Abstract. We lift the Kostant-Sekiguchi correspondence for classical groups to an equivariant homeomorphism between real and symmetric nilpotent cones.

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1. Introduction

1.1. Main results.

1.1.1. Real-symmetric homeomorphism. Let us first illustrate our main results with a notable case accessible to a general audience.

Let $M_n(\mathbb{C}) \simeq \mathbb{C}^{n^2}$ denote the space of $n \times n$ complex matrices, and $N_n(\mathbb{C}) \subset M_n(\mathbb{C})$ the nilpotent matrices, i.e. matrices whose eigenvalues are all zero.

Let $N_n(\mathbb{R}) \subset N_n(\mathbb{C})$ denote the real nilpotent matrices, i.e. those with real entries, and $N_n^{sym}(\mathbb{C}) \subset N(\mathbb{C})$ the symmetric nilpotent matrices, i.e. those equal to their transpose.

The real general linear group $GL_n(\mathbb{R})$ and complex orthogonal group $O_n(\mathbb{C})$ act by conjugation on $N_n(\mathbb{R})$ and $N_n^{sym}(\mathbb{C})$ respectively. In both cases, the orbits are determined by Jordan form, and so indexed by partitions of $n$. The real orthogonal group $O_n(\mathbb{R}) = GL_n(\mathbb{R}) \cap O_n(\mathbb{C})$ acts on both $N_n(\mathbb{R})$ and $N_n^{sym}(\mathbb{C})$, and we have $\dim_{\mathbb{R}} N_n(\mathbb{R}) = n^2 - n$ and $\dim_{\mathbb{C}} N_n^{sym}(\mathbb{C}) = (n^2 - n)/2$.

Here is a notable case of our general results.

**Theorem 1.1.** There is an $O_n(\mathbb{R})$-equivariant homeomorphism

\begin{equation}
N_n(\mathbb{R}) \simeq N_n^{sym}(\mathbb{C})
\end{equation}

that induces a poset isomorphism

\begin{equation}
|GL_n(\mathbb{R}) \backslash N_n(\mathbb{R})| \simeq |O_n(\mathbb{C}) \backslash N_n^{sym}(\mathbb{C})|
\end{equation}

Furthermore, the homeomorphism restricts to a real analytic isomorphism between individual $GL_n(\mathbb{R})$-orbits and $O_n(\mathbb{C})$-orbits.

**Example 1.2** (n=2). When $n = 2$, the real nilpotent cone $N_2(\mathbb{R})$ is a real two-dimensional cone; the symmetric nilpotent cone $N_2^{sym}$ is a nodal pair of complex lines (in Figure 1, we are only able to draw its real points). The theorem provides an $O_2(\mathbb{R})$-equivariant homeomorphism $N_2(\mathbb{R}) \simeq N_2^{sym}(\mathbb{C})$.

We deduce Theorem 1.1 from the following more fundamental structure of linear algebra.
\[ N_2(\mathbb{R}) = \{ x^2 + y^2 = z^2 \} \subset \mathbb{R}^3 \quad N_{2}^{\text{sym}}(\mathbb{C}) = \{ uv = 0 \} \subset \mathbb{C}^2 \]

**Figure 1.**

**Theorem 1.3.** There is a continuous one-parameter family of maps

(1.3) \[ \alpha_a : N_n(\mathbb{C}) \longrightarrow N_n(\mathbb{C}) \quad a \in [0, 1] \]

satisfying the properties:

1. \( \alpha_a^2 \) is the identity, for all \( a \in [0, 1] \).
2. \( \alpha_a \) takes each \( \text{GL}_n(\mathbb{C}) \)-orbit real analytically to a \( \text{GL}_n(\mathbb{C}) \)-orbit, for all \( a \in [0, 1] \).
3. At \( a = 0 \), we recover conjugation: \( \alpha_0(A) = \bar{A} \).
4. At \( a = 1 \), we recover transpose: \( \alpha_1(A) = A^T \).

1.1.2. Kostant-Sekiguchi correspondence. To state a general version of our main results, we next recall some standard constructions in Lie theory, in particular in the study of real reductive groups.

Let \( G \) be a complex reductive Lie group with Lie algebra \( \mathfrak{g} \) and nilpotent cone \( N \subset \mathfrak{g} \).

Let \( G_\mathbb{R} \subset G \) be a real form, defined by a conjugation \( \eta : G \to G \), with Lie algebra \( \mathfrak{g}_\mathbb{R} \subset \mathfrak{g} \) and nilpotent cone \( N_\mathbb{R} = N \cap \mathfrak{g}_\mathbb{R} \).

Choose a Cartan conjugation \( \delta : G \to G \) that commutes with \( \eta \), and let \( G_c \subset G \) be the corresponding maximal compact subgroup.

Introduce the involution \( \theta = \delta \circ \eta : G \to G \), and let \( K \subset G \) be the fixed subgroup of \( \theta \) with Lie algebra \( \mathfrak{k} \subset \mathfrak{g} \). Let \( \mathfrak{p} \subset \mathfrak{g} \) be the \(-1\)-eigenspace of \( \theta \) so that \( \mathfrak{g} \simeq \mathfrak{k} \oplus \mathfrak{p} \), and introduce the \( \mathfrak{p} \)-nilpotent cone \( N_\mathfrak{p} = N \cap \mathfrak{p} \).

One can organize the above groups into the diagram:

(1.4)

Here \( K_c \) is the fixed subgroup of \( \theta, \delta, \) and \( \eta \) together (or any two of the three) and the maximal compact subgroup of \( G_\mathbb{R} \) with complexification \( K \).
The Kostant-Sekiguchi correspondence is an isomorphism of orbit posets
\[ |G_\mathbb{R}\backslash N_\mathbb{R}| \simeq |K\backslash N_p| \]

The bijection was proved by Kostant (unpublished) and Sekiguchi \([S]\). Vergne \([V]\), using Kronheimer’s instanton flow \([K]\), showed the corresponding orbits are diffeomorphic. Schmid-Vilonen \([SV]\) gave an alternative proof and further refinements using Ness’ moment map. Barbasch-Sepanski \([BS]\) deduced the bijection is a poset isomorphism from Vergne’s results.

We have the following lift of the Kostant-Sekiguchi correspondence.

**Theorem 1.4** (Theorem 7.8 below). Suppose all simple factors of the complex reductive Lie algebra \(g\) are of classical type. There is a \(K_c\)-equivariant homeomorphism
\[ N_\mathbb{R} \simeq N_p \]
that induces the Kostant-Sekiguchi correspondence \((1.5)\) on orbit posets. Furthermore, it restricts to real analytic isomorphisms between individual \(G_\mathbb{R}\)-orbits and \(K\)-orbits.

**Remark 1.5.** To see the homeomorphism \((1.6)\) indeed induces the Kostant-Sekiguchi correspondence \((1.5)\), we check its compatibility with embeddings of \(\mathfrak{sl}_2\)-triples. See Section 7.3.

**Remark 1.6.** Our methods also provide interesting but incomplete results for the exceptional types \(E_6, E_7, E_8, F_4, G_2\). Namely, we obtain Kostant-Sekiguchi homeomorphisms for the minimal nilpotent orbits and the closed subcone \(N_{sm} \subset N\) of “Reeder pieces” in the terminology of \([AH]\). We conjecture it is possible to extend these homeomorphisms to all of \(N\). See Remark 1.14.

We deduce Theorem 1.4 from the following.

**Theorem 1.7** (Theorem 7.9 below). Under the same assumption as Theorem 1.4 there is a continuous one-parameter family of maps
\[ \alpha_a : N \longrightarrow N \quad a \in [0, 1] \]
satisfying the properties:
1. \(\alpha_a^2\) is the identity, for all \(a \in [0, 1]\).
2. \(\alpha_a\) takes each \(G\)-orbit real analytically to a \(G\)-orbit, for all \(a \in [0, 1]\).
3. At \(a = 0\), we recover the conjugation: \(\alpha_0 = \eta\).
4. At \(a = 1\), we recover the anti-symmetry: \(\alpha_1 = -\theta\).

**Remark 1.8.** The special case of Theorems 1.4 and 1.7 stated in Theorems 1.1 and 1.3 is when \(G = GL_n(\mathbb{C}), \ g \cong M_n(\mathbb{C}), \ G_\mathbb{R} = GL_n(\mathbb{R}), \ K = O_n(\mathbb{C}), \) and \(K_c = O_n(\mathbb{R})\). In general, we first establish Theorems 1.4 and 1.7 for split type \(A\), then check the constructions are compatible with inner automorphisms and Cartan involutions to deduce them for all classical types.

**Remark 1.9.** To deduce Theorem 1.4 from Theorem 1.7, we consider the product \(N \times [0, 1]\) with the horizontal vector field \(\partial_s\) where \(s\) is the coordinate on \([0, 1]\). By averaging \(\partial_s\) with respect to the \(\mathbb{Z}/2\mathbb{Z}\)-action given by \(\alpha_s\), we obtain a new horizontal vector field on \(N \times [0, 1]\). Its integral gives a new trivialization of \(N \times [0, 1]\) taking \(N_\mathbb{R} \times \{0\}\) to \(N_p \times \{1\}\). It would be interesting to compare this construction with Kronheimer’s instanton flow \([K]\).
1.1.3. Derived categories. Let \( D_{G_R}(N_R) \), \( D_{K}(N_p) \) denote the respective equivariant derived categories of sheaves (over any commutative ring). Since \( K_c \to G_R, K_c \to K \) are homotopy equivalences, the forgetful functors \( D_{G_R}(N_R) \to D_{K_c}(N_R), D_{K}(N_p) \to D_{K_c}(N_p) \) to \( K_c \)-equivariant complexes are fully faithful with essential image those complexes constructible along the respective orbits of \( G_R \) and \( K \).

Transport along the homeomorphism of Theorem 1.4 immediately provides:

**Corollary 1.10.** Pushforward along the homeomorphism (1.6) provides an equivalence of equivariant derived categories

\[
(1.8) \quad D_{G_R}(N_R) \simeq D_{K}(N_p)
\]

1.1.4. Intersection cohomology. Theorem 1.4 implies that the singularities of symmetric nilpotent orbit closures \( \tilde{\mathcal{O}}_p \subset N_p \) are homeomorphic to the singularities of the corresponding real nilpotent orbit closures \( \tilde{\mathcal{O}}_R \subset N_R \). Thus we can deduce results about one from the other.

Here is a notable example. Let \( \mathcal{IC}(\tilde{\mathcal{O}}_R, \mathcal{L}_R) \) be the intersection cohomology sheaf of a real nilpotent orbit \( \tilde{\mathcal{O}}_R \subset N_R \) with coefficients in a \( G_R \)-equivariant local system \( \mathcal{L}_R \). (Recall that all nilpotent orbits \( \mathcal{O} \subset N \) have even complex dimension, so all real nilpotent orbits \( \mathcal{O}_R \subset N_R \) have even real dimension, hence middle perversity makes sense.)

**Corollary 1.11.** The cohomology sheaves \( H^i(\mathcal{IC}(\mathcal{O}_R, \mathcal{L}_R)) \) vanish for \( i - \dim_{\mathbb{C}} \mathcal{O}_p/2 \) odd.

**Proof.** Using the equivalence (1.8), it suffices to prove the asserted vanishing for the intersection cohomology sheaf \( \mathcal{IC}(\mathcal{O}_p, \mathcal{L}_p) \) of a symmetric nilpotent orbit \( \mathcal{O}_p \subset N_p \) with coefficients in a \( K \)-equivariant local system \( \mathcal{L}_p \), and \( i - \dim_{\mathbb{C}} \mathcal{O}_p \) odd. This is proved in [LY, Theorem 14.10]. \( \square \)

**Remark 1.12.** The proof of [LY, Theorem 14.10] makes use of Deligne’s theory of weights and the theory of canonical bases, and hence does not have an evident generalization to a real algebraic setting.

1.1.5. Real and symmetric arcs. We deduce Theorem 1.4 from a geometric result of independent interest. To reduce notation, we will assume in this section that \( G \) is simply-connected; our goal is to state Theorem 1.13 which only depends on the isogeny class of \( G \).

Let us first recall some constructions inspired by the geometric Langlands program.

Let \( T \subset G \) be a Cartan subgroup, stable under \( \eta \) and \( \theta \), maximally split with respect to \( \eta \), with split subtorus \( S \subset T \). Let \( \Lambda_T = \text{Hom}(\mathbb{C}^\times, T) \) be the coweight lattice, and \( \Lambda_S = \text{Hom}(\mathbb{C}^\times, S) \) the split coweight lattice. Let \( B \subset G \) be a Borel subgroup containing \( T \), and \( \Lambda_T^+ \subset \Lambda_T \), \( \Lambda_T^- = \Lambda_S \cap \Lambda_T^+ \) the respective cones of dominant coweights.

Let \( \mathcal{K} = \mathbb{C}((z)) \) denote the field of Laurent series, \( \mathcal{O} = \mathbb{C}[[z]] \) the ring of power series, and \( \mathcal{O}^- = \mathbb{C}[z^{-1}] \) the ring of Laurent poles.

Let \( \text{Gr} = G(\mathcal{K})/G(\mathcal{O}) \) be the affine Grassmannian of \( G \).

Via the natural inclusion \( \Lambda_T \to T(\mathcal{K}) \subset G(\mathcal{K}) \), any coweight \( \lambda \in \Lambda_T \) defines a point \( [\lambda] \in \text{Gr} \). For a dominant coweight \( \lambda \in \Lambda_T^+ \), introduce the \( G(\mathcal{O}) \)-orbit \( S^\lambda = G(\mathcal{O}) \cdot [\lambda] \subset \text{Gr} \).

\( \text{In fact, [LY] establishes the odd vanishing in the more general setting of graded Lie algebras.} \)
Consider the arc group $G = G(\mathcal{O}^-)$, its evaluation $ev : G \to G$, $ev(g) = g(\infty)$, and the subgroup of based arcs $G^* = ev^{-1}(e)$.

The action of $G^*$ on the base-point $[e] \in Gr$ is free and gives isomorphisms
\begin{equation}
G^* \simeq G^* \cdot [e] \simeq T^0
\end{equation}

We stratify $G^*$ by its intersection with the spherical strata $G^{*\lambda} = G^* \cap S^\lambda$, $\lambda \in \Lambda^+_T$.

On the one hand, we can repeat the above constructions over the real numbers. In particular, the conjugation $\eta : G \to G$, with real form $G_R \subset G$, induces a conjugation $\eta : G(\mathcal{K}) \to G(\mathcal{K})$, $\eta(g(z)) = \eta(g(\bar{z}))$. We denote by $Gr_R$ the corresponding real affine Grassmannian, and $G_R \subset G$ the corresponding real arc group. We also have the subgroup of based arcs along with its spherical strata
\begin{equation}
G^* = G^* \cap G^* \cap G^{*\lambda} = G^* \cap G^{*\lambda}
\end{equation}

Here one finds that $G^{*\lambda}_R$ is non-empty if and only if $\lambda \in \Lambda^+_S$.

On the other hand, let us return to the involution $\theta : G \to G$, with fixed subgroup $K \subset G$, and consider the symmetric space $X = K \setminus G$. Via the map $X \to G$, $x = Kg \mapsto \theta(g^{-1})g$, we will identify $X$ with the neutral component of the subspace of $g \in G$ such that $\theta(g) = g^{-1}$.

We have the arc space $X = X(\mathcal{O}^-)$, along with the based arc space and its spherical strata
\begin{equation}
X^* = X \cap G^* \quad X^{*\lambda} = X^* \cap G^{*\lambda}
\end{equation}

Again one finds that $X^{*\lambda}$ is non-empty if and only if $\lambda \in \Lambda^+_S$.

**Theorem 1.13** (Corollary 6.6 below). There is a $K_c$-equivariant homeomorphism
\begin{equation}
G^*_R \simeq X^*
\end{equation}

that restricts to real analytic isomorphisms on spherical strata
\begin{equation}
G^{*\lambda}_R \simeq X^{*\lambda} \quad \lambda \in \Lambda^+_S
\end{equation}

**Remark 1.14.** There is no restriction on type in Theorem 1.13. The restriction appearing in Theorems 1.4 and 1.7 arises when relating the nilpotent cone $N$ to the based arc group $G^*$. We expect it is possible to remove this and extend Theorems 1.4 and 1.7 to the exceptional types $E_6, E_7, E_8, F_4, G_2$.

**Remark 1.15.** In contrast to Theorem 1.13, the closure $G^{*\lambda}_R \subset Gr_R$, for $\lambda \in \Lambda^+_S$, is not in general homeomorphic to a complex projective variety. For example, for the rank one group $G_R = SL_2(\mathbb{H})$, and the generator $\lambda \in \Lambda^+_S$, one finds that $G^{*\lambda}_R \subset Gr_R$ is isomorphic to the one-point compactification of the cotangent bundle $T^*\mathbb{H}P^1$ of the quaternionic projective line. Thus its intersection cohomology Poincaré polynomial is $1 + t^4 + t^8$ so does not satisfy the Hard Lefschetz Theorem.
1.2. Degenerations of quasimaps. We outline here the geometry underlying our proof of Theorem 1.13. (We in turn deduce Theorem 1.14 from Theorem 1.13 by invoking constructions detailed in Section 7.)

Let \( \mathbb{P}^1 \) be the complex projective line with coordinate \( t \).

For the symmetric variety \( X = K \setminus G \), and subset \( S = \{0, \infty\} \subset \mathbb{P}^1 \), consider the ind-stack of quasi-maps with poles \( QM_{G,K}(\mathbb{P}^1, S) \) classifying a \( G \)-bundle \( \mathcal{E} \to \mathbb{P}^1 \) and a meromorphic section

\[
\text{(1.15)} \quad s : \mathbb{P}^1 \setminus S \longrightarrow X|_{\mathbb{P}^1 \setminus S}
\]

into the associated \( X \)-bundle \( X_\mathcal{E} = \mathcal{E} \times^G X \).

The conjugation \( c : \mathbb{P}^1 \to \mathbb{P}^1 \), \( c(t) = \overline{t} - 1 \), with real form \( S^1 \subset \mathbb{P}^1 \), and conjugation \( \eta : G \to G \), with real form \( G_\mathbb{R} \subset G \), together induce a conjugation of \( QM_{G,X}(\mathbb{P}^1, S) \), and we denote its real points by \( QM_{G,X}(\mathbb{P}^1, S)_\mathbb{R} \).

There is a union of components \( QM_{G,X}(\mathbb{P}^1, S)_\mathbb{R, L} \subset QM_{G,X}(\mathbb{P}^1, S)_\mathbb{R} \) that admits the uniformization

\[
\text{(1.16)} \quad LK_c / \text{Gr} \cong QM_{G,X}(\mathbb{P}^1, S)_\mathbb{R, L}
\]

where \( LK_c \subset K(\mathbb{C}[t, t^{-1}]) \) is the subgroup of polynomial maps that take the unit circle \( S^1 = \{|t| = 1\} \subset \mathbb{P}^1 \) into \( K_c \subset K \).

Now we will consider two degenerations of \( QM_{G,X}(\mathbb{P}^1, S)_\mathbb{R} \) associated to degenerations of the curve \( \mathbb{P}^1 \) and points \( S \subset \mathbb{P}^1 \).

1.2.1. Real degeneration. On the one hand, we can fix the curve \( \mathbb{P}^1 \) and allow the points \( S = \{0, \infty\} \) to move in the family \( S_a = \{1 - a, (1 - a)^{-1}\} \), for \( a \in \mathbb{A}^1 \). At the special fiber \( a = 0 \), the subset \( S_0 = \{1\} \) consists of a single real point. The special fiber of the corresponding family of real quasi-maps admits the uniformization

\[
\text{(1.17)} \quad K_c(\mathbb{R}[z^{-1}]) / \text{Gr} \cong QM_{G,X}(\mathbb{P}^1, \{1\})_\mathbb{R}
\]

Since \( K_c \) is compact, the real arc group \( K_c(\mathbb{R}[z^{-1}]) \) reduces to constant maps \( K_c \). (This is a simple instance of the remarkable factorization appearing in Theorem 1.18 below.) Restricting as well to the open spherical costratum, we thus obtain an open embedding of the left hand side of (1.13):

\[
\text{(1.18)} \quad K_c \setminus S_\mathbb{R} \hookrightarrow QM_{G,X}(\mathbb{P}^1, \{1\})_\mathbb{R}
\]
1.2.2. **Nodal degeneration.** On the other hand, we can allow the curve $\mathbb{P}^1$ to vary in the nodal family $C_b = \{ xy = b \}$, for $b \in \mathbb{A}^1$, while keeping the points $S = \{ 0, \infty \}$ fixed. At the special fiber $b = 0$, the curve $C_0 = \mathbb{P}^1 \cup \mathbb{P}^1$ is nodal with conjugation exchanging the components. The special fiber of the corresponding family of real quasi-maps admits an open embedding

$$K(\mathbb{C}[t^{-1}])_c \simeq QM_{G,X}(\mathbb{P}^1 \cup \mathbb{P}^1, S)_{\mathbb{R}}$$

where $K(\mathbb{C}[t^{-1}])_c \subset K(\mathbb{C}[t^{-1}])$ is the inverse-image of $K_c \subset K$ under the evaluation $\text{ev} : K(\mathbb{C}[t^{-1}]) \to K$, $g(t^{-1}) \mapsto g(\infty)$. This admits a reinterpretation as an open embedding of the right hand side of (1.13):

$$K_c \simeq QM_{G,X}(\mathbb{P}^1 \cup \mathbb{P}^1, S)_{\mathbb{R}}$$

To deduce Theorem 1.13, we show that the above families of real quasi-maps are topologically trivial along the open subspaces of interest. This is straightforward for a complex group regarded as a real form using the factorization of Beilinson-Drinfeld Grassmannians. Thus our primary efforts are focused on insuring such trivializations can be suitably averaged under Galois symmetries.

1.3. **Further developments.** Let us sketch a natural extension of our results to appear in a sequel [CN2] with further applications to Springer theory and character sheaves.

Our goal is to extend Theorem 1.4 beyond nilpotent cones to include all conjugacy classes in the real Lie algebra $\mathfrak{g}_{\mathbb{R}}$ and symmetric subspace $\mathfrak{p}$ whose eigenvalues are real but not necessarily zero.

Let us return to the Cartan subgroup $T \subset G$, stable under $\eta$ and $\theta$, and maximally split with respect to $\eta$. Let $\mathfrak{t} \subset \mathfrak{g}$ denote its Lie algebra, $W = N_G(\mathfrak{t})/Z_G(\mathfrak{t})$ the Weyl group, and introduce the affine quotient

$$\mathfrak{c} = \mathfrak{t} / W = \text{Spec}(\mathcal{O}(\mathfrak{t})^W) \simeq \mathfrak{g} / G = \text{Spec}(\mathcal{O}(\mathfrak{g})^G)$$

The conjugation $\eta$ descends to a real structure on $\mathfrak{c}$, and we denote its real points by $\mathfrak{c}_{\mathbb{R}}$. We also have the real characteristic polynomial map $\mathfrak{g}_{\mathbb{R}} \to \mathfrak{c}_{\mathbb{R}}$ from real matrices to their unordered eigenvalues.

Next, let $\mathfrak{a} = \mathfrak{t} \cap \mathfrak{p}$ be the $-1$-eigenspace of $\theta$, and write $\mathfrak{a}_{\mathbb{R}} = \mathfrak{a} \cap \mathfrak{g}_{\mathbb{R}}$ for the real form of $\mathfrak{a}$ with respect to $\eta$. Let $W_0 = N_K(\mathfrak{a})/Z_K(\mathfrak{a})$ be the “baby Weyl group”, and introduce the affine quotient

$$\mathfrak{c}_p = \mathfrak{a} / W_0 = \text{Spec}(\mathcal{O}(\mathfrak{a})^{W_0}) \simeq \mathfrak{p} / K = \text{Spec}(\mathcal{O}(\mathfrak{p})^{K})$$

Consider the natural maps $\mathfrak{a}_{\mathbb{R}} \to \mathfrak{c}_{\mathbb{R}}$, $\mathfrak{a}_{\mathbb{R}} \to \mathfrak{c}_{\mathbb{R}}$ from ordered, split, real eigenvalues to symmetric and real characteristic polynomials. Introduce the fiber products

$$\mathfrak{p}' = \mathfrak{p} \times_{c_p} \mathfrak{a}_{\mathbb{R}} \quad \mathfrak{g}'_{\mathbb{R}} = \mathfrak{g}_{\mathbb{R}} \times_{c_p} \mathfrak{a}_{\mathbb{R}}$$

Note that $K$ and $G_{\mathbb{R}}$ naturally act on $\mathfrak{p}'$ and $\mathfrak{g}'_{\mathbb{R}}$ respectively, and the actions are along the fibers of the natural projections $\mathfrak{p}' \to \mathfrak{a}_{\mathbb{R}}$, $\mathfrak{g}'_{\mathbb{R}} \to \mathfrak{a}_{\mathbb{R}}$.

In the sequel [CN2], we will establish the following extension of Theorem 1.4. As a corollary, we mention one application to Springer theory also established in [CN2].
Theorem 1.16. Suppose all simple factors of the complex reductive Lie algebra $\mathfrak{g}$ are of classical type. There is a $K_c$-equivariant homeomorphism

$$g'_R \simeq p'$$

compatible with the natural projections to $a_R$. Furthermore, the homeomorphism restricts to real analytic isomorphisms between individual $G_R$-orbits and $K$-orbits.

Corollary 1.17. Under the same assumption as Theorem 1.16, the nearby cycles complex along the special fiber $N_p \subset p$ in the family $p \to c_p$ is equivalent to the Springer complex on $N_R \subset g_R$ constructed from the Springer resolution $\tilde{N}_R \to N_R$.

We deduce Theorem 1.16 from an analogous extension of Theorem 1.13. We will not state it here, but mention that it again involves real families of real quasi-maps, but now with additional pole points.

A key ingredient of independent interest is a beautiful factorization generalizing the well-known “Gramm-Schmidt” homomorphism

$$\Omega G_c \times G(\mathbb{C}[t]) \sim \sim G(\mathbb{C}[t,t^{-1}])$$

(1.25)

Here $\Omega G_c \subset G(\mathbb{C}[t,t^{-1}])$ is the based loop group of polynomial maps that take the unit circle $S^1 = \{|t| = 1\} \subset \mathbb{P}^1$ into the compact form $G_c \subset G$, and $1 \in S^1$ to $1 \in G_c$. The map (1.25) is multiplication inside of $G(\mathbb{C}[t,t^{-1}])$.

To state the generalization, let $\mathbb{H}^+ = \{|t| < 1\} \subset \mathbb{P}^1$ denote the open disk, and $\mathbb{H}^- = \{|t| > 1\} \subset \mathbb{P}^1$ its conjugate with respect to the conjugation $c(t) = \bar{t}^{-1}$. Given $S^+ \subset \mathbb{H}^+$ a finite set of points, let $S^- \subset \mathbb{H}^-$ denote its conjugate, and $S = S^+ \cup S^-$ their union. Then the generalization of (1.25) arising in the proof of Theorem 1.16 is the following factorization. We expect it to have many other interesting applications.

Theorem 1.18. Multiplication provides a homeomorphism

$$\left(\prod_{s \in S^+} \Omega G_c\right) \times G(\mathbb{C}[\mathbb{P}^1 \setminus S^-]) \sim \rightarrow G(\mathbb{C}[\mathbb{P}^1 \setminus S])$$

(1.26)

1.4. Organization. We briefly summarize here the main goals of each section. In Sect. 2 immediately to follow, we collect standard notation and constructions from Lie theory, in particular of loop groups and affine Grassmannians. In Sect. 3 we define the moduli spaces of real quasi-maps with poles that we will study in the rest of the paper. The notions for a fixed curve and poles are standard, and we spell out similar notions for families of curves and poles parameterized by a base. In the next two sections, we specialize the quasi-map constructions of Sect. 3 to two cases: first, in Sect. 4 for fixed curve $\mathbb{P}^1$ with a pair of colliding poles; and second, in Sect. 5 for a nodal degeneration of $\mathbb{P}^1$ with a pair of non-interacting poles. We focus on complex groups regarded as symmetric spaces where the first family and an open locus of the second may be trivialized. We also explain how to identify the two families over generic parameters. In Sect. 6 we apply the preceding results to construct stratification-preserving trivializations of quasi-map families interpolating between real affine Grassmannians and loop spaces of symmetric spaces. When $G$ is of type $A$, where one may embed the nilpotent cone in the affine Grassmannian respecting stratifications,
this immediately provides Kostant-Sekiguchi homeomorphisms between real and symmetric nilpotent cones. Finally, in Sect. 7 we develop a characterization of the trivializations of quasi-map families sufficient to confirm their compatibility with Cartan involutions. This allows us to deduce Kostant-Sekiguchi homeomorphisms for all classical groups.

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2. Group data

We collect here notation and standard constructions used throughout the rest of the paper (for further discussion, see for example [N1, N2, CN1]).

Let \( \eta : G \to G \) be a conjugation of \( G \) with real form \( G_R \). Let \( \theta : G \to G \) be the corresponding Cartan involution and \( K \subset G \) the fixed-point subgroup. We write \( \delta = \eta \circ \theta = \theta \circ \eta \) for the compact conjugation on \( G \) with compact real form \( G_c \). We write \( X = K \setminus G \) for the corresponding symmetric variety, and identify \( X \) with the neutral component of the subspace \( \{ g \in G \mid \theta(g) = g^{-1} \} \) via the map \( X \to G, x = K g \mapsto \theta(g^{-1})g \).

**Example 2.1** (Complex groups). An important special case is when the real form is itself a complex group. To avoid potential confusion in this case, we will use the alternative notation \( \text{sw} : G \times G \to G \times G, \text{sw}(h, g) = (h, g) \) for the Cartan involution with fixed-point subgroup the diagonal \( G \subset G \times G \). The corresponding conjugation is the composition \( \text{sw}_\delta = \text{sw} \circ (\delta \times \delta) \), and the compact conjugation is the product \( \delta \times \delta \). The corresponding symmetric variety is again \( G \simeq G' \setminus (G \times G) \) via the left factor, which we also identify with the subspace \( \{ (g, h) \in G \times G \mid h = g^{-1} \} \) via the left factor.

Let \( T \subset G \) be a Cartan subgroup, stable under \( \eta \) and \( \theta \), maximally split with respect to \( \eta \), and let \( \Lambda_T = \text{Hom}(\mathbb{C}^*, T) \) be its coweight lattice. Let \( B \subset G \) be a Borel subgroup containing \( T \), and write \( R_G \subset \Lambda_T \) for the coroot lattice. Let \( \Lambda_T^+ \subset \Lambda_T \) be the cone of coweights dominant with respect to \( B \), and set \( R_G^+ = R_G \cap \Lambda_T^+ \).

Let \( K = \mathbb{C}(\!(z)\!) \) denote the field of Laurent series, \( \mathbb{O} = \mathbb{C}[\![z]\!] \) the ring of power series, and \( \mathbb{O}^- = \mathbb{C}[z^{-1}] \) the ring of Laurent poles. Let \( \text{Gr} = G(K)/G(\mathbb{O}) \) be the affine Grassmannian of \( G \). For any \( g \in G(K) \) we denote by \( [g] \in \text{Gr} \) the corresponding coset.

Via the natural inclusion \( \Lambda_T \to T(K) \subset G(K) \), any coweight \( \lambda \in \Lambda_T \) defines a point \([\lambda]\) \( \in \text{Gr} \). For a dominant coweight \( \lambda \in \Lambda_T^+ \), introduce the \( G(\mathbb{O})\)-orbit \( S^\lambda = G(\mathbb{O}) \cdot [\lambda] \subset \text{Gr} \) (spherical stratum) and \( G(\mathbb{O}^-)\)-orbit \( T^\lambda = G(\mathbb{O}^-) \cdot [\lambda] \subset \text{Gr} \) (cospherical stratum). Recall the disjoint union decompositions

\[
\text{Gr} = \bigsqcup_{\lambda \in \Lambda_T^+} S^\lambda \quad \text{Gr} = \bigsqcup_{\lambda \in \Lambda_T^+} T^\lambda
\]

We have the subgroup \( G(\mathbb{C}[z, z^{-1}]) \subset G(K) \) of polynomial loops, and the natural map \( G(\mathbb{C}[z, z^{-1}])/G(\mathbb{C}[z]) \to \text{Gr} \) is an isomorphism.

\footnote{Our concerns in this paper will be exclusively topological, and we will ignore any non-reduced structure throughout.}
Let $\Omega G_c \subset G(\mathbb{C}[z, z^{-1}])$ be the subgroup of maps that take the unit circle $S^1 \subset \mathbb{P}^1$ to the compact form $G_c \subset G$, and the base-point $1 \in \mathbb{P}^1$ to the identity $e \in G_c$. Recall the factorization $\Omega G_c \times G(\mathbb{C}[z]) \simeq G(\mathbb{C}[z, z^{-1}])$ given by group multiplication, and hence the action of $\Omega G_c$ on the base-point $[e] \in Gr$ defines a homeomorphism

$$\Omega G_c \xrightarrow{\sim} Gr$$

(2.2)

For $\lambda \in \Lambda_+^+$, we also write $S^\lambda, T^\lambda \subset \Omega G_c$ for the respective preimages of $S^\lambda, T^\lambda \subset Gr$ under the homeomorphism (2.2).

We can similarly repeat the above constructions over the real numbers. Let $\mathcal{K}_\mathbb{R} = \mathbb{R}((t))$ denote the field of real Laurent series, and $\mathcal{O}_\mathbb{R} = \mathbb{R}[[t]]$ the ring of real power series. Let $Gr_\mathbb{R} = G_\mathbb{R}(\mathcal{K}_\mathbb{R})/G_\mathbb{R}(\mathcal{O}_\mathbb{R})$ be the real affine Grassmannian of the real form $G_\mathbb{R}$.

Note the conjugation $\eta$ of $G$ extends to a conjugation $\eta$ of $G(\mathcal{K})$ defined by $\eta(g)(z) = \eta(g(z))$ with real form $G_\mathbb{R}(\mathcal{K}_\mathbb{R})$. The conjugation $\eta$ of $G(\mathcal{K})$ preserves $G(\mathcal{O})$ with real form $G_\mathbb{R}(\mathcal{O}_\mathbb{R})$, and descends to $Gr$ with real form $Gr_\mathbb{R}$.

Let $S \subset T$ be the maximally split subtorus, and $\Lambda_S = \text{Hom}(\mathbb{C}^x, S)$ its lattice of coweights. For a dominant coweight $\lambda \in \Lambda_+^+ = \Lambda_S \cap \Lambda_T^+$, set $S_\mathbb{R}^\lambda = S^\lambda \cap Gr_\mathbb{R}$ (real spherical stratum) and $T_\mathbb{R}^\lambda = T^\lambda \cap Gr_\mathbb{R}$ (real cospHERiCAL stratum). We have the disjoint union decompositions

$$Gr_\mathbb{R} = \bigsqcup_{\lambda \in \Lambda_+^+} S_\mathbb{R}^\lambda$$

(3.1)

$$Gr_\mathbb{R} = \bigsqcup_{\lambda \in \Lambda_+^+} T_\mathbb{R}^\lambda$$

Finally, let $LG_\mathbb{R} \subset G(\mathbb{C}[z, z^{-1}])$ denote the subgroup of maps that take the unit circle $S^1 \subset \mathbb{P}^1$ to the real form $G_\mathbb{R} \subset G$.

3. Quasi-maps

3.1. Definitions. Let $A$ be a complex base-scheme (we will only use the case when $A$ is a point $pt = \text{Spec}(\mathbb{C})$ or affine line $A^1 = \text{Spec}(\mathbb{C}[s])$).

Let $\pi : \mathcal{Z} \to A$ be a family of curves, with fibers denoted $\mathcal{Z}_a = \pi^{-1}(a)$.

Let $\text{Bun}_G(\mathcal{Z}/A)$ denote the moduli stack of a point $a \in A$ and a $G$-bundle $E$ on the fiber $\mathcal{Z}_a$. More precisely, an $S$-point consists of an $S$-point $a : \text{Spec}S \to A$ and a $G$-bundle $E$ on the corresponding fiber product $\mathcal{Z} \times_A \text{Spec}S$. Denote by $p : \text{Bun}_G(\mathcal{Z}/A) \to A$ the evident projection with fibers $p^{-1}(a) = \text{Bun}_G(\mathcal{Z}_a)$.

Let $\sigma = (\sigma_1, \ldots, \sigma_n) : A \to \mathcal{Z}^n$ be an ordered $n$-tuple of sections of $\pi$ (we will only use the case when $n = 2$, but crucially allow the points to intersect).

For an affine $G$-variety $X$ (we will only use the case of a symmetric variety $X = K\setminus G$), let $QM_{G,X}(\mathcal{Z}/A, \sigma)$ denote the ind-stack of quasi-maps classifying a point $a \in A$, a $G$-bundle $E$ on the fiber $\mathcal{Z}_a$, and a section

$$s : \mathcal{Z}_a \setminus \{\sigma_1(a), \ldots, \sigma_n(a)\} \longrightarrow X_E$$

(3.1)

to the associated $X$-bundle over the complement of the points $\sigma_1(a), \ldots, \sigma_n(a) \in \mathcal{Z}_a$. We have the evident forgetful maps

$$q : QM_{G,X}(\mathcal{Z}/A, \sigma) \longrightarrow \text{Bun}_G(\mathcal{Z}/A) \longrightarrow A$$

(3.2)
with fibers $q^{-1}(a) = QM_{G,X}(Z_a, \sigma(a))$.

Let $\xi : A \to Z$ be another section of $\pi$ such that $\xi(a) \neq \sigma_i(a)$, for all $a \in A$, and $i = 1, \ldots, n$.

Let $Q_{\xi : Z/A, \sigma, \xi}$ denote the ind-stack of rigidified quasimaps classifying quadruples $(a, E, s, \iota)$ where $(a, E, s)$ is a quasi-map as above and $\iota : E_{\xi} |_{\xi(a)} \simeq K$ is a trivialization, where $E_{\xi}$ is the $K$-reduction of $E$ on $Z_a \setminus \{\sigma_1(a), \ldots, \sigma_n(a)\}$ given by the section $s$. We have the evident forgetful maps

$$r : QM_{G,X}(Z/A, \sigma, \xi) \to \text{Bun}_G(Z/A) \to A$$

with fibers $r^{-1}(a) = QM_{G,X}(Z_a, \sigma(a), \xi(a))$.

Suppose given conjugations $c_z : Z \to Z$ and $c_A : A \to A$ intertwined by $\pi : Z \to A$.

The conjugation $\eta$ of $G$ induces a natural conjugation $\eta$ of $\text{Bun}_G(Z/A)$ defined by $\eta(a, E) = (c_A(a), c^*_AE_{\eta})$ where we write $c^*_AE_{\eta}$ for the bundle $c^*_AE$ with its $\eta$-twisted $G$-action. We denote by $\text{Bun}_G(Z/A)_R$ the corresponding real points.

Suppose $\sigma = (\sigma_1, \sigma'_1, \ldots, \sigma_n, \sigma'_n) : A \to Z^{2n}$ is an ordered $2n$-tuple of sections of $\pi$ organized into pairs $(\sigma_i, \sigma'_i)$, for $i = 1, \ldots, n$. Suppose further that $c_z(\sigma_i(c_A(a))) = \sigma'_i(a)$, for $i = 1, \ldots, n$, and $c_z(\xi(c_A(a))) = \xi(a)$.

Then the conjugation $\eta$ of $G$ similarly induces natural conjugations $\eta$ of $QM_{G,X}(Z/A, \sigma)$ and $\eta$ of $QM_{G,X}(Z/A, \sigma, \xi)$. We denote by $QM_{G,X}(Z/A, \sigma)_R$ and $QM_{G,X}(Z/A, \sigma, \xi)_R$ the respective real points.

### 3.2. Uniformizations

Let $\text{Gr}_{G,Z,\sigma}$ (resp. $\text{Gr}_{G,Z,\sigma}$) denote the Beilinson-Drinfeld Grassmannian of a point $a \in A$, a $G$-bundle $E$ on $Z_a$, and a section $s : Z_a \setminus \sigma_i(a) \to E$ (resp. $s : Z_a \setminus \{\sigma_1(a), \ldots, \sigma_n(a)\} \to E$).

Let $G[\mathbb{Z}, \sigma_i]$ denote the group scheme of a point $a \in A$ and a section $\hat{D}_{\sigma_i(a)} \to G$, where $\hat{D}_{\sigma_i(a)}$ is the formal disk around $\sigma_i(a)$.

Let $G[\mathbb{Z}, \sigma_i]$ (resp. $G[\mathbb{Z}, \sigma]$) denote the group ind-scheme of a point $a \in A^1$ and a section $s : Z_a \setminus \{\sigma_i(a)\} \to G$ (resp. $s : Z_a \setminus \{\sigma_1(a), \ldots, \sigma_n(a)\} \to G$).

Let $G[\mathbb{Z}, \sigma, \xi]$ (resp. $G[\mathbb{Z}, \sigma, \xi]$) denote the subgroup ind-scheme of $G[\mathbb{Z}, \sigma_i]$ (resp. $G[\mathbb{Z}, \sigma]$) consisting of $(a, s) \in G[\mathbb{Z}, \sigma_i]$ (resp. $(a, s) \in G[\mathbb{Z}, \sigma]$) such that $s(\xi(a)) = e$.

For any $a \in A$, we write $\text{Gr}_{G,Z,\sigma,a}, G[\mathbb{Z}, \sigma, a]$, etc., for the respective fibers over $a$.

The conjugations $c_z, c_A, \eta$ induce conjugations on $\text{Gr}_{G,Z,\sigma}, G[\mathbb{Z}, \sigma]$, etc., and we denote by $\text{Gr}_{G,Z,\sigma,R}, G[\mathbb{Z}, \sigma,R]$, etc., the respective real points.

For any $a \in A(\mathbb{R})$, we write $\text{Gr}_{G,Z,\sigma,a,R}, G[\mathbb{Z}, \sigma,a,R]$, etc., for the respective fibers over $a$.

The group ind-scheme $G[\mathbb{Z}, \sigma]$ (resp. $G[\mathbb{Z}, \sigma_i]$) naturally acts on $\text{Gr}_{G,Z,\sigma}$ (resp. $\text{Gr}_{G,Z,\sigma_i}$) and we have uniformizations morphisms

$$G[\mathbb{Z}, \sigma] \to \text{Bun}_G(Z/A) \quad G[\mathbb{Z}, \sigma_i] \to \text{Bun}_G(Z/A)$$

with fibers $q^{-1}(a) = QM_{G,K}(Z_a, \sigma)$.
sections \( \sigma \) on real points:

\[
\text{(4.1) } \begin{array}{c}
\Gr^1 \quad & \quad \text{The isomorphism followed by } \text{pr}_1 \text{ provides an isomorphism } \\
\text{Thus } \quad & \quad \text{uniformization morphisms, uniformization morphisms, and real forms of those are } \\
\text{constant conjugations } & \text{functorial with respect to homomorphism } f : G_1 \to G_2 \text{ that intertwine } \eta_1, \eta_2 \text{ and } \theta_1, \theta_2.
\end{array}
\]

4. REAL DEGENERATION

We apply here the constructions of the previous section in the following situation. Set \( A = \mathbb{A}^1 \) to be the affine line with coordinate \( a \). Let \( \mathbb{P}^1 \) be the projective line with local coordinate \( z \).

We will consider the trivial family of projective lines \( \pi : Z = \mathbb{P}^1 \times \mathbb{A}^1 \to A = \mathbb{A}^1 \) with sections \( \sigma = (\sigma_1, \sigma_2) : \mathbb{A}^1 \to \mathbb{A}^2 \), \( \sigma(a) = (\sigma_1(a), \sigma_2(a)) = (-a, a) \), \( \xi(a) = (\infty, a) \), and conjugations \( c_{\mathbb{A}^1}(a) = -\bar{a} \) and \( c_{\mathbb{A}^1}(x, a) = (\bar{x}, -\bar{a}) \). Note that \( \sigma_1(a) = \sigma_2(a) \) if and only if \( a = 0 \), and \( \mathbb{A}^1(\mathbb{R}) = i\mathbb{R}, \mathbb{Z}(\mathbb{R}) = \mathbb{RP}^1 \times i\mathbb{R} \).

4.1. Complex groups. Recall \( \delta = \theta \circ \eta = \eta \circ \theta \) denotes the Cartan conjugation of \( G \) with compact real form \( G_c \). Equip \( G \times G \) with the swap involution \( \text{sw}(g, h) = (h, g) \) and the conjugation \( \text{sw}_\delta(g, h) = (\delta(h), \delta(g)). \) The fixed-point subgroup of \( \text{sw} \) is the diagonal \( G \subset G \times G \), and the corresponding symmetric space is isomorphic to the group \( G \setminus (G \times G) \simeq G \).

The projection maps \( \text{pr}_1, \text{pr}_2 : G \times G \to G \) provide an isomorphism

\[
\text{(4.1) } \begin{array}{c}
\Gr_{G \times G, \sigma} \quad & \quad \text{is given by} \\
\simeq \quad & \quad (\mathcal{E}, s, \mathcal{E}', s') \mapsto (c_z^* \mathcal{E}_\delta, c_z^* (s'), c_z^* \mathcal{E}_\delta, c_z^* (s))
\end{array}
\]

Thus the isomorphism (4.1) followed by \( \text{pr}_1 \) provides an isomorphism

\[
\text{(4.3) } \begin{array}{c}
\text{of real analytic spaces over } \mathbb{A}^1(\mathbb{R}) \simeq i\mathbb{R}.
\end{array}
\]

Lemma 4.1. The uniformization morphisms (4.6), (4.9) are isomorphisms:

(1) \( QM_{G \times G,G}(\mathbb{Z}/\mathbb{A}^1, \sigma, \xi) \simeq G[\mathbb{Z}, \sigma, \xi] \setminus \Gr_{G \times G, \sigma} \)
\( Q_{M,G,G}(\mathbb{Z}/\mathbb{A}^1, \sigma, \xi)_\mathbb{R} \simeq G[\mathbb{Z}, \sigma, \xi]_\mathbb{R} \setminus \text{Gr}_{G \times G, \sigma, \xi, \mathbb{R}} \simeq G[\mathbb{Z}, \sigma, \xi]_\mathbb{R} \setminus \text{Gr}_{G, \sigma, \xi, \mathbb{R}} \)

Proof. Part (1) is standard. Part (2) follows from [CN1 Proposition 6.5] and the fact that the fixed-point subgroup \( G \subset G \times G \) is connected and \( H^1(\text{Gal}(\mathbb{C}/\mathbb{R}), G \times G) \) is trivial for the Galois-action given by \( s \omega \).

**Lemma 4.2.** The restriction of the isomorphisms of Lemma 4.1 to fibers give isomorphisms:

1. \( a \neq 0 \in \mathbb{A}^1(\mathbb{C}) \simeq \mathbb{C} \):

\( Q_{M,G,G}(\mathbb{Z}_a, \sigma(a), \xi(a)) \simeq G[\mathbb{Z}, \sigma, \xi, a] \setminus (\text{Gr}_{G, \sigma, \xi, a} \times \text{Gr}_{G, \sigma, \xi, a} \times \text{Gr}_{G, \sigma, \xi, a} \times \text{Gr}_{G, \sigma, \xi, a}) \)

2. \( a = 0 \in \mathbb{A}^1(\mathbb{C}) \simeq \mathbb{C} \):

\( Q_{M,G,G}(\mathbb{Z}_0, \sigma(0), \xi(0)) \simeq G[\mathbb{Z}, \sigma, \xi, 0] \setminus (\text{Gr}_{G, \sigma, \xi, 0} \times \text{Gr}_{G, \sigma, \xi, 0}) \)

3. \( a \neq 0 \in \mathbb{A}^1(\mathbb{R}) \simeq i\mathbb{R} \):

\( Q_{M,G,G}(\mathbb{Z}_a, \sigma(a), \xi(a))_\mathbb{R} \simeq G[\mathbb{Z}, \sigma, \xi, a]_\mathbb{R} \setminus (\text{Gr}_{G, \sigma, \xi, a} \times \text{Gr}_{G, \sigma, \xi, a}) \simeq \text{Gr}_{G, \sigma, \xi, a} \)

4. \( a = 0 \in \mathbb{A}^1(\mathbb{R}) \simeq i\mathbb{R} \):

\( Q_{M,G,G}(\mathbb{Z}_0, \sigma(0), \xi(0))_\mathbb{R} \simeq G[\mathbb{Z}, \sigma, \xi, 0]_\mathbb{R} \setminus \text{Gr}_{G, \sigma, \xi, 0}_\mathbb{R} \simeq \text{Gr}_{G, \sigma, \xi, 0} \)

Proof. Parts (1) and (2) are Lemma 4.1 and the factorization of Beilinson-Drinfeld Grassmannians. For part (3), observe that

\[(4.4) \quad G[\mathbb{Z}, \sigma, \xi, a]_\mathbb{R} = \{g : \mathbb{P}^1(\mathbb{C}) \setminus \{-a, a\} \to G | g(\mathbb{P}^1(\mathbb{R})) \subset G_c, g(\infty) = e\} \]

Consider the alternative coordinate \( t = \frac{z - a}{z + a} \) on \( \mathbb{P}^1 \) taking \( z = \infty, a, -a \) to \( t = 1, 0, \infty \) respectively. The conjugation \( z \mapsto \bar{z} \) takes the form \( t \mapsto \bar{t}^{-1} \), hence we have \( G[\mathbb{Z}, \sigma, \xi, a]_\mathbb{R} \simeq \Omega G_c \subset G([t, t^{-1}]) \), and part (3) follows from the factorization (2.2). For part (4), observe that

\[(4.5) \quad G[\mathbb{Z}, \sigma, \xi, a]_\mathbb{R} = \{g : \mathbb{P}^1(\mathbb{C}) \setminus \{0\} \to G | g(\mathbb{P}^1(\mathbb{R}) \setminus \{0\}) \subset G_c, g(\infty) = e\} \]

This group is trivial by [CN1 Lemma 4.3], and part (4) immediately follows.

4.2. Splitting. Let \( \ell : G \to G \times G, \ell(g) = (g, e) \) denote the left factor embedding, where \( e \in G \) is the identity. It induces a map

\[(4.6) \quad \ell : \text{Gr}_{G, \sigma, a} \to \text{Gr}_{G, \sigma, a} \quad \ell(a, \mathcal{E}, s) = (a, \mathcal{E}, s|_{\mathbb{P}^1 \setminus \{a, -a\}}) \]

given by forgetting that the section \( s : \mathbb{P}^1 \setminus \{-a\} \to \mathcal{E} \) extends over \( a \in \mathbb{P}^1 \).

By Lemma 4.2 (3), (4), the composition

\[(4.7) \quad \ell \circ \text{Gr}_{G, \sigma, a} \to \text{Gr}_{G, \sigma, a} \to G[\mathbb{Z}, \sigma, \xi]_\mathbb{R} \setminus \text{Gr}_{G, \sigma, a} \simeq Q_{M,G,G}(\mathbb{Z}/\mathbb{A}^1, \sigma, \xi)_\mathbb{R} \]

is an isomorphism, equivariant for the natural \( G_c \)-actions.

Let us transport some standard structures on the Beilinson-Drinfeld Grassmannian \( \text{Gr}_{G, \sigma, a} \) across the isomorphism (4.7).

First, the choice of any global coordinate on \( \mathbb{A}^1 \) provides a trivialization \( \text{Gr}_{G, \sigma, a} \simeq \text{Gr} \times \mathbb{A}^1 \).

The spherical and cospherical strata \( S^\lambda, T^\lambda \subset \text{Gr} \), for \( \lambda \in \mathbb{A}^1 \), extend to spherical and
cospherical strata $S_{z}^{λ} = S^{λ} \times \mathbb{A}^{1}, T_{z}^{λ} = T^{λ} \times \mathbb{A}^{1} \subset \text{Gr}_{G,z,σ_{1}}$, respectively. Fiberwise, these are given by $G[z,σ_{1}]-$orbits and $G[z,σ_{1}]-$orbits respectively, and hence are independent of the coordinate and resulting trivialization.

We will also denote by $\delta_{z}^{λ}, T_{z}^{λ} \subset QM_{G×G,G}(\mathbb{Z}/\mathbb{A}^{1}, σ, ξ)_{\mathbb{R}},$ for $λ \in \Lambda_{T}^{+},$ the transport of the respective strata under the isomorphism (4.7).

Next, let $\text{Bun}_{G}(\mathbb{Z}/\mathbb{A}^{1}) \subset \text{Bun}_{G}(\mathbb{Z}/\mathbb{A}^{1})$ denote the open sub-stack of a point $a \in \mathbb{A}^{1}$ and a trivializable $G$-bundle on $\mathbb{Z}_{a}$. Let $\text{Gr}_{G,z,σ_{1}}^{0} \simeq QM_{G×G,G}(\mathbb{Z}/\mathbb{A}^{1}, σ, ξ)_{\mathbb{R}}$ denote the base-changes to $\text{Bun}_{G}(\mathbb{Z}/\mathbb{A}^{1}) \subset \text{Bun}_{G}(\mathbb{Z}/\mathbb{A}^{1})$. Note that each coincides with the open stratum $T_{z}^{λ} \subset \text{Gr}_{G,z,σ_{1}}^{0}$.

Equip $T_{z}^{λ} \simeq \text{Gr}_{G,z,σ_{1}}^{0} \simeq QM_{G×G,G}(\mathbb{Z}/\mathbb{A}^{1}, σ, ξ)_{\mathbb{R}}$ with the strata $S_{z}^{λ} \cap T_{z}^{0},$ for $λ \in \Lambda_{T}^{+}$. Note these are nonempty if and only if $λ \in R_{G}^{1} \subset \Lambda_{T}^{+}$.

For easy reference, we summarize the preceding discussion in the following:

**Proposition 4.3.**

1. $\text{Gr}_{G,z,σ_{1}} \simeq QM_{G×G,G}(\mathbb{Z}/\mathbb{A}^{1}, σ, ξ)_{\mathbb{R}},$ equipped with either its spherical or strata $S_{z}^{λ},$ for $λ \in \Lambda_{T}^{+},$ or cospherical strata $T_{z}^{λ},$ for $λ \in \Lambda_{T}^{+},$ is a real analytic trivializable stratified family.

2. The open locus $T_{z}^{0} \simeq \text{Gr}_{G,z,σ_{1}}^{0} \simeq QM_{G×G,G}(\mathbb{Z}/\mathbb{A}^{1}, σ, ξ)_{\mathbb{R}},$ equipped with its spherical strata $T_{z}^{0} \cap S_{z}^{λ},$ for $λ \in R_{G}^{1},$ is a real analytic trivializable stratified family.

4.3. $\mathbb{Z}/2 \times \mathbb{Z}/2$-action $(δ_{z}, β_{z})$.

4.3.1. **Involution $δ_{z}$**. Recall $δ = θ \circ η = η \circ θ$ denotes the Cartan conjugation of $G$ with compact real form $G_{c}$. Recall $sw$ denotes the swap involution of $G \times G$ with fixed-point subgroup the diagonal $G$, and $sw_{δ} = sw \circ (δ \times δ) = (δ \times δ) \circ sw$.

Note that the conjugation $δ \times δ$ of $G \times G$ commutes with $sw_{a}$, and hence together with the conjugation $c_{z}$ of $\mathbb{Z}$ induces an involution $δ_{z}$ of $QM_{G×G,G}(\mathbb{Z}/\mathbb{A}^{1}, σ, ξ)_{\mathbb{R}}$.

At $a = 0 \in \mathbb{A}^{1}(\mathbb{R}) \simeq i\mathbb{R},$ the standard coordinate $z$ provides an isomorphism

$$(4.8) \quad v_{0} : \text{Gr} \xrightarrow{\sim} \text{Gr}_{G, z_{0}, σ_{1}}^{0} \xrightarrow{\text{Len}(4.2)} QM_{G×G,G}(\mathbb{Z}_{0}, σ(0), ξ(0))_{\mathbb{R}}$$

and the involution $δ_{z}$ satisfies

$$(4.9) \quad δ_{z} \circ v_{0} = v_{0} \circ δ,$$

where as usual the conjugation $δ$ of $\text{Gr}$ is given by $δ([g(z)]) = [δ(\bar{g}(z))]$.

Let us describe the involution $δ_{z}$ on the fiber $QM_{G×G,G}(\mathbb{Z}_{a}, σ(a), ξ(a))_{\mathbb{R}},$ for $a \neq 0 \in \mathbb{A}^{1}(\mathbb{R}) \simeq i\mathbb{R}$. Consider the alternative coordinate $t = \frac{z-a}{z+a}$ on $\mathbb{Z}_{a} = \mathbb{P}^{1}$ taking $z = \infty, a, -a$ to $t = 1, 0, \infty$ respectively. The conjugation $z \mapsto \bar{z}$ takes the form $t \mapsto \bar{t}^{-1}$, and hence the coordinate $t$ provides isomorphisms

$$(4.10) \quad \text{Gr}_{G,z,σ_{1}}^{0, a} \simeq G(\mathbb{C}[t,t^{-1}])/G(\mathbb{C}[t^{-1}]) \quad \text{Gr}_{G,z,σ_{2}}^{0, a} \simeq G(\mathbb{C}[t,t^{-1}])/G(\mathbb{C}[t])$$

$$(4.11) \quad G[z, σ, τ, a]_{\mathbb{R}} \simeq ΩG_{c} \subset G(\mathbb{C}[t,t^{-1}])$$
With the above identifications, the isomorphism of Lemma 4.2(3) becomes an isomorphism

\[(4.12) \quad QM_{G \times G, G}(Z_a, \sigma(a), \tau(a))_{\mathbb{R}} \simeq \Omega G_{c}(G(\mathbb{C}[t, t^{-1}]) / G(\mathbb{C}[t^{-1}]) \times G(\mathbb{C}[t, t^{-1}]) / G(\mathbb{C}[t])) \]

\[\simeq G(\mathbb{C}[t, t^{-1}]) / G(\mathbb{C}[t^{-1}]).\]

Moreover, the upper map in (4.12) intertwines the involution \(\delta_Z\) of \(QM_{G \times G, G}(Z_a, \sigma(a), \tau(a))_{\mathbb{R}}\) and the involution of the right hand side of (4.12) given by

\[(4.13) \quad ([g_1], [g_2]) \mapsto ([\delta(g_2)], [\delta(g_1)])\]

where \(\tau : G(\mathbb{C}[t, t^{-1}]) \to G(\mathbb{C}[t, t^{-1}]), \tau(g)(t) = g(t^{-1}).\)

Now consider the composite isomorphism

\[(4.14) \quad \nu_a : \Omega G_{c} \xrightarrow{\sim} G(\mathbb{C}[t, t^{-1}]) / G(\mathbb{C}[t^{-1}]) \xrightarrow{4.12} QM_{G \times G, G}(Z_a, \sigma(a), \tau(a))_{\mathbb{R}}\]

where the first map is itself the composite isomorphism

\[(4.15) \quad \Omega G_{c} \xrightarrow{2.2} G(\mathbb{C}[t, t^{-1}]) / G(\mathbb{C}[t]) \xrightarrow{\tau} G(\mathbb{C}[t, t^{-1}]) / G(\mathbb{C}[t^{-1}])\]

Then it is a simple diagram chase to check the following:

**Lemma 4.4.** For \(\text{inv} : \Omega G_{c} \to \Omega G_{c}\) the group-inverse, we have

\[(4.16) \quad \delta_Z \circ \nu_a = \nu_a \circ (\text{inv} \circ \delta \circ \tau) = \nu_a \circ \text{inv}\]

That is, the isomorphism (4.14) intertwines the involution \(\delta_Z\) and the group-inverse \(\text{inv}\).

4.3.2. *Involution \(\beta_Z\).* Consider the involution \(\beta_Z : Z \to Z, \beta_Z(z, a) = (-z, a)\). It naturally induces an involution

\[(4.17) \quad \beta_Z : QM_{G \times G, G}(Z, \sigma, \tau)_{\mathbb{R}} \longrightarrow QM_{G \times G, G}(Z, \sigma, \tau)_{\mathbb{R}}\]

In terms of our prior constructions, at \(a = 0 \in A^1(\mathbb{R}) \simeq i\mathbb{R}\), the involution \(\beta_Z\) satisfies

\[(4.18) \quad \beta_Z \circ \nu_0 = \nu_0 \circ \text{neg}\]

where \(\text{neg} : \text{Gr} \to \text{Gr}\) is the coordinate automorphism \(\text{neg}([g(z)]) = [g(-z)]\).

At \(a \neq 0 \in A^1(\mathbb{R}) \simeq i\mathbb{R}\), in terms of the coordinate \(t = \frac{z-a}{z+a}\) on \(Z_a = \mathbb{P}^1\), the involution \(z \mapsto -z\) takes the form \(t \mapsto t^{-1}\), and hence under the isomorphism of Lemma 4.2(3), the involution \(\beta_Z\) takes the form \(([g_1], [g_2]) \mapsto ([\tau(g_2)], [\tau(g_1)])\). It follows that \(\beta_Z\) satisfies

\[(4.19) \quad \beta_Z \circ \nu_a = \nu_a \circ (\text{inv} \circ \tau) = \nu_a \circ (\text{inv} \circ \delta)\]

Here we use that \(\delta(\tau(\gamma)) = \gamma\) for \(\gamma \in \Omega G_{c}\), hence \(\tau(\gamma) = \delta(\gamma)\).
4.3.3. Compatibility with stratifications. Recall the spherical and cospherical stratifications $S^\lambda; \mathcal{T}_z^\lambda \subset QM_{G \times G,G}(\mathbb{Z}/\mathbb{A}^1, \sigma, \xi)\mathbb{R}$, for $\lambda \in \Lambda^+_G$. At $a = 0 \in \mathbb{A}^1(\mathbb{R}) \simeq i\mathbb{R}$, the isomorphism $v_0$ of (4.8) takes the spherical stratum $S^\lambda$ (resp. cospherical stratum $T^\lambda$) of $\text{Gr}$ isomorphically to the fiber $S^\lambda_{a|0}$ (resp. $T^\lambda_{a|0}$). At $a \neq 0 \in \mathbb{A}^1(\mathbb{R}) \simeq i\mathbb{R}$, the isomorphism $v_a$ of (4.14) takes the transported spherical stratum $S^\lambda$ (resp. cospherical stratum $T^\lambda$) of $\Omega G_c$ isomorphically to the fiber $S^\lambda_{a|0}$ (resp. $T^\lambda_{a|0}$).

**Lemma 4.5.** Let $w_0 \in W$ denote the longest element of the Weyl group.

For any $\lambda \in \Lambda^+_G$, we have:

1. The involution $\delta_z$ of $QM_{G \times G,G}(\mathbb{Z}/\mathbb{A}^1, \sigma, \xi)\mathbb{R}$ maps $S^\lambda_z$ (resp. $T^\lambda_z$) to $S^{-v_0(\lambda)}_z$ (resp. $T^{-v_0(\lambda)}_z$).

2. The involution $\beta_z$ of $QM_{G \times G,G}(\mathbb{Z}/\mathbb{A}^1, \sigma, \xi)\mathbb{R}$ maps $S^\lambda_z$ (resp. $T^\lambda_z$) to itself.

**Proof.** Note the conjugation $\delta$ of $\text{Gr}$ maps $S^\lambda$ (resp. $T^\lambda$) to $S^{-v_0(\lambda)}$ (resp. $T^{-v_0(\lambda)}$), and the involution neg of $\text{Gr}$ maps $S^\lambda$ (resp. $T^\lambda$) to itself. Similarly, the involution inv of $\Omega G_c$ maps $S^\lambda$ (resp. $T^\lambda$) to $S^{-v_0(\lambda)}$ (resp. $T^{-v_0(\lambda)}$), and hence the composition inv $\delta$ maps $S^\lambda$ (resp. $T^\lambda$) to itself. From here, the lemma is an elementary verification from the formulas (4.9), (4.16), (4.18), and (4.19).

The lemma implies that the $\mathbb{Z}/2 \times \mathbb{Z}/2$-action ($\delta_z, \beta_z$) on $QM_{G \times G,G}(\mathbb{Z}/\mathbb{A}^1, \sigma, \xi)\mathbb{R}$ preserves the open stratum $\mathcal{T}^0 \simeq QM^0_{G \times G,G}(\mathbb{Z}/\mathbb{A}^1, \sigma, \xi)\mathbb{R}$ and its spherical stratification is preserved.

(3) At $a = 0$, the isomorphism $v_0 : \text{Gr} \simeq QM_{G \times G,G}(\mathbb{Z}_0, \sigma(0), \xi(0))\mathbb{R}$ of (4.8) intertwines the action with the $\mathbb{Z}/2 \times \mathbb{Z}/2$-action on $\text{Gr}$ given by $(\delta, \text{neg})$.

(4) At $a \neq 0$, the isomorphism $v_a : \Omega G_c \simeq QM_{G \times G,G}(\mathbb{Z}_a, \sigma(a), \xi(a))\mathbb{R}$ of (4.14) intertwines the action with the $\mathbb{Z}/2 \times \mathbb{Z}/2$-action on $\Omega G_c$ given by $(\text{inv}, \text{inv} \circ \delta)$.

5. Nodal degeneration

In this section, we apply the constructions of Section 3 to a nodal degeneration of $\mathbb{P}^1$. Set $A = \mathbb{A}^1$ to be the affine line with coordinate $a$. Consider the product $\mathbb{P}^1 \times \mathbb{P}^1$ with respective homogeneous coordinates $[x_0, x_1], [y_0, y_1]$, local coordinates $x = x_1/x_0, y = y_1/y_0$, and projections $p_x, p_y : \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^1$.

Introduce the surface $\mathcal{Y} \subset \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{A}^1$ cut out by $x_1 y_1 = a^2 x_0 y_0$. We will regard $\mathcal{Y}$ as a family of curves via the evident projection $p : \mathcal{Y} \to \mathbb{A}^1$. We denote the fibers by $\mathcal{Y}_a = p^{-1}(a)$, for $a \in \mathbb{A}^1$. When $a \neq 0$, projection along $p_x$ or $p_y$ provides an isomorphism $\mathcal{Y}_a \simeq \mathbb{P}^1$. When $a = 0$, the image of the inclusion $\mathcal{Y}_0 \subset \mathbb{P}^1 \times \mathbb{P}^1$ is the nodal curve

$$\mathbb{P}^1 \cup \mathbb{P}^1 := (\mathbb{P}^1 \times \{0\}) \cup (\{0\} \times \mathbb{P}^1)$$
There are three canonical sections \( \sigma = (\sigma_x, \sigma_y) : \mathbb{A}^1 \to \mathbb{Y}^2 \) and \( \xi : \mathbb{A}^1 \to \mathbb{Y} \) given by
\[
\sigma_x(a) = (\infty, 0, a) \quad \sigma_y(t) = (0, \infty, a) \quad \xi(a) = (a, a, a).
\]
Note that the section \( \xi \) never intersects \( \sigma_x, \sigma_y \), and \( x^{-1}, y^{-1} \) provide local coordinates along \( \mathbb{Y} \) based at \( \sigma_x, \sigma_y \) respectively.

There is also an involution \( \beta_y : \mathbb{Y} \to \mathbb{Y} \), \( \beta_y(x, y, a) = (y, x, a) \). Note that \( \beta_y \) fixes the section \( \xi \) and intertwines the other sections \( \beta_y \circ \sigma_x = \sigma_y \) as well as the local coordinates \( x^{-1} \circ \beta_y = y^{-1} \).

Set \( c_{\mathbb{A}^1}(a) = \bar{a} \) to be the standard conjugation. Similarly, we have the standard real structure on \( \mathbb{Y} \) given by \( (x, y, t) \mapsto (\bar{x}, y, \bar{t}) \), but we will work with its twist by \( \text{sw} \) which we denote by \( c_y(x, y, t) = (\bar{y}, \bar{x}, \bar{t}) \). Note that \( \mathbb{A}^1(\mathbb{R}) \simeq \mathbb{R} \). When \( a \neq 0 \in \mathbb{A}^1(\mathbb{R}) \), under the identification \( \mathbb{Y} \simeq \mathbb{P}^1 \) given by \( p_x \), we have \( c_y(x) = a^2 / \bar{x} \). Thus when \( a \neq 0 \), the real points of \( \mathbb{Y}_a \) are non-empty and isomorphic to \( \mathbb{P}^1(\mathbb{R}) \). When \( a = 0 \in \mathbb{A}^1(\mathbb{R}) \), the components of \( \mathbb{Y}_0 \) are exchanged by \( c_y \), and there is a unique real point \( x = 0, y = 0 \).

5.1. **Relation to real degeneration.** Before studying the nodal degeneration \( p : \mathbb{Y} \to \mathbb{A}^1 \) further, let us record here how it can be “glued” to the real degeneration \( \pi : \mathbb{Z} \to \mathbb{A}^1 \) of the previous section.

Consider the map
\[
\kappa : \mathbb{Z} \to \mathbb{Y} \quad \kappa(z, a) = (x, y, -ia) \quad x = (-ia)^{\frac{z-a}{z+a}}, \quad y = (-ia)^{\frac{z+a}{z-a}}
\]
Note that \( \kappa \) fits into a commutative diagram
\[
\begin{array}{ccc}
\mathbb{Z} & \xrightarrow{\kappa} & \mathbb{Y} \\
\downarrow & & \downarrow \\
\mathbb{A}^1 & \xrightarrow{r} & \mathbb{A}^1
\end{array}
\]
where \( r(a) = -ia \). Furthermore, it is elementary to check \( \kappa \) intertwines the conjugations \( \kappa \circ c_{\mathbb{Z}} = c_y \circ \kappa \), and together \( \kappa \) and \( r \) intertwine the sections
\[
\kappa \circ \xi = \xi \circ r \quad \kappa \circ \sigma_1 = \sigma_x \circ r \quad \kappa \circ \sigma_2 = \sigma_y \circ r
\]
Moreover, \( \kappa \) intertwines involutions \( \kappa \circ \beta_{\mathbb{Z}} = \text{sw}_{\mathbb{Y}} \circ \kappa \).

A direct verification, left to the reader, shows that:

**Lemma 5.1.** The base-change of \( \kappa \) over the open locus \( \mathbb{A}^1 \setminus \{0\} \) in the base is an isomorphism.

**Corollary 5.2.** For \( a \in \mathbb{A}^1(\mathbb{R}) \setminus \{0\} \simeq \mathbb{R}^\times \), if we set \( a' := -ia \in i\mathbb{R}^\times \), then pullback along \( \kappa \) provides isomorphisms
\[
\kappa_a^* : QM_{G,X}(\mathbb{Y}_a, \sigma(a), \xi(a))_\mathbb{R} \xrightarrow{\sim} QM_{G,X}(\mathbb{Z}_{a'}, \sigma(a'), \xi(a'))_\mathbb{R}
\]

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5.2. Nodal uniformizations. We continue here with the constructions of Section 3 applied to the nodal degeneration $p : Y \to \mathbb{A}^1$, with sections $\sigma = (\sigma_x, \sigma_y)$, $\xi(a) = (a, a, a)$, and conjugations $c(a) = \bar{a}$, $c(y, y, a) = (\bar{y}, \bar{x}, \bar{a})$.

Note that, since $\sigma_x(a) \neq \sigma_y(a)$, for all $a \in \mathbb{A}^1$, we have the factorization

\[(5.14) \quad \text{Gr}_{G,Y,\sigma} \simeq \text{Gr}_{G,Y,\sigma_x} \times \text{Gr}_{G,Y,\sigma_y}\]

and the projection to the factor $\text{Gr}_{G,Y,\sigma_x}$ restricts to an isomorphism

\[(5.15) \quad \text{Gr}_{G,Y,\sigma \in \mathbb{R}} \simeq \text{Gr}_{G,Y,\sigma_x}\]

of real analytic varieties over $\mathbb{A}^1(\mathbb{R}) = \mathbb{R}$.

In terms of the isomorphisms \((5.7), (5.8)\), the uniformization morphisms of \((3.6), (3.9)\) take the respective forms

\[(5.9) \quad K[y, \sigma, \xi] / (\text{Gr}_{G,Y,\sigma_x} \times \text{Gr}_{G,Y,\sigma_y}) \longrightarrow \text{QM}_{G,X}(Y/\mathbb{A}^1, \sigma, \xi)\]

\[(5.10) \quad K[y, \sigma, \xi]_{\mathbb{R}} / \text{Gr}_{G,Y,\sigma_x} \longrightarrow \text{QM}_{G,X}(Y/\mathbb{A}^1, \sigma, \xi)_{\mathbb{R}}\]

We can describe them more concretely in terms of our coordinates. Note $K[y, \sigma, \xi]$ is the group scheme $K(\mathbb{C}[xy = a^2])_{x \to e}$ of maps $g : \text{Spec}(\mathbb{C}[x, y, a]/(xy - a^2)) \to K$ such that $g(a, a, a) = e$. Similarly, the real form $K[y, \sigma, \xi]_{\mathbb{R}}$ is the subgroup scheme $K(\mathbb{C}[xy = a^2])_{x \to e, \mathbb{R}}$ of those maps $g$ such that $\eta(g(\bar{y}, \bar{x}, \bar{a})) = g(x, y, a)$.

Going further, we can consider the projection $p_x : Y \to \mathbb{P}^1 \times \mathbb{A}^1$ to the first $\mathbb{P}^1$-factor. For $a \neq 0$, let $K(\mathbb{C}[x, x^{-1}])_{a \to e} \subset K(\mathbb{C}[x, x^{-1}])$ denote the subgroup of maps $g : \mathbb{A}^1 \setminus \{0\} \to K$ such that $g(a) = e$. When $a = 0$, let $K(\mathbb{C}[x])_{0 \to e} \subset K(\mathbb{C}[x])$ denote the subgroup of maps $g : \mathbb{A}^1 \to K$ such that $g(0) = e$. Then pullback gives isomorphisms

\[(5.11) \quad p_x^* : K(\mathbb{C}[x, x^{-1}])_{a \to e} \longrightarrow K(\mathbb{C}[xy = a^2])_{x \to e} \simeq K[y, \sigma, \xi, a]\]

\[(5.12) \quad p_x^* : K(\mathbb{C}[x])_{0 \to e} \longrightarrow K(\mathbb{C}[xy = 0])_{x \to e} \simeq K[y, \sigma, \xi, 0]\]

Passing to real forms, for $a \neq 0 \in \mathbb{A}^1(\mathbb{R}) \simeq \mathbb{R}$, let $K(\mathbb{C}[x, x^{-1}])_{a \to e, \mathbb{R}}$ denote those maps such that $\eta(g(\bar{x}^{-1})) = g(x)$. Note this is simply the based loop group $\Omega_a K_e$ at the base point $x = a$. Again pullback gives isomorphisms

\[(5.13) \quad p_x^* : \Omega_a K_e \longrightarrow K(\mathbb{C}[xy = a^2])_{x \to e, \mathbb{R}} \simeq K[y, \sigma, \xi, a]_{\mathbb{R}}\]

\[(5.14) \quad p_x^* : K(\mathbb{C}[x])_{0 \to e} \longrightarrow K(\mathbb{C}[xy = 0])_{x \to e, \mathbb{R}} \simeq K[y, \sigma, \xi, 0]_{\mathbb{R}}\]

Note that there is no appearance of real structure on the left hand side of isomorphism \((5.14)\).

Thus we can rewrite the real uniformization morphism along fibers in the form

\[(5.15) \quad \Omega_a K_e \setminus \text{Gr}_{G,Y,\sigma_x, a} \longrightarrow \text{QM}_{G,X}(Y/\mathbb{A}^1, \sigma, \xi, a)_{\mathbb{R}} \quad a \neq 0\]

\[(5.16) \quad K(\mathbb{C}[x])_{0 \to e} \setminus \text{Gr}_{G,Y,\sigma_x, 0} \longrightarrow \text{QM}_{G,X}(Y/\mathbb{A}^1, \sigma, \xi, 0)_{\mathbb{R}} \quad a = 0\]
Finally, before continuing on, observe that the isomorphism over \( \mathbb{A}^1 \setminus \{0\} \) recorded in Corollary 5.5.1 is compatible with uniformizations. For \( a \neq 0 \in \mathbb{A}^1(\mathbb{R}) \cong \mathbb{R} \), if we set \( a' := -ia \in i\mathbb{R} \), pullback along the isomorphism \( \kappa \) of Lemma 5.1 provides isomorphisms (5.17)

\[
\text{Gr}_{G,Y,\sigma_x,a} \cong \text{Gr}_{G,Z,\sigma_1,a'} \quad \text{Gr}_{G,Y,\sigma_y,a} \cong \text{Gr}_{G,Z,\sigma_2,a'} \quad K[y,\sigma,\xi,a]_{\mathbb{R}} \cong K[Z,\sigma,\xi,a']_{\mathbb{R}}
\]

so that we have a commutative diagram of uniformization morphisms (5.18)

\[
\begin{array}{c}
K[y,\sigma,\xi,a]_{\mathbb{R}} \setminus \text{Gr}_{G,Y,\sigma_x,a} \xrightarrow{\sim} K[Z,\sigma,\xi,a']_{\mathbb{R}} \setminus \text{Gr}_{G,Z,\sigma_1,a'} \\
\downarrow \quad \quad \quad \downarrow \\
\text{QM}_{G,X}(y_a,\sigma(a),\xi(a))_{\mathbb{R}} \xrightarrow{\sim} \text{QM}_{G,X}(Z_{a'},\sigma(a'),\xi(a'))_{\mathbb{R}}
\end{array}
\]

5.3. Complex groups. Recall \( \delta = \theta \circ \eta = \eta \circ \theta \) denotes the Cartan conjugation of \( G \) with compact real form \( G_c \). Equip \( G \times G \) with the swap involution \( \text{sw}(g,h) = (h,g) \) and the conjugation \( \text{sw}_\delta(g,h) = (\delta(h),\delta(g)) \). The fixed-point subgroup of \( \text{sw} \) is the diagonal \( G \subset G \times G \), and the corresponding symmetric space is isomorphic to the group \( G \setminus (G \times G) \cong G \).

The uniformization morphisms (5.9), (5.10) then take the form (5.19)

\[
G[y,\sigma,\xi]\setminus(\text{Gr}_{G,Y,\sigma_x} \times \text{Gr}_{G,Y,\sigma_x} \times \text{Gr}_{G,Y,\sigma_y} \times \text{Gr}_{G,Y,\sigma_y}) \longrightarrow \text{QM}_{G\times G,G}(y/\mathbb{A}^1,\sigma,\xi)
\]

(5.20)

\[
G[y,\sigma,\xi]\setminus(\text{Gr}_{G,Y,\sigma_x} \times \text{Gr}_{G,Y,\sigma_x}) \longrightarrow \text{QM}_{G\times G,G}(y/\mathbb{A}^1,\sigma,\xi)
\]

Similarly, the concrete descriptions (5.15), (5.16) of the real uniformization morphisms along the fibers can be rewritten in the form (5.21)

\[
\text{Gr}_{G,Y,\sigma_x,a} \cong \Omega_a G_c \setminus(\text{Gr}_{G,Y,\sigma_x,a} \times \text{Gr}_{G,Y,\sigma_x,a}) \longrightarrow \text{QM}_{G\times G,G}(y_a,\sigma(a),\xi(a))_{\mathbb{R}} \quad a \neq 0
\]

(5.22)

\[
G(\mathbb{C}[x])_{0 \rightarrow }\setminus(\text{Gr}_{G,Y,\sigma_x,0} \times \text{Gr}_{G,Y,\sigma_x,0}) \longrightarrow \text{QM}_{G\times G,G}(y_0,0,\sigma(0),\xi(0))_{\mathbb{R}} \quad a = 0
\]

where the first isomorphism uses the factorization of (2.2).

5.4. Splitting over trivial bundle. Let \( \text{Bun}_G^0(\mathbb{Y}/\mathbb{A}^1) \subset \text{Bun}_G(\mathbb{Y}/\mathbb{A}^1) \) denote the open sub-stack of a point \( a \in \mathbb{A}^1 \) and a trivializable \( G \)-bundle on \( \mathbb{Y}_a \). Denote by \( \text{Gr}_{G,Y,\sigma_x}^{0} \rightarrow \text{QM}_{G\times G,G}^0(Z/\mathbb{A}^1,\sigma,\xi)_{\mathbb{R}} \) the base-change to \( \text{Bun}_G^0(\mathbb{Y}/\mathbb{A}^1) \subset \text{Bun}_G(\mathbb{Y}/\mathbb{A}^1) \) along the natural projections \( \text{Gr}_{G,Y,\sigma_x} \rightarrow \text{Bun}_G(\mathbb{Y}/\mathbb{A}^1) \), \( \text{QM}_{G\times G,G}^0(Z/\mathbb{A}^1,\sigma,\xi)_{\mathbb{R}} \rightarrow \text{Bun}_{G\times G}(\mathbb{Y})_{\mathbb{R}} \cong \text{Bun}_G(\mathbb{Y}) \), where the last isomorphism is given by projection to the left factor.

Consider the composition (5.23)

\[
u : \text{Gr}_{G,Y,\sigma_x} \longrightarrow G[y,\sigma,\xi]\setminus(\text{Gr}_{G,Y,\sigma_x} \times \text{Gr}_{G,Y,\sigma_x}) \longrightarrow \text{QM}_{G\times G,G}(y/\mathbb{A}^1,\sigma,\xi)_{\mathbb{R}}
\]

where the first arrow is the left factor embedding and the second is the uniformization morphism. Note that the maps in (5.23) are equivariant for the natural \( G_c \)-actions.
Lemma 5.3. The restriction of the map $u$ of \((5.23)\) is a $G_c$-equivariant isomorphism \begin{equation}
(5.24) \quad \Gr^0_{G,y,\sigma_x} \sim QM^0_{G \times G,G}(\mathbb{Z}/\mathbb{A}^1, \sigma, \xi)_{\mathbb{R}}
\end{equation}
of real analytic varieties over $\mathbb{A}^1(\mathbb{R}) \simeq \mathbb{R}$.

Proof. When $a \neq 0 \in \mathbb{A}^1(\mathbb{R}) \simeq \mathbb{R}$, the isomorphism $\kappa$ of \((5.5.1)\) identifies the map \((5.23)\) with the map \((4.7)\), hence is an isomorphism by Proposition 4.3.

When $a = 0 \in \mathbb{A}^1(\mathbb{R}) \simeq \mathbb{R}$, using the description \((5.22)\) of the real uniformization map, the restriction of the map $u$ of \((5.23)\) to the open locus $\Gr^0_{G,y,\sigma_x,0}$ takes the form\begin{equation}
(5.25) \quad \Gr^0_{G,y,\sigma_x,0} \to G(\mathbb{C}[x])_{0 \to e} \setminus (\Gr^0_{G,y,\sigma_x,0} \times \Gr^0_{G,y,\sigma_x,0}) \to QM^0_{G \times G,G}(y_0, \sigma(0), \xi(0))_{\mathbb{R}}
\end{equation}
where the first arrow is the left factor embedding and the second is the uniformization morphism. Since the $G(\mathbb{C}[x])_{0 \to e}$-action on $\Gr^0_{G,y,\sigma_x,0}$ is free and transitive, the first arrow is an isomorphism. It is a standard fact that the uniformization morphism of the second arrow is an isomorphism. Hence their composition is an isomorphism and the lemma is proved. \[\square\]

The coordinate $x^{-1}$ provides a trivialization $\Gr_{G,y,\sigma_x} \simeq G \times \mathbb{A}^1$. The spherical and cospherical strata $S^\lambda, T^\lambda \subset Gr, \lambda \in \Lambda^+_T$, extend to spherical and cospherical strata $S^\lambda_0 = S^\lambda \times \mathbb{A}^1, T^\lambda_0 = T^\lambda \times \mathbb{A}^1 \subset Gr_{G,y,\sigma_x}$, respectively. Fiberwise, these are given by $G[y, \sigma_x]$-orbits and $G[y, x \sigma]$-orbits respectively, and hence are independent of the coordinate and resulting trivialization.

By construction, the trivialization of the family $\mathcal{T}^0_y \simeq \Gr^0_{G,y,\sigma_x} \simeq QM^0_{G \times G,G}(\mathbb{Z}/\mathbb{A}^1, \sigma, \xi)_{\mathbb{R}}$ preserves the natural strata $S^\lambda_0 \cap \mathcal{T}^0_y, \lambda \in R^+_G$. Thus we may summarize the situation in the following proposition:

**Proposition 5.4.** The open locus $\mathcal{T}^0_y \simeq \Gr^0_{G,y,\sigma_x} \simeq QM^0_{G \times G,G}(\mathbb{Z}/\mathbb{A}^1, \sigma, \xi)_{\mathbb{R}}$, equipped with its spherical strata $S^\lambda_0 \cap \mathcal{T}^0_y, \lambda \in R^+_G$, is a real analytic trivializable stratified family.

5.5. $\mathbb{Z}/2 \times \mathbb{Z}/2$-action $(\delta_y, \beta_y)$.

5.5.1. Involutions $\delta_y$. Recall $\delta = \theta \circ \eta = \eta \circ \theta$ denotes the Cartan conjugation of $G$ with compact real form $G_c$. Recall $sw$ denotes the swap involution of $G \times G$ with fixed-point subgroup the diagonal $G$, and $sw=sw \circ (\delta \times \delta) = (\delta \times \delta) \circ sw$.

Note that the conjugation $\delta \times \delta$ of $G \times G$ commutes with $sw$, and hence together with the conjugation $c_y$ of $y$ induces an involution $\delta_y$ of $QM_{G \times G,G}(y/\mathbb{A}^1, \sigma, \xi)_{\mathbb{R}}$.

At $a = 0 \in \mathbb{A}^1(\mathbb{R}) \simeq \mathbb{R}$, the uniformization map \((5.22)\)\begin{equation}
u : G(\mathbb{C}[x])_{0 \to e} \setminus (\Gr_{G,y,\sigma_x,0} \times \Gr_{G,y,\sigma_x,0}) \to QM_{G \times G,G}(y_0, \sigma(0), \xi(0))_{\mathbb{R}}
\end{equation}satisfies\begin{equation}
(5.27) \quad \delta_y(\nu([\gamma_1], [\gamma_2])) = \nu([\gamma_2], [\gamma_1])
\end{equation}
Set $\mathcal{G}^* = G(\mathbb{C}[x^{-1}])_{0 \to e}$, and consider the isomorphism\begin{equation}
(5.28) \quad \mathcal{G}^* = G(\mathbb{C}[x^{-1}])_{0 \to e} \simeq G(\mathbb{C}[x])_{0 \to e} \simeq \Gr^0_{G,y,\sigma_x,0}
\end{equation}
It is elementary to check the composite isomorphism
\[ u_0 : G^* \overset{(5.28)}{=} G_{G,y,\sigma_x,0} \overset{\text{Lem 5.3}}{=} QM_{G \times G,G}^0(Y_0, \sigma(0), \xi(0)) \] satisfies
\[ \delta_y \circ u_0 = u_0 \circ \text{inv} \] where inv : $G^* \rightarrow G^*$ is the group-inverse.

For $a \neq 0 \in A^1(\mathbb{R}) \simeq \mathbb{R}$, set $a' := -ia \in i\mathbb{R}$, and recall the isomorphism of Corollary
\[ \kappa_a^* : QM_{G,X}^0(Y_a, \sigma(a), \xi(a)) \longrightarrow QM_{G,Z,a'}(Z_{a'}, \sigma(a'), \xi(a')) \] It is again elementary to check that
\[ \delta_z \circ \kappa_a^* = \kappa_a^* \circ \delta_y \]
From this, we see that the isomorphism
\[ u_a : \Omega G_e \overset{(5.33)}{=} QM_{G \times G,G}(Z_{a'}, \sigma(a'), \xi(a')) \] satisfies
\[ \delta_y \circ u_a = u_a \circ \text{inv} \] where inv : $\Omega G_e \rightarrow \Omega G_e$ is the group-inverse.

5.5.2. *Involution $\beta_y$.* Recall the involution $\beta_y : Y \rightarrow Y, \beta_y(x,y,a) = (y,x,a)$. It induces an involution
\[ \beta_y : QM_{G \times G,G}(Y, \sigma, \xi) \rightarrow QM_{G \times G,G}(Y, \sigma, \xi). \]
Over $a = 0$, the uniformization map
\[ u : G(\mathbb{C}[x])_1 \setminus (Gr_{G,Y,\sigma_x} \times Gr_{G,Y,\sigma_x}) \rightarrow QM_{G \times G,G}(Y_0, \sigma(0), \xi(0)) \] satisfies
\[ \beta_y \circ u([\gamma_1], [\gamma_2]) = u([\delta(\gamma_2), [\delta(\gamma_1)]]. \] Thus the isomorphism $u_0$ in (5.29) satisfies
\[ \beta_y \circ u_0 = u_0 \circ (\text{inv} \circ \delta) \] 

Over $a \neq 0$, as the isomorphism $\kappa$ in Lemma 5.1 satisfies $\kappa(-z, a) = sw_y \circ \kappa(z, a)$, the isomorphism $\kappa_a$ in (5.31) intertwines the involution $\beta_z$ in (4.17) and $\beta_y$. From this we see that the involution $\beta_y$ satisfies
\[ \beta_y \circ u_a = \beta_y \circ \kappa_a \circ v_{-ia} = \kappa_a \circ \beta_z \circ v_{-ia} = \kappa_a \circ v_{-ia} \circ \text{inv} \circ \delta = u_a \circ \text{inv} \circ \delta. \]
5.5.3. **Compatibility with stratifications.** Recall the spherical stratification $S_\lambda^G \cap T_y^0$, $\lambda \in R_G^+$, of the open locus $QM_{G \times G, G}^0(\mathbb{Y}/\mathbb{A}^1, \sigma, \xi)_\mathbb{R}$. At $a = 0 \in \mathbb{A}^1(\mathbb{R}) = \mathbb{R}$, the isomorphism $u_0$ of (5.29) takes the spherical stratum $S^* \cdot \lambda = S^* \cap S^* \subset S^*$ to the fiber $(S_\lambda^G \cap T_y^0)|_0 \subset QM_{G \times G, G}^0(\mathbb{Y}_0, \sigma(0), \xi(0))_\mathbb{R}$. At $a \neq 0 \in \mathbb{A}^1(\mathbb{R}) = \mathbb{R}$, the isomorphism $u_a$ of (5.33) takes the stratum $S^\lambda \cap T^0 \subset \Omega_{G_c}$ to the fiber $(S_\lambda^G \cap T_y^0)|_a \subset QM_{G \times G, G}^0(y_a, \sigma(a), \xi(a))_\mathbb{R}$.

**Lemma 5.5.** Let $w_0 \in W$ denote the longest element of the Weyl group. For any $\lambda \in R_G^+$, we have:

1. The involution $\delta_y$ of $QM_{G \times G, G}^0(\mathbb{Y}/\mathbb{A}^1, \sigma, \xi)_\mathbb{R}$ maps the stratum $S_\lambda^G \cap T_y^0$ to the stratum $S_{-w_0(\lambda)}^G \cap T_y^0$.
2. The involution $\beta_y$ of $QM_{G \times G, G}^0(\mathbb{Y}/\mathbb{A}^1, \sigma, \xi)_\mathbb{R}$ preserves the stratum $S_\lambda^G \cap T_y^0$.

**Proof.** The proof is the same as the proof of Lemma 4.5 using the formulas in (5.30), (5.34), (5.36), and (5.37). □

We have arrived at the following analogue of Proposition 4.6.

**Proposition 5.6.** The involution $(\delta_y, \beta_y)$ provide a $\mathbb{Z}/2 \times \mathbb{Z}/2$-action on $QM_{G \times G, G}^0(\mathbb{Y}/\mathbb{A}^1, \sigma, \xi)_\mathbb{R}$.

1. The action commutes with the natural $G_c$-action.
2. The action preserves the spherical stratification.
3. At $a = 0$, the isomorphism $u_0 : S^* \simeq QM_{G \times G, G}^0(\mathbb{Y}_0, \sigma(0), \xi(0))_\mathbb{R}$ intertwines the action with the $\mathbb{Z}/2 \times \mathbb{Z}/2$-action on $S^*$ given by $(\text{inv}, \text{inv} \circ \delta)$.
4. At $a \neq 0$, the isomorphism $u_a : \Omega_{G_c} \simeq QM_{G \times G, G}^0(y_a, \sigma(a), \xi(a))_\mathbb{R}$ intertwines the action with the $\mathbb{Z}/2 \times \mathbb{Z}/2$-action on $\Omega_{G_c}$ given by $(\text{inv}, \text{inv} \circ \delta)$.

6. **Stratified homeomorphisms**

Recall $\theta$ denotes the Cartan involution of $G$ with fixed-point subgroup $K$, and $\delta = \theta \circ \eta = \eta \circ \theta$ the Cartan conjugation of $G$ with compact real form $G_c$. Since $\eta$ and $\theta$ commute with $\delta$, the conjugation $\eta \times \eta$ and involution $\theta \times \theta$ of $G \times G$ define involutions of the real moduli $QM_{G \times G, G}(\mathbb{Z}, \sigma, \xi)_\mathbb{R}$ (resp. $QM_{G \times G, G}(\mathbb{Y}, \sigma, \xi)_\mathbb{R}$) of rigidified quasi-maps which we respectively denote by $\eta_Z$ and $\theta_Z$ (resp. $\eta_y$ and $\theta_y$). Note that $\eta_Z = \delta_Z \circ \theta_Z$ (resp. $\eta_y = \delta_y \circ \theta_y$).

Recall the $\mathbb{Z}/2 \times \mathbb{Z}/2$-action $(\delta_Z, \beta_Z)$ (resp. $(\delta_y, \beta_y)$) on the real moduli $QM_{G \times G, G}(\mathbb{Z}, \sigma, \xi)_\mathbb{R}$ (resp. $QM_{G \times G, G}(\mathbb{Y}, \sigma, \xi)_\mathbb{R}$). The composition of this $\mathbb{Z}/2 \times \mathbb{Z}/2$-action with the involution $\theta_Z$ (resp. $\theta_y$) defines a new $\mathbb{Z}/2 \times \mathbb{Z}/2$-action

\[(\eta_Z, \beta_Z \circ \theta_Z) \text{ (resp. } (\eta_y, \beta_y \circ \theta_y))\]
on $QM_{G \times G, G}(\mathbb{Z}, \sigma, \xi)_\mathbb{R}$ (resp. $QM_{G \times G, G}(\mathbb{Y}, \sigma, \xi)_\mathbb{R}$).

Propositions 4.6 and 5.6 immediately imply:

**Proposition 6.1.** (1) The $\mathbb{Z}/2 \times \mathbb{Z}/2$-action on $QM_{G \times G, G}(\mathbb{Z}/\mathbb{A}^1, \sigma, \xi)_\mathbb{R}$ given by the involutions $(\eta_Z, \beta_Z \circ \theta_Z)$ satisfies:

(a) The action commutes with the natural $G_c$-action.
(b) The action preserves the spherical and cospherical stratifications. In particular, the open locus $\mathcal{T}_Z^0 \simeq QM^0_{G,G,G}(\mathbb{A}^1,\sigma,\xi)\mathbb{R}$ and its spherical stratification are preserved.

(c) At $a = 0$, the isomorphism $\nu_0 : \mathcal{G} \simeq QM^0_{G,G,G}(\mathcal{Y}^0,\sigma(0),\xi(0))\mathbb{R}$ of (4.14) intertwines the action with the $\mathcal{Z}/2 \times \mathbb{Z}/2$-action on $\mathcal{Y}^*$ given by $(\eta,\neg \circ \theta)$.

(d) At $a \neq 0$, the isomorphism $\nu_a : \mathcal{Y}^* \simeq QM^0_{G,G,G}(\mathcal{Y}^0,\sigma(a),\xi(a))\mathbb{R}$ of (4.14) intertwines the action with the $\mathcal{Z}/2 \times \mathbb{Z}/2$-action on $\mathcal{G}$ given by $(\eta \circ \theta, \eta \circ \eta)$. 

6.1. Trivializations of fixed-points. Our aim is to trivialize the fixed-point family of the involution $\eta_Z$ (resp. $\eta$). To that end, we will invoke the following lemma:

**Lemma 6.2.** Let $f : M \to \mathbb{R}$ be a stratified real analytic submersion of a real analytic Whitney stratified ind-variety $M$ (where $\mathbb{R}$ is equipped with the trivial stratification).

1. Assume there is a compact group $H \times \mathbb{Z}/2$ acting real analytically on $M$ such that the action preserves the stratifications and $f$ is $H \times \mathbb{Z}/2$-invariant. Then the $\mathbb{Z}/2$-fixed-point ind-variety $M^{\mathbb{Z}/2}$ is Whitney stratified by the fixed-points of the strata and the induced map $f^{\mathbb{Z}/2} : M^{\mathbb{Z}/2} \to \mathbb{R}$ is an $H$-equivariant stratified submersion.

2. Assume further that $f$ is ind-proper and there is an $H$-equivariant stratified trivialization of $f^{\mathbb{Z}/2} : M^{\mathbb{Z}/2} \to \mathbb{R}$ that is real analytic on each stratum. Then there is an $H$-equivariant stratified trivialization of $f^{\mathbb{Z}/2} : M^{\mathbb{Z}/2} \to \mathbb{R}$ that is real analytic on each stratum.

**Proof.** Part (1) is proved in [N2, Lemma 4.5.1]. For part (2), the $H$-equivariant stratified trivialization of $f : M \to \mathbb{R}$ provides a horizontal lift of the coordinate vector field $\partial_t$ on $\mathbb{R}$ to a continuous $H$-invariant vector field $v$ on $M$ that is tangent to and real analytic along each stratum. Let $w$ be the average of $v$ with respect to the $\mathbb{Z}/2\mathbb{Z}$-action. As $f$ is ind-proper and the $\mathbb{Z}/2\mathbb{Z}$-action is real analytic, the vector field $w$ is complete and the integral curves of $w$ define an $H$-equivariant stratified trivialization of $f^{\mathbb{Z}/2} : M^{\mathbb{Z}/2} \to \mathbb{R}$ that is real analytic along each stratum.

**Remark 6.3.** In fact, to find an $H$-equivariant trivialization as in part (2) of the lemma, one only needs the assumption that $f$ is ind-proper. Indeed, the Thom-Mather theory shows that $\partial_t$ admits a lift to a controlled vector field $v$ such that the integral curves of $v$ define a trivialization of $f$. The integral curves of the average of $v$ with respect to the $H \times \mathbb{Z}/2\mathbb{Z}$-action gives rise to an $H$-equivariant trivialization of $f^{\mathbb{Z}/2\mathbb{Z}} : M^{\mathbb{Z}/2\mathbb{Z}} \to \mathbb{R}$. In our applications, we

---

3A vector vector field is called complete if each of its integral curves exists for all time.
will have the assumed $\mathcal{H}$-equivariant trivialization of part (2), and thus not need to invoke the Thom-Mather theory.

Now let us apply the above lemma to the family

\[ QM_{G \times G,G}(\mathbb{Z}/\mathbb{A}^1, \sigma, \xi)_\mathbb{R} \longrightarrow \mathbb{A}^1(\mathbb{R}) \simeq i\mathbb{R} \]

(\text{resp. } QM_{G \times G,G}(\mathbb{Y}/\mathbb{A}^1, \sigma, \xi)_\mathbb{R} \longrightarrow \mathbb{A}^1(\mathbb{R}) \simeq \mathbb{R})

with its spherical stratification. We will consider two possible cases for the $\mathcal{H} \times \mathbb{Z}/2$-action: either $K_c \times \langle \eta \rangle$ (resp. $K_c \times \langle \eta_y \rangle$) or $K_c \times \langle \beta_2 \circ \theta_2 \rangle$ (resp. $K_c \times \langle \beta_g \circ \theta_2 \rangle$).

**Proposition 6.4.** (1) There is a $K_c$-equivariant topological trivialization of the fixed-points of $\eta$ (resp. $\beta_2 \circ \theta_2$) on $QM_{G \times G,G}(\mathbb{Z}/\mathbb{A}^1, \sigma, \xi)_\mathbb{R}$. The trivialization induces a $K_c$-equivariant stratified homeomorphism between the fixed-points of $\eta$ (resp. $\beta_2 \circ \theta_2$) on $\text{Gr}$ and the fixed points of $\text{inv} \circ \theta$ (resp. $\text{inv} \circ \eta$) on $\Omega G_c$.

(2) The trivialization in (1) restricts to a $K_c$-equivariant topological trivialization of the fixed-points of $\eta$ (resp. $\beta_2 \circ \theta_2$) on the open locus $QM_{G \times G,G}^0(\mathbb{Z}/\mathbb{A}^1, \sigma, \xi)_\mathbb{R}$, and induces a homeomorphism between the fixed-points of $\eta$ (resp. $\beta_2 \circ \theta_2$) on the open locus $T^0 \subset \text{Gr}$ and the fixed points of $\text{inv} \circ \theta$ (resp. $\text{inv} \circ \eta$) on $T^0 \subset \Omega G_c$.

(3) There is a $K_c$-equivariant topological trivialization of the fixed-points of $\eta_y$ (resp. $\beta_y \circ \theta_2$) on $QM_{G \times G,G}^0(\mathbb{Y}/\mathbb{A}^1, \sigma, \xi)_\mathbb{R}$. The trivialization induces a $K_c$-equivariant stratified homeomorphism between the fixed-points of $\text{inv} \circ \theta$ (resp. $\text{inv} \circ \eta$) on the open locus $S^* \simeq T^0$ and the fixed points of $\text{inv} \circ \theta$ (resp. $\text{inv} \circ \eta$) on $T^0 \subset \Omega G_c$.

In addition, the restrictions of the homeomorphisms in (1), (2), and (3) to strata are real analytic.

**Proof.** For part (1), by Proposition 4.3, there is a $K_c$-equivariant stratified trivialization of

\[ QM_{G \times G,G}(\mathbb{Z}/\mathbb{A}^1, \sigma, \xi)_\mathbb{R} \longrightarrow \mathbb{A}^1(\mathbb{R}) \simeq i\mathbb{R} \]

Applying Lemma 6.2 with $\mathcal{H} \times \mathbb{Z}/2 = K_c \times \langle \eta \rangle$ (resp. $K_c \times \langle \beta_2 \circ \theta_2 \rangle$), we obtain part (1). Part (2) is immediate by the invariance of the open locus $QM_{G \times G,G}^0(\mathbb{Z}/\mathbb{A}^1, \sigma, \xi)_\mathbb{R}$ under all constructions.

For part (3), by Lemma 5.3, there is a $K_c$-equivariant stratified trivialization of

\[ QM_{G \times G,G}^0(\mathbb{Y}/\mathbb{A}^1, \sigma, \xi)_\mathbb{R} \longrightarrow \mathbb{A}^1(\mathbb{R}) \simeq \mathbb{R} \]

Following the proof of Lemma 6.2, consider the averaged vector field $w$ with respect to the $\mathbb{Z}/2\mathbb{Z}$-action given by $\langle \eta \rangle$ (resp. $\langle \beta_y \circ \theta_2 \rangle$). We claim that $w$ is complete, hence the integral curves of $w$ provide the desired trivialization.

To prove the claim, observe that, over the open locus $\mathbb{A}^1(\mathbb{R}) \setminus \{0\} \simeq \mathbb{R} \setminus \{0\}$ in the base, the $K_c$-equivariant trivialization provided by Lemma 5.3 extends to a $K_c$-equivariant trivialization of the ind-proper family

\[ QM_{G \times G,G}(\mathbb{Y}/\mathbb{A}^1, \sigma, \xi)_\mathbb{R} \longrightarrow \mathbb{A}^1(\mathbb{R}) \setminus \{0\} \simeq \mathbb{R} \setminus \{0\} \]
Thus for any $a > 0$ (resp. $a < 0$), any integral curve $p(t)$ for $w$ with initial point $p(a) \in QM_{G \times G,G}(y_a,\sigma(a),\xi(a))_\mathbb{R}$ exists for $t \geq a$ (resp. $t \leq a$). Together with the local existence of integral curves with initial point in the special fiber $QM^0_{G \times G,G}(y_0,\sigma(0),\xi(0))_\mathbb{R}$, this implies $p(t)$ exists for all $t \in \mathbb{R}$. Hence $w$ is complete and we have proved the claim. \hfill $\square$

6.2. Real and symmetric spherical strata. Let us summarize here the results obtained by the preceding considerations.

Recall we write $\mathcal{G} = G(\mathbb{C}[x^{-1}])_{\infty} \to e$ for the group of maps $g: \mathbb{P}^1(\mathbb{C}) \setminus \{0\} \to G$ such that $g(\infty) = e$. Recall the open cospherical stratum $T^0 \simeq \mathcal{G} \cdot [e] \subset \text{Gr}$, and the transported spherical strata $\mathcal{G}^* = T^0 \cap S^\lambda$, for $\lambda \in R^+_G$.

Proposition 6.4 immediately implies:

**Theorem 6.5.** (1) There is a $K_e$-equivariant stratified homeomorphism between the fixed-points of $\eta$ and $\text{inv} \circ \theta$ on $\mathcal{G}^*$. The homeomorphism restricts to a $K_e$-equivariant real analytic isomorphism between the fixed-points of $\eta$ and $\text{inv} \circ \theta$ on $\mathcal{G}^*\lambda$.

(2) There is a $K_e$-equivariant stratified homeomorphism between the fixed-points of $\text{neg} \circ \theta$ and $\text{inv} \circ \eta$ on $\mathcal{G}^*$. The homeomorphism restricts to a $K_e$-equivariant real analytic isomorphism between fixed points of $\text{neg} \circ \theta$ and $\text{inv} \circ \eta$ on stratum $\mathcal{G}^*\lambda$.

Observe that the fixed-points $(\mathcal{G}^*)^n$ coincides with the group $\mathcal{G}_{\mathbb{R}}$ of maps $g: \mathbb{P}^1 \setminus \{0\} \to G$ such that $g(\mathbb{P}^1(\mathbb{R}) \setminus \{0\}) \subset G_{\mathbb{R}}$ and $g(\infty) = e$. Denote by $\mathcal{G}_{\mathbb{R}}^* = \mathcal{G}^* \cap \mathcal{G}_{\mathbb{R}}$, $\lambda \in R^+_G \cap \Lambda_S$, its spherical strata. Similarly, observe that the fixed-points $(\mathcal{G}^*)^{\text{inv} \circ \theta}$ coincides with the space $X^* \cap \Lambda_S$ of maps $g: \mathbb{P}^1 \setminus \{0\} \to X \subset G$ such that $g(\infty) = e$. and $X^* = X \cap \mathcal{G}^*$. Denote by $X_{\mathbb{R}}^* = \mathcal{G}_{\mathbb{R}}^* \cap X$, $\lambda \in R^+_G \cap \Lambda_S$, its spherical strata.

We can restate part (1) of the above theorem in the form:

**Corollary 6.6.** There is a $K_e$-equivariant stratified homeomorphism

\begin{equation}
\mathcal{G}_{\mathbb{R}}^* \simeq X^*
\end{equation}

that restricts to real analytic isomorphisms on strata

\begin{equation}
\mathcal{G}_{\mathbb{R}}^{*\lambda} \simeq X^{*\lambda} \quad \lambda \in R^+_G \cap \Lambda_S
\end{equation}

6.3. Examples.

6.3.1. Type A nilpotent cones. Consider the case $G = \text{SL}_n(\mathbb{C})$. Let $N_n$ be the nilpotent cone. The set of $G$-orbits on $N_n$ is parametrized by the set $P(n) = \{(k_1 \geq k_2 \geq \cdots \geq k_n)|k_i \geq 0, \sum k_i = n\}$ of partition of $n$. For $\lambda \in P(n)$, we write $\mathcal{O}_\lambda$ the corresponding nilpotent orbit. We will identify the set of dominant co-weights $\Lambda^+_T$ with

$$\Lambda^+_T = \{(k_1 \geq k_2, \ldots, \geq k_n)|k_i \in \mathbb{Z}, \sum k_i = n\}$$

such that

$$S^\lambda = G(\mathcal{O}) \cdot \text{diag}(t^{k_1-1}, \ldots, t^{k_n-1}) \text{ mod } G(\mathcal{O})$$

for $\lambda = (k_1 \geq k_2 \geq \cdots \geq k_n) \in \Lambda^+_T$. Note $P(n)$ is a subset of $\Lambda^+_T$. Recall the Lusztig embedding

$$N_n \to \mathcal{G}^*, \quad M \to 1 - t^{-1}M.$$
It induces an isomorphism

\[ N_n \simeq G^* \cap \overline{S}^\beta \]

here \( \beta \) the real dominant co-weight \( \beta = (n \geq 0, \ldots, \geq 0) \in \Lambda^+_\mathfrak{t}. \) In addition, for \( \lambda \in P(n) \), the isomorphism above restricts to an isomorphism

\[ O_\lambda \simeq G^* \cap S^\lambda. \]

A direct computation shows that, in the cases \( G_\mathbb{R} = \text{SL}_n(\mathbb{R}), \text{SL}_m(\mathbb{H}) \) (resp. \( \text{SU}_{p,q} \)), the isomorphism above restricts to isomorphisms

\[ N_n \cap g_\mathbb{R} \simeq (G^* \cap \overline{S}^\beta)^{\eta} \quad \text{(resp. } N_n \cap i g_\mathbb{R} \simeq (G^* \cap \overline{S}^\beta)^{\text{inv} \circ \eta} \text{)} \]

and

\[ N_n \cap p \simeq (G^* \cap \overline{S}^\beta)^{\text{inv} \circ \theta} \quad \text{(resp. } N_n \cap p \simeq (G^* \cap \overline{S}^\beta)^{\text{neg} \circ \theta} \text{)} \]

Thus, by Theorem 6.5, there is a \( K_c \)-equivariant homeomorphism

\[ (6.8) \quad N_n \cap g_\mathbb{R} \simeq N_n \cap p \]

between the real and symmetric nilpotent cone in type A.

In Section 7, we will extend the homeomorphism above to all classical types. Moreover, we will further show that the homeomorphism restricts to real analytic isomorphisms between symmetric and real nilpotent orbits and induces the Kostant-Sekiguchi bijection between orbit posets.

6.3.2. Orbit correspondence for \( \text{SL}_2(\mathbb{R}) \). Let us pin down the precise map on orbits induced by the homeomorphism in the case of \( G = \text{SL}_2(\mathbb{C}) \).

We regard the Lie algebra \( \mathfrak{sl}_2(\mathbb{C}) \) as traceless \( 2 \times 2 \) complex matrices

\[ (6.9) \quad A = \begin{pmatrix} a & b \\ c & -a \end{pmatrix} \quad a, b, c \in \mathbb{C} \]

Thus the nilpotent cone \( N_2 \subset \mathfrak{sl}_2(\mathbb{C}) \) is cut out by the vanishing of the determinant

\[ (6.10) \quad N_2 = \{ A \in \mathfrak{sl}_2(\mathbb{C}) \mid \det(A) = -a^2 - bc = 0 \} \]

The real nilpotent cone \( N_{2,\mathbb{R}} = N \cap \mathfrak{sl}_2(\mathbb{R}) \) consists of nilpotent real matrices and thus is the two-dimensional real quadric

\[ (6.11) \quad N_{2,\mathbb{R}} = \{ (a, b, c) \in \mathbb{R}^3 \mid -a^2 - bc = 0 \} \]

There are three \( \text{SL}_2(\mathbb{R}) \)-orbits in \( N_{2,\mathbb{R}} \) passing through the respective elements

\[ (6.12) \quad A_0 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \quad A_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad A_- = \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix} \]

The symmetric nilpotent cone \( N_{2,p} = N_2 \cap p \) consists of nilpotent symmetric matrices and thus is the one-dimensional complex quadric

\[ (6.13) \quad N_{2,p} = \{ (a, b) \in \mathbb{C}^2 \mid -a^2 - b^2 = 0 \} \]
There are three $\text{SO}_2(\mathbb{C})$-orbits in $N_{2,p}$ passing through the respective elements

\begin{align*}
(6.14) \quad A_0 &= \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \quad A_i = \begin{pmatrix} 1 & i \\ i & -1 \end{pmatrix} \quad A_{-i} = \begin{pmatrix} 1 & -i \\ -i & -1 \end{pmatrix}
\end{align*}

Observe that in this case there are two possible Kostant-Sekiguchi orbit poset isomorphisms determined by the two possible assignments

\begin{align*}
(6.15) \quad \text{SL}_2(\mathbb{R}) \cdot A_+ &\longrightarrow \text{SO}_2(\mathbb{C}) \cdot A_{\pm i}
\end{align*}

Without further choices, there is no reason to prefer one over the other as they are related by Galois symmetry. Our geometric constructions are independent of any choices, and it is only in a preference for the fiber at $-i$ of the real degeneration rather than at $i$ that we break symmetry.

**Lemma 6.7.** *On orbits, the homeomorphism* (6.8) *satisfies*

\begin{align*}
(6.16) \quad \text{SL}_2(\mathbb{R}) \cdot A_+ &\longrightarrow \text{SO}_2(\mathbb{C}) \cdot A_{-i}
\end{align*}

*Proof.* Recall that we write $z$ for the standard coordinate on $\mathbb{P}^1$, and also work with the alternative coordinate $x = \frac{z - i}{z + i}$. Observe that the nodal degeneration and the Lusztig embedding for the coordinate $x$ are evidently compatible. Thus to prove the lemma, it suffices to analyze the real degeneration.

Let us start with the element

\begin{align*}
(6.17) \quad A_+ &= \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \in N_{2,\mathbb{R}}
\end{align*}

and its image at $z = 0$ under the Lusztig embedding

\begin{align*}
(6.18) \quad a_0 &= 1 - z^{-1} A_+ = \begin{pmatrix} 1 & -z^{-1} \\ 0 & 1 \end{pmatrix} \in \text{Gr}_0
\end{align*}

Consider its parallel transport to $z = -i$ under the usual translational trivialization

\begin{align*}
(6.19) \quad a_{-i} &= 1 - (z + i)^{-1} A_+ = \begin{pmatrix} 1 & -(z + i)^{-1} \\ 0 & 1 \end{pmatrix} \in \text{Gr}_{-i}
\end{align*}

Then consider the conjugation of $a_{-i}$ with respect to the split conjugation of $G$ and the usual coordinate conjugation

\begin{align*}
(6.20) \quad \bar{a}_{-i} &= 1 - (z - i)^{-1} A_+ = \begin{pmatrix} 1 & -(z - i)^{-1} \\ 0 & 1 \end{pmatrix} \in \text{Gr}_{i}
\end{align*}

We seek to express the quasi-map represented by $\bar{a}_{-i} \in \text{Gr}_i$ once again in terms of an element $b_{-i} \in \text{Gr}_{-i}$. Let us first rewrite $\bar{a}_{-i} \in \text{Gr}_i$ in terms of the alternative local coordinate $x$. Using the formula $x = \frac{z + 1}{z + i}$, an elementary calculation shows that we have the equality

\begin{align*}
(6.21) \quad \bar{a}_{-i} &= \begin{pmatrix} 1 & \frac{z - 1}{z + 1} x^{-1} \\ 0 & 1 \end{pmatrix} \in \text{Gr}_i
\end{align*}

It is work emphasizing that, here and below, we are not asserting equalities of matrices only of the points they represent in the Grassmannian.
Going further, a similar an elementary calculation shows we may then express \( \overline{a}_{-i} \in \text{Gr}_i \) in the form
\[
(6.22) \quad \overline{a}_{-i} = \omega \omega_1 \in \text{Gr}_i
\]
\[
(6.23) \quad \omega = \begin{pmatrix} x^{-1} & 0 \\ 0 & 1 \end{pmatrix}, \quad \omega_1 = k \begin{pmatrix} 1 & 0 \\ 0 & x \end{pmatrix} k^{-1} \in \Omega G_c
\]
\[
(6.24) \quad k = \begin{pmatrix} \cos(\rho) & -i \sin(\rho) \\ -i \sin(\rho) & \cos(\rho) \end{pmatrix} \in G_c \quad \rho = \arctan(2)
\]

The quasi-map represented by \( \overline{a}_{-i} \in \text{Gr}_i \) is alternatively represented by the element \( b_{-i} = \iota(\overline{a}_{-i}) \in \text{Gr}_i \) where we write \( \iota : \Omega G_c \to \Omega G_c \) for the map \( \iota(g(t)) = g(t^{-1})^{-1} \). Note that \( \iota \) fixes the individual elements \( \omega, \omega_1 \in \Omega G_c \), and thus we have
\[
(6.25) \quad b_{-i} = \iota(\overline{a}_{-i}) = \iota(\omega \omega_1) = \omega_1 \omega \in \text{Gr}_{-i}
\]

Finally, we would like to find \( B \in N_2 \) with image \( b_{-i} \in \text{Gr}_{-i} \) under the Lusztig embedding
\[
(6.26) \quad b_{-i} = 1 - (z + i)^{-1} B \in \text{Gr}_i
\]

We could calculate \( B \) explicitly, but the following coarser understanding will suffice. Consider \( N_2^0 = N_2 \setminus \{0\} \), the projection \( \pi : N_2^0 \to \mathbb{P}^1 \), \( \pi(A) = \ker(A) \), and its restrictions \( \pi_R : N_2^0 = N_{2,R} \cap N_2^0 \to \mathbb{P}_R^1 \), \( \pi_p : N_2^0 = N_{2,p} \cap N_2^0 \to \{[1, \pm i]\} \).

It is elementary to check that
\[
(6.27) \quad \pi(B) = k \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \cos(\rho) \\ -i \sin(\rho) \end{bmatrix} = \begin{bmatrix} 1 \\ -2i \end{bmatrix} \in \mathbb{P}^1
\]
where as before \( \rho = \arctan(2) \).

On the other hand, note that
\[
(6.28) \quad \pi(A_{-i}) = \begin{bmatrix} 1 \\ -i \end{bmatrix} \in \mathbb{P}^1
\]

Finally, by evident monotonicity (the orbit correspondence will not change if we repeat the above constructions at \( i \epsilon \) in place of \( i \), for any \( \epsilon > 0 \)), we conclude that the averaged invariant trivialization must map the orbit \( \text{SL}_2(\mathbb{R}) \cdot A_+ \) as asserted:
\[
(6.29) \quad \text{SL}_2(\mathbb{R}) \cdot A_+ \longrightarrow \text{SO}_2(\mathbb{C}) \cdot A_{-i}
\]

\[\square\]

6.3.3. Minimal nilpotent orbits. We assume \( G \) is simple. Let \( \mathcal{O}_{\min} \) be the minimal nilpotent orbit in \( \mathfrak{g} \), i.e., \( \mathcal{O}_{\min} \) is the \( G \)-orbit of a vector \( v \in \mathfrak{g}_{\alpha_{\max}} \) where \( \alpha_{\max} \) is the maximal root. Note that we have \( \overline{\mathcal{O}}_{\min} = \mathcal{O}_{\min} \cup \{0\} \). Let \( \beta := \alpha_{\max} \in \Lambda_+^T \) be the coroot corresponding to \( \alpha_{\max} \). In \([\text{MOV}]\), the authors show that the map
\[
\overline{\mathcal{O}}_{\min} \to \mathfrak{g}^*, \quad x \mapsto \exp(t^{-1}x) = 1 + t^{-1} \text{ad}(x)
\]
induces an isomorphism of algebraic varieties \( \overline{O}_{\min} \simeq \mathfrak{g}^* \cap S^\beta \). A direct computation shows that the isomorphism above restricts to isomorphisms

\[ \overline{O}_{\min} \cap g_R \simeq \left( \mathfrak{g}^* \cap S^\beta \right)^\eta, \quad \overline{O}_{\min} \cap p \simeq \left( \mathfrak{g}^* \cap S^\beta \right)^{\text{inv} \circ \theta}, \]

and Theorem 6.5 implies that there is a \( K_c \)-equivariant homeomorphism

\[ \overline{O}_{\min} \cap g_R \simeq \overline{O}_{\min} \cap p \]

between the real and symmetric minimal nilpotent orbits closures that restricts to a real analytic isomorphism

\[ O_{\min} \cap g_R \simeq O_{\min} \cap p. \]

### 6.4. Quasi-maps for symmetric spaces

In this section, we use the results obtained in previous sections to study quasi-maps for general symmetric spaces \( X = K \backslash G \) (as opposed to our prior focus on the case of complex groups \( G = G \backslash (G \times G) \)). We show that (a certain union of components of) the family of real rigidified quasi-maps

\[
(6.30) \quad QM_{G,X}(\mathbb{Z}/A^1, \sigma, \xi) \rightarrow \mathbb{A}^1(\mathbb{R}) \simeq i\mathbb{R}
\]

in fact admits a topological trivialization. As applications, we show that there is a \( K_c \)-equivariant homeomorphism between (a certain union of components of) the real affine Grassmannian \( \text{Gr}_R \) and the quotient \( \Omega K_c \backslash \text{Gr} \) providing real analytic isomorphisms between real spherical orbits (resp. real co-spherical orbits) of \( \text{Gr}_R \) and \( \Omega K_c \)-quotients of \( K(K) \)-orbits (resp. \( LG_R \)-orbits) on \( \text{Gr} \). The results of this section are not used in the rest of this paper, but are used in [CN1].

Recall the uniformization map (3.9):

\[
(6.31) \quad K[\mathbb{Z}, \sigma, \xi] \backslash \text{Gr}_{G,\mathbb{Z},\sigma,R} \rightarrow QM_{G,X}(\mathbb{Z}/A^1, \sigma, \xi)_R
\]

Unlike the case of a complex group \( G = G \backslash (G \times G) \), the map (6.31) is not surjective for general symmetric space \( X = K \backslash G \).

We shall introduce a certain union of connected components of \( QM_{G,X}(\mathbb{Z}/A^1, \sigma, \xi)_R \) in the image of the map (6.31).

Consider the map \( \pi : G \rightarrow G, \pi(g) = \theta(g)^{-1}g \). It factors through \( X = K \backslash G \) and induces an isomorphism \( X \simeq G^{\text{inv} \circ \theta,0} \), where \( G^{\text{inv} \circ \theta,0} \) denotes the neutral component of the fixed points of \( \text{inv} \circ \theta \) on \( G \). Consider the induced map \( \pi_\ast : \pi_1(G) \rightarrow \pi_1(X) \) between fundamental groups (based at \( e \)), and define

\[
(6.32) \quad \mathcal{L} \subset \Lambda_S^+\mathbb{R}
\]

to be the inverse image of \( \pi_\ast(\pi_1(G)) \subset \pi_1(X) \) under \( \Lambda_S^+ \rightarrow \pi_1(X) \). Here the map \([-]\) assigns a based loop its homotopy class. We have \( \mathcal{L} = \Lambda_S^+ \) if and only if \( K \) is connected.

By [CN1, Lemma 2.10], the union of real spherical strata

\[
(6.33) \quad \text{Gr}_R^0 = \coprod_{\mathcal{L} \in \mathcal{L}} S^\lambda_{\mathbb{R}}
\]
is a union of connected components of $Gr_R$. Let $Bun_G(\mathbb{P}^1)_R^0$ be the image of $Gr_R^0$ under the uniformization map $Gr_R \to Bun_G(\mathbb{P}^1)_R$. Introduce the unions of connected components of $QM_{G,X}(\mathbb{Z}/\mathbb{A}^1, \sigma, \xi)_R$ and $Gr_{G,\lambda, \sigma, R, \xi}$ respectively:

\begin{equation}
QM_{G,X}(\mathbb{Z}/\mathbb{A}^1, \sigma, \xi)_{R, \xi} := QM_{G,X}(\mathbb{Z}/\mathbb{A}^1, \sigma, \xi)_R \times_{Bun_G(\mathbb{P}^1)_R} Bun_G(\mathbb{P}^1)_R^0
\end{equation}

\begin{equation}
Gr_{G,\lambda, \sigma, R, \xi} := Gr_{G,\lambda, \sigma, R} \times_{Bun_G(\mathbb{P}^1)_R} Bun_G(\mathbb{P}^1)_R^0
\end{equation}

**Remark 6.8.** According to [CN1], $Bun_G(\mathbb{P}^1)_R^0$ is a union of components of $Bun_G(\mathbb{P}^1)_R$, and parametrizes real bundles on $\mathbb{P}^1$ that admit complex uniformizations. Assume $K$ is connected. Then $Gr_R = Gr_R^0$, and $Bun_G(\mathbb{P}^1)_R^0 = \text{Im}(Gr_R \to Bun_G(\mathbb{P}^1)_R)$ parametrizes real bundles on $\mathbb{P}^1$ that are trivializable at the real point $\infty \in \mathbb{P}^1(\mathbb{R})$.

By [CN1] Proposition 6.2], the uniformization map (6.31) restricts to an isomorphism

\begin{equation}
K[\mathbb{Z}, \sigma, \xi]_R \setminus Gr_{G,\lambda, \sigma, R, \xi} \simeq QM_{G,X}(\mathbb{Z}/\mathbb{A}^1, \sigma, \xi)_{R, \xi}
\end{equation}

In addition, over $0 \in i\mathbb{R}$, it specializes to an isomorphism

\begin{equation}
Gr_R^0 \simeq K[\mathbb{Z}, \sigma, \xi, 0]_R \setminus Gr_{G,\lambda, \sigma, 0, R, \xi} \simeq QM_{G,X}(\mathbb{Z}_0, \sigma(0), \xi(0))_{R, \xi},
\end{equation}

and over $a \neq 0 \in i\mathbb{R}$, it specializes to

\begin{equation}
\Omega K_c \setminus Gr \simeq K[\mathbb{Z}, \sigma, \xi, a]_R \setminus Gr_{G,\lambda, \sigma, a, R, \xi} \simeq QM_{G,X}(\mathbb{Z}_a, \sigma(a), \xi(a))_{R, \xi}.
\end{equation}

Here the isomorphism $\Omega K_c \setminus Gr \simeq K[\mathbb{Z}, \sigma, \xi, a]_R \setminus Gr_{G,\lambda, \sigma, a, R, \xi}$ is induced by the coordinate transformation $t = \frac{z-a}{z+a}$ on $\mathbb{Z}_a = \mathbb{P}^1$.

Let us compare $QM_{G,X}(\mathbb{Z}_a, \sigma(a), \xi(a))_{R, \xi}$ and the fixed-points

\begin{equation}
\Omega := (QM_{G\times G,G}(\mathbb{Z}/\mathbb{A}^1, \sigma, \xi)_R)^{\eta_c}
\end{equation}

of the involution $\eta_c$ as studied in Section 6.1.

Set $Q_c = S^\lambda_c \cap \Omega$ and $Q^\lambda = T^\lambda_c \cap \Omega$. By Propositions 6.1 and 6.4 we have natural isomorphisms

\begin{equation}
v_0 : \Omega|_0 \xrightarrow{\sim} (Gr)^{\eta} = Gr_R
\end{equation}

\begin{equation}
v_a : \Omega|_{a \neq 0} \xrightarrow{\sim} (OG_c)^{\text{inv} \circ \theta}
\end{equation}

which restrict to

\begin{equation}
\Omega|_0 \xrightarrow{\sim} S^\lambda_R, \quad \Omega^\lambda|_0 \xrightarrow{\sim} T^\lambda_R
\end{equation}

\begin{equation}
\Omega|_{a \neq 0} \xrightarrow{\sim} (S^\lambda)^{\text{inv} \circ \theta}, \quad \Omega^\lambda|_{a \neq 0} \xrightarrow{\sim} (T^\lambda)^{\text{inv} \circ \theta}.
\end{equation}

Moreover, as $S^\lambda_R$ (resp. $T^\lambda_R$) is non-empty if and only if $\lambda \in \Lambda^+_S$, the stratum $\Omega_\lambda$ (resp. $\Omega^\lambda$) is non-empty if and only if $\lambda \in \Lambda^+_S$, and the family $\Omega \to i\mathbb{R}$ equipped with the strata $\Omega_\lambda$ (resp. $\Omega^\lambda$), $\lambda \in \Lambda^+_S$, is topologically trivial.

Define

\begin{equation}
Q_c := \Omega \times_{Bun_G(\mathbb{P}^1)_R} Bun_G(\mathbb{P}^1)_R^0
\end{equation}
where \( Q \to \text{Bun}_G(\mathbb{P}^1)_\mathbb{R} \) is the natural forgetful map. As \( Q \to i\mathbb{R} \) is topologically trivial with special fiber \( Q|_0 \simeq G_\mathbb{R} \), we have \( Q_\mathbb{L} = \prod_{\lambda \in \mathbb{L}} Q_\lambda \) and the isomorphisms in (6.40) restrict to

\[
(6.42) \quad v_0 : Q_\mathbb{L}|_0 \xrightarrow{\sim} \text{Gr}^0_\mathbb{R} \subset \text{Gr}_\mathbb{R}
\]

\[
(6.43) \quad v_a : Q_\mathbb{L}|_a \xrightarrow{\sim} \prod_{\lambda \in \mathbb{L}} (S^\lambda)^{\text{inv} \circ \eta} = \prod_{\lambda \in \mathbb{L}} (T^\lambda)^{\text{inv} \circ \eta} \subset (\Omega G_c)^{\text{inv} \circ \eta}
\]

Consider the morphism \( f : G \to G \times G, \ g \to (g, \theta(g)) \). It is equivariant for the conjugations and Cartan involutions on \( G \) and \( G \times G \) hence, by the functoriality noted in Section 3.3, we obtain a map

\[
(6.44) \quad Q_{M,G,X}(\mathbb{Z}/\mathbb{A}^1, \sigma, \xi)_\mathbb{R} \longrightarrow Q_{M,G \times G,G}(\mathbb{Z}/\mathbb{A}^1, \sigma, \xi)_\mathbb{R}.
\]

**Lemma 6.9.** The map (6.43) restricts to an isomorphism

\[
Q_{M,G,X}(\mathbb{Z}/\mathbb{A}^1, \sigma, \xi)_{\mathbb{R}, \mathbb{L}} \xrightarrow{\sim} Q_{\mathbb{L}}
\]

**Proof.** We have the following commutative diagram

\[
\begin{array}{ccc}
K[\mathbb{Z}, \sigma, \xi]_\mathbb{R} \setminus \text{Gr}_{G,\mathbb{Z},\sigma,\mathbb{L}, \mathbb{L}} & \xrightarrow{f_1} & G[\mathbb{Z}, \sigma, \xi]_\mathbb{R} \setminus \text{Gr}_{G \times G,\mathbb{Z},\sigma, \mathbb{R}, \mathbb{L}} \\
\downarrow u_2 & & \downarrow u_1 \\
Q_{M,G,X}(\mathbb{Z}/\mathbb{A}^1, \sigma, \xi)_{\mathbb{R}, \mathbb{L}} & \xrightarrow{f_2} & Q_{M,G \times G,G}(\mathbb{Z}/\mathbb{A}^1, \sigma, \xi)_\mathbb{R}
\end{array}
\]

where the vertical arrows are the uniformization isomorphisms and the horizontal arrows are the maps induced by \( f \). So it suffices to show that the composition

\[
q = u_1 \circ f_1 : K[\mathbb{Z}, \sigma, \xi]_\mathbb{R} \setminus \text{Gr}_{G,\mathbb{Z},\sigma,\mathbb{L}, \mathbb{L}} \longrightarrow Q_{M,G \times G,G}(\mathbb{Z}/\mathbb{A}^1, \sigma, \xi)_\mathbb{R}
\]

maps \( K[\mathbb{Z}, \sigma, \xi]_\mathbb{R} \setminus \text{Gr}_{G,\mathbb{Z},\sigma,\mathbb{L}, \mathbb{L}} \) isomorphically onto \( Q_{\mathbb{L}} \subset Q_{M,G \times G,G}(\mathbb{Z}/\mathbb{A}^1, \sigma, \xi)_\mathbb{R} \). Recall the isomorphisms \( \text{Gr}^0_{\mathbb{R}} \simeq K[\mathbb{Z}, \sigma, \xi, 0]_\mathbb{R} \setminus \text{Gr}_{G,\mathbb{Z},\sigma,0,\mathbb{R}, \mathbb{L}}, \ \Omega K_c \setminus \text{Gr} \simeq K[\mathbb{Z}, \sigma, \xi, a]_\mathbb{R} \setminus \text{Gr}_{G,\mathbb{Z},\sigma,a,\mathbb{R}, \mathbb{L}} \) in (6.37) and (6.38). Then under the above isomorphisms, the map \( q \) specializes to

\[
(6.44) \quad q_0 : \text{Gr}^0_{\mathbb{R}} \xrightarrow{v_0} \text{Gr} \simeq Q_{M,G \times G,G}(\mathbb{Z}_0, \sigma(0), \xi(0))_\mathbb{R} \quad a = 0
\]

and to

\[
(6.45) \quad q_{a \neq 0} : \Omega K_c \setminus \text{Gr} \xrightarrow{\pi} \Omega G_c \xrightarrow{\rho_a} Q_{M,G \times G,G}(\mathbb{Z}_a, \sigma(a), \xi(a))_\mathbb{R} \quad a \neq 0
\]

where \( \pi \) is given by

\[
(6.46) \quad \pi : \Omega K_c \setminus \text{Gr} \simeq \Omega K_c \setminus \Omega G_c \longrightarrow \Omega G_c \quad \pi(\gamma) = \theta(\gamma)^{-1} \gamma
\]

It is proved in [CNI] Proposition 2.8, that the image of \( \pi \) is equal to \( \prod_{\lambda \in \mathbb{L}} (S^\lambda)^{\text{inv} \circ \eta} \). Thus, by (6.42), the map \( q_0 \) and \( q_{a \neq 0} \) induce isomorphisms

\[
q_0 : \text{Gr}^0_{\mathbb{R}} \xrightarrow{\sim} Q_{\mathbb{L}}|_0, \quad q_{a \neq 0} : \Omega K_c \setminus \text{Gr} \xrightarrow{\sim} Q_{\mathbb{L}}|_a.
\]

This concludes the proof of the lemma. \( \square \)
For any $\lambda \in \mathcal{L}$, let $\tilde{Q}_\lambda$ (resp. $\tilde{Q}^\lambda$) be the image of the stratum $Q_\lambda$ (resp. $Q^\lambda$) under the isomorphism in Lemma 6.9. The collection of strata $\tilde{Q}_\lambda$, (resp. $\tilde{Q}^\lambda$), $\lambda \in \mathcal{L}$, forms a stratification of $QM_{G,X}(\mathbb{Z}/\mathbb{A}^1, \sigma, \xi)_{R,L}$.

We shall relate the strata $\tilde{Q}_\lambda$ and $\tilde{Q}^\lambda$ with $K(\mathcal{K})$- and $LG_R$-orbits on $Gr$. According to [N1], there is a bijection between $K(\mathcal{K})$-orbits (resp. $LG_R$-orbits) on $Gr$ and the set $L \subset \Lambda^+_S$. Write $O_K^\lambda$ (resp. $O_R^\lambda$) for the orbit corresponding to $\lambda \in L$.

**Proposition 6.10.** We have the following:

1. There is $K_c$-equivariant trivialization of the stratified family
   
   $QM_{G,X}(\mathbb{Z}/\mathbb{A}^1, \sigma, \xi)_{R,L} \to i\mathbb{R}$
   
   which is real analytic on strata.
2. The uniformization isomorphisms in (6.37) and (6.38) restrict to isomorphisms
   
   $S^\lambda_K \sim \tilde{Q}_\lambda|_0$, $T^\lambda_K \sim \tilde{Q}^\lambda|_0$
   
   $\Omega K_c \setminus O^\lambda_K \sim \tilde{Q}_\lambda|_{a \neq 0}$, $\Omega K_c \setminus O^\lambda_R \sim \tilde{Q}^\lambda|_{a \neq 0}$.
3. The trivialization in (1) induces a $K_c$-equivariant strata-preserving homeomorphism
   
   $\Omega K_c \setminus Gr \sim Gr^0_R$
   
   which restricts to $K_c$-equivariant real analytic isomorphisms
   
   $\Omega K_c \setminus O^\lambda_K \sim S^\lambda_K$, $\Omega K_c \setminus O^\lambda_R \sim T^\lambda_K$.

**Proof.** Part (3) follows from parts (1) and (2). Part (1) follows from Lemma 6.9 and Proposition 6.4. For part (2), the statement when $a = 0$ is clear. Assume $a \neq 0$. We have the following commutative diagram

$$
\begin{array}{ccc}
\Omega K_c \setminus Gr & \xrightarrow{\pi} & \Omega G_c \\
\downarrow \text{(6.38)} & & \downarrow v_a \\
QM_{G,K}(\mathbb{Z}_a, \sigma(a), \xi(a))_{R,L} \xrightarrow{f_a} QM_{G \times G, G}(\mathbb{Z}_a, \sigma(a), \xi(a))_{R,L}
\end{array}
$$

Here $f_a$ is the the restriction of (6.43) to fibers over $a \neq 0$. As $f_a(\tilde{Q}_\lambda|_a) = \tilde{Q}_\lambda|_a$ (resp. $f_a(\tilde{Q}^\lambda|_a) = \tilde{Q}^\lambda|_a$) and $v_a((S^\lambda)^{\text{inv} \circ \theta}) = \tilde{Q}_\lambda|_a$ (resp. $v_a((T^\lambda)^{\text{inv} \circ \theta}) = \tilde{Q}^\lambda|_a$), it suffices to show that that $\pi : \Omega K_c \setminus Gr \to \Omega G_c$ maps $\Omega K_c \setminus O^\lambda_K$ (resp. $\Omega K_c \setminus O^\lambda_R$) isomorphically onto $(S^\lambda)^{\text{inv} \circ \theta}$ (resp. $(T^\lambda)^{\text{inv} \circ \theta}$). This is proved in [CN1, Proposition 2.8]. The proof is complete. $\square$

7. Kostant-Sekiguchi homeomorphisms

In this section, we further analyze our preceding constructions to obtain Kostant-Sekiguchi homeomorphisms for all classical groups.
7.1. **Family of involutions.** In this section, we assume \( G = \text{SL}_n(\mathbb{C}) \) with the split real form \( G_\mathbb{R} = \text{SL}_n(\mathbb{R}) \). Let \( N_n \) be the nilpotent cone. Note that we have \( R_G = \Lambda_T = \Lambda_S \).

Recall the Beilinson-Drinfeld grassmannian \( \text{Gr}_{G, z, \sigma} \), classifying a point \( a \in \mathbb{A}^1 \), a \( G \)-bundle \( \mathcal{E} \) on \( \mathbb{P}^1 \), and a section \( \mathbb{P}^1 \setminus \{-a\} \rightarrow \mathcal{E} \). The coordinate function \( z \) of \( \mathbb{A}^1 = \mathbb{P}^1 \setminus \{\infty\} \simeq \text{Spec}(\mathbb{C}[z]) \) provides a trivialization

\[
\text{Gr} \times \mathbb{A}^1 \simeq \text{Gr}_{G, z, \sigma_1}. 
\]

Recall the Lusztig embedding

\[
e^{-} : N_n \rightarrow S^* = G(\mathbb{C}[z^{-1}])_{\infty \rightarrow e} \simeq T^0, \quad M \rightarrow [1 - z^{-1}M] \in T^0.
\]

It induces an embedding

\[
e_z : N_n \times i\mathbb{R} \xrightarrow{e^{-} \times \text{id}} T^0 \times i\mathbb{R} \xrightarrow{\text{Lemma 7.1}} \text{Gr}^0_{G, z, \sigma_1} \simeq QM^0_{G \times G, G}(\mathbb{Z}/\mathbb{A}^1, \sigma, \xi)_\mathbb{R},
\]

where the last isomorphism is the uniformization map. We have \( e_z(N_n \times i\mathbb{R}) = T^0 \cap S_z^\beta \).

For any \( a \in i\mathbb{R} \), we denote by

\[
e_{z,a} : N_n \rightarrow QM^0_{G \times G, G}(\mathbb{Z}/\mathbb{A}^1, \sigma, \xi)_\mathbb{R}
\]

the restriction of \( e_z \) to \( N_n \simeq N_n \times \{a\} \).

Recall the involution \( \eta_z \) on \( QM^0_{G \times G, G}(\mathbb{Z}/\mathbb{A}^1, \sigma, \xi)_\mathbb{R} \).

**Lemma 7.1.** The involution \( \eta_z \) preserves the stratum \( T^0_z \cap S_\lambda_z \), \( \lambda \in \Lambda_T^+ \).

**Proof.** Since, by Proposition \( \text{Lemma 6.1} \), \( \eta_z \) preserves the stratification \( \{T^0_z \cap S_\lambda_z \}_{\lambda \in \Lambda_T^+} \), it suffices to check that \( \eta_z \) preserves the special fiber \( T^0_z \cap S^1_z \) of the stratum \( T^0_z \cap S^1_z \). This is clear, since by Proposition \( \text{Lemma 6.1} \), the isomorphism \( QM^0_{G \times G, G}(\mathbb{Z}/\mathbb{A}^1, \sigma, \xi)_\mathbb{R} \) maps the stratum \( T^0_z \cap S^1_z \) to \( T^0 \cap S^\lambda \) and intertwines \( \eta_z \) and the complex conjugation \( \eta([\gamma(z)]) = [(\overline{\gamma(z)})] \) on \( T^0 \).

It follows from the lemma above that there exists a unique continuous families of involution \( \alpha_{z,a} : N_n \rightarrow N_n, \ a \in i\mathbb{R} \) satisfying

\[
e_{z,a} \circ \alpha_{z,a}(M) = \eta_z \circ e_{z,a}(M)
\]

We have the following characterization of \( \alpha_{z,a} \).

**Lemma 7.2.** Let \( M \in N_n \). The image \( N = \alpha_{z,a}(M) \) is the unique nilpotent matrix satisfying the following equation:

\[
(1 - uM^t)(1 - v\bar{M}) = (1 - v\bar{N}^t)(1 - uN).
\]

Here \( u = (z + a)^{-1}, \ v = (z - a)^{-1} \) and we regard the equation above as an equality of maps in \( G[\mathbb{P}^1 \setminus \{\pm a\}] \).
Proof. Let $G[\mathbb{P}^1 \setminus \{\pm a\}]_{\infty \to e}$ be the group of maps from $\mathbb{P}^1 \setminus \{\pm a\}$ to $G$ sending $\infty$ to $e$ and let $G[\mathbb{P}^1 \setminus \{-a\}]_{\infty \to e} \subset G[\mathbb{P}^1 \setminus \{\pm a\}]_{\infty \to e}$ be the subgroup consisting of maps that are regular on $a \in \mathbb{P}^1$. Let $G[\mathbb{P}^1 \setminus \{\pm a\}]_{\infty \to e, \mathbb{R}}$ be the real form with respect to the conjugation sending $\gamma(z) \to \eta_z(\gamma(z)) = (\gamma(\bar{z}))^{-1}$. We have the factorization isomorphism

$$G[\mathbb{P}^1 \setminus \{\pm a\}]_{\infty \to e, \mathbb{R}} \times G[\mathbb{P}^1 \setminus \{-a\}]_{\infty \to e} \simeq G[\mathbb{P}^1 \setminus \{\pm a\}]_{\infty \to e},$$

and it induces an isomorphism

$$\nu_\gamma : G[\mathbb{P}^1 \setminus \{-a\}]_{\infty \to e} \simeq G[\mathbb{P}^1 \setminus \{\pm a\}]_{\infty \to e, \mathbb{R}} \setminus G[\mathbb{P}^1 \setminus \{\pm a\}]_{\infty \to e}.$$ 

Consider the involution $\nu_\gamma$ on $G[\mathbb{P}^1 \setminus \{\pm a\}]_{\infty \to e, \mathbb{R}} \setminus G[\mathbb{P}^1 \setminus \{\pm a\}]_{\infty \to e}$ given by $\nu_\gamma(\gamma(z)) = \gamma(\bar{z})$ and let $\tilde{\nu}_\gamma := \nu_\gamma^{-1} \circ \nu_\gamma$ be the induced involution on $G[\mathbb{P}^1 \setminus \{-a\}]_{\infty \to e}$. Then the image $\tilde{\nu}_\gamma(\gamma(z))$ is characterized by the property that

$$\nu_\gamma(\gamma(z)) \nu_\gamma(\gamma(z))^{-1} \in G[\mathbb{P}^1 \setminus \{\pm a\}]_{\infty \to e, \mathbb{R}}.$$

Note that the composed isomorphism

$$l_\gamma : G[\mathbb{P}^1 \setminus \{-a\}]_{\infty \to e} \overset{\nu_\gamma}{\longrightarrow} G[\mathbb{P}^1 \setminus \{\pm a\}]_{\infty \to e, \mathbb{R}} \setminus G[\mathbb{P}^1 \setminus \{\pm a\}]_{\infty \to e} \simeq QM^0_{G \times G, \mathbb{R}}(\mathbb{Z}_a, \sigma(a), \xi(a)),$$

where the last map is the uniformization isomorphism, intertwines the involutions $\tilde{\nu}_\gamma$ and $\eta_\gamma$. Consider the embedding

$$e_a^- : N_\gamma \to G[\mathbb{P}^1 \setminus \{-a\}]_{\infty \to e}, \quad M \mapsto 1 - (z + a)^{-1} M.$$ 

It is elementary to check that the composition

$$N_\gamma \overset{e_a^-}{\longrightarrow} G[\mathbb{P}^1 \setminus \{-a\}]_{\infty \to e} \overset{l_\gamma}{\longrightarrow} QM^0_{G \times G, \mathbb{R}}(\mathbb{Z}_a, \sigma(a), \xi(a)) \simeq N_\gamma$$

is exactly the morphism $e_{z,a}$ in (7.2). Thus we have

$$l_\gamma \circ e_a^- \circ (\alpha_{z,a}(M)) = e_{z,a} \circ \alpha_{z,a}(M) \overset{\text{by } \eta_\gamma \circ e_{z,a}(M) = l_\gamma \circ \tilde{\nu}_\gamma \circ (e_a^-(M))}\longrightarrow$$

and it implies

$$e_a^-(N) = \tilde{\nu}_\gamma(e_a^-(M)), \quad M \in N_\gamma.$$ 

Using the characterization of $\tilde{\nu}_\gamma$ in (7.5), we see that that $N = \alpha_{z,a}(M)$ is the unique nilpotent matrix satisfying

$$\nu_\gamma(e_a^-(M)) \nu_\gamma(N) = \in G[\mathbb{P}^1 \setminus \{\pm a\}]_{\infty \to e, \mathbb{R}},$$

that is,

$$(1 - (z - a)^{-1} \bar{M})(1 - (z + a)^{-1} N)^{-1} \in G[\mathbb{P}^1 \setminus \{\pm a\}]_{\infty \to e, \mathbb{R}}.$$ 

Since the above equation is equivalent to

$$(1 - (z - a)^{-1} \bar{M})(1 - (z + a)^{-1} N)^{-1} = (1 - (z + a)^{-1} \bar{M}^{-1})(1 - (z - a)^{-1} \bar{N}^{-1}),$$

the desired equation (7.4) follows.

\[ \square \]

Note that when $a = 0$, we have $u = v = z^{-1}$ and the equations above become

$$(1 - z^{-1} \bar{M}^{-1})(1 - z^{-1} \bar{N})^{-1} = (1 - z^{-1} \bar{N})^{-1}(1 - z^{-1} \bar{N})^{-1}$$

Thus we have $N = \alpha_{z,0}(M) = \bar{M}$ and

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7.1.1. Recall the Beilinson-Drinfeld grassmianian \( \text{Gr}_{G,Y,\sigma_x} \) classifying a point \( a \in \mathbb{A}^1 \), a \( G \)-bundle on \( Y_a \), and a section of \( \mathcal{E} \) over \( Y_a \setminus \{ \sigma_x(a) \} \). Recall that

\[
Y_a \setminus \{ \sigma_x(a) \} \overset{p_x}{\cong} \mathbb{P}^1 \setminus \{ \infty \}, \quad a \neq 0 \in \mathbb{R},
\]

\[
Y_0 \setminus \{ \sigma_x(0) \} = \mathbb{A}^1 \times \{ 0 \} \cup \{ 0 \} \times \mathbb{P}^1, \quad a = 0 \in \mathbb{R}.
\]

The local coordinate \( x^{-1} \) for \( \sigma_x(a) = (\infty, 0, a) \in Y_a \) provides a trivialization

\[
(7.6) \quad \text{Gr}_\infty \times \mathbb{A}^1 \cong \text{Gr}_{G,Y,\sigma_x}
\]

where \( \text{Gr}_\infty := G(\mathbb{C}[x, x^{-1}])/G(\mathbb{C}[x^{-1}]) \). Let \( T_\infty^0 \) be the open \( G(\mathbb{C}[x]) \)-orbit on \( \text{Gr}_\infty \). Then the trivialization above restricts to an isomorphism

\[
T_\infty^0 \times \mathbb{A}^1 \cong \text{Gr}_{G,Y,\sigma_x}^0.
\]

For any \( a \in \mathbb{A}^1 \), let \( \text{ev}_a : G(\mathbb{C}[x]) \rightarrow G \), \( \gamma(x) \rightarrow \gamma(a) \) be the evaluation map at \( x = a \) and we define \( G(\mathbb{C}[x])_{a \rightarrow e} = \text{ev}_a^{-1}(e) \). The action of \( G(\mathbb{C}[x])_{a \rightarrow e} \) on the based point of \( \text{Gr}_\infty \) defines an isomorphism \( G(\mathbb{C}[x])_{a \rightarrow e} \cong T_\infty^0 \) and we write \([\gamma(z)] \in T_\infty^0 \) for the image of \( \gamma(z) \in G(\mathbb{C}[x])_{a \rightarrow e} \).

Consider the family of embeddings

\[
e_a^+ : N_n \rightarrow G(\mathbb{C}[x])_{a \rightarrow e} \cong T_\infty^0, \quad M \rightarrow [1 - \frac{1}{2}(x - a)M] \in T_\infty^0, \quad a \in \mathbb{A}^1.
\]

We claim that

\[
(7.7) \quad e_a^+(N_n) = T_\infty^0 \cap S_\infty^0,
\]

here \( S_\infty^0 \) is the \( G(\mathbb{C}[x^{-1}]) \)-orbit of \( [x^{-\beta}] \in \text{Gr}_\infty \). Consider the family of \( G \)-equivariant isomorphism

\[
f_a : N_n \rightarrow N_n, \quad M \rightarrow M_a := M(1 + \frac{i}{2}aM)^{-1}.
\]

Then a direct computation shows that

\[
e_a^+ = e_0^+ \circ f_a : N_n \rightarrow G(\mathbb{C}[x])_{0 \rightarrow e} \cong T_\infty^0, \quad M \rightarrow [1 - \frac{i}{2}xM_a] \in T_\infty^0,
\]

As \( e_0^+(N_n) = T_\infty^0 \cap S_\infty^0 \), we obtain \( e_a^+(N_n) = e_0^+ \circ f_a(N_n) = e_0^+(N_n) = T_\infty^0 \cap S_\infty^0 \). The claim follows.

As \( \lambda \) varies, we obtain an embedding

\[
e_y : N_n \times \mathbb{R} \rightarrow T_\infty^0 \times \mathbb{R} \cong \text{Gr}_{G,Y,\sigma_x}^0 \cong \mathcal{Q}M_{G \times G,G}^0(Y/\mathbb{A}^1, \sigma, \xi)_\mathbb{R}
\]

where the last isomorphism is the uniformization map. It follows from (7.7) that \( e_y(N_n \times \mathbb{R}) = T_\infty^0 \cap S_\infty^0 \). For any \( a \in \mathbb{R} \), we denote by

\[
e_{y,a} : N_n \rightarrow \mathcal{Q}M_{G \times G,G}^0(Y_a, \sigma(a), \xi(a))_\mathbb{R}
\]

the restriction of \( e_y \) to \( N_n \cong N_n \times \{ a \} \).

Recall the \( \eta_\gamma \) on \( \mathcal{Q}M_{G \times G,G}^0(Y/\mathbb{A}^1, \sigma, \xi)_\mathbb{R} \).

**Lemma 7.3.** The involution \( \eta_\gamma \) preserves the stratum \( T_\gamma^0 \cap S_\gamma^0 \), \( \lambda \in \Lambda_T^\pm \).
Proof. Since, by Proposition 6.1, the involution \( \eta_y \) preserves the stratification \( \{ \mathcal{T}_y^0 \cap S^\lambda_y \}_{\lambda \in \Lambda^+} \), it suffices to check that \( \eta_y \) preserves the special fiber \( \mathcal{T}_y^0 \cap S^\lambda_y \). This is clear, since by Proposition 6.1, the isomorphism \( QM^0_{G \times G}(y_0, \sigma(0), \xi(0))_R \cong \mathbb{G}^* \) maps the stratum \( \mathcal{T}_y^0 \cap S^\lambda_y \) to \( \mathbb{G}^* \cap S^\lambda \) and intertwines \( \eta_y \) and the transpose action \( \text{inv} \circ \theta(z) = (\gamma(z))^t \) on \( \mathbb{G}^* \).

It follows from the lemma above that there exists a unique continuous families of involution
\[
(7.9) \quad \alpha_{y,a} : \mathcal{N}_n \to \mathcal{N}_n, \quad a \in \mathbb{R}
\]
satisfying
\[
(7.10) \quad e_{y,a} \circ \alpha_{y,a}(M) = \eta_y \circ e_{y,a}(M)
\]
We have the following characterization of \( \alpha_{y,a} \).

**Lemma 7.4.** Let \( M \in \mathcal{N}_n \) and \( a \in \mathbb{R} \). The image \( N = \alpha_{y,a}(M) \) is the unique nilpotent matrix satisfying the following equation:
\[
(7.11) \quad (1 - uM^t)(1 - vM) = (1 - vN^t)(1 - uN).
\]
Here \( u = \frac{t}{2}(x - a) \), \( v = \frac{v}{2}(y - a) \) and we regard the equation above as an equality of maps in \( G(\mathbb{C}[xy = a^2]) \).

**Proof.** The proof is similar to the one for Lemma 7.2. Let
\[
G[\mathcal{Y}_a \setminus \{ \sigma_x(a), \sigma_y(a) \}]_{\xi \to e} = G(\mathbb{C}[xy = a^2])_{\xi \to e}
\]
be the group consisting of maps \( \gamma(x, y) : \mathcal{Y}_a \setminus \{ \sigma_x(a), \sigma_y(a) \} = \{ xy = a^2 \} \to G \) such that \( \gamma(a, a) = 1 \) and let \( G[\mathcal{Y}_a \setminus \{ \sigma_x(a), \sigma_y(a) \}]_{\xi \to e} \subset G[\mathcal{Y}_a \setminus \{ \sigma_x(a), \sigma_y(a) \}]_{\xi \to e} \) be the subgroup consisting of maps that are regular on \( \sigma_y(a) \). Let
\[
G[\mathcal{Y}_a \setminus \{ \sigma_x(a), \sigma_y(a) \}]_{\xi \to e, \mathbb{R}} = G(\mathbb{C}[xy = a^2])_{\xi \to e, \mathbb{R}}
\]
be the real form with respect to the conjugation sending \( \gamma(x, y) \to \eta_e(\gamma(\bar{y}, \bar{x})) \). We have the factorization isomorphism
\[
G[\mathcal{Y}_s \setminus \{ \sigma_x(s), \sigma_y(s) \}]_{\xi \to e, \mathbb{R}} \times G[\mathcal{Y}_s \setminus \{ \sigma_x(s) \}]_{\xi \to e} \cong G[\mathcal{Y}_s \setminus \{ \sigma_x(s), \sigma_y(s) \}]_{\xi \to e}.
\]
Since \( \mathcal{Y}_s \setminus \{ \sigma_x(s) \} \simeq \mathbb{P}^1 \setminus \{ \infty \} = \text{Spec}(\mathbb{C}[x]) \), \( s \neq 0 \) and \( \mathcal{Y}_0 \setminus \{ \sigma_x(0) \} = (\text{Spec}(\mathbb{C}[x]) \times \{ 0 \}) \cup (\{ 0 \} \times \mathbb{P}^1) \), we have a canonical isomorphism
\[
G[\mathcal{Y}_s \setminus \{ \sigma_x(s) \}]_{\xi \to e} = G(\mathbb{C}[x])_{a \to e},
\]
and the factorization isomorphism above can be identified with
\[
G(\mathbb{C}[xy = a^2])_{\xi \to e, \mathbb{R}} \times G(\mathbb{C}[x])_{a \to e} \cong G(\mathbb{C}[xy = a^2])_{\xi \to e}.
\]
From it we obtain an isomorphism
\[
t_y : G(\mathbb{C}[x])_{a \to e} \cong G(\mathbb{C}[xy = a^2])_{\xi \to e, \mathbb{R}} \setminus G(\mathbb{C}[xy = a^2])_{\xi \to e}.
\]
Consider the involution $\nu_y$ on $G(\mathbb{C}[xy = a^2])_{t \to e, \mathbb{R}} \setminus G(\mathbb{C}[xy = a^2])_{t \to e}$ given by $\nu_y(\gamma(x, y)) = \bar{\gamma}(\bar{y}, \bar{x})$ and let $\tilde{\nu}_y := i_y^{-1} \circ \nu_y \circ \nu_y$ be the induced involution on $G(\mathbb{C}[x])_{a \to e}$. Then the image $\tilde{\nu}_y(\gamma(x))$ is characterized by the property that

$$\nu_y(\gamma(x))\tilde{\nu}_y(\gamma(x))^{-1} \in G(\mathbb{C}[xy = a^2])_{t \to e, \mathbb{R}}.$$  

Note that the composed isomorphism

$$l_y : G(\mathbb{C}[x])_{a \to e} \xrightarrow{\eta_y} G(\mathbb{C}[xy = a^2])_{t \to e, \mathbb{R}} \setminus G(\mathbb{C}[xy = a^2])_{t \to e} \simeq QM^0_{G \times G, G}(\mathbb{Z}_a, \sigma(a), \xi(a))_{\mathbb{R}},$$

where the last map is the uniformization isomorphism, intertwines the involutions $\tilde{\nu}_y$ with $\eta_y$. Moreover, it is elementary to check that the map

$$N_n \xrightarrow{e_a^+} G(\mathbb{C}[x])_{a \to e} \xrightarrow{l_y} QM^0_{G \times G, G}(\mathbb{Z}_a, \sigma(a), \xi(a))_{\mathbb{R}}$$

is equal to the morphism $e_y.a$ in (7.8). Thus we have

$$l_y \circ e_a^+ \circ (\alpha_{y,a}(M)) = e_{y,a} \circ \alpha_{y,a}(M) \quad \text{by (7.10)}$$

and it implies

$$e_a^+(N) = \tilde{\nu}_y \circ e_a^+(M), \quad M \in N_n.$$  

Using the characterization of $\tilde{\alpha}_y$ in (7.12), we see that $N = \alpha_{y,s}(M)$ is the unique nilpotent matrix satisfying

$$\nu_y(e_a^+(M))e_a^+(N)^{-1} \in G(\mathbb{C}[xy = a^2])_{t \to e, \mathbb{R}},$$

that is,

$$(1 + \frac{i}{2}(y - a)M)(1 - \frac{i}{2}(x - a)N)^{-1} \in G(\mathbb{C}[xy = a^2])_{t \to e, \mathbb{R}}.$$  

Since the above equation is equivalent to

$$(1 + \frac{i}{2}(y - a)\bar{M})(1 - \frac{i}{2}(x - a)\bar{N})^{-1} = (1 - \frac{i}{2}(x - a)\bar{M}t)^{-1}(1 + \frac{i}{2}(y - a)^{-1}Nt),$$

the desired equation (7.11) follows.

Note that when $s = 0$, we have $u = \frac{i}{2}x$, $v = \frac{-i}{2}y$ and the equations above become

$$(1 - \frac{i}{2}xMt)(1 + \frac{i}{2}y\bar{M}) = (1 + \frac{i}{2}y\bar{N})t(1 - \frac{i}{2}xN)$$

as elements in $G[xy = 0]$, and we see that $N = \alpha_{y,0}(M) = Mt$.

7.1.2. Under the change of coordinate $x = \frac{z-1}{\bar{y}+1}$ the function $(z + i)^{-1}$ becomes $\frac{i}{2}(x - 1)$ and $(z - i)^{-1}$ becomes $\frac{i}{2}(y - 1)$, thus by Lemma 7.2 and Lemma 7.4 we have

$$\alpha_{z,a} = 1 : \mathbb{R} \to \mathbb{R},$$

Consider the following families of maps

$$\alpha_{n,a} : N_n \to N_n \quad \text{a} \in [0, 1]$$

where $\alpha_{n,a} = \alpha_{z,2ia}$ if $a \in [0, \frac{1}{2}]$ and $\alpha_{n,a} = \alpha_{y,2a-2}$ if $a \in [\frac{1}{2}, 1]$.  

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Proposition 7.5. \( \alpha_{n,a} : N_n \to N_n, \ a \in [0,1] \) is a continuous one-parameter family of maps satisfying the following:

1. \( \alpha_{n,a}^2 = \text{the identity}, \) for all \( a \in [0,1] \).
2. \( \alpha_{n,a} \) is \( K_t \)-equivariant and take \( G \)-orbits real analytically to a \( G \)-orbits.
3. At \( a = 0 \), we have \( \alpha_{n,0}(M) = M \).
4. At \( a = 1 \), we have \( \alpha_{n,1}(M) = M^t \).

7.2. Compatibility with inner and Cartan involutions. In this section we show that the family of involutions \( \alpha_{n,a} : N_n \to N_n \) are compatible with inner and Cartan involutions.

Lemma 7.6. Let \( g \in \text{GL}_n(\mathbb{C}) \) and let \( \text{Ad} : N_n \to N_n \) be the conjugation action. Assume \( \text{Ad} g \) commutes with \( \alpha_{n,0} \) and \( \alpha_{n,1} \). Then we have \( \text{Ad} \circ \alpha_{n,a} = \alpha_{n,a} \circ \text{Ad} \) for all \( a \in [0,1] \).

Proof. Let \( M \in N_n \) and \( N = \alpha_{n,s}(M) \). By Lemma 7.2 and 7.4, it suffices to show that the pair \( (A = \text{Ad} g(M), B = \text{Ad} g(N)) \) satisfies

\[
(1 - uA^t)(1 - vA) = (1 - vB^t)(1 - uB).
\]

This is clear since \((M, N)\) satisfies

\[
(1 - uM^t)(1 - vM) = (1 - vN^t)(1 - uN)
\]

and it implies, as \( \text{Ad} g \) commutes with \( \alpha_{n,0} \) and \( \alpha_{n,1} \),

\[
(1 - uA^t)(1 - vA) = \text{Ad} g((1 - uM^t)(1 - vM)) = \text{Ad} g((1 - vN^t)(1 - uN)) = (1 - vB^t)(1 - uB).
\]

Lemma 7.7. The family of involutions \( \alpha_{n,a} : N_n \to N_n, \ a \in [0,1] \) commutes with the Cartan involution \( \omega : N_n \to N_n, \omega(M) = -M^t \).

Proof. It suffices to show that both \( \alpha_{z,a} \) and \( \alpha_{y,a} \) commute with \( \omega \). It is clear when \( a = 0 \). We now assume \( a \neq 0 \). It suffices to prove the following claim: Let \( M \in N \) and \( N = \alpha_{z,a}(M) \) \( a \in i\mathbb{R}^\times \) (resp. \( N = \alpha_{y,a}(M), a \in \mathbb{R}^\times \)). Then the pair \((\omega(M), \omega(N))\) satisfies

\[
(1 - u\omega(M)^t)(1 - v\omega(M)) = (1 - v\omega(N)^t)(1 - u\omega(N))
\]

as maps in \( G[\mathbb{P}^1 \setminus \{\pm a\}] \), where \( u = (z + a)^{-1} \) and \( v = (z - a)^{-1} \) and \( z \) is the coordinate at \( 0 \in \mathbb{P}^1 \) (resp. as maps in \( G(\mathbb{C}[xy = a^2]) \)), where \( u = \frac{i}{2}(x - a) \) and \( v = \frac{1}{2}(y - a) \). Note that (7.14) is equal to

\[
(1 + uM)(1 + vM^t) = (1 + vN)(1 + uN^t).
\]

To prove the claim, we observe that, by Lemma 7.2 (resp. Lemma 7.4), the pair \((M, N)\) satisfies

\[
(1 - uM^t)(1 - vM) = (1 - vN^t)(1 - uN).
\]

Recall the conjugation \( \eta \) on \( G[\mathbb{P}^1 \setminus \{\pm a\}] \) (resp. \( G(\mathbb{C}[xy = a^2]) \)) given by \( \eta(\gamma(z)) = \overline{\gamma(z)} \) (resp. \( \eta(\gamma(x, y)) = \overline{\gamma(y, x)} \)). Applying \( \text{inv} \cdot \eta \) to the equation above, we get

\[
(1 - uM)^{-1}(1 - vM^t)^{-1} = (1 - vN)^{-1}(1 - uN^t)^{-1}.
\]
As $M$ and $N$ are nilpotent, the above equation is the same as
\[
(1 + uM + (vM)^2 + \cdots + (uM)^{n-1})(1 + v\bar{M}t + (v\bar{M}t)^2 + \cdots + (v\bar{M}t)^{n-1}) = \\
(1 + v\bar{N} + (v\bar{N})^2 + \cdots + (v\bar{N})^{n-1})(1 + uNt + (uNt)^2 + \cdots + (uNt)^{n-1}),
\]
and one can rewrite it as
\[
(7.16) \quad (1 + uM + v\bar{M}t + uvM\bar{M}t) - (1 + v\bar{N} + u\bar{N}t + uv\bar{N}Nt) = \sum_{k \geq 2 \text{ or } l \geq 2} u^k v^j P_{kl}
\]
here $P_{kl}$ are certain $n \times n$ matrices. As $z \to -a$ (resp. $x \to \infty$), we have $u \to \infty$ and $v \to \frac{-1}{2a}$ (resp. $v \to \frac{1}{2a}$), thus we must have $P_{kl} = 0$ for $k \geq 2$. Similarly, as $z \to a$ (resp. $x \to 0$), we have $u \to \frac{1}{2a}$ (resp. $u \to \frac{-1}{2a}$) and $v \to \infty$, thus we must have $P_{kl} = 0$ for $l \geq 2$. So we have $P_{kl} = 0$ and (7.16) becomes
\[
(7.17) \quad (1 + uM + v\bar{M}t + uvM\bar{M}t) - (1 + v\bar{N} + u\bar{N}t + uv\bar{N}Nt) = 0
\]
The desired equation (7.15) follows. □

7.3. Classical groups. In this section we shall establish the Kostant-Sekiguchi homeomorphism for classical groups:

**Theorem 7.8.** Suppose all simple factors of the complex reductive Lie algebra $\mathfrak{g}$ are of classical type. There is a $K_c$-equivariant homeomorphism
\[
(7.18) \quad N_R \simeq N_p
\]
that induces the Kostant-Sekiguchi correspondence on orbit posets. Furthermore, it restricts to real analytic isomorphisms between individual $G_R$-orbits and $K$-orbits.

We will deduce the theorem above from the following:

**Theorem 7.9.** There is a continuous one-parameter families of maps
\[
\alpha_a : N \to N, \quad a \in [0, 1]
\]
satisfying the following:

1. $\alpha_a^2$ is the identity, for all $s \in [0, 1]$.
2. $\alpha_a$ is $K_c$-equivariant and take $G$-orbits real analytically to $a$ $G$-orbits.
3. $\alpha_a$ is compatible with strictly normal morphism $\rho : \mathfrak{s} \to \mathfrak{g}$ (see Lemma 7.12).
4. At $a = 0$, we have $\alpha_0(M) = \eta(M)$.
5. At $a = 1$, we have $\alpha_1(M) = -\theta(M)$.

7.4. Proof of Theorem 7.9. Recall the classification of real forms of classical types:

**Lemma 7.10.** [OV] Section 4 | Here is the complete list of all possible quadruple $(\mathfrak{g}_R, \mathfrak{k}, \eta, \theta)$ (up to isomorphism):

- (a) $\mathfrak{g} = \mathfrak{sl}_n(\mathbb{C})$:
  1. $\mathfrak{g}_R = \mathfrak{sl}_n(\mathbb{R})$, $\mathfrak{k} = \mathfrak{so}_n(\mathbb{C})$, $\eta(g) = \bar{g}$, $\theta(g) = 0$.
  2. $\mathfrak{g}_R = \mathfrak{sl}_m(\mathbb{H})$, $\mathfrak{k} = \mathfrak{sp}_m(\mathbb{C})$, $\eta(g) = \Ad S_m(g)$, $\theta(g) = -\Ad S_m(g^t)$ ($n = 2m$).
  3. $\mathfrak{g}_R = \mathfrak{su}_{p,n-p}$, $\mathfrak{k} = (\mathfrak{gl}_p(\mathbb{C}) \oplus \mathfrak{gl}_{n-p}(\mathbb{C})) \cap \mathfrak{g}$, $\eta(g) = -\Ad I_{p,n-p}(g^t)$, $\theta(g) = \Ad I_{p,n-p}(g)$. 

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(b) \( g = \mathfrak{so}_n(\mathbb{C}) \):

\( g_\mathbb{R} = \mathfrak{so}_{p,n,p}, \mathfrak{t} = \mathfrak{so}_p(\mathbb{C}) \oplus \mathfrak{so}_{n-p}(\mathbb{C}), \) \( \eta(g) = \text{Ad} I_{p,n-p}(\bar{g}), \theta(g) = \text{Ad} I_{p,n-p}(g). \)

(2) \( g_\mathbb{R} = \mathfrak{u}^*_m(\mathbb{H}), \mathfrak{t} = \mathfrak{gl}_m(\mathbb{C}), \) \( \eta(g) = \text{Ad} S_m(\bar{g}), \theta(g) = \text{Ad} S_m(g) \) \( (n = 2m). \)

(c) \( g = \mathfrak{sp}_{2m}(\mathbb{C}) \):

\( g_\mathbb{R} = \mathfrak{sp}_{2m}(\mathbb{R}), \mathfrak{t} = \mathfrak{gl}_m(\mathbb{C}), \) \( \eta(g) = \bar{g}, \theta(g) = \text{Ad} S_m(g). \)

(2) \( g_\mathbb{R} = \mathfrak{sp}_{p,m-p}, \mathfrak{t} = \mathfrak{sp}_{2p}(\mathbb{C}) \oplus \mathfrak{sp}_{2m-2p}(\mathbb{C}), \) \( \eta(g) = -\text{Ad} K_{p,m-p}(\bar{g}^t), \theta(g) = \text{Ad} K_{p,m-p}(g). \)

Here \( S_m = \begin{pmatrix} 0 & -I_{dm} \\ I_{dm} & 0 \end{pmatrix}, I_{p,n-p} = \begin{pmatrix} I_d & 0 \\ 0 & -I_{n-p} \end{pmatrix}, \) and \( K_{p,m-p} = \begin{pmatrix} I_{p,m-p} & 0 \\ 0 & I_{p,m-p} \end{pmatrix} \)

Note that the inner automorphisms \( \text{Ad} S_m, \text{Ad} I_{p,n-p}, \) and \( \text{Ad} K_{p,m-p} \) commute with \( \alpha_{n,0} \) and \( \alpha_{n,1}. \) Using the fact that \( \alpha_{n,s} : \mathcal{N}_n \to \mathcal{N}_n \) is compatible with such inner automorphisms and Cartan involution \( \omega \) (see Lemma 7.6 and Lemma 7.7), a direct computation shows that the family of maps

\[ \alpha_{n,s} \circ \theta \circ \omega : \mathcal{N}_n \to \mathcal{N}_n \]

preserves \( \mathcal{N} \subseteq \mathcal{N}_n \) and defines a family of maps

\[ \alpha_s : \mathcal{N} \to \mathcal{N} \]

satisfying properties (1), (2), (4), and (5) of Theorem 7.9. Here is the explicit formula for \( \alpha_s \) for each classical type:

(a) \( g = \mathfrak{sl}_n(\mathbb{C}) : \)

\( 1 \) \( g_\mathbb{R} = \mathfrak{sl}_n(\mathbb{R}), \alpha_s = \alpha_{n,s}. \)

(2) \( g_\mathbb{R} = \mathfrak{sl}_m(\mathbb{H}), \alpha_s = \alpha_{n,s} \circ \text{Ad}(S_m) \) \( (n = 2m). \)

(3) \( g_\mathbb{R} = \mathfrak{su}_{p,n-p}, \alpha_s = \alpha_{n,s} \circ \text{Ad} I_{p,m-p} \circ \omega. \)

(b) \( g = \mathfrak{so}_n(\mathbb{C}) : \)

\( 1 \) \( g_\mathbb{R} = \mathfrak{so}_p(\mathbb{R}), \alpha_s = \alpha_{n,s} \circ \text{Ad} I_{p,n-p} \)

(2) \( g_\mathbb{R} = \mathfrak{so}(\mathbb{H}), \alpha_s = \alpha_{n,s} \circ \text{Ad} S_m \) \( |N \) \( (n = 2m). \)

(c) \( g = \mathfrak{sp}_{2m}(\mathbb{C}) : \)

\( 1 \) \( g_\mathbb{R} = \mathfrak{sp}_{2m}(\mathbb{R}), \alpha_s = \alpha_{2m,s} \)

(2) \( g_\mathbb{R} = \mathfrak{sp}_{p,m-p}, \alpha_s = \alpha_{2m,s} \circ \text{Ad} K_{p,m-p} \circ \omega |N. \)

We shall show that \( \alpha_s \) is compatible with strictly normal morphisms. Let me first recall the definition of strictly normal morphisms. Let \( s = \mathfrak{sl}_2(\mathbb{C}). \) For \( M \in s, \) set \( \eta_s(M) = \bar{M} \) and \( \theta_s(M) = -M^t, \) that is, \( \eta_s \) is the split conjugation and \( \theta_s \) is the corresponding Cartan involution.

**Definition 7.11.** A Lie algebra morphism \( \rho : s \rightarrow g \) is called **strictly normal** if \( \rho \) intertwines \( \eta_s \) with \( \eta \) and \( \theta_s \) with \( \theta. \)

We have the following compatibility of \( \alpha_s \) with strictly normal morphism:
Lemma 7.12. Let $\rho : s \to g$ be a strictly normal morphism and let $\rho : N_2 \to N$ be the induced map between nilpotent cones. We have the following commutative diagram

\[
\begin{array}{ccc}
N_2 & \xrightarrow{\alpha_s} & N_2 \\
\downarrow{\rho} & & \downarrow{\rho} \\
N & \xrightarrow{\alpha_s} & N
\end{array}
\]

Proof. Let $M_s \in N_2$ and $N_s = \alpha_s(M_s)$. Let $M = \rho(M_s)$ and $N = \rho(N_s)$. We want to show that

\[
N = \alpha_a(M) = \alpha_{n,a} \circ \theta \circ \omega(M).
\]

Set $A = \theta \circ \omega(M)$. By Lemma 7.2 and Lemma 7.4, it suffices to show that the pair $(A, N)$ satisfies the equation

\[
(1 - uA^t)(1 - v\bar{A}) = (1 - v\bar{N}^t)(1 - uN).
\]

We have

\[
(1 - uM_s^t)(1 - v\bar{M}_s) = (1 - v\bar{N}_s^t)(1 - uN_s).
\]

Let $\bar{\rho} : SL_2(\mathbb{C}) \to G \subset SL_n(\mathbb{C})$ be a group homomorphism such that $d\bar{\rho} = \rho$. Applying $\bar{\rho}$ to the equations above and using the facts that $(1 - uY) = \exp(-uY), (1 - vY) = \exp(-vY)$ for $Y \in N_2$ and $\exp \circ \rho = \bar{\rho} \circ \exp$, we obtain

\[
\exp(-u\rho(M_s^t)) \exp(-v\rho(\bar{M}_s)) = \exp(-v\rho(\bar{N}_s)) \exp(-u\rho(N_s)).
\]

Since $\rho$ is strictly normal we have $\rho(M_s^t) = -\theta(M), \rho(\bar{M}_s) = \eta(M), \rho(\bar{N}_s^t) = -\eta_c(N)$, and the equation above becomes

\[
(1 - u(-\theta(M)) + \cdots + (-1)^{n-1} \frac{(u(-\theta(M))^{n-1}}{(n-1)!})(1 - v\eta(M) + \cdots + (-1)^{n-1} \frac{(v\eta(M))^{n-1}}{(n-1)!}) = \\
(1 - v(-\eta_c(N)) + \cdots + (-1)^{n-1} \frac{(v(-\eta_c(N)))^{n-1}}{(n-1)!})(1 - uN + \cdots + (-1)^{n-1} \frac{(uN)^{n-1}}{(n-1)!}).
\]

Using the equalities

\[-\theta(M) = (\theta \circ \omega(M))^t = A^t, \quad \eta(M) = \overline{\theta \circ \omega(M)} = \bar{A}, \quad -\eta_c(N) = N^t,
\]

one can rewrite the equation above as

\[
(1 - uA^t - v\bar{A} + uvA^t\bar{A}) - (1 - vN^t - u\bar{N} + uvN^t\bar{N}) = \sum_{k \geq 2 \text{ or } l \geq 2} u^k v^l Q_{kl}
\]

here $Q_{kl}$ are certain $n \times n$ matrices. Now the same argument as in Lemma 7.7 shows that $Q_{kl}$ are zero and we have

\[
(1 - uA^t - v\bar{A} + uvA^t\bar{A}) - (1 - vN^t - u\bar{N} + uvN^t\bar{N}) = 0.
\]

The equation \((7.21)\) follows. We are done. \[
\]

We have shown that $\alpha_s$ is compatible with strictly normal morphisms. This completes the proof of Theorem 7.9.
7.5. **Proof of Theorem 7.8.** Consider the product $N \times [0,1]$ with the horizontal vector field $\partial_s$. By averaging $\partial_s$ with respect to the $\mathbb{Z}/2\mathbb{Z}$-action given by $\alpha_s$ we obtain a new horizontal vector field $w$ on $N \times [0,1]$. It follows from Proposition 6.4 and Example 6.3.1 that $w$ is complete and the integral curves of $w$ defines a homeomorphism $N \simeq N$ intertwines the $\alpha_0 = \eta$ action and the $\alpha_1 = -\theta$ action, hence gives rise to a homeomorphism

$$h_{KS} : N_R \simeq N_p$$

between the real and symmetric nilpotent cones. Property (2) in Theorem 7.9 implies that the homeomorphism $h_{KS}$ is $K_c$-equivariant and, for any nilpotent $G$-orbit $\mathcal{O}$, it restricts to a $K_c$-equivariant real analytic isomorphism

$$\mathcal{O} \cap g_R \simeq \mathcal{O} \cap p.$$ 

Write $\mathcal{O} \cap g_R = \bigcup_j \mathcal{O}_{R,j}$ as a union $G_R$-orbits, and similarly, $\mathcal{O} \cap p = \bigcup_j \mathcal{O}_{p,j}$ as a union $K$-orbits. Since each $\mathcal{O}_{R,j}$ (resp. $\mathcal{O}_{p,j}$) orbit is a union of connected components of $\mathcal{O} \cap g_R$ (resp. $\mathcal{O} \cap p$) and $K_c$-acts transitively on the component group of $\mathcal{O}_{R,j}$ (resp. $\mathcal{O}_{p,j}$), there exists a bijection

$$(7.22) \quad |G_R \setminus \mathcal{O} \cap g_R| \leftrightarrow |K \setminus \mathcal{O} \cap p|, \quad [\mathcal{O}_{R,j}] \leftrightarrow [\mathcal{O}_{p,j}]$$

such that $h_{KS}$ restricts to a $K_c$-equivariant real analytic isomorphism

$$\mathcal{O}_{R,j} \simeq \mathcal{O}_{p,j}.$$ 

We shall show that the bijection (7.22) is equal to the Kostant-Sekiguchi bijection [S]. Recall the following characterization of Kostant-Sekiguchi bijection: a nilpotent $G_R$-orbit $\mathcal{O}_R$ corresponds to a $K$-orbit $\mathcal{O}_p$ if and only if there exists a strictly normal morphism $\rho : s \to g$ such that

$$\rho \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \in \mathcal{O}_R, \quad \rho \begin{pmatrix} 1 & -i \\ -i & -1 \end{pmatrix} \in \mathcal{O}_p.$$ 

By [S], there exists a strictly normal morphism $\rho : s \to g$ such that $\rho \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \in \mathcal{O}_{R,j}$. The morphism $\rho$ restricts to maps $N_{2,R} \to N_{R}, N_{2,p} \to N_{p}$ which we still denote by $\rho$. It follows from Lemma 7.12 that we have the following commutative diagram

$$\begin{array}{ccc}
N_{2,R} & \xrightarrow{h_{KS}} & N_{2,p} \\
\downarrow \rho & & \downarrow \rho \\
N_R & \xrightarrow{h_{KS}} & N_p
\end{array}$$

By Example 6.3.2 we know that both

$$h_{KS} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & -i \\ -i & -1 \end{pmatrix}$$

are in the same $SO_2(\mathbb{C})$-orbit. It implies

$$h_{KS} \circ \rho \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \rho \circ h_{KS} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad \rho \begin{pmatrix} 1 & -i \\ -i & -1 \end{pmatrix}$$
are in the same $K$-orbit. As $h_{KS} \circ \rho \left( \begin{array}{cc} 0 & 1 \\ 0 & 0 \end{array} \right) \in h_{KS}(O_{\mathbb{R},j}) = O_{p,j}$, we conclude that

$$\rho \left( \begin{array}{cc} 0 & 1 \\ 0 & 0 \end{array} \right) \in O_{\mathbb{R},j}, \quad \rho \left( \begin{array}{cc} 1 & -i \\ -i & -1 \end{array} \right) \in O_{p,j}.$$  

By the characterization above we see that the bijection (7.22) coincides with the Kostant-Sekiguchi bijection. The finishes the proof of Theorem 7.8.

REFERENCES

[AH] P. Achar, A. Henderson. Geometric Satake, Springer correspondence, and small representations, Sel. Math. New Ser. (2013) 19: 949-986.

[BS] D. Barbasch, M.R. Sepanski. Closure ordering and the Kostant-Sekiguchi correspondence, Proc. Amer. Math. Soc. 126 (1998) 311-317.

[CN1] T.H. Chen, D. Nadler. Affine Matsuki correspondence for sheaves, preprint.

[CN2] T.H. Chen, D. Nadler. Real and symmetric Springer theory, preprint.

[K] P. B. Kronheimer. Instantons and the geometry of the nilpotent variety, Differ. Geom. 32 (1990), p. 473-490.

[LY] G. Lusztig, Z. Yun. $\mathbb{Z}/m$-graded Lie algebras and perverse sheaves, II, Represent. Theory 21 (2017), 322-353.

[MV] A. Malkin, V. Ostrik, and M. Vybornov. The minimal degeneration singularities in the affine Grassmannians, Duke Math. J. Volume 126, Number 2 (2005), 233-249.

[N1] D. Nadler. Matsuki correspondence for the affine Grassmannian, Duke Math. J. 124 (2004), no. 3, 421-457.

[N2] D. Nadler. Perverse Sheaves on Real Loop Grassmannians, Invent. Math. 159 (2005), no. 1, 1-73.

[OV] A.L. Onishchik, E.B. Vinberg (Eds.) Lie Groups and Lie Algebras III: Structure of Lie Groups and Lie Algebras, Encyclopaedia of Mathematical Sciences, Springer, 1994.

[S] J. Sekiguchi. Remarks on nilpotent orbits of a symmetric pair, J. Math. Soc. Japan Volume 39, Number 1 (1987), 127-138.

[SV] K. Vilonen, W. Schmid. On the geometry of nilpotent orbits, Asian J. Math. Vol 3, (1999), 233-274.

[V] M. Vergne. Instantons et correspondance de Kostant-Sekiguchi, C.R. Acad. Sci. Paris., 320 (1995), 901-906.

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