Einstein locally conformal calibrated $G_2$-structures

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Abstract We study locally conformal calibrated $G_2$-structures whose underlying Riemannian metric is Einstein, showing that in the compact case the scalar curvature cannot be positive. As a consequence, a compact homogeneous 7-manifold cannot admit an invariant Einstein locally conformal calibrated $G_2$-structure unless the underlying metric is flat. In contrast to the compact case, we provide a non-compact example of homogeneous manifold endowed with a locally conformal calibrated $G_2$-structure whose associated Riemannian metric is Einstein and non Ricci-flat. The homogeneous Einstein metric is a rank-one extension of a Ricci soliton on the 3-dimensional complex Heisenberg group endowed with a left-invariant coupled $SU(3)$-structure $(\omega, \Psi)$, i.e., such that $d\omega = c\text{Re}(\Psi)$, with $c \in \mathbb{R} - \{0\}$. Nilpotent Lie algebras admitting a coupled $SU(3)$-structure are also classified.

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1 Introduction

We recall that a seven-dimensional smooth manifold $M$ admits a $G_2$-structure if the structure group of the frame bundle reduces to the exceptional Lie group $G_2$. The existence of a $G_2$-structure is equivalent to the existence of a non-degenerate 3-form $\varphi$ defined on the whole manifold (see for example [26]) and using this 3-form it is possible to define a Riemannian metric $g_{\varphi}$ on $M$. 

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If $\varphi$ is parallel with respect to the Levi–Civita connection, i.e., $\nabla^{LC}\varphi = 0$, then the holonomy group is contained in $G_2$, the $G_2$-structure is called parallel and the corresponding manifolds are called $G_2$-manifolds. In this case, the induced metric $g_\varphi$ is Ricci-flat. The first examples of complete metrics with holonomy $G_2$ were constructed by Bryant and Salamon [6]. Compact examples of manifolds with holonomy $G_2$ were obtained first by Joyce [24–26] and then by Kovalev [28] and by Corti, Haskins, Nordström, Pacini [12]. Incomplete Ricci-flat metrics of holonomy $G_2$ with a 2-step nilpotent isometry group $N$ acting on orbits of codimension 1 were obtained in [9,20]. It turns out that these metrics are locally isometric (modulo a conformal change) to homogeneous metrics on solvable Lie groups, which are obtained as rank one extensions of a six-dimensional nilpotent Lie group endowed with an invariant SU(3)-structure of a special kind, known in the literature as half-flat [10].

Examples of compact and non-compact manifolds endowed with non-parallel $G_2$-structures were given for instance in [1,7–9,14,15,17,19,27,38]. In particular, in [9] conformally parallel $G_2$-structures on solvmanifolds, i.e., on simply connected solvable Lie groups, were studied. More in general, in [23] it was shown that a seven-dimensional compact Riemannian manifold $M$ admits a locally conformal parallel $G_2$-structure if and only if it has as covering a Riemannian cone over a compact nearly Kähler 6-manifold such that the covering transformations are homotheties preserving the corresponding parallel $G_2$-structure.

By [5,11,18], it is evident that the Riemannian scalar curvature of a $G_2$-structure may be expressed in terms of the 3-form $\varphi$ and its derivatives. More precisely, in [5] an expression of the Ricci curvature and the scalar curvature in terms of the four intrinsic torsion forms $\tau_i$, $i = 0, \ldots, 3$, and their exterior derivatives was given. Moreover, using this it is possible to show that the scalar curvature has a definite sign for certain classes of $G_2$-structures.

If $d\varphi = 0$, the $G_2$-structure is called calibrated or closed. The geometry of this family of $G_2$-structures was studied in [11]. Furthermore, Bryant proved in [5] that if the scalar curvature of a closed $G_2$-structure is non-negative then the $G_2$-structure is parallel.

We say that a $G_2$-structure $\varphi$ is Einstein if the underlying Riemannian metric $g_\varphi$ is Einstein. In [5,11] it was proved, as an analogous of Goldberg conjecture for almost-Kähler manifolds, that on a compact manifold an Einstein (or, more in general, with divergence-free Weyl tensor [11]) calibrated $G_2$-structure has holonomy contained in $G_2$. In the non-compact case, Cleyton and Ivanov showed that the same result is true with the additional assumption that the $G_2$-structure is $\ast$-Einstein, but it still an open problem to see if there exist (even incomplete) Einstein metrics underlying calibrated $G_2$-structures. Recently, some negative results were proved in the case of non-compact homogeneous spaces in [16]. In particular, the authors showed that a seven-dimensional solvmanifold cannot admit any left-invariant calibrated $G_2$-structure inducing an Einstein metric $g_\varphi$ unless $g_\varphi$ is flat.

In the present paper, we are mainly interested in the geometry of locally conformal calibrated $G_2$-structures, i.e., $G_2$-structures whose associated metric is conformally equivalent (at least locally) to the metric induced by a calibrated $G_2$-structure.

In Sect. 3, we prove that a compact manifold endowed with an Einstein locally conformal calibrated $G_2$-structure has non-positive scalar curvature (and then has either zero or negative curvature if it is also connected) and we show that a compact homogeneous 7-manifold cannot admit an invariant Einstein locally conformal calibrated $G_2$-structure unless the underlying metric is flat.

In the last section, we give a non-compact example of a homogeneous manifold endowed with an Einstein locally conformal calibrated $G_2$-structure. The homogeneous manifold is a solvmanifold, thus this example and the aforementioned result of [16] highlight a different behaviour of calibrated and locally conformal calibrated $G_2$-structures. Moreover, the homogeneous Einstein metric is a rank-one extension of a Ricci soliton on the complex
Heisenberg group induced by a coupled SU(3)-structure \((\omega, \Psi)\) such that \(d\omega = -\text{Re}(\Psi)\). Recall that a half-flat SU(3)-structure is said to be coupled if \(d\omega\) is proportional to \(\text{Re}(\Psi)\) at each point (see [37]). Finally, we classify nilpotent Lie groups admitting a left-invariant coupled SU(3)-structure, showing that the complex Heisenberg group is, up to isomorphisms, the only nilpotent Lie group admitting a coupled SU(3)-structure \((\omega, \Psi)\) whose associated metric is a Ricci soliton.

2 Preliminaries on \(G_2\) and SU(3)-structures

Let \((e_1, \ldots, e_7)\) be the standard basis of \(\mathbb{R}^7\) and \((e^1, \ldots, e^7)\) be the corresponding dual basis. We set

\[
\varphi = e^{123} + e^{145} + e^{167} + e^{246} - e^{257} - e^{347} - e^{356},
\]

where for simplicity \(e^{ijk}\) stands for the wedge product \(e^i \wedge e^j \wedge e^k\) in \(\Lambda^3((\mathbb{R}^7)^*)\). The subgroup of \(\text{GL}(7, \mathbb{R})\) fixing \(\varphi\) is \(G_2\). The basis \((e_1, \ldots, e_7)\) is an oriented orthonormal basis for the underlying metric and the orientation is determined by the inclusion \(G_2 \subset \text{SO}(7)\). The group \(G_2\) also fixes the 4-form \(\ast \varphi = e^{4567} + e^{2367} + e^{2345} + e^{1357} - e^{1346} - e^{1256} - e^{1247}\), where \(\ast\) denotes the Hodge star operator determined by the associated metric and orientation.

We recall that a \(G_2\)-structure on a 7-manifold \(M\) is characterized by a positive 3-form \(\varphi\). Indeed, it turns out that there is a \(1-1\) correspondence between \(G_2\)-structures on a 7-manifold \(M\) and 3-forms for which the bilinear form \(B_\varphi\) defined by

\[
B_\varphi(X, Y) = \frac{1}{6} i_X \varphi \wedge i_Y \varphi \wedge \varphi
\]

is positive definite, where \(i_X\) denotes the contraction by \(X\). A 3-form \(\varphi\) for which \(B_\varphi\) is positive definite defines a unique Riemannian metric \(g_\varphi\) and volume form \(dV_\varphi\) such that for any couple of vectors \(X\) and \(Y\) on \(M\) the following relation holds

\[
g_\varphi(X, Y) dV_\varphi = \frac{1}{6} i_X \varphi \wedge i_Y \varphi \wedge \varphi.
\]

As in [11], we let

\[
\varphi = \frac{1}{6} \varphi_{ijk} e^{ijk}
\]

and define the \(\ast\)-Ricci tensor of the \(G_2\)-structure as

\[
\rho^\ast_{sm} = R_{ijkl} \varphi_{js} \varphi_{klm}.
\]

A \(G_2\)-structure is said to be \(\ast\)-Einstein if the traceless part of the \(\ast\)-Ricci tensor vanishes, i.e., if \(\rho^\ast = \frac{s^\ast}{7} g\), where \(s^\ast\) is the trace of \(\rho^\ast\).

On a 7-manifold endowed with a \(G_2\)-structure, the action of \(G_2\) on the tangent spaces induces an action of \(G_2\) on the exterior algebra \(\Lambda^p(M)\), for any \(p \geq 2\). In [4], it was shown that there are irreducible \(G_2\)-module decompositions

\[
\Lambda^2((\mathbb{R}^7)^*) = \Lambda^2_7((\mathbb{R}^7)^*) \oplus \Lambda^2_{14}((\mathbb{R}^7)^*),
\]

\[
\Lambda^3((\mathbb{R}^7)^*) = \Lambda^3_1((\mathbb{R}^7)^*) \oplus \Lambda^3_7((\mathbb{R}^7)^*) \oplus \Lambda^3_{27}((\mathbb{R}^7)^*),
\]

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where $\Lambda^k_p((\mathbb{R}^7)^*)$ denotes an irreducible $G_2$-module of dimension $k$. Using the previous decomposition of $p$-forms, in [5] a simple expression of $d\varphi$ and $d\ast\varphi$ was obtained, where $\ast$ denotes the Hodge operator defined by the metric $g_{\varphi}$ and the volume form $dV_{\varphi}$. More precisely, for any $G_2$-structure $\varphi$ there exist unique differential forms $\tau_0 \in \Lambda^0(M)$, $\tau_1 \in \Lambda^1_1(M)$, $\tau_2 \in \Lambda^2_{14}(M)$, $\tau_3 \in \Lambda^3_{27}(M)$, such that

\[
\begin{align*}
    d\varphi &= \tau_0 \ast \varphi + 3\tau_1 \wedge \varphi + \ast\tau_3, \\
    d\ast\varphi &= 4\tau_1 \wedge \ast\varphi + \tau_2 \wedge \varphi,
\end{align*}
\]

where $\Lambda^k_p(M)$ denotes the space of sections of the bundle $\Lambda^k_p(T^*M)$.

In the case of a closed $G_2$ structure we have

\[
\begin{align*}
    d\varphi &= 0, \\
    d\ast\varphi &= \tau_2 \wedge \varphi.
\end{align*}
\]

By the results of [5], the scalar curvature is given by

\[
\text{Scal}(g_{\varphi}) = -\frac{1}{2}|\tau_2|^2
\]

and from this it is clear that it cannot be positive.

For a locally conformal calibrated $G_2$-structure $\varphi$ one has $\tau_0 \equiv 0$ and $\tau_3 \equiv 0$, so

\[
\begin{align*}
    d\varphi &= 3\tau_1 \wedge \varphi, \\
    d\ast\varphi &= 4\tau_1 \wedge \ast\varphi + \tau_2 \wedge \varphi,
\end{align*}
\]

and taking the exterior derivative of the former it is easy to show that $\tau_1$ is a closed 1-form. Moreover, in this case the scalar curvature has not a definite sign as one can check from its expression

\[
\text{Scal}(g_{\varphi}) = 12\delta\tau_1 + 30|\tau_1|^2 - \frac{1}{2}|\tau_2|^2,
\]

where $\delta$ denotes the adjoint of the exterior derivative $d$ with respect to the metric $g_{\varphi}$.

If the only nonzero intrinsic torsion form is $\tau_1$, we have the so called locally conformal parallel $G_2$-structures. They are named in this way since a conformal change of the metric $g_{\varphi}$ associated to a $G_2$-structure of this kind gives (at least locally) the metric induced by a parallel $G_2$-structure. In this case

\[
\begin{align*}
    d\varphi &= 3\tau_1 \wedge \varphi, \\
    d\ast\varphi &= 4\tau_1 \wedge \ast\varphi.
\end{align*}
\]

We will give an example of such a structure at the end of Sect. 4.

We recall that a six-dimensional smooth manifold admits an SU(3)-structure if the structure group of the frame bundle can be reduced to SU(3). It is possible to show that the existence of an SU(3)-structure is equivalent to the existence of an almost Hermitian structure $(h, J, \omega)$ and a unit $(3, 0)$-form $\Psi$.

Since SU(3) is the stabilizer of the transitive action of $G_2$ on the 6-sphere $S^6$, it follows that a $G_2$-structure on a 7-manifold induces an SU(3)-structure on any oriented hypersurface. If the $G_2$-structure is parallel, then the SU(3)-structure is half-flat [10]. In terms of the forms $(\omega, \Psi)$ this means $d(\omega \wedge \omega) = 0$, $d(\text{Re}(\Psi)) = 0$.

In our computations we will use another characterization of SU(3)-structures which follows from the results of [22,36]. We describe it here. Consider a six-dimensional oriented real vector space $V$, a $k$-form on $V$ is said to be stable if its $\text{GL}(V)$-orbit is open. Let
A : \Lambda^5(V^*) \to V \otimes \Lambda^6(V^*) denote the canonical isomorphism given by \( A(\gamma) = w \otimes \Omega \), where \( i_w \Omega = \gamma \), and define for a fixed 3-form \( \sigma \in \Lambda^3(V^*) \)

\[ K_\sigma : V \to V \otimes \Lambda^6(V^*), \quad K_\sigma(w) = A((i_w \sigma) \wedge \sigma) \]

and

\[ \lambda : \Lambda^3(V^*) \to (\Lambda^6(V^*))^{\otimes 2}, \quad \lambda(\sigma) = \frac{1}{6} \text{tr} K_\sigma^2. \]

A 3-form \( \sigma \) is stable if and only if \( \lambda(\sigma) \neq 0 \) and whenever this happens it is possible to define a volume form by \( \sqrt{\lambda(\sigma)} \in \Lambda^6(V^*) \), where the positively oriented root is chosen, and an endomorphism

\[ J_\sigma = \frac{1}{\sqrt{\lambda(\sigma)}} K_\sigma, \]

which is a complex structure when \( \lambda(\sigma) < 0 \).

A pair of stable forms \((\omega, \sigma) \in \Lambda^2(V^*) \times \Lambda^3(V^*)\) is called compatible if \( \omega \wedge \sigma = 0 \) and normalized if \( J_\sigma^* \sigma \wedge \sigma = 2 \omega^3 \) (the latter identity is non-zero since a 2-form \( \omega \) is stable if and only if \( \omega^3 \neq 0 \)). Such a pair defines a (pseudo) Euclidean metric \( h(\cdot, \cdot) = \omega(J_\sigma \cdot, \cdot) \). As a consequence, on a six-dimensional smooth manifold \( N \) there is a one to one correspondence between SU(3)-structures and pairs \((\omega, \sigma) \in \Lambda^2(N) \times \Lambda^3(N)\) such that for each point \( p \in N \) the pair of forms defined on \( T_p N \) \((\omega_p, \sigma_p)\) is stable, compatible, normalized, has \( \lambda(\sigma_p) < 0 \) and induces a Riemannian metric \( h_p(\cdot, \cdot) = \omega_p(J_{\sigma_p} \cdot, \cdot) \). In this case we have \( \Psi = \sigma + iJ_{\sigma}^* \sigma \) and, then, \( \sigma = \text{Re}(\Psi) \). We refer to \( h \) as the associated Riemannian metric to the SU(3)-structure \((\omega, \sigma)\).

An SU(3)-structure \((\omega, \sigma)\) on a 6-manifold \( N \) is called coupled if \( d\omega = c \sigma \), with \( c \) a non-zero real number. Note that in particular a coupled SU(3)-structure is half-flat since \( d(\omega^2) = 0 \) and \( d\sigma = 0 \) and its intrinsic torsion belongs to the space \( \mathcal{W}_1^- \oplus \mathcal{W}_2^- \), where \( \mathcal{W}_1^- \cong \mathbb{R} \) and \( \mathcal{W}_2^- \cong \text{su}(3) \) (see [10]).

It is interesting to notice that the product manifold \( N \times \mathbb{R} \), where \( N \) is a 6-manifold endowed with a coupled SU(3)-structure \((\omega, \sigma)\), has a natural locally conformal calibrated \( G_2 \)-structure defined by

\[ \varphi = \omega \wedge dt + \sigma. \]

Indeed,

\[ d\varphi = c\sigma \wedge dt = c\varphi \wedge dt, \]

since in local coordinates the components of \( \sigma \) are functions defined on \( N \) and thus they do not depend on \( t \). Then, \( \tau_0 = 0 \), \( \tau_3 = 0 \) and \( \tau_1 = (-\frac{1}{3}c) \, dt \).

### 3 Einstein locally conformal calibrated \( G_2 \)-structures on compact manifolds

We will show now that a seven-dimensional, compact, smooth manifold \( M \) endowed with an Einstein locally conformal calibrated \( G_2 \)-structure \( \varphi \) has \( \text{Scal}(g_\varphi) \leq 0 \). It is worth observing here that, up to now, there are no known examples of smooth manifolds endowed with a locally conformal calibrated \( G_2 \)-structure whose associated metric is Ricci-flat (and then has zero scalar curvature).

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First of all recall that given a Riemannian manifold \((M, g)\) of dimension \(n \geq 3\) it is possible to define the so called conformal Yamabe constant \(Q(M, g)\) in the following way: set \(a_n := \frac{4(n-1)}{n-2}\), \(p_n := \frac{2n}{n-2}\) and let \(C_c^\infty(M)\) denote the set of compactly supported smooth real valued functions on \(M\). Then

\[
Q(M, g) := \inf_{u \in C_c^\infty(M), u \neq 0} \left\{ \frac{\int_M (a_n |du|^2_g + u^2 \text{Scal}(g))dV_g}{(\int_M |u|^{p_n}dV_g)^{\frac{2n}{pn}}} \right\}.
\]

The sign of \(Q(M, g)\) is a conformal invariant, in particular the following characterization holds:

**Proposition 3.1** If \((M, g)\) is a compact Riemannian manifold of dimension \(n \geq 3\), then \(Q(M, g)\) is negative/zero/positive if and only if \(g\) is conformal to a Riemannian metric of negative/zero/positive scalar curvature.

Using the conformal Yamabe constant it is possible to prove the following

**Theorem 3.2** Let \(M\) be a seven-dimensional, compact, smooth manifold endowed with an Einstein locally conformal calibrated \(G_2\)-structure \(\varphi\). Then \(\text{Scal}(g_\varphi) \leq 0\). Moreover, if \(M\) is connected, \(\text{Scal}(g_\varphi)\) is either zero or negative.

**Proof** Suppose that \(\text{Scal}(g_\varphi) > 0\), then the 1-form \(\tau_1\) is exact. Indeed, since \(d\tau_1 = 0\), we can consider the de Rham class \([\tau_1] \in H^1_{\text{DR}}(M)\) and take the harmonic 1-form \(\xi\) representing \([\tau_1]\), that is, \(\tau_1 = \xi + df\), where \(\Delta \xi = 0\) and \(f \in C^\infty(M)\). \(\xi\) has to vanish everywhere on \(M\) since it is compact, oriented and has positive Ricci curvature. Then \(\tau_1 = df\). Let us consider \(\tilde{\varphi} := e^{-3f} \varphi\), it is clear that \(\tilde{\varphi}\) is a \(G_2\)-structure defined on \(M\). Moreover

\[
d\tilde{\varphi} = d(e^{-3f}\varphi) = -3e^{-3f}df \wedge \varphi + e^{-3f}d\varphi = -3e^{-3f}\tau_1 \wedge \varphi + e^{-3f}(3\tau_1 \wedge \varphi) = 0,
\]

so \(\tilde{\varphi}\) is a closed \(G_2\)-structure and \(\text{Scal}(g_\varphi) \leq 0\) by \([5]\). We have \(g_\tilde{\varphi} = e^{-2f}g_\varphi\), that is, \(g_\tilde{\varphi}\) is conformal to the Riemannian metric \(g_\varphi\) of positive scalar curvature, then the conformal Yamabe constant \(Q(M, g_\varphi)\) is positive by the previous characterization.

Since \(M\) is compact, it has finite volume and is complete as a consequence of the well known Hopf–Rinow Theorem. Then, by \([34, \text{Corollary 2.2}]\) we have that \(Q(M, g_\varphi) \leq 0\), which is in contrast with the previous result.

As a consequence of the previous proposition we have the

**Corollary 3.3** A seven-dimensional, compact, homogeneous, smooth manifold \(M\) cannot admit an invariant locally conformal calibrated Einstein \(G_2\)-structure \(\varphi\), unless the underlying metric \(g_\varphi\) is flat.

**Proof** Recall that a homogeneous Einstein manifold with negative scalar curvature is not compact \([3]\). Thus, every seven-dimensional, compact, homogeneous, smooth manifold \(M\) with an invariant \(G_2\)-structure \(\varphi\) whose associated metric is Einstein has \(\text{Scal}(g_\varphi) \geq 0\). Combining this result with the previous proposition we have \(\text{Scal}(g_\varphi) = 0\) and, in particular, \(g_\varphi\) is Ricci-flat. The statement then follows recalling that in the homogeneous case Ricci flatness implies flatness \([2]\).
4 Noncompact homogeneous examples and coupled SU(3)-structures

In this section, after recalling some facts about noncompact homogeneous Einstein manifolds, we first study the classification of coupled SU(3)-structures on nilmanifolds and then we construct an example of a locally conformal calibrated $G_2$-structure $\varphi$ inducing an Einstein (non Ricci-flat) metric on a noncompact homogeneous manifold.

All the known examples of noncompact homogeneous Einstein manifolds are solvmanifolds, i.e., simply connected solvable Lie groups $S$ endowed with a left-invariant metric (see for instance the recent survey [32]). D. Alekseevskii conjectured that these might exhaust the class of non-compact homogeneous Einstein manifolds (see [3, 7.57]).

Lauret in [33] showed that every Einstein solvmanifold is standard, i.e., it is a solvable Lie group $S$ endowed with a left-invariant metric such that the orthogonal complement $\mathfrak{a} = [\mathfrak{s}, \mathfrak{s}]^\perp$, where $\mathfrak{s}$ is the Lie algebra of $S$, is abelian. We recall that given a metric nilpotent Lie algebra $\mathfrak{n}$ with an inner product $\langle \cdot, \cdot \rangle_{\mathfrak{n}}$, a metric solvable Lie algebra $(\mathfrak{s} = \mathfrak{n} \oplus \mathfrak{a}, \langle \cdot, \cdot \rangle_{\mathfrak{s}})$ is called a metric solvable extension of $(\mathfrak{n}, \langle \cdot, \cdot \rangle_{\mathfrak{n}})$ if $[\mathfrak{s}, \mathfrak{s}] = \mathfrak{n}$ and the restrictions to $\mathfrak{n}$ of the Lie bracket of $\mathfrak{s}$ and of the inner product $\langle \cdot, \cdot \rangle_{\mathfrak{s}}$ coincide with the Lie bracket of $\mathfrak{n}$ and with $\langle \cdot, \cdot \rangle_{\mathfrak{n}}$, respectively. The dimension of $\mathfrak{a}$ is called the algebraic rank of $\mathfrak{s}$.

In [21, 4.18], it was proved that the study of standard Einstein metric solvable Lie algebras reduces to the rank-one metric solvable extension of a nilpotent Lie algebra (i.e., those for which $\dim(\mathfrak{a}) = 1$). Indeed, by [21] the metric Lie algebra of any $(n + 1)$-dimensional rank-one solvmanifold can be modelled on $(\mathfrak{s} = \mathfrak{n} \oplus \mathbb{R} \mathfrak{h}, \langle \cdot, \cdot \rangle_{\mathfrak{s}})$ for some nilpotent Lie algebra $\mathfrak{n}$, with the inner product $\langle \cdot, \cdot \rangle_{\mathfrak{s}}$ such that $\langle \mathfrak{h}, \mathfrak{n} \rangle_{\mathfrak{s}} = 0$, $\langle \mathfrak{h}, \mathfrak{h} \rangle_{\mathfrak{s}} = 1$ and the Lie bracket on $\mathfrak{s}$ given by

$$[H, X]_{\mathfrak{s}} = DX, \quad [X, Y]_{\mathfrak{s}} = [X, Y]_{\mathfrak{n}},$$

where $[\cdot, \cdot]_{\mathfrak{n}}$ denotes the Lie bracket on $\mathfrak{n}$ and $D$ is some derivation of $\mathfrak{n}$. By [30], a left-invariant metric $h$ on a nilpotent Lie group $N$ is a Ricci soliton if and only if the Ricci operator satisfies $\text{Ric}(h) = \mu I + D$, for some $\mu \in \mathbb{R}$ and some derivation $D$ of $\mathfrak{n}$, when $h$ is identified with an inner product on $\mathfrak{n}$ or, equivalently, if and only if $(N, h)$ admits a metric standard extension whose corresponding standard solvmanifold is Einstein. The inner product $h$ is also called nilsoliton.

Using the results of [29,31], in [39] all the seven-dimensional rank-one Einstein solvmanifolds were determined, proving that each one of the 34 nilpotent Lie algebras $\mathfrak{n}$ of dimension 6 admits a rank-one solvable extension which can be endowed with an Einstein inner product.

Six-dimensional nilpotent Lie algebras admitting a half-flat SU(3)-structure were classified in [13]. For coupled SU(3)-structures we can show the following

**Theorem 4.1** Let $\mathfrak{n}$ be a six-dimensional, non-abelian, nilpotent Lie algebra admitting a coupled SU(3)-structure. Then $\mathfrak{n}$ is isomorphic to one of the following

$$\mathfrak{n}_9 = (0, 0, 0, e^{12}, e^{14} - e^{23}, e^{15} + e^{34}), \quad \mathfrak{n}_{28} = (0, 0, 0, 0, e^{13} - e^{24}, e^{14} + e^{23}),$$

where for instance $\mathfrak{n}_9 = (0, 0, 0, e^{12}, e^{14} - e^{23}, e^{15} + e^{34})$ means that there exists a basis $(e^1, \ldots, e^6)$ of $\mathfrak{n}_9^*$ such that

$$de^1 = 0, \quad j = 1, 2, 3, \quad de^4 = e^{12}, \quad de^5 = e^{14} - e^{23}, \quad de^6 = e^{15} + e^{34}.$$
By the results in [13], the generic nilpotent Lie algebra $n$ admitting a half-flat SU(3)-structure is isomorphic to one of the 24 Lie algebras described in Table 1. Consider on $n$ a generic 2-form

$$\omega = b_1 e^{12} + b_2 e^{13} + b_3 e^{14} + b_4 e^{15} + b_5 e^{16} + b_6 e^{23} + b_7 e^{24} + b_8 e^{25} + b_9 e^{26} + b_{10} e^{34} + b_{11} e^{35} + b_{12} e^{36} + b_{13} e^{45} + b_{14} e^{46} + b_{15} e^{56},$$

where $b_i \in \mathbb{R}, i = 1, \ldots, 15$, and the 3-form

$$\sigma = c(d\omega), \quad c \in \mathbb{R} - \{0\}.$$

The expression of $\lambda(\sigma)$ for each nilpotent Lie algebra considered is given in Table 1.

We observe that among the 24 nilpotent Lie algebras admitting a half-flat SU(3)-structure we have:
• 1 case (n\textsubscript{28}) for which \(\lambda(\sigma) < 0\) if \(b_{15} \neq 0\),
• 2 cases (n\textsubscript{4} and n\textsubscript{9}) for which the sign of \(\lambda(\sigma)\) depends on \(\omega\),
• 21 cases for which \(\lambda(\sigma)\) cannot be negative.

Therefore, the 21 algebras having \(\lambda(\sigma) \geq 0\) do not admit any coupled SU(3)-structure.

Consider n\textsubscript{4}, it has structure equations
\[
(0, 0, e^{12}, e^{13}, e^{14} + e^{23}, e^{24} + e^{15}).
\]
First of all, observe that if \(b_{15} = 0\) then \(\lambda(\sigma) = 0\). So if we want to find an SU(3)-structure we have to look for \(\omega\) with \(b_{15} \neq 0\). Moreover, \(\sigma\) induces an almost complex structure if and only if \(\lambda(\sigma)\) is negative, then we have to suppose in addition that \(b_{15}(b_{12} + b_{13}) > b_{14}^2\).
Since we want \(\omega\) to be the 2-form associated to an SU(3)-structure, it must be a form of type (1, 1) and this happens if and only if \(\omega(\cdot, \cdot) = \omega(J \cdot, J \cdot)\), where \(J = J_\sigma\). Computing the previous identity with respect to the considered frame, we have that the following equations have to be satisfied by the components of \(\omega\):
\[
\omega_{ab} = \sum_{k, m=1}^{6} J^k_a J^m_b \omega_{km}, \quad 1 \leq a < b \leq 6
\]
(observe that \(\omega_{12} = b_1, \omega_{13} = b_2\) and so on). Using these equations it is possible to write four of the \(b_i\) in terms of the remaining and obtain a new expression for \(\omega\). We can now compute the matrix associated to \(h(\cdot, \cdot) = \omega(J \cdot, \cdot)\) with respect to the basis \((e_1, \ldots, e_6)\) and observe that for the nonzero vector \(v = e_4 - \frac{b_{12}}{b_{15}} e_5 + \frac{b_{13}}{b_{15}} e_6\) we have \(h(v, v) = 0\). Therefore, \(h\) cannot be positive definite and, as a consequence, it is not possible to find a coupled SU(3)-structure on n\textsubscript{4}.

For the Lie algebras n\textsubscript{9} and n\textsubscript{28} we can give an explicit example of coupled SU(3)-structure.

Consider n\textsubscript{9} the forms
\[
\omega = -\frac{3}{2} e^{12} - \frac{1}{4} e^{14} - e^{15} - e^{24} + \frac{1}{2} e^{26} - \frac{1}{2} e^{35} - e^{36} + e^{56},
\]
\[
\sigma = \frac{\sqrt{15} \sqrt{2}}{4} e^{123} + \frac{\sqrt{15} \sqrt{2}}{8} e^{234} - \frac{\sqrt{15} \sqrt{2}}{8} e^{125} - \frac{\sqrt{15} \sqrt{2}}{8} e^{134} \\
+ \frac{\sqrt{15} \sqrt{2}}{4} e^{135} - \frac{\sqrt{15} \sqrt{2}}{4} e^{146} + \frac{\sqrt{15} \sqrt{2}}{4} e^{236} + \frac{\sqrt{15} \sqrt{2}}{4} e^{345}.
\]
We have
\[
\omega \wedge \sigma = 0, \quad \omega^3 \neq 0, \quad \lambda(\sigma) = -\frac{225}{64}, \quad d\omega = -\frac{4}{\sqrt{15} \sqrt{2}} \sigma,
\]
in particular \((\omega, \sigma)\) is a compatible pair of stable forms. The associated almost complex structure \(J = J_\sigma\) has the following matrix expression with respect to the basis \((e_1, \ldots, e_6)\):
\[
J = \begin{bmatrix}
0 & 0 & -\sqrt{2} & 0 & 0 & 0 \\
\sqrt{\frac{3}{2}} & 0 & 0 & -\sqrt{2} & 0 & 0 \\
\frac{\sqrt{3}}{2} & 0 & 0 & 0 & 0 & 0 \\
0 & \frac{\sqrt{3}}{2} & -\sqrt{2} & 0 & 0 & 0 \\
\frac{\sqrt{3}}{2} & 0 & \frac{\sqrt{3}}{2} & \frac{\sqrt{3}}{2} & 0 & \sqrt{2} \\
-\frac{\sqrt{3}}{4} & -\frac{\sqrt{3}}{4} & \frac{3\sqrt{3}}{2} & 0 & -\frac{\sqrt{3}}{2} & 0
\end{bmatrix}
\]
and it is easy to check that \(J^* \sigma \wedge \sigma = \frac{2}{3} \omega^3\), i.e., the pair \((\omega, \sigma)\) is normalized.
The inner product \( h(\cdot, \cdot) = \omega(J\cdot, \cdot) \) is given with respect to the basis \((e_1, \ldots, e_6)\) by
\[
h = \begin{bmatrix}
5\sqrt{2} & \sqrt{2} & \sqrt{2} & -\sqrt{2} & 0 & \sqrt{2} \\
\sqrt{2} & 5\sqrt{2} & \sqrt{2} & 0 & \sqrt{2} & 0 \\
\sqrt{2} & \sqrt{2} & 7\sqrt{2} & \sqrt{2} & -\sqrt{2} & \sqrt{2} \\
-\sqrt{2} & 0 & \sqrt{2} & \sqrt{2} & 0 & 0 \\
0 & \sqrt{2} & -\sqrt{2} & 0 & \sqrt{2} & 0 \\
\sqrt{2} & 0 & \sqrt{2} & 0 & 0 & \sqrt{2}
\end{bmatrix}.
\]
and it is positive definite. Therefore, we can conclude that \((\omega, \sigma)\) is a coupled SU(3)-structure on \(n_9\).

For \(n_{28}\) consider the pair of compatible, normalized, stable forms
\[
(\omega = e^{12} + e^{34} - e^{56}, \quad \sigma = e^{136} - e^{145} - e^{235} - e^{246}).
\]
This pair defines a coupled SU(3)-structure with \(d\omega = -\sigma\). Moreover, the associated inner product
\[
h = (e^1)^2 + \cdots + (e^6)^2
\]
is a nilsoliton with
\[
\text{Ric}(h) = -3I + 2 \text{diag}(1, 1, 1, 1, 2, 2).
\]
Summarizing our results, we can conclude that \(n_0\) and \(n_{28}\) are, up to isomorphisms, the only six-dimensional nilpotent Lie algebras admitting a coupled SU(3)-structure.

We have just provided a coupled SU(3)-structure on \(n_{28}\) whose associated inner product is a nilsoliton, we claim that this is the unique case among all six-dimensional nilpotent Lie algebras. It is clear that to prove the previous assertion it suffices to show that \(n_9\) does not admit any coupled SU(3)-structure inducing a nilsoliton inner product. In order to do this, we consider an orthonormal basis \((e_1, \ldots, e_6)\) of \(n_9\) whose dual basis satisfies the structure equations
\[
\left(0, 0, 0, \frac{\sqrt{5}}{2} e^{12}, e^{14} - e^{23}, \frac{\sqrt{5}}{2} e^{15} + e^{34}\right)
\]
(by the results of [30] and [39], these are, up to isomorphisms, the structure equations for which the considered inner product on \(n_9\) is a nilsoliton). As we did before, consider a generic 2-form \(\omega\), the 3-form \(\sigma = c(d\omega)\), evaluate \(\lambda(\sigma)\) and impose that it is negative. Then compute \(J_\sigma\) and the matrix associated to \(h(\cdot, \cdot) = \omega(J_\sigma \cdot, \cdot)\) with respect to the considered basis. Since \(h\) has to be the restriction to \(n_9\) of an Einstein inner product defined on \(n_9 \oplus \mathbb{R}e_7\) and since the latter is unique up to scaling, we have to impose that the symmetric matrix associated to \(h\) is a multiple of the identity. Solving the associated equations we find that \(\lambda(\sigma)\) has to be zero, which is a contradiction.

Starting from a six-dimensional nilpotent Lie algebra \(n\) endowed with a coupled SU(3)-structure, it is possible to construct a locally conformal calibrated \(G_2\)-structure on the rank-one solvable extension \(s = n \oplus \mathbb{R}e_7\) under some extra hypothesis. Let \(\hat{d}\) denote the exterior derivative on \(n\) and \(d\) denote the exterior derivative on \(s\). Observe that given a \(k\)-form \(\theta \in \Lambda^k(n^*)\) we have

\[\square\]
\[
d\theta = \hat{d}\theta + \rho \wedge e^7
\]
for some \( \rho \in \Lambda^k(n^*) \).

**Proposition 4.2** Let \( n \) be a six-dimensional, nilpotent Lie algebra endowed with a coupled SU(3)-structure \((\omega, \sigma)\) with \( \hat{d}\omega = c\sigma \), \( c \in \mathbb{R} - \{0\} \). Consider on its rank one solvable extension \( s = n \oplus \mathbb{R}e_7 \) the \( G_2 \)-structure defined by \( \phi = \omega \wedge e^7 + \sigma \), where the closed 1-form \( e^7 \) is the dual of \( e_7 \). Then the \( G_2 \)-structure is locally conformal calibrated with \( \tau_1 = \frac{1}{3}ce^7 \) if and only if \( d\sigma = -2c\sigma \wedge e^7 \).

**Proof** Suppose that \( d\sigma = -2c\sigma \wedge e^7 \), we can write \( d\omega = \hat{d}\omega + \gamma \wedge e^7 \) for some 2-form \( \gamma \in \Lambda^2(n^*) \). We obtain \( d\phi = ce^7 \wedge \phi \). Then, \( \phi \) is locally conformal calibrated with \( \tau_1 = \frac{1}{3}ce^7 \).

Conversely, suppose that \( \phi \) is locally conformal calibrated with \( \tau_1 = \frac{1}{3}ce^7 \). Then \( \phi \) is locally conformal calibrated with \( \tau_1 = \frac{1}{3}ce^7 \).

Now we will construct an Einstein locally conformal calibrated \( G_2 \)-structure on a rank-one extension of the Lie algebra \( n_{28} \) (Lie algebra of the 3-dimensional complex Heisenberg group) endowed with the coupled SU(3)-structure (1).

**Example 4.3** Consider \( n_{28} \) and the metric rank-one solvable extension \( s = n_{28} \oplus \mathbb{R}e_7 \) with structure equations

\[
\left( \frac{1}{2}e^{17}, \frac{1}{2}e^{27}, \frac{1}{2}e^{37}, \frac{1}{2}e^{47}, e^{13} - e^{24} + e^{57}, e^{14} + e^{23} + e^{67}, 0 \right).
\]

The associated solvable Lie group \( S \) is not unimodular and so it does not admit any compact quotient [35]. Consider on \( n_{28} \) the coupled SU(3)-structure \((\omega, \sigma)\) given by (1) with the nilsoliton associated inner product

\[
h = (e^1)^2 + \cdots + (e^6)^2.
\]

Then the inner product on \( s \)

\[
g = (e^1)^2 + \cdots + (e^7)^2
\]
is Einstein with Ricci tensor \( \text{Ric}(g) = -3g \).

Since \( d\sigma = 2\sigma \wedge e^7 \), by the previous proposition we have a locally conformal calibrated \( G_2 \)-structure on \( s \) given by

\[
\phi = \omega \wedge e^7 + \sigma = e^{127} + e^{347} - e^{567} + e^{136} - e^{145} - e^{235} - e^{246}
\]

and it is easy to show that \( g\phi = g \). Then the corresponding solvmanifold \((S, \phi)\) is an example of non-compact homogeneous manifold endowed with an Einstein (non-flat) locally conformal calibrated \( G_2 \)-structure.
Observe that the $G_2$-structure $\varphi$ satisfies the conditions
\[
d\varphi = -e^7 \wedge \varphi,
\]
\[
d^* \varphi = -e^7 \wedge (3e^{1256} + 2e^{1234} + 3e^{3456}).
\]

Then
\[
\tau_1 = -\frac{1}{3} e^7,
\]
as we expected from Proposition 4.2, and
\[
\tau_2 = - \left( \frac{5}{3} e^{12} + \frac{5}{3} e^{34} + \frac{10}{3} e^{56} \right).
\]

Moreover, the $G_2$-structure is not $\ast$-Einstein, since by direct computation with respect to the orthonormal basis $(e_1, \ldots, e_7)$ one has
\[
\rho^* = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 22 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 22 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -6
\end{pmatrix}.
\]

It is worth emphasizing here that, by [16], on seven-dimensional solvmanifolds there are no left-invariant calibrated $G_2$-structures inducing an Einstein non-flat metric. The previous example shows that the situation is different in the case of locally conformal calibrated $G_2$-structures.

We provide now a non-compact example of homogeneous manifold admitting an Einstein (non-flat) locally conformal parallel $G_2$-structure.

**Example 4.4** The Einstein rank-one solvable extension of the six-dimensional abelian Lie algebra is the solvable Lie algebra with structure equations
\[
( ae^{17}, ae^{27}, ae^{37}, ae^{47}, ae^{57}, ae^{67}, 0),
\]
where $a$ is a nonzero real number. The Riemannian metric
\[
g = (e^1)^2 + \cdots + (e^7)^2
\]
is Einstein with Ricci tensor given by $\text{Ric}(g) = -6a^2 g$.

The 3-form $\varphi = -e^{125} - e^{136} - e^{147} + e^{237} - e^{246} + e^{345} - e^{567}$ has stabilizer $G_2$, is such that $g_\varphi = g$ and satisfies the conditions
\[
d\varphi = -3ae^{2467} + 3ae^{3457} - 3ae^{1257} - 3ae^{1367},
\]
\[
d^* \varphi = 4ae^{23567} + 4ae^{12347} - 4ae^{14567}.
\]

It is immediate to show that $\tau_1 = -ae^7$ and $\tau_0 \equiv 0$, $\tau_2 \equiv 0$, $\tau_3 \equiv 0$, that is, the $G_2$-structure $\varphi$ is locally conformal parallel.
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