The Mathematical Effects of Visco-elasticity in Quasi-static Biot Models

Lorena Bociu∗ Boris Muha† Justin T. Webster‡
August 25, 2022

Abstract

We investigate and clarify the mathematical properties of linear poro-elastic systems (quasistatic Biot) with the addition of classical visco-elasticity. We precisely demonstrate the time-regularization and dissipative effects of visco-elasticity in this context. We use the full coupled presentation of such systems, as well as the framework of implicit, degenerate evolution equations, to demonstrate these effects and fully characterize poro-visco-elastic systems. We consider a simple presentation of the original displacement-pressure system (with convenient boundary conditions, etc.) for clarity in exposition across several physically-relevant parameter ranges. Clear well-posedness results are provided, with associated a priori estimates on the solutions. In addition, precise statements of admissible initial conditions in each scenario are given. This work should be of interest across several applied and mathematical communities, as described in detail below.

Keywords: poroelasticity, implicit evolution equations, strong damping, viscoelasticity

2010 AMS: 74F10, 76S05, 35M13, 35A01, 35B65, 35Q86, 35Q92

Acknowledged Support: Author 1 was supported by NSF-DMS 1555062 (CAREER). Author 2 was supported by the Croatian Science Foundation project IP-2019-04-1140. Author 3 was partially supported by NSF-DMS 1907620.

1 Introduction

In the past 10 years, there has been a rapid and intense growth of work in theoretical and numerical studies invoking the equations of poroelasticity [3, 4, 6, 8, 10, 13, 14, 21, 23, 34, 35] (to name a few). While the initial development of the mathematical theory of poro-elasticity was driven by geophysical applications [5, 18, 19, 33, 37], the recent interest in this field seems largely due to the fact that deformable porous media models describe the behavior of biological tissues, such as organs, cartilages, bones and engineered tissue scaffolds [4, 7–9, 11, 26–28, 30]. The mechanics of biological tissues typically exhibit both elastic and visco-elastic behaviors, resulting from the combined action of both elastin and collagen [25, 27, 28]. These effects can change over time, and the loss of tissue visco-elasticity is relevant to the study of several age-related diseases such as glaucoma, atherosclerosis, and Alzheimer’s disease [30]. Mathematically-oriented studies have invoked or utilized visco-elastic effects in the dynamics, owing to their analytically and numerically regularizing and dissipative properties [6, 8, 34]. Thus, there is both mathematically-driven and application-driven motivation to consider a comprehensive mathematical investigation of poro-visco-elastic systems.

In the case of purely hyperbolic-like dynamics (bulk, plate, shell, or beam elasticity), the abstract effects of visco-elasticity are well-understood and fully characterized at the abstract level [2, 15, 24]. Additionally, for traditional Biot-type systems of porous-elastic dynamics, the abstract theory with clear estimates on solutions is established, as discussed in detail below. Yet, for poro-visco-elasticity models, there seems to be no clear presentation in the literature demarcating which parabolic behaviors (dissipation and regularization) are present at an abstract level, with clear estimates on solutions. There have been recent numerical investigations into the effects of visco-elasticity in Biot type models [9, 38]. Therefore, in this article, we investigate the general...
3D linear quasi-static poro-visco-elastic systems, and clarify the time-regularization and dissipation effects of structural visco-elasticity. We fundamentally build on the constructions in [8], where weak solutions were constructed in a particular 3D case, and the subsequent 1D investigations focusing on regularity in [9,35]. Here, we provide clear well-posedness results, with estimates on the solutions and a discussion of their construction (when they are illustrative). In addition, we make precise an appropriate notion of initial conditions in each scenario. In cases where it is appropriate, we relate the system (abstractly) to an associated semigroup framework [24,29]. We believe the results below will be of interest across several applied and mathematical communities.

In particular, in this work we focus on the popular quasi-static formulation (neglecting elastic inertia) of Biot’s equations. The “full” inertial Biot system is formally equivalent to thermo-elasticity, which is well-studied [22,24]. Latter foundational works for the mathematical theory of linear poro-elasticity can be found in [1,31,39], and culminating in the more modern works [32,33]. In this traditional framework, visco-elastic effects can be considered in the displacement dynamics by invoking the so called secondary consolidation [20][34], typical for studies of clays. More recently, as described above, a growing interest in biologically-based Biot systems can also be observed [4,10,12,13]. In these bio-Biot models, visco-elastic effects are incorporated into traditional linear Biot dynamics by taking into account the visco-elastic strain and adjusting the formula for the local fluid content accordingly (depending on the specific scenario considered, either focusing on incompressible or compressible constituents). We will address both fluid content adjustments across several parameter regimes of physical interest. We do this in the spirit of the analysis in the aforementioned seminal reference [33]; we also include a remark on secondary consolidation at the end of the work in Section 4. Although we focus on linear models with constant coefficients, recent applications—which inspired the consideration of these models—are in fact nonlinear or taken with time-dependent coefficients [7,3,19].

A main focus of this work is to introduce the appropriate constituent operators to visco-elastic dynamics which have been used in abstractly describing the quasi-static Biot dynamics for some time [10,12,13]. Following this, we can “reduce” or frame the poro-visco-elastic dynamics in the context of these operators to apply, when possible, existing theory. Where it is instructive, we will provide a priori estimates and mention their justification. We will clearly exposit the mathematical features of visco-elasticity, including time regularization and dissipation, when considered in the framework of poro-elasticity. Interestingly, in some cases below, the abstract presentation of these systems reveals central features of poro-visco-elastic dynamics which are not immediately obvious in the full presentation of solutions. This is particularly true in considering which type of initial conditions are warranted for each configuration of interest, and connects the analysis thereof to appropriate ODE or semigroup frameworks.

2 Quasi-static Poro-Visco-elastic Dynamics

Let $\Omega \subset \mathbb{R}^n$ for $n=2,3$ be a bounded, smooth domain. In the traditional fully-saturated Biot system we have a pressure equation and a momentum equation; these are given in the variable $u$, describing the displacements of the solid matrix, and the homogenized pore pressure $p$. The pressure equation (resulting from mass balance) reads:

$$\zeta_t - \nabla \cdot [K \nabla p] = S. \tag{2.1}$$

The quantity $\zeta$ is the fluid-content, and in the standard Biot model of poro-elasticity it is given by

$$\zeta = c_0p + \alpha \nabla \cdot u. \tag{2.2}$$

The constant $c_0 \geq 0$ represents compressibility of the constituents, and will be considered in two regimes here: $c_0 = 0$ (incompressible) and $c_0 > 0$ (compressible) [4,10]. The coupling constant $\alpha > 0$ is referred to as the Biot-Willis constant, and, in the case of incompressible constituents $c_0 = 0$, we have $\alpha = 1$ [19]. The quantity $K(x, t)$ is the permeability tensor of the porous-elastic structure. We present it generally here (see for instance [10,13]), but for the analysis below, we will take $k = \text{const.}$—this will provide clarity as we demonstrate the mathematical structures of poro-visco-elastic systems. (This is in line with the convention in the seminal reference [33].) The fluid source function $S$ can depend on $x$ and $t$.

The momentum equation for the fluid-solid mixture is given as an elliptic $(\mu, \lambda)$-Lamé system, as driven by the pressure gradient and a source $F$:

$$-\mu \Delta u - (\lambda + \mu) \nabla \cdot u + \alpha \nabla p = F. \tag{2.3}$$

\footnote{We recall here that incompressibility of each component means that the volumetric deformation of the solid constituent corresponds to the variation of fluid volume per unit volume of porous material, i.e., $\zeta = \nabla \cdot u.$}
We will consider the body force $F$ to be spatially and temporally dependent.

**Remark 2.1.** The primal, inertial form of the elasticity equation is

$$\rho u_{tt} - \mu \Delta u - (\lambda + \mu) \nabla \nabla \cdot u + \nabla p = F.$$ 

It is instructive to remember that the Biot dynamics begin here, then neglect $\rho u_{tt} \approx 0$ to obtain the quasi-static equations of poro-elasticity [18,33].

### 2.1 Addition of Visco-elasticity: $\delta_1 > 0$

The addition of visco-elasticity into the system may be achieved through the momentum equation [8]. We refer to this here as full visco-elasticity, and follow the Kelvin-Voigt approach of including strong (or “structural”) damping $\varepsilon(v) = \frac{1}{2} \nabla \nabla + \nabla v^T$, we have the standard linear elastic strain $\sigma(v) = 2\mu \varepsilon(v) + \lambda (\nabla \cdot v) I$, we obtain the structural term

$$-\text{div} \sigma(u + \delta_1 u_t) = -[\mu \Delta + (\lambda + \mu) \nabla \text{div}](u + \delta_1 u_t)$$

in the system, where $\delta_1$ is a parameter measuring the presence of visco-elasticity (i.e., $\delta_1 = 0$ represents the standard Biot dynamics).

**Remark 2.2.** There are, of course, many ways to incorporate visco-elastic effects into the modeling of poro-elastic systems. See, for example, [35], where a viscoelastic strain is considered in the case of compressible constituents. The other components of the system are also updated there, including the formula for the fluid content.

A central point here is that the presence of visco-elasticity may affect the choice of defining the fluid content $\zeta$; there are two choices for $\zeta$ which are relevant, encapsulated by $\delta_2 = 0$ or $\delta_2 > 0$:

$$\zeta = c_0 p + \alpha \nabla \cdot u + \delta_2 \nabla \cdot u_t.$$  

When $\delta_2 = 0$, this represents the standard Biot definition of the fluid content, which prevalently appears in the literature for linearized poro-elastic systems with and without visco-elastic effects, e.g., [8,12,33]. A derivation of the model, obtained by heterogeneous mixture approach, can be found in [30], Section 12.2, for instance. In the seminal modeling reference [18], compressible constituents are permitted, and viscous effects are considered ($\delta_1 \geq 0$) with a modified fluid content ($\delta_2 > 0$). In this note, we take the approach of classifying the system and its solutions in both regimes: $\delta_2 = 0$ and $\delta_2 > 0$, noting that the application of interest can inform which selection is made.

**Remark 2.3.** Below, we discuss the possible relationship of $\delta_2 > 0$ to other physical parameters involved in the modeling as well as the construction of solutions.

In conclusion, taking $\alpha, \lambda, \mu > 0$, we consider the following poro-visco-elastic systems with $\delta_1 > 0$:

$$\begin{cases}
-\text{div} \sigma(u + \delta_1 u_t) + \alpha \nabla p = F \\
[c_0 p + \alpha \nabla \cdot u + \delta_2 \nabla \cdot u_t], - \nabla \cdot [k \nabla p] = S
\end{cases}$$

where we accommodate all possible regimes dependent on $c_0 \geq 0$, $\delta_2 = 0$, or $\delta_2 > 0$.

To (2.4) we associate the following boundary conditions:

$$u = 0, \text{ and } k \nabla p \cdot n = 0 \text{ on } \Gamma \equiv \partial \Omega.$$  

At this juncture, we note that the natural initial conditions to be specified are those quantities appearing under the time derivatives above; namely, $\delta_1 u_t(0) = u_t$ and $[c_0 p + \alpha \nabla \cdot u + \delta_2 \nabla \cdot u_t](0) = d_0$. It is immediately clear that the regime of interest and the parameter values affect the independence of these quantities. In what follows, we will precisely specify the initial quantities, their relationships, and their spatial regularities, as each are dependent on the regime under consideration and the type of solution sought.

**Remark 2.4.** A model of partial visco-elasticity, known as secondary consolidation (for soils and clays [26,33]), has appeared in the literature. In that case, for $\lambda^* > 0$, the displacement equation reads:

$$-(\mu + \lambda) \nabla \nabla \cdot u - \mu \Delta u - \lambda^* \nabla \nabla \cdot u_t + \alpha \nabla p = F$$

We remark on secondary consolidation briefly in Section 4 at the end of this note.

**For the remainder of the paper, we consider $\alpha, \mu, \lambda > 0$ to be fixed** and do not explicitly name them in theorems and subsequent discussions.
2.2 Notation

The Sobolev space of order \( s \) defined on a domain \( D \) will be denoted by \( H^s(D) \), with \( H^0_0(D) \) denoting the closure of test functions \( C_0^\infty(D) := \mathcal{D}(D) \) in the standard \( H^s(D) \) norm (which we denote by \( \| \cdot \|_{H^s(D)} \) or \( \| \cdot \|_{s,D} \)). When \( s = 0 \) we may further abbreviate the notation to \( \| \cdot \| \) denoting \( \| \cdot \|_{L^2(D)} \). Vector-valued spaces will be denoted as \( L^2(\Omega) = [L^2(\Omega)]^n \) and \( H^s(\Omega) = [H^s(\Omega)]^n \). We make use of the standard notation for the trace of a function \( w \) as \( \gamma[w] \), namely \( \gamma[\cdot] \) as the map from \( H^1(D) \) to \( H^{1/2}(\partial D) \). We will make use of the spaces \( L^2(0,T;U) \) and \( H^s(0,T;U) \), when \( U \) is a Hilbert space. Associated norms (and inner products) will be denoted with the appropriate subscript, e.g., \( \| \cdot \|_{L^2(0,T;U)} \), though we will simply denote \( L^2 \) inner products by \( \langle \cdot, \cdot \rangle \) when the context is clear. For estimates on solutions, we utilize the notation that \( A \lesssim B \) means there exists a constant \( c > 0 \) not depending on critical constants (made clear by context) so that \( A \leq cB \).

2.3 Operators, Spaces, and Solutions

Let \( V = H^1_0(\Omega) \) and \( V = H^1(\Omega) \cap L^2_0(\Omega) \). We will topologize \( V \) through the inner-product
\[
a(p, q) = \langle k \nabla p, \nabla q \rangle_{L^2(\Omega)},
\]
which gives rise to the gradient norm on \( V \); by the Poincaré-Wirtinger inequality,
\[
\| \cdot \|_V = \| k^{1/2} \nabla \cdot \|,
\]
we have a norm equivalent to the standard \( H^1(\Omega) \). Through Korn’s inequality, as well as Poincaré, we topologize \( V \) through the bilinear form:
\[
e(u, v) = \langle \sigma(u), \varepsilon(v) \rangle_{L^2(\Omega)},
\]
with \( \sigma, \varepsilon \) defined as above, leading to the norm
\[
\| \cdot \|_V = \varepsilon(\cdot, \cdot)^{1/2}.
\]

We define two central differential operator associated to the bilinear forms \( a(\cdot, \cdot) \) and \( e(\cdot, \cdot) \), with actions given by
\[
\mathcal{E}u = -\mu \Delta u - (\lambda + \mu) \nabla \nabla \cdot u; \quad Ap = -\nabla \cdot [k \nabla p] = -k \Delta p. \tag{2.9}
\]

Invoking the smoothness of \( \Omega \) (and standard elliptic regularity), we characterize the domain \( \mathcal{D}(\mathcal{E}) = H^2(\Omega) \cap V \), which corresponds to homogeneous Dirichlet conditions for the elastic displacements. Similarly, we take \( \mathcal{D}(A) = \{ p \in H^2(\Omega) \cap L^2_0(\Omega) : \gamma_1[p] = 0 \} \) (where \( \gamma_1 \) is the normal trace). Here, \( \mathcal{E} \) realizes an isomorphism in two contexts: \( \mathcal{D}(\mathcal{E}) \to L^2(\Omega) \) and \( V \to V' \), where \( \mathcal{E}^{-1} \) is interpreted appropriately (i.e., through its natural coercive bilinear form \( e(\cdot, \cdot) \)) \[10,17,33\]. Similarly, \( A : \mathcal{D}(A) \to L^2(\Omega) \) or \( V \to V' \) is an isomorphism. See \[10\] for precise details.

Lastly, we define the nonlocal, zeroth order pressure-to-dilation mapping as follows:
\[
B = -\nabla \cdot \mathcal{E}^{-1} \nabla. \tag{2.10}
\]

As a mapping on \( L^2(\Omega) \), \( B \) is central to many abstract analyses of Biot \[8,13,14\]. We state its relevant properties as a lemma coming from \[10\] in the specific context of \( L^2_0(\Omega) \) and \( V = H^1(\Omega) \cap L^2_0(\Omega) \):

**Lemma 2.1.** The operator \( B \in \mathfrak{L}(L^2_0(\Omega)) \cap \mathfrak{L}(V) \). \( B \) is an isomorphism on \( L^2_0(\Omega) \) and is injective on \( V \). Finally, we have that \( B \) is a self-adjoint, monotone operator when considered on \( L^2_0(\Omega) \).

Finally, we conclude with a definition of weak solutions for \( 2.6 \) which will be valid in all parameter regimes, and is consistent with the abstract definition given in the Appendix for \( 6.1 \). Recall that the fluid content is given by \( \zeta \equiv c_0 p + a \nabla \cdot u + \delta_2 \nabla \cdot u \), which is of course dependent on the nonnegative parameters \( c_0 \) and \( \delta_2 \).

**Definition 1.** Let \( c_0, \delta_1, \delta_2 \geq 0 \). We say that \( (u, p) \in L^2(0,T;V \times V) \), such that \( \delta_1 u \in H^1(0,T;V) \) and \( \zeta \in L^2(0,T;V') \), is a weak solution to problem \( 2.6 \) if:

- For every pair of test functions \( (v, q) \in V \times V \), the following equality holds in the sense of \( \mathcal{D}'(0,T) \):
\[
e(u, v) + \frac{d}{dt} e(u, v) + (\alpha \nabla p, v)_{L^2(\Omega)} + \frac{d}{dt} \langle \zeta, q \rangle_{L^2(\Omega)} + a(p, q) = \langle F, v \rangle_{V' \times V} + \langle S, q \rangle_{V' \times V}. \tag{2.11}
\]
• The initial conditions \( \zeta(0) = d_0 \) and \( \delta_1 \mathbf{u}(0) = \mathbf{u}_0 \) are satisfied in the sense of \( C([0, T]; \mathbb{V}') \) and \( C([0, T]; \mathbb{V}) \), respectively.

There are many notions of “stronger” solutions to Biot-type systems in the literature (see [33], for instance). To avoid confusion with notions of strong or classical solutions coming from other references, notions of stronger solution will be discussed in this paper in the sense of weak solutions with additional regularity. Depending on the regularity of the sources in various cases, we will comment on when the PDEs hold in a point-wise sense.

\subsection{Biot Solutions: \( \delta_1 = \delta_2 = 0 \)}

We begin with a discussion of classical Biot dynamics in order to establish a baseline for comparison with our results below on porous-visco-elastic dynamics. Consider now the quasi-static Biot dynamics—in the absence of visco-elastic effects—given in the operator-theoretic form by

\begin{equation}
\begin{aligned}
\mathcal{E} \mathbf{u} + \alpha \nabla p &= \mathbf{F} \in H^1(0, T; \mathbb{V}) \\
[\mathcal{E} \mathbf{u}]_t + Ap &= S \in L^2(0, T; \mathbb{V}') \\
[\mathcal{E} \mathbf{u}]_t(0) &= d_0 \in \mathbb{V}.
\end{aligned}
\tag{2.12}
\end{equation}

From the established theory (see the Appendix), one seeks weak solutions in the class

\[ (\mathbf{u}, p) \in L^2(0, T; \mathbb{V} \times \mathbb{V}) \]

Formally solving the elasticity equation a.e. \( t \) as \( \mathbf{u} = -\alpha \mathcal{E}^{-1} \nabla p + \mathcal{E}^{-1} \mathbf{F} \), and relabeling the source

\[ S \mapsto S - \nabla \cdot \mathcal{E}^{-1} \mathbf{F}_t \equiv \tilde{S}, \]

we obtain the reduced, implicit equation:

\[ [\mathcal{B} p]_t + Ap = \tilde{S} \in L^2(0, T; \mathbb{V}'), \quad [\mathcal{B} p](0) = d_0 \in \mathbb{V}', \quad \text{where} \quad \mathcal{B} = (\alpha_0 I + \alpha^2 B). \tag{2.13} \]

The above system can, in principle, degenerate if \( \alpha_0 = 0 \) and \( B \) has a non-trivial kernel [10]—though this will not be the case here. Indeed, in this work, \( \mathcal{B} \) is invertible on \( L^2_0(\Omega) \) independent of \( \alpha_0 \).

\textbf{Remark 2.5}. We note that the temporal regularity of \( \mathbf{F} \) is directly invoked in the reduction step. Namely, to consider \( \tilde{S} \) as a given RHS for the abstract degenerate equation, we must at least require that \( \nabla \cdot \mathcal{E}^{-1} \mathbf{F}_t \in L^2(0, T; \mathbb{V}') \); this provides consistency with the original source, \( S \). Additionally, to “solve” the elasticity equation for \( \mathbf{u} \) (given \( p \) and \( \mathbf{F} \)) we will require \( \mathbf{F} \in \mathbb{V}' \) a.e. \( t \) to obtain \( \mathbf{u} \in \mathbb{V} \), accordingly. (Smotherer considerations below will require additional spatial regularity for \( \mathbf{F} \) and \( \mathbf{F}_t \).) Regularity of \( \mathbf{F}_t \) is at issue for the analysis of Biot’s dynamics [10] and the analysis below.

We provide the application of the general theory developed in [1] and [33] for Biot’s dynamics, and adapted recently in [10][13]. This provides contrast with our results in the sequel for \( \delta_1 > 0 \).

\textbf{Theorem 2.2}. Let \( d_0 \in L^2_0(\Omega) \), \( \mathbf{F} \in H^1(0, T; \mathbb{V}') \), \( S \in L^2(0, T; \mathbb{V}') \), and \( \alpha \geq 0 \). Then there exists a unique weak solution with \( (\mathbf{u}, p) \in C([0, T]; \mathbb{V}) \times L^2(0, T; \mathbb{V}) \) to (2.12). Moreover, any weak solution satisfies the following energy estimate:

\[ \|\mathbf{u}\|_{L^\infty(0, T; \mathbb{V})}^2 + c_0 \|\mathbf{u}\|_{L^2(0, T; L^2(\Omega))}^2 + \|p\|_{L^2(0, T; \mathbb{V})}^2 \lesssim \|d_0\|_{L^2(\Omega)}^2 + \|S\|_{L^2(0, T; \mathbb{V}')}, \quad C(T) \|\mathbf{F}\|_{H^1(0, T; \mathbb{V}')}. \tag{2.14} \]

In all cases \( \alpha \geq 0 \) the dynamics are parabolic. In particular, if \( S \equiv 0 \) and \( \mathbf{F} \equiv 0 \), we have:

\[ \|Ap\|_{L^\infty(0, T; L^2(\Omega))} + \|\mathcal{E} \mathbf{u}\|_{L^\infty(0, T; L^2(\Omega))} + \|\nabla \cdot [\mathcal{E} \mathbf{u}]\|_{L^\infty(0, T; L^2(\Omega))} \lesssim \frac{\|d_0\|_{L^2(\Omega)}}{T}, \tag{2.15} \]

\textit{to which elliptic regularity for \( A \) and \( \mathcal{E} \) can then be applied (as in [33]).}

\textbf{Proof Sketch of Theorem 2.2}. We remark that the theorem above can be obtained in two steps: First, weak solutions can be constructed directly (for instance through Galerkin method or the theory in the Appendix—see also [10][13][14][33].) The particular constructed solution will satisfy the energy identity [10][32]. Then, using a classical argument which invokes a test function of the form \( \int p \, d s \) in the reduced pressure equation (2.13) (see [32] pp.116–117), we can conclude that weak solutions are unique.

\[ \blacksquare \]
Remark 2.6. The issue of uniqueness is much more subtle in the case of time-dependent coefficients, as the above approach depends on the ability to perform temporal integration by parts on the term $Ap$. See the detailed discussion in [10].

Remark 2.7. Above, we have the ability to specify only the quantity $d_0 \in L^2_0(\Omega)$—rather than a pair $(u_0, d_0)$ or $(p_0, d_0)$, so that $c_0 p_0 + \nabla \cdot u_0 = d_0$. Indeed, given $d_0 \in L^2_0(\Omega)$ in this framework and recalling that $Bp(0) = [(c_0 I + B)p](0) = c_0 p_0 + \nabla \cdot u_0$:

$$d_0 \in L^2_0(\Omega) \implies Bp(0) \in L^2_0(\Omega) \implies p(0) \in L^2_0(\Omega) \implies \mathcal{E}u(0) \in V' \implies u(0) \in V.$$

In the case when the operator $B$ is not invertible on a chosen state space and $c_0 = 0$ (e.g. $L^2(\Omega)$), the issue can be more subtle. See [10] for more discussion, as well as the original papers [1] [53]. The equivalence of $p_0$ and $d_0$ will not necessarily be available when $\delta_1 > 0$ and an additional time derivative is present in the equations.

We now briefly describe the notion of smooth solution for the classical Biot dynamics above, when the data are smooth. These results can be obtained through elliptic regularity for $\mathcal{E}$ and $A$ on $L^2(\Omega)$, and formal a priori estimates via the weak form, or via the implicit semigroup formulation as applied to Biot’s dynamics in [53] Theorems 3.1 and 4.1. We first invoke the properties of $B$ in the context of [53] to obtain the chain:

$$d_0 \in V \implies Bp(0) \in V \implies p(0) \in V \implies \mathcal{E}u(0) \in L^2(\Omega) \implies u(0) \in \mathcal{D}(\mathcal{E}).$$

Via the standard methodology for parabolic dynamics equation, choosing stronger initial data yields additional regularity.

**Theorem 2.3.** If $d_0 \in V$, with $S \in H^1(0, T; V')$ and $F \in H^2(0, T; V')$, there exists a unique weak solution with the additional regularity:

$$p \in H^1(0, T; L^2_0(\Omega)) \cap L^\infty(0, T; V) \text{ and } u \in H^1(0, T; V).$$

If, in addition, $S \in L^2(0, T; L^2(\Omega))$ and $F \in H^1(0, T; L^2(\Omega))$, this solution satisfies (2.12) a.e. $t$ and a.e. $x$ and we obtain, in addition, the regularities:

$$u \in L^2(0, T; \mathcal{D}(\mathcal{E})), \quad p \in L^\infty(0, T; \mathcal{D}(A)).$$

If $F \in L^\infty(0, T; L^2(\Omega))$, then $u \in L^\infty(0, T; \mathcal{D}(\mathcal{E})).$

**Remark 2.8.** We note that, solutions of higher regularity—for instance considering $d_0$ or $p_0 \in \mathcal{D}(A)$—can be considered; however, one must address certain commutators associated to boundary conditions encoded in $A$ and $B$. This can be seen, for instance, in attempting to test (2.13) with $Ap_t$.

For completeness, we provide the formal identities which give rise to the solutions above.

**Proof Sketch of Theorem 2.3.** Consider the reduced form of the Biot equation,

$$[Bp]_t + Ap = \tilde{S} \in H^1(0, T; V'),$$

with $B = [c_0 I + \alpha^2 B]$, as defined in Section 2.3 and

$$\tilde{S} \equiv S - \nabla \cdot \mathcal{E}^{-1} F_t \in H^1(0, T; V').$$

Consider a smooth solution (as for finite dimensional approximants) and test with $p_t$ to obtain:

$$\frac{k}{2} \|\nabla p(T)\|^2 + \int_0^T (Bp_t, p_t) dt = \frac{k}{2} \|\nabla p(0)\|^2 + \langle \tilde{S}(0), p(0) \rangle_{V' \times V}$$

$$+ \langle \tilde{S}(T), p(T) \rangle_{V' \times V} + \int_0^T \langle \tilde{S}_t, p \rangle_{V' \times V} dt.$$ (2.16)

Alternatively, if we assume that $\tilde{S} \in L^2(0, T; L^2_0(\Omega))$—which follows from the alternative assumptions above—then the identity is similar:

$$\frac{k}{2} \|\nabla p(T)\|^2 + \int_0^T (Bp_t, p_t) dt = \frac{k}{2} \|\nabla p(0)\|^2 + \int_0^T \langle \tilde{S}, p_t \rangle dt$$ (2.17)
In both situations, the assumed regularity of the data is sufficient to estimate the RHS and obtain an estimate on \( p \).

With regularity of the pressure \( p \) in hand, we consider the full system in (2.12) and formally differentiate the elasticity equation (2.12). This yields:

\[
\begin{align*}
\mathcal{E} \mathbf{u}_t + \alpha \nabla p_t &= \mathbf{F}_t \in L^2(0, T; \mathbf{V}') \\
[c_0 p + \alpha \nabla \cdot \mathbf{u}]_t + Ap &= S \in L^1(0, T; L^2_0(\Omega)) \quad \text{or} \quad H^1(0, T; \mathbf{V}')
\end{align*}
\]  

(2.18)

We can test the first equation by \( \mathbf{u} \) and the second by \( p_t \) and add to obtain the identity:

\[
\epsilon(\mathbf{u}_t, \mathbf{u}_t) + c_0 ||p_t||^2 + \frac{k}{2} \frac{d}{dt} ||\nabla p||^2 = \langle \mathbf{F}_t, \mathbf{u}_t \rangle_{\mathbf{V}' \times \mathbf{V}} + \langle S, p_t \rangle_{L^2(\Omega)},
\]  

(2.19)

where, of course, we have assumed the case \( S \in L^2(0, T; L^2_0(\Omega)) \). (The appropriate modifications are clear for the other case, as in (2.16) and (2.17) above.) The RHS can be estimated, with the assumed regularities of \( \mathbf{F}_t \) and \( S \). The additional regularities in the theorem are then read off from the individual equations in (2.12).

2.5 Visco-elastic Cases of Interest

In considering full poro-visco-elasticity, we will take \( \delta_1 > 0 \), and retain the parameter to track its dependencies. We consider the independent cases:

- compressible \( c_0 > 0 \) and incompressible \( c_0 = 0 \) constituents;
- standard fluid content \( \delta_2 = 0 \), as well as adjusted fluid content \( \delta_2 > 1 \).

This yields four cases of interest for:

\[
\begin{align*}
\mathcal{E} \mathbf{u} + \delta_1 \mathcal{E} \mathbf{u}_t + \alpha \nabla p &= \mathbf{F} \in H^1(0, T; \mathbf{V}') \\
[c_0 p + \alpha \nabla \cdot \mathbf{u} + \delta_2 \nabla \cdot \mathbf{u}_t]_t + Ap &= S \in H^1(0, T; \mathbf{V}')
\end{align*}
\]  

(2.20)

In line with the analysis in the sequel, we will possibly require additional temporal regularity on the volumetric source \( S \) here and additional regularity for \( \mathbf{F} \) beyond what is specified above. It is natural (and analogous to (2.12)) to take initial conditions of the form

\[ [c_0 p + \alpha \nabla \cdot \mathbf{u} + \delta_2 \nabla \cdot \mathbf{u}_t](0) = d_0 \in \mathbf{V}', \quad \text{and} \quad \delta_1 \mathcal{E}(\mathbf{u}(0)) \in \mathbf{V}'. \]  

(2.21)

However, as alluded above, we will discuss initial conditions more precisely on a case by case basis. In fact, a main feature of our subsequent analysis is in addressing this point. Which initial quantities can be specified depends on the specific parameter regime \( (\delta_1, \delta_2, c_0 \geq 0) \), of course being mindful of possible over-specification. Owing to the time-derivatives present in both equations—in contrast to Biot’s traditional equations (2.12)—we will be unable to circumvent the need to specify two initial quantities; however the relationship between them will be an interesting question below.

Remark 2.9 (Summary of Initial Conditions). We now provide a summary of some proper specifications of initial conditions, with justifications to follow in the appropriate sections. Of course, there are questions of scaling and regularity of these conditions. Such matters will translate into the sought-after notion of solution. Though a natural quantity, as discussed above and present in the definition of weak solution, is the fluid content. Here, we relegate our summary to the two primal variables: \( (p, \mathbf{u}) \) with possible initial conditions \( (p(0), \mathbf{u}(0)) \), of course possibly related through the quantity \( d_0 = [c_0 p + \alpha \nabla \cdot \mathbf{u} + \delta_2 \nabla \cdot \mathbf{u}_t](0) \).

The proper initial conditions for (2.20) are given below. We take \( \delta_1 > 0 \) and consider \( c_0 \geq 0, \delta_2 \geq 0 \).

| \( \delta_2 = 0 \) | \( c_0 = 0 \) | \( \delta_2 > 0 \) | \( c_0 > 0 \) |
|-----------------|-----------------|-----------------|-----------------|
| \( p(0) \) or \( p(0), \mathbf{u}(0) \) | \( (p(0), \mathbf{u}(0)) \) or \( (p(0), p_t(0)) \) | \( p(0) \) or \( (p(0), \mathbf{u}(0)) \) |

Remark 2.10. There are physical restrictions about the permissibility of certain parameter combinations. For instance, in the application to biological tissues, when \( \alpha = 1 \) and \( c_0 = 0 \) [30], the parameter \( \delta_2 \) should be nullified [35]. A derivation of the relevant model, obtained by heterogeneous mixture approach, can be found in [30] Section 12.2. This is to say, the combination \( \delta_1, \delta_2 > 0, c_0 = 0 \) may not be physically relevant, however, in this mathematically-oriented work we accommodate all mathematically permissible combinations and describe the features of the resulting dynamics.
In each case described in Section 3, we provide: a discussion of the dynamics, a state reduction, a discussion of initial conditions, and, finally, a well-posedness theorem with estimates.

3 Poro-Visco-elastic System and Reduction

We now consider (2.20) with \( \delta_1 > 0 \) and note that the boundary conditions are embedded in the operators \( A \) and \( \mathcal{E} \). In addition to analyzing the full system, we will reformulate the model abstractly to apply established results and obtain well-posedness of the dynamics in a variety of functional frameworks. When it is instructive, we provide corresponding estimates on solutions and describe the resulting constructions of weak and strong solutions.

3.1 Case 1: Traditional Fluid Content, \( \delta_2 = 0 \)

Take \( \delta_2 = 0 \) in (2.20). We begin with formal discussions of weak solutions for the full system, and the proceed to the abstract system reduction. After both of these, we proceed to rigorous statements concerning solutions.

Let us begin by describing the energy identity for the full dynamics, before any system reduction is made. Recall the system:

\[
\begin{cases}
\mathcal{E} u + \delta_1 \mathcal{E} u_t + \alpha \nabla p = F \in H^1(0, T; \mathcal{V}'), \\
[\alpha p + \alpha \nabla \cdot u]_t + Ap = S \in H^1(0, T; \mathcal{V}').
\end{cases}
\] (3.1)

From this, we have (following the construction in [8]) the estimate:

\[
\| u \|_{L^2(0, T; \mathcal{V})} + c_0 \| p \|_{L^2(0, T; L^2(\Omega))} + \delta_1 \| u_t \|_{L^2(0, T; \mathcal{V})} + \| p \|_{L^2(0, T; \mathcal{V})} \\
\lesssim c_0 \| p_0 \| + \| u_0 \| + \| S \|_{L^2(0, T; L^2(\Omega))} + \| F \|_{L^2(0, T; L^2(\Omega))}.
\] (3.2)

Several comments are in order:

- We have not yet invoked any additional regularity of the sources \( S \) or \( F \); indeed this will be done later.
- Even above, one can replace the norm on \( F \) as follows: \( \| F \|_{L^2(0, T; L^2(\Omega))} \rightarrow \| F \|_{H^1(0, T; \mathcal{V})} \). In this case, the constant on the RHS corresponding to \( \lesssim \) becomes dependent on time, i.e., \( \text{LHS} \leq C(T) \text{RHS} \).
- Note, from the RHS, \( d_0 = [\nabla \cdot u + c_0 p](0) \) does not explicitly appear; on the other hand, any two of the three of \( u(0) = u_0, p(0) = p_0, \) or \( d_0 \) can be specified, and the third obtained immediately.

Now, we proceed to obtain an abstract reduction of (3.1). From the elasticity equation, we have:

\[
\delta_1 u_t + u = \mathcal{E}^{-1} F - \alpha \mathcal{E}^{-1}(\nabla p),
\] (3.3)

which can be explicitly solved for \( u \) as an ODE in \( t \) for \( a.e. \) \( x \) (see Lemma 3.3). We may then differentiate the pressure equation:

\[
c_0 p_t + \alpha \nabla \cdot u + Ap = S_t \in L^2(0, T; \mathcal{V}').
\]

We can then rewrite \( \nabla \cdot u_t \) through the time derivative of (2.20)_1:

\[
\delta_1 \nabla \cdot u_t = -\nabla \cdot u_t + \nabla \cdot \mathcal{E}^{-1}(F_t) + \alpha Bp_t = \alpha^{-1} [c_0 p_t + Ap] - \alpha^{-1} S + \nabla \cdot \mathcal{E}^{-1}(F_t) + \alpha Bp_t.
\]

Recalling \( B = -\nabla \cdot \mathcal{E}^{-1} \nabla \), and taking

\[
\hat{S} = \delta_1^{-1} [S - \alpha \nabla \cdot \mathcal{E}^{-1}(F_t)] + S_t,
\] (3.4)

we finally have the reduced pressure equation:

\[
c_0 p_t + [A + \delta_1^{-1}(c_0 I + \alpha^2 B)] p_t + \delta_1^{-1} Ap = \hat{S}.
\] (3.5)

We again make several observations:
When \(c_0 > 0\), we note *strong damping* in a hyperbolic-type equation \([24]\) (and references therein). The damping operator \(D : V \to V'\) (or from \(D(A) \to L^2(\Omega)\)) is given by

\[
D \equiv A + \delta_1^{-1}(c_0 I + \alpha^2 B) = A + \delta_1^{-1}B.
\]  

(3.6)

This operator is nonlocal but has the property of being \(A\)-bounded in the sense of \([13, p.17]\) (see also \([24]\)). Formally, \(D\) being \(A\)-bounded means that \(D\) acts as \(A\) does in its dissipative properties. We provide an explicit choice of definition for weak solutions to \((3.5)\) below.

- There is clear singular behavior in the equation when \(\delta_1 \searrow 0\).
- In obtaining the reduced equation, we have differentiated the fluid source, \(S\). We then see the requirement that \(S \in H^1(0,T; \mathcal{V}')\) and also, consequently, again need \(F \in H^1(0,T; \mathcal{V}')\) when invoking the *abstract form in \((3.5)\).*
- It is not obvious at this stage what the connection is between the primally specified initial quantities, \(u(0)\) and \(d_0\), and those required for the strongly damped wave equation, \(p(0)\) and \(p_t(0)\), when \(c_0 > 0\); we resolve this clearly below, with different theorems for different frameworks.
- The formal energy identity for the above “reduced” dynamics can be written:

\[
\frac{1}{2} \frac{d}{dt} \left[ c_0 \|p_t\|^2 + \frac{1}{\delta_1} a(p,p) \right] + a(p_t,p_t) + \delta_1^{-1} \langle \mathcal{B}p_t,p_t \rangle_{L^2(\Omega)} = \langle \mathcal{S}, p_t \rangle_{V' \times V}. \tag{3.7}
\]

Now, we present a well-posedness and regularity theorem (in two parts) which is independent of \(c_0 \geq 0\). In Theorem 3.1 we approach the problem through the full formulation and provide a priori estimates and various regularity assumptions on the data. Secondly, in the following sections, we will consider \(c_0 > 0\) and \(c_0 = 0\) separately. For the full system with \(c_0 > 0\), we have Theorem 3.2 For the reduced formulation, we provide Theorem 3.3 which is achieved through the second-order semigroup theory, and requires specification of both \((p(0), p_t(0))\); from the obtained solution, we infer regularity about the “natural” initial quantities. Following these theorems, we may then compare the resulting regularity of the produced solutions.

**Theorem 3.1.** Consider \(c_0 \geq 0\) in (3.1). Let \(S \in L^2(0,T; \mathcal{V}'), \ F \in H^1(0,T; \mathcal{V}') \cup L^2(0,T; L^2(\Omega))\). Take \(u_0 \in \mathcal{V}\) and \(c_0 p_0 \in L^2_0(\Omega)\).

[Part 1] Then there exists unique (finite energy, as in 3.2 above) weak solution \(p \in L^2(0,T; \mathcal{V}), \ u \in H^1(0,T; \mathcal{V}), \) and \([c_0 p + \alpha \nabla \cdot u]_t \in L^2(0,T; \mathcal{V}')\).

From these regularities we infer

- \(u \in C([0,T]; \mathcal{V})),\)
- \(c_0 p_t \in L^2(0,T; \mathcal{V}'),\)
- \(c_0 p \in C([0,T]; L^2_0(\Omega)).\)

[Part 2] Now take \(p_0 \in \mathcal{V}\) and \(u_0 \in \mathcal{D}(\mathcal{E}).\) Assume, in addition, that \(F \in H^1(0,T; L^2(\Omega))\) and \(S \in H^1(0,T; \mathcal{V}') \cap L^2(0,T; L^2_0(\Omega)).\) Then the above weak solution has the additional regularity that

\(p \in H^1(0,T; L^2(\Omega)) \cap L^\infty(0,T; \mathcal{V}') \cap L^2(0,T; \mathcal{D}(A))\) and \(u_t \in L^{\infty}(0,T; \mathcal{V}).\)

Such solutions satisfy the system (3.1) a.e. \(x\) a.e. \(t\).

Finally, if \(F \in L^2(0,T; \mathcal{V}),\) then we have \(\partial_t \mathcal{E} u \in H^1(0,T; \mathcal{V})\) and \(u \in H^2(0,T; \mathcal{V}).\)

**Remark 3.1.** In Part 1 of the Theorem above, there is not enough regularity to infer any spatial regularity of \(p_t(0)\) from the equation. In Part 2, however, we can read off the regularity of \(p_t(0) \in \mathcal{V}'\) directly from the pressure equation, since \(S(0) \in \mathcal{V}'\) is defined for \(S \in H^1(0,T; \mathcal{V}'):\)

\[
c_0 p_t(0) = S(0) - Ap(0) - \alpha \nabla \cdot u_t(0) \in \mathcal{V}'.\]

We also obtain an initial condition for \(u_t(0) \in \mathcal{D}(\mathcal{E})\) in Part 2 of the theorem via the elasticity equation and the regularity of \(p_0, u_0,\) and \(F.\)
Remark 3.2. As can be seen from the proof below, when \( c_0 > 0 \), we do not require \( S \in H^1(0,T;V') \). It would be sufficient, to obtain estimates, to only assume \( S \in L^2(0,T;L^2_0(\Omega)) \).

Proof of Theorem 3.7. The construction of solutions in Part 1 of the theorem follows a standard approach through the baseline energy inequality in (3.2). (For instance, see the construction given through full spatio-temporal discretization given in [8].) Uniqueness is obtained straightforwardly, as, for a weak solution, the function \( u_t \in L^2(0,T;V) \) can be used as a test function in the elasticity equation (3.1). This implies that any weak solution satisfies the energy inequality (3.2). As the problem is linear, uniqueness follows.

To obtain Part 2, we simply point to the requisite a priori estimate for higher regularity. This estimate can be obtained in the discrete or semi-discrete framework (for instance on Galerkin approximants, as in [13]), and the constructed solution will satisfy the resulting estimate. Uniqueness at the level of weak solutions remains.

To obtain the a priori estimate, differentiate test (3.1) in time and test with \( u_t \) and (3.1) with \( p_t \). This produces the identities

\[
\frac{\delta_1}{2} \frac{d}{dt} e(u_t, u_t) + e(u_t, u_t) + \alpha(\nabla p_t, u_t) = (F_t, u_t) \quad (3.8)
\]

\[
c_0 ||p_t||^2 + \alpha(\nabla \cdot u_t, p_t) + \frac{k}{2} \frac{d}{dt} ||\nabla p||^2 = (S, p_t). \quad (3.9)
\]

We note that, from the second identity, we can treat the term \((S, p_t)\) as an inner product (and absorb it) when \( c_0 > 0 \). However, to have a result which is independent of \( c_0 \geq 0 \), we relax the regularity below. Doing so, adding, and integrating in time, we obtain:

\[
\frac{\delta_1}{2} e(u(T), u(T)) + \frac{k}{2} ||\nabla p(T)||^2 + \int_0^T [e(u_t, u_t) + c_0 ||p_t||^2] dt
\]

\[
= \frac{\delta_1}{2} e(u(0), u(0)) + \frac{k}{2} ||\nabla p_0||^2 + \int_0^T (F_t, u_t) dt - \int_0^T \langle S, p \rangle V \cdot V dt + \langle S, p \rangle V \cdot V \bigg|_{t=0}^{t=T}.
\]

The RHS and the data \( S, F, \) and \( p_0 \) have appropriate regularity to control the LHS. We note, of course, that at \( t = 0 \) we have:

\[ E(u(0)) = -\alpha \nabla p_0 - E(u_0) + F(0); \]

which is bounded in \( V' \). Since \( E: V \to V' \) is identified as the Riesz Isomorphism, this gives that

\[ ||u_t(0)||_V \lesssim ||p_0||_V + ||u_0||_V + ||F||^2_{H^1(0,T;V')} \]

From the above, we obtain that solutions are bounded in the sense of \( u_t \in L^\infty(0,T;V) \) and \( p \in L^\infty(0,T;V) \cap H^1(0,T;L^2_0(\Omega)) \). The remaining regularities are read off from the equations (3.1) for the data as prescribed respectively in the statement of Theorem 3.1. \( \square \)

Remark 3.3. Another way to obtain regularity here—which yields equivalent results—is to invoke the multiplier \( E(u_t) \) in the elasticity equation (3.1). Noting that the divergence operator commutes with the Laplacian, we obtain the needed cancellation in cross-terms to observe \( u \in H^1(0,T;D(E)) \), \( p \in L^\infty(0,T;V) \cap L^2(0,T;D(A)) \), having specified initial conditions \( p_0 \in V, u_0 \in D(E) \).


3.1.1 Compressible Constituents, \( c_0 > 0 \)

The equation (3.5) above, with \( c_0 > 0 \), is hyperbolic-like with strong damping; this renders the entire system parabolic [15, 24], with associated parabolic estimates. The damping operator \( D \) defined in (3.6) is \( A \)-bounded in the sense of [15]. We elaborate below through several additional theorems. Before doing so, let us provide a clear definition of weak solutions to the reduced wave equation (3.5). Namely, when \( c_0 > 0 \), we define \( p \) to be a weak solution to

\[ c_0 p_{tt} + [A + \delta_1^{-1} B] p_t + \delta_1^{-1} A p = \tilde{S}, \]

with \( \tilde{S} = \delta_1^{-1} [S - \alpha \nabla \cdot E^{-1}(F_t)] + S_t \) and \( B = c_0 I + \alpha^2 B \), through the following definition.
Definition 2 (Weak Solution of Reduced, Strongly-Damped Wave). Let $\hat{S} \in L^2(0, T; V')$. A weak solution of $S$ is a function $p \in H^1(0, T; V) \cap H^2(0, T; V')$ such that for a.e. $t > 0$ and all $q \in V$, one has

$$\langle c_0 p_t, q \rangle_{V' \times V} + \langle D p_t, q \rangle_{V' \times V} + \delta_1^{-1} a(p, q) = \langle \hat{S}, q \rangle_{V' \times V},$$

where we interpret $D = A + \delta_1^{-1}(c_0 I + \alpha^2 B) : V \to V'$ through the properties of the operators $A$ and $B$ given in Section 2.3.

Remark 3.4. Above, we note that the natural weak space for the undamped wave equation would be:

$$p \in L^\infty(0, T; V) \cap W^{1,\infty}(0, T; L^2(\Omega)) \cap H^2(0, T; V').$$

In addition, we are making use of the embedding: $H^1(0, T; V) \to L^\infty(0, T; V)$.

Remark 3.5. Note that $S \in H^1(0, T; V')$ and $F \in H^1(0, T; V')$ is sufficient to guarantee that $\hat{S} \in L^2(0, T; V')$.

In anticipation of the use of semigroup theory applied to the abstract presentation of the dynamics in (3.5), we now address the following question: What regularity can be obtained from the full system when specifying, as an initial state, $p_t(0) = p_0$? The next theorem addresses this directly through a priori estimates on the full system (3.1), before we move to the semigroup theory for (3.5) in the later Theorem 3.3.

Theorem 3.2. Let $S \in H^1(0, T; L^2(\Omega))$ and $F \in H^1(0, T; V')$. Consider initial data of the form $p(0) = p_0 \in V$ and $p_t(0) = p_1 \in L^2(\Omega)$.

Then there exists unique, finite energy weak solution $p$ (in the sense of Definition 2) to the reduced problem (3.3), with the identity (3.7) holding in $\mathcal{D}'(0, T)$.

Assuming, in addition, that $u_0 \in \mathcal{D}(\mathcal{E})$ and $F \in L^2(0, T; L^2(\Omega))$, one obtains a unique weak solution to the full system (3.1) (in the sense of Definition 2). The following additional statements hold:

- The unique solution $u$ to (3.3) has $u \in H^1(0, T; \mathcal{D}(\mathcal{E}))$.
- $p \in L^\infty(0, T; \mathcal{D}(A))$.
- $S \in L^2(0, T; V) \implies Ap \in L^2(0, T; V)$.

Proof of Theorem 3.2. Since (3.3) has the form of the strongly damped wave equation, the construction of solutions is standard; we omit the details. Solution can be constructed, for instance, via the Galerkin method and approximants satisfy energy identity (3.7). Therefore, one obtains a weak solution as weak/weak-* limits of the approximations. Moreover, uniqueness follows directly from the energy identity and linearity of the system—the regularity of $p_t$ is sufficient for it to be used as a test function for an arbitrary weak solution (unlike the case for the undamped wave equation (20)).

The displacement $u$ is recovered by using obtained regularity of $p$ and solving ODE (3.3) in time, in the space $\mathcal{D}(\mathcal{E})$ (see Lemma 3.5 below). Additional regularity of $p$ is obtained by using equation (3.1)2:

$$Ap = S - c_0 p_t - \alpha \nabla \cdot u_t \in L^\infty(0, T; L^2(\Omega)).$$

The last statement also follows by noticing $p_t \in L^2(0, T; V)$ and $\nabla \cdot u_t \in L^2(0, T; V)$. \qed

Remark 3.6. Here we emphasize that in the original formulation (2.20) it is not necessary to specify $p_t(0)$. However, $p_t(0)$ can be formally obtained from $u_0$ and $d_0$ as we outline below. Let us consider

$$d_0 = c_0 p(0) + \nabla \cdot u(0) \in V.$$

And, correspondingly, assume that $F \in H^1(0, T; L^2(\Omega))$ and take $\mathcal{E} u(0) \in L^2(\Omega)$ to be fully specified. As $\mathcal{E} : \mathcal{D}(\mathcal{E}) \to L^2(\Omega)$ is an isomorphism, this provides $u(0) \in H^2(\Omega) \cap V$, and we can back solve from $d_0 \in V$ to obtain $p(0) \in V$, providing $\nabla p(0) \in L^2(\Omega)$. Then, again from (2.20)1, we read off $\mathcal{E}[u + \delta_1 u_t](0) \in L^2(\Omega)$ and infer that $u_t(0) \in H^2(\Omega) \cap V$ since

$$\delta_1 \mathcal{E} u_t(0) = F(0) - \alpha \nabla p(0) - \mathcal{E} u(0) \in L^2(\Omega).$$

Finally, $p_t(0)$ can be read off from the pressure equation, when the time trace $S(0) \in L^2(\Omega)$ is well-defined:

$$c_0 p_t(0) = S(0) - Ap(0) - \alpha \nabla \cdot u_t(0) \in V'. \quad (3.11)$$
Thus, we observe that the energy methods (and standard solutions, as in Theorem 3.1 Part 2) applied to the original system gives different (lower) regularity of solutions than obtained in Theorem 3.2. Thus, by prescribing $p_t(0) \in L^2_0(\Omega)$ (in place of prescribing $u_0$) and invoking the wave structure, we obtain an improved regularity result in Theorem 3.2.

Now, we proceed to invoke the semigroup theory for second-order abstract equations with strong damping. Our primary semigroup reference will be [29], and for the strongly damped wave equation [2, 15]. In this case, our damper $D$ is $A$-bounded, where $A$ is the principal spatial operator. This is typically written as $A \leq B \leq A$, which we rewrite here as: There exists appropriate constants such that

$$(Aq, q)_{L^2(\Omega)} \lesssim (Bq, q)_{L^2(\Omega)} \lesssim (Aq, q)_{L^2(\Omega)}.$$

We do not provide an in-depth discussion of the correspondence of the existence of a $C_0$-semigroup on a given state space and associated solutions, instead referring to the classic reference [29 Chapter 4].

We now provide the framework for Theorem 3.3.

- The strongly damped wave equation in (3.6) has a first order formulation considering the states
  
  $$y = [p, p_t]^T \in Y \equiv V \times L^2_0(\Omega)$$

  written as
  
  $$\dot{y} = \begin{bmatrix} 0 & I \\ -[c_0\delta_1]^{-1}A & -c_0^{-1}D \end{bmatrix} y + F, \quad y(0) = [p_0, p_1]. \tag{3.12}$$

  - The operator $\mathcal{A} \equiv \begin{bmatrix} 0 & I \\ -[c_0\delta_1]^{-1}A & -c_0^{-1}D \end{bmatrix}$ is taken with domain
    
    $$\mathcal{D}(\mathcal{A}) \equiv \mathcal{D}(A) \times \mathcal{D}(A),$$

    and $D$ as given in (3.6) with $F \equiv [0, c_0^{-1}\hat{S}]^T$.

  The theorem below will provide the existence of a semigroup $e^{\mathcal{A}t} \in \mathcal{L}(Y)$. In this context, we will obtain a solution $y(t) = e^{\mathcal{A}t}y_0$ to the first order formulation in two standard contexts:

  - When $y_0 \in \mathcal{D}(\mathcal{A})$, the resulting solution lies in $C^1([0, T]; Y) \cap C^0([0, T]; \mathcal{D}(\mathcal{A}))$ and satisfies (3.12) pointwise.

  - When $y_0 \in Y$, the resulting solution is $C^0([0, T]; Y)$ and satisfies a time-integrated version of (3.12); these solutions are sometimes called generalized or mild [29], and are, in fact, $C^0([0, T]; Y)$-limits of solutions from the previous category. Namely, we can approximate the data $y_0 \in Y$ by $y_0^t \in \mathcal{D}(\mathcal{A})$, and obtain the generalized solution as a $C^0([0, T]; Y)$-limit of the solutions emanating from the $y_0^t$.

  - It is standard (in this linear context) to obtain weak solutions (in the sense of Definition 2) by considering smooth solutions emanating from $y_0^t \in \mathcal{D}(\mathcal{A})$ as approximants.

**Theorem 3.3 (Damped Semigroup Theorem).** In the above framework, with $\delta_1, c_0 > 0$, the operator $\mathcal{A}$ generates a $C_0$-semigroup of contractions $e^{\mathcal{A}t} \in \mathcal{L}(Y)$, which is analytic (in the sense of [29 Chapter 2.5]) on $Y \equiv V \times L^2_0(\Omega)$.

This is to say:

- For $[p_0, p_1] \in Y$ and $\hat{S} \in L^2(0, T; L^2_0(\Omega))$, we obtain a unique (mild) solution $[p(\cdot), p_t(\cdot)] \in C([0, T]; Y)$ (as described above).

- For $[p_0, p_1] \in \mathcal{D}(A) \times \mathcal{D}(A)$ and $\hat{S} \in H^1(0, T; L^2(\Omega))$, we obtain a unique solution $[p(\cdot), p_t(\cdot)] \in C([0, T]; \mathcal{D}(A) \times \mathcal{D}(A)) \cap C^1((0, T); Y)$ that satisfies the system in a point-wise sense.

Lastly, one may select the state space $Z = L^2_0(\Omega) \times L^2_0(\Omega)$ with $\mathcal{D}(\mathcal{A})$ the same as before. In this context, $\mathcal{A}$ still generates a $C_0$-semigroup $e^{\mathcal{A}t} \in \mathcal{L}(Z)$. This semigroup is again analytic, though it is not a contraction semigroup. In this case, with $[p_0, p_1] \in (L^2_0(\Omega))^2$ we obtain solutions in the sense of $C^0([0, T]; Z)$.

**Proof of Theorem 3.3.** The proof of this theorem follows immediately from the application of [24 pp.292–293] to the present framework to obtain the semigroup. For the case of taking the state space $Z$, see also [2]. In passing from the semigroup to solutions—taking into account the inhomogeneity $F$—we invoke [24 Chapter 4.2].
Remark 3.7. Note that $S \in H^1(0, T; L_0^2(\Omega))$ and $F \in H^1(0, T; V')$ implies $\tilde{S} \in L^2(0, T; L_0^2(\Omega))$. Moreover, stronger assumptions $S \in H^2(0, T; L_0^2(\Omega))$ and $F \in H^2(0, T; V')$ imply $\tilde{S} \in H^1(0, T; L_0^2(\Omega))$.

We can say more, since the strongly damped wave equation is known to be exponentially stable.

**Theorem 3.4** (Exponential Decay of Solutions). Consider the above framework, and take $F \equiv 0$ and $S \equiv 0$ in (3.1) (so $\mathcal{F} = \{0, 0^T\}$). Consider $c_0, \delta_1 > 0$. Then the analytic semigroup $e^{\lambda t}$ generated by $A : Y \supset \mathcal{F}(A) \to Y$ is uniformly exponentially stable. That is, there exists $\gamma, M_0, M_k > 0$ (each depending on $c_0, \delta_1 > 0$) so that:

$$||e^{\lambda t}||_Y \leq M_0 e^{-\gamma t}, \quad \text{and, more generally,} \quad ||A^k e^{\lambda t}||_Y \leq M_k t^{-k} e^{-\gamma t}, \quad t \geq 0, \quad k \in \mathbb{N}. \quad (3.13)$$

**Proof of Theorem 3.4.** As the above semigroup is analytic, and the requisite spectral criteria are satisfied, exponential decay is inferred from the abstract theory \cite[Theorem 1.1(b), pp.20–21]{2}.

In the above semigroup approach, we have constructed solutions in the variable $p$ via the semigroup approach; we now comment on how to pass from the variable $p$ to $u$ via the ODE in (3.1). This observation is central to subsequent sections, but we suffice to say here that, when a given pressure function $p$ is obtained in a regularity class for which the ODE

$$\delta_1 u_t + u = \mathcal{E}^{-1} F - \alpha \mathcal{E}^{-1} (\nabla p),$$

can be readily interpreted, we obtain a mapping $\nabla p \mapsto u$. When one has decay estimates, for instance as above in Theorem 3.4, these can be pushed from $p$ to $u$ in the solution to the full system (3.1).

**Lemma 3.5.** Consider the ODE in (3.3). Letting

$$Q = \delta^{-1} [-\mathcal{E}^{-1} (\nabla p) + \mathcal{E}^{-1} (F)],$$

we have $u_t + \delta^{-1} u = Q$, which can be solved by the integrating factor as:

$$u(x, t) = e^{-t/\delta_1} u(x, 0) + \int_0^t e^{(\tau - t)/\delta_1} Q(x, \tau) d\tau. \quad (3.14)$$

From Lemma 3.5, one can easily pass Theorems 3.3 and 3.4 (as well as Theorem 3.2) from the variables $p$ to $u$ (and $u_t$). We provide an example for the state space norm below.

Take $F \equiv 0$ and $S = 0$. We will obtain a decay result for the displacement $u$ through the ODE. From (3.13) we have that

$$\|y(t)\|_Y \leq M_0 e^{-\gamma t} \|y_0\|_Y \quad \implies \quad \|\nabla p(t)\|_{L^2(\Omega)} \leq M_0 e^{-\gamma t} \|y_0\|_Y \quad \implies \quad \|\mathcal{E}^{-1} (\nabla p(t))\|_{D(\mathcal{E})} \leq M_0 e^{-\gamma t} \|y_0\|_Y. \quad (3.15)$$

Using (3.14) we obtain

$$\|u(t)\|_{D(\mathcal{E})} \leq e^{-\gamma t} \|u_0\|_{D(\mathcal{E})} + \frac{M_0}{1 - \gamma \delta_1} \left(e^{-\gamma t} - e^{-\delta_1 t}\right) \|y_0\|_Y$$

Of course, the estimate above can be readily adjusted to accommodate the space in which $u_0$ is specified.

**3.1.2 Incompressible Constituents**

We now take $c_0 = \delta_2 = 0$ in (2.20), which, following the calculations of Section 3.1, yields the abstract dynamics:

$$[\delta_1 A + \alpha^2 B] p_t + A p = \overline{S}, \quad \text{with} \quad \overline{S} = S - \alpha \nabla : \mathcal{E}^{-1} (F_t) + \delta_1 \overline{S}_t. \quad (3.16)$$

Note that $\overline{S} = \delta_1 \overline{S}$ from the previous section. In this case, we make a simple change of variable:

$$q = [\alpha^2 B + \delta_1 A] p \quad (3.17)$$

and proceed in the variable $q$. In this scenario, $\alpha^2 B + \delta_1 A \in \mathcal{L}(V, V')$ and is boundedly invertible in this sense, following the properties of $A$ and $B$ in Section 2.3. Indeed, this follows immediately from: (i) Lax-Millgram on the strength of $A$ and (ii) the continuity and self-adjointness of $B$ on $L_0^2(\Omega)$.

Under the change of variable, our abstract equation (3.16) can be written as an ODE in the variable $q$ as:

$$q_t + A [(\alpha^2 B + \delta_1 A)^{-1}] q = \overline{S} \in L^2(0, T; V'); \quad q(0) = q_0 \in V'. \quad (3.18)$$
The operator $Z$ has that
\[ Z \equiv A[(\alpha^2 B + \delta_1 A)^{-1}] \in \mathcal{L}(V^\prime) \cap \mathcal{L}(L^2_0(\Omega)); \]
it is zeroth-order with $L^2(\Omega)$-adjoint
\[ Z^* = [(\alpha^2 B + \delta_1 A)^{-1}]^* A. \]
The ODE in (3.18) can be interpreted either in the sense of $C(0, T; V^\prime)$ or $C(0, T; L^2_0(\Omega))$, depending on the regularity of the data which comprise $\overline{S}$—namely, whether we require $S, S_t$ and $F_t$ to take values in $L^2$ or $H^1$ type spaces. In any case, it can thus be solved in the context of a uniformly continuous semigroup [29]. Here, the semigroup is $e^{Zt} \in \mathcal{L}(X)$, where $X$ can be chosen $V^\prime$ or $L^2_0(\Omega)$.

For $S \in L^2(0, T; V^\prime)$ and $q_0 \in V^\prime$, the classical variation of parameters formula [29] yields that $q \in H^1(0, T; V^\prime)$. However, $p \in H^1(0, T; V)$ is immediately obtained through inverting the change of variables (3.17) a.e. $t$. Then, the elasticity ODE can be explicitly solved in time as in Lemma 3.5, providing $u \in H^1(0, T; V)$. From this, we observe that $p(0) = p_0 \in V$ must be specified at the outset in order to possess an initial condition of the form $q(0) = q_0 \in V^\prime$. This reflects the fact, that since this cases reduces to an ODE, there is no spatial regularization provided by the pressure dynamics.

In this case, one only needs to specify $p(0)$ to obtain a solution to the abstract ODE in (3.18). This comes through the appearance of the combination $\delta_1 u + u$ in the elasticity equation and the structure of the solution to the ODE in $u$. To recover the displacement variable $u$ from $p$, one must additionally specify $u(0)$ as well. Lastly, after solving the ODE in $q$ (and therefore for $p$), one can revert back to the ODE for $u$ to obtain additional temporal regularity (as a function of the regularity of $F$), since both sides of the equality below can be time-differentiated
\[ \delta_1 u + u = \mathcal{E}^{-1}[F - \alpha \nabla p]. \]

Through the discussion above, we have arrived at the following theorem.

**Theorem 3.6 (ODE Theorem).** Let $S \in H^1(0, T; V^\prime)$ and $F \in H^1(0, T; V^\prime)$ (so that $\overline{S} = S - \alpha \nabla \cdot \mathcal{E}^{-1}(F_t) + \delta_1 S_t \in L^2(0, T; V^\prime)$) and take $p_0 \in V$.

Then there exists a unique (ODE) solution $p \in H^1(0, T; V)$ to (3.18), given by
\[ p(t) = [\alpha^2 B + \delta_1 A]^{-1} e^{Zt}[\alpha^2 B + \delta_1 A]p_0. \]

- If, in addition $S \in L^2(0, T; L^2_0(\Omega))$, then $p \in L^2(0, T; D(A)).$
- If, in addition, $u_0 \in D(\mathcal{E})$ and $F \in L^2(0, T; L^2(\Omega))$, then there exists a unique weak solution $(p, u)$ to (2.20), with $p$ as before and $u \in H^1(0, T; D(\mathcal{E}))$.
- The formal energy equality in (5.7) holds for the weak solution $(p, u)$ with $c_0 = 0$.

**Theorem 3.7 (Regularity for ODE).** Let $m \in \mathbb{N}$, $S \in H^1(0, T; H^{m+1}(\Omega) \cap L^2_0(\Omega))$ and $F \in H^1(0, T; H^{m+1}(\Omega))$ (so that $\overline{S} = S - \alpha \nabla \cdot \mathcal{E}^{-1}(F_t) + \delta_1 S_t \in L^2(0, T; H^{m}(\Omega)))$, and take $p(0) \in H^{m+2}(\Omega) \cap V$.

Then the ODE solution $p \in L^2(0, T; H^{m+3}(\Omega))$ has $p \in H^1(0, T; H^{m+3}(\Omega))$.

- If, in addition $S \in L^2(0, T; H^{m+1}(\Omega))$, then $p \in L^2(0, T; H^{m+3}(\Omega))$.
- If, in addition, $u_0 \in H^{m+1} \cap V$, then the weak solution $(p, u)$ to (2.20), satisfies an additional regularity property $u \in H^1(0, T; H^{m+1}(\Omega))$.
- If, in addition, $u(0) \in H^{m+3} \cap V$ and $F \in L^2(0, T; H^{m+1}(\Omega))$, then the weak solution $(p, u)$ to (2.21), satisfies an additional regularity property $u \in H^1(0, T; H^{m+3}(\Omega))$.

**Remark 3.8.** Note that by time-differentiating to obtain the abstract formulation we wish to invoke, we are, in some sense, lowering the regularity of the solution. However, some additional regularity is obtained a posteriori through elliptic regularity.

Finally, we comment on decay in this case. First, the energy estimate (3.12) holds for $c_0 = 0$ solutions:
\[ \|u\|^2_{L^\infty(0, T; V)} + \delta_1 \|u\|^2_{L^2(0, T; V)} + \|p\|^2_{L^2(0, T; V)} \lesssim \|u_0\|^2_{H^1} + \|S\|^2_{L^2(0, T; V^\prime)} + \|F\|^2_{L^2(0, T; L^2(\Omega))}. \]
The above, of course, indicates dissipation in both variables $p$ and $u$, yet the $\|p(t)\|^2_{L^2_0(\Omega)}$ has, of course, disappeared. As this case yields an ODE for $q$ (on either $V^\prime$ or $L^2_0(\Omega)$), the spectral properties of $Z$ (on the respective space) dictate decay in $q$; we do not pursue this here. Estimates on the solution variable $p$ are then obtained through (3.17), and passing estimates from $p(t)$ to $u(t)$ can be done precisely as in the previous section.
### 3.2 Case 2: Adjusted Fluid Content, \( \delta_2 > 0 \)

We now consider \( \delta_1, \delta_2 > 0 \) in (2.20), which is to say, we allow for visco-elastic effects in the structural equation and we modify the definition of the fluid content of (2.20). Thus, in this section, the fluid content will be given by

\[
\zeta = c_0 p + \alpha \nabla \cdot \mathbf{u} + \delta_2 \nabla \cdot \mathbf{u}_t,
\]

where again we retain the coefficients \( \delta_2, \delta_1 \) to observe their presence in the reduced dynamics. We will note explicitly below that for the dynamics to admit proper energy estimates, *we must observe the identity*

\[
\alpha \delta_1 = \delta_2.
\]

*Remark 3.9.* Alternatively, one obtains this relation by formally mapping \( \mathbf{u} \mapsto \mathbf{u} + \delta_1 \mathbf{u}_t \) in the original Biot dynamics (2.20).

Note that *we will not make a distinction between \( c_0 \geq 0 \) in this section.* Indeed, as we will see, upon making the abstract reduction, we will obtain precisely the same system (under different state variables) as the original Biot dynamics. (The elasticity state \( \mathbf{u} + \delta_1 \mathbf{u}_t \) will replace \( \mathbf{u} \).) The one difference is that it will not be adequate to only specify \( d_0 \) or \( p_0 \) in the initial conditions. Indeed, we will require \( \mathbf{u}_0 \) and one of \( p_0, d_0 \). Even considering the pressure equation alone, it will not be adequate to specify \( d_0 \) alone, as we will see.

Now, invoking the elasticity equation as in (3.3), we can write, as before:

\[
\delta_1 \mathbf{u}_t + \mathbf{u} = \mathcal{E}^{-1} \mathbf{F} - \alpha \mathcal{E}^{-1} (\nabla p).
\]

Taking the divergence, enforcing the condition \( \alpha \delta_1 = \delta_2 \), and plugging into the fluid content expression above in (3.19), we obtain a pressure equation from (2.20) of the form:

\[
[c_0 p + \alpha^2 B p + \nabla \cdot \mathcal{E}^{-1} \mathbf{F}]_t + A p = S,
\]

which is rewritten as

\[
[(c_0 I + \alpha^2 B) p]_t + A p = \bar{S},
\]

where \( \bar{S} \) is as before in Section 2.4. We note that this pressure equation coming from (2.20) with \( \delta_1, \delta_2 > 0 \) has the exact same structure as the original implicit degenerate system without visco-elasticity, as presented in (2.13). Thus, the inclusion of viscous effects in both the fluid content and displacement equation recovers the same implicit, degenerate (parabolic) equation given by Biot’s poro-elastic dynamics. Although we have the same system abstractly, it is worth noting the estimates and relevant quantities in this case.

We note explicitly that, for the above dynamics in \( p \), it is sufficient to specify either \( p(0) = p_0 \) or \( B p(0) = [(c_0 I + \alpha^2 B)p](0) \)—indeed, these are equivalent. However, unlike the case of pure Biot dynamics (\( \delta_1 = \delta_2 = 0 \)), we cannot get from \( d_0 = \zeta(0) = [c_0 p + \alpha \nabla \cdot \mathbf{u} + \delta_2 \nabla \cdot \mathbf{u}_t](0) \) to \( p_0 \) directly, as we must pass through the ODE for \( \mathbf{u} \) in (3.3). Said differently: by moving to the abstract framework we can solve for \( p \) in (3.21). Then, solving the ODE in (3.3)—with a given initial condition \( \mathbf{u}_0 \)—we then obtain the corresponding displacement solution \( \mathbf{u} \). As in previous cases, it is clear that given \( p_0 \) and \( \mathbf{u}_0 \) the quantity \( \mathbf{u}_t(0) \) can be recovered through

\[
\delta_1 \mathbf{u}_t(0) = -\mathbf{u}_0 - \alpha \mathcal{E}^{-1} \nabla p_0 + \mathcal{E}^{-1} \mathbf{F}(0) \in \mathbf{V}.
\]

This produces associated estimates in a roundabout way. One can immediately obtain energy estimates through the multiplier method, as in previous sections. However, to obtain a priori estimates (e.g., on approximants), one will test the pressure equation with \( p \) in (2.20) and the displacement equation (2.20) with \( \delta_1 \mathbf{u}_t + \mathbf{u}_t \). In this step, *one again observes the necessary requirement* that \( \alpha \delta_1 = \delta_2 \). The resulting (formal) identities are:

\[
\frac{1}{2} \frac{d}{dt} \left( \delta_1 \mathbf{u}_t + \mathbf{u}_t, \delta_1 \mathbf{u}_t + \mathbf{u}_t \right) - \alpha (p, \delta_1 \nabla \cdot \mathbf{u}_t + \nabla \cdot \mathbf{u}_t) = - \langle \mathbf{F}_t, [\delta_1 \mathbf{u}_t + \mathbf{u}_t] \rangle_{\mathbf{V}' \times \mathbf{V}} + \langle \mathbf{F}(\tau), [\delta_1 \mathbf{u}_t(\tau) + \mathbf{u}(\tau)] \rangle_{\mathbf{V}' \times \mathbf{V}} \bigg|_{\tau=t}^{\tau=0},
\]

\[
(\nabla \cdot [\delta_2 \mathbf{u}_t + \alpha \mathbf{u}_t], p) + \frac{c_0}{2} \frac{d}{dt} ||p||^2 + a(p, p) = \langle S, p \rangle_{\mathbf{V}' \times \mathbf{V}}.
\]

These identities, along with the application of the abstract theorem in Section 2.4 produce the baseline theorem for this case.
\textbf{Theorem 3.8} (Visco-elastic with Modified Fluid Content). Suppose that $S \in L^2(0,T;V')$ and $F \in H^1(0,T;V')$. Take $p_0 \in L^2(\Omega)$. Suppose $\delta_2 > 0$ with $c_0 \geq 0$ and enforce the condition that $\delta_2 = \alpha_0 \lambda^1$. Then there exists a unique weak solution $p \in L^2(0,T;V)$ satisfying (3.21) (in the sense of the Appendix).

If $u_0 \in V$, then (with $p$ the same as above) there exists a unique weak solution $(u,p) \in C([0,T];V) \times L^2(0,T;V)$ to (2.20) satisfying the energy inequality

$$\|u|_{L^2(0,T;V)} + c_0\|p\|^2_{L^2(0,T;V')} \lesssim \|u_0\|^2_{T^*} + c_0\|p_0\|^2_{L^2(\Omega)} + \|S\|^2_{L^2(0,T;V')} + C(T)||F||_{H^1(0,T;V')}.$$  

(3.22)

\textbf{Remark 3.10.} In this above framework—as we have reduced to the same abstract theory for classical Biot with $\delta_1 = \delta_2 = 0$—we can accordingly discuss parabolic estimates and smooth solutions. We do not repeat the statements here, but point back to Theorem 2.2 and Theorem 2.4.

\textbf{Remark 3.11.} If one allows for the possibility that the coefficient $\delta_2$ is independent, we obtain a system like:

$$\begin{align*}
\mathcal{E}u + \delta_1 \mathcal{E}u_t + \alpha \nabla p &= F \\
[c_0 p + \alpha \nabla \cdot u + \alpha \nabla \cdot u_t]_t + Ap &= S.
\end{align*}$$

(3.23)

In full generality, we obtain the equation (not closed in $p$):

$$[c_0 p + \frac{\alpha \tilde{\alpha}}{\delta_1} Bp]_t + \left(\alpha - \frac{\alpha \tilde{\alpha}}{\delta_1}\right) \frac{\alpha}{\delta_1} Bp - \left(\alpha - \frac{\alpha \tilde{\alpha}}{\delta_1}\right) \delta_1^{-1} \nabla \cdot u + Ap = S - \frac{\alpha \tilde{\alpha}}{\delta_1} \nabla \cdot \mathcal{E}^{-1} F - \delta_1^{-1} \left(\alpha - \frac{\alpha \tilde{\alpha}}{\delta_1}\right) \nabla \cdot \mathcal{E}^{-1} F$$

Solving the ODE for $u$ (as before) yields:

$$\begin{align*}
&\left[\left(c_0 I + \frac{\alpha \tilde{\alpha}}{\delta_1} \frac{\alpha}{\delta_1} B\right)p + \left(\alpha \tilde{\alpha} - \frac{\alpha \tilde{\alpha}}{\delta_1}\right) Bp - \left(\alpha \tilde{\alpha} - \frac{\alpha \tilde{\alpha}}{\delta_1}\right) \int_0^t e^{-(t-\tau)/\delta_1} Bp(\tau)d\tau \right]_t \\
&\quad = S - \frac{\alpha \tilde{\alpha}}{\delta_1} \nabla \cdot \mathcal{E}^{-1} F + \frac{\alpha \tilde{\alpha} - \alpha}{\delta_1} e^{-t/\delta_1} \nabla \cdot u_0 + \frac{\tilde{\alpha} \alpha - \alpha}{\delta_1} \nabla \cdot \mathcal{E}^{-1} F - \frac{\tilde{\alpha} \alpha - \alpha}{\delta_1} \int_0^t e^{-(t-\tau)/\delta_1} \nabla \cdot \mathcal{E}^{-1} F(\tau)d\tau.
\end{align*}$$

(3.24)

In this case we see the emergence of additional terms which vanish when we enforce $\tilde{\alpha} = \delta_1 \alpha$.

\section{Remarks on Secondary Consolidation}

In some sense, the effects of secondary consolidation of soils (i.e., creep) can be thought of as partial visco-elasticity \cite{29}. Therefore, for completeness, we add some remarks here on the nature of solutions and associated estimates which mirror the remarks in \cite{33}. In the case of secondary consolidation, as it is described in \cite{33}, we “regularize” only the divergence term in the momentum equation (omitting $\Delta u$ from the full visco-elastic terms). We here consider the Biot system with secondary consolidation; as before, we take both definitions of the fluid content based strictly on mathematical grounds—ignoring physical interpretations.

Consider:

$$\begin{align*}
\begin{cases}
-\lambda^* \nabla \cdot u_t - \mu \Delta u - (\lambda + \mu) \nabla \nabla \cdot u + \alpha \nabla p = F \\
[c_0 p + \alpha \nabla \cdot u + \delta_2 \nabla \cdot u]_t + Ap = S
\end{cases}
\end{align*}$$

(4.1)

Below, we track the impact of this “partial viscoelastic” $\lambda^*$ term.

\textbf{Remark 4.1.} Note that there is no need to consider the case with both full visco-elasticity $\delta_1 > 0$ and secondary consolidation, as the latter would be redundant; the effects of secondary consolidation would be incorporated into the full visco-elastic model.

\subsection{Traditional Fluid Content}

Here we consider the secondary consolidation model as explicitly discussed in depth in \cite{33}. We take $\delta_2 = 0$, so our system is:

$$\begin{align*}
\begin{cases}
-\lambda^* \nabla \cdot u_t - \mu \Delta u - (\lambda + \mu) \nabla \nabla \cdot u + \alpha \nabla p = F \\
[c_0 p + \alpha \nabla \cdot u]_t + Ap = S
\end{cases}
\end{align*}$$

(4.2)
The well-posedness of weak solutions is given in [33]. We simply mention here that, using the standard multipliers for weak solutions (justified, for instance, by the approach in [10]) one obtains the following estimate on solutions:

\[ ||u||_{L^\infty_2(0,T,V)}^2 + c_0||p||_{L^\infty_2(0,T,L^2(\Omega))}^2 + \lambda^*||\nabla \cdot u||_{L^2_2(0,T,L^2(\Omega))}^2 \lesssim ||u_0||_{V^2} + ||S||_{L^2_2(0,T,V')}^2 + ||F||_{L^2_2(0,T,V')}^2 \]  (4.3)

A partial “visco-elastic” effect of secondary consolidation is immediate obviated: we note the additional damping/dissipation term above for \( \nabla \cdot u \); upon temporal integration, we will obtain the additional property of solutions (from the standard weak solution to Biot dynamics as in [10, 33]) that the \( \lambda^* > 0 \) term represents a certain “smoothing” as well, as solutions from the standard weak solution to Biot dynamics as in [10, 33] can now extract that the presence of \( \lambda^* > 0 \) is lost, since we cannot decouple the two terms in the sum for the fluid content.

4.1.1 Incompressible Constituents

In the case of \( c_0 = 0 \), we observe some partial regularization of the dynamics for \( \lambda^* > 0 \); this is explicitly mentioned in [33], and we expound it here.

Note that from the pressure equation, we can write:

\[ Ap = S - \alpha \nabla \cdot u, \]

from which elliptic regularity can be applied—for weak solutions—when \( \Omega \) is sufficiently regular and \( S \in L^2(0,T;L^2(\Omega)) \). Then, with \( \nabla \cdot u \in L^2(0,T;L^2(\Omega)) \) as described above, we observe a boost \( p \in L^2(0,T;V) \rightarrow p \in L^2(0,T;D(A)) \) through elliptic regularity applied a.e. \( t \). But this cannot be pushed on the momentum equation, owing to the addition of the secondary consolidation term:

\[ E(u) = F - \alpha \nabla p + \lambda^* \nabla \cdot u. \]

This is to say that the regularity gain in \( p \) is not realized for \( u \) through the momentum equation.

One further observation, in this case, is a particular representation of the system which is not available in other cases. Noting that \( p = A^{-1}[S - \alpha \nabla \cdot u] \), one can plug this into the elasticity equation to obtain:

\[ -\alpha^2 \nabla S^{-1} \nabla \cdot u + \lambda^* \nabla \cdot u + E(u) = F - \alpha \nabla A^{-1} S. \]

This is an implicit equation directly in \( u \)—see [33].

4.1.2 Compressible Constituents

In the case of \( c_0 > 0 \), [33] observes that the effect of secondary consolidation is de-regularizing. This is observed in hindering the discussion in the previous section, namely the pressure equation now reads as:

\[ Ap = S - \alpha \nabla \cdot u - c_0 p. \]

The effect of secondary consolidation through \( \lambda^* \) (boosting \( \nabla \cdot u \) to \( L^2(0,T;L^2(\Omega)) \)) is lost, since we can only conclude that \( c_0 p \in L^2(0,T;V) \) rather than \( L^2(0,T;L^2(\Omega)) \). Thus there is neither smoothing in \( p \), nor \( u \) in this case.

4.2 Adjusted Fluid Content

Finally, we observe that in the case of adjusted fluid content, we obtain the natural analog to our earlier discussions. Taking \( \delta_2 > 0 \), we consider the system:

\[
\begin{aligned}
-\lambda^* \nabla \cdot u + E(u) + \alpha \nabla p &= F \\
[c_0 p + \alpha \nabla \cdot u + \delta_2 \nabla \cdot u] + Ap &= S
\end{aligned}
\]  (4.4)

As above, we invoke (as before) the test function \( \delta_2 u \cdot u + u \cdot u \) in the elasticity equation, and \( p \) in the pressure equation. This provides an identical estimate as that in Theorem 3.8 with the additional property that \( \nabla \cdot u \in L^2(0,T;L^2(\Omega)) \), and the associated term

\[ \lambda^* ||\nabla \cdot u||_{L^2_2(0,T;L^2(\Omega))}^2 \]

appears on the LHS of [33] in Theorem 3.8. Again, then, we see the effect of secondary consolidation as being that of partial damping.
5 Summary and Conclusions

In this note we characterized poro-visco-elastic systems across several physical regimes:

\begin{equation}
\begin{aligned}
\mathcal{L}u + \delta_1 \mathcal{L}u_t + \alpha \nabla p &= F \\
[\mathcal{Q}_0 t] + \alpha \nabla \cdot u + \delta_2 \nabla \cdot u_t &= Ap = S.
\end{aligned}
\end{equation}

We began with the traditional Biot system ($\delta_1 = \delta_2 = 0$), i.e., no visco-elastic effects, and recapitulated existence and regularity results for weak solutions, as well as solutions with higher regularity in Section 2.3. Using this as a jumping off point, we considered the addition of Kelvin-Voigt type (strong) dissipation in the Lamé system. Our central focuses were in the well-posedness and regularity of solutions across all parameter regimes, as well as the clear determination of the abstract structure of the problem, including the determination of appropriate initial quantities. Our approach includes providing clear a priori estimates on solutions, where they are illuminating. We employed an operator-theoretic framework inspired by [1,33] and developed in [10] which we introduced in Section 2.3. The central operators were $A$ (a Neumann Laplacian) and $B$ (zeroth order, nonlocal), the so-called pressure to divergence operator.

We then considered the model with $\delta_1 > 0$ and $\delta_2 = 0$, which is to say we left the fluid-content unaltered in our addition of strong damping. We first gave a well-posedness result which was valid for both compressible $c_0 > 0$ and incompressible $c_0 = 0$ constituents in Section 2.1. We then distinguished between compressible and incompressible cases. We determined that for $c_0 > 0$, the system constitutes a strongly damped hyperbolic-type system. It was important, in this case, to distinguish results based on which initial quantities were specified. Regardless, the regularity of solutions was made clear, and the parabolicity of the system was detailed in several ways. In the case when $c_0 = 0$, we observed that the abstract, reduced version of the dynamics constituted an ODE in a Hilbert space of our choosing (either $L^2_0(\Omega)$ or an $H^{-1}(\Omega)$ type space, $V'$). We exploited the ODE nature of the dynamics to produce a clear well-posedness and regularity result.

In the case when $\delta_1, \delta_2 > 0$ (i.e., the adjusted fluid content), we observed that the abstract reduction of the adjusted fluid-content system brings the dynamics back to a traditional Biot-structure. In other words, by adding visco-elasticity to the displacement equation ($\delta_1 > 0$) as well as adjusting the fluid content ($\delta_2 > 0$), we do not observe additional effects from the damping—rather, we obtain the same qualitative results for the solution as we had for Biot’s original dynamics. Noting a small difference in which quantities must be prescribed, we presented a well-posedness theorem, with relevant a priori estimates.

Finally, in Section 4, we provide some small remarks on partial visco-elasticity, known in soil consolidation as secondary consolidation. The main focus of this section was to provide clear a priori estimates on solutions, indicating precisely how dissipation is introduced into the system through secondary consolidation effects. Additionally, we elaborate on certain remarks in [33] concerning the extent to which partial visco-elasticity can be partially regularizing (when $c_0 = 0$) and de-regularizing ($c_0 > 0$).

The appendix serves to provide a small overview of the standard theory of weak solutions for implicit, degenerate evolution equations, and is taken from [10,32].

We believe that the work presented here, as it is in the spirit of [1,31,33], will be of interest to any researchers working on applied problems in poro-elasticity. In particular, as the effects of visco-elasticity are prominent in biological sciences, those who work on biologically motivated Biot models may find the results presented herein interesting. Indeed, to the best of our knowledge, we have provided the first elucidation of the mathematical effects of visco-elasticity, when added into poro-elastic dynamics. This includes giving well-posedness and regularity results, with clear a priori estimates on solutions, as well as the clear specification of initial quantities.

6 Appendix: Abstract Framework for Weak Solutions

Let $V$ be a separable Hilbert space with dual $V'$ (not identified with $V$). Assume $V$ densely and continuously includes into another Hilbert space $H$, which is identified with its dual: $V \hookrightarrow H \equiv H' \hookrightarrow V'$. We denote the inner-product in $H$ simply as $(\cdot, \cdot)$, with $(h, h) = ||h||_H^2$ for each $h \in H$. Similarly, we denote the $V' \times V$ duality pairing as $(\cdot, \cdot)$. (For $h \in H$, we identify $(h, h) = ||h||_V^2$ as well.) Assume that $A \in \mathcal{L}(V, V')$ and $B \in \mathcal{L}(H)$. Finally, suppose that $d_0 \in V'$ and $S \in L^2(0, T; V')$ are the specified data.

In this setup, we can define the weak (implicit-degenerate) Cauchy problem to be solved as:
Find \( w \in L^2(0,T;V) \) such that

\[
\begin{aligned}
\frac{d}{dt}[Bw] + Aw &= S \in L^2(0,T;V') \\
\lim_{t \to 0} [Bw(t)] &= d_0 \in V'.
\end{aligned}
\]  

(6.1)

The time derivative above is taken in the sense of \( V' \), and since such a solution would have \( Bu \in H^1(0,T;V') \) (with the natural inclusion \( V \to V' \) holding), \( Bu \) has point-wise (in time) values into \( V' \) and the initial conditions make sense through the boundedness of \( B \) with \( H \to V' \).

The following generation theorem is adapted from [32 III.3, p.114–116] for weak solutions, and produces weak solvability of (6.1) in a straight-forward way:

**Theorem 6.1.** Let \( A, B \) be as above, and assume additionally that they are self-adjoint and monotone (in the respective sense, \( A : V \to V' \) and \( B : H \to H \)). Assume further that there exists \( \lambda, c > 0 \) so that

\[
2\langle Av, v \rangle + \lambda |Bv, v| \geq c|v|^2, \quad \forall \ v \in V.
\]

Then, given \( Bw(0) = d_0 \in H \) and \( S \in L^2(0,T;V') \), there exists a unique weak solution to (6.1) satisfying

\[
||w||^2_{L^2(0,T;V)} \leq C(\lambda, c) \left[ ||S||^2_{L^2(0,T;V')} + (d_0, w(0))_H \right].
\]  

(6.2)

The assumption in this theorem is that there exists a \( w(0) \in V \) so that \( Bw(0) = d_0 \), the given initial data. In more recent work applying this theorem [10], we need not assume the existence of such \( w(0) \). See Theorem 2.3.

One may also consult the implicit semigroup theory presented in [33, Section 5] and [32 IV.6], in particular for a discussion of smoother solutions.

**References**

[1] Auriault, J.L., 1980. Dynamic behaviour of a porous medium saturated by a Newtonian fluid. International Journal of Engineering Science, 18(6), pp.775-785.

[2] Balakrishnan, A.V. and Triggiani, R., 1993. Lack of generation of strongly continuous semigroups by the damped wave operator on \( H \times H \) (or: The little engine that couldn’t). *Applied Mathematics Letters*, 6(6), pp.33-37.

[3] Banks, H.T., Bekele-Maxwell, K., Bociu, L., Noorman, M., and Guidoboni, G., 2019. Sensitivity Analysis via the complex-step derivative approximation for 1-D poro-elastic and poro-visco-elastic models, *Mathematical Control and Related Fields* 9(4), pp.623–642.

[4] Banks, H.T., Bekele-Maxwell, K., Bociu, L., Noorman, M., and Guidoboni, G., 2017. Sensitivity Analysis in Poro-Elastic and Poro-Visco-Elastic Models with Respect to Boundary Data, *Quarterly of Applied Mathematics* 75, pp.697–735.

[5] Biot, M.A., 1941. General theory of three-dimensional consolidation, *J. Appl. Phys.* 12(2), pp.155–164.

[6] Both, J.W., Borregales, M., Nordbotten, J.M., Kumar, K. and Radu, F.A., 2017. Robust fixed stress splitting for Biot’s equations in heterogeneous media. Applied Mathematics Letters, 68, pp.101-108.

[7] Bociu, L., Canić, S., Muha, B. and Webster, J.T., 2021. Multilayered poroelasticity interacting with stokes flow. *SIAM Journal on Mathematical Analysis*, 53(6), pp.6243-6279.

[8] Bociu, L., Guidoboni, G., Sacco, R., and Webster, J.T., 2016. Analysis of Nonlinear Poro-Elastic and Poro-Visco-Elastic Models, *ARMA*, 222, pp.1445–1519.

[9] Bociu, L., Guidoboni, G., Sacco, R., and Verri, M., 2019. On the role of compressibility in poroviscoelastic models, *Mathematical Biosciences and Engineering*, 16(5), 6167–6208.

[10] Bociu, L., Muha, B. and Webster, J.T., 2022. Weak solutions in nonlinear poroelasticity with incompressible constitutents. *Nonlinear Analysis: Real World Applications*, 67, p.103563.

[11] Bociu, L. and Noorman, M., 2019. Poro-Visco-Elastic Models in Biomechanics: Sensitivity Analysis, *Communications in Applied Analysis*, 23(1), pp.61–77.

[12] Bociu, L. and Strikwerda, S., 2022. Poro-Visco-Elasticity in Biomechanics - Optimal Control, *Springer AWM Volume on Research in the Mathematics of Materials Science*, to appear.

[13] Bociu, L. and Webster, J.T., 2021. Nonlinear quasi-static poroelasticity. Journal of Differential Equations, 296, pp.242–278.
[14] Cao, Y., Chen, S., and Meir, A.J., 2014. Steady flow in a deformable porous medium, *Math. Meth. Appl. Sci.*, 37, pp.1029–1041.

[15] Chen, S.P. and Triggiani, R., 1989. Proof of extensions of two conjectures on structural damping for elastic systems. Pacific Journal of Mathematics, 136(1), pp.15-55.

[16] Chen, G. and Russell, D.L., 1982. A mathematical model for linear elastic systems with structural damping. Quarterly of Applied Mathematics, 39(4), pp.433-454.

[17] P.G. Ciarlet, 1988. *Three-dimensional elasticity, Vol. 1*, Elsevier.

[18] Coussy, O., 2004. *Poromechanics*. John Wiley & Sons.

[19] Detournay, E. and Cheng, A.H.-D., 1993. Fundamentals of poroelasticity. Chapter 5 in *Comprehensive Rock Engineering: Principles, Practice and Projects, Vol. II, Analysis and Design Method*, ed. C. Fairhurst, Pergamon Press, 113–171.

[20] Evans, L., 2010. Partial Differential Equations, 2nd Ed., *AMS*, Graduate Studies in Mathematics, 19.

[21] Gurvich, E. and Webster, J.T., 2021. Weak solutions for a Poro-elastic Plate System. *Applicable Analysis*, 101(5), pp.1617–1636.

[22] Henry DB, Perissinitto A Jr, Lopes O., 1988. On the essential spectrum of a semigroup of thermoelasticity. *Nonlinear Analysis: Theory Methods Appl.*, 21, 65–75.

[23] Hosseinkhan, A. and Showalter, R.E., 2021. Biot-pressure system with unilateral displacement constraints. Journal of Mathematical Analysis and Applications, 497(1), p.124882.

[24] Lasiecka, I. and Triggiani, R., 2000. Control theory for partial differential equations (Vol. 1). Cambridge: Cambridge University Press.

[25] Mow, V.C., Kuei, S.C., Lai, W.M., and Armstrong, C.G., 1980. Biphasic creep and stress relaxation of articular cartilage in compression: Theory and experiments, *ASME J. Biomech. Eng.*, 102, pp.73–84.

[26] Murad, M.A. and Cushman, J.H., 1996. Multiscale flow and deformation in hydrophilic swelling porous media. *International Journal of Engineering Science*, 34(3), pp.313-338.

[27] Nia, H.T., Han, Li, Li, Y., Ortiz, C., and Grodzinsky, A., 2011. Poroelasticity of cartilage at the nanoscale, *Biophys. J.*, 101, pp.2304–2313.

[28] Ozkaya, N., Nordin, M., Goldsheyder, D., and Leger, D., 1999. *Fundamentals of Biomechanics. Equilibrium, Motion, and Deformation*, Springer, New York.

[29] Pazy, A., 2012. Semigroups of linear operators and applications to partial differential equations (Vol. 44). Springer Science & Business Media.

[30] Sacco, R., Guidoboni, G., and Mauri, A.G., 2019. *A Comprehensive Physically Based Approach to Modeling in Bioengineering and Life Sciences*, Elsevier Academic Press.

[31] Showalter, R.E., 1974. Degenerate evolution equations and applications. Indiana University Mathematics Journal, 23(8), pp.655-677.

[32] Showalter, R.E., 1996. *Monotone Operators in Banach Space and Nonlinear Partial Differential Equations*, AMS, Mathematical Surveys and Monographs, 49.

[33] Showalter, R.E., 2000. Diffusion in poro-elastic media, *JMAA*, 251, pp. 310–340.

[34] Storvik, E., Both, J.W., Nordbotten, J.M. and Radu, F.A., 2022. A Cahn-Hilliard-Biot system and its generalized gradient flow structure. *Applied Mathematics Letters*, 126, p.107799.

[35] Storvik, E., Both, J.W., Kumar, K., Nordbotten, J.M. and Radu, F.A., 2019. On the optimization of the fixed-stress splitting for Biot’s equations. International Journal for Numerical Methods in Engineering, 120(2), pp.179-194.

[36] Temam, R., 2001. Navier-Stokes equations: theory and numerical analysis (Vol. 343). American Mathematical Soc..

[37] Terzaghi, K., 1925. *Principle of Soil Mechanics*, Eng. News Record, A Series of Articles.

[38] Verri, M., Guidoboni, G., Bocia, L., and Sacco, R., 2018. *The Role of Structural Viscoelasticity in Deformable Porous Media with Incompressible Constituents: Applications in Biomechanics*, Mathematical Biosciences and Engineering, Volume 15, Number 4, 933–959.

[39] Zenisek, A., 1984. The existence and uniqueness theorem in Biot’s consolidation theory, *Appl. Math.*, 29., pp.194-211.