LIPSCHITZ GEOMETRY AT INFINITY OF COMPLEX PLANE ALGEBRAIC CURVES

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1. Introduction

Abstract. We present a complete classification of the Lipschitz geometry at infinity of complex plane algebraic curves.

Definition 1.1. Let $(M, d)$ and $(M', d')$ be two metric spaces. A map $f : M \to M'$ is Lipschitz if there exists a real constant $c > 0$ such that

$$d'(f(x), f(y)) \leq cd(x, y) \text{ for all } x, y \in M.$$  

A Lipschitz map $f : M \to M'$ is called bilipschitz if its inverse exists and it is Lipschitz.

In this paper, all the subsets of $\mathbb{R}^n$ or $\mathbb{C}^n$ are considered equipped with the induced Euclidean metric.

Definition 1.2. Let $X \subset \mathbb{R}^n$ and $Y \subset \mathbb{R}^m$ be two subsets. We say that $X$ and $Y$ are bilipschitz equivalent at infinity if there exist compact subsets $K \subset \mathbb{R}^n$ and $\tilde{K} \subset \mathbb{R}^m$, and a bilipschitz map $\Phi : X \setminus K \to Y \setminus \tilde{K}$. The equivalence class of $X$ in this relation is called the Lipschitz geometry at infinity of $X$.

The above definition may be found in the article [1], where the authors proved among other things that a pure dimensional complex algebraic subset of $\mathbb{C}^n$ with the same Lipschitz geometry at infinity as an Euclidean space must be an affine linear space of $\mathbb{C}^n$.

The aim of this paper is to study the Lipschitz geometry of complex plane algebraic curves at infinity. The problem of classification of germ of complex analytic sets up to bilipschitz change of coordinates has been intensively studied in the last years. One of the recent works on this subject, Neumann and Pichon (see [9]) proved that two germs of plane complex curve are bilipschitz homeomorphic if only if they have the same topological type. For previous contributions, see [8] and [2].

Definition 1.3. Let $(C_1, p_1) \subset (S_1, p_1)$ and $(C_2, p_2) \subset (S_2, p_2)$ be two germs of complex curves on smooth surfaces. We say that $(C_1, p_1)$ and $(C_2, p_2)$ have the same topology type if there is a homeomorphism of germs $h : (S_1, p_1) \to (S_2, p_2)$ such that $h(C_1) = C_2$.

We denoted by $\mathbb{P}^2$ the projective plane. Let $[x : y : z] \in \mathbb{P}^2$ denote the subspace spanned by $(x, y, z)$, and let $\iota : \mathbb{C}^2 \hookrightarrow \mathbb{P}^2$ be the parametrization given by $\iota(x, y) = [x : y : 1]$. The line at infinity, denoted by $L_\infty$, is the complement of $\iota(\mathbb{C}^2)$ in $\mathbb{P}^2$. 


**Definition 1.4.** Let $f \in \mathbb{C}[x, y]$ be a polynomial of degree $n$. The *homogenization* of $f$ is the homogeneous polynomial $	ilde{f} \in \mathbb{C}[x, y, z]$ defined by

$$
\tilde{f}(x, y, z) = z^n f \left( \frac{x}{z}, \frac{y}{z} \right).
$$

Let $C$ be the algebraic curve with equation $f(x, y) = 0$. The projective curve $\tilde{C} = \{ [x : y : z] \in \mathbb{P}^2 : \tilde{f}(x, y, z) = 0 \}$ is called the *homogenization* of $C$. The *points at infinity* of $C$ are the elements of the intersection $\tilde{C} \cap L_\infty$.

Our main result is the following:

**Theorem 1.5.** Let $C$ be an algebraic complex plane curve. The Lipschitz geometry at infinity of $C$ determines and is determined by:

A) the number of points at infinity of $C$;

B) the embedded topological type of the germ of the curve $\tilde{C} \cup L_\infty$ at each point at infinity of $C$.

We organized the paper in the following way. In Section 2, we present definitions of Eggers-Wall and carousel tree. We also describe how one gets the Eggers-Wall tree from the carousel tree. Section 3 is devoted to prove that the Lipschitz geometry at infinity of an algebraic curve gives us the data in A) and B). The idea of the prove is a version at infinity of the so-called bubble trick argument developed by Neumann and Pichon in the paper [9]. In the last section, we prove that the data A) and B) determine the Lipschitz geometry at infinity of an algebraic plane curve. In order to do that, we consider two algebraic plane curves with the same data in A) and B). By using their Newton-Puiseux parametrization at infinity we provide a bilipschitz map between them.

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2. Plane curve germs and their Eggers-Wall and carousel trees

In this section we explain the basic notations and conventions used throughout the paper about reduced germs $C$ of curves on smooth surfaces. Then we define the Eggers-Wall tree and the carousel tree of such a germ relative to a smooth branch contained in it. The definition of Eggers-Wall tree which are given in this paper are the same present in [6]. Finally, we will describe how one gets the Eggers-Wall tree from the carousel tree. This process is also described in [9].

We recall some definitions and conventions about power series with positive rational exponents. Let $n$ be a positive integer, the ring $\mathbb{C}[[x^{1/n}]]$ consists of sequence $(A_k)_{k \in \mathbb{N}}$ of elements of $\mathbb{C}$. Let $\eta = (A_k)_{k \in \mathbb{N}} \in \mathbb{C}[[x^{1/n}]]$, we denote this element by

$$
\eta = \sum_{k=0}^{\infty} A_k x^{k/n}.
$$
The **exponents** of \( \eta \) are the numbers \( k/n \) such that \( A_k \neq 0 \). We denote the set of exponents of \( \eta \) by \( \mathcal{E}(\eta) \). The **order** of \( \eta \neq 0 \), denoted by \( \text{ord}_x \eta \), is the smallest exponent of \( \eta \). For technical reasons it is convenient to define the order of the zero to be \(+\infty\). The subgroup of \( n \)-th roots of \( 1 \) acts on \( \mathbb{C}[[x^{1/n}]] \) by the rule

\[
(\rho, \eta) \mapsto \eta(\rho \cdot x^{1/n}) := \sum_{k=0}^{\infty} A_k \rho^k x^{k/n}, \text{ where } \rho \text{ is a } n \text{-th root of } 1.
\]

All over this section, \( S \) denotes a complex manifold of dimension two. We fix a point \( O \in S \). All coordinate charts of this section are defined in a neighborhood of \( O \), moreover, the point \( O \) always has coordinate \((0,0) \in \mathbb{C}^2 \). A curve germ in \((S,O)\) is the zero set of a non-constant holomorphic function germ from \((S,O)\) to \((\mathbb{C},0)\). We denote by \((C,O)\) the germ of \( C \) at \( O \) and by \( \mathcal{O}_O \) the ring of holomorphic function germs at \( O \).

Any chart of \( S \) induces an isomorphism between \( \mathcal{O}_O \) and \( \mathbb{C}\{x,y\} \). Since \( \mathbb{C}\{x,y\} \) is factorial, \( \mathcal{O}_O \) is factorial. Let \( C \) be a complex curve with equation \( f = 0 \). Then \( f \) can be written as a product \( g_1^{n_1} \ldots g_k^{n_k} \), with \( g_1, \ldots, g_k \) irreducible, and the \( n_j \)'s are positive integers. The zero set of \( g_j \)'s are the **branches** of \( C \). When \( k = 1 \), we say that \( C \) is **irreducible**. The holomorphic function \( f \) is **reduced** if each \( n_j = 1 \).

We will always suppose all equations for curves are reduced. The curve \( C \) is said to be **smooth at** \( O \) if there is a neighborhood \( U \) of \( O \) in \( S \) such that \( C \cap U \) is a complex submanifold of \( U \).

The next definitions of this section depend on the choice of a smooth curve \( L \) at \( O \). In this section, we always choose a coordinate system \((x,y)\) such that \( L = \{x = 0\} \). Assume that a coordinate system \((x,y)\) is fixed. Let \( C \) be a curve on \( S \) and assume that \( A \) is a branch of \( C \) different from the curve \( L \). Relative to the system \((x,y)\), the branch \( A \) may be defined by a Weierstrass polynomial \( f_A \in \mathbb{C}\{x\}\{y\} \), which is monic, and of degree \( d_A \). Note that the degree \( d_A \) does not depend on the system of coordinates.

By the Newton-Puiseux Theorem, there exists a parametrization of \( A \) of the form \( \gamma_A(w) = (w^{d_A}, \eta_A(w)) \) where \( \eta_A(w) = \sum_{k>0} a_k w^k \in \mathbb{C}\{w\} \). Let \( n \) be the product of the degrees of the Weierstrass polynomials of the branches of \( C \) different from \( L \). We consider the formal power series \( \sum_{k=0} A_k x^{k/n} \in \mathbb{C}[[x^{1/n}]] \) where

\[
A_k = \begin{cases} a_{k+d_A}, & \text{if } n \text{ divides } kd_A \\ 0, & \text{otherwise.} \end{cases}
\]

We still denote by \( \eta_A \) the formal power series \( \sum_{k=0} A_k x^{k/n} \). The **Newton-Puiseux roots relative to** \( L \) of the branch \( A \) are the formal power series \( \eta_A(\rho \cdot x^{1/n}) \in \mathbb{C}[[x^{1/n}]] \), for \( \rho \) running through the \( n \)-th roots of \( 1 \).

Let \( \rho \in \mathbb{C} \) be a primitive \( n \)-root of unity, notice that there are only \( d_A \) Newton-Puiseux roots relative to \( L \) of the branch \( A \), namely

\[
\eta_A(\rho \cdot x^{1/n}), \ldots, \eta_A(\rho^{d_A} \cdot x^{1/n}).
\]

All the Newton-Puiseux roots relative to \( L \) of the curve \( A \) have the same exponents. Some of those exponents may be distinguished by looking at the differences of roots:

**Definition 2.1.** The **characteristic exponents relative to** \( L \) of the curve \( A \) are the \( x \)-orders \( \text{ord}_x(\eta_A - \eta_A') \) of the differences between distinct Newton-Puiseux roots relative to \( L \) of \( A \).
The characteristic exponents relative to $L$ of $A$ consist of exponents of $\eta_A$ which, when written as quotient of integers, need a denominator strictly bigger than the lowest common denominator of the previous exponents. That is: $\frac{1}{m}$ is characteristic exponent relative to $L$ of $A$ if and only if $N|\frac{1}{m} \not\in \mathbb{Z}$ where $N = \min\{N \in \mathbb{Z} : \mathcal{E}(\eta_A) \cap \{0, \frac{1}{2}\} \in \frac{1}{m}\mathbb{Z}\}$. By [6, Proposition 3.10] the characteristic exponents relative to $L$ do not depend on the coordinate system $(x, y)$, but only on the branch $L$.

The Newton-Puiseux roots relative to $L$ of the curve $C$ are the Newton-Puiseux roots relative to $L$ of its branches different from $L$. Let us denote by $\mathcal{I}_C$ the set of branches of $C$ which are different from $L$. Therefore, $C$ has $d_C := \sum_{A \in \mathcal{I}_C} d_A$ Newton-Puiseux roots relative to $L$.

**Example 2.2.** Let $L$ be the $y$-axis. Consider a plane curve $C$ whose branches $A$ and $B$ are parametrized by

$$\gamma_A(w) = (w^4, w^6 + w^7), \quad \gamma_B(w) = (w^2, w),$$

respectively. The Newton-Puiseux roots relative to $L$ of $A$ are

$$\eta_A(x^{1/8}) = x^{12/8} + x^{14/8}, \quad \eta_A(\rho x^{1/8}) = \rho^4 x^{12/8} + \rho^6 x^{14/8},$$

$$\eta_A(\rho^2 x^{1/8}) = x^{12/8} + \rho^4 x^{14/8}, \quad \eta_A(\rho^3 x^{1/8}) = \rho^4 x^{12/8} + \rho^2 x^{14/8},$$

where $\rho$ is a primitive 8-th root of unity. While the Newton-Puiseux roots relative to $L$ of $B$ are

$$\eta_B(x^{1/8}) = x^{4/8}, \quad \eta_B(\rho x^{1/8}) = \rho^4 x^{4/8}.$$  

The characteristic exponents relative to $y$-axis of $A$ are $3/2, 7/4$. The characteristic exponent of $B$ relative to $y$-axis is $1/2$.

We keep assuming that $A$ is a branch of $C$ different from $L$. The Eggers-Wall tree of $A$ relative to $L$ is a geometrical way of encoding the set of characteristic exponents, as well as the sequence of their successive common denominators:

**Definition 2.3.** The Eggers-Wall tree $\Theta_L(A)$ of the curve $A$ relative to $L$ is a compact oriented segment endowed with the following supplementary structures:

- an increasing homeomorphism $e_{L,A} : \Theta_L(A) \to [0, \infty]$, the exponent function;
- marked points, which are by definition the points whose values by the exponent function are the characteristic exponents of $A$, as well as the smallest end of $\Theta_L(A)$, labeled by $L$, and the greatest end, labeled by $A$.
- an index function $\mathcal{i}_{L,A} : \Theta_L(A) \to \mathbb{N}$, which associates to each point $P \in \Theta_L(A)$ the smallest common denominator of the exponents of a Newton-Puiseux root of $A$ which are strictly less than $e_{L,A}(P)$.

Let us consider now the case of a curve with several branches. In order to construct the Eggers-Wall tree in this case, one needs to know not only the characteristic exponents of its branches, but also the exponent of coincidence of its pairs of branches:

**Definition 2.4.** If $A$ and $B$ are two distinct branches of $C$, then their exponent of coincidence relative to $L$ is defined by:

$$k_L(A, B) := \max\{\text{ord}_x(\eta_A - \eta_B)\},$$
where η_A, η_B ∈ ℂ[[x^{1/n}]] vary among the Newton-Puiseux roots of A and B, respectively.

**Definition 2.5.** Let C be a germ of curve on (S, O). Let us denote by IC the set of branches of C which are different from L. The Eggers-Wall tree Θ_L(C) of C relative to L is the rooted tree obtained as the quotient of the disjoint union of the individual Eggers-Wall trees Θ_L(A), A ∈ IC, by the following equivalence relation. If A, B ∈ IC, then we glue Θ_L(A) with Θ_L(B) along the initial segments e_{L,A}^{-1}([0, k_L(A, B)]) and e_{L,B}^{-1}([0, k_L(A, B)]) by:

e_{L,A}^{-1}(α) ∼ e_{L,B}^{-1}(α), for all α ∈ [0, k_L(A, B)].

One endows Θ_L(C) with the exponent function e_L : Θ_L(C) → [0, ∞] and the index function i_L : Θ_L(C) → ℕ induced by the initial exponent functions e_{L,A} and i_{L,A} respectively, for A varying among the irreducible components of C different from L. The tree Θ_L(L) is the trivial tree with vertex set a singleton, whose element is labelled by L. If L is an irreducible component of C, then the marked point L ∈ Θ_L(L) is identified with the root of Θ_L(L) for any A ∈ IC. The set of marked points of Θ_L(C) is the union of the set of marked points of the Eggers-Wall tree of the branches of C and of the set of ramification points of Θ_L(C).

Again, the fact that in the previous notations Θ_L(C), e_L, i_L we mentioned only the dependency on L, and not on the coordinate system (x, y), comes from [6, Proposition 3.10].

**Example 2.6.** Consider again the curve of Example 2.2. One has K_L(A, B) = 1/2.

![Figure 1](image.png)

The carousel tree is a variant of the Eggers-Wall tree, but using all the Newton-Puiseux roots of C, not only one root for each branch. The name was introduced in [9] and it is inspired by the carousel geometrical model for the link of the curve C described in [3, Section 5.3].

**Definition 2.7.** Let C be a germ of curve on S. Let us denote by [d_C] the set {1, ..., d_C} and let η_j, j ∈ [d_C] the Newton-Puiseux roots relative to L of C. Consider the map ord_x : [d_C] × [d_C] → ℚ ∪ {∞}, (j, k) ↦ ord_x(η_j - η_k). The map ord_x
have the property that \(\text{ord}_{\mathcal{C}}(j,l) \geq \min\{\text{ord}_{\mathcal{C}}(j,k), \text{ord}_{\mathcal{C}}(k,l)\}\) for any triple \(j,k,l\). So for any \(q \in \mathbb{Q} \cup \{\infty\}\), the relation on the set \([d_C]\) given by \(j \sim_q k \iff \text{ord}_{\mathcal{C}}(j,k) \geq q\) is an equivalence relation. Name the elements of the set \(\text{ord}_{\mathcal{C}}([d_C])\) in ascending order: \(0 = q_0 < q_1 < \cdots < q_r = \infty\). For each \(i = 0, \ldots, r\) let \(G_{i,1}, \ldots, G_{i,\mu_i}\) be the equivalence classes for the relation \(\sim_{q_i}\). So \(\mu_r = d_C\) and the sets \(G_{r,j}\) are singletons while \(\mu_0 = \mu_1 = 1\) and \(G_{0,1} = G_{1,1} = [d_C]\). We form a tree with these equivalence classes \(G_{i,j}\) as vertices and edges given by inclusion relations: there is an edge between \(G_{i,j}\) and \(G_{i+1,k}\) if \(G_{i+1,k} \subseteq G_{i,j}\). The vertex \(G_{0,1}\) is the root of this tree and the singleton sets \(G_{r,j}\) are the leaves. We weight each vertex with its corresponding \(q_i\). The carousel tree relative to \(L\) is the tree obtained from this tree by suppressing valence 2 vertices: we remove each such vertex and amalgamate its two adjacent edges into one edge.

We will describe how one gets the Eggers-Wall tree from the carousel tree. This process is essentially the same process described in [9, Lemma 3.1]. At any vertex \(v\) of the carousel tree we have a weight \(q_v\) which is one of the \(q_i\)'s. Let \(d_v\) be the denominator of the \(q_v\) when \(q_v\) is written as a quotient of coprime integers.

The process of obtaining the Eggers-Wall tree from the carousel tree is an induction process in \(i\). First, we label the edge between \(G_{0,1}\) and \(G_{1,1}\) by 1. The subtrees cut off above \(G_{1,1}\) consist of groups of \(d_{G_{1,1}}\) isomorphic trees, with possibly one additional tree. We label the edge connecting \(G_{1,1}\) to this additional tree, if it exists, with 1, and then delete all but one from each group of \(d_{G_{1,1}}\) isomorphic trees. Finally, we label the remain edges contain \(G_{1,1}\) with \(\text{lcm}\{d_{G_{1,1}}, 1\}\).

Inductively, let \(v\) vertex with weight \(q_v\). Let \(v'\) be the adjacent vertex below \(v\) along the path from \(v\) up to the root vertex and let \(l_{v,v'}\) the label of the edge between \(v\) and \(v'\). The subtrees cut off above \(v\) consist of groups of \(\frac{\text{lcm}(d_v,l_{v,v'})}{l_{v,v'}}\) isomorphic trees, with possibly one additional tree. We label the edge connecting \(v\) to this additional tree, if it exists, with \(l_{v,v'}\), and then delete all but one from each group of \(\frac{\text{lcm}(d_v,l_{v,v'})}{l_{v,v'}}\) isomorphic trees below \(v\). Finally, we label the remain edges contain \(v\) with \(\text{lcm}\{d_v, l_{v,v'}\}\).

The resulting tree, with the \(q_v\) labels at vertices and the extra label on the edges is easily recognized as the Eggers-Wall tree relative to \(L\) of \(C\).

**Example 2.8.** We illustrate the above process using the Example

![Figure 2](image-url)
3. Lipschitz geometry at infinity determines A) and B)

In this section, we prove one direction of Theorem [4,5] which is: Lipschitz geometry of an algebraic curve determines A) and B). We introduce the asymptotic notations of Bachmann-Landau which are convenient for study of Lipschitz geometry (see [3] for a historical survey about these notation).

Definition 3.1. Let \( f, g : (\mathbb{R}, \infty) \to \mathbb{R}_+ \). We say

(1) \( f \) is big-Theta of \( g \), and we write \( f(t) = \Theta(g(t)) \), if there exists \( R_0 > 0 \) and a constant \( c > 0 \) such that \( \frac{1}{c} g(t) \leq f(t) \leq cg(t) \) for all \( t > R_0 \).

(2) \( f \) is small-o of \( g \), and we write \( f(t) = o(g(t)) \), if \( \limsup_{t \to \infty} \frac{f(t)}{g(t)} = 0 \).

Let \([a : b : 0]\) be a point at infinity of an algebraic complex plane curve \( C \). The linear subspace spanned by \((a, b)\) in \( \mathbb{C}^2 \) is the tangent line at infinity to \( C \) associated with \([a : b : 0]\) (see [1] and [4]).

Example 3.2. Consider the polynomial \( f(x, y) = y^2x - y \), and let \( C_\lambda \) the algebraic curve with equation \( f(x, y) + \lambda = 0 \) for \( \lambda \in \mathbb{C} \). One has \( \tilde{f}(x, y, z) = y^2x - yz^2 + \lambda z^3 \), and the points at infinity of \( C_\lambda \) are \([1 : 0 : 0]\) and \([0 : 1 : 0]\) and the tangent lines at infinity to \( C_\lambda \) are the coordinates axis.

Lemma 3.3. Let \( C \) be an algebraic complex plane curve, and let \( P : \mathbb{C}^2 \to \mathbb{C} \) be a linear projection whose kernel does not contain any tangent line at infinity to \( C \). Then there exist a compact set \( K \) and a constant \( M > 1 \) such that for each \( u, u' \in C \setminus K \), there is an arc \( \tilde{a} \) in \( C \setminus K \) joining \( u \) to a point \( u'' \) with \( P(u'') = P(u') \) and

\[
d(u, u') \leq \text{length}(\tilde{a}) + d(u'', u') \leq Md(u, u').
\]

Proof. After a linear change of coordinates if necessary, we may assume that \( P \) is the projection on the first coordinate and that the \( y \)-axis is not a tangent line at infinity to \( C \). Let \([1 : a_1 : 0], \ldots, [1 : a_m : 0]\) be the points at infinity of \( C \). For each \( i \), let \( B_{1i}, \ldots, B_{ki} \) be the branches of \((\tilde{C}, [1 : a_i : 0])\).

The open set \( U = \{[x : y : z] \in \mathbb{P}^2 : x \neq 0\} \) contains all the points at infinity of \( C \), so we can use the the coordinate chart \( \varphi : U \to \mathbb{C}^2 \) defined by \( \varphi([x : y : z]) = (z/x, y/x) \) to obtain Newton-Puiseux parametrization of the branch \( \varphi(B_{ij}) \) for each \( i \). Let \( \epsilon > 0 \) sufficiently small such that there exists Newton-Puiseux parametrization \( \gamma_{ij} : D_\epsilon \to \mathbb{C}^2 \) of \( \varphi(B_{ij}) \) given by

\[
\gamma_{ij}(w) = (w^{d_{ij}}, a_i + v_{ij}(w)),
\]

where \( D_\epsilon \) is the open disk of radius \( \epsilon \) centered at the origin and \( v_{ij} \in \mathbb{C}\{w\} \), \( v_{ij}(0) = 0 \). Let \( \Gamma_{ij} : D_\epsilon \setminus \{0\} \to \mathbb{C}^2 \) given by

\[
\Gamma_{ij}(w) = (\varphi^{-1} \circ \gamma_{ij})(w) = \left( \frac{1}{w^{d_{ij}}} \cdot \frac{a_i + v_{ij}(w)}{w^{d_{ij}}} \right).
\]

We will prove that the compact \( K = C \setminus \bigcup_{ij} \Gamma_{ij}(D_\epsilon \setminus \{0\}) \) satisfies the desired conditions.

We claim that there exists a constant \( c > 0 \) such that \( C \setminus K \) is a subset of the cone \( \{(x, y) \in \mathbb{C}^2 : |y| \leq c|x|\} \). Moreover, \( c \) may be chosen such that tangent space of \( C \setminus K \) at a point \( p \), denoted by \( T_pC \), is also a subset of the same cone.
a covering map. Moreover, Remark 3.4. In the above lemma, we prove that 
\( \alpha \) is bounded above and below by positive constants. In particular, for a non-constant arc 
and the constant 
where \( \eta_{ij} = \sup \frac{\|w_{ij}'(w) - d_{ij}v_{ij}(w)\|}{d_{ij}} \). Now, putting \( c = \max_{ij} \{\eta_{ij} + |a_i|\} \) we have 
\[ T_pC \subset \{ (x, y) \in \mathbb{C}^2 ; |y| \leq c|x| \} \] for all \( p \in C \setminus K \), 
as claimed.

Suppose \( u, u' \in C \setminus K \) are arbitrary. Let \( i_0, j_0, i_0', j_0' \) such that \( u \in \Gamma_{i_0,j_0}(D_r \setminus \{0\}) \) and \( u' \in \Gamma'_{i_0,j_0}(D_r \setminus \{0\}) \) and suppose that \( 1/e^{d_{i_0,j_0}} \leq 1/e^{d_{i_0,j_0'}} \). Let \( R = 1/e^{d_{i_0,j_0}} \) and choose a path \( \alpha : [0, 1] \to \mathbb{C} \setminus D_R \) such that \( \alpha(0) = P(u), \alpha(1) = P(u') \) and 
length\((\alpha) \leq \pi R|P(u) - P(u')| \). Consider the lifting \( \tilde{\alpha} \) of \( \alpha \) by \( P|\Gamma_{i_0,j_0}(D_r \setminus \{0\}) \) with origin \( u \) and let \( u'' \) be its end. We obviously have 
\[ d(u, u') \leq \text{length}(\tilde{\alpha}) + d(u', u'') \].

On the other hand, since \( P \) is linear, \( dP_p = P|_{T_pC} \). Thus 
\[ \frac{1}{\sqrt{1 + c^2}} \leq ||dP_p|| \leq 1 \] for all \( p \in C \setminus K \).

In particular, 
\[ \text{length}(\tilde{\alpha}) \leq \sqrt{1 + c^2} \text{length}(\alpha) \leq \pi R \sqrt{1 + c^2} |P(u) - P(u')| , \] as 
\[ |P(u) - P(u')| \leq d(u, u') \], we obtain 
\[ \text{length}(\tilde{\alpha}) \leq \pi R \sqrt{1 + c^2} d(u, u') . \]

If we join the segment \([u, u']\) to \( \tilde{\alpha} \) at \( u \), we have a curve from \( u' \) to \( u'' \), so 
\[ d(u', u'') \leq (1 + \pi R \sqrt{1 + c^2})d(u, u') . \] Finally, 
\[ \text{length}(\tilde{\alpha}) + d(u', u'') \leq (1 + 2\pi R \sqrt{1 + c^2})d(u, u') , \] and the constant \( M = 1 + 2\pi R \sqrt{1 + c^2} \) satisfies the desired conditions. \( \square \)

Remark 3.4. In the above lemma, we prove that \( P|_{C \setminus K} : C \setminus K \to \mathbb{C} \setminus P(K) \) is a covering map. Moreover, \( P|_{C \setminus K} \) has derivative bounded above and below by positive constants. In particular, for a non-constant arc \( \alpha \) the quotient 
\[ \frac{\text{length}(\tilde{\alpha})}{\text{length}(\alpha)} \] is bounded above and below by positive constants.

The demonstration technique of the Theorem 1.5 is similar to the case of germ of analytic curves in \( \mathbb{C} \). In particular, it is based on a so-called bubble trick argument.

Proof of the first part of Theorem 1.5. We first prove that the Lipschitz geometry at infinity gives A). Let \( f \in \mathbb{C}[x, y] \) be a polynomial that defines \( C \) which does not have multiple factors. Let \( n = \deg f \), then by a linear change of coordinates if necessary, we can assume that the monomial \( y^n \) has coefficient equal to 1 in \( f \).

The points at infinity of \( C \) are the points \( [x : y : 0] \in \mathbb{P}^2 \) satisfying \( f_n(x, y) = 0 \), where \( f_n \) denotes the homogeneous polynomial composed by the monomials in \( f \) of degree \( n \), so \([0 : 1 : 0]\) is not a point at infinity of \( C \).

We claim that there are constant \( c > 0 \) and an open Euclidean ball \( B_{R_0}(0) \) of radius \( R_0 \) centered at origin such that \( |y| \leq c|x| \) for all \( (x, y) \in C \setminus B_{R_0}(0) \). Indeed,
otherwise, there exists a sequence \( \{ z_k = (x_k, y_k) \} \subset C \) such that \( \lim_{k \to +\infty} \| z_k \| = +\infty \) and \( |y_k| > k|x_k| \). Thus, taking a subsequence, one can suppose that \( \lim_{k \to +\infty} \frac{y_k}{|y_k|} = y_0 \) for some \( y_0 \) such that \( |y_0| = 1 \). Since \( \frac{|x_k|}{|y_k|} < 1 \), \( \lim_{k \to +\infty} \frac{z_k}{\| z_k \|} = (0, y_0) \). On the other hand,

\[
0 = f(z_k) = f \left( \frac{z_k}{\| z_k \|} \right) = \| z_k \|^n \sum_{i=0}^{n} \frac{1}{\| z_k \|^{n-i}} f_i \left( \frac{z_k}{\| z_k \|} \right),
\]

where \( f_i \) denotes the homogeneous polynomial composed by the monomials in \( f \) of degree \( i \). This implies that

\[
0 = f(z_k) = \sum_{i=0}^{n} \frac{1}{\| z_k \|^{n-i}} f_i \left( \frac{z_k}{\| z_k \|} \right),
\]

Letting \( k \to \infty \) yields \( f_n(0, y_0) = 0 \), which implies that \([0 : 1 : 0] \) is a point at infinity of \( C \), this is a contradiction. Therefore, the claim is true.

Now, let \([1 : a_j : 0], j = 1, \ldots, m \leq n \) be the points at infinity of \( C \). We define cones

\[
V_j := \{ (x, y) \in \mathbb{C}^2 : |y - a_j x| \leq \epsilon |x| \}
\]

where \( \epsilon > 0 \) is small enough that the cones are disjoint except at 0. Then increasing \( R_0 > 0 \), if necessary,

\[
C \setminus B_{R_0}(0) \subset \bigcup_{j=1}^{m} V_j.
\]

Indeed, otherwise, there exists a sequence \( \{ z_k = (x_k, y_k) \} \subset C \) such that \( \lim_{k \to +\infty} \| z_k \| = +\infty \) and \( |y_k - a_j x_k| > \epsilon |x_k| \) for all \( j = 1, \ldots, m \). Again, since \( \| z_k \| \to +\infty \) as \( k \to \infty \), we have

\[
\lim_{k \to +\infty} f_n \left( \frac{z_k}{\| z_k \|} \right) = 0.
\]

On the another hand, writing \( f_n(x, y) = \prod_{j=1}^{m} (y - a_j x)^{d_j} \), where \( d_j \) is a positive integer such that \( n = \sum_{1 \leq j \leq m} d_j \), we have

\[
\left\| f_n \left( \frac{z_k}{\| z_k \|} \right) \right\| = \prod_{j=1}^{m} \left| \frac{y_k - a_j x_k^{d_j}}{\| z_k \|^n} \right| \geq \left( \frac{\epsilon |x_k|}{\| z_k \|} \right)^n.
\]

But, because of the first claim, we have

\[
\frac{|x_k|}{\| z_k \|} = \sqrt{\frac{1}{1 + \| \frac{z_k}{x_k} \|^2}} \geq \frac{1}{\sqrt{1 + c^2}},
\]

which derives a contradiction.

We denote by \( C'_j \) the part of \( C \setminus B_{R_0}(0) \) inside \( V_j \). Now, let \( C' \) be a second plane curve with the same Lipschitz geometry at infinity as \( C \) and \( K, K' \subset \mathbb{C}^2 \) compact sets such that there is a bilipschitz map \( \Phi : C \setminus K \to C' \setminus K' \). Let \([1 : a'_j : 0], j = 1, \ldots, m \) be a sequence
$1, \ldots, m'$ be the points at infinity of $C'$. We repeat the above arguments for $C'$, then increasing $R_0 > 0$, if necessary,

$$C' \setminus B_{R_0}(0) \subset \bigcup_{j=1}^{m'} V_j', \text{ where } V_j' := \{ (x, y) \in \mathbb{C}^2 : |y - a_j'x| \leq \epsilon |x| \}.$$ 

Likewise, denote by $C_j'$ the set $(C' \setminus B_{R_0}(0)) \cap V_j'$. We have $\Phi(C \setminus B_R(0)) \subset C' \setminus B_{h(R)}(0)$ with $h(R) = \Theta(R)$. Since $\text{dist}(C_j \setminus B_R(0), C_k \setminus B_R(0)) = \Theta(R)$ we have

$$\text{dist}(\Phi(C_j \setminus B_R(0)), \Phi(C_k \setminus B_R(0))) = \Theta(R).$$

Notice that the sets $C_j'$, $l = 1, \ldots, m'$ have the following property: the distance between any two connected component of $C_j'$ outside a ball of radius $h(R)$ around $0$ is $o(R)$. Then, we cannot have

$$\Phi(C_j \setminus B_R(0)) \subset C_k' \setminus B_{h(R)}(0) \text{ and } \Phi(C_k \setminus B_R(0)) \subset C_j' \setminus B_{h(R)}(0)$$

for $k \neq j$ then $m \leq m'$ and using the inverse $\Phi^{-1}$ we get $m = m'$.

Now, we deal with b). Without loss of generality, we can suppose that $[1 : a_1 : 0] = [1 : 0 : 0]$ is a point at infinity for $C$. We extract the characteristic and the coincidence exponents relative to $L_{\infty}$ of the curve $(\overline{C} \cup L_{\infty}, [1 : 0 : 0])$ using the coordinate system and the induced Euclidean metric $d$ on $C_1$. Next, we prove that these data determine the embedded topology type of $(\overline{C} \cup L_{\infty}, [1 : 0 : 0])$. Finally, we prove that these data can be obtained without using the chosen coordinate system and even using the equivalent metric $d'$ induced by $\Phi$, for this we operate the “bubble trick”.

Let $U = \{ [x : y : z] \in \mathbb{P}^2 : x \neq 0 \}$ and consider the coordinate chart $\varphi : U \to \mathbb{C}^2$ defined by $\varphi([x : y : z]) = (z/x, y/x) = (u, v)$. In this local coordinates, $\varphi([1 : 0 : 0])$ is the origin and we have $\text{ord}_x(f \circ \varphi^{-1})(0, v) = d_1$. Let $B_1, \ldots, B_{k_1}$ be the branches of $(\varphi(\overline{C} \cap U), 0)$. Every branch of the curve $(\varphi(\overline{C} \cap U), 0)$ has a Newton-Puiseux parametrization of the form

$$\gamma_s(w) = \left( \sum_{k>0} a_{sk} w^k \right),$$

where $d_{1s}$ are positive integers such that $\sum_{s=1}^{k_1} d_{1s} = d_1$. Then, increasing $R_0 > 0$ if necessary, the images of the maps

$$\Gamma_s(w) = (t^{-1} \circ \varphi^{-1} \circ \gamma)(w) = \left( \sum_{k>0} a_{sk} w^k \right), s = 1, \ldots, k_1$$

cover $C_1$. Therefore, the lines $x = t$ for $t \in (R_0, \infty)$ intersect $C_1$ for $d_1$ points $p_1(t), \ldots, p_{d_1}(t)$ which depend continuously on $t$. Denote by $[d_1]$ the set $\{1, \ldots, d_1\}$. For each $j, k \in [d_1]$ with $j < k$, the distance $d(p_j(t), p_k(t))$ has the form $\Theta(t^1-q(j,k))$, where $q(j, k) = q(k, j)$ is either a characteristic Puiseux exponent relative to $L_{\infty}$ for a branch of the plane curve $(\overline{C} \cup L_{\infty}, [1 : 0 : 0])$ or a coincidence exponent relative to $L_{\infty}$ between two branches of $(\overline{C} \cup L_{\infty}, [1 : 0 : 0])$. For $j \in [d_1]$ define $q(j, j) = \infty$.

**Lemma 3.5.** The map $q : [d_1] \times [d_1] \to \mathbb{Q} \cup \{\infty\}, (j, k) \mapsto q(j, k)$, determines the embedded topology type of $(\overline{C} \cup L_{\infty}, [1 : 0 : 0])$.

**Proof.** The topological type of $(\overline{C} \cup L_{\infty}, [1 : 0 : 0])$ is encoded by its Eggens-Wall tree relative to a smooth branch $L$ transversal to $(\overline{C} \cup L_{\infty}, [1 : 0 : 0])$ (see
Wall [3 Proposition 4.3.9 and Theorem 5.5.9]). To prove the lemma we notice that the function \( q \) is the same as the function \( \text{ord}_x \) of Definition 2.7. By the process described in Section 2 one obtains the Eggers-Wall tree relative to \( L_\infty \) of \((\tilde{C} \cup L_\infty, [1 : 0 : 0])\). By applying the inversion theorem for Eggers-Wall tree [6 Theorem 4.5] to \( \Theta_{L_\infty}(\tilde{C} \cup L_\infty \cup L, [1 : 0 : 0]) \), one gets the Eggers-Wall tree \( \Theta_L(\tilde{C} \cup L_\infty, [1 : 0 : 0]) \).

\[ \blacksquare \]

As already noted, this discovery of the embedded topology type involved the chosen coordinate system and the metric \( d \). We must show we can discover it using \( d' \) and without use of the chosen coordinate system.

The points \( p_1(t), \ldots, p_{d}(t) \) that we used to find the numbers \( q(j, k) \) were obtained by intersecting \( C_1 \) with the line \( x = t \). The arc \( t \in (R_0, \infty) \mapsto p_1(t) \) satisfies
\[
d(0, p_1(t)) = \Theta(t).
\]

Moreover, the other points \( p_2(t), \ldots, p_{d}(t) \) are in the disk of radius \( \eta t \) centered at \( p_1(t) \) in the plane \( x = t \). Here, \( \eta > 0 \) can be as small as we like, so long as \( R_0 \) is then chosen sufficiently big.

Instead of a disk of radius \( \eta t \), we can use a ball \( B(p_1(t), \eta t) \) of radius \( \eta t \) centered at \( p_1(t) \). This ball \( B(p_1(t), \eta t) \) intersects \( C_1 \) in \( d_1 \) disks \( D_1(\eta t), \ldots, D_{d_1}(\eta t) \), named such that \( p_1(t) \in D_j(\eta t), l = 1, \ldots, d_1 \) and thus \( \text{dist}(D_j(\eta t), D_k(\eta t)) \leq d(p_j(t), p_k(t)) \). On the other hand, let \( \tilde{p}_l(t) \in D_l(\eta t), l = 1, \ldots, d_1 \) such that
\[
\text{dist}(D_l(\eta t), D_k(\eta t)) = d(\tilde{p}_l(t), \tilde{p}_k(t)).
\]

Consider the projection \( P : \mathbb{C}^2 \to \mathbb{C} \) given by \( P(x, y) = x \) and let \( \alpha_t \) be the segment in \( \mathbb{C} \) joining \( P(\tilde{p}_l(t)) \) to \( P(\tilde{p}_k(t)) \) and let \( \tilde{\alpha}_t \) be the lifting of \( \alpha_t \) by the restriction \( P|_{C \setminus B_{R_0}(0)} \) with origin \( \tilde{p}_l(t) \). Applying Lemma 3.3 to \( P \) with \( u = \tilde{p}_l(t) \) and \( u' = \tilde{p}_k(t) \), we then obtain
\[
d(\tilde{p}_l(t), \tilde{p}_k(t)) \geq \frac{1}{M}(\text{length}(\tilde{\alpha}_t) + d(\tilde{p}_l(t), \tilde{\alpha}_t(1))) \geq \frac{1}{M}d(\tilde{p}_l(t), \tilde{\alpha}_t(1)).
\]

But \( d(\tilde{p}_l(t), \tilde{\alpha}_t(1)) = \Theta(t^{1-q(j,k)}) \) since \( P(\tilde{p}_l(t)) = P(\tilde{\alpha}_t(1)) \) and \( |P(\tilde{p}_l(t))| = \Theta(t) \).

We now replace the arc \( p_1 \) by any continuous arc on \( C_1 \) satisfying (11) and we still denote this new arc by \( p_1 \). If \( \eta \) is sufficiently small it is still true that \( B_{C_1}(p_1(t), \eta t) := C_1 \cap B(p_1(t), \eta t) \) consists of \( d_1 \) disks \( D_1(\eta t), \ldots, D_{d_1}(\eta t) \) with \( \text{dist}(D_j(\eta t), D_k(\eta t)) = \Theta(t^{1-q(j,k)}) \). So at this point, we have gotten rid of the dependence on the chosen of coordinate system in discovering the topology, but not yet dependence on the metric \( d \).

A \( L \)-bilipschitz change to the metric may make the components of \( B_{C_1}(p_1(t), \eta t) \) disintegrate into many pieces, so we can no longer simply use distance between all pieces. To resolve this, we consider \( B_{C_1}(p_1(t), \eta t/L) \) and \( B_{C_1}(p_1(t), \eta Lt) \). Note that
\[
B_{C_1}(p_1(t), \eta t/L) \subset B_{C_1}(p_1(t), \eta t) \subset B_{C_1}(p_1(t), \eta Lt),
\]
where \( B' \) means we are using the modified metric \( d' \).

Denote by \( D_j(\eta t/L) \) and \( D_j(\eta Lt), j = 1, \ldots, d_1 \) the disk of \( B_{C_1}(p_1(t), \eta t/L) \) and \( B_{C_1}(p_1(t), \eta Lt) \), respectively, so that \( D_j(\eta t/L) \subset D_j(\eta Lt) \) for \( j = 1, \ldots, d_1 \). Thus \( B_{C_1}(p_1(t), \eta t) \) has \( d_1 \) components such that each one contains at most one component of \( B_{C_1}(p_1(t), \eta t/L) \). Therefore, exactly \( d_1 \) components of \( B_{C_1}(p_1(t), \eta t) \) intersect \( B_{C_1}(p_1(t), \eta t/L) \). Naming these components \( D'_j(\eta t), \ldots, D'_{d_1}(\eta t) \), such that
This implies that \( B \)
We have let
\[
\text{construct a bilipschitz map between algebraic curves with the same data in A) and }
\]
one of the curves have the point \([0 : 1 : 0]\) as a point at infinity.

Up to now, we have used the metric \( d \) to select the components \( D_j'(\eta t), j = 1, \ldots, d_1 \).
We have
\[
\text{So we can use only the metric } d' \text{ to select these components and we are done.}
\]

\[
\text{4. Lipschitz geometry at infinity is determined by A) and B)}
\]

In this section, we prove the other direction of Theorem 1.5. For this, we will construct a bilipschitz map between algebraic curves with the same data in A) and B).

**Proof of the second part of Theorem 1.5** Let \( C_1 \) and \( C_2 \) plane algebraic curves with the same data described by A) and B). Choose \((x, y)\) coordinates in such way that none of the curves have the point \([0 : 1 : 0]\) as a point at infinity.

Let \([1 : a_1^1 : 0], \ldots, [1 : a_m^1 : 0]\) be the points at infinity of \( C_l, l = 1, 2 \), denoted in such a way that \((\tilde{C}_1, [1 : a_1^2 : 0])\) has the same topological type as \((\tilde{C}_2, [1 : a_1^2 : 0])\).

Then, by [3, Theorem 5.5.9] and [3, Proposition 4.3.9], for any smooth branch \( L_1 \) (resp. \( L_2 \)) through \([1 : a_1^1 : 0]\) (resp. \([1 : a_2^1 : 0]\)) transversal to \((C_1 \cup L_\infty, [1 : a_1^1 : 0])\) (resp. \((C_1 \cup L_\infty, [1 : a_2^1 : 0])\)) the Eggers-Wall trees \( \Theta_{L_1}(C_1 \cup L_\infty, [1 : a_1^1 : 0]) \) and \( \Theta_{L_2}(C_1 \cup L_\infty, [1 : a_1^1 : 0]) \) are isomorphic. Then, we apply the inversion theorem for Eggers-Wall tree [6, Theorem 4.5] to both and we get that \( \Theta_{L_1}(C_1 \cup L_\infty, [1 : a_1^1 : 0]) \) and \( \Theta_{L_\infty}(C_2 \cup L_\infty, [1 : a_1^2 : 0]) \) are isomorphic.

For each \( i \), let \( B_{i1}, \ldots, B_{iK_l} \) be the branches of \((\tilde{C}_l, [1 : a_1^1 : 0]), l = 1, 2\). Again, we denoted in such a way that \((B_{i1}, [1 : a_1^1 : 0])\) has the same topological type as \((B_{i1}'[1 : a_1^1 : 0])\). From what has been said above, we have that \( B_{ij}^1 \) and \( B_{ij}^2 \) have the same characteristic exponents relative to \( L_\infty \) and \( k_{L_\infty}(B_{ij}^1, B_{ij}^2) = k_{L_\infty}(B_{ij}^1, B_{ij}^2) \).

The open set \( U = \{[x : y : z] \in \mathbb{P}^2 : x \neq 0\} \) contains all the points at infinity of \( C_l, l = 1, 2 \). We can use the coordinate chart \( \varphi : U \to \mathbb{C}^2 \) defined by \( \varphi([x : y : z]) = (z/x, y/z) \) to obtain a Newton-Puiseux parametrization of the branches \( \varphi(B_{ij}^1) \).

Let \( D_{e_0} \) be the open disk of radius \( e_0 > 0 \) centered at the origin with \( e_0 \) sufficiently small such that there exist Newton-Puiseux parametrization \( \gamma_{ij}^l : D_{e_0} \to \mathbb{C}^2 \) of \( \varphi(B_{ij}^l) \) given by
\[
\gamma_{ij}^l(w) = \left( w^{d_{ij}}, a_1^l + \sum_{k > 0} a_{ijk}^l w^k \right).
\]

Let \( \Gamma_{ij}^l : D_{e_0} \setminus \{0\} \to \mathbb{C}^2 \) given by
\[
\Gamma_{ij}^l(w) = \left( \varphi^{-1} \circ \varphi^{-1} \circ \gamma_{ij}^l(w) \right) = \left( \frac{1}{w^{d_{ij}}} \frac{\sum_{k > 0} a_{ijk}^l w^k}{w^{d_{ij}}} \right) \right), l = 1, 2.
Consider the compact set \( K_l^i = C \setminus \bigcup_{j} \Gamma_{ij}^l(D_{\epsilon} \setminus \{0\}) \), \( l = 1, 2 \). We will prove that there exists \( \epsilon > 0 \) that the map

\[
\Phi : C_1 \setminus K_1^i \rightarrow C_2 \setminus K_2^i \\
\Gamma_{ij}^l(w) \rightarrow \Gamma_{ij}^l(w)
\]

is bilipschitz.

**Claim.** Consider the projection \( P : \mathbb{C}^2 \rightarrow \mathbb{C} \) given by \( P(x, y) = x \). In order to check that \( \Phi \) is a Lipschitz map it is enough to consider points in \( C_1 \setminus K_1^i \) with the same \( x \) coordinate. That is, there exists a constant \( c > 0 \) such that

\[
d(\Gamma_{ij}^l(w'), \Gamma_{ij}^l(w'')) \leq cd(\Gamma_{ij}^2(w'), \Gamma_{ij}^2(w''))
\]

for all \( w', w'' \) such that \( P(\Gamma_{ij}^1(w')) = P(\Gamma_{ij}^1(w'')) \).

Indeed, let \( \Gamma_{ij}^1(w) \) and \( \Gamma_{ij}^1(w') \) be any two elements of \( C_1 \setminus K_1^i \) and suppose that \( 1/\epsilon^{d_{ij}} \leq 1/\epsilon^{d_{ij'}} \). Let \( \alpha \) be a curve in \( C \setminus D_{1/\epsilon^{s_{ij}}} \) joining \( P(\Gamma_{ij}^1(w)) \) to \( P(\Gamma_{ij}^1(w')) \) as in the Lemma 3.3. Let \( \tilde{\alpha}_1 \) (resp. \( \tilde{\alpha}_2 \)) be the lifting of \( \alpha \) by the restriction \( P|_{\Gamma_{ij}^1(D_{1/\epsilon^{s_{ij}}} \setminus \{0\})} \) (resp. \( P|_{\Gamma_{ij}^1(D_{1/\epsilon^{s_{ij}}} \setminus \{0\})} \)) with origin \( \Gamma_{ij}^1(w) \) (resp. \( \Gamma_{ij}^1(w) \)). Consider the unique \( w'' \in D_{\epsilon} \) such that \( \Gamma_{ij}^1(\alpha) \) and \( \Gamma_{ij}^1(\Gamma_{ij}^1(w'')) \) is the end of \( \tilde{\alpha}_1 \). Notice that \( P \circ \Gamma_{ij}^1 = P \circ \Gamma_{ij}^2 \) and by uniqueness of lifts \( \tilde{\alpha}_2 = \Gamma_{ij}^2 \circ (\Gamma_{ij}^1)^{-1} \circ \tilde{\alpha}_1 \) which implies that \( \Gamma_{ij}^2(\alpha) \) is the end of \( \tilde{\alpha}_2 \).

We have

\[
d(\Gamma_{ij}^1(w), \Gamma_{ij}^1(w')) \leq \text{length}(\tilde{\alpha}_1) + d(\Gamma_{ij}^1(\alpha), \Gamma_{ij}^1(w'))
\]

According to the Remark 3.4 there are constant, say \( c_1 \) and \( c_2 \) such that \( \text{length}(\tilde{\alpha}_1) \leq c_1 \text{length}(\alpha) \leq c_1 c_2 \text{length}(\tilde{\alpha}_2) \). By hypothesis, there exists a constant \( c > 0 \) such that

\[
d(\Gamma_{ij}^1(\alpha), \Gamma_{ij}^1(w')) \leq cd(\Gamma_{ij}^1(\alpha), \Gamma_{ij}^1(w'))
\]

Therefore setting \( C = \max\{c_1, c_2, c\} \), we obtain

\[
d(\Gamma_{ij}^1(w), \Gamma_{ij}^1(w')) \leq C\left(\text{length}(\tilde{\alpha}_2) + d(\Gamma_{ij}^1(\alpha), \Gamma_{ij}^1(w'))\right)
\]

Applying Lemma 3.3 to \( C_2 \) with \( u = \Gamma_{ij}^2(w) \) and \( u' = \Gamma_{ij}^2(w') \), we then have

\[
d(\Gamma_{ij}^1(w), \Gamma_{ij}^1(w')) \leq CMd(\Gamma_{ij}^2(w), \Gamma_{ij}^2(w'))
\]

This proves \( \Phi \) is Lipschitz and the claim.

Now, let \( B_{ij}^l \) and \( B_{ij'}^l \) be branches of \( \tilde{C}_1 \) and \( \tilde{C}_2 \), respectively, with \( i \neq i' \). Let \( s \in (0, 1) \rightarrow \Gamma_{ij}^1(ws^{1/d_{ij}}) \) and \( s \in (0, 1) \rightarrow \Gamma_{ij'}^1(w's^{1/d_{ij'}}) \) the two real arcs with \( w^{d_{ij}} = (w')^{d_{ij'}}. \) Then we have

\[
d(\Gamma_{ij}^1(ws^{1/d_{ij}}), \Gamma_{ij'}^1(w's^{1/d_{ij'}})) = \frac{1}{s|w^{d_{ij}}|} \left| a_{ij}^1 - a_{ij'}^1 + \sum_{k>0} a_{ijk}w^ks^{k/d_{ij}} - \sum_{k>0} a_{ij'k}(w')^ks^{k/d_{ij'}} \right|.
\]
and
\[
d(\Phi(\Gamma_{ij}^1(ws^{1/d_{ij}})), \Phi(\Gamma_{i'j'}^1(w's^{1/d_{i'j'}}))) = \frac{1}{s|w|^{d_{ij}}} \left| a_{ij}^2 - a_{i'j'}^2 + \sum_{k>0} a_{ij,k}^2 w^k s^{k/d_{ij}} - \sum_{k>0} a_{i'j',k}^2 (w')^k s^{k/d_{i'j'}} \right|
\]
Hence the ratio
\[
(2) \quad d(\Gamma_{ij}^1(ws^{1/d_{ij}}), \Gamma_{i'j'}^1(w's^{1/d_{i'j'}}))/d(\Phi(\Gamma_{ij}^1(ws^{1/d_{ij}})), \Phi(\Gamma_{i'j'}^1(w's^{1/d_{i'j'}})))
\]
tends to the non-zero constant $\frac{|a_{ij} - a_{i'j'}|}{|a_{ij}^2 - a_{i'j'}^2|}$ as $s$ tends to 0 for every such pairs $(w, w')$. So there exists $\epsilon > 0$ such that for each such $(w, w')$ with $|w| = 1$ and each $s < \epsilon$, the quotient $\frac{2}{s}$ belongs to $[1/c, c]$ where $c > 0$.

Now, consider the branches $B^1_{ij}$ and $B^2_{ij}$. Let $s \in (0, 1] \rightarrow \Gamma_{ij}^1(ws)$ and $s \in (0, 1] \rightarrow \Gamma_{i'j'}^1(w's)$ the two real arcs with $w^{d_{ij}} = (w')^{d_{ij}}$. Then we have
\[
d(\Gamma_{ij}^1(ws), \Gamma_{ij}^1(w's)) = \frac{1}{s|w|^{d_{ij}}} \left| \sum_{k>0} a_{ij,k}^1 (w^k - (w')^k) s^k \right|
\]
and
\[
d(\Phi(\Gamma_{ij}^1(ws)), \Phi(\Gamma_{ij}^1(w's))) = \frac{1}{s|w|^{d_{ij}}} \left| \sum_{k>0} a_{ij,k}^2 (w^k - (w')^k) s^k \right|
\]
Let $k_0$ be the minimal element of $\{k; a_{ij,k}^1 \neq 0$ and $w^k \neq (w')^k\}$. Then $k_0/d_{ij}$ is an characteristic exponent for $B^1_{ij}$ relative to $L_{\infty}$, so $a_{ij,k_0}^1$ and $a_{ij,k_0}^2$ are non-zero. Hence the ratio
\[
(3) \quad d(\Gamma_{ij}^1(ws), \Gamma_{ij}^1(w's))/d(\Phi(\Gamma_{ij}^1(ws)), \Phi(\Gamma_{ij}^1(w's)))
\]
tends to the non-zero constant $c_{ij,k_0} = \frac{|a_{ij,k_0}^1|}{|a_{ij,k_0}^2|}$ as $s$ tends to 0.

Notice that the integer $k_0$ depends on the pair of points $(w, w')$. But $k_0/d_{ij}$ is a characteristic exponent relative to $L_{\infty}$ of $B^1_{ij}$. Therefore there is a finite number of values for $k_0$ and $c_{ij,k_0}$. Moreover, the set of pairs $(w, w')$ such that $w \neq w'$ and $w^{d_{ij}} = (w')^{d_{ij}}$ consists of a disjoint union of $d_{ij} - 1$ lines, say $L_l = \{(w, \exp(2\pi l/d_{ij})w), w \in \mathbb{C}^*\}, l = 1, \ldots, d_{ij} - 1$. Observe that for any $(w, w') \in L_l$ the quotient $\frac{2}{s}$ tends to positive constant as $s \rightarrow 0$ which does not depend on the pair $(w, w')$. So there exists $\epsilon_1 > 0$ such that for each such $(w, w')$ with $|w| = 1$ and each $s \leq \epsilon_1$, the quotient $\frac{2}{s}$ belongs to $[1/c, c]$ where $c > 0$, as claimed.

For the case of branches $B^1_{ij}$ and $B^2_{ij}$ with $j \neq j'$, the same arguments work taking into account their coincidence exponent relative to $L_{\infty}$.

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\square
\]

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