Lie–Poisson pencils related to semisimple Lie algebras: towards classification

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1 Introduction

Since its invention by Magri [Mag78] the bihamiltonian property proved to be very useful in the study of integrable systems. The second hamiltonian structure was discovered for majority of important examples. In most of the cases it has been first found “by hand” but afterwards it was seen as a subcase of some general algebraic scheme. For instance, the bihamiltonian structure of the KdV [Mag78] later was understood as the so-called argument shift method [AK98, Ch. VI], [Ke03] on the Virasoro algebra.

Recall that a bihamiltonian structure is a pair of Poisson structures on a manifold which are compatible, i.e. such that their sum is again a Poisson structure. There is a lot of the above mentioned algebraic schemes of appearing bihamiltonian structures. However, using a simple characteristic, the order of the coefficients of the corresponding Poisson bivectors, one can organize these schemes in the following hierarchy:

- constant+constant;
- linear+constant;
- linear+linear;
- quadratic+linear;
- etc.

The first case (when the coefficient of both the brackets are constant) is rather not interesting. The second one (here one of the bivectors is the Lie–Poisson structure on a Lie algebra and another one is related to a cocycle on this Lie algebra) is already very important, for instance, due to the argument shift method mentioned, see also [Boi92]. The third case is the subject of this paper and we will comment it below. The higher steps in the hierarchy are usually related to different forms of the classical Yang–Baxter equations [KSM89], [KM93 §1–2], [DG93], [GP94], [GR95] (however it is
worth mentioning that the argument shift of arbitrary quadratic Poisson structure in the vanishing direction always gives a ”quadratic+linear” pair).

The ”linear+linear” case is represented by pairs of compatible Lie–Poisson structures, or, equivalently, pairs of compatible structures of Lie algebras on a vector space (cf. Definition 2.1). The situation here is essentially more complicated than in the ”linear+constant” case (note that it is easy to construct examples of pairs related to the last case on any Lie algebra considering a trivial cocycle). There are no ”automatic” ”linear+linear” examples related to arbitrary Lie algebra (with the exception of the Lie bracket being the antisymmetrization of an associative multiplication \( x \cdot y \); for any fixed element \( a \) the operation \( x \cdot_a y := x \cdot a \cdot y \) is a new associative product and the corresponding commutator is compatible with the initial one).

One has to distinguish two main fields on which ”linear+linear” pairs appeared. In finite-dimensional context and within the ”purely” bihamiltonian theory the pair \( [\cdot,\cdot], [\cdot, A] \) of compatible Lie brackets on \( \mathfrak{so}(n) \) (here \( [\cdot,\cdot] \) is the standard commutator and \( [\cdot, A] \) is the modified one, \( xAy - yAx \), with a diagonal matrix \( A \)) was applied by Bolsinov [Bol92, Bol98] to prove the integrability of the Euler–Manakov top and to show its relation to the so-called Clebsch–Perolomov case (see also [MP96], [Yan00]). Another Lie–Poisson pencil on \( \mathfrak{so}(n) \times \mathfrak{so}(n) \) was used for study of generalized Steklov–Lyapunov systems [BF92]. A Poisson pencil which incorporate, on one hand, a Lie–Poisson pair (either the \( \mathfrak{so}(n) \) pair mentioned, or another one related to a \( \mathbb{Z}_2 \)-grading on a Lie algebra), and, on the other hand, the argument shift method, was used in the theory of a series of systems (in particular for building Lax representations) such as Brun–Bogoyavlenskii and Zhukovsky–Volterra systems, the Kovalevski top, etc. [BB02, BM03 Ch. 2].

Another area where pairs of compatible Lie brackets play important role is the classical \( r \)-matrix formalism. The main idea (which in implicit form is contained already in the paper of Holod [Hol87]) that explains this role is as follows. We say that a Lie algebra \( (\mathfrak{g}, [\cdot,\cdot]) \) with a decomposition \( \mathfrak{g} = \oplus_{n \in \mathbb{Z}} \mathfrak{g}_n \) is quasigraded of degree 1 if \( [\mathfrak{g}_i, \mathfrak{g}_j] \subset \mathfrak{g}_{i+j} \oplus \mathfrak{g}_{i+j+1} \) for any \( i, j \in \mathbb{Z} \). Given such a Lie algebra, one observes that \( \mathfrak{g}_+ := \oplus_{n \geq 0} \mathfrak{g}_n \) and \( \mathfrak{g}_- := \oplus_{n < 0} \mathfrak{g}_n \) are subalgebras, so the kernel of the difference of the corresponding projectors \( P_- - P_+ \) is the classical \( r \)-matrix and one can apply the standard Adler–Costant scheme [RSTS94]. On the other hand, if \( \mathfrak{g} \) is a vector space with two compatible Lie brackets \( [\cdot,\cdot]_0, [\cdot,\cdot]_1 \), one obtains a quasigraded Lie algebra of degree 1 by putting \( \mathfrak{g} := \mathfrak{g}[\lambda, 1/\lambda], [\cdot,\cdot] := [\cdot,\cdot]_0 + \lambda [\cdot,\cdot]_1 \) and extending this bracket by bilinearity to \( \mathfrak{g} \) (the grading is usual, by the degree of a Laurent polynomial). This idea was consequently developed by Skrypnyk [Skr02, Skr04, Skr06], and, in a slightly different form, by Golubchik and Sokolov [GS02, GS03] generating many new results on finite- and infinite-dimensional integrable systems (see also [LM05], [GVY08 Ch. 16]). Note that the correspondence \{compatible Lie brackets on \( \mathfrak{g} \)\} \( \rightarrow \) \{Lie algebra structures \( (\mathfrak{g}[\lambda, 1/\lambda], [\cdot,\cdot]) \) such that \( \mathfrak{g}_+, \mathfrak{g}_- \) are subalgebras\} is one-to-one under some reasonable restrictions on the bracket \( [\cdot,\cdot] \).

The above mentioned results are enough motivating for considering the classification matter of pairs of compatible Lie brackets on finite dimensional spaces. However, this matter (like many questions of Lie theory) is itself a beautiful mathematical problem which probably inspired Kantor and Persits in their pioneering work in this direction. In [KP88] they announced (unfortunately, the proof never appeared) the following result. Let \( \mathfrak{g}, V \) be finite-dimensional vector spaces and let \( \{[\cdot,\cdot]^v\}_{v \in V} \) be a family of Lie brackets on \( \mathfrak{g} \) parametrized by \( V \) (note that any pair of compatible Lie brackets generates such a family with \( V \) 1- or 2-dimensional). We say that such a family is irreducible if the Lie algebras \( (\mathfrak{g}, [\cdot,\cdot]^v) \) do not have common ideals and is closed if for any \( a \in \mathfrak{g}, v, w \in V \) there exists \( z \in V \) such that \( [\cdot,\cdot]^v_{ad^wa} = [\cdot,\cdot]^z \), where \( ad^wa(b) := [a,b]^w \) and we used notation \((3.1.1)\).
result of Kantor and Persits gives a complete list of closed irreducible $\mathbb{C}$-vector spaces of Lie brackets of dimension greater than one:

1. $\mathfrak{g} = \mathfrak{so}(n, \mathbb{C})$ is the space of $n \times n$ antisymmetric matrices, $V$ is the space of $n \times n$ symmetric matrices, the commutator $[,]^X, X \in V,$ is given by $[a, b]^X = aXb - bXa =: [a, X b];$

2. $\mathfrak{g}$ is the space of $n \times n$ symmetric matrices, $V = \mathfrak{so}(n, \mathbb{C}), [.,.]^X = [., X ];$

3. $\mathfrak{g}$ is the space of $n \times m$ matrices, $V$ is the space of $m \times n$ matrices, $[,]^X = [., X ];$

4. $\mathfrak{g} = V$ is an even-dimensional vector space with a nondegenerate skew-symmetric form $(,),$ the commutator is given by $[a, b]^v := (v, a)b - (v, b)a - (a, b)v;$

Compatible Lie brackets from this list were studied in [TF95, §44] and [Yan00].

Another result which should be mentioned here is due to Odesski and Sokolov [OS06] who classified associative multiplications on the complex $n \times n$ matrices compatible with the usual one (in particular, by antisymmetrization one can deduce from this result a lot of new examples of compatible Lie brackets on $\mathfrak{gl}(n, \mathbb{C})$, see also [CGM00]).

The main objective of this paper is to outline a systematic approach to classification of pairs of compatible complex Lie brackets (we call such pairs bi-Lie structures in this paper) one of which is semisimple. As a byproduct we obtain new examples of such pairs which are not contained in the following list of known examples (KE) of this type (in Items 1,2 $[,]$ stands for the standard commutator of matrices).

**KE1** $(\mathfrak{so}(n, \mathbb{C}), [., [, .]], [., .])$, where $A$ is a symmetric $n \times n$-matrix (cf. the first example from the Kantor–Persits list and Example 2.3).

**KE2** $(\mathfrak{sp}(n, \mathbb{C}), [., [, .]], [., .])$, see Example 2.3 (this example can be derived from the second item of the Kantor–Persits list in the case when the matrix $X$ is the standard antisymmetric matrix of maximal rank equal to $2n$ [TF95, §44]).

**KE3** Let $(\mathfrak{g}, [., .])$ be semisimple. There exists a bi-Lie structure related to any $\mathbb{Z}_m$-grading on $(\mathfrak{g}, [., .])$ and to decomposition of the subalgebra $\mathfrak{g}_0$ to two subalgebras, see Examples 4.10, 12.12, Theorem 13.15 (in general case first appeared in [GS02], for $m = 2$ known much earlier, see discussion above).

**KE4** Let $(\mathfrak{g}, [., .])$ be semisimple. There exists a bi-Lie structure related to any parabolic subalgebra $\mathfrak{g}_0 \subset \mathfrak{g}$, see Example 4.9 (in case when $\mathfrak{g}_0$ is a Borel subalgebra it appeared in [Pan06]).

**KE5** Examples on $\mathfrak{sl}(3, \mathbb{C}), \mathfrak{so}(4, \mathbb{C})$ related to $\mathbb{Z}_2 \times \mathbb{Z}_2$-gradings [GS02].

Another byproduct is establishing of isomorphisms between some of bi-Lie structures from Items KE3 and KE4, which are not obvious.

It is known [KSM90, GS02] that, given a semisimple Lie algebra $(\mathfrak{g}, [., .])$ and a bilinear bracket $[., .]'$, the triple $(\mathfrak{g}, [., .], [., .]')$ is a bi-Lie structure if and only if $[., .]'$ is of the form $[,]' = [,]_W$ (see 3.1.1) for some linear operator $W \in \text{End}(\mathfrak{g})$ such that there is another operator $P \in \text{End}(\mathfrak{g})$ satisfying the "main identity" $T_W(., .) = [., .]_P$, where $T_W$ is the Nijenhuis torsion of $N$ (see Section 4). We call an operator $W$ satisfying $T_W(., .) = [., .]_P$ for some $P$ a weak Nijenhuis operator (WNO for short) and $P$ itself a primitive of $W$. The problem is that these operators are defined by the bi-Lie structure nonuniquely,
up to adding a differentiation of the bracket \([,]\) (which is inner). Our first main result (see Section 7) proposes a special fixing of these two operators and establishes some important properties of them. Namely, among all the WNOs corresponding to a bi-Lie structure \((\mathfrak{g}, [,], [,]')\) we choose that (uniquely defined) orthogonal to \(\text{ad} \mathfrak{g}\) in \(\text{End}(\mathfrak{g})\) with respect to the "trace form" (we call it \textit{principal}). We further prove that two bi-Lie structures are isomorphic (see Definition 2.6) if and only if so are the corresponding principal WNOs (Theorem 7.10). We also show that the principal operator \(W\) of a bi-Lie structure \((\mathfrak{g}, [,], [,]')\) possesses the unique primitive (called \textit{principal}) which is symmetric with respect to the Killing form on \((\mathfrak{g}, [,])\) (Theorem 7.5). Thus the problem of classification of bi-Lie structures with semisimple \((\mathfrak{g}, [,])\) is reduced to the problem of classification of the pairs \((W, P)\), where \(W\) is a principal WNO on \((\mathfrak{g}, [,])\) and \(P\) is its principal primitive, with respect to the natural action of the automorphisms of \((\mathfrak{g}, [,])\).

The last problem in general seems to be very complicated. We are trying to solve it under the following reasonable restriction: a principal WNO preserves some \(\Gamma\)-grading on \((\mathfrak{g}, [,])\), where \(\Gamma\) is an abelian group. This restriction is motivated by the fact that the majority of examples from the list above are in fact of this type (for Examples KE3, KE5 this is obvious, one needs a while to see a \(\mathbb{Z}_2 \times \cdots \times \mathbb{Z}_2\)-grading in Examples KE1, KE2 if \(A\) is diagonalizable, for Example KE4 see Theorem 13.15). We first prove (see Theorem 9.2) that if a principal WNO preserves a grading, then so does its principal primitive. Further on, we make one more restriction and consider principal WNOs \(W\) such that:

1. \(W\) preserves the root grading \(\mathfrak{g} = \mathfrak{h} + \sum_{\alpha \in \mathcal{R}} \mathfrak{g}_\alpha\) related to some Cartan subalgebra \(\mathfrak{h}\) in \((\mathfrak{g}, [,]);\)

2. \(W|_{\mathfrak{h}}\) is semisimple.

For such operators the "main identity" \(T_N(, ) = [,]_P\) becomes a system of 1-dimensional algebraic equations of second order (see Theorem 10.3), that is why is in principle solvable. Next we propose a method of solving it.

More precisely, this system consists of two parts. First is a system of quadratic equations indexed by the roots and relating the eigenvalues of the operators \(W, P\) with another invariants, which we call the \textit{times} of the corresponding bi-Lie structure \((\mathfrak{g}, [,], [,]')\). These are the complex numbers \(t_i\) such that the Lie algebra \((\mathfrak{g}, [,'] - t_i[,])\) is not semisimple (in the pencil \([[,'] - t_i[,]]\) the generic bracket is isomorphic to \([,]\) but for a finite number of values of \(t\) the corresponding brackets are nonsemisimple; we call them \textit{exceptional}). The second part, indexed by positive roots (with respect to any basis of roots) relates the eigenvectors and eigenvalues of \(W|_{\mathfrak{h}}\) with the times. As a result we obtain a family of unordered pairs \(\{t_{i,\alpha}t_{j,\alpha}\}_{\alpha \in \mathcal{R}}\) of complex numbers indexed by the roots (we call such a family a \textit{pairs diagram}) and a system of restrictions on the operator \(W|_{\mathfrak{h}}\) (see Section 10). It turns out that the "main identity" implies some rules of behavior of the pairs \(\{t_{i,\alpha}t_{j,\alpha}\}\) with respect to the addition of the roots (see Theorem 10.7 Item 1). This relates bi-Lie structures with the geometry of the corresponding root systems and makes it possible to classify these first.

Our next result (Theorem 11.1) says that the set of pairs diagrams splits in two disjoint classes, I and II. Further on, we study bi-Lie structures corresponding to these classes.

The rules of behavior of unordered pairs \(\{t_{i,\alpha}t_{j,\alpha}\}\) in the pairs diagrams of Class I with respect to the addition of roots remind very much the operation rules of the famous pair groupoid and induce (together with the operator \(W|_{\mathfrak{h}}\) subject to the restrictions mentioned) on \((\mathfrak{g}, [,])\) a special type of grading which we call a \textit{toral symmetric pairoid quasigrading of Class I} (see Section 12). It turns out that this quasigrading together with the grading set \(\{t_1, \ldots, t_n\}\) contains all the information...
about the corresponding bi-Lie structure (see Theorem 12.7) note that the corresponding WNO has
the symmetric restriction to $h^\perp = \sum_{\alpha \in R} g_\alpha$. Thus the classification problem of bi-Lie structures of
Class I is equivalent to the classification problem of toral symmetric pairoid quasigradings of Class I.
The last remains open, however we present a list of examples (which contains known and new ones)
and conjecture that this list is in a sense complete (see Conjecture 12.20). We remark that any such
quasigrading is related to a specific type of $\mathbb{Z}_2 \times \cdots \times \mathbb{Z}_2$-grading on $(\mathfrak{g}, [\cdot, \cdot])$ (see Lemma 12.9).

In the case of pairs diagrams of Class II the corresponding bi-Lie structures also induce on $(\mathfrak{g}, [\cdot, \cdot])$
a special type of gradings which we call a toral symmetric pairoid quasigradings of Class II. They are
much simpler than that of Class I, however, they contain only a part of information about the initial
bi-Lie structures. The crucial difference here, in contrast to Class I structures, lies in the fact that the
corresponding WNOs have also antisymmetric parts on $h^\perp$ which contain the rest of the information
about the bi-Lie structures (in Theorem 13.7 we show how to rebuild the bi-Lie structure from the
corresponding quasigrading and the antisymmetric part of the WNO on $h^\perp$). These antisymmetric
parts are subject to some restrictions implied by the ”main identity” (see Theorem 10.7 Item 2) and
in principle can be classified. The problem of classification of bi-Lie structures of Class II is now
reduced to the problem of classification of toral symmetric pairoid quasigradings of Class II and the
corresponding antisymmetric operators. Although this last is not solved in this paper in full extent,
we present a list of examples (containing known and new ones) and conjecture the completeness
of this list (see Conjecture 13.18). We also prove this conjecture for $\mathfrak{g} = \mathfrak{s}(n, \mathbb{C})$ thus obtaining a
complete classification of bi-Lie structures of Class II in this case (Theorem 13.19). Note that bi-Lie
structures corresponding to pairs diagrams of Class II are also related to some gradings on $(\mathfrak{g}, [\cdot, \cdot])$
which are coarsenings of the root grading (see Theorem 13.7).

Let us exhibit the examples which we get within our theory and compare them with that from
the list above.

1. Class I

(a) We recover Example KE1 in the particular case of $A = \text{diag}(t_1, t_1, \ldots, t_n, t_n, t_n+1)$ for
$\mathfrak{so}(2n+1, \mathbb{C})$ and $A = \text{diag}(t_1, t_1, \ldots, t_n, t_n)$ for $\mathfrak{so}(2n, \mathbb{C})$ (see Examples 12.13, 12.14).

(b) We recover Example KE2 in the particular case of $A = \text{diag}(t_1, \ldots, t_n, t_1, \ldots, t_n)$ (see
Example 12.15).

(c) We obtain new examples of bi-Lie structures, which are analogues of the examples above,
for $\mathfrak{sl}(n, \mathbb{C})$ and which, surprisingly, were not known in the literature (see Examples 12.16
12.17).

(d) We recover Example KE3 for $m = 2$ (see Example 12.12).

2. Class II

(a) We recover Example KE3 for $m > 2$ in the case when the $\mathbb{Z}_m$-grading is related to an
inner automorphism (see Theorem 13.15).

(b) We recover and generalize Example KE4 (see Theorem 13.12).

(c) We obtain a new example of a bi-Lie structure on the exceptional Lie algebra $\mathfrak{e}_7$ related
to $\mathbb{Z}_3 \times \mathbb{Z}_3$-grading (see Example 13.17).
Moreover, we prove that all the bi-Lie structures from Example KE4 are isomorphic to special bi-Lie structures from Example KE3 (see Theorem 13.13). We also deduce from Examples 12.16, 12.17 new examples of real bi-Lie structures on \( \mathfrak{su}(n) \) and, in particular, on \( \mathfrak{so}(6, \mathbb{R}) \cong \mathfrak{su}(4) \) (see Appendix II).

Remark that a specific form of the matrix \( A \) which appeared in Example Class I (a) is related to the fact that the above mentioned restrictions (1), (2) on the WNO imply the following condition on the corresponding bi-Lie structure: the sum of the centres of the exceptional brackets contains the Cartan subalgebra \( \mathfrak{h} \) (see Theorem 10.1). In the case of generic \( A \) (diagonalizable with simple spectrum) in Example KE1 these centres are trivial and this case does not fit our theory. However, this case is related to a specific pairoid quasigrading which is not toral (see Remark 12.22).

Summarizing, we can say that although the theory of the second part of the paper (Sections 10–13) recover only examples related to toral gradings and quasigradings (cf. Appendix I and Definition 12.4), or in other words, to inner automorphisms, the results of the first part give a hope that similar theory can be also built in general, i.e. including gradings coming from outer automorphisms too.

The author also hopes that some of these results (for instance Theorems 7.3, 7.5, 7.10, 8.3, 8.5, 9.2), which deal with general properties of bi-Lie structures, are new and useful.

Now let us overview the content of the paper in details. In Section 2 we give basic definitions (of bi-Lie structures and their isomorphisms) and examples. We also prove a lemma, which will be used in the proof of Theorem 7.3. In Section 3 we recall basic formulae from the theory of cohomology of Lie algebras, mainly for fixing notations.

Section 4 is devoted to WNOs and their primitives. We give definitions, basic facts and examples. We also prove a result (Theorem 4.7) which shows how the primitive changes when we add a differentiation to the WNO.

In Section 5 we introduce a notion of a \textit{parial Nijenhuis operator} (PNO for short) which in fact provides (besides the WNO) an alternative way of description of bi-Lie structures (in the semisimple case). The notion of a PNO is auxiliary but convenient for understanding the relation between bi-Lie structures and WNOs. The relation between PNOs and WNOs is considered in Section 6.

This relation is exhibited in the full extent in Section 7, where we prove the existence of a WNO related to any bi-Lie structure \( (\mathfrak{g}, [\cdot, \cdot], [\cdot, \cdot']) \) such that \( (\mathfrak{g}, [\cdot, \cdot]) \) is semisimple, see Theorem 7.3 (note that this proof is independent of the Whitehead lemmata on the triviality of the cohomology of semisimple Lie algebras). We also introduce the notion of a principal WNO and prove the existence of a symmetric primitive for the principal WNO (again without a reference to the Whitehead lemmata, see Theorem 7.5). Finally, we show how the principal WNO determines the centre of the bracket \( [\cdot, \cdot]' = [\cdot, \cdot]_N \) (Corollary 7.7) and prove the equivalence of the category of bi-Lie structures \( (\mathfrak{g}, [\cdot, \cdot], [\cdot, \cdot']) \) with the same semisimple Lie algebra \( (\mathfrak{g}, [\cdot, \cdot]) \) and the category of principal WNOs on \( (\mathfrak{g}, [\cdot, \cdot]) \) (Theorem 7.10).

In Section 8 we study the pencil of brackets \( [\cdot, \cdot]' - t[\cdot, \cdot], t \in \mathbb{C}, \) in particular, the kernels of the Killing forms of the exceptional Lie brackets. We introduce two important subalgebras in \( (\mathfrak{g}, [\cdot, \cdot]) \), a \textit{central} subalgebra \( \mathfrak{z} \subset \mathfrak{g} \), which is the sum of the centres of the exceptional brackets, and a subalgebra \( \mathfrak{z} \supset \mathfrak{z} \), which roughly speaking is the maximal subspace in \( \mathfrak{g} \) such that \( \mathfrak{W}|_{\mathfrak{z}} \) is a Nijenhuis operator, i.e. such that its Nijenhuis torsion vanishes.

Section 9 is devoted to WNOs preserving a group grading on \( (\mathfrak{g}, [\cdot, \cdot]) \). The main result (Item 1 of Corollary 9.3) says that if a principal WNO preserves a grading, then so does its principal primitive. We also present Theorem 9.4 which, given a WNO \( W \) preserving the root grading with respect to a Cartan subalgebra, describes the principal WNO \( \text{pr}(W) \) with the property \( [\cdot, \cdot]'_W = [\cdot, \cdot]_{\text{pr}(W)} \). Although
we apply this result only partially (in the proof of Theorem 13.7), it is useful in calculations and we retain it in full extent.

Section 10 is central. Here we study the bi-Lie structures related to the WNOs satisfying conditions (1), (2). In Theorem 10.3 we obtain the system of equations on the operators $W, P$ implied by the "main identity" $T_W(., .) = [., .]_P$, describe the central subalgebra and the set of times. We introduce the notions of admissible operator on a Cartan subalgebra implied by the preceding theorem (Definition 10.4). Further on, we introduce a notion of a pairs diagram (see Definition 10.6), and show that conditions (1), (2) imply existence of a pairs diagram (Item 1 of Theorem 10.7). In this theorem we also obtain restrictions on the antisymmetric part of $W|_{h^⊥}$ (Item 2) and describe an important subalgebra of $(\mathfrak{g}, [., .])$ called the basic subalgebra (Items 3–6) of a bi-Lie structure (the last is related to the above mentioned subalgebra $\hat{\mathfrak{z}}$, see Remark 10.8). Finally we formalize the restrictions on the antisymmetric part of $W|_{h^⊥}$ in a notion of an operator obeying the triangle rule (Definitions 10.12, 10.13).

In Section 11 we prove that the set of pairs diagrams splits into two disjoint parts: Class I and Class II.

In Section 12 we study bi-Lie structures related to pairs diagrams of Class I. We introduce a notion of toral symmetric pairoid quasigrading of Class I on $(\mathfrak{g}, [., .])$ (Definition 12.4) and show that it is equivalent to the pair consisting of a pairs diagram and the corresponding admissible operator (Lemma 12.6). Theorem 12.7 shows how to construct a WNO by a quasigrading and asserts that all WNOs related with bi-Lie structures of Class I are of this kind. It also gives necessary and sufficient conditions for two such bi-Lie structures to be isomorphic. Then we show that the structure of a toral symmetric pairoid quasigrading of Class I implies a specific type $\mathbb{Z}/2 \times \cdots \times \mathbb{Z}/2$-grading on $(\mathfrak{g}, [., .])$ called admissible (Lemma 12.9). We further present examples of gradings which are not admissible (Example 12.13), of admissible grading not related with toral symmetric pairoid quasigradings of Class I (Example 12.10), and a series of examples of these last (Examples 12.12–12.18). In Lemma 12.19 we propose a method of constructing a toral symmetric pairoid quasigrading of Class I from a given one. Finally, we conjecture that all bi-Lie structures of Class I are obtained from the listed examples or as reductions from them with the help of Lemma 12.19 (see Conjecture 12.20).

Section 13 is devoted to bi-Lie structures related to pairs diagrams of Class II. Similarly to the previous case we start from introducing of the notion of a toral symmetric pairoid quasigrading of Class II and showing its equivalence to the corresponding pair consisting of an admissible operator and a pairs diagram (Lemma 13.5). In Theorem 13.7 we show how to construct a WNO $W$ by a quasigrading (which in fact is determined by two reductive subalgebras, see Remark 13.6) and an antisymmetric operator on $h^⊥$ obeying the triangle rule. Note that from the quasigrading itself we get $W|_h$ and the symmetric part of $W|_{h^⊥}$. Again, all bi-Lie structures of Class II can be obtained by this construction (Item 2 of Theorem 13.7). Then we deal with the question of uniqueness (the answer is given by Theorem 13.10). In Theorem 13.12 we construct a series of examples generalizing Example KE4. In Theorem 13.15 we obtain a series of examples recovering Example KE3 and prove the existence of some isomorphisms between them and with examples from the previous series. Then we present an example of bi-Lie structure of Class II on the Lie algebra $\mathfrak{e}_7$ using a construction combining that from Theorem 13.7 and an observation related to the "triangle rule" from preceding examples. We conclude the section by conjecturing that this construction is universal, i.e. all the bi-Lie structures of Class II can be obtained by means of it (Conjecture 13.18). We also prove this conjecture for $\mathfrak{g} = \mathfrak{a}_n$, in particular obtaining the classification of bi-Lie structures of Class II for this Lie algebra.
The paper contains two appendices. In Appendix I we discuss toral gradings of semisimple Lie algebras \( g \) corresponding to regular reductive subalgebras \( g_0 \). In particular, we recall [Ost] that the family of irreducible components of the natural representation of \( g_0 \) in \( g \) forms a group grading, which we call the toral irreducible grading related to \( g_0 \). We also recall [Dyn52], [DO02] the classification of regular reductive subalgebras.

Appendix II is devoted to examples of real bi-Lie structures related to the complex ones of Class I. We show that the WNO constructed in Theorem 12.7 under some additional conditions can be restricted to the compact real form of the complex semisimple Lie algebra \( (g,[,]) \), see Theorem 15.1. We also deduce, by means of this theorem, examples of real bi-Lie structures from complex ones, in particular for the Lie algebra \( su(n) \).

2 Preliminaries and basic examples

We assume throughout the paper that the ground field is \( \mathbb{C} \) (with the exception of Appendix II).

2.1. Definition A bi-Lie structure is a triple \((g,[,],[,]')\), where \( g \) is a vector space and \([,],[,]'\) are two Lie algebra structures on \( g \) which are compatible, i.e. so that any their linear combination \([,]:=\lambda[,] + \lambda'[,]'\) is a Lie algebra structure.

Note that, given Lie brackets \([,],[,]'\) on \( g \), they are compatible if and only if \([,]+[,]'\) is a Lie bracket.

2.2. Example Let \( g=\mathfrak{gl}(n,\mathbb{C}) \), \( A \in g \) be a fixed matrix. Put \([x,y]'=xAy-yAx=[x,Ay]\). Then it is easy to see that the triple \((g,[,],[,]')\), where \([,]\) is the standard commutator, is a bi-Lie structure.

2.3. Example Let \( g=\mathfrak{so}(n,\mathbb{C}) \), \( A \in \text{Symm}(n,\mathbb{C}) \), a fixed symmetric matrix. Then again \((g,[,],[,]')\), where \([,]'\) is given by the same formula as above, is a bi-Lie structure. Note that symmetric matrices form the orthogonal complement to \( \mathfrak{so}(n,\mathbb{C}) \) in \( \mathfrak{gl}(n,\mathbb{C}) \) with respect to the trace form \( \text{Tr}(XY) \).

2.4. Example Let \( g=\mathfrak{sp}(n,\mathbb{C}) = \{X \in \mathfrak{gl}(2n) \mid XJ+JX^T=0\} \), here \( J = \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix} \) is the matrix of the standard symplectic form, and let \( A \) belong to the orthogonal complement \( \mathfrak{sp}(n,\mathbb{C})^\perp \) to the symplectic subalgebra in \( \mathfrak{gl}(2n) \) with respect to the trace form. This subspace can be described by the formula \( \{X \in \mathfrak{gl}(2n) \mid XJ-JX^T=0\} \). Then again \((g,[,],[,A])\), is a bi-Lie structure.

The following lemma will be used later.

2.5. Lemma Let \((g,[,])\) and \((g,[,]')\) be Lie algebras and let \( \text{ad}, \text{ad}' \) be the corresponding \( \text{ad} \)-operators, \( \text{ad} (x(y) = [x, y], \text{ad}' (x(y) = [x, y]'). The triple \((g,[,],[,]')\) is a bi-Lie structure if and only if
\[
\text{ad} [x,y]' + \text{ad}' [x,y] = [\text{ad}', \text{ad} y] + [\text{ad} x, \text{ad}' y], x, y \in g
\]
Indeed, $[,]$ and $[,]'$ are compatible if and only if $[,]'' := [,]+[,]'$ is a Lie bracket if and only if $\text{ad}''[x,y]'' = [\text{ad}''x,\text{ad}''y]$, i.e. $(\text{ad} + \text{ad}'')[x,y]'' = [\text{ad}x + \text{ad}'x, \text{ad}y + \text{ad}'y]$. Simplifying the last condition with the account of $\text{ad}[x,y] = [\text{ad}x,\text{ad}y]$ and $\text{ad}'[x,y]' = [\text{ad}'x,\text{ad}'y]$ gives the result. □

2.6. Definition. We say that bi-Lie structures $(g, [,],[,]'')$ and $(g_1, [,],[,]'')$ are strongly isomorphic (isomorphic) if there exists a linear map $\varphi : g \to g_1$ transforming isomorphically the bracket $[,]$ to $[,]_1$ and $[,]'$ to $[,]'_{1'}$ (transforming isomorphically the brackets $[,]$ to $[,]_1$ to some linear combinations of the brackets $[,]'$ to $[,]'_{1'}$).

2.7. Remark. The notion of isomorphism of bi-Lie structures is more natural in the context of studying the pencils $\{[,]^\lambda\}$ and $\{[,]_{1'}^\lambda\}$ of Lie brackets generated by the pairs $([,],[,]')$ and $([,]_1,[,]'_{1'})$.

3 Lie algebra cohomology

In this section we recall basic definitions from the theory of (low-dimensional) cohomology of a Lie algebra $(g, [,])$ with coefficients in a $g$-module $m$. The space of $j$-cochains $C^j(g, m)$ consists of $j$-linear skew-symmetric maps $g^j \to m$. By definition $C^0(g, m) = m$.

The coboundary operator $\partial^j : C^j(g, m) \to C^{j+1}(g, m)$ is defined by

$$
\partial \alpha(h_1) := h_1 \alpha \quad \text{for } j = 0;
$$

$$
\partial \alpha(h_1, h_2) := h_1 \alpha(h_2) - h_2 \alpha(h_1) - \alpha([h_1, h_2]) \quad \text{for } j = 1;
$$

$$
\partial \alpha(h_1, h_2, h_3) := \sum_{c.p.} (h_i \alpha(h_j, h_k) + \alpha(h_i, [h_j, h_k])) \quad \text{for } j = 2
$$

(here $\alpha \in C^j(g, m)$, $h_i \in g$). The cohomology space is defined standardly: $H^j(g, m) := \ker \partial^j / \im \partial^{j-1}$. The Whitehead lemmata assert that $H^j(g, m) = 0$ if $g$ is semisimple and $j = 1, 2$.

For the adjoint module $m = g$ we will also use notations $L(g) := C^1(h, m)$ (later we will also write $\text{End}(g)$ for $L(g)$), $L^2(g) := C^2(g, m)$, and

$$
[,]_W := \partial W(\cdot, \cdot) = [W, \cdot] + [, W \cdot] - W[, \cdot], W \in L(g). \tag{3.1.1}
$$

4 Weak Nijenhuis operators

Given two Lie algebras $(g, [,])$ and $(g, [,]'')$, we observe that $(g, [,]', [',']')$ is a bi-Lie structure if and only if $[,]'$ is a 2-cocycle with respect to $[,]$ with coefficients in the adjoint module. In particular, if $(g, [,], [,]'')$ is a bi-Lie structure such that $(g, [,])$ is semisimple, then $[,]' = [,]_W$ for some $W \in L(g)$ (note that in Theorem 7.3 we will prove existence of such $W$ without using the Whitehead lemmata).

Vice versa, let an operator $W \in L(g)$ be given such that $[,]_W$ is a Lie bracket. Then, since $[,]_W$ is a cocycle, it is automatically compatible with $[,]$ and $(g, [,], [,]'')$ is a bi-Lie structure.

We come to the following definitions.

4.1. Definition. Let $(g, [,])$ be a Lie algebra. An operator $W \in L(g)$ is called a weak Nijenhuis operator (WNO for short) if the bracket $[,]_W$ defined by formula (3.1.1) is a Lie bracket on $g$ (the last will be called the modified bracket on $g$).
4.2. Remark A similar structure in the context of Lie algebroids appeared in [CGM04], we borrowed the terminology from this paper.

4.3. Definition The Nijenhuis torsion of an operator $W \in L(\mathfrak{g})$ is defined as

$$T_W(x, y) := [Wx, Wy] - W[x, y]_W.$$  

The following lemma justifies the term ”WNO” (in the sense that it generalizes the well known notion of a Nijenhuis operator, i.e. an operator $W$ with vanishing $T_W$).

4.4. Lemma [KSM90] An operator $W \in L(\mathfrak{g})$ is a WNO if and only if $T_W$ is a 2-cocycle on $\mathfrak{g}$ with coefficients in the adjoint module.

4.5. Definition Let $W \in L(\mathfrak{g})$ be a WNO. An operator $P \in L(\mathfrak{g})$ is said to be a primitive of $W$ if $T_W(x, y) = [x, y]_P, x, y \in \mathfrak{g}$ (in other words $T_W = \partial P$ in the sense of the cohomology with coefficients in the adjoint module).

If $(\mathfrak{g}, [\cdot, \cdot])$ is semisimple, its second cohomology is trivial, hence any WNO possesses a primitive. Below we will prove the existence of such an operator directly (see Theorem 7.5). Note that the primitive of a WNO is defined up to addition of a derivation of the Lie algebra $(\mathfrak{g}, [\cdot, \cdot])$.

Similarly, given a WNO $W \in L(\mathfrak{g})$ and any derivation $d \in \text{Der}(\mathfrak{g})$ of the Lie algebra $(\mathfrak{g}, [\cdot, \cdot])$, one observes that $W + d$ is again a WNO (since $[\cdot, \cdot]_{W+d} = [\cdot, \cdot]_W$). Assuming $W$ has a primitive $P$, what is a primitive of $W + d$? Below we will need to know how to express a primitive of $W + d$ in terms of $P$ in a particular case of inner derivation. To find such an expression let us introduce another useful notion.

4.6. Definition A formal power series $R(t) = I + tR_1 + t^2 R_2 + \cdots \in L(\mathfrak{g})[[t]]$ is called a formal resolving function of order $n$ for a WNO $W$ if

$$[R(t)\cdot, R(t)\cdot] \stackrel{n}{=} R(t)(\cdot, \cdot) + t[\cdot, \cdot]_W, \quad (4.6.1)$$

where $\stackrel{n}{=}$ stands for the equality modulo terms of order $> n$ in $t$ (here we understand the left hand side and the right hand side of the equality as elements of the space $L^2(\mathfrak{g})[[t]]$ of the formal power series with coefficients in $L^2(\mathfrak{g})$ and assume that $[\cdot, \cdot]$ is bilinear with respect to $t$).

We say that $\Phi(t) = I + t\Phi_1 + t^2 \Phi_2 + \cdots \in L(\mathfrak{g})[[t]]$ is a formal automorphism of the Lie algebra $(\mathfrak{g}, [\cdot, \cdot])$ if

$$[\Phi(t)\cdot, \Phi(t)\cdot] = \Phi(t)(\cdot, \cdot).$$

Clearly, if $\Phi(t) = I + t\Phi_1 + \cdots$ is a formal automorphism, then $\Phi_1 \in \text{Der}(\mathfrak{g})$.

4.7. Theorem 1. A series $R(t) = I + tR_1 + \cdots$ is a formal resolving function of order 1 for a WNO $W$ if and only if $R_1 = W + d$ for some $d \in \text{Der}(\mathfrak{g})$.

2. A formal series $R(t) = I + t(W + d) + t^2 R_2 + \cdots$ is a formal resolving function of order 2 for a WNO $W$ if and only if $-R_2$ is a primitive of the WNO $W + d$.  

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3. Let $R(t)$ be a formal resolving function of order $n$ for a WNO $W$. If $\Phi(t) = I + t\Phi_1 + t^2\Phi_2 + \cdots$ is a formal automorphism of the Lie algebra $(g, [\cdot, \cdot])$, then $\tilde{R}(t) = \Phi(t)R(t)$ is also a formal resolving function of order $n$ for $W$.

4. In particular, if $P$ is a primitive for $W$, then $P - \Phi_1W - \Phi_2$ is a primitive for $W + \Phi_1$.

Ad 1, 2. The proof follows by comparing of coefficients of $t$ and $t^2$ in the left hand side and the right hand side of (14.6.1).

Ad 3. The proof follows from the definitions.

Ad 4. If $R(t) = I + tW + t^2R_2 + \cdots$, then $\tilde{R}(t) = I + t(W + \Phi_1) + t^2(R_2 + \Phi_1W + \Phi_2) + \cdots$ and the result follows from Item 2. \qed

4.8. COROLLARY If $P$ is a primitive of a WNO $W$, the operator $P - \text{ad} x_0 W - (\text{ad}^2 x_0)/2$ is a primitive of the WNO $W + \text{ad} x_0,$ $x_0 \in g$. In particular, the WNO $\text{ad} x_0$ has a primitive $-(\text{ad}^2 x_0)/2$.

Use the theorem and the fact that $\Phi(t) := \exp(t\text{ad} x_0) = I + t\text{ad} x_0 + t^2(\text{ad}^2 x_0)/2 + \cdots$ is a formal automorphism of the Lie algebra $(g, [\cdot, \cdot])$. \qed

4.9. EXAMPLE Let $g$ be decomposed to a sum of two subalgebras $g = g_1 \oplus g_2$. Put $W|_{g_i} = \lambda_i \text{Id}_{g_i},$ $i = 1, 2$, where $\lambda_{1,2}$ are any scalars. Then it is easy to see that the Nijenhuis torsion of $W$ vanishes (cf. [Pan06]). This way one gets a lot of examples of bi-Lie structures (for instance, taking $g$ to be semisimple, $g_1$ to be a parabolic subalgebra and $g_2$ its "complement", cf. Theorem 13.12).

4.10. EXAMPLE Let $g = g_0 \oplus \cdots \oplus g_{n-1}$ be a $\mathbb{Z}_n$-grading on $g$. Put $W|_{g_i} = i \text{Id}_{g_i},$ $i = 0, \ldots, n - 1$. We will show that $W$ is a WNO calculating explicitly its primitive $P$.

Put $P|_{g_i} = \frac{1}{2}i(n - i) \text{Id}_{g_i}$. Show that it is indeed a primitive of $W$. Let $x_i \in g_i, x_j \in g_j$. Then, if $i + j \leq n - 1$, we have $T_W(x_i, x_j) = [Wx_i, Wx_j] = ij[x_i, x_j]$ (the term $W[x_i, x_j]_W$ vanishes). On the other hand, we have $[x_i, x_j] = \frac{1}{2}(i(n-i)+j(n-j)-(i+j)(n-i-j))[x_i, x_j] = ij[x_i, x_j]$. If $i + j \geq n$, we have $T_W(x_i, x_j) = [Wx_i, Wx_j] - W[x_i, x_j]_W = (ij -(n-i-j)(i+j-(n-i-j))[x_i, x_j] = (ij + (n-i-j)n)[x_i, x_j]$ and $[x_i, x_j] = \frac{1}{2}(i(n-i)+j(n-j)-(i+j-n)(2n-i-j))[x_i, x_j] = (ij - (i+j-n)n)[x_i, x_j]$.

4.11. EXAMPLE Consider $g = \mathfrak{gl}(n, \mathbb{C})$ and the operator $L_A, L_A x := Ax$, of the left multiplication by a fixed matrix $A \in g$. Then the Nijenhuis torsion of $L_A$ vanishes, i.e. it is a WNO, and the modified bracket $[\cdot, \cdot]_{L_A}$ coincides with the bracket $[\cdot, \cdot]_A$ from Example 2.2.

4.12. EXAMPLE Consider $g = \mathfrak{so}(n, \mathbb{C})$ and $A \in \text{Symm}(n, \mathbb{C})$. The operator $L_A$ is not correctly defined on $g$ (since $L_{A^2} \not\subseteq g$). However, the operator $W := W'|g_0$, where $W' := (1/2)(L_A + R_A) \in \text{End}(\mathfrak{gl}(n, \mathbb{C}))$ and $R_A$ stands for the operator of the right multiplication by $A$, is a WNO and it easy to see that again $[\cdot, \cdot]_W = [\cdot, \cdot]_A$ (see Example 2.3). The torsion of $W$ does not vanish, but it is a cocycle. Indeed, since $W' = L_A - (1/2)(L_A - R_A) = L_A - \text{ad}(A/2)$ and $0$ is a primitive of $L_A$, by Corollary 4.8 we conclude that the operator $\text{ad}(A/2) \circ L_A - (1/8)\text{ad}^\circ A$ is a primitive of $W'$. Thus $T_W'$ is a cocycle and so is $T_W$.\[12]
5 Partial Nijenhuis operators

Let us start from a different look at Example 4.12. Put $G := \mathfrak{gl}(n, \mathbb{C})$ and consider the operator $L_A$ as an operator acting from $g = \mathfrak{so}(n, \mathbb{C})$ to $G$. It is easy to see that the formula similar to (3.1.1), i.e.
\[
\{ [\cdot, \cdot] - L_A \cdot, [\cdot, \cdot] \} + \{ [\cdot, \cdot], L_A \cdot \} - L_A \{ [\cdot, \cdot] \}
\]
still defines a bilinear operation on $g$ (in fact $[x, y] = [x, A y]$) and, moreover, $T_{L_A}(x, y) = [L_A x, L_A y] - L_A [x, y]_{L_A}$ vanishes for any $x, y \in g$. This motivates the following definition.

5.1. Definition Let $G$ be a Lie algebra and $g \subset G$ a Lie subalgebra. We say that a pair $(g, N)$, where $N : g \to G$ is a linear operator, is a partial Nijenhuis operator on $G$ (PNO for short) if the following two conditions hold:

(i) $[x, y]_N \in g$ for any $x, y \in g$;

(ii) $T_N(x, y) = 0$ for any $x, y \in g$.

(Here $[,]_N$ and $T_N$ are given by the same formulae as in Definitions 4.1, 4.3; note that it follows from condition (i) that the term $N[x, y]_N$ which appears in the definition of $T_N$ is correctly defined.)

5.2. Remark A similar structure in the context of Lie algebroids appeared in [CGM04] under the name “outer Nijenhuis tensor”.

5.3. Lemma If $(g, N)$ is a PNO on $G$, then:

1. $Ng$ is a Lie subalgebra in $G$;

2. $(g, N^t), N^t := N - tI$, is a partial Nijenhuis operator on $G$ for any $t \in \overline{\mathbb{C}} := \mathbb{C} \cup \{\infty\}$, here $I : g \to G$ is the natural embedding and $N^\infty = I$ by definition;

3. $N^t g$ is a Lie subalgebra in $G$ for any $t$;

4. for any $t \in \overline{\mathbb{C}}$ the operation $[,]_{N^t}$ is a Lie algebra structure on $g$ and $N^t : g \to G$ is a homomorphism between Lie algebras $(g, [,]_{N^t})$ and $(G, [\cdot, \cdot])$;

5. the Lie bracket $[,]_N$ is compatible with the Lie bracket $[,]$.

Indeed, Item 1 follows from the definition of a PNO. Item 2 is due to the equality $[,]_{N^t} = [,]_N - t[,]$ and to the equality $T_{N - tI} = T_N$. Item 3 follows from Item 2.

Now Item 4 follows easily from the equality $[x, y]_{N - tI} = (N - tI)^{-1}[(N - tI)x, (N - tI)y]$, which makes sense for almost all $t$, and Item 5 is a consequence of Item 4. □

5.4. Example The basic example here is $L_A : \mathfrak{so}(n, \mathbb{C}) \to \mathfrak{gl}(n, \mathbb{C}), A \in \text{Symm}(n, \mathbb{C}), [\cdot, ]_{L_A} = [\cdot, A]$.

Below we will associate a PNO with any bi-Lie structure $(g, [,], [\cdot, \cdot]^t)$ with a semisimple bracket $[,]$. 

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6 PNO and WNO in the presence of orthogonal decomposition

Let $\mathcal{G}$ be a Lie algebra, $\tilde{\mathfrak{g}} \subset \mathcal{G}$ a Lie subalgebra and let $B$ be an invariant form on $\mathcal{G}$ (we put tilde over $\mathfrak{g}$ to be consistent with the notations of the next section).

6.1. Lemma $[\mathfrak{g}_0, \mathfrak{g}_1] \subset \mathfrak{g}_1$, here $\mathfrak{g}_0 := \tilde{\mathfrak{g}}, \mathfrak{g}_1 := \tilde{\mathfrak{g}}^\perp$ is the orthogonal complement to $\tilde{\mathfrak{g}}$ with respect to $B$.

Let $x, z \in \mathfrak{g}_0, y \in \mathfrak{g}_1$. Then $B([x, y], z) = -B(y, [x, z]) = 0$. Since $z \in \mathfrak{g}_0$ is arbitrary, we get the result. □

From now on we will assume that

$$\mathfrak{g}_0 \oplus \mathfrak{g}_1 = \mathcal{G}. \quad (6.1.1)$$

Such decomposition occurs for instance when $B$ is nondegenerate and so is $B|_{\tilde{\mathfrak{g}} \times \tilde{\mathfrak{g}}}$. The main example which we have in mind is the pair $\tilde{\mathfrak{g}}$, such decomposition occurs for instance when $\mathfrak{g}_0, \mathfrak{g}_1$ are the operators $\mathfrak{g}_0$, $\mathfrak{g}_1$ on $\mathfrak{g}_0$ and $\mathfrak{g}_1$ is a WNO on $\mathfrak{g}_1$ (see Section 7.4 cf. also Example 6.3 for an instance of different kind).

Now let $N : \tilde{\mathfrak{g}} \to \mathcal{G}$ be a linear operator. Put $p_i : \mathcal{G} \to \mathfrak{g}_i$ for the corresponding orthogonal projection, $N_i := p_i N, i = 1, 2$. Then $N = N_0 + N_1$ and $[\cdot, \cdot]_N = [\cdot, \cdot]_{N_0} + [\cdot, \cdot]_{N_1}$. We obviously have $[\mathfrak{g}_0, \mathfrak{g}_0]_{N_0} \subset \mathfrak{g}_0$. Assuming that $[\mathfrak{g}_0, \mathfrak{g}_0]_N \subset \mathfrak{g}_0$ (the first condition of the definition of PNO), we conclude from the lemma that

$$[\mathfrak{g}_0, \mathfrak{g}_0]_{N_0} = \{ [N_1 x, y] + [x, N_1 y] + N_1 [x, y] \mid x, y \in \mathfrak{g}_0 \} \subset \mathfrak{g}_1$$

and $[\cdot, \cdot]_N = [\cdot, \cdot]_{N_0}$ on $\mathfrak{g}_0$. Hence, if also the second condition of the definition of PNO holds for $N$, the operator $N_0$ is a WNO on $\tilde{\mathfrak{g}}$ by Item 4 of Lemma 5.3 which is uniquely (!) defined by $N$ and which gives the same modified bracket on $\tilde{\mathfrak{g}}$ as $N$. Let us summarize this information in the following lemma.

6.2. Lemma Let $N : \tilde{\mathfrak{g}} \to \mathcal{G}$ be a PNO. Assume that an invariant form $B$ on $\mathcal{G}$ is given and condition (6.1.1) is satisfied, where $\mathfrak{g}_0 = \tilde{\mathfrak{g}}, \mathfrak{g}_1 = \tilde{\mathfrak{g}}^\perp$. Then the operator $N_0 := p_0 N$ (here $p_0 : \mathcal{G} \to \mathfrak{g}_0$ is the orthogonal projection) is a WNO on $\tilde{\mathfrak{g}}$ and moreover $[\cdot, \cdot]_N = [\cdot, \cdot]_{N_0}$.

6.3. Example Let $\tilde{\mathfrak{g}} = \mathfrak{g}_0 = \mathfrak{so}(n, \mathbb{C}), \mathcal{G} = \mathfrak{gl}(n, \mathbb{C}), \mathfrak{g}_1 = \text{Symm}(n, \mathbb{C}), N = L_A|_{\tilde{\mathfrak{g}}}, A \in \text{Symm}(n, \mathbb{C})$ (see Example 5.4), and let $B$ be the trace form on $\mathcal{G}$. Then $N_0 = (1/2)(L_A + R_A)|_{\tilde{\mathfrak{g}}}, N_1 = (1/2)(L_A - R_A)|_{\tilde{\mathfrak{g}}}$. Thus we get the WNO $W = N_0$ from Example 4.12.

Let us study the pair of operators $(N, N_0)$, where $N$ is a PNO, in more details. First, we are able to express the Nijenhuis torsion of the WNO $N_0$ by means of the operator $N_1$. Indeed, using Lemma 6.4 we get $0 = p_0 T_N(x, y) = p_0 ([N x, N y] - N[x, y]_N) = [N_0 x, N_0 y] + p_0 [N_1 x, N_1 y] - N_0 ([x, y]_{N_0}) = T_{N_0} + p_0 [N_1 x, N_1 y]$ for $x, y \in \mathfrak{g}_0$, hence

$$T_{N_0}(x, y) = -p_0 [N_1 x, N_1 y], \quad x, y \in \tilde{\mathfrak{g}}. \quad (6.3.1)$$

Second, the condition $[\cdot, \cdot]_N = [\cdot, \cdot]_{N_0}$, shows that $[\cdot, N_1] = 0$ on $\tilde{\mathfrak{g}}$. In other words, the map $N_1 : \tilde{\mathfrak{g}} \to \mathfrak{g}_1$ is a 1-cocycle on $\tilde{\mathfrak{g}}$ with coefficients in the $\tilde{\mathfrak{g}}$-module $\mathfrak{g}_1$. Thus, if we assume additionally that $\tilde{\mathfrak{g}}$ is semisimple, by the Whitehead lemma $N_1 = \partial w$ for some (uniquely defined) element $w \in \mathfrak{g}_1$, i.e.

$$N_1 = -(\text{ad}_{\tilde{\mathfrak{g}}} w)|_{\tilde{\mathfrak{g}}}. \quad (6.3.2)$$
By Corollary 4.8 we have \([\text{ad } w(x), \text{ad } w(y)] = T_{\text{ad } w}(x, y) = [x, y] - (\text{ad}^2 w)/2\), hence
\[
T_{N_0}(x, y) = -p_0[x, y] - (\text{ad}^2 w)/2 = [x, y]p_0(\text{ad}^2 w)/2, \quad x, y \in \tilde{g}.
\]
In other words, \(P := p_0(\text{ad}^2 w)/2\) is a primitive of the WNO \(N_0\). Note that \(P\) is a symmetric operator with respect to the form \(B|_{\tilde{g}^2}\). Indeed, \(\text{ad}_w g\) is antisymmetric, hence, given \(x, y \in \tilde{g}\), we have \(B(Px, y) = (1/2)B((\text{ad}^2_w w)(x), y) = (1/2)B(x, (\text{ad}^2_w w)(y)) = B(x, Py)\). Let us summarize this in

6.4. LEMMA Retain the hypotheses of Lemma 6.2 and assume moreover that \(\tilde{g}\) is semisimple. Then there exists a uniquely defined element \(w \in g_1\) such that \(N_1 := p_1N = -\text{ad}_w w\). Moreover the operator \(P := (1/2)p_0(\text{ad}^2_w w) : \tilde{g} \rightarrow \tilde{g}\) is symmetric with respect to \(B|_{\tilde{g}^2}\) and is a primitive of the WNO \(N_0\).

7 Semisimple bi-Lie structures and their principal WNO

7.1. DEFINITION We say that a bi-Lie structure \((\tilde{g}, [\cdot, \cdot], [\cdot, \cdot])\) is semisimple if \((\tilde{g}, [\cdot, \cdot])\) is a semisimple Lie algebra (which will be called the underlying Lie algebra of the bi-Lie structure).

We want to apply the construction from Lemma 6.2 to a situation in which:

1. \((\tilde{g}, [\cdot, \cdot], [\cdot, \cdot])\) is a semisimple bi-Lie structure;
2. \(G := \text{End}(g)\);
3. \(\tilde{g} := \text{ad } g \subset G\);
4. \(N := \text{ad}'(\text{ad}^{-1}) : \tilde{g} \rightarrow G\);
5. \(B(a, b) := \text{Tr}(ab), \quad a, b \in G\);

(here \(\text{ad } x(y) := [x, y], \text{ad }' x(y) = [x, y]', x, y \in g\)). It is easy to see that \(B|_{\tilde{g}^2}\) is the Killing form of \((\tilde{g}, [\cdot, \cdot])\) (here [.,.] is the commutator of endomorphisms). It is nondegenerate (due to the semisimplicity of \(g\)) and, moreover, \(B\) is nondegenerate itself, hence condition (6.1.1) is satisfied.

The following lemma shows that \(N\) is a PNO on \(G\).

7.2. LEMMA \(1. [\text{ad } x, \text{ad } y]_N = [x, y]', x, y \in g;\)

2. \(T_N(\text{ad } x, \text{ad } y) = 0, x, y \in g.\)

\(Ad\ 1.\) By definition, \(N(\text{ad } x) = \text{ad }' x.\) Thus \([\text{ad } x, \text{ad } y]_N = [N(\text{ad } x), \text{ad } y] + [\text{ad } x, N(\text{ad } y)] - N[\text{ad } x, \text{ad } y] = [\text{ad }' x, \text{ad } y] + [\text{ad } x, \text{ad }' y] - N\text{ad }[x, y] = [\text{ad }' x, \text{ad } y] + [\text{ad } x, \text{ad }' y] - \text{ad }'[x, y].\) The last expression is equal to \(\text{ad } [x, y]'\) due to the compatibility of the brackets \([\cdot, \cdot], [\cdot, \cdot]'\) (see Lemma 2.5).

\(Ad\ 2.\) \(T_N(\text{ad } x, \text{ad } y) = [\text{Nad } x, \text{Nad } y] - N[\text{ad } x, \text{ad } y]_N = [\text{ad }' x, \text{ad }' y] - \text{Nad }[x, y]' = [\text{ad }' x, \text{ad }' y] - \text{ad }'[x, y]' = 0.\)

The construction from the previous section gives a WNO \(\tilde{W} : \tilde{g} \rightarrow \tilde{g}, \tilde{W} := N_0 = p_0 \circ N.\) Let us denote by \(W\) the corresponding WNO on \(g\) induced by \(\tilde{W}\) and by the isomorphism of Lie algebras \(\text{ad} : (g, [\cdot, \cdot]) \rightarrow (\tilde{g}, [\cdot, \cdot]),\) in other words, \(W(\text{ad } x) = \text{ad } W x, x \in g.\) It follows from the lemma above that under the identification ”\(\text{ad}\)” the second bracket \([\cdot, \cdot]'\) corresponds to the bracket \([\cdot, N] = [\cdot, \tilde{W}]\). Hence \([\cdot, \cdot]' = [\cdot, \cdot]_W\).

Summarizing we get the first assertion of the following theorem.
7.3. **Theorem**  
1. Given a semisimple bi-Lie structure \((\mathfrak{g}, [\cdot, \cdot], [\cdot, \cdot]')\), the operator \(W = \text{ad}^{-1} \circ p_0 \circ \text{ad}': \mathfrak{g} \to \mathfrak{g}\), where \(p_0: \mathcal{G} \to \tilde{\mathfrak{g}}\) is the orthogonal projection with respect to \(B\), is a WNO on the Lie algebra \((\mathfrak{g}, [\cdot, \cdot])\) such that

\([\cdot, \cdot]' = [\cdot, \cdot]W\).

2. Among all WNOs \(V \in \text{End}(\mathfrak{g})\) such that \([\cdot, \cdot]' = [\cdot, \cdot]V\) there is a unique WNO \(V'\) satisfying the condition \(V' \in \tilde{\mathfrak{g}}^\perp\).

3. \(W = V'\).

We have to justify only items 2,3. If \(V_1: \mathfrak{g} \to \mathfrak{g}\) is a WNO with the property \([\cdot, \cdot]_{V_1} = [\cdot, \cdot]V\), then the difference \(V - V_1\) obviously should be a derivation of the bracket \([\cdot, \cdot]\). Since all the derivations of \([\cdot, \cdot]\) are inner, \(V\) and \(V_1\) should differ by some \(\text{ad}x, x \in \mathfrak{g}\), i.e. by an element of \(\tilde{\mathfrak{g}} = \mathfrak{g}_0\). Now the decomposition property \((6.1.1)\) implies the uniqueness of WNO belonging to \(\tilde{\mathfrak{g}}^\perp = \mathfrak{g}_1\).

Item 3 follows from Corollary \(7.8\) below. \(\square\)

7.4. **Definition**  
Let \((\mathfrak{g}, [\cdot, \cdot])\) be a semisimple Lie algebra. An operator \(V \in \text{End}(\mathfrak{g})\) is said to be principal if \(V \in \tilde{\mathfrak{g}}^\perp\), here \(\tilde{\mathfrak{g}} = \text{ad} \mathfrak{g}\).

Given a semisimple bi-Lie structure \((\mathfrak{g}, [\cdot, \cdot], [\cdot, \cdot]')\), any \(W \in \text{End}(\mathfrak{g})\) such that \([\cdot, \cdot]' = [\cdot, \cdot]W\) will be called a WNO of the bi-Lie structure and the unique (in view of Theorem \(7.3\)) WNO of the bi-Lie structure belonging to \(\mathfrak{g}^\perp\) will be called the principal WNO of this bi-Lie structure.

The next result gives an explicit expression of the orthogonal decomposition \(N = N_0 + N_1\) (cf. Section \(6\)) for \(N = \text{ad}'(\text{ad})^{-1}\) in terms of the operator \(W\) and calculates one of the primitives of the WNO \(W\).

7.5. **Theorem**  
1. The action of the operator \(N : \tilde{\mathfrak{g}} \to \mathcal{G}\) can be given by the following expression: \(N(\text{ad}x) = \text{ad}Wx + [\text{ad}x, W]\); moreover, \(N_0(\text{ad}x) = \tilde{W}(\text{ad}x) = \text{ad}Wx, N_1(\text{ad}x) = [\text{ad}x, W], x \in \mathfrak{g}\) (in other words \(N_1 = L|_{\tilde{\mathfrak{g}}}\), where \(L : \mathcal{G} \to \mathcal{G}\) is equal to \(-\text{ad} \mathfrak{g} W\)).

2. The operator \(\tilde{P} := (1/2)p_0 \circ \text{ad}^2 \tilde{\mathfrak{g}} : \tilde{\mathfrak{g}} \to \tilde{\mathfrak{g}}\) is symmetric with respect to the Killing form \(B_\tilde{\mathfrak{g}}\) and is a primitive of the WNO \(\tilde{W}\). The formula \(P = \text{ad}^{-1} \circ \tilde{P} \circ \text{ad}\) gives a primitive of the WNO \(W\), which is symmetric with respect to the Killing form \(B_\mathfrak{g}\).

By the definition of \(N\) for any \(x, y \in \mathfrak{g}\) we have \((N(\text{ad}x))y = \text{ad}'x(y) = [x, y]'\). Moreover, we know that \([\cdot, \cdot]' = [\cdot, \cdot]W\), hence \((N(\text{ad}x))y = [Wx, y] + [x, Wy] - W[x, y]\). Rewrite this equality as \((N(\text{ad}x))y = \text{ad}Wx(y) + [\text{ad}x, W]y\). Since the first term of this expression is equal to \((\tilde{W}(\text{ad}x))y\) (by the definition of \(W\)), the second one has to be equal \(((N - \tilde{W})(\text{ad}x))y = (N_1(\text{ad}x))y\). The rest of the proof follows from Lemma \(6.4\) \(\square\)

7.6. **Definition**  
The operator \(P\) defined in Theorem \(7.5\) will be called the principal primitive of the principal WNO \(W\) (note that the condition of being symmetric distinguishes the principal primitive uniquely among all primitives of the WNO \(W\)).

7.7. **Corollary**  
An element \(x \in \mathfrak{g}\) belongs to the centre of a Lie bracket \([\cdot, \cdot]'\) if and only if \(x \in \ker W\) and \([\text{ad}x, W] = 0\), where \(W\) is the principal WNO of the bi-Lie structure \((\mathfrak{g}, [\cdot, \cdot], [\cdot, \cdot]')\).
Indeed, by definition the images of the operators \(\widetilde{W}, N_1\) are mutually orthogonal. Thus \(0 = \text{ad}' x = N(\text{ad} x)\) if and only if \(\widetilde{W}(\text{ad} x) = 0\) and \(N_1(\text{ad} x) = 0\) if and only if \(Wx = 0\) and \([\text{ad} x, W]\) = 0. □

7.8. **Corollary**  The endomorphism \(W \in \text{End}(\mathfrak{g})\) defined before Theorem 7.3 belongs to \(\mathfrak{g}_1 = \mathfrak{g}^\perp\).

We know from Theorem 7.5 that \([\text{ad} x, W] \in \mathfrak{g}_1\) for any \(x \in \mathfrak{g}\), i.e. \(B(\text{ad} y, [\text{ad} x, W]) = 0\) for any \(x, y \in \mathfrak{g}\). By the invariance property of \(B\) we have \(B(W, [\text{ad} x, \text{ad} y]) = 0\). However, \((\mathfrak{g}, [\cdot, \cdot])\) is semisimple and coincides with its commutant, hence the last equality means that \(W \in \mathfrak{g}_1\). □

7.9. **Corollary**  Let \((\mathfrak{g}, [\cdot, \cdot], \mathfrak{g}_1)\) and \((\mathfrak{g}, [\cdot, \cdot], \mathfrak{g}_2)\) be two bi-Lie structures with semisimple \((\mathfrak{g}, [\cdot, \cdot])\) and let \(W_1, W_2\) be the corresponding principal WNOs on \(\mathfrak{g}\). Then the bi-Lie structures are strongly isomorphic (see Definition 2.6) if and only if there exists an automorphism \(\varphi\) of the Lie algebra \((\mathfrak{g}, [\cdot, \cdot])\) with the property \(\varphi \circ W_1 = W_2 \circ \varphi\).

Assume first that \(\varphi\) is an automorphism with the prescribed property. Then direct calculation shows that it transforms the bracket \([\cdot, \cdot]_{W_1}\) to the bracket \([\cdot, \cdot]_{W_2}\).

Now assume \(\varphi\) is an automorphism of \((\mathfrak{g}, [\cdot, \cdot])\) with the property \([\cdot, \cdot]_{W_1} = [\varphi \circ [\cdot, \cdot]]_{W_2}\).

Note that, given Lie algebras \((\mathfrak{g}, [\cdot, \cdot]), (\mathfrak{g}, [\cdot, \cdot])\), a linear automorphism \(\varphi \in \text{End}(\mathfrak{g})\) transforms the bracket \([\cdot, \cdot]\) to \([\cdot, \cdot]\) if and only if \(\varphi \circ \text{ad} x \circ \varphi^{-1} = \text{ad} \varphi x\) for any \(x \in \mathfrak{g}\), here \(\text{ad}_1 x(y) := [x, y]\).

Thus, by the assumption, \(\varphi \circ \text{ad} x \circ \varphi^{-1} = \text{ad} \varphi x\) and

\[
\varphi \circ \text{ad} W_1 x \circ \varphi^{-1} = \text{ad} W_2 \varphi x, \tag{7.9.1}
\]

here \(\text{ad} W_1 x(y) := [x, y]_{W_1}\).

We know that \(\text{ad} W_2 \varphi x = \text{ad} W_2 \varphi x + [\text{ad} \varphi x, W_2]\), where the first term belongs to \(\mathfrak{g}\) and the second one to \(\mathfrak{g}^\perp\).

On the other hand, \(\varphi \circ \text{ad} W_1 x \circ \varphi^{-1} = \varphi \circ \text{ad} W_1 x \circ \varphi^{-1} + \varphi \circ [\text{ad} x, W_1] \circ \varphi^{-1} = \text{ad} \varphi W_1 x + [\text{ad} \varphi x, \varphi \circ W_1 \circ \varphi^{-1}]\). We claim that the first term belongs to \(\mathfrak{g}\) (which is obvious) and the second one to \(\mathfrak{g}^\perp\).

Indeed \(B(\text{ad} x, \varphi \circ W_1 \circ \varphi^{-1}) = \text{Tr}(\text{ad} x \circ \varphi \circ W_1 \circ \varphi^{-1}) = \text{Tr}(\varphi^{-1} \circ \text{ad} x \circ \varphi \circ W_1) = B(\varphi^{-1} x, W_1)\).

The last expression is equal to zero by Corollary 7.8. Thus \(\varphi \circ W_1 \circ \varphi^{-1} \in \mathfrak{g}^\perp\) and by Lemma 6.1 \([\text{ad} \varphi x, \varphi \circ W_1 \circ \varphi^{-1}] \in \mathfrak{g}^\perp\).

Hence equality (7.9.1) implies the equality of 0-components, \(\text{ad} W_2 \varphi x = \text{ad} \varphi W_1 x\), for any \(x \in \mathfrak{g}\), which, in view of the injectivity of \(\text{ad}\), completes the proof. □

Let us summarize the discussion above in the following theorem.

7.10. **Theorem**  There is a one-to-one correspondence between semisimple bi-Lie structures \((\mathfrak{g}, [\cdot, \cdot], [\cdot, \cdot])\) and principal WNOs \(W \in \text{End}(\mathfrak{g})\) given by \([\cdot, \cdot]' = [\cdot, \cdot]_W\). Two semisimple bi-Lie structures \((\mathfrak{g}, [\cdot, \cdot], [\cdot, \cdot])\) and \((\mathfrak{g}, [\cdot, \cdot], [\cdot, \cdot])\), with the same underlying Lie algebra are (strongly) isomorphic (see Definition 2.6) if and only if the corresponding principal WNOs \(W_1, W_2\) are (strongly) equivalent in the sense of the following definition.

7.11. **Definition**  Let \((\mathfrak{g}, [\cdot, \cdot])\) be a semisimple Lie algebra and let \(W, W_1 \in \text{End}(\mathfrak{g})\) be two principal WNOs. We say that they are strongly equivalent (equivalent) if there exists an automorphism \(\varphi\) of the Lie algebra \((\mathfrak{g}, [\cdot, \cdot])\) such that \(\varphi \circ W \circ \varphi^{-1} = W_1\) (respectively \(\varphi \circ W \circ \varphi^{-1} = \lambda W_1 + \lambda' \text{Id}_\mathfrak{g}\) for some \(\lambda, \lambda' \in \mathbb{C}\)).
We conclude this section by an example of a principal WNO.

7.12. Example Consider the operator $W$ from Example 4.12. We claim that it is the principal WNO for the corresponding bi-Lie structure $(\mathfrak{so}(n, \mathbb{C}), [,], [,])$. Indeed $\text{Tr}(W(X)Y) = (1/2) \text{Tr}((AX + XA)Y) = (1/2) \text{Tr}(X(AY + YA)) = \text{Tr}(XW(Y))$, i.e. the operator $W$ is symmetric with respect to the Killing form $B_{\mathfrak{so}(n, \mathbb{C})}$ which is proportional to the trace form. It remains to use the following lemma.

7.13. Lemma Let an operator $W \in \text{End}(\mathfrak{g})$ be given on a Lie algebra $(\mathfrak{g}, [,])$ such that $W$ is symmetric with respect to the Killing form $B_{\mathfrak{g}}$. Then $W$ is orthogonal to any antisymmetric operator with respect to the trace form $B$ on $\text{End}(\mathfrak{g})$, in particular $W$ is principal.

Let $A : \mathfrak{g} \to \mathfrak{g}$ be an antisymmetric operator. If $e_1, \ldots, e_n$ is an orthonormal with respect to $B_{\mathfrak{g}}$ basis in $\mathfrak{g}$, the trace of any operator $L$ can be calculated as $\text{Tr}(L) = \sum_{i=1}^{n} B_{\mathfrak{g}}(Le_i, e_i)$. Now, $\text{Tr}(WA) = \sum_{i=1}^{n} B_{\mathfrak{g}}(WA e_i, e_i) = - \sum_{i=1}^{n} B_{\mathfrak{g}}(e_i, AW e_i) = - \text{Tr}(AW) = - \text{Tr}(WA)$. Hence $\text{Tr}(WA) = 0$. □

8 Kernels of the Killing forms of the exceptional brackets and their centres

In this section we assume that $(\mathfrak{g}, [,], [\cdot, \cdot])$ is a semisimple bi-Lie structure and $W : \mathfrak{g} \to \mathfrak{g}$ is its principal WNO. As usual, we will write $B$ for the trace form on $\mathcal{G} := \text{End}(\mathfrak{g})$ and $B_{\mathfrak{g}}$ for the Killing form of the Lie algebra $(\mathfrak{g}, [,])$.

Consider the bracket $[[\cdot, \cdot]] := [[\cdot], -]\cdot = [[\cdot]]_{\mathfrak{g}^t}$, $t \in \mathbb{C}$, here $W^t := W - t \text{Id}_{\mathfrak{g}}$. Let $\text{ad}^t : \mathfrak{g} \to \mathfrak{g}$ stand for the corresponding bi-Lie structure, $\text{ad}^t(x) = x - [x, \cdot]_{\mathfrak{g}^t}$. This operator can be also written in the form $\text{ad}^t x = \text{ad} W^t x + [\text{ad} x, W]$ (since $[\text{ad} x, W^t] = [\text{ad} x, W]$ for any $x \in \mathfrak{g}$, see Theorem 7.5).

8.1. Theorem The Killing form $B^t$ of the Lie algebra $(\mathfrak{g}, [\cdot, \cdot])$ is given by the formula

$$B^t(x, y) = B_{\mathfrak{g}}(W^t x, W^t y) + b(x, y), \quad x, y \in \mathfrak{g},$$

where $b$ is a symmetric form on $\mathfrak{g}$ equal to

$$b(x, y) = B([\text{ad} x, W], [\text{ad} y, W]) = - 2 B_{\mathfrak{g}}(Px, y);$$

here $P$ is the principal primitive of the principal WNO $W$ (see Definition 7.6).

By definition $B^t(x, y) = B(\text{ad}^t x, \text{ad}^t y) = B(\text{ad} W^t x + [\text{ad} x, W], \text{ad} W^t y + [\text{ad} x, W]) = B(\text{ad} W^t x, \text{ad} W^t y) + B([\text{ad} x, W], [\text{ad} y, W]) = B_{\mathfrak{g}}(W^t x, W^t y) + b(x, y)$. Here we used the fact that the endomorphisms $\text{ad} W^t z, z \in \mathfrak{g}$, and $[\text{ad} z, W]$ are mutually orthogonal with respect to $B$ (see Corollary 7.8 and Lemma 6.1). The equality $b(x, y) = - 2 B_{\mathfrak{g}}(P x, y)$ follows from the definition of $P$ and from the invariance of the form $B$. □

The theorem above implies (due to the nondegeneracy of $B_{\mathfrak{g}}$) that an element $x \in \mathfrak{g}$ belongs to the kernel $\ker B^t$ of the Killing form $B^t$ for some $t \in \mathbb{C}$ if and only if $x$ belongs to the kernel of the operator $\hat{B}^t := (W^t)^* W^t - 2P \in \text{End}(\mathfrak{g})$, or in more details,

$$\hat{B}^t = W^* W - 2P - t(W + W^*) + t^2 \text{Id}_{\mathfrak{g}};$$

(8.1.1)

here $L^*$ stands for the operator adjoint to an operator $L$ with respect to $B_{\mathfrak{g}}$. In particular, the set $T = \{t \in \mathbb{C} \mid \ker B^t \neq \{0\}\}$ is finite.
8.2. Definition The elements of the set \( T = \{ t_1, t_2, \ldots \} \) will be called the \textit{times} of a bi-Lie structure \((g, [\cdot, \cdot], [\cdot, \cdot]')\) and the corresponding brackets \([\cdot, \cdot]'\) will be called \textit{exceptional}.

The following result describes the centres of the exceptional brackets.

8.3. Theorem 1. An element \( x \in g \) belongs to the centre \( z \) of the bracket \([\cdot, \cdot]' := [\cdot, \cdot]' - t[\cdot, \cdot] = [\cdot, \cdot]_W'\) if and only if \( x \) is an eigenvector of the principal WNO \( W \) corresponding to the eigenvalue \( t \) and \([\text{ad} x, W] = 0\).

2. The subset \( z \) is a subalgebra in \((g, [\cdot, \cdot])\) for any \( t \) (this result is known [TF95, §44]);

3. \( z_{t_1} \cap z_{t_2} = \{ 0 \} \) if \( t_1 \neq t_2 \);

4. \([z_{t_1}, \ker W_{t_2}] \subset \ker W_{t_2} \) for any \( t_1, t_2 \in \mathbb{C} \);

5. \([z_{t_1}, z_{t_2}] = 0 \) if \( t_1 \neq t_2 \); in particular, the set \( z := z_{t_1} \oplus \cdots \oplus z_{t_m} \) is a subalgebra in \((g, [\cdot, \cdot])\) which is a direct sum of its ideals \( z_{t_i} \), here \( \Theta := \{ \theta_1, \ldots, \theta_m \} = \{ \theta \in \mathbb{C} \mid z_{\theta} \neq \{ 0 \} \} \subset T \).

Ad 1. The proof is a direct consequence of Corollary [77] and the fact that \([\text{ad} x, W_t] = [\text{ad} x, W] \).

Ad 2. For any \( x, y \in z \) we have \( 0 = [x, y]_{W_{t_1}} = -W_{t_1} [x, y] \), hence \([x, y] \in \ker W_{t_1} \). On the other hand, since \([\text{ad} x, W] = 0 = [\text{ad} y, W] \), then \([\text{ad} x, y], W] = [[\text{ad} x, y], W] = -[[W, \text{ad} x], y] - \text{[ad} y, W], ax = 0 \). Thus by Item 1 \([x, y] \in z \).

Ad 3. The proof follows from the inclusion \( z_{t_1} \subset \ker W_{t_2} \).

Ad 4. Let \( x \in z_{t_1}, y \in \ker W_{t_2} \). Then \( 0 = [x, y]_{W_{t_1}} = [x, W_{t_1} y] = [x, W_{t_2} y] + (t_2 - t_1)[x, y] = W_{t_2} [x, y] = W_{t_2} [x, y] = W_{t_1} [x, y] = W_{t_1} [x, y] = W_2 [x, y] - W_2 [x, y] \). Hence \([x, y] \in \ker W_{t_2} \).

Ad 5. We have \([z_{t_1}, z_{t_2}] \subset [z_{t_1}, \ker W_{t_2}] \subset \ker W_{t_2} \) by Item 4 and, analogously, \([z_{t_1}, z_{t_2}] \subset [\ker W_{t_1}, z_{t_2}] \subset \ker W_{t_1} \). Hence \([z_{t_1}, z_{t_2}] \subset \ker W_{t_1} \cap \ker W_{t_2} = \{ 0 \} \). The inclusion \( \Theta \subset T \) is obvious, since the centre of any Lie algebra is contained in the kernel of the Killing form. \( \square \)

8.4. Definition The subalgebra \( \mathfrak{z} \subset g \) equal to the sum of the centres of the exceptional brackets will be called the \textit{central subalgebra} of a bi-Lie structure \((g, [\cdot, \cdot], [\cdot, \cdot]')\).

We conclude this section by introducing a series of another subalgebras of \((g, [\cdot, \cdot])\).

8.5. Theorem Let \((g, [\cdot, \cdot], [\cdot, \cdot]')\) be a semisimple bi-Lie structure, \( W : g \to g \) be its principal WNO, and \( P \) its pricipal primitive.

1. The sets \( g_{P} := \{ x \in g \mid [\text{ad} x, P] = 0 \} \), \( \hat{z} := g_{P} \cap \ker P \) are subalgebras in \((g, [\cdot, \cdot])\) and \( \hat{z} \subset \mathfrak{z} \subset g_{P} \).

2. If \( W \) is diagonalizable, then \( \hat{z} := \hat{z} \cap \ker W_{t} \) also is a subalgebra for any \( t \in \mathbb{C} \), \( \hat{z} \subset \hat{z} \) and \( \hat{z}_{t_1} \cap \hat{z}_{t_2} = \{ 0 \} \) if \( t_1 \neq t_2 \). If, moreover, \( W_{\hat{z}} \subset \hat{z} \), then \( \hat{z} = \hat{z}_{t_1} \oplus \cdots \oplus \hat{z}_{t_k} \) for some \( t_i \in T \) and \( \hat{z}_{t_i} + \hat{z}_{t_j} \) is a subalgebra for any \( i, j \).
Ad 1. By the Jacobi identity \([\text{ad } [x,y], P] = [[\text{ad } x, P], \text{ad } y] - [[\text{ad } y, P], \text{ad } x]\). Thus \(x, y \in \mathfrak{g}_P\) implies \([x, y] \in \mathfrak{g}_P\). If \(x, y \in \mathfrak{j}\), then \(P[x, y] = \text{Pad } x(y) = \text{ad } x(Py) = 0\) and \([x, y] \in \mathfrak{j}\).

To prove the inclusion \(\mathfrak{j} \subset \mathfrak{j}\) rewrite the equality \(T_W(x, y) = [x,y]_P\) in a different way. Namely, fix \(x\) and consider left and right hand sides of this equality as operators on the second argument. Then we get \(T_W(x, \cdot) = [Wx, W]\) - \(W([x, \cdot]_W) = (\text{ad } Wx \circ W - W \circ (\text{ad } Wx + [ad x, W]))(\cdot) = ([ad Wx, W] - W \circ [ad x, W])(\cdot)\). Analogously, \([x, \cdot]_P = (P x + [ad x, P])(\cdot)\). Finally,

\[
[\text{ad } Wx, W] - W \circ [ad x, W] = \text{ad } P x + [ad x, P].
\]

(8.5.1)

Note also that \(T_W = T_W^t\) for any \(t \in \mathbb{C}\), hence \([\text{ad } W^t x, W] - W^t \circ [ad x, W] = \text{ad } P x + [ad x, P]\). If \(x \in \mathfrak{j}\), then by Theorem 8.3 \(x \in \ker W^t\) for some \(t\) and \([ad x, W] = 0\), whence \(\text{ad } P x + [ad x, P] = 0\). Now it is enough to use Lemmata 6.1, 7.1 to deduce that \(\text{ad } P x \in \mathfrak{g}, [ad x, P] \in (ad \mathfrak{g})^\perp\) have to be zero and that \(x \in \mathfrak{j}\). The same argument proves the inclusion \(\mathfrak{j}^t \subset \mathfrak{j}\).

Ad 2. Let \(x, y \in \mathfrak{j}\). Then \(T_W(x, y) = t^2[x,y] - W(2t[x,y] - W[x,y]) = (W^t)^2[x,y]\) (since \(x, y \in \ker W^t\)). On the other hand, \(T_W(x, y) = \text{ad } P x(y) + [ad x, P]y = 0\) (since \(x \in \mathfrak{j}\)). The diagonalizability of \(W\) implies \([x, y] \in \ker W^t\).

The equality \(\mathfrak{j}^t_1 \cap \mathfrak{j}^t_2 = \{0\}\) follows from the equality \(\ker W^t_1 \cap \ker W^t_2 = \{0\}\).

If \(\mathfrak{j}\) is \(W\)-invariant, we have a direct decomposition \(\mathfrak{j} = \mathfrak{j}^t_1 \oplus \cdots \oplus \mathfrak{j}^t_k\). To prove the last assertion it is enough to observe that \(W|\mathfrak{j}\) due to formula (8.5.1) has zero torsion, i.e. is Nijenhuis. The needed property is a characteristic property of any diagonalizable Nijenhuis operator [Pan06]. □

9 Operators preserving a grading

We will consider \(\Gamma\)-gradings of a Lie algebra \(\mathfrak{g}\), i.e. direct decompositions \(\mathfrak{g} = \bigoplus_{i \in \Gamma} \mathfrak{g}_i\) such that \([\mathfrak{g}_i, \mathfrak{g}_j] \subset \mathfrak{g}_{i+j}\) for any \(i, j \in \Gamma\), here \(\Gamma\) is an abelian group and we use the additive notation.

9.1. Definition We say that a linear operator \(W \in \text{End}(\mathfrak{g})\) preserves a \(\Gamma\)-grading \(\mathfrak{g} = \bigoplus_{i \in \Gamma} \mathfrak{g}_i\) if \(W \mathfrak{g}_i \subset \mathfrak{g}_i\) for any \(i \in \Gamma\).

The first two items of the following theorem are standard. Let \(\mathfrak{g}\) be a semisimple Lie algebra with the Killing form \(B_\mathfrak{g}\) and let \(\mathfrak{g} = \bigoplus_{i \in \Gamma} \mathfrak{g}_i\) be a \(\Gamma\)-grading.

9.2. Theorem 1. \(B_\mathfrak{g}(\mathfrak{g}_i, \mathfrak{g}_j) = 0\) if \(i + j \neq 0\).

2. The restriction of \(B_\mathfrak{g}\) to \(\mathfrak{g}_i \times \mathfrak{g}_j\) is a nondegenerate pairing for all \(i, j \in \Gamma\), \(i + j = 0\). In particular so is the restriction of \(B_\mathfrak{g}\) to \(\mathfrak{g}_0\).

3. If \(W \in \mathcal{G} = \text{End}(\mathfrak{g})\) preserves the grading, then the operator \(\tilde{\mathcal{P}} \in \text{End}(\tilde{\mathcal{G}}), \tilde{\mathcal{G}} := \text{ad } \mathfrak{g}\), given by the formula \(\tilde{\mathcal{P}} := (1/2)p_0 \circ \text{ad } \mathfrak{g}^2 W\) preserves the induced grading on \(\tilde{\mathcal{G}}\), here \(p_0 : \mathcal{G} \rightarrow \tilde{\mathcal{G}}\) is the orthogonal projection onto \(\text{ad } \mathfrak{g}\) with respect to the trace form \(B\). Consequently, the operator \(\mathcal{P} := \text{ad }^{-1} \circ \tilde{\mathcal{P}} \circ \text{ad}\) preserves the initial grading on \(\mathfrak{g}\).

4. If \(W \in \text{End}(\mathfrak{g})\) and, moreover, \(W|\mathfrak{g}_i = \lambda_i \text{Id } \mathfrak{g}_i\), \(\lambda_i \in \mathbb{C}\), for any \(i \in \Gamma\), then the adjoint with respect to \(B_\mathfrak{g}\) operator \(W^*\) is given by \(W^*|\mathfrak{g}_i = \lambda_{-i} \text{Id } \mathfrak{g}_i\).
Ad 1. Let $x_i \in g_i$, $x_j \in g_j$. Consider a basis of $g$ which can be divided to parts each of which forms a basis of one of the subspaces $g_i$. Let $e^1, \ldots, e^m$ be such a part which is a basis of $g_k$. Then for $A := \text{ad} x_i \circ \text{ad} x_j$ we have $A(e^i) \notin g_k$ as soon as $i + j \neq 0$. Hence the matrix of $A$ in such a basis has zero diagonal entries and $\text{Tr}(A) = B_g(x_i, x_j) = 0$.

Ad 2. Now let $x \in g_i$ and $i + j = 0$. Then there exists an element $y \in g_j$ such that $B_g(x, y) \neq 0$. Indeed, otherwise $x$ would be orthogonal to all the space $g$ by Item 1.

Ad 3. Let $x_{i'} \in g_i$, $x_{i''} \in g_{i''}$. Since $W$ preserves the grading, we have $\text{ad}_{g}^2 W(\text{ad} x_{i'})(x_{i''}) = (W^2 \circ \text{ad} x_{i'} - 2W \circ \text{ad} x_{i'} \circ W + \text{ad} x_{i'} \circ W^2)(x_{i''}) \in g_{i'+i''}$.

By Items 1, 2 we can choose a hyperbolic basis $e_1^1, \ldots, e_{m_i}^i, e_{m_i}^{-1}, \ldots, e_{m_i}^{-m_i} \in g_i$, in each $g_i + g_{-i}$ such that $2i \neq 0$. In $g_i$ with $2i = 0$ we will choose an orthonormal basis $e_1^i, \ldots, e_{m_i}^i$. The nonzero diagonal entries of the matrix of the endomorphism $\text{ad} e_{i_1}^j \circ X_{i_2}^{j'} \in g_{-i_1 + i_2 + i'}$. The nonzero diagonal entries of the matrix of the endomorphism $\text{ad} e_{i_1}^j \circ X_{i_2}^{j'}$ in the basis $\{e_i^j\}$ can appear only if $i' = i$. Thus $x_{i_1''}^{j''} = B(\text{ad} e_{i_1}^j, X_{i_2}^{j'}) = 0$ if $i' \neq i$.

Summarizing, we get $\tilde{P}(\text{ad} e_{i_1}^j) = (1/2) \sum_{i, j} x_{i_1''}^{j''} \text{ad} e_i^j$, where $x_{i_1''}^{j''} = 0$ for $i \neq i'$, which shows the invariance of $g_i$ with respect to $\tilde{P}$.

Ad 4. It is enough to consider the operator $W$ in the basis built in the proof of Item 3. □

9.3. COROLLARY Let $W$ be a principal WNO of a semisimple bi-Lie structure (see Definition 7.4).

1. Assume that $W$ preserves a $\Gamma$-grading on $g$. Then its principal primitive $P$ (see Definition 7.6) also preserves this grading.

2. If the initial grading is the root decomposition grading with respect to a Cartan subalgebra $h \subset g$, $g = h + \sum_{\alpha \in R} g_{\alpha}$, (9.3.1)

(here $R$ stands for the set of roots) and $W$ preserves this grading, then $P|_{\mathfrak{g}_{\alpha} + \mathfrak{g}_{-\alpha}} = \pi_{\alpha} \text{Id}_{\mathfrak{g}_{\alpha} + \mathfrak{g}_{-\alpha}}$, $\alpha \in R$, for some $\pi_{\alpha} \in \mathbb{C}$ and $P h \subset h$.

9.4. THEOREM Let $W \in \mathcal{G} = \text{End}(g)$ preserve the root grading (9.3.1), $W|_{\mathfrak{g}_{\alpha}} = \lambda_{\alpha} \text{Id}_{\mathfrak{g}_{\alpha}}, \kappa_{\alpha} := (1/2)(\lambda_{\alpha} - \lambda_{-\alpha})$. Then the operator $\text{pr}(W) := p_1(W) \in \text{End}(g)$, where $p_1 : \mathcal{G} \to \mathfrak{g}^\perp$ is the orthogonal projection onto $\mathfrak{g}^\perp$ along $\mathfrak{g}$, is given by the formula

$$\text{pr}(W) = W - \text{ad} \left( \sum_{\alpha \in R} \lambda_{\alpha} H_{\alpha} \right) = W - \text{ad} \left( 2 \sum_{\alpha \in R^+} \kappa_{\alpha} H_{\alpha} \right),$$

here $R^+$ stands for the set of positive roots with respect to any basis and $H_{\alpha} \in h$ is given by $B_g(H_{\alpha}, H) = \alpha(H), H \in h$. The operator $\text{pr}(V)$ will be called the principal projection of $V$. In particular, $\text{pr}(W) = W$ (i.e. $W$ is the principal operator in the sense of Definition 7.4) if and only if $\sum_{\alpha \in R^+} \kappa_{\alpha} \alpha = 0$. 

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We have to prove that the operator \( \text{ad}(2 \sum_{\alpha \in R^+} \kappa_\alpha H_\alpha) \) is equal to \( p_0(W) \), where \( p_0 : G \to \tilde{g} \) is the orthogonal projection. To this end choose a basis in \( g \) as in the proof of Item 3 of Theorem 9.2 i.e. an orthonormal basis \( e_1, \ldots, e_n \) in \( h \) and an element \( E_\alpha \in g_\alpha \) for any \( \alpha \in R \) such that \( B_\theta(E_\alpha, E_{-\alpha}) = 1 \). Moreover, let \( f_1, \ldots, f_k \) be any basis in \( \tilde{g}^\perp \). Then we have to calculate the coefficients \( a_i, b_\alpha \) of the decomposition \( W = \sum_i a_i \text{ad} e_i + \sum_\alpha b_\alpha \text{ad} E_\alpha + \sum_j f_j \). Since \( \text{ad} e_i, \text{ad} E_\alpha \) are mutually orthogonal, we have \( a_i = B(\text{ad} e_i, W), b_\alpha = B(\text{ad} E_\alpha, W) \). The operator \( \text{ad} E_\alpha \) does not preserve any of the subspaces \( h, g_\alpha \), hence \( b_\alpha = \text{Tr}(\text{ad} E_{-\alpha} \circ W) = 0 \).

To calculate \( a_i \), remark that \( \text{ad} e_i \mid_{g_\alpha} = 0, \text{ad} e_i \mid_{g_\alpha} = \alpha(e_i) \text{Id}_{g_\alpha} \), whence \( a_i = \text{Tr}(\text{ad} e_i \circ W) = \sum_{\alpha \in R} \alpha(e_i) \lambda_\alpha \). Thus \( p_0(W) = \sum_i a_i \text{ad} e_i = \sum_i \sum_{\alpha \in R} \alpha(e_i) \lambda_\alpha \text{ad} e_i = \text{ad} \sum_{\alpha \in R} \lambda_\alpha H_\alpha = 2\text{ad} \sum_{\alpha \in R^+} \kappa_\alpha H_\alpha \). The last equality is due the fact that \( H_\alpha = -H_{-\alpha} \). □

## 10 Regular semisimple bi-Lie structures

Let \((g, [\cdot, \cdot], [\cdot, \cdot]')\) be a semisimple bi-Lie structure, \( W \in \text{End}(g) \) its principal WNO (see Definition 7.4). Let \( h \subset g \) be a Cartan subalgebra of the semisimple Lie algebra \((g, [\cdot, \cdot])\) and \( R = R(g, h) \subset h_\mathbb{R} \) be the corresponding root system.

### 10.1. Theorem

The following two conditions are equivalent:

1. The operator \( W \in \text{End}(g) \) preserves the root grading (9.3.1) and the operator \( W \mid_h \) is diagonalizable.

2. \( h \subset \mathfrak{z} \), where \( \mathfrak{z} \) is the central subalgebra (see Definition 8.4).

If the conditions above are satisfied the central subalgebra \( \mathfrak{z} \subset g \) is homogeneous with respect to the decomposition (9.3.7), i.e.

\[ \mathfrak{z} = h + \sum_{\alpha \in R} \mathfrak{z} \cap g_\alpha. \]

Assume first that \( h \subset \mathfrak{z} \). Item 1 of Theorem 8.3 implies that \([\text{ad} x, W] = 0 \) for any \( x \in h \). Let \( R = \{ \alpha_1, \ldots, \alpha_k \} \) (arbitrary ordering of nonzero roots). Choose a basis in \( g \) of the form \( e_1, \ldots, e_l \), \( E_{\alpha_1}, \ldots, E_{\alpha_k} \), \( E_{\alpha_i} \in g_{\alpha_i} \), where \( e_1, \ldots, e_l \) is any basis in \( h \). The matrix of the endomorphism \( \text{ad} x \) in this basis has the block form

\[
\begin{bmatrix}
0 & 0 & 0 \\
0 & D(x) & 0 \\
0 & 0 & M
\end{bmatrix},
\]

where \( D(x) := \text{diag}(\alpha_1(x), \ldots, \alpha_k(x)) \). For generic \( x \in h \) the numbers \( \alpha_1(x), \ldots, \alpha_k(x) \) are pairwise distinct, hence the matrix of \( W \) in the same basis is of the form

\[
\begin{bmatrix}
M & 0 \\
0 & D
\end{bmatrix},
\]

where \( M \) is an \( l \times l \) matrix and \( D \) is a diagonal \( k \times k \) matrix.

On the other hand, Item 1 of Theorem 8.3 implies that \( \mathfrak{z}^\theta_i \subset \ker W^\theta_i \), i.e. \( \mathfrak{z} = \mathfrak{z}^\theta_1 \oplus \cdots \oplus \mathfrak{z}^\theta_m \) is a sum of eigenspaces of \( W \) containing a \( W \)-invariant subspace \( h \). Hence \( h \) also is a direct sum of eigenspaces of \( W \).

Vice versa, assume condition 1 holds. Then, obviously, \([\text{ad} x, W] = 0 \) for any \( x \in h \) and \( h \) is a direct sum of eigenspaces of \( W \), hence by Theorem 8.3 \( x \in \mathfrak{z} \).

The proof of the last statement also follows from this theorem. □
10.2. **Definition**  A semisimple bi-Lie structure \((\mathfrak{g},[,],[-])\) with the central subalgebra \(\mathfrak{z}\) will be called **regular** if there exists a Cartan subalgebra \(\mathfrak{h}\) of the semisimple Lie algebra \((\mathfrak{g},[\cdot,\cdot])\) such that \(\mathfrak{h} \subset \mathfrak{z}\). (This terminology is motivated by the fact that the central subalgebra of a regular bi-Lie structure is a regular reductive Lie subalgebra, see Theorem 10.3.)

Fix a Cartan subalgebra \(\mathfrak{h} \subset \mathfrak{g}\) and assume that the equivalent conditions of Theorem 10.1 are satisfied for a semisimple bi-Lie structure \((\mathfrak{g},[,],[-])\) with the principal WNO \(W\).

Choose \(E_\alpha \in \mathfrak{g}\), for any \(\alpha \in R\) such that \(B_\mathfrak{g}(E_\alpha,E_-\alpha) = 1\), in particular, \([E_\alpha,E_-\alpha] = H_\alpha\), where \(B_\mathfrak{g}(H_\alpha,H) = \alpha(H)\) for any \(H \in \mathfrak{h}\). Let \(\lambda_\alpha \in \mathbb{C}\) be such that \(W(E_\alpha) = \lambda_\alpha E_\alpha\). We conclude from Corollary 9.3 that for any \(\alpha \in R\) there exists \(\pi_\alpha \in \mathbb{C}\) such that \(\pi_\alpha = \pi_-\alpha\) and \(P(E_\alpha) = \pi_\alpha E_\alpha\), here \(P\) is the principal primitive of the principal WNO \(W\). Moreover, since \([\text{ad} x,W] = 0\) for any \(x \in \mathfrak{z}\), the definition of \(P\) implies that \(P|_\mathfrak{h} = 0\), in particular \(P|_\mathfrak{h} = 0\).

If \(W^*\) stands for the operator adjoint to \(W\) with respect to the Killing form \(B_\mathfrak{g}\), then \(W^*E_\alpha = \lambda_-\alpha E_\alpha\) for any \(\alpha \in R\) by Item 4 of Theorem 9.2. Put \(\sigma_\alpha := (1/2)(\lambda_\alpha + \lambda_-\alpha), \kappa_\alpha := (1/2)(\lambda_\alpha - \lambda_-\alpha)\).

10.3. **Theorem**  Retain the assumptions and notations mentioned. Then

1. For any \(\alpha \in R\) the set of times \(\{t \in T|E_\alpha \in \ker B^t\}\) (see Definition 8.2) coincides with the set \(T_\alpha = \{t_{1,\alpha}, t_{2,\alpha}\}\) of the solutions of the quadratic equation
   \[
   (t - \lambda_\alpha)(t - \lambda_-\alpha) - 2\pi_\alpha = 0.
   \]
   Moreover, \(T_\alpha = T_-\alpha\).

2. For any \(\alpha \in R\)
   \[
   \sigma_\alpha = (1/2)(t_{1,\alpha} + t_{2,\alpha}), \kappa_\alpha \in \sqrt{\zeta_\alpha - 2\pi_\alpha},
   \]
   where \(\zeta_\alpha := ((t_{1,\alpha} - t_{2,\alpha})/2)^2\) and \(\sqrt{a}\) stands for the set of square roots of \(a\).

3. For any \(\alpha \in R\) the following formula is valid:
   \[
   W^{t_{1,\alpha}} W^{t_{2,\alpha}} H_\alpha = 0
   \]
   (recall \(W^t := W - t\text{Id}_\mathfrak{g}\)). Consequently, \(H_\alpha\) is either an eigenvector of \(W\) corresponding to the eigenvalue \(t_{1,\alpha}\), or an eigenvector of \(W\) corresponding to the eigenvalue \(t_{2,\alpha}\), or a sum of such eigenvectors.

4. The centre \(\mathfrak{z}^{\theta_1}\) (see Section 5) of the exceptional bracket \([\cdot,\cdot]^{\theta_1}\) is a regular with respect to \(\mathfrak{h}\) (i.e. stabilized by \(\text{ad} \mathfrak{h}\)) subalgebra reductive in \((\mathfrak{g},[,])\), in particular, \(E_\alpha \in \mathfrak{z}^{\theta_1} \implies E_-\alpha \in \mathfrak{z}^{\theta_1}\).

Moreover, \(\mathfrak{z}^{\theta_1} = \mathfrak{h} \cap \ker W^{\theta_1} + \bigoplus_{\theta \in R^{\theta_1}} E_\alpha\), where \(R^{\theta_1} \subset R\) is the set of roots \(\alpha\) such that \(\lambda_\alpha = \lambda_-\alpha = t_{1,\alpha} = t_{2,\alpha} = \theta_1\) and \(\alpha(z^{\theta_1} \cap \mathfrak{h}) = 0\) for any \(\theta_1 \in \Theta, \theta_j \neq \theta_1\).

In particular, the central subalgebra \(\mathfrak{z} = \mathfrak{z}^{\theta_1} \oplus \cdots \oplus \mathfrak{z}^{\theta_m}\) is a regular with respect to \(\mathfrak{h}\) reductive subalgebra in \((\mathfrak{g},[,])\).

5. The set \(\Theta = \{\theta_1,\ldots,\theta_m\}\) of the times corresponding to the nontrivial centres is equal to the spectrum of the operator \(W|_\mathfrak{h}\).

6. The operator \(W\) preserves the irreducible components of the representation \(x \mapsto \text{ad}_\mathfrak{g} x\) of the Lie algebra \(\mathfrak{z}\) in \(\mathfrak{z}^\perp \subset \mathfrak{g}\) and the restriction of \(W\) to any of them is a scalar operator (here \(\mathfrak{z}^\perp\) is the orthogonal complement in \(\mathfrak{g}\) to \(\mathfrak{z}\) with respect to the Killing form).
Recall the basic equality Ad 3. Indeed, since $WE_α = λ_α E_α$, $W^* E_α = λ_{-α} E_α; PE_α = π_α E_α$, then $\dot{B}^t E_α = [λ_{-α} λ_α - 2π_α - t(λ_α + λ_{-α}) + t^2] E_α = [(t - λ_α)(t - λ_{-α}) - 2π_α] E_α$ (see formula (8.1.1)).

Ad 2. The Viète formulae and Item 1 give

$$t_{1,α} + t_{2,α} = λ_α + λ_{-α}, \quad t_{1,α} t_{2,α} = λ_α λ_{-α} - 2π_α,$$

which implies the formula for $σ_α$ and the formula $(λ_α - λ_{-α})^2 = (t_{1,α} - t_{2,α})^2 - 8π_α$, hence the formula for $κ_α$.

Ad 3. Recall the basic equality $T_W(\cdot, \cdot) = [\cdot, \cdot]_P$ relating the WNO $W$ to its primitive $P$. Calculate this expression on the pair $E_α, E_{-α}$:

$$λ_α λ_{-α} H_α - W(λ_α + λ_{-α} - W) H_α = 2π_α H_α$$

(we used the fact that $PH_α = 0$). Taking into account the Viète formulae above we get $W(t_{1,α} + t_{2,α} - W) H_α = t_{1,α} t_{2,α} H_α$, or finally $(W - t_{1,α})(W - t_{2,α}) H_α = 0$.

Ad 4. Recall that by Theorem 8.3 $E_α \in \mathfrak{z}$ if and only if $[\text{ad } E_α, W] = 0$. To show that $\mathfrak{z}$ is reductive in $\mathfrak{g}$ in view of Bon75 Prop. 2, §3, Ch. VIII it is sufficient to prove that $E_{-α} \in \mathfrak{z}$. Since $WE_{-α} = λ_α E_{-α}$, this last is equivalent to $[\text{ad } E_{-α}, W] = 0$ again by Theorem 8.3.

The equality $[\text{ad } E_α, W] = 0$ is equivalent to the following list of conditions:

a) $0 = [\text{ad } E_α, W] E_β = N_{α,β}(λ_β - λ_{α+β}) E_{α+β}$ for any $β ∈ R$ such that $α + β ∈ R$;

b) $0 = [\text{ad } E_α, W] E_{-α} = (λ_{-α} - W) H_α = -W^λ H_α$;

c) $0 = [\text{ad } E_α, W] H = (λ_α α(H) - α(W H)) E_α = -α(W^λ H) E_α$ for any $H ∈ \mathfrak{h}$;

here $N_{α,β} ≠ 0$ is given by $[E_α, E_β] = N_{α,β} E_{α+β}$.

On the other hand, the equality $[\text{ad } E_{-α}, W] = 0$ is equivalent to a similar list of conditions:

a’) $0 = [\text{ad } E_{-α}, W] E_β' = N_{-α,β'}(λ_β' - λ_{-α+β'}) E_{-α+β'}$ for any $β' ∈ R$ such that $-α + β' ∈ R$;

b’) $0 = [\text{ad } E_{-α}, W] E_α = W^λ H_α$;

c’) $0 = [\text{ad } E_{-α}, W] H = α(W^λ H) E_α$ for any $H ∈ \mathfrak{k}$;

It is clear that a’ follows from a) (by putting $β := -α + β'$. Now we will prove that b), c) imply b’), c’).

Indeed, by b) we have $WH_α = λ_{-α} H_α$, whence $W^λ H_α = (λ_{-α} - λ_α) H_α$, from which by c) we deduce that $λ_α = λ_{-α}$ (since $α(H_α) ≠ 0$). Thus b), c) give $W^λ H_α = W^λ H$ and b’), c’) follow and we have proven that $\mathfrak{z}$ is reductive in $\mathfrak{g}$.

Now let $E_α ∈ \mathfrak{z}^θ$, i.e. $E_α ∈ \mathfrak{z} \cap \ker W^θ$, in particular, $θ_1 = λ_α$. We have shown that $λ_α = λ_{-α}$ and $E_α, E_{-α}, H_α ∈ \ker W^λ$. Since $π_α = 0$, Item 1 implies also that $T_α = \{λ_α\}$. Condition $α(\mathfrak{z}^θ \cap \mathfrak{h}) = 0, θ_j ≠ θ_i, follows from c).
Ad 5. The fact that the spectrum of $W|_{\mathfrak{h}}$ lies in $\Theta$ follows from Item 1 of Theorem 8.3 and the inclusion $\mathfrak{h} \subset \{ x \mid [\text{ad} x, W] = 0 \}$. Conversely, any $\mathfrak{z}^{\Theta}$ is a regular with respect to $\mathfrak{h}$ reductive subalgebra, hence $\mathfrak{z}^{\Theta} \cap \mathfrak{h} = \ker W^{\Theta} \cap \mathfrak{h} \neq \{0\}$.

Ad 6. Note that any irreducible component of the representation $\text{ad}_{\mathfrak{g}|\mathfrak{h}}$ in $\mathfrak{z}^{\perp}$ is spanned by some root spaces $\mathfrak{g}_{\alpha}$ hence is preserved by $W$. Moreover, due to the equality $[\text{ad}_x W] = 0, x \in \mathfrak{z}$, (see Theorem 8.3) the operator $W$ is intertwining for the $\text{ad}_{\mathfrak{g}}$ representation of $\mathfrak{z}$. Now the result follows by the Schur lemma (another argument is Item 4 of Theorem 8.3 saying that the eigenspaces of $W$ are invariant for $\mathfrak{z}^{\Theta}$).

Ad 7. Taking into account the block structure of the operators $W, P$ and the fact that $P|_{\mathfrak{h}} = 0$ we have $\det \tilde{B}^t = \det((W^*)^t W^t - 2P) = \det(W|_{\mathfrak{h}} - t\text{Id}_h)^2 \prod_{\alpha \in R}((t - \lambda_\alpha)(t - \lambda_{-\alpha}) - 2\pi_\alpha)$. It remains to use Item 5. □

Identify for a moment $\mathfrak{h}$ with $\mathfrak{h}^*$ by means of the Killing form. Then each vector $H_\alpha$ will be identified with the root $\alpha$ itself and the root system $R$ induces a reduced root system on $\mathfrak{h}$ which will be denoted also by $R$. Item 3 of the theorem above shows that $W|_{\mathfrak{h}}$ is an $R$-admissible operator in the following sense.

10.4. Definition Let $V$ be a vector space over $\mathbb{R}$ and let $R \subset V$ be a reduced root system in $V$. A diagonalizable linear operator $U : V^C \to V^C$ (here $V^C$ stands for the complexification of the vector space $V$) will be called $R$-admissible if for any $\alpha \in R \subset V^C$ either

1. there exist two eigenvectors $w_{1,\alpha}, w_{2,\alpha} \in V^C$ corresponding to different eigenvalues $t_{1,\alpha}, t_{2,\alpha}$ of the operator $U$ such that

   $$\alpha = w_{1,\alpha} + w_{2,\alpha};$$

   or

2. $\alpha$ itself is an eigenvector of $U$ corresponding to an eigenvalue $t_\alpha$.

We say that a root $\alpha \in R$ is complete with respect to an $R$-admissible operator $U$ if condition 1 holds. An $R$-admissible operator $U$ is called complete if any $\alpha \in R$ is complete with respect to $U$.

10.5. Remark Note that, since the eigenvectors corresponding to different eigenvalues are linearly independent, the eigenvectors $w_{1,\alpha}, w_{2,\alpha}$ if exist are defined uniquely. Put $V_\alpha := \langle w_{1,\alpha}, w_{2,\alpha} \rangle, U_\alpha := \{t_{1,\alpha}, t_{2,\alpha}\}$ if $\alpha$ is complete and $V_\alpha := \langle \alpha \rangle, U_\alpha := \{t_\alpha\}$ otherwise (here $\langle \cdot \rangle$ stands for the linear span).

We see that in order to classify semisimple bi-Lie structures $(\mathfrak{g}, [\cdot], [\cdot],[\cdot])$ with condition $\mathfrak{z} \supset \mathfrak{h}$ one needs, in particular, to classify $R$-admissible operators on $\mathfrak{h}$, where $\tilde{R} = \tilde{R}(\mathfrak{g}, \mathfrak{h})$ is the root system of the semisimple Lie algebra $(\mathfrak{g}, [\cdot])$ transported to $\mathfrak{h}$ by means of the Killing form, up to conjugations by the restrictions to $\mathfrak{h}$ of the automorphisms of $(\mathfrak{g}, [\cdot])$.

Another piece of data which can be extracted from the principal WNO of a semisimple bi-Lie structure with the help of Theorem 10.3 and which will be important in the classification matters is a prescription of the pair of times $T_\alpha$ to any root $\alpha \in R$. This will be formalized in the following definition (the ”times selection rules” will be justified below in Theorem 10.7).
10.6. Definition Let \( R \) be a reduced root system on a vector space \( V \). A collection \( \{ T_\alpha \}_{\alpha \in R} \) of unordered pairs \( T_\alpha = \{ t_{1,\alpha}, t_{2,\alpha} \} \) of complex numbers is called a diagram of pairs of times (or simply a pairs diagram) if \( T_\alpha = T_{-\alpha} \) for any \( \alpha \in R \) and for any triple \( \alpha, \beta, \gamma \in R \) such that \( \alpha + \beta + \gamma = 0 \) (such triples will be called triangles) the pairs \( T_\alpha, T_\beta, T_\gamma \) obey the following "times selection rules":

1. either there exist \( t_1, t_2, t_3 \in \mathbb{C} \) such that
   \[
   T_\alpha = \{ t_1 t_2 \}, T_\beta = \{ t_2 t_3 \}, T_\gamma = \{ t_3 t_1 \};
   \]
2. or there exist \( t_1, t_2 \in \mathbb{C}, t_1 \neq t_2 \), such that
   \[
   T_\alpha = T_\beta = T_\gamma = \{ t_1 t_2 \}.
   \]

We write \((T_\alpha)\) if we understand \( T_\alpha \) as a set. A pair \((U, T)\), where \( U : V^C \to V^C \) is an \( R \)-admissible operator and \( T := \{ T_\alpha \}_{\alpha \in R} \) is a pairs diagram such that \( U_\alpha \subset (T_\alpha) \) (see Remark 10.5) for any \( \alpha \in R \), will be called admissible too.

We put \( T_T := \bigcup_{\alpha \in R} (T_\alpha) \) and call \( T_T \) the set of times of the diagram \( T \).

Let \((U, T)\) be an admissible pair. Note that, if \( \alpha \in R \) is complete, the set of eigenvalues \( U_\alpha \) coincides with the set of times \((T_\alpha)\). In general, a root \( \alpha \) itself can be an eigenvector of \( U \) and the corresponding eigenvalue \( t_\alpha \) is one of the elements of \((T_\alpha)\). The second element is called virtual and can not be read from \( U \) (cf. Examples 12.13, 12.23).

The next theorem elucidates further properties of regular semisimple bi-Lie structures, in particular, shows that the set of the pairs \( T_\alpha \) from Theorem 10.3 forms a pairs diagram. We will refer to the "times selection rules" of Definition 10.6 as to TSR 1 and TSR 2.

10.7. Theorem Retain the assumptions and notations of Theorem 10.3. Then

1. Given a triangle \( \alpha, \beta, \gamma \in R \), the sets \( T_\alpha, T_\beta, T_\gamma \) from Item 1 of Theorem 10.3 understood as unordered pairs obey the "times selection rules" of Definition 10.6.

2. If the triangle is such that TSR 1 (respectively TSR 2) occurs, the following equality holds:
   \[
   \kappa_\alpha + \kappa_\beta + \kappa_\gamma = 0;
   \]
   respectively:
   \[
   \kappa_\alpha + \kappa_\beta + \kappa_\gamma = \pm (t_1 - t_2)/2.
   \]

3. The set \( g_0^t := h \cap \ker W^t + \bigoplus_{\alpha \in R_0^t} g_\alpha \), where \( R_0^t \subset R \) is the set of roots \( \alpha \) with \( t_{1,\alpha} = t_{2,\alpha} = t \), forms a regular with respect to \( h \) subalgebra reductive in \((g, [\cdot, \cdot])\) such that \( g_0^t \supset \mathfrak{z}^t \), where \( \mathfrak{z}^t \) is the centre of the corresponding bracket \([\cdot, \cdot]^t\). The subalgebra \( g_0^t \) is nontrivial if and only if so is the subalgebra \( \mathfrak{z}^t \).

4. (In Items 4–7 assume additionally that \( R \) is irreducible, i.e. that \((g, [\cdot, \cdot])\) is simple.) For any \( \alpha \in R_0^t \) we have \( \kappa_\alpha = 0 \) and \( \pi_\alpha = 0 \).

5. For any \( t \in \mathbb{C} \) we have \( g_0^t \subset \ker W^t \). In particular, \( g_0^{t_1} \cap g_0^{t_2} = \{0\} \) for \( t_1 \neq t_2 \).
6. The set $g_0 = g_0^1 \oplus \cdots \oplus g_0^m$ (here $\{\theta_1, \ldots, \theta_m\} = \Theta$ is the set of times for which the centre $s^{\theta_1}$ of the corresponding bracket $[\cdot , \cdot ]^{\theta_1}$ is nontrivial, see Theorems 8.3 10.3) is a subalgebra reductive in $(g, [\cdot , \cdot ])$ such that $g_0 \supset \mathfrak{j}$. The subspace $g_0^i \oplus g_0^j$ is a subalgebra in $(g, [\cdot , \cdot ])$ for any $i, j$. The subalgebra $g_0$ will be called the basic subalgebra of a regular semisimple bi-Lie structure.

7. The antisymmetric part $W_\alpha$ of the operator $W|_{h^\perp}$ preserves the irreducible components of the representation $x \mapsto \text{ad}_g x$ of the Lie algebra $g_0$ in $g_0^+ \subset g$ and the restriction of $W_\alpha$ to any of them is a scalar operator, here the orthogonal complement is taken with respect to the Killing form.

10.8. REMARK It can be shown that $g_0 \subset \mathfrak{j}$, where $\mathfrak{j}$ is the subalgebra from Theorem 8.3. We conjecture that in fact $g_0 = \mathfrak{j}$.

10.9. REMARK The assumption that $(g, [\cdot , \cdot ])$ is simple in Items 4–7 reduces complexity of formulations. It can be substituted by a less restrictive assumption that $(g, [\cdot , \cdot ])$ is semisimple and the subalgebras $g_0^i$ do not contain any of its simple components.

The proof of the theorem will use the following lemma.

10.10. LEMMA Retain the assumptions of Item 1 of the theorem. Let $T_\alpha := \{t_1t_2\}, T_\beta := \{t_3t_4\}, T_\gamma := \{t_5t_6\}$. Then the following equalities hold:

\[
\begin{align*}
(t_5 + t_6)(t_1 + t_2 - t_3 - t_4) + (t_3^2 + t_4^2 - t_1^2 - t_2^2) &= 0 \\
(t_1 + t_2)(t_3 + t_4 - t_5 - t_6) + (t_5^2 + t_6^2 - t_3^2 - t_4^2) &= 0 \\
(t_3 + t_4)(t_5 + t_6 - t_1 - t_2) + (t_5^2 + t_6^2 - t_3^2 - t_4^2) &= 0.
\end{align*}
\]

The proof will be deduced from the basic equality $T_W(\cdot , \cdot ) = [\cdot , \cdot ]_P$. Taking $E_\alpha, E_\beta$ as the arguments, we get $\lambda_\alpha \lambda_\beta - \lambda_\gamma (\lambda_\alpha + \lambda_\beta - \lambda_\gamma) = \pi_\alpha + \pi_\beta - \pi_\gamma$, hence $\lambda_\gamma (\lambda_\gamma - \lambda_\alpha)(\lambda_\gamma - \lambda_\beta) = \pi_\alpha + \pi_\beta - \pi_\gamma$. Substituting here $-\alpha, -\beta, -\gamma$ instead of $\alpha, \beta, \gamma$ respectively and recalling that $\pi_\alpha = \pi_\gamma$ for any $\alpha \in R$ we obtain

$$\lambda_\gamma (\lambda_\gamma - \lambda_\alpha)(\lambda_\gamma - \lambda_\beta) = (\lambda_\alpha - \lambda_\gamma)(\lambda_\beta - \lambda_\gamma)$$

Recalling that $\lambda_\pm = \sigma_\alpha \pm \kappa_\alpha$ (and $\sigma_\gamma = \sigma_\alpha, \kappa_\gamma = \pm \kappa_\alpha$) we get

$$((\sigma_\gamma - \sigma_\alpha) - (\kappa_\gamma + \kappa_\alpha))((\sigma_\gamma - \sigma_\beta) - (\kappa_\gamma + \kappa_\beta)) = ((\sigma_\gamma - \sigma_\alpha) + (\kappa_\gamma + \kappa_\alpha))((\sigma_\gamma - \sigma_\beta) + (\kappa_\gamma + \kappa_\beta)).$$

Put $A := \sigma_\gamma - \sigma_\alpha, B := \kappa_\gamma + \kappa_\alpha, C := \sigma_\gamma - \sigma_\beta, D := \kappa_\gamma + \kappa_\beta$. Then the above equality can be rewritten as $(A - B)(C - D) = (A + B)(C + D)$, whence $AD + BC = 0$ and $(A - B)(C - D) = AC + BD$. Thus

\[
\begin{align*}
(\sigma_\gamma - \sigma_\alpha)(\kappa_\gamma + \kappa_\beta) + (\kappa_\gamma + \kappa_\alpha)(\sigma_\gamma - \sigma_\beta) &= 0, \quad \text{(10.10.1)} \\
(\sigma_\gamma - \sigma_\alpha)(\sigma_\gamma - \sigma_\beta) + (\kappa_\gamma + \kappa_\alpha)(\kappa_\gamma + \kappa_\beta) &= \pi_\alpha + \pi_\beta - \pi_\gamma. \quad \text{(10.10.2)}
\end{align*}
\]

Now we note that the formula from Item 2 of Theorem 10.3 implies the formula $\pi_\alpha = (1/2)(\zeta_\alpha - \kappa_\alpha^2)$ in account of which equality (10.10.2) gives

\[
(\sigma_\gamma - \sigma_\alpha)(\sigma_\gamma - \sigma_\beta) + (\kappa_\gamma + \kappa_\alpha)(\kappa_\gamma + \kappa_\beta) = \frac{1}{2}(\zeta_\alpha + \zeta_\beta - \zeta_\gamma - (\kappa_\alpha^2 + \kappa_\beta^2 - \kappa_\gamma^2)).
\]
Indeed, the substitution two of the roots, say Case c):

\[(\sigma_\gamma - \sigma_\alpha)(\sigma_\gamma - \sigma_\beta) + \frac{1}{2}(\kappa_\alpha + \kappa_\beta + \kappa_\gamma)^2 = \frac{1}{2}(\zeta_\alpha + \zeta_\beta - \zeta_\gamma).\]  (10.10.3)

Cyclic permutations of \(\alpha, \beta, \gamma\) give

\[(\sigma_\beta - \sigma_\gamma)(\sigma_\beta - \sigma_\alpha) + \frac{1}{2}\kappa = \frac{1}{2}(\zeta_\alpha + \zeta_\beta - \zeta_\gamma)\]  (10.10.4)

\[(\sigma_\alpha - \sigma_\beta)(\sigma_\alpha - \sigma_\gamma) + \frac{1}{2}\kappa = \frac{1}{2}(\zeta_\beta + \zeta_\gamma - \zeta_\alpha),\]  (10.10.5)

where we put \(\kappa := (\kappa_\alpha + \kappa_\beta + \kappa_\gamma)^2\).

Adding respectively equalities (10.10.3) and (10.10.4), (10.10.3) and (10.10.5), (10.10.4) and (10.10.5) we get the following equalities:

\[(\sigma_\beta - \sigma_\gamma)^2 + \kappa = \zeta_\alpha\]  (10.10.6)

\[(\sigma_\gamma - \sigma_\alpha)^2 + \kappa = \zeta_\beta\]

\[(\sigma_\alpha - \sigma_\beta)^2 + \kappa = \zeta_\gamma.\]

From this we get \((\sigma_\beta - \sigma_\gamma)^2 - \zeta_\alpha = (\sigma_\gamma - \sigma_\alpha)^2 - \zeta_\beta\), or, equivalently (using formulae from Item 2 of Theorem 10.3),

\[(t_5 + t_6 - (t_3 + t_4))^2 - (t_1 - t_2)^2 = (t_5 + t_6 - (t_1 + t_2))^2 - (t_3 - t_4)^2.\]

Elementary calculations show that this is equivalent to the first equality of the lemma. The remaining equalities are proved analogously. □

Proof of Theorem 10.7. Ad 1. We have to consider several cases which exhaust all possible ones. Recall that \(W|_h\) is \(R\)-admissible in the sense of Definition 10.4 and use notation from Remark 10.5. Recall that we identify \(h\) with \(h^*\) (in particular, \(H_\alpha\) with \(\alpha\)) by means of the Killing form \(B_0\).

Case a): \(\alpha, \beta, \gamma\) are complete. Assume first that \(V_\alpha \cap V_\beta = \{0\}\). Then, necessarily, \(T_\alpha = T_\beta = \{t_1t_2\}\), where \(t_1 \neq t_2\) (otherwise the vectors \(w_{1,\alpha}, w_{2,\alpha}, w_{1,\beta}, w_{2,\beta}\) would be linearly independent and \(\gamma\) would depend on more than two eigenvectors), and \(T_\gamma = \{t_1t_2\}\) (TSR 2). The cases \(V_\alpha \cap V_\gamma = \{0\}, V_\gamma \cap V_\beta = \{0\}\) are analogous.

Now assume that \(V_\alpha \cap V_\beta \neq \{0\}, V_\alpha \cap V_\gamma \neq \{0\}, V_\gamma \cap V_\beta \neq \{0\}\). Then either \(V_\alpha = V_\beta = V_\gamma = \langle w_1, w_2\rangle\), where \(w_i\) is an eigenvector corresponding to an eigenvalue \(t_i, i = 1, 2\), and \(T_\alpha = T_\beta = T_\gamma = \{t_1, t_2\}, t_1 \neq t_2\) (TSR 2), or \(\alpha = w_1 - w_2, \beta = w_2 - w_3, \gamma = w_3 - w_1\), where \(w_i\) is an eigenvector corresponding to an eigenvalue \(t_i, i = 1, 2, 3\), and \(T_\alpha = \{t_1t_2\}, T_\beta = \{t_2t_3\}, T_\gamma = \{t_3t_1\}\) with \(t_1 \neq t_2, t_2 \neq t_3, t_3 \neq t_1\) (TSR 1).

Case b): one of the roots, say \(\gamma\), is not complete, but the other two are complete. Then \(V_\alpha = V_\beta = \langle w_1, w_2\rangle\), where \(w_i\) is an eigenvector corresponding to an eigenvalue \(t_i, i = 1, 2, t_1 \neq t_2\), and \(\alpha = w_1 + w_2, \beta = -w_1 + aw_2\) for some \(a \neq -1\), i.e. \(V_\gamma = \langle w_3\rangle, T_\alpha = T_\beta = \{t_1t_2\}, T_\gamma = \{t_2(t_6)\}\) for some \(t_6 \in \mathbb{C}\) (here the brackets \(\langle \rangle\) mean that the corresponding element of the unordered pair is virtual, see discussion after Definition 10.6). A priori \(t_6\) can be arbitrary, but in fact is not so. Indeed, the substitution \(t_3 = t_1, t_4 = t_2, t_5 = t_2\) reduces the equalities of Lemma 10.10 to one equality \((t_6 - t_1)(t_2 - t_6) = 0\). Hence \(t_6 = t_1\) (TSR 2) or \(t_6 = t_2\) (TSR 1).

Case c): two of the roots, say \(\alpha, \beta\), are not complete, but the remaining one is complete. Then \(V_\alpha = \langle \alpha\rangle, V_\beta = \langle \beta\rangle, V_\gamma = \langle \alpha, \beta\rangle\), the roots \(\alpha, \beta\) are the eigenvectors corresponding to eigenvalues
$t_1, t_3$ respectively with $t_1 \neq t_3$. Moreover $T_\alpha = \{t_1(t_2)\}, T_\beta = \{t_3(t_4)\}, T_\gamma = \{t_1t_3\}$ for some $t_2, t_4 \in \mathbb{C}$. The second and third of the equalities from the lemma give $(t_4-t_1)(t_2-t_4) = 0$ and $(t_3-t_2)(t_4-t_2) = 0$ respectively. Thus either $t_2 = t_4$ (TSR 1), or $t_2 \neq t_4$, but $t_4 = t_1$ and $t_3 = t_2$ (TSR 2).

**Case d):** all the three roots $\alpha, \beta, \gamma$ are not complete. Then $V_\alpha = \langle \alpha \rangle, V_\beta = \langle \beta \rangle, V_\gamma = \langle \gamma \rangle$ and the roots $\alpha, \beta, \gamma$ are the eigenvectors corresponding to same eigenvalue $t_1$. Then $T_\alpha = \{t_1(t_2)\}, T_\beta = \{t_1(t_3)\}, T_\gamma = \{t_1(t_4)\}$. The first two equalities of the lemma give:

\[
(t_4 - t_2)(t_4 + t_2 - t_1 - t_6) = 0 \\
(t_6 - t_4)(t_6 + t_4 - t_1 - t_2) = 0.
\]

Hence, either $t_4 = t_2$, which implies $(t_6 - t_2)(t_6 - t_1) = 0$ and $t_6$ equal to $t_1$ (TSR 1) or $t_2$ (TSR 2 or 1 with all times equal), or $t_4 = -t_2 + t_1 + t_6$, which implies $(t_6 - t_4)(t_6 - t_2) = 0$. In the last case we have either $t_6 = t_4$ which implies $t_1 = t_2$ (TSR 1), or $t_6 = t_2$ which implies $t_4 = t_1$ (TSR 1).

This finishes the list of possible cases and the proof of the first item of the theorem.

**Ad 2.** Item 2 is proved by direct check using formulae (10.10.6) and the definition of $\sigma_\alpha, \zeta_\alpha$ (given before Theorem 10.3).

**Ad 3.** The fact that $R'_0$ is a closed set of roots follows from the times selection rules (TSR 1). The symmetry of $R'_0$ (which will imply the reductivity), is due to the property $T_\alpha = T_{-\alpha}$. The fact that $g'_0$ is a subalgebra follows from the condition $H_\alpha = [E_\alpha, E_{-\alpha}] \in \ker W^\tau, \alpha \in R'_0$, which is due to Item 3 of Theorem 10.3 and diagonalizability of $W$.

The inclusion $g'_0 \supset z^i$ follows from Item 4 of Theorem 10.3. Thus the nontriviality of $z^i$ implies that of $g'_0$. Finally, if $g'_0$ is nontrivial, then $\mathfrak{h} \cap \ker W^\tau \neq \{0\}$ and $z^i$ is nontrivial by Item 5 of Theorem 10.3.

**Ad 4.** Assume $R'_0 \neq \emptyset$. We will first prove that

\[
\forall \alpha \in R'_0 \exists \beta \in R \setminus R'_0 : \alpha + \beta \in R. \tag{10.10.7}
\]

Let $R_{\text{max}}$ be the maximal symmetric closed proper root set containing $R'_0$. We will show that

\[
\forall \alpha \in R_{\text{max}} \exists \beta, \gamma \in R \setminus R_{\text{max}} : \alpha = \beta + \gamma, \tag{10.10.8}
\]

which will prove assertion (10.10.7).

All maximal symmetric closed root sets, which are the root sets of maximal reductive regular subalgebras of simple Lie algebras, are known [GOV94, Ch. 6]. It appears that each of these subalgebras is a fixed point subalgebra $g_{\text{fix}}$ of an inner automorphism of order 2, 3 or 5. Obviously, the assertion (10.10.8) is equivalent to the following one: the subspace $[m, m] \cap g_{\text{fix}}$ coincides with $g_{\text{fix}}$, here $m = (g_{\text{fix}})^\perp$ is the orthogonal complement to $g_{\text{fix}}$ with respect to the Killing form.

The last assertion follows from the fact that $[m, m] + m$ is an ideal in the initial simple Lie algebra, which can be proved directly using the commutation relations of the corresponding $\mathbb{Z}_2, \mathbb{Z}_3$ or $\mathbb{Z}_5$ gradings.

Now let $\alpha \in R'_0, \beta \in R \setminus R'_0$ be such that $-\gamma := \alpha + \beta \in R$. Then by the times selection rules there exists $t' \in \mathbb{C}, t' \neq t$, such that $T_\beta = T_\gamma = \{tt'\}$ (recall $T_\alpha = \{tt\}$). By formula (10.10.1) we have $((t' - t)/2)(\kappa_\gamma + \kappa_\beta) = 0$, whence $\kappa_\gamma = -\kappa_\beta$. Using Item 2, we conclude that $\kappa_\alpha = 0$.

The equality $\pi_\alpha = 0$ now follows from the formula $\kappa_\alpha^2 = \zeta_\alpha - 2\pi_\alpha$ (see Item 2 of Theorem 10.3).
Ad 5. The previous item and Item 1 of Theorem 10.3 imply that \( \lambda_\alpha = \lambda_{-\alpha} = t \) for any \( \alpha \in R_0^t \), hence \( E_\alpha, E_{-\alpha} \in \ker W \).

Ad 6. The proof of the fact that \( g_0^\beta \oplus g_0^\gamma \) is a subalgebra in \( (g, [,]) \) follows from the definition of \( g_0^\beta \). The rest of the proof follows from Items 3 and 5.

Ad 7. Any irreducible component of the representation \( \text{ad}_g|_{g_0} \) in \( g_0^\perp \) is spanned by some root spaces \( g_\alpha \), hence is preserved by \( W \). If \( E_\alpha \in g_0, E_\beta \in g_0^\perp \), we have \( [\text{ad} E_\alpha, W_a]E_\beta = N_{\alpha,\beta}(\kappa_\beta - \kappa_{\alpha + \beta})E_{\alpha + \beta} = 0 \) (cf. the proof of Item 4 of Theorem 10.3), where the last equality is due to the first equality of Item 2 and the fact that \( \kappa_\alpha = 0 \) (Item 4). Moreover, \( [\text{ad} H, W_a]E_\beta = \beta(H)\kappa_\beta E_\beta - \kappa_\beta \beta(H)E_\beta = 0 \) for any \( H \in h \). Thus the operator \( W_a \) is intertwining for the corresponding representation and the Schur lemma can be applied. \( \square \)

Items 2 and 4 of Theorem 10.7 imply the following result in case when \( (g, [,]) \) is simple (cf. Remark 10.9).

10.11. COROLLARY Assume \( (g, [,]) \) is simple. Then the antisymmetric part of the operator \( W|_{h^\perp} \) obeys the \( ((t_1 - t_2)/2) \)-triangle rule subject to the root set \( R_0 \subset R \) in the sense of the following definition, here \( R_0 := R_0^{\beta_1} \cup \cdots \cup R_0^{\beta_m} \) (see Items 3, 6 of Theorem 10.7).

10.12. DEFINITION Let \( R_0 \subset R \) be a closed symmetric root set (see Section 13) and let \( S \in \text{End}(h^\perp) \) be an antisymmetric operator preserving the root spaces \( g_\alpha \) of the corresponding root decomposition, \( S|_{g_\alpha} = \kappa_\alpha \text{Id}_{g_\alpha}, \kappa_\alpha \in \mathbb{C}, \kappa_2 = -\kappa_{-2} \) (the last equality follows from Item 4 of Theorem 9.2). Given a constant \( a \in \mathbb{C} \), we say that \( S \) obeys the \( a \)-triangle rule subject to the root set \( R_0 \), if \( \kappa_\alpha = 0 \) for any \( \alpha \in R_0 \) and for any triangle \( \alpha, \beta, \gamma \in R \) (see Definition 10.6):

1. \( \kappa_\alpha + \kappa_\beta + \kappa_\gamma = 0 \) whenever two of three roots belong to \( R \setminus R_0 \) and the third one to \( R_0 \) or all the three roots belong to \( R_0 \);
2. \( \kappa_\alpha + \kappa_\beta + \kappa_\gamma = \pm a \) whenever \( \alpha, \beta, \gamma \in R \setminus R_0 \).

We say that \( 0, + \) or \( - \) is a label of the corresponding triangle. The whole family of labels is called the system of labels corresponding to the operator \( S \) and is denoted by \( \mathcal{L}_S \).

In fact due to Items 5–7 of Theorem 10.7 we can say more about the antisymmetric part \( W_a \) of the operator \( W|_{h^\perp} \).

10.13. DEFINITION Let a \( \Gamma \)-grading \( g = \bigoplus_{i \in \Gamma} g_i \) of a Lie algebra \( g \), where \( \Gamma \) is an abelian group, be given. Put \( \overline{\Gamma} := \{ i \in \Gamma \setminus \{0\} \mid g_i \neq \{0\} \} \) and call the elements of this set quasiroots. We call a triangle any triple \( i, j, k \in \overline{\Gamma} \) such that \( i + j + k = 0 \). Given an antisymmetric operator \( S \in \text{End}(g) \) which is scalar on each \( g_i \), i.e. such that \( S|_{g_i} = \kappa_i \text{Id}_{g_i} \), for some \( \kappa_i \in \mathbb{C} \) (necessarily satisfying equalities \( \kappa_i = -\kappa_{-i} \), cf. Item 4 of Theorem 9.2), and a constant \( a \in \mathbb{C} \), we say that \( S \) obeys the \( a \)-triangle rule subject to the grading, if

\[
\kappa_i + \kappa_j + \kappa_k = \pm a
\]

for any triangle \( i, j, k \). We say that \( + \) or \( - \) is a label of the corresponding triangle. The whole family of labels is called the system of labels corresponding to the operator \( S \) and is denoted by \( \mathcal{L}_S \).
10.14. REMARK  Note that if an endomorphism $S \in \text{End}(h^\perp)$ satisfies the conditions of Definition 10.12, the extended by zero on $h$ endomorphism will obey the $a$-triangle rule subject to the root grading in the sense of Definition 10.13.

On the other hand, let $g_0 \subset g$ be a subalgebra reductive in $g$ of maximal rank, let $h \subset g_0$ be some Cartan subalgebra and let $g = \bigoplus_{i \in I} g_i$ be the irreducible toral grading (see Appendix 14). Then an endomorphism $S \in \text{End}(g)$ satisfies the conditions of Definition 10.13 if and only if its restriction to $h^\perp$ obeys the $a$-triangle rule subject to the root set $R_0$ in the sense of Definition 10.6, where $R_0 \subset \tilde{R}(g, h)$ is the root set corresponding to the subalgebra $g_0$.

Items 2, 4–7 of Theorem 10.7 imply the following result.

10.15. COROLLARY  Let $(g, [\cdot, \cdot])$ be simple. The antisymmetric part of the operator $W|_{h^\perp}$ extended by zero on $h$ obeys the $((t_1 - t_2)/2)$-triangle rule subject to the irreducible toral $\Gamma(g_0)$-grading on $(g, [\cdot, \cdot])$ (see Definition 14.3) corresponding to the reductive Lie subalgebra $g_0 \subset g$ (the basic subalgebra).

11  Two classes of pairs diagrams

The aim of this section is to prove the following theorem.

11.1. THEOREM  Let $R$ be a reduced irreducible root system in a vector space $V$ over $\mathbb{R}$ and let $\mathcal{T} := \{T_\alpha\}_{\alpha \in R}$ be a pairs diagram. Assume that there exist $\alpha, \beta, \gamma \in R$ such that $\alpha + \beta + \gamma = 0$ and

$$T_\alpha = T_\beta = T_\gamma = \{t_1t_2\}$$

for some $t_1, t_2 \in \mathbb{C}, t_1 \neq t_2$. Then the set of times $T_\mathcal{T}$ is equal to $\{t_1, t_2\}$ (see Definition 10.6).

11.2. DEFINITION  Let $R$ be a reduced root system. We say that a pairs diagram $\{T_\alpha\}_{\alpha \in R}$ is of Class II, if there exist $\alpha, \beta, \gamma \in R$ satisfying the hypotheses of the theorem, and of Class I, if such roots do not exist.

We start from auxiliary results.

Let $R$ be a reduced root system. We define on $R$ an equivalence relation by putting $\alpha \sim \alpha$ and $\alpha \sim -\alpha$ for any $\alpha \in R$. Write $\tilde{R} := R/\sim$. Obviously, any pairs diagram is in fact indexed by the set $\tilde{R}$. An unordered triple $\{a, b, c\} \subset \tilde{R}$ is called a triangle if there exist $\alpha \in a, \beta \in b, \gamma \in c$ such that $\alpha + \beta + \gamma = 0$. Given a pairs diagram $\{T_\alpha\}_{\alpha \in \tilde{R}}$, a triangle $\{a, b, c\}$ is said to be a $\{t_1t_2\}$-triangle, if

$$T_a = T_b = T_c = \{t_1t_2\}$$

for some $t_1, t_2 \in \mathbb{C}, t_1 \neq t_2$.

11.3. LEMMA  Let $R$ be a reduced irreducible root system in a vector space $V$ and let $\{T_\alpha\}_{\alpha \in \tilde{R}}$ be a pairs diagram. Let $a, b, c, a', b', c' \in \tilde{R}$ be pairwise distinct elements such that $\{a, b, c\}, \{a, c, b'\}, \{b, c, a'\}, \{a', b', c'\}$ are triangles (see the figure below). Then, if $\{a, b, c\}$ is a $\{t_1t_2\}$-triangle, the pairs of times $T_a, T_b, T_c$ are equal to one of the pairs

$$\{t_1t_2\}, \{t_1t_1\}, \{t_2t_2\}$$

and moreover at least one of the triangles $\{a, c, b'\}, \{b, c, a'\}, \{a', b', c'\}$ is a $\{t_1t_2\}$-triangle.
The times selection rules (TSR for short) applied to the triangle \(a,c,b'\) imply, that each of the pairs \(T_c,T_{b'}\) should contain at least one of the times \(t_1,t_2\). So, a priori, we have the following possibilities:

1. \(T_a = \{t_1t_2\}, T_c = \{t_1t_2\}, T_{b'} = \{t_1t_2\};\)
2. \(T_a = \{t_1t_2\}, T_c = \{t_2t_1\}, T_{b'} = \{t_1t_1\};\)
3. \(T_a = \{t_2t_1\}, T_c = \{t_1t_1\}, T_{b'} = \{t_1t_2\};\)
4. \(T_a = \{t_2t_1\}, T_c = \{t_1t_2\}, T_{b'} = \{t_2t_2\};\)
5. \(T_a = \{t_1t_2\}, T_c = \{t_2t_2\}, T_{b'} = \{t_2t_1\};\)
6. \(T_a = \{t_1t_2\}, T_c = \{t_2t_3\}, T_{b'} = \{t_3t_1\} \text{ for some } t_3 \neq t_1,t_2;\)
7. \(T_a = \{t_2t_1\}, T_c = \{t_1t_3\}, T_{b'} = \{t_3t_2\} \text{ for some } t_3 \neq t_1,t_2.\)

However the last two do not occur. Indeed, assume possibility 6 occurs. Then the TSR for the triangle \(b,c,a'\) would imply \(T_{a'} = \{t_1t_3\}\) which contradicts to the TSR for the triangle \(c',b',a'\). Analogous considerations show impossibility of 7.

Thus we have proved that pairs of times \(T_c,T_{b'}\) are equal to one of the pairs \(\{t_1t_2\}, \{t_1t_1\}, \{t_2t_2\}\). By the symmetry of the triangles \(a,c,b\) and \(b,c,a'\) one comes to the same conclusion about \(T_{a'}\).

To finish the proof note that the TSR imply that the pairs \(\{t_1t_1\}\) or \(\{t_2t_2\}\) can appear only once among \(T_c,T_{b'},T_{a'}\) and the rest should be equal to \(\{t_1t_2\}\). \(\square\)

11.4. Theorem Theorem [11.4] holds for the root system \(R := a_n\).

We will give a "geometrical" proof of this theorem. The elements of the set \(\tilde{R}\) will be represented as points of the \(xy\)-coordinate plane with integer coordinates \((l,m)\) with \(l,m \geq 0, l+m \leq n\). The points \((n,0),(n-1,1),\ldots,(0,n)\) represent the elements \([\alpha_1],\ldots,[\alpha_n]\), where \(B := \{\alpha_1,\ldots,\alpha_n\}\) is a basis of the root system \(a_n\). Recall [Bou75] Ch. VI.§1, Cor. 3 of Prop. 19] that, \(B\) is identified with the set of vertices of the graph of corresponding Coxeter system (which will be denoted by \(\Gamma(R)\) and called the Coxeter graph of \(R\)) and that for any subset \(Y \subset B\) which is connected as a subset in \(\Gamma(R)\) we have \(\sum_{\beta \in Y} \beta \in R\). On the other hand, for the root system \(a_n\) all the positive roots are obtained this way. From this we can get the following lemma.
11.5. **Lemma**  Let \( a = (l_0, m_0) \in \mathfrak{a}_n \). Then the elements \( b \in \mathfrak{a}_n \) such that there exist a triangle \( a, b, c \) for some \( c \in \mathfrak{a}_n \) lie on the ”X-shape” \( X(a) \), i.e. the union of lines \( x = l_0, y = m_0, x = n - m_0 + 1, y = n - l_0 + 1 \) intersected with \( \mathfrak{a}_n \) (see the figure below) with the crossing at \( a \).

Given \( a \in \mathfrak{a}_n \) and \( b \in X(a) \), the element \( c \in \mathfrak{a}_n \) such that \( a, b, c \) is a triangle is uniquely defined by \( c = X(a) \cap X(b) \).

![X-shape diagram](image.png)

11.6. **Lemma**  Let \( R \neq \mathfrak{g}_2 \), \( T \) and \( \alpha, \beta, \gamma \in R \) satisfy hypotheses of Theorem 11.1. Then there exists a basis \( \alpha_1, \ldots, \alpha_n \) of the root system \( R \) and two its elements \( \alpha_i, \alpha_j \) such that \( [\alpha_i], [\alpha_j] \) belong to the triangle \( \{[\alpha_i], [\alpha_j], [\alpha_i + \alpha_j]\} \) (in particular, \( \alpha_i, \alpha_j \) form a connected subgraph of the Coxeter graph of \( R \)) and \( T[\alpha_i] = T[\alpha_j] = T[\alpha_i + \alpha_j] = \{t_1t_2\} \).

Let \( V' \subset V \) be a (2-dimensional) subspace generated by \( \alpha, \beta, \gamma \) and let \( R' := R \cap V' \). Then \( R' \) is a root system in \( V' \) by [Bou75, Cor. of Prop. 4, §1, Ch. VI] and is equal to one of the root systems \( \mathfrak{a}_2, \mathfrak{b}_2 \). Indeed, \( R' \) is an irreducible reduced root system of rank 2; the list of all such systems is \( \mathfrak{a}_2, \mathfrak{b}_2, \mathfrak{g}_2 \) (cf. [Hel00, Ch. X, Exc. B1]). However the case \( \mathfrak{g}_2 \) is excluded by the assumption \( R \neq \mathfrak{g}_2 \) as we will show below.

Let \( B' \) be a basis of \( R' \). Then by [Bou75, Prop. 24, §1, Ch. VI] there exists a basis \( B \) of \( R \) such that \( B' \subset B \). In particular, this shows that \( R' \neq \mathfrak{g}_2 \), since neither of the Coxeter graphs of reduced irreducible root systems \( R \) contains the graph of \( \mathfrak{g}_2 \) except \( R = \mathfrak{g}_2 \) itself.

Now, the direct inspection of the systems \( \mathfrak{a}_2, \mathfrak{b}_2 \) shows that among the roots \( \alpha, \beta, \gamma, -\alpha, -\beta, -\gamma \) there exist two roots \( \alpha', \beta' \) forming a basis \( B' \) of \( R' \) and the proposition cited gives the result. \( \square \)

Now we are able to prove Theorem 11.4. Let \( [\alpha_i], [\alpha_j] \) be as in Lemma 11.6. Then they have to be neighbours in the first row of the coordinate representation of \( \tilde{R} \). Denote them \( a, b \) and let \( a = (l, m - 1), b = (l - 1, m), l + m - 1 = n \). The element \( c' := (l - 1, m - 1) \) correspond to the third element \( [\alpha_i + \alpha_j] \) of the \( \{t_1t_2\} \)-triangle, which will be called *basic* for a moment.
We claim that for any \( e \in X(a) \cup X(b) \cup X(c') \) (the union of the "X-shapes" with crossings at \( a, b, c' \), see Lemma [11.5] the pair of times \( T_e \) is equal to one of the pairs \( \{t_1t_2\}, \{t_1t_1\}, \{t_2t_2\} \). Indeed, let \( e \) lie on the line \( x = k \) with \( 0 \leq k \leq l - 2 \), i.e. \( e \) is one of the points \( b' := (k, m - 1), c := (k, m), a' := (k, m + 1) \). Then the elements \( a, b, c, a', b', c' \) satisfy hypotheses of Lemma [11.3] and our claim follows (analogous arguments work if \( e \) lies on the line \( y = j \) with \( 0 \leq j \leq m - 2 \)).

Moreover, among the triangles \( \{a, c, b'\}, \{b, c, a'\}, \{a', b', c'\} \) at least one is a \( \{t_1t_2\} \)-triangle. In any of these cases we can repeat the above considerations taking this triangle as a basic one. Irrespective to which of these cases is considered we will obtain new points \( e \) such that \( T_e \) is equal to one of the pairs \( \{t_1t_2\}, \{t_1t_1\}, \{t_2t_2\} \). These points will lie on on \( X(a') \cap X(b') \cap X(c) \), i.e. on the intersection of the lines \( x = k, y = n - k + 1 \) with \( \tilde{R} \), see the figure below (which corresponds to the new basic triangle \( \{a, c, b'\} \)).

Varying \( k \) from 0 to \( l - 2 \) and \( j \) from 0 to \( m - 2 \) we will cover the whole set \( \tilde{R} \). □

Proof of Theorem [11.7] Let \( R \neq g_2 \) be a reduced irreducible root system with the Coxeter graph \( \Gamma(R) \) (see the beginning of the proof of Theorem [11.4]). We say that a connected subgraph \( \Gamma' \subset \Gamma(R) \) is a
chain if it does not have ramifying points (i.e. vertices connected with at least three other vertices, see [Bou75 Ch. IV, App.]). In particular, the graphs of $\mathfrak{a}_n, \mathfrak{b}_n, \mathfrak{c}_n, \mathfrak{f}_4$ are chains in themselves.

Let $\alpha_i, \alpha_j$ be as in Lemma 11.6. We can consider any chain $\Gamma' \subset \Gamma(R)$ containing $\alpha_i, \alpha_j$ and proceed as in the proof of Theorem 11.4 to prove that $T_e$ is equal to one of the pairs $\{t_1 t_2\}, \{t_1 t_1\}, \{t_2 t_2\}$ for any $e$ of the form $\left[\sum_{\beta \in Y} \beta\right]$, where $Y$ is a subset of vertices of $\Gamma'$ connected as a subgraph. Taking all maximal chains $\Gamma'$ containing $\alpha_i, \alpha_j$ we will prove in this way that $T_{[a]}$ is equal to one of the pairs $\{t_1 t_2\}, \{t_1 t_1\}, \{t_2 t_2\}$ for all the elements $\alpha$ of the basis $B$ (and also for all positive roots which are the combinations of the simple roots with coefficients not greater than 1).

To prove this for all positive roots we will use the fact that any such root can be written as $\beta = \beta_1 + \cdots + \beta_k$, where all $\beta_j$ (not necessarily distinct) belong to $B$ and each partial sum $\beta_1 + \cdots + \beta_j \in R$ (see [Hel00 Lem. 3.10, Ch. X]). Proceed by induction. We already know that $T_{[\beta_j]}$ is equal to one of the pairs $\{t_1 t_2\}, \{t_1 t_1\}, \{t_2 t_2\}$. Assume this is true for $T_{[\beta_1 + \cdots + \beta_{j-1}]}$. Then the times selection rules applied to the triangle $\{[\beta_1 + \cdots + \beta_j - 1], [\beta_j], [\beta_1 + \cdots + \beta_j]\}$ imply that also $T_{[\beta_1 + \cdots + \beta_j]}$ is equal to one of the pairs $\{t_1 t_2\}, \{t_1 t_1\}, \{t_2 t_2\}$.

To complete the proof consider the case $R = \mathfrak{g}_2$ and proceed by direct inspection. □

12 Bi-Lie structures of Class I

Let $(\mathfrak{g}, [\cdot, \cdot], [\cdot, \cdot]'')$ be a semisimple bi-Lie structure with a simple Lie algebra $(\mathfrak{g}, [\cdot, \cdot])$ and let $W : \mathfrak{g} \to \mathfrak{g}$ be the corresponding principal WNO. Assume that there exists a Cartan subalgebra $\mathfrak{h}$ contained in the central subalgebra $\mathfrak{z}$. Then the results of Section 10 show that $W$ defines the time diagram $\mathcal{T} = \mathcal{T}_W$ and that $(W|_{\mathfrak{h}}, \mathcal{T})$ is an admissible pair (see Definition 10.6).

12.1. DEFINITION We say that a regular semisimple bi-Lie structure $(\mathfrak{g}, [\cdot, \cdot], [\cdot, \cdot]'')$ with the principal WNO $W$ satisfying Condition 1 of Theorem 10.1 is of Class I or II depending on the class of the pairs diagram $\mathcal{T}_W$ (see Definition 11.2).

An algebraic structure which will be introduced below is essentially what we obtain from the structure of the pair groupoid (see eg. [Wei96]) if we substitute ordered pairs by unordered ones.

12.2. DEFINITION Let $X$ be a finite set and let $X^2 := \{\{xy\} \mid x, y \in X\}$ be the set of unordered pairs of elements from $X$. Introduce a partial binary operation $X^2 \times X^2 \to X^2, (\{xy\}, \{zu\}) \mapsto \{xy\} \circ \{zu\}$ by the following rules:

1. $\{xy\} \circ \{zu\}$ is defined if and only if the sets $\{x, y\}, \{z, u\}$ have a common element;
2. $\{xy\} \circ \{yu\} := \{xu\}$ if $x \neq u$;
3. $\{xy\} \circ \{xy\}$ is either equal to $\{xx\}$ or to $\{yy\}$.

The pair $(X^2, \circ)$ will be called a pairoid of Class I with the base $X$.

12.3. REMARK Note that unlike the pair groupoid structure on the $X \times X$ a pairoid structure is not defined uniquely due to uncertainty of Condition 3.
Let \((\mathfrak{g}, [\cdot, \cdot])\) be a semisimple Lie algebra, \(\mathfrak{h} \subset \mathfrak{g}\) a fixed Cartan subalgebra. Let \(R = R(\mathfrak{g}, \mathfrak{h}) \subset \mathfrak{h}_R^*\) be the corresponding reduced irreducible root system and \(\mathfrak{g} = \mathfrak{h} + \sum_{\alpha \in R} \mathfrak{g}_\alpha\) the root grading.

12.4. Definition Let \(X\) be a finite set. A decomposition \(\mathfrak{g} = \bigoplus_{(xy) \in X^2} \mathfrak{g}(xy)\) such that \(\mathfrak{g}(xy) \neq \{0\}\) for any pair \(\{xy\} \in X^2, x \neq y\), is a pairoid quasigrading on \(\mathfrak{g}\) of Class I with the base \(X\) if

1. \([\mathfrak{g}(xy), \mathfrak{g}(yu)] \subset \mathfrak{g}(zu)\) whenever \(x \neq u\);
2. \([\mathfrak{g}(xy), \mathfrak{g}(yx)] \subset \mathfrak{g}(xx) \oplus \mathfrak{g}(yy)\);
3. \([\mathfrak{g}(xy), \mathfrak{g}(zy)] = \{0\}\) whenever \(\{xy\} \circ \{zu\}\) is not defined.

A pairoid quasigrading is toral (with respect to \(\mathfrak{h}\)) if: each \(\mathfrak{g}(xy)\) is the sum of the spaces \(\mathfrak{g}_\alpha\) and some subspaces of \(\mathfrak{h} \subset \bigoplus_{x \in X} \mathfrak{g}_{xx}\) (cf. Definition 14.1). A toral pairoid quasigrading is symmetric if \(\mathfrak{g}_{-\alpha} \subset \mathfrak{g}(xy)\) whenever \(\mathfrak{g}_\alpha \subset \mathfrak{g}(xy)\).

The elements of the set \(X\) are called the times of the quasigrading. A time \(t \in X\) is virtual if \(\mathfrak{g}(tt) = \{0\}\). Given an unordered pair \(\{xy\} \in X^2\), the element \(x\) is called a virtual element of the pair if \([\mathfrak{g}(xy), \mathfrak{g}(yx)] \subset \mathfrak{g}(xy)\). In particular, a virtual time is a virtual element of any pair containing it (cf. Examples 12.13, 12.23). If we want to distinguish virtual elements we denote them in brackets (cf. the proof of Theorem 10.7 and examples below).

Two pairoid quasigradings of Class I, \(\mathfrak{g} = \bigoplus_{(xy) \in X^2} \mathfrak{g}(xy)\) and \(\mathfrak{g} = \bigoplus_{(x'y') \in (X')^2} \mathfrak{g}'(x'y')\), with the bases \(X\) and \(X'\) are equivalent if there exists \(\varphi \in \text{Aut}(\mathfrak{g})\) and a bijection \(\chi : X \rightarrow X'\) such that \(\varphi(\mathfrak{g}(ab)) = \mathfrak{g}'(\chi(a)\chi(b))\) for any \(a, b \in X\) (here \(\text{Aut}(\mathfrak{g})\) denotes the set of automorphisms of the Lie algebra \((\mathfrak{g}, [\cdot, \cdot])\)).

For the sake of simplicity we will omit the braces in the notations of unordered pairs indexing grading subspaces.

12.5. Example Let \(X := \{x, y\}\), where \(x \neq y\), and let \(\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1\) be a \(\mathbb{Z}_2\)-grading on \(\mathfrak{g}\). Putting \(\mathfrak{g}_{xx} := \mathfrak{g}_0 \oplus \mathfrak{g}_0, \mathfrak{g}_{yy} := \{0\}, \mathfrak{g}_{x(y)} := \mathfrak{g}_1\) we get a pairoid quasigrading on \(\mathfrak{g}\) with the base \(X\) and a virtual time \(y\). An equivalent quasigrading can be given by \(\mathfrak{g}_{xx} := \{0\}, \mathfrak{g}_{yy} := \mathfrak{g}_0, \mathfrak{g}_{(x)y} := \mathfrak{g}_1\). An example of quasigrading nonequivalent to that from previous examples is obtained when \(\mathfrak{g}_0\) is a sum of two nontrivial subalgebras, \(\mathfrak{g}_0 = \mathfrak{g}_0^{1} \oplus \mathfrak{g}_0^{2}\). \(\mathfrak{g}_{xx} := \mathfrak{g}_0^{1} \oplus \mathfrak{g}_0^{2}, \mathfrak{g}_{yy} := \mathfrak{g}_0^{1} \oplus \mathfrak{g}_0^{2}, \mathfrak{g}_{xy} := \mathfrak{g}_0^{1}\). These examples exhaust up to equivalence all nontrivial pairoid quasigradings of Class I with the base \(X\).

Let us mention some properties of pairoid quasigradings of Class I. Obviously, the notion of pairoid quasigrading of Class I on \(\mathfrak{g}\) with the base \(X\) is independent of the choice of pairoid structure on \(X^2\) (see Remark 12.3). Moreover, the subspaces \(\mathfrak{g}_{xx}\) are subalgebras and if the pairoid quasigrading is toral symmetric these subalgebras are reductive in \(\mathfrak{g}\) (by [Bou75, §3, Ch. VII], see also the beginning of Section 13). The following lemma is a direct consequence of the definitions.

12.6. Lemma Let \(X := \{t_1, \ldots, t_n\}\) be a set of pairwise distinct complex numbers.

Let \((U, \mathcal{T}), \mathcal{T} = \{T_\alpha\}_{\alpha \in R}\) be an admissible pair (see Definition 10.6) such that \(T_T = X, U \in \text{End}(\mathfrak{h})\) and the pairs diagram is of Class I (we transport the root system \(R\) to \(\mathfrak{h}\) by means of the Killing form, see the discussion before Definition 10.4). Write \(\mathfrak{h}_{t_i}\) for the eigenspace of the operator \(U\) corresponding to the eigenvalue \(t_i\).
Put \( \mathfrak{g}_{t_i,t_j} := \bigoplus_{\alpha \in T_T \setminus \{t,t_j\}} \mathfrak{g}_\alpha \) if \( i \neq j \) and \( \mathfrak{g}_{t_i} := \mathfrak{h}_i \bigoplus \bigoplus_{\alpha \in T_T \setminus \{t,t_i\}} \mathfrak{g}_\alpha \). Then \( \mathfrak{g} = \bigoplus_{t_i,t_j \in X} \mathfrak{g}_{t_i,t_j} \) is a toral symmetric pairoid quasigrading of class I with the base \( X \).

Vice versa, given a toral symmetric pairoid quasigrading \( \mathfrak{g} = \bigoplus_{t_i,t_j \in X} \mathfrak{g}_{t_i,t_j} \) of class I with the base \( X \), one obtains an admissible pair \( (U,T) \) with \( T_T = X \) and the pairs diagram of class I by putting \( T_\alpha := \{ t,t_j \} \) for all \( \alpha \in R \) such that \( \mathfrak{g}_\alpha \subset \mathfrak{g}_{t_i,t_j} \) and \( U|_{\mathfrak{g}_{t_i,t_j} \cap \mathfrak{h}} = t_i \text{Id}_{\mathfrak{g}_{t_i,t_j} \cap \mathfrak{h}} \).

The built correspondence is one-to-one.

In the following theorem we will build a WNO from a given toral symmetric pairoid quasigrading of class I, or, in view of the lemma above, from an admissible pair. This theorem shows that the necessary conditions for an operator to be a WNO obtained in Section 10 are in fact sufficient in the case of Class I bi-Lie structures.

12.7. Theorem Let \( (\mathfrak{g},[,]) \) be a semisimple Lie algebra.

1. Let \( X := \{ t_1, \ldots, t_n \} \) be a set of pairwise distinct complex numbers and let \( \mathfrak{g} = \bigoplus_{t_i,t_j \in X} \mathfrak{g}_{t_i,t_j} \) be a toral symmetric pairoid quasigrading of class I with the base \( X \). Define an operator \( W \in \text{End}(\mathfrak{g}) \) by
   \[
   W|_{\mathfrak{g}_{t_i,t_j}} := \frac{t_i + t_j}{2} \text{Id}_{\mathfrak{g}_{t_i,t_j}}.
   \]

   Then the triple \( (\mathfrak{g},[,],[,])_W \) is a semisimple bi-Lie structure of class I such that
   (a) \( X \) is its set of times, the subalgebras \( \mathfrak{g}_{t_i} \) coincide with the subalgebras \( \mathfrak{g}_{t_i}^i \) from Theorem 10.7, in particular, the basic subalgebra is equal to \( \mathfrak{g}_0 := \bigoplus_{i=1}^n \mathfrak{g}_{t_i}^i \);
   (b) its central subalgebra \( \mathfrak{z} \) contains \( \mathfrak{h} \) and, moreover, \( \mathfrak{z} = \bigoplus_{i=1}^n \mathfrak{z}_{t_i}^i \) (\( \mathfrak{z}_{t_i}^i \) being the centre of the exceptional bracket \( (\mathfrak{g},[,]) - t_i([,]) \)), \( \mathfrak{z}_{t_i}^i = \mathfrak{g}_{t_i} \cap \mathfrak{h} + \bigoplus_{\alpha \in \tilde{R}^i} \mathfrak{g}_\alpha \), where \( \tilde{R}^i := \{ \alpha \in R^i \mid \alpha(\bigoplus_{j \neq i} \mathfrak{g}_{t_j} \cap \mathfrak{h}) = 0 \} \), here \( R^i \) is a closed symmetric root set corresponding to the subalgebra \( \mathfrak{g}_{t_i} \) (see the beginning of Section 13);
   (c) the operator \( W|_{\mathfrak{z}_{t_i}} \) is symmetric, in particular, \( W \) is the principal WNO of the constructed bi-Lie structure.

2. Any semisimple bi-Lie structure \( (\mathfrak{g},[,],[,])' \) of class I is of the form above.

3. If \( W, W' \) are the operators built by two toral symmetric pairoid quasigradings of class I with the bases \( X, X' \), then the corresponding bi-Lie structures are strongly isomorphic if and only if the quasigradings are equivalent, \( X = X' \), and \( \chi(x) = x \) for any \( x \in X \), here \( \chi \) is the bijection from Definition 12.4. The bi-Lie structures are isomorphic if and only if the quasigradings are equivalent and there exist \( \lambda, \lambda' \in \mathbb{C} \) such that \( \chi(x) = \lambda x + \lambda' \) for any \( x \in X \).

Ad 1. We will first prove that the operator \( W \) satisfies the basic equality \( T_W(x,y) = [x,y]_P \), where \( P \in \text{End}(\mathfrak{g}) \) is a symmetric operator preserving the root grading, \( P|_{\mathfrak{h}} \subset \mathfrak{h}, P|_{\mathfrak{g}_\alpha} = \pi_\alpha \), where \( \pi_\alpha \) are some complex numbers, \( \pi_\alpha = \pi_{-\alpha} \).

Let \( \{ t_{1,0}, t_{2,0} \} = \{ t_i,t_j \} \) if \( \mathfrak{g}_0 \subset \mathfrak{g}_{t_i,t_j} \). Put \( \lambda_\alpha := (t_{1,0} + t_{2,0})/2, P|_{\mathfrak{h}} = 0, \pi_\alpha := (t_{1,0} - t_{2,0})^2/8 \) (the last choice is suggested by Item 2 of Theorem 10.3 since \( \lambda_\alpha = \lambda_{-\alpha} \) and \( \kappa_\alpha = (\lambda_\alpha - \lambda_{-\alpha})/2 = 0 \)). Then, obviously,
\[
\lambda_0 + \lambda_{-\alpha} = t_{1,0} + t_{2,0}; \lambda_\alpha \lambda_{-\alpha} = t_{1,0} t_{2,0} + 2\pi_\alpha.
\]
We claim that
\[(\lambda_\alpha - \lambda_\gamma)(\lambda_\beta - \lambda_\gamma) = \pi_\alpha + \pi_\beta - \pi_\gamma\] (12.7.2)
whenever \(\alpha + \beta = \gamma\) for some \(\alpha, \beta, \gamma \in R\). To prove this take into account the commutation relations of pairoid quasigrading of Class I and observe that the equality \(\alpha + \beta = \gamma\) implies the inclusions
\[g_\alpha \subset g_{t_i t_j}, g_\beta \subset g_{t_j t_k}, g_\gamma \subset g_{t_k t_i},\]
where some of the indices \(i, j, k\) can be equal. Now we have \(\lambda_\alpha = (t_i + t_j)/2, \lambda_\beta = (t_j + t_k)/2, \lambda_\gamma = (t_k + t_i)/2, \pi_\alpha = (t_i - t_j)^2/8, \pi_\beta = (t_j - t_k)^2/8, \pi_\gamma = (t_k - t_i)^2/8\) and we can make a direct inspection.

To prove the main equality we have to consider several cases. Let \(x, y \in h\). Then, obviously,
\[T_W(x, y) = 0 = [x, y]_p.\]
If \(x \in h, y \in g_\alpha\) (recall \([x, y] = \alpha(x)y\)), we have \(T_W(x, y) = [Wx, \lambda_\alpha y] - W([Wx, y] + [x, \lambda_\alpha y] - W(\alpha(x)y)) = \lambda_\alpha \alpha(Wx)y - W(\alpha(Wx)y + \lambda_\alpha \alpha(x)y - \lambda_\alpha \alpha(x)y) = 0\). On the other hand, \([x, y]_p = [x, \pi_\alpha y] - P[x, y] = \pi_\alpha \alpha(x)y - P\alpha(x)y = 0\).

Now, let \(x \in g_\alpha, y \in g_{-\alpha}\) be such that \(B(x, y) = 1\). Then (recall \([x, y] = H_\alpha\) \(T_W(x, y) = (\lambda_\alpha \lambda_\alpha \text{Id} - (\lambda_\alpha + \lambda_\alpha)W - W^2)H_\alpha = (W - \lambda_\alpha \text{Id})(W - \lambda_\alpha \text{Id})H_\alpha\). By formula (12.7.1), the last expression is equal to \((W - t_\alpha \text{Id})(W - t_\alpha \text{Id})H_\alpha + 2\pi_\alpha H_\alpha = 2\pi_\alpha H_\alpha\) (here we used the fact that \(H_\alpha\) is a sum of eigenvectors of \(W\) corresponding to the eigenvalues \(t_\alpha, t_\alpha\) or is an eigenvector corresponding to one of them). On the other hand, \([x, y]_p = [\pi_\alpha x, y] + [x, \pi_\alpha y] - P\alpha = 2\pi_\alpha H_\alpha\).

If \(x \in g_\alpha, y \in g_\beta\) with \(\alpha + \beta \not\in R\cup\{0\}\), then \(T_W(x, y) = 0 = [x, y]_p\). Finally, assume \(x \in g_\alpha, y \in g_\beta\) with \(\alpha + \beta = \gamma \in R\). Then (cf. the proof of Lemma [10.10]) \(T_W(x, y) = (\lambda_\alpha \lambda_\beta - \lambda_\gamma (\lambda_\alpha + \lambda_\beta - \lambda_\gamma))[x, y] = (\lambda_\alpha - \lambda_\gamma)(\lambda_\beta - \lambda_\gamma)[x, y] = (\pi_\alpha + \pi_\beta - \pi_\gamma)[x, y] = [x, y]_p\), where we used formula (12.7.2).

We have proven that \((g, [\cdot, \cdot], [\cdot, \cdot]_W)\) is a (semisimple) bi-Lie structure which by construction \((g, [\cdot, \cdot], [\cdot, \cdot]_W)\) is of Class I.

Properties (a), (b) follow from Theorems [10.3] [10.7]. Property (c) is a consequence of Item 4 of Theorem [9.2] and the symmetric property of the quasigrading.

Ad 2. Item 2 follows from Theorems [10.3] [10.7].

Ad 3. Item 3 is a consequence of Theorem [7.10]. □

Now we will show that any pairoid quasigrading of class I induces a special type of \(Z_2^{m-1}\)-grading (here \(Z_p := Z/pZ\) on \((g, [\cdot, \cdot])\)). Recall that a basis of a finite abelian group \(G\) is a set of elements \(e_1, \ldots, e_n \in G\) such that any element \(g \in G\) has a unique decomposition of the form \(g = k_1 e_1 + \cdots + k_n e_n\) (we use the additive notation) with \(0 \leq k_i < o_i\), where \(o_i\) is the order of the element \(e_i\). Clearly, given a basis as above we can define an isomorphism \(\varphi : G \rightarrow Z_{o_1} \times \cdots \times Z_{o_n}\) by the formula \(g \mapsto (k_1, \ldots, k_n)\). If \(G = G_{m-1}\) is a group isomorphic to \(Z^{m-1}_2\), the cardinality of all the bases of \(G\) is the same.

Let \(g = \bigoplus_{i \in G_{m-1}} g_i\) be a \(G_{m-1}\)-grading. An element \(i \in G_{m-1}\) will be called a quasiroot if \(g_i \neq \{0\}\).

12.8. Definition. Given \(k, l \in \{1, \ldots, m\}, k < l\), put \(I_{kl} := (i_1, \ldots, i_{m-1})\), where \(i_1 = \cdots = i_{k-1} = i_l = \cdots = i_{m-1} = 0, i_k = \cdots = i_{l-1} = 1\).

We say that a toral (see Appendix [14]) \(G_{m-1}\)-grading \(g = \bigoplus_{i \in G_{m-1}} g_i\) is admissible if there exists a basis of \(G_{m-1}\) such that the element \((0, \ldots, 0)\) and all the elements \(I_{kl}, k \in \{1, \ldots, m-1\}, l \in \{2, \ldots, m\}, k < l\), are the only quasiroots of the induced \(Z_2^{m-1}\)-grading
\[g = \bigoplus_{(i_1, \ldots, i_{m-1}) \in Z_{m-1}^2} g(i_1, \ldots, i_{m-1}).\]
The following lemma is a direct consequence of the definition.

12.9. Lemma Let \( X := \{t_1, \ldots, t_n\} \) be a set of pairwise distinct complex numbers and let \( g = \bigoplus_{i,j \in X} g_{t_it_j} \) be a toral symmetric pairoid quasigrading of class I with the base \( X \). Then the formula

\[
g(i_1, \ldots, i_{m-1}) := \begin{cases} \bigoplus_{i=1}^{n} g_{t_it_i} & \text{if } (i_1, \ldots, i_{m-1}) = (0, \ldots, 0); \\ g_{t_k} & \text{if } (i_1, \ldots, i_{m-1}) = I_{kl}; \\ \{0\} & \text{in other cases.} 
\end{cases}
\]

gives an admissible \( \mathbb{Z}_2^{m-1} \)-grading on \( g \).

Note that in fact the grading above is determined by the corresponding pairs diagram \( T \) (without participation of the \( R \)-admissible operator \( U \), cf. Lemma 12.6).

Below we will construct a series of examples of pairoid qusigradings of Class I. In order to do this one should start from constructing admissible \( \mathbb{Z}_2^{m-1} \)-gradings. By definition any \( \mathbb{Z}_2^{m-1} \)-grading with \( m = 2, 3 \) is so, hence in principle this is a nontrivial problem only for \( m > 3 \). However the first of the following examples is in a sense a counterexample since it shows that the correspondence from the preceding lemma is not one-to-one, i.e. finding an admissible toral \( \mathbb{Z}_2^{m-1} \)-grading could be insufficient for building a pairoid quaternion grading (even a pairs diagram).

It is clear that any toral symmetric \( \mathbb{Z}_2^{m-1} \)-grading is defined by \( m - 1 \) commuting inner automorphisms of order 2. In the examples below we will use the model automorphisms of different types (see [Hc00 Ch. X] and discussion after Remark 13.13). We will also use the notations from Tables I-IV of [Bou68].

12.10. Example Let \( g = \mathfrak{d}_4 \) and let \( \alpha_1, \alpha_2, \alpha_3, \alpha_4 \) be the basis of the root system. Here \( \alpha_1, \alpha_3, \alpha_4 \) have labels 1 and \( \alpha_2 \) has label 2. Consider \( \mathbb{Z}_2 \)-grading defined by the model automorphisms of type \((0, 1, 0, 0, 1, 1), (0, 0, 0, 1, 1, 1)\). Then we have \( g_{\alpha_1} \subset g_{(1,0)}, g_{\alpha_2} \subset g_{(0,0)}, g_{\alpha_3} \subset g_{(0,1)}, g_{\alpha_4} \subset g_{(1,1)} \), or, by the hypothetical inverse correspondence to that of Lemma 12.9, \( g_{\alpha_1} \subset g_{t_1t_2}, g_{\alpha_3} \subset g_{t_2t_3}, g_{\alpha_4} \subset g_{t_1t_3} \). However, there is a problem: in which of three components \( g_{t_1t_2}, g_{t_2t_3} \) or \( g_{t_1t_3} \) should lie \( g_{\alpha_2} \) in order that the axioms of the pairoid quaternion grading be satisfied? This problem is related to the ramifying point of the Dynkin diagram.

The next example shows that nonadmissible toral \( \mathbb{Z}_2^{m-1} \)-gradings with \( m > 3 \) exist.

12.11. Example Let \( g = \mathfrak{d}_4 \). Consider the model automorphisms of types \((1, 1, 0, 0, 0, 1; 1), (1, 0, 0, 1, 0, 1), (1, 0, 0, 0, 1, 1)\). They define a \( \mathbb{Z}_2 \)-grading on \( g \) which is not admissible. Indeed, it has eight quasiroots: \( g_{\alpha_1} \subset g_{(100)}, g_{\alpha_2} \subset g_{(000)}, g_{\alpha_3} \subset g_{(010)}, g_{\alpha_4} \subset g_{(001)}, g_{\alpha_1+a_2+a_3} \subset g_{(110)}, g_{\alpha_2+a_3+a_4} \subset g_{(111)}, g_{\alpha_1+a_2+a_4} \subset g_{(101)}, g_{\alpha_1+a_2+a_3+a_4} \subset g_{(111)} \). However, an admissible \( \mathbb{Z}_2^3 \)-grading has only seven quasiroots. Again the ramifying point plays crucial role.

Below we give a series of examples of toral symmetric pairoid quaternion grading of Class I. The first of these examples is universal, i.e. it appears in any semisimple Lie algebra.

12.12. Example Consider pairoid quaternion gradings from Example 12.5 such that the corresponding \( \mathbb{Z}_2 \)-grading is induced by an inner automorphism. Then the quaternion gradings are toral and symmetric and by Theorem 12.7 we get a number of regular semisimple bi-Lie structures of Class I with two times.
Now we will construct a series of (in a sense canonical, see Conjecture 12.20) toral symmetric pairroid quasigrading of Class I on the classical Lie algebras. First three series were known in the literature. The forth an fifth one, related to the \( a_n \)-series, are new.

We will describe the quasigradings by means of admissible pairs \((U, T)\) (cf. Lemma 12.16).

12.13. Example Let \( g = h_n = \mathfrak{so}(2n + 1, \mathbb{C}) \), the roots are \( \pm \varepsilon_i, \pm \varepsilon_i \pm \varepsilon_j, 1 \leq i < j \leq n \), where \( \varepsilon_i \) are the elements of an orthonormal basis in \( \mathfrak{h}_n^* \). Put \( U(H_{\varepsilon_i}) := t_i H_{\varepsilon_i}, T_{\pm \varepsilon_i \pm \varepsilon_j} := \{ t_i t_j \} \) and \( T_{\pm \varepsilon_i} := \{ t_i(t_{n+1}) \} \) (recall that the brackets mean that the corresponding element is virtual, cf. Definition 10.6, discussion after it, and Definition 12.4). The set of times is \( \{ t_1, \ldots, t_{n+1} \} \) and the time \( t_{n+1} \) is virtual. The labels of the standard basis \( a_1, \ldots, a_n \) are 1, 2, \ldots, 2. The corresponding admissible \( \mathbb{Z}_2^2 \)-grading is defined by the model automorphisms of types \((1, 1, 0, 0, \ldots, 0, 1), (0, 0, 1, 0, \ldots, 0, 0, 1), (0, 0, 1, \ldots, 0, 0; 1), (0, 0, 1, \ldots, 0, 1; 1), \ldots, (0, 0, 0, 0, \ldots, 1, 1; 1)\). This example corresponds to Example 2.23 with the diagonal matrix \( A = \text{diag}(t_1, t_1, t_2, \ldots, t_n, t_n, t_{n+1}) \).

12.14. Example Let \( g = d_n = \mathfrak{so}(2n, \mathbb{C}) \), the roots are \( \pm \varepsilon_i \pm \varepsilon_j, 1 \leq i < j \leq n \). Put \( U(H_{\varepsilon_i}) := t_i H_{\varepsilon_i}, T_{\pm \varepsilon_i \pm \varepsilon_j} := \{ t_i t_j \} \). The set of times is \( \{ t_1, \ldots, t_n \} \) and there no virtual times (and elements). The labels of the standard basis \( a_1, \ldots, a_n \) are 2, 2, \ldots, 2, 1. The corresponding admissible \( \mathbb{Z}_2^{n-1} \)-grading is defined by the model automorphisms of types \((0, 1, 0, \ldots, 0, 1), (0, 0, 1, \ldots, 0, 0, 1), (0, 0, \ldots, 0, 1), (0, 1, \ldots, 0, 1; 1), \ldots, (1, 0, 0, \ldots, 1, 0; 1)\). This example corresponds to Example 2.24 with the diagonal matrix \( A = \text{diag}(t_1, t_2, \ldots, t_n, t_1, t_2, \ldots, t_n) \).

12.15. Example Let \( g = d_n = \mathfrak{sp}(n, \mathbb{C}) \), then the roots are \( \pm 2 \varepsilon_i, \pm \varepsilon_i \pm \varepsilon_j, 1 \leq i < j \leq n \). Put \( U(H_{\varepsilon_i}) := t_i H_{\varepsilon_i}, T_{\pm 2 \varepsilon_i \pm \varepsilon_j} := \{ t_i t_j \} \) and \( T_{\pm 2 \varepsilon_i} := \{ t_i t_i \} \). The set of times is \( \{ t_1, \ldots, t_n \} \) and there no virtual times (and elements). The labels of the standard basis \( a_1, \ldots, a_n \) are 2, 2, \ldots, 2, 1. The corresponding admissible \( \mathbb{Z}_2^{n-1} \)-grading is defined by the model automorphisms of types \((0, 1, 0, \ldots, 0, 1), (0, 0, 1, \ldots, 0, 0, 1), (0, 0, \ldots, 0, 1), (0, 1, \ldots, 0, 1; 1), \ldots, (1, 0, 0, \ldots, 1, 0; 1)\). This example corresponds to Example 2.24 with the diagonal matrix \( A = \text{diag}(t_1, t_2, \ldots, t_n, t_1, t_2, \ldots, t_n) \).

12.16. Example Let \( g = a_n = \mathfrak{sl}(n+1, \mathbb{C}) \), then the roots are \( \pm \varepsilon_i - \varepsilon_j, 1 \leq i < j \leq n + 1 \), where \( \varepsilon_i \) are the elements of an orthonormal basis in a \((n+1)\)-dimensional euclidean space in which \( \mathfrak{h}_n^* \) is embedded as the hyperplane orthogonal to the vector \((1, 1, \ldots, 1)\). Put \( U(H_{\varepsilon_i - \varepsilon_{n+1}}) := t_i H_{\varepsilon_i - \varepsilon_{n+1}}, i = 1, \ldots, n \), \( T_{\pm (\varepsilon_i - \varepsilon_j)} := \{ t_i t_j \}, \) if \( i < j < n + 1 \) and \( T_{\pm (\varepsilon_i - \varepsilon_{n+1})} = \{ t_i(t_{n+1}) \} \). The set of times is \( \{ t_1, \ldots, t_{n+1} \} \) and the time \( t_{n+1} \) is virtual. The labels of the standard basis \( a_1, \ldots, a_n \) are 1, 1, \ldots, 1. The corresponding admissible \( \mathbb{Z}_2^n \)-grading is defined by the model automorphisms of types \((1, 1, 0, \ldots, 0; 1), (1, 0, 1, \ldots, 0; 1), \ldots, (1, 0, 0, \ldots, 1; 1)\). The WNO constructed via Theorem 12.7 by these data has the form \( WX = (1/2)(L_A + R_A)X - \text{Tr}(1/2)(L_A + R_A)X)B \), where \( X \in \mathfrak{sl}(n+1), A = \text{diag}(t_1, t_2, \ldots, t_{n+1}), B = \text{diag}(0, 0, \ldots, 0, 1) \) (cf. Example 11.12). Explicitly, if \( X = ||x_{ij}|| \in \mathfrak{sl}(n+1, \mathbb{C}) \), then \( WX = ||y_{ij}|| \), where \( y_{ij} = x_{ij}(t_i + t_j)/2 \) for \( i \neq j \), \( y_{ii} = x_{ii}t_i \) for \( i = 1, \ldots, n \), and \( y_{n+1,n+1} = -\sum_{j=1}^{n} x_{ij}t_j \).

The proof of Theorem 12.7 suggests that the WNO \( W \) will have the principal primitive equal to that of the operator \( Z := (1/2)(L_A + R_A) \) (considered as an element of \( \text{End}(\mathfrak{gl}(n+1)) \)). In other words, the following fact is true: \( T_W = T_Z \) (here also \( W \) is considered as an operator on \( \mathfrak{gl}(n+1) \)). Its direct proof surprisingly is not evident, and we present it below.
We have to prove that $T_{Z-V} = T_Z$, where $VX := (Z-W)X = \text{Tr}(ZX)B$. We will use the following simple observations (which are true for any $X,Y \in \mathfrak{gl}(n+1)$): 1) $[B,X] = [B,ZX]$ (since the matrices $A$ and $B$ commute); 2) $Z[ VX,Y] = [ VX,ZY]$ (as a consequence of 1)); 3) $V^2 X = t_{n+1} VX$ and $Z VX = t_{n+1} VX$ (by definitions); 4) $\text{Tr}(Z[X,Y]Z) = 0$ (this follows from the main identity $T_Z(X,Y) = [X,Y]_p$, where $P = \text{ad}(A/2) \circ L_A - (1/8)\text{ad}^3(A)$, see Example 12.12 since $Z[X,Y]_p = [ ZX, ZY] - \langle [ PX, PY] + [ X, PY] - [ A/2, A[X,Y]] + (1/8)[ A, A[X,Y]] \rangle$ is the combination of commutators); 5) $[ VX,VY] = 0$ (obvious); 6) $\text{Tr}(Z[B,X]) = 0$ (the matrices $[B,X]$ and $Z[B,X]$ has zeroes on the diagonal); 7) $\text{Tr}(Z[X,Y]) = 0$ (as a consequence of 6)).

Now we have $T_{Z-V}(X,Y) = T_Z(X,Y) - [Z,X,VY] - [VX,ZY] + [VY,XV] + Z[X,Y] + V[X,Y] + V[X,Y] + \text{Tr}(Z[AX,Y]Z) B - (\text{Tr}(Z[AX,Y]) B + \text{Tr}(Z[VX,Y]) B) - t_{n+1} V[X,Y] = T_Z(X,Y) - [Z,X,VY] - [VX,ZY] + [VY,XV] + ([VX,ZY] + [ZV,Y] - t_{n+1} V[X,Y]) + \text{Tr}(Z[X,Y]Z) B - (\text{Tr}(Z[AX,Y]) B + \text{Tr}(Z[VX,Y]) B) - t_{n+1} V[X,Y] = T_Z(X,Y)$.

The next two examples contain an additional continuous parameter (i.e. moduli of bi-Lie structures with the fixed set of times are appearing here).

12.17. Example Let $g = \mathfrak{a}_n = \mathfrak{sl}(n+1,\mathbb{C})$. Put $T_{\pm(\varepsilon_i-\varepsilon_j)} := \{ t_i t_j \}$, if $i < j < n + 1$, and $T_{\pm(\varepsilon_i-\varepsilon_{i+1})} = \{ t_i t_n \}, i = 1, \ldots, n$. The corresponding admissible $\mathbb{Z}^n_{-1}$-grading is defined by the model automorphisms of types $(1,1,0,\ldots,0;1), (1,0,1,\ldots,0;1), \ldots, (1,0,0,\ldots,1;0)$. Put $w_n := a H_{\alpha_n}, w_{n-1} := w_n + H_{\alpha_{n-1}}, \ldots, w_1 := w_2 + H_{\alpha_1}$, where $\alpha_i := \varepsilon_i - \varepsilon_{i+1}$. Put $w_i := t_i w_i$. It is easy to see that any root vector is a linear combination of not more than two eigenvectors of $U$, i.e. the operator $U$ is $R$-admissible, and that the pair $(U,T)$ is admissible too. By Theorem 12.7 we get a family of semisimple bi-Lie structures with the set of times $X := \{ t_1, \ldots, t_n \}$ (there are no virtual times) depending on the parameter $a$. It turns out that these structures are pairwise nonisomorphic if the parameter $a$ is taken from small vicinity of 1.

Indeed, in order that two such structures related with quasigradings $g = \bigoplus_{i,j \in X} \mathfrak{g}_{(i,j)}; g = \bigoplus_{i,j \in X} \mathfrak{g}_{(i,j)}'$, corresponding to parameters $a$ and $a'$, were isomorphic it is necessary that there existed an automorphism $\varphi \in \text{Aut}(g)$ such that $\varphi(\mathfrak{g}_{(i,j)}a) = \mathfrak{g}_{(i,j)}a$ and $\varphi(\mathfrak{g}_{(i,j)}') = \mathfrak{g}_{(i,j)}'$, $i = 1, \ldots, n - 1$, for some permutation $\sigma \in S_{n-1}$. By the construction the subalgebra $\mathfrak{g}_{(i,j)}$ is the 3-dimensional subalgebra generated by the subspace $\mathfrak{g}_{\alpha_n}$, while the remaining subalgebras $\mathfrak{g}_{\beta_{\alpha}}$ are one dimensional subalgebras in $\mathfrak{h}$ of specific form. It can be shown that such $\varphi$ has to preserve not only the subalgebra $\mathfrak{h} + \mathfrak{g}_{\alpha_n} + \mathfrak{g}_{-\alpha_n}$ (which is the zero subalgebra of the admissible $\mathbb{Z}^n_{-1}$-grading), but also the Cartan subalgebra $\mathfrak{h}$ itself. Thus the automorphisms which could realize an isomorphism between these bi-Lie structures belong to discrete group, while the parameter $a$ gives a ”continuous degree of freedom”.

12.18. Example Let $g = \mathfrak{b}_n = \mathfrak{so}(2n+1)$. Consider the $\mathbb{Z}_2$-grading is defined by the model automorphism of type $(1,1,0,0,\ldots,0;1)$. Its zero subalgebra $\mathfrak{g}_0$ is the direct sum of the subalgebra $\mathfrak{so}(2n-1 := \mathfrak{g}_{2t_2}$ embedded standardly in the lower right corner and arbitrary 1-dimensional subalgebra $\mathfrak{s} := \mathfrak{g}_{t_1}$ such that $\mathfrak{s}$ is not contained in $\mathfrak{g}_{2t_2} \cap \mathfrak{h}$. It remains to put $\mathfrak{g}_{t_1 t_2} := \mathfrak{g}_1$ as in Example 12.7. Situation here is similar to the preceding example. In order that the bi-Lie structures corresponding to subalgebras $\mathfrak{s}, \mathfrak{s}'$ were isomorphic an automorphism $\varphi \in \text{Aut}(g)$ such that $\varphi(\mathfrak{g}_{2t_2}) = \mathfrak{g}_{t_1 t_2}$ and $\varphi(\mathfrak{s}) = \mathfrak{s}'$ should exist. However such an automorphism should preserve the Cartan subalgebra $\mathfrak{h}$ and should belong to a discrete group.

Similar examples exist for the Lie algebras $\mathfrak{d}_n, \mathfrak{e}_6, \mathfrak{e}_7$ (cf. [GOV94] Table 6).
The following simple observation the proof of which follows from the definition of toral symmetric pa iroid quasi-gradation of Class I allows to produce new regular semisimple bi-Lie structures of class I from known ones.

12.19. Lemma Let \( X := \{t_1, \ldots, t_n\} \) and let \( g = \bigoplus_{t_i, t_j \in X} g_{t_it_j} \) be a toral symmetric pa iroid quasi-gradation of Class I on \((g, [\cdot,\cdot])\) with the base \( X \). Then for any surjective map \( \mu : X \rightarrow Y := \{u_1, \ldots, u_m\} \) the formula \( g = \bigoplus_{u_i, u_j \in X} g^\mu_{u_iu_j} \), where \( g^\mu_{u_iu_j} := \bigoplus_{x \in \mu^{-1}(u_i), y \in \mu^{-1}(u_j)} g_{xy} \), gives a toral symmetric pa iroid quasi-gradation of Class I on \((g, [\cdot,\cdot])\) with the base \( Y \).

In particular, if \( \mu_i : X \rightarrow Y := \{u_1, u_2\} \) is given by \( \mu_i(t_i) := u_1, \mu_i(t_j) := u_2 \) for any \( j \neq i \), we get quasigradings \( g = g^\mu_{u_1u_1} \oplus g^\mu_{u_2u_2} \oplus g^\mu_{u_1u_2} \).

The second part of this result gives a hope that toral symmetric pa iroid quasi-gradation of Class I can be completely classified since they are ”weaved” from quasigradings with two times, the structure of which is understood (cf. Example 12.5).

12.20. Conjecture Any toral symmetric pa iroid quasi-gradation of Class I on a simple Lie algebra is equivalent to one of that from Examples 12.12–12.17 or to their reductions by means of procedure from Lemma 12.19.

12.21. Remark All WNOs built in Theorem 12.7 by toral symmetric quasigradings of Class I with the base \( \{t_1, \ldots, t_n\} \) linearly depend on these parameters. Hence all the examples above are in fact examples of families of Lie brackets parametrized by the linear space \( \mathbb{C}^n \) with corresponding \( n \) (cf. the result of Kantor and Persits discussed in Introduction).

12.22. Remark The WNO of the bi-Lie structure from Example 2.3 in the case when the matrix \( A \) is diagonal with a simple spectrum is also related to a specific pa iroid quasigrading of Class I on \( g = so(n, \mathbb{C}) \) which, however, is not toral. Let \( E_{ij}, i < j \), be the ”antisymmetric matrix unit”; put \( X = \{t_1, \ldots, t_n\} \) and \( g = \bigoplus_{i < j} g_{t_it_j} \), where \( g_{t_it_j} := \mathbb{C}E_{ij} \). Now one can proceed as in Theorem 12.7 for constructing the WNO (and the formulae from the proof give the primitive).

We conclude this section by one more example explaining the subtlety of the notions of virtual time and virtual element (cf. Definition 12.4).

12.23. Example consider the pa iroid quasigrading from Example 12.13 and apply to it the procedure of Lemma 12.19 with \( \mu : \{t_1, \ldots, t_{n+1}\} \rightarrow \{t_1, \ldots, t_n\} \) given by the identification of \( t_{n+1} \) with \( t_n \) (the corresponding bi-Lie structure is that from Example 2.3 with \( A = \text{diag}(t_1, t_1, t_2, t_2, \ldots, t_n, t_n, t_n) \)). Then the time \( t_n \) is not virtual (since \( g_{t_nt_n} \) is generated by \( g_{\alpha_\alpha} \)) but \( t_n \) is a virtual element of all the pairs \( \{t_i(t_n)\}, i = 1, \ldots, n - 1 \).
13 Bi-Lie structures of Class II

In this section we will build a theory related to pairs diagrams of Class II (see Definitions 10.6, 11.2) similar to that from the previous section.

We start from auxiliary results. Let $(g, [\cdot, \cdot])$ be a simple Lie algebra (cf. Remark 10.9), $g_0 \subset g$ a reductive in $g$ Lie subalgebra of maximal rank. In particular, there exists a Cartan subalgebra $h$ in $g$ such that $h \subset g_0$. In what follows we fix a Cartan subalgebra $h \subset g$.

Let $R = R(g, h) \subset h^*_{\mathbb{R}}$ be the corresponding reduced irreducible root system. Recall [Bou68] that a set $R_0 \subset R$ is closed if for any $\alpha, \beta \in R_0$ such that $\gamma := \alpha + \beta \in R$ the root $\gamma$ belongs to $R_0$. The set $R_0$ is symmetric if $R_0 = -R_0$. An empty set is closed symmetric by definition.

In terms of the root system $R$ any reductive in $g$ Lie subalgebra $g_0 \supset h$ can be described as follows. If $g = h + \bigoplus_{\alpha \in R} g_\alpha$ is the corresponding root decomposition, then $g_0 = h + \bigoplus_{\alpha \in R_0} g_\alpha$, where $R_0 \subset R$ is some closed symmetric root set [Bou68, VII, §3].

13.1. Definition We say that a subalgebra $g_0 \subset g, g_0 \supset h$, is admissible if it is a reductive subalgebra in $g$ and it is not the fixed point subalgebra of an inner automorphism of order 2. Given a pair of reductive in $g$ subalgebras such that $g_0^1 \oplus g_0^2 = g_0$, we say that $(g_0^1, g_0^2)$ is an admissible pair of subalgebras (trivial cases $g_0^1 = \{0\}$ or $g_0^2 = \{0\}$ are admitted). The closed symmetric root set $R_0$ corresponding to an admissible subalgebra $g_0$ is called admissible too.

13.2. Remark The fixed point subalgebras of an inner automorphism of order 2 of simple Lie algebras and their root systems are well known [Hel00, Chapter X].

13.3. Definition Let $X = \{x, y\}$ be a 2-element set of complex numbers. A decomposition $g = g_{\{xx\}} \oplus g_{\{yy\}} \oplus g_{\{xy\}} \neq \{0\}$ (recall that $\{xy\}$ denotes the unordered pair of the elements $x, y \in X$), is a pairoid quasigrading on $g$ of Class II with the base $X$ if

1. $[g_{\{xx\}}, g_{\{xx\}}] \subset g_{\{xx\}}; [g_{\{yy\}}, g_{\{yy\}}] \subset g_{\{yy\}}; [g_{\{xx\}}, g_{\{yy\}}] = \{0\}$;
2. $[g_{\{xx\}}, g_{\{xy\}}] \subset g_{\{xy\}}; [g_{\{yy\}}, g_{\{xy\}}] \subset g_{\{xy\}}$;
3. $[g_{\{xy\}}, g_{\{xy\}}] \cap g_{\{xy\}} \neq \{0\}$.

Given a pairoid quasigrading of Class II with a base $X = \{x, y\}$, the opposite pairoid quasigrading is obtained by interchanging $x, y$.

A pairoid quasigrading is toral if each $g_{\{zu\}}$ is the sum of the spaces $g_\alpha$ and some subspaces of $h$ (cf. Definition 14.1). A toral pairoid quasigrading is symmetric if $g_{-\alpha} \subset g_{\{zu\}}$ whenever $g_\alpha \subset g_{\{zu\}}$.

Two pairoid quasigradings of Class II $g = g_{\{xx\}} \oplus g_{\{yy\}} \oplus g_{\{xy\}}$ and $g = g'_{\{x'x'\}} \oplus g'_{\{y'y'\}} \oplus g'_{\{x'y'\}}$ with the bases $X = \{x, y\}$ and $X' = \{x', y'\}$ are equivalent if there exists $\varphi \in \text{Aut}(g)$ and a bijection $\chi : X \to X'$ such that $\varphi(g_{\{ab\}}) = g'_{\{\chi(a)\chi(b)\}}$ for any $a, b \in X$ (here $\text{Aut}(g)$ denotes the set of automorphisms of the Lie algebra $(g, [\cdot, \cdot])$). Two pairoid quasigradings of Class II are antiequivalent if one of them is equivalent to the opposite to another.

The following lemma relates the definitions above.
13.4. Lemma Let a toral symmetric pairoid quasigrading \( \mathfrak{g} = \bigoplus_{t_i, t_j \in X} \mathfrak{g}_{(t_i, t_j)} \) of Class II with the base \( X = \{t_1, t_2\} \) be given. Put \( \mathfrak{g}_0^i := \mathfrak{g}_{(t_i, t_1)}, \mathfrak{g}_0^2 := \mathfrak{g}_{(t_2, t_2)} \). Then \( (\mathfrak{g}_0^1, \mathfrak{g}_0^2) \) is an admissible pair of subalgebras.

Vice versa, given a set \( X = \{t_1, t_2\} \) of distinct complex numbers and an admissible pair \( (\mathfrak{g}_0^1, \mathfrak{g}_0^2) \) of subalgebras, the formulae \( \mathfrak{g}_{(t_1, t_1)} := \mathfrak{g}_0^1, \mathfrak{g}_{(t_2, t_2)} := \mathfrak{g}_0^2, \mathfrak{g}_{(t_1, t_2)} := \mathfrak{g}_0^+ \), where \( \mathfrak{g}_0 = \mathfrak{g}_0^1 \oplus \mathfrak{g}_0^2 \), define a toral symmetric pairoid quasigrading of Class II with the base \( X \).

The built correspondence is one-to-one up to the transposition \( (t_1 t_2) \).

For the proof we only have to mention that the definition of a pairs diagram of Class II excludes the case when \( \mathfrak{g}_0 \) is the fixed point subalgebra of an inner automorphism of order 2. Indeed, assuming the contrary, we have \( (R \setminus R_0) + (R \setminus R_0) \subset R_0 \) and there does not exist any triangle (see Definition \( 10.6) \) \( \alpha, \beta, \gamma \in R \setminus R_0 \), i.e. such that \( T_\alpha = T_\beta = T_\gamma = \{t_1 t_2\} \). It is also easy to see that the case of the fixed point subalgebra of an automorphism of order 2 is the only which should be excluded. \( \square \)

The next lemma follows from definitions.

13.5. Lemma Let \( X := \{t_1, t_2\}, t_1 \neq t_2, t_i \in \mathbb{C} \).

Let \( (U, \mathcal{T}) \), \( \mathcal{T} = \{T_\alpha\}_{\alpha \in R} \) be an admissible pair (see Definition \( 10.6) \), where \( T_X = X \), \( \mathcal{T} \) is the pairs diagram of Class II, and \( U \in \text{End}(\mathfrak{h}) \) (we transport the root system \( R \) to \( \mathfrak{h} \) by means of the Killing form, see the discussion before Definition \( 10.4) \). Write \( \mathfrak{h}_i \) for the eigenspace of the operator \( U \) corresponding to the eigenvalue \( t_i \).

Put \( \mathfrak{g}_{t_i t_j} := \bigoplus_{\alpha, \alpha \in R} \mathfrak{g}_\alpha \) if \( i \neq j \) and \( \mathfrak{g}_{(t_i, t_i)} := \mathfrak{h}_i \oplus \bigoplus_{\alpha, T_\alpha = \{t_i t_i\}} \mathfrak{g}_\alpha \). Then \( \mathfrak{g} = \bigoplus_{t_i, t_j \in X} \mathfrak{g}_{(t_i, t_j)} \) is a toral symmetric pairoid quasigrading of Class II with the base \( X \).

Vice versa, given a toral symmetric pairoid quasigrading \( \mathfrak{g} = \bigoplus_{t_i, t_j \in X} \mathfrak{g}_{(t_i, t_j)} \) of Class II with the base \( X \), one obtains an admissible pair \( (U, \mathcal{T}) \) with \( T_X = X \) and the pairs diagram of Class II by putting \( T_\alpha := \{t_i t_j\} \) for all \( \alpha \in R \) such that \( \mathfrak{g}_\alpha \subset \mathfrak{g}_{(t_i, t_i)} \) and \( U|_{\mathfrak{g}_{(t_i, t_i)} \cap \mathfrak{h}} = t_i \mathbb{I}|_{\mathfrak{g}_{(t_i, t_i)} \cap \mathfrak{h}} \).

The built correspondence is one-to-one.

13.6. Remark We introduced the notion of a toral symmetric pairoid quasigrading of Class II to show the parallelism with the theory built in the previous section. However, since by Lemma \( 13.4) \) this notion is almost equivalent to the notion of an admissible pair of subalgebras, we will formulate the further results entirely in terms of this last (see also Remark \( 13.11) \).

The following theorem says that in the particular case of the bi-Lie structures of Class II the necessary conditions obtained in Section \( 10) \) are in fact sufficient for an operator to be a WNO. This theorem takes into account the information about the restriction \( W|_\mathfrak{h} \) of this operator to the Cartan subalgebra \( \mathfrak{h} \) and the symmetric part of the operator \( W|_{\mathfrak{h}^\perp} \) as well as the information about the antisymmetric part of this operator described in Corollary \( 10.15) \).

13.7. Theorem Let \( (\mathfrak{g}, [\cdot, \cdot]) \) be a simple Lie algebra.

1. Let an ordered pair \( (t_1, t_2) \) of distinct complex numbers be given and an admissible pair of Lie subalgebras \( (\mathfrak{g}_0^1, \mathfrak{g}_0^2) \). Let \( \mathfrak{g}_0 = \mathfrak{g}_0^1 \oplus \mathfrak{g}_0^2 \) and let

\[
\mathfrak{g} = \bigoplus_{i \in \Gamma(\mathfrak{g}_0)} \mathfrak{g}_i \quad (13.7.1)
\]
be the toral irreducible $\Gamma(g_0)$-grading corresponding to the reductive Lie subalgebra $g_0$ (see Definition [10.4]). Assume that an operator $W \in \text{End}(g)$, which is scalar on the subspaces $g_i, i \neq 0,$ and $g_0^1, j = 1, 2,$ satisfies the following conditions:

(a) $W|_{g_0} = t_j \text{Id}_{g_0}, j = 1, 2;$

(b) the symmetric part $W_s$ of the operator $W|_{g_0^1}$ is equal to $((t_1 + t_2)/2)\text{Id}_{g_0^1}$, here $g_0^1 = \bigoplus_{i \in \Gamma, i \neq 0} g_i$;

(c) The extension by zero of the antisymmetric part $W_a$ of the operator $W|_{g_0^1}$, which we denote by $W_a$, obeys the $((t_1 - t_2)/2)$-triangle rule subject to the grading (see Definition [10.15]).

Then the principal projection $\text{pr}(W)$ (see Theorem [9.4]) of the operator $W$ also satisfies Conditions (a),(b),(c), and the triple $(g, [\cdot, \cdot], [\cdot, \cdot]_W) = (g, [\cdot, \cdot], [\cdot, \cdot]_{\text{pr}(W)})$ is a semisimple bi-Lie structure of Class II such that

(a') $\{t_1, t_2\}$ is its set of times, the subalgebras $g_0^1$ coincide with the subalgebras $g_0^{t_i}$ from Theorem [10.7]; in particular, its basic subalgebra is $g_0$;

(b') its central subalgebra $z$ contains $h$ and, moreover, $z = z^{t_1} \oplus z^{t_2}$ ($z^{t_i}$ being the centre of the exceptional bracket $(g, [\cdot, \cdot], [\cdot]_{t_i})$), $z^{t_1} = g_0^1 \cap h + \bigoplus_{\alpha \in R} g_\alpha$, where $R^1 := \{\alpha \in R \mid \alpha(g_0^1 \cap h) = 0\}$, $R^2 := \{\alpha \in R^2 \mid \alpha(g_0^1 \cap h) = 0\}$, here $R^i$ is a closed symmetric root set corresponding to the subalgebra $g_0^i$.

2. Any semisimple bi-Lie structure of Class II is of the form above.

Ad 1. We will first prove that the operator $W$ satisfies the basic equality $T_W(x, y) = [x, y]_P$, where $P \in \text{End}(g)$ is a symmetric operator preserving the root decomposition (9.3.1), $P\h \subset \h, P|_{g_\alpha} = \pi_{\alpha}\text{Id}_{g_\alpha}$, where $\pi_{\alpha}$ are some complex numbers, $\pi_{\alpha}$ are some complex numbers, $\pi_{\alpha}$ are some complex numbers.

Let $t_{1,\alpha} = t_{2,\alpha} = t_i$ if $\alpha \in R^i$ and $t_{1,\alpha} = t_1, t_{2,\alpha} = t_2$ if $\alpha \notin R^1 \cup R^2$. Let $\lambda_\alpha$ be the eigenvalue of $W$ corresponding to the eigenspace $g_\alpha$ and let $\sigma_\alpha := (1/2)(\lambda_\alpha + \lambda_{-\alpha}), \kappa_\alpha := (1/2)(\lambda_\alpha - \lambda_{-\alpha})$ be the corresponding eigenvalues of $W_1, W_2$, in particular, $\sigma_\alpha = (t_{1,\alpha} + t_{2,\alpha})/2$ for any $\alpha$ by condition (b), $\kappa_\alpha = 0$ for $\alpha \in R^1 \cup R^2$ by condition (c).

Put $P|_{h} = 0, \pi_{\alpha} := ((t_{1,\alpha} - t_{2,\alpha})/2)^2 - \kappa_{\alpha}^2)/2$ (the last choice is suggested by Item 2 of Theorem [10.3]). One checks the following formulæ:

$$\lambda_\alpha = t_{1,\alpha} + t_{2,\alpha}, \lambda_\alpha\kappa_\alpha = \sigma_\alpha^2 - \kappa_\alpha^2 = t_{1,\alpha}t_{2,\alpha} + 2\pi_{\alpha}.$$  

We claim that

$$(\lambda_\alpha - \lambda_{-\alpha})(\lambda_\beta - \lambda_{-\beta}) = \pi_{\beta} + \pi_{\beta} - \pi_{-\gamma}$$

whenever $\alpha + \beta = \gamma$ for some $\alpha, \beta, \gamma \in R$. This can be proven by direct inspection taking into account conditions (a), (b) and considering the following cases

- $\alpha, \beta, \gamma \in R^i \Rightarrow \sigma_\alpha = \sigma_\beta = \sigma_\gamma = t_i, \kappa_\alpha = \kappa_\beta = \kappa_\gamma = 0, \pi_{\alpha} = \pi_{\beta} = \pi_{-\gamma} = 0$;
- $\alpha \in R^i, \beta, \gamma \notin R^1 \cup R^2 \Rightarrow \sigma_\alpha = t_i, \sigma_\beta = \sigma_\gamma = (t_1 + t_2)/2, \kappa_\alpha = 0, \kappa_\beta = \kappa_\gamma, \pi_{\alpha} = 0, \pi_{\beta} = \pi_{\gamma} = (t_1 - t_2)^2 - \kappa_{\gamma}^2)/2$;
- $\alpha, \beta \notin R^1 \cup R^2, \gamma \in R^i \Rightarrow \sigma_\alpha = \sigma_\beta = (t_1 + t_2)/2, \sigma_\gamma = t_i, \kappa_\alpha = -\kappa_\beta, \kappa_\gamma = 0, \pi_{\alpha} = \pi_{\beta} = (t_1 - t_2)^2 - \kappa_{\gamma}^2)/2, \pi_{\gamma} = 0;$

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\( \bullet \alpha, \beta, \gamma \not\in R^1 \cup R^2, \Rightarrow \sigma_\alpha = \sigma_\beta = \sigma_\gamma = (t_1 + t_2)/2, \kappa_\alpha + \kappa_\beta - \kappa_\gamma = \pm(t_1 - t_2)/2, \pi_\delta = ((t_1 - t_2)/2)^2 - \kappa_\delta^2)/2, \) where \( \delta = \alpha, \beta, \gamma. \)

Now the proof of the main equality \( T_W(x, y) = [x, y]_P \) is literally the same as in the proof of Theorem 12.7.

Hence we have proven that \( (g, [\cdot, \cdot], [\cdot, \cdot])_W \) is a semisimple bi-Lie structure. Now let us show that \( \text{pr}(W) \) satisfies conditions (a), (b), (c). Indeed, since the projecting does not affect \( W|_h \) and \( W_s \), conditions (a), (b) follow from the assumptions. The antisymmetric parts of \( W|_h \) and \( \text{pr}(W)|_h \) differ by an operator of the form \( \text{ad} H, H \in h \), due to Theorem 9.4, hence obey the same \( ((t_1 - t_2)/2) \)-triangle rule by Lemma 13.8 below. Finally, the condition \( \kappa_\alpha' = 0, \alpha \in R^1 \cup R^2, \) where \( \kappa_\alpha' \) is the eigenvalue of the antisymmetric part of \( \text{pr}(W)|_h \) corresponding to \( g_\alpha \), is a consequence of Item 4 of Theorem 10.7 and the fact that \( [\cdot, \cdot]_W = [\cdot, \cdot]_{\text{pr}(W)}. \)

It is clear that \( (g, [\cdot, \cdot], [\cdot, \cdot])_W \) is of Class II. Property (a’) follows from Theorem 10.3. Property (b’) is a consequence of Item 4 of this theorem.

**Ad 2.** Item 2 follows from Theorems 10.3 10.7 and Corollary 10.15. □

Later we will use the theorem above to construct a series of examples of bi-Lie structures of Class II. In order to formulate the second our next theorem (13.10), dealing with the uniqueness questions, we need two more lemmata.

13.8. **Lemma** Let \( h \subset g \) be a Cartan subalgebra, \( g = h + \bigoplus_{\alpha \in R} g_\alpha \) the corresponding root decomposition. Assume \( R_0 \subset R \) is an admissible root set (see Definition 13.7) and \( S : h^\perp \to h^\perp \) is an antisymmetric operator preserving the root spaces \( g_\alpha \) and obeying the a-triangle rule subject to \( R_0 \) (see Definition 10.12). Then

1. given any basis \( B = \{\alpha_1, \ldots, \alpha_n\} \) of the root system \( R \), the operator \( S \) can be reconstructed from \( a \), its system of labels \( L_S \), and its eigenvalues \( \kappa_{\alpha_1}, \ldots, \kappa_{\alpha_n} \) corresponding to the eigenspaces \( g_{\alpha_1}, \ldots, g_{\alpha_n}; \)

2. an antisymmetric operator \( S' : h^\perp \to h^\perp \) preserving the root spaces \( g_\alpha \) obeys the a-triangle rule subject to the same root set with \( L_{S'} = L_S \) if and only if \( S' = S + (\text{ad} H)|_h \) for some \( H \in h \) such that \( \alpha(H) = 0 \) for any \( \alpha \in R_0; \)

3. in the case when \( S \) is principal (i.e. if \( S = L|_h \) for some principal operator \( L \in \text{End}(g) \), see Definition 7.4), any other principal antisymmetric operator \( S' \in \text{End}(h^\perp) \) preserving the root spaces \( g_\alpha \) and obeying the a-triangle rule subject to the same root set with \( L_{S'} = L_S \) coincides with \( S; \) moreover, \( S \) can be uniquely reconstructed from \( a \) and its system of labels \( L_S \).

**Ad 1.** Proceeding by induction assume that \( \kappa_{\alpha} \), the eigenvalue of \( S \) corresponding to the eigenspace \( g_{\alpha} \), is already reconstructed for all positive roots of height \( \text{ht}(\alpha) = k \). Since any positive root \( \alpha \) of height \( k+1 \) can be decomposed as a sum \( \alpha = \beta + \gamma \), where \( \beta, \gamma \) are positive roots with \( \text{ht}(\beta), \text{ht}(\gamma) \leq k \), we can put \( \kappa_\alpha = \kappa_\beta + \kappa_\gamma \) or \( \kappa_\alpha = \kappa_\beta + \kappa_\gamma \pm a \) depending of the label of the triangle \( \alpha, -\beta, -\gamma. \)

**Ad 2.** Obviously, for any \( H \in h \) the operator \( \text{ad} H|_h \) is diagonal: \( \text{ad} H|_{g_\alpha} = \alpha(H)|_{g_\alpha}. \) Since for any triangle \( \alpha, \beta, \gamma \) we have \( \alpha(H) + \beta(H) + \gamma(H) = 0 \) (i.e. \( \text{ad} H \) satisfies the 0-triangle rule), adding of \( \text{ad} H \) to \( S \) will not affect the system of labels \( L_S \). On the other hand, the condition \( \alpha(H) = 0, \alpha \in R_0, \) guarantees that \( \text{ad} H|_{g_\alpha} = 0 \) and \( (S + \text{ad} H)|_{g_\alpha} = 0 \) for any \( \alpha \in R_0. \) The other implication will be proven below.
We will first show that for the operator $S := (\mathrm{ad} \, H)|_{\mathfrak{h} \perp}$ the vector $(\kappa_{\alpha_1}, \ldots, \kappa_{\alpha_n})$ runs through all $(z_1, \ldots, z_n) \in \mathbb{C}^n$ as $H$ runs through $\mathfrak{h}$. Indeed, we have $\kappa_{\alpha_i} = \alpha_i(H)$ and in order to find $H \in \mathfrak{h}$ such that $\alpha_i(H) = z_i, i = 1, \ldots, n$, we have to solve a system of linear equations with the matrix $\alpha_i(H_{\alpha_i})$ the determinant of which is proportional to that of the Cartan matrix of the base $B$, hence is nonzero.

We can now prove the second implication. The operator $S' := S' - S$ obeys the 0-triangle rule. Let $B = \{\alpha_1, \ldots, \alpha_n\}$ be a basis of the root system $R$ and let $\kappa''$ be the corresponding eigenvalues of $S'$. The considerations above show that there exists $H \in \mathfrak{h}$ such that $(\mathrm{ad} \, H)|_{\mathfrak{g}_{\alpha_i}} = \kappa''_{\alpha_i}, i = 1, \ldots, n$. However by Item 1 the operators $S''$, $\mathrm{ad} \, H$ coincide. Moreover $\alpha(H) = 0$ for any $\alpha \in R_0$ since $S''|_{\mathfrak{g}_{\alpha}} = 0$.

Ad 3. By Item 2 we have $S' = S + (\mathrm{ad} \, H)|_{\mathfrak{h} \perp}$ for some $H \in \mathfrak{h}$. However, both $S, S'$ are principal and by definition $H = 0$.

Now let $S = L|_{\mathfrak{h} \perp}$ for some principal operator $L \in \mathrm{End}(\mathfrak{g})$. Starting the induction process of the proof of Item 1 from arbitrary vector $(c_1, \ldots, c_n)$ we will obtain one of the operators $S + (\mathrm{ad} \, H)|_{\mathfrak{h} \perp}, H \in \mathfrak{h}$. To get $S$ one has to project $L + \mathrm{ad} \, H$ onto $\mathfrak{g} \perp$ along $\mathfrak{g} = \mathfrak{g} \cap \mathrm{End}(\mathfrak{g})$ and to restrict the result to $\mathfrak{h} \perp$ (cf. Theorem 9.4). □

Let $\varphi \in \mathrm{Aut}(\mathfrak{g})$ and let $\mathfrak{g}_0 \subset \mathfrak{g}$ be an admissible subalgebra (see Definition 13.1). Put $\mathfrak{g}_0' = \varphi(\mathfrak{g}_0)$. If $\mathfrak{h} \subset \mathfrak{g}_0$ is a Cartan subalgebra, $Q \subset \mathfrak{h}^*_R$ the corresponding root lattice, and $Q_0 \subset Q$ the sublattice related with the subalgebra $\mathfrak{g}_0$, put also $\mathfrak{h}' := \varphi(\mathfrak{h}), Q' := \varphi(Q)$ and $Q_0' := \varphi(Q_0)$, here $\varphi := ((\varphi|_{\mathfrak{h}})^*)^T = ((\varphi|_{\mathfrak{h}})^{-1})^T$ is the operator transposed to the conjugate with respect to the Killing form to $\varphi|_{\mathfrak{h}}$. Clearly $\mathfrak{h}' \subset \mathfrak{g}_0'$ is a Cartan subalgebra, $Q' \subset (\mathfrak{h}'_R)^*$ is the corresponding root lattice, and $Q_0'$ is the sublattice related to the subalgebra $\mathfrak{g}_0'$.

It is easy to see that $\varphi$ induces the isomorphism of groups $\bar{\varphi} : \Gamma \to \Gamma'$, where $\Gamma := \Gamma(\mathfrak{h}, \mathfrak{g}_0) = Q/Q_0$ and $\Gamma' := \Gamma(\mathfrak{h}', \mathfrak{g}_0') = Q'/Q_0'$ (see Appendix 13) and the operator $\varphi$ preserves the corresponding toral irreducible gradings $\mathfrak{g} = \bigoplus_{i \in \Gamma} \mathfrak{g}_i = \bigoplus_{i \in \Gamma'} \mathfrak{g}_i'$, or, more precisely, $\varphi(\mathfrak{g}_i) = \mathfrak{g}_{\varphi(i)}$.

If $i, j, k \in \Gamma$ is a triangle, the triple $\varphi(i), \varphi(j), \varphi(k)$ is obviously again a triangle (see Definition 10.13). Moreover, if $\mathcal{L}_S$ is the system of labels of an operator $S \in \mathrm{End}(\mathfrak{g})$ diagonal with respect to the grading $\mathfrak{g} = \bigoplus_{i \in \Gamma} \mathfrak{g}_i$, we can endow the triangle $\bar{\varphi}(i), \bar{\varphi}(j), \bar{\varphi}(k)$ with the same label as the label of $i, j, k$ from the family $\mathcal{L}_S$ and obtain a new system of labels, which will be denoted $\bar{\varphi}\mathcal{L}_S$. Also, by $-\mathcal{L}_S$ we will denote the system of labels opposite to $\mathcal{L}_S$ (i.e. with interchanged pluses and minuses).

13.9. Lemma Let $\mathfrak{g}_0, \mathfrak{g}_0' \subset \mathfrak{g}$ be admissible subalgebras such that $\varphi(\mathfrak{g}_0) = \mathfrak{g}_0'$ for some $\varphi \in \mathrm{Aut}(\mathfrak{g})$. Let $S, S' \in \mathrm{End}(\mathfrak{g})$ be principal (see Definition 7.4) operators diagonal with respect to the toral irreducible gradings $\mathfrak{g} = \bigoplus_{i \in \Gamma(\mathfrak{g}_0)} \mathfrak{g}_i, \mathfrak{g} = \bigoplus_{i \in \Gamma(\mathfrak{g}_0')} \mathfrak{g}_i'$ and obeying the $a, a'$-triangle rule subject to these gradings respectively, where $a \neq 0, a' \neq 0$.

Then the following conditions are equivalent

1. $S' \circ \varphi := \varphi \circ S$;
2. either $a = a'$ and $\mathcal{L}_{S'} = \bar{\varphi}\mathcal{L}_S$ or $a = -a'$ and $\mathcal{L}_{S'} = -\bar{\varphi}\mathcal{L}_S$.

Assume $S' \circ \varphi := \varphi \circ S$. Let $\kappa_i, \kappa_i'$ be the eigenvalues of the corresponding operators related to the eigenspaces $\mathfrak{g}_i, \mathfrak{g}_i'$. Since $\varphi(\mathfrak{g}_i) = \mathfrak{g}_{\varphi(i)}$, then $\kappa_i = \kappa'_{\varphi(i)}$ and, given any triangle $i, j, k$, we have $\kappa_i + \kappa_j + \kappa_k = \kappa_{\varphi(i)} + \kappa_{\varphi(j)} + \kappa_{\varphi(k)}$. Thus either $a = a'$ and the labels of the corresponding triangles coincide, or $a = -a'$ and they are opposite.
Conversely, let condition 2 hold. Put $S'' := \varphi \circ S \circ \varphi^{-1}$ and let $\kappa''_i$ be the corresponding eigenvalues. Then $\kappa_i = \kappa''_i$ and in both cases $S''$ obeys the $a'$-triangle rule with the same labels as $S'$. Now we have to choose a Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}_0$ and observe that the operators $S', S''$ preserve the root decomposition of $\mathfrak{g}$ with respect to the Cartan subalgebra $\mathfrak{h}' := \varphi(\mathfrak{h})$ and their restrictions $S'|_{(\mathfrak{h}')^+}, S''|_{(\mathfrak{h}')^+}$ obey the $a'$-triangle rule subject to the admissible root set $R_0'$ corresponding to the subalgebra $\mathfrak{g}_0'$ (see Definition 10.12).

On the other hand, it is easy to see that $S''$ is principal (indeed, $\text{Tr}(\varphi \circ S \circ \varphi^{-1} \circ \text{ad} x) = \text{Tr}(S \circ \varphi^{-1} \circ \text{ad} x \circ \varphi) = \text{Tr}(S \circ \text{ad} (\varphi^{-1} x)) = 0$ since $\text{Tr}(S \circ \text{ad} x) = 0$ for any $x \in \mathfrak{g}$). By Item 3 of Lemma 13.8 we conclude that $S' = S''$ (cf. Remark 10.14). □

13.10. Theorem Two bi-Lie structures, $(\mathfrak{g},[,][,]_W)$ and $(\mathfrak{g},[,][,]_{W'})$, each of which is constructed as in Theorem 13.7 by data $(t_1, t_2, g_0, g_0^2, \mathcal{L})$ and $(t'_1, t'_2, (g_0^1)', (g_0^2)', \mathcal{L}')$, where $\mathcal{L}, \mathcal{L}'$ denote the systems of labels related to the antisymmetric parts of the corresponding operators, are strongly isomorphic (see Definition 2.6) if and only if the following conditions hold:

1a. there exists $\varphi \in \text{Aut}(\mathfrak{g})$ such that $\varphi(g_0^1) = (g_0^1)', \varphi(g_0^2) = (g_0^2)'$ and $\varphi\mathcal{L} = \mathcal{L}'$, see notations introduced before Lemma 13.9

2a. $t_1 = t'_1, t_2 = t'_2$.

or

1b. there exists $\varphi \in \text{Aut}(\mathfrak{g})$ such that $\varphi(g_0^1) = (g_0^1)', \varphi(g_0^2) = (g_0^2)'$ and $\varphi\mathcal{L} = -\mathcal{L}'$

2b. $t_1 = t'_2, t_2 = t'_1$.

The bi-Lie structures $(\mathfrak{g},[,][,]_W), (\mathfrak{g},[,][,]_{W'})$ are isomorphic if and only if either Condition 1a or 1b is satisfied.

The "strong" case follows from the construction of the operators $W, W'$, Lemma 13.9 and Theorem 7.10. To manage the general case we have only to prove that the principal WNOs $W$ and $W'$ built by data $(t_1, t_2, g_0, g_0^2, \mathcal{L})$ and $(t'_1, t'_2, (g_0^1)', (g_0^2)', \mathcal{L}')$ are related by the formula $W = \lambda W' + \mu \text{Id}_\mathfrak{g}$ for some $\lambda, \mu \in \mathbb{C}$ if and only if either $g_0 = (g_0^1)', g_0^2 = (g_0^2)'$ and $\mathcal{L} = \mathcal{L}'$, or $g_0^1 = (g_0^2)', g_0^2 = (g_0^1)'$ and $\mathcal{L} = -\mathcal{L}'$.

Indeed, if $W = \lambda W' + \mu \text{Id}_\mathfrak{g}$, the operators $W$ and $W'$ should have the same eigenspaces $g_0^1, g_0^2, \mathfrak{g}_j, j \in \Gamma, j \neq 0$. In particular, either 1) $t_i = \lambda t'_i + \mu$ and $g_0^i = (g_0^i)', i = 1, 2$; or 2) $t_i = \lambda t'_i + \mu$ and $g_0^i = (g_0^i)'$, where $i = 1, 2, i' = 2, 1$. Since $\text{Id}_\mathfrak{g}$ does not affect the antisymmetric part of an operator, the systems of labels of the corresponding antisymmetric operators $\overline{W}_a, \overline{W}'_a$ (see Theorem 13.7) should coincide in Case 1) or be opposite in Case 2).

Conversely, assume principal WNOs $W, W'$ are built by data $(t_1, t_2, g_0^1, g_0^2, \mathcal{L})$ and $(t'_1, t'_2, g_0^1, g_0^2, \mathcal{L})$. Since $t'_1 \neq t'_2$, the system of equations $t_i = \lambda t'_i + \mu, i = 1, 2$, has a unique solution $\lambda_0, \mu_0$ with $\lambda_0 \neq 0$. We have decompositions $W = W|_{g_0} + W_a, W' = W'|_{g_0} + W'_a$ and the obvious equalities $W|_{g_0} = \lambda_0 W'|_{g_0} + \mu_0 \text{Id}_{g_0}, W_a = \lambda_0 W'_a + \mu_0 \text{Id}_{\mathfrak{g}_0}$. Thus it remains only to show that $W_a = \lambda_0 W'_a$.

To this end consider the operator $W'' \in \text{End}(\mathfrak{h}^1)$ which is the extension by zero to $\mathfrak{h}^1 \setminus g_0^0$ of the operator $(1/\lambda_0)W_a \in \text{End}(g_0^0)$. Then $W''$ is the restriction to $\mathfrak{h}^1$ of the principal operator $(1/\lambda_0)\overline{W}_a$, obeys the $((t_1 - t'_2)/2)$-rule subject to the same admissible root set with the same labels as $W'$, hence $W'' = W'_a$ on $g_0^0$ by Item 3 of Lemma 13.8.

The case when $W, W'$ are built by data $(t_1, t_2, g_0^1, g_0^2, \mathcal{L})$ and $(t'_1, t'_2, g_0^1, -\mathcal{L})$ is similar. □
13.11. Remark One can reformulate Theorem [13.7] by saying that the corresponding bi-Lie structure is built by a toral symmetric pairoid quasigrading of Class II with the base $X = \{t_1, t_2\}$ and by a system of labels $\mathcal{L}$. Theorem [13.10] can be also reformulated in this spirit. For instance, the second part of it says that the bi-Lie structures built by two pairoid quasigradings and systems of labels $\mathcal{L}, \mathcal{L}'$ are isomorphic if and only if either the pairoid quasigradings are equivalent (see Definition [13.3]) and $\varphi \mathcal{L} = \mathcal{L}'$, where $\varphi$ is the automorphism realizing the equivalence, or antiequivalent and $\varphi \mathcal{L} = -\mathcal{L}'$. 

Now we will use Theorem [13.7] to construct a series of examples of bi-Lie structures of Class II. Let $\mathfrak{g}_0 \subset \mathfrak{g}$ be a Levi subalgebra. Choose a Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$ such that $\mathfrak{h} \subset \mathfrak{g}_0$ and a basis $B$ of the root system $R(\mathfrak{g}, \mathfrak{h})$ such that the corresponding root system $R_0$ of $\mathfrak{g}_0$ is generated over $\mathbb{Z}$ by some subset $B_0 \subset B$ (see Appendix [14]). Put $R^\pm$ for the set of positive (negative) roots with respect to $B$. Then $\mathfrak{g}_0^\pm = \mathfrak{g}_0^0 \oplus \mathfrak{g}_0^\perp$, where $\mathfrak{g}_0^\pm := \sum_{\alpha \in R^\pm \setminus R_0} \mathfrak{g}_0^\alpha$. The subspace $\mathfrak{p}^\pm(\mathfrak{g}_0, B) := \mathfrak{g}_0 \oplus \mathfrak{g}_0^\pm$ is a parabolic subalgebra.

In the following theorem we generalize Example [4.9]

13.12. Theorem Let $\mathfrak{g}_0$ be an admissible subalgebra which is a Levi subalgebra and let $(\mathfrak{g}_0^1, \mathfrak{g}_0^2), \mathfrak{g}_0^1 \perp \mathfrak{g}_0^2 = \mathfrak{g}_0$, be an admissible pair of subalgebras. Choose a Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$ such that $\mathfrak{h} \subset \mathfrak{g}_0$ and a basis $B$ of the root system $R(\mathfrak{g}, \mathfrak{h})$ such that the corresponding root system $R_0$ of $\mathfrak{g}_0$ is generated over $\mathbb{Z}$ by some subset $B_0 \subset B$. Let $\mathfrak{g} = \mathfrak{g}_0^+ \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_0^\perp$ be the corresponding decomposition and let $t_1, t_2 \in \mathbb{C}, t_1 \neq t_2$. Define an operator $W \in \text{End}(\mathfrak{g})$ by:

1. $W|_{\mathfrak{g}_0^j} = t_j \text{Id}_{\mathfrak{g}_0^j}, j = 1, 2$;
2. $W|_{\mathfrak{g}_0^1} = t_1 \text{Id}_{\mathfrak{g}_0^1}, W|_{\mathfrak{g}_0^2} = t_2 \text{Id}_{\mathfrak{g}_0^2}$.

Then the operator $W$ satisfies assumptions of Item 1 of Theorem [13.7] and the triple $(\mathfrak{g}, [], [\cdot, \cdot], W)$ is a semisimple bi-Lie structure of Class II satisfying conditions 1 (a'), (b') of this theorem.

Moreover, two bi-Li structures $(\mathfrak{g}, [], [\cdot, \cdot], W)$ and $(\mathfrak{g}, [], [\cdot, \cdot], W')$ constructed by data $(t_1, t_2, \mathfrak{g}_0^1, \mathfrak{g}_0^2, \mathfrak{h}, B)$ and $(t_1', t_2', (\mathfrak{g}_0^1)', (\mathfrak{g}_0^2)', \mathfrak{h}', B')$, respectively, are isomorphic if and only if either there exists $\varphi \in \text{Aut}(\mathfrak{g})$ such that $\varphi(\mathfrak{g}_0^1) = (\mathfrak{g}_0^1)', j = 1, 2$, and $\varphi(\mathfrak{p}^+(\mathfrak{g}_0 \oplus \mathfrak{g}_0^2, B)) = \mathfrak{p}^+((\mathfrak{g}_0^1)' \oplus (\mathfrak{g}_0^2)', B')$ or there exists $\varphi \in \text{Aut}(\mathfrak{g})$ such that $\varphi(\mathfrak{g}_0^1) = (\mathfrak{g}_0^2)'$, $\varphi(\mathfrak{g}_0^2) = (\mathfrak{g}_0^1)',$ and $\varphi(\mathfrak{p}^+(\mathfrak{g}_0 \oplus \mathfrak{g}_0^2, B)) = \mathfrak{p}^+((\mathfrak{g}_0^1)' \oplus (\mathfrak{g}_0^2)', B')$.

First note that the subspaces $\mathfrak{g}_0^\pm$ are invariant with respect to the action of the subalgebra $\mathfrak{g}_0$ on $\mathfrak{g}_0^\pm = \mathfrak{g}_0^+ \oplus \mathfrak{g}_0^\perp$, i.e. $\mathfrak{g}_0^\pm$ are direct sums of the components of the toral irreducible $\Gamma(\mathfrak{g}_0)$-grading. Obviously, the operator $W$ is scalar on the components of this grading and the subalgebras $\mathfrak{g}_0^\pm$ and satisfies conditions 1 (a), (b) of Theorem [13.7].

Thus we have only to show that the operator $\mathcal{W}_a$ obeys the $((t_1 - t_2)/2)$-triangle rule subject to the grading.

To prove the last fact observe that the triangles $i, j, k \in \Gamma(\mathfrak{g}_0)$ are of two kinds:

- Two of three quasiroots (see Definition [10.13]), say $i, j$, are such that $\mathfrak{g}_k, \mathfrak{g}_j \subset \mathfrak{g}_0^+$, for the third one, $k$, we have $\mathfrak{g}_k \subset \mathfrak{g}_0^{-}$. Then $\kappa_i = \kappa_j = (t_1 - t_2)/2 = -\kappa_k$, the label is $+$. 
- Two of three quasiroots, say $i, j$, are such that $\mathfrak{g}_k, \mathfrak{g}_j \subset \mathfrak{g}_0^{-}$, for the third one, $k$, we have $\mathfrak{g}_k \subset \mathfrak{g}_0^{+}$. Then $\kappa_i = \kappa_j = -(t_1 - t_2)/2 = -\kappa_k$, the label is $-$. 

The last statement of the theorem follows from Theorem [13.10].
13.13. Remark It is easy to see that the eigenspaces of the operator $W$ built in Theorem 13.12 are the subalgebras in $(g, [,])$. Hence the operator $W$ is a Nijenhuis operator, i.e. its torsion $T_W$ vanishes (cf. Example 1.9).

Another series of examples (generalizing Example 1.10 see also [GS02 Example 3]) will be related to inner automorphisms of finite order and to more general admissible subalgebras.

Recall [Hel00 Ch. X] that any such automorphism of a simple Lie algebra $g$ is conjugated to a model one which can be described as follows. Choose a Cartan subalgebra $h \subset g$ and let $R$ stand for the corresponding root system. Pick up a basis $B = \{\alpha_1, \ldots, \alpha_n\}$ of $R$ and let $a_0 = 1, a_1, \ldots, a_n$ be the labels of the Dynkin diagram $\tilde{\Pi}$ of the extended system $\{\alpha_0, \ldots, \alpha_n\}$ (here $\alpha_0$ denotes the lowest root and $-\alpha_0 = \sum_{i=1}^n a_i \alpha_i$). Let $s_0, \ldots, s_n$ be nonnegative integers without nontrivial common factor. Then there exists a system of canonical generators $X_0, \ldots, X_n, X_k \in g_{\alpha_k}$, of the Lie algebra $g$ such that the formula $\sigma(X_k) = e^{\alpha_k(2\pi/m)}X_k$, where $m = \sum_{k=0}^n a_k s_k$, defines uniquely an inner automorphism of $m$-th order, called a model automorphism of type $(s_0, \ldots, s_n; 1)$. Any conjugated automorphism is said to be an automorphism of type $(s_0, \ldots, s_n; 1)$.

The fixed point subalgebra $g_0$ of the automorphism $\sigma$ contains $h$ and is a direct sum of its $(n-p)$-dimensional centre and the semisimple Lie algebra $[g_0, g_0]$ whose Dynkin diagram $\pi$ is the subdiagram of $\tilde{\Pi}$ consisting of the vertices $i_1, \ldots, i_p$ for which $s_{i_1} = \cdots = s_{i_p} = 0$. In particular, if $s_0 \neq 0$, the subalgebra $g_0$ is a Levi subalgebra, and if $s_0 = 0$, the subalgebra $g_0$ is a regular subalgebra of Class 2) (see Appendix 13).

13.14. Remark Any automorphism mentioned induces a $\mathbb{Z}_m$-grading $g = g_0 \oplus \cdots \oplus g_{m-1}$, where $g_j$ is the eigenspace of $\sigma$ corresponding to the eigenvalue $e^{(2\pi/m)j}$ (in particular $g_0 = g_0$). This grading is toral with respect to $h$. On the other hand, this grading is a coarsening of the toral irreducible $\Gamma(g_0)$-grading (13.7.1) and there exists an additive epimorphism $\psi: \Gamma(g_0) \to \mathbb{Z}_m$ such that $g_j = \bigoplus_{i \in \psi^{-1}(j)} g_i$, $j = 1, \ldots, m-1$ (recall that we denote by $\Gamma$ the set of quasiroots of a $\Gamma$-grading, see Definition 10.13).

13.15. Theorem 1. Let $g_0$ be the fixed point subalgebra of an inner automorphism of order $m > 2$ of type $(s_0, \ldots, s_n; 1)$ (in particular $g_0$ is admissible) and let $(g_0^1, g_0^2)$ be an admissible pair of subalgebras such that $g_0^1 \oplus g_0^2 = g_0$. Let $g = g_0 \oplus \cdots \oplus g_{m-1}$ be the corresponding grading. Given $t_1, t_2 \in \mathbb{C}, t_1 \neq t_2$, define an operator $W \in \text{End}(g)$ by

(a) $W|_{g_i^1} = t_1 \text{Id}_{g_i^1}$, $i = 1, 2$;
(b) $W|_{g_i^1} = (((m-j)t_1 + jt_2)/m)\text{Id}_{g_i^1}$ if $j > 0$.

Then the operator $W$ satisfies assumptions of Item 1 of Theorem 13.7; consequently, the triple $(g, [,], [.,])$ is a semisimple bi-Lie structure of Class II satisfying conditions (a’), (b’) of this theorem.

2. If $g_0$ is a Levi subalgebra (i.e. $s_0 \neq 0$), the bi-Lie structure $(g, [,], [.,])$ is isomorphic to that constructed by data $(t_1, t_2, g_0^1, g_0^2, h, B)$ in Theorem 13.12.

3. Two bi-Lie structures constructed by data $(t_1, t_2, h, B, (s_0, \ldots, s_n), (g_0^1, g_0^2))$ and $(t_1’, t_2’, h’, B’, (s_0’, \ldots, s_n’), (g_0^1’, g_0^2’))$ are isomorphic if and only if either there exists $\varphi \in \text{Aut}(g)$ such that $\varphi(g_0^1) = (g_0^1’), \varphi(g_0^2) = (g_0^2’)$ and $\varphi([.,]) = [.,]$ (in particular $\varphi$ is an automorphism of type $(s_0, \ldots, s_n; 1)$) or there exists $\varphi \in \text{Aut}(g)$ such that $\varphi(g_0^1) = (g_0^1’), \varphi(g_0^2) = (g_0^2’)$ and $\varphi([.,]) = [.,]$ (in particular $\varphi$ is an automorphism of type $(s_0, \ldots, s_n; 1)$).
13.16. Remark The formulae (a), (b) appeared in [GS02, Example 3].

Ad 1. Clearly it is enough consider a model automorphism of type \((s_0, \ldots, s_n; 1)\). Since the \(\mathbb{Z}_m\)-grading is a coarsening of the toral irreducible \(\Gamma(\mathfrak{g}_0)\)-grading, the operator \(W\) is scalar on the components of the latter grading. By Item 4 of Theorem 13.2 we have \(W_s|_{g_j} = ((t_1 + t_2)/2)\text{Id}_{g_j}\) for any \(j > 0\) and Conditions 1 (a), (b) of Theorem 13.7 are satisfied.

To check Condition 1 (c) due to Remark 13.14 it is enough to check the \(((t_1 - t_2)/2)\)-triangle rule subject to the \(\mathbb{Z}_m\)-grading. Observe that \(W_a|_{g_j} = \kappa_j\text{Id}_{g_j}\), where \(\kappa_j = ((-j + m/2)(t_1 - t_2)/m)\), and that triangles \(\bar{i}, \bar{j}, \bar{k} \in \mathbb{Z}_m\), are of two kinds:

- \(i + j + k = m\) (then \(\kappa_i + \kappa_j + \kappa_k = (t_1 - t_2)/2\), the label is +);
- \(i + j + k = 2m\) (then \(\kappa_i + \kappa_j + \kappa_k = -(t_1 - t_2)/2\), the label is -).

Ad 2. This item follows from Theorem 13.10 and from the fact that the corresponding operators have identical systems of labels with respect to the toral irreducible \(\Gamma(\mathfrak{g}_0)\)-grading.

To prove this fact notice that, if \(\bar{i}, \bar{j}, \bar{k}\) is a triangle, then there exist roots \(\alpha, \beta, \gamma\) such that \(\mathfrak{g}_0 \subset \mathfrak{g}_i, \mathfrak{g}_\beta \subset \mathfrak{g}_j, \mathfrak{g}_\gamma \subset \mathfrak{g}_k\) and \(\alpha + \beta + \gamma = 0\). We will show that in the case \(i + j + k = m\) two of these roots are positive (and the third negative) and vice versa in the case \(i + j + k = 2m\) (cf. the proof of Theorem 13.12).

Note that any root \(\alpha\) can be uniquely represented as \(\alpha = \sum_{i=0}^{n} k_i^\alpha \alpha_i\), where \(0 \leq k_i^\alpha \leq a_i\) \((k_0^\alpha = 0\) if \(\alpha > 0\) and \(k_0^\alpha = 1\) if \(\alpha < 0\)). Moreover, \(\mathfrak{g}_p\) consists of those roots \(\alpha\) for which \(\sum_{i=0}^{n} k_i^\alpha s_i = p\) \([GOV94, \S 3.7]\).

Let \(i + j + k = 2m\) and assume that two of the three roots, say \(\alpha, \beta\), are positive. Then \(\alpha = \sum_{i=1}^{n} k_i^\alpha \alpha_i, \beta = \sum_{i=1}^{n} k_i^\beta \alpha_i\) and \(\alpha + \beta = \sum_{i=1}^{n} (k_i^\alpha + k_i^\beta) \alpha_i\), in particular \(k_i^\alpha + k_i^\beta \leq a_i\) for any \(l\). On the other hand, \(i + j = \sum_{i=1}^{n} (k_i^\alpha + k_i^\beta) s_i\) has to be greater than \(m = s_0 + \sum_{i=1}^{n} a_is_i\), whence \(k_i^\alpha + k_i^\beta > a_i\) for some \(l \in \{1, \ldots, n\}\), a contradiction.

Now let \(i + j + k = m\) and assume that two of the three roots, say \(\alpha, \beta\), are negative. Then \(\alpha = \alpha_0 + \sum_{i=1}^{n} k_i^\alpha \alpha_i, \beta = \alpha_0 + \sum_{i=1}^{n} k_i^\beta \alpha_i\) and \(\alpha + \beta = \alpha_0 + \sum_{i=1}^{n} (k_i^\alpha + k_i^\beta - a_i) \alpha_i\), in particular \(k_i^\alpha + k_i^\beta \geq a_i\) for any \(l\). On the other hand, \(i + j = 2s_0 + \sum_{i=1}^{n} (k_i^\alpha + k_i^\beta) s_i\) has to be less than \(m = s_0 + \sum_{i=1}^{n} a_i s_i\), whence \(k_i^\alpha + k_i^\beta < a_i\) for some \(l \in \{1, \ldots, n\}\), a contradiction.

Ad 3. The proof of the previous item shows that the system of labels of the antisymmetric part of the operator \(W\) depends only on the subalgebra \(\mathfrak{g}_0\) and the decomposition of \(\mathfrak{g}_0\) to subspaces corresponding to positive and negative roots. Now we can use Theorem 13.10. \(\square\)

The proofs of Theorems 13.12 \& 13.15 show that the corresponding systems of labels have the following simple property (up to a sign):

- any triangle with nonzero label and two positive (negative) roots has label + (−).

Using this observation we can build new examples of a bi-Lie structures of Class II, in particular, having the most general reductive subalgebras of maximal rank as their basic subalgebras.

13.17. Example Let \(\mathfrak{g} = \mathfrak{e}_7\). We will use notations from [Bon68, Table VI]. Let \(R_0 = \text{Span}_\mathbb{Z}\{\alpha_0, \alpha_1, \alpha_2, \alpha_4, \alpha_6, \alpha_7\} \cap R = \text{Span}_\mathbb{Z}\{\alpha_1, \alpha_2, 3\alpha_3, 4\alpha_4, 3\alpha_5, \alpha_6, \alpha_7\} \cap R\), here \(\alpha_0 = -(2\alpha_1 + 2\alpha_2 + 3\alpha_3 + 4\alpha_4 + 3\alpha_5 + 2\alpha_6 + \alpha_7)\) is the lowest root. Then the corresponding regular reductive Lie subalgebra \(\mathfrak{g}_0\) is
of Class 3) (see Appendix 14) since \( \Gamma(\mathfrak{g}_0) = Q/Q_0 \cong \mathbb{Z}_3 \times \mathbb{Z}_3 \). The subalgebra \( \mathfrak{g}_0 \) is the intersection of the fixed point subalgebras of two commuting inner automorphisms of order 3 and is not a fixed point subalgebra of any single inner automorphism of finite order (i.e. \( \mathfrak{g}_0 \) is not of Class 2)).

To build the antisymmetric part of a WNO we will use the induction process from the proof of Lemma 13.8. Starting this process from any vector \((c_1, \ldots, c_7)\) we will obtain a diagonal antisymmetric operator \( S \in \text{End}(\mathfrak{h}^+) \), \( S|_{\mathfrak{h}_0} = \kappa_{\alpha} \text{Id}_{\mathfrak{g}_0} \). We have to take care of the condition \( \kappa_{\alpha} = 0, \alpha \in R_0 \) (see Definitions 10.12, 10.13), which gives a system of linear equations on the variables \((c_1, \ldots, c_7)\). Five equations from this system are obvious: \( c_1 = 0, c_2 = 0, c_4 = 0, c_6 = 0, c_7 = 0 \). Thus in fact we can start the induction process from the vector \((0, 0, c_3, 0, c_5, 0, 0)\). In the result the nonzero entries will correspond only to the roots containing the combinations of the roots \( \alpha_3, \alpha_5 \):

\[
\begin{array}{c|c|c|c|c|c|c|c}
\alpha_3 & c_3 & \alpha_3 + \alpha_5 & c_3 + c_5 + a & 2\alpha_3 + \alpha_5 & 2c_3 + c_5 + 2a & 2\alpha_3 + 3\alpha_5 & 2c_3 + 3c_5 + 4a \\
\hline
\alpha_5 & c_5 & \alpha_5 + 2\alpha_5 & c_2 + 2c_3 + 2a & 2\alpha_5 + 2\alpha_5 & 2c_3 + 2c_5 + 3a \\
\end{array}
\]

Note that the combination \( 3\alpha_3 + 2\alpha_5 \) do not entry in any root and the combination \( 3\alpha_3 + 3\alpha_5 \) gives the only relation on the constants \((c_3, c_5)\): \( 3c_3 + 3c_5 + 4a = 0 \) (the last equation from the above mentioned system). Putting \( x := c_3 \) we get \( c_5 = -x - 4a/3 \) and the following table:

\[
\begin{array}{c|c|c|c|c|c|c|c}
\alpha_3 & x & \alpha_3 + \alpha_5 & -a/3 & 2\alpha_3 + \alpha_5 & x + 2a/3 & 2\alpha_3 + 3\alpha_5 & -x \\
\hline
\alpha_5 & -x - 4a/3 & \alpha_5 + 2\alpha_5 & -a/3 & 2\alpha_5 + 2\alpha_5 & a/3 \\
\end{array}
\]

To complete the construction we need to: 1) extend this table by the skew symmetry to negative roots; 2) split \( \mathfrak{g}_0 \) to an admissible pair \((\mathfrak{g}_0^1, \mathfrak{g}_0^3)\) (remark that the subalgebra \([\mathfrak{g}_0, \mathfrak{g}_0] \) is a sum of three simple subalgebras, so such splitting can be done in several ways); 3) choose \( t_1, t_2 \in \mathbb{C}, t_1 \neq t_2 \), and put \( W_s|_{\mathfrak{g}_0} = t_1 \text{Id}_{\mathfrak{g}_0} \) for \( i = 1, 2 \) and \( W_s|_{\mathfrak{g}_0} = ((t_1 + t_2)/2) \text{Id}_{\mathfrak{g}_0} \) for \( \alpha \in R_0 \); 4) recall that \( a = (t_1 - t_2)/2 \). We get a one-parameter family of bi-Lie structures of Class II with the set of times \( \{t_1, t_2\} \).

This example can be generalized to the case of arbitrary reductive subalgebra \( \mathfrak{g}_0 \). We can now formulate

13.18. CONJECTURE  For any bi-Lie structure \((\mathfrak{g}, [], [\cdot, \cdot])\) of Class II there exists a Cartan subalgebra \( \mathfrak{h} \) and a basis of the root system \( R = R(\mathfrak{g}, \mathfrak{h}) \) such that the system of labels of the antisymmetric part of corresponding principal WNO restricted to \( \mathfrak{h}^\perp \) has the property mentioned before Example 13.17. In particular,

1. any bi-Lie structure of Class II with the basic subalgebra \( \mathfrak{g}_0 \) being a Levi subalgebra (i.e. a regular reductive subalgebra of Class 1), see Appendix 14) is isomorphic to one of that built in Theorem 13.12.

2. any bi-Lie structure of Class II with the basic subalgebra \( \mathfrak{g}_0 \) being a regular reductive subalgebra of Class 2) is isomorphic to one of that built in Theorem 13.13.

We conclude this section by proving this conjecture in a particular case \((\mathfrak{g}, [\cdot, \cdot]) = \mathfrak{a}_n \) thus obtaining a complete classification of bi-Lie structures of Class II in this case.

13.19. THEOREM  Let \((\mathfrak{g}, [\cdot, \cdot]) = \mathfrak{a}_n \). Then any semisimple regular bi-Lie structure \((\mathfrak{g}, [\cdot, \cdot], [\cdot, \cdot])\) of Class II is isomorphic to one of that built in Theorem 13.12.
Let \( \{t_1, t_2\} \) be the set of times of the bi-Lie structure \((\mathfrak{g}, [\cdot, \cdot], [\cdot, \cdot])\). Let \( \{\alpha_1, \ldots, \alpha_n\} \) be a basis of roots of the corresponding root system \( R(\mathfrak{g}, \mathfrak{h}) \) and let \( R_0 \subseteq R \) be the root set corresponding to the basic subalgebra \( \mathfrak{g}_0 \). By Corollary 10.4.1 the corresponding principal WNO obeys the \( a \)-triangle rule subject to \( R_0 \), where \( a = (t_1 - t_2)/2 \). We will first prove that there exists a WNO \( W \) of this bi-Lie structure such that all the eigenvalues of \( W_{\mathfrak{a}^+} \) are equal to zero or \( \pm a \).

Consider the triangle \( \{\beta_{i-1}, \alpha_i, -\beta_i\}, i = 2, \ldots, n \), where we write \( \beta_i := \alpha_1 + \cdots + \alpha_i \), and put \( d_i = 0, \pm 1 \) for the corresponding label. Consider the vector \( (d_1a, d_2a, \ldots, d_na) \), where by definition \( d_1 = 0 \) if \( \alpha_1 \in R_0 \) and \( d_1 = 1 \) otherwise, and start the induction process of the proof of Lemma 13.8 from this vector. We claim that the eigenvalues \( \kappa_\alpha \) of the resulting antisymmetric operator will satisfy the required condition.

Indeed, let \( \alpha := \alpha_k + \alpha_{k+1} + \cdots + \alpha_{k+m} = \beta_{k+m} - \beta_{k-1} \) for some \( 1 \leq k \leq n-1, m \geq 0 \). Then it belongs to the triangle \( \{\alpha, \beta_{k-1}, -\beta_{k+m}\} \) with the label \( d = 0, \pm 1 \) and

\[
\kappa_\alpha = -\kappa_{\beta_{k-1}} + \kappa_{\beta_{k+m}} + da. \tag{13.19.1}
\]

On the other hand, the induction process gives \( \kappa_{\beta_{i-1}} = \kappa_{\alpha_i} = d_1a, \kappa_{\beta_i} = \kappa_{\beta_{i-1}} + d_2a = d_1a + d_2a - d_2a = d_1a, \ldots, \kappa_{\beta_{k-1}} = \kappa_{\beta_{k-2}} + \kappa_{\alpha_{k-1}} - d_{k-1}a = d_1a + d_{k-1}a - d_{k-1}a = d_1a, \ldots, \kappa_{\beta_{k+m}} = d_1a. \) Hence by (13.19.1) we get \( \kappa_\alpha = da \).

In the next step of the proof consider the operator just built and notice that the set \( Y := \{\alpha \in R \mid \kappa_\alpha = a\} \) is closed: if \( \alpha, \beta \in Y \) and \( \gamma := \alpha + \beta \in R \), then \( \kappa_\gamma = \kappa_\alpha + \kappa_\beta - d'\alpha = a \), here \( d' \) is the corresponding label, which necessarily equals \( +1 \). Now remark that \( Y \cap (-Y) = \emptyset \) and use [Bou68, Prop. 22, Ch.VI] to deduce that there exists a basis of \( R \) such that \( Y \) is contained in the set of positive roots \( R^+ \) with respect to this basis.

Finally, observing that any triangle with a nonzero label and with two positive roots by construction has label \( +1 \) we prove Conjecture 13.18.

The rest of the proof follows from Theorems 13.7 13.10 and Remark 14.4. □

14 Appendix I: Toral gradings and regular reductive Lie subalgebras

14.1. Definition Let \( \mathfrak{g} \) be a semisimple Lie algebra and let \( \mathfrak{g} = \mathfrak{h} + \sum_{\alpha \in R} \mathfrak{g}_\alpha \) be the root decomposition of \( \mathfrak{g} \) with respect to some Cartan subalgebra \( \mathfrak{h} \subset \mathfrak{g} \). We say that a \( \Gamma \)-grading \( \mathfrak{g} = \bigoplus_{i \in \Gamma} \mathfrak{g}_i \), where \( \Gamma \) is an abelian group, is toral with respect to \( \mathfrak{h} \) if the subspaces \( \mathfrak{g}_i \) are spanned by the subspaces \( \mathfrak{h}, \mathfrak{g}_0 \) and \( \mathfrak{g}_0 \supseteq \mathfrak{h} \).

Let \( \mathfrak{g}_0 \) be a reductive subalgebra in \( \mathfrak{g} \) such that \( \mathfrak{g}_0 \supseteq \mathfrak{h} \). In particular the representation \( x \mapsto \text{ad}_\mathfrak{g} x \) of the Lie algebra \( \mathfrak{g}_0 \) in \( \mathfrak{g}_0^\perp \) (here \( \mathfrak{g}_0^\perp \) is the orthogonal complement to \( \mathfrak{g}_0 \) in \( \mathfrak{g} \) with respect to the Killing form) is semisimple.

Let \( R_0 := \{\alpha \in R \mid \mathfrak{g}_\alpha \subset [\mathfrak{g}_0, \mathfrak{g}_0]\} \subseteq R \subseteq \mathfrak{h}_R^* \) be the root system of the subalgebra \( [\mathfrak{g}_0, \mathfrak{g}_0] \) considered as the subsystem of \( R \) and let \( Q_0 \subset Q \subset \mathfrak{h}_R^* \) be the corresponding root lattices of \( [\mathfrak{g}_0, \mathfrak{g}_0] \) and \( \mathfrak{g} \). Given \( \alpha \in Q \), put \( \mathfrak{g}_{\alpha+Q_0} := \bigoplus_{\beta \in (\alpha+Q_0) \cap R} \mathfrak{g}_\beta \).

14.2. Theorem [Os, Section 6.2] The decomposition \( \mathfrak{g} = \bigoplus_{\alpha+Q_0 \in Q/Q_0} \mathfrak{g}_{\alpha+Q_0} \) is a toral \( Q/Q_0 \)-grading. Moreover, the subspaces \( \mathfrak{g}_{\alpha+Q_0}, \alpha \notin Q_0 \), are the irreducible components of the representation of \( \mathfrak{g}_0 \) in \( \mathfrak{g}_0^\perp \).
Let $\mathfrak{h}'$ be another Cartan subalgebra such that $\mathfrak{h}' \subset \mathfrak{g}_0$ and let $Q', Q'_0$ stand for the corresponding root lattices. Then there exists $\varphi \in \text{Aut}(\mathfrak{g})$ such that $\varphi(\mathfrak{h}) = \mathfrak{h}'$. It is easy to see that $(\varphi|_\mathfrak{h})^T(Q) = Q$ and $(\varphi|_\mathfrak{h})^T(Q'_0) = Q_0$, here $(\varphi|_\mathfrak{h})^T$ is the operator transposed to $\varphi|_\mathfrak{h}$. In particular, the groups $\Gamma(\mathfrak{h}, \mathfrak{g}_0) := Q/Q_0$ and $\Gamma(\mathfrak{h}', \mathfrak{g}_0) := Q'/Q'_0$ are isomorphic.

14.3. Definition Let $\mathfrak{g}_0 \subset \mathfrak{g}$ be a reductive in $\mathfrak{g}$ subalgebra of maximal rank (call such a subalgebra regular) and let $\Gamma(\mathfrak{g}_0)$ be an abstract group isomorphic to one of the groups $\Gamma(\mathfrak{h}, \mathfrak{g}_0)$, where $\mathfrak{h}$ is a Cartan subalgebra such that $\mathfrak{h} \subset \mathfrak{g}_0$. The grading constructed above will be called the irreducible toral $\Gamma(\mathfrak{g}_0)$-grading corresponding to the subalgebra $\mathfrak{g}_0$.

Regular subalgebras can be divided to the following three classes depending on the structure of the group $\Gamma(\mathfrak{g}_0)$ (cf. [DO02]): 1) $\Gamma(\mathfrak{g}_0)$ is free; 2) the torsion component $\text{Tor}(\Gamma(\mathfrak{g}_0))$ of $\Gamma(\mathfrak{g}_0)$ is cyclic; 3) $\text{Tor}(\Gamma(\mathfrak{g}_0))$ is not cyclic.

In the first case the subalgebra $\mathfrak{g}_0$ is the so-called Levi subalgebra, which can be also characterized by the following equivalent conditions ([GOV94, Th. 1.3, Ch. 6]):

1. $\mathfrak{g}_0$ is a centralizer $\mathfrak{j}_0(\mathfrak{c})$ of some subspace $\mathfrak{c} \subset \mathfrak{h}$, where $\mathfrak{h}$ is some Cartan subalgebra.

2. $\mathfrak{g}_0$ contains a Cartan subalgebra $\mathfrak{h}$ and the corresponding system $R_0 \subset R$ of roots of $\mathfrak{g}_0$ is generated over $\mathbb{Z}$ by some subset $B_0$ of some system $B(\mathfrak{g})$ of simple roots of $\mathfrak{g}$.

To describe the other two cases recall [Dyn52, GOV94] that any regular subalgebra $\mathfrak{g}_0 \subset \mathfrak{g}$ can be obtained as the last element in a nested sequence of subalgebras $\mathfrak{g} = \mathfrak{g}_0 \supset \mathfrak{g}_1 \supset \cdots \supset \mathfrak{g}_k = \mathfrak{g}_0$, where each $\mathfrak{g}_j$, $j = 1, \ldots, k - 1$, is obtained from $\mathfrak{g}_{j-1}$ by an elementary transformation and $\mathfrak{g}_0$ is a Levi subalgebra in $\mathfrak{g}_{j-1}$. We say that a semisimple subalgebra $\mathfrak{k} \subset \mathfrak{g}$ of a semisimple Lie algebra $\mathfrak{g} = \bigoplus_{j \in J} \mathfrak{g}_j$, where $\mathfrak{g}_j$ are the simple components, is obtained by an elementary transformation from $\mathfrak{g}$ if $\mathfrak{k} = \bigoplus_{j \neq j'} \mathfrak{g}_j \oplus \mathfrak{g}_{j'}$ for some $j' \in J$. Here $\mathfrak{g}_{j'}$ is a subalgebra in $\mathfrak{g}_{j'}$ such that its system of simple roots $B(\mathfrak{g}_{j'})$ is obtained by eliminating of some root $\alpha_i$, $i > 0$, from the system $B(\mathfrak{g}_{j'}) \cup \{\alpha_0\}$ of simple roots of $\mathfrak{g}_{j'}$ extended by the lowest root $\alpha_0$. In other words, $\mathfrak{g}_{j'}$ is the fixed point subalgebra of an (inner) automorphism $\sigma \in \text{Aut}(\mathfrak{g}_{j'})$ of order $a_i$ of type $(0, \ldots, 0, 1, 0, \ldots, 0; 1)$ (the first unit is on the $i$-th place and $i > 0$) in terminology of [Hel00, Ch. X] (see also the discussion after Remark 13.13), here $a_i$ is the label (i.e. the coefficient in the highest root) of the root $\alpha_i$.

The equivalent characterization of regular subalgebras of Class 2) (respectively 3)) is, [DO02], that an elementary transformation in the sequence $\mathfrak{g} \supset \cdots \supset \mathfrak{g}_0$ is made only once (respectively more than once).

14.4. Remark The structure of the extended Dynkin diagram of the Lie algebra $\mathfrak{g} = \mathfrak{a}_n$ implies that any regular subalgebra in $\mathfrak{g}$ is a Levi subalgebra.

15 Appendix II: bi-Lie structures of Class I on compact real forms of complex semisimple Lie algebras

15.1. Theorem Let $(\mathfrak{g}, [\cdot, \cdot])$ be a complex semisimple Lie algebra, $\mathfrak{h} \subset \mathfrak{g}$ a Cartan subalgebra, $\mathfrak{g} = \mathfrak{h} + \sum_{\alpha \in R} \mathfrak{g}_\alpha$ the corresponding root grading. Let $\mathfrak{g} = \bigoplus_{t_i, t_j \in X} \mathfrak{g}_{t_i, t_j}$ a toral (with respect to $\mathfrak{h}$) symmetric pair quasigrading of class I with the base $X = \{t_1, \ldots, t_n\}$, where $t_i \in \mathbb{R}$, such that the subspaces $\mathfrak{g}_{t_i, t_i} \cap \mathfrak{h}$ are the complexifications of subspaces in $\mathfrak{h}_\mathbb{R}$. 
For any $\alpha \in \mathbb{R}$ choose $E_\alpha \in \mathfrak{g}_\alpha$ such that $B_\mathfrak{g}(E_\alpha, E_{-\alpha}) = 1$ and put $H_\alpha := [E_\alpha, E_{-\alpha}]$ as usual. Let $u = \sum_{\alpha \in \mathbb{R}} \mathbb{R}(iH_\alpha) + \sum_{\alpha \in \mathbb{R}} \mathbb{R}(E_\alpha - E_{-\alpha}) + \sum_{\alpha \in \mathbb{R}} \mathbb{R}(i(E_\alpha + E_{-\alpha})) \subset \mathfrak{g}$ be the compact real form of $\mathfrak{g}$ related with the root decomposition and the choice of $E_\alpha$ (see [Hel00, Th. 6.3, Ch. III])

Then the WNO $W$ built by the quasigrading (see Theorem 12.7) has a correct restriction to $u$ which is a real WNO and induces a real bi-Lie structure on $u$.

The fact that $W$ preserves $u$ follows from the assumptions and from the construction of $W$ (recall $W_{|\mathfrak{h}^\bot}$ is symmetric: $W_{|\mathfrak{g}_\alpha} = W_{|\mathfrak{g}_{-\alpha}}$). The principal primitive $P$ of $W$ is symmetric too (see Definition 7.6 or check this directly using the formulae from the proof of Theorem 12.7). Thus the main identity $T_N(,) = [,]_P$ implies $T_N|_u(,) = [,]_P|_u$. □

15.2. Example From Example 12.16 we get the following example of WNO on $\mathfrak{su}(n + 1)$: $WX := (1/2) (LA + RA)X - \text{Tr}(1/2) (LA + RA)X B$, where $A = \text{diag}(t_1, t_2, \ldots, t_{n+1})$, $B = \text{diag}(0, 0, \ldots, 0, 1)$ and $t_i \in \mathbb{R}$. Explicitly, if $X = ||x_{ij}|| \in \mathfrak{su}(n + 1)$, then $WX = ||y_{ij}||$, where $y_{ij} = x_{ij}(t_i + t_j)/2$ for $i \neq j$, $y_{ii} = x_{ii}t_i$ for $i = 1, \ldots, n$, and $y_{n+1,n+1} = -\sum_{j=1}^n x_{jj}t_j$.

Analogous example can be obtained from Example 12.17 taking the parameter $a$ to be real.

15.3. Example Here we will only mention the existence of a nonstandard bi-Lie structure on $\mathfrak{so}(6, \mathbb{R})$ coming from the isomorphism $\mathfrak{so}(6, \mathbb{C}) \cong \mathfrak{sl}(4, \mathbb{C})$ and Examples 12.16 12.17

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