Two kinds of conditionings for stable Lévy processes

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Abstract

Two kinds of conditionings for one-dimensional stable Lévy processes are discussed via $h$-transforms of excursion measures: One is to stay positive, and the other is to avoid the origin.

1 Introduction

It is well-known that a one-dimensional Brownian motion conditioned to stay positive is a three-dimensional Bessel process. As an easy consequence, it follows that the former conditioned to avoid the origin is a symmetrized one of the latter.

The aim of the present article is to give a brief survey on these two different kinds of conditionings for one-dimensional stable Lévy processes via $h$-transforms of Itô’s excursion measures.

The organization of this article is as follows. In Section 2, we recall the conditionings for Brownian motions. In Section 3, we give a review on the conditioning for stable Lévy processes to stay positive. In Section 4, we present results on the conditioning for symmetric stable Lévy processes to avoid the origin.

2 Conditionings for Brownian motions

We recall the conditionings for Brownian motions. For the details, see, e.g., [8, §III.4.3] and [10, §VI.3 and Chap.XII].

Let $(X_t)$ denote the coordinate process on the space of càdlàg functions and $(\mathcal{F}_t)$ its natural filtration. Set $\mathcal{F}_\infty = \sigma(\cup_{t>0}\mathcal{F}_t)$. For $t \geq 0$, we write $\theta_t$ for the shift operator: $X_s \circ \theta_t = X_{t+s}$. For $0 < t < \infty$, a functional $Z_t$ is called $\mathcal{F}_t$-nice if $Z_t$ is of the form $Z_t = f(X_{t_1}, \ldots, X_{t_n})$ for some $0 < t_1 < \ldots < t_n < t$ and some continuous function $f : \mathbb{R}^n \rightarrow \mathbb{R}$ which vanishes at infinity.

Let $W_x$ denote the law of the one-dimensional Brownian motion starting from $x \in \mathbb{R}$.

2.1 Brownian motions conditioned to stay positive

For any fixed $t > 0$, we define a probability law $W^{\uparrow,t}(\cdot)$ on $\mathcal{F}_t$ as

$$ W^{\uparrow,t}(\cdot) = \frac{n^+; (\zeta > t)}{n^+(\zeta > t)} \quad (2.1) $$
where \( n^+ \) stands for the excursion measure of the reflecting Brownian motion (see, e.g., \([8, \S III.4.3]\) and \([10, \S\S XII.4]\)) and \( \zeta \) for the lifetime. The process \((X_s : s \leq t)\) under \( W_{\uparrow,t} \) is called the Brownian meander. Durrett–Iglehart–Miller \([7, \text{Thm.2.1}]\) have proved that

\[
W_{\uparrow,t} = \lim_{\varepsilon \to 0+} W_0 \left[ Z_t \bigg| \forall u \leq t, \ X_u \geq -\varepsilon \right] \tag{2.2}
\]

for any bounded continuous \( \mathcal{F}_t \)-measurable functional \( Z_t \); in particular, for any \( \mathcal{F}_t \)-nice functional \( Z_t \). We may represent (2.2) symbolically as

\[
W_{\uparrow,t}(\cdot) = W_0 \left( \cdot \bigg| \forall u \leq t, \ X_u \geq 0 \right) \tag{2.3}
\]

that is, the Brownian meander is the Brownian motion conditioned to stay positive until time \( t \). As another interpretation of (2.3), we have

\[
W_{\uparrow,t} = \lim_{\varepsilon \to 0+} W_0 \left[ Z_t \circ \theta_{\varepsilon} \bigg| \forall u \leq t, \ X_u \circ \theta_{\varepsilon} \geq 0 \right] \tag{2.4}
\]

for any \( \mathcal{F}_t \)-nice functional \( Z_t \). The proof of (2.4) will be given in Theorem 3.5 in the settings of stable Lévy processes.

We write \( W_0 \) for the law of a three-dimensional Bessel process, that is, the law of the radius \( \sqrt{(B_t^{(1)})^2 + (B_t^{(2)})^2 + (B_t^{(3)})^2} \) of a three-dimensional Brownian motion \((B_t^{(1)}, B_t^{(2)}, B_t^{(3)})\).

We remark that \( W_0 \) is locally equivalent to \( n^+ \):

\[
dW_0|_{\mathcal{F}_t} = \frac{1}{C^t} X_t \, dn^+|_{\mathcal{F}_t} \tag{2.5}
\]

where \( C^t = n^+[X_t] \) is a constant independent of \( t > 0 \). We say that \( W_0 \) is the \textit{h-transform of the excursion measure} \( n^+ \) with respect to the function \( h(x) = x \). Then it holds (see, e.g., Theorem 3.6) that

\[
W_0[Z] = \lim_{t \to \infty} W_{\uparrow,t}[Z] \tag{2.6}
\]

for any \( \mathcal{F}_t \)-nice functional \( Z \) with \( 0 < t < \infty \). We may represent (2.6) symbolically as

\[
W_0^\uparrow(\cdot) = W_0 \left( \cdot \bigg| \forall u, \ X_u \geq 0 \right) \tag{2.7}
\]

that is, \( W_0^\uparrow \) is the Brownian motion conditioned to stay positive during the whole time.

### 2.2 Brownian motions conditioned to avoid the origin

For any fixed \( t > 0 \), we define a probability law \( W_{\times,t} \) on \( \mathcal{F}_t \) as

\[
W_{\times,t} = \frac{W_{\uparrow,t} + W_{\downarrow,t}}{2} \tag{2.8}
\]
where $W^{\downarrow, (t)}$ stands for the law of the process $(-X_s : s \leq t)$ under $W^{\downarrow, (t)}$. The law $W^{\times, (t)}$ may be represented as

$$W^{\times, (t)}(\cdot) = \frac{n(\cdot ; \zeta > t)}{n(\zeta > t)}$$  \hspace{1cm} (2.9)

where $n$ stands for the excursion measure of the Brownian motion; in fact, $n = \frac{n^+ + n^-}{2}$ where $n^-$ is the image measure of the process $(-X_t)$ under $n^+$. Immediately from (2.4) and continuity of paths, we have

$$W^{\times, (t)}[Z_t] = \lim_{\varepsilon \to 0^+} W_0 \left[ Z_t \circ \theta_{\varepsilon} \mid \forall u \leq t, X_u \circ \theta_{\varepsilon} \neq 0 \right]$$  \hspace{1cm} (2.10)

for any $\mathcal{F}_t$-nice functional $Z_t$. We may represent (2.10) symbolically as

$$W^{\times, (t)}(\cdot) = W_0 \left( \cdot \mid \forall u \leq t, X_u \neq 0 \right) ;$$  \hspace{1cm} (2.11)

that is, $W^{\times, (t)}$ is the Brownian motion conditioned to avoid the origin until time $t$.

We define

$$W_0^\times = \frac{W_0^\uparrow + W_0^\downarrow}{2}$$  \hspace{1cm} (2.12)

where $W_0^\downarrow$ stands for the law of the process $(-X_t)$ under $W_0^\uparrow$. In other words, the law $W_0^\downarrow$ is the symmetrization of the three-dimensional Bessel process. We also remark that $W_0^\times$ is locally equivalent to $n$:

$$dW_0^\times|_{\mathcal{F}_t} = \frac{1}{C^\times} |X_t| dW|_{\mathcal{F}_t}$$  \hspace{1cm} (2.13)

where $C^\times = n[|X_t|]$ is a constant independent of $t > 0$. We say that $W_0^\times$ is the $h$-transform with respect to the function $h(x) = |x|$. Then it is immediate from (2.6) that

$$W_0^\times[Z] = \lim_{t \to \infty} W^{\times, (t)}[Z]$$  \hspace{1cm} (2.14)

for any $\mathcal{F}_t$-nice functional $Z$ with $0 < t < \infty$. We may represent (2.14) symbolically as

$$W_0^\times(\cdot) = W_0 \left( \cdot \mid \forall u, X_u \neq 0 \right) ;$$  \hspace{1cm} (2.15)

that is, $W_0^\times$ is the Brownian motion conditioned to avoid the origin during the whole time.

### 3 Stable Lévy processes conditioned to stay positive

Let us review the theory of strictly stable Lévy processes conditioned to stay positive. For concise references, see, e.g., [1] and [6]. We refer to these textbooks also about the theory of conditioning to stay positive for spectrally negative Lévy processes, where we do not go into the details.
For a Borel set $F$, we denote the first hitting time of $F$ by
\[ T_F = \inf\{t > 0 : X_t \in F\}. \tag{3.1} \]
Define
\[ X_t = \inf_{s \leq t} X_s, \quad R_t = X_t - X_t \tag{3.2} \]
and call the process $(R_t)$ the reflected process.

Let $(P_x)$ denote the law of a strictly stable Lévy process of index $0 < \alpha \leq 2$, that is, a process with càdlàg paths and with stationary independent increments satisfying the following scaling property:
\[ (k^{-\frac{\alpha}{\alpha'}} X_{kt} : t \geq 0) \overset{\text{law}}{=} (X_t : t \geq 0) \text{ under } P_0 \tag{3.3} \]
for any $k > 0$. Note that the Brownian case corresponds to $\alpha = 2$. From the scaling property (3.3), it is immediate that the quantity
\[ \rho := P_0(X_t \geq 0) \tag{3.4} \]
does not depend on $t > 0$, which is called the positivity parameter. The possible values of $\rho$ range over $[0, 1]$ if $0 < \alpha < 1$, $(0, 1)$ if $\alpha = 1$, and $[1 - \frac{1}{\alpha}, \frac{1}{\alpha}]$ if $1 < \alpha \leq 2$. Let $(Q_x : x > 0)$ denote the law of the process killed at $T_{(-\infty,0)}$:
\[ Q_x(\Lambda_t; t < \zeta) = P_x(\Lambda_t; t < T_{(-\infty,0)}), \quad x > 0, \Lambda_t \in \mathcal{F}_t. \tag{3.5} \]
Note that the function
\[ (x, t) \mapsto Q_x(t < \zeta) = P_x(t < T_{(-\infty,0)}) \tag{3.6} \]
is jointly continuous in $x > 0$ and $t > 0$.

Let us exclude the case where $|X|$ is a subordinator, i.e.,
\[ 0 < \alpha < 1 \quad \text{and} \quad \rho = 0, 1. \tag{3.7} \]
Then the reflected process $(R_t)$ under $(P_x)$ is a Feller process where the origin is regular for itself, and hence there exists the continuous local time process $(L_t)$ at level 0 of the reflected process $(R_t)$ such that
\[ P_0 \left[ \int_0^\infty e^{-qt} dL_t \right] = q^{\rho - 1}, \quad q > 0. \tag{3.8} \]
Let $n^\uparrow$ denote the corresponding excursion measure away from 0 of the reflected process $(R_t)$. The Markov property of $n^\uparrow$ may be expressed as
\[ n^\uparrow(1_{\Lambda \circ \theta_t}; \Lambda_t, t < \zeta) = n^\uparrow[Q_{X_t}(\Lambda); \Lambda_t, t < \zeta] \tag{3.9} \]
for any $\Lambda \in \mathcal{F}_\infty$ and $\Lambda_t \in \mathcal{F}_t$. We introduce the following function:
\[ h^\uparrow(x) = P_0 \left[ \int_0^\infty 1\{X_t \geq -x\} dL_t \right], \quad x \geq 0. \tag{3.10} \]
Since the ladder height process $H := X \circ L^{-1}$ is a stable Lévy process of index $\alpha(1 - \rho)$, we see that

$$h^i(x) = P_0 \left[ \int_0^\infty 1\{H_u \geq -x\} \, du \right] = C_1^i x^{\alpha(1-\rho)}$$

(3.11)

for some constant $C_1^i > 0$ independent of $x \geq 0$. The following theorem is due to Silverstein [14, Thm.2]; another proof can be found in Chaumont–Doney [5, Lem.1] (see also Doney [6, Lem.10 in §8.3]).

**Theorem 3.1** ([14]). It holds that

$$Q_x[(X_t)^{\alpha(1-\rho)}; \, t < \zeta] = x^{\alpha(1-\rho)}, \quad x > 0, \quad t > 0,$$

(3.12)

$$n^i[(X_t)^{\alpha(1-\rho)}; \, t < \zeta] = C_2^i, \quad t > 0$$

(3.13)

for some constant $C_2^i$ independent of $t > 0$.

By virtue of this theorem, we may define the $h$-transform by

$$dP^i_x|_{\mathcal{F}_t} = \begin{cases} 
\frac{(X_t)^{\alpha(1-\rho)}}{C_2^i} dQ_x|_{\mathcal{F}_t} & \text{if } x > 0, \\
\frac{1}{C_2^i} (X_t)^{\alpha(1-\rho)} d\mathcal{N}^i|_{\mathcal{F}_t} & \text{if } x = 0; 
\end{cases}$$

(3.14)

indeed, the family $(P^i_x|_{\mathcal{F}_t}: t \geq 0)$ is proved to be consistent by the Markov property of $n^i$.

**Theorem 3.2** ([3]). The process $((X_t), (P^i_x))$ is a Feller process.

This theorem is due to Chaumont [3, Thm.6], where he proved weak convergence $P^i_x \to P^i_0$ as $x \to 0+$ in the càdlàg space equipped with Skorokhod topology. Bertoin–Yor [2, Thm.1] proved the weak convergence for general positive self-similar Markov processes. Tanaka [15, Thm.4] proved the Feller property for quite a general class of Lévy processes.

Now let us discuss the conditionings (2.3) and (2.7) for stable Lévy processes. The following theorem is an immediate consequence of Chaumont [4, Lem.1] and of the continuity of (3.6).

**Theorem 3.3** ([4]). Let $t > 0$ be fixed. Then the function

$$[0, \infty) \ni x \mapsto P^i_x \left[ (X_t)^{-\alpha(1-\rho)} \right]$$

(3.15)

is continuous and vanishes at infinity.

Define a probability law $M^{i,(t)}$ on $\mathcal{F}_t$ as

$$M^{i,(t)}(\Lambda_t) = \frac{n^i(\Lambda_t; \zeta > t)}{n^i(\zeta > t)} = \frac{P^i_0[(X_t)^{-\alpha(1-\rho)}; \Lambda_t]}{P^i_0[(X_t)^{-\alpha(1-\rho)}]}$$

(3.16)

for $\Lambda_t \in \mathcal{F}_t$. The following theorem, which generalizes (2.2), can be found in Bertoin [1, Thm.VIII.18].
Theorem 3.4 \((\text{II})\). For any \(t > 0\), it holds that
\[
M^{\uparrow,(t)}[Z_t] = \lim_{\varepsilon \to 0^+} P_0 \left[ Z_t \mid \forall u \leq t, \ X_u \geq -\varepsilon \right] \tag{3.17}
\]
for any \(\mathcal{F}_t\)-nice functional \(Z_t\).

Now we give the following version of (2.4) for stable Lévy processes.

Theorem 3.5. For any \(t > 0\), it holds that
\[
M^{1,(t)}[Z_t] = \lim_{\varepsilon \to 0^+} P_0 \left[ Z_t \circ \theta_\varepsilon \mid \forall u \leq t, \ X_u \circ \theta_\varepsilon \geq 0 \right] \tag{3.18}
\]
for any \(\mathcal{F}_t\)-nice functional \(Z_t\).

Proof. By the Markov property, the expectation of the right hand side of (3.18) is equal to
\[
\frac{P_0[X_\varepsilon \mid \forall u \leq t, \ X_u \geq 0]}{P_0[X_\varepsilon \circ \theta_\varepsilon \mid \forall u \leq t, \ X_u \geq 0]} = \frac{P_0[(X_\varepsilon)^{-\gamma}P^{\uparrow}_{X_\varepsilon} [Z_t(X_t)^{-\gamma}]]}{P_0[(X_\varepsilon)^{-\gamma}P^{\uparrow}_{X_\varepsilon} [(X_t)^{-\gamma}]]}, \tag{3.19}
\]
where we put \(\gamma = \alpha(1 - \rho)\). By the scaling property, this is equal to
\[
\frac{P_0[(X_1)^{-\gamma}P^{\uparrow}_{\varepsilon/\alpha \cdot X_1} [Z_t(X_t)^{-\gamma}]]}{P_0[(X_1)^{-\gamma}P^{\uparrow}_{\varepsilon/\alpha \cdot X_1} [(X_t)^{-\gamma}]]}. \tag{3.20}
\]
By Theorems 3.2 and 3.3 and by the dominated convergence theorem, we see that this quantity converges as \(\varepsilon \to 0^+\) to
\[
\frac{P_0^\uparrow [Z_t(X_t)^{-\gamma}]}{P_0^\uparrow [(X_t)^{-\gamma}]}, \tag{3.21}
\]
which coincides with \(M^{1,(t)}[Z_t]\) by the definition (3.16). \(\square\)

The following theorem generalizes (2.6).

Theorem 3.6. It holds that
\[
P_0^\uparrow [Z] = \lim_{t \to \infty} M^{1,(t)}[Z] \tag{3.22}
\]
for any \(\mathcal{F}_t\)-nice functional \(Z\) with \(0 < t < \infty\).

The proof can be done in the same way as Theorem 3.5.
4 Symmetric stable Lévy processes conditioned to avoid the origin

Let us discuss the conditioning for symmetric stable Lévy processes to avoid the origin. This has been introduced by Yano–Yano–Yor [18] in order to extend some of the penalisation problems for Brownian motions by Roytette–Vallois–Yor [11], [12] and Najnudel–Roytette–Yor [9], to symmetric stable Lévy processes.

We assume that \( \rho = \frac{1}{2} \), i.e., the process \((X_t, (P_x))\) is symmetric, and that the index \( \alpha \) satisfies \( 1 < \alpha \leq 2 \). For simplicity, we assume that \( P_0[e^{i\lambda X_1}] = e^{-|\lambda|^{\alpha}} \). Then the origin is regular for itself, and there exists the continuous resolvent density:

\[
    u_q(x) = \frac{1}{\pi} \int_0^\infty \frac{\cos x\lambda}{q + \lambda^\alpha} d\lambda, \quad q > 0, \quad x \in \mathbb{R}. \tag{4.1}
\]

Moreover, there exists the local time process at level 0 of the process \((X_t)\), which we denote by \((L_t)\), such that

\[
    P_0 \left[ \int_0^\infty e^{-qt} dL_t \right] = u_q(0) = \frac{q^{\frac{1}{\alpha}-1}}{\alpha \sin \frac{\pi}{\alpha}}, \quad q > 0. \tag{4.2}
\]

The corresponding excursion measure away from 0 will be denoted by \( n^\times \). We introduce the following function:

\[
    h^\times(x) = \lim_{q \to 0^+} \left\{ u_q(0) - u_q(x) \right\} = \frac{1}{C^\times} |x|^{\alpha-1}, \quad x \in \mathbb{R}; \tag{4.3}
\]

see, e.g., [17, Appendix] for the exact value of the constant \( C^\times \). Note that the function \( h^\times(x) \) may also be represented as

\[
    h^\times(x) = P_0[L_{T_{\{x\}}}], \quad x \in \mathbb{R}; \tag{4.4}
\]

see, e.g., [11, Lem.V.11]. Let \((P^0_x)\) denote the law of the process \((X_t, (P_x))\) killed at \( T_{\{0\}} \):

\[
    P^0_x(\Lambda_t; t < \zeta) = P_x(\Lambda_t; t < T_{\{0\}}), \quad x \neq 0, \quad \Lambda_t \in \mathcal{F}_t. \tag{4.5}
\]

By [18, Thm.3.5], we see that the function

\[
    (x, t) \mapsto P^0_x(t < \zeta) = P_x(t < T_{\{0\}}) \tag{4.6}
\]

is jointly continuous in \( x \in \mathbb{R} \setminus \{0\} \) and \( t > 0 \). The following theorem is due to Salminen–Yor [13, eq.(3)] and Yano–Yano–Yor [18, Thm.4.7].

**Theorem 4.1** ([13], [18]). It holds that

\[
    P^0_x(|X_t|^{\alpha-1}) = |x|^{\alpha-1}, \quad x \neq 0, \quad t > 0, \tag{4.7}
\]

\[
    n^\times(|X_t|^{\alpha-1}, t < \zeta) = C^\times, \quad t > 0. \tag{4.8}
\]
By virtue of this theorem, we may define the \( h \)-transform by
\[
\frac{dP_x^\gamma}{d\mathcal{F}_t} = \begin{cases} 
|X_t|^\alpha \frac{dP_x^0}{d\mathcal{F}_t} & \text{if } x \neq 0, \\
\frac{1}{|x|} |X_t|^\alpha \frac{d\mathbb{P}_x}{d\mathcal{F}_t} & \text{if } x = 0;
\end{cases}
\] (4.9)
indeed, the family \((P_x^\gamma : t \geq 0)\) is proved to be consistent by the Markov property of \( n^x \). The following theorem is due to [16, Thm.1.5].

**Theorem 4.2** ([16]). *Suppose that \( 1 < \alpha < 2 \). Then the process \(((X_t), (P_x^\gamma))\) is a Feller process.*

**Remark 4.3.** Yano [16, Thm.1.4 and Cor.1.9] obtained the following long-time behavior of paths: If \( 1 < \alpha < 2 \), then
\[
P_0^\gamma \left( \limsup_{t \to \infty} X_t = \limsup_{t \to \infty} (-X_t) = \lim_{t \to \infty} |X_t| = \infty \right) = 1.
\] (4.10)

**Remark 4.4.** In the Brownian case (\( \alpha = 2 \)), the process \(((X_t), (P_x^\gamma))\) is not a Feller process. Indeed, \( P_0^\gamma \) is not irreducible (see (2.12)). Contrary to (4.10), the long-time behavior in this case is as follows:
\[
P_0^\gamma \left( \lim_{t \to \infty} X_t = \infty \right) = P_0^\gamma \left( \lim_{t \to \infty} X_t = -\infty \right) = \frac{1}{2}.
\] (4.11)

Now let us discuss the conditionings (2.11) and (2.15) for symmetric stable Lévy processes. The following theorem is an immediate consequence of Yano–Yano–Yor [18, Lem.4.10] and of the continuity of (1.6).

**Theorem 4.5** ([18]). *Let \( t > 0 \) be fixed. Then the function
\[
\mathbb{R} \ni x \mapsto P_x^\gamma \left[ |X_t|^{-(\alpha-1)} \right]
\] (4.12)
is continuous and vanishes at infinity.*

Define a probability law \( M^{\gamma,(t)} \) on \( \mathcal{F}_t \) as
\[
M^{\gamma,(t)}(\Lambda_t) = \frac{n^\gamma(\Lambda_t; \zeta > t)}{n^\gamma(\zeta > t)} = \frac{P_0^\gamma [|X_t|^{-(\alpha-1)}; \Lambda_t]}{P_0^\gamma [|X_t|^{-(\alpha-1)}]}
\] (4.13)
for \( \Lambda_t \in \mathcal{F}_t \). The following theorem generalizes (2.10).

**Theorem 4.6.** *For any \( t > 0 \), it holds that
\[
M^{\gamma,(t)}[Z_t] = \lim_{\epsilon \to 0^+} P_0 \left[ Z_t \circ \theta_\epsilon \right] \quad \forall u \leq t, \ X_u \circ \theta_\epsilon \neq 0
\] (4.14)
for any \( \mathcal{F}_t \)-nice functional \( Z_t \).*

The proof can be done in the same way as Theorem 3.5 by virtue of Theorems 4.2 and 4.5. The following theorem generalizes (2.14).

**Theorem 4.7** ([18]). *It holds that
\[
P_0^\gamma[Z] = \lim_{t \to \infty} M^{\gamma,(t)}[Z]
\] (4.15)
for any \( \mathcal{F}_t \)-nice functional \( Z \) with \( 0 < t < \infty \).*

This is a special case of [18, Thm.4.9].

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