A study for system subjected to random shocks with accumulated damage

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Abstract. System could be continuously operating over an indefinitely long operation cycle, where each operation causes a random amount of damage to the system, and these damages are accumulated. We also propose a maintenance policy to optimize the expected cost rate for an infinite time span. Some useful properties and result discussions are presented, which indicate that the optimal maintenance policy is to perform preventive maintenance only depend on the level of accumulated damage, it is unnecessary to depend on the number of operation. Several special cases of such a maintenance policy are also presented and discussed.

1. Introduction
Maintenance policies for systems that are subject to stochastic failures have been treated extensively in literature. A comprehensive set of models review and discussion on the maintenance policies are included in Nakagawa (2005, 2007). The main aim of optimal maintenance policies is to provide optimum system reliability, availability and safety performance at the lowest possible maintenance cost. Maintenance can also be further divided into several categories according to the degree to which the operating conditions of an item are restored by maintenance.
A perfect maintenance means a maintenance action that restores system operating conditions to as good as new. That is, upon perfect maintenance, a system has the same operating conditions as a new one. A complete overhaul of an engine with a broken connecting rod is an example of perfect maintenance.
Generally, the replacement of a failed system with a new one is a perfect maintenance or perfect repair. A minimal maintenance means that after it is conducted, the system’s operating state is as bad as old. Changing a flat tire on a car, or changing a flat or broken fan belt on an engine are examples of minimal maintenance, as the overall state of the operating condition is essentially unchanged.
An imperfect maintenance is an action that does not make a system like new, but less. It is usually assumed that imperfect maintenance restores the system operating state to somewhere between as good as new and as bad as old.
Clearly, imperfect maintenance is a general maintenance that can include two extreme cases, namely, minimal and perfect. An engine tune-up is an example of imperfect maintenance, as a tune-up may not make an engine as good as new, but its performance would be greatly improved. For more details about the classification of maintenance degree, please refer to Pham and Wang (1996). Most optimum maintenance problems are based on a continuous time process where a system is operating.
However, in failure studies, the time to item failure is often measured by the number of operational (cycles) to failure, and therefore, operating a system in a discrete time process might be more appropriate. Also, cumulative damage models play an important role in reliability and maintenance theory. These models are considered as a sequence of shocks which occur randomly in time as an event in accordance with a stochastic process and give some amount of damage to a system.

A discrete-time order replacement model was first considered by Kaio & Osaki (1979). Later, they analyzed discrete-time ordering policies in Kaio & Osaki (1981) by taking account of minimal repair. Dohi et al. (2004) treated a generalized model with the similar complex cost structure to Dohi et al. (2004).

The damage suffered for the system is accumulated to the current damage level and weakens the system gradually. The system fails when the total damage exceeds a failure level. A variety of maintenance models subjected to shocks were studied extensively in Wortman et al. (1994), Sheu et al. (1996), Sheu (1998) and Sheu et al. (2002). The reliability properties and optimal maintenance policies for various damage models were also summarized sufficiently in Nakagawa (2007).

Besides, another application of the cumulative process in the maintenance related problems is the cumulative repair-cost limit policy, which concept uses the information of all repair costs to decide whether a system is repaired or replaced. For more details about the cumulative repair-cost limit policy on the maintenance model, please refer to Chien et al. (2009, 2010).

It is important to avoid failures of a complex system during the actual operation when such an event is costly or dangerous. In this paper, we consider a system continuously operating over an indefinitely long operational cycle. Also, the cumulative damage model is applied. A maintenance policy, where the timing to perform PM is depends on the operational number as well as the accumulated damage level, is determined and investigated. The expected cost rate for an infinite time span is adopted as criterion of optimality.

2. Model Development

Consider a system operating over the time with an indefinitely long operation cycle \( n \) \((n = 1, 2, \cdots)\) and each operation causes a random amount of damage to the system. These damages are accumulated to the current damage to the system.

2.1. CM and PM

A system fails when the total damage exceeds failure level, then a corrective maintenance (CM) is immediately conducted. Preventive maintenance (PM) action should be performed to prevent such a failure. The maintenance policy that considered and investigated for such a system is that the PM is carried out when the accumulated damage exceeds a pre-specified PM level \( \delta \) (but less than the failure level), or it is performed at the completion of \( N \)th \((N = 1, 2, \cdots)\) operation after the system installation, whichever comes first.

That is, the system undergoes CM when the total damage exceeds a failure level \( \zeta \), and undergoes PM at damage \( \delta \) \((0 \leq \delta \leq \zeta)\) or at operation \( N \) \((N = 1, 2, \cdots)\), whichever occurs first.

In addition to the CM and PM, regular maintenance (RM) actions also should be performed at every completion of operation in order to maintain the system for the next use.

2.2. Damage Amount

Let a random variable \( Y_j \) \((j = 1, 2, \cdots)\) be the amount of damage due to \( j \)th operation that has an distribution function

\[
G(y) = P(Y_j \leq y).
\]

These damages are accumulated to the current damage level.

Then, the total damage \( Z_j \) up to the \( j \)th operation has a distribution function
The above distribution function can be represented the $j$-fold Stieltjes of $G(w)$ with itself, and

$$G^{(j)}(w) = \Pr\left(\sum_{i=1}^{j} Y_i \leq w\right).$$

Under the maintenance policy that described above, it is obviously that the probability that a PM is performed at the completion of $N$th operation is $G^{(N)}(\delta)$. If $N = \infty$, then the probability that a PM is performed at the completion of $j$th operation is given by

$$\int_{0}^{\delta} G(\zeta - y) dG^{(j-1)}(y) - G^{(j)}(\delta)$$

and the probability that a CM is performed at $j$th operation is given by

$$G^{(j-1)}(\delta) - \int_{0}^{\delta} G(\zeta - y) dG^{(j-1)}(y).$$

### 2.3. Minimal Maintenance

CM and PM are assumed to be a perfect maintenance, while RM is assumed to be minimal maintenance. That is, a system is as good as new after CM or PM, and after RM the system is as bad as old.

Therefore, the perfect maintenance scheme for such a system is that it undergoes scheduled PM at operation $N$th, or unscheduled PM at damage $\delta$, or unscheduled CM at failure, whichever occurs first.

The time between successive perfect maintenance (i.e., CM or PM) can be regarded as a renewal cycle, and the behaviour in each renewal cycle repeats.

From the renewal reward theorem, the expected cost rate for an infinite time span is the expected total cost per renewal cycle, divided by expected renewal cycle length (Ross, 1970).

### 2.4. Cost Rate and expected total cost

To develop the expected cost rate for the maintenance policy in the model framework, the following costs are introduced.

Let $c_o$ be the fixed operating cost for each operation, $c_{rm}$ be the fixed cost for each RM, $C_{PM}$ be the fixed PM cost, and $C_{CM}$ be the fixed CM cost, where $C_{CM} > C_{PM} > c_{rm}$ is assumed.

For a renewal cycle, we have the expected total cost $ETC$

$$c_o \cdot \sum_{j=1}^{N} G^{(j)}(\delta) + c_{rm} \cdot \sum_{j=1}^{N} G^{(j)}(\delta) + C_{PM} \cdot \left\{ \sum_{j=1}^{N} P_{j}^{PM} + G^{(N)}(\delta) \right\} + C_{CM} \cdot \sum_{j=1}^{N} P_{j}^{CM}$$

and the expected renewal cycle length $ECL$

$$\sum_{j=1}^{N} j \cdot \left[ G^{(j-1)}(\delta) - G^{(j)}(\delta) \right] + N \cdot G^{(N)}(\delta).$$

The expected cost rate $CR(\delta, N)$ is given by

$$CR(\delta, N) = \frac{ETC}{ECL}.$$
3. Optimization

For further analytical study, we consider that the amount of damage due to each operation has an exponential distribution with a mean of $\mu$. That is,

$$G(y) = 1 - \exp\left(-\frac{y}{\mu}\right),$$

$$G^{(j)}(y) = \sum_{i=j}^{\infty} \frac{(y/\mu)^i}{i!} \cdot \exp\left(-\frac{y}{\mu}\right).$$

The expected total cost $ETC$ in (3) can be reduced to

$$(c_0 + c_m) \sum_{j=0}^{N-1} G^{(j)}(\delta) + (C_{PM} - c_m) + (C_{CM} - C_{PM}) \cdot \bar{G}(\xi - \delta) \cdot G^{(N)}(\delta).$$

The following theorem regarding the properties of the optimal pair $\left(\delta^*, N^*\right)$ that satisfies

$$\min_{\delta \in [\xi, N], N=1, 2, \ldots} CR(\delta, N) = CR(\delta^*, N^*).$$

The above result is presented based on the expected cost rate given in (5).

**Theorem 1.** For the cost rate $CR(\delta, N)$ given in (5) where $0 \leq \delta \leq \xi$ and $N = 1, 2, \ldots$. If

$$\frac{C_{CM} - C_{PM}}{C_{PM} - c_m} \leq \frac{\xi}{\mu},$$

then

$$\left(\delta^*, N^*\right) = (\xi, \infty).$$

Otherwise, $0 < \delta^* < \xi$ and $N^* = \infty$.

The theorem indicates that the problem of determining the optimal pair $\left(\delta^*, N^*\right)$ to minimize the expected cost rate $CR(\delta, N)$ is equivalent to only determining the $\delta^*$ to minimize $CR(\delta, \infty)$.

4. Conclusion

A preventive maintenance policy for a continuously running system over an indefinitely long operation cycle are proposed in this study. Each operation causes a random amount of damage to the system, and these damages are accumulated to the current damage to a system, which fails when the total damage exceeds a pre-specified failure level, and then corrective maintenance should be conducted.

To prevent such a failure, a PM action should be carried out at suitable time. In the maintenance policy, the timing to perform PM depends on the operational number as well as the accumulated damage level.

Avoid failures of an operating system during the actual operation when such an event is costly or dangerous is importance. In this study, we have considered a continuously operating system over an indefinitely long operational cycle. The cumulative damage model is also applied.

A maintenance policy, where the timing to perform PM is depends on the operational number as well as the accumulated damage level, is determined and investigated in this research. The expected cost
rate for an infinite time span is adopted as criterion of optimality. Some useful properties and result discussions are also presented, which indicate that the optimal maintenance policy is to perform preventive maintenance only depend on the level of accumulated damage. And, it is unnecessary to depend on the number of operation.

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