Curvature and Tachibana numbers

S. E. Stepanov

Abstract. The aim of this paper is to define the $r$th Tachibana number $t_r$ of an $n$-dimensional compact oriented Riemannian manifold as the dimension of the space of conformally Killing $r$-forms, for $r = 1, 2, \ldots, n - 1$. We also describe properties of these numbers, by analogy with properties of the Betti numbers $b_r$ of a compact oriented Riemannian manifold.

Bibliography: 25 titles.

Keywords: compact Riemannian manifold, differential forms, elliptic operator, scalar invariants.

§ 1. Introduction

1.1. In this paper we define the Tachibana numbers of a compact oriented Riemannian manifold. Using this definition, we interpret some well-known results of the theory of Riemannian manifolds, including those obtained by ourselves. The work relies on the ideas of the classical monograph [1].

The paper is based on the author’s talk at the International Conference “Differential equations and topology” dedicated to the Centenary of Lev Semionovich Pontryagin, which was held in Moscow on 17–22 June 2008 (see [2]).

1.2. We shall consider four subspaces of the vector space $\Omega^r(M, \mathbb{R})$ of exterior differential $r$-forms on a Riemannian manifold $(M, g)$, namely, the subspace $\mathbf{H}^r(M, \mathbb{R})$ of harmonic $r$-forms, the subspace $\mathbf{T}^r(M, \mathbb{R})$ of conformally Killing $r$-forms, the subspace $\mathbf{K}^r(M, \mathbb{R})$ of coclosed conformally Killing $r$-forms, and the subspace $\mathbf{P}^r(M, \mathbb{R})$ of closed conformally Killing $r$-forms, for all $r = 1, 2, \ldots, n - 1$.

For a compact oriented manifold $(M, g)$, we shall refer to $t_r = \dim \mathbf{T}^r(M, \mathbb{R})$ as the Tachibana number, to $k_r = \dim \mathbf{K}^r(M, \mathbb{R})$ as the Killing number, and to $p_r = \dim \mathbf{P}^r(M, \mathbb{R})$ as the planar number of $(M, g)$, by analogy with the Betti number $b_r = \dim \mathbf{H}^r(M, \mathbb{R})$.

We shall show that the Tachibana numbers $t_r$ are conformal scalar invariants, while the Killing numbers $k_r$ and the planar numbers $p_r$ are projective scalar invariants of a compact oriented Riemannian manifold $(M, g)$. We shall also establish the following duality properties: $t_r = t_{n-r}$ and $k_r = p_{n-r}$ for all $r = 1, 2, \ldots, n - 1$.

Furthermore, we shall establish the identity $t_r = k_r + p_r$ for an $n$-dimensional compact oriented Riemannian manifold $(M, g)$ with the positive curvature operator, and prove the vanishing theorem for the Tachibana numbers $t_r$, $1 \leq r \leq n - 1$.

AMS 2010 Mathematics Subject Classification. Primary 53C21, 58A10.
§ 2. Seven vector spaces of differential $r$-forms

In this section we define seven subspaces of the space of exterior differential $r$-forms on an $n$-dimensional Riemannian manifold $(M, g)$. These subspaces will arise from the theory of natural first-order differential operators on exterior differential $r$-forms, $1 \leq r \leq n - 1$.

2.1. More than thirty years ago Bourguignon (cf. [3]) considered the space $\text{Diff}_1(\Lambda^r M, \Psi(M))$ of natural (with respect to isometric diffeomorphisms) first-order differential operators on the space of $C^\infty$-sections of the vector bundle $\Lambda^r M$ of exterior differential $r$-forms with values in the space $\Psi(M)$ of homogeneous tensors on $(M, g)$.

Bourguignon proved the existence of a basis consisting of three operators $D_1$, $D_2$ and $D_3$ in the space $\text{Diff}_1(\Lambda^r M, \Psi(M))$, and defined explicitly two of these three operators (see [3]). The operator $D_1$ is the exterior derivation
\[ d: C^\infty \Lambda^r M \to C^\infty \Lambda^{r+1} M, \]
and $D_2$ its formally adjoint co-differential operator
\[ d^*: C^\infty \Lambda^r M \to C^\infty \Lambda^{r-1} M \]
defined by the following identity (see [4], Ch. I, 1.55 and [5], Ch. 8, § 11):
\[ d^* = (-1)^{nr+n+1} d^*, \tag{2.1} \]
where $*$ is the Hodge operator which takes each exterior differential $r$-form to an $(n - r)$-form in such a way that \( *: \Omega^r(M, \mathbb{R}) \to \Omega^{n-r}(M, \mathbb{R}) \) is an isomorphism and \( *^2 = (-1)^{(n-r)} \text{Id}_{\Lambda^r M} \) for a fixed choice of a local orientation on the manifold $(M, g)$.

As for the third natural basis differential operator $D_3$, it was said in [3] that it does not have any geometric interpretation for $r > 1$. It was also pointed out that in the case $r = 1$ the kernel of $D_3$ consists of infinitesimal conformal transformations of $(M, g)$.

2.2. By way of specification of Bourguignon’s result, we showed in [6] that the basis of natural first order differential operators on exterior differential $r$-forms consists of three operators of the following form:
\[ D_1 = \frac{1}{r+1} d, \quad D_2 = \frac{1}{n-r+1} g \wedge d^*, \quad D_3 = \nabla - \frac{1}{r+1} d - \frac{1}{n-r+1} g \wedge d^*, \]
where
\[ (g \wedge d^* \omega)(X_0, X_1, \ldots, X_r) \]
\[ = \sum_{\alpha=2}^{r} (-1)^{\alpha} g(X_0, X_\alpha)(d^* \omega)(X_1, \ldots, X_{\alpha-1}, X_{\alpha+1}, \ldots, X_r) \]
for any $r$-form $\omega$ and vector fields $X_1, X_2, \ldots, X_r$ on $(M, g)$. 

The kernel of \( D_1 \) consists of closed exterior differential \( r \)-forms, the kernel of \( D_2 \) consists of coclosed exterior differential \( r \)-forms, and the kernel of \( D_3 \) consists of conformally Killing exterior differential \( r \)-forms (or, in other terminology, conformally Killing skew-symmetric tensors of order \( r \)), see [7] and [8]. These three kernel subspaces will be denoted by \( D^r(M, \mathbb{R}) \), \( F^r(M, \mathbb{R}) \) and \( T^r(M, \mathbb{R}) \), respectively.

**Remark.** Conformally Killing skew-symmetric tensors were introduced by Tachibana more than forty years ago (see [8]) as a generalization of conformally Killing vector fields (or, in other terminology, infinitesimal conformal transformations). Some properties of these tensors were also described by Tachibana. The theory of conformally Killing tensors has been further developed in a series of subsequent papers (see [6], [7], [9], [10] and others).

2.3. If the compactness of \( (M, g) \) is not assumed, the condition \( \omega \in \text{Ker} \ D_1 \cap \text{Ker} \ D_2 \) characterizes harmonic \( r \)-forms \( \omega \) (see [1], Ch. II.9 and [5], Ch. 8, §11). Therefore, the space of harmonic \( r \)-forms can be defined as

\[
H^r(M, \mathbb{R}) = D^r(M, \mathbb{R}) \cap F^r(M, \mathbb{R}).
\]

The condition \( \omega \in \text{Ker} \ D_3 \cap \text{Ker} \ D_2 \) characterizes coclosed conformally Killing \( r \)-forms \( \omega \). These forms are also known as *Killing tensors* (see, for example, [7], [8] and [11]). The vector space of coclosed conformally Killing \( r \)-forms is therefore defined as

\[
K^r(M, \mathbb{R}) = F^r(M, \mathbb{R}) \cap T^r(M, \mathbb{R}).
\]

The condition \( \omega \in \text{Ker} \ D_3 \cap \text{Ker} \ D_1 \) characterizes closed conformally Killing \( r \)-forms \( \omega \). These forms are sometimes called *planar*. The vector space of closed conformally Killing \( r \)-forms can therefore be defined as

\[
P^r(M, \mathbb{R}) = D^r(M, \mathbb{R}) \cap T^r(M, \mathbb{R}).
\]

2.4. We continue to denote the vector space of all exterior differential \( r \)-forms on \( (M, g) \) by \( \Omega^r(M, \mathbb{R}) \), and denote by \( C^r(M, \mathbb{R}) \) the vector space of \( r \)-forms which are parallel with respect to the Levi-Civit\`{a} connection \( \nabla \). We have the following diagram of inclusions of subspaces:

\[
\begin{array}{ccc}
D^r(M, \mathbb{R}) & \longrightarrow & P^r(M, \mathbb{R}) \\
\Omega^r(M, \mathbb{R}) & \longrightarrow & T^r(M, \mathbb{R}) & \longrightarrow & H^r(M, \mathbb{R}) & \longrightarrow & C^r(M, \mathbb{R}) \\
F^r(M, \mathbb{R}) & \longrightarrow & K^r(M, \mathbb{R})
\end{array}
\]

Here, for instance, the arrow \( F^r(M, \mathbb{R}) \to K^r(M, \mathbb{R}) \) means that the vector space \( K^r(M, \mathbb{R}) \) is a subspace of \( F^r(M, \mathbb{R}) \).
§ 3. Betti numbers

In this section we summarize the known information on the Betti numbers of a compact Riemannian manifold.

3.1. Let \((M, g)\) be an \(n\)-dimensional compact oriented Riemannian manifold. We identify the space \(\Omega^r(M, \mathbb{R})\) of differential \(r\)-forms \(\omega\) on \((M, g)\) with the space of \(C^\infty\)-sections of the vector bundle \(\Lambda^r M\).

An exterior differential \(r\)-form \(\omega\) is called harmonic if \(\Delta \omega = 0\), where \(\Delta = d^*d + dd^*\) is the Hodge-de Rham Laplacian operator on exterior differential forms (see \([4]\), Ch. I, 1.57, \([5]\), Ch. 8, §11 and \([12]\), Ch. V, §25).

Since \((M, g)\) is compact oriented, the kernel \(\text{Ker} \, \Delta\) is finite dimensional, and the condition \(\omega \in \text{Ker} \, \Delta\) is equivalent to \(\omega \in \text{Ker} \, d \cap \text{Ker} \, d^*\) (see \([12]\), Ch. V, §25). Therefore, harmonic \(r\)-forms, which are determined by the condition \(\omega \in \text{Ker} \, \Delta\), form the vector space \(H^r(n, \mathbb{R})\).

An important property of the Hodge-de Rham Laplacian is that it commutes with the Hodge operator \(*\), that is, \(*\Delta = \Delta *\) (see \([12]\), Ch. V, §25). Therefore, if \(\omega\) is a harmonic \(r\)-form, then the \((n-r)\)-form \(*\omega\) is also harmonic, that is, the map \(*: H^r(n, \mathbb{R}) \cong H^{n-r}(n, \mathbb{R})\) is an isomorphism. This implies the identity \(b_r = b_{n-r}\), which is known as the Poincaré duality theorem for the Betti numbers.

3.2. For a Riemannian manifold \((M, g)\), a conformal transformation \(\tilde{g} = e^{2f}g\) of the metric, given by a differentiable function \(f\) on \((M, g)\), maps harmonic \(r\)-forms \(\omega \in H^r(n, \mathbb{R})\) to harmonic forms if \(n = 2r\) (see \([4]\), Ch. I, 1.162). This implies that the Betti numbers \(b_r\) are conformal scalar invariants of the Riemannian manifold \((M, g)\) of dimension \(n = 2r\).

3.3. The rough Bochner Laplacian is defined as \(\nabla^* \nabla\), where \(\nabla\) denotes the Levi-Cività connection on the bundle \(\Lambda^r M\) of exterior differential \(r\)-forms, and \(\nabla^*\) is the formally adjoint operator of \(\nabla\) (see \([4]\), Ch. I, 1.135). The rough Bochner Laplacian and the Hodge-de Rham Laplacian are related by the classical formula of Bochner-Weitzenböck:

\[
\Delta \omega = \nabla^* \nabla \omega + F_r(\omega),
\]

where \(F_r(\omega)\) is the quadratic form whose coefficients are the components of the curvature tensor and the Ricci tensor of the manifold \((M, g)\) (see \([4]\), Ch. 1, 1.136 and \([13]\), Ch. 7, 3.3).

The Bochner-Weitzenböck formula in combination with the Green theorem

\[
\int_M \text{div} \, X \, dv = 0
\]

(see \([1]\), Ch. 2.2) gives the integral formula

\[
\int_M g(\Delta \omega, \omega) \, dv = \int_M ||\nabla \omega||^2 \, dv + \int_M g(F_r(\omega), \omega) \, dv.
\]

If the \(r\)-form \(\omega\) is harmonic, then the integral formula above implies that (see \([13]\), p. 221)

\[
0 = \int_M ||\nabla \omega||^2 \, dv + \frac{1}{4} \int_M \sum_{\alpha} \lambda_{\alpha} ||[\Theta_{\alpha}, \omega]||^2 \, dv. \tag{3.1}
\]
Here $\lambda_\alpha$ are the eigenvalues of the standard symmetric Riemannian curvature operator $\overline{R}: \Lambda^2(TM) \to \Lambda^2(TM)$ given by the identity
\[ g(\overline{R}(X \wedge Y), Z \wedge U) = g(R(X, Y)U, Z) \]
for arbitrary vector fields $X$, $Y$, $Z$ and $U$. Furthermore, $R$ denotes the curvature tensor of $(M, g)$ and the forms $\Theta_\alpha$ dual to the eigenvectors of the curvature operator $\overline{R}$.

We say that $(M, g)$ has positive curvature operator if all eigenvalues of $\overline{R}$ are positive. In this case we shall write $\overline{R} > 0$.

Suppose that $(M, g)$ has positive curvature operator; then $\lambda_\alpha > 0$ and formula (3.1) implies that $\nabla \omega = 0$ and $[\Theta_\alpha, \omega] = 0$ for all $\alpha$. In this case $\omega$ must vanish for all $r$ except 0 and $n$ (see [13], p. 221). Therefore, $b_r = 0$ for all $r = 1, 2, \ldots, n - 1$.

If $\overline{R} > 0$, then it follows from (3.1) that $\nabla \omega = 0$; and therefore,
\[ b_r = \dim H_r(n, \mathbb{R}) \leq \frac{n!}{r!(n-r)!}, \]
where the equality is achieved only in the case of a flat Riemannian $n$-torus (see [13], p. 212). In particular, for a compact Riemannian manifold of constant positive sectional curvature, or for a compact conformally flat oriented Riemannian manifold with positive-definite Ricci tensor, we have $b_r = 0$ for all $r = 1, 2, \ldots, n - 1$.

These are the best known results of the ‘Bochner technique’ (see, for example, [1] and [13], Ch. 7).

§ 4. Tachibana numbers

4.1. Let $(M, g)$ be an $n$-dimensional compact Riemannian manifold. In [14] we defined the operator $D_3^*$, formally adjoint to the natural basis operator $D_3$, by the formula
\[ D_3^* = \nabla^* - \frac{1}{r+1}d^* - \frac{1}{n-r+1}d \circ \text{trace}. \]
We also constructed the rough Laplacian (see [4], Ch. I, 1.35 and [3], § 3) in the form
\[ D_3^*D_3 = \frac{1}{r(r+1)}(\nabla^*\nabla - \frac{1}{r+1}d^*d - \frac{1}{n-r+1}dd^*) \]
(4.1)
and described its properties. In particular, we obtained the following:
\[ \omega \in T^r(M, \mathbb{R}) \iff \omega \in \text{Ker}(D_3^*D_3), \]
\[ \omega \in K^r(M, \mathbb{R}) \iff \omega \in \text{Ker}(D_3^*D_3) \cap \text{Ker} d^*, \]
\[ \omega \in P^r(M, \mathbb{R}) \iff \omega \in \text{Ker}(D_3^*D_3) \cap \text{Ker} d. \]

The operator $D_3^*D_3$ is a typical example of the so-called Stein-Weiss operator considered by Branson in [15]. In particular, it was proved in [15] that such operators are elliptic. This implies that $D_3^*D_3$ is an elliptic operator (see also [16]), and its kernel on the compact Riemannian manifold $(M, g)$ has finite dimension (this follows from the general theory, see [4], Appendix). We therefore conclude that $\dim T^r(M, \mathbb{R}) = \dim_{\mathbb{R}}(\text{Ker} D_3^*D_3) < \infty$. Furthermore, the latter inequality also implies that $\dim K^r(M, \mathbb{R}) < \infty$ and $\dim P^r(M, \mathbb{R}) < \infty$. 
Definition. The Tachibana number $t_r$ of a compact $n$-dimensional Riemannian manifold $(M, g)$ is the dimension of the space of conformally Killing $r$-forms $(1 \leq r \leq n - 1)$ on $(M, g)$.

The numbers $k_r = \dim K^r(M, \mathbb{R})$ and $p_r = \dim P^r(M, \mathbb{R})$ will be called the Killing number and the planar number of a Riemannian manifold $(M, g)$, respectively.

4.2. Fix an orientation of the Riemannian manifold $(M, g)$, and consider the Hodge operator $\ast : \Omega^r(M, \mathbb{R}) \cong \Omega^{n-r}(M, \mathbb{R})$. There are obvious isomorphisms

\[ \ast : \Omega^r(M, \mathbb{R}) \cong \Omega^{n-r}(M, \mathbb{R}), \quad \ast : C^r(M, \mathbb{R}) \cong C^{n-r}(M, \mathbb{R}). \]

By (2.1), we also have an isomorphism

\[ \ast : \mathcal{F}^r(M, \mathbb{R}) \cong \mathcal{D}^{n-r}(M, \mathbb{R}). \quad (4.3) \]

It implies the well-known isomorphism $\ast : \mathcal{H}^r(M, \mathbb{R}) \cong \mathcal{H}^{n-r}(M, \mathbb{R})$ of the vector spaces of harmonic forms. The isomorphism

\[ \ast : \mathcal{T}^r(M, \mathbb{R}) \cong \mathcal{T}^{n-r}(M, \mathbb{R}) \quad (4.4) \]

of the vector spaces of conformally Killing forms is also well-known (see [10] and [17]). Then (4.3) and (4.4) imply the following isomorphism:

\[ \ast : \mathcal{P}^r(M, \mathbb{R}) \cong \mathcal{K}^{n-r}(M, \mathbb{R}). \quad (4.5) \]

(see also [6]). The isomorphisms (4.4) and (4.5) imply that the numbers $t_r$, $k_r$ and $p_r$ satisfy the duality properties of the form $t_r = t_{n-r}$ and $k_r = p_{n-r}$. These identities are analogues of the Poincaré duality theorem for the Betti numbers. We have therefore proved the following result.

Theorem 1. Let $(M, g)$ be a compact oriented Riemannian manifold of dimension $n \geq 2$. The Tachibana numbers $t_r$, the Killing numbers $k_r$, and the planar numbers $p_r$ of the manifold $(M, g)$ satisfy the duality properties of the form $t_r = t_{n-r}$ and $k_r = p_{n-r}$ for all $r = 1, 2, \ldots, n - 1$.

For a compact oriented Riemannian manifold $(M, g)$, the subspaces $\text{Im} d$ and $\text{Ker} d^*$ are orthogonal complements with respect to the scalar product

\[ \langle \omega, \omega' \rangle = \int_M \omega \wedge \ast \omega' \]

(see [5], Ch. 8, § 11 and [13], Ch. 7.2). Furthermore, the orthogonal complement of the subspace $\text{Im} d$ in $\text{Ker} d$ coincides with the space $\text{Ker} d \cap \text{Ker} d^*$ (see [5], Ch. 8, § 11). Therefore, in the case of the positive curvature operator $\overline{R}$, the vector spaces $\text{Im} d$ and $\text{Ker} d$ coincide due to the absence of harmonic forms on $(M, g)$.

We consider the vector space of conformally Killing $r$-forms $\mathcal{T}^r(n, \mathbb{R})$ on a compact Riemannian oriented manifold $(M, g)$ with positive-definite curvature operator. This subspace is finite-dimensional and is endowed with the scalar product $\langle \cdot, \cdot \rangle$. It is clear from the discussion above that the orthogonal complement to the vector
subspace $K^r(n, \mathbb{R})$ of coclosed conformally Killing $r$-forms in the space $T^r(n, \mathbb{R})$ is the vector subspace $P^r(n, \mathbb{R})$ of exact conformally Killing $r$-forms:

$$T^r(M, \mathbb{R}) = K^r(M, \mathbb{R}) \oplus P^r(M, \mathbb{R}).$$  \hspace{1cm} (4.6)

As a result, we obtain the following theorem:

\textbf{Theorem 2.} Let $(M, g)$ be an $n$-dimensional oriented Riemannian manifold with positive-definite curvature operator ($n \geq 2$). Then the Tachibana numbers $t_r$ of the manifold are decomposed as follows: $t_r = k_r + p_r$, for all $r = 1, 2, \ldots, n-1$.

We note that decomposition (4.6) was obtained in [7] and [8] for a Riemannian manifold of constant sectional curvature, without assuming compactness and orientability.

4.3. One of the important properties of conformally Killing forms is their conformal invariance (see [7], [18]). Namely, if we consider the identity conformal transformation $\text{id}: (M, g) \to (M, \tilde{g})$ such that $\tilde{g} := e^{2f}g$ for a differentiable function $f$ on $(M, g)$, then, for any conformally Killing $r$-form $\omega$, the form $\tilde{\omega} := e^{(r+1)f}\omega$ will be a conformally Killing $r$-form on the Riemannian manifold $(M, \tilde{g})$ with metric $\tilde{g} := e^{2f}g$.

On the other hand, if we consider the identity projective transformation $\text{id}: (M, g) \to (M, \tilde{g})$, that is, a map preserving geodesics, then, for any closed (respectively, coclosed) conformally Killing $r$-form $\omega$, the form $\tilde{\omega} := e^{-(r+1)f}\omega$, where $f = (n+1)^{-1}\ln \sqrt{\det g/\det \tilde{g}}$, will be a closed (respectively, coclosed) conformally Killing $r$-form on the manifold $(M, \tilde{g})$ with the projectively equivalent metric $\tilde{g}$ (see [11], [18]).

As a result, we obtain the following theorem.

\textbf{Theorem 3.} Let $(M, g)$ be a compact oriented Riemannian manifold of dimension $n \geq 2$. The Tachibana numbers $t_r$ are conformal scalar invariants, while the Killing numbers $k_r$ and the planar numbers $p_r$ are projective scalar invariants of $(M, g)$.

This theorem is an analogue of the statement about the conformal invariance of the Betti numbers.

§ 5. On the existence of the Tachibana numbers

5.1. Let $E^{n+1}$ be a Euclidean space with an orthogonal coordinate system $\{x^1, x^2, \ldots, x^{n+1}\}$. Then an arbitrary coclosed conformally Killing 2-form $\omega$ on $E^{n+1}$ has components $\omega_{i_1i_2} = A_{k_{i_1}i_2}x^k + B_{i_1i_2}$, where $A_{k_{i_1}i_2}$ and $B_{i_1i_2}$ are the components of skew-symmetric constant tensors, and $k, i_1, i_2 = 1, 2, \ldots, n+1$ (see [19]).

Let $S^n: (x^1)^2 + (x^2)^2 + \cdots + (x^{n+1})^2 = 1$ be the unit $n$-dimensional hypersphere in $E^{n+1}$. It was proved by Tachibana in [19] that $\omega_{i_1i_2} = A_{k_{i_1}i_2}x^k$ are the components of a Killing 2-form defined globally on $S^n$. This implies that the number $k_2$ is nonzero for any unit hypersphere $S^n$ in the Euclidean $(n+1)$-dimensional
space \( E^{n+1} \). Furthermore, the identity \( t_2 = k_2 + p_2 \) implies that \( t_2 \geq k_2 > 0 \), and therefore, \( t_2 \neq 0 \).

In [6] we proved the following. Let \((M, g)\) be an \( n \)-dimensional locally flat Riemannian manifold with an orthogonal coordinate system \( x^1, \ldots, x^n \). Then an arbitrary coclosed conformally Killing \( r \)-form \( \omega \) has components \( \omega_{i_1 i_2 \ldots i_r} = A_{k_1 k_2 \ldots i_r} x^k + B_{i_1 i_2 \ldots i_r} \), where \( A_{k_1 k_2 \ldots i_r} \) and \( B_{i_1 i_2 \ldots i_r} \) are the components of constant skew-symmetric tensors, and \( k, i_1, i_2, \ldots, i_r = 1, 2, \ldots, n \). Following Tachibana, we constructed in [20] an example of a coclosed conformally Killing \( r \)-form \( \omega_{i_1 i_2 \ldots i_r} = A_{k_1 k_2 \ldots i_r} x^k \) \((1 \leq r \leq n-1)\) which is defined globally on the unit hypersphere \( S^n \). As a result, we obtain that the Killing numbers \( k_r \) are nonzero on any unit hypersphere \( S^n \) for \( 1 \leq r \leq n-1 \). Furthermore, we obtain that \( t_r \neq 0 \), since \( t_r = k_r + p_r > 0 \) for all \( r = 1, \ldots, n-1 \).

**Theorem 4.** The Tachibana numbers \( t_r, 1 \leq r \leq n-1 \), are nonzero on the unit \( n \)-dimensional sphere \( S^n \) in a Euclidean space \( E^{n+1} \).

**5.2.** Let \((M_1, g_1), (M_2, g_2)\) be Riemannian manifolds such that \( \dim M_1 = 1 \) and \( \dim M_2 = n-1 \). Let \( f \) be a positive smooth function on \( M_1 \). The twisted product \( M_1 \times_f M_2 \) of Riemannian manifolds \((M_1, g_1)\) and \((M_2, g_2)\) (see [4], Ch. 9, 9.9–9.11) is defined as the underlying manifold of the topological product \( M = M_1 \times M_2 \) with the metric \( g = g_1 \times_f g_2 \) defined by the identity

\[
g_1 \times_f g_2 = \pi_1^* g_1 + (f \circ \pi_1) \pi_2^* g_2,
\]

where \( \pi_1: M_1 \times M_2 \rightarrow M_1 \) and \( \pi_2: M_1 \times M_2 \rightarrow M_2 \) are the natural projections onto \( M_1 \) and \( M_2 \), respectively. If both \((M_1, g_1)\) and \((M_2, g_2)\) are compact oriented Riemannian manifolds, then their twisted product \( M_1 \times_f M_2 \) is also a compact oriented Riemannian manifold.

For any nonvanishing differentiable periodic function \( \lambda: S^1 \rightarrow \mathbb{R} \), we have that \( \omega = \text{grad} \lambda \) is a globally defined closed conformally Killing 1-form on \( S^1 \times_f S^{n-1} \) (or, in other terminology, a concircular vector field); see [21].

We therefore obtain that \( p_1 = k_{n-1} \geq 1 \).

**Theorem 5.** Both the planar number \( p_1 \) and the Killing number \( k_{n-1} \) are greater than or equal to one on the twisted product \( S^1 \times_f S^{n-1} \).

**5.3.** It is known (see [22]) that any Hamiltonian form on a Kähler manifold defines a conformally Killing 2-form. Using the (global) classification of Kähler manifolds admitting a Hamiltonian 2-form (see [23]), it is possible to give a list of Kähler manifolds with \( t_2 \geq 1 \).

**§ 6. The vanishing theorem for Tachibana numbers**

In this section we prove the vanishing theorem for the Tachibana numbers of a compact oriented Riemannian manifold with negative-definite curvature operator, using the ‘Bochner technique’ (see, for example, [1] and [13], Ch. 7).

**6.1.** Let \((M, g)\) be an \( n \)-dimensional compact Riemannian manifold. For any exterior differential \( r \)-form \( \omega \) we have the identity

\[
D_3^* D_3 \omega = \frac{1}{r(r+1)} \left( \nabla^* \nabla \omega - \frac{1}{r+1} d^* d \omega - \frac{1}{n-r+1} d d^* \omega \right),
\]
which together with the Green theorem implies the integral formula
\[ r(r + 1) \int_M \|D_3 \omega\|^2 \, dv \]
\[ = \int_M \|\nabla \omega\|^2 \, dv - \frac{1}{r + 1} \int_M \|d \omega\|^2 \, dv - \frac{1}{n - r + 1} \int_M \|d^* \omega\|^2 \, dv. \] (6.1)

On the other hand, using the classical Bochner-Weitzenböck formula
\[ \Delta \omega = \nabla^* \nabla \omega + F_r(\omega) \]
we obtain the second integral formula (see [13], Ch. 7.4) by integrating over the manifold \((M, g)\):
\[ \int_M \|d \omega\|^2 \, dv + \int_M \|d^* \omega\|^2 \, dv = \int_M \|\nabla \omega\|^2 \, dv + \frac{1}{4} \int_M \sum_{\alpha} \lambda_\alpha ||[\Theta_\alpha, \omega][\|^2 \, dv. \] (6.2)

Finally, identities (6.1) and (6.2) imply the following integral formula:
\[ r(r + 1) \int_M \|D_3 \omega\|^2 \, dv \]
\[ = \frac{r}{r + 1} \int_M \|d \omega\|^2 \, dv + \frac{n - r}{n - r + 1} \int_M \|d^* \omega\|^2 \, dv - \frac{1}{4} \int_M \sum_{\alpha} \lambda_\alpha ||[\Theta_\alpha, \omega][\|^2 \, dv. \] (6.3)

6.2. Consider a compact Riemannian manifold \((M, g)\) with negative-definite curvature operator \(\overline{R}\) (the latter means that \(\lambda_\alpha < 0\) for all \(\alpha\)). We shall further assume that the \(r\)-form \(\omega\) in (6.3) is conformally Killing. Then it follows from (6.3) that \(\nabla \omega = 0\), and \([\Theta_\alpha, \omega] = 0\) for all \(\alpha\). In this case the \(r\)-form \(\omega\) vanishes, unless \(r\) is not equal to 0 or \(n\) (see [13], pp. 205–213). This implies that there are no nonzero conformally Killing \(r\)-forms \((1 \leq r \leq n - 1)\) on an \(n\)-dimensional compact oriented Riemannian manifold with negative-definite curvature operator (see also [24]). Therefore, the Tachibana numbers \(t_r\) must vanish for all \(r = 1, \ldots, n - 1\). Thus, we have proved the following result.

**Theorem 6.** The Tachibana numbers \(t_r, 1 \leq r \leq n - 1\), vanish for an \(n\)-dimensional compact oriented Riemannian manifold \((M, g)\) with negative-definite curvature operator.

If \(\overline{R} \leq 0\), then (6.3) implies that \(\nabla \omega = 0\), and we obtain
\[ t_r = \dim T^r(n, \mathbb{R}) \leq \frac{n!}{r!(n - r)!}, \]
since any parallel \(r\)-form is completely determined by its components at any point of the manifold (see [13], p. 212).

Theorem 6 implies in particular that for a compact Riemannian manifold of constant negative sectional curvature, or for a compact conformally flat oriented Riemannian manifold with negative-definite Ricci tensor, we have \(t_r = 0\) for all \(r = 1, 2, \ldots, n - 1\) (see [1], [7], [9], [10] and [25]).

This theorem is an analogue of the vanishing theorem for the Betti numbers.
Bibliography

[1] K. Yano and S. Bochner, Curvature and Betti numbers, Ann. of Math. Stud., vol. 32, Princeton Univ. Press, Princeton, NJ 1953.
[2] S. E. Stepanov, “On analogue of the Poincaré duality theorem for Betti numbers”, Abstracts of the International Conference “Differential equations and topology” (Moscow 2008), Moscow 2008, pp. 456–457.
[3] Géométrie riemannienne en dimension 4, Séminaire Arthur Besse 1978/79, Textes Math., vol. 3, CEDIC, Paris 1981.
[4] A. L. Besse, Einstein manifolds, Ergeb. Math. Grenzgeb. (3), vol. 10, Springer-Verlag, Berlin 1987.
[5] D. V. Alekseev, A. M. Vinogradov and V. V. Lychagin, “Basic ideas and concepts of differential geometry”, Geometry – 1, Sovrem. Probl. Mat. Fund. Naprav., vol. 28, VINITI, Moscow 1988, pp. 5–289; English transl., Geometry, vol. I, Encyclopaedia Math. Sci., vol. 28, Springer-Verlag, Berlin 1991, pp. 1–264.
[6] S. E. Stepanov, “On conformal Killing 2-form of the electromagnetic field”, J. Geom. Phys. 33:3–4 (2000), 191–209.
[7] T. Kashiwada, “On conformal Killing tensor”, Natur. Sci. Rep. Ochanomizu Univ. 19:2 (1968), 67–74.
[8] Sh. Tachibana, “On conformal Killing tensor in a Riemannian space”, Tohoku Math. J. (2) 21:1 (1969), 56–64.
[9] U. Semmelmann, “Conformal Killing forms on Riemannian manifolds”, Math. Z. 245:3 (2003), 503–527.
[10] M. Kora, “On conformal Killing forms and the proper space of \( \Delta \) for \( p \)-forms”, Math. J. Okayama Univ. 22:2 (1980), 195–204.
[11] S. E. Stepanov, “The Killing–Yano tensor”, Teoret. Mat. Fiz. 134:3 (2003), 382–387; English transl. in Theoret. and Math. Phys. 134:3 (2003), 333–338.
[12] G. de Rham, Variétés différentiables, Hermann, Paris 1955.
[13] P. Petersen, Riemannian geometry, Grad. Texts in Math., vol. 171, Springer-Verlag, New York 1998.
[14] S. E. Stepanov, “A new strong Laplacian on differential forms”, Mat. Zametki 76:3 (2004), 452–458; English transl. in Math. Notes 76:3 (2004), 420–425.
[15] T. Branson, “Stein–Weiss operators and ellipticity”, J. Funct. Anal. 151:2 (1997), 334–383.
[16] S. E. Stepanov, “New theorem of duality and its applications”, Current problems of field theory 1999–2000, Kazan’ State University Publishing House, Kazan’ 2000, pp. 373–376.
[17] S. E. Stepanov, “Isomorphism of spaces of conformally Killing forms”, Differentsial’naya Geom. Mnogoobrazii Figur 31 (2000), 81–84. (Russian)
[18] S. E. Stepanov, “Some conformal and projective scalar invariants of Riemannian manifolds”, Mat. Zametki 80:6 (2006), 902–907; English transl. in Math. Notes 80:6 (2006), 848–852.
[19] S. Tachibana, “On Killing tensors in a Riemannian space”, Tohoku Math. J. (2) 20:2 (1968), 257–264.
[20] S. E. Stepanov and V. M. Isaev, “Examples of a Killing and a conformally Killing forms”, Differentsial’naya Geom. Mnogoobrazii Figur 32 (2001), 52–57. (Russian)
[21] J. Mikeš, “On existence of nontrivial global geodesic mappings of \( n \)-dimensional compact surfaces of revolution”, Differential geometry and its applications (Brno, Czechoslovakia 1989), World Sci. Publ., Singapore 1990, pp. 129–137.
[22] V. Apostolov, D. M. J. Calderbank and P. Gauduchon, “Hamiltonian 2-forms in Kähler geometry. I. General theory”, *J. Differential Geom.* **73**:3 (2006), 359–412.

[23] V. Apostolov, D. M. J. Calderbank and P. Gauduchon, “Hamiltonian 2-forms in Kähler geometry. II. Global classification”, *J. Differential Geom.* **68**:2 (2004), 277–345.

[24] S. E. Stepanov, “On a generalization Kashiwada’s theorem”, *Webs and quasigroups*, Tver State Univ. Press, Tver 1999, pp. 162–167.

[25] S. E. Stepanov, “The vector space of conformal Killing forms on a Riemannian manifold”, *Geometry and topology*. 4, Zap. Nauchn. Sem. S.-Peterburg. Otdel. Mat. Inst. Steklov., vol. 261, POMI, St.-Petersburg 1999, pp. 240–265; English transl. in *J. Math. Sci. (New York)* **110**:4 (2002), 2892–2906.

**S. E. Stepanov**  
Financial University under the Government of the Russian Federation, Moscow  
*E-mail: s.e.stepanov@mail.ru*