Second Quantized Scalar QED in Homogeneous Time-Dependent Electromagnetic Fields

Sang Pyo Kim

Department of Physics, Kunsan National University, Kunsan 573-701, Korea
Yukawa Institute for Theoretical Physics, Kyoto University, Kyoto 606-8502, Japan and
Center for Relativistic Laser Science, Institute for Basic Science, Gwangju 500-712, Korea

(Dated: March 26, 2014)

We formulate the second quantized scalar quantum electrodynamics in homogeneous, time-dependent electromagnetic fields, in which the Hamiltonian for a charged scalar field is an infinite system of decoupled time-dependent oscillators for electric fields but of coupled time-dependent oscillators for magnetic fields. We then employ the quantum invariant method to find various quantum states for the charged field. For time-dependent electric fields, a pair of quantum invariant operators for each oscillator plays the role of the time-dependent annihilation and creation operators, constructs the exact quantum states, and gives the vacuum persistence amplitude as well as the pair-production rate. We also find the quantum invariants for the coupled oscillators for the charged field in time-dependent magnetic fields and advance a perturbation method when the magnetic fields change adiabatically. Finally the quantum state and pair production is discussed when a time-dependent electric field is present in parallel to the magnetic field.

PACS numbers: 11.15.Tk, 12.20.Ds, 12.20.-m, 13.40.-f

I. INTRODUCTION

Recent development of intense strong laser sources has brought intensive study of quantum motions of charged particles in electromagnetic fields [1]. In astrophysics, magnetars, strongly magnetized neutron stars, have magnetic fields over the critical strength [2]. The quantum states of charged particles in such strong electromagnetic fields provide an essential ingredient to understand the vacuum structure and pair production. The interaction of virtual pairs of the Dirac sea with a strong external electromagnetic field makes the vacuum polarized and leads to a nonlinear effective action beyond the Maxwell theory. In particular, the complex effective action in an electric field implies the vacuum instability due to pair production of charged particles. Hence, intense lasers have been proposed a feasible tool to probe the vacuum structure via pair production and vacuum polarization (for review and references, see Refs. [3, 4]).

In this paper we formulate the second quantized scalar quantum electrodynamics (QED) in homogeneous, time-dependent electromagnetic fields and employ the invariant operator method to find the quantum states of charged scalars. The Hamiltonian from the field action in a homogeneous, time-dependent electric field is a system of infinite number of time-dependent, decoupled oscillators, while it is equivalent to another system of infinite number of coupled oscillators in a time-dependent magnetic field with or without a parallel electric field. At the level of the first quantization, the Klein-Gordon equation in the vector potential separates into each Fourier mode for a time-dependent electric field [5], while it does not separate into each harmonic wave function corresponding to an instantaneous Landau state as shown in Ref. [6]. In fact, the Klein-Gordon equation in a time-dependent magnetic field in the two-component first order formalism [6] is equivalent to the Wheeler-DeWitt equation for the Friedmann-Robertson-Walker universe minimally coupled to a massive scalar field [7, 8]. Interestingly, the third quantized universe coupled to a massless scalar is equivalent to a system of infinite number of decoupled oscillators with time-dependent mass and frequency [9].

The functional Schrödiner picture provides a second quantized field theory, which extends the time-dependent Schrödinger equation for quantum mechanical systems to quantum fields [10, 11]. Thus, in the second quantized scalar QED, the quantum theory is the functional Schrödinger equation with time-dependent Hamiltonians for a charged scalar field in external electromagnetic fields. In the first case of time-dependent electric fields, the Gaussian

*Electronic address: sangkim@kunsan.ac.kr
wave functional can be directly found in terms of the covariance [12]. In the Fourier decomposition, the functional Schrödinger equation becomes the Schrödinger equation for decoupled, time-dependent oscillators and the quantum state of the field is the product of the time-dependent wave function for each oscillator. It has long been known that an oscillator with time-dependent frequency and/or mass has a quantum invariant, known as the Lewis-Riesenfeld invariant, whose eigenstate provides an exact solution of the Schrödinger equation up to a time-dependent phase factor [13]. Hence, in the former case of electric fields, we may employ the time-dependent annihilation and creation operators, also quantum invariants, and construct not only excited states for each time-dependent oscillator [14,15] but also a thermal state [17,18]. In the second case of magnetic fields, however, the Hamiltonian for the functional Schrödinger equation is equivalent to coupled, time-dependent oscillators and hence, we should employ the invariants for coupled, time-dependent oscillators [19,20].

The quantum invariant method for time-dependent oscillators is very useful in constructing various quantum states from the vacuum state, from excited states to coherent states and even to thermal states (for review and references, see Refs. [23,24]). By analogy with time-independent oscillators, the time-dependent annihilation and creation operators for time-dependent oscillators, quantum invariants linear in momentum and position operators, may be used to find the Fock space of all the number states, which are the exact solutions for the time-dependent Schrödinger equation [14,17]. Furthermore, it can be readily used in constructing thermal states and coherent thermal states, which are also the exact quantum states [17,18]. In scalar QED in time-dependent electric fields, the time-dependent vacuum state leads to the pair-production rate [23,24] and also to the in-out scattering matrix between the remote past and future, which yields the renormalized one-loop effective action [5]. The quantum invariant method can also be used to describe the quantum motion of charged particles in strong electromagnetic fields. In the case of the adiabatically changing magnetic fields, the invariants may be found for coupled, time-dependent oscillators by treating the off-diagonal Hamiltonian as perturbation for the diagonal Hamiltonian, decoupled time-dependent oscillators. We advance the time-dependent perturbation theory to find the improved states. The method is valid for the time-dependent magnetic fields parallel to electric fields.

The organization of this paper is as follows. In Sec. II we formulate the second quantized scalar QED in homogeneous, time-dependent electric fields or magnetic fields prescribed by vector potentials. The Hamiltonian from the field action in the electric fields is decomposed by Fourier modes and expressed as a sum of decoupled, time-dependent oscillators. In the time-dependent magnetic fields the Hamiltonian is decomposed both by Fourier modes along the longitudinal direction and by Landau labels in the transverse plane that diagonalize the transverse part of the Hamiltonian. In Sec. III we study the time-dependent Schrödinger equation for the quadratic Hamiltonian in the extended phase of positions and momenta and search invariants for the field. In Sec. IV we find the time-dependent annihilation and creation operators and construct the Fock space for the charged scalar field in time-dependent electric fields and discuss pair production and the vacuum persistence amplitude. In Sec. V we extend the quantum invariant method to find the time-dependent annihilation and creation operators for the charged field in time-dependent magnetic fields. We advance perturbation method to find the improved quantum states beyond the Landau states when the magnetic fields change adiabatically. It is shown that the Hamiltonian has the same algebraic structure even when electric field is added in parallel to the magnetic field and the same form of invariants. In Sec. VI we discuss the physical implications of the result of this paper.

II. SECOND QUANTIZED SCALAR QED

We formulate the second quantized QED for a charged, spinless scalar field in homogeneous, time-dependent electromagnetic fields, first by expressing the Hamiltonian from the field action through a spectral method and then by finding quantum states via the functional Schrödinger equation. For that purpose, we consider the action for a spinless scalar with charge $q$ and mass $m$ [in units of $\hbar = c = 1$ and the spacetime signature $\eta_{\mu\nu} = (+, -, -, -)$]

$$S = \int dtd^3x \left[ \eta^{\mu\nu} \left( \partial_\mu + iqA_\mu \right) \phi^* \left( \partial_\mu - iqA_\mu \right) \phi - m^2 \phi^* \phi \right]$$

in an electromagnetic field given by the four-vector

$$E = \nabla A_0 - \frac{\partial A}{\partial t}, \quad B = \nabla \times A.$$  

Introducing the conjugate momenta for each $\phi$ and $\phi^*$

$$\pi = \frac{\partial L}{\partial \dot{\phi}} = \dot{\phi}^* + iqA^0 \phi^*, \quad \pi^* = \frac{\partial L}{\partial \dot{\phi}^*} = \dot{\phi} - iqA^0 \phi,$$

and the Fock space of all the number states, which are the exact solutions for the time-dependent Schrödinger equation up to a time-dependent phase factor [13]. Hence, in the former case of electric fields, we may employ the time-dependent annihilation and creation operators, also quantum invariants, and construct not only excited states for each time-dependent oscillator [14,15] but also a thermal state [17,18]. In the second case of magnetic fields, however, the Hamiltonian for the functional Schrödinger equation is equivalent to coupled, time-dependent oscillators and hence, we should employ the invariants for coupled, time-dependent oscillators [19,20].

The quantum invariant method for time-dependent oscillators is very useful in constructing various quantum states from the vacuum state, from excited states to coherent states and even to thermal states (for review and references, see Refs. [23,24]). By analogy with time-independent oscillators, the time-dependent annihilation and creation operators for time-dependent oscillators, quantum invariants linear in momentum and position operators, may be used to find the Fock space of all the number states, which are the exact solutions for the time-dependent Schrödinger equation [14,17]. Furthermore, it can be readily used in constructing thermal states and coherent thermal states, which are also the exact quantum states [17,18]. In scalar QED in time-dependent electric fields, the time-dependent vacuum state leads to the pair-production rate [23,24] and also to the in-out scattering matrix between the remote past and future, which yields the renormalized one-loop effective action [5]. The quantum invariant method can also be used to describe the quantum motion of charged particles in strong electromagnetic fields. In the case of the adiabatically changing magnetic fields, the invariants may be found for coupled, time-dependent oscillators by treating the off-diagonal Hamiltonian as perturbation for the diagonal Hamiltonian, decoupled time-dependent oscillators. We advance the time-dependent perturbation theory to find the improved states. The method is valid for the time-dependent magnetic fields parallel to electric fields.

The organization of this paper is as follows. In Sec. II we formulate the second quantized scalar QED in homogeneous, time-dependent electric fields or magnetic fields prescribed by vector potentials. The Hamiltonian from the field action in the electric fields is decomposed by Fourier modes and expressed as a sum of decoupled, time-dependent oscillators. In the time-dependent magnetic fields the Hamiltonian is decomposed both by Fourier modes along the longitudinal direction and by Landau labels in the transverse plane that diagonalize the transverse part of the Hamiltonian. In Sec. III we study the time-dependent Schrödinger equation for the quadratic Hamiltonian in the extended phase of positions and momenta and search invariants for the field. In Sec. IV we find the time-dependent annihilation and creation operators and construct the Fock space for the charged scalar field in time-dependent electric fields and discuss pair production and the vacuum persistence amplitude. In Sec. V we extend the quantum invariant method to find the time-dependent annihilation and creation operators for the charged field in time-dependent magnetic fields. We advance perturbation method to find the improved quantum states beyond the Landau states when the magnetic fields change adiabatically. It is shown that the Hamiltonian has the same algebraic structure even when electric field is added in parallel to the magnetic field and the same form of invariants. In Sec. VI we discuss the physical implications of the result of this paper.

$$S = \int dtd^3x \left[ \eta^{\mu\nu} \left( \partial_\mu + iqA_\mu \right) \phi^* \left( \partial_\mu - iqA_\mu \right) \phi - m^2 \phi^* \phi \right]$$

in an electromagnetic field given by the four-vector

$$E = \nabla A_0 - \frac{\partial A}{\partial t}, \quad B = \nabla \times A.$$  

Introducing the conjugate momenta for each $\phi$ and $\phi^*$

$$\pi = \frac{\partial L}{\partial \dot{\phi}} = \dot{\phi}^* + iqA^0 \phi^*, \quad \pi^* = \frac{\partial L}{\partial \dot{\phi}^*} = \dot{\phi} - iqA^0 \phi,$$
we obtain the Hamiltonian for the field
\[ H(t) = \int d^3x \left[ \pi^* \pi + i q A^0 (\pi \phi - \pi^* \phi) + (\partial_k + i q A_k^\parallel) \phi^* (\partial_k - i q A_k^\parallel) \phi + m^2 \phi^* \phi \right]. \]  
(4)

Then, the quantum dynamics is governed by the functional Schrödinger equation
\[ i \frac{\partial}{\partial t} \Psi(t, \phi, \phi^*) = \hat{H}(t) \Psi(t, \phi, \phi^*). \]  
(5)

In this paper we consider only the vector potential for a homogeneous, time-dependent electric and magnetic field along a fixed direction of the form
\[ A_\parallel(t) = - \int_{-\infty}^t dt' E_\parallel(t'), \quad A_\perp(t, \mathbf{x}_\perp) = \frac{1}{2} B(t) \times \mathbf{x}. \]  
(6)

We further assume that \( E_\parallel(-\infty) = 0 \) and \( B(-\infty) = B_0 \) so that the initial state is either the Minkowski vacuum or the Landau state, respectively.

Firstly, in the case of time-dependent electric field, we decompose the fields by Fourier modes as
\[ \phi(t, \mathbf{x}) = \int \frac{d^3k}{(2\pi)^3} \phi_k(t) e^{i \mathbf{k} \cdot \mathbf{x}}, \quad \phi^*(t, \mathbf{x}) = \int \frac{d^3k}{(2\pi)^3} \phi_k^*(t) e^{-i \mathbf{k} \cdot \mathbf{x}} \]  
(7)

and obtain the time-dependent Hamiltonian
\[ H(t) = \int \frac{d^3k}{(2\pi)^3} \left[ \pi^*_k \pi_k + \omega_k^2(t) \phi_k^* \phi_k \right], \]  
(8)

where \( \pi_k = \dot{\phi}_k^*, \pi_k^* = \dot{\phi}_k \), and the time-dependent frequencies
\[ \omega_k^2(t) = (k_\parallel - q A_\parallel(t))^2 + k_\perp^2 + m^2. \]  
(9)

The Heisenberg equation
\[ \ddot{\phi}_k(t) + \omega_k^2(t) \phi_k(t) = 0, \]  
(10)

is the corresponding Fourier mode of the Klein-Gordon equation, which explains the equivalence between the quantum invariant method in this paper and the conventional canonical quantum field theory. As mentioned in Introduction the quantum invariant method has the merit of diversity of quantum states.

Secondly, in the case of homogeneous, time-dependent magnetic fields along the \( z \)-direction, after Fourier-decomposing the fields along the longitudinal direction, we find the Lagrangian
\[ L(t) = \int d^2x_\perp \left[ \hat{\phi}^*_k (x_\perp) \dot{\phi}_k (x_\perp) - \phi_k^* (x_\perp) \left( H_\perp(t) + k_\perp^2 + m^2 \right) \phi_k (x_\perp) \right], \]  
(11)

where
\[ H_\perp(t) = \mathbf{p}_\perp^2 + \left( \frac{q B(t)}{2} \right) \mathbf{x}_\perp^2 - q B(t) L_z \]  
(12)

is the Hamiltonian transverse to the magnetic field and \( L_z \) is the angular momentum from the orbital motion of the charge. Following Ref. [6], we diagonalize the Hamiltonian [12] as
\[ \hat{H}_\perp(t) = q B(t) \left[ 2 \hat{c}_\perp^\dagger (t) \hat{c}_\perp (t) + 1 \right], \]  
(13)

where the time-dependent annihilation and creation operators
\[ \hat{c}_\perp(t) = \frac{1}{\sqrt{2}} (\hat{a}_x(t) - i \hat{a}_y(t)), \quad \hat{c}_\perp^\dagger(t) = \frac{1}{\sqrt{2}} (\hat{a}_x^\dagger(t) + i \hat{a}_y^\dagger(t)) \]  
(14)

are constructed by the annihilation operators \( \hat{a}_x(t) \) and \( \hat{a}_y(t) \) for the \( x \)-component and \( y \)-component of the oscillator
\[ H_{\perp0}(t) = \mathbf{p}_\perp^2 + \left( \frac{q B(t)}{2} \right) \mathbf{x}_\perp^2. \]  
(15)
and introducing the extended phase space variables the Hamiltonian can be rewritten in the quadratic form where the Hamiltonian matrix is 

\[ H(t) = \sum_{m} \sum_{n} \Omega_{mn}(t) \phi_{m} \phi_{n} \]

Note that the coupling matrix is antisymmetric, \( \Omega^{T} = -\Omega \).

Thus, omitting the longitudinal momentum for simplicity, we have the expansion

\[ \Phi_{k_{\perp}}(x_{\perp}, t) = \sum_{n} \phi_{n}(t) \Phi_{n}(x_{\perp}, t), \]

where \( \Phi_{n}(x_{\perp}, t) = (x_{\perp}|n, t) \) is the wave function for the \( n \)-th Landau level. Finally, we obtain the Hamiltonian

\[ H(t) = \int \frac{dk_{\perp}}{2\pi} \left[ \sum_{n} (\pi_{n}^{*} \pi_{n} + \omega_{n}^{2} \phi_{n}^{*} \phi_{n}) + \sum_{m,n} (\pi_{m}^{*} \Omega_{mn} \phi_{n}^{*} + \pi_{m} \Omega_{mn} \phi_{n}) \right], \]

where

\[ \pi_{n} = \dot{\phi}_{n} + \sum_{m} \phi_{m}^{*} \Omega_{mn} = \dot{\phi}_{n} - \sum_{m} \Omega_{mn} \phi_{m}^{*}, \]

\[ \pi_{n}^{*} = \dot{\phi}_{n}^{*} + \sum_{m} \phi_{m} \Omega_{mn} = \dot{\phi}_{n}^{*} - \sum_{m} \Omega_{nm} \phi_{m}, \]

and the time-dependent Landau energies

\[ \omega_{n}^{2}(t) = |qB(t)|(2n + 1) + m^{2} + k_{z}^{2}. \]

III. TIME-DEPENDENT SCHRODINGER EQUATION

The Hamiltonian (8) or (18) is an infinite sum of time-dependent oscillators. Using a compact notation, in which \( \alpha, \beta \) stand for \( k, k' \) for electric fields or \( (m, k_{z}), (n, k'_{z}) \) for magnetic fields such that \( \sum_{\alpha} = \int d^{3}k/(2\pi)^{3} \) or \( \sum_{n} = \int dk_{z}/(2\pi) \), and introducing the extended phase space variables

\[ Z_{\alpha} = \left( \begin{array}{c} \pi_{\alpha} \\ \phi_{\alpha}^{*} \end{array} \right), \quad Z_{\alpha}^{*} = \left( \begin{array}{c} \pi_{\alpha}^{*} \\ \phi_{\alpha} \end{array} \right), \]

the Hamiltonian can be rewritten in the quadratic form

\[ H(t) = \sum_{\alpha \beta} Z_{\alpha}^{*} \mathbf{H}_{\alpha \beta}(t) Z_{\beta}, \]

where the Hamiltonian matrix is

\[ \mathbf{H}_{\alpha \beta}(t) = \left( \begin{array}{cc} \delta_{\alpha \beta} & 0 \\ 0 & \omega_{\alpha}^{2} \delta_{\alpha \beta} \end{array} \right). \]
for the electric field and

\[ H_{\alpha\beta}(t) = \begin{pmatrix} \delta_{\alpha\beta} & \Omega_{\alpha\beta} \\ -\Omega_{\alpha\beta} & \omega^2_{\alpha}\delta_{\alpha\beta} \end{pmatrix} \tag{25} \]

for the magnetic field. Both matrices (24) and (25) are Hermitian, \( H^\dagger = H \), and also make the Hamiltonian (23) Hermitian.

The quantization of the Hamiltonian (23) follows from the commutation relations

\[ [\hat{\phi}_\alpha, \hat{\pi}_\beta] = i\delta_{\alpha\beta}, \quad [\hat{\phi}_\alpha^*, \hat{\pi}_\beta^*] = i\delta_{\alpha\beta}, \tag{26} \]

and all other commutators vanish. In other words, the commutation relations hold in the extended phase space such that

\[ [\hat{Z}_\alpha, \hat{Z}_\beta^*] = \begin{pmatrix} 0 & -i\delta_{\alpha\beta} \\ i\delta_{\alpha\beta} & 0 \end{pmatrix}, \quad [\hat{Z}_\alpha, \hat{Z}_\beta] = [\hat{Z}_\alpha^*, \hat{Z}_\beta^*] = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}. \tag{27} \]

Then, the quantum evolution is governed by the time-dependent Schrödinger equation

\[ i\frac{\partial}{\partial t}\ket{\Psi(t)} = \sum_{\alpha\beta} \hat{Z}_{\alpha}^\dagger H_{\alpha\beta}(t) \hat{Z}_\beta \ket{\Psi(t)}. \tag{28} \]

In order to find the quantum states for Eq. (28), we employ the quantum invariant method, which satisfies the Liouville-von Neumann equation

\[ i\frac{\partial \hat{I}(t)}{\partial t} + [\hat{I}(t), \hat{H}(t)] = 0. \tag{29} \]

Then, the exact solution to Eq. (28) is given by an eigenstate of the invariant up to a time-dependent phase factor

\[ \ket{\Psi(t)} = \sum_\lambda C_\lambda e^{-i\int^t dt' \langle \lambda, t' | \hat{H}(t') - i\partial/\partial t' | \lambda, t' \rangle} \ket{\lambda, t}, \tag{30} \]

where \( C_\lambda \) is a constant and \( \lambda \) is a constant eigenvalue corresponding to the eigenvalue problem of the invariant operator

\[ \hat{I}(t)|\lambda, t\rangle = \lambda|\lambda, t\rangle. \tag{31} \]

Considering the symmetric form of the Hamiltonian (23) from two fields \( \phi \) and \( \phi^* \) and their conjugate momenta, we may search for the invariants in the extended phase space (22) of the form

\[ \hat{I}(t) = (U(t), V(t)) \frac{\hat{Z} + \hat{Z}^*}{\sqrt{2}}, \tag{32} \]

where \( U(t) \) and \( V(t) \) are matrix-valued functions, carrying indices of \( \alpha, \beta \), and are determined by Eq. (29).

**IV. INVARIANT OPERATORS IN TIME-DEPENDENT ELECTRIC FIELDS**

In the case of electric fields the Hamiltonian (8) is a system of decoupled, time-dependent oscillators, for which we may use a pair of invariants for each oscillator as the time-dependent annihilation and creation operators and construct the Fock space of number states [14–18]. From Eq. (32) we introduce the first class invariant for each \( \alpha \)

\[ \hat{A}_\alpha(t) = \frac{i}{\sqrt{2}} \left[ \varphi_\alpha^*(\hat{\pi}_\alpha^* + \hat{\pi}_\alpha) - \varphi_\alpha^* (\hat{\phi}_\alpha^* + \hat{\phi}_\alpha) \right], \tag{33} \]

and the second class invariant

\[ \hat{A}_\alpha^\dagger(t) = -\frac{i}{\sqrt{2}} \left[ \varphi_\alpha (\hat{\pi}_\alpha^* + \hat{\pi}_\alpha) - \varphi_\alpha^* (\hat{\phi}_\alpha^* + \hat{\phi}_\alpha) \right]. \tag{34} \]
Here, $\varphi_{\alpha}$ is a complex solution to the mode equation \((10)\) and satisfies the Wronskian condition for quantization

$$\varphi_{\alpha}(t)\dot{\varphi}_{\alpha}^*(t) - \varphi_{\alpha}^*(t)\dot{\varphi}_{\alpha}(t) = i. \quad (35)$$

Note that the invariants \((33)\) and \((34)\)

$$\hat{A}_{\alpha}(t) = \frac{1}{\sqrt{2}}(\hat{a}_{\alpha}(t) + \hat{b}_{\alpha}(t)),$$

$$\hat{A}_{\beta}^\dagger(t) = \frac{1}{\sqrt{2}}(\hat{a}_{\beta}^\dagger(t) + \hat{b}_{\beta}^\dagger(t)), \quad (36)$$

can be expressed in terms of the annihilation and creation operators for particles

$$\hat{a}_{\alpha}(t) = i(\varphi_{\alpha}^*(t)\hat{\pi}^*_{\alpha} - \varphi_{\alpha}(t)\hat{\phi}_{\alpha}),$$

$$\hat{a}_{\alpha}^\dagger(t) = -i(\varphi_{\alpha}(t)\hat{\pi}^*_{\alpha} - \varphi_{\alpha}^*(t)\hat{\phi}_{\alpha}), \quad (37)$$

and those for antiparticles

$$\hat{b}_{\alpha}(t) = i(\varphi_{\alpha}^*(t)\hat{\pi}^*_{\alpha} - \varphi_{\alpha}(t)\hat{\phi}_{\alpha}^*),$$

$$\hat{b}_{\alpha}^\dagger(t) = -i(\varphi_{\alpha}(t)\hat{\pi}^*_{\alpha} - \varphi_{\alpha}^*(t)\hat{\phi}_{\alpha}^*). \quad (38)$$

The commutators hold

$$[\hat{A}_{\alpha}(t), \hat{A}_{\beta}^\dagger(t)] = [\hat{a}_{\alpha}(t), \hat{a}_{\beta}^\dagger(t)] = [\hat{b}_{\alpha}(t), \hat{b}_{\beta}^\dagger(t)] = \delta_{\alpha\beta}, \quad (39)$$

while other commutators vanish. Thus $\hat{A}_{\alpha}(t)$ annihilates one particle-antiparticle pair while $\hat{A}_{\alpha}^\dagger(t)$ creates the pair with the same quantum $\alpha$, and the field has the canonical representation

$$\hat{\phi}(t, x) = \sum_{\alpha} \left[ \varphi_{\alpha}(t)\hat{a}_{\alpha}(t) + \varphi_{\alpha}^*(t)\hat{b}_{\alpha}^\dagger(t) \right]. \quad (40)$$

Using the commutation relations \((39)\), we may construct the time-dependent particle-antiparticle states

$$|k_{\alpha}, \bar{l}_{\beta}, t\rangle = \frac{(\hat{a}_{\alpha}^\dagger(t))^{k_{\alpha}}(\hat{b}_{\beta}^\dagger(t))^{l_{\beta}}}{\sqrt{k_{\alpha}!l_{\beta}!}}|0_{\alpha}, 0_{\beta}, t\rangle, \quad (41)$$

where bars denote antiparticles and the vacuum state is given by

$$\hat{a}_{\alpha}(t)|0_{\alpha}, 0_{\beta}, t\rangle = \hat{b}_{\beta}(t)|0_{\alpha}, 0_{\beta}, t\rangle = 0. \quad (42)$$

These states are orthonormal among themselves at the equal time

$$\langle m_{\alpha}, \bar{n}_{\beta}, t|k_{\gamma}, \bar{l}_{\delta}, t\rangle = \delta_{m_{\alpha}k_{\gamma}}\delta_{\bar{n}_{\beta}l_{\delta}}. \quad (43)$$

However, each multiple particle-antiparticle state is not a solution to Eq. \((28)\), since $\hat{a}_{\alpha}(t)$ and $\hat{b}_{\alpha}(t)$ cannot separately become invariants for Eq. \((29)\). Instead, the particle-antiparticle pair constitutes invariants $\hat{A}_{\alpha}(t)$ and $\hat{A}_{\alpha}^\dagger(t)$. Hence, the vacuum state for a given $\alpha$ should be annihilated by the zero particle-antiparticle operator

$$\hat{A}_{\alpha}(t)|0_{\alpha}, t\rangle = 0, \quad (44)$$

and at the same time have the zero particle and antiparticle number, respectively,

$$\langle 0_{\alpha}, t|\hat{a}_{\alpha}^\dagger(t)\hat{a}_{\alpha}(t)|0_{\alpha}, t\rangle = \langle 0_{\alpha}, t|\hat{b}_{\alpha}^\dagger(t)\hat{b}_{\alpha}(t)|0_{\alpha}, t\rangle = 0. \quad (45)$$

The zero particle and antiparticle content \((43)\) excludes another null state \((44)\) with one particle or antiparticle

$$|l_{\alpha}, t\rangle_{\text{spurious}} = \sqrt{\frac{1}{2}} \left(|l_{\alpha}, \bar{l}_{\alpha}, t\rangle - |0_{\alpha}, \bar{l}_{\alpha}, t\rangle \right). \quad (46)$$
Indeed, \(|0, t\rangle := |0, \tilde{0}, t\rangle\) is the unique state satisfying Eqs. (41) and (45) and constructs the time-dependent vacuum state

\[
|0, t\rangle = \prod_{\alpha} |0, \tilde{0}, t\rangle.
\]

(47)

The first excited state is obtained by acting \(\hat{A}_\alpha^\dagger(t)\) on the vacuum

\[
|1_\alpha, t\rangle := \hat{A}_\alpha^\dagger(t)|0, t\rangle = \sqrt{\frac{1}{2}} \left(|1_\alpha, \tilde{0}, t\rangle + |0_\alpha, \tilde{1}, t\rangle\right).
\]

(48)

The one particle-antiparticle state (48) is the superposition of a particle with \(k\) and an antiparticle with \(-k\) from Eq. (7) but is not \(|1_\alpha, \tilde{1}, t\rangle\), as expected naïvely. Similarly, acting \(n\)-times \(\hat{A}_\alpha^\dagger(t)\) excites \(n\) particle-antiparticle state for the quantum \(\alpha\)

\[
|n_\alpha, t\rangle = \frac{\hat{A}_\alpha^\dagger(t)}{\sqrt{n_\alpha!}} |0, t\rangle = \sum_{k_\alpha=0}^{n_\alpha} \frac{n_\alpha!}{2^n(n_\alpha - k_\alpha)!k_\alpha!} |n_\alpha - k_\alpha, \tilde{k}_\alpha, t\rangle,
\]

(49)

in which the statistical weight comes from the number of ways for arranging indistinguishable \(n_\alpha\) particle-antiparticles. Any general particle-antiparticle state is a linear combination of (49).

It is one of the merits of the invariant operators that the exact quantum state can also be given by the eigenstates of another invariant \(\hat{N}_\alpha(t) = \hat{A}_\alpha^\dagger(t)\hat{A}_\alpha(t)\) (10). Hence, the total number of initial particle-antiparticle with the quantum \(\alpha\) contained in the vacuum state at \(t\) is the same as that of particle-antiparticle at \(t\) contained in the initial vacuum state at \(t_0\):

\[
\hat{N}_\alpha(t) = \langle 0, t | \hat{A}_\alpha^\dagger(t_0)\hat{A}_\alpha(t_0) | 0, t\rangle = \langle 0, t_0 | \hat{A}_\alpha^\dagger(t_0)\hat{A}_\alpha(t_0) | 0, t_0\rangle = |\hat{\phi}_\alpha(t)|^2|\phi_\alpha(t_0)|^2 + |\phi_\alpha(t)|^2|\hat{\phi}_\alpha(t_0)|^2.
\]

(50)

Here, \(\hat{A}_\alpha(t_0)\) and \(\hat{A}_\alpha^\dagger(t_0)\) are the operators (37) and (38) evaluated at \(t_0\). Note that \(\hat{A}_\alpha^\dagger(t)\hat{A}_\alpha(t)\) and \(\hat{A}_\alpha(t)\hat{A}_\alpha^\dagger(t)\) form an S(1,1) algebra, which leads to the quantum master equation (21) and the evolution operator and therefrom the pair-production rate (26). Furthermore, the one-loop effective action is obtained from the scattering matrix between the out-vacuum and the in-vacuum under the action of the electric field modulo the scattering matrix in the absence of the field (5).

\[
\pi^s \frac{d}{d\epsilon} \mathcal{L}_{\text{eff}}(E) = \prod_{\alpha,T=\infty} \frac{(0_\alpha, T/2) |0_\alpha, -T/2\rangle(E)}{(0_\alpha, T/2) |0_\alpha, -T/2\rangle(E = 0)}.
\]

(51)

V. INVARIANT OPERATORS IN TIME-DEPENDENT MAGNETIC FIELDS

The charged scalar field in a time-dependent magnetic field along the fixed direction has the Hamiltonian (13) or (20), which may be written in the form

\[
H_\epsilon(t) = H_D(t) + \epsilon H_O(t).
\]

(52)

Here, \(\epsilon\) is an expansion parameter for the perturbation theory and set \(\epsilon = 1\), and

\[
H_D = \sum_{\alpha} \pi^s_\alpha \pi_\alpha + \omega^2_\alpha(t) \phi^2_\alpha \phi_\alpha
\]

(53)

is the diagonal Hamiltonian while

\[
H_O = \sum_{\alpha, \beta} \pi^s_\alpha \pi^s_\beta + \pi^s_\alpha \Omega_{\alpha \beta} \phi^2_\beta + \pi_\alpha \Omega_{\alpha \beta} \phi_\beta
\]

(54)

is the off-diagonal Hamiltonian. The Hamiltonian (52) is an infinite system of coupled oscillators due to the off-diagonal Hamiltonian (54), which induces continuous transitions among Landau levels as shown in Eq. (16). In the non-relativistic theory a charged particle in time-dependent magnetic fields has similarly a finite number of coupled oscillators, to which the quantum invariant method has been applied (13, 27) (for coupled time-dependent oscillators, see also Refs. (19, 22). The infinite degrees of freedom for the charged field necessarily involve renormalizing physical quantities, in strong contrast to finite degrees of the freedom for the particle.
The quantum motion of the charged field may be classified according to the relative ratio \(||H_D||/||H_O||\) for some appropriate measure. For instance, a measure has been introduced that the integrated rate of the change of Landau levels to the dynamical phase for any time interval \(\mathcal{R}_\alpha = \frac{\int_{t_0}^{t_0+\Delta t} dt' |\Omega_{\alpha\beta}(t')|}{\int_{t_0}^{t_0+\Delta t} dt' \omega_{\alpha}(t')}\)\(^{(55)}\) classifies the quantum motions (states) into the adiabatic change, the sudden change, and the nonadiabatic change. Now, these motions are classified by the relative ratio: (i) the adiabatic change when \(||H_D|| \gg ||H_O||\), static fields being an extreme limit of \(||H_O|| = 0\), (ii) the sudden change when \(||H_D|| \ll ||H_O||\), in which Landau levels change more rapidly than Landau energies, and (iii) the nonadiabatic change when \(||H_D|| \approx ||H_O||\), in which the change of Landau states is comparable to the dynamical phase of Landau energies. The exact quantum motion may not be found for general time-dependent magnetic fields, except for some limiting cases such as a constant magnetic field and a suddenly changing field from one constant value to another. Hence, we propose another scheme which applies perturbation theory to the Hamiltonian \((52)\), by treating \(H_D\) as the unperturbed Hamiltonian and \(H_O\) as the perturbed one when \(||H_D|| \gg ||H_O||\) or vice versa.

From Eq. \((54)\) the pair of invariants \((52)\) in the extended phase space \((52)\)
\[
\hat{I}(\pm)(t) = \frac{1}{\sqrt{2}} \left[ U(\pm)\alpha\beta(\hat{\pi}_\beta^* + \hat{\pi}_\beta) + V(\pm)\alpha\beta(\hat{\phi}_\beta^* + \hat{\phi}_\beta) \right],
\] may be found by solving the matrix-valued differential equations
\[
\dot{U}(\pm) + U(\pm)\Omega + V(\pm) = 0, \quad \dot{V}(\pm) - U(\pm)\omega^2 + V(\pm)\Omega = 0.
\] Here, the positive (negative) sign denotes the positive (negative) frequency solution. The two equations \((57)\) equal to the transpose of the mode equation in the vector form \((51)\).
\[
\dot{U}(\pm) + 2U(\pm)\Omega + U(\pm)(\omega^2 + \Omega^2) = 0.
\] However, the merit of the quantum invariant method is not solving the classical field equation but the diversity of quantum states to be constructed. In the two-component first order formalism the Cauchy problem has been reduced to solving Eq. \((21)\), whose solutions may not be known for generic magnetic fields, and requires a perturbation method. In a similar manner, finding the exact invariants \((50)\) is equivalent to solving Eq. \((58)\) or \((21)\). Therefore, we need a systematic method such as the perturbation theory to find the invariants or directly solve the time-dependent Schrödinger equation.

### A. Adiabatic Change

The field \(\phi\) is complex due to the longitudinal motion along the magnetic field in the four-dimensional spacetime, so we may quantize the time-dependent oscillators \((53)\) as for electric fields in Sec. \([V]\). Hence, the invariant operators are
\[
\hat{\phi}_\alpha(t) = \frac{i}{\sqrt{2}} \left[ \varphi_\alpha^*(\hat{\pi}_\alpha + \hat{\pi}_\alpha^*) - \varphi_\alpha(\hat{\phi}_\alpha^* + \hat{\phi}_\alpha^*) \right] = \frac{1}{\sqrt{2}} (\hat{a}_\alpha(t) + \hat{b}_\alpha(t)),
\]
\[
\hat{\phi}_\alpha^+(t) = -\frac{i}{\sqrt{2}} \left[ \varphi_\alpha^*(\hat{\pi}_\alpha + \hat{\pi}_\alpha^*) - \varphi_\alpha(\hat{\phi}_\alpha^* + \hat{\phi}_\alpha^*) \right] = \frac{1}{\sqrt{2}} (\hat{a}_\alpha^+(t) + \hat{b}_\alpha^+(t)).
\] Here, the auxiliary field \(\varphi_\alpha\) for \(\alpha = (n, k_z)\) is the complex solution to the mode equation
\[
\varphi_\alpha + \omega_\alpha^2 \varphi_\alpha = 0
\] with \(\omega_\alpha^2 = |qB| (2n + 1) + m^2 + k_z^2\) and \(\text{Wr}[\varphi_\alpha, \varphi_\alpha^*] = i\). Then, the diagonal Hamiltonian becomes
\[
H_D = \sum_\alpha \left[ (\varphi_\alpha^* \varphi_\alpha + \omega_\alpha^2 \varphi_\alpha^* \varphi_\alpha) (\hat{a}_\alpha^+ \hat{a}_\alpha + \hat{b}_\alpha^+ \hat{b}_\alpha) + (\varphi_\alpha^* \varphi_\alpha^* + \omega_\alpha \varphi_\alpha^* \varphi_\alpha) \hat{a}_\alpha^+ \hat{a}_\alpha + (\varphi_\alpha^* \varphi_\alpha^* + \omega_\alpha \varphi_\alpha^* \varphi_\alpha) \hat{b}_\alpha^+ \hat{b}_\alpha \right],
\] and the off-diagonal Hamiltonian takes the form
\[
H_O = \sum_{\alpha\beta} \left[ (\varphi_\alpha^* \varphi_\beta - \varphi_\alpha \varphi_\beta^*) \Omega_{\alpha\beta} \hat{a}_\alpha^+ \hat{b}_\beta + (\varphi_\alpha \varphi_\beta^* - \varphi_\alpha^* \varphi_\beta) \Omega_{\alpha\beta} \hat{a}_\alpha \hat{b}_\beta \right].
\]
The invariant operators (59) satisfy Eq. (29) for the diagonal Hamiltonian (61) when \( \varphi_{\alpha} \) is the solution to Eq. (60).

In the adiabatic change, we find the quantum states for the unperturbed Hamiltonian (61) with the aid of the quantum invariant method and then improve the unperturbed states by the Hamiltonian (62). The unperturbed states consist of the time-dependent vacuum state

\[
\hat{a}_\alpha(t)|0_\alpha, t\rangle_0 = \hat{b}_\alpha(t)|0_\alpha, t\rangle_0 = 0,
\]

and the excited states

\[
|n_\alpha, t\rangle_0 = \frac{(\hat{A}_\alpha(t))^{n_\alpha}}{\sqrt{n_\alpha!}}|0_\alpha, t\rangle_0.
\]

For each \( \alpha \) the time-dependent vacuum state (63) leads to the Landau state and thereby the time-dependent vacuum state for the field

\[
|0, t\rangle_0 = \prod_\alpha |0_\alpha, t\rangle_0.
\]

Furthermore, the general excited states are

\[
|N, t\rangle_0 = \prod_\mathcal{N} \hat{A}^\dagger_\mathcal{N}(t)\sqrt{|N\rangle_0!}|0, t\rangle_0,
\]

where \( \{\mathcal{N}\} = (n_0, \ldots, n_\alpha, \ldots) \), \( \hat{A}^{\dagger_\mathcal{N}} = \hat{A}^{\dagger_0}_{n_0} \cdots \hat{A}^{\dagger_\alpha}_{n_\alpha} \cdots \), and \( |\mathcal{N}\rangle_0! = \prod_\alpha n_\alpha! \).

The quantum state of the field in a constant magnetic field \( B_0 \), though looks trivial since \( \Omega_{\alpha\beta} = 0 \), requires a careful understanding in contrast to that of a charged particle. Each Landau level has the solution

\[
\varphi_{\alpha} = \frac{1}{\sqrt{2\omega_{\alpha}}} e^{-i\omega_{\alpha}t}
\]

with the Landau energy (20). Then, the exact vacuum state (30) for the time-dependent Schrödinger equation is given by

\[
|0, t\rangle = \prod_\alpha e^{-i\omega_{\alpha}t}|0_\alpha\rangle.
\]

Note that \( \prod_\alpha |0_\alpha\rangle \) is the vacuum state associated with \( \Phi_\alpha(x_\perp, t) \) in Eq. (17), in which each Landau state is equally occupied. The scattering matrix between the out-vacuum and the in-vacuum gives the one-loop effective action

\[
(0, \frac{T}{2})|0, -\frac{T}{2}\rangle = e^{iV_\perp \sum_\alpha \omega_{\alpha}(B_0)} = e^{iV \mathcal{L}_{\text{eff}}(B_0)},
\]

where \( V \) and \( V_\perp \) are the three- and two-dimensional volumes of the problem. In fact, the summation of all the Landau energies equals to the one-loop effective action after renormalization.

For adiabatically changing magnetic fields, the scattering matrix between the out-vacuum and the in-vacuum provides the vacuum persistence amplitude

\[
(0)|0, \infty\rangle(0) = e^{i \int dx \mathcal{L}_{\text{eff}}},
\]

The non-unimodular amplitude \( |(0)|0, \infty\rangle(0)|^2 < 1 \) implies that the initial vacuum state is not stable against pair production and that the effective action obtains an imaginary part. The instability can be expected from the induced electric field \( E = -\partial A / \partial t \). However, the adiabatic vacuum state (65) is no longer a good approximation for rapidly changing magnetic fields because \( ||H_D|| \ll ||H_0|| \).

**B. Perturbation beyond Adiabatic Change**

In the time-dependent perturbation theory [28], the time-dependent vacuum state (65) does not lead to any quantum correction since

\[
(0)|0, t|\tilde{H}_0|0, t\rangle(0) = 0.
\]
It would be interesting to compare the Hamiltonian (52) with the Fourier-decomposed one for an interacting scalar field, for instance, the $\Phi^4$-theory [29]. We may look for the improved state

$$|N, t⟩ = \hat{U}(\hat{A}^\dagger(t), \hat{A}(t), t)|N, t⟩_0,$$

and solve the time-dependent Schrödinger equation

$$i \frac{\partial}{\partial t} \hat{U}(t) = \hat{H}_O(t) \hat{U}(t). \quad (73)$$

It is the property of the invariant operators that the perturbation $\hat{H}_O(t)$ is invariant in the interaction-like picture [29]. Thus, the evolution operator for Eq. (73) has the formal solution by the time-ordered integral

$$\hat{U}(t) = T \exp \left[-i \int_{t_0}^t dt' \hat{H}_O(t') \right], \quad (74)$$

and the leading approximation is

$$\hat{U}(1)(t) = e^{-i \int_{t_0}^t dt' \hat{H}_O(t')} \quad (75)$$

The vacuum persistence amplitude between the out-vacuum and the in-vacuum improved by Eq. (75) when the magnetic field changes slowly during a large period $T$ from one value at $-T/2$ to another at $T/2$ is

$$e^{iV \int dt' \mathcal{L}_{\text{det}}} = \langle 0, \frac{T}{2} | e^{i \int_{-T/2}^{T/2} dt' \hat{H}_O(t')} | 0, -\frac{T}{2} \rangle_0.$$ \quad (76)

### C. Electric Field Parallel to Magnetic Field

When a time-dependent electric field parallel to the magnetic field is present, the vector potential (6) may be used. Then, the Hamiltonian after the spectral decomposition (17) is given by

$$H(t) = \sum_\alpha (\pi^*_\alpha \pi_\alpha + \omega^2_\alpha \phi^*_\alpha \phi_\alpha) + \sum_{\alpha \beta} (\pi_\alpha \Omega_{\alpha \beta} \phi_\beta + \pi^*_\alpha \Omega_{\alpha \beta} \phi^*_\beta), \quad (77)$$

where $\alpha = (n, k_z)$ and

$$\omega^2_\alpha(t) = |qB(t)|(2n + 1) + m^2 + (k_z - qA_z(t))^2.$$ \quad (78)

The nonstationary nature of quantum states does not essentially change as far as the magnetic field is time-dependent or electric field is present. Note that the quantum motion becomes nonstationary even for a constant magnetic field due to the electric field, which is the reasoning behind pair production due to the electric field. In the case of the adiabatic change of the magnetic field, we may use the quantum states in Sec. VI A and the improved states in Sec. VI B, which will not be repeated here.

### VI. CONCLUSION

In this paper we have formulated the second quantized scalar QED in a homogeneous, time-dependent electromagnetic field, in which the quantum law is the functional Schrödinger equation with a time-dependent Hamiltonian from the field action. The Hamiltonian obtained from the spectral method is quadratic in position and momentum operators, which is an infinite sum of decoupled time-dependent oscillators in the time-dependent electric field but is another infinite sum of coupled time-dependent oscillators in the time-dependent magnetic field due to continuous transitions of Landau levels. The quantum law is thus the time-dependent Schrödinger equation for an infinite system of time-dependent oscillators.

We have then employed the quantum invariant method to find the quantum states for the charged scalar field, which makes use quantum invariants that satisfy the Liuoville-von Neumann equation with respect to the time-dependent Hamiltonian [13]. We have further used the time-dependent annihilation and creation operators, quantum invariants, and constructed the Fock space of all the excited states [14, 17]. In the case of time-dependent electric fields the quantum law reduces to finding the wave functions for each time-dependent oscillator, while in the case of
time-dependent magnetic fields the task is equivalent to solving the Schrödinger equation for coupled oscillators with time-dependent frequencies and couplings among different oscillators. We have sought the quantum invariants for scalar QED in the time-dependent magnetic fields.

The quantum invariants directly lead to the exact quantum states for decoupled or coupled time-dependent oscillators up to time-dependent phase factors. In particular, the time-dependent annihilation and creation operators construct not only the Fock of all the excited states but also generalized correlated states and thermal states. The two-dimensional Landau states have been studied in time-dependent magnetic fields in the non-relativistic theory [13, 27]. The second quantized scalar QED is a relativistic theory and is equivalent to an infinite number of time-dependent oscillators as explored in this paper. We have advanced a perturbation method based on the quantum invariants for the time-dependent magnetic fields since the solutions for the classical mode equations are not known in general. In particular, when the magnetic fields change adiabatically in the sense of the rate of the change of the Landau levels is smaller than the corresponding dynamical phases, the off-diagonal Hamiltonian responsible for the transitions among the time-dependent Landau levels could be treated as perturbation to the adiabatic, diagonal Hamiltonian from the Landau levels. The perturbation improves the time-dependent vacuum state of Landau levels.

The quantum state or wave functional for a field always involves some infinite quantity which should be regulated away through renormalization of physical quantities, such as the mass and charge of the particle. For instance, the one-loop effective action via the scattering matrix between the out-vacuum at the remote future and the in-vacuum in the remote past is given by the sum over all the quantum numbers, the three-momenta or the Landau levels and the longitudinal momenta. The renormalized action follows from an appropriate regularization scheme as for the Sauter-type electric field [5]. The renormalization problem also occurs for a scalar field in an expanding universe [30]. The renormalization problem in the second quantized QED may be studied model by model, which goes beyond the scope of this paper.

There are some interesting issues not handled in this paper but requiring further study. The external electromagnetic fields that have both temporal and spatial distribution require more advanced spectral method, which goes beyond the scope of this paper. Another interesting problem is a time-dependent magnetic field whose direction rotates around a fixed direction [31], and which may be used as a model for highly magnetized neutron stars. It would be also interesting to compare the second quantized QED in rotating electric fields with the recent Wigner function formalism [32]. Still another problem is the second quantized formulation of charged spin-1/2 fermions in the time-dependent magnetic fields, which will be addressed in future publication.

Acknowledgments

The author thanks Hyun Kyu Lee and Yongsung Yoon for helpful discussions on Landau states and the adiabatic theorem in magnetic fields and Lee Lindblom for useful discussions on the spectral method during the AP School on Gravitation and Cosmology at Academia Sinica, Taiwan, 2014. He also would like to thank Misao Sasaki and Takahiro Tanaka for the warm hospitality at Yukawa Institute for Theoretical Physics, Kyoto University, Don N. Page for the warm hospitality at University of Alberta, Jeremy S. Heyl for the warm hospitality at Pacific Institute for Theoretical Physics, University of British Columbia, and Toshiki Tajima for the warm hospitality at University of California, Irvine, where parts of this paper were done, respectively. This work was supported by the Research Center Program of IBS (Institute for Basic Science) in Korea.

[1] A. Di Piazza, C. Müller, K. Z. Hatsagortsyan and C. H. Keitel, “Extremely high-intensity laser interactions with fundamental quantum systems,” Rev. Mod. Phys. 84, 1177 (2012).
[2] A. K. Harding and D. Lai, “Physics of strongly magnetized neutron stars,” Rep. Prog. Phys. 69, 2631 (2006).
[3] G. V. Dunne, “New strong-field QED effects at extreme light infrastructure,” Eur. Phys. J. D 55, 327 (2009).
[4] H. Gies, “Strong laser fields as a probe for fundamental physics,” Eur. Phys. J. D 55, 311 (2009).
[5] S. P. Kim, H. K. Lee and Y. Yoon, “Effective action of QED in electric field backgrounds,” Phys. Rev. D 78, 105013 (2008).
[6] S. P. Kim, “Landau Levels of Scalar QED in Time-Dependent Magnetic Fields,” Ann. Phy. 344, 1 (2014).
[7] S. P. Kim, “Quantum mechanics of conformally and minimally coupled Friedmann-Roberston-Walker cosmology,” Phys. Rev. D 46, 3403 (1992).
[8] S. P. Kim, “Massive Scalar Field Quantum Cosmology,” The Universe, Vol. 1, 11 (2013) [arXiv:1304.7439].
[9] S. P. Kim, “Third Quantization and Quantum Universes,” Nucl. Phys. Proc. Suppl. 246-247, 68-75 (2014) [arXiv:1212.5355].
[10] K. Freese, C. T. Hill, and M. Mueller, “Covariant functional Schrödinger formalism and application to the Hawking effect,” Nucl. Phys. B255, 693 (1985).
[11] A. Guth and S.-Y. Pi, “Quantum mechanics of the scalar field in the new inflationary universe,” Phys. Rev. D 32, 1899 (1985).
[12] C. Kiefer, “Functional Schrödinger equation for scalar QED,” Phys. Rev. D 45, 2044 (1992).
[13] H. R. Lewis and W. B. Riesenfeld, “An Exact Quantum Theory of the Time-Dependent Harmonic Oscillator and of a Charged Particle in a Time-Dependent Electromagnetic Field,” J. Math. Phys. 10, 1458 (1969).
[14] I. A. Malkin, V. I. Man’ko and D. A. Trifonov, “Coherent States and Transition Probabilities in a Time-Dependent Electromagnetic Field,” Phys. Rev. D 2, 1371 (1970).
[15] J. K. Kim and S. P. Kim, “One-parameter squeezed Gaussian states of a time-dependent harmonic oscillator and the selection rule for vacuum states,” J. Phys. A 32, 2711 (1999).
[16] S. P. Kim and D. N. Page, “Classical and quantum action-phase variables for time-dependent oscillators,” Phys. Rev. A 64, 012104 (2001).
[17] S. P. Kim and C. H. Lee, “Nonequilibrium quantum dynamics of second order phase transitions,” Phys. Rev. D 62, 125020 (2000).
[18] S. P. Kim and D. N. Page, “Exact quantum-statistical dynamics of time-dependent generalized oscillators,” Phys. Lett. B 723, 393 (2013).
[19] P. G. Leach, “On the theory of time-dependent linear canonical transformations as applied to Hamiltonians of the harmonic oscillator type,” J. Math. Phys. 18, 1608 (1977).
[20] J-Y. Ji and J. Hong, “Heisenberg picture approach to the invariants and the exact quantum motions for coupled parametric oscillators,” J. Phys. A: Math. Gen. 31, L689 (1998).
[21] V. V. Dodonov, “Universal integrals of motion and universal invariants of quantum systems,” J. Phys. A: Math. Gen. 33, 7721 (2000).
[22] M-H. Lee, “Exact Schrödinger wavefunctions of N-coupled time-dependent harmonic oscillators,” J. Phys. A: Math. Gen. 34, 9475 (2001).
[23] V. V. Dodonov and V. I. Man’ko, “Invariants and the evolution of nonstationary quantum systems,” Proc. Lebedev Physics Institute Vol 183 (Commack: Nova Science, 1989)
[24] M. A. Lohe, “Exact time dependence of solutions to the time-dependent Schrödinger equation,” J. Phys. A: Math. Theor. 42, 035307 (2009).
[25] S. P. Kim and C. Schubert, “Nonadiabatic quantum Vlasov equation for Schwinger pair production,” Phys. Rev. D 84, 125028 (2011).
[26] S. P. Kim, H. W. Lee, and R. Ruffini, “Schwinger Pair Production in Pulsed Electric Fields,” arXiv:1207.5213 [hep-th].
[27] G. Fiore and L. Gouba, “Class of invariants for the two-dimensional time-dependent Landau problem and harmonic oscillator in a magnetic field,” J. Math. Phys. 52, 103509 (2011).
[28] A. Messiah, Quantum Mechanics (North-Holland Publishing, Amsterdam, 1970).
[29] S. P. Kim and F. C. Khanna, “Non-Gaussian Effects on Domain Growth,” arXiv:hep-ph/0011115 (unpublished).
[30] A. Ringwald, “Evolution Equation for the Expectation Values of a Scalar Field in Spatially Flat RW Universes,” Ann. Phys. 177, 129 (1987).
[31] A. Di Piazza and G. Calucci, “Pair production in a rotating strong magnetic field,” Phys. Rev. D 65, 125019 (2002).
[32] A. Blinne and H. Gies “Production in Rotating Electric Fields,” arXiv:1311.1678 [hep-ph].