RADIAL STABILITY IN STRATIFIED STARS

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Received 2014 November 12; accepted 2014 December 29; published 2015 February 26

ABSTRACT

We formulate within a generalized distributional approach the treatment of the stability against radial perturbations for both neutral and charged stratified stars in Newtonian and Einstein’s gravity. We obtain from this approach the boundary conditions connecting any two phases within a star and underline its relevance for realistic models of compact stars with phase transitions, owing to the modification of the star’s set of eigenmodes with respect to the continuous case.

Key words: stars: fundamental parameters – stars: interiors – stars: neutron – stars: oscillations

1. INTRODUCTION

There is theoretical evidence that compact stars, such as neutron stars, are made up of several matter phases (Shapiro & Teukolsky 1986). This is a consequence of the astonishingly high density excursion they could attain in their inner regions. For instance, a neutron star is in general thought to be composed of at least two different regions: the crust and the core. Starting from very low densities of a few grams per cubic centimeter close to their surfaces, and up to densities on the order of the nuclear saturation value, $\rho_{\text{nuc}} \approx 2.7 \times 10^{14}$ g cm$^{-3}$, the crust of a neutron star is thought to be in a solid-like state. The core, with densities that might be greater by orders of magnitude than $\rho_{\text{nuc}}$, is instead thought to be in a liquid-like state. The details of the treatment of the thermodynamic transition (Maxwell or Gibbs phase construction), as well as the conditions of density and pressure at which such a transition occurs, are still a matter of debate. The application of the Gibbs construction with more than one conserved charge leads to the appearance of mixed phases in between the pure phases, with an equilibrium pressure that varies with the density, leading to a spatially extended phase-transition region of nonnegligible thickness with respect to the star’s radius (Glendenning 1992, 2001; Glendenning & Pei 1995; Christiansen & Glendenning 1997; Glendenning & Schaffner-Bielich 1999; Christiansen et al. 2000). In contrast, in the traditional Maxwell construction the phases are in “contact” with each other. It is worth mentioning that in these treatments the pure phases are subject to the condition of local charge neutrality, so they do not account for the possible interior Coulomb fields. Indeed, the complete equilibrium of the multicomponent fluid in the cores of compact stars needs the presence of a Coulomb potential formed by electric charge separation due to gravito-polarization effects (Rotondo et al. 2011; Rueda et al. 2011), favoring a sharp core–crust transition that ensures the global, but not the local, charge neutrality (see Belvedere et al. 2012, 2014, and references therein). Other than the core–crust transition, additional phase transitions, such as the ones allowed by quantum chromo-dynamics, could occur within the core of the star itself (see, e.g., Glendenning 1996, and references therein). From all of the above we can conclude that ultradense stars such as neutron stars necessarily show a nontrivial stratification. Between any two phases, which can be very different, it is reasonable to investigate the situation where some quantities are discontinuous, such as the energy density and the pressure. Such discontinuities can be harnessed by appropriate surface tensions. These surface quantities influence the stability of a system, adding new boundary conditions to the problem that, as we shall show here, modify the set of eigenfrequencies and eigenmodes of a star.

In this work we analyze the problem of perturbations in systems constituted of various phases that are split by surfaces that host nontrivial degrees of freedom. This analysis is thought to be a generalization of the treatment for continuous systems (see Herrera & Santos 1997 and references therein for a comprehensive analysis of properties, types, and stability of continuous anisotropic fluids also in the presence of radiation and heat flux). By investigating the dynamics of perturbations, we are automatically probing the stability of systems. We shall restrict ourselves to the simplest possible case: spherically symmetric extended bodies where radial perturbations take place. In order to model the problem, we shall assume that these surfaces of discontinuity separating two arbitrary phases are very thin, and a generalized distributional approach (Poisson 2004; Raju 1982b) shall be adopted. We start our analysis in the Newtonian case in order to gain some intuition into the relevant aspects of the problem, and finally we generalize it to general relativity. Our purpose is solely to expound the problem and seek to solve it as generically as we can. Our analysis is far from complete and must be considered as a first step toward deeper investigations, and scrutinies of specific cases will be the object of later studies.

We show in this work that phase transitions in the presence of surface degrees of freedom can be enclosed in additional boundary conditions on the problem. Our formalism also tells us that such boundary conditions are only self consistent when the set of eigenfrequencies of the perturbation modes is related to the global system, not with individual phases. This is consistent with the well-known results from coupled springs, where there are only global frequencies. The presence of further boundary conditions naturally modifies the possible set of eigenfrequencies because we are inserting further restrictive aspects to the physical oscillation modes. Therefore, measurements on the pulsation modes in a star could tell us very precisely about its internal structure, being a sort of fingerprint that could help us understand better the nature of these systems.
2. STABILITY OF CLASSICAL SYSTEMS WITH PHASE TRANSITIONS

Assume a continuous classical astrophysical system with spherical symmetry. When its volume elements are perturbed radially, it is well known (see, e.g., Shapiro & Teukolsky 1986) that the evolution of perturbations of the form

$$\xi(t, r) = \xi(r) e^{i \omega t},$$

with \(\omega\) an arbitrary constant, is described by

$$\frac{d}{dr} \left[ \Gamma \frac{1}{r^2} \frac{d}{dr} (r^2 \xi) \right] - \frac{4}{r} \frac{dP}{dr} \xi + \omega^2 \rho \xi = 0,$$

where \(P(r)\) is the pressure of the background system under hydrostatic equilibrium and

$$\Gamma \doteq \frac{\rho}{P} \frac{\delta P}{\delta \rho},$$

with \(\rho(r)\) the mass density of the system. Formulated in this way, we have at hand an eigenvalue problem. For continuous systems, the boundary conditions to be added to Equation (2) are quite simple. They are directly related to the spherical symmetry of the system, as well as to the vanishing of its pressure on its border even in the presence of perturbations (other situations where a surface tension is present could also be envisaged, and we shall attempt to elaborate on them below). In other words, we impose that

$$\xi(0) = 0, \quad \text{and} \quad \xi(R) = \text{finite},$$

where \(R\) is defined as the radius of the star, such that \(P(R) = 0\). For further details about these boundary conditions see Shapiro & Teukolsky (1986). Equation (2) supplemented with Equation (4) constitutes a Sturm–Liouville problem, where the aspects of its solutions are already known. Concerning the eigenfrequencies, \(\omega^2\), they are all real and form a discrete hierarchical set. When one seeks stable solutions to Equation (2), one seeks solutions with positive \(\omega^2\), especially for its fundamental mode. As can be seen from Equation (1), negative values of \(\omega^2\) indicate instabilities in the assumed background system, which leads to the conclusion that they do not linger on in time. They would either implode or explode.

We now turn to the more involved problem of permitting the system to be stratified and harboring surface degrees of freedom on the interface of two given phases. Such degrees of freedom have themselves a dynamic, described generically by the thin-shell formalism or Darmois–Israel formalism (Israel 1966, 1967; Lobo & Crawford 2005; Poisson 2004). We are here, however, particularly interested in another aspect of the problem, namely understanding the role such degrees of freedom play in the stability of the system when its parts (defined naturally by the hypersurface that hosts the aforesaid degrees of freedom) are perturbed. Therefore, before anything, it is assumed that, for it to be meaningful to talk about this scenario, one has that the hypersurfaces of discontinuity themselves are stable (see Pereira et al. 2014 for further details). If this is not the case, any displacement of the hypersurface of discontinuity would trigger a cataclysmic set of events that would result in the disruption of the system.

One expects that the stratified problem could be accounted for additional boundary conditions to the system. The reason for this is that the perturbations in the upper and lower regions with respect to a given surface of discontinuity would be described by the same physics (for example Equation (2)), as well as the totality of the matches. The only missing points would be their connection (allowing combinations of solutions to also be solutions to the physical equations involved) and generalization by means of surface quantities. For example, the existence of a surface tension would account for an extra surface force term. The same ensues with the presence of a surface mass (enclosed by a surface mass density) and the associated presence of a surface gravitational force. Therefore, in order to properly describe stratified systems, which need the addition of surface boundary conditions to match different regions, one must make use of distributions (Poisson 2004; Raju 1982b).

We proceed now with the distributional generalization of the equations describing continuous fluids under gravitational fields. Assume that a surface harboring surface degrees of freedom in a system in equilibrium is at \(r = R\). The first equation to be generalized in terms of distributions in this case is the equation of hydrostatic equilibrium. This should now read

$$\frac{dP}{dr} + \rho g(r) - \frac{2P}{R} \delta(r - R) = 0,$$

where \(g(r)\) has been defined as the norm of the gravitational field, the solution to (the distributional) Poisson’s equation

$$\nabla \cdot \mathbf{g} = -4\pi G \rho, \quad \mathbf{g} = -g(r) \hat{r},$$

which can always be written as

$$g(r) = \frac{GM(r)}{r^2}, \quad M(r) \doteq 4\pi \int_0^r \rho(\tilde{r}) \tilde{r}^2 d\tilde{r}.$$
of discontinuity of the system (at $R$):
\[ [g(R)]^+_- = 4\pi G \sigma, \]
where we have introduced the convention $[A]^+_- \equiv A^+ - A^-$, the jump of $A(r)$ across $r = R$. Therefore, $g(r)$ could be represented distributionally as
\[ g = g^+(r)\theta(r - R) + g^-(r)\theta(R - r). \]

The above equation means that the associated distributional gravitational potential $\phi(r) (g = -\nabla\phi, g = -g\phi)$ is always a continuous function, although not differentiable at the surface of a discontinuity.

We will also assume that the pressure can be discontinuous at $r = R$ and hence written as
\[ P(r) = P^-(r)\theta(R - r) + P^+(r)\theta(r - R). \]

The reason this is so will be clarified when we deal with our original problem in the scope of general relativity. The heuristic argument corroborating the validity of Equation (10) is that it is meaningless to colligate a surface term to the radial pressure because its associated force would necessarily be normal to it and therefore would not lie on the surface. Only tangential pressures should be tied within their surface terms. From Equations (10), (7), (6), (8), and (9) we have
\[ P = \frac{R}{2}[P(R)]^+_- + \frac{G}{16\pi R^5}[M^2(R)]^+_- \]
and
\[ \frac{dP^\pm}{dr} + \rho^\pm g^\pm(r) = 0. \]

The arithmetic average present in Equation (11) (see the definition of $g(r)$ and the value of $\sigma$) is a general consequence of the product of delta functions with Heaviside ones in the generalized sense of distributions (Raju 1982a, 1982b). Note from Equation (11) that its first term is the known Young-Laplace equation for spherical surfaces at equilibrium (see, e.g., Rodríguez-Valverde et al. 2003; Peters 2013), where only geometric aspects are taken into account for the surface tension. Its second term, though, is the gravitational surface tension, uniquely due to the nonzero surface mass. If the surface tension were null, the pressure jump could not be arbitrary but is proportional to the surface mass density and must be a monotonically decreasing function of the radial coordinate (see Equations (8) and (11)). From Equation (11), one sees further that the force per unit area associated with the surface tension is exactly the one necessary to counterbalance both of the forces coming from the pressure gradient at $R$ and the surface gravitational force, as it should be.

One sees that the above procedure generalizes our notion of hydrostatic equilibrium in each phase the stratified system has (see Equation (12)) and automatically gives the surface tension at $R$ that guarantees the hydrostatic equilibrium for arbitrary pressure jumps and surface masses. We will keep the same philosophy now concerning the generalization of Equation (2). From our generalized hydrostatic equilibrium equation, we have that an important term for the deduction of the equation governing radial perturbations would be the application of the Lagrangian operator $\Delta (\Delta A \equiv A(r, r + \xi) - A_0(t, r), A_0$ and $A$ being a physical quantity in the equilibrium and perturbed cases, respectively) on the surface force in Equation (5) (see Equation (20)):
\[ \Delta \left[ \frac{P}{R} \delta(r - R) \right] = \frac{\Delta P}{R} \delta(r - R) - \frac{\sigma}{R^2} \frac{\xi}{R} \delta(r - R) \]
because $\delta \xi = 0$ and $\Delta \xi = \xi$. Now we assume that $P = \rho \sigma$. This means that we are endowing the fluid at the surface of a discontinuity with adiabatic properties, and the underlying microphysics is not contemplated in this procedure. For continuous media, the total mass in the interface of two phases is generally not a constant. This means that mass fluxes are allowed to take place. This generically would render the mass of each phase not constant, an aspect not taken into account in Equation (2). Nevertheless, if the displacements of the surface of a discontinuity are small and oscillatory, we have that on average the masses on each phase are conserved (here it becomes clear why the surface of a discontinuity should be stable). For adiabatic processes, we have
\[ \Delta \rho \equiv \eta_1^2 \Delta \sigma, \quad \eta_1 \equiv \frac{\partial P}{\partial \sigma}, \]
with $\eta^2$ the square of the speed of sound in the fluid at the surface of a discontinuity. The missing term $\Delta \sigma$ can be found via the thin-shell formalism when the classical limit is taken there. Generically, in the static and spherically symmetric case, $\sigma$ can be written as (Lobo & Crawford 2005)
\[ \sigma = \frac{c^2}{4\pi G R}[\epsilon^{\beta(R)}]_+^-, \]
with the classical limit $\beta(r) \approx GM(r)/(rc^2) \ll 1$. It is easy to check that Equation (15) reduces to Equation (8) in the aforementioned limit. When perturbed, it can be shown that $\beta \rightarrow \beta + \delta \beta$, with $\delta \beta = -4\pi G R_0 \epsilon^{2\beta} / c^2$ (Misner et al. 1973), $\rho_0$ here meaning the mass density in the hydrostatic background solution. Hence,
\[ \Delta \sigma = \delta \sigma + \sigma_0 \xi = -[\rho_0]_+^-, \quad \xi \equiv \frac{\sigma_0}{\rho_0}, \]
where we also considered $\sigma_0$ the background solution (Equation (9)).

Another simpler way of obtaining Equation (16) would be through the dynamics of $\delta g = -\delta \phi$. In the spherically symmetric case we have $\delta g = -4\pi G \rho_0 \epsilon^{2\beta}$ (Shapiro & Teukolsky 1986), and because $\delta g \equiv g - g_0$, with $g_0$ the norm of the gravitational field without the perturbation $\xi$, from Equation (8), we finally obtain
\[ \Delta \sigma = \frac{[\rho_0]_+^-}{4\pi G} + \sigma_0 \xi = \sigma_0 \xi - [\rho_0 \xi]_+^-. \]

In addition, from Equations (6) and (8), for the case when the jump of $\xi$ is null at the surface of a discontinuity (which will be justified below), one shows that the above equation can be further simplified to
\[ \Delta \sigma = -\frac{2}{r} \sigma \xi. \]
Now we show the general equation governing the propagation of radial perturbations. For $P$ defined as in Equation (10), $\rho$ in terms of Equation (7), $g$ as given by Equation (9), and finally
\[ \xi(r, t) = \xi^+(r, t)\theta(r - R) + \xi^- \theta(R - r), \]
the equation governing the evolution of perturbations in a given volume element of the fluid is

$$\Delta \left\{ \frac{d v_r}{d t} + \frac{\partial P}{\partial r} + \rho g(r) - \left[ \frac{2P}{R} + \frac{\sigma}{2} (\dot{v}_r^2 + v_r^2) \right] \delta(r - R) \right\} = 0. \tag{20}$$

Note that we assumed that $\dot{v}_r$ is a distribution like $\tilde{\xi}$. Physically this must be taken because the phases are always “localizable” and do not mix. In other words, this constraint reflects the intuitive fact that the surface of a discontinuity should be well defined. The mathematical reason for this will be given below, and it is related to the well-posedness of the problem.

When developed, taking into account the hydrostatic equilibrium equation (see Equation(5)), it can be simplified to

$$\frac{\partial}{\partial r} \left[ \frac{\Gamma P}{r^2} \frac{\partial (r^2 \tilde{\xi})}{\partial r} \right] - 4 \frac{\partial P}{r \partial r} - \frac{\sigma}{2} \left( \frac{\partial^2 \tilde{\xi}^+}{\partial r^2} + \frac{\partial^2 \tilde{\xi}^-}{\partial r^2} \right) = 2 \frac{\tilde{\xi}}{R^2} \left( 2n^2 \sigma - 3P \right) - \frac{\sigma}{2} \left( \frac{\partial^2 \tilde{\xi}^+}{\partial r^2} + \frac{\partial^2 \tilde{\xi}^-}{\partial r^2} \right) \delta(r - R). \tag{21}$$

First note that coefficients multiplied by $\delta^2(r - R) \text{ or } \delta'(r - R)$ in Equation (21) must all be null. Taking into account Equation (19), this means that

$$[\tilde{\xi}]^\pm = 0. \tag{22}$$

This automatically warrants $\dot{\tilde{\xi}}$ as a distribution without Dirac deltas, as we have advanced previously. In order to obtain Equation (21), we used the results that for a distribution $A(r) = A^+(r) \theta (r - R) + A^-(r) \theta (R - r)$,

$$\Delta A = \delta A + \frac{\partial A}{\partial r} \xi - [A]^\pm \xi \delta(r - R), \tag{23}$$

$$\Delta \left( \frac{\partial A}{\partial r} \right) = \frac{\partial}{\partial r} (\Delta A) - \frac{\partial \tilde{A}}{\partial r} \frac{\partial A}{\partial r} + \frac{1}{2} \left[ \frac{\partial \tilde{A}^+}{\partial r} + \frac{\partial \tilde{A}^-}{\partial r} \right] [A(R)]^\pm \delta(r - R). \tag{24}$$

In addition to the above mathematical properties, we have also made use of

$$\Delta \rho = - \rho \frac{\partial}{\partial r} (r^2 \tilde{\xi}) + \rho \frac{\partial}{\partial r} \left( \frac{\partial \tilde{\xi}^+}{\partial r} + \frac{\partial \tilde{\xi}^-}{\partial r} \right) \delta(r - R), \tag{25}$$

which is a direct consequence of assuming that the total mass in each phase is constant, even in the presence of perturbations. This is only guaranteed if the surface of a discontinuity is stable, a prime hypothesis for having a well-posed stability problem. We have also assumed that $P = P(\rho)$, which implies that $\Delta P^\pm = \Gamma^0 P^\pm \Delta \rho^\pm / \rho^\pm$. In deriving Equation (21), we further took into account Equation (11). We finally stress that a simpler way to obtain Equation (21) is to recall that $\Delta M = 0$, which guarantees that $\Delta g = - 2g \tilde{\xi} / r$. The fact that $\Delta M = 0$ means that observers comoving with the fluid do not note a mass change. The aforementioned result can also be directly shown by Equations (6), (24), and (25).

It can be seen that only solutions of the type $\tilde{\xi}^\pm (r, t) = e^{\omega T} \tilde{\xi}^\pm (r)$ for Equation (21) are meaningful if

$$\omega^+ = \omega^- = \omega. \tag{26}$$

This is the only way to eliminate the time dependence above in Equation (21) and also to guarantee that the jump of $\tilde{\xi}$ is null for any surface of discontinuity at any time. Therefore, we arrive at the important conclusion that even a stratified system where oscillatory perturbations take place should be described by a sole set of frequencies. Each member of this set describes the eigenfrequency of a whole system, instead of one or another phase. Nevertheless, we recall that at the surface of a discontinuity the frequencies are in principle not defined. Bearing in mind the above conclusions, we have that, using Equation (1),

$$\tilde{\xi}(r) = \tilde{\xi}^- (r) \theta (R - r) + \tilde{\xi}^+ (r) \theta (r - R) \tag{27}$$

and the boundary condition

$$[\tilde{\xi}(R)]^\pm = 0, \quad \text{or} \quad \tilde{\xi}^+ (R) = \tilde{\xi}^- (R) \hat{\xi} (R), \tag{28}$$

and therefore the only meaningful $\Gamma$ are given by

$$\Gamma (r) = \Gamma^- (r) \theta (R - r) + \Gamma^+ (r) \theta (r - R). \tag{29}$$

Gathering the above equations in Equation(21), we obtain

$$\frac{d}{dr} \left[ \frac{\Gamma P}{r^2} \frac{\partial (r^2 \tilde{\xi})}{\partial r} \right] - 4 \frac{d P}{r \partial r} \tilde{\xi} + \omega^2 \rho \tilde{\xi} = - \tilde{\xi} (R) \left[ \frac{2}{R^2} \left( 3P - 2n^2 \sigma \right) - \omega^2 \sigma \right] \delta (r - R). \tag{30}$$

One sees from Equation (30) that in the case where $P$ and $\sigma$ are null, the classical expression, Equation (2), is recovered.

Summing up, substituting Equations (27) and (7) into Equation (30), one sees that the only way to satisfy such an equation is by imposing that

$$\frac{d}{dr} \left[ \frac{\Gamma P}{r^2} \frac{\partial (r^2 \tilde{\xi})}{\partial r} \right] - 4 \frac{d P}{r \partial r} \tilde{\xi} + \omega^2 \rho \tilde{\xi} = 0, \tag{31}$$

and for completeness, condition (28). Equation (31) is obtained here as a consequence of our distributional search for solutions to the radial Lagrangian displacements. This is exactly what one expects under physical arguments. Equations (32) and (28) are our desired boundary conditions to be further taken into account (besides Equation (4)) at the interface of any two phases.

For the case where $[P(R)]^\pm = [\Gamma(R)]^\pm = \sigma = \rho = 0$, we have that the derivative of $\xi$ is also continuous, and therefore $\xi$ is a differentiable function anywhere, as it should be because we are defining here a continuous system. Nevertheless, whenever the aforementioned conditions do not take place, richer scenarios arise. Even in the case of a phase transition at constant pressure and negligible surface mass, the discontinuity of $\Gamma$ and the existence of $\rho$ generally render the derivative of $\xi$ discontinuous.

3. A SPECIFIC EXAMPLE: UNIFORM-DENSITY STARS

We would like to stress that the boundary condition we have derived previously is actually very restrictive. This is because the physically acceptable cases (solutions to Equations (31) associated with a surface of discontinuity at $r = R$) are only the ones that deliver enough arbitrary constants of integration
to satisfy Equations (27) and (32). This should be taken into account together with the physical requirement of only admitting finite \( \xi \) everywhere and that are null at the origin (see Equation (4)). In the following, we shall see a particular example where all of these aspects are evidenced.

Let us now investigate a star made of two phases, each with a uniform mass density. Let us assume also, for the sake of simplicity and example, that the associated \( \Gamma \) for each region is an arbitrary constant. As we will see, although this can be considered only as a first academic example, it already evidences some aspects that stratified systems should have. For this case it is straightforward to solve Poisson’s equation and the equation of hydrostatic equilibrium (see, e.g., Shapiro & Teukolsky 1986, for further details), and we have for \( r < R \)

\[
P^-(r) = \frac{2\pi G \rho^2}{3}(R^2 - r^2), \quad R \pm \frac{3 p_0^-}{2\pi G \rho^2},
\]

(33)

where \( p_0^- \) is an arbitrary constant that corresponds to the pressure of the system at the origin. For \( r > R \), instead

\[
P^+(r) = \frac{2\pi G \rho^2}{3}(R^2 - r^2).
\]

(34)

The constant mass density in the inner and outer regions has been defined as \( \rho^+ \) and \( \rho^- \), respectively. The pressure at the origin \( p_0 \) could always be chosen such that it matches the pressure at the base of the outer phase, and as can be seen from Equation (34), we have also introduced the condition of having a null pressure at the star’s surface. Substituting Equations (33) and (34) into Equation (31), we are led to

\[
(1 - x_s^2) \frac{d^2 x_s^\pm}{dx_s^2} + \left( \frac{2}{x_s^\pm} - 4x_s \right) \frac{dx_s^\pm}{dx_s} + \left( A_\pm - \frac{2}{x_s^\pm} \right) x_s^\pm = 0,
\]

(35)

where we assumed that \( x_s \approx r/R_s, x_- \approx r/R \) and

\[
A_\pm \approx \frac{3\omega_s^2}{2\pi G \rho^2 \Gamma^\pm} \pm \frac{8}{\Gamma^\pm} - 2.
\]

(36)

We now solve Equation (35) by the method of Frobenius. For the sake of simplicity, we drop the \( \pm \) notation. We therefore assume solutions of the form

\[
\xi = \sum_{n=0}^{\infty} a_n x_s^{n+1},
\]

(37)

where \( a_n \) and \( s \) are arbitrary constants to be fixed by primarily demanding that the first condition of Equation (4) is satisfied, as well as \( \xi(x) \) always being finite. By substituting Equation (37) into Equation (35), it can be checked that the solutions to \( s \) are either \( s = 1 \) or \( s = -2 \). The associated recurrence relation obtained generally is

\[
a_{m+2} = \frac{(m + s)(m + s + 1) - A}{(m + s + 2)(m + s + 3) - 2} a_m,
\]

(38)

with \( m = 0, 2, 4..., \) and \( a_1 = d_3 = d_5 = ... = 0 \). Let us analyze first the inner region. It is clear in this case that the associated \( a_0 \) for \( s = -2 \) must be null, as a consequence of one of our boundary conditions. From Equation (38), one clearly sees that the power series given by Equation (37) does not converge. Therefore, in order to satisfy the finiteness anywhere of \( \xi \), we have to impose that the series be truncated somewhere, rendering it actually a polynomial. Hence

\[
A_{m,s=1} = (m + 1)(m + 4).
\]

(39)

From Equation (36), one sees that only discrete frequencies (given by Equation (39)) are possible in this region. From Equations (36) and (39), to have the frequency of the fundamental mode \( (m = 0) \) positive, one should have \( \Gamma^- \geq 4/3 \). Summing up, the physically relevant solution to this case just leaves out an arbitrary constant of integration, as required due to the scaling law present for \( \xi \) from Equation (31).

Let us now analyze the outer region. This is the most physically interesting region because the problems at the star’s center are absent, and therefore in principle one could have two linearly independent solutions to \( \xi \). Because of the finiteness of \( \xi \) in this region, the outer counterpart of Equation (39) must again take place. Nevertheless, for \( s = -2 \), one should also impose

\[
A_{m,s=-2} = (m - 2)(m + 1).
\]

(40)

From Equation (36), one sees from this case that its associated fundamental mode \( (m = 0) \) is unstable. This means in principle that this solution to the outer region should be excluded, leaving out just the one from the case \( s = 1 \), where we should consider

\[
A_{m,s=1} = (m + 1)(m + 4).
\]

(41)

Still, our previous analysis exhibits clear problems: there are not enough arbitrary constants to fix Equations (28) and (32), and the eigenfrequencies in each region are different. However, we shall show that the condition of having a same eigenfrequency for the whole system, as required by our formalism, addresses all of the problems. Obviously, the stable eigenfrequencies of the star are only related to the solution \( s = 1 \). However, they could arise here from aspects of either the inner or outer phases of the star. Let us see what ensues from this conclusion. Assume initially that the only possible \( \omega \) are given by Equation (39), associated with the modes \( m_{s=1} \). So, for having finite \( \xi^+ \) related to \( s = 1 \), one must impose that there exists a \( m_{s=1}^+ \) to the outer phase such that the numerator of the associated recurrence relation is null. It can be shown that this is only the case if

\[
m_{s=1}^+ = \frac{-5 + \sqrt{9 + 4A_{s=1}^+(m_{s=1}^-)}}{2}.
\]

(42)

Therefore, Equation (42) demands that

\[
9 + 4A_{s=1}^+ = (2p + 1)^2, \quad p \geq 2, \quad p \in N.
\]

(43)

For the case \( s = -2 \), it can be shown that the condition for the existence of a \( m_{s=-2}^+ \) related to \( a_{s=-2}^+ = 0 \) is exactly given by Equation (43). The mode itself is

\[
m_{s=-2}^+ = \frac{1 + \sqrt{9 + 4A_{s=-1}^+(m_{s=-1}^-)}}{2}.
\]

(44)

Summarizing: if Equation (43) is satisfied for any natural \( p \geq 2 \), there always exist modes, characterized by Equations (42) and (44), that guarantee the finiteness of \( \xi^+ \) as a linear combination of solutions for \( s = 1 \) and \( s = -2 \), associated with a given eigenfrequency \( a_{s=1}^- \) that only takes into account aspects of the inner phase of the system. In this case,
one is able to come up with two arbitrary constants of integration, which would then guarantee that the additional boundary conditions raised by the stratification, Equations (28) and (32), are satisfied. It is immediately apparent that a reasoning similar to the above ensues if one now chooses \( a_m \) as coming from aspects of the outer region, given now by Equations (36) and (41). For this case, we will now find an \( m_{s_{-1}} \) and an \( m_{s_{-2}} \) associated with \( a_m \), as given by Equations (42) and (44), with the condition given by Equation (43), replacing \( A_{s_{1}} (m_{s_{-1}}) \) by \( A_{s_{1}} (m_{s_{-2}}) \). Because \( \Gamma^\pm \) and \( \delta^\pm \) are given quantities, one sees that the only possible eigenfrequencies for a system should satisfy \( 9 + 4A_{s_{1}}^2 = (2p + 1)^2 \). This constraint is uniquely imposed because of the extra boundary conditions to the problem and is very restrictive. We have just shown a simple example where some of the aspects imprinted by stratification arise. Whenever there are two arbitrary solutions to \( \xi \) in a given phase, it will be always possible to satisfy the constraints (28) and (32).

4. SYSTEMS WITH AN ELECTROMAGNETIC STRUCTURE

Now we attempt to take a further step in our classical generalization, by endowing the phases (as well as the surface of a discontinuity) with an electromagnetic structure. Just for clarity, let us work with a system that exhibits just an electric field. The first point to be taken into account is the additional electric force present in the system. This would have the same structure as the gravitational force, and therefore its generalization is straightforward. Now one should also define a distributional solution to the charge density. The surface force associated with the surface tension should have the same form as previously, but now it should also take into account the present electric aspects. The pressure in this case would also change because of the presence of the electric field, and its jump over a surface of discontinuity could still be kept free.

From the (distributional) Maxwell equations in the spherically symmetric case, one has that

\[
E(r) = \frac{Q(r)}{r^2}, \quad Q(r) = 4\pi \int_0^r \rho_e(\tilde{r})\tilde{r}^2 d\tilde{r},
\]

(45)

where \( \rho_e(r) \) is the charge density at \( r \). The associated “force density” is \( dF/dv = \rho_e E \tilde{r} \). Therefore, the equation of hydrostatic equilibrium now reads

\[
\frac{dP}{dr} + \rho(r)g(r) - \rho_e(r)E(r) - \frac{2\rho_Q}{R} \delta(r - R) = 0.
\]

(46)

Therefore, like the gravitational field, the electric field also presents a jump at any surface of a discontinuity (at \( r = R \)) endowed with surface charges. We write the charge density as

\[
\rho_e(r) = \rho^-(r)\bar{\theta}(r - R) + \rho^+(r)\bar{\theta}(r + R) + \sigma_e \delta(r - R).
\]

(47)

and the distributional electric field is

\[
E(r) = \bar{E}^-(r)\bar{\theta}(r - R) + \bar{E}^+(r)\bar{\theta}(r - R), \quad \bar{E}(r) = 4\pi \sigma_e.
\]

(48)

By substituting now Equations (10), (45), (47), and (48) into Equation (46), we have that the surface tension at equilibrium should read

\[
\rho_Q = \frac{R}{2} \left[ P(R) \right]^- + \frac{G}{16\pi R^3} \left[ M^2(R) \right]^- \left( e^{2\beta_0(r)} d\tau_\pm^2 - r_\pm d\Omega_\pm^2 \right).
\]

(49)

Note that the existence of a surface mass would lead to \( [M^2(R)]^- > 0 \), and \( [Q^2(R)]^- \) could in principle be any. The appearance of the last term in Equation (49) is consistent with the expected and long ago known contribution of electric double layers to the surface tension and surface energy of metals, as recalled by Frenkel (1917) in his seminal work. The existence of such surface electric fields is well known in materials science, and it has been determined experimentally from the photoelectric phenomenon by measuring the amount of work done by electrons to escape from a metal’s surface. There is a vast literature on the role of electric double layers on surface phenomena in metals and contact surfaces, and we refer the reader for instance to Huang & Wyllie (1949) and Israelachvili (2011), and references therein, for further details on this subject.

Because in the presence of an electric field the hydrostatic equilibrium equation and the surface tension changes, it can be checked that Equation (30) keeps the same functional form. In drawing this conclusion, it was also assumed that the total charge of the system is a constant. This also means that \( \Delta Q = 0 \). One also sees immediately that the main results concerning the stability of the stratified charged case are totally analogous to the neutral one, obtained by simply making the replacement \( \mathcal{P} \rightarrow \mathcal{P}_Q \).

5. STRATIFIED SYSTEMS IN GENERAL RELATIVITY

Now we generalize the analysis of stratified systems to general relativity. From the classical analysis, we have learned that surface quantities must also be inserted into the generalized equation of hydrostatic equilibrium. Therefore, in a certain sense, we must find the proper generalization of the surface forces in general relativity. This will not be difficult bearing in mind the thin-shell formalism, as we shall see below. Such a formalism states that in order to search for distributional solutions to general relativity, one has to consider an energy-momentum tensor at a surface of discontinuity, which we shall name \( \Sigma \). It is precisely this surface content that leads to the jump of quantities that are related to physical observables, such as the extrinsic curvature. We now outline the formalism succinctly. Let us work just in the spherically symmetric case, where \( \Sigma \) is defined as \( \Phi = r - R(\tau) = 0 \), with \( \tau \) the proper time of an observer on the aforesaid hypersurface. Assume that the metrics in the regions above and below \( \Sigma \) (with respect to the normal vector to it), described by the coordinate systems \( x_\pm = (t_\pm, r_\pm, \theta_\pm, \varphi_\pm) \), respectively, are given by

\[
ds^2_\pm = e^{2\alpha_\pm(r_\pm)} d\tau_\pm^2 - e^{2\beta_\pm(r_\pm)} dr_\pm^2 - r_\pm^2 d\Omega_\pm^2.
\]

(50)

where

\[
d\Omega_\pm^2 = d\theta_\pm^2 + \sin^2 \theta_\pm d\varphi_\pm^2.
\]

(51)

Assume that the (three-dimensional) hypersurface \( \Sigma \) is described by the (intrinsic) coordinates \( y^0 = (\tau, \theta, \varphi) \) such that at the hypersurface \( t_\pm = t_\pm(\tau) \), \( \theta_\pm = \theta \) and \( \varphi_\pm = \varphi \) and obviously \( r_\pm = R(\tau) \). In order to render the procedure consistent, one has to impose primarily that the intrinsic metric to \( \Sigma \) is unique. This fixes the coordinate transformations \( x_\pm = x_\pm(y^0) \). This is the generalization of the continuity of the gravitational potential across a surface harboring surface degrees of freedom. Now, if the jump of the extrinsic curvature is nullnonn, the existence of a surface energy-momentum tensor (Poisson 2004) is automatically guaranteed that in the spherically symmetric case can always be cast as \( S^a{}_{b} = \text{diag}(\sigma, -\mathcal{P}, -\mathcal{P}) \), with
appears in this formalism. First of all, we know that the static and stable (upon radial displacements of the discontinuity of the gravitational field across a surface with discontinuity of the extrinsic curvature is the generalization of the energy-momentum tensor given by Equation (54), we also have 

\( \alpha \) and \( \rho \) are differentiable at \( \Sigma \), where the labels \( \pm \) for each term in the above equation were taken to evaluate it, because it must be unique. Let us constrain ourselves first to the case of perfect fluids (locally neutral) on each side of \( \Sigma \). One sees from Equation (54) and the coordinate transformations at \( \Sigma \) that

\[
T^{0\pm}_\mu = \rho' \theta(r - R) + \rho \theta(r - R) + \sigma \delta(r - R),
\]

(55)

\[
T^{1\pm}_\mu = -P' \theta(r - R) - P \theta(r - R),
\]

(56)

\[
T^{2\pm}_2 = T^3_2 \equiv -P, -P - P \delta(r - R).
\]

(57)

From Equation (56), we note that there are no associated surface stresses. This is exactly what we advanced in the classical limit, Equation (5), and thus it is its proper generalization. First of all, note that in such a limit, \( \alpha = \phi(r) \), and \( \phi(r) \) the gravitational potential. It can be also shown that in such a case

\[
\sigma = \frac{1}{4\pi R^2} [M(R)]^+, \quad \text{(63)}
\]

where the above quantities are in cgs units. In Equation (63) one recognizes the jump of the gravitational field \( g(r) = M/R^2 \) at \( r = R \), as exactly given by Equation (8). Therefore,

\[
\frac{\sigma}{2} [\alpha'(R) + \alpha''(R)] \approx \frac{G [M^2(R)]^+}{8\pi R^4}. \quad \text{(64)}
\]

Substituting Equation (64) into Equation (62), we see that the term inside the curly brackets of the latter equation is null (see Equations (53) and (11). Hence, the remaining term in front of the delta function is exactly \( 2P/R \), as we already anticipated and expected (see Equation (5)).

Now we are in a position to talk about perturbations in the general relativistic scenario. When they take place, metric and fluid quantities change at a given spacetime point from their static counterparts. It is customary to assume that such departures are small, which allows us to work perturbatively. The primary task is to find such changes from the system of equations coming from relativistic hydrodynamics and general relativity. Nevertheless, these solutions are already very well known (Misner et al. 1973). Our ultimate task is to generalize them to the distributional case.

The equation governing the evolution of the fluid displacements on each side of \( \Sigma \) is the general relativistic Euler equation, related to the orthogonal projection of \( T^\mu_{\nu \pm} \) (perfect fluids) onto \( u^\mu \):

\[
(\rho + P)u^\mu_{\nu \pm} (\rho + P)u^\nu = (g^\mu_{\nu \pm} - u^\mu u^\nu)P_{\mu \nu}, \quad \text{(65)}
\]

where the labels \( \pm \) for each term in the above equation were omitted just to not overload the notation.
In the hydrostatic case, we have only that \( u'_{\pm} = e^{-\alpha_{\pm} \xi} \). When perturbations are present (Misner et al. 1973),
\[
u'_{\pm} = e^{-\alpha_{\pm} \xi} = e^{-\alpha_{\pm} (1 - \delta \alpha_{\pm}^{\prime})} \tag{66}
\]
where \( \delta \alpha \) is the change of the static solution \( \alpha_{0} \) in the presence of perturbations at a given spacetime point. For the \( u'_{\pm} \) component, using the normalization condition \( u'^{\mu}_{\pm}u'^{\mu}_{\pm} = 1 \), one shows that (Misner et al. 1973)
\[
u'_{\pm} = e^{-\alpha_{\pm} \xi_{\pm}} \tag{67}
\]
with \( \xi_{\pm} = \partial \xi_{\pm}/\partial t_{\pm} \). Just for completeness, \( u'^{\mu}_{\pm} = u_{\pm}^{\mu} = 0 \). For the components of \( u'^{\mu}_{\pm} \) given by Equations (66) and (67), the left-hand side of Equation (65) gives as the only nontrivial component \( a'_{\pm}^{\nu} \), with
\[
-a'_{\pm}^{\nu} = -a_{0,\pm}^{\nu} + e^{2(\beta_{0}^{\nu} - \alpha_{0}^{\nu}) \xi_{\pm}} \tag{68}
\]
and the associated equation of motion
\[
(\rho'_{\pm} + P'_{\pm})(-a_{\pm}^{\nu}) = -\frac{\partial P'_{\pm}}{\partial r_{\pm}} \tag{69}
\]
From Equation (68), Equation (69) can be cast as
\[
(\rho_{0} + P_{0})e^{2(\beta_{0}^{\nu} - \alpha_{0}^{\nu}) \xi_{\pm}} = -\frac{\partial P'_{\pm}}{\partial r_{\pm}} - (\rho'_{\pm} + P'_{\pm})a'_{\pm}^{\nu} \tag{70}
\]
Therefore, in terms of distributions, Equation (70) reads
\[
(\rho_{0} + P_{0})e^{2(\beta_{0}^{\nu} - \alpha_{0}^{\nu}) \xi_{\pm}} = -\frac{\partial P'_{\pm}}{\partial r_{\pm}} - (\rho'_{\pm} + P'_{\pm})a'_{\pm}^{\nu} \tag{71}
\]
where \( \rho_{0} \) and \( P_{0} \) are given by Equations (55) and (56), respectively, and now
\[
\tilde{\xi}(r, t) = \tilde{\xi}^{+}(r^{+}, t^{+})\theta(r^{+} - R) + \tilde{\xi}^{-}(r^{-}, t^{-})\theta(R - r^{-}) \tag{72}
\]
Note that in Equation (71) we are considering jumps and symmetrizations of quantities defined in the presence of perturbations. As we stated previously, such perturbations change slightly the value of the physical quantities with respect to their hydrostatic values. The square brackets term of Equation (71) is the proper generalization of the curly brackets term in Equation (20). Naturally, the reasoning for the eigenfrequencies of \( \tilde{\xi} \) in the general relativistic case is the same as in the classical case. The same can be said about the continuity, though not differentiability, of \( \tilde{\xi} \) at any surface of a discontinuity.

Now, in order to have the proper generalization of Equation (30), we should evaluate Equation (71) at \( r + \tilde{\xi} \) and then subtract it from its evaluation at \( r \) concerning the static solution. In order to do it properly, one should take into account the general results for the case of Lagrangian displacements coming from the standard procedure (see, e.g., Misner et al. 1973), but now in the sense of distributions, by recalling that \( \Delta \Lambda \equiv A(r + \tilde{\xi}, t) - A_{0}(r, t) \), where \( A_{0} \) concerns the quantity \( A \) at equilibrium.

It is not difficult to see that we have the following results in the general relativistic distributional case (see, e.g., Misner et al. 1973, for the treatment of a continuous system):
\[
\Delta \beta = -\alpha_{0}^{\nu} \tilde{\xi} \tag{73}
\]
\[
\Delta \alpha' = 4\pi r e^{2\beta_{0}} [\Delta P + 2\delta \beta P_{0}] + \frac{e^{2\beta_{0}}}{r} \delta \beta + \alpha_{0}^{\nu} \tilde{\xi} \tag{74}
\]
\[
\Delta P = -\gamma \rho_{0} \left[ e^{-\beta_{0}} (r^{2} e^{2\beta_{0}}) \tilde{\xi} \right] / r^{2} + \delta \beta \tag{75}
\]
\[
\Delta \rho = -\rho_{0} + P_{0} \left[ \frac{1}{r^{2}} \frac{\partial}{\partial r} (r^{2} \tilde{\xi}) \right] + \frac{\alpha_{0}^{\nu}}{2} \left[ \frac{\partial \tilde{\xi}^{+}}{\partial r} + \frac{\partial \tilde{\xi}^{-}}{\partial r} \right] \tag{76}
\]
\[
\Delta \sigma = -\frac{2\sigma_{0}^{\nu} \tilde{\xi}}{R} - \tilde{\xi} \left[ e^{\beta_{0}} P_{0}^{\nu} + \frac{[\cosh \beta_{0}]^{+}}{4\pi R^{2}} \right] \tag{77}
\]
We stress that Equation (75) assumes adiabatic processes, in which one considers \( P = P(\rho) \), and we have made use of \( [\tilde{\xi}]_{\nu}^{+} = 0 \) for the above equations.

We shall seek solutions to the perturbations as \( \tilde{\xi} = e^{i\omega_{\nu} \xi}(r) \) with \( \omega_{\nu} \) the same for all of the phases the system may have. We just need to worry about the Dirac delta function term because it gives us the desired boundary condition valid for the separation of each two phases. The terms in front of the Heaviside functions by default will be the ones found in continuous media. It is not hard to see that the surface terms at the end should satisfy the condition
\[
\Delta \sigma \Delta (2\eta^{2} + 1) - \frac{\Delta (\alpha' e^{\beta_{0}})^{+}}{4\pi R} = 0 \tag{79}
\]
where we have used Equation (24). When the last term on the left-hand side of the above equation is expanded by using Equations (60) and (73), Equation (79) can be further simplified to
\[
\frac{2\eta^{2}}{R} \Delta \sigma - \frac{\Delta [\cosh \beta_{0}^{\nu}]}{4\pi R^{2}} = \frac{2\rho e^{\beta_{0}^{\nu}}}{R^{2}} - \left[ \frac{P e^{\beta_{0}^{\nu}}}{R^{2}} \right] \tilde{\xi} + \Delta [P e^{\beta_{0}^{\nu}}]^{+} \tag{80}
\]
One sees from Equation (80) that Equation (32) is recovered in the classical limit by recalling that \( \beta = M(r)/r \), which implies that the last term on the left-hand side of the above equation is \( 4\pi (g^{++} - g^{+}/(2R)) \). In this limit we take \( P \to 0 \) and \( e^{\beta} \to 1 \) for the remaining terms.

The case where electromagnetic interactions are also present is also of interest because its associated energy-momentum
tensor is anisotropic. This naturally influences the equation of hydrostatic equilibrium, because it now becomes
\[ \alpha'(P + \rho) = -P' - \frac{2}{r}(P_t - P), \] (81)
where \( P, P_t, \) and \( \rho \) are the resultant radial pressure, tangential pressure, and energy density of the fluid, respectively. For the electromagnetic fields, clearly \((P_t - P)\) is solely related to them. Because of the aforementioned aspects, the latter should also influence the dynamics of the radial perturbations, as we will show in the next section.

6. ELECTROMAGNETIC INTERACTIONS IN STRATIFIED SYSTEMS WITHIN GENERAL RELATIVITY

We consider now the inclusion of electromagnetic interactions within the scope of stratified systems in general relativity. An important comment at this level is in order. Because we are dealing with electromagnetic fields in stars, it would be more reasonable to assume the Maxwell equations in material media. Nevertheless, because our knowledge of the structures constituting the stars is not yet precise, it is difficult to assess their realistic dielectric properties. Because working with Maxwell equations in the absence of material media gives us upper limits to the fields under normal circumstances, this seems to be a good first tool to evaluate the relevance and effects of electromagnetism in stars. For the time being we will follow this approach. The energy-momentum tensor of each layer of the system we are now interested in should also have the electromagnetic one\(^5\):
\[ 4\pi T^\text{em}_{\mu\nu} = -F_{\mu\alpha}b^\alpha + b_{\mu\nu}F^{\mu\nu}/4, \] (82)
where we defined \( F_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu \). Solving Einstein–Maxwell equations on each layer of a stratified system leads us to the following equilibrium condition (Bekenstein 1971):
\[ \frac{\partial P}{\partial r} = \frac{Q(r)Q'(r)}{4\pi r^4} - \alpha'_Q(\rho + P), \] (83)
where
\[ Q(r) = \int_0^r 4\pi r'^2 \rho_c e^{-\beta_0} dr, \quad E(r) = e^{\varphi_0+\beta_0} \frac{Q(r)}{r^2}, \] (84)
\[ e^{-\beta_0} = 1 - \frac{2m_Q(r)}{r} + \frac{Q^2(r)}{r^2}, \] (85)
\[ \alpha'_Q = \frac{e^{2\beta_0}}{r^2} \left[ 4\pi r^3 P + m_Q(r) - \frac{Q^2(r)}{r} \right] \] (86)
and
\[ m_Q(r) \equiv \int_0^r 4\pi r'^2 \rho_c dr' + \frac{Q^2}{2r} + \frac{1}{2} \int_0^r \frac{Q^2}{r'^2} dr'. \] (87)
Equations (84), (85), (86), and (87) are the charge, radial, and time components of the metric and the energy of the system up to a radial coordinate \( r \), respectively. In Equation (84), we also showed the electric field \( E(r) \) in the context of general relativity, obtained by means of the definition \( F_{\mu\nu} = E(r) = -\partial_\mu A_\nu \). We stress that \( \rho_c \) is the physical charge density of the system, defined in terms of the four-current by \( j^\mu = e^{-\varphi_0} u^\mu \), where \( u^\mu \) is the four-velocity of the fluid with respect to the coordinate system \((t, r, \theta, \phi)\) (see Landau & Lifshitz 1975, for further details).

From Equation (83) one sees that in the scope of general relativity the effect of the charge is not merely to counterbalance the gravitational pull. For certain cases, it could even contribute to it. This is due to the contribution of the electromagnetic energy to the final mass of the system, as clearly given by Equation (87). Note from Equation (87) that we have assumed that the mass at the origin is null, in order to avoid singularities there. More generically, one could assume point or surface mass contributions in Equation (87) by conveniently adding Dirac delta functions in \( \rho \). Finally, we stress that Equation (84) can indeed be seen as the generalization of the charge in general relativity because it takes into account the nontrivial contribution coming from the spacetime warp due to its energy-momentum content.

Note that the classical limit to Equation (83) can be shown to coincide with Equation (46), by recalling that \( E_\text{clas}(r) = Q_\text{clas}(r)/r^2 \) and, from Equation (84), \( Q_\text{clas}(r) = 4\pi r^2 \rho_c r \). In addition, we recall that when converted to cgs units, the term \( Q^2/r \) (here in geometric units) becomes \( Q^2/(e^2 r) \), which is null in the classic nonrelativistic limit, as well as any pressure term on the right-hand side of the aforementioned equation.

Now, consider the analysis of a charged system constituted of two parts, connected by a surface of discontinuity (at \( r^\pm = R \)) that hosts surface degrees of freedom, such as an energy density, a charge density, and a surface tension. Its generalization to an arbitrary number of layers is immediate because each surface of discontinuity is only split by two phases. The proper description of the charge density in this case would be given by the generalization of Equation (47). Therefore, one would have at equilibrium that
\[ Q(r) = Q^-(r^-)\theta(R - r^-) + Q^+(r^+)\theta(r^+ - R), \] (88)
and for \( Q'(r) \), a Dirac delta will rise due to \( \rho_c \).

Let us define the distribution
\[ \tilde{\rho}_c = \rho_c e^{\beta_0} \equiv \tilde{\rho}_c^-(r - R) + \tilde{\rho}_c^+ (R - r) + \tilde{\sigma} c (r - R) \] (89)
where \( \tilde{\rho}_c^\pm = \tilde{\rho}_c^0 (r^\pm) \). From the above definition, we have that \( Q' = 4\pi r^2 \tilde{\rho}_c \). It implies that the total charge is the same as the one associated with \( \tilde{\rho}_c \) in a Euclidean space. Therefore, all classical results apropos of the charge densities and total charges that we deduced in the previous sections ensue here for \( \tilde{\rho}_c \).

We seek now the distributional generalization of Equation (83). This can be easily done by following the same reasoning from the previous section, which finally leads us to
\[ \frac{dP}{dr} = \frac{Q(r)\tilde{\rho}_c}{r^2} - (\rho + P)\alpha'_Q + \frac{2\tilde{\sigma}Q}{R} \delta(r - R) \]
\[ + \left\{ \left[ P(R) \right]^{\tilde{\rho}_c^+} + \frac{\tilde{\sigma}Q}{2} \left[ \tilde{\rho}_c^+(R) + \alpha'_Q(R) \right] + \frac{\tilde{\sigma}Q}{R} \right. \]
\[ - \frac{\left[ \tilde{\rho}_c^-(r^+) \right]}{4\pi R} - \frac{\tilde{\sigma}Q}{2R^2} \left[ \tilde{Q}_+(r^+) + \tilde{Q}_-(R) \right] \delta(r - R) \] (90)
where we are assuming that surface quantities with the subindex \( Q \) are related to the charged versions of Equations (52) and (53) (see also Equations (85) and (86)). It is easy to show that in the classical limit Equation (49) naturally rises, implying that in such a limit the curly brackets in Equation (90) are null.

\(^5\) We restrict our analyses to the Maxwell Lagrangian, \(-F_{\mu\nu}F_{\mu\nu}/4 \equiv -F/F/4\).
We consider now the case where radial perturbations take place in our charged system. This case is more involved than the neutral case because the charged particles also feel an electric force. The equation describing the evolution of the displacements can be shown to be generalized to (see Anninos & Rothman 2002, for the dynamics of the radial perturbations in a given phase)

\[ (\rho_0 + P_0)e^{2(\beta_0 - \alpha_0)} \xi' = -\frac{\partial P}{\partial r} - (\rho + P)\alpha'_Q + \frac{Q(r)}{r^2} + \frac{2P}{R} + \left[ \frac{\sigma_Q}{R} - \frac{[\alpha'_Q e^{-\beta_0}]^+}{4\pi R} \right] \]

For the change in \( Q(r) \), it can be shown (see Bekenstein 1971) that in the comoving frame there are no currents. This means that the Lagrangian displacements of \( Q \) are null, \( \Delta Q = 0 \). Equation (91) takes into account the values of the physical quantities in the presence of perturbations at \( r \). In order to obtain the generalization of Equation (30), we should evaluate Equation (91) at \( r + \xi \) and subtract it from Equation (90). This is due to the definition of the Lagrangian displacement of a given physical quantity, intrinsically related to the notion of observers comoving with the fluid, who naturally could describe its thermodynamics.

In order to simplify Equation (91), we have in the generalized charged case (see Anninos & Rothman 2002 for the treatment in a phase of a charged system)

\[ \Delta \beta_Q = -\alpha'_Q \xi, \quad (92) \]

\[ \Delta \alpha'_Q = 4\pi e^{2\beta_0} \left[ \delta P + 2\delta \beta_Q \left( P_0 - \frac{Q_0^2}{8\pi r^4} \right) \right] + \frac{e^{2\beta_0}}{r} \delta \beta_Q + \alpha'_Q \xi + \frac{4\pi e^{2\beta_0} \rho_0 Q_0}{r} \xi - \frac{2\pi \sigma_c \xi}{R} \times (Q_0^* e^{2\beta_0} + Q_0 e^{2\beta_0}) \delta(r - R), \quad (93) \]

\[ \Delta P = -\gamma P_0 \left[ \frac{e^{-\beta_0} \left( r^2 e^{2\beta_0} \xi' \right)}{r^2} + \Delta \beta_Q \right], \quad (94) \]

\[ \Delta \rho = -\left( \rho_0 + P_0 \right) \left[ \frac{1}{r^2} \frac{\partial}{\partial r} (r^2 \xi) \right] - \left[ \frac{dP}{dr} - \frac{Q \xi}{r^2} \right] \xi - \left( \frac{\sigma_Q}{2} \frac{\partial \xi^+}{\partial r} + \frac{\partial \xi^-}{\partial r} + \Delta \sigma_Q + \frac{2\sigma_Q \xi}{R} + [P_0] \xi \right) \delta (r - R), \quad (95) \]

\[ \gamma = \frac{\rho_0 + P_0}{P_0} \left( \frac{\partial P}{\partial \rho} \right)_{\rho=\text{const}}, \quad (96) \]

\[ \Delta \sigma_Q = -\left[ \frac{2\sigma_Q}{R} + [e^{\beta_0} P_0]^- \cdot \frac{[\cosh \beta_0]^-}{4\pi R^2} \right] \xi. \quad (97) \]

The additional 0 subindex in a physical quantity means that its value at equilibrium was taken. We just stress that Equation (96) is the general relativistic definition of the adiabatic index and assumes the existence of an equation of state linking the pressure and density of the system, \( P = P(\rho) \). In this sense, it generalizes \( \Gamma \) as defined by Equation (3).

By seeking solutions for \( \xi = e^{\omega t} \xi(r) \), one can see that Equation (91) only gives meaningful boundary conditions when a frequency \( \omega \) is the same for all of the phases present in the system. We emphasize that this is a universal property of the approach developed here, due to the surface degrees of freedom and the well-posedness of the problem of radial perturbations in stratified systems. The associated boundary condition arising from this analysis leads us to the conclusion that generically \( \xi(r) \) is not differentiable at a surface of discontinuity, though continuous. It can be shown that the associated boundary condition to be taken into account here is functionally the same as Equation (79) (or Equation (80)), where now the metric and surface quantities should be related to the charged case.

7. CONCLUSIONS

In this article we have developed a formalism for assessing the stability of a stratified star against radial perturbations. We have derived the relevant equations defining this boundary-value problem for both neutral and charged stars and in Newtonian and Einstein’s gravity. It makes use of the generalized theory of distributions: we assumed that the surfaces of discontinuity are thin, that they host surface degrees of freedom, and that the phases separated by them do not mix. We showed that although the phases may be very different among themselves, when perturbations take place, they lead to the notion of a set of eigenfrequencies describing the whole system, instead of an independent set for each phase. As a consequence, our formalism also gave us the proper additional boundary conditions to take into account when working with stratified systems. Such boundary conditions encompass surface degrees of freedom in the surfaces of discontinuities and generically modify the set of eigenfrequencies with respect to their continuous counterpart. This should be a generic fingerprint of stratified systems with nontrivial surface degrees of freedom. Our analyses are relevant for the assessment of the stability of realistic star models because they ensue the precise notion of boundary conditions. It was not our objective to systematically apply our formalism here, but simply to derive and expound it. It is clear that for precise and realistic numerical stability calculations, it would be ideal to have the microphysical knowledge of the properties of the interfacial surfaces. However, this a difficult problem that has been elusive even in the most advanced fields of materials science, where laboratory data of material surface properties are accessible, but there is still a lack of a complete physical theory for their explanation (Israelachvili 2011). Thus, the measurement of the star’s eigenmodes becomes of major relevance because it could give information not only on the star’s bulk structure but also on the possible existence of interior interfaces and their associated microphysical and electromagnetic phenomena.

The radial instabilities shown in our analyses should be interpreted analogously for continuous stars because, even in the stratified case, a global set of eigenfrequencies arises. Thus,
stratified stars would either implode or explode when they are radially unstable. Although we were concerned only with radial perturbations, it is of interest to investigate the additional oscillation modes owing to nonradial perturbations. The only point to be added, with respect to continuous stars, is the proper redifinition of the surfaces of discontinuity when such perturbations take place. This is clearly a richer scenario that inserts additional degrees of freedom into the system, leading to the appearance of additional modes such as the gravitational g modes (see, e.g., Reisenegger & Goldreich 1992). Such an analysis, however, is a second step that goes beyond the goal of the present work and that we are planning to investigate elsewhere.

We are grateful to Professor Thibault Damour for discussions on various occasions at the International Relativistic Astrophysics (IRAP) PhD-Erasmus Mundus Joint Doctorate Schools held in Nice. We are likewise grateful to Professor Luis Herrera. J.P.P. acknowledges the support given by the Erasmus Mundus Joint Doctorate Program within the IRAP PhD, under Grant Number 2011-1640 from EACEA of the European Commission. J.A.R. acknowledges the support by the International Cooperation Program CAPES-ICRANet financed by CAPES–Brazilian Federal Agency for Support and Evaluation of Graduate Education within the Ministry of Education of Brazil.

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