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HOROCYCLIC INVARIANCE OF RUELLE RESONANT STATES
FOR CONTACT ANOSOV FLOWS IN DIMENSION 3

COLIN GUILLARMOU AND FRÉDÉRIC FAURE

Abstract. We show that for contact Anosov flows in dimension 3 the resonant
states associated to the first band of Ruelle resonances are distributions that are
invariant by the unstable horocyclic flow.

1. Introduction

By the work of Liverani [Li], Butterley-Liverani [BuLi], Faure-Sjöstrand [FaSj] or
Dyatlov-Zworski [DyZw2], one can define an intrinsic discrete spectrum for the vector
field $X$ generating a smooth Anosov flow on a compact manifold $M$. More precisely,
one view $P := -X$ as a first order differential operator and we can construct
appropriate anisotropic Sobolev spaces $\mathcal{H}^N$ (depending on parameter $N > 0$) related to
the stable/unstable splitting of the flow, on which the first order differential operator
$P - \lambda$ is an analytic family of Fredholm operators of index 0 in the complex half-plane
$\{\text{Re}(\lambda) > C_0 - \mu N\}$ for some $C_0 \geq 0$ and $\mu > 0$ depending on $X$; here $N > 0$ can be
taken as large as we like. The eigenvalues and the eigenstates of $P$ are independent
of $N$, they are called resonances and resonant states. The operator is not self-adjoint
on $\mathcal{H}^N$ and there can be Jordan blocks. We say that $u \in \mathcal{H}^N$ is a generalized resonant
state with resonance $\lambda_0 \in \{\text{Re}(\lambda) > C_0 - \mu N\}$ if $(P - \lambda_0)^j u = 0$ for some $j \in \mathbb{N}$.
An equivalent way to define resonances for $P$ is through the resolvent: the resolvent
$R_P(\lambda) := (P - \lambda)^{-1}$ is an analytic family of bounded operators on $L^2(M, dm)$ (for
some fixed Lebesgue type measure $dm$) in $\{\text{Re}(\lambda) > C_0\}$ for some $C_0 \geq 0$, there exists
a meromorphic continuation of $R_P(\lambda)$ to $\lambda \in \mathbb{C}$ as a map

$$R_P(\lambda) : C^\infty(M) \rightarrow D'(M)$$

and the polar part of the Laurent expansion of $R_P(\lambda)$ at a pole $\lambda_0$ is a finite rank
operator. The resonances are the poles of $R_P(\lambda)$ and the generalized resonant states
are the elements in the range of the residue

$$\Pi_{\lambda_0} := -\text{Res}_{\lambda_0}R_P(\lambda)$$

which turns out to be a projector.

We will now assume that $M$ is a closed oriented manifold with dimension 3 and
that $X$ generates a contact Anosov flow, i.e there is a smooth one-form $\alpha$ such that
$d\alpha$ is symplectic on $\ker \alpha$, $\alpha(X) = 1$ and $i_X d\alpha = 0$. We fix a smooth metric $G$ on $\mathcal{M}$ and we denote by $E_s$ and $E_u$ the stable and unstable bundles, the tangent bundle has a flow-invariant continuous splitting

$$T\mathcal{M} = \mathbb{R}X \oplus E_s \oplus E_u$$

such that there is $C > 1$ and $\nu > 0$ such that for all $z \in \mathcal{M}$

$$\forall \xi \in E_s(z), \forall t \geq 0, \ |d\varphi_t(z) . \xi|_G \leq Ce^{-\nu t} |\xi|_G;$$

$$\forall \xi \in E_u(z), \forall t \geq 0, \ |d\varphi_{-t}(z) . \xi|_G \leq Ce^{-\nu t} |\xi|_G.$$

We define the minimal/maximal expansion rates of the flow

$$\mu_{\min} := \lim_{t \to +\infty} \sup_{z \in \mathcal{M}} \frac{1}{t} \log \left| d\varphi_t(z) \right|_{E_u(z)} G = \lim_{t \to +\infty} \sup_{z \in \mathcal{M}} \frac{1}{t} \log \left| d\varphi_{-t}(z) \right|_{E_u(z)} G,$$

$$\mu_{\max} := \lim_{t \to +\infty} \inf_{z \in \mathcal{M}} \frac{1}{t} \log \left| d\varphi_t(z) \right|_{E_u(z)} G = \lim_{t \to +\infty} \inf_{z \in \mathcal{M}} \frac{1}{t} \log \left| d\varphi_{-t}(z) \right|_{E_u(z)} G \leq 1$$

that satisfy $0 < \mu_{\min} < \mu_{\max}$. The equality of the limits in (1.3) is due to the fact that we work with a contact flow. We assume that $E_u$ is an orientable bundle and let $U_-$ be a global non-vanishing section of $E_u$, called an unstable horocyclic vector field.

Since the flow is contact, we know by Hurder-Katok [HuKa] that $U_-$ is a vector field that can be chosen with regularity $C^{2-\epsilon}(\mathcal{M})$ for all $\epsilon > 0$. We will show in Lemma 2.2 that $U_-$ satisfies a commutation relation

$$[X, U_-] = -r_- U_-$$

for some function $r_- \in C^{2-\epsilon}(\mathcal{M})$. There is a preserved smooth measure $dm := \alpha \wedge d\alpha$, thus $P$ is skew-adjoint on $L^2(\mathcal{M}, dm)$ and $R_P(\lambda)$ is analytic in $\text{Re}(\lambda) > 0$, the $L^2$-spectrum being the whole imaginary line. The operator $U_-$ can be viewed as acting on the negative Sobolev space $H^{-s}(\mathcal{M})$ for $s < 1$ as follows: for $u \in H^{-s}(\mathcal{M})$, for all $f \in C^\infty(\mathcal{M})$,

$$\langle U_- u, f \rangle := \langle u, -U_- f - \text{div}(U_-) f \rangle$$

where $\text{div}(U_-)$ is the divergence of $U_-$ with respect to $dm$.

**Theorem 1.** Let $\mathcal{M}$ be a smooth 3-dimensional oriented compact manifold and let $X$ be a smooth vector field generating a contact Anosov flow. Assume that the unstable bundle is orientable. For $P = -X$, if $\lambda_0$ is a resonance of $P$ with $\text{Re}(\lambda_0) > -\mu_{\min}$ and if $u$ is a generalized resonant state of $P$ with resonance $\lambda_0$, then $U_- u = 0$.

We shall see in Corollary 1.1 that for flows with pinched Lyapunov exponents, there are infinitely many resonances in the region $\text{Re}(\lambda_0) > -\mu_{\min}$, thus infinitely many resonant states are killed by $U_-$. We expect the result to hold more generally if $\text{Re}(\lambda_0) > \text{Pr}(-2r_-)$ if $r_-$ is the function appearing in (1.4) and $\text{Pr}$ denotes the topological pressure. The problem to reach that bound is that we do not know if a resonant
state \( u \) with resonance \( \lambda_0 \in \{ \text{Re}(\lambda) \leq -\mu_{\text{min}} \} \) is sufficiently regular to be able to define \( U_- u \).

In view of the regularity of the stable/unstable foliation in our case, we have locally near each point \( x_0 \in \mathcal{M} \) a decomposition of \( \mathcal{M} \) as a product \( W_u \times W_s \times (-\epsilon, \epsilon) \), using the stable/unstable foliation, where \( W_{u/s} \) are diffeomorphic to \((\epsilon, \epsilon)\). The flow is \( X = \partial_t \) is those coordinates, and Theorem 1 says that a resonant state \( w \) with resonance \( \lambda_0 \) (if \( \text{Re}(\lambda_0) > -\mu_{\text{min}} \)) is of the form

\[
w(u, s, t) = e^{-\lambda_0 t} \omega(s)
\]

for some distribution \( \omega \) on \( W_s \). In fact, due to the wave-front set analysis of resonant states in [FaSj], a resonant state \( w \) can be restricted locally to each piece of local stable leaf (which is an embedded smooth submanifold), or alternatively the lift of \( w \) to the universal cover \( \tilde{\mathcal{M}} \) of \( \mathcal{M} \) can be restricted to the stable leaves in \( \tilde{\mathcal{M}} \).

The horocyclic invariance of the first band of resonances was shown for geodesic flows in constant negative curvature in any dimension by Dyatlov-Faure-Guillarmou [DFG]. For hyperbolic surfaces, this follows also from the work of Flaminio-Forni [FlFo]. We find quite stricking that this type of properties still holds for general contact Anosov flows in dimension 3. We also notice that for an Anosov diffeomorphism on \( \mathbb{T}^2 \), the first resonant state for a certain transfer operator associated to an Anosov diffeomorphism on \( \mathbb{T}^2 \) has shown to be horocyclic invariant by Giulietti-Liverani [GiLi]. There are other related cases which appeared in the work of Dyatlov [Dy] for resonances of semi-classical operators with \( r \)-normally hyperbolic trapped set, but the resonant states are only microlocally killed by some smooth pseudo-differential operator playing the role of \( U_- \).

In Theorem 2, we prove a more general result which applies to the operator \( P := -X + V \) where \( V \) is a smooth potential, and where the unstable derivative \( U_- \) is replaced by \( U_- + \alpha V \) for some appropriate function \( \alpha V \) depending on \( V \). After Corollary 3.6, we discuss some interesting particular cases of potentials, namely \( V = cr_- \) with \( c \in \mathbb{R} \), that do not technically fit our assumptions due to smoothness issues but could still be considered without problems using the works [BuLi, GoLi].

Using Faure-Tsujii [FaTs1], we deduce the following result about existence of an infinite dimensional space of horocyclic invariant distributions:

**Corollary 1.1.** Let \( \mathcal{M} \) be a smooth 3-dimensional oriented manifold and let \( X \) be a smooth vector field generating a contact Anosov flow. Assume that \( E_u \) is orientable and that \( \mu_{\text{max}} < 2\mu_{\text{min}} \). Then, for each \( \epsilon > 0 \) small, there exist infinitely many resonant

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\(^1\)Note that the result [FaTs1] is an announcement and has been proved only in the case \( V = \frac{1}{2} r_- \) in [FaTs2]. The general case will appear in [FaTs3].
states in $\ker U_-$ with associated resonances contained in the band
\[ \{ \text{Re}(\lambda) \in [-\frac{1}{2}\mu_{\text{max}} - \epsilon, -\frac{1}{2}\mu_{\text{min}} + \epsilon] \} . \]

These resonant states belong to the Sobolev space $H^{-\frac{1}{2}\mu_{\text{max}} + 2\epsilon}_{-\mu_{\text{min}}}(\mathcal{M})$.

The proof of Theorem 1 comes from the following commutation relation
\[ U_-( -X - \lambda)^{-1} = ( -X - \lambda - r_- )^{-1} U_- \]
between the resolvents of $-X$ and of $-X - r_-$ where $r_-$ is the function obtained in (1.4).

The main part of the argument is to show that such relation holds and makes sense as a map $C^\infty(\mathcal{M}) \to \mathcal{D}'(\mathcal{M})$ in a certain region of the complex $\lambda$-plane. This is not completely obvious since the resolvents $(-X - \lambda)^{-1}$ and $(-X - \lambda - r_-)^{-1}$ a priori map smooth functions to distributions and $U_-$ is only a $C^{2-\epsilon}(\mathcal{M})$ vector field for all $\epsilon > 0$.

We notice that our proof would apply similarly in higher dimension under pinching conditions on the Lyapunov exponents, except that one needs to use a covariant derivative in the unstable direction instead of just a vector field $U_-$. The horocyclic invariance of resonant states would typically apply only to finitely many resonant states, for there is only finitely many resonances in the complex region where our result would hold, by a result of Tsujii [Ts]. We have thus decided to focus only on the case of dimension 3, where in addition the regularity of $E_u$ is known to be better.

We finally emphasize that an alternative proof of Theorem 1 using more advanced tools is given in the preprint version [FaGu] of this paper.

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2. Stable/unstable bundles

2.1. Anosov flows and the regularity of stable/unstable bundles. Let $\mathcal{M}$ be a smooth compact 3-dimensional oriented manifold and let $X$ be a smooth vector field with Anosov flow denoted by $\varphi_t$. We fix a smooth metric $G$ on $\mathcal{M}$ and we denote by $E_s$ and $E_u$ the stable and unstable bundles so that one has the flow-invariant continuous splitting (1.1) with (1.2). Let $\alpha$ be the continuous flow-invariant 1-form on $\mathcal{M}$ defined by $\ker \alpha = E_u \oplus E_s$ and $\alpha(X) = 1$. By Hurder-Katok [HuKa, Theorem 2.3], if $\alpha \in C^1(\mathcal{M}; T^*\mathcal{M})$ then $\alpha \in C^\infty(\mathcal{M}; T^*\mathcal{M})$ and either $\alpha \wedge d\alpha = 0$ or it is a nowhere vanishing 3-form and $\varphi_t$ is a contact flow, i.e $i_Xd\alpha = 0$ and $d\alpha$ is symplectic on $\ker \alpha$. For simplicity, we shall assume in the whole paper that we are in case of a
contact flow. Let us now define the dual Anosov decomposition

\[ T^*\mathcal{M} = \mathbb{R} \alpha \oplus E_u^* \oplus E_s^*, \quad \text{with } E_s^*(E_s \oplus \mathbb{R} X) = 0, \quad E_u^*(E_u \oplus \mathbb{R} X) = 0. \]

In [HuKa], Hurder-Katok proved the following regularity statement on the unstable/stable bundles.

**Lemma 2.1** (Hurder-Katok). For a smooth contact flow in dimension 3, the regularity of the bundles \( E_u \) and \( E_s \) is as follows:

\[ \forall \epsilon > 0, \ E_u \in C^{2-\epsilon}, \quad \forall \epsilon > 0, \ E_s \in C^{2-\epsilon}. \quad (2.1) \]

By regularity \( C^r \) of a bundle, it is meant that the bundle is locally spanned by vector fields which have \( C^r \) coefficients in smooth charts on \( \mathcal{M} \). For what follows, we will write \( f \in C^{2-}(\mathcal{M}) \) to mean that a function/vector field belongs to \( \cap_{\epsilon>0} C^{2-\epsilon}(\mathcal{M}) \).

Anosov [An] proved that there exist local stable and unstable smooth submanifolds \( W_u(z), W_s(z) \) of \( \mathcal{M} \) at each point \( z \) such that \( T_z W_u(z) = E_u(z) \) and \( T_z W_s(z) = E_s(z) \).

As in Lemma 2.1 and [HuKa], the dependence in \( z \) is in fact \( C^{2-} \), but it is never \( C^2 \) except for geodesic flows in constant negative curvature.

The submanifolds \( W_u \) form a foliation near \( z \) and by [HPS, Theorem 6.1] (see also [DMM, Lemma 2.1]), there are continuous maps

\[ \Lambda : V_1 \times V_2 \to \mathcal{M}, \quad V_1 \subset \mathbb{R}, V_2 \subset \mathbb{R}^2 \]

such that \( \Lambda_y : V_1 \to \mathcal{M} \) defined by \( \Lambda_y(x) = \Lambda(x, y) \) is a \( C^\infty \) embedding with image an unstable local submanifold \( W_u(z) \) for some \( z \) and the derivatives \( \partial_x^\beta \Lambda \) are continuous on \( V_1 \times V_2 \) for all \( \beta \in \mathbb{N} \). The same holds for the stable foliation.

Next, we want to make sense of unstable derivatives.

**Lemma 2.2.** Assume that \( X \) generates a smooth contact flow on an orientable 3-dimensional manifold \( \mathcal{M} \) and that \( E_u \) is an orientable bundle. There exists a non-vanishing vector field \( U_- \) on \( \mathcal{M} \) with regularity \( C^{2-}(\mathcal{M}; TM) \) such that \( U_-(z) \in E_u(z) \) for all \( z \in \mathcal{M} \), and there exists a function \( r_- \) with regularity \( C^{2-}(\mathcal{M}) \) such that

\[ [X, U_-] = -r_- U_. \quad (2.2) \]

The function \( r_- \) satisfies for \( t \geq 0 \)

\[ d\varphi_{-t}(z) . U_-(z) = e^{-\int_0^t r_- (\varphi_s(z)) ds} U_-(\varphi_{-t}(z)). \]

If \( a_i \) are the coefficients of \( U_- \) in a smooth coordinates system, then \( U_k^k(a_i) \) are continuous for all \( k \in \mathbb{N} \). The same properties hold with \( U_+ \) replacing \( U_- \), \( E_s \) replacing \( E_u \), \( r_+ \) replacing \( r_- \), with \( d\varphi_t(z) . U_+(z) = e^{\int_0^t r_+ (\varphi_s(z)) ds} U_+(\varphi_{t}(z)) \), and \( U_+ \) is a \( C^{2-}(\mathcal{M}; TM) \) section of \( E_s \) with local coefficients \( b_i \) such that \( U_k^k(b_i) \) are continuous for all \( k \in \mathbb{N} \).
Proof. The orientability of $E_u$ insures that there exists a non-vanishing vector field $U$ which is a section of $E_u$, and we normalize it so that its $G$-norm is $|U|_G = 1$. It can be chosen to be globally $C^2$-($\mathcal{M}$) by Lemma 2.1. By the remark following the Lemma (which describes the unstable foliation regularity), we also have that the coefficients $a_i$ of $U$ in local coordinates are such that $U^n(a_i)$ are continuous for all $n \in \mathbb{N}$. We approximate $U$ by a smooth vector field $U_\epsilon$ in a way that $|U - U_\epsilon|_G \leq \epsilon$ for $\epsilon > 0$ small. Since $\mathcal{M}$ is oriented and 3-dimensional (thus parallelizable), we can find a smooth vector field $S$ so that $(X, U_\epsilon, S)$ is a global smooth basis of $T\mathcal{M}$, and we write $U = a_\epsilon U_\epsilon + b_\epsilon X + c_\epsilon S$ with $|a_\epsilon - 1| = \mathcal{O}(\epsilon)$ and $a_\epsilon, b_\epsilon, c_\epsilon \in C^2$($\mathcal{M}$). Let us define $U_- := (1/a_\epsilon)U$ which is also a $C^2$-($\mathcal{M}$) non-vanishing section of $E_u$ for $\epsilon > 0$ fixed small enough. Since $d\varphi_t(z).E_u(z) = E_u(\varphi_t(z))$, we have $d\varphi_t(z).U_-(z) = f(t, z)U_-(\varphi_t(z))$ for some $f(t, z) \in C^2(\mathbb{R} \times \mathcal{M})$ with $f(t, z) > 0$, and $\partial_t f(t, z) \in C^1(\mathbb{R} \times \mathcal{M})$. We also have $f(s + t, z) = f(s, z)f(t, \varphi_s(z))$. We differentiate at $t = 0$ and get (2.2) with $r_-(z) := \partial_t f(0, z)/f(0, z)$ and more generally $\partial_s f(s, z)/f(s, z) = r_-(\varphi_s(z))$. A priori $r_- \in C^1(\mathcal{M})$ but a small computation using $[X, U_-] = -r_-U_-$ implies that

$$-r_- = h + \frac{c_\epsilon}{a_\epsilon} k$$

where $h, k \in C^\infty(\mathcal{M})$ are the $U_\epsilon$ components of $[X, U_\epsilon]$ and $[X, S]$ in the basis $(X, U_\epsilon, S)$. Thus $r_- \in C^2(\mathcal{M})$. The regularity of the coefficients of $U_-$ when differentiated twice in the direction $U_-$ follows from the same property as for $U$. By definition of $r_-$ we also have that

$$|d\varphi_{-t}(z)U_-(z)|_G = e^{-\int_{-t}^0 r_-(\varphi_s(z))ds} |U_-(\varphi_{-t}(z))|_G$$

and this completes the proof. \hfill \Box

Remark 1. We notice that $U_{\pm}$ are not uniquely defined: one can always multiply $U_\pm$ by a positive smooth function $f$, and $fU_\pm$ would satisfy all the same properties as $U_\pm$ described in Lemma 2.2. On the other hand, the kernel of $U_-$ is independent of the choice of non-vanishing section $U_-$ of $E_u$.

It is interesting to give the following interpretation to (2.2), which explains why the operator $P = -X - r_-$ appears naturally: the flow acts on the bundle $E_s^*$, and if $\omega$ is a non-vanishing section of $E_s^*$ defined by $\omega|_{E_s^* \otimes \mathbb{R}X} = 0$ and $\omega(U_-) = 1$, we have $\mathcal{L}_X \omega = r_- \omega$; thus for each $f \in C^2(\mathcal{M})$,

$$\mathcal{L}_X(f \omega) = (-Xf - r_- f) \omega.$$ 

The map $\pi : C^2(\mathcal{M}; E_s^*) \to C^2(\mathcal{M})$ defined by $\pi(h) := h(U_-)$ is an isomorphism with inverse $e : C^2(\mathcal{M}) \to C^2(\mathcal{M}; E_s^*)$ given by $e(f) = f \omega$, one has $\pi \mathcal{L}_X e = -X - r_-$. 

\footnote{It is probably known from experts that $\partial_t f(t, z) \in C^2(\mathbb{R} \times \mathcal{M})$, from which $r_- \in C^2(\mathcal{M})$ would follow, but we haven’t found references for such a fact, which is the reason why we use the approximation argument involving $U_\epsilon$.}
and (2.2) can be reinterpreted as the identity: for each $f \in C^\infty(M)$
\[ \mathcal{L}_-f = df \]
where $df : C^\infty(M) \to C^2(-M,E_s^*)$ is the operator defined by $df := df|_{E_u}$. We refer to [FaTs2, Section 3.3.2] for a related discussion.

To conclude this section, we define the minimal and maximal expansion rates by
\[ \mu_{\min} := \lim_{t \to +\infty} \inf_{z \in M} \frac{1}{t} \int_0^t r_-(\varphi_s(z))ds, \quad \mu_{\max} := \lim_{t \to +\infty} \sup_{z \in M} \frac{1}{t} \int_0^t r_-(\varphi_s(z))ds. \quad (2.3) \]

First, we remark that the two limits exist as $t \to +\infty$ by Fekete’s lemma since $F_1(t) := \sup_{z \in M} \int_0^t r_-(\varphi_s(z))ds$ is easily seen to be a subadditive function and $F_2(t) := \inf_{z \in M} \int_0^t r_-(\varphi_s(z))ds$ is superadditive. By Lemma (2.2), for each $\epsilon > 0$, there is $C_\epsilon$ such that for all $t \geq 0$ and all $z \in M$
\[ C_\epsilon^{-1}e^{-t(\mu_{\max}+\epsilon)} \leq |d\varphi_{-t}(z)|_{E_u} \leq C_\epsilon e^{-t(\mu_{\min}-\epsilon)}. \quad (2.4) \]

2.2. The case of geodesic flow. To illustrate the discussion above, let us discuss the special case of the geodesic flow of negatively curved surfaces. Let $(M,g)$ be a smooth oriented compact Riemannian surface with Gauss curvature $K(x) < 0$ and let $SM$ be its unit tangent bundle with the projection $\pi_0 : SM \to M$. We define $M = SM$ and the geodesic flow at time $t \in \mathbb{R}$ is denoted by $\varphi_t : SM \to SM$, its generating vector field is denoted by $X$ as above. The generator of rotations $R_s(x,v) := (x,e^{is}v)$ in the fibers of $SM$ is a smooth vertical vector field denoted by $V$. Let $X_\perp := [X,V]$, this is a horizontal vector field and $(X,X_\perp,V)$ is an orthonormal basis for the Sasaki metric $G$ on $SM$. We have the commutator formulas (see for example [PSU])
\[ [X,X_\perp] = -KV, \quad [V,X_\perp] = X. \quad (2.5) \]

The Jacobi equation along a geodesic $x(t) = \pi_0(\varphi_t(x,v))$ is
\[ \ddot{y}(t) + K(x(t))y(t) = 0. \quad (2.6) \]

For $(x,v) \in SM$ and $a,b \in \mathbb{R}$, one has
\[ d\varphi_t(x,v).(-aX_\perp + bV) = -y(t)X_\perp(\varphi_t(x,v)) + \dot{y}(t)V(\varphi_t(x,v)) \quad (2.7) \]
if $y(t)$ solves the Jacobi equation with $y(0) = a, \dot{y}(0) = b$. Notice that the function $r(t) = \dot{y}(t)/y(t)$ solves the Riccati equation
\[ \dot{r}(t) + r(t)^2 + K(x(t)) = 0 \quad (2.8) \]
for the times so that $y(t) \neq 0$. For $T \in \mathbb{R}$, let $y_T(t,x,v)$ be the solution of the Jacobi equation (2.6) along the geodesic $x(t) = \pi_0(\varphi_t(x,v))$ with conditions
\[ y_T(0,x,v) = 1, \quad y_T(T,x,v) = 0. \]
Since $g$ has no conjugate points, $y_T(t, x, v) \neq 0$ when $t \neq T$. Let $r_T(t, x, v) := \dot{y}_T(t, x, v)/y_T(t, x, v)$ which solves (2.8), it is defined for $t < T$ and $r_T(t, x, v) \to -\infty$ as $t \to T$. By Hopf [Ho], the following limits exist for all $t, x, v$

$$r_+(t, x, v) := -\lim_{T \to +\infty} r_T(t, x, v), \quad r_-(t, x, v) := \lim_{T \to +\infty} r_T(t, x, v).$$

We denote $r_\pm(t, x, v) := r_\pm(0, x, v)$ and we see that $r_\pm(t, x, v) = r_\pm(\varphi_t(x, v))$. We have $r_\pm > 0$ and they solve the Riccati equation on $SM$

$$\mp Xr_\pm + r_\pm^2 + K = 0. \quad (2.9)$$

The functions $r_\pm(t, x, v)$ are smooth in the $X$ direction and are globally Hölder, they are called the stable (for $r_+$) and unstable (for $r_-$) Riccati solutions. We define the vector fields

$$U_- := X_\perp - r_- V, \quad U_+ := X_\perp + r_+ V$$

Lemma 2.3. The following commutation relations hold

$$[X, U_-] = -r_- U_- , \quad [X, U_+] = r_+ U_+ ,$$

the function $r_\pm$ are in $C^2-(M)$ and

$$d\varphi_t(x, v)U_-(x, v) = e^{\int_0^t r_-(\varphi_s(x, v))ds} U_-(\varphi_t(x, v)),$$

$$d\varphi_t(x, v)U_+(x, v) = e^{\int_0^t r_+(\varphi_s(x, v))ds} U_+(\varphi_t(x, v)).$$

Proof. We just compute, using (2.5) and the fact that $r_\pm$ solves (2.9),

$$[X, X_\perp - r_- V] = -KV - X(r_-)V - r_- X_\perp = -r_-(X_\perp - r_- V) = -r_- U_-$$

and similarly for $[X, U_+]$. By (2.7), we have for each $(x, v) \in SM$

$$d\varphi_t(x, v).U_- = -y(t)X_\perp + \dot{y}(t)V$$

where $\ddot{y} + Ky = 0$ and $y(0) = -1$ and $\dot{y}(0) = -r_-(x, v)$. Clearly we have $w := \dot{y}/y$ which satisfies the Riccati equation (2.8) with $w(0) = r_-(x, v)$, thus $w(t) = r_-(\varphi_t(x, v))$. This implies

$$d\varphi_t(x, v).U_-(x, v) = -y(t)U_-(\varphi_t(x, v)).$$

and $y(t) = e^{\int_0^t r_-(\varphi_s(x, v))ds}$, and it shows $U_\pm$ are sections of $E_u$ and $E_s$. By Lemma 2.1, since $X_\perp, V$ is a smooth frame, we deduce that $r_\pm$ are in $C^2-(M)$. \hfill \Box

We remark that by Klingenberg [Kl], if the Gauss curvature satisfies $-k_0^2 \leq K(x) \leq -k_1^2$ for some $k_0 > k_1 > 0$, then there exists $C > 0$ (depending only on $k_0/k_1$) so that for each $z \in SM$

$$\forall \xi \in E_s(z), \forall t \geq 0, \quad Ce^{-k_0 t}|\xi|_G \leq |d\varphi_t(z).\xi|_G \leq Ce^{-k_1 t}|\xi|_G.$$ 

In particular this implies the bounds

$$k_0 \geq \mu_{\max} \geq \mu_{\min} \geq k_1. \quad (2.10)$$
3. Resonant states and horocyclic invariance

3.1. Analytic preliminaries. We first recall basic facts about microlocal analysis. Let \( dm := \alpha \wedge d\alpha \) be the contact measure on \( \mathcal{M} \) associated to the contact form \( \alpha \), that is invariant by the flow. We use the notation \( H^s(\mathcal{M}) \) for the \( L^2 \)-based Sobolev space of order \( s \in \mathbb{R} \), the space \( C^\gamma(\mathcal{M}) \) denotes the Banach space of Hölder functions with order \( \gamma \in \mathbb{R}^+ \setminus \mathbb{N} \); for \( k \in \mathbb{N}_0 \) we shall write \( C^k(\mathcal{M}) \) for the space of functions \( k \)-times differentiable and with continuous \( k \)-derivatives. We will write \( (C^\gamma(\mathcal{M}))' \) for their dual spaces and \( C^{\gamma-}(\mathcal{M}) = \cap_{\epsilon>0} C^{\gamma-\epsilon}(\mathcal{M}) \). We recall the embedding (see [Hö, Chapter 7.9])

\[
\text{if } \gamma \notin \mathbb{N}, \quad C^\gamma(\mathcal{M}) \subset H^s(\mathcal{M}) \text{ for } s < \gamma; \quad \text{if } k \in \mathbb{N}_0, \quad C^k(\mathcal{M}) \subset H^k(\mathcal{M}).
\]

We denote by \( \Psi^s(\mathcal{M}) \) the space of pseudo-differential operators of order \( s \in \mathbb{R} \), which have Schwartz kernel that can be written in local coordinates as

\[
K(x, x') = \frac{1}{(2\pi)^3} \int_{\mathbb{R}^3} e^{i(x-x')\xi} \sigma(x, \xi) d\xi
\]

where \( \sigma(x, \xi) \) is smooth and satisfies the following symbolic estimates of order \( s \)

\[
\forall \alpha, \beta \in \mathbb{N}^3, \exists C_{\alpha, \beta} > 0, \quad |\partial_\xi^\alpha \partial_x^\beta \sigma(x, \xi)| \leq C_{\alpha, \beta} |\xi|^{s-|\beta|}.
\]

For \( A \in \Psi^s(\mathcal{M}) \), there is a homogeneous symbol \( \sigma_p \) on \( T^*\mathcal{M} \) of order \( s \), called principal symbol, so that in local coordinates \( \sigma - \sigma_p \) is a symbol of order \( s - 1 \) outside \( \xi = 0 \). We say that \( A \) is elliptic in a conic set \( W \subset T^*\mathcal{M} \) if there is \( C > 0 \) such that \( |\sigma_p(x, \xi)| \geq C|\xi|^s \) in \( W \) for \( |\xi| > 1 \). The wave-front set of a distribution \( u \in \mathcal{D}'(\mathcal{M}) \) is the closed conic subset \( \text{WF}(u) \subset T^*\mathcal{M} \setminus \{\xi = 0\} \) defined by: \( (x_0, \xi_0) \notin \text{WF}(u) \) if and only if there is \( A \in \Psi^0(\mathcal{M}) \) elliptic in a conic open set \( W \) containing \( (x_0, \xi_0) \) such that \( Au \in C^\infty(\mathcal{M}) \). The wave-front set \( \text{WF}(A) \) of \( A \in \Psi^s(\mathcal{M}) \) is the conic closed set in \( T^*\mathcal{M} \) whose complement is the conic region where the symbol of \( A \) and its derivatives decay to infinite order as \( |\xi| \to \infty \).

3.2. Discrete spectrum in Sobolev anisotropic spaces. We recall the results of Butterley-Liverani [BuLi] and Faure-Sjöstrand [FaSj] (see also Dyatlov-Zworski [DyZw1] for similar results).

**Proposition 3.1** (Faure-Sjöstrand). Let \( X \) be a smooth vector field generating an Anosov flow on a compact manifold \( \mathcal{M} \), let \( V \in C^\infty(\mathcal{M}) \) and let \( P = -X + V \) be the associated first-order differential operator.

1) There exists \( C_0 \geq 0 \) such that the resolvent \( R_P(\lambda) := (P - \lambda)^{-1} : L^2(\mathcal{M}) \to L^2(\mathcal{M}) \) of \( P \) is defined for \( \text{Re}(\lambda) > C_0 \) and extends meromorphically to \( \lambda \in \mathbb{C} \) as a family of bounded operators \( R_P(\lambda) : C^\infty(\mathcal{M}) \to \mathcal{D}'(\mathcal{M}) \). The poles are called Ruelle resonances, the operator \( \Pi_{\lambda_0} := -\text{Res}_{\lambda=\lambda_0} R_P(\lambda) \) at a pole \( \lambda_0 \) is a finite rank projector and there exists \( p \geq 1 \) such that \( (P - \lambda_0)^p \Pi_{\lambda_0} = 0 \). The distributions in \( \text{Ran} \Pi_{\lambda_0} \) are called generalized...
resonant states and those in \( \text{Ran} \Pi_{\lambda_0} \cap \ker(P - \lambda_0) \) are called resonant states.

2) There exists \( c > 0 \) such that for each \( N \in [0, \infty) \), there exists a Hilbert space \( \mathcal{H}^N \) so that \( C^\infty(\mathcal{M}) \subset \mathcal{H}^N \subset H^{-N}(\mathcal{M}) \) and such that \( R_P(\lambda) : \mathcal{H}^N \to \mathcal{H}^N \) is a meromorphic family of bounded operators in \( \text{Re}(\lambda) > C_0 - cN \), and \( (P - \lambda) : \text{Dom}(P) \cap \mathcal{H}^N \to \mathcal{H}^N \) is an analytic family of Fredholm operators\(^3\) in that region with inverse given by \( R_P(\lambda) \).

3) For each \( N_0 > 0 \) large enough and each conic neighborhood \( W \) of \( E^*_u \), \( \mathcal{H}^N \) can be chosen in such a way that \( H^{N_0}(\mathcal{M}) \subset \mathcal{H}^N \), and for each \( A \in \Psi^0(\mathcal{M}) \) microsupported outside \( W \), one has \( Au \in H^{N_0}(\mathcal{M}) \) for all \( u \in \mathcal{H}^N \). For a resonance \( \lambda_0 \), the wave-front set of each generalized resonant state \( u \in \text{Ran}(\Pi_{\lambda_0}) \) is contained in \( E^*_u \).

The space \( \mathcal{H}^N \) is called an anisotropic Sobolev space. The statement in \cite{FaSj} is only for the case with no potential (i.e. \( V = 0 \)), but their proof applies as well to the case \( P = -X + V \) as long as \( V \in C^\infty(\mathcal{M}) \). It also follows readily from the proof of \cite{FaSj} that, if the flow of \( X \) preserves a smooth measure \( dm \) and \( V = 0 \), then one can take \( C_0 = 0 \). Indeed one has \( P^* = -P \) in \( L^2(\mathcal{M}, dm) \) in that case, thus \( \lambda \mapsto (P - \lambda)^{-1} \) is analytic in \( \text{Re}(\lambda) > 0 \) as a family of bounded operators on \( L^2(\mathcal{M}, dm) \). Moreover, the proof of \cite[Lemma 3.4]{FaSj} can be done with the constant \( C \) of that Lemma to be 0 since the operator \( \hat{P}_2 \) appearing in \cite[Section 3.3.]{FaSj} has principal symbol given by \( X(G_m) \) by using that \( P^* = -P \) with respect to the \( L^2(\mathcal{M}, dm) \) product; here \( G_m \) is the escape function of \cite{FaSj}.

For a general potential and a flow preserving a smooth measure, we will give an estimate on \( C_0 \). Let us first define the quantity

\[
V_{\text{max}} := \lim_{t \to -\infty} \sup_{z \in \mathcal{M}} \frac{1}{t} \int_t^0 V(\varphi_s(z))ds.
\]

**Lemma 3.2.** Let \( V \in C^\infty(\mathcal{M}) \) be real-valued and assume that \( X \) is a smooth vector field generating an Anosov flow preserving a smooth measure \( dm \). The resolvent \( R_P(\lambda) \) of Proposition 3.1 is analytic in \( \lambda \) as an \( L^2(\mathcal{M}) \) bounded operator in \( \text{Re}(\lambda) > V_{\text{max}} \). For each \( N > 0 \), there is \( N_0 > 0 \) such that \( R_P(\lambda) : H^{N_0}(\mathcal{M}) \to H^{-N}(\mathcal{M}) \) is a meromorphic family of bounded operators in the region \( \text{Re}(\lambda) > V_{\text{max}} - N\mu_{\text{min}} \).

**Proof.** The resolvent of \( P = -X + V \) for \( \text{Re}(\lambda) \gg 1 \) large enough is given by the expression

\[
R_P(\lambda)f = -\int_{-\infty}^0 e^{M + \int_0^s V \circ \varphi_t \, ds} f \circ \varphi_t \, dt.
\]

We see that it converges in \( L^2(\mathcal{M}, dm) \) in the region \( \{ \text{Re}(\lambda) > V_{\text{max}} \} \) by using first the estimate \( ||f \circ \varphi_t||_{L^2(dm)} = ||f||_{L^2(dm)} \) and the pointwise bounds (following from

\(^3\)Here \( \text{Dom}(P) := \{ u \in \mathcal{H}^N ; Pu \in \mathcal{H}^N \} \) is the domain of \( P \) equipped with the graph norm.
Cauchy-Schwarz)

\[ |R_P(\lambda)f(z)|^2 \leq C_{\lambda,\epsilon} \int_{-\infty}^{0} e^{\text{Re}(\lambda)t - t(V_{\max} + \epsilon)} |f(\varphi_t(z))|^2 \, dt \]

for some constant \(C_{\lambda,\epsilon}\) depending on \(\text{Re}(\lambda)\) and \(\epsilon > 0\), where \(\epsilon > 0\) can be chosen as small as we want. We next prove the second statement. First, we know from Proposition 3.1 that there is \(N_0 > 0\) such that \(R_P(\lambda)f \in H^{N_0} \subset H^{-N_0}(\mathcal{M})\) if \(f \in H^{N_0}\) and \(\text{Re}(\lambda) > V_{\max} - \mu_{\min} N\). By 3) of Proposition 3.1 and choosing appropriately the space \(H^{N_0}\), we also know that \(BR_P(\lambda)f \in H^{N_0}(\mathcal{M})\) for some large \(N_0 > 0\) if \(B \in \Psi^0(\mathcal{M})\) is elliptic outside an arbitrarily small fixed conic neighborhood of \(E_u^*\). The desired statement will be a consequence of the radial point estimates proved in Dyatlov-Zworski [DyZw2, Theorem E.56] (see also [DyZw1]). Let \(\Phi_t(z,\xi) := (\varphi_t(z), (d\varphi_t(z)^{-1})^T\xi)\) be the symplectic lift of the flow \(\varphi_t\) to \(T^*\mathcal{M}\) and let \(X\) be its vector field on \(T^*\mathcal{M}\). In [DyZw2, Theorem E.56 and the following remark], it is shown that if for \(T > 0\) large enough and for some \(s \in \mathbb{R}\), one has

\[ \int_0^T ((V - \text{Re}(\lambda)) \circ \varphi_t + sX(\log\langle\xi\rangle) \circ \Phi_t) dt < 0 \]

on \(E_u^*\) for all \(|\xi|\) large, then there is \(A, B \in \Psi^0(\mathcal{M})\) with \(WF(B) \subset T^*\mathcal{M} \setminus E_u^*\) and \(A\) elliptic near \(E_u^*\) such that, if \(u \in H^{-N_0}(\mathcal{M})\), \((P - \lambda)u \in H^s(\mathcal{M})\) and \(Bu \in H^s(\mathcal{M})\), then \(Au \in H^s(\mathcal{M})\). The quantity above can be rewritten as

\[ \int_0^T V(\varphi_t(z)) \, dt - \text{Re}(\lambda)T + s \log \left( \frac{\langle \Phi_T(z,\xi) \rangle}{\langle \xi \rangle} \right) \]

and, if \(s < 0\), this is negative on \(E_u^*\) for large \(|\xi|\) if \(\text{Re}(\lambda) > V_{\max} + s\mu_{\min}\) by using the bounds (2.4). Taking \(s = -N_0, N_0 > 0\) large enough so that \(H^{N_0}(\mathcal{M}) \subset H^{N_0}\) and applying this with \(u = R_P(\lambda)f\) and \(f \in H^{N_0}(\mathcal{M})\), we obtain \(R_P(\lambda)f \in H^{-N}(\mathcal{M})\).

3.3. Horocyclic invariance of resonant states for contact flows. In this section, we shall assume that \(\mathcal{M}\) is a 3-dimensional oriented compact manifold and \(X\) is a smooth vector field generating a contact Anosov flow, with oriented unstable bundle. Here \(dm\) will denote the contact measure and \(V \in C^1(\mathcal{M})\) is a potential. Due to the \(C^2\) regularity of \(U_-,\) for \(u \in H^{-1+\epsilon}(\mathcal{M})\) we can define \(\omega = U_-u\) as a distribution by the expression

\[ \forall f \in C^\infty(\mathcal{M}), \quad \langle U_-u, f \rangle := \langle u, -(U_-f + \text{div}(U_-)f) \rangle; \]

here \(\text{div}\) denotes the divergence with respect to \(dm\) and \(-U_- - \text{div}(U_-)\) is the adjoint to \(U_-\) with respect to \(dm\). The quantity \(\text{div}(U_-)\) is in \(C^1(\mathcal{M})\), thus if \(u\) is a generalized resonant state of \(-X, U_-u\) is well-defined as long as \(\text{Re}(\lambda) > -\mu_{\min}\) since \(u \in H^{-1+\epsilon}(\mathcal{M})\) for some \(\epsilon > 0\) in that case by Lemma 3.2.
We define the transfer operator
\[ \mathcal{L}^t : C^\infty(\mathcal{M}) \to C^\infty(\mathcal{M}), \quad (\mathcal{L}^t f)(x) := f(\varphi_t(x)) . \]
It extends as a bounded operator on \( L^2(\mathcal{M}, dm) \) with norm \( ||\mathcal{L}^t||_{L^2 \to L^2} = 1 \). If \( V \in C^1(\mathcal{M}) \) and \( P = -X + V \), we also define the operator
\[ e^{-tP} : C^1(\mathcal{M}) \to C^1(\mathcal{M}), \quad e^{-tP} f := e^{-\int_0^t \mathcal{L}^sV ds} \mathcal{L}^t f \]
satisfying \( \partial_t(e^{-tP} f) = -Pe^{-tP} f \). Let us first prove an easy Lemma.

**Lemma 3.3.** For each \( \epsilon > 0 \), there exists \( C_\epsilon > 0 \) such that for each \( s \in [-1, 1] \) and each \( t \in \mathbb{R} \), the operator \( \mathcal{L}^t \) is bounded on \( C^1(\mathcal{M}) \) with norm
\[ ||\mathcal{L}^t||_{C^1 \to C^1} \leq C_\epsilon e^{(\mu_{\text{max}} + \epsilon)|t|} \tag{3.2} \]
and on \( H^s(\mathcal{M}) \) with norm
\[ ||\mathcal{L}^t||_{H^s \to H^s} \leq C_\epsilon e^{|s|(\mu_{\text{max}} + \epsilon)|t|} . \tag{3.3} \]

**Proof.** The \( C^1 \) bound follows from the definition of \( \mu_{\text{max}} \). We have \( ||\mathcal{L}^t||_{L^2 \to L^2} = 1 \) and for each \( \epsilon > 0 \), there is \( C_\epsilon > 0 \) such that for all \( u \in C^\infty(\mathcal{M}) \) and \( x \in \mathcal{M} \)
\[ |d\mathcal{L}^t u|_{G_z} = |du(\varphi_t(x))d\varphi_t(x)|_{G_z} \leq C_\epsilon e^{(\mu_{\text{max}} + \epsilon)|t|}|du(\varphi_t(x))|_{G_{\varphi_t(x)}} \]
thus by integrating the square of this inequality on \( \mathcal{M} \) and using that \( \varphi_t \) preserves \( dm \), we get \( ||d\mathcal{L}^t u||_{L^2} \leq C_\epsilon e^{(\mu_{\text{max}} + \epsilon)|t|}||du||_{L^2} \) and \( ||\mathcal{L}^t||_{H^1 \to H^1} \leq C_\epsilon e^{(\mu_{\text{max}} + \epsilon)|t|} \). Interpolating between \( H^1 \) and \( L^2 \) we get the result for \( s \geq 0 \) and using that \((\mathcal{L}^t)^* = \mathcal{L}^{-t}\) we obtain the desired result for \( s \leq 0 \). \( \square \)

As a direct corollary, we get

**Corollary 3.4.** Let \( V \in C^1(\mathcal{M}) \) be real-valued. If \( \text{Re}(\lambda) > \mu_{\text{max}} + V_{\text{max}} \), then the resolvent \( R_P(\lambda) \) of \( P = -X + V \) is bounded as a map
\[ R_P(\lambda) : C^1(\mathcal{M}) \to C^1(\mathcal{M}) . \]

**Proof.** The resolvent of \( P = -X + V \) for \( \text{Re}(\lambda) > 0 \) is given by the expression
\[ R_P(\lambda)f = -\int_{-\infty}^0 e^{\lambda t + \int_0^t \mathcal{L}^sV ds} \mathcal{L}^t f \, dt \]
and (3.2) shows that the integral converges in \( C^1 \) norm if \( \text{Re}(\lambda) > \mu_{\text{max}} + V_{\text{max}} \). \( \square \)

Next, define the potential \( W := V - r_- \) and the quantities
\[ W_{\text{max}} := \lim_{t \to -\infty} \sup_{z \in \mathcal{M}} \frac{1}{|t|} \int_t^0 W(\varphi_s(z)) \, ds , \]
which in turn are bounded by \( W_{\text{max}} \leq V_{\text{max}} - \mu_{\text{min}} \). We obtain
Lemma 3.5. Let \( r_- \) be the function of Lemma 2.2, \( V \in C^1(M) \) and \( W := V - r_- \). The operator \( P' = -X + W \) has an analytic resolvent \( R_{P'}(\lambda) : C^0(M) \to C^0(M) \) in the region \( \{ \Re(\lambda) > W_{\text{max}} \} \), given by the convergent expression

\[
R_{P'}(\lambda) f := -\int_{-\infty}^{0} e^{\lambda t + \int_{t}^{0} L^* W ds} (L^t f) dt
\]  

(3.4)

and satisfying \( (P' - \lambda) R_{P'}(\lambda) = \text{Id} \) in the distribution sense. If \( f \in C^1(M) \), then for \( \Re(\lambda) > W_{\text{max}} + s\mu_{\text{max}} \) with \( s \in (0, 1] \), we have

\[
R_{P'}(\lambda) f \in C^{s-}(M).
\]  

(3.5)

Finally, there is no \( C^0(M) \) solution \( \omega \) to \( (P' - \lambda) \omega = 0 \) in the region \( \{ \Re(\lambda) > W_{\text{max}} \} \).

Proof. The proof of the first statement is straightforward using that for each \( \epsilon > 0 \) small, we have for \( t < 0 \) large enough and uniformly on \( M \)

\[
\int_{t}^{0} L^* W ds \leq (W_{\text{max}} + \epsilon) |t|.
\]

For the regularity (3.5), we observe that for \( s = 1 \) this follows directly from the expression (3.4) and the bound (3.2). To obtain the \( s < 1 \) case, it suffices to use interpolation (i.e Hadamard three line theorem) between the line \( \Re(\lambda) = W_{\text{max}} + 2\epsilon \) where we have \( C^0 \) bounds and the line \( \Re(\lambda) = W_{\text{max}} + \mu_{\text{max}} + \epsilon \) where we have \( C^1 \) bounds.

To prove that \( (P' - \lambda) \) is injective on \( C^0(M) \), assume \( (P' - \lambda) \omega = 0 \) and let \( \omega(t) = L^t \omega \in C^0(M) \). We have in the weak sense

\[
\partial_t \omega(t) = L^t X \omega = -L^t (r_- - V + \lambda) \omega = (L^t(W) - \lambda) \omega(t)
\]

and therefore \( \omega(t) = \omega e^{-\lambda t + \int_{t}^{0} L^* W ds} \). Since \( ||\omega(t)||_{C^0} \leq ||\omega||_{C^0} \), we can let \( t \to -\infty \) and we obtain a contradiction if \( \omega \neq 0 \). \( \Box \)

A first consequence of Lemma 3.5 is that for each \( V \in C^{2-}(M) \) there exists a function \( \alpha_V := R_{-X - r_-}(0)U_-(V) \) satisfying

\[
\alpha_V \in C^{\mu_{\text{min}}}_{\text{max}}(M), \quad (-X - r_-) \alpha_V = U_-(V).
\]  

(3.6)

This will be useful for what follows. Note also that the operator \( U_- \) is not a priori skew-adjoint with respect to the measure \( dm \): one has \( U_-^* = -U_- - \text{div}(U_-) \) where \( \text{div}(U_-) \in C^{1-}(M) \) is the divergence of \( U_- \) with respect to \( dm \). We observe that

\[
V = r_- \implies \alpha_V = \text{div}(U_-).
\]  

(3.7)

Indeed, taking the adjoint of (2.2), we have the identity of operators

\[
(-X - r_-)U_-^* = U_-^*(-X + r_-) - r_- U_-^*
\]
and therefore

\((-X - r_-(\text{div}(U_-))) = -(X - r_-)U_+^*(1) = -U_+^*(r_-) + r_+U_+^*(1) = U_-(r_-).\)

which shows (3.7). In particular we see that \(\alpha_V \in C^1(M)\) in that case and more generally for \(V = kr_-\) with \(k \in \mathbb{R}\), the regularity of \(\alpha_V\) is better than the regularity expected in (3.6). We can view the first order differential operator \(U_+ + \alpha_V\) as a connection along the unstable leaves associated to the potential \(V\).

Now we can give a short proof of the following

**Theorem 2.** Let \(V \in C^\infty(M)\), \(W := V - r_-\), \(P := -X + V\) and \(P' := -X + W\). Let \(\alpha_V\) be the function of (3.6) and let \(s \in [\mu_{\text{min}}/\mu_{\text{max}}, 1]\) be the largest number so that \(\alpha_V \in C^s(M)\). In the region \(\{\Re(\lambda) > W_{\text{max}}\}\), the operator \(R_{P'}(\lambda)(U_+ + \alpha_V) : C^\infty(M) \to D'(M)\) is analytic and one has the identity

\[(U_+ + \alpha_V)R_P(\lambda) = R_{P'}(\lambda)(U_+ + \alpha_V)\]  

in \(\{\Re(\lambda) > V_{\text{max}} - s\mu_{\text{min}}\}\). For each generalized resonant state \(u\) of \(P\) with resonance \(\lambda_0\) contained in \(\{\Re(\lambda) > V_{\text{max}} - s\mu_{\text{min}}\}\), we have \((U_+ + \alpha_V)u = 0\).

**Proof.** It suffices to prove (3.8) for \(\Re(\lambda)\) large enough and then use meromorphic continuation in \(\lambda\). Let \(u \in C^\infty(M)\) and assume that \(\Re(\lambda) > \mu_{\text{max}} + V_{\text{max}}\). By Lemma 2.2, we have

\[[-X + V, U_+ + \alpha_V] = r_-(U_+ + \alpha_V) - U_-(V) - (X + r_-)(\alpha_V) = r_-(U_+ + \alpha_V)\]

and thus

\[(-X + V - r_- - \lambda)((U_+ + \alpha_V)R_P(\lambda)u - R_{P'}(\lambda)(U_+ + \alpha_V)u) = (U_+ + \alpha_V)(-X + V - \lambda)R_P(\lambda)u - (U_+ + \alpha_V)u = 0.\]  

(3.9)

To make sense of this identity, we use Corollary 3.4 that shows that \((U_+ + \alpha_V)R_P(\lambda)u \in C^0(M)\) and Lemma 3.5 that proves that \(R_{P'}(\lambda)U_-u \in C^0(M)\). Thus

\[\omega := (U_+ + \alpha_V)R_P(\lambda)u - R_{P'}(\lambda)(U_+ + \alpha_V)u \in \text{ker}(P' - \lambda).\]

By Lemma 3.5 again, we know that there is no \(C^0\) solution to \((P' - \lambda)\omega = 0\) in \(\{\Re(\lambda) > -\mu_{\text{min}} + V_{\text{max}}\}\) thus \(\omega = 0\) and the proof of (3.8) in \(\{\Re(\lambda) > \mu_{\text{max}} + V_{\text{max}}\}\) is complete. Among the terms in (3.8), all have meromorphic extension to \(\{\Re(\lambda) > -\mu_{\text{min}}s + V_{\text{max}}\}\) as operators mapping \(C^\infty(M)\) to \(D'(M)\): indeed, if \(f \in C^\infty(M)\) then by Lemma 3.5 \(R_{P'}(\lambda)(U_+ + \alpha_V)f \in C^0(M)\) for all \(\Re(\lambda) > W_{\text{max}}\) thus for all \(\Re(\lambda) > V_{\text{max}} - s\mu_{\text{min}}\), while by Lemma 3.2 we have \(R_P(\lambda)f \in (C^s(M))'\) if \(\Re(\lambda) > V_{\text{max}} - s\mu_{\text{min}}\) is well defined. Taking the residue at a resonance \(\lambda_0 \in \{\Re(\lambda) > V_{\text{max}} - s\mu_{\text{min}}\}\) in the identity (3.8), we obtain

\[(U_+ + \alpha_V)\Pi_{\lambda_0} = 0\]
that generalized resonant states are in $\ker(U)$ if $\Pi \Pi = 1$.

The operator $\Pi$.

**Corollary 3.6.** The operator $R_{-X-r}(\lambda)U_ - : C^\infty(M) \to \mathcal{D}'(M)$ is analytic in the region $\{\text{Re}(\lambda) > -\mu_{\min}\}$ and one has in that region

$$U_- R_{-X}(\lambda) = R_{-X-r}(\lambda)U_-.$$ (3.10)

Each generalized resonant state $u$ of $-X$ with resonance $\lambda_0$ contained in the region $\{\text{Re}(\lambda) > -\mu_{\min}\}$ satisfies $U_- u = 0$.

A particular case of interest is when $V = 0$.

There are other natural cases of interest, namely when $V = k r_-$ for some $k \in \mathbb{R}$. This potential is not smooth, thus one would need a theory of resonances for operators with non-smooth coefficients. This has been developed by Butterley-Liverani [BuLi] for non-smooth flows. Even though it is not explicitly written in their paper, their technique should allow to deal with potentials $V \in C^{1+q}(M)$ for $q \in (0, 1)$. In fact, the analysis with potentials has been done carefully by Gouëzel-Liverani [GoLi] for Anosov diffeomorphisms using the same technique. Combining the methods of [BuLi, Theorem 1] for flows with the arguments of [GoLi, Proposition 4.4. and Theorem 6.4.] (taking $p = 1, q < 1$ and $\nu = 0$ in their notations, since our flow is $C^\infty(M) \subset C^{p+q+1}(M)$), one would in principle obtain the following result: for $V \in C^{1+q}(M)$ for some $0 < q < 1$ and $X$ a smooth vector field generating an Anosov flow preserving a smooth measure $dm$ in dimension 3, there exist a Banach space $B_{1,q}$ satisfying that for each $q' > q$, one has $C^1(M) \subset B_{1,q} \subset (C^q(M))^\prime$, the operator $P = -X + V$ has discrete spectrum in the region $\text{Re}(\lambda) > \rho - q\mu_{\min}$, the resolvent $R_P(\lambda) = (P - \lambda)^{-1} : B_{1,q} \to B_{1,q}$ is meromorphic there and analytic in $\text{Re}(\lambda) > \rho$. Here $\rho := \text{Pr}(V - r_-)$ is the topological pressure of the potential $V - r_-$ and $r_-$ is the function of Lemma 2.2. Since writing the proof of such result would be long, technical and not really in the scope of the paper, we just mention the expected results provided one accepts the combination of [BuLi] and [GoLi] works out: that result combined with the proof of Theorem 2 would give the following statement: for each $k \in \mathbb{R}$, the operator $R_{-X+(k-1)r_-}(\lambda)(U_- + k \text{div}(U_-)) : C^\infty(M) \to \mathcal{D}'(M)$ is analytic in the region $\{\text{Re}(\lambda) > \text{Pr}((k-2)r_-)\}$ and the following identity holds

$$(U_- + k \text{div}(U_-))R_{-X+kr_-}(\lambda) = R_{-X+(k-1)r_-}(\lambda)(U_- + k \text{div}(U_-))$$ (3.11)

in $\{\text{Re}(s) > \text{Pr}((k-1)r_-) - \mu_{\min}\}$. Each generalized resonant state of $-X + kr_-$ in that region is killed by $U_- + k \text{div}(U_-)$.

An interesting particular case is when $k = 1$ and when $k = 1/2$. In the first case, this gives that $R_{-X}(\lambda)U^* : C^\infty(M) \to \mathcal{D}'(M)$ is analytic in the region $\{\text{Re}(\lambda) > 0\}$
and one has in $\{\text{Re}(s) > h_{\text{top}} - \mu_{\text{min}}\}$

$$U^* R_{-X+r_-} (\lambda) = R_{-X} (\lambda) U^*, \quad (3.12)$$

where $h_{\text{top}} = \text{Pr}(0)$ is the topological entropy of the flow of $X$. Each generalized resonant state $u$ of $-X+r_-$ with resonance $\lambda_0$ contained in $\{\text{Re}(\lambda) > h_{\text{top}} - \mu_{\text{min}}\}$ satisfies $U^* u = 0$. When $k = 1/2$, the operator $R_{-X-\frac{1}{2}r_-} (\lambda)(U_- + \frac{1}{2}\text{div}(U_-)) : C^\infty(M) \to D'(M)$ is analytic in the region $\{\text{Re}(\lambda) > \text{Pr}(-\frac{3}{2}r_-)\}$ and the following identity holds

$$(U_- + \frac{1}{2}\text{div}(U_-)) R_{-X+\frac{1}{2}r_-} (\lambda) = R_{-X-\frac{1}{2}r_-} (\lambda)(U_- + \frac{1}{2}\text{div}(U_-)), \quad (3.13)$$

in $\{\text{Re}(\lambda) > \text{Pr}(-\frac{1}{2}r_-) - \mu_{\text{min}}\}$. Each generalized resonant state $u$ of $-X + \frac{1}{2}r_-$ with resonance $\lambda_0$ contained in $\{\text{Re}(\lambda) > \text{Pr}(-\frac{1}{2}r_-) - \mu_{\text{min}}\}$ satisfies $(U_- + \frac{1}{2}\text{div}(U_-)) u = 0$. It can be noticed that the horocyclic derivative $U_- := U_- + \frac{1}{2}\text{div}(U_-)$ is skew-adjoint with respect to the contact measure $dm$. The study of the spectrum in the case $V = \frac{1}{2}r_-$ has been done in details by Faure-Tsujii [FaTs2] using the Grassmanian extension. It is particularly interesting since the first band of resonances concentrate near $\{\text{Re}(\lambda) = 0\}$.

### 3.4. Invariant distributions for $U_-$. We recall the result of Faure-Tsujii [FaTs1] describing the localisation of Ruelle resonances. For a potential $V \in C^\infty(M)$ let us define the quantities for $k = 0, 1$

$$\gamma_k^+ := \lim_{t \to +\infty} \sup_{z \in M} \frac{1}{t} \int_0^t (V - (\frac{1}{2} + k)r_-) \circ \varphi_s(z) ds,$$

$$\gamma_k^- := \lim_{t \to +\infty} \inf_{z \in M} \frac{1}{t} \int_0^t (V - (\frac{1}{2} + k)r_-) \circ \varphi_s(z) ds.$$

In particular, when $V = 0$, this gives

$$\gamma_0^+ = -\frac{1}{2}\mu_{\text{min}}, \quad \gamma_0^- = -\frac{1}{2}\mu_{\text{max}}, \quad \gamma_1^+ = -\frac{3}{2}\mu_{\text{min}}, \quad \gamma_1^- = -\frac{3}{2}\mu_{\text{max}}.$$

**Theorem 3** (Faure-Tsujii [FaTs1]). *Let $\mathcal{M}$ be a 3-dimensional oriented manifold and let $X$ be a smooth vector field generating a contact Anosov flow and $V \in C^\infty(M)$. Then for each $\epsilon > 0$ small, there exists only finitely many resonances of $P = -X + V$ in the region

$$\{\text{Re}(\lambda) > \gamma_0^+ + \epsilon\} \setminus \{\text{Re}(\lambda) \in [\gamma_0^- - \epsilon, \gamma_0^+ + \epsilon]\}.$$

If $\gamma_1^+ < \gamma_0^-$, then there are infinitely many resonances in $\{\text{Re}(\lambda) \in [\gamma_0^- - \epsilon, \gamma_0^+ + \epsilon]\}$, with a Weyl type asymptotics. In the case $V = 0$, the condition $\gamma_1^+ < \gamma_0^-$ can be rewritten as the pinching condition $3\mu_{\text{min}} > \mu_{\text{max}}$.*

By Lemma 3.2, the generalized resonant states in $\{\text{Re}(\lambda) > -\frac{1}{2}\mu_{\text{max}} - \epsilon\}$ have the regularity $H^{-\frac{1}{2}\mu_{\text{min}} - \epsilon'}(M)$ with $\epsilon' = \epsilon/\mu_{\text{min}}$. The proof of Corollary 1.1 about the existence of infinitely many distributions in that Sobolev space that are horocyclic
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invariant (for all $\epsilon > 0$) is a direct consequence of Theorems 1 and 3, applied with $V = 0$. We notice that for general flows (not necessarily contact), existence of infinitely many resonances in some strip is proved by Jin-Zworski [JiZw] but we do not know if the associated resonant states would be regular enough to apply $U_-$. 

REFERENCES

[An] D. V. Anosov, Tangent fields of transversal foliations in $U$-systems, Mathematical Notes of the Academy of Sciences of the USSR 2 (1967), no. 5, 818–823
[BuLi] O. Butterley, C. Liverani, Smooth Anosov flows: correlation spectra and stability. J. Mod. Dyn. 1 (2007), 301–322.
[DMM] R. de la Llave, J.M. Marco, R. Moriyón, Canonical perturbation theory of Anosov systems and regularity results for the Livsic cohomology equation. Ann. of Math. (2) 123 (1986), no. 3, 537–611,
[Dy] S. Dyatlov, Spectral gaps for normally hyperbolic trapping, Ann. Inst. Fourier 66 (2016), 55–82.
[DFG] S. Dyatlov, F. Faure, C. Guillarmou, Power spectrum of the geodesic flow on hyperbolic manifolds, Analysis and PDE 8 (2015), 923–1000.
[DyZw1] S. Dyatlov, M. Zworski, Dynamical zeta functions for Anosov flows via microlocal analysis, Ann. Sci. Ec. Norm. Supér. 49 (2016), 543–577.
[DyZw2] S. Dyatlov, M. Zworski, Mathematical theory of scattering resonances, book in preparation, available at https://math.berkeley.edu/~zworski/
[FaGu] F. Faure, C. Guillarmou, Horocyclic invariance of Ruelle resonant states for contact Anosov flows in dimension 3, arXiv 1705.07965.
[FaSj] F. Faure, J. Sjöstrand, Upper bound on the density of Ruelle resonances for Anosov flows, Comm. Math. Phys. 308 (2011), no. 2, 325–364,
[FaTs1] F. Faure, M. Tsujii, Band structure of the Ruelle spectrum of contact Anosov flows, Comptes rendus Mathématique 351 (2013), 385–391.
[FaTs2] F. Faure, M. Tsujii, The semiclassical zeta function for geodesic flows on negatively curved manifolds, Invent. Math. (2016). Doi:10.1007/s00222-016-0701-5
[FaTs3] F. Faure, M. Tsujii, Band structure of the Ruelle spectrum of contact Anosov flows, in preparation.
[FlFo] L. Flaminio, G. Forni, Invariant distributions and time average of horocycle flows, Duke Math. J. 119 (2003), no 3, 465–525.
[GiLi] P. Giulietti, C. Liverani, Parabolic dynamics and Anisotropic Banach spaces. preprint arXiv:1412.7181.
[GoLi] S. Gouëzel, C. Liverani, Compact locally hyperbolic sets for smooth maps: fine statistical properties, J. Diff. Geom. 79 (2008) 433–477.
[Ha] B. Hasselblatt, Regularity of the Anosov splitting and of horospheric foliations, Ergod. Th. Dynam. Sys. 14 (1994), 645–666.
[HPS] M.W. Hirsch, C.C. Pugh, M. Shub, Invariant manifolds , Bull. Am. Math. Soc. 76 (1970), 1015–1019.
[Ho] E. Hopf, Closed surfaces without conjugate points, Proc. Nat. Acad. Sci. USA 34 (1948), 47–51.
[Hö] L. Hörmander, The Analysis of Linear Partial Differential Operators I. Distribution Theory and Fourier Analysis, Springer, 1983.
[HuKa] S. Hurder, A. Katok, Differentiability, rigidity and Godbillon-Vey classes for Anosov flows, Publications IHES, 72 (1990), 5–61.
[JiZw] L. Jin, M. Zworski, *A local trace formula for Anosov flows* (with an appendix by Frédéric Naud), Ann. Inst. Henri Poincaré (A), 18 (2017), 1–35.

[KL] W. Klingenberg, *Riemannian Geometry*, De Gruyter, Berlin-New York 1982.

[Li] C. Liverani, *On contact Anosov flows*, Ann. of Math. (2) 159 (2004), 1275–1312.

[PSU] G.P. Paternain, M. Salo, G. Uhlmann, *Tensor tomography on surfaces*. Invent. Math. 193 (2013), no. 1, 229–247.

[Ta2] M. Taylor, *Partial differential equations II*, qualitative studies of linear equations, Applied Mathematical Sciences 116, Springer.

[Ts] M. Tsujii, *Quasi-compactness of transfer operators for contact Anosov flows*. Nonlinearity 23 (2010), no. 7.

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