to some criterion. Centrality measures are typically employed in such cases. Examples of such measures include degree, betweenness, closeness, and eigenvector centrality. While the aforementioned notions of centrality are related they can vary in practice. This is particularly troublesome when two centrality measures identify different vertices as being ‘the most central’. Famously, the Krakhardt kite is an example of a graph where different vertices have the greatest degree, betweenness, and closeness centrality [1]. In a similar vein we construct a hypergraph whose most central vertex (by eigenvector centrality) changes
depending on the order of the associated matrix (see Figure 2). We explore this phenomenon by examining its underlying spectral properties.

In recent years, the principal eigenvector of the (normalized) adjacency hypermatrix of a $k$-uniform hypergraph (see [2,3,4]) has received increasing attention as a way to model multi-relational data which faithfully analogizes the graph case [5,6,7,8,9,10]. Despite these developments, the term ‘hypergraph’ has been historically employed in various contexts. For example, in [6] the authors define the adjacency matrix of a hypergraph to be $A = HWT - D$ where $H$ is the $|V| \times |E|$ incidence matrix, $W$ is a square matrix of edge-weights, and $D$ is the diagonal degree matrix. Note that $A$ is precisely the co-occurrence matrix of $H$ when $W$ is the identity matrix. This approach of using the co-occurrence matrix, or some variation thereof, as a stand-in for the adjacency hypermatrix has been the basis for a longstanding corpus of work. To facilitate the adoption of spectral hypergraph theory in practice the field needs to overcome this historical momentum. In particular spectral hypergraph theory needs to establish its computational feasibility and analytical novelty compared to traditional graph methods.

There have been great strides in the computation of the principal eigenvector of a hypergraph as a constrained optimization problem [11,12]. One can also consider the problem from an algebraic approach via the Lu-Man Method which was introduced in [13] and further developed in [14,15]. Herein we consider the question of novelty. That is, how ‘different’ is the principal eigenvector of a hypergraph from the graph formed from its co-occurrence matrix? We take a practical approach by addressing the following questions. To what extent can the spectral ranking of a hypergraph and its co-occurrence matrix differ? Moreover, how much can these vectors vary coordinate-wise?

We begin by presenting the necessary background for our discussion in the following section. In Section 3 we describe a property of the principal eigenvector of a $k$-partite $k$-uniform hypergraph. We leverage this property to construct a hypergraph whose most central vertex depends on the order of the accompanying hypermatrix in Section 4. In Section 5 we pivot to a structural approach and show how the loss of information incurred from depreciating data can lead to variations in the spectral ranking. Finally, in Section 6 we answer the second question by providing an upper bound on the Chebyshev distance between the aforementioned vectors. We further provide a family of hypergraphs which achieves this bound in the limit.

2 Preliminaries

A $k$-uniform hypergraph, abbreviated $k$-graph, is an ordered pair $H = ([n], E)$ where $E \subseteq \binom{[n]}{k}$. Throughout we will assume that all hypergraphs are uniform and we reserve the language of “hypergraph” specifically for $k$-graphs where $k > 2$. We maintain the notation of [4]. The (normalized) adjacency hypermatrix of a $k$-graph $H$ is an order $k$ and dimension $n$ hypermatrix, denoted $A(H)$, which is a collection
of \( n^k \) elements where,

\[
a_{i_1, i_2, \ldots, i_k} = \frac{1}{(k-1)!} \begin{cases} 
1 & : \{i_1, i_2, \ldots, i_k\} \in E(H) \\
0 & : \text{otherwise}.
\end{cases}
\]

Let \( H \) be a simple \( k \)-uniform hypergraph and \( x \in \mathbb{C}^{||V||} \). For \( e \in E \) we denote \( x^e = \prod_{v \in e} x_v \). The hypergraph \( H \) defines a polynomial form,

\[
F_{A(H)}(x) = k \cdot \sum_{e \in H} x^e. \tag{1}
\]

**Lemma 1** ([4]) In the case when \( H \) is connected there is a unique strictly positive eigenpair \((\lambda, x)\) where \( ||x||_k = 1 \) and

\[
\lambda = \max_{y : ||y||_k = 1} F_{A(H)}(y).
\]

Moreover \( x \) is the only strictly positive eigenvector which satisfies Equation 1.

We refer to the eigenpair in Lemma 1 as the **principal eigenpair of** \( H \). More generally we have that \((\lambda, x)\) is an **eigenpair** of \( H \) if it satisfies the **eigenequations**

\[
\lambda x_{i_{k-1}} = \sum_{e \in E} x^e \setminus i \quad \text{for} \quad i \in [n].
\]

The enterprise of this paper is to motivate the use of \( k \)-order hypergraphs to model \( k \)-relational data. We do so by comparing the principal eigenvector of the normalized adjacency hypermatrix of \( H \) with its co-occurrence matrix.

**Definition 1** For a hypergraph \( H = ([n], E) \), we define the clique-shadow of \( H \) as the multigraph

\[
\partial^* H = ([n], \{uv : uv \in e \in E(H)\})
\]

where \( \mu(uv) = |\{e \in E(H) : u, v \in e\}| \) is the number of edges in \( H \) containing \( u \) and \( v \).

An example of a 3-graph and its clique-shadow is given in Figure 1. The term ‘clique-expansion’ or ‘2-shadow’ is sometimes used for Definition 1. The nomenclature of ‘clique-shadow’ was chosen instead to synthesize the language. We also note that the **shadow** of a hypergraph, denoted \( \partial H \), replaces each \( k \)-edge of a hypergraph with all possible \((k-1)\)-subedges. To avoid confusion we adopt the notation of \( \partial^* \).

**Remark 1** The adjacency matrix of \( \partial^*(H) \) is the co-occurrence matrix of \( H \). Note that the co-occurrence matrix of \( H \) preserves multiplicity.

Throughout we consider a hypergraph \( H \) and its clique-shadow \( \partial^* H \). For clarity, we reserve \((\rho, y)\) and \((\lambda, x)\) for the principal eigenpair of a hypergraph and a graph, respectively.

We adopt the language of **spectral ranking** as in [10] so that ‘the most central vertices by eigenvector centrality’ have spectral rank 1. We say that a hypergraph is **opaque**
if its spectral ranking differs from that of its clique-shadow. We further define the umbral index of a hypergraph $u(H)$ to be the least index for which the spectral ranking of $H$ and $\partial^*(H)$ differ. In the case when $H$ is not opaque (i.e., $H$ and $\partial^*(H)$ have the same spectral ranking) we write $u(H) = 0$.

3 Principle Eigenvector of a $k$-partite $k$-graph

The following elegant result, given in [17], provides insight into how vertices in an independent set ‘compete’ for centrality in a bipartite graph.

**Theorem 2** ([17]) If $S$ is an independent set of a connected graph $G$ and $x$ is the principal eigenvector of $G$, then

$$\sum_{i \in S} x_i^2 \leq \frac{1}{2}.$$  

Equality happens if and only if $G$ is bipartite having $S$ as one color class.

We extend Theorem 2 to hypergraphs. A $k$-graph is $k$-partite, or a $k$-cylinder, if its vertices can be partitioned into $k$ sets so that every edge uses exactly one vertex from each set [4]. Our proof is similar to that of Theorem 2. We include it as it succinctly highlights the mechanisms underpinning this phenomenon.

**Theorem 3** If $S$ is an independent set of a connected $k$-uniform hypergraph $H$ and $y$ is the principal eigenvector of $H$, then

$$\sum_{i \in S} y_i^k \leq \frac{1}{k}.$$  

Equality occurs if and only if $H$ is a $k$-cylinder having $S$ as one color class.

**Proof:** For each $i \in S$ we have

$$\rho y_i^{k-1} = \sum_{e \in E, i \in e} y_e \setminus i.$$  

Multiplying each equation by $y_i$ and summing over $i \in S$ yields

$$\rho \sum_{i \in S} y_i^k = \sum_{i \in S} \sum_{e \in E, i \in e} y_e \setminus i.$$  

Since $S$ is independent and the entries of $y$ are positive we have that

$$\sum_{i \in S} \sum_{e \in E, i \in e} y_e \setminus i \leq F_H(y) = \frac{\rho}{k}.$$  

The desired inequality follows whence $\rho > 0$.

Moreover, equality occurs if and only if every edge $e$ which does not have a vertex in $S$ has $y_e = 0$. Since $y$ is strictly positive this implies that every edge must have a
vertex in $S$. Whence $S$ is an independent set it follows that each edge of $H$ contains exactly one node from $S$. Thus $S$ is a color class of $H$. 

Theorem 3 shows that vertices in the same color class in a $k$-partite $k$-graph 'compete for centrality' like in a zero-sum game. However, this relationship breaks down when the number of color classes exceeds the order of the (hyper)graph. Note that the clique-shadow operation preserves the color classes of a hypergraph so that vertices in a $k$-partite $k$-graph 'compete' in a stricter sense than in the clique-shadow. This observation forms the basis of our constructions moving forward.

4 Generalized Bowtie

We construct a hypergraph which has the property that it and its clique-shadow identify different vertices as being the most central under eigenvector centrality. Consider the following $t$-pleated bowtie 3-graph,

$$B_t = \{[c, \ell_1, \ell_2], [c, r_1, r_2]\} \cup \bigcup_{i=1}^{t}\{[c, \ell_1, \ell_{i+1}], [r_1, r_2, r_{i+2}]\}.$$ 

A drawing of $B_1$ and $\partial^* B_1$ is given in Figure 1.

We establish a spectral characterization for the spectral rank 1 vertices of $B_t$ and $\partial^* B_t$, respectively. In particular we show that $r_1$ and $r_2$ are the most central vertices in $B_t$ precisely when $\rho$ is large (compared to $t$). We further show that vertex $c$ is the most central in $\partial^* B_t$ precisely when $\lambda$ is big (compared to $t$). We conclude by showing that both $\rho$ and $\lambda$ are sufficiently large when $t = 8$. A drawing of $B_8$ is given in Figure 2.

Figure 1: The one-pleated bowtie, $B_1$ where edges are drawn as triangular faces (left), and its clique-shadow (right).

**Lemma 4** Fix $t$ and let $(\rho, y)$ be the principal eigenvector of $B_t$. Then the spectral rank 1 vertices of $B_t$ are $\{r_1, r_2\}$ if and only if $t < \rho \sqrt{\rho - 1} - 1$.

**Proof:** For simplicity we write

$$\alpha = y(c), \beta = y(r_1) = y(r_2), \gamma = y(r_i) \text{ for } i > 2, \delta = y(\ell_1) \text{ and } \varepsilon = y(\ell_j) \text{ for } j > 1.$$
As such, the eigenequations of $B_t$ are

$$\begin{align*}
\rho\alpha^2 &= \beta^2 + (t+1)\delta\varepsilon \\
\rho\beta^2 &= \alpha\beta + t\beta\gamma \\
\rho\gamma^2 &= \beta^2 \\
\rho\delta^2 &= (t+1)\alpha\varepsilon \\
\rho\varepsilon^2 &= \alpha\delta
\end{align*}$$

We show that $\beta > \alpha > \gamma, \delta, \varepsilon$ if and only if $t < \rho\sqrt{\rho - 1}$. From the last two eigenequations we have

$$\frac{\rho\delta^2}{(t+1)\varepsilon} = \alpha = \frac{\rho\varepsilon^2}{\delta}.$$ 

It follows that $\delta^3 = (t+1)\varepsilon^3$. We further have

$$\begin{align*}
\rho\alpha^2 &= \beta^2 + (t+1)\delta\varepsilon = \beta^2 + (t+1)^{4/3}\varepsilon^2 \\
&= \beta^2 + (t+1)^{4/3}\left(\frac{\alpha^2(t+1)^{2/3}}{\rho^2}\right) = \beta^2 + \frac{\alpha^2(t+1)^2}{\rho^2}.
\end{align*}$$

Thus

$$\left(\frac{\alpha}{\beta}\right)^2 = \frac{\rho^2}{\rho^3 - (t+1)^2}.$$ 

Hence $\beta > \alpha$ if and only if $t < \rho\sqrt{\rho - 1} - 1$.

The difference of first and third eigenequation yield $\alpha > \gamma$. We now show $\alpha > \delta$.

Rearranging the fourth eigenequation yields

$$\varepsilon = \frac{\rho\delta^2}{(t+1)\alpha}.$$ 

Substitution into fifth eigenequation yields

$$\left(\frac{\alpha}{\delta}\right)^3 = \frac{\rho^3}{(t+1)^2}.$$ 

Indeed, $\alpha > \delta$ if and only if $\rho^3 > (t+1)^2$. This inequality is satisfied when $\rho\sqrt{\rho - 1} - 1 > t$. Finally we take the ratio of fourth eigenequation multiplied by $\delta$ and the fifth eigenequation multiplied by $\varepsilon$ to conclude

$$\left(\frac{\delta}{\varepsilon}\right)^3 = t + 1$$

so that $\delta/\varepsilon > \sqrt{t+1}$ implying that $\delta > \varepsilon$. It follows that $\alpha > \varepsilon$ concluding the proof.

We now establish a similar characterization for $\partial^*B_t$.

**Lemma 5** Fix $t$ and let $(\lambda, x)$ be the principal eigenvector of $\partial^*B_t$. We have that the only vertex of spectral rank 1 is $c$ if and only if $t < (2\lambda - \lambda^2)/(\lambda + 1)$ or equivalently

$$\lambda > \frac{t + 2 + \sqrt{t^2 + 12t + 4}}{2}.$$
**Proof:** For simplicity, we write 
\[ a = x(c), b = x(r_1) = x(r_2), c = x(r_i) \text{ for } i > 2, d = x(\ell_1), \text{ and } e = x(\ell_j) \text{ for } j > 1. \]

As such, the eigenequations of \( \partial^* B_t \) are
\[
\begin{align*}
\lambda a &= 2b + (t + 1)(d + e) \\
\lambda b &= a + (t + 1)b + tc \\
\lambda c &= 2b \\
\lambda d &= (t + 1)(a + e) \\
\lambda e &= a + d
\end{align*}
\]

We prove our claim by showing that \( a > b, c, d, e \) when \( \lambda \) is sufficiently large.

Substituting the third eigenequation into the second yields
\[
\lambda b = a + (t + 1)b + tc = a + (t + 1)b + t(2b/\lambda)
\]
so that
\[
\frac{a}{b} = \frac{\lambda^2 - \lambda(t + 1) - 2t}{\lambda}.
\]

Indeed \( a > b \) if and only if \( t < (\lambda^2 - 2\lambda)/(\lambda + 2) \) or equivalently
\[
\lambda > \frac{t + 2 + \sqrt{t^2 + 12t + 4}}{2}.
\]

Taking the difference between the first and third eigenequations yields
\[
\lambda(a - c) = (t + 1)(d + e).
\]

Whence \( \lambda, t, d, e > 0 \) we have that \( a > c \).

Now consider the difference of the first and fourth eigenequations,
\[
\lambda(a - d) = 2b + (t + 1)(d - a).
\]

Suppose to the contrary that \( a \leq d \). Then \( \lambda(a - d) \leq 0 \). Moreover, we have that \( 2b + (t + 1)(d - a) > 0 \) since \( b, t > 0 \) and \( d - a \geq 0 \). This implies that \( 0 < 0 \) which is a contradiction. It must be the case that \( a > d \).

We conclude by showing \( d > e \). Substituting the fifth eigenequation into the fourth yields
\[
\lambda d = (t + 1)(\lambda e - d + e)
\]
so that
\[
d = e \left( \frac{(t + 1)(\lambda + 1)}{\lambda - (t + 1)} \right).
\]

Since
\[
(t + 1)(\lambda + 1) > \lambda - (t + 1)
\]
we have that \( d > e \) as desired.

We now prove our main result.
Figure 2: The 8-pleated bowtie, $B_8$. 

**Theorem 6** $u(B_8) = 1$. That is, the spectral rank 1 vertices of $B_8$ and its clique-shadow $\partial^* B_8$ are distinct.

**Proof:** We adhere to the notation of Lemmas 4 and 5. From Theorem 3 we have that $0 < \alpha, \beta < (1/3)^{1/3}$ and

\[
\frac{1}{3} = \alpha^3 + t\gamma^3 \\
\frac{1}{3} = \beta^3 + (t + 1)\varepsilon^3 \\
\frac{1}{3} = \beta^3 + \delta^3.
\]

It follows that

\[
\gamma = \left(\frac{1/3 - \alpha^3}{t}\right)^{1/3} \\
\delta = (1/3 - \beta^3)^{1/3} \\
\varepsilon = \left(\frac{1/3 - \beta^3}{t + 1}\right)^{1/3}.
\]

Appealing to Equation 1, we find that the polynomial form $F_{B_t}$ is a function only of $t, \alpha, \text{and } \beta$. Consider $y'$ where $\alpha = \beta = (1/3)^{1/3} - 1/8$. We have then that

\[
\rho \geq F_{B_8}(y') \geq 4.68949
\]

so that $\rho \sqrt{\rho - 1} - 1 > 8.007 > t$.

Further we find that the minimal polynomial of $\lambda$ is

\[
m_\lambda(x) = x^5 - 9x^4 - 98x^3 + 592x^2 + 2448x + 2048
\]

so that $\lambda > 11.097$ and thus

\[
\lambda > \frac{t + \sqrt{t^2 + 12t + 4}}{2} \approx 10.403.
\]

The conclusion follows immediately from Lemmas 4 and 5. 

**Conjecture 1** We have $u(B_t) = 1$ for $t \geq 8$. 

\[\blacksquare\]
5 Modified Octahedron

Consider the modified octahedron given in Figure 3. Let $O_R$ denote the 3-graph formed by taking the red faces of the octahedron and the green edge. To be precise,

$$E(O_R) = \{[t, p, q], [t, r, s], [b, q, r], [b, p, s], [u, p, q]\}.$$

Similarly define $O_B$ to be the 3-graph formed from the blue faces of the octahedron and the green edge,

$$E(O_B) = \{[t, p, r], [t, p, s], [b, p, q], [b, r, s], [u, p, q]\}.$$

An approximation of the principal eigenvectors of $O_R, O_B, \partial^* O_R,$ and $\partial^* O_B$ is given in Table 1 and the corresponding spectral rankings are given in Table 2.

**Definition 2** We say that two hypergraphs $H_1$ and $H_2$ are co-umbral mates if $H_1 \neq H_2$ but $\partial^*(H_1) = \partial^*(H_2)$.

Co-umbral mates demonstrate the loss of information incurred by using the co-occurrence matrix of a hypergraph instead of its adjacency hypermatrix. Observe that $O_R$ and $O_B$ are co-umbral mates as $O_R \neq O_B$ and $\partial^* O_R = \partial^* O_B$. Moreover, $t$ are $b$ are identically situated in the clique-shadow while occupying distinct positions in $O_R$ (and $O_B$). We thus expect the spectral ranking of the red/blue modified octahedron and its clique-shadow to differ.

**Theorem 7** $O_R$ and $O_B$ are opaque.

**Proof:** Consider $O_R$. Let $(\rho, y)$ and $(\lambda, x)$ be the principal eigenvector of $O_R$ and $\partial^*(O_R)$, respectively. We remark that $x_t = x_b$ by symmetry of $\partial^*(O_R)$. It remains to be shown that $y_t \neq y_b$. We will abuse notation and write $v$ for $y_v$, the value of the principal eigenvector of the vertex $v$. By symmetry of $O_R$ we have $p = q$ and $r = s$. First suppose to the contrary that $q = r$. Taking the difference of their eigenequations yields

$$\rho q^2 - \rho r^2 = (rb + tq + uq) - (tr + qb)$$
Table 1: An approximation of the principal eigenvector of $O_R, O_B, \partial^* O_R$ and $\partial^* O_B$. Approximations were computed via Sage [18].

| Vertex | $y(O_R)$ | $y(O_B)$ | $x(\partial^*(O_R)) = x(\partial^*(O_B))$ |
|--------|----------|----------|------------------------------------------|
| $p$    | 0.5938   | 0.5938   | 0.4871                                   |
| $q$    | 0.5938   | 0.5938   | 0.4871                                   |
| $t$    | 0.5159   | 0.5121   | 0.3579                                   |
| $b$    | 0.5121   | 0.5159   | 0.3579                                   |
| $r$    | 0.4986   | 0.4986   | 0.3348                                   |
| $s$    | 0.4986   | 0.4986   | 0.3348                                   |
| $u$    | 0.3951   | 0.3951   | 0.2121                                   |

Table 2: The spectral ranking of $O_R, O_B, \partial^* O_R$ and $\partial^* O_B$.

| Rank | $O_R$ | $O_B$ | $\partial^*(O_R) = \partial^*(O_R)$ |
|------|-------|-------|--------------------------------------|
| 1    | $p, q$| $p, q$| $p, q$                               |
| 2    | $t$   | $b$   | $t, b$                               |
| 3    | $b$   | $t$   | $r, s$                               |
| 4    | $r, s$| $r, s$| $u$                                  |
| 5    | $u$   | $u$   | $\emptyset$                         |

so that $0 = uq$. It follows that $u = 0$ or $q = 0$. This cannot be the case as the principal eigenvector is non-negative. Indeed $r \neq q$.

Now consider the difference of the eigenequations of $t$ and $b$,

$$\rho t^2 - \rho b^2 = rs + pq - (rq + ps) = r^2 + q^2 - 2rq = (r - q)^2.$$  

Since $r \neq q$ we have that $(r - q)^2 > 0$ which implies that $t > b$.

\[\blacksquare\]

6 Hyperstars and Windmills

Consider now the $k$-order star graph $S(\eta, k)$ which consists of $\eta$ $k$-edges all sharing a common vertex. That is,

$$E(S(\eta, k)) = \{[i, (i - 1)(k - 1) + 2, \ldots, (i - 1)(k - 1) + k] : 1 \leq i \leq \eta\}.$$  

The clique-shadow of a $k$-star is a windmill graph (aka fan or friendship graph). In [19], the author determined the spectrum and network properties of windmill graphs. Adhering to their notation, the windmill graph $W(\eta, k)$ consists of $\eta$ copies of the complete graph $K_k$ joined at a single vertex. We provide a drawing of $S(3, 3)$ and $W(3, 2)$ in Figure 4. With this notation we have $\partial^*(S(\eta, k)) = W(\eta, k - 1)$.  

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Figure 4: $S(3, 3)$ and its shadow, $W(3, 2)$, respectively.

**Theorem 8** Let $y$ and $x$ be the principal eigenvectors of $H$ and $\partial^*(H)$, respectively. Consider $\hat{y} := (y^k_v)_{v \in V}$. Then the Chebyshev distance between

$$D(\hat{y}, \hat{x}) = \max_v |y^k_v - x^2_v| \leq 1/2.$$  

Moreover, for

$$\Delta_k := \max_{H \in \mathcal{H}(k)} D(\hat{y}, \hat{x}),$$

where the maximum is taken over all connected $k$-graphs, we have

$$\lim_{k \to \infty} \Delta_k = 1/2.$$  

**Proof:** From Theorem 3 we have that $D(\hat{y}, \hat{x}) \leq 1/2$. Let $(\lambda, x)$ be the principal eigenpair of $W(\eta, k)$. From [19] we have

$$\lambda = \frac{k - 1}{2} + \sqrt{\left(\frac{k - 1}{2}\right)^2 + \eta k}$$

where $\lambda x_1 = 1 - x_1$ and $x_i = (1 - x_1)/(\eta k)$ for $i > 1$. Solving for $x_1$ and normalizing such that $||x||_2 = 1$ yields

$$x_1 = \frac{2}{\zeta \sqrt{\left(\frac{\zeta - 1}{k \eta}\right)^2 + \frac{4}{\zeta^2}}}$$

for $\zeta = k + \sqrt{(k - 1)^2 + 4k \eta + 1}$.  

For fixed $k$, $\lim_{\eta \to \infty} \hat{x}_1(\eta) = 1/2$. Now consider $S(n, k)$ which is a $k$-cylinder where vertex-1 forms a color class. Appealing to Theorem 3 we have that $\hat{y}_1 = 1/k$. We have shown

$$\Delta_k \geq \lim_{\eta \to \infty} |\hat{y}_1(S(\eta, k)) - \hat{x}_1(W(\eta, k - 1)| = 1/2 - 1/k.$$  

Indeed $\lim_k \Delta_k = 1/2$ as desired.  

**Conjecture 2** $\Delta_k = 1/2 + o(1)$ for all $k \geq 3$.  

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