NONEXISTENCE OF SEMIORTHOGONAL DECOMPOSITIONS AND SECTIONS OF THE CANONICAL BUNDLE

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Abstract. We investigate semiorthogonal decompositions (SODs) of the derived category of coherent sheaves on a smooth proper variety. We prove that global/local sections of the canonical bundle give a strong constraint on the supports of objects in one of the semiorthogonal summands. We also show that SODs are rigid under the action of topologically trivial autoequivalences. As applications of these results, we prove the non-existence of non-trivial SODs for various minimal models.

Contents

1. Introduction 1
2. Preliminaries 4
  2.1. Deligne-Mumford stacks 4
  2.2. (Semi)orthogonal decompositions 6
  2.3. Picard scheme 7
3. Results in arbitrary dimensions 8
  3.1. Proof of Theorem 1.2 8
  3.2. Local situation 9
  3.3. Rigidity of Semiorthogonal decompositions 10
4. Results for surfaces 12
  4.1. $\kappa = 0$ 12
  4.2. $\kappa = 1$ 12
  4.3. $\kappa = 2$ 13
5. Toward higher dimensions 14
6. Generalization to twisted sheaves 16
7. Concluding comments 18
References 18

1. Introduction

Let $X$ be a smooth proper Deligne-Mumford stack (DM stack for short) defined over a field $k$. The bounded derived category $D(X) = D^b\text{coh} X$ of coherent sheaves on $X$, as a $k$-linear triangulated category, has been acquiring considerable attentions from both mathematical and physical points of view.

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One of the most fundamental notions about triangulated categories is \textit{semiorthogonal decomposition} (SOD for short). Although quite a few interesting examples are known, their classification is far from being fully understood. The purpose of this paper is to partially clarify the nature of SODs of the derived category of coherent sheaves.

It is conjectured in general and has been verified in some cases that each step of the \textit{minimal model program} (MMP for short) induces a non-trivial SOD of $D(X)$ (see [Kaw09]). Therefore if $D(X)$ admits no non-trivial SODs we expect that the variety is minimal, i.e. its canonical bundle is nef.

In addition to them, there is another way of producing SODs. Let $X$ be a smooth projective variety satisfying
\[ H^*(X, \mathcal{O}_X) \simeq k \in D^b(\text{Spec } k). \] (1.1)
Then any line bundle $L$ on $X$ is an \textit{exceptional object} ([Huy06, Definition 1.57]), so that it induces a non-trivial SOD $D(X) = \left< \langle L \rangle^\perp, \langle L \rangle \right>$. Here $\langle L \rangle$ denotes the smallest triangulated subcategory containing $L$, and $\langle L \rangle^\perp$ is its right orthogonal complement (see [Huy06, Definition 1.42]). The condition (1.1) is satisfied by an arbitrary variety of Fano type in characteristic zero due to the Kawamata-Viehweg vanishing theorem.

There are minimal varieties, such as (classical) Enriques surfaces, which also satisfy (1.1). Because of these examples, the correspondence between MMP and SODs is not perfect: i.e. SODs are (conjecturally) finer than MMP\(^1\).

In this paper we give two kinds of constraints which should be fulfilled by any SOD of the derived category of a smooth proper DM stack. As an application, we prove the non-existence of non-trivial SODs on various varieties which are (as expected) minimal.

One of the constraints on SODs is provided by global/local sections of the canonical bundle. The following is a prototype result in this direction.

\textbf{Theorem 1.1.} Let $X$ be a smooth proper DM stack and $D(X) = \left< \mathcal{A}, \mathcal{B} \right>$ an SOD. Then

1. At least one of the followings holds.
   (a) $k(x) \in \mathcal{A}$ for any closed point $x \notin Bs|\omega_X| \cup S$.
   (b) $k(x) \in \mathcal{B}$ for any closed point $x \notin Bs|\omega_X| \cup S$.

2. When (1a) (resp. (1b)) is satisfied, the support of any object in $\mathcal{B}$ (resp. $\mathcal{A}$) is contained in $Bs|\omega_X| \cup S$.

See Definition 2.2 for the definition of the stable base locus $Bs|\omega_X|$ and Definition 2.3 for that of the stacky points $S \subset X$.

As an immediate corollary we obtain the following sufficient condition for the non-existence of SODs.

\textbf{Theorem 1.2.} Let $X$ be a smooth proper DM stack satisfying the following properties:

1. there exists a non-stacky closed point.
2. $X$ satisfies the resolution property: i.e., every coherent sheaf on $X$ admits a surjective morphism from a locally free sheaf.

\(^1\)Recently several groups worked on exceptional objects of the derived categories of surfaces of general type satisfying (1.1), and discovered several examples of (quasi-)phantom categories ([AO13], [BvBS13], [BvBKS12], and [GS13], to name a few). Another remarkable discovery was that one can also produce a counter-example for the Jordan-Hölder property of SODs on such a surface (see [BGvBS14]). Note that later an easier example was found on a rational threefold in [Kuz13].
There exists an open neighborhood of \( S \cup Bs |\omega_X| \) on which \( \omega_X \) is trivial. Then \( D(X) \) has no non-trivial SOD.

If the coarse moduli space of \( X \) is projective, or more generally a scheme with affine diagonal, then the condition (2) is always satisfied by [Kaw04, Theorem 4.2] and [Tot04, Theorem 1.2].

In order to establish the correspondence between MMP and SOD for varieties with quotient singularities, we should think of the derived category of the smooth DM stack which is obtained by replacing neighborhoods of singular points by the corresponding quotient stacks (see [Kaw05]). This is one of the reasons why we should think of stacks, not only schemes.

A special case of Theorem 1.2 is

**Corollary 1.3.** Let \( X \) be a smooth proper variety such that \( Bs |\omega_X| \) is a finite set. Then \( D(X) \) has no non-trivial SOD.

In particular the global generation of the canonical bundle implies the non-existence of non-trivial SODs. Examples of such varieties are submanifolds of abelian varieties and complete intersections with non-negative Kodaira dimensions in projective spaces. This is a far generalization of [Oka11, Theorem 1.1], in which only 1-dimensional case was discussed.

The other constraint on SODs is the rigidity under the action of topologically trivial autoequivalences.

**Theorem 1.4** (= Theorem 3.7). Let \( X \) be a projective scheme over \( k \) and \( D(X) = \langle \mathcal{A}, \mathcal{B} \rangle \) an SOD. Then for any line bundle \( L \) satisfying \([L] \in \text{Pic}_{X/k}^0\), we have the equality of subcategories \( \mathcal{A} \otimes L = \mathcal{A} \subset D(X) \).

Immediately we see

**Corollary 1.5** (= Corollary 3.8). Let \( X \) be a smooth projective variety satisfying \([\omega_X] \in \text{Pic}_{X/k}^0\). Then \( D(X) \) admits no SOD.

**Corollary 1.6.** Let \( X \) be the product of a bielliptic surface with an abelian variety. Then \( D(X) \) admits no SOD.

Since the canonical bundle of the variety \( X \) in Corollary 1.6 admits no global section (in another word \( Bs |\omega_X| = X \), the converse of Corollary 1.3 does not hold at all in dimensions at least two.

In Section 4 we intensively study SODs of minimal surfaces. We obtain satisfactory results for the cases \( \kappa(X) = 0 \) and 1, where \( \kappa(X) \) is the Kodaira dimension of \( X \).

**Theorem 1.7.** Let \( X \) be a smooth projective minimal surface.

1. If \( \kappa(X) = 0 \) and \( X \) is not a classical Enriques surface, then \( D(X) \) admits no SOD.
2. If \( \kappa(X) = 1 \) and \( p_g(X) > 0 \), then \( D(X) \) admits no SOD.

In the study of SODs of (quasi-)elliptic fibrations (Section 4.2), Theorem 3.7 will be effectively used on multiple fibers.

For the case \( \kappa(X) = 2 \) we have to put a rather strong assumption to prove the non-existence of SODs. We suspect that it should be considerably weakened.

**Theorem 1.8** (= Theorem 4.4). Let \( X \) be a minimal smooth projective surface of general type with \( \dim_k H^0(X, \omega_X) > 1 \). Assume the following condition (*).
For any one-dimensional connected component \( Z \subset Bs|\omega_X| \), its intersection matrix is negative definite.

Then \( D(X) \) admits no SOD.

If there exists a local section of \( \omega_X \) defined on an infinitesimal neighborhood of \( Bs|\omega_X| \), similar arguments as in the proof of Theorem 1.1 work. See Section 3.2 and Corollary 3.5 for the precise statements. This will be effectively used in the proof of Theorem 4.4.

In Section 5, we briefly treat varieties of dimensions greater than 2. There is nothing new in the case \( \kappa = 0 \), and we mainly discuss the case \( \kappa = 1 \). Then a difficulty which does not appear in dimension 2 shows up. We illustrate this issue with an example due to Keiji Oguiso.

Finally, in Section 6 we generalize our results to the category of twisted sheaves. It turns out that our arguments go through without essential change, though there are a couple of technical issues to be settled.

**Notations and conventions.** The base field \( \mathbf{k} \) will be assumed to be algebraically closed. Note that this assumption is not so restrictive. In fact if \( \mathbf{k} \) is not algebraically closed, any SOD of \( D(X) \) induces an SOD of \( D(X \otimes_{\mathbf{k}} \bar{\mathbf{k}}) \), where \( \bar{\mathbf{k}} \) is the algebraic closure of \( \mathbf{k} \), and this SOD is invariant under the action of \( \text{Aut}(\bar{\mathbf{k}}/\mathbf{k}) \) ([Kuz11, Proposition 5.1]). Since \( X \) is connected, \( \text{Aut}(\bar{\mathbf{k}}/\mathbf{k}) \) acts transitively on \( \pi_0(X \otimes_{\mathbf{k}} \bar{\mathbf{k}}) \). Therefore in order to show the non-existence of SOD for \( D(X) \), it is enough to show the same for \( D(\bar{X}) \), where \( \bar{X} \subset X \otimes_{\mathbf{k}} \bar{\mathbf{k}} \) is a connected component.

Any Deligne-Mumford stack in this paper will be assumed to be connected and smooth over \( \mathbf{k} \), unless otherwise stated.

The following standard symbols will be used.

\[ p_g = p_g(X) = \dim H^0(X, \omega_X) = \dim \text{Hom}_{D(X)}(\mathcal{O}_X, \omega_X) \]
\[ q = q(X) = \dim H^1(X, \mathcal{O}_X) \]

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2. **Preliminaries**

2.1. **Deligne-Mumford stacks.** We begin with a couple of notions about Deligne-Mumford (DM for short) stacks.
Definition 2.1. Let $X$ be a DM stack. A **closed point** of $X$ is an irreducible reduced closed substack of dimension zero. For a closed point $\iota: x \hookrightarrow X$, we denote by $k(x)$ the sheaf $\iota_*O_x$. This situation will be simply denoted as $x \in X$.

Definition 2.2. Let $X$ be a smooth DM stack with the canonical bundle $\omega_X$. The **base locus of the canonical complete linear system** is the closed substack of $X$ defined by the ideal $\text{Im} \left( \text{Hom}(\omega^{-1}_X, O_X) \otimes \omega^{-1}_X \rightarrow O_X \right) \subset O_X$ and will be denoted by $\text{Bs} |_{\omega_X}$.

See [LMB00, Application (14.2.7)] for the correspondence between closed substacks and quasi-coherent ideal sheaves.

Definition 2.3. Let $X$ be a smooth proper DM stack. By [KM97, Corollary 1.3.(1)], $X$ admits a coarse moduli algebraic space $\pi: X \rightarrow |X|$. The **stacky locus**, which will be denoted by $S \subset X$, is the complement of the maximal open substack of $X$ on which $\pi$ is an isomorphism to its image. A closed point $x \in X$ is said to be **non-stacky** if and only if the inclusion factors through $X \setminus S$.

Definition 2.4. Let $X$ be a smooth DM stack, and $E \in D(X)$ a bounded complex of coherent sheaves on $X$. The **support of $E$** is the union of the supports of its cohomology sheaves: i.e., $\text{Supp} E = \bigcup_i \text{Supp} H^i(E)_{\text{red}}$. By definition, $\text{Supp} E$ is a closed substack of $X$. If $\text{Supp} E$ is contained in a closed substack $Z \subset X$, we say that $E$ is **supported in $Z$**.

Remark 2.5. The support of $E$ can be alternatively defined as the complement of the maximal open substack of $X$ on which $E$ is zero. If $E$ is a coherent sheaf, we can introduce the natural stack structure on $\text{Supp} E$ (which is not necessarily reduced) in such a way that $E$ is isomorphic to the pushforward of a coherent sheaf on $\text{Supp} E$ (see [HL10, Section 1.1]). In general, let $Z \subset X$ be a reduced closed substack and $E \in D(Z)$ a bounded complex of coherent sheaves supported in $Z$. Then by [Rou08, Lemma 7.41] there exists a positive integer $n > 0$ and a bounded complex of coherent sheaves $E' \in D(nZ)$ such that $E \cong \iota_* E'$, where $\iota: nZ \hookrightarrow X$ is the natural closed immersion. The proof uses the Artin-Rees lemma in an essential way, so that we do not have a control over the value of $n$. If we could introduce a sufficiently thin stack structure on the support of complexes, that would be useful to improve the results of this paper.

Lemma 2.6. Let $X$, $E$, and $Z$ be as in Remark 2.3. Suppose that $L$ is a line bundle on $X$ which is trivial on an open neighborhood of $Z$. Then $E \cong E \otimes L$.

Proof. Let $\iota: U \hookrightarrow X$ be an open neighborhood of $Z$ on which $L$ is trivial. Since $E$ is supported in $Z$, the natural morphism $E \rightarrow \iota_* \iota^* E$ is an isomorphism. The same holds for $E \otimes L$, and hence

$$E \cong \iota_* \iota^* E \cong \iota_* (\iota^* E \otimes L|_U) \cong \iota_* \iota^* (E \otimes L) \cong E \otimes L. \quad (2.1)$$

Next we recall the Serre duality for DM stacks. In this paper the dualizing sheaf of $X$ will be denoted by $\omega_X$.

Fact 1 (= [Nir08, Theorem 2.22]). Let $X \rightarrow \text{Spec} k$ be a smooth proper DM stack of dimension $n$ over a field $k$. Then $\omega_X[n]$ is a dualizing complex of $X$.

We will use the following useful lemma, which first appeared in [BO95, Proposition 1.5].
Lemma 2.7. Let $X$ be a DM stack and $x \in X$ a non-stacky closed point. Take a bounded complex of coherent sheaves $E \in D(X)$. If $x \in \text{Supp } E$, then $\text{Hom}(E, k(x)[i]) \neq 0$ for some $i \in \mathbb{Z}$. Moreover if $X$ is smooth and proper, $\text{Hom}(k(x)[j], E) \neq 0$ also holds for some $j \in \mathbb{Z}$.

Proof. Since the support of an object does not change by taking a tensor product with an invertible sheaf, by applying the Serre duality, the second assertion can be reduced to the first.

Set $i = \max\{j | x \in \text{Supp } H^j(E)\}$. Then we get a sequence of morphisms

$$\begin{align*}
E &\to E_{\geq i} \cong H^i(E)[-i] \to H^i(E) \otimes k(x)[-i] \to k(x)[-i],
\end{align*}$$

(2.2)

where $\otimes k(x)$ is the underived tensor product. To construct the last morphism, one has to choose a surjective morphism of the $k(x)$-vector spaces; since $x$ is not a stacky point, there exists at least one such. The symbol $\bullet_{\geq i}$ is the upper truncation at degree $i$ of the complex $\bullet$ with respect to the standard $t$-structure.

Since the $i$th cohomology of the morphisms in (2.2), seen locally at $x$, are shifts of surjections of sheaves, their composition is also non-trivial. □

For the sake of completeness, here we include the following lemma.

Lemma 2.8 (Nakayama-Azumaya-Krull(NAK) lemma). Let $X$ be a scheme, $\iota: x \hookrightarrow X$ a closed point, and $E \in D^{-}(\text{coh } X)$ a complex of coherent sheaves on $X$ bounded above. If $\mathbb{L}\iota^{*}E = 0$, then $x \notin \text{Supp } E$.

Proof. See [Huy06, Lemma 3.29 and Exercise 3.30]. □

2.2. (Semi)orthogonal decompositions. We recall the notion of (semi)orthogonal decompositions and show the non-existence of orthogonal decompositions for stacks which admits a non-stacky closed point. This is well known for varieties (see, e.g., [Huy06, Proposition 3.10]). The proof given below was suggested by Yujiro Kawamata. Recall that a full subcategory $A \subset C$ is said to be strict if any object in $C$ which is isomorphic to an object in $A$ is already contained in $A$.

Definition 2.9. A pair of strictly full triangulated subcategories $A, B$ of a triangulated category $T$ is a semiorthogonal decomposition if the following conditions are satisfied:

- $\text{Hom}_{T}(b, a) = 0$ for all $a \in A$ and $b \in B$.
- Any object $x \in T$ is decomposed into a pair of objects $a \in A$ and $b \in B$ by a distinguished triangle

$$b \to x \to a \to b[1].$$

(2.3)

This situation will be denoted by the symbol $T = \langle A, B \rangle$. If $\text{Hom}_{T}(A, B) = 0$ also holds, the decomposition is called an orthogonal decomposition (OD for short) and denoted by $T = A \oplus B$. In this case the triangle (2.3) splits and we obtain the direct sum decomposition $x \cong a \oplus b$.

Remark 2.10.

1. If $T = \langle A, B \rangle$ is an SOD of $T$, then $A = B^\perp$ and $B = A^\perp$. Here $\bullet^\perp$ (resp. $\perp \bullet$) denotes the right (resp. left) orthogonal complement of the subcategory $\bullet$ ([Huy06, Definition 1.42]).

2. For a strictly full triangulated subcategory $\iota: C \subset T$, the pair $C, \perp C$ (resp. $C^\perp, C$) gives an SOD of $T$ if and only if the inclusion functor $\iota$ admits a left (resp. right) adjoint (see [Bon89, Lemma 3.1]).
Lemma 2.11. Let $X$ be a connected locally separated DM stack which admits a non-stacky point. Assume that $X$ satisfies the resolution property. Then $D(X)$ has no non-trivial OD.

Proof. Let $D(X) = \mathcal{A} \oplus \mathcal{B}$ be an OD of $D(X)$. Take a non-stacky point $x \in X$. Since $\text{End}(k(x))$ is a field, $k(x)$ is indecomposable and hence is contained in either $\mathcal{A}$ or $\mathcal{B}$. Let us assume it is contained in $\mathcal{A}$.

Let $E$ be any locally free sheaf, and consider the decomposition $E = E_\mathcal{A} \oplus E_\mathcal{B}$ provided by the OD. If $E_\mathcal{B} \neq 0$, then $E_\mathcal{B} \otimes k(x)$ is isomorphic to $k(x)^{\oplus r}$ with $r$ the rank of $E_\mathcal{B}$. Here we used the facts that $X$ is connected and the closed substack $x$ is a scheme by [Knu71, Chapter II, Proposition 5.9]. Then we get a surjective morphism $E_\mathcal{B} \twoheadrightarrow E_\mathcal{B} \otimes k(x) \rightarrow k(x)$ and it is a contradiction. Hence $E$ belongs to $\mathcal{A}$. Since locally free sheaves form a spanning class of $D(X)$ by the resolution property, $\mathcal{B}$ should be trivial by the orthogonality. □

Remark 2.12. If there is no non-stacky point on $X$, $D(X)$ may have an OD even if it is smooth and irreducible. For example consider the quotient stack $X = [\text{Spec} \mathbb{C}/(\mathbb{Z}/2)]$. A coherent sheaf on $X$ is nothing but a finite dimensional representation of $\mathbb{Z}/2$ over $\mathbb{C}$. The derived category $D(X)$ is orthogonally decomposed by the trivial representation and the non-trivial character of $\mathbb{Z}/2$.

Lemma 2.11 provides us with the following useful criterion for the triviality of an SOD.

Corollary 2.13. Let $X$ be a smooth proper DM stack which has a non-stacky point and satisfies the resolution property. Consider an SOD $D(X) = \langle \mathcal{A}, \mathcal{B} \rangle$. Then it is trivial, i.e. $\mathcal{A} = 0$ or $\mathcal{B} = 0$, if and only if $\mathcal{A} \otimes \omega_X \subset \mathcal{A} \subset D(X)$ holds.

The following argument is well known to experts (see [BK89, Proposition 3.6]), but we include it here because of its importance in this paper.

Proof. By Lemma 2.11, it is enough to show that it is an OD. Given $a \in \mathcal{A}$ and $b \in \mathcal{B}$, by applying the Serre duality and the assumption $a \otimes \omega_X[\text{dim} X] \in \mathcal{A}$, we see $\text{Hom}(a, b) \simeq \text{Hom}(b, a \otimes \omega_X[\text{dim} X])^\vee = 0$. Hence we see that $\mathcal{A}$ is also the right orthogonal of $\mathcal{B}$, concluding the proof. □

2.3. Picard scheme. We recall basics of Picard schemes from [FGI+05, Chapter 9]. Let $X \to S$ be a morphism of finite type between locally Noetherian schemes. The relative Picard functor, which will be denoted by Pic$_{X/S}$, is a contravariant functor from the category of locally Noetherian $S$-schemes to the category of abelian groups defined by

$$\text{Pic}_{X/S}(T) = \text{Pic}(X \times_S T)/\text{Pic}(T),$$

where $T$ is a locally Noetherian $S$-scheme. The functor Pic$_{X/S}$ is a presheaf, and the associated sheaf on the fppf site will be denoted by Pic$_{(X/S)}(\text{fppf})$. If Pic$_{(X/S)}(\text{fppf})$ is represented by a scheme, it will be denoted by Pic$_{X/S}$. A line bundle $L$ on $X$ naturally defines an $S$-valued point $[L] \in \text{Pic}_{X/S}(S)$.

The following existence result for Picard schemes is enough for us.

Theorem 2.14 (= [FGI+05, Corollary 9.4.18.3]). Let $S$ be the spectrum of a field and $X$ a proper scheme over $S$. Then Pic$_{X/S}$ exists and is a disjoint union of open quasi-projective subschemes.
If $\text{Pic}_{X/S}$ exists and $S$ is the spectrum of a field, its identity component (i.e., the connected component containing the identity) will be denote by $\text{Pic}_{X/S}^0$. The following theorem characterizes the $k$-valued points of $\text{Pic}_{X/k}^0$.

**Theorem 2.15** (=[FGI+05, Corollary 9.5.10]). Assume that $S$ is the spectrum of a field and $\text{Pic}_{X/S}$ exists. Let $L$ be an invertible sheaf on $X$. Then $L$ is algebraically equivalent to $\mathcal{O}_X$ if and only if $[L] \in \text{Pic}_{X/S}^0(S)$.

The notion of algebraic equivalence is defined as follows. For simplicity, we assume that the base field is algebraically closed.

**Definition 2.16** ([FGI+05, Definition 9.5.9]). Assume $S$ is the spectrum of an algebraically closed field $k$. Let $L$ and $N$ be invertible sheaves on $X$. Then $L$ is said to be algebraically equivalent to $N$ if, for some $n$ and all $i$ with $1 \leq i \leq n$, there exist a connected $k$-schemes of finite type $T_i$, closed points $s_i, t_i \in T_i$, and an invertible sheaf $M_i$ on $X \times_k T_i$ such that

$$ L \cong M_{1,s_1}, M_{1,t_1} \cong M_{2,s_2}, \ldots, M_{n-1,t_{n-1}} \cong M_{n,s_n}, M_{n,t_n} \cong N. \quad (2.5) $$

3. Results in arbitrary dimensions

### 3.1. Proof of Theorem 1.2

**Proof of Theorem 1.2.** Take an arbitrary closed point $x \in X \setminus (Bs |\omega_X| \cup S)$. We first show that $k(x)$ is contained in either $A$ or $B$. Let us include $k(x)$ in the triangle provided by the SOD:

$$ b \rightarrow k(x) \rightarrow a \xrightarrow{f} b[1]. \quad (3.1) $$

Take a global section $s \in H^0(X, \omega_X)$ which is not vanishing at $x$, and set $U = X \setminus Z(s)$. If $f|_U \neq 0$, we see $(\otimes s \circ f)|_U = \otimes s|_U \circ f|_U$ is also non-trivial. This contradicts

$$ \text{Hom}(a, b \otimes \omega_X[1]) \simeq \text{Hom}(b, a[\dim X - 1])^\vee = 0. \quad (3.2) $$

Thus we see $f|_U = 0$. This implies the decomposition $k(x) \simeq a|_U \oplus b|_U$ and hence we obtain either $a|_U = 0$ or $b|_U = 0$. If $a|_U = 0$, the morphism $k(x) \rightarrow a$ is zero. Then we obtain the decomposition $b \simeq k(x) \oplus a[-1]$. By the semiorthogonality we see $a = 0$ and hence $k(x) \in B$. If we instead assume $b|_U = 0$, similarly we obtain $k(x) \in A$.

If $k(x) \in B$ (respectively $k(x) \in A$) holds for some closed point $x \notin S$, then by Lemma 2.7 any object $E \in A$ should satisfy $x \notin \text{Supp} E$ (resp. any $E \in B$). This in particular implies that the closed substack $\text{Supp} E \subset X$ is strictly smaller than $X$.

Finally assume for a contradiction that $A$ and $B$ both contain non-stacky closed points. As said in the previous paragraph, the support of any object in $A$ or $B$ is a strictly smaller closed subset of $X$. On the other hand, consider the decomposition of the structure sheaf

$$ b \rightarrow \mathcal{O}_X \rightarrow a \rightarrow b[1]. \quad (3.3) $$

From this triangle we obtain the equality $X = \text{Supp} \mathcal{O}_X = \text{Supp} a \cup \text{Supp} b$, which contradicts the irreducibility of $X$. Hence we see that all the non-stacky closed points outside of $Bs |\omega_X|$ are contained simultaneously in $A$, or otherwise in $B$. This concludes (1) of Theorem 1.1. In the former case, the support of any object in $B$ is contained in $S \cup Bs |\omega_X|$ as we saw above, concluding the proof of (2).
Example 3.1. Let $Y$ be a smooth projective surface such that $\omega_Y$ is globally generated. Let $f: X \to Y$ be the blow-up of $Y$ at a closed point $y$, with the exceptional divisor $E \subset X$. Then we obtain the SOD $D(X) = \langle (\mathcal{O}_E(E)), \mathbb{L}f^*D(Y) \rangle$ (see [Orl92]). Observe that the objects in $\langle \mathcal{O}_E(E) \rangle$ are supported in $E = Bs|\omega_X|$, and that all the closed points $x \notin Bs|\omega_X|$ are contained in $\mathbb{L}f^*D(Y)$.

Proof of Theorem 1.2. Take an SOD $D(X) = \langle \mathcal{A}, \mathcal{B} \rangle$. Write $U = X \setminus (\mathcal{S} \cup Bs|\omega_X|)$. By Theorem 1.1, closed points of $U$ are simultaneously contained in either $\mathcal{A}$ or $\mathcal{B}$. Let us assume they are in $\mathcal{B}$.

By Theorem 1.1 then the support of any object $a \in \mathcal{A}$ is contained in $\mathcal{S} \cup Bs|\omega_X|$. Since $\omega_X$ is trivial on an open neighborhood of this set, we see $a \otimes \omega_X \simeq a$ by Lemma 2.6. Therefore we can apply Corollary 2.13 to conclude the proof.

Example 3.2. Let $X$ be the surface (of general type) discussed in [Zuc03, Proposition 3]. We can easily check that $Bs|\omega_X|$ consists of 4 points. Hence Corollary 1.3 tells us that the derived category $D(X)$ admits no SOD.

3.2. Local situation. We refine the arguments in the proof of Theorem 1.1 so as to make it applicable to local situations. This will be applied later to surfaces of general type. For simplicity we restrict ourselves to varieties.

Let $X$ be a variety and $Z$ a closed subset. Consider the strict full subcategory $C_Z = \{ E \in D(X) \mid \text{Supp } E \subset Z \} \subset D(X)$. An immediate corollary of Theorem 1.1 is

Lemma 3.3. Let $X$ be a smooth proper variety with $p_g(X) > 0$. Suppose that for any connected component $Z$ of $Bs|\omega_X|$, the category $C_Z$ admits no SOD. Then $D(X)$ has no SOD.

Proof. Take an SOD $D(X) = \langle \mathcal{A}, \mathcal{B} \rangle$. By Corollary 2.13 it is enough to show $\mathcal{A} \otimes \omega_X = \mathcal{A}$. Assume that the conclusion (1) of Theorem 1.1 holds, so that $\mathcal{A}$ is a triangulated subcategory of $C_{Bs|\omega_X|}$. For any connected component $Z \subset Bs|\omega_X|$, set $\mathcal{A}_Z = \{ a \in \mathcal{A} \mid \text{Supp } a \subset Z \}$ so as to obtain the orthogonal decomposition $\mathcal{A} = \bigoplus Z \mathcal{A}_Z$ by the triangulated subcategories $\mathcal{A}_Z \subset C_Z$.

Let $p: D(X) \to \mathcal{A}$ be the left adjoint of the inclusion functor $\mathcal{A} \hookrightarrow D(X)$. Composing $p$ with the obvious functors, we get the left adjoint of the inclusion $\mathcal{A}_Z \hookrightarrow C_Z$ and hence obtain the SOD $C_Z = \langle A_Z, \frac{1}{Z}A_Z \rangle$. Since $C_Z$ admits no SOD by the assumption, $A_Z$ is either 0 or $C_Z$ itself. In any case we obtain the equality $A_Z \otimes \omega_X = A_Z \subset D(X)$. Summing up over $Z$, we obtain the conclusion.

Now we show the local version of Theorem 1.1 for $C_Z$.

Proposition 3.4. Let $X$ be a smooth proper variety and $Z \subset X$ a closed subscheme. Assume that for each $m \geq 1$ we have a section $s_m \in H^0(mZ, \omega_X | mZ)$ such that $s_{m+1}|mZ = s_m$ and $s_1$ is generically non-vanishing on an irreducible component $W \subset Z$. Write the projective limit as $s = (s_m)_{m \geq 1} \in \varprojlim H^0(mZ, \omega_X | mZ)$.

Then for any SOD $C_Z = \langle \mathcal{A}, \mathcal{B} \rangle$, one and only one of the followings holds.

(a) For any closed point $x \in W$ at which $s$ does not vanish, $k(x) \in \mathcal{A}$.
(b) For any closed point $x \in W$ at which $s$ does not vanish, $k(x) \in \mathcal{B}$.
Proof. We follow essentially the same line as that of the proof of Theorem 1.1. Take any closed point \( x \in W \) at which \( s \) does not vanish. Consider the decomposition
\[
b \to k(x) \to a \xrightarrow{f} b[1].
\]
(3.5)
Since \( b \) is supported in \( Z \), by Remark 2.5 there exists \( m \geq 1 \) and \( b' \in D(mZ) \) together with an isomorphism \( b \xrightarrow{\sim} \iota_*b' \), where \( \iota : mZ \hookrightarrow X \) is the natural immersion. Hence we can define the “multiplication by \( s \)” as the composition of morphisms
\[
b \xrightarrow{\sim} \iota_*b' \xrightarrow{\iota_*\otimes s_m} \iota_*b' \otimes \omega_X \xrightarrow{\sim} b \otimes \omega_X.
\]
(3.6)
Arguing as in the proof of Theorem 1.1, we can show that the morphism \( f \) vanishes at \( x \).
Thus we see that \( k(x) \) is contained in \( A \) or \( B \).

Finally, by looking at the decomposition of \( \mathcal{O}_W \) instead of \( \mathcal{O}_X \), we see that all such closed points are simultaneously contained in \( A \) or \( B \). \( \square \)

The next statement provides us with an inductive way of proving the non-existence of SODs.

**Corollary 3.5.** Let \( X \) be a smooth proper variety and \( Z \) a connected closed subscheme. Define the closed subset \( B \subset Z \) by
\[
B = \bigcap_{s \in \lim_{\leftarrow} H^0(mZ, \omega_X|_{mZ})} V(s_1).
\]
(3.7)
Assume \( B \neq Z \), and take an irreducible component \( Z_1 \subset Z \) which is not contained in \( B \). Then \( \mathcal{C}_Z \) admits no SOD if the same holds for all \( \mathcal{C}_W \), where \( W \) runs through all the connected components of \( (Z \setminus Z_1) \cup (B \cap Z_1) \).

**Proof.** Note first that \( \mathcal{C}_Z \) admits no OD, since it is connected; proof is essentially the same as that of Lemma 2.11 once one replaces ‘locally free sheaves’ with ‘locally free sheaves on thickenings of \( Z \)’. Hence it is enough to show that any SOD of \( \mathcal{C}_Z \) is in fact an OD.

Take an SOD \( \mathcal{C}_Z = (\mathcal{A}, \mathcal{B}) \). By Proposition 3.4, one and only one of the followings holds:
(a) For any closed point \( x \in Z_1 \setminus B \), \( k(x) \in \mathcal{A} \).
(b) For any closed point \( x \in Z_1 \setminus B \), \( k(x) \in \mathcal{B} \).

Let us assume (a) holds. Then, as before, for any \( b \in \mathcal{B} \) we see \( \text{Supp} \ b \subset (Z \setminus (Z_1 \setminus B)) = (Z \setminus Z_1) \cup (B \cap Z_1) \). The rest of the proof is completely analogous to that of Lemma 3.3. \( \square \)

**Remark 3.6.** In fact, the arguments above work under weaker assumptions. It is enough to find infinitely many integers \( m > 0 \) such that for each \( m \) one can find \( s_m \in H^0(mZ, \omega_X|_{mZ}) \) which does not vanish at the generic point of an irreducible component of \( Z \).

### 3.3. Rigidity of Semiorthogonal decompositions

We show that SODs are rigid under the action of topologically trivial autoequivalences.

**Theorem 3.7.** Let \( X \) be a projective scheme over a field \( k \), and \( D(X) = D^b \text{coh} \ X = (\mathcal{A}, \mathcal{B}) \) an SOD. Take any line bundle \( L \) satisfying \( [L] \in \text{Pic}^0_{X/k} \). Then we have the equality of subcategories \( \mathcal{A} \otimes L = \mathcal{A} \subset D(X) \).

We immediately obtain
Corollary 3.8. Let \( X \) be a smooth projective variety whose canonical bundle is contained in \( \text{Pic}^0_{X/k} \). Then \( D(X) \) has no SOD.

Proof. By Corollary 2.13 it is enough to show that any SOD \( D(X) = \langle \mathcal{A}, \mathcal{B} \rangle \) satisfies \( \mathcal{A} \otimes \omega_X = \mathcal{A} \). By Theorem 3.7, this follows from the assumption \( [\omega_X] \in \text{Pic}^0_{X/k} \). \( \square \)

Before showing Theorem 3.7 we prepare some lemmas.

Lemma 3.9. Let \( X \) be a projective scheme and \( a, b \in D(X) \) bounded complexes of coherent sheaves. Let \( T \) be a scheme of finite type over \( k \) with a point \( 0 \in T(k) \), and \( M \) a line bundle on \( X \times_k T \) such that \( \mathbb{R}\text{Hom}(b, a \otimes M_0) = 0 \). Then there exists an open neighborhood \( 0 \in U \subset T \) such that for any \( t \in U(k) \) \( \mathbb{R}\text{Hom}(b, a \otimes M_t) = 0 \).

Proof. Consider the sequence of isomorphisms
\[
\mathcal{L}_t \mathcal{L} \mathcal{P}_T \mathcal{R} \text{Hom}(p_X^* b, p_X^* a \otimes M) \xrightarrow{\cong} \mathcal{L}_t \mathcal{L} \mathcal{P}_T (p_X^* \mathbb{R} \text{Hom}(b, a) \otimes M) \\
\xrightarrow{\cong} \mathbb{R} \Gamma (X, \mathbb{R} \text{Hom}(b, a) \otimes M_t) \xrightarrow{\cong} \mathbb{R} \text{Hom}(b, a \otimes M_t).
\]
(3.8)

Here \( \mathcal{L}_t : \{t\} \xrightarrow{} T \) is the natural inclusion. The second isomorphism follows from the base change theorem for flat morphisms ([Kuz06, Corollary 2.23]).

From this and by Lemma 2.3 we see that the closed subset
\[
S = \text{Supp} (\mathbb{R} \text{P}_T \mathcal{L} \mathbb{R} \text{Hom}(p_X^* b, p_X^* a \otimes M)) \subset T
\]
does not contain \( 0 \). Now we can define \( U \) as the complement of \( S \). \( \square \)

We keep the notations of Lemma 3.9. For any SOD \( D(X) = \langle \mathcal{A}', \mathcal{B}' \rangle \), consider the subset \( U(\mathcal{A}') = \{t \in T(k) | \mathcal{A} \otimes M_t = \mathcal{A}' \} \). Then

Lemma 3.10. \( U(\mathcal{A}') \subset T(k) \) is an open subset.

Proof. We check the following two claims separately for each point \( 0 \in U(\mathcal{A}') \).

1. There exists an open neighborhood \( 0 \in U \) such that \( \forall t \in U \mathcal{A} \otimes M_t \subset \mathcal{A}' \).
2. There exists an open neighborhood \( 0 \in U \) such that \( \forall t \in U \mathcal{A} \otimes M_t^{-1} \subset \mathcal{A}' \).

We give a proof only for the first one; the second follows from this by replacing \( M \) with \( M^{-1} \).

By [Rou08, Theorem 7.39], \( D(X) \) admits a classical generator. Then its images in \( \mathcal{A} \) and \( \mathcal{B}' \) under the projection functors, which will be denoted by \( a \) and \( b' \) respectively, are again classical generators. Then we have the useful criterion
\[
\mathcal{A} \otimes M_t \subset \mathcal{A}' \iff \mathbb{R} \text{Hom}(b', a \otimes M_t) = 0.
\]
(3.10)

Since the latter condition on \( t \) is known to be open by Lemma 3.9 we are done. \( \square \)

Proof of Theorem 3.7. By Theorem 2.15 and Definition 2.16, it is enough to show the following

Claim 3.11. Under the same assumptions as in Theorem 3.7, let \( T \) be a connected scheme of finite type over \( k \), and \( M \) a line bundle on \( X \times_k T \). If \( \mathcal{A} \otimes M_0 = \mathcal{A} \) holds for \( 0 \in T(k) \), then \( \mathcal{A} \otimes M_t = \mathcal{A} \) holds for all \( t \in T(k) \).

In order to show the claim, set \( S = \{ \mathcal{A} \otimes M_t | t \in T(k) \} \). By Lemma 3.10 we obtain a decomposition \( T(k) = \bigsqcup_{\mathcal{A}' \in S} U(\mathcal{A}') \) of \( T(k) \) into disjoint open subsets. Since \( T(k) \) is connected by the assumption \( k = \overline{k} \), this implies that \( U(\mathcal{A}) = T(k) \). \( \square \)
Let $X$ be a smooth projective variety over $k$. Using similar arguments, we can also show $g^*A = A \subset D(X)$ for any automorphism $g \in \text{Aut}^0_{X/k}$. Here $\text{Aut}^0_{X/k}$ is the identity component of the group scheme $\text{Aut}_{X/k}$ (see [FGi+05 p.133 Exercise] for the definition and the existence of $\text{Aut}_{X/k}$). Thus we obtain

**Corollary 3.12.** For any SOD $D(X) = \langle A, B \rangle$ and $g \in \text{Pic}^0_{X/k} \times \text{Aut}^0_{X/k}$, we have $gA = A \subset D(X)$.

As proven in [Ros09 Theorem 2.12], the group scheme $\text{Pic}^0_{X/k} \times \text{Aut}^0_{X/k}$ coincides with the identity component of the group scheme of autoequivalences of $D(X)$.

4. Results for Surfaces

We consider smooth projective minimal surfaces with non-negative Kodaira dimensions and establish as many non-existence results for SODs as possible. Readers can refer to [CD89] for notions of surfaces in positive characteristics such as quasi-bielliptic surfaces, non-classical Enriques surfaces, quasi-elliptic fibrations and wild fibers.

4.1. $\kappa = 0$. Since classical Enriques surface satisfies $p_g = q = 0$, any line bundle on it is exceptional. Hence the derived category always admits a non-trivial SOD (see [IK15] and [HT15] for further results on this topic). For non-classical Enriques, abelian, and K3 surfaces we have no SOD by Corollary 2.13 since their canonical bundles are trivial.

The most non-trivial is the following

**Proposition 4.1.** Let $X$ be an (quasi-)bielliptic surface. Then $D(X)$ admits no SOD.

**Proof.** Take an (quasi-)elliptic fibration $f: X \to C$ to a smooth elliptic curve $C$ without multiple fibers. By the Kodaira-Bombieri-Mumford canonical bundle formula (4.1), there exists a line bundle $L$ on $C$ such that $\omega_X \simeq f^*L$. Since $\omega_X$ is a torsion line bundle, so is $L$ by the projection formula. Thus $[L] \in \text{Pic}^0_{C/k}$. Finally, since $f^*: \text{Pic}_{C/k} \to \text{Pic}_{X/k}$ preserves the identity components, we obtain $[\omega_X] \in \text{Pic}^0_{X/k}$. Now we can use Corollary 3.8 to conclude the proof. $\square$

4.2. $\kappa = 1$. Let $f: X \to C$ be a relatively minimal (quasi-)elliptic surface with multiple fibers $X_{c_1} = m_1F_1, \ldots, X_{c_k} = m_kF_k$. Then we have the Kodaira-Bombieri-Mumford canonical bundle formula (see [BM77 Theorem 2])

$$\omega_X \simeq f^*(\omega_C \otimes \mathbb{R}^1 f_*\mathcal{O}_X/T) \otimes \mathcal{O}_X \left( \sum_i a_iF_i \right),$$

(4.1)

where $T \subset \mathbb{R}^1 f_*\mathcal{O}_X$ is the torsion part and $0 < a_i \leq m_i - 1$ are some integers. It is known that $\omega_C \otimes \mathbb{R}^1 f_*\mathcal{O}_X/T$ is a line bundle of degree $2g(C) - 2 + \chi(\mathcal{O}_X) + \text{length } T$.

**Theorem 4.2.** Let $f: X \to C$ be a relatively minimal elliptic or quasi-elliptic surface. If $p_g(X) > 0$, then $D(X)$ has no SOD. In particular, if $X$ is a minimal surface of Kodaira dimension 1 with $p_g(X) > 0$, $D(X)$ has no SOD.

**Proof.** Since the contribution from the multiple fibers in the RHS of (4.1) is fixed as a linear system and $f$ is an algebraic fiber space, if $p_g(X) > 0$, then $\text{Bs } [\omega_X]$ is a union of finitely many fibers of $f$. The finite set $f(\text{Bs } [\omega_X]) \subset C$ will be denoted by $S$. 

12
Take any SOD $D(X) = \langle A, B \rangle$. By Theorem 1.1, either $A$ or $B$ is supported in $f^{-1}(S)$. This implies that the SOD under consideration is $C$-linear in the sense of [Kuz11]; to see this, note that the pull-back of any locally free sheaf on $C$ is trivial on an open neighborhood of $f^{-1}(S)$. Also since $f$ is flat, we can apply [Kuz11] Theorem 5.6 to any base change of $f$. In particular, for any closed point $s \in S$ we obtain the SOD $D^b \text{coh } X_s = \langle A_{X_s}, B_{X_s} \rangle$ (following the notations used in [Kuz11]). On the other hand, since $\omega_{X_s}$ is torsion and any torsion line bundle on a complete curve over a field is contained in its $\text{Pic}^0$ (see [CD89, Chapter 0 Section 7]), we can use Theorem 3.7 to see $A_{X_s} \otimes \omega_{X_s} = A_{X_s} \subset D^b \text{coh } X_s$. Therefore either $A_{X_s} = 0$ or $B_{X_s} = 0$ should hold for any $s \in S$.

Without loss of generality, let us assume that $A$ is supported in $f^{-1}(S)$ for the rest of proof. Assume for a contradiction $A \neq 0$. By the construction of $A_{X_s}$ given in [Kuz11] Section 5.4, we see that for any object $a \in A$, its base change $a_s = \tilde{a} \otimes \mathcal{O}_s$ is contained in $A_{X_s}$ (note $D^b \text{coh } X = \mathcal{D}^\text{perf } X$). Therefore, by Lemma 2.8 there should be a point $s \in S$ for which $A_{X_s} \neq 0$. Then we obtain $B_{X_s} = 0$, so that any object $b \in B$ satisfies $\text{Supp } b \cap f^{-1}(s) = \emptyset$ by Lemma 2.8 again. Since $f$ is proper, this implies that any object $b \in B$ is supported in a union of finitely many fibers. Thus we conclude that any object in either $A$ or $B$ should be supported in a union of finitely many fibers, and it clearly contradicts the assumption that $D(X)$ is generated by $A$ and $B$.

\begin{remark}
(1) Our method is also applicable to other situations in which we have a sufficiently nice canonical bundle formula (see Section 5).

(2) (Assume $k = \mathbb{C}$ for simplicity.) Let $X$ be a minimal projective surface with $\kappa(X) = 1$. If $p_g(X) = 0$, $h^1(\mathcal{O}_X)$ should be either 0 or 1 since $\chi(\mathcal{O}_X) \geq 1$ holds (see [BHPVdV04, Chapter V, §12]). If $h^1(\mathcal{O}_X) = 0$, as we saw before, any line bundle is exceptional. If $h^1(\mathcal{O}_X) = 1$ (and hence $\chi(\mathcal{O}_X) = 0$), although we can restrict the nature of the fibration as follows, we do not know if $D(X)$ can admit an SOD or not.

- $g(C) = 0$ or 1 by the canonical bundle formula and the assumption $p_g(X) = 0$.
- Smooth fibers are all isomorphic to one another and the multiple fibers are of type $m I_0$ for some $m > 0$ (see [BHPVdV04, Chapter III, §18]).
\end{remark}

4.3. $\kappa = 2$. We apply the results of Section 3.2 to smooth projective minimal surfaces with $\kappa = 2$ (i.e., of general type). There are some examples of minimal surfaces of general type on which the connected components of fixed part of the canonical linear system can be birationally contracted to points (in the category of algebraic spaces). This property turns out to ensure the non-existence of SOD.

**Theorem 4.4.** Let $X$ be a smooth projective minimal surface of general type with $p_g \geq 2$. Assume the following condition (*).

\begin{itemize}
\item[(*)] For any one-dimensional connected component $Z$ of $\text{Bs } |\omega_X|$, its intersection matrix is negative definite.
\end{itemize}

Then $D(X)$ has no SOD.

**Proof.** By Corollary 3.3 it is enough to show that for any one-dimensional connected component $Z$ of $\text{Bs } |\omega_X|$, the category $\mathcal{C}_Z$ has no SOD. We prove this for more general $\mathcal{C}_W$, where $W$ is any reduced connected one cycle contained in $\text{Bs } |\omega_X|$, by an induction on the number of irreducible components of $W$. If $W$ is empty, there is nothing to show. In general we can use the following
Lemma 4.5. Under the assumptions of Theorem 4.4, let $W$ be a reduced connected one-cycle which is contained in $B \cdot \omega_X |_W$. Then there exists $s = (s_m)_{m \geq 1} \in \lim H^0(mW, \omega_X |_{mW})$ which does not vanish at the generic point of an irreducible component of $W$.

This implies the strict inequality $B \subset W$ (see Corollary 3.5 for notations). Since $W$ has pure dimension 1, we can pick an irreducible curve $Z_1 \subset W$ which is not contained in $B$. By Corollary 3.5 it is then enough to show the non-existence of SOD for

$W$, where $W'$ are connected components of $W \setminus Z_1 \cup (B \cap Z_1)$. Since $W'$ is either a point or a connected one cycle whose number of irreducible components is strictly less than that of $W$, we can apply the induction hypothesis.

Proof of Lemma 4.5. By the Riemann-Roch ([BHPvdV04, Chapter II, Theorem 3.1.]) and the adjunction formula, we see $h^0(\omega_X |_W) = h^1(\omega_X |_W) + \frac{1}{2} \cdot (K_X - W)$. Since we assumed $p_g(X) > 1$, $K_X - W$ is linearly equivalent to a non-zero effective divisor. Hence by the 2-connectedness of the canonical divisor of minimal surfaces of general type ([BHPvdV04, Chapter VII, Proposition 6.2. (ii)]), we see $W \cdot (K_X - W) \geq 2$. Thus we obtain $h^0(\omega_X |_W) > 0$.

Take any non-zero global section $s_1$ of $\omega_X |_W$. Since $W$ is reduced, $s_1$ is generically non-vanishing on at least one irreducible component of $W$. For each $m > 1$ we show that the global section $s_{m-1}$ of $\omega_X |_{m-1}Z$ lifts to a global section $s_m$ of $\omega_X |_{mW}$, so as to obtain the desired $s = (s_m)_{m \geq 1} \in \lim H^0(mW, \omega_X |_{mW})$. Consider the exact sequence

$$0 \to O_W(K_X - (m - 1)W) \to O_{mW}(K_X) \to O_{(m-1)W}(K_X) \to 0 \quad (4.2)$$

and the associated cohomology long exact sequence. This yields an exact sequence

$$H^0(mW, O_{mW}(K_X)) \to H^0((m-1)W, O_{(m-1)W}(K_X)) \to H^1(W, O_W(K_X - (m - 1)W)), \quad (4.3)$$

and hence it is enough to show the vanishing of the third term. By the adjunction formula and the Serre duality for embedded curves (see [BHPvdV04, Chapter II, Section 1]), its dimension can be rewritten as $h^1(W, O_W(K_X - (m - 1)W)) = h^0(W, O_W(mW))$. Finally, the vanishing of the RHS follows from the assumption $W^2 < 0$. □

Example 4.6. Minimal surfaces $X$ of general type with $p_g = K^2 = 2$ and $q = 0$ were investigated in [Hor79]. Among them, those of type III (see [Hor79, page 104]) satisfy the assumption of Theorem 4.4. In fact the fixed part consists of a $(-2)$-curve. In this example the moving part of the canonical linear system is base point free and defines a genus two pencil over the projective line ([Hor79, Theorem 1.3]).

5. Toward higher dimensions

Most of the arguments of Section 4.2 can be generalized to higher dimensions, except one point.

Theorem 5.1. Let $X$ be a non-singular projective $n$-fold defined over $k$ such that $X$ is a minimal model with $\kappa(X) = 1$ and $p_g(X) > 0$. Suppose $\omega_X$ is semi-ample so that the canonical morphism $f : X \to C$ exists. Suppose that for any scheme-theoretic fiber $X_c$ of $f$ we have $\omega_X |_{X_c} \in \text{Pic}^0_{X_c}$. Then $D(X)$ admits no SOD.

Proof. The assumption $p_g(X) > 0$ implies that $p_g(\omega_{X_c}) > 0$ holds for a general fiber $X_c$. Since $X$ is irreducible and $C$ is a non-singular curve, the morphism $f$ is flat ([Har77, Chapter III, Proposition 9.7]). Combined with the torsion-freeness [Kol86, Theorem 2.1] and the theory
of cohomology and base change [Har77, Chapter III, Theorem 12.11], we see that the direct image \( f_*\omega_X \) is an invertible sheaf. The natural injective morphism \( f^*f_*\omega_X \to \omega_X \) provides us with an effective divisor \( E \) on \( X \) which fits in the canonical bundle formula

\[
\omega_X \simeq f^*f_*\omega_X \otimes \mathcal{O}_X(E).
\] (5.1)

Arguing as in [BHPVdV04, Proof of Theorem 12.1, Chapter V], we see that the morphism \( f^*f_*\omega_X \to \omega_X \) is an isomorphism on smooth fibers: in fact, for a smooth fiber \( X_c \) we can find a section \( \tilde{s} \in (f_*\omega_X)_c \) which, under the isomorphism \( f_*\omega_X \otimes \mathcal{O}_c \to H^0(X_c, \omega_X|_{X_c}) \), corresponds to the global trivialization of \( \omega_X|_{X_c} \simeq \omega_{X_c} \). Since \( f \) is projective, there exists an open neighborhood \( U \ni c \) such that \( \tilde{s} \) is well-defined and vanishes nowhere on \( f^{-1}(U) \). This shows that the morphism \( f^*f_*\omega_X \to \omega_X \) is surjective (and hence is an isomorphism) on \( f^{-1}(U) \).

Thus we conclude that \( E \) is contained in the union of the singular fibers of \( f \) and hence \( \text{Bs} |\omega_X| \) is contained in the union of finitely many fibers of \( f \). Since we assumed \( \omega_X|_{X_c} \in \text{Pic}_{X_c/k}^0 \), the rest of the arguments are the same as those in Section 4.2. \( \square \)

Let \( \text{Pic}^r \subset \text{Pic} \) be the subscheme of numerically trivial line bundles (see [FGI+05, Section 9.6]). If \( \text{Pic}^0_{X_c} = \text{Pic}_{X_c}^0 \) holds for any singular fiber \( X_c \), since the morphism \( f \) is defined by some multiple of \( \omega_X \), the last assumption of Theorem 5.1 will be automatically satisfied. This is always the case if \( \dim X_c = 1 \), but not in general. Actually, even worse, the following example due to Keiji Oguiso satisfies all the assumptions of Theorem 5.1 but the last one. The authors are not sure if its derived category admits a non-trivial SOD or not.

**Example 5.2.** We assume \( k = \mathbb{C} \) for simplicity. Fix an integer \( n \geq 4 \) such that \( n+1 \) is a prime number. We construct a minimal \( n \)-fold \( X \) with \( \kappa(X) = 1 \) and \( p_g(X) > 0 \), together with the canonical morphism \( f: X \to C \) for which \( \omega_X|_{X_c} \in \text{Pic}^0_{X_c} \) does not hold for some singular fiber \( X_c \).

Consider the Fermat hypersurface \( Y = (\sum_{i=0}^n X_i^{n+1} = 0) \subset \mathbb{P}^n \). It is smooth projective and a Calabi-Yau \((n-1)\)-fold, in the sense \( h^i(O_Y) = 0 \) for \( i = 1, 2, \ldots, n-2 \) and \( \omega_Y \simeq O_Y \).

Note that it admits the following nowhere vanishing residue top form

\[
\psi = \text{Res}_{Y} \left( \sum_{i=0}^{n} (-1)^{i} X_i dX_0 \wedge \cdots \wedge d\widehat{X}_i \wedge \cdots \wedge dX_n \right).
\] (5.2)

Consider the cyclic group \( G = \mathbb{Z}/(n+1)\mathbb{Z} \) and its action on \( Y \) defined by \( X_i \mapsto \zeta^i X_i \), where \( \zeta \) is a primitive \((n+1)\)th root of unity. Since \( n+1 \) is a prime number, this action is free and hence we obtain the non-singular quotient \( Z = Y/G \). Since the top form \( \psi \) is easily seen to be \( G \)-invariant, it follows that \( Z \) is also a Calabi-Yau \((n-1)\)-fold.

Let \( C' \) be the smooth projective model of the affine curve \((y^2 - x^{4(n+1)} + 1 = 0) \subset \mathbb{A}^2 \), and let \( G \) act on \( C' \) by \((x, y) \mapsto (\zeta x, y) \). This action is effective but not free. As can easily be seen, \( g(C') = 2(n+1) \) and \( \gamma = (y^{-1} x^n) dx \) defines an \( G \)-invariant regular 1-form on \( C' \).

Now consider the étale quotient \( \pi: Y \times C' \to (Y \times C')/G =: C \). Since \( \psi \otimes \gamma \in H^0(Y \times C', \omega_{Y \times C'}) \simeq H^0(X, \omega_X) \), we see \( p_g(X) > 0 \). Also since \( \omega_{Y \times C'} \simeq \pi^*\omega_X \), \( X \) is minimal and \( \omega_X \) is not trivial. Combined with the inequalities \( 0 \leq \kappa(X) \leq \kappa(Y \times C') = 1 \), we see \( \kappa(X) = 1 \). Also it is easily seen that the algebraic fiber space \( f: X \to C'/G =: C \) is induced by the pluri-canonical linear system of \( X \).
Let \( c \in C \) be a branch point of \( C' \to C \). By an abuse of notation we write \( (X_c)_{\text{red}} = Z \), so that \( X_c = (n+1)Z \). Here we claim that \( \mathcal{O}_{X_c}(K_X) \not\in \text{Pic}^0_{X_c/k} \). To see this we prove \( \mathcal{O}_Z(K_X) \not\in \text{Pic}^0_{Z/k} \), which in turn is equivalent to \( \mathcal{O}_Z(Z) \not\neq \mathcal{O}_Z \).

Take a Stein open neighborhood \( c \in V \subset C \) such that \( X_c = (n+1)Z \) is a deformation retract of \( U = f^{-1}(V) \subset X \) (see [BHPVdV04, Chapter I, Theorem 8.8]). Consider the following commutative diagram with exact rows.

\[
\begin{array}{cccc}
H^1(U, \mathcal{O}_U) & \longrightarrow & H^1(U, \mathcal{O}_U^c) & \longrightarrow & H^2(U, \mathcal{O}_U) \\
\downarrow & & \downarrow & & \downarrow \\
H^1(X_c, \mathcal{O}_{X_c}) & \longrightarrow & H^1(X_c, \mathcal{O}_{X_c}^c) & \longrightarrow & H^2(X_c, \mathcal{O}_{X_c}) \\
\downarrow & & \downarrow & & \downarrow \\
H^1(Z, \mathcal{O}_Z) & \longrightarrow & H^1(Z, \mathcal{O}_Z^c) & \longrightarrow & H^2(Z, \mathcal{O}_Z)
\end{array}
\] (5.3)

We check that the terms in the 1st and the 4th columns in the diagram (5.3) all vanish. First, since \( Z \) is a Calabi-Yau \((n-1)\)-fold with \( n \geq 4 \), \( H^i(Z, \mathcal{O}_Z) = 0 \) for \( i = 1, 2 \). For the remaining four terms, note first that the higher direct image sheaves \( \mathbb{R}^i f_* \mathcal{O}_X \) are locally free for all \( i \geq 0 \) due to the torsion-freeness theorem [Kol86, Theorem 2.1], relative duality, and the fact that the base space \( C \) is a non-singular curve. From this we obtain the isomorphisms \( \mathbb{R}^i f_* \mathcal{O}_X \otimes \mathbb{C}(t) \cong H^i(X_t, \mathcal{O}_{X_t}) \) ([Har77, Chapter III, Theorem 12.11]). Since \( H^i(X_t, \mathcal{O}_{X_t}) = 0 \) for \( i = 1, 2 \) and general \( t \), we obtain \( \mathbb{R}^i f_* \mathcal{O}_X = 0 \) for \( i = 1, 2 \) and hence the desired vanishings.

As a result, it turns out that the six terms in the 2nd and the 3rd columns in (5.3) are isomorphic to one another. Since we can easily check that \( \mathcal{O}_U(Z) \in H^1(U, \mathcal{O}_U^c) \) is a \((n+1)\)-torsion non-trivial line bundle, so is \( \mathcal{O}_Z(Z) \).

**Remark 5.3.** The structure sheaf \( \mathcal{O}_Z \) is *not* an exceptional object. Actually one can easily derive from the calculations above that \( \text{Ext}_{X}^{n-1}(\mathcal{O}_Z, \mathcal{O}_Z) \neq 0 \).

### 6. Generalization to twisted sheaves

Most of the results we have established so far can be generalized to derived categories of *twisted* coherent sheaves without essential change.

**Definition 6.1.** A *cohomological Brauer class* of a scheme \( X \) is an element \( \alpha \in \text{Br}^t(X) := H^2_{\text{ét}}(X, \mathcal{O}_X^*) \). A pair \((X, \alpha)\) will be called a *cohomological Brauer pair*.

Given such a pair \((X, \alpha)\), we can define the abelian category \( \text{coh}(X, \alpha) \) of \( \alpha \)-twisted coherent sheaves. When \( X \) is defined over a field \( k \), it comes with the structure of a \( k \)-linear category.

Fix an étale cover \( U = (U_i)_{i \in I} \) of \( X \) on which the cohomology class \( \alpha \) is represented by a Čech cocycle

\[
\alpha = (\alpha_{ijk})_{i,j,k \in I} \in \check{Z}^2(U, \mathcal{O}^*) = \prod_{i,j,k \in I} H^0(U_{ijk}, \mathcal{O}^*),
\]

where \( U_{ijk} = U_i \times X U_j \times X U_k \) (by an abuse of notation, we used the same symbol \( \alpha \) to describe its representative). Then an \( \alpha \)-twisted coherent sheaf \( F \) is a collection of coherent
sheaves $F_i \in \text{coh } U_i$ and isomorphisms $\varphi_{ij} : F_j|_{U_{ij}} \sim \rightarrow F_i|_{U_{ij}}$ which satisfy the $\alpha$-twisted cocycle conditions
\[ \varphi_{ij} \varphi_{jk} \varphi_{ki} = \alpha_{ijk} \cdot \text{id}_{U_{ijk}} : F_i|_{U_{ijk}} \sim \rightarrow F_i|_{U_{ijk}}. \] (6.2)

A morphism between such data is a collection of $O_{U_i}$-homomorphisms which satisfy the obvious consistency. Then we can check that thus obtained category is abelian and is independent of the choice of a representative of $\alpha$ (see [Cal00, Lemma 1.2.3]). We write $D(X, \alpha) = \text{Db coh}(X, \alpha)$, so that $D(X, 0) = D(X)$.

**Definition 6.2.** For an $\alpha$-twisted coherent sheaf $F \in \text{coh } (X, \alpha)$, its support $\text{Supp } F$ is defined as the closed subscheme $\text{Spec}_X \left( \text{Im}(O_X \rightarrow \text{End}(F)) \right) \subset X$. For an object $F \in D(X, \alpha)$, its support is defined as $\text{Supp } F = \bigcup_i \text{Supp } H^i(F)$.

For $\alpha$-twisted sheaves $F$ and $G$, we can define the untwisted coherent sheaf of homomorphisms $\text{Hom}(F, G) \in \text{coh } (X)$. The following fact is an easy consequence of this observation.

**Lemma 6.3.** Let $(X, \alpha)$ be a smooth proper cohomological Brauer pair. Then $\otimes \omega_X \left[ \dim X \right]$ is the Serre functor of $D(X, \alpha)$.

**Proof.** See [NJ10, Example 1.4.3]

The next lemma is a direct consequence of the definition of twisted coherent sheaves.

**Lemma 6.4.** For any closed point $x \in X$, its structure sheaf $k(x)$ is an $\alpha$-twisted sheaf for any cohomological Brauer class $\alpha$.

Although some general arguments work for cohomological Brauer classes, it is sometimes convenient to restrict ourselves to Brauer classes. Brauer classes $\alpha$ forms a subgroup $\text{Br}(X) \subset \text{Br}^r(X)$, and they are characterized by either of the following properties (see [Cal00, Theorem 1.3.5]).

- There exists a sheaf of Azumaya algebras which represents the class $\alpha$.
- There exists a non-zero locally free $\alpha$-twisted sheaf of finite rank.

The difference of these two notions are very subtle. In fact, it is shown in [dJ03, Theorem 1.1] that $\text{Br} = \text{Br}^r$ holds on any projective scheme.

Now that we have prepared basic lemmas, we can show

**Theorem 6.5.** Let $(X, \alpha)$ be a smooth proper Brauer pair. Assume that $\omega_X$ is trivial on an open neighborhood of the canonical base locus $\text{Bs } |K_X|$. Then $D(X, \alpha)$ admits no SOD.

**Proof.** Since we assumed $X$ is proper, it satisfies the resolution property by [Tot04, Theorem 1.2]. Combined with the assumption $\alpha \in \text{Br}(X)$, [Cal00, Lemma 2.1.4] implies that any coherent $\alpha$-twisted sheaf on $X$ is a quotient of a locally free $\alpha$-twisted sheaf of finite rank. Hence those sheaves form a spanning class of $D(X, \alpha)$, and the original proof of Lemma 2.11 can be used without change to show the non-existence of OD. Then the original proof of Theorem 1.2 works with a minor modification, by replacing the corresponding lemmas with Lemma 6.3 and Lemma 6.4. In order to show that both of the SOD summands can not contain closed points at the same time, one can use a locally free twisted sheaf instead of $O_X$. □

Similarly, by using [Cal00, Lemma 2.1.4], the original proof of Theorem 4.4 works without change and we obtain
Theorem 6.6. Let \((X, \alpha)\) be a smooth projective Brauer pair such that \(X\) is a minimal surface of general type satisfying \(\dim_k H^0(X, \omega_X) > 1\) and the condition (*). Then \(D(X, \alpha)\) admits no SOD.

Remark 6.7. We expect that the other results can be generalized as well with a bit more effort. For Theorem 3.7, all we need is the existence of classical generators in the derived category of twisted coherent sheaves. Similarly, for Theorem 5.1, we have to check that the base change theorem \([Kuz11, \text{Theorem 5.6}]\) works for Brauer pairs as well.

7. Concluding comments

(1) Our results for minimal surfaces apparently indicate that the existence/non-existence of SOD corresponds to \(p_g > 0/\le 0\), respectively. This is quite similar to the Bloch conjecture \([\text{Blo75}]\) and a theorem of Mumford \([\text{Mum68}]\) on the finite generation of the Chow group of zero-cycles.

(2) Let \(X\) be a smooth projective surface whose minimal model does not admit SODs. It is natural to ask if all the SODs of \(D(X)\), up to autoequivalences, are of Orlov type (associated to contractions of \((-1)\)-curves). Also it would be interesting to ask if the Jordan-Hölder property holds or not for SODs of such \(X\).

(3) Having worked on this subject, the authors now have an impression that minimal models which admit non-trivial SODs are rather rare. A major obstacle to obtaining further results in this direction, as can be observed in the proof of Theorem 4.4, is the lack of keen understanding of the support of objects in the derived category \(D(X)\) (see also Remark 2.5).

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