Thurston has claimed (unpublished) that central extensions of word hyperbolic groups by finitely generated abelian groups are automatic. We show that they are in fact biautomatic. Further, we show that every 2-dimensional cohomology class on a word hyperbolic group can be represented by a bounded 2-cocycle. This lends weight to the claim of Gromov that for a word hyperbolic group, the cohomology in every dimension is bounded.

Our results apply more generally to virtually central extensions. We build on the ideas presented in [NR], where the general problem was reduced to the case of central extensions by \( \mathbb{Z} \) and was solved for Fuchsian groups. Some special cases of automaticity or biautomaticity in this case had previously been proved in [ECHLPT], [Sha], and [Ge].

The new ingredient is a maximising technique inspired by work of Epstein and Fujiwara. Beginning with an arbitrary finite generating set for a central extension by \( \mathbb{Z} \), this maximising process is used to obtain a section which, in the language of [NR], corresponds to a “regular 2-cocycle” on the hyperbolic group \( G \), and can be used to obtain a biautomatic structure for the extension. Since central extensions correspond to 2-dimensional cohomology classes, it follows that every such class can be represented by a regular 2-cocycle. Using the geometric properties of \( G \), we then further modify this cocycle to obtain a bounded representative for the original cohomology class.

We also discuss the relations between various concepts of “weak boundedness” of a 2-cocycle on an arbitrary finitely generated group, related to quasi-isometry properties of central extensions. For cohomology classes these weak boundedness concepts are shown to be equivalent to each other. We do not know if a weakly bounded cohomology class must be bounded.

1. Preliminaries

Let \( G \) be a finitely generated group and \( X \) a finite set which maps to a monoid generating set of \( G \). The map of \( X \) to \( G \) can be extended in the obvious way to give a monoid homomorphism of \( X^* \) onto \( G \) which will be denoted by \( w \mapsto \overline{w} \). For convenience of exposition we will always assume our generating sets are symmetric, that is, they satisfy \( \overline{X} = \overline{X}^{-1} \). If \( L \subset X^* \) then the pair consisting of \( L \) and the evaluation map \( L \to G \) will be called a language on \( G \). Abusing terminology, we will often suppress the evaluation map and just call \( L \) the language on \( G \) (but therefore, we may use two letters, say \( L \) and \( L' \), to represent the same language \( L \subset X^* \) with two different evaluation maps to two different groups). A language on \( G \) is a normal form if it surjects to \( G \).

A rational structure for \( G \) is a normal form \( L \subset X^* \) for \( G \) which is a regular language (i.e., the set of accepted words for some finite state automaton). A subset \( S \) of \( G \) is then \( L \)-rational if the set of \( w \in L \) with value in \( S \) is a regular language.
The Cayley graph $\Gamma_X(G)$ is the directed graph with vertex set $G$ and a directed edge from $g$ to $g\mathbf{x}$ for each $g \in G$ and $x \in X$; we give this edge a label $x$.

Each word $w \in X^*$ defines a path $[0, \infty) \to \Gamma$ in the Cayley graph $\Gamma = \Gamma_X(G)$ as follows (we denote this path also by $w$): $w(t)$ is the value of the $t$-th initial segment of $w$ for $t = 0, \ldots, \text{len}(w)$, is on the edge from $w(s)$ to $w(s + 1)$ for $s < t < s + 1 \le \text{len}(w)$ and equals $\mathbf{w}$ for $t \ge \text{len}(w)$. We refer to the translate by $g \in G$ of a path $w$ by $gw$.

Let $\delta \in \mathbb{N}$. Two words $v, w \in X^*$ synchronously $\delta$-fellow-travel if the distance $d(w(t), v(t))$ never exceeds $\delta$. They asynchronously $\delta$-fellow-travel if there exist non-decreasing proper functions $t \mapsto t', t \mapsto t'': [0, \infty) \to [0, \infty)$ such that $d(v(t''), w(t'')) \le \delta$ for all $t$.

A rational structure $L$ for $G$ is a synchronous resp. asynchronous automatic structure if there is a constant $\delta$ such that any two words $u, v \in L$ with $d(\mathbf{u}, \mathbf{v}) \le 1$ synchronously resp. asynchronously fellow-travel. A synchronous automatic structure $L$ is synchronously biautomatic if there is a constant $\delta$ such that if $v, w \in L$ satisfy $\mathbf{w} = \mathbf{v} \mathbf{w}$ with $x \in X$ then $\mathbf{w}v$ and $w$ synchronously $\delta$-fellow-travel. See [NS1] for a discussion of the relationship of these definitions with those of [ECHLPT]. In particular, as discussed there, if $L \to G$ is finite-to-one, then the definitions are equivalent; by going to a sublanguage of $L$ this can always be achieved.

Two rational structures $L_1$ and $L_2$ on $G$ are equivalent if there is a $\delta$ such that each $w \in L_1$ asynchronously $\delta$-fellow-travels some $v \in L_2$ with $\mathbf{w} = \mathbf{v}$ and vice versa. A subset of $G$ is then $L_1$-rational if and only if it is $L_2$-rational (see [NS2]). A word-hyperbolic group has a unique (bi)-automatic structure up to equivalence, given by the language of geodesic words for any finite generating set (see [NS1]).

If $G$ is a group and $A$ an abelian group then an extension

$$0 \to A \to E \to G \to 1$$

is called virtually central if the induced map $G \to \text{Aut}(A)$ has finite image (we say the action of $G$ on $A$ is finite). In [NR] the problem of finding a biautomatic structure on $E$ is reduced to the case of central extensions by $A = \mathbb{Z}$. We describe this reduction now.

We write $A$ additively and we denote the action of an element $g \in G$ on $A$ by $a \mapsto a^g$. Choose a section $s : G \to E$. Then a general element of $E$ has the form $s(g)\iota(a)$ with $g \in G$ and $a \in A$ and the group structure in $E$ is given by a formula

$$s(g_1)\iota(a_1)s(g_2)\iota(a_2) = s(g_1g_2)\iota(a_1^{g_2} + a_2 + \sigma(g_1, g_2)),$$

where $\sigma : G \times G \to A$ is a 2-cocycle on $G$ with coefficients in $A$. Changing the choice of section changes the cocycle $\sigma$ by a coboundary. Conversely, given a cocycle $\sigma$, that is, a function $G \times G \to A$ which satisfies the cocycle relation

$$\sigma(g, h_1h_2) = \sigma(g, h_1)^{h_2} + \sigma(gh_1, h_2) - \sigma(h_1, h_2),$$

the above multiplication rule defines a virtually central extension of $G$ by $A$. 


**Definition.** Suppose $G$ has finite generating set $X$ and $L \subset X^*$ is an asynchronous automatic structure on $G$. We say a 2-cocycle $\sigma$ as above is *weakly bounded* if

1. The sets $\sigma(g, G)$ and $\sigma(G, g)$ are finite for each $g \in G$ (equivalently, $\sigma(X, G)$ and $\sigma(G, X)$ are both finite — this equivalence follows from the cocycle relation); and is $L$-regular if in addition
2. For each $h \in G$ and $a \in A$ the subset \( \{ g \in G : \sigma(g, h) = a \} \) is an $L$-rational subset of $G$ (it suffices to require this for $h \in X$ by [NR, Lemma 2.1]).

A cohomology class in $H^2(G; A)$ is $L$-regular if it can be represented by an $L$-regular cocycle. The term “weakly bounded” reflects the standard terminology of *bounded* for a cocycle that satisfies $\sigma(G, G)$ finite.

The following lemma is elementary:

**Lemma 1.1.**

1. $L$-regular cohomology classes form a subgroup $H^2_L(G; A) \subset H^2(G; A)$ which only depends on the equivalence class of $L$.
2. If $A \to B$ is an equivariant map of finitely generated abelian groups with finite $G$-actions then the induced map $H^2(G; A) \to H^2(G; B)$ maps $H^2_L(G; A)$ to $H^2_L(G; B)$.

The following is proved in [NR].

**Theorem 1.2.** Let $0 \to A \to E \to G \to 1$ be a virtually central extension with $A$ finitely generated. Then $E$ has a biautomatic structure if and only if $G$ has a biautomatic structure $L$ for which the cohomology class of the extension is in $H^2_L(G; A)$.

Our first main result can thus be formulated:

**Theorem 1.3.** Suppose $G$ is word hyperbolic and $L$ is the biautomatic structure on $G$ (which is unique up to equivalence). If $A$ is any finitely generated abelian group with finite $G$-action then $H^2_L(G; A) = H^2(G; A)$. In particular, any virtually central extension of $G$ by $A$ is biautomatic.

In the remainder of this section we show that this theorem follows from the special case that $A = \mathbb{Z}$ with trivial $G$-action. Assume therefore that it is always true in this case.

Recall (e.g., [NS1]) that if $H < G$ is a subgroup of finite index and $L$ is a biautomatic structure on $G$ then there is an induced biautomatic structure $L_H$ on $H$ which is unique up to equivalence.

**Lemma 1.4.** ([NR, Corollaries 2.5 and 2.7]). Let $L$ be a biautomatic structure on $G$. Suppose $\alpha \in H^2(G; A)$.

1. Let $H < G$ be a subgroup of finite index and suppose $\beta \in H^2(H; A)$ is the restriction of $\alpha$ to $H$. Then $\alpha$ is $L$-regular if and only if $\beta$ is $L_H$-regular.
2. A non-zero multiple of $\alpha$ is $L$-regular if and only if $\alpha$ is.

We restrict to the kernel $H$ of the $G$-action on $A$. For $\alpha \in H^2(G; A)$ we denote its restriction by $\beta \in H^2(H; A)$. By part 1 of the lemma it suffice to show $\beta$ is regular. Write $A$ as a direct sum of a finite group $F$ and copies of $\mathbb{Z}$ as follows: $A = F \oplus \bigoplus_{i=1}^n \mathbb{Z}$. Then $H^2(H; A) = H^2(H; F) \oplus \bigoplus_{i=1}^n H^2(H; \mathbb{Z})$. For each $j = 1, \ldots, n$ we can form $Z_j = A/(F \oplus \bigoplus_{i \neq j} \mathbb{Z})$ and the induced class $\beta_j \in H^2(H; Z_j)$ is regular by assumption. We can
embed \( \mathbb{Z}_j \) in \( A \) as the \( j \)-th summand \( \mathbb{Z} \) and thus lift \( \beta_j \) to a regular class \( \beta'_j \in H^2(H; A) \). Now \( \beta \) differs from the regular class \( \beta' := \sum_{i=1}^n \beta'_j \) by torsion, so \( \beta \) and \( \beta' \) have a common nontrivial multiple. Thus \( \beta \) is regular by part 2 of the Lemma.

2. Central Extensions of Hyperbolic Groups are Biautomatic

In view of the preceding discussion we only need show that it \( G \) is a hyperbolic group and

\[
0 \rightarrow \mathbb{Z} \xrightarrow{i} E \xrightarrow{\pi} G \rightarrow 1
\]

is a central extension of \( G \) then \( E \) is biautomatic.

Let \( X \) be a finite set which maps to a symmetric generating set for \( E \). Then \( X \) also maps to a generating set for \( G \). We use the notations \( \overline{x} \) and \( \pi(\overline{x}) \) for evaluation of \( x \) into \( E \) and \( G \). Fix some finite presentation \( \langle X | R \rangle \) for \( G \).

For each \( g \in G \) the fibre \( \pi^{-1}(g) \) has a total ordering determined by the usual ordering on \( \mathbb{Z} \). It therefore makes sense to talk about existence of a “maximum” for a subset of this fiber.

Lemma 2.1. There exists \( C > 0 \) such that for all \( g \in G \)

\[
\max\{\overline{w}i(-C \text{len}(w)) : w \in X^*, \pi(\overline{w}) = g\}
\]

exists. Moreover, there then exists \( \lambda > 0 \) so that any word \( w \) which achieves the maximum defines a \((\lambda,0)\)-quasigeodesic in \( G \).

Proof. Let \( T \in \mathbb{Z} \) be given by \( i(T) = \max\{\overline{r} : r^{\pm 1} \text{ is a relator in } G\} \), and let \( K \) be the constant in the linear isoperimetric inequality for \( G \), that is, if \( \pi(\overline{w}) = 1 \) then the combinatorial area bounded by \( w \) is less than \( K \text{len}(w) \). In particular, we then have \( \overline{w} \leq i(T \text{area}(w)) \leq i(TK \text{len}(w)) \). Choose \( C \in \mathbb{Z} \) with \( C > TK \).

Fix an element \( g \in G \), and suppose that \( w, \gamma \in X^* \) satisfy \( \pi(\overline{w}) = \pi(\overline{\gamma}) = g \). We have

\[
\overline{\gamma}^{-1}\overline{w} \leq i(TK \text{len}(\gamma^{-1}w)) = i(TK \text{len}(\gamma))i(TK \text{len}(w))
\]

So,

\[
\overline{w}i(-C \text{len}(w)) \leq \overline{w}i(-TK \text{len}(w)) \\
\leq \overline{\gamma}i(TK \text{len}(\gamma))
\]

and it follows that the maximum exists in the lemma. We shall call a word which achieves the maximum in Lemma 2.1 a maximising word.

Now assume that \( w \) is maximising and that \( \gamma \) defines a geodesic in \( G \). Then

\[
\overline{\gamma}i(-C \text{len}(\gamma)) \leq \overline{w}i(-C \text{len}(w))
\]

so

\[
i(C(\text{len}(w) - \text{len}(\gamma))) \leq \overline{\gamma}^{-1}\overline{w}
\]

Combining (1) and (2) gives \( C(\text{len}(w) - \text{len}(\gamma)) \leq TK(\text{len}(\gamma) - \text{len}(w)) \), and hence

\[
\text{len}(w) \leq ((C + TK)/(C - TK))\text{len}(\gamma).
\]

Since a subword of a maximising word is itself a maximising word, this says that \( w \) is \((\lambda,0)\)-quasigeodesic in \( G \) with \( \lambda = ((C + TK)/(C - TK)) \).
Define a new section $\rho: G \to E$ by setting $\rho(g) = \overline{\pi}(\overline{-C\text{len}(w)})$, where $w$ is a maximising word such that $\pi(\overline{w}) = g$. The following is the main result of this section.

**Proposition 2.2.** The cocycle defined by the section $\rho$ is a regular cocycle.

**Proof.** For elements $h_1, h_2$ in a common fibre of $E \to G$ we shall denote by $h_1 - h_2$ the integer with $\nu(h_1 - h_2) = h_1h_2^{-1}$. The cocycle is then defined by the formula

$$\sigma(g_1, g_2) = \rho(g_1)\rho(g_2) - \rho(g_1g_2).$$

The weak boundedness of $\sigma$ is therefore the statement that $\rho(g)\rho(x) - \rho(gx)$ and $\rho(x)\rho(g) - \rho(xg)$ are bounded functions of $(g, x) \in G \times X$. The following lemma clearly implies this, since $X$ is finite.

**Lemma 2.3.** Suppose $g \in G$ and $x \in X$. Then
1. $|\rho(g)x - \rho(gx)| \leq C$.
2. $|\overline{\pi}(\rho(g)) - \rho(xg)| \leq C$.

**Proof.** We prove part 1. Choose maximising words $w_1$ and $w_2$ with $\pi(\overline{w_1}) = g$ and $\pi(\overline{w_2}) = gx$. Since $w_2$ is maximising and $\pi(\overline{w_1x}) = \pi(\overline{w_2})$ we have

$$\overline{w_1}\nu(-C\text{len}(w_1))\overline{\pi}(-C\nu(h)) = \overline{w_2}\nu(-C\text{len}(w_2)),$$

for some $h \geq 0$. By symmetry, since $\pi(\overline{w_1}) = \pi(\overline{w_2})$, we have

$$\overline{w_2}\nu(-C\text{len}(w_2))\overline{\pi}(-C\nu(k)) = \overline{w_1}\nu(-C\text{len}(w_1)),$$

for some $k \geq 0$. Comparing, we see $-C \leq h - C = C - k \leq C$ and the result follows. Part 2 is completely analogous.

Let $L \subset X^*$ be the language of maximising words. Since $L$ consists of $(\lambda, 0)$ quasi-geodesics in $G$ and quasigeodesics fellow-travel geodesics (see [G] or [Sho]), the language $L$ has the asynchronous fellow traveller property. We shall show next that it is a regular language, so it is an asynchronous automatic structure which is equivalent to the geodesic automatic structure on $G$. (It is not necessarily itself synchronously automatic — for example it fails to be synchronous if $G$ is a surface group and $E$ a non-trivial extension.) Let $\delta$ be the fellow-traveller constant for $L$.

We first note that if $u \in X^*$ is a word that is not in $L$ then this fact can be recognised via a $\delta$-fellow-traveller. That is, there is a $v \in X^*$ with $\pi(u) = \pi(v)$ such that $v$ and $u$ $\delta$-fellow-travel and $v$ is “better” than $u$ in that $\overline{\nu}(-C\text{len}(u)) < \overline{\nu}(-C\text{len}(v))$. Indeed, we obtain $v$ by taking the shortest initial segment $u_1$ of $u$ that fails to be in $L$ (i.e., is not maximising) and replacing it by a maximising word $v_1$. This $u_1$ and $v_1$ $\delta$-fellow travel because $u_1$ has the form $u_0x$ with $u_0 \in L$ and $x \in X$.

Given this observation, called “falsification by fellow-traveller” in [NS2], the regularity of $L$ is a standard argument, originally due to Cannon (see [NS2, Proposition 4.1]). We sketch it here for completeness.

To test if a path $u$ is maximising, as we move along $u$ we must keep track at each time $t$ of what points $u(t)/h$ in a $\delta$-neighborhood of $u(t)$ have been reached by paths $v$ that
asynchronously $\delta$-fellow travel $u$. Moreover, for each such $g$ we must record the difference in “height” of $u(t)\epsilon(-Ct)$ and $\overline{vw}u(-C\text{len}(vw))$ where $w$ is a geodesic with value $h^{-1}$. If this difference is larger than $2\delta C$ then $v$ is not a viable competitor to $u$ and we can ignore it (following $u$ to $u(t)$ and then $w^{-1}$ to $\pi(v)$ does better than $v$). If this difference is negative then the path $vw$ does better than $u$, so $u$ is not in $L$. We are thus only interested in differences that lie in the interval $[0, 2\delta C]$. At each point $h$ of the ball $B(\delta)$ of radius $\delta$ about 1 in the Cayley graph we record the “best” value of this difference for paths $v$ ending at $u(t)h$. Thus the information that we must keep track of is a function $\phi: B(\delta) \to \{0, 1, \ldots, 2\delta C\}$. We can build a finite state automaton with the set of such maps as states plus one “fail state,” since, as one moves along $u$, the state at any point of the path $u$ can clearly be computed from its value at the previous point and the last letter of $u$ read. Once we reach the end of $u$, we know $u$ is in $L$ if and only if we have not yet reached the fail state. Thus $L$ is recognised by a finite state automaton so it is a regular language.

We now lift $L$ via the section $\rho$ as follows. For $x \in X$ denote $x' := \overline{\pi}(\delta^C)$. Denote by $L'$ the language $L$ with the evaluation map $x \mapsto x'$ instead of $x \mapsto \overline{\pi}$. If $v = x_1x_2\ldots x_n \in L'$ then the initial segments of $v$ have values $\rho(\pi(x_1)), \rho(\pi(x_1x_2)), \ldots, \rho(\pi(x_1\ldots x_n))$. $L'$ is a regular language mapping onto the image of our section $\rho$ with asynchronous fellow-traveller property. As in the proof of Proposition 2.2 of [NR], this implies that the cocycle $\sigma$ defined by $\rho$ is regular. Indeed, for any $x \in X$ and $a \in \mathbb{Z}$ the set $\{(w_1, w_2) : w_1, w_2 \in L, \rho(\overline{w_1})\rho(\overline{w_2}) = \rho(\overline{a})\}$ is the language of an asynchronous two-tape automaton so its projection to the first factor is a regular language. But that projection is $\{g : \sigma(g, x) = a\}$, so this set is rational, as required.

We have thus proved the biautomaticity of $E$. It is worth recalling from [NR] that a biautomatic structure is found by taking a finite-to-one automatic structure on $G$ (e.g., the geodesic language) and lifting it using the section $\sigma$ to get a language $M$ evaluating onto $\sigma$ (the alphabet for $M$ consists of elements of the form $\overline{\pi}(\delta^C)$ with $g \in G$ and $x \in X$). Then $M\{(t)^*, \{t^{-1}\}\}$ is the desired biautomatic structure, where $t$ is a generator of $\mathbb{Z}$. The language $L'$ of lifted maximising words cannot be used instead of $M$ since it is not generally synchronous.

3. Bounded Cohomology

The main result of this section is

**Theorem 3.1.** Suppose $G$ is word hyperbolic and $A$ is any finitely generated abelian group with finite $G$-action then any class in $H^2(G; A)$ can be represented by a bounded cocycle.

As in section 1, we can reduce to the case that $A = \mathbb{Z}$ with trivial $G$-action. Given a 2-cocycle, and hence some central extension, in the previous section we defined a section $\rho$ determining a cohomologous cocycle which was weakly bounded. In this section we modify $\rho$ to define a section for which the corresponding cocycle is bounded. That is, given a weakly bounded 2-cocycle on a word hyperbolic group we produce a cohomologous bounded 2-cocycle.
Lemma 3.3. Suppose that \( k \) is a fixed constant. Then there is a constant \( K(k) \) such that if \( g, h \in G \) are such that \( g \) lies within distance \( k \) of a geodesic path from 1 to \( gh \), then 
\[
d_E(\rho(gh), \rho(g)\rho(h)) \leq K(k).
\]

Proof. Our section \( \rho \) has the property that if \( g \) lies on the maximising path from 1 to \( gh \) then \( \rho(gh) = \rho(g)\rho(h) \). The weak boundedness of the cocycle associated to our section thus implies that if \( g \) lies a bounded distance from the maximising path to \( gh \) then 
\[
d_E(\rho(gh), \rho(g)\rho(h)) \text{ is bounded. Since maximising paths fellow travel geodesics, the lemma follows.}
\]

To show boundedness of our cocycle \( \sigma \), we shall need that the section is symmetric. To ensure this we define a new section \( q \) by taking the average of \( \rho(g) \) and \( \rho(g^{-1})^{-1} \), that is, 
\[
q(g) = \frac{1}{2}(\rho(g) + \rho(g^{-1})^{-1}).
\]

This section is defined in the pushout extension 
\[
0 \to \mathbb{Q} \to E_\mathbb{Q} \to G \to 1
\]
determined by the embedding \( \mathbb{Z} \to \mathbb{Q} \). Note that although the fibres of \( E_\mathbb{Q} \to G \) only have an affine structure, the average of two points in a fibre is nevertheless well defined. We shall show that the cocycle defined by this section is bounded. It then follows easily that the cocycle defined by the section \([q]: G \to E \) (integral part of \( q \)) is bounded.

It is clear from the definition that

Lemma 3.3. For all \( g \in G \), \( q(g^{-1}) = q(g)^{-1} \).

Proposition 3.4. The cocycle determined by \( q \) is bounded.

Proof. We need to show that there is a bound on the values attained by \( q(gh)^{-1}q(g)q(h) \), for \( g, h \in G \). For \( g \in G \) denote by \( w_g \) a geodesic path from 1 to \( g \). Consider the geodesic triangle in \( G \) with sides labelled by \( w_{gh}, w_g \), and \( gw_h \). Since \( G \) is hyperbolic there is a constant \( k \) such that for all such triangles there is a \( c \in G \) such that \( c \) lies within distance \( k \) of each of the sides. Notice that \( q \) inherits from \( \rho \) the property described in Lemma 3.2. That is replacing \( q(g) \) by \( q(c)q(c^{-1}g) \) alters the value by a bounded amount. Similarly we replace \( q(h) \) by \( q(g^{-1}c)q(c^{-1}gh) \), and \( gh \) by \( q(c)q(c^{-1}gh) \). So, \( q(gh)^{-1}q(g)q(h) \) differs from \( (q(c)q(c^{-1}gh))^{-1}q(c)q(c^{-1}g)q(g^{-1}c)q(c^{-1}gh) = 1 \) by a bounded amount.

Remark. It is not hard to verify that the averaging process described in this section to make our cocycle bounded does not destroy regularity, so, in fact, any class in \( H^2(G; A) \) has a bounded regular representative.

4. Weakly bounded cocycles

In this section we do not assume \( G \) is word hyperbolic. We do assume \( G \) is finitely generated. We shall call a 2-cocycle \( \sigma \) on \( G \) left weakly bounded if \( \sigma(g,G) \) is bounded for all \( g \in G \) and right weakly bounded if \( \sigma(G,g) \) is bounded for all \( g \in G \). So a cocycle is weakly bounded if and only if it is both left and right weakly bounded.
In [NR] we pointed out that if a central extension $0 \to A \to E \to G \to 1$ is given by a right weakly bounded cocycle then $E$ is quasi-isometric to $G \times A$. In fact, the corresponding section $s: G \to E$ is quasi-isometric and induces a quasi-isometry $G \times A \to E$ by $(g, a) \mapsto s(g) \varsigma(a)$. Conversely, if one has a quasi-isometric section then the corresponding cocycle is right weakly bounded. We asked there about the relation between the concepts right weakly bounded, weakly bounded, and bounded. We still do not know if a weakly bounded cohomology class is always bounded, but we have:

**Theorem 4.1.** A cocycle which is either right or left weakly bounded is always cohomologous to one which is both right and left weakly bounded.

**Proof.** The same argument as in section 1 can be used to reduce to the case that $A = Z$. We assume therefore that we have a central extension $0 \to Z \to E \to G \to 1$ given by a right weakly bounded cocycle (the left weakly bounded case is completely analogous). As usual, we assume we have a symmetric generating set $X$ for $E$, and we also consider it as a generating set for $G$. Let $s: G \to E$ be a section whose cocycle $\sigma$ is right weakly bounded. Since changing $\sigma$ at finitely many points does not destroy right weak boundedness, we may assume $s(1) = 1$ and $s(\pi(x)) = \pi$ for $x \in X$. Let $K$ be a bound with $|\sigma(g, \pi(x))| \leq K$ for $g \in G$ and $x \in X$. Then if $w = x_1 x_2 \ldots x_n$ is any word we have $s(\pi(w(t))) = s(\pi(w(t-1)))\pi t(-\sigma(w(t-1), \pi(x_t)))$ for each $t$. Thus if $w$ represents 1 in $G$ we have by induction $1 = s(\pi(w)) = \bar{x}_1 \bar{x}_2 \ldots \bar{x}_n \varsigma(-\sum_{t=1}^{n} \sigma(w(t-1), \pi(x_t)))$. Thus $\overline{w} = \overline{x}_1 \ldots \overline{x}_n$ is bounded by $nK$ in absolute value. Thus if we choose $C > K$ the proof of Lemma 2.1 goes through. The section defined by maximising paths gives, as in section 2, a weakly bounded cocycle. $lacksquare$

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