Wilson–’t Hooft lines as transfer matrices

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Based on joint work with Kazunobu Maruyoshi and Toshihiro Ota
Various connections between

supersymmetric QFTs $\leftrightarrow$ quantum integrable systems

discovered in the past 10 years or so:

- Bethe/gauge correspondence (2d & 4d) [Nekrasov–Shatashvili]
- Bazhanov–Sergeev model from 4d $\mathcal{N} = 1$ quiver gauge theories [Spiridonov, Yamazaki]
- Surface defects as transfer matrices [Maruyoshi–Yagi]
- 4d Chern–Simons ($= \Omega$-deformed 6d SYM [Costello–Y])
  [Costello, Costello–Yamazaki–Witten]
- ...

Many of them are related by string dualities [Costello–Y].
Quantization of Donagi–Witten integrable system

- $\mathcal{N} = 2$ theory on $\mathbb{R}^3 \times S^1$ on Coulomb branch
- IR: $\mathcal{N} = 4$ sigma model on $\mathbb{R}^3$
- Target $\mathcal{M}$ is the phase space of a classical complex integrable system [Donagi–Witten]
- $\Omega$-deformation on $\mathbb{R}^2 \subset \mathbb{R}^3$ quantizes $\mathcal{M}$ [Nekrasov–Shatashvili, Nekrasov–Witten, Y]
- For class-S theories, $\mathcal{M}$ is a Hitchin system.
Surface defects as transfer matrices [Maruyoshi–Y, Y]

- $\mathcal{N} = 1$ theory constructed by “brane tiling” or of class $S_k$
- Place it on $S^3 \times S^1$
- Insert surface defects on $S^1 \times S^1$
- Surface defects act on SUSY index as difference operators, shifting flavor fugacities [Gadde–Gukov, Gaiotto–Rastelli–Razamat]
- Coincide with transfer matrices of elliptic QIS [Maruyoshi–Y, Y]
- Simplest case: elliptic Ruijsenaars–Schneider system [GRR, Bullimore–Fluder–Hollands–Richmond]
We found a new correspondence:

Wilson–’t Hooft lines = transfer matrices

- $\mathcal{N} = 2$ circular quiver theory (class-$S$)
- Place it on $S^1 \times \mathbb{R}^3$
- Wind a Wilson–’t Hooft line $T$ around $S^1$
- $\langle T \rangle$ is a function of Coulomb branch parameters
- Quantization of $\langle T \rangle$ coincides with transfer matrix of trigonometric QIS

Related to other correspondences
Consider a periodic spin chain

![Diagram of a periodic spin chain with spins labeled a_1, a_2, ..., a_8.]

Spins $a^1, \ldots, a^n \in \mathfrak{h}^*$, $\mathfrak{h} = \text{Cartan of } \mathfrak{sl}_N$:

$$a^r = \text{diag}(a^r_1, \ldots, a^r_N), \quad \sum_{i=1}^{N} a^r_i = 0$$

Local Hilbert space:

$$\mathcal{M}_{\mathfrak{h}^*} = \{ \text{meromorphic functions on } \mathfrak{h}^* \}$$

Total Hilbert space

$$\mathcal{H} = \mathcal{M}_{\mathfrak{h}^*} \otimes \cdots \otimes \mathcal{M}_{\mathfrak{h}^*}_{n}$$
Equivalent lattice model

Spins live between double lines:

$a^r$ are called **dynamical parameters**.
Transfer matrix $T(z)$ is horizontal loop operator:

![Diagram](image)

Solid line = worldline of particle whose Hilbert space is $\mathbb{C}^N$.

The particle’s state changes when it crosses other lines.

Solid line also has spectral parameter $z \in \mathbb{C}$.

$T(z)$ consists of $n$ copies of L-operator

$$L(z) = z$$
Dynamical parameters jump across solid lines:

\[ L(z; a^1, a^2)_i^j = z \]

\[ a^1 - \epsilon h_i \]

\[ a^2 - \epsilon h_j \]

\( \epsilon \in \mathbb{C} \): fixed parameter (Planck constant)

\( h_i \) are the weights of the vector rep \( \mathbb{C}^N \):

\[ h_1 = \text{diag}(1 - \frac{1}{N}, -\frac{1}{N}, -\frac{1}{N}, \ldots, -\frac{1}{N}) \],

\[ h_2 = \text{diag}(-\frac{1}{N}, 1 - \frac{1}{N}, -\frac{1}{N}, \ldots, -\frac{1}{N}) \],

\[ \vdots \]

\[ h_N = \text{diag}(-\frac{1}{N}, -\frac{1}{N}, -\frac{1}{N}, \ldots, 1 - \frac{1}{N}) \].
Matrix elements $L(z)^j_i$ are difference operators on $\mathcal{M}_{\mathfrak{h}^*} \otimes \mathcal{M}_{\mathfrak{h}^*}$:

$$L(z) = \sum_{i,j} L(z; a^1, a^2)^j_i \Delta^1_i \Delta^2_j ,$$

$$\Delta^r_i : a^r \mapsto a^r - \epsilon h_i .$$

Transfer matrix

$$T(z) = \sum_{i^1, \ldots, i^n} \prod_{r=1}^{n} L(z; a^r, a^{r+1})^{i^r+1}_{i^r} \prod_{s=1}^{n} \Delta^s_{i^s} , \quad i^{n+1} = i^1$$

is a difference operator on $\mathcal{H} = \mathcal{M}_{\mathfrak{h}^*}^{\otimes n}$.
Crossing solid lines give R-matrix

\[ R(z - z'; a)_{ij}^{kl} = z \]

R-matrix satisfies dynamical Yang–Baxter equation

\[ R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12} \]

but with shifts in the dynamical parameters.
L-operator and R-matrix satisfy RLL relation

\[ z' \frac{dz}{dz} = z \frac{dz}{dz'} \]

It follows that transfer matrices commute:

\[ T(z)T(z') = T(z')T(z) \]
Proof:

By RLL relation

\[
\begin{align*}
\text{Integrable system side} & \quad \text{Gauge theory side} \\
\begin{array}{c}
\uparrow \\
\uparrow \\
\uparrow \\
\vdots \\
\uparrow \\
\uparrow \\
\uparrow \\
\uparrow \\
\end{array} & \quad \begin{array}{c}
\text{Red} \\
\text{Red} \\
\text{Blue} \\
\text{Blue} \\
\vdots \\
\text{Red} \\
\text{Red} \\
\text{Blue} \\
\end{array} \\
\begin{array}{c}
R \\
R \\
R \\
\cdots \\
R \\
R \\
R \\
R \\
\end{array} & \quad \begin{array}{c}
\text{Red} \\
\text{Red} \\
\text{Blue} \\
\text{Blue} \\
\vdots \\
\text{Red} \\
\text{Red} \\
\text{Blue} \\
\end{array} \\
\begin{array}{c}
\text{Blue} \\
\text{Blue} \\
\text{Red} \\
\text{Red} \\
\vdots \\
\text{Blue} \\
\text{Blue} \\
\text{Red} \\
\end{array} & \quad \begin{array}{c}
\text{Blue} \\
\text{Blue} \\
\text{Red} \\
\text{Red} \\
\vdots \\
\text{Blue} \\
\text{Blue} \\
\text{Red} \\
\end{array} \\
\begin{array}{c}
\uparrow \\
\uparrow \\
\uparrow \\
\vdots \\
\uparrow \\
\uparrow \\
\uparrow \\
\uparrow \\
\end{array} & \quad \begin{array}{c}
\text{Red} \\
\text{Red} \\
\text{Blue} \\
\text{Blue} \\
\vdots \\
\text{Red} \\
\text{Red} \\
\text{Blue} \\
\end{array} \\
\begin{array}{c}
R^{-1} \\
R^{-1} \\
R^{-1} \\
\cdots \\
R^{-1} \\
R^{-1} \\
R^{-1} \\
R^{-1} \\
\end{array} & \quad \begin{array}{c}
\text{Red} \\
\text{Red} \\
\text{Blue} \\
\text{Blue} \\
\vdots \\
\text{Red} \\
\text{Red} \\
\text{Blue} \\
\end{array}
\end{align*}
\]

Multiply both sides by $R^{-1}$:

\[
\begin{align*}
\text{Integrable system side} & \quad \text{Gauge theory side} \\
\begin{array}{c}
\uparrow \\
\uparrow \\
\uparrow \\
\vdots \\
\uparrow \\
\uparrow \\
\uparrow \\
\uparrow \\
\end{array} & \quad \begin{array}{c}
\text{Red} \\
\text{Red} \\
\text{Blue} \\
\text{Blue} \\
\vdots \\
\text{Red} \\
\text{Red} \\
\text{Blue} \\
\end{array} \\
\begin{array}{c}
\text{Blue} \\
\text{Blue} \\
\text{Red} \\
\text{Red} \\
\vdots \\
\text{Blue} \\
\text{Blue} \\
\text{Red} \\
\end{array} & \quad \begin{array}{c}
\text{Blue} \\
\text{Blue} \\
\text{Red} \\
\text{Red} \\
\vdots \\
\text{Blue} \\
\text{Blue} \\
\text{Red} \\
\end{array} \\
\begin{array}{c}
\uparrow \\
\uparrow \\
\uparrow \\
\vdots \\
\uparrow \\
\uparrow \\
\uparrow \\
\uparrow \\
\end{array} & \quad \begin{array}{c}
\text{Red} \\
\text{Red} \\
\text{Blue} \\
\text{Blue} \\
\vdots \\
\text{Red} \\
\text{Red} \\
\text{Blue} \\
\end{array} \\
\begin{array}{c}
R^{-1} \\
R^{-1} \\
R^{-1} \\
\cdots \\
R^{-1} \\
R^{-1} \\
R^{-1} \\
R^{-1} \\
\end{array} & \quad \begin{array}{c}
\text{Red} \\
\text{Red} \\
\text{Blue} \\
\text{Blue} \\
\vdots \\
\text{Red} \\
\text{Red} \\
\text{Blue} \\
\end{array}
\end{align*}
\]

Take the trace, making the horizontal direction periodic.
Since

$$[T(z), T(z')] = 0 ,$$

coefficients of Laurent expansion

$$T(z) = \sum_{m=-\infty}^{\infty} T_m z^m$$

are commuting difference operators on $\mathcal{H}$:

$$[T_m, T_n] = 0 .$$

This is integrability.
Trigonometric L-operator [Hasegawa]

\[ \mathcal{L}_{w,m}(z)^j_i = \sum_{i,j} (\Delta^1_i \Delta^2_j)^{\frac{1}{2}} \frac{\sin \pi (z - w + a^2_j - a^1_i)}{\sin \pi (z - w)} \ell_m(a^1, a^2)^j_i (\Delta^1_i \Delta^2_j)^{\frac{1}{2}} \]

satisfies RLL relation with a trigonometric dynamical R-matrix (a limit of the 8vSOS R-matrix).

\[ \ell_m(a^1, a^2)^j_i = \left( \frac{\prod_{k(\neq i)} \sin \pi (a^1_k - a^2_j - m) \prod_{l(\neq j)} \sin \pi (a^1_i - a^2_l - m)}{\prod_{k(\neq i)} \sin \pi (a^1_{ki} - \frac{1}{2} \epsilon) \sin \pi (a^1_{ik} - \frac{1}{2} \epsilon)} \right)^{\frac{1}{2}} \]

\( w, m \in \mathbb{C} \) are spectral parameters assigned to the double line:

\[ \mathcal{L}_{w,m}(z) = z \rightarrow \rightarrow \cdot \]

\( w, m \)
Introduce fundamental L-operators

\[ L_{\pm,m} = \lim_{w \to \pm i \infty} L_{w,m}. \]

Then

\[ (L_{\pm,m})_{ij}^j = \sum_{i,j} \left( \Delta_i^1 \Delta_j^2 \right)^{1/2} e^{\pm \pi i (a_j^2 - a_i^1)} \ell_m (a_1^1, a_2^2)_i (\Delta_i^1 \Delta_j^2)^{1/2} \]

and

\[ L_{w,m}(z) = \frac{e^{\pi i (z-w)} L_{+,m} - e^{-\pi i (z-w)} L_{-,m}}{\sin \pi (z - w)}. \]

We may as well consider \( L_{\pm,m} \) without loss of generality.
Pick \( n \)-tuple of signs

\[ \sigma = (\sigma^1, \ldots, \sigma^n) \in \{\pm\}^n \]

and \( n \)-tuple of complex numbers

\[ m = (m^1, \ldots, m^n) \in \mathbb{C}^n. \]

Let \( T_{\sigma,m} \) be the transfer matrix constructed from \( n \) L-operators

\[ L_{\sigma^1,m^1}, \ldots, L_{\sigma^n,m^n}. \]

\[ T_{\sigma,m} = \sum_{i^1, \ldots, i^n} \left( \prod_{s=1}^{n} \Delta_{i_s}^s \right)^{1/2} \prod_{r=1}^{n} e^{\pi i \sigma^r (a_{i_{r+1}} - a_{i_r})} \ell_{m^r} (a^r, a^r_{i_r+1})^{i_r+1} \left( \prod_{s=1}^{n} \Delta_{i_s}^s \right)^{1/2}. \]

This is the main character from the integrable system.
\( \mathcal{N} = 2 \) gauge theories have half-BPS Wilson–’t Hooft lines.

Worldlines of very massive dyonic particles

Charge of WH line

\[
(m, e) \in (\Lambda_{\text{coweight}} \times \Lambda_{\text{weight}})/\text{Weyl}.
\]

Wilson line has \( m = 0 \) and is labeled by representation of \( g \).

’t Hooft line has \( e = 0 \) and is labeled by representation of \( Lg \).

Wilson–’t Hooft

= (’t Hooft) + (Wilson for subgroup of \( G \) leaving \( m \) invariant)
\( \mathcal{N} = 2 \) gauge theory described by \( n \)-node circular quiver

Each node is \( \text{SU}(N) \) (more precisely, \( \text{PSU}(N) \)).

Edges are bifundamental hypers with masses \( m^1, \ldots, m^n \).

Compactification of 6d \( \mathcal{N} = (2, 0) \) SCFT on \( n \)-punctured torus

WH lines = surface defects wrapping 1-cycles of the torus
Consider Wilson–’t Hooft line $T_{\Box, \sigma}$ corresponding to

$$\gamma_\sigma = b + \sum_r \frac{1 - \sigma^r}{2} c^r.$$ 

If $\sigma^r = +1 (-1)$, the cycle passes above (below) $r$th puncture.

$$m = \Box \oplus \cdots \oplus \Box \text{ under } su_N \oplus \cdots \oplus su_N$$

$e$ specified by $\sigma \in \{\pm\}^n$
Put the theory on twisted product

\[ S^1 \times \epsilon \mathbb{R}^2 \times \mathbb{R}. \]

Wrap \( T_{\square, \sigma} \) around \( S^1 \times \{0\} \times \{t\} \).

Ito–Okuda–Taki tell us how to compute the vev by localization:

\[
\langle T_{\square, \sigma} \rangle = \sum_{i_1, \ldots, i_n} \prod_{r=1}^{n} e^{2\pi i b^r_{i_r}} e^{\pi i \sigma^r (a^r_{i_r+1} - a^r_{i_r})} \ell_{m r} (a^r, a^{r+1})_{i_r}^{i_{r+1}}
\]

in complexified Fenchel–Nielsen coordinates on Seiberg–Witten moduli space:

\[
a = \frac{\theta_e}{2\pi} + i \beta \text{Re} \phi + \cdots, \quad b = \frac{\theta_m}{2\pi} - \frac{4\pi i \beta}{g^2} \text{Im} \phi + i \frac{\vartheta}{2\pi} \beta \text{Re} \phi + \cdots.
\]

Alternatively, we can compute it from Toda theory by AGT.
Compare

\[ \langle T_{\Box, \sigma} \rangle = \sum_{i^1, \ldots, i^n} \prod_{r=1}^{n} e^{2\pi i b^r_{ir}} e^{\pi i \sigma^r (a^{r+1}_{ir} - a^r_{ir})} \ell_{mr}(a^r_{ir}, a^{r+1}_{ir})^{i^r+1}, \]

\[ T_{\sigma, m} = \sum_{i^1, \ldots, i^n} \left( \prod_{s=1}^{n} \Delta_{i_s}^s \right)^{1/2} \prod_{r=1}^{n} e^{\pi i \sigma^r (a^{r+1}_{ir} - a^r_{ir})} \ell_{mr}(a^r_{ir}, a^{r+1}_{ir})^{i^r+1} \left( \prod_{s=1}^{n} \Delta_{i_s}^s \right)^{1/2}. \]

If we quantize \( a^r, b^r \) so that

\[ [\hat{a}^r_i, \hat{b}^s_j] = -i \frac{\epsilon}{2\pi} \delta^{rs} \left( \delta_{ij} - \frac{1}{N} \right), \]

then

\[ T_{\sigma, m} = \text{Weyl quantization of } \langle T_{\Box, \sigma} \rangle. \]
M-theory setup

Spacetime \( R_0 \) \( R_{12}^2 \) \( S_3^1 \) \( R_{45}^2 \) \( S_6^1 \) \( R_7 \) \( R_8 \) \( R_9 \) \( S_{10}^1 \)
\( N \) M5 \( R_0 \) \( R_{12}^2 \) \( S_3^1 \) \( - \) \( S_6^1 \) \( - \) \( - \) \( - \) \( S_{10}^1 \)
\( n \) M5' \( R_0 \) \( R_{12}^2 \) \( S_3^1 \) \( - \) \( - \) \( - \) \( R_8 \) \( R_9 \) \( - \)
M2 \( - \) \( - \) \( S_3^1 \) \( - \) \( S_6^1 \) \( - \) \( R_8^{\geq 0} \) \( - \) \( - \)

12345 directions: twisted product \( R_{12}^2 \times \epsilon S_3^1 \times -\epsilon R_{45}^2 \)

M5: 6d \( \mathcal{N} = (2, 0) \) SCFT on \( R_0 \times R_{12}^2 \times \epsilon S_3^1 \times S_6^1 \times S_{10}^1 \)

M5': \( n \) punctures on \( S_6^1 \times S_{10}^1 \)

M2: surface defect

Reduction on \( S_6^1 \times S_{10}^1 \) gives the 4d setup with \( \sigma = (+, \ldots, +) \).
Compactify $\mathbb{R}_9 \rightarrow S^1_9$:

| Intergable system side | Gauge theory side | Correspondence | Brane realization | Summary |
|------------------------|------------------|----------------|-------------------|---------|
| $\mathbb{R}_9$         | $S^1_9$          |                |                   |         |
| $\mathbb{R}_0 \times \mathbb{R}_{12}$ | $S^1_3$   | $\mathbb{R}_{45}$ | $S^1_6$ | $R_7$ | $R_8$ | $S^1_9$ | $S^1_{10}$ |
| $N$ M5                 | $R_0 \times \mathbb{R}_{12}$ | $S^1_3$ |                | $S^1_6$ |                | $R_8$ | $S^1_{10}$ |
| $n$ M5$'$             | $R_0 \times \mathbb{R}_{12}$ | $S^1_3$ |                | $S^1_6$ |                | $R_8 \geq 0$ | $S^1_9$ |
| M2                     |                 | $S^1_3$ |                | $S^1_6$ |                | $R_8 \geq 0$ | $-$ |

Reduce on $S^1_3$:

| Intergable system side | Gauge theory side | Correspondence | Brane realization | Summary |
|------------------------|------------------|----------------|-------------------|---------|
| $\mathbb{R}_9$         | $S^1_9$          |                |                   |         |
| $\mathbb{R}_0 \times \mathbb{R}_{12}$ | $S^1_6$   | $\mathbb{R}_{45}$ | $S^1_6$ | $R_7$ | $R_8$ | $S^1_9$ | $S^1_{10}$ |
| $N$ D4                 | $R_0 \times \mathbb{R}_{12}$ | $S^1_6$ |                | $S^1_6$ |                | $R_8$ | $S^1_{10}$ |
| $n$ D4                 | $R_0 \times \mathbb{R}_{12}$ | $S^1_6$ |                | $S^1_6$ |                | $R_8 \geq 0$ | $S^1_9$ |
| F1                     |                 | $S^1_6$ |                | $S^1_6$ |                | $R_8 \geq 0$ | $-$ |
| Spacetime | $\mathbb{R}_0$ | $\mathbb{R}^2_{12}$ | $\mathbb{R}^2_{45}$ | $S^1_6$ | $R_7$ | $R_8$ | $S^1_9$ | $S^1_{10}$ |
|-----------|---------------|---------------------|---------------------|--------|------|------|---------|---------|
| $N$ D4    | $\mathbb{R}_0$ | $\mathbb{R}^2_{12}$ | $-$                | $S^1_6$ | $-$  | $-$  | $-$     | $S^1_{10}$ |
| $n$ D4    | $\mathbb{R}_0$ | $\mathbb{R}^2_{12}$ | $-$                | $-$    | $-$  | $R_8$ | $S^1_9$ | $S^1_{10}$ |
| F1        | $-$            | $-$                 | $-$                | $S^1_6$ | $-$  | $R^\geq_8$ | $-$      |

Apply T-duality $S^1_9 \rightarrow \check{S}^1_9$:

| Spacetime | $\mathbb{R}_0$ | $\mathbb{R}^2_{12}$ | $\mathbb{R}^2_{45}$ | $S^1_6$ | $R_7$ | $R_8$ | $\check{S}^1_9$ | $S^1_{10}$ |
|-----------|---------------|---------------------|---------------------|--------|------|------|----------------|---------|
| $N$ D5    | $\mathbb{R}_0$ | $\mathbb{R}^2_{12}$ | $-$                | $S^1_6$ | $-$  | $-$  | $\check{S}^1_9$ | $S^1_{10}$ |
| $n$ D3    | $\mathbb{R}_0$ | $\mathbb{R}^2_{12}$ | $-$                | $-$    | $-$  | $R_8$ | $-$            |
| F1        | $-$            | $-$                 | $-$                | $S^1_6$ | $-$  | $R^\geq_8$ | $-$      |
Spacetime $\mathbb{R}_0 \mathbb{R}^2_{12} \mathbb{R}^2_{45} S^1_6 R_7 R_8 \check{S}^1_9 S^1_{10}$

$N$ D5 $\mathbb{R}_0 \mathbb{R}^2_{12} - S^1_6 - - \check{S}^1_9 S^1_{10}$

$n$ D3 $\mathbb{R}_0 \mathbb{R}^2_{12} - - - R^\geq_8 -$

F1 $- - - S^1_6 - R^\geq_8 -$ 

D5: 6d $\mathcal{N} = (1, 1)$ SYM on $\mathbb{R}_0 \times \mathbb{R}^2_{12} \times S^1_6 \times \check{S}^1_9 \times S^1_{10}$

D3: codim-3 operator on $\mathbb{R}_0 \times \mathbb{R}^2_{12}$

F1: Wilson line on $S^1_6$

$\Omega$-deformation on $\mathbb{R}^2_{12}$ from nontrivial background, due to the initial twisted product in 12345 directions [Hellerman–Orland–Reffert].
**Ω-deformed 6d \( \mathcal{N} = (1, 1) \) SYM on** \( \mathbb{R}_0 \times \mathbb{R}^2_{12} \times S^1_6 \times \tilde{S}^1_9 \times S^1_{10} \)

\( \rightsquigarrow \) **Costello’s 4d Chern–Simons on** \( \mathbb{R}_0 \times S^1_6 \times \tilde{S}^1_9 \times S^1_{10} \) [Costello–Y]

**Codim-3 operators on** \( \mathbb{R}_0 \times \mathbb{R}^2_{12} \)

\( \rightsquigarrow \) **line operators on** \( \mathbb{R}_0 \)

**Wilson line on** \( S^1_6 \)

\( \rightsquigarrow \) **Wilson line on** \( S^1_6 \)
Topological on $\mathbb{R}_0 \times S^1_6$, holomorphic on $\tilde{S}^1_9 \times S^1_{10}$

$2d$ TQFT + line defects $\implies$ lattice model

$\text{TQFT} + \text{extra dimensions} \implies \text{integrability}$ [Costello]
Wilson line gives transfer matrix of elliptic QIS with

\[ \tau = \frac{i R_{10}}{\tilde{R}_9}. \]

Now, decompactify \( S^1_9 \to \mathbb{R}_9 \). Take \( R_9 \to \infty \), or \( \tilde{R}_9 \to 0 \).

This is the **trigonometric limit** \( \tau \to i\infty \).

Dependence on the position on the torus (spectral parameter) is gone in this limit.
For Nekrasov–Shatashvili, apply S-duality:

| Spacetime | $\mathbb{R}_0$ | $\mathbb{R}_{12}^2$ | $\mathbb{R}_{45}^2$ | $S_6^1$ | $R_7$ | $R_8$ | $\tilde{S}_9^1$ | $S_{10}^1$ |
|-----------|----------------|----------------------|----------------------|---------|-------|-------|--------------|-------------|
| $N$ $D5$  | $\mathbb{R}_0$ | $\mathbb{R}_{12}^2$ | $S_6^1$ | $-$ | $-$ | $-$ | $\tilde{S}_9^1$ | $S_{10}^1$ |
| $n$ $D3$  | $\mathbb{R}_0$ | $\mathbb{R}_{12}^2$ | $-$ | $-$ | $-$ | $R_8$ | $-$ | $-$ |
| $F1$      | $-$ | $-$ | $-$ | $S_6^1$ | $-$ | $R_{8}^{\geq 0}$ | $-$ | $-$ |
| Spacetime | $\mathbb{R}_0$ | $\mathbb{R}^2_{12}$ | $\mathbb{R}^2_{45}$ | $S^1_6$ | $R^7$ | $R^8$ | $\tilde{S}^1_9$ | $S^1_{10}$ |
|----------|---------------|-----------------|-----------------|-------|------|------|--------|--------|
| $N$ NS5  | $\mathbb{R}_0$ | $\mathbb{R}^2_{12}$ | $S^1_6$ | $-$ | $-$ | $-$ | $\tilde{S}^1_9$ | $S^1_{10}$ |
| $n$ D3   | $\mathbb{R}_0$ | $\mathbb{R}^2_{12}$ | $-$ | $-$ | $-$ | $R^8$ | $-$ | $-$ |
| D1       | $-$ | $-$ | $S^1_6$ | $-$ | $-$ | $R^8_{\geq 0}$ | $-$ | $-$ |

Then T-duality on $S^1_6$:

| Spacetime | $\mathbb{R}_0$ | $\mathbb{R}^2_{12}$ | $\mathbb{R}^2_{45}$ | $\tilde{S}^1_6$ | $R^7$ | $R^8$ | $\tilde{S}^1_9$ | $S^1_{10}$ |
|----------|---------------|-----------------|-----------------|--------|------|------|--------|--------|
| $N$ NS5  | $\mathbb{R}_0$ | $\mathbb{R}^2_{12}$ | $\tilde{S}^1_6$ | $-$ | $-$ | $-$ | $\tilde{S}^1_9$ | $S^1_{10}$ |
| $n$ D4   | $\mathbb{R}_0$ | $\mathbb{R}^2_{12}$ | $\tilde{S}^1_6$ | $-$ | $R^8$ | $-$ | $-$ | $-$ |
| D0       | $-$ | $-$ | $-$ | $-$ | $-$ | $R^8_{\geq 0}$ | $-$ | $-$ |
Spacetime  \( R_0 \quad R_{12}^2 \quad R_{45}^2 \quad \hat{S}_6^1 \quad R_7 \quad R_8 \quad \hat{S}_9^1 \quad S_{10}^1 \)

N NS5  \( R_0 \quad R_{12}^2 \quad - \quad \hat{S}_6^1 \quad - \quad - \quad \hat{S}_9^1 \quad S_{10}^1 \)

n D4  \( R_0 \quad R_{12}^2 \quad - \quad \hat{S}_6^1 \quad - \quad R_8 \quad - \quad - \)

D0  \( - \quad - \quad - \quad - \quad - \quad - \quad R_8^{\geq 0} \quad - \quad - \)

D4–NS5: 4d \( \mathcal{N} = 2 \) theory for \( (N + 1) \)-node linear quiver

\[
\begin{array}{c}
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\bullet \\
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\bullet \\
\bullet \\
\bullet
\end{array}
\]

placed on \( R_0 \times R_{12}^2 \times \hat{S}_6^1 \).

\( \Omega \)-deformation quantizes DW system (trigonometric Gaudin)

\[ \Rightarrow \text{noncompact XXX spin chain} \]

D0 is a local operator, acting as a transfer matrix.

Actuality, 9 & 10 directions are compact, so it's a 6d lift. We get the elliptic version of the integrable system.
Go back to M-theory

| Spacetime    | $R_0$ | $R_{12}^2$ | $S_3^1$ | $R_{45}^2$ | $S_6^1$ | $R_7$ | $R_8$ | $S_9^1$ | $S_{10}^1$ |
|--------------|-------|------------|---------|------------|---------|-------|-------|---------|-----------|
| $N M5$       | $R_0$ | $R_{12}^2$ | $S_3^1$ | $-$         | $S_6^1$ | $-$   | $-$   | $-$     | $S_{10}^1$ |
| $n M5'$      | $R_0$ | $R_{12}^2$ | $S_3^1$ | $-$         | $-$     | $-$   | $-$   | $R_8^0$ | $S_9^1$   |
| $M2$         | $-$   | $-$        | $S_3^1$ | $-$         | $S_6^1$ | $-$   | $R_8^0$| $-$     | $-$       |

Reduce on $S_{10}^1$:

| Spacetime    | $R_0$ | $R_{12}^2$ | $S_3^1$ | $R_{45}^2$ | $S_6^1$ | $R_7$ | $R_8$ | $\check{S}_9^1$ |
|--------------|-------|------------|---------|------------|---------|-------|-------|-----------------|
| $N D4$       | $R_0$ | $R_{12}^2$ | $S_3^1$ | $-$         | $S_6^1$ | $-$   | $-$   | $-$              |
| $n NS5$      | $R_0$ | $R_{12}^2$ | $S_3^1$ | $-$         | $-$     | $-$   | $R_8^0$| $\check{S}_9^1$ |
| $D2$         | $-$   | $-$        | $S_3^1$ | $-$         | $S_6^1$ | $-$   | $R_8^0$| $-$              |
|                | $\mathbb{R}$ | $\mathbb{R}^2$ | $S^1_{3}$ | $\mathbb{R}^2$ | $S^1_{6}$ | $\mathbb{R}$ | $\mathbb{R}$ | $\mathbb{R}$ |
|----------------|-------------|---------------|-----------|-------------|-----------|-------------|-------------|-------------|
| **Spacetime**  | $\mathbb{R}_0$ | $\mathbb{R}^2_{12}$ | $S^1_3$ | $\mathbb{R}^2_{45}$ | $S^1_6$ | $\mathbb{R}_7$ | $\mathbb{R}_8$ | $\mathbb{S}^1_9$ |
| **$N$ D4**     | $\mathbb{R}_0$ | $\mathbb{R}^2_{12}$ | $S^1_3$ | $-$ | $S^1_6$ | $-$ | $-$ | $-$ |
| **$n$ NS5**    | $\mathbb{R}_0$ | $\mathbb{R}^2_{12}$ | $S^1_3$ | $-$ | $-$ | $-$ | $\mathbb{R}_8$ | $\mathbb{S}^1_9$ |
| **D2**         | $-$ | $-$ | $S^1_3$ | $-$ | $S^1_6$ | $-$ | $\mathbb{R}_8 \geq 0$ | $-$ |

Apply T-duality on $S^1_9$:

|                | $\mathbb{R}$ | $\mathbb{R}^2$ | $S^1_{3}$ | $\mathbb{R}^2$ | $S^1_{6}$ | $\mathbb{R}$ | $\mathbb{R}$ | $\mathbb{R}$ |
|----------------|-------------|---------------|-----------|-------------|-----------|-------------|-------------|-------------|
| **Spacetime**  | $\mathbb{R}_0$ | $\mathbb{R}^2_{12}$ | $S^1_3$ | $\mathbb{R}^2_{45}$ | $S^1_6$ | $\mathbb{R}_7$ | $\mathbb{R}_8$ | $\mathbb{S}^1_9$ |
| **$N$ D5**     | $\mathbb{R}_0$ | $\mathbb{R}^2_{12}$ | $S^1_3$ | $-$ | $S^1_6$ | $-$ | $-$ | $\mathbb{S}^1_9$ |
| **$n$ NS5**    | $\mathbb{R}_0$ | $\mathbb{R}^2_{12}$ | $S^1_3$ | $-$ | $-$ | $-$ | $\mathbb{R}_8$ | $\mathbb{S}^1_9$ |
| **D3**         | $-$ | $-$ | $S^1_3$ | $-$ | $S^1_6$ | $-$ | $\mathbb{R}_8 \geq 0$ | $\mathbb{S}^1_9$ |
| Spacetime   | $\mathbb{R}_0$ | $\mathbb{R}^2_{12}$ | $S^1_3$ | $S^1_{45}$ | $S^1_6$ | $\mathbb{R}_7$ | $\mathbb{R}_8$ | $\tilde{S}^1_9$ |
|------------|----------------|----------------------|--------|------------|--------|----------------|------------|----------------|
| $N$ D5     | $\mathbb{R}_0$ | $\mathbb{R}^2_{12}$ | $S^1_3$ | $-$        | $S^1_6$ | $-$            | $-$        | $\tilde{S}^1_9$ |
| $n$ NS5    | $\mathbb{R}_0$ | $\mathbb{R}^2_{12}$ | $S^1_3$ | $-$        | $-$    | $-$            | $\mathbb{R}^\geq_8$ | $\tilde{S}^1_9$ |
| D3         | $-$             | $-$                  | $S^1_3$ | $-$        | $S^1_6$ | $-$            | $\mathbb{R}^\geq_8$ | $\tilde{S}^1_9$ |

**D5–NS5**: 5d circular quiver theory on $\mathbb{R}_0 \times \mathbb{R}^2_{12} \times \epsilon S^1_3 \times \tilde{S}^1_9$

**D3**: surface defect on $S^1_3 \times \tilde{S}^1_9$

We can add more NS5s, preserving 4d $\mathcal{N} = 1$ SUSY on $\mathbb{R}^2_{12} \times \epsilon S^1_3 \times \tilde{S}^1_9$. This leads to the brane tiling story [Maruyoshi–Y].
Summary

• We considered a class of Wilson–’t Hooft lines in 4d $\mathcal{N} = 2$ circular quiver theories.

• We found that they can be identified with transfer matrices of trigonometric QIS.

• By embedding into string theory, this correspondence can be related to other known correspondences via dualities.
Further directions

- Surface defects in 5d circular quiver theory correspond to transfer matrices of elliptic QIS.
- Variations of the present setup
- Circular quiver theories deconstruct 6d $\mathcal{N} = (2, 0)$ SCFT. Integrability is behind surface operators in 6d theory.