A Statistical Model of Current Loops and Magnetic Monopoles

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We formulate a natural model of current loops and magnetic monopoles for arbitrary planar graphs, which we call the monopole-dimer model, and express the partition function of this model as a determinant. We then extend the method of Kasteleyn and Temperley-Fisher to calculate the partition function exactly in the case of rectangular grids. This partition function turns out to be a square of the partition function of an emergent monomer-dimer model when the grid sizes are even. We use this formula to calculate the local monopole density, free energy and entropy exactly. Our technique is a novel determinantal formula for the partition function of a model of vertices and loops for arbitrary graphs.

INTRODUCTION

The dimer model on a planar graph $G$ is a statistical mechanical model which idealises the adsorption of diatomic molecules on $G$. The associated combinatorial problem is the weighted enumeration of all dimer covers of $G$, also known as perfect matchings or 1-factors. This problem was solved in a beautiful and explicit way by Kasteleyn [1] and by Temperley-Fisher [2]. The related monomer-dimer model, which idealises the adsorption of both monoatomic as well as diatomic molecules on $G$, has not had as much success. In this case, one considers the weighted enumeration of all possible matchings of $G$ with separate fugacities for both kinds of molecules. There is some indirect evidence that it is not likely to be exactly solvable [3]. The only solutions so far are obtained by perturbative expansions (see the review in [4], for example). The asymptotics of the free energy has been studied by various authors, see [5–7] for instance. There have also been several numerical studies [8–10], as well as study of monomer correlations in a sea of dimers [11]. We note that there has been some success in solving restricted versions of the classical monomer-dimer model exactly, either for finite size or in the limit of infinite size. Such is the case for a single monomer on the boundary [12, 13], arbitrary monomers on the boundary in the scaling limit [14] and a single monomer in the bulk in the thermodynamic limit [15].

Our primary aim in this article is to solve an integrable version of the monomer-dimer model on a planar graph. We will consider a model where, for every fixed choice of monomers, we will consider all possible ways of superposing two dimer model configurations on top of each other, generalizing the so-called double-dimer model [16]. The configurations of this model are thus single vertices, doubled edges and even oriented loops on the graph. The crucial aspect of the model, which relates it to current loops and magnetic monopoles, is that the weight of each double-dimer loop is given a sign which is the parity of the number of monomers. Inspired by Dirac’s insight into charge quantization [17], we call this the monopole-dimer model.

We note in passing that signed dimer models and signed loop models have gained attention recently, the former in the context of spin liquids [18] and the latter as an approach towards solving the Ising model [19].

We summarise the main results of this paper. We will first express the partition function of the monopole-dimer model on planar graphs as a determinant in [4]. This property is extremely useful because one can obtain a large amount of information about the model using nothing more than linear algebra. This approach has been extremely fruitful in studying the Ising model in one-dimension, the sandpile model [20] and the dimer model for general graphs [1], for instance. The infinite-dimensional version known as the Fredholm determinant has proven very useful in studying time-dependent probability distributions for the TASEP [21]. We will use this determinant formula to solve the model on the finite two-dimensional grid and express the partition function as a product formula in (8). This gives a natural generalisation of Kasteleyn’s and Temperley-Fisher’s formula for the dimer model on the rectangular grid. To the best of our knowledge, such a formula has not appeared in the literature before. We will then derive an exact formula for the free energy in the limit and compare it with existing results, both rigorous and numerical. We will also calculate the entropy and the monomer density. The starting point, namely the determinant formula, is a consequence of a more general model of oriented loops, doubled edges and vertices on a general graph, which we will first explain.

The statements of the paper can be verified using the Maple program file Monopole.maple available from the author’s webpage or as an ancillary file from the arXiv source. An article exploring further consequences of these models and with more details of soem of the calculations here will appear separately [22].
A LOOP-VERTEX MODEL ON GENERAL GRAPHS

Our input data is an arbitrary simple (not necessarily planar) vertex- and edge-weighted labelled graph $G$ on $n$ vertices and an acyclic orientation $O$ on $G$. We will denote vertex weights by $x(v)$ for $v$ a vertex in $G$ and edge weights as $a(v, v') \equiv a(v', v)$ whenever $v$ and $v'$ are joined by an edge and the orientation is from $v$ to $v'$ in $O$. Graphs which are not simple can be absorbed in this framework by modifying the vertex weights and removing self-loops, and modifying edge weights and removing multiple edges. Any labelled graph comes with a canonical acyclic orientation, the one got by directing edges from a lower vertex to a higher one. See Figure 1 for an illustrative example.

![Figure 1](image)

**FIG. 1.** A nonplanar graph $G$ with its natural acyclic orientation on the left along with a particular loop-vertex configuration on the right.

We will now define the main objects in the loop-vertex model. A **loop-vertex configuration** $C$ consists of oriented loops of even length, doubled edges (to be thought of as loops of length 2) and vertices with the property that every vertex belongs to exactly zero or two edges.

We first define the signed weight of a loop in such a configuration. First, the sign of an edge $(v_1, v_2)$, denoted $\text{sgn}(v_1, v_2)$ is +1 if the orientation is from $v_1 \to v_2$ in $O$ and −1 otherwise. Then, given an even oriented loop $c = (c_1, \ldots, c_{2t}, c_1)$, the weight of the loop is

$$w(c) = -\prod_{i=1}^{2t} \text{sgn}(c_i, c_{i+1}) a(c_i, c_{i+1}),$$

with the understanding that $c_{2t+1} = c_1$. The reason for the overall minus sign in the weight of a loop is that we want the weight of a cycle with an odd number of "wrongly-directed" edges to contribute with a positive sign (the so-called clockwise-odd condition of Kasteleyn [1]). It is for this reason that the loop-vertex model is a direct generalisation of the dimer model. Note also that the weight of a doubled edge is always $+a(c_1, c_2)^2$.

Lastly, to each vertex $v$, we associate the weight $x(v)$. The weight $w(C)$ of a configuration $C$ is then

$$w(C) = \prod_{c \text{ a loop}} w(c) \prod_{v \text{ a vertex}} x(v).$$

For example, the weight of the configuration in Figure 1 is

$$+x(1) a(6, 9)^2 a(2, 3)a(3, 5)a(5, 8)a(7, 8)a(4, 7)a(2, 4)$$

The partition function of the loop-vertex model on $(G, O)$ is then

$$Z_{G, O} = \sum_{C \text{ a loop-vertex configuration}} w(C).$$

Whenever the acyclic orientation is canonically defined by the labelling on the graph, we will denote the partition function simply as $Z_G$.

To state the main result of this section, we now define the **modified Kasteleyn matrix** $K$ associated to the pair $(G, O)$ to be the matrix indexed by the vertices of $G$ as

$$K(v, v') = \begin{cases} x(v) & v' = v \\ a(v, v') & \text{orientation is from } v \text{ to } v' \in O \\ -a(v, v') & \text{orientation is from } v' \text{ to } v \in O. \end{cases}$$

Note that $K$ differs from the original Kasteleyn matrix by a diagonal matrix when $G$ is a planar graph and $O$ is a Kasteleyn orientation on $G$. Our main result is the following.

The partition function of the loop-vertex model on $(G, O)$ is given by

$$Z_{G, O} = \det K.$$

To see why this is true, look at the determinant expansion of $K$. We will consider the permutations in $S_n$ according to their cycle decomposition. The first observation is that the sign of a non-trivial odd cycle $c = (c_1, c_2, \ldots, c_{2t+1}, c_1)$ is the opposite of its reverse $c' = (c_1, c_{2t+1}, \ldots, c_2, c_1)$, but the weights are the same. Therefore, such terms cancel out. The only odd cycles which appear are cycles of length one, also known as fixed points.

It is then clear that the terms in the determinant expansion of $K$ are in bijection with loop-vertex configurations of $G$. We now need to show that the signs are the same. Therefore we decompose the permutation $\pi$ into $k$ fixed points and cycles of lengths $2m_1, \ldots, 2m_k$. This ensures that $k$ has the same parity as $n$.

A well-known combinatorial result states that if $n$ is odd (resp. even), $\pi$ is odd if and only if the number of cycles is even (resp. odd) in its cycle decomposition. In our case, the number of cycles is $k+c$. A short tabulation shows that the sign of $\pi$ is always the same as $(-1)^c$. In other words, the sign of a loop is precisely the product of all the corresponding terms in $K$ plus one extra sign. But this is precisely what we have in (4).

One class of examples is the loop-vertex model on the complete graph $K_n$ with its natural orientation, with
vertex-weights $x$ and edge-weights $a$. In that case, it is not too difficult to show that the partition function is given by
\begin{equation}
Z_{K_n} = \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n}{2k} x^{n-2k} a^{2k} = \frac{(x + a)^n + (x - a)^n}{2}.
\end{equation}

THE MONOPOLE-DIMER MODEL ON PLANAR GRAPHS

We now consider the case where $G$ is a planar graph. We will use $G$ to mean both the graph and its planar embedding. As before, $G$ will be a labelled graph with vertex weights $x(v)$ and edge weights $a(v, v')$ whenever $v$ and $v'$ are connected by an edge.

The monopole-dimer model on planar graphs is a model of oriented loops of even length, doubled edges and single vertices on $G$ where the weight of an even loop $c = (c_1, \ldots, c_{2\ell}, c_1)$ is given by
\begin{equation}
w(c) = (-1)^{\text{number of vertices in } G \text{ enclosed by } c} \prod_{i=1}^{2\ell} a(c_i, c_{i+1}),
\end{equation}
with $c_{2\ell+1} \equiv c_1$, the weight of vertex $v$ is $x(v)$, and the weight $w(C)$ of a monopole-dimer configuration $C$ is
\begin{equation}
w(C) = \prod_{c \text{ a loop}} w(c) \prod_{v \text{ a vertex}} x(v).
\end{equation}

The partition function of the monopole-dimer model on $G$ is then
\begin{equation}Z_G = \sum_{C \text{ a monopole-dimer configuration}} w(C).
\end{equation}

Note that the definition of the model on planar graphs is independent of any orientation.

We recall the notion of a Kasteleyn orientation for a planar graph. We will consider the case of bipartite graphs for simplicity; the general case is similar. In this case, Kasteleyn showed that there exists an orientation $O$ on $G$ such that every loop enclosing a face has an odd number of clockwise oriented edges. This is sometimes called the clockwise-odd property. Using this orientation $O$, Kasteleyn showed that the dimer partition function on $G$ can be written as a Pfaffian of an even antisymmetric matrix.

The main result of this section can now be stated. Let $O$ be a Kasteleyn orientation on the planar graph $G$ and let $K$ be the modified Kasteleyn matrix defined as $[K]$. Then the partition function of the monopole-dimer model on $G$ can be written as a determinant $|K|$.

To prove this, we have to show that the weight of a loop in a planar graph with a Kasteleyn orientation defined by $[K]$ is the same as that defined in $[O]$. Suppose the loop is of length $2\ell$ and there are $v$ internal vertices, $e$ internal edges and $f$ faces. Since the Kasteleyn orientation is such that there are $o_i$ clockwise edges in face $i$, where each $o_i$ is odd. The total number of clockwise edges on the loop is therefore $\sum_{i=1}^{f} o_i - e$ since each internal edge contributes twice to the count, once clockwise and once counter-clockwise. By Euler’s formula $v - e + f = 1$. Since the parity of $\sum_{i=1}^{f} o_i - e$ is the same as that of $f - e$, which equals $n - 1$, we have shown that the total number of clockwise edges on the loop is odd if and only if $n$ is even. This shows that the weights in $[K]$ and $[O]$ match.

This leads to the following corollary. In the absence of vertex weights $x(v) = 0$ for all $v \in V$, the loop-vertex model becomes a superposition of two dimer models and the partition function $Z_G$ becomes the square of the dimer model partition function $|\text{Pf } K|$.

THE MONOPOLE-DIMER MODEL ON THE RECTANGULAR GRID

We will now state the result for the partition function for the case of the grid, thereby generalising the famous product formula of Temperley-Fisher [2] and Kasteleyn [1]. Consider the $m \times n$ grid $Q_{m,n}$ with horizontal edge weights $a$, vertical edge weights $b$ and vertex weights $z$, and the Kasteleyn orientation $O$ prescribed by Fisher [2] and Kasteleyn [1]. We then obtain a remarkable generalisation of their result. Define the function
\begin{equation}Y_m(b; z) = \prod_{i=1}^{[m/2]} \left( z^2 + 4b^2 \cos^2 \left( \frac{j\pi}{m+1} \right) \right).
\end{equation}

The partition function of the loop-vertex model on $Q_{m,n}$ is given by
\begin{equation}Z_{m,n} = \prod_{j=1}^{[m/2]} \prod_{k=1}^{[n/2]} \left( z^2 + 4b^2 \cos^2 \left( \frac{j\pi}{m+1} \right) + 4a^2 \cos^2 \left( \frac{k\pi}{n+1} \right) \right)^2 \times \begin{cases} 1 & \text{if } m \text{ and } n \text{ are even,} \\ Y_m(b; z) & \text{if } m \text{ is even and } n \text{ is odd,} \\ Y_n(a; z) & \text{if } m \text{ is odd and } n \text{ is even,} \\ zY_m(b; z)Y_n(a; z) & \text{if } m \text{ and } n \text{ are odd.} \end{cases}
\end{equation}
The matrix $K_{m,n}$ is exactly the matrix of Fisher or Kasteleyn plus $z$ times the identity matrix of size $mn$. Therefore, the inversion technique described in either of these papers works identically. The case when both $m$ and $n$ are odd is a special case, which has to worked out separately, and can be done without too much difficulty.

An example of this result, consider the case of $m = n = 2$. Figure 2 shows $Q_{2,2}$ with a Kasteleyn orientation and a particular configuration.

FIG. 2. The graph of $Q_{2,2}$ along with its Kasteleyn orientation on the left and a loop configuration which can represent both $(1, 2, 4, 3)$ and $(1, 3, 4, 2)$ with weight $+a^2b^2$ on the right.

$$K_{(c,d),(e,f)}^{-1} = \frac{4\varepsilon_{c+d}(-i)^{c+f}}{(m+1)(n+1)} \sum_{(g,b)\in Q_{m,n}} \sin \frac{\pi b}{m+1} \sin \frac{\pi b}{n+1} \sin \frac{\pi d}{m+1} \sin \frac{\pi d}{n+1} \left( z + 2ia \cos \frac{\pi b}{m+1} - 2(-1)^f \sin \frac{\pi d}{n+1} \cos \frac{\pi g}{m+1} \right)$$

This can be used to calculate the local monopole density at $(c, d)$ as $K_{(c,d),(c,d)}^{-1}$; see, for example, Figure 3. One can also compute monopole correlations this way. For instance, the two-point correlation of monopoles at positions $(c, d)$ and $(e, f)$ far apart is given by

$$\det \begin{pmatrix} K_{(c,d),(c,d)}^{-1} & K_{(c,d),(e,f)}^{-1} \\ K_{(e,f),(c,d)}^{-1} & K_{(e,f),(e,f)}^{-1} \end{pmatrix}.$$
and they satisfy, as they must,

$$\rho_a + \rho_b + \rho_z = \frac{1}{2}$$

Surprisingly, it turns out $\rho_a$ can be concisely expressed in terms of a known elliptic function, the Heuman’s Lambda function $\Lambda_0(\theta, k)$ [24, 17.4.39] as

$$\rho_a = \frac{1}{2} (1 - \Lambda_0(\theta_a, k)) \quad (10)$$

where

$$\theta_a = \tan^{-1} \left( \frac{\sqrt{4b^2 + z^2}}{4a^2} \right), \quad k = \frac{4ab}{\sqrt{(4a^2 + z^2)(4b^2 + z^2)}} \quad (11)$$

The Heuman Lambda function has come up in various physical problems such as gravitational problems [25], vibrational studies [26], crack propagation [27] and inductance calculations in solenoids [28]. The monopole density can be written, using a nontrivial addition formula for $\Lambda$ [29, Formula 153.01] as

$$\rho_z = \frac{K(k) k z^2}{4 \pi a b} \quad (12)$$

where $K(k)$ is the complete elliptic integral of the first kind. The simplest expression we have found for $F(a, b, z)$ in general is an infinite sum involving $2F1$–hypergeometric functions,

$$F(a, b, z) = \sum_{n=0}^{\infty} \frac{(2n - 1)!!}{2^n n!} \left( \frac{4\theta^2}{a^2 - b^2}(a^2 + b^2 + z^2) \right) \frac{4^2 b^2}{8ab(2n - 1)} \begin{pmatrix} 1/2 - n \quad 1/2 + n \mid 4\theta^2 + z^2 \\ 3/2 - n \mid 4b^2 - 4a^2 \end{pmatrix} \quad (13)$$

As expected from general considerations, $F(1, 1, z)$ grows monotonically in $z$ and is concave [6] as seen in Figure 4. In accordance with the intuition developed for the monomer-dimer model [3], $F(1, 1, z)$ is smooth. One can verify that $F(1, 1, 0) = G/\pi$, where $G$ is Catalan’s constant, as expected from the dimer model [2].

The result also compares favourably with rigorous bounds for the free energy of the classical monomer-dimer model in the literature, although it is not close to numerical data; see Figure 5. Note that the plot here is as a function of dimer density $\rho = \rho_a + \rho_b$, not $z$. The transformation is a classic exercise in demonstrating equivalence of ensembles. See [10, Appendix A] for details.

One can also calculate the entropy using standard thermodynamic relations,

$$S(a, b, z) = F(a, b, z) + z\ln z \frac{\partial}{\partial z} F(a, b, z)$$

$$= F(a, b, z) - \rho_z \ln z.$$

In the special case of equal dimer weights, this leads to

$$S(1, 1, z) = F(1, 1, z) - z\ln z \frac{\pi K}{\pi(4 + z^2)} K \left( \frac{4}{4 + z^2} \right).$$

Using (18) and (12), one can show that the entropy is maximum when $z = 1$, at which point the monopole density is

$$\rho_z = \frac{1}{5\pi} K \left( \frac{4}{5} \right) \approx 0.1270.$$

FIG. 4. A plot of $F(1, 1, z)$ for $z$ varying between 0 and 1000.
Many qualitative properties of the emergent monomer-dimer model on grids are similar to those of the classical monomer-dimer model, which is of much interest to scientists in various fields. The exact formulas for grids presented here might be used to gain further insight about the monomer-dimer model. The determinantal character of the partition function for the loop-vertex model on general graphs and the monopole-dimer model on planar graphs might also prove useful in other contexts.

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