Supergravity in $2 + \epsilon$ Dimensions

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Abstract

Supergravity theory in $2 + \epsilon$ dimensions is studied. It is invariant under supertransformations in 2 and 3 dimensions. One-loop divergence is explicitly computed in the background field method and a nontrivial fixed point is found. In quantizing the supergravity, a gauge fixing condition is devised which explicitly isolates conformal and superconformal modes. The renormalization of the gravitationally dressed operators is studied and their anomalous dimensions are computed. Problems to use the dimensional reduction are also examined.

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1. Introduction

Quantum theory of gravity has been an outstanding challenge for many years. It is power-counting renormalizable at two spacetime dimensions. It has been proposed to study the quantum theory of gravity at \( d = 2 + \epsilon \) dimensions and to expand it in powers of \( \epsilon \). After paying due attention to subtleties associated with a conformal mode, a nontrivial fixed point has been found at one-loop order \([4]\). More recently, renormalization properties of the \( 2 + \epsilon \) dimensional quantum gravity have been further studied using a convenient choice of variables and gauge fixing conditions \([5]\).

Quantum gravity in two spacetime dimensions is useful not only as a theoretical laboratory for higher dimensional gravity theories, but also as a basis for string theory. Two-dimensional supergravity is especially important for string theories, although it is also interesting from the point of view of quantum gravity theories. Recent progress in matrix models provides possibilities for a nonperturbative treatment of the two-dimensional quantum gravity \([6]\). (For a review of matrix models, see ref. \([7]\).) However, so far it has been difficult to incorporate supersymmetry in the two-dimensional spacetime in such a discretized approach. At the moment we need to develop continuum approaches to study supergravity in two dimensions. Therefore it is useful to explore a computational scheme to deal with the supergravity theory at and near two-dimensional spacetime, in order to understand the superstring theory as well.

The purpose of this paper is to formulate a supergravity theory in \( d = 2 + \epsilon \) dimensions and to compute the beta function up to one-loop order. We find a nontrivial fixed point of order \( \epsilon \), analogous to the nonsupersymmetric case. We also study the \( \epsilon \to 0 \) limit of the theory and find that the result is in agreement with the usual continuum approach using conformal field theory \([8]\)–\([11]\).

We shall present an action for supergravity multiplet and supersymmetric matter multiplet in \( d = 2 + \epsilon \) dimensions, which is exactly invariant under the supersymmetry transformation at 2 and 3 dimensions. Our theory smoothly interpolates between \( d = 2 \) and \( d = 3 \) supergravity theories \([12]\)–\([14]\), but is not exactly invariant at noninteger dimensions. This is primarily because the proof of invariance requires Fierz identities which are valid only at integer dimensions. We shall attempt to use dimensional reduction \([15]\) from \( D \) to \( d \) dimensions, for instance, \( D = 3 \) to \( d = 2 + \epsilon \). However, we find that the proof of invariance is plagued by inconsistencies which
are similar to those encountered in the case of \( D = 4 \) dimensions \([16]\).

We shall use the background field method \([17]\), \([18]\) to compute one-loop counter terms. We introduce a two-parameter family of gauge fixing conditions which is convenient to separate conformal and superconformal modes from the rest of the supergravity multiplet. By choosing the gauge parameters, we can obtain a gauge in which (super)conformal modes have no mixing with the non(super)conformal modes, and moreover the propagators of non(super)conformal modes have no \( 1/\epsilon \) pole. This gauge choice facilitates computations of one-loop counter terms significantly. This gauge can be understood as a supersymmetric generalization of the gauge adopted for the \( 2 + \epsilon \) dimensional gravity \([5]\).

One-loop counter terms are obtained by computing contributions from the superconformal modes (spin \( 1/2 \) components) and those from the non-superconformal modes (spin \( 3/2 \) components) of the gravitino field separately. They are combined with contributions from superghosts and Nakanishi-Lautrup fields. The contributions from the graviton sector are also computed and found to agree with refs. \([4]\), \([5]\). By combining all these one-loop counter terms, we obtain beta function for the gravitational coupling constant \( G \). The resulting beta function turns out to have a nontrivial fixed point at

\[
G^* = \frac{\epsilon}{9 - \hat{c}}, \tag{1.1}
\]

where \( \hat{c} \) denotes the central charge of the superconformal matter multiplet such as \( \hat{c} \) free scalar and spinor fields. By taking the \( \epsilon \to 0 \) limit, we can define a two-dimensional quantum gravity theory. We shall construct physical operators in such a limit and compute their scaling dimensions. The results are found to agree with the conformal field theory approach \([8]\)–\([11]\).

In the next section, supergravity theory in \( d = 2 + \epsilon \) dimensions is presented together with supersymmetric matter field theories. In sect. 3, the supergravity theory is quantized by introducing gauge fixing conditions which separate (super)conformal and non(super)conformal modes. One-loop divergences are computed in sect. 4. In sect. 5, a nontrivial fixed point is shown to follow from the beta function and the two-dimensional limit is also worked out. In sect. 6, dimensional reduction from \( D \) to \( d \) dimensions is examined and is shown to possess inconsistencies if we want to use it to prove invariance under supersymmetry transformations. In appendix A one-loop counterterms of a vector gauge field are computed. In appendices B and C details of dimensional reductions of supergravity action are worked out.
2. Supergravity in $2 + \epsilon$ dimensions

The $N = 1$ supergravity multiplet in two- and three-dimensional spacetime consists of a vielbein $e_{\mu}^\alpha$, a Majorana Rarita-Schwinger field $\psi_{\mu}$ and a real scalar auxiliary field $S$ \cite{12} - \cite{14}. The local Lorentz indices are denoted by Greek letters starting $\alpha, \beta, \cdots$, and the world indices are denoted by middle Greek letters starting $\mu, \nu, \cdots$. Both indices run from 0 to $d - 1$. A signature of the metric is $(-,+,\cdots,+)$.

We shall use the same field content as a supergravity multiplet in $d = 2 + \epsilon$ dimensions. The action in $d = 2 + \epsilon$ dimensions is an interpolation between two- and three-dimensional ones \cite{13}, \cite{14}

\[ S_{SG} = \frac{1}{16\pi G_0} \int d^d x \ e \left[ R + i \bar{\psi}_{\mu} \gamma^{\mu\rho} D_{\nu} \psi_{\rho} - \frac{d - 2}{d - 1} S^2 \right], \tag{2.1} \]

where $e = \det e_{\mu}^\alpha$ and $G_0$ is the bare gravitational coupling constant. The multi-index matrices $\gamma^{\mu\nu\cdots}$ are the antisymmetrized products of gamma matrices such as $\gamma^{\mu\nu\rho} = \frac{1}{3!} (\gamma^{\mu} \gamma^{\nu} \gamma^{\rho} \pm \text{permutations})$. The scalar curvature in the Einstein term is given by

\[ R = e_{\alpha}^\mu e_{\nu}^\nu \left( \partial_{\mu} \omega_{\nu}^{\alpha\beta} + \omega_{\mu}^{\alpha} \omega_{\nu}^{\beta} - (\mu \leftrightarrow \nu) \right). \tag{2.2} \]

Let us note that the kinetic term of the gravitino $\psi_{\rho}$ is invariant under the general coordinate transformation without the affine connection (Christoffel symbol $\Gamma$) in the covariant derivative $D_{\nu}$

\[ D_{\nu} \psi_{\rho} = \left( \partial_{\nu} + \frac{1}{4} \omega_{\nu}^{\alpha\beta} \gamma_{\alpha\beta} \right) \psi_{\rho}. \tag{2.3} \]

In the spirit of the so-called second order formalism \cite{19}, the spin connection $\omega_{\mu}^{\alpha\beta}$ is chosen such that it satisfies the equation of motion derived from the above action

\[ D_{\mu} e_{\nu}^\alpha - D_{\nu} e_{\mu}^\alpha = -\frac{1}{2} i \bar{\psi}_{\mu} \gamma^{\alpha} \psi_{\nu} \equiv T_{\mu\nu}^{\alpha}, \tag{2.4} \]

where $T_{\mu\nu}^{\alpha}$ is a torsion. Again there is no affine connection in the covariant derivatives. The solution of the above equation of motion for the spin connection is given by

\[ \omega_{\mu}^{\alpha\beta} = \omega_{\mu}^{\alpha\beta}(e) + \kappa_{\mu}^{\alpha\beta}, \tag{2.5} \]

\[ \kappa_{\mu}^{\alpha\beta} = \frac{1}{2} \epsilon_{\mu\nu\rho} \left( \bar{\psi}_{\nu} \gamma^{\rho\alpha} \psi_{\rho} - \bar{\psi}_{\rho} \gamma^{\rho\beta} \psi_{\rho} \right) . \]
\[ \omega_{\mu\alpha\beta}(e) \text{ is the usual spin connection without torsion} \]

\[ \omega_{\mu\alpha\beta} = \frac{1}{2} e_{\alpha}^{\rho} e_{\beta}^{\sigma} \left[ e_{\sigma}^{\gamma} (\partial_{\mu} e_{\rho\gamma} - \partial_{\rho} e_{\mu\gamma}) - e_{\mu}^{\gamma} \partial_{\rho} e_{\sigma\gamma} - (\rho \leftrightarrow \sigma) \right] \quad (2.6) \]

and the torsion part is explicitly separated as

\[ \kappa_{\mu\alpha\beta} = -\frac{1}{4} i \left( \bar{\psi}_{\mu} \gamma_{\alpha} \psi_{\beta} - \bar{\psi}_{\mu} \gamma_{\beta} \psi_{\alpha} + \bar{\psi}_{\alpha} \gamma_{\mu} \psi_{\beta} \right) . \quad (2.7) \]

The supertransformations of the supergravity multiplet are given in terms of an infinitesimal anticommuting parameter \( \varepsilon(x) \)

\[ \delta e_{\mu}^{\alpha} = -i \bar{\varepsilon} \gamma^{\alpha} \psi_{\mu}, \]

\[ \delta \psi_{\mu} = 2 \left( D_{\mu} + \frac{1}{2(d-1)} S_{\gamma\mu} \right) \varepsilon; \]

\[ \delta S = \frac{1}{2} i S \bar{\varepsilon} \gamma^{\mu} \psi_{\mu} - \frac{1}{2} i \bar{\varepsilon} \gamma^{\mu\nu} \psi_{\mu\nu}, \quad (2.8) \]

where \( \psi_{\mu\nu} \) is the antisymmetrized covariant derivative of the Rarita-Schwinger field without the affine connection

\[ \psi_{\mu\nu} = D_{\mu} \psi_{\nu} - D_{\nu} \psi_{\mu}. \quad (2.9) \]

The transformation of the spin connection turns out to be

\[ \delta \omega_{\mu\alpha\beta} = \frac{1}{2} i (\bar{\varepsilon} \gamma_{\alpha} \psi_{\mu\beta} - \bar{\varepsilon} \gamma_{\beta} \psi_{\mu\alpha} + \bar{\varepsilon} \gamma_{\mu} \psi_{\alpha\beta}) + \frac{1}{2(d-1)} i S (\bar{\varepsilon} \psi_{\beta} e_{\mu\alpha} - \bar{\varepsilon} \psi_{\alpha} e_{\mu\beta} + \bar{\varepsilon} \gamma_{\alpha\beta} \psi_{\mu}). \quad (2.10) \]

We do not need this transformation of the spin connection to prove the invariance of the supergravity action since we have chosen the spin connection such that it satisfies the equation of motion.

Let us show the invariance of the supergravity action \((2.1)\) for \( d = 2 \) and \( d = 3 \) \cite{13}, \cite{14}. For \( d = 2 \) it is trivial since the action is vanishing. To prove the invariance for \( d = 3 \), it is convenient to rewrite the Rarita-Schwinger action as

\[ S_{RS} = -\frac{1}{16\pi G_{0}} \int d^{d} x i e^{\mu\nu\rho} \bar{\psi}_{\mu} D_{\nu} \psi_{\rho}. \quad (2.11) \]
Under the supertransformation, each term in the action \((2.1)\) is transformed into

\[
\begin{align*}
\delta S_E &= \frac{1}{16\pi G_0} \int d^3x \, e (-i) \bar{\epsilon} \gamma^\alpha \psi_\mu (Re_\alpha^{\mu} - 2R_\alpha^{\mu}), \\
\delta S_{RS} &= \frac{1}{16\pi G_0} \int d^3x \, e \left[ i\bar{\epsilon} \gamma^\alpha \psi_\mu (Re_\alpha^{\mu} - 2R_\alpha^{\mu}) - \frac{1}{2} i S \bar{\epsilon} \gamma^{\mu\nu} \psi_{\mu\nu} \right], \\
\delta S_{aux} &= \frac{1}{16\pi G_0} \int d^3x \, e \frac{1}{2} i S \bar{\epsilon} \gamma^{\mu\nu} \psi_{\mu\nu}.
\end{align*}
\]

We see that the supergravity action is invariant under the supertransformation. For \(d \neq 2, 3\), we can show that the action \((2.1)\) is invariant under the supertransformations \((2.8)\) neglecting terms of order (fermi fields)^3.

As a matter multiplet in \(N = 1\) supergravity theory we take a scalar supermultiplet consisting of a real scalar field \(X\), a Majorana spinor field \(\lambda\) and a real scalar auxiliary field \(F\). The matter action is an interpolation between the two- and three-dimensional ones \([12]–[14]\)

\[
S_M = \int d^d x \, e \left[ -\frac{1}{2} g^{\mu\nu} \partial_\mu X \partial_\nu X + \frac{1}{2} i \bar{\lambda} \gamma^\mu D_\mu \lambda + \frac{1}{2} FF \\
+ \frac{1}{2} i \bar{\psi}_\mu \gamma^\nu \gamma^\mu \lambda \partial_\nu X - \frac{d - 2}{4(d - 1)} i S \bar{\lambda} \lambda - \frac{1}{16} \bar{\psi}_\mu \gamma^\nu \gamma^\mu \psi_\nu \bar{\lambda} \lambda \right],
\]

where the spin connection contains the torsion part \(\kappa_{\mu\alpha\beta}\) as is defined in eq. \((2.7)\). The matter supermultiplet is noninteracting if we switch off the gravitational interactions in the flat spacetime limit. We shall consider \(c\) of such matter supermultiplets: \((X^i, \lambda^i, F^i)\) \((i = 1, \cdots, c)\).

Supertransformations for the matter multiplet are given by

\[
\begin{align*}
\delta X &= i \bar{\epsilon} \lambda, \\
\delta \lambda &= -\gamma^\mu \bar{\epsilon} D^P_\mu X - \bar{\epsilon} F, \\
\delta F &= i \bar{\epsilon} \gamma^\mu D^P_\mu \lambda - \frac{d - 2}{2(d - 1)} i S \bar{\epsilon} \lambda,
\end{align*}
\]

where \(D^P_\mu\) is the supercovariant derivative:

\[
\begin{align*}
D^P_\mu X &= \partial_\mu X - \frac{1}{2} i \bar{\psi}_\mu \lambda, \\
D^P_\mu \lambda &= D_\mu \lambda + \frac{1}{2} \bar{\gamma}^\nu \psi_\mu D^P_\nu X + \frac{1}{2} \psi_\mu F.
\end{align*}
\]
Supertransformation of the matter action gives residual terms which are of order \((\text{fermi fields})^3\) and vanish for \(d = 2, 3\) once we use the Fierz identities. The Fierz identities in two and three dimensions are

\[
\begin{align*}
\text{d} = 2 & : \bar{\chi}_1 \chi_2 \bar{\chi}_3 \chi_4 = -\frac{1}{2} \left[ \bar{\chi}_1 \chi_4 \bar{\chi}_3 \chi_2 + \bar{\chi}_1 \gamma^\alpha \chi_4 \bar{\chi}_3 \gamma_\alpha \chi_2 - \frac{1}{2} \bar{\chi}_1 \gamma^{\alpha \beta} \chi_4 \bar{\chi}_3 \gamma_{\alpha \beta} \chi_2 \right], \\
\text{d} = 3 & : \bar{\chi}_1 \chi_2 \bar{\chi}_3 \chi_4 = -\frac{1}{2} \left[ \bar{\chi}_1 \chi_4 \bar{\chi}_3 \chi_2 + \bar{\chi}_1 \gamma^\alpha \chi_4 \bar{\chi}_3 \gamma_\alpha \chi_2 \right],
\end{align*}
\]  
(2.16)

where \(\chi_1, \ldots, \chi_4\) are arbitrary anticommuting spinors. Let us note that the Fierz identities imply that we are assuming spinors to be two-component and the gamma matrices \(\gamma^\alpha\) to be \(2 \times 2\) matrices. In proving the invariance of the matter action it is necessary to use the transformation of the spin connection (2.10). The order \((\text{fermi field})^3\) terms remain for \(d \neq 2, 3\).

By using a superfield formalism [13], [9], we can construct physical operators in two dimensions, which are invariant under all the local gauge symmetries. We consider only the simplest operator \(O_p\), which corresponds to the tachyon vertex operator with momentum \(p\) in the Neveu-Schwarz sector of string theories. In terms of the component fields it is given by

\[
O_p = \int d^d x O_p(x),
\]

\[
O_p(x) = e \left( ip_i \bar{\Lambda}_i \lambda^i p_i - 2ip_i F^i + \bar{\psi}_\mu \gamma^\mu \lambda^i p_i - 2S + \frac{1}{2} i \bar{\psi}_\mu \gamma^{\mu \nu} \psi_\nu \right) e^{ip \cdot X}.
\]  
(2.17)

We now consider this operator in general \(d\) dimensions. By using the gamma matrix identities in two or three dimensions, we find the supertransformation of the integrand

\[
\delta O_p(x) = \partial_\mu \left[ 2e(\bar{\varepsilon} \gamma^\mu \lambda^i p_i + i \bar{\varepsilon} \gamma^{\mu \nu} \psi_\nu) e^{ip \cdot X} \right].
\]  
(2.18)

Therefore \(O_p\) is invariant not only in two dimensions but also in three dimensions. For three dimensions, we have used the following identity

\[
e^{\mu \nu \rho} \bar{\varepsilon} \gamma^\alpha \psi_\rho \bar{\psi}_\mu \gamma_\alpha \psi_\nu = 0,
\]  
(2.19)

which is valid because of the Fierz identity. For \(d \neq 2, 3\), we can show that the operator (2.17) is invariant under the supertransformations neglecting terms of order \((\text{fermi fields})^3\). The above operator can be considered as a bare operator and we need to consider its renormalization properties. In particular we should consider
the dressing due to the conformal mode. We shall describe the case of the two-dimensional limit in detail in sect. 5.

3. Gauge fixing and quantization

To find one-loop counterterms we shall compute one-loop divergences using the action (2.1) with the bare gravitational constant \( G_0 \) replaced by \( G/\mu^\epsilon \), where \( G \) and \( \mu \) are the renormalized gravitational constant and the renormalization scale respectively. We use the background field method \([17]\), \([18]\). Fields \( \Phi \) are written as a sum of background fields \( \hat{\Phi} \) and quantum fields \( \Phi_q \): \( \Phi = \hat{\Phi} + \Phi_q \). At one-loop level, the effective action \( \Gamma[\hat{\Phi}] \), i.e., the generating functional of one-particle-irreducible diagrams is given by a path integral

\[
e^{i\Gamma[\hat{\Phi}]} = e^{iS[\hat{\Phi}]} \int \mathcal{D}\Phi_q \, e^{iS^{(2)}[\Phi_q; \hat{\Phi}]},
\]

(3.1)

where \( S^{(2)} \) is a part of the action which is quadratic in the quantum fields \( \Phi_q \).

We shall denote the background vielbein as \( \hat{e}_\mu^\alpha \) and use a parametrization for quantum fields to separate conformal mode (trace part) \( \phi \) from nonconformal mode (traceless part) \( h_{\alpha\beta} \) \([5]\)

\[
e_\mu^\alpha = \hat{e}_\mu^\beta (e^{\frac{1}{2\kappa}h})^\alpha_\beta \, e^{-\frac{1}{2\kappa}\phi}, \quad h_{\alpha\beta} \equiv h_{\alpha}^\gamma \eta_{\gamma\beta} = h_{\beta\alpha}, \quad h_{\alpha}^\alpha = 0,
\]

(3.2)

where \( \kappa^2 = 16\pi G/\mu^\epsilon \). We have fixed the local Lorentz symmetry such that \( h_{\alpha\beta} \) is symmetric. The field \( \phi \) is called Liouville field. For the Rarita-Schwinger field, we also introduce a parametrization to separate superconformal mode (spin 1/2 part) \( \eta \) from the nonsuperconformal mode (spin 3/2 part) \( \phi_\alpha \)

\[
\psi_\mu = \kappa e_\mu^\alpha (\phi_\alpha + \gamma_\alpha \eta), \quad \gamma^\alpha \phi_\alpha = 0.
\]

(3.3)

We have chosen background fields other than that of the vielbein to vanish. In terms of these parametrizations, a part of the supergravity action which is quadratic in the quantum fields is given by

\[
S_{SG}^{(2)} = \int d^d x \hat{e} \left[ -\frac{1}{4} \hat{D}_{\mu} h_{\alpha\beta} \hat{D}^\mu h^{\alpha\beta} + \frac{1}{4} (d-2)(d-1) \tilde{g}^{\mu\nu} \partial_\mu \phi \partial_\nu \phi \right]
\]
\[
+ \frac{1}{2} \hat{D}_\alpha h^{\alpha \gamma} \hat{D}_\beta h^{\beta \gamma} - \frac{1}{2} h^{\alpha \beta} h^{\gamma \delta} \hat{R}_{\alpha \beta \gamma \delta} + \frac{1}{8} (d-2)^2 \phi^2 \hat{R} \\
+ \frac{1}{2} (d-2) \phi h^{\alpha \beta} \hat{R}_{\alpha \beta} - \frac{1}{2} (d-2) \phi \hat{D}_\alpha \hat{D}_\beta h^{\alpha \beta} + i \bar{\phi} \hat{D} \phi + \\
+ (d-2) i \left( \bar{\eta} \hat{D}_\alpha \phi - \bar{\phi} \hat{D} \eta \right) - (d-2) (d-1) i \bar{\eta} \hat{D} \eta - \frac{d-2}{d-1} S^2 \right],
\]

where \( \hat{D} = \gamma^\alpha \hat{D}_\alpha \) and we have rescaled the auxiliary field \( S \) by \( \kappa \).

To fix the gauge symmetries of general coordinate and local supersymmetry transformations, we use the method of ref. [20]. In this method the gauge symmetry is fixed by adding a BRST exact term \(-i \delta_B (bF)\) to the lagrangian, where \( b \) is the anti-ghost field and \( F \) is a gauge function. The gauge function \( F \) is an arbitrary function of the fields. This method is equivalent to the procedure of refs. [21], [19]. At one-loop level we can discuss the gauge fixing of general coordinate symmetry and local supersymmetry separately.

To fix general coordinate symmetry we introduce fermionic Faddeev-Popov ghost and anti-ghost fields \( c^\alpha, b_\alpha \) and a bosonic Nakanishi-Lautrup auxiliary field \( B_\alpha \), all of which are vector fields. The BRST transformations of \( h_{\alpha \beta}, \phi, b_\alpha \) and \( B_\alpha \) are given by

\[
\delta_B h_{\alpha \beta} = \hat{D}_\alpha c_\beta + \hat{D}_\beta c_\alpha - \frac{2}{d} \eta_{\alpha \beta} \hat{D}_\gamma c^\gamma + \cdots,
\]

\[
\delta_B \phi = -\frac{2}{d} \hat{D}_\alpha c^\alpha + \cdots,
\]

\[
\delta_B b_\alpha = iB_\alpha, \quad \delta_B B_\alpha = 0,
\]

where the dots represent terms quadratic and higher order in the quantum fields. We will use the following gauge function with two parameters \( \alpha, \beta \) to fix the general coordinate symmetry \( ^* \)

\[
(F_{GC})_\alpha = \hat{e} \left( \hat{D}^\beta h_{\beta \alpha} + \frac{1}{2} \beta \hat{D}_\alpha \phi + \frac{1}{2} \alpha B_\alpha \right).
\]

Then the gauge fixing term and the ghost action are given by

\[
S_{GC} = \int d^d x \delta_B \left[ -i b^\alpha (F_{GC})_\alpha \right]
\]

\[ ^* \text{Our gauge parameters can be compared to those used in other papers such as ref. [2] which is denoted by a suffix KN: } \beta_{ours} = \beta_{KN} d - 2, \alpha_{ours} = 1/\alpha_{KN}. \]
\[ \int d^d x \hat{e} \left[ \frac{1}{2\alpha} B'^\alpha B'_\alpha - \frac{1}{2\alpha} \left( \hat{D}^\beta h_{\alpha\beta} + \frac{\beta}{2} \hat{D}_\alpha \phi \right)^2 + ib^\alpha \left( \hat{D}^\beta (\hat{D}_\alpha c_\beta + \hat{D}_\beta c_\alpha) - \frac{\beta + 2}{d} \hat{D}_\alpha \hat{D}_\beta \phi \right) \right], \] 

(3.7)

where \( B'_\alpha \) is a shifted auxiliary field.

For local supersymmetry gauge fixing, we introduce bosonic Faddeev-Popov ghost and antighost fields \( \gamma, \beta \), and a fermionic Nakanishi-Lautrup field \( B \), all of which are spinor fields. The BRST transformations are

\[
\begin{align*}
\delta_B \phi_\alpha &= \frac{2}{d} ((d - 1) \eta_{\alpha\beta} - \gamma_{\alpha\beta}) \hat{D}^\beta \gamma + \cdots, \\
\delta_B \eta &= \frac{2}{d} \hat{D} \gamma + \cdots, \\
\delta_B \beta &= B, \quad \delta_B B = 0.
\end{align*}
\]

(3.8)

We use the gauge fixing condition with two parameters \( a \) and \( b \)

\[ F_{\text{SUSY}} = 2 \hat{e} \left( \hat{D}_\alpha \phi_\alpha - \hat{\bar{D}} \left( b\eta - \frac{a}{2} B \right) \right). \]

(3.9)

The gauge fixing term and the ghost action are

\[ S_{\text{SUSY}} = \int d^d x \delta_B \left[ -i \bar{\beta} F_{\text{SUSY}} \right] \]

\[ = \int d^d x i \hat{e} \left[ - \frac{1}{a} \bar{\phi}^\alpha \hat{D}_\alpha \hat{\bar{D}}^{-1} \hat{D}^\beta \phi_\beta - \frac{b}{a} (\bar{\eta} \hat{D}_\alpha \phi_\alpha - \bar{\phi}^\alpha \hat{D}_\alpha \eta) + \frac{b^2}{a} \bar{\eta} \hat{\bar{D}} \eta \right. \]

\[ - \frac{4}{d} \bar{\beta} \left( (d - 1) \hat{D}^\alpha \hat{D}_\alpha - \gamma_{\alpha\beta} \hat{D}_\alpha \hat{D}_\beta - b \hat{\bar{D}}^2 \right) \gamma - a B' \hat{\bar{D}} B' + \cdots \],

(3.10)

where we have shifted the Nakanishi-Lautrup field to eliminate mixing with other fields

\[ B' = B + \frac{1}{a} (\hat{D}^{-1} \hat{D}_\alpha \phi_\alpha - b\eta). \]

(3.11)

In our gauge condition, the Nakanishi-Lautrup field \( B' \) is free but is propagating unlike ordinary auxiliary fields, whereas the nonsuperconformal mode (spin 3/2 part) \( \phi_\alpha \) acquires a nonlocal term.

So far we treated the gauge fixing for the general coordinate symmetry and the local supersymmetry separately. Since the gauge fixing condition for one symmetry
can actually violate the other symmetry too, we need to consider the combined
BRST transformation of the total gauge conditions $-i\delta_B \left( b^\alpha (F_{GC})_\alpha + \beta F_{SUSY} \right)$ as the
gauge fixing and ghost terms, where the BRST transformation refers to a combined
BRST transformations of (3.5) and (3.8). The resulting cross terms do not contribute
at one-loop level, since they are higher orders in quantum fields. After the gauge
fixing, the quadratic part of the total action is

$$S_{tot}^{(2)} = S_{SG}^{(2)} + S_{GC}^{(2)} + S_{SUSY}^{(2)}$$

$$= \int d^d x \hat{e} \left[ -\frac{1}{4} \hat{D}_\mu h_{\alpha\beta} \hat{D}^\mu h^{\alpha\beta} + \frac{1}{4} \left( (d - 2)(d - 1) - \frac{\beta^2}{2\alpha} \right) \hat{\gamma}^{\mu\nu} \partial_\mu \phi \partial_\nu \phi 
+ \frac{\alpha - 1}{2\alpha} \hat{D}_\alpha h^{\alpha\gamma} \hat{D}_\beta h^{\beta\gamma} - \frac{1}{2} h^{\alpha\beta} h^{\gamma\delta} \hat{R}_{\alpha\gamma\beta\delta} + \frac{1}{8} (d - 2)^2 \hat{\phi}^2 \hat{R} 
+ \frac{1}{2} (d - 2) \phi \hat{h}^{\alpha\beta} \hat{R}_{\alpha\beta} - \frac{1}{2} \left( (d - 2)(d - 1) - \frac{\beta}{\alpha} \right) \phi \hat{D}_\alpha \hat{D}_\beta h^{\alpha\beta} 
+ i \hat{\phi}^\alpha \hat{D} \phi_\alpha - \frac{1}{a} i \hat{\phi}^\alpha \hat{D}_\alpha \hat{D}^{-1} \hat{D}^\beta \phi_\beta - \left( (d - 2)(d - 1) - \frac{b^2}{a} \right) i \hat{\eta} \hat{D} \eta 
+ \left( d - 2 - \frac{b}{a} \right) i (\hat{\eta} \hat{D}_\alpha \phi^\alpha - \hat{\phi}^\alpha \hat{D}_\alpha \eta) + \frac{1}{2} \alpha B^{\alpha\beta} B'_\alpha - a i \bar{B}' \hat{D} B' 
- \frac{d - 2}{d - 1} S^2 + \text{the ghost terms} \right].$$ (3.12)

We choose the gauge parameters such that propagators have no mixing

$$\langle h_{\alpha\beta}(x)\phi(y) \rangle = 0, \quad \langle \phi_\alpha(x)\bar{\eta}(y) \rangle = 0. \quad (3.13)$$

From eq. (3.12) we find that these conditions require

$$\beta = (d - 2)\alpha, \quad b = (d - 2)a. \quad (3.14)$$

It is most convenient to use propagators in the flat background metric by separating
the background metric into the flat metric and the fluctuations $\hat{h}_{\alpha\beta}, \hat{\phi}$ as

$$\hat{e}_\mu^\alpha = \delta_\mu^\beta (e^{\frac{b}{a}})^\beta_\beta \hat{e}^{-\frac{1}{2}\hat{\phi}} = \delta_\mu^\alpha + \frac{1}{2} \hat{h}_\mu^\alpha - \frac{1}{2} \delta_\mu^\alpha \hat{\phi} + \cdots. \quad (3.15)$$

When the relations (3.14) are satisfied, the propagators on the flat background
metric are given by

$$\langle h_{\alpha\beta}(x)h_{\gamma\delta}(y) \rangle = -i \int \frac{d^d p}{(2\pi)^d} \left[ \eta_{\alpha\gamma} \eta_{\beta\delta} + \eta_{\alpha\delta} \eta_{\beta\gamma} - \frac{2(2 - \alpha)}{2 + 2\epsilon - \alpha \epsilon} \eta_{\alpha\beta} \eta_{\gamma\delta} \right].$$
\(- (1 - \alpha) \frac{\eta_{\alpha\gamma} p_{\beta} p_{\delta} + \eta_{\beta\delta} p_{\alpha} p_{\gamma} + \eta_{\alpha\delta} p_{\beta} p_{\gamma} + \eta_{\beta\gamma} p_{\alpha} p_{\delta}}{p^2}
+ \frac{4(1 - \alpha)}{2 + 2\epsilon - \alpha\epsilon} \frac{\eta_{\alpha\beta} p_{\gamma} p_{\delta} + \eta_{\gamma\delta} p_{\alpha} p_{\beta}}{p^2}
+ \frac{4\epsilon(1 - \alpha)^2}{2 + 2\epsilon - \alpha\epsilon} \frac{p_{\alpha} p_{\beta} p_{\gamma} p_{\delta}}{p^4} \frac{1}{p^2} e^{ip \cdot (x - y)} \),

\langle \phi(x) \phi(y) \rangle = i \int \frac{d^d p}{(2\pi)^d} \frac{4}{\epsilon(2 + 2\epsilon - \alpha\epsilon)} \frac{1}{p^2} e^{ip \cdot (x - y)} , \tag{3.16}

\langle \phi_\alpha(x) \bar{\phi}_\beta(y) \rangle = - \frac{1}{2(1 + \epsilon - a\epsilon)} i \int \frac{d^d p}{(2\pi)^d} \left[ (1 - a) \gamma_{\alpha\gamma} \beta \gamma + a(\gamma_{\alpha} p_{\gamma} + \gamma_{\beta} p_{\alpha}) \right.
+ (a + (1 - a)\epsilon) \eta_{\alpha\beta} p - (4a + \epsilon) \frac{p_{\alpha} p_{\beta}}{p^2} \left] \frac{1}{p^2} e^{ip \cdot (x - y)} , \right.

\langle \eta(x) \bar{\eta}(y) \rangle = \frac{1}{2\epsilon(1 + \epsilon - a\epsilon)} i \int \frac{d^d p}{(2\pi)^d} \frac{p}{p^2} e^{ip \cdot (x - y)} . \tag{3.17}

We see that the propagators of \( \phi \) and \( \eta \) have a factor \( \epsilon^{-1} \). This is due to the vanishing of the supergravity action in \( d = 2 \). In addition, there are gauge parameter dependent factors such as \( 1/(2 + 2\epsilon - \alpha\epsilon) \). They can become singular, since the gauge parameters can depend on \( \epsilon \). It is convenient to choose gauge parameters such that these factors do not diverge in the limit \( \epsilon \to 0 \). This is achieved by choosing \( \alpha \) and \( a \) finite in the limit \( \epsilon \to 0 \). The propagators (3.16) in the gravity sector are simplified by choosing \( \alpha = 1, \beta = \epsilon \). This is the gauge used in ref. 5. The Rarita-Schwinger propagators (3.17) are simplified by choosing \( a = 1, b = \epsilon \). We will use this gauge to compute one-loop divergences of the Rarita-Schwinger field in the next section.

The gauge function of the local supersymmetry (3.9) can be guessed from the supertransformation of the gauge function (3.6) of the general coordinate symmetry. It is useful to consider a rigid supersymmetry in flat space by taking the transformation parameter \( \varepsilon \) in eq. (2.8) to be covariantly constant \( D_{\mu}\varepsilon = 0 \) and to supplement with the transformation for the Nakanishi-Lautrup fields \( \delta B_{\alpha} = -i\varepsilon \partial_{\alpha} B \). Then we find that these two gauge functions are related each other, under the following identification of the gauge parameters

\[ a = \frac{\alpha}{d + 2}, \quad b = \frac{\beta}{d + 2} . \tag{3.18} \]

The conditions (3.14) are consistent with these identifications.
4. One-loop divergences

Let us first illustrate our methods taking the matter supermultiplet as an example. The action for the matter multiplet in eq. (2.13) can be expanded in powers of the quantum fields. In order to compute the one-loop divergences from the matter loop, we only need to consider quadratic terms in propagating quantum fields. By the general covariance for the background field and the dimensional analysis, divergences in the effective action (3.1) are proportional to the Einstein action. To compute the coefficients we expand the background field in powers of fluctuation on the flat metric as in eq. (3.15). We define a unit of divergence as

\[ I = -\frac{1}{24\pi\epsilon} \int d^d x \hat{e} \hat{R} = -\frac{1}{24\pi\epsilon} \int d^d x \left[ -\frac{1}{4} \partial_\mu \hat{h}_{\nu\rho} \partial^\mu \hat{h}^{\nu\rho} + \cdots \right], \tag{4.1} \]

where we have explicitly exhibited a term in the right-hand side which survives when we impose conditions

\[ \partial_\mu \hat{h}^{\mu\nu} = 0, \quad \hat{h}^{\mu\nu} = \hat{h}^{\nu\mu}, \quad \hat{\phi} = 0. \tag{4.2} \]

The coefficients of divergences can be obtained by computing terms quadratic in \( \hat{h}^{\mu\nu} \) which satisfies eq. (4.2). These conditions simplify the loop calculations in many cases.

For the scalar field in the matter supermultiplet, we consider the following decomposition of the action quadratic in the quantum field \( X \)

\[ S_{\text{scalar}}^{(2)} = -\frac{1}{2} \int d^d x \hat{e} \hat{g}^{\mu\nu} \partial_\mu X \partial_\nu X = -\frac{1}{2} \int d^d x \left[ \eta^{\mu\nu} \partial_\mu X \partial_\nu X + S^{\mu\nu} \partial_\mu X \partial_\nu X \right]. \tag{4.3} \]

The first term gives the propagator on the flat background, and the second term gives an interaction with background metric through the vertex \( S^{\mu\nu} \)

\[ S^{\mu\nu} = \hat{e} \hat{g}^{\mu\nu} - \eta^{\mu\nu} = -\hat{h}^{\mu\nu} + O(\hat{h}^2), \tag{4.4} \]

where we have used the conditions (4.2). As shown in Fig. [1], the tadpole diagram does not diverge at one-loop order, whereas the diagram with two \( S \) vertices gives
the following one-loop divergence in the effective action $\Gamma$ defined in (3.1)

$$\Gamma_{\text{scalar}} = 1 \times I.\quad (4.5)$$

Similarly the action quadratic in the Majorana spinor field can be decomposed into free part and interaction vertices

$$S_{\text{spinor}}^{(2)} = \frac{1}{2} \int d^d x i \hat{\bar{\lambda}} \hat{D}\lambda$$

$$= \frac{1}{2} \int d^d x i \left[ \hat{\bar{\lambda}} \hat{\partial} \lambda + S_{\alpha}^{\mu} \hat{\bar{\lambda}} \gamma^{\alpha} \partial_{\mu} \lambda + \frac{1}{4} \Omega_{\alpha\beta\gamma} \hat{\bar{\lambda}} \gamma^{\alpha} \gamma^{\beta\gamma} \lambda \right],\quad (4.6)$$

$$S_{\alpha}^{\mu} = \hat{\epsilon}_{\alpha}^{\mu} - \delta_{\alpha}^{\mu} = -\frac{1}{2} \hat{h}_{\alpha}^{\mu} + O(\hat{h}^2), \quad \Omega_{\alpha\beta\gamma} = \hat{\epsilon}_{\alpha}^{\mu} \hat{\omega}_{\mu\beta\gamma}.\quad (4.7)$$

As shown in Fig. 2, only the diagram with two $S$ vertices gives the one-loop divergence

$$\Gamma_{\text{spinor}} = \frac{1}{2} \times I.\quad (4.8)$$

As another example of our methods, we compute one-loop divergences of a vector gauge field in appendix A.
Next let us consider contributions given by the Rarita-Schwinger field. For simplicity, we choose \( a = 1, b = \epsilon \) gauge. The quadratic part of the action for the Rarita-Schwinger field with the gauge fixing term and the ghost term can be decomposed into free part and interaction vertices

\[
S_{RS}^{(2)} + S_{SUSY}^{(2)} = \int d^d x \left[ \bar{\phi} \gamma^{\alpha} \partial_{\alpha} \phi + \partial_{\alpha} \bar{\phi} \gamma^{\alpha} \partial_{\alpha} \phi - \epsilon \hat{e} \bar{\eta} \hat{D} \eta \right] + S_1 + S_2 + S_{SUSY-\text{FP}},
\]

(4.9)

\[
S_1 = \int d^d x \left[ S_2 \mu \partial_{\mu} \bar{\phi} \gamma^{\alpha} \partial_{\alpha} \phi + \frac{1}{4} \Omega_{\beta\gamma\delta} \bar{\phi} \gamma^{\beta} \gamma^{\gamma} \gamma^{\delta} \phi - \frac{1}{4} \Omega_{\alpha} \gamma^{\alpha} \gamma^{\beta} \phi \right],
\]

(4.10)

\[
S_2 = \int d^d x \left[ (S_2 \mu \partial_{\mu} \bar{\phi} \gamma^{\alpha} \gamma^{\alpha} \gamma^{\beta} \partial_{\alpha} \phi + \frac{1}{4} \Omega_{\alpha} \gamma^{\alpha} \gamma^{\beta} \phi) \left( \frac{1}{\gamma} - \frac{1}{\gamma} V \frac{1}{\gamma} \right) \partial_{\beta} \phi^\beta \right.
\]

\[
+ \partial_{\beta} \bar{\phi} \gamma^{\beta} \left( \frac{1}{\gamma} - \frac{1}{\gamma} V \frac{1}{\gamma} \right) \left( S_2 \mu \partial_{\mu} \bar{\phi} \gamma^{\alpha} \gamma^{\beta} \partial_{\alpha} \phi + \frac{1}{4} \Omega_{\alpha} \gamma^{\alpha} \gamma^{\beta} \phi \right)
\]

\[
\left. + \left( S_2 \mu \partial_{\mu} \bar{\phi} \gamma^{\alpha} \gamma^{\beta} \partial_{\alpha} \phi + \frac{1}{4} \Omega_{\alpha} \gamma^{\alpha} \gamma^{\beta} \phi \right) \right] (S_2 \nu \partial_{\nu} \phi^\beta + \frac{1}{4} \Omega_{\delta} \gamma^{\delta} \phi^\beta),
\]

(4.11)

\[
S_{SUSY-\text{FP}} = \int d^d x \left[ \frac{4}{d} \left( \eta_{\mu \nu} \partial_{\mu} \bar{\beta} \partial_{\nu} \gamma + S_{\mu \nu} \partial_{\mu} \bar{\beta} \partial_{\nu} \gamma \right.ight.
\]

\[
\left. + \frac{1}{4} \Omega_{\mu \alpha \beta} \partial_{\mu} \bar{\beta} \gamma_{\alpha \beta} \partial_{\mu} \gamma - \frac{1}{16} \Omega_{\mu \alpha \beta} \Omega_{\mu \gamma \delta} \bar{\beta} \gamma_{\alpha \beta} \gamma_{\gamma \delta} \right)
\]

\[
\left. - \frac{1}{4} \hat{e} \hat{R} \gamma \right) - i \hat{e} \hat{B} \hat{B} \right],
\]

(4.12)

where we have used the condition (4.2) and

\[
V = (\hat{e}_\alpha^\mu - \delta_\alpha^\mu) \gamma^\alpha \partial_{\mu} + \frac{1}{4} \hat{e}_\alpha^\mu \hat{\omega}_\mu \gamma^\alpha \gamma^\delta.
\]

(4.13)

The action (4.13) shows that the superconformal mode \( \eta \) gives the same contribution as that of the spin \( \frac{1}{2} \) Majorana spinor \( \lambda \)

\[
\Gamma_\eta = \frac{1}{2} \times I.
\]

(4.14)

Diagrams with a \( \phi_\alpha \) internal loop are constructed with the vertices involved in \( S_1 \)
in eq. (4.10) and/or $S_2$ in eq. (4.11). First we consider vertices $S$ and $\Omega$ in $S_1$. As shown in Fig. 3, tadpole diagrams with vertices from $S_1$ do not diverge. The one-loop diagrams with two vertices from $S_1$ diverge individually, but they cancel among them. Therefore we find that the divergences due to diagrams with vertices from $S_1$ cancel

$$\Gamma_{S_1} = 0.$$  \hspace{1cm} (4.15)

We next consider diagrams with vertices from $S_2$. Since $S_2$ contains $\varphi^{-1}$, it gives nonlocal vertices involving $\varphi^{-1}$. Tadpole diagrams with vertices from $S_2$ give divergences as exhibited in Fig. 4. The $\phi_\alpha$-loop diagrams with two vertices from the $S_2$ contribute divergences as shown in Fig. 5. Combining them together we find contributions of $\phi_\alpha$-loop diagrams with vertices from $S_2$ as
\[ S \quad S = -2I \quad \Omega \quad \Omega = 6I \quad V \quad V = \frac{1}{2}I \]

\[ S \quad \Omega = -3I \quad \Omega \quad V = -6I \quad V \quad S = 6I \]

Figure 5: The \( \phi_\alpha \)-loop diagrams with two vertices from \( S_2 \).

\[ S \quad S = 0 \quad \Omega \quad \Omega = 0 \quad S \quad V = 0 \]

\[ S \quad \Omega = -6I \quad \Omega \quad S = -6I \quad \Omega \quad V = 12I \]

Figure 6: The \( \phi_\alpha \)-loop diagrams with one vertex from \( S_1 \) (left vertex) and the other from \( S_2 \) (right vertex).

\[ \Gamma_{S_2} = \frac{1}{2} \times I. \quad (4.16) \]

We also have \( \phi_\alpha \)-loop diagrams with one vertex from \( S_1 \) and the other from \( S_2 \), as shown in Fig. 6. Their divergences cancel each other

\[ \Gamma_{S_1S_2} = 0. \quad (4.17) \]

Therefore we find one-loop divergences from \( \phi_\alpha \)-loop diagrams sum up to

\[ \Gamma_\phi = \Gamma_{S_1} + \Gamma_{S_2} + \Gamma_{S_1S_2} = \frac{1}{2} \times I. \quad (4.18) \]

Let us now discuss the one-loop divergence of ghost loops with vertices from \( S_{\text{SUSY-FF}} \) in eq. (4.12). Tadpole diagrams with \( S, \Omega \) or \( \hat{R} \) vertices gives divergences as shown in Fig. 7. The remaining tadpole diagram with the \( \Omega\Omega \) vertex (the fourth
term in eq. (4.12) should be combined with the one-loop diagram with two $\Omega$ vertices in order to give a generally covariant result, as illustrated in Fig. 7. There are also one-loop diagrams with two vertices from $S_{\text{SUSY-FP}}$. Therefore contributions of loops of the supersymmetry ghosts $\beta, \gamma$ sum up to

$$\Gamma_{\beta\gamma} = 10 \times I. \quad (4.19)$$

Contribution from the Nakanishi-Lautrup field $B'$ is the same as that of the Majorana spinor $\lambda$

$$\Gamma_{\text{NL}} = \frac{1}{2} \times I. \quad (4.20)$$

Combining eqs. (4.14), (4.18), (4.19) and (4.20), we obtain the one-loop divergence from the gravitino-ghost system as

$$\Gamma_{\text{gravitino}} = \Gamma_{\eta} + \Gamma_{\phi} + \Gamma_{\beta\gamma} + \Gamma_{\text{NL}} = \frac{23}{2} \times I. \quad (4.21)$$

Now we turn to discuss the one-loop divergence of graviton loops in our method. For the gauge parameters $\alpha = 1, \beta = \epsilon$, the action for graviton-ghost system contains the following quadratic terms in the quantum fields $h_{\alpha\beta}, \phi$

$$S_{E}^{(2)} + S_{GC}^{(2)} = \int d^dx \left[ -\frac{1}{4} \eta^{\mu\nu} \partial_{\mu} h_{\alpha\beta} \partial_{\nu} h^{\alpha\beta} + \frac{\epsilon(\epsilon + 2)}{8} \eta^{\mu\nu} \partial_{\mu} \phi \partial_{\nu} \phi \right]$$

Figure 7: The loop diagrams of the supersymmetry ghosts $\beta$ and $\gamma$. 
As shown in Fig. 8, tadpole diagram with the vertex $T$ should be combined with the one-loop diagram with two vertices of $\Omega$ to give a general coordinate invariant result. Summing up the diagrams in Fig. 8, we find the contributions from the graviton ($h_{\alpha\beta}, \phi$) one-loop diagrams as

$$\Gamma_{h\phi} = -9 \times I. \quad (4.26)$$

Similarly contributions from the ghosts $b^\alpha$ and $c_\alpha$ for the general coordinate symmetry are given as shown in Fig. 8 as

$$\Gamma_{bc} = -16 \times I. \quad (4.27)$$
There is no contribution from the auxiliary field $B'^\alpha$. The graviton and the general coordinate ghosts add up to give

$$\Gamma_{\text{graviton}} = \Gamma_{h\phi} + \Gamma_{bc} = -25 \times I.$$  \hspace{1cm} (4.28)

This result agrees with that in ref. [4]. By combining eqs. (4.21) and (4.28) with the supersymmetric matter multiplets $(X^i, \lambda^i)$ $(i = 1, \cdots, \hat{c})$ in eqs. (4.5) and (4.8), we find the total one-loop divergence as

$$\Gamma_{\text{total}} = -\frac{3}{2}(9 - \hat{c}) I.$$  \hspace{1cm} (4.29)

This result may be expected from the conformal anomaly for the supergravity coupled to the supersymmetric matter multiplets in two dimensions.

\section{5. Fixed point and two-dimensional limit}

In order to define a renormalized gravitational coupling constant $G$, we need to specify a reference scale for the metric $g_{\mu\nu}$ [4]. The choice of the reference scale affects the definition of the renormalized coupling constant and consequently the associated beta function in general. In order to obtain a scale invariant result in the limit of two dimensions, it is most appropriate to choose the reference scale as the coefficient for a spinless operator $\Psi$ which has the canonical dimension $2\Delta_0 = 1$ in two dimensions [4]. It has been found that such a reference operator does not suffer from divergences at one-loop order in the parametrization (3.2) [5]. This fact allows
us to renormalize the gravitational coupling constant alone without considering the
renormalization of the reference operator $\Psi$, since it is automatically renormalized in
our parametrization at least at one-loop order. Therefore the bare coupling constant
$G_0$ is related to the renormalized coupling constant $G$ through

$$
\frac{1}{16\pi G_0} = \frac{\mu^\epsilon}{16\pi} \left(\frac{1}{G} - \frac{9 - \hat{c}}{\epsilon}\right),
$$

(5.1)

The second term in the right hand side is the one-loop counter term which cancels
the one-loop divergence (4.29).

The beta function defined by $\beta(G) = \mu \frac{\partial}{\partial \mu} G$ is determined by $\mu \frac{\partial}{\partial \mu} G_0 = 0$

$$
\beta(G) = \epsilon G - (9 - \hat{c})G^2.
$$

(5.2)

We see that quantum supergravity in $2 + \epsilon$ dimensions exhibits a nontrivial ultraviolet
fixed point $G^*$ at

$$
G^* = \frac{\epsilon}{9 - \hat{c}}.
$$

(5.3)

Let us now turn to consider renormalizing the physical operators which are ob-
tained in sect. 2. We are especially interested in taking the two-dimensional limit.
Following ref. [3], the two-dimensional theory can be obtained as a limit $\epsilon \to 0$ of
the $2 + \epsilon$ gravity in a strong coupling region $G \gg \epsilon$. In that limit, one can obtain the
anomalous dimension in powers of $1/(25 - c)$. Moreover, by assuming the dominance
of conformal mode, one can derive a result which exactly reproduces the conformal
field theory approach [8]. We can apply their reasoning to our supergravity case as
well. In this paper, we restrict ourselves to operators $\Psi_{NS}$ in the Neveu-Schwarz
sector.

Although we have obtained the bare physical operators (2.17), which are in-
variant under all the local gauge symmetries classically, we need to find an extra
dependence on the supergravity multiplet induced by quantum effects of the matter
fields. As an example let us consider a physical operator for the matter operator
$e^{ip \cdot X}$ in the bosonic theory. Classically, the integrand of the physical operator is
$e^{e^{ip \cdot X}} = e^{-\frac{\Delta_0}{2}(1 - \Delta_0)} e^{ip \cdot X}$. Quantum effects of the matter field $X$ changes the conformal
dependence to $e^{-\frac{\Delta_0}{2}(1 - \Delta_0)}$, where $2\Delta_0 = p^2$ is the conformal dimension of the
matter operator [23].
To find out the gravitational dressing in the supersymmetric case we use the superfield formalism. Superfields $X(x, \theta)$ for a matter supermultiplet and $\Phi(x, \theta)$ for the (super)conformal modes can be written in terms of a Grassmann number spinor $\theta$ [13]

\[ X(x, \theta) = X(x) - i\theta \lambda(x) - \frac{1}{2} i\bar{\theta} F(x), \]
\[ \Phi(x, \theta) = \phi(x) - i\bar{\theta} \eta(x) + \frac{1}{d} i\bar{\theta} S'(x). \]  

(5.4)

The auxiliary field $S'$ is related to the auxiliary field $S$ in eq. (2.1) as

\[ S' = S + \frac{1}{8\kappa_0} i\bar{\psi}_\mu \gamma^\nu \gamma^\rho \psi_\nu, \]
\[ = S + \frac{1}{4} i\kappa_0 \bar{\phi}_\alpha \phi^\alpha + \frac{1}{8} i d(d-2) \kappa_0 \bar{\eta} \eta. \]  

(5.5)

As in ref. [3] we use the supergravity action with the bare gravitational constant $G_0$ to compute the anomalous dimension of the operators. The fields $\phi, \phi_\alpha$ and $\eta$ in eq. (5.4) are defined as in eqs. (3.2) and (3.3) with $\kappa$ replaced by $\kappa_0 = \sqrt{16\pi G_0}$. The field $S$ has also been rescaled by $\kappa_0$. The physical operator in eq. (2.17) is given in superspace

\[ O_p = \int d^d x d^2 \theta \hat{e} e^{-\frac{i}{2\kappa_0} \Phi(x, \theta)} e^{ip \cdot X(x, \theta)}, \]  

(5.6)

We now quantize the matter fields and find the conformal dimension $2\Delta_0$ for the matter part of the operator. In the case of the momentum eigenstate (5.6), we find that $\Delta_0 = \frac{1}{2} p^2$. For the spinless operator, it is enough to consider the dressed operator by multiplying the appropriate factor of the exponential of the superfield $\Phi(x, \theta)$ for the conformal mode in eq. (5.4) \[ O_p \text{dressed}(x) = \int d^2 \theta \hat{e} e^{-\frac{i}{2(1-\Delta'_0)} \kappa_0 \Phi(x, \theta)} e^{ip \cdot X(x, \theta)} \]
\[ = \hat{e} e^{-\frac{i}{4(1-\Delta'_0)} \kappa_0 \phi} \left[ i p \cdot \bar{\lambda} \lambda \cdot p - 2 i p \cdot F - 2(1 - \Delta'_0) \kappa_0 \bar{\eta} \eta \right] e^{ip \cdot X} + \frac{d^2}{4} i (1 - \Delta'_0)^2 \kappa_0 \bar{\eta} \eta - d(1 - \Delta'_0) \kappa_0 \bar{\eta} \lambda \cdot p \right] e^{ip \cdot X}, \]  

(5.7)

where

\[ \Delta'_0 = \frac{1}{2} + \Delta_0, \quad \Delta_0 = \frac{1}{2} p^2. \]  

(5.8)

The conformal dimension $\Delta'_0$ of the operator $O_p(x)$ is $\frac{1}{2}$ larger than the conformal dimension $\Delta_0$ for the operator in superspace because of the $\theta$ integration.

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To compute one-loop renormalization due to quantum effects of the supergravity multiplet, we give nonzero background values to the matter fields $\hat{X} \neq 0$, $\hat{\lambda} \neq 0$, $\hat{\psi} \neq 0$, $\hat{S} = -i\kappa_0 \bar{\hat{\lambda}} \hat{\lambda}$.

The background of $S$ has been chosen such that it satisfies the equation of motion. In this background the dressed operator (5.7) becomes

$$\hat{O}^{\text{dressed}}_p(x) = \hat{e} \left[ i p \cdot \bar{\hat{\lambda}} \cdot p - 2 i \bar{\hat{p}} \cdot \hat{\tilde{F}} + \frac{1}{4} i (1 - \Delta_0') \kappa_0^2 \bar{\hat{\lambda}} \hat{\lambda} \right] e^{ip \cdot \hat{X}}. \quad (5.9)$$

We have to compute quantum effects of the supergravity multiplet by introducing their fluctuations $h_{\alpha \beta}$, $\phi$, $\phi_\alpha$, $\eta$. We only need to compute $O(\epsilon^{-2})$ singularities since $\kappa_0^2 = O(\epsilon)$ as we will see below. At one-loop order the expectation value of the dressed operator (5.7) becomes

$$\langle O^{\text{dressed}}_p(x) \rangle = \left[ 1 + \frac{(1 - \Delta_0')^2 \kappa_0^2}{2\pi \epsilon^2} + O(\epsilon^{-1}) \right] \hat{O}^{\text{dressed}}_p(x), \quad (5.10)$$

The divergence can be removed by defining a renormalized operator

$$O^{\text{ren}}_p(x) = Z_{\Delta_0} O^{\text{dressed}}_p(x), \quad Z_{\Delta_0} = 1 - \frac{(1 - \Delta_0')^2 \kappa_0^2}{2\pi \epsilon^2} \mu^\epsilon. \quad (5.11)$$

Consequently the anomalous dimension is given by

$$\gamma_{\Delta_0} = \mu \frac{\partial}{\partial \mu} \ln Z_{\Delta_0} = -8(1 - \Delta_0')^2 \frac{G_0 \mu^\epsilon}{\epsilon}. \quad (5.12)$$

If we consider the strong coupling region $G \gg \epsilon$,

$$\frac{1}{G_0 \mu^\epsilon} = \frac{1}{G} - \frac{9 - \hat{c}}{\epsilon} \approx -\frac{9 - \hat{c}}{\epsilon}, \quad (5.13)$$

we obtain the anomalous dimension to the first order in $1/(9 - \hat{c})^2$

$$\gamma_{\Delta_0} = \frac{4(1 - \Delta_0')^2}{Q^2} = \frac{(1 - 2\Delta_0)^2}{Q^2}. \quad (5.14)$$
where $Q$ is the source charge for the Liouville field in the case of superconformal field theory

$$Q = \sqrt{\frac{9 - \hat{c}}{2}}. \quad (5.15)$$

In the conformal field theory approach, the gravitational dressing of the operator with the conformal dimension $(\Delta_0, \Delta_0)$, $2\Delta_0 = \frac{1}{2}p^2$ for the matter part

$$\int d^2x d^2\theta \hat{E} e^{i\Phi} e^{ip \cdot X(x, \theta)} \quad (5.16)$$

is determined by requiring that they must have conformal dimension $(\frac{1}{2}, \frac{1}{2})$.

$$\Delta_0 - \frac{1}{2}\beta(\beta + Q) = \frac{1}{2}. \quad (5.17)$$

The choice of the two solution can be made by resorting to the classical limit

$$\beta = -\frac{Q}{2} \left(1 - \sqrt{1 - \frac{4(1 - 2\Delta_0)}{Q^2}}\right). \quad (5.18)$$

We can compare our result with the conformal field theory approach by using the scaling argument. Let us call the dressing exponent $\beta$ for the operator with the conformal dimension $(0, 0)$ as $\alpha$. The scaling exponent of the operator insertion $O_p$ is given by the ratio between the dressing exponent $\beta$ and $\alpha$

$$\frac{\beta}{\alpha} = \frac{\frac{1}{2} - \Delta_0 + \frac{1}{2}\left(\frac{1}{4} - \Delta_0\right)^2 + O\left(\frac{4}{Q^2}\right)^2 + O\left(\frac{4}{Q^2}\right)^4}{\frac{1}{2} + \frac{4}{8Q^2} + O\left(\frac{4}{Q^2}\right)^2}. \quad (5.19)$$

The scaling exponent can be given in terms of the anomalous dimension $\gamma_{\Delta_0}$ as

$$\frac{\beta}{\alpha} = \frac{2\left(\frac{1}{2} - \Delta_0\right) + \gamma_{\Delta_0}}{1 + \gamma_{\Delta_0=0}}. \quad (5.20)$$

Our result in eq. (5.14) is nothing but the first nontrivial term of the expansion in powers of $1/Q^2$. Moreover, we can show that the result of the conformal field theory
approach can be reproduced to all orders of $1/Q^2$ in a way precisely analogous to ref. \cite{5}. In renormalizing the operators, we see that the fields other than the conformal mode $\phi$ does not play a role. Let us assume that the divergences at higher orders are also dominated by the one-loop counter term in the bare lagrangian and consider only the conformal mode. Then the divergent part can be recast into a zero-dimensional path-integral. We find that the argument in ref. \cite{5} is valid in our case with the replacement of $1 - \Delta_0$ by $1 - \Delta'_0$ and $Q^2 = (25 - c)/3$ by $Q^2 = (9 - \hat{c})/2$. Therefore we see that our result is fully consistent with the conformal field theory approach.

6. A method of dimensional reduction

So far we have been discussing $d = 2 + \epsilon$ dimensional supergravity using the action (2.1) interpolating between two- and three-dimensional ones. A problem of this approach is that the action is invariant under the local supersymmetry transformations only up to terms of order (fermi fields)$^3$ in general $d$ dimensions. This noninvariance is due to the fact that the bosonic and fermionic fields have different numbers of components in general $d$ dimensions while supersymmetry requires the same number. Since these noninvariant terms can only affect higher loop orders, our computations of divergences should be valid at one loop level. Nevertheless, it is desirable to have a manifestly supersymmetric regularization method in noninteger dimensions.

Another way to consider supergravity in $2 + \epsilon$ dimensions is to use a method of dimensional reduction \cite{22}. This method was proposed by Siegel \cite{15} as a regularization which preserves rigid supersymmetry for four-dimensional theories. In this regularization one starts from a supersymmetric theory in $D = 4$ dimensions and supposes that fields depend on only $4 - \epsilon$ coordinates while keeping the number of components of the fields unchanged. In contrast to the ordinary dimensional regularization \cite{24}, one may hope that supersymmetry is preserved in the resulting $d = 4 - \epsilon$ dimensional theory since the number of components of bosonic and fermionic fields remain the same.

We can use this idea of dimensional reduction to construct a theory in $d = 2 + \epsilon$ dimensions starting from the $D = 2$ or $D = 3$ dimensional supergravity. We denote the $D$-dimensional fields as $E_M^A$, $\Psi_M (= E_M^A \Psi_A)$, $\hat{S}$, where $M, N, \cdots =$
0, 1, · · · , D − 1 and A, B, · · · = 0, 1, · · · , D − 1 are D-dimensional world indices and local Lorentz indices respectively. The spinor fields are two-component ones. To distinguish quantities in D dimensions and those in d dimensions, we put a hat on D-dimensional ones, if necessary. The action of these fields has the form in eq. (2.1) and is shown to be invariant under the supertransformations (2.8) using the D-dimensional gamma matrix identities. We split the indices as \( M = (\mu, m) \) and \( A = (\alpha, a) \), where \( \mu, \alpha = 0, \cdots, d - 1 \) and \( m, a = d, \cdots, D - 1 \). To reduce the D-dimensional theory to \( d = 2 + \epsilon \) we suppose that all the fields are independent of \( D - d \) coordinates \( x^m \) and parametrize them as

\[
E^A_M = \Delta^{1\over (D-2)2} \left( \begin{array}{c} e_\mu^\alpha A_{\mu}^m e_m^a \\ 0 \\ e_m^a \end{array} \right),
\]

\[
\Psi_A = \Delta^{1\over (D-2)} \left( \psi_A + i{\bar{\gamma}}_{\alpha}^b \psi_b \right),
\]

\[
\hat{S} = \Delta^{1\over (D-2)} S, \quad \Delta \equiv (\det \ e_m^a)^2. \tag{6.1}
\]

We have used a part of the D-dimensional local Lorentz symmetry to put \( E^A_m = 0 \). We have also made (super) Weyl rescalings to obtain the standard kinetic terms in the d-dimensional action. The field content of the d-dimensional theory is a vielbein \( e_\mu^\alpha \), \( D - d \) vector fields \( A_{\mu}^m \), \( (D - d)^2 \) scalar fields \( e_m^a \), a Rarita-Schwinger field \( \psi_\mu = e_\mu^\alpha \psi_\alpha \), \( D - d \) spin 1/2 spinor fields \( \psi_a \) and a scalar field \( S \). Because of the local symmetry SO\((D - d)\), which is a part of the D-dimensional local Lorentz symmetry, not all components of \( e_m^a \) are physical. There are only \( (D - d)(D - d + 1)/2 \) physical scalar fields in \( e_m^a \). The spinor fields \( \psi_\mu \) and \( \psi_a \) remain two-component. The action and the supertransformations in d dimensions can be obtained from eqs. (2.1) and (2.8). We give the results for them in appendix B. One can also introduce matter fields in d dimensions by dimensionally reducing D-dimensional matter multiplets (2.13). The (super) Weyl rescalings in eq. (6.1) is singular at \( D = 2 \). This is a reflection of the fact that the Weyl rescaling in D dimensions cannot absorb the spacetime dependent factor multiplying the Einstein term. This singular behavior can be avoided if we start from \( D = 3 \), for instance. However, we still have a singularity at \( d = 2 \). For \( D = 3, d = 2 \), eq. (6.1) gives \( E^a_m = 1, \Psi_m = 0 \) and these degrees of freedom are not represented by the d-dimensional fields. Therefore, it is better to use a parametrization without (super) Weyl rescalings for \( d = 2 \) case. The resulting two-dimensional theory turns out to be a supersymmetric version of so-called dilaton gravity [25]. We present the action and the supertransformations in
this case in appendix C.

It was later pointed out that the regularization by dimensional reduction is mathematically inconsistent \[16\]. The inconsistency discussed in ref. \[16\] uses the antisymmetric epsilon tensor \( \epsilon_{\mu\nu\rho\sigma} \). One might think that this is due to a difficulty to define chiral quantities such as the epsilon tensor and \( \gamma_5 \) in general non-integer dimensions. Such a difficulty is well known in the ordinary dimensional regularization \[24\]. Chiral quantities are essential for four-dimensional theories since the definition of chiral scalar supermultiplets uses \( \gamma_5 \). Therefore one might hope that the regularization by dimensional reduction could be consistent in vector like theories, such as supergravity theories we are considering, which do not use chiral quantities. Unfortunately, this is not the case as we will show below.

Let us consider a reduction from \( D = 3 \) to \( d \) dimensions. According to the definition of this regularization the gamma matrices \( \gamma^A = (\gamma^\alpha, \gamma^a) \) are \( 2 \times 2 \) matrices and satisfy

\[
\{\gamma^A, \gamma^B\} = 2\eta^{AB},
\]

\[
\gamma^{A_1 \cdots A_n} = 0 \quad (n > 3).
\]

These equations are also required to prove supersymmetry of the action. Let us assume

\[
\eta_{\alpha\beta} \eta^{\beta\gamma} = \delta^\gamma_\alpha, \quad \delta^\alpha_\alpha = d,
\]

(6.4)

and define \( \gamma_\alpha \equiv \eta_{\alpha\beta} \gamma^\beta \). Since \( \gamma^{\alpha\beta} \gamma^{cd} = \gamma^{\alpha\beta cd} \), from eq. (6.3) we have

\[
\gamma^{\alpha\beta} \gamma^{cd} = 0.
\]

(6.5)

Multiplying this equation by \( \gamma_{\alpha\beta} \) and \( \gamma_{cd} \), and using eqs. (6.2) and (6.4), we obtain

\[
0 = \gamma_{\alpha\beta} \gamma^{\alpha\beta} \gamma^{cd} \gamma_{cd} = d(d-1)(d-2)(d-3).
\]

(6.6)

Therefore this regularization can be consistent only for integer dimensions \( d = 0, 1, 2, 3 \).

The essential point of this discussion is eq. (6.3). To emphasize this point let us consider another difficulty. From eqs. (6.2), (6.4) we can show that

\[
\gamma_{\alpha} \gamma^{\alpha\beta_1 \cdots \beta_n} = (d - n) \gamma^{\beta_1 \cdots \beta_n}.
\]

(6.7)
When $d$ is not an integer, using eqs. (6.3), (6.7) we find that $\gamma^{\alpha_1 \cdots \alpha_n} = 0$ for all non-negative integers $n$. In particular, we have $\gamma^\alpha = 0$.

In the above discussions we have not used the antisymmetric epsilon tensor or $\gamma^5$. The above difficulties do not arise in the ordinary dimensional regularization [24] since eq. (6.3) need not be satisfied. If one drops the requirement (6.3), however, supersymmetry of the $(2 + \epsilon)$-dimensional theory will not be guaranteed. To construct supergravity in $2 + \epsilon$ dimensions in this approach, one has to find out a modification of the regularization by dimensional reduction such that it is consistent and preserves supersymmetry.

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**Appendix A. One-loop divergences of a vector field**

In this appendix we compute one-loop divergences of a vector gauge field $A_\mu$ in a background gravitational field $\hat{g}_{\mu\nu}$. The gauge invariant action is

\[ S_V = -\frac{1}{4} \int d^d x \hat{e} \hat{g}^{\mu\rho} \hat{g}^{\nu\sigma} F_{\mu\nu} F_{\rho\sigma}, \]

\[ = -\frac{1}{2} \int d^d x \hat{e} \left[ \hat{g}^{\mu\nu} \hat{D}_\mu A_\nu \hat{D}^\rho A_\rho - (\hat{D}^\mu A_\mu)^2 + \hat{R}^{\mu\nu} A_\mu A_\nu \right]. \quad \text{(A.1)} \]

To fix the gauge symmetry we introduce Faddeev-Popov (anti-)ghost fields $b, c$ and a Nakanishi-Lautrup auxiliary field $B$. Their BRST transformations are

\[ \delta_B A_\mu = \partial_\mu c, \quad \delta_B c = 0, \quad \delta_B b = i B, \quad \delta_B B = 0. \quad \text{(A.2)} \]
We use a gauge function
\[ F_{YM} = \hat{e} \left( \hat{D}^\mu A_\mu + \frac{1}{2} \xi B \right), \] (A.3)
where \( \xi \) is a constant gauge parameter. Then, the gauge fixing term and the ghost action are
\[ S_{YM} = \int d^d x \delta_B (-ib F_{YM}) = \int d^d x \hat{e} \left[ \frac{1}{2} \xi B^2 - \frac{1}{2\xi} (\hat{D}^\mu A_\mu)^2 + ib \hat{D}^\mu \partial_\mu c \right], \] (A.4)
where \( B' \) is a shifted auxiliary field. The total gauge fixed action is a sum of eqs. (A.1) and (A.4): \( S_{tot} = S_V + S_{YM} \). We choose the gauge parameter as \( \xi = 1 \), which simplifies loop calculations.

To compute one-loop counterterms we expand the background gravitational field as in eq. (3.15). The total action is decomposed as
\[ S_{tot} = -\frac{1}{2} \int d^d x \left[ \eta^{\mu\nu} \partial_\mu A_\alpha \partial_\nu A^\alpha + S^{\mu\nu} \partial_\mu A_\alpha \partial_\nu A^\alpha + 2 \Omega^{\mu\alpha\beta} A_\beta \partial_\mu A_\alpha + T'^{\alpha\beta} A_\alpha A_\beta - 2i \eta^{\mu\nu} \partial_\mu b \partial_\nu c - 2i S^{\mu\nu} \partial_\mu b \partial_\nu c + \frac{1}{2} \hat{e} B^2 \right], \] (A.5)
where we have used a field with a local Lorentz index \( A_\alpha = \hat{e}_\alpha^\mu A_\mu \). The functions \( S^{\mu\nu}, \Omega^{\mu\alpha\beta} \) are defined in eqs. (4.4), (4.25) and
\[ T'^{\alpha\beta} = \hat{e} \left( \hat{R}^{\alpha\beta} + \hat{\omega}_{\mu\gamma}^{\alpha} \hat{\omega}^{\mu\gamma\beta} \right). \] (A.6)
Divergent \( A_\alpha \) one-loop diagrams are shown in Fig. [10]. The tadpole diagram with the vertex \( T' \) should be combined with the diagram with two \( \Omega \) vertices to give a general coordinate invariant result. The contribution from the ghost loop diagrams

Figure 10: The \( A_\alpha \)-loop diagrams.
is the same as that of a complex scalar field of fermionic statistics and is given by $-2I$. Therefore the total one-loop divergence of a vector gauge field is

$$\Gamma_{\text{vector}} = -6 \times I.$$  \hfill (A.7)

This result is consistent with eq. (16.75) in ref. [1].

**Appendix B. Dimensional reduction to \(d\) dimensions**

In this appendix we work out a dimensional reduction of supergravity in \(D\) dimensions to \(d\) dimensions. The \(D\)-dimensional fields are taken to be independent of \(D-d\) coordinates \(x^m\) and are parametrized by \(d\)-dimensional fields as in eq. (6.1). We obtain the \(d\)-dimensional action up to four-fermi terms and the supertransformations to this order.

It is convenient to describe the \(d\)-dimensional scalar fields \(e_m{}^a\) as a \(G/H\) nonlinear \(\sigma\)-model, where \(G = \text{GL}(D-d)\) and \(H = \text{SO}(D-d)\). The scalar fields \(e_m{}^a(x) \in G\) transform under \(G_{\text{rigid}} \times H_{\text{local}}\) as

$$e_m{}^a(x) \rightarrow L_m{}^n(x) e_n{}^b(x) O_b{}^a(x), \quad L \in G, \quad O(x) \in H.$$  \hfill (B.1)

We decompose a derivative of the scalar fields into two parts:

$$e_a{}^m \partial_\mu e_{mb} = P_{\mu ab} + Q_{\mu ab},$$

$$P_{\mu ab} \equiv \frac{1}{2}(e_a{}^m \partial_\mu e_{mb} + e_b{}^m \partial_\mu e_{ma}) = P_{\mu ab},$$

$$Q_{\mu ab} \equiv \frac{1}{2}(e_a{}^m \partial_\mu e_{mb} - e_b{}^m \partial_\mu e_{ma}) = -Q_{\mu ba}. \hfill (B.2)$$

\(Q_{\mu ab}\) is in the Lie algebra of \(H\), while \(P_{\mu ab}\) is in the orthogonal complement of \(H\) in \(G\). They are invariant under \(G_{\text{rigid}}\). \(Q_{\mu ab}\) transforms as an \(H\) gauge field under \(H_{\text{local}}\) and can be used to define covariant derivatives. \(P_{\mu ab}\) can be expressed as a covariant derivative of \(e_m{}^a\):

$$P_{\mu ab} = e_a{}^m (\partial_\mu e_{mb} + Q_{\mu b c} e_{mc}) \equiv e_a{}^m D_\mu e_{mb} \hfill (\text{B.3})$$

30
and transforms covariantly under $H_{\text{local}}$. The kinetic terms of the scalar fields will be written by using $P_{\mu ab}$. The torsionless spin connection in $D$ dimensions $\hat{\omega}_{ABC} = E_{A}^{M} \hat{\omega}_{MBC}(E)$ becomes

$$\hat{\omega}_{\alpha \beta \gamma} = \Delta^{\frac{1}{2}(D-2)} \left[ \omega_{\alpha \beta \gamma} - \frac{1}{D-2} (\eta_{\alpha \beta} e_{\gamma}^{\mu} - \eta_{\alpha \gamma} e_{\beta}^{\mu}) P_{\mu d} \right],$$

$$\hat{\omega}_{\alpha \beta c} = \frac{1}{2} \Delta^{\frac{1}{2}(D-2)} F_{a \beta}^{m} e_{mc},$$

$$\hat{\omega}_{abc} = \Delta^{\frac{1}{2}(D-2)} Q_{abc},$$

$$\hat{\omega}_{\alpha \beta \gamma} = -\frac{1}{2} \Delta^{\frac{1}{2}(D-2)} F_{m}^{\beta \gamma} e_{ma},$$

where $\omega_{\alpha \beta \gamma} = e_{\alpha}^{\mu} \omega_{\mu \beta \gamma}(\epsilon)$ is the torsionless spin connection in $d$ dimensions defined by $e_{\mu \alpha}$.  

Using the above formulae the $D$-dimensional action of the form (2.1) becomes

$$S_{SG} = \frac{1}{16\pi G_{0}} \int d^{D}x \left[ R - \frac{1}{4} g_{mn} F_{\mu \nu}^{m} F_{\mu \nu}^{n} - P_{\mu ab} P_{\mu ab} + \frac{1}{D-2} (P_{\mu a}^{\, a})^{2} \right.$$

$$+ i \bar{\psi}_{a} e_{\alpha}^{\beta} \gamma^{\beta} D_{\beta} \psi_{a} + i \bar{\psi}_{a} e_{\alpha}^{\beta} D_{\beta} \psi^{a} + \frac{1}{D-2} i \bar{\psi}_{a} e_{\alpha}^{\beta} \gamma^{\beta} \gamma^{c} D_{\beta} \psi_{c} + \frac{1}{8} i F_{a \beta}^{m} e_{ma} \times \left( \bar{\psi}_{a} e_{\alpha}^{\beta} \gamma^{\beta} \gamma_{\alpha} D_{\beta} \psi_{a} + \bar{\psi}_{a} e_{\alpha}^{\beta} \gamma^{\beta} \psi_{a} + \bar{\psi}_{b} e_{\alpha}^{\beta} \gamma^{\beta} \gamma_{\alpha} D_{\beta} \psi_{b} + \frac{1}{D-2} \bar{\psi}_{b} e_{\alpha}^{\beta} \gamma^{\beta} \gamma_{\alpha} \psi_{b} \right)$$

$$- i P_{a bc} \bar{\psi}_{a} e_{\alpha}^{\beta} \gamma^{\beta} \gamma_{\alpha} \psi_{c} + \frac{1}{D-2} i P_{a bc} \bar{\psi}_{c} e_{\alpha}^{\beta} \gamma^{\beta} \gamma_{\alpha} \psi_{b} - \frac{D-2}{D-1} S^{2} + O(\psi^{4}) \right],$$

where $g_{mn} = e_{m}^{\, a} e_{n}^{\, b} \delta_{ab}$. The covariant derivatives on the spinor fields are given by

$$D_{\mu} \psi_{a} = \left( \partial_{\mu} + \frac{1}{4} \omega_{\mu \beta} \gamma_{\beta \gamma} + \frac{1}{4} Q_{\mu \beta}^{bc} \gamma_{bc} \right) \psi_{a} + \omega_{\mu a} \beta \psi_{a} \beta,$$

$$D_{\mu} \psi^{a} = \left( \partial_{\mu} + \frac{1}{4} \omega_{\mu \beta} \gamma_{\beta \gamma} + \frac{1}{4} Q_{\mu \beta}^{bc} \gamma_{bc} \right) \psi^{a} + Q_{\mu a}^{\, b} \psi_{b}. \quad \text{ (B.6)}$$

The coefficient of each term of the action is $d$-independent.

Supertransformations in $d$ dimensions is defined as a sum of $D$-dimensional supertransformations (2.8) and particular local Lorentz transformations:

$$\delta_{Q}(\epsilon) = \delta_{Q}(\eta) + \delta_{L}(\lambda(\epsilon)), \quad \eta = \Delta^{\frac{1}{2}(D-2)} \epsilon,$$
\[ \lambda_{\alpha\beta}(\varepsilon) = -\frac{1}{D - 2} i \bar{\varepsilon} \gamma_{\alpha\beta} \gamma^c \psi_c, \]
\[ \lambda_{ab}(\varepsilon) = -\lambda_{ba}(\varepsilon) = \bar{\varepsilon} \gamma_\alpha \left( \psi_b - \frac{1}{D - 2} \gamma_{b\gamma} \gamma^c \psi_c \right), \]
\[ \lambda_{ab}(\varepsilon) = -\frac{1}{D - 2} i \bar{\varepsilon} \gamma_{ab} \gamma^c \psi_c. \] (B.7)

The parameter \( \lambda_{ab} \) has been chosen to preserve the condition \( E_m^a = 0 \), while \( \lambda_{\alpha\beta}, \lambda_{ab} \) have been chosen to simplify transformations of \( e_\mu^a, e_m^a \) respectively. We obtain the dimensionally reduced supertransformations as

\[ \delta Q e_\mu^a = -i \bar{\varepsilon} \gamma^a \psi_\mu, \]
\[ \delta Q A_\mu^m = -i \bar{\varepsilon} \gamma_\mu \psi_\mu e_a^m - i \bar{\varepsilon} \gamma^a \psi_e_a^m, \]
\[ \delta Q e_m^a = -i \bar{\varepsilon} \gamma^a \psi_b e_m^b, \]
\[ \delta Q \psi_\mu = 2D_\mu \varepsilon + \frac{D - 2}{(d - 2)(D - 1)} S \gamma_\mu \varepsilon \]
\[ + \frac{1}{2} F_{\alpha\beta}^m e_{mc} \left( e_\mu^{a\gamma} \gamma^c - \frac{1}{2(d - 2)} \gamma_\mu \gamma_\alpha \gamma^c \right) \gamma^c \varepsilon + O(\psi^2), \]
\[ \delta Q \psi_a = P_{\beta ac} \gamma^c \gamma^\beta \varepsilon \]
\[ + \frac{D - 2}{(d - 2)(D - 1)} S \gamma_a \varepsilon \]
\[ - \frac{1}{4} \left( \delta_a^b + \frac{1}{d - 2} \gamma_a \gamma^b \right) F_{\beta a}^m e_{mb} \gamma^\beta \varepsilon + O(\psi^2), \]
\[ \delta Q S = \frac{1}{2} i S \bar{\varepsilon} \gamma^a \psi_a - \frac{1}{2} i \bar{\varepsilon} \gamma_\alpha \gamma^a \psi_\alpha \gamma_\beta \psi_\beta + \frac{1}{D - 2} i \bar{\varepsilon} \gamma^a \gamma^b D_\alpha \psi_b \]
\[ + \frac{1}{8} i F_{\alpha\beta}^m e_{mc} \left( \gamma_\alpha \gamma^b \gamma^c \psi_\delta - \frac{D - 3}{D - 2} \varepsilon \gamma_\alpha \gamma^c \psi_\delta + \frac{1}{D - 2} \varepsilon \gamma_\alpha \gamma^c \gamma^{cd} \psi_d \right) \]
\[ - \frac{1}{2(D - 2)} i P_{\alpha c} \epsilon \bar{\varepsilon} \gamma^c \gamma^\alpha \psi_\beta - \frac{1}{2} i P_{arb} \bar{\varepsilon} \gamma^\alpha \gamma^b \psi_c \]
\[ + \frac{1}{2(D - 2)} i P_{ac} \epsilon \bar{\varepsilon} \gamma^c \gamma^\alpha \psi_b + O(\psi^3), \] (B.8)

where the covariant derivative of the parameter is

\[ D_\mu \varepsilon = \left( \partial_\mu + \frac{1}{4} \omega_{\mu}^{\alpha\beta} \gamma_{\alpha\beta} + \frac{1}{4} Q_{\mu}^{ab} \gamma_{ab} \right) \varepsilon. \] (B.9)
Appendix C. Dimensional reduction to two dimensions

A dimensional reduction from $D = 3$ to two dimensions requires a special treatment. The resulting theory in two dimensions turns out to be a supersymmetric version of dilaton gravity \cite{25}. We obtain the supertransformations and identify two-dimensional supergravity multiplet and a matter supermultiplet.

Since eq. (6.1) is singular for $d = 2$, we use a different parametrization without (super) Weyl rescalings. Three-dimensional fields $E_M^A, \Psi_M, \hat{S}$ are parametrized in terms of two-dimensional fields $e_\mu^\alpha, A_\mu, X, \psi_\mu, \lambda, S$ as

\[
E_M^A = \begin{pmatrix} e_\mu^\alpha & A_\mu e^{-X} \\ 0 & e^{-X} \end{pmatrix},
\]

\[
E_\alpha^M \Psi_M = e_\alpha^\mu \psi_\mu, \quad E_2^M \Psi_M = \gamma_2 \lambda,
\]

\[
\hat{S} = 2S - F' + i\bar{\lambda}\lambda,
\]

where $F'$ is the supercovariantized field strength

\[
F' = -\frac{1}{2} \epsilon^{\alpha\beta} \left( F_{\alpha\beta} e^{-X} + i\bar{\psi}_\alpha \gamma_\beta \gamma_2 \lambda + \frac{1}{2} i\bar{\psi}_\alpha \gamma_2 \psi_\beta \right).
\]

Two-dimensional supertransformations are defined by

\[
\delta_Q(\varepsilon) = \delta_Q(\varepsilon) + \delta_L(\lambda(\varepsilon)),
\]

\[
\lambda_{\alpha2}(\varepsilon) = i\bar{\varepsilon}\gamma_\alpha \gamma_2 \lambda, \quad \lambda_{\alpha\beta}(\varepsilon) = 0,
\]

where a local Lorentz transformation is added to preserve the condition $E_\mu^\alpha = 0$. We find the supertransformations of the fields as

\[
\delta_Q e_\mu^\alpha = -i\bar{\varepsilon}\gamma_\alpha \psi_\mu,
\]

\[
\delta_Q \psi_\mu = 2 \left( D_\mu + \frac{1}{2} S \gamma_\mu \right) \varepsilon,
\]

\[
\delta_Q S = \frac{1}{2} i S \bar{\varepsilon} \gamma^\mu \psi_\mu - \frac{1}{2} i \bar{\varepsilon} \gamma^{\mu\nu} \psi_\mu \psi_\nu,
\]

\[
\delta_Q X = i\bar{\varepsilon}\lambda,
\]

\[
\delta_Q A_\mu = -i\bar{\varepsilon}\gamma_\mu \gamma_2 \lambda e^X - i\bar{\varepsilon}\gamma_2 \psi_\mu e^X,
\]

\[
\delta_Q \lambda = -\gamma^\mu \varepsilon D_\mu^\rho X - \varepsilon \left( F' - S - \frac{1}{2} i\bar{\lambda}\lambda \right),
\]

\[
(C.4)
\]
where $D^\mu_P$ is the supercovariant derivative (2.13). To obtain the transformation of $S$ we have used

$$
\delta_Q F' = \frac{1}{2} i F' \varepsilon^\mu \psi_\mu - i (F' - S) \varepsilon \lambda - \frac{1}{2} i \bar{\varepsilon} \gamma^{\mu\nu} \psi_{\mu\nu} + i \bar{\varepsilon} \gamma^\mu D_\mu \lambda
+ \frac{1}{2} i D^\mu_P X \varepsilon \gamma^\nu \gamma^\mu \psi_\nu + i D^\mu_P X \varepsilon \gamma^\mu \lambda + \frac{1}{4} \bar{\varepsilon} \gamma^\mu \psi_\mu \bar{\lambda} \lambda.
$$

(C.5)

Comparing eq. (C.4) with eq. (2.8), we find that the fields $e^\mu_\alpha, \psi_\mu, S$ transform as the two-dimensional supergravity multiplet. The other fields $X, \lambda, A_\mu$ form a matter supermultiplet. This multiplet is similar to the scalar supermultiplet in eq. (2.14) but contains a vector field $A_\mu$ instead of a scalar auxiliary field $F$. Notice that the fields $A_\mu$ and $S$ have the same off-shell degrees of freedom. Actually we can relate these two supermultiplets. If we define

$$
F = F' - S - \frac{1}{2} i \bar{\lambda} \lambda,
$$

(C.6)

the supertransformations of $X, \lambda, F$ derived from eq. (C.4) become exactly the same as eq. (2.14) with $d = 2$.

Two-dimensional action dimensionally reduced from the three-dimensional action (2.1) with $d = 3$ is found to be

$$
S_{SG} = \frac{1}{16 \pi G_0} \int d^2 x \, e^{-X} \left[ R + 2FS - iS \bar{\lambda} \lambda - i \bar{\lambda} \gamma^{\mu\nu} \psi_{\mu\nu} \right].
$$

(C.7)

This is the action of a supersymmetric dilaton gravity [24]. Strictly speaking, our theory is not exactly the same as the theories in ref. [24], which use a scalar supermultiplet with a fundamental scalar auxiliary field. In the action (C.7) the field $F$ is not fundamental but is constructed from other fields as in eq. (C.6).
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