1 Introduction

1.1 Motivation

Solving equations is one of the main themes in mathematics. A large part of the combinatorial group theory revolves around the word and conjugacy problems - particular types of equations in groups. Whether a given equation has a solution in a given group is, as a rule, a non-trivial problem. A more general and more difficult problem is to decide which formulas of the first-order logic hold in a given group.

Around 1945 A. Tarski put forward two problems on elementary theories of free groups that served as a motivation for much of the research in group theory and logic for the last sixty years. A joint effort of mathematicians of several generations culminated in the following theorems, solving these Tarski’s conjectures.

**Theorem 1** (Kharlampovich and Myasnikov [43], Sela [59]). The elementary theories of non-abelian free groups coincide.

**Theorem 2** (Kharlampovich and Myasnikov [43]). The elementary theory of a free group $F$ (with constants for elements from $F$ in the language) is decidable.

We recall that the elementary theory $Th(G)$ of a group $G$ is the set of all first order sentences in the language of group theory which are true in $G$. A discussion of these conjectures can be found in several textbooks on logic, model and group theory (see, for example, [13], [21], [28]).

The work on the Tarski conjectures was rather fruitful - several areas of group theory were developed along the way. It was clear from the beginning that to deal with the Tarski’s conjectures one needs to have at least two principal things done: a precise description of solution sets of systems of equations over free groups and a robust theory of finitely generated groups which satisfy the same universal (existential) formulas as a free non-abelian group. In the classical case, algebraic geometry provides a unifying viewpoint on polynomial equations, while commutative algebra and the elimination theory give the required decision tools. Around 1998 three papers appeared almost simultaneously that address analogous issues in the group case. Basics of algebraic (or Diophantine) geometry over groups has been outlined by Baumslag, Miasnikov and Remeslennikov in [5], while the fundamentals of the elimination theory and the theory of fully residually free groups appeared in the works by Kharlampovich and Miasnikov [50], [57]. These two papers contain results that became fundamental for the proof of the above two theorems, as well as in the theory of fully residually free groups. The goal of these lectures is to explain why these results are important and to give some ideas of the proof.
1.2 Milestones of the theory of equations in free groups

The first general results on equations in groups appeared in the 1960’s \[30\]. About this time Lyndon (a former student of Tarski) came up with several extremely important ideas. One of these is to consider completions of a given group \( G \) by extending exponents into various rings (analogous to extensions of ring of scalars in commutative algebra) and use these completions to parameterize solutions of equations in \( G \). Another idea is to consider groups with free length functions with values in some ordered abelian group. This allows one to axiomatize the classical Nielsen technique based on the standard length function in free groups and apply it to "non-standard" extensions of free groups, for instance, to ultrapowers of free groups. A link with the Tarski’s problems comes here by the Keisler-Shelah theorem, that states that two groups are elementarily equivalent if and only if their ultrapowers (with respect to a non-principal ultrafilter) are isomorphic. The idea to study freely discriminated (fully residually free) groups in connection to equations in a free group also belongs to Lyndon. He proved \[29\] that the completion \( F_{\mathbb{Z}[[t]]} \) of a free group \( F \) by the polynomial ring \( \mathbb{Z}[[t]] \) (now it is called the Lyndon’s completion of \( F \)) is discriminated by \( F \). At the time the Tarski’s problems withstood the attack, but these ideas gave birth to several influential theories in modern algebra, which were instrumental in the recent solution of the problems. One of the main ingredients that was lacking at the time was a robust mechanism to solve equations in free groups and a suitable description of the solution sets of equations. The main technical goal of these lectures is to describe a host of methods that altogether give this mechanism, that we refer to as Elimination Processes.

Also in 1960’s Malcev \[33\] described solutions of the equation\[ zxyz^{-1}y^{-1}z^{-1} = aba^{-1}b^{-1} \] in a free group, which is the simplest non-trivial quadratic equation in groups. The description uses the group of automorphisms of the coordinate group of the equation and the minimal solutions relative to these automorphisms - a very powerful idea, that nowadays is inseparable from the modern approach to equations. The first breakthrough on Tarski’s problem came from Merzljakov (who was a part of Malcev’s school in Novosibirsk). He proved \[49\] a remarkable theorem that any two nonabelian free groups of finite rank have the same positive theory and showed that positive formulas in free groups have definable Skolem functions, thus giving quantifier elimination of positive formulas in free groups to existent formulas. Recall that the positive theory of a group consists of all positive (without negations in their normal forms) sentences that are true in this group. These results were precursors of the current approach to the elementary theory of a free group.

In the eighties new crucial concepts were introduced. Makanin proved \[47\] the algorithmic decidability of the Diophantine problem over free groups, and showed that both, the universal theory and the positive theory of a free group are algorithmically decidable. He created an extremely powerful technique (the Makanin elimination process) to deal with equations over free groups.

Shortly afterwards, Razborov (at the time a PhD student of Steklov’s Institute, where Makanin held a position) described the solution set of an arbitrary system of equations over a free group in terms of what is known now as Makanin-Razborov diagrams \[53\], \[54\].

A few years later Edmunds and Commerford \[17\] and Grigorchuk and Kurchanov \[26\] described solution sets of arbitrary quadratic equations over free groups. These equations came to group theory from topology and their role in group theory was not altogether clear then. Now they form one of the corner-stones of the theory of equations in groups due to their relations to JSJ-decompositions of groups.

1.3 New age

These are milestones of the theory of equations in free groups up to 1998. The last missing principal component in the theory of equations in groups was a general geometric point of view similar to the classical affine algebraic geometry. Back to 1970’s Lyndon (again!) was musing on this subject \[31\].
but for no avail. Finally, in the late 1990’s Baumslag, Kharlampovich, Myasnikov, and Remeslen-
nikov developed the basics of the algebraic geometry over groups [3, 38, 37, 38, 35], introducing
analogous of the standard algebraic geometry notions such as algebraic sets, the Zariski topology,
Noetherian domains, irreducible varieties, radicals and coordinate groups, rational equivalence, etc.

With all this machinery in place it became possible to make the next crucial step and tie the
algebraic geometry over groups, Makanin-Razborov process for solving equations, and Lyndon’s
free \( Z[t] \)-exponential group \( F^{\mathbb{Z}[t]} \) into one closely related theory. The corner stone of this theory
is Decomposition Theorem from [37] (see Section 4.2 below) which describes the solution sets of
systems of equations in free groups in terms of non-degenerate triangular quasi-quadratic (NTQ)
systems. The coordinate groups of the NTQ systems (later became also known as residually free
towers) play a fundamental role in the theory of fully residually free groups, as well as in the
elementary theory of free groups. The Decomposition Theorem allows one to look at the processes
of the Makanin-Razborov’s type as non-commutative analogs of the classical elimination processes
from algebraic geometry. With this in mind we refer to such processes in all their variations as
Elimination Processes (EP).

In the rest of the notes we discuss more developments of the theory, focusing mostly on the
elimination processes, fully residually free (limit) groups, and new techniques that appear here.

## 2 Basic notions of algebraic geometry over groups

Following [3] and [38] we introduce here some basic notions of algebraic geometry over groups.

Let \( G \) be a group generated by a finite set \( A, F(X) \) be a free group with basis \( X = \{ x_1, x_2, \ldots, x_n \} \),
we defined \( G[X] = G \ast F(X) \) to be a free product of \( G \) and \( F(X) \). If \( S \subset G[X] \) then the expression
\( S = 1 \) is called a system of equations over \( G \). As an element of the free product, the left side of every
equation in \( S = 1 \) can be written as a product of some elements from \( X \cup X^{-1} \) (which are
called variables) and some elements from \( A \) (constants). To emphasize this we sometimes write
\( S(X,A) = 1 \).

A solution of the system \( S(X) = 1 \) over a group \( G \) is a tuple of elements \( g_1, \ldots, g_n \in G \) such
that after replacement of each \( x_i \) by \( g_i \) the left hand side of every equation in \( S = 1 \) turns into the
trivial element of \( G \). To study equations over a given fixed group \( G \) it is convenient to consider
the category of \( G \)-groups, i.e., groups which contain the group \( G \) as a distinguished subgroup. If \( H 
and \( K \) are \( G \)-groups then a homomorphism \( \phi : H \rightarrow K \) is a \( G \)-homomorphism if \( g^\phi = g \) for every
\( g \in G \), in this event we write \( \phi : H \rightarrow_G K \). In this category morphisms are \( G \)-homomorphisms;
subgroups are \( G \)-subgroups, etc. A solution of the system \( S = 1 \) over \( G \) can be described as a
\( G \)-homomorphism \( \phi : G[X] \rightarrow G \) such that \( \phi(S) = 1 \). Denote by \( \mathrm{ncl}(S) \) the normal closure of \( S \) in
\( G[X] \), and by \( G_S \) the quotient group \( G[X]/\mathrm{ncl}(S) \). Then every solution of \( S(X) = 1 \) in \( G \) gives rise
to a \( G \)-homomorphism \( G_S \rightarrow G \), and vice versa. By \( V_G(S) \) we denote the set of all solutions in \( G \)
of the system \( S = 1 \), it is called the algebraic set defined by \( S \). This algebraic set \( V_G(S) \) uniquely
corresponds to the normal subgroup

\[
R(S) = \{ T(x) \in G[X] \mid \forall A \in G^n(S(A) = 1 \Rightarrow T(A) = 1) \}
\]

of the group \( G[X] \). Notice that if \( V_G(S) = \emptyset \), then \( R(S) = G[X] \). The subgroup \( R(S) \) contains \( S \),
and it is called the radical of \( S \). The quotient group

\[
G_{R(S)} = G[X]/R(S)
\]

is the coordinate group of the algebraic set \( V(S) \). Again, every solution of \( S(X) = 1 \) in \( G \) can be
described as a \( G \)-homomorphism \( G_{R(S)} \rightarrow G \).

By \( \text{Hom}_G(H,K) \) we denote the set of all \( G \)- homomorphisms from \( H \) into \( K \). It is not hard to
see that the free product \( G \ast F(X) \) is a free object in the category of \( G \)-groups. This group is called
a free $G$-group with basis $X$, and we denote it by $G[X]$. A $G$-group $H$ is termed finitely generated $G$-group if there exists a finite subset $A \subset H$ such that the set $G \cup A$ generates $H$. We refer to [3] for a general discussion on $G$-groups.

A group $G$ is called a CSA group if every maximal abelian subgroup $M$ of $G$ is malnormal, i.e., $M^g \cap M = 1$ for any $g \in G - M$. The abbreviation CSA means conjugacy separability for maximal abelian subgroups. The class of CSA-groups is quite substantial. It includes all abelian groups, all torsion-free hyperbolic groups, all groups acting freely on $A$-trees and many one-relator groups (see, for example, [25]).

We define a Zariski topology on $G^n$ by taking algebraic sets in $G^n$ as a sub-basis for the closed sets of this topology. Namely, the set of all closed sets in the Zariski topology on $G^n$ can be obtained from the set of algebraic sets in two steps:
1) take all finite unions of algebraic sets;
2) take all possible intersections of the sets obtained in step 1).

If $G$ is a non-abelian CSA group and we in the category of $G$-groups, then the union of two algebraic sets is again algebraic. Indeed, if \{\(w_i = 1, i \in I\)\} and \{\(u_j = 1, j \in J\)\} are systems of equations, then in a CSA group their disjunction is equivalent to a system
\[
[w_i, u_j] = [w_i, u_j^a] = [w_i, u_j^b] = 1, \ i \in I, \ j \in J
\]
for any two non-commuting elements $a, b$ from $G$. Therefore the closed sets in the Zariski topology on $G^n$ are precisely the algebraic sets.

A group $G$ is called equationally Noetherian if every system $S(X) = 1$ with coefficients from $G$ is equivalent over $G$ to a finite subsystem $S_0 = 1$, where $S_0 \subset S$, i.e., $V_G(S) = V_G(S_0)$. It is known that linear groups (in particular, freely discriminated groups) are equationally Noetherian (see [24], [10], [5]). If $G$ is equationally Noetherian then the Zariski topology on $G^n$ is Noetherian for every $n$, i.e., every proper descending chain of closed sets in $G^n$ is finite. This implies that every algebraic set $V$ in $G^n$ is a finite union of irreducible subsets (they are called irreducible components of $V$), and such decomposition of $V$ is unique. Recall that a closed subset $V$ is irreducible if it is not a union of two proper closed (in the induced topology) subsets.

### 3 Fully residually free groups

#### 3.1 Definitions and elementary properties

Finitely generated fully residually free groups (limit groups) play a crucial role in the theory of equations and first-order formulas over a free group. It is remarkable that these groups, which have been widely studied before, turn out to be the basic objects in newly developing areas of algebraic geometry and model theory of free groups. Recall that a group $G$ is called fully residually free (or freely discriminated, or $\omega$-residually free) if for any finitely many non-trivial elements $g_1, \ldots, g_n \in G$ there exists a homomorphism $\phi$ of $G$ into a free group $F$, such that $\phi(g_i) \neq 1$ for $i = 1, \ldots, n$. The next proposition summarizes some simple properties of fully residually free groups.

**Proposition 1.** Let $G$ be a fully residually free group. Then $G$ possesses the following properties.

1. $G$ is torsion-free;
2. Each subgroup of $G$ is a fully residually free group;
3. $G$ has the CSA property;
4. Each Abelian subgroup of $G$ is contained in a unique maximal finitely generated Abelian subgroup, in particular, each Abelian subgroup of $G$ is finitely generated;
5. $G$ is finitely presented, and has only finitely many conjugacy classes of its maximal Abelian subgroups.

6. $G$ has solvable word problem;

7. $G$ is linear;

8. Every 2-generated subgroup of $G$ is either free or abelian;

9. If rank $(G)=3$ then either $G$ is free of rank 3, free abelian of rank 3, or a free rank one extension of centralizer of a free group of rank 2 (that is $G = \langle x, y, t | [u(x, y), t] = 1 \rangle$, where the word $u$ is not a proper power).

Properties 1 and 2 follow immediately from the definition of an $F$-group. A proof of property 3 can be found in [4]; property 4 is proven in [37]. Properties 3 and 4 are proved in [37]. Solvability of the word problem follows from [48] or from residual finiteness of a free group. Property 9 is proved in [22]. Property 7 follows from linearity of $F$ and property 6 in the next proposition. The ultraproduct of $SL_2(\mathbb{Z})$ is $SL_2(\ast \mathbb{Z})$, where $\ast \mathbb{Z}$ is the ultrapower of $\mathbb{Z}$.

Proposition 2. (no coefficients) Let $G$ be a finitely generated group. Then the following conditions are equivalent:

1) $G$ is freely discriminated (that is for finitely many non-trivial elements $g_1, \ldots, g_n \in G$ there exists a homomorphism $\phi$ from $G$ to a free group such that $\phi(g_i) \neq 1$ for $i = 1, \ldots, n$);

2) [Remeslennikov] $G$ is universally equivalent to $F$ (in the language without constants);

3) [Baumslag, Kharlampovich, Myasnikov, Remeslennikov] $G$ is the coordinate group of an irreducible variety over a free group.

4) [Sela] $G$ is a limit group (to be defined in the proof of proposition 3).

5) [Champetier and Guirardel] $G$ is a limit of free groups in Gromov-Hausdorff metric (to be defined in the proof of proposition 3).

6) $G$ embeds into an ultrapower of free groups.

Proposition 3. (with coefficients) Let $G$ be a finitely generated group containing a free non-abelian group $F$ as a subgroup. Then the following conditions are equivalent:

1) $G$ is $F$-discriminated by $F$;

2) [Remeslennikov] $G$ is universally equivalent to $F$ (in the language with constants);

3) [Baumslag, Kharlampovich, Myasnikov, Remeslennikov] $G$ is the coordinate group of an irreducible variety over $F$.

4) [Sela] $G$ is a restricted limit group.

5) [Champetier and Guirardel] $G$ is a limit of free groups in Gromov-Hausdorff metric.

6) $GF$-embeds into an ultrapower of $F$.
We will prove Proposition 3, the proof of Proposition 2 is very similar. We will first prove the equivalence $1)\iff 2)$. Let $L_A$ be the language of group theory with generators $A$ of $F$ as constants. Let $G$ be a f.g. group which is $F$-discriminated by $F$. Consider a formula
\[ \exists X(U(X, A) = 1 \land W(X, A) \neq 1). \]
If this formula is true in $F$, then it is also true in $G$, because $F \leq G$. If it is true in $G$, then for some $X \in G^n$ holds $U(X, A) = 1$ and $W(X, A) \neq 1$. Since $G$ is $F$-discriminated by $F$, there is an $F$-homomorphism $\phi: G \to F$ such that $\phi(W(X, A)) \neq 1$, i.e. $W(X^\phi, A) \neq 1$. Of course $U(X^\phi, A) = 1$. Therefore the above formula is true in $F$. Since in $F$-group a conjunction of equations [inequalities] is equivalent to one equation [resp., inequality], the same existential sentences in the language $L_A$ are true in $G$ and in $F$.

Suppose now that $G$ is $F$-universally equivalent to $F$. Let $G = \langle X, A \mid S(X, A) = 1 \rangle$, be a presentation of $G$ and $w_1(X, A), \ldots, w_k(X, A)$ nontrivial elements in $G$. Let $Y$ be the set of the same cardinality as $X$. Consider a system of equations $S(Y, A) = 1$ in variables $Y$ in $F$. Since the group $F$ is equationally Noetherian, this system is equivalent over $F$ to a finite subsystem $S_1(Y, A) = 1$. The formula
\[ \Psi = \forall Y(S_1(Y, A) = 1 \rightarrow (w_1(Y, A) = 1 \lor \cdots \lor w_k(Y, A) = 1)). \]
is false in $G$, therefore it is false in $F$. This means that there exists a set of elements $B$ in $F$ such that $S_1(B, A) = 1$ and, therefore, $S(B, A) = 1$ such that $w_1(B, A) \neq 1 \land \cdots \land w_k(B, A) \neq 1$. The map $X \to B$ that is identical on $F$ can be extended to the $F$-homomorphism from $G$ to $F$.

1)$\iff 3)$ Let $H$ be an equationally Noetherian CSA-group. We will prove that $V(S)$ is irreducible if and only if $H_{R(S)}$ is discriminated in $H$ by $H$-homomorphisms.

Suppose $V(S)$ is not irreducible and $V(S) = V(S_1) \cup V(S_2)$ is its decomposition into proper subvarieties. Then there exist $s_i \in R(S_i) \setminus R(S_j), j \neq i$. The set $\{s_1, s_2\}$ cannot be discriminated in $H$ by $H$-homomorphisms.

Suppose now $s_1, \ldots, s_n$ are elements such that for any retract $f: H_{R(S)} \to H$ there exists $i$ such that $f(s_i) = 1$: then $V(S) = \bigcup_{i=1}^n V(S \cup s_i)$. \hfill \Box

Sela defined limit groups as follows. Let $G$ be a f.g. group and let $\{\phi_j\}$ be a sequence of homomorphisms from $G$ to a free group $F$ belonging to distinct conjugacy classes (distinct $F$-homomorphisms belong to distinct conjugacy classes).

$F$ acts by isometries on its Cayley graph $X$ which is a simplicial tree. Hence, there is a sequence of actions of $G$ on $X$ corresponding to $\{\phi_j\}$.

By rescaling metric on $X$ for each $\phi_j$ one obtains a sequence of simplicial trees $\{X_j\}$ and a corresponding sequence of actions of $G$. $\{X_j\}$ converges to a real tree $Y$ (Gromov-Hausdorff limit) endowed with an isometric action of $G$. The kernel of the action of $G$ on $Y$ is defined as
\[ K = \{g \in G \mid gy = y, \forall y \in Y\}. \]

Finally, $G/K$ is said to be the limit group (corresponding to $\{\phi_j\}$ and rescaling constants). We will prove now the equivalence $1)\iff 4)$. A slight modification of the proof below should be made to show that limit groups are exactly f.g. fully residually free groups.

Suppose that $G = \langle g_1, \ldots, g_k \rangle$ is f.g. and discriminated by $F$. There exists a sequence of homomorphisms $\phi_n : G \to F$, so that $\phi_n$ maps the elements in a ball of radius $n$ in the Cayley graph of $G$ to distinct elements in $F$. By rescaling the metric on $F$, we obtain a subsequence of homomorphisms $\phi_m$ which converges to an action of a limit group $L$ on a real tree $Y$. In general, $L$ is a quotient of $G$, but since the homomorphisms were chosen so that $\phi_n$ maps a ball of radius $n$ monomorphically into $F$, $G$ is isomorphic to $L$ and, therefore, $G$ is a limit group.

To prove the converse, we need the fact (first proved in [50]) that a f.g. limit group is finitely presented. We may assume further that a limit group $G$ is non-abelian because the statement is,
obviously, true for abelian groups. By definition, there exists a f.g. group $H$, an integer $k$ and a sequence of homomorphisms $h_k : H \to F$, so that the limit of the actions of $H$ on the Cayley graph of $F$ via the homomorphisms $h_k$ is a faithful action of $G$ on some real tree $Y$. Since $G$ is finitely presented for all but finitely many $n$, the homomorphism $h_n$ splits through the limit group $G$, i.e. $h_n = \phi\psi_n$, where $\phi : H \to G$ is the canonical projection map, and the $\psi_n$’s are homomorphisms $\psi_n : G \to F$. If $g \neq 1$ in $G$, then for all but finitely many $\psi_n$’s $g^{\psi_n} \neq 1$. Hence, for every finite set of elements $g_1, \ldots, g_m \neq 1$ in $G$ for all but finitely many indices $n$, $g_1^{\psi_n}, \ldots, g_m^{\psi_n} \neq 1$, so $G$ is $F$-discriminated. □

The equivalence 2) $\iff$ 6) is a particular case of general results in model theory (see for example [4] Lemma 3.8 Chap.9). □

5) $\iff$ 6). Champetier and Guirardel [15] used another approach to limit groups.

A marked group $(G, S)$ is a group $G$ with a prescribed family of generators $S = (s_1, \ldots, s_n)$.

Two marked groups $(G, (s_1, \ldots, s_n))$ and $(G', (s'_1, \ldots, s'_n))$ are isomorphic as marked groups if the bijection $s_i \leftrightarrow s'_i$ extends to an isomorphism. For example, $((a), (1, a))$ and $((a), (a, 1))$ are not isomorphic as marked groups. Denote by $\mathcal{G}_n$ the set of groups marked by $n$ elements up to isomorphism of marked groups.

One can define a metric on $\mathcal{G}_n$ by setting the distance between two marked groups $(G, S)$ and $(G', S')$ to be $e^{-N}$ if they have exactly the same relations of length at most $N$ (under the bijection $S \leftrightarrow S'$).

Finally, a limit group in their terminology is a limit (with respect to the metric above) of marked free groups in $\mathcal{G}_n$.

It is shown in [15] that a group is a limit group if and only if it is a finitely generated subgroup of an ultraproduct of free groups (for a non-principal ultrafilter), and any such ultraproduct of free groups contains all the limit groups. This implies the equivalence 5) $\iff$ 6). □

Notice that ultrapowers of a free group have the same elementary theory as a free group by Los’ theorem.

First non-free finitely generated examples of fully residually free groups, that include all non-exceptional surface groups, appeared in [2], [3]. They obtained fully residually free groups as subgroups of free extensions of centralizers in free groups.

### 3.2 Lyndon’s completion $F^{\mathbb{Z}[t]}$

Studying equations in free groups Lyndon [32] introduced the notion of a group with parametric exponents in an associative unitary ring $R$. It can be defined as a union of the chain of groups

$$F = F_0 < F_1 < \cdots < F_n < \cdots,$$

where $F = F(X)$ is a free group on an alphabet $X$, and $F_k$ is generated by $F_{k-1}$ and formal expressions of the type

$$\{w^\alpha \mid w \in F_{k-1}, \alpha \in R\}.$$

That is, every element of $F_k$ can be viewed as a parametric word of the type

$$w_1^{\alpha_1}w_2^{\alpha_2}\cdots w_m^{\alpha_m},$$

where $m \in \mathbb{N}$, $w_i \in F_{k-1}$, and $\alpha_i \in R$. In particular, he described free exponential groups $F^{\mathbb{Z}[t]}$ over the ring of integer polynomials $\mathbb{Z}[t]$. Notice that ultrapowers of free groups are operator groups over ultraproducts of $\mathbb{Z}$.

In the same paper Lyndon proved an amazing result that $F^{\mathbb{Z}[t]}$ is fully residually free. Hence all subgroups of $F^{\mathbb{Z}[t]}$ are fully residually free. Lyndon showed that solution sets of one variable equations can be described in terms of parametric words. Later it was shown in [4] that coordinate groups of irreducible one-variable equations are just extensions of centralizers in $F$ of rank one (see the definition in the second paragraph below). In fact, this result is not entirely accidental,
extensions of centralizers play an important part here. Recall that Baumslag \[2\] already used them in proving that surface groups are freely discriminated.

Now, breaking the natural march of history, we go ahead of time and formulate one crucial result which justifies our discussion on Lyndon’s completion $F^{Z[t]}$ and highlights the role of the group $F^{Z[t]}$ in the whole subject.

**Theorem** (The Embedding Theorem \([37,38]\)) Given an irreducible system $S = 1$ over $F$ one can effectively embed the coordinate group $F_{R(S)}$ into $F^{Z[t]}$.

A modern treatment of exponential groups was done by Myasnikov and Remeslennikov \([33]\) who proved that the group $F^{Z[t]}$ can be obtained from $F$ by an infinite chain of HNN-extensions of a very specific type, so-called *extensions of centralizers*:

$$F = G_0 < G_1 < \ldots < \ldots \cup G_i = F^{Z[t]}$$

where

$$G_{i+1} = \langle G_i, t_i | [C_{G_i}(u_i), t_i] = 1 \rangle.$$ (extension of the centralizer $C_{G_i}(u_i)$, where $u_i \in G_i$).

This implies that finitely generated subgroups of $F^{Z[t]}$ are, in fact, subgroups of $G_i$. Since $G_i$ in an HNN-extension, one can apply Bass-Serre theory to describe the structure of these subgroups. In fact, f.g. subgroups of $G_i$ are fundamental groups of graphs of groups induced by the HNN structure of $G_i$. For instance, it is routine now to show that all such subgroups $H$ of $G_i$ are finitely presented. Indeed, we only have to show that the intersections $G_{i-1} \cap H^G$ are finitely generated. Notice, that if in the amalgamated product amalgamated subgroups are finitely generated and one of the factors is not, then the amalgamated product is not finitely generated (this follows from normal forms of elements in the amalgamated products). Similarly, the base group of a f.g. HNN extension with f.g. associated subgroups must be f.g. Earlier Pfander \([51]\) proved that f.g. subgroups of the free $Z[t]$-group on two generators are finitely presented. Description of f.g. subgroups of $F^{Z[t]}$ as fundamental groups of graphs of groups implies immediately that such groups have non-trivial abelian splittings (as amalgamated product or HNN extension with abelian edge group which is maximal abelian in one of the base subgroups). Furthermore, these groups can be obtained from free groups by finitely many free constructions (see next section).

The original Lyndon’s result on fully residual freeness of $F^{Z[t]}$ gives decidability of the Word Problem in $F^{Z[t]}$, as well as in all its subgroups. Since any fully residually free group given by a finite presentation with relations $S$ can be presented as the coordinate group $F_{R(S)}$ of a coefficient-free system $S = 1$. The Embedding Theorem then implies decidability of WP in arbitrary f.g. fully residually free group.

The Conjugacy Problem is also decidable in $F^{Z[t]}$ - but this was proved much later, by Ribes and Zalesski in \([38]\). A similar, but stronger, result is due to Lyutikova who showed in \([36]\) that the Conjugacy Problem in $F^{Z[t]}$ is residually free, i.e., if two elements $g, h$ are not conjugate in $F^{Z[t]}$ (or in $G_i$) then there is an $F$-epimorphism $\phi : F^{Z[t]} \to F$ such that $\phi(g)$ and $\phi(h)$ are not conjugate in $F$. Unfortunately, this does not imply immediately that the CP in subgroups of $F^{Z[t]}$ is also residually free, since two elements may be not conjugated in a subgroup $H \leq F^{Z[t]}$, but conjugated in the whole group $F^{Z[t]}$. We discuss CP in arbitrary f.g. fully res. free groups in Section 5.

4 **Main results in \([37]\)**

4.1 **Structure and embeddings**

In 1996 we proved the converse of the Lyndon’s result mentioned above, every finitely generated fully residually free group is embeddable into $F^{Z[t]}$. 

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Theorem 3. [37, 38] Given an irreducible system $S = 1$ over $F$ one can effectively embed the coordinate group $F_{R(S)}$ into $F^{Z[t]}$ i.e., one can find $n$ and an embedding $F_{R(S)} \to G_n$ into an iterated centralizer extension $G_n$.

Corollary 1. For every freely indecomposable non-abelian finitely generated fully residually free group one can effectively find a non-trivial splitting (as an amalgamated product or HNN extension) over a cyclic subgroup.

Corollary 2. Every finitely generated fully residually free group is finitely presented. There is an algorithm that, given a presentation of a f.g. fully residually free group $G$ and generators of the subgroup $H$, finds a finite presentation for $H$.

Corollary 3. Every finitely generated residually free group $G$ is a subgroup of a direct product of finitely many fully residually free groups; hence, $G$ is embeddable into $F^{Z[t]} \times \ldots \times F^{Z[t]}$. If $G$ is given as a coordinate group of a finite system of equations, then this embedding can be found effectively.

Indeed, there exists a finite system of coefficient free equations $S = 1$ such that $G$ is a coordinate group of this system, and $\text{acl}(S) = R(S)$. If $V(S) = \bigcup_{i=1}^{n} V(S_i)$ is a representation of $V(S)$ as a union of irreducible components, then $R(S) = \bigcap_{i=1}^{n} R(S_i)$ and $G$ embeds into a direct product of coordinate groups of systems $S_i = 1$, $i = 1, \ldots, n$.

This allows one to study the coordinate groups of irreducible systems of equations (fully residually free groups) via their splittings into graphs of groups. This also provides a complete description of finitely generated fully residually free groups and gives a lot of information about their algebraic structure. In particular, they act freely on $\mathbb{Z}^n$-trees, and all these groups, except for abelian and surface groups, have a non-trivial cyclic JSJ-decomposition.

Let $K$ be an HNN-extension of a group $G$ with associated subgroups $A$ and $B$. $K$ is called a separated HNN-extension if for any $g \in G$, $A^g \cap B = 1$.

Corollary 4. Let a group $G$ be obtained from a free group $F$ by finitely many centralizer extensions. Then every f. g. subgroup $H$ of $G$ can be obtained from free abelian groups of finite rank by finitely many operations of the following type: free products, free products with abelian amalgamated subgroups at least one of which is a maximal abelian subgroup in its factor, free extensions of centralizers, separated HNN-extensions with abelian associated subgroups at least one of which is maximal.

Corollary 5. (Groves, Wilton [27]) One can enumerate all finite presentations of fully residually free groups.

Theorem 3 is proved as a corollary of Theorem 6 below.

Corollary 6. Every f.g. fully residually free group acts freely on some $\mathbb{Z}^n$-tree with lexicographic order for a suitable $n$.

Hence, a simple application of the change of the group functor shows that $H$ also acts freely on an $\mathbb{R}^n$-tree. Recently, Guirardel proved the latter result independently using different techniques [23]. It is worthwhile to mention here that free group actions on $\mathbb{Z}^n$-trees give a tremendous amount of information on the group and its subgroups, especially with regard to various algorithmic problems (see Section 5).

Notice, that there are f.g. groups acting freely on $\mathbb{Z}^n$-trees which are not fully residually free (see conjecture (2) from Sela’s list of open problems). The simplest example is the group of closed non-orientable surface of genus 3. In fact, the results in [44, 45] show that there are very many groups like that - the class of groups acting freely on $\mathbb{Z}^n$-trees is much wider than the class of fully residually free groups. This class deserves a separate discussion, for which we refer to [44, 45]. Combining Corollary 4 with the results from [39] or [7] we proved in [37] that f.g. fully residually
free groups without subgroups $\mathbb{Z} \times \mathbb{Z}$ (or equivalently, with cyclic maximal abelian subgroups) are hyperbolic. We will see in Section 4.2 that this has some implication on the structure of the models of the $\forall \exists$-theory of a given non-abelian free group. Later, Dahmani [20] proved a generalization of this, namely, that an arbitrary f.g. fully residually free group is hyperbolic relative to its maximal abelian non-cyclic subgroups.

Recently N. Touikan described coordinate groups of two-variable equations [60].

4.2 Triangular quasi-quadratic systems

We use an Elimination Process to transform systems of equations. Elimination Process EP is a symbolic rewriting process of a certain type that transforms formal systems of equations in groups or semigroups. Makanin (1982) introduced the initial version of the EP. This gives a decision algorithm to verify consistency of a given system - decidability of the Diophantine problem over free groups. He estimates the length of the minimal solution (if it exists). Makanin introduced the fundamental notions: generalized equations, elementary and entire transformations, notion of complexity. Razborov (1987) developed EP much further. Razborov’s EP produces all solutions of a given system in $F$. He used special groups of automorphisms, and fundamental sequences to encode solutions.

We obtained in 1996 [37] an effective description of solutions of equations in free (and fully residually free) groups in terms of very particular triangular systems of equations. First, we give a definition.

**Triangular quasi-quadratic (TQ) system** is a finite system that has the following form

\[
S_1(X_1, X_2, \ldots, X_n, A) = 1, \\
S_2(X_2, \ldots, X_n, A) = 1, \\
\ldots \\
S_n(X_n, A) = 1
\]

where either $S_i = 1$ is quadratic in variables $X_i$, or $S_i = 1$ is a system of commutativity equations for all variables from $X_i$ and, in addition, equations $[x, u] = 1$ for all $x \in X_i$ and some $u \in F_{R(S_{i+1}, \ldots, S_n)}$ or $S_i$ is empty.

A TQ system above is non-degenerate (NTQ) if for every $i$, $S_i(X_1, \ldots, X_n, A) = 1$ has a solution in the coordinate group $F_{R(S_{i+1}, \ldots, S_n)}$, i.e., $S_i = 1$ (in algebraic geometry one would say that a solution exists in a generic point of the system $S_{i+1} = 1, \ldots, S_n = 1$).

We proved in [38] (see also [39]) that NTQ systems are irreducible and, therefore, their coordinate groups (NTQ groups) are fully residually free. (Later Sela called NTQ groups $\omega$-residually free towers [56].)

We represented a solution set of a system of equations canonically as a union of solutions of a finite family of NTQ groups.

**Theorem 4.** [37], [38] One can effectively construct EP that starts on an arbitrary system

\[
S(X, A) = 1
\]

and results in finitely many NTQ systems

\[
U_1(Y_1) = 1, \ldots, U_m(Y_m) = 1
\]

such that

\[
V_F(S) = P_1(V(U_1)) \cup \ldots \cup P_m(V(U_m))
\]

for some word mappings $P_1, \ldots, P_m$. ($P_i$ maps a tuple $Y_i \in V(U_i)$ to a tuple $X \in V(S)$. One can think about $P_i$ as an $A$-homomorphism from $F_{R(U_i)}$ into $F_{R(S)}$, then any solution $\psi : F_{R(U_i)} \to F$ pre-composed with $P_i$ gives a solution $\phi : F_{R(S)} \to F$. )
Our elimination process can be viewed as a non-commutative analog of the classical elimination process in algebraic geometry.

Hence, going “from the bottom to the top” every solution of the subsystem $S_n = 1, \ldots S_i = 1$ can be extended to a solution of the next equation $S_{i-1} = 1$.

**Theorem 5.** [37], [38] All solutions of the system of equations $S = 1$ in $F(A)$ can be effectively represented as homomorphisms from $F_{R(S)}$ into $F(A)$ encoded into the following finite canonical Hom-diagram. Here all groups, except, maybe, the one in the root, are fully residually free, (given by a finite presentation) arrows pointing down correspond to epimorphisms (defined effectively in terms of generators) with non-trivial kernels, and loops correspond to automorphisms of the coordinate groups.

A family of homomorphisms encoded in a path from the root to a leaf of this tree is called a fundamental sequence or fundamental set of solutions (because each homomorphism in the family is a composition of a sequence of automorphisms and epimorphisms). Later Sela called such family a resolution. Therefore the solution set of the system $S = 1$ consists of a finite number of fundamental sets. And each fundamental set ”factors through” one of the NTQ systems from Theorem 4. If $S = 1$ is irreducible, or, equivalently, $G = F_{R(S)}$ is fully residually free, then, obviously, one of the fundamental sets discriminates $G$. This gives the following result.

**Theorem 6.** [37], [38] Finitely generated fully residually free groups are subgroups of coordinate groups of NTQ systems. There is an algorithm to construct an embedding.

This corresponds to the extension theorems in the classical theory of elimination for polynomials. In [36] we have shown that an NTQ group can be embedded into a group obtained from a free group by a series of extensions of centralizers. Therefore Theorem 3 follows from Theorem 6.

Since NTQ groups are fully residually free, fundamental sets corresponding to different NTQ groups in Theorem 4 discriminate fully residually free groups which are coordinate groups of irreducible components of system $S(X, A) = 1$. This implies

**Theorem 7.** [37], [38] There is an algorithm to find irreducible components for a system of equations over a free group.
Now we will formulate a technical result which is the keystone in the proof of Theorems 4 and 5. An elementary abelian splitting of a group is the splitting as an amalgamated product or HNN-extension with abelian edge group. Let \( G = A \ast_C B \) be an elementary abelian splitting of \( G \). For \( c \in C \) we define an automorphism \( \phi_c : G \to G \) such that \( \phi_c(a) = a \) for \( a \in A \) and \( \phi_c(b) = b^c = c^{-1}bc \) for \( b \in B \).

If \( G = A \ast_C = \langle A, a, c \rangle = \langle c', c \in C \rangle \) then for \( c \in C \) define \( \phi_c : G \to G \) such that \( \phi_c(a) = a \) for \( a \in A \) and \( \phi_c(t) = ct \).

We call \( \phi_c \) a Dehn twist obtained from the corresponding elementary abelian splitting of \( G \). If \( G \) is an \( F \)-group, where \( F \) is a subgroup of one of the factors \( A \) or \( B \), then Dehn twists that fix elements of the free group \( F \leq A \) are called canonical Dehn twists.

If \( G = A \ast_C B \) and \( B \) is a maximal abelian subgroup of \( G \), then every automorphism of \( B \) acting trivially on \( C \) can be extended to the automorphism of \( G \) acting trivially on \( A \). The subgroup of \( \text{Aut}(G) \) generated by such automorphisms and canonical Dehn twists is called the group of canonical automorphisms of \( G \).

Let \( G \) and \( K \) be \( H \)-groups and \( A \leq \text{Aut}_H(G) \) a group of \( H \)-automorphisms of \( G \). Two \( H \)-homomorphisms \( \phi \) and \( \psi \) from \( G \) into \( K \) are called \( A \)-equivalent (symbolically, \( \sim_A \)) if there exists \( \sigma \in A \) such that \( \phi = \sigma \psi \) (i.e., \( g^\phi = g^{\sigma \psi} \) for \( g \in G \)). Obviously, \( \sim_A \) is an equivalence relation on \( \text{Hom}_H(G, K) \).

Let \( G \) be a fully residually \( F \)-group \( (F = F(A) \leq G) \) generated by a finite set \( X \) (over \( F \)) and \( A \) the group of canonical \( F \)-automorphisms of \( G \). Let \( \bar{F} = F(A \cup Y) \) a free group with basis \( A \cup Y \) (here \( Y \) is an arbitrary set) and \( \phi_1, \phi_2 \in \text{Hom}_F(G, \bar{F}) \). We write \( \phi_1 < \phi_2 \) if there exists an automorphism \( \sigma \in A \) and an \( F \)-endomorphism \( \pi \in \text{Hom}_F(\bar{F}, \bar{F}) \) such that \( \phi_2 = \sigma^{-1} \phi_1 \pi \) and

\[
\sum_{x \in X} |x^{\phi_1}| < \sum_{x \in X} |x^{\phi_2}|.
\]

Figure 1: \( \phi_1 < \phi_2 \)

An \( F \)-homomorphism \( \phi : G \to \bar{F} \) is called minimal if there is no \( \phi_1 \) such that \( \phi_1 < \phi \). In particular, if \( S(X, A) = 1 \) is a system of equations over \( F = F(A) \) and \( G = F_{R(S)} \) then \( X \cup A \) is a generating set for \( G \) over \( F \). In this event, one can consider minimal solutions of \( S = 1 \) in \( \bar{F} \).

**Definition 1.** Denote by \( R_A \) the intersection of the kernels of all minimal (with respect to \( A \)) \( F \)-homomorphisms from \( \text{Hom}_F(G, \bar{F}) \). Then \( G/R_A \) is called the maximal standard quotient of \( G \) and the canonical epimorphism \( \eta : G \to G/R_A \) is the canonical projection.

**Theorem 8.** [37] The maximal standard quotient of a finitely generated fully residually free group is a proper quotient and can be effectively constructed.

This result (without the algorithm) is called the "shortening argument" in Sela's approach.
5 Elimination process

Given a system \( S(X) = 1 \) of equations in a free group \( F(A) \) one can effectively construct a finite set of generalized equations

\[ \Omega_1, \ldots, \Omega_k \]

(systems of equations of a particular type) such that:

- given a solution of \( S(X) = 1 \) in \( F(A) \) one can effectively construct a reduced solution of one of \( \Omega_i \) in the free semigroup with basis \( A \cup A^{-1} \).

- given a solution of some \( \Omega_i \) in the free semigroup with basis \( A \cup A^{-1} \) one can effectively construct a solution of \( S(X) = 1 \) in \( F(A) \).

This is done as follows. First, we replace the system \( S(X) = 1 \) by a system of equations, such that each of them has length 3. This can be easily done by adding new variables. For one equation of length 3 we can construct a generalized equation as in the example below. For a system of equations we construct it similarly (see [37]).

Example. Suppose we have the simple equation \( xyz = 1 \) in a free group. Suppose that we have a solution to this equation denoted by \( x^\phi, y^\phi, z^\phi \) where \( \phi \) is a given homomorphism into a free group \( F(A) \). Since \( x^\phi, y^\phi, z^\phi \) are reduced words in the generators \( A \) there must be complete cancellation. If we take a concatenation of the geodesic subpaths corresponding to \( x^\phi, y^\phi \) and \( z^\phi \) we obtain a path in the Cayley graph corresponding to complete cancellation. This is called a cancellation tree. Then \( x^\phi = \lambda_1 \circ \lambda_2, \ y^\phi = \lambda_2^{-1} \circ \lambda_3 \) and \( z^\phi = \lambda_3^{-1} \circ \lambda_1^{-1} \), where \( u \circ v \) denotes the product of reduced words \( u \) and \( v \) such that there is no cancellation between \( u \) and \( v \). In the case when all the words \( \lambda_1, \lambda_2, \lambda_3 \) are non-empty, the generalized equation would be the interval in Fig. 2.

![Cancellation Tree](image)

Figure 2: From the cancellation tree for the equation \( xyz = 1 \) to the generalized equation \((x^\phi = \lambda_1 \circ \lambda_2, \ y^\phi = \lambda_2^{-1} \circ \lambda_3, \ z^\phi = \lambda_3^{-1} \circ \lambda_1^{-1})\).

Given a generalized equation \( \Omega \) one can apply elementary transformations (there are only finitely many of them) and get a new generalized equation \( \Omega' \). If \( \sigma \) is a solution of \( \Omega \), then elementary
transformation transforms \( \sigma \) into \( \sigma' \).

\[
(\Omega, \sigma) \rightarrow (\Omega', \sigma').
\]

Elimination Process is a branching process such that on each step one of the finite number of elementary transformations is applied according to some precise rules to a generalized equation on this step.

\[
\Omega_0 \rightarrow \Omega_1 \rightarrow \ldots \rightarrow \Omega_k.
\]

From the group theoretic viewpoint the elimination process tells something about the coordinate groups of the systems involved.

This allows one to transform the pure combinatorial and algorithmic results obtained in the elimination process into statements about the coordinate groups.

### 5.1 Generalized equations

**Definition 2.** A combinatorial generalized equation \( \Omega \) consists of the following components:

1. A finite set of bases \( BS = BS(\Omega) \). The set of bases \( M \) consists of \( 2n \) elements \( M = \{\mu_1, \ldots, \mu_{2n}\} \). The set \( M \) comes equipped with two functions: a function \( \varepsilon : M \rightarrow \{1, -1\} \) and an involution \( \Delta : M \rightarrow M \) (that is, \( \Delta \) is a bijection such that \( \Delta^2 = 1 \) is an identity on \( M \)). Bases \( \mu \) and \( \Delta(\mu) \) (or \( \bar{\mu} \)) are called dual bases. We denote bases by letters \( \mu, \lambda, \) etc.

2. A set of boundaries \( BD = BD(\Omega) \). \( BD \) is a finite initial segment of the set of positive integers \( BD = \{1, 2, \ldots, p + 1 + m\} \), where \( m \) is the cardinality of the basis \( A = \{a_1, \ldots, a_m\} \) of the free group \( F = F(A) \). We use letters \( i, j, \) etc. for boundaries.

3. Two functions \( \alpha : BS \rightarrow BD \) and \( \beta : BS \rightarrow BD \). We call \( \alpha(\mu) \) and \( \beta(\mu) \) the initial and terminal boundaries of the base \( \mu \) (or endpoints of \( \mu \)). These functions satisfy the following conditions for every base \( \mu \in BS \): \( \alpha(\mu) < \beta(\mu) \) if \( \varepsilon(\mu) = 1 \) and \( \alpha(\mu) > \beta(\mu) \) if \( \varepsilon(\mu) = -1 \).

4. The set of boundary connections \( (p, \lambda, q) \), where \( p \) is a boundary on \( \lambda \) (between \( \alpha(\lambda) \) and \( \beta(\lambda) \)) and \( q \) is a boundary on \( \Delta(\lambda) \).

For a combinatorial generalized equation \( \Omega \), one can canonically associate a system of equations in variables \( h_1, \ldots, h_p \) over \( F(A) \) (variables \( h_i \) are sometimes called items). This system is called a **generalized equation**, and (slightly abusing the language) we denote it by the same symbol \( \Omega \).

The generalized equation \( \Omega \) consists of the following three types of equations.

1. Each pair of dual bases \( (\lambda, \Delta(\lambda)) \) provides an equation

   \[
   [h_{\alpha(\lambda)}h_{\alpha(\lambda)+1}\cdots h_{\beta(\lambda)-1}]^{\varepsilon(\lambda)} = [h_{\Delta(\lambda)}h_{\Delta(\lambda)+1}\cdots h_{\beta(\lambda)-1}]^{\varepsilon(\Delta(\lambda))}.
   \]

   These equations are called **basic equations**.

2. Every boundary connection \( (p, \lambda, q) \) gives rise to a **boundary equation**

   \[
   [h_{\alpha(\lambda)}h_{\alpha(\lambda)+1}\cdots h_{p-1}] = [h_{\alpha(\Delta(\lambda))}h_{\alpha(\Delta(\lambda))+1}\cdots h_{q-1}],
   \]

   if \( \varepsilon(\lambda) = \varepsilon(\Delta(\lambda)) \) and

   \[
   [h_{\alpha(\lambda)}h_{\alpha(\lambda)+1}\cdots h_{p-1}] = [h_q h_{q+1} \cdots h_{\beta(\Delta(\lambda))-1}]^{-1},
   \]

   if \( \varepsilon(\lambda) = -\varepsilon(\Delta(\lambda)) \).
3. Constant equations: \( h_{\rho+1} = a_1, \ldots, h_{\rho+1+m} = a_m \).

**Remark 1.** We assume that every generalized equation comes associated with a combinatorial one.

Denote by \( F_{R(\Omega)} \) the coordinate group of the generalized equation.

**Definition 3.** Let \( \Omega(h) = \{ L_1(h) = R_1(h), \ldots, L_s(h) = R_s(h) \} \) be a generalized equation in variables \( h = (h_1, \ldots, h_\rho) \). A sequence of reduced nonempty words \( U = (U_1(Z), \ldots, U_\rho(Z)) \) in the alphabet \( (A \cup Z)^{\pm 1} \) is a solution of \( \Omega \) if:

1. all words \( L_i(U), R_i(U) \) are reduced as written,
2. \( L_i(U) = R_i(U), \ i \in [1, s] \).

If we specify a particular solution \( \delta \) of a generalized equation \( \Omega \) then we use a pair \( (\Omega, \delta) \).

It is convenient to visualize a generalized equation \( \Omega \) as follows.

5.2 Elementary transformations

In this section we describe elementary transformations of generalized equations. Let \( \Omega \) be a generalized equation. An elementary transformation \( (ET) \) associates to a generalized equation \( \Omega \) a family of generalized equations \( ET(\Omega) = \{ \Omega_1, \ldots, \Omega_k \} \) and surjective homomorphisms \( \pi_i : F_{R(\Omega)} \to F_{R(\Omega_i)} \) such that for any solution \( \delta \) of \( \Omega \) and corresponding epimorphism \( \pi_\delta : F_{R(\Omega)} \to F \) there exists \( i \in \{1, \ldots, k\} \) and a solution \( \delta_i \) of \( \Omega_i \) such that the following diagram commutes.

\[
\begin{array}{c}
F_{R(\Omega)} \xrightarrow{\pi_i} F_{R(\Omega_i)} \\
\downarrow \pi_\delta \downarrow \pi_{\delta_i} \\
F \\
\end{array}
\]

**ET1** (Cutting a base (see Fig. 3)). Let \( \lambda \) be a base in \( \Omega \) and \( p \) an internal boundary of \( \lambda \) with a boundary connection \( (p, \lambda, q) \). Then we cut the base \( \lambda \) in \( p \) into two new bases \( \lambda_1, \lambda_2 \) and cut \( \lambda \) in \( q \) into the bases \( \bar{\lambda}_1, \bar{\lambda}_2 \).

**ET2** (Transfering a base (see Fig. 4)). If a base \( \lambda \) of \( \Omega \) contains a base \( \mu \) (that is, \( \alpha(\lambda) \leq \alpha(\mu) < \beta(\mu) \leq \beta(\lambda) \)) and all boundaries on \( \mu \) are \( \lambda \)-tied by boundary connections then we transfer \( \mu \) from its location on the base \( \lambda \) to the corresponding location on the base \( \bar{\lambda} \).

**ET3** (Removal of a pair of matched bases (see Fig. 5)). If the bases \( \lambda, \bar{\lambda} \) are matched (that is, \( \alpha(\lambda) = \alpha(\bar{\lambda}), \beta(\lambda) = \beta(\bar{\lambda}) \)) then we remove \( \lambda, \bar{\lambda} \) from \( \Omega \).

**Remark 2.** Observe, that for \( i = 1, 2, 3 \), \( ET_i(\Omega) \) and \( \Omega \) have the same set of variables \( H \), and the identity map \( F[H] \to F[H] \) induces an isomorphism \( F_{R(\Omega)} \to F_{R(\Omega')} \). Moreover, \( \delta \) is a solution of \( \Omega \) if and only if \( \delta \) is a solution of \( \Omega' \).
Figure 3: Elementary transformation (ET1).

(ET4) (Removal of a lone base (see Fig. 6)). Suppose, a base $\lambda$ in $\Omega$ does not intersect any other base, that is, the items $h_{\alpha(\lambda)}, \ldots, h_{\beta(\lambda)-1}$ are contained only on the base $\lambda$. Suppose also that all boundaries in $\lambda$ are $\lambda$-tied, i.e., for every $i$ ($\alpha(\lambda) \leq i \leq \beta - 1$) there exists a boundary $b(i)$ such that $(i, \lambda, b(i))$ is a boundary connection in $\Omega$. Then we remove the pair of bases $\lambda$ and $\bar{\lambda}$ together with all the boundaries $\alpha(\lambda) + 1, \ldots, \beta(\lambda) - 1$ (and rename the rest $\beta(\lambda) - \alpha(\lambda) - 1$ boundaries correspondingly).

We define the homomorphism $\theta : F_{R(\Omega)} \to F_{R(\Omega')} \,$ as follows:

$$\theta(h_j) = h_j \text{ if } j < \alpha(\lambda) \text{ or } j \geq \beta(\lambda)$$

$$\theta(h_i) = \begin{cases} h_{b(i)} \ldots h_{b(i+1)-1}, & \text{if } \varepsilon(\lambda) = \varepsilon(\bar{\lambda}), \\ h_{\bar{b}(i)} \ldots h_{\bar{b}(i+1)-1}, & \text{if } \varepsilon(\lambda) = -\varepsilon(\bar{\lambda}) \end{cases}$$

for $\alpha(\lambda) \leq i \leq \beta(\lambda) - 1$. It is not hard to see that $\theta$ is an isomorphism.

(ET5) (Introduction of a boundary (see Fig. 7)). Suppose a boundary $p$ in a base $\lambda$ is not $\lambda$-tied. The transformation (ET5) $\lambda$-ties it. To this end, suppose $\delta$ is a solution of $\Omega$. Denote $\lambda^\delta$ by $U_\lambda$. 
and let $U'_\lambda$ be the beginning of this word ending at $p$. Then we perform one of the following transformations according to where the end of $U'_\lambda$ on $\lambda$ might be situated:

(a) If the end of $U'_\lambda$ on $\lambda$ is situated on the boundary $q$, we introduce the boundary connection $\langle p, \lambda, q \rangle$. In this case the corresponding homomorphism $\theta_q : F_{R(\Omega)} \rightarrow F_{R(\Omega_q)}$ is induced by the identity isomorphism on $F[H]$. Observe that $\theta_q$ is not necessary an isomorphism.

(b) If the end of $U'_\lambda$ on $\lambda$ is situated between $q$ and $q + 1$, we introduce a new boundary $q'$ between $q$ and $q + 1$ (and rename all the boundaries); introduce a new boundary connection $\langle p, \lambda, q' \rangle$. Denote the resulting equation by $\Omega'_{q'}$. In this case the corresponding homomorphism $\theta_{q'} : F_{R(\Omega)} \rightarrow F_{R(\Omega_{q'})}$ is induced by the map $\theta_{q'}(h) = h$, if $h \neq h_q$, and $\theta_{q'}(h_q) = h_q h_{q' + 1}$. Observe that $\theta_{q'}$ is an isomorphism.

Obviously, the is only a finite number of possibilities such that for any solution $\delta$ one of them takes place.

5.3 Derived transformations and auxiliary transformations

In this section we define complexity of a generalized equation and describe several useful “derived” transformations of generalized equations. Some of them can be realized as finite sequences of elementary transformations, others result in equivalent generalized equations but cannot be realized by finite sequences of elementary moves.

A boundary is open if it is an internal boundary of some base, otherwise it is closed. A section $\sigma = [i, \ldots, i + k]$ is said to be closed if boundaries $i$ and $i + k$ are closed and all the boundaries between them are open.

Sometimes it will be convenient to subdivide all sections of $\Omega$ into active and non-active sections. Constant section will always be non-active. A variable $h_q$ is called free is it meets no base. Free variables are transported to the very end of the interval behind all items in $\Omega$ and become non-active.
Figure 5: Elementary transformation (ET3).

(D1) (Deleting a complete base). A base $\mu$ of $\Omega$ is called complete if there exists a closed section $\sigma$ in $\Omega$ such that $\sigma = [\alpha(\mu), \beta(\mu)]$. Suppose $\mu$ is a complete base of $\Omega$ and $\sigma$ is a closed section such that $\sigma = [\alpha(\mu), \beta(\mu)]$. In this case using ET5, we transfer all bases from $\mu$ to $\bar{\mu}$; using ET4, we remove the lone base $\mu$ together with the section $\sigma(\mu)$.

**Complexity.** Denote by $\rho$ the number of variables $h_i$ in all (active) sections of $\Omega$, by $n = n(\Omega)$ the number of bases in (active) sections of $\Omega$, by $n(\sigma)$ the number of bases in a closed section $\sigma$.

The complexity of an equation $\Omega$ is the number

$$\tau = \tau(\Omega) = \sum_{\sigma \in A\Sigma_{\Omega}} \max\{0, n(\sigma) - 2\},$$

where $A\Sigma_{\Omega}$ is the set of all active closed sections.

(D2) (Linear elimination). Let $\gamma(h_i)$ denote the number of bases met by $h_i$. A base $\mu \in BS(\Omega)$ is called eliminable if at least one of the following holds:

(a) $\mu$ contains an item $h_i$ with $\gamma(h_i) = 1$,

(b) at least one of the boundaries $\alpha(\mu), \beta(\mu)$ is different from $1, \rho + 1$, does not touch any other base (except $\mu$) and is not connected by any boundary connection.

We denote this boundary by $\epsilon$. A linear elimination for $\Omega$ works as follows.

Suppose the base $\mu$ is removable because it satisfies condition (b). We first cut $\mu$ at the nearest to $\epsilon$ $\mu$-connected boundary and denote it by $\tau$. If there is no such a boundary we denote by $\tau$ the other boundary of $\mu$. Then we remove the base obtained from $\mu$ between $\epsilon$ and $\tau$ together with its dual (maybe this part is the whole base $\mu$), and remove the boundary $\epsilon$. Denote the new equation by $\Omega'$. 
Suppose the base is removable because it satisfies condition (a).

Suppose first that $\gamma(h_i) = 1$ for the leftmost item $h_i$ on $\mu$. Denote by $\epsilon$ the left boundary of $h_i$. Let $\tau$ be the nearest to $\epsilon$ $\mu$-connected boundary (or the other terminal boundary of $\mu$ if there are no $\mu$-connected boundaries). We remove the base obtained from $\mu$ between $\epsilon$ and $\tau$ together with its dual (maybe this part is the whole base $\mu$), and remove $h_i$.

We make a mirror reflection of this transformation if $\gamma(h_i) = 1$ for the rightmost item $h_i$ on $\mu$.

Suppose now that $h_i$ is not the leftmost or the rightmost item on $\mu$. Let $\epsilon$ and $\tau$ be the nearest to $h_i$ $\mu$-connected boundaries on the left and on the right of $h_i$ (each of them can be a terminal boundary of $\mu$). We cut $\mu$ at the boundaries $\epsilon$ and $\tau$, remove the base between $\epsilon$ and $\tau$ together with its dual and remove $h_i$.

Lemma 1. Linear elimination does not increase the complexity of $\Omega$, and the number of items decreases. Therefore the linear elimination process stops after finite number of steps.

Proof. The input of the closed sections not containing $\mu$ into the complexity does not change. The section than contained $\mu$ could be divided into two. In all cases except the last one the total number of bases does not increase, therefore the complexity cannot increase too. In the last case the number of bases is increased by two, but the section is divided into two closed sections, and each section contains at least two bases. Therefore the complexity is the same. The number of items every time is decreased by one.

We repeat linear elimination until no eliminable bases are left in the equation. The resulting generalized equation is called a kernel of $\Omega$ and we denote it by $\text{Ker}(\Omega)$. It is easy to see that $\text{Ker}(\Omega)$ does not depend on a particular linear elimination process. Indeed, if $\Omega$ has two different eliminable bases $\mu_1, \mu_2$, and deletion of a part of $\mu_i$ results in an equation $\Omega_i$ then by induction (on the number of eliminations) $\text{Ker}(\Omega_i)$ is uniquely defined for $i = 1, 2$. Obviously, $\mu_1$ is still eliminable in $\Omega_2$, as well as $\mu_2$ is eliminable in $\Omega_1$. Now eliminating $\mu_1$ and $\mu_2$ from $\Omega_2$.
and $\Omega_1$ we get one and the same equation $\Omega_0$. By induction $\text{Ker}(\Omega_1) = \text{Ker}(\Omega_0) = \text{Ker}(\Omega_2)$ hence the result.

The following statement becomes obvious.

**Lemma 2.** The generalized equation $\Omega$ (as a system of equations over $F$) has a solution if and only if $\text{Ker}(\Omega)$ has a solution.

So linear elimination replaces $\Omega$ by $\text{Ker}(\Omega)$.

Let us consider what happens on the group level in the process of linear elimination. This is necessary only for the description of all solutions of the equation.

We say that a variable $h_i$ belongs to the kernel ($h_i \in \text{Ker}(\Omega)$), if either $h_i$ belongs to at least one base in the kernel, or it is constant.

Also, for an equation $\Omega$ by $\overline{\Omega}$ we denote the equation which is obtained from $\Omega$ by deleting all free variables. Obviously,

$$F_{R(\Omega)} = F_{R(\overline{\Omega})} \ast F(\overline{Y})$$

where $\overline{Y}$ is the set of free variables in $\Omega$.

We start with the case when a part of just one base is eliminated. Let $\mu$ be an eliminable base in $\Omega = \Omega(h_1, \ldots, h_p)$. Denote by $\Omega_1$ the equation resulting from $\Omega$ by eliminating $\mu$.

(a) Suppose $h_i \in \mu$ and $\gamma(h_i) = 1$. Let $\mu = \mu_1 \ldots \mu_k$, where $\mu_1, \ldots, \mu_k$ are the parts between $\mu$-connected boundaries. Let $h_i \in \mu_j$. Replace the basic equation corresponding to $\mu$ by the equations corresponding to $\mu_1, \ldots, \mu_k$. Then the variable $h_i$ occurs only once in $\Omega$ - precisely in the equation $s_{\mu_i} = 1$ corresponding to $\mu_j$. Therefore, in the coordinate group $F_{R(\Omega)}$ the relation $s_{\mu_i} = 1$ can be written as $h_i = w$, where $w$ does not contain $h_i$. Using
Tietze transformations we can rewrite the presentation of $F_{R(\Omega)}$ as $F_{R(\Omega')}$, where $\Omega'$ is obtained from $\Omega$ by deleting $s_{\mu_j}$ and the item $h_i$. It follows immediately that

$$F_{R(\Omega_1)} \simeq F_{R(\Omega')} \ast \langle h_i \rangle$$

and

$$F_{R(\Omega)} \simeq F_{R(\Omega')} \simeq F_{R(\Omega_1)} \ast F(B) \quad (1)$$

for some free or trivial group $F(B)$.

(b) Suppose now that $\mu$ satisfies case b) above with respect to a boundary $i$. Let $\mu = \mu_1 \ldots \mu_k$. Replace the equation $s_{\mu_k} = 1$ and the boundary equations corresponding to the boundary connections through $\mu$ by the equations $s_{\mu_i}$, $i = 1, \ldots, k$. Then in the equation $s_{\mu_k} = 1$ the variable $h_{i-1}$ either occurs only once or it occurs precisely twice and in this event the second occurrence of $h_{i-1}$ (in $\bar{\mu}$) is a part of the subword $(h_{i-1}h_i)^{\pm 1}$. In both cases it is easy to see that the tuple $(h_1, \ldots, h_{i-2}, s_{\mu_k}, h_{i-1}h_i, h_{i+1}, \ldots, h_{\rho})$ forms a basis of the ambient free group generated by $(h_1, \ldots, h_{\rho})$ and constants from $A$.

Therefore, eliminating the relation $s_{\mu_k} = 1$, we can rewrite the presentation of $F_{R(\Omega)}$ in generators $\bar{Y} = (h_1, \ldots, h_{i-2}, h_{i-1}h_i, h_{i+1}, \ldots, h_{\rho})$. Observe also that any other basic or boundary equation $s_{\lambda} = 1$ ($\lambda \neq \mu$) of $\Omega$ either does not contain variables $h_{i-1}, h_i$ or it contains them as parts of the subword $(h_{i-1}h_i)^{\pm 1}$, that is, any such a word $s_{\lambda}$ can be expressed as a word $w_{\lambda}(\bar{Y})$ in terms of generators $\bar{Y}$. This shows that

$$F_{R(\Omega)} \simeq G(\bar{Y})R_{R(w_{\lambda}(\bar{Y})) \lambda \neq \mu} \simeq F_{R(\Omega')} ,$$

where $\Omega'$ is a generalized equation obtained from $\Omega_1$ by deleting the boundary $i$. Denote by $\Omega'$ an equation obtained from $\Omega'$ by adding a free variable $z$ to the right end of $\Omega'$. It follows now that

$$F_{R(\Omega_1)} \simeq F_{R(\Omega')} \simeq F_{R(\Omega)} \ast \langle z \rangle$$

and

$$F_{R(\Omega)} \simeq F_{R(\Omega')} \ast F(Z) \quad (2)$$

or some free group $F(Z)$. Notice that all the groups and equations which occur above can be found effectively.

By induction on the number of steps in a cleaning process we obtain the following lemma.

**Lemma 3.** $F_{R(\Omega)} \simeq F_{R(\text{Ker}(\Omega))} \ast F(Z)$, where $F(Z)$ is a free group on $Z$.

**Proof.** Let

$$\Omega = \Omega_0 \rightarrow \Omega_1 \rightarrow \ldots \rightarrow \Omega_l = \text{Ker}(\Omega)$$

be a linear elimination process for $\Omega$. It is easy to see (by induction on $l$) that for every $j \in [0, l - 1]$

$$\overline{\text{Ker}(\Omega_j)} = \overline{\text{Ker}(\Omega_j)}.$$

Moreover, if $\Omega_{j+1}$ is obtained from $\Omega_j$ as in the case 2 above, then (in the notation above)

$$\overline{\text{Ker}(\Omega_j)} = \overline{\text{Ker}(\Omega_j)}.$$

Now the statement of the lemma follows from the remarks above and equalities $[1]$ and $[2]$.

$\square$
5.4 Rewriting process for $\Omega$

In this section we describe a rewriting process for a generalized equation $\Omega$.

5.4.1 Tietze Cleaning and Entire Transformation

In the rewriting process of generalized equations there will be two main sub-processes:

1. **Titze cleaning.** This process consists of repetition of the following four transformations performed consecutively:
   
   - (a) Linear elimination,
   - (b) deleting all pairs of matched bases,
   - (c) deleting all complete bases,
   - (d) moving all free variables to the right and declare them non-active.

2. **Entire transformation.** This process is applied if $\gamma(h_i) \geq 2$ for each $h_i$ in the active sections. We need a few further definitions. A base $\mu$ of the equation $\Omega$ is called a leading base if $\alpha(\mu) = 1$. A leading base is said to be maximal (or a carrier) if $\beta(\lambda) \leq \beta(\mu)$, for any other leading base $\lambda$. Let $\mu$ be a carrier base of $\Omega$. Any active base $\lambda \neq \mu$ with $\beta(\lambda) \leq \beta(\mu)$ is called a transfer base (with respect to $\mu$).

Suppose now that $\Omega$ is a generalized equation with $\gamma(h_i) \geq 2$ for each $h_i$ in the active part of $\Omega$. An entire transformation is a sequence of elementary transformations which are performed as follows. We fix a carrier base $\mu$ of $\Omega$. We transfer all transfer bases from $\mu$ onto $\mu'$. Now, there exists some $i < \beta(\mu)$ such that $h_1, \ldots, h_i$ belong to only one base $\mu$, while $h_{i+1}$ belongs to at least two bases. Applying (ET1) we cut $\mu$ along the boundary $i + 1$. Finally, applying (ET4) we delete the section $[1, i + 1]$.

Notice that neither process increases complexity.

5.4.2 Solution tree

Let $\Omega$ be a generalized equation. We construct a solution tree $T(\Omega)$ (with associated structures), as a rooted tree oriented from the root $v_0$, starting at $v_0$ and proceeding by induction on the distance $n$ from the root.

If

$$v \rightarrow v_1 \rightarrow \cdots \rightarrow v_s \rightarrow u$$

is a path in $T(\Omega)$, then by $\pi(v, u)$ we denote composition of corresponding epimorphisms

$$\pi(v, u) = \pi(v, v_1) \cdots \pi(v_s, u).$$

If $v \rightarrow v'$ is an edge then there exists a finite sequence of elementary or derived transformations from $\Omega_v$ to $\Omega_{v'}$ and the homomorphism $\pi(v, v')$ is composition of the homomorphisms corresponding to these transformations. We also assume that active [non-active] sections in $\Omega_{v'}$ are naturally inherited from $\Omega_v$, if not said otherwise.

Suppose a path in $T(\Omega)$ is constructed by induction up to a level $n$, and suppose $v$ is a vertex at distance $n$ from the root $v_0$. We describe now how to extend the tree from $v$.

We apply the Tietze cleaning at the vertex $v_n$ if it is possible. If it is impossible ($\gamma(h_i) \geq 2$ for any $h_i$ in the active part of $\Omega_v$), we apply the entire transformation. Both possibilities involve either creation of new boundaries and boundary connections or creation of new boundary connections without creation of new boundaries, and, therefore, addition of new relations to $F_{\mathbb{R}(\Omega_v)}$. The boundary connections can be made in few different ways, but there is a finite number of possibilities.
According to this, different resulting generalized equations are obtained, and we draw edges from \( v \) to all the vertices corresponding to these generalized equations.

**Termination condition:**
1. \( \Omega_v \) does not contain active sections. In this case the vertex \( v \) is called a *leaf* or an *end vertex*. There are no outgoing edges from \( v \).
2. \( \Omega_v \) is inconsistent. There is a base \( \lambda \) such that \( \bar{\lambda} \) is oriented the opposite way and overlaps with \( \lambda \), or the equation implies an inconsistent constant equation.

### 5.4.3 Quadratic case

Suppose \( \Omega_v \) satisfies the condition \( \gamma_i = 2 \) for each \( h_i \) in the active part. Then \( F_{R(\Omega_v)} \) is isomorphic to the free product of a free group and a coordinate group of a standard quadratic equation (to be defined below) over the coordinate group \( F_{R(\Omega')} \) of the equation \( \Omega' \) corresponding to the non-active part. In this case entire transformation can go infinitely along some path in \( T(\Omega) \), and, since the number of bases is fixed, there will be vertices \( v \) and \( w \) such that \( \Omega_v \) and \( \Omega_w \) are the same. Then the corresponding epimorphism \( \pi : F_{R(\Omega_v)} \to F_{R(\Omega_w)} \) is an automorphism of \( F_{R(\Omega_v)} \) that decreases the total length of the interval. For a minimal solution of \( F_{R(\Omega_v)} \) the process will stop.

**Definition 4.** A standard quadratic equation over the group \( G \) is an equation of the one of the following forms (below \( d, c_i \) are nontrivial elements from \( G \)):

\[
\prod_{i=1}^{n} [x_i, y_i] = 1, \quad n > 0; \tag{3}
\]

\[
\prod_{i=1}^{n} [x_i, y_i] \prod_{i=1}^{m} z_i^{-1} c_i z_i d = 1, \quad n, m \geq 0, m + n \geq 1; \tag{4}
\]

\[
\prod_{i=1}^{n} x_i^2 = 1, \quad n > 0; \tag{5}
\]

\[
\prod_{i=1}^{n} x_i^2 \prod_{i=1}^{m} z_i^{-1} c_i z_i d = 1, \quad n, m \geq 0, n + m \geq 1. \tag{6}
\]

Equations (3), (4) are called orientable, equations (5), (6) are called non-orientable. Number \( n \) is called a *genus* of the equation (notation \( \text{gen}(S) \)).

The proof of the following fact can be found in [28].

**Lemma 4.** Let \( W \) be a strictly quadratic word over \( G \). Then there is a \( G \)-automorphism \( f \in \text{Aut}_G(G[X]) \) such that \( W^f \) is a standard quadratic word over \( G \).

### 5.4.4 Entire transformation goes infinitely

Let now \( \gamma(h_i) \geq 2 \) for all \( h_i \) in the active part, and for some \( h_i \) the inequality is strict. Let \( (\Omega, \delta) \) be a generalized equation with a solution. Define the **excess** \( \psi \) of \( (\Omega, \delta) \):

\[
\psi = \Sigma_{\lambda}(\lambda^4) - 2|I^\delta|,
\]

where \( \lambda \) runs through the set of bases participating in entire transformation and \( I^\delta \) is the segment between the initial point of the interval and the leftmost point of the base that never participates (as carrier or transfer base).

It is possible that the cleaning after the entire transformation decreases complexity. This occurs if some base is transferred onto its dual and removed by (ET3). Otherwise, we use the same name for a base of \( \Omega_i \) and the reincarnation of this base in \( \Omega_{i+1} \). If we cannot apply Tietze cleaning after
the entire transformation, then we successively apply entire transformation. It is possible that the entire transformation sequence for \( \Omega \) goes infinitely, and the complexity does not decrease. If we apply the entire transformation to \((\Omega, \delta)\) and the complexity does not decrease, then \( \psi \) does not change.

We say that bases \( \mu \) and its dual of the equation \( \Omega \) form an overlapping pair if \( \mu \) intersects with its dual \( \bar{\mu} \).

If \( \phi_1 \) and \( \phi_2 \) are two solutions of a generalized equation \( \Omega \) in \( F(A,Y) \), then we define \( \phi_1 < \phi_2 \) if \( \phi_2 = \sigma \phi_1 \pi \), where \( \sigma \) is a canonical automorphism of \( F_{R(\Omega)} \) and \( \pi \) is an endomorphism of \( F(A,Y) \), and \( \sum_{i=1}^{\rho}(h_i)^{\phi_1} < \sum_{i=1}^{\rho}(h_i)^{\phi_2} \). Then we can define minimal solutions of a generalized equation.

**Theorem 9.** [37] Let \((\Omega, \delta)\) be a generalized equation with a minimal solution. Suppose \((\Omega, \delta) = (\Omega_0, \delta_0), (\Omega_1, \delta_1), \ldots \) be the generalized equations (with solutions) formed by the entire transformation sequence. Then one can construct a number \( N = N(\Omega) \) such that the sequence ends after at most \( N \) steps.

We will prove the key lemmas.

**Lemma 5.** If \( \delta \) is a solution minimal with respect to the subgroup of \( A \) generated by the canonical Dehn twists corresponding to the quadratic part of \( \Omega \), then one can construct a recursive function \( f = f(\Omega) \) such that \( |I^{\delta_i}| \leq f(\psi) \).

This lemma shows that for a minimal solution the length of the participating part of the interval is bounded in terms of the excess. And the excess does not change in the sequence of entire transformations when the complexity does not decrease.

**Proof.** We can temporarily change generalized equation \( \Omega \) such a way that it consists of one or several quadratic closed sections (such that \( \gamma(h_i) = 2 \) for any \( h_i \)) and non-quadratic sections (such that \( \gamma(h_i) > 2 \) for any \( h_i \)). Indeed, if \( \sigma \) is a quadratic section of \( \Omega \), we can cut all bases in \( \Omega \) through the end-points of \( \sigma \). Moreover, we will put the non-quadratic sections on the right part of the interval. Denote by \( \Omega_1 \) this new generalized equation. We apply the entire transformation to the pair \((\Omega_1, \delta_1)\), where \( \delta_1 \) is obtained from \( \delta \), and, therefore, minimal. We can find a number \( k(\Omega) \) such that after \( k \) transformations \((\Omega_1, \delta_1) \to \ldots \to (\Omega_k, \delta_k)\) all bases situated on the quadratic part will either form matched pairs or will be transferred to the non-quadratic part. Indeed, while we transforming the quadratic part we notice that:

1) two equations \( \Omega_i \) and \( \Omega_j \) for \( i < j \) cannot be the same, because then \( \delta_j \) would be shorter than \( \delta_i \), contradicting the minimality.

2) there is only a finite number of possibilities for the quadratic part since the number of items and complexity does not increase.

The sequence of consecutive quadratic carrier bases is bounded. Therefore after a bounded number of steps, a quadratic coefficient base is carrier, and we transfer a transfer base to the non-quadratic part. For a minimal solution, the length of a free variable corresponding to a matching pair is 1. And for each base \( \lambda \) transferred to the non-quadratic part, \( \lambda^\delta \) is shorter than the interval corresponding to the non-quadratic part, and, therefore, shorter than \( \psi \). This gives a hint how to compute a function \( f(\Omega) \). We can now return to the generalized equation \( \Omega \) and replace its solution by a minimal solution \( \delta \).

The **exponent of periodicity** of a family of reduced words \( \{w_1, \ldots, w_k\} \) in a free group \( F \) is the maximal number \( t \) such that some \( w_i \) contains a subword \( u^t \) for some simple cyclically reduced word \( u \). The exponent of periodicity of a solution \( \delta \) is the exponent of periodicity of the family \( \{h_i^\delta, \ldots, h_k^\delta\} \).

We call a solution of a system of equations in the group \( F(A,Y) \) strongly minimal if it is minimal and cannot be obtained from a shorter solution by a substitution.
Lemma 6. (Bulitko’s lemma). Let $S$ be a system of equations over a free group. The exponent of periodicity of a strongly minimal solution can be effectively bounded.

Proof. Let $P$ be a simple cyclically reduced word. A $P$-occurrence in a word $w$ is an occurrence in $w$ of a word $P^t$, $t \geq 1$. We call a $P$-occurrence $v_1 \cdot P^t \cdot v_2$ stable if $v_1$ ends with $P^e$ and $v_2$ starts with $P^e$. Clearly, every stable $P$-occurrence lies in a maximal stable $P$-occurrence. Two distinct maximal stable $P$-occurrences do not intersect.

A $P$-decomposition $D_P(w)$ of a word $w$ is the unique representation of $w$ as a product

$$v_0 \cdot P^e r_1 \cdot v_1 \cdot \ldots \cdot P^e r_m \cdot v_m$$

where the occurrences of $P^e r_i$ are all maximal stable $u$-occurrences in $w$. If $w$ has no stable $P$-occurrences then, by definition, its $P$-decomposition is trivial, that is, it has one factor which is $w$ itself.

By adding new variables we can transform the system $S$ is the triangular form, namely, such that each equation has length 3. If we have equation $xyz = 1$ with solution $x^\phi, y^\phi, z^\phi$, then the cancellation table for this solution looks as the triangle in Fig. 2.

Let

$$x^\phi = v_{10} \cdot P^{e_1 r_1} \cdot v_{11} \cdot \ldots \cdot P^{e_1, m r_1, m} \cdot v_{1, m},$$
$$y^\phi = v_{20} \cdot P^{e_2 r_2} \cdot v_{21} \cdot \ldots \cdot P^{e_2, n r_2, n} \cdot v_{2, n},$$
$$z^\phi = v_{30} \cdot P^{e_3 r_3} \cdot v_{31} \cdot \ldots \cdot P^{e_3, k r_3, k} \cdot v_{3, k}$$

be corresponding $P$-decompositions. From the cancellation table we will have a system of equations on the natural numbers $r_{ij}$, $i = 1, 2, 3, j = 1, \ldots, \max \{k, m, n\}$. All equations except, maybe, one will have form $r_{ij} = r_{st}$ for some pairs $i, j$ and $s, t$ and one equation may correspond to the middle of the triangle. If the middle of the triangle is inside a stable $P$-occurrence in $z^\phi$, then the equation would either have form $r_{1j} + r_{2j} + 2 = r_{3j}$ or $r_{1j} + r_{2j} + 3 = r_{3j}$. Notice that since $x^\phi, y^\phi, z^\phi$ are reduced words, the middle of the triangle cannot be inside a stable $P$-occurrence for more than one variable.

If we replace a solution $r_{ij}$, $i = 1, 2, 3, j = 1, \ldots, \max \{k, m, n\}$ of this system of equations by another positive solution, say $q_{ij}$, $i = 1, 2, 3, j = 1, \ldots, \max \{k, m, n\}$ and replace in the solution $x^\phi, y^\phi, z^\phi$ stable $P$-occurrences $P^{e_{ij}}$ by $P^{q_{ij}}$ we will have another solution of the equation $xyz = 1$.

Now, instead of one equation $xyz = 1$ we take a system of equations $S$. We obtain a corresponding linear system for natural numbers $r_{ij}$'s. Let $R$ be the family of variables $r_{ij}$'s that occur in the linear equations of length 3. The number of such equations is not larger than the number of triangles, that is the number of equations in the system $S$. Therefore $R$ is a finite family. Consider a system of all linear equations on $R$. It depends on the particular solution of $S$, but there is a finite number of possible such systems. We now can replace values of variables from $R$ by a minimal positive solution, say $\{q_{ij}\}$, of the same linear system (if $r_{ij}$ does not appear in any linear equation we replace it by $q_{ij} = 1$) and replace in the solution of the system $S$ stable $P$-occurrences $P^{e_{ij}}$ by $P^{q_{ij}}$. We obtain another solution of the system $S$. The length of a minimal positive solution $\{q_{ij}\}$ of the linear system is bounded as in the formulation of the lemma. The lemma is proved.

Lemma 7. Suppose $F_{R(\Omega)}$ is not a free product with an abelian factor, and there are solutions of $\Omega$ with unboundedly large exponent of periodicity. One can effectively find a number $M$ and an abelian splitting of $F_{R(\Omega)}$ (or a quotient obtained from $F_{R(\Omega)}$ by adding commutation transitivity condition for certain subgroups) as an amalgamated product with abelian vertex group or as an HNN-extension (or both), such that the exponent of periodicity of a minimal solution of $\Omega$ with respect to the group of canonical automorphisms corresponding to this splitting and the quadratic part (if exists) is bounded by $M$.
The proof of this lemma uses the notion of a periodic structure and can be found in ([40], Lemma 22) or in [37].

Consider an infinite path in \( T(\Omega) \) corresponding to an infinite sequence in entire transformation

\[ r = v_1 \rightarrow v_2 \rightarrow \cdots \rightarrow v_m. \]  

(7)

Let \( \delta \) be a solution of \( \Omega \). The following lemma gives the way to construct a function \( f_1 \) depending on \( \Omega \) such that for any number \( M \), if the sequence of entire transformations for \( (\Omega, \delta) \) has \( f_1(M) \) steps, then \( |P^\delta| > M\psi. \)

Denote by \( \mu_i \) the carrier base of the equation \( \Omega_{v_i} \). The path (7) will be called \( \mu \)-reducing if \( \mu_1 = \mu \) and one of the following holds:

1. \( \mu_2 \) does not overlap with its double and \( \mu \) occurs in the sequence \( \mu_1, \ldots, \mu_{m-1} \) at least twice.
2. \( \mu_2 \) overlaps with its double and \( \mu \) occurs in the sequence \( \mu_1, \ldots, \mu_{m-1} \) at least \( M + 2 \) times, where \( M \) is the exponent of periodicity of \( \delta \).

The following lemma is just an easy exercise.

**Lemma 8.** In a \( \mu \)-reducing path the length of \( I^\delta \) decreases at least by \( |\mu^\delta|/10 \).

**Proof.** Case 1. \( \mu = \mu_1 \neq \mu_2 \), and not more than half of \( \mu_2 \) overlaps with its double. Then after two steps the leftmost boundary of the reincarnation of \( \mu \) will be to the right of the middle of \( \mu_2 \). Therefore by the time when the reincarnation of \( \mu \) becomes a carrier, the part from the beginning of the interval to the middle of \( \mu_2 \) will be cut and removed. This part is already longer than half of \( \mu \).

Case 2. \( \mu = \mu_1 = \mu_2 \), and \( \mu_2 \) (second reincarnation of \( \mu \)) does not overlap with its double. Then on the first step we cut the part of the interval that is longer than half of \( \mu \).

Case 3. \( \mu_2 \) overlaps with its double. Denote by \( \mu^{\delta(i)} \) the value of the reincarnation of \( \mu^\delta \) on step \( i \) and by \( [1, \sigma]^{\delta(i)} \) the word corresponding to the beginning of the interval until boundary \( \sigma \) on step \( i \). Then \( [1, \alpha(\mu_2)]^{\delta(2)} = P^\delta \) for some cyclically reduced word \( P \) which is not a proper power and \( \mu^{\delta(2)} \), \( \mu_2^{\delta(2)} \) are beginnings of \( [1, \alpha(\mu_2)]^{\delta(2)} \) which is a beginning of \( P^\infty \).

We have

\[ \mu^{\delta(2)} = P^\delta P_1, r \leq M \]  

(8)

Let \( \mu_{i_1} = \mu_{i_2} = \mu \) for \( i_1 < i_2 \) and \( \mu_i \neq \mu \) for \( i_1 < i < i_2 \). If

\[ |\mu^{\delta(i_1+1)}| \geq 2|P| \]  

(9)

and \( [1, \rho_{i_1+1} + 1]^{\delta(i_1+1)} \) begins with a cyclic permutation of \( P^\delta \), then

\[ ||1, \alpha(\mu_{i_1+1})|^{\delta(i_1+1)}| \geq |P|. \]

The base \( \mu \) occurs in the sequence \( \mu_1, \ldots, \mu_{m-1} \) at least \( r + 1 \) times, so either \( \square \) fails for some \( i_1 \leq m-1 \) or the part of the interval that was removed after \( m-1 \) steps is longer than \( \max\{|r-3||P|, |P|\} \).

If \( \square \) fails, then \( |[1, \alpha(\mu_{i_1})]^{\delta(i_1)}| \geq (r-2)||P| \). So everything is reduced to the case when the part of the interval that was removed after \( m - 1 \) steps is longer than \( \max\{|r-3||P|, |P|\} \). Together with \( \square \) this implies that in \( m-1 \) steps the length of the interval was reduced at least by \( \frac{1}{10}|\mu^{\delta(2)}| \) which is not less than \( \frac{1}{10}|\mu^\delta| \).

We can now finish the proof of Theorem 9. Let \( L \) be the family of bases such that every base \( \mu \in L \) occurs infinitely often as a leading base. Suppose a number \( m \) is so big that for every base \( \mu \in L \), a \( \mu \)-reducing path occurs more than \( 20m \) times during these \( m \) steps. Since \( \sum|\mu^{\delta_m}| \geq \psi \), we have over all the participating bases, at least for one base \( \lambda \in L \), \( |\lambda^{\delta_m}| \geq \psi/2m \). Moreover, \( |\lambda^{\delta_i}| \geq |\lambda^{\delta_m}| \geq \psi/2m \) for all \( i \leq m \). Since a \( \lambda \)-reducing path occurs more than \( 20m \) times, the length of the interval would be decreased in \( m \) steps by more than it initially was. This gives a
Lemma 9. Let $\Omega_0, \Omega_1, \ldots$ be the generalized equations formed by the entire transformation sequence. Then one of the following holds.
1. the sequence ends,

2. for some $i$ we obtain the quadratic case on the interval $I$,

3. we obtain an overlapping pair $\lambda, \bar{\lambda}$ such that $\lambda$ is a leading base, $\lambda^\delta$ begins with some $n$th power of the word $[\alpha(\lambda), \alpha(\bar{\lambda})]^{\delta}$ and there are solutions $\delta$ of $\Omega_i$ with number $n$ arbitrary large (with arbitrary large exponent of periodicity).

Proof. We assume cases 1 and 2 do not hold. Then, our sequence is infinite and we may assume that every base that is carried is carried infinitely often, and that every base that carries does so infinitely often. So every base that participates does so infinitely often. We also assume the complexity does not change and that no base is moved off the interval.

Let $\Omega$ be a generalized equation (with solution) and let 

$$ B = \Omega_1, \ldots, \Omega_n, \ldots $$

be an infinite branch. Let $\delta_1, \delta_2, \ldots$ be a set of solutions of $\Omega$ such that $\delta_i$ "factors" through $\Omega_i$.

If we rescale the metric so that $\Omega_i$ has length 1, then each $\delta_i$ puts a length function on the items of $\Omega$, in particular we assume that each base has length and midpoint between 0 and 1. This means that for each $\delta_i$ there is a point $x_i$ in $[0, 1]^m$, where this point represents the lengths of the bases and the items as well as their midpoints in the normalized metric.

We pass to a subsequence of $\delta_i$ (omit the double subscript) such that the $x_i$ converge to a point $x$ in $[0, 1]^m$. We call the limit a metric on $(\Omega, \delta^*)$.

Normed excess denoted $m(\psi)$ is a constant and we can apply the Bestvina, Feighn argument (toral case) on the generalized equation $\Omega$ with lengths given by $\delta^*$.

The argument goes as follows. Entire transformation is moving bases to the right and shortening them, and $m(\psi)$ is a constant. During the process the initial point of every base is only moved towards the final point of $I$, and the length of a base is never increased, therefore, every base has a limiting position. Since $m(\psi)$ is a constant, there is a base $\lambda$ of length not going to zero that participates infinitely often. If $\lambda$ is eventually the only carrier, then we must have case 3 for the process to go unboundedly long. Suppose $\lambda$ is carried infinitely often. Whenever $\lambda$ is the carrier, the midpoint of some base moves the distance between the midpoints of $\lambda$ and its dual. Since every base has a limiting position, it follows that $\lambda$ and its dual have the same limiting position.

The argument shows that after some finite number of steps we get an overlapping initial section i.e. carrier and dual have high length, but midpoints are close.

It follows that for $n$ sufficiently large doing the process with $(\Omega, \delta_n)$ will give a similar picture.

This implies case 3.

Case 3 can only happen is there are solutions of an arbitrary large exponent of periodicity. If we consider only minimal solutions, then the exponent of periodicity can be effectively bounded, and entire transformation always stops after a bounded number of steps.

On the group level, case 2 corresponds to the existence a QH vertex group in the JSJ decomposition of $F_{R(\Omega)}$ and case 3 corresponds to the existence of an abelian vertex group in the abelian JSJ decomposition of $F_{R(\Omega)}$.

6 Elementary free groups

If an NTQ group does not contain non-cyclic abelian subgroups we call it regular NTQ group. We have shown in [37] that regular NTQ groups are hyperbolic. (Later Sela called these groups hyperbolic $\omega$-residually free towers [56].)

Theorem 10. [13], [59] Regular NTQ groups are exactly the f.g. models of the elementary theory of a non-abelian free group.

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7 Stallings foldings and algorithmic problems

A new technique to deal with $\mathbb{F}_Z[t]$ became available when Myasnikov, Remeslennikov, and Serbin showed that elements of this group can be viewed as reduced infinite words in the generators of $\mathbb{F}$. It turned out that many algorithmic problems for finitely generated fully residually free groups can be solved by the same methods as in the standard free groups. Indeed, they introduce analog of the Stallings’ folding for an arbitrary finitely generated subgroup of $\mathbb{F}_Z[t]$, which allows one to solve effectively the membership problem in $\mathbb{F}_Z[t]$, as well as in an arbitrary finitely generated subgroup of it.

Theorem 11. (Myasnikov-Remeslennikov-Serbin)[50] Let $G$ be a f.g. fully residually free group and $G \hookrightarrow G^*$ the effective Nielsen completion. For any f.g. subgroup $H \leq G$ one can effectively construct a finite graph $\Gamma_H$ that in the group $G^*$ accepts precisely the normal forms of elements from $H$.

Theorem 12. [42] The following algorithmic problems are decidable in a f.g. fully residually free group $G$:

- the membership problem,
- the intersection problem (the intersection of two f.g. subgroups in $G$ is f.g. and one can find a finite generated set effectively),
- conjugacy of f.g. subgroups,
- malnormality of subgroups,
- finding the centralizers of finite subsets.

It was proved by Chadas and Zalesski [14] that finitely generated fully residually free groups are conjugacy separable.

Notice that the decidability of conjugacy problem also follows from the results of Dahmani and Bumagin. Indeed, Dahmani showed that $G$ is relatively hyperbolic and Bumagin proved that the conjugacy problem is decidable in relatively hyperbolic groups. We prove that for finitely generated subgroups $H, K$ of $G$ there are only finitely many conjugacy classes of intersections $H^g \cap K$ in $G$. Moreover, one can find a finite set of representatives of these classes effectively. This implies that one can effectively decide whether two finitely generated subgroups of $G$ are conjugate or not, and check if a given finitely generated subgroup is malnormal in $G$. Observe, that the malnormality problem is decidable in free groups, but is undecidable in torsion-free hyperbolic groups - Bridson and Wise constructed corresponding examples. We provide an algorithm to find the centralizers of finite sets of elements in finitely generated fully residually free groups and compute their ranks. In particular, we prove that for a given finitely generated fully residually free group $G$ the centralizer spectrum $\text{Spec}(G) = \{\text{rank}(C) \mid C = C_G(g), g \in G\}$, where $\text{rank}(C)$ is the rank of a free abelian group $C$, is finite and one can find it effectively.

Theorem 13. [12] The isomorphism problem is decidable in f.g. fully residually free groups.

We also have an algorithm to solve equations in fully residually free groups and to construct the abelian JSJ decomposition for them.

Recently Dahmani and Groves [18] proved

Theorem 14. The isomorphism problem is decidable in relatively hyperbolic groups with abelian parabolics.

Dahmani [20] proved the decidability of the existential theory of a torsion free relatively hyperbolic group with virtually abelian parabolic subgroups. This implies our result in [31] about the decidability of the existential theory of f.g. fully residually free groups.
8 Residually free groups

Any f.g. residually free group can be effectively embedded into a direct product of a finite number of fully residually free groups [37].

Important steps towards the understanding of the structure of finitely presented residually free groups were recently made in [8] [9].

There exists finitely generated subgroups of $F \times F$ (this group is residually free but not fully residually free) with unsolvable conjugacy and word problem (Miller).

In finitely presented residually free groups these problems are solvable [8].

**Theorem 15.** [9] Let $G < \Gamma_0 \times \ldots \times \Gamma_n$ be the subdirect product of limit groups. Then $G$ is finitely presented iff it satisfies the virtual surjection to pairs (VSP) property:

\[ \forall \ 0 \leq i < j \leq n \ \ |\Gamma_i \times \Gamma_j : P_{ij}(G)| < \infty. \]

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