Trace Anomaly in Geometric Discretization

Bartłomiej Czech

David Rittenhouse Laboratories, University of Pennsylvania, Philadelphia, PA 19104, U.S.A.

Abstract

I develop the simplest geometric-discretized analogue of two dimensional scalar field theory, which qualitatively reproduces the trace anomaly of the continuous theory. The discrete analogue provides an interpretation of the trace anomaly in terms of a non-trivial transformation of electric-magnetic duality-invariant modes of resistor networks that accommodate both electric and magnetic charge currents.

*czech@sas.upenn.edu
1 Introduction

Finding new formulations of familiar concepts frequently proves beneficial in physics. This may be motivated by computational needs, as in lattice field theory, or by a hope to gain new fundamental insights. Since the 1960s the idea of geometric discretization has lured physicists with promises of new ways of looking at field theory and quantum gravity. After Regge calculus [1], a discretized theory of gravity, a discrete understanding of Abelian Chern-Simons theory [2], $BF$ field theory [3] and chiral fermions [4] has been attained. Not surprisingly, geometric discretization has been most successful in modelling topological concepts and theories. Meanwhile, progress in discretizing non-topological objects has been less convincing. Although Desbrun et al. [5] provided explicit discrete analogues of geometric constructs including the Hodge star and the Lie derivative, the complexity of their formalism makes it difficult to derive insights from it. For example, at present there is no discrete derivation of the well-known trace anomaly in two dimensions.

Here I develop a new, simple discretization of scalar field theory and use it to derive the form of the trace anomaly from first principles. I treat the discrete theory as defined independently of its continuous counterpart and work only with constructs which are intrinsic to the discrete theory on a triangulation. The starting point of the discretization is the identification of $p$-forms on a manifold with $p$-cochains on a triangulation, and thereafter the calculation proceeds along a path which is essentially selected by consistency requirements. As such, the present work exhibits the simplest discrete analogue of two-dimensional scalar field theory on a metric space.

At the end of the calculation a new way of thinking about the trace anomaly emerges. The manifold (triangulation) is viewed as an electric-magnetic network and conformal transformations are found to act non-trivially on the relevant charge densities. Appealing to the correspondence with the continuous case, the final form of the discrete trace anomaly may be viewed as singling out a preferred, smeared out notion of curvature on a finite triangulation of a two-dimensional manifold [3]. More on this in the discussion, which follows the review of Sec. 2 and the calculations of Sec. 3.

2 Review of the continuous theory

I begin with a brief review of the scalar theory on a smooth compact orientable two-dimensional manifold $M$ without boundary, based on [7]. This will help to fix all relevant conventions and provide a necessary reference point for setting up a dictionary between the continuous and discrete constructs.

A manifold $M$ is endowed with the de Rham complex $\Omega^r(M)$

$$0 \overset{i}{\rightarrow} \Omega^0(M) \overset{d_0}{\rightarrow} \Omega^1(M) \overset{d_1}{\rightarrow} \Omega^2(M) \overset{d_2}{\rightarrow} 0,$$

where the maps $d_r$ satisfy $d_{r+1}d_r = 0$, and the wedge product

$$\wedge^r: \Omega^r(M) \times \Omega^{2-r}(M) \rightarrow \mathbb{C}$$

$$(\alpha^r, \gamma^{2-r}) = \int_M \alpha^r \wedge^r \gamma^{2-r} = (-1)^r \int_M \gamma^{2-r} \wedge^r \alpha^r.$$
The introduction of a metric $g_{\mu \nu}$ induces the Hodge star, a natural isomorphism between $\Omega^r(M)$ and $\Omega^{2-r}(M)$, which in turn induces a canonical inner product on $\Omega^r(M)$:

$$
* : \Omega^r(M) \rightarrow \Omega^{2-r}(M)
$$

$$
\langle \ldots, \ldots \rangle^{(r)} : \langle \alpha, \beta \rangle = \int_M \alpha \wedge * \beta,
$$

where in the second line I suppressed the superscripts indicating the degrees of the forms. I take the manifold $M$ to be Riemannian, so the Hodge star satisfies:

$$
*_{2-r} \circ * = (-1)^r.
$$

The Hodge star allows one to introduce operators $\delta_{r+1}$ adjoint to the exterior derivatives $d_r$:

$$
\delta_{r+1} : \Omega^{r+1}(M) \rightarrow \Omega^r(M)
$$

$$
\delta_{r+1} = *_{2-r} \circ d_{1-r} \circ *_{r+1}.
$$

The resulting structures, the de Rham complex together with its adjoint, are summarized in the diagram below:

$$
\begin{array}{ccccccc}
0 & \xrightarrow{\delta_0} & \Omega^0(M) & \xrightarrow{d_0} & \Omega^1(M) & \xrightarrow{d_1} & \Omega^2(M) & \xrightarrow{d_2} & 0.
\end{array}
$$

A 0-form field $\Phi$ coupled to a gravitational background on a 2-dimensional manifold $M$ without boundary has the action:

$$
S = \frac{1}{2} \int_M d_0 \Phi \wedge^1 \delta_1 d_0 \Phi = \frac{1}{2} \langle d_0 \Phi, d_0 \Phi \rangle^{(1)}.
$$

The appearance of $*_1$ in (7) breeds a dependence on $g_{\mu \nu}$ and is the ultimate reason for the theory not being topological. The effective action

$$
W[g] = \log \int [d\Phi] e^{-S[\Phi, g]}.
$$

suffers from an anomaly:

$$
g_{\mu \nu}(x) \frac{\delta W[g]}{\delta g_{\mu \nu}(x)} = -\frac{1}{2} \sqrt{g} \lim_{\epsilon \rightarrow 0} \langle x| e^{\Delta} | x \rangle.
$$

The small time diagonal heat kernel takes the form

$$
\lim_{\epsilon \rightarrow 0} \langle x| e^{\Delta} | x \rangle = \frac{1}{4 \pi \epsilon} + \frac{1}{24 \pi} R(x) + O(\epsilon),
$$

which after the addition of a local counterterm $S \rightarrow S - A/8 \pi \epsilon$ yields

$$
g_{\mu \nu}(x) \frac{\delta W[g]}{\delta g_{\mu \nu}(x)} = -\frac{1}{48 \pi} \sqrt{g} R(x).
$$

The anomaly, integrated over $M$, gives:

$$
\int_M g_{\mu \nu} \frac{\delta W[g]}{\delta g_{\mu \nu}} = \frac{1}{6} \chi.
$$
3 The discrete theory

I now set out to find the analogues of eqs. (7-12) for a theory defined on a finite triangulation $G$ of the manifold $M$. $G$ is most easily thought of as a graph $G(V, E)$ containing $|V|$ vertices and $|E|$ edges, embedded in $M$ in such a way that no two edges cross. For technical convenience I shall assume that $G$ does not contain loops and the number of edges $E$ is even. The embedding partitions $M$ into $|F|$ faces, which are (homeomorphic to) triangles when $G$ corresponds to a triangulation but may be (homeomorphic to) arbitrary polygons for a more general $G$. The number of faces is such that the Euler characteristics agree:

$$\chi_G = |V| + |F| - |E| = \chi_M. \quad (13)$$

It is natural to associate $p$-forms on $M$ with $p$-cochains of $G$. Because we are interested in modelling a real scalar $\Phi$, I shall take $\Omega^0_{x \in M} \rightarrow \Omega^0_{v \in V} \equiv \mathbb{R}$. Thus:

$$\Omega^0(M) \longleftrightarrow \mathbb{R}^V$$
$$\Omega^1(M) \longleftrightarrow \mathbb{R}^E$$
$$\Omega^2(M) \longleftrightarrow \mathbb{R}^F \quad (14)$$

This set-up suggests a canonical correspondence between the exterior derivative on $M$ and the co-boundary operator on $G$. To define the latter, assign to each edge $e \in E$ an arbitrary orientation, that is to say, think of one of the two vertices incident to $e$ as “initial” and of the other one as “final”. Then the $|E| \times |V|$ directed incidence matrix $M^G$ given by

$$M^G_{ve} = \begin{cases} -1 & \text{if } v = \text{init}(e) \\ 1 & \text{if } v = \text{fin}(e) \\ 0 & \text{otherwise} \end{cases} \quad (15)$$

acts on $\mathbb{R}^V$ as a coboundary operator $[S]$ on the space of 0-cochains:

$$M^G: \mathbb{R}^V \rightarrow \mathbb{R}^E \longleftrightarrow d_0: \Omega^0(M) \rightarrow \Omega^1(M) \quad (16)$$

The image of $M^G$ in $\mathbb{R}^E$ is called the bond space of $G$, denoted $\mathcal{B}(G)$. If we think of $G$ as specifying an electrical network, $\mathcal{B}(G)$ is the space of all possible potential differences. The orthogonal complement of $\mathcal{B}(G)$ in $\mathbb{R}^E$ is the cycle space $\mathcal{C}(G)$, the space of all circular flows in $G$. The analogy with an electrical network will be useful to us later.

If $G$ represents a triangulation, the correspondence (16) is canonically extended to $d_1$ as follows. Denote every edge $e \in E$ by $[ij]$, where $v_i = \text{init}(e)$ and $v_j = \text{fin}(e)$. In a similar way, denote every face $f \in F$ by $[ijk]$, where $v_i, v_j, v_k$ are incident to $f$ and their ordering agrees with the orientation of the underlying manifold $M$. Permutations act on $[ij], [ijk]$ like parity, e.g. $[ij] = -[ji]$ and $[ijk] = [jki] = -[jik]$. We define the linear operator $N^G: \mathbb{R}^E \rightarrow \mathbb{R}^F$ via:

$$(N^G \omega)([ijk]) = \omega([ij]) + \omega([jk]) + \omega([ki]) \quad \text{for } \omega \in \mathbb{R}^E \quad (17)$$

Then by construction $N^G M^G = 0$, which mimics $d_1 d_0 = 0$ under the correspondence:

$$N^G: \mathbb{R}^E \rightarrow \mathbb{R}^F \longleftrightarrow d_1: \Omega^1(M) \rightarrow \Omega^2(M). \quad (18)$$
It is easy to see that the image of $N^G$ spans all of $\mathbb{R}^F$ except the constant mode. This agrees with the Hodge decomposition theorem, which stipulates that

$$\dim \Omega^2(M) = \dim d_1 \Omega^1(M) + b^2.$$ \hfill (19)

The only remaining non-trivial consistency condition is the antisymmetry of the wedge product $\wedge^1$, which is made possible by the evenness of $|E|$. Consider any matrix of wedge products $\Lambda^1(G)$:

$$\Lambda^1(G)_{\omega_1 \omega_2} = (\omega_1, \omega_2)$$ \hfill (20)

Then the analogous $\Lambda^0(G)$ is determined by enforcing Stokes’ theorem:

$$\int_M d_0 \alpha^{(0)} \wedge^1 \gamma^{(1)} = - \int_M \alpha^{(0)} \wedge^0 d_1 \gamma^{(1)} \quad \Longleftrightarrow \quad (M^G)^T \Lambda^1(G) = -\Lambda^0(G) \, N^G.$$ \hfill (21)

The above definition is supplemented by requiring that the matrix element of $\Lambda^0(G)$ evaluated in the constant modes $\mathbb{R}^F \setminus \text{Im}(N^G)$ and $\ker(M^G)$ be non-zero, so as to ensure non-vanishing area of the manifold. In summary, the topological data of the manifold $M$ find well-defined analogues in the triangulation $G$.

However, the introduction of a metric interrupts this complacent state of affairs. The isomorphism requires $\Omega^r(M)$ and $\Omega^{2-r}(M)$ to be of the same dimension. Under the correspondence this translates into $|V| = |F|$ and the only triangulation which satisfies this is the tetrahedron triangulating a sphere. If we relax the condition of working with a triangulation, the map will be lost; meanwhile, $|V| = |F|$ remains a cumbersome and non-trivial condition. A more general possibility is to replace $G$ with the union of the graph and its dual, $G \cup \tilde{G}$, with the understanding that the resulting $p$-cochains should correspond to a reduplicated space $\Omega^p(M) \oplus \Omega^p(M)$ \footnote{This program was pursued in \cite{9}.}. In this paper I follow the latter strategy but abstain from the onerous detail.

Consider two copies of the scalar field theory defined in \cite{7}. The fundamental scalars $\Phi_1$ and $\Phi_2$ are naturally associated with the 0-cochains of $G \cup \tilde{G}$ whose space is isomorphic to $\mathbb{R}^{V \oplus F}$. The previous notion of the exterior derivative $d_0 \leftrightarrow M^G$ naturally extends to $\tilde{G}$. First one identifies the sets $E$ and $\tilde{E}$, that is, $\mathbb{R}^F \ni \tilde{e} = \hat{e} \in \mathbb{R}^E$ and $\mathbb{R}^E = \mathbb{R}^E$. Then one extends the orientation on $G$ to $\tilde{G}$ in the following fashion: for an edge $e \in G$ separating faces $f_i, f_j \in F$ we say that $\text{init}(\hat{e}) = \tilde{f}_i$ if in crossing from $f_i$ to $f_j$ in $G$ one passes $\text{fin}(e)$ on the right and $\text{init}(e)$ on the left; otherwise $\tilde{f}_i = \text{fin}(\hat{e})$. This prescription is well-defined by virtue of the orientability of the manifold $M$ in which $G$ is embedded. Then one forms an $|E| \times |\tilde{F}|$ (i.e. $|E| \times |F|$) directed incidence matrix $M^{\tilde{G}}$ as in \cite{15}:

$$M_{\hat{e}j}^{\tilde{G}} = \begin{cases} -1 & \text{if } \tilde{f} = \text{init}(\hat{e}) \\ 1 & \text{if } \tilde{f} = \text{fin}(\hat{e}) \\ 0 & \text{otherwise} \end{cases}$$ \hfill (22)

The matrices $M^G$ and $M^{\tilde{G}}$ can be concatenated to form the collective incidence matrix $M^{G \cup \tilde{G}}$, whose dimensions are $|E| \times (|V| + |F|)$. It is easy to see that $M^{\tilde{G}}$ sends $\mathbb{R}^\tilde{F}$ into $\mathcal{C}(G)$, that is, $\mathcal{B}(G) \perp \mathcal{B}(\tilde{G})$. In other words, a potential difference in the network $G$ is
necessarily a circular flow in the network $\tilde{G}$. However, the converse is not true for non-zero genera of $M$. There are $2g$ linearly independent flows which are circular in both $G$ and $\tilde{G}$, i.e. $\dim C(G) \cap C(\tilde{G}) = 2g$. This observation will be of essential importance later.

Because we are working in $G \cup \tilde{G}$, the triangular character of the underlying graph is in general lost and we do not have the luxury of extending \([17]\) to the present case. However, since the action for two scalar fields reads

$$S_{12} = \frac{1}{2} \int_M d\Phi_1 \wedge *d\Phi_1 + \frac{1}{2} \int_M d\Phi_2 \wedge *d\Phi_2 = \frac{1}{2} \langle d\Phi_1, d\Phi_1 \rangle + \frac{1}{2} \langle d\Phi_2, d\Phi_2 \rangle,$$

finding an explicit operator representing $d_1$ is not necessary. Similarly, an explicit representation of $\wedge^1$ is redundant as the only combination that shows up in the action is $\wedge^1 \wedge^1$.

Thus, in the following I shall not be concerned with searching for disc rete analogues of all the standard machinery of algebraic topology and geometry. Instead, I shall limit myself to making a simple ansatz for the discrete representation

$$\int_M \ldots \wedge^1 \ldots = \langle \ldots, \ldots \rangle(1).$$

Complemented with a notion of area and the correspondence $M^{G,e_G} \leftrightarrow d_0$, will be sufficient for determining the trace anomaly.

The introduction of a metric is in order. This is done by assigning to each edge $e \in E$ a length $g_e$. I then take the inner product on $R^E$ to be:

$$J_{ee'} = g_e \delta_{ee'} \quad (24)$$

To complete the mapping, a notion of area is necessary. This is defined as:

$$A = \langle 1, 1 \rangle(0) = \int_M \sqrt{g} \leftrightarrow \sum_{e \in E} g_e = \frac{1}{2} \sum_{v \in V} d_v = \frac{1}{2} \sum_{f \in F} d_f = \text{vol}(G), \quad (25)$$

where the degree of each vertex $d_v$ is the sum of the lengths of the edges incident to $v$. The correspondence \((25)\) has been considered in mathematical literature (viz. Sec. 3.4 in \([10]\)), although it is not unique. Because differences between the present approach and others (for example, \([6]\)) may ultimately be traced to the choice of metric, eqs. \((24-25)\) represent a key step in the present development. I shall return to this point in the discussion.

The correspondence \((24-25)\) is well-motivated. From an abstract point of view, no prior restrictions constrain the discrete analogue of the Hodge star so the initial choice of mapping $A \leftrightarrow \text{vol}(G)$ is unrestrained. After that, the inner product $\langle \ldots, \ldots \rangle(1)$ is expected to have the same scaling as $\langle \ldots, \ldots \rangle(0)$ and exhibit an appropriate metric-related notion of locality in $R^E$. This essentially determines $\langle \ldots, \ldots \rangle(1)$ in a unique way.

On a more intuitive level, the analogy with an electrical network is helpful. The zero modes in $R^V$ and $R^F$, the respective constants, can be thought of as specifying constant densities of excess $\Phi_1$ and $\Phi_2$ charge in the network. If we envision the charges as residing along the wires of the network, then it is a canonical choice to demand:

$$Q_{\text{total}} = \sigma_0 A \leftrightarrow \lambda_0 \sum_{e \in E} g_e = Q_{\text{total}}, \quad (26)$$

where $\sigma_0$ and $\lambda_0$ denote the respective constant charge densities. The vectors in $R^E$ specify flows in the network. Since $\langle 1, 1 \rangle(0) \propto Q_{\text{total}}$, it is natural to require that $\langle 1_e, 1_{e'} \rangle(1)$ be
We have:

The second determinant is not primed as it includes the regularized zero mode contributions.

bounded by:

relevant zero modes are flows which are circular both with respect to $\tau_i$. This motivates (21).

With the definitions of $M^{G,\tilde{G}} \leftrightarrow d_0$ and $\langle \ldots, \ldots \rangle^{(1)}$ in place, the action (28) reads:

$$S_{12} = \frac{1}{2} \Psi^T (M^{G,\tilde{G}})^T J M^{G,\tilde{G}} \Psi,$$

(27)

where $\Psi \in \mathbb{R}^{V \oplus \tilde{F}}$ represents both $\Phi_1$ and $\Phi_2$ and lives in the direct sum of their respective field spaces. The effective action is given by

$$W_{12}[g] = -\log \int [d\Psi] e^{-S_{12}}.$$  

(28)

There are two zero modes which need to be regulated, one for each of $\Phi_1, \Phi_2$. They are given by the constant modes on $V$ and $\tilde{F}$, respectively. Using the electrical networks analogy, it is natural to pick the following regularization scheme. Suppose the network is capable of sustaining static charge densities only up to a maximal value $\lambda_{i_{\text{max}}}$, $i = 0, 1$. In terms of the normalized zero mode $\psi_0^i = (\text{vol}(G))^{-1/2}$, the charge density is given by $0 \leq \lambda_0^i = c_0^i \psi_0^i = c_0^i (\text{vol}(G))^{-1/2} \leq \lambda_{i_{\text{max}}}^i$. Then the integral over each zero mode returns the value of $(c_0^i)_{\text{max}} = \lambda_{i_{\text{max}}}^i (\text{vol}(G))^{1/2}$. Thus, eq. (28) yields

$$W_{12}[g] = -\log \text{vol}(G) + \log \det' \left\{ (M^{G,\tilde{G}})^T J M^{G,\tilde{G}} \right\}$$

(29)

up to constant terms involving $\lambda_{i_{\text{max}}}$.

The second term in (29) cannot be split into logarithms of determinants of the factor matrices because the rank of $M^{G,\tilde{G}}$ is lower than that of $J$. The obstruction to doing so lies, therefore, in the zero modes of $(M^{G,\tilde{G}})^T: \mathbb{R}^E \to (\mathbb{R}^{V \oplus \tilde{F}})^{\vee}$. I proceed to regularize these modes.

The strategy will be analogous to the regularization of the zero modes of $M^{G,\tilde{G}}$. The relevant zero modes are flows which are circular both with respect to $G$ and $\tilde{G}$. There are $|E| - (|V| + |F| - 2) = 2 - \chi$ linearly independent such flows. Suppose that for a particular zero mode $\tau_i \in C(G) \cap C(\tilde{G})$ the underlying electrical network can only support charge fluxes not exceeding a maximal value $T_i^{\text{max}}$. This means that the flux density over an edge $e$ is bounded by:

$$0 \leq c_i \tau_i(e) \left( \sum_{e \in E} \tau_i(e)^2 g_e \right)^{-1/2} \leq T_i^{\text{max}} \tau_i(e)$$

(30)

and the volume of integration over $c_i$ is $(\sum_{e \in E} \tau_i(e)^2 g_e)^{1/2} T_i^{\text{max}}$. The regularization allows us to treat the determinant in (29) as if it were non-singular:

$$\det' \left\{ (M^{G,\tilde{G}})^T J M^{G,\tilde{G}} \right\} = \det J \det \left\{ M^{G,\tilde{G}} (M^{G,\tilde{G}})^T \right\}.$$  

(31)

The second determinant is not primed as it includes the regularized zero mode contributions. We have:

$$\det \left\{ M^{G,\tilde{G}} (M^{G,\tilde{G}})^T \right\} = \det' \left\{ M^{G,\tilde{G}} (M^{G,\tilde{G}})^T \right\} \prod_{i=1}^{2-\chi} T_i^{\text{max}} \left( \sum_{e \in E} \tau_i(e)^2 g_e \right)^{1/2}$$

(32)
The index $i$ runs over an orthonormal basis $\tau_i$ of $\mathcal{C}(G) \cap \mathcal{C}(\bar{G})$.

Although the first factor is independent of the metric and will merely contribute a constant term to $W[g]$, it is amusing to note that it is blessed with a beautiful combinatorial interpretation. If neither $G$ nor $\bar{G}$ contains loops, then the determinant factorizes over $G$, $\bar{G}$ and according to the matrix tree theorem [11]

$$\det \{ M^G (M^G)^T \} = |E| \# \{ \text{spanning trees of } G \}$$

with the analogous equality holding for $\bar{G}$. This result, first demonstrated by Kirchhoff in 1847, is useful in calculating total resistances in electric networks which are neither parallel nor in series.

Putting all the results together and dropping constant terms, the effective action takes the form

$$W[g] = -\frac{1}{2} \log \left( \sum_{e \in E} g_e \right) + \frac{1}{4} \sum_{i=1}^{2-\chi} \log \left( \sum_{e \in E} (\sum_{\tau_i(e)^2} g_e)^2 \right) + \frac{1}{2} \sum_{e \in E} \log g_e .$$

(34)

Here I have restored the factor of $1/2$ coming from the fact that $W_{12}$ contains contributions from two scalars $\Phi_1$, $\Phi_2$. The conformal variation of $W$ is given by

$$\frac{\delta W[g]}{\delta g_e} = -\frac{1}{2} \frac{g_e}{\sum_{e \in E} g_e} + \frac{1}{4} \sum_{i=1}^{2-\chi} \frac{\tau_i(e)^2 g_e}{\sum_{e \in E} (\tau_i(e)^2)^2} + \frac{1}{2} .$$

(35)

As in the continuous case, the last term may be cancelled by introducing a local $\Phi$-independent counterterm $S \rightarrow S - 1/2 \text{ Tr } \log J$. The first two terms, however, form a topological invariant:

$$\sum_{e \in E} g_e \frac{\delta W[g]}{\delta g_e} = -\frac{1}{4} \chi .$$

(36)

The correspondence with the discrete result (11, 12) is imperfect - the coefficients do not agree. Nevertheless, eq. (35) singles out a preferred discretized notion of curvature, distinct from that of [11]:

$$\sqrt{g} R(x) \leftrightarrow g_e \left( \frac{1}{2} \sum_{i=1}^{2-\chi} \frac{\tau_i(e)^2}{\sum_{e \in E} (\tau_i(e)^2)^2} - \frac{1}{\sum_{e \in E} g_e} \right) .$$

(37)

4 Discussion

This paper presents the shortest route to discussing the trace anomaly in discretized settings. As a starting point, the canonical correspondence between $p$-forms and $p$-cochains on a triangulation was adopted, along with the natural definition of the $0^\text{th}$ exterior derivative. After that, the metric was introduced via the Hodge star, leading to a doubling of the field space for consistency. Every construct introduced in this development is strictly necessary for a discussion of the trace anomaly, and conversely, the construction is free of redundancies.
It is in this sense that the presented theory is the simplest possible in the equivalence class of consistent discrete two-dimensional scalar field theories exhibiting trace anomalies.

The derivation involved one arbitrary choice - that of the discrete Hodge star $\ast_1$ (and therefore the metric) in eqs. (24-25). The correspondence adopted in this paper was motivated by an effort to treat the objects intrinsic to the triangulation - vertices, edges and edge lengths - as fundamental, and avoid making recourse to an explicit embedding. It should be borne in mind, however, that other equally motivated choices are possible. Of these, a notable one is that employed by Ko and Roček in a related paper [6], which follows the tenets of Regge calculus. In it, the discrete theory is implicitly assumed to live on a piecewise linear space consisting of a union of flat triangles, so that the area of the manifold does not conform with eq. (25). This leads to an effective action, whose conformal variation features the conical curvature and which differs from eq. (34). However, the general approach presented in this work is robust. With a different input metric to replace eqs. (24-25), it should in principle reproduce the result of [6]. The numeric disagreement between the discrete anomaly (36) and its continuous counterpart (12) is likewise easily fixed by a trivial redefinition of the metric $g_e \rightarrow g_e^{-3/2}$ in eq. (24). It would be interesting to understand the exponent $-3/2$ in detail, as it provides a quantitative bridge between two-dimensional scalar theory and electric-magnetic resistor networks.

The latter analogy is a nice bonus of the calculation presented in Sec. 3. We interpret the zero modes of $\mathbf{M}$, $\psi^i_0$ as charge densities and the zero modes of $\mathbf{M}^T$, $\tau^i$'s as circular flows in a network of resistors. The network is electro-magnetic as both electric and magnetic charges propagate in it. This is reflected in the fact that the field $\Psi$ has two independent zero modes, one living on $G$ and the other on $\tilde{G}$. Indeed, upon the exchange $G \leftrightarrow \tilde{G}$ we have:

\begin{align}
\mathcal{B}(G) &\leftrightarrow \mathcal{C}(	ilde{G}) \\
\mathcal{B}(\tilde{G}) &\leftrightarrow \mathcal{C}(G).
\end{align}

(38)

This operation is easily seen as an interchange of the two-dimensional divergence and curl, which effects an electric-magnetic duality. For $g > 0$ the network contains flows which are cyclic both from the electric and the magnetic points of view, given by $\mathcal{C}(\tilde{G}) \cap \mathcal{C}(G)$. Populating such a circular flow diminishes the ability of the network to respond to differences in potentials because it takes away from the pool of propagating charges. Hence the contribution of the modes $\tau_i \in \mathcal{C}(\tilde{G}) \cap \mathcal{C}(G)$ to the effective action is opposite from that of the zero modes of the fundamental field $\psi^i_0$, which correspond to free excess charges. As $\tau_i$, $\psi^i_0$ are zero modes, this effect is not countered by a corresponding rescaling of the Laplacian eigenvalues, as is the case for all other modes. As a result, $\tau_i$, $\psi^i_0$ entirely capture the effect of conformal transformations on the effective action. This provides a natural interpretation of the familiar result

$$
\int_M g_{\mu \nu} \frac{\delta W[g]}{\delta g_{\mu \nu}} \propto \text{ind} \, D \quad \text{with} \quad D = d_1 + \delta_1,
$$

(39)

where $D$ acts on the space of 1-forms $\Omega^1(M)$. In this language, the need for a doubling of the degrees of freedom, which is ubiquitous in geometric discretization [2], is understood as

\[3\]In particular, the theory presented herein is consistent even if the edge lengths defining the underlying discrete space do not satisfy the triangle inequality.

8
arising from the requirement that electric and magnetic degrees of freedom be treated in a
duality-symmetric way. Geometric discretization without doubling treats the two kinds of
degrees of freedom asymmetrically, viz. the paragraph below equation (21), which in itself is
not inconsistent. Invariance under the electric-magnetic duality becomes necessary, however,
after the introduction of the metric and the Hodge star. It is satisfying to have geometric
discretization provide an intuitive understanding of the trace anomaly in both physical and
algebro-geometric terms.

The route toward discretization followed in Sec. 3 may also provide attractive settings
for modelling emergent locality. The discrete counterpart of curvature proposed in (37) is
not local in that it depends on the detailed form of the zero modes, which is determined by
the full structure of the triangulation. This is again to be contrasted with the approach of
Ko and Roček [6], who exhibited the effective action fixed by demanding agreement with the
conical (and thereby local) notion of curvature. A study of the discrepancies between the
two results may reveal some lessons about the emergence of locality.

5 Acknowledgements

I am particularly grateful to Vijay Balasubramanian for encouragement and critical com-
ments on the manuscript and to Martin Roček for an illuminative correspondence. Dis-
cussions with and comments of Tamaz Brelidze, Peng Gao, Klaus Larjo, Tao Liu, Robert
Richter, Michael Schulz, and Joan Simón are kindly acknowledged. This work was supported
in part by DOE grant DE-FG02-95ER40893.

References

[1] T. Regge, “General Relativity Without Coordinates,” Nuovo Cim. 19, 558 (1961)

[2] D. H. Adams, “R-torsion and linking numbers from simplicial abelian gauge theories,”
[arXiv:hep-th/9612009].

D. H. Adams, “A doubled discretisation of abelian Chern-Simons theory,” Phys. Rev.
Lett. 78, 4155 (1997) [arXiv:hep-th/9704150].

[3] P. Mnëv, “Notes on simplicial BF theory,” [arXiv:hep-th/0610326].

[4] V. de Beauce and S. Sen, “Chiral Dirac fermions on the lattice using geometric discreti-
sation,” [arXiv:hep-th/0305125].

[5] M. Desbrun, A. Hirani, M. Leok, J. Marsden, “Discrete Exterior Calculus,”
[arXiv:math.DG/0508341].

[6] A. Ko and M. Roček, “A gravitational effective action on a finite triangulation,” JHEP
0603, 021 (2006) [arXiv:hep-th/0512293].

A. Ko and M. Roček, “A gravitational effective action on a finite triangulation as a
discrete model of continuous concepts,” [arXiv:hep-th/0605022].
[7] P. Di Francesco, P. Mathieu, D. Sénéchal, “Conformal Field Theory”, Springer-Verlag, New York, 1997.

M. Nakahara, “Geometry, Topology and Physics”, Institute of Physics Publishing, Bristol and Philadelphia, 1990.

[8] R. P. Stanley, “Topics in Algebraic Combinatorics,” Course notes for Mathematics 192 (Algebraic Combinatorics), Harvard University, Fall 2000.

[9] S. Sen, “A Cubic Whitney and Further Developments in Geometric Discretisation,” arXiv:hep-th/0307166.

[10] F. R. K. Chung, “Spectral Graph Theory,” American Mathematical Society, Providence, RI, 1997.

[11] G. Kirchhoff, “Über die Auflösung der Gleichungen, auf welche man bei der untersuchung der linearen verteilung galvanischer Ströme geführt wird,” Ann. Phys. Chem. 72, 497-508 (1847)