The extended Lorentz force

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Abstract

The Lorentz force equations provide a partial description of the geodesic motion of a charged particle on a four-manifold. Under the hypothesis that Maxwell’s equations express symmetry properties of the Ricci tensor, the full electromagnetic connection is determined. From this connection, the fourth equation of the geodesic is derived. The validity of this fourth equation can be determined by studying the decay of charged particles in an electric field. Time will accelerate or decelerate relative to the proper time of a charged particle moving in an electric field. Unstable charged particles moving in opposite directions parallel to an electric field should exhibit different decay rates.

It has been proposed before that the motion of a charged particle in an electromagnetic field on the four-manifold of our universe is described by the geodesic equation. (See Kaluza 1921, Klein 1926, Fock 1927, which are translated and reprinted in O’Raifeartaigh 1997.) At this time, it remains unclear as to whether such motion does follow a geodesic path.

The geodesic equation on a four-manifold is four equations giving the second derivatives of the four spatial and temporal coordinates:

\[
\frac{d^2 x}{ds^2}, \quad \frac{d^2 y}{ds^2}, \quad \frac{d^2 z}{ds^2}, \quad \frac{d^2 t}{ds^2},
\]

where \( s \) is the geodesic length parameter. The Lorentz force equations are an empirical solution for the motion of a charged particle in three-space; they account for only three of the second derivatives. It is possible that \( d^2 t/ds^2 \) is zero. However, there is no theoretical justification for this assumption. The Lorentz force equations were constructed to explain experimental results. At the time the equations were developed, there was no motivation to look beyond three dimensions.

This analysis develops a geometric model based on the hypothesis that Maxwell’s equations express symmetry properties of the Ricci tensor. The full electromagnetic connection is derived. The connection components determine the geodesic equation. Defining the proper time \( \tau \) in
terms of the length parameter, $c\tau = s$, the model determines the fourth equation of the Lorentz force to be

$$\frac{d^2 t}{d\tau^2} = 2 \frac{q}{mc^2} \left( \mathbf{E} \cdot \frac{d\mathbf{x}}{d\tau} \right) \frac{dt}{d\tau}$$

This equation can be approximated as

$$\frac{d^2 t}{d\tau^2} = 2 \frac{q}{mc^2} \mathbf{E} \cdot \mathbf{v}$$

The new “force” equation can be subjected to experimental verification by studying the decay of charged particles in an electric field. Time will accelerate or decelerate relative to the proper time of a charged particle moving in an electric field. Unstable charged particles moving in opposite directions parallel to an electric field should exhibit different decay rates.

## 1 Development of the Connection

The classical Lorentz force equation

$$m \mathbf{a} = q \mathbf{E} + \frac{q}{c} \mathbf{v} \times \mathbf{B}$$

provides an initial source of information about the connection components. The form of the Lorentz force appears to contradict its interpretation as a geodesic equation, which must be uniformly quadratic in the components of the 4-velocity. However, the fourth component of the 4-velocity, $d(ct)/ds$, would be very close to 1 and essentially constant in classical experiments; it would not have been observable. In presenting the Lorentz force as a first approximation to the geodesic path, we add the fourth velocity component to the classical terms by writing them as

$$\frac{d^2 x_i}{ds^2} = \frac{q}{mc^2} E_x \frac{d(ct)}{ds} \frac{dx_i}{ds} + \frac{q}{mc^2} B_z \frac{dy}{ds} \frac{d(ct)}{ds} - \frac{q}{mc^2} B_y \frac{dz}{ds} \frac{d(ct)}{ds}$$

$$\frac{d^2 y}{ds^2} = \frac{q}{mc^2} E_y \frac{d(ct)}{ds} \frac{dy}{ds} + \frac{q}{mc^2} B_z \frac{dz}{ds} \frac{d(ct)}{ds} - \frac{q}{mc^2} B_x \frac{dx}{ds} \frac{d(ct)}{ds}$$

$$\frac{d^2 z}{ds^2} = \frac{q}{mc^2} E_z \frac{d(ct)}{ds} \frac{dz}{ds} + \frac{q}{mc^2} B_y \frac{dx}{ds} \frac{d(ct)}{ds} - \frac{q}{mc^2} B_x \frac{dy}{ds} \frac{d(ct)}{ds}$$

This step creates the quadratic structure that allows the interpretation of the Lorentz force as a geodesic.

The electromagnetic equations presented here use Gaussian units, as described in Jackson (1962) p. 616. The geodesic equation in terms of the connection components $\Gamma^i_{jk}$ is

$$\frac{d^2 x_i}{ds^2} + \sum_{j,k} \Gamma^i_{jk} \frac{dx_j}{ds} \frac{dx_k}{ds} = 0$$
The connection components appear in equation [1] as the negative of the coefficients of the
derivative terms on the right-hand side.

There is a certain degree of ambiguity in the initial placement of the $B$-valued components.
After studying different placements, the components from the classical Lorentz force equation
were placed in the following manner:

|   |   |   |   |   |   |   |
|---|---|---|---|---|---|---|
| $i$ | $j$ | $k$ | $\Gamma^i_{jk}$ | $i$ | $j$ | $k$ | $\Gamma^i_{jk}$ | $i$ | $j$ | $k$ | $\Gamma^i_{jk}$ |
| 1 | 0 | 0 | $-qE_x/mc^2$ | 2 | 0 | 0 | $-qE_y/mc^2$ | 3 | 0 | 0 | $-qE_z/mc^2$ |
| 2 | 0 | 3 | $-qB_x/mc^2$ | 3 | 1 | 0 | $-qB_y/mc^2$ | 1 | 2 | 0 | $-qB_z/mc^2$ |
| 3 | 0 | 2 | $qB_x/mc^2$ | 1 | 0 | 3 | $qB_y/mc^2$ | 2 | 0 | 1 | $qB_z/mc^2$ |

The time dimension is denoted by 0, while 1, 2, 3 are the spatial dimensions $x$, $y$, $z$.

The following alternative placement has identical properties in all areas considered in this
paper.

|   |   |   |   |   |   |   |
|---|---|---|---|---|---|---|
| $i$ | $j$ | $k$ | $\Gamma^i_{jk}$ | $i$ | $j$ | $k$ | $\Gamma^i_{jk}$ | $i$ | $j$ | $k$ | $\Gamma^i_{jk}$ |
| 1 | 0 | 0 | $-qE_x/mc^2$ | 2 | 0 | 0 | $-qE_y/mc^2$ | 3 | 0 | 0 | $-qE_z/mc^2$ |
| 2 | 0 | 3 | $-qB_x/mc^2$ | 3 | 1 | 0 | $-qB_y/mc^2$ | 1 | 2 | 0 | $-qB_z/mc^2$ |
| 3 | 2 | 0 | $qB_x/mc^2$ | 1 | 3 | 0 | $qB_y/mc^2$ | 2 | 1 | 0 | $qB_z/mc^2$ |

Other alternative placements of the $B$-valued components (including one which is a linear
combination of the two examples above) give good results at this early stage. However, they
have problems generating the fully developed Maxwell’s equations.

The full connection is produced using a geometric model based on the hypothesis that
Maxwell’s equations express symmetry properties of the Ricci tensor. The Ricci tensor is
commonly assumed to be symmetric. However, this assumed symmetry depends on the Torsion
being zero. In the present calculation, the Torsion is not required to be zero. In fact, the
Torsion is shown to be related to the magnetic field. Therefore, the Ricci tensor does not have
a simple symmetry.

The Ricci tensor is developed to have the following symmetry:

1) the trace of the Ricci tensor is zero, $R_{ii} = 0$, and
2) the off-diagonal components of the Ricci tensor have the mixed symmetry

\[ R_{0i} + R_{i0} = 0 \]
\[ R_{ij} - R_{ji} = 0 \]

where 0 denotes the time dimension and \( i, j \) denote spatial dimensions. This hypothesis is not coordinate invariant. The mixed symmetry simply indicates that the universe does distinguish between time and space. This symmetry has the flavor of the Minkowski metric. Note that the Minkowski metric has not been used in the calculation of the electromagnetic connection. No metric was assumed.

The components of the curvature tensor in terms of the connection components are

\[ R_{ijkl} = \left( \partial \Gamma^i_{lj} / \partial x_k - \partial \Gamma^i_{kj} / \partial x_l \right) + \sum_m \left( \Gamma^m_{lj} \Gamma^i_{km} - \Gamma^m_{kj} \Gamma^i_{lm} \right). \]  

(2)

The Ricci tensor is defined in terms of the curvature as

\[ R_{ij} = \sum_k R_{jkl}. \]

On evaluating the Ricci tensor expressions with the connection components from the Lorentz force equation in Table 1, the results are

\[ R_{ii} \implies q/mc^2 B^2 - \nabla \cdot E \]
\[ R_{0i} + R_{i0} \implies \nabla \times B \]
\[ R_{ij} - R_{ji} \implies 0 \]

where a factor of \( q/mc^2 \) has been dropped. This result led to the expectation that the trace \( R_{ii} \) would generate Coulomb’s law and that \( R_{0i} + R_{i0} \) would generate Ampere’s law. Note that these expressions have been evaluated with only the components that can be read from the Lorentz force equations.

Using these observations as guides, the connection was completed. The manner in which Maxwell’s equations are satisfied is described below. The natural coordinate system for electrodynamics is Cartesian coordinates \((ct, x, y, z)\). The full connection is shown in Table 2, with the indices \((0, 1, 2, 3)\) representing the four coordinates.
Table 2

Components of the Electromagnetic connection $\Gamma$

| $i$ | $j$ | $k$ | $\Gamma_{ijk}$ |
|-----|-----|-----|----------------|
| 0   | 2   | 1   | $-qE_x/mc^2$ |
| 2   | 3   | 0   | $-qB_x/mc^2$ |
| 3   | 0   | 2   | $qB_x/mc^2$  |
| 0   | 3   | 2   | $-\frac{1}{2}qB_x/mc^2$ |
| 0   | 2   | 3   | $\frac{1}{2}qB_x/mc^2$ |
| 0   | 0   | 1   | $-qE_x/mc^2$ |
| 0   | 1   | 0   | $-qE_x/mc^2$ |
| 2   | 1   | 2   | $i\sqrt{5/6}qE_x/mc^2$ |
| 1   | 2   | 3   | $i\sqrt{5/6}qE_x/mc^2$ |
| 3   | 1   | 2   | $-i\sqrt{5/6}qE_x/mc^2$ |
| 1   | 2   | 2   | $1 + i\sqrt{5/6}qE_x/mc^2$ |
| 1   | 3   | 3   | $1 - i\sqrt{5/6}qE_x/mc^2$ |

No metric was used to construct the electromagnetic connection. A metric consistent with this connection almost certainly exists. At this time the metric is not known.

1.1 $R_{ii}$: Coulomb’s Law

The trace of the Ricci tensor $R_{ii}$ produces

$$\left(\frac{q}{mc^2}\right)^2 (B_x^2 + B_y^2 + B_z^2 + E_x^2 + E_y^2 + E_z^2) + 2 \frac{q}{mc^2} \left(\frac{\partial}{\partial x} E_x + \frac{\partial}{\partial y} E_y + \frac{\partial}{\partial z} E_z\right).$$

Setting $R_{ii} = 0$ and using the definition for the charge density $\rho$, one finds

$$4\pi \rho = \nabla \cdot E = -\frac{q}{mc^2} (E^2 + B^2)$$

$$\rho = -\frac{q}{mc^2} \left(\nabla \cdot E\right)$$

where $u$ is the electromagnetic energy density $(E^2 + B^2)/8\pi$.

Note that this equation does not say that charge density is present whenever there is electromagnetic energy. In relativity theory, rest mass is generated by energy; but there can be energy in a form other than rest mass. In this geometric electromagnetic theory, too, there can be electromagnetic energy without charge density. But the geometry says that whenever there is charge density there is underlying electromagnetic energy.
1.2 $R_{0i} + R_{i0}$: Ampere’s Law with Displacement Current

The following three sums of off-diagonal components of the Ricci tensor $(R_{01} + R_{10})$, $(R_{02} + R_{20})$, and $(R_{03} + R_{30})$ produce the three expressions

\[
\left(\frac{q}{mc^2}\right)^2 (E_y B_z - E_z B_y) + \frac{q}{mc^2} \left( \frac{\partial}{\partial y} B_z - \frac{\partial}{\partial z} B_y - \frac{1}{c} \frac{\partial}{\partial t} E_x \right),
\]

\[
\left(\frac{q}{mc^2}\right)^2 (E_z B_x - E_x B_z) + \frac{q}{mc^2} \left( \frac{\partial}{\partial z} B_x - \frac{\partial}{\partial x} B_z - \frac{1}{c} \frac{\partial}{\partial t} E_y \right),
\]

\[
\left(\frac{q}{mc^2}\right)^2 (E_x B_y - E_y B_x) + \frac{q}{mc^2} \left( \frac{\partial}{\partial x} B_y - \frac{\partial}{\partial y} B_x - \frac{1}{c} \frac{\partial}{\partial t} E_z \right).
\]

Setting each of the three expressions to 0 and using the definition for the current density $J$, one finds

\[
\frac{4\pi}{c} J = \nabla \times B - \frac{1}{c} \frac{\partial}{\partial t} E = -\frac{q}{mc^2} E \times B
\]

\[
J = -\frac{q S}{mc^2}
\]

where $S$ is Poynting’s vector for the electromagnetic energy flow $(c/4\pi) (E \times B)$.

As in the discussion of Coulomb’s law above, this equation does not assert that whenever there is a propagating electromagnetic wave or a non-zero Poynting vector there must be a charge density or a current. Rather, the geometry says that a current is generated by an underlying Poynting vector.

1.3 $R_{ij} - R_{ji}$: Faraday’s Law

If we evaluate the differences in off-diagonal components $(R_{12} - R_{21})$, $(R_{31} - R_{13})$, and $(R_{23} - R_{32})$ and set each expression equal to 0, then we produce

\[
\nabla \times E + \frac{1}{c} \frac{\partial}{\partial t} B = 0.
\]

2 Other Equations of Electromagnetism

Although the hypothesis relating three of Maxwell’s equations to symmetry properties of the Ricci tensor served as the central guide to this calculation, other equations were reviewed. The expression $\nabla \cdot B$ was not found in the Ricci tensor. A separate geometric source was found for this last member of Maxwell’s equations. The continuity equation connects Coulomb’s law with Ampere’s law. No geometric source for the continuity equation has been identified. The transformation properties of the connection were studied. With sufficient restrictions, the Lorentz transformation can be seen.
2.1 \( \nabla \cdot \mathbf{B} = 0 \)

The expression \( \nabla \cdot \mathbf{B} \) was found in the Torsion tensor. The Torsion is calculated directly from the connection as \( T^i_{jk} = \Gamma^i_{jk} - \Gamma^i_{kj} \). The Torsion tensor, shown in Table 3, depends on the magnetic field. If the Torsion were required to be zero, all connection components which depend on the magnetic field would be zero.

| Table 3 |
|---|
| Components of the Torsion \( T \) |
| showing only \( j < k \) |
| \( i \) | \( j \) | \( k \) | \( T^i_{jk} \) |
| 2 | 0 | 3 | \( qB_x/mc^2 \) |
| 3 | 0 | 2 | \( qB_x/mc^2 \) |
| 0 | 2 | 3 | \( qB_x/mc^2 \) |
| \( i \) | \( j \) | \( k \) | \( T^i_{jk} \) |
| 3 | 0 | 1 | \( qB_y/mc^2 \) |
| 1 | 0 | 3 | \( qB_y/mc^2 \) |
| 0 | 1 | 3 | \(-qB_y/mc^2 \) |
| \( i \) | \( j \) | \( k \) | \( T^i_{jk} \) |
| 1 | 0 | 2 | \( qB_z/mc^2 \) |
| 2 | 0 | 1 | \( qB_z/mc^2 \) |
| 0 | 1 | 2 | \( qB_z/mc^2 \) |

Define the Kronecker \( \epsilon \) function as

\[
\epsilon_{ijkl} = \begin{cases} 
0 : \{i, j, k, l\} \text{ is not a permutation of } \{0, 1, 2, 3\} \\
+1 : \{i, j, k, l\} \text{ is an even permutation of } \{0, 1, 2, 3\} \\
-1 : \{i, j, k, l\} \text{ is an odd permutation of } \{0, 1, 2, 3\}
\end{cases}
\]

Then the Kronecker-signed sum of the components of the covariant derivative of the Torsion produces

\[
\sum_{ijkl} \epsilon_{ijkl} T^i_{jk,l} = 2 \frac{q}{mc^2} \left( \frac{\partial}{\partial x} B_x + \frac{\partial}{\partial y} B_y + \frac{\partial}{\partial z} B_z \right).
\]

Setting the sum to 0 yields the fourth of Maxwell’s equations, \( \nabla \cdot \mathbf{B} = 0 \).

2.2 Continuity Equation

The continuity equation is \( \partial \rho/\partial t + \nabla \cdot \mathbf{J} = 0 \). With the expressions for \( \rho \) and \( \mathbf{J} \) from Coulomb’s law and Ampere’s law, the equation becomes \( \partial u/\partial t + \nabla \cdot \mathbf{S} = 0 \). Poynting’s Theorem can be formulated as \( \partial u/\partial t + \nabla \cdot \mathbf{S} = -\mathbf{J} \cdot \mathbf{E} \), but \( \mathbf{J} \cdot \mathbf{E} \propto (\mathbf{E} \times \mathbf{B}) \cdot \mathbf{E} = 0 \) in this model. So \( \partial u/\partial t + \nabla \cdot \mathbf{S} = 0 \) and the continuity equation is satisfied.

2.3 Transformation properties of the connection: the Lorentz Transformation

The Lorentz coordinate transformation for motion parallel to the \( z \)-axis is given by

\[
x' = x, \quad y' = y, \quad z' = \gamma z - \gamma vt, \quad t' = \gamma t - \gamma vz/c^2,
\]
where $\gamma = (1 - v^2/c^2)^{-1/2}$. The formula for the new connection components under such a coordinate transformation is complicated by a second-derivative term. If we approximate the Lorentz transformation as a linear transformation with $\gamma$ and $v$ constant, then the second-derivative term goes away, and the connection transforms as a tensor. Even with this simplification, the expected transformation of the electric and magnetic fields is not observed for every connection component.

Looking at the classical Lorentz force converted to a geodesic format in equation [1], there is one component based on $E_x$ and potentially four components based on $B_x$. We call components of this type “observable” components and conjecture that under a coordinate transformation the four “observable” $B$-components would not be seen individually, but rather as an average value. For example, the transformed $E_x$ would be seen in the transformed individual component $-\Gamma_{100}$, while the transformed $B_x$ would be seen in the component average $(\Gamma_{302} + \Gamma_{320} - (\Gamma_{230} + \Gamma_{203})) / 2$. This approach was found to be successful.

The effect of a transformation along the $z$-axis was calculated for the $x$-, $y$-, and $z$-field components. Transformations for motion in the $x$ and $y$ directions give equivalent results. Terms of order $v^2/c^2$ are ignored in order to generate the expected transformed components. Note that within this approximation $\gamma \approx 1$; excess powers of $\gamma$ were eliminated. The result of the Lorentz coordinate transformation along the $z$-axis on the connection expressions is given in Table 4, where the six “observable” component averages are displayed in the center column. The result agrees with the transformed electromagnetic fields displayed in Jackson (1962) p. 380.

| Initial Field | Average “Observed” Component | Transformed Field |
|---------------|------------------------------|-------------------|
| $B_x$         | $(\Gamma_{302} + \Gamma_{320} - (\Gamma_{303} + \Gamma_{303})) / 2$ | $\gamma B_x + \gamma \frac{v}{c} E_y$ |
| $B_y$         | $(\Gamma_{103} + \Gamma_{130} - (\Gamma_{101} + \Gamma_{101})) / 2$ | $\gamma B_y - \gamma \frac{v}{c} E_x$ |
| $B_z$         | $(\Gamma_{201} + \Gamma_{210} - (\Gamma_{210} + \Gamma_{201})) / 2$ | $B_z$ |
| $E_x$         | $-\Gamma_{100}$               | $\gamma E_x - \gamma \frac{v}{c} B_y$ |
| $E_y$         | $-\Gamma_{200}$               | $\gamma E_y + \gamma \frac{v}{c} B_x$ |
| $E_z$         | $-\Gamma_{300}$               | $E_z$ |
3 The first Chern class

The curvature tensor with components defined in equation (2) can be interpreted as a curvature 2-form matrix with elements

$$\Theta^i_j = \sum_{k,l>k} R^i_{jkl} dx_k \wedge dx_l.$$

Neglecting a constant multiplier, the trace of the curvature 2-form $\Theta^i_i$ is the first Chern class. Further ignoring the common multiplier $q/mc^2$, the first Chern class is calculated to be

$$\frac{\partial E_x}{\partial t} dx \wedge dt + \frac{\partial E_y}{\partial t} dy \wedge dt + \frac{\partial E_z}{\partial t} dz \wedge dt$$

$$- \left( \frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z} \right) dy \wedge dz + \left( \frac{\partial E_z}{\partial x} - \frac{\partial E_x}{\partial z} \right) dx \wedge dz - \left( \frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} \right) dx \wedge dy. \quad (3)$$

With Faraday’s law, $\nabla \times E = -\frac{1}{c} \frac{\partial B}{\partial t}$, and the definitions of the electromagnetic vector potential $A$ and scalar potential $\Phi$, this 2-form can be shown to be the coboundary of a 1-form. Therefore, the 2-form in equation (3) is a member of the zero cohomology class and does not offer any interesting information about the universe.

4 Lorentz Force

The observed motion of a test particle follows the geodesic equation. This equation in terms of the connection components is

$$\frac{d^2 x_i}{ds^2} + \sum_{j,k} \Gamma^i_{jk} \frac{dx_j}{ds} \frac{dx_k}{ds} = 0$$

where $s$ is the parameter of the path. Taking the real parts of the connection components from Table 2, the three “space-like” equations are

$$m \frac{d^2 x}{ds^2} = E_x q \frac{dt}{ds} + B_z \frac{q dy}{c ds} \frac{dt}{ds} - B_y \frac{q dz}{c ds} \frac{dt}{ds} - E_x \frac{q}{c^2} \frac{dy}{ds}^2 - E_x \frac{q}{c^2} \frac{dz}{ds}^2$$

$$m \frac{d^2 y}{ds^2} = E_y q \frac{dt}{ds} + B_x \frac{q dy}{c ds} \frac{dt}{ds} - B_z \frac{q dx}{c ds} \frac{dt}{ds} - E_y \frac{q}{c^2} \frac{dx}{ds}^2 - E_y \frac{q}{c^2} \frac{dz}{ds}^2$$

$$m \frac{d^2 z}{ds^2} = E_z q \frac{dt}{ds} + B_y \frac{q dx}{c ds} \frac{dt}{ds} - B_x \frac{q dy}{c ds} \frac{dt}{ds} - E_z \frac{q}{c^2} \frac{dx}{ds}^2 - E_z \frac{q}{c^2} \frac{dy}{ds}^2$$

Making the change to proper time and defining

$$s = c \tau, \quad \frac{d^2 x_i}{d \tau^2} = a_i, \quad \frac{dx_i}{d \tau} = v_i, \quad \frac{dt}{d \tau} \approx 1.$$
this simplifies to well-known vector equations. In a magnetic field, the equation has the classical form

$$ F = \frac{q}{c} v \times B. $$

In a parallel electric field, it is the expected

$$ F_{\parallel} = qE_{\parallel}. $$

However, in a transverse electric field, the force is different from the classical form at relativistic velocities with the value

$$ F_{\perp} = qE_{\perp} \left(1 - \frac{v^2}{c^2}\right). $$

A new extension of the Lorentz force is

$$ \frac{d^2 t}{d\tau^2} = 2 \frac{q}{mc^2} \left( E \cdot \frac{dx}{d\tau} \right) \frac{dt}{d\tau}. $$

Time will accelerate or decelerate relative to the proper time of a charged particle moving in an electric field. Unstable charged particles moving in opposite directions parallel to an electric field should exhibit different decay rates.

5 References

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