Yamabe Solitons on \((LCS)_n\)-manifolds

By
Soumendu Roy\(^1\), Santu Dey\(^2\) and Arindam Bhattacharyya\(^3\)

Department of Mathematics
Jadavpur University,
Kolkata-700032, India.
E-mail\(^1\): soumendu1103mtma@gmail.com
E-mail\(^2\): santu.mathju@gmail.com
E-mail\(^3\): battachar1968@yahoo.co.in

Abstract
The object of the present paper is to study some properties of \((LCS)_n\)-manifolds whose metric is Yamabe soliton. We establish some characterization of \((LCS)_n\)-manifolds when the soliton becomes steady. Next we have studied some certain curvature conditions of \((LCS)_n\)-manifolds admitting Yamabe solitons. Lastly we construct a 3-dimensional \((LCS)_n\)-manifold satisfying the results.

Key words: Yamabe soliton, Einstein manifold, \((LCS)_n\)-manifold.

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1. Introduction
In 2003, the notion of Lorentzian concircular structure manifolds (briefly, \((LCS)_n\)-manifolds) was introduced by Shaikh \([12]\) with an example, that generalize the notion of LP-Sasakian manifolds which was introduced by Matsumoto \([7]\) and also by Mihai and Rosca \([8]\). In 2005 and 2006, the application of \((LCS)_n\)-manifolds to the general theory of relativity and cosmology was studied by Shaikh and Baishya \([13]\), \([14]\).

Many other authors M. Ateceken\([1]\), S. K. Hui\([5]\), D. Narain\([9]\), S. Yadav(\([18]\), \([19]\), \([20]\), \([21]\)), A. Shaikh(\([15]\), \([16]\)) also studied the \((LCS)_n\)-manifolds.

The concept of Yamabe flow was first introduced by Hamilton \([4]\) to construct Yamabe metrics on compact Riemannian manifolds. On a Riemannian or pseudo-Riemannian manifold \(M\), a time-dependent metric \(g(\cdot, t)\) is said to evolve by the Yamabe flow if the metric \(g\) satisfies the given equation,

\[
\frac{\partial}{\partial t} g(t) = -rg(t), \quad g(0) = g_0,
\]  
(1.1)
where $r$ is the scalar curvature of the manifold $M$.

In 2-dimension the Yamabe flow is equivalent to the Ricci flow (defined by $\frac{\partial}{\partial t}g(t) = -2S(g(t))$, where $S$ denotes the Ricci tensor). But in dimension $> 2$ the Yamabe and Ricci flows do not agree, since the Yamabe flow preserves the conformal class of the metric but the Ricci flow does not in general.

A Yamabe soliton correspond to self-similar solution of the Yamabe flow, is defined on a Riemannian or pseudo-Riemannian manifold $(M, g)$ by a vector field $\xi$ satisfying the equation
\[ \frac{1}{2}L_\xi g = (r - \lambda)g, \quad (1.2) \]
where $L_\xi g$ denotes the Lie derivative of the metric $g$ along the vector field $\xi$, $r$ is the scalar curvature and $\lambda$ is a constant. Moreover a Yamabe soliton is said to be expanding if $\lambda < 0$, steady if $\lambda = 0$ and shrinking if $\lambda > 0$.

Yamabe solitons on a three-dimensional Sasakian manifold were studied by R. Sharma [17].

Again,
\[ R(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X, Y]}Z, \quad (1.3) \]
\[ H(X, Y)Z = R(X, Y)Z - \frac{1}{(n-2)}[g(Y, Z)QX - g(X, Z)QY + S(Y, Z)X - S(X, Z)Y], \quad (1.4) \]
\[ P(X, Y)Z = R(X, Y)Z - \frac{1}{(n-1)}[g(QY, Z)X - g(QX, Z)Y], \quad (1.5) \]
\[ \tilde{C}(X, Y)Z = R(X, Y)Z - \frac{r}{n(n-1)}[g(Y, Z)X - g(X, Z)Y], \quad (1.6) \]
\[ W_2(X, Y)Z = R(X, Y)Z + \frac{1}{n-1}[g(X, Z)QY - g(Y, Z)QX], \quad (1.7) \]
are the Riemannian-Christoffel curvature tensor $R$ [10], the conharmonic curvature tensor $H$ [6], the projective curvature tensor $P$ [22], the concircular curvature tensor $\tilde{C}$ [11] and the $W_2$-curvature tensor [11] respectively in a Riemannian manifold $(M^n, g)$, where $Q$ is the Ricci operator, defined by $S(X, Y) = g(QX, Y)$, $S$ is the Ricci tensor, $r = tr(S)$ is the scalar curvature, where $tr(S)$ is the trace of $S$ and $X, Y, Z \in \chi(M)$, $\chi(M)$ being the Lie algebra of vector fields of $M$.

In the present paper we study Yamabe soliton on $(LCS)_n$-manifolds. The paper is organized as follows:
After introduction, section 2 is devoted for preliminaries on $(LCS)_n$-manifolds. In section 3, we have studied Yamabe soliton on $(LCS)_n$-manifolds. Here we examined when $(LCS)_n$-manifold admits Yamabe soliton, then the manifold becomes $K - (LCS)_n$-manifold and Ricci symmetric. In this section we have also
shown that, \((LCS)_n\)-manifold admits Yamabe soliton is \(\xi\)-projectively flat, \(\xi\)-concircularly flat and \(\xi\)-conharmonically flat iff the soliton becomes steady. Section 4 deals with curvature properties on \((LCS)_n\)-manifold admitting Yamabe soliton. Here we obtained some results on Yamabe soliton satisfying the conditions of the following type:

\[ S(\xi, X) \cdot R = 0, S(\xi, X) \cdot W_2 = 0, \]

where \(W_2\) is the \(W_2\)-curvature tensor and \(S\) is the Ricci tensor. Also we have found that if the manifold admits Yamabe soliton then \(R(\xi, X) \cdot S = 0\) and \(W_2(\xi, X) \cdot S = 0\).

In the last section, we gave an example of 3-dimensional \((LCS)_n\)-manifold on which we can easily verify our results.

2. Preliminaries

Let \((M, g)\) be an \(n\)-dimensional Lorentzian manifold admitting a unit timelike concircular vector field \(\xi\). Then the vector field satisfying \(g(\xi, \xi) = -1\). Since \(\xi\) is a unit concircular vector field, it follows that there exists a nonzero 1-form \(\eta\) such that, \(g(X, \xi) = \eta(X)\). Also \(\xi\) satisfies \(\nabla \xi = \alpha(I + \eta \otimes \xi)\) with a nowhere zero smooth function \(\alpha\) on \(M\) verifying the equation \(\nabla_X \alpha = (X \alpha) = d\alpha(X) = \rho \eta(X)\), for \(\rho \in \mathcal{C}^\infty(M)\), where \(\nabla\) is the Levi-Civita connection of \(g\) and \(X\) is a vector field. Here also \(\phi\) is the \((1,1)\) tensor field, denoted by, \(\phi := \frac{1}{\alpha} \nabla \xi\).

The notion of Lorentzian para-Sasakian manifold was introduced by K. Matsumoto [7]. More general the Lorentzian manifold \(M\) together with the unit timelike concircular vector field \(\xi\), an 1-form \(\eta\), and an \((1,1)\) tensor field \(\phi\) is said to be a Lorentzian concircular structure manifold \((M, g, \xi, \eta, \phi, \alpha)\) (briefly, \((LCS)_n\)-manifold), which was introduced by A. A. Shaikh [12].

In an \(n\)-dimensional \((LCS)_n\)-manifold the following relations hold:

\[
\phi^2 = I + \eta \otimes \xi, \quad \eta(\xi) = -1, \quad \phi \xi = 0, \quad \eta \circ \phi = 0, \tag{2.1}
\]

\[
g(\phi X, \phi Y) = g(X, Y) + \eta(X)\eta(Y) \quad \text{and} \quad g(\phi X, Y) = g(X, \phi Y), \tag{2.2}
\]

\[
(\nabla_X \phi) Y = \alpha [g(X, Y)\xi + 2\eta(X)\eta(Y)\xi + \eta(Y)X], \tag{2.3}
\]

for any \(X, Y \in \chi(M)\).

\[
\phi X = \frac{1}{\alpha} \nabla_X \xi, \tag{2.4}
\]

\[
\eta(\nabla_X \xi) = 0, \quad \nabla_X \xi = 0, \tag{2.5}
\]

\[
R(X, Y)Z = (\alpha^2 - \rho)[g(Y, Z)X - g(X, Z)Y], \tag{2.6}
\]

\[
R(X, Y)\xi = (\alpha^2 - \rho)[\eta(Y)X - \eta(X)Y], \tag{2.7}
\]

\[
R(\xi, X)Y = (\alpha^2 - \rho)[g(X, Y)\xi - \eta(Y)X], \tag{2.8}
\]

\[
\eta(R(X, Y)Z) = (\alpha^2 - \rho)[\eta(X)g(Y, Z) - \eta(Y)g(X, Z)], \tag{2.9}
\]

for any \(X, Y, Z \in \chi(M)\).

\[
\eta(R(X, Y)\xi) = 0, \tag{2.10}
\]

\[
S(X, Y) = (\alpha^2 - \rho)(n - 1)g(X, Y), \tag{2.11}
\]

\[
r = n(n - 1)(\alpha^2 - \rho), \tag{2.12}
\]

\[
\nabla \eta = \alpha (g + \eta \otimes \eta), \quad \nabla \xi \eta = 0, \tag{2.13}
\]

\[
\mathcal{L}_\xi \phi = 0, \quad \mathcal{L}_\xi \eta = 0, \quad \mathcal{L}_\xi g = 2\nabla \eta = 2\alpha(g + \eta \otimes \eta), \tag{2.14}
\]
where $R$ is the Riemannian Curvature tensor, $S$ is the Ricci tensor, $r$ is the scalar curvature, $\nabla$ is the Levi-Civita connection associated with $g$ and $\mathcal{L}_\xi$ denotes the Lie derivative along the vector field $\xi$.

3. Yamabe soliton on $(LCS)_n$ manifold

Let $(M, g, \xi, \eta, \phi, \alpha)$ be an $n$-dimensional $(LCS)_n$ manifold. Consider the Yamabe soliton on $M$ as:
\[
\frac{1}{2} \mathcal{L}_\xi g = (r - \lambda) g.
\] (3.1)
Then from (2.14), we get,
\[
\alpha [g(X,Y) + \eta(X)\eta(Y)] = (r - \lambda) g(X,Y),
\] (3.2)
for all vector fields $X, Y$ on $M$.
which implies that,
\[
(r - \lambda - \alpha) g(X,Y) - \alpha \eta(X)\eta(Y) = 0.
\] (3.3)
Taking $Y = \xi$ in the above equation and using (2.1), we get,
\[
(r - \lambda - \alpha) \eta(X) + \alpha \eta(X) = 0.
\] (3.4)
Then we have,
\[
(r - \lambda) \eta(X) = 0.
\] (3.5)
Since $\eta(X) \neq 0$, so we get,
\[
r = \lambda.
\] (3.6)
Using the above equation in (3.1), we have,
\[
\mathcal{L}_\xi g = 0.
\] (3.7)
Thus $\xi$ is a Killing vector field and consequently $M$ is a $K-(LCS)_n$ manifold. Moreover, since $\lambda$ is constant, the scalar curvature $r$ is constant.

So we can state the following theorem:
Theorem 3.1. If an $(LCS)_n$ manifold $(M, g, \xi, \eta, \phi, \alpha)$ admits a Yamabe soliton $(g, \xi)$, $\xi$ being the Reeb vector field of the Lorentzian concircular structure, then the scalar curvature is constant and the manifold is a $K-(LCS)_n$ manifold.

Now from (2.12) and (3.6), we get,
\[
\lambda = n(n - 1)(\alpha^2 - \rho).
\] (3.8)
Then using (2.11) and (3.8), we obtain,
\[
S(X,Y) = \frac{\lambda}{n} g(X,Y),
\] (3.9)
for all vector fields $X, Y$ on $M$.

This leads to the following:
Proposition 3.2. If an $(LCS)_n$ manifold $(M, g, \xi, \eta, \phi, \alpha)$ admits a Yamabe soliton $(g, \xi)$, $\xi$ being the Reeb vector field of the Lorentzian concircular structure, then the manifold becomes Einstein manifold.
Now replacing the expression of $S$ from (3.9) in
\[(\nabla_X S)(Y, Z) = X(S(Y, Z)) - S(\nabla_X Y, Z) - S(Y, \nabla_X Z)\]
we get,
\[(\nabla_X S)(Y, Z) = \frac{\lambda}{n}(\nabla_X g)(Y, Z),\] (3.10)
which implies that,
\[\nabla S = 0.\] (3.11)

This leads to the following:

**Proposition 3.3.** If an $(LCS)_n$ manifold $(M, g, \xi, \eta, \phi, \alpha)$ admits a Yamabe soliton $(g, \xi)$, $\xi$ being the Reeb vector field of the Lorentzian concircular structure, then the manifold becomes Ricci symmetric.

Again let the Ricci tensor $S$ of the $(LCS)_n$ manifold is $\eta$-recurrent i.e
\[\nabla S = \eta \otimes S,\]
which implies that,
\[(\nabla_X S)(Y, Z) = \eta(X)S(Y, Z),\] (3.12)
for all vector fields $X, Y, Z$ on $M$.

Then using (3.11) and (3.9), we get
\[\frac{\lambda}{n}\eta(x)g(Y, Z) = 0.\] (3.13)
Taking $Y = \xi, Z = \xi$ in the above equation we obtain,
\[\lambda\eta(X) = 0.\] (3.14)
As $\eta(X) \neq 0$, hence $\lambda = 0$. Also from (3.6), we get $r = 0$.

This leads to the following:

**Proposition 3.4.** Let $(M, g, \xi, \eta, \phi, \alpha)$ be an $(LCS)_n$ manifold, admitting a Yamabe soliton $(g, \xi)$, $\xi$ being the Reeb vector field of the Lorentzian concircular structure. If the Ricci tensor $S$ of the manifold is $\eta$-recurrent (i.e $\nabla S = \eta \otimes S$), then the soliton is steady and the manifold becomes flat.

Let us assume that a symmetric $(0, 2)$ tensor field $h = \mathcal{L}_\xi g - 2rg$ is parallel with respect to the Levi-Civita connection associated to $g$.

Then
\[h(\xi, \xi) = \mathcal{L}_\xi g(\xi, \xi) - 2rg(\xi, \xi) = 2\lambda,\] (3.15)
implies that,
\[\lambda = \frac{1}{2}h(\xi, \xi).\] (3.16)

Now as $h$ is parallel with respect to $g$, then from [3], we get,
\[h(X, Y) = -h(\xi, \xi)g(X, Y)\] (3.17)
for all vector fields $X, Y$ on $M$.
which leads to,

$$\mathcal{L}_\xi g(X, Y) = 2(r - \lambda)g(X, Y).$$  \hfill (3.18)

So we can state the following theorem:

**Theorem 3.5.** Let $(M, g, \xi, \eta, \phi, \alpha)$ be an $(LCS)_n$ manifold. Assume that a symmetric $(0, 2)$ tensor field $h = \mathcal{L}_\xi g - 2rg$ is parallel with respect to the Levi-Civita connection associated to $g$. Then $(g, \xi)$ yields an Yamabe soliton on $M$.

We know,

$$(\nabla_\xi Q)X = \nabla_\xi QX - Q(\nabla_\xi X),$$  \hfill (3.19)

and

$$(\nabla_\xi S)(X, Y) = \xi S(X, Y) - S(\nabla_\xi X, Y) - S(X, \nabla_\xi Y),$$  \hfill (3.20)

for any vector fields $X, Y$ on $M$.

Now using (3.9) we obtain,

$$QX = \frac{\lambda}{n}X,$$  \hfill (3.21)

for any vector field $X$ on $M$.

Then in view of (3.9) and (3.21), the equations (3.19) and (3.20) become

$$(\nabla_\xi Q)X = 0,$$  \hfill (3.22)

$$(\nabla_\xi S)(X, Y) = 0,$$  \hfill (3.23)

respectively, for any vector fields $X, Y$ on $M$.

This leads to the following:

**Theorem 3.6.** Let $(M, g, \xi, \eta, \phi, \alpha)$ be an $(LCS)_n$ manifold, admitting a Yamabe soliton $(g, \xi)$, $\xi$ being the Reeb vector field of the Lorentzian concircular structure. Then $Q$ and $S$ are parallel along $\xi$, where $Q$ is the Ricci operator, defined by $S(X, Y) = g(QX, Y)$ and $S$ is the Ricci tensor of $M$.

Also in view of (3.21), we obtain,

$$(\nabla_X Q)Y = \nabla_X QY - Q(\nabla_X Y) = 0,$$  \hfill (3.24)

for any vector fields $X, Y$ on $M$.

Then we have,

**Corollary 3.7.** Let $(M, g, \xi, \eta, \phi, \alpha)$ be an $(LCS)_n$ manifold, admitting a Yamabe soliton $(g, \xi)$, $\xi$ being the Reeb vector field of the Lorentzian concircular structure. Then $Q$ is parallel to any arbitrary vector field on $M$.

Let a Yamabe soliton is defined on an $n$-dimensional $(LCS)_n$ manifold $M$ as,

$$\frac{1}{2}L_V g = (r - \lambda)g$$  \hfill (3.25)

where $L_V g$ denotes the Lie derivative of the metric $g$ along a vector field $V$ and $r, \lambda$ are defined as (1.2).
Let $V$ be pointwise co-linear with $\xi$ i.e. $V = b\xi$ where $b$ is a function on $M$.

Then the equation (3.25) becomes,

$$\mathcal{L}_{b\xi} g(X, Y) = 2(r - \lambda)g(X, Y), \quad (3.26)$$

for any vector fields $X, Y$ on $M$.

Applying the property of Lie derivative and Levi-Civita connection we have,

$$bg(\nabla_X \xi, Y) + (Xb)\eta(Y) + bg(\nabla_Y \xi, X) + (Yb)\eta(X) = 2(r - \lambda)g(X, Y). \quad (3.27)$$

Using (2.4), the above equation reduces to,

$$b\alpha g(\phi X, Y) + (Xb)\eta(Y) + b\alpha g(\phi Y, X) + (Yb)\eta(X) = 2(r - \lambda)g(X, Y). \quad (3.28)$$

Taking $Y = \xi$ in the above equation and using (2.1), we get,

$$-Xb + (\xi b)\eta(X) = 2(r - \lambda)\eta(X). \quad (3.29)$$

Again putting $X = \xi$ in the above equation, we obtain,

$$\xi b = r - \lambda. \quad (3.30)$$

Then using (3.30), (3.29) becomes,

$$Xb = -(r - \lambda)\eta(X). \quad (3.31)$$

Applying exterior differentiation in (3.31), we have,

$$(r - \lambda)d\eta = 0. \quad (3.32)$$

Now in an $n$-dimensional $(LCS)_n$ manifold we have,

$$(d\eta)(X, Y) = X(\eta(Y)) - Y(\eta(X)) - \eta([X, Y]),$$

which implies

\[
\begin{align*}
(d\eta)(X, Y) & = g(Y, \nabla_X \xi) - g(X, \nabla_Y \xi) \\
& = \alpha g(Y, X) + \eta(X)\eta(Y) - \alpha g(X, Y) + \eta(X)\eta(Y) \\
& = 0. \quad (3.33)
\end{align*}
\]

Hence the 1-form $\eta$ is closed.

Then using the above equation, (3.32) implies that, either $r \neq \lambda$ or $r = \lambda$.

Now if $r \neq \lambda$ then from (3.25), we have,

$$\mathcal{L}_V g = 2(r - \lambda)g, \quad (3.34)$$

which implies $V$ is a conformal Killing vector field.

Again if $r = \lambda$ then from (3.31), we get,

$$Xb = 0, \quad (3.35)$$

implies that $b$ is constant.

So we can state the following theorem:

**Theorem 3.8.** Let $(M, g, \xi, \eta, \phi, \alpha)$ be an $(LCS)_n$ manifold, admitting a Yamabe soliton $(g, V)$, $V$ being a vector field on $M$. If $V$ is pointwise co-linear with $\xi$ then either $V$ is a conformal Killing vector field, provided $r \neq \lambda$, or $V$ is a constant multiple of $\xi$, where $\xi$ being the Reeb vector field of the Lorentzian concircular structure, $r$ is the scalar curvature and $\lambda$ is a constant.
Also if \( r = \lambda \) then from (3.25), we obtain,

\[
\mathcal{L}_V g = 0, \tag{3.36}
\]

implies that \( V \) is a Killing vector field.

Then we have,

**Corollary 3.9.** Let \((M, g, \xi, \eta, \phi, \alpha)\) be an \((LCS)_n\) manifold, admitting a Yamabe soliton \((g, V)\), \(V\) being a vector field on \(M\). If \(V\) is pointwise co-linear with \(\xi\) and \(r = \lambda\) then \(V\) becomes a Killing vector field, where \(\xi\) being the Reeb vector field of the Lorentzian concircular structure, \(r\) is the scalar curvature and \(\lambda\) is a constant.

From the definition of projective curvature tensor (1.5), defined on an \(n\)-dimensional \((LCS)_n\) manifold, we have,

\[
P(X, Y)Z = R(X, Y)Z - \frac{1}{(n - 1)}[S(Y, Z)X - S(X, Z)Y], \tag{3.37}
\]

for any vector fields \(X, Y, Z\) on \(M\).

Putting \(Z = \xi\), we get,

\[
P(X, Y)\xi = R(X, Y)\xi - \frac{1}{(n - 1)}[S(Y, \xi)X - S(X, \xi)Y]. \tag{3.38}
\]

Using (2.7) and (3.9), we obtain,

\[
P(X, Y)\xi = [\left(\alpha^2 - \rho\right) - \frac{\lambda}{n(n - 1)}][\eta(Y)X - \eta(X)Y]. \tag{3.39}
\]

Again using (3.8), we get,

\[
P(X, Y)\xi = 0. \tag{3.40}
\]

This leads to the following:

**Proposition 3.10.** An \((LCS)_n\) manifold \((M, g, \xi, \eta, \phi, \alpha)\), admitting a Yamabe soliton \((g, \xi)\), \(\xi\) being the Reeb vector field of the Lorentzian concircular structure, is \(\xi\)-projectively flat.

From the definition of concircular curvature tensor (1.6), defined on an \(n\)-dimensional \((LCS)_n\) manifold, we have,

\[
\tilde{C}(X, Y)Z = R(X, Y)Z - \frac{r}{n(n - 1)}[g(Y, Z)X - g(X, Z)Y], \tag{3.41}
\]

for any vector fields \(X, Y, Z\) on \(M\).

Putting \(Z = \xi\), we get,

\[
\tilde{C}(X, Y)\xi = R(X, Y)\xi - \frac{r}{n(n - 1)}[g(Y, \xi)X - g(X, \xi)Y]. \tag{3.42}
\]

Using (2.7) and (3.6), we obtain,

\[
\tilde{C}(X, Y)\xi = [\left(\alpha^2 - \rho\right) - \frac{\lambda}{n(n - 1)}][\eta(Y)X - \eta(X)Y]. \tag{3.43}
\]
Again using (3.8), we get,
\[ \tilde{C}(X, Y)\xi = 0. \] (3.44)

This leads to the following:

**Proposition 3.11.** An \((LCS)_n\) manifold \((M, g, \xi, \eta, \phi, \alpha)\), admitting a Yamabe soliton \((g, \xi)\), \(\xi\) being the Reeb vector field of the Lorentzian concircular structure, is \(\xi\)-concircularly flat.

From the definition of conharmonic curvature tensor (1.4), defined on an \(n\)-dimensional \((LCS)_n\) manifold, we have,
\[
H(X, Y)Z = R(X, Y)Z - \frac{1}{(n-2)}[g(Y, Z)QX - g(X, Z)QY
+ S(Y, Z)X - S(X, Z)Y],
\] (3.45)
for any vector fields \(X, Y, Z\) on \(M\).

Putting \(Z = \xi\), we get,
\[
H(X, Y)\xi = R(X, Y)\xi - \frac{1}{(n-2)}[g(Y, \xi)QX - g(X, \xi)QY
+ S(Y, \xi)X - S(X, \xi)Y],
\] (3.46)
Using (2.7), (3.9) and (3.21), we obtain,
\[
H(X, Y)\xi = [(\alpha^2 - \rho) - \frac{2\lambda}{n(n-2)}][\eta(Y)X - \eta(X)Y].
\] (3.47)

Again using (3.8), we get,
\[
H(X, Y)\xi = -\frac{\lambda}{(n-1)(n-2)}[\eta(Y)X - \eta(X)Y].
\] (3.48)

This implies that \(H(X, Y)\xi = 0\) iff \(\lambda = 0\).

This leads to the following:

**Proposition 3.12.** An \((LCS)_n\) manifold \((M, g, \xi, \eta, \phi, \alpha)\), admitting a Yamabe soliton \((g, \xi)\), \(\xi\) being the Reeb vector field of the Lorentzian concircular structure, is \(\xi\)-conharmonically flat iff the soliton is steady.

4. **Curvature properties on \((LCS)_n\) manifold admitting Yamabe soliton**

We know,
\[
R(\xi, X) \cdot S = S(R(\xi, X)Y, Z) + S(Y, R(\xi, X)Z),
\] (4.1)
for any vector fields \(X, Y, Z\) on \(M\).

Using (2.8), we obtain,
\[
R(\xi, X) \cdot S = S((\alpha^2 - \rho)(g(X, Y)\xi - \eta(Y)X), Z) + S(Y, (\alpha^2 - \rho)(g(X, Z)\xi - \eta(Z)X)).
\] (4.2)
Then using (3.9), we get,
\[ R(\xi, X) \cdot S = \frac{\lambda}{n}(\alpha^2 - \rho)[g(\xi, X)\eta(Z) - g(X, Z)\eta(Y) + g(X, Z)\eta(Y) - g(X, Y)\eta(Z)], \quad (4.3) \]
which implies that,
\[ R(\xi, X) \cdot S = 0. \]
So we can state the following theorem:
**Theorem 4.1.** If an \((LCS)\) manifold \((M, g, \xi, \eta, \phi, \alpha)\) admits a Yamabe soliton \((g, \xi)\), \(\xi\) being the Reeb vector field of the Lorentzian concircular structure, then \(R(\xi, X) \cdot S = 0\), i.e. the manifold is \(\xi\)-semi symmetric, where \(R\) is the Riemannian curvature tensor and \(S\) is the Ricci tensor.

Again the condition \(S(\xi, X) \cdot R = 0\) implies that,
\[ S(X, R(Y, Z)W)\xi - S(\xi, R(Y, Z)W)X + S(X, Y)R(\xi, Z)W - S(\xi, Y)R(X, Z)W + S(X, Z)R(Y, \xi)W - S(\xi, Z)R(Y, X)W + S(X, W)R(Y, Z)\xi - S(\xi, W)R(Y, Z)X = 0, \quad (4.4) \]
for any vector fields \(X, Y, Z, W\) on \(M\).

Taking the inner product with \(\xi\), the above equation becomes,
\[ -S(X, R(Y, Z)W) - S(\xi, R(Y, Z)W)\eta(X) + S(X, Y)\eta(R(\xi, Z)W) - S(\xi, Y)\eta(R(X, Z)W) - S(\xi, Z)\eta(R(Y, X)W) + S(X, W)\eta(R(Y, Z)\xi) - S(\xi, W)\eta(R(Y, Z)X) = 0. \quad (4.5) \]
Replacing the expression of \(S\) from (3.9) and taking \(Z = \xi, W = \xi\), we get,
\[ \frac{\lambda}{n}[\eta(Y)\eta(R(X, \xi)\xi) - \eta(R(Y, \xi)\xi)\eta(X) + g(X, Y)\eta(R(\xi, \xi)\xi)] = 0. \quad (4.6) \]
Now using (2.7), (2.9), (2.10), we get on simplification,
\[ \frac{\lambda}{n}(\alpha^2 - \rho)[g(X, Y) + \eta(x)\eta(Y)] = 0. \quad (4.7) \]
Using (2.2), the above equation becomes,
\[ \frac{\lambda}{n}(\alpha^2 - \rho)g(\phi X, \phi Y) = 0, \quad (4.8) \]
for any vector fields \(X, Y\) on \(M\).
This implies that,
\[ \frac{\lambda}{n}(\alpha^2 - \rho) = 0. \quad (4.9) \]
Then using (3.8), we get,
\[ \frac{\lambda^2}{n^2(n - 1)} = 0, \]
implies that $\lambda = 0$.

Hence using (3.6), $r = 0$.

So we can state the following theorem:

**Theorem 4.2.** If an $(LCS)_n$ manifold $(M, g, \xi, \eta, \phi, \alpha)$ admits a Yamabe soliton $(g, \xi)$, $\xi$ being the Reeb vector field of the Lorentzian concircular structure and satisfies $S(\xi, X) \cdot R = 0$ then the manifold becomes flat and the soliton is steady, where $R$ is the Riemannian curvature tensor and $S$ is the Ricci tensor.

We know,

$$W_2(\xi, X) \cdot S = S(W_2(\xi, X)Y, Z) + S(Y, W_2(\xi, X)Z), \quad (4.10)$$

for any vector fields $X, Y, Z$ on $M$.

Replacing the expression of $S$ from (3.9) and using the definition of $W_2$-curvature tensor from (1.7), we get,

$$W_2(\xi, X) \cdot S = \frac{\lambda}{n}g(R(\xi, X)Y + \frac{1}{n-1}[g(\xi, Y)QX - g(X, Y)Q\xi], Z)$$

$$+ \frac{\lambda}{n}g(Y, R(\xi, X)Z + \frac{1}{n-1}[g(\xi, Z)QX - g(X, Z)Q\xi]).$$

$$(4.11)$$

Now using (2.8) and the property $g(QX, Y) = S(X, Y)$, we obtain on simplification,

$$W_2(\xi, X) \cdot S = \frac{\lambda}{n(n-1)}[\eta(Y)S(X, Z) - S(\xi, Z)g(X, Y)$$

$$+ \eta(Z)S(X, Y) - S(\xi, Y)g(X, Z)].$$

$$(4.12)$$

Then using (3.9), the above equation becomes,

$$W_2(\xi, X) \cdot S = 0.$$

$$(4.13)$$

So we can state the following theorem:

**Theorem 4.3.** If an $(LCS)_n$ manifold $(M, g, \xi, \eta, \phi, \alpha)$ admits a Yamabe soliton $(g, \xi)$, $\xi$ being the Reeb vector field of the Lorentzian concircular structure, then $W_2(\xi, X) \cdot S = 0$, where $W_2$ is the $W_2$-curvature tensor and $S$ is the Ricci tensor.

Again the condition $S(\xi, X) \cdot W_2 = 0$ implies that,

$$S(X, W_2(Y, Z)V)\xi - S(\xi, W_2(Y, Z)V)X + S(X, Y)W_2(\xi, Z)V$$

$$- S(\xi, Y)W_2(X, Z)V + S(X, Z)W_2(Y, \xi)V - S(\xi, Z)W_2(Y, X)V$$

$$+ S(X, V)W_2(Y, Z)\xi - S(\xi, V)W_2(Y, Z)X = 0.$$  

$$(4.14)$$
for any vector fields $X, Y, Z, V$ on $M$.

Taking the inner product with $\xi$, the above equation becomes,

$$
-S(X, W_2(Y, Z)V) - S(\xi, W_2(Y, Z)V)\eta(X) + S(X, Y)\eta(W_2(\xi, Z)V)
-S(\xi, Y)\eta(W_2(X, Z)V) + S(X, Z)\eta(W_2(Y, \xi)V) - S(\xi, Z)\eta(W_2(Y, X)V)
+ S(X, V)\eta(W_2(Y, Z)V) - S(\xi, V)\eta(W_2(Y, Z)V) - S(\xi, Z)\eta(W_2(Y, X)V)
= 0. \quad (4.15)
$$

Replacing the expression of $S$ from (3.9) and taking $Z = \xi, V = \xi$, we get,

$$
\frac{\lambda}{n}[-g(X, W_2(Y, \xi)\xi) - \eta(W_2(Y, \xi)\xi)\eta(X) + g(X, Y)\eta(W_2(\xi, \xi)\xi)
- \eta(Y)\eta(W_2(X, \xi)\xi) + \eta(X)\eta(W_2(Y, \xi)\xi) - \eta(\xi)\eta(W_2(Y, X)\xi) + \eta(X)\eta(W_2(Y, \xi)\xi)
- \eta(\xi)\eta(W_2(Y, \xi)V)] = 0. \quad (4.16)
$$

Now using (1.7), (2.7), (2.9), (2.11), we obtain on simplification,

$$
\frac{\lambda}{n}[g(X, Y) + \eta(Y)\eta(Y) - (\alpha^2 - \rho)g(X, Y) - (\alpha^2 - \rho)\eta(X)\eta(Y)] = 0, \quad (4.17)
$$

implies that,

$$
\frac{\lambda}{n}(1 - \alpha^2 + \rho)[g(X, Y) + \eta(x)\eta(Y)] = 0. \quad (4.18)
$$

Using (2.2), the above equation becomes,

$$
\frac{\lambda}{n}(1 - \alpha^2 + \rho)g(\phi X, \phi Y) = 0. \quad (4.19)
$$

for any vector fields $X, Y$ on $M$.

This implies that,

$$
\lambda(1 - \alpha^2 + \rho) = 0.
$$

Then either $\lambda = 0$, or $\alpha^2 - \rho = 1$.

Now if $\alpha^2 - \rho = 1$, then from (2.12), we have,

$$
r = n(n - 1).
$$

So we can state the following theorem:

**Theorem 4.4.** If an $(LCS)_n$ manifold $(M, g, \xi, \eta, \phi, \alpha)$ admits a Yamabe soliton $(g, \xi)$, $\xi$ being the Reeb vector field of the Lorentzian concircular structure and satisfies $S(\xi, X)\cdot W_2 = 0$ then either the soliton is steady, or $r = n(n - 1)$, where $W_2$ is the $W_2$-curvature tensor and $r$ is the scalar curvature.

5. Example of an $(LCS)_3$ manifold:

We consider the 3-dimensional manifold $M = \{(x, y, z) \in \mathbb{R}^3, z \neq 0\}$, where $(x, y, z)$ are standard coordinates in $\mathbb{R}^3$. Let $e_1, e_2, e_3$ be a linearly independent system of vector fields on $M$ given by,

$$
e_1 = z \frac{\partial}{\partial x}, \quad e_2 = z \frac{\partial}{\partial y}, \quad e_3 = z \frac{\partial}{\partial z}.
$$
Let $g$ be the Riemannian metric defined by,

\[ g(e_1, e_1) = g(e_2, e_2) = 1, \quad g(e_3, e_3) = -1, \]
\[ g(e_1, e_2) = g(e_2, e_3) = g(e_3, e_1) = 0. \]

Let $\eta$ be the 1-form defined by $\eta(Z) = g(Z, e_3)$, for any $Z \in \chi(M)$, where $\chi(M)$ is the set of all differentiable vector fields on $M$ and $\phi$ be the $(1, 1)$-tensor field defined by,

\[ \phi e_1 = e_2, \quad \phi e_2 = e_1, \quad \phi e_3 = 0. \]

Then, using the linearity of $\phi$ and $g$, we have

\[ \eta(e_3) = -1, \phi^2(Z) = Z + \eta(Z)e_3 \]

and

\[ g(\phi Z, \phi W) = g(Z, W) + \eta(Z)\eta(W), \]

for any $Z, W \in \chi(M)$.

Let $\nabla$ be the Levi-Civita connection with respect to the Riemannian metric $g$. Then we have,

\[ [e_1, e_2] = 0, [e_2, e_3] = -e_2, [e_1, e_3] = -e_1. \]

The connection $\nabla$ of the metric $g$ is given by,

\[ 2g(\nabla_X Y, Z) = Xg(Y, Z) + Yg(Z, X) - Zg(X, Y) \]
\[ - g([X, Y], Z) - g(Y, [X, Z]) + g(Z, [X, Y]), \]

which is known as Koszul's formula.

Using Koszul's formula, we can easily calculate,

\[ \nabla_{e_1} e_3 = -e_1, \quad \nabla_{e_2} e_3 = -e_2, \quad \nabla_{e_3} e_3 = 0, \]
\[ \nabla_{e_1} e_1 = -e_3, \quad \nabla_{e_2} e_1 = 0, \quad \nabla_{e_3} e_1 = 0, \]
\[ \nabla_{e_1} e_2 = 0, \quad \nabla_{e_2} e_2 = -e_3, \quad \nabla_{e_3} e_2 = 0. \]

Hence in this case $(g, \xi, \eta, \phi, \alpha)$ is an $(LCS)_3$-structure on $M$, where $\alpha = -1$.

Also as $\alpha = -1$ then $\rho = 0$ and consequently, $(M, g, \xi, \eta, \phi, \alpha)$ is an $(LCS)_n$ manifold of dimension 3.

On this manifold $(M, g, \xi, \eta, \phi, \alpha)$, we can easily verify our results.

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