A Non-Technical Introduction to Confinement and $N = 2$
Globally Supersymmetric Yang-Mills Gauge Theories

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Abstract

The aim of this talk is to give a brief introduction to the problem of confinement in QCD and to $N = 2$ globally supersymmetric Yang-Mills gauge theories (SYM). While avoiding technicalities as much as possible I will try to emphasize the physical ideas which lie behind the picture of confinement as a consequence of the vacua of QCD to be a dual superconductor. Finally I review the implementation of this picture in the framework of $N = 2$ SYM.
1 Introduction

The last two decades have witnessed the successes of the perturbative approach to the study of QCD. The discovery of asymptotic freedom has marked an era of fruitful theoretical work which has been thoroughly confirmed by experimental data. Unfortunately one of the oldest experimental evidence in QCD, the confinement of quarks and gluons inside hadrons, has not yet received a satisfactory theoretical explanation. The main reason for this fact is that confinement is a non-perturbative effect which does not allow the use of a ”standard” computational technique. In spite of these difficulties great progresses have been done in understanding non-perturbative phenomena in the past years. I think that recalling few facts under a ”historical” perspective will help the reader follow my reasoning. The first big progress has come with the discovery of two non-trivial classical solutions of the equations of motion of the Georgi-Glashow model and of pure gauge non-abelian Yang-Mills theory: they are called magnetic monopole and instanton respectively and will play a major role in my future discussion \[1\]. A magnetic monopole is a field configuration of finite energy (and infinite action) representing a particle with zero electric charge and magnetic charge different from zero. As we will see later on, its presence is badly needed in the description I will give of the vacuum of QCD. An instanton is a field configuration of finite action. Its importance rests on the fact that it allows the evaluation of the contribution of tunnelling among different vacua: loosely speaking instantons are the generalization to the path integral formalism of the WKB method in quantum mechanics. Soon after the instanton solution discovery a first attempt to evaluate non-perturbative effects in QCD was carried out \[2\]. In spite of the great technical difficulties overcome in this work, the final result is disappointing, since it is infrared divergent. Some time earlier that this work it had appeared the proposal of confinement as a dual Meissner effect \[3\] which I will review in the next chapter. Even though this mechanism is extremely appealing,
its actual implementation is problematic since the evaluation of the QCD effective action is a formidable task. The interest in the work that I will review in the third chapter rests on it being the first example of a non-trivial four dimensional gauge theory in which the computation of the effective potential can be successfully carried out. Let me now conclude this introduction by telling you the last part of what I have called before a ”historical” path. The last development of interest for our story came afterwards the introduction of supersymmetry (SUSY). It was in fact very soon realized [4] that SUSY greatly simplifies the evaluation of effective actions as it is a very constraining symmetry. In its presence, miraculous cancellations [4] avoid the divergences found in [2]. Indeed SUSY is so powerful that the final result of these computations can be guessed in advance, by using Ward Identities, to be a pure number times its physical dimension as it is later verified by explicit computations [6]. Other important novelties will come out in the presence of SUSY as we will see later. But now our ”historical” diversion is over and it is time to start telling our story.

2 Confinement as a Dual Meissner effect

For a generic conductor, the current, \( \vec{j} \), that flows into it, is proportional to the electric field, \( \vec{E} \) in which it is immersed. The proportionality constant is called the resistivity, \( \sigma \)
\[
\vec{j} = \sigma \vec{E}. \tag{2.1}
\]
In a superconductor \( \sigma \to \infty \) and, to keep \( \vec{j} \) constant, \( \vec{E} \to 0 \). As a consequence
\[
\frac{\partial \vec{B}}{\partial t} = \vec{\nabla} \times \vec{E} \to 0. \tag{2.2}
\]
Thus a superconductor expels the magnetic field in which it is immersed. This is the Meissner effect and such a superconductor is called of type I. If the external magnetic field is very strong, it will penetrate into the superconductor (which will
now be called of type II), but it will be squeezed into narrow flux tubes. If we imagine
the QCD vacuum to behave like a superconductor of type II, at the end of the flux
tubes we will have magnetic monopoles. It is well-known that the phenomenon that
sets the onset of the superconducting phase is the creation of Cooper’s pairs. For
a field theorist the way to describe this is via the Higgs mechanism: the inverse of
the mass thus generated, will be the size of the flux tube. What I have described
until now is the creation of magnetic flux tubes: color, though, is an electric type
of charge. The only mechanism at our disposal to exchange electric and magnetic
charges has to be of the type of the dual symmetry observed in abelian theories.
Maxwell’s equation of motion are in fact invariant under the transformation
\[ E \rightarrow -B, \]
\[ B \rightarrow E. \]  
(2.3)

Then color confinement can be achieved only if in the QCD vacuum we see a kind of
dual Meissner effect. Appealing to duality is the only way out even if this is going
to raise a host of problems, since non-abelian gauge theories are not invariant under
a duality transformation.

As we have seen, the condensation of a pair of electrically charged particles (via
the Higgs effect), leads to the creation of a magnetic flux tube. The dual of this effect,
will require a magnetic Higgs effect implying that the microscopic fields entering the
Lagrangian have to be magnetically charged: the description of the vacuum of QCD
thus imposes the existence of at least two different phases. How can I characterize
these two phases? The electric phase is usually studied by introducing the Wilson
line
\[ W(C') = P e^{ig \oint_{C'} A_\mu dx^\mu}, \]  
(2.4)

which ”measures” the electric flux going through the surface bounded by \( C' \). In the
above formula, \( g \) is the coupling constant and \( A_\mu \) is the non-abelian gauge connection
of the theory. For the magnetic phase we can think of introducing an analogous
order parameter, $M(C)$, ”measuring” the magnetic flux going through $C$. The main feature of the operator $M(C)$ is to produce a singular gauge transformation $\Omega(C)$, upon acting on the vacuum of the theory, $|A_\mu>$

$$M(C)|A_\mu> = |A_\mu^{\Omega(C)}>.$$  \hspace{1cm} (2.5)

An explicit expression for $M(C)$ does not exist in the framework of the second order formalism that consists in writing the functional integral in terms of the field strength $F_{\mu\nu}$ and with a functional measure given in terms of $A_\mu$. Upon introducing an auxiliary antisymmetric tensor field (first order formalism) it is possible to write a non-local expression for $M(C)$ \footnote{That is to introduce a unity of magnetic flux or analogously a monopole.}; in reality, the property of (2.5) is sufficient for the purpose I have in mind (to characterize the phases of QCD) so I will not delve into this subject any longer. Before undertaking our next computation, I remind the reader of the fact that a field transforming in the adjoint representation of a gauge group (let’s say $SU(N)$) is blind to the center of the group so that

$$\Omega(C) : \Omega(2\pi) = e^{2\pi i \frac{\Phi}{N}}\Omega(0).$$ \hspace{1cm} (2.6)

I now compute

$$M(C)W(C')|A_\mu> = M(C)Pe^{ig\oint_{C'} A_\mu dx^\mu}|A_\mu> =$$

$$Tr\{\Omega(2\pi)Pe^{ig\oint_{C'} A_\mu dx^\mu}\Omega^{-1}(0)|A_\mu^{\Omega(C)}>\} = e^{2\pi i \frac{\Phi}{N}}Pe^{ig\oint_{C'} A_\mu dx^\mu}|A_\mu^{\Omega(C)}>,$$ \hspace{1cm} (2.7)

and

$$W(C')M(C)|A_\mu> = Pe^{ig\oint_{C'} A_\mu dx^\mu}|A_\mu^{\Omega(C)}>.$$ \hspace{1cm} (2.8)

Putting (2.7) and (2.8) together, I finally get

$$W(C')M(C) = e^{ig\Phi}M(C)W(C'),$$ \hspace{1cm} (2.9)

where $\Phi = 2\pi n/N$. (2.3) represents a kind of braiding relation taking in account the linkage between $C$ and $C'$. Up to now all is well, since I have worked in the
Hamiltonian formalism at constant time. But going to a four dimensional Lagrangian description I get into troubles since the linking of two curves is well defined only in three dimensions. Using the fourth dimension the two curves can be unlinked. The way out of this is to replace the linkage between the two curves with the linkage between one of the two curves and the surface, $\Sigma$, bounded by the other curve, being this linkage well defined in four dimension. The consequence is that the surface acquires a physical meaning and that the effective action gets proportional to it. This is the origin of the well-known area and perimeter laws in QCD. The options at our disposal for the phases of QCD are

- Higgs phase
  \[
  \langle W(C') \rangle \approx e^{-L(C')} \quad ; \quad \langle M(C) \rangle \approx e^{-\Sigma(C)}.
  \]

- Confinement phase
  \[
  \langle W(C') \rangle \approx e^{-\Sigma(C')} \quad ; \quad \langle M(C) \rangle \approx e^{-L(C)}.
  \]

- Partial Higgs phase
  \[
  \langle W(C) \rangle \approx \langle M(C) \rangle \approx e^{-\Sigma(C)}.
  \]

- Coulomb phase
  \[
  \langle W(C) \rangle \approx \langle M(C) \rangle \approx e^{-L(C)}.
  \]

3 $N = 2$ Four Dimensional SYM

In this section I will briefly review some recent work on $N = 2$ SYM trying not to burden my account with too many details which can be found in one of the many excellent reviews that have appeared so far in literature. The starting point is
the Lagrangian density of $N = 2$ SUSY. In terms of $N = 1$ superfields, the $N = 2$ multiplet contains a vector and a chiral multiplet

$$V = (\lambda^a_{\alpha_1}, A^a_{\mu}, D^a) \quad ; \quad \Phi = (\phi^a, \lambda^a_{\alpha_2}, F^a).$$

(3.1)

The Lagrangian density of $N = 2$ SYM (I choose $SU(2)$ as the gauge group out of simplicity) is the same of that of the $N = 1$ theory coupled to matter in the adjoint representation of the gauge group

$$\mathcal{L} = 2Tr\left\{\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + i\bar{\lambda}A\lambda + (D_\mu\phi)^\dagger(D^\mu\phi) + g^2[\phi, \phi^\dagger]^2 + \left(\frac{ig}{\sqrt{2}}[\phi^\dagger, \lambda_A]\lambda_B e^{AB} + h.c.\right)\right\}. \quad (3.2)$$

The terms in the Lagrangian can be labelled as kinetic terms (those containing covariant derivatives), couplings of Yukawa type (those trilinear in the fields) and those giving rise to the potential

$$V_{pot} = g^2[\phi, \phi^\dagger]^2. \quad (3.3)$$

The potential $V_{pot}$ has a very peculiar property: it has flat directions that is a continuum of points for which the potential attains its minimum, which has to be zero for the theory to be SUSY. This is a characteristic of SUSY theories at large. In field theory minima are points in general like those in the two wells of a Higgs like potential. In (3.3) every gauge rotated scalar field will give a minimum: the minima of the theory are a manifold and not isolated points. The condition $V_{pot} = 0$ is satisfied by a normal field, which is conveniently chosen to be

$$\phi^b = a\delta^{b3}, \quad (3.4)$$

where $b$ is a gauge index. Different values of the expectation value $a$, will lead to different values for the masses of the gauge bosons and hence to theories with different vacua. The space of these different vacua is called the classical moduli space. One of our tasks will be of giving a full description of the quantum moduli
space, that is, to compute the quantum potential of the theory. A manifold is conveniently described upon the introduction of an appropriate set of coordinates. It is very natural to take as a coordinate of the classical moduli space

$$u = \frac{1}{2} Tr \phi^2 = \frac{1}{2} a^2,$$  \hspace{1cm} (3.5)$$

the simplest gauge invariant combination of the expectation value. The quantum version of the moduli space will obviously use as a coordinate the quantum expectation value of $Tr \phi^2$.

I need now some more $N=2$ technicalities: in first place let us write down a $N=2$ superfield, $\Psi$, in terms of $N=1$ superfields $\Phi, \mathcal{W}, \mathcal{G}$

$$\Psi = \Phi(\bar{y}, \theta) + \sqrt{2} + \bar{\theta} \mathcal{W}(\bar{y}, \theta) + \frac{1}{2} \bar{\theta}^2 \mathcal{G}(\bar{y}, \theta),$$  \hspace{1cm} (3.6)$$

where $y = x + i \bar{\theta} \sigma \bar{\theta} + i \theta \sigma \theta$. A superfield is a convenient way to bookkeep the components of a SUSY multiplet: each member of the multiplet will be a component of the superfield. As in the multiplet there will appear fields of different naive dimensions (fermions and bosons) to write down an expansion of the superfield.

I introduce a fermionic basis for each supersymmetry. The advantage of this way of thinking is that a supersymmetric transformation is a translation in this new space (called superspace) in which there are bosonic and fermionic (as many as supersymmetries) directions $(\theta, \bar{\theta})$ in (3.6). A supersymmetric Lagrangian is now a Lagrangian which is translationally invariant in the superspace. The most generic SUSY $N=2$ Lagrangian is

$$L = \frac{1}{16 \pi} \Im \left[ \int d^2 \theta d^2 \bar{\theta} \mathcal{F}(\Psi) \right].$$  \hspace{1cm} (3.7)$$

(3.2) is now recovered setting $\mathcal{F} = \frac{\mathcal{G}}{2} \Psi^a \Psi^a$ as it can be checked by an explicit computation. Expanding $\Psi$ around the $N=1$ superfield $\Phi$, (3.7) becomes

$$L = \frac{1}{16 \pi} \Im \left[ \int d^2 \theta d^2 \bar{\theta} \left( e^{2gV} \Phi^a \Phi^a \left( e^{-2gV} \Phi^a \right)^a \frac{\partial \mathcal{F}}{\partial \Phi^a} \right) \right].$$  \hspace{1cm} (3.8)$$

\footnote{Remember that all the terms of the expansion must have the same naive dimension.}
where $V$ is the $N = 1$ vector field. The vacuum expectation value (3.4) gives, via the Higgs mechanism, a mass to the $W^{\pm}$ bosons of the theory, while the $Z^{0}$ stays massless because the scalar is in the adjoint representation of the gauge group. It is then possible to decouple the massive and massless particles of the theory to study the effective Lagrangian

$$L = \frac{1}{16\pi^3} \left[ \int d^2 \theta \frac{\partial^2 F}{\partial \Phi \partial \Phi} WW + \int d^2 \theta d^2 \bar{\Phi} i \frac{\partial F}{\partial \Phi} \right], \quad (3.9)$$

which represents the contribution of the massless states only. As our gauge group was $SU(2)$ the only massless degree of freedom has a $U(1)$ symmetry.

Before proceeding further I now take a little time to explain the physical meaning of the effective action (3.9). Take, as an example, QED with a cut-off $\Lambda_0$. The Lagrangian is

$$L_0 = \bar{\psi} (i\partial \cdot \gamma - e_0 A \cdot \gamma - m_0) \psi - \frac{1}{2} (F_{\mu \nu})^2. \quad (3.10)$$

The idea is that if I use a new cut-off $\Lambda \ll \Lambda_0$ the new Lagrangian can be obtained from that of (3.10) by adding a certain number of terms which represent new interactions [11]. For example, from the one loop diagram, arising from the scattering of two fermions, of momenta $p, p' \ll \Lambda$, off the external field, I get

$$T = -e_0 \int_{\Lambda}^{\Lambda_0} \frac{d^4 k}{(2\pi)^4} \frac{1}{k^2} \bar{u}(p') \gamma^\mu \frac{1}{(p' - k) \cdot \gamma - m_0} A(p' - p) \cdot \gamma \frac{1}{(p - k) \cdot \gamma - m_0} \gamma^\mu u(p)$$

$$\simeq -\frac{ie_0^3}{24\pi^2} \ln \left( \frac{\Lambda}{\Lambda_0} \right) \bar{u}(p') A(p' - p) \cdot \gamma u(p). \quad (3.11)$$

This contribution can be taken care of by adding to the Lagrangian, the term

$$\delta L = -\frac{e_0^3}{24\pi^2} \ln \left( \frac{\Lambda}{\Lambda_0} \right) \bar{\psi} A \cdot \gamma \psi, \quad (3.12)$$

which amounts to a renormalization of the electric charge. Other terms of higher order in $p/\Lambda$ arising from (3.11) and coming from other diagrams may be added to the effective Lagrangian (in fact there is an infinite number of such terms) [11]. Some...
of them will be non renormalizable, given their naive dimensions. In this approach this does not raise any concern. The Lagrangian represents the theory at a certain energy scale given by the cut-off. This is the same thing of what happens in strong interactions when, to describe the theory, I use a $\sigma$-model, where the fundamental fields are the mesons, instead of using the standard QCD Lagrangian. Coming back to our problem, (3.2) is the analogous of (3.10) that I have written as (3.8) to allow for the next step. Going to an energy scale (cut-off) much lower than the masses of the particles corresponds to use the effective Lagrangian (3.9). The task is now to explicitly build it. Trying to do it by computing the relevant diagrams as in the previous example, is hopeless. Luckily enough the symmetries of the theory and an educated ansatz will be enough to do the job as we will see later on.

I now want to convince the reader that the functional $F(\Phi)$ is in reality a multi-valued function so that, to build it, all I have to do is to know its monodromies around its singular points. In terms of its component fields, (3.9) is

$$L = \frac{1}{4\pi} 3 \left[ F'' | \partial_\mu \phi |^2 - i F'' \psi \sigma^\mu \partial_\mu \bar{\psi} - \frac{1}{4} F'' F_{\mu\nu} F^{\mu\nu} + \ldots \right],$$

(3.13)

where I have written only the terms which will be necessary for our discussion. From (3.13) I see that $\tau = F''$ behaves as the metric of a $\sigma$-model and that it has to be positive as it multiplies the kinetic terms. Then $\tau$ is a holomorphic function which must be positive. From a well-known theorem in complex analysis it then has to be a constant. This happens in the classical case, but I have seen that in the effective action, (3.9), it is not true anymore: therefore the coupling $\tau$ which appears in (3.9) must be a multi-valued function. The monodromies of this function will be dictated by the type of physics to be described.

We have seen in the second chapter that the picture of confinement as a dual Meissner effect, requires at least an electric and a magnetic phase which are connected by a duality transformation. Let us build now this transformation in our case. The commutation relations of the SUSY charges of the $N = 2$ algebra lead to
the remarkable formula \[12\]

\[ M = a|q + ig|, \tag{3.14} \]

where \( M \) is the mass of the particle and \( q, g \) its electric and magnetic charge. As this relation comes out of the algebra, it is valid both at the classical and quantum level. Moreover we see that using the Dirac quantization rule, \( qg = 4\pi \), (3.14) is left invariant. In reality the dual transformation I am after is more involved that the Dirac rule. It is in fact possible to show \[13\] that the appropriate electric coupling of the theory is

\[ \tau = \left( \frac{4\pi i}{q_0} + \frac{\theta}{2\pi} \right), \tag{3.15} \]

where \( \theta \) is the coupling I have to add to the theory to take into account the effect of instantons. If I define \( q = nq_0, g = mg_0 \), where \( q_0, g_0 \) are the fundamental units of charge, and redefine \( a = aq_0, a_D = \tau a \), then (3.14) becomes

\[ M = |an + a_D m|. \tag{3.16} \]

(3.16) describes a lattice on whose sites are the charges. The symmetry group of this lattice is \( SL(2, \mathbb{Z}) \), whose most general transformation is

\[ \tau' = \frac{a\tau + b}{c\tau + d}, \tag{3.17} \]

with \( a, b, c, d \in \mathbb{Z} \). The generators of \( SL(2, \mathbb{Z}) \) are

\[ T : \quad \tau \mapsto \tau + 1 \]

\[ S : \quad \tau \mapsto -\frac{1}{\tau}. \tag{3.18} \]

The transformation \( S \) is the analogous of the dual transformation given by the Dirac rule in which strong and weak coupling are exchanged. In order to implement these transformations in our Lagrangian (3.9), I need the quantum expression for \( a_D \). If classically \( \mathcal{F} = 1/2\tau a^2 \) then

\[ a_D = \tau a = \frac{\partial \mathcal{F}}{\partial a}, \tag{3.19} \]

\[ ^4 \text{Electric here as to be interpreted as the coupling of the theory appearing in the Lagrangian describing the electric phase.} \]
which I take to be valid in the quantum domain once $F$ is substituted by its quantum expression. If I extend the definition (3.19) to the entire superfield of which $a$ is the scalar component, I get $\Phi_D = F'(\Phi)$. At the level of the Lagrangian, the transformations $S$ is given by

$$
\begin{pmatrix}
\Phi_D \\
\Phi
\end{pmatrix} \mapsto \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} \Phi_D \\
\Phi
\end{pmatrix} = \begin{pmatrix} \Phi \\
\Phi_D
\end{pmatrix}.
$$

(3.20)

Transforming the Lagrangian according to (3.20) and using the Legendre transform

$$
F_D(\Phi_D) = F(\Phi) - \Phi \Phi_D,
$$

(3.21)

I recover a new Lagrangian which has the same functional form of the original one with the electric fields exchanged with the magnetic ones and a new magnetic coupling $\tau_D = -1/\tau$. The $T$ transformation is given by

$$
\begin{pmatrix}
\Phi_D \\
\Phi
\end{pmatrix} \mapsto \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \Phi_D + b\Phi \\
\Phi
\end{pmatrix} = \begin{pmatrix} \Phi \\
\Phi_D
\end{pmatrix},
$$

(3.22)

where $b \in \mathbb{Z}$, and it leaves the Lagrangian invariant.

I am now ready for the computation of the effective action. The strategy is very simple. I give an ansatz for the moduli space to make it obey the type of physics I want to describe. I then compute the monodromies of $F$ around the singularities of the moduli space. Knowing these singularities I find $a, a_D$. So what is the minimum number of singularities I can have? The moduli space is described by a complex variable, so it is two-dimensional. Then the simplest Riemann surface I can build, on which $F$ is single-valued is the thrice punctured sphere or the torus. One of the singularities will be given by the weakly coupled (that is around $a = \infty$) Higgs phase; in this case I can exhibit the form of $F$. One other phase will be the weakly coupled magnetic phase (around $a_D \simeq 0$). The theory around this point has to look like a magnetic QED, that is as a $U(1)$ theory in which the fundamental fields are the monopole and a dual photon. This is implicit in the idea of duality for which to create a magnetic condensate I need a magnetic Higgs mechanism. It is remarkable
that in this description the monopole naturally comes to have the right mass for the Higgs mechanism, without having to introduce special gauges [14]. In the weakly coupled electric phase $\mathcal{F}$ looks like

$$\mathcal{F} = \frac{i}{2\pi} a^2 \ln \frac{a^2}{\Lambda^2} + \sum_{k=1}^{\infty} \mathcal{F}_k \left( \frac{\Lambda}{a} \right)^4 k a^2. \quad (3.23)$$

The first term in the r.h.s. is the perturbative contribution due to the $U(1)$ anomaly, while the sum is given by the instanton contribution. In the $a \to \infty$ limit, the perturbative sector dominates the non-perturbative one. Then if $u \to \exp\{2\pi i\}u$, the monodromy is given by

$$\left( \begin{array}{c} a_D \\ a \end{array} \right) = \left( \begin{array}{cc} -1 & 2 \\ 0 & -1 \end{array} \right) \left( \begin{array}{c} a_D \\ a \end{array} \right) = M_{\infty} \left( \begin{array}{c} a_D \\ a \end{array} \right) = \left( \begin{array}{c} -a_D + 2a \\ -a \end{array} \right). \quad (3.24)$$

In the $U(1)$ magnetic phase I will have that the $\beta$ function of the theory is

$$\beta(g) = -\frac{g^3}{4\pi^2}, \quad (3.25)$$

which goes to zero for $g \to 0$. The monodromy around this point is then

$$\left( \begin{array}{c} a_D \\ a \end{array} \right) = \left( \begin{array}{cc} 1 & 0 \\ -2 & 1 \end{array} \right) \left( \begin{array}{c} a_D \\ a \end{array} \right) = M_1 \left( \begin{array}{c} a_D \\ a \end{array} \right) = \left( \begin{array}{c} a_D \\ a - 2a_D \end{array} \right). \quad (3.26)$$

The monodromy around the third singularity (that I postulate to be a dyon with one unit of electric and magnetic charge) is given by the group relation

$$M_{\infty} = M_1 M_{-1}. \quad (3.27)$$

Finding the $a, a_D$ with these monodromy is now an exercise in complex analysis, whose solution is

$$a = \frac{\sqrt{2}}{\pi} \int_{-1}^{1} \frac{\sqrt{x_u}}{\sqrt{x^2 - 1}} dx, \quad \quad \quad a_D = \frac{\sqrt{2}}{\pi} \int_{1}^{u} \frac{\sqrt{x_u}}{\sqrt{x^2 - 1}} dx. \quad (3.28)$$

Before concluding this chapter, I want to show that in the magnetic phase, the monopole field develops a vacuum expectation value. As I have already said, in this
phase the Lagrangian is given by a QED type Lagrangian

\[ L = \frac{1}{16\pi} \mathcal{I} \left[ \int d^2\theta \partial^2 F_D \partial^2 \Phi_D + \int d^2\theta d^2\bar{\theta} \Phi_D \partial^2 F_D \right] + \int d^2\theta d^2\bar{\theta} [\bar{\Theta} e^{-2gV_0} \Theta + \Theta e^{-2gV_0} \bar{\Theta}] + \int d^2\theta [\sqrt{2}\Phi_D Q\bar{Q} + mU(\Phi)]. \] (3.29)

All the fields with the subscript \( D \) are the dual of those appearing in (3.9). The second line contains the monopole superfields \( Q = (q, \psi_Q, F_Q) \), \( \tilde{Q} = (\tilde{q}, \tilde{\psi}_Q, \tilde{F}_Q) \) with the typical matter interaction. At last, the third line contains the only possible form of an interaction which is compatible with \( N = 2 \) SUSY \([15]\) and a mass term which breaks \( N = 2 \) into \( N = 1 \). As we will see shortly, this term is necessary to develop a vacuum expectation value. As (3.29) is an effective Lagrangian, according to our previous discussion, multiplying the mass term there is a generic non-renormalizable function \( U(\Phi) \) instead of the standard \( \Phi^2 \) term. The potential of this theory in components is

\[ V = \sqrt{2}[F_D q\tilde{q} + a_D F_Q q\bar{q} + a_D q F\tilde{Q}] + mU'(a_D)F_D + h.c. \]
\[ + F_Q F_D + F_Q F\tilde{Q} + |q|^2 D + |\tilde{q}|^2 D + 3\tau_2 F_D F_D + \frac{1}{2} D^2. \] (3.30)

To find the minimum of the potential I just minimize with respect to the auxiliary fields. As the potential has to be zero to conserve SUSY I see that \( F_Q = F_{\tilde{Q}} = F_D = D = 0 \). Plugging this value back into the minimization with respect to the auxiliary fields, I find the minimum of the potential for

\[ q = \left( \frac{m}{\sqrt{2}} \frac{dU(a_D)}{da_D} \right)^2 |_{a_D=0} \neq 0, \] (3.31)

which is the sought for result. I point out that the expectation value for the monopole, goes to zero for \( m \to 0 \), as I remarked before.

\[ ^5 \text{I remind the reader who is not expert in SUSY, that there are two Weyl fields in the description of SUSY QED to reproduce the Dirac spinor appearing in the standard QED Lagrangian.} \]
4 Conclusions and Overlook

The work I have just described, has triggered an intensive activity in the last years. Before attempting a rough description of these developments, I would like to comment on the physical relevance of the results I have discussed: this is the first example of a four dimensional gauge theory in which an almost complete analysis of the perturbative and non-perturbative sector is carried out. In the past similar analysis were performed on $O(N)$ or $Z_N$ Ising type models, whose similarity with QCD was much less evident than for SUSY YM models. In this lecture, all the ingredients that seem to be important for confinement are at work: duality, massless monopoles, phases of QCD etc. Moreover, for the first time, an extensive control of the non-perturbative sector has been achieved. All of this has been possible because of SUSY and of the peculiar place that $N = 2$ has among all SUSY models. The $N = 1$ model has in fact a perturbative sector which is by far more complicated than the one discussed here. At the same time $N = 4$ has a trivial non-perturbative sector. $N = 2$ seems to have that right mixture that makes it rich enough to show the interesting phenomena but not too constrained to become trivial. On the relevance of SUSY for physics I won’t comment at all for lack of time. I just want to say that if hints of SUSY are not going to be found in the next experiments, most of the work done in the last twenty years is in peril, since SUSY is now obiquitous in most of theoretical physics.

The developments of the ideas exposed in this lecture, may be divided into two main streams: studies of duality and applications to string theory, on which I will not comment, and further studies in global $N = 2$. In first place the results presented here have been extended to generic gauge groups [16]. Then the generality of the ansatzes used for the solution presented [17] here and its relation with microscopic instanton calculus [18] have been checked. It is remarkable that the quantum symmetry of the theory is sufficient to make the solution
I have discussed here the only possible one and to give results which are in agreement with instanton computations. Also the non-holomorphic sector of the theory has been recently investigated and the preliminary results seem to be encouraging [19]. Finally there have also been attempts to extend the results to $N = 0$ [20].

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