ON THE $z$-DEGREE OF THE KAUFFMAN POLYNOMIAL OF A TANGLE DECOMPOSITION

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0. Introduction.

In 1987, the elder author produced [2] an upper bound on the degree of the then-new Brandt-Lickorish-Millett-Ho polynomial in terms of the crossing number and the length of the longest bridge in a link diagram. This result extends immediately to the $z$-degree of the two-variable Kauffman polynomial (in any of its forms; we shall use the Dubrovnik version $D(\lambda, z)$).

The length of the longest bridge (that is, the longest consecutive string of overcrossings) in a diagram represents some measure of how nonalternating the diagram is. For a fixed crossing number, the greatest $z$-degree will occur when the diagram is prime and alternating, which is when the longest bridge (and every bridge) has length 1. A bridge of length $> 1$ contributes to a quicker unravelling of the link under the skein relations, and lowers the $z$-degree. More precisely, it is shown in [2] that the $z$-degree of the polynomial is less than or equal to $N - B$, where $N$ is the crossing number and $B$ is the length of the longest bridge in a given link diagram. It is furthermore shown that if the link diagram is prime, reduced, and alternating, then the $z$-degree is $N - 1$.

For a diagram with more than one long bridge, one would like to find an inequality that considers more than just the single longest bridge. For example, it was pointed out by Thistlethwaite [6] that if a link diagram is composite, then the longest bridge in each factor counts toward lowering the $z$ degree of $D$. On the other hand, there are numerous examples (such as $8_{19}$–$8_{21}$ from the table in Rolfsen [5]) of non-alternating diagrams with $N$ crossings and two or more bridges of length 2 and with $z$-degree (of the Kauffman polynomial) of $N - 2$. Thus it is necessary to find some method of keeping the bridges separate from each other while computing the skein tree. We will accomplish this by cutting the link diagram into tangles, and considering the longest bridge in each tangle. We obtain an upper bound on the $z$-degree in terms of the number of crossings in the diagram, and the lengths of each of these longest separated bridges.

The Dubrovnik polynomial of a tangle may be defined as a linear combination, over an appropriate ring, of simple tangles. We bound the $z$-degree of the polynomials in this linear combination in terms of the crossing number and length of the longest bridge in each tangle. Our result for links follows by closing up the tangles.

Yokota [7] has shown that if $L$ is represented by a reduced alternating diagram with $n$ crossings, then the span of the Kauffman polynomial in the other variable ($\lambda$ or $a$) is
equal to $n$. Our result concerns not the span of $z$ but the degree. (The highest negative power of $z$ that occurs is always one less than the number of components in the link.)

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1. **The Bound.**

By a *tangle* we mean a planar rectangular diagram, with overcrossings and undercrossings labeled in the usual way, containing any number of closed “circle” components and exactly two “arc” components. Two of the arc endpoints are on the top of the rectangle and two on the bottom. Equivalence of tangles is up to regular isotopy. A *bridge* is a maximal segment of a tangle containing no undercrossings. The *length* of a bridge is the number of crossings at which it overcrosses.

Following Morton and Traczyk [4], we shall work over the ring $\Lambda'$ generated over $\mathbb{Z}$ by $\lambda^{\pm 1}, z$, and $\delta$ with the single relation

$$\lambda^{-1} - \lambda = z(\delta - 1) \tag{*}$$

At the end of our calculations, we will use this relation to eliminate $\delta$ at the expense of introducing $z^{-1}$ (but not lowering the $z$-degree).

The module $M_2$ is defined to be the set of all $\Lambda'$-linear combinations of tangles modulo the following local relations, where $T \bigcup$ unknot means the addition of a single unknotted and unlinked circle component to $T$, and the meaning of the rest of the symbols is indicated in Figure 1.

(i) $T^+ - T^- = z(T^0 - T^\infty)$

(ii) $T^\text{right} = \lambda^{-1}T; T^\text{left} = \lambda T$

(iii) $T \bigcup$ unknot = $\delta T$

![Figure 1](image)

Figure 1

Morton and Traczyk [4] prove that the tangles called $P, Q$, and $R_1$ in Figure 2 form a free $\Lambda'$-basis for $M_2$. We find it easier to control the $z$-degree by working with $P, Q, R_1,$ and $R_2$. For $f \in \Lambda'$, we understand the $z$-degree of $f$ to be the minimum $z$-degree over all polynomials in the equivalence class of $f$. We shall also sometimes refer to the $z$-degree of the Kauffman polynomial of a link or diagram as simply the $z$-degree of the link or diagram.
As in [2], there are a number of bad situations that must be dealt with before we can proceed to the main argument. Call a bridge $b$ in a tangle $T$ improper if any of the following is true. Examples are shown in Figure 3.

(a) The bridge $b$ consists of a full circle with length $B > 0$.
(b) One or both of the crossings at which the bridge ends also has $b$ as an overcrossing.
(c) The bridge $b$ starts and ends at the same crossing and has length $B > 1$.
(d) The bridge begins and ends at an endpoint of the tangle and has length $B > 1$.

Lemma 1.1. If a given diagram of a tangle $T$ has $N$ crossings and contains an improper bridge $b$ of length $B$, then the diagram can be altered by type II and III Reidemeister
moves to a diagram with \( N' \) crossings and a bridge \( b' \) of length \( B' \) in such a way that \( N' < N \) and \( N' - B' \leq N - B \).

**Proof:** For improper bridges in (a), (b), and (c) above, the proof is identical to Lemma 0 in [2]. In the case of an overcrossing arc, we may move the arc to the far right of the tangle, leaving at most one crossing on that arc. The two possible cases are shown in Figure 4. The reduction in crossing number is matched by a reduction in the length of the bridge. □

\[ \begin{array}{c}
\text{Figure 4}
\end{array} \]

**Theorem 1.2.** Let \( T \) be a tangle with \( N \) crossings and a bridge of length \( B \). Considered as an element of \( M_2 \), \( T \) can be written as a \( \Lambda' \)-linear combination

\[
f_1(z, \lambda^{\pm 1}, \delta)P + f_2(z, \lambda^{\pm 1}, \delta)Q + f_3(z, \lambda^{\pm 1}, \delta)R_1 + f_4(z, \lambda^{\pm 1}, \delta)R_2
\]

where the \( z \)-degree of each \( f_i \) is less than or equal to \( N - B \).

**Proof:** The theorem is true for any tangle with crossing number 0 or 1 simply by replacing each disjoint circle with a \( \delta \) and a loop with \( \lambda^{\pm 1} \). If the set of counterexamples to the theorem is nonempty, let \( T \) be such a tangle diagram with minimal crossing number \( N \); among all such tangles with minimal crossing number, let \( T \) be one with maximal longest bridge length \( B \). Let \( b \) be one of the longest bridges in \( T \). By Lemma 1.1, \( b \) cannot be improper; thus it must have at least one endpoint \( c \) in the interior of \( T \). If we perform the skein operation (i) at \( c \), the two smoothings \( T^0 \) and \( T^\infty \) will have smaller crossing number than \( T \), while the new \( T^{\pm 1} \) will have a longer bridge. None of these tangles can be counterexamples to the theorem. Thus the \( z \)-degree of any coefficient of \( T^0 \) or \( T^\infty \) is at most \((N - 1) - B \), and the \( z \)-degree of any coefficient of the changed tangle \( T^{\pm 1} \) is at most \( N - (B + 1) \). Thus there can be no coefficient of \( zT^\infty \) or \( zT^0 \) or \( T^{\pm 1} \) with \( z \)-degree high enough for \( T \) to be a counterexample to the theorem. □

If one should want to write \( T \) as a \( \Lambda' \)-linear combination of the basis tangles \( P, Q, R_1 \), one can do so at the expense of adding 1 to the \( z \)-degree of the \( P \) and \( Q \) coefficients. Note that \( N - B = 0 \) in each of the tangles \( P, Q, R_1 \), so performing a skein move to replace \( R_2 \) with \( zP - zQ + R_1 \) adds 1 to the \( z \)-degree and accomplishes nothing as far as crossings and bridges are concerned.

We now consider a wiring diagram in the plane as defined in Morton [3]. The endpoints of several disjoint tangles are joined by non-crossing arcs. See for example Figure 5. One obtains a link by inserting a tangle into every box of a wiring diagram.
Theorem 1.3. Let $L$ be a link diagram written as a wiring diagram with $k$ tangles \( \{T_i\}_i^k \). Let tangle $T_i$ have $N_i$ crossings and a longest bridge of length $B_i$. Then the Dubrovnik polynomial $D_L(\lambda^{\pm 1}, \delta, z)$ has $z$-degree less than or equal to $k - 1 + \sum_{i=1}^{k} (N_i - B_i)$.

Proof: Apply the skein relations inside of each tangle $T_i$ to obtain a linear combination of $P, Q, R_1,$ and $R_2$ with $z$-degree bounded by $N_i - B_i$. The result is that the Dubrovnik polynomial has been written as a $\Lambda'$-linear combination of the polynomials of at most $4^k$ links, with each coefficient of the combination having $z$-degree at most $\sum_{i=1}^{k} (N_i - B_i)$. Each of these $4^k$ links has at most $k$ crossings and, unless it has no crossings at all, a bridge of length at least 1. By the main theorem of [2], each link has a Dubrovnik polynomial with $z$-degree at most $k - 1$. □

It is customary to use relation (*) to replace the variable $\delta$ in $\Lambda'$ with $(\lambda^{-1} - \lambda)z^{-1} + 1$. Since this replacement introduces no positive powers of $z$ and leaves one constant term, it has no effect on the $z$-degree of a polynomial in $\Lambda'$. It is also irrelevant to the $z$-degree of $D$, whether one uses the regular isotopy invariant or the ambient isotopy invariant version of the polynomial, since these differ only by a power of $\lambda$.

2. Examples from Rational Tangles.

Consider the closed chain $L$ represented by the two different diagrams in Figure 6. The seven-component link $L$ must have crossing number at least 14, since each component has linking number $\pm 1$ with exactly two other components and so each component must have at least four crossings. Therefore the diagrams in Figure 6 are minimal crossing
diagrams. If we apply Theorem 1.3 to the top diagram, we find two bridges of length 2 which can be separated into two separate tangles, and we therefore obtain an upper bound of $14 - 2 - 1 = 11$ for the $z$-degree of this link. However, the bottom diagram may be decomposed into three tangles, each containing a bridge of length 2, and so we get an upper bound of $14 - 3 - 1 = 10$. The $z$-degree of $L$ is in fact 10.

Figure 6

More generally, let $L$ be a chain similar to the one in Figure 6 but with $p + q$ components. Let $p$ be the number of positive linkages (consecutive components “positively” linked) and $q$ be the number of negative linkages. For example, the top diagram of Figure 6 has four positive linkages followed by three negative linkages, whereas the bottom diagram alternates between positive and negative. Arranging the linkages in positive-negative pairs, we find that Theorem 1.3 gives a bound of $N - \min (p, q) - 1$ for the $z$-degree of $L$ (where $\min (p, q)$ is the smaller of the two numbers $p$ and $q$). This bound is in fact exact. However, grouping all the positive linkages together and all the negative linkages together, we get a bound of $N - 3$, which is far from exact if $\min (p, q)$ is large.

If $p = 0$ or $q = 0$, then $L$ is an alternating link, and $N - \min (p, q) - 1$ reduces to the $N - 1$ of [2]. Consider the case $p > q > 0$, so that $N - \min (p, q) = N - q - 1 = 2(p + q) - q - 1 = 2p + q - 1$. Apply the skein relation (i) to one of the crossings in a positive linkage, with $L = L^+$. Then $L^-$ is a connected sum of $p + q$ Hopf links, and so has degree $p + q - 1 < 2p + q - 1$. $L^\infty$ is isotopic (picking up a factor of $\lambda$, of course) to a chain with $N - 2$ crossings, $p - 1$ positive linkages, and $q$ negative linkages, so it has
Applying the move in Figure 7, \( L^0 \) is isotopic (picking up \( \lambda^{-1} \)) to a chain with \( p \) positive linkages and \( q - 1 \) negative linkages. Inductively, we see that the \( L^0 \) term has the largest \( z \)-degree, and that the \( z \)-degree of \( L \) is therefore \( N - q - 1 \).

The case \( p < q \) follows similarly. The case \( p = q \) is more complicated to do by this method because in applying the skein relation (i), one encounters the case that more than one of the three replacement terms have maximal degree. We will see that the formula holds in the \( p = q \) case below. First, however, we want to point out that the chains described so far are all prime links, so that the reduction in \( z \)-degree obtained (reduction below the crossing number) is not just the connected sum effect (as mentioned in the introduction) in disguise. For suppose that such a link \( L \) has a two-sphere \( S \) which intersects \( L \) in exactly two points. Those two points must be on the same component \( K \). \( L - K \) cannot be split, no matter what \( K \) is (linking number considerations again), so every component of \( L - K \) lies on the same side of \( S \). On the other side of \( S \) must be only a single arc of \( K \), which cannot be knotted because none of the components of \( L \) are individually knotted.

More examples may be obtained by considering chains where two consecutive components may be twisted together any even number of times. Such a chain with \( k \) components may be indexed by \( k \) nonzero even integers \((m_1, m_2, \ldots, m_k)\). The integers describe the linking between consecutive components. For example, the diagram indexed by \((2, 4, -4, 2)\) is shown in Figure 8. The same arguments used above show that such a diagram has minimal crossing number and represents a prime link. If the \( m_i \) are permuted, then even if the represented link changes, the Kauffman polynomial does not, since permutations may be accomplished by mutations. Let \( p \) be the number of positive \( m_i \) (which correspond to positive linkages) and let \( q \) be the number of negative \( m_i \) (which correspond to negative linkages), so that \( k = p + q \). If the positive and negative linkages are arranged in alternating fashion, then once again Theorem 1.3 gives us a bound of \( N - \min(p, q) - 1 \). Again, we shall see that this bound is exact. Arranging the positive linkages all together, however, again gives a bound of \( N - 3 \).
We now define $R_1$ to be a positive rational tangle. Moreover, we will say that any tangle built up out of a positive rational tangle $T$ and $R_1$ in either of the two ways shown in Figure 9 is also a positive rational tangle. We define a negative rational tangle similarly, building with $R_2$ instead of $R_1$. The term rational comes from Conway [1], and in fact all rational tangles are either positive or negative (in our sense), depending on whether their associated continued fractions (in Conway’s sense) represent positive or negative numbers. It is clear that a positive or negative rational tangle is alternating. A positive tangle that can be written as in Figure 9a will be called vertical, and one that can be written as in Figure 9b will be called horizontal. We define $R_1$ to be neither horizontal nor vertical. We define horizontal and vertical negative tangles similarly.

Another way of stating Theorem 1.2 is that any tangle may be written as a polynomial in $z$ of degree $\leq N - B$, where the coefficients of the polynomial are themselves polynomials in $\lambda^{\pm 1}$, $P$, $Q$, $R_1$, and $R_2$. For rational tangles, we have the following:
Theorem 2.1. A positive rational tangle with $N$ crossings may be uniquely written as a polynomial in $z$ of degree $N - 1$, where the coefficients are polynomials in $\lambda^{\pm 1}$, $P$, $Q$, and $R_1$. If $N > 1$, then the $z^{N-1}$ coefficient of a vertical tangle is $\pm(R_1 - \lambda Q)$, and the $z^{N-1}$ coefficient of a horizontal tangle is $\pm(R_1 - \lambda^{-1} P)$. A negative rational tangle may be uniquely written as a polynomial in $z$ of degree $N - 1$, where the coefficients are polynomials in $\lambda^{\pm 1}$, $P$, $Q$, and $R_2$. If $N > 1$, then the $z^{N-1}$ coefficient of a vertical tangle is $\pm(R_2 - \lambda^{-1} Q)$, and the $z^{N-1}$ coefficient of a horizontal tangle is $\pm(R_2 - \lambda P)$.

Proof: The uniqueness follows from the result of Morton and Traczyk [4], that $P$, $Q$, and $R_1$ form a free basis for $M_2$. (It is clear that $R_1$ may be replaced by $R_2$ without affecting this statement.) Beyond that, it is a simple induction argument, applying the skein relation (i) to $R_1$ in Figure 9 for the positive case (and similarly for the negative case).

Now it follows easily that if $L$ is constructed by wiring $p$ positive and $q$ negative vertical rational tangles together in a horizontal line, the $z$-degree of $L$ is $N - \min(p, q) - 1$. We replace each rational tangle with a linear combination as in the proof of Theorem 1.3. If $p > q$, then the largest $z$-degree in the result is obtained from the $R_1$ term in each positive tangle and the $Q$ term in each negative tangle. The leading $z$-coefficient will be $\pm \lambda^{-q}$ times the leading $z$-coefficient of the $(2, p)$ torus link (which is built up of $p$ copies of $R_1$ and $q$ copies of $Q$). If $q < p$, then the largest $z$-degree is obtained from the $R_2$ term in each negative tangle and the $Q$ term in each positive tangle. The leading $z$-coefficient will be $\pm \lambda^{p}$ times the leading $z$-coefficient of the $(2, p)$ torus link. If $p = q$, then both of the terms just described are of maximal degree, but because they differ by $\lambda^{2p}$ they cannot cancel.

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