LOCALIZATION LENGTHS FOR SCHRÖDINGER OPERATORS ON $\mathbb{Z}^2$ WITH DECAYING RANDOM POTENTIALS

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Abstract. We study a class of Schrödinger operators on $\mathbb{Z}^2$ with a random potential decaying as $|x|^{-\sigma}$, $0 < \sigma \leq \frac{1}{2}$, in the limit of small disorder strength $\lambda$. For the critical exponent $\sigma = \frac{1}{2}$, we prove that the localization length of eigenfunctions is bounded below by $2\lambda^{-\frac{1}{4}} + \eta$, while for $0 < \sigma < \frac{1}{2}$, the lower bound is $\lambda^{-\frac{1}{2}} - \eta$, for any $\eta > 0$. These estimates "interpolate" between the lower bound $\lambda^{-\frac{1}{2}} + \eta$ due to recent work of Schlag-Shubin-Wolff for $\sigma = 0$, and pure a.c. spectrum for $\sigma > \frac{1}{2}$ demonstrated in recent work of Bourgain.

1. Introduction

We study the discrete random Schrödinger operator

$$H_\omega = \Delta + \lambda V_\omega$$

on $l^2(\mathbb{Z}^2)$, where $\Delta$ is the (centered) nearest neighbor Laplacian, with spectrum $[-4, 4]$, and $\lambda$ is a small parameter (the disorder strength). The random potential is given by $V_\omega(x) = v_\sigma(x)\omega_x$, where $v_\sigma(x) \sim |x|^{-\sigma}$ and $\{\omega_x\}_{x \in \mathbb{Z}^2}$ are Gaussian i.i.d. random variables. The restriction to Gaussian randomness has expository advantages, but is not essential for our techniques to apply. Extension of our methods to non-Gaussian random potentials can be accessed along the lines demonstrated in [3]. The purpose of this paper is to derive lower bounds on the localization lengths of eigenfunctions of $H_\omega$.

In the supercritical case $\sigma > \frac{1}{2}$, it was proven by Bourgain in [11] that with large probability, $H_\omega$ (with Bernoulli or Gaussian randomness) has, for small $\lambda$, pure a.c. spectrum in $(-4 + \tau, -\tau) \cup (\tau, 4 - \tau)$ ($\tau > 0$ arbitrary, but fixed); moreover, the wave operators were constructed, and asymptotic completeness was established. The (generalized) eigenfunctions are therefore delocalized. Certain other classes of lattice Schrödinger operators with decaying random potentials have been proven to exhibit a.c. spectrum, scattering, and asymptotic completeness by Bourgain in [2], and by Rodnianski and Schlag in [10]. We also note the contextually related work of Denissov in [5].

In the case $\sigma = 0$, Schlag, Shubin and Wolff have proven lower bounds on the localization length of eigenfunctions of the form $\lambda^{-2 + \eta}$, for any $\eta > 0$, [11]. For $\sigma = 0$ and $d = 3$, lower bounds of the form $\lambda^{-2|\log \lambda|^{-1}}$ were derived in [4].

We shall here address the case $0 < \sigma \leq \frac{1}{2}$ in dimension two. Our main results are as follows.
For the critical decay exponent \( \sigma = \frac{1}{2} \), the problem is marginal in the language of renormalization group theory. Accordingly, we obtain a comparison of the logarithm of the localization length to powers of \( \lambda \), yielding lower bounds on the localization length that are exponential in \( \frac{1}{\lambda} \), of the form \( 2^{\lambda^{-\frac{1}{2} + \eta}} \) (\( \eta > 0 \) arbitrary).

In the subcritical case \( 0 < \sigma < \frac{1}{2} \), it is suspected that the model exhibits a significant component of point spectrum. In the language of renormalization group theory, the potential scales like a relevant perturbation, whereby we obtain a comparison of the localization length to powers of \( \lambda \). Consequently, our lower bounds on the localization lengths are polynomial in \( \frac{1}{\lambda} \) for \( 0 < \sigma < \frac{1}{2} \), of the form \( \lambda^{-\frac{1}{2} - \eta} \) (\( \eta > 0 \) arbitrary).

On the one hand, our strategy employs graph expansion methods due to Erdős and Yau \([7, 8]\), and further elaborated on by the author \([3, 4]\). On the other hand, we use a smoothing of resolvent multipliers by dyadic restriction, inspired by Bourgain’s approach in \([1]\). Our methods can be extended to higher dimensions, but we will here only focus on the case \( d = 2 \).

The following works, which determine macroscopic hydrodynamic limits of the quantum dynamics in the Anderson model at small disorders (without spatial decay, i.e. \( \sigma = 0 \)), are closely related to the topics discussed here. In an important early work, Spohn proved in \([12]\) that the kinetic macroscopic scaling and low coupling limit is determined by a linear Boltzmann equation, locally in macroscopic time. Erdős and Yau proved the corresponding global in macroscopic time result for the continuum model in \( \mathbb{R}^d, d = 2, 3 \), and Gaussian randomness, \([8]\), which was extended by Erdős to the case of a Schrödinger electron interacting with a phonon heat bath, \([7]\). The author derived the corresponding result for the lattice \( \mathbb{Z}^3 \) and non-Gaussian randomness, \([3]\), and proved that the mode of convergence can be extended to \( r \)-th mean, for any \( r \in \mathbb{R}_+ \) (the previous works proved convergence in expectation), \([4]\). Eng and Erdős proved the corresponding result for the kinetic macroscopic and low density limit, \([6]\). Very recently, Erdős, Salmhofer and Yau established the breakthrough result that beyond kinetic scaling, the macroscopic dynamics is governed by a diffusion equation, \([9]\).

2. Definition of the model and statement of the main results

We consider the discrete random Schrödinger operator

\[
H_\omega = \Delta + \lambda V_\omega
\]

on \( l^2(\mathbb{Z}^2) \), with a radially decaying potential function

\[
V_\omega(x) = v_\sigma(x) \omega_x ,
\]

where \( \{ \omega_x \}_{x \in \mathbb{Z}^2} \) are independent, identically distributed Gaussian random variables normalized by \( \mathbb{E}[\omega_x] = 0, \mathbb{E}[\omega_x^2] = 1 \), for all \( x \in \mathbb{Z}^2 \). Expectations of higher powers of \( \omega_x \) satisfy Wick’s theorem, see \([8]\), and our discussion below.
We shall use the convention
\[ \mathcal{F}(f)(k) \equiv \hat{f}(k) = \sum_{x \in \mathbb{Z}^2} e^{-2\pi ikx} f(x) \]
(4)
\[ \mathcal{F}^{-1}(g)(x) \equiv \tilde{g}(x) = \int_{\mathbb{T}^2} dk \ e^{2\pi ikx} g(k) \]
for the Fourier transform and its inverse, where \( T := [-\frac{1}{2}, \frac{1}{2}] \).

We introduce a partition of unity \( \sum_{j=0}^{\infty} P_j = 1 \) on \( \mathbb{Z}^2 \), where \( P_j \sim \chi(2^j < |x| \leq 2^{j+1}) \), \( j \in \mathbb{N}_0 \), is an approximate characteristic functions for a dyadic shell of scale \( 2^j \). We require that
\[ |\mathcal{F}(P_j P_{j'} v_\sigma^2)| \leq C 2^{-2\sigma j} |\mathcal{F}(P_j P_{j'})| \sim C 2^{-2\sigma j} |\mathcal{F}(P_j^2)| \]
(5)
for a constant \( C \) independent of \( j, j' \).

The centered nearest neighbor lattice Laplacian \( \Delta \) defines the Fourier multiplier
\[ \mathcal{F}(\Delta f)(k) = e_{\Delta}(k) \hat{f}(k) , \]
where
\[ e_{\Delta}(k) = 2 \cos(2\pi k_1) + 2 \cos(2\pi k_2) \]
is the quantum mechanical kinetic energy of the electron.

For almost every realization of \( V_\omega \), \( H_\omega \) is a selfadjoint operator on \( \ell^2(\mathbb{Z}^2) \).

We shall use the same argument for the determination of the localization length of eigenfunctions of \( H_\omega \) as in \[3\]. Let \( L > e^{-\lambda^{-2}} \), and
\[ \Lambda_L := [-L, L]^{2} \cap \mathbb{Z}^2 . \]
(10)
For \( \ell \ll L \) and \( x \in \Lambda_L \), let
\[ R_{x,\delta,\ell} \sim \chi( \{ y \in \mathbb{Z}^2 | |y| \leq |x| \leq |x| + \ell \} ) \]
(11)
denote an approximate characteristic function supported on a cubical shell centered at \( x \), of outer and inner side lengths \( \ell \) and \( \delta \ell \), respectively. We shall adopt the choice for \( R_{x,\delta,\ell} \) from \[3\], which is a product of differences of Fejér kernels with
\[ ||R_{x,\delta,\ell}||_{L^\infty(\Lambda_L)} = 1 . \]
(12)
It is not necessary here to specify \( R_{x,\delta,\ell} \) in more detail, as its explicit form only enters a result that can be straightforwardly adapted from \[3\] (Eq. (45)).
Given a fixed realization of the random potential for which \(H_\omega\) is selfadjoint on \(\ell^2(\mathbb{Z}^2)\), let \(H_\omega^{(\Lambda_L)}\) denote the restriction of \(H_\omega\) to \(\Lambda_L\). Moreover, let \(\{\psi_\alpha^{(L)}\}_{\alpha \in \mathbb{A}_L}\) denote an orthonormal \(H_\omega^{(\Lambda_L)}\)-eigenbasis in \(\ell^2(\Lambda_L)\), satisfying Dirichlet boundary conditions
\[
\psi_\alpha^{(L)}(x) = 0 \quad (x \in \partial \Lambda_L := \Lambda_{L+1} \setminus \Lambda_L).
\]
The number of eigenfunctions is given by
\[
|\mathbb{A}_L| = |\Lambda_L|.
\]

Let, for \(\tau > 0\) arbitrary but fixed, and independent of \(\lambda\) and \(\sigma\),
\[
I_\tau := (-4 + \tau, -\tau) \cup (\tau, 4 - \tau).
\]
Let
\[
\mathbb{A}_L(I_\tau) := \{\alpha \in \mathbb{A}_L | e_\alpha^{(L)} \in I_\tau\},
\]
and similarly as in [3], let for \(\varepsilon\) small
\[
\mathbb{A}_L^{(\omega)}(\varepsilon, \delta, \ell; I_\tau) := \{ \alpha \in \mathbb{A}_L(I_\tau) | \sum_{x \in \Lambda_L} |\psi_\alpha^{(L)}(x)| \|R_{x,\delta,\ell}\psi_\alpha^{(L)}\|_{\ell^2(\Lambda_L)} < \varepsilon \}.
\]

As pointed out in [3], the key observation is that \(\{\psi_\alpha^{(L)}\}_{\alpha \in \mathbb{A}_L^{(\omega)}(\varepsilon, \delta, \ell; I_\tau)}\) contains the class of localized eigenstates with energies in \(I_\tau\) that are concentrated in balls of radius \(O(\delta \ell \log \ell)\), with \(\delta\) independent of \(\ell\).

Our main result is the following theorem.

**Theorem 2.1.** For \(\delta > 0\) sufficiently small, \(0 < \delta \ll \delta\), any fixed \(\tau\) with \(\lambda \ll \tau < \delta\), and any arbitrary \(\eta > 0\),
\[
\liminf_{L \to \infty} E \left[ \frac{|\mathbb{A}_L(\delta^{\frac{1}{2}}, \delta, \ell; \lambda); I_\tau)|}{|\mathbb{A}_L|} \right] \geq 1 - \delta^\star.
\]
The lower bound on the localization length \(\ell_\sigma(\lambda)\) satisfies the following estimates:

- In the subcritical case \(0 \leq \sigma < \frac{1}{2}\), there exist positive constants \(\lambda_0(\sigma, \eta) \ll 1\) and \(C_\sigma\) for every fixed \(0 < \sigma < \frac{1}{3}\) such that
\[
\ell_\sigma(\lambda) \geq C_\sigma \lambda^{-\frac{2 - \eta}{1 - 2\sigma}}
\]
for all \(\lambda < \lambda_0(\sigma, \eta)\).
- In the critical case \(\sigma = \frac{1}{2}\), there exists a positive constant \(\lambda_0(\eta) \ll 1\) such that
\[
\ell_{\sigma = \frac{1}{2}}(\lambda) \geq 2^{\lambda^{1 + \eta}}
\]
for all \(\lambda < \lambda_0(\eta)\).

We add the following remarks.
• (19) trivially implies

\[
P \left[ \liminf_{L \to \infty} \frac{\mathcal{A}_L \setminus \mathcal{A}_L (\delta^\frac{1}{2}, \delta, \ell_\sigma (\lambda); I_\tau)}{|\mathcal{A}_L|} > 1 - \delta^\frac{1}{2} \right] > 1 - \delta^\frac{1}{2}.
\]

• Spectral restriction to the interval \( I_\tau \) suppresses infrared singularities, and enables one to apply certain smoothing procedures to \( \frac{1}{\delta \log \lambda} \).

• Only a slight modification of the bounds used in our analysis of the subcritical case along the lines of [3] is necessary to yield the lower bound \( \lambda^{-2 + \eta} \) for \( \sigma = 0 \). Inclusion of a classification of graphs argument as in [8, 3] would improve the lower bound to \( \lambda^{-2} \frac{1}{\log \lambda} \). We shall not further discuss these matters here, since the argument is the same as the one presented in [3] for the 3-D problem.

3. Proof of Theorem 2.1

Our starting point is the following key lemma. It is an extension of a joint result with L. Erdös and H.-T. Yau in [3].

**Lemma 3.1.** Let \( \epsilon, \delta > 0 \) be small and \( \lambda \ll 1 \). Assume that there exists \( t^\ast (\delta, \ell) > 0 \), such that

\[
\mathbb{E} \left[ \frac{1}{|\mathcal{A}_L|} \sum_{x \in \mathcal{A}_L} \| R_x, \delta \chi \mathcal{A}_L (\mathcal{A}_L) e^{-it^\ast (\delta, \ell) H_\omega (\mathcal{A}_L)} \delta_x \|_{\ell^2 (\mathcal{A}_L)}^2 \right] \geq 1 - \epsilon - \mathbb{E} \left[ \frac{|\mathcal{A}_L (I_\tau)|}{|\mathcal{A}_L|} \right] - C \frac{\ell}{L}.
\]

Then,

\[
\liminf_{L \to \infty} \mathbb{E} \left[ \frac{\mathcal{A}_L \setminus \mathcal{A}_L (\epsilon, \delta, \ell; I_\tau)}{|\mathcal{A}_L|} \right] \geq 1 - 4 \epsilon^\frac{1}{2}.
\]

**Proof.** The proof follows closely a line of arguments presented in [3], but comprises key modifications due to the restriction of the energy range to \( I_\tau \).

We expand \( \delta_x \) in the eigenbasis \( \{ \psi^{(L)}_\alpha \} \),

\[
\delta_x = \sum_\alpha a_\alpha^{(L)} \psi^{(L)}_\alpha
\]

so that in particular,

\[
\| \delta_x \|_{\ell^2 (\mathcal{A}_L)}^2 = \sum_\alpha |a_\alpha^{(L)}|^2 = 1.
\]

Applying the Schwarz inequality,

\[
\| R_x, \delta \chi \mathcal{A}_L (\mathcal{A}_L) e^{-it H_\omega (\mathcal{A}_L)} \delta_x \|_{\ell^2 (\mathcal{A}_L)}^2 \leq (1 + \epsilon^{-\frac{1}{2}}) (A) + (1 + \epsilon^\frac{1}{2}) (B),
\]

where

\[
A = \sum_\alpha |a_\alpha^{(L)}|^2 \quad \text{and} \quad B = \sum_\alpha \left| \langle \delta_x, \psi^{(L)}_\alpha \rangle \right|^2.
\]
where

\[
(A) := \left\| R_{x,\delta,t} e^{-it\hat{H}^{(L)}} \sum_{\alpha \in \mathfrak{A}_L^{(\omega)}(\varepsilon, \delta, \ell; I_x)} a^\alpha_x \psi^{(L)}_\alpha \right\|_{\ell^2(\Lambda_L)}^2 \\
\leq \left\| R_{x,\delta,t} \sum_{\alpha \in \mathfrak{A}_L^{(\omega)}(\varepsilon, \delta, \ell; I_x)} e^{-it\epsilon^{(L)}_\alpha} a^\alpha_x \psi^{(L)}_\alpha \right\|_{\ell^2(\Lambda_L)}^2 \\
\leq \sum_{\alpha \in \mathfrak{A}_L^{(\omega)}(\varepsilon, \delta, \ell; I_x)} \left| \psi^{(L)}_\alpha(x) \right| \left\| R_{x,\delta,t} \psi^{(L)}_\alpha \right\|_{\ell^2(\Lambda_L)},
\]

(27)

using the a priori bound

\[
(A) \leq \left\| \sum_{\alpha \in \mathfrak{A}_L^{(\omega)}(\varepsilon, \delta, \ell; I_x)} e^{-it\epsilon^{(L)}_\alpha} a^\alpha_x \psi^{(L)}_\alpha \right\|_{\ell^2(\Lambda_L)}^2 \\
= \sum_{\alpha \in \mathfrak{A}_L^{(\omega)}(\varepsilon, \delta, \ell; I_x)} |a^\alpha_x|^2 \leq 1,
\]

(28)

which follows from \( \| R_{x,\delta,t} \|_{\infty} = 1 \), orthonormality of \( \{ \psi^{(L)}_\alpha \}_{\alpha \in \mathfrak{A}_L} \), and (25).

Moreover,

\[
(B) := \left\| R_{x,\delta,t} e^{-it\hat{H}^{(L)}} \sum_{\alpha \in \mathfrak{A}_L(I_x) \setminus \mathfrak{A}_L^{(\omega)}(\varepsilon, \delta, \ell; I_x)} a^\alpha_x \psi^{(L)}_\alpha \right\|_{\ell^2(\Lambda_L)}^2 \\
\leq \left\| \sum_{\alpha \in \mathfrak{A}_L(I_x) \setminus \mathfrak{A}_L^{(\omega)}(\varepsilon, \delta, \ell; I_x)} e^{-it\epsilon^{(L)}_\alpha} a^\alpha_x \psi^{(L)}_\alpha \right\|_{\ell^2(\Lambda_L)}^2 \\
= \sum_{\alpha \in \mathfrak{A}_L(I_x) \setminus \mathfrak{A}_L^{(\omega)}(\varepsilon, \delta, \ell; I_x)} |a^\alpha_x|^2 \\
= \sum_{\alpha \in \mathfrak{A}_L(I_x) \setminus \mathfrak{A}_L^{(\omega)}(\varepsilon, \delta, \ell; I_x)} \left| \psi^{(L)}_\alpha(x) \right|^2,
\]

(29)

Summing over \( x \in \Lambda_L \),

\[
\sum_{x \in \Lambda_L} \left\| R_{x,\delta,t} e^{-it\hat{H}^{(L)}} \delta_x \right\|_{\ell^2(\Lambda_L)}^2 \leq (1 + \varepsilon^{-\frac{1}{2}}) \left| \mathfrak{A}_L(I_x) \setminus \mathfrak{A}_L^{(\omega)}(\varepsilon, \delta, \ell; I_x) \right| \\
+ \varepsilon(1 + \varepsilon^{-\frac{1}{2}}) \left| \mathfrak{A}_L^{(\omega)}(\varepsilon, \delta, \ell; I_x) \right|,
\]

(30)

using the definition of \( \mathfrak{A}_L^{(\omega)}(\varepsilon, \delta, \ell; I_x) \).
Lemma 3.2. Let $I^c_\tau := \mathbb{R} \setminus I_\tau$. We thus get
\[
\frac{|\mathcal{A}_L \setminus \mathcal{A}_L^{(\omega)}(\varepsilon, \delta, \ell; I_\tau)|}{|\mathcal{A}_L|} = \frac{|\mathcal{A}_L(I^c_\tau)|}{|\mathcal{A}_L|} + \frac{|\mathcal{A}_L(I_\tau) \setminus \mathcal{A}_L^{(\omega)}(\varepsilon, \delta, \ell; I_\tau)|}{|\mathcal{A}_L|} \geq \frac{|\mathcal{A}_L(I^c_\tau)|}{|\mathcal{A}_L|} + \frac{1 - \varepsilon^\frac{1}{2}}{|\mathcal{A}_L|} \sum_{x \in \Lambda_L} \| R_{x, \delta, \ell, \tau} (H^{(\omega)}_{A_L}) e^{-i t^*(\delta, \lambda) A} \|_{L^2(A_L^2)}^2.
\]
Taking expectations and using (23),
\[
\mathbb{E} \left[ \frac{|\mathcal{A}_L \setminus \mathcal{A}_L^{(\omega)}(\varepsilon, \delta, \ell; I_\tau)|}{|\mathcal{A}_L|} \right] \geq 1 - \varepsilon^\frac{1}{2} \mathbb{E} \left[ \frac{|\mathcal{A}_L(I^c_\tau)|}{|\mathcal{A}_L|} \right] - 3 \varepsilon^\frac{1}{2} - C \frac{\ell}{L}.
\]
Since $|\mathcal{A}_L(I^c_\tau)|/|\mathcal{A}_L| \leq 1$, this implies the claim. \hfill \Box

Our strategy therefore is to find large values for $\ell$ and $t^*(\delta, \ell)$ such that (23) is satisfied.

The following lemma controls the free Schrödinger evolution.

Lemma 3.2. Let for $\lambda$ small and $0 < \delta < 1$
\[
t^*(\delta, \lambda) := \delta^\frac{1}{2} \ell.
\]
Then, the free evolution satisfies
\[
\frac{1}{|\mathcal{A}_L|} \sum_{x \in \Lambda_L} \| R_{x, \delta, \ell, \tau} (H^{(\omega)}_{A_L}) e^{-it^*(\delta, \lambda) A} \|_{L^2(A_L^2)}^2 \geq 1 - \delta^\frac{1}{2} - C \frac{\ell}{L}.
\]

Proof. We note that
\[
\sum_{x \in \Lambda_L} \| R_{x, \delta, \ell, \tau} (H^{(\omega)}_{A_L}) e^{-it^*(\delta, \lambda) A} \|_{L^2(A_L^2)}^2 \geq (I) - (II)
\]
where
\[
(I) := \sum_{x \in \Lambda_L} \| R_{x, \delta, \ell, \tau} (H^{(\omega)}_{A_L}) e^{-it^*(\delta, \lambda) A} \|_{L^2(A_L)}^2,
\]
\[
(II) := \sum_{x \in \Lambda_L} \| \chi_{I^c_\tau} (H^{(\omega)}_{A_L}) e^{-it^*(\delta, \lambda) A} \|_{L^2(A_L)}^2.
\]
This follows from $\chi R^2 \chi = \chi^2 - \chi R^2 \chi = \chi - \chi = \chi^2 - \chi^2 = R^2 - \chi^2$, where $R \equiv R_{x, \delta, \ell, \tau}(\lambda)$, $\chi \equiv \chi_{I^c_\tau} (H^{(\omega)}_{A_L})$, and $\bar{A} : = 1 - A$ (so that $\bar{R} = \chi^2 (H^{(\omega)}_{A_L}))$. 7
Replacing $\| \cdot \|_{\ell^2(\Lambda_L)}$ by $\| \cdot \|_{\ell^2(\mathbb{Z}^2)}$ in (I) costs a boundary term of size $O(\ell L)$ or smaller. Since $|\mathfrak{A}_L| \sim L^2$,

$$\frac{1}{|\mathfrak{A}_L|} \sum_{x \in \Lambda_L} \| R_{x, \delta} e^{it^*(\delta, \lambda) \Delta} \delta_x \|_{\ell^2(\Lambda_L)}$$

(37)

$$= \frac{1}{|\mathfrak{A}_L|} \sum_{x \in \Lambda_L} \| R_{x, \delta} e^{it^*(\delta, \lambda) \Delta} \delta_x \|_{\ell^2(\mathbb{Z}^2)} + O(\frac{\ell}{L}).$$

We then find

$$\| R_{x, \delta} e^{it^*(\delta, \lambda) \Delta} \delta_x \|_{\ell^2(\mathbb{Z}^2)} \geq 1 - \delta^\frac{10}{11},$$

(38)

from a related argument in [3], adapted to the present case.

On the other hand,

$$(II) \leq \sum_{x \in \Lambda_L} \| \chi_I^x (H_\omega^{(\Lambda_L)}) e^{it^*(\delta, \lambda) \Delta} \delta_x \|_{\ell^2(\Lambda_L)}^2$$

$$= \text{Tr} \left[ e^{it^*(\delta, \lambda) \Delta} \chi_I^x (H_\omega^{(\Lambda_L)}) e^{it^*(\delta, \lambda) \Delta} \right]$$

$$= \text{Tr} \left[ \chi_I^x (H_\omega^{(\Lambda_L)}) \right]$$

$$= |\mathfrak{A}_L(I_x^c)|.$$  

(39)

Recalling that $|\Lambda_L| = |\mathfrak{A}_L|$, this completes the proof. □

Our result is implied by the following key lemma. It controls the interaction of the electron with the impurity potential over a time $t^*$ comparable to the lower bound on the localization length $\ell_{\sigma}(\lambda)$.

Lemma 3.3. Let for $0 < \delta < 1$

$$t_{\sigma, \lambda}^\dagger = \delta^\frac{5}{4} \ell_{\sigma}(\lambda).$$

(40)

Then, for any arbitrary, but fixed $\tau > 0$,

$$\limsup_{L \to \infty} \mathbb{E} \left[ \frac{1}{|\mathfrak{A}_L|} \sum_{x \in \Lambda_L} \| \chi_I^x (H_\omega^{(\Lambda_L)}) (e^{-it_{\sigma, \lambda}^{\dagger} H_\omega^{(\Lambda_L)}} - e^{-it_{\sigma, \lambda} H_\omega^{(\Lambda_L)}}) \|_{\ell^2(\Lambda_L)}^2 \right] \leq C\tau^\frac{7}{8} + \lambda^n.$$  

(41)

The definition of $\ell_{\sigma}(\lambda)$ is given in Theorem 2.1.

To establish Lemma 3.3, it suffices to prove the following estimate.

Lemma 3.4. Under the assumptions of Lemma 3.3

$$\sup_{\| \phi \|_{\ell^2(\mathbb{Z}^2)} = 1} \mathbb{E} \left[ \| \chi_I^x (H_\omega)(e^{-it_{\sigma, \lambda}^{\dagger} H_\omega} - e^{-it_{\sigma, \lambda} H_\omega}) \phi \|_{\ell^2(\mathbb{Z}^2)}^2 \right] < C\tau^\frac{7}{8} + \lambda^n.$$  

(42)

The rest of this paper is devoted to the proof of Lemma 3.4.
4. Resolvent expansion

Let henceforth \( t \equiv t_{\delta, \sigma, \lambda}^\ast \). We write
\[
\phi_t = \chi_{I_\tau}(H_\omega) e^{-itH_\omega} \phi_0
\]
with \( \phi_0 \in \ell^2(\mathbb{Z}^2) \) in resolvent representation
\[
\phi_t = \frac{1}{2\pi i} e^{\varepsilon t} \int_{\mathbb{R}} d\alpha e^{-it\alpha} \chi_{I_\tau}(H_\omega) \frac{H_\omega - \alpha - i\varepsilon}{H_\omega - \alpha - i\varepsilon} \phi_0
\]
where we will use the choice
\[
\varepsilon = \frac{1}{t}
\]
in all that follows. Due to the spectral restriction of \( H_\omega \) to the disjoint union of intervals \( I_\tau \), the \( \alpha \)-integration contour can be deformed into
\[
\phi_t = \frac{1}{2\pi i} e^{\varepsilon t} \int_{C_- \cup C_+} d\alpha e^{-it\alpha} \chi_{I_\tau}(H_\omega) \frac{H_\omega - \alpha - i\varepsilon}{H_\omega - \alpha - i\varepsilon} \phi_0
\]
where the loops
\[
C_- := [-4 + \tau/2, -\tau/2] \cup (-4 + \tau/2 - 2i\varepsilon[0,1]) \cup ([-4 + \tau/2, -\tau/2 - 2i\varepsilon] \cup (-\tau/2 - 2i\varepsilon[0,1])
\]
\[
C_+ := [\tau/2, 4 - \tau/2] \cup (4 - \tau/2 - 2i\varepsilon[0,1]) \cup ([\tau/2, 4 - \tau/2 - 2i\varepsilon] \cup (\tau/2 - 2i\varepsilon[0,1])
\]
are taken in the clockwise direction. \( C_- \) and \( C_+ \) each enclose one of the components of \( I_\tau - i\varepsilon \).

Let \( C^{(v)} := \{ C_j^{(v)} \}_{j=1}^4 \) denote the four vertical, and \( C^{(h)} := \{ C_j^{(h)} \}_{j=1}^4 \) the four horizontal segments in \( C_- \) and \( C_+ \). Each segment carries an orientation accounting for the direction in which the contour integration is taken.

Then,
\[
\left| \frac{1}{2\pi i} e^{\varepsilon t} \int_{C_j^{(v)}} d\alpha e^{-it\alpha} \chi_{I_\tau}(H_\omega) \frac{H_\omega - \alpha - i\varepsilon}{H_\omega - \alpha - i\varepsilon} \phi_0 \right| < \frac{1}{4} |C_j^{(v)}| \sup_{z' \in I_\tau - i\varepsilon} |z - z'| = \varepsilon \tau^{-1},
\]
as \( \text{dist}(C_j^{(v)}, I_\tau - i\varepsilon) = \tau/2 \), and \( |C_j^{(v)}| = 2\varepsilon \).

Henceforth, we shall omit the subscript "\( \omega \)" in the random potential \( V_\omega \equiv V \).

Defining
\[
\phi^{(h)}_t := \frac{1}{2\pi i} e^{\varepsilon t} \int_{C^{(h)}} d\alpha e^{-it\alpha} \frac{1}{H_\omega - \alpha - i\varepsilon} \phi_0
\]
we have
\[
\| \phi_t \|_{\ell^2(\mathbb{Z}^2)}^2 \leq 2 \left( \frac{\varepsilon}{\tau} \right)^2 + 2 \| \chi_{I_\tau}(H_\omega) \phi^{(h)}_t \|_{\ell^2(\mathbb{Z}^2)}^2.
\]
Next, we expand $\phi_t^{(h)}$ into

$$
\phi_t^{(h)} = \sum_{n=0}^{N} \phi_{n,t} + R_{N,t},
$$

where the $n$-th term is given by

$$
\phi_{n,t} := \frac{e^{\varepsilon t}}{2\pi i} \int_{C(h)} d\alpha e^{-i\tau_0 \tilde{\phi}_{n,\varepsilon}(\alpha)},
$$

with

$$
\tilde{\phi}_{n,\varepsilon}(\alpha) := (-\lambda)^n \frac{1}{\Delta - \alpha - i\varepsilon} \left( V \frac{1}{\Delta - \alpha - i\varepsilon} \right)^n \phi_0.
$$

In frequency space,

$$
\mathcal{F}(\phi_{n,t})(k_0) = \frac{1}{2\pi} e^{\varepsilon t} \int_{C(h)} d\alpha e^{-i\tau_0 \mathcal{F}(\tilde{\phi}_{N,\varepsilon}(\alpha))(k_0)}
$$

where

$$
\mathcal{F}(\tilde{\phi}_{N,\varepsilon}(\alpha))(k_0) = (-\lambda)^n \int_{(\mathbb{T}^3)^n} dk_1 \cdots dk_n \frac{1}{e^\Delta(k_0) - \alpha - i\varepsilon}
$$

$$
\times \left[ \prod_{j=1}^{n} e^\Delta(k_j) - \alpha - i\varepsilon \right] \tilde{\phi}_0(k_n),
$$

and $\mathbb{T} = [-\frac{1}{2}, \frac{1}{2}]$. We will refer to the Fourier multiplier $\frac{1}{e^\Delta(k) - \alpha - i\varepsilon}$ as a particle propagator.

The remainder term is given by

$$
R_{N,t} = -\lambda e^{\varepsilon t} \frac{1}{2\pi} \int_{C(h)} d\alpha e^{-i\tau_0} \frac{1}{H_\omega - \alpha - i\varepsilon} V \tilde{\phi}_{N,\varepsilon}(\alpha).
$$

The depth of the expansion $N$ remains to be optimized.

We remark that due to the truncation of the integration contour, $\phi_{n,t}$ and $R_{N,t}$ cannot be written as time integrals of the form

$$
\phi_{n,t} \leftrightarrow (-i\lambda)^n \int_{\mathbb{R}_+^{n+1}} \delta(t - \sum_{j=0}^{n} s_j) e^{-s_0 \Delta V e^{-s_1 \Delta} \cdots V e^{-s_n \Delta} \phi_0}
$$

$$
R_{N,t} \leftrightarrow -i\lambda \int_0^t ds e^{-i(t-s)H_\omega V} \phi_{N,s}
$$

as in the Duhamel expansions used in [3] [4] [5]. While for $\phi_{n,t}$, this is not essential in the present work (because we admit a polynomial error $O(\lambda^\eta)$, $\eta > 0$, in our bounds), our methods require an expression of the above form for $R_{N,t}$ (because we will apply the time partitioning trick used in [8] and [3]).

To this end, we claim that

$$
R_{N,t} = R_{N,t}^{(0)} + R_{N,t}^{(1)}
$$
where\n\[ R_{N,t}^{(0)} := e^{-itH} - \frac{\lambda}{2\pi i} \int_{C^{(h)}} d\alpha \frac{1}{H_\omega - \alpha - i\varepsilon} V\hat{\phi}_{N,t}(\alpha) \]
\[ R_{N,t}^{(1)} := -i\lambda \int_0^t ds e^{-i(t-s)H_\omega} V\phi_{N,s} \cdot \]

To see this, we note that \((56)\) implies \((60)\) which has a distance \(O\) necessary to control \(e\) is bounded by its length \(O\) and \(\tilde{\omega}\) as there is no obstructing phase factor \(e\). We note that \((-1, 0, 1)\) would vanish if \(C^{(h)}\) were connected arcs, while \(\tilde{\omega}\) and \(\tilde{\omega}\) are connected arcs, while \(\tilde{\omega}\) consists of two disjoint, parallel lines, all of length \(O(\tau)\). We claim that
\[ \int_{\tilde{\omega} \cup \tilde{\omega} \cup \tilde{\omega}} d\alpha \frac{1}{e\Delta(p) - \alpha - i\varepsilon} \]
\[ \leq C \left[ \chi(|e\Delta(p) + 4| < 2\tau) + \chi(|e\Delta(p) + 4| < 2\tau) \right. \]
\[ + \chi(|e\Delta(p)| < 4\tau) + \left. \frac{\varepsilon}{\tau} \right] . \]

For fixed \(p\), the size of
\[ \int_{\tilde{\omega} - \tilde{\omega}} d\alpha \frac{1}{e\Delta(p) - \alpha - i\varepsilon} \]
can be estimated as follows.

If \(|e\Delta(p) - 4| < 2\tau\), we deform \(\tilde{\omega}\) into a loop that encloses \(e\Delta(p) - i\varepsilon\), and a disjoint arc of length \(O(\varepsilon)\) connecting the endpoints of \(\tilde{\omega}\). The resolvent at \(e\Delta(p) - i\varepsilon\), due to the loop, yields a factor \(e^{-i(e\Delta(p) - i\varepsilon)}\). The integral over the arc is bounded by its length \(O(\varepsilon)\), multiplied with the bound \(\frac{1}{\varepsilon}\) on the resolvent. Both contributions are \(O(1)\).

If \(|e\Delta(p) + 4| > 2\tau\), we deform \(\tilde{\omega}\) into a line of length \(2\varepsilon\) connecting its endpoints, which has a distance \(\geq \tau\) from \(e\Delta(p)\). The modulus of the resolvent is therefore \(\leq O(\frac{1}{\varepsilon})\), and integrating, we get an error bound of order \(O(\frac{1}{\varepsilon})\).
The cases $\tilde{C}_0$ and $\tilde{C}_+$ are similar.

Thus,

$$\begin{align*}
(62) &< C \left[ \text{mes}\{|e_\Delta(p) + 4| < 2\tau\} + \text{mes}\{|e_\Delta(p)| < 4\tau\} \\
&\quad + \text{mes}\{|e_\Delta(p) - 4| < 2\tau\} + \frac{\varepsilon}{\tau} \right] \\
(66) &< C \tau^{\frac{1}{2}},
\end{align*}$$

as $\varepsilon$ will be chosen $\ll \tau$ in the end.

The Schwarz inequality thus yields

$$\begin{align*}
\mathbb{E} \left[ \|\chi_I(H_\omega) (\phi_t^{(h)} - e^{-it\Delta} \phi_0) \|_{L^2_\omega(\mathbb{Z})}^2 \right] \\
&\leq C \tau^{\frac{1}{2}} + 2 \mathbb{E} \left[ \| \sum_{n=1}^N \phi_{n,t} \|_2^2 \right] + 2 \mathbb{E} \left[ \| \chi_I(H_\omega) R_{N,t} \|_2^2 \right] \\
(67) &= C \tau^{\frac{1}{2}} + 2 \sum_{n,n'=1}^N \mathbb{E} \left[ \langle \phi_{n',t}, \phi_{n,t} \rangle \right] + 2 \mathbb{E} \left[ \| \chi_I(H_\omega) R_{N,t} \|_2^2 \right].
\end{align*}$$

Clearly, if $n + n' \not\in 2N$, $\mathbb{E} \langle \phi_{n',t}, \phi_{n,t} \rangle = 0$.

We partition $V$ into dyadic shells,

$$V = \sum_{j=0}^{J+1} V_j ,$$

where

$$V_j(x) = P_j(x) v_\sigma(x) \omega_x$$

for $0 \leq j \leq J$. The cutoff functions $P_j$ are defined at the beginning of section 2.

For $j > J$, we rename $P_j \rightarrow \tilde{P}_j$, and define

$$P_{J+1} := \sum_{j=J+1}^\infty \tilde{P}_j$$

Hence, the functions $V_j$ are supported on dyadic annuli of radii and thicknesses $\sim 2^j$ centered at the origin, $j = 1, \ldots, J$, while $V_{J+1}$ is the part of $V$ supported in regions with a distance larger than $2^{J+1}$ from the origin.

Let

$$R_\pm := \frac{1}{\Delta - z}.$$ 

Then, we have

$$\begin{align*}
\mathbb{E} \left[ \langle \phi_{n',t}, \phi_{n,t} \rangle \right] &= \sum_{j_1, \ldots, j_{2n}=1}^{J+1} e^{2it\chi_{n'}} \frac{(2\pi)^2}{\int_{C^{(h)} \times C^{(n)}} \mathrm{d}x \mathrm{d}y e^{-it(\alpha - \beta)}} \\
&\quad \cdot \mathbb{E} \left[ \langle \phi_0, R_{\alpha+i\varepsilon} V_{j_1} R_{\beta-i\varepsilon} V_{j_2} R_{\beta-i\varepsilon} \cdots \right. \\
&\quad \left. \cdots V_{j_n} R_{\beta-i\varepsilon} R_{\alpha+i\varepsilon} V_{j_{n+2}} \cdots V_{2n} R_{\alpha+i\varepsilon} \phi_0 \rangle \right] \\
(72) &= \cdots V_{j_n} R_{\beta-i\varepsilon} R_{\alpha+i\varepsilon} V_{j_{n+2}} \cdots V_{2n} R_{\alpha+i\varepsilon} \phi_0 \rangle
\end{align*}$$

$$\left(\begin{array}{c}
\end{array}\right)$$
for $1 \leq n, n' \leq N$, and $\tilde{n} := \frac{n+n'}{2} \in \mathbb{N}$, $C^{(h)}$ denotes the complex conjugate of $C^{(h)}$, and is taken in the counterclockwise direction by the variable $\beta$.

For $1 \leq n, n' \leq N$, and $\tilde{n} := \frac{n+n'}{2} \in \mathbb{N}$, let

$$p = (p_0, \ldots, p_n, p_{n+1}, \ldots, p_{2\tilde{n}+1})$$

and

$$(\alpha_j, \sigma_j) = \begin{cases} (\alpha, 1) & 0 \leq j \leq n \\ (\beta, -1) & n < j \leq 2n + 1 \end{cases}$$

Then, in frequency space representation,

$$\begin{align*}
(73) & = \sum_{j_1, \ldots, j_{2\tilde{n}} = 1}^{J+1} \frac{e^{2\pi i \tilde{n}^2}}{(2\pi)^2} \int_{C^{(h)} \times C^{(h)}} d\alpha d\beta e^{-it(\alpha - \beta)} \\
& \int_{\mathbb{T}^{2\tilde{n}+2}} dp \delta(p_n - p_{n+1}) F(\phi_0)(p_0) F(\phi_0)(p_{2\tilde{n}+1}) \\
& \prod_{l=0}^{2\tilde{n}+1} \frac{1}{\cos(\theta_l) - \sigma_l} \\
& E \left[ \prod_{i=1}^{2n+1} F(V_{j_i})(p_i - p_{i-1}) \right]
\end{align*}$$

(noting that $F(V)(k) = F(V)(-k)$).

5. Graph expansion

We systematize the evaluation of the expectation value of product s of random potentials by use of (Feynman) graphs, which we represent as follows.

We consider two parallel, horizontal solid lines, which we refer to as particle lines, joined at a distinguished vertex which accounts for the $L^2$-inner product (henceforth referred to as the "$L^2$-vertex").

The particle line to the left of the $L^2$ vertex shall contain $n$, and the one its right shall contain $n'$ vertices, accounting for copies of the random potential $V$ (henceforth referred to as "$V$-vertices").

The $n + 1$ edges on the left of the $L^2$-vertex correspond to the propagators in $\hat{\psi}_{n,t}$, while the $n' + 1$ edges on the right correspond to those in $\hat{\psi}_{n',t}$. We shall refer to those edges as propagator lines.

The expectation produces a sum over the products of $\tilde{n} = \frac{n+n'}{2} \in \mathbb{N}$ contractions between all possible pairs of random potentials. We insert an edge referred to as a contraction line between every pair of mutually contracted random potentials. We then identify the contraction type with the corresponding graph.

We let $\Pi_{n,n'}$ denote the set of all graphs comprising $n + n'$ $V$-vertices, one $L^2$-vertex, two particle lines, $\tilde{n}$ contraction lines, and $2\tilde{n} + 2$ propagator lines as defined above.
5.1. Dyadic Wick expansion. We shall next discuss the expectation of products of dyadically resolved random potentials in detail.

It is evident that

\[ E[V_j(x)V_j'(x')] = \delta_{|j-j'|\leq 1} P_j(x) P_j'(x) v^2_\sigma(x) \delta_{x,x'} \]

(76)

and

\[ E[V_{j+1}(x)V_{j+1}(x')] \leq C 2^{-2\sigma j} \delta_{x,x'} . \]

(77)

The expectation of products \( \prod_i \omega_{x_i} \) satisfies Wick’s theorem, and the same is true for the expectation of products \( \prod_i V_{j_i}(x_i) \). This can be formulated as follows.

There are \( \bar{n} \) pairing contraction lines joining pairs of \( \hat{V}_\omega \)-vertices in \( \pi \). We enumerate the contraction lines in an arbitrary, but fixed order by \( \{1, \ldots, \bar{n}\} \).

We write \( i \sim_m i' \) to express that the \( i \)-th and the \( i' \)-th \( V \)-vertex are connected by the \( m \)-th contraction line.

Given

\[ j := (j_1, \ldots, j_{2\bar{n}}) \]

(78)

\[ x := (x_0, \ldots, x_{2\bar{n}+1}) ; \]

let

\[ \delta_{\pi}(j,x) := \prod_{m=1}^{\bar{n}} \left[ \delta_{|j_i-j'_i|\leq 1} \delta_{x_i,x'_i} \right]_{i \sim_m i'} . \]

(79)

Then, in position space,

\[ E \left[ \prod_{i=1}^{2\bar{n}} V_{j_i}(x_i) \right] = \sum_{\pi \in \Pi_{n,n'}} \delta_{\pi}(j,x) \prod_{i=1}^{2\bar{n}} v_{\sigma}(x_i) . \]

(80)

On the other hand, we arrive at the frequency space picture as follows.

Let

\[ p := (p_0, \ldots, p_n, p_{n+1}, \ldots, p_{2\bar{n}+1}) . \]

(81)

If \( i \sim_m i' \), contraction of \( \mathcal{F}(P_{j_i}, V)(p_{i+1} - p_i) \) with \( \mathcal{F}(P_{j_i'}, V)(p_{i'+1} - p_{i'}) \) yields

\[ E \left[ \mathcal{F}(P_{j_i}, V)(p_{i+1} - p_i) \mathcal{F}(P_{j_i'}, V)(p_{i'+1} - p_{i'}) \right] = \delta_{|j_i-j_i'|\leq 1} \mathcal{F}(P_{j_i} P_{j_i'}, v^2_\sigma) \delta(p_{i+1} - p_i + p_{i'+1} - p_{i'}) . \]

(82)

We define

\[ \delta_{\pi}(j,p; v_{\sigma}) := \]

\[ \prod_{m=1}^{\bar{n}} \left[ \delta_{|j_i-j_i'|\leq 1} \mathcal{F}(P_{j_i} P_{j_i'}, v^2_\sigma) \delta(p_{i+1} - p_i + p_{i'+1} - p_{i'}) \right]_{i \sim_m i'} . \]

(83)
Then,

\[
E \left[ \prod_{i=1}^{2n+1} F(V_{j_i}) (p_i - p_{i-1}) \right] = \sum_{\pi \in \Pi_{n,n'}} \delta_{\pi}(j_i, p_i; v_\sigma).
\]

We emphasize that the products (79) and (83) vanish unless the scales of the contracted dyadic potentials pairwise coincide (up to overlap errors). That is, \(|j_i - j_{i'}| \leq 1\) (where \(|j_i - j_{i'}| = 1\) accounts for overlap errors) for every pair \(i \sim m, i'\).

Expanding the expectation of the product of random potentials,

\[
E[(\phi_{n',t}, \phi_{n,t})] = \sum_{\pi \in \Pi_{n,n'}} \text{Amp}(\pi)
\]

where

\[
\text{Amp}(\pi) = \frac{J+1}{(2\pi)^2} \int_{C(h) \times C(h)} \int_{(\mathbb{T}^2)^{2n+2}} dp \delta(p_n - p_{n+1}) \delta_{\pi}(j_i; p_i; v_\sigma) \delta(p_n - p_{n+1}) \delta_{\pi}(j_i; p_i; v_\sigma) F(\phi_0)(p_n) F(\phi_0)(p_{2n+1})
\]

\[
\prod_{l=0}^{2n+1} e^{\Delta(p_l) - \alpha_l - \sigma_l \varepsilon}.
\]

Here, \(\delta(p_n - p_{n+1})\) corresponds to the \(L^2\)-vertex.

6. Bounds on pairing graphs

We shall use an analogy of the frequency space \(L^1 - L^\infty\) estimates on the resolvents adapted to a spanning tree of \(\pi\) from [3, 3].

**Lemma 6.1.** Assume that \(\alpha \in C(h)\). Then, for the assumptions (2) on \(P_j\),

\[
\left\| \frac{1}{\varepsilon_\Delta - \alpha - \varepsilon} * |F(P_jP_{j'}v_\sigma^2)| \right\|_{L^\infty(\mathbb{T}^2)} \leq \left\{
\begin{array}{ll}
C_\tau 2^{(1-2\sigma)j} & \text{if } j \leq J \\
C\sigma^{-1} 2^{-2\sigma J} & \text{if } j, j' = J + 1
\end{array}\right.
\]

where the constant \(C_\tau\) only depends on \(\tau\). Furthermore,

\[
\left\| \frac{1}{\varepsilon_\Delta - \alpha - \varepsilon} * |F(P_jP_{j'}v_\sigma^2)| \right\|_{L^1(\mathbb{T}^2)} \leq C \log \frac{1}{\varepsilon}
\]

for \(0 \leq j, j' \leq J + 1\).

**Proof.** We recall that by (5),

\[
|F(P_jP_{j'}v_\sigma^2)(p)| \leq C 2^{-2\sigma j} |F(P_jP_{j'})(p)| \sim C 2^{-2\sigma j} |F(P_j^2)(p)|
\]

for \(|j - j'| \leq 1\), and any \(j\). It thus suffices to discuss the diagonal term \(j = j'\).
For \( \alpha \in C^{(h)} \), it is shown in [1] that given our assumptions on \( P_j \), convolution with \( |F(P_j^2)| \) acts like a smoothing operator on \( \frac{1}{e^\Delta - \alpha - i\varepsilon} \), on the scale dual to \( 2^j \), to the effect that

\[
\left| \frac{1}{e^\Delta - \alpha - i\varepsilon} \right| \ast |F(P_j^2)| \leq \frac{C_\tau}{|e^\Delta - \alpha| + \varepsilon + 2^{-j}}.
\]

The \( L^\infty \)-bounds (87) for \( 0 \leq j \leq J \) then follow immediately. For \( j = J + 1 \),

\[
\left| \frac{1}{e^\Delta - \alpha - i\varepsilon} \right| \ast |F(P_{j+1}^2)| \leq \left\| \frac{1}{e^\Delta - \alpha - i\varepsilon} \right\|_{L^\infty(T^2)} \sum_{i=J+1}^\infty \left\| F(\tilde{P}_i^2) \right\|_{L^1(T^2)}
\]

\[
\leq C\varepsilon^{-1} \sum_{i=J+1}^\infty 2^{-2\sigma i} \left\| F(\tilde{P}_i^2) \right\|_{L^1(T^2)}
\]

\[
\leq C\varepsilon^{-1} 2^{-2\sigma J},
\]

as \( \left\| F(P_j^2) \right\|_{L^1(T^2)} \sim 1 \) (\( \tilde{P}_i \) is defined in (80)).

The \( L^1 \)-bound (88) has been proven in [3].

Lemma 6.2. For \( 1 \leq n, n' \leq N \), \( \tau > 0 \) and \( \pi \in \Pi_{n,n'} \), there exists a finite constant \( C_\tau \) depending only on \( \tau \) such that defining

\[
A_{\sigma,\tau,J,\lambda,\varepsilon} := C_\tau (K_\sigma(J)\lambda^2 \log \frac{1}{\varepsilon} + \varepsilon^{-1} 2^{-2\sigma J} \lambda^2 \log \frac{1}{\varepsilon})
\]

and

\[
K_\sigma(J) := \begin{cases} 
J + 1 & \text{if } \sigma = \frac{1}{2} \\
\frac{\sigma^{1-2\sigma} J + 1}{2^{1-2\sigma} J^2 - 1} & \text{if } 0 < \sigma < \frac{1}{2},
\end{cases}
\]

one gets

\[
|\text{Amp}(\pi)| < (\log \frac{1}{\varepsilon})^2 (A_{\sigma,\tau,J,\lambda,\varepsilon})^n.
\]

Proof. We choose a spanning tree \( T \) on \( \pi \) that contains all contraction lines between the pairs of random potentials, and \( n \) out of all particle lines. In addition, \( T \) shall include those particle lines labeled by the momenta \( p_n, p_{2n+1} \), but not those labeled by \( p_0, p_{n+1} \). We then call \( T \) admissible. Momenta (resolvents) supported on \( T \) are referred to as tree momenta (resolvents), and momenta (resolvents) supported on its complement \( T^c \) are called loop momenta (resolvents). We shall then group together every tree resolvent with one adjacent contraction line carrying a factor \( F(P_j, P_{j'}, v^2_j) \), \( |j - j'| \leq 1 \), and estimate the corresponding convolution integral of the form (102) below. All loop resolvents supported on \( T^c \) are estimated in \( L^1(T^2) \).
We recall that

\[
\text{Amp}(\pi) = \sum_{j_0, \ldots, j_{2n+1}} \frac{e^{2it\lambda^{2n}}}{(2\pi)^2} \int_{\mathcal{C}(\lambda) \times \mathcal{C}(\beta)} d\alpha d\beta e^{-it(\alpha-\beta)}
\]

\[
\int_{(T^2)^{2n}} dp \, \delta(p_n - p_{n+1}) \delta(j; \pi; v_\sigma)
\]

\[
\bar{F}(\phi_0)(p_n) F(\phi_0)(p_{2n+1}) \prod_{l=0}^{2n+1} \frac{1}{e_\Delta(p_l) - \alpha_l - i\sigma_l \varepsilon}
\]

(95)

for \( j = (j_1, \ldots, j_{2n}) \).

We integrate out the variable \( p_{n+1} \), and apply the coordinate transformation \( p_j \mapsto p_j + p_n \), for all \( j = 0, \ldots, n-1, n+2, \ldots, 2n+1 \). It is easy to see that thereby, \( \delta_\pi(j; \pi; v_\sigma) \) becomes independent of \( p_n \) and \( p_{n+1} \). We obtain

\[
\text{Amp}(\pi) = \sum_{j_0, \ldots, j_{2n+1}} \frac{e^{2it\lambda^{2n}}}{(2\pi)^2} \int_{\mathcal{C}(\lambda) \times \mathcal{C}(\beta)} d\alpha d\beta e^{-it(\alpha-\beta)}
\]

\[
\int_{(T^2)^{2n}} dp' \delta'_\pi(j; p'; v_\sigma)
\]

\[
\int_{T^2} dp_n \frac{1}{e_\Delta(p_n) - \alpha - i\varepsilon} \frac{1}{e_\Delta(p_n) - \beta + i\varepsilon}
\]

\[
\bar{F}(\phi_0)(p_0 + p_n) F(\phi_0)(p_{2n+1} + p_n) \prod_{l=0}^{2n+1} \frac{1}{e_\Delta(p_l + p_n) - \alpha_l - i\sigma_l \varepsilon}
\]

(96)

where

\[
p' := (p_0, \ldots, p_{n-1}, p_{n+2}, \ldots, p_{2n+1})
\]

and

\[
\delta'_\pi(j; p'; v_\sigma) := \delta_\pi(j; p'; v_\sigma) \bigg|_{p_{n+1}, p_n \to 0}
\]

Clearly,

\[
|\text{Amp}(\pi)| \leq \sum_{j_0, \ldots, j_{2n+1}} \frac{e^{2it\lambda^{2n}}}{(2\pi)^2} \left[ \sup_{q, q' \in T^2} \int_{\mathcal{C}(\lambda) \times \mathcal{C}(\beta)} |d\alpha| |d\beta| \right]
\]

\[
\int_{T^2} dp_n \frac{1}{|e_\Delta(p_n) - \alpha - i\varepsilon|} \frac{1}{|e_\Delta(p_n) - \beta + i\varepsilon|}
\]

\[
\left| \bar{F}(\phi_0)(p_0 + q) F(\phi_0)(p_{2n+1} + q') \right|
\]

\[
\sup_{\alpha \in \mathcal{C}(\lambda)} \sup_{\beta \in \mathcal{C}(\beta)} \sup_{p_n \in T^2} \int_{(T^2)^{2n}} dp' \delta'_\pi(j; p'; v_\sigma)
\]

\[
\prod_{j=0}^{2n+1} \frac{1}{|e_\Delta(p_l + p_n) - \alpha_l - i\sigma_l \varepsilon|}
\]

(99)
Thus, dividing the resolvents into tree and loop terms and defining

\[ \delta_\pi(j) := \prod_{m=1}^{\bar{n}} \delta_{|j_i - j_{i'}| \leq 1} \bigg|_{i \sim i'} \]

(see also (S3), one gets

\[ |\text{Amp}(\pi)| \leq \sum_{j_0 \ldots j_{2n+1} = 0}^{J+1} \frac{e^{2\pi t}2^{2\bar{n}}}{(2\pi)^2} \delta_\pi(j) \]

\[
\begin{align*}
& \left[ \sup_{q,q' \in T^2} \int_{T^2} dp_n |\phi_0(p_n + q)| \phi_0(p_n + q') \right] \\
& \cdot \left[ \sup_{p_n \in T^2} \int_{C(h)} |d\alpha| \frac{1}{|e_\Delta(p_n) - \alpha - i\epsilon|} \right] \\
& \cdot \left[ \sup_{p_n \in T^2} \int_{C(h)} |d\beta| \frac{1}{|e_\Delta(p_n) - \beta + i\epsilon|} \right] \\
& \sup_{\alpha \in C(h)} \sup_{\beta \in C(h)} \sup_{p_n \in T^2} \left\{ \prod_{T, c} \left| \frac{1}{e_\Delta - \alpha_i \pm i\epsilon} \right|_{L^1(T^2)} \right\},
\end{align*}
\]

(101)

where \( i \sim i' \) implies that the vertices indexed by \( i \) and \( i' \) are linked by a contraction line. \( \prod_T \) and \( \prod_{T,c} \) denote the products over all resolvents supported on \( T \) and \( T^c \), respectively. Assuming (S9), we can bound the off-diagonal terms \( |j_i - j_{i'}| = 1 \) by the diagonal terms \( j_i = j_{i'} \), and due to Lemma 6.1 we have

\[ \sup_{q \in T^2} \int_{T^2} dp \left| \frac{1}{e_\Delta(p) - \alpha - i\epsilon} \right| \left| \mathcal{F}(P_j^2 \phi_\sigma^2)(p - q) \right| \leq C \cdot 2^{(1-2\sigma)j} \]

if \( 0 \leq j \leq J \), and

\[ \sup_{q \in T^2} \int_{T^2} dp \left| \frac{1}{e_\Delta(p) - \alpha - i\epsilon} \right| \left| \mathcal{F}(P_j^2 \phi_\sigma^2)(p - q) \right| \leq \epsilon^{-1} \sigma^{-1} 2^{-2\sigma j} \]

if \( j = J + 1 \). Hence,

\[ |\text{Amp}(\pi)| \leq (C \log \frac{1}{\epsilon})^2 \left\| \phi_0 \right\|_{L^2(T^2)}^2 \sum_{j_0 \ldots j_{2n+1} = 0}^{J+1} \delta_\pi(j) (C \log \frac{1}{\epsilon})^{|T^c|} \]

\[
\prod_{i=1}^{2\bar{n}} \left( 2^{(1-2\sigma)j_i} \chi(j \leq J) + \sigma^{-1} \epsilon^{-1} 2^{-2\sigma j} \delta_{j_i,t+1} \right)^{1/2},
\]

(104)

where we have used

\[ \sup_{p \in T^2} \int_{C(h)} |d\alpha| \left| \frac{1}{|e_\Delta(p) - \alpha - i\epsilon|} \right| < C \log \frac{1}{\epsilon}. \]

(105)

The power \( \frac{1}{2} \) on the last line in (104) arises because the product extends over all random potentials, while \( T \) accounts only for the contraction lines (each adjoining to two random potentials). We note also that \( \delta_\pi(j) \) forces elements of \( j \) to be pairwise equal, up to overlap terms.
Therefore,
\begin{equation}
|\text{Amp}(\pi)| \leq (C \log \frac{1}{\varepsilon})^{2+|T^c|}\left(\sum_{j=0}^{J} 2^{(1-2\sigma)j} + \varepsilon^{-1}2^{-2J}\right)^{|T|},
\end{equation}
where $|T|$ and $|T^c|$ denote the numbers of resolvents supported on $T$ and $T^c$, respectively. From
\begin{equation}
\sum_{j=0}^{J} 2^{(1-2\sigma)j} = \begin{cases} 
J + 1 & \text{if } \sigma = \frac{1}{2} \\
\frac{2^{(1-2\sigma)(J+1)} - 1}{2^{(1-2\sigma)} - 1} & \text{if } 0 < \sigma < \frac{1}{2}
\end{cases}
\end{equation}
and $|T| = |T^c| = \bar{n}$, the assertion of the lemma follows. □

7. Estimating the remainder term

The remainder term of the resolvent expansion is given by
\begin{equation}
R_{N,t} = -\lambda e^{it} \frac{1}{2\pi i} \int_{C(h)} d\alpha e^{-it\alpha} \frac{1}{H_\omega - \alpha - i\varepsilon} V\tilde{\phi}_{N,\varepsilon}(\alpha),
\end{equation}
as we recall from (56). The trivial bound
\begin{equation}
\mathbb{E}[\|R_{N,t}\|_2^2(z,\mathbb{Z})] \leq C\lambda^2 \varepsilon^{-2} \mathbb{E}[\|V\phi_{N,t}\|_2^2(z,\mathbb{Z})]
\end{equation}
is insufficient in the subcritical case $0 < \sigma < \frac{1}{2}$. We shall instead apply the time partitioning trick used in [8] and [3]. In the critical case $\sigma = \frac{1}{2}$, the time partitioning trick is not effective, but the trivial bound (109) suffices.

7.1. The subcritical case $0 < \sigma < \frac{1}{2}$. We have
\begin{equation}
R_{N,t} = R_{N,t}^{(0)} + R_{N,t}^{(1)}
\end{equation}
with
\begin{equation}
R_{N,t}^{(0)} := e^{-itH_\omega} \frac{-\lambda}{2\pi i} \int_{C(h)} d\alpha \frac{1}{H_\omega - \alpha - i\varepsilon} V\tilde{\phi}_{N,\varepsilon}(\alpha)
\end{equation}
and
\begin{equation}
R_{N,t}^{(1)} := -i\lambda \int_{0}^{t} ds e^{-i(t-s)H_\omega} V\phi_{N,s},
\end{equation}
as was shown in (58).

Lemma 7.1.
\begin{equation}
\mathbb{E}[\|R_{N,t}^{(0)}\|_2^2(z,\mathbb{Z})] \leq N! \lambda^2 \left(\frac{1}{\varepsilon}\right)^2 (A_{\sigma,\tau,J,\lambda,\varepsilon})^N,
\end{equation}
where $A_{\sigma,\tau,J,\lambda,\varepsilon}$ is defined in (107).

Proof. We can deform the contour $C^{(h)}$ of the $\alpha$-integration in (111) into
\begin{equation}
\hat{C}^{(h)} := (-4 + \tau/2 + i[0,1]) \cup ([-4 + \tau/2, -\tau/2] + i) \cup (-\tau/2 + i[0,1]) \cup (\tau/2 + i[0,1]) \cup ([4 - \tau/2, \tau/2] - i) \cup (4 - \tau/2 + i[0,1]),
\end{equation}
where $|\hat{C}^{(h)}|$ denotes the length of the contour $\hat{C}^{(h)}$. This is done by cutting the upper and lower lines of the integration at the points $-\tau/2 + i[0,1]$ and $4 - \tau/2 + i[0,1]$. The faces of $\hat{C}^{(h)}$ are
\begin{itemize}
\item[-] $\{\alpha < -\tau/2 - i\}$
\item[-] $\{\alpha > 4 - \tau/2 + i\}$
\item[-] $\{\alpha = -\tau/2 + i\}$
\item[-] $\{\alpha = 4 - \tau/2 + i\}$
\end{itemize}
Thus, the contribution of the $\alpha$-integration in $\mathbb{R}$ to the remainder term
\begin{equation}
\int_{\hat{C}^{(h)}} \frac{1}{H_\omega - \alpha - i\varepsilon} V\tilde{\phi}_{N,\varepsilon}(\alpha)
\end{equation}
is bounded by
\begin{equation}
\frac{N! \lambda^2}{\varepsilon^2} (A_{\sigma,\tau,J,\lambda,\varepsilon})^N,
\end{equation}
where $A_{\sigma,\tau,J,\lambda,\varepsilon}$ is defined in (107). \hfill \Box
as there is no obstructing phase factor $e^{-i\alpha}$. One then immediately sees that
\begin{equation}
\mathbb{E}[\| \chi_{I_\tau}(H_\omega) R_{N,t}^{(0)} \|_{l^2(Z^2)}^2] \leq \frac{\epsilon \lambda^2}{\tau^2} \mathbb{E}[\| V \phi_{N,t} \|_{l^2(Z^2)}^2],
\end{equation}
since almost surely,
\begin{equation}
\left\| \chi_{I_\tau}(H_\omega) \frac{1}{H_\omega - \alpha - i\epsilon} \right\|_{op} < c\tau^{-1},
\end{equation}
for any $\alpha \in \tilde{C}^{(h)}$. We note that by the effect of the infrared regularization, use of unitarity of $e^{itH}$ in estimating (111) is not penalized by the usual factor $t^2 = \epsilon^{-2}$.

Using unitarity in bounding the corresponding quantity for $R_{N,t}^{(1)}$, however, costs a factor $\epsilon^{-2}$, and we shall use the time partitioning trick of [8] to account for it.

**Lemma 7.2.** For $1 \ll \kappa \ll \epsilon^{-1}$, and $0 < \sigma < \frac{1}{2}$,
\begin{equation}
\mathbb{E}[\| R_{N,t}^{(1)} \|_{l^2(Z^2)}] \leq (3\kappa N)^2 (\log \frac{1}{\epsilon})^2 \sum_{n=N+1}^{4N-1} n! (A_{\sigma,\tau,J,L,\epsilon})^n
\end{equation}
\begin{equation}
+ (4N)! \frac{1}{\epsilon^2 \kappa (1-2\sigma)N} (\log \frac{1}{\epsilon})^2 C^{4N} (A_{\sigma,\tau,J,L,\epsilon})^{4N}
\end{equation}

**Proof.** The asserted estimate is obtained from application of the time partitioning trick introduced in [8]. The details for the lattice model are presented in [3], and we shall here only sketch the strategy.

We choose $\kappa \in \mathbb{N}$ with $1 \ll \kappa \ll \epsilon^{-1}$, and partition $[0,t]$ into $\kappa$ subintervals
\begin{equation}
[0, t] = [0, \theta_1] \cup_{j=1}^{\kappa-1} (\theta_j, \theta_{j+1}]
\end{equation}
with $\theta_j = \frac{j \kappa}{\kappa}$, $j = 1, \ldots, \kappa$. Thereby,
\begin{equation}
R_{N,t}^{(1)} = -i\lambda \sum_{j=0}^{\kappa-1} e^{-i(t-\theta_{j+1})H_\omega} \int_{\theta_j}^{\theta_{j+1}} ds \ e^{-isH_\omega} V \phi_{N,s}.
\end{equation}

Let
\begin{equation}
\phi_{n,N,\theta}(s) = (-i\lambda)^n \int_{\mathbb{R}^{n-N}} ds_0 \cdots ds_{n-N} \delta\left(\sum_{j=0}^{n-N} s_j - (s - \theta)\right)
\end{equation}
\begin{equation}
\times e^{-is_0 \Delta} \cdots V e^{-is_{n-N} \Delta} V \phi_{N,\theta}.
\end{equation}
That is, the first $N$ out of $n$ collisions happen in the time interval $[0,\theta]$, while the remaining $n-N$ collisions occur in the time interval $(\theta,s]$.

Expanding $e^{-isH_\omega}$ in (119) into a Duhamel series with $3N$ terms and remainder, we find
\begin{equation}
R_{N,t}^{(1)} = \tilde{R}_{N,t}^{(4N)} + \tilde{R}_{N,t}^{(4N)}
\end{equation}
where

\begin{equation}
\tilde{R}^{(\leq 4N)}_{N,t} = \sum_{n=N+1}^{4N-1} \tilde{\phi}_{n,N,t} ,
\end{equation}

\begin{equation}
\tilde{\phi}_{n,N,t} := -i\lambda \sum_{j=1}^{K} e^{-i(t-\theta_j)H_\omega} V \phi_{n,N,\theta_j-1}(\theta_j)
\end{equation}

and

\begin{equation}
\tilde{R}^{(4N)}_{N,t} = -i\lambda \sum_{j=1}^{K} e^{-i(t-\theta_j)H_\omega} \int_{\theta_j-1}^{\theta_j} ds \ e^{-i(\theta_j-s)H_\omega} V \phi_{4N,N,\theta_j-1}(s).
\end{equation}

By the Schwarz inequality,

\begin{equation}
\| \tilde{R}^{(\leq 4N)}_{N,t} \|_{L^2(\mathbb{R})} \leq (3N\kappa) \sup_{N<n<4N,1 \leq j \leq K} \| \lambda V \phi_{n,N,\theta_j-1}(\theta_j) \|_{L^2(\mathbb{R})}
\end{equation}

and

\begin{equation}
\| \tilde{R}^{(4N)}_{N,t} \|_{L^2(\mathbb{R})} \leq t \sup_{1 \leq j \leq K} \sup_{s \in [\theta_j-1,\theta_j]} \| \lambda V \phi_{4N,N,\theta_j-1}(s) \|_{L^2(\mathbb{R})} .
\end{equation}

The functions $\phi_{n,N,\theta_j-1}(\theta_j)$ and $\phi_{4N,N,\theta_j-1}(s)$ have the following properties.

The expected value of $|122|^2$ is bounded by the first term after the inequality sign in (117). This is a straightforward consequence of Lemma 6.2. For the detailed argument, see [4, 8].

It remains to estimate (126). With $\theta' - \theta = \frac{1}{\pi}$, we find

\begin{equation}
(\hat{\phi}_{n,N,\theta}(\theta'))(k_0) = \frac{i(-\lambda)^{n-N}e^{\frac{i\pi}{\alpha}}}{2\pi} \int d\alpha e^{-\frac{i\alpha}{\kappa\varepsilon}} \int_{(\mathbb{Z})^{n-N+1}} dk_1 \cdots dk_{n-N} \times \frac{1}{e_\Delta(k_0) - \alpha - i\kappa\varepsilon} \hat{V}(k_1 - k_0) \cdots \frac{1}{e_\Delta(k_{N-N}) - \alpha - i\kappa\varepsilon} \hat{V}(k_{N-N+1} - k_{n-N})
\end{equation}

\begin{equation}
\times \hat{\phi}_{N,\theta}(k_{n-N+1}),
\end{equation}

where we recall that

\begin{equation}
\hat{\phi}_{N,\theta}(k_{n-N+1}) = \frac{i(-\lambda)^{n}e^{\theta}}{2\pi} \int_{C(\rho)} d\alpha e^{-i\alpha} \int_{(\mathbb{Z})^{n-N+1}} \prod_{j=n-N+1}^{n+1} dk_j \times \frac{1}{e_\Delta(k_{n-N+1}) - \alpha - \frac{i\pi}{\kappa\varepsilon}} \hat{V}(k_{n-N+2} - k_{n-N+1}) \cdots \frac{1}{e_\Delta(k_{n+1}) - \alpha - \frac{i\pi}{\kappa\varepsilon}} \hat{\phi}_{0}(k_{n+1}).
\end{equation}

The key observation here is that there are $n-N+1$ propagators with imaginary parts $\pm i\kappa\varepsilon$ in the denominator, where $\kappa\varepsilon \gg \varepsilon$ (and $N+1$ propagators whose denominators have an imaginary part $-\frac{i\pi}{\kappa\varepsilon}$, where $\frac{1}{\kappa}$ and $\varepsilon$ can have a comparable size). For those $n-N+1$ propagators, we have a bound

\begin{equation}
\frac{1}{|e_\Delta - \alpha - i\kappa\varepsilon|} |f(P_j^2\nu_\sigma^2)| \leq C2^{-2\sigma j} \frac{1}{|e_\Delta(p) - \alpha| + \kappa\varepsilon + 2^{-j}}.
\end{equation}
We now separate the dyadic scales of the random potential into

\[ 0 \leq j \leq J' + 1, \quad 2^J' \sim \frac{1}{\kappa} 2^J. \]

Using

\[ \left\| \frac{1}{|e_\Delta - \alpha - i\kappa\epsilon|} * |\mathcal{F}(P^2_j v_\sigma^2)| \right\|_{L^\infty(\mathbb{T}^2)} \leq 2^{(1-2\sigma)j} \]

(130)

for \( j \leq J' \), we have

\[ \sum_{j=0}^{J'} \left\| \frac{1}{|e_\Delta - \alpha - i\kappa\epsilon|} * |\mathcal{F}(P^2_j v_\sigma^2)| \right\|_{L^\infty(\mathbb{T}^2)} \leq \frac{2^{(1-2\sigma)(J'+1)} - 1}{2^{(1-2\sigma)} - 1} \]

(131)

Furthermore,

\[ \sum_{j=J'+1}^{J'+1} \left\| \frac{1}{|e_\Delta - \alpha - i\kappa\epsilon|} * |\mathcal{F}(P^2_j v_\sigma^2)| \right\|_{L^\infty(\mathbb{T}^2)} \leq \frac{1}{\kappa^{1-2\sigma}} \left( \frac{1}{\epsilon} - 1 \right) \left( 1 - 2^{(1-2\sigma)J'} \right) \]

(132)

for \( j = J' + 1 \).

Therefore, the estimates for resolvents with \( \pm i\kappa\epsilon \) in the denominators are by a factor \( \frac{1}{\kappa(1-2\sigma)} \) smaller than those for resolvents with \( \pm i\epsilon \) derived above.

\[ \sum_{j=0}^{J'+1} \left\| \frac{1}{|e_\Delta(p) - \alpha - i\kappa\epsilon|} * |\mathcal{F}(P^2_j v_\sigma^2)| \right\|_{L^\infty(\mathbb{T}^2)} \leq \frac{1}{\kappa^{1-2\sigma}} \left( K_\sigma(J) + \sigma^{-1} 2^{(1-2\sigma)J} \right) \]

(133)

As before, we systematize the evaluation of

\[ \mathbb{E} \left[ \|\lambda V\phi_{4N,N,\theta_j-1}(s)\|_{\ell_2(\mathbb{Z}^2)}^2 \right] \]

by invoking a graph expansion with \( \pi \in \Pi_{4N,AN} \).

For every graph, we again introduce an admissible spanning tree \( T \), as in the proof of Lemma 6.2 and use the estimate (130) for tree propagators with \( \pm i\kappa\epsilon \) in the denominators. By the pigeonhole principle, there are at least \( N \) of those for every \( \pi \), and any admissible spanning tree \( T \) for \( \pi \). This gains a factor of at least \( \frac{1}{\kappa(1-2\sigma)N} \) in comparison to the bound in Lemma 6.2. The \( L^1(\mathbb{T}^2) \)-bounds on loop resolvents are estimated by \( C \log \frac{1}{\epsilon} \), as before. Observing that the number of tree propagators is \( \tilde{n} \), and that there are \( \tilde{n} + 2 \) propagators estimated in \( L^1 \), one concludes that the expected value of \( \|\mathcal{F}(v_\sigma^2)\|^2 \) is bounded by the second term after the inequality sign in (117). A detailed exposition is given in [8] and [3].
7.2. The critical case $\sigma = 1/2$. The time partitioning only provides a logarithmic improvement in $\kappa$,

$$
\sum_{j=0}^{J'} \left\| \frac{1}{|e^r - \alpha - i\kappa \xi|} \ast |F(P_j u_j^2)| \right\|_{L^\infty(\mathbb{T}^2)} \leq J' + 1 \sim \frac{1}{\log \kappa} J
$$

which is too small to produce a significant effect. However, the trivial estimate (109) is sufficient for our analysis, because the large factor $2^J$ enters $A_{\sigma,\tau,J,\lambda,\varepsilon}$ only logarithmically.

8. Conclusion of the proof of Lemma 3.4

To conclude the proof of Lemma 3.4 we make the following choices for $\varepsilon, J, N, \kappa$ as functions of $\sigma, \lambda$ and $\eta$ (depending implicitly on $\tau$).

8.1. The subcritical case $0 < \sigma < 1/2$. Recalling (50), (67), and summarizing the estimates formulated in Lemmata 6.2 and 7.1, our analysis infers that

$$
l.h.s. of (12) < C\tau^{\frac{1}{2}} + 2 \left( \frac{\varepsilon}{\tau} \right)^2 + \sum_{n=1}^{N} n! (\log \frac{1}{\varepsilon})^2 (A_{\sigma,\tau,J,\lambda,\varepsilon})^n + \lambda^2 (3\kappa N)^2 (\log \frac{1}{\varepsilon})^2 \sum_{n=N+1}^{4N-1} n! (A_{\sigma,\tau,J,\lambda,\varepsilon})^n + (4N)! \frac{\lambda^2}{\varepsilon^2 K(1-2\sigma)N} (\log \frac{1}{\varepsilon})^2 (A_{\sigma,\tau,J,\lambda,\varepsilon})^{4N},
$$

where we recall from (92) that

$$
A_{\sigma,\tau,J,\lambda,\varepsilon} = C\tau (K_{\sigma}(J)) \lambda^2 \log \frac{1}{\varepsilon} + \varepsilon^{-1} \sigma^{-1} 2^{-2\sigma J} \lambda^2 \log \frac{1}{\varepsilon}.
$$

We have

$$
K_{\sigma}(J) = \frac{2^{(1-2\sigma)(J+1)} - 1}{2^{1-2\sigma} - 1}.
$$

Let $\eta > 0$ be arbitrary but fixed. Setting

$$
\varepsilon = 2^{-J},
$$

$$
JK_{\sigma}(J) = \lambda^{-2+2\eta}
$$

we find

$$
K_{\sigma}(J) \lambda^2 \log \frac{1}{\varepsilon} = JK_{\sigma}(J) \lambda^2 \leq \lambda^{2\eta}
$$

$$
\varepsilon^{-1} \sigma^{-1} 2^{-2\sigma J} \lambda^2 \log \frac{1}{\varepsilon} = \sigma^{-1} 2^{(1-2\sigma)J} \lambda^2 \log \frac{1}{\varepsilon} = \sigma^{-1} JK_{\sigma}(J),
$$

so that

$$
A_{\sigma,\tau,J,\lambda,\varepsilon} < \lambda^{1.9\eta},
$$

for $\lambda$ sufficiently small (depending on $\sigma$).
Choosing

\[ N = \frac{\eta \log \frac{1}{\lambda}}{10 \log \log \frac{1}{\lambda}}, \]

one gets (noting that \( \varepsilon > \lambda^2 \))

\[ \sum_{n=1}^{N} n! \left( \log \frac{1}{\varepsilon} \right)^2 (A_{\sigma, \tau, I, n})^n < C \left( \log \frac{1}{\lambda} \right)^2 \sum_{n=1}^{N} (N A_{\sigma, \tau, I, n})^n \]

(144)

\[ < C \left( \log \frac{1}{\lambda} \right)^2 \sum_{n=1}^{N} \lambda^{1.5n} < \lambda^{1.1n} \]

and

\[ N! \frac{\lambda^2}{\tau^2} \left( \log \frac{1}{\varepsilon} \right)^2 (A_{\sigma, \tau, I, n})^N < C \left( \log \frac{1}{\lambda} \right)^2 (N A_{\sigma, \tau, I, n})^N < \lambda \]

for \( \tau \gg \lambda \). Choosing

\[ \kappa = (\log \frac{1}{\lambda})^{\frac{30}{4^{N-1}}} \]

(145)

one gets

\[ \lambda^2 (3\kappa N)^2 \left( \log \frac{1}{\varepsilon} \right)^2 \sum_{n=N+1}^{4N-1} n! (A_{\sigma, \tau, I, n})^n < C \lambda^2 \left( \log \frac{1}{\lambda} \right)^{\frac{100}{4^{N-1}}} (4N A_{\sigma, \tau, I, n})^N \]

(146)

\[ < \lambda^{2\eta}. \]

Furthermore, since

\[ \kappa (1-2\sigma)^N > \lambda^{-3}, \]

one finds

\[ (4N)! \frac{\lambda^2}{\varepsilon^2 \kappa (1-2\sigma)^N} \left( \log \frac{1}{\varepsilon} \right)^2 (A_{\sigma, \tau, I, n})^{4N} < (4N A_{\sigma, \tau, I, n})^{4N} < \lambda^{2\eta}. \]

(147)

Thus, for \( \lambda \) sufficiently small (depending on \( \sigma \) and \( \eta \)),

\[ \ell_\sigma(\lambda) \geq C_\sigma \lambda^{-\frac{2-\sigma}{2}}. \]

(148)

Moreover, (149), (150) and (151) combined imply that for every fixed \( 0 < \sigma < \frac{1}{2} \), there exists a positive constant \( C_\sigma \) such that

\[ \ell_\sigma(\lambda) \geq C_\sigma \lambda^{-\frac{2-\sigma}{2}}. \]

(152)

This proves the assertion of Lemma 8.2 for \( 0 < \sigma < \frac{1}{2} \).

### 8.2. The critical case \( \sigma = \frac{1}{2} \).

Using (149), (151), Lemma 8.2 and (151),

\[ l.h.s. \ of \ (142) < C \tau^2 + 2 \left( \frac{\varepsilon}{7} \right)^2 + \sum_{n=1}^{N} n! (\log \frac{1}{\varepsilon})^2 (A_{\sigma, \tau, I, n})^n \]

\[ + N! \frac{\lambda^2}{\tau^2} \left( \log \frac{1}{\varepsilon} \right)^2 (A_{\sigma, \tau, I, n})^N \]

\[ + N! \lambda^2 \varepsilon^{-2} (\log \frac{1}{\varepsilon})^2 (A_{\sigma, \tau, I, n})^N. \]

(153)
We have
\[ K_{\frac{1}{2}}(J) = J + 1 . \]  
(154)

Let \( \eta > 0 \) be arbitrary (small) but fixed. Setting
\[ J = N = \lambda^{-\frac{1}{4}} + \eta \]
\[ \varepsilon = 2^{-\lambda^{-\frac{1}{4}} + \eta} = 2^{-N} = 2^{-J} , \]
we get, for sufficiently small \( \lambda > 0 \),
\[ A_{\sigma,\tau,J,\lambda,\varepsilon} = C_{\tau} \left( J \lambda^2 \log \frac{1}{\varepsilon} + 2 \varepsilon^{-1} 2^{-J} \log \frac{1}{\varepsilon} \right) \]
\[ < 2C_{\tau} N^2 \lambda^2 \]
(156)

and
\[ N^2 A_{\sigma,\tau,J,\lambda,\varepsilon} < \lambda^{3\eta} . \]
(157)

Then,
\[ \sum_{n=1}^{N} n!(\log \frac{1}{\varepsilon})^2 (A_{\sigma,\tau,J,\lambda,\varepsilon})^n < N^2 A_{\sigma,\tau,J,\lambda,\varepsilon} + \sum_{n=2}^{N} N^2 (NA_{\sigma,\tau,J,\lambda,\varepsilon})^n \]
\[ < \lambda^{2\eta} \]
(158)

and
\[ N! \frac{\lambda^2}{\tau^2} (\log \frac{1}{\varepsilon})^2 (A_{\sigma,\tau,J,\lambda,\varepsilon})^N < \frac{\lambda^2}{\tau^2} N^2 (NA_{\sigma,\tau,J,\lambda,\varepsilon})^N < \lambda . \]
(159)

Furthermore,
\[ N! \varepsilon^{-3} \lambda^2 (\log \frac{1}{\varepsilon})^2 (A_{\sigma,\tau,J,\lambda,\varepsilon})^N < \lambda^2 N^2 (4 NA_{\sigma,\tau,J,\lambda,\varepsilon})^N \]
\[ < \lambda (4 \lambda^{2\eta}) \lambda^{\frac{1}{4} + \eta} < \lambda . \]
(160)

In conclusion,
\[ l.h.s. \ of \ (132) < C_{\tau} \frac{\lambda}{2} + \lambda^{3\eta} . \]
(161)

From (130) and (155), we infer that
\[ \ell_{\sigma}(\lambda) \geq 2^{-\lambda^{\frac{1}{4}} + \eta} . \]
(162)

This concludes our proof of Lemma 3.4 for \( \sigma = \frac{1}{2} \).

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Figure 1. A contraction graph $\pi \in \Pi_{n,n'}$ with $n = 5$, $n' = 7$ and $\bar{n} = 6$. The particle lines are solid, the contraction lines dashed. The $L^2$-vertex is black, while the $V$-vertices are not filled.

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