Supercurrent Flow in NJL$_{2+1}$ at High Baryon Density

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Abstract: We present results of numerical simulations of the 2+1$N$ Nambu – Jona-Lasinio model with non-zero baryon chemical potential $\mu$ and spatially-varying complex diquark source strength $j$. By choosing $\text{arg}(j)$ to vary smoothly through $2\pi$ across the spatial extent of the lattice, a baryon number current is induced which in the high density phase remains non-vanishing as $|j| \to 0$; we are hence able to extract a quantity characteristic of a superfluid known as the helicity modulus. We also study supercurrent flow at non-zero temperature and estimate the critical temperature at which the normal phase is restored, which is consistent with the conventional picture for thin-film superfluids in which the transition is viewed in terms of vortex – anti-vortex unbinding.

1 Introduction

There are unfortunately rather few quantum field theories amenable to study using lattice Monte Carlo techniques in the presence of a non-zero chemical potential $\mu$, or more specifically with $\mu/T \gg 1$. Many important theories, including QCD, cannot be studied because their path integral measure with $\mu \neq 0$ is not real on analytic continuation to Euclidean metric, making Monte Carlo importance sampling inoperative. Of those with positive definite measure, the Nambu Jona-Lasinio (NJL) model with $N_f = 2$ quark flavors [1] is one of the most interesting. At $\mu = 0$ the theory exhibits dynamical chiral symmetry breaking, with the generation of a constituent quark mass scale $\Sigma$ much larger than the bare mass $m$. For $\mu > \mu_c \approx \Sigma$, chiral symmetry is restored, and the ground state is a degenerate fermi system with $\mu = E_F \simeq k_F [2]$. In $d+1$ dimensions the baryon density in this case is $n_B = 4N_f \mu^d \theta(\mu - \mu_c)/(4\pi)^{\frac{d}{2}}d\Gamma(\frac{d}{2})$.

The precise nature of the ground state at high density depends on $d$. For the realistic case $d = 3$, lattice simulations suggest that condensation of diquark pairs at the Fermi surface takes place leading to spontaneous breakdown of the $U(1)_B$ baryon number symmetry [3]. An energy gap $\Delta > 0$ to excite fermionic quasiparticles develops; for phenomenologically-motivated lattice parameters the simulations predict $\Delta/\Sigma \simeq 0.15$, in good agreement with self-consistent model calculations of the gap in superconducting quark matter [4]. In this case the NJL model appears to behave as an orthodox BCS superfluid; there is long-range ordering of the ground state signalled by the non-vanishing
condensate $\langle qq \rangle \neq 0$, and a dynamically-generated mass scale $\Delta$. Since a $U(1)_B$ symmetry has been spontaneously broken, there is a massless scalar $qq$ bound state in the spectrum, which is associated with long-range interactions between vortex excitations in the superfluid, and with a collective propagating mode for $T > 0$ known as second sound.

However, both for obvious numerical convenience, and for a more formal reason, namely the existence of an interacting continuum limit, lattice simulations were first applied to the NJL model with $\mu \neq 0$ in 2+1 dimensions [5, 6, 7]. In this case the physics appears radically different. Whilst there is evidence for long-range coherence of diquark correlation functions [5], there is no long-range order, and apparently no gap. Rather, the condensate vanishes non-analytically with the diquark source strength, $\langle qq \rangle \propto j^\alpha$, with $0 < \alpha(\mu) < 1$ [6]. The results were interpreted in terms of a critical phase for all $\mu > \mu_c$, in which the diquark correlator decays algebraically, $\langle qq(0)qq(\vec{r}) \rangle \propto r^{-\eta}$ [7]. Since all simulations are performed on finite systems, and therefore necessarily at $T > 0$, the absence of long-range order is consistent with the Coleman-Mermin-Wagner theorem for 2$d$ systems [8]. The situation is analogous to the low-$T$ phase of the 2$d$ X-Y model, one of whose physical applications is the description of superfluidity in thin films [9].

The defining property of a superfluid is that the flux density of conserved charge, or supercurrent $J$, is related to the spatial variation of the phase angle $\theta$ of the $U(1)$-valued order parameter field (in this case $\langle qq \rangle$) via

$$\vec{J} = \Upsilon \vec{\nabla} \theta.$$  \hfill (1)

The constant of proportionality $\Upsilon$ is known as the *helicity modulus*. For a textbook non-relativistic superfluid such as $^4$He it is given by

$$\Upsilon = \frac{\hbar}{M} n_s$$  \hfill (2)

where $M$ is the mass of the helium atom and $n_s$ is a parameter called the *superfluid density*, which need not coincide with the charge density of the condensate. For a relativistic system $\Upsilon$ is best thought of as a phenomenological parameter in its own right, rather like $f_\pi^2$ in $(d+1)$-dimensional chiral model [10]. One way of understanding superfluidity in the absence of long-range order in a 2$d$ system is to observe that the only way to change the quantised circulation $\oint \vec{J}.d\vec{\ell}$ around one direction of a finite torus is to excite a vortex – anti-vortex pair, and transport one member of the pair around the other direction of the torus before re-annihilation. The energy required to do this scales as $\ln L$ where $L$ is the size of the system [9]: hence in the thermodynamic limit circulation patterns are topologically stable.

This Letter will present further support for this scenario in NJL$_{2+1}$ by extracting $\Upsilon$ via a calculation of the induced baryon number current $\vec{J} = \langle \bar{\psi} \gamma \psi \rangle$ in response to a spatially varying diquark source. As well as providing direct verification of superfluid behaviour in a fermionic model, we will also study the behaviour of $\vec{J}$ as temperature $T$ is increased, and find the transition to “normal” behaviour at a critical $T_c$ consistent with analytic expectations.
2 Method

The lattice NJL model studied is identical to that of [5, 7]:

\[ S_{NJL} = \sum \bar{\chi}_x M[\Phi]_{xy} \chi_y + j \bar{\chi}_x \tau_2 \chi_x + j \bar{\chi}_x \bar{\tau}_2 \chi_x + \frac{1}{g^2} \sum_x \text{tr}\Phi_x^\dagger \Phi_x, \]  

(3)

where \( \chi, \bar{\chi} \) are isodoublet staggered lattice fermion fields, \( \Phi = \sigma_1 \mathbb{1} + i \vec{\pi}.\vec{\tau} \) is an auxiliary bosonic field defined on the dual lattice sites \( \bar{x} \), and the matrix \( M \) is

\[ M_{pq}^{xy} = \delta_{pq} \sum_{\nu=0,1,2} \eta_{\mu x} \frac{1}{2} [e^{\mu\delta_{\nu,0}} \delta_{y,x+\nu} - e^{-\mu\delta_{\nu,0}} \delta_{y,x-\nu}] + \delta_{xy} \left\{ m \delta_{pq} + \frac{1}{8} \sum_{\bar{x},x} [\sigma_{\bar{x}} \delta_{pq} + i \varepsilon_{x} \vec{\pi}_{\bar{x},\bar{\tau}} \tau^{pq}] \right\}. \]  

(4)

Here \( \langle \bar{x}, x \rangle \) denotes the set of 8 dual sites \( \bar{x} \) surrounding \( x \), \( \eta_{\mu x} = (-1)^{x_0+\cdots+x_{\mu-1}} \) is the Kawamoto-Smit phase required for a Lorentz covariant continuum limit, and \( \varepsilon_x = (-1)^{x_0+x_1+x_2} \). A full description of the symmetries of (3) and the numerical simulation method is given in [7]. The only novelty in the current study is that the diquark source strengths \( j, \bar{j} \) are now specified to be spatially varying, or “twisted”:

\[ j = j_0 \exp(i\theta_x); \quad \bar{j} = j_0 \exp(-i\theta_x) \]  

(5)

with \( j_0 \) a real constant. To ensure homogeneity and single-valuedness on an \( L^2_s \times L_t \) lattice we demand

\[ \theta = \frac{2\pi}{L_s} (n_1 x_1 + n_2 x_2) \Rightarrow \bar{\nabla} \theta = \frac{2\pi}{L_s} (n_1, n_2). \]  

(6)

A constant supercurrent of the form (1) is therefore specified by a pair of integers \( (n_1, n_2) \). It remains to define the conserved baryon number current \( J_\nu \):

\[ J_\nu x = \frac{1}{2} \langle e^{\mu\delta_{\nu,0}} \bar{\chi}_x \chi_{x+\nu} + e^{-\mu\delta_{\nu,0}} \bar{\chi}_x \chi_{x-\nu} \rangle. \]  

(7)

The timelike component of (7) is none other than the baryon charge density \( n_B \) reported in [5, 7]. Here we shall use the same stochastic technique to estimate the quantum expectation value of the spacelike components \( \bar{J}(j, \mu) = (L^2_s L_t)^{-1} \sum_{x} \bar{J}_x(j, \mu) \) to demonstrate behaviour of the form (1). The strategy will be to compute \( \bar{J} \) for fixed \( (n_1, n_2) \) for a range of \( j_0 \) and extrapolate \( j_0 \to 0 \). Behaviour consistent with (1) in this limit is deemed to be superfluid.

We used the same simulation parameters as [5, 7], namely \( g^2 = 2.0a, m a = 0.01 \), which at \( \mu = 0 \) yields a dynamically-generated constituent mass, which in effect sets the scale, of \( \Sigma a = 0.71 \). As \( \mu \) is raised, there is a sharp transition from a chirally broken vacuum with \( \langle \bar{\chi}\chi \rangle \simeq \frac{2}{g^2} \Sigma, n_B \simeq 0 \) to a chirally restored phase with \( n_B > 0 \) at \( \mu_c a \simeq 0.65 \). Studies of the fermion dispersion relation in the phase \( \mu > \mu_c \) are consistent with a sharp Fermi surface with \( k_F \lesssim \mu \) and vanishing gap \( \Delta \simeq 0 \) [5, 7].
3 Results

3.1 $T = 0$

Figure 1: $J_2$ vs. $j_0$ on a $16^2 \times L_t$ lattice for two different values of $\mu$.

In Fig. 1 we plot $J_2$ (strictly its imaginary part) as a function of $j_0$ for lattices of various temporal extent $L_t$ at two representative values of $\mu$: $\mu a = 0.2$ lies in the chirally-broken low density phase, and $\mu a = 0.8$ in the high density phase, where $n_{BA} a^2 \simeq 0.25$ [7]. In all the plots shown here we have chosen $(n_1, n_2) = (0, 1)$ to minimise lattice artifacts, and from now on we set the lattice spacing $a = 1$.

The contrast between the two phases is quite dramatic. For $\mu = 0.2$ $J_2$ appears to vary approximately quadratically with $j_0$, and extrapolate to zero as $j_0 \rightarrow 0$. There is no significant effect as $L_t \rightarrow \infty$, or alternatively as $T \rightarrow 0$. At $\mu = 0.8$ the small-$j_0$ behaviour depends very sensitively on $L_t$; as $T \rightarrow 0$ the data accumulate on a straight line which clearly extrapolates to a non-zero value as $j_0 \rightarrow 0$.

This behaviour is readily explained by writing the order parameter (diquark) field as $\phi = \phi_0 e^{i\theta}$, with $\phi_0$ approximately constant. A natural effective Hamiltonian for long wavelength order parameter fluctuations at low temperature is then

$$H_{eff} = \frac{1}{2} (\vec{\nabla} \phi)^* (\vec{\nabla} \phi) \simeq \frac{\phi_0^2}{2} (\vec{\nabla} \theta)^2.$$ (8)

The corresponding Noether current is $\vec{J} = -\frac{i}{2} [\phi^* \vec{\nabla} \phi - (\vec{\nabla} \phi^*) \phi] \simeq \phi_0^2 \vec{\nabla} \theta$. For $\mu < \mu_c$, 

4
it is natural to postulate $\phi$ proportional to $j$, leading to $J_2(j_0) \propto j_0^2$. For $\mu > \mu_c$, if we assume that $\lim_{j_0 \to 0} \phi_0 \neq 0$ we recover (11) with $\Upsilon = \phi_0^2$.

Figure 2: $J_2$ vs. $j_0$ on a $16^2 \times 64$ lattice for various $\mu$.

With confidence that $L_t = 64$ suffices to determine the phase, in Fig. 2 we plot $J_2(j_0)$ for various $\mu$. There is a sharp change between values of $\mu \leq 0.65$, which display the low density quasi-quadratic behaviour and smoothly extrapolate to zero as $j_0 \to 0$, and $\mu \geq 0.68$ which show a negative curvature characteristic of the high density phase. We thus determine the critical chemical potential for the onset of superfluidity $0.65 < \mu_c < 0.68$, in good agreement with the critical value for chiral symmetry restoration. Since as yet we have no systematic method of extrapolating to $j_0 \to 0$ for $\mu \geq \mu_c$ to obtain an estimate for $J_2(\mu)$ as an “order parameter”, we can make no decisive statement about the nature of the transition, but note that the behaviour of $J_2(j_0)$ varies much more sharply across the transition than the diquark condensate $\langle qq_+ (j) \rangle$, either in this model (Cf. Figure 2 of [7]), or even in NJL$_{3+1}$ (Cf. Figure 4 of [3]). This matches the sharp drop in the chiral order parameter $\langle \bar{\chi} \chi \rangle$ and corresponding rise in $n_B$ at $\mu = \mu_c$ [5, 7], and is consistent with the analytic prediction of a strong first order transition in the large-$N_f$ limit [11].

From now on we work exclusively at $\mu = 0.8$ in an attempt to understand the superfluid phase further. In Fig. 3 we plot $\Upsilon = J_2 L_s / 2\pi$ versus $j_0$ on $L_s^2 \times 64$ lattices. Just as for the $\langle qq_+ (j) \rangle$ data [7], it turns out that the data are well-fitted by $\Upsilon(j_0) = A + B / L_s$, resulting in the $L_s \to \infty$ extrapolation shown in the plot. Recall that as well as genuine finite-size effects in this case there may also be some discretisation artifacts, since as $L_s$ increases the gradient operator in (6) becomes better-approximated by the finite
difference. Finally the $\Upsilon(j_0)$ data in the thermodynamic limit are extrapolated to $j_0 \to 0$ with a remarkably simple linear fit, resulting in $\Upsilon = 0.1413(14)$. We thus quote a result for the helicity modulus of $\Upsilon/\Sigma = 0.200(2)$ at $\mu a = 0.8$.

It is interesting to pause and ask what value might be expected for $\Upsilon$ in a conventional symmetry-breaking scenario. Let us define diquark operators $qq^\pm = \frac{1}{2}(\chi^\text{tr} \tau_2 \chi \pm \bar{\chi}_\tau \tilde{\chi}^\text{tr})$ and source strengths $j^\pm = j \pm \bar{j}$, so that the diquark terms in the action (3) read $j^\pm qq^\pm$. In the limit $j^- = 0$ the equation of motion for the current is then

$$\Delta^- j_\mu = 2j^+ qq^-.$$  \hspace{1cm} (9)

In the same limit the $\text{U}(1)_B$-equivalent form of the axial Ward identity reads

$$\langle qq^+ \rangle = j^+ \sum_x \langle qq^-(0)qq^-(x) \rangle = \frac{j^+}{M^2} |\langle 0|qq^-|\rangle|^2.$$  \hspace{1cm} (10)

where the second equality assumes that the correlation function is dominated by a pseudo-Goldstone pole of the form $(k^2 + M^2)^{-1}$, and $|\rangle$ denotes a one-Goldstone state. We now introduce the $\text{U}(1)_B$-equivalent form of the PCAC hypothesis:

$$\langle 0|\Delta^- j_\mu|\rangle = \sqrt{\Upsilon} M^2 = 2j^+ \langle 0|qq^-|\rangle$$  \hspace{1cm} (11)

where we have used the relation $\Upsilon = f_2^2$ derived in [10], and the second equality follows from (9). Combining (10) and (11) leads to the equivalent of the “Gell-Mann-Oakes-Renner” relation:

$$\Upsilon_{\text{GMOR}} M^2 = 8j \langle qq^+ \rangle.$$  \hspace{1cm} (12)
This can be compared with numerical data for \( \langle qq_+(j) \rangle \) and \( M_- (j) \) in [7]. At \( j = 0.3, M_- = 0.95, \langle qq_+ \rangle = 0.72 \) yielding \( \Upsilon_{GMOR} \approx 1.9 \); at \( j = 0.1, M_- = 0.4, \langle qq_+ \rangle = 0.52 \) yielding \( \Upsilon_{GMOR} \approx 2.6 \). We conclude \( \Upsilon \ll \Upsilon_{GMOR} \), consistent with our hypothesis that no symmetry breaking occurs, but that the dynamics are dominated by long-range phase fluctuations of the order parameter field, described by a strongly-interacting scalar diquark field rather than a weakly-interacting Goldstone mode.

### 3.2 \( T > 0 \)

In this section for the first time we explore the superfluid phase at non-zero temperature. We expect a restoration to the normal phase at some critical \( T_c \). In a comparable numerical study of NJL\(_{3+1} \) which exhibits superfluidity via orthodox symmetry breaking [3], the value of \( T_c \) could be estimated from the zero temperature gap \( \Delta \) via the BCS prediction \( \Delta / T_c \approx 1.76 \). Since this implied that \( L_t \) had to exceed 35\( a \) for the system to be superfluid, an unambiguous extrapolation \( j \to 0 \) to permit a systematic study of \( T > 0 \) was not possible. In the current case we shall see that although the \( j \to 0 \) extrapolation still remains a problem, attaining \( T < T_c \) is well within reach of the simulation.

First let us review a heuristic argument for the expected value of \( T_c \), starting from the Hamiltonian \( H_{eff} \) with \( \phi_0^2 = \Upsilon \) [9]. The phase field \( \theta(\vec{x}) \) may be disrupted by topologically non-trivial vortex excitations of the form \( \theta = q\psi, |\vec{\nabla}\theta| = q/r \), where \( q \) is integer and \( \vec{x} \) is written \((r, \psi)\). The energy of a single vortex is thus

\[
E \approx \frac{\Upsilon}{2} \int_a^{L_s} 2\pi r dr \left( \frac{q}{r} \right)^2 = \pi \Upsilon q^2 \ln \left( \frac{L_s}{a} \right) .
\]

Since a vortex can be located on any of \( L_s^2 \) lattice sites, the entropy

\[
S = 2 \ln \left( \frac{L_s}{a} \right) .
\]

The free energy \( F = E - TS \) thus changes sign for \( q = 1 \) vortices at a critical temperature

\[
T_c = \frac{\pi}{2} \Upsilon .
\]

The interpretation is that a phase transition separates a low-\( T \) superfluid phase in which vortices are confined to bound dipole pairs, and a high-\( T \) normal phase in which the vortex anti-vortex plasma screens the long-range interactions responsible for the divergent energy in (13). The relation (15) remains valid in a more sophisticated renormalisation group treatment, except that \( \Upsilon \) is now \( T \)-dependent and should be replaced by its value \( \Upsilon(T_c) \) exactly at the transition [12].

Combining our result for \( \Upsilon \) with (15) yields an estimate \( L_t \approx 4.5 \) for the temporal lattice extent where a transition to the normal phase might be expected at \( \mu = 0.8 \). Fig. 4 shows \( J_2(j_0) \) on \( 32^2 \times L_t \) lattices with \( L_t \) ranging from 64 all the way down to 2. At the extremes \( L_t \geq 32, L_t \leq 4 \) the data are reminiscent of those characterising respectively the
Figure 4: $J_2$ vs. $j_0$ on a $32^2 \times L_t$ lattice at $\mu = 0.8$ for various $L_t$.

high and low baryon density phases in Fig. 1. For intermediate temperatures, however, $J_2(j_0)$ shows positive curvature near the origin followed by negative curvature at larger $j_0$, and once again the means of extrapolating $j_0 \to 0$ to determine whether superfluidity

Figure 5: $J_2$ vs. $L_t$ on a $32^2 \times L_t$ lattice at $\mu = 0.8$ for fixed $j_0 = 0.025$. 

8
persists is unclear.

In Fig. 5 we try a different tactic, plotting $J_2$ for every even $L_t \in [2, 64]$ for fixed $j_0 = 0.025$. A linear fit $J_0 = a_0 L_t + a_1$ through data with $L_t \leq 42$ seems quite reasonable, yielding $a_0 = 0.00212(25)$, $a_1 = -0.01537(9)$. If we identify the intercept on the $L_t$-axis with the transition, we deduce $L_{tc} = 7.25(95)$ and hence $\Upsilon/T_c = 1.02(13)$, to be compared with the theoretical value 0.637 from \cite{15}.

4 Summary

In this short study of the response of the system to a twisted diquark source forcing a baryon number current, we have provided direct evidence for the superfluid nature of the ground state of NJL$_{2+1}$ at high baryon density, and quantified it at one representative value of $\mu$ via the helicity modulus $\Upsilon$. It should be stressed that the “physical” value $\Upsilon/\Sigma \simeq 0.2$ quoted is still to be extrapolated to the continuum limit. It is probably more important to note that the numerical value of $\Upsilon$ is an order of magnitude smaller than might be expected in an orthodox symmetry breaking scenario, and is consistent with the non-Goldstone, strongly self-interacting nature of the scalar diquark excitations above the ground state.

We also studied the response of the system to non-zero temperature. Whilst we were unable to extrapolate to the zero source limit in a controlled way, by studying fixed $j_0$ we were able to estimate a critical temperature $T_c$ for breakdown of superfluidity of the same order of magnitude as, and only slightly smaller than, the Kosterlitz-Thouless prediction for a 2d system with U(1) global symmetry, which follows from characterising the superfluid/normal transition as arising from vortex pair unbinding. More refined simulations would be required to determine whether $T_c$ actually has the KT value, or whether NJL$_{2+1}$, which in addition to the scalar diquark excitations contains massless fermion degrees of freedom, actually lies in a different universality class, as suggested by estimates of the critical exponent $\delta$ \cite{14}.

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