A COMBINATORIAL SUBSTITUTE FOR THE DEGREE THEOREM IN AUTER SPACE

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Abstract. Auter space $A_n$ is contractible. A. Hatcher and K. Vogtmann constructed a stratification of $A_n$ into subspaces $A_{n,k}$ such that $A_{n,k}$ is $k$-connected. Their argument that $A_{n,k}$ is $(k-1)$-connected, the Degree Theorem and its proof, is somewhat global in nature. Here we present a combinatorial substitute for the Degree Theorem that uses only local considerations to show that $A_{n,k}$ is $(k-1)$-connected.

Let $R_0$ be a (topological) connected graph with one vertex and $n$ edges and an identification $\pi_1(R_0) \cong F_n$, the free group on $n$ generators. If $\Gamma$ is a metric graph with basepoint $p$, then a homotopy equivalence $\rho : R_0 \to \Gamma$ sending the basepoint of $R_0$ to $p$ is called a marking on $\Gamma$. There is an equivalence relation on the set of markings where two markings are considered equivalent if there is a basepoint-preserving homotopy between them. The space of all marked graphs for which the underlying metric graph has fundamental group of rank $n$ and edge lengths sum to 1 is denoted $A_n$. In this way we can identify $\text{Aut}(F_n)$ with the group of basepoint-preserving homotopy equivalences of $R_0$. Thus there is a right action of $\text{Aut}(F_n)$ on $A_n$ as follows: If $A \in \text{Aut}(F_n)$ and $(\Gamma, p, \rho)$ is a point in $A_n$ then $A(\Gamma, p, \rho) = (\Gamma, p, \rho \circ A)$. The spine of Auter space, denoted here by $L_n$, is a deformation retract of $A_n$ where the metric data is ignored and only the combinatorial data of the graph and the marking are considered. For a more complete description of the construction of the analogous space for the outer automorphism group of $F_n$ see [1].

In [2], Hatcher and Vogtmann use a function denoted here by $d_0$ to stratify the space $A_n$ into subspaces $A_{n,k} := \{ (\Gamma, p, \rho) \mid d_0(\Gamma) \leq k \}$. These subspaces are invariant under the action of $\text{Aut}(F_n)$ since the action only affects markings. The main technical result of [2] is:

**Degree Theorem.** A piecewise linear map $f_0 : D^k \to A_n$ is homotopic to a map $f_1 : D^k \to A_{n,k}$ by a homotopy $f_1$ during which $d_0$ decreases monotonically, that is, if $t_1 < t_2$ then $d_0(f_{t_1}(s)) \geq d_0(f_{t_2}(s))$ for all $s \in D^k$.

Using the Degree Theorem, they show that the pair $(A_n, A_{n,k})$ is $k$-connected. Since Auter space is contractible, this is equivalent to $A_{n,k}$ being $(k-1)$-connected. This result is used in the main theorems of [2] to show ranges for integral and rational homological stability. The proof of the Degree Theorem is of global nature. Here we present an alternate proof that

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\( A_{n,k} \) is \((k-1)\)-connected using combinatorial Morse theory and working only with the spine \( L_n \) and the subcomplexes \( L_{n,k} := L_n \cap A_{n,k} \).

Let \( \Gamma \in L_n \) have \( V \) vertices and \( E \) edges, and all non-basepoint vertices at least tri-valent. Let \( d(p, v) \) denote the minimum length of a path in \( \Gamma \) from a vertex \( v \) to \( p \), and call \( d(p, v) \) the level of \( v \). Define \( \Lambda_i(\Gamma) := \{ v \in \Gamma \mid d(p, v) = i \} \), \( n_i(\Gamma) := -|\Lambda_i(\Gamma)| \), and \( d_i(\Gamma) := \sum_{v \notin \Lambda_i(\Gamma)} \text{degree}(v) \) for \( i \geq 1 \), where \( \text{degree}(v) \) is the number of half-edges beginning or terminating at \( v \). Finally, define \( h(\Gamma) := (d_0(\Gamma), n_1(\Gamma), d_1(\Gamma), n_2(\Gamma), d_2(\Gamma), ...) \) to be the height of the graph \( \Gamma \). Note that this height function is a refinement of the \( d_0 \) function used in [2]. We will use this height as our Morse function. We compare the height using the lexicographic order, e.g.: \((2, -5, 4) > (1, -3, 6) > (1, -3, 3)\). With this Morse function to stratify each \( L_{n,i} \), we consider the descending links of points \((\Gamma, p, \rho)\) with \( d_0(\Gamma) \geq i \) in \( L_n \). The descending link can be described as the simplicial join of the descending blow-up complex, which we call the up-link, and descending blow-down complex of \( \Gamma \), which we call the down-link.

**Proposition 1.4.** The down-link of \( \Gamma \) in \( L_n \) is either contractible or homotopy equivalent to \( \bigvee S^{V-2} \).

See Section 1 for the proof of this proposition.

**Lemma 2.5.** For \( \Gamma \) with \( V \) vertices and \( d_0(\Gamma) = k \), if the down-link in \( L_n \) is not contractible, then the up-link in \( L_n \) is homotopy equivalent to \( \bigvee S^{k-V} \).

See Section 2 for the proof of this lemma.

**Corollary.** For \( \Gamma \) with \( d_0(\Gamma) = k \), the descending link of \( \Gamma \) in \( L_n \) is either homotopy equivalent to \( \bigvee S^{k-1} \) or contractible.

**Proof.** If the down-link of \( \Gamma \) is contractible, then joining it with the up-link still yields a contractible space, and so the descending link is contractible. If the down-link is not contractible, then joining the up-link and down-link yields \( \bigvee S^{k-V} \ast \bigvee S^{V-2} \simeq \bigvee S^{(k-V)+(V-2)+1} \simeq \bigvee S^{k-1} \). \( \square \)

**Theorem.** \( L_{n,k} \) and \( A_{n,k} \) are \((k-1)\)-connected.

**Proof.** Let \((\Gamma, p, \rho)\) be a point in \( L_{n,k} \), so in particular \( d_0(\Gamma) = k \). By the previous corollary, \((\Gamma, p, \rho)\) has a descending link which is greater than or equal to \( k \) spherical, and thus \((k-1)\)-connected. Since \( L_n \) is itself contractible and so \((\Gamma, p, \rho)\) has a descending link which is greater than or equal to \( k \) spherical, and thus \((k-1)\)-connected, we have that \( L_{n,k} \) must be \((k-1)\)-connected. Since \( A_{n,k} \) deformation retracts onto \( L_{n,k} \), we have that \( A_{n,k} \) is also \((k-1)\)-connected. \( \square \)

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1. Connectivity of the down-link

Let \( \epsilon \) be an edge and \( F \) a forest in \( \Gamma \) and define \( D(F) := \min \{ i \mid F \text{ has a vertex in } \Lambda_i \} \) to be the height of the forest \( F \). Let \( \Gamma/F \) denote the quotient graph when each component of \( F \) is collapsed to a point.

**Lemma 1.1.** If \( F \) connects two vertices of height \( D(F) \), then \( h(\Gamma/F) > h(\Gamma) \). If \( F \) does not connect two vertices of height \( D(F) \), then \( h(\Gamma/F) < h(\Gamma) \).

**Proof.** Assume \( F \) connects two vertices of level \( D(F) \). Since \( D(F) \) is the minimum distance of vertices in \( F \) from the basepoint \( p \), blowing down \( F \) will not change \( n_i \) or \( d_i \) for \( i < D(F) \). As \( F \) connects at least two vertices of level \( D(F) \), we have \( n_{D(F)} \) strictly greater in \( \Gamma/F \) than in \( \Gamma \). Thus \( h(\Gamma/F) > h(\Gamma) \).

Assume \( F \) does not connect two vertices of height \( D(F) \), so blowing down \( F \) will not change \( n_{D(F)} \). However, since each non-basepoint vertex of \( \Gamma \) is at least trivalent, \( d_{D(F)} \) will be smaller in \( \Gamma/F \) than in \( \Gamma \). Thus \( h(\Gamma/F) < h(\Gamma) \). \( \square \)

Note that the Lemma 1.1 shows that blow-downs either strictly increase or strictly decrease the height of graphs in \( L_n \), and so no two graphs related by a forest blow-down can have the same height.

We call an edge \( \epsilon \), with vertices \( v_1 \) and \( v_2 \), horizontal if \( d(p, v_1) = d(p, v_2) \). If \( d(p, v_1) \neq d(p, v_2) \), call \( \epsilon \) a vertical edge. For a vertical edge \( \epsilon \) with vertices \( v_1 \) and \( v_2 \) assume that \( d(p, v_1) > d(p, v_2) \). We say that the edge \( \epsilon \) descends from its initial vertex \( v_1 \) to its terminal vertex \( v_2 \). We say that \( \Gamma \) has a unique descending edge if there is some vertex \( v \) in \( \Gamma \) from which exactly one edge descends.

**Lemma 1.2.** The complex of descending forest blow-downs of \( \Gamma \), denoted \( P(\Gamma) \), is homotopy equivalent to a (possibly empty) wedge of spheres of dimension \( V - 2 \).

**Proof.** The proof mimics the proof that the unrestricted forest complex of a graph is homotopy equivalent to \( \bigvee S^{V-2} \) presented in [4].

We proceed by induction on \( V + E \). Note that loops in \( \Gamma \) do not affect the forest complex, so we assume that \( \Gamma \) has no loops. Note also that if \( \Gamma \) has a separating edge, which necessarily connects two vertices on different levels, then this edge serves as a cone point for the descending forest complex, making it contractible and thus an empty wedge of spheres.

Base Case: If \( V + E = 1 \), then \( \Gamma \) is a point and so has an empty blow-down complex, which is \( S^{-1} \) as desired.

Now consider \( \Gamma \) with \( V \) vertices and \( E \) edges and let \( \epsilon \) be an edge in \( \Gamma \) farthest away from the basepoint \( p \).

Let \( P_I(\Gamma) \) be the poset of all descending forests in \( \Gamma \) except the forest just consisting of the edge \( \epsilon \) and \( P_0(\Gamma) \subseteq P_1(\Gamma) \) be the poset of all descending forests which do not contain \( \epsilon \). Then \( m : P_1(\Gamma) \to P_1(\Gamma) \) defined by \( m(F) = \)
$F - \{\epsilon\}$ is a poset map which is the identity on $P_0(\Gamma)$ and thus a poset retraction onto $P_0(\Gamma)$. Since $m(F) \leq F$ for all $F \in P_1(\Gamma)$, paragraph 1.3 of [3] implies that $m$ induces a homotopy equivalence between $P_1(\Gamma)$ and image $P_0(\Gamma)$. Since $P_0(\Gamma)$ is isomorphic to $P(\Gamma - \epsilon)$ we have by induction that $P_1(\Gamma) \simeq \sqrt{S^{V-2}}$.

If $\epsilon$ is a horizontal edge in $\Gamma$ then $\epsilon$ is not a descending forest and so $P_1(\Gamma) = P(\Gamma)$, and thus is a wedge of spheres of the appropriate dimension. Assume instad that $\epsilon$ is a vertical edge in $\Gamma$, so $P(\Gamma) = P_1(\Gamma) \cup \text{(star}(\epsilon))$ with $P_1(\Gamma) \cap \text{(star}(\epsilon)) = \text{link}(\epsilon)$ with link and star as defined for simplicial complexes. Using the Mayer-Vietoris sequence and the Van Kampen Theorem, it only remains to show that $\text{link}(\epsilon) \simeq \sqrt{S^{V-3}}$.

Assume that $F$ is a forest in $\text{link}(\epsilon)$. Note that $D(F) = D(F/\epsilon)$ as $\epsilon$ has maximal distance to $p$ and also that if $F$ does not connect vertices on level $D(F)$, then $F/\epsilon$ does not connect vertices on level $D(F/\epsilon)$. Hence, $F \mapsto F/\epsilon$ induces a map $c : \text{link}(\epsilon) \to P(\Gamma/\epsilon)$. Note that $c$ is injective.

Consider a forest $f$ in $P(\Gamma/\epsilon)$. Blowing-up $\epsilon$ in $\Gamma/\epsilon$ and adding $\epsilon$ to $f$ leaves $D(f \cup \{\epsilon\}) = D(f)$; and $f \cup \{\epsilon\}$ does not connect any vertices of $\Gamma/\epsilon$ not already connected by $f$. Thus $f \cup \{\epsilon\} \in P(\Gamma)$ with $c(f \cup \{\epsilon\}) = f$, and so $c$ is surjective and thus an isomorphism, whence $\text{link}(\epsilon) \simeq \sqrt{S^{V-3}}$ as desired.

Lemma 1.3. $P(\Gamma)$ is contractible if $\Gamma$ has a unique descending edge.

Proof. The proof is almost the same as the proof of the previous lemma.

We proceed again by induction on $V + E$. If $V + E = 1$, then $\Gamma$ is a point, has no edges, and in particular does not have unique descending edges. Thus the claim is vacuously true.

Now assume that $V + E > 1$ and that $\Gamma$ has a unique descending edge $\eta$. If $\eta$ has maximum distance to the base point among edges in $\Gamma$ then it is separating and $P(\Gamma)$ is contractible with the forest consisting just of $\eta$ serving as a cone point. Otherwise, let $\epsilon \neq \eta$ be an edge in $\Gamma$ that has maximum distance to the basepoint.

As in the previous proof, $P_1(\Gamma) \simeq P_0(\Gamma) \simeq P(\Gamma - \epsilon)$, which is contractible since $\Gamma - \epsilon$ still contains the unique descending edge $\eta$. We again reduce the proof to showing that $\text{link}(\epsilon)$ has the appropriate homotopy type, which is a point.

We still conclude that $D(F) = D(F/\epsilon)$ and in the same manner as the previous proof get an isomorphism $c : \text{link}(\epsilon) \to P(\Gamma/\epsilon)$. Since collapsing the vertical edge $\epsilon$ in $\Gamma$ can only make top-level vertical edges become horizontal or top-level horizontal edges become vertical in $\Gamma/\epsilon$, $\eta$ is left unchanged and so is still a unique descending edge in $\Gamma/\epsilon$, whence $\text{link}(\epsilon)$ is contractible. □

Proposition 1.4. The down-link of $\Gamma$ in $L_n$ is either contractible or homotopy equivalent to $\sqrt{S^{V-2}}$.

Proof. This is a direct consequence of Lemma 1.2 and Lemma 1.3. □
2. Connectivity of the up-link

Let $B_v$ denote a blow-up at a vertex $v$ in $\Gamma$. Define $B := (B_v \mid v$ is a vertex in $\Gamma)$ and $D(B) := \min\{d(p, v) \mid B_v$ is nontrivial}$ and let $\Gamma^B$ be the result of blowing-up $B$ in $\Gamma$. Let $e_1$ and $e_2$ be two edges which descend from $v$. If after the blow-up $B_v$ we have that $e_1$ and $e_2$ no longer share their initial vertex $v$, we say that $B_v$ separates or that $B$ separates at $v$.

**Lemma 2.1.** If $B$ separates at a vertex on level $D(B)$, then $h(\Gamma^B) < h(\Gamma)$. If $B$ does not separate at a vertex on level $D(B)$, then $h(\Gamma^B) > h(\Gamma)$.

**Proof.** Assume $B$ separates at a vertex on level $D(B)$. Then $B$ leaves $n_i$ and $d_i$ unchanged for $i < D(B)$, but increases $n_{D(B)}$ in $\Gamma^B$. Thus $h(\Gamma^B) < h(\Gamma)$.

Assume $B$ does not separate at a vertex on level $D(B)$. Then $B$ leaves $n_i$ unchanged in $\Gamma^B$ for $i \leq D(B)$ and $d_i$ is unchanged in $\Gamma^B$ for $i < D(B)$. However, $d_{{D(B)}}$ is smaller in $\Gamma^B$, and so $h(\Gamma^B) > h(\Gamma)$. \hfill $\Box$

Define $BU(v)$ to be the poset of blow-ups at the vertex $v$ and $SBU(v)$ the poset of blow-ups at $v$ that separate at $v$. We can describe $SBU(v)$ using the combinatorial framework for graph blow-ups described in [1] as the poset of compatible partitions of the set of incident edges, with the poset ordered by inclusion. Assume that the vertex $v$ has $n$ incident half-edges labelled 1 through $n$. We will consider only partitions of the set $\{1, 2, \ldots, n\}$ into two blocks with the size of each block at least two. We denote a partition by $\{a, \bar{a}\}$, where $a$ is the block containing 1. Recall that distinct partitions $\{a, \bar{a}\}$ and $\{b, \bar{b}\}$ are said to be compatible if one of the following inclusions holds:

$$a \subset b \text{ or } \bar{a} \subset \bar{b}.$$ 

Let $\Sigma(n)$ denote the simplicial complex of partitions, where each vertex is a partition and a $j$ simplex is a collection of $j + 1$ distinct, pairwise-compatible partitions. We will say that a partition $\{a, \bar{a}\}$ splits a set $S$ if $S \not\subseteq a$ and $S \not\subseteq \bar{a}$. Define $\Sigma(n, k)$ to be the subcomplex of $\Sigma(n)$ spanned by vertices $\{a, \bar{a}\}$ where the set $\{1, \ldots, k\}$ is split by $\{a, \bar{a}\}$. We think of 1, $\ldots$, $k$ as the labels for the half-edges descending from a vertex in $\Gamma$. Let $v = \{a, \bar{a}\}$ be a partition, and let the size of $v$ be the cardinality of $\bar{a}$. Define $\Sigma(n, k)_{<m}$ to be the subcomplex of $\Sigma(n, k)$ spanned by $\Sigma(n, k - 1)$ and all vertices in $\Sigma(n, k)$ of size less than $m$.

Note that there is the following filtration on $\Sigma(n)$ in terms of these subcomplexes:

$$
\Sigma(n, 2) = \Sigma(n, 3)_{<2} \subseteq \Sigma(n, 3)_{<3} \subseteq \cdots \subseteq \Sigma(n, 3)_{<n-1} \subseteq \Sigma(n, 3)_{<n} = \Sigma(n, 3) = \Sigma(n, 4)_{<2} \subseteq \Sigma(n, 4)_{<3} \subseteq \cdots \subseteq \Sigma(n, n-1)_{<n-2} \subseteq \Sigma(n, n-1)_{<n-1} \subseteq \Sigma(n, n-1) = \Sigma(n, n) = \Sigma(n).
$$

We denote by $(a_1, a_2, a_3, \ldots, a_{j-1})$ the vertex $v = \{a, \bar{a}\}$ with $a = \{1, a_1, \ldots, a_{j-1}\}$ and $\bar{a} = \{a_j, a_{j+1}, \ldots, a_n\}$, and with $1 < a_2 < a_3 < \ldots < a_{j-1}$ and $a_j < a_{j+1} < \ldots < a_n$. 

Note that there is the following filtration on $\Sigma(n)$ in terms of these subcomplexes:
Lemma 2.2. $\Sigma(n,k)_{\leq m}$ is $n-4$ spherical for $2 \leq k \leq n-1$ and $2 \leq m \leq n$. Note that this generalizes Theorem 2.4 in [4].

Proof. We proceed by building the complex using the filtration.

To start, $\Sigma(n,2)$ is the complex spanned by partitions $\{a,\bar{a}\}$ in which the set $\{1,2\}$ is split, and so each $a$ will be $\{1\} \cup$ a proper subset of $\{3,4,\ldots,n\}$. Thus the complex is the surface of a barycentrically-divided $n-3$ simplex with vertices $3,4,\ldots,n$, and so is homeomorphic to a single $n-4$ sphere.

Assume $\Sigma(n,k)_{<m}$ is $n-4$ spherical. We will adjoin vertices of size $m$ to $\Sigma(n,k)_{<m}$ and show that the resulting complex $\Sigma(n,k)_{<m+1}$, the next complex in the filtration, is $n-4$ spherical.

Let $v = (1, a_2, \ldots, a_{j-1} \mid k, a_{j+1}, \ldots, a_n)$ and $w = (1, b_2, \ldots, b_{j-1} \mid k, b_{j+1}, \ldots, b_n)$ be distinct vertices of size $m$ to be adjoined to $\Sigma(n,k)_{<m}$, so there is some $i$ with $j < i \leq n$ and $a_i \neq b_i$. Since $v$ and $w$ are not in $\Sigma(n,k-1)$, we have that $\{1,2,\ldots,k-1\} \subset \{1, a_2, \ldots, a_{j-1}\}$ and $\{1,2,\ldots,k-1\} \subset \{1, b_2, \ldots, b_{j-1}\}$. Note that $v$ and $w$ are not compatible, and so $v$ and $w$ are not connected by an edge in $\Sigma(n)$. Thus we may adjoin $v$ and $w$ to $\Sigma(n,k)<m$ in any order.

Note that the relative link of $v = (1, a_2, \ldots, a_{j-1} \mid k, a_{j+1}, \ldots, a_n)$ in $\Sigma(n,k)_{<m}$ is spanned by the following two types of vertices: vertices where subsets of $\{1, a_2, \ldots, a_{j-1}\}$ with size at most $n-m-2$ splitting the block $\{1, \ldots, k-1\}$ are moved to the right of the $\mid$, and vertices where subsets of $\{a_{j+1}, \ldots, a_n\}$ with size at most $m-2$ are moved to the left of the $\mid$. Call these types left-to-right vertices and right-to-left vertices respectively.

Let $\{c,\bar{c}\}$ be a left-to-right vertex and $\{d,\bar{d}\}$ be a right-to-left vertex in the relative link of $v$ in $\Sigma(n,k)_{<m}$. Note that $c$ and $d$ are compatible since $\bar{d} \subset \bar{c}$, so that $\{c,\bar{c}\}$ and $\{d,\bar{d}\}$ are connected by an edge in the relative link of $v$ in $\Sigma(n,k)<m$. Thus the relative link of $v$ in $\Sigma(n,k)_{<m}$ is the join of the subcomplex spanned by the left-to-right vertices and the subcomplex spanned by the right-to-left vertices.

The subcomplex spanned by the right-to-left vertices is the complex of non-empty proper subsets of an $m-1$ element set, which is the surface of a barycentrically-divided $m-2$ simplex and thus homeomorphic to an $m-3$ sphere.

In the subcomplex spanned by left-to-right vertices, a vertex is created from by moving subsets that split $\{1,\ldots,k\}$ from the left of the $\mid$ to the right. Note that as a result, the right hand side in any left-to-right vertex will always contain the size $m$ set $\{k+1, a_{j+1}, \ldots, a_n\}$, and so we can imagine collapsing this set to a single object. Let $k'$ be the object representing the collapsed set and assign this object the numerical value of $k$ so that it can be compared to elements of the set $\{1, a_2, a_3, \ldots, a_{j-1}\}$. Then the subcomplex of left-to-right vertices of $v$ is homotopy equivalent to the complex of partitions of the size $n-(m-1)$ set $\{1, a_2, a_3, \ldots, a_{j-1}, k'\}$ such that the block $\{1, \ldots, k-1\}$ is split, which is $\Sigma(n-(m-1),k-1)$, and so is $(n-m+1-4)$-spherical by an outer induction on $n$. 
When we adjoin \( v \) along its \((n - 5)\)-spherical relative link to the \((n - 4)\)-spherical \( \Sigma(n,k) \), the resulting complex is \((n - 4)\)-spherical by combinatorial Morse theory. Since any two arbitrary size \( m \) vertices are not compatible, we may adjoin all vertices at once to get that \( \Sigma(n,k) < m + 1 \) is \((n - 4)\)-spherical. Thus by induction \( \Sigma(n) \) is \((n - 4)\)-spherical.

**Lemma 2.3.** If there are at least two edges descending from \( v \), then \( SBU(v) \simeq \bigvee S^{\text{degree}(v) - 4} \).

**Proof.** Let \( d \) represent the number of descending half-edges incident to \( v \). Assign to each of the \( \text{degree}(v) \) incident half-edges at \( v \) number from the set \( \{1, ..., \text{degree}(v)\} \) beginning with the \( d \) descending half-edges. Then \( SBU(v) \) is isomorphic to \( \Sigma(\text{degree}(v), d) \) and is thus homotopy equivalent to \( \bigvee S^{\text{degree}(v) - 4} \) by the previous lemma.

**Corollary 2.4.** Let \( A := \ast_{v \in \Gamma - \{p\}} SBU(v) \). Suppose that \( \Gamma \) has no unique descending edges, let \( k = d_0(\Gamma) \), and let \( V \) denote the number of vertices in \( \Gamma \). Then \( A \simeq \bigvee S^{k - V} \).

**Proof.** Recall that \( d_0(\Gamma) := \sum_{v \in \Gamma - \{p\}} \text{degree}(v) - 2 \). This gives that \( A \simeq \ast_{v \in \Gamma - \{p\}} \bigvee S^{(\text{degree}(v) - 2) - 2} = \bigvee S^{(k+V-2) - 2(V-1)} = \bigvee S^{k-V} \) since \( SBU(v) \) is a wedge of spheres of dimension \( \text{degree}(v) - 4 \) for all vertices in \( \Gamma \).

If \( \Gamma \) has a unique descending edge from a vertex \( v \) of degree greater than three, then \( SBU(v) \) will be empty rather than a wedge of spheres of dimension \( \text{degree}(v) - 4 \), leaving \( A \) homotopy equivalent to spheres of dimension strictly less than \( k - V \).

**Lemma 2.5.** Suppose that \( \Gamma \) has no unique descending edges, let \( k = d_0(\Gamma) \), and let \( V \) denote the number of vertices in \( \Gamma \). Then the up-link in \( L_n \) is homotopy equivalent to \( \bigvee S^{k-V} \).

**Proof.** For a poset \( Y \) define \( Y \) to be \( Y \sqcup \{\perp\} \) with \( \perp \) a formal minimum element. Then \( Y \ast Z \simeq Y \times Z^{\perp} \).

Define \( X := \{ f \in \prod_v BU(v) \mid \exists v \in \Gamma - \{p\} \text{ with } f_v \in SBU(v) \} \) for \( f_v \) the blow-up at vertex \( v \) in the tuple. In order to turn \( X \) into the correct poset we settle on a fixed but arbitrary order of factors in the product. Define a map \( r : X \to A \) by

\[
(f_v)_{v \in \Gamma - \{p\}} \mapsto \left( \begin{array}{l}
\{ f_v \text{ for } f_v \in SBU(v) \\
\perp \text{ for } f_v \not\in SBU(v)
\end{array} \right)_{v \in \Gamma - \{p\}}
\]

Note that \( r \) is clearly a poset map and the identity map on elements of \( A \), so \( r \) is a poset retraction onto \( A \). Since \( r(f) \leq f \) for all \( f \in X \), paragraph 1.3 in [3] implies that \( r \) induces a homotopy equivalence between their geometric realizations.
Define $L := \{ f \in \prod \mathbb{B}U(v) \mid \exists v \in \Lambda_D(f) \text{ with } f_v \in S\mathbb{B}U(v) \} \subseteq X$. Note that when $r$ is restricted to $L$ that $r$ is still a poset retraction onto $A$ with $r(f) \leq f$ and so still induces a homotopy equivalence between the geometric realizations of $L$ and $A$. Since the geometric realization of $L$ is the up-link of $\Gamma$, the up-link is homotopy equivalent to $A$, which is itself homotopy equivalent to $\sqcup S^{k-V}$ by Corollary 2.4 when $\Gamma$ has no unique descending edges. If $\Gamma$ has a unique descending edge at a vertex, then the up-link is homotopy equivalent to spheres of dimension strictly less than $k-V$. □

REFERENCES

[1] M. Culler and K. Vogtmann. Moduli of graphs and automorphisms of free groups. Invent. Math., 84:91–119, 1986.
[2] A. Hatcher and K. Vogtmann. Cerf theory for graphs. Journal of the London Mathematical Society, pages 633–655, 1998.
[3] D. Quillen. Homotopy properties of the poset of non-trivial $p$-subgroups of a group. Advances in Math., 28:101–128, 1978.
[4] K. Vogtmann. Local structure of some $\text{Out}(F_n)$-complexes. Proc. Edinburgh Math Society, 33:367–369, 1990.