A NOTE ON STABILITY OF ELIASSON-KUKSIN’S KAM TORI FOR THE NONLINEAR SCHRÖDINGER EQUATION

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Abstract. Eliasson and Kuksin developed a KAM approach to study the persistence of the invariant tori for nonlinear Schrödinger equation on $\mathbb{T}^d$. In this note, we improve Eliasson and Kuksin’s KAM theorem by using Kolmogorov’s iterative scheme and obtain a local normal form for the transformed Hamiltonian. As a consequence, we are able to derive the time $\delta^{-1}$ stability of the obtained KAM tori.

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1. Introduction

In this paper, we consider the existence and the long time stability of the invariant tori for the $d$-dimensional nonlinear Schrödinger equation (NLS)

\begin{equation}
-\frac{i}{\hbar} \dot{u} = -\Delta u + V(x) \ast u + \varepsilon \frac{\partial}{\partial u}(x, u, \bar{u}), \quad u = u(t, x)
\end{equation}

under the periodic boundary condition $x \in \mathbb{T}^d$. The convolution function $V : \mathbb{T}^d \to \mathbb{C}$ is analytic and the Fourier coefficient $\hat{V}(a)$ takes real value, when expanding $V$ into Fourier series $V(x) = \sum_{a \in \mathbb{Z}^d} \hat{V}(a) e^{i(a,x)}$. The nonlinearity $F$ is real analytic in $x, \Re u, \Im u$.

The NLS equation (1.1) can be written as an infinite dimensional Hamiltonian system. The KAM theory for Hamiltonian PDEs started in late 1980’s and originally applied to the one dimensional PDEs, which is now pretty well understood (see for example [10,11]). However, the space-multidimensional KAM theory for Hamiltonian PDEs is at its early stage. The first breakthrough in this respect was made by Bourgain [1] on the two dimensional NLS equation, in which he developed further Craig and Wayne’s scheme on the persistency problem of periodic solutions. Later, Bourgain and his collaborators developed new techniques of Green’s function estimates for linear problems, based on which he proved the persistence of invariant tori for

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space-multidimensional NLS and NLW equations. This method is now known as the Craig-Wayne-Bourgain (CWB) method. However, Bourgain’s original proof gives no information on the linear stability of the obtained invariant tori. Recently, there are some progress in applying the CWB method to obtain the linear stability of KAM tori for finitely dimensional Hamiltonian system, but it remains an open question for the infinitely dimensional case.

The classical KAM approach for space-multidimensional Hamiltonian PDEs is developed by Eliasson and Kuksin in their paper on NLS equation. They take a sequence of symplectic transformations such that the transformed Hamiltonian guarantees the existence of the invariant torus. Moreover, the KAM approach in provides also the reducibility and linear stability of the obtained invariant tori. However, they have to pay for the price that the number of the second Melnikov conditions becomes infinite when solving the block-diagonal homological equation, which is far more complicated than that in the one-dimensional case. To reduce those infinitely many conditions, they analyze carefully the functions with the Töplitz-Lipschitz property. Moreover, due to the size of the blocks grows much faster than quadratically along the iterations, they need to take sufficiently many normal form computations at each KAM step to obtain a much faster iteration scheme. See also and for the KAM approach on the space-multidimensional beam equation and some shallow-water equations, respectively. But the problem on the linear stability of the KAM tori for space-multidimensional nonlinear wave equation is still open and requires special attention.

Once the existence of the invariant tori of the Hamiltonian PDEs is established, one naturally concerns the stability problem of the obtained KAM tori. It is known that there exists transfer of energy in the cubic defocusing NLS equation, which shows the instability phenomenon in Hamiltonian PDEs. In the present paper, however, we concentrate on the long time stability of the KAM tori for the NLS equation.

Consider a nonlinear differential equation

\[ \dot{x} = X(x), \]

which has an invariant tori \( T \) carrying the quasi-periodic flow \( x(t) = x_0(t) \). We say that the invariant tori \( T \) is linearly stable if the equilibrium of the linearized equation

\[ \dot{y} = DX(x_0(t))y \]

of (1.2) along \( T \) is Lyapunov stable. However, in general, we cannot determine the stability of the nonlinear system (1.2) from the linear stability of the linearized equation. This prompts us to study the nonlinear stability, among which the long time stability plays an important role in PDEs.

There are lots of literatures devoting to the long time stability of the equilibrium for the Hamiltonian PDEs. Fortunately, it does not cause too much trouble to generalize the results in one dimensional to the multidimensional space (see for example ), which is due to the fact that frequency shift does not come into small divisors during the normal form computations. But not so for the long time stability of KAM tori. In, the long time stability of KAM tori for one dimensional NLS equation (i.e., \( d = 1 \)) is proved by using the tame property and the Birkhoff normal form. More precisely, given any \( M > 0 \), there exists small \( \delta_0 > 0 \) such that for any \( 0 < \delta < \delta_0 \) and any solution \( u(x, t) \) of (1.1) with its initial value \( u(x, 0) \) staying in a \( \delta \)-neighborhood (under the Sobolev norm) of some KAM tori \( T \), the solution \( u(x, t) \) will stay inside the \( C\delta \)-neighborhood of \( T \) for all \( 0 < |t| < \delta^{-M} \).

When it comes to \( d \geq 2 \) for equation (1.1), we are not able to apply the results in directly. The reason is that the domain (of the action variable and the normal coordinate) of the transformed Hamiltonian decreases to zero after the KAM iteration, which does not affect the existence and linear stability of invariant tori, but is not helpful to study the long...
time stability. To this end, we improve Eliasson-Kuksin’s KAM theorem, by employing Kolmogorov’s iteration scheme (see Remark 2.12 below) and solving the modified homological equation as in [3], to obtain a uniform domain for the transformed Hamiltonian, which enables us to get a local normal form around the obtained KAM tori. Moreover, based on the local normal form, it is easy to show the time $\delta^{-1}$ stability of KAM tori. However, due to the much more complicated small divisor problems and the frequency shift, we are not able to establish the time $\delta^{-M}$ (with large $M$) stability in this note, although the tame property for space-multidimensional NLS equation can be preserved during the KAM iteration.

The main result of this paper is as follows.

**Theorem 1.1.** Under the assumptions for equation (1.1), if $\varepsilon > 0$ is sufficiently small, then for most $V$ (in the sense of measure), the $d$-dimensional nonlinear Schrödinger equation (1.1) has a quasi-periodic solution. Moreover, assume $u_0(t, x)$ with initial value $u_0(0, x)$ is a quasi-periodic solution for the equation (1.1). Then, for any solution $u(t, x)$ with initial value $u(0, x)$ satisfying

$$\|u(0, \cdot) - u_0(0, \cdot)\|_{H^p(\mathbb{T}^d)} < \delta, \forall 0 < \delta \ll 1,$$

we have

$$\|u(t, \cdot) - u_0(t, \cdot)\|_{H^p(\mathbb{T}^d)} < C\delta, \forall 0 < |t| < \delta^{-1}.$$  

In order to prove Theorem 1.1, we write the nonlinear Schrödinger equation (1.1) as an infinite dimensional Hamiltonian system. We keep the notations consistent with those in [8]. Write

$$u(x) = \sum_{a \in \mathbb{Z}^d} u_a e^{i(a, x)}, \quad u(x) = \sum_{a \in \mathbb{Z}^d} v_a e^{i(-a, x)},$$

and let

$$\zeta_a = \begin{pmatrix} \xi_a \\ \eta_a \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} u_a + v_a \\ -i(u_a - v_a) \end{pmatrix}.$$  

Then the nonlinear Schrödinger equation (1.1) becomes a real Hamiltonian system with the symplectic structure $d\xi \wedge d\eta$ and the Hamiltonian

$$\frac{1}{2} \sum_{a \in \mathbb{Z}^d} (|a|^2 + \hat{V}(a)) (\xi_a^2 + \eta_a^2) + \varepsilon \int_{\mathbb{T}^d} F(x, u(x), \overline{u(x)}) dx.$$  

Let $\mathcal{A}$ be a finite subset of $\mathbb{Z}^d$ and $\mathcal{L} = \mathbb{Z}^d \setminus \mathcal{A}$. Introduce action angle variables $(\varphi_a, r_a)$, $a \in \mathcal{A}$,

$$\xi_a = \sqrt{2}(r_a + q_a) \cos \varphi_a, \quad \eta_a = \sqrt{2}(r_a + q_a) \sin \varphi_a, \quad q_a > 0.$$  

Let

$$\omega_a = |a|^2 + \hat{V}(a), \quad a \in \mathcal{A}, \quad \Omega_a = |a|^2 + \hat{V}(a), \quad a \in \mathcal{L}.$$  

We have the Hamiltonian

$$h + f = \sum_{a \in \mathcal{A}} \omega_a r_a + \frac{1}{2} \sum_{a \in \mathcal{L}} \Omega_a (\xi_a^2 + \eta_a^2) + \varepsilon \int_{\mathbb{T}^d} F(x, u(x), \overline{u(x)}) dx.$$  

Assume $f$ is real analytic on

$$D(\rho, \mu, \sigma) = \{(\varphi, r, \zeta) \in (\mathbb{C}/2\pi\mathbb{Z})^d \times \mathbb{C}^d \times \mathbb{L}^2 : |3\varphi| \leq \rho, |r| \leq \mu, \|\zeta\|_p \leq \sigma\},$$

where

$$\|\zeta\|_p = \sum_{a \in \mathcal{L}} (|\xi_a|^2 + |\eta_a|^2)^{2p}, \quad \langle a \rangle = \max(|a|, 1).$$  

In the KAM iteration, we have symplectic maps

$$\Phi_j : D(\rho_{j+1}, \mu_{j+1}, \sigma_{j+1}) \rightarrow D(\rho_j, \mu_j, \sigma_j).$$
such that \((h_j + f_j) \circ \Phi_j = h_{j+1} + f_{j+1}\). The existence of the KAM tori follows from the transformed Hamiltonian vector field on the uniform domain \(\cap_{j \geq 0} D(\rho_j, \mu_j, \sigma_j)\). In Eliasson-Kuksin [8], there is \(\rho_j \to \frac{\epsilon}{2}, \mu_j \to 0, \sigma_j \to 0\), which causes some trouble to study the long time stability of the obtained KAM tori. In this paper, we use Kolmogorov’s iterative procedure as in [8] such that \(\rho_j \to \rho_2, \mu_j \to \mu_4, \sigma_j \to \sigma_2\). This is the main improvement on Eliasson and Kuksin’s paper. As a consequence, it is easy to show the time \(\delta^{-1}\) stability of the obtained KAM tori.

The paper is organized as follows. In Section 2, we formulate and solve the homological equation in the KAM iteration. Some definitions and notations are introduced at the beginning. In Section 3 following Eliasson-Kuksin [8], we prove a KAM theorem for \(d\)-dimensional NLS, based on which our main Theorem 1.1 is an immediate result. In this paper, \(\| \cdot \|\) is an operator norm or \(l^2\) norm. \(| \cdot |\) will in general denote a sup norm. For \(a \in \mathbb{Z}^d\), we use \(|a|\) for the \(l^2\) norm. The dimension \(d\) will be fixed and \(p > \frac{d}{2}\). Let \(A\) be a finite subset of \(\mathbb{Z}^d\) and \(\mathcal{L} = \mathbb{Z}^d \setminus A\). Denote \(\langle \xi, \eta \rangle = \sum (\xi_a \xi'_{a} + \eta_a \eta'_{a})\) and \(J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\).

2. Homological equation

In this section, we formulate and solve the homological equation in the KAM iteration. To obtain an open and uniform domain for the transformed Hamiltonian, we apply Kolmogorov’s iteration scheme. As a result, the homological equation is complicated than that in [8], but it can be solved by the method developed in [8]. To begin with, we introduce some notations and definitions.

2.1. Notations and definitions. Recalling the Hamiltonian formulation of (1.1) in Section 1, we define the phase space and the associated norms. For \(\gamma \geq 0\), we denote \(l^2_{p,\gamma} = \{ \zeta = (\xi, \eta) \in \mathbb{C}^\mathcal{L} \times \mathbb{C}^\mathcal{L} : \| \zeta \|_{p,\gamma} < \infty \}\), where

\[
\| \zeta \|_{p,\gamma} = \sum_{a \in \mathcal{L}} (|\xi_a|^2 + |\eta_a|^2) e^{2\gamma|a|} \langle a \rangle^{2p}, \quad \langle a \rangle = \max(|a|, 1).
\]

When \(\gamma = 0\), we write \(l^2_p\) and \(\| \zeta \|_p\) for simplicity. The phase space of the Hamiltonian dynamical system is defined by

\[
\mathcal{P}^p = (\mathbb{C}/2\pi\mathbb{Z})^A \times \mathbb{C}^A \times l^2_p.
\]

Let \(U \subset [-1, 1]^A\) be a parameter set with positive measure.

2.1.1. Töplitz-Lipschitz property. We recall here some definitions and notations of the Töplitz-Lipschitz property in [8]. For more heuristic formulation and explanation, we refer to [8, Part I].

Let \(\text{gl}(2, \mathbb{C})\) be the space of all complex \(2 \times 2\)-matrices. For \(A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{gl}(2, \mathbb{C})\), denote \(\pi A = \frac{1}{2} \begin{pmatrix} a + d & b - c \\ c - b & a + d \end{pmatrix}\) and \([A] = \begin{pmatrix} |a| & |b| \\ |c| & |d| \end{pmatrix}\). Now consider an infinite-dimensional \(\text{gl}(2, \mathbb{C})\)-valued matrix

\[
A : \mathcal{L} \times \mathcal{L} \to \text{gl}(2, \mathbb{C}), \quad (a, b) \mapsto A^a_b.
\]

For reader’s convenience, we collect the following notations and definitions for the infinite-dimensional matrix \(A\).
(1) |A|_{\mathcal{D}}. For any $\mathcal{D} \subset \mathcal{L} \times \mathcal{L}$, define

$$|A|_{\mathcal{D}} = \sup_{(a,b) \in \mathcal{D}} \|A_{a}^{b}\|,$$

where $\| \cdot \|$ is the operator norm.

(2) $|A|_{\gamma}$. Letting $(\pi A)_{a}^{b} = \pi A_{a}^{b}$ and $(E_{\gamma}^{\pm})_{a}^{b} = [A_{a}^{b}]_{\gamma}[a \pm b]$, we define the norm

$$|A|_{\gamma} = \max(|E_{\gamma}^{+}\pi A|_{\mathcal{L} \times \mathcal{L}}, \ |E_{\gamma}^{-}(1 - \pi)A|_{\mathcal{L} \times \mathcal{L}}).$$

(3) $T_{\Delta}^{A}$ and $T_{\Delta}A$. We also define the truncation operators

$$T_{\Delta}^{A} = |A|_{\{(a,b) \in \mathcal{L} \times \mathcal{L} : [a \pm b] \leq \Delta\}}, \quad T_{\Delta}A = T_{\Delta}^{A} + T_{\Delta}^{\pm}(1 - \pi)A.$$

(4) $A_{a}^{b}(\pm, c)$. A matrix $A : \mathcal{L} \times \mathcal{L} \to \text{gl}(2, \mathbb{C})$ is \emph{Töplitz} at $\infty$, if for all $a, b, c$, the two limits

$$\lim_{t \to +\infty} A_{a}^{b}(\pm, c) = A_{a}^{b}(\pm, c).$$

(5) $D_{\Delta}^{A}(c)$. For $\Delta \geq 0$, define the \emph{Lipschitz domain} $D_{\Delta}^{A}(c) \subset \mathcal{L} \times \mathcal{L}$ to be the set of all $(a, b)$ such that there exist $a', b' \in \mathbb{Z}^{d}$, $t \geq 0$ such that

$$|a| = |a' + tc| \geq \Lambda(|a'| + |c|)|c|, \quad |b| = |b' + tc| \geq \Lambda(|b'| + |c|)|c|, \quad \frac{|a|}{|c|}, \frac{|b|}{|c|} \geq 2\Lambda^{2}.$$

Define $(a, b) \in D_{\Delta}^{-}(c)$ if and only if $(a, -b) \in D_{\Delta}^{+}(c)$.

(6) $\text{Lip}_{\Delta}^{A}(\pm)$. For $c \neq 0$, we let $(\mathcal{M}, A)_{a}^{b} = \left(\max(\frac{|a|}{|c|}, \frac{|b|}{|c|}) + 1\right) [A_{a}^{b}]$ and define the \emph{Lipschitz constants}

$$\text{Lip}_{\Delta}^{\pm}A = \sup |E_{\gamma}^{\pm}\mathcal{M}(A - A_{a}^{b}(\pm, c))|_{D_{\Delta}^{\pm}(c)}.$$

(7) $\langle A \rangle_{\Lambda, \gamma}$. For $c \neq 0$, the \emph{Lipschitz norm} is given by

$$\langle A \rangle_{\Lambda, \gamma} = \text{Lip}_{\Lambda, \gamma}^{\pm}A, \quad \text{Lip}_{\Lambda, \gamma}^{-}(1 - \pi)A + |A|_{\gamma}.$$

(8) $|A|_{\gamma, U}$. Let $A(w) : \mathcal{L} \times \mathcal{L} \to \text{gl}(2, \mathbb{C})$ be $C^{1}$ (in the sense of Whitney) in $w \in U$, define

$$|A|_{\gamma, U} = \sup_{w \in U} (|A(w)|_{\gamma}, |\partial_{w}A(w)|_{\gamma}), \quad |A|_{U} = |A|_{0, U}.$$

(9) $\langle A \rangle_{\Lambda, \gamma, U}$. Let $A(w) : \mathcal{L} \times \mathcal{L} \to \text{gl}(2, \mathbb{C})$ be $C^{1}$ (in the sense of Whitney) in $w \in U$. If $A(w), \partial_{w}A(w)$ are Töplitz at $\infty$ for all $w \in U$, define

$$\langle A \rangle_{\Lambda, \gamma, U} = \sup_{w \in U} (\langle A(w) \rangle_{\Lambda, \gamma}, \langle \partial_{w}A(w) \rangle_{\Lambda, \gamma}), \quad \langle A \rangle_{\Lambda, U} = \langle A \rangle_{\Lambda, 0, U}.$$

(10) $d$-Töplitz. For $d = 2$, the matrix $A$ is Töplitz-Lipschitz if it is Töplitz at $\infty$ and $\langle A \rangle_{\Lambda, \gamma} < \infty$ for some $\Lambda, \gamma$. For $d > 2$, we can define Töplitz-Lipschitz matrices inductively (see Section 2.4 in [4]).

Based on Töplitz-Lipschitz matrices, we define the Töplitz-Lipschitz property for functions.

\textbf{Definition 2.1.} Let $D^{0}(\sigma) = \{\zeta \in l_{p, \gamma}^{2} : \|\zeta\|_{p, \gamma} \leq \sigma\}$, and the function $f : D^{0}(\sigma) \to \mathbb{C}$ be real analytic. We say $f$ is \emph{Töplitz} at $\infty$ if $\partial_{\zeta}^{2}f(\zeta)$ is Töplitz at $\infty$ for all $\zeta \in D^{0}(\sigma)$. The norm $[f]_{\Lambda, \gamma, \sigma}$ is defined by the smallest $C$ such that

$$|f(\zeta)| \leq C, \quad \forall \zeta \in D^{0}(\sigma),$$

and

$$\|\partial_{\zeta}f(\zeta)\|_{p, \gamma'} \leq \frac{C}{\sigma^{2}}, \quad \langle \partial_{\zeta}^{2}f(\zeta) \rangle_{\Lambda, \gamma'} \leq \frac{C}{\sigma^{2}}, \quad \forall \zeta \in D^{0}(\sigma), \quad \forall \gamma' \leq \gamma.$$

For more quantitative estimates of Töplitz-Lipschitz functions, we refer to [4] Section 3.
2.1.2. The tame property. Comparing with Eliasson-Kuksin [8], we shall also prove the preservation of the tame property of the Hamiltonian in the KAM iteration. To begin with, we define the norms of analytic functions on the phase space.

Definition 2.2. Let $$D(\rho) = \{ \varphi \in \mathbb{C}/(2\pi\mathbb{Z})^A : |3\varphi| \leq \rho \},$$  
$$f : D(\rho) \times U \to \mathbb{C}$$ be analytic in $$\varphi \in D(\rho)$$ and $$C^1$$ in $$w \in U,$$  
$$f(\varphi; w) = \sum_{k \in \mathbb{Z}^A} \hat{f}(k; w)e^{i(k; \varphi)}.$$  

Define the norm $$\|f\|_{D(\rho) \times U} = \sup_{w \in U} \sum_{k \in \mathbb{Z}^A} \left( |\hat{f}(k; w)| + |\partial_w \hat{f}(k; w)| \right) e^{k|\rho|}.$$  

Definition 2.3. Let $$D(\rho, \mu) = \{ (\varphi, r) \in \mathbb{C}/(2\pi\mathbb{Z})^A \times \mathbb{C}^A : |3\varphi| \leq \rho, |r| \leq \mu \},$$  
$$f : D(\rho, \mu) \times U \to \mathbb{C}$$ be analytic in $$(\varphi, r) \in D(\rho, \mu)$$ and $$C^1$$ in $$w \in U,$$  
$$f(\varphi, r; w) = \sum_{\alpha \in \mathbb{N}^A} f^\alpha(\varphi; w)r^\alpha.$$  

Define the norm $$\|f\|_{D(\rho, \mu) \times U} = \sum_{\alpha \in \mathbb{N}^A} \|f^\alpha(\varphi; w)\|_{D(\rho) \times U \mu|\alpha|}.$$  

Definition 2.4. Let $$D(\rho, \mu, \sigma) = \{ (\varphi, r, \zeta) \in \mathbb{C}/(2\pi\mathbb{Z})^A \times \mathbb{C}^A \times \mathbb{C}^\mathcal{L} : |3\varphi| \leq \rho, |r| \leq \mu, ||\zeta||_p \leq \sigma \},$$  
$$f : D(\rho, \mu, \sigma) \times U \to \mathbb{C}$$ be analytic in $$(\varphi, r, \zeta) \in D(\rho, \mu, \sigma)$$ and $$C^1$$ in $$w \in U,$$  
$$f(\varphi, r, \zeta; w) = \sum_{\alpha \in \mathbb{N}^A, \beta \in \mathbb{N}\mathcal{L}} f^{\alpha\beta}(\varphi; w)r^\alpha\zeta^\beta,$$  

where $$\mathcal{L} = \mathcal{L}_{-1} \cup \mathcal{L}, \mathcal{L}_{-1} = \mathcal{L}.$$ For a $$a \in \mathcal{L}_{-1}, \zeta_a = \xi_a,$$ for $$a \in \mathcal{L}, \zeta_a = \eta_a.$$ Define the modulus $$|f|_{D(\rho, \mu) \times U}(\zeta) = \sum_{\beta \in \mathbb{N}\mathcal{L}} \|f^\beta(\varphi, r; w)\|_{D(\rho, \mu) \times U \zeta^\beta},$$  

where $$f^\beta(\varphi, r; w) = \sum_{\alpha \in \mathbb{N}^A} f^{\alpha\beta}(\varphi; w)r^\alpha.$$  

For a homogeneous polynomial $$f(\zeta)$$ of degree $$h > 0,$$ it is associated with a symmetric $$h$$-linear form $$\tilde{f}(\zeta^{(1)}, \ldots, \zeta^{(h)})$$ such that $$\tilde{f}(\zeta^{(1)}, \ldots, \zeta^{(h)}) = f(\zeta).$$ For a monomial $$f(\zeta) = f^\beta\zeta^\beta = f^\beta\zeta_{j_1} \cdots \zeta_{j_h},$$  

we define $$\tilde{f}(\zeta^{(1)}, \ldots, \zeta^{(h)}) = \tilde{f}_{j_1}^{\beta}(\zeta^{(1)}, \ldots, \zeta^{(h)}) = \frac{1}{h!} \sum_{\tau_h} f^\beta(\zeta_{\tau_h(1)}, \ldots, \zeta_{\tau_h(h)}),$$  

where $$\tau_h$$ is an $$h$$-permutation. For a homogeneous polynomial $$f(\zeta) = \sum_{|\beta| = h} f^\beta\zeta^\beta,$$
we have
\[ \tilde{f}(\zeta^{(1)}, \ldots, \zeta^{(h)}) = \sum_{|\beta| = h} f^{\beta} \zeta^\beta. \]

Now we can define \textit{p-tame norm} of a Hamiltonian vector field. We first consider a Hamiltonian
\[ f(\varphi, r, \zeta; w) = f_h = \sum_{\alpha \in \mathbb{N}^A, \beta \in \mathbb{N}^L, |\beta| = h} f^\alpha_\beta (\varphi; w) r^\alpha \zeta^\beta, \]
which is homogenous polynomial in \( \zeta \) of order \( h \). Letting \( f_\zeta = (f_\eta, -f_\xi) \), the Hamiltonian vector field \( X_f \) is given by \((f_r, -f_\varphi, f_\zeta)\). For \( h \geq 1 \), denote
\[ \| (\zeta^h) \|_{p,1} = \frac{1}{h} \sum_{j=1}^h \| \zeta^{(1)} \|_1 \cdots \| \zeta^{(j-1)} \|_1 \| \zeta^{(j)} \|_p \| \zeta^{(j+1)} \|_1 \cdots \| \zeta^{(h)} \|_1. \]

Recall the domains \( D(\rho, \mu) \) and \( D(\rho, \mu, \sigma) \) in (2.1) and (2.2), respectively.

**Definition 2.5.** For \( f \) in (2.3), we define
\[ ||| f^\xi |||_{p,D(\rho,\mu) \times U}^T = \sup_{0 \neq (\zeta) \in \mathbb{L}^h, 1 \leq j \leq h-1} \frac{||| \tilde{f}^\xi |||_{D(\rho,\mu) \times U}(\zeta^{(1)}, \ldots, \zeta^{(h-1)}) |||_p}{||| \zeta^{(h)} |||_{1,1}}, h \geq 2, \]
\[ ||| f^\xi |||_{p,D(\rho,\mu) \times U}^T = ||| \tilde{f}^\xi |||_{D(\rho,\mu) \times U} |||_p, h = 0,1. \]

Then we define the \( p \)-tame norm of \( f^\xi \) by
\[ ||| f^\xi |||_{p,D(\rho,\mu,\sigma) \times U}^T = \max( ||| f^\xi |||_{p,D(\rho,\mu) \times U}^T, ||| f^\xi |||_{1,D(\rho,\mu) \times U}^T ) \sigma^h. \]

**Definition 2.6.** For \( f \) in (2.3), we define
\[ ||| f^\varphi |||_{D(\rho,\mu) \times U}^T = \sup_{0 \neq (\zeta) \in \mathbb{L}^h, 1 \leq j \leq h} \frac{||| \tilde{f}^\varphi |||_{D(\rho,\mu) \times U}(\zeta^{(1)}, \ldots, \zeta^{(h)}) |||_1}{||| \zeta^{(h)} |||_{1,1}}, h \geq 1, \]
\[ ||| f^\varphi |||_{D(\rho,\mu,\sigma) \times U}^T = ||| \tilde{f}^\varphi |||_{D(\rho,\mu) \times U} |||_1, h = 0. \]

Then we define the norm of \( f^\varphi \) by
\[ ||| f^\varphi |||_{D(\rho,\mu,\sigma) \times U} = ||| f^\varphi |||_{D(\rho,\mu) \times U} \sigma^h. \]

The norm of \( f^\varphi \) is defined as that of \( f^\varphi \).

**Definition 2.7.** For \( f \) in (2.3), the \( p \)-tame norm of the Hamiltonian vector field \( X_f \) is defined by
\[ ||| X_f |||_{p,D(\rho,\mu,\sigma) \times U}^T = ||| f^\varphi |||_{D(\rho,\mu,\sigma) \times U} + \frac{1}{\mu} ||| f^\xi |||_{D(\rho,\mu,\sigma) \times U} + \frac{1}{\sigma} ||| f^\zeta |||_{p,D(\rho,\mu,\sigma) \times U}. \]

**Definition 2.8.** For a general Hamiltonian
\[ f(\varphi, r, \zeta; w) = \sum_{h \geq 0} f_h, f_h = \sum_{\alpha \in \mathbb{N}^A, \beta \in \mathbb{N}^L, |\beta| = h} f^\alpha_\beta (\varphi; w) r^\alpha \zeta^\beta, \]
define the \( p \)-tame norm of the Hamiltonian vector field \( X_f \) by
\[ ||| X_f |||_{p,D(\rho,\mu,\sigma) \times U}^T = \sum_{h \geq 0} ||| X^h_f |||_{p,D(\rho,\mu,\sigma) \times U}. \]
Remark 2.9. The $p$-tame norm can also be defined in complex coordinates
\[ z = \begin{pmatrix} u \\ v \end{pmatrix} = C^{-1} \begin{pmatrix} \xi \\ \eta \end{pmatrix}, C = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -i & i \end{pmatrix}. \]

Following the proof of Theorem 3.1 in [5], we have the following proposition.

Proposition 2.10. If $0 < \tau < \rho, 0 < \tau' < \frac{\sigma}{2}$, then
\[ |||X_{\{f,g\}}|||_{p,D(\rho-\tau, (\sigma-\tau')^2, \sigma-\tau')} \times U \leq C \max \left( \frac{1}{\tau}, \frac{\sigma}{\tau'} \right) |||X_f|||_{p,D(\rho, \sigma^2, \sigma)} \times U \times |||X_g|||_{p,D(\rho, \sigma^2, \sigma)} \times U, \]
where $C > 0$ is a constant depending on $\#A$.

We define the weighted norm of the Hamiltonian vector field $X_f$ by
\[ |||X_f|||_{p,D(\rho, \mu, \sigma)} \times U = \sup_{(\varphi, r, \xi, w) \in D(\rho, \mu, \sigma)} \|X_f\|_{p,D(\rho, \mu, \sigma)}, \]
where
\[ \|X_f\|_{p,D(\rho, \mu, \sigma)} = |f_r| + \frac{1}{\mu} |f_{\varphi}| + \frac{1}{\sigma} |f_{\xi}|. \]

Following the proof of Theorem 3.5 in [5], we have
\[ |||X_f|||_{p,D(\rho, \mu, \sigma)} \times U \leq |||X_f|||_{p,D(\rho, \mu, \sigma)} \times U. \]

2.1.3. Normal form matrices. We introduce the blocks in [8]. For $\Delta \geq 0$, define an equivalence relation on $\mathcal{L} = \mathbb{Z}^d \setminus \mathcal{A}$ generated by the pre-equivalence relation
\[ a \sim b \iff |a| = |b|, |a - b| \leq \Delta. \]
Let $[a]_\Delta$ be the equivalence class (block) of $a$ and $\mathcal{E}_\Delta$ be the set of equivalence classes. Let $d_\Delta$ be the supremum of all block diameters, then by Proposition 4.1 in [8], there is $d_\Delta \leq \Delta^{\frac{d+1}{2}}$.

A matrix $A : \mathcal{L} \times \mathcal{L} \to \text{gl}(2, \mathbb{C})$ is on normal form, denoted $\mathcal{NF}_\Delta$, if $A$ is real valued, symmetric, $\pi A = A$ and block-diagonal over $\mathcal{E}_\Delta$, i.e., $A^0 = 0, \forall [a]_\Delta \neq [b]_\Delta$. A matrix $Q : \mathcal{L} \times \mathcal{L} \to \mathbb{C}$ is on normal form, denoted $\mathcal{NF}_\Delta$, if $Q$ is Hermitian and block-diagonal over $\mathcal{E}_\Delta$.

For a normal form matrix $A$, we write
\[ \frac{1}{2} \langle \xi, A \xi \rangle = \frac{1}{2} \langle \xi, A_1 \xi \rangle + \langle \xi, A_2 \eta \rangle + \frac{1}{2} \langle \eta, A_1 \eta \rangle, \]
where $A_1 + iA_2$ is a Hermitian matrix. Let
\[ z = \begin{pmatrix} u \\ v \end{pmatrix} = C^{-1} \begin{pmatrix} \xi \\ \eta \end{pmatrix}, C = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -i & i \end{pmatrix} \]
and define $CTAC : \mathcal{L} \times \mathcal{L} \to \text{gl}(2, \mathbb{C})$ by $(CTAC)_a^b = CT A^c_b C$. If $A$ is on normal form, then
\[ \frac{1}{2} \langle z, CTAC z \rangle = \langle u, Qv \rangle, \]
where $Q$ is the normal form matrix associated to $A$. 
2.2. The homological equations. In this part, we formulate and solve the homological equations in the KAM iteration, from which we derive the quantitative estimates of the symplectic transformation and the new Hamiltonian vector field. See Proposition 2.11. To begin with, we make some basic assumptions (H1)-(H3) below.

Assumption (H1): Consider the unperturbed Hamiltonian
\[ h(r, \zeta; w) = \langle \omega(w), r \rangle + \frac{1}{2} \langle \zeta, (\Omega(w) + H(w)) \zeta \rangle, \]
where \( \Omega(w) \) is a real diagonal matrix with diagonal elements \( \Omega_a(w)I \). \( H(w) \), \( \partial_w H(w) \) are Töplitz at \( \infty \) and \( \mathcal{N} \mathcal{F}_A \) for all \( w \in U \).

Assume
\[
\begin{align*}
\partial_{w_a} \omega_b(w) &= \delta_{ab}, \quad a \in \mathcal{A}, b \in \mathcal{A}, w \in U, \\
\partial_{w_a} \Omega_b(w) &= \delta_{ab}, \quad a \in \mathcal{A}, b \in \mathcal{L}, w \in U.
\end{align*}
\]
Assume further that there exist constants \( c_1, c_2, c_3, c_4, c_5 > 0 \) such that
\[
\begin{align*}
|\Omega_a(w) - |a|^2| &\leq c_1 e^{-c_2 |a|}, \quad a \in \mathcal{L}, w \in U, \\
|\Omega_a(w)| &\geq c_3, \quad a \in \mathcal{L}, w \in U, \\
|\Omega_a(w) + \Omega_b(w)| &\geq c_4, \quad a, b \in \mathcal{L}, w \in U, \\
|\Omega_a(w) - \Omega_b(w)| &\geq c_5, \quad |a| \neq |b|, \quad a, b \in \mathcal{L}, w \in U, \\
\|H(w)\| &\leq \frac{c_3}{4}, \quad w \in U, \\
\|\partialHH(w)\| &\leq c_4, \quad w \in U.
\end{align*}
\]
\[
\langle H \rangle_{A; U} \leq c_5.
\]
Assumption (H2): Consider the perturbation \( f : D(\rho, \mu, \sigma) \times U \to \mathbb{C} \) and write
\[ f(\varphi, r, \zeta; w) = f^{low} + f^{high}, \]
where
\[
\begin{align*}
f^{low} &= f^\varphi + f^0 + f^1 + f^2 = F^\varphi(\varphi; w) + \langle F_0(\varphi; w), r \rangle + \langle F_1(\varphi; w), \zeta \rangle + \frac{1}{2} \langle F_2(\varphi; w) \zeta, \zeta \rangle.
\end{align*}
\]
Assume that
\[
\begin{align*}
|||X_{f^{low}}|||_{T, D(\rho, \mu, \sigma) \times U}^T &\leq \varepsilon, \\
|||X_{f^{high}}|||_{T, D(\rho, \mu, \sigma) \times U}^T &\leq 1,
\end{align*}
\]
Let
\[
D^\gamma(\rho, \mu, \sigma) = \{(\varphi, r, \zeta) \in (\mathbb{C}/2\pi\mathbb{Z})^A \times \mathbb{C}^A \times \mathbb{R}^2_{\rho, \gamma} : |3\varphi| \leq \rho, |r| \leq \mu, \|\zeta\|_{p, \gamma} \leq \sigma\},
\]
and suppose that \( f : D^\gamma(\rho, \mu, \sigma) \times U \to \mathbb{C} \) is real analytic in \( (\varphi, r, \zeta) \in D^\gamma(\rho, \mu, \sigma) \) and \( C^1 \) in \( w \in U \). Define
\[
[f]_{A, \gamma, \sigma; U, \rho, \mu} = \sup_{(\varphi, r) \in D(\rho, \mu)} [f(\varphi, r, \cdot)]_{A, \gamma, \sigma; U},
\]
in which the norm inside the supremum is defined in Definition 2.7. Assume further that
\[
\begin{align*}
[f^{low}]_{A, \gamma, \sigma; U, \rho, \mu} &\leq \varepsilon, \\
[f^{high}]_{A, \gamma, \sigma; U, \rho, \mu} &\leq 1.
\end{align*}
\]
Assumption (H3): Let $\Delta' > 1$ and $0 < \kappa < 1$. Assume there exists $U' \subset U$ such that for all $w \in U'$, $0 < |k| \leq \Delta'$,

(2.16) \quad |\langle k, \omega(w) \rangle| \geq \kappa,

(2.17) \quad |\langle k, \omega(w) \rangle + \alpha(w)| \geq \kappa, \quad \forall \alpha(w) \in \text{spec}(((\Omega + H)(w))_{[\alpha]}), \forall [\alpha]_{\Delta},

(2.18) \quad |\langle k, \omega(w) \rangle + \alpha(w) + \beta(w)| \geq \kappa, \quad \forall \left\{ \begin{array}{l} \alpha(w) \in \text{spec}(((\Omega + H)(w))_{[\alpha]}), \\ \beta(w) \in \text{spec}(((\Omega + H)(w))_{[\beta]}), \end{array} \right\}, \forall [\alpha]_{\Delta}, [\beta]_{\Delta},

and

(2.19) \quad |\langle k, \omega(w) \rangle + \alpha(w) - \beta(w)| \geq \kappa, \quad \forall \left\{ \begin{array}{l} \alpha(w) \in \text{spec}(((\Omega + H)(w))_{[\alpha]}), \\ \beta(w) \in \text{spec}(((\Omega + H)(w))_{[\beta]}), \end{array} \right\},

for $\text{dist}([\alpha]_{\Delta}, [\beta]_{\Delta}) \leq \Delta' + 2d_{\Delta}$.

Recalling $f_{\text{low}}$ in (2.13), we define the truncation operator

$$T_{\Delta'} f_{\text{low}} = \sum_{|k| \leq \Delta'} \left( \hat{F}_{\varphi}(k; w) + \langle \hat{F}_0(k; w), r \rangle + \langle \hat{F}_1(k; w), \zeta \rangle + \frac{1}{2} \langle T_{\Delta'} \hat{F}_2(k; w), \zeta \rangle \right) e^{i(k, \varphi)}.$$

Now we state our main results on the homological equations, whose proof is delayed to the next subsection.

Proposition 2.11. Under the assumptions (H1)-(H3), if $\gamma, \sigma, \rho, \mu < 1$, $\Lambda, \Delta \geq 3$, $\rho = \sigma$, $\mu = \sigma^2$, $d_{\Delta} \gamma < 1$, then for all $w \in U'$, the homological equation

(2.20) \quad \{ h, s \} = -T_{\Delta'} f_{\text{low}} - T_{\Delta'} \{ f_{\text{high}}, s \}_{\text{low}} + h_1

has solutions

(2.21) \quad s(\varphi, r, \zeta; w) = s_{\text{low}} = s^0 + s^1 + s^2,

(2.22) \quad h_1(r, \zeta; w) = a_1(w) + \langle \chi_1(w), r \rangle + \frac{1}{2} \langle \zeta, H_1(w) \zeta \rangle

with the estimates

\[
\begin{align*}
||| X_{s^0} |||_{p, D(\rho - \tau, \sigma^2) \times U'} &< a, \\
||| X_{s^1} |||_{p, D(\rho - 3\tau, (\sigma - 3\tau)^2 \times U'} &< \frac{d_{\Delta}^3 \varepsilon}{\tau^3 k^4}, \\
||| X_{s^2} |||_{p, D(\rho - 5\tau, (\sigma - 5\tau)^2 \times U'} &< \frac{d_{\Delta}^5 \varepsilon}{\tau^3 k^4}, \\
\end{align*}
\]

(2.23)

\[
\begin{align*}
||| X_{s^3} |||_{p, D(\rho - 3\tau, (\sigma - 3\tau)^2 \times U'} &< \frac{d_{\Delta}^3 \varepsilon}{\tau^5 k^6}, \\
||| X_{s^4} |||_{p, D(\rho - 5\tau, (\sigma - 5\tau)^2 \times U'} &< \frac{d_{\Delta}^5 \varepsilon}{\tau^5 k^6}, \\
||| X_{s^5} |||_{p, D(\rho - 5\tau, (\sigma - 5\tau)^2 \times U'} &< \frac{d_{\Delta}^7 \varepsilon}{\tau^5 k^6}, \\
\end{align*}
\]

where $0 < \tau < \frac{\rho}{100}$, $a \leq b$ means there exists a constant $c > 0$ depending on $d$, $\#A, p, c_1, c_2, c_3, c_4, c_5$ such that $a \leq cb$.

The new Hamiltonian

(2.24) \quad (h + f) \circ X_{s^1}^{\tau} |||_{t=1} = h + h_1 + f_1,
with the estimates we shall not indicate the dependence on the parameter $w$ is not repeated here. Since the proof is too long, we divide it into five steps. In what follows,

\[ f_1 = (1 - T_\Delta) f^{\text{low}} + f^{\text{high}} + (1 - T_\Delta) \{ f^{\text{high}}, s \}^{\text{low}} + \{ f^{\text{high}}, s \}^{\text{high}} \]

\[
+ \int_0^1 (1 - t) \{ \{ h, s \}, s \} \circ X_t^s dt + \int_0^1 \{ f^{\text{low}}, s \} \circ X_t^s dt + \int_0^1 (1 - t) \{ \{ f^{\text{high}}, s \}, s \} \circ X_t^s dt
\]

with the estimates

\[ |||X^{f_{t,w}} |||^2_{p, D(\rho - 8\tau, (\sigma - 8\tau)^2, \sigma - 8\tau) \times U'} \leq \frac{d^{d-\varepsilon} e^{-\frac{1}{2} T_\Delta'}}{\kappa^{d-p} A^{4+\varepsilon}} + \frac{(\Delta_\Delta')^{\exp} e^{-\frac{1}{2} \gamma \Delta'}}{\kappa^{d-p} A^{4+\varepsilon}} + \frac{d^{d\varepsilon} e^{2}}{\tau^{12} \kappa^{12}}, \]

\[ |||X^{f_{t,w}} |||^T_{p, D(\rho - 8\tau, (\sigma - 8\tau)^2, \sigma - 8\tau) \times U'} \leq 1 + \frac{d^{d\varepsilon} e^{2}}{\tau^{12} \kappa^{12}}, \]

where the exponent $\varepsilon$ depends on $d, \# A, p$.

Moreover, the following estimates hold:

1. The functions $s$ and $h_1$ satisfy

\[ [s] N + d + 2, \gamma, \sigma', U', \rho, \rho' \leq \frac{1}{\kappa^{d}} (\Delta_\Delta')^{\exp} \frac{1}{\rho - \rho'} \left( \frac{1}{\sigma - \sigma'} + \frac{1}{\sigma - \rho'} \frac{1}{\mu - \mu'} \right) \frac{1}{\mu} \varepsilon, \]

\[ [h_1] N + d + 2, \gamma, \sigma', U', \rho, \rho' \leq \frac{1}{\kappa^{d}} (\Delta_\Delta')^{\exp} \frac{1}{\rho - \rho'} \left( \frac{1}{\sigma - \sigma'} + \frac{1}{\sigma - \rho'} \frac{1}{\mu - \mu'} \right) \frac{1}{\mu} \varepsilon, \]

where $\rho' < \rho, \mu' < \mu, \sigma' < \sigma, N' \geq \max \{ \Lambda, d_\Delta, d_{\Delta'} \}$, the constant $c_{\text{te}}$ is the one in Proposition 6.7 in [8].

2. There is measure estimate

\[ \text{meas}(U \setminus U') \leq \max(\Lambda, \Delta, \Delta')^{\exp} \kappa^{(\frac{1}{\kappa} - 1)^d}. \]

**Remark 2.12.** In Eliasson-Kuksin [8], the homological equation therein is

\[ \{ h, s \} = -T_\Delta f^{\text{low}} + h_1. \]

As we see, the term $\{ f^{\text{high}}, s \}^{\text{low}}$ is not eliminated and is put into the new perturbation along the iteration. To obtain a fast convergent iteration, they require the radius of domain for the variables $r$ and $\zeta$ to decrease to zero. However, in the present paper, we follow Kolmogorov’s iterative scheme, which eliminates also the lower order term $\{ f^{\text{high}}, s \}^{\text{low}}$ (up to the truncation). This certainly complicates the homological equation when expressing in Fourier modes (see [2.31], [2.31] below), but enables us to obtain a uniform domain after the KAM iteration. Based on the uniform domain, we can establish a normal form of the Hamiltonian vector field around the obtained KAM tori and then study its local stability.

### 2.3. Proof of Proposition 2.11

The whole subsection is devoted to the proof of Proposition 2.11. Note that the measure estimate (2.29) follows from Proposition 6.6 and 6.7 in [8], which is not repeated here. Since the proof is too long, we divide it into five steps. In what follows, we shall not indicate the dependence on the parameter $w$ of functions when it is known from the text.

#### 2.3.1. Step 1: Write the homological equation in Fourier modes.

Recall that

\[ f^{\text{low}} = f^\varphi + f^0 + f^1 + f^2 = F^\varphi(\varphi; w) + \langle F_0(\varphi; w), r \rangle + \langle F_1(\varphi; w), \zeta \rangle + \frac{1}{2} \langle F_2(\varphi; w), \zeta \rangle, \]

\[ s = s^{\text{low}} = s^\varphi + s^0 + s^1 + s^2 = S^\varphi(\varphi; w) + \langle S_0(\varphi; w), r \rangle + \langle S_1(\varphi; w), \zeta \rangle + \frac{1}{2} \langle S_2(\varphi; w), \zeta \rangle. \]
By the calculation of Section 4.1.2 in [5], we obtain
\[
\{f^{\text{high}}, s\}^0 = \{f^{\text{high}}, s\}^0 + \{f^{\text{high}}, s\}^1 + \{f^{\text{high}}, s\}^2,
\]
(2.30)
\[
\{f^{\text{high}}, s\}^1 = \{f^{\text{high}}, s^+\}^1,
\]
\[
\{f^{\text{high}}, s\}^0 = \{f^{\text{high}}, s^+ + s\}^0, \quad \{f^{\text{high}}, s\}^2 = \{f^{\text{high}}, s^+ + s\}^2.
\]
Let \(g = \{f^{\text{high}}, s\}\) and write
\[
g^{\text{low}} = g^0 + g^1 + g^2 = \langle G_0(\varphi; w), r \rangle + \langle G_1(\varphi; w), \zeta \rangle + \frac{1}{2} \langle G_2(\varphi; w)\zeta, \zeta \rangle.
\]

In Fourier modes, the homological equation (2.20) decomposes into
\[
-i\langle k, \omega(w)\rangle \hat{S}^\varphi(k; w) = -\hat{F}^\varphi(k; w) + \delta_0^k a_1(w),
\]
(2.31)
\[
-i\langle k, \omega(w)\rangle \hat{S}_1(k; w) + J(\Omega(w) + H(w)) \hat{S}_1(k; w) = -\hat{F}_1(k; w) - \hat{G}_1(k; w),
\]
(2.32)
\[
-i\langle k, \omega(w)\rangle \hat{S}_0(k; w) = -\hat{F}_0(k; w) - \hat{G}_0(k; w) + \delta_0^k \chi_1(w),
\]
(2.33)
\[
-i\langle k, \omega(w)\rangle \hat{S}_2(k; w) + \Omega(\omega(w) + H(w)) \hat{S}_2(k; w) - \hat{S}_2(k; w) J(\Omega(w) + H(w))
\]
\[
= -\hat{F}_2(k; w) - \hat{G}_2(k; w) + \delta_0^k \hat{H}_1(w).
\]

2.3.2. Step 2: Solve the homological equations (2.31)-(2.34). Since equations (2.31)-(2.34) are coupled, we solve them in the following order
\[
(2.31) \rightarrow (2.32) \rightarrow (2.33) \rightarrow (2.34).
\]

Solution of (2.31). The homological equation (2.31) is very standard in KAM theory. We obtain from (2.31) that
\[
a_1 = \hat{F}^\varphi(0), \quad \hat{S}^\varphi(k) = \frac{\hat{F}^\varphi(k)}{\iota(k, \omega)}, \quad k \neq 0.
\]
By (2.16), we have
\[
|\hat{S}^\varphi(k)| \leq \frac{1}{\kappa} |\hat{F}^\varphi(k)|.
\]
Differentiating (2.31) with respect to the parameter \(w\), we derive a similar homological equation
\[
-i\partial_w \langle \langle k, \omega(w)\rangle \hat{S}^\varphi(k; w) - i\langle k, \omega(w)\rangle \partial_w \hat{S}^\varphi(k; w) = -\partial_w \hat{F}^\varphi(k; w)
\]
for \(\partial_w \hat{S}^\varphi(k; w)\) and there is also
\[
|\partial_w \hat{S}^\varphi(k)| \leq \frac{1}{\kappa} |\hat{S}^\varphi(k)| + |\partial_w \hat{F}^\varphi(k)|,
\]
which together with (2.35) implies
\[
|\hat{S}^\varphi(k)| + |\partial_w \hat{S}^\varphi(k)| \leq \frac{|k| + 1}{\kappa^2} (|\hat{F}^\varphi(k)| + |\partial_w \hat{F}^\varphi(k)|).
\]
It then follows that
\[
\|S^\varphi\|_{D(\rho - \tau) \times U} = \sum_{k \in \mathbb{Z}^d} \left( |\hat{S}^\varphi(k)| + |\partial_w \hat{S}^\varphi(k)| \right) e^{k|\rho - \tau|}
\]
\[
\leq \sum_{k \in \mathbb{Z}^d} \frac{|k| + 1}{\kappa^2} (|\hat{F}^\varphi(k)| + |\partial_w \hat{F}^\varphi(k)|) e^{k|\rho - \tau|} \leq \frac{1}{\tau \kappa^2} \|F^\varphi\|_{D(\rho) \times U}.
\]
Consequently, we have
\[ \|X_{x^*}\|_{p, D(\rho-\tau, \sigma^2, \sigma) \times U'} \leq \frac{1}{\tau K^2} \|X_{f^*}\|_{p, D(\rho, \sigma^2, \sigma) \times U'}. \]

**Solution of (2.32).** Now we consider the equation (2.32) and write it simply, we as
\[ i(k, \omega(w))S + J(\Omega + H)S = F + G. \]

We change to complex coordinates
\[ z = \left( \frac{u}{v} \right) = C^{-1} \left( \begin{array}{c} \xi \\ \eta \end{array} \right), \quad C = \frac{1}{\sqrt{2}} \left( \begin{array}{cc} 1 & 1 \\ -1 & i \end{array} \right). \]

Let \( S' = C^{-1} S = \left( \begin{array}{c} S_1' \\ S_2' \end{array} \right), \quad F' = C^{-1} F = \left( \begin{array}{c} F_1' \\ F_2' \end{array} \right), \quad G' = C^{-1} G = \left( \begin{array}{c} G_1' \\ G_2' \end{array} \right). \] The equation (2.32) becomes
\[ i(k, \omega(w))S' - i(\Omega + HT)S' = F' + G'. \]

We only solve \( S'_1 \) in (2.38) since \( S'_2 \) can be solved accordingly. By (2.17), we have
\[ \|S'_1[\alpha]\| \leq \frac{1}{K} \|F'_1[\alpha] + G'_1[\alpha]\|. \]

Using similar arguments to that of (2.39), we get
\[ \|S'_1[\alpha]\| + \|\partial_x S'_1[\alpha]\| \leq \frac{1}{K^2} \left( \|F'_1[\alpha] + G'_1[\alpha]\| + \|\partial_x F'_1[\alpha] + \partial_x G'_1[\alpha]\| \right). \]

For \( z = (u, v) \), define \( \tilde{z} = (\tilde{u}, \tilde{v}) \) such that for all \( a \in [\alpha], \quad \tilde{u}_a = \|u[a]\|, \quad \tilde{v}_a = \|v[a]\|. \) It follows from (2.39) that
\[ \left| \sum_{a \in \mathcal{L}} \|S'_{1a}(\varphi)\|_{D(\rho - 3\tau) \times U'v_a} \right| \]
\[ \leq \sum_{a \in \mathcal{L}} \sum_{k \in \mathbb{Z}^A} \sum_{\alpha \in [\alpha]} \left( |S'_{1a}(k)| + |\partial_x S'_{1a}(k)| \right) e^{(\rho - 3\tau)|k|} \left| u_a \right| \]
\[ = \sum_{k \in \mathbb{Z}^A} \sum_{\alpha \in [\alpha]} \sum_{a \in \mathcal{L}} \left( |S'_{1a}(k)| + |\partial_x S'_{1a}(k)| \right) \left| u_a \right| e^{(\rho - 3\tau)|k|} \]
\[ \leq \sum_{k \in \mathbb{Z}^A} \sum_{\alpha \in [\alpha]} \sum_{a \in \mathcal{L}} \left( |F'_{1a}(\varphi) + \tilde{G}'_{1a}(\varphi)|, (k) \right) \left| u_a \right| e^{(\rho - 3\tau)|k|} \]
\[ \leq \frac{1}{\tau K^2} \sum_{a \in \mathcal{L}} \|F'_{1a}(\varphi) + \tilde{G}'_{1a}(\varphi)\|_{D(\rho - 2\tau) \times U'v_a}. \]

There is a similar estimate for that of \( \sum_{a \in \mathcal{L}} \|S'_{2a}(\varphi)\|_{D(\rho - 3\tau) \times U'v_a}. \)

Noticing that
\[ s^1 = \sum_{a \in \mathcal{L}} (S'_{1a}(\varphi)u_a + S'_{2a}(\varphi)v_a), \]
and
\[ |s^1|_{D(\rho - 3\tau, \sigma^2) \times U'} = \sum_{a \in \mathcal{L}} (|S'_{1a}(\varphi)|_{D(\rho - 3\tau) \times U'u_a} + |S'_{2a}(\varphi)|_{D(\rho - 3\tau) \times U'v_a}). \]
we get

\[ ||s_\phi^1|||D_{(\rho-3\tau,\sigma^2)}\times U'|| z|| \leq \frac{1}{\tau K^2} \left( \sum_{k \in \mathbb{Z}^2} |s_\phi^1(k) + g_\phi^1| \right) D_{(\rho-2\tau,\sigma^2)} \times U' \].

It then follows from

\[ \sup_{\theta \neq z \in \mathbb{T}_1} ||s_\phi^1|||D_{(\rho-3\tau,\sigma^2)}\times U'|| z|| = \frac{d_\Delta^2}{\tau K^2} \left( \sum_{k \in \mathbb{Z}^2} |s_\phi^1(k) + g_\phi^1| \right) D_{(\rho-3\tau,\sigma^2)} \times U' \]

and \( \|z\|_1 \leq d_\Delta^2 \|z\|_1 \) that

\[ ||s_\phi^1|||D_{(\rho-3\tau,\sigma^2)}\times U'|| = \frac{d_\Delta^2}{\tau K^2} \left( \sum_{k \in \mathbb{Z}^2} |s_\phi^1(k) + g_\phi^1| \right) D_{(\rho-2\tau,\sigma^2)} \times U'. \]

Next we estimate

\[ ||s_\phi^1|||D_{(\rho-3\tau,\sigma^2)}\times U'|| = \frac{d_\Delta^2}{\tau K^2} \left( \sum_{k \in \mathbb{Z}^2} |s_\phi^1(k) + g_\phi^1| \right) D_{(\rho-2\tau,\sigma^2)} \times U'. \]

By (2.33), we see from the Minkowski’s inequality that

\[ \left( \sum_{\alpha \in [\Delta]} \||S_{1\alpha}'(\phi)|||D_{(\rho-3\tau,\sigma^2)}\times U'| \right)^\frac{2}{p} \leq \sum_{\alpha \in [\Delta]} \left( \sum_{k \in \mathbb{Z}^2} |(s_\phi^1(k) + g_\phi^1)| e^{(\rho-3\tau)|k|} \right)^\frac{2}{p} \]

\[ \leq \sum_{\alpha \in [\Delta]} \left( \sum_{k \in \mathbb{Z}^2} |(s_\phi^1(k) + g_\phi^1)| e^{(\rho-3\tau)|k|} \right)^\frac{2}{p} \]

\[ \leq \frac{1}{\tau K^2} \sum_{\alpha \in [\Delta]} \||F_{1\alpha}'(\phi) + G_{1\alpha}'(\phi)|||D_{(\rho-2\tau)}\times U'|| \]

As a result, there is

\[ \sum_{\alpha \in [\Delta]} \||S_{1\alpha}'(\phi)|||D_{(\rho-3\tau,\sigma^2)}\times U'| \leq \frac{d_\Delta^2}{\tau K^2} \sum_{\alpha \in [\Delta]} \||F_{1\alpha}'(\phi) + G_{1\alpha}'(\phi)|||D_{(\rho-2\tau)}\times U'|| \]

and similar estimate of \( \sum_{\alpha \in [\Delta]} \||S_{2\alpha}'(\phi)|||D_{(\rho-3\tau,\sigma^2)}\times U'| \leq \frac{d_\Delta^2}{\tau K^2} \sum_{\alpha \in [\Delta]} \||F_{2\alpha}'(\phi) + G_{2\alpha}'(\phi)|||D_{(\rho-2\tau)}\times U'|| \]

which together with (2.40) implies

\[ ||X_{s_\phi^1}|||D_{(\rho-3\tau,\sigma^2,\sigma)}\times U'|| = \frac{d_\Delta^2}{\tau K^2} \left( \sum_{k \in \mathbb{Z}^2} |s_\phi^1(k) + g_\phi^1| \right) D_{(\rho-2\tau,\sigma^2)} \times U'. \]

Solution of (2.33). Solving equation (2.33) as equation (2.31), we obtain

\[ ||X_{s_\phi^1}|||D_{(\rho-5\tau,\sigma^2,\sigma)}\times U'|| \leq \frac{1}{\tau K^2} \left( \sum_{k \in \mathbb{Z}^2} |s_\phi^1(k) + g_\phi^1| \right) D_{(\rho-4\tau,\sigma^2,\sigma)} \times U'. \]

Solution of (2.34). For simplicity, we write (2.34) as

\[ i(k, \omega(w))S + (\Omega + H)JS - SJ(\Omega + H) = F + G - H_1. \]
We change to complex coordinates $z = C^{-1} \zeta$. Let $S' = C^T S C = \begin{pmatrix} S'_1 & S'_2 & S'_3 \end{pmatrix}$ and $F' = C^T FC, G' = C^T GC, H'_1 = C^T H_1 C$. Then equation (2.31) becomes

\begin{align}
(2.43) & \quad i(k, \omega(w)) S'_1 + i(\Omega + H) S'_1 + i S'_1 (\Omega + H^T) = F'_1 + G'_1, \\
(2.44) & \quad i(k, \omega(w)) S'_2 + i(\Omega + H) S'_2 - i S'_1 (\Omega + H) = F'_2 + G'_2 - H'_{12}, \\
(2.45) & \quad i(k, \omega(w)) S'_3 T - i(\Omega + H^T) S'_3 + i S'_1 (\Omega + H^T) = F'_3 + G'_3.
\end{align}

For $k \neq 0, H'_{12} = 0$. By (2.19), we have

$$
\| S'^{[b, \Delta]}_{2[a, \Delta]} \| \leq \frac{1}{\kappa} \| F'^{[b, \Delta]}_{2[a, \Delta]} + G'^{[b, \Delta]}_{2[a, \Delta]} \|.
$$

Using similar argument to that of (2.36), we get

$$
(2.47) \quad \| S'^{[b, \Delta]}_{2[a, \Delta]} \| + \| \partial_w S'^{[b, \Delta]}_{2[a, \Delta]} \| \leq \frac{|k| + 1}{\kappa^2} \| F'^{[b, \Delta]}_{2[a, \Delta]} + G'^{[b, \Delta]}_{2[a, \Delta]} \| + \| \partial_w F'^{[b, \Delta]}_{2[a, \Delta]} + \partial_w G'^{[b, \Delta]}_{2[a, \Delta]} \|.
$$

For $k = 0, H'_{12} = (F'_2 + G'_2)_{(a,b) \in L \times L : |a| = |b|}$ and estimate (2.44) still holds.

Using (2.18), we also have

$$
(2.48) \quad \| S'^{[b, \Delta]}_{1[a, \Delta]} \| + \| \partial_w S'^{[b, \Delta]}_{1[a, \Delta]} \| \leq \frac{|k| + 1}{\kappa^2} \| F'^{[b, \Delta]}_{1[a, \Delta]} + G'^{[b, \Delta]}_{1[a, \Delta]} \| + \| \partial_w F'^{[b, \Delta]}_{1[a, \Delta]} + \partial_w G'^{[b, \Delta]}_{1[a, \Delta]} \|,
$$
$$
\| S'^{[b, \Delta]}_{3[a, \Delta]} \| + \| \partial_w S'^{[b, \Delta]}_{3[a, \Delta]} \| \leq \frac{|k| + 1}{\kappa^2} \| F'^{[b, \Delta]}_{3[a, \Delta]} + G'^{[b, \Delta]}_{3[a, \Delta]} \| + \| \partial_w F'^{[b, \Delta]}_{3[a, \Delta]} + \partial_w G'^{[b, \Delta]}_{3[a, \Delta]} \|.
$$

Recall that

$$
S^2 = \frac{1}{2} \sum_{a,b \in L} (S_{1a}^{ab}(\varphi) u_a u_b + 2 S_{2a}^{ab}(\varphi) u_a v_b + S_{3a}^{ab}(\varphi) v_a v_b).
$$

We consider first $S^2_\varphi$

$$
[ S^2_\varphi D_{(\rho-5\tau, \sigma^2)} \times U^r(z) ] = \frac{1}{2} \sum_{a,b \in L} (\| S_{1a\varphi}^{ab}(\varphi) \| D_{(\rho-5\tau)} \times U^r u_a u_b \\
+ 2 \| S_{2a\varphi}^{ab}(\varphi) \| D_{(\rho-5\tau)} \times U^r v_a v_b + \| S_{3a\varphi}^{ab}(\varphi) \| D_{(\rho-5\tau)} \times U^r v_a v_b ),
$$

and its associated multilinear form $\tilde{S}^2_\varphi$

$$
\tilde{S}^2_\varphi D_{(\rho-5\tau, \sigma^2)} \times U^r(z^{(1)}, z^{(2)}) = \frac{1}{2} \sum_{a,b \in L} (\| S_{1a\varphi}^{ab}(\varphi) \| D_{(\rho-5\tau)} \times U^r v_a^{(1)} u_b^{(2)} \\
+ \| S_{2a\varphi}^{ab}(\varphi) \| D_{(\rho-5\tau)} \times U^r (v_a^{(1)} v_b^{(2)} + v_a^{(2)} v_b^{(1)}) + \| S_{3a\varphi}^{ab}(\varphi) \| D_{(\rho-5\tau)} \times U^r v_a^{(1)} v_b^{(2)} ).
$$
By (2.48), we know that $$\left| \sum_{a \in [a], b \in [b]} \| S^b_{1a\varphi}(\varphi) \|_{D(\rho^{-5} \times U')} u_a^{(1)} u_b^{(2)} \right|$$ is less than

$$\sum_{k \in \mathbb{Z}^A} \left( \| \tilde{\zeta}_k \| + \| \tilde{\varphi}_k \| \right) \| w \| D(\rho^{-5} \times U') u_a^{(1)} u_b^{(2)} |.$$ 

Hence, we have

$$\| \tilde{\zeta}_k \| + \| \tilde{\varphi}_k \| \leq \sum_{k \in \mathbb{Z}^A} \left( \| \tilde{\zeta}_k \| + \| \tilde{\varphi}_k \| \right) \| w \| D(\rho^{-5} \times U') u_a^{(1)} u_b^{(2)} |.$$ 

which implies

$$\| \tilde{\zeta}_k \| + \| \tilde{\varphi}_k \| \leq \sum_{k \in \mathbb{Z}^A} \left( \| \tilde{\zeta}_k \| + \| \tilde{\varphi}_k \| \right) \| w \| D(\rho^{-5} \times U') u_a^{(1)} u_b^{(2)} |.$$ 

There are similar estimates for the other two summation in the R.H.S of (2.49). Then we obtain

$$\| \tilde{\zeta}_k \| + \| \tilde{\varphi}_k \| \leq \sum_{k \in \mathbb{Z}^A} \left( \| \tilde{\zeta}_k \| + \| \tilde{\varphi}_k \| \right) \| w \| D(\rho^{-5} \times U') u_a^{(1)} u_b^{(2)} |.$$ 

Since

$$\| \tilde{\zeta}_k \| + \| \tilde{\varphi}_k \| \leq \sum_{k \in \mathbb{Z}^A} \left( \| \tilde{\zeta}_k \| + \| \tilde{\varphi}_k \| \right) \| w \| D(\rho^{-5} \times U') u_a^{(1)} u_b^{(2)} |.$$ 

we see from $$\| \tilde{\zeta}_k \| + \| \tilde{\varphi}_k \| \leq \sum_{k \in \mathbb{Z}^A} \left( \| \tilde{\zeta}_k \| + \| \tilde{\varphi}_k \| \right) \| w \| D(\rho^{-5} \times U') u_a^{(1)} u_b^{(2)} |.$$ 

Next we estimate

$$\| s(z) \|_{D(\rho^{-5} \times U')} \leq \sup_{0 \neq z \in \mathbb{F}} \left( \| s(z) \|_{D(\rho^{-5} \times U')} \right)_p,$$

in which

$$\| s(z) \|_{D(\rho^{-5} \times U')} = \sup_{0 \neq z \in \mathbb{F}} \left( \| s(z) \|_{D(\rho^{-5} \times U')} \right)_p.$$ 

By (2.48), we see that $$\left( \sum_{a \in [a], b \in [b]} \| S^b_{1a\varphi}(\varphi) \|_{D(\rho^{-5} \times U') U_a^{(1)} u_b^{(2)} \|} \right)^2$$ equals to

$$\left( \sum_{a \in [a], b \in [b]} \left( \| S^b_{1a\varphi}(\varphi) \|_{D(\rho^{-5} \times U')} U_a^{(1)} u_b^{(2)} \| \right)^2 \right)^{\frac{1}{2}}.$$
which is less than
\[
\sum_{k \in \mathbb{Z}^d} \left[ \sum_{a \in [a]} \left( \sum_{b \in \mathcal{L}} \left( |\hat{S}^{tb}_{1a}(k)| + |\partial \hat{S}^{tb}_{1a}(k)| \right) e^{(\rho-5\tau)|k|} \right)^2 \right]^{\frac{1}{2}}.
\]

In conclusion, we have
\[
\sum_{a \in [a]\Delta} \left( \sum_{b \in \mathcal{L}} \left( s_{1a}^{tb}(\varphi) \right)_{D(\rho-5\tau),\sigma^2} U' \right)^2 \leq \left( \frac{d\Delta}{\tau K^2} \right)^2 \sum_{a \in [a]\Delta} \left( \sum_{b \in \mathcal{L}} \left( F_{1a}^{tb}(\varphi) + \hat{G}_{1a}^{tb}(\varphi) \right)_{D(\rho' \tau),\sigma^2} U' \right)^2.
\]

Similar estimates hold for the other three summations in R.H.S of (2.51). Then we have
\[
\|s_{1a}^\varphi\|_{D(\rho-5\tau),\sigma^2} U' \leq \frac{d\Delta}{\tau K^2} \|f_{\varphi}^2 + g_{\varphi}^2\|_{D(\rho' \tau),\sigma^2} U'.
\]

which together with \(\|\hat{z}\|_p \leq d\Delta \|\hat{z}\|_p\) implies
(2.52) \[
\|s_{1a}^\varphi\|_{D(\rho-5\tau),\sigma^2} U' \leq \frac{d\Delta}{\tau K^2} \|f_{\varphi}^2 + g_{\varphi}^2\|_{D(\rho' \tau),\sigma^2} U'.
\]

Combining (2.51) and (2.52), we have
(2.53) \[
\|X_{f_{h_{\varphi}},\sigma_{\varphi}}\|_{D(\rho-2\tau),\sigma^2} U' \leq \frac{d\Delta}{\tau K^2} \|X_{f_{\varphi}^2 + g_{\varphi}^2}\|_{D(\rho' \tau),\sigma^2} U'.
\]

2.3.3. **Step 3: Estimate of the vector fields** \(X_s\) **and** \(X_{h_1}\). **In this part**, we shall verify the estimates in (2.23). **We only estimate** \(X_s\) **since** \(X_{h_1}\) **is easier.**

Recall that \(s = s^0 + s^1 + s^2\). **By** (2.14) **and** (2.39), we have
\[
\|X_{s^0}\|_{p,D(\rho-\tau),\sigma^2} U' \leq \frac{1}{\tau K^2} \|X_{f_{\varphi}^2 + g_{\varphi}^2}\|_{p,D(\rho',\sigma^2)} U'.
\]

It then follows from Proposition (2.10) and (2.14) that
(2.53) \[
\|X_{f_{h_{\varphi},s_{\varphi}}}\|_{p,D(\rho-2\tau),\sigma^2} U' \leq \frac{d\Delta}{\tau K^2} \|X_{f_{\varphi}^2 + g_{\varphi}^2}\|_{p,D(\rho',\sigma^2)} U'.
\]

Using (2.14), (2.30) and (2.42), we have
\[
\|X_{f_{\varphi}}\|_{p,D(\rho-3\tau),\sigma^2} U' \leq \frac{d\Delta}{\tau K^2} \|X_{f_{\varphi}^2 + g_{\varphi}^2}\|_{p,D(\rho',\sigma^2)} U'.
\]


Similar to that of $X_{s^1}$, we can estimate $X_{s^0}, X_{s^2}$ in sequence and finally get

\begin{equation}
|||X_s|||^T_{p,D(\rho-5\tau, (\sigma-5\tau)^2, \sigma-5\tau)\times U'} \leq \frac{d_{3d}^s}{\tau^5 R^6}.
\end{equation}

Moreover, the vector field $X_{h_1}$ satisfies

\begin{equation}
|||X_{h_1}|||^T_{p,D(\rho-5\tau, (\sigma-5\tau)^2, \sigma-5\tau)\times U'} \leq |||X_{f^0+g^0}|||^T_{p,D(\rho-5\tau, (\sigma-5\tau)^2, \sigma-5\tau)\times U'}
\end{equation}

\begin{equation}
+ |||X_{f^2+g^2}|||^T_{p,D(\rho-5\tau, (\sigma-5\tau)^2, \sigma-5\tau)\times U'} \leq \frac{d_{3d}^s}{\tau^5 R^6}.
\end{equation}

### 2.3.4. Step 4: Estimate of the vector field $X_{f_1}$

In this step, we shall verify the estimates \((2.20) - (2.27)\). Using Taylor’s formula, we obtain from the homological equation \((2.20)\) that

\[
(h + f) \circ X^i_s |_{t=1} = (h + f)\text{low} + f\text{high} \circ X^i_s |_{t=1}
\]

\[
= h + \{h, s\} + \int_0^1 (1-t)\{h, s\} \circ X^i_s dt + f\text{low} + \int_0^1 \{f\text{low}, s\} \circ X^i_s dt
\]

\[
+ f\text{high} + \{f\text{high}, s\} + \int_0^1 (1-t)\{f\text{high}, s\} \circ X^i_s dt
\]

\[
= h + h_1 + (1 - T_{\Delta'}) f\text{low} + f\text{high} + (1 - T_{\Delta'}) \{f\text{high}, s\} \text{low} + \{f\text{high}, s\} \text{high}
\]

\[
+ \int_0^1 (1-t)\{h, s\} \circ X^i_s dt + \int_0^1 \{f\text{low}, s\} \circ X^i_s dt + \int_0^1 (1-t)\{f\text{high}, s\} \circ X^i_s dt.
\]

Then we have $f_1 = f_1\text{low} + f_1\text{high}$, where

\begin{equation}
f_1\text{low} \equiv (1 - T_{\Delta'}) f\text{low} + (1 - T_{\Delta'}) \{f\text{high}, s\} \text{low} + \left(\int_0^1 (1-t)\{h, s\} \circ X^i_s dt\right)\text{low}
\end{equation}

\[
+ \left(\int_0^1 \{f\text{low}, s\} \circ X^i_s dt\right)\text{low} + \left(\int_0^1 (1-t)\{f\text{high}, s\} \circ X^i_s dt\right)\text{low},
\]

\begin{equation}
f_1\text{high} \equiv f\text{high} + \{f\text{high}, s\} \text{high} + \left(\int_0^1 (1-t)\{h, s\} \circ X^i_s dt\right)\text{high}
\end{equation}

\[
+ \left(\int_0^1 \{f\text{low}, s\} \circ X^i_s dt\right)\text{high} + \left(\int_0^1 (1-t)\{f\text{high}, s\} \circ X^i_s dt\right)\text{high}.
\]

Firstly, we consider the vector field generated by $(1 - T_{\Delta'}) f\text{low}$. Observing that

\[
\| (1 - T_{\Delta'}) F_{\varphi}^0\|_{D(\rho-\tau)\times U'} = \sum_{|k| > \Delta'} \left( |F_{\varphi}^0(k; w)| + |\partial_w F_{\varphi}^0(k; w)| \right) e^{|k| \rho - \tau}
\]

\[
\leq \sum_{|k| > \Delta'} e^{-\tau |k|} \| F_{\varphi}^0\|_{D(\rho)\times U'} \leq \frac{1}{\tau \# A} e^{-\frac{\tau}{4} \Delta'} \| F_{\varphi}^0\|_{D(\rho)\times U'},
\]

we obtain

\[
|||X_{(1 - T_{\Delta'})f^*}|||^T_{p,D(\rho-\tau, \sigma^2, \sigma)\times U'} \leq \frac{1}{\tau \# A} e^{-\frac{\tau}{4} \Delta'} \|X_{f^*}\|_{D(\rho, \sigma^2, \sigma)\times U'} \leq \frac{1}{\tau \# A} e^{-\frac{\tau}{4} \Delta'} \varepsilon.
\]

and the same estimates hold for $|||X_{(1 - T_{\Delta'})f^2}|||^T_{p,D(\rho-\tau, \sigma^2, \sigma)\times U'}$ and $|||X_{(1 - T_{\Delta'})f^4}|||^T_{p,D(\rho-\tau, \sigma^2, \sigma)\times U'}$. Then we turn to

\begin{equation}
(1 - T_{\Delta'}) f^2 = (1 - T_{\Delta'}) f^4 + (1 - T_{\Delta'}) f^2,
\end{equation}
(1 - T_{\Delta'})f^2 = \frac{1}{2}\langle \zeta, (1 - T_{\Delta'})F_2(\varphi; w)\rangle, \quad (1 - T_{\Delta'}^2)f^2 = \frac{1}{2}\sum_{|k| > \Delta'} \langle \zeta, T_{\Delta'}\hat{F}_2(k; w)\zeta \rangle e^{i(k; \varphi)} \tag{2.62}

It is easy to see

\begin{equation}
\|||X(1 - T_{\Delta'}^2)f^2|||^T_{p, D(\rho - \tau, \sigma^2, \sigma) \times U'} \leq \frac{1}{\tau^{\#A}} e^{-\frac{1}{2}\gamma\Delta'} \varepsilon. \tag{2.58}
\end{equation}

By (2.15), we have

\begin{equation}
\sup_{(\varphi, w) \in D(\rho) \times U} (\|F_2(\varphi; w)\|_{\gamma_1}, \|\partial_w F_2(\varphi; w)\|_{\gamma}) \leq \frac{\varepsilon}{\sigma^2}.
\end{equation}

Hence

\[\|\hat{F}_{2\varphi}(k; w)\| \leq \frac{\varepsilon}{\sigma^2} e^{-\gamma|a-b| - \rho|k|},\]

which implies

\[\|F_{2\varphi}(\varphi; w)\|_{D(\rho - \tau) \times U} = \sum_{k \in \mathbb{Z}^A} (\|F_{2\varphi}(k; w)\| + \|\partial_w F_{2\varphi}(k; w)\|) e^{(\rho - \tau)|k|} \leq \sum_{k \in \mathbb{Z}^A} |k| e^{-\tau|k|} \frac{\varepsilon}{\sigma^2} \leq \frac{1}{\tau^{\#A+1}} \frac{\varepsilon}{\sigma^2} e^{-\gamma|a-b|} \tag{2.59}.
\]

Using Young’s inequality (2) in [8], we obtain

\[\|||X(1 - T_{\Delta'}^2)f^2|||^T_{p, D(\rho - \tau, \sigma^2, \sigma) \times U'} \leq \frac{1}{\gamma_{d+p}} \frac{1}{\tau^{\#A+1}} \frac{\varepsilon}{\sigma^2} e^{-\gamma\Delta'},\]

which together with (2.58) leads to

\[\|||X(1 - T_{\Delta'})f^2|||^T_{p, D(\rho - \tau, \sigma^2, \sigma) \times U'} \leq \frac{1}{\tau^{\#A}} e^{-\frac{1}{2}\gamma\Delta'} \varepsilon + \frac{1}{\gamma_{d+p}} \frac{1}{\tau^{\#A+1}} \frac{\varepsilon}{\sigma^2} e^{-\gamma\Delta'} \tag{2.60}.
\]

In conclusion, we have

\[\|||X(1 - T_{\Delta'})f^{low}|||^T_{p, D(\rho - \tau, \sigma^2, \sigma) \times U'} \leq \frac{1}{\tau^{\#A}} e^{-\frac{1}{2}\gamma\Delta'} \varepsilon + \frac{1}{\gamma_{d+p}} \frac{1}{\tau^{\#A+1}} \frac{\varepsilon}{\sigma^2} e^{-\gamma\Delta'}.\]

Secondly, we consider the vector field generated by \((1 - T_{\Delta'})g^{low} = (1 - T_{\Delta'})\{f^{high}, s\}^{low}.

From equation (2.31), we obtain

\begin{equation}
[s^{\varphi}]_{\Lambda, \gamma, \sigma; U', \rho, \mu} \leq \frac{1}{\kappa^2} (\Delta')^{\exp \varepsilon}. \tag{2.61}
\end{equation}

Applying Proposition 6.6 in [8] to the equation (2.32), we obtain

\[\||\hat{S}(k; \cdot)||_{p, \gamma; U'} \leq \frac{1}{\kappa^2} (\Delta')^{\exp \varepsilon} \||F_1(k; \cdot) + G_1(k; \cdot)||_{p, \gamma; U'}.\]

Noticing that

\[\{f^{high}, s^{\varphi}\} = -\langle \partial_s f^{high}, \partial_s s^{\varphi} \rangle ,\]

it follows from (2.15), (2.61) and in [8], Equation (42) that

\[\{[f^{high}, s^{\varphi}]\}_{\Lambda, \gamma, \sigma; U', \rho^{(1)}, \mu^{(1)}} \leq \frac{1}{\rho - \rho^{(1)}} \frac{1}{\mu - \mu^{(1)}} \frac{1}{\kappa^2} (\Delta')^{\exp \varepsilon},\]

which together with (2.15) and (2.61) implies

\[\{s^{1}\}_{\Lambda, \gamma, \sigma; U', \rho^{(1)}, \mu^{(1)}} \leq \frac{1}{\kappa^2} (\Delta')^{\exp \varepsilon} \frac{1}{\rho - \rho^{(1)}} \frac{1}{\mu - \mu^{(1)}} \varepsilon.\]
Since $s^1$ is independent of $r$, there is
\begin{equation}
[s^1]_{\Lambda, \gamma, \sigma; U', \rho^{(1)}, \mu} \leq \frac{1}{K^4} (\Delta \Delta')^{\exp} \frac{1}{\rho - \rho^{(1)}} \frac{1}{\mu} \varepsilon.
\end{equation}

Next we estimate
\begin{equation*}
\{ f^{\text{high}}, s^1 \} = -\langle \partial_x f^{\text{high}}, \partial_x s^1 \rangle + \langle \partial_x f^{\text{high}}, J \partial_x s^1 \rangle.
\end{equation*}
By (2.15), (2.63), and Cauchy estimates (42) in [8], we have
\begin{equation*}
\left[ \langle \partial_x f^{\text{high}}, \partial_x s^1 \rangle \right]_{\Lambda, \gamma, \sigma; U', \rho^{(1)}, \mu} \leq \frac{1}{K^4} (\Delta \Delta')^{\exp} \frac{1}{\rho - \rho^{(1)}} \frac{1}{\sigma - \rho^{(1)}} \frac{1}{\mu} \varepsilon.
\end{equation*}
Applying further Proposition 3.1 (ii) in [8], we have
\begin{equation*}
\left[ \langle \partial_x f^{\text{high}}, J \partial_x s^1 \rangle \right]_{\Lambda, \gamma, \sigma; U', \rho^{(1)}, \mu} \leq \frac{1}{K^4} (\Delta \Delta')^{\exp} \frac{1}{\rho - \rho^{(1)}} \frac{1}{\sigma - \rho^{(1)}} \frac{1}{\mu} \varepsilon,
\end{equation*}
which implies
\begin{equation*}
\left[ \left\{ f^{\text{high}}, s^1 \right\} \right]_{\Lambda, \gamma, \sigma; U', \rho^{(1)}, \mu} \leq \frac{1}{K^4} (\Delta \Delta')^{\exp} \frac{1}{\rho - \rho^{(1)}} \left( \frac{1}{\sigma - \rho^{(1)}} \frac{1}{\mu} \right) \frac{1}{\mu} \varepsilon.
\end{equation*}
By (2.30) and (2.64), we have
\begin{equation*}
\left[ G_2(\varphi; w) \right]_{\gamma, \Lambda, \gamma, \sigma; U', \rho^{(1)}, \mu} \leq \frac{1}{K^4} (\Delta \Delta')^{\exp} \frac{1}{\rho - \rho^{(1)}} \left( \frac{1}{\sigma - \rho^{(1)}} \frac{1}{\mu} \right) \frac{1}{\mu} \varepsilon,
\end{equation*}
which implies
\begin{equation*}
\left\| [G_2(\varphi; w)]_{\gamma, \Lambda, \gamma, \sigma; U', \rho^{(1)}, \mu} \right\| \leq \frac{1}{K^4} (\Delta \Delta')^{\exp} \frac{1}{\rho - \rho^{(1)}} \left( \frac{1}{\sigma - \rho^{(1)}} \frac{1}{\mu} \right) \frac{1}{\mu} \varepsilon.
\end{equation*}
By (2.30) and (2.65), we have
\begin{equation*}
\left\| [X_{g^{\text{low}}}]_{\gamma, \Lambda, \gamma, \sigma; U', \rho^{(1)}, \mu} \right\| \leq \frac{1}{K^4} (\Delta \Delta')^{\exp} \frac{1}{\rho - \rho^{(1)}} \left( \frac{1}{\sigma - \rho^{(1)}} \frac{1}{\mu} \right) \frac{1}{\mu} \varepsilon.
\end{equation*}
Following the proof of (2.60), using (2.65), (2.66), we have
\begin{equation*}
\left\| X_{1-t_A} g^{\text{low}} \right\|_{p, D(\rho - 5 \tau, (\sigma - 5 \tau)^2, (\sigma - 4 \tau) \times U')} \leq \frac{1}{\tau^4 A^4} e^{-\frac{1}{\tau^2} (\Delta \Delta')^{\exp} \frac{1}{\rho - \rho^{(1)}} \left( \frac{1}{\sigma - \rho^{(1)}} \frac{1}{\mu} \right) \frac{1}{\mu} \varepsilon},
\end{equation*}
where $\rho^{(1)} = \rho - \tau, \rho^{(2)} = \rho - 2 \tau, \sigma^{(1)} = \sigma - \tau$.

Thirdly, by (2.14), (2.21), (2.52), (2.66), we have
\begin{equation*}
\left\| X_{\{h, s\}} \right\|_{p, D(\rho - 6 \tau, (\sigma - 5 \tau)^2, (\sigma - 5 \tau) \times U')} \leq \frac{1}{\tau^4 K^4} \frac{d^{l_A}}{\mu^2} \varepsilon,
\end{equation*}
which together with Proposition 2.10 and (2.54) implies
\begin{equation*}
\left\| X_{\{h, s\}} \right\|_{p, D(\rho - 6 \tau, (\sigma - 6 \tau)^2, (\sigma - 6 \tau) \times U')} \leq \frac{d^{l_A} \varepsilon}{\tau^4 K^{10}}.
\end{equation*}
By Proposition 2.10, 2.13, 2.54, we have

\[(2.69)\]

\[
|||X_{\{f_{low},s\}}|||_{T}^{p,D(\rho-6\tau,\sigma-6\tau) \times U'} \lesssim \frac{d_{A}^{3} \varepsilon^{2}}{\tau^{6} K^{6}},
\]

which implies

\[
|||X_{\{f_{high},s\}}|||_{T}^{p,D(\rho-6\tau,\sigma-6\tau) \times U'} \lesssim \frac{d_{A}^{3} \varepsilon^{2}}{\tau^{6} K^{6}},
\]

Then applying Theorem 3.3 in [3], we have

\[(2.70)\]

\[
|||X_{f_{low}}^{0}(1-\tau)\{h,s\} \circ X_{j}^{\mu}d|||_{T}^{p,D(\rho-7\tau,\sigma-7\tau) \times U'} \lesssim \frac{d_{A}^{3} \varepsilon^{2}}{\tau^{10} K^{10}},
\]

By (2.60), (2.67), (2.70) and (2.56), we have

\[(2.71)\]

\[
|||X_{f_{low}}^{0}\|\|_{T}^{p,D(\rho-8\tau,\sigma-8\tau) \times U'} \lesssim \frac{d_{A}^{3} \varepsilon^{2}}{\tau^{12} K^{12}}.
\]

Finally, by (2.14), (2.69), (2.70) and (2.56), we have

\[(2.72)\]

\[
|||X_{f_{low}}^{0}||\|_{p,D(\rho-8\tau,\sigma-8\tau) \times U'} \lesssim \frac{d_{A}^{3} \varepsilon^{2}}{\tau^{12} K^{12}}.
\]

2.3.5. **Step 5: Estimate of the functions s and f_{1}.** In this step, we shall verify the estimates \(2.22-2.24\). From the equation \(2.32\), using \(2.15\) and \(2.64\), we obtain

\[
[s]^{0}_{A,\gamma,\sigma;U',\rho(2)_{\mu(1)}} \lesssim \frac{1}{k^{6}}(\Delta \Delta')^{\exp} \frac{1}{\rho - \rho(1)} \left( \frac{1}{\sigma - \sigma(1)} \frac{1}{\sigma} + \frac{1}{\rho(1)} \frac{1}{\rho(1) - \rho(2)} \frac{1}{\mu - \mu(1)} \right) \frac{1}{\mu} \varepsilon.
\]

Since \([s]^{0}\) is independent of \(\zeta\), we obtain

\[(2.73)\]

\[
[s]^{0}_{A,\gamma,\sigma;U',\rho(2)_{\mu(1)}} \lesssim \frac{1}{k^{6}}(\Delta \Delta')^{\exp} \frac{1}{\rho - \rho(1)} \left( \frac{1}{\sigma - \sigma(1)} \frac{1}{\sigma} + \frac{1}{\rho(1)} \frac{1}{\rho(1) - \rho(2)} \frac{1}{\mu - \mu(1)} \right) \frac{1}{\mu} \varepsilon.
\]

Applying Proposition 6.7 in [8] to the equation \(2.34\), it follows from \(2.15\) and \(2.64\) that

\[(2.74)\]

\[
[s]\left\{ \Lambda_{U',\rho(2)_{\mu(1)}}^{+} \gamma_{\sigma(1)_{\mu(1)}} \right\} \lesssim \frac{1}{k^{6}}(\Delta \Delta')^{\exp} \frac{1}{\rho - \rho(1)} \left( \frac{1}{\sigma - \sigma(1)} \frac{1}{\sigma} + \frac{1}{\rho(1)} \frac{1}{\rho(1) - \rho(2)} \frac{1}{\mu - \mu(1)} \right) \frac{1}{\mu} \varepsilon.
\]

Using \(2.61\), \(2.63\), \(2.73\) and \(2.74\), we obtain

\[
[s]\left\{ \Lambda_{U',\rho(2)_{\mu(1)}}^{+} \gamma_{\sigma(1)_{\mu(1)}} \right\} \lesssim \frac{1}{k^{6}}(\Delta \Delta')^{\exp} \frac{1}{\rho - \rho(1)} \left( \frac{1}{\sigma - \sigma(1)} \frac{1}{\sigma} + \frac{1}{\rho(1)} \frac{1}{\rho(1) - \rho(2)} \frac{1}{\mu - \mu(1)} \right) \frac{1}{\mu} \varepsilon.
\]

This completes the proof of Proposition 2.11.
3. KAM theorem

In this section, we will prove the following KAM theorem, upon which our main result Theorem 1.1 is an immediate result.

**Theorem 3.1.** Consider the Hamiltonian $h + f$, where

$$ h(r, \zeta; w) = \langle \omega(w), r \rangle + \frac{1}{2} \langle \zeta, (\Omega(w) + H(w)) \zeta \rangle $$

satisfy (2.1)-(2.12), $H(w), \partial_w H(w)$ are Töplitz at $\infty$ and $\mathcal{N}_F(\Delta)$ for all $w \in U$,

$$ |||X_f|||_{p,D(\rho,\mu,\sigma)} \leq \varepsilon, $$

(3.1)

$$ |||f|||_{p,D(\rho,\mu,\sigma)} \leq \varepsilon. $$

(3.2)

Assume $0 < \gamma, \sigma, \rho, \mu < 1$, $\Lambda, \Delta \geq 3$, $\rho = \sigma$, $\mu = \sigma^2$, $d_\Delta \gamma \leq 1$. Then there is a subset $U_\infty \subset U$ such that if

$$ \varepsilon \leq c \min \left( \gamma, \rho, \frac{1}{\Delta}, \frac{1}{\Lambda} \right)^{\exp}, $$

then for all $w \in U_\infty$, there is a real analytic symplectic map

$$ \Phi : D\left(\frac{\rho}{2}, \frac{\mu}{4}, \frac{\sigma}{2}\right) \to D(\rho, \mu, \sigma) $$

such that

$$ (h + f) \circ \Phi = h_\infty + f_\infty, $$

where

$$ h_\infty = \langle \omega_\infty(w), r \rangle + \frac{1}{2} \langle \zeta, (\Omega(w) + H_\infty(w)) \zeta \rangle, $$

$$ f_\infty = O(|r|^2 + |r||\zeta|_p + ||\zeta||_p^3) $$

with the estimates

$$ |||\Phi - id|||_{p,D(\frac{\rho}{2}, \frac{\mu}{4}, \frac{\sigma}{2})} \leq c\varepsilon^{\frac{2}{3}}, $$

(3.3)

$$ |||X_{f_\infty}|||_{p,D(\frac{\rho}{2}, \frac{\mu}{4}, \frac{\sigma}{2}) \times U_\infty} \leq c\varepsilon^{\frac{2}{3}}, $$

(3.4)

$$ |\omega_\infty(w) - \omega(w)| + |\partial_w (\omega_\infty(w) - \omega(w))| \leq c\varepsilon^{\frac{2}{3}}, $$

(3.5)

$$ ||H_\infty(w) - H(w)|| + ||\partial_w(H_\infty(w) - H(w))|| \leq c\varepsilon^{\frac{2}{3}}, $$

(3.6)

$$ \text{meas}(U \setminus U_\infty) \leq c\varepsilon^{\exp'} $$

(3.7)

where the exponents $\exp, \exp'$ depend on $d, \#A, p$, the constant $c$ depends on $d, \#A, p, c_1, c_2, c_3, c_4, c_5$.

Based on the KAM theorem, we prove Theorem 1.1 on the existence and time $\delta^{-1}$ stability of the invariant tori.

**Proof of Theorem 1.1.** Recall the Hamiltonian formulation of NLS equation (1.1) in Section 1. Let $\omega_a = |a|^2 + \tilde{V}(a), a \in A, \Omega_a = |a|^2 + \tilde{V}(a), a \in \mathcal{L}$, and take $w_a = \tilde{V}(a), w \in U = [-1, 1]^d$. Then we have

$$ h = \sum_{a \in A} \omega_a r_a + \frac{1}{2} \sum_{a \in \mathcal{L}} \Omega_a (\xi_a^2 + \eta_a^2), \quad f = \varepsilon \int_{\mathbb{T}^d} F(x, u(x), \overline{u(x)}) dx. $$

The Töplitz-Lipschitz property of $f$ follows from Theorem 7.2 in [8] and the tame property follows from Section 3.5 in [3]. By Theorem 3.1 if $\varepsilon > 0$ is sufficiently small, then for most $V$
(in the sense of measure), the 1-dimensional nonlinear Schrödinger equation (1.1) has a quasi-periodic solution. As done in [5], assume \( u_0(t, x) \) with initial value \( u_0(0, x) \) is a quasi-periodic solution for the equation (1.1), then for any solution \( u(t, x) \) with initial value \( u(0, x) \) satisfying
\[
\|u(0, \cdot) - u_0(0, \cdot)\|_{H_p(T^d)} < \delta, \forall 0 < \delta \ll 1,
\]
we have
\[
\|u(t, \cdot) - u_0(t, \cdot)\|_{H_p(T^d)} < C\delta, \forall 0 < |t| < \delta^{-1}.
\]
In other words, the obtained KAM tori for the nonlinear Schrödinger equation (1.1) are of long time stability. □

As remarked in Eliasson-Kuksin [8], the size of the blocks grows much faster than quadratically along the KAM iteration, we need to take sufficiently many normal form computations at each step to obtain a much faster iteration scheme.

3.1. Normal form computations. For \( \rho_+ < \rho, \gamma_+ < \gamma \), let \( \Delta' = 80(\log \frac{1}{\varepsilon})^2 \min(\gamma_+, \rho - \rho_+) \), \( n = \lfloor \log \frac{1}{\varepsilon} \rfloor \). Assume \( \rho = \sigma, \mu = \sigma^2, d \Delta \gamma \leq 1 \). For \( 1 \leq j \leq n \), let
\[
\varepsilon_j = \frac{\varepsilon}{\kappa^{20}} \varepsilon_{j-1}, \varepsilon_0 = \varepsilon, \\
\gamma_j = \gamma - j \frac{\gamma - \gamma_+}{n}, \gamma_0 = \gamma, \\
\rho_j = \rho - j \frac{\rho - \rho_+}{n}, \rho_0 = \rho, \\
\sigma_j = \sigma - j \frac{\sigma - \sigma_+}{n}, \sigma_0 = \sigma, \\
\mu_j = \sigma_j^2, \mu_0 = \mu,
\]
\[
\Lambda_j = \Lambda_{j-1} + d \Delta + 30, \Lambda_0 = \text{cte.} \max(\Lambda, d \Delta, d \Delta'),
\]
where the constant cte. is the one in Proposition 6.7 in [8].

We have the following lemma.

Lemma 3.2. For \( 0 \leq j < n \), consider the Hamiltonian \( h + h_1 + \cdots + h_j + f_j \), where
\[
h(r, \zeta; w) = \langle \omega(w), r \rangle + \frac{1}{2} \langle \zeta, (\Omega(w) + H(w))\zeta \rangle
\]
satisfy (2.23)-(2.12), \( H(w), \partial_w H(w) \) are Töplitz at \( \infty \) and \( NF_\Delta \) for all \( w \in U \). Let \( U' \subset U \) satisfy (2.16)-(2.19). For all \( w \in U' \),
\[
h_j = a_j(w) + \langle \chi_j(w), r \rangle + \frac{1}{2} \langle \zeta, H_j(w)\zeta \rangle,
\]
\[
f_j = f_j^{\text{low}} + f_j^{\text{high}}
\]
satisfy
\[
\|X_{f_j^{\text{low}}}\|_{L_p(D\rho, \mu, \sigma_j) \times U'} \leq \beta^j \varepsilon_j, \|X_{f_j^{\text{high}}}\|_{L_p(D\rho, \mu, \sigma_j) \times U'} \leq 1,
\]
\[
[f_j^{\text{low}}]_{\Lambda_j, \gamma_j, \sigma_j; U'}, [f_j^{\text{high}}]_{\Lambda_j, \gamma_j, \sigma_j; U'} \leq \beta^j \varepsilon_j, \leq \beta^j \varepsilon_j
\]
for some
\[
\beta \leq \max \left( \frac{1}{\gamma - \gamma_+}, \frac{1}{\rho - \rho_+}, \Delta, \Lambda, \log \frac{1}{\varepsilon} \right)^{\text{exp}_1},
\]
Then there exists an exponent \( \text{exp}_2 \) such that if
\[
\varepsilon \leq \kappa^{20} \min \left( \frac{1}{\gamma - \gamma_+}, \frac{1}{\rho - \rho_+}, \frac{1}{\Delta}, \frac{1}{\Lambda}, \frac{1}{\log \frac{1}{\varepsilon}} \right)^{\text{exp}_2},
\]
then for all \( w \in U' \), there is a real analytic symplectic map \( \Phi_j \) such that
\[
(h + h_1 + \cdots + h_j + f_j) \circ \Phi_j = h + h_1 + \cdots + h_{j+1} + f_{j+1},
\]
with the estimates
\[
\|X_{j+1} \|^T_{p,D(\rho_{j+1},\mu_{j+1},\sigma_{j+1}) \times U'} \leq \beta^{j+1} \varepsilon_{j+1},
\]
\[
\|X_{j+1} \|^T_{p,D(\rho_{j+1},\mu_{j+1},\sigma_{j+1}) \times U'} \leq 1 + \frac{1}{\kappa^6} \beta^{j+1} \varepsilon_j + \beta^{j+1} \varepsilon_{j+1},
\]
\[
\left[ \sum_{j=1}^{j+1} 1 \right]_{p,D(\rho_{j+1},\mu_{j+1},\sigma_{j+1}) \times U'} \leq \beta^{j+1} \varepsilon_{j+1},
\]
\[
\left[ \sum_{j=1}^{j+1} 1 \right]_{p,D(\rho_{j+1},\mu_{j+1},\sigma_{j+1}) \times U'} \leq 1 + \frac{1}{\kappa^6} \beta^{j+1} \varepsilon_j + \beta^{j+1} \varepsilon_{j+1},
\]
where the exponents \( \exp_1, \exp_2 \) depend on \( d, \#A, p \).

Proof. By Proposition 2.11, we can solve the homological equation
\[
\{h, s_j\} = -T_{\Delta'} f_{j}^{\text{low}} - T_{\Delta'} \{f_{j}^{\text{high}}, s_j\}^{\text{low}} + h_{j+1}
\]
with the estimates
\[
\|s_j\|_{\Lambda_t + d_A + 2\gamma_j, \sigma_j^{(1)}, \sigma_j^{(1)}}, \|h_{j+1}\|_{\Lambda_t + d_A + 2\gamma_j, \sigma_j^{(1)}, \sigma_j^{(1)}} \leq \frac{1}{\kappa^4} (\Delta \Delta') \exp \frac{1}{\rho_j - \rho_j^{(1)}} \left( \frac{1}{\sigma_j - \sigma_j^{(1)}} \frac{1}{\sigma_j - \sigma_j^{(1)}} \right) \frac{1}{\mu_j} \beta^{j} \varepsilon_j.
\]

Using Taylor's formula, by the homological equation (3.14), we obtain
\[
(h + h_1 + \cdots + h_j + f_j) \circ X_{s_j}^{t} |_{t=1} = (h + h_1 + \cdots + h_j + f_j^{\text{low}} + f_j^{\text{high}}) \circ X_{s_j}^{t} |_{t=1}
\]
\[
= h + \{h, s_j\} + \int_0^1 \{h, s_j\} \circ X_{s_j}^{t} dt + h_1 + \cdots + h_j + \int_0^1 \{h_1 + \cdots + h_j, s_j\} \circ X_{s_j}^{t} dt
\]
\[
+ f_j^{\text{low}} + \int_0^1 \{f_j^{\text{low}}, s_j\} \circ X_{s_j}^{t} dt + f_j^{\text{high}} + \{f_j^{\text{high}}, s_j\} + \int_0^1 \{f_j^{\text{high}}, s_j\} \circ X_{s_j}^{t} dt
\]
\[
= h + h_1 + \cdots + h_{j+1} + (1 - T_{\Delta'}) f_j^{\text{low}} + f_j^{\text{high}} + (1 - T_{\Delta'}) \{f_j^{\text{high}}, s_j\}^{\text{low}} + \{f_j^{\text{high}}, s_j\}^{\text{high}} + \int_0^1 \{h, s_j\} \circ X_{s_j}^{t} dt + \int_0^1 \{h_1 + \cdots + h_j, s_j\} \circ X_{s_j}^{t} dt
\]
\[
+ \int_0^1 \{f_j^{\text{low}}, s_j\} \circ X_{s_j}^{t} dt + \int_0^1 \{f_j^{\text{high}}, s_j\} \circ X_{s_j}^{t} dt.
\]

It follows from (3.19) that
\[
(1 - T_{\Delta'}) f_j^{\text{low}} \{\Lambda_t + \gamma_j^{(1)}, \sigma_j^{(1)}\}_{\rho_j, \mu_j} = \left[ \frac{1}{\rho_j - \rho_j^{(1)}} \right]_{\#A} e^{-\frac{1}{2}(\rho_j - \rho_j^{(1)}) \Delta'} + e^{-(\gamma_j - \gamma_j^{(1)}) \Delta'} \beta^{j} \varepsilon_j.
\]
By (2.64), we have

\[ [(f_j^{\text{high}}, s_j)_{\text{low}}; \{\Lambda_j, \gamma_j, \sigma_j^{(1)}; U^s, \rho_j^{(1)}; \mu_j^{(1)}\}] \leq \frac{1}{\kappa^4} \exp \frac{1}{\rho_j - \rho_j^{(1)}} \left( \frac{1}{\sigma_j^{(1)} - \sigma_j^{(1)}} \frac{1}{\sigma_j^{(1)}} + \frac{1}{\rho_j - \rho_j^{(1)}} \frac{1}{\mu_j - \mu_j^{(1)}} \right) \frac{1}{\mu_j^{(2)}} \beta \varepsilon_j. \]

Hence

\[ (1 - T_{\Delta'}) [(f_j^{\text{high}}, s_j)_{\text{low}}; \{\Lambda_j, \gamma_j, \sigma_j^{(1)}; U^s, \rho_j^{(1)}; \mu_j^{(1)}\}] \leq \left[ \frac{1}{\rho_j - \rho_j^{(1)}} \right] \#A \exp \frac{1}{\rho_j^{(1)} - \rho_j^{(2)}} \Delta' \left( e^{-\rho_j^{(1)} \Delta'} + e^{-(\gamma_j - \gamma_j^{(1)}) \Delta'} \right) \]

\[ \times \frac{1}{\kappa^4} \exp \frac{1}{\rho_j - \rho_j^{(1)}} \left( \frac{1}{\sigma_j^{(1)} - \sigma_j^{(1)}} \frac{1}{\sigma_j^{(1)}} + \frac{1}{\rho_j - \rho_j^{(1)}} \frac{1}{\mu_j - \mu_j^{(1)}} \right) \frac{1}{\mu_j^{(2)}} \beta \varepsilon_j. \]

By (3.15), (3.16), using Proposition 3.3 in [8], we have

\[ [(f_j^{\text{high}}, s_j)]_{\Lambda_j + d_{\Delta} + 5, \gamma_j^{(1)}, \sigma_j^{(2)}; U^s, \rho_j^{(2)}; \mu_j^{(2)}} \]

\[ \leq \left( \Lambda_j + d_{\Delta} + 2 \right)^{d+1} \left( \frac{1}{\gamma_j - \gamma_j^{(1)}} \right)^{d+1} \left( \frac{1}{\sigma_j^{(1)} - \sigma_j^{(1)}} \frac{1}{\sigma_j^{(1)}} + \frac{1}{\rho_j^{(1)} - \rho_j^{(2)}} \frac{1}{\mu_j^{(1)} - \mu_j^{(2)}} \right) \]

\[ \times \frac{1}{\kappa^4} \exp \frac{1}{\rho_j - \rho_j^{(1)}} \left( \frac{1}{\sigma_j^{(1)} - \sigma_j^{(1)}} \frac{1}{\sigma_j^{(1)}} + \frac{1}{\rho_j^{(1)} - \rho_j^{(2)}} \frac{1}{\mu_j^{(1)} - \mu_j^{(2)}} \right) \frac{1}{\mu_j^{(2)}} \beta \varepsilon_j^2. \]

By (3.14), (3.18), using Proposition 3.3 in [8], we have

\[ [(f_j^{\text{high}}, s_j)]_{\Lambda_j + d_{\Delta} + 5, \gamma_j^{(1)}, \sigma_j^{(2)}; U^s, \rho_j^{(2)}; \mu_j^{(2)}} \]

\[ \leq \left( \Lambda_j + d_{\Delta} + 2 \right)^{d+1} \left( \frac{1}{\gamma_j - \gamma_j^{(1)}} \right)^{d+1} \left( \frac{1}{\sigma_j^{(1)} - \sigma_j^{(1)}} \frac{1}{\sigma_j^{(1)}} + \frac{1}{\rho_j^{(1)} - \rho_j^{(2)}} \frac{1}{\mu_j^{(1)} - \mu_j^{(2)}} \right) \]

\[ \times \frac{1}{\kappa^{11}} \exp \frac{1}{(\rho_j^{(1)} - \rho_j^{(1)})^2} \left( \frac{1}{\sigma_j^{(1)} - \sigma_j^{(1)}} \frac{1}{\sigma_j^{(1)}} + \frac{1}{\rho_j^{(1)} - \rho_j^{(2)}} \frac{1}{\mu_j^{(1)} - \mu_j^{(2)}} \right)^2 \frac{1}{\mu_j^{(2)}} \beta \varepsilon_j^2. \]

By (3.15), (3.20), using Proposition 3.3 in [8], we have

\[ [(f_j^{\text{high}}, s_j)]_{\Lambda_j + d_{\Delta} + 8, \gamma_j^{(2)}, \sigma_j^{(3)}; U^s, \rho_j^{(3)}; \mu_j^{(3)}} \]
\[
\begin{align*}
&\leq \left(\Lambda_j + d_\Delta + 5\right)^2 \left(\frac{1}{\gamma_j^{(1)} - \gamma_j^{(2)}}\right)^{d+1} \frac{1}{\sigma_j^{(2)} - \sigma_j^{(3)}} \frac{1}{\sigma_j^{(2)} - \sigma_j^{(3)}} + \frac{1}{\rho_j^{(2)} - \rho_j^{(3)}} \frac{1}{\mu_j^{(2)} - \mu_j^{(3)}} \\
&\times \left(\Lambda_j + d_\Delta + 2\right)^2 \left(\frac{1}{\gamma_j - \gamma_j^{(1)}}\right)^{d+1} \frac{1}{\sigma_j^{(1)} - \sigma_j^{(1)}} \frac{1}{\sigma_j^{(1)} - \sigma_j^{(1)}} + \frac{1}{\rho_j^{(1)} - \rho_j^{(2)}} \frac{1}{\mu_j^{(1)} - \mu_j^{(2)}} \\
&\times \frac{1}{\kappa_4}(\Delta \Delta')^{d+1} \frac{1}{\rho_j - \rho_j^{(1)}} \left(\frac{1}{\sigma_j^{(1)} - \sigma_j^{(1)}} \frac{1}{\rho_j - \rho_j^{(1)}} \frac{1}{\mu_j^{(1)}} \right)^{1/2} \frac{1}{\mu_j^{(1)}} \beta^j \epsilon_j^{2}. 
\end{align*}
\]

By \([3.16]\) and \([8\) Proposition 3.3\], we have \([\{h_{l+1}, s_j\}]_{\Lambda_j + d_\Delta + 5, \gamma_j^{(1)}, \sigma_j^{(2)}, \rho_j^{(2)}, \mu_j^{(3)}}\) is less than

\[
\begin{align*}
&\leq \left(\Lambda_j + d_\Delta + 2\right)^2 \left(\frac{1}{\gamma_j - \gamma_j^{(1)}}\right)^{d+1} \frac{1}{\sigma_j^{(1)} - \sigma_j^{(1)}} \frac{1}{\sigma_j^{(1)} - \sigma_j^{(1)}} + \frac{1}{\rho_j^{(1)} - \rho_j^{(2)}} \frac{1}{\mu_j^{(1)} - \mu_j^{(2)}} \\
&\times \frac{1}{\kappa_4}(\Delta \Delta')^{d+1} \frac{1}{\rho_j - \rho_j^{(1)}} \left(\frac{1}{\sigma_j^{(1)} - \sigma_j^{(1)}} \frac{1}{\rho_j - \rho_j^{(1)}} \frac{1}{\mu_j^{(1)}} \right)^{1/2} \frac{1}{\mu_j^{(1)}} \beta^j \epsilon_j^{2}. 
\end{align*}
\]

Take \(\rho_j^{(l)} = \rho_j - \frac{l}{d}(\rho_j - \rho_j^{(l+1)}), \gamma_j^{(l)} = \gamma_j - \frac{l}{d}(\gamma_j - \gamma_j^{(l+1)}), \sigma_j^{(l)} = \sigma_j - \frac{l}{d}(\sigma_j - \sigma_j^{(l+1)}), \mu_j^{(l)} = (\sigma_j^{(l)})^2, \)

\(l = 1, 2, 3, 4.\) By \([3.17]\), we have

\[(3.21) \quad [(1 - T_{\Delta'}) f_{\text{low}}]\Lambda_j, \gamma_j^{(1)}, \sigma_j^{(2)}, \rho_j^{(2)}, \mu_j^{(3)} \leq \beta \epsilon \beta^j \epsilon_j \leq \beta^j \epsilon_j^{2}.\]

By \([3.19]\), we have

\[(3.22) \quad [(1 - T_{\Delta'}) f_{\text{low}}]\Lambda_j, \gamma_j^{(1)}, \sigma_j^{(2)}, \rho_j^{(2)}, \mu_j^{(3)} \leq \beta \epsilon \beta^j \epsilon_j \leq \beta^j \epsilon_j^{2}.\]
By (3.13), (3.16), we obtain
\[
f_{j+1} = (1 - T_{\Delta'}) f_{j}^{\text{low}} + f_{j}^{\text{high}} + (1 - T_{\Delta'}) \{ f_{j}^{\text{high}}, s_j \}^{\text{low}} + \{ f_{j}^{\text{high}}, s_j \}^{\text{high}} \\
+ \int_0^1 (1-t) \{-T_{\Delta'} f_{j}^{\text{low}}, s_j \} \circ X_{s_j}^t dt + \int_0^1 (1-t) \{-T_{\Delta'} \{ f_{j}^{\text{high}}, s_j \}^{\text{low}}, s_j \} \circ X_{s_j}^t dt \\
+ \int_0^1 \{ h_1 + \cdots + h_j + (1-t) h_{j+1}, s_j \} \circ X_{s_j}^t dt + \int_0^1 \{ f_{j}^{\text{low}}, s_j \} \circ X_{s_j}^t dt \\
+ \int_0^1 (1-t) \{ f_{j}^{\text{high}}, s_j, s_j \} \circ X_{s_j}^t dt.
\]

Hence
\[
(3.23) \quad f_{j+1}^{\text{low}} = (1 - T_{\Delta'}) f_{j}^{\text{low}} + (1 - T_{\Delta'}) \{ f_{j}^{\text{high}}, s_j \}^{\text{low}} + \left( \int_0^1 (1-t) \{-T_{\Delta'} f_{j}^{\text{low}}, s_j \} \circ X_{s_j}^t dt \right)^{\text{low}} \\
+ \left( \int_0^1 (1-t) \{-T_{\Delta'} \{ f_{j}^{\text{high}}, s_j \}^{\text{low}}, s_j \} \circ X_{s_j}^t dt \right)^{\text{low}} \\
+ \left( \int_0^1 \{ f_{j}^{\text{low}}, s_j \} \circ X_{s_j}^t dt \right)^{\text{low}} + \left( \int_0^1 \{ h_1 + \cdots + h_j + (1-t) h_{j+1}, s_j \} \circ X_{s_j}^t dt \right)^{\text{low}}.
\]

(3.24) and 
\[
(3.25) \quad \left[ \int_0^1 (1-t) \{-T_{\Delta'} f_{j}^{\text{low}}, s_j \} \circ X_{s_j}^t dt \right]_{\Lambda_{j+1, \gamma_{j+1}, \sigma_{j+1}; U', \rho_{j+1}, \mu_{j+1}}} \leq \beta^{j+1} \varepsilon_{j+1},
\]

(3.26) 
\[
\left[ \int_0^1 \{ f_{j}^{\text{low}}, s_j \} \circ X_{s_j}^t dt \right]_{\Lambda_{j+1, \gamma_{j+1}, \sigma_{j+1}; U', \rho_{j+1}, \mu_{j+1}}} \leq \beta^{j+1} \varepsilon_{j+1},
\]

(3.27) 
\[
\left[ \int_0^1 (1-t) \{-T_{\Delta'} \{ f_{j}^{\text{high}}, s_j \}^{\text{low}}, s_j \} \circ X_{s_j}^t dt \right]_{\Lambda_{j+1, \gamma_{j+1}, \sigma_{j+1}; U', \rho_{j+1}, \mu_{j+1}}} \leq \beta^{j+1} \varepsilon_{j+1},
\]

(3.28) 
\[
\left[ \int_0^1 \{ h_1 + \cdots + h_j + (1-t) h_{j+1}, s_j \} \circ X_{s_j}^t dt \right]_{\Lambda_{j+1, \gamma_{j+1}, \sigma_{j+1}; U', \rho_{j+1}, \mu_{j+1}}} \leq \beta^{j+1} \varepsilon_{j+1},
\]

(3.29) 
\[
\left[ \int_0^1 (1-t) \{ f_{j}^{\text{high}}, s_j, s_j \} \circ X_{s_j}^t dt \right]_{\Lambda_{j+1, \gamma_{j+1}, \sigma_{j+1}; U', \rho_{j+1}, \mu_{j+1}}} \leq \beta^{j+1} \varepsilon_{j+1}.
\]

By (3.22), (3.25), (3.28), (3.26), (3.29), we have
\[
(3.30) \quad [ f_{j+1}^{\text{low}} ]_{\Lambda_{j+1, \gamma_{j+1}, \sigma_{j+1}; U', \rho_{j+1}, \mu_{j+1}}} \leq \beta^{j+1} \varepsilon_{j+1}.
\]
By (3.9), (3.22), (3.24), (3.25)-(3.29), we have
\[ f^{|j|}_{j+1} \leq 1 + \frac{1}{K^1} \beta^{j+1} \varepsilon_j + \beta^{j+1} \varepsilon_{j+1}. \]

By (3.54), we have
\[ \|X_{s_1}\|_T^T \leq \frac{1}{K^{\beta}} \Delta \exp \left( \frac{1}{\rho_j - \rho_j^{(1)}} \right)^5 \beta^j \varepsilon_j, \]
(3.31)
\[ \|X_{h_{j+1}}\|_T^T \leq \frac{1}{K^{\beta}} \Delta \exp \left( \frac{1}{\rho_j - \rho_j^{(1)}} \right)^4 \beta^j \varepsilon_j. \]

By (2.68), we have
\[ \|X_{(1-\mathcal{T}) f_{j}^{f_{I_{I_{I}}}}(j_{I_{I_{I}}})} \|_T^T \leq \frac{1}{K^{\beta}} \Delta \exp \left( \frac{1}{\rho_j - \rho_j^{(1)}} \right)^{A+1} \beta^j \varepsilon_j \leq \beta^{j+1} \varepsilon_{j+1}. \]

By (2.66), (2.67), we have
\[ \|X_{(1-\mathcal{T}) f_{j}^{f_{I_{I_{I}}}}(j_{I_{I_{I}}})} \|_T^T \leq \frac{1}{K^{\beta}} \Delta \exp \left( \frac{1}{\rho_j - \rho_j^{(1)}} \right)^{A+4} \left[ e^{-\frac{1}{2} \Delta'} + \frac{1}{\rho_j - \rho_j^{(1)}} e^{-\frac{1}{2} \Delta'} \right] \beta^j \varepsilon_j \leq \beta^{j+1} \varepsilon_{j+1}. \]

By (3.58), (3.31), using Proposition 2.10, we have
\[ \|X_{f_{j_{I_{I_{I}}}}(j_{I_{I_{I}}})} \|_T^T \leq \frac{1}{K^{\beta}} \Delta \exp \left( \frac{1}{\rho_j - \rho_j^{(1)}} \right)^5 \beta^{2j} \varepsilon_j, \]
(3.35)
\[ \|X_{f_{j_{I_{I_{I}}}}(j_{I_{I_{I}}})} \|_T^T \leq \frac{1}{K^{\beta}} \Delta \exp \left( \frac{1}{\rho_j - \rho_j^{(1)}} \right)^5 \beta^{2j} \varepsilon_j, \]
(3.36)
\[ \|X_{h_{j_{I_{I_{I}}}}(j_{I_{I_{I}}})} \|_T^T \leq \frac{1}{K^{\beta}} \Delta \exp \left( \frac{1}{\rho_j - \rho_j^{(1)}} \right)^4 \beta^j \varepsilon_j. \]

By (3.38), (3.39), using Proposition 2.10, we have
\[ \|X_{h_{j_{I_{I_{I}}}}(j_{I_{I_{I}}})} \|_T^T \leq \frac{1}{K^{\beta}} \Delta \exp \left( \frac{1}{\rho_j - \rho_j^{(1)}} \right)^9 \beta^{2j} \varepsilon_j, \]
(3.38)
\[ \|X_{h_{j_{I_{I_{I}}}}(j_{I_{I_{I}}})} \|_T^T \leq \frac{1}{K^{\beta}} \Delta \exp \left( \frac{1}{\rho_j - \rho_j^{(1)}} \right)^{10} \beta^{2j} \varepsilon_j, \]
(3.39)
(3.40)\[ |||X_{\{h_{j+1},s_j\}}|||^T_{p,D,p_j^{(h_j)},\rho_j^{(s_j)}} \times U' \leq \frac{1}{\rho_j^{(1)}} \frac{1}{\rho_j^{(2)}} \Delta^{10} \exp \left( \frac{1}{\rho_j^{(1)}} \right) \frac{1}{\rho_j^{(1)}} \beta^{j+1} \varepsilon_j \varepsilon_j. \]

By (3.16), we obtain

(3.41)\[ f_{j+1} = (1 - \mathcal{T}_{\Delta^j}) f_j^{low} + f_j^{high} + (1 - \mathcal{T}_{\Delta^j}) \{ f_j^{high}, s_j \}^{low} + \{ f_j^{high}, s_j \}^{high} \]

\[ + \int_0^1 \left( h_1 + \cdots + h_{j+1} \right) \circ X_{s_j}^t dt + \int_0^1 \left( f_j^{high}, s_j \right) \circ X_{s_j}^t dt \]

\[ + \int_0^1 \{ f_j^{low}, s_j \} \circ X_{s_j}^t dt + \int_0^1 (1 - t) \{ f_j^{high}, s_j \} \circ X_{s_j}^t dt. \]

Hence

(3.42)\[ f_{j+1}^{low} = (1 - \mathcal{T}_{\Delta^j}) f_j^{low} + (1 - \mathcal{T}_{\Delta^j}) \{ f_j^{high}, s_j \}^{low} + \left( \int_0^1 (1 - t) \{ f_j^{high}, s_j \} \circ X_{s_j}^t dt \right)^{low} \]

\[ + \left( \left( \int_0^1 \{ h_1 + \cdots + h_{j+1} \} \circ X_{s_j}^t dt \right)^{low} + \left( \int_0^1 \{ f_j^{low}, s_j \} \circ X_{s_j}^t dt \right)^{low} \right), \]

(3.43)\[ f_{j+1}^{high} = f_j^{high} + \{ f_j^{high}, s_j \}^{high} + \left( \int_0^1 (1 - t) \{ f_j^{high}, s_j \} \circ X_{s_j}^t dt \right)^{high} \]

\[ + \left( \left( \int_0^1 \{ h_1 + \cdots + h_{j+1} \} \circ X_{s_j}^t dt \right)^{high} + \left( \int_0^1 \{ f_j^{low}, s_j \} \circ X_{s_j}^t dt \right)^{high} \right), \]

By (3.32), (3.34), (3.40), using Theorem 3.3 in [3], we have

(3.44)\[ |||X_{f_j^{low}\{1-(h_j,s_j)\} \circ X_{s_j}^t dt}|||^T_{p,D,p_j^{(h_j)},\rho_j^{(s_j)}} \times U' \leq \beta^{j+1} \varepsilon_j. \]

(3.45)\[ |||X_{f_j^{high}\{1-(h_j,s_j)\} \circ X_{s_j}^t dt}|||^T_{p,D,p_j^{(h_j)},\rho_j^{(s_j)}} \times U' \leq \beta^{j+1} \varepsilon_j. \]

(3.46)\[ |||X_{f_j^{low}\{f_j^{low},s_j\} \circ X_{s_j}^t dt}|||^T_{p,D,p_j^{(h_j)},\rho_j^{(s_j)}} \times U' \leq \beta^{j+1} \varepsilon_j. \]

(3.47)\[ |||X_{f_j^{high}\{f_j^{high},s_j\} \circ X_{s_j}^t dt}|||^T_{p,D,p_j^{(h_j)},\rho_j^{(s_j)}} \times U' \leq \beta^{j+1} \varepsilon_j. \]

By (3.32), (3.34), (3.40), (3.44)- (3.47), we have

(3.48)\[ |||X_{f_j^{low}}|||^T_{p,D,p_j^{(h_j)},\rho_j^{(s_j)}} \times U' \leq \beta^{j+1} \varepsilon_j. \]

By (3.8), (3.30), (3.43), (3.44)- (3.47), we have

(3.49)\[ |||X_{f_j^{high}}|||^T_{p,D,p_j^{(h_j)},\rho_j^{(s_j)}} \times U' \leq 1 + \frac{1}{\kappa^6} \beta^{j+1} \varepsilon_j. \]

□
3.2. Iterative Lemma. Assume $\rho = \sigma$, $\mu = \sigma^2$, $d_{\Delta} \gamma \leq 1$. For $m \geq 0$, let
\[
\varepsilon_m = e^{-\frac{1}{20}(\log \frac{1}{\varepsilon_0})^2}, \quad \varepsilon_0 = \varepsilon,
\]
\[
\vartheta_m = \frac{1}{2} \sum_{j=1}^{\infty} \frac{1}{\varepsilon_j}, \quad \vartheta_0 = 0,
\]
\[
\rho_m = (1 - \vartheta_m) \rho, \quad \rho_0 = \rho,
\]
\[
\sigma_m = (1 - \vartheta_m) \sigma, \quad \sigma_0 = \sigma,
\]
\[
\mu_m = \sigma_m^2, \quad \mu_0 = \mu,
\]
\[
\gamma_m = d_{\Delta_m}^{-1}, \quad \gamma_0 = \min(\gamma, d_{\Delta}^{-1}),
\]
\[
\Delta_m = 80(\log \frac{1}{\varepsilon_m})^2 \frac{1}{\min(\gamma_m, \rho_m - \rho)}, \quad \Delta_0 = \Delta,
\]
\[
\Lambda_m = \text{cte} \cdot d_{\Delta_m}^2,
\]
where the constant $\text{cte.}$ is the one in Proposition 6.7 in [8].

We have the following iterative lemma.

Lemma 3.3. For $m \geq 0$, consider the Hamiltonian $h_m + f_m$, where
\[
h_m = \langle \omega_m(w), r \rangle + \frac{1}{2} \langle \zeta, (\Omega(w) + H_m(w)) \zeta \rangle,
\]
$H_m(w), \partial_w H_m(w)$ are Töplitz at $\infty$ and $\mathcal{N}\mathcal{F}_{\Delta_m}$ for all $w \in U_m$,
\[
f_m = f_m^{\text{low}} + f_m^{\text{high}}
\]
satisfy
\[
|||X_{f_m^{\text{low}}-m}|||_{p,D(\rho_m, \mu_m, \sigma_m) \times U_m} \leq \varepsilon_m, \quad |||X_{f_m^{\text{high}}-m}|||_{p,D(\rho_m, \mu_m, \sigma_m) \times U_m} \leq \varepsilon + \sum_{j=1}^{m} \varepsilon_j^{\frac{2}{3}},
\]
\[
[f_m^{\text{low}}]_{\Lambda_m, \gamma_m, \sigma_m; U_m, \rho_m, \mu_m} \leq \varepsilon_m, \quad [f_m^{\text{high}}]_{\Lambda_m, \gamma_m, \sigma_m; U_m, \rho_m, \mu_m} \leq \varepsilon + \sum_{j=1}^{m} \varepsilon_j^{\frac{2}{3}}.
\]
Assume for all $w \in U_m$,
\[
|\omega_m(w) - \omega(w)| + |\partial_w(\omega_m(w) - \omega(w))| \leq \sum_{j=1}^{m} \varepsilon_j^{\frac{2}{3}},
\]
\[
||H_m - H||_{U_m} + \langle H_m - H \rangle_{\Lambda_m, U_m} \leq \sum_{j=1}^{m} \varepsilon_j^{\frac{2}{3}}.
\]
Then there is a subset $U_{m+1} \subset U_m$ such that if
\[
\varepsilon \leq \min \left( \gamma, \rho, \frac{1}{\Delta}, \frac{1}{\Lambda} \right)^{\text{exp}},
\]
then for all $w \in U_{m+1}$, there is a real analytic symplectic map $\Phi_m$ such that
\[
(h_m + f_m) \circ \Phi_m = h_{m+1} + f_{m+1}
\]
with the following estimates

\[ |||X_{f_{m+1}}^{low}|||^T_{\rho,D(p_{m+1},\mu_{m+1},\sigma_{m+1}) \times U_{m+1}} \leq \varepsilon_{m+1},\]
\[ |||X_{f_{m+1}}^{high}|||^T_{\rho,D(p_{m+1},\mu_{m+1},\sigma_{m+1}) \times U_{m+1}} \leq \varepsilon + \sum_{j=1}^{m+1} \varepsilon_{j-1}^2,\]
\[ |\omega_{m+1}(w) - \omega(w)| + |\partial_w(\omega_{m+1}(w) - \omega(w))| \leq \sum_{j=1}^{m+1} \varepsilon_{j-1}^2,\]
\[ \|H_{m+1} - H\|_{U_{m+1}} + \langle H_{m+1} - H, \Lambda_{m+1} \rangle_{U_{m+1}} \leq \sum_{j=1}^{m+1} \varepsilon_{j-1}^2,\]
\[ \text{meas}(U_m \setminus U_{m+1}) \leq \varepsilon_{m+1}^{\exp'},\]

where the exponents \( \exp, \exp' \) depend on \( d, \#A, p \).

**Proof.** Take \( \varepsilon^{20} = \varepsilon^{\frac{1}{20}} \) in Lemma 3.2 there is a real analytic symplectic map \( \Phi \) such that

\[(h + f) \circ \Phi = h + h_1 + \cdots + h_n + f_n.\]

Let \( h_+ = h + h_1 + \cdots + h_n, f_+ = f_n. \) Using Lemma 3.2 we prove the iterative lemma. \( \square \)

Now Theorem 3.1 follows from Lemma 3.3 and we omit its proof here.

**References**

[1] J. Bourgain. Quasi-periodic solutions of Hamiltonian perturbations of 2D linear Schrödinger equations. *Ann. of Math. (2)*, 148(2):363–439, 1998.

[2] J. Bourgain. *Green’s function estimates for lattice Schrödinger operators and applications.* Annals of Mathematics Studies. 158. Princeton University Press, Princeton, NJ, 2005.

[3] D. Bambusi and B. Grébert. Birkhoff normal form for partial differential equations with tame modulus. *Duke Math. J.*, 135(3):507–567, 2006.

[4] J. Colliander, M. Keel, G. Staffilani, H. Takaoka, and T. Tao. Transfer of energy to high frequencies in the cubic defocusing nonlinear Schrödinger equation. *Invent. Math.*, 181(1):39–113, 2010.

[5] H. Cong, J. Liu, and X. Yuan. Stability of KAM tori for nonlinear Schrödinger equation. *Mem. Amer. Math. Soc.*, 239(1134), 2016.

[6] L. H. Eliasson, B. Grébert, and S. B. Kuksin. KAM for the nonlinear beam equation. *Geom. Funct. Anal.*, 26(6):1588–1715, 2016.

[7] L. H. Eliasson and S. B. Kuksin. Infinite Töplitz-Lipschitz matrices and operators. *Z. Angew. Math. Phys.*, 59(1):24–50, 2008.

[8] L. H. Eliasson and S. B. Kuksin. KAM for the nonlinear Schrödinger equation. *Ann. of Math. (2)*, 172(1):371–435, 2010.

[9] X. He, J. Shi, Y. Shi and X. Yuan. On linear stability of KAM tori via the Craig-Wayne-Bourgain method. 2020. arXiv:2003.01487.

[10] S. B. Kuksin. *Nearly integrable infinite-dimensional Hamiltonian systems.* Lecture Notes in Mathematics. 1556. Springer-Verlag, Berlin, 1993.
[11] S. B. Kuksin and J. Pöschel. Invariant Cantor manifolds of quasi-periodic oscillations for a nonlinear Schrödinger equation. *Ann. of Math. (2)*, 143(1):149–179, 1996.

[12] J. Pöschel. A KAM-theorem for some nonlinear partial differential equations. *Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4)*, 23(1):119–148, 1996.

[13] J. Pöschel. Quasi-periodic solutions for a nonlinear wave equation. *Comment. Math. Helv.*, 71(2):269–296, 1996.

[14] C. E. Wayne. Periodic and quasi-periodic solutions of nonlinear wave equations via KAM theory. *Comm. Math. Phys.*, 127(3):479–528, 1990.

[15] X. Yuan. KAM theorem with normal frequencies of finite limit-points for some shallow water equations. *Commun. Pure Appl. Math.*, 74(6):1193–1281, 2021.

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