WELLPOSEDNESS OF BOUNDED SOLUTIONS
OF THE NON-HOMOGENEOUS INITIAL BOUNDARY
VALUE PROBLEM FOR THE OSTROVSKY-HUNTER EQUATION

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Abstract. The Ostrovy-Hunter equation provides a model for small-amplitude long waves in
a rotating fluid of finite depth. It is a nonlinear evolution equation. In this paper the wellposed-
ness of bounded solutions for a non-homogeneous initial boundary value problem associated to
this equation is studied.

1. Introduction

The non-linear evolution equation
\begin{equation}
\partial_x(\partial_t u + u\partial_x u - \beta \partial_x^3 u) = \gamma u,
\end{equation}
with $\beta, \gamma \in \mathbb{R}$, was derived by Ostrovsky [21] to model small-amplitude long waves in
a rotating fluid of finite depth. This equation generalizes the Korteweg-deVries equation (that corresponds to $\gamma = 0$) by the additional term induced by the Coriolis force. Mathematical properties of the Ostrovsky equation (1.1) were studied recently in many
details, including the local and global well-posedness in energy space [8, 14, 17, 26], sta-
bility of solitary waves [12, 15, 18], convergence of solutions in the limit, $\gamma \to 0$, of the
Korteweg-deVries equation [13, 18], and convergence of solutions in the limit, $\beta \to 0$, of
no high-frequency dispersion [4].

We shall consider the limit of no high-frequency dispersion $\beta = 0$, therefore (1.1) reads
\begin{equation}
\partial_x(\partial_t u + u\partial_x u) = \gamma u, \quad t > 0, \quad x > 0.
\end{equation}
It is deduced considering two asymptotic expansions of the shallow water equations, first
with respect to the rotation frequency and then with respect to the amplitude of the
waves (see [7, 10]). It is known under different names such as the reduced Ostrovsky
equation [22, 24], the Ostrovsky-Hunter equation [2], the short-wave equation [9], and the
Vakhnenko equation [19, 23].

We augment (1.2) with the boundary condition
\begin{equation}
u(t, 0) = g(t), \quad t > 0,
\end{equation}
and the initial datum
\begin{equation}
u(0, x) = u_0(x), \quad x > 0,
\end{equation}
on which we assume that
\begin{equation}
u_0 \in L^\infty(0, \infty) \cap L^1(0, \infty).
\end{equation}
On the function
\[ P_0(x) = \int_0^x u_0(y)dy, \]
we assume that
\[ \|P_0\|_{L^2(0,\infty)} = \int_0^{\infty} \left( \int_0^x u_0(y)dy \right)^2 dx < \infty. \]

On the boundary datum \( g(t) \), we assume that
\[ g(t) \in W^{1,\infty}(0,\infty), \quad g(0) = 0. \]
Moreover, we assume that
\[ \gamma > 0. \]

Integrating (1.2) on \((0, x)\) we gain the integro-differential formulation of the initial-boundary value problem (1.2), (1.3), (1.4) (see [16])
\[ \begin{cases} 
\partial_t u + u\partial_x u = \gamma \int_0^x u(t, y)dy, & t > 0, \quad x > 0, \\
u(t, 0) = g(t), & t > 0, \\
u(0, x) = u_0(x), & x > 0, 
\end{cases} \]
that is equivalent to
\[ \begin{cases} 
\partial_t u + u\partial_x u = \gamma P, & t > 0, \quad x > 0, \\
\partial_x P = u, & t > 0, \quad x > 0, \\
u(t, 0) = g(t), & t > 0, \\
P(t, 0) = 0, & t > 0, \\
u(0, x) = u_0(x), & x > 0. 
\end{cases} \]

Due to the regularizing effect of the \( P \) equation in (1.11) we have that
\[ u \in L^\infty((0, T) \times (0, \infty)) \implies P \in L^\infty((0, T); W^{1,\infty}(0, \infty)), \quad T > 0. \]
Therefore, if a map \( u \in L^\infty((0, T) \times (0, \infty)), \, T > 0, \) satisfies, for every convex map \( \eta \in C^2(\mathbb{R}), \)
\[ \partial_t \eta(u) + \partial_x q(u) - \gamma \eta'(u)P \leq 0, \quad q(u) = \int^u f'(\xi)\eta'(\xi) \, d\xi, \]
in the sense of distributions, then [6, Theorem 1.1] provides the existence of strong trace \( u_0^\tau \) on the boundary \( x = 0 \).

We give the following definition of solution (see [1]):

**Definition 1.1.** We say that \( u \in L^\infty((0, T) \times (0, \infty)), \, T > 0, \) is an entropy solution of the initial-boundary value problem (1.2), (1.3), and (1.4) if for every nonnegative test function \( \phi \in C^2(\mathbb{R})^2 \) with compact support, and \( c \in \mathbb{R} \)
\[ \begin{align*}
\int_0^\infty \int_0^\infty \left( |u - c|\partial_t \phi + \text{sign} \, (u - c) \left( \frac{u^2}{2} - \frac{c^2}{2} \right) \partial_x \phi \right) dt \, dx \\
+ \gamma \int_0^\infty \int_0^\infty \text{sign} \, (u - c) P\phi dt \, dx \\
+ \int_0^\infty \text{sign} \, (g(t) - c) \left( \frac{(u_0^\tau(t))^2}{2} - \frac{c^2}{2} \right) \phi(t, 0) dt \\
+ \int_0^\infty |u_0(x) - c|\phi(0, x) \, dx \geq 0,
\end{align*} \]
where \( u^0(t) \) is the trace of \( u \) on the boundary \( x = 0 \).

The main result of this paper is the following theorem.

**Theorem 1.1.** Assume (1.3), (1.4), (1.5), (1.6), (1.7), (1.8) and (1.9). The initial-boundary value problem (1.2), (1.3), and (1.4) possesses an unique entropy solution \( u \) in the sense of Definition 1.1. Moreover, if \( u \) and \( v \) are two entropy solutions (1.2), (1.3), (1.4) in the sense of Definition 1.4 the following inequality holds

\[
\| u(t, \cdot) - v(t, \cdot) \|_{L^1(0, R)} \leq e^{C(T)t} \| u(0, \cdot) - v(0, \cdot) \|_{L^1(0, R + C(T)t)},
\]

for almost every \( 0 < t < T, R > 0 \), and some suitable constant \( C(T) > 0 \).

A similar result has been proved in [3, 7] in the context of locally bounded solutions under the assumption \( g \equiv 0 \).

The paper is organized as follows. In Section 2 we prove several a priori estimates on a vanishing viscosity approximation of (1.11). Those play a key role in the proof of our main result, that is given in Section 3.

2. Vanishing viscosity approximation

Our existence argument is based on passing to the limit in a vanishing viscosity approximation of (1.11).

Fix a small number \( 0 < \varepsilon < 1 \), and let \( u_\varepsilon = u_\varepsilon(t, x) \) be the unique classical solution of the following mixed problem [5]

\[
\begin{align*}
\partial_t u_\varepsilon + u_\varepsilon \partial_x u_\varepsilon &= \gamma P_\varepsilon + \varepsilon \partial^2_{xx} u_\varepsilon, & t > 0, \quad x > 0, \\
-\varepsilon \partial_{xx}^2 P_\varepsilon + \partial_x P_\varepsilon &= u_\varepsilon, & t > 0, \quad x > 0, \\
u_\varepsilon(t, 0) &= g_\varepsilon(t), & t > 0, \\
P_\varepsilon(t, 0) &= 0, & t > 0, \\
u_\varepsilon(0, x) &= u_{\varepsilon, 0}(x), & x > 0,
\end{align*}
\]

where \( u_{\varepsilon, 0} \) is a \( C^\infty \) approximation of \( u_0 \) such that

\[
\begin{align*}
\| u_{\varepsilon, 0} \|_{L^2(0, \infty)} &\leq \| u_0 \|_{L^2(0, \infty)}, & \| u_{\varepsilon, 0} \|_{L^\infty(0, \infty)} &\leq \| u_0 \|_{L^\infty(0, \infty)}, \\
\| P_{\varepsilon, 0} \|_{L^2(0, \infty)} ^2 &\leq \| P_0 \|_{L^2(0, \infty)} ^2, & \varepsilon^2 \| \partial_x P_{\varepsilon, 0} \|_{L^2(0, \infty)} &\leq C_0, \\
\| g_\varepsilon \|_{L^\infty(0, \infty)} &+ \| g_\varepsilon' \|_{L^\infty(0, \infty)} &\leq C_0, & g_\varepsilon(0) &= 0,
\end{align*}
\]

and \( C_0 \) is a constant independent on \( \varepsilon \).

Let us prove some a priori estimates on \( u_\varepsilon \) and \( P_\varepsilon \), denoting with \( C_0 \) the constants which depend on the initial data, and \( C(T) \) the constants which depend also on \( T \).

**Lemma 2.1.** For each \( t \in (0, \infty) \),

\[
P_\varepsilon(t, \infty) = \partial_x P_\varepsilon(t, \infty) = 0.
\]

Moreover,

\[
\varepsilon^2 \| \partial_{xx} P_\varepsilon(t, \cdot) \|_{L^2(0, \infty)} ^2 + \varepsilon (\partial_x P_\varepsilon(t, 0))^2 \\
+ \| \partial_x P_\varepsilon(t, \cdot) \|_{L^2(0, \infty)} ^2 = \| u_\varepsilon(t, \cdot) \|_{L^2(0, \infty)} ^2.
\]

**Proof.** We begin by proving that (2.3) holds true.

Differentiating the first equation of (2.1) with respect to \( t \), we have

\[
\partial_x (\partial_t u_\varepsilon + u_\varepsilon \partial_x u_\varepsilon - \varepsilon \partial_{xx}^2 u_\varepsilon) = \gamma \partial_x P_\varepsilon.
\]
For the the smoothness of $u_\varepsilon$, it follows from (2.1) and (2.5) that
\[
\lim_{x \to \infty} \partial_t u_\varepsilon + u_\varepsilon \partial_x u_\varepsilon - \varepsilon \partial_{xx}^2 u_\varepsilon = \gamma P_\varepsilon(t, \infty) = 0,
\]
\[
\lim_{x \to \infty} \partial_x (\partial_t u_\varepsilon + u_\varepsilon \partial_x u_\varepsilon - \varepsilon \partial_{xx}^2 u_\varepsilon) = \gamma \partial_x P_\varepsilon(t, \infty) = 0,
\]
which gives (2.3).

Let us show that (2.4) holds true. Squaring the equation for $P_\varepsilon$ in (2.1), we get
\[
\varepsilon^2 (\partial_{xx}^2 P_\varepsilon)^2 + (\partial_x P_\varepsilon)^2 - \varepsilon \partial_x ((\partial_x P_\varepsilon)^2) = u_\varepsilon^2.
\]
Therefore, (2.4) follows from (2.1), (2.3) and an integration on $(0, \infty)$.

**Lemma 2.2.** For each $t \in (0, \infty)$,
\[
(2.6) \quad \int_{0}^{\infty} u_\varepsilon(t,x)dx = \varepsilon \partial_x P_\varepsilon(t,0),
\]
\[
(2.7) \quad \sqrt{\varepsilon} \| \partial_x P_\varepsilon(t, \cdot) \|_{L^\infty(0,\infty)} \leq \| u(t, \cdot) \|_{L^2(0,\infty)},
\]
\[
(2.8) \quad \int_{0}^{\infty} u_\varepsilon(t,x)P_\varepsilon(t,x)dx \leq \| u(t, \cdot) \|_{L^2(0,\infty)}^2.
\]

**Proof.** Integrating on $(0, \infty)$ the equation for $P_\varepsilon$ in (2.1), for (2.3), we have
\[
\int_{0}^{\infty} u_\varepsilon(t,x)dx = \varepsilon \partial_x P_\varepsilon(t,0),
\]
that is (2.6).

Let us show that (2.7) holds true. Observe that
\[
0 \leq -\varepsilon \partial_{xx}^2 P_\varepsilon + \partial_x P_\varepsilon = \varepsilon^2 (\partial_{xx}^2 P_\varepsilon)^2 + (\partial_x P_\varepsilon)^2 - \varepsilon \partial_x ((\partial_x P_\varepsilon)^2),
\]
that is,
\[
(2.9) \quad \varepsilon \partial_x ((\partial_x P_\varepsilon)^2) \leq \varepsilon^2 (\partial_{xx}^2 P_\varepsilon)^2 + (\partial_x P_\varepsilon)^2.
\]

Integrating (2.9) in $(0, x)$, we have
\[
\varepsilon (\partial_x P_\varepsilon(t,0))^2 - \varepsilon (\partial_x P_\varepsilon(t,0))^2 \leq \varepsilon^2 \int_{0}^{x} (\partial_{xx}^2 P_\varepsilon)^2 dx \quad \text{(2.10)}
\]
\[
\leq \varepsilon^2 \int_{0}^{\infty} (\partial_{xx}^2 P_\varepsilon)^2 dx + \int_{0}^{\infty} (\partial_x P_\varepsilon)^2 dx.
\]

It follows from (2.4) and (2.10) that
\[
\varepsilon (\partial_x P_\varepsilon)^2 \leq \varepsilon^2 \int_{0}^{\infty} (\partial_{xx}^2 P_\varepsilon)^2 dx + \int_{0}^{\infty} (\partial_x P_\varepsilon)^2 dx + \varepsilon (\partial_x P_\varepsilon(t,0))^2 \leq \| u_\varepsilon(t, \cdot) \|_{L^2(0,\infty)}^2.
\]

Therefore,
\[
\sqrt{\varepsilon} |\partial_x P_\varepsilon(t,x)| \leq \| u_\varepsilon(t, \cdot) \|_{L^2(0,\infty)},
\]
which gives (2.7).

Finally, we prove (2.8). Multiplying by $P_\varepsilon$ the equation for $P_\varepsilon$ of (2.1), we get
\[
-\varepsilon P_\varepsilon \partial_{xx}^2 P_\varepsilon + P_\varepsilon \partial_x P_\varepsilon = u_\varepsilon P_\varepsilon.
\]

An integration on $(0, \infty)$ and (2.3) give
\[
\int_{0}^{\infty} u_\varepsilon P_\varepsilon dx = \frac{1}{2} \int_{0}^{\infty} \partial_x (P_\varepsilon)^2 dx - \varepsilon \int_{0}^{\infty} P_\varepsilon \partial_{xx}^2 P_\varepsilon dx
\]
\[
= -\varepsilon \int_{0}^{\infty} P_\varepsilon \partial_{xx}^2 P_\varepsilon dx = \varepsilon \int_{0}^{\infty} (\partial_x P_\varepsilon)^2 dx,
\]
Therefore, due to the Young’s inequality,
\[ \int_0^\infty u_\varepsilon P_\varepsilon dx = \varepsilon \int_0^\infty (\partial_x P_\varepsilon)^2 dx. \]
Since \( 0 < \varepsilon < 1 \), for (2.4), we have (2.8). \( \square \)

Let us consider the following function
\[ v_\varepsilon(t, x) = u_\varepsilon(t, x) - g_\varepsilon(t)\chi(x), \]
where \( \chi \in C^\infty(0, \infty) \) is a cut-off function such that
\[ \chi(0) = 1, \]
\[ \|\chi\|_{L^\infty(0, \infty)}; \|\chi'\|_{L^\infty(0, \infty)} \leq C_0, \]
\[ \|\chi\|^2_{L^2(0, \infty)}; \|\chi'|^2_{L^2(0, \infty)} \leq C_0. \]
Therefore, it follows from (2.11), (2.11) and (2.12) that
\[ v_\varepsilon(t, 0) = g_\varepsilon(t) - g_\varepsilon(t) = 0. \]

For (2.2),
\[ v_\varepsilon(0, x) = v_\varepsilon,0(x) = u_\varepsilon(0, x) = u_\varepsilon,0(x). \]

Therefore, again by (2.2),
\[ \|v_\varepsilon,0\|_{L^2(0, \infty)} = \|u_\varepsilon,0\|_{L^2(0, \infty)}. \]

Moreover,
\[ \partial_t u_\varepsilon = \partial_t v_\varepsilon + g_\varepsilon'(t)\chi, \]
\[ \partial_x u_\varepsilon = \partial_x v_\varepsilon + g_\varepsilon(t)\chi', \]
\[ \partial^2_{xx} u_\varepsilon = \partial^2_{xx} v_\varepsilon + g_\varepsilon(t)\chi''. \]

Thus, for (2.11), (2.11) and (2.13), we have
\[ \partial_t v_\varepsilon + g_\varepsilon'(t)\chi + (v_\varepsilon + g_\varepsilon(t)\chi)(\partial_x v_\varepsilon + g_\varepsilon(t)\chi') = \gamma P_\varepsilon + \varepsilon(\partial^2_{xx} v_\varepsilon + g_\varepsilon(t)\chi''), \]
that is,
\[ \partial_t v_\varepsilon + v_\varepsilon \partial_x v_\varepsilon + g_\varepsilon(t)v_\varepsilon \chi' + g_\varepsilon(t)\chi \partial_x v_\varepsilon = \gamma P_\varepsilon + \varepsilon(\partial^2_{xx} v_\varepsilon + g_\varepsilon(t)\chi'' - g_\varepsilon'(t)\chi - g_\varepsilon^2(t)\chi'). \]

**Lemma 2.3.** For each \( t > 0 \), we have that
\[ \|u_\varepsilon(t, \cdot)\|^2_{L^2(0, \infty)} \leq 2 \|v_\varepsilon(t, \cdot)\|^2_{L^2(0, \infty)} + C_0, \]
\[ \|\partial_x P_\varepsilon(t, \cdot)\|^2_{L^2(0, \infty)} \leq 2 \|v_\varepsilon(t, \cdot)\|^2_{L^2(0, \infty)} + C_0, \]
\[ \int_0^\infty P_\varepsilon(t, x)v_\varepsilon(t, x)dx \leq C_0 \|v_\varepsilon(t, \cdot)\|^2_{L^2(0, \infty)} + C_0. \]

**Proof.** We begin by observing that, for (2.11), we get
\[ u_\varepsilon = v_\varepsilon + g_\varepsilon(t)\chi. \]
Squaring (2.20), we have
\[ u_\varepsilon^2 = v_\varepsilon^2 + 2g_\varepsilon(t)v_\varepsilon\chi + g_\varepsilon^2(t)\chi^2. \]
Due to the Young’s inequality,
\[ 2|g_\varepsilon(t)v_\varepsilon\chi| \leq v_\varepsilon^2 + g_\varepsilon^2(t)\chi^2. \]
Therefore,
\[ u_\varepsilon^2 \leq 2v_\varepsilon^2 + 2g_\varepsilon^2(t)\chi^2. \]
In particular, we have
\[ \int_0^\infty P_x v_x dx = \int_0^\infty P_x u_x dx - g_\varepsilon(t) \int_0^\infty P_x \chi dx 
= \int_0^\infty P_x u_x dx + g_\varepsilon(t) \int_0^\infty \partial_x P_x \chi' dx. \]

Thanks to (2.2), (2.12) and Young’s inequality,
\[
\left| g_\varepsilon(t) \int_0^\infty \partial_x P_x \chi' dx \right| \leq |g_\varepsilon(t)| \int_0^\infty |\partial_x P_x \chi'| dx \leq \frac{C_0}{2} \|\partial_x P_x(t, \cdot)\|_{L^2(0, \infty)}^2 + \frac{C_0}{2} \|\chi'\|_{L^2(0, \infty)}^2 
\leq C_0 \|\partial_x P_x(t, \cdot)\|_{L^2(0, \infty)}^2 + C_0.
\]

Hence, for (2.8), (2.17), (2.18) and (2.21),
\[
\int_0^\infty P_x v_x dx \leq 2 \|v_\varepsilon(t, \cdot)\|_{L^2(0, \infty)}^2 + C_0 + C_0 \|v_\varepsilon(t, \cdot)\|_{L^2(0, \infty)}^2 + C_0 
\leq C_0 \|v_\varepsilon(t, \cdot)\|_{L^2(0, \infty)}^2 + C_0,
\]
that is (2.19). \hfill \square

**Lemma 2.4.** For each \( t > 0 \), the inequality holds
\[ \|v_\varepsilon(t, \cdot)\|_{L^2(0, \infty)}^2 + \varepsilon e^{C_0 t} \int_0^t e^{-C_0 s} \|\partial_x v_\varepsilon(s, \cdot)\|_{L^2(0, \infty)}^2 ds \leq C_0 e^{C_0 t}(1 + t). \]

In particular, we have
\[ \|u_\varepsilon(t, \cdot)\|_{L^2(0, \infty)}^2 \leq C_0 \left( e^{C_0 t}(1 + t) + 1 \right), \]
\[ \varepsilon \int_0^t \|\partial_x u_\varepsilon(s, \cdot)\|_{L^2(0, \infty)}^2 ds \leq C_0 \left( e^{C_0 t}(1 + t) + t \right). \]

Moreover,
\[ \varepsilon \|\partial_{xx} P_\varepsilon(t, \cdot)\|_{L^2(0, \infty)}, \|\partial_x P_\varepsilon(t, \cdot)\|_{L^2(0, \infty)} \leq \sqrt{C_0 \left( e^{C_0 t}(1 + t) + 1 \right)}, \]
\[ \sqrt{\varepsilon} |\partial_x P_\varepsilon(t, 0)|, \sqrt{\varepsilon} \|\partial_x P_\varepsilon(t, \cdot)\|_{L^\infty(0, \infty)} \leq \sqrt{C_0 \left( e^{C_0 t}(1 + t) + 1 \right)}. \]

**Proof.** Let \( t > 0 \). Multiplying (2.10) by \( v_\varepsilon \), we have
\[
v_\varepsilon \partial_t v_\varepsilon + v_\varepsilon^2 \partial_x v_\varepsilon + g_\varepsilon(t)v_\varepsilon \chi' + g_\varepsilon(t)v_\varepsilon \chi \partial_x v_\varepsilon = \gamma P_x v_\varepsilon + \varepsilon v_\varepsilon \partial_{xx} v_\varepsilon + \varepsilon g_\varepsilon(t)v_\varepsilon \chi'' - g_\varepsilon(t)v_\varepsilon \chi - g_\varepsilon(t)v_\varepsilon \chi'.
\]

Since,
\[
\int_0^\infty v_\varepsilon \partial_t v_\varepsilon dx = \frac{1}{2} \frac{d}{dt} \|v_\varepsilon(t, \cdot)\|_{L^2(0, \infty)}^2, \\
g_\varepsilon(t) \int_0^\infty v_\varepsilon \chi \partial_x v_\varepsilon dx = -\frac{g_\varepsilon(t)}{2} \int_0^\infty v_\varepsilon^2 \chi' dx, \\
\varepsilon \int_0^\infty v_\varepsilon \partial_{xx} v_\varepsilon dx = -\varepsilon \|\partial_x v_\varepsilon(t, \cdot)\|_{L^2(0, \infty)}^2, \\
\varepsilon g_\varepsilon(t) \int_0^\infty v_\varepsilon \chi'' dx = -\varepsilon g_\varepsilon(t) \int_0^\infty \partial_x v_\varepsilon \chi' dx,
\]
where integrating (2.26) on \((0, \infty)\),

\[
\frac{1}{2} \frac{d}{dt} \| v_\varepsilon(t, \cdot) \|_{L^2(0, \infty)}^2 + \varepsilon \| \partial_x v_\varepsilon(t, \cdot) \|_{L^2(0, \infty)}^2 = -g_\varepsilon(t) \int_0^\infty v_\varepsilon^2 \chi \, dx + \frac{g_\varepsilon(t)}{2} \int_0^\infty v_\varepsilon^2 \chi' \, dx
\]

\[
+ \gamma \int_0^\infty P_\varepsilon v_\varepsilon \, dx - \varepsilon g_\varepsilon(t) \int_0^\infty \partial_x v_\varepsilon \chi' \, dx
\]

\[
- g_\varepsilon'(t) \int_0^\infty v_\varepsilon \chi \, dx - g_\varepsilon^2(t) \int_0^\infty v_\varepsilon \chi' \, dx.
\]

(2.27)

Due to (2.2), (2.12) and Young’s inequality,

\[
\varepsilon \left| g_\varepsilon(t) \int_0^\infty \partial_x v_\varepsilon \chi' \, dx \right| \leq \varepsilon |g_\varepsilon(t)| \int_0^\infty \left| \frac{\partial_x v_\varepsilon}{D_1} \right| |\chi' D_1| \, dx
\]

\[
\leq \varepsilon \frac{C_0}{2D_1^2} \| \partial_x v_\varepsilon(t, \cdot) \|_{L^2(0, \infty)}^2 + \frac{D_1^2}{2} \| \chi' \|_{L^2(0, \infty)}^2
\]

\[
\leq \varepsilon \frac{C_0}{2D_1^2} \| \partial_x v_\varepsilon(t, \cdot) \|_{L^2(0, \infty)}^2 + D_1^2 C_0,
\]

\[
|g_\varepsilon'(t) \int_0^\infty v_\varepsilon \chi \, dx| \leq |g_\varepsilon'(t)| \int_0^\infty |v_\varepsilon| |\chi| \, dx
\]

\[
\leq \frac{C_0}{2} \| v_\varepsilon(t, \cdot) \|_{L^2(0, \infty)}^2 + \frac{C_0}{2} \| \chi \|_{L^2(0, \infty)}^2
\]

\[
\leq C_0 \| v_\varepsilon(t, \cdot) \|_{L^2(0, \infty)}^2 + C_0,
\]

\[
g_\varepsilon^2(t) \left| \int_0^\infty v_\varepsilon \chi' \, dx \right| \leq g_\varepsilon^2(t) \int_0^\infty |v_\varepsilon| |\chi'| \, dx
\]

\[
\leq \frac{C_0 \| \chi' \|_{L^\infty(0, \infty)}^2 \left( \| v_\varepsilon(t, \cdot) \|_{L^2(0, \infty)}^2 + \| \chi \|_{L^2(0, \infty)}^2 \right)}{2}
\]

\[
\leq C_0 \| v_\varepsilon(t, \cdot) \|_{L^2(0, \infty)}^2 + C_0,
\]

where \(D_1\) is a positive constant that will be specified later.

Moreover, again by (2.2) and (2.12),

\[
\left| g_\varepsilon(t) \int_0^\infty v_\varepsilon^2 \chi \, dx \right| \leq |g_\varepsilon(t)| \int_0^\infty v_\varepsilon^2 |\chi| \, dx
\]

\[
\leq C_0 \| \chi \|_{L^\infty(0, \infty)} \| v_\varepsilon(t, \cdot) \|_{L^2(0, \infty)}^2 \leq C_0 \| v_\varepsilon(t, \cdot) \|_{L^2(0, \infty)}^2,
\]

\[
\left| \frac{g_\varepsilon(t)}{2} \int_0^\infty v_\varepsilon^2 \chi' \, dx \right| \leq \frac{g_\varepsilon(t)}{2} \int_0^\infty v_\varepsilon^2 |\chi'| \, dx
\]

\[
\leq C_0 \| \chi' \|_{L^\infty(0, \infty)} \| v_\varepsilon(t, \cdot) \|_{L^2(0, \infty)}^2 \leq C_0 \| v_\varepsilon(t, \cdot) \|_{L^2(0, \infty)}^2.
\]

It follows from (2.19) and (2.27) that

\[
\frac{d}{dt} \| v_\varepsilon(t, \cdot) \|_{L^2(0, \infty)}^2 + \varepsilon \left( 2 \frac{C_0}{D_1^2} \right) \| \partial_x v_\varepsilon(t, \cdot) \|_{L^2(0, \infty)}^2
\]

\[
\leq \gamma C_0 \| v_\varepsilon(t, \cdot) \|_{L^2(0, \infty)}^2 + 8 C_0 \| v_\varepsilon(t, \cdot) \|_{L^2(0, \infty)}^2
\]

\[
+ 2 \gamma C_0 + C_0 + D_1^2 C_0.
\]
that is
\[
\frac{d}{dt} \|v_\varepsilon(t, \cdot)\|_{L^2(0, \infty)}^2 + \varepsilon \left( 2 - \frac{C_0}{D_1^2} \right) \|\partial_x v_\varepsilon(t, \cdot)\|_{L^2(0, \infty)}^2 \\
\leq C_0 \|v_\varepsilon(t, \cdot)\|_{L^2(0, \infty)}^2 + C_0 + D_1^2 C_0.
\]
Choosing \(D_1^2 = C_0\), we get
\[
\frac{d}{dt} \|v_\varepsilon(t, \cdot)\|_{L^2(0, \infty)}^2 + \varepsilon \|\partial_x v_\varepsilon(t, \cdot)\|_{L^2(0, \infty)}^2 \leq C_0 \|v_\varepsilon(t, \cdot)\|_{L^2(0, \infty)}^2 + C_0.
\]
Gronwall's Lemma and (2.14) give
\[
\|v_\varepsilon(t, \cdot)\|_{L^2(0, \infty)}^2 + \varepsilon e^{C_0 t} \int_0^t e^{-C_0 s} \|\partial_x v_\varepsilon(s, \cdot)\|_{L^2(0, \infty)}^2 ds \\
\leq \|u_0\|_{L^2(0, \infty)} e^{C_0 t} + C_0 e^{C_0 t} \int_0^t e^{-C_0 s} ds \\
\leq \|u_0\|_{L^2(0, \infty)} e^{C_0 t} + C_0 te^{C_0 t},
\]
which gives (2.22).

Let us show that (2.24) holds true. We begin by observing that, (2.15) and an multiplication by \(\sqrt{\varepsilon}\) give
\[
\sqrt{\varepsilon} \partial_x u_\varepsilon = \sqrt{\varepsilon} \partial_x v_\varepsilon + \sqrt{\varepsilon} g_\varepsilon(t) \chi'.
\]
Squaring (2.24), we have
\[
\varepsilon (\partial_x u_\varepsilon)^2 = \varepsilon (\partial_x v_\varepsilon)^2 + 2 \varepsilon g_\varepsilon(t) \partial_x v_\varepsilon \chi' + \varepsilon g_\varepsilon^2(t) (\chi')^2.
\]
Due to Young’s inequality,
\[
2 \varepsilon |g_\varepsilon(t) \partial_x v_\varepsilon \chi'| \leq \varepsilon (\partial_x v_\varepsilon)^2 + \varepsilon g_\varepsilon^2(t) (\chi')^2.
\]
Therefore, since 0 < \(\varepsilon < 1\),
\[
\varepsilon (\partial_x u_\varepsilon)^2 \leq 2 \varepsilon (\partial_x v_\varepsilon)^2 + 2 g_\varepsilon^2(t) (\chi')^2.
\]
An integration on \((0, \infty)\), (2.2) and (2.12) give
\[
\varepsilon \|\partial_x u_\varepsilon(t, \cdot)\|_{L^2(0, \infty)}^2 \leq 2 \varepsilon \|\partial_x v_\varepsilon(t, \cdot)\|_{L^2(0, \infty)}^2 + C_0.
\]
Integrating (2.29) on \((0, t)\), we get
\[
\varepsilon \int_0^t \|\partial_x u_\varepsilon(s, \cdot)\|_{L^2(0, \infty)}^2 ds \leq 2 \varepsilon \int_0^t \|\partial_x v_\varepsilon(s, \cdot)\|_{L^2(0, \infty)}^2 ds + C_0 t
\]
(2.30)
\[
\leq 2 \varepsilon e^{C_0 t} \int_0^t e^{-C_0 s} \|\partial_x v_\varepsilon(s, \cdot)\|_{L^2(0, \infty)}^2 ds + C_0 t.
\]
(2.24) follows from (2.22) and (2.30).

Finally, (2.25) follows from (2.4), (2.7) and (2.24). \hfill \Box

**Lemma 2.5.** Let us consider the following function
\[
F_\varepsilon(t, x) = \int_0^x P_\varepsilon(t, y) dy \quad t > 0, \; x > 0.
\]
We have that
\[
\lim_{x \to \infty} F_\varepsilon(t, x) = \int_0^\infty P_\varepsilon(t, x) dx = \frac{\varepsilon}{\gamma} \partial_x^2 P_\varepsilon(t, 0) + \frac{\varepsilon}{\gamma} \partial_x u_\varepsilon(t, 0) - \frac{1}{2\gamma} g_\varepsilon^2(t).
\]
Proof. Integrating on \((0,x)\) the first equation of (2.1), we get
\[
\int_0^x \partial_t u_\varepsilon(t,y) dy + \frac{1}{2} u_\varepsilon^2(t,x) - \frac{1}{2} g_\varepsilon^2(t) - \varepsilon \partial_x u_\varepsilon(t,x) + \varepsilon \partial_x u_\varepsilon(t,0) = \gamma \int_0^x P_\varepsilon(t,y) dy.
\]
It follows from the regularity of \(u_\varepsilon\) that
\[
\lim_{x \to \infty} \left( \frac{1}{2} u_\varepsilon^2(t,x) - \varepsilon \partial_x u_\varepsilon(t,x) \right) = 0.
\]
For (2.33), we have that
\[
\lim_{x \to \infty} \int_0^x \partial_t u_\varepsilon(t,y) dy = \int_0^\infty \partial_t u_\varepsilon(t,x) dx = \frac{d}{dt} \int_0^\infty u_\varepsilon(t,x) dx = \varepsilon \partial_{t,x}^2 P_\varepsilon(t,0).
\]
(2.33), (2.34) and (2.35) give (2.32).

Lemma 2.6. Let \(0 < t < T\). There exists a function \(C(T) > 0\), independent on \(\varepsilon\), such that
\[
\|P_\varepsilon\|_{L^\infty(I_T)} \leq C(T),
\]
\[
\|P_\varepsilon(t,\cdot)\|_{L^2(0,\infty)} \leq C(T),
\]
\[
\varepsilon \|\partial_x P_\varepsilon(t,\cdot)\|_{L^2(0,\infty)} \leq C(T),
\]
\[
e^{2\gamma t} \int_0^t e^{-2\gamma s} \left( \varepsilon \partial_{t,x}^2 P_\varepsilon(s,0) + \varepsilon \partial_x u_\varepsilon(s,0) - \frac{1}{2} g_\varepsilon^2(s) \right)^2 ds \leq C(T),
\]
where
\[
I_T = (0,T) \times (0,\infty).
\]
In particular, we have
\[
\varepsilon \left| \int_0^t \int_0^\infty P_\varepsilon \partial_{t,x}^2 P_\varepsilon ds dx \right| \leq C(T), \quad 0 < t < T.
\]
Proof. Let \(0 < t < T\). We begin by observing that, integrating in \((0,x)\) the second equation of (2.1), we get
\[
P_\varepsilon(t,x) = \int_0^x u_\varepsilon(t,y) dy + \varepsilon \partial_x P_\varepsilon(t,x) - \varepsilon \partial_x P_\varepsilon(t,0).
\]
Differentiating with respect to \(t\), we have that
\[
\partial_t P_\varepsilon(t,x) = \frac{d}{dt} \int_0^x u_\varepsilon(t,y) dy + \varepsilon \partial_{t,x}^2 P_\varepsilon(t,x) - \varepsilon \partial_{t,x}^2 P_\varepsilon(t,0)
\]
\[
= \int_0^x \partial_t u_\varepsilon(t,x) + \varepsilon \partial_{t,x}^2 P_\varepsilon(t,x) - \varepsilon \partial_{t,x}^2 P_\varepsilon(t,0).
\]
It follows from (2.31) and (2.33) that
\[
\partial_t P_\varepsilon(t,x) = \gamma F_\varepsilon(t,x) - \frac{1}{2} u_\varepsilon^2(t,x) + \frac{1}{2} g_\varepsilon^2(t) + \varepsilon \partial_x u_\varepsilon(t,x)
\]
\[
- \varepsilon \partial_x u_\varepsilon(t,0) + \varepsilon \partial_{t,x}^2 P_\varepsilon(t,x) - \varepsilon \partial_{t,x}^2 P_\varepsilon(t,0).
\]
Multiplying (2.43) by \( P_\varepsilon - \varepsilon \partial_x P_\varepsilon \), we have that
\[
(P_\varepsilon - \varepsilon \partial_x P_\varepsilon) \partial_t P_\varepsilon = \gamma (P_\varepsilon - \varepsilon \partial_x P_\varepsilon) F_\varepsilon - \frac{1}{2} (P_\varepsilon - \varepsilon \partial_x P_\varepsilon) u_\varepsilon^2
\]
\[
+ \frac{1}{2} (P_\varepsilon - \varepsilon \partial_x P_\varepsilon) g_\varepsilon^2(t) - \varepsilon (P_\varepsilon - \varepsilon \partial_x P_\varepsilon) \partial_x u_\varepsilon(t, 0)
\]
\[
+ \varepsilon (P_\varepsilon - \varepsilon \partial_x P_\varepsilon) \partial_x u_\varepsilon + \varepsilon (P_\varepsilon - \varepsilon \partial_x P_\varepsilon) \partial_{xx}^2 P_\varepsilon
\]
\[
- \varepsilon (P_\varepsilon - \varepsilon \partial_x P_\varepsilon) \partial_{xx}^2 P_\varepsilon(t, 0).
\]

Integrating (2.44) on \((0, x)\), for (2.1), we get
\[
\int_0^x P_\varepsilon \partial_t P_\varepsilon dy - \varepsilon \int_0^x \partial_x P_\varepsilon \partial_t P_\varepsilon dy
\]
\[
= - \varepsilon \int_0^x \partial_x P_\varepsilon \partial_t P_\varepsilon dy = - \varepsilon \partial_x P_\varepsilon \partial_t P_\varepsilon + \varepsilon \int_0^x P_\varepsilon \partial_{xx}^2 P_\varepsilon dy.
\]

We observe that, for (2.1),
\[
- \varepsilon \int_0^x \partial_x P_\varepsilon \partial_t P_\varepsilon dy = - \varepsilon \partial_x P_\varepsilon \partial_t P_\varepsilon + \varepsilon \int_0^x P_\varepsilon \partial_{xx}^2 P_\varepsilon dy.
\]

Therefore, (2.45) and (2.46) give
\[
\int_0^x P_\varepsilon \partial_t P_\varepsilon dy + \varepsilon^2 \int_0^x \partial_x P_\varepsilon \partial_{xx}^2 P_\varepsilon dy
\]
\[
= \varepsilon P_\varepsilon \partial_t P_\varepsilon + \gamma \int_0^x P_\varepsilon F_\varepsilon dy - \varepsilon \int_0^x P_\varepsilon \partial_x P_\varepsilon dy
\]
\[
- \frac{1}{2} \int_0^x P_\varepsilon u_\varepsilon^2 dy + \frac{\varepsilon}{2} \int_0^x \partial_x P_\varepsilon u_\varepsilon^2 dy + \frac{1}{2} g_\varepsilon^2(t) \int_0^x P_\varepsilon dy
\]
\[
- \frac{\varepsilon}{2} g_\varepsilon^2(t) P_\varepsilon - \varepsilon \partial_x u_\varepsilon(t, 0) \int_0^y P_\varepsilon dx + \varepsilon^2 \partial_x u_\varepsilon(t, 0) P_\varepsilon
\]
\[
+ \varepsilon \int_0^x P_\varepsilon \partial_x u_\varepsilon dy - \varepsilon^2 \int_0^x \partial_x P_\varepsilon \partial_x u_\varepsilon dy - \varepsilon \partial_{xx}^2 P_\varepsilon(t, 0) \int_0^x P_\varepsilon dy
\]
\[
+ \varepsilon^2 \partial_{xx}^2 P_\varepsilon(t, 0) P_\varepsilon.
\]

Since
\[
\int_0^\infty P_\varepsilon \partial_t P_\varepsilon dx = \frac{1}{2} \frac{d}{dt} \int_0^\infty P_\varepsilon^2 dx,
\]
\[
\varepsilon^2 \int_0^\infty \partial_{xx}^2 P_\varepsilon \partial_x P_\varepsilon dx = \frac{\varepsilon^2}{2} \frac{d}{dt} \int_0^\infty (\partial_x P_\varepsilon)^2 dx,
\]
when \( x \to \infty \), for (2.3) and (2.47), we have that
\[
\frac{1}{2} \frac{d}{dt} \int_0^\infty P_\varepsilon^2 dx + \frac{\varepsilon^2}{2} \frac{d}{dt} \int_0^\infty (\partial_x P_\varepsilon)^2 dx \\
= \gamma \int_0^\infty P_\varepsilon F_\varepsilon dx - \varepsilon \gamma \int_0^\infty \partial_x P_\varepsilon F_\varepsilon dx - \frac{1}{2} \int_0^\infty P_\varepsilon u_\varepsilon^2 dx \\
+ \frac{\varepsilon}{2} \int_0^\infty \partial_x P_\varepsilon u_\varepsilon^2 dx + \frac{1}{2} \gamma^2(t) \int_0^\infty P_\varepsilon dx - \varepsilon \partial_x u_\varepsilon(t,0) \int_0^\infty P_\varepsilon dx \\
+ \varepsilon \int_0^\infty P_\varepsilon \partial_x u_\varepsilon dx + \varepsilon^2 \int_0^\infty \partial_x P_\varepsilon \partial_x u_\varepsilon dx - \varepsilon \partial_x^2 P_\varepsilon(t,0) \int_0^\infty P_\varepsilon dx.
\] (2.48)
Due to (2.31) and (2.32),
\[
2\gamma \int_0^\infty P_\varepsilon F_\varepsilon dx = 2\gamma \int_0^\infty F_\varepsilon \partial_\varepsilon F_\varepsilon dx = \gamma (F_\varepsilon(t,\infty))^2 \\
= \frac{1}{\gamma} \left( \varepsilon \partial_{tx}^2 P_\varepsilon(t,0) + \varepsilon \partial_x u_\varepsilon(t,0) - \frac{1}{2} \gamma g_\varepsilon^4(t) \right)^2,
\]
that is
\[
2\gamma \int_0^\infty P_\varepsilon F_\varepsilon dx = \frac{\varepsilon^2}{\gamma} (\partial_{tx}^2 P_\varepsilon(t,0))^2 + \frac{2\varepsilon^2}{\gamma} \partial_{tx}^2 P_\varepsilon(t,0) \partial_x u_\varepsilon(t,0) + \frac{\varepsilon^2}{\gamma} (\partial_x u_\varepsilon(t,0))^2 \\
+ \frac{1}{4\gamma} g_\varepsilon^4(t) - \frac{\varepsilon}{\gamma} \partial_{tx}^2 P_\varepsilon(t,0) g_\varepsilon^2(t) - \frac{\varepsilon}{\gamma} \partial_x u_\varepsilon(t,0) g_\varepsilon^2(t).
\] (2.49)
Again by (2.32),
\[
-2\varepsilon \partial_x u_\varepsilon(t,0) \int_0^\infty P_\varepsilon dx = -2 \frac{\varepsilon^2}{\gamma} (\partial_{tx}^2 P_\varepsilon(t,0)) \partial_x u_\varepsilon(t,0) \\
-2 \frac{\varepsilon^2}{\gamma} (\partial_x u_\varepsilon(t,0))^2 + \frac{\varepsilon}{\gamma} \partial_x u_\varepsilon(t,0) g_\varepsilon^2(t),
\] (2.50)
\[
-2\varepsilon \partial_{tx}^2 P_\varepsilon(t,0) \int_0^\infty P_\varepsilon dx = -2 \frac{\varepsilon^2}{\gamma} (\partial_{tx}^2 P_\varepsilon(t,0))^2 \\
-2 \frac{\varepsilon^2}{\gamma} \partial_{tx}^2 P_\varepsilon(t,0) \partial_x u_\varepsilon(t,0) + \frac{\varepsilon}{\gamma} \partial_{tx}^2 P_\varepsilon(t,0) g_\varepsilon^2(t),
\]
\[
g_\varepsilon^2(t) \int_0^\infty P_\varepsilon dx = \frac{\varepsilon}{\gamma} \partial_{tx}^2 P_\varepsilon(t,0) g_\varepsilon^2(t) + \frac{\varepsilon}{\gamma} \partial_x u_\varepsilon(t,0) g_\varepsilon^2(t) - \frac{1}{2\gamma} g_\varepsilon^4(t).
\]
Therefore, (2.48), (2.49) and (2.50) give
\[
\frac{d}{dt} \left( \int_0^\infty P_\varepsilon^2 dx + \varepsilon^2 \int_0^\infty (\partial_x P_\varepsilon)^2 dx \right) \\
= \frac{\varepsilon^2}{\gamma} (\partial_{tx}^2 P_\varepsilon(t,0))^2 + \frac{2\varepsilon^2}{\gamma} \partial_{tx}^2 P_\varepsilon(t,0) \partial_x u_\varepsilon(t,0) + \frac{\varepsilon^2}{\gamma} (\partial_x u_\varepsilon(t,0))^2 \\
+ \frac{1}{4\gamma} g_\varepsilon^4(t) - \frac{\varepsilon}{\gamma} \partial_{tx}^2 P_\varepsilon(t,0) g_\varepsilon^2(t) - \frac{\varepsilon}{\gamma} \partial_x u_\varepsilon(t,0) g_\varepsilon^2(t) \\
- 2\varepsilon \int_0^\infty \partial_x P_\varepsilon F_\varepsilon dx - \int_0^\infty P_\varepsilon u_\varepsilon^2 dx + \varepsilon \int_0^\infty \partial_x P_\varepsilon u_\varepsilon^2 dx \\
+ \frac{\varepsilon}{\gamma} \partial_{tx}^2 P_\varepsilon(t,0) g_\varepsilon^2(t) + \frac{\varepsilon}{\gamma} \partial_x u_\varepsilon(t,0) g_\varepsilon^2(t) - \frac{1}{2\gamma} g_\varepsilon^4(t)
\].
Hence, (2.51), (2.52) and (2.53) give

\[
\begin{align*}
&\frac{d}{dt} \left( \int_0^\infty P_\varepsilon(t,0)^2 \, dx + \varepsilon^2 \int_0^\infty (\partial_x P_\varepsilon(t,0))^2 \, dx \right) + \frac{1}{\gamma} \left( \varepsilon \partial_t^2 P_\varepsilon(t,0) + \varepsilon \partial_t u_\varepsilon(t,0) - \frac{1}{2} g_\varepsilon^2(t) \right)^2 \\
&\leq 2\gamma \| P_\varepsilon(t, \cdot) \|_{L^2(0, \infty)}^2 + \int_0^\infty P_\varepsilon u_\varepsilon^2 \, dx + \varepsilon \int_0^\infty \partial_x P_\varepsilon u_\varepsilon^2 \, dx \\
&+ 2\varepsilon \int_0^\infty u_\varepsilon \partial_x P_\varepsilon \, dx + 2\varepsilon^2 \int_0^\infty \partial_x P_\varepsilon \partial_x u_\varepsilon \, dx.
\end{align*}
\]

That is,

\[
(2.51)
\]

\[
\begin{align*}
-2 \varepsilon & \int_0^\infty \partial_x P_\varepsilon \, dx + 2\varepsilon \int_0^\infty \partial_x P_\varepsilon u_\varepsilon \, dx = \int_0^\infty \partial_x P_\varepsilon \, dx + \varepsilon \int_0^\infty \partial_x P_\varepsilon u_\varepsilon^2 \, dx \\
&+ 2\varepsilon \int_0^\infty u_\varepsilon \partial_x P_\varepsilon \, dx + 2\varepsilon^2 \int_0^\infty \partial_x P_\varepsilon \partial_x u_\varepsilon \, dx.
\end{align*}
\]

Thanks to (2.1), (2.3), (2.31) and (2.32),

\[
(2.52)
\]

while, for (2.1) and (2.3),

\[
(2.53)
\]

Hence, (2.51), (2.52) and (2.53) give

\[
\begin{align*}
&\frac{d}{dt} \left( \| P_\varepsilon(t, \cdot) \|_{L^2(0, \infty)}^2 + \varepsilon^2 \| \partial_x P_\varepsilon(t, \cdot) \|_{L^2(0, \infty)}^2 \right) \\
&+ \frac{1}{\gamma} \left( \varepsilon \partial_t^2 P_\varepsilon(t,0) + \varepsilon \partial_t u_\varepsilon(t,0) - \frac{1}{2} g_\varepsilon^2(t) \right)^2 \\
&\leq 2\gamma \| P_\varepsilon(t, \cdot) \|_{L^2(0, \infty)}^2 + \int_0^\infty P_\varepsilon u_\varepsilon^2 \, dx + \varepsilon \int_0^\infty \partial_x P_\varepsilon u_\varepsilon^2 \, dx \\
&+ 2\varepsilon \int_0^\infty u_\varepsilon \partial_x P_\varepsilon \, dx + 2\varepsilon^2 \int_0^\infty \partial_x P_\varepsilon \partial_x u_\varepsilon \, dx.
\end{align*}
\]

Thus,

\[
(2.54)
\]
For Young’s inequality,
\[
2\varepsilon \int_0^\infty |\partial_x P_\varepsilon||u_\varepsilon|dx = \int_0^\infty \left| \frac{u_\varepsilon}{\sqrt{\varepsilon}} \right| |2\varepsilon \sqrt{\gamma} \partial_x P_\varepsilon|dx \\
\leq 2\gamma \varepsilon^2 \|\partial_x P_\varepsilon(t,\cdot)\|_{L^2(0,\infty)}^2 + \frac{1}{2\gamma} \|u_\varepsilon(t,\cdot)\|_{L^2(0,\infty)}^2 , \\
2\varepsilon^2 \int_0^\infty |\partial_x P_\varepsilon||\partial_x u_\varepsilon| \leq \varepsilon^2 \|\partial_x P_\varepsilon(t,\cdot)\|_{L^2(0,\infty)}^2 + \varepsilon^2 \|\partial_x u_\varepsilon(t,\cdot)\|_{L^2(0,\infty)}^2 .
\]
Thus,
\[
\frac{d}{dt} G(t) + \frac{1}{\gamma} \left( \varepsilon \partial_t^2 P_\varepsilon(t,0) + \varepsilon \partial_x u_\varepsilon(t,0) - \frac{1}{2} g_\varepsilon^2(t) \right)^2 \\
\leq 2\gamma G(t) + \frac{1}{\gamma} \|u_\varepsilon(t,\cdot)\|_{L^2(0,\infty)}^2 + \int_0^\infty |P_\varepsilon|u_\varepsilon^2dx + \varepsilon \int_0^\infty |\partial_x P_\varepsilon|u_\varepsilon^2dx \\
+ \varepsilon^2 \int_0^\infty (\partial_x u_\varepsilon)^2dx + \varepsilon^2 \|\partial_x P_\varepsilon(t,\cdot)\|_{L^2(0,\infty)}^2 + \varepsilon^2 \|\partial_x u_\varepsilon(t,\cdot)\|_{L^2(0,\infty)}^2 ,
\]
that is
\[
\frac{d}{dt} G(t) - 2\gamma G(t) + \frac{1}{\gamma} \left( \varepsilon \partial_t^2 P_\varepsilon(t,0) + \varepsilon \partial_x u_\varepsilon(t,0) - \frac{1}{2} g_\varepsilon^2(t) \right)^2 \\
\leq \frac{1}{2\gamma} \|u_\varepsilon(t,\cdot)\|_{L^2(0,\infty)}^2 + \int_0^\infty |P_\varepsilon|u_\varepsilon^2dx \\
+ \varepsilon \int_0^\infty |\partial_x P_\varepsilon|u_\varepsilon^2dx + \varepsilon^2 \|\partial_x P_\varepsilon(t,\cdot)\|_{L^2(0,\infty)}^2 \\
+ \varepsilon^2 \|\partial_x u_\varepsilon(t,\cdot)\|_{L^2(0,\infty)}^2 ,
\]
where
\[
G(t) = \|P_\varepsilon(t,\cdot)\|_{L^2(0,\infty)}^2 + \varepsilon^2 \|\partial_x P_\varepsilon(t,\cdot)\|_{L^2(0,\infty)}^2 .
\]
We observe that, for (2.23),
\[
\int_0^\infty |P_\varepsilon|u_\varepsilon^2dx \leq C_0 \left( e^{C_0 t} + 1 \right) \|P_\varepsilon\|_{L^\infty(I_T)} ,
\]
where $I_T$ is defined in (2.40).
Since $0 < \varepsilon < 1$, it follows from (2.23) and (2.25) that
\[
\varepsilon \int_0^\infty |\partial_x P_\varepsilon|u_\varepsilon^2dx \leq \varepsilon \|\partial_x P_\varepsilon(t,\cdot)\|_{L^\infty(0,\infty)} \|u_\varepsilon(t,\cdot)\|_{L^2(0,\infty)}^2 \\
\leq \sqrt{\varepsilon} C_0 \left( e^{C_0 t} + 1 \right)^{\frac{3}{4}} \leq C_0 \left( e^{C_0 t} + 1 \right)^{\frac{3}{4}} .
\]
Again by $0 < \varepsilon < 1$ and (2.25), we have that
\[
\varepsilon^2 \int_0^\infty (\partial_x P_\varepsilon)^2dx \leq \|\partial_x P_\varepsilon(t,\cdot)\|_{L^2(0,\infty)}^2 \leq C_0 \left( e^{C_0 t} + 1 \right) .
\]
Therefore, (2.23), (2.55), (2.57), (2.58) and (2.59) give
\[
\frac{d}{dt} G(t) - 2\gamma G(t) + \frac{1}{\gamma} \left( \varepsilon \partial_t^2 P_\varepsilon(t,0) + \varepsilon \partial_x u_\varepsilon(t,0) - \frac{1}{2} g_\varepsilon^2(t) \right)^2 \\
\leq \theta_1(t) + \theta_2(t) \|P_\varepsilon\|_{L^\infty(I_T)} + \varepsilon^2 \|\partial_x u_\varepsilon(t,\cdot)\|_{L^2(0,\infty)}^2 .
\]
where
\[ \theta_1(t) = 2C_0 \left( e^{C_0t} (1 + t) + 1 \right) + C_0 \left( e^{C_0t} (1 + t) + 1 \right)^{3/2}, \]
\[ \theta_2(t) = C_0 \left( e^{C_0t} (1 + t) + 1 \right), \]
are two continuous functions in \( t \).

Gronwall’s Lemma, (2.22) and (2.50) give
\[
\begin{align*}
\| P_\epsilon(t, \cdot) \|^2_{L^2(0, \infty)} &+ \epsilon^2 \| \partial_x P_\epsilon(t, \cdot) \|^2_{L^2(0, \infty)} \\
&+ \frac{e^{2\gamma t}}{\gamma} \int_0^t e^{-2\gamma s} \left( \epsilon \partial_{x_2}^2 P_\epsilon(s, 0) + \epsilon \partial_x u_\epsilon(s, 0) - \frac{1}{2} g_\epsilon^2(s) \right)^2 ds \\
&\leq \| P_0 \|^2_{L^2(0, \infty)} e^{2\gamma t} + \epsilon e^{2\gamma t} \int_0^t e^{-2\gamma s} \theta_1(s) ds + \| P_\epsilon \|_{L^\infty(I_T)} e^{2\gamma t} \int_0^t e^{-2\gamma s} \theta_2(s) ds \\
&+ \epsilon^2 e^{2\gamma t} \int_0^t e^{-2\gamma s} \| \partial_x u_\epsilon(s, \cdot) \|^2_{L^2(0, \infty)} ds \\
&\leq \| P_0 \|^2_{L^2(0, \infty)} e^{2\gamma t} + \| P_\epsilon \|_{L^\infty(I_T)} \gamma t e^{2\gamma t} + \| P_\epsilon \|_{L^\infty(I_T)} \| \theta_2 \|_{L^\infty(0, T)} \gamma t e^{2\gamma t} \\
&+ \epsilon^2 e^{2\gamma t} \int_0^t e^{-2\gamma s} \| \partial_x u_\epsilon(s, \cdot) \|^2_{L^2(0, \infty)} ds.
\end{align*}
\]

For (2.22),
\[
\epsilon^2 e^{2\gamma t} \int_0^t e^{-2\gamma s} \| \partial_x u_\epsilon(s, \cdot) \|^2_{L^2(0, \infty)} ds \\
\leq \epsilon e^{2\gamma t} \int_0^t \| \partial_x u_\epsilon(s, \cdot) \|^2_{L^2(0, \infty)} ds \leq \theta_3(t) \leq \| \theta_3 \|_{L^\infty(0, T)},
\]
where
\[ \theta_3(t) = C_0 e^{2\gamma t} \left( e^{C_0t} (1 + t) + t \right). \]

Hence,
\[
\begin{align*}
\| P_\epsilon(t, \cdot) \|^2_{L^2(0, \infty)} &+ \epsilon^2 \| \partial_x P_\epsilon(t, \cdot) \|^2_{L^2(0, \infty)} \\
&+ \frac{e^{2\gamma t}}{\gamma} \int_0^t e^{-2\gamma s} \left( \epsilon \partial_{x_2}^2 P_\epsilon(s, 0) + \epsilon \partial_x u_\epsilon(s, 0) - \frac{1}{2} g_\epsilon^2(s) \right)^2 ds \\
&\leq \| P_0 \|^2_{L^2(0, \infty)} e^{2\gamma t} + \| P_\epsilon \|_{L^\infty(I_T)} \gamma t e^{2\gamma t} + \| P_\epsilon \|_{L^\infty(I_T)} \| \theta_2 \|_{L^\infty(0, T)} \gamma t e^{2\gamma t} + \| \theta_3 \|_{L^\infty(0, T)}
\end{align*}
\]
that is
\[
\begin{align*}
\| P_\epsilon(t, \cdot) \|^2_{L^2(0, \infty)} &+ \epsilon^2 \| \partial_x P_\epsilon(t, \cdot) \|^2_{L^2(0, \infty)} \\
&+ \frac{e^{2\gamma t}}{\gamma} \int_0^t e^{-2\gamma s} \left( \epsilon \partial_{x_2}^2 P_\epsilon(s, 0) + \epsilon \partial_x u_\epsilon(s, 0) - \frac{1}{2} g_\epsilon^2(s) \right)^2 ds \\
&\leq C(T) \left( \| P_\epsilon \|_{L^\infty(I_T)} + 1 \right).
\end{align*}
\]

Due to (2.1), (2.25), (2.61) and the Hölder inequality,
\[
\begin{align*}
P_\epsilon^2(t, x) &\leq \int_0^\infty | P_\epsilon | | \partial_x P_\epsilon | dx \\
&\leq \| P_\epsilon \|_{L^\infty(I_T)} \| \partial_x P_\epsilon(t, \cdot) \|^2_{L^2(0, \infty)} + \| P_\epsilon \|_{L^\infty(I_T)} \| \partial_x P_\epsilon(t, \cdot) \|^2_{L^2(0, \infty)} \\
&\leq 2 \sqrt{C(T) \left( \| P_\epsilon \|_{L^\infty(I_T)} + 1 \right)} \sqrt{C_0 \left( e^{C_0t} (1 + t) + 1 \right)} \\
&\leq C(T) \left( \| P_\epsilon \|_{L^\infty(I_T)} + 1 \right).
\end{align*}
\]
Therefore,
\begin{equation}
\|P_\varepsilon\|_{L^\infty(I_T)}^2 - C(T) \|P_\varepsilon\|_{L^\infty(I_T)} - C(T) \leq 0,
\end{equation}
which gives (2.36).

Let us show that (2.44) holds true. Multiplying (2.43) by \(P_\varepsilon\), an integration on \((0, \infty)\) gives
\begin{align*}
2\varepsilon \int_0^\infty P_\varepsilon \partial^2_{tx} P_\varepsilon dx &= \frac{d}{dt} \|P_\varepsilon(t, \cdot)\|_{L^2(0, \infty)}^2 \left(\|P_\varepsilon(t, \cdot)\|_{L^2(0, \infty)}^2 - 2\gamma \int_0^\infty P_\varepsilon F_\varepsilon dx + \int_0^\infty P_\varepsilon u_\varepsilon^2 dx \right) \\
&\quad - \varepsilon^2 \gamma \left(\int_0^{\infty} \int_0^{\infty} P_\varepsilon dx - 2\varepsilon \int_0^\infty P_\varepsilon \partial_x u_\varepsilon dx \right) \\
&\quad + 2\varepsilon \partial_x u_\varepsilon(t, 0) \int_0^\infty P_\varepsilon dx + 2\varepsilon \partial^2_{tx} P_\varepsilon(t, 0) \int_0^\infty P_\varepsilon dx.
\end{align*}

It follows from (2.31), (2.32), (2.49) and (2.50) that
\begin{align*}
2\varepsilon \int_0^\infty P_\varepsilon \partial^2_{tx} P_\varepsilon dx &= \frac{d}{dt} \|P_\varepsilon(t, \cdot)\|_{L^2(0, \infty)}^2 \left(\|P_\varepsilon(t, \cdot)\|_{L^2(0, \infty)}^2 - \frac{\varepsilon^2}{\gamma} \left(\partial^2_{tx} P_\varepsilon(t, 0)\right)^2 \right) \\
&\quad - \frac{2\varepsilon^2}{\gamma} \partial^2_{tx} P_\varepsilon(t, 0) \partial_x u_\varepsilon(t, 0) - \frac{\varepsilon^2}{\gamma} \left(\partial_x u_\varepsilon(t, 0)\right)^2 \\
&\quad - \frac{1}{4\gamma} g_\varepsilon^4(t) + \frac{\varepsilon}{\gamma} \partial^2_{tx} P_\varepsilon(t, 0) g_\varepsilon^2(t) + \frac{\varepsilon}{\gamma} \partial_x u_\varepsilon(t, 0) g_\varepsilon^2(t) \\
&\quad + \varepsilon \gamma \left(\partial^2_{tx} P_\varepsilon(t, 0)\right)^2 + 2\varepsilon \gamma \partial^2_{tx} P_\varepsilon(t, 0) \partial_x u_\varepsilon(t, 0) - \frac{\varepsilon}{\gamma} \partial^2_{tx} P_\varepsilon(t, 0) g_\varepsilon^2(t),
\end{align*}
that is,
\begin{align*}
2\varepsilon \int_0^\infty P_\varepsilon \partial^2_{tx} P_\varepsilon dx &= \frac{d}{dt} \|P_\varepsilon(t, \cdot)\|_{L^2(0, \infty)}^2 \\
&\quad + \frac{1}{\gamma} \left(\varepsilon \partial^2_{tx} P_\varepsilon(t, 0) + \varepsilon \partial_x u_\varepsilon(t, 0) - \frac{\gamma}{2} g_\varepsilon^2(t)\right)^2 \\
&\quad + \int_0^\infty P_\varepsilon u_\varepsilon^2 dx - 2\varepsilon \int_0^\infty P_\varepsilon \partial_x u_\varepsilon dx.
\end{align*}

An integration on \((0, t)\) gives
\begin{align*}
2\varepsilon \int_0^t \int_0^\infty P_\varepsilon \partial^2_{tx} P_\varepsilon ds dx &= \|P_\varepsilon(t, \cdot)\|_{L^2(0, \infty)}^2 - \|P_\varepsilon_{t, 0}\|_{L^2(0, \infty)}^2 \\
&\quad + \frac{1}{\gamma} \int_0^t \left(\varepsilon \partial^2_{tx} P_\varepsilon(s, 0) + \varepsilon \partial_x u_\varepsilon(s, 0) - \frac{\gamma}{2} g_\varepsilon^2(s)\right)^2 ds \\
&\quad + \int_0^t \int_0^\infty P_\varepsilon u_\varepsilon^2 dx - 2\varepsilon \int_0^t \int_0^\infty P_\varepsilon \partial_x u_\varepsilon ds dx.
\end{align*}
It follows from (2.2), (2.37) and (2.57) that
\[
2\varepsilon \int_0^{t} \int_0^{\infty} P_\varepsilon \partial_{xx}^2 P_\varepsilon \, ds \, dx \leq \|P_\varepsilon(t, \cdot)\|_{L^2(0, \infty)}^2 + \|P_\varepsilon(0, \cdot)\|_{L^2(0, \infty)}^2 \\
+ \frac{1}{\gamma} \int_0^t \left( \varepsilon \partial_{xx}^2 P_\varepsilon(s, 0) + \varepsilon \partial_x u_\varepsilon(s, 0) - \gamma(s) \right)^2 \, ds \\
+ 2\varepsilon \int_0^t \int_0^{\infty} |P_\varepsilon| |\partial_x u_\varepsilon| \, dx \, dx + C(T) \\
\leq \|P_0\|_{L^2(0, \infty)}^2 \\
+ \frac{e^{2\gamma t}}{\gamma} \int_0^t \left( \varepsilon \partial_{xx}^2 P_\varepsilon(s, 0) + \varepsilon \partial_x u_\varepsilon(s, 0) - \frac{1}{2} \gamma(s) \right)^2 \, ds \\
+ 2\varepsilon \int_0^t \int_0^{\infty} |P_\varepsilon| |\partial_x u_\varepsilon| \, ds \, dx + C(T) \\
\leq \|P_0\|_{L^2(0, \infty)}^2 + 2\varepsilon \int_0^t \int_0^{\infty} |P_\varepsilon| |\partial_x u_\varepsilon| \, ds \, dx + C(T)
\]

Due to (2.37) and Young’s inequality,
\[
2\varepsilon \int_0^\infty |P_\varepsilon| |\partial_x u_\varepsilon| \, dx = 2 \int_0^\infty |P_\varepsilon| |\varepsilon \partial_x u_\varepsilon| \\
\leq \|P_\varepsilon(t, \cdot)\|_{L^2(0, \infty)}^2 + \varepsilon^2 \|\partial_x u_\varepsilon(t, \cdot)\|_{L^2(0, \infty)}^2 \\
\leq C(T) + \varepsilon^2 \|\partial_x u_\varepsilon(t, \cdot)\|_{L^2(0, \infty)}^2
\]

Thus, for (2.24) and (2.63), we have that
\[
2\varepsilon \int_0^t \int_0^{\infty} |P_\varepsilon| |\partial_x u_\varepsilon| \, ds \, dx \leq \int_0^t \|P_\varepsilon(s, \cdot)\|_{L^2(0, \infty)}^2 \, ds + \varepsilon^2 \int_0^t \|\partial_x u_\varepsilon(s, \cdot)\|_{L^2(0, \infty)}^2 \, ds \leq C(T).
\]

Therefore,
\[
2\varepsilon \int_0^t \int_0^{\infty} P_\varepsilon \partial_{xx}^2 P_\varepsilon \, ds \, dx \leq \|P_0\|_{L^2(0, \infty)}^2 + C(T),
\]

which gives (2.41).

\[\square\]

**Lemma 2.7.** Let \(T > 0\). Then,
\[
\|u_\varepsilon\|_{L^\infty(I_T)} \leq \|u_0\|_{L^\infty(0, \infty)} + C(T),
\]
where \(I_T\) is defined in (2.40).

**Proof.** Due to (2.1) and (2.36),
\[
\partial_t u_\varepsilon + u_\varepsilon \partial_x u_\varepsilon - \varepsilon \partial_{xx}^2 u_\varepsilon \leq \gamma C(T).
\]

Since the map
\[
\mathcal{F}(t) := \|u_0\|_{L^\infty(0, \infty)} + \gamma C(T) t,
\]
solves the equation
\[
\frac{d\mathcal{F}}{dt} = \gamma C(T)
\]
and
\[
\max\{u_\varepsilon(0, x), 0\} \leq \mathcal{F}(t), \quad (t, x) \in I_T,
\]
the comparison principle for parabolic equations implies that
\[
u_\varepsilon(t, x) \leq \mathcal{F}(t), \quad (t, x) \in I_T.
\]
In a similar way we can prove that
\[ u_\varepsilon(t, x) \geq -F(t), \quad (t, x) \in I_T. \]
Therefore,
\[ |u_\varepsilon(t, x)| \leq \|u_0\|_{L^\infty(0, \infty)} + \gamma C(T)t \leq \|u_0\|_{L^\infty(0, \infty)} + C(T), \]
which gives (2.64).

\[ \square \]

3. PROOF OF THEOREM 1.1

This section is devoted to the proof of Theorem 1.1. Let us begin by proving the existence of a distributional solution to (1.12), (1.13), (1.14) satisfying (1.14).

Lemma 3.1. Let \( T > 0 \). There exists a function \( u \in L^\infty((0, T) \times (0, \infty)) \) that is a distributional solution of (1.11) and satisfies (1.14).

We construct a solution by passing to the limit in a sequence \( \{u_\varepsilon\}_{\varepsilon > 0} \) of viscosity approximations (2.1). We use the compensated compactness method [25].

Lemma 3.2. Let \( T > 0 \). There exists a subsequence \( \{u_{\varepsilon_k}\}_{k \in \mathbb{N}} \) of \( \{u_\varepsilon\}_{\varepsilon > 0} \) and a limit function \( u \in L^\infty((0, T) \times (0, \infty)) \) such that
\[ u_{\varepsilon_k} \to u \text{ a.e. and in } L^p_{loc}((0, T) \times (0, \infty)), \quad 1 \leq p < \infty. \]

Moreover, we have
\[ P_{\varepsilon_k} \to P \text{ a.e. and in } L^p_{loc}(0, T; W^{1,p}_{loc}(0, \infty)), \quad 1 \leq p < \infty, \]
where
\[ P(t, x) = \int_0^x u(t, y)dy, \quad t \geq 0, \quad x \geq 0, \]
and (1.14) holds true.

Proof. Let \( \eta : \mathbb{R} \to \mathbb{R} \) be any convex \( C^2 \) entropy function, and \( q : \mathbb{R} \to \mathbb{R} \) be the corresponding entropy flux defined by \( q' = f_0' \eta' \). By multiplying the first equation in (2.1) with \( \eta'(u_\varepsilon) \) and using the chain rule, we get
\[
\partial_\varepsilon \eta(u_\varepsilon) + \partial_x q(u_\varepsilon) = \varepsilon \partial_{xx} \eta(u_\varepsilon) - \varepsilon \eta''(u_\varepsilon) (\partial_x u_\varepsilon)^2 + \gamma \eta'(u_\varepsilon) P_{\varepsilon},
\]
where \( \mathcal{L}_{1, \varepsilon}, \mathcal{L}_{2, \varepsilon}, \mathcal{L}_{3, \varepsilon} \) are distributions.

Let us show that
\[ \mathcal{L}_{1, \varepsilon} \to 0 \text{ in } H^{-1}((0, T) \times (0, \infty)), \quad T > 0. \]

Since
\[ \varepsilon \partial_{xx} \eta(u_\varepsilon) = \partial_\varepsilon (\varepsilon \eta'(u_\varepsilon) \partial_x u_\varepsilon), \]
for (2.24) and Lemma 2.7,
\[
\| \varepsilon \eta'(u_\varepsilon) \partial_x u_\varepsilon \|_{L^2((0, T) \times (0, \infty))} \leq \varepsilon \| \eta' \|_{L^\infty(J_T)} \int_0^T \| \partial_x u_\varepsilon(s, \cdot) \|_{L^2(0, \infty)} ds \\
\leq \varepsilon \| \eta' \|_{L^\infty(J_T)} C(T) \to 0,
\]
where
\[ J_T = \left( -\|u_0\|_{L^\infty(0, \infty)} - C(T), \|u_0\|_{L^\infty(0, \infty)} + C(T) \right). \]

We claim that
\[ \{\mathcal{L}_{2, \varepsilon}\}_{\varepsilon > 0} \text{ is uniformly bounded in } L^1((0, T) \times (0, \infty)), \quad T > 0. \]
Again by (2.24) and Lemma 2.7
\[
\|\varepsilon\eta''(u_\varepsilon)(\partial_x u_\varepsilon)^2\|_{L^1((0,T) \times (0,\infty), J_T)} \leq \|\eta''\|_{L^\infty(J_T)} \varepsilon \int_0^T \|\partial_x u_\varepsilon(s,\cdot)\|^2_{L^2(0,\infty)} ds
\]
\[
\leq \|\eta''\|_{L^\infty(J_T)} C(T).
\]
We have that
\[
\{L_{3,\varepsilon}\}_{\varepsilon>0} \text{ is uniformly bounded in } L^1_{loc}((0,T) \times (0,\infty)), \ T > 0.
\]
Let \(K\) be a compact subset of \((0,T) \times (0,\infty)\). For Lemmas 2.6 and 2.7,
\[
\|\gamma\eta'(u_\varepsilon)P_\varepsilon\|_{L^1(K)} = \gamma \int_K |\eta'(u_\varepsilon)||P_\varepsilon|dt dx
\]
\[
\leq \gamma \|\eta''\|_{L^\infty(J_T)} \|P_\varepsilon\|_{L^\infty(J_T)} |K|.
\]
Therefore, Murat’s lemma [20] implies that
\[
(3.4) \quad \{\partial_t \eta(u_\varepsilon) + \partial_x \eta(u_\varepsilon)\}_{\varepsilon>0} \text{ lies in a compact subset of } H^{-1}_{loc}((0,T) \times (0,\infty)).
\]
The \(L^\infty\) bound stated in Lemma 2.7 (3.3), and the Tartar’s compensated compactness method [25] give the existence of a subsequence \(\{u_{\varepsilon_k}\}_{k \in \mathbb{N}}\) and a limit function \(u \in L^\infty((0,T) \times (0,\infty)), \ T > 0, \) such that (3.1) holds.
Let us prove that (3.2) holds true.
We show that
\[
(3.5) \quad \varepsilon \partial_x P_\varepsilon(t,x) \to 0 \text{ in } L^\infty(0,T;L^\infty(0,\infty)), \ T > 0.
\]
It follows from (2.24) that
\[
\varepsilon \|\partial_x P_\varepsilon\|_{L^\infty(0,T;L^\infty(0,\infty))} \leq \sqrt{\varepsilon} \sqrt{C_0 (e^{C_0 T} (1 + T) + 1)} = \sqrt{\varepsilon} C(T) \to 0,
\]
that is (3.3). Then, (2.32), (3.1), (3.5) and the Hölder inequality give (3.2).
Finally, we prove (1.14).
Let \(k \in \mathbb{N}, \ c \in \mathbb{R}\) be a constant, and \(\phi \in C^\infty(\mathbb{R}^2)\) be a nonnegative test function with compact support. Multiplying the first equation of (2.1) by \(\text{sign}(u_\varepsilon - c)\), we have
\[
\partial_t |u_{\varepsilon_k} - c| + \partial_x \left( \text{sign} (u_{\varepsilon_k} - c) \left( \frac{u_{\varepsilon_k}^2}{2} - \frac{c^2}{2} \right) - \gamma \text{sign} (u_{\varepsilon_k} - c) P_{\varepsilon_k} - \varepsilon_k \partial_{xx} |u_{\varepsilon_k} - c| \right) \leq 0.
\]
Multiplying by \(\phi\) and integrating over \((0,\infty)^2\), we get
\[
\int_0^\infty \int_0^\infty \left( |u_{\varepsilon_k} - c| \partial_t \phi + \left( \text{sign} (u_{\varepsilon_k} - c) \left( \frac{u_{\varepsilon_k}^2}{2} - \frac{c^2}{2} \right) \right) \partial_x \phi \right) dtdx
\]
\[
+ \gamma \int_0^\infty \int_0^\infty \text{sign} (u_{\varepsilon_k} - c) P_{\varepsilon_k} dtdx - \varepsilon_k \int_0^\infty \int_0^\infty \partial_x |u_{\varepsilon_k} - c| \partial_x \phi dtdx
\]
\[
+ \int_0^\infty |u_0(x) - c| \phi(0,x) dx + \int_0^\infty \text{sign} (g_{\varepsilon_k}(t) - c) \left( \frac{g_{\varepsilon_k}^2(t)}{2} - \frac{c^2}{2} \right) \phi(t,0) dt
\]
\[
- \varepsilon_k \int_0^\infty \partial_x |u_{\varepsilon_k}(t,0) - c| \phi(t,0) dt \geq 0.
\]
Since
\[
g_{\varepsilon_k}(t) \to g(t) \text{ in } W^{1,\infty}(0,\infty),
\]
thanks to Lemmas 2.4 2.6 and 2.7 when \( k \to \infty \), we have
\[
\int_0^\infty \int_0^\infty \left( |u - c| \partial_t \phi + \left( \text{sign} (u - c) \left( \frac{u^2}{2} - \frac{c^2}{2} \right) \right) \partial_x \phi \right) dt \, dx \\
+ \gamma \int_0^\infty \int_0^\infty \text{sign} (u - c) \, P \, dt \, dx + \int_0^\infty |u_0(x) - c| \phi(0,x) \, dx \\
- \lim_{\varepsilon_k \to 0} \varepsilon_k \int_0^\infty \partial_x |u_{\varepsilon_k}(t,0)| - c|\phi(t,0)| \, dt \\
= \int_0^\infty \text{sign} (g(t) - c) \left( \frac{g^2(t)}{2} - \frac{(u_0(t))^2}{2} \right) \phi(t,0) \, dt.
\]

We have to prove that (see \( \| \) )
\[
\lim_{\varepsilon_k} \varepsilon_k \int_0^\infty \partial_x |u_{\varepsilon_k}(t,0)| - c|\phi(t,0)| \, dt
\]
\[(3.6) = \int_0^\infty \text{sign} (g(t) - c) \left( \frac{g^2(t)}{2} - \frac{(u_0(t))^2}{2} \right) \phi(t,0) \, dt.
\]

Let \( \{ \rho_{\nu} \}_{\nu \in \mathbb{N}} \subset C^\infty (\mathbb{R}) \) be such that
\[
0 \leq \rho_{\nu} \leq 1, \quad \rho_{\nu}(0) = 1, \quad |\rho_{\nu}'| \leq 1, \quad x \geq \frac{1}{\nu} \implies \rho_{\nu}(x) = 0.
\]

Using \( (t,x) \to \rho_{\nu}(x) \phi(t,x) \) as test function for the first equation of (2.1) we get
\[
\int_0^\infty \int_0^\infty \left( u_{\varepsilon_k} \partial_t \phi_{\rho_{\nu}} + \frac{u_{\varepsilon_k}^2}{2} \partial_x \phi_{\rho_{\nu}} + \frac{u_{\varepsilon_k}^2}{2} \phi_{\rho_{\nu}}' \right) dt \, dx + \gamma \int_0^\infty \int_0^\infty P_{\varepsilon_k} \phi_{\rho_{\nu}} dt \, dx \\
- \varepsilon_k \int_0^\infty \int_0^\infty \partial_x u_{\varepsilon_k} \left( \partial_x \phi_{\rho_{\nu}} + \phi_{\rho_{\nu}}' \right) dt \, dx + \int_0^\infty u_0(x) \phi(0,x) \rho_{\nu}(x) \, dx \\
+ \int_0^\infty \frac{g_{\varepsilon_k}^2(t)}{2} \phi(t,0) \, dt - \varepsilon_k \int_0^\infty \partial_x u_{\varepsilon_k}(t,0) \phi(t,0) \, dt = 0.
\]

As \( k \to \infty \), we obtain that
\[
\int_0^\infty \int_0^\infty \left( u \partial_t \phi_{\rho_{\nu}} + \frac{u^2}{2} \partial_x \phi_{\rho_{\nu}} + \frac{u^2}{2} \phi_{\rho_{\nu}}' \right) dt \, dx + \gamma \int_0^\infty \int_0^\infty P \phi_{\rho_{\nu}} dt \, dx \\
+ \int_0^\infty u_0(x) \phi(0,x) \rho_{\nu} \, dx + \int_0^\infty \frac{g^2(t)}{2} \phi(t,0) \, dt \\
= \lim_{\varepsilon_k} \varepsilon_k \int_0^\infty \partial_x u_{\varepsilon_k}(t,0) \phi(t,0) \, dt.
\]

Sending \( \nu \to \infty \), we get
\[
\lim_{\varepsilon_k} \varepsilon_k \int_0^\infty \partial_x u_{\varepsilon_k}(t,0) \phi(t,0) \, dt = \int_0^\infty \left( \frac{g^2(t)}{2} - \frac{(u_0(t))^2}{2} \right) \phi(t,0) \, dt.
\]

Therefore, due to the strong convergence of \( g_{\varepsilon_k} \) and the continuity of \( g \) we have
\[
\lim_{\varepsilon_k} \varepsilon_k \int_0^\infty \partial_x |u_{\varepsilon_k}(t,0)| - c|\phi(t,0)| \, dt \\
= \lim_{\varepsilon_k} \int_0^\infty \partial_x u_{\varepsilon_k}(t,0) \text{sign} (u_{\varepsilon_k}(t,0) - c) \phi(t,0) \, dt \\
= \lim_{\varepsilon_k} \int_0^\infty \partial_x u_{\varepsilon_k}(t,0) \text{sign} (g_{\varepsilon_k}(t) - c) \phi(t,0) \, dt.
By arguing as in [1, 3, 7, 11], using the fact that the two solutions satisfy the same boundary conditions, we prove that

\[
\begin{align*}
\text{(3.9)}
\end{align*}
\]

\[
\int_0^\infty \text{sign} (g(t) - c) \left( \frac{g^2(t)}{2} - \frac{(u_0(t))^2}{2} \right) \phi(t,0) dt,
\]

that is (3.6).

**Proof of Theorem 1.1.** Lemma (3.2) gives the existence of entropy solution \(u(t,x)\) of (1.10), or equivalently (1.11).

Let us show that \(u(t,x)\) is unique, and that (1.15) holds true. Fixed \(T > 0\), since our solutions are bounded in \(L^\infty((0,T) \times \mathbb{R})\), we use the doubling of variables method.

Let \(u,v \in L^\infty((0,T) \times \mathbb{R})\) be two entropy solutions of (1.10), or equivalently of (1.11).

By arguing as in [1, 3, 7, 11], using the fact that the two solutions satisfy the same boundary conditions, we prove that

\[
\begin{align*}
\text{(3.10)}
\end{align*}
\]

\[
\partial_t(|u - v|) + \partial_x \left( \frac{u^2}{2} - \frac{v^2}{2} \right) \text{sign} (u - v) - \gamma \text{sign} (u - v) (P_u - P_v) \leq 0
\]

holds in sense of distributions in \((0,\infty) \times (0,\infty)\), where

\[
\begin{align*}
\text{(3.11)}
P_u(t,x) &= \int_0^x u(t,y) dy, \quad P_v = \int_0^x v(t,y) dy.
\end{align*}
\]

Let \(\phi(t,\tau,x,y) \in C^{\infty}(\mathbb{R}^4)\) be a non-negative test function such that \(\text{supp}(\phi) \subset (0,\infty)^4\). Since \(u,v\) are entropy solutions of (1.10), we have

\[
\begin{align*}
\text{(3.10)}
\end{align*}
\]

\[
\int_0^\infty \int_0^\infty |u(t,x) - v(\tau,y)| \partial_t \phi(t,\tau,x,y)
\]

\[
+ \left( \frac{u^2(t,x)}{2} - \frac{v^2(\tau,y)}{2} \right) \text{sign} (u(t,x) - v(\tau,y)) \cdot \partial_x \phi(t,\tau,x,y)
\]

\[
+ \gamma \text{sign} (u(t,x) - v(\tau,y)) P_u(t,x) \phi(t,\tau,x,y) dt dx \geq 0,
\]

\[
\int_0^\infty \int_0^\infty |v(\tau,y) - u(t,x)| \partial_t \phi(t,\tau,x,y)
\]

\[
+ \left( \frac{v^2(\tau,y)}{2} - \frac{u^2(t,x)}{2} \right) \text{sign} (v(\tau,y) - u(t,x)) \cdot \partial_x \phi(t,\tau,x,y)
\]

\[
+ \gamma \text{sign} (v(\tau,y) - u(t,x)) P_v(\tau,y) \phi(t,\tau,x,y) d\tau dy \geq 0.
\]

Integrating (3.10) with respect to \(\tau, y\), (3.11) with respect to \(t, x\), and adding these two results, we obtain

\[
\begin{align*}
\int_0^\infty \int_0^\infty \int_0^\infty \int_0^\infty |u(t,x) - v(\tau,y)| &| \partial_t \phi(t,\tau,x,y) + \partial_x \phi(t,\tau,x,y) |
\]

\[
+ \left( \frac{u^2(t,x)}{2} - \frac{v^2(\tau,y)}{2} \right) \text{sign} (u(t,x) - v(\tau,y)) \cdot \left( \partial_x \phi(t,\tau,x,y) + \partial_y \phi(t,\tau,x,y) \right)
\]

\[
+ \gamma \text{sign} (u(t,x) - v(\tau,y)) (P_u(t,x) - P_v(\tau,y)) \cdot \phi(t,\tau,x,y) dt d\tau dx dy \geq 0.
\]

Now, we choose a sequence of functions \(\{\delta_k\}_{k \geq 1}\), approximating the Dirac mass at the origin. More precisely, let \(\delta : \mathbb{R} \to [0,1]\) be a \(C^\infty\) function such that

\[
\int_{\mathbb{R}} \delta(z) dz = 1, \quad \delta(z) = 0, \quad \text{for all } z \notin [-1,1],
\]
We observe that (3.16) is positive if $c \not\in I$ where
\[ t \quad \text{since the maps} \quad \delta \quad \text{is (3.8)}. \]

Let us consider the following test function
\[ \phi_h(t, \tau, x, y) = \psi\left(\frac{t + \tau}{2}, \frac{x + y}{2}\right)\delta_h\left(\frac{\tau - t}{2}\right)\delta_h\left(\frac{y - x}{2}\right), \]
where $\psi \in C^\infty(\mathbb{R}^2)$ is a non-negative test function such that $\text{supp}(\psi) \subset (0, \infty)^2$.

Using (3.13) as test function in the previous inequality, we have
\[
\int_0^\infty \int_0^\infty \int_0^\infty \left\{ \delta_h\left(\frac{\tau - t}{2}\right)\delta_h\left(\frac{-x}{2}\right) [u(t, x) - v(t, y)] \partial_t \psi\left(\frac{t + \tau}{2}, \frac{x + y}{2}\right) \right. \\
+ \left(\frac{u^2(t, x)}{2} - \frac{v^2(t, y)}{2}\right) \text{sign} \left( u(t, x) - v(t, y) \right) \partial_x \psi\left(\frac{t + \tau}{2}, \frac{x + y}{2}\right) \\
+ \gamma \psi\left(\frac{t + \tau}{2}, \frac{x + y}{2}\right) \delta_h\left(\frac{\tau - t}{2}\right) \delta_h\left(\frac{y - x}{2}\right) \cdot \text{sign} \left( u(t, x) - v(t, y) \right) (P_u(t, x) - P_v(t, y)) \right\} dt d\tau dx dy \geq 0.
\]

We observe that $\delta_h \to \delta_0$ when $h \to 0$, where $\delta_0$ is Dirac mass centered in $\{0\}$. Therefore, since the maps $t \to u(t, \cdot)$, $t \to v(t, \cdot)$ are continuous from $[0, \infty)$ into $L^1_{\text{loc}}(0, \infty)$, and $t \to P_u(t, \cdot)$, $t \to P_v(t, \cdot)$ are continuous from $[0, \infty)$ into $L^\infty_{\text{loc}}(0, \infty)$, it follows from the previous inequality that
\[
\int_0^\infty \int_0^\infty (|u - v| \partial_t \psi + \left(\frac{u^2}{2} - \frac{v^2}{2}\right) \text{sign} \left( u - v \right) \partial_x \psi dt dx \\
+ \gamma \int_0^\infty \int_0^\infty \text{sign} \left( u - v \right) (P_u - P_v) \psi dt dx \geq 0,
\]
that is (3.18).

Let us show that (1.15) holds true. Since $u$ is an entropy solution of (1.10), then it satisfies the inequality (1.14). We write the boundary condition in this way (see [1]):

\[
\min_{c \in I(u_0^\tau(t), g(t))} \left\{ \text{sign} \left( u_0^\tau(t) - g(t) \right) \left( \frac{(u_0^\tau(t))^2}{2} - \frac{c^2}{2} \right) \right\} = 0,
\]
where $I(u_0^\tau(t), g(t))$ is the closed interval $[\min\{u_0^\tau(t), g(t)\}, \max\{u_0^\tau(t), g(t)\}]$.

Let us consider, now, the following product:
\[
\left(\frac{(u_0^\tau(t))^2}{2} - \frac{c^2}{2}\right) \text{sign} \left( u_0^\tau(t) - c \right) + \text{sign} (c), \quad c \in \mathbb{R}.
\]

We observe that (3.16) is positive if $c \not\in I(u_0^\tau(t), g(t))$. Instead, if we consider $c \in I(u_0^\tau(t), g(t))$, (3.16) coincides with (3.15). Therefore, for each $c \in \mathbb{R}$, we have that
\[
\left(\frac{(u_0^\tau(t))^2}{2} - \frac{c^2}{2}\right) \text{sign} \left( u_0^\tau(t) - c \right) + \text{sign} (c) \geq 0.
\]
Since (3.8) holds in the sense of distributions in \((0, \infty)^2\), we have that
\[
\int_0^\infty \int_0^\infty (|u - v| \partial_t \psi + \left( \frac{u^2}{2} - \frac{v^2}{2} \right) \text{sign} (u - v) \partial_x \psi) dt dx \\
+ \gamma \int_0^\infty \int_0^\infty \text{sign} (u - v) (P_u - P_v) \psi dt dx \\
\geq \int_0^\infty \text{sign} (u^*_0(t) - v^*_0(t)) \cdot
\left( \frac{(u^*_0(t))^2}{2} - \frac{(v^*_0(t))^2}{2} \right) \psi(t, 0) dt,
\]
where \(\psi \in C^\infty(\mathbb{R}^2)\) is a non-negative test function with compact support, and \(v^*_0(t)\) is the trace of \(v\) at \(x = 0\).

To determine the sign of the right-hand side of (3.18), for each \(t > 0\), we define the real number \(c(t)\) in the following way:
\[
c(t) = \begin{cases} 
    u^*_0(t) & \text{if } u^*_0(t) \in I(g(t), v^*_0(t)), \\
    g(t) & \text{if } g(t) \in I(u^*_0(t), v^*_0(t)), \\
    v^*_0(t) & \text{if } v^*_0(t) \in I(u^*_0(t), g(t)).
\end{cases}
\]
From (3.19), it follows that
\[
\text{sign} (u^*_0(t) - v^*_0(t)) \left( \frac{(u^*_0(t))^2}{2} - \frac{(v^*_0(t))^2}{2} \right) \\
= \text{sign} (u^*_0(t) - c(t)) \left( \frac{(u^*_0(t))^2}{2} - \frac{c^2(t)}{2} \right) \\
+ \text{sign} (v^*_0(t) - c(t)) \left( \frac{(v^*_0(t))^2}{2} - \frac{c^2(t)}{2} \right).
\]

For (3.17), we get that the right-hand side of (3.18) is non-negative. Therefore, we have (3.14).

Let \(T, R > 0\), and let us consider the sets
\[
\Omega := \{(t, x) \in [0, T] \times [-R, R]; \quad 0 \leq s \leq t, \quad |x| \leq R + C(T)(t - s)\},
\]
\[
\Omega^+ := \Omega \cap (0, \infty)^2,
\]
where
\[
C(T) = \sup_{(0, T) \times \mathbb{R}} \left\{|u| + |v|\right\}.
\]
We define the following test function
\[
\phi_h(t, x) = [\alpha_h(s) - \alpha_h(s - t)] [1 - \alpha_h(|x| - R + C(T)(t - s))] \geq 0,
\]
where \(\alpha_h(z)\) is defined in (3.12).

We observe that the function \([\alpha_h(s) - \alpha_h(s - t)] [1 - \alpha_h(|x| - R + C(T)(t - s))\]

is an approximation of the characteristic function of \(\Omega\). Moreover, since \(u\) and \(v\) are in \(L^\infty((0, T) \times \mathbb{R})\), we have that
\[
|u^2(t, x) - v^2(t, x)| \leq C(T)|u(t, x) - v(t, x)|, \quad (t, x) \in \Omega^+.
\]
From (3.12), \( \alpha_h' = \delta_h \geq 0 \). Using \( \phi_h \) as test function in (3.14), we have
\[
\int_0^\infty \int_0^\infty \{|u - v| - \delta_h(s) - \delta_h(s - t)|[1 - \alpha_h(|x| - R + C(T)(t - s))]
+ (\alpha_h(s) - \alpha_h(s - t))\delta_h(|x| - R + C(T)(t - s)).
\cdot \text{sign}(u - v) \left( \frac{u^2}{2} - \frac{v^2}{2} \right) \text{sign}(x) - C(T)|u - v|] \cdot \gamma \text{sign}(u - v) (P_u - P_v)
\cdot [\alpha_h(s) - \alpha_h(s - t)] \geq \alpha_h(|x| - R + C(T)(t - s))] \} dsdx \geq 0.
\]
Therefore, it follows from (3.22) and the previous inequality that
\[
\int_0^\infty \int_0^\infty \{|u - v| - \delta_h(s) - \delta_h(s - t)|[1 - \alpha_h(|x| - R + C(T)(t - s))]
- \gamma \text{sign}(u - v) (P_u - P_v)
\cdot [\alpha_h(s) - \alpha_h(s - t)] \geq \alpha_h(|x| - R + C(T)(t - s))\}
\cdot (C(T)|u - v| - \text{sign}(u - v) \left( \frac{u^2}{2} - \frac{v^2}{2} \right) \text{sign}(x)) dsdx \geq 0.
\]
Since
\[
\delta_h \to \delta_0,
\]
\[
[\alpha_h(s) - \alpha_h(s - t)] \to \chi_{[\alpha_0, \alpha_1]},
\]
when \( h \to 0 \), where \( \delta_0 \) is Dirac mass, the continuity of \( u(t, \cdot), v(t, \cdot) \) from \( [0, \infty) \) into \( L^1_{\text{loc}}(0, \infty) \), the continuity of \( P_u(t, \cdot), P_v(t, \cdot) \) from \( [0, \infty) \) into \( L^1_{\text{loc}}(0, \infty) \), and the previous inequality give
\[
\|u(t, \cdot) - v(t, \cdot)\|_{L^1(0, R)} \leq \|u_0 - v_0\|_{L^1(0, R + C(T)t)}
+ \gamma \int_{\Omega^+} \text{sign}(u - v) (P_u - P_v) dsdx
\]
\[
\leq \|u_0 - v_0\|_{L^1(0, R + C(T)t)}
+ \gamma \int_0^t \int_{I(s)} \text{sign}(u - v) (P_u - P_v) dsdx,
\]
where
\[
I(s) = [0, R + C(T)(t - s)].
\]
In particular, we have
\[
I(t) = [0, R], \quad I(0) = [0, R + C(T)t].
\]
Therefore, it follows from (3.23) that
\[
\|u(t, \cdot) - v(t, \cdot)\|_{I(t)} \leq \|u_0 - v_0\|_{I(0)}
+ \gamma \int_0^t \int_{I(s)} \text{sign}(u - v) (P_u - P_v) dsdx.
\]
We observe that, for (3.9),
\[ \gamma \int_{0}^{t} \int_{I(s)} \text{sign} (u - v) (P_{u} - P_{v}) dsdx \]
\[ \leq \gamma \int_{0}^{t} \int_{I(s)} |P_{u} - P_{v}| dsdx \]
(3.27)
\[ \leq \gamma \int_{0}^{t} \int_{I(s)} \left( \int_{0}^{x} |u - v| dy \right) dsdx \]
\[ \leq \gamma \int_{0}^{t} \int_{I(s)} \left( \int_{I(s)} |u - v| dy \right) dsdx \]
\[ = \gamma \int_{0}^{t} |I(s)| \| u(s, \cdot) - v(s, \cdot) \|_{L^{1}(I(s))} ds. \]

Thanks to (3.24), we have
(3.28)
\[ |I(s)| = R + C(T)(t - s) \leq R + C(T)t \leq R + C(T). \]

Let us consider the following continuous function:
(3.29)
\[ G(t) = \| u(t, \cdot) - v(t, \cdot) \|_{L^{1}(I(t))}, \quad t \geq 0. \]

Therefore, it follows from (3.26), (3.27), and (3.28) that
\[ G(t) \leq G(0) + C(T) \int_{0}^{t} G(s) ds. \]

Gronwall’s Lemma, (3.25), and (3.29) give
\[ \| u(t, \cdot) - v(t, \cdot) \|_{L^{1}(0, R)} \leq e^{C(T)t} \| u_{0} - v_{0} \|_{L^{1}(0, R+C(T)t)}, \]
that is (1.15). \qed

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