Concentration estimates for the isoperimetric constant of the super critical percolation cluster

Eviatar B. Procaccia∗, Ron Rosenthal†
November 9, 2011

Abstract
We consider the Cheeger constant \( \phi(n) \) of the giant component of supercritical bond percolation on \( \mathbb{Z}^d/n\mathbb{Z}^d \). We show that the variance of \( \phi(n) \) is bounded by \( \xi n^d \), where \( \xi \) is a positive constant that depends only on the dimension \( d \) and the percolation parameter.

1 Introduction
Let \( \mathbb{T}^d(n) \) be the \( d \) dimensional torus with side length \( n \), i.e., \( \mathbb{Z}^d/n\mathbb{Z}^d \), and denote by \( \mathbb{E}_d(n) \) the set of edges of the graph \( \mathbb{T}^d(n) \). Let \( p_c(\mathbb{Z}^d) \) denote the critical value for bond percolation on \( \mathbb{Z}^d \), and fix some \( p_c(\mathbb{Z}^d) < p \leq 1 \). We apply a \( p \)-bond Bernoulli percolation process on the torus \( \mathbb{T}^d(n) \) and denote by \( C_d(n) \) the largest open component of the percolated graph (In case of two or more identically sized largest components, choose one by some arbitrary but fixed method). Let \( \Omega = \Omega_n = \{0, 1\}^{\mathbb{E}_d(n)} \) be the space of configurations for the percolation process and \( P = P_p \) is the probability measure associated with the percolation process. For a subset \( A \subset C_d(n) \) we denote by \( \partial C_d(n)A \) the boundary of the set \( A \) in \( C_d(n) \), i.e, the set of edges \( (x, y) \in \mathbb{E}_d(n) \) such that \( \omega((x, y)) = 1 \) and with either \( x \in A \) and \( y \notin A \) or \( x \notin A \) and \( y \in A \). Throughout this paper \( c, C \) and \( c_i \) denote positive constants which may depend on the dimension \( d \) and the percolation parameter \( p \) but not on \( n \). The value of the constants may change from one line to the next.

Next we define the Cheeger constant

Definition 1.1. For a set \( \emptyset \neq A \subset C_d(n) \) we denote,

\[
\psi_A = \frac{|\partial C_d(n)A|}{|A|},
\]

where \( |\cdot| \) denotes the cardinality of a set. The Cheeger constant of \( C_d(n) \) is defined by:

\[
\phi = \phi(n) := \min_{\emptyset \neq A \subset C_d(n)} \psi_A.
\]

∗Weizmann Institute of Science
†Hebrew University of Jerusalem
In [BM03] Benjamini and Mossel studied the robustness of the mixing time and Cheeger constant of $\mathbb{Z}^d$ under a percolation perturbation. They showed that for $p_c(\mathbb{Z}^d) < p < 1$ large enough $n\phi(n)$ is bounded between two constants with high probability. In [MR04], Mathieu and Remy improved the result and proved the following on the Cheeger constant:

**Theorem 1.2.** There exist constants $c_2, c_3, c > 0$ such that for every $n \in \mathbb{N}$

$$\mathbb{P}\left(\frac{c_2}{n} \leq \phi(n) \leq \frac{c_3}{n}\right) \geq 1 - e^{-c\log^2 n}.$$

Recently, Marek Biskup and Gábor Pete brought to our attention that better bounds on the Cheeger constant exist. In [Pet07] and [BBHK08] it is shown that:

**Theorem 1.3 ([Pet07]).** For $d \geq 2$ and $p > p_c(\mathbb{Z}^d)$, there are constants $\alpha(d, p) > 0$ and $\beta(d, p) > 0$ such that

$$\mathbb{P}\left(\exists S \text{ connected} : 0 \in S \subset C_\infty, M \leq |S| < \infty, \frac{|\partial S|}{|S|^{(d-1)/d}} \leq \alpha\right) \leq \exp\left(-\beta M^{(d-1)/d}\right)$$

The improved bounds don’t improve our result, thus we kept the original [MR04] bounds in our proofs.

**Conjecture 1.4.** The limit $\lim_{n \to \infty} n\phi(n)$ exists.

Even though the last conjecture is still open, and the expectation of the Cheeger constant is quite evasive, we managed to give a good bound on the variance of the Cheeger constant. This is given in the main Theorem of this paper:

**Theorem 1.5.** There exists a constant $\xi = \xi(p, d) > 0$ such that

$$\text{Var}(\phi) \leq \frac{\xi}{n^d}.$$

A major ingredient of the proof is Talagrand’s inequality for concentration of measure on product spaces. This inequality is used by Benjamini, Kalai and Schramm in [BKS03] to prove concentration of first passage percolation distance. A related study that uses another inequality by Talagrand is [AKV02], where Alon, Krivelevich and Vu prove a concentration result for eigenvalues of random symmetric matrices.

## 2 The Cheeger constant

Before turning to the proof of Theorem 1.5, we give the following definitions:

**Definition 2.1.** For a function $f : \Omega \to \mathbb{R}$ and an edge $e \in E(n)$ we define $\nabla_e f : \Omega \to \mathbb{R}$ by

$$\nabla_e f(\omega) = f(\omega) - f(\omega^e)$$
where
\[ \omega'(\omega') = \begin{cases} 
\omega(\omega') & \omega' \neq \omega \\
1 - \omega(\omega') & \omega' = \omega.
\end{cases} \]

In addition, for a configuration \( \omega \in \Omega \) and an edge \( e \in \mathcal{E}_d(n) \), let \( \bar{\omega} = \min\{\omega, \omega^e\} \) and \( \bar{\omega}^e = \max\{\omega, \omega^e\} \).

**Definition 2.2.** For \( n \in \mathbb{N} \) we define the following events:

\[
H^1_n(c_1) = \{ \omega \in \Omega : |C_d(n)(\omega)| > c_1n^d \}
\]

\[
H^2_n(c_2, c_3) = \left\{ \omega \in \Omega : \frac{c_2}{n} < \phi(n)(\omega) < \frac{c_3}{n} \right\}
\]

\[
H^3_n = \{ \omega \in \Omega : \forall e \in \mathcal{E}_d(n) \quad |C_d(n)(\omega) \Delta C_d(n)(\omega^e)| \leq \sqrt{n} \}, \tag{2.1}
\]

\[
H^4_n(c_4) = \{ \omega \in \Omega : \exists A : |A| > c_4n^d, \psi_A(\omega) = \phi(n)(\omega) \}
\]

\[
H^5_n(c_5) = \{ \omega \in \Omega : \exists A : |A| > c_5n^d, \psi_A(\omega^e) = \phi(n)(\omega^e) \}
\]

and
\[
H_n = H^1_n(c_1, c_2, c_3, c_4, c_5) = H^1_n(c_1) \cap H^2_n(c_2, c_3) \cap H^3_n \cap H^4_n(c_4) \cap H^5_n(c_5). \tag{2.2}
\]

We start with the following deterministic claim:

**Claim 2.3.** Given \( c_1, c_2, c_3, c_4, c_5 > 0 \), there exists a constant \( C = C(c_1, c_2, c_3, c_4, c_5, d, p) > 0 \) such that if \( \omega \in H_n(c_1, c_2, c_3, c_4, c_5) \) then for every \( e \in \mathcal{E}_d(n) \)
\[
|\nabla_e \phi(\omega)| \leq \frac{C}{n^d}.
\]

In order to prove Claim 2.3 we will need the following two lemmas:

**Lemma 2.4.** Fix a configuration \( \omega \in \Omega \) and an edge \( e \in \mathcal{E}_d(n) \). Let \( A \subset C_d(n)(\bar{\omega}^e) \) be a subset such that \( |A| = \alpha n^d \). Then
\[
|\nabla_e \psi_A| \leq \frac{1}{\alpha n^d}.
\]

**Proof.** Since \( A \) is a subset of \( C_d(n)(\bar{\omega}^e) \) it follows that the size of \( A \) doesn’t change between the configurations \( \bar{\omega}^e \) and \( \bar{\omega} \) and the size of \( \partial C_d(n)A \) is changed by at most 1. It therefore follows that
\[
|\nabla_e \psi(A)| = |\psi_A(\omega) - \psi_A(\omega^e)| = |\psi_A(\bar{\omega}^e) - \psi_A(\bar{\omega})| \leq \frac{|\partial A|}{|A|} + \frac{1}{|A|} = \frac{1}{|A|}. \tag{2.3}
\]

**Lemma 2.5.** Let \( G \) be a finite graph, and let \( A, B \subset G \) be disjoint such that there exists a unique edge \( e = (x, y) \), such that \( x \in A \) and \( y \in B \), then
\[
\psi_{A \cup B} \geq \min\{\psi_A, \psi_B\} - \frac{2}{|A| + |B|}.
\]
Proof. From the assumptions on $A$ and $B$ it follows that

$$\psi_{A\cup B} = \frac{|\partial (A \cup B)|}{|A \cup B|} = \frac{|\partial A| + |\partial B| - 2}{|A| + |B|} \geq \min \left\{ \frac{|\partial A|}{|A|}, \frac{|\partial B|}{|B|} \right\} - \frac{2}{|A| + |B|},$$

and so the lemma follows. \hfill \Box

Proof of Claim 2.3. We separate the proof into six different cases according to the following table:

| $e=(x,y)$ | $\omega(e) = 0$ | $\omega(e) = 1$ |
|---|---|---|
| $x, y \notin C_d(n)$ | 1 | 2 |
| $x, y \in C_d(n)$ | 3 | 4 |
| $x \in C_d(n), y \notin C_d(n)$ or $y \in C_d(n), x \notin C_d(n)$ | 5 | 6 |

- **Cases 1 and 2:** In those cases the set $C_d(n)$ and the edges available from it is the same for both configurations $\omega$ and $\omega^e$. It therefore follows that $\nabla_e \phi(\omega) = 0$. See Figure 2.1a and 2.1b.

- **Case 3:** In this case the set $C_d(n)$ is the same for both configurations $\omega$ and $\omega^e$, however the set of edges available from $C_d(n)$ is increased by one when moving to the configuration $\omega^e$, see figure 2.1c. Fix a set $A \subset C_d(n)(\omega)$ of size bigger than $c_4 n^d$ which realize the Cheeger constant. It follows that

$$\psi_A(\omega) = \phi(\omega) \leq \phi(\omega^e) \leq \psi_A(\omega^e),$$

and therefore by Lemma 2.4 we have

$$|\phi(\omega^e) - \phi(\omega)| \leq \psi_A(\omega^e) - \psi_A(\omega) \leq \frac{1}{c_4 n^d},$$

as required.

- **Case 4:** We separate this case into two subcases according to the fact whether $C_d(n)(\omega) \setminus C_d(n)(\omega^e)$ is an empty set or not. If $C_d(n)(\omega) \setminus C_d(n)(\omega^e) = \emptyset$ then we are in the same situation as in Case 3, see Figure 2.1d, and so the same argument gives the desired result. So, let us assume that $C_d(n)(\omega) \setminus C_d(n)(\omega^e) \neq \emptyset$, see Figure 2.1e. Since $\omega \in H_n$ we know that

$$|C_d(n)(\omega) \setminus C_d(n)(\omega^e)| \leq \sqrt{n}. \tag{2.5}$$

Since $\omega \in H_n^4$ there exists a set $A \subset C_d(n)(\omega)$ of size bigger than $c_4 n^d$ realizing the Cheeger constant in the configuration $\omega$. We denote $A_1 = A \cap C_d(n)(\omega^e)$ and $A_2 = A \cap (C_d(n)(\omega) \setminus C_d(n)(\omega^e))$. Applying Lemma 2.3 to $A_1$ and $A_2$ we see that

$$\psi_A(\omega) = \psi_{A_1 \cup A_2}(\omega) \geq \min \{ \psi_{A_1}(\omega), \psi_{A_2}(\omega) \} - \frac{2}{|A|}. \tag{2.6}$$
Figure 2.1: Illustrations of the different cases
From [2.5] it follows that \(|A_2| \leq \sqrt{n}\) and therefore \(\psi_{A_2}(\omega) \geq \frac{1}{\sqrt{n}}\) which gives us that\(\min\{\psi_{A_1}(\omega), \psi_{A_2}(\omega)\} = \psi_{A_1}(\omega)\). Indeed, if the last equality doesn’t hold then
\[
\frac{c_2}{n} \geq \psi_A(\omega) \geq \psi_{A_2}(\omega) - \frac{2}{|A|} \geq \frac{1}{\sqrt{n}} - \frac{2}{c_4 n^d},
\]
which for large enough \(n\) yields a contradiction. Consequently from [2.6] we get that
\[
\psi_{A_1}(\omega) - \frac{2}{c_4 n^d} \leq \phi(\omega) \leq \psi_{A_1}(\omega),
\]
and so
\[
\phi(\omega^e) - \frac{2}{c_4 n^d} \leq \psi_{A_1}(\omega^e) - \frac{2}{c_4 n^d} \leq \psi_{A_1}(\omega) - \frac{2}{c_4 n^d} \leq \phi(\omega),
\]
i.e, \(\phi(\omega^e) - \phi(\omega) \leq \frac{2}{c_4 n^d}\).

For the other direction, since \(\omega \in H^5_n\) there exists a set \(B \subset C_d(n)(\omega^e)\) of size bigger than \(c_5 n^d\) realizing the Cheeger constant in \(\omega^e\), then
\[
\phi(\omega) \leq \psi_B(\omega) \leq \psi_B(\omega^e) + \frac{1}{|B|} = \phi(\omega^e) + \frac{1}{|B|} \leq \phi(\omega^e) + \frac{1}{c_5 n^d}.
\]

Consequently,
\[
|\phi(\omega) - \phi(\omega^e)| \leq \max \left\{ \frac{2}{c_4 n^d}, \frac{1}{c_5 n^d} \right\},
\]
as required.

- **Case 5:** This case is similar to **Case 4**, see Figure 2.11. The proof of this case follows the proof of **Case 4** above.

- **Case 6:** This case is impossible by the definition of the set \(C_d(n)(\omega)\).

Next we turn to estimate the probability of the event \(H_n\).

**Claim 2.6.** There exist constants \(c_1, c_2, c_3, c_4, c_5 > 0\) and a constant \(c > 0\) such that for large enough \(n \in \mathbb{N}\) we have
\[
P(H_n^c) \leq e^{-c \log^3 n}. \tag{2.7}
\]

**Proof.** Since \(P(H_n^c) \leq \sum_{i=1}^5 P((H_i^1)^c)\), it’s enough to bound each of the last probabilities. The proof of the exponential decay of \(P((H_n^1)^c)\) for appropriate constant is presented in the Appendix.

By [MR04] Theorem 3.1 and section 3.4, there exists a \(c > 0\) such that for \(n\) large enough, \(P((H_n^2)^c) \leq e^{-c \log^{1/2} n}\) for some constants \(c_2, c_3 > 0\).
By [MR04] Appendix B and [Gri99] Theorem 8.61, for large enough $n$ we have that, the occurrence of this event implies the existence of an open cluster of size bigger than $\sqrt{n}$ which is not connected to $C_d(n)$, and therefore its probability is bounded by

$$
\Pr((H_n^3)^c) = \frac{1}{1-p} \Pr \left( \{ \omega \in \Omega : \exists e \in E_d(n) \ |C_d(n)(\omega)\triangle C_d(n)(\omega^c)| \geq \sqrt{n}, \ e \text{ is closed} \} \right)
$$

$$
\leq \frac{1}{1-p} \Pr \left( \{ \omega \in \Omega : \exists e \in E_d(n) \ |C_d(n)(\omega)\triangle C_d(n)(\omega^c)| \geq \sqrt{n}, \ e \text{ is closed} \} \cap H_n^1 \right)
$$

$$
+ \frac{1}{1-p} \Pr((H_n^4)^c).
$$

(2.8)

We already gave appropriate bound for the last term and therefore we are left to bound the probability of $\{ \omega \in \Omega : \exists e \in E_d(n) \ |C_d(n)(\omega)\triangle C_d(n)(\omega^c)| \geq \sqrt{n}, \ e \text{ is closed} \} \cap H_n^1$. Notice that the occurrence of this event implies the existence of an open cluster of size bigger than $\sqrt{n}$ which is not connected to $C_d(n)$.

By [MR04] Appendix B and [Gri99] Theorem 8.61, for large enough $n$ we have that,

$$
\Pr(F \cap H_n^1) := \Pr(\{ \exists B, |B| \geq \sqrt{n}, B \text{ is an open cluster that is not connected to } C_d(n) \} \cap H_n^1).
$$

By (2.8) we define one last event

$$
G_n = \left\{ I_{\epsilon(n)}(C_d(n)) \geq c_6 n^{d(\frac{4}{\xi(n)})-1} \right\},
$$

where $\epsilon(n) = d + 2d \frac{\log \log n}{\log n}$ and

$$
I_{\epsilon}(C_d(n)) = \min_{\emptyset \neq A \in C_d(n)} \frac{|\partial C_d(n)A|}{|A|^{(\epsilon-1)/\epsilon}}.
$$

(2.10)

By [MR04] there exists a constant $c > 0$ such that for large enough $n \in \mathbb{N}$ $\Pr(G_n^c) < e^{-c \log^2 n}$. As before we write

$$
\Pr((H_n^4)^c) \leq \Pr((H_n^4)^c \cap H_n^1 \cap H_n^2 \cap G_n) + \Pr((H_n^1)^c \cup (H_n^2)^c \cup G_n^c),
$$

and by the probability bound mentioned so far it’s enough to bound the probability of the first event $(H_n^4)^c \cap H_n^1 \cap H_n^2 \cap G_n$. What we will actually show is that for appropriate choice of $0 < c_4 < \frac{1}{2}$ we have $(H_n^4)^c \cap H_n^1 \cap H_n^2 \cap G_n = \emptyset$. Indeed, since we assumed the event $G_n$ occurs we have that for large enough $n \in \mathbb{N}$ and every set $A \subset C_d(n)(\omega)$ of size smaller than $c_4 n^d$

$$
|\partial C_d(n)A| \geq c_6 n^{d(\frac{4}{\xi(n)})-1}|A|^{(\epsilon(n)-1)/\epsilon(n)}.
$$
It follows that
\[ \psi_A \geq c_6 n^{d-1} \frac{1}{|A|^{1/\epsilon(n)}} \geq c_6 n^{\epsilon(n)-1} \frac{1}{c_4^{1/\epsilon(n)} n^{d/\epsilon(n)}} = \frac{c_6}{c_4^{1/\epsilon(n)} n}. \quad (2.11) \]
Choosing \( c_4 > 0 \) such that for large enough \( n \in \mathbb{N} \) we have \( \frac{c_6}{c_4^{1/\epsilon(n)}} > c_3 \), we get a contradiction to the event \( H_n^2 \), which proves that the event is indeed empty.

Finally we turn to deal with the event \((H_n^5)^c\). As before it’s enough to bound the probability of the event \((H_n^5)^c \cap H_n^1 \cap H_n^2 \cap H_n^3 \cap H_n^4 \cap G_n \). We divide the last event into two disjoint events according to the status of the edge \( e \), namely
\[
\begin{align*}
V_n^0 &= (H_n^5)^c \cap H_n^1 \cap H_n^2 \cap H_n^3 \cap H_n^4 \cap G_n \cap \{ \omega(e) = 0 \} \\
V_n^1 &= (H_n^5)^c \cap H_n^1 \cap H_n^2 \cap H_n^3 \cap H_n^4 \cap G_n \cap \{ \omega(e) = 1 \},
\end{align*}
\]
and will show that for right choice of \( c_5 \) both \( V_n^0 \) and \( V_n^1 \) are empty events.

Let us start with \( V_n^0 \). Going back to the proof of Claim 2.3 one can see that under the event \( H_n^1 \cap H_n^2 \cap H_n^3 \cap H_n^4 \) there exists a constant \( c > 0 \) such that
\[ \phi(e) \leq \frac{c}{n^d} \leq \frac{c_3}{n^d}, \]
and therefore \( \phi(\omega^c) \leq \tilde{c}_3 > c_3 \) and \( n \in \mathbb{N} \) large enough. If \( \emptyset \neq A \subset C_d(n)(\omega^e) \) is a set of size smaller than \( \frac{n}{\tilde{c}_3} \) then
\[ \psi_A(\omega^e) \geq \frac{1}{|A|} > \frac{\tilde{c}_3}{n}, \]
and therefore \( A \) cannot realize the Cheeger constant. On the other hand, if \( A \subset C_d(n)(\omega^c) \) satisfy \( \frac{n}{\tilde{c}_3} \leq |A| \leq c_5 n^d \) then
\[ |\partial_{C_d(n)(\omega^e)} A| \geq |\partial_{C_d(n)(\omega^c)} (A \cap C_d(n)(\omega))| - 1 \geq |\partial_{C_d(n)(\omega)} (A \cap C_d(n)(\omega))| - 2, \]
and therefore (Since we assumed the event \( G \) occurs)
\[ \psi_A(\omega^e) \geq \frac{|\partial_{C_d(n)(\omega)} (A \cap C_d(n)(\omega))| - 2}{|A|} \geq \frac{c_6 n^{d/\epsilon(n)-1}}{|A|} - 2 \frac{1}{|A|} = \frac{c_6}{2c_5^{1/\epsilon(n)} n} - \frac{2\tilde{c}_3}{n}. \quad (2.15) \]
Taking \( c_5 > 0 \) small enough such that \( \frac{c_6}{2c_5^{1/\epsilon(n)} n} - 2\tilde{c}_3 > \tilde{c}_3 \) we get a contradiction to (2.13). It follows that no set \( A \subset C_d(n)(\omega^c) \) of size smaller than \( c_5 n^d \) can realize the Cheeger constant which contradicts \((H_n^5)^c\), i.e., \( V_n^0 = \emptyset \).

Finally, for \( V_n^1 \). The case \( A \subset C_d(n)(\omega^e) \) such that \( |A| < \frac{n}{\tilde{c}_3} \) is the same as for the event \( V_n^0 \). If \( A \subset C_d(n)(\omega^c) \) satisfy \( \frac{n}{\tilde{c}_3} \leq |A| \leq c_5 n^d \) then
\[ |\partial_{C_d(n)(\omega^c)} A| \geq |\partial_{C_d(n)(\omega)} A| - 1. \]
and therefore as in the case of $V_n^0$

$$\psi_A(e^n) \geq \frac{|\partial C_d(n)(\omega)| - 1}{|A|} \geq c_6 n^{d/\epsilon(n) - 1} |\partial C_d(n)(\omega)| A| - 1 \geq c_6 \frac{2c_5^{1/2} n}{n} - \frac{c_3}{n}. \tag{2.16}$$

Choosing $c_5$ small enough, we again get a contradiction to (2.13), and as before this yields that $V_n^1 = \emptyset$.

**Proof of theorem 1.5.** By [Tal94] (Theorem 1.5) the following inequality holds for some $K = K(p)$,

$$\text{Var}(\phi) \leq K \cdot \sum_{e \in E(C_n(n))} \frac{||\nabla_e \phi||_2^2}{1 + \log (||\nabla_e \phi||_2/||\nabla_e \phi||_1)}. \tag{2.17}$$

Where $||\nabla_e \phi||_2^2 = E[(\nabla_e \phi)^2]$ and $||\nabla_e \phi||_1 = E[|\nabla_e \phi|]$. Observe that $||\nabla_e \phi||_1 = ||\nabla_e \phi 1_{\{\nabla_e \phi \neq 0\}}||_1 \leq ||\nabla_e \phi||_2 1_{\{\nabla_e \phi \neq 0\}}||_2$, and therefore

$$\frac{||\nabla_e \phi||_2}{||\nabla_e \phi||_1} \geq \frac{1}{\sqrt{\mathbb{P}(\nabla_e \phi \neq 0)}} \geq 1.$$

Consequently, if we fix some edge $e_0 \in E_d(n)$,

$$\text{Var}(\phi) \leq K \cdot \sum_{e \in E(C_n(n))} ||\nabla_e \phi||_2^2 = K|E_d(n)| \cdot ||\nabla_{e_0} \phi||_2^2 = Kdn^d \cdot ||\nabla_{e_0} \phi||_2^2. \tag{2.18}$$

where the first equality follows from the symmetry of $T_d(n)$.

$$||\nabla_{e_0} \phi||_2^2 = E[|\nabla_{e_0} \phi|^2 1_{H_n}] + E[|\nabla_{e_0} \phi|^2 1_{H_n^c}]. \tag{2.19}$$

Notice that since $|\nabla_{e_0} \phi| \leq 4d$ we have $E[|\nabla_{e_0} \phi|^2 1_{H_n}] \leq 16d^2 \mathbb{P}(H_n^c)$. Thus applying Lemma 2.6,

$$E[|\nabla_{e_0} \phi|^2 1_{H_n}] \leq 16d^2 e^{-c \log^2 2(n)}, \tag{2.20}$$

and by Lemma 2.3

$$E[|\nabla_{e_0} \phi|^2 1_{H_n}] \leq \frac{C^2}{n^{2d}}. \tag{2.21}$$

Thus combining equations (2.20) and (2.21) with equation (2.18) the result follows.

3 Appendix

In this Appendix for completeness and future reference we sketch a proof of the exponential decay of $\mathbb{P}((H_n^c)^c)$. The proof follows directly from two papers [DP96] by Deuschel and Pistorza and [AP96] by Antal Pisztora, which together gives a proof by a renormalization argument. We borrow the terminology of [AP96] without giving here the definitions.
Lemma 3.1. Let $p > p_c(\mathbb{Z}^d)$. There exists a $c_1, c > 0$ such that for $n$ large enough
\[
P_p(|C_d(n)(\omega)| < c_1 n^d) < e^{-cn}.
\]

Proof. By [DP96] Theorem 1.2, there exists a $p_c(\mathbb{Z}^d) < p^* < 1$ such that for every $p > p^*$, $P_p(|C_d(n)(\omega)| < \tilde{c}_1 n^d) < e^{-cn}$. Since $\{|C_d(n)(\omega)| < \tilde{c}_1 n^d\}^c$ is an increasing event, by Proposition 2.1 of [AP96] for $N \in \mathbb{N}$ large enough, i.e such that $\bar{p}(N) > p^*$,
\[
P_N(|C_d(n)(\omega)| < \tilde{c}_1 n^d) \leq P^*_{\bar{p}(N)}(|C_d(n)(\omega)| < \tilde{c}_1 n^d) < e^{-cn},
\]
where $P_N$ is the probability measure of the renormalized dependent percolation process and $P^*_{\bar{p}(N)}$ is the probability measure of standard bond percolation with parameter $\bar{p}(N)$. From the definition of the event $R_i^{(N)}$, the crossing clusters of all the boxes $B'_i$ that admit $R_i^{(N)}$ are connected, thus
\[
P_p\left(|C_d(n)(\omega)| < \frac{\tilde{c}_1}{N^d} n^d\right) < e^{-cn}.
\]

References

[AKV02] N. Alon, M. Krivelevich, and V.H. Vu. On the concentration of eigenvalues of random symmetric matrices. *Israel Journal of Mathematics*, 131(1):259–267, 2002.

[AP96] P. Antal and A. Pisztora. On the chemical distance for supercritical bernoulli percolation. *The Annals of Probability*, pages 1036–1048, 1996.

[BBHK08] N. Berger, M. Biskup, C.E. Hoffman, and G. Kozma. Anomalous heat-kernel decay for random walk among bounded random conductances. In *Annales de l’Institut Henri Poincaré, Probabilités et Statistiques*, volume 44, pages 374–392. Institut Henri Poincaré, 2008.

[BKS03] I. Benjamini, G. Kalai, and O. Schramm. First passage percolation has sublinear distance variance. *Annals of probability*, 31(4):1970–1978, 2003.

[BM03] I. Benjamini and E. Mossel. On the mixing time of a simple random walk on the super critical percolation cluster. *Probability Theory and Related Fields*, 125(3):408–420, 2003.

[DP96] J.D. Deuschel and A. Pisztora. Surface order large deviations for high-density percolation. *Probability Theory and Related Fields*, 104(4):467–482, 1996.

[Gri99] G. Grimmett. *Percolation*. Springer Verlag, 1999.

[MR04] P. Mathieu and E. Remy. Isoperimetry and heat kernel decay on percolation clusters. *The Annals of Probability*, 32(1):100–128, 2004.
[Pet07] G. Pete. A note on percolation on zd: Isoperimetric profile via exponential cluster repulsion. *Preprint (arxiv: math. PR/0702474)*, 2007.

[Tal94] M. Talagrand. On russo’s approximate zero-one law. *The Annals of Probability*, pages 1576–1587, 1994.