SEPARATION IN SIMPLY LINKED
NEIGHBOURLY 4-POLYTOPES

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Abstract. The Separation Problem asks for the minimum number \( s(O, K) \) of hyperplanes required to strictly separate any interior point \( O \) of a convex body \( K \) from all faces of \( K \). The Conjecture is \( s(O, K) \leq 2^d \) in \( \mathbb{R}^d \), and we verify this for the class of simply linked neighbourly 4-polytopes.

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1 INTRODUCTION

We recall that the Separation Problem is the polar version of the Gohberg-Markus-Hadwiger Covering Problem for convex bodies, and refer to [2], [6] and [9] for an overview of the topic.

For convex \( d \)-polytopes \( P \), the Conjecture has been verified in the case that \( P \) is cyclic or a type of neighbourly 4-polytope (totally-sewn or with at most ten vertices). We refer to [3] for an overview of these results.

In the following, we assume that \( P \) is a neighbourly 4-dimensional polytope in \( \mathbb{R}^4 \). Then \( P \) is convex and any two distinct vertices determine an edge of \( P \). We refer to [8] and [12] for the basic geometric and combinatorial properties of \( P \).

With formal definitions to follow; we note only that cyclic polytopes are neighbourly and totally-sewn, and that totally-sewn \( P \) are linked. Thus, we verify the Conjecture for a new class of \( P \).

As for organization: Section 2 contains definitions and conventions. In Section 3, we examine the inner structure of \( P \). In Section 4, we determine some separation properties of \( P \). We introduce simply linked \( P \) and present our separation results in Section 5 and 6.

2 DEFINITIONS

Let \( Y \) be a set of points in \( \mathbb{R}^d \). Then \( \text{conv} \ Y \) and \( \text{aff} \ Y \) denote, respectively, the convex hull and the affine hull of \( Y \). For sets \( Y_1, Y_2, \ldots, Y_k \), let
\[
[Y_1, Y_2, \ldots, Y_k] = \text{conv}(Y_1 \cup Y_2 \cup \ldots \cup Y_k)
\]
and \( \langle Y_1, Y_2, \ldots, Y_k \rangle = \text{aff} (Y_1 \cup Y_2 \cup \ldots \cup Y_k) \). For a point \( y \), let \([y] = \{y\}\) and \( \langle y \rangle = \langle \{y\} \rangle \).

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Let $Q \in \mathbb{R}^d$ denote a (convex) d-polytope with $\mathcal{V}(Q), \mathcal{E}(Q)$ and $\mathcal{F}(Q)$ denoting, respectively, its sets of vertices, edges and facets. For $x \in \mathcal{V}(Q)$, $Q/x$ denotes the vertex figure of $Q$ at $x$. For $E = [x, y] \in \mathcal{E}(Q)$, $Q/E$ denotes the quotient polytope $(Q/y)/x$. We note that $Q/E$ is a $(d−2)$-polytope.

Let $d = 4$. As a simplification, we assume always that $Q/x$ is contained in a hyperplane $H \subset \mathbb{R}^4$ that strictly separates $x$ from each $y \in \mathcal{V}(Q)/\{x\}$, and denote $H \cap [x, y] = H \cap \{x, y\}$ also by $y$. Then of importance here are the following:

2.1. For $y_1 \in \mathcal{V}(Q)/\{x\}$, a plane $\langle y_1, y_2, y_3 \rangle$ separates $y_4$ and $y_5$ in $(Q/x)$ if, and only if, the hyperplane $\langle x, y_1, y_2, y_3 \rangle$ separates $y_4$ and $y_5$ in $\mathbb{R}^4$, and

2.2. For $y_1 \in \mathcal{V}(Q)/E$; a line $\langle z_1, z_2 \rangle$ separates $z_3$ and $z_4$ in $(Q/E)$ if, and only if, the hyperplane $\langle E, z_1, z_2 \rangle$ separates $z_3$ and $z_4$ in $\mathbb{R}^4$.

Let $S \subset \mathbb{R}^3$ be a 3-polytope with $s \geq 4$ vertices. Then $S$ is stacked if either $s = 4$ or $S$ is the convex hull of a stacked 3-polytope with $s − 1$ vertices and a point in $\mathbb{R}^3$ that is beyond exactly one facet of $S$.

Let $S$ be stacked, $\{x, y, z\} \subset \mathcal{V}(S)$ and $C = [x, y, z]$ be a triangle. We say that $C$ is a cut of $S$ if $\mathcal{E}(C) \subset \mathcal{E}(S)$ but $C \notin \mathcal{F}(S)$. All the cuts of $S$ decompose $S$ into components, each of which is a 3-simplex. We note that $|\mathcal{V}(S)| = s$ yields that $S$ has $s − 4$ cuts and $s − 3$ components.

Let $\mathcal{N}_m$ denote the family of combinatorially distinct neighbourly 4-polytopes with $m \geq 5$ vertices, $P \in \mathcal{N}_{m+1}, x \in \mathcal{V}(P)$ and $Q = [\mathcal{V}(P)\{x\}]$. We note that $Q \in \mathcal{N}_m$.

The relevance of stacked 3-polytopes here is the following result in [1]:

2.3. $P/x$ is a stacked 3-polytope with $m$ vertices; furthermore, $[y_1, y_2, y_3, y_4]$ is a component of $P/x$ if, and only if, $[y_1, y_2, y_3, y_4] \in \mathcal{F}(Q)$. Hence, $x$ is beyond exactly $m − 3$ facets of $Q$.

Next, let $E = [x, y] \in \mathcal{E}(P)$. Then $E$ is a universal edge of $P$ if $[E, z]$ is a 2-face of $P$ for each $z \in \mathcal{V}(P)/\{x, y\}$. Let $\mathcal{U}(P)$ denote the set of universal edges of $P$. We observe from [12] and [13] that

2.4. $E = [x, y] \in \mathcal{U}(P)$ if, and only if, $x$ and $y$ lies on the same side of every hyperplane determined by the vertices of $P$. From the same sources: if $|\mathcal{V}(P)| \geq 7$ then any vertex of $P$ is on at most two members of $\mathcal{U}(P)$, and $|\mathcal{U}(P)| \leq |\mathcal{V}(P)|$.

We recall that a cyclic 4-polytope $C_m$ with $m$ vertices is combinatorially equivalent to the convex hull of $m$ points on the moment curve in $\mathbb{R}^4$. From [7], [8] and [12], we note that $C_m \in \mathcal{N}_m$, $\mathcal{N}_0 = \{C_0\}$, $|\mathcal{U}(C_0)| = 9$, $\mathcal{N}_7 = \{C_7\}$, $|\mathcal{U}(C_m)| = m$ for $m \geq 7$, and any 4-subpolytope of $C_m$ is again cyclic. For $m \geq 6$, there is a natural ordering (Gale’s Evenness Condition) of $\mathcal{V}(P_m)$ that corresponds to the order of appearance of equivalent points on the moment curve.

Let $m \geq 8$. Most of our knowledge about members of $\mathcal{N}_m$ is based upon various construction techniques: given $Q \in \mathcal{N}_{m−1}$, find a point $\bar{x} \in \mathbb{R}^4/Q$ such that $\bar{Q} = [Q, \bar{x}] \in \mathcal{N}_m$. It is noteworthy that, at present, known constructions
such as Shemer Sewing, Extended Sewing and Gale Sewing (cf. [12], [10] and [11]) yield that $\mathcal{U}(Q) \setminus \mathcal{U}(Q) \neq \emptyset$. We introduce a class of polytopes to reflect this fact.

Let $n \geq 7$ and $P_n \in \mathcal{N}_n$. We say that $P_n$ is linked if for $m = n - 1, \cdots, 6$, there is a $P_m \in \mathcal{N}_m$ with the property that

$$P_{m+1} \supset P_m \text{ and } \mathcal{U}(P_{m+1}) \setminus \mathcal{U}(P_m) \neq \emptyset$$

We say that $P_n$ is linked under the (vertex) array $x_n > x_{n-1} > \cdots > x_1$ if for $m = n - 1, \cdots, 6$,

$$P_m = [x_m, x_{m-1}, \cdots, x_1] \text{ and } \mathcal{U}(P_{m+1}) \setminus \mathcal{U}(P_m) \neq \emptyset$$

For $x_t \in \{x_7, \cdots, x_n\}$ and $x_r \in \{x_1, \cdots, x_{t-1}\}$, we say that $x_t$ is linked to $x_r$ ($x_t \rightarrow x_r$) if $[x_t, x_r] \in \mathcal{U}(P_t)$ and $[x_t, x_r] \notin \mathcal{U}(P_t)$ for $j > \max\{6, r\}$.

By way of clarification for requiring that $t \geq 7$; we note that

2.5. $P_6$ is cyclic and there are disjoint three element subsets $Y$ and $Z$ of $\mathcal{V}(P_6)$ such that $\mathcal{U}(P_6) = \{[y, z] | y \in Y \text{ and } z \in Z\}$. Thus, there is no meaningful labeling of a greatest or a least vertex of $P_6$.

3 THE INNER STRUCTURE OF $P$

Let $v \in \mathcal{V}(P)$, $Q \supset P$, $v \notin Q$, $Q \in \mathcal{N}_m$ and $R = [Q, v]$. We recall that $R^* = R/v$ is a stacked 3-polytope and that $y \in \mathcal{V}(Q)$ denotes also $\{y^*\} = \langle v, y \rangle \cap \langle R^* \rangle$. We describe $R^*$.

Let

$$Y_a = \{y_1, y_2, \cdots, y_a\}, \ z \in \mathcal{V}(Q) \setminus Y_a,$$

$$Z_t = \{z \mid (v, y_1, y_t, y_{t+1}) \text{ strictly separates } y_2 \text{ and } z\}$$

and

$$Z'_t = \{z \mid (v, y_2, y_t, y_{t+1}) \text{ strictly separates } y_1 \text{ and } z\}$$

From 2.1, we have that

$\bullet$ $Z_t \neq \emptyset / Z'_t \neq \emptyset$ if and only if, $[y_1, y_t, y_{t+1}]$ (or $[y_2, y_t, y_{t+1}]$) is a cut of $R^*$.

Hence, we have a generic description of $R^*$; cf. the Schlegel diagram in Figure 1.

Next, we observe from 2.3 and 2.1 that

$\bullet$ $\langle y_1, y_2, y_t, y_{t+1} \rangle$ separates $v$ and $Q$ for $t = 3, \cdots, a - 1$, and

$\bullet$ $\langle v, y_1, y_2, y_t \rangle$ separates $Z_r \cup Z'_r$ for $r < t$ and $Z_s \cap Z'_t$ for $t = 4, \cdots, a - 1$.

From 2.2, we depict these separation properties with respect to $Q/[y_1, y_2]$ and $R/[y_1, y_2]$ in Figure 2.

REMARKS Let $R^* = R/v$ be labeled as above.
Fig. 1.

Fig. 2.
3.1. If \( a = m \) then \( \mathcal{F}(Q) \setminus \mathcal{F}(R) = \{ [y_t, y_{t+1}] | t = 3, \ldots, a-1 \} \), \( [y_{1}, y_2] \in \mathcal{U}(Q) \) and \( \{ [v, y_1], [v, y_2] \} \subset \mathcal{U}(R) \).

There is such a labeling of \( R^* \) if \( R \) is cyclic or if \( R \) is constructed by a Shenar sewing of \( v \) onto \( Q \).

3.2. If \( w \in \mathcal{V}(P) \setminus \mathcal{V}(Q) \) and \( [w, v] \in \mathcal{U}([Q, v, w]) \) then \( \mathcal{F}(Q) \setminus \mathcal{F}([Q, w]) = \mathcal{F}(Q) \setminus \mathcal{F}(R) \) by 2.4.

3.3. Let \( 3 < t < a \). If \( \langle v, y_1, y_2, y_t \rangle \) strictly separates vertices \( p_i \) and \( s_t \) of \( Q \) then \( [p_i, s_t] \) is not an edge of \( R^* \), and \( [v, p_i, s_t] \) is not a face of \( R \).

We note from Figure 2 that under the hypotheses of 3.3, each hyperplane through \( \langle y_1, y_2, y_t \rangle \) strictly separates some two of \( v, p_i \) and \( s_t \). Thus, the following is the more general result; cf. [5].

3.4. If \( \{ x_a, x_b, x_c, x_e, x_f, x_g \} \) is a set of six vertices of \( P \) and each hyperplane of \( \mathbb{R}^4 \) though \( \{ x_a, x_b, x_c \} \) strictly separates two of \( x_e, x_f \) and \( x_g \), then \( \{ x_a, x_b, x_c \} \) and \( \{ x_e, x_f, x_g \} \) are not faces of \( P \).

4 GENUINE SEPARATION PROPERTIES OF \( P \)

Let \( P \in \mathcal{N}_m \), \( m \geq 6 \), and \( O \) be an interior point of \( P \). We determine hyperplanes \( H \in \mathbb{R}^4 \) that strictly separate \( O \) from facets of \( P \). As a simplification, we determine \( H \) that do not contain \( O \). We consider first \( F \in \mathcal{F}(P) \) that either are contained in a subpolytope \( Q \) such that \( O \notin \text{int} \ Q \) or have a common vertex \( w \).

**Lemma A.** (cf. [4]) Let \( O \in \text{bd}(Q) \). Then \( O \) is strictly separated from any \( F \in \mathcal{F}(P) \cap \mathcal{F}(Q) \) by one of at most three hyperplanes.

**Lemma B.** Let \( w \in \mathcal{V}(P), R \in \mathcal{N}_{m-1}, P = [R, w] \) and \( F \in \mathcal{F}(P) \) such that \( w \in F \). Then \( O \) is strictly separated from any such \( F \) by one of at most four (six) hyperplanes in case \( O \) is (not) an interior point of \( R \).

**Proof.** Since \( P^* = P \setminus w \) is stacked and \( O \in \text{int} P \), it follows that \( O^* \in \langle w, O \rangle \cap P^* \) is in a component \( A^* = [x^*, y^*, z^*, v^*] \) of \( P^* \). If \( O^* \in \text{relint} A^* \), then \( O \) is separated from \( F \) by one of \( \langle w, x, y, z \rangle, \langle w, v, x, y, v \rangle \) and \( \langle w, x, y, v \rangle \) and \( \langle w, y, z, v \rangle \).

Let \( O^* \in B^* = [x^*, y^*, z^*] \), say. Then \( B^* \) is a cut of \( P^* \), \( O \in \langle w, x, y, z \rangle \) and there are subpolytopes \( P' \) and \( P'' \) of \( P \) such that \( P' \cap P'' = [w, x, y, z], [P', P''] = P \) and (since \( w \in F \)) either \( F \subset P' \) or \( F \subset P'' \).

We recall from 2.3 that \( [x, y, z, v] \in \mathcal{F}(R) \setminus \mathcal{F}(P) \). If \( O \in \text{int} R \) then it is clear that \( O \notin [w, x, y, z] \); that is, \( O \notin P' \cap P'' \) and \( O \) is separated from \( F \) by one of two hyperplanes. If \( O \in [w, x, y, z] \) then \( O \in \text{bd}(P') \cap \text{bd}(P'') \) and we apply **LEMMA A**.

**REMARKS** Let \( Q \) be a subpolytope of \( P \) such that \( O \notin \text{int} Q \).

4.1. If \( Q \in \mathcal{N}_{m-1} \) then \( O \) is strictly separated from any \( F \in \mathcal{F}(P) \) by one of at most nine (three from **LEMMA A**, six from **LEMMA B**) hyperplanes.
4.2. If \( Q \in \mathcal{N}_{m-3} \) then \( O \) is strictly separated from any \( F \in \mathcal{F}(P) \) by one of at most fifteen hyperplanes.

For 4.2, we apply \textit{Lemma B} under the assumption that \( O \) is an interior point of any \( Q' \in \mathcal{N}_{m-1} \) such that \( Q' \subset P \)

5 SIMPLY LINKED P

Let \( n \geq 7 \) and \( P = P_n \in \mathcal{N}_n \) be linked under the array \( x_n > x_{n-1} > \cdots > x_1 \).

Let \( W = \{w_s, w_{s-1}, \cdots, w_1\} \) be an \( s \) element subset of \( \mathcal{V}(P) \) with the induced array \( w_s > w_{s-1} > \cdots > w_1 \) in the case \( s > 1 \). Then \( \mathcal{V} \) is a \textit{chain} if either \( s = 1 \) or

\[
w_s \to w_{s-1} \to \cdots \to w_1.
\]

For \( x_k \in \mathcal{V}(P) \), let \( \mathcal{V}^k \) denote the union of all chains of \( P \) with \( x_k \) as the least vertex.

Finally, we say that \( P_n \) is \textit{simply linked} if for \( k = 7, \cdots, n \):

- \( \mathcal{V}^k \) is a chain, and
- for disjoint chains \( \mathcal{V}^c \) and \( \mathcal{V}^d \), there are \( x_i \neq x_j \) in \( \mathcal{V}(P_b) \) such that \( x_c \to x_i, x_d \to x_j \) and \( [x_i, x_j] \notin \mathcal{U}(P_b) \).

Henceforth, we assume that \( P_n \) is simply linked. Then it follows from 2.5 that \( \{x_7, \cdots, x_n\} \) is the union of at most three pairwise disjoint maximal chains.

\textbf{Lemma C.} Let \( 6 \leq m < n, x_m < x_t, x_k < x_t \) and \( x_t \notin \mathcal{V}^k \).

\textbf{C.1} \( H \cap [\mathcal{V}] = \emptyset \) for any hyperplane \( H \) spanned from \( \{x_1, \cdots, x_m\} \).

\textbf{C.2} Let \( H_h = \langle x_a, x_b, x_c, x_h \rangle \) be a hyperplane with \( \{x_a, x_b, x_c\} \subset \{x_1, \cdots, x_m\} \) and \( H_h \cap \mathcal{V}^k = \{x_h\} \). Then \( H_h \cap [\mathcal{V}^k] = \emptyset \).

\textbf{C.3} Let \( x_t \to \{x_j, x_j \notin \{x_a, x_b, x_c\} \) and \( H_h \) be defined as above. Then \( H_h \cap [\mathcal{V}^k] = \emptyset \).

\textit{Proof.} Since \( P \) is simplicial, it follows from \( H \cap [\mathcal{V}] \neq \emptyset \) that \( H \) strictly separates some \( x_v \) and \( x_u \) in \( \mathcal{V}^t \) such that \( x_v \to x_u \). Then \( [x_v, x_u] \in \mathcal{U}(P_v) \) and \( P_m \subset P_t \subset P_v \) yield a contradiction by 2.4.

As above, \( H_h \cap [\mathcal{V}^k] \neq \emptyset \) implies that \( H_h \) strictly separates some \( x_s \) and \( x_q \) in \( \mathcal{V}^t \) such that \( x_s \to x_q \). Thus, C.1 yields that \( x_s < x_h \) and \( x_t \in P_s \subset P_h \). From \( x_h \in \mathcal{V}^k \) and \( x_k < x_t < x_h \), there is an \( x_g \in \mathcal{V}^k \) such that \( x_g \to x_q \). Then \( [x_g, x_h] \in \mathcal{U}(P_h), x_m < x_t < x_h, \mathcal{V}^k \cap \{x_a, x_b, x_c\} = \emptyset \) and 2.4 yield that in the pencil of hyperplanes containing \( \langle x_a, x_b, x_c \rangle \):

\[
\langle x_a, x_b, x_c, x_s \rangle \cap [x_g, x_h] = \emptyset = \langle x_a, x_b, x_c, x_q \rangle \cap [x_g, x_h].
\]

Hence, \( \langle x_a, x_b, x_c, x_q \rangle \) also strictly separates \( x_r \) and \( x_q \), and \( x_t \in P_s \subset P_q \subset P_h \). It now follows from \( x_h \to x_g \to \cdots \to x_k \) that \( x_t \in P_s \subset P_k \subset P_h \); a contradiction.
We note that \( V^j = \{x_j\} \cup V^i \) and that if \( H_h \cap |V^j| \neq \emptyset \) then \( H_h \) strictly separates \( x_i \) and \( x_j \) by C.2, and \( x_t < x_h \) by C.1. We now argue on above and obtain a contradiction.

### REMARKS

We recall that \( P_m = [x_m, x_{m-1}, \ldots, x_1] \) for \( m = n, \ldots, 6 \). Let \( P_6 \) denote any 4-subpolytope of \( P_6 \). In view of 2.5,

5.1. there is a labeling of \( V(P_6) \), which we may denote by \( x_1, x_2, \ldots, x_6 \), such that

- \( P_6 \) satisfies Gale’s Evenness Condition with \( x_1 < x_2 < \cdots < x_6, Y = \{x_1, x_3, x_5\}, Z = \{x_2, x_4, x_6\} \)
- \( P_5 = [x_1, x_2, \ldots, x_5] \), and
- any hyperplane through \( \langle Y \rangle \) strictly separates two elements of \( Z \).

We recall that \( P = P_n \) is simply linked under \( x_n > x_{n-1} > \cdots > x_1 \) and \( P_m = [x_m, \ldots, x_1] \) for \( m \geq 5 \). Let \( O \) be an interior point of \( P, 6 \leq m \leq n - 1 \) and \( O \in P_m \setminus P_{m-1} \). We note that a vertex of \( [x_{m+1}, \ldots, x_n] \) is linked to a vertex of \( P_m \).

With \( v = x_w > x_m, Q = P_m \) and \( R = [Q, v] \), we label \( Q \) and \( R^* = R/v \) as in Section 3 so that \( x_w \to y_1 \) (hence, each \( Z_i \) is empty) and \( \langle x_w, O \rangle \cap \{y_1, y_2, y_t, y_{t+1}\} \neq \emptyset \) for some \( 3 \leq t \leq a - 1 \). We let \( T = [y_1, y_2, y_t, y_{t+1}] \).

\[
Z_i^- = \{y_3, \ldots, y_{t-1}\} \cup Z_3 \cup \cdots \cup Z_i - 1, \quad Z_i^+ = \{y_{t+2}, \ldots, y_a\} \cup Z_{t+1} \cup \cdots \cup Z_{a-1}
\]

and note that

\( T \in \mathcal{F}(Q) \setminus \mathcal{F}(R) \) and \( \mathcal{V}(P_m) = Y_a \cup Z_3 \cup \cdots \cup Z_{a-1} = \mathcal{V}(T) \cup Z_i^- \cup Z_i \cup Z_i^+ \).

From the Schlegel diagram of \( R^* \) on \( [y_1, y_2, y_a] \) in Figure 1, we readily obtain diagrams of \( R^* \) on 2-faces containing \( [y_1, y_t] \) or \( [y_2, y_t] \). In Figure 3, 4 and 5, we depict associated polygons \( R/[y_1, y_2], R/[y_1, y_t] \) and \( R/[y_2, y_t] \) that include \( \mathcal{V}^w \) (as per 3.2 and C.1) and hyperplanes \( H_3, H_5 \) that separate \( O \) and \( \mathcal{V}^w \). We note that each of \( H_2, H_3, H_4 \) and \( H_5 \) intersects and supports \( \mathcal{V}^w \). For \( i = 1, \ldots, 5 \), let \( H_i^- \) and \( H_i^+ \) denote the open half-spaces of \( \mathbb{R}^3 \) determined by \( H_i \) with \( \mathcal{V}^w \subset H_i \cup H_i^+ \).

### REMARKS

Let \( F \in \mathcal{F}(P) \) and assume by 4.2 that \( m \leq n - 3 \). From \( \langle x_w, O \rangle \cap T \neq \emptyset \), we have the following:

5.2. \( O \) is separated from all \( F \) with a common vertex by one of at most four hyperplanes; cf. LEMMA B.

5.3. \( O \in [x_m, P_{m-1}] \setminus P_{m-1} \) is separated from any \( F \in \mathcal{F}(P_m) \) by one of at most five hyperplanes.

5.4. If \( F \cap \mathcal{V}^w \neq \emptyset \) then \( F \) intersects at most one of \( Z_i^- \), \( Z_i \) and \( Z_i^+ \); cf. 3.4.

5.5. If \( O \notin T \) then \( O \) is separated from any \( F \) such that \( F \cap \mathcal{V}^w \neq \emptyset \) and \( \mathcal{V}(F) \subset \mathcal{V}(P_m) \cup \mathcal{V}^w \) by one of \( H_2, H_3, H_4, \) and \( H_5 \) in the case \( F \cap (Z_i^- \cup Z_i \cup Z_i^+) = \emptyset \).
5.6. Let \( x_m \notin T \). Then \( O \) is separated from any \( F \) such that \( V(F) \subset V(P_{m-1}) \cup V^w \) by one hyperplane.

Regarding 5.6: let \( \langle x_w, O \rangle \cap T \neq \emptyset \) with \( T = [y_1, y_2, y_t, y_t+1] \notin F(P_m, x_w) \), \( \langle x, O \rangle \cap K \neq \emptyset \) with \( K = [u_1, u_2, u_k, u_k+1] \notin F(P_m, x_s) \).

We simplify the notation and let \( y_j = \hat{y}_j, U_j = \hat{Z}_j, r_j = \bar{y}_j \) and \( V_j = \bar{Z}_j \). With reference to Figure 3, 4 and 5, we assume that \( \{T, L, I\} \subset F(P_m) \) and that
\[\langle x_r, O \rangle \cap I \neq \emptyset \text{ with } I = [v_1, v_2, v_i, v_i+1] \notin F([P_m, x_r]) .\]

- \(O\) is separated from \([V^w]/([V^w])\) by \(H_1, \ldots, H_5(H_1, \ldots, H_5)\) and \(V^w \subset \tilde{H}_j \cup \tilde{H}_j^+([V^w] \subset \tilde{H}_j \cup \tilde{H}_j^+)\) for \(j = 1, \ldots, 5\).

**REMARK** We refer to Figure 4, and consider any hyperplane \(H'\) through \(\langle y_1, y_t, y_{t+1} \rangle\) in the case that \(x_s \rightarrow u_1 \rightarrow y_2\). Then \([x_s, y_2] \in U([P_m, x_r])\), and it follows from LEMMA C that if \(H' \cap Z_t \neq \emptyset\), then \(H' \cap [y_2, x_s] = \emptyset, H' \cap [V^w] = \emptyset\) and \(H'\) strictly separates \([V^w]\) and \([V^w]\). The following now follows from 3.4:

5.7. If \(u_1 = y_2\) and \(F \cap V^w \neq \emptyset \neq F \cap V^w\) then \(F \cap Z_t = \emptyset\).

**Lemma D.** Let \(F \in F(P)\). Then \(F\) intersects at most two of \(V^2, V^4\) and \(V^6\), and at most two of \(V^w, V^v\) and \(V^w\).

**Proof.** The existence of \(V^w, V^v\) and \(V^v\) imply that \(\{x_7, \ldots, x_n\}\) is the union of pairwise disjoint chains \(V^e, V^f\) and \(V^g\), say. Since \(P_6\) is cyclic with \(x_1 < x_2 < \cdots < x_6\), we may assume by 2.5 and 5.1 that \(x_e \rightarrow x_2, x_f \rightarrow x_4\) and \(x_g \rightarrow x_6\).

From 5.1 and LEMMA C, we obtain that any hyperplane \(H\) through \(\langle x_1, x_3, x_5 \rangle\) strictly separates two of \(V^2, V^4\) and \(V^6\). Hence, no face of \(P\) intersects each of \(V^2, V^4\) and \(V^6\) by 3.4.

**REMARK** We refer to Figure 3, 4 and 5, and consider a \(v \in V(P)\) with the property that \(v \in H_1^-, v \notin H_2^+ \cup H_3^+ \cup H_4^+\) and no hyperplane spanned from \(v, y_1, y_2, y_t, y_{t+1}\) intersects \([V^w]\).

Then \(\langle x_w, y_1, y_2, y_t\rangle\) separates \(v\) and \(Z_t^-, \langle x_w, y_1, y_2, y_{t+1}\rangle\) separates \(v\) and \(Z_t^+, \langle x_w, y_1, y_2, y_{t+1}\rangle\) separates \(v\) and \(Z_t^+\), and \(\langle x_w, y_1, y_2, y_{t+1}\rangle\) separates \(v\) and \(Z_t^+\). From \([P_m, x_w]/[x_w, y_1]\), it now follows that \([x_w, y_1, v]\) is not a 2-face of \([P_m, x_w, v]\) or \([x_w, y_1, y_2, y_t, y_{t+1}, v]\). Since the latter polytope is cyclic, we obtain from 5.1 that
5.8. \((v, y_2, y_1, y_{t+1})\) strictly separates \(x_w\) and \(y_1\).

**Lemma E.** Let \(x_w < x_s, F \in \mathcal{F}(P)\) and \(F \cap \mathcal{V}^w \neq \emptyset \neq F \cap \mathcal{V}^s\). Then \(O\) is separated from any such \(F\) by

- **E.1** at most three hyperplanes \((\hat{H}_2, \hat{H}_3, \hat{H}_4)\) in the case \(x_w \in \hat{H}_1^-\) and \(O \notin \bd(K)\),
- **E.2** one hyperplane \((H_i, 2 \leq i \leq 4)\) in the case \(u_1 \notin T\) and \(O \notin \bd(T)\),
- **E.3** one hyperplane \((H_i, 2 \leq i \leq 5)\) in the case \(x_w \in \hat{H}_1^+, u_1 \in T, x_s \in H_i^-\) and \(O \notin \bd(T)\), and
- **E.4** at most two hyperplanes from \(H_2, H_3, H_2, \hat{H}_3\) in the case \(x_w \in \hat{H}_1^+, x_s \in H_i^+, O \notin \bd(K) \cup \bd(T)\) and either \(T \neq K\), or \(T = K\) and \(F \cap H_i^- \neq \emptyset \neq F \cap H_i^+\).

**Proof.** We refer to Figure 3, 4 and 5, and the analogous figures with \(\hat{y}_{j}\) and \(\hat{H}_{j}\) for the location of \(O\), and note \(\mathcal{V}(F) \subset \mathcal{V}(P_m) \cup \mathcal{V}^w \cup \mathcal{V}^f\) by LEMMA D.

**E.1** Let \(x_w \in \hat{H}_1^-\). Then \(\mathcal{V}^w \subset \hat{H}_1^-\) by C.1, and either \((\mathcal{V}(F) \cap \hat{H}_1^-) \cap \hat{H}_1^+ \neq \emptyset\) for some \(j \in \{2, 3, 4\}\) or there is a \(v \in \mathcal{V}^w\) such that \(v \notin \hat{H}_2^+ \cup \hat{H}_3^+ \cup \hat{H}_4^+\). In case of the former, \(O\) is separated from \(F\) by \(\hat{H}_j\); cf. 5.4. In case of the latter, it follows from \(x_w < x_s\) and C.3 that \((v, u_2, u_k, u_{k+1} \cap [\mathcal{V}^s \cup \{u_1\}] = \emptyset\); a contradiction of 5.8.

**E.2** Let \(u_1 \notin T\). Then \(x_s - u_1 \in P_m\) yields that \(u_1 \in Z_i^- \cup Z_t \cup Z_{i+1}^+ \subset H_i^+ \cup H_t^+ \cup H_{i+1}^+\) and \(\mathcal{V}^s \subset H_1^-\). Now \(x_w < x_s\) and C.1 yield that if \(u_1 \subset H_j^+\) then \(\mathcal{V}^s \subset H_j^+\) and \(O\) is separated from \(F\) by \(H_j\).

**E.3** Let \(x_w \in \hat{H}_1^+\) and \(u_1 \in T\). Then \(u_1 \in \{y_2, y_{y_1+1}\}, y_1 \in \{u_2, u_k, u_{k+1}\}\) and may assume that \(u_1 = y_2\) and \(y_1 = u_2\). From 5.7, we obtain that \(F \cap Z_i = \emptyset = F \cap U_k\).

Let \(x_s \in H_1^-\). Then \((T) = H_1 \neq \hat{H}_1 = \langle K \rangle\) and \(T \neq K\); cf. Figure 6 with \(x_s \in H_3^+,\) say, and \(\mathcal{V}^s \subset H_1^- \cap H_3^+.\) We consider the hyperplanes through
\[\langle y_1, y_2, y_{t+1} \rangle\] and obtain from 3.4 that \(F \cap Z_t^- = \emptyset\). Hence, \(O\) is separated from \(F\) by \(H_3\) in this case.

**E.4** Let \(x_w \in H_1^+\), \(x_s \in H_1^+\) and \(O \notin bd(K) \cup bd(T)\). Then \(u_1 \in T\), we assume that \((y_1, y_2) = (u_2, u_1)\) and note that \(F \cap (Z_t \cup U_k) = \emptyset\).

If \(T \neq K\) then we may assume also that \(V^\circ \subset H_1^+ \cup H_3^+\) as in Figure 6. If there is an \(F' \in F(P)\) such that \(O\) is not separated from \(F'\) by \(H_3\) then \(F' \cap Z_t^- = \emptyset\) by 5.4. From \(F' \cap Z_t = \emptyset\), it now follows that \(F' \cap (Z_t^- \cup \{y_t\}) \neq \emptyset\).

Let \(Z_t^- (s) = \{z \in Z_t^- \mid \langle y_1, y_2, y_{t+1}, z \rangle\ does\ not\ separate\ x_w\ and\ x_s\} \). We apply C.1 and 3.4, and obtain that \(F' \cap Z_t^- \subset Z_t^- (s)\). Thus, either \(O\) is separated from \(F'\) by \(H_1\) (and so \(H_2\)), or there is an \(F'\) such that \(F' \cap Z_t^- \neq \emptyset\). In the latter case, it is easy to check (cf. Figure 6) that \(O\) is separated from any such \(F'\) by \(H_2\) or \(\hat{H}_2\).

Finally, let \(T = K\) with \((y_1, y_2, y_{t+1}) = (u_2, u_1, u_k, u_{k+1})\) and, say, \(V^\circ \subset H_3^+\). We argue now as above that if \(O\) is not separated from \(F' \in F(P)\) by \(H_3\) or \(H_1 = \hat{H}_1\) then \(O\) is separated from \(F'\) by \(H_2\) or \(\hat{H}_2\).

**REMARK** We observe that under the hypotheses of **LEMMA E**, it follows from **LEMMA A** that

**5.9.** If \(O \in bd(k) \cup bd(T)\) and \(H\) is a separating hyperplane through \(O\) then we may replace \(H\) by three strictly separating hyperplanes.

![Fig. 6.](image-url)

### 6 SEPARATION RESULTS

With \(O \in int P\), let \(s(O)\) denote the minimum number of hyperplanes required to strictly separate \(O\) from any facet of \(P\). We prove that \(s(O) \leq 16\) under the
assumption that $P = P_n$ is simply linked under the array $x_n > x_{n-1} > \cdots > x_1, P_m = [x_m, \cdots, x_2, x_1]$ for $m = n - 1, \cdots, 5$ and $\mathcal{V}(P) = \{x_1, x_3, x_5\} \cup \mathcal{V}^2 \cup \mathcal{V}^4 \cup \mathcal{V}^6$.

6.1. We consider first the case of $O \in P_m \setminus P_{m-1}$ for some $6 \leq m \leq n$. As noted in Sections 4 and 5, we may assume that $m \leq n - 3$ and that $\{x_{m+1}, \cdots, x_n\}$ is the union of non-empty chains $\mathcal{V}^r, \mathcal{V}^s$ and $\mathcal{V}^w$ described in Section 5.

Our arguments are based upon

- the location of $x_m$ with respect to $T$ and $K$,
- the order of $x_r$ with respect to $x_w < x_s$, and
- the location of $O$ with respect to $T, K$ and $I$.

For each location of $O$, we present the separation result

- $\{k\}$: property: rationale

to indicate that at most $k$ separating hyperplanes suffice for $F \in \mathcal{F}(P)$ with the indicated property due to the specified reasons. $\{-\}$ indicates that the separating hyperplanes for this case have already counted.

I. $x_m \notin T \cup K$

Then $T \cup K \subset P_{m-1}$ and $O \notin T \cup K$. From $x_m \in H_1^T \cap \hat{H}_1^T, x_r \to x_m$ and 2.4, we have that $x_r \in H_1^T \cap \hat{H}_1^T$. Next, Lemma D and its proof yield that any $F$ intersects at most two of $\mathcal{V}^w, \mathcal{V}^r$ and $\mathcal{V}^s$, and that $[v_1, u_1, y_1]$ is not a 2-face of $P_m$. Hence, $x_m \in I$ implies that $\{u_1, y_1\} \notin I$.

I.1 $O \notin bd(I)$

We apply our Lemmas and Remarks. Then

- $\{4\}: x_m \in F : 5.2$
- $\{1\}: \mathcal{V}(F) \subset \mathcal{V}(P_{m-1}) \cup \mathcal{V}^w : 5.6$
- $\{1\}: \mathcal{V}(F) \subset \mathcal{V}(P_{m-1}) \cup \mathcal{V}^s : 5.6$
- $\{3\}: F \cap \mathcal{V}^w \neq \emptyset \neq F \cap \mathcal{V}^s : E.1$, and
- $\{4\}: \mathcal{V}(F) \subset \mathcal{V}(P_m) \cup \mathcal{V}^r, F \cap \mathcal{V}^r \neq \emptyset : 5.5$ with $\bar{H}_2, \bar{H}_3, \bar{H}_4, \bar{H}_5$ (as $v_1 = x_m \notin F$).

It remains to consider $F$ that intersect $\mathcal{V}^r$ and $\mathcal{V}^w \cup \mathcal{V}^s$. Here, we apply $\{u_1, y_1\} \notin I$ and Lemma E with relabeling as necessary.

I.1.1 $x_r < x_w < x_s$

As $x_w$ and $x_s$ are interchangeable with respect to $x_r$, we assume that $u_1 \notin I$, say. Then

- $\{-\}: F \cap \mathcal{V}^r \neq \emptyset \neq F \cap \mathcal{V}^s : x_r < x_s, u_1 \notin I$ and E.2 with $\bar{H}_2, \bar{H}_3$ and $\bar{H}_4$ already counted, and.
II.1 \( \{3\} : F \cap V^r \neq \emptyset \neq F \cap V^w : x_r < x_w, x_r \in H_1^r \) and E.1.

I.2 \( x_w < x_r < x_s \)

If \( u_1 \notin I \) then one case is above, and

\[ \{1\} : F \cap V^r \neq \emptyset \neq F \cap V^w : x_r < v_1 = x_m \notin T \) and E.2

If \( u_1 \in I \) and \( y_1 \notin I \) then

\[ \{-\} : F \cap V^r \neq \emptyset \neq F \cap V^w : x_r < x_w \in H_1^r \) and E.1, and

\[ \{3\} : F \cap V^r \neq \emptyset \neq F \cap V^s : x_r < x_s, x_r \in H_1^r \) and E.1.

I.3 \( x_w < x_s < x_r \)

Then \( v_1 = x_m \notin T \cup K \) and E.2 yield \( \{1\} \) for \( F \cap V^w \neq \emptyset \neq F \cap V^r \), and

\[ \{1\} \) for \( F \cap V^s \neq \emptyset \neq F \cap V^r \).

I.2 \( O \in bd(I) \)

We recall that \( \langle x_w, O \rangle \cap T \neq \emptyset \) and \( O \notin T \). Hence, \( H_1 = \langle T \rangle \) strictly separates \( O \) and \( x_w \), and \( x_w \) is necessarily beneath any facet of \( P_m \) that contains \( O \). Thus \( x_w \in H_1^r \) and, similarly, \( x_s \in H_1^r \); whence \( V^w \cup V^s \subset H_1^r \). Since \( v_1 = x_m \) implies that \( H_1 \cap H_5 = [v_2, v_1, v_i+1] \subset bd(P_m-1) \), it follows from \( O \in H_1 \setminus P_m-1 \) that \( O \notin [v_2, v_1, v_i+1] \). From these observations, we have that

\[ \{3\} : F \cap V^r = \emptyset : (L E M M A A) A. \]

\[ \{8\} : V(F) \subset V(P_m) \cup V^r, F \cap V^r \neq \emptyset : 5.5, A \) and 5.9 with \( H_1, H_2, H_3, H_4 \)

as separating hyperplanes and \( O \in H_1 \cap H_j \) for some \( j \in \{2, 3, 4\} \). We apply LEMMA A and replace \( H_1 \) and \( H_j \) as per 5.9. We indicate these eight hyperplanes by \( 2H_i + 3 + 3 \).

We now argue as in I.1.1, I.1.2 and I.3 with 5.9 applied for \( H_1 \) and \( H_j \), and obtain the same counts. Thus, \( s(O) \leq 16 \) in each of these cases.

II. \( x_m \in K \) and \( x_m \notin T \).

Then \( T \subset P_m-1, O \notin T, x_r \in H_1^r \) and we let \( x_m = u_2 \). We have again that \( \{y_1, u_1\} \notin I \); and from \( \{v_1, u \} \subset K \), it follows that \( y_1 \notin K \) and \( x_w \in H_1^r \). We note that \( x_m = u_2 \) yields that \( H_4 \) separates \( O \) from any \( F \) with \( x_m \notin F \) and \( V(F) \subset V^s \cup \{u_1, u_2, u_X, u_k+1\} \).

II.1 \( O \notin bd(I) \cup bd(K) \)

Similarly to I.1, we obtain

\[ \{4\} : x_m \in F : 5.2, \]

\[ \{1\} : V(F) \subset V(P_m-1) \cup V^s : 5.6, \]

\[ \{3\} : V(F) \subset V(P_m) \cup V^s, F \cap V^s \neq \emptyset : 5.5 \) with \( H_2, H_3, H_4 \).

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• \{4\} : \mathcal{V}(F) \subset \mathcal{V}(P_m) \cup \mathcal{V}, F \cap \mathcal{V}^r \neq \emptyset : 5.5 with \bar{H}_2, \bar{H}_3, \bar{H}_4, \bar{H}_5 and
• \{-\} : F \cap \mathcal{V}^w \neq \emptyset \neq F \cap \mathcal{V}^s : x_w < x_s, x_w \in \bar{H}_r^-, and E.1.

II.1.1 \ x_r < x_w < x_s

If \( u_1 \notin I \) then we recall that \( x_r \in \bar{H}_1^- \) and argue as in I.1. If \( y_1 \notin I \) then

• \{\} : F \cap \mathcal{V}^r \neq \emptyset \neq F \cap \mathcal{V}^w : x_r < x_w and E.2.

For \( F \cap \mathcal{V}^r \neq \emptyset \neq F \cap \mathcal{V}^s \); we obtain from \( x_r < x_s \) and \( u_1 \in I \) that one of E.1, E.3 or E.4 is applicable. We note that E.1 and E.3 yield \{-\}, and E.4 yields either \{-\} or \{3\} with \( O \in I = K \) and 5.9 applied to \( \bar{H}_1 = \bar{H}_1 \).

Henceforth, as a simplification, we list only “worst case scenario” results. In that regard, it is noteworthy that the assertion of E.4 is the same if \( x_s \) and \( x_w \) are interchanged.

II.1.2 \ x_w < x_r < x_s \ or \ x_w < x_s < x_r

Then as worst case scenarios, we have

• \{1\} : F \cap \mathcal{V}^w \neq \emptyset \neq F \cap \mathcal{V}^r : x_m = v_1 \notin T \text{ and E.2, and}
• \{3\} : F \cap \mathcal{V}^s \neq \emptyset \neq F \cap \mathcal{V}^r : E.4, 5.9 with \( O \in \bar{H}_1 = \bar{H}_1 \).

II.2 \( O \in bd(I) \)

We note that as in I.2; \( x_w \in \bar{H}_1^- \) and

• \{8\} : \mathcal{V}(F) \subset \mathcal{V}(P_m) \cap \mathcal{V}, F \cap \mathcal{V}^r \neq \emptyset : 2 \bar{H}_1 + 3 + 3 with \( O \in \bar{H}_1 \cap \bar{H}_j \) for some \( j \in \{2, 3, 4\} \).

If \( x_s \in \bar{H}_1^- \) then \( \mathcal{V}^w \cup \mathcal{V}^s \subset \bar{H}_1^- \).

• \{3\} : F \cap \mathcal{V}^r = \emptyset : \text{LEMMA A}

and, as worst case scenario, E.2 and \( \{y_1, u_1\} \notin I \) yield \( x_r < x_w < x_s \) and \( u_1 \in I \).

Then

• \{-\} : F \cap \mathcal{V}^r \neq \emptyset \neq F \cap \mathcal{V}^w : y_1 \notin I \text{ and E.2, and}
• \{5\} : F \cap \mathcal{V}^r \neq \emptyset \neq F \cap \mathcal{V}^s : x_r \in \bar{H}_r^+, E.1, 5.9 with \( O \in \bar{H}_i \) for some \( i \in \{2, 3, 4\} \)

Let \( x_s \in \bar{H}_r^+ \). Then \( u_1 \in I \) with \( u_1 = v_2 \), say, and \( \langle x_s, O \rangle \cap bd(P_m) \subset I \).

Hence, we choose \( K = I \) with \( u_2 = v_1, u_k = v_i \) and \( u_{k+1} = v_{i+1} \). Then \( \bar{H}_1 = \bar{H}_1 \) and

• \{5\} : \mathcal{V}(F) \subset \mathcal{V}(P_m) \cup \mathcal{V}, F \cup \mathcal{V}^s \neq \emptyset : 2 \bar{H}_i + 3 \text{ with } O \in \bar{H}_1 \cap \bar{H}_j \text{ and } \{\bar{H}_j, \bar{H}_1, \bar{H}_i\} = \{\bar{H}_2, \bar{H}_3, \bar{H}_4\}.

From \( \{y_1, u_1\} \notin I \), we obtain that \( y_1 \notin I \) and \( x_w \in \bar{H}_1^- = \bar{H}_1^- \) and

• \{3\} : \mathcal{V}(F) \subset \mathcal{V}(P_m) \cup \mathcal{V}^w : \text{LEMMA A},
• \{-\} : F \cap V^w \neq \emptyset \neq F \cap V^s : x_w < x_s, x_w \in \hat{H}_1^- and E.1,
• \{-\} : F \cap V^w \neq \emptyset \neq F \cap V^r : either x_w < x_r and E.1, or x_r < x_w, y_1 \notin I and E.2, and
• \{-\} : F \cap V^r \neq \emptyset \neq F \cap V^s : E.4 with \hat{H}_1 = \hat{H}_1, \hat{H}_2, \hat{H}_3, \hat{H}_2, \hat{H}_3.

II.3 O \in bd(K) and O \notin bd(I)

We recall that x_w \in \hat{H}_1^- , and note that I \neq K implies that x_r \in \hat{H}_1^- . Then

• \{3\} : F \cap V^s = \emptyset : LEMMA A,
• \{8\} : \forall(F) \subset \forall(P_m) \cup \forall^s, F \cap V^s \neq \emptyset : 2\hat{H}_1 + 3 + 3 with O \in \hat{H}_1 \cap \hat{H}_j for some j = \{2,3,4\},
• \{-\} : F \cap V^w \neq \emptyset \neq F \cap V^s : x_w < x_s, x_w \in \hat{H}_1^- and E.1, and as worst case scenario,
• \{3\} : F \cap V^s \neq \emptyset \neq F \cap V^r : x_s < x_r, x_s \in \hat{H}_1^- and E.1.

III. x_m \in T and x_m \notin K.

As \{y_1,v_1 = x_m\} \subset T and K \subset P_m, we have that u_1 \in \hat{H}_1^- , V^s \subset \hat{H}_1^- , V^r \subset \hat{H}_1^- and O \notin K. We let v_1 = y_2 and note that H_4 separates O from any F with x_m \notin F \subset [V^w \cup \{y_1, y_1, y_1+1\}].

III.1. O \notin bd(I) \cup bd(T)

As in II.1, we obtain that

• \{4\} : x_m \in F : 5.2,
• \{1\} : \forall(F) \subset \forall(P_m) \cup \forall^s : 5.6,
• \{3\} : \forall(F) \subset \forall(P_m) \cup \forall^w, F \cap \forall^w \neq \emptyset : 5.5 with H_2, H_3, H_4,
• \{4\} : \forall(F) \subset \forall(P_m) \cup \forall^r, F \cap \forall^r \neq \emptyset : 5.5 with H_2, H_3, H_4, H_5, and
• \{-\} : F \cap \forall^w \neq \emptyset \neq F \cap \forall^s : x_w < x_s, u_1 \neq T and E.2.

We note that our repetitive arguments are dependent upon Lemmas A and E, and \{u_1, v_1\} \subset I. Also that we present only worst case scenarios.

If x_r < x_s then with u_1 \in I and y_1 \notin I, we have

• \{3\} : F \cap \forall^r \neq \emptyset \neq F \cap \forall^s : x_r \in \hat{H}_1^- with E.1, and
• \{-\} : F \cap \forall^r \neq \emptyset \neq F \cap \forall^w : either x_r < x_w with E.2, or x_w < x_r with E.1.

Let x_w < x_s < x_r. By E.1, we may assume that \{x_w,x_s\} \notin \hat{H}_1^- . With x_s \in \hat{H}_1^+ and x_r \in \hat{H}_1, we have u_1 \in I, y_1 \notin I, x_w \in \hat{H}_1^- ,

• \{-\} : F \cap \forall^w \neq \emptyset \neq F \cap \forall^r : E.1, and
• \{1\} : F \cap \forall^s \neq \emptyset \neq F \cap \forall^r : E.2 or E.3.

With x_w \in \hat{H}_1^+, we have x_s \in \hat{H}_1^- ,

• \{-\} : F \cap \forall^s \neq \emptyset \neq F \cap \forall^r : E.1, and
• \{3\} : F \cap \forall^w \neq \emptyset \neq F \cap \forall^r : E.4 and 5.9 with O \in T = I.
III.2 $O \in \text{bd}(I)$.

We note as in I.2 that $x_s \in \hat{H}_1^-$ follows from $O \notin K$. Next, we obtain the same separating hyperplanes for $F$ with $F \cap V' \neq \emptyset$ and $V(F) \subset V(P_m) \cup V'$ as in II.2, and with $x_w$ and $x_s$ interchanged, the corresponding worst case scenario for $x_w \in \hat{H}_1^-$. Let $x_w \in \hat{H}_1^+$. Then we choose $T = I$ and, similarly to II.2, obtain that

- $\{5\} : \mathcal{V}(P) \subset \mathcal{V}(P_m) \cup \mathcal{V}^w, F \cap \mathcal{V}^w \neq \emptyset : 2H_1 + 3$ with $H_1 = \hat{H}_1$, and $O \in H_1 \cap H_j$ for some $j \in \{2, 3, 4\}$,
- $\{3\} : \mathcal{V}(P) \subset \mathcal{V}(P_m) \cup \mathcal{V}^s$: LEMMA A,
- $\{\cdot\} : F \cap \mathcal{V}^w \neq \emptyset \neq F \cap \mathcal{V}^s : x_w < x_s, u_1 \notin T$ and E.2,
- $\{\cdot\} : F \cap \mathcal{V}^r \neq \emptyset \neq F \cap \mathcal{V}^s$: either $x_r < x_s, u_1 \notin I$ and E.2, or $x_s < x_r, x_s \in \hat{H}_1^-$ and E.1, and
- $\{\cdot\} : F \cap \mathcal{V}^r \neq \emptyset \neq F \cap \mathcal{V}^s$: E.4.

III.3 $O \in \text{bd}(T)$ and $O \notin \text{bd}(I)$.

Then $I \neq T$ and $x_r \in H_1^-$. We recall that $y_1 \notin T$ and $x_s \in H_1^-$. Hence,

- $\{8\} : \mathcal{V}(F) \subset \mathcal{V}(P_m) \cup \mathcal{V}^w, F \cap \mathcal{V}^w \neq \emptyset : 2H_1 + 3$ with $O \in H_1 \cap H_j$ and $\{H_j, H_1, H_1\} = \{H_2, H_3, H_4\}$,
- $\{3\} : F \cap \mathcal{V}^w = \emptyset$: LEMMA A,
- $\{\cdot\} : F \cap \mathcal{V}^w \neq \emptyset \neq F \cap \mathcal{V}^s : x_w < x_s, u_1 \notin T$ and E.2, and as worst case scenario,
- $\{3\} : F \cap \mathcal{V}^w \neq \emptyset \neq F \cap \mathcal{V}^r : x_w < x_r, x_w \in \hat{H}_1^-$ and E.1.

IV $x_m \in T \cap K$.

We let $v_1 = x_m = y_2 = u_2$, and note that $\{v_1, y_1\} \subset T$ implies that $u_1 \notin T$ and $x_s \in H_1^-$; and $\{v_1, u_1\} \subset K$ implies that $y_1 \notin K$ and $x_w \in \hat{H}_1^-$.

IV.1 $O \notin \text{bd}(K) \cup \text{bd}(T) \cup \text{bd}(I)$.

We recall that $O \in P_m \setminus P_m - 1$. Then

- $\{5\} : x_m \in F$ or $F \subset P_m$; 5.3,
- $\{3\} : \mathcal{V}(F) \subset \mathcal{V}(P_m) \cup \mathcal{V}^w \neq \emptyset, y_2 = x_m \notin F$: 5.5 with $H_2, H_3$, and $H_4$ for $F \cap Z_i \neq \emptyset$ or $F \cap (Z_i \cup Z_t \cup Z_i^1) = \emptyset$,
- $\{3\} : \mathcal{V}(F) \subset \mathcal{V}(P_m) \cup \mathcal{V}^s, F \cap \mathcal{V}^s \neq \emptyset, u_2 = x_m \notin F$: 5.5 with $\hat{H}_2, \hat{H}_3, \hat{H}_4$,
- $\{4\} : \mathcal{V}(F) \subset \mathcal{V}(P_m) \cup \mathcal{V}^r, F \cap \mathcal{V}^r \neq \emptyset$: 5.5 with $\hat{H}_2, \hat{H}_3, \hat{H}_4, \hat{H}_5$, and
- $\{\cdot\} : F \cap \mathcal{V}^w \neq \emptyset \neq F \cap \mathcal{V}^s : x_w < x_s, x_w \in \hat{H}_1^-$ and E.1.

We observe that for $x_r$ and $x_s$: E.1 and E.2 yield $\{\cdot\}$ for $F \cap \mathcal{V}^r \neq \emptyset \neq F \cap \mathcal{V}^s$, and E.3 and E.4 yield $u_1 \in I, v_1 \in K$ and the worst case scenario

- $\{1\} : F \cap \mathcal{V}^r \neq \emptyset \neq F \cap \mathcal{V}^s$: either $x_r < x_r$ with $\hat{H}_5$, or $x_r < x_s$ with $\hat{H}_1 = H_1$. 

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The corresponding observation for $x_r$ and $x_w$, and $\{u_1, y_1\} \notin I$, now yield

- $(1) : F \cap V^r \neq \emptyset \neq F \cap (V^w \cup V^s)$.

**IV.2** $O \in \text{bd}(K)$.

Then $O \notin P_{m-1}$ and $u_2 = x_m$ imply that $O \notin [u_1, u_k, u_{k+1}]$ and

- $(8) : V(F) \subset V(P_m) \cup V^w, F \cap V^r \neq \emptyset : 2\bar{H}_i + 3 + 3$ with $O \in \bar{H}_1 \cap \bar{H}_j$ and
  $\{\bar{H}_j, \bar{H}_i, \check{H}_i\} = \{\check{H}_2, \check{H}_3, \check{H}_4\}$

We recall that $x_w \in \check{H}_1^-$. If $x_r \in \check{H}_1^-$ then

- $(3) : F \cap V^s = \emptyset$: LEMMA A,
- $(\emptyset) : F \cap V^w \neq \emptyset \neq F \cap V^s : x_w < x_s$ and E.1, and
- $(3) : F \cap V^r \neq \emptyset \neq F \cap V^s : x_s < x_r$ and either $x_s \in \check{H}_1^-$ and E.1, or $x_s \in \check{H}_1^-, v_1 \in K, x_r \in \check{H}_1^-$ and E.3, 5.9 with $O \in \check{H}_5$.

Let $x_r \in \check{H}_1^+$. Then we choose $I = K$ with $(v_1, v_2, v_t, v_{t+1}) = (u_2, u_1, u_k, u_{k+1})$, and note that $y_1 \in I = K$ and $x_w \in \check{H}_1^- = \check{H}_1^-$. Now

- $(3) : V(F) \subset V(P_m) \cup V^w$: LEMMA A with $\check{H}_1^-$,
- $(5) : V(F) \subset V(P_m) \cup V^w, F \cap V^r \neq \emptyset : 2\bar{H}_i + 3$ with $O \in \bar{H}_1 \cap \bar{H}_j$ and
  $\{\bar{H}_j, \bar{H}_i, \check{H}_i\} = \{\check{H}_2, \check{H}_3, \check{H}_4\}$,
- $(\emptyset) : F \cap V^w \neq \emptyset \neq F \cap V^s : x_w < x_s$ and E.1,
- $(\emptyset) : F \cap V^r \neq \emptyset \neq F \cap V^s$: either $x_w < x_r$ and E.1, or $x_r < x_w$ and E.2, and
- $(\emptyset) : F \cap V^s \neq \emptyset \neq F \cap V^r$: E.4.

**IV.3** $O \in \text{bd}(T)$

We argue as in IV.2 with

- $(8) : V(F) \subset V(P_m) \cup V^w, F \cap V^w \neq \emptyset : 2\bar{H}_i + 3 + 3$,
- $(\emptyset) : F \cap V^w \neq \emptyset \neq F \cap V^s : x_w < x_s$ and E.2, and the cases $x_r \in \check{H}_1^-$ and $x_r \in \check{H}_1^+$.

**IV.4** $O \in \text{bd}(I)$ and $O \notin \text{bd}(K) \cup \text{bd}(T)$

Since we choose $K = I(T = I)$ if $x_s \in \check{H}_1^+(x_w \in \check{H}_1^+)$, we may assume that $\{x_w, x_s\} \subset \check{H}_1^-$. Then

- $(8) : V(F) \subset V(P_m) \cup V^w, F \cap V^w \neq \emptyset : 2\bar{H}_i + 3 + 3$ with $O \in \bar{H}_1 \cap \bar{H}_j$ and
  $\{\bar{H}_j, \bar{H}_i, \check{H}_i\} = \{\check{H}_2, \check{H}_3, \check{H}_4\}$, and
- $(3) : F \cap V^r = \emptyset$: LEMMA A
  If $u_1 \notin I$, then
- $(\emptyset) : F \cap V^w \neq \emptyset \neq F \cap V^r$: either $x_r < x_s$ and E.2, or $x_s < x_r$ and E.1, and
- $(3) : F \cap V^w \neq \emptyset \neq F \cap V^r : x_r < x_w$ and E.1.
Let \( u_1 \in I \). Then \( y_1 \notin I \) and we argue as above with \( x_4 \) and \( x_6 \) interchanged. 
\\
\[ 6.2. \text{It remains to determine } s(O) \text{ in the case that } O \in P_6 \text{ and (in view of 5.1) } O \text{ is contained in every 4-subpolytope of } P_6. \text{ Since } P_6 \text{ satisfies Gale's Evenness Condition with } x_1 < x_2 < x_3 < x_4 < x_5 < x_6 \text{ and } O \text{ is not contained in any facet of } P_6, \text{ it follows that } \\
O \in [x_1, x_3, x_5] \cap [x_2, x_4, x_6]. \\
\]

From LEMMA D and its proof, we have
- \( \mathcal{V}(P) = \{x_1, x_3, x_5\} \cup \mathcal{V}^2 \cup \mathcal{V}^4 \cup \mathcal{V}^6 \),
- any hyperplane through \( \langle x_1, x_3, x_5 \rangle \) intersects at most two of \( \mathcal{V}^2, \mathcal{V}^4 \) and \( \mathcal{V}^6 \), and
- any \( F \in \mathcal{F}(P) \) intersects at most two of \( \mathcal{V}^2, \mathcal{V}^4 \) and \( \mathcal{V}^6 \).

Let \( \mathcal{W}^{ij} = [\{x_1, x_3, x_5\} \cup \mathcal{V}^i \cup \mathcal{V}^j], i \neq j \in \{2, 4, 6\} \). Then \( [x_1, x_3, x_5] \subset bd(\mathcal{W}^{ij}) \), any \( F \in \mathcal{F}(P) \) is contained in some \( \mathcal{W}^{ij} \), and \( s(O) \leq 9 \) by LEMMA A. 
\\
We conclude with the observation that any linked \( P \) with \( |\mathcal{V}(P)| \leq 11 \) is simply linked, and the problem: Is every linked \( P \) also simply linked?

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