What you need to know to train recurrent neural networks to make Flip Flops memories and more

Cecilia G. Jarne

Universidad Nacional de Quilmes, Departamento de Ciencia y Tecnología -CONICET

(Dated: September 8, 2021)
Abstract

Training neural networks to perform different tasks is relevant across various disciplines that go beyond Machine Learning. In particular, Recurrent Neural Networks (RNN) are of great interest to different scientific communities, for example, Computational Neuroscience research and Dynamical Systems among others. Open-source frameworks dedicated to Machine Learning such as Tensorflow [1] and Keras [2] has produced significant changes in the development of technologies that we currently use. One relevant problem that can be approached is how to build the models for the study of dynamical systems, and how to extract the relevant information to be able to answer the scientific questions of interest. The purpose of the present work is to contribute to this aim by using a temporal processing task, in this case, a 3-bit Flip Flop memory, to show the modeling procedure in every step: from equations to the software code using Tensorflow and Keras. The obtained networks are analyzed to describe the dynamics and to show different visualization and analysis tools. The code developed in this work is provided to be used as a base for model other systems.

INTRODUCTION

Machine learning methods and Deep Learning, in particular, have demonstrated remarkable success in a wide range of tasks in multiple domains [3]. In recent years, the emergence of open-source frameworks dedicated to Machine Learning such as Pytorch[4], or Tensorflow [1] and Keras [2] has produced huge changes in the development of technologies that we use every day. Sometimes the novelty and the complexity of those frameworks make that takes some time to be fully exploited in other related fields of science such as Computational Neuroscience, or different areas of Engineering.

Most of the current tutorials and papers, which refer to the mentioned libraries, focus on the use of these techniques for Machine Learning. That enforces the need to develop good tutorials or premieres to explain how to implement the algorithms that we need to ask the scientific questions of interest, and also to be able to solve different problems using the newly available tools.

One relevant problem is how to build the models for the study of dynamical systems and how to extract the relevant information to answer the scientific questions of interest.

Neural Networks are algorithms that allow us to model different systems. According
to Universal Approximation Theorem, a neural network with one hidden layer containing a sufficient but finite number of neurons can approximate any continuous function to a reasonable accuracy under certain conditions for activation functions [5]. In particular, RNNs allow modeling dynamical systems since a dynamical system can be approximated by continuous-time RNN [6].

The problem of training neural networks to perform different tasks is relevant across various disciplines that go beyond Machine Learning. In particular, RNNs are of great interest in different scientific communities. They are widely used in Neurosciences to describe the behavior of the cortex, an area that presents great recurrence in its connections [7]. These systems also have great relevance concerning control systems and other areas such as electronics [8-10].

In the present work, the modeling of a complex system is presented, using Tensorflow and Keras. The reason for this selection is that these new scientific libraries are open source, and their use is rapidly growing. Good documentation can be found for software development [11-13], and also new tools such as Google Colaboratory allow implementing and testing directly online network implementation, with the focus on ML.

An RNN was chosen because it is relevant in various fields. It was trained to perform a temporal processing task, that was inspired in Neuroscience tasks such as [14]. The implementation of the network, the training, and the tools are presented, as well as different forms to obtain the information that allows describing the system appropriately.

Training an RNN to perform temporal tasks has many difficulties and can be done through various paradigms. Here it is proposed to approach the problem through supervised learning. The entire procedure is described in detail using open-source software. Gradient descendant minimization was used to take advantage of different optimized implementations of the current algorithms available.

The Flip Flop task was chosen as a case example. A 3-bit memory, which is a task composed of Flip Flops. The parametrization of the task is as described in [15], but also is revisited here. The novelty is also that it has been trained here using the gradient minimization technique.

All steps will be described in detail, from the parameterization of the task to the description of the dynamics of the trained networks. This example is used to show how the problem of training networks can be studied using these new computing tools applied in any
temporal task. In general, are also discussed the limitations that networks have and the alternatives to solve them.

The motivation of this work is to provide a step-by-step tutorial based on a basic example where the previous results have been capture with new techniques. It aims to be used to develop useful code for the researchers working in the field of Dynamical Systems, Complex Systems, and Neurosciences, but it could be used in other areas also.

As with other works such as [16], the set of methods presented here is not exhaustive, although it is representative to discuss the scientific questions and try to solve the technical challenges in the field by showing one possible application.

The rest of the paper is organized as follows. First, in Section , the description of the dynamics of the RNN and the discretization is presented. In Section , the task parametrization is explained. Then, in Section , the training protocol is shown. In Section , the results and different analyses of the network and tools are discussed. Finally, in Section final remarks are presented.

MODEL

The dynamics of the units in the RNN model in terms of the activity, \( h_i(t) \) is described by Equation 1 based on the Hopfield model [17], where units have index \( i \), with \( i = 1, 2..., n \).

\[
\frac{dh_i(t)}{dt} = -\frac{h_i(t)}{\tau} + \sigma \left( \sum_j w_{ij}^{\text{Rec}} h_j(t) + \sum_j w_{ij}^{\text{in}} x_j \right)
\]

In Equation 1, the matrix elements \( w_{ij}^{\text{Rec}} \) are the synaptic connection strengths of the matrix \( \mathbf{W}^{\text{Rec}} \) and \( w_{ij}^{\text{in}} \) the matrix elements of \( \mathbf{W}^{\text{in}} \) from the input units. \( x_j \) are the components of the vector \( \mathbf{X} \) of the input signal. \( \tau \) represents the time constant of the system and \( \sigma \) is a non-linear activation function.

The network is fully connected, and matrices have weights given by a certain parametrization of interest. In our case, a normal distribution with zero mean and variance \( \frac{1}{N} \).

The network has three layers: the input, the recurrent hidden layer, and the output layer. The readout output in terms of the matrix elements \( w_{ij}^{\text{out}} \) from \( \mathbf{W}^{\text{out}} \) is described by Equation ...
\[ Z(t) = \sum_j w_{ij}^\text{out} h_j(t) \]  \hspace{2cm} (2)

For this work, it was considered \( \sigma() = \tanh() \) and \( \tau = 1 \), without loss of generality. The model is discretized using the Euler method. A simple schema of the model is presented in Figure 1 in this case, with three inputs and three outputs corresponding to the inputs and memory states of the 3-bit Flip Flop.

As described in Section , the model is implemented in Python using Keras and Tensorflow [1, 2], which allows making use of all current algorithms and optimizations developed and maintained by a massive research community.

In vector form, the equations 1 and 2 can be written as:

\[ \frac{d\mathbf{H}(t)}{dt} = -\frac{\mathbf{H}(t)}{\tau} + \sigma(W^\text{Rec}\mathbf{H}(t) + W^\text{in}\mathbf{X}(t)) \]  \hspace{2cm} (3)

and respectively:

\[ \text{FIG. 1. Recurrent Neural Network model schema to represent the network described by equations 1 and 2. In this case, with 3 inputs and 3 outputs corresponding to the inputs and memory states of the 3-bit Flip Flop task.} \]
The system represented by Equation 1 is approximated using Euler’s method with a step time $\delta t$. A value of $\tau = 1$ for the model time constant was considered. Then the dynamics of the discrete-time RNN is done through:

$$
H(t + \delta t) = H(t) + (-H(t) + \sigma(W^{\text{Rec}}H(t) + W^{\text{in}}X(t)))\delta t,
$$

(5)

The value of the time step is $\delta t = 1$ mS, for the time evolution. Then, from equation 5, the activity of the recurrent units at the next time step is given by Equation 6.

$$
H(t + 1) = \sigma(W^{\text{Rec}}H(t) + W^{\text{in}}X(t))
$$

(6)

Tensorflow has a recurrent layer implemented to represent Equation 6, where it is possible to choose the initialization of parameters, units, and activation function.

```python
import tensorflow as tf

tf.keras.layers.SimpleRNN( units, activation="tanh",
  kernel_initializer="glorot_uniform",
  recurrent_initializer="orthogonal",**kwargs)
```

Code for a Recurrent layer defined in Tensor Flow.

To define a Recurrent Network with the topology of Figure 1 it is possible to build a sequential model with layers such as:

```python
model = Sequential()
model.add(SimpleRNN(units=N, input_shape=(None,3),activation="tanh"))
model.add(Dense(units=3,input_dim=N))
```

Code for the sequential model defined for Figure 1.
Where \( \text{input\_shape} = (None, 3) \) is the shape of the vector input, \( \text{activation} = \tanh \) corresponds to the definition of activation function, and \( \text{Dense} \) is the output layer. In this way, we have completed the first step that consists of the model definition in terms of the code.

**TASK PARAMETRIZATION**

Once the task was chosen according to the hypothesis to be tested or the model to be explored, the requirements must be translated into the algorithm to generate the training set. In this case, a Flip Flop task was considered, meaning an 8-bit memory task, where we have 8 different possible output states corresponding to the \( 2^3 \) binary states, given by the 3 flip flops that compose the "memory".

To parameterize the task, the following criterion was considered for the present work:

- The possible states of the Flip Flop are represented in such a way that a positive pulse represents a set and a negative pulse represents a reset.
- The state of the output will change corresponding to the input command.
- A certain delay was considered after the falling edge of the input signal.

The training data set consists of time series with pulses of fixed duration that represent SET and RESET. Those signals can be activated randomly and are separated by a random time interval. In all time series, a certain noise level has been superimposed on the input. Each input elicitate a target output according to the Flip Flop rule: if we have a set signal or positive pulse, the output is in high-state. If we have a reset signal or negative pulse output is in low-state, otherwise, the output remains in the previous state.

The number of inputs in the network corresponds to the number of bits that can be stored. A Flip Flop is a one-bit of memory. In this case, we have registers formed by three Flip Flops (a 3 bits memory).

To generate the complete training set, it is necessary to create sets of tensors of size \( \text{sample\_size} \) with the input time series of length \( \text{time\_series\_length} \), for each of the three inputs and outputs. To do that efficiently, we can use Numpy arrays [18]. With present work, sample code is provided to generate the Flip Flop data set. Three of components
of the full set $x_{train} - y_{train}$ are shown in Figure 2. The target output, $y_{train}$, was simulated with a time delay answer of 20 mS. Each row (and color) corresponds to one of the inputs and each column to a different sample. In this way, each training sample consists of a Numpy array.

$x_{train}[\text{sample\_size, time\_series\_length, 3}]$

$y_{train}[\text{sample\_size, time\_series\_length, 3}]$

Training data set pairs defined as Numpy arrays.

FIG. 2. Three samples of the data set data for each input channel. Each row (and color) corresponds to one of the inputs and each column to a different sample. The gray line in each case represents the expected output.
TRAINING PROTOCOL AND PARAMETER SELECTION

The training methods for neural networks can be unsupervised or supervised. We will focus on applying a supervised method.

Studies, where one kind of gradient descent method is applied, stand out in the literature. An example is the paradigm of Reservoir Computing paradigm with liquid- or echo-state networks [19], where the modifications of the network are made in the weights of the output layer, \( W_{\text{out}} \).

Other outstanding methods were developed by Sussillo and Abbot. They have developed a method called FORCE that allows them to reproduce complex output patterns, including human motion-captured data [20]. Modifications to the algorithm have also been applied successfully in various applications [21].

The other method used for the estimation of the gradient in RNNs is called Backpropagation Through Time (BPTT), and then some optimization method for minimizing the gradient. Given the recent advances in the implementation of this method with the Open-source libraries previously mentioned, this is the method explored here. There are also other back propagation-based methods been published recently for example in [22], authors propose to use fractional calculus to improve the conventional BPTT.

In this work, supervised learning was used, with standard backpropagation through time implementing an Adaptive SGD training method provided by Keras framework.

First, recurrent weights were obtained using a random normal distribution with the orthogonal condition on the matrix. Noisy square pulse signals, as presented in Figure 2, were used at the inputs.

The appropriate loss function to train the model is the mean square error between the target function and the output of the network. It is defined as:

\[
E(w) = \frac{1}{2} \sum_{t=1}^{M} \sum_{j=1}^{L} |Z_j(t) - Z_j^{\text{target}}(t)|^2,
\]

where \( Z_j^{\text{target}}(t) \) is the desired target function and \( Z_j(t) \) is the actual output. The training sample consisted of more than 15000 samples.

In terms of the code, the previously defined steps are: compile the model and train it. In the first step, the loss function and the optimizer algorithm are selected. In the training
stage, different detail on the training data set, epochs, and other characteristics of training are chosen.

```python
model.compile(loss = "mse", optimizer=ADAM)
model.fit(x_train[50:sample_size,:,:], y_train[50:sample_size,:,:],
        epochs=epochs, batch_size=128, shuffle=True)
```

Code for the compiling and training stages.

The parameters of a neural network are the weights of the connections. In this case, these parameters are learned during the training stage. The hyperparameters are parameters of your neural network that cannot be learned via gradient descent or some other training method. These include learning rate, number of layers, number of neurons in a given layer.

Tuning the hyperparameters means the process of choosing the best values. Typically this is done by evaluating the performance of the network on a validation set. Then, we have to change the hyperparameters and re-evaluate, choosing the values that give the best performance on the validation set.

How do we choose these values? Often there is good standard initialization related to each particular task of interest, here an example of the criteria is provided for the Flip Flop case in Table I. In our example, the parameters are also given in the table.

Another aspect to consider is the regularization of the model. Regularization refers to train our model well enough that it can generalize over data it hasn’t seen before.

To summarize, in the training stage is where we have to consider the size of the network, data set, noise, regularization terms that are appropriate for the considered task, in our case, the Flip Flop.

**ANALYZING THE RESULTS**

Finally, after training, we can obtain a set of RNN that can perform the tasks. Now we search the aspects to analyze regarding the network’s collective behavior. Also, we choose a way for visualization of the model obtained and tools that allow us to extract the relevant information.
| Parameter/criteria | Value                  |
|--------------------|------------------------|
| Units              | 400                    |
| Input Weight       | $3 \times 400$        |
| Recurrent Weights  | $400 \times 400$      |
| Output Weight      | $400 \times 3$        |
| Training algorithm | BPTT ADAM             |
| Initialization     | Random Orthogonal      |
| Regularization     | None                   |

**TABLE I. Model’s parameters and criteria for the network’s implementation and training.**

For example, it is possible to visualize the connectivity matrix (recurrent weight matrix), as it is shown on the left side of Figure 3. The columns represent the output connection of the i-neuron, and the rows are the input connection. The color bar represents the intensity of the connection considered. With such visualization, it is not clear which is the relevant information, apart from the fact that after training, most of the weights remain close to zero. In case that we have imposed constraints on the connections, such as Dales’ Law, or any particular constraint they will be visible in this stage, and this representation will be more useful.

Different Linear Algebra operations are available in the Numpy that are optimized to be used with the array structures. If we perform, for example, a decomposition of $W^{\text{rec}}$ in their eigenvectors and eigenvalues, we can obtain the eigenvalue distribution. This is shown on the right side of Figure 3.

```python
from numpy import linalg as LA
eigenvalues, eigenvectors= LA.eig(Matrix)
```

How to use linear algebra library from Numpy for eigenvalue decomposition.

We can visualize that, except for a small group of eigenvalues that migrated out of the unit circle, the rest remain on the unitary circle (orthogonal initial condition). This situation is repeated in all the obtained simulations. Eigenvalues outside the unitary circle seem to be related to the behavior (or modes) observed for the different stimuli. Other additional
information, related to the connectivity matrix, could also be obtained.

![Connectivity matrix and eigenvalue distribution](image)

**FIG. 3.** Left: Example of the connectivity matrix for a trained RNN. Right: eigenvalue distribution of $W^{\text{rec}}$.

Other possible studies are related to the response of the network when applying the different stimuli.

Since we have a large number of units, and for each and activity vector, dimensionally reduction methods are appropriate to analyze such behavior, and it has been used widely in different works related to large-scale neural recordings [23, 24].

Scikit-learn [25] is, for example, a python open source library based on Numpy that allows us to perform dimensionality reduction, feature extraction, and normalization, among other efficient methods for predictive data analysis. Such decomposition could be, for example, Principal Component Analysis (PCA) or Single Value Decomposition (SVD).

```python
from sklearn.decomposition import PCA
from sklearn.decomposition import TruncatedSVD
```

How to import Scikit-learn libraries to perform single value decomposition and principal component analysis.

We can use such tools to extract relevant features of our system.

It is well known that the different memory states in a 3-bit memory distribute in the vertex of a cube-like form in the space state [15]. It was shown by them when they explored
the hypothesis that fixed points, both stable and unstable, and the linearized dynamics around them. This can reveal aspects of how RNNs implement their computations.

Here, a data set was built for testing. It generates eight different memory states, as it is shown on the left side of Figure 4. Time series of 600 mS were considered to generate all the possible different memory states of the 3-bit memory, by choosing the correct commutation for the inputs. Input and output response is shown.

The testing set is injected into the network (right upper panel of the Figure), and then we can analyze the activity of the units using SVD applied on the activity vector $H(t)$. The behavior of the system was represented into the three axes of the greatest variance. The bottom right part of Figure 4 shows the activity in the reduced state space (3-dimensional). Each vertex corresponding to each memory state is shown in different colors.

As it is well known, differences in the realizations, in terms of weight distribution and dynamical behavior, are possible when training networks for the same task [26, 27]. Such is the case is shown in Figure 5, where four different realizations for the trained networks elicited with the same testing data set are shown. Vertices are distributed in different positions, and a cube-like structure is rotated in different spatial directions. We can study and classify the behavior of the obtained systems by comparing different network realizations.

Additional analysis could be considered depending on the interest in certain aspects to be studied. Here a minimal analysis was proposed by describing in detail the steps, visualization tool, criteria, and implementation. The code is provided for training and analysis, but also to be a framework open for different tasks or additional studies.

DISCUSSION

Researchers in the dynamical modeling field rarely realize that they could be using well-supported and efficient open-source code such as Keras, Tensorflow, which remain mainly in the area of machine learning for applications, thus the applications presented here aim to contribute to expanding the knowledge to be implemented in such areas. More and
FIG. 4. Left side: data set for testing. Each panel corresponds to one input and output of the Flip Flop. The top-right figure is a schema of such network. Top-bottom corresponds to the Single Value Decomposition applied activity vector \( \mathbf{H}(t) \).

more such libraries are being used in the modeling of physical systems, such as the case of Lagrangian Neural Networks, or Hamiltonian networks [28–30].

Building and studying different realizations that give rise to the same response of this non-linear system can be a good motivation and excuse to implement and discover this new framework and use it for different problems.

CONCLUSIONS

In this work, all steps to build and analyze an RNN have been presented using Keras and TensorFlow. We started from the model description in terms of the equations, discretization, and code implementation. Then we described the task parametrization and network
FIG. 5. Four different realizations of RNN for the same task. A cube-like structure is rotated in different spatial directions.

training protocol. We also presented a set of tools to analyze the results using open-source scientific libraries, making use of the different visualization tools that allow extracting relevant features. We could reproduce the previous results obtained in [15] for the flip flop, but using gradient minimization, also finding that the 8 possible memory states are distributed in a cube, whose characteristics depend on the particular realization.

We use the 3 bit-Flip Flop task as an example, but other relevant tasks could be considered. The use of the new open-source scientific tools that are designed and maintained for large communities, such as the tools used here, allows enhancing research. This is why we are currently using tools that are more transparent in terms of code and documentation because they are open to being modified and improved for thousands of users.
CODE AVAILABILITY

Code is provided in an open repository after paper publication and with a MIT License.
https://github.com/katejarne/3-bit-FF-tutorial

ACKNOWLEDGMENTS

The present work was supported by CONICET and UNQ. This research did not receive any additional specific grants from funding agencies. I want to thanks Gabriel Lio, and to the editor and anonymous reviewers who contributed to improving the manuscript.

*cecilia.jarne@unq.edu.ar*

[1] F. Chollet et al., Keras, https://keras.io (2015).
[2] M. Abadi, A. Agarwal, P. Barham, E. Brevdo, Z. Chen, C. Citro, G. S. Corrado, A. Davis, J. Dean, M. Devin, S. Ghemawat, I. Goodfellow, A. Harp, G. Irving, M. Isard, Y. Jia, R. Joulev, L. Kaiser, M. Kudlur, J. Levenberg, D. Mané, R. Monga, S. Moore, D. Murray, C. Olah, M. Schuster, J. Shlens, B. Steiner, I. Sutskever, K. Talwar, P. Tucker, V. Vanhoucke, V. Vasudevan, F. Viégas, O. Vinyals, P. Warden, M. Wattenberg, M. Wicke, Y. Yu, and X. Zheng, TensorFlow: Large-scale machine learning on heterogeneous systems (2015), software available from tensorflow.org.
[3] T. Ogunfunmi, R. P. Ramachandran, R. Togneri, Y. Zhao, and X. Xia, Circuits, Systems, and Signal Processing 38, 3406 (2019).
[4] A. Paszke, S. Gross, F. Massa, A. Lerer, J. Bradbury, G. Chanan, T. Killeen, Z. Lin, N. Gimelshein, L. Antiga, A. Desmaison, A. Kopf, E. Yang, Z. DeVito, M. Raison, A. Tejani, S. Chilamkurthy, B. Steiner, L. Fang, J. Bai, and S. Chintala, in Advances in Neural Information Processing Systems 32, edited by H. Wallach, H. Larochelle, A. Beygelzimer, F. d’ Alché-Buc, E. Fox, and R. Garnett (Curran Associates, Inc., 2019) pp. 8024–8035.
[5] K. Hornik, Neural Networks 4, 251 (1991).
[6] K. ichi Funahashi and Y. Nakamura, Neural Networks 6, 801 (1993).
[7] B. K. Murphy and K. D. Miller, Neuron 61, 635 (2009)
[8] J. Deng, Engineering Applications of Artificial Intelligence 26, 281 (2013).

[9] N. Mohajerin and S. L. Waslander, in 2017 International Joint Conference on Neural Networks (IJCNN) (2017) pp. 2330–2337.

[10] H. Dinh, R. Kamalapurkar, S. Bhasin, and W. Dixon, Neural Networks 60, 44 (2014).

[11] A. Gulli and S. Pal, Deep Learning with Keras: Implementing Deep Learning Models and Neural Networks with the Power of Python (Packt Publishing, 2017).

[12] B. Ramsundar and R. B. Zadeh, TensorFlow for Deep Learning: From Linear Regression to Reinforcement Learning, 1st ed. (O’Reilly Media, Inc., 2018).

[13] P. Singh and A. Manure, Learn TensorFlow 2.0: Implement Machine Learning and Deep Learning Models with Python (Apress, 2019).

[14] D. Sussillo, Current Opinion in Neurobiology 25, 156 (2014), theoretical and computational neuroscience.

[15] D. Sussillo and O. Barak, Neural Computation 25, 626 (2013).

[16] G. Montavon, W. Samek, and K.-R. Müller, Digital Signal Processing 73, 1 (2018).

[17] J. J. Hopfield, Proceedings of the National Academy of Sciences 81, 3088 (1984).

[18] C. R. Harris, K. J. Millman, S. J. van der Walt, R. Gommers, P. Virtanen, D. Cournapeau, E. Wieser, J. Taylor, S. Berg, N. J. Smith, R. Kern, M. Picus, S. Hoyer, M. H. van Kerkwijk, M. Brett, A. Haldane, J. F. del Río, M. Wiebe, P. Peterson, P. Gérard-Marchant, K. Sheppard, T. Reddy, W. Weckesser, H. Abbasi, C. Gohlke, and T. E. Oliphant, Nature 585, 357 (2020).

[19] W. Maass, T. Natschläger, and H. Markram, Neural Computation 14, 2531 (2002).

[20] D. Sussillo and L. Abbott, Neuron 63, doi: 10.1016/j.neuron.2009.07.018 (2009).

[21] B. DePasquale, C. J. Cueva, K. Rajan, G. S. Escola, and L. F. Abbott, PLOS ONE 13, 1 (2018).

[22] S. Khan, J. Ahmad, I. Naseem, and M. Moinuddin, Circuits, Systems, and Signal Processing 37, 593 (2018).

[23] A. H. Williams, T. H. Kim, F. Wang, S. Vyas, S. I. Ryu, K. V. Shenoy, M. Schnitzer, T. G. Kolda, and S. Ganguli, Neuron 98, 1099 (2018).

[24] J. P. Cunningham and B. M. Yu, Nature Neuroscience 17, 1500 (2014).

[25] F. Pedregosa, G. Varoquaux, A. Gramfort, V. Michel, B. Thirion, O. Grisel, M. Blondel, P. Prettenhofer, R. Weiss, V. Dubourg, J. Vanderplas, A. Passos, D. Cournapeau, M. Brucher,
M. Perrot, and E. Duchesnay, Journal of Machine Learning Research 12, 2825 (2011).

[26] H. F. Song, G. R. Yang, and X.-J. Wang, PLOS Computational Biology 12, 1 (2016).

[27] C. Jarne, Journal of Physics: Complexity (2021).

[28] M. Cranmer, S. Greydanus, S. Hoyer, P. Battaglia, D. Spergel, and S. Ho, Lagrangian neural networks (2020), arXiv:2003.04630 [cs.LG].

[29] C.-D. Han, B. Glaz, M. Haile, and Y.-C. Lai, Phys. Rev. Research 3, 023156 (2021).

[30] S. Greydanus, M. Dzamba, and J. Yosinski, Hamiltonian neural networks (2019), arXiv:1906.01563 [cs.NE].