Geometry of Regular Algebras of Global Dimension 4 related to Graded Skew Clifford Algebras of Global Dimension 3

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Abstract

We compute point schemes of some regular algebras in [9] using (Wolfram) Mathematica. These algebras are Ore extensions of regular graded skew Clifford algebras of global dimension 3 (c.f., [chapter 4, [8]]).
1 Introduction

M. Artin, W. Schelter, J. Tate, and M. Van den Bergh introduced the notion of non-commutative regular algebras and invented new methods in algebraic geometry in the late 1980s to study them ([1], [2], [3]). Such algebras are viewed as non-commutative analogues of polynomial rings; indeed, polynomial rings are examples of regular algebras.

By the 1980s, a lot of algebras had arisen in quantum physics, specifically quantum groups, and many traditional algebraic techniques failed on these new algebras. In physics, quantum groups are viewed as algebras of non-commuting functions acting on some “non-commutative space” ([4]). In the early 1980s, E. K. Sklyanin, a physicist, constructed a family of graded algebras on four generators ([5]). These algebras were later proved to depend on an elliptic curve and an automorphism ([6]). By the late 1980s, it was known that many of the algebras in quantum physics are regular algebras; in particular, the family of algebras constructed by Sklyanin consists of regular algebras.

In 2010, T. Cassidy and M. Vancliff introduced a class of algebras that provide an “easy” way to write down some quadratic regular algebras of global dimension $n$ where $n \in \mathbb{N}$ ([4]). In fact, they generalized the notion of a graded Clifford algebra and called it a graded skew Clifford algebra (see [Definition 3.1, [7]]).

In this lecture notes, we compute point schemes of some regular algebras in [9] using (Wolfram) Mathematica. These algebras are Ore extensions of regular graded skew Clifford algebras of global dimension 3 (c.f., [chapter 4, [8]]).

Remark: We compute the coordinate $(\alpha_1, \alpha_2, \alpha_3, \alpha_4) \in \mathbb{P}^3$. The computation of $(\beta_1, \beta_2, \beta_3, \beta_4) \in \mathbb{P}^3$ will be added in the near future.

2 Acknowledgement

I would like to thank Michaela Vancliff (my Ph.D. advisor during my studies 2007-2011 at the University of Texas at Arlington) for her invaluable advice and for explaining essential concepts on non-commutative algebraic geometry.
3 Geometry of Graded Skew Clifford Algebras of Global Dimension 4

Throughout this lecture notes, \( \mathbb{K} \) denotes an algebraically closed field, char(\( \mathbb{K} \)) \( \neq 2 \), and \( \mathbb{K}^\times \) denotes \( \mathbb{K} \setminus \{0\} \).

3.1 Proposition 1:

Suppose \( q \in \mathbb{K}^\times \), and \( q^2 \neq 1 \). If the algebra
\[
A = \frac{\mathbb{K}\langle x_1, x_2, x_3, x_4 \rangle}{\langle g_1, \ldots, g_6 \rangle}
\]
where
\[
g_1 = x_2x_1 - qx_1x_2, \quad g_2 = x_2x_3 - x_3x_2, \\
g_3 = x_3x_1 - qx_1x_3, \quad g_4 = x_4x_1 - x_1x_4 - (q - q^{-1})x_2x_3, \\
g_5 = x_4x_2 - q_2x_4, \quad g_6 = x_4x_3 - q_2x_4,
\]
then \( A \) has point scheme given by \( \mathcal{V}(x_2, x_3) \cup \mathcal{V}(x_2x_3 - x_1x_4) \) (see Figure 1).

![Figure 1: Depiction of the Point Scheme in Proposition 1](image)

Proof:

Suppose \( p = ((\alpha_1, \alpha_2, \alpha_3, \alpha_4), (\beta_1, \beta_2, \beta_3, \beta_4)) \in \mathbb{P}^3 \times \mathbb{P}^3 \).

To find the point scheme \( \mathcal{P} \) of \( A \), we solve
\[
0 = g_1(p) = \alpha_2\beta_1 - qa_1\beta_2, \\
0 = g_2(p) = \alpha_2\beta_3 - \alpha_3\beta_2, \\
0 = g_3(p) = \alpha_3\beta_1 - qa_1\beta_3, \\
0 = g_4(p) = \alpha_4\beta_1 - \alpha_1\beta_4 - (q - q^{-1})\alpha_2\beta_3, \\
0 = g_5(p) = \alpha_4\beta_2 - qa_2\beta_4, \\
0 = g_6(p) = \alpha_4\beta_3 - qa_3\beta_4,
\]
which yields \( DE = F \), where
\[
D = \begin{bmatrix}
\alpha_2 & -qa_1 & 0 & 0 \\
0 & -\alpha_3 & \alpha_2 & 0 \\
\alpha_3 & 0 & -qa_1 & 0 \\
\alpha_4 & 0 & -(q - q^{-1})\alpha_2 & -a_1 \\
0 & \alpha_4 & 0 & -qa_2 \\
0 & 0 & \alpha_4 & -qa_3
\end{bmatrix}, \quad E = \begin{bmatrix}
\beta_1 \\
\beta_2 \\
\beta_3 \\
\beta_4
\end{bmatrix}, \text{ and } F = \begin{bmatrix}
0 \\
0 \\
0 \\
0
\end{bmatrix}.
\]
We find all $4 \times 4$ minors of $D$ (using Wolfram Mathematica). They are:

\begin{align*}
\alpha_3^2(\alpha_2\alpha_3 - \alpha_1\alpha_4)(-1 + q)(1 + q) \\
\alpha_2\alpha_3(\alpha_2\alpha_3 - \alpha_1\alpha_4)(-1 + q)(1 + q) \\
-\alpha_2\alpha_4(\alpha_2\alpha_3 - \alpha_1\alpha_4)(-1 + q)(1 + q) \\
-\alpha_3\alpha_4(\alpha_2\alpha_3 - \alpha_1\alpha_4)(-1 + q)(1 + q) \\
\alpha_1\alpha_2(\alpha_2\alpha_3 - \alpha_1\alpha_4)(-1 + q)q(1 + q) \\
-\alpha_2^2(\alpha_2\alpha_3 - \alpha_1\alpha_4)(-1 + q)(1 + q) \\
\alpha_1\alpha_3(\alpha_2\alpha_3 - \alpha_1\alpha_4)(-1 + q)q(1 + q)
\end{align*}

Consider the first equation, therefore we have $\alpha_3 = 0$ or $\alpha_2\alpha_3 - \alpha_1\alpha_4 = 0$. If $\alpha_3 = 0$, then by Mathematica, we have

\[ \alpha_1\alpha_2^2\alpha_4(1 + q)(1 + q), \quad -\alpha_1^2\alpha_2\alpha_4(-1 + q)q(1 + q), \quad \alpha_1\alpha_2\alpha_4^2(-1 + q)(1 + q). \]

Therefore in this case, solutions are

\[ \{ (0, \beta, 0, \delta) \in \mathbb{P}^3 : (\beta, \delta) \in \mathbb{P}^1 \} \cup \{ (\alpha, 0, 0, \delta) \in \mathbb{P}^3 : (\alpha, \delta) \in \mathbb{P}^1 \} \cup \{ (\alpha, \beta, 0, 0) \in \mathbb{P}^3 : (\alpha, \beta) \in \mathbb{P}^1 \}. \]

If $\alpha_3 \neq 0$ and $\alpha_2\alpha_3 = \alpha_1\alpha_4$, then we obtain

\[ \{ (\alpha, \beta, \gamma, \delta) \in \mathbb{P}^3 : \beta \gamma = \alpha \delta \} \]

So, in general, the solutions are:

\[ \{ (\alpha, 0, 0, \delta) \in \mathbb{P}^3 : (\alpha, \delta) \in \mathbb{P}^1 \} \cup \{ (\alpha, \beta, \gamma, \delta) \in \mathbb{P}^3 : \beta \gamma = \alpha \delta \} \]

and the point scheme is $\mathcal{V}(x_2, x_3) \cup \mathcal{V}(x_2x_3 - x_1x_4)$.

\section{Proposition 2:}

Suppose $q \in K^\times$. If the algebra

\[ A = \frac{K[x_1, x_2, x_3, x_4]}{(g_1, \ldots, g_6)} \]

where

\begin{align*}
g_1 &= x_1x_2 - x_2x_1, \\
g_2 &= x_3x_2 - x_2x_3, \\
g_3 &= x_1x_3 - x_3x_1, \\
g_4 &= x_1x_1 - x_1x_4 + q(x_4x_3 - x_1x_2), \\
g_5 &= x_4x_2 - x_2x_4, \\
g_6 &= x_1x_3 - x_3x_1,
\end{align*}

then $A$ has point scheme given by $\mathcal{V}(x_2(x_1x_2 - x_3x_4), x_3(x_1x_2 - x_3x_4))$ which contains the double line $\mathcal{V}(x_2, x_3)$ (see Figure 2).

\begin{figure}[h]
\centering
\includegraphics[width=0.2\textwidth]{point_scheme.png}
\caption{Depiction of the Point Scheme in Proposition 2}
\end{figure}
Proof:
Suppose \( p = ((\alpha_1, \alpha_2, \alpha_3, \alpha_4), (\beta_1, \beta_2, \beta_3, \beta_4)) \in \mathbb{P}^3 \times \mathbb{P}^3 \).

To find the point scheme \( P \) of \( A \), we solve

\[
\begin{align*}
0 &= g_1(p) = \alpha_1 \beta_2 - \alpha_2 \beta_1, \\
0 &= g_2(p) = \alpha_3 \beta_2 - \alpha_2 \beta_3, \\
0 &= g_3(p) = \alpha_1 \beta_3 - \alpha_3 \beta_1, \\
0 &= g_4(p) = \alpha_4 \beta_1 - \alpha_1 \beta_4 + q(\alpha_3 \beta_3 - \alpha_1 \beta_2), \\
0 &= g_5(p) = \alpha_4 \beta_2 - \alpha_2 \beta_4, \\
0 &= g_6(p) = \alpha_4 \beta_3 - \alpha_3 \beta_4,
\end{align*}
\]

which yields \( DE = F \), where

\[
D = \begin{bmatrix}
-\alpha_2 & \alpha_1 & 0 & 0 \\
0 & \alpha_3 & -\alpha_2 & 0 \\
-\alpha_3 & 0 & \alpha_1 & 0 \\
\alpha_4 & -q \alpha_1 & q \alpha_4 & -\alpha_1 \\
0 & \alpha_4 & 0 & -\alpha_2 \\
0 & 0 & \alpha_4 & -\alpha_3
\end{bmatrix},
E = \begin{bmatrix}
\beta_1 \\
\beta_2 \\
\beta_3 \\
\beta_4
\end{bmatrix},
\text{and } F = \begin{bmatrix}
0 \\
0 \\
0 \\
0
\end{bmatrix}.
\]

we find all \( 4 \times 4 \) minors of \( D \) (using Wolfram Mathematica). They are:

\[
\begin{align*}
\alpha_3^2(\alpha_1 \alpha_2 - \alpha_3 \alpha_4)q \\
-\alpha_2^2(\alpha_1 \alpha_2 - \alpha_3 \alpha_4)q \\
-\alpha_2 \alpha_3(\alpha_1 \alpha_2 - \alpha_3 \alpha_4)q \\
\alpha_1 \alpha_2(\alpha_1 \alpha_2 - \alpha_3 \alpha_4)q \\
\alpha_1 \alpha_3(\alpha_1 \alpha_2 - \alpha_3 \alpha_4)q \\
\alpha_2 \alpha_4(\alpha_1 \alpha_2 - \alpha_3 \alpha_4)q \\
\alpha_3 \alpha_4(\alpha_1 \alpha_2 - \alpha_3 \alpha_4)q
\end{align*}
\]

Consider the first equation, therefore we have \( \alpha_3 = 0 \) or \( \alpha_1 \alpha_2 - \alpha_3 \alpha_4 = 0 \). If \( \alpha_3 = 0 \), then by Mathematica, we have

\[-\alpha_1 \alpha_2 \alpha_3 q, \quad \alpha_1^2 \alpha_2^2 q, \quad \alpha_1 \alpha_2^2 q, \quad \alpha_1 \alpha_2 \alpha_4 q.\]

Therefore in this case, solutions are

\[
\{(0, \beta, 0, \delta) \in \mathbb{P}^3 : (\beta, \delta) \in \mathbb{P}^1\} \cup \{(\alpha, 0, 0, \delta) \in \mathbb{P}^3 : (\alpha, \delta) \in \mathbb{P}^1\}.
\]

If \( \alpha_3 \neq 0 \) and \( \alpha_1 \alpha_2 = \alpha_3 \alpha_4 \), then we obtain

\[
\{(\alpha, \beta, \gamma, \delta) \in \mathbb{P}^3 : \alpha \beta = \gamma \delta\}
\]

So, in general, the solutions are:

\[
\{(\alpha, \beta, \gamma, \delta) \in \mathbb{P}^3 : \alpha \beta = \gamma \delta\}
\]
and the point scheme is \( \mathcal{V}(x_2(x_1x_2 - x_3x_4), x_3(x_1x_2 - x_3x_4)) \) which contains the double line \( \mathcal{V}(x_2, x_3) \).

Notice that
\[
\langle x_1x_2 - x_3x_4 \rangle \langle x_2^2, x_2x_1, x_1x_3, x_2x_4, x_3^2, x_3x_4 \rangle = \langle x_1x_2 - x_3x_4 \rangle \langle x_2, x_3 \rangle \langle x_1, x_2, x_3, x_4 \rangle
\]

Therefore, the point scheme \( \mathcal{P} \) of \( A \) is
\[
\mathcal{V}(x_1x_2 - x_3x_4) \cup \mathcal{V}(x_2, x_3) \cup \mathcal{V}(x_1, x_2, x_3, x_4)
\]
\[
= \mathcal{V}(x_1x_2 - x_3x_4) \cup \mathcal{V}(x_2, x_3)
\]
where \( \mathcal{V}(x_2, x_3) \subset \mathcal{V}(x_1x_2 - x_3x_4) \).

\[\blacksquare\]

### 3.3 Proposition 3:
Suppose \( q \in K^\times \setminus \{-1\} \). If the algebra
\[
A = \frac{K \langle x_1, x_2, x_3, x_4 \rangle}{\langle g_1, \ldots, g_6 \rangle}
\]
where
\[
\begin{align*}
g_1 &= x_1x_2 - x_3x_1, \\
g_2 &= x_2x_3 - x_3x_2, \\
g_3 &= x_1x_3 - x_3x_1, \\
g_4 &= x_1x_4 - x_4x_1, \\
g_5 &= x_2x_4 - x_4x_2 - q(x_1^2 - x_2x_4), \\
g_6 &= x_4x_3 - x_3x_4,
\end{align*}
\]
, then \( A \) has point scheme given by \( Q \cup L \) where \( Q = \mathcal{V}(x_1^2 - x_2x_4) \) and \( L = \mathcal{V}(x_1, x_3) \) (see Figure 3).

![Figure 3: Depiction of the Point Scheme in Proposition 3](image)

### Proof:
Suppose
\[
p = ((\alpha_1, \alpha_2, \alpha_3, \alpha_4), (\beta_1, \beta_2, \beta_3, \beta_4)) \in \mathbb{P}^3 \times \mathbb{P}^3.
\]

To find the point scheme \( \mathcal{P} \) of \( A \), we solve
\[
\begin{align*}
0 &= g_1(p) = \alpha_1\beta_2 - \alpha_2\beta_1, \\
0 &= g_2(p) = \alpha_2\beta_3 - \alpha_3\beta_2, \\
0 &= g_3(p) = \alpha_1\beta_3 - \alpha_3\beta_1, \\
0 &= g_4(p) = \alpha_1\beta_4 - \alpha_4\beta_1, \\
0 &= g_5(p) = \alpha_2\beta_4 - \alpha_4\beta_2 - q(\alpha_1\beta_1 - \alpha_2\beta_4), \\
0 &= g_6(p) = \alpha_4\beta_3 - \alpha_3\beta_4,
\end{align*}
\]
which yields $DE = F$, where

$$D = \begin{bmatrix} -α_2 & α_1 & 0 & 0 \\ 0 & -α_3 & α_2 & 0 \\ -α_3 & 0 & α_1 & 0 \\ -α_4 & 0 & 0 & α_1 \\ -qα_1 & -α_4 & 0 & (q + 1)α_2 \\ 0 & 0 & α_4 & -α_3 \end{bmatrix}, \quad E = \begin{bmatrix} β_1 \\ β_2 \\ β_3 \\ β_4 \end{bmatrix}, \quad \text{and} \quad F = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$ 

We find all $4 \times 4$ minors of $D$ (using Wolfram Mathematica). They are:

$$\begin{align*}
α_1^2(α_1^2 - α_2α_4)q \\
-α_3^2(α_1^2 - α_2α_4)q \\
α_1α_2(α_1^2 - α_2α_4)q \\
α_2α_3(α_1^2 - α_2α_4)q \\
α_1α_3(α_1^2 - α_2α_4)q \\
α_1α_4(α_1^2 - α_2α_4)q \\
-α_3α_4(α_1^2 - α_2α_4)q
\end{align*}$$

Consider the first equation, therefore we have $α_1 = 0$ or $α_1^2 - α_2α_4 = 0$. If $α_1 = 0$, then by Mathematica, we have

$$-α_2^2α_3α_4q, \quad α_2α_3^2α_4q, \quad α_2α_3α_4^2q.$$ 

Therefore in this case, solutions are

$$\{(0, 0, γ, δ) ∈ P^3 : (γ, δ) ∈ P^1\} ∪ \{(0, β, 0, δ) ∈ P^3 : (β, δ) ∈ P^1\} ∪ \{(0, β, γ, 0) ∈ P^3 : (β, γ) ∈ P^3\}.$$ 

If $α_3 ≠ 0$ and $α_1^2 = α_2α_4$, then we obtain

$$\{(α, β, γ, δ) ∈ P^3 : α^2 = βδ\}$$ 

So, in general, the solutions are:

$$\{(0, β, 0, δ) ∈ P^3 : (β, δ) ∈ P^1\} ∪ \{(α, β, γ, δ) ∈ P^3 : α^2 = βδ\},$$

and the point scheme is $V(x_1^2 - x_2x_4) ∪ V(x_1, x_3)$

3.4 Proposition 4:

If the algebra

$$A = \frac{K(x_1, x_2, x_3, x_4)}{⟨g_1, \ldots, g_6⟩}$$

where

$$\begin{align*}
g_1 &= x_1x_2 - x_2x_1, \\
g_2 &= x_3x_2 - x_2x_3, \\
g_3 &= x_1x_3 - x_3x_1, \\
g_4 &= x_1x_4 - x_4x_1 - x_1^2 + x_4x_3, \\
g_5 &= x_2x_4 - x_4x_2, \\
g_6 &= x_3x_4 - x_4x_3,
\end{align*}$$

then $A$ has point scheme given by $Q ∪ L$ where $Q = V(x_1^2 - x_3x_4)$ and $L = V(x_2, x_3)$ (so the line $L$ is tangential to the quadric $Q$ at a nonsingular point of $Q$)(see Figure 4).
Proof:

Suppose

\[ p = ((\alpha_1, \alpha_2, \alpha_3, \alpha_4), (\beta_1, \beta_2, \beta_3, \beta_4)) \in P^3 \times P^3. \]

To find the point scheme \( P \) of \( A \), we solve

\[
\begin{align*}
0 &= g_1(p) = \alpha_1 \beta_2 - \alpha_2 \beta_1, \\
0 &= g_2(p) = \alpha_3 \beta_2 - \alpha_2 \beta_3, \\
0 &= g_3(p) = \alpha_1 \beta_3 - \alpha_3 \beta_1, \\
0 &= g_4(p) = \alpha_1 \beta_4 - \alpha_4 \beta_1 + \alpha_1 \beta_3, \\
0 &= g_5(p) = \alpha_2 \beta_4 - \alpha_4 \beta_2, \\
0 &= g_6(p) = \alpha_3 \beta_4 - \alpha_4 \beta_3,
\end{align*}
\]

which yields \( DE = F \), where

\[
D = \begin{bmatrix}
-\alpha_2 & \alpha_1 & 0 & 0 \\
0 & -\alpha_2 & 0 & 0 \\
-\alpha_3 & 0 & \alpha_1 & 0 \\
-\alpha_4 - \alpha_1 & 0 & \alpha_4 & \alpha_1 \\
0 & -\alpha_4 & 0 & \alpha_2 \\
0 & 0 & -\alpha_4 & \alpha_3
\end{bmatrix},
E = \begin{bmatrix}
\beta_1 \\
\beta_2 \\
\beta_3 \\
\beta_4
\end{bmatrix},
\text{and } F = \begin{bmatrix}
0 \\
0 \\
0 \\
0
\end{bmatrix}.
\]

We find all \( 4 \times 4 \) minors of \( D \) (using Wolfram Mathematica). They are:

\[
\begin{align*}
&-\alpha_3^2(\alpha_1^2 - \alpha_3 \alpha_4) \\
&\alpha_2^2(\alpha_1^2 - \alpha_3 \alpha_4) \\
&\alpha_2 \alpha_3(\alpha_1^2 - \alpha_3 \alpha_4) \\
&-\alpha_1 \alpha_2(\alpha_1^2 - \alpha_3 \alpha_4) \\
&-\alpha_1 \alpha_3(\alpha_1^2 - \alpha_3 \alpha_4) \\
&\alpha_2 \alpha_4(\alpha_1^2 - \alpha_3 \alpha_4) \\
&\alpha_3 \alpha_4(\alpha_1^2 - \alpha_3 \alpha_4)
\end{align*}
\]

Consider the first equation, therefore we have \( \alpha_3 = 0 \) or \( \alpha_1^2 - \alpha_3 \alpha_4 = 0 \). If \( \alpha_3 = 0 \), then by Mathematica, we have

\[
\alpha_1^2 \alpha_2^2, \quad -\alpha_1^3 \alpha_2, \quad \alpha_1^2 \alpha_2 \alpha_4.
\]
Therefore in this case, solutions are
\[
\{(0, \beta, 0, \delta) \in \mathbb{P}^3 : (\beta, \delta) \in \mathbb{P}^1\} \cup \{(\alpha, 0, 0, \delta) \in \mathbb{P}^3 : (\alpha, \delta) \in \mathbb{P}^1\}.
\]
If \(\alpha_3 \neq 0\) and \(\alpha_1^2 = \alpha_3 \alpha_4\), then we obtain
\[
\{(\alpha, \beta, \gamma, \delta) \in \mathbb{P}^3 : \alpha^2 = \gamma \delta\}
\]
So, in general, the solutions are:
\[
\{(\alpha, 0, 0, \delta) \in \mathbb{P}^3 : (\alpha, \delta) \in \mathbb{P}^1\} \cup \{(\alpha, \beta, \gamma, \delta) \in \mathbb{P}^3 : \alpha^2 = \gamma \delta\}
\]
, and the point scheme is \(V(x_1^2 - x_3 x_4) \cup V(x_2, x_3)\).

3.5 Proposition 5:
If the algebra
\[
A = \mathbb{K}(x_1, x_2, x_3, x_4) / (g_1, \ldots, g_6)
\]
where
\[
g_1 = x_1 x_2 - x_2 x_1, \quad g_2 = x_2 x_3 - x_3 x_2, \\
g_3 = x_1 x_3 - x_3 x_1, \quad g_4 = x_1 x_4 - x_4 x_1 - x_1^2 + x_2 x_3, \\
g_5 = x_2 x_4 - x_4 x_2, \quad g_6 = x_3 x_4 - x_4 x_3,
\]
, then \(A\) has point scheme given by \(Q \cup L\) where \(Q = V(x_1^2 - x_2 x_3)\) and \(L = V(x_2, x_3)\) (so the line \(L\) is tangential to the quadric \(Q\) at a singular point of \(Q\)) (see Figure 5).

![Figure 5: Depiction of the Point Scheme in Proposition 5](image)

Proof:
Suppose
\[
p = ((\alpha_1, \alpha_2, \alpha_3, \alpha_4), (\beta_1, \beta_2, \beta_3, \beta_4)) \in \mathbb{P}^3 \times \mathbb{P}^3.
\]
To find the point scheme \(P\) of \(A\), we solve
\[
0 = g_1(p) = \alpha_1 \beta_2 - \alpha_2 \beta_1, \\
0 = g_2(p) = \alpha_2 \beta_3 - \alpha_3 \beta_2, \\
0 = g_3(p) = \alpha_1 \beta_3 - \alpha_3 \beta_1, \\
0 = g_4(p) = \alpha_1 \beta_4 - \alpha_4 \beta_1 - \alpha_1 \beta_1 + \alpha_2 \beta_3, \\
0 = g_5(p) = \alpha_2 \beta_4 - \alpha_4 \beta_2, \\
0 = g_6(p) = \alpha_3 \beta_4 - \alpha_4 \beta_3,
\]

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which yields $DE = F$, where
\[
D = \begin{bmatrix}
-\alpha_2 & \alpha_1 & 0 & 0 \\
0 & -\alpha_3 & \alpha_2 & 0 \\
-\alpha_3 & 0 & \alpha_1 & 0 \\
-\alpha_4 & -\alpha_1 & 0 & \alpha_2 \\
0 & -\alpha_4 & 0 & \alpha_2 \\
0 & 0 & -\alpha_4 & \alpha_3
\end{bmatrix},
E = \begin{bmatrix}
\beta_1 \\
\beta_2 \\
\beta_3 \\
\beta_4
\end{bmatrix},
\text{and } F = \begin{bmatrix}
0 \\
0 \\
0 \\
0
\end{bmatrix}.
\]

We find all $4 \times 4$ minors of $D$ (using Wolfram Mathematica). They are:
\[
\begin{align*}
\alpha_3^2(\alpha_1^2 - \alpha_2 \alpha_3) \\
-\alpha_2^2(\alpha_1^2 - \alpha_2 \alpha_3) \\
\alpha_2 \alpha_3(\alpha_1^2 - \alpha_2 \alpha_3) \\
-\alpha_1 \alpha_2(\alpha_1^2 - \alpha_2 \alpha_3) \\
-\alpha_1 \alpha_3(\alpha_1^2 - \alpha_2 \alpha_3) \\
\alpha_2 \alpha_4(\alpha_1^2 - \alpha_2 \alpha_3) \\
\alpha_3 \alpha_4(\alpha_1^2 - \alpha_2 \alpha_3)
\end{align*}
\]

Consider the first equation, therefore we have $\alpha_3 = 0$ or $\alpha_1^2 - \alpha_2 \alpha_3 = 0$. If $\alpha_3 = 0$, then by Mathematica, we have
\[-\alpha_1^2 \alpha_2^2, \quad -\alpha_1^3 \alpha_2, \quad \alpha_1^2 \alpha_2 \alpha_4.
\]
Therefore in this case, solutions are
\[
\{(0, \beta, 0, \delta) \in \mathbb{P}^3 : (\beta, \delta) \in \mathbb{P}^1\} \cup \{(\alpha, 0, 0, \delta) \in \mathbb{P}^3 : (\alpha, \delta) \in \mathbb{P}^1\}.
\]
If $\alpha_3 \neq 0$ and $\alpha_1^2 = \alpha_2 \alpha_3$, then we obtain
\[
\{(\alpha, \beta, \gamma, \delta) \in \mathbb{P}^3 : \alpha^2 = \beta \gamma\}
\]
So, in general, the solutions are:
\[
\{(\alpha, 0, 0, \delta) \in \mathbb{P}^3 : (\alpha, \delta) \in \mathbb{P}^1\} \cup \{(\alpha, \beta, \gamma, \delta) \in \mathbb{P}^3 : \alpha^2 = \beta \gamma\}
\]
, and the point scheme is $V(x_1^2 - x_2 x_3) \cup V(x_2, x_3)$.

### 3.6 Proposition 6:
If the algebra
\[
A = \frac{\mathbb{K}(x_1, x_2, x_3, x_4)}{\langle g_1, \ldots, g_6 \rangle}
\]
where
\[
\begin{align*}
g_1 &= x_1 x_2 - x_2 x_1, \\
g_2 &= x_2 x_3 - x_3 x_2, \\
g_3 &= x_1 x_3 - x_3 x_1, \\
g_4 &= x_1 x_4 - x_4 x_1, \\
g_5 &= x_2 x_4 - x_4 x_2, \\
g_6 &= x_3 x_4 - x_4 x_3 - x_1^2 + x_2 x_3,
\end{align*}
\]
, then $A$ has point scheme given by $V(x_1(x_1^2 - x_2 x_3), x_2(x_1^2 - x_2 x_3))$, which contains the double line $V(x_1, x_2)$ (see Figure 6).
Figure 6: Depiction of the Point Scheme in Proposition 6

Proof:

Suppose 

\[ p = ((\alpha_1, \alpha_2, \alpha_3, \alpha_4), (\beta_1, \beta_2, \beta_3, \beta_4)) \in \mathbb{P}^3 \times \mathbb{P}^3. \]

To find the point scheme \( \mathcal{P} \) of \( A \), we solve

\[
\begin{align*}
0 &= g_1(p) = \alpha_1 \beta_2 - \alpha_2 \beta_1, \\
0 &= g_2(p) = \alpha_2 \beta_3 - \alpha_3 \beta_2, \\
0 &= g_3(p) = \alpha_1 \beta_3 - \alpha_3 \beta_1, \\
0 &= g_4(p) = \alpha_1 \beta_4 - \alpha_4 \beta_1, \\
0 &= g_5(p) = \alpha_2 \beta_4 - \alpha_4 \beta_2, \\
0 &= g_6(p) = \alpha_3 \beta_4 - \alpha_4 \beta_3 - \alpha_1 \beta_1 + \alpha_2 \beta_3,
\end{align*}
\]

which yields \( DE = F \), where

\[
\begin{bmatrix}
-\alpha_2 & \alpha_1 & 0 & 0 \\
0 & -\alpha_3 & \alpha_2 & 0 \\
-\alpha_3 & 0 & \alpha_1 & 0 \\
-\alpha_4 & 0 & 0 & \alpha_1 \\
0 & -\alpha_4 & 0 & \alpha_2 \\
-\alpha_1 & 0 & \alpha_2 - \alpha_4 & \alpha_3
\end{bmatrix},
\begin{bmatrix}
\beta_1 \\
\beta_2 \\
\beta_3 \\
\beta_4
\end{bmatrix},
\text{ and }
\begin{bmatrix}
0 \\
0 \\
0 \\
0
\end{bmatrix}.
\]

We find all \( 4 \times 4 \) minors of \( D \) (using Wolfram Mathematica). They are:

\[
\begin{align*}
\alpha_1^2(\alpha_1^2 - \alpha_2 \alpha_3) \\
\alpha_1 \alpha_2(\alpha_1^2 - \alpha_2 \alpha_3) \\
\alpha_2^2(\alpha_1^2 - \alpha_2 \alpha_3) \\
-\alpha_1 \alpha_3(\alpha_1^2 - \alpha_2 \alpha_3) \\
-\alpha_2 \alpha_3(\alpha_1^2 - \alpha_2 \alpha_3) \\
-\alpha_2 \alpha_4(\alpha_1^2 - \alpha_2 \alpha_3) \\
-\alpha_1 \alpha_4(\alpha_1^2 - \alpha_2 \alpha_3)
\end{align*}
\]

Consider the first equation, therefore we have \( \alpha_1 = 0 \) or \( \alpha_1^2 - \alpha_2 \alpha_3 = 0 \). If \( \alpha_1 = 0 \), then by Mathematica, we have

\[-\alpha_2 \alpha_3, \quad \alpha_2 \alpha_3^2, \quad \alpha_2^2 \alpha_3 \alpha_4.\]
Therefore in this case, solutions are

\[
\{(0, 0, \gamma, \delta) \in \mathbb{P}^3 : (\gamma, \delta) \in \mathbb{P}^1\} \cup \{(0, \beta, 0, \delta) \in \mathbb{P}^3 : (\beta, \delta) \in \mathbb{P}^1\}.
\]

If \( \alpha_1 \neq 0 \) and \( \alpha_1^2 = \alpha_2 \alpha_3 \), then we obtain

\[
\{(\alpha, \beta, \gamma, \delta) \in \mathbb{P}^3 : \alpha^2 = \beta \gamma\}
\]

So, in general, the solutions are:

\[
\{(\alpha, \beta, \gamma, \delta) \in \mathbb{P}^3 : \alpha^2 = \beta \gamma\}
\]

, and the point scheme is \( \mathcal{V}(x_1(x_1^2 - x_2 x_3), x_2(x_1^2 - x_2 x_3)) \), which contains the double line \( \mathcal{V}(x_1, x_2) \).

Notice that

\[
\langle x_1^2 - x_2 x_3 \rangle \langle x_1 x_2, x_2^2, x_1^2, x_1 x_3, x_2 x_3, x_2 x_4, x_1 x_4 \rangle = \langle x_1^2 - x_2 x_3 \rangle \langle x_1, x_2 \rangle \langle x_1, x_2, x_3, x_4 \rangle
\]

Therefore, the point scheme \( \mathcal{P} \) of \( A \) is

\[
\mathcal{V}(x_1^2 - x_2 x_3) \cup \mathcal{V}(x_1, x_2) \cup \mathcal{V}(x_1, x_2, x_3, x_4)
\]

\[
= \mathcal{V}(x_1^2 - x_2 x_3) \cup \mathcal{V}(x_1, x_2)
\]

where \( \mathcal{V}(x_1, x_2) \subset \mathcal{V}(x_1^2 - x_2 x_3) \).

\[\square\]
4 Appendix
Mathematica code for Proposition 1.

\[ M = \begin{pmatrix} a_2 & -a_1 & 0 & 0 \\ 0 & a_3 & -a_1 & 0 \\ a_4 & 0 & -a_2 & a_1 \\ 0 & a_4 & 0 & -a_2 \end{pmatrix} \]

\[ M \rightarrow \text{MatrixForm} \]

\[ \text{Out[8]} \text{\texttt{\#/MatrixForm}} = \begin{pmatrix} a_2 & -a_1 & 0 & 0 \\ 0 & a_3 & -a_1 & 0 \\ a_4 & 0 & -a_2 & a_1 \\ 0 & a_4 & 0 & -a_2 \end{pmatrix} \]

\[ P = \text{Minors}[M, 4] \rightarrow \text{Flatten} \rightarrow \text{Factor} \rightarrow \text{TableForm} \]

\[ \text{Out[9]} \text{\texttt{\#/TableForm}} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ -a_2^3 (a_2 a_3 - a_1 a_4) (-1 - q) (1 + q) \\ a_2 a_3 (a_2 a_3 - a_1 a_4) (-1 - q) (1 + q) \\ -a_1 a_2 (-a_2 a_3 - a_1 a_4) (-1 - q) q (1 + q) \\ -a_1 a_3 (-a_2 a_3 - a_1 a_4) (-1 - q) q (1 + q) \\ -a_2 a_4 (a_2 a_3 - a_1 a_4) (-1 - q) (1 + q) \\ a_2 a_3 (a_2 a_3 - a_1 a_4) (-1 - q) (1 + q) \\ 0 \\ 0 \\ a_3 a_4 (a_2 a_3 - a_1 a_4) (-1 - q) (1 + q) \end{pmatrix} \]

\[ P \rightarrow (a_3 \rightarrow 0) \rightarrow \text{Expand} \rightarrow \text{Factor} \rightarrow \text{TableForm} \]

\[ \text{Out[10]} \text{\texttt{\#/TableForm}} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ a_1 a_2 a_3 (-1 - q) (1 + q) \\ 0 \\ -a_1 a_2 a_4 (-1 - q) q (1 + q) \\ 0 \\ a_1 a_2 a_4 (-1 - q) (1 + q) \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \]
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