Torsion Degrees of Freedom in the Regge Calculus as Dislocations on the Simplicial Lattice

Jürgen Schmidt∗
Institut für Physik
Lehrstuhl Nichtlineare Dynamik
Universität Potsdam
D-14469 Potsdam, Germany

Christopher Kohler
Institut für Theoretische und Angewandte Physik
Universität Stuttgart
D-70550 Stuttgart, Germany

Proposed running title:
Torsion in the Regge Calculus

March 24, 2022

Abstract

Using the notion of a general conical defect, the Regge Calculus is generalized by allowing for dislocations on the simplicial lattice in addition to the usual disclinations. Since disclinations and dislocations correspond to curvature and torsion singularities, respectively, the method we propose provides a natural way of discretizing gravitational theories with torsion degrees of freedom like the Einstein-Cartan theory. A discrete version of the Einstein-Cartan action is given and field equations are derived, demanding stationarity of the action with respect to the discrete variables of the theory.

∗Corresponding author, e-mail:j.schmidt@agnld.uni-potsdam.de
1 Introduction

The theory we now call Regge Calculus was proposed in 1961 by T. Regge [1] as a discrete version of General Relativity formulated within the framework of Riemannian geometry. In spite of the experimental success of General Relativity, a number of authors started to put gravitational theory on the grounds of non-Riemannian geometry: The notion of Riemannian curvature was first generalized by Cartan [2], introducing torsion degrees of freedom, and later this concept was included in a formulation of gravitation as a gauge theory of the Poincaré group by Sciama [3] and Kibble [4] and worked out by Hehl et al. [5] (the theory presented in the latter reference will be referred to as Einstein-Cartan theory in the following). An even more general geometry was proposed by Hehl et al. [6], allowing for nonmetricity in addition to curvature and torsion degrees of freedom. An alternative approach to gravity are the so called teleparallel theories [7], based on the Weitzenböck geometry, working with torsion and vanishing curvature.

Despite this development in the continuum theory, only few attempts have been made to include concepts of non–Riemannian geometry into the Regge Calculus. Caselle et al. [10] formulated Regge Calculus as a lattice gauge theory of the Poincaré group and pointed out the possibility of including torsion as closure failures of the building blocks of the simplicial manifold (see also [9]). Drummond [8] described torsion on the d-dimensional Regge lattice as a piecewise constant tensor field, i.e. within every d-simplex the torsion field was assigned a constant value, which in general changes discontinuously at the hypersurface between two neighboring simplices. Thus, in Drummond’s approach the geometric quantities curvature and torsion are treated in a different way: While curvature appears on the lattice as a conical defect (a disclination) of the underlying simplicial manifold, torsion does not correspond to the simplicial structure itself, but is dealt with in a way similar to Sorkin’s treatment of the electromagnetic field on the simplicial lattice [11].

On the other hand the notion of torsion singularities, appearing as a conical defect (a dislocation), has recently been discussed in the literature again [12]. Its application to the theory of crystal defects has been known for a long time [13], [14], [15], [16], [17], and the connection to gravitation is pointed out in [18], [19], [20], and [21]. This suggests a natural way of incorporating torsion degrees of freedom into the Regge Calculus much in the same way as curvature: treating it as a conical defect of the simplicial manifold.

In this paper, we will apply this idea in order to find a discrete version of the Einstein-Cartan theory, i.e. we will construct the discrete analogue of the Einstein-Cartan action,
choose appropriate sets of discrete variables, and compute the corresponding field equations.

2 Simplicial Torsion

In the presence of torsion, infinitesimal parallelograms in space-time generally do not close, i.e. to a surface element $dx^\mu \wedge dx^\nu$ there belongs a closure failure $dG^\alpha$ proportional to the torsion tensor

$$dG^\alpha = T^\alpha_{\mu\nu} dx^\mu \wedge dx^\nu.$$  \hspace{1cm} (1)

Here, greek indices take on the values $0, \ldots, d-1$ where $d$ is the dimension of space-time.

We assume that the tensor of the torsion density is of the form

$$T^\alpha_{\mu\nu} \sim b^\alpha S_{\mu\nu},$$  \hspace{1cm} (2)

where the Burgers vector $b^\alpha$ gives the strength and direction of the associated closure failure and the antisymmetric tensor $S_{\mu\nu}$ the orientation of the support of the distributional torsion field. This can be motivated in a heuristic way similar to Regge’s [1] construction of his simplicial curvature tensor: We take a bundle of parallel dislocations in three dimensions (e.g. in an elastic medium, generated by a Volterra Process [22]) each with the same Burgers vector $b^\alpha$ (see Fig. 1), their orientation given by the unit vector $U^\alpha$. Now we encircle the bundle by a small loop with normal vector $dF^\alpha$. 

![Figure 1: (a) Loop around a bundle of dislocations and (b) the same loop in defect free medium.](image-url)
Transferred to defect free space, this loop does not close, the closure failure $dG^\alpha$ being proportional to the Burgers vector $b^\alpha$. If $dN$ denotes the number of dislocations enclosed by the loop, we have

$$dG^\alpha = dN b^\alpha = \rho U^\lambda dF_\lambda b^\alpha,$$

where $\rho$ is the density of dislocations in a surface perpendicular to the bundle. Inserting the dual quantities

$$U^\lambda = \frac{1}{2} \epsilon^{\lambda\mu\nu} S_{\mu\nu},$$
$$dF_\lambda = \frac{1}{2} \epsilon_{\lambda\beta\gamma} dF^{\beta\gamma}$$

leads to

$$dG^\alpha = \frac{1}{2} \rho b^\alpha S_{\mu\nu} dF^{\mu\nu}. \quad (5)$$

Comparison with equation (3) yields

$$T^\alpha_{\mu\nu} = \frac{1}{2} \rho b^\alpha S_{\mu\nu}. \quad (6)$$

In $d$ dimensions, the defect has codimension two, i.e. its orientation is given by $d-2$ orthogonal unit vectors $U_{\alpha_1}^{\alpha_1}, \ldots, U^{\alpha_{d-2}}_{\alpha_{d-2}}$. The antisymmetric tensor $S_{\mu\nu}$ is then defined by

$$S_{\mu\nu} = \epsilon_{\mu\nu\alpha_1\ldots\alpha_{d-2}} U_{\alpha_1}^{\alpha_1} \ldots U^{\alpha_{d-2}}_{\alpha_{d-2}}. \quad (7)$$

### 3 Discrete Action, Dynamical Variables, and Field Equations

In the geometry of a continuous manifold, torsion enters the scene when we allow the connection $\Gamma^\alpha_{\mu\nu}$ to be nonsymmetric, its antisymmetric part defining the torsion tensor

$$T^\alpha_{\mu\nu} \equiv \Gamma^\alpha_{\mu\nu} - \Gamma^\alpha_{\nu\mu}. \quad (8)$$

Assuming the condition of metricity $g_{\mu\nu,\lambda} = 0$, the connection reads

$$\Gamma^\alpha_{\mu\nu} = \left\{ \begin{array}{c} \alpha \\ \mu
\end{array} \right\} + K^\alpha_{\mu\nu} \quad (9)$$
where

\[ K^\alpha_{\mu\nu} \equiv \frac{1}{2}(T^\alpha_{\mu\nu} - T^\alpha_{\mu\nu} - T^\alpha_{\nu\mu}) \]  

(10)
defines the contortion tensor and

\[ \left\{ \frac{\alpha}{\mu\nu} \right\} = \frac{1}{2}g^{\alpha\rho}(\partial_\mu g_{\nu\rho} + \partial_\nu g_{\mu\rho} - \partial_\rho g_{\mu\nu}) \]  

(11)
is the Levi-Civita connection of General Relativity. In terms of the connection \( \Gamma^\alpha_{\mu\nu} \), the curvature tensor is defined as

\[ R^\alpha_{\beta\mu\nu} \equiv \partial_\mu \Gamma^\alpha_{\beta\nu} - \partial_\nu \Gamma^\alpha_{\beta\mu} + \Gamma^\alpha_{\gamma\mu} \Gamma^\gamma_{\beta\nu} - \Gamma^\alpha_{\gamma\nu} \Gamma^\gamma_{\beta\mu}. \]  

(12)

Using equation (9) we can now split the curvature scalar into a part depending only on \( g_{\mu\nu} \) and on its first derivatives and a part depending only on the torsion. The result is

\[ R \equiv R^\alpha_{\beta\mu\nu} = g^{\alpha\beta} \left( \left\{ \frac{\mu}{\alpha\nu} \right\} \left\{ \frac{\nu}{\beta\mu} \right\} - \left\{ \frac{\mu}{\alpha\beta} \right\} \left\{ \frac{\nu}{\mu\nu} \right\} \right) \]
\[ + \frac{1}{4}T^{\alpha\mu\nu}T_{\alpha\mu\nu} + \frac{1}{2}T^{\alpha\mu\nu}T_{\nu\mu\alpha} + T^{\alpha\mu\nu}T_{\nu\mu\alpha} \]
\[ + \frac{1}{\sqrt{-g}} \text{divergence}, \]  

(13)

where the last term does not contribute to the action integral when we compute it for a manifold without boundary.

With the splitting (13) of the curvature scalar, the Einstein-Cartan action reads

\[ S_{EC} \equiv \frac{1}{2} \int d^d x \sqrt{-g} R(g, T) \]
\[ = S_{EH} + \frac{1}{2} \int d^d x \sqrt{-g} \left( \frac{1}{4}T^{\alpha\mu\nu}T_{\alpha\mu\nu} + \frac{1}{2}T^{\alpha\mu\nu}T_{\nu\mu\alpha} + T^{\alpha\mu\nu}T_{\nu\mu\alpha} \right) \]  

(14)

where we have identified the Einstein–Hilbert action

\[ S_{EH} = \frac{1}{2} \int d^d x \sqrt{-g}g^{\alpha\beta} \left( \left\{ \frac{\mu}{\alpha\nu} \right\} \left\{ \frac{\nu}{\beta\mu} \right\} - \left\{ \frac{\mu}{\alpha\beta} \right\} \left\{ \frac{\nu}{\mu\nu} \right\} \right). \]  

(15)

Discretization of the Einstein–Hilbert action leads to the Regge action [1], so we are concerned here with the second term in equation (14).
In Regge Calculus, the curvature scalar is defined as a distribution with support on the d–2 dimensional hypersurfaces of the lattice (in Regge Calculus called the bones of the lattice). In the same way we treat the non-Riemannian curvature scalar, in particular its non-Riemannian part, i.e. we define squares of the torsion tensor as a distribution.

Using equation (6) we obtain for the contribution of the \(i\)-th bone to a typical term appearing under the Einstein-Cartan action integral (eq. (14)) [no sum over repeated latin indices]

\[
T^{\alpha\mu\nu}_{(i)} T_{(i)\alpha\mu\nu} = \frac{1}{4} b^\alpha_{(i)} b_{(i)\alpha} S^\mu_{(i)} S_{(i)\mu\nu} \int ds_1^{(i)} \ldots ds_{d-2}^{(i)} \delta(d) \left\{ x - y_{(i)}(s_1^{(i)}, \ldots, s_{d-2}^{(i)}) \right\}.
\]

Here, \(y_{(i)}(s_1^{(i)}, \ldots, s_{d-2}^{(i)})\) is the set of points of the \(i\)-th bone, parameterized by \(s_1^{(i)}, \ldots, s_{d-2}^{(i)}\).

With

\[
S_{\mu\nu} S^{\mu\nu} = 2, \quad S_{\mu\nu} S_{\mu\alpha} = \delta^\nu_\alpha - (U_1 U_1^\nu + \ldots + U_{d-2} U_{d-2}^\nu),
\]

we obtain

\[
\frac{1}{4} T^{\alpha\mu\nu}_{(i)} T_{(i)\alpha\mu\nu} + \frac{1}{2} T^{\alpha\mu\nu}_{(i)} T_{(i)\alpha\nu\alpha} + T^\alpha_{(i)\alpha\mu} T^{\nu\mu}_{(i)\nu} = \frac{1}{8} b^2_{(i)} \int ds_1^{(i)} \ldots ds_{d-2}^{(i)} \delta(d) \left\{ x - y_{(i)}(s_1^{(i)}, \ldots, s_{d-2}^{(i)}) \right\},
\]

with

\[
b^2_{(i)} = \left( (b_{(i)\alpha} U_1^\alpha)^2 + \ldots + (b_{(i)\alpha} U_{d-2}^\alpha)^2 \right),
\]

where \(b_{(i)}\) denotes the part of the Burgers vector that is parallel to the bone, i.e. only screw dislocations contribute to the action.

Summing equation (18) over all the bones of the lattice and integrating over the whole manifold yields

\[
\int dV \left( \frac{1}{4} T^{\alpha\mu\nu} T_{\alpha\mu\nu} + \frac{1}{2} T^{\alpha\mu\nu} T_{\nu\mu\alpha} + T^\alpha_{\alpha\mu} T^{\nu\mu}_{\nu} \right)
\]

\[
= \frac{1}{8} \sum_i b^2_{(i)} \int dV \int ds_1^{(i)} \ldots ds_{d-2}^{(i)} \delta(d) \left\{ x - y_{(i)}(s_1^{(i)}, \ldots, s_{d-2}^{(i)}) \right\}
\]

\[
= \frac{1}{8} \sum_i b^2_{(i)} A_{(i)},
\]

where \(A_{(i)}\) is the \((d–2)\) dimensional area of the \(i\)-th bone. This is the simplicial analogue to the second term in (14). For the Einstein-Hilbert term we substitute the Regge action
and obtain the lattice action

\[ S = \sum_i \left( \varphi(i) + \frac{1}{16} b^2(i) \right) A(i), \]  

(21)

where the \( \varphi(i) \) are the deficit angles of the bones.

As the first set of dynamical variables of the theory, we take the link lengths \( l(i) \) of the lattice (corresponding to the components of the metric tensor of the continuum theory) as in the usual Regge Calculus. In the Einstein-Cartan theory, the components of the torsion tensor itself become the second set of variables of the action, so the Burgers vectors \( b(i) \) of the bones are the most appropriate choice (see eq. (6)) of lattice variables in a discretized Einstein-Cartan theory. Since only the projections of the Burgers vectors onto the bones contribute to the action, we choose the \( b(i) \) defined in (19) as (scalar) variables representing torsion on the simplicial lattice.

Adding a matter term \( L(\{l(j)\}, \{b(j)\}) \) to the action, variation with respect to \( l(i) \) and \( b(j) \) gives two sets of field equations,

\[ \sum_i \left( \varphi(i) + \frac{1}{16} b^2(i) \right) \frac{\partial A(i)}{\partial l(j)} = -\frac{\partial L}{\partial l(j)}, \]  

(22)

and

\[ b(j) = -\frac{1}{A(j)} \frac{\partial L}{\partial b(j)}. \]  

(23)

In four dimensions we have

\[ \frac{\partial A(i)}{\partial l(j)} = \cot \theta_{(ij)}, \]  

(24)

where \( \theta_{(ij)} \) is in the \( i \)-th bone (triangle of the lattice) the angle opposite to the \( j \)-th link. So, the first field equation reads

\[ \sum_i \left( \varphi(i) + \frac{1}{16} b^2(i) \right) \cot \theta_{(ij)} = -\frac{\partial L}{\partial l(j)}, \]  

(25)

where the sum now extends over all the bones having the link \( l(j) \) in common.

The Burgers vector couples algebraically to the matter term \( \partial L/\partial b(j) \), i.e. we have no dislocations on the lattice in vacuum. Without this matter term, which is interpreted in continuum theory as the spin density of matter, the field equations reduce to the ordinary Regge equations. This is in parallel to the result of the continuum theory, where the Einstein-Cartan field equations reduce to the Einstein equation in vacuum.
4 Conclusion

From the application of non-Riemannian geometry to the theory of defects in solids [13], [14], we know that the torsion field can be interpreted as a continuous distribution of dislocations. Vice versa, we can express the torsion induced by a single dislocation as a delta like distribution with support on the dislocation line. This suggests the idea of regarding torsion degrees of freedom within the Regge Calculus as dislocations of the lattice, the bones carrying singular torsion in addition to a curvature singularity. Thus, to a loop around a bone, we assign not only a rotation (of a test vector), but also a translation, which we call the Burgers vector of the bone. Discretizing the Einstein-Cartan action, we find a generalization of the Regge action where the Burgers vector of a particular bone leads to a shift of the deficit angle. Variation of the action with respect to the link lengths and the Burgers vectors leads to two sets of field equations that show fundamental properties of the continuum equations.

Fermion fields, defined on the lattice, would lead to a nonvanishing spin density and act as a source of torsion [23] localized at the matter. Although classically in vacuum not possible, in simplicial quantum gravity configurations with nonvanishing Burgers vectors could be important as quantum fluctuations of the lattice [10].

Acknowledgment
We have benefited from discussions with A. Holz and T. Filk.

References

[1] T. Regge, General Relativity without Coordinates, Nuovo Cimento 19, 558 (1961)

[2] É. Cartan, Sur une généralisation de la notion de courbure de Riemann et les espaces à torsion, Comptes Rendus hebdomadaires des Séances de l'Academie des Sciences

[3] D. W. Sciama, On a Non-symmetric Theory of the Pure Gravitational Field, Proc. Camb. Philos. Soc. 54, 72 (1958)

[4] T. W. B. Kibble, Lorentz Invariance and the Gravitational Field, J. Math. Phys. 2, 212 (1961)

[5] F. W. Hehl, P. von der Heyde, G. D. Kerlick and J. M. Nester, General Relativity with Spin and Torsion: Foundations and Prospects, Rev. Mod. Phys. 48, 393 (1976)
[6] F. W. Hehl, J. D. McCrea, E. Mielke and Y. Ne’eman, Metric-affine Gauge Theory of Gravity: Field Equations, Noether Identities, World Spinors, and Breaking of Dilation Invariance, Phys. Rep. 258, 1 (1995)

[7] K. Hayashi and T. Shirafuji, New general relativity, Phys. Rev. D 19, 3524 (1979)

[8] I. T. Drummond, Regge-Palatini Calculus, Nucl. Phys. B 273, 125 (1985)

[9] F. Gronwald, Nonriemannian parallel transport in Regge Calculus, Class. Quantum Grav. 12, 1181 (1995)

[10] M. Caselle, A. D’Adda and L. Magnea, Regge Calculus as a Local Theory of the Poincaré Group, Phys. Lett. B 232, 457 (1989)

[11] R. Sorkin, The electromagnetic field on a simplicial net, J. Math. Phys. 12, 2432 (1975)

[12] K. P. Tod, Conical Singularities and Torsion, Class. Quantum Grav. 11, 1331 (1994)

[13] K. Kondo, On the Geometrical and Physical Foundations of the Theory of Yielding, in Proceedings of the 2nd Japan National Congress for applied mechanics 41, (1952)

[14] B. A. Bilby, R. Bullough and E. Smith, Continous Distributions of Dislocations: A New Application of the Methods of nonriemannian Geometry, Proc. Roy. Soc. Lond. A 231, 263 (1955)

[15] E. Kröner, Allgemeine Kontinuumstheorie der Versetzungen und Eigenspannungen, Arch. Ration. Mech. Anal. 4, 467 (1960)

[16] B. K. D. Gairola, Nonlinear Elastic Problems, in Dislocations in Solids, vol. 1 ed. F. R. N. Nabarro, North–Holland

[17] A. Holz Geometry and Action of Arrays of Disclinations in Crystals and Relation to (2 + 1)-dimensional Gravitation, Class. Quantum Grav. 5, 1259 (1988)

[18] A. Holz Topological Properties of Linked Disclinations and Dislocations in Solid Continua, J. Phys. A: Math. Gen. 25, L1 (1992)

[19] M. O. Katanaev and I. V. Volovich, Theory of Defects in Solids and Three-dimensional Gravity, Ann. Phys. 216, 1 (1992)
[20] C. Kohler, *Point Particles in (2 + 1) Dimensional Gravity as Defects in Solid Continua*, Class. Quantum Grav. **12**, L11 (1995)

[21] C. Kohler, *Line Defects in Solid Continua and Point Particles in (2+1) Dimensional Gravity*, Class. Quantum Grav. **12**, 2988 (1995)

[22] R. A. Puntigam and H. H. Soleng, *Volterra Distorions, Spinning Strings, and Cosmic Defects*, Class. Quantum Grav. **14**, 1129 (1997)

[23] H. C. Ren, *Matter Fields in Lattice Gravity*, Nucl. Phys. **B301**, 661 (1988)