A NEW SUBADDITIVITY FORMULA FOR TEST IDEALS

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ABSTRACT. We exhibit a new subadditivity formula for test ideals on singular varieties using an argument similar to [DEL00] and [HY03]. Any subadditivity formula for singular varieties must have a correction term that measures the singularities of that variety. Whereas earlier subadditivity formulas accomplished this by multiplying by the Jacobian ideal, our approach is to use the formalism of Cartier algebras [Bli13]. We also show that our subadditivity containment is sharper than ones shown previously in [Tak06] and [Eis10]. The first of these results follows from a Noether normalization technique due to Hochster and Huneke. The second of these results is obtained using ideas from [Tak08] and [Eis10] to show that the adjoint ideal $\mathfrak{J}_X(A, Z)$ reduces mod $p$ to Takagi’s adjoint test ideal, even when the ambient space is singular, provided that $A$ is regular at the generic point of $X$. One difficulty of using this new subadditivity formula in practice is the computational complexity of computing its correction term. Thus, we discuss a combinatorial construction of the relevant Cartier algebra in the toric setting.

1. Introduction

Test ideals are an important measure of singularity in characteristic-$p$ commutative algebra. The test ideals in a regular ambient ring $R$ enjoy a property called subadditivity. Namely, if $a$ and $b$ are ideals of $R$ and $s, t \geq 0$ are real numbers, then

$$\tau(R, a^s b^t) \subseteq \tau(R, a^s) \tau(R, b^t).$$

Here, and throughout this paper, we refer only to the big/non-finitistic test ideal. The analogous formula was originally proven for multiplier ideals in characteristic 0 in [DEL00]. Later, Hara and Yoshida defined the test ideal of a pair $(R, a^s)$ and showed the above relationship of test ideals holds in characteristic $p$ [HY03]. Together, these two formulas have numerous applications. Perhaps the most striking one is Ein, Lazarsfeld, and Smith’s proof that smooth varieties satisfy the so-called “Uniform Symbolic Topologies Property” [ELS01].

One can ask to what extent the subadditivity property holds for non-regular rings. The first result in this direction came from [Tak06] Theorem 2.7, where Takagi showed

$$\text{Jac}(R) \tau(R, a^s b^t) \subseteq \tau(R, a^s) \tau(R, b^t)$$

for any equidimensional reduced affine algebra over a perfect field of positive characteristic. Since multiplier ideals are test ideals mod $p \gg 0$ [Smi00, Har01, HY03] and inclusion mod $p \gg 0$ implies inclusion in characteristic 0, the same formula holds for multiplier ideals.

Our proof of subadditivity is similar to the original ones in [DEL00] and [HY03]. There, the idea is to notice that, if $k$ is a field of positive characteristic and $R$ is a $k$-algebra essentially of finite type, then $\tau(R \otimes_k R, (a \otimes_k R)^s (R \otimes_k b)^t) \subseteq \tau(R, a^s) \otimes_k \tau(R, b^t)$. Then it’s enough to check that

$$\tau(R, a^s b^t) \subseteq \mu(\tau(R \otimes_k R, (a \otimes R)^s (R \otimes_k b)^t))$$

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where \( \mu: R \otimes_k R \to R \) is the multiplication map. This is accomplished by the following restriction theorem:

**Theorem** (c.f. [HY03 Theorem 4.1]). Let \( R \) be a normal \( \mathbb{Q} \)-Gorenstein ring, and that suppose \( S = R/(x) \) is normal and \( \mathbb{Q} \)-Gorenstein as well, for some regular element \( x \in R \). Then \( \tau(S, (aS)^s) \subseteq \tau(R, a^s)S \).

If \( R \) is smooth then the kernel of \( \mu \) is generated by a regular sequence, so repeated application of the restriction theorem yields the desired containment.

Eisenstein [Eis10] uses a similar argument to obtain his subadditivity theorem for multiplier ideals. Namely, he proves a more general restriction theorem for multiplier ideals and carefully studies its implications in the case where one restricts to the diagonal subscheme of \( \text{Spec}(R \otimes_k R) \).

Our approach is to follow this diagonal argument. To get around the restriction theorem, we will use the formalism of Cartier algebras [Sch11, Bh13]. In particular, for all rings \( R \) and ideals \( I \subseteq R \), we construct a Cartier algebra \( \mathcal{C} \) on \( R/I \) such that \( \tau(R, a^s)R/I \supseteq \tau(R/I, \mathcal{C}, (aR/I)^s) \). Applying this construction to the quotient \( R \cong (R \otimes_k R)/I_\Delta \), we call this Cartier algebra \( \mathcal{D}^{(2)}(R) \) and we get

**Proposition 1** (Theorem 3.11). Let \( R \) be a \( k \)-algebra of positive characteristic. Then for all ideals \( a, b \subseteq R \) and all real numbers \( s, t \geq 0 \), we have \( \tau(R, \mathcal{D}^{(2)}(R), a^s b^t) \subseteq \tau(R, a^s \tau(R, b^t)). \)

We show in Theorem 3.12 and Corollary 5.6 that this containment is sharper than the previously-known subadditivity results for test ideals:

**Theorem A** (Theorem 3.12). Let \( k \) be a perfect field and \( R \) a \( k \)-algebra essentially of finite type. Suppose also that \( R \) is equidimensional and reduced. Then

\[
\text{Jac}(R) \tau \left( R, \prod_i a_i^{t_i} \right) \subseteq \tau \left( R, \mathcal{D}^{(2)}, \prod_i a_i^{t_i} \right)
\]

for all formal products of ideals \( \prod_i a_i^{t_i} \) such that each \( a_i \) contains a regular element.

**Theorem B** (Corollary 5.6). Let \( R \) be a \( \mathbb{Q} \)-Gorenstein ring of finite type over a field of characteristic 0 and let \( a, b \subseteq R \) be ideals. Let \( R_p \) denote the mod-\( p \) reduction of \( R \), and similarly for \( a_p \) and \( b_p \). Then

\[
\text{Jac}(R_p) \tau(R_p, (a_p)^s (b_p)^t) \subseteq \tau(R, \mathcal{D}^{(2)}(R), (a_p)^s (b_p)^t)
\]

for all \( p \gg 0 \).

The main barrier to applying this new subadditivity formula in practice is the computation of \( \mathcal{D}^{(2)}(R) \). We give a method for doing so in the affine toric case in section 6. Recall that, for any toric ring \( R \) over a field \( k \), one can construct the so-called anti-canonical polytope \( P_{-K_X} \), where \( X = \text{Spec} R \). Then the set \( \text{Hom}_R(F^*_e R, R) \) is generated as a \( k \)-vector space by maps \( \pi_a \) corresponding to fractional lattice points \( a \in \frac{1}{p^e} \mathbb{Z}^n \cap \text{int}(P_{-K_X}) \). We determine which of these generators belong to \( \mathcal{D}^{(2)}(R) \) in Theorem 6.4

**Theorem C** (Theorem 6.4). Let \( R \) be a toric ring with anticanonical polytope \( P_{-K_X} \). Then \( \mathcal{D}^{(2)}(R) \) is generated as a \( k \)-vectorspace by the maps \( \pi_a \) where \( a \in \frac{1}{p^e} \mathbb{Z}^n \cap \text{int}(P_{-K_X}) \) and

\(^1\text{For instance, if } R \text{ is regular and } k \text{ is perfect.} \)
the interior of $P_{-K_X} \cap (a - P_{-K_X})$ contains a representative of each equivalence class in $\frac{1}{p^n} \mathbb{Z}/\mathbb{Z}^n$.

In Section 2, we review some background information on test ideals and Cartier algebras. In Section 3 we introduce our new subadditivity formula and prove Theorem A. In Section 4, we discuss adjoint ideals in characteristic 0 and introduce a characteristic-$p$ analog which generalizes earlier constructions found in \cite{Tak10}, \cite{Sch09}, and \cite{BSTZ10}. We use these constructions in Section 5 to prove Theorem B. This is accomplished by showing that adjoint ideals in characteristic 0 reduce modulo $p > 0$ to our analogous test ideal \( \mathcal{Q}^{(2)}(R) \). Finally, in Section 6, we give a combinatorial criterion for computing $\mathcal{Q}^{(2)}(R)$ in the toric setting.

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## 2. Background on Cartier algebras and test ideals

In this section, we provide the basic definitions and results from the theory of test ideals and Cartier algebras. A more complete account of these theories may be found in the surveys, \cite{ST11 \cite{BS13}}.

Let $R$ be a Noetherian ring of characteristic $p > 0$. We let $F: R \to R$ denote the Frobenius map and let $F^e$ denote the $e$th iterate of $F$. In particular, $F^e(x) = x^{p^e}$ for all $x \in R$. We define $F^e_* R$ to be the $R$-module given by restriction of scalars via $F^e$. In other words, $F^e_* R := \{ F^e_* r \mid r \in R \}$ as a set, and the $R$-module structure on $F^e_* R$ is given by $s F^e_* r = F^e_* s^{p^e} r$ for all $r, s \in R$. We give $\text{Hom}_R(F^e_* R, R)$ an $F^e_* R$ module structure defined by pre-multiplication. That is, we define:

$$(F^e_* r \cdot \varphi)(x) := \varphi(r x)$$

Similarly, we define a right $R$-module structure on $\text{Hom}_R(F^e_* R, R)$ by setting $\varphi \cdot r := F^e_* r \varphi$. The left $R$-module structure on $\text{Hom}_R(F^e_* R, R)$ is given by post-multiplication, namely $r \cdot \varphi(F^e_* x) := r \varphi(F^e_* x)$. Note that $r \cdot \varphi = \varphi \cdot r^{p^e}$ for all $r$ and $\varphi$. Note that the Frobenius map gives us a map of $R$-modules $F^e: R \to F^e_* R$.

**Remark 2.1.** In the literature, people often use the notation $R^{1/p^e}$ instead of $F^e_* R$. If $R$ is a domain, we can identify $R^{1/p^e}$ with the set of $(p^e)$th roots of elements of $R$ in a fixed algebraic closure of $\text{frac}(R)$. Even if $R$ is just reduced, the Frobenius map is injective and we have $R \subseteq R^{1/p^e}$.

**Definition 2.2.** A ring $R$ is said to be $F$-finite if $F^e_* R$ is a finitely-generated $R$-module for some (equivalently, all) $e > 0$.

For instance, perfect fields are $F$-finite. Further, any algebra essentially of finite type over an $F$-finite ring is $F$-finite.

**Global Setting.** We will assume in this paper that all of our rings are $F$-finite.

Perhaps the central idea in the study of so-called “$F$-singularities” is that the nonregularity of a ring $R$ in positive characteristic can be understood by studying the modules $F^e_* R$. This theory was initiated by the following result of Kunz.
Theorem 2.3 ([Kun69]). Let $R$ be Noetherian a ring of positive characteristic. Then $R$ is regular if and only if $F^e_* R$ is a flat $R$-module for some (equivalently, all) $e > 0$.

Definition 2.4. Let $R$ be a reduced ring of positive characteristic. The test ideal of $R$, denoted $\tau(R)$, is the unique, smallest ideal $J$ containing a regular element such that $\varphi(F^e_* J) \subseteq J$ for all $e$ and all $\varphi \in \text{Hom}_R(F^e_* R, R)$.

Proving this ideal exists is non-trivial. Whenever $R$ is regular, we have $\tau(R) = R$. The converse is not true, however. Thus, we say that a ring $R$ is $F$-regular if $\tau(R) = R$. One philosophy in the study of $F$-singularities is that rings with milder singularities have larger test ideals.

We can obtain variants of the test ideal by restricting the set of maps $\varphi$ under consideration. One of the first instances of this idea was the construction of test ideals of pairs $(R, a^t)$. Here, $a$ is an ideal of $R$ and $t \geq 0$ is a real number. We don’t actually define the power $a^t$ for arbitrary real numbers $t$. However, this formal notation for the test ideal of a pair turns out to be quite useful. Indeed, it is unambiguous when $t$ is an integer.

Definition 2.5 ([HY03]). Let $R$ be reduced, $a \subseteq R$ an ideal containing a regular element, and $t$ a positive real number. Then we define $\tau(R, a^t)$ to be the unique, smallest ideal $J$ containing a regular element such that $\varphi(F^e_* J) \subseteq J$ for all $e$ and all $\varphi \in F^e_* (a^{t(p^e-1)}) \text{Hom}_R(F^e_* R, R)$.

People have also considered test ideals of triples $(R, a^t, \Delta)$, where one further restricts the maps $\varphi$ under consideration using a divisor $\Delta$ on $\text{Spec} R$. This leads one to ask, more generally, for which sets of maps $F^e_* R \to R$ can one define the test ideal of that set? The answer is that this set of maps must form a Cartier Algebra.

Definition 2.6 ([Bli13, Sch11]). A Cartier algebra on $R$ is an additive abelian group $\mathcal{C} = \bigoplus_{e} \mathcal{C}_e$, with $\mathcal{C}_e \subseteq \text{Hom}_R(F^e_* R, R)$ for all $e$, that is closed under multiplication on the left and right by elements in $R$ and closed under composition. In other words, given $\varphi_1, \varphi_2 \in \mathcal{C}_e$, $\psi \in \mathcal{C}_d$, and $r \in R$, we have: $\varphi_1 + \varphi_2 \in \mathcal{C}_e$, $r \cdot \varphi \in \mathcal{C}_e$, $\varphi \cdot r \in \mathcal{C}_e$, and $\varphi \cdot \psi := \varphi \circ (F^e_* \psi) \in \mathcal{C}_{d+e}$.

By convention, we also assume that $\mathcal{C}_0 = R$. The full Cartier algebra on $R$ is the Cartier algebra $\mathcal{C}^R := \bigoplus_{e \geq 0} \text{Hom}_R(F^e_* R, R)$.

Here, $F^e_* \psi$ denotes the map $F^{e+d}_* R \to F^e_* R$ given by

$$(F^e_* \psi)(F^{e+d}_* x) := F^e_* \psi(F^d_* x)$$

We note that Cartier algebras are typically not commutative rings. Further, they’re not necessarily $R$-algebras, in the sense that $R$ is typically not in the center of a given Cartier algebra.

Let $\Psi = \sum_i \psi_i \in \mathcal{C}$, where the sum is finite, each $\psi_i$ is nonzero, and $\psi_i \in \mathcal{C}_e$, for each $i$. Then $R$ has a natural left $\mathcal{C}$-module structure given by

$$\Psi \cdot r := \sum_i \psi_i (F^e_* r)$$

for each $r \in R$. Further, we say $\Psi$ has minimal degree $e_0$ if $e_0 = \min_i \{ e_i \}$.

If $\mathcal{C}$ is a Cartier algebra on a reduced ring $R$, then the test ideal $\tau(R, \mathcal{C})$, if it exists, is defined to be the smallest ideal $J \subseteq R$ containing a regular element such that $\varphi(F^e_* J) \subseteq J$ for all $e$ and for all $\varphi \in \mathcal{C}_e$. Schwede showed in [Sch11] that these test ideals exist whenever
\(\mathcal{C}\) is nondegenerate\(^2\). This is a very weak condition. For instance, if \(R\) is a domain, then a Cartier algebra \(\mathcal{C}\) on \(R\) is nondegenerate whenever \(\mathcal{C}_e \neq 0\) for some \(e > 0\).

We can also define \(\tau(R, \mathcal{C}, \prod a_i^{t_i})\) for any nonzero ideals \(a_i \subseteq R\) and real numbers \(t_i \geq 0\): this is the smallest nonzero ideal \(J \subseteq R\) such that \(\varphi(F_e^*J) \subseteq J\) for all \(e\) and for all \(e \in F_e^* \left( \prod a_i^{t_i(p^e-1)\} \right) \mathcal{C}_e\). As before, this product \(\prod a_i^{t_i}\) is just formal (though certainly useful) notation.

**Notation 2.7.** Given a cartier algebra \(\mathcal{C}\) on \(R\), a collection of nonzero ideals \(a_i \subseteq R\), and rational numbers \(t_i \geq 0\), we define
\[
\mathcal{C}_R, \prod a_i^{t_i} := \bigoplus_{e \geq 0} F_e^* \left( \prod a_i^{t_i(p^e-1)} \right) \mathcal{C}_e.
\]

Further, we define \(\tau(R, \prod a_i^{t_i}) := \tau(R, \mathcal{C}_R, \prod a_i^{t_i})\). In other words, when we omit \(\mathcal{C}\) from the test ideal notation, \(\mathcal{C}\) is understood to be the full Cartier algebra \(\mathcal{C}_R\). Note that \(\tau(R, \mathcal{C}, \prod a_i^{t_i}) = \tau \left( R, \mathcal{C}_R, \prod a_i^{t_i} \right) \). We will stick to the first notation, mainly for historical reasons.

**Theorem 2.8** (C.f. [Sch11]). Let \(R\) be reduced, \(\mathcal{C}\) a non-degenerate Cartier algebra on \(R\), and \(c \in \tau(R, \mathcal{C})\) a regular element. Then
\[
\tau(R, \mathcal{C}) = \sum_{e \geq 0} \sum_{\varphi \in \mathcal{C}_e} \varphi(F_e^*c).
\]

Given any map \(\varphi : F_e^*R \to R\) and any multiplicative set \(W \subseteq R\), we get an induced map \(W^{-1} \varphi : W^{-1}F_e^*R \to W^{-1}R\). Note that \(W^{-1}F_e^*R = F_e^*(W^{-1}R)\). Thus, any Cartier algebra \(\mathcal{C}\) on \(R\) induces a Cartier algebra \(W^{-1}\mathcal{C}\) on \(W^{-1}R\).

**Notation 2.9.** Let \(\mathcal{C}\) be a Cartier algebra on \(R\) and \(\gamma \in R\) a regular element. Then \(\mathcal{C}\) induces a Cartier algebra on the localization \(R, \gamma\). We denote this induced Cartier algebra by \(\mathcal{C}_\gamma\).

An important notion in the study of test ideals is that of compatibility of ideals and Cartier algebras.

**Definition 2.10.** Let \(S\) be a ring, \(J\) an ideal of \(S\), and let \(\varphi : F_e^*S \to S\). We say that \(J\) is compatible with \(\varphi\), or \(\varphi\)-compatible, or that \(\varphi\) is compatible with \(J\), if \(\varphi(F_e^*J) \subseteq J\). Let \(\mathcal{D}\) a Cartier algebra on \(S\). Similarly, we say that \(J\) is compatible with \(\mathcal{D}\), or \(\mathcal{D}\)-compatible, or that \(\mathcal{D}\) is compatible with \(J\), if \(J\) is compatible with each map in \(\mathcal{D}\).

Finally, we collect some well-known and useful properties of test ideals.

**Lemma 2.11** (C.f. [HH94, HY03, Sch11]). Let \(R\) be a reduced \(F\)-finite ring.

(a) Let \(\mathcal{C} \subseteq \mathcal{D}\) be two non-degenerate Cartier algebras on \(R\). Then \(\tau(R, \mathcal{C}) \subseteq \tau(R, \mathcal{D})\).

(b) Let \(a_i\) be a collection of ideals of \(R\) and \(t_i \geq 0\) a collection of rational numbers. For each \(i\), let \(b_i = \bar{a}_i\) be the integral closure of \(a_i\). Then \(\tau(R, \prod a_i^{t_i}) = \tau(R, \prod b_i^{t_i})\).

(c) Let \(W \subseteq R\) be a multiplicative set consisting of regular elements. Then \(W^{-1}\tau(R, \mathcal{C}) = \tau(W^{-1}R, W^{-1}\mathcal{C})\).

(d) The Cartier algebra \(\mathcal{C}_R\) is non-degenerate.

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\(^2\)A map \(\varphi : F_e^*R \to R\) is called nondegenerate if \(\varphi(F_e^*R)\eta \neq 0\) for all minimal primes \(\eta \in \text{Spec } R\). A Cartier algebra \(\mathcal{C}\) is called nondegenerate if \(\mathcal{C}_e\) contains a nondegenerate map for some \(e > 0\).
3. Diagonal Cartier Algebras and Subadditivity

Let \( R \) be a noetherian ring in characteristic \( p \) and \( I \subseteq R \) an ideal. Then we have 
\[
F^e_*(R/I) = F^e_*R/F^e_*I.
\]
Thus, each map \( \varphi: F^e_*R \to R \) satisfying \( \varphi(F^e_*I) \subseteq I \) induces a map 
\[
\overline{\varphi}: F^e_*(R/I) \to R/I.
\]
We can do something similar for Cartier algebras:

Definition 3.1. Let \( \mathcal{D} \) be a Cartier algebra on \( R \) compatible with an ideal \( I \subseteq R \). We define the restriction of \( \mathcal{D} \) to \( R/I \), denoted \( \mathcal{D}|_{R/I} \), to be the set of maps \( \bigoplus_{e \geq 0} \mathcal{D}_e|_{R/I} \), where 
\[
\mathcal{D}_e|_{R/I} := \{ \overline{\varphi}: F^e_*(R/I) \to R/I \mid \varphi \in \mathcal{D}_e \}.
\]

Proposition 3.2. Let \( \mathcal{D} \) be a Cartier algebra on \( R \) compatible with an ideal \( I \subseteq R \). Then \( \mathcal{D}|_{R/I} \) is a Cartier algebra on \( R/I \).

Proof. Let \( \varphi_1, \varphi_2 \in \mathcal{D}|_{R/I}, \psi \in \mathcal{D}|_{R/I}, \) and \( r \in R/I \). Then there exist some \( \varphi_1', \varphi_2' \in \mathcal{D} \) with \( \overline{\varphi_1'} = \varphi_1 \) and \( \overline{\varphi_2'} = \varphi_2 \). As \( \mathcal{D} \) is a Cartier algebra, we have \( \varphi_1' + \varphi_2' \in \mathcal{D} \), and we see that 
\[
\varphi_1 + \varphi_2 = \overline{\varphi_1'} + \overline{\varphi_2'} \in \mathcal{D}|_{R/I}.
\]
A similar argument shows that \( r\psi, \psi r \in \mathcal{D}|_{R/I} \) and \( \varphi_1 \circ F^d_*\psi \in \mathcal{D}|_{R^+eR/I} \). It follows from the definitions that \( \mathcal{D}|_{R^0/I} = R/I \) provided that \( \mathcal{D}_0 = R \). □

We define another useful operation on Cartier algebras.

Definition 3.3. Let \( \mathcal{C} \) be a Cartier algebra on \( R \) and let \( I \subseteq R \) be an ideal. We define the subalgebra of maps compatible with \( I \), denoted \( \mathcal{C}^{I\oslash} \), to be the set of maps \( \bigoplus_{e \geq 0} \mathcal{C}_e^{I\oslash} \), where 
\[
\mathcal{C}_e^{I\oslash} := \{ \varphi \mid \varphi \in \mathcal{C}_e, (F^e_*I) \subseteq I \}.
\]

Proposition 3.4. Let \( \mathcal{C} \) be a Cartier algebra on \( R \) and \( I \subseteq R \) an ideal. Then \( \mathcal{C}^{I\oslash} \) is a Cartier algebra.

Proof. Suppose \( \varphi \in \mathcal{C}_e \) and \( \psi \in \mathcal{C}_d \) are two maps satisfying \( \varphi(F^e_*I) \subseteq I \) and \( \psi(F^d_*I) \subseteq I \). Clearly, for all \( x \in R, x\varphi(F^e_*I) \subseteq I \) and \( \varphi(F^e_*xI) \subseteq I \). It’s also clear that \( \varphi(F^e_*I) + \psi(F^d_*I) \subseteq I \). Further, 
\[
\varphi \circ F^e_*\psi \left( F^{e+d}_*I \right) \subseteq (F^e_*I) \subseteq I.
\]
Finally, note that \( \mathcal{C}_0^{I\oslash} = R \) whenever \( \mathcal{C}_0 = R \). □

The next lemma is the key ingredient in proving our new subadditivity formula.

Lemma 3.5. For any reduced ring \( R \), Cartier algebra \( \mathcal{C} \) on \( R \), and radical ideal \( I \subseteq R \) we have 
\[
\tau \left( R/I, \mathcal{C}^{I\oslash}|_{R/I} \right) \subseteq \tau \left( R, \mathcal{C} \right) R/I,
\]
provided that the right-hand side contains a regular element of \( R/I \) and the test ideal on the left, \( \tau \left( R/I, \mathcal{C}^{I\oslash}|_{R/I} \right) \), exists.

Proof. Let \( \varphi \in \mathcal{C}_e|_{R/I} \). By definition there exists a lifting \( \hat{\varphi} \in \mathcal{C}_e \), so that the diagram commutes:
\[
\begin{array}{ccc}
F^e_*R & \xrightarrow{\hat{\varphi}} & R \\
F^e_*R \downarrow \quad \varphi & \quad \downarrow \pi \\
F^e_*(R/I) & \xrightarrow{\varphi} & R/I
\end{array}
\]
This means that
\[ \varphi(F^e \tau(R, C) R/I) = \widehat{\varphi(F^e \tau(R, C)) R/I} \]
By definition of \( \tau(R, C) \), we see the right hand side is contained in \( \tau(R, C) R/I \). Then we are done by the minimality of \( \tau(R/I, C|_{R/I}) \). \( \square \)

Note that the above lemma does not apply when \( C \) is compatible with \( I \), for then we would have \( \tau(R, C) \subseteq I \). This motivates Definition 4.3, c.f. Proposition 4.9.

**Proposition 3.6.** For all reduced rings \( R \), Cartier algebras \( C \) on \( R \), formal products \( \prod_i a_i^{n_i} \) of ideals on \( R \), and radical ideals \( I \), we have
\[
\tau\left( R/I, C^{I \cap} |_{R/I}, \prod_i (a_i R/I)^{n_i} \right) \subseteq \tau\left( R, C, \prod_i a_i^{n_i} \right) R/I,
\]
provided that the right-hand side contains a regular element of \( R/I \) and the test ideal on the left-hand side exists.

**Proof.** Let \( \mathcal{D} \) be a Cartier algebra on \( R \) compatible with \( I \). Then we have
\[
(D|_{R/I})^{\prod_i (a_i R/I)^{n_i}} \subseteq (D|_{R/I})^{\prod_i a_i^{n_i}}
\]
It follows that
\[
\tau\left( R/I, C^{I \cap} |_{R/I}, \prod_i (a_i R/I)^{n_i} \right) \subseteq \tau\left( (C^{I \cap})^{\prod_i a_i^{n_i}} |_{R/I} \right),
\]
by Lemma 2.11. Similarly, we note that
\[
(C^{I \cap})^{\prod_i a_i^{n_i}} \subseteq (C^{\prod_i a_i^{n_i}})^{I \cap}.
\]
Then we get
\[
\tau\left( R/I, (C^{I \cap})^{\prod_i a_i^{n_i}} |_{R/I} \right) \subseteq \tau\left( R/I, (C^{\prod_i a_i^{n_i}})^{I \cap} |_{R/I} \right) \subseteq \tau\left( R, C, \prod_i a_i^{n_i} \right) R/I,
\]
where the second containment follows from Lemma 3.5. \( \square \)

We obtain our subadditivity formula by applying Proposition 3.6 to the case where we consider the ideal \( I_\Delta := \ker(R \otimes_k R \xrightarrow{\mu} R) \). First, we introduce some notation which will be used throughout the rest of this paper:

**Notation 3.7.** Let \( R \) be a \( k \)-algebra essentially of finite type, where \( k \) is a perfect field of positive characteristic.

- \( \mu: R \otimes_k R \to R \) is the map given by \( x \otimes y \mapsto xy \).
- \( I_\Delta \subseteq R \otimes_k R \) denotes the kernel of \( \mu \). Geometrically, \( I_\Delta \) cuts out the diagonal embedding \( \text{Spec } R \subseteq \text{Spec } R \times_{\text{Spec } k} \text{Spec } R \). Note that, in terms of generators, \( I_\Delta = \langle x \otimes 1 - 1 \otimes x \mid x \in R \rangle \).
- We let \( C^{R \otimes_k R, I_\Delta \cap} := (C^{R \otimes_k R})^{I_\Delta \cap} \) denote the Cartier algebra on \( R \otimes_k R \) of all maps compatible with \( I_\Delta \). We say that such maps are **compatible with the diagonal**.
We define the second diagonal Cartier algebra on $R$ to be
\[
\mathcal{D}^{(2)}(R) := \mathcal{O}^{R \otimes_k R, I_\Delta} |_{(R \otimes_k R)/I_\Delta}.
\]

If the ring $R$ is understood from context, we will denote this Cartier algebra simply as $\mathcal{D}^{(2)}$.

**Remark 3.8.** In particular, $\mathcal{D}^{(2)}(R)$ is the set of maps $\varphi : F^e_\ast R \to R$ that admit a lifting to the tensor product $R \otimes_k R$:
\[
\begin{array}{ccc}
F^e_\ast (R \otimes_k R) & \longrightarrow & R \\
\downarrow F^e_\ast \mu & & \downarrow \mu \\
F^e_\ast R & \longrightarrow & R \\
\varphi & &
\end{array}
\]

The notation $\mathcal{D}^{(2)}$ is meant to suggest that one can define $\mathcal{D}^{(n)}$ as the Cartier algebra of maps on $R$ that lift to the $n$-fold tensor product $R \otimes_k R^n$. This theory will be developed, along with applications to symbolic powers, in future work joint with Javier Carvajal-Rojas [CRS18].

Before we may proceed, we need to recall a general fact about modules:

**Lemma 3.9.** Let $R$ and $S$ be commutative algebras over a field $k$. Let $M$ and $N$ be $R$-modules and let $U$ and $V$ be $S$-modules. Suppose also that $M$ and $U$ are finitely presented over their respective rings. Then the canonical map:
\[
\Theta : \text{Hom}_R(M, N) \otimes_k \text{Hom}_S(U, V) \to \text{Hom}_{R \otimes_k S}(M \otimes_k U, N \otimes_k V)
\]
is an isomorphism.

**Proof.** We have the following chain of natural isomorphisms:
\[
\begin{align*}
(1) \quad \text{Hom}_R(M, N) \otimes_k \text{Hom}_S(U, V) & \cong \text{Hom}_R(M, \text{Hom}_S(U, N \otimes_k V)) \\
(2) & \cong \text{Hom}_R(M, \text{Hom}_S(U, \text{Hom}_{R \otimes_k S}(R \otimes_k S, N \otimes_k V))) \\
(3) & \cong \text{Hom}_R(M, \text{Hom}_{R \otimes_k S}(R \otimes_k U, N \otimes_k V)) \\
(4) & \cong \text{Hom}_{R \otimes_k S}(M \otimes R \otimes_k U, N \otimes_k V) \\
(5) & \cong \text{Hom}_{R \otimes_k S}(M \otimes_k U, N \otimes_k V) \\
(6) & \cong \text{Hom}_{R \otimes_k S}(M \otimes_k U, N \otimes_k V)
\end{align*}
\]
The isomorphism in (1) follows from the facts that $M$ is finitely presented and $\text{Hom}_S(U, V)$ is a flat $k$-module (c.f. [Lan05, Chapter XVI, Exercise 11]). The isomorphism in (2) follows by the same argument. The isomorphisms in (4) and (5) follow from Hom-Tensor adjunction. \qed

**Corollary 3.10.** Let $k$ be a perfect field of characteristic $p$ and let $R$ and $S$ be $k$-algebras essentially of finite type. Then the canonical map
\[
\Theta : \text{Hom}_R(F^e_\ast R, R) \otimes_k \text{Hom}_S(F^e_\ast S, S) \to \text{Hom}_{R \otimes_k S}(F^e_\ast (R \otimes_k S), R \otimes_k S)
\]
is an isomorphism.

**Proof.** As $R$ is $F$-finite and Noetherian, we see that $F^e_\ast R$ is a finitely-presented $R$-module. The same goes for $S$. Then this result follows from the above lemma. Note that, since $k$ is perfect, we have $F^e_\ast R \otimes_k F^e_\ast R = F^e_\ast (R \otimes_k R)$ and similarly for $S$. \qed
Theorem 3.11. Let $k$ be a perfect field of positive characteristic and let $R$ be a reduced $k$-algebra essentially of finite type. Then

$$\tau(R, \mathcal{O}^{(2)}(a^s b^t)) \subseteq \tau(R, a^s) \tau(R, b^t)$$

for all ideals $a, b \subseteq R$ and real numbers $s, t \geq 0$, provided that neither $a$ nor $b$ consists of zero-divisors.

Proof. From Corollary 3.10 it’s easy to see that

$$\tau(R \otimes_k R, (a \otimes_k R)^s \cdot (R \otimes_k b)^t) \subseteq \tau(R, a^s) \otimes_k \tau(R, b^t)$$

by the minimality of the test ideal on the left. If we mod out the above equation by $I_\Delta$, we get $\tau(R, a^s) \tau(R, b^t)$ on the right-hand side. By Proposition 3.6 we’re done if we can show that

$$\tau(R \otimes_k R, (a \otimes_k R)^s \cdot (R \otimes_k b)^t) / I_\Delta$$

contains a regular element and that $\mathcal{O}^{(2)}(R)$ is non-degenerate. To that end, let $f \in a$ and $g \in b$ be regular elements such that $R_f$ and $R_g$ are regular. As we know $(R \otimes R)_{f \otimes g} = R_f \otimes R_g$, we see that

$$\tau(R \otimes_k R, (a \otimes_k R)^s \cdot (R \otimes_k b)^t)(R \otimes R)_{f \otimes g} = \tau(R_f \otimes_k R_g, (aR_f \otimes_k R_g)^s \cdot (R_f \otimes_k bR_g)^t)$$

$$= \tau(R_f \otimes_k R_g) = R_f \otimes_k R_g,$$

where the last equality follows from the regularity of $R_f \otimes_k R_g$. Thus, we have

$$(f \otimes g)^n \in \tau(R \otimes_k R, (a \otimes_k R)^s \cdot (R \otimes_k b)^t),$$

for some $n$, and so $(fg)^n \in \tau(R \otimes_k R, (a \otimes_k R)^s \cdot (R \otimes_k b)^t) / I_\Delta$.

It remains to check that $\mathcal{O}^{(2)}(R)$ is non-degenerate. We know that $\mathcal{O}^{(e)}_e R$ contains a non-degenerate map $\varphi$ for some $e > 0$, by Lemma 2.11. Further, since $R$ is reduced, we know that $R_q$ is regular for all minimal primes $q \in \text{Spec } R$ and that every zero-divisor of $R$ is contained in a minimal prime. As the singular locus of $\text{Spec } R$ is Zariski-closed, it follows by prime avoidance that there exists a regular element $f \in R$ such that $R_f$ is regular. As $R_f \otimes_k R_f$ is regular and $F$-finite, we have $\mathcal{O}_e^c(R_f \otimes_k R_f)$ is a projective $R \otimes_k R$-module, by Theorem 2.3. It follows that there exists some $\hat{\varphi} \in \text{Hom}_{R \otimes_k R}(\mathcal{O}_e^c(R \otimes_k R), R \otimes_k R)_{f \otimes f}$ such that the diagram,

$$
\begin{array}{ccc}
\mathcal{O}_e^c(R_f \otimes_k R_f) & \xrightarrow{\hat{\varphi}} & R_f \otimes_k R_f \\
F_e^c R_f & \xrightarrow{\mu} & R_f \\
\end{array}
$$

commutes. Thus there exists some $N$ with $(f \otimes_k f)^N \hat{\varphi} \in \text{Hom}_{R \otimes_k R}(\mathcal{O}_e^c(R \otimes_k R), R \otimes_k R)$. We see that $f^{2N} \varphi$ is a non-degenerate element of $\mathcal{O}_e^{(2)}(R)$, as desired. \qed

The next theorem shows that this subadditivity formula is sharper than the one found in $\text{Tak06}$.

3Note that this relies on the fact that $k$ is perfect. Indeed, $R \otimes_k R$ need not be regular if $k$ is not perfect, see $\text{Izi}$. On the other hand, since $k$ is perfect, we know that $R_f$ is in fact smooth, meaning $\Omega_{R_f/k}$ is a free $R_f$-module. Thus $\Omega_{R_f \otimes_k R_f/k} = (\Omega_{R_f/k} \otimes_k R_f) \oplus (R_f \otimes_k \Omega_{R_f/k})$ is a free $R_f \otimes_k R_f$-module and $R_f \otimes_k R_f$ is also smooth, and therefore regular.
**Theorem 3.12.** Let $k$ be a perfect field of positive characteristic and let $R$ be a $k$-algebra essentially of finite type. Suppose also that $R$ is equidimensional and reduced. Then we have $\text{Jac}(R)\mathcal{C}^R \subseteq \mathcal{D}^{(2)}(R)$. In particular,

$$\text{Jac}(R)\tau \left( R, \prod_i a_i^t \right) \subseteq \tau \left( R, \mathcal{D}^{(2)}, \prod_i a_i^t \right)$$

for all formal products of ideals $\prod a_i^t$ such that each $a_i$ contains a regular element.

**Proof.** For any multiplicative subset $W \subseteq R$, one checks that

$$W^{-1}(\text{Jac}(R)\mathcal{C}^R) = \text{Jac}(W^{-1}R)\mathcal{C}^{W^{-1}R}$$

and $W^{-1}\mathcal{D}^{(2)}(R) \subseteq \mathcal{D}^{(2)}(W^{-1}R)$. Thus we may assume that $R$ is a finitely generated $k$-algebra.

Next, we reduce to the case that $k$ is infinite. Suppose that $k$ is finite, let $t$ be an indeterminate over $k$, and let $L = k(t^{1/p^q})$ be the perfection of $k(t)$. Set $R_L = R \otimes_k L$ and suppose $\text{Jac}(R_L/L)\mathcal{C}^{R_L} \subseteq \mathcal{D}^{(2)}(R_L)$. Let $e \geq 0$ and set $q = p^e$. As $L$ is perfect, we have $L^{1/q} \subseteq L$. Let $\iota: L^{1/q} \to L$ denote the inclusion map. Note that $\iota$ is a map of $L$-modules. In other words, $\iota$ is an element of $\mathcal{C}_e^{R_L}$. Then any map $\varphi \in \mathcal{C}_e^R$ induces a map,

$$\varphi \otimes_k \iota := (R \otimes_k L)^{1/q} = R^{1/q} \otimes_k L^{1/q} \to R \otimes_k L,$$

in $\mathcal{C}_e^{R_L}$. Further, any $x \in \text{Jac} R$ gives an element $x \otimes_k 1 \in \text{Jac}(R_L/L)$. By assumption, we have a lifting,

$$R_L^{1/q} \otimes_L R_L^{1/q} \xrightarrow{\hat{\varphi}} R_L \otimes_L R_L \xrightarrow{\mu_L} R_L$$

As $R \subseteq R_L$ and $R \otimes_k R \subseteq R_L \otimes_L R_L$, it suffices to show that $\hat{\varphi} (R^{1/q} \otimes_k R^{1/q}) \subseteq R \otimes_k R$. But this follows from the fact that $\mu_L^{-1}(R) = R \otimes_k R$.

Now assume that $k$ is infinite. Let $x \in \text{Jac}(R)$ and let $\varphi \in \mathcal{C}_e^R$. As $k$ is infinite, we can use [HH02, Theorem 3.4] (c.f. [HH99, Corollary 1.5.4]) to say there exists a (generically separable) Noether normalization $A \subseteq R$ such that $xR^{1/q} \subseteq A^{1/q}[R]$ and $A^{1/q}[R] \cong A^{1/q} \otimes_A R$. As $A$ is a polynomial ring, we have $A^{1/q}$ is a free $A$-module, and so $A^{1/q} \otimes_A S$ is a free $S$-module for any $A$-algebra $S$. In particular, we have that $A^{1/q}[R] \otimes_k R^{1/q} \cong A^{1/q} \otimes_A R \otimes_k R^{1/q}$ is a free $R \otimes_k R^{1/q}$-module.

Further, the usual multiplication map

$$\mu: R^{1/q} \otimes_k R^{1/q} \to R^{1/q}$$

induces an $R^{1/q} \otimes_k R^{1/q}$-module structure on $R^{1/q}$. By definition we have that $\mu$ is $R^{1/q} \otimes_k R^{1/q}$-linear, and in particular $R \otimes_k R^{1/q}$-linear. As $A^{1/q}[R] \otimes_k R^{1/q}$ and $R \otimes_k R^{1/q}$ are contained in $R^{1/q} \otimes_k R^{1/q}$, the map $\mu$ restricts to $R \otimes_k R^{1/q}$-linear maps,

$$\mu: A^{1/q}[R] \otimes_k R^{1/q} \to R^{1/q}$$

$$\mu: R \otimes_k R^{1/q} \to R^{1/q}.$$  

It follows that there exists an $R \otimes_k R^{1/q}$-linear (and, *a fortiori*, $R \otimes_k R$-linear) map,

$$\Psi: A^{1/q}[R] \otimes_k R^{1/q} \to R \otimes_k R^{1/q},$$
making the following diagram commute:

\[
\begin{array}{ccc}
A^{1/q}[R] \otimes_k R^{1/q} & \xrightarrow{\Psi} & R \otimes_k R^{1/q} \\
\mu & \downarrow & \mu \\
R^{1/q} & \xrightarrow{id} & R^{1/q}
\end{array}
\]

The fact that \( \varphi \) is \( R \)-linear means that the diagram

\[
\begin{array}{ccc}
R \otimes_k R^{1/q} & \xrightarrow{1 \otimes \varphi} & R \otimes_k R \\
\mu & \downarrow & \mu \\
R^{1/q} & \xrightarrow{\varphi} & R
\end{array}
\]

commutes. The map \( 1 \otimes \varphi \) is \( R \otimes_k R \)-linear as \( \varphi \) is \( R \)-linear. Finally, we have a commuting diagram

\[
\begin{array}{ccc}
R^{1/q} \otimes_k R^{1/q} & \xrightarrow{x \otimes 1} & A^{1/q}[R] \otimes_k R^{1/q} \\
\mu & \downarrow & \mu \\
R^{1/q} & \xrightarrow{x} & R^{1/q}
\end{array}
\]

where the horizontal maps are given by multiplication. Putting these three diagrams together, we have a commutative diagram,

\[
\begin{array}{ccc}
R^{1/q} \otimes_k R^{1/q} & \xrightarrow{x \otimes 1} & A^{1/q}[R] \otimes_k R^{1/q} \\
\mu & \downarrow & \mu \\
R^{1/q} & \xrightarrow{x} & R^{1/q}
\end{array}
\]

where each of the maps in the top row is \( R \otimes_k R \)-linear. This proves the first assertion. The second assertion follows from Theorem 2.8. \( \square \)

4. Test ideals and multiplier ideals along a closed subscheme

In this section we introduce two new definitions. The first is a notion of test ideals “along a closed subscheme”. This is a generalization of Takagi’s generalized test ideal along \( I \) \([\text{Tak10, BSTZ10}]\). We also define a similar generalization of Takagi’s adjoint ideal. In the next section, we show that the expected reduction theorem holds: the adjoint ideal reduces to the test ideal mod \( p \gg 0 \). We use this result to show that our subadditivity formula for test ideals is sharper than the one obtained by reducing \([\text{Eis10, Theorem 6.5}]\) mod \( p \gg 0 \).

Remark 4.1. In this section and the next, we will make heavy use of the floor and ceiling functions, \( \lfloor \cdot \rfloor \) and \( \lceil \cdot \rceil \). It will be helpful to keep in mind the following inequalities: for any \( a, b \in \mathbb{R} \) we have \( \lfloor a \rfloor + \lfloor b \rfloor \leq \lfloor a + b \rfloor \) and \( \lfloor a - b \rfloor \leq \lfloor a \rfloor - \lfloor b \rfloor \). Similarly, \( \lceil a + b \rceil \leq \lceil a \rceil + \lceil b \rceil \) and \( \lceil a \rceil - \lceil b \rceil \leq \lceil a - b \rceil \).

4.1. Test Ideals along a Closed Subscheme. For the rest of this subsection, we will be working in the following setting.

Setting 4.2. \( R \) is a Noetherian \( F \)-finite domain, \( I \subseteq R \) is a prime ideal, and \( \mathcal{C} \) a nonzero Cartier algebra on \( R \) such that \( I \) is compatible with \( \mathcal{C} \). We assume there exists \( c > 0 \) and \( \psi \in \mathcal{C}_e \) with \( \psi(F^e_* R) \subsetneq I \).
Definition 4.3. Let $R, I, \mathcal{C}$ be as in Setting 4.2. Then we define the test ideal of $\mathcal{C}$ along $I$, denoted $\tau_I(R, \mathcal{C})$, to be the unique smallest ideal of $R$ not contained in $I$ that is compatible with $\mathcal{C}$. We also call this the test ideal of $\mathcal{C}$ along the closed subscheme $\text{Spec}(R/I)$.

The proof that $\tau_I(R, \mathcal{C})$ exists is a standard though technical argument. We relegate it to Appendix A. For now, we just note some examples of interest where the conditions of Setting 4.2 are satisfied.

Example 4.4. Suppose $R$ is a domain essentially of finite type over a field $k$. Then $I = I_\Delta \subseteq R \otimes_k R$ and $\mathcal{C} = \mathcal{C}^{R \otimes_k R, I_\Delta \subseteq}$ satisfy Setting 4.2. Indeed, we have that $\mathcal{C}$ is compatible with $I$ by construction, so we just need to check that there exists $e > 0$ and $\varphi \in \mathcal{C}_e$ with $\varphi(R) \not\subseteq I$. This is equivalent to checking that $D_e^{(2)}(R) \neq 0$ for some $e > 0$, which follows from Theorem 2.3.

Notation 4.5. Let $\mathfrak{a}_i$ be a collection of ideals and $t_i$ a set of non-negative rational numbers. Then we denote, for all $e$,

$$\mathfrak{a}^{[t([p^e - 1])] : = \prod_i \mathfrak{a}_i^{[t_i([p^e - 1])]}.}$$

Notation 4.6. Work in Setting 4.2. If $\mathcal{C} = \mathcal{C}^{R, \prod, a_i^{t_i}}$ for some ideals $\mathfrak{a}_i$ and non-negative numbers $t_i$, then we denote

$$\tau_I \left(R, \prod_i \mathfrak{a}_i^{t_i} \right) := \tau_I(R, \mathcal{C}).$$

Example 4.7. Suppose $R_I$ is regular, and $\dim R_I = c$. Let $\{b_i\}$ be a set of ideals, none of which is contained in $I$, and let $\{t_i\}$ be some collection of non-negative rational numbers. Then $I^{e} \prod_i b_i^{t_i} \cdot \mathcal{C}^R$ satisfies the conditions of Setting 4.2. In this case, $\tau_I \left(R, I^{e} \prod_i b_i^{t_i} \right)$ is what Takagi calls $\tau_I(R, \prod b_i^{t_i})$. The Cartier algebras $\mathcal{C}^{R, I^{e} \prod, b_i^{t_i}}$ and $\mathcal{C}^{R, \prod, b_i^{t_i}}$ satisfy the conditions of Setting 4.2 as well.

More generally, let $\{a_i\}$ be any collection of ideals and $\{u_i\}$ any collection of non-negative rational numbers, satisfying the following: for each $i$, suppose there exists some natural number $n_i$ with $a_i R_I = I^{n_i} R_I$, and suppose $\sum_i n_i t_i = c$. Further, for all $i$ such that $a_i \subseteq I$, we assume the denominator of $t_i$ is not divisible by $p$. Then $\mathcal{C}^{R, \prod, a_i^{t_i}}$ satisfies the conditions of Setting 4.2. To see this, we need to check that

(a) $\mathcal{C}^{R, \prod, a_i^{t_i}}$ is compatible with $I$, and

(b) There exist $e > 0$ and $\psi \in F_* \left(\mathfrak{a}^{[t([p^e - 1])] : \text{Hom}_R(F^e_* R, R) with } \psi(F^e_* R) \not\subseteq I.\right.$

To show condition (a), let $x \in \mathfrak{a}^{[t([p^e - 1])]$ and let $\varphi \in \text{Hom}_R(F^e_* R, R)$. We wish to show that $\varphi(F^e_* x I) \subseteq I$. Whether or not $\varphi(F^e_* x I)$ is contained in $I$ is not affected by localizing at $I$, so we localize at $I$. Then $IR_I$ is generated by $c$ elements, since $R_I$ is regular, and $x \in I^{cp^{e-c} R_I}$, since

$$\mathfrak{a}^{[t([p^e - 1])] R_I = \prod_i I^{n_i t_i([p^e - 1])} \subseteq \prod_i I^{n_i t_i([p^e - 1])} \subseteq I^{[\sum_i n_i t_i([p^e - 1])] = I^{cp^{e-c} R_I}}.$$

Thus $xIR_I \subseteq I^{cp^{e-c+1} R_I} \subseteq I^{[p^e] R_I}$. This shows that (a) is satisfied. To see condition (b) is satisfied, we notice again that this question can be checked locally at $I$. As $R_I$ is regular local and $F$-finite, it follows from Theorem 2.3 that $F^e_* R_I$ is a free $R_I$ module for all $e$. Then it follows that $\varphi(F^e_* x) \in I$ for all $\varphi \in \text{Hom}_{R_I}(F^e_* R_I, R_I)$ if and only if $x \in I^{[p^e] R_I}$. As the denominator of each $t_i$ is not divisible by $p$, we know there exists some $e$ so that $t_i([p^e - 1]$ is an integer for all $i$. For this $q$, we have $\mathfrak{a}^{[t([p^e - 1])] R_I = I^{cp^{e-c} \not\subseteq I^{[p^e]}}$, so we’re done.
4.2. **Basic properties of** $\tau_I(R, \mathcal{C})$. Here we explore the basic theory of $\tau_I(R, \mathcal{C})$. With the exception of the restriction theorem found in Proposition 4.9, the following are properties satisfied by all objects deserving of the name “test ideal.” The restriction theorem suggests that $\tau_I(R, \mathcal{C})$ is a good candidate for a positive-characteristic analog to the adjoint ideals of birational geometry.

**Lemma 4.8.** Suppose that $\mathcal{C} \subseteq \mathcal{D}$ are two Cartier algebras on $R$ satisfying Setting 4.2. Then $\tau_I(R, \mathcal{C}) \subseteq \tau_I(R, \mathcal{D})$.

**Proof.** The ideal $\tau_I(R, \mathcal{C})$ is the minimum element of the set

$$S_\mathcal{C} := \{a \in R \mid a \not\in I \text{ and } a \text{ is } \mathcal{C}\text{-compatible}\}$$

Similarly, $\tau_I(R, \mathcal{D})$ is the minimum element of the set

$$S_\mathcal{D} := \{a \in R \mid a \not\in I \text{ and } a \text{ is } \mathcal{D}\text{-compatible}\}$$

We see that $S_\mathcal{C} \supseteq S_\mathcal{D}$, whence the minimum of $S_\mathcal{C}$ is smaller than the minimum of $S_\mathcal{D}$. □

**Proposition 4.9** (Restriction theorem). Let $R$, $I$, and $\mathcal{C}$ be as in Setting 4.2. Then $\tau_I(R, \mathcal{C})|_{R/I} = \tau(R/I, \mathcal{C}|_{R/I})$.

**Proof.** The proof is very similar to that of Proposition 3.6. Let $\varphi \in \mathcal{C}|_{R/I}$. By definition, there exists some $\tilde{\varphi} \in \mathcal{C}_e$ such that the following diagram commutes:

$$
\begin{array}{ccc}
F_*^eR & \stackrel{\tilde{\varphi}}{\longrightarrow} & R \\
\downarrow F_*^e\pi & & \downarrow \pi \\
F_*^e(R/I) & \stackrel{\varphi}{\longrightarrow} & R/I
\end{array}
$$

We see that

$$\varphi(F_*^e\tau_I(R, \mathcal{C})|_{R/I}) = \tilde{\varphi}(F_*^e\tau_I(R, \mathcal{C}))|_{R/I} \subseteq \tau_I(R, \mathcal{C})|_{R/I}.$$ 

By the minimality of $\tau(R/I, \mathcal{C}|_{R/I})$, it follows that $\tau_I(R, \mathcal{C})|_{R/I} \supseteq \tau(R/I, \mathcal{C}|_{R/I})$. To get the reverse inclusion, it suffices to show that $\tau_I(R, \mathcal{C}) \subseteq \pi^{-1}(\tau(R/I, \mathcal{C}|_{R/I}))$. By definition, $\tau(R/I, \mathcal{C}|_{R/I}) \neq 0$, which means that $\pi^{-1}(\tau(R/I, \mathcal{C}|_{R/I})) \not\subseteq I$. Thus it suffices to check that $\pi^{-1}(\tau(R/I, \mathcal{C}|_{R/I}))$ is compatible with $\mathcal{C}$. To that end, let $\psi \in \mathcal{C}_e$ be arbitrary. As $\mathcal{C}$ is compatible with $I$, there exists some $\tilde{\psi} \in \mathcal{C}_e|_{R/I}$ such that the following diagram commutes:

$$
\begin{array}{ccc}
F_*^eR & \stackrel{\psi}{\longrightarrow} & R \\
\downarrow F_*^e\pi & & \downarrow \pi \\
F_*^e(R/I) & \stackrel{\bar{\psi}}{\longrightarrow} & R/I
\end{array}
$$

It follows from the above diagram and the $\mathcal{C}|_{R/I}$-compatibility of $\tau(R/I, \mathcal{C}|_{R/I})$ that:

$$\pi \circ \psi(F_*^e\pi^{-1}(\tau(R/I, \mathcal{C}|_{R/I})) = \tilde{\psi}(F_*^e\pi^{-1}(\tau(R/I, \mathcal{C}|_{R/I})) = \tilde{\psi}(F_*^e\tau(R/I, \mathcal{C}|_{R/I})) \\
\subseteq \tau(R/I, \mathcal{C}|_{R/I}).$$

In other words,

$$\psi(F_*^e\pi^{-1}(\tau(R/I, \mathcal{C}|_{R/I})) \subseteq \pi^{-1}(\tau(R/I, \mathcal{C}|_{R/I})),$$

as desired. □
Definition 4.10. We say that an element $b$ in $\tau_I(R, C) \setminus I$ is a $C$-test element along $I$. If $C = C^R \prod a_i$, then we say that $b$ is an $\prod a_i$-test element along $I$.

Lemma 4.11. Let $c$ be a $C$-test element along $I$. Then $\tau_I(R, C) = \sum_{e \geq 0} \sum_{\varphi \in C_e} \varphi(F_e^c)$.  

Proof. For ease of notation, set 
\[
\tau_I(R, C; c) = \sum_{e \geq 0} \sum_{\varphi \in C_e} \varphi(F_e^c).
\]
As $c \notin I$, have that $\tau_I(R, C; c) \notin I$ by Proposition A.4. Now let $J \notin I$ be an ideal compatible with $C$. By definition of $\tau_I(R, C)$, we know that $c \in J$. It follows that $\tau_I(R, C; c) \subseteq J$, since $J$ is compatible with $C$. 

Lemma 4.12. Let $c$ be a $C$-test element along $I$ and let $e' \geq 0$. Then 
\[
\tau_I(R, C) = \sum_{e \geq e'} \sum_{\varphi \in C_e} \varphi(F_e^c).
\]
Proof. Set $J = \sum_{e \geq e'} \sum_{\varphi \in C_e} \varphi(F_e^c)$. By Lemma 4.11, we see that $J \subseteq \tau_I(R, C)$. Thus it suffices to show that $J \notin I$ and that $J$ is compatible with $C$. The latter is obvious. To see that $J \notin I$, note that there exists some $e''$ and $\varphi \in C_{e''}$ with $\varphi(F_{e''}^c) \notin I$. This follows, for instance, from Lemma 4.11. Then by Lemma A.3, we have $\varphi^n(F_{e''}^c \tau) \notin I$ for all $n$. In particular, we get the desired result by taking $n > e''/e'$.

Lemma 4.13. Work in Setting 4.2 and let $W \subseteq R \setminus I$ be a multiplicative set. Then 
\[
W^{-1}\tau_I(R, C) = \tau_{IW^{-1}R}(W^{-1}R, W^{-1}C).
\]
Proof. As $R$ is a domain and $W \cap I = \emptyset$, the ideal $\tau_{IW^{-1}R}(W^{-1}R, W^{-1}C)$ is well-defined. As $R$ is $F$-finite and Noetherian, we have $W^{-1}\text{Hom}_R(F_e^c R, R) = \text{Hom}_{W^{-1}R}(F_e^c W^{-1}R, W^{-1}R)$ for all $e$. Then this lemma follows quickly from Theorem A.5.

4.3. Multiplier ideals along a closed subscheme.

Definition 4.14. Let $A$ be a $\mathbb{Q}$-Gorenstein scheme of finite type over a field of characteristic 0 and let $X$ be a reduced subscheme of $A$ of pure codimension $c$. Suppose also that $A$ is smooth at the generic points of $X$. Let $Z$ be a formal $\mathbb{Q}$-sum of subschemes of $A$ such that $Z$ equals $cX$ at the generic points of $X$. We define $\mathcal{J}_X(A, Z)$ as follows: let $\pi : \overline{A} \rightarrow A$ be a factorizing resolution\footnote{See [Eis10] Section 2] for the definition of a factorizing resolution.} of $X \subseteq A$ such that $\pi^{-1}Z \cup \text{supp}(\text{Exc}(\pi))$ is a simple normal crossings variety and the components of $Z$ not vanishing along $X$ lift to divisors. This is possible by [Eis10] Corollary 3.2]. Let $\overline{X} \subseteq \overline{A}$ be the strict transform of $X$ in $\overline{A}$ and let $\psi : A' \rightarrow \overline{A}$ be the blow up of $\overline{A}$ along $\overline{X}$. We get the following diagram: 

\[
\begin{array}{ccc}
X' & \subseteq & A' \\
\downarrow & & \downarrow \\
X & \subseteq & \overline{A} \\
\downarrow & & \downarrow \\
X & \subseteq & A.
\end{array}
\]
Then \( X' := \text{Exc}(\psi) \) is a prime divisor dominating \( X \). Let \( f = \pi \circ \psi \). We define the multiplier ideal of \((A, Z)\) along \( X \) to be the ideal:

\[
\mathcal{J}_X(A, Z) := f_* \mathcal{O}_X \left( \left[ K_{A'/A} - f^{-1}Z \right] + X' \right).
\]

The following lemma shows that \( \mathcal{J}_X(A, Z) \) is a generalization of Takagi’s adjoint ideal.

**Lemma 4.15** ([Eis10, Proof of proposition 3.5]). If \( X \) is a subscheme of \( A \) with pure codimension \( c \), and \( A \) is smooth at the generic points of \( X \), and none of the \( Z_i \) vanish at the generic points of \( X \), then, using Takagi’s definition for \( \text{adj}_X(A, Z) \),

\[
\text{adj}_X(A, Z) = \mathcal{J}_X(A, Z + cX).
\]

5. Comparison with Eisenstein’s Subadditivity theorem

Our main technical result in this section is that the adjoint ideal equals the test ideal along a subscheme mod \( p \gg 0 \), even when \( A \) is singular. This extends earlier results by Takagi in the setting where \( X \) is a divisor \([\text{Tak08}]\) and in the setting where \( A \) is regular \([\text{Tak13}]\). We will mainly work in the following setting.

**Setting 5.1.** \( R \) is a \( \mathbb{Q} \)-Gorenstein ring essentially of finite type over a perfect field \( k \). \( I \subseteq R \) is a prime ideal of height \( c \) and \( R_I \) is regular. \( a_i \subseteq R \) is a collection of ideals, \( 1 \leq i \leq N \), and \( t_i \geq 0 \) a collection of rational numbers. We further assume that \( a_i R_I = I^{n_i} R_I \) for each \( i \) and also \( \sum_i n_i t_i = c \). Set \( \mathcal{C} = \mathcal{C}^{R/I, a_i^i} \).

Further, set \( A = \text{Spec} R, X = \text{Spec}(R/I) \subseteq A, \) and \( Z_i = \text{Spec}(R/a_i) \subseteq A. \) Let \( Z \) denote the formal sum \( Z = \sum_i t_i Z_i. \) If \( R \) has characteristic 0, let \( S \subseteq k \) be a descent datum and let \( s \in \text{MaxSpec} S. \) Set \( \kappa := \kappa(s) \) and \( p = \text{char} \kappa. \) Then we write \( R_\kappa \) to denote the mod-\( p \) reduction of \( R \) at \( s \), and similarly for \( I, a_i, A, X, \) and \( Z_i. \) We further denote \( \sum_i t_i (Z_i)_\kappa \) by \( Z_\kappa \) and we denote \( \prod_i (a_i)_\kappa \) by \( a_i^\kappa. \)

Note that this setting is agnostic to the characteristic of \( R. \)

**Theorem 5.2.** Work in **Setting 5.1** and assume that \( \text{char} R = 0. \) Then \( (\mathcal{J}_X(A, Z))_\kappa = \tau_{I_\kappa}(R_\kappa, a_i^\kappa) \) for all \( s \in \text{MaxSpec} S \) sufficiently general.

**Remark 5.3** (Reduction mod \( p \)). Here we briefly review the process of reducing rings and schemes modulo \( p. \) See [HH99 Chapter 2] for a detailed reference and [Tak13 Section 2] for a succinct reference.

Let \( R \) be an algebra essentially of finite type over a field \( k \) of characteristic 0. We can find a finitely generated \( \mathbb{Z} \)-subalgebra \( B \) of \( k \) and a \( B \)-subalgebra \( R_B \) of \( R \) such that the inclusion \( R_B \subseteq R \) induces an isomorphism \( R = R_B \otimes_B k. \) This algebra \( B \) is called descent datum for \( R, \) and the algebra \( R_B \) is called a model for \( R \) over \( B. \) For instance, if

\[
R = \frac{\mathbb{C}[x_1, \ldots, x_n]}{(f_1, \ldots, f_m)},
\]

\[\text{See Remark 5.3} \]
we can choose \( B \) to be the \( \mathbb{Z} \)-subalgebra of \( \mathbb{C} \) generated by all of the coefficients of all of the polynomials \( f_i \), and we can set

\[
R_B = \frac{B[x_1, \ldots, x_n]}{(f_1, \ldots, f_m)}.
\]

For any maximal ideal \( \mathfrak{m} \subseteq B \), the residue field \( B/\mathfrak{m} \) will have positive characteristic \( p \). Then \( R_{\kappa(\mathfrak{m})} := R_B \otimes_B B/\mathfrak{m} \) is a \( \text{mod } p \) reduction of \( R \). More specifically, we call \( R_{\kappa(\mathfrak{m})} \) the \( \text{mod } p \) reduction of \( R \) at \( \mathfrak{m} \). For any finitely generated \( R \)-module \( M \), we can find a model \( M_B \) for \( M \) over \( B \) in the same manner. The choice of descent datum \( B \) is not unique. However, given two descent data \( B \) and \( B' \), we can find a third descent datum \( C \) containing \( B \) and \( B' \), such that \( R_B \otimes_B C = R_{B'} \otimes_B C \). Given any descent datum \( B \), many properties of \( R \) will be preserved by \( R_{\kappa(\mathfrak{m})} \) for \( \mathfrak{m} \) sufficiently general, that is, for all \( \mathfrak{m} \) in a dense open subset of \( \text{MaxSpec } B \). For instance, if \( R \) is regular, then so is \( R_{\kappa(\mathfrak{m})} \) for all \( \mathfrak{m} \) sufficiently general.

Note that, for all \( N \), the subset of \( \mathfrak{m} \) in \( \text{MaxSpec } B \) such that \( \text{char } \kappa(\mathfrak{m}) > N \) forms a dense open set. Thus we have \( \text{char } R_{\kappa(\mathfrak{m})} \gg 0 \) for \( \mathfrak{m} \) sufficiently general.

It will be useful for us later to note that reduction modulo \( p \) commutes with tensor products, in the following sense. If \( B \) is a descent datum for \( R \), then \( B \) is also a descent datum for \( R \otimes_k R \), and we have \( R_B \otimes_B B = (R \otimes_k R)_B \). Then we compute:

\[
R_{\kappa(\mathfrak{m})} \otimes_{\kappa(\mathfrak{m})} R_{\kappa(\mathfrak{m})} = R_B \otimes_{\kappa(\mathfrak{m})} \kappa(\mathfrak{m}) \otimes_{\kappa(\mathfrak{m})} R_B \otimes_{\kappa(\mathfrak{m})} \kappa(\mathfrak{m})
\]

\[
= R_B \otimes_{\kappa(\mathfrak{m})} R_B \otimes_{\kappa(\mathfrak{m})} \kappa(\mathfrak{m}) = (R \otimes_k R)_{\kappa(\mathfrak{m})}
\]

An important tool in the study of reduction modulo \( p \) is the generic freeness lemma \([\text{HH}99\text{, (2.1.4)}]\). By this lemma, we can always enlarge our choice of descent datum \( B \) to ensure that, for any finite collection \( \{M_i\} \) of finitely generated \( R \)-modules, the models \( (M_i)_B \) will be free.

Given any map \( \phi : M \to M' \) of finitely generated \( R \)-modules, we can find a suitable descent datum \( B \) so that \( \phi(M_B) \subseteq M'_B \). Then we say \( \phi_B := \phi|_{M_B} \) is a model for \( \phi \), and we have \( \phi_B \otimes_B k = \phi \). Given a bounded exact sequence of finitely generated \( R \)-modules, we can choose our descent datum \( B \) so that the models of these maps over \( B \) form an exact sequence of \( B \)-modules.

Similarly, given any scheme \( X \) of finite type over \( k \), we can find a descent datum \( B \subseteq k \) and a \( B \)-scheme \( X_B \) of finite type such that \( X = X_B \times_{\text{Spec } B} \text{Spec } k \). We can perform a similar construction for coherent sheaves on \( X \), morphisms \( f : Y \to X \), and divisors on \( X \). Given any closed point \( \mu \in \text{Spec } B \), the residue field \( \kappa(\mu) \) will have positive characteristic \( p \). We call the fiber \( X_{\kappa(\mu)} := (X_B)_\mu \) a \( \text{mod } p \) reduction of \( X \). Given a morphism of \( k \)-schemes \( f : Y \to X \), we get an induced morphism \( f_{\kappa(\mu)} : Y_{\kappa(\mu)} \to X_{\kappa(\mu)} \). If \( f \) is projective, then so is \( f_{\kappa(\mu)} \) for all \( \mu \) sufficiently general. If \( K_X \) is the canonical divisor of \( X \), then \( K_X_{\kappa(\mu)} \) is the canonical divisor of \( X_{\kappa(\mu)} \) for all \( \mu \) sufficiently general.

**Remark 5.4.** Working in the setting of \( \text{[Definition 4.14]} \) we can find a descent datum \( B \subseteq k \) and reduce the maps \( f : A' \xrightarrow{\psi} A \xrightarrow{\pi} A \) modulo \( p \). For \( \mu \in \text{MaxSpec } B \) sufficiently general, \( \pi_{\mu} : A_{\mu} \to A_{\mu} \) will still be a factorizing resolution of \( X_{\mu} \subseteq A_{\mu} \). In this way, we can define the multiplier ideal \( \mathcal{J}_{X_{\mu}}(A_{\mu}, Z_{\mu}) \) for \( \mu \) sufficiently general. By generic freeness, we can choose our descent datum so that \( \mathcal{O}_{A_{\mu}}[K_{A_{\mu}}/A_{\mu} - f_{\mu}^{-1}Z_{B}] + X'_{B} \), as well as all of its cohomology sheaves, are flat over \( B \). It follows from \([\text{Har}98\text{, Lemma 4.1]}\) that the \( \text{mod } p \) reduction of \( \mathcal{J}_X(A, Z) \) at \( \mu \) equals \( \mathcal{J}_{X_{\kappa(\mu)}}(A_{\kappa(\mu)}, Z_{\kappa(\mu)}) \) for \( \mu \) sufficiently general. Thus, to prove
Theorem 5.2. it suffices to show that \( \mathcal{J}_{X_t}(A_\kappa, Z_\kappa) = \tau_{I_t}(R_\kappa, J^{(s)}) \) for all \( s \in \text{MaxSpec} \ S \) sufficiently general.

Before proving [Theorem 5.2] we show how this theorem allows us to compare our subadditivity formula with the one obtained in [Eis10]. Consider the following setting:

**Setting 5.5.** \( R \) is a \( \mathbb{Q} \)-Gorenstein ring essentially of finite type over a field \( k \) of characteristic 0 and \( a, b \subseteq R \) are ideals. Set \( X = \text{Spec} \ R, Z_1 = V(a) \), and \( Z_2 = V(b) \). Let \( s, t \geq 0 \) be rational numbers. Let \( S \subseteq k \) be a descent datum and \( s \in \text{MaxSpec} \ S \). Set \( \kappa = \kappa(s) \).

In [Eis10], Eisenstein derives a new formula for the restriction of multiplier ideals to a closed subscheme. By carefully studying the case \( \Delta = X \times_k X \), where \( \Delta = X \) is the diagonal, he arrives at the containment:

\[
(7) \quad \text{adj}_\Delta (X \times_k X, sp^*_1 Z_1 + tp^*_2 Z_2) \cdot \mathcal{O}_\Delta \supseteq \overline{\text{Jac}_X \mathcal{J}(X, sZ_1 + tZ_2)}
\]

where \( \rho_i: X \times_k X \to X \) are the projection maps [Eis10]. The left-hand side of (7) is easily seen to be contained in the product of multiplier ideals, \( \mathcal{J}(X, Z_1) \mathcal{J}(X, Z_2) \). Now, the containment:

\[
\text{adj}_\Delta (X \times_k X, sp^*_1 Z_1 + tp^*_2 Z_2) = \mathcal{J}_\Delta (X \times_k X, sp^*_1 Z_1 + tp^*_2 Z_2 + d \Delta)
\]

where \( d = \dim X \). By [HY03], we have

\[
\left( \text{Jac}_X \mathcal{J}(X, sZ_1 + tZ_2) \right)_k = \overline{\text{Jac}(R_\kappa) \tau (R_\kappa, (a_\kappa)^s(b_\kappa)^t)}
\]

for \( s \) sufficiently general. So we see, combining Theorem 5.2 with (7) that

\[
\tau(I_\Delta)_\kappa \left( (R \otimes_k R)_\kappa, (a \otimes_k R)^s(R \otimes_k b)^t \right) \cdot (I_\Delta)^d \kappa \supseteq \overline{\text{Jac}(R_\kappa) \tau (R_\kappa, (a_\kappa)^s(b_\kappa)^t)}
\]

for all \( s \) sufficiently general. As \( \mathcal{O}^{(R \otimes_k R)_\kappa}(I_\Delta)^d \) is compatible with \( (I_\Delta)^d \kappa \) (c.f. Example 4.7), it follows from Lemma 4.8 that

\[
\tau(I_\Delta)_\kappa \left( (R \otimes_k R)_\kappa, (a \otimes_k R)^s(R \otimes_k b)^t \right) \cdot (I_\Delta)^d \kappa
\]

Here we’re using the fact that \( R_\kappa \otimes_k R_\kappa = (R \otimes_k R)_\kappa \). By Proposition 4.9, we have

\[
\tau(I_\Delta)_\kappa \left( (R \otimes_k R)_\kappa, (a \otimes_k R)^s(R \otimes_k b)^t \right) \cdot (I_\Delta)^d \kappa
\]

Thus we have shown, assuming [Theorem 5.2]

**Corollary 5.6.** Work in Setting 5.5. Then

\[
\text{Jac}(R_\kappa) \tau (R_\kappa, (a_\kappa)^s(b_\kappa)^t) \subseteq \tau(R_\kappa, \mathcal{O}^{(2)}(a_\kappa)^s(b_\kappa)^t)
\]

for all \( s \) sufficiently general.

Namely, our subadditivity formula,

\[
\tau(R_\kappa, \mathcal{O}^{(2)}(a_\kappa)^s(b_\kappa)^t) \subseteq \tau(R_\kappa, (a_\kappa)^s(b_\kappa)^t)
\]

is a sharper containment than the previously known formula,

\[
\text{Jac}(R_\kappa) \tau (R_\kappa, (a_\kappa)^s(b_\kappa)^t) \subseteq \tau(R_\kappa, (a_\kappa)^s(b_\kappa)^t).
\]
5.1. **Proof that** \( (\mathcal{O}_X(A, Z))_\kappa \cong \tau_\kappa(R_\kappa, \mathcal{O}_X) \). We prove Theorem 5.2 in two parts. The first part is easier and just uses the minimality of \( \tau_\kappa \). The second part follows an argument similar to [Tak08 Theorem 5.3] and requires a variant of Har‘a’s surjectivity theorem [Har98].

Recall that for any normal \( F \)-finite scheme \( X \) of characteristic \( p \) and any map \( \varphi: F_\kappa \mathcal{O}_X \to \mathcal{O}_X \) we can associate an effective \( \mathbb{Q} \)-divisor \( \Delta_\varphi \) on \( X \) such that \( K_X + \Delta_\varphi \) is \( \mathbb{Q} \)-Cartier with Cartier index not divisible by \( p \) (c.f. [BS13, Sch09 Section 3]). If \( h \) is a global section of \( \mathcal{O}_X \) and we set \( \varphi_h = \varphi(F_\kappa h \cdot -) \), then \( \Delta_{\varphi_h} = \Delta_\varphi + 1/(p^\ell - 1) \text{ div } h \).

Now suppose that \( X \) is an \( F \)-finite integral scheme with fraction field sheaf \( \mathcal{K}(X) \). As localization commutes with the \( F_\kappa \) functor, any map \( \varphi: F_\kappa \mathcal{O}_X \to \mathcal{O}_X \) induces a map \( \hat{\varphi}: F_\kappa \mathcal{K}(X) \to \mathcal{K}(X) \).

**Lemma 5.7** (C.f. [Sch08, HW02]). Let \( R \) be a normal \( F \)-finite domain and set \( X = \text{Spec } R \). Suppose \( \pi: Y \to X \) is a log-resolution of the ideals \( \mathfrak{a}_i \subseteq R \) and set \( \mathfrak{a}_i \mathcal{O}_Y = \mathcal{O}_Y(-G_i) \). Let \( t_i > 0 \) be a collection of rational numbers. Then for any map \( \varphi: F_\kappa \mathcal{O}_X \to \mathcal{O}_X \) we have

\[
\hat{\varphi} \left( F_\kappa \mathfrak{a}^{[\ell(p^\ell - 1)]} \mathcal{O}_Y \left( \left[ K_Y - \pi^* (K_X + \Delta) - \sum_i t_i G_i \right] + E \right) \right) \\
\subseteq \mathcal{O}_Y \left( \left[ K_Y - \pi^* (K_X + \Delta) - \sum_i t_i G_i \right] + E \right),
\]

where \( \hat{\varphi} \) is the induced map \( F_\kappa \mathcal{K}(X) \to \mathcal{K}(X) \) and \( \Delta \) is any divisor on \( X \) such that \( \Delta \leq \Delta_\varphi \) and \( K_X + \Delta \) is \( \mathbb{Q} \)-Cartier.

**Proof.** We work locally, so that \( \pi_* K_Y = K_X \). Let \( h \in \mathfrak{a}^{[\ell(p^\ell - 1)]} \). Then \( \Delta_{\varphi_h} = \Delta_\varphi + 1/(p^\ell - 1) \text{ div } h \). By the proof of [Sch08 Theorem 6.7], we have

\[
\hat{\varphi}_h \left( F_\kappa \mathcal{O}_Y \left( \left[ K_Y - \pi^* \left( K_X + \Delta_\varphi + \frac{1}{p^\ell - 1} \text{ div } h \right) \right] + F \right) \right) \\
\subseteq \mathcal{O}_Y \left( \left[ K_Y - \pi^* \left( K_X + \Delta_\varphi + \frac{1}{p^\ell - 1} \text{ div } h \right) \right] + F \right)
\]

for any integral effective divisor \( F \). Set

\[
F = \left[ K_Y - \pi^* (K_X + \Delta_\varphi) - \sum_i t_i G_i \right] - \left[ K_Y - \pi^* \left( K_X + \Delta_\varphi + \frac{1}{p^\ell - 1} \text{ div } h \right) \right]
\]

Then \( F \) is integral. Also \( F \) is effective, as \( \text{ div } h \geq \sum_i \ell(p^\ell - 1) G_i \), which means \( 1/(p^\ell - 1) \text{ div } h \geq \sum_i t_i G_i \). Thus we have

\[
\hat{\varphi}_h \left( F_\kappa \mathcal{O}_Y \left( \left[ K_Y - \pi^* (K_X + \Delta_\varphi) - \sum_i t_i G_i \right] \right) \right) \subseteq \mathcal{O}_Y \left( \left[ K_Y - \pi^* (K_X + \Delta_\varphi) - \sum_i t_i G_i \right] \right)
\]

Then for any effective divisor \( E \) we have

\[
\hat{\varphi}_h \left( F_\kappa \mathcal{O}_Y \left( \left[ K_Y - \pi^* (K_X + \Delta_\varphi) - \sum_i t_i G_i \right] + E \right) \right) \\
\subseteq \mathcal{O}_Y \left( \left[ K_Y - \pi^* (K_X + \Delta_\varphi) - \sum_i t_i G_i \right] + E \right)
\]

\[^6\text{Let } X \text{ be a scheme of positive characteristic. Then the Frobenius map on each affine chart of } X \text{ induces a morphism of schemes } F: X \to X \text{ called the absolute Frobenius morphism. We say } X \text{ is } F\text{-finite if } F_\kappa \mathcal{O}_X \text{ is a coherent } \mathcal{O}_X\text{-module for some (equivalently, all) } e > 0.\]
using the projection formula, the fact that \((F^e)^*(\mathcal{O}_Y(E)) = \mathcal{O}_Y(p^eE)\), and the fact that \(E \leq p^eE\).

Similarly, for any \(\Delta \leq \Delta_\varphi\) with \(K_X + \Delta\ \mathbb{Q}\)-Cartier, we have
\[
\left[ K_Y - \pi^*(K_X + \Delta) - \sum t_iG_i \right] - \left[ K_Y - \pi^*(K_X + \Delta_\varphi) - \sum t_iG_i \right] \geq 0,
\]
and so
\[
\hat{\varphi}_h \left( F^e \mathcal{O}_Y \left( \left[ K_Y - \pi^*(K_X + \Delta) - \sum t_iG_i \right] + E' \right) \right) 
\leq \mathcal{O}_Y \left( \left[ K_Y - \pi^*(K_X + \Delta) - \sum t_iG_i \right] + E' \right),
\]
for any effective divisor \(E'\), as desired.

**Theorem 5.8.** Work in Setting 5.1 and assume that \(\text{char } R = 0\). Then \(\mathcal{J}_{X_n}(A_\kappa, Z_n) \supseteq \tau_{I_\kappa}(R_\kappa, a_\kappa^l)\) for \(s\) sufficiently general.

**Proof.** For \(s\) sufficiently general, we have that \(p = \text{char } \kappa\) does not divide the denominator of any \(t_i\) and thus \(\tau_{I_\kappa}(R_\kappa, a_\kappa^l)\) is well-defined. Fix such an \(s\). We just need to prove two things:

- \(\varphi \left( F^e a_\kappa^{[p-1]} \mathcal{J}_{X_n}(A_\kappa, Z_\kappa) \right) \subseteq \mathcal{J}_{X_n}(A_\kappa, Z_\kappa)\), for all \(e > 0\) and \(\varphi \in \mathcal{O}_e^R\), and
- \(\mathcal{J}_{X_n}(A_\kappa, Z_\kappa) \nsubseteq I_\kappa\).

Set \(a_\kappa \mathcal{O}_{A'} = \mathcal{O}_{A'}(-F_i)\). Then, by definition,
\[
\mathcal{J}_{X_n}(A_\kappa, Z_\kappa) = (f_\kappa)_* \mathcal{O}_{A'_\kappa} \left( \left[ K_{A'_\kappa} - f_\kappa^*(K_{A_\kappa} - \sum t_i(F_i)_{\kappa}) \right] + X'_\kappa \right).
\]
We see that the first assertion follows from Lemma 5.7 using \(\Delta = 0\).

The second assertion is something we can check locally at \(I_\kappa\), so now we assume that \(R_\kappa\) is a local ring with maximal ideal \(I_\kappa\). But then, by assumption, \(\mathcal{O}_{A'_\kappa}(-\sum t_i(F_i)_{\kappa}) = \mathcal{O}_{A'_\kappa}(-cX'_\kappa)\). So we see
\[
\mathcal{J}_{X_n}(A_\kappa, Z_\kappa) = (f_\kappa)_* \mathcal{O}_{A'_\kappa} \left( \left[ \psi^*K_{A'_\kappa} - f_\kappa^*(K_{A_\kappa}) \right] \right)
\]
and \(\psi^*K_{A'_\kappa} - f_\kappa^*(K_{A_\kappa})\) has no support along \(X'_\kappa\). \(\square\)

5.2. **Proof that** \((\mathcal{J}_X(A, Z))^\kappa \subseteq \tau_{I_\kappa}(R_\kappa, a_\kappa^l)\). For the other containment, we use a similar argument to the one in [Tak08]. First, we recall Haras's surjectivity theorem. The following statement is slightly stronger than the one found in [Har98]; in [Har98], the author assumes that \(X\) is the blow up of an ideal sheaf on \(Y\). However, the same proof actually shows the following statement, where \(X\) is just assumed to be projective over \(Y\) and smooth.

**Theorem 5.9** ([Har98 Section 4.3]). Let \(Y = \text{Spec } R\), where \(R\) is finitely generated over a field \(k\) of characteristic 0, and let \(X\) be a smooth Noetherian scheme projective over \(Y\). Suppose \(E\) is a reduced simple normal crossings divisor on \(X\) and \(D\) an ample divisor with \(\text{supp}(D - [D]) \subseteq \text{supp } E\). Choose some finitely generated \(\mathbb{Z}\)-subalgebra \(B\) of \(k\), over which we do our reduction mod \(p\). For any closed point \(s \in S = \text{Spec } B\) with residue field \(\kappa = \kappa(s)\), let \(Y_\kappa, X_\kappa, E_\kappa, D_\kappa\) be the fibers of the corresponding objects over \(s\). Then, for sufficiently general closed points \(s\),

\(\begin{align*}
(a) \quad & H^j(X_\kappa, \Omega_{X_\kappa/\kappa}^\infty(\log E_\kappa)(-E_\kappa - \lceil -p^eD_\kappa \rceil)) = 0, \quad i + j > d, e \geq 0 \\
(b) \quad & H^j(X_\kappa, \Omega_{X_\kappa/\kappa}^\infty(\log E_\kappa)(-E_\kappa - \lceil -p^{e+1}D_\kappa \rceil)) = 0, \quad i + j > 0, e \geq 0
\end{align*}\)

where \(d = \text{dim } X\) and \(p = \text{char } \kappa(s)\).
Combined with [Har98] Proposition 3.6 (as stated), we obtain the following result:

**Corollary 5.10** (c.f. [Har98] Section 4.4). Using notation as in Theorem 5.9 the map 
\[(F^e)^\gamma : H^0(X_\kappa, F^e_\kappa \omega_{X_\kappa}([p^e D_\kappa])) \to H^0(X_\kappa, \omega_{X_\kappa}([D_\kappa]))\]
is surjective for \(e > 0\) and for sufficiently general \(s \in S\), where \(\kappa = \kappa(s)\).

We will also need the following lemmas:

**Lemma 5.11.** Work in Setting 5.1 and assume that \(R\) has characteristic 0. There exists \(d \in R \setminus I\) such that, for all sufficiently general \(s\), a power of \(d_\kappa\) (depending on \(s\)) is an \(\mathfrak{a}_s^e\)-test element along \(I_\kappa\) in \(R_\kappa\).

**Proof.** If \(s\) is sufficiently general, then \(p = \text{char} \kappa(s)\) will not divide the denominator of any \(\tau_i\) and \(C_\kappa := \mathcal{C}^{R_\kappa, \mathfrak{a}_s^e}_{R_\kappa\mathfrak{a}_s^e}\) will satisfy the conditions of Setting 4.2. As \(R_I\) is regular, we can find a regular sequence \((x_1, \ldots, x_c)\) in \(R\) such that \((x_1, \ldots, x_c)R_I = IR_I\). Set \(J = (x_1, \ldots, x_c)\). Then there exists an element \(d \in R \setminus I\) such that \(R_d\) is regular, \(R_d/IR_d\) is regular, and \(J^m R_d \subseteq a_s R_d\) for all \(i\). Note that \((R_d)_\kappa\) is the same as \(R_\kappa\) localized at \(d_\kappa\). We set \(R_{d,\kappa} := (R_\kappa)_d\) and \(C_{d,\kappa} := (C_\kappa)_d\). Then we have

\[\tau_{J_\kappa}(R_\kappa, C_\kappa)R_{d,\kappa} = \tau_{J_\kappa R_{d,\kappa}}(R_{d,\kappa}, C_{d,\kappa}) \supseteq \tau_{J_\kappa R_{d,\kappa}}(R_{d,\kappa}, J^e R_{d,\kappa}),\]

where the containment follows quickly from the minimality of \(\tau_{J_\kappa R_{d,\kappa}}(R_{d,\kappa}, J^e R_{d,\kappa})\). By Lemma 5.12, we see that \(\tau_{J_\kappa}(R_\kappa, C_\kappa)R_{d,\kappa} = R_{d,\kappa}\). Thus, there exists some \(N\) such that \(d_\kappa^N \in \tau_{J_\kappa}(R_\kappa, C_\kappa)\). As \(d_\kappa^N \in R_\kappa \setminus I_\kappa\), we’re done.

**Lemma 5.12.** Let \(k\) be a perfect field of characteristic \(p\). Let \(R\) be a regular \(k\)-algebra essentially of finite type and \(I\) a prime ideal generated by a regular sequence of length \(c\). Suppose also that \(R/I\) is regular. Then \(\tau_I(R, I^e) = R\).

**Proof.** This fact is well-known to experts, and is essentially shown in [Tak13] Theorem 3.2.

**Lemma 5.13.** Let \(I \subseteq R\) be a prime ideal such that \(R_I\) is regular. Let \(a \subseteq R\) be an ideal such that \(a R_I = I^m R_I\) for some \(m \geq 0\). Then there exists some \(\xi \in R \setminus I\) such that \(\xi \mathfrak{a}^m \subseteq a^n\) for all integers \(n\).

**Proof.** By [SH06] Proposition 5.3.4], there is some integer \(k\), such that \(\mathfrak{a}^m = a^{n-k} \mathfrak{a}^k\) for all \(n \geq k\). As \(R_I\) is regular, we have \(IR_I\) generated by a regular sequence and is therefore a normal ideal, meaning \(I^n R_I\) is integrally closed for all \(n\) (see, for instance, [SH06] Exercise 5.7]). As integral closure of ideals commutes with localization, we have

\[\mathfrak{a}^m R_I = a^n R_I = I^{nm} R_I = I^{nm} R_I = a^n R_I,\]

for all \(n\). Thus, for \(n = 1, \ldots, k\) there exist elements \(\xi_n \in R \setminus I\) satisfying \(x_i \mathfrak{a}^m \subseteq a^n\). Then we can set \(\xi = \xi_1 \cdots \xi_k\).

**Lemma 5.14.** Work in Setting 5.1 and assume that \(\text{char} R = 0\). There exists some \(\xi \in R \setminus I\) such that \(\xi^{-1} \mathfrak{a}^m \subseteq a^n\) for all \(p > 0\) and all \(e > 0\) sufficiently divisible, where \(q = p^e\).

**Proof.** For each \(i\), write \(t_i = a_i/b_i\). Set \(m\) to be the least common multiple of the \(b_i\), so that for each \(i\) there exists an integer \(a_i'\) such that \(t_i = a_i'/m\). For \(p\) sufficiently large, we have \(p\)
does not divide \( b_i \) for each \( i \). Then for \( e \) sufficiently divisible, we have \( m \mid (p^e - 1) \). Thus

\[
\prod_i a_i^{[t_i(p^e-1)]} = \prod_i a_i^{\alpha'_i(p^e-1)/m} = \left( \prod_i a_i^{\alpha'_i} \right)^{\frac{p^e-1}{m}}.
\]

Then we can find the desired element \( \xi \) by applying Lemma 5.13 to the ideal

\[
a = \prod_i a_i^{\alpha'_i}.
\]

This completes the proof. □

**Theorem 5.15.** Work in Setting 5.1 and assume that \( \text{char} R = 0 \). Then \( I(X, Z) \subseteq \tau_{I_0}(R, a'_s) \) for all \( s \) sufficiently general.

**Proof.** Similarly to Definition 4.14, let \( \pi: \overline{A} \to A \) be a factorizing resolution of \( X \subseteq A \) such that \( \pi^{-1}Z \cup \text{supp}(\text{Exc}(\pi)) \cup \text{supp}(\pi^*K_A) \) is a simple normal crossings scheme. Let \( X \subseteq A \) be the strict transform of \( X \) in \( A \) and let \( \psi: A' \to \overline{A} \) be the blow up along \( X \). Set \( f = \pi \circ \psi \) and \( X' = \text{Exc}(\psi) \). We have the following diagram:

\[
\begin{array}{ccc}
X' & \subseteq & A' \\
\downarrow g & & \downarrow \psi \\
\overline{X} & \subseteq & \overline{A} \\
\downarrow h & & \downarrow \pi \\
X & \subseteq & A.
\end{array}
\]

Let \( \mathcal{O}_{A'} = \mathcal{O}_{A'}(-F_i) \) and set \( F = \sum t_i F_i \). Note that \( F_i \geq 0 \) for all \( i \). Note that we can construct \( \pi \) by taking \( A \) and successively blowing up along closed subschemes contained in the singular locus of the previous blow up. As \( A \) is normal, we can assume that \( \pi \) is constructed by successively blowing up along closed subschemes of codimension at least 2.

As \( A \) is smooth at the generic point of \( \overline{X} \), we can assume these closed subschemes are disjoint from the strict transforms of \( \overline{X} \). Thus, we can find an exceptional \( \pi \)-ample anti-effective \( \mathbb{Q} \)-divisor on \( \overline{A} \) which is disjoint from \( \overline{X} \), c.f. [Sta, Tag 01OF]. Call this divisor \( \overline{H} \). After possibly performing more blow ups, we may assume that \( \overline{H} \) has simple normal crossings support. Then \( H := \psi^*\overline{H} - X' \) is an exceptional \( f \)-ample anti-effective \( \mathbb{Q} \) divisor with simple normal crossings support on \( A' \). We may assume that

\[
[f^*K_A + F - H] = [f^*K_A + F],
\]

possibly after multiplying \( H \) by a sufficiently small, positive rational number.

Employing Lemma 5.11, choose some \( d \in R \setminus I \) such that, for \( s \) sufficiently general, there is some \( N \) such that \( d^N \) is an \( a'_s \)-test element along \( I_0 \) in \( R_0 \). Choose also an element \( \xi \in R \setminus I \) as in Lemma 5.14. Fix a canonical divisor \( K_A \) on \( A \) so that \( K_A - \pi^*K_A \) is \( \pi \)-exceptional. Then there exists \( \eta \in R \setminus I \) such that

\[
\psi^*K_A - [mf^*K_A] + \text{div} \eta \leq f^*((1-m)K_A)
\]
for all \( m \) such that \((1 - m)K_A\) is Cartier. Indeed, we have
\[
f^* ((1 - m)K_A) - \psi^* K_X + [mf^* K_A] = -\psi^* K_X + [mf^* K_A + f^* ((1 - m)K_A)] \\
= -\psi^* K_X + [f^* K_A] \\
= -[\psi^* K_X - f^* K_A].
\]

Since the support of \( K_X - \pi^* K_A \) is disjoint from \( X \), the support of
\[
\psi^* K_X - f^* K_A = \psi^* (K_X - \pi^* K_A)
\]
is disjoint from \( X' \). Further, this divisor is \( f \)-exceptional. Thus we have
\[
H^0 \left( A, f_* \mathcal{O}_A \left( f^* ((1 - m)K_A) - \psi^* K_X + [mf^* K_A] \right) \right) \subseteq R
\]
and also
\[
H^0 \left( A, f_* \mathcal{O}_A \left( f^* ((1 - m)K_A) - \psi^* K_X + [mf^* K_A] \right) \right) \nsubseteq I,
\]
so we can find the desired element \( \eta \) by taking
\[
\eta \in H^0 \left( A, f_* \mathcal{O}_A \left( f^* ((1 - m)K_A) - \psi^* K_X + [mf^* K_A] \right) \right) \setminus I.
\]

Next, we define
\[
B = f^* K_A + F + \varepsilon \text{div}_{A'}(d\xi \eta) - H,
\]
where \( \varepsilon > 0 \) is chosen to be small enough such that
\[
[B] = [f^* K_A + F].
\]

Note that \( B \) is \( f \)-anti-ampl, since \( f^* K_A \) is \( f \)-numerically trivial, \( F \) and \( \text{div}_{A'}(d\xi \eta) \) are \( f \)-anti-nef, and \(-H\) is \( f \)-anti-ampl. For all \( m \in \mathbb{N}_{>0} \), we have a short exact sequence of sheaves on \( A' \),
\[
0 \to \mathcal{O}_{A'}(K_{A'} - [mB]) \to \mathcal{O}_{A'}(K_{A'} - [mB] + X') \to \mathcal{O}_{X'} \left( ((K_{A'} - [mB] + X')|_{X'}) \right) \to 0.
\]

Then we get an exact sequence:
\[
0 \to H^0(A', \mathcal{O}_{A'}(K_{A'} - [mB])) \to H^0(A', \mathcal{O}_{A'}(K_{A'} - [mB] + X')) \\
\to H^0(X', \mathcal{O}_{X'}(K_{X'} - [mB]|_{X'})) \to H^1(A', \mathcal{O}_{A'}(K_{A'} - [mB])),
\]
noting that \((K_{A'} + X')|_{X'} = K_{X'}\) by the adjunction formula. Now, \(-mB\) is an \( f \)-ample divisor whose fractional part has simple normal crossings support. By Kawamata-Viehweg vanishing [Laz04, Corollary 9.1.20], we have
\[
H^1(A', \mathcal{O}_{A'}(K_{A'} - [mB])) = H^1(A', \mathcal{O}_{A'}(K_{A'} + [-mB])) = 0.
\]

Thus, we have the short exact sequence,
\[
0 \to H^0(A', \mathcal{O}_{A'}(K_{A'} - [mB])) \to H^0(A', \mathcal{O}_{A'}(K_{A'} - [mB] + X')) \\
\to H^0(A', \mathcal{O}_{X'}(K_{X'} - [mB]|_{X'})) \to 0.
\]

Set \( p = \text{char} \, k(s) \). For \( s \) sufficiently general, we have:

- The map
  \[
  (F_{\kappa}^e)^\vee : H^0 \left( A'_\kappa, F^e_* \omega_{A'_\kappa} \left( -[p^e B_\kappa] \right) \right) \to H^0 \left( A'_\kappa, \omega_{A'_\kappa} \left( -[B_\kappa] \right) \right)
  \]
is a surjection for all \( e > 0 \). This is possible by [Corollary 5.10]
The map
\[(F^e_{X_k})^\vee: H^0(X'_k, F^e_{X_k} \omega_{X_k}(-[p^eB_k|_{X_k}])) \to H^0(X'_k, \omega_{X_k}(-[B_k|_{X_k}]))\]

is a surjection for all \(e\). This is also possible by Corollary 5.10. Note that \(X'\) is a projective bundle over \(X\), so the original statement of Hara’s surjectivity theorem would not apply to \(X' \to X\).

\(p\) does not divide the Cartier index of \(K_A\).
\(p\) does not divide the denominator of any \(t_i\).
Some power of \(t\) is an \(a^t\)-test element along \(I_\kappa\) in \(R_\kappa\).
\(H^0(A'_\kappa, \omega_{A'_\kappa}(-[mB_\kappa])) = H^0(A'_\kappa, \omega_{A'_\kappa}(-[mB_\kappa]))\) for all \(m \in \mathbb{N}_{>0}\). This is possible because \(B\) is a \(\mathbb{Q}\)-divisor. Indeed, let \(v\) be the least common multiple of the denominators of the coefficients appearing in \(B\). Then
\[
\{[mB] \mid m \geq 0\} = \{uvB + [mB] \mid 0 \leq m < v, u \geq 0\}.
\]

By generic freeness, we can choose our decent datum \(S\) to ensure that the coherent sheaves \(\omega_{A'_\kappa}(-[mB_\kappa])\) and \(\mathcal{O}_{A'_\kappa}(-vB_\kappa)\), as well as their cohomologies, are flat for \(0 \leq m < v\). Then the result follows from [Har98, Lemma 4.1].

Similarly, we can ensure that
\[H^0(A'_\kappa, \omega_{A'_\kappa}(X'_\kappa - [mB_\kappa])) = H^0(A'_\kappa, \omega_{A'_\kappa}(X'_\kappa - [mB_\kappa]))\]
for all \(m \in \mathbb{N}_{>0}\), and
\[H^0(X'_\kappa, \omega_{X'_\kappa}(-[mB_\kappa])) = H^0(X'_\kappa, \omega_{X'_\kappa}(-[mB_\kappa]))\]
for all \(m \in \mathbb{N}_{>0}\).

Fix such an \(s\). Fix also a number \(N\) so that \(d^N_\kappa\) is an \(a^t\)-test element along \(I_\kappa\) in \(R_\kappa\). Then for all \(e \in \mathbb{N}\) sufficiently divisible we have:

- \(\xi a^{t(p^e - 1)} \subseteq a^{t(p^e - 1)}\),
- \(p^e \varepsilon > N\),
- \((1 - p^e)K_A\) is Cartier, and
- \(t_i(p^e - 1) \in \mathbb{Z}\) for all \(i\).

Fix such an \(e\). With that taken care of, reduce the whole setup modulo \(p\) at \(\kappa\) and set \(q = p^e\).

We get the following diagram:

\[
\begin{array}{c}
0 \to H^0(A'_\kappa, F^e_{\omega_{A'_\kappa}}(-[qB_\kappa])) \to H^0(A'_\kappa, F^e_{\omega_{A'_\kappa}}(X'_\kappa - [qB_\kappa])) \to H^0(X'_\kappa, F^e_{\omega_{X'_\kappa}}(-[qB_\kappa])) \to 0 \\
\downarrow(F^e_{A'_\kappa})^\vee \quad \downarrow(F^e_{\omega_{A'_\kappa}})^\vee \quad \downarrow(F^e_{\omega_{X'_\kappa}})^\vee \\
0 \to H^0(A'_\kappa, \omega_{A'_\kappa}(-[B_\kappa])) \to H^0(A'_\kappa, \omega_{A'_\kappa}(X'_\kappa - [B_\kappa])) \to H^0(X'_\kappa, \omega_{X'_\kappa}(-[B_\kappa])) \to 0
\end{array}
\]

By the five lemma, as well as our assumptions on \(p\), we see that
\[(F^e_{A'_\kappa})^\vee: H^0(A'_\kappa, F^e_{\omega_{A'_\kappa}}(X'_\kappa - [qB_\kappa])) \to H^0(A'_\kappa, \omega_{A'_\kappa}(X'_\kappa - [B_\kappa]))\]
is a surjection. But the right-hand side is exactly (the global sections of) \(\mathcal{J}_{X_\kappa}(A_\kappa, Z_\kappa)\) by Lemma 4.15. Thus it’s enough to show that the image of this map is contained in \(I_{\kappa}(R_\kappa, a^t\)).

It follows from a straightforward computation that
\[H^0(A'_\kappa, F^e_{\omega_{A'_\kappa}}(X'_\kappa - [qB_\kappa])) \subseteq R_\kappa((1 - q)K_A) \prod_i (a_i)_\kappa^{t_i(q-1)} a^N_\kappa \xi^N_\kappa.\]
variety over a perfect field $k$.

We let $\Sigma$ be a strictly convex cone in $\mathbb{R}^n$. For each $i$, let $\varphi_i = (1 - q)K_{\Sigma_i}$, see [Sch09]. So we have shown that

$$\mathcal{I}_{X_k}(A_k, Z_k) \subseteq \varphi \left( F^{e \cdot [t(p^{e-1})]} \cdot d \right)$$

for some $\varphi \in \mathcal{C}_e$. \hfill $\Box$

It’s worth noting the following analog to Lemma 2.11(b). This proposition follows from Lemma 5.13, and it provides further evidence that $\tau_I(R, \mathcal{C})$ is well-behaved in Setting 5.1

**Proposition 5.16.** Work in Setting 5.1 and assume that $\text{char } R = p$. For each $i$, let $b_i = \overline{a_i}$ be the integral closure of $a_i$. Then

$$\tau_I \left( R, \prod_i a_i^t \right) = \tau_I \left( R, \prod_i b_i^t \right).$$

**Proof.** The $\subseteq$ inclusion follows from Lemma 4.8, so we prove the reverse inclusion. As $b_i^n \subseteq \overline{a_i^n}$ for all $i$ and $n$, it follows from Lemma 5.13 that there exists some $\xi \in R \setminus I$ such that

$$\xi \prod_i b_i^t(p^{e-1}) \subseteq \prod_i b_i^t(p^{e-1}).$$

Let $d \in \tau_I(R, \prod_i a_i) \setminus I$ be arbitrary. Then Lemma 4.11 tells us:

$$\tau_I \left( R, \prod_i b_i^t \right) = \sum_{e \geq 0} \sum_{\varphi \in \mathcal{C}_e} \varphi \left( F^{e \cdot d} \xi \prod_i b_i^t(p^{e-1}) \right)$$

$$\subseteq \sum_{e \geq 0} \sum_{\varphi \in \mathcal{C}_e} \varphi \left( F^{e \cdot d} \prod_i a_i^t(p^{e-1}) \right) = \tau_I \left( R, \prod_i a_i^t \right).$$

\hfill $\Box$

6. **Computing $\mathcal{D}^{(2)}(R)$ for Affine Toric Varieties**

Our next goal is to find a nice description of $\mathcal{D}^{(2)}$ that allows us to compute test ideals $\tau(R, \mathcal{D}^{(2)}(\prod_i a_i^t))$. The case where $R$ is a normal affine semigroup ring over a field $k$ (equivalently, Spec $R$ is an affine toric variety over $k$) turns out to be quite tractable.

**Setting 6.1.** We let $\Sigma$ be a strictly convex cone in $\mathbb{R}^n$ and $X = X(\Sigma)$ the associated toric variety over a perfect field $k$ of characteristic $p > 0$. We let $R$ be the coordinate ring of $X$. In particular, $R$ is a toric ring, that is, a normal affine semigroup ring. For all rays $\rho \subseteq \Sigma$, we let $v_{\rho}$ denote the primitive generator of $\rho$. That is, $v_{\rho}$ is the shortest nonzero vector in $\mathbb{Z}^n \cap \rho$.

Choose $e > 0$ and set $q = p^e$. In this section, it will be more convenient to use the notation $R^{1/q}$ rather than $F^{e \cdot R}$. Working in Setting 6.1, we have a nice $k$-basis for $\text{Hom}_R(R^{1/q}, R)$. In particular, we let

$$k[T] = k \left[ x_1^{\pm 1}, \ldots, x_n^{\pm 1} \right]$$
be the coordinate ring of the \( n \)-dimensional torus \( T \). Similarly, we let
\[
k[T \times T] = k[T] \otimes_k k[T]
\]
be the coordinate ring of the \( 2n \)-dimensional torus, \( T \times T \). We let \( q = p^e \), and we adopt the notation,
\[
\frac{1}{q} \mathbb{Z}^n := \left\{ \left( \frac{a_1}{q}, \ldots, \frac{a_n}{q} \right) \mid a_1, \ldots, a_n \in \mathbb{Z} \right\}.
\]
For any vector \( u \in \frac{1}{q} \mathbb{Z}^n \), we adopt the shorthand notation
\[
x^u := \prod_{i=1}^n x_i^{u_i}.
\]
Then for all \( a \in \frac{1}{q} \mathbb{Z}^n \) we define a map \( \pi_a \) in terms of its action on monomials: for all \( u \in \frac{1}{q} \mathbb{Z}^n \), set
\[
\pi_a(x^u) = \begin{cases} x^{a+u}, & a + u \in \mathbb{Z}^n \\ 0, & \text{otherwise} \end{cases}
\]
It’s not hard to see that these maps \( \pi_a \) generate all the maps \( k[T]^{1/q} \rightarrow k[T] \):

**Lemma 6.2 ([Pay09]).** In the notation above, the set \( \{ \pi_a \mid a \in \frac{1}{q} \mathbb{Z}^n \} \) is a \( k \)-vector space basis of \( \text{Hom}_{k[T]}(k[T]^{1/q}, k[T]) \).

Further, every map \( R^{1/q} \rightarrow R \) extends to a unique map \( k[T]^{1/q} \rightarrow k[T] \). Thus, each map in \( \text{Hom}(R^{1/q}, R) \) is just a map \( k[T]^{1/q} \rightarrow k[T] \) that happens to send \( R^{1/q} \) into \( R \). Payne has characterized such maps: we define the anticanonical polytope of \( R \),
\[
P_{-K_X} = \{ u \in \mathbb{R}^n \mid \langle u, v_\rho \rangle \geq -1 \text{ for all rays } \rho \subset \Sigma \}
\]
Then we have:

**Proposition 6.3 ([Pay09]).** Work in Setting 6.1. Then the set of maps \( \pi_a \) where \( a \) is in \( \text{int}(P_{-K_X}) \cap \frac{1}{q} \mathbb{Z}^n \) forms a \( k \)-vector space basis for \( \text{Hom}_R(R^{1/q}, R) \).

Our characterization of \( \mathcal{D}^{(2)} \) is as follows:

**Theorem 6.4.** Work in Setting 6.1. Then \( \mathcal{D}^{(2)}_e(R) \) is generated as a \( k \)-vector space by the maps \( \pi_a \) where \( a \in \frac{1}{p^e} \mathbb{Z}^n \cap \text{int}(P_{-K_X}) \) and the interior of \( P_{-K_X} \cap (a - P_{-K_X}) \) contains a representative of each equivalence class in \( \frac{1}{p^e} \mathbb{Z}^n / \mathbb{Z}^n \).

First, we must prove a lemma. This lemma is similar to [CHP+16, Theorem 7.3]. Note however that we do not assume that \( \varphi \) is a splitting.

**Lemma 6.5.** Let \( \varphi = \sum c_{a,a'} \pi_a \otimes \pi_{a'} \) be a map in \( \text{Hom}_{k[T \times T]}(k[T \times T]^{1/q}, k[T \times T]) \). Then \( \varphi \) is compatible with \( I_\Delta \) if and only if for all equivalence classes \( [u_1], [u_2] \in \frac{1}{q} \mathbb{Z}^n / \mathbb{Z}^n \), we have, for all \( d \in \frac{1}{q} \mathbb{Z}^n \),
\[
\sum_{a \in [u_1]} c_{a,d-a} = \sum_{b \in [u_2]} c_{b,d-b}
\]
Proof. Note that the ideal $I_{\Delta}^{1/q} \subseteq k[T \times T]^{1/q}$ is generated by the elements
\[
\left\{ x^u \otimes x^{-u} - 1 \mid u \in \frac{1}{q} \mathbb{Z}^n \right\}.
\]
Since $k[T \times T]$ is a smaller ring than $k[T \times T]^{1/q}$, we need more elements to generate $I_{\Delta}^{1/q}$ as a $k[T \times T]$-module. However, elements of the form $x^v \otimes x^{v'}$, where $v$ and $v'$ are vectors in $\frac{1}{q} \mathbb{Z}^n$, generate $k[T \times T]^{1/q}$ as a $k[T \times T]$-module (indeed, as a $k$-vector space). Thus, the set
\[
\left\{ x^u \otimes x^{v'} (x^u \otimes x^{-u} - 1) \mid u, v, v' \in \frac{1}{q} \mathbb{Z}^n \right\}
\]
generates $I_{\Delta}^{1/q}$ as a module over $k[T \times T]$.

Suppose $\varphi = \sum_{a,a'} c_{a,a'} \pi_a \otimes \pi_{a'}$ is compatible with the diagonal. This is equivalent to asserting that
\[
(8) \quad \varphi \left( x^v \otimes x^{v'} (x^u \otimes x^{-u} - 1) \right) \equiv 0 \mod I_{\Delta}
\]
for all $u, v, v' \in \frac{1}{q} \mathbb{Z}^n$. Set $\varphi_{v,v'} := \varphi(x^v \otimes x^{v'} \cdot \cdot \cdot )$. Then the condition in (8) is equivalent to saying
\[
\varphi_{v,v'} (x^u \otimes x^{-u} - 1) \equiv 0 \mod I_{\Delta},
\]
for all $u, v, v' \in \frac{1}{q} \mathbb{Z}^n$, or in other words,
\[
(9) \quad \varphi_{v,v'} (x^u \otimes x^{-u}) \equiv \varphi_{v,v'} (1) \mod I_{\Delta}.
\]
Now, it’s easy to see that $\pi_a \otimes \pi_{a'} (x^v \otimes x^{v'} \cdot \cdot \cdot ) = \pi_{a+v} \otimes \pi_{a'+v'}$. Thus
\[
\varphi_{v,v'} = \sum_{a,a' \in \frac{1}{q} \mathbb{Z}^n} c_{a,a'} \pi_a \otimes \pi_{a'} = \sum_{a,a' \in \frac{1}{q} \mathbb{Z}^n} c_{a-a',v-v'} \pi_{a} \otimes \pi_{a'}
\]
This means that (9) is equivalent to saying
\[
\sum_{a \in \frac{1}{q} \mathbb{Z}^n, a' \in \frac{1}{q} \mathbb{Z}^n} c_{a-a',v-v'} x^{a+a'} \otimes x^{a'-a} \equiv \sum_{b,b' \in \mathbb{Z}^n} c_{b-b',v-v'} x^b \otimes x^{b'} \mod I_{\Delta},
\]
and this is the case if and only if
\[
(10) \quad \sum_{a \in \frac{1}{q} \mathbb{Z}^n, a' \in \frac{1}{q} \mathbb{Z}^n} c_{a-a',v-v'} x^{a+a'} = \sum_{b,b' \in \mathbb{Z}^n} c_{b-b',v-v'} x^b x^{b'}.
\]
Now, the above is an equality of Laurent polynomials, so it holds if and only if the corresponding coefficients for each exponent of $x$ are the same. So our initial assertion (8) holds if and only if, for all $d \in \mathbb{Z}^n$ and all $u, v, v' \in \frac{1}{q} \mathbb{Z}^n$, we have
\[
\sum_{d \in \frac{1}{q} \mathbb{Z}^n} c_{a-a',d-d} = \sum_{b \in \mathbb{Z}^n} c_{b-b',v-v'}
\]
(In other words, we’re setting $d = a + a' = b + b'$ in (10)). By setting $U = -u - v$ and $D = d - v' - v$, the above is equivalent to
\[
\sum_{a \in U \cup \mathbb{Z}^n} c_{a,D-a} = \sum_{b \in -v + \mathbb{Z}^n} c_{b,D-b}
\]
where $U, v, D$ independently range over $\frac{1}{q} \mathbb{Z}^n$. This completes the proof. \qed
Corollary 6.6. Let $R$ be an affine semigroup ring. The Cartier algebra $C^{R\otimes_k R, I\Delta_\circ}$ is “graded”, in the sense that the map $\sum_{a,a'} c_{a,a'} \pi_a \otimes \pi_{a'}$ is compatible with the diagonal if and only if, for each $d \in \frac{1}{q} \mathbb{Z}^n$, we have $\sum_{a+a'=d} c_{a,a'} \pi_a \otimes \pi_{a'}$ is compatible with the diagonal. It follows that $D(2)(R)$ is generated over $k$ by the maps $\pi_d$ in $D(2)(R)$.

Proof. First, we focus on the case that $R = k[T]$. The first part follows immediately from the above lemma: a map $\sum_{a,a'} c_{a,a'} \pi_a \otimes \pi_{a'}$ satisfies the condition in Lemma 6.5 if and only if the maps $\sum_{a+a'=d} c_{a,a'} \pi_a \otimes \pi_{a'}$ satisfy the condition in Lemma 6.5 for all $d$.

For the second assertion, let $\psi \in C^{R\otimes_k R, I\Delta_\circ}$ be arbitrary. Then $\tilde{\psi} := \psi|_{R\otimes_k R/I\Delta}$ is an arbitrary element of $D(2)(R)$. If

$$\psi = \sum_{a,a'} b_{a,a'} \pi_a \otimes \pi_{a'},$$
then by the first assertion, we have $\psi' := \sum_{a+a'=u} b_{a,a'} \pi_a \otimes \pi_{a'}$ is also in $C^{R\otimes_k R, I\Delta_\circ}$ for all $u \in \frac{1}{q} \mathbb{Z}^n$. Now let $v \in \frac{1}{q} \mathbb{Z}^n$ be arbitrary. We compute:

$$\psi' \mid_\Delta (x^v) = \left( \sum_{a} b_{a,u-a} \pi_a (x^v) \otimes \pi_{u-a}(1) \right)\bigg|_{R\otimes_k R/I\Delta}$$

$$= \left( \sum_{a \in v+\mathbb{Z}^n \text{ where } u-a \in \mathbb{Z}^n} b_{a,u-a} x^{a+v} \otimes x^{u-a} \right)\bigg|_{R\otimes_k R/I\Delta}$$

$$= \begin{cases} \left( \sum_{a \in v+\mathbb{Z}^n} b_{a,u-a} \right) x^{u+v}, & u + v \in \mathbb{Z}^n \\ 0, & \text{otherwise} \end{cases}$$

Note that, a priori, it looks like the coefficient of $\pi_u(x^v)$ in (14) depends on $v$ (or even worse, on our choice of lifting of $x^v$ to $R \otimes_k R$), but by Lemma 6.5 this is not the case.

Proof of Theorem 6.4. Any map in $\psi \in \text{Hom}_{R\otimes_k R}(R \otimes_k R) \rightarrow k[T \times T]$ extends to a unique map $k[T \times T]^{1/q} \rightarrow k[T \times T]$, and one of these maps is compatible with the diagonal whenever the other map is (c.f. [BK07, Lemma 1.1.7]).

Note that, a priori, it looks like the coefficient of $\pi_u(x^v)$ in (14) depends on $v$ (or even worse, on our choice of lifting of $x^v$ to $R \otimes_k R$), but by Lemma 6.5 this is not the case.
compatible with \( I_\Delta \) that restricts to a map \( (R \otimes_k R)^{1/q} \to R \otimes_k R \), and such that the sum \( \sum_{a+a'=d} c_{a,a'} \) is non-zero. By Lemma 6.5, this means for all \([u_1], [u_2] \in \frac{1}{q} \mathbb{Z}^n / \mathbb{Z}^n\), we have
\[
\sum_{a \in [u_1]} c_{a,d-a} = \sum_{b \in [u_2]} c_{b,d-b}.
\]
Further, if \( \sum_{a+a'=d} c_{a,a'} \neq 0 \), then there exists some \([u] \in \frac{1}{q} \mathbb{Z}^n / \mathbb{Z}^n\) such that \( \sum_{a \in [u]} c_{a,d-a} \neq 0 \). This is just because
\[
\sum_{a+a'=d} c_{a,a'} = \sum_{[u] \in \frac{1}{q} \mathbb{Z}^n / \mathbb{Z}^n} \sum_{a \in [u]} c_{a,d-a} 
\]
Using Lemma 6.5 again, this means that for all \([u] \in \frac{1}{q} \mathbb{Z}^n / \mathbb{Z}^n\), the sum \( \sum_{a \in [u]} c_{a,d-a} \) is nonzero. In particular, for all \([u] \in \frac{1}{q} \mathbb{Z}^n / \mathbb{Z}^n\), there is some \( a \in [u] \) such that \( c_{a,d-a} \neq 0 \). Since \( \varphi \) restricts to a map \( F^e_c(R \otimes_k R) \to R \otimes_k R \), this means \( a, d - a \in \text{int}(P_{-K_X}) \), by Proposition 6.3. In other words, \( a \) is in the interior of \( P_{-K_X} \cap (d - P_{-K_X}) \).

Conversely, given some \( d \), suppose that each equivalence class \([u] \) has a representative in the interior of \( P_{-K_X} \cap (d - P_{-K_X}) \). Then we can label these representatives \( a_1, \ldots, a_N \). Then \( \sum_{i} \pi_{a_i}^e \otimes \pi_{d-a_i} \) is a map compatible with the diagonal, and its restriction to the diagonal is \( \pi_d \).

This is just an application of equation (14) in this case, there is only one nonzero coefficient \( b_{a,d-a} \) where \( a \) is in any particular equivalence class of \( \frac{1}{q} \mathbb{Z}^n / \mathbb{Z}^n \).

\[ \square \]

Remark 6.7. Theorem 6.4 can be seen as a generalization of [Pay09, Theorem 1.2]. Indeed, the following lemma shows that an affine toric variety is diagonally \( F \)-split if and only if \( \pi_0 \in \mathcal{D}^{(2)} \). Thus we recover [Pay09, Theorem 1.2] by setting \( a = 0 \) in the statement of Theorem 6.4.

Lemma 6.8. Let \( R \) be a toric ring. The following are equivalent:

(i) \( R \) is diagonally \( F \)-split
(ii) \( \pi_0 \in \mathcal{D}^{(2)}(R) \) for some \( e > 0 \)
(iii) \( \pi_0 \in \mathcal{D}^{(2)}(R) \).

Proof. To see that (i) implies (ii), suppose that \( R \) is diagonally \( F \)-split. By Pay09 Proposition 4.5], there exists some \( e > 0 \) and some map
\[
\varphi = \sum_{a \in \frac{1}{q} \mathbb{Z}^n} c_a \pi_a \in \mathcal{D}^{(2)}(R).
\]
with \( c_0 \neq 0 \). By Corollary 6.6, we have \( \pi_0 \in \mathcal{D}^{(2)}(R) \).

To see that (ii) implies (iii), suppose that \( \pi_0 \in \mathcal{D}^{(2)}(R) \) and set \( q = p^e \). As \( R^{1/p} \subseteq R^{1/q} \), the map \( \pi_0 \) restricts to a map \( R^{1/p} \to R \). One checks that this restriction is in \( \mathcal{D}^{(2)}(R) \), for instance by using Theorem 6.4.

Finally, (iii) implies (ii) by definition.

\[ \square \]

Example 6.9. Consider the case \( R = k[x, y, z]/(xy - z^2) \), and assume that \( \text{char} \, k > 2 \). To use the techniques in this section, we use the presentation \( R = k[y, xy, xy^2] \). Then Figure 1 shows the polytope \( P_{-K_X} \). Using Theorem 6.4, one can compute \( \mathcal{D}^{(2)}(R) \):

\[
\mathcal{D}^{(2)}(R) = \bigoplus_{e \geq 0} \langle x^{p+1} y^e, \frac{y^{p+1}}{x}, \frac{x^{p+1}}{y}, \frac{y^{p+1}}{z}, \frac{z^{p+1}}{x}, \frac{z^{p+1}}{y}, \frac{z^{p+1}}{z}, \frac{z^{p+1}}{z} \rangle \text{Hom}_R(F^e_c R, R)
\]
In terms of the more familiar presentation, \( R \cong k[x, xy, y^2] \), this formula becomes
\[
\mathcal{G}^{(2)}(R) = \bigoplus_{e \geq 0} F_*^e \langle x^{p^e+1}, x^{p^e} y, x^{p^e-1} y^{p^e-1}, x y^{p^e}, y^{p^e+1} \rangle \text{Hom}_R \left( F_*^e R, R \right)
\]

To see this, we will prove the following:

Figure 1. The polytope \( P_{-K_X} \) for the quadric cone \( R = k[y, xy, x^2y] \), along with the fractional lattice \( \frac{1}{5} \mathbb{Z}^2 \). The area in red denotes the set of maps not in \( \mathcal{G}^{(2)}_1(R) \). The points in blue denote the generators of \( \mathcal{G}^{(2)}_1(R) \) over \( F_*^1 R \).

(i) If \( a > -1 \) and \( b > a/2 \), then \( \pi_{(a,b)} \in \mathcal{G}^{(2)}(R) \)

(ii) If \( a > 0 \) and \( b > (a-1)/2 \), then \( \pi_{(a,b)} \in \mathcal{G}^{(2)}(R) \)

(iii) \( \pi_{(0,0)} \in \mathcal{G}^{(2)}(R) \)

(iv) The maps \( \pi_{(0,-1/q)}, \pi_{(-1/q,-1/q)} \), and \( \pi_{(-2/q,-2/q)} \) are not in \( \mathcal{G}^{(2)}(R) \).

Because \( \mathcal{G}^{(2)}(R) \) is a Cartier algebra and \( \pi(a, x^{b}) = \pi_{a+b} \) for all \( a, b \in \frac{1}{q} \mathbb{Z}^n \), it follows from (iv) that the maps described in (i)–(iii) are the only maps in \( \mathcal{G}^{(2)}(R) \). (Another way to see this is to notice that, for any map \( \pi_v \) not among those described in (i)–(iii), the corresponding polytope \( P_{-K_X} \cap (v - P_{-K_X}) \) is contained in the polytope corresponding to one of the maps described in (iv)). Consequently we see that \( \mathcal{G}^{(2)}_e \) is generated over \( F_*^e R \) by the maps \( \pi_v \), where
\[
v \in \left\{ \left( \frac{1}{q} \left( \frac{q+1}{2} \right), \frac{2-q}{q}, \frac{(q+1)/2}{q} \right), (0,0), \left( \frac{1}{q}, \frac{q+1}{2} \right), \left( \frac{2}{q}, \frac{(q+1)/2}{q} \right) \right\}
\]

As \( \text{Hom}_R(F_*^e R, R) \) is generated as an \( F_*^e R \)-module by \( \pi_{\left( \frac{1-q}{q}, \frac{1-q}{q} \right)} \), we get that \( \mathcal{G}^{(2)}(R) \) has the description given in equation (15). So, let \( (a, b), (\alpha, \beta) \in \mathbb{R}^2 \). Then
\[
P_{(a,b)} := \text{int} \left( P_{-K_X} \cap ((a, b) - P_{-K_X}) \right)
= \{(x, y) \mid -1 < x < a + 1, -1 < 2y - x < 1 - a + 2b\}
We wish to find an integer translation of \((\alpha, \beta)\) in \(P_{(a,b)}\), where \((a,b)\) is as in (i) or (ii). Let \(\overline{\alpha} = \alpha - [\alpha]\). If \((a,b)\) is as in (i), then we have
\[
\begin{aligned}
(\overline{\alpha}, \beta - [\beta]) &\in P_{(a,b)}, & 2(\beta - [\beta]) - \overline{\alpha} &\leq 1 \\
(\overline{\alpha}, \beta - [\beta] - 1) &\in P_{(a,b)}, & 2(\beta - [\beta]) - \overline{\alpha} &> 1
\end{aligned}
\]
If \((a,b)\) is as in (ii), then we have
\[
\begin{aligned}
(\alpha + 1, \beta - [\beta] - 1) &\in P_{(a,b)}, & \frac{1}{2} \overline{\alpha} + \frac{1}{2} &< \beta - [\beta] \\
(\overline{\alpha}, \beta - [\beta] - 1) &\in P_{(a,b)}, & \frac{1}{2} \overline{\alpha} - \frac{1}{2} &< \beta - [\beta] \leq \frac{1}{2} \overline{\alpha} + \frac{1}{2} \\
(\alpha + 1, \beta - [\beta]) &\in P_{(a,b)}, & 0 &\leq \beta - [\beta] \leq \frac{1}{2} \overline{\alpha} - \frac{1}{2}
\end{aligned}
\]
To check (iii), we note that
\[
\begin{aligned}
(\overline{\alpha}, \beta - [\beta]) &\in P_{(0,0)}, & 2(\beta - [\beta]) - \overline{\alpha} &< 1 \\
(\overline{\alpha}, \beta - [\beta] - 1) &\in P_{(0,0)}, & 2(\beta - [\beta]) - \overline{\alpha} &> 1
\end{aligned}
\]
Here we’re using the fact that \(\text{char} \, k \neq 2\) to see that \(\overline{\alpha} < 0\) if \(2(\beta - [\beta]) - \overline{\alpha} = 1\). Indeed, the point \((0, \frac{1}{2})\) has no integer translation in \(P_{(0,0)}\), so \(\pi_0 \notin \mathcal{D}(2)(R)\) if \(\text{char} \, k = 2\).

Finally, to check (iv), we note that \((0, \frac{(q-1)/2}{q})\) has no integer translations in \(P_{(0,-1/q)}\). The polytope \(P_{(-1/q, -1/q)}\) has no integer translations of \((0, \frac{(q-1)/2}{q})\) nor, for that matter, of \((-\frac{1}{q}, \frac{(q-1)/2}{q})\). The polytope \(P_{(-2/q, -1/q)}\) has no integer translations of \((-\frac{1}{q}, \frac{(q-1)/2}{q})\).

We can use this calculation to compute the \(F\)-signature of \(\mathcal{D}(2)(R)\) in the sense of \[BST12\]. Indeed, we see that the only splitting of \(R \to R^{1/q}\) contained in \(\mathcal{D}(2)(R)\) is \(\pi_{(0,0)}\). Thus \(s(\mathcal{D}(2)(R)) = 0\).

**Example 6.10.** Let \(R = k[x, y, z, xyz^{-1}] \cong k[s, t, u, v]/(st - uv)\). The cone \(\sigma\) of \(R\) is given by the extremal rays:
\[
(1, 0, 0), \quad (0, 1, 0), \quad (1, 0, 1), \quad (0, 1, 1)
\]
We will show the following:

**Claim 6.11.** Let \(e > 0\). If \((a, b, c) \in P_{-K} \cap \frac{1}{q} \mathbb{Z}^3\) and \(a + b + c > -1\), then \(\pi_{(a,b,c)}\) is in \(\mathcal{D}^2_e\).

To see this claim, it’s enough to consider the case \(-1 < a, b \leq 0\), since \(\varphi(F^e_x \cdot -) \in \mathcal{D}^2\) whenever \(\varphi \in \mathcal{D}^2\) and \(x \in R\). The key point is that then
\[
\begin{aligned}
2 + a + c &> 1 - b \geq 1, \\
2 + b + c &> 1 - a \geq 1.
\end{aligned}
\]
Set \(\vec{d} = (a, b, c)\). Let \((\alpha, \beta, \gamma) \in \mathbb{R}^3\). We wish to find an integer translation of \((\alpha, \beta, \gamma)\) in \(P \cap (\vec{d} - P)\). We start by translating the polytope \(P \cap (\vec{d} - P)\) by \((1, 1, 0)\): the resulting polytope is described as the set of \((x, y, z)\) satisfying the inequalities
\[
0 < x < 2 + a \\
0 < y < 2 + b \\
0 < x + z < 2 + a + c \\
0 < y + z < 2 + b + c
\]
We may assume, without loss of generality, that $0 < \alpha, \beta \leq 1$ and $0 \leq \gamma < 1$. Note that we automatically have

$$0 < \alpha < 2 + a, \quad 0 < \beta < 2 + b, \quad 0 < \alpha + \gamma, \quad 0 < \beta + \gamma$$

If we happen to have $\alpha + \gamma < 2 + a + c$ and $\beta + \gamma < 2 + b + c$, then we’re done. So suppose otherwise. Without loss of generality, we may assume that $\alpha + \gamma \geq 2 + a + c$. If we also have $\beta + \gamma \geq 2 + b + c$, then the point $(\alpha, \beta, \gamma - 1)$ is in $P \cap (\tilde{d} - P)$; since $\alpha + \gamma < 2$, we have $\alpha + \gamma - 1 < 1 < 2 + a + c$ by equation (16). On the other hand, $\alpha + \gamma \geq 2 + b + c > 1$, so $\alpha + \gamma - 1 > 0$. Similarly, we have $0 < \beta + \gamma - 1 < 2 + b + c$.

Now suppose that $\alpha + \gamma \geq 2 + a + c$ but $\beta + \gamma < 2 + b + c$. If $\beta + \gamma > 1$, then the point $(\alpha, \beta, \gamma - 1)$ is again in $P \cap (\tilde{d} - P)$, as clearly we have $0 < \beta + \gamma - 1 < 2 + b + c$. On the other hand, if $\beta + \gamma \leq 1$, then $(\alpha, \beta + 1, \gamma - 1)$ is in $P \cap (\tilde{d} - P)$. Indeed, we just have to check that $\beta + 1 < 2 + b$, or in other words, that $b > \beta - 1$. We know, by assumption, that $b > -1 - a - c$, so it suffices to check that $-a - c \geq \beta$. As $\gamma \geq 2 + a + c - \alpha$, we have

$$\beta \leq 1 - \gamma \leq 1 - (2 + a + c - \alpha) = -a - c + \alpha - 1 < -a - c$$

The last inequality comes from the assumption that $0 \leq \alpha < 1$. This proves the claim.

By [vK11], the splittings of $R^{1/q}$ correspond to the points in $\frac{1}{q} \mathbb{Z}^3 \cap P_{\text{sig}}$, where $P_{\text{sig}}$ is the polytope given by

$$P_{\text{sig}} := \left\{ x \in \mathbb{R}^3 \mid \begin{array}{l}
-1 < \langle 1, 0, 0 \rangle \cdot x \leq 0 \\
-1 < \langle 0, 1, 0 \rangle \cdot x \leq 0 \\
-1 < \langle 1, 0, 1 \rangle \cdot x \leq 0 \\
-1 < \langle 0, 1, 1 \rangle \cdot x \leq 0
\end{array} \right\}.$$ 

This polytope is depicted in Figure 2. As seen in the figure, the plane $x + y + z = -1$ cuts this polytope in half. This shows that $s(\mathcal{D}(R)) \leq s(R)/2 = 1/3$. Calculations in Macaulay2 [GS] suggest that there are no further maps in $\mathcal{D}(R)$ and that $s(\mathcal{D}(R)) = 1/3$.

![Figure 2](image-url) Comparison of maps in $\mathcal{D}(R)$ and a polytope whose volume is $s(R)$, according to [vK11]. The plane is given by $x + y + z = -1$. All fractional lattice points lying above the plane correspond to maps in $\mathcal{D}(R)$. 
7. Classifying singularities in terms of $\mathcal{D}^{(2)}$

Theorem 3.12 suggests that rings with milder singularities have larger Cartier algebras $\mathcal{D}^{(2)}$. We wonder whether the singularities of a ring can be well understood just by considering, in some sense, the size of $\mathcal{D}^{(2)}$. The following conjecture would be a natural place to start in order to develop such a theory:

**Conjecture 7.1.** Let $R$ be a finitely-generated algebra over a perfect field of positive characteristic. Then $\mathcal{D}^{(2)}(R) = \mathcal{C}^R$ if and only if $R$ is regular.

One direction is clear: if $R$ is regular then so is $R \otimes_k R$, meaning that $F^e_*(R \otimes_k R)$ is a projective $R \otimes_k R$-module for all $e > 0$. As $R \otimes_k R \to R$ is a surjective map of $R \otimes_k R$ modules, the universal property of projective modules tells us that any map $F^e_* R \to R$ will lift:

$$
\begin{array}{ccc}
F^e_*(R \otimes_k R) & \xrightarrow{\phi} & R \\
\downarrow & & \downarrow \\
F^e_* R & \xrightarrow{\mu} & R
\end{array}
$$

Here, we’re thinking of $R$ and $F^e_* R$ as $R \otimes_k R$-modules via $\mu$. We have the following partial converses.

**Proposition 7.2.** Suppose $k$ is a perfect field of positive characteristic and $R$ is a reduced $k$-algebra essentially of finite type. If $\mathcal{D}^{(2)}(R) = \mathcal{C}^R$, then $R$ is strongly $F$-regular.

**Proof.** Apply Corollary 5.6 to the case where $a = b = R$. We see that $0 \neq \tau(R) \subseteq \tau(R)^2$, so $\tau(R) = \tau(R)^2$. As $\tau(R)$ contains a regular element of $R$, it follows from Nakayama’s lemma that $\tau(R) = R$. \hfill $\square$

We also know that the converse of Conjecture 7.1 holds in the $\mathbb{Q}$-Gorenstein toric case, using von Korff’s characterisation of the $F$-signature in that setting. The point of the $\mathbb{Q}$-Gorenstein condition is just so that we know $P_{-K_X}$ is a translation of the dual cone $\sigma^\vee$ of $R$. Note that, by [CLS11, Proposition 4.2.7], this includes the case when the cone $\sigma$ of $R$ is simplicial, and in particular all toric surfaces.

**Proposition 7.3.** Work in Setting 6.1. Suppose also that $R$ is $\mathbb{Q}$-Gorenstein and $\mathcal{D}^{(2)}(R) = \mathcal{C}^R$. Then $R$ is regular.

**Proof.** It follows from Lemma 7.4 that $P_{-K_X} = C + v$ for some $v \in \mathbb{R}^n$ such that $\langle v, \rho \rangle = -1$ for all $\rho \in \Sigma(1)$. Let $Q = \{x \mid \forall \rho \in \Sigma(1): 0 < \langle x, \rho \rangle < 1\}$. By [vK11], we know that $s(R) = \text{vol}(Q)$. The key point is to notice that

$$
P_{-K_X} \cap (v - P_{-K_X}) = (C + v) \cap (-C) = \{x \mid \forall \rho \in \Sigma(1): -1 < \langle x, \rho \rangle < 0\} = -Q.
$$

Thus $s(R) = \text{vol}(P_{-K_X} \cap (v - P_{-K_X}))$.

Now, the function $\mathbb{R}^n \to \mathbb{R}$ given by $d \mapsto \text{vol}(P_{-K_X} \cap (d - P_{-K_X}))$ is continuous, as this intersection is always compact. By taking $e$ sufficiently large, we can find a lattice point $d \in \frac{1}{p^e} \mathbb{Z}^n \cap P_{-K_X}$ arbitrarily close to $v$. Thus we can make $\text{vol}(P_{-K_X} \cap (d - P_{-K_X}))$ arbitrarily close to $\text{vol}(P_{-K_X} \cap (v - P_{-K_X}))$. Since $\mathcal{D}^{(2)}(R) = \mathcal{C}^R$, we have that $\pi_d \in \mathcal{D}^{(2)}(R)$ and also that $R$ is diagonally $F$-split. Then $\text{vol}(P_{-K_X} \cap (v - P_{-K_X})) \geq 1$ by Lemma 7.5.

Thus $s(R) = 1$ and $R$ is regular. \hfill $\square$
Lemma 7.4. Work in Setting 6.1 and suppose that $R$ is $\mathbb{Q}$-Gorenstein. Then $P_{-K_X} = \sigma^\vee + v$ where $v$ is a vector satisfying $\langle v, \rho \rangle = -1$ for all $\rho \in \sigma(1)$.

Proof. This can be seen in a few different ways, but here’s one. Let $r$ be the Cartier index of $R$, so that $rK_X$ is Cartier. As $-rK_X = \sum_{\rho \in \sigma(1)} rD_\rho$, we have by [CLST11, Theorem 4.2.8] that there exists $w$ such that $\langle w, \rho \rangle = -r$ for all $\rho \in \sigma(1)$. Then we certainly have $\frac{1}{r}w + \sigma^\vee \subseteq P_{-K}$. On the other hand, for any $x \in P_{-K}$ we have $\langle x - \frac{1}{r}w, \rho \rangle > 0$ for all $\rho$, meaning $\frac{1}{r}w + \sigma^\vee = P_{-K}$. So we set $v = \frac{1}{r}w$. \hfill $\square$

Lemma 7.5. Let $R$ be a diagonally split $n$-dimensional affine toric variety. For all $e$ and all $d \in \frac{1}{p} \mathbb{Z}^n$, if $\pi_d: F_e^eR \to R$ is in $\mathcal{D}_c^{(2)}(R)$ then $\text{vol}(P_{-K_X} \cap (d - P_{-K_X})) \geq 1$.

Proof. For all $e' > e$, let $\pi_{d}^{e'} = \pi_{d} \cdot (\pi_{0})^{e'-e} \in \mathcal{C}_c^R$. This is the map $F_{e'}^eR \to R$ corresponding to the lattice point $d \in \frac{1}{p} \mathbb{Z}^n$. The map $\pi_0$ is in $\mathcal{D}_c^{(2)}(R)$ by (6.8) so we have $\pi_{d}^{e'} \in \mathcal{D}_c^{(2)}(R)$ since $\mathcal{D}_c^{(2)}(R)$ is a Cartier algebra. By Theorem 6.4 the polytope $P_{-K_X} \cap (d - P_{-K_X})$ contains at least $p^e_\mathbb{Z}^{n}$ fractional lattice points in $\frac{1}{p} \mathbb{Z}^n$. Then we’re done, as for any polytope $\mathcal{P} \subseteq \mathbb{R}^n$ we have

$$\text{vol}(\mathcal{P}) = \lim_{m \to \infty} \frac{\# \{ \frac{1}{m} \mathbb{Z}^n \cap \mathcal{P} \}}{m^n}$$

This is a well-known fact; see for instance [MS06, Theorem 2.2]. \hfill $\square$

Appendix A. Proof that Test Ideals Along Closed Subschemes Exist

In this appendix, we show there’s a notion of test elements for test ideals along closed subschemes, working in Setting 4.2. Consequently, these test ideal exist. We remark that these proofs are essentially the same as those in [Sch09, §6]. The salient difference between our setting and the one in [Sch09] is that here we’re not assuming that $I$ is an $F$-pure center of $R$. Instead, we’re just assuming that $\mathcal{C}$ is compatible with $I$.

Lemma A.1. Work in Setting 4.2. There exists some $\gamma \in R \setminus I$ such that

(a) All proper ideals of $R_\gamma$ compatible with $\mathcal{C}_\gamma$ are contained in $IR_\gamma$, and

(b) The Cartier algebra $\mathcal{C}_\gamma$ is $F$-pure

Proof. Let $\pi: R \to R/I$ be the quotient map. It follows from our assumptions in Setting 4.2 that $\mathcal{C} |_{R/I}$ is a nondegenerate Cartier algebra on $R/I$, so $\tau (R/I, \mathcal{C} |_{R/I})$ is well-defined (and, in particular, nonzero) [Sch11]. Choose $\gamma_1 \in R$ so that $\pi(\gamma_1) \in \tau (R/I, \mathcal{C} |_{R/I})$ and $\pi(\gamma_1) \neq 0$. Then all proper ideals of $R$ compatible with $\mathcal{C}_\gamma$ are contained in $IR_\gamma$. Indeed, we have the following diagram for all $\varphi \in \mathcal{C}_\gamma$:

$$\begin{array}{ccc}
F_e^e R_{\gamma_1} & \xrightarrow{\varphi} & R_{\gamma_1} \\
\downarrow \pi & & \downarrow \pi \\
F_e^e (R/I)_{\gamma_1} & \xrightarrow{\pi} & (R/I)_{\gamma_1}
\end{array}$$

If $\varphi(J) \subseteq J$, then $\pi(\varphi(J)) \subseteq \pi(J)$. Note that as $\varphi$ runs through all maps in $\mathcal{C}_{\gamma_1}$, $\varphi$ will run through all maps in $(\mathcal{C} |_{R/I})_{\gamma_1}$. So if $J$ is a proper ideal of $R_\gamma$ compatible with $\mathcal{C}_\gamma$, then $\pi(J)$ is compatible with $(\mathcal{C} |_{R/I})_{\gamma_1}$. But we have that $\tau (R/I, (\mathcal{C} |_{R/I})_{\gamma_1}) = \tau (R/I, (\mathcal{C} |_{R/I})_{\gamma_1}) = (R/I)_{\gamma_1}$, so $(\mathcal{C} |_{R/I})_{\gamma_1}$ is $F$-regular. This means that $\pi(J) = 0$, meaning $J \subseteq IR_{\gamma_1}$. 

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We note that all proper ideals of $R_{\gamma_1\gamma_2}$ compatible with $\mathcal{C}_{\gamma_1\gamma_2}$ are contained in $IR_{\gamma_1\gamma_2}$, for all $\gamma_2 \in R \setminus I$. So choose $e > 0$ and $\psi \in \mathcal{C}_e$ to be some map whose image is not contained in $I$ and let $\gamma_2 \in \psi(F_e^c) \setminus I$. Then the element $\gamma = \gamma_1\gamma_2$ satisfies the conclusion of the lemma.

\[ \square \]

**Proposition A.2** (C.f. [Sch09] Lemma 6.12]). Work in Setting 4.2. There exists an element $\gamma \in R \setminus I$ such that, for all $d \in R \setminus I$, there exists an integer $m$ and a map $\Psi \in \mathcal{C}$ of minimal degree greater than 0 such that $\gamma^m = \Psi \cdot d$.

**Proof.** Choose $\gamma$ as in Lemma A.1. It suffices to prove that

\[ J := \sum_{e > 0} \sum_{\varphi \in \mathcal{C}_e} \varphi(F_e^c)R_\gamma = R_\gamma \]

for all $d \in R \setminus I$. By definition of $\gamma$, it suffices to show that $J$ is compatible with $\mathcal{C}_e$ and not contained in $IR_\gamma$. It’s clear that $J$ is compatible with $\mathcal{C}_\gamma$, so we’ll just show $J$ is not contained in $IR_\gamma$. Let $\pi : R \to R/I$ be the natural surjection. As $(\mathcal{C} |_{R/I})_\gamma$ is $F$-regular and $\pi(d) \neq 0$, there exist $e$ and $\varphi \in (\mathcal{C} |_{R/I})_\gamma$ such that $\varphi(F_e^c)\pi(d) = 1$. This means that $\varphi(F_e^c) = 1 + x$ for some $x \in I$, where $\varphi$ is any map in $\mathcal{C}_\gamma$ that induces $\varphi$. But then $1 + x \in J$, so $J$ is not contained in $IR_\gamma$. \[ \square \]

**Lemma A.3** (C.f. [Sch09] Lemma 6.13]). Suppose $\varphi \in \mathcal{C}^R$, $c \in R$, and $b \in \varphi \cdot (cR)$. Then $b^2 \in \varphi^n \cdot (cR)$ for all $n > 0$.

**Proof.** This proof is essentially the same as that of Lemma 6.13 of [Sch09]. We include it here for completeness.

We proceed by induction. The base case is given by the hypothesis. Suppose that $\varphi = \sum_i \varphi_i$, where $\varphi_i \in \mathcal{C}^R_i$. Then we compute:

\[
\begin{align*}
 b^2 \in b \varphi \cdot (cR) &= b \sum_i \varphi_i (F_e^c) cR = \sum_i \varphi_i (F_e^c b^e cR) \subseteq \sum_i \varphi_i (F_e^c b^e cR) \\
 &= \varphi \cdot (b^2 cR) \subseteq \varphi \cdot ((\varphi^n \cdot (cR)) c) \subseteq \varphi^{n+1} \cdot (cR)
\end{align*}
\]

\[ \square \]

**Proposition A.4** (C.f. [Sch09] Proposition 6.14]). There is an element $b \in R \setminus I$ such that for all $d \in R \setminus I$, there exists $\Psi \in \mathcal{C}$ such that $b = \Psi \cdot d$.

**Proof.** This proof is essentially the same as that of Proposition 6.14 of [Sch09]. We include it here for completeness.

Choose $\gamma$ as in Proposition A.2. Then there exists $m$ and $\Psi$, of minimal degree $e_0 > 0$, such that $\gamma^m = \Psi \cdot 1$. By Lemma A.3, $\gamma^{2m} \in \Psi^n \cdot R$ for all $n > 0$. We will show that $b = \gamma^{3m}$ works.

Let $d \in R \setminus I$ be arbitrary. Then there exists $\Psi_1$ and $m_1$ such that $\gamma^m = \Psi_1 \cdot d$. If $m_1 < 3m$, then we’re done. Otherwise, choose $n$ such that $m_1 < mp^{n e_0}$ and write $\Psi^n = \sum_i \psi_i$ with $\psi_i \in \mathcal{C}_{e_i}$ for all $i$. Note that $e_i \geq n e_0$ for all $i$. Then we have:

\[
\begin{align*}
\gamma^{3m} &= \gamma^m \gamma^{2m} \in \gamma^m \Psi^n \cdot (R) = \gamma^m \sum_i \psi_i (F_e^c) R = \sum_i \psi_i (F_e^c \gamma^m \psi_i R) \subseteq \sum_i \psi_i (F_e^c \gamma^m \psi_i R) \\
\quad \subseteq \sum_i \psi_i (F_e^c \gamma^m R) = \Psi^n \cdot \gamma^m R \subseteq \Psi^n \Psi_1 \cdot (dR),
\end{align*}
\]

as desired. \[ \square \]
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**Theorem A.5** (C.f. [Sch09, Lemma 6.17 and Theorem 6.18]). Let \( b \) be as in Proposition A.4. Then

\[
\tau_I(R, \mathcal{C}) = \sum_{e \geq 0} \sum_{\varphi \in \mathcal{C}} \varphi(F_e b)
\]

**Proof.** This proof is essentially the same as Lemma 6.17 and Theorem 6.18 of [Sch09]. We include it here for completeness.

Let \( \tau_I(R, \mathcal{C}; b) \) denote the ideal

\[
\sum_{e \geq 0} \sum_{\varphi \in \mathcal{C}} \varphi(F_e b),
\]

and note that we have

\[
\tau_I(R, \mathcal{C}; b) = \sum_{\varphi \in \mathcal{C}} \varphi \cdot b.
\]

Then we need to show:

(a) \( \tau_I(R, \mathcal{C}; b) \not\subset I \),
(b) \( \tau_I(R, \mathcal{C}; b) \) is compatible with \( \mathcal{C} \), and
(c) \( \tau_I(R, \mathcal{C}; b) \) is contained in any other ideal satisfying (a) and (b).

For (a), it’s enough to show that \( b \in \tau_I(R, \mathcal{C}; b) \). This follows from Proposition A.4 using \( d = b \). Assertion (b) is clear from the construction of \( \tau_I(R, \mathcal{C}; b) \).

For the final assertion, let \( J \) be some ideal satisfying (a) and (b). Choose some \( d \in J \setminus I \). Then

\[
\sum_{\varphi \in \mathcal{C}} \varphi \cdot d \subseteq \sum_{\varphi \in \mathcal{C}} \varphi \cdot J \subseteq J.
\]

By Proposition A.4 we have \( b \in J \). But then

\[
\tau_I(R, \mathcal{C}; b) = \sum_{\varphi \in \mathcal{C}} \varphi \cdot b \subseteq \sum_{\varphi \in \mathcal{C}} \varphi \cdot J \subseteq J.
\]

\[\square\]

**Appendix B. Multiplier Ideal Computations**

This appendix is devoted to proving the following containment, in the context of the proof of Theorem 5.15:

\[
H^0(A'_\kappa, F_e \omega_{A'_\kappa}(X'_\kappa - [qB_{\kappa}])) \subseteq R \left((1 - q)K_{A_{\kappa}}\right) \prod_i (a_i)^{t_i(q-1)} d_{\kappa}^N \varepsilon^N.
\]

Recall the following notation: first, we work in Setting 5.1 with char \( R = 0 \). Further, \( \pi : \overline{A} \to A \) is a factorizing resolution of \( X \subseteq A \). We denote by \( \overline{X} \) the strict transform of \( X \) in \( \overline{A} \) and we let \( \psi : A' \to \overline{A} \) be the blow up along \( X \). We set \( X' = \text{Exc} \psi \). For each \( i \), we set \( a_i \mathcal{O}_{A'} = \mathcal{O}_{A'}(-F_i) \) and we set \( F = \sum_i t_i F_i \). We have that \( Z = cX \) at the generic point of \( X \), where \( c \) is the codimension of \( X \) in \( A \). Further, \( d \) and \( \varepsilon \in R \), and \( \varepsilon > 0 \). We also have that \( \eta \in R \setminus I \) is an element satisfying

\[
\psi^* K_X - [qf^* K_A] - \text{div } \eta \leq f^* ((1 - q)K_A),
\]

and \( H \) is an anti-effective divisor on \( A' \). We define

\[
B = f^* K_A + F + \varepsilon \text{div}_{A'}(d\xi \eta) - H.
\]
We also choose \( e > 0 \) so that \( t_i(q - 1) \in \mathbb{Z} \) and \((1 - q)K_A\) is Cartier, where \( q = p^e \). We also have an integer \( N > 0 \) and we assume that \( qe > N \). From now on, we work exclusively modulo \( p \) at \( \kappa \) and we abuse notation by omitting the subscripts \( \kappa \). Then we compute:

\[
\begin{align*}
H^0 (A', F^e A'(X' - |qB|)) \\
= H^0 (A', F^e A'(K_A + X' - |qf^* K_A + qF + qe \div A'(d\xi) - qH|)) \\
\subseteq H^0 (A, f_* F^e A' (K_A + X' - |qf^* K_A| - |qF| - |qe| \div A'(d\xi) - [-qH|)) \\
\subseteq H^0 (A, f_* F^e A' (K_A + X' - |qf^* K_A| - |qF| - N \div A'(d\xi))).
\end{align*}
\]

Here, we’re using the fact that \( H \) is anti-effective. Note that \( K_A' = \psi^* K_A + (c - 1)X' \) by [Har77, Exercise II.8.5].

Next, we examine the term \(-|qF|\). For each \( i \), we can write \( F_i = F'_i + a_iX' \) for some \( a_i \in \mathbb{N} \), where \( F'_i \) is not supported along \( X' \). Since we assumed \( Z = cX \) at the generic point of \( X \), we have \( \sum_i t_i a_i = c \). Then we see

\[
-qF = -\left[ \sum_i q t_i F'_i + q t_i a_i X' \right] = -\left[ \sum_i q t_i F'_i \right] - qcX'.
\]

Thus we have:

\[
\begin{align*}
H^0 (A, f_* F^e A' (K_A + X' - |qf^* K_A| - |qF| - N \div A'(d\xi))) \\
\subseteq H^0 \left( A, f_* F^e A' \left( \psi^* K_A - |qf^* K_A| - \sum_i q t_i F'_i + (c - qc) X' - N \div A'(d\xi) \right) \right) \\
\subseteq H^0 \left( A, f_* F^e A' \left( \psi^* K_A - |qf^* K_A| - \sum_i (q - 1) t_i F'_i - (q - 1)cX' - N \div A'(d\xi) \right) \right) \\
\subseteq H^0 \left( A, f_* F^e A' \left( \psi^* K_A - |qf^* K_A| - \sum_i ((q - 1) t_i F'_i + (q - 1) t_i a_i X') - N \div A'(d\xi) \right) \right) \\
\subseteq H^0 \left( A, f_* F^e A' \left( \psi^* K_A - |qf^* K_A| - \sum_i (q - 1) t_i F_i - N \div A'(d\xi) \right) \right) \\
\subseteq H^0 \left( A, f_* F^e A' \left( f^* \left( (1 - q) K_A \right) - \sum_i (q - 1) t_i F_i - N \div A'(d\xi) \right) \right)
\]

\[
\subseteq R \left( (1 - q) K_A \right) \prod_i a_i^{t_i(q - 1)} d^N \xi^N,
\]

as desired.
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