Estimates for the $L^q$-mixed problem in $C^{1,1}$-domains

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Abstract

We consider the $L^q$-mixed problem in domains in $\mathbb{R}^n$ with $C^{1,1}$-boundary. We assume that the boundary between the sets where we specify Neumann and Dirichlet data is Lipschitz. With these assumptions, we show that we may solve the $L^q$-mixed problem for $q$ in the range $1 < q < n/(n - 1)$.

1 Introduction

The goal of this note is to establish a regularity result for the $L^q$-mixed problem. Our work builds on an earlier result of Ott and Brown [7] which establishes existence and uniqueness for the $L^q$-mixed problem for $q$ near 1. In this paper, we consider a more restrictive class of domains than was considered in Ott and Brown, but we are able to give an explicit range of exponents $q$ for which we can solve the mixed problem. This range is easily seen to be sharp in two dimensions. The new ingredient in this work compared to Ott and Brown’s work is a result of Savaré [8]. Savaré’s result is a regularity result for solutions of the mixed problem in a Besov space. We use his result to prove a reverse Hölder inequality. This inequality then feeds into the machinery of Ott and Brown to obtain our main theorem.

We let $\Omega \subset \mathbb{R}^n$ be a bounded open set and suppose the boundary $\partial \Omega$ is partitioned into two sets $D$ and $N$. We assume that we are given functions $f_D$ and $f_N$ defined on $D$ and $N$, respectively. By the $L^q$-mixed problem, we mean the problem of finding a

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function $u$ which satisfies
\[
\begin{aligned}
\Delta u &= 0, & \text{in } \Omega \\
u &= f_D, & \text{on } D \\
\partial u \big/ \partial \nu &= f_N, & \text{on } N \\
\nabla u^* &\in L^q(\partial \Omega)
\end{aligned}
\]  
(1.1)

Our assumptions on the domain are below. In particular, our hypotheses will imply that the surface measure on $\partial \Omega$ is defined. We use $\nabla u^*$ to denote the non-tangential maximal function and this will be defined in section 2 below.

Our main result is the following theorem. See section 2 for definitions of several of the objects appearing in the theorem.

**Theorem 1.2.** Suppose that $\Omega \subset \mathbb{R}^n$, $N$, and $D$ is a standard $C^{1,1}$-domain for the mixed problem as defined in section 2. Suppose that $q \in (1, n/(n-1))$, that $f_N$ is in $L^q(N)$ and $f_D$ is in the Sobolev space $W^{1,q}(D)$. Under these assumptions there exists a unique solution of the $L^q$-mixed problem for the Laplacian (1.1) and the solution satisfies the estimate

$$
\lVert \nabla u^* \rVert_{L^q(\partial \Omega)} \leq C [\lVert f_N \rVert_{L^q(N)} + \lVert f_D \rVert_{W^{1,q}(D)}].
$$

We next recall a well-known example that shows that, at least in two dimensions, the range of exponents in Theorem 1.2 is sharp.

**Example.** We let $\Omega \subset \{(x_1, x_2) : x_2 > 0\}$ be a smooth domain with $[-1, 1] \times \{0\} \subset \partial \Omega$. We define $N$ and $D$ by $D = [0, 1] \times \{0\}$ and $N = \partial \Omega \setminus D$. Consider the function $u$ defined in polar coordinates by

$$
u(r, \theta) = r^{1/2} \cos(\theta/2).$$

(1.3)

The function $u$ will solve the mixed problem in $\Omega$ with $f_N$ bounded and $f_D = 0$. However, we have $|\nabla u(x)| = c|x|^{-1/2}$ and thus we have $\nabla u^* \in L^q(\partial \Omega)$ precisely if $q < 2$.  

2 Definitions and preliminary results

In this section we give the main definitions used in the statement and proof of our main result. We begin by defining the domains we will use. We assume that $\Omega \subset \mathbb{R}^n$ is a bounded, connected, and open set and that the boundary is $C^{1,1}$. This will mean that there exists $r_0$ and $M$ so that for each $x \in \partial \Omega$ we may find coordinates $(y', y_n) \in \mathbb{R} \times \mathbb{R}^{n-2} \times \mathbb{R}$ and a $C^{1,1}$-function $\phi : \mathbb{R}^{n-1} \to \mathbb{R}$ so that we have

\[
\begin{aligned}
\Omega \cap B_{100r_0}(x) &= \{(y', y_n) : y_n > \phi(y')\} \cap B_{100r_0}(x) \\
\partial \Omega \cap B_{100r_0}(x) &= \{(y', y_n) : y_n = \phi(y')\} \cap B_{100r_0}(x).
\end{aligned}
\]  
(2.1)
Here, we are using $B_r(x)$ to denote a ball with radius $r$ and center $x$. To prove our regularity result, we will need to impose conditions on the boundary between $D$ and $N$. We let $\Lambda$ denote the boundary of $D$ relative to $\partial \Omega$ and for each $x \in \Lambda$, we assume that with the coordinate system and $\phi$ as above, we also have a Lipschitz function $\psi : \mathbb{R}^{n-2} \to \mathbb{R}$ so that

$$D \cap B_{100r_0}(x) = \{(y_1, y'' : y_n) \in \mathbb{R} \times \mathbb{R}^{n-2} \times \mathbb{R} : y_n = \phi(y'), y_1 \geq \psi(y'')\} \cap B_{100r_0}(x)$$

$$N \cap B_{100r_0}(x) = \{(y_1, y'' : y_n) \in \mathbb{R} \times \mathbb{R}^{n-2} \times \mathbb{R} : y_n = \phi(y'), y_1 < \psi(y'')\} \cap B_{100r_0}(x).$$

In both (2.1) and (2.2), we require that the coordinate system be a rigid motion of the standard coordinates on $\mathbb{R}^n$ and that the functions $\phi$ and $\psi$ satisfy the conditions

$$\|\nabla \phi\|_{L^\infty(\mathbb{R}^{n-1})} + r_0 \|\nabla^2 \phi\|_{L^\infty(\mathbb{R}^{n-1})} \leq M, \quad \|\nabla \psi\|_{L^\infty(\mathbb{R}^{n-2})} \leq M. \quad (2.3)$$

We will call $\Omega$, $N$, and $D$ a standard $C^{1,1}$-domain for the mixed problem. We will use $r_0$ as a characteristic length for the domain. Our goal is provide results which are scale-invariant and this is the reason for the appearance of $r_0$ in (2.3).

Next, we define Sobolev spaces on $\Omega$. For $1 \leq p < \infty$, we let $W^{1,p}(\Omega)$ be the standard Sobolev space of functions with one derivative in $L^p(\Omega)$. For $D \subset \partial \Omega$ and $p < \infty$, we define $W^{1,p}_D(\Omega)$ to be the closure of $C^\infty_D(\Omega)$ in $W^{1,p}(\Omega)$. Here, $C^\infty_D(\Omega)$ denotes the collection of functions in $C^\infty(\Omega)$ which vanish on a neighborhood of $D$, the closure of $D$. In order to make our estimates scale-invariant, we will define the norm on $W^{1,p}_D(\Omega)$ as

$$\|u\|_{W^{1,p}(\Omega)} = \left(\int_\Omega |\nabla u|^p + r_0^{-p}|u|^p \, dx\right)^{1/p}.$$ 

We will use $W^{-1,2}_D(\Omega)$ to denote the dual of the space $W^{1,2}_D(\Omega)$.

We will use $W^{1,q}_D(D)$ to denote the Sobolev space of functions in $L^q(D)$ which also have one derivative in $L^q(D)$. This space will be defined as the restriction of $W^{1,q}(\partial \Omega)$ to the closed set $D$. In order to make our estimates scale correctly, we will use the norm

$$\|u\|_{L^q_D(D)} = \left(\int_D |\nabla_{\text{tan}} u|^q + r_0^{-q}|u|^q \, d\sigma\right)^{1/q}.$$ 

See [7] p. 1337 or [12] p. 580 for a definition of this space and the tangential gradient, $\nabla_{\text{tan}}$.

We let $W^{1/2,2}(\partial \Omega)$ denote the image of $W^{1,2}(\Omega)$ under the trace map. Similarly, $W^{1/2,2}_D(\partial \Omega)$ will denote the image of $W^{1,2}_D(\Omega)$ under the trace map. We denote the dual of $W^{1/2,2}_D(\partial \Omega)$ by $W^{-1/2,2}_D(\partial \Omega)$. The space $W^{-1/2,2}_D(\partial \Omega)$ is a natural space for Neumann data for the weak mixed problem. The Dirichlet data in the mixed problem will be the restriction to $D$ of an element in $W^{1/2,2}_D(\partial \Omega)$.

While our main result is for the Laplacian, at one point in the argument of Savaré it is convenient to flatten the boundary using a $C^{1,1}$-diffeomorphism. Pulling back
a harmonic function in \( \Omega \) to a domain with a flat boundary will produce a function which solves an elliptic equation with Lipschitz coefficients.

We let \( L \) denote an operator \( L = \text{div} A \nabla \) where the coefficient matrix \( A \) is symmetric, Lipschitz, and elliptic. We will have the quantitative assumptions

\[
|A(x) - A(y)| \leq M|x - y|, \quad \text{for all } x, y \in \Omega,
\]

\[
M^{-1}|\xi|^2 \leq A(x)\xi \cdot \xi \leq M|\xi|^2, \quad \xi \in \mathbb{R}^n.
\]

We will consider the problem

\[
\begin{aligned}
\text{div } A \nabla u &= F, & \quad & \text{in } \Omega, \\
A \nabla u \cdot \nu &= f_N, & \quad & \text{on } N, \\
u &= f_D, & \quad & \text{on } D.
\end{aligned}
\]

We will generally work with the weak formulation of the problem (2.6) which is

\[
\begin{cases}
\int_{\Omega} A \nabla u \cdot \nabla \phi \, dx + \langle F, \phi \rangle_{\Omega} = \langle f_N, \phi \rangle_{\partial \Omega}, & \phi \in W^{1,2}_D(\Omega) \\
u - f_D \in W^{1,2}_D(\Omega).
\end{cases}
\] (2.7)

In the statement (2.7), we are using \( f_D \) to denote a function in \( W^{1,2}(\Omega) \) as well as the boundary values in \( W^{1/2,2}(\partial \Omega) \). The forcing term \( F \) will lie in \( W^{-1,2}_D(\Omega) \) and the Neumann data will come from \( W^{-1/2,2}_D(\partial \Omega) \). We use \( \langle \cdot, \cdot \rangle_{\Omega} \) to denote the pairing between \( W^{-1,2}_D(\Omega) \times W^{1,2}_D(\Omega) \to \mathbb{R} \) and \( \langle \cdot, \cdot \rangle_{\partial \Omega} \) for the pairing \( W^{-1/2,2}_D(\partial \Omega) \times \mathbb{R} \to W^{1/2,2}_D(\partial \Omega) \to \mathbb{R} \). We observe that we have the Sobolev embedding of \( W^{1/2,2}_D(\partial \Omega) \) into \( L^{2(n-1)/(n-2)}(\partial \Omega) \) when \( n \geq 3 \). Thus, passing to the dual implies \( L^p(N) \subset W^{-1/2,2}_D(\partial \Omega) \) if \( p \geq 2(n-1)/n \). When \( n = 2 \), we have that \( 2(n-1)/(n-2) = \infty \) and the embedding of \( W^{1/2,2}_D(\partial \Omega) \) into \( L^{\infty}(\partial \Omega) \) fails. However, we do have the embedding \( W^{1/2,2}_D(\partial \Omega) \subset L^p(\partial \Omega) \) for \( p < \infty \) and thus we still obtain the embedding \( L^p(N) \subset W^{-1/2,2}_D(\partial \Omega) \) for \( p > 1 \).

To estimate solutions of the mixed problem when \( f_N \) comes from \( L^p(N) \), we will use the non-tangential maximal function. For a function \( u \) on \( \Omega \) taking values in \( \mathbb{R}^d \), we define the non-tangential maximal function \( u^\ast \) by

\[
u^\ast(x) = \sup_{y \in \Gamma(x)} |u(y)|.
\]

In this definition, \( \Gamma(x) \) is a non-tangential approach region defined by

\[
\Gamma(x) = \{ y : |x - y| < (1 + \alpha) \text{dist}(y, \partial \Omega) \}, \quad x \in \partial \Omega
\]

where \( \alpha > 0 \) is fixed. While \( u^\ast \) depends on \( \alpha \), the \( L^p \)-norms of non-tangential maximal functions defined using different values of \( \alpha \) will be comparable. Thus we suppress the value of \( \alpha \) in our notation.
Our main argument will consider a number of local estimates. For these estimates, we will use surface balls \( \Delta_r(x) = B_r(x) \cap \partial \Omega \). We will also need local domains \( \Omega_r(x) = \Omega \cap B_r(x) \). Both objects will be defined for \( x \in \partial \Omega \) and \( 0 < r < r_0 \). In our estimates, we will allow the constants to depend on \( M \), the constant which appears in our definition of the domain, and the \( L^q \)-exponents.

### 3 A reverse Hölder inequality at the boundary

The new ingredient in this work as compared to the earlier work of Ott and Brown [7] is a reverse Hölder inequality with a larger range of exponents. This inequality follows from a regularity estimate of Savaré [8] for the mixed problem. The example in (1.3) shows that Savaré’s regularity result is sharp in the scale of Sobolev spaces when \( n = 2 \). In addition, it shows that the upper bound on the exponent for the mixed problem is sharp as well.

To prove a local estimate, it will be helpful to have a definition of a weak solution with boundary data specified on only part of the boundary. Thus, if \( \Omega \) is a domain and \( D, N \subset \partial \Omega \) are a decomposition of the boundary, we say that \( u \) solves the local mixed problem

\[
\begin{aligned}
\text{div} A \nabla u &= 0, & \text{in } \Omega_r(x) \\
u &= 0, & \text{on } D \cap \Omega_r(x) \\
\frac{\partial u}{\partial \nu} &= f_N, & \text{on } N \cap \Omega_r(x)
\end{aligned}
\]

if

\[
\int_\Omega A \nabla u \cdot \nabla \phi \, dy = \langle f_N, \phi \rangle_{\partial \Omega}, \quad \phi \in W^{1,2}_D(\Omega_r(x))
\]

where \( D' = (D \cap B_r(x)) \cup (\partial B_r(x) \cap \Omega) \). Note that if \( \phi \in W^{1,2}_D(\Omega_r(x)) \), then Following Stein [10, p. 152], we introduce the Besov spaces on \( \mathbb{R}^n \), \( B^s_{p,q} \), \( 1 \leq p, q \leq \infty \) and \( 0 < s < 1 \). We let

\[
\Delta_h(x) = u(x + h) - u(x), \quad \text{for } x, h \in \mathbb{R}^n.
\]

The norm for \( B^s_{p,q} \) is defined by

\[
\|u\|_{B^s_{p,q}} = \|u\|_{L^p} + \left( \int_{\mathbb{R}^n} \left( \frac{\|\Delta_h u\|_{L^p}}{|h|^s} \right)^q \, dh \right)^{1/q}
\]

for \( q < \infty \). When \( q = \infty \), we set

\[
\|u\|_{B^s_{p,\infty}} = \|u\|_{L^p} + \sup_{h \in \mathbb{R}^n} \frac{\|\Delta_h u\|_{L^p}}{|h|^s}
\]

For a domain \( \Omega \), we let \( B^s_{p,q}(\Omega) \) be the image of \( B^s_{p,q}(\mathbb{R}^n) \) under the restriction map, \( u \to u|_\Omega \).
We will localize a solution to a neighborhood of a point on the boundary and apply a change of variables to obtain a problem in a half-space. It will be an important point that we have uniform estimates for the family of problems that arise from this procedure. We will use $M$ in the quantitative estimates for the inputs to these problems and obtain estimates which depend on the problem through $M$.

We let $A(x)$ be a symmetric matrix which satisfies the Lipschitz and ellipticity conditions, (2.4) and (2.5). We let $\psi : \mathbb{R}^{n-2} \to \mathbb{R}$ be a Lipschitz function with constant $M$ and assume that

$$D = \{(x_1, x'', 0) : x_1 \geq \psi(x'')\}$$

$$N = \{(x_1, x'', 0) : x_1 < \psi(x'')\}.$$

With $A$, $N$, and $D$ as above, we consider the mixed problem on $\mathbb{R}^n_+$,

$$\begin{cases}
\text{div } A \nabla u - u = F, & \text{in } \mathbb{R}^n_+ \\
u = 0, & \text{on } D \\
A \nabla u \cdot \nu = f_N, & \text{on } N
\end{cases} \quad (3.2)$$

and recall a regularity result for this problem.

**Theorem 3.3 (Savaré).** Let $u$ solve (3.2), then we have a constant $C$ so that

$$\|\nabla u\|_{B^{1/2}_\infty(\mathbb{R}^n_+)} \leq C[\|F\|_{L^2(\mathbb{R}^n_+)} + \|f_N\|_{W^{1/2,2}_D(\mathbb{R}^{n-1})}].$$

The constant in this estimate depends only on $M$ and the dimension $n$.

Note that in this theorem we are assuming that $f_N \in W^{1/2,2}_D(\mathbb{R}^{n-1})$. The function $f_N$ defines an element of $W^{-1/2,2}_D(\mathbb{R}^{n-1})$ by

$$\phi \to \int_N f_N \phi d\sigma.$$

Theorem 3.3 is a small extension of the result stated by Savaré [8]. The difference is that we allow a more general separation between $D$ and $N$. Our condition that $\psi$ is Lipschitz implies that we have that $h + D \subset D$ for $h$ in an open cone in $\mathbb{R}^{n-1} \times \{0\}$. Since a cone in $\mathbb{R}^{n-1} \times \{0\}$, contains a basis of $\mathbb{R}^{n-1} \times \{0\}$ we are able to carry through the argument on page 882 of Savaré’s work [8]. The dissertation of Croyle [3, p. 16] provides more details.

We are now ready to state the reverse Hölder inequality.

**Theorem 3.4.** Suppose $\Omega$, $N$, and $D$ is a standard $C^{1,1}$-domain for the mixed problem. Fix $0 < r < r_0$ and let $u$ be a solution to

$$\begin{cases}
\Delta u = 0, & \text{in } \Omega_{2r}(x) \\
u = 0, & \text{on } D \cap \Omega_{2r}(x) \\
\frac{\partial u}{\partial \nu} = 0, & \text{on } N \cap \bar{\Omega}_{2r}(x)
\end{cases}$$
as in (3.1).

Then for $1 < p < 2n/(n - 1)$, we have

$$\left(\int_{\Omega_r(x)} |\nabla u|^p \, dy\right)^{1/p} \leq C \int_{\Omega_r(x)} |\nabla u| \, dy.$$ 

The constant in this estimate depends on $M$ and $p$.

**Proof.** We may rescale and translate the coordinates so that $r = 1$ and $x = 0$. We choose a cutoff function $\eta$ which is supported in $B_2(0)$ and is equal to one on $B_1(0)$. We assume that $\partial \Omega = \{y : y_n = \phi(y')\}$ in a neighborhood of $B_2(0)$ and let $\Phi(y) = (y', \phi(y') + y_n)$ map a neighborhood of 0 in $\mathbb{R}^n_+$ onto $B_2(0) \cap \Omega = \Omega_2(0)$. Furthermore, we set $\tilde{N} = (\mathbb{R}^{n-1} \times \{0\}) \cap \{y : y_1 < \psi(y'')\}$ and $\tilde{D} = (\mathbb{R}^{n-1} \times \{0\}) \cap \{y : y_1 \geq \psi(y'')\}$.

We define $v$ by

$$v(y) = (\eta(u - \bar{u})) \circ \Phi(y)$$

where

$$\bar{u} = \begin{cases} 0, & \text{if } B_2(0) \cap D \neq \emptyset \\ \int_{B_2(0) \cap \Omega} u \, dy, & \text{if } B_2(0) \cap D = \emptyset. \end{cases}$$

With this definition, we have the following variant of the Poincaré inequality (see the dissertation of Croyle [3, pp. 38–39]). There exists a constant $C$ so that

$$\|u - \bar{u}\|_{L^2(\Omega_2(0) \cap \Omega)} \leq C \|
abla u\|_{L^2(\Omega_4(0) \cap \Omega)}, \quad u \in W^{1,2}_D(B_4(0) \cap \Omega) \quad (3.5)$$

and that $v \in W^{1,2}_D(\mathbb{R}^n_+)$. Furthermore, $v$ satisfies the mixed boundary value problem on $\mathbb{R}^n_+$,

$$\begin{cases} \text{div } A \nabla v - v = G, & \text{in } \mathbb{R}^n_+ \\ v = 0, & \text{on } \tilde{D} \\ A \nabla v \cdot \nu = g_N, & \text{on } \tilde{N} \end{cases}$$

for some $G \in L^2(\mathbb{R}^n_+)$ and $g_N \in W^{1/2,2}(\mathbb{R}^{n-1})$. A tedious calculation gives that

$$\|G\|_{L^2(\mathbb{R}^n_+)} \leq C \|
abla u\|_{L^2(\Omega_4(0))}.$$ 

This estimate makes use of the Poincaré inequality (3.5). The Neumann data $g_N$ is given by

$$g_N(y) = ((u - \bar{u}) \frac{\partial \eta}{\partial \nu} \sqrt{1 + |\nabla \phi|^2}) \circ \Phi.$$ 

Thus $g_N$ satisfies

$$\|g_N\|_{W^{1/2,2}(\mathbb{R}^{n-1})} \leq C \|
abla u\|_{L^2(\Omega_4(0))}$$

where have used trace theorem and the Poincaré inequality (3.5).
Since we assume that $0 < r < r_0$, we may assume a uniform bound on the $C^{1,1}$-norm of $\phi$. Hence we may apply Theorem 3.3 to estimate \( \nabla v \) in $B_{2,\infty}^{1/2}(\mathbb{R}^n_+)$ and then a Sobolev embedding theorem for Besov spaces to conclude
\[
\left( \frac{1}{\Omega t} |\nabla v|_{t}^{p} \right)^{1/p} \leq C \left( \frac{1}{\Omega t} |\nabla u|^{2}_{t} \right)^{1/2}, \quad 1 \leq p < 2n/(n - 1).
\]
Finally, a change of variable leads to the estimate
\[
\left( \frac{1}{\Omega t} |\nabla u|_{t}^{p} \right)^{1/p} \leq \left( \frac{1}{\Omega t} |\nabla u|^{2}_{t} \right)^{1/2}.
\]
From here, the techniques found in Giaquinta [6, pp. 80–82], for example, allow us to establish the inequality with an $L^1$-average on the left-hand side.

### 4 Proof of Theorem 1.2

We first observe that it is known that we may solve the Dirichlet problem with data in $W^{1,q}(\partial \Omega)$ (commonly known as the regularity problem) and obtain non-tangential maximal function estimates for the gradient for a larger range of indices than we are considering here. This will allow us to reduce to the case when $f_D = 0$. See Dahlberg and Kenig [4] where results are given for $n \geq 3$ and $1 < q < 2 + \epsilon$. However, the result for $C^{1,1}$-domains is much easier and is covered by the results for $C^{1}$-domains of Fabes, Jodeit, and Riviére [5] as well as classical results such as Kellogg.

Thus, we restrict our attention to the case $f_D = 0$. The proof of our main theorem in this case relies on a real-variable technique of Caffarelli and Peral [1] which Shen [9, Theorem 3.2] adapted to the study of boundary value problems. We quote the result of Shen that is a key part of our argument.

**Theorem 4.1** ([9, Theorem 3.2]). Let $Q_0$ be a cube in $\mathbb{R}^n$ and $F \in L^{q_0}(2Q_0)$. Suppose that $q_0 < q < q_1$ and $f \in L^{q_0}(2Q_0)$. For each subcube $Q$ with $Q \subset Q_0$ and $|Q| < \beta |Q_0|$, there exist functions $F_Q$ and $R_Q$ on $Q$ so that

\[
|F| \leq C(|F_Q| + |R_Q|) \quad (4.2)
\]
\[
\left( \int_{Q} |R_Q|^{q_1} \right)^{1/q_1} \leq C \left[ \left( \int_{2Q} |F|^{q_0} \, dx \right)^{1/q_0} + \sup_{Q' \supset Q} \left( \int_{Q'} |f|^q \, dx \right)^{1/q_0} \right] \quad (4.3)
\]
\[
\left( \int_{Q} |F_Q|^{q_0} \, dy \right)^{1/q_0} \leq C \sup_{Q' \supset Q} \left( \int_{Q'} |f|^{q_0} \, dy \right)^{1/q_0}. \quad (4.4)
\]

With these assumptions, we have
\[
\left( \int_{Q_0} |F|^q \, dy \right)^{1/q} \leq C \left[ \left( \int_{2Q_0} |F|^{q_0} \, dy \right)^{1/q_0} + \left( \int_{2Q_0} |f|^q \, dy \right)^{1/q} \right].
\]
Here, \( \beta < 1 \) and the constants \( C \) in (4.2) are independent of \( f \) and \( Q \).

To apply this result, we will need to work on a set in \( \partial \Omega \) which can be mapped to a cube in \( \mathbb{R}^{n-1} \). We will call these sets surface cubes and give a precise definition. We recall our covering of \( \partial \Omega \) by balls as in (2.1). If we fix a ball \( B = B_{r_0}(x) \) so that \( \partial \Omega \) is given by the graph of \( \phi \) in \( B \), we define a surface cube to be the image of a cube in \( \mathbb{R}^{n-1} \) under the map \( x' \rightarrow (x', \phi(x')) \). We also may define dilations of boundary cubes \( rQ \) (at least for \( r \) small) by dilating the cube in \( \mathbb{R}^{n-1} \).

Our next step towards applying Theorem 4.1 is the following reverse Hölder inequality at the boundary.

**Lemma 4.5.** Let \( \Omega, N \) and \( D \) be a standard \( C^{1,1} \)-domain for the mixed problem. Let \( u \) be a weak solution of the mixed problem (2.7) in \( \Omega \). Assume that \( \nabla u^* \in L^{q_0}(\partial \Omega) \) for some \( q_0 > 1 \).

If \( u = 0 \) on \( D \cap B_{2r}(x) \) and \( \partial u/\partial \nu = 0 \) on \( N \cap B_{2r}(x) \), then for \( q \) with \( 1 < q < n/(n-1) \), we have

\[
\left( \frac{1}{\Delta_r(x)} \int |\nabla u|^q \, d\sigma \right)^{1/q} \leq C \left( \int_{\Omega_{2r}} |\nabla u|^p \, dy \right)^{1/p}.
\]

**Proof.** Given \( q \) with \( 1 < q < n/(n-1) \), we may choose \( s \in (0,1) \), but close to 1, so that \( p = (1 + s)q/s \) satisfies \( 2 < p < 2n/(n-1) \). We fix \( x \in \partial \Omega \) and \( r > 0 \) and let \( \Delta_r = \Delta_r(x) \). We begin by showing

\[
\left( \frac{1}{\Delta_r} \int |\nabla u|^q \, d\sigma \right)^{1/q} \leq C \left( \frac{1}{\Omega_{2r}} \int |\nabla u|^p \, dy \right)^{1/p}.
\]

The first step to proving (4.6) is to use Hölder’s inequality to obtain that

\[
\left( \frac{1}{\Delta_r} \int |\nabla u|^q \, d\sigma \right)^{1/q} \leq C \left( \frac{1}{\Delta_r} \int |\nabla u|^2 \delta^\alpha \, d\sigma \right)^{1/2} \left( \frac{1}{\Delta_r} \int \delta^{-\alpha q/(2-q)} \, d\sigma \right)^{1/q-1/2}
\]

where \( \alpha \) is chosen so that \( \alpha q/(2-q) = s \). Next, we use estimate of Ott and Brown [7 Lemma 4.9], that \( \partial u/\partial \nu = 0 \) on \( \Delta_r \cap N \) and then Hölder’s inequality to obtain

\[
\left( \int_{\Delta_r} |\nabla u|^2 \delta^\alpha \, d\sigma \right)^{1/2} \leq C r^{1/2} \left( \int_{\Omega_{2r}} |\nabla u|^2 \delta^{\alpha-1} \, dy \right)^{1/2}
\]

\[
\leq C r^{1/2} \left( \int_{\Omega_{2r}} |\nabla u|^p \, dy \right)^{1/p} \cdot \left( \int_{\Omega_{2r}} \delta^{(\alpha-1)p/(p-2)} \, dy \right)^{1/2-1/p}.
\]

Using our definitions of \( p, q, \) and \( \alpha \), a calculation gives that \( (1 - \alpha)p/(p-2) = 1 + s \). Thus, we arrive at the estimate

\[
\left( \frac{1}{\Delta_r} \int |\nabla u|^q \, d\sigma \right)^{1/q} \leq C r^{1/2} \left( \int_{\Delta_r} \delta^{-s} \, d\sigma \right)^{1/q-1/2}
\]

\[
\times \left( \int_{\Omega_{2r}} \delta^{-s-1} \, dy \right)^{1/2-1/p} \left( \int_{\Omega_{2r}} |\nabla u|^p \, dy \right)^{1/p}.
\]

(4.7)
From Lemmata 2.4 and 2.5 in Taylor et al. [11] we have
\[
\int_{\Delta_r(x)} \delta^{-a} \, d\sigma \approx \max(r, \delta(x))^{-a}, \quad a < 1.
\]
\[
\int_{\Omega_r(x)} \delta^{-b} \, d\sigma \approx \max(r, \delta(x))^{-b}, \quad b < 2.
\]
Recalling that \(s/q - (1 + s)/p = 0\), we have
\[
\frac{r^{1/2}}{a} \left( \int_{\Delta_r} \delta^{-a} \, d\sigma \right)^{1/q - 1/2} \left( \int_{\Omega_{2r}} \delta^{-s} \, d\sigma \right)^{1/2 - 1/p} \approx r^{1/2} \max(r, \delta(x))^{-1/2} \leq C.
\]
Using this, (4.7) and Theorem 3.4 we obtain the conclusion of the Lemma, except with \(\Omega_{4r}(x)\) on the right. A simple covering argument allows us to obtain the result as stated.

Before continuing, we introduce several truncated maximal functions. One appears in the next Lemma and the remaining ones will be needed the proof of our main theorem. The use of these auxiliary functions is needed to repair an error in the work of Ott and Brown. The estimate (7.4) of [7] is not correct. A correction is being prepared which uses a version of the argument presented below. Thus, the results of Brown and Ott are correct.

We fix a small constant \(c\) and a parameter \(r > 0\). In applications, the value of \(r\) will be clear from the context. The truncated non-tangential maximal functions are defined by
\[
u^{\triangle}(x) = \sup_{y \in \Gamma(x), |x-y| > cr} |u(y)|, \quad \nu^{\nabla}(x) = \sup_{y \in \Gamma(x), |x-y| < cr} |u(y)|. \tag{4.8}
\]
We will also need to introduce the Hardy-Littlewood maximal function on \(\partial\Omega\) which we define as
\[
M(f)(x) = \sup_{s > 0} \int_{\Delta_s(x)} |f| \, d\sigma.
\]
In analogy with the truncated non-tangential maximal functions defined in (4.8), we will also define runcated versions of the Hardy-Littlewood maximal function using the parameter \(r\). The truncated maximal functions are defined by:
\[
M_{0}(f)(x) = \sup_{0 < s < r} \int_{\Delta_s(x)} |f| \, d\sigma \quad M_{\infty}(f)(x) = \sup_{r \leq s} \int_{\Delta_s(x)} |f| \, d\sigma.
\]

The next Lemma gives the value of \(c\) that we will use in (4.8).

**Lemma 4.9.** Suppose that \(u\) is a local solution of the mixed problem
\[
\begin{cases}
\Delta u = 0, & \text{in } \Omega_{2r}(x) \\
u = 0, & \text{on } D \cap B_{2r}(x) \\
\frac{\partial u}{\partial \nu} = 0, & \text{on } N \cap B_{2r}(x)
\end{cases}
\]
Then given \( q \) in \((1, \infty)\), there exists \( c > 0 \) so that with \( \nabla u^\ast \) as in (4.8) we have

\[
\left( \int_{\Delta_r(x)} (\nabla u^\ast)^q \, d\sigma \right)^{1/q} \leq C \int_{\Delta_{2r}(x)} \nabla u^\ast \, dy.
\]

**Proof.** We establish a representation formula for \( \nabla u \) and apply the result of Coifman, McIntosh and Meyer [2] as in the work of Ott and Brown [7, Section 6] to conclude that

\[
\left( \int_{\Delta_r(x)} (\nabla u^\ast)^q \, d\sigma \right)^{1/q} \leq C \left[ \int_{\Omega_{2r}(x)} |\nabla u| \, dy + \left( \int_{\Delta_{2r}(x)} |\nabla u|^q \, d\sigma \right)^{1/q} \right]. \tag{4.10}
\]

We may use Lemma 4.5 to bound the second term on the right of (4.10) and a standard argument gives that there is a constant \( C \) so that

\[
\int_{\Omega_{2r}(x)} |\nabla u| \, dy \leq C r \int_{\Delta_{C_r}(x)} \nabla u^\ast \, d\sigma. \tag{4.11}
\]

Combining (4.10) and (4.11), we obtain the desired result with \( \Delta_{C_r}(x) \) rather than \(\Delta_{2r}(x)\) on the right. We may obtain the stated result by a simple covering argument. This may require us to decrease the value of the constant \( c \) used in the definition of \( \nabla u^\ast \).

We now give two Lemma related to the truncated maximal functions.

**Lemma 4.12.** Suppose that \( x, y \) are in \( \partial\Omega \) and \( |x - y| < Ar \), then we have

\[
M_\infty(f)(x) \leq C_A M_\infty(f)(y).
\]

**Proof.** By the triangle inequality, we have \( \Delta_s(x) \subset \Delta_{s+Ar}(y) \). Thus it follows that

\[
\int_{\Delta_s(x)} |f| \, d\sigma \leq \frac{\sigma(\Delta_{s+Ar}(y))}{\sigma(\Delta_s(x))} \int_{\Delta_{s+Ar}(y)} |f| \, d\sigma.
\]

If we require that \( s \geq r \), then we have a constant so that \( \sigma(\Delta_{s+Ar}(y))/\sigma(\Delta_s(x)) \leq C_A \) which gives the Lemma.

**Lemma 4.13.** We have

\[
u^\hat{\ast}(x) \leq CM_\infty(u^\ast)(x).
\]

The constant depends on the value of \( c \) entering into the definition of \( u^\ast \).

**Proof.** Fix \( x \in \partial\Omega \) and suppose that \( y \in \Gamma(x) \). Fix \( \hat{y} \) so that \( |y - \hat{y}| = d(y) = \text{dist}(y, \partial\Omega) \) and observe that if \( |z - \hat{y}| < \alpha d(y) \), we have \( y \in \Gamma(z) \). This implies
\[ |u(y)| \leq u^*(z) \text{ for } z \in \Delta_{ad(y)}(\hat{y}). \] By the triangle inequality \[ |x - \hat{y}| \leq |x - y| + |y - \hat{y}| \leq (2 + \alpha) d(y). \] Hence we have that \[ \Delta_{ad(y)}(\hat{y}) \subset \Delta_{(2+2\alpha)d(y)}(x). \] It follows that
\[ |u(y)| \leq \frac{\sigma(\Delta_{(2+2\alpha)d(y)}(x))}{\sigma(\Delta_{ad(y)}(\hat{y}))} \int_{\Delta_{(2+2\alpha)d(y)}(x)} u^* \, d\sigma. \]

If, in addition, we assume that \[ |x - y| > cr, \] then we will have \[ d(y) > cr/(1 + \alpha) \] and we obtain
\[ u(x) \leq CM_{\infty}(u^*)(x). \]

**Proof of Theorem 1.2.** We fix \( f_N \in L^q(N) \) with \( 1 < q < n/(n - 1) \) and let \( u \) the solution of
\[
\begin{cases}
\Delta u = 0, & \text{in } \Omega \\
u = 0, & \text{on } D \\
\frac{\partial u}{\partial \nu} = f_N, & \text{on } N \\
\nabla u^* \in L^{q_0}(\partial \Omega)
\end{cases}
\]

According to Theorem 1.2 of Ott and Brown [7] there is an index \( q_0 \) with \( 1 < q_0 < q \) for which we can solve this boundary value problem and find \( u \).

We fix a surface cube \( Q_0 \subset \partial \Omega \) and suppose \( \partial \Omega \) is given by a graph in \( 2Q_0 \). We will show that
\[
\int_{Q_0} M(\nabla u^*)^q \, d\sigma \leq C \int_{\partial \Omega} |f_N|^q \, d\sigma. \tag{4.14}
\]
If we cover \( \partial \Omega \) by a finite collection of surface cubes and sum the resulting estimates we will obtain an estimate for the maximal function \( \nabla u^* \).

Thus, we turn to the proof of (4.14). To verify the hypotheses of Theorem 4.1 we fix a cube \( Q \subset Q_0 \) and define \( v \) and \( w \) in \( \Omega \) as the solutions of the boundary value problems
\[
\begin{cases}
\Delta v = 0, & \text{in } \Omega \\
v = 0, & \text{on } D \\
\frac{\partial v}{\partial \nu} = g, & \text{on } N \\
\nabla v^* \in L^{q_0}(\partial \Omega)
\end{cases}
\quad \begin{cases}
\Delta w = 0, & \text{in } \Omega \\
w = 0, & \text{on } D \\
\frac{\partial w}{\partial \nu} = h, & \text{on } N \\
\nabla w^* \in L^{q_0}(\partial \Omega)
\end{cases}
\]

where \( g = \chi_{2Q} f_N \) and \( h = f_N - g \). In preparation for using Theorem 4.1 we put \( F = M(\nabla u^*), \ F_Q = M(\nabla v^*) \) and \( R_Q = M(\nabla w^*) \). By uniqueness for the \( L^{q_0} \)-mixed problem [7, Theorem 5.1], we have that \( u = v + w \) and it follows that (4.12) holds on \( Q \).
To prove (4.4) we use Theorem 7.7 of Ott and Brown [7] and the Hardy-Littlewood maximal Theorem to conclude that
\[
\int_{\partial \Omega} M(\nabla v^*)^q_0 \, d\sigma \leq C \int_{2Q} |f_N|^q_0 \, d\sigma
\]  
(4.15)
The estimate (4.4) follows easily from (4.15).

The proof of (4.3) will require a bit more work. We begin by choosing \( r > c \) diam(\( Q \)) so that if \( x \in Q \), then \( \Delta_{4r}(x) \subset 2Q \). This will be the value of \( r \) we use in defining our truncated maximal functions. We claim that for \( q_1 < n/(n-1) \), we have
\[
\left( \int_{\Delta_r(x)} M(\nabla w^*)^{q_1} \, d\sigma \right)^{1/q_1} \leq C \left[ \int_{\Delta_r(x)} M(\nabla u^*) \, d\sigma + \left( \int_{2Q} |f_N|^{q_0} \, d\sigma \right)^{1/q_0} \right] 
\]  
(4.16)
We may obtain (4.3) from (4.16) by covering \( Q \) with a finite number of surfaces balls.

To prove (4.16) we begin by observing that
\[
M(\nabla w^*) \leq M_\infty(\nabla w^*) + M_0(\nabla w^\Delta) + M_0(\nabla w^\triangle). 
\]
According to Lemma 4.12 we have
\[
M_\infty(\nabla w^*)(y) \leq C \int_{\Delta_r(x)} M(\nabla w^*) \, d\sigma, \quad y \in \Delta_r(x). 
\]  
(4.17)
The estimate
\[
\left( \int_{\Delta_r(x)} M_0(\nabla w^\Delta)^{q_1} \, d\sigma \right)^{1/q_1} \leq C \int_{\Delta_r(x)} M(\nabla w^*) \, d\sigma 
\]  
(4.18)
follows from Lemma 4.3.

To estimate \( M(\nabla w^\Delta) \) we use Lemma 4.12 and Lemma 4.13 to conclude that
\[
\nabla w^\Delta(y) \leq C \int_{\Delta_{2r}(x)} M_\infty(\nabla w^*) \, d\sigma, \quad y \in \Delta_{2r}(x). 
\]
From this, we conclude that
\[
M_0(\nabla w^\Delta)(y) \leq \int_{\Delta_{2r}(x)} M(\nabla w^*) \, d\sigma, \quad y \in \Delta_{2r}(x). 
\]  
(4.19)
Combining (4.17)-(4.19), we conclude
\[
\left( \int_{\Delta_r(x)} M(\nabla w^*)^{q_1} \, d\sigma \right)^{1/q_1} \leq C \left[ \int_{\Delta_r(x)} M(\nabla u^*) \, d\sigma + \left( \int_{2Q} |f_N|^{q_0} \, d\sigma \right)^{1/q_0} \right]. 
\]  
(4.20)
Since \( w = u - v \), it follows that \( M(\nabla w^*) \leq M(\nabla u^*) + M(\nabla v^*) \) we may use (4.15) to obtain
\[
\int_{\Delta_{4r}(x)} M(\nabla w^*) \, d\sigma \leq C \left[ \int_{\Delta_{4r}(x)} M(\nabla u^*) \, d\sigma + \left( \int_{2Q} |f_N|^{q_0} \, d\sigma \right)^{1/q_0} \right]. 
\]  
(4.21)
The estimate (4.16) follows from (4.20) and (4.21).
We close with several questions for further investigation.

1. As observed above, the range of exponents is sharp when the dimension $n = 2$. In dimensions $n \geq 3$, there is a gap between the result of Theorem 1.2 and our example (extended to higher dimensions by adding extra variables).

2. In principle, the argument here should extend to systems. However, we have not written out the details.
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