Abstract: Both geometric and wave optical models, as well as classical and quantum mechanics, realize linear transformations with matrices; for plane optics, these are $2 \times 2$ and of unit determinant. Students and some researchers could assume that the structure of this matrix group is fair enough and hardly interesting. However, the properties and applications even of this lowest $2 \times 2$ case are already unexpectedly rich. While in mechanics they cover classical angular momentum, quantum spin, and represent ‘2+1’ relativity, in optical models they lead from the geometrical description of light propagation in the paraxial regime to wave optics via linear canonical transforms requiring a more penetrating view of their manifold structure and multiple covers. The purpose of this review article is to highlight the topological space of $2 \times 2$ matrices as it applies to classical versus quantum and wave models, to underline how the latter requires the double cover of the former, thus using $2 \times 2$ matrices as an alternative viewpoint of the quantization process, beside the traditional characterization by commutation and non-commutations of position and momentum.

Keywords: lie groups; representation theory; phase space

1. Introduction: Linear Transformations

We are familiar with the use of vectors and matrices, both as a convenient shorthand to write related sets of values and equations, and as a transparent way of expressing physical quantities whose description depends on position, space orientation, or time, through linear transformations. The study of matrices *per se* is generally credited to Cayley [1] and several of his contemporaries and followers, including the encyclopedic work of Muir [2] on determinants. The theory of finite groups, developed in the second half of the 19th century, particularly resulting from Schur’s lemma [3], binds the number of group elements with the number of entries of all their matrix representations. The golden age for group theory, turning on continuous groups, was started by Sophus Lie with his doctoral thesis at the University of Christiania (Oslo) [4]; the continuous groups were finally classified by Cartan [5] into four classical families: Unitary, symplectic, and odd and even-dimensional orthogonal groups—plus the five exceptional groups.

The 20th century witnessed Heisenberg’s matrix version of quantum mechanics, the recognition of spin as a fundamental non-classical property of matter, Wigner’s development of angular momentum couplings, Dirac’s relativistic electron equation, and more recently, linear optics and canonical transforms [6–8]. Most of the applications in quantum angular momentum mechanics involve large matrices that nevertheless depend on only a few ‘active’ parameters, which are actually represented by $2 \times 2$ (or special $4 \times 4$) matrices. By the mid-20th century, the mathematical theory of Hilbert spaces and Gel’fand triplets [9] allowed the consideration of ‘matrices’ with a continuum of rows and columns, which are properly called integral kernels. To underscore these constructs and methods, we consider sufficient to work with the lowest $2 \times 2$ case, pointing out some features which are generic in higher dimensions. For plane geometric optics or mechanics, the manifold of canonical transforms is the group of $2 \times 2$ real matrices.
The fundamental property of the set of square matrices is its closure under multiplication. In the $2 \times 2$ case, the product of $\mathbf{M}_1 = \begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix}$ and $\mathbf{M}_2 = \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix}$, is:

$$\begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix} \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix} = \begin{pmatrix} a_1a_2 + b_1c_2 & a_1b_2 + b_1d_2 \\ c_1a_2 + d_1c_2 & c_1b_2 + d_1d_2 \end{pmatrix},$$

(1)

where the entries $a_1, \ldots, d_2$ are numbers in some field: Real $\mathbb{R}$ or complex $\mathbb{C}$ (other fields, such as quaternions or quadratic extensions $x + y\sqrt{2}$, etc., are seldom considered). This product is not generally commutative ($\mathbf{M}_1\mathbf{M}_2 \neq \mathbf{M}_2\mathbf{M}_1$), but it is associative: $(\mathbf{M}_1\mathbf{M}_2)\mathbf{M}_3 = \mathbf{M}_1(\mathbf{M}_2\mathbf{M}_3)$, and it has a single unit element $\mathbf{1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ that satisfies $\mathbf{M}\mathbf{1} = \mathbf{1} = \mathbf{1}\mathbf{M}$ for all $\mathbf{M}$. These are three of the four axioms these matrices need to fit in a mathematical structure called a group. The fourth axiom is the existence of a unique inverse for every matrix $\mathbf{M} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, which is given by:

$$\mathbf{M}^{-1} = \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}, \quad \text{provided } \det \mathbf{M} = ad - bc \neq 0,$$

(2)

and such that $\mathbf{M}\mathbf{M}^{-1} = \mathbf{1} = \mathbf{M}^{-1}\mathbf{M}$. This last axiom restricts the matrices belonging to a group to have a non-zero determinant. Thus defined, the group is called the general linear group of $2 \times 2$ matrices, indicated as $\text{GL}(2,\mathbb{R})$ or $\text{GL}(2,\mathbb{C})$ for real or complex matrices respectively. Since the determinant of a product of matrices is the product of their determinants, a handier three-parameter subgroup is formed by the subset of those matrices that have unit determinant, $\det \mathbf{M} = ad - bc = 1$, which is called the group of special linear matrices, indicated $\text{SL}(2,\mathbb{R})$ or $\text{SL}(2,\mathbb{C})$, whose elements are identified as points in a manifold of 3 or 6 real parameters respectively.

The purpose of this review is, first, to explore the topology of the manifold and, second, show their efficacy for models of mechanical and optical systems such as those cited above. In Section 2, we present the concept of manifold covers through the examples of the two-fold cover that spin matrices provide over rotation matrices, and the two-fold cover of $\text{SL}(2,\mathbb{R})$ over the group of $3 \times 3$ matrices of ‘2+1’ relativity. Section 3 will identify $2 \times 2$ matrices with the basic elements of plane paraxial geometric optical setups (free displacements and lenses) to form imagers, classical fractional Fourier transformers, and negative displacements. In Section 4 we detail the infinite cover of $2 \times 2$ matrices for wave optics and quantum oscillator mechanics. Thus we place the two-fold cover that is necessary to parametrize the manifold of canonical integral transforms. Finally, in the concluding Section 4 we add comments on all the irreducible representations of $2 \times 2$ matrices, whose recount is beyond application in geometric optics.

### 2. $2 \times 2$ Matrices Covering $3 \times 3$ Matrices

The best known example of one manifold covering another manifold is provided by the earth’s daily rotation over a clock whose hour hand rotates twice over the circle in that period, or whose minute hand rotation is covered 24 times over the same, etc., so that every point of the latter, with a non-vanishing neighborhood, is mapped on two or 24 points on the former. Thus the question arises whether a manifold less trivial than a circle, can also cover another similar manifold more than once.
2.1. Unitary Matrices Cover Orthogonal Matrices

To see and prove the cover provided by spin over rotations, it is sufficient to provide a map between a subset of complex $2 \times 2$ and a subset of $3 \times 3$ orthogonal matrices:

$$
\mathbf{U}_\pm = \pm \begin{pmatrix} \alpha & \beta \\ -\beta^* & \alpha^* \end{pmatrix}, \quad \text{where } \det \mathbf{U}_\pm = |\alpha|^2 - |\beta|^2 = 1,
$$

$$
\mathbf{O} = \mathbf{D}^{3 \times 3}(\mathbf{U}_\pm) = \begin{pmatrix}
\text{Re} (\alpha^2 - \beta^2) & \text{Im} (\alpha^2 + \beta^2) & -2 \text{Re} (\alpha \beta) \\
\text{Im} (\beta^2 - \alpha^2) & \text{Re} (\alpha^2 + \beta^2) & 2 \text{Im} (\alpha \beta) \\
2 \text{Re} (\alpha^* \beta) & 2 \text{Im} (\alpha^* \beta^*) & |\alpha|^2 - |\beta|^2 
\end{pmatrix}.
$$

Upon writing out the complex numbers $a = x_1 + ix_2$ and $\beta = x_3 + ix_4$, we see that $\det \mathbf{U} = \sum_{i=1}^{3} x_i^2 = 1$ is a 3-sphere or, if we write $\sum_{i=1}^{3} x_i^2 = 1 - x_4^2 \leq 1$, we recognize it as the boundary (for $x_4 = 0$), and twice the center (for $x_4 = \pm 1$) and the interior (for $0 < |x_4| < 1$) of a 2-sphere. The 2:1 map (3) is quadratic and covers twice that set of real $3 \times 3$ matrices. The former is the set of all $2 \times 2$ unitary matrices of unit determinant: $\mathbf{U} \mathbf{U}^\dagger = 1$, that form the group of special (det $\mathbf{U} = +1$) unitary matrices denoted SU(2), covering 2:1 the group of special (det $\mathbf{O} = +1$) orthogonal matrices $\mathbf{O} \mathbf{O}^\dagger = 1$, denoted SO(3). (We indicate complex conjugation by $^\star$, matrix transposition by $^\dagger$, and matrix adjunction by $^\ast$.) These matrices will apply to and rotate 3-vectors $\bar{u} = (xyz)^T$ respecting their magnitude $|\bar{u}|^2 = \bar{u}^\dagger \bar{u}$.

Most importantly, the map (3) preserves the group properties between the $2 \times 2$ and $3 \times 3$ matrices:

$$
\mathbf{U}_1 \mathbf{U}_2 = \mathbf{U}_3 \implies \mathbf{D}^{3 \times 3}(\mathbf{U}_1) \mathbf{D}^{3 \times 3}(\mathbf{U}_2) = \mathbf{D}^{3 \times 3}(\mathbf{U}_3),
$$

while the unit is $\mathbf{D}^{3 \times 3}(\pm 1) = 1^{3 \times 3}$, all inverses $\mathbf{D}^{3 \times 3}(\pm \mathbf{U}^{-1}) = \mathbf{D}^{3 \times 3}(\pm \mathbf{U})^{-1}$ exist, and associativity holds as it does for all matrices.

The SU(2) manifold is a 3-sphere which is simply connected, as all spheres are; this means that any closed path in the manifold can be contracted to a point. The SO(3) manifold on the other hand can be visualized as the interior of a 2-sphere corresponding to all rotation axes $\bar{n}(\theta, \phi)$ with radii given by the rotation angles $0 \leq \phi \leq \pi$. However since a rotation by $\pi$ around $\bar{n}$ is the same as a rotation by $\pi$ around the antipodal $-\bar{n}$ point on the surface of that sphere, the two transformations are identified as the same group element. This manifold is thus doubly connected: Trajectories which cross the sphere ‘surface’ an even number of times can be contracted to a point; those that cross it an odd number can contract only to a single-crossing trajectory.

2.2. Pseudo-Unitary Matrices Cover Pseudo-Orthogonal Matrices

Closely related to the cover afforded by SU(2) over SO(3) in (3), is the cover given by:

$$
\mathbf{V}_\pm = \pm \begin{pmatrix} \alpha & \beta \\ -\beta^* & \alpha^* \end{pmatrix}, \quad \text{where } \det \mathbf{V}_\pm = |\alpha|^2 - |\beta|^2 = 1,
$$

$$
\mathbf{P} = \mathbf{D}^{2 \times 2}(\mathbf{V}_\pm) = \begin{pmatrix}
\text{Re} (\alpha^2 - \beta^2) & \text{Im} (\alpha^2 + \beta^2) & 2 \text{Im} (\alpha \beta) \\
\text{Im} (\beta^2 - \alpha^2) & \text{Re} (\alpha^2 + \beta^2) & 2 \text{Re} (\alpha \beta) \\
-2 \text{Re} (\alpha^* \beta) & 2 \text{Im} (\alpha^* \beta^*) & \alpha^2 + |\beta|^2 
\end{pmatrix}.
$$

The $\mathbf{V}$’s form a group of matrices that satisfy $\mathbf{V} \mathbf{V}^\dagger = \mathbf{\Omega} \mathbf{V}^\dagger$, with $\mathbf{\Omega} \mathbf{\Omega}^\dagger := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, thus called special (det $\mathbf{V} = +1$) pseudo-unitary and denoted SU(1, 1), while the $\mathbf{P}$’s also form a group of real matrices that satisfy $\mathbf{P} \mathbf{\Omega} \mathbf{P}^\dagger = \mathbf{\Omega} \mathbf{P}^\dagger$, where $\mathbf{\Omega} \mathbf{\Omega}^\dagger := \text{diag} (1, 1, -1)$ characterizes the ‘2+1’ special pseudo-orthogonal or Lorentz group of three-dimensional relativity, denoted SO(2, 1). In this case, the $3 \times 3$ matrices apply to 3-vectors $\bar{v} = (xyt)^T$, and respect the ‘space-time’ magnitude $|\bar{v}|^2 = \bar{v}^\dagger \mathbf{\Omega} \bar{v}$.

To examine their manifolds, we again write the entries of $\mathbf{V}$ with $a = x_1 + ix_2$ and $\beta = x_3 + ix_4$, so that now the determinant is $\det \mathbf{V} = x_1^2 + x_2^2 - x_3^2 - x_4^2 = 1$. Written as $x_1^2 + x_2^2 - x_3^2 = 1 + x_4^2 \geq 1$ in three-dimensional space, this is the surface (for $x_4 = 0$) and
twice the exterior (for \(|\pm x_4| > 0\)) of a one-sheeted hyperboloid. We have thus an \(\mathbb{R}^3\) manifold with a hyperbolic ‘hole’, which is multiply connected: Trajectories that wind a different number of loops around the hole are distinct, and cannot be deformed one onto another. The map (5) is 2:1, but leaves the SU(1, 1) group to cover further manifolds where an angular coordinate around the hyperboloid axis can be counted any times 2\(\pi\) before returning to the initial value.

While the SU(2, C) matrices do not have a real form alternative to the complex (3), we note that the complex SU(1, 1) matrices \(\mathbf{V}\) in (5) are equivalent to the set of real \(2 \times 2\) matrices \(\mathbf{M} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\) of unit determinant \(ad - bc = 1\), denoted SL(2, R). Indeed, with the unitary matrix \(\mathbf{W} := \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -i \\ 1 & i \end{pmatrix} = \mathbf{W}^{-1}\), we have:

\[
\mathbf{M} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \mathbf{W}^{-1} \mathbf{V} \mathbf{W} = \begin{pmatrix} \text{Re}(a + \beta) & -\text{Im}(a - \beta) \\ \text{Im}(a + \beta) & \text{Re}(a - \beta) \end{pmatrix}, \tag{6}
\]

\[
\mathbf{V} = \begin{pmatrix} a & \beta \\ \beta^* & a^* \end{pmatrix} = \mathbf{W} \mathbf{M} \mathbf{W}^{-1} = \frac{1}{2} \begin{pmatrix} (a + d) - i(b - c) & (a - d) + i(b + c) \\ (a - d) - i(b + c) & (a + d) + i(b - c) \end{pmatrix}. \tag{7}
\]

This brings in the realization that finding coverings is related to finding coordinates appropriate to the task. The real coordinates of \(\mathbf{M}\) hide the clear description of the SL(2, R) manifold as a 3-space with a ‘hole’. The real parameters of \(\mathbf{P} \in \text{SO}(2, 1)\) in (5) are better in showing that when the matrix acts on column 3-vectors \((x \ y \ t)^\top\), it contains trigonometric rotations in the \(x\)-\(y\) plane, and hyperbolic accelerations (‘boosts’) in the \(x\)-\(t\) and \(y\)-\(t\) planes. Before addressing in the next section the question of the group covers afforded by these representation matrices, we shall mention one more group that is also realized by \(2 \times 2\) real matrices.

### 2.3. Real SL(2N, R) and Symplectic Sp(2N, R) Matrices

Real \(2N \times 2N\) symplectic matrices \(\mathbf{S} \in \text{Sp}(2N, R)\) are even-dimensional and defined in a manner similar to that defining unitary and orthogonal matrices, but with a distinct skew-symmetric metric [10],

\[
\mathbf{S} \tilde{\Omega} \mathbf{S}^\top = \tilde{\Omega}, \quad \tilde{\Omega} := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad \det \mathbf{S} = 1, \tag{8}
\]

where \(0\) and \(1\) are \(N \times N\) null and unit submatrices. The Greek root of the peculiar name ‘symplectic’ means interwoven, imbricate, and is rarely used outside Hamiltonian mechanics and mathematics.

In the \(2 \times 2\) case \((N = 1)\) all matrices of unit determinant are also symplectic, \(\tilde{\Omega} \mathbf{S}^\top = \Omega\left(\begin{array}{cc} a & \tilde{c} \\ b & \tilde{d} \end{array}\right) = \left(\begin{array}{cc} -b & -d \\ a & -c \end{array}\right) \tilde{\Omega} = \mathbf{S}^{-1} \tilde{\Omega}, \ i.e., \ \text{Sp}(2, R) = \text{SL}(2, R)\). This is an accident among low-dimensional groups.

Symplectic matrices are indispensable for Hamiltonian mechanics and geometric optics because they embody linear phase space evolution. Positions \(\mathbf{q}(z)\) and momenta \(\mathbf{p}(z)\) in a system ruled by a Hamiltonian \(\hbar(\mathbf{q}, \mathbf{p})\) obey the \(2N\) Hamilton equations, written as a single vector equation with the symplectic metric matrix as:

\[
\frac{\mathbf{d}\mathbf{q}}{dz} = \frac{\partial h}{\partial \mathbf{p}}, \quad \frac{\mathbf{d}\mathbf{p}}{dz} = -\frac{\partial h}{\partial \mathbf{q}} \iff \frac{d}{dz} \left(\begin{array}{c} \mathbf{q} \\ \mathbf{p} \end{array}\right) + \tilde{\Omega} \left(\begin{array}{c} \nabla_{\mathbf{q}} \\ \nabla_{\mathbf{p}} \end{array}\right) h(\mathbf{q}, \mathbf{p}) = 0, \tag{9}
\]

where the evolution parameter \(z\) is time in the case of mechanics or, in optics, distance along the setup axis. Let the phase space \(2N\)-vectors be written as \(\mathbf{w} := \begin{pmatrix} \mathbf{q} \\ \mathbf{p} \end{pmatrix}\) and the phase space gradient as \(\nabla_{\mathbf{w}} := \left(\begin{array}{c} \nabla_{\mathbf{q}} \\ \nabla_{\mathbf{p}} \end{array}\right)\). A permissible linear transformation of phase space by a matrix \(\mathbf{S}\) acting as \(\mathbf{w} \mapsto \mathbf{w}' = \mathbf{S} \mathbf{w}\), is such that it conserves the Hamilton Equation (9). Since the phase space gradient transforms as \(\nabla_{\mathbf{w}} \mapsto \nabla_{\mathbf{w}'} = \mathbf{S}^{-1} \tilde{\Omega} \nabla_{\mathbf{w}}\), this conservation condition is
evinced multiplying (9) by $S$ from the left and introducing $1 = S^\top S^{-1\top}$ between $\tilde{\Omega}$ and $\nabla_w$ to write:

$$\frac{d}{dz} S w + S \tilde{\Omega} S^\top S^{-1\top} \nabla_w H(w) = 0 \quad \Rightarrow \quad \frac{d}{dz} w' + \tilde{\Omega} \nabla_{w'} H(w') = 0,$$

(10)

provided that the matrix $S$ satisfies (8), i.e., is symplectic. Transformations of phase space (linear and also nonlinear) that preserve the Hamilton equations of motion are called **canonical**. Basically equivalent definitions of canonicity are provided by transformations that keep invariant the volume and orientation of phase space elements, and those that keep invariant the Poisson brackets between the phase space coordinates \([11] [\text{Ch. 3}]\). All linear canonical transformations are produced through this action of real symplectic matrices. The outcome of the last two subsections leads us to write $\text{Sp}(2, R) \cong \text{SO}(2, 1)$. In fact, there is one more repetition of this relation in one higher dimension $N = 2$: $\text{Sp}(4, R) \cong \text{SO}(3, 2)$. This applies for two-dimensional mechanical systems with correspondingly two-dimensional momenta, or to three-dimensional optics with two-dimensional screens, where the group of $4 \times 4$ symplectic matrices covers the group of $5 \times 5$ pseudo-orthogonal matrices with metric $\Omega^{22} = \text{diag}(1, 1, 1, -1, -1) \ [11] [\text{Ch. 12}]$. These two homomorphisms are accidental; there are no more systematic cover relations between higher symplectic and pseudo-orthogonal groups.

### 2.4. Matrices in Paraxial Geometric Optics

Geometric optics uses symplectic matrices to keep track of linear transformations between ray positions, registered as the points $q$ where the ray crosses the standard plane screen $z = 0$, and the ray momentum, which is the projection $p$ on the screen plane of a vector $n$ along the ray, whose magnitude is the refraction index $n := |n|$ ($n = 1$ in vacuum).

In plane optics $N = 1$, these are coordinates $q \in \mathbb{R}$ and $p = n \sin \theta \in n[-1, 1]$, where $\theta$ is the angle between the ray $n\hat{n}$ and the normal to the screen. The coordinates $(q, p)$ are said to be **canonically conjugate**; they enter the Hamilton Equation (9) with a Hamiltonian $h = n \cos \theta$, where we note that $p^2 + h^2 = n^2$ lies on the so-called **Descartes sphere** (for $N = 1$ a circle). We thus interpret $h$ as the component of the ray vector that is normal to the screen, in the same sense that $z$ is the coordinate normal to it and becomes the evolution parameter of the ray $q(z)$ through a setup of free spaces and lenses.

Symplectic matrices are used to describe linear canonical transformations of phase space in a model of geometric optics, called **paraxial**, where the angle $\theta$ is assumed to be conveniently small, so that $\sin \theta \approx \theta$ and $\cos \theta \approx 1 + \frac{1}{2} \theta^2$, but extending $\theta \in \mathbb{R}$. For $N = 1$, paraxial optical systems have thus phase space coordinates in a plane $(q, p) \in \mathbb{R}^2$ that can be subject to linear transformations. A basis for all linear transformations in this paraxial model are ‘free’ spaces for displacements by $z \geq 0$, and ‘flat’ lenses of Gaussian power $G$ (i.e., focal distance $1/G$), which are represented through the following matrices and their actions,

$$\text{displacement } z, \quad D_z := \begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} q' \\ p' \end{pmatrix} = D_z \begin{pmatrix} q \\ p \end{pmatrix} = \begin{pmatrix} q +zp \\ p \end{pmatrix},$$

(11)

$$\text{lens power } G, \quad L_G := \begin{pmatrix} 1 & 0 \\ -G & 1 \end{pmatrix}, \quad \begin{pmatrix} q' \\ p' \end{pmatrix} = L_G \begin{pmatrix} q \\ p \end{pmatrix} = \begin{pmatrix} q \\ p - Gq \end{pmatrix}.$$

(12)

To corroborate these identifications: Displacement skews phase space, translating the point on the screen $q$ to $q'$ by $zn\theta = zp$, and leaving the ray direction $\theta$ invariant. A lens of power $G > 0$ is convex and convergent by redirecting rays of inclination $p$ to $p'$ towards the optical axis by angles $-Gq/n$ to concentrate them at the focal distance $F = 1/G$, while if $G < 0$, the lens is concave and divergent.

Representing the optical elements of the paraxial model by matrices allows us to design compound systems with little effort \([12, 13]\). As we picture light rays advancing
from left to right, one lens \( G \) between a free space \( z_1 \geq 0 \) at its left, and another \( z_2 \geq 0 \) to its right, three-element setups are represented:

\[
D_2L_GD_{z_1} = \begin{pmatrix}
1 & z_2 \\
0 & 1 \\
\end{pmatrix} \begin{pmatrix}
1 & 0 \\
G & 1 \\
\end{pmatrix} \begin{pmatrix}
1 & z_1 \\
0 & 1 \\
\end{pmatrix} = \begin{pmatrix}
1 & z_2Gz_2 + z_1 - z_2Gz_1 \\
-G & 1 - Gz_1 \\
\end{pmatrix}.
\]  \tag{13}

The order is important because the matrices act on \( \begin{pmatrix} w' \\
p' \end{pmatrix} \), the phase space vector to their right: First the light ray is acted by the \( z_1 \) flight, then by the lens, and last by the \( z_2 \) flight.

When the 1–2 element of the product matrix (13) is zero, the image \( q'(q) = \mu q \) is independent of the incoming ray direction, so the system satisfies the focal condition: \( z_2 + z_1 = z_2Gz_1 \) (i.e., \( 1/F = 1/z_1 + 1/z_2 \)), and is an imager with magnification \( \mu := 1 - z_2G = (1 - Gz_1)^{-1} = -z_2/z_1 \), being negative, the image will be inverted. On the other hand, when \( z_1 = z_2 =: \tan \frac{\pi}{4a} \) and \( G = \sin \frac{\pi}{4a} \), the matrix (13) becomes \( F^t = \begin{pmatrix}
\cos \frac{\pi}{2a} & -\sin \frac{\pi}{2a} \\
\sin \frac{\pi}{2a} & \cos \frac{\pi}{2a} \\
\end{pmatrix} \), namely a rotation of phase space \((q, p)\) by \( \alpha \), which is a fractional Fourier transform of power \( a := -2\alpha/\pi \); for \( a = 1 \) this is the standard Fourier transform [14,15],

\[
F := \begin{pmatrix}
0 & 1 \\
-1 & 0 \\
\end{pmatrix}, \quad \text{that acts as } F : (q, p) \mapsto (p, -q), \tag{14}
\]

and the inverse is \( F^{-1} = -F = \begin{pmatrix}
0 & -1 \\
1 & 0 \\
\end{pmatrix} \). Fractional Fourier transformers (for an integer or non-integer) can be concatenated with products within the set; so \( F^2 = -1 \) and \( F^4 = 1 \) brings us back to an ‘identity’ geometric optical system (there will be more on this in the next section). With a Fourier transformer piece, its inverse, and one lens, we can build:

\[
F L_G F^{-1} = \begin{pmatrix}
0 & 1 \\
-1 & 0 \\
\end{pmatrix} \begin{pmatrix}
1 & 0 \\
0 & 1 \\
\end{pmatrix} \begin{pmatrix}
1 & -1 \\
0 & 1 \\
\end{pmatrix} = \begin{pmatrix}
1 & G \\
0 & 1 \\
\end{pmatrix} = D_G. \tag{15}
\]

When the lens is divergent (concave, \( G < 0 \)), this is negative free flight. An even better setup for negative free flight is:

\[
L_{3/z}D_2L_{3/z}D_2L_{3/z} = D_{-z}, \tag{16}
\]

because its total length is only \( 2z \); it is also back-forth-symmetric so it minimizes aberrations.

In (14) we saw that four concatenated geometrical Fourier transformers bring us back to the identity transformation, while the setup (15) cannot reach back to \( z = 0 \). A third setup to produce the identity can be ‘built’ with repeated blocks of one lens and a displacement,

\[
B_{G,z} := L_G D_z = \begin{pmatrix}
1 & 0 \\
-G & 1 \\
\end{pmatrix} \begin{pmatrix}
1 & z \\
0 & 1 \\
\end{pmatrix} = \begin{pmatrix}
1 & z \\
-G & 1 - Gz \\
\end{pmatrix}. \tag{17}
\]

\[
B_{G,z}^2 = \begin{pmatrix}
1 - Gz & z(2 - Gz) \\
-G(2 - Gz) & (1 - Gz)^2 - Gz \\
\end{pmatrix} \quad \text{is } \begin{pmatrix}
-1 & 0 \\
0 & -1 \\
\end{pmatrix} \text{ for } Gz = 2, \tag{18}
\]

\[
B_{G,z}^3 = \begin{pmatrix}
(1 - Gz)^2 - Gz & z(1 - Gz)(3 - Gz) \\
-G(1 - Gz)(3 - Gz) & (1 - Gz)^3 - Gz(3 - 2Gz) \\
\end{pmatrix} \quad \text{is } \begin{pmatrix}
1 & 0 \\
0 & 1 \\
\end{pmatrix} \text{ for } Gz = 3, \tag{19}
\]

Hence, the geometric optical transformation of inversion \( \begin{pmatrix}
-1 & 0 \\
0 & -1 \\
\end{pmatrix} \) can be obtained with \( F^2 \), \( B_{2/z,z}^2 \), or \( B_{1/z,z}^3 \) setups, while the identity \( \begin{pmatrix}
1 & 0 \\
0 & 1 \\
\end{pmatrix} \) of linear transformations is built with \( F^4 \), or \( B_{3/z,z}^3 \) setups. This will be contrasted below with the inversion and unit integral canonical transformations in the linear wave model.

It is finally a natural question whether any \( 2 \times 2 \) matrix \( \begin{pmatrix} a & b \\ c & d \end{pmatrix} \) of unit determinant can be ‘represented’ by a (paraxial) optical system. Part of the difficulty in proving this, is that the decomposition of generic matrices into left- and right-triangular ones is seldom used,
compared with other decompositions such as the Euler angle, polar rotation, Iwasawa, or Bargmann, which are mathematically richer. Yet, the answer is in the affirmative: It requires up to three lenses separated by two free flights to realize all $2 \times 2$ symplectic matrices, except that we only miss a subset of zero measure as seen in (16), which can be bridged by just allowing one last displacement at either end; see [11] (Sect. 10.5) and [16]. The case $N = 2$ of three-dimensional setups with generally cylindrical lenses is addressed in Ref. [11] (Chs. 10 and 12), although the matter of minimal arrangements has been eschewed—as far as we know.

3. Canonical Integral Transforms and the Universal $2 \times 2$ Matrix Covers

Two-by-two matrices serve to present and exhaust all one-dimensional linear integral canonical transforms, including the fractional Fourier transforms that were introduced into the mathematical and physical literature hardly 50 years ago (indeed, this strangely late extension of Fourier analysis [14] has been itself the subject of a scientometric study [17]). To explore the subject in a more fundamental way we shall present, after the transforms themselves, the need of the double cover of the $N = 1$ group $\text{SL}(2,\mathbb{R}) = \text{Sp}(2,\mathbb{R})$ of $2 \times 2$ real matrices. This covering pattern has been absent from several works that address the subject, the need of the double cover of the $N = 1$ group $\text{SL}(2,\mathbb{R}) = \text{Sp}(2,\mathbb{R})$ of $2 \times 2$ real matrices. This covering pattern has been absent from several works that address the passage from the paraxial geometric optical model to its wave-optical counterpart.

3.1. Transforms of Functions and of Operators

All $2 \times 2$ matrices of unit determinant, such as the displacement and lens transformations (11) and (12), will now act on phase space Schrödinger operators of position and momentum, $\hat{q}$ and $\hat{p} = -i\partial_q$, which in turn are applied to a space of functions, $f(q) \in L^2(\mathbb{R})$, that stand for wavefield (or ‘signal’) amplitudes on a one-dimensional screen. These functions are assumed to belong for convenience to the Hilbert space of square-integrable functions over the real line, under the usual inner product $\langle f_1, f_2 \rangle = \int_\mathbb{R} dq \, f_1(q)^* f_2(q) < \infty$.

In this context, the well-known [18] Fourier transform $\mathcal{F}$ that maps functions $\mathcal{F} : f(q) \mapsto \tilde{f}(q)$, will map $\hat{q} f(q) = q f(q)$ and $\hat{p} f(q) = -i\partial_q f(q)$ as:

$$\tilde{f}(q) = \mathcal{F} : f(q) := \frac{1}{\sqrt{2\pi}} \int_\mathbb{R} dq' e^{-i q' q} f(q'), \quad \mathcal{F} : \left( \begin{array}{c} \hat{q} f(q) \\hat{p} f(q) \end{array} \right) = \left( \begin{array}{c} -i \hat{p} \tilde{f}(q) \\hat{q} \tilde{f}(q) \end{array} \right). \quad (20)$$

On the other hand, the canonical transform operator $\mathcal{C}_\mathbf{F}$ corresponding to the Fourier matrix $\mathbf{F} = \left( \begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix} \right)$ in (14), acts through its inverse $\mathbf{F}^{-1} = \left( \begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix} \right)$, as:

$$\mathcal{C}_\mathbf{F} \left( \begin{array}{c} \hat{q} \\ \hat{p} \end{array} \right) f(q) = \mathcal{C}_\mathbf{F} \left( \begin{array}{c} \hat{q} \\ \hat{p} \end{array} \right) \mathbf{C}_\mathbf{F}^{-1} \mathbf{C}_\mathbf{F} f(q) = \mathbf{F}^{-1} \left( \begin{array}{c} \hat{q} \\ \hat{p} \end{array} \right) \tilde{f}(q) = \left( \begin{array}{c} -\hat{p} \\ \hat{q} \end{array} \right) \tilde{f}(q). \quad (21)$$

The action of $\mathcal{C}_\mathbf{M}$ on the Schrödinger space operators $\hat{q}, \hat{p}$ is bilateral as shown, i.e., $\mathcal{C} \cdot \mathcal{C}^{-1}$, so it preserves commutation relations, such as Heisenberg’s $[\hat{q}, \hat{p}] := \hat{q} \hat{p} - \hat{p} \hat{q} = i\hbar$, where we can write $\mathcal{C} \hat{q} \mathcal{C}^{-1} = \hat{q} \mathcal{C} \mathcal{C}^{-1} \mathcal{C} \hat{p} \mathcal{C}^{-1}$, etc.

For generic $2 \times 2$ symplectic matrices $\mathbf{M} = \left( \begin{smallmatrix} a & b \\ c & d \end{smallmatrix} \right)$, with $ad - bc = 1$, we expect that the operators $\mathcal{C}_\mathbf{M}$ will transform the signals $f(q) \in L^2(\mathbb{R})$ on the screen, through generally integral transforms, because that is the case for the Fourier transform (20), but with a general integral kernel $\mathcal{C}_\mathbf{M}(\hat{q}, \hat{q}')$, as:

$$\mathcal{C}_\mathbf{M} f(q) = \int_\mathbb{R} dq' \, \mathcal{C}_\mathbf{M}(\hat{q}, \hat{q}') f(q'), \quad \mathcal{C}_\mathbf{M} \left( \begin{array}{c} \hat{q} \\ \hat{p} \end{array} \right) \mathbf{C}_\mathbf{M}^{-1} \mathbf{C}_\mathbf{M} \left( \begin{array}{c} \hat{q} \\ \hat{p} \end{array} \right) = \mathbf{M}^{-1} \left( \begin{array}{c} \hat{q} \\ \hat{p} \end{array} \right) \mathbf{M} \left( \begin{array}{c} \hat{q} \\ \hat{p} \end{array} \right) = \left( \begin{array}{c} d\hat{q} - b\hat{p} \\ -c\hat{q} + d\hat{p} \end{array} \right), \quad (22)$$

where the kernel $\mathcal{C}_\mathbf{M}(\hat{q}, \hat{q}')$ will be found below.
The appearance of the inverse matrix is mandated because we need the canonical transforms to form a group, so that the application of one transform \( C_{M_1} \) followed by another \( C_{M_2} \), compose as:

\[
C_{M_2} C_{M_1} = C_{M_2}^{-1} C_{M_1}^{-1} = C_{M_2}^{-1} M_1^{-1} C_{M_2}^{-1} \left( \frac{q}{p} \right) C_{M_1}^{-1} \left( \frac{q}{p} \right) = M_1^{-1} M_2^{-1} \left( \frac{q}{p} \right) = \left( M_2 M_1 \right)^{-1} \left( \frac{q}{p} \right) = C_{M_2 M_1} \left( \frac{q}{p} \right) \quad \text{(23)}
\]

Composition then follows matrix multiplication, left to right, as \( C_{M_2} C_{M_1} = \pm C_{M_2 M_1} \) for the double sign appears because the bilateral action \( C \cdots C^{-1} \) is quadratic. It also follows that \( C_{M}^{-1} = \pm C_{M^{-1}} \), and that the identity canonical transform operator \( I = \pm C_1 \), corresponding to two 'unit' transformations that commute between themselves and all other \( C_M \)'s—the first and second metaplectic units. Thus we find a fundamental upgrading of classical to quantum formalism that occurs between geometric and wave optics.

### 3.2. The Canonical Transform Kernel

The canonical transform integral kernel, \( C_M(q, q') \) in (22), can be found straightforwardly applying the integral action of \( C_M \) to \( q f(q) \) and \( \hat{p} f(q) \), the latter subject to integration by parts, that yield the following two simultaneous linear differential equations of first degree for the two arguments of the kernel [7,8],

\[
\begin{align*}
-2i \partial_q C_M(q, q') & = (cq + ia \delta_q) C_M(q, q'), \\
q' C_M(q, q') & = (dq + iab \delta_q) C_M(q, q'), \\
\Rightarrow \quad C_M(q, q') & = K_M \exp \left[ \frac{i}{2}(aq^2 + 2qq')/2b \right].
\end{align*}
\]

Here \( K_M \) is an \( M \)-dependent normalization factor that makes \( C_M(q, q') \) the kernel of unitary transformations, whose inverse is the transpose conjugate: \( C_M^{-1}(q, q') = C_M(q', q)^* \) under the \( L^2(\mathbb{R}) \) inner product, provided that its scale constant obeys \( K_{M^{-1}} = K_M^* \). In the limit to the identity transformation \( I \), the unit element under the integral transform group, the kernel becomes a Dirac \( \delta \), as:

\[
\lim_{M \to I} C_M(q, q') = \delta(q - q').
\]

This manifold of integral kernels \( C_M(q, q'), M \in \text{SL}(2, \mathbb{R}) \) form another, distinct representation of the group of \( 2 \times 2 \) matrices. Due to (23) this representation inherits the group product law as:

\[
M_2 M_1 = M_3 \quad \Rightarrow \quad \int_{\mathbb{R}} dq' C_{M_2}(q, q') C_{M_1}(q', q'') = \pm C_{M_3}(q, q''),
\]

with the inevitable sign ambiguity. Performing the integration of the \( q' \)-dependent factor in (28), we come to shift contours in the complex plane, resulting in:

\[
I(r, s) := \int_{\mathbb{R}} dq' \exp i(r^2 q'^2 + sq') = \varphi(r) \frac{\exp i \pi/4}{\sqrt{r}} \exp \left( \frac{-is^2}{4r^2} \right),
\]

where \( \varphi(r) := \begin{cases} 1 & \text{when } \arg r \in [0, \frac{1}{2} \pi] \mod 2\pi, \\ -1 & \text{when } \arg r \in [-\pi, -\frac{1}{2} \pi] \mod 2\pi, \end{cases} \)

for \( r = a_2/2b_2 + d_1/2b_1 = b_{21}/2b_2b_1 \) with specified phases \( \arg r > 0 \) for \( r > 0 \) or \( \arg r < 0 \) for \( r < 0 \), and \( s = -q/b_2 - q'/b_1 \). The factors in \( q, q'' \) then reconstitute the kernel of the product (28), and the sign ambiguity is resolved by \( \varphi(b_{21}/2b_2b_1) \). Appor-
tioning the factors in (29) to (28) determines the normalization and phase of the coefficient function in (26) that should be written with a carefully determined phase [7, 18] §9.1 as,

$$K_M = \frac{1}{\sqrt{2\pi i b}} \cdot \frac{e^{-i\frac{1}{2} \pi} \exp(-i\frac{1}{2} \arg b)}{\sqrt{2\pi |b|}}. \quad (31)$$

The singular limit $b \to 0$ of the kernel of $M := \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is an oscillating Gaussian with a Dirac-delta limit that has a stationary point at $q = aq', \ a = 1/d$. Under the integral this is:

$$\lim_{|b| \to 0} C_M(q, q') = \frac{1}{\sqrt{a}} \exp \left( \frac{c q^2}{2 a} \right) \delta \left( q' - \frac{q}{a} \right). \quad (32)$$

The paraxial lens transformation $L_G = \begin{pmatrix} 1 & 0 \\ -G & 1 \end{pmatrix}$ in (12) thus multiplies the signal by a quadratic phase $\sim \exp \left( -\frac{1}{2} G q^2 \right)$. The free flight transformation $D_2 := \begin{pmatrix} 1 & \infty \\ 0 & 1 \end{pmatrix}$ in (11) yields the Fresnel transform [19] which is a convolution of the signal at $z = 0$ with an oscillating Gaussian $e^{q^2/4} \exp \left( (q - q')^2 / 2z \right) / \sqrt{2\pi z}$.

The complex normalization constant (31) clearly depends on the phase that we care to assign to the element $b$ in the matrix: When $b > 0$ we can choose $\arg b \in \{0, \pm 2\pi, \ldots \}$ or, when $b < 0$, $\arg b \in \{\pm \pi, \pm 3\pi, \ldots \}$. This is where the double valuation, inevitable in the square root, shows that the group of canonical integral transforms covers twice the group of $2 \times 2$ real matrices. When $b = 0$ this is the (true and only) group unit, while when $b = 2\pi$ we obtain the ‘second metaplectic unit’. For $\arg b = \pi$ or $3\pi$ we have the first and second negative unit of matrices $-1$. This double cover of $\text{Sp}(2, \mathbb{R})$ is called the metaplectic group $\text{Mp}(2, \mathbb{R})$, using the Greek root $\mu \epsilon \tau \alpha$, signifying beyond symplectic. The metaplectic group does not have a faithful finite-dimensional matrix representation. For each matrix $M \in \text{Sp}(2, \mathbb{R})$ there are thus two operators, $\pm C_M$. Of course, the sign of $C_M$ must be defined consistently, the unit operator and the inverse $C^{-1}_M$ of an operator $C_M$ are unique, but the labeling of $C$’s by $2 \times 2$ matrices is not. This is referred to as the ‘metaplectic sign problem’, and reflects the fact that the set of $C_M$’s actually realizes the metaplectic group $\text{Mp}(2, \mathbb{R})$. The sign problem is controlled with the help of (29)–(30).

To the point, consider the Fourier transform $F$ in (21), and the canonical transform $C_F$ parametrized by the matrix $F = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ with $b = 1$ ($\arg b = 0$), whose kernel coincide but whose constant $K_F$ in (31) is $e^{-\frac{1}{4} \pi} / \sqrt{2\pi}$. Clearly $F$ and $C_F$ differ by this phase that distinguishes them as:

$$C_F = e^{-\frac{1}{4} \pi} \cdot F, \quad (33)$$

hence $C_F^4 = I$ but $C_F^8 = -I$, while only $C_F^8 = I$. On the other hand, when we consider $F' = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ also with $b' = 1$ but $\arg b' = 2\pi$; on the second Riemann sheet the corresponding relation is $C_{F'} = e^{-\frac{1}{4} \pi} \cdot F$. Thus, $-C_{F'}$ is the ‘second metaplectic unit,’ while $C_{F'}^8 = I$ returns the first, that is the true group unit.

3.3. Bargmann Parameters for the Metaplectic Group

We now introduce appropriate coordinates to show that real $2 \times 2$ matrices are multiply covered in the same way that the real line covers circles. We first note that the upper-left $2 \times 2$ blocks of the $3 \times 3$ matrices in (3) and (5) represent rotations in the $x$-$y$ plane in the three-dimensional vector spaces where they act. Their familiar form corresponds to the parameters $a = \exp(i\frac{1}{2} \theta)$ and $b = 0$ in the $2 \times 2$ complex matrices, and through (6), to the subgroup of rotation matrices $R(\theta) := \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix}$, with $\theta$ modulo $2\pi$.

The polar decomposition of complex numbers $z \in \mathbb{C}$ refers to their factorization into a phase and a positive radius as $z = e^{i\zeta} |z|$, $\zeta = \arg z$. For matrices, a corresponding
decomposition is into a unitary and a positive definite symmetric matrix [9] [Sect. 15]. So let us write what we shall call Bargmann coordinates [20] for $M \in \text{SL}(2,\mathbb{R})$,

$$
\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix} \begin{pmatrix} \lambda + Re \mu & Im \mu \\ Im \mu & \lambda - Re \mu \end{pmatrix} = \begin{pmatrix} \lambda \cos \theta + Re(e^{i\theta} \mu) & -\lambda \sin \theta + Im(e^{i\theta} \mu) \\ \lambda \sin \theta + Im(e^{i\theta} \mu) & \lambda \cos \theta - Re(e^{i\theta} \mu) \end{pmatrix},$

(34)

where $\theta \in \mathbb{R}$ modulo $2\pi$, $\lambda := +\sqrt{|\mu|^2 + 1} \geq 1$, $\mu \in \mathbb{C}$. (35)

When $ad - bc = 1$ and $\theta$ is in the range (35), one can straightforwardly check that there is a $1:1$ relation between the three independent coordinates $\{a, b, c\}$ and $\{\theta, Re \mu, Im \mu\}$, and conversely,

$$
\theta = \arg [a + d + i(b - c)], \quad \mu = \frac{1}{2} e^{-i\theta}[a - d + i(b + c)].$

(36)

Upon writing out the product of two such matrices $M = M_2 M_1$ with coordinates, after some algebra one arrives at:

$$
e^{i\theta} \lambda = e^{i(\theta_2 + \theta_1)} \lambda_2 \lambda_1 + e^{i(\theta_2 - \theta_1)} \mu_2 \mu_1 =: e^{i(\theta_2 + \theta_1)} \lambda_2 v \lambda_1,$n

(37)

$$
e^{i\theta} \mu = e^{i(\theta_2 + \theta_1)} \lambda_2 \mu_1 + e^{i(\theta_2 - \theta_1)} \mu_2 \lambda_1,$n

(38)

where $v := 1 + e^{-2i\theta} \mu_2 \mu_1^* / \lambda_2 \lambda_1$. (39)

Now, since $|\mu|/\lambda < 1$ from (35), it follows that in the complex plane the auxiliary quantity $v$ in (39) satisfies $|v - 1| < 1$, so it is strictly inside a disk of center in $v = 1$ and unit radius that excludes the origin, and this means that the phase of $v$ is uniquely defined. The phase of (37) and (38) thus lead to the composition law for two real $2 \times 2$ in these Bargmann coordinates, as given by:

$$
\theta = \theta_2 + \theta_1 + \arg v, \quad \arg v \in (-\frac{1}{2} \pi, +\frac{1}{2} \pi),$

(40)

$$
\mu = e^{-i\arg v}(\lambda_2 \mu_1 + e^{-2i\theta_1} \mu_2 \lambda_1), \quad \lambda = \lambda_2 |\mu| \lambda_1 \geq 1.$n

(41)

We draw attention to Equation (40) because it allows $\theta$ to evade the $2\pi$ modularity condition by rendering a $\theta$-line, $\theta \in \mathbb{R}$. This coordinate will uniquely parametrize the universal cover group $\text{SL}(2,\mathbb{R})$ of $2 \times 2$ matrices. And depending on the modularity condition we may choose for $\theta$, it serves to define and parametrize any multiple cover as well, delivering the sequence of covers:

$$
\text{SL}(2,\mathbb{R}) \overset{\text{c}}{\overset{\theta \in \mathbb{R}}{\overset{1 \text{c}}{\overset{\theta = 1}{\overset{2 \text{c}}{\overset{\theta = 1}{\overset{m \text{c}}{\overset{\theta = \frac{1}{m}}{\overset{1 \text{c}}{\overset{\theta = \frac{1}{m}}{\overset{\theta = 1}{\overset{1 \text{c}}{\overset{\theta = 1}{\overset{2 \text{c}}{\overset{\theta = \frac{1}{2}}{\theta \in (-\frac{1}{2}, \frac{1}{2})}}}}}}}}}}}}}}}}
$$

(42)

The metaplectic group $\text{Mp}(2,\mathbb{R})$, the two-fold cover of the group of $2 \times 2$ matrices, is the group of linear canonical transforms in quantum mechanics and wave optics.

In higher dimensions $N$, the symplectic groups $\text{Sp}(2N,\mathbb{R})$ will follow a pattern similar to the $N = 1$ case reported here. The Bargmann decomposition (34) of $\text{Sp}(2N,\mathbb{R})$ is now into unitary $\text{U}(N)$ and positive definite $N \times N$ symmetric matrices [21]; the former are further decomposed as $\text{U}(1) \otimes \text{SU}(N)$, with the $\text{U}(1)$ circle covered by a parameter range over chosen periodicity intervals from the universal cover $\text{Sp}(2N,\mathbb{R})$ to the metaplectic $\text{Mp}(2N,\mathbb{R})$, and down to its double covering of the pseudo-orthogonal group $\text{SO}(3,2)$.

3.4. Mielnik’s Geometric Triangle and Wave Hexagon

A memorable line of work by Bogdan Mielnik concerns those quantum systems which, after a given cycle of transformations, return the input states of the system to their original values [22]. In particular, he found that six oscillator-potential jolts spaced by six
appropriate periods of free drift, will comply with this requirement. Here we have the same requirement for a wave optical signal. In (17)–(19) we introduced blocks composed of a lens and a free flight, \( B_{Gz} = \begin{pmatrix} 1 & -z \\ -G & 1 \end{pmatrix} \), constructing a cycle of up to three such elements to build the identity geometric transformation, \( B_{3/2}^3 = 1 \).

We now build the corresponding integral canonical transforms of those matrices and their first few powers with the Bargmann parameters, (34) and (36) for the parameter values \( Gz \in \{1, 2, 3\} \) with \( z = 1 \), whose powers yield the \( \pm 1 \) unity matrices [23]. We can proceed using numerical computation and recognize the \( \pm \pi \) values and its multiples, to write:

\[
B_{1,1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \phi = -1.10715, \quad \mu = 0.223607 + 0.447214 i, \quad (43)
\]

\[
B_{1,1}^3 = -1, \quad \phi = -\pi, \quad \mu = 0,
\]

\[
B_{1,1}^6 = +1, \quad \phi = -2\pi, \quad \mu = 0;
\]

\[
B_{2,1} = \begin{pmatrix} 1 & -1 \\ -2 & 1 \end{pmatrix}, \quad \phi = -\frac{1}{2}\pi, \quad \mu = \frac{1}{2} + i \quad (44)
\]

\[
B_{2,1}^2 = -1, \quad \phi = -\pi, \quad \mu = 0,
\]

\[
B_{2,1}^4 = +1, \quad \phi = -2\pi, \quad \mu = 0,
\]

\[
B_{2,1}^6 = -1, \quad \phi = -3\pi, \quad \mu = 0;
\]

\[
B_{3,1} = \begin{pmatrix} 1 & -2 \\ -3 & 1 \end{pmatrix}, \quad \phi = -1.81577, \quad \mu = 0.606339 + 1.697750 i, \quad (45)
\]

\[
B_{3,1}^3 = +1, \quad \phi = -2\pi, \quad \mu = 0,
\]

\[
B_{3,1}^6 = +1, \quad \phi = -4\pi, \quad \mu = 0.
\]

For \( \phi = \pm \pi \), the \( \phi \)-factor in (34) is the first inversion matrix \(-1\) while the \( \mu \)-factor, for \( \mu = 0 \), is the unit matrix; when \( \phi = \pm 2\pi \) the first factor is the unit matrix which is the symplectic unit that is the second metaplectic unit. Then, \( \phi = \pm 3\pi \) is again an inversion (metaplectically distinct from the previous one), while only \( \phi = \pm 4\pi \) returns the ‘true’ symplectic and metaplectic unit. Hence, while \( B_{1,1}^1, B_{2,1}^1, \) and \( B_{3,1}^1 \) yield the identity geometric system, in wave optics it will change the signal values by a minus sign; only the six-block setup \( B_{3,1}^1 \) truly reproduces the initial state as the symplectic and metaplectic unit. This is the optical realization of Mielnik’s hexagon.

One topic that we must skip is that of complex symplectic \( 2 \times 2 \) matrices [24] which are used to describe linear systems that are diffusive [18] [Ch. 9]. Two cases of interest are diffusion itself, \( D_\tau := \begin{pmatrix} 1 & -i \tau \\ 0 & 1 \end{pmatrix} \), \( \tau \geq 0 \), which has a widening Gaussian integral kernel that corresponds to the Gauss–Weierstrass transform, and the Bargmann transform \( B := \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & i \\ -i & 1 \end{pmatrix} \) that intertwines the Schrödinger oscillator energy states and their corresponding Schwinger power-function states [25]. The integral kernel (26) and its coefficient (31) present no problem as they are extended into the complex \((q, q')\)-plane, but the unitarity—implying reversibility—of the transform does require a specific measure over the now-complex plane. This extension of the \( 2 \times 2 \) group of matrices does not seem to apply directly to the geometric model of optics, so we mention it only for general interest.

4. Conclusions

We want to conclude this essay on \( 2 \times 2 \) matrices by pointing out that we have viewed them as elements in the manifold of a group. The theory of classical Lie groups provides a complementary panoramic point of view; its representations. For the rotation group \( SO(3) \) that second viewpoint looks for multipoles and represents them by spherical harmonic functions \( Y_{\ell m}(\Omega) \), identified by two eigenvalues, \( |\ell, m| \), with integer \( \ell \geq 0 \) and bounded \( |m| \leq \ell \) spaced by units as restricted by periodicity conditions.

The \( 2 \times 2 \) matrices \( \mathbf{M} \in SL(2, \mathbb{R}) \overset{21}{\supseteq} SO(2,1) \) are thus represented by unitary square matrices \( \mathbf{D}_{m,\mu}^\mathbf{M}(\mathbf{M}) \). These representation matrix elements were also analyzed by Valentine.
Bargmann, who called them the spin or vector three-dimensional Lorentz groups \cite{20}. They have also two representation labels $|k, m\rangle$, and three essentially different representation series: The continuous series where $k = \frac{1}{2} + i\sigma$, $\sigma \in \mathbb{R}$, and $m$ is integer in $\text{SO}(2,1)$, and is integer or half-integer in $\text{SL}(2,\mathbb{R})$, yielding infinite matrix representations of these groups, with no upper or lower limits. There are also the discrete representation series for integer or half-integer $k \geq 0$ respectively, where the representation matrices are ‘half-infinite’ and bounded from below $m \geq k$ or from above $m \leq -k$. Finally, the interval $0 \leq k \leq 1$ contains superposed the exceptional representation series, where $m$ has no bounds and contains a one-parameter family of unitary self-adjoint extensions. An independent analysis by Gel’fand and Graev names these representation series as principal, complementary, and supplementary. The representations of the universal cover group $\text{SL}(2,\mathbb{R})$ simply extend the discrete and exceptional series with continuous values of $k \geq 0$.

The group view and the representation view of $2 \times 2$ matrices are, in a sense, Fourier transforms of each other in the larger theory of harmonic analysis on Lie groups. This was already known as Schur’s Lemma \cite{3} for finite groups, and is mostly valid for continuous groups. The $2 \times 2$ case is interesting in its own right for its complexity, but also because it foreshadows generic features of matrix groups, and is realized in Hamiltonian geometric and classical, quantum and wave, mechanical, and optical systems.

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