Adaptive Elastic Net Method for Cox Model

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Abstract

In this paper, we study the Adaptive Elastic Net method for the Cox model. We prove the grouping effect and oracle property of its estimators. Finally, we show these two properties by an empirical analysis and a numerical simulation, respectively.

Keywords: Adaptive Elastic Net; Cox model; grouping effect; oracle property

1. Introduction

The aim of survival analysis is usually to identify risk factors and their risk contributions. Often, many covariates are collected and then a large parametric model is built. Hence, to efficiently select a subset of significant variables upon which the hazard function depends becomes an important and challenging task. Recently, many scholars used the variable selection techniques in studying linear regression models to deal with this kind of problems. For more information on this, see Fan and Li [6] and the references therein. However the treatment of using these methods directly also causes some problems. To overcome these drawbacks, statisticians have recently proposed a family of penalized partial likelihood methods to study the survival data, such as the Lasso method and so on.

The Lasso method introduced by Tibshirani [10] is a penalized least squares method imposing a penalty on the regression coefficients. Due to the nature of the penalty, the Lasso method does both continuous shrinkage and automatic variable selection simultaneously. However, the Lasso estimator does not possess the oracle property and instability with high-dimensional data. Hence, Zou [12] proposed the Adaptive Lasso method, which has the oracle property. Namely, the true regression coefficients that are zero are automatically estimated as zero, and the remaining coefficients are estimated as well as if the correct submodel were known in advance. Contrast to the Lasso and Adaptive Lasso, the Elastic Net method proposed by Zou and Hastie [13] is particularly useful when the number of predictors is much bigger than that of observations. In addition, the Elastic Net method encourages a grouping effect, which means that strongly correlated predictors tend to be in or out of the model together. However, Fan and Li [5,6] stated that the estimator of the Elastic Net method does not have the oracle property. Hence, Zou and Zhang [14] proposed the Adaptive Elastic Net method, which has the oracle property and grouping effect.

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The Cox model \[2\] is a classical method to deal with survival data. It is well-known that the Elastic Net method for the Cox model has grouping effect and can deal with the highly correlated data. Inspired by these facts and Zou and Zhang \[14\], in this paper, we study the Adaptive Elastic Net method for the Cox model and show that it has the grouping effect and oracle property.

The rest of this paper is organized as follows. In Section 2, we introduce the Adaptive Elastic Net method for the Cox model. Section 3 is devoted to studying the grouping effect. The oracle property is discussed in Section 4. In Section 5, we show these two properties by an empirical analysis and a numerical simulation.

2. Adaptive Elastic Net method

In this section, we give the definition of the Adaptive Elastic Net method for the Cox model. We first recall some known facts about the Cox model. Recall that the hazard function for an individual at the failure time \( t \) is

\[
h(t) = h_0(t) \exp\{\beta^T X\},
\]

(2.1)

where \( h_0(t) \) is a baseline hazard function, \( \beta = (\beta_1, \cdots, \beta_p)^T \) is the regression vector of unknown coefficients, \( X \) is the covariate of an individual. Let \( R_i \) denote the risk set at time \( t_i - 0 \), that is the set of individuals who have not failed or been censored by that time. Furthermore, let \( X_j = (x_{j1}, \cdots, x_{jp})^T \) denote the value of \( X \) for the \( j \)th individual and \( X_i \) the value for the individual failing at time \( t_i \). Suppose a random sample of \( n \) individuals is chosen, then the likelihood function for inference about \( \beta \) is given by:

\[
L(\beta) = \prod_{i=1}^{n} \frac{\exp(\sum_{k=1}^{p} \beta_k x_{ik})}{\sum_{j \in R_i} \exp(\sum_{k=1}^{p} \beta_k x_{jk})}.
\]

Therefore, the log-likelihood function is

\[
l(\beta) = \sum_{i=1}^{n} \left\{ \sum_{k=1}^{p} \beta_k x_{ik} - \ln \left[ \sum_{j \in R_i} \exp(\sum_{k=1}^{p} \beta_k x_{jk}) \right] \right\}.
\]

(2.2)

By maximizing (2.2), we can get the estimator of \( \beta \).

From Tibshirani \[10\], and Fan and Li \[6\], by minimizing the opposite number of (2.2) first, and then adding the appropriate penalty, we can get the Elastic Net estimator for the Cox model:

\[
\hat{\beta}(EN) = \arg \min \left\{ \frac{1}{n} \sum_{i=1}^{n} \left\{ - \beta^T X_i + \ln \left[ \sum_{j \in R_i} \exp(\beta^T X_j) \right] \right\} + \lambda_1 \|\beta\|_1 + \lambda_2 \|\beta\|_2^2 \right\},
\]

(2.3)

where \( \lambda_1 \geq 0 \) and \( \lambda_2 \geq 0 \) are regularization parameters, and

\[
\|\beta\|_2^2 = \sum_{j=1}^{p} \beta_j^2, \text{ and } \|\beta\|_1 = \sum_{j=1}^{p} |\beta_j|.
\]
Moreover, (2.3) can be rewritten as

\[
\hat{\beta}_{(EN)} = \arg \min \left\{ \frac{1}{n} \sum_{i=1}^{n} \left\{ -\sum_{k=1}^{p} \beta_k x_{ik} + \ln \left[ \sum_{j \in R_i} \exp \left( \sum_{k=1}^{p} \beta_k x_{jk} \right) \right] \right\} + \lambda_1 \sum_{k=1}^{p} |\beta_k| + \lambda_2 \sum_{k=1}^{p} \beta_k^2 \right\}.
\] (2.4)

Following Zou and Zhang [14], we introduce the Adaptive Elastic Net estimator for the Cox model as follows.

**Definition 2.1** The Adaptive Elastic Net estimator \( \hat{\beta}_{(AEN)} = \left( \hat{\beta}_{(AEN)}^1, \ldots, \hat{\beta}_{(AEN)}^p \right)^T \) for the Cox model is defined by

\[
\hat{\beta}_{(AEN)} = \arg \min \left\{ \frac{1}{n} \sum_{i=1}^{n} \left\{ -\sum_{k=1}^{p} \beta_k x_{ik} + \ln \left[ \sum_{j \in R_i} \exp \left( \sum_{k=1}^{p} \beta_k x_{jk} \right) \right] \right\} + \lambda^* \sum_{k=1}^{p} \hat{\omega}_k |\beta_k| + \lambda_2 \sum_{k=1}^{p} \beta_k^2 \right\},
\] (2.5)

where \( \hat{\omega}_k = (|\hat{\beta}_{(EN)}^k|)^{-\gamma} \) with \( \gamma > 0 \).

3. **Grouping effect**

In this section, we study the grouping effect of the Adaptive Elastic Net method for the Cox model. Before we state the main result, we need the following notation. Let \( x_a = (x_{1a}, \ldots, x_{na}) \) and \( x_b = (x_{1b}, \ldots, x_{nb}) \), \( a, b = 1, \ldots, p \), be highly correlated, and \( \hat{\beta}_a(\lambda_1^*, \lambda_2) \) and \( \hat{\beta}_b(\lambda_1^*, \lambda_2) \) denote the estimators of the \( a \)th variable \( \tilde{X}_a \) and the \( b \)th variable \( \tilde{X}_b \) in the covariate, respectively. Following the notation in Andersen and Gill [1], let \( Y_i(t) = I(T_i \geq t, C_i \geq t) \), \( N_i(t) = I(T_i \leq t, T_i \leq C_i) \), \( \delta_i = I(T_i \leq C_i) \) and \( Z_i = \min\{T_i, C_i\} \). Then,

**Theorem 3.1** For the Cox model, given the data \((Z_i, \delta_i, X_i)\) and parameter \((\lambda_1^*, \lambda_2)\), the responses are centered and the predictors are standardized. Let \( \hat{\beta}(\lambda_1^*, \lambda_2) \) be the Adaptive Elastic Net estimator. Suppose that

\[
\hat{\beta}_a(\lambda_1^*, \lambda_2) \hat{\beta}_b(\lambda_1^*, \lambda_2) > 0.
\] (3.1)

Define

\[
D_{\lambda_1^*, \lambda_2}(a, b) = \left| \hat{\beta}_a(\lambda_1^*, \lambda_2) - \hat{\beta}_b(\lambda_1^*, \lambda_2) \right|.
\]

then

\[
D_{\lambda_1^*, \lambda_2}(a, b) \to 0,
\]

which means that \( D_{\lambda_1^*, \lambda_2}(a, b) \) approximates to 0.
Proof: By (3.1), we have
\[ \text{sgn}\{\hat{\beta}_a(\lambda_1^*, \lambda_2)\} = \text{sgn}\{\hat{\beta}_b(\lambda_1^*, \lambda_2)\} \] (3.2)
and
\[ \hat{\beta}_a(\lambda_1^*, \lambda_2) \neq 0 \text{ and } \hat{\beta}_b(\lambda_1^*, \lambda_2) \neq 0. \]

Now, let \( \hat{\beta}_m(\lambda_1^*, \lambda_2) \neq 0 \) and at the point \( \hat{\beta}(\lambda_1^*, \lambda_2) \),
\[ \frac{\partial L(\lambda_1^*, \lambda_2, \beta)}{\partial \beta_m} = 0, \]
where
\[ L(\lambda_1^*, \lambda_2, \beta) = \frac{1}{n} \sum_{i=1}^{n} \{ -\sum_{k=1}^{p} \beta_k x_{ik} + \ln \left[ \sum_{j \in R_i} \exp \left( \sum_{k=1}^{p} \beta_k x_{jk} \right) \right]\} + \lambda_2^* \sum_{k=1}^{p} \hat{\omega}_k |\beta_k| + \lambda_2 \sum_{k=1}^{p} \beta_k^2. \]

Then,
\[ -\frac{1}{n} \sum_{i=1}^{n} x_{ia} + \frac{1}{n} \sum_{i=1}^{n} \sum_{j \in R_i} x_{ja} \exp \left( \sum_{k=1}^{p} \beta_k x_{jk} \right) \]
\[ - \frac{1}{n} \sum_{i=1}^{n} \sum_{j \in R_i} x_{ia} \exp \left( \sum_{k=1}^{p} \beta_k x_{jk} \right) \]
\[ + \lambda_1^* \hat{\omega}_a \text{sgn}\{\hat{\beta}_a(\lambda_1^*, \lambda_2)\} + 2 \lambda_2 \hat{\beta}_a(\lambda_1^*, \lambda_2) = 0. \]

Therefore,
\[ \hat{\beta}_a(\lambda_1^*, \lambda_2) = \]
\[ \frac{1}{2\lambda_2} \left\{ \frac{1}{n} \sum_{i=1}^{n} x_{ia} - \frac{1}{n} \sum_{i=1}^{n} \sum_{j \in R_i} x_{ja} \exp \left( \sum_{k=1}^{p} \beta_k x_{jk} \right) \sum_{j \in R_i} \exp \left( \sum_{k=1}^{p} \beta_k x_{jk} \right) - \lambda_2^* \hat{\omega}_a \text{sgn}\{\hat{\beta}_a(\lambda_1^*, \lambda_2)\} \right\}. \] (3.3)

Similar to (3.3), we get
\[ \hat{\beta}_b(\lambda_1^*, \lambda_2) = \]
\[ \frac{1}{2\lambda_2} \left\{ \frac{1}{n} \sum_{i=1}^{n} x_{ib} - \frac{1}{n} \sum_{i=1}^{n} \sum_{j \in R_i} x_{jb} \exp \left( \sum_{k=1}^{p} \beta_k x_{jk} \right) \sum_{j \in R_i} \exp \left( \sum_{k=1}^{p} \beta_k x_{jk} \right) - \lambda_2^* \hat{\omega}_b \text{sgn}\{\hat{\beta}_b(\lambda_1^*, \lambda_2)\} \right\}. \] (3.4)

It follows from (3.2), (3.3) and (3.4) that
\[ \hat{\beta}_a(\lambda_1^*, \lambda_2) - \hat{\beta}_b(\lambda_1^*, \lambda_2) = \]
\[ \frac{1}{2\lambda_2} \left\{ \frac{1}{n} \sum_{i=1}^{n} \left[ x_{ia} - x_{ib} + \sum_{j \in R_i} x_{ja} \exp \left( \sum_{k=1}^{p} \beta_k x_{jk} \right) \sum_{j \in R_i} \exp \left( \sum_{k=1}^{p} \beta_k x_{jk} \right) \right] - \lambda_1^* \text{sgn}\{\hat{\beta}_b(\lambda_1^*, \lambda_2)\} (\hat{\omega}_b - \hat{\omega}_a) \right\}. \] (3.5)
Adaptive Elastic Net Method

From Schoenfeld [9], we have

$$\hat{r}_{ir} := x_{ir} - \mathbb{E}(x_{ir}|R_i) = x_{ir} - \frac{\sum_{i \in R_i} x_{ir} \exp(x_{ir}/\beta)}{\sum_{i \in R_i} \exp(x_{ir}/\beta)},$$  \hspace{1cm} (3.6)

where \( r = 1, 2, \cdots, p \). Therefore, (3.5) is equivalent to

$$\hat{\beta}_a(\lambda_1^*, \lambda_2) - \hat{\beta}_b(\lambda_1^*, \lambda_2) = \frac{1}{2n\lambda_2} \sum_{i=1}^{n} (\hat{r}_{ia} - \hat{r}_{ib}) + \frac{\lambda_1^*}{2\lambda_2} \text{sgn}\{\hat{\beta}_b(\lambda_1^*, \lambda_2)\} (\hat{\omega}_b - \hat{\omega}_a).$$

Hence,

$$\left| \hat{\beta}_a(\lambda_1^*, \lambda_2) - \hat{\beta}_b(\lambda_1^*, \lambda_2) \right| \leq \frac{1}{2n\lambda_2} \sum_{i=1}^{n} |\hat{r}_{ia} - \hat{r}_{ib}| + \frac{\lambda_1^*}{2\lambda_2} |\hat{\omega}_b - \hat{\omega}_a|. \hspace{1cm} (3.7)$$

Since \( x_a \) and \( x_b \) are highly correlated, i.e.,

$$\mathbb{E}[x_a x_b^T] \rightarrow 1.$$

Then, we have for an individual \( i \)

$$|x_{ia} - x_{ib}| \rightarrow 0, \text{ and } \left| \mathbb{E}(x_{ia}) - \mathbb{E}(x_{ib}) \right| \rightarrow 0.$$

Hence,

$$\left| [x_{ia} - \mathbb{E}(x_{ia}|R_i)] - [x_{ib} - \mathbb{E}(x_{ib}|R_i)] \right| \rightarrow 0. \hspace{1cm} (3.8)$$

By (3.6) and (3.8), we have

$$|\hat{r}_{ia} - \hat{r}_{ib}| \rightarrow 0 \text{ and } \left| \mathbb{E}(\hat{r}_{ia}) - \mathbb{E}(\hat{r}_{ib}) \right| \rightarrow 0. \hspace{1cm} (3.9)$$

Since

$$\hat{\omega}_a = \left( |\hat{\beta}_{(EN)a}| \right)^{-\gamma} \text{ and } \hat{\omega}_b = \left( |\hat{\beta}_{(EN)b}| \right)^{-\gamma},$$

we have

$$|\hat{\omega}_a - \hat{\omega}_b| \rightarrow 0. \hspace{1cm} (3.10)$$

It follows from (3.7), (3.9) and (3.10) that

$$D_{\lambda_1^*, \lambda_2}(a, b) \rightarrow 0.$$

The lemma holds. \( \square \)
4. Oracle property

In this section, we study the oracle property of the Adaptive Elastic Net method for the time-dependent Cox model. To emphasize the time dependence, we rewrite the model (2.1) as follows

\[ h(t|X) = h_0(t) \exp \{ \beta^T X(t) \}, \]  

where the covariate \( X(t) \) is time-dependent. Hence, the Adaptive Elastic Net estimator for this Cox model is:

\[ \hat{\beta}_{(AEN)} = \arg \min \left\{ -\frac{1}{n} l_n(\beta) + \lambda_1 \sum_{k=1}^{p} \hat{\omega}_k |\beta_k| + \lambda_2 \sum_{k=1}^{p} \beta_k^2 \right\}, \]  

where \( l_n(\beta) \) is the partial log-likelihood function.

Next we consider

\[ Q_n(\beta) = l_n(\beta) - n\lambda_1 \sum_{k=1}^{p} \hat{\omega}_k |\beta_k| - n\lambda_2 \sum_{k=1}^{p} \beta_k^2. \]

We suppose that the real parameter \( \beta_0 = (\beta_{01}, \cdots, \beta_{0p})^T \) is sparse, and \( A = \{ k : \beta_{0k} \neq 0 \} = \{1, \cdots, p_0\} \) with \( p_0 < p \). Hence \( \beta_0 = (\beta_A, \beta_{Ac})^T \) with \( \beta_{Ac} = 0 \), where \( 0 \) is the zero vector. Hence the corresponding estimator \( \hat{\beta}_0 \) takes the form of

\[ \hat{\beta}_0 = (\hat{\beta}_A, \hat{\beta}_{Ac})^T. \]

Let \( I(\hat{\beta}_0) \) be the Fisher information matrix. We also introduce the following notation. For any matrix \( B \),

\[ \|B\| = \sup_{ij} |b_{ij}|, \]

and for any vector \( a \)

\[ a^\otimes 0 = 1, \ a^\otimes 1 = a, \ a^\otimes 2 = a \otimes a, \ \|a\| = \sup_i |a_i|, \ \text{and} \ |a| = \left( \sum_i a_i^2 \right)^{\frac{1}{2}}, \]

where \( a \otimes b \) is the \( p \times p \) matrix \( ab^T \) for any vectors \( a, b \in \mathbb{R}^p \). Before we state our main result in this section, similar to Fan and Li \[3\], we need the following conditions:

(a) \[ \int_0^1 h_0(t)dt < \infty, n\lambda_1^* \rightarrow \infty, \text{and} \ \frac{\lambda_2}{\sqrt{n}} \rightarrow 0, \text{as} \ n \rightarrow \infty; \]

(b) There exists a neighborhood \( \Omega \) of \( \beta_0 \) such that for any \( \beta \in \Omega \),

\[ \mathbb{E} \left[ \sup_{t \in [0, 1], \beta \in \Omega} Y(t)X^T(t)X(t) \exp \{ \beta^T X(t) \} \right] < \infty; \]

Moreover,
(c) for any $\beta \in \Omega$, we have
\[
\begin{align*}
    s_0(\beta, t) &= \mathbb{E}[Y(t) \exp \{\beta^T X(t)\}], \\
    s_1(\beta, t) &= \mathbb{E}[Y(t)X(t) \exp \{\beta^T X(t)\}], \text{ and} \\
    s_2(\beta, t) &= \mathbb{E}[Y(t)X(t)X^T(t) \exp \{\beta^T X(t)\}],
\end{align*}
\]
where $s_i(\beta, t), i = 0, 1, 2$, satisfies the following:

(I) $s_0(\beta, t), s_1(\beta, t)$ and $s_2(\beta, t)$ are uniformly continuous in $t \in [0, 1]$ for $\beta \in \Omega$.

(II) $s_2(\beta, t)$ and $s_1(\beta, t)$ are bounded on $\Omega \times [0, 1]$. Moreover assume $s_0(\beta, t)$ is positive and bounded away from zero on $\Omega \times [0, 1]$.

(d) Define:
\[
v(\beta, t) = \frac{s_2(\beta, t)}{s_0(\beta, t)} - e \otimes 2
\]
with
\[
e = \frac{s_1(\beta, t)}{s_0(\beta, t)}.
\]

Then, the fisher information matrix
\[
I(\beta_0) = \int_0^1 v(\beta_0, t)s_0(\beta_0, t)h_0(t)dt
\]
is finite positive definite.

In addition, the regularity conditions in Anderson and Gill [1] are assumed in the whole section. Then, we have the following theorem.

**Theorem 4.1** If (a)-(d) hold, then the Adaptive Elastic Net estimator $\hat{\beta}$ for the time-dependent Cox model has the sparsity, i.e.,
\[
\mathbb{P}(\hat{\beta}_{A^c} = 0) \rightarrow 1.
\]

**Proof:** We note that the partial log-likelihood function of the time-dependent Cox model is:
\[
l_n(\beta) = \sum_{i=1}^n \int_0^1 \beta^T X_i(t) dN_i(t) - \int_0^1 \log \left[\sum_{i=1}^n Y_i(t) \exp \{\beta^T X_i(t)\}\right] d\tilde{N}(t) \quad (4.5)
\]
where $\tilde{N}(t) = \sum_{i=1}^n N_i(t)$.

Then, for each $\beta$ in the neighborhood $\Omega$ of $\beta_0$, we have
\[
\frac{1}{n} \{l_n(\beta) - l_n(\beta_0)\} = f(\beta) + O_P\left(\frac{\|\beta - \beta_0\|}{\sqrt{n}}\right), \quad (4.6)
\]
where $O_P(\cdot)$ denotes convergence rate, and
\[
f(\beta) = \int_0^1 \left[ (\beta - \beta_0)^T s_1(\beta_0, t) - \log \left\{ \frac{s_0(\beta, t)}{s_0(\beta_0, t)} \right\} s_0(\beta_0, t) \right] h_0(t) dt. \quad (4.7)
\]
Let $\beta = \beta_0 + \frac{u}{\sqrt{n}}$, where $\|u\| \leq C$ for some large enough constant $C$. According to the theorem 3.1 in Fan and Li [5], we know that for any $\epsilon > 0$,

$$
\mathbb{P}\left\{ \sup_{\|u\| = C} Q_n(\beta) < Q_n(\beta_0) \right\} \geq 1 - \epsilon.
$$

Then

$$
\left\| \hat{\beta}_0 - \beta_0 \right\| = O_p(n^{-\frac{1}{2}}). \quad (4.8)
$$

By the Taylor expansion of $\ln(\beta)$, we have

$$
\ln(\beta) = \ln(\beta_0) + \frac{\partial \ln(\beta)}{\partial \beta} |_{\beta = \beta_0} (\beta - \beta_0) + O_p(\sqrt{n}\|\beta - \beta_0\|).
$$

(4.9)

By (4.6) and (4.9), we obtain

$$
\ln(\beta) = \ln(\beta_0) + nf(\beta) + O_p(\sqrt{n}\|\beta - \beta_0\|),
$$

(4.10)

where $f(\beta)$ is given by (4.7).

On the other hand, by the condition (c), we have

$$
\frac{\partial f(\beta)}{\partial \beta} = \int_0^1 \left[ \frac{s_1(\beta_0, t)}{s_0(\beta_0, t)} - \frac{s_1(\beta, t)}{s_0(\beta, t)} \right] s_0(\beta_0, t)h_0(t)dt,
$$

and

$$
- \frac{\partial^2 f(\beta)}{\partial \beta \partial \beta^T} = \int_0^1 \left[ \frac{s_2(\beta, t)s_0(\beta, t) - s_1(\beta, t)s_1^T(\beta, t)}{[s_0(\beta, t)]^2} \right] s_0(\beta_0, t)h_0(t)dt.
$$

Hence

$$
f(\beta_0) = 0, \quad \frac{\partial f(\beta)}{\partial \beta} |_{\beta = \beta_0} = 0, \text{ and } - \frac{\partial^2 f(\beta)}{\partial \beta \partial \beta^T} = I(\beta_0).
$$

(4.11)

From (4.11) and the Taylor expansion, we have

$$
f(\beta) = - \frac{1}{2} (\beta - \beta_0)^T \left\{ I(\beta_0) + o(1) \right\} (\beta - \beta_0).
$$

(4.12)

Next we study (4.10). Since

$$
\frac{\partial l_n(\beta)}{\partial \beta_k} = O_p(\sqrt{n}), \quad k = p_0 + 1, \cdots, p,
$$

we have

$$
\frac{\partial Q_n(\beta)}{\partial \beta_k} = O_p(\sqrt{n}) - n\lambda_k^2 \sum_k \omega_k \text{sgn}(\beta_k) - 2n\lambda_2 \sum_k \beta_k.
$$

(4.13)

Noting

$$
n \frac{2}{\gamma} (|\beta_k|^\gamma - 0) = O_p(1),
$$
we obtain
\[
\frac{\partial Q_n(\beta)}{\partial \beta_k} = n^\frac{3}{2} \left\{ O_p(n^{-1}) - n \lambda_1 \text{sgn}\{\beta_k\} - 2\lambda_2 \sqrt{n} \sum_k \beta_k \right\}. \tag{4.14}
\]

Combing (4.3) and (4.14), we get that the sign of \(\frac{\partial Q_n(\beta)}{\partial \beta_k}\) is determined only by \(\beta_k\), where
\[\beta_k \in (-Cn^{-\frac{1}{2}}, Cn^{-\frac{1}{2}}), \quad k = p_0 + 1, \cdots, p.\]
Therefore,
\[
\|\hat{\beta}_A - \beta_A\| = O_p(n^{-\frac{1}{2}}). \tag{4.15}
\]
By (4.8) and (4.15),
\[Q_n(\beta_A, 0) = \max_{\|\hat{\beta}_A\| \leq Cn^{-\frac{1}{2}}} Q_n(\hat{\beta}_A, \hat{\beta}_A^c).\]
Therefore
\[P(\hat{\beta}_A^c = 0) \to 1.\]
The proof is completed. \(\square\)

Next we study its asymptotic normality.

**Theorem 4.2** Suppose that the conditions (a)-(d) hold. Then the Adaptive Elastic Net estimator \(\hat{\beta}\) for the time-dependent Cox model has the following asymptotic normality:
\[\sqrt{n}(\hat{\beta}_A - \beta_A) \overset{D}{\to} N(0, I_1^{-1}(\beta_A)),\]
where \(\overset{D}{\to}\) denotes convergence in distributions.

**Proof:** According to the proof of Theorem 4.1, we know that there exists \(\hat{\beta}_A\) such that for \(k = 1, \cdots, p_0\)
\[
\frac{\partial Q_n(\beta)}{\partial \beta_k} \bigg|_{\beta = (\hat{\beta}_A, 0)^T} = 0. \tag{4.16}
\]
Let \(U_n(\beta)\) be the score function of \(I_n(\beta)\), that is
\[
U_n(\beta) = \sum_{i=1}^n \int_0^1 X_i(t) dN_i(t) - \int_0^1 \sum_{i=1}^n Y_i(t)X_i(t) \exp\{\beta^T X_i(t)\} \sum_{i=1}^n Y_i(t) \exp\{\beta^T X_i(t)\} d\tilde{N}(t). \tag{4.17}
\]
Moreover, define
\[
\hat{I}(\beta) = \int_0^1 \left[ \sum_{i=1}^n Y_i(t)X_i(t)X_i^T(t) \exp\{\beta^T X_i(t)\} \left( \sum_{i=1}^n Y_i(t) \exp\{\beta^T X_i(t)\} \right) \right. \\
\left. - \frac{\sum_{i=1}^n Y_i(t)X_i(t) \exp\{\beta^T X_i(t)\} \left( \sum_{i=1}^n Y_i(t)X_i(t) \exp\{\beta^T X_i(t)\} \right)^T}{\left( \sum_{i=1}^n Y_i(t) \exp\{\beta^T X_i(t)\} \right)^2} \right] d\tilde{N}(t). \tag{4.18}
\]
On the other hand, we have

\[
\frac{\partial Q_n(\beta)}{\partial \beta_k} \bigg|_{\beta=(\hat{\beta}_A,0)^T} = \frac{\partial l_n(\beta)}{\partial \beta_k} \bigg|_{\beta=(\hat{\beta}_A,0)^T} - n\lambda^*_1 \sum_k \omega_k \text{sgn}\{\beta_k\} - 2n\lambda_2 \sum_k \beta_k = 0. \tag{4.19}
\]

According to the Taylor expansion, (4.19) is equivalent to

\[
U_A(\beta_0) - \hat{I}_1(\hat{\beta})(\hat{\beta}_A - \beta_A) - n\lambda^*_1 \sum_k \omega_k \text{sgn}\{\beta_k\} - 2n\lambda_2 \sum \beta_k = 0, \tag{4.20}
\]

where \(\hat{\beta} = (\hat{\beta}_0, \beta_0)^T\).

According to (4.3) and Andersen and Gill [1], we have

\[
\frac{1}{\sqrt{n}} U_A(\beta_0) \to N(0, I_1(\beta_A)), \tag{4.21}
\]

and

\[
\frac{1}{n} \hat{I}_1(\hat{\beta}) \to I_1(\beta_A), \tag{4.22}
\]

where \(U_A(\beta_0)\) is composed by the first \(p_0\) elements of \(U(\beta_0)\), and \(I_1(\beta_A)\) is a \(p_0 \times p_0\) submatrix of \(I(\beta_0)\).

On the other hand, if we assume

\[
\sqrt{n}\lambda^*_1 \to \lambda_0, \tag{4.23}
\]

then

\[
\frac{1}{\sqrt{n}} (\hat{\beta}_A - \beta_A) = \hat{I}_1^{-1}(\hat{\beta}) \left[ \frac{1}{\sqrt{n}} U_A(\beta_0) - \sqrt{n}\lambda^*_1 \sum_k \omega_k \text{sgn}\{\beta_k\} \right] + O_p(1). \tag{4.24}
\]

Note that from (4.22),

\[
n\hat{I}_1^{-1}(\hat{\beta}) \to I_1^{-1}(\beta_A). \tag{4.25}
\]

Then (4.24) can be rewritten as:

\[
\sqrt{n}(\hat{\beta}_A - \beta_A) = I_1^{-1}(\beta_A) \left[ \frac{1}{\sqrt{n}} U_A(\beta_0) - \lambda_0 b_1 \right] + O_p(1), \tag{4.26}
\]

where

\[
b_1 = \sum_k \omega_k \text{sgn}\{\beta_k\}, \quad k = 1, \ldots, p_0.
\]

According to the Slutsky’s Theorem, we have as \(n \to \infty\),

\[
\sqrt{n}(\hat{\beta}_A - \beta_A) \to N \left( -\lambda_0 I_1^{-1}(\beta_A)b_1, I_1^{-1}(\beta_A) \right). \tag{4.27}
\]

Hence, if \(\lambda_0 = 0\), then (4.26) and (4.27) can be simplified to

\[
\sqrt{n}(\hat{\beta}_A - \beta_A) = I_1^{-1}(\beta_A) \left[ \frac{1}{\sqrt{n}} U_A(\beta_0) \right] + O_p(1), \tag{4.28}
\]

and

\[
\sqrt{n}(\hat{\beta}_A - \beta_A) \to N \left( 0, I_1^{-1}(\beta_A) \right), \quad \text{as } n \to \infty, \tag{4.29}
\]

respectively. The proof of the theorem is finished. \(\Box\)
5. Empirical analysis

Theorem 3.1 reveals that the Adaptive Elastic Net estimator for the Cox model enjoys the grouping effect. Next, we do an empirical analysis.

The data come from a questionnaire, which studies the mobile phone cards’ usage of college students. $x_1, \ldots, x_9$ and $x_{10}$ denote Sex, Grade, Position, Nation, Registered residence, School address, Operators, The average monthly telephone charges, The quality of service and The average monthly living expenses, respectively. In addition, the study was recorded by years, and from 2007 to 2014. We finally got the 380 effective questionnaires. The training set number is 300, and the test set number is 80. Part of the data is listed in the following table 1.

| No. | Study times | Status    | $x_1$ | $x_2$ | $x_3$ | $x_4$ | $x_5$ | $x_6$ | $x_7$ | $x_8$ | $x_9$ | $x_{10}$ |
|-----|-------------|-----------|-------|-------|-------|-------|-------|-------|-------|-------|-------|----------|
| 1   | 6           | censored  | 2     | 6     | 1     | 1     | 2     | 1     | 1     | 3     | 2     | 2        |
| 2   | 1           | loss      | 2     | 3     | 1     | 1     | 1     | 1     | 2     | 2     | 1     | 1        |
| 3   | 1           | loss      | 1     | 3     | 2     | 2     | 1     | 1     | 2     | 2     | 2     | 1        |
| 4   | 1           | loss      | 1     | 6     | 1     | 1     | 1     | 2     | 2     | 1     | 1     | 1        |
| 5   | 4           | censored  | 2     | 4     | 2     | 2     | 1     | 1     | 5     | 2     | 4     | 1        |
| \vdots | \vdots | \vdots | \vdots | \vdots | \vdots | \vdots | \vdots | \vdots | \vdots | \vdots | \vdots | \vdots   |
| 380 | 2           | loss      | 2     | 2     | 2     | 2     | 2     | 1     | 1     | 2     | 3     | 1        |

Since $x_8$ and $x_{10}$ are medium correlation, we did the variable selection by the Lasso method, the Adaptive Lasso method(ALasso), the Elastic Net method(EN) and the Adaptive Elastic Net method(AEN), respectively. The selected variables are in Table 2 and the coefficient estimators are in Table 3.

| Variables | selected in the model |
|-----------|-----------------------|
| Lasso     | $x_2$ $x_4$ $x_5$ $x_6$ $x_9$ $x_{10}$ |
| ALasso    | $x_2$ $x_5$ $x_9$ $x_{10}$ |
| EN        | $x_2$ $x_4$ $x_5$ $x_6$ $x_8$ $x_9$ $x_{10}$ |
| AEN       | $x_2$ $x_5$ $x_6$ $x_8$ $x_9$ $x_{10}$ |

| $x_1$ | $x_2$ | $x_3$ | $x_4$ | $x_5$ | $x_6$ | $x_7$ | $x_8$ | $x_9$ | $x_{10}$ |
|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|
| Lasso | 0    | -0.092 | 0.005 | -0.011 | 0.125 | -0.246 | 0 | 0 | 0.277 | 0.012 |
| ALasso | 0    | 0.042 | 0 | 0.096 | 0 | 0 | 0 | 0.018 | 0.047 |
| EN | 0    | -0.112 | 0.004 | -0.023 | 0.124 | -0.284 | 0 | 0.019 | 0.313 | 0.026 |
| AEN | 0    | -0.120 | 0.002 | 0 | 0.125 | -0.301 | 0 | 0.030 | 0.329 | 0.036 |

From Table 3, we obtain the following:

(1) These four methods do not select $x_1$ and $x_7$ into the model, which shows that the respondents’ gender and operators have no effect to the usages of mobile phone card.
(2) $x_2$ is negative, which shows the senior students have a low probability of loss. $x_9$ is positive, which shows that the higher the students’ average monthly living expenses is, the greater the loss probability is. Similarly, $x_{10}$ is positive, which means that the worse the operator’s customer service quality is, the greater the loss probability is. It is consistent with the actual situation.

(3) The coefficient estimators obtained by the AEN are the most close to the true model.

(4) For $x_8$ and $x_{10}$, both the Lasso and ALasso select $x_8$ only, while the EN and AEN can select both into the model, suggesting that these two methods can select the all strongly correlated variables into the model, and their estimators of coefficients are almost the same, which reflects the grouping effect.

Theorems 3.1, 4.1 and 4.2 reveal that the Adaptive Elastic Net estimator for the Cox model enjoys the grouping effect and oracle property. Next, we show these properties through a numerical simulation.

Let $x_i \sim N(0, 1)$, $i = 1, 2, 5, 6, 8, 9, 10$. Moreover, let $x_2 = x_3, x_6 = x_7$ and $x_4 = 2x_1 + \frac{1}{2}x_2 + \frac{1}{2}x_3$. Then $x_2$ and $x_3$ are strongly correlated, so as $x_6$ and $x_7$. Moreover, there exists the linear relationship between $x_1, x_2, x_3$ and $x_4$. Consider the following Cox model,

$$h(t) = h_0(t) \exp \left( \sum_{i=1}^{10} \beta_i x_i \right),$$

where $t \sim U[0, 1]$. The real parameter $\beta$ is $(-1, 2, 2, 0, \frac{1}{2}, 1, 0, 0, 0)^T$. Then we did the simulation with $n = 1000$ and $p = 10$.

We used the Lasso, ALasso, EN and AEN to do variable selection, respectively. Let $\lambda_2 = \frac{1}{3}$, $\gamma = 3$, and the other parameters be selected by the cross validation method \[15\]. By using the Lars algorithm \[4\], we obtain the coefficient estimators. See Table 4.

| Variable | Lasso | ALasso | EN | AEN |
|----------|-------|--------|----|-----|
| $x_1$    | -0.99553 | -0.99796 | -0.99509 | -0.99947 |
| $x_2$    | 3.89255  | 3.89967 | 1.99726 | 1.99951 |
| $x_3$    | 0.09627  | 0.09901 | 1.99726 | 1.99951 |
| $x_4$    | 0       | 0       | 0.50021 | 0.49992 |
| $x_5$    | 0.49304 | 0.49936 | 0.49936 | 0.49992 |
| $x_6$    | 1.88214 | 1.90981 | 0.99854 | 0.99976 |
| $x_7$    | 0.05014 | 0.03938 | 0.99854 | 0.99976 |
| $x_8$    | 0.00794 | 0.00261 | 0.00261 | 0.00010 |
| $x_9$    | 0.00261 | 0.00016 | 0.00261 | 0.00000 |
| $x_{10}$ | 0.00385 | 0.00013 | 0.00248 | 0.00010 |

From Table 4, we get:

(1) The coefficient estimators obtained by the AEN are the most close to the true model.
(2) None of the four methods selects \( x_4 \) into the model, which implies that these four methods are all able to deal with the collinearity problems.

(3) We look at grouped variables \((x_2, x_3)\) and \((x_6, x_7)\). The AEN and EN can select all the strongly correlated variables into the model, and the coefficient estimators of these two groups are the same. While the Lasso and ALasso select \( x_2 \) and \( x_6 \), respectively. This indicates the AEN and EN enjoy the grouping effect.

(4) We focus on the variables \( x_8, x_9 \) and \( x_{10} \). The ALasso and AEN can get more accurate estimators for the estimation of zero variables than the other two methods do. This indicates that the AEN has the oracle property.

6. Conclusion

In this paper, we study the Adaptive Elastic Net method for the Cox model. We show that it has the grouping effect and oracle property. These two properties are showed by an empirical analysis and a numerical simulation. In these examples, the Adaptive Elastic Net and Elastic Net can make up for the lack of the Lasso and Adaptive Lasso, and can select all the strongly correlated variables into the model, i.e., the Adaptive Elastic Net method for the Cox model enjoys the grouping effect. In addition, the Adaptive Elastic Net method for the Cox model has the oracle property.

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