Stein’s lemma, Malliavin calculus, and tail bounds, with application to polymer fluctuation exponent

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Abstract

We consider a random variable $X$ satisfying almost-sure conditions involving $G := \langle DX, -DL^{-1}X \rangle$ where $DX$ is $X$’s Malliavin derivative and $L^{-1}$ is the inverse Ornstein-Uhlenbeck operator. A lower-(resp. upper-) bound condition on $G$ is proved to imply a Gaussian-type lower (resp. upper) bound on the tail $P[X > z]$. Bounds of other natures are also given. A key ingredient is the use of Stein’s lemma, including the explicit form of the solution of Stein’s equation relative to the function $1_{x>z}$, and its relation to $G$. Another set of comparable results is established, without the use of Stein’s lemma, using instead a formula for the density of a random variable based on $G$, recently devised by the author and Ivan Nourdin. As an application, via a Mehler-type formula for $G$, we show that the Brownian polymer in a Gaussian environment which is white-noise in time and positively correlated in space has deviations of Gaussian type and a fluctuation exponent $\chi = 1/2$. We also show this exponent remains $1/2$ after a non-linear transformation of the polymer’s Hamiltonian.

Key words and phrases: Malliavin calculus, Wiener chaos, sub-Gaussian, Stein’s lemma, polymer, Anderson model, random media, fluctuation exponent.

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1 Introduction

1.1 Background and context

Ivan Nourdin and Giovanni Peccati have recently made a long-awaited connection between Stein’s lemma and the Malliavin calculus: see \cite{9}, and also \cite{10}. Our article uses crucial basic elements from their work, to investigate the behavior of square-integrable random variables whose Wiener chaos expansions are not finite. Specifically we devise conditions under which the tail of a random variable is bounded below by Gaussian tails, by using Stein’s lemma and the Malliavin calculus. Our article also derives similar lower bounds by way of a new formula for the density of a random variable, established in \cite{12}, which uses Malliavin calculus, but not Stein’s lemma. Tail upper bounds are also derived, using both methods.

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Stein’s lemma has been used in the past for Gaussian upper bounds, e.g. in [4] in the context of exchangeable pairs. Malliavin derivatives have been invoked for similar upper bounds in [21]. In the current paper, the combination of these two tools yields a novel criterion for a Gaussian tail lower bound. We borrow a main idea from Nourdin and Peccati [9], and also from [12]: to understand a random variable $Z$ which is measurable with respect to a Gaussian field $W$, it is fruitful to consider the random variable

$$G := \langle DZ, -DL^{-1}Z \rangle,$$

where $D$ is the Malliavin derivative relative to $W$, $\langle \cdot, \cdot \rangle$ is the inner product in the canonical Hilbert space $H$ of $W$, and $L$ is the Ornstein-Uhlenbeck operator. Details on $D$, $H$, $L$, and $G$, will be given below.

The function $g(z) = E[G|Z = z]$ has already been used to good effect in the density formula discovered in [12]; this formula implied new lower bounds on the densities of some Gaussian processes’ suprema. The article [11], in preparation, contains some very sharp Gaussian supremum density formulas, also based on $g$. These results are made possible by fully using the Gaussian property, and in particular by exploiting both upper and lower bounds on the process’s covariance. The authors of [12] noted that, if $Z$ has a density and an upper bound is assumed on $G$, in the absence of any other assumption on how $Z$ is related to the underlying Gaussian process $W$, then $Z$’s tail is sub-Gaussian. On the other hand, the authors of [12] tried to discard any upper bound assumption, and assume instead that $G$ was bounded below, to see if they could derive a Gaussian lower bound on $Z$’s tail; they succeeded in this task, but only partially, as they had to impose some additional conditions on $Z$’s function $g$, which are of upper-bound type, and which may not be easy to verify in practice.

The techniques used in [12] are well adapted to studying densities of random variables under simultaneous lower and upper bound assumptions, but less so under single-sided assumptions. The point of the current paper is to show that, while the quantitative study of densities via the Malliavin calculus seems to require two-sided assumptions as in [11] and [12], single-sided assumptions on $G$ are in essence sufficient to obtain single sided bounds on tails of random variables, and there are two strategies to this end: Nourdin and Peccati’s connection between Malliavin calculus and Stein’s lemma, and exploiting the Malliavin-calculus-based density formula in [12].

The key new component in our work, relative to the first strategy, may be characterized by saying that, in addition to a systematic exploitation of the Stein-lemma–Malliavin-calculus connection (via Lemma 3.4 below), we carefully analyze the behavior of solutions of the so-called Stein equation, and use them profitably, rather than simply use the fact that there exist bounded solutions with bounded derivatives. The novelty in our second strategy is simply to note that the difficulties inherent to using the density formula of [12] with only one-sided assumptions, tend to disappear when one passes to tail formulas.

Our work follows in the footsteps of Nourdin and Peccati’s. One major difference between our work and their’s, and indeed between ours and the main use of Stein’s method since its inception in [18] to the most recent results (see [3], [5], [16], and references therein) is that Stein’s method is typically concerned with convergence to the normal distribution while we are only interested in rough bounds of Gaussian or other types for single random variables (not sequences), without imposing conditions which would lead to normal or any other convergence. As an exception to this statement, [9] implies that a bound on the variance of a single $G$ has clear implications for the distance from $Z$’s distribution to the normal law [see for instance Remark 3.6 therein]; Nourdin and Peccati in [9] did not make systematic use of this result, because their motivations did not require it.

One other difference between our work and theirs is that we do not consider the case of a single Wiener chaos. This last specificity of our work (see however Remark 3.8 in [9]), that we systematically consider random variables with infinitely many non-zero Wiener chaos components, comes from the application which we also consider in this article, to the so-called fluctuation exponent $\chi$ of a polymer in a random environment. Details on this application, where we show that $\chi = 1/2$ for a certain class of environments, are in Section 5.
There is a more fundamental obstacle to seeking upper or lower Gaussian tail bounds on an r.v. in a single Wiener chaos: unlike convergence results for sequences of r.v.’s, such as [15], a single qth chaos r.v. has a tail of order \(\exp\left(-\frac{x}{c^q}\right)\) (see [2]), it never has a Gaussian behavior; our lower-bound results below (e.g. Theorem 1.3 Point 3) does apply to such an r.v., but the result cannot be sharp.

### 1.2 Summary of results

We now describe our main theoretical results. All stochastic analytic concepts used in this introduction are described in Section 2. Let \(W\) be an isonormal Gaussian process relative to a Hilbert space \(H = L^2(T, \mathcal{B}, \mu)\) (for instance if \(W\) is the Wiener process on \([0, 1]\), then \(T = [0, 1]\) and \(\mu\) is the Lebesgue measure). The norm and inner products in \(H\) are denoted by \(|\cdot|\) and \(<\cdot, \cdot>\). Let \(L^2(\Omega)\) be the set of all random variables which are square-integrable and measurable with respect to \(W\). Let \(D\) be the Malliavin derivative with respect to \(W\) (see Paul Malliavin’s or David Nualart’s texts [8], [13]). Thus \(DX\) is a random element in \(L^2(\Omega)\) with values in the Hilbert space \(H\). The set of all \(X \in L^2(\Omega)\) such that \(\|DX\|\in L^2(\Omega)\) is called \(D^{1,2}\).

Let \(\Phi\) be the tail of the standard normal distribution

\[\Phi(u) := \int_u^\infty e^{-x^2/2}dx/\sqrt{2\pi}.\]

The following result, described in [21] as an elementary consequence of a classical stochastic analytic inequality found for instance in Üstünel’s textbook [20, Theorem 9.1.1], makes use of a condition based solely on the Malliavin derivative of a given r.v. to guarantee that its tail is bounded above by a Gaussian tail.

**Proposition 1.1** For any \(X \in D^{1,2}\), if \(\|DX\|\) is bounded almost surely by 1, then \(X\) is a standard sub-Gaussian random variable, in the sense that \(P[|X - E[X]| > u] \leq 2e^{-u^2/2}\).

**Remark 1.2** The value 1 in this proposition, and indeed in many places in this paper, has the role of a dispersion coefficient. Since the Malliavin derivative \(D\) is linear, the above proposition implies that for any \(X \in D^{1,2}\) such that \(\|DX\| \leq \sigma\) almost surely, then \(P[|X - E[X]| > u] \leq 2e^{-u^2/(2\sigma^2)}\). This trivial normalization argument can be used throughout this paper, because our hypotheses are always based on linear operators such as \(D\). We use this argument in our application in Section 2.

The question of whether a lower bound on \(\|DX\|^2\) gives rise to an inequality in the opposite direction as in the above proposition arises naturally. However, we were unable to find any proof of such a result. Instead, after reading Eulalia Nualart’s article [14] where she finds a class of lower bounds by considering exponential moments on the divergence (Skorohod integral) of a covering vector field of \(X\), we were inspired to look for other Malliavin calculus operations on \(X\) which would yield a Gaussian lower bound on \(X\)’s tail.

We turned to the quantity \(G = \langle DX; -DL^{-1}X\rangle\), identified in [9], and used profitably in [11] and [12]. Here \(L^{-1}\), the inverse of the so-called Ornstein-Uhlenbeck operator, is defined in Section 2. This article’s first theoretical result is that a lower (resp. upper) bound on \(G\) can yield a lower (resp. upper) bound similar to the upper bound in Proposition 1.1. For instance, summarizing the combination of some consequences of our results and Proposition 1.1 we have the following.

**Theorem 1.3** Let \(X\) be a random variable in \(D^{1,2}\). Let \(G := \langle DX; -DL^{-1}X\rangle\).

1. If \(G \geq 1\) almost surely, then

\[Var[X] \geq K_u := \frac{1}{\pi^2} \left(2\sqrt{1 + 2\sqrt{2\pi} - 1}\right)^2 \simeq 0.21367.\]
2. If $G \geq 1$ almost surely, and if for some $c > 2$, $E[X^c] < \infty$, then
\[
\limsup_{z \to \infty} \frac{P[X - E[X] > z]}{\Phi(z)} \geq \frac{c - 2}{c}.
\] (1)

3. If $G \geq 1$ almost surely, and if there exist $c' < 1$ and $z_0 > 0$, such that and $G \leq c'X^2$ almost surely when $X \geq z_0$, then for $z > z_0$,
\[
P[X - E[X] > z] \geq \frac{1}{2c' + 1} \Phi(z).
\]

4. If $G \leq 1$ almost surely, then for every $z > 0$
\[
P[X - E[X] > z] \leq \left(1 + \frac{1}{z^2}\right) \Phi(z).
\] (2)

5. If $\|DX\|^2 \leq 1$ almost surely, then $Var[X] \leq (\pi/2)^2$ and for $z > 0$,
\[
P[X - E[X] > z] \leq e^{-z^2/2}.
\] (3)

**Remark 1.4** Point 1 in this theorem is Corollary 1.4 Point 1. Point 2 here comes from Corollary 1.4 Point 3. Point 3 here follows from Corollary 1.4 Point 1. Inequality (3) in Point 5 here is equivalent to Proposition 1.1. The variance upper bound in Point 5 here follows from [22, Theorem 9.2.3 part (iii)]. Other, non-Gaussian comparisons are also obtained in this article: see Corollary 3.5.

The results in Theorem 1.3 point to basic properties of the Malliavin derivative and Ornstein-Uhlenbeck operator when investigating tail behavior of random variables. The importance of the relation of $G$ to the value 1 was already noticed in [9, Theorem 3.1] where its $L^2$-convergence to 1 for a sequence of r.v.’s was a basic building block for convergence to the standard normal distribution. Here we show what can still be asserted when the condition is significantly relaxed. An attempt was made to prove a version of the theorem above in [12, Section 4]; here we significantly improve that work by: (i) removing the unwieldy upper bound conditions made in [12, Theorem 4.2] to prove lower bound results therein; and (ii) improving the upper bound in [12, Theorem 4.1] while using a weaker hypothesis.

Our results should have applications in any area of pure or applied probability where Malliavin derivatives are readily expressed. In fact, Nourdin and Peccati [9, Remark 1.4, point 4] already hint that $G$ is not always as intractable as one may fear. We present such an application in this article, in which the deviations of random polymer models in some random media are estimated, and its fluctuation exponent is calculated to be $\chi = 1/2$, a result which we prove to be robust to non-linear changes in the polymer’s Hamiltonian.

The structure of this article is as follows. Section 2 presents all necessary background information from the theory of Wiener chaos and the Malliavin calculus needed to understand our statements and proofs. Section 3 recalls Stein’s lemma and equation, presents the way it will be used in this article, and recalls the density representation results from [12]. Section 4 states and proves our main lower and upper bound results. Section 5 gives a construction of continuous random polymers in Gaussian environments, and states and proves the estimates on its deviations and its fluctuation exponent under Gaussian and non-Gaussian Hamiltonians, when the Gaussian environment has infinite-range correlations. Several interesting open questions are described in this section as well. Section 6, the Appendix, contains the proofs of some lemmas.

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2 Preliminaries: Wiener chaos and Malliavin calculus

For a complete treatment of this topic, we refer the reader to David Nualart’s textbook [13]. We use an abstract Wiener space given by an isonormal Gaussian process \( W \): it is defined as a Gaussian field \( W = L^2(T, \mathcal{B}, \mu) \) where \( \mu \) is a \( \sigma \)-finite measure that is either discrete or without atoms, and the covariance of \( W \) coincides with the inner product in \( H \). This forces \( W \) to be linear on \( H \); consequently, it can be interpreted as an abstract Wiener integral. For instance, if \( T = [0, 1] \) and \( \mu \) is the Lebesgue measure, then \( W(f) \) represents the usual Wiener stochastic integral \( \int_0^1 f(s) \, dW(s) \) of a square-integrable non-random function \( f \) with respect to a Wiener process also denoted by \( W \); i.e. we confuse the notation \( W(t) \) and \( W(1_{[0,t]}) \). In general for \( \{ f_i : i = 1, \cdots, n \} \in H^n \), \( (W(f_i) : i = 1, \cdots, n) \) is a centered Gaussian vector, with covariance matrix given by \( \sigma_{i,j}^2 = \langle f_i; f_j \rangle \). The set \( \mathcal{H}_1 \) of all Wiener integrals \( W(f) \) when \( f \) ranges over all of \( H \) is called the first Wiener chaos of \( W \). To construct higher-order chaoses, one may for example use iterated Itô integration in the case of standard Brownian motion, where \( H = L^2([0,1]) \).

If we denote \( I_0(f) = f \) for any non-random constant \( f \), then for any integer \( n \geq 1 \) and any symmetric function \( f \in H^n \), we let

\[
I_n(f) := n! \int_0^1 \int_0^{s_1} \cdots \int_0^{s_{n-1}} f(s_1, s_2, \cdots, s_n) \, dW(s_n) \cdots dW(s_2) \, dW(s_1).
\]

This is the \( n \)-th iterated Wiener integral of \( f \) w.r.t. \( W \).

**Definition 2.1** The set \( \mathcal{H}_n := \{ I_n(f) : f \in H^n \} \) is the \( n \)-th Wiener chaos of \( W \).

We refer to [13, Section 1.2] for the general definition of \( I_n \) and \( \mathcal{H}_n \) when \( W \) is a more general isonormal Gaussian process.

**Proposition 2.2** \( L^2(\Omega) \) is the direct sum - with respect to the inner product defined by expectations of products of r.v.’s - of all the Wiener chaoses. Specifically for any \( X \in L^2(\Omega) \), there exists a sequence of non-random symmetric functions \( f_n \in H^n \) with \( \sum_{n=0}^{\infty} \| f_n \|_{H^n}^2 < \infty \) such that \( X = \sum_{n=0}^{\infty} I_n(f_n) \). Moreover \( \mathbb{E}[X] = f_0 = I_0(f_0) \) and \( \mathbb{E}[I_n(f_n)] = 0 \) for all \( n \geq 1 \), and \( \mathbb{E}[I_n(f_n) I_m(g_m)] = \delta_{m,n} n! \langle f_n, g_n \rangle_{H^n} \), where \( \delta_{m,n} \) equals 0 if \( m \neq n \) and 1 if \( m = n \). In particular \( \mathbb{E}[X^2] = \sum_{n=0}^{\infty} n! \| f_n \|_{H^n}^2 \).

The Malliavin derivative operator is usually constructed via an extension starting from so-called simple random variables which are differentiable functions of finite-dimensional vectors from the Gaussian space \( \mathcal{H}_1 \). The reader can consult Nualart’s textbook [13]. We recall the properties which are of use to us herein.

1. The Malliavin derivative operator \( D \) is defined from \( \mathcal{H}_1 \) into \( H \) by the formula: for all \( r \in T \),

\[
D_r W(f) = f(r).
\]

The Malliavin derivative of a non-random constant is zero. For any \( m \)-dimensional Gaussian vector \( G = (G_i)_{i=1}^m = (I_1(g_i))_{i=1}^m \in (\mathcal{H}_1)^m \), for any \( F \in C^1(\mathbb{R}^m) \) such that \( X = F(G) \in L^2(\Omega) \), we have

\[
D_r X = \sum_{i=1}^m \frac{\partial F}{\partial r_i}(G) g_i(r).
\]

2. The Malliavin derivative of an \( n \)-th Wiener chaos r.v. is particularly simple. Let \( X_n \in \mathcal{H}_n \), i.e. let \( f_n \) be a symmetric function in \( H^n \) and \( X_n = I_n(f_n) \). Then

\[
D_r X = D_r I_n(f_n) = n I_{n-1}(f_n(r, \cdot)).
\] (4)

The Malliavin derivative being linear, this extends immediately to any random variable \( X \in L^2(\Omega) \) by writing \( X \) as its Wiener chaos expansion \( \sum_{n=0}^{\infty} I_n(f_n) \), which means that, using the covariance formulas in Proposition 2.2, \( DX \in L^2(\Omega \times T) \) if and only if

\[
\mathbb{E}\left[\| DX \|^2 \right] := \sum_{n=1}^{\infty} n! \| f_n \|_{H^n}^2 < \infty.
\] (5)
The set of all \( X \in L^2(\Omega) \) such that \( DX \in L^2(\Omega \times T) \) is denoted by \( D^{1,2} \).

**Remark 2.3** The general chain rule of point 1 above generalizes to \( D [h(X)] = h'(X) DX \) for any \( X \in D^{1,2} \) such that \( X \) has a density, and any function \( h \) which is continuous and piecewise differentiable with a bounded derivative. This is an immediate consequence of \([13, \text{Proposition 1.2.3}]\).

In the special case of the standard Wiener space \( (H = L^2[0,1]) \) we have the Clark-Ocone representation formula \([13, \text{Proposition 1.3.5}]\).

**Proposition 2.4** For any \( X \in D^{1,2} \),

\[
X = \mathbb{E}[X] + \int_0^1 \mathbb{E}[D_s X | \mathcal{F}_s] dW(s).
\]

### 3 Tools: using Stein’s lemma and Malliavin derivatives

#### 3.1 Stein’s lemma and equation

The version of Stein’s lemma which we use can be found in \([9]\). Let \( Z \) be a standard normal random variable and \( \Phi(z) = \mathbb{P}[Z > z] \) its tail. Let \( h \) be a measurable function of one real variable. Stein’s equation poses the following question: to find a continuous and piecewise differentiable function \( f \) such that for all \( x \in \mathbb{R} \) where \( f' \) exists,

\[
h(x) - \mathbb{E}[h(Z)] = f'(x) - xf(x).
\]

The precise form of the solution to this differential equation, given in the next lemma, was derived in Stein’s original work \([13]\); a recent usage is found in equalities (1.5), (2.20), and (2.21) in \([10]\).

**Lemma 3.1** Fix \( z \in \mathbb{R} \). Let \( h = 1_{(-\infty,z]} \). Then Stein’s equation (7) has at least one solution \( f \) satisfying \( \|f'\|_\infty := \sup_{x \in \mathbb{R}} |f'(x)| \leq 1 \). One such solution is the following:

- for \( x \leq z \), \( f(x) = \sqrt{2\pi}e^{x^2/2} (1 - \Phi(x)) \Phi(z) \),
- for \( x > z \), \( f(x) = \sqrt{2\pi}e^{x^2/2} (1 - \Phi(z)) \Phi(x) \).

**Corollary 3.2** Let \( X \in L^2(\Omega) \). Setting \( x = X \) in Stein’s equation (7) and taking expectations we get

\[
\mathbb{P}[X > z] = \Phi(z) - \mathbb{E}[f'(X)] + \mathbb{E}[xf(X)].
\]

The next section gives tools which will allow us to combine this corollary with estimates of the random variable \( G = \langle DX; -DL^{-1}X \rangle \) in order to get tail bounds. It will also show that \( G \) can be used, as in \([12]\), to express the density of \( X \) without using Stein’s lemma.

#### 3.2 Malliavin derivative tools

**Definition 3.3** The Ornstein-Uhlenbeck operator \( L \) is defined as follows. Let \( X = \sum_{n=1}^\infty I_n(f_n) \) be a centered r.v. in \( L^2(\Omega) \). If \( \sum_{n=1}^\infty n^2! |f_n|^2 < \infty \), then we define a new random variable \( LX \) in \( L^2(\Omega) \) by \( -LX = \sum_{n=1}^\infty n I_n(f_n) \). The inverse of \( L \) operating on centered r.v.’s in \( L^2(\Omega) \) is defined by the formula \( -L^{-1}X = \sum_{n=1}^\infty 1/n I_n(f_n) \).

The following formula will play an important role in our proofs where we use Stein’s lemma. It was originally noted in \([9]\). We provide a self-contained proof of this result in the Appendix, which does not use the concept of divergence operator (Skorohod integral).
Lemma 3.4 For any centered $X \in \mathbf{D}^{1,2}$ with a density and any deterministic continuous and piecewise differentiable function $h$ such that $h'$ is bounded,

$$
E[Xh(X)] = E[h'(X) \langle DX; -DL^{-1}X \rangle].
$$

(8)

On the other hand, the next result and its proof (see [12]), make no reference to Stein’s lemma. Let the function $g$ be defined almost everywhere by

$$
g(z) := E[\langle DX; -DL^{-1}X \rangle | X = z].
$$

(9)

Proposition 3.5 Let $X \in \mathbf{D}^{1,2}$ be centered with a density $\rho$ which is supported on a set $I$. Then $I$ is an interval $[a, b]$ and, with $g$ as above, we have for almost all $z \in (a, b)$,

$$
\rho(z) = \frac{E[Z]}{2g(z)} \exp \left( -\int_0^z \frac{ydy}{g(y)} \right).
$$

Strictly speaking, the proof of this proposition is not contained in [12], since the authors there use the additional assumption that $g(x) \geq 1$ everywhere, which implies that $\rho$ exists and that $I = \mathbb{R}$. However, the modification of their arguments to yield the proposition above is straightforward, and we omit it: for instance, that $I$ is an interval follows from $X \in \mathbf{D}^{1,2}$ as seen in [13, Proposition 2.1.7].

As one can see from this proposition, and the statement of Theorem 1.3, it is important to have a technique to be able to calculate $DL^{-1}X$. We will use a device which can be found for instance in a different form in the proof of Lemma 1.5.2 in [13], and is at the core of the so-called Mehler formula, also found in [13]. It requires a special operator which introduces a coupling with an independent Wiener space. This operator $R_\theta$ replaces $W$ by the linear combination $W\cos \theta + W'\sin \theta$ where $W'$ is an independent copy of $W$. For instance, if $W$ is Brownian motion and one writes the random variable $X$ as $X = F(W)$ where $F$ is a deterministic Borel-measurable functional on the space of continuous functions, then

$$
R_\theta X := F(W\cos \theta + W'\sin \theta).
$$

(10)

We have the following formula (akin to the Mehler formula, and proved in the Appendix), where $\text{sgn}(\theta) = \theta/|\theta|$, where $E'$ represents the expectation w.r.t. the randomness in $W'$ only, i.e. conditional on $W$, and where $D'$ is the Malliavin derivative w.r.t. $W'$ only.

Lemma 3.6 For any $X \in \mathbf{D}^{1,2}$, for all $s \in T$,

$$
-D_s (L^{-1}X) = \frac{1}{2} \int_{-\pi/2}^{\pi/2} E'[D'_s (R_\theta X)] \text{sgn} (\theta) d\theta.
$$

4 Main results

All results in this section are stated and discussed in the first two subsections, the first one dealing with consequences of Stein’s lemma, the second with the function $g$. All proofs are in the third subsection.

4.1 Results using Stein’s lemma

Our first result is tailored to Gaussian comparisons.

Theorem 4.1 Let $X \in \mathbf{D}^{1,2}$ be centered. Assume that almost surely,

$$
G := \langle DX; -DL^{-1}X \rangle \geq 1.
$$

(11)
Then for every $z > 0$,
\[
P[X > z] \geq \Phi(z) - \frac{1}{1 + z^2} \int_z^\infty 2xP[X > x] \, dx
\]

Assume instead that one has the reverse of inequality (11), then for every $z > 0$,
\[
P[X > z] \leq \left(1 + \frac{1}{z^2}\right)\Phi(z).
\]

Before proving this theorem, we record some consequences of its lower bound result in the next Corollary.

In order to obtain a more precise lower bound result on the tail $S(z) := P[X > z]$, it appears to be necessary to make some regularity and integrability assumptions on $S$. This is the aim of the second point in the next corollary. The first and third points show what can be obtained by using only an integrability condition, with no regularity assumption: we may either find a universal lower bound on such quantities as $X$’s variance (the constant we find there may not be of any special significance), or an asymptotic statement on $S$ itself.

**Corollary 4.2** Let $X \in D^{1,2}$ be centered. Let $S(z) := P[X > z]$. Assume that condition (11) holds.

1. We have
\[
\text{Var}[X] \geq K_u := \frac{1}{\pi^2} \left(2\sqrt{1 + 2\sqrt{2\pi} - 1}\right)^2 \simeq 0.21367.
\]

2. Assume there exists a constant $c > 2$ such that $|S'(z)|/S(z) \leq c/z$ holds for large $z$. Then for large $z$,
\[
P[X > z] \geq \frac{(c - 2)(1 + z^2)}{c - 2 + cz^2} \Phi(z) \simeq \left(\frac{c - 2}{c}\right)\Phi(z).
\]

3. Assume there exists a constant $c > 2$ such that $S(z) < z^{-c}$ holds for large $z$. Then, for large $z$,
\[
\sup_{x \geq z} x^c P[X > x] \geq \frac{c - 2}{c} z^c \Phi(z).
\]

Consequently,
\[
\limsup_{z \to \infty} \frac{P[X > z]}{\Phi(z)} \geq \frac{c - 2}{c}.
\]

Let us discuss the assumptions and results in the corollary from a quantitative standpoint. The assumption of point 2, $|S'(z)|/S(z) \leq c/z$, when integrated, implies no more than existence of a moment of order larger than 2; it does, however, represent an additional monotonicity condition since it refers to $S'$. The assumption of point 3, which is weaker because it does not require any monotonicity, also implies the same moment condition. This moment condition is little more than the integrability required from $X$ belonging to $D^{1,2}$. If $c$ can be made arbitrarily large (for instance in point 3, this occurs when $X$ is assumed to have moments of all orders), asymptotically $(c - 2)/c$ can be replaced by 1, yielding the sharpest possible comparison to the normal tail. If indeed $S$ is close to the normal tail, it is morally not a restriction to assume that $c$ can be taken arbitrarily large: it is typically easy to check this via a priori estimates.

**4.2 Results using the function $g$**

We now present results which do not use Stein’s lemma, but refer only to the random variable $G := \langle DX; -DL^{-1}X \rangle$ and the resulting function $g(z) := E[G | X = z]$ introduced in (9). We will prove the theorem below using the results in [12] on representation of densities. Its corollary shows how to obtain quantitatively explicit upper and lower bounds on the tail of a random variable, which are as sharp as the upper and lower bounds one might establish on $g$. A description of the advantages and disadvantages of using $g$ over Stein’s lemma follows the statements of the next theorem and its corollary.

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Theorem 4.3 Let \( X \in \mathbb{D}^{1,2} \) be centered. Let \( G := \langle DX; -DL^{-1}X \rangle \) and \( g(z) := \mathbb{E}[G|X = z] \). Assume that \( X \) has a density which is positive on the interior of its support \((a, +\infty)\). For \( x \geq 0 \), let

\[
A(x) := \exp \left( -\int_0^x \frac{ydy}{g(y)} \right).
\]

Then for all \( x > 0 \),

\[
\mathbb{P}[X > x] = \frac{\mathbb{E}[|X|]}{2} \left( \frac{A(x)}{x} - \int_x^\infty \frac{A(y)}{y^2}dy \right).
\] (12)

Remark 4.4 The density formula in Proposition (3.5) shows that \( g \) must be non-negative. Assuming our centered \( X \in \mathbb{D}^{1,2} \) has a density \( \rho \), we have already noted that \( \rho \) must be positive on \((a, b)\) and zero outside. To ensure that \( b = +\infty \), as is needed in the above theorem, it is sufficient to assume that \( g \) is bounded below on \([0, b)\) by a positive constant. If in addition we can assume, as in (11), that this lower-boundedness of \( g \) holds everywhere, then \( X \) has a density, and its support is \( R \).

Corollary 4.5 Assume that for some \( c' \in (0, 1) \) and some \( z_0 > 1 \), we have for all \( x > z_0 \), \( g(x) \leq c'x^2 \).

Then, with \( K := \frac{\mathbb{E}[X]}{2} \frac{(c')^x}{(1+c')^{x^2}}, \) for \( x > z_0 \),

\[
\mathbb{P}[X > x] \geq K \frac{A(x)}{x}.
\] (13)

1. Under the additional assumption (11), \( g \geq 1 \) everywhere, and we have

\[
\mathbb{P}[X > z] \geq K^{-\frac{1}{x}} \exp \left( -\frac{x^2}{2} \right) \simeq \sqrt{2\pi}K \Phi(z).
\]

2. If we have rather the stronger lower bound \( g(x) \geq c''x^2 \) for some \( c'' \in (0, c'] \) and all \( x > z_0 \), then for \( x > z_0 \), and with some constant \( K' \) depending on \( g \), \( c'' \) and \( z_0 \),

\[
\mathbb{P}[X > z] \geq K' x^{-1-1/c''}.
\]

3. If we have instead that \( g(x) \geq c_1x^p \) for some \( c_1 > 0 \), \( p < 2 \), and for all \( x > z_0 \), then for \( x > z_0 \), and with some constant \( K'' \) depending on \( g \), \( c_1 \), \( p \), and \( z_0 \),

\[
\mathbb{P}[X > z] \geq K'' \exp \left( -\frac{x^{2-p}}{(2-p)c_1} \right).
\]

4. In the last two points, if the inequalities on \( g \) in the hypotheses are reversed, the conclusions are also reversed, without changing any of the constants.

The tail formula (12) in Theorem [4.3] readily implies asymptotic estimates on \( S \) of non-Gaussian type if one is able to compare \( g \) to a power function. Methods using Stein’s lemma, at least in its form described in Section 3.1, only work efficiently for comparing \( S \) to the Gaussian tail. Arguments found in Nourdin and Peccati’s articles (e.g. [9]) indicate that Stein’s method may be of use in some specific non-Gaussian cases, which one could use to compare tails to the Gamma tail, and perhaps to other tails in the Pearson family, which would correspond to polynomial \( g \) with degree at most 2. The flexibility of our method of working directly with \( g \) rather than Stein’s lemma, is that it seems to allow any type of tail. Stein’s method has one important advantage, however: it is not restricted to having a good control on \( g \); Theorem [4.1] establishes Gaussian lower bounds on tails by only assuming (11) and mild conditions on the tail itself. This is to be
correlated compared to the lower bound \[12\] Theorem 4.2 proved via the function \( g \) alone, where it required growth conditions on \( g \) which may not be that easy to check.

There is one intriguing, albeit perhaps technical, fact regarding the use of Stein’s method: in Point 1 of the above Corollary \[4.3\] since the comparison is made with a Gaussian tail, one may wonder what the usage Stein’s lemma via Theorem \[4.1\] may produce when assuming, as in Point 1 of Corollary \[4.3\] that \( g (x) \geq 1 \) and \( g \) grows slower than \( x^2 \). As it turns out, Stein’s method is not systematically superior to Corollary \[4.6\] as we now see.

**Corollary 4.6 (Consequence of Theorem \[4.1\])** Assume that \( g (x) \geq 1 \) and, for some \( c' < 1 \) and large \( x > z_0 \), \( g(x) \leq c' x^2 \). Then for \( x > z_0 \),

\[
P [X > z] \geq \frac{1 + z^2}{1 + (2c' + 1) z^2} \Phi (z) \simeq \frac{1}{2c' + 1} \Phi (z).
\]

When this corollary and Point 1 in Corollary \[4.5\] are used in an efficient situation, this means that \( \varepsilon \) is presumably “subgaussian” as well as being “supergaussian” as a consequence of assumption \[11\]. For illustrative purposes, we can translate this roughly as meaning that \( g(x) \) is in the interval, say, \([1, 1 + \varepsilon]\) for all \( x \). This implies that we can take \( c' \rightarrow 0 \) in both Corollaries \[4.3\] and \[4.6\] as a consequence, the first corollary yields \( P [X > z] \geq \Phi (z) \), while the second gives \( P [X > z] \geq (\sqrt{2\pi} E |X|/2) \Phi (z) \). The superiority of one method over another then depends on how \( \sqrt{2\pi} E |X|/2 \) compares to 1. It is elementary to check that, in “very sharp” situations, which means that \( \varepsilon \) is quite small, \( \sqrt{2\pi} E |X|/2 \) will be close to 1, from which one can only conclude that both methods appear to be equally efficient.

### 4.3 Proofs

We now turn to the proofs of the above results.

**Proof of Theorem \[4.1\]** Step 1: exploiting the negativity of \( f' \). From lemma \[3.1\] we are able to calculate the derivative of the solution \( f \) to Stein’s equation:

- for \( x \leq z \), \( f'(x) = \Phi(z) \left( 1 + \sqrt{2\pi} (1 - \Phi(x)) xe^{x^2/2} \right) \);
- for \( x > z \), \( f'(x) = \left( 1 - \Phi(z) \right) \left( -1 + \sqrt{2\pi} \Phi(x) xe^{x^2/2} \right) \).

We now use the standard estimate, valid for all \( x > 0 \):

\[
\frac{x}{(x^2 + 1) \sqrt{2\pi} e^{-x^2/2}} \leq \Phi(x) \leq \frac{1}{x \sqrt{2\pi} e^{-x^2/2}}. \tag{14}
\]

In the case \( x > z \), since \( z > 0 \), the upper estimate yields \( f'(x) \leq \left( 1 - \Phi(z) \right) (-1 + 1) = 0 \). Now by the expression for \( P [X > z] \) in Corollary \[3.2\] the negativity of \( f' \) on \( \{ x > z \} \) implies for all \( z > 0 \),

\[
P [X > z] = \Phi(z) - E [1_{X \leq z} f'(X)] - E [1_{X > z} f'(X)] + E [X f(X)]
\]

\[
\geq \Phi(z) - E [1_{X \leq z} f'(X)] + E [X f(X)].
\]

**Step 2: Exploiting the positivities and the smallness of \( f' \).** Using Step 1, we have

\[
P [X > z] \geq \Phi(z) - E [1_{X \leq z} f'(X)] + E [1_{X \leq z} X f(X)] + E [1_{X > z} X f(X)].
\]
We apply Lemma 3.4 to the function $h(x) = (f(x) - f(z))1_{x \leq z}$; $h$ is continuous everywhere; it is differentiable everywhere with a bounded derivative, equal to $f'(x)1_{x \leq z}$, except at $x = z$. Thus we get

$$\mathbb{P} [X > z] \geq \Phi(z) - \mathbb{E} [1_{X \leq z}f'(X)] + \mathbb{E} [Xh(X)] + \mathbb{E} [1_{X \leq z}f(z)] + \mathbb{E} [1_{X > z}Xf(X)]$$

$$\geq \Phi(z) + \mathbb{E} [1_{X \leq z}f'(X)(-1 + \langle DX; -DL^{-1}X \rangle)] + \mathbb{E} [1_{X \leq z}f(z)] + \mathbb{E} [1_{X > z}Xf(X)]. \quad (15)$$

When $x \leq z$, we can use the formula in Step 1 to prove that $f'(x) \geq 0$. Indeed this is trivial when $x \geq 0$, while when $x < 0$, it is proved as follows: for $x = -y < 0$, and using the upper bound in (14)

$$f'(x) = \Phi(z) \left(1 + \sqrt{2\pi} \left(1 - \Phi(x)\right) xe^{x^2/2}\right)$$

$$= \Phi(z) \left(1 - \sqrt{2\pi} \Phi(y)e^{y^2/2}\right) \geq 0.$$ 

By the lower bound hypothesis (11), we also have positivity of $-1 + \langle DX; -DL^{-1}X \rangle$. Thus the second term on the right-hand side of (15) is non-negative. In other words we have

$$\mathbb{P} [X > z] \geq \Phi(z) + \mathbb{E} [1_{X \leq z}f(z)] + \mathbb{E} [1_{X > z}Xf(X)] \quad (16)$$

$$= \Phi(z) + A. \quad (17)$$

The sum of the last two terms on the right-hand side of (16), which we call $A$, can be rewritten as follows, using the fact that $\mathbb{E} [X] = 0$:

$$A := \mathbb{E} [1_{X \leq z}f(z)] + \mathbb{E} [1_{X > z}Xf(X)]$$

$$= \mathbb{E} [1_{X \leq z}f(z)] + \mathbb{E} [1_{X > z}X(f(X) - f(z))] + f(z) \mathbb{E} [1_{X > z}X]$$

$$= \mathbb{E} [1_{X > z}X(f(X) - f(z))].$$

This quantity $A$ is slightly problematic since, $f$ being decreasing on $[z, +\infty)$, we have $A < 0$. However, we can write $f(X) - f(z) = f'(\xi)(X - z)$ for some random $\xi > z$. Next we use the lower bound in (14) to get that for all $\xi > z$,

$$|f'(\xi)| = f'(\xi) = (1 - \Phi(z)) \left(1 - \sqrt{2\pi} \Phi(\xi)\xi e^{\xi^2/2}\right) \leq 1 \cdot \left(1 - \frac{\xi^2}{1 + \xi^2}\right) = \frac{1}{1 + \xi^2}. \quad (18)$$

This upper bound can obviously be further bounded above uniformly by $(1 + z^2)^{-1}$, which means that

$$|A| \leq \mathbb{E} [1_{X > z}X(X - z)] \frac{1}{1 + z^2} \leq \mathbb{E} [1_{X > z}X^2] \frac{1}{1 + z^2}.$$ 

By using this estimate in (17) we finally get

$$\mathbb{P} [X > z] \geq \Phi(z) - \frac{1}{1 + z^2} \mathbb{E} [1_{X > z}X^2]. \quad (19)$$

**Step 3: integrating by parts.** For notational compactness, let $S(z) := \mathbb{P} [X > z]$. We integrate the last term in (19) by parts with respect to the positive measure $-dS(x)$. We have, for any $z > 0$,

$$\mathbb{E} [1_{X > z}X^2] = -\int_z^{\infty} x^2 dS(x) = \int_z^{\infty} 2xS(x) dx$$

$$\leq z^2 S(z) + \int_z^{\infty} 2xS(x) dx.$$ 

The conclusion (19) from the previous step now implies

$$S(z) \geq \Phi(z) - \frac{z^2}{1 + z^2} S(z) - \frac{1}{1 + z^2} \int_z^{\infty} 2xS(x) dx.$$
which finishes the proof of the theorem’s lower bound.

Step 4: Upper bound. The proof of the upper bound is similar to, not symmetric positivity with, and less delicate than, the proof of the lower bound. Indeed, we can take advantage of a projective positivity result on the inner product of $DX$ and $-DL^{-1}X$, namely [9] Proposition 3.9 which says that $E \langle DX; -DL^{-1}X \rangle |X| \geq 0$. This allows us to avoid the need for any additional moment assumptions. Using Lemma 3.4 directly with the function $h = f$, which is continuous, and differentiable everywhere except at $x = z$, we have

$$P \{ X > z \} = \bar{\Phi}(z) - E[f'(X)] + E[f'(X) \langle DX; -DL^{-1}X \rangle]$$

$$= \bar{\Phi}(z) + E[1_{X \leq z}f'(X)(-1 + \langle DX; -DL^{-1}X \rangle)] + E[1_{X > z}f'(X)(-1 + \langle DX; -DL^{-1}X \rangle)]$$

$$\leq \bar{\Phi}(z) + E[1_{X > z}f'(X)(-1 + \langle DX; -DL^{-1}X \rangle)]$$

(20)

where the last inequality simply comes from the facts that by hypothesis $-1 + \langle DX; -DL^{-1}X \rangle$ is negative, and when $x \leq z$, $f'(x) \geq 0$ (see previous step for proof of this positivity). It remains to control the term in (20): since $E[\langle DX; -DL^{-1}X \rangle |X|] \geq 0$, and using the negativity of $f'$ on $x > z$,

$$E[1_{X > z}f'(X)(-1 + \langle DX; -DL^{-1}X \rangle)] = E[1_{X > z}] E[(-1 + \langle DX; -DL^{-1}X \rangle) |X|]$$

$$\leq -E[1_{X > z}] f'(X) = E[1_{X > z} |f'(X)|]$$

This last inequality together with the bound on $f'$ obtained in (18) imply

$$E[1_{X > z}] f'(X)(-1 + \langle DX; -DL^{-1}X \rangle) \leq P \{ X > z \} \frac{1}{1 + z^2}.$$

Thus we have proved that

$$P \{ X > z \} \leq \bar{\Phi}(z) + P \{ X > z \} \frac{1}{1 + z^2}$$

which implies the upper bound of the theorem, finishing its proof. ■

Proof of Corollary 4.2. Proof of Point 2. One notes first that by a result in [12], condition (11) implies that $X$ has a density, so that $S'$ is defined. Then we get

$$F(z) := \int_{z}^{\infty} xP \{ X > z \} dx = \int_{z}^{\infty} xS(x) dx \leq \frac{1}{c} \int_{z}^{\infty} x^2 |S'(x)| dx$$

$$= \frac{1}{c} \left( z^2 S(z) - \lim_{x \to \infty} x^2 S(x) + \int_{z}^{\infty} 2xS(x) dx \right)$$

$$\leq \frac{1}{c} \left( z^2 S(z) + 2F(z) \right)$$

which implies

$$F(z) \leq \frac{1}{c - 2z^2} S(z).$$

With the lower bound conclusion of Theorem 4.1 we obtain

$$S(z) \geq \bar{\Phi}(z) - \frac{2z^2}{1 + z^2} \frac{1}{c - 2} S(z)$$

which is equivalent to the statement of Point 2.

Proof of Point 3. From Theorem 4.1 we have for large $z$,

$$S(z) \geq \bar{\Phi}(z) - \frac{1}{1 + z^2} \int_{z}^{\infty} 2x^{1-c}x^cS(x) dx$$

$$\geq \bar{\Phi}(z) - \frac{1}{1 + z^2} \sup_{x > z} [x^c S(x)] \int_{z}^{\infty} 2x^{1-c} dx = \bar{\Phi}(z) - \frac{z^{2-c}2}{1 + z^2} \sup_{x > z} [x^c S(x)]$$

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which implies
\[ \sup_{x > z} [x^c S(x)] \left( \frac{2}{c - 2} + 1 \right) \geq z^c \Phi(z) \]
which is equivalent to the first part of the statement of Point 3, the second part following from the fact that \( z^c \Phi(z) \) is decreasing for large \( z \).

**Proof of Point 1.** As in Point 2, we define \( F(z) := \int_{z}^{\infty} xP[X > z] \, dx \) but this time, we do not need to use the density of \( X \). Instead, we note that by integration by parts,
\[ E[X^2 1_{X > z}] = z^2 S(z) - \lim_{x \to \infty} x^2 S(x) + 2F(z), \]
Since \( X \in L^2(\Omega), \lim_{x \to \infty} x^2 S(x) \leq \lim_{x \to \infty} E[X^2 1_{X > z}] = 0 \) and therefore \( E[X^2 1_{X > 0}] = 2F(0) \). Since our hypothesis is invariant with respect to changing \( X \) into \(-X\), we also get \( E[X^2 1_{X < 0}] = 2F(0) \). Therefore
\[ Var[X] = 4F(0). \]
Thus we only need to find a lower bound on \( F(0) \).

Now let \( p = 1, 2 \), and define
\[ a_p := E[X^p 1_{X > 0}]. \]
Thus we have \( a_2 = F(0) \) and \( a_1 = E[X^+] \). Now, by integration by parts, \( a_1 \geq \int_0^\infty S(x) \, dx \). Using Theorem 4.1, we thus get
\[ a_1 \geq \int_0^\infty \Phi(x) \, dx - 2 \int_0^\infty \frac{1}{1+x^2} F(x) \, dx \geq \int_0^\infty \Phi(x) \, dx - 2F(0) \int_0^\infty \frac{1}{1+x^2} \, dx = \frac{1}{\sqrt{2\pi}} - \pi a_2. \]
Since \( a_2 \geq a_1^2 \), this proves that \( F(0) = a_2 \geq \left( \sqrt{1 + 2\sqrt{2\pi}} - 1 \right)^2 / (2\pi)^2 \) and the conclusion follows. \( \blacksquare \)

**Proof of Theorem 4.3.** By Proposition 3.5 with \( L = E|X|/2 \), for \( x \in (a, b) \),
\[ \rho(x) = LA(x)/g(x). \]
By definition we also get \( A'(x) = -xA(x)/g(x) = -xL^{-1}\rho(x) \), and thus
\[ P[X > x] =: S(x) = L \int_x^{+\infty} \frac{-A'}{y} \, dy = L \left( \frac{A(x)}{x} - \lim_{y \to \infty} \frac{A(y)}{y} - \int_x^{+\infty} \frac{A(y)}{y^2} \, dy \right). \]
Since \( g \) is non-negative, \( A \) is bounded, and the term \( \lim_{y \to \infty} A(y)/y \) is thus zero. Equality (12) follows immediately, proving the theorem. \( \blacksquare \)

**Proof of Corollary 4.5.** *Proof of inequality (13).* From Theorem 4.3 with \( L = E|X|/2 \), and \( k > 1 \), and using the fact that \( A \) is decreasing, we can write
\[
S(x) =: P[X > x] = L \left( \frac{A(x)}{x} - \int_x^{kx} \frac{A(y)}{y^2} \, dy - \int_{kx}^{+\infty} \frac{A(y)}{y^2} \, dy \right) \\
\geq L \left( \frac{A(x)}{x} - \frac{A(x)}{x} \left( 1 - \frac{1}{k} \right) - \frac{A(kx)}{kx} \right) \\
= L \frac{A(x)}{x} \frac{1}{k} \left( 1 - \frac{A(kx)}{A(x)} \right).
\]
Similarly, for Point 2, we have

\[ \frac{A(kx)}{A(x)} = \exp \left( - \int_x^{kx} \frac{ydy}{g(y)} \right) \leq \exp \left( - \frac{1}{c'} \log k \right) = k^{-1/c'} . \]

This proves

\[ S(x) \geq L \frac{A(x)}{x} 1 \left( 1 - k^{-1/c'} \right) . \]

The proof is completed simply by optimizing this over the values of \( k > 1 \): the function \( k \mapsto \left( 1 - k^{-1/c'} \right) / k \) reaches its maximum of \((c')^{c'} (1 + c')^{-c' - 1} \) at \((1 + 1/c')^{c'}\).

**Proof of Points 1, 2, 3, and 4.** Point 1 is immediate since \( g(x) \geq 1 \) implies \( A(x) \geq \exp \left( -x^2/2 \right) \). Similarly, for Point 2, we have

\[ A(x) \geq \exp \left( - \int_0^{z_0} \frac{ydy}{g(y)} \right) \exp \left( - \frac{1}{c''} \int_x^{z_0} \frac{dy}{y} \right) = \text{cst } x^{-1/c''} , \]

and Point 3 follows in the same fashion. Point 4 is shown identically by reversing all inequalities, concluding the proof of the corollary.

**Proof of Corollary 4.6** This is in fact a corollary of the proof of Theorem 4.1. At the end of Step 2 therein, in (19), we prove that (11), the lower bound assumption \( \langle DX; -DL^{-1}X \rangle =: G \geq 1 \), implies

\[ S(x) \geq \Phi(z) - E[1_{X>z} X (X - z)] \frac{1 + z^2}{1 + z^2} . \]

(21)

Let us investigate the term \( B := E[1_{X>z} X (X - z)] \). Using Lemma 3.4 with the function \( h(x) = (x - z) 1_{x > z} \), we have

\[ B = E[1_{X>z} G] . \]

Now use the upper bound assumption on \( G \): we get, for all \( z \geq z_0 \),

\[ B \leq c' E[1_{X>z} X^2] = c' E[1_{X>z} X (X - z)] + c' z E[1_{X>z} X] \]

\[ = c'B + c' z E[1_{X>z} X] = c'B + c' z \left( zS(z) + \int_z^\infty S(x) dx \right) , \]

(22)

where we used integration by parts for the last inequality. Integration by parts also directly shows:

\[ B = 2 \int_z^\infty xS(s) - z \int_z^\infty S(x) dx . \]

Introducing the following additional notation: \( D := z \int_z^\infty S(x) dx \) and \( E := 2 \int_z^\infty xS(s) \), we see that \( B = E - D \) and also that \( E \geq 2D \). Moreover, in (22), we also recognize the appearance of \( D \). Therefore we have

\[ (E - D) (1 - c') \leq c' D + c' z^2 S(z) \leq (c'/2) E + c' z^2 S(z) \]

which easily implies

\[ B \leq E \leq 2c' z^2 S(z) . \]

From (21), we now get

\[ S(z) \geq \Phi(z) - \frac{2c' z^2}{1 + z^2} S(z) \]

from which we obtain, for \( z \geq z_0 \)

\[ S(z) \geq \frac{1 + z^2}{1 + (2c' + 1) z^2} \Phi(z) , \]

finishing the proof of the corollary.

\[ \square \]
5 Fluctuation exponent and deviations for polymers in Gaussian environments

Lemma 5.6 provides a way to calculate $\langle DX; -DL^{-1}X \rangle$ in order to check, for instance, whether it is bounded below by a positive constant $c^2$. If $c^2 \neq 1$, because of the bilinearity of Condition (11), one only needs to consider $X/c$ instead of $X$ in order to apply Theorem 4.1 say. To show that such a tool can be applied with ease in a non-trivial situation, we have chosen the issue of fluctuation exponents for polymers in random environments.

We can consider various polymer models in random environments constructed by analogy with the so-called stochastic Anderson models (see [17] and [6]). A polymer’s state space $\mathcal{R}$ with the letter $M$ is a slight possibility of notational confusion since now the underlying isonormal Gaussian process is called an isonormal Gaussian process whose Hilbert space is $H$. We can equip $\mathcal{R}$ with a Markov process $b$ on $[0, \infty)$ whose infinitesimal generator, under the probability measure $P_b$, is the Laplace(-Beltrami) operator or the discrete Laplacian. Thus for instance, $b$ is Brownian motion when $\mathcal{R} = \mathbb{R}^d$, or is the simple symmetric random walk when $\mathcal{R} = \mathbb{Z}^d$; it is the image of Brownian motion by the imaginary exponential map when $\mathcal{R} = S^1$. To simplify our exposition, we can and will typically assume, unless explicitly stated otherwise, that $\mathcal{R} = \mathbb{R}$, but our constructions and proofs can be adapted to any of the above choices.

5.1 The random environment

Let $W$ be a Gaussian model on $\mathbb{R}_+ \times \mathbb{R}$ which is homogeneous in space and is Brownian in time for fixed space parameter: the covariance of $W$ is thus

$$E [W(t, x)W(s, y)] = \min (s, t) Q(x - y),$$

for some homogeneous covariance function $Q$ on $\mathbb{R}$. We assume that $Q$ is continuous and that its Fourier transform $\hat{Q}$ is a measure with a density also denoted by $\hat{Q}$. Note that $\hat{Q}$ is positive and $|Q|$ is bounded by $Q(0)$. The field $W$ can be represented using a very specific isonormal Gaussian process: there exists a white noise measure $M$ on $\mathbb{R}_+ \times \mathbb{R}$ such that

$$W(t, x) = \int_0^t \int_{\mathbb{R}} M(ds, d\lambda) \sqrt{\hat{Q}(\lambda)} e^{i\lambda x},$$

where the above integral is the Wiener integral of $(s, \lambda) \mapsto 1_{[0, t]}(s) \sqrt{\hat{Q}(\lambda)} e^{i\lambda x}$ with respect to $M$. This $M$ is an isonormal Gaussian process whose Hilbert space is $H = L^2(\mathbb{R}_+ \times \mathbb{R})$. Malliavin derivatives relative to $M$ will take their parameters $(s, \lambda)$ in $\mathbb{R}_+ \times \mathbb{R}$, and inner products and norms are understood in $H$. There is a slight possibility of notational confusion since now the underlying isonormal Gaussian process is called $M$, with the letter $W$ – the traditional name of the polymer potential field – being a linear transformation of $M$.

The relation between $D$ and $W$ is thus that $D_{s, \lambda} W(t, x) = e^{i\lambda x} \sqrt{\hat{Q}(\lambda)} 1_{[0, t]}(s)$. We will make use of the following similarly important formulas: for any measurable function $f$:

$$D_{s, \lambda} \int_0^t \int_{\mathbb{R}} M(ds, d\lambda) \sqrt{\hat{Q}(\lambda)} e^{i\lambda f(s)} = \sqrt{\hat{Q}(\lambda)} e^{i\lambda f(s)};$$ (23)

$$\int_0^t \int_{\mathbb{R}} ds \hat{Q}(\lambda) d\lambda e^{i\lambda f(s)} = \int_0^t Q(f(s)) ds$$ (24)

Quantitatively, this calculation will be particularly useful as a key to easy upper bounds by noting the fact that $\max_{x \in \mathbb{R}} Q(x) = Q(0)$ is positive and finite. On the other hand, if $Q$ is positive and non-degenerate, lower bounds will easily follow.
In order to use the full strength of our estimates in Section 4, we will also allow $Q$ to be inhomogeneous, and in particular, unbounded. This is easily modeled by specifying that

$$W(t, x) = \int_0^t \int_\mathbb{R} M(ds, d\lambda) q(\lambda, x)$$

where $\int_\mathbb{R} q(\lambda, x) q(\lambda, y) = Q(x, y)$. Calculations similar to (23) and (24) then ensue.

We may also devise polymer models in non-Gaussian environments by considering $W$ as a mixture of Gaussian fields. This means that we consider $Q$ to be random itself, with respect to some separate probability space. We will place only weak restrictions on this randomness: under a probability measure $\mathcal{P}$, we assume $Q$ is a non-negative random field on $\mathbb{R}$, integrable on $\mathbb{R}$, with $Q(0) = \int_\mathbb{R} Q(\lambda) d\lambda$ integrable with respect to $\mathcal{P}$.

### 5.2 The polymer and its fluctuation exponent

Let the Hamiltonian of a path $b$ in $\mathbb{R}$ under the random environment $W$ be defined, up to time $t$, as

$$H^W_t(b) = \int_0^t \int_\mathbb{R} M(ds, d\lambda) e^{i\lambda b_s}.$$  

Since $W$ is a symmetric field, we have omitted the traditional negative sign in front of the definition of $H^W_t$. For fixed path $b$, this Hamiltonian $H^W_t(b)$ is a Gaussian random variable w.r.t $W$.

The polymer $P_b$ based on $b$ in the random Hamiltonian $H^W$ is defined as the law whose Radon-Nykodym derivative with respect to $P_b$ is $Z_t(b)/E_b[Z_t(b)]$ where

$$Z_t(b) := \exp H^W_t(b).$$

We use the notation $u$ for the partition function (normalizing constant) for this measure:

$$u(t) := E_b[Z_t(b)].$$

The process $u(t)$ is of special importance: its behavior helps understand the behavior of the whole measure $P_b$. When $b_0 = x$ instead of 0, the resulting $u(t, x)$ is the solution of a stochastic heat equation with multiplicative noise potential $W$, and the logarithm of this solution solves a so-called stochastic Burgers equation.

It is known that $t^{-1} \log u(t)$ typically converges almost surely to a non-random constant $\chi$ called the almost sure Lyapunov exponent of $u$ (see [17] and references therein for instance; the case of random $Q$ is treated in [7]; the case of inhomogeneous $Q$ on compact space is discussed in [6]). The speed of concentration of $\log u(t)$ around its mean has been the subject of some debate recently. One may consult [1] for a discussion of the issue and its relation to the so-called wandering exponent in non-compact space. The question is to evaluate the asymptotics of $\log u(t) - E[\log u(t)]$ for large $t$, or to show that it is roughly equivalent to $t^{\chi}$, where $\chi$ is called the fluctuation exponent. The most widely used measure of this behavior is the asymptotics of $\text{Var}[\log u(t)]$. Here we show that if space is compact with positive correlations, or if $W$ has infinite spatial correlation range, then $\text{Var}[\log u(t)]$ behaves as $t$, i.e. the fluctuation exponent $\chi$ is 1/2. This result is highly robust to the actual distribution of $W$, since it does not depend on the law of $Q$ under $\mathcal{P}$ beyond its first moment. We also provide a class of examples in which $H^W$ is replaced by a non-linear functional of $W$, and yet the fluctuation exponent, as measured by the power behavior of $\text{Var}[\log u(t)]$, is still 1/2.

We hope that our method will stimulate the study of this problem for other correlation structures not covered by the theorem below, in particular in infinite space when the correlation range of $W$ is finite or decaying at a certain speed at infinity, or in the case of space-time white-noise in discrete space, i.e. when the Brownian motions $\{W(\cdot, x) : x \in \mathbb{Z}^d\}$ form an IID family. We conjecture that $\chi$ will depend on the decorrelation speed of $W$. It is at least believed by some that in the case of space-time white noise, $\chi < 1/2$. 

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The starting point for studying \( \text{Var} \left[ \log u(t) \right] \) is the estimation of the function \( g \) relative to the random variable \( \log u(t) = \log E_b \left[ \exp H_t^W(b) \right] \). Here because the integral \( H_t^W(b) = \int_0^t W(ds, b_s) \) has to be understood as \( \int_0^t \int_R M(ds, d\lambda) \sqrt{Q(\lambda)} e^{i\lambda b_s} \), we must calculate the Malliavin derivative with parameters \( r \) and \( \lambda \). We will use the consequence of Mehler’s formula described in Lemma 3.6 of Section 3.2. More specifically, we have the following.

**Lemma 5.1** Assume \( Q \) is homogeneous. Let

\[
Y := \log u(t) - \log u(t) = \frac{1}{\sqrt{t}} \left[ \int_0^t ds Q(b_s - \bar{b}_s) \frac{\exp H^W_t(b)}{u(t)} \frac{\exp H^W_t(b)}{\bar{b}_s} \right].
\]

Then

\[
D_{\lambda, \lambda} Y = \frac{1}{\sqrt{t}} \left[ \int_0^t ds Q(b_s - \bar{b}_s) \frac{\exp H^W_t(b)}{u(t)} \frac{\exp H^W_t(b)}{\bar{b}_s} \right],
\]

and

\[
\langle DY, -DL^{-1}Y \rangle = \frac{1}{2t} \left[ \int_{-\pi/2}^{\pi/2} \frac{\sin \theta}{d\theta} E_{\theta, \theta} \left[ \int_0^t ds Q(b_s - \bar{b}_s) \frac{\exp H^W_t(b)}{u(t)} \frac{\exp H^W_t(b)}{\bar{b}_s} \right] \right].
\]

where \( E_{b, \bar{b}} \) is the expectation w.r.t. two independent copies \( b \) and \( \bar{b} \) of Brownian motion, and \( R_{\theta}W \) was defined in [10]. When \( Q \) is inhomogeneous, the above formula still holds, with \( Q(b_s - \bar{b}_s) \) replaced by \( Q(\bar{b}_s, \bar{b}_s) \).

**Proof.** By formula (23) and the chain rule for Malliavin derivatives, we have for fixed \( b \),

\[
D_{\lambda, \lambda} \left( e^{H_t^W(b)} \right) = \sqrt{Q(\lambda)} e^{i\lambda b_s} e^{H_t^W(b)}
\]

and therefore by linearity of the expectation \( E_b \), and the chain rule again, the first statement of the lemma follows immediately.

Now we investigate \( DL^{-1}Y \). To use Lemma 3.6 relative to \( W \), we note that the expression for \( R_{\theta}Y \) is straightforward, since \( Y \) is defined as a non-random non-linear functional of an expression involving \( b \) and \( W \) with the latter appearing linearly via \( H_t^W(b) \); in other words, \( R_{\theta}Y \) is obtained by replacing \( H_t^W(b) \) by \( H_t^{R_{\theta}W(b)} \), so we simply have

\[
R_{\theta}Y = \frac{\log E_b \left[ \exp \left( \frac{H^W_t(b) \cos \theta + H^W_t(b) \sin \theta}{u(t)} \right) \right] - \log u(t)}{\sqrt{t}}.
\]

Thus by Lemma 3.6

\[
-D_{\lambda, \lambda} L^{-1} Y = \int_{-\pi/2}^{\pi/2} \frac{\sin \theta}{2\sqrt{t}} E_{\theta, \theta} \left[ \sqrt{Q(\lambda)} e^{i\lambda b_s} \sin \theta \frac{\exp H^{R_{\theta}W}(b)}{R_{\theta}u(t)} \right].
\]

We may thus calculate explicitly the inner product \( \langle DX, -DL^{-1}X \rangle \), using equation (24), obtaining the second announced result (25). The proof of the first statement is identical in structure to the above arguments. The last statement is obtained again using identical arguments. 

It is worth noting that a similar expression as for \( \langle DY, -DL^{-1}Y \rangle \) can be obtained for \( \|DY\|^2 \). Using the same calculation technique as in the above proof, we have

\[
\|DY\|^2 = \|DY\|^2_{L^2([0,t] \times R)} = \frac{1}{t} E_{b, \bar{b}} \left[ \frac{e^{H_t^W(b) + H_t^W(\bar{b})}}{u^2(t)} \int_0^t ds Q(b_s, \bar{b}_s) \right]
\]

\[
= \frac{1}{t} \hat{E}_{b, \bar{b}} \left[ \int_0^t ds Q(b_s, \bar{b}_s) \right],
\]

(26)
where the last expression involves the expectation w.r.t. the polymer measure \( \hat{P} \) itself, or rather w.r.t. the product measure \( d\hat{P}_{b,\tilde{b}} = e^{H_{\hat{W}}(b)}e^{H_{\hat{W}}(\tilde{b})}u^{-2}(t)\,d\hat{P}_b \times d\hat{P}_\tilde{b} \) of two independent polymers \( (b, \tilde{b}) \) in the same random environment \( W \). This measure is called the two-replica polymer measure, and the quantity \( \hat{E}_{b,\tilde{b}} \left[ \int_0^t ds Q(b, \tilde{b}) \right] \) is the so-called replica overlap for this polymer. This notion should be familiar to those studying spin glasses such as the Sherrington-Kirkpatrick model (see \[19\]). The strategy developed in this article suggests that the expression \( \langle DY, -DL^{-1}Y \rangle \) may be better suited than the rescaled overlap \( \|DY\|^2 \) in seeking lower bounds on \( \log u \)'s concentration.

**Notation 5.2** In order to simplify the notation in the next theorem, when \( Q \) is not homogeneous, we denote \( Q(0) = \max_{x \in \mathbb{R}} Q(x, x) \). We then have, in all cases, \( Q(0) \geq |Q(x, y)| \) for all \( x, y \in \mathbb{R} \). Similarly we denote \( Q_m = \min_{x,y \in \mathbb{R}} Q(x, y) \). In the homogeneous case \( Q_m \) thus coincides with \( \min_{x \in \mathbb{R}} Q(x) \). When \( Q \) is random, assumptions about \( Q \) below are to be understood as being required \( \mathcal{P} \)-almost surely.

**Definition 5.3** To make precise statements about the fluctuation exponent, it is convenient to use the following definition:

\[
\chi := \lim_{t \to \infty} \frac{\log \text{Var} \left[ \log u(t) \right]}{2 \log t}
\]

**Theorem 5.4**

1. Assume \( Q(0) \) is finite. We have for all \( a, t > 0 \),

\[
P \left[ \left| \log u(t) - E \left[ \log u(t) \right] \right| > a \sqrt{t} \right] \leq 1 \wedge \frac{2Q(0)^{1/2}}{a \sqrt{2\pi}} \exp \left( -\frac{a^2}{2Q(0)} \right). \tag{27}
\]

If \( Q \) is random, one only needs to take an expectation \( E_\mathcal{P} \) of the above right-hand side.

2. Assume \( Q(0) \) is finite. Then for all \( t \),

\[
\text{Var} \left[ \log u(t) \right] \leq \left( \frac{\pi}{2} \right)^2 E_\mathcal{P} [Q(0)] \, t. \tag{28}
\]

3. Assume \( Q_m \) is positive. Then for all \( t \),

\[
\text{Var} \left[ \log u(t) \right] \geq K_u E_\mathcal{P} [Q_m] \, t, \tag{29}
\]

where the universal constant \( K_u \simeq 0.21367 \) is defined in Point 1 of Corollary 4.2

4. Assume \( Q_m \) is positive and \( Q(0) \) is finite. Then, in addition to \( \tag{27} \), we have for any \( K < 1 \) and all \( a \) large,

\[
P \left[ \left| \log u(t) - E \left[ \log u(t) \right] \right| > a \sqrt{t} \right] \geq K \frac{Q_m^{1/2}}{a} \exp \left( -\frac{a^2}{2Q_m} \right) \tag{30}
\]

Moreover, the conclusions \( \tag{28} \) and \( \tag{29} \) hold simultaneously, so that the fluctuation exponent is \( \chi = 1/2 \) as soon as \( Q(0) \in L^1 [\mathcal{P}] \).

The hypotheses in Points 3 and 4 of this theorem are satisfied if the state space \( \mathbb{R} \) is replaced by a compact set such as \( S^1 \), or a finite set, and \( Q \) is positive everywhere: then indeed \( Q_m > 0 \). Although the hypothesis of uniform positivity of \( Q \) can be considered as restrictive for non-compact state space, one notes that there is no restriction on how small \( Q_m \) can be compared to \( Q(0) \); in this sense, the slightest persistent correlation of the random environment at distinct sites results in a fluctuation exponent \( \chi = 1/2 \). In sharp contrast is the case of space-time white noise in discrete space, which is not covered by our theorem, since

\[18\]
then $Q(x) = 0$ except if $x = 0$; the main open problem in discrete space is to prove that $\chi < 1/2$ in this white noise case.

In relation to the overlap $\|DY\|^2$, we see that under the assumptions of Point 4 above, $\|DY\|$ is also bounded above and below by non-random multiples of $t^{1/2}$. Hence, while our proofs cannot use $\|DY\|$ directly to prove $\chi = 1/2$, the situation in which we can prove $\chi = 1/2$ coincides with a case where the overlap has the same rough large-time behavior as $Var[\log u(t)]$. We believe this is in accordance with common intuition about related spin glass models.

More generally, we consider it an important open problem to understand the precise deviations of $\log u(t)$. The combination of the sub-Gaussian and super-Gaussian estimates (27) and (30) are close to a central limit theorem statement, except for the fact that the rate is not sharply pinpointed. Finding a sharper rate is an arduous task which will require a finer analysis of the expression (25), and should depend heavily and non-trivially on the correlations of the covariance function, just as the obtention of a non-trivially on the correlations of the covariance function, just as the obtention of a

The combination of the sub-Gaussian and super-Gaussian estimates (27) and (30) are close to a central limit

Proof of Theorem 5.4
Proof of Point 1. Since $Q(x,y) \leq Q(0)$ for all $x,y$, from Lemma 5.1 we have

$$\langle DY, DL^{-1}Y \rangle \leq \frac{Q(0)}{2t} \int_{-\pi/2}^{\pi/2} |\sin \theta| d\theta \ E_{b,b} \left[ \frac{\exp H^W_t(b) \exp H^{R_0W}_t(b)}{u(t)} \frac{R_0u(t)}{R_0u(t)} \right] = Q(0),$$

where we used the trivial facts that $E_{b,b} [\exp H^W_t(b)] = u(t)$ and $E_{b,b} [\exp H^{R_0W}_t(b)] = R_0u(t)$. The upper bound result in Theorem 4.1 applied to the random variable $X = Y/\sqrt{Q(0)}$, now yields

$$P[Y > z] = P[X > zQ(0)^{-1/2}] \leq \left( 1 + \frac{Q(0)}{z^2} \right) \Phi \left( \frac{z}{Q(0)^{1/2}} \right)$$

and the upper bound statement (27).

Proof of Points 2 and 3. Now we note that, since all terms in the integrals in Lemma 5.1 are positive, our hypothesis that $Q(x,y) \geq Q_m > 0$ for all $x,y$ implies

$$\langle DX, DL^{-1}X \rangle \geq \frac{Q_m}{2t} \int_{-\pi/2}^{\pi/2} |\sin \theta| d\theta \ E_{b,b} \left[ \frac{\exp H^W_t(b) \exp H^{R_0W}_t(b)}{u(t)} \frac{R_0u(t)}{R_0u(t)} \right] = \frac{Q_m}{2} \int_{-\pi/2}^{\pi/2} |\sin \theta| d\theta \ E_{b,b} \left[ \frac{\exp H^W_t(b)}{u(t)} \frac{R_0u(t)}{R_0u(t)} \right] = Q_m.$$  

Applying Point 1 in Corollary 4.6 to the random variable $X = Y/\sqrt{Q_m}$, the lower bound of (29) in Point 3 follows. The upper bound (28) of Point 2 can be proved using the result (27) of Point 1, although one obtains a slightly larger constant than the one announced. The constant $(\pi/2)^2$ is obtained by using the bound $\|DY\|^2 \leq Q(0)$ which follows trivially from (20), and then applying the classical result $Var[Y] \leq (\pi/2)^2 E[\|DY\|^2]$, found for instance in [20, Theorem 9.2.3].

Proof of Point 4. Since $Q(0)$ is finite and $Q_m$ is positive, using $X = Y/\sqrt{Q_m}$ in Corollary 4.6, we have that $g(x) \geq 1$ and $g(x) \leq Q(0)/Q_m$, so that we may use any value $c' > 0$ in the assumption of that
corollary, with thus \( K = 1/(2c' + 1) \) arbitrarily close to 1; the corollary’s conclusion is the statement of Point 4. This finishes the proof of the theorem.

5.3 Robustness of the fluctuation exponent: a non-Gaussian Hamiltonian

The statements of Point 4 of Theorem 5.3 show that if the random environment’s spatial covariance is bounded above and below by positive constants, then the partition function’s logarithm \( \log u(t) \) is both sub-Gaussian and super-Gaussian, in terms of its tail behavior (tail bounded respectively above and below by Gaussian tails). We now provide an example of a polymer subject to a non-Gaussian Hamiltonian, based still on the same random environment, whose logarithmic partition function may not be sub-Gaussian, yet still has a fluctuation exponent equal to 1/2. It is legitimate to qualify the persistence of this value 1/2 in a non-Gaussian example as a type of robustness.

Let

\[
X_t^W(b) := \int_0^t W(ds, b_s).
\]

With \( F(t, x) = x + x|x|/t \), we define our new Hamiltonian as

\[
H_t^W(b) := F(t, X_t^W(b)).
\]

Similarly to Lemma 5.1 using the Chain Rule for Malliavin derivatives, we can prove that

\[
D_{s, \lambda}Y = \frac{1}{\sqrt{t \ u(t)}} E_b \left[ \sqrt{\tilde{Q}(\lambda)e^{i\lambda b_s}e^{H_t^W(b)}} \left( 1 + \frac{|X_t^W(b)|}{t} \right) \right],
\]

and

\[
\langle DY, -DL^{-1}Y \rangle = \frac{1}{2t} \int_{-\pi/2}^{\pi/2} |\sin \theta| d\theta \ E_b \left[ \int_0^t \left( \frac{\exp H_t^W(b) \ exp H_t^{R_uW}(\tilde{b})}{u(t)} \right) \left( 1 + \frac{|X_t^W(b)|}{t} \right) \left( 1 + \frac{|X_t^{R_uW}(\tilde{b})|}{t} \right) \right].
\]

Theorem 5.5 Consider \( u(t) = E_b \left[ \exp H_t^W(b) \right] \) where the new Hamiltonian \( H_t^W \) is given in (31). The random environment \( W \) is as it was defined in Section 5.1 and \( Q(0) \) and \( Q_m \) are given in Notation 5.2.

1. Assume \( Q(0) < 1/9 \). Then \( \text{Var} \ |\log u(t)| \leq 2^8 (\pi/2)^2 E \left[ Q^3(0) \right] t + o(t) \).

2. Assume \( Q_m \) is positive. Then \( \text{Var} \ |\log u(t)| \leq K_u E_P \left[ Q_m \right] t \).

If both assumptions of Points 1 and 2 hold, the fluctuation exponent of Definition 5.5 is \( \chi = 1/2 \), and the conclusion of Point 4 in Theorem 5.4 holds.

We suspect that the logarithmic partition function \( \log u(t) \) given by the non-Gaussian Hamiltonian in (31) is eminently non-Gaussian itself; in fact, the form of its derivative in (32), with the additional factors of the form \( (1+X(b))/t \), can presumably be compared with \( Y \). We conjecture, although we are unable to prove it, that the corresponding \( g(y) \) grows linearly in \( y \). This would show, via Corollary 4.5 Point 3, that \( \log u(t) \) has exponential tails. Other examples of non-Gaussian Hamiltonians can be given, using the formulation (31) with other functions \( F \), such as \( F(t, x) = x + x|x|^p/t^{(1+p)/2} \) for \( p > 0 \). It should be noted, however, that in our Gaussian environment, any value \( p > 1 \) results in a partition function \( u(t) \) with infinite
first moment, in which case the arguments we have given above for proving that \( \chi = 1/2 \) will not work. This does not mean that the logarithmic partition function cannot be analyzed using finer arguments; it can presumably be proved to be non-Gaussian with heavier-than-exponential tails when \( p > 1 \).

**Proof of Theorem 5.5** Since the additional terms in \( \chi \), compared to Lemma 5.1, are factors greater than 1, the conclusion of Point 2 follows immediately using the proof of Points 3 and 4 of Theorem 5.4.

To prove that Point 1 holds, we will use the again the classical fact \( \text{Var}[Y] \leq (\pi/2)^2 \mathbb{E}[\|DY\|^2] \). Here from (82) we have immediately

\[
\|DY\|^2 \leq Q(0) \left( 1 + E_b \left[ \frac{e^{H^W(b)}|X_t^W(b)|}{u(t)} \right] \right)^2.
\]

Therefore, to get an upper bound on the variance of \( Y \) uniformly in \( t \) we only need to show that the quantity

\[
B := \mathbb{E} \left[ \left( E_b \left[ \frac{e^{H^W(b)}|X_t^W(b)|}{u(t)} \right] \right)^2 \right]
\]

is bounded in \( t \). We see that, using Jensen’s inequality w.r.t. the polymer measure, and then w.r.t. the random medium’s expectation,

\[
B = \frac{1}{t^2} \mathbb{E} \left[ \left( \log e^{\chi t^2/9} \right)^2 \right] \leq \frac{1}{t^2} \log \left( 1 + \mathbb{E} \left[ \left( \log e^{\chi t^2/9} \right)^2 \right] \right).
\]

Now we evaluate

\[
\mathbb{E} \left[ E_b \left[ \frac{e^{H^W(b)+|X_t^W(b)|^2/9} |X_t^W(b)|}{u(t)} \right] \right] = E_b \left[ \frac{e^{X_t^W(b)+|X_t^W(b)|^2/9} |X_t^W(b)|}{u(t)} \right] \leq E^{1/2} \left[ u(t)^{-2} e^{4(X_t^W(b)+|X_t^W(b)|^2/9)} \right] \leq E^{1/2} \left[ e^{-4H^W(b)} \right] E_b \left[ e^{4|X_t^W(b)|+2|X_t^W(b)|^2/9} \right].
\]

The first term in the above product is actually less than the second. For the second, we note that for any fixed \( b \), the random variable \( X_t^W(b) \) is Gaussian centered, with a variance bounded above by \( Q(0) t \leq t/9 \). Therefore we have that \( \mathbb{E} \left[ e^{4|X_t^W(b)|^2/9} \right] \) is bounded by the finite universal constant \( \mathbb{E} \left[ \exp \left( 4Z^2/9 \right) \right] \). This proves, via another application of Schwartz’s inequality, that for some universal constant \( K_u \),

\[
B \leq \frac{1}{t^2} \log^2 \left( 1 + K_u \exp (16Q(0) t) \right) \leq \frac{1 + \log^2 K_u}{t^2} + 16^2 Q^2(0) = 2^8 Q^2(0) + o(t),
\]

where \( o(t) \) is non-random, proving Point 1, and the theorem.
6 Appendix

To prove Lemma 3.4 we begin with an intermediate result in the \( n \)th Wiener chaos.

**Lemma 6.1** Let \( n \in \mathbb{N} \) and \( f_n \in H^n \) be a symmetric function. Let \( Y \in D^{1,2} \). Then

\[
E[I_n(f_n)Y] = \frac{1}{n}E[\langle D, (I_n(f_n)) ; DY \rangle] = E[(I_{n-1}(f_n(\ast, \cdot)) ; DY)],
\]

where we used the notation \( I_{n-1}(f_n(\ast, \cdot)) \) to denote the function \( r \mapsto I_{n-1}(f_n(\ast, r)) \) where \( I_{n-1} \) operates on the \( n-1 \) variables “\( \ast \)” of \( f_n(\ast, r) \).

**Proof.** This is an immediate consequence of formula (1) and the famous relation \( \delta D = -L \) (where \( \delta \) is the divergence operator (Skorohod integral), adjoint of \( D \), see [13, Proposition 1.4.3]).

Here, however, we present a direct proof. Note that, because of the Wiener chaos expansion of \( Y \) in Proposition 2.2 and the fact that all chaos terms of different orders are orthogonal, without loss of generality, we can assume \( Y = I_n(g_n) \) for some symmetric \( g_n \in H^n \); then, using the formula for the covariance of two \( n \)th-chaos r.v.’s in Proposition 2.2 we have

\[
E[(I_{n-1}(f_n(\ast, \cdot)) ; DY)] = E[(I_{n-1}(f_n(\ast, \cdot)) ; nI_{n-1}(g_n(\ast, \cdot)))]
\]

\[
= n \int_T E[I_{n-1}(f_n(\ast, r)) I_{n-1}(g_n(\ast, r))] \mu (dr)
\]

\[
= n \int_T (n-1)! \langle f_n(\ast, r) ; g_n(\ast, r) \rangle_{L^2(B_2, \mu^{\otimes n-1})} \mu (dr)
\]

\[
= n! \langle f_n ; g_n \rangle_{L^2(B_2, \mu^{\otimes n})} = E[I_n(f_n)Y].
\]

which, together with formula (1), proves the lemma. \( \blacksquare \)

**Proof of Lemma 3.4.** Since \( X \in D^{1,2} \) and is centered, it has a Wiener chaos expansion \( X = \sum_{n=1}^{\infty} I_n(f_n) \). We calculate \( E[Xh(X)] \) via this expansion and the Malliavin calculus, invoking Remark 2.3 and using Lemma 6.1

\[
E[Xh(X)] = \sum_{n=1}^{\infty} E[I_n(f_n)h(X)]
\]

\[
= \sum_{n=1}^{\infty} \frac{1}{n} E \left[ \int_T D_r I_n(f_n) D_r h(X) \mu (dr) \right]
\]

\[
= E \left[ h'(X) \int_T D_r \left( \sum_{n=1}^{\infty} \frac{1}{n} I_n(f_n) \right) D_r X \mu (dr) \right]
\]

which by the definition of \( -L \) is precisely the statement (6). \( \blacksquare \)

**Proof of Lemma 3.6** The proof goes exactly as that of Lemma 1.5.2 in [13], with only the following change: the point there was to represent the Malliavin derivative of the operator \( (-C)^{-1} \) which changes \( I_n(f_n) \) into \( n^{-1/2} I_n(f_n) \), whereas here \( -L^{-1} \) changes \( I_n(f_n) \) into \( n^{-1} I_n(f_n) \); in [13, Lemma 1.5.2], the function \( \varphi \) was introduced with the property that \( \int_{-\pi/2}^{\pi/2} \sin^a(\theta) \varphi(\theta) \cos^a(\theta) d\theta = (n+1)^{-1/2} \); here we therefore only need to replace that \( \varphi \) by a function \( \phi \) such that \( \int_{-\pi/2}^{\pi/2} \sin(\theta) \cos^a(\theta) \phi(\theta) d\theta = (n+1)^{-1} \). It is clear that this function \( \phi \) is \( \phi(\theta) = 2^{-1} \text{sgn} (\theta) \). Our lemma follows by the proof of [13, Lemma 1.5.2]. \( \blacksquare \)
For completeness, we finish with a short, self-contained proof of the upper bound in Theorem 1.3, which is equivalent to Proposition 1.1.

**Proof of Proposition 1.1.** Assume $X$ is centered and $W$ is the standard Wiener space. By the Clark-Ocone representation formula (6), we can define a continuous square-integrable martingale $M$ with $M(1) = X$, via the formula $M(t) := \int_0^t \mathbb{E}[D_s X | \mathcal{F}_s] dW(s)$. The quadratic variation of $M$ is equal to $\langle M \rangle_t = \int_0^t \mathbb{E}[|D_s X|^2] ds$; therefore, by hypothesis, $\langle M \rangle_t \leq t$. Using the Doleans-Dade exponential martingale $\mathbb{E}(\lambda M)$ based on $\lambda M$, defined by $\mathbb{E}(\lambda M)_t = \exp(\lambda M_t - \frac{\lambda^2}{2} \langle M \rangle_t)$ we now have

$$\mathbb{E}[\exp \lambda X] = \mathbb{E} \left[ \mathbb{E}(\lambda M)_1 \exp \left( \frac{\lambda^2}{2} \langle M \rangle_1 \right) \right] \leq \mathbb{E}[\mathbb{E}(\lambda M)_1] e^{\frac{\lambda^2}{2}} = e^{\frac{\lambda^2}{2}}.$$

The proposition follows using a standard optimization calculation and Chebyshev’s inequality. Theorem 9.1.1 in [20] can be invoked to prove the same estimate in the case of a general isonormal Gaussian process $W$. ■

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