Decay $D^+ \rightarrow K^- \pi^+ \pi^+$: chiral symmetry and scalar resonances

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Abstract

The low-energy S-wave component of the decay $D^+ \rightarrow K^- \pi^+ \pi^+$ is studied by means of a chiral $SU(3) \times SU(3)$ effective theory. As far as the primary vertex is concerned, we allow for the possibility of either direct production of three pseudoscalar mesons or a meson and a scalar resonance. Special attention is paid to final state interactions associated with elastic meson-meson scattering. The corresponding two-body amplitude is unitarized by resumming s-channel diagrams and can be expressed in terms of the usual phase shifts $\delta$. This procedure preserves the chiral properties of the amplitude at low-energies. Final state interactions also involve another phase $\omega$, which describes intermediate two-meson propagation and is theoretically unambiguous. This phase is absent in the $K$-matrix approximation. Partial contributions to the decay amplitude involve a real term, another one with phase $\delta$ and several others with phases $\delta + \omega$. Our main result is a simple and almost model independent chiral generalization of the usual Breit-Wigner expression, suited to be used in analyses of production data involving scalar resonances.

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I. INTRODUCTION

Decays of heavy mesons have been recognized recently as important sources of information about scalar resonances, since the E791 experiment has produced solid evidence for a broad scalar-isoscalar state in $D^+ \rightarrow \pi^+ \pi^- \pi^+$ [1], known as the $\sigma$. In the case of the reaction we are concerned with, namely $D^+ \rightarrow K^- \pi^+ \pi^+$, the E791 group included in their fit another scalar state, known as the $\kappa$ (or $K^*(800)$), and concluded that this resonance was the dominant source of $S$-wave $\pi^+K^-$ pairs[2]. This represented a turn point in our understanding of this reaction, which was hitherto thought to be dominated by the non-resonant background. The existence of the $\kappa$ was then confirmed in a different approach by FOCUS [3] and by the new high statistics results from CLEO [4]. Empirical information about the parameters of these resonances, especially the position of their poles in the complex energy plane, can only be obtained after analyses of large amounts of data organized into Dalitz plots.

Low-energy mesons correspond to gentle deviations from the QCD vacuum and tend to be highly collective states, as indeed happens with pions or kaons, which are large objects. In the case of low-energy scalar resonances, one is entitled to expect that the influence of the vacuum will be a truly overwhelming one. This is probably one of the reasons why they are so broad and prove to be so elusive.

Dalitz plots of $D$-meson decays into three pseudoscalars usually contain several $S$- and $P$-wave resonances[1, 5], which couple by means of final state interactions (FSIs). This makes the disentangling of individual resonance properties both very involved and strongly dependent on particular forms of trial functions adopted. Theoretical ansätze employed in data interpretation usually rely on Breit-Wigner expressions, which proved to work well in many instances. A popular ansatz for the trial function has the form

$$\mathcal{A} = a_0 e^{i\phi_0} \mathcal{A}_0 + \sum_{S\text{-wave}} a_n^S e^{i\phi_n^S} \mathcal{A}_n^S + \sum_{P\text{-wave}} a_n^P e^{i\phi_n^P} \mathcal{A}_n^P + \cdots,$$

(1)

where the first term represents a non-resonant background and the amplitudes $\mathcal{A}_n^\ell$ are Breit-Wigner expressions for each resonance present in the final state. The masses and widths of well established states are used as input, whereas those of low-lying resonances, as well as the free parameters $a_n^\ell$ and $\phi_n^\ell$, are fitted to data. As a consequence, these adjusted parameters acquire the status of empirical quantities.

In the case of $\pi\pi$ scattering, the fact is well established that the lowest pole of the
amplitude is located roughly at $\sqrt{s} \approx (0.47 - i0.29)\text{ GeV}$, whereas the $S$-wave phase shift reaches $\pi/2$ around $\sqrt{s_{\pi/2}} \approx 0.92\text{ GeV}$. These findings are clearly at odds with the traditional use of Breit-Wigner expressions. An explanation for these seemingly paradoxical results was provided by Colangelo, Gasser and Leutwyler[6], who have shown that chiral symmetry requires a compromise between the polynomial nature of the amplitude at very low-energies and the vanishing of its real part at $s_{\pi/2}$, which is responsible for a large shift in the pole position. This kind of feature is inherent to the isoscalar channel. The resonance, which is a non-leading chiral effect, must always coexist with an important polynomial in $s$. In the framework of chiral symmetry, the first two terms in eq.(1), namely $[a_0 e^{i\phi_0} A_0 + a_1^S e^{i\phi_1} A_1^S]$, are not suited for describing low-energy interactions. The use of this kind of trial function is problematic and may give rise to results which are not reliable.

The extraction of information from experiments involving scalar resonances must be performed in the best theoretical framework possible, as the quality of results for masses and coupling constants depend on the ansätze employed. In this work we propose an alternative form for the low-energy sector of the trial function, which can be used as a tool in analyses of the decay $D^+ \rightarrow K^- \pi^+ \pi^+$. Our motivation for choosing this particular reaction is two-fold. The first one is that scalar resonances in the final state occur just in the $\pi^+K^-$ subsystem, the number of possible couplings is relatively small, and the problem is simplified. The second is the availability of recent data analyses on this process, which could allow the testing of our results. Nevertheless, the general lines adopted here can be extended to other systems in a straightforward manner. Our theoretical model is based on standard $SU(3) \times SU(3)$ effective chiral lagrangians incorporating scalar resonances[7, 8].

We are also concerned with the presence of phases in theoretical ansätze, which are sometimes employed with no visible physical meaning. In the realm of two-body systems, the dynamical origin of phases is well understood. In particular, it is known that two-body rescattering gives rise to the elastic phase shift $\delta$. When resonances are present, this phase is shared with the so called production amplitude, as dictated by Watson’s theorem [9]. We show, in the sequence, that another phase is also present in the $D^+$ decay considered here, associated with two-meson intermediate states.

The theoretical description of a heavy-meson decay into three pseudoscalars is necessarily complex. Nevertheless, we have made an effort to produce final expressions which incorporate a compromise between reliability and simplicity, so that they could be employed.
directly in data analyses. Our paper is organized as follows. In section II, we review the
basic lagrangians, which are used in the description of both the primary weak vertex and
final state interactions. The former is discussed in section III, whereas the latter are dis-
cussed in sections IV, V and VI. These results are assembled in section VII, where our main
results are presented and analyzed. In section VIII, we display individual predictions from
the various components of the decay amplitude in Dalitz plots, so as to produce a feeling of
their dynamical content. Finally a summary and comprehensive conclusions are presented
in section IX. Details concerning kinematical variables, the form of the two-body propagator
and background interactions are left to appendices.

II. DYNAMICS

The reaction $D^+ \rightarrow K^-\pi^+\pi^+$ involves both weak and strong processes. The former
are associated with the isospin conserving quark transition $c \rightarrow s W^+$, whereas the latter
occur in final state interactions involving both pseudoscalar mesons and their resonances.
In order to keep approximations under control, we remain, as much as possible, within the
single theoretical framework provided by a chiral effective field theory. This choice is also
motivated by the fact that we are concerned mostly with the low-energy sector of the trial
function.

• strong interactions:

Meson-meson interactions are described by the chiral $SU(3) \times SU(3)$ lagrangian at $O(q^2)$
given by Gasser and Leutwyler[7], whereas couplings of scalar resonances to pseudoscalar
mesons are taken from the work of Ecker, Gasser, Pich and De Rafael[8]. Keeping only
relevant terms, one has

$$\mathcal{L}^{(2)} = \frac{F^2}{4} \langle \nabla_\mu U^\dagger \nabla^\mu U + \chi^\dagger U + \chi U^\dagger \rangle$$

$$+ c_d \langle S u_\mu u^\mu \rangle + c_m \langle S \chi_+ \rangle$$

$$+ \tilde{c}_d S_1 \langle u_\mu u^\mu \rangle + \tilde{c}_m S_1 \langle \chi_+ \rangle, \quad (2)$$

where $\langle \cdots \rangle$ indicates the trace, $F$, $c_d$, $c_m$, $\tilde{c}_d$ and $\tilde{c}_m$ are constants, $S$ and $S_1$ represent
scalar resonances and $U$ is the pseudoscalar field. Using the definition $u = U^{1/2}$, one has

$$\nabla_\mu U = \partial_\mu U,$$

$$u_\mu = i u^\dagger \nabla_\mu U u^\dagger,$$

$$\chi = 2 B \sigma,$$

$$\chi_+ = u^\dagger \chi u^\dagger + u \chi^\dagger u. \quad (3)$$
The field $\sigma$ incorporates the quark masses as external scalar sources and, in the isospin limit, is written in terms of the usual Gell-Mann matrices as $\sigma = \sigma_0 I + \sigma_8 \lambda_8$, with $\sigma_0 = (2\hat{m} + m_s)/3$, $\sigma_8 = (\hat{m} - m_s)/\sqrt{3}$ and $m_u = m_d \equiv \hat{m}$. We neglect $\eta$-$\eta'$ mixing and use $B\sigma_0 = (2M_K^2 + M_\pi^2)/6$ and $B\sigma_8 = -(M_K^2 - M_\pi^2)/\sqrt{3}$. The meson field is written as $U \equiv e^{i\Phi/F}$, $\Phi \equiv \lambda_i \phi_i$, and the leading order strong lagrangian becomes

$$L^{(2)} = -\frac{1}{6F^2} f_{ijkl} \phi_i \partial_{\mu} \phi_j \partial_{\nu} \phi_k + \frac{B}{2AF^2} \left[ \sigma_0 \left( \frac{4}{3} \delta_{ij} \delta_{kl} + 2d_{ij}d_{kl} \right) \right] \phi_i \phi_j \phi_k \phi_l$$

$$+ \sigma_8 \left( \frac{4}{3} \delta_{ij} d_{kl8} + \frac{4}{3} d_{ijkl} \delta_{kl} + 2d_{ijm}d_{klm} \right) \phi_i \phi_j \phi_k \phi_l$$

$$+ \frac{2c_d}{F^2} S_0 \partial_{\mu} \phi_i \partial_{\nu} \phi_i - \frac{4c_m}{F^2} B \left[ \sigma_0 \delta_{ij} + \sigma_8 d_{ij} \right] S_0 \phi_i \phi_j$$

$$+ \frac{2c_d}{F^2} d_{bij} S_0 \partial_{\mu} \phi_i \partial_{\nu} \phi_j - \frac{c_m}{F^2} B \left[ 3 \sigma_0 d_{aij} + \sigma_8 (2\delta_{bi} \delta_{j} + 4d_{bjs} \delta_{js}) \right] S_0 \phi_i \phi_j,$$

(4)

where $f_{ijkl}$ and $d_{ijkl}$ are the usual $SU(3)$ constants. We use the conventions of ref.[10] for meson fields and $\bar{\kappa}^0 \equiv (S_6+iS_7)/\sqrt{2}$ for the scalar resonance. In the sequence, the $\bar{\kappa}^0$ state is called $\kappa$ for simplicity and, in isospin space, one has

$$|\pi^+ K^- \rangle = \sqrt{1/3} |3/2, 1/2 \rangle + \sqrt{2/3} |1/2, 1/2 \rangle,$$

$$|\pi^0 K^0 \rangle = \sqrt{2/3} |3/2, 1/2 \rangle - \sqrt{1/3} |1/2, 1/2 \rangle.$$  

(5)

**weak interactions:**

![Diagram](image)

FIG. 1: (Color online) Weak amplitudes involving pseudoscalar mesons (dashed lines) and the scalar resonance (continuous line).

Effective weak vertices contain a propagating $W$ and are based on the processes shown in fig.1, which involve the following combinations of reactions:

- type (a): $(D^+ \to \pi K W^+)_a$ and $(W^+ \to \pi)_a$, 
- type (b): $(D^+ \to K W^+)_v$ and $(W^+ \to \pi \pi)_v$, 

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type (c): \((D^+ \rightarrow \bar{K} W^+)\) and \((W^+ \rightarrow \pi)\),

where the labels \(v\) and \(a\) refer to either vector or axial currents in \(W\) couplings. In the want of a comprehensive theory, these weak vertices can be derived by means of appropriate hadronic currents and \textit{ad hoc} phenomenological coupling constants. The latter can be estimated in semi-leptonic processes, by replacing the top \(\pi^+\) in the figure by \((\ell^+ \nu_\ell)\). Although feasible, the piecemeal implementation of this program is cumbersome, owing to the large number of possible isospin couplings and phase conventions.

Fortunately a rather economic alternative is available, based on the group \(SU(4)\). Before proceeding, we would like to make clear that we are quite aware that \(SU(4)\) is not a good symmetry and therefore we are not advocating its use here. On the other hand, the inclusion of \(D\) mesons into the pseudoscalar multiplet does give rise, by means of the external source technique, to an effective lagrangian in which all currents and isospin couplings are generated automatically. We obtain \(W^+\) couplings by using a \(SU(4) \times SU(4)\) mesonic lagrangian for the weak sector and, at the end, break any commitments with a symmetry, by allowing phenomenological coupling constants into the Feynman rules.

In the standard model lagrangian, weak couplings are given by

\[
\mathcal{L}_w = \bar{q} \gamma_\mu \left[ v_\mu + \gamma_5 a_\mu \right] q,
\]

where the external sources are written in terms of \(SU(4)\) matrices as \(v_\mu = v_\mu^a \lambda_a/2\) and \(a_\mu = a_\mu^a \lambda_a/2\). The charged currents are

\[
v_\mu + a_\mu = 0, \quad v_\mu - a_\mu = -\frac{g}{\sqrt{2}} \left[ W^+_\mu V_{SU(4)} + W^-_\mu V^\dagger_{SU(4)} \right],
\]

where \(g\) is the weak coupling constant, \(W\) is the gauge field and \(V_{SU(4)}\) is the \(SU(4)\) sector of the Cabibbo-Kobayashi-Maskawa mixing matrix. Its explicit form, in terms of the Cabibbo angle \(\theta_C\), reads\(^1\)

\[
V_{SU(4)} = \frac{1}{2} \left[ \cos \theta_C (\lambda_1 + i \lambda_2) + \sin \theta_C (\lambda_4 + i \lambda_5) - \sin \theta_C (\lambda_{11} + i \lambda_{12}) + \cos \theta_C (\lambda_{13} + i \lambda_{14}) \right].
\]

The weak effective lagrangian is obtained by replacing \(\nabla_\mu\) in eqs.\((2)\) and \((3)\) with \(\nabla_\mu U = \partial_\mu U - i (v_\mu + a_\mu) U + i U (v_\mu - a_\mu)\). The term that interests is then given by

\[
\mathcal{L}_w^{(2)} = v_\mu^a \left[ f_{ajk} \phi_j \partial_\mu \phi_k \right] - a_\mu^a \left[ F \delta_{al} \partial_\mu \phi_l + \frac{2}{3 F} f_{ajl} f_{kla} \phi_j \phi_k \partial_\mu \phi_l + \frac{4 c_d}{F} d_{abl} S_b \partial_\mu \phi_l \right],
\]

where the factors within square brackets are respectively vector and axial hadronic currents.

\(^1\) We use the \(SU(4)\) conventions of ref.\cite{11}.
III. WEAK VERTICES

In this section we display the tree level amplitudes given in fig.1 and include final state interactions afterwards. The weak vertices in which the $D^+$ participates involve the transformation of a $c$-quark into a $s$-quark and are Cabibbo allowed. As final state interactions allow the transition $\pi^0 K^0 \rightarrow \pi^+ K^-$, one also needs to consider vertices involving neutral mesons. The production of a single pion in $W^+ \mu \rightarrow \pi^+ (q')$ is represented by

$$ T^\pi_w = i \left( F g / 2 \right) \cos \theta_C \ q'_\mu \ . \ (9) $$

Decays of the type (a) are based on the processes $D^+(P) \rightarrow W^+_\mu K^-(k) \ \pi^+(q)$ and $D^+(P) \rightarrow W^+_\mu K^0(k) \ \pi^0(q)$, given respectively by the amplitudes $\sqrt{2/3} \ T^{D\pi K}_w$ and $-\sqrt{1/3} \ T^{D\pi K}_w$, with

$$ T^{D\pi K}_w = -i \left( g / 2 \sqrt{6} F \right) \cos \theta_C \ (P - k)_\mu \ . \ (10) $$

In the case of vector couplings, one needs the vertices $D^+(P) \rightarrow W^+_\mu K^0(k)$ and $W^+_\mu \rightarrow \pi^+ (q') \ \pi^0(q)$, which read

$$ T^{DK}_w = -(g / 2 \sqrt{2}) \cos \theta_C \ (P + k)_\mu \ , \ (11) $$
$$ T^{\pi\pi}_w = (g / 2) \cos \theta_C \ (q - q')_\mu \ . \ (12) $$

Finally, for the vertex $D^+(P) \rightarrow W^+_\mu \bar{\kappa}$, involving the scalar resonance, one has

$$ T^{D\bar{\kappa}}_w = i (c_d g \sqrt{2} / F) \cos \theta_C \ P_\mu \ . \ (13) $$

Using $\Delta^{\mu\nu}_W = i \left( g^{\mu\nu} / M^2_W \right)$ for the $W$ propagator and the definition $G_F \equiv \sqrt{2} g^2 / 8 M^2_W = 1.166 \times 10^{-5} \text{GeV}^2$[12] , one finds the following weak amplitudes

**type (a):**

$$ D^+(P) \rightarrow K^-(k) \ \pi^+(q) \ \pi^+(q') : \ \sqrt{2/3} \ \mathcal{W}_a \ , \ (14) $$
$$ D^+(P) \rightarrow \bar{K}^0(k) \ \pi^0(q) \ \pi^+(q') : \ -\sqrt{1/3} \ \mathcal{W}_a \ , \ (15) $$
$$ \mathcal{W}_a = [\delta_a] \left( G_F / \sqrt{3} \right) \cos^2 \theta_C \ (P - k) \cdot q' , \ (16) $$

**type (b):**

$$ D^+(P) \rightarrow \bar{K}^0(k) \ \pi^0(q) \ \pi^+(q') : \ \mathcal{W}_b = -[\delta_b] G_F \cos^2 \theta_C \ (P + k) \cdot (q - q') \ , \ (17) $$
type (c):

\[ D^+(P) \rightarrow \bar{\kappa}(q_\kappa) \pi^+(q') : \quad \mathcal{W}_c = -[\delta_c] 4 G_F c_d \cos^2 \theta_C P \cdot q' , \]  

(18)

where ad hoc factors \([\delta_i]\) were introduced so to freeing results from any constraints imposed by \(SU(4)\) symmetry.

IV. FSI: KERNEL

Final state interactions are essential to structures observed in Dalitz plots, since they promote couplings among various channels and, in particular, give rise to widths of resonances. The final state considered here contains three mesons and a complete treatment of the problem is not possible. One is forced to employ approximations and we adopt the quasi-two-body approach, in which one of the final mesons acts as a mere spectator. As emerging pions have isospin 2 and no resonance is known in this channel, their interactions can be safely neglected and strong interactions are restricted to the \(\pi K\) subsystem.

These assumptions lead to the model given by fig.2, in which all strong processes are incorporated into the amplitudes \(T\) and \(\Pi\), representing respectively elastic scattering and production. As the isospin of the \(\pi K\) system can be either 1/2 or 3/2 and only the former couples with the \(\kappa\), one needs to consider three amplitudes, namely \(T_{1/2}^1, T_{3/2}^3\) and \(\Pi_{1/2}^{1/2}\). The basic building blocks of both \(T_I\) and \(\Pi_I\) are kernels \(K_I\), which describe elastic S-wave \(\pi K\) scattering at tree level. These kernels are obtained by projecting out S-wave components from tree-level amplitudes \(\bar{T}_I\), given by the diagrams of fig.3. As far as chiral symmetry is concerned, the leading term is given by the \(O(q^2)\) contact term whereas diagrams involving resonances are \(O(q^4)\) corrections.

We consider the process \(\pi(Q) K(K) \rightarrow \pi(q) K(k)\) and the tree amplitudes are explicitly written in terms of the Mandelstam variables as

\[ \bar{T}_{1/2} = \frac{1}{4 F^2} \left[ 4 s + 3 t - 4 \left( M_\pi^2 + M_K^2 \right) \right] \]

\[ - \frac{3}{4} \frac{1}{s-m_\kappa^2} \frac{4}{F^4} \left[ c_d (s-M_\pi^2-M_K^2) + c_m \left( 5 M_\pi^2+4 M_K^2 \right) / 6 \right]^2 \]

\[ + \bar{T}_t^0 - \bar{T}_t^8/6 - \bar{T}_u^\kappa/4 , \]  

(19)

\[ \bar{T}_{3/2} = - \frac{1}{2 F^2} \left[ s - (M_\pi^2+M_K^2) \right] + \bar{T}_t^0 - \bar{T}_t^8/6 + \bar{T}_u^\kappa/2 , \]  

(20)
FIG. 2: (Color online) Diagrams contributing to the decay $D^+ \to K^- \pi^+ \pi^+$; (a) corresponds to a direct process, (b-e) involve the $\pi K$ scattering amplitude $T$, and (f) depends on the production amplitude $\Pi$.

\[ \bar{T}_0^\ell = -\frac{1}{t-m_0^2} \frac{4}{F^4} \{ [\tilde{c}_d (t-2 M^2_{\pi}) + 2 \tilde{c}_m M^2_{\pi}] [\tilde{c}_d (t-2 M^2_K) + 2 \tilde{c}_m M^2_K] \} , \tag{21} \]

\[ \bar{T}_8^\ell = -\frac{1}{t-m_8^2} \frac{4}{F^4} \{ [c_d (t-2 M^2_{\pi}) - c_m (2 M^2_K - 11 M^2_{\pi})/6] \} , \tag{22} \]
× \left\{ c_d \left( t - 2M_K^2 \right) + c_m \left( 10M_K^2 - M^2 \right) / 6 \right\} ,
\]
(23)

\[ T_u^c = - \frac{1}{u - m^2} \frac{4}{F^4} \left[ c_d \left( u - M^2 - M_K^2 \right) + c_m \left( 4M_K^2 + 5M^2 \right) / 6 \right] ,
\]
(24)

The projection into S-waves is performed using results from appendix A and one finds

\[ K_{1/2} = \frac{1}{4F^2} \left[ \left( 4 - 3\rho^2 / 2 \right) s - 4 \left( M^2 + M_K^2 \right) \right]
\]

\[ - \frac{1}{s - m^2} \frac{3}{F^4} \left[ c_d \left( M^2 - M^2 - M_K^2 \right) + c_m \left( 5M^2 + 4M_K^2 \right) / 6 \right] + B_{1/2} ,
\]
(25)

\[ K_{3/2} = - \frac{1}{2F^2} \left[ s - (M^2 + M_K^2) \right] + B_{3/2} ,
\]
(26)

with \( \rho = \sqrt{1 - 2(M_K^2 + M^2) / s + (M_K^2 - M^2)^2 / s^2} \). The functions \( B_I \) are smooth backgrounds given explicitly in appendix C. As discussed there, all \( t \) and \( u \) channel are small and can be either treated as four-point contact interactions or neglected. This gives rise to the effective structures shown in fig.4, where contact terms in the kernels include both leading chiral contributions and non-resonant backgrounds.

FIG. 4: (Color online) Effective structures of the kernels \( K_I \); the cross hatched bubbles represent effective contact interactions, which include both the leading \( O(q^2) \) contribution and background terms.

In the isospin \( 1/2 \) channel, it is convenient to emphasize the role of the resonance by factorizing the \( s \)-channel denominator and writing[13]

\[ K_{1/2} = - \frac{\gamma^2}{s - m^2} ,
\]
(27)

\[ \gamma^2 = \left( 3/F^4 \right) \left[ c_d \left( m_K^2 + M^2 - M_K^2 \right) + c_m \left( 5M^2 + 4M_K^2 \right) / 6 \right] - \left\{ \left( 1/4F^2 \right) \left[ (4 - 3\rho^2 / 2) s - 4 \left( M^2 + M_K^2 \right) \right] + B_{1/2} \right\} \left( s - m^2 \right) .
\]

Of course, in spite of differences in form, eqs.(25) and (27) have exactly the same content.

As discussed in the introduction, we are interested in mapping low-energy degrees of freedom of the amplitude \( D^+ \rightarrow K^+\pi^+\pi^+ \). This means that masses and coupling constants must be kept free, so that their values can be extracted from experiment. On the other hand,
in discussing qualitative features of our results, we need to fix somehow these free parameters. In this case, we choose: $m_\kappa = 1.2$ GeV, $(c_d, c_m) = (3.2, 4.2) \times 10^{-2}$ GeV; $(\tilde{c}_d, \tilde{c}_m) = (1.8, 2.4) \times 10^{-2}$ GeV[8]. It is worth stressing that we are by no means recommending these values.

![Graph](image)

**FIG. 5:** (Color online) Kernels $K_I$ (continuous lines) and the leading $O(q^2)$ contact chiral contribution (dashed lines).

In fig.5 we display the full kernels $K_I$, together with their leading $O(q^2)$ chiral components, and it is possible to note an important isospin dependence of the results. The dynamical structure of $K_{1/2}$ involves three different regimes. At low energies, for $s$ between threshold and $\sim 0.6$ GeV$^2$, it is determined by chiral constraints whereas, as $s$ increases, it becomes dominated by the first resonance pole. For larger values of $s$, effects associated with other resonances, not considered in this work, do show up. Therefore, with the choice $m_\kappa = 1.2$ GeV, the upper limit of validity for our results is assumed to be $\sqrt{s} \sim 1.3$ GeV. The kernel $K_{3/2}$ is repulsive and monotonic.

**V. FSI: SCATTERING AMPLITUDE**

The elastic $\pi K$ scattering amplitudes $T_I$ are derived by using the two-body irreducible kernels $K_I$ into the Bethe-Salpeter equation, written schematically as

$$ T_I = K_I + i \int d^4\ell \frac{\Delta_{\pi K}(\ell)}{(2\pi)^4} \ K_I(\ell) \ \Delta_{\pi K}(\ell) \ T_I(\ell) \ , $$

where $\Delta_{\pi K}$ is the two-meson propagator. The diagramatic representation of this equation, together with its perturbative solution, are shown in fig.6. In the case of low-energy in-
interactions, the treatment of the Bethe-Salpeter equation can be enormously simplified, as pointed out by Oller and Oset[14]. The fact that the kernels $K_I$ involve only effective contact interactions and $s$-channel resonances makes the two-meson propagator $\Delta_{\pi K}$ to depend just on $s$ and eq.(28) can be rewritten as

$$T_I = K_I - K_I \Omega T_I, \quad (29)$$

$$\Omega = i \int \frac{d^4 \ell}{(2\pi)^4} \frac{1}{[(q+k)/2+\ell]^2 - M_{\pi}^2} \frac{1}{[(q+k)/2-\ell]^2 - M_{K}^2}. \quad (30)$$

The function $\Omega$ diverges and has to be regularized. As discussed in appendix B, this brings into the problem one free parameter for each isospin channel. The corresponding regular functions are denoted by $\bar{\Omega}_I$ and the solutions of eq.(29) become

$$T_I = \frac{\mathcal{K}_I}{1 + \bar{\Omega}_I \mathcal{K}_I}. \quad (31)$$

Above threshold, the functions $\bar{\Omega}_I$ are complex and written as $\bar{\Omega}_I = \bar{R}_I + i I$. They involve a loop phase $\omega_I \equiv \tan^{-1}[I/\bar{R}_I]$ and their explicit analytic forms are given in appendix B.

The amplitudes $T_I$ do respect unitarity and, below the first inelastic threshold, can always be written as

$$T_I = \frac{16\pi}{\rho} \sin \delta_I e^{i\delta_I}, \quad (32)$$

$$\tan \delta_I = -\frac{I \mathcal{K}_I}{1 + \bar{R}_I \mathcal{K}_I}, \quad (33)$$

where the real phase shifts $\delta_I$ incorporate the dynamical content of the interaction. These results allow the kernels to be expressed as

$$\mathcal{K}_I = \frac{16\pi}{\rho} \frac{\tan \delta_I}{(1 + \tan \delta_I/\tan \omega_I)}. \quad (34)$$

$^2$ The amplitude $T$ is relativistic and we employ the conventions of Refs. [15] and [13].
The unitarization procedure employed in the derivation of $T_I$ generalizes that based on the on-shell iteration of the $K$-matrix [15, 16], which amounts to neglecting the real part of $\bar{\Omega}_I$ and to assuming $K_I \simeq [16\pi/\rho] \tan \delta_I$.

The behaviors of the amplitudes $T_{1/2}$ and $T_{3/2}$ are very different, owing to the presence of a resonance in the former. In this case, we use the kernel (27) into eq.(31) and finds

$$T_{1/2} = \frac{16\pi}{\rho} \frac{m_\kappa \Gamma_\kappa}{\mathcal{M}_\kappa^2 - s - i m_\kappa \Gamma_\kappa}, \quad (35)$$

where the running mass and width are defined by

$$\mathcal{M}_\kappa^2 \equiv m_\kappa^2 + \gamma^2 \bar{R}_{1/2}, \quad (36)$$

$$m_\kappa \Gamma_\kappa \equiv \gamma^2 \rho/(16\pi), \quad (37)$$

and the free parameter in $\bar{R}_{1/2}$ was chosen so that $\mathcal{M}_\kappa^2(m_\kappa^2) = m_\kappa^2$. This yields a unitary amplitude $T_{1/2}$ which becomes purely imaginary at $s = m_\kappa^2$. Therefore, we call $m_\kappa$ the *nominal* kappa mass. Predictions for phase shifts can also be expressed as

$$\tan \delta_{1/2} = \frac{m_\kappa \Gamma_\kappa}{\mathcal{M}_\kappa^2 - s}, \quad (38)$$

and the relationship between nominal and running masses is determined by

$$\frac{m_\kappa^2 - s}{\mathcal{M}_\kappa^2 - s} = 1 + \tan \delta_{1/2}/\tan \omega_{1/2}. \quad (39)$$

$K$-matrix unitarization corresponds to making $\mathcal{M}_\kappa^2 \rightarrow m_\kappa^2$ and deviations between both approaches are quantified by the factor $(1 + \tan \delta_{1/2}/\tan \omega_{1/2})$.

Real and imaginary parts of the amplitude $T_{1/2}$, eq.(35), together with the corresponding $K$-matrix approximation, are given in fig.7, for the choice $m_\kappa = 1.2$ GeV. In both cases, amplitudes become purely imaginary at $s = m_\kappa^2$ and peaks occur at lower energies. As one discusses in section IX, these shifts in the peaks are a direct consequence of chiral symmetry. The figure shows that the $K$-matrix approximation is a crude one and that the role played by the running mass is important.

In figs.8 and 9 we display the isospin dependence of $\delta_I$ and $|T_I|^2$. It is worth noting that, by construction, $\delta_{1/2}$ passes through $90^0$ at $s = m_\kappa^2$ and, again, the $K$-matrix approximation is crude. In the case of the $I = 3/2$ channel, the free parameter in $\bar{R}_{3/2}$ was fixed by imposing that the predicted phase shifts around $s = 0.9$ GeV agree roughly with those given in Ref.[17]. This gives rise to huge differences between full and $K$-matrix results.
FIG. 7: (Color online) Real (full lines) and imaginary (dashed lines) components of the amplitude $T_{1/2}$ obtained by means of eq.(35) (blue) and in the $K$-matrix approximation (red).

FIG. 8: (Color online) Predicted phase shifts for $T_{1/2}$ and $T_{3/2}$ amplitudes, together with $K$-matrix results.

In the decay $D^+ \rightarrow K^-\pi^+\pi^+$, elastic amplitudes $T_I$ contribute to final state interactions only. They are always accompanied by the two-body propagator $\Omega_I$, as indicated in fig.2 [please see also eqs.(44-47)]. Real and imaginary parts of the products $\Omega_I T_I$ are displayed in fig.10. In order to clarify the meaning of this figure, one notes that, in the $K$-matrix approximation, $\Omega \rightarrow -i \rho/(16\pi)$, and $[\Re \Omega T, \Im \Omega T]$ is given by $[\rho/(16\pi) (\Im T, -\Re T)]$. 

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FIG. 9: (Color online) Dependence of $|T_{1/2}|^2$ and $|T_{3/2}|^2$ on $s$, together with the $K$-matrix approximation.

When the real part of $\bar{\Omega}$ is turned on, a shift in the curves occur and the phase of $\bar{\Omega}_I T_I$ becomes $\delta_I + \omega_I$. The magnitudes of the $\omega_I$ may be inferred by noting that, in the $K$-matrix case, $\Re \bar{\Omega} T = \Im T = 0$ at threshold. Loop phases are, therefore, explicit ingredients of the decay amplitude. With future purposes in mind, one notes that the condition $-1 \leq \bar{\Omega}_{1/2} T_{1/2} \leq 1$ holds for both the real and imaginary components of this quantity.

FIG. 10: (Color online) Real (continuous line) and imaginary (dashed line) components of the functions $\bar{\Omega}_I T_I$. 

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VI. FSI: PRODUCTION AMPLITUDE

\[ i \Pi_{1/2} = \frac{\sqrt{3}/F^2}{s - m_k^2} \left[ c_d (s - M_\pi^2 - M_K^2) + c_m (4M_K^2 + 5M_\pi^2)/6 \right] \left[ 1 - \bar{\Omega}_{1/2} T_{1/2} \right] . \] \hspace{1cm} (40)

This function is complex, owing to the factor \( [1 - \bar{\Omega}_{1/2} T_{1/2}] \), and can be cast in various fully equivalent forms\[18\], namely

\[ i \Pi_{1/2} = g \frac{\cos \delta_{1/2}}{M_K^2 - s} \frac{\cos \delta_{1/2}}{\tan \omega_{1/2}} \left[ 1 + \frac{\tan \delta_{1/2}}{\tan \omega_{1/2}} \right] e^{i\delta_{1/2}} \]

\[ = g \frac{\cos \delta_{1/2}}{M_K^2 - s} e^{i\delta_{1/2}} = g \frac{\sin \delta_{1/2}}{m_k \Gamma_k} e^{i\delta_{1/2}} , \] \hspace{1cm} (41)

\[ g = -\left(\sqrt{3}/F^2\right) \left[ c_d (s - M_\pi^2 - M_K^2) + c_m (4M_K^2 + 5M_\pi^2)/6 \right] . \] \hspace{1cm} (42)

Results have very little model dependence and show that, as predicted by Watson’s theorem, the phase in the production amplitude is \( \delta_{1/2} \), the same as in the elastic process. On the other hand, the magnitude of \( \Pi_{1/2} \) is determined by both \( \delta_{1/2} \) and \( \omega_{1/2} \). It is important to stress that the full equivalence among these different forms holds only for the running mass and width given by eqs.(36) and (37). If other forms for these functions are employed, consistency is lost and results are no longer under control. As in the elastic case, the function \( \Pi_{1/2} \) contains the \( K \)-matrix approximation as a particular case. In Fig.12 we show the behavior of the function \( i\Pi_{1/2}(s) \), together with the corresponding \( K \)-matrix approximation, and notes that the differences between both sets of results are important. The figure also indicates that bumps in the former are more pronounced.
Fig. 12: (Color online) Modulus (full line), real and imaginary components (dashed and dot-dashed lines) of the function $i \Pi_{1/2}(s)$, eqs.(40-42), together with the $K$-matrix approximation.

VII. Amplitude for $D^+ \rightarrow \pi^+\pi^+K^-$

We display here individual contributions to the $D^+$ decay width from the diagrams of fig.2, using both the scattering and production amplitudes derived previously. They are covariant and expressed in terms of the invariant masses $\mu_{ij}$, defined in appendix A. The sum $A = [A_a + \cdots + A_f]$ is designed to replace the background plus $S$-wave factor $[a_0 e^{i\phi_0} A_0 + a_1^S e^{i\phi_1} A_1^S]$ mentioned in the introduction.

diagram (a):

$$A_a(\mu_{\pi\pi}^2) = \frac{1}{3\sqrt{2}} [\delta_a] G_F \cos^2 \theta_C \mu_{\pi\pi}^2 ;$$ (43)

diagram (b):

$$A_b(\mu_{K\pi'}^2) = -\frac{1}{3\sqrt{2}} [\delta_a] G_F \cos^2 \theta_C \left[ (P \cdot q + M^2_{\pi}) - \frac{(P \cdot q - M^2_{\pi}) (M^2_K - M^2_{\pi})}{M^2_D + M^2_{\pi} - 2P \cdot q} \right]$$

$$\times \left[ \frac{2}{3} \bar{\Omega}_{1/2} T_{1/2}(\mu_{K\pi'}^2) + \frac{1}{3} \bar{\Omega}_{3/2} T_{3/2}(\mu_{K\pi'}^2) \right] ;$$ (44)
diagrams (c+d):

\[ \mathcal{A}_{c+d}(\mu_{K\pi}^2) = -\frac{1}{3\sqrt{2}} [\delta_a] G_F \cos^2 \theta_C \left[ (P \cdot q' + M_{\pi}^2) - \frac{(P \cdot q' - M_{\pi}^2) (M_{K}^2 - M_{\pi}^2)}{M_D^2 + M_{\pi}^2 - 2 P \cdot q'} \right] \times \bar{\Omega}_{1/2} T_{1/2}(\mu_{K\pi}^2) ; \]  

\hspace{1cm} (45)

diagram (e):

\[ \mathcal{A}_e(\mu_{K\pi}^2) = -\frac{\sqrt{2}}{3} [\delta_b] G_F \cos^2 \theta_C \left[ (M_D^2 - 3 P \cdot q') - \frac{(M_D^2 - P \cdot q') (M_{K}^2 - M_{\pi}^2)}{M_D^2 + M_{\pi}^2 - 2 P \cdot q'} \right] \times \left[ \bar{\Omega}_{1/2} T_{1/2}(\mu_{K\pi}^2) - \bar{\Omega}_{3/2} T_{3/2}(\mu_{K\pi}^2) \right] ; \]  

\hspace{1cm} (46)

diagram (f):

\[ \mathcal{A}_f(\mu_{K\pi}^2) = -4\sqrt{3} [\delta_c] G_F \cos^2 \theta_C \frac{P \cdot q'}{\mu_{K\pi}^2 - m_{\pi}^2} \left[ c_d/F^2 \right] \times \left[ c_d (\mu_{K\pi}^2 - M_{\pi}^2 - M_{K}^2) + c_m (4 M_{K}^2 + 5 M_{\pi}^2)/6 \right] \left[ 1 - \bar{\Omega}_{1/2} T_{1/2}(\mu_{K\pi}^2) \right] . \]  

\hspace{1cm} (47)

The process \( D^+ \to K^- \pi^+ \pi^+ \) is Cabibbo allowed and, as expected, amplitudes share the factor \( G_F \cos^2 \theta_C \). As far as phases are concerned, one finds three kinds of structures. The amplitude \( \mathcal{A}_a \) comes from a tree diagram and is necessarily real. The phases of the amplitudes \( \mathcal{A}_b, \mathcal{A}_{c+d} \) and \( \mathcal{A}_e \), on the other hand, are contained in the products \( \bar{\Omega}_IT_I \) and given by \( (\delta_I + \omega_I) \). Finally, as discussed in \([18]\), the phase of \( \mathcal{A}_f \) is \( \delta_{1/2} \), the same of free scattering.

In the evaluation of Dalitz plots, the amplitude \( \mathcal{A} \) has to be symmetrized with respect to the variables \( \mu_{K\pi}^2 \) and \( \mu_{K'\pi'}^2 \), since outgoing pions are identical. Allowed values for these invariant masses lie in the interval \( 0.40 \text{ GeV}^2 \leq \mu_{K\pi}^2, \mu_{K'\pi'}^2 \leq 2.99 \text{ GeV}^2 \), whereas the diagonal of Dalitz plot corresponds to \( 0.94 \text{ GeV}^2 \leq \mu_{K\pi}^2 = \mu_{K'\pi'}^2 \leq 1.85 \text{ GeV}^2 \). As discussed in section V, we assume our results to be valid for \( s \leq 1.6 \text{ GeV}^2 \) in the two-body channel.

Isospin 3/2 contributions are present in \( \mathcal{A}_b \) and \( \mathcal{A}_e \) only. Their dependence on isospin is displayed in figs.13, where it is possible to see that, for \( I = 3/2 \), just the real part is relevant.

Real and imaginary components of the amplitudes \( \mathcal{A}_i \), as functions of \( \mu_{K\pi}^2 \), are shown in fig.14, for the choice \( m_{\pi} = 1.2 \text{ GeV} \) and \( \delta_a = \delta_b = \delta_c = 1 \). As the amplitude \( \mathcal{A}_a \) depends on \( \mu_{\pi\pi}^2 \), we rewrote it as

\[ \mathcal{A}_a(\mu_{\pi\pi}^2) = [\bar{\mathcal{A}}_a(\mu_{K\pi}^2) + \bar{\mathcal{A}}_a(\mu_{K'\pi'}^2)]/2 , \]

\[ \bar{\mathcal{A}}_a(x) = \frac{1}{3\sqrt{2}} [\delta_a] G_F \cos^2 \theta_C \left( M_D^2 + 2M_{\pi}^2 + M_{K}^2 - x \right) \]  

\hspace{1cm} (48)
and just $\mathcal{A}_a(\mu_{K\pi})$ was included in the figure. All contributions have comparable magnitudes in this range, with a dominance of diagrams (a) and (f) in fig.2, which represent the non-resonant background and the direct production of the resonance at the weak vertex. We note, however, that the latter is rather sensitive to the resonance coupling constants $c_d$ and $c_m$ and recall that the values adopted here are just illustrative.

VIII. DALITZ PLOTS

In order to produce a feeling for the $\mathcal{A}_i$ given by eqs.(43-47), we display here their predictions for Dalitz plots. The plotted quantity is $|\mathcal{A}_i(\mu_{K\pi})+\mathcal{A}_i(\mu_{K\pi}^\prime)|^2$ and individual contributions correspond to diagrams in fig.2. Our description for the amplitudes $\mathcal{A}_i$ is valid for invariant masses below 1.6 GeV$^2$. As, in the plots, this condition must hold simultaneously for both amplitudes, a reliable region around the lower side of the diagonal is selected.

Diagram (2.a) gives rise to fig.15. It describes the non-resonating background and one learns that it is not evenly distributed along the plot, as sometimes assumed in the literature. We show, in fig.16, individual contributions from diagrams (2.b-f), which involve final state interactions. Amplitudes $\mathcal{A}_b$ and $\mathcal{A}_{c+d}$ share same weak vertices, but the latter is based on just the isospin $1/2$ kernel, whereas the former contains an admixture of isospins. However, their Dalitz plots are very similar, indicating that isospin $3/2$ contributions are small. The plot corresponding to $\mathcal{A}_e$ is very different from the other ones because its weak vertices
FIG. 14: (Color online) Full (black) and partial contributions to the real (continuous line) and imaginary (dashed line) components of the amplitudes $A_i$; the vertical scale has to be multiplied by the weak factor $G_F \cos^2 \theta_C$.

are of the vector type and the $W$ is coupled to two mesons. The direct production of the resonance at the weak vertex is associated with $A_f$ and produces a plot with a rather broad peak around $1.3 \text{GeV}^2$. Finally, the sum $[A_a + \cdots + A_f]$ is given in fig.17, where a typical interference pattern can be noted.

IX. SUMMARY AND CONCLUSIONS

The low-energy components of the amplitude $D^+ \to K^-\pi^+\pi^+$ are studied in the framework of a rather conservative $SU(3) \times SU(3)$ chiral effective theory and special attention is paid to the resonance $\kappa$. For practical reasons, the derivation of weak vertices is performed using the group $SU(4)$, but without any commitment with the corresponding symmetry. In dealing with final state interactions, proper three-meson processes are neglected and we remain within the quasi-two body approximation. Our main results are summarized by eqs.(43-47), which represent individual contributions from the diagrams in fig.2. At low-
energies, the decay amplitude is represented by $\mathcal{A} = [A_a + \cdots + A_f]$ and symmetrization with respect to final pions is required. Conclusions are presented in the sequence.

1. **degrees of freedom:** The amplitudes $A_i$ contain both fixed and adjustable parameters. The former class encompasses pseudoscalar masses $M_i$, their decay constant $F = F_\pi = 0.093$ GeV and the weak constants $G_F = 1.166 \times 10^{-5}$ GeV$^{-2}$ and $\cos \theta_C = 0.9745$ [12]. Adjustable parameters involve resonance masses and coupling constants. In principle, three scalar states should be considered, but two of them appear just in the $t$-channel background. As discussed in appendix C, contributions from $t$- and $u$-channel interactions are very small and can be safely neglected. In this approximation, the only free parameters in our results are $m_\kappa$, the $\kappa\pi K$ coupling constants $c_d$ and $c_m$, and three $SU(4)$ breaking factors $[\delta_i]$.

2. **background:** The non-resonating background is represented by $A_a$, which is a *real* function. It important to note that this does not happen by chance. The process shown in fig.2a is the simplest possible in $D^+$ decays and given by a tree-level diagram. In field theory, tree diagrams are real and imaginary components are produced by loops. Of course, it would
be possible to dress the primary weak vertex with mesonic loops, but this would amount to including higher order chiral corrections. Therefore, at low-energies, the background is necessarily real and should not be represented by trial functions of the form $[a_0 e^{i\phi_0} A_0]$. Inspecting figs. 14 and 15, one learns that the magnitude of $A_\alpha$ is comparable to other contributions and that its distribution over the Dalitz plot is not uniform, as sometimes assumed.

**3. phases:** Our calculation begins with a lagrangian, which yields real vertices only. Loops are introduced in a subsequent step, and only then amplitudes become complex. This construction process is systematic and one has full control over all imaginary terms and

![Graphs showing contributions from processes involving FSIs from various diagrams: (b) top-left; (c+d) top-right; (e) bottom-left; (f) bottom-right.](image-url)
understand clearly their dynamical origins. Complex amplitudes are due to final state interactions and encoded into the functions $\Omega_I$ and $T_I$, representing respectively two-meson propagators and elastic $K\pi$ scattering. As one deals with two isospin channels, in principle, four phases need to be considered. However, the $I = 3/2$ channel is repulsive and the corresponding phases are small. One is then left with just $\omega_{1/2}$ and $\delta_{1/2}$. Both of them are present in the phases of $A_b$, $A_{c+d}$ and $A_e$, which are identical and given by $(\delta_{1/2} + \omega_{1/2})$. Diagram 2f represents the direct production of the $\kappa$-resonance at the weak vertex and involves the subamplitude $\Pi_{1/2}$, represented in fig.11. As pointed out in ref.[18], this structure gives rise to the same phase as in $T_{1/2}$, namely $\delta_{1/2}$. Finally, one notes that another phase is produced when the complex $I = 1/2$ amplitudes are added to the almost real $I = 3/2$ counterparts. In summary, the low-energy decay amplitude contains several different energy-dependent phases and cannot be well represented by a trial function such as $[a_1^S e^{i\phi_1^S} A_1^S]$. 

4. two-body isospin channels: Our numerical results indicate that final state interactions are very important in $D^+ \rightarrow K^-\pi^+\pi^+$. They are present in five of the diagrams shown in fig.2. Three of them involve just the isospin 1/2 channel, whereas the other two depend
on both components. These cases were studied in fig.13 and one notes that isospin $3/2$ is relatively important in $A_e$ only.

5. **$K$-matrix approximation:** This problem is addressed in section V. The main advantage of the $K$-matrix is its simplicity. Our expressions encompass the $K$-matrix approximation, since it amounts to neglecting the real part of the two-loop propagator and setting $\omega_I = \pi/2$. In the isospin $3/2$ channel, this procedure is disastrous. For $I = 1/2$, on the other hand, it gives rise to reasonable qualitative predictions. One should bear in mind, however, that the phase $\omega_{1/2}$ influences final result in two different ways, since it both helps shaping the $K\pi$ amplitude and enters directly into the expressions for $A_b, \cdots, A_f$.

6. **dynamics:** We assume our results to be reliable for $K\pi$ invariant masses between 0.4 GeV$^2$ and 1.6 GeV$^2$. The magnitudes of the $A_i$ are comparable in this range, as shown in fig.14. There, it is also possible to see that the non-resonant background and the direct resonance production dominate. However, numerical results are particularly sensitive to the coupling constant $c_d$ and, for the time being, we take this conclusion as provisional.

7. **Breit-Wigner expressions:** Our results involve functions which are akin to the usual Breit-Wigner ones. However, as we discuss in the sequence, differences between them are very important. These functions are hidden in the two-body amplitudes $T_{1/2}$, present in diagrams (2 b, ..., e), and also contribute to the $\Pi_{1/2}$ in (f). Eq.(35) can be rewritten as

$$T_{1/2} = \frac{\gamma^2}{[(m^2 - s) + \gamma^2 R_{1/2}] - i [\gamma^2 \rho/(16\pi)]},$$

where $R_{1/2}$ represents off-shell effects in the two-meson propagator. The function $\gamma^2$ is given by eq.(27) and can be rewritten as

$$\gamma^2 = 3 h^2/F^2 + [\alpha \Lambda + c_d \beta] (m^2_s - s),$$

$$h = [c_d (m^2 - M^2 - M^2_K) + c_m (4 M^2_K + 5 M^2_\pi)/6],$$

$$\Lambda = (1/4F^2) [(4 - 3 \rho^2/2) s - 4 (M^2_\pi + M^2_K)],$$

$$\beta = [3 c_d (m^2_s - s) - 6 h]/F^3.$$

In this expression, $h^2$ is a $O(q^4)$ effective coupling constant, $\Lambda$ is the $O(q^2)$ leading term in the chiral amplitude and $\beta$ is a $O(q^4)$ background. A parameter $\alpha$ has been introduced, so that the leading chiral contribution could be turned on or off. The usual Breit-Wigner
expression can be recovered from eq.(49), by going to the \( K \)-matrix approximation \( (\bar{R}_{1/2} \to 0) \) and by choosing \((\alpha = 0, c_d = 0)\). This yields the curve \( BW \) in fig.18, with its well known shape. The reintroduction of \( \bar{R}_{1/2} \) produces no visible effects. The choices \((\alpha = 1, c_d = 0)\) and \((\alpha = 0, c_d = 0.032 \text{ GeV})\) give rise respectively to curves \( \alpha \) and \( \beta \). In both cases, one notes huge enhancements in the region \( 0.8 \text{ GeV}^2 < s < m_\kappa^2 \). At threshold, on the other hand, the chiral hierarchy is respected, since \( \alpha \to O(q^2) \) and \( \beta \to O(q^4) \). Finally, the curve \( \chi BW \) is produced by \((\alpha = 1, c_d = 0.032 \text{ GeV})\) and is directly related with those discussed in section V.

![Diagram](image-url)

**FIG. 18**: (Color online) Relationship between a usual Breit-Wigner (\( BW \), dashed line) and its chiral generalization (\( \chi BW \), continuous line); the other curves correspond to the choices \( \alpha : (\alpha = 1, c_d = 0) \) and \( \beta : (\alpha = 0, c_d = 0.032 \text{ GeV}) \) in eq.(50).

8. **chiral symmetry**: The phase shifts predicted by eq.(49) pass through \( \pi/2 \) at \( s = m_\kappa^2 \). This feature defines the *nominal* mass of the resonance and is completely independent of parameters adopted. On the other hand, the implementation of chiral theorems requires that amplitudes be represented by polynomials which have well known values at threshold and grow close by. The form of the curve \( \chi BW \) in fig.18 corresponds to a compromise between those two features. It is important to note that this kind of behavior cannot be obtained
by adding polynomials to usual Breit-Wigner expressions. Rather, it derives directly from the unitarization of contact interactions added to resonance poles, as discussed in sections IV and V. A study of pole movements induced by this procedure is in progress and will be presented elsewhere.

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APPENDIX A: KINEMATICS

Initial and final momenta are uniformly represented by capital and low-case letters.

**two-body system:** In the description of the two-body reaction $\pi(Q)K(K) \rightarrow \pi(q)K(k)$, Mandelstam variables are defined by $s = (Q+K)^2 = (q+k)^2$, $t = (Q-q)^2 = (K-k)^2$, $u = (Q-k)^2 = (K-q)^2$ and satisfy the condition $s+t+u = 2(M_K^2+M_\pi^2)$. In the center of mass, one has

$$t = -(s \rho^2/2) (1-\cos \theta), \quad (A1)$$
$$u = (M_K^2-M_\pi^2)^2/s - (s \rho^2/2) (1+\cos \theta), \quad (A2)$$
$$\rho = \sqrt{|1 - 2 (M_K^2+M_\pi^2)/s + (M_K^2-M_\pi^2)^2/s^2|}. \quad (A3)$$

In performing $S$-wave projections, one uses

$$T_0 = (1/2) \int_0^\pi d \cos \theta \ T \quad (A4)$$

and finds

$$t \rightarrow t_0 = -s \rho^2/2, \quad u \rightarrow u_0 = (M_K^2-M_\pi^2)^2/s - (s \rho^2/2), \quad (A5)$$
$$\left[1/(x-m^2)\right]_0 = -(1/s \rho^2) \ln[1+s \rho^2/(m^2-x_0 - s \rho^2/2)], \quad (A6)$$
with \( x = t, u \).

**three-body system:** In the \( D^+ \) decay, momentum variables are defined as \( D^+(P) \to \pi^+(q) \pi^+(q') K^-(k) \). The \( W \) propagator splits diagrams into two parts and the momentum \((q)\) is associated with the pion in the same sector as the \( D^+ \). Invariant masses are described by

\[
\mu_{\pi\pi}^2 = (q+q')^2, \quad \mu_{K\pi}^2 = (q+k)^2, \quad \mu_{K\pi'}^2 = (q'+k)^2, \quad (A7)
\]

and one has \( \mu_{\pi\pi}^2 + \mu_{K\pi}^2 + \mu_{K\pi'}^2 = M_D^2 + 2M_\pi^2 + M_K^2 \). Scalar products are then given by \( P \cdot q = (M_D^2 + M_\pi^2 - \mu_{K\pi'}^2)/2 \) and \( P \cdot q' = (M_D^2 + M_\pi^2 - \mu_{K\pi}^2)/2 \).

**APPENDIX B: TWO-MESON PROPAGATOR**

\[ \text{FIG. 19: (Color online) Two-meson propagator.} \]

The two-meson propagator is shown in fig. 19, for a system with total momentum \( X = Q + K = q + k \). We also use the combination \( \ell = (Q-K)/2 \) and define

\[
[I_{\pi K}; I^\mu_{\pi K}] = \int \frac{d^4\ell}{(2\pi)^2} \frac{[1; \ell^\mu]}{[((\ell+X/2)^2-M_\pi^2)[((\ell-X/2)^2-M_K^2]]}. \quad (B1)
\]

In terms of Feynman parameters, these integrals read

\[
[I_{\pi K}; I^\mu_{\pi K}] = \frac{i}{(4\pi)^2} \left[ \Pi^{00}_{\pi K} X^\mu \frac{\Pi^{10}_{\pi K}}{2} (\Pi^{10}_{\pi K} - \Pi^{01}_{\pi K}) \right],
\]

\[
\Pi^{mn}_{\pi K} = -\int_0^1 da \left[ -(1-a)^m \right] \left[ -a^n \right] \ln[D_{\pi K}/\Lambda^2] + \cdots,
\]

\[
D_{\pi K} = (1-a)M_\pi^2 + aM_K^2 - a(1-a)X^2. \quad (B2)
\]

where the ellipsis indicate an infinite quantity associated with dimensional regularization. Multiplying \( I^\mu_{\pi K} \) by \( X^\mu \) in eq. (B1) and manipulating the integrand, one finds the following useful result

\[
(\Pi^{10}_{\pi K} - \Pi^{01}_{\pi K}) = \frac{M_\pi^2-M_K^2}{X^2}\Pi^{00}_{\pi K} + \cdots, \quad (B3)
\]

The two-meson propagator given by eq. (30) is written as \( \Omega = -(1/16\pi^2)[L(s) + \Lambda_{\infty}] \), where the function \( L(s) \) is explicitly given below and \( \Lambda_{\infty} \) is an infinite constant that has to
be removed by renormalization. In this procedure, the function $\Omega$ is replaced by

$$\bar{\Omega}_I = -(1/16\pi^2) [L(s) + c_I] ,$$  \hfill (B4)

where the $c_I$ are constants which depend on the isospin channel. They are chosen by tuning the predicted phase shifts $\delta_I(s)$, eq.(33), to experimental results at a given point $s = s_I$ and one imposes $\delta_I(s_I) \equiv \delta_I^{\exp}(s_I)$. When a resonance is present, a rather convenient choice[18] for $s_I$ is the point at which the experimental phase is $\pi/2$.

The function $L(s)$ entering eq.(B4) is given by

- for $s < (M_K - M_\pi)^2$ :
  $$L(s) = -\rho \log \left[ \frac{\sigma - 1}{\sigma + 1} \right] - \eta ,$$  \hfill (B5)

- for $(M_K - M_\pi)^2 < s < (M_K + M_\pi)^2$ :
  $$L(s) = \rho \left[ \tan^{-1} \sigma - \pi/2 \right] - \eta ,$$  \hfill (B6)

- for $s > (M_K + M_\pi)^2$ :
  $$L(s) = \rho(s) \log \left[ \frac{1 - \sigma}{1 + \sigma} \right] - \eta + i\pi \rho ,$$  \hfill (B7)

$$\sigma = \sqrt{|s-(M_K+M_\pi)^2|/|s-(M_K-M_\pi)^2|} ,$$  \hfill (B8)

$$\eta = 2 - [(M_K^2 - M_\pi^2)/s] \log(M_K/M_\pi) .$$  \hfill (B9)

These results allow the renormalized two-loop propagator to be written as

$$\bar{\Omega}_I = -\frac{1}{16\pi^2} \{ \Re [L(s) - c_I] + i \Im L(s) \}$$

$$\equiv \bar{R}_I(s) + i \theta [s-(M_K+M_\pi)^2] I(s) .$$  \hfill (B10)

The imaginary component is very simple and reads $I(s) = -\rho/(16\pi)$. We define a loop phase $\omega_I$ by

$$\tan \omega_I \equiv I/\bar{R}_I .$$  \hfill (B11)

For the channel $I = 1/2$, the constant $c_{1/2}$ is chosen so that

$$\bar{R}_{1/2}(s) = -\frac{1}{16\pi^2} \Re [L(s) - L(m_K^2)]$$  \hfill (B12)

and, by construction, $\bar{R}_{1/2}(m_K^2) = 0$. In the $I = 3/2$ channel, fit to data requires $c_{3/2} \sim 140$ GeV. The functions $\bar{R}_I$, $I$ and $\omega_I$ are shown in fig.20.
FIG. 20: (Color online) Left: real (continuous line) and imaginary (dashed line) components of the two-meson propagator; right: propagator phases.

APPENDIX C: BACKGROUND AMPLITUDES

The resonance-exchange amplitudes calculated in section IV have the general form

\[ T^a_x = -\frac{4}{F^4} \left\{ \frac{1}{x-m^2_a} \left[ (c_d m_a^2 + C^a_{\pi}) (c_d m_a^2 + C^a_K) \right] + c_d^2 (x+m_a^2) + c_d \left( C^a_{\pi} + C^a_K \right) \right\} , \]  

(C1)

where \( x = t, u, \) and

\[ C^x_{\pi} = C^x_K = -c_d (M_{\pi}^2 + M_K^2) + c_m (4 M_K^2 + 5 M_{\pi}^2) / 6 , \]  

(C2)

\[ C^0_{\pi} = -2 (\tilde{c}_d - \tilde{c}_m) M_{\pi}^2 , \quad C^0_K = -2 (\tilde{c}_d - \tilde{c}_m) M_K^2 , \]  

(C3)

\[ C^8_{\pi} = -2 c_d M_{\pi}^2 - c_m (2 M_K^2 - 11 M_{\pi}^2) / 6 , \]  

(C4)

\[ C^8_K = -2 c_d M_K^2 + c_m (10 M_K^2 - M_{\pi}^2) / 6 . \]  

(C5)

Using the results for \( S\)-wave projection given in appendix A, the background amplitudes entering the kernels of section IV are written as

\[ B_{1/2} = 3 B_8^s / 4 + B_t^0 - B_t^8 / 6 - B_u^s / 4 , \]  

(C6)

\[ B_{3/2} = B_t^0 - B_t^8 / 6 + B_u^s / 2 , \]  

(C7)

\[ B_8^s = -\frac{4}{F^4} \left\{ c_d^2 (s+m_{\pi}^2) + c_d \left( C^s_{\pi} + C^s_K \right) \right\} , \]  

(C8)

\[ B_8^a = -\frac{4}{F^4} \left\{ -\frac{1}{s \rho^2} \ln \left[ 1 + \frac{s \rho^2}{m_a^2 - x_0 - s \rho^2 / 2} \right] \left[ (c_d m_a^2 + C^a_{\pi}) (c_d m_a^2 + C^a_K) \right] \right. \]  

\[ + c_d^2 (x_0+m_a^2) + c_d \left( C^a_{\pi} + C^a_K \right) \} . \]  

(C9)
FIG. 21: Background amplitudes.

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