**Bouncing Ball Problem: Numerical Behavior Characterization**

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**Abstract.** This paper gives an overview of the simple yet fundamental bouncing ball problem, which consists of a ball bouncing vertically on a sinusoidally vibrating table under the action of gravity. The dynamics is modeled on the basis of a discrete map of difference equations, which numerically solved fully reveals a rich variety of nonlinear behaviors, encompassing irregular non-periodic orbits, subharmonic and chaotic motions, chattering mechanisms, and also unbounded non-periodic orbits.

1. Introduction

Consisting of a ball bouncing vertically under the action of gravity on a massive sinusoidally vibrating platform, such a deterministic system exhibits large families of irregular non-periodic solutions, fully developed chaos in addition to harmonic and subharmonic motions \([1]\) depending upon the amplitude and frequency of the driving platform and also on the coefficient of restitution \(0 \leq \varepsilon \leq 1\), which accounts for the amount of energy dissipated during each collision, with the elastic limit \(\varepsilon = 1\) giving rise to unbounded non-periodic or stochastic motions \([2]\). This system shows to be meaningful in modeling relevant nonlinear systems subject to periodic excitation \([3-7]\).

In past studies, the bouncing-ball problem has been examined in many of its ramifications. The first systematic study is attributed to Holmes \([8]\), upon constructing a discrete map whereby the conditions for stability and bifurcation of periodic trajectories are determined on the assumption the jumps of the ball are larger compared to the overall displacement of the table. Other studies based on the differential equation of motion of the ball \([9]\), or else using a mapping approach similar to Holmes’s investigated chaotic response and manifold collisions \([10]\), period doubling regime \([11]\), noise-induced chaotic motion \([12]\), the completely inelastic case \([13]\), rate of energy input into the system \([14, 15]\), and chattering mechanisms through which the ball gets locked on the vibrating table \([16,17]\).

However, there still remains a lack of information on how to access the appropriate driving parameters and also the starting conditions so as to drive the ball into a prescribed oscillation mode at a given coefficient of restitution. To supplement this kind of information and extend past studies, the present paper embraces numerical methods based on computer simulation to examine in a unified way the rich phase behavior of the bouncing ball with emphasis on the driving and launching parameters so
that the ball dropped at zero initial velocity might evolve to a desired periodic orbit and keep bouncing there.

2. Bouncing-ball behavior under various starting conditions

Let us consider an elastic ball, with coefficient of restitution $\varepsilon$, which is kept continually bouncing off a vertically oscillating base. Infinitely massive, the platform is fixed to a rigid frame that vibrates sinusoidally as $S(t) = A \sin \omega t$ so as to maintain the motion of the ball, whose dynamics is governed by a gravitational field $g$ and the impacts with the base (Fig. 1). The next collision time after the departure time $t_i$ from the platform is the smallest solution $t_{i+1} > t_i$ of the discrete-time dynamics map

$$A(\sin \omega t_{i+1} - \sin \omega t_i) = V_i(t_{i+1} - t_i) - \frac{1}{2} g (t_{i+1} - t_i)^2$$

where $V_i$ is the post impact velocity (Fig. 1), which relates to the pre-impact velocity $U_{i+1}$ at time $t_{i+1}$ through

$$U_{i+1} = V_i - g(t_{i+1} - t_i)$$

As far as the collision is partially elastic, the ball bounces back instantaneously at $t_{i+1}$ with a relative velocity

$$V_{i+1} - \dot{S}(t_{i+1}) = -\varepsilon[U_{i+1} - \dot{S}(t_{i+1})]$$

where the relative landing velocity $U_{i+1} - \dot{S}(t_{i+1})$ is always negative. Physically, the coefficient $\varepsilon$ (defined as the ratio of the relative velocities before and after the collision) gives a measure through the quantity $(1 - \varepsilon^2)$ of the energy lost in the collision. Combining (1)-(3) and non-dimensionalizing the time and velocity variables according to $t_i \to \omega t_i \equiv \phi_i$, and $v_i \to V_i(\omega / g)$ gives the phase and velocity maps

$$\phi_{i+1} = \phi_i + \tau_i$$

$$\Gamma [\sin(\phi_i + \tau_i) - \sin \phi_i] = v_i \tau_i - \frac{1}{2} \tau_i^2$$

$$v_{i+1} = -\varepsilon(v_i - \tau_i) + (1 + \varepsilon) \Gamma \cos(\phi_i + \tau_i)$$

where $\Gamma = A \omega^2 / g$ is the dimensionless shaking acceleration. With state characterized by the phase $\phi_i$ and the post-velocity $v_i$, the above discrete map describes the complete bouncing ball dynamics, which is controlled by two parameters, namely, $\Gamma$ and $\varepsilon$.

Figure 1 One-dimensional ball bouncing off a sinusoidally vibrating table. The table and ball trajectories are depicted by solid lines over time.
We now proceeding to derive the initial starting conditions for the ball to execute a periodic motion upon collision with the vibrating platform. Then we discuss the various characters of the bouncing-ball trajectories determined numerically from the full system (4). The numerical solutions are obtained by using an event driven procedure \[18\] that consists in monitoring a sequence of events for which the force and trajectory equations (4) are solved without resorting to any approximation.

Let us consider a ball that drops at zero initial velocity form the height \(H_0\) at initial phase \(\phi_0\). Changing to the nondimensionalized coordinates, \(g = H_0 w^2 / g\), the free flight is described by

\[
h = h_0 - (\phi - \phi_0)^2 / 2,
\]

such that at the collision phase \(\phi\), the height is \(h = \Gamma \sin \phi\), and since for a periodic motion of subharmonic index \(n\) the initial phase \(\phi_0\) is symmetrically spaced from the phases for the \(n-1\) and \(nth\) impacts, the following relation

\[
\phi = \phi_0 + n \pi
\]

holds, and therefore

\[
h_0 = \frac{1}{2} (n \pi)^2 + \Gamma \sin \phi
\]

The exact phase at which impact occurs is such that the upward movement of the platform compensates for the energy loss from the inelastic collisions so as the ball lands and departs from the platform at the same speed \(v = n \pi\) (relative to the laboratory frame), which is consistent with the final velocity the ball reaches after the time interval \(\phi - \phi_0 = n \pi\) given in (9). Recalling that the coefficient of restitution \(\varepsilon\) relates to \(\Gamma\) and \(\phi\) through

\[
\Gamma \cos \phi = n \pi \frac{1 - \varepsilon}{1 + \varepsilon}
\]

then for a given \(\varepsilon\) and period index \(n\) the initial height \(h_0\) and phase \(\phi_0\) are calculated from (5)-(7) provided the constraint \(0 < \Gamma \sin \phi < 1\) is fulfilled to ensure stability of the periodic orbits. Assuring the existence of periodic orbits, Eq. (11) has two solutions, one of which is unstable \[19\]. To this end, we set \(\Gamma \sin \phi = 1/2\) at \(\varepsilon = 0.85\) and \(n = 1\), this resulting in a complex pair of eigenvalues \(\lambda_{1,2} = \pm i \sqrt{\varepsilon}\). Using(11) gives \(\Gamma = 0.5611\) and \(\phi = \pm 0.3501 \pi\), which combined with Eqs (5)-(7) yields two solutions with the corresponding starting conditions \((\phi_0 = -0.6449 \pi, h_0 = (\pi^2 + 1)/2)\) and \((\phi_0 = 0.6449 \pi, h_0 = (\pi^2 - 1)/2)\). If dropped at such consistent starting parameters the ball immediately enters the 1-periodic mode, i.e., without overshoot or transient as shown in Fig. 2. Similar behavior is exhibited by the second (and unstable) solution indicated by the dotted line. Concerning the first solution, it shows a robust stability with respect to initial heights ranging in the interval as wide as \([1.823, 11.510]\).
At $\phi_0 = 65001\pi$, if we now drop the ball from initial height $h_0=1.822$, which is located just on the left limit of the stability range for $h_0$, now the the ball is unable to sustain its motion and then comes to a permanent contact with the platform by executing a convergent sequence of decaying jumps. As shown in Fig. 3, after the 10th collision the impact position commutes to a descending phase when the base, moving downward, has a negative velocity. The ball loses energy and in the next collision the ball, upon rising to a lower height, arrives further delayed; the lost synchronism cannot be restored and the ball rests immobile on the platform.

To drive a higher-order $n$th-subharmonic periodic-1 mode from required specifications, for instance $n=5$ at $\varepsilon = 0.85$ and $\Gamma \sin \phi = 0.8$ we use Eqs. (5)-(7) to consistently obtain the starting parameters $\phi_0 = 4.821\pi$, $h_0 = \left(\Gamma \sin \phi + (5\pi)^2/2\right) = 124.17$ and the drive amplitude $\Gamma = 1.504$, by noting that the collision phase $\phi = 0.1785\pi$, relates with the initial phase by $\phi = \phi_0 + 5\pi$. Instead of dropping the ball from the calculated height ($h_0=124.17$), at which the ball would enter the $n=5$, period-$I$ motion without transient, let us drop the ball from a larger height $h_0=132.0$. Preceded by a persistent sequence of somehow period-tripling oscillations, as shown in Fig. 4, the ball ultimately reaches its steady state of motion past the time span of nearly $3500/2\pi$ oscillation periods through exponentially damped oscillations, as in this case the resulting eigenvalues $\lambda_{1,2} = -0.5500 \pm 0.7362i$. 

[Figure 2] For $\varepsilon = 0.85$, period-1 modes driven at $\Gamma = 0.5611$ with collision phases $\phi = 0.3501 \pi$ ($h_0 = (\pi^2 + 1)/2$, solid line) and $\phi = -0.3501 \pi$ ($h_0 = (\pi^2 - 1)/2$, dashed line).

[Figure 3] the ball dropped from $h_0=1.822$ is unable to enter the period-1 mode. Transient motion is zoomed in the inset.
typify a stable focus. After the transient is finished, the ball reaches the collision velocity $v = 5\pi$ at the phase $\phi = 0.1785\pi$ as portrayed in the phase-space plot in Fig. 5. Detailed in Fig. 6, the steady-state oscillations consist of equal jumps separated by a periodic time span of $5\pi$; each jump in 25 times as high as the single jump in the $n=1$ periodic mode discussed in Fig. 2. Here we note that for a $nth$-subharmonic periodic mode the maximum height relative to the impact point is just $(n\pi)^2/2$, irrespective of the drive amplitude $\Gamma$. In spite of increasing $\Gamma$ the relative height remains constant, otherwise the ball would start jumping higher, thus leading to longer flights which would not be synchronized with the oscillation period of the platform. In preserving its relative height to the collision point, the wavetrain of parabolic jumps shifts as a whole by searching a new footpoint so as to keep both the landing and departure velocities matched at the synchronous value $v = n\pi$ [19].

**Figure 4** Dropped from $h_0=132.0$, at $\varepsilon = 0.85$ and with $\Gamma =1.504$, the ball reaches the 5-period mode

**Figure 5** Phase-space plot of the ball motion shown in Fig. 4. The system evolves to the converging point $(\phi/\pi, v/\pi)=(0.1775, 5)$ identifying a $n=5$ periodic mode; the three clusters of points displays the period-tripling transient time series displayed in Fig. 4

**Figure 6** Periodic mode with subharmonic index $n=5$: both the landing and departure velocities are $5\pi$ and two consecutive collision phases are spaced by $5\pi$.

Without changing the previous control parameters $\Gamma$ and $\varepsilon$, if now the ball is dropping from larger height, namely $h_0=135.0$, chaotic oscillations are generated (Fig. 7). Chaotic trajectories do not fill the phase space in a random like manner. Instead, as seen in the map map (Fig. 8) encompassing many unstable orbits which remain in the system, the trajectories fall onto a complex but well defined and bounded object (chaotic attractor) which is cut at the bottom by a cosine-shaped boundary rendered by the velocity time profile ($\Gamma\cos \omega t$, with $\omega t=\phi$) of the table oscillation. At $\Gamma=1.504$ and $\varepsilon=0.85$ one finds $|v|=10.027$, which is in good agreement with the velocity range $[-1.504, 9.291]$ portrayed in Fig. 8 [19].
At $\varepsilon = 0.85$ and $\Gamma \sin \phi = 0.8$ with $\Gamma = 1.504$, but dropped from $h_0 = 135.0$ the ball enters a chaotic motion.

Figure 7

Figure 8 Phase-space plot of the chaotic motion in Fig. 7.

Discussing now the elastic case $\varepsilon = 1$, by considering first the equation $v_2 = \varepsilon v_1 + (1 + \varepsilon) \Gamma \cos \phi$ that relates the pre- and post-collisional velocities of the ball upon impact against the moving platform, where $v_1 > 0$ denotes the incoming velocity. Then, at $\varepsilon = 1$, and for head-on collisions, in which the base moves upwards, the post-collisional reduces to $v_2 = v_1 + 2 \Gamma \cos \phi$, while for overtaking collisions (with the base moving downwards) the post-velocity turns into $v_2 = v_1 - 2 \Gamma \cos \phi$, where $\phi$ is the collision phase. If the post-collision velocity is negative (particle still moving down) then a second impact will occur, provided that $V < v_1 < 2V$ where $V = |2\Gamma \cos \phi|$. But for both types of collisions, when $\phi = n\pi / 2$, $n$ integer, the ball bounces back after collision with a departure velocity which is simply the reverse of the velocity before the bounce, neither gaining nor losing energy on the collision. For the 1-periodic mode, at $\phi_0 = -\pi / 2$, $\Gamma = 0.5$, and $h_0 = \pi^2 / 2 + \Gamma$, this situation is shown in Fig. 9. At such starting conditions, the time length between collisions is matched with the period of the platform motion, with the ball performing a sequence of perfectly equal jumps and seeing the base as it were static. Accordingly, by dropping the ball from $h_0 = (6\pi)^2 / 2 + \Gamma$ at $\phi_0 = -11\pi / 2$, the ball will execute a $n=6$ subharmonic motion (Fig. 10), in which two consecutive collisions are spaced by six periods of the base oscillation.

Figure 9 Period-1 mode at $\varepsilon = 1$ ($\Gamma = 0.5, \phi = \pi / 2$).

Figure 10 6th subharmonic mode at $\varepsilon = 1$. The inset shows that two consecutive collisions are spaced by six periods of the base oscillation.
Releasing now the ball from $h_0=0.5$ at $\Gamma = 0.5$ and $\phi_0 = -\pi/2$, we see in Fig. 11 that the jumps start increasing without limit, an example that exhibits Fermi acceleration—a process in which the particle gains energy by collision against a moving scatterer. Gain or loss of energy occurs on head-on (base moves towards the incoming particle) and overtaking collisions (base moves away from the particle), but the net result will be an average gain by the reason that increasing velocities make head-on collisions more frequent. In fact, averaging from $v_2 = v_1 + 2\Gamma \cos \phi$ the square velocity $v_2$ over an oscillation period leads to $\left\langle v_2^2 \right\rangle = \left\langle v_1^2 \right\rangle + 2\Gamma^2$, which describes a constant net energy gain per collision, thus meaning that the particle’s energy (or height) tends to increases linearly with time. It means that there is an increase of the particle energy roughly follows a linear growth, thus characterizing a random walking particle, for which the average velocity scales with the collision number as $\langle v \rangle \sim \sqrt{N}$ [5]. In the phase-space plot (Fig. 12) of this motion, there appear stochastic layers which, separated from each other, seem to be limited in their width.

![Figure 11](image1.png)  ![Figure 12](image2.png)

**Figure 11** At the start conditions $h_0=0.5$, $\phi_0=-\pi/2$, and $\Gamma=0.5$ the amplitude of the period-1 mode shows a boundless increase.

**Figure 12** Phase-space plot of the motion in Fig. 22

### 3. Conclusion
Through numerical examples from computer simulations guided by analytical considerations, this paper has presented a quantitative description of the bouncing ball problem. The dynamics is modeled on the basis of a discrete map of difference equations for the trajectory (describing the ball free fall under gravitational acceleration), velocity, and phase of the ball’s motion on assuming a constant restitution coefficient and the collisions to be instantaneous.

Once iterated numerically the equations fully reveal a rich variety of nonlinear behaviors, including non periodic motions as well as chaotic and stochastic phenomena. Numerical examples have demonstrated that the bouncing-ball behavior is strongly dependent on the control parameters ($\Gamma$ and $\varepsilon$) and also on the initial conditions.

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