SPACETIME FOAM AND THE CASIMIR ENERGY

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Abstract

We conjecture that the neutral black hole pair production is related to the vacuum fluctuation of pure gravity via the Casimir-like energy. A generalization of this process to a multi-black hole pair is considered. Implications on the foam-like structure of spacetime and on the cosmological constant are discussed.

I. INTRODUCTION

It was J. A. Wheeler who first conjectured that spacetime could be subjected to topology fluctuation at the Planck scale \[1\]. This means that spacetime undergoes a deep and rapid transformation in its structure. This changing spacetime is best known as “\textit{spacetime foam}”, which can be taken as a model for the quantum gravitational vacuum. Some authors have investigated the effects of such a \textit{foamy} space on the cosmological constant, one example is the celebrated Coleman mechanism involving wormholes \[4\]. Nevertheless, how to realize such a foam-like space is still unknown as too is whether this represents the real quantum gravitational vacuum. For this purpose, we begin to consider the “simplest” quantum process that could approximate the foam structure in absence of matter fields, that is the black hole
pair creation. Different examples are known on this subject. The first example involves the study of black hole pair creation in a background magnetic field represented by the Ernst solution [3] which asymptotically approaches the Melvin metric [4]. Another example is the Schwarzschild-deSitter metric (SdS) which asymptotically approaches the deSitter metric. The extreme version is best known as the Nariai metric [5]. In this case the background is represented by the cosmological constant $\Lambda$ acting on the neutral black hole pair produced, accelerating the components away from each other. Finally another example is given by the Schwarzschild metric which asymptotically approaches the flat metric and depends only on the mass parameter $M$. Metrics of this type are termed asymptotically flat (A.F.). Another metric which has the property of being A.F. is the Reissner-Nordström metric, which depends on two parameters: the mass $M$ and the charge $Q$ of the electromagnetic field. Nevertheless, all the cases mentioned above introduce an external background field like the magnetic field or the cosmological constant to produce the pair and accelerate the components far away. In this letter, we would like to consider the same process without the contribution of external fields, except gravity itself and consider the possible implications on the foam-like structure of spacetime. This choice linked to the vacuum Einstein’s equations leads to the Schwarzschild and the flat metrics, where only the simplest case is considered, i.e. metrics which are spherically symmetric. Since the A.F. spacetimes are non-compact a subtraction scheme is needed to recover the correct equations under the constraint of fixed induced metrics on the boundary [6,7].

**II. QUASILOCAL ENERGY AND ENTROPY IN PRESENCE OF A BIFURCATION SURFACE**

Although it is not necessary for the forthcoming discussions, let us consider the maximal analytic extension of the Schwarzschild metric, i.e., the Kruskal manifold whose spatial slices $\Sigma$ represent Einstein-Rosen bridges with wormhole topology $S^2 \times R^1$. Following Ref. [8], the complete manifold $\mathcal{M}$ can be taken as a model for an eternal black hole composed of
two wedges $\mathcal{M}_+$ and $\mathcal{M}_-$ located in the right and left sectors of a Kruskal diagram. The hypersurface $\Sigma$ is divided in two parts $\Sigma_+$ and $\Sigma_-$ by a bifurcation two-surface $S_0$. On $\Sigma$ we can write the gravitational Hamiltonian

$$H_p = H - H_0 = \frac{1}{2\kappa} \int_{\Sigma} d^3 x (N \mathcal{H} + N^i \mathcal{H}_i)$$

$$+ \frac{1}{\kappa} \int_{S_+} d^2 x N \sqrt{\sigma} \left( k - k^0 \right) - \frac{1}{\kappa} \int_{S_-} d^2 x N \sqrt{\sigma} \left( k - k^0 \right),$$

(1)

where $\kappa = 8\pi G$. The Hamiltonian has both volume and boundary contributions. The volume part involves the Hamiltonian and momentum constraints

$$\mathcal{H} = (2\kappa) G_{ijkl} \pi^{ij} \pi^{kl} - \sqrt{3gR}/(2\kappa) = 0,$$

$$\mathcal{H}_i = -2\pi^j_{ij} = 0,$$

(2)

where $G_{ijkl} = (g_{ik}g_{jl} + g_{il}g_{jk} - g_{ij}g_{kl}) / (2\sqrt{g})$ and $R$ denotes the scalar curvature of the surface $\Sigma$. The volume part of the Hamiltonian (1) is zero when the Hamiltonian and momentum constraints are imposed. However, for the flat and the Schwarzschild space, constraints are immediately satisfied, then in this context the total Hamiltonian reduces to

$$H_p = \frac{1}{\kappa} \int_{S_+} d^2 x N \sqrt{\sigma} \left( k - k^0 \right) - \frac{1}{\kappa} \int_{S_-} d^2 x N \sqrt{\sigma} \left( k - k^0 \right).$$

(3)

Quasilocal energy is defined as the value of the Hamiltonian that generates unit time translations orthogonal to the two-dimensional boundaries, i.e.

$$E_{tot} = E_+ - E_-,$$

$$E_+ = \frac{1}{\kappa} \int_{S_+} d^2 x \sqrt{\sigma} \left( k - k^0 \right)$$

$$E_- = -\frac{1}{\kappa} \int_{S_-} d^2 x \sqrt{\sigma} \left( k - k^0 \right),$$

(4)

where $|N| = 1$ at both $S_+$ and $S_-$. $E_{tot}$ is the quasilocal energy of a spacelike hypersurface $\Sigma = \Sigma_+ \cup \Sigma_-$ bounded by two boundaries $^3S_+$ and $^3S_-$ located in the two disconnected
regions $M_+$ and $M_-$ respectively. We have included the subtraction terms $k^0$ for the energy. $k^0$ represents the trace of the extrinsic curvature corresponding to embedding in the two-dimensional boundaries $^2S_+$ and $^2S_-$ in three-dimensional Euclidean space. Let us consider the case of the static Einstein-Rosen bridge whose metric is defined as:

$$ds^2 = -N^2 dt^2 + g_{yy} dy^2 + r^2 (y) d\Omega^2,$$

(5)

where $N$, $g_{yy}$, and $r$ are functions of the radial coordinate $y$ continuously defined on $\mathcal{M}$, with $dy = dr/\sqrt{1 - 2m/r}$. If we make the identification $N^2 = 1 - 2m/r$, the line element (3) reduces to the S metric written in another form. The boundaries $^2S_+$ and $^2S_-$ are located at coordinate values $y = y_+$ and $y = y_-$ respectively. The normal to the boundaries is $n^\mu = (h^{yy})^{1/2} \delta^\mu_y$. Since this normal is defined continuously along $\Sigma$, the value of $k$ depends on the function $r,y$, which is positive for $^2B_+$ and negative for $^2B_-$. The application of the quasilocal energy definition gives

$$E = E_+ - E_-$$

$$= (r |_{r,y} \left[ 1 - (h^{yy})^{1/2} \right]_{y=y_+} - (r |_{r,y} \left[ 1 - (h^{yy})^{1/2} \right]_{y=y_-}).$$

(6)

It is easy to see that $E_+$ and $E_-$ tend individually to the ADM mass $M$ when the boundaries $^3B_+$ and $^3B_-$ tend respectively to right and left spatial infinity. It should be noted that the total energy is zero for boundary conditions symmetric with respect to the bifurcation surface, i.e.,

$$E = E_+ - E_- = M + (-M) = 0,$$

(7)

where the asymptotic contribution has been considered. The same behaviour appears in the entropy calculation for the physical system under examination. Indeed

$$S_{tot} = S_+ - S_- = \exp \left( \frac{A^+ - A^-}{4} \right) \approx \exp \left( \frac{A_H - A_H}{4} \right) = \exp (0),$$

(8)

where $A^+$ and $A^-$ have the same meaning as $E_+$ and $E_-$. Note that for both entropy and energy this result is obtained at zero loop. We can also see Eqs. (7) and (8) from a different
point of view. In fact these Eqs. say that flat space can be thought of as a composition of two pieces: the former, with positive energy, in the region $\Sigma^+$, and the latter, with negative energy, in the region $\Sigma^-$, where the positive and negative concern the bifurcation surface (hole) which is formed due to a topology change of the manifold. The most appropriate mechanism to explain this splitting seems to be a black hole pair creation.

III. BLACK HOLE PAIR CREATION

The formation of neutral black hole pairs with the two holes residing in the same universe is believed to be a highly suppressed process, at least for $\Lambda \gg 1$ in Planck’s units. The metric which describes such pair creation is the Nariai metric. When the cosmological constant is absent the SdS metric is reduced to the Schwarzschild metric which concerns a single black hole. However, one could regard each single Schwarzschild black hole in our universe as a mere part of a neutral pair, with the partner residing in the other universe. In this case the whole spacetime can be regarded as a black-hole pair formed up by a black hole with positive mass $M$ in the coordinate system of the observer and an anti black-hole with negative mass $-M$ in the system where the observer is not present. From the instantaneous point of view, one can represent neutral black hole pairs as instantons with zero total energy. An asymptotic observer in one universe would interpret each such pair as being formed by one black hole with positive mass $M$. What such an observer would actually observe from the pair is only either a black hole with positive energy or a wormhole mouth opening to the observer’s universe, interpreting that the black hole in the “other universe” has negative mass without violating the positive-energy theorems. This scenario gives spacetime a different structure. Indeed it is well known that flat spacetime cannot spontaneously generate a black hole, otherwise energy conservation would be violated. In other terms we cannot compare spacetimes with different asymptotic behaviour. The different boundary conditions reflect on the fact that flat space is not periodic in euclidean time which means that the temperature is zero. On the other hand a black hole with an imaginary time
necessitates periodicity, but this implies a temperature different from zero. Then, unless flat spacetime has a temperature $T$ equal to the black hole temperature, there is no chance for a transition from flat to curved spacetime. This transition is a decay from the false vacuum to the true one \cite{10, 13}. However, taking account a pair of neutral black holes living in different universes, there is no decay and more important no temperature is necessary to change from flat to curved space. This could be related with a vacuum fluctuation of the metric which can be measured by the Casimir energy.

IV. CASIMIR ENERGY

One can in general formally define the Casimir energy as follows

$$E_{\text{Casimir}} [\partial M] = E_0 [\partial M] - E_0 [0], \quad (9)$$

where $E_0$ is the zero-point energy and $\partial M$ is a boundary. For zero temperature, the idea underlying the Casimir effect is to compare vacuum energies in two physical distinct configurations. We can recognize that the expression which defines quasilocal energy is formally of the Casimir type. Indeed, the subtraction procedure present in Eq.\,(3) describes an energy difference between two distinct situations with the same boundary conditions. However, while the expression contained in Eq.\,(3) is only classical, the Casimir energy term has a quantum nature. One way to escape from this disagreeable situation is the induced gravity point of view discussed in Ref.\,\cite{17} and Refs. therein. However, in those papers the subtraction procedure in the energy term is generated by the zero point quantum fluctuations of matter fields. Nevertheless, we are working in the context of pure gravity, therefore quasilocal energy has to be interpreted as the zero loop or tree level approximation to the true Casimir energy. To this end it is useful to consider a generalized subtraction procedure extended to the volume term up to the quadratic order. This corresponds to the semiclassical approximation of quasilocal energy. What are the possible effects on the foam-like scenario? Suppose we enlarge this process from one pair to a large but fixed number of
such pairs, say $N$. What we obtain is a multiply connected spacetime with $N$ holes inside the manifold, each of them acting as a single bifurcation surface with the sole condition of having symmetry with respect to the bifurcation surface even at finite distance. Let us see in which way such a spacetime can be modeled.

V. SPACETIME FOAM: THE MODEL

In the one-wormhole approximation we have used an eternal black hole, to describe a complete manifold $\mathcal{M}$, composed of two wedges $\mathcal{M}_+$ and $\mathcal{M}_-$ located in the right and left sectors of a Kruskal diagram. The spatial slices $\Sigma$ represent Einstein-Rosen bridges with wormhole topology $S^2 \times R^1$. Also the hypersurface $\Sigma$ is divided in two parts $\Sigma_+$ and $\Sigma_-$ by a bifurcation two-surface $S_0$. We begin with the line element

$$ds^2 = -N^2(r)\,dt^2 + \frac{dr^2}{1 - \frac{2m}{r}} + r^2 \left(d\theta^2 + \sin^2 \theta d\phi^2\right)$$

and we consider the physical Hamiltonian defined on $\Sigma$

$$H_p = H - H_0 = \frac{1}{16\pi l_p^2} \int_{\Sigma} d^3x \left(N \mathcal{H} + N_i \mathcal{H}^i\right) + \frac{2}{l_p^2} \int_{S_+} d^2x \sqrt{\sigma} (k - k^0) - \frac{2}{l_p^2} \int_{S_-} d^2x \sqrt{\sigma} (k - k^0),$$

where $l_p^2 = G$. The volume term contains two constraints

$$\begin{cases} 
\mathcal{H} = G_{ijkl} \pi^i \pi^j \pi^k \pi^l \left(\frac{r^2}{\sqrt{g}}\right) - \left(\frac{\sqrt{\sigma}}{l_p^2}\right) R^{(3)} = 0, \\
\mathcal{H}^i = -2\pi^i_j = 0
\end{cases}$$

both satisfied by the Schwarzschild and Flat metric respectively. The supermetric is $G_{ijkl} = \frac{1}{2} (g_ikg_{jl} + g_ilg_{jk} - g_{ij}g_{kl})$ and $R^{(3)}$ denotes the scalar curvature of the surface $\Sigma$. By using the expression of the trace

$$k = -\frac{1}{\sqrt{h}} \left(\sqrt{h}n^\mu\right)_\mu,$$

7
with the normal to the boundaries defined continuously along $\Sigma$ as $n^\mu = (h^{yy})^{\frac{1}{2}} \delta^\mu_y$. The value of $k$ depends on the function $r_y$, where we have assumed that the function $r_y$ is positive for $S_+$ and negative for $S_-$. We obtain at either boundary that

$$k = \frac{-2r_y}{r}.$$  \hspace{2cm} (14)

The trace associated with the subtraction term is taken to be $k^0 = -2/r$ for $B_+$ and $k^0 = 2/r$ for $B_-$. Then the quasilocal energy with subtraction terms included is

$$E_{\text{quasilocal}} = l_p^2 (E_+ - E_-) = l_p^2 \left[ (r [1 - |r_y|]_{y=y_+}) - (r [1 - |r_y|])_{y=y_-} \right].$$  \hspace{2cm} (15)

Note that the total quasilocal energy is zero for boundary conditions symmetric with respect to the bifurcation surface $S_0$ and this is the necessary condition to obtain instability with respect to the flat space. In this sector satisfy the constraint equations (3). Here we consider perturbations at $\Sigma$ of the type

$$g_{ij} = \bar{g}_{ij} + h_{ij},$$  \hspace{2cm} (16)

where $\bar{g}_{ij}$ is the spatial part of the Schwarzschild and Flat background in a WKB approximation. In this framework we have computed the quantity

$$\Delta E (M) = \frac{\langle \Psi | H^{\text{Schw.}} - H^{\text{Flat}} | \Psi \rangle}{\langle \Psi | \Psi \rangle} + \frac{\langle \Psi | H_{\text{quasilocal}} | \Psi \rangle}{\langle \Psi | \Psi \rangle},$$  \hspace{2cm} (17)

by means of a variational approach, where the WKB functionals are substituted with trial wave functionals. This quantity is the natural extension to the volume term of the subtraction procedure for boundary terms and it is interpreted as the Casimir energy related to vacuum fluctuations. By restricting our attention to the graviton sector of the Hamiltonian approximated to second order, hereafter referred as $H_{(2)}$, we define

$$E_{(2)} = \frac{\langle \Psi^\perp | H_{(2)} | \Psi^\perp \rangle}{\langle \Psi^\perp | \Psi^\perp \rangle},$$

where

$$\Psi^\perp = \Psi \left[ h_{ij} \right] = \mathcal{N} \exp \left\{ -\frac{1}{4l_p^2} \left[ \langle (g - \bar{g}) K^{-1} (g - \bar{g}) \rangle_x^\perp \right] \right\}.$$
After having functionally integrated $H_{|2}$, we get

$$H_{|2} = \frac{1}{4L_p^4} \int_\Sigma d^3x \sqrt{g} G^{ijkl} \left[ K^{-1\perp} (x, x)_{ijkl} + (\Delta_2)^a_j K^\perp (x, x)_{ialk} \right]$$

(18)

The propagator $K^\perp (x, x)_{ialk}$ comes from a functional integration and it can be represented as

$$K^\perp (\vec{x}', \vec{y})_{ialk} := \sum_N \frac{h^\perp_{ia} (\vec{x}') h^\perp_{kl} (\vec{y})}{2\lambda_N (p)},$$

(19)

where $h^\perp_{ia} (\vec{x}')$ are the eigenfunctions of

$$(\Delta_2)^a_j := -\Delta \delta^a_j + 2R^a_j.$$

(20)

This is the Lichnerowicz operator projected on $\Sigma$ acting on traceless transverse quantum fluctuations and $\lambda_N (p)$ are infinite variational parameters. $\Delta$ is the curved Laplacian (Laplace-Beltrami operator) on a Schwarzschild background and $R^a_j$ is the mixed Ricci tensor whose components are:

$$R^a_j = diag \left\{ \frac{-2MG}{r^3}, \frac{MG}{r^3}, \frac{MG}{r^3} \right\}. $$

(21)

The minimization with respect to $\lambda$ and the introduction of a high energy cutoff $\Lambda$ give to the Eq. (7) the following form

$$\Delta E (M) \sim -\frac{V}{32\pi^2} \left( \frac{3MG}{r^3_0} \right)^2 \ln \left( \frac{\Lambda^2}{3MG} \right),$$

(22)

where $V$ is the volume of the system and $r_0$ is related to the minimum radius compatible with the wormhole throat. We know that the classical minimum is achieved when $r_0 = 2MG$. However, it is likely that quantum processes come into play at short distances, where the wormhole throat is defined, introducing a quantum radius $r_0 > 2MG$. We now compute the minimum of $\Delta E (M)$, after having rescaled the variable $M$ to a scale variable $x = 3MG/(r^3_0 \Lambda^2)$. Thus

$$\Delta E (M) \rightarrow \Delta E (x, \Lambda) = -\frac{V}{32\pi^2} \Lambda^4 x^2 \ln x.$$
We obtain two values for \( x \): \( x_1 = 0 \), i.e. flat space and \( x_2 = e^{-\frac{1}{2}} \). At the minimum we obtain

\[
\Delta E (x_2) = -\frac{V}{64\pi^2} \frac{\Lambda^4}{e}.
\]  

(23)

Nevertheless, there exists another part of the spectrum which has to be considered: the discrete spectrum containing one mode. This gives the energy an imaginary contribution, namely we have discovered an unstable mode \([18,19]\). Let us briefly recall, how this appears.

The eigenvalue equation

\[
(\Delta_2)^a h_{aj} = \alpha h_{ij}
\]

(24)

can be studied with the Regge-Wheeler method. The perturbations can be divided in odd and even components. The appearance of the unstable mode is governed by the gravitational field component \( h_{11}^{\text{even}} \). Explicitly

\[
-E^2 H (r)
\]

\[
= - \left( 1 - \frac{2MG}{r} \right) \frac{d^2 H (r)}{dr^2} + \left( \frac{2r - 3MG}{r^2} \right) \frac{dH (r)}{dr} - \frac{4MG}{r^3} H (r),
\]

(25)

where

\[
h_{11}^{\text{even}} (r, \vartheta, \phi) = \left[ H (r) \left( 1 - \frac{2m}{r} \right)^{-1} \right] Y_{00} (\vartheta, \phi)
\]

(26)

and \( E^2 > 0 \). Eq. (25) can be transformed into

\[
\mu = \frac{\int_0^y dy \left[ \left( \frac{dh(y)}{dy} \right)^2 - \frac{3}{2\rho(y)} h(y) \right]}{\int_0^y dy h^2 (y)},
\]

(27)

where \( \mu \) is the eigenvalue, \( y \) is the proper distance from the throat in dimensionless form. If we choose \( h (\lambda, y) = \exp (-\lambda y) \) as a trial function we numerically obtain \( \mu = -0.701626 \). In terms of the energy square we have

\[
E^2 = -0.17541 / (MG)^2
\]

(28)
to be compared with the value $E^2 = -.19/ (MG)^2$ of Ref. [18]. Nevertheless, when we compute the eigenvalue as a function of the distance $y$, we discover that in the limit $\bar{y} \to 0$,

$$
\mu \equiv \mu (\lambda) = \lambda^2 - \frac{3}{2} + \frac{9}{8} \left[ \bar{y}^2 + \frac{\bar{y}}{2\lambda} \right].
$$

(29)

Its minimum is at \(\tilde{\lambda} = \left( \frac{9}{32}\bar{y} \right)^{\frac{1}{4}}\) and

$$
\mu (\tilde{\lambda}) = 1.287 \: 8\bar{y}^{\frac{9}{2}} + \frac{9}{8} \bar{y}^2 - \frac{3}{2}.
$$

(30)

It is evident that there exists a critical radius where $\mu$ turns from negative to positive. This critical value is located at $\rho_c = 1.113 \: 4$ to be compared with the value $\rho_c = 1.445$ obtained by B. Allen in [20]. What is the relation with the large number of wormholes? As mentioned in I, when the number of wormholes grows, to keep the coherency assumption valid, the space available for every single wormhole has to be reduced to avoid overlapping of the wave functions. If we fix the initial boundary at $R_{\pm}$, then in presence of $N_w$ wormholes, it will be reduced to $R_{\pm}/N_w$. This means that boundary conditions are not fixed at infinity, but at a certain finite radius and the ADM mass term is substituted by the quasilocal energy expression under the condition of having symmetry with respect to each bifurcation surface.

The effect on the unstable mode is clear: as $N_w$ grows, the boundary radius reduces more and more until it will reach the critical value $\rho_c$ below which no negative mode will appear corresponding to a critical wormholes number $N_{wc}$. To this purpose, suppose to consider $N_w$ wormholes and assume that there exists a covering of $\Sigma$ such that $\Sigma = \bigcup_{i=1}^{N_w} \Sigma_i$, with $\Sigma_i \cap \Sigma_j = \emptyset$ when $i \neq j$. Each $\Sigma_i$ has the topology $S^2 \times R^1$ with boundaries $\partial \Sigma_i^\pm$ with respect to each bifurcation surface. On each surface $\Sigma_i$, quasilocal energy gives

$$
E_{i \text{ quasilocal}} = \frac{2}{l_p^2} \int_{S_i^+} d^2 x \sqrt{\sigma} \left( k - k^0 \right) - \frac{2}{l_p^2} \int_{S_i^-} d^2 x \sqrt{\sigma} \left( k - k^0 \right).
$$

(31)

Thus if we apply the same procedure of the single case on each wormhole, we obtain

$$
E_{i \text{ quasilocal}} = l_p^2 (E_{i+} - E_{i-}) = l_p^2 (r \left[ 1 - |r_{i\gamma} | \right])_{y=y_i+} - l_p^2 (r \left[ 1 - |r_{i\gamma} | \right])_{y=y_i-}.
$$

(32)

Note that the total quasilocal energy is zero for boundary conditions symmetric with respect to each bifurcation surface $S_{0,i}$. We are interested in a large number of wormholes, each of
them contributing with a term of the type \((\ref{7})\). If the wormholes number is \(N_w\), we obtain (semiclassically, i.e., without self-interactions)\(^\dagger\)

\[
H_{tot}^{N_w} = H^1 + H^2 + \ldots + H^{N_w}.
\]  

(33)

Thus the total energy for the collection is

\[
E_{tot}^{1/2} = N_w H_{1/2}.
\]

The same happens for the trial wave functional which is the product of \(N_w\) t.w.f. Thus

\[
\Psi_{tot}^{\perp} = \Psi_{1}^{\perp} \otimes \Psi_{2}^{\perp} \otimes \ldots \otimes \Psi_{N_w}^{\perp} = \mathcal{N} \exp N_w \left\{ -\frac{1}{4 l_p^2} \left\langle \left( g - \bar{g} \right) K^{-1} (g - \bar{g}) \right\rangle_{x,y}^{\perp} \right\} 
\]

\[
= \mathcal{N} \exp \left\{ -\frac{1}{4} \left\langle \left( g - \bar{g} \right) K^{-1} (g - \bar{g}) \right\rangle_{x,y}^{\perp} \right\},
\]

(34)

where we have rescaled the fluctuations \(h = g - \bar{g}\) in such a way to absorb \(N_w / l_p^2\). The propagator \(K^{\perp}(x,x)_{takt}\) is the same one for the one wormhole case. Thus, repeating the same steps of the single wormhole, but in the case of \(N_w\) wormholes, one gets

\[
\Delta E_{N_w}(x, \Lambda) \sim N_w^2 \frac{V}{32 \pi^2} \Lambda^4 x^2 \ln x,
\]

(35)

where we have defined the usual scale variable \(x = 3MG / (r_0^2 \Lambda^2)\). Then at one loop the cooperative effects of wormholes behave as one macroscopic single field multiplied by \(N_w^2\), but without the unstable mode. At the minimum, \(\bar{x} = e^{-\frac{1}{2}}\)

\[
\Delta E(\bar{x}) = -N_w^2 \frac{V}{64 \pi^2} \frac{\Lambda^4}{e}.
\]

(36)

\(^\dagger\)Note that at this approximation level, we are in the same situation of a large collection of \(N\) harmonic oscillators whose Hamiltonian is

\[
H = \frac{1}{2} \sum_{n \neq 0}^{\infty} \left[ \pi_n^2 + n^2 \omega^2 \phi_n^2 \right].
\]
VI. THE COSMOLOGICAL CONSTANT

Einstein introduced his cosmological constant $\Lambda_c$ in an attempt to generalize his original field equations. The modified field equations are

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + \Lambda_c g_{\mu\nu} = 8\pi G T_{\mu\nu}. \tag{37}$$

By redefining

$$T_{\mu\nu}^{\text{tot}} \equiv T_{\mu\nu} - \frac{\Lambda_c}{8\pi G} g_{\mu\nu}, \tag{38}$$

one can regain the original form of the field equations

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 8\pi G T_{\mu\nu}, \tag{39}$$

at the prize of introducing a vacuum energy density and vacuum stress-energy tensor

$$\rho_\Lambda = \frac{\Lambda_c}{8\pi G}, \quad T_{\mu\nu}^\Lambda = -\rho_\Lambda g^{\mu\nu}. \tag{40}$$

If we look at the Hamiltonian in presence of a cosmological term, we have the expression

$$H = \int_{\Sigma} d^3 x (N (H + \rho_\Lambda \sqrt{g}) + N^i \mathcal{H}_i) + b.t., \tag{41}$$

where $\mathcal{H}$ is the usual Hamiltonian density defined without a cosmological term. We know that the effect of vacuum fluctuation is to inducing a cosmological term. Indeed by looking at Eq. (36), we have that

$$\langle \Delta H \rangle_V = -N^2 \frac{\Lambda^4}{64\pi^2}. \tag{42}$$

The WDW equation in presence of a cosmological constant is

$$\left[ G_{ijkl} \pi^{ij} \pi^{kl} - \frac{\sqrt{g}}{2\kappa} (R - 2\Lambda_c) \right] \Psi [g_{ij}] = 0. \tag{43}$$

By integrating over the hypersurface $\Sigma$ and looking at the expectation values, we can write

$$\int_{\Sigma} d^3 x \left[ \frac{1}{\sqrt{g}} G_{ijkl} \pi^{ij} \pi^{kl} - \frac{\sqrt{g}}{2\kappa} R \right] = -\frac{\Lambda_c}{\kappa} \int_{\Sigma} d^3 x \sqrt{g} = -\frac{\Lambda_c}{\kappa} V_c. \tag{44}$$
$V_c$ is the cosmological volume. The first term of Eq. (44) is the same that generates the vacuum fluctuation (42). Thus, by comparing the second term of Eq. (44) with Eq. (42), we have

$$- \frac{\Lambda_c}{\kappa} V_c = -N_w^2 \frac{\Lambda^4}{64\pi^2} V_w. \quad (45)$$

Therefore

$$\Lambda_c = N_w^2 \frac{\Lambda^4 \kappa}{V_c 64\pi^2} V_w. \quad (46)$$

Since $\kappa = 8\pi G = 8\pi l_p^2$ and $\Lambda \to 1/l_p$, we obtain

$$\Lambda_c = N_w^2 \frac{l_p^{-2}}{V_c 8\pi^2} V_w. \quad (47)$$

The cosmological volume has to be rescaled in terms of the wormhole radius, in such a way to obtain that $V_c \to N_w^3 V_w$. This is the direct consequence of the boundary rescaling, namely $R_\pm \to R_+/N_w$. Thus

$$\Lambda_c = \frac{l_p^{-2}}{N_w 8\pi} \Lambda_c = \tilde{c} \frac{1}{N_w l_p^2}. \quad (48)$$

Due to the uncertainty relation

$$\Delta E \propto \frac{A}{L^4} \propto -N_w^2 \frac{V}{64\pi^2} \frac{\Lambda^4}{e} \propto A \Lambda^4. \quad (49)$$

The fourth power of the cutoff (or the inverse of the fourth power of the region of dimension L) is a clear signal of a Casimir-like energy generated by vacuum fluctuations. As a consequence a positive cosmological constant is induced by such fluctuations. The probability of this process in a Euclidean time (not periodically identified) is

$$P \sim e^{-(\Delta E)(\Delta t)} \sim \exp \left( N_w^2 \frac{\Lambda^4}{64\pi^2} (V \Delta t) \right)^2. \quad (50)$$

From Eq. (45), we obtain

$$P \sim \exp \left( \frac{\Lambda_c}{\kappa} V_c \right) (\Delta t)^2. \quad (51)$$
As an application we consider a periodically identified Euclidean time

\[ \Delta t = 2\pi \sqrt{\frac{3}{\Lambda}} \quad (52) \]

and we admit that a cosmological volume is given by

\[ V_c = \frac{4\pi}{3} \left( \sqrt{\frac{3}{\Lambda}} \right)^3, \quad (53) \]

namely

\[ \exp \left( \frac{3\pi}{l_p^2 \Lambda_c} \right). \quad (54) \]

Thus we recover the Hawking result about the cosmological constant approaching zero. Note that the vanishing of \( \Lambda_c \) is related to the growing of the holes. Even if this assemblage of coherent wormholes seems to have the right trend to describe both a spacetime foam and a quantum gravitational vacuum, we have a lot of problems to solve and other corrections to include:

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REFERENCES

[1] J.A. Wheeler, Ann. Phys. 2 (1957) 604.

[2] S. Coleman, Nucl. Phys. B 310 (1988) 643.

[3] F.J. Ernst, J. Math. Phys., 17 (1976) 515.

[4] M.A. Melvin, Phys. Lett. 8 (1964) 65.

[5] S. Nariai, *On some static solutions to Einstein’s gravitational field equations in a spherically symmetric case*. Science Reports of the Tohoku University, 34 (1950) 160; S. Nariai, *On a new cosmological solution of Einstein’s field equations of gravitation*. Science Reports of the Tohoku University, 35 (1951) 62.

[6] V.P. Frolov and E.A. Martinez, *Action and Hamiltonian for Eternal Black Holes*, Class.Quant.Grav.13 :481-496,1996, gr-qc/9411001.

[7] S. W. Hawking and G. T. Horowitz, *The Gravitational Hamiltonian, Action, Entropy and Surface Terms*, Class. Quant. Grav. 13 1487, (1996), gr-qc/9501014

[8] R. Bousso and S.W. Hawking, Phys. Rev. D 52, 5659 (1995), gr-qc/9506047. R. Bousso and S.W. Hawking, Phys.Rev. D 54 6312 (1996), gr-qc/9606052

[9] E.A. Martinez, *Entropy of eternal black holes*. To appear in the proceedings of the Sixth Canadian Conference on General Relativity and Relativistic Astrophysics, gr-qc/9508057.

[10] S. Coleman, Nucl. Phys. B 298 (1988), 178.

[11] P. Ginsparg and M.J. Perry, Nucl. Phys. B 222 (1983) 245.

[12] D.J. Gross, M.J. Perry and L.G. Yaffe, Phys. Rev. D 25, (1982) 330.

[13] P.O. Mazur Mod. Phys. Lett. A 4, (1989) 1497.

[14] R. Garattini, Nucl. Phys. B (Proc. Suppl.) 57 (1997) 316-319, gr-qc/9701060; R. Garat-
[15] P.F. Gonzalez-Diaz, *The Schwarzschild Black Hole Pair*. In *Topics in Quantum Field Theory. Modern Methods in Fundamental Physics*, ed. D.H. Tchrakian (World Scientific, Singapore, 1995) p. 142, gr-qc/9503011; P.F. Gonzalez-Diaz, *Quantum black-hole kinks*, Grav. Cosmol. 2 (1996) 122-142, gr-qc/9606036.

[16] E. Witten, Nucl. Phys. B 195 (1982) 481.

[17] F. Belgiorno and S. Liberati, *Black Hole Thermodynamics, Casimir Effect and Induced Gravity*, Gen. Rel.Grav (29) 1997 1181-1194, gr-qc/9612024.

[18] D.J. Gross, M.J. Perry and L.G. Yaffe, Phys. Rev. D 25, (1982) 330.

[19] R. Garattini, *Probing foamy spacetime with variational methods*, to appear in Int. J. Mod. Phys. A, gr-qc/9801043.

[20] B. Allen, Phys. Rev. D 30 (1984) 1153.