Rational expansion method of exponent function for solving exact solutions to nonlinear differential-difference equations

Chengshi Liu
Department of Mathematics
Daqing petroleum Institute
Daqing 163318, China
Email: chengshiliu-68@126.com

October 1, 2018

Abstract
A new method named rational expansion method of exponent function is presented to find exact traveling wave solutions of differential-difference equations. This method generalizes the so-called tanh-method and other similar methods. Some examples are dealt with and their abundant exact solutions which include solitary solutions and periodic solutions are obtained. Among them, many solutions are new.

Keywords: differential-difference equation, exact solution, traveling wave solution, rational expansion method of exponent function

PACS: 02.70.Wz; 02.30.Ik; 02.30.Jr; 02.90.+p

1 Introduction
In some physical fields such as solid physics and biophysics, differential-difference equations (DDEs) are used to model some physical phenomena ([12]) such as particle vibrations in lattices, currents in electrical networks, pulses in biological chains, etc. One also use DDEs to simulate numerically soliton dynamics in high energy physics and other problems where they arise as approximations of continuum models. DDEs are semi-discretized with some (or all) of their special variables discretized while time is usually kept continuous. Many aspects of DDEs such as integrability criteria, the computation of densities and symmetries and so on have been investigated([3456]). Some works have been done to investigate the exact solutions of DDEs. Qian et al([7]) have extended multilinear variable separation approach to a special DDE. In particular, some symbolic computation methods have been applied to this problem. For example, recently some expansive methods such as tanh-method([8]) and its generalizations([9])
and elliptic function expansive method ([10]) have been proposed to solve the traveling wave solutions of DDEs. Using symbolic computation software such as Maple or mathematica, some exact solutions of some DDEs have been obtained. In present paper, we propose a new method named rational expansion method of exponent function for solving traveling wave solutions of DDEs. Our method unify tanh-method and its some generalizations. For illustration, we apply our proposed method to some examples such as Langniuir lattice, discrete mKdV lattice equation and Hybrid lattice equation, and derive their abundant exact solutions. Among them, many solutions are new. Those solutions include abundant solitary solutions and periodic solutions, thus they will be helpful for physical investigation further.

2 Rational expansion method of exponent function

Let us consider rather general lattice equation formed as:

\[ P(u_n'(x), u_n''(x), \cdots, u_n^{(m_1)}(x), u_{n+k_1}(x), \cdots, u_{n+k_2}(x)) = 0, \] (1)

where \( P \) is a polynomial of its entries, \( u, x \) and \( n \) all represent multi-components, and \( u^{(r)} \) denotes the collection of mixed derivative terms of order \( r \). We take traveling wave transformation

\[ \xi = \sum d_i n_i + \sum c_j x_j + \delta, \] (2)

where \( d_i, c_j \) and \( \delta \) are constants to be determined. Under the transformation, the equation (1) becomes

\[ P(u_n'(\xi), u_n''(\xi), \cdots, u_n^{(m_1)}(\xi), u_{n+k_1}(\xi), \cdots, u_{n+k_2}(\xi)) = 0, \] (3)

where the prime is derivative with aspect to \( \xi \). A crucial step of our method is to assume that the solutions of the equation (3) has the following form

\[ u(\xi) = F_{n_1}(\exp(\xi)) = a_0 + a_1 \exp(\xi) + \cdots + a_{n_1} \exp(n_1 \xi), \] (4)

and

\[ u(\xi) = F_{n_2}(\exp(i\xi)) = a_0 + a_1 \exp(i\xi) + \cdots + a_{n_2} \exp(n_2 \xi), \] (5)

where \( a_0, \cdots, a_{n_1} \) and \( b_0, \cdots, b_{n_2} \) are constants to be determined, \( i^2 = -1 \). Instituting the assumed solutions (4) and (5) respectively into the equation (3) and according to balance principle we will derive a relation of \( n_1 \) and \( n_2 \), and furthermore clearing the denominators we will get two polynomials of \( \exp(\xi) \) and \( \exp(i\xi) \) respectively. In order to determine the values of parameters we set the coefficients of those two polynomials to be zeros, thus we get systems of algebraic equations. Solving the algebraic equations systems we will determine
those parameters and obtain the corresponding exact solutions. Of course we can use software such as *Mathematica* or *maple* to solve corresponding algebraic equations system. But our results in present paper are computed by hand. If computation is more complicated we will need *Maple* or *Mathematica*.

It is obvious that our method includes the tanh-method and its generalizations as special cases.

### 3 Applications to DDFs

**Example 1.** Langmuir chains equation

\[
\frac{du_n(t)}{dt} = u_n(u_{n+1} - u_{n-1}),
\]

which arise in the study of langmuir oscillations in plasmas, population dynamics, quantum field theory and polymer science\(^{[11,12,13]}\). It is also named Volterra lattice. Under the wave transformation \(u_n(t) = u(\xi), \xi = dn + ct + \delta\), the Eq.(6) becomes

\[
cu'(\xi) = u(\xi)[u(\xi + d) - u(\xi - d)].
\]

Instituting the expression (4) and (5) respectively into the Eq.(6) and according to the balance principle, we have \(n_1 = n_2\). Through detail computation, we find all solution are trivial solution \(u \equiv \text{constant}\) in cases \(0 \leq n_1 = n_2 \leq 3\). In order to give nontrivial solutions, we take \(n_1 = n_2 = 4\). Thus for convenience we can take the solution assumed as following form

\[
u(\xi) = \frac{A(a_0 + a_1 \exp \xi + a_2 \exp(2\xi) + a_3 \exp(3\xi) + \exp(4\xi))}{b_0 + b_1 \exp \xi + b_2 \exp(2\xi) + b_3 \exp(3\xi) + \exp(4\xi)}, (8)
\]

and

\[
u(\xi) = \frac{A(a_0 + a_1 \exp(i\xi) + a_2 \exp(2i\xi) + a_3 \exp(3i\xi) + \exp(4i\xi))}{b_0 + b_1 \exp(i\xi) + b_2 \exp(2i\xi) + b_3 \exp(3i\xi) + \exp(4i\xi)}, (9)
\]

where \(A, a_i\) and \(b_i\) are constants to be determined, \(i = 0, 1, 2, 3\). According to the algorithm in section 2, in real exponent function case, we obtain the following results:

\[
A = -\frac{2c}{\exp 2d - \exp(-2d)}, \quad a_0 = b_0 \neq 0, \quad a_1 = b_1 = a_3 = b_3 = 0,
\]

\[
a_2 = (\exp(2d) + \exp(-2d) - 1)b_2, \quad b_2 = \pm \sqrt{\frac{a_0(\exp(4d) + \exp(-4d) - 2)}{\exp(2d) + \exp(-2d) - 2}}; (10)
\]

in complex exponent function case, we have

\[
A = -\frac{c}{\sin(2d)}, \quad a_0 = b_0 \neq 0, \quad a_1 = b_1 = a_3 = b_3 = 0,
\]
\[ a_2 = (2 \cos(2d) - 1)b_2, \quad b_2 = \pm 2 \cos(d) \sqrt{a_0}. \]  

Thus we obtain four kinds of traveling wave solutions to Langmuir chains equation

\[ u_n(t) = -c \sech 2d \times \frac{a_0 + 2 \cosh(d)(2 \cosh(2d) - 1) \sqrt{a_0} \exp(2\xi) + \exp(4\xi)}{a_0 + 2 \cosh(d) \sqrt{a_0} \exp(2\xi) + \exp(4\xi)}; \quad (12) \]

\[ u_n(t) = -c \sech 2d \times \frac{a_0 - 2 \cosh(d)(2 \cosh(2d) - 1) \sqrt{a_0} \exp(2\xi) + \exp(4\xi)}{a_0 - 2 \cosh(d) \sqrt{a_0} \exp(2\xi) + \exp(4\xi)}; \quad (13) \]

\[ u_n(t) = -c \frac{\mp 2 \cos(d)(2 \cos(d) - 1) \sqrt{a_0} \exp(2\xi) + \exp(4i\xi)}{\cos(2\xi) \pm \cos(d)(2 \cos(2d) - 1)}; \quad (a_0 > 0); \quad (14) \]

and

\[ u_n(t) = -c \frac{\mp 2 \cos(d)(2 \cos(d) - 1) \sqrt{a_0} \exp(2\xi) + \exp(4i\xi)}{\sin(2\xi) \pm \cos(d)(2 \cos(2d) - 1)}; \quad (a_0 < 0), \quad (15) \]

where \( \xi = dn + ct + \delta \), and \( d, c, \delta \) are arbitrary constants. In special, the parameter \( \delta \) in the Eqs.(14) and (15) have been rescaled.

**Remark 1**: In the case \( n_1 = n_2 > 4 \), the computation is more complicated, we will consider it in the future.

**Example 2**: A general lattice equation

\[ u'_n(t) = (\alpha + \beta u_n(t) + \gamma u^2_n(t))(u_{n+1}(t) - u_{n-1}(t)) \]  

This general lattice equation includes some lattice equations such as Hybrid lattice(14), discretized mKdV lattice(15 8) as special cases. For examples, if \( \alpha = -1 \), we get Hybrid lattice

\[ u'_n(t) = (1 - \beta u_n(t) - \gamma u^2_n(t))(u_{n+1}(t) - u_{n+1}(t)); \quad (17) \]

if \( \beta = 0, \gamma = -1 \), we get discretized mKdV lattice(14 8)

\[ u'_n(t) = (\alpha - u^2_n(t))(u_{n+1}(t) - u_{n-1}(t)); \quad (18) \]

Thus we consider the general lattice (16). Under the wave transformation \( u_n(t) = u(\xi), \xi = dn + ct + \delta \), this equation becomes

\[ \frac{c}{\gamma} u'(\xi) = \left( \frac{\alpha}{\gamma} + \frac{\beta}{\gamma} u(\xi) + u^2(\xi) \right) \{ u(\xi + d) - u(\xi - d) \}. \]  

(19)
For simplicity, sometimes we make a transformation for $u$ as follows

$$u = v - \frac{\beta}{2\gamma},$$

(20)

then the Eq.(14) becomes

$$\frac{c}{\gamma}v' (\xi) = \left( \frac{\alpha}{\gamma} - \frac{\beta^2}{4\gamma^2} + v^2 (\xi) \right) \{v(\xi + d) - v(\xi - d)\}.$$  

(21)

Instituting the assumed solution (4) and (5) into the Eq.(19) and according to the balance principle, we get

$$n_1 \leq n_2.$$  

If we take $n_2 = 1$, we find there is only trivial solution $u(\xi) \equiv \text{constant}$, therefore we take $n_2 \geq 2$. Here we only consider the case $n_2 = 2$ for simplicity, other cases will be dealt with in the future. When $n_1 = 0$, we get only trivial solution $u = \text{constant}$. When $n_1 = 1$, for convenience we take the assumed solution as follows

$$u(\xi) = A(a_0 + \exp \xi) + b_0 + b_1 \exp(i\xi) + \exp(2i\xi),$$

(22)

and

$$u(\xi) = A(a_0 + \exp(i\xi)) + b_0 + b_1 \exp(i\xi) + \exp(2i\xi),$$

(23)

where $A \neq 0$ and $a_0, b_0, b_1$ are constants to be determined. According to the algorithm in section 2, if $\alpha = 0$, then no solution is obtained, so we let $\alpha \neq 0$. In real exponent function case we have

$$c = \alpha \{\exp d - \exp(-d)\}, \quad a_0 = 0, \quad b_1 = \frac{A\beta}{\alpha \{\exp d + \exp(-d) - 2\}}, \quad b_0 = \frac{A^2 \{\frac{\beta^2}{\alpha (2 \cos d - 2)} + \gamma\}}{\alpha \{\exp(2d) + \exp(-2d) - 2\}},$$

(24)

in complex exponent function case we have

$$c = 2\alpha \sin(d), \quad a_0 = 0, \quad b_1 = \frac{A\beta}{\alpha (2 \cos d - 2)}, \quad b_0 = \frac{A^2 \{\frac{\beta^2}{\alpha (2 \cos d - 2)} + \gamma\}}{\alpha (2 \cos(2d) - 2)},$$

(25)

and here $\alpha$ and $A$ are arbitrary nonzero constants. Thus the corresponding solutions of the Eq.(16) are as follows:

$$u_n(t) = \frac{A \exp(dn + 2\alpha \sinh(d)t + \delta)}{\frac{A^2 (\frac{\beta^2}{4\alpha \sinh^2(d/2)} + \gamma)}{4\alpha \sinh^2(d/2)} + \frac{A\beta \exp(dn + 2\alpha \sinh(d)t + \delta)}{4\alpha \sinh^2(d/2)} + \exp(2(dn + 2\alpha \sinh(d)t + \delta))};$$

(26)

$$u_n(t) =$$

5
\[
\begin{align*}
A \exp(i(dn + 2\alpha \sin(d)t + \delta)) & = \frac{A^2 \exp(i(dn + 2\alpha \sin(d)t + \delta))}{4\alpha \sin^2(d/2)} + \exp(2i(dn + 2\alpha \sin(d)t + \delta)) + \frac{1}{\pm 2\sqrt{\frac{A^2}{4\alpha \sin^2(d/2)} + \frac{\alpha}{4\alpha \sin^2(d/2)}}}, \\
\text{and} \\
A^2 \left\{ \frac{\beta^2}{4\alpha \sin^2(d/2)} + \frac{\gamma}{4\alpha \sin^2(d/2)} \right\} & = \frac{1}{\pm 2\sqrt{\frac{A^2}{4\alpha \sin^2(d/2)} + \frac{\alpha}{4\alpha \sin^2(d/2)}}}, \\
\text{and} \\
A \exp(i(dn + 2\alpha \sin(d)t + \delta)) & = \frac{1}{\pm 2\sqrt{\frac{A^2}{4\alpha \sin^2(d/2)} + \frac{\alpha}{4\alpha \sin^2(d/2)}}}.
\end{align*}
\]

Correspondingly if \( \alpha \neq 0, \beta = 0, \gamma = -1 \), we obtain the exact solutions to discretized mKdV lattice equation (18) as follows:

\[
\begin{align*}
\text{when } n_1 = 2, \text{ for convenience we take the assumed solution to the Eq.}(14) & \text{ as follows:} \\
& \text{as follows} \\
u_n(t) = \pm \sinh(d) \sqrt{\alpha} \sech (dn + 2\alpha \sinh(d)t + \delta), (\alpha > 0); \\
u_n(t) = \pm \sinh(d) \sqrt{-\alpha} \sec (dn + 2\alpha \sinh(d)t + \delta), (\alpha < 0); \\
u_n(t) = \pm \sinh(d) \sqrt{-\alpha} \cosh (dn + 2\alpha \sinh(d)t + \delta), (\alpha < 0); \\
\end{align*}
\]

where \( \delta \) have been rescaled. If \( \alpha = -1 \), we obtain the exact solution to Hybrid lattice equation (17) as follows:

\[
\begin{align*}
\text{and} \\
& \text{as follows} \\
u_n(t) = \pm \sinh(d) \sqrt{\alpha} \cosh (dn + 2\alpha \sinh(d)t + \delta), (\alpha < 0), \\
u_n(t) = \pm \sinh(d) \sqrt{\alpha} \cosh (dn + 2\alpha \sinh(d)t + \delta), (\alpha < 0), \\
\end{align*}
\]

To our knowledge, the solutions (29-35) all are new.
where $A \neq 0$ and $a_0, a_1, b_0, b_1$ are constants to be determined. According to the algorithm in section 2, we have the following three families of solutions:

**Family 1:** for real exponent function,

$$c = \frac{4\gamma \alpha - \beta^2}{2\gamma} \tanh(d), \quad a_1 = b_1 = 0,$$

$$A = -\frac{\beta}{2\gamma} \pm \tanh(d) \times \frac{\sqrt{\beta^2 - 4\alpha \gamma}}{2\gamma}, \quad a_0 = \frac{(\beta \pm \sqrt{\beta^2 - 4\alpha \gamma})b_0}{\beta \pm \sqrt{\beta^2 - 4\alpha \gamma}}; \quad (38)$$

for complex exponent function,

$$c = \frac{4\gamma \alpha - \beta^2}{2\gamma} \tan(d), \quad a_1 = b_1 = 0,$$

$$A = -\frac{\beta}{2\gamma} \pm \tan(d) \times \frac{\sqrt{\beta^2 - 4\alpha \gamma}}{2\gamma}, \quad a_0 = \frac{(\beta \pm \sqrt{\beta^2 - 4\alpha \gamma})b_0}{\beta \pm \sqrt{\beta^2 - 4\alpha \gamma}}; \quad (39)$$

where $\beta^2 - 4\alpha \gamma > 0$. Thus the corresponding solutions are as follows:

$$u_n(t) = \frac{\beta}{2\gamma} \pm \sqrt{\frac{\beta^2 - 4\alpha \gamma}{2\gamma}} \tanh(d) \frac{-b_0 + \exp(dn + \frac{4\alpha \gamma - \beta^2}{2\gamma} \tanh(d)t + \delta)}{b_0 + \exp(dn + \frac{4\alpha \gamma - \beta^2}{2\gamma} \tanh(d)t + \delta)}, \quad (40)$$

and

$$u_n(t) = \frac{\beta}{2\gamma} \pm \sqrt{\frac{\beta^2 - 4\alpha \gamma}{2\gamma}} \tan(d) \frac{-b_0 + \exp(i(dn + \frac{4\alpha \gamma - \beta^2}{2\gamma} \tan(d)t + \delta))}{b_0 + \exp(i(dn + \frac{4\alpha \gamma - \beta^2}{2\gamma} \tan(d)t + \delta))}, \quad (41)$$

According to the cases $b_0 > 0$ and $b_0 < 0$, there are four solutions,

$$u_n(t) = \frac{\beta}{2\gamma} \pm \sqrt{\frac{\beta^2 - 4\alpha \gamma}{2\gamma}} \tanh(d) \tanh(dn + \frac{4\alpha \gamma - \beta^2}{2\gamma} \tanh(d)t + \delta); \quad (42)$$

$$u_n(t) = \frac{\beta}{2\gamma} \pm \sqrt{\frac{\beta^2 - 4\alpha \gamma}{2\gamma}} \tanh(d) \coth(dn + \frac{4\alpha \gamma - \beta^2}{2\gamma} \tanh(d)t + \delta); \quad (43)$$

$$u_n(t) = \frac{\beta}{2\gamma} \pm \sqrt{\frac{\beta^2 - 4\alpha \gamma}{2\gamma}} \tanh(d) \tan(dn + \frac{4\alpha \gamma - \beta^2}{2\gamma} \tanh(d)t + \delta); \quad (44)$$

and

$$u_n(t) = \frac{\beta}{2\gamma} \pm \sqrt{\frac{\beta^2 - 4\alpha \gamma}{2\gamma}} \tanh(d) \cot(dn + \frac{4\alpha \gamma - \beta^2}{2\gamma} \tanh(d)t + \delta). \quad (45)$$

From above results, if we take $\alpha \neq 0, \beta = 0, \gamma = -1$, then we obtain two exact solutions to discretized mKdV lattice equation (13) as follows:

$$u_n(t) = \pm \sqrt{\alpha} \tanh(d) \tanh(dn + 2\alpha \tanh(d)t + \delta); \quad (46)$$
and
\[ u_n(t) = \pm \sqrt{\alpha} \cot(d) \cot(dn + 2\alpha \tan(d)t + \delta). \quad (49) \]

If we take \( \alpha = -1 \), we obtain the exact solutions to Hybrid lattice equation (12) as follows:
\[ u_n(t) = \frac{\beta}{2\gamma} \pm \frac{2\sqrt{\beta^2 + 4\gamma}}{2\gamma} \tanh(d) \tanh(dn - \frac{4\gamma + \beta^2}{2\gamma} \tanh(d)t + \delta); \quad (50) \]
\[ u_n(t) = \frac{\beta}{2\gamma} \pm \frac{2\sqrt{\beta^2 + 4\gamma}}{2\gamma} \tanh(d) \coth(dn - \frac{4\gamma - \beta^2}{2\gamma} \tanh(d)t + \delta); \quad (51) \]
\[ u_n(t) = \frac{\beta}{2\gamma} \pm \frac{2\sqrt{\beta^2 + 4\gamma}}{2\gamma} \tan(d) \tan(dn - \frac{4\gamma + \beta^2}{2\gamma} \tan(d)t + \delta); \quad (52) \]
and
\[ u_n(t) = \frac{\beta}{2\gamma} \pm \frac{2\sqrt{\beta^2 + 4\gamma}}{2\gamma} \tan(d) \cot(dn - \frac{4\gamma - \beta^2}{2\gamma} \tan(d)t + \delta); \quad (53) \]

To our knowledge, the solutions (47-49) and (51-53) are new. solutions (46) and (50) have been given in Ref.([8]).

**Family 2:** for real exponent function,
\[ c = \frac{4\alpha\gamma - \beta^2}{2\gamma} \sinh(d), \ A = -\frac{\beta}{2\gamma}, a_0 = b_0, b_1 = 0, a_1 = \frac{4(4\alpha\gamma - \beta^2)}{\beta^2} \sinh^2(d)b_0; \quad (54) \]
and for complex exponent function,
\[ c = \frac{4\alpha\gamma - \beta^2}{2\gamma} \sin(d), \ A = -\frac{\beta}{2\gamma}, a_0 = b_0, b_1 = 0, a_1 = \frac{4(4\alpha\gamma - \beta^2)}{\beta^2} \sin^2(d)b_0; \quad (55) \]

Hence, if \( 4\alpha\gamma - \beta^2 > 0 \), the corresponding solutions to the Eq.(16) are as follows:
\[ u_n(t) = \frac{\beta}{2\gamma}(1 \pm \frac{2\sqrt{4\alpha\gamma - \beta^2}}{\beta} \sinh(d) \exp(dn + \frac{4\alpha\gamma - \beta^2}{2\gamma} \sinh(d)t + \delta)) \exp(2(dn + \frac{4\alpha\gamma - \beta^2}{2\gamma} \sinh(d)t + \delta) + 1) \]
\[ = -\frac{\beta}{2\gamma}(1 \pm \frac{2\sqrt{4\alpha\gamma - \beta^2}}{\beta} \sinh(d) \cosh(dn + \frac{4\alpha\gamma - \beta^2}{2\gamma} \sinh(d)t + \delta)); \quad (56) \]
and
\[ u_n(t) = \frac{\beta}{2\gamma}(1 \pm \frac{2\sqrt{4\alpha\gamma - \beta^2}}{\beta} \sinh(d) \exp(i(dn + \frac{4\alpha\gamma - \beta^2}{2\gamma} \sin(d)t + \delta)) \exp(2i(dn + \frac{4\alpha\gamma - \beta^2}{2\gamma} \sin(d)t + \delta) + 1) \]
\[ = -\frac{\beta}{2\gamma}(1 \pm \frac{2\sqrt{4\alpha\gamma - \beta^2}}{\beta} \sinh(d) \csc(dn + \frac{4\alpha\gamma - \beta^2}{2\gamma} \sin(d)t + \delta)); \quad (57) \]
if $4\alpha \gamma - \beta^2 > 0$, the corresponding solutions to the Eq.(16) are as follows:

$$u_n(t) = -\frac{\beta}{2\gamma}(1 \pm \frac{2\sqrt{\beta^2 - 4\alpha \gamma}}{\beta} \sinh(d) \exp(\frac{4\alpha \gamma - \beta^2}{2\gamma} \sinh(d)t + \delta))$$

$$= -\frac{\beta}{2\gamma}(1 \pm \sqrt{\frac{4\alpha \gamma - \beta^2}{\beta}} \sinh(d) \sech(d + \frac{4\alpha \gamma - \beta^2}{2\gamma} \sinh(d)t + \delta)).$$

(58)

and

$$u_n(t) = -\frac{\beta}{2\gamma}(1 \pm \frac{2\sqrt{\beta^2 - 4\alpha \gamma}}{\beta} \sin(d) \exp(\frac{4\alpha \gamma - \beta^2}{2\gamma} \sin(d)t + \delta))$$

$$= -\frac{\beta}{2\gamma}(1 \pm \sqrt{\frac{4\alpha \gamma - \beta^2}{\beta}} \sin(d) \sec(d + \frac{4\alpha \gamma - \beta^2}{2\gamma} \sin(d)t + \delta)).$$

(59)

When $\alpha = -1$, we have the corresponding solution to hybrid lattice equation as follows:

$$u_n(t) = -\frac{\beta}{2\gamma}(1 \pm \frac{2\sqrt{\beta^2 - 4\alpha \gamma}}{\beta} \sin(d) \exp(i(\frac{4\alpha \gamma - \beta^2}{2\gamma} \sin(d)t + \delta))$$

$$= -\frac{\beta}{2\gamma}(1 \pm \sqrt{\frac{4\alpha \gamma - \beta^2}{\beta}} \sin(d) \sec(d + \frac{4\alpha \gamma - \beta^2}{2\gamma} \sin(d)t + \delta)).$$

(60)

and

$$u_n(t) = -\frac{\beta}{2\gamma}(1 \pm \frac{2\sqrt{\beta^2 - 4\alpha \gamma}}{\beta} \sin(d) \exp(\frac{4\alpha \gamma - \beta^2}{2\gamma} \sin(d)t + \delta))$$

$$= -\frac{\beta}{2\gamma}(1 \pm \sqrt{\frac{4\alpha \gamma - \beta^2}{\beta}} \sin(d) \sec(d + \frac{4\alpha \gamma - \beta^2}{2\gamma} \sin(d)t + \delta)).$$

(61)

To our knowledge the solutions (60) and (61) are new.

**Family 3:** for real exponent function,

$$A = \frac{\beta(\frac{1}{2} \sech(d) - 1) \pm \sqrt{\beta^2(1 - \frac{1}{2} \sech(d)) - 4\alpha \gamma(1 - \sech(d))}}{2\gamma}, a_0 = b_0,$$

$$a_1 = 0, b_1 = -\frac{4\alpha + 2\beta A}{\gamma A^2 + \beta A} \sech(d) \sin^2(d)a_0, c = 2(\alpha + \beta A + \gamma A^2) \sin(d);$$

(62)

for complex exponent function,

$$A = \frac{\beta(\frac{1}{2} \sec(d) - 1) \pm \sqrt{\beta^2(1 - \frac{1}{2} \sec(d)) - 4\alpha \gamma(1 - \sec(d))}}{2\gamma}, a_0 = b_0,$$

$$a_1 = 0, b_1 = -\frac{4\alpha + 2\beta A}{\gamma A^2 + \beta A} \sec(d) \sin^2(d)a_0, c = 2(\alpha + \beta A + \gamma A^2) \sin(d).$$

(63)

Hence we have solutions to the Eq.(16) as follows:

$$u_n(t) = \frac{A(a_0 + \exp(2\xi))}{a_0 \pm \sqrt{-\frac{4\alpha + 2\beta A}{\gamma A^2 + \beta A} \sech(d)a_0 \sinh(d) \exp(\xi) + \exp(2\xi)}}.$$  

(64)
and
\[ u_n(t) = \frac{A(a_0 + \exp(2i\xi))}{a_0 \pm \sqrt{\frac{-4\alpha + 2\beta A}{\gamma A^2 + \beta A} \sec(d)a_0 \sin(d) \exp(i\xi) + \exp(2i\xi)}}. \tag{65} \]

If \( a_0 > 0 \), the above solutions become
\[ u_n(t) = \frac{A(1 + \exp(2\xi))}{1 \pm \sqrt{\frac{-4\alpha + 2\beta A}{\gamma A^2 + \beta A} \sec(d) \sinh(d) \exp(\xi) + \exp(2\xi)}}; \tag{66} \]
and
\[ u_n(t) = \frac{A(1 + \exp(2i\xi))}{1 \pm \sqrt{\frac{-4\alpha + 2\beta A}{\gamma A^2 + \beta A} \sec(d) \sin(d) \exp(i\xi) + \exp(2i\xi)}}. \tag{67} \]

If \( a_0 < 0 \), the above solutions become
\[ u_n(t) = \frac{A(-1 + \exp(2\xi))}{-1 \pm \sqrt{\frac{-4\alpha + 2\beta A}{\gamma A^2 + \beta A} \sec(d) \sinh(d) \exp(\xi) + \exp(2\xi)}}; \tag{68} \]
and
\[ u_n(t) = \frac{A(-1 + \exp(2i\xi))}{-1 \pm \sqrt{\frac{-4\alpha + 2\beta A}{\gamma A^2 + \beta A} \sec(d) \sin(d) \exp(i\xi) + \exp(2i\xi)}}. \tag{69} \]

Furthermore, when \( \alpha \neq 0, \beta = 0, \gamma = -1 \), respectively we have the solutions to the discretized KdV equation as follows:
\[ u_n(t) = \frac{A(1 + \exp(2\xi))}{1 \pm \frac{\sqrt{\alpha \sec(d) \sinh(d)}}{A} \sqrt{-\alpha \sec(d) \sinh(d) \exp(\xi) + \exp(2\xi)}}; \tag{70} \]
where \( \alpha \sec(d) < 0 \). And
\[ u_n(t) = \frac{A(-1 + \exp(2\xi))}{-1 \pm \frac{\sqrt{\alpha \sec(d) \sinh(d)}}{A} \sqrt{-\alpha \sec(d) \sinh(d) \exp(\xi) + \exp(2\xi)}}; \tag{71} \]
where \( \alpha \sec(d) > 0 \).
\[ u_n(t) = \frac{A \cos \xi}{\cos \xi \pm \frac{\sqrt{-\alpha \sec(d) \sinh(d)}}{A} \sqrt{-\alpha \sec(d) \sinh(d) \sin(d)}}; \tag{72} \]
where \( \alpha \sec(d) < 0 \). And
\[ u_n(t) = \frac{A \sin \xi}{\sin \xi \pm \frac{\sqrt{-\alpha \sec(d) \sinh(d)}}{A} \sqrt{-\alpha \sec(d) \sinh(d) \sin(d)}}; \tag{73} \]
where \( \alpha \sec(d) < 0 \).

When \( \alpha = -1 \), we have the solutions to hybrid Lattice equation as follows:
\[ u_n(t) = \frac{A(1 + \exp(2\xi))}{1 \pm \sqrt{\frac{-4\alpha + 2\beta A}{\gamma A^2 + \beta A} \sec(d) \sinh(d) \exp(\xi) + \exp(2\xi)}}; \tag{74} \]
\[ u_n(t) = \frac{A(-1 + \exp(2\xi))}{-1 \pm \sqrt{-\frac{4 + 2\beta A}{\gamma A + \beta A}} \text{sech}(d) \sinh(d) \exp(\xi) + \exp(2\xi)}; \quad (75) \]

\[ u_n(t) = \frac{2A \cos \xi}{2 \cos \xi \pm \sqrt{-\frac{4 + 2\beta A}{\gamma A + \beta A}} \sec(d) \sin(d)}; \quad (76) \]

\[ u_n(t) = \frac{2A \sin \xi}{2 \sin \xi \pm \sqrt{-\frac{4 + 2\beta A}{\gamma A + \beta A}} \sec(d) \sin(d)}. \quad (77) \]

To our knowledge all these solutions given in family 3 are new.

4 Discussions and conclusions

We proposed a new method named rational expansion method of exponent function to solve exact solution of nonlinear differential-difference equations. This method generalized Tanh-method and some other similar methods. Using our method, abundant exact solutions to Langmiuir lattice equation, Hybrid lattice equation and discretized mKdV lattice equation. Among them, many solutions are new. Because of using the property of exponent function

\[ \exp(\xi + d) = \exp(\xi) \exp(d), \exp(i(\xi + d)) = \exp(i\xi) \exp(id), \quad (78) \]

we transfer the addition into the multiplication, so corresponding computation becomes relative simple. For a lot of DDEs such as Toda lattice, discrete KdV lattice and Ablowitz-Ladik lattice and so on, our method is also applicable. Of course our method can be applied to lattice equations system.

We must point out that there are also some problems for our method to study further. An open problem is listed in the following:

**Open problem:** whether or not our method can always give new solutions when the orders of polynomials \( F \) and \( G \) in the Eqs.(4) and (5) increase step by step.

Here I have an example to illustrate above problem. We consider the relativistic Toda lattice system

\[ u_n'(t) = (1 + \alpha u_n)(v_n - v_{n-1}), \quad (79) \]
\[ v_n'(t) = v_n(u_{n+1} - u_n + \alpha v_{n+1} - \alpha v_{n-1}). \quad (80) \]

Take a transformation \( v_n = -\frac{1}{\alpha} u_n - \frac{1}{\alpha^2} \), we have

\[ u_n'(t) = (u_n + \frac{1}{\alpha})(u_{n-1} - u_n). \quad (81) \]

Under the traveling wave transformation \( u_n(t) = u(\xi), \xi = dn + ct + \delta \), we have

\[ cu'(\xi) = (u(\xi) + \frac{1}{\alpha})(u(\xi - d) - u(\xi)). \quad (82) \]
Instituting the Eq.(4) into the Eq.(82) and according to the balance principle yield $n_1 = n_2$. Moreover we can easily prove the following conclusion:

For any positive integer $k$, the equation (81) has solutions as follows:

$$u_n(t) = \frac{A(a_0 + \exp(kd + kc + k\delta))}{b_0 + \exp(kd + kc + k\delta)},$$  \hspace{1cm} (83)

where $A = \frac{kc}{\exp(kd) - 1} - \frac{1}{\alpha}$, $a_0 = \frac{(A + \frac{1}{\alpha})\exp(kd) - 1}{A}b_0$, $c$ and $d$ are arbitrary constants. For the complex case, we have a similar result.

From above conclusion, although $k$ is arbitrary, if we rescale $d, c$ and $\delta$, it is easy to see that we can’t give new solutions. This result implicates that our open problem is difficult to answer.

References

[1] M. Toda, Theory of nonlinear lattices, springer Verlag, Berlin, Germany, 1981.

[2] E. Fermi, J. Pasta and S. Ulam, Collected Papers of Enrico Fermi, 1965 Chicago: University of Chicago Press. 978.

[3] D. Levi and O. Ragnisco, Lett. Nuovo Cimento 1978 22:691-696.

[4] D. Levi and R. I. Yamilov, J. Math. Phys. 1997 38:6648-6674.

[5] R. I. Yamilov, J. Phys. A: Math. Gen. 1994 27:6839-6851.

[6] Yu. B. Suris, J. Phys. A: Math. Gen. 1997 30:1745-1761.

[7] Xian-Min Qian, Sen-Yue Lou and Xing-Biao Hu, J. Phys. A: Math. Gen. 2004 37:2401

[8] D. Baldwin, Ü. Göktaş and W. Hereman, Comput. Phys. Commun. 2004 163:203.

[9] Jia-Min Zhu, Chinese Physics, 2005 14:1290-1295.

[10] Chao-Qin Dai and Jie-Fang Zhang, 2005 Int. J. Mod. Phys. B 19 2129

[11] M. Kac and Van Moerbeke, Adv. in Math. 1975 16:160.

[12] M. Wadati, Prog. Theor. Phys. Suppl. 1976 59:36

[13] I. Y. Cherdantsev and R. I. Yamilov, Physica D 1995 87:140.

[14] M. J. Ablowitz and J. F. Ladik, Stud. Appl. Math. 1977 57 1-12.

[15] R. Hirota and M. Iwao, Time-discretization of solitonequations, in: D. Levi, O. Ragnisco (Eds.), SIDE III-Symmetries and Integrability of difference equations, CRM Proc. and Lect. Notes 25, AMS, Providence, Rhode Island, 2000, pp. 217-229.