ASYMMETRIC ANISOTROPIC FRACTIONAL
SOBOLEV NORMS

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Abstract. Bourgain, Brezis & Mironescu showed that (with suitable scaling) the fractional Sobolev \( s \)-seminorm of a function \( f \in W^{1,p}(\mathbb{R}^n) \) converges to the Sobolev seminorm of \( f \) as \( s \to 1^- \). Ludwig introduced the anisotropic fractional Sobolev \( s \)-seminorms of \( f \) defined by a norm on \( \mathbb{R}^n \) with unit ball \( K \), and showed that they converge to the anisotropic Sobolev seminorm of \( f \) defined by the norm whose unit ball is the polar \( L_p \) moment body of \( K \), as \( s \to 1^- \). The asymmetric anisotropic \( s \)-seminorms are shown to converge to the anisotropic Sobolev seminorm of \( f \) defined by the Minkowski functional of the polar asymmetric \( L_p \) moment body of \( K \).

1. Introduction

Let \( \Omega \) be an open set in \( \mathbb{R}^n \). For \( p \geq 1 \) and \( 0 < s < 1 \), Gagliardo introduced the fractional Sobolev spaces

\[
W^{s,p}(\Omega) = \left\{ f \in L^p(\Omega) : \frac{|f(x) - f(y)|}{|x - y|^{\frac{n}{p} + s}} \in L^p(\Omega \times \Omega) \right\},
\]

and the fractional Sobolev \( s \)-seminorm of a function \( f \in L^p(\Omega) \)

\[
\|f\|_{W^{s,p}(\Omega)}^p = \int_{\Omega} \int_{\Omega} \frac{|f(x) - f(y)|^p}{|x - y|^{n+ps}} dxdy
\]

(see [8]). They have found many applications in pure and applied mathematics (see [3, 5, 24]).

Although \( \|f\|_{W^{s,p}(\Omega)} \to \infty \) as \( s \to 1^- \), Bourgain, Brezis and Mironescu showed in [2] that

\[
\lim_{s \to 1^-} (1 - s) \|f\|_{W^{s,p}(\Omega)}^p = \frac{K_{n,p}}{p} \|f\|_{W^{1,p}(\Omega)}^p,
\]

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for $f \in W^{1,p}(\Omega)$ and $\Omega \subset \mathbb{R}^n$ a smooth bounded domain, where

$$K_{n,p} = \frac{2\Gamma((p + 1)/2)\pi^{(n-1)/2}}{\Gamma((n + p)/2)}$$

is a constant depending on $n$ and $p$,

$$\|f\|^p_{W^{1,p}} = \int_{\Omega} |\nabla f(x)|^p \, dx$$

is the Sobolev seminorm of $f$, and $\nabla f : \mathbb{R}^n \to \mathbb{R}^n$ denotes the $L^p$ weak derivative of $f$.

If instead of the Euclidean norm $|\cdot|$, we consider an arbitrary norm $\|\cdot\|_K$ with unit ball $K$, we obtain the anisotropic Sobolev seminorm,

$$\|f\|^p_{W^{1,p},K} = \int_{\mathbb{R}^n} \|\nabla f(x)\|^p_{K} \, dx,$$

where $K^* = \{v \in \mathbb{R}^n : v \cdot x \leq 1 \text{ for all } x \in K\}$ is the polar body of $K$, and $v \cdot x$ denotes the inner product between $v$ and $x$. Anisotropic Sobolev seminorms and the corresponding Sobolev inequalities attracted a lot of attentions in recent years (see [1, 4, 7, 10]).

Anisotropic $s$-seminorms, introduced very recently by Ludwig [17], reflect a fine structure of the anisotropic fractional Sobolev spaces. She established that

$$\lim_{s \to 1^-} (1 - s) \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|f(x) - f(y)|^p}{\|x - y\|^{n+ps}_K} \, dx \, dy = \frac{2}{p} \int_{\mathbb{R}^n} \|\nabla f(x)\|^p_{Z^*_{p}K} \, dx,$$

for $f \in W^{1,p}(\mathbb{R}^n)$ with compact support, where the norm associated with $Z^*_{p}K$, the polar $L_p$ moment body of $K$, is defined as

$$\|v\|^p_{Z^*_{p}K} = \frac{n + p}{2} \int_{K} |v \cdot x|^p \, dx,$$

for $v \in \mathbb{R}^n$, and a convex body $K \subset \mathbb{R}^n$. Several different other cases were considered in [16, 17, 29].

In this paper, by replacing the absolute value $|\cdot|$ by the positive part $(\cdot)_+$, for $x \in \mathbb{R}$, where $(x)_+ = \max\{0, x\}$, we obtain the following generalization. Note that here it is no longer required that $K$ is origin-symmetric. As a consequence, for $K \subset \mathbb{R}^n$ a convex body containing the origin in its interior and $x \in \mathbb{R}^n$,

$$\|x\|_K = \min\{\lambda \geq 0 : x \in \lambda K\}$$

just defines the Minkowski functional of $K$ and no longer a norm.
Theorem 1. If \( f \in W^{1,p}(\mathbb{R}^n) \) has compact support, then
\[
\lim_{s \to 1^-} (1 - s) \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{(f(x) - f(y))^p}{\|x - y\|^p_{K^s}} dxdy = \frac{1}{p} \int_{\mathbb{R}^n} \|\nabla f(x)\|^p_{Z^+,\cdot K} dx,
\]
where \( Z^+,\cdot K \) is the polar asymmetric \( L_p \) moment body of \( K \).

For a convex body \( K \subset \mathbb{R}^n \), the polar asymmetric \( L_p \) moment body is the unit ball of the Minkowski functional defined by
\[
\|v\|^p_{Z^+,\cdot K} = (n + p) \int_K (v \cdot x)^p dx,
\]
for \( v \in \mathbb{R}^n \), \( Z^+ K = Z^+(-K) \). For \( p > 1 \), in [14], Ludwig introduced and characterized the two-parameter family
\[
c_1 \cdot Z^+ K + c_2 \cdot Z^- K
\]
as all possible \( L_p \) analogs of moment bodies, including the symmetric case
\[
Z_p K = \frac{1}{2} \cdot Z^+ K + \frac{1}{2} \cdot Z^- K,
\]
where \( \|\cdot\|^{p}_{(\cdot+K_p+\beta \cdot L^*)} = \alpha \|\cdot\|^p_{K_p} + \beta \|\cdot\|^p_{L^*} \), for \( \alpha, \beta \geq 0 \), defines the \( L_p \) Minkowski combination. In recent years, this family of convex bodies have found important applications within convex geometry, probability theory, and the local theory of Banach spaces (see [9, 11–15, 18–23, 25–28, 31]).

The proof given in this paper makes use of an asymmetric version of the one-dimensional case of result (1.1) by Bourgain, Brezis and Mironescu and an asymmetric decomposition of Blaschke-Petkantschin type.

2. Proof of the main result

First, we need the asymmetric one-dimensional analogue of (1.1). For its proof we require the following result from [2].

Lemma 2. Let \( \rho \in L^1(\mathbb{R}^n) \) and \( \rho \geq 0 \). If \( f \in W^{1,p}(\mathbb{R}^n) \) is compactly supported and \( 1 \leq p < \infty \), then
\[
\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|f(x) - f(y)|^p}{|x - y|^p} \rho(x - y) dxdy \leq C \|f\|^p_{W^{1,p}} \|\rho\|_{L^1},
\]
where \( C \) depends only on \( p \) and the support of \( f \).

Let \( \Omega \subset \mathbb{R} \) be a bounded domain.
Proposition 3. If \( f \in W^{1,p}(\Omega) \), then

\[
\lim_{s \to 1^-} \left[ (1 - s) \int_{\Omega} \int_{\Omega \cap \{ x > y \}} \frac{(f(x) - f(y))^p}{|x - y|^{1 + ps}} \, dx \, dy \right] = \frac{1}{p} \int_{\Omega} (f'(x))^p \, dx.
\]

Proof. Take a sequence \((\rho_\varepsilon)\) of radial mollifiers, i.e., \( \rho_\varepsilon(x) = \frac{\rho_\varepsilon(|x|)}{\varepsilon^p} \); \( \rho_\varepsilon \geq 0 \); \( \int_0^\infty \rho_\varepsilon(x) \, dx = 1 \); \( \lim_{\varepsilon \to 0} \int_\delta^\infty \rho_\varepsilon(r) \, dr = 0 \) for every \( \delta > 0 \). Let \( F_\varepsilon(x, y) = \frac{(f(x) - f(y))^p}{|x - y|^{1 + p\varepsilon}} \rho_\varepsilon(x - y), \) for \( x > y \). It suffices to prove that

\[
(2.2) \quad \lim_{\varepsilon \to 0} \int_{\Omega} \int_{\Omega \cap \{ x > y \}} F_\varepsilon(x, y) \, dx \, dy = \int_{\Omega} (f'(x))^p \, dx.
\]

Indeed, as in \([30]\), let \( R > \max \{|x - y| : x, y \in \Omega\} \), \( \varepsilon = 1 - s \) and

\[
\rho_\varepsilon(x) = \frac{\chi_{[0, R]}(|x|)}{R^p} \frac{p\varepsilon}{|x|^{1 - p\varepsilon}},
\]

where \( \chi_A \) is the indicator function of \( A \). Then one obtains (2.1) from (2.2) as desired.

By Lemma 2, we have, for any \( \varepsilon > 0 \) and \( f, g \in W^{1,p}(\Omega) \)

\[
\left| \| F_\varepsilon \|_{L^p(\Omega \times \Omega)} - \| G_\varepsilon \|_{L^p(\Omega \times \Omega)} \right| \leq \| F_\varepsilon - G_\varepsilon \|_{L^p(\Omega \times \Omega)} \leq C \| f - g \|_{W^{1,p}},
\]

for some constant \( C \) dependent on \( \varepsilon, f \) and \( g \). Therefore, it suffices to establish (2.2) for \( f \) in some dense subset of \( W^{1,p}(\Omega) \), e.g., for \( f \in C^2(\bar{\Omega}) \), where \( \bar{\Omega} \) is the closure of \( \Omega \).

Fix \( f \in C^2(\bar{\Omega}) \). Since for \( t \in \mathbb{R} \) and \( \lambda > 0 \), \( (\lambda t)_+ = \lambda (t)_+ \), there exists \( \delta > 0 \), such that for \( y < x < y + \delta \) and a constant \( c \),

\[
\left| \frac{(f(x) - f(y))^p}{|x - y|^p} - (f'(y))^p_+ \right| \leq c(x - y).
\]

We have

\[
\int_{\Omega \cap \{ x > y \}} \frac{(f(x) - f(y))^p}{|x - y|^p} \rho_\varepsilon(x - y) \, dx = \int_{\Omega \cap \{ y < x < y + \delta \}} \frac{(f(x) - f(y))^p}{|x - y|^p} \rho_\varepsilon(x - y) \, dx + \int_{\Omega \cap \{ x \geq y + \delta \}} \frac{(f(x) - f(y))^p}{|x - y|^p} \rho_\varepsilon(x - y) \, dx,
\]
yet, only the former integral on the right hand side need be considered, as the latter vanishes. In fact, for each fixed \( y \in \Omega \), since
\[
\left| \int_y^{y+\delta} \left( \frac{(f(x) - f(y))^p_+}{|x-y|^p} - (f'(y))^p_+ \right) \rho_\epsilon(x-y) dx \right| \\
\leq \int_y^{y+\delta} \frac{(f(x) - f(y))^p_+}{|x-y|^p} - (f'(y))^p_+ \rho_\epsilon(x-y) dx \\
\leq c \int_y^y (x-y) \rho_\epsilon(x-y) dx \\
= c \int_0^{\delta} r \rho_\epsilon(r) dr 
\to 0, \quad \text{as} \quad \epsilon \to 0,
\]
we have
\[
\lim_{\epsilon \to 0} \int_y^{y+\delta} \frac{(f(x) - f(y))^p_+}{|x-y|^p} \rho_\epsilon(x-y) dx \\
= (f'(y))^p_+ \lim_{\epsilon \to 0} \int_y^{y+\delta} \rho_\epsilon(x-y) dx \\
= (f'(y))^p_+ \lim_{\epsilon \to 0} \int_0^{\delta} \rho_\epsilon(r) dr \\
= (f'(y))^p_+. 
\]
Therefore,
\[
\begin{align*}
(2.3) \quad \lim_{\epsilon \to 0} \int_{\Omega \cap \{x>y\}} \frac{(f(x) - f(y))^p_+}{|x-y|^p} \rho_\epsilon(x-y) dx &= (f'(y))^p_+. 
\end{align*}
\]
Since \( f \in C^2(\Omega) \), there exists \( L > 0 \) is such that \( |f(x) - f(y)| < L|x-y| \), for every \( x, y \in \Omega \), then
\[
\begin{align*}
(2.4) \quad \int_{\Omega} \frac{|f(x) - f(y)|^p}{|x-y|^p} \rho_\epsilon(x-y) dx &\leq L^p, \quad \text{for each} \quad y \in \Omega.
\end{align*}
\]
Hence, for \( f \in C^2(\Omega) \), (2.2) follows by dominated convergence theorem from (2.3) and (2.4). \qed
Now, for \( u \in S^{n-1} \), the Euclidean unit sphere, let \( [u] = \{ \lambda u : \lambda \in \mathbb{R} \} \) and \( [u]^+ = \{ \lambda u : \lambda > 0 \} \). Denote the \( k \)-dimensional Hausdorff measure on \( \mathbb{R}^n \) by \( H^k \). For \( f \in W^{1,p}(\mathbb{R}^n) \), we denote by \( \bar{f} \) its precise representative (see [6, Section 1.7.1]). We require the following result. For every \( u \in S^{n-1} \), the precise representative \( \bar{f} \) is absolutely continuous on the lines \( L = \{ x + \lambda u : \lambda \in \mathbb{R} \} \) for \( H^{n-1} \)-a.e. \( x \in u^\perp \) and its first-order (classical) partial derivatives belong to \( L^p(\mathbb{R}^n) \) (see [6, Section 4.9.2]). Hence, we have for the restriction of \( \bar{f} \) to \( L \),

\begin{equation}
\bar{f}|_L \in W^{1,p}(L)
\end{equation}

for a.e. line \( L \) parallel to \( u \).

**Proof of Theorem 1.** By the polar coordinate formula and Fubini’s theorem, we have

\begin{equation}
\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{(f(x) - f(y))^p}{\|x - y\|_K^{n+sp}} dH^n(x) dH^n(y)
\end{equation}

\begin{align*}
&= \int_{\mathbb{R}^n} \int_{\mathbb{S}^{n-1}} \|u\|_K^{-(n+ps)} \int_0^\infty \frac{(f(y + ru) - f(y))^p}{r^{1+sp}} dH^1(r) d\sigma(u) dH^n(y) \\
&= \int_{\mathbb{S}^{n-1}} \|u\|_K^{-(n+ps)} \int_0^\infty \int_{u^\perp [u]^+ z} \frac{(f(w + ru) - f(w))^p}{r^{1+sp}} dH^1(w) dH^{n-1}(z) dH^1(r) d\sigma(u) \\
&= \int_{\mathbb{S}^{n-1}} \|u\|_K^{-(n+ps)} \int_0^\infty \int_{u^\perp [u]^+ z} \int_{u^\perp [u]^+ w} \frac{(f(t) - f(w))^p}{|t - w|^{1+sp}} dH^1(t) dH^1(w) dH^{n-1}(z) d\sigma(u),
\end{align*}

where \( \sigma \) denotes the standard surface area measure on \( S^{n-1} \). By Proposition 3 and (2.5), we obtain

\begin{equation}
\lim_{s \to 1^-} (1 - s) \int_{[u]^+ [u]^+ w} \frac{(f(t) - f(w))^p}{|t - w|^{1+sp}} dH^1(t) dH^1(w)
\end{equation}

\begin{equation}
= \frac{1}{p} \int_{[u]^+ z} (\nabla f(t) \cdot u)^p dH^1(t).
\end{equation}
By Fubini’s theorem and the polar coordinate formula, we get

\[
\frac{1}{p} \int_{S^{n-1}} \left\| u \right\|^{(n+p)}_K \int_{u^+ [u+z]} (\nabla f(t) \cdot u)^p_+ dH^1(t) dH^{n-1}(z) d\sigma(u)
\]

\[= \frac{1}{p} \int_{S^{n-1}} \int_{\mathbb{R}^n} \left\| u \right\|^{(n+p)}_K (\nabla f(x) \cdot u)^p_+ dH^n(x) d\sigma(u)
\]

\[= \frac{n + p}{p} \int_{\mathbb{R}^n} (\nabla f(x) \cdot y)^p_+ dH^n(x) dH^n(y)
\]

Using Fubini’s theorem and the definition of the asymmetric \(L_p\) moment body of \(K\), we obtain

\[
\int_{S^{n-1}} \int_{u^+ [u+z]} (\nabla f(t) \cdot u)^p_+ dH^1(t) dH^{n-1}(z) d\sigma(u)
\]

\[= \int_{\mathbb{R}^n} \left\| \nabla f(x) \right\|^p_{Z^{+, \ast}_p K} dH^n(x).
\]

So, in particular, we have

\[
\int_{S^{n-1}} \int_{u^+ [u+z]} (\nabla f(t) \cdot u)^p_+ dH^1(t) dH^{n-1}(z) d\sigma(u)
\]

\[= \frac{n + p}{4} K_{n,p} \int_{\mathbb{R}^n} |\nabla f(x)|^p dH^n(x) < +\infty.
\]

Using the dominated convergence theorem with Lemma 2 and (2.9), we obtain from (2.6), (2.7) and (2.8) that

\[
\lim_{s \to 1^-} (1 - s) \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{(f(x) - f(y))^p}{\|x - y\|^{n+sp}_K} dxdy = \frac{1}{p} \int_{\mathbb{R}^n} \left\| \nabla f(x) \right\|^p_{Z^{+, \ast}_p K} dx.
\]

\[\square\]

Remark 4. In Theorem 1, let \(g = -f\) and \((x)_- = -\min \{0, x\} = (-x)_+,\) for \(x \in \mathbb{R}\). Then, we get

\[
\lim_{s \to 1^-} (1 - s) \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{(f(x) - f(y))^p}{\|x - y\|^{n+sp}_K} dxdy = \frac{1}{p} \int_{\mathbb{R}^n} \left\| \nabla f(x) \right\|^p_{Z^{-, \ast}_p K} dx.
\]

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References

[1] A. Alvino, V. Ferone, G. Trombetti, and P.-L. Lions, Convex symmetrization and applications, Ann. Inst. H. Poincaré Anal. Non Linéaire 14 (1997), 275–293.

[2] J. Bourgain, H. Brezis, and P. Mironescu, Another look at Sobolev spaces, Optimal Control and Partial Differential Equations, A volume in honor of A. Bensoussan’s 60th birthday (Amsterdam) (J. L. Menaldi, E. Rofman, and A. Sulem, eds.), IOS Press, 2001, pp. 439–455.

[3] ______, Limiting embedding theorems for $W^{s,p}$ when $s \uparrow 1$ and applications, J. Anal. Math. 87 (2002), 77–101, Dedicated to the memory of Thomas H. Wolff.

[4] D. Cordero-Erausquin, B. Nazaret, and C. Villani, A mass-transportation approach to sharp Sobolev and Gagliardo-Nirenberg inequalities, Adv. Math. 182 (2004), 307–332.

[5] E. Di Nezza, G. Palatucci, and E. Valdinoci, Hitchhiker’s guide to the fractional Sobolev spaces, Bull. Sci. Math. 136 (2012), 521–573.

[6] L. Evans and R. Gariepy, Measure theory and fine properties of functions, Studies in Advanced Mathematics, CRC Press, Boca Raton, FL, 1992.

[7] A. Figalli, F. Maggi, and A. Pratelli, Sharp stability theorems for the anisotropic Sobolev and log-Sobolev inequalities on functions of bounded variation, Adv. Math. 242 (2013), 80–101.

[8] E. Gagliardo, Caratterizzazioni delle tracce sulla frontiera relative ad alcune classi di funzioni in $n$ variabili, Rend. Sem. Mat. Univ. Padova 27 (1957), 284–305.

[9] R. J. Gardner, Geometric tomography, 2nd ed., Cambridge Univ. Press, New York, 2006.

[10] M. Gromov, Isoperimetric inequalities in Riemannian manifolds, Asymptotic Theory of Finite-dimensional Normed Spaces (V. D. Milman and G. Schechtman, eds.), Springer-Verlag, Berlin Heidelberg, 1986, pp. 114–129.

[11] C. Haberl, Minkowski valuations intertwining the special linear group, J. Eur. Math. Soc. 14 (2012), 1565–1597.

[12] C. Haberl and F. Schuster, General $L_p$ affine isoperimetric inequalities, J. Differential Geom. 83 (2009), 1–26.

[13] M. Ludwig, Ellipsoids and matrix valued valuations, Duke Math. J. 119 (2003), 159–188.

[14] ______, Minkowski valuations, Trans. Amer. Math. Soc. 357 (2005), 4191–4213.

[15] ______, Minkowski areas and valuations, J. Differential Geom. 86 (2010), 133–161.

[16] ______, Anisotropic fractional perimeters, J. Differential Geom. 96 (2014), 77–93.

[17] ______, Anisotropic fractional Sobolev norms, Adv. Math. 252 (2014), 150–157.
[18] E. Lutwak, *Centroid bodies and dual mixed volumes*, Proc. London Math. Soc. **60** (1990), 365–391.
[19] E. Lutwak, D. Yang, and G. Zhang, *$L_p$ affine isoperimetric inequalities*, J. Differential Geom. **56** (2000), 111–132.
[20] ______, *A new ellipsoid associated with convex bodes*, Duke Math. J. **104** (2000), 375–390.
[21] ______, *The Cramer-Rao inequality for star bodies*, Duke Math. J. **112** (2002), 59–81.
[22] ______, *Moment-entropy inequalities*, Ann. Probab. **32** (2004), 757–774.
[23] ______, *Orlicz centroid bodies*, J. Differential Geom. **84** (2010), 365–387.
[24] V. G. Maz’ya, *Sobolev spaces with applications to elliptic partial differential equations*, augmented ed., Grundlehren der Mathematischen Wissenschaften, vol. 342, Springer-Verlag, Berlin Heidelberg, 2011.
[25] G. A. Paouris, *Concentration of mass on convex bodies*, Geom. Funct. Anal. **16** (2006), 1021–1049.
[26] G. A. Paouris and E. Werner, *Relative entropy of cone measures and $L_p$ centroid bodies*, Proc. Lond. Math. Soc. **104** (2012), no. 2, 253–286.
[27] L. Parapatits, *$SL(n)$-contravariant $L_p$-Minkowski valuations*, Trans. Amer. Math. Soc. **366** (2014), 1195–1211.
[28] ______, *$SL(n)$-covariant $L_p$-Minkowski valuations*, J. London Math. Soc. **89** (2014), 397–414.
[29] A. Ponce, *A new approach to Sobolev spaces and connections to $\Gamma$-convergence*, Calc. Var. Partial Differential Equations **19** (2004), 229–255.
[30] D. Spector, *Characterization of Sobolev and BV spaces*, Ph.D. thesis, Carnegie Mellon University, 2011.
[31] T. Wannerer, *GL(n) equivariant Minkowski valuations*, Indiana Univ. Math. J. **60** (2011), 1655–1672.

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