The Reductive Subgroups of $G_2$

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Abstract. Let $G := G_2(K)$ be a simple algebraic group of type $G_2$ defined over an algebraically closed field $K$ of characteristic $p > 0$. Let $\sigma$ denote a standard Frobenius automorphism of $G$ such that $G_{\sigma} \cong G_2(q)$ with $q \geq 4$. In this paper we find all reductive subgroups of $G$ and quasi-simple subgroups of $G_{\sigma}$ in the defining characteristic. Our results extend the complete reducibility results of [13, Thm 1].

1 Introduction

Recall that $G_2$ has maximal rank subgroups of type $A_1 \tilde{A}_1$ and $A_2$ (also $\tilde{A}_2$ generated by all short root groups of $G$ when $p = 3$). When $p = 2$ we define $Z_1$ to be the subgroup of type $A_1$ obtained from the embedding

$$A_1(K) \to A_1(K) \circ A_1(K) \leq G; \quad x \mapsto (x, x).$$

Also when $p = 2$, we define $Z_2$ to be the subgroup of type $A_1$ obtained from the embedding

$$A_1(K) \to A_2(K) \leq G$$

where $A_1 \cong PSL_2(K)$ embeds in $A_2$ by its action on the three-dimensional space $\text{Sym}^2 V$ for $V$ the standard module for $SL_2(K)$. It is shown later that these subgroups are contained in the long root parabolic of $G$, that is, $P = \langle B, x_{-r}(t) : t \in K \rangle$ where $r$ is the long simple root associated with the choice of Borel subgroup $B$.

Let $\bar{L}$, (resp. $\bar{L}$) denote the standard Levi subgroup of the standard long root (resp. short root) parabolic subgroup of $G$ containing the Borel subgroup $B$. Let $\bar{L}_0$ (resp $\bar{L}_0$) denote the subgroup of $\bar{L}$ (resp $\bar{L}$) generated by the unipotent elements. Observe that $\bar{L}_0 \cong \bar{L}_0 \cong A_1$. 

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The main theorem is:

**Theorem 1.** Let $X \cong A_1(K)$ be a subgroup of a parabolic subgroup in $G = G_2(K)$.

If $p > 2$ then $X$ is conjugate to precisely one of $\bar{L}_0$ and $\tilde{L}_0$.

If $p = 2$ then $X$ is conjugate to precisely one of $\bar{L}_0$, $\tilde{L}_0$, $Z_1$ and $Z_2$.

Recall Serre’s notion of $G$-complete reducibility \cite{17}. A subgroup is said to be $G$-completely reducible or $G$-cr if, whenever it is a subgroup of a parabolic subgroup of $G$, it is contained in a Levi subgroup of that parabolic subgroup.

**Corollary 2.** All connected reductive subgroups of $G$ are $G$-cr unless $p = 2$, in which case there are precisely two classes of non $G$-cr subgroups.

This extends the result \cite[Thm 1]{13} which states that all subgroups of $G$ are $G$-cr provided $p > 3$.

**Corollary 3.** Let $X$ denote a closed, connected semisimple subgroup of $G$. Then up to conjugacy, $(X, p, V_7 \downarrow X)$ is precisely one entry in the following table where $V_7 \downarrow X$ denotes the restriction of the seven-dimensional Weyl module $W_G(\lambda_1)$ to $X$.

| $X$         | $p$   | $V_7 \downarrow X$               |
|-------------|-------|-----------------------------------|
| $A_2$       | any   | $10 \oplus 01 \oplus 0$          |
| $\tilde{A}_2$ | $p = 3$ | 11                               |
| $A_1A_1$    | any   | $1 \otimes 1 \oplus 0 \otimes \tilde{W}(2)$ |
| $\bar{L}_0$ | any   | $1 \oplus 1 \oplus 0^3$         |
| $\tilde{L}_0$ | any   | $1 \oplus 1 \oplus W(2)$        |
| $Z_1$       | $p = 2$ | $T(2) \oplus W(2)$              |
| $Z_2$       | $p = 2$ | $W(2) \oplus W(2)^* \oplus 0$   |
| $A_1 \hookrightarrow A_1\tilde{A}_1$; $x \mapsto (x^{(p^r)}, x^{(p^s)})$ $r \neq s$ | any      | $(1^{(p^r)} \otimes 1^{(p^s)}) \oplus W(2)^{(p^r)}$ |
| $A_1 \hookrightarrow A_2$, irreducible | $p > 2$ | $2 \oplus 2 \oplus 0$            |
| $A_1$, max  | $p \geq 7$ | 6                                |

The subgroup denoted $\tilde{A}_2$ exists only when $p = 3$ and is generated by the short root subgroups of $G$. (The above table appears not to contain the
irreducible $A_1 \leq \tilde{A}_2$. It is shown later that this subgroup is conjugate to the subgroup $A_1 \hookrightarrow A_1 \tilde{A}_1$ where $r = 1, s = 0$.)

Some remarks on notation. In the above table and elsewhere we refer to an irreducible module by its high weight $\lambda$. When $X$ is of type $A_1$, $\lambda$ is given as an integer; by a module $ab$ for a group of type $A_2$ we mean the irreducible module with high weight $a \lambda_1 + b \lambda_2$ where $\lambda_i$ is the fundamental dominant weight corresponding to the simple root $\alpha_i$. By $V^{(p^r)}$ we mean the Frobenius twist of the module $V$ induced by the Frobenius morphism $x \mapsto x^{(p^r)}$. The notation $\mu_1 | \mu_2 | \cdots | \mu_n$ indicates a module with the same composition factors as the module $\mu_1 \oplus \mu_2 \oplus \cdots \oplus \mu_n$. The notation $\mu_1/\mu_2/\cdots/\mu_n$ indicates an indecomposable module with composition factors of high weights $\mu_i$ for some dominant weights $\mu_i$ and is given in the order in which the factors occur so that there is a submodule $\mu_i/\cdots/\mu_n$ and a quotient $\mu_1/\cdots/\mu_{i-1}$. By $W(2)$ we denote the Weyl module for $A_1$ of high weight 2; when $p > 2$ this is irreducible and when $p = 2$ it is indecomposable of type $1^{2(2)}/0$. Lastly when $p = 2$ we denote by $T(2)$ the four-dimensional tilting module for $A_1$ which is indecomposable of type $0/1^{2(2)}/0$.

Now let $\sigma$ denote a standard Frobenius automorphism of $G$ such that $G_\sigma = G_2(q)$ with $q \geq 4$. We use the proof of Theorem 1 and its corollaries to prove a result about the quasi-simple subgroups of Lie type of $G_\sigma$ in the defining characteristic. (A quasi-simple group of Lie type is a perfect central extension of a simple group of Lie type.)

**Theorem 2.** Let $X(q_0) \leq G_\sigma$ where $X(q_0)$ is a quasi-simple group of Lie type over $\mathbb{F}_{q_0}$, a field of the same characteristic as $\mathbb{F}_q$. Then there exists a $\sigma$-stable simple algebraic subgroup $\bar{X}$ of $G$ of the same type as $X(q_0)$ containing $X(q_0)$.

**Remark 1.1.** Using [10, 5.1], it follows that $X(q_0)$ is unique up to conjugacy in $\bar{X}$. Since Corollary 3 determines $\bar{X}$, it follows that we have found, up to $G_\sigma$-conjugacy, all quasi-simple subgroups of Lie type of $G_\sigma$ with the same defining characteristic as $G$.

**Remark 1.2.** The only non-simple semisimple subgroups of $G_\sigma$ are of the form $SL_2(q_1) \circ SL_2(q_2)$ with $q_1, q_2 \geq 4$, since any such group must have rank 2. Since we have found all the quasi-simple groups using the above theorem, we have also found all semisimple subgroups of Lie type of $G_\sigma$ in the defining characteristic. (A semisimple subgroup of Lie type, $H$ is a subgroup such
that \( H' = H \) and \( H/Z(H) \) is a direct product of simple subgroups of Lie type.)

\section{Preliminaries}

Let \( X \cong A_1(K) \) with \( |K| \geq 4 \) finite or \( K \) algebraically closed of characteristic \( p > 0 \). Let \( V := V_X(\lambda) \) denote an irreducible rational \( KX \)-module of high weight \( \lambda \). To prove Theorem 1 we require some information about \( H^1(X, V) \), the first cohomology group of \( X \) with coefficients in \( V \). We recall that \( H^1(X, V) \) is a \( K \)-vector space and is in bijection with the \( V \)-classes of closed complements to \( V \) in the semidirect product \( XV \). Recall also the standard fact that \( H^1(X, V) \cong \text{Ext}_X^1(K, V) \) (see \cite[p50]{9}).

\begin{lemma}
\text{Ext}_X^1(K, V_X(\lambda)) \text{ is non-zero if and only if } \lambda \text{ is a Frobenius twist of the module } (p-2) \otimes 1^{(p)}. \text{ When it is non-zero it is one-dimensional unless } |K| = 9 \text{ and } V_X(\lambda) = 1 \otimes 1^{(3)} \text{ where it is two-dimensional.}
\end{lemma}

\textbf{Proof.} This follows from setting \( \mu = 0 \) in \cite[4.5]{2} with the small correction given in \cite[1.2]{15}.

Recall that a parabolic subgroup \( P \) has a decomposition as a semidirect product \( LQ \) of a Levi subgroup \( L \) with unipotent radical \( Q \). We employ the above result to investigate complements to \( Q \) in \( P \). The next result shows how \( Q \) admits a filtration by \( KL \)-modules. We recall the notions of height, shape and level of a root from \cite{1}. Take a root system \( \Phi \) for \( G(K) \) with fixed base of simple roots \( \Pi \). Let \( J \subset \Pi \) be a subset of the simple roots and define the parabolic subgroup \( P_J \) by \( P_J = \langle B, x_{-\alpha}(t) : \alpha \in J \rangle \). Let \( \Phi_J = ZJ \cap \Phi \). Fix a root \( \beta \in \Phi^+ - \Phi_J \). We write \( \beta = \beta_J + \beta'_J \) where \( \beta_J = \sum_{\alpha_i \in J} c_i \alpha_i \) and \( \beta'_J = \sum_{\alpha_i \in \Pi - J} d_i \alpha_i \). Define

\begin{align*}
\text{height}(\beta) &= \sum c_i + \sum d_i \\
\text{shape}(\beta) &= \beta'_J \\
\text{level}(\beta) &= \sum d_i.
\end{align*}

Now define \( Q(i) := \langle x_{\beta}(t) : t \in K, \text{level}(\beta) \geq i \rangle \) and define \( V_S = \langle x_{\beta}(t) : t \in K, \text{shape}(\beta) = S \rangle \).
Lemma 2.2. Let $G(K)$ be a split Chevalley group. For each $i \geq 1$, $Q(i)/Q(i + 1)$ has the structure of a $KL$-module with decomposition $Q(i)/Q(i + 1) = \prod V_S$, the product over all shapes $S$ of level $i$. Furthermore, each $V_S$ is a $KL$-module with highest weight $\beta$ where $\beta$ is the unique root of maximal height and shape $S$.

Proof. This is the main result of [1], noting the Remark 1 at the end of the paper which gives the result even in the case $G(K)$ is special. \qed

Throughout the paper we will need the restrictions $V_7 \downarrow X$ of the seven-dimensional Weyl module $V_7 := W_{G_2}(\lambda_1)$ to various subgroups $X$ of $G = G_2(K)$. We calculate these now.

Lemma 2.3. The entries in the table following Corollary 3 have the restrictions $V_7 \downarrow X$ as stated.

Proof. The restriction $V_7 \downarrow X$ for the maximal $A_1$ when $p \geq 7$ is well known and is listed in [19, Main Theorem].

Consider $G_2$ embedded in $D_4$ as the fixed points of the triality automorphism. We consider the restriction of the natural 8-dimensional module $V_8$ for $D_4$. Recall that $V_8 \downarrow G_2 = 0/V_7$. For $p = 2$, $V_7$ becomes reducible and $V_8 \downarrow G_2 = 0/V_6/0$.

Recall that $\tilde{L}_0, \bar{L}_0$ are the simple, connected subgroups of the long and short Levi subgroups respectively. We first consider $V_7 \downarrow \tilde{L}_0, \bar{L}_0$ and $A_1\tilde{A}_1$.

We can see that $A_1\tilde{A}_1 \leq A^4_1 \leq D_4$. It is clear that the $A^4_1$ subsystem in $D_4$ is realised as $A_1 \otimes A_1 \perp A_1 \otimes A_1 \cong SO_4 \perp SO_4$. Take the long $A_1$ to be the first of the four and the short $A_1$ to be embedded diagonally in the other three.

Now it follows that we have $V_8 \downarrow \tilde{L}_0 = 0^2 \otimes 1 \perp 1 \otimes 1 = 1 \oplus 1 \perp T(2)$ for $p = 2$ and $1 \oplus 1 \perp 2 \oplus 0$ for $p > 2$. This gives $V_7 \downarrow \bar{L}_0 = 1 \oplus 1 \perp W(2)$ for $p = 2$ and $V_7 \downarrow \bar{L}_0 = 1 \oplus 1 \perp 2$ for $p > 2$.

We also have $V_8 \downarrow \bar{L}_0 = 1 \otimes 0^2 \perp 0^2 \otimes 0^2 = 1 \oplus 1 \perp 0^4$. Hence $V_7 \downarrow L_0 = 1 \oplus 1 \perp 0^3$. It follows also that $V_7 \downarrow A_1\tilde{A}_1 = 1 \otimes \bar{1} \oplus 0 \otimes \bar{W}(2)$.

Next we establish $V_7 \downarrow A_2$. As the $A_2$ is a subsystem subgroup of $G_2$, it is in a subsystem of the $D_4$. It is therefore contained in an $A_3$. We can see

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easily that $\lambda_1$ for $D_4$ restricts to $A_3$ as $\lambda_1 + \lambda_3 = \lambda_1 + \lambda_1^*$ (see e.g. [13.3.4]). Since $A_2$ sits inside $A_3$ such that the natural module for $A_3$ restricts to $A_2$ as $\lambda_1 \oplus 0$ we see that $V_7 \downarrow A_2 = \lambda_1 + \lambda_1^* \oplus 0$.

Using this we can restrict to the irreducible $A_1 \leq A_2$ for $p > 2$, and to $Z_2 \leq A_2$, when $p = 2$. In this case the natural module for $A_2$, $\lambda_1 \downarrow A_1 = 2$ for $p > 2$ and $\lambda_1 \downarrow Z_2 = W(2)$. Hence $V_7 \downarrow A_1 = 2 \oplus 2 \oplus 0$ and $V_7 \downarrow Z_2 = W(2) \oplus W(2)^* \oplus 0$.

Now we compute $V_7 \downarrow X$ for $X := A_1 \hookrightarrow A_1 \tilde{A}_1$ twisted by $p^r$ on the first factor and $p^s$ on the second. Using the decomposition above, we read off $V_7 \downarrow X = 1^{(p^r)} \otimes 1^{(p^s)} \oplus 2^{(p^s)}$. For $s = r = 0$ when $p = 2$, this gives $V_7 \downarrow Z_1 = T(2) \oplus 2/0$.

Lastly let $X = \tilde{A}_2$ ($p = 3$). One checks that a base of simple roots $\{\beta_1, \beta_2\}$ for $G$ is expressed in terms of the roots of $D_4$ as $\frac{1}{3}(\alpha_1 + \alpha_3 + \alpha_4), \alpha_2\}$. On these two elements, the weight $\lambda_1$ for $D_4$ has $\lambda_1(\beta_1) = 1$ and $\lambda_1(\beta_2) = 1$ implying $V_8 \downarrow \tilde{A}_2$ has composition factors $11|00$ so that $V_7 \downarrow \tilde{A}_2 = 11$.

### 3 Complements in parabolics: proof of Theorem 1

Let $G = G_2(K)$ with $K$ algebraically closed of characteristic $p$ and let $X \cong A_1(K)$ be a subgroup of $G$ contained in a parabolic subgroup $P = LQ$ of $G$. Then $X$ is a complement to $Q$ in $L_0Q$, where $L_0$ denotes the simple subgroup of $L$ generated by the unipotent elements. In the cases we are considering $L_0 = L'$.

**Lemma 3.1.** If $X$ is not conjugate to $L_0$, then $p = 2$ and $X$ is contained in the long root parabolic subgroup of $G$.

**Proof.** Using 2.2, for the short root parabolic one calculates that there are two levels in $Q$ and they have the structure of $KL_0$ modules with high weights 0 and 3 respectively. For $p > 3$ they are restricted and thus irreducible. For $p = 3$ they are the modules 0 and $1^{(3)}/1$; for $p = 2$ they are 0 and $1^{(2)}/1$. 

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For the long parabolic one calculates that there are three levels with high weights 1, 0, and 1 respectively. These are restricted and irreducible for all characteristics.

As \( \breve{L}_0 \) (resp. \( \check{L}_0 \)) has some odd weights on the modules in \( Q \), it is simply connected and hence admits a morphism \( \phi \) to \( X \). Composing this with the projection \( \pi \) to the Levi factor, we have the morphism \( \pi \circ \phi : L_0 \to L_0 \). It follows that \( \pi \circ \phi \) is an isogeny. We may assume that this is the standard Frobenius morphism corresponding to \( x \mapsto x^{(q)} \), say. This has the effect of twisting the modules found for \( \breve{L}_0 \) or \( \check{L}_0 \) above. Comparing these weights with \( 2.1 \), we see that none of the modules admitting a non-trivial \( H^1 \) is present unless \( p = 2 \), \( q \) is non-trivial, and \( X \) is in the long parabolic, a complement to \( Q \) in \( \breve{L}_0 \).

From this point we assume that \( p = 2 \), \( X \leq P \) the long root parabolic, a complement to \( Q \) in \( \breve{L}_0 \) and \( X \) is not conjugate to \( \breve{L}_0 \). As \( H^1(X,1^{(q)}) \) is 1-dimensional for all \( q > 1 \) we may assume that \( q = 2 \), observing that we can obtain any other complement to \( Q \) by applying a Frobenius map to an appropriate complement we get for \( q = 2 \).

Some notation is necessary for the next part of the paper. Recall the notation from \([4]\) which uses \( x_r(t) \) to refer to the root element with parameter \( t \) corresponding to the root \( r \). Since we are working entirely within \( G \), we will use \( x_i(t) \) for \( i \in \{ \pm 1, \ldots, \pm 6 \} \). If we write \((a,b)\) for \( a\alpha_1 + b\alpha_2 \) with \( \alpha_1 \) the short fundamental root and \( \alpha_2 \) the long fundamental root of \( G \), then

\[
[x_1, x_2, x_3, x_4, x_5, x_6] = [x_{(1,0)}, x_{(0,1)}, x_{(1,1)}, x_{(2,1)}, x_{(3,1)}, x_{(3,2)}]
\]

Under this notation and that of Lemma \(2.2\)

\[
Q = Q(1) = \langle x_i(t) : i \in \{1, 3, 4, 5, 6\} \rangle
\]
\[
Q(2) = \langle x_i(t) : i \in \{4, 5, 6\} \rangle
\]
\[
Q(3) = \langle x_i(t) : i \in \{5, 6\} \rangle.
\]

We see then that \( Q/Q(2) \), \( Q(2)/Q(3) \) and \( Q(3) \) are modules for \( X \) of high weights 2, 0 and 2, respectively.

**Lemma 3.2.** Let \( k, l \in K \). The groups \( X_{k,l} \) generated by

\[
x_+(t) = x_2(t^2)x_3(kt)x_6(k^3t + lt) \quad \text{and} \quad x_-(t) = x_2(t^2)x_1(kt)x_5(lt)
\]
for all \( t \in K \) are closed complements to \( Q \) in \( \bar{L}_0 Q \).

**Proof.** We certainly have \( X_{k,l} Q = \bar{L}_0 Q \) as \( \bar{L}_0 Q \) is generated by \( \{ x_i(t) \} \) for \( i \in \{ 1, 2, 3, 4, 5, 6, -2 \} \). It remains to show that \( X_{k,l} \) is isomorphic to \( A_1(K) \), and it follows that \( X_{k,l} \cap Q = \{ 1 \} \) as required.

To show this we will check the generators and relations given in [4, 12.1.1 & Rk. p198], leaving us to show the following three statements hold:

(i) \( x_\pm(t_1)x_\pm(t_2) = x_\pm(t_1 + t_2) \),
(ii) \( h_+(t)h_+(u) = h_+(tu) \) and
(iii) \( n_+(t)x_+(t_1)n_+(t)^{-1} = x_-(-t^{-2}t_1) \),

for all \( t_1, t_2 \in K \) and \( t, u \in K^\times \) where \( n_+(t) = x_+(t)x_-(t^{-1})x_+(t) \) and \( h_+(t) = n_+(t)n_+(-1) \). We will abbreviate \( n_{\alpha_i}(t) \) to \( n_i(t) \), similarly for \( h_{\alpha_i}(t) \).

Using the commutator relations for \( G_2 \) given in [4, 5.2.2] we show that these relations hold.

Write

\[
\begin{bmatrix} \alpha \\ t \end{bmatrix} := x_i(t).
\]

Firstly, item (i) is easily checked: no positive linear combination of roots \( \alpha_2 \), \( \alpha_3 \) and \( \alpha_6 \) is a root except for the roots themselves, so

\[
\begin{bmatrix} 2 \\ t^2 \end{bmatrix}, \begin{bmatrix} 3 \\ kt \end{bmatrix} \text{ and } \begin{bmatrix} 6 \\ k^3t + lt \end{bmatrix},
\]

all commute with each other. The same argument follows for \( x_-(t) \).

For (ii), we first calculate \( n_+(t) \). So we must simplify

\[
\begin{bmatrix} 2 \\ t^2 \end{bmatrix} \begin{bmatrix} 3 \\ kt \end{bmatrix} \begin{bmatrix} 6 \\ k^3t + lt \end{bmatrix} \begin{bmatrix} -2 \\ t^{-2} \end{bmatrix} \begin{bmatrix} 1 \\ kt^{-1} \end{bmatrix} \begin{bmatrix} 5 \\ lt^{-1} \end{bmatrix} \begin{bmatrix} 2 \\ t^2 \end{bmatrix} \begin{bmatrix} 3 \\ kt \end{bmatrix} \begin{bmatrix} 6 \\ k^3t + lt \end{bmatrix}
\]

We will move all \( \pm \alpha_2 \) root elements to the left. The result of this calculation is

\[
n_+(t) = \begin{bmatrix} 2 \\ t^2 \end{bmatrix} \begin{bmatrix} -2 \\ t^{-2} \end{bmatrix} \begin{bmatrix} 2 \\ t^2 \end{bmatrix} \begin{bmatrix} 4 \\ k^2 \end{bmatrix} = n_2(t^2)x_4(k^2).
\]
Now it is easy to write down $h_+(t)$. Since $x_{\pm 2}(t)$ commute with $x_4(u)$ as $\alpha_4 \pm \alpha_2$ are not roots we have

$$h_+(t) = \begin{bmatrix} 2 & -2 \\ t^2 & t^{2} \end{bmatrix} \begin{bmatrix} 2 & 4 \\ t^2 & k \end{bmatrix} \begin{bmatrix} 2 & -2 \\ t^2 & 1 \end{bmatrix} \begin{bmatrix} 2 & 4 \\ t^2 & 1 \end{bmatrix} = \begin{bmatrix} 2 & -2 \\ t^2 & 1 \end{bmatrix} \begin{bmatrix} 2 & 4 \\ t^2 & 1 \end{bmatrix} = n_2(t^2)n_2(1) = h_2(t^2).$$

It is then immediate that (ii) follows, since it holds for $h_2(t)$. Part (iii) is similar. $\square$

Notice that $X_{0,0} = \bar{L}_0$ and so $X$ is not conjugate to $X_{0,0}$ by our standing assumption. The next two lemmas are necessary to show that the groups $X_{k,l}$ exhaust all closed complements to $Q$ in $\bar{L}_0Q$.

**Lemma 3.3.** The groups $X_{k,0}Q(2)$ are distinct up to $Q/Q(2)$-conjugacy in $XQ/Q(2)$ and so form a space isomorphic to $H^1(XQ(2)/Q(2), Q/Q(2))$.

**Proof.** $X_{k,0}Q(2)/Q(2)$ is generated by root groups $x_{+,k}(t) = x_2(t^2)x_3(kt)Q(2)$ and $x_{-,k}(t) = x_{-2}(t^2)x_1(kt)Q(2)$. Take a fixed, arbitrary element of $Q/Q(2)$, $g := x_1(c_1)x_3(c_2)Q(2)$. Conjugating $x_{+,k}(t)$ by $g$ we get

$$x_{+,k}(t)^g = x_2(t^2)x_3(c_1t^2 + kt)Q(2)$$

and accordingly for $x_{-,k}(t)^g$. Suppose these generate $X_{k',0}Q(2)/Q(2)$. Then we have an automorphism of $X_{k',0}Q(2)/Q(2) \cong PSL_2(K)$ extending the map $x_{+,k}(t) \to x_{+,k}(t) \to x_{+,k}(t)^g$. This is an inner automorphism. So we must have both root groups $x_{+,k'}(t)$ and $x_{+,k}(t)^g$ conjugate, say

$$x_{+,k'}(t)^{hQ(2)} = (x_2(t^2)x_3(k't)Q(2))^{hQ(2)} = x_2(t^2)x_3(c_1t^2 + kt)Q(2) = x_{+,k}(t)^g$$

for some $hQ(2) \in X_{k',0}Q(2)/Q(2)$. In particular they are conjugate modulo $Q$ in $X_{k',0}Q/Q$ by $hQ$. Then since $x_{+,k'}(t)Q = x_{+,k}(t)^gQ = x_2(t^2)Q$, $hQ$ must centralise $x_2(t^2)Q$ in $X_{k',0}Q/Q$. It follows that

$$hQ = x_2(u_1)Q \quad (\ast)$$
Now, using the canonical form of any element of \(X_{k,0}\) is uniquely expressible as either
\[
h = x_{+,(k')}(v_1)h_2(v_2)Q(2) \quad \text{or} \quad h = x_{+,(k')}(v_1)h_2(v_2)n_2x_{+,(k')}(v_3)Q(2)
\]
where \(n_2\) is a representative of the non-identity element of the Weyl group of \(X_{k',0}\). In the latter case, observe that modulo \(Q\) we have \(h = x_2(u_1^2)h_2(u_2)n_2x_2(u_3^2)Q\) which does not centralise \(x_2(t^2)\) as it is not of the (unique) form (**) - a contradiction. In the former case, observe that \(v_2 = 1\) by (**) and so \(hQ(2)\) centralises \(x_{+,(k')}(t)\). So \(c_1t^2 + kt = k't\) for all \(t \in K\). As there are at least four elements \(t \in K\) this is impossible unless \(c_1 = 0\) and \(k = k'\).

Lastly, to see that these complements form a space isomorphic to the space \(H^1(X_{Q(2)}, Q/Q(2))\), observe that \(X_{k,0}\) is the closed complement corresponding to a rational cocycle \(\gamma_k\), and we can define an addition \(\gamma_k + \gamma_{k'} = \gamma_{k+k'}\) which is evidently well-defined on equivalence classes making the collection into a one-dimensional vector space as required.

**Lemma 3.4.** The group \(X_{k,l}\) is not conjugate to \(X_{k,l'}\) by \(Q(3)\) for \(l \neq l'\). Thus for a fixed \(k\), the groups \(X_{k,l}\) form a space isomorphic to \(H^1(X, Q(3))\).

**Proof.** The proof is similar to that of the previous lemma.

**Proposition 3.5.** \(X\) is \(Q\)-conjugate to \(X_{k,l}\) for some \(k, l \in K\), \(k, l\) not both 0.

**Proof.** Firstly, observe that \(XQ(2)/Q(2)\) must also be a complement to \(Q/Q(2)\) in \(XQ/Q(2)\). As \(Q/Q(2)\) is a module for \(X\) of high weight 2, \(H^1(X, Q/Q(2)) = K\) and \(XQ/Q(2)\) admits a one-dimensional collection of complements to \(Q/Q(2)\). By 3.3 these are represented by \(X_{k,0}\). Replace \(X\) by a \(Q\)-conjugate to have \(XQ(2) = X_{k,0}Q(2)\).

Now observe \(XQ(3)/Q(3)\) is a complement to \(Q(2)/Q(3)\) in \(X_{k,0}Q(2)/Q(3)\). As \(Q(2)/Q(3)\) is a trivial module for \(X\), we have \(H^1(X, Q(2)/Q(3)) = 0\) and we may replace \(X\) by a \(Q\)-conjugate to have \(XQ(3) = X_{k,0}Q(3)\).
Finally, observe that $X$ is a complement to $Q(3)$ in $X_{k,0}Q(3)$. As $Q(3)$ is a module for $X$ of high weight 2, $H^1(X, Q(3)) = K$ and $X_{k,0}Q(3)$ admits a one-dimensional collection of complements to $Q(3)$. By 3.4 these are represented by $X_{k,l}$. Thus we may replace $X$ by a $Q$-conjugate to have $X = X_{k,l}$.

Now, if $k = l = 0$ then visibly $X_{k,l} \leq \bar{L}_0$ which we had earlier assumed was not the case.

**Lemma 3.6.** The group $X_{k,l}$ is $P$-conjugate to one of $X_{1,0}$ or $X_{0,1}$.

**Proof.** If $k \neq 0$, we can conjugate the generators of $X_{k,l}$ by the fixed element $x_4(l/k)$ by repeated use of Chevalley’s commutator formula to get that

$$x_4(l/k)X_{k,l}x_4(l/k)^{-1} = X_{k,0}.$$ 

For instance,

$$x_4(l/k)x_+(t)x_4(l/k)^{-1} = x_4(l/k)x_2(t^2)x_3(kt)x_6(k^3t + lt)x_4(l/k)$$

$$= x_2(t^2)x_3(kt)x_6(kt^2 + lt)$$

$$= x_2(t^2)x_3(kt)x_6(k^3t),$$

and the analogous calculation holds for the negative root group. Similarly we calculate that

$$h_4(k)^{-1}X_{k,0}h_4(k) = X_{1,0}.$$ 

If $k = 0$, and $l \neq 0$, again a calculation on the generators shows

$$h_4(c)^{-1}X_{0,l}h_4(c) = X_{0,1},$$

where $c$ is any cube root of $l$. 

**Lemma 3.7.** The groups $X_{1,0}$ and $X_{0,1}$ are conjugate to $Z_1$ and $Z_2$ respectively. The subgroups $Z_1$, $Z_2$ and $\bar{L}_0$ are pairwise non-conjugate in $G$.

**Proof.** The construction of $Z_2$ as a subgroup of $A_2$ acting on the symmetric square representation allows us to calculate its root groups in terms of those of $A_2$. As the $A_2$ is a subsystem of $G$ it is easy to write these generators in terms of the root groups of $G$. Choosing the embeddings appropriately, one sees that $X_{0,1}$ has precisely the same generators, hence is conjugate to $Z_2$. 

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Next, for $p = 2$, the module $V_7 = W(\lambda_1)$ for $G$ is reducible and has a trivial submodule, so $V_7 = V_6/0$ with $G \leq Sp(V_6)$. From the restriction $V_6 \downarrow Z_1 = W(2) \oplus W(2)^*$ in 2.3 we see that $Z_1$ stabilises a 1-space of $V_6$. Since the stabiliser of a 1-space is parabolic, and $G$ acts transitively on all such by [12 Thm B], it follows that $Z_1$ is in a parabolic subgroup of $G$. Since it has a different restriction to $\tilde{L}_0$ it follows that from 3.1 that it is in the long parabolic of $G$.

Now examine all the restrictions $V_7 \downarrow Z_1, Z_2$ and $\tilde{L}_0$ given by 2.3. One sees that they are all distinct. It follows that they are all distinct up to $G$-conjugacy. It now follows from 3.1 that $X_{1,0}$ is in a long parabolic, not conjugate to $Z_2$ or $\tilde{L}_0$ and so must be conjugate to $Z_1$. \[\square\]

In conclusion we have established that a complement $X$ to $Q$ in $\tilde{L}_0Q$ must be conjugate to precisely one of the subgroups $Z_1, Z_2$ or $\tilde{L}_0$. Together with 3.1, this completes the proof of Theorem 1, and Corollary 2.

4 Classification of semisimple subgroups of $G_2$: proof of Corollary 3

In the proof of Corollary 3 we need the classification of maximal subgroups of the algebraic group $G = G_2(K)$, from [14].

**Lemma 4.1.** Let $M$ be a maximal closed connected subgroup of $G$. Then $M$ is one of the following:

(i) a maximal parabolic subgroup;

(ii) a subsystem subgroup of maximal rank;

(iii) $A_1$ with $p \geq 7$.

**Proof of Corollary 3:**

Firstly, a semisimple subgroup in a parabolic of $G_2$ must be of type $A_1$ and we have determined these by Theorem 1. Secondly, the subsystem subgroups of $G_2$ are well known and can be determined using the algorithm of Borel-de
Siebenthal. They are $A_2$, $A_1\tilde{A}_1$ and $\tilde{A}_2$ ($p = 3$) where the $\tilde{A}_2$ is generated by the short roots of $G_2$.

Subgroups of maximal rank are unique up to conjugacy so to verify Corollary 3 it remains to check that we have listed all subgroups of type $A_1$ in subsystem subgroups in the table. If $X \cong A_1$ is a subgroup of $A_2$ or $\tilde{A}_2$ it must be irreducible or else it is in a parabolic; we have listed these in the table in Corollary 3. If $X \leq A_1\tilde{A}_1$, let the projection to the first (resp. second) factor be an isogeny induced by a Frobenius morphism $x \to x^{(p^r)}$ (resp. $x \to x^{(p^s)}$).

We note some identifications amongst these subgroups:

When $p \neq 2$ and $r = s$ (without loss of generality $r = s = 0$), $V_7 \downarrow X = 2 \oplus 2 \oplus 0$ which is the same as $V_7 \downarrow Y$ where $Y := A_1 \hookrightarrow A_2$ where $Y$ acts irreducibly on the natural module for $A_2$. Indeed these are conjugate since $G$ acts transitively on non-singular 1-spaces (see [12, Thm B]). When $p = 2$ we get the subgroup $Z_1$. When $r = s + 1$ and $p = 3$, we have $V_7 \downarrow X$ is a twist of $V_7 \downarrow Y$ where $Y$ is similarly irreducible in $\tilde{A}_2$, and we have actually $X$ conjugate to $Y$ up to twists: the long word in the Weyl group $w_0$ induces a graph automorphism on $\tilde{A}_2$ and it is easy to see that we can arrange the embedding $Y \leq \tilde{A}_2$ such that $Y \leq C_G(w_0)$. Now $C_G(w_0) = A_1\tilde{A}_1$ as there is only one class of involutions in $G$ when $p \neq 2$ by [7, p288]. The restriction $V_7 \downarrow X, Y$ then gives the identification required.

Finally one can see that all other subgroups listed in the table of Corollary 3 are pairwise non-conjugate as the restrictions of $V_7$ in the table are all distinct.

This proves Corollary 3.

5 Quasi-simple subgroups of $G_2 = G_2(q)$: proof of Theorem 2

Let $X(q_0)$ be a finite quasi-simple subgroup of $G_\sigma = G(q)$, defined over a field of the same characteristic as $G$, where $q, q_0 \geq 4$. We classify all such $X(q_0)$. For this we use the classification of maximal subgroups of $G_\sigma$. The following table is obtained from [10, 1.3A] for $p > 2$ and [5] for $p = 2$.

**Lemma 5.1.** Let $M$ be a maximal subgroup of $G_\sigma = G_2(q)$ where $q = p^n \geq 4$. 

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Then $M$ is conjugate to one of the following groups.

| ID | Group | Structure | Remarks |
|----|-------|-----------|---------|
| (i) | $P_a$ | $[q^5] : GL_2(q)$ | parabolic |
| (ii) | $P_b$ | $[q^5] : GL_2(q)$ | parabolic |
| (iii) | $C_G(s_2)$ | $SL_2(q) \circ SL_2(q). (q - 1, 2)$ | involution centraliser |
| (iv) | $I$ | $2^3.L_3(2)$ | $q = p$, odd |
| (v) | $K_+$ | $SL_3(q) : 2$ | long |
| (vi) | $K_+$ | $SL_3(q) : 2$ | $p = 3$, short |
| (vii) | $K_-$ | $SU_3(q) : 2$ | long |
| (viii) | $K'_-$ | $SU_3(q) : 2$ | $p = 3$, short |
| (ix) | $C_G(\phi)$ | $G_2(q_1)$ | $q = q_1^\alpha$, $\alpha$ a prime |
| (x) | $C_G(\phi)$ | $2G_2(q)$ | $p = 3$, $n$ odd |
| (xi) | $PGL_2(q)$ | | |
| (xii) | $L_2(8)$ | | |
| (xiii) | $L_2(13)$ | | $p \neq 13$, $GF(q) = GF(p)[\sqrt{13}]$ or $q = 4$ |
| (xiv) | $G_2(2)$ | | $q = p \geq 5$ |
| (xv) | $J_1$ | | $q = 11$ |
| (xvi) | $J_2$ | | $q = 4$. |

Proof of Theorem 2:

If $X(q_0)$ has rank 2 then it is $2G_2(q_0)$, $G_2(q_0)$ or $A_2(q_0)$ and one can see that $X(q_0) \leq M$ where $M$ has ID (v)-(x) of the same type as $X(q_0)$: it is obvious for $X(q_0)$ of rank 2, $M$ cannot be as in cases (i)-(iv) and (xi); for cases (xii)-(xvi) one checks the appropriate pages in the Atlas [6]. Such subgroups are unique up to $G_\sigma$-conjugacy by [16, 5.1]. Therefore we have $X(q_0) \leq X$ a $\sigma$-stable subgroup of $G$ of the same type.

We now consider the case where $X(q_0)$ has rank 1. Here $X(q_0) \cong A_1(q_0)$. We show that each of these is contained in a $\sigma$-stable connected subgroup of type $A_1 \leq G$. Let $X(q_0) \leq M$, a maximal subgroup of $G_\sigma$. Firstly, if $M$ is case (i) or (ii), $X(q_0) \leq P_a$ or $P_b$ and we can use the proof of Theorem 1 to show that $X(q_0)$ is conjugate to a subgroup of a Levi or, when $p = 2$, to a subgroup of one of the $\sigma$-stable subgroups $X_{k,l} \cong A_1$ defined above: 2.1 implies the groups $H^1(X, V)$ are still the same for all $q$ and $V$ being considered, 2.2 still
applies for finite groups, and so 3.5 goes through to show that \(X(q_0) \leq X_{k,l}\), a \(\sigma\)-stable subgroup of \(G\) as required.

If \(M\) is as in case (iii), \(X(q_0)\) is embedded in \(SL_2(q) \circ SL_2(q)\), twisted by \(p^r\) on the first factor and \(p^s\) on the second. We may assume \(p^r, p^s < q\). Since \(\sigma\) commutes with the twists on each factor, we have \(X(q_0) \leq A_1\) where \(A_1 \hookrightarrow A_1\tilde{A}_1; x \mapsto (x^{p^r}, x^{p^s})\) and is clearly \(\sigma\)-stable.

If \(X(q_0) \leq M\) where \(M\) has ID (iv) then \(X(q_0) = L_2(7) \cong L_3(2)\). Checking [6, p60] one sees that the subgroup \(2^3.L_3(2)\) is a non-split extension with normal subgroup \(2^3\) so does not contain a subgroup of type \(L_3(2)\).

We cannot have \(X(q_0) \leq M\) if \(M\) has ID (xii) or (xiii) as these do not contain subgroups of type \(A_1(q_0)\), which is easily seen using [6, p6 and p8].

If \(M\) has ID (xi), an \(A_1(q_0) = L_2(q_0)\) in the \(\text{PGL}_2(q)\) above is unique up to conjugacy and thus in the \(\sigma\)-stable maximal \(A_1\).

**Lemma 5.2.** Let \(M\) have ID (xiv) or (xv). Then \(X(q_0) = L_2(7)\) or \(L_2(11)\) respectively and it is conjugate to the subgroup \(L_2(7) \leq \text{PGL}_2(7)\) or \(L_2(11) \leq \text{PGL}_2(11)\) respectively with ID (xi) in the above list.

**Proof.** Pages 36 and 60 respectively of the Atlas substantiate the fact that we must have \(X(q_0) = L_2(7)\) or \(L_2(11)\) (rather than \(L_2(7^2)\) or \(L_2(11^2)\) for example). Examining the 7-dimensional Brauer characters in the Modular Atlas [8] of \(L_2(7) \leq G_2(2)\) and \(L_2(7) \leq \text{PGL}_2(7)\) one sees that they are irreducible and therefore conjugate in \(\text{GL}_7(7)\). Similarly, the Brauer characters of \(L_2(11) \leq J_1\) and \(L_2(11) \leq \text{PGL}_2(11)\) in \(G_2(11)\) are the same irreducible representation and therefore conjugate in \(\text{GL}_7(11)\). The result [10, 1.5.11] then implies that they are conjugate in \(G(q_0)\). Thus in each case, the subgroup \(X(q_0)\) is in the \(\sigma\)-stable maximal \(A_1\) of \(G\).

**Lemma 5.3.** Let \(M\) have ID (v)-(viii). If \(X(q_0)\) is a subgroup of \(SL_3(q)\) or \(SU_3(q)\) and is distinct from those already considered, then \(q_0\) is odd and \(X(q_0)\) is irreducible on the standard modules in each case. Moreover, each is contained in a \(\sigma\)-stable subgroup of \(\tilde{A}_1 \leq G\).

**Proof.** The action of \(A_1(q_0)\) on the standard module \(V\) for \(SL_3(q)\) or \(SU_3(q)\) must be irreducible or else it is in a parabolic and already considered. It follows that \(q_0\) is odd.
The fact that $A_1(q_0)$ is irreducible gives the restriction of the three-dimensional standard module as a high weight 2. Thus it is unique up to conjugacy in $GL_3(q)$ (or $GU_3(q)$) by [11, 2.10.4(iii)]. Hence it is contained in a $\sigma$-stable $A_1 \leq A_2$.

**Lemma 5.4.** Let $M$ have ID (xvi). Then $X(q_0) = L_2(4)$ and $X(q_0)$ is contained in a subsystem subgroup and is thus already considered.

**Proof.** Checking the maximal subgroups of $J_2$ in the Atlas, one establishes that $A_1(q_0) = L_2(4) \cong A_5$. A simple Magma [3] calculation in $G_2(4)$ shows all of these lie within subsystem subgroups.

Observe finally that if $X(q_0) \leq M$ for $M$ with ID (ix) or (x) then $X(q_0)$ is in $G_2(q_0)$ or $^2G_2(q_0)$. It is thus in one of its maximal subgroups and has already been considered, completing the proof of Theorem 2 and this paper.

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**References**

[1] H. Azad, M. Barry and G. Seitz, On the structure of parabolic subgroups *Comm. Algebra* 18 (1990), 551-562.

[2] H. Andersen, J. Jørgensen and P. Landrock, The projective indecomposable modules of $SL(2,p^n)$, *Proc. London Math. Soc. (3)* 46 (1983), no.1, 38-52.

[3] W. Bosma, J. Cannon, and C. Playoust, The Magma algebra system. I. The user language. *J. Symbolic Comput.* 24 (1997), no. 3-4, 235-265.

[4] R. W. Carter, Simple groups of Lie type, *Wiley-Interscience*, London, 1972.

[5] B. N. Cooperstein, Maximal subgroups of $G_2(2^n)$, *J. Algebra* 70 (1981), 23-26.
[6] J. H. Conway, R. T. Curtis, S. P. Norton, R. A. Parker and R. A. Wilson, Atlas of finite groups. Maximal subgroups and ordinary characters for simple groups. With computational assistance from J. G. Thackray Oxford University Press, Eynsham, (1985).

[7] N. Iwahori, Centralizers of involutions in finite Chevalley groups, Lecture Notes in Math. 131, Springer, Berlin, (1970).

[8] C. Jansen, K. Lux, R. Parker and R. Wilson, An Atlas of Brauer Characters, London Mathematical Society Monographs. New Series, Vol. 11, Appendix 2 by T. Breuer and S. Norton, Oxford Science Publications, The Clarendon Press Oxford University Press, New York, (1995).

[9] J. C. Jantzen, Representations of algebraic groups. Second edition. Mathematical Surveys and Monographs 107, American Mathematical Society, Providence, RI, (2003).

[10] P. B. Kleidman, The maximal subgroups of the Chevalley groups $G_2(q)$ with q odd, of the Ree groups $^2G_2(q)$, and of their automorphism Groups. J. Algebra 117 (1988), 30-71.

[11] P. B. Kleidman and M. W. Liebeck. The Subgroup Structure of the Finite Classical Groups, LMS Lecture Note Series 129, Cambridge Univ. Press (1990).

[12] M. W. Liebeck, J. Saxl, and G. M. Seitz, Factorizations of simple algebraic groups, Trans. Amer. Math. Soc. 348 (1996), no. 2, 799822.

[13] M.W. Liebeck and G.M. Seitz, Reductive subgroups of exceptional algebraic groups, Mem. Amer. Math. Soc. 121 (1996), no. 580.

[14] M.W. Liebeck and G.M. Seitz, The maximal subgroups of positive dimension in exceptional algebraic groups, Mem. Amer. Math. Soc. 169 (2004), no. 802.

[15] M.W. Liebeck and G.M. Seitz, On the subgroup structure of exceptional groups of Lie type, Trans. Amer. Math. Soc. 350 (1998), 3409-3482.

[16] M.W. Liebeck and G.M. Seitz, Subgroups generated by root elements in groups of Lie type, Annals of Math. 139 (1994), 293-361.

[17] J.-P. Serre, Moursund Lectures, University of Oregon Mathematics Department, (1998), arXiv:math/0305257v1.
[18] G. M. Seitz, The maximal subgroups of classical algebraic groups, *Mem. Amer. Math. Soc.* **67** (1987), no. 365.

[19] D. M. Testerman, Irreducible subgroups of exceptional algebraic groups, *Mem. Amer. Math. Soc.* **75** (1988), no. 390.

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