On the Cauchy problem of the Boltzmann equation with a very soft potential

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Abstract

The Cauchy problem for the Boltzmann equation with soft potential is studied by Ukai in [9] when \( \gamma \in [0,1) \) in the framework of small perturbation of an equilibrium state. By extending the estimate on linearized collision operator \( L \) in [1] from \( \gamma \in [0,1) \) to \( \gamma \in [0,d) \), we obtain a global existence result for \( \gamma \in [0,d) \). For soft potential, the spectrum structure of \( \hat{B}(y) \) couldn’t give spectral gap, so we use the method of integration by parts and consider a weighted velocity space in order to obtain algebraic decay in time.

Keywords: Boltzmann equation, linearized collision operator, global existence, soft potential, interpolation.

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1 Introduction

Consider the Cauchy problem of the Boltzmann equation in $d$-dimension:

$$f_t + \xi \cdot \nabla_x f = Q(f, f), \quad f|_{t=0} = f_0. \tag{1.1}$$

Here $f = f(x, \xi, t)$ is the distribution function of particle at position $x \in \mathbb{R}^d$ with velocity $\xi \in \mathbb{R}^d$ at time $t \geq 0$, $Q(f, g)$ is the bilinear collision operator defined by

$$Q(f, g) := \int_{\mathbb{R}^d} \int_{S^{d-1}} (f'g' + f'g - f'g + fg') q(\xi - \xi, \theta) \, d\omega d\xi, \tag{1.2}$$

where

$$f' = f(x, \xi', t), \quad f' = f(x, \xi', t), \quad f = f(x, \xi, t), \quad f = f(x, \xi, t),$$

and similarly for $g$, and

$$\xi' = \xi - ((\xi - \xi) \cdot \omega) \omega, \quad \xi' = \xi + ((\xi - \xi) \cdot \omega) \omega, \tag{1.3}$$

where $\omega \in S^{d-1}$. Here ($\xi, \xi$) are the velocities of two gas particles before collision and ($\xi', \xi'$) are the velocities after collision, while $\theta \in [0, \pi]$ is the angle between the first variable of $q$ and $\omega$. For example, if $q = q(\xi - \xi, \theta)$, then

$$\cos \theta = \frac{(\xi - \xi) \cdot \omega}{|\xi - \xi|}. \tag{1.4}$$

The function $q$ is the collision kernel determined by the interaction potential model between two colliding particles. In this paper, we will only consider the Grad’s angular cut-off assumption. That is,

$$q(\xi - \xi, \theta) \leq q_0 |\xi - \xi|^{-\gamma} |\cos \theta|, \tag{1.5}$$

where $q_0 > 0$ is a constant and also $q$ is almost everywhere positive. It’s called hard potential if $\gamma > 0$ and soft potential if $\gamma < 0$.

We are looking for a solution $f$ near the equilibrium, that is the global Maxwellian $M$. Suppose the solution has the form

$$f = M + M^2 g. \tag{1.6}$$

By a translation and scaling of the velocity variable $\xi$, without loss of generality, we will take

$$M = \frac{1}{(2\pi)^{d/2}} \exp\left(-\frac{||\xi||^2}{2}\right). \tag{1.7}$$

Substitute (1.6) into (1.1), and notice a global Maxwellian is collision invariant, we have

$$g_t + \xi \cdot \nabla_x g = Lg + M^{-1/2}Q(M^{1/2}g, M^{1/2}g), \tag{1.8}$$

where $L$ is a linear operator defined by

$$Lg := M^{-1/2} \left(Q(M^{1/2}g, M) + Q(M, M^{1/2}g)\right).$$

Also we define

$$\Gamma(f, g) = M^{-1/2}Q(M^{1/2}f, M^{1/2}g).$$
We introduce a weighted normed space for finding the solution as follows. Define \( H^1(\mathbb{R}^d) \) to be the standard Sobolev space and
\[
\|f\|_{L^p_\beta} := \|(1+|\xi|)^\beta f\|_{L^p},
\]
\[
L^p_\beta(\mathbb{R}^d) := \{f: \mathbb{R}^d \rightarrow \mathbb{C} \mid \|f\|_{L^p_\beta} < \infty\}.
\]
Also we denote some multi-variable space.
\[
L^p_\beta(H^1) := L^p_\beta(\mathbb{R}^d; (H^1(\mathbb{R}^d))),
\]
\[
L^\infty_\alpha (L^p_\beta(H^1)) := L^\infty_\alpha (\mathbb{R}^d; L^p_\beta(\mathbb{R}^d; (H^1(\mathbb{R}^d)))).
\]
In addition, for any linear operator \( T \) acting on some normed space \( X \), we denote its resolvent set, spectrum and point spectrum respectively by
\[
\rho(T) := \{\lambda \in \mathbb{C} : \lambda I - T \text{ is bijective and } (\lambda I - T)^{-1} \text{ is continuous on } X\},
\]
\[
\sigma(T) := \mathbb{C} \setminus \rho(T),
\]
\[
\sigma_p(T) := \{\lambda \in \sigma(T) : \lambda \text{ is an eigenvalue of } T\}.
\]
Also we define the a half space in \( \mathbb{C} \) by
\[
\mathbb{C}_+ := \{\lambda \in \mathbb{C} : \text{Re} \lambda > 0\}.
\]

The goal of present paper is to find a solution to the Cauchy problem of equation (1.8) by applying semigroup theory to operator
\[
B = -\xi \cdot \nabla_x + L,
\]
where we regard \( B \) as an operator acting on \( L^p_\beta(H^1) \). The properties of \( L \) has been well studied by Caflisch in [1] and by Ukai and Asano in [9]. The linearized collision operator has expression \( L = -\nu + K \), where \( \nu \) is a function satisfying \( \nu(\xi) \sim (1 + |\xi|)^\gamma \) and \( K \) is an integral operator with kernel \( k \).

In the present paper, we extend the estimate of \( K \) given by Caflisch, Ukai and Asano from \( \gamma \in [0,1) \) to \( \gamma \in [0,d) \). They are:
\[
\|Kf\|_{L^p_{\beta+\gamma+2}} \leq C_{\gamma,d,q,p}\|f\|_{L^p_\beta},
\]
\[
\|Kf\|_{L^p_{\beta+\gamma+1}} \leq C_{\gamma,d,q,p}\|f\|_{L^p_\beta},
\]
where \( \frac{1}{q_0} = \frac{\theta}{\infty} + \frac{1-\theta}{p} \), \( \frac{1}{p_0} = \frac{\theta}{\infty} + \frac{1-\theta}{1} \). We found that these estimates are still valid for \( \gamma \in [0,d) \), by using a slightly different technique. Essentially, the generalized property is given by \( k \) as
\[
\int_{\mathbb{R}^d} (1 + |\xi|)^{\beta}|k(\xi,\xi_*)|^p d\xi_* \leq C_{\gamma,d,q} \frac{1}{(1 + |\xi|)^{\beta+p(\gamma+1)+1}},
\]
which is valid for \( p \in [1,\max(\frac{\beta}{\gamma+2},\frac{\beta}{\gamma})] \), \( \beta \in \mathbb{R} \), and \( \gamma \in [0,d] \). Because we only need the term \( |\cdot - \xi|^{-\gamma} \) is locally integrable, so we just require \( \gamma \in [0,d) \).

With the good properties of \( K \), we can establish the estimate on semigroup \( e^tB \). To do this, we will define \( P : L^2 \rightarrow \text{Ker}L \) to be the orthogonal projection from \( L^2 \) to the kernel of \( L \) and
\[
\tilde{B}(y) := -2\pi iy \cdot \xi + L,
\]
\[
\tilde{B}_0(y) := -2\pi iy \cdot \xi + L - P
\]
as two linear unbounded operators acting on \( L^2_\beta(\mathbb{R}^d) \). In order to obtain the estimate of semigroup \( e^{\tilde{B}(y)} \), we need to analyze the behavior of the resolvent \( (\lambda I - \tilde{B}(y))^{-1} \) of \( \tilde{B}(y) \). When \( |y| \) is large, the resolvent of \( \tilde{B}(y) \) has a good
property obtained in [9]. While \((\lambda I - \hat{B}(y))^{-1}\) has a singular behavior near \(|y| = 0\). Using resolvent identity, we have

\[
(\lambda I - \hat{B}(y))^{-1} = (\lambda I - \hat{B}_0(y))^{-1} + (\lambda I - \hat{B}_0(y))^{-1} P(I - P(\lambda I - \hat{B}_0(y))^{-1} P)^{-1} P(\lambda I - \hat{B}_0(y))^{-1}.
\]

So the singularity near \(y = 0\) comes from the operator \((I - P(\lambda I - \hat{B}_0(y))^{-1} P)^{-1}\) acting on \(\text{Ker}L\). In this paper, we will follow Ukai and Asano’s idea in [9] but use a different approach to obtain the eigenvalues of \(P(\lambda I - \hat{B}_0(y))^{-1} P\) and its asymptotic behavior. The method in this paper is similar to [7]. Write \(r = |y|\). We will find in section 4 that the singular point of \((I - P(\lambda I - \hat{B}_0(y))^{-1} P)^{-1}\) near \(y = 0\) is \(\lambda(r) = \sigma_j(r) + i\tau_j(r) \in C^\infty(B(0,r_2))\) for some small \(r_2\) and their asymptotic behavior near \(r = 0\) are

\[
\begin{align*}
\sigma_j(r) &= -\sigma_j^{(2)} r^2 + O(r^3), \\
\tau_j(r) &= \tau_j^{(1)} r + O(r^3),
\end{align*}
\]

as \(r \to 0\), where \(\sigma_j^{(1)} \in \mathbb{R}\), \(\sigma_j^{(2)} < 0\). They have the same asymptotic behaviors as the eigenvalues of \((\lambda I - \hat{B}(y))^{-1}\) given in [3, 10]. So it turns out that the singular behavior of \((I - P(\lambda I - \hat{B}_0(y))^{-1} P)^{-1}\) is exactly the same as \((\lambda I - \hat{B}(y))^{-1}\) near \(y = 0\).

With the well-studied properties on operator \((\lambda I - \hat{B}(y))^{-1}\), one can get the estimate on semigroup \(e^{t \hat{B}(y)}\) by using the inversion formula of semigroup. For the soft potential, we can’t use the spectral gap to get a good decay on time \(t\) as in the hard potential case. However, we can use the method of integration by parts to construct an algebraic decay on time \(t\) with a stronger assumption on initial data \(f_0\). So we will use the weighted normed space \(L^2_\beta\) on velocity to find our solution. Combining the inverse Fourier transform formula as well as the Duhamel formula, we can get a good boundedness on semigroup \(e^{tB}\).

Once we have the estimate on semigroup \(e^{tB}\), we can get our global existence result.

**Theorem 1.1.** Assume the cross-section \(q\) satisfies the angular cut-off assumption (1.5). Assume \(d \geq 3\), \(\gamma \in (0,d), \alpha \in [\frac{1}{d}, \min(\frac{d}{2}, 1)]\). \(t > \frac{d}{2}, \beta > \frac{d}{2} - \gamma + \alpha \gamma\). There exists constants \(A_0, A_1\) such that if the initial data \(f_0 \in L^\infty_{\beta + \alpha \gamma}(H^1) \cap L^2_\alpha(L^1)\) satisfies

\[
\|f_0\|_{L^\infty_{\beta + \alpha \gamma}(H^1)} + \|f_0\|_{L^2_\alpha(L^1)} \leq A_0.
\]

Let \(X = \{f \in L^\infty_\alpha(L^\infty_\beta(H^1)) : \|f\|_{L^\infty_\alpha(L^\infty_\beta(H^1))} \leq A_1\}\). Then the Cauchy problem to Boltzmann equation

\[
\begin{cases}
ft + \xi \cdot \nabla f = Q(f,f), \\
f|_{t=0} = f_0.
\end{cases}
\]

possesses a unique solution \(f = f(t) \in X \cap C^0([0,\infty); L^\infty_\beta(H^1))\) and

\[
\|f\|_{L^\infty_{\beta}(L^\infty_\beta(H^1))} \leq C_{\nu,\gamma,\alpha,\beta,d} \left(\|f_0\|_{L^\infty_{\beta + \alpha \gamma}(H^1)} + \|f_0\|_{L^2_\alpha(L^1)}\right).
\]

The uniqueness is taken in the sense that \(f \in X\).

Finally, we present the main strategy of analysis in this paper. In section 2, we prove the estimate on \(K\) when \(\gamma \in (0,d)\). Section 3 presents some boundedness and invertibility result on resolvents of \(\hat{B}(y), \hat{B}_0(y)\). In section 4, we will analyze the singular behavior of \((\lambda I - \hat{B}(y))^{-1}\) near \(y = 0\). Section 4 and 5 give the main estimate on our semigroup \(e^{t\hat{B}(y)}\) and \(e^{tB}\) as well as the global existence result. In appendix, we list some basic properties on linearized collision operator \(L\). Also we extend two useful results on the Hilbert-Schmidt operator and interpolation theory in order to make our arguments valid.
2 Properties of the Linearized Collision Operator

In this section, we should firstly list some properties of the linearized collision operator $L$ and derive the new estimate on operator $K$. Suppose that the collision kernel satisfies the Grad’s angular cut-off assumption:

$$q(\xi - \xi^*, \theta) = |\xi - \xi^*|^{-\gamma}b(\cos \theta),$$

where $\gamma \in \mathbb{R}$, $|b(\cos \theta)| \leq b_0 |\cos \theta|$, $\int_{S^{d-1}} b(\cos \theta) \, d\omega = q_0$, with some constants $b_0, q_0 > 0$.

Define a sup norm and a Banach space by

$$\|f\|_{L^\infty_{\beta}} := \sup_{\xi \in \mathbb{R}^d} (1 + |\xi|)^{\beta}|f(\xi)|,$$

$$L^\infty_{\beta}(\mathbb{R}^d) : = \{ f : \mathbb{R}^d \to \mathbb{C} \mid \|f\|_{L^\infty_{\beta}} < \infty \}.$$

The space $L^\infty_{\beta}$ consists of functions $f$ having algebraic decay on velocity variable.

Denote $P_{\xi^*, -\xi}$ to be the hyperplane in $\mathbb{R}^d$ which is orthogonal to the vector $\xi^* - \xi$ and contains the origin. Denote

$$a = \frac{\xi + \xi^*}{2} - \left( \frac{\xi + \xi^*}{2}, \frac{\xi^* - \xi}{|\xi^* - \xi|} \right) \frac{\xi^* - \xi}{|\xi^* - \xi|},$$

$$b = \left( \frac{\xi + \xi^*}{2}, \frac{\xi^* - \xi}{|\xi^* - \xi|} \right) \frac{\xi^* - \xi}{|\xi^* - \xi|},$$

where the vector $a$ is the projection of $\frac{\xi + \xi^*}{2}$ onto the hyperplane $P_{\xi^*, -\xi}$, while $b$ is its projection onto the direction $\xi^* - \xi$.

The following theorem shows the basic properties of linearized collision operator $L$. Here I will only give proof of estimate (2.3), (2.4) and (2.5), since the other estimate has been well studied in [1] and I will put them in appendix for the sake of completeness.

**Theorem 2.1. (Properties of $L$). Assume $\gamma \in [0, d)$, The linear operator $L$ has expression**

$$L f = K f - \nu f,$$

where $K$ is a linear continuous operator on $L^2(\mathbb{R}^d)$, and $\nu(\xi) = \int_{\mathbb{R}^d} \int_{S^{d-1}} M^{1/2}(\xi^*) q(\xi - \xi^*, \theta) d\omega d\xi^*$ is a real positive function. $K$ and $\nu$ satisfy the following properties.

1. For $\gamma \in [0, d)$, there exist constants $\nu_0, \nu_1 > 0$ depending on $\gamma, q, d$ s.t.

   $$\nu_0 (1 + |\xi|)^{-\gamma} \leq \nu(\xi) \leq \nu_1 (1 + |\xi|)^{-\gamma}.$$

2. The operator $K$ is an integral operator with kernel $k(\xi, \xi^*)$. That is

   $$K f(\xi) = \int_{\mathbb{R}^d} k(\xi, \xi^*) f(\xi^*) \, d\xi^*.$$

The kernel $k$ can be divided as $k(\xi, \xi^*) = k_1(\xi, \xi^*) + k_2(\xi, \xi^*)$, with

$$k_1(\xi, \xi^*) = \frac{1}{|\xi^* - \xi|^{d-1}} \int_{P_{\xi^*, -\xi}} (2\pi)^{-d/2} e^{-\frac{|x|^2}{2}} q(x - a + \xi^* - \xi, \theta) \, dx \exp \left( - \frac{|b|^2}{2} - \frac{|\xi^* - \xi|^2}{8} \right),$$

$$k_2(\xi, \xi^*) = - \int_{S^{d-1}} M^{1/2}(\xi^*) M^{1/2}(\xi) q(\xi^* - \xi, \theta) \, d\omega.$$

Here $k_1, k_2$ are symmetric functions. For $0 < \varepsilon < 1$, they satisfy:

$$|k_1(\xi, \xi^*)| \leq C_{\gamma, \varepsilon, d, q} |\xi^* - \xi|^{d-2(1 + |\xi^*| + |\xi|)\gamma} \exp \left( - (1 - \varepsilon) \left( \frac{|b|^2}{2} + \frac{|\xi^* - \xi|^2}{8} \right) \right),$$

$$|k_2(\xi, \xi^*)| \leq C_{\gamma, \varepsilon, d, q} |\xi^* - \xi|^{d-2(1 + |\xi^*| + |\xi|)\gamma} \exp \left( - (1 - \varepsilon) \left( \frac{|b|^2}{2} + \frac{|\xi^* - \xi|^2}{8} \right) \right).$$


\[ |k_2(\xi, \xi_*)| \leq C_{\gamma, \varepsilon, d, q} \frac{1}{|\xi_* - \xi|^\gamma (1 + |\xi| + |\xi_*|)^\gamma + 1} \exp \left( -(1 - \varepsilon) \frac{|\xi_*|^2 + |\xi|^2}{4} \right). \] (2.3)

Consequently, for \( p \in [1, \max\{ \frac{d}{d-2}, \frac{d}{\gamma} \}] \), \( \beta \in \mathbb{R} \), we have

\[ |k(\xi, \xi_*)| \leq C_{\gamma, \varepsilon, d, q} \left( \frac{1}{|\xi_* - \xi|^{d-2} + |\xi_* - \xi|^{\gamma}} \right) \exp \left( -(1 - \varepsilon) \frac{|b|^2 + |\xi_* - \xi|^2}{2} \right) \frac{1}{(1 + |\xi| + |\xi_*|)^{\gamma + 1}}, \] (2.4)

\[ \int_{\mathbb{R}^d} (1 + |\xi_*|)^\beta |k(\xi, \xi_*)|^p \, d\xi_* \leq C_{\gamma, \varepsilon, d, q} \frac{1}{(1 + |\xi|)^{\beta + p(\gamma + 1) + 1}}. \] (2.5)

**Proof.** 1. The expression of \( K \) can be found in appendix. For any \( \varepsilon \in (0, 1) \),

\[
|k_2(\xi, \xi_*)| = \left| \int_{S^{d-1}} M^{1/2}(\xi_*) M^{1/2}(\xi) q(\xi_* - \xi, \theta) \, d\omega \right|
= \left( 2\pi \right)^{-d/2} \exp\left( - \frac{|\xi_*|^2 + |\xi|^2}{4} \right)|\xi_* - \xi|^{-\gamma} \int_{S^{d-1}} b(\cos \theta) \, d\omega
= C_{d, q, 0, \varepsilon} \exp\left( -(1 - \varepsilon) \frac{|\xi_*|^2 + |\xi|^2}{4} \right)|\xi_* - \xi|^{-\gamma} (1 + |\xi| + |\xi_*|)^{-\gamma - 1}
\times \sup_{\xi, \xi_* \in \mathbb{R}^d} \exp\left( -\varepsilon (|\xi_*|^2 + |\xi|^2) \right)(1 + |\xi| + |\xi_*|)^{\gamma + 1}
= C_{d, \gamma, q, 0, \varepsilon} \exp\left( -(1 - \varepsilon) \frac{|\xi_*|^2 + |\xi|^2}{4} \right)|\xi_* - \xi|^{-\gamma} (1 + |\xi| + |\xi_*|)^{-\gamma - 1}.
\]

This proves (2.3).

2. Once we get the estimate (2.2) and (2.3), by using a trivial inequality that

\[
\frac{|\xi_*|^2 + |\xi|^2}{4} - \frac{|\xi_* - \xi|^2}{8} = \frac{|\xi_* + \xi|^2}{8} \geq \frac{|b|^2}{2},
\]

we will have

\[
|k(\xi, \xi_*)| \leq C_{\gamma, \varepsilon, d, q} \left( \frac{1}{|\xi_* - \xi|^{d-2} + |\xi_* - \xi|^{\gamma}} \right) \exp\left( -(1 - \varepsilon) \frac{|b|^2 + |\xi_* - \xi|^2}{8} \right) \frac{1}{(1 + |\xi| + |\xi_*|)^{\gamma + 1}}.
\]

Therefore, by (7.8) in lemma 7.6, for \( p \in [1, \max\{ \frac{d}{d-2}, \frac{d}{\gamma} \}] \),

\[
\int_{\mathbb{R}^d} (1 + |\xi_*|)^\beta |k(\xi, \xi_*)|^p \, d\xi_* \leq C_{\gamma, \varepsilon, d, q} \int_{\mathbb{R}^d} \left( \frac{1}{|\xi_* - \xi|^{d-2} + |\xi_* - \xi|^{\gamma}} \right) \frac{(1 + |\xi_*|)^\beta}{(1 + |\xi| + |\xi_*|)^{p(\gamma + 1) + 1}}
\times \exp\left( -p(1 - \varepsilon) \frac{|b|^2 + |\xi_* - \xi|^2}{8} \right) \, d\xi_*
\leq C_{\gamma, \varepsilon, d, q} \frac{1}{(1 + |\xi|)^{-\beta + p(\gamma + 1) + 1}}.
\]

This completes the proof. \( \square \)

**Remark 2.2.** Similar estimates are valid for \( \gamma \in [-1, 0] \), which is the hard potential case. The only difference is that for hard potential, the term \( \frac{b}{|\xi - \xi_*|^2} \) in (2.3) is not singular any more.

**Theorem 2.3.** (Properties of \( K \)). Let \( d \geq 3 \) be the dimension, \( \gamma \in [0, d) \), \( \beta \in \mathbb{R} \). Then the followings are valid.

1. For \( p \in [1, \infty] \),

\[
\| Kf \|_{L^p_{\beta+2}} \leq C_{\gamma, d, q, p} \| f \|_{L^p_{\beta}}.
\] (2.6)
(2) The linear operator $K : L^2_\alpha \rightarrow L^2_{\beta+\gamma+2}$ is compact for $\alpha > \beta$.

(3) Let $p > \max\left(\frac{d}{\alpha-\gamma}, \frac{2}{\beta-\gamma}\right)$. Then

$$\|Kf\|_{L^p_{\beta+\gamma+2-1/p}} \leq C_{\gamma, d, q} \|f\|_{L^p_\alpha}. \tag{2.7}$$

(4) Pick $p_0 > \max\left(\frac{d}{3-\gamma}, \frac{2}{\beta-\gamma}\right)$. For $\theta \in (0, 1)$, $K$ is a linear bounded operator from $L^{p_0}$ to $L^{q_0}$ with estimate

$$\|Kf\|_{L^{p_0}_{\beta+\gamma+1}} \leq C_{\gamma, d, q, p_0, \theta} \|f\|_{L^{p_0}_\beta}, \tag{2.8}$$

where

$$\frac{1}{q_0} = \frac{\theta}{\infty} + \frac{1 - \theta}{1}, \quad \frac{1}{p_0} = \frac{\theta}{\beta} + \frac{1 - \theta}{1}. \tag{2.9}$$

Consequently, if $f$ lies in the space $L^p_\beta(H^1)$ or $L^{p_0}_\beta(H^1)$, then we have the following estimate respectively.

$$\|Kf\|_{L^p_{\beta+\gamma+2-1/p}(H^1)} \leq C_{\gamma, d, q} \|f\|_{L^p_\beta(H^1)},$$

$$\|Kf\|_{L^{p_0}_{\beta+\gamma+1}(H^1)} \leq C_{\gamma, d, q, p_0, \theta} \|f\|_{L^{p_0}_\beta(H^1)}.$$

**Remark 2.4.** All the estimates are true on $\int_{\mathbb{R}^d} |k(\xi, \xi_*)|f(\xi_*) \, d\xi_*$ instead of $\int_{\mathbb{R}^d} k(\xi, \xi_*)f(\xi_*) \, d\xi_*$ as well. This property is useful in some special situation.

**Proof.** 1. Let $r \in \mathbb{R}$ to be arbitrary, $\beta \in \mathbb{R}$, $p \in [1, \infty)$. Applying Hölder’s inequality, (2.5) and noticing $1/p + 1/p' = 1$, we have

$$|Kf(\xi)| \leq \int_{\mathbb{R}^d} |k(\xi, \xi_*)||(1 + |\xi_*|)^{-\theta}(1 + |\xi_*|)^{\gamma}f(\xi_*)| \, d\xi_* \leq \left(\int_{\mathbb{R}^d} |k(\xi, \xi_*)||(1 + |\xi_*|)^{-\theta}d\xi_* \right)^{1/p'} \left(\int_{\mathbb{R}^d} |k(\xi, \xi_*)||(1 + |\xi_*|)^{\theta}f(\xi_*)|^p \, d\xi_* \right)^{1/p} = \frac{C_{\gamma, d, q, r, p}}{(1+|\xi|)^{r+(\gamma+2)(p-1)/p}} \left(\int_{\mathbb{R}^d} |k(\xi, \xi_*)||(1 + |\xi_*|)^{\theta}f(\xi_*)|^p \, d\xi_* \right)^{1/p}. \tag{2.8}$$

Thus using (2.5) again,

$$\|Kf\|_{L^p_\beta(\mathbb{R}^d)} \leq C_{\gamma, d, q, r, p} \int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^d} |(1 + |\xi_*|)^{\theta}f(\xi_*)|^p \, d\xi_* \right)^{1/p} \leq C_{\gamma, d, q, r, p} \|f\|_{L^p_{\beta+\gamma+2}}. \tag{2.9}$$

If $p = \infty$, then

$$\|Kf\|_{L^\infty_\beta} \leq \sup_{\xi \in \mathbb{R}^d} (1 + |\xi|)^{\beta} \int_{\mathbb{R}^d} (1 + |\xi_*|)^{\gamma+2-\beta} |k(\xi, \xi_*)| \, d\xi_* \|f\|_{L^\infty_{\beta+\gamma+2-2}} \leq C_{d, q, p} \|f\|_{L^\infty_{\beta+\gamma+2-2}}. \tag{2.10}$$

2. Now we prove that $K : L^2_\alpha \rightarrow L^2_\beta$ is compact for $\alpha > \beta - \gamma - 2$. Similar to the estimates in step 1, for $R > 1/2 > \varepsilon$, $r \geq 0$, we have

$$|K(\chi_{|\xi_*| \leq \varepsilon} \chi_{|\xi| \geq R}f)(\xi)| \leq \frac{C_{\gamma, d, q, r, p}}{(1 + |\xi|)^{r+(\gamma+2)(p-1)/p}} \left(\int_{|\xi_*| \geq R \atop |\xi - \xi_*| \leq \varepsilon} |k(\xi, \xi_*)||(1 + |\xi_*|)^{\theta}f(\xi_*)|^p \, d\xi_* \right)^{1/p}. \tag{2.11}$$

We claim that $K$ is the limit of compact operators $K(\chi_{|\xi_*| \leq \varepsilon} \chi_{|\xi| \geq R})$ under the operator norm $\|\cdot\|_{L^p_\alpha \rightarrow L^p_\beta}$. The term $\chi_{|\xi_*| \leq \varepsilon}$ is used to eliminate the singularity of $k(\xi, \xi_*)$ near $\xi = \xi_*$. Pick $r \in \mathbb{R}$ and notice that $|\xi - \xi_*| \leq \varepsilon < 1/2$ implies $\frac{1}{2}(1 + |\xi_*|) \leq (1 + |\xi|) \leq \frac{3}{2}(1 + |\xi_*|)$. Thus we have

$$\|K(\chi_{|\xi_*| \leq \varepsilon} \chi_{|\xi| \leq R}f)(\xi) - K(\chi_{|\xi| \leq R}f)(\xi)\|_{L^p_\beta} \tag{2.12}$$
\[ \leq C_{\gamma,d,q,r,p} \int_{|\xi| \leq R} \left( \int_{\mathbb{R}^d} (1 + |\xi|^p)^{\beta - pr - (\gamma + 2)(p-1)} |k(\xi, \xi^*)| \chi_{|\xi| \leq \varepsilon} d\xi \right) (1 + |\xi|) p \| f(\xi^*) \| d\xi^* \]
\[ \leq C_{\gamma,d,q,r,p,\beta} \int_{|\xi| \leq R} \left( \int_{|\xi - \xi^*| \leq \varepsilon} \left( \frac{1}{|\xi - \xi^*|} + \frac{1}{|\xi - \xi^*|^{d-2}} \right) d\xi \right) (1 + |\xi|)^{\beta - (\gamma + 2)(p+1)} |f(\xi^*)| d\xi^* \]
\[ \leq C_{\gamma,d,q,r,p,\beta} \int_{|\xi| \leq \varepsilon} \left( \frac{1}{|\xi|^\gamma} + \frac{1}{|\xi|^{d-2}} \right) d\xi \min\{1, (1 + R)^{\beta - (\gamma + 2)p + 1 - p\alpha}\} \| f \|_{L_\beta^p}^p \to 0, \]
as \varepsilon \to 0, for any fixed \( R > 1/2 \). On the other hand,
\[ \| K(\chi_{|x| \leq R} f)(\xi) - K f(\xi) \|_{L_\beta^p}^p \]
\[ \leq C_{\gamma,d,q,r,p} \int_{\mathbb{R}^d} \left( \int_{\mathbb{R}^d} (1 + |\xi|^p)^{\beta - pr - (\gamma + 2)(p-1)} |k(\xi, \xi^*)| d\xi \right) (1 + |\xi|) p \| f(\xi^*) \| d\xi^* \]
\[ \leq C_{\gamma,d,q,r,p,\beta} \int_{\mathbb{R}^d} (1 + |\xi|^p)^{\beta - (\gamma + 2)p} |f(\xi^*)| d\xi^* \]
\[ = C_{\gamma,d,q,r,p}(1 + R)^{\beta - (\gamma + 2)p - p\alpha} \| f \|_{L_\beta^p}^p, \]
provided \( \beta - (\gamma + 2) - \alpha \leq 0 \). Thus
\[ \| K(\chi_{|x| \geq R} f)(\xi) \|_{L_\beta^p}^p \leq C_{\gamma,d,q,r,p}(1 + R)^{\beta - (\gamma + 2) - \alpha} \| f \|_{L_\beta^p}^p \to 0, \]
if \( \beta - (\gamma + 2) - \alpha < 0 \). This proves that \( K \) can be approximated by \( K(\chi_{|x| \geq R} f)(\xi) \) under the operator norm \( \| \cdot \|_{L_\beta^p} \to L_\beta^p \).

Let \( p = 2 \), it remains to show that \( K \chi_{|x| \geq R} \) is compact in \( L^2(L_\alpha^2, L_\beta^3) \) when \( \beta - (\gamma + 2) - \alpha < 0 \). By 7.2 in appendix, it suffices to prove that \( K \chi_{|x| \geq R} \) is a Hilbert-Schmidt operator. That is to show that \( k(\xi, \xi^*) \chi_{|\xi - \xi^*| \geq \varepsilon} \chi_{|\xi| \leq R} \in L^2(\mathbb{R}^d \times \mathbb{R}^d, (1 + |\xi|^p) \gamma d\xi \otimes (1 + |\xi|)^{2\beta} d\xi) \).
\[
\int_{\mathbb{R}^d} \int_{\mathbb{R}^d} |k(\xi, \xi^*)|^2 \chi_{|\xi - \xi^*| \geq \varepsilon} \chi_{|\xi| \leq R} (1 + |\xi|^p) \gamma d\xi \otimes (1 + |\xi|)^{2\beta} d\xi \]
\[
\leq C_{\gamma,d,q} \int_{|\xi| \leq R} \int_{|\xi - \xi^*| \geq \varepsilon} \left( \frac{1}{|\xi - \xi^*|^\gamma} + \frac{1}{|\xi - \xi^*|^{d-2}} \right) \exp \left( -2(1 - \varepsilon) \left( |b|^2 / 2 + |\xi^*| - |\xi|^2 / 4 \right) \right) d\xi d\xi^* \]
\[
\leq C_{\gamma,d,q} \left( \frac{1}{\varepsilon^\gamma} + \frac{1}{\varepsilon^{d-2}} \right) \int_{|\xi| \leq R} (1 + |\xi|)^{2\gamma + 2\beta - 2\gamma - 3} d\xi^* \]
\[
< \infty. \]
This shows that \( K \chi_{|x| \geq R} \) is compact for \( \beta > \alpha \), and thus \( K : L_\alpha^2 \to L_\beta^{2+\gamma+2} \) is compact. 3. For \( f \in L_\beta^p, \beta \in \mathbb{R}, p \in (1, \infty), \)
\[ |Kf(\xi)| \leq \int_{\mathbb{R}^d} |k(\xi, \xi^*)| \| f(\xi^*) \| d\xi^* \]
\[ \leq \left( \int_{\mathbb{R}^d} |k(\xi, \xi^*)|^{p'} (1 + |\xi|^p) \gamma \right)^{1/p'} \left( \int_{\mathbb{R}^d} (1 + |\xi|^p) \| f(\xi^*) \|^{p'} d\xi^* \right)^{1/p'} \]
\[ \leq C_{\gamma,d,q} \| f \|_{L_\beta^p} \| f \|_{L_\beta^p}, \]
provided \( p' \gamma < d \) and \( (d-2)p' < d \), that is \( p > \frac{d}{d - 2} \) and \( p > \frac{d}{2} \). Thus for \( p \in (\max(\frac{d}{d - 2}, \frac{d}{2}), \infty) \), we have
\[ \| Kf \|_{L_\beta^{\infty,\gamma+1+p'}} \leq C_{\gamma,d,q,p} \| f \|_{L_\beta^p}, \quad (2.10) \]
4. To prove (4), we only need a weaker result then (2.10). That is for \( p \in (\max(\frac{d}{\beta - \gamma}, \frac{d}{2}), \infty) \), \( \beta \in \mathbb{R} \),
\[
\|Kf\|_{L^{p}_{\beta + \gamma + 1}} \leq C_{\gamma,d,q,p}\|f\|_{L^{p}_{\beta}}.
\]
Also step 1 gives that for \( \beta \in \mathbb{R} \), \( \|Kf\|_{L^{p}_{\beta + \gamma + 1}} \leq C_{\gamma,d,0}\|f\|_{L^{p}_{\beta}} \). Pick \( p_0 > \max(\frac{d}{\beta - \gamma}, \frac{d}{2}) \), \( p_1 = 1 \). Applying Riesz-Thorin interpolation theorem to \( p_0 \) and \( p_1 \), we obtain that for \( \theta \in (0,1) \), \( K \) is a linear bounded operator from \( L^{p_0} \) to \( L^{p_1} \) with
\[
\|Kf\|_{L^{p_{\theta}}_{\beta + \gamma + 1}} \leq C_{\gamma,d,q,p,\theta}\|f\|_{L^{p_{\theta}}_{\beta}},
\]
where \( \frac{1}{p_{\theta}} = \frac{\theta}{\infty} + \frac{1-\theta}{1} \). For the last assertion, it suffices to notice that if \( f \in L^{p_0}_{\beta}(H^s) \),
\[
\|Kf\|_{H^{s'}} \leq \int_{\mathbb{R}^d} |f(\xi,\xi_\ast)| \|\xi_\ast\|_{H^{s'}} \, d\xi_\ast.
\]
This completes the theorem. \( \Box \)

The following theorem is well studied in many literature such as [2] and I will put the proof in appendix.

**Theorem 2.5. (Properties of \( L \)).** Assume \( \gamma \in [0,d) \). Then \( L : L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d) \) is a linear unbounded operator satisfying the following properties.
(a). \( L \) is a self-adjoint non-positive linear operator on \( D(L) \).
(b). \( \text{Ker}L = \text{Span}\{M^{1/2}, \xi_1M^{1/2}, \ldots, \xi_dM^{1/2}, |\xi|^2M^{1/2}\} \).

By orthogonal decomposition, we can decompose \( L^2(\mathbb{R}^d) \) by
\[
L^2(\mathbb{R}^d) = \text{Ker}L \oplus (\text{Ker}L)^\perp.
\]
(2.12)
Denote \( \varphi_0 = M^{1/2}, \varphi_i = \xi_iM^{1/2} (i = 1,\ldots,d) \), \( \varphi_{d+1} = |\xi|^2M^{1/2} \). Then we can define projection \( P \) from \( L^2(\mathbb{R}^d) \) onto \( \text{Ker}L \) by
\[
Pf := \sum_{i=0}^{d+1} (f,\varphi_i)\varphi_i.
\]
(2.13)

3 Estimate on \( \hat{B}(y) \)

In this section, we will compute some basic estimate on operator \( \hat{B}(y) \), \( \hat{B}_0(y) \), \( \hat{A}(y) \) as well as their resolvents. In order to make the subsequent arguments rigorous, we need to verify the existence of the resolvent in some specific space.

Suppose \( \gamma \in [0,d) \), \( \beta \in \mathbb{R} \). Define
\[
\hat{B}(y) := -2\pi iy \cdot \xi + L,
\]
\[
\hat{B}_0(y) := -2\pi iy \cdot \xi + L - P,
\]
\[
\hat{A}(y) := -2\pi iy \cdot \xi - \nu,
\]
with domain depending on \( \beta \):
\[
D_{\beta}(\hat{B}(y)) = D_{\beta}(\hat{B}_0(y)) = D_{\beta}(\hat{A}(y)) := \{ f \in L^2_\beta(\mathbb{R}^d) : y \cdot \xi f \in L^2_\beta(\mathbb{R}^d) \}.
\]
Also we define
\[
K_0 = K - P.
\]
Remark 3.1. It’s important to notice that the resolvent set and spectrum of these operators depend on the whole space that we are considering.

The following theorem gives some spectrum structures of operators $\tilde{B}(y)$ and $\tilde{B}_0(y)$.

**Theorem 3.2.** Assume $\gamma \in [0, d)$, $\beta \in \mathbb{R}$, $y \in \mathbb{R}^d$. Then the following statements are valid.

1. $\tilde{B}(y)$ generates a contraction semigroup on $L^2(\mathbb{R}^d)$. Consequently,

   $$\rho(\tilde{B}(y)) \supset \{ \text{Re} \lambda > 0 \},$$

   and

   $$e^{t\lambda} = \lim_{a \to \infty} \frac{1}{2\pi i} \int_{\sigma - it} e^{\lambda t} (\lambda I - \tilde{B}(y))^{-1} u \, d\lambda,$$

   for $u \in D_0(\tilde{B}(y))$, $\sigma > 0$, where the limit is taken with respect to $L^2(\mathbb{R}^d)$ norm.

2. $\tilde{B}(y)$ generates a semigroup on $L^2_\beta$ with

   $$||e^{t\tilde{B}(y)}||_{L(L^2_\beta)} \leq e^{t||K||_{L(L^2_\beta)}}.$$  

3. There are two cases about $\sigma_p(\tilde{B}(y)) \cap \{ \text{Re} \lambda = 0 \}$, where $\tilde{B}(y)$ is considered acting on $L^2$.

   $$\sigma_p(\tilde{B}(y)) \cap \{ \text{Re} \lambda = 0 \} = \begin{cases} \emptyset, & \text{if } y \neq 0, \\ \{0\}, & \text{if } y = 0. \end{cases}$$

4. One can delete the kernel of $L$ by defining $\tilde{B}_0(y) := \tilde{B}(y) - P$. Then on $L^2$, for $y \in \mathbb{R}^d$,

   $$\sigma_p(\tilde{B}(y) - P) \subset \{ \text{Re} \lambda < 0 \}.$$  

**Proof.** 1. Noticing that $\tilde{B}(y)$ and its adjoint are dissipative on $D_0(\tilde{B}(y))$ and $(\tilde{B}(y), D_0(\tilde{B}(y)))$ is closed on $L^2$, so $\tilde{B}(y)$ generates a contraction semigroup on $L^2$.

2. For $\beta \in \mathbb{R}$, the semigroup generated by $\tilde{A}(y)$ on $L^2_\beta$ is defined by

   $$e^{t\tilde{A}(y)} u := e^{-2\pi i y \cdot \xi - \nu t} u,$$

   for any $u \in L^2_\beta$ and so $||e^{t\tilde{A}(y)}||_{L(L^2_\beta)} \leq 1$. Applying the theory of bounded perturbation of semigroup in [4, 6], we know that $\tilde{B}(y) = \tilde{A}(y) + K$ generates a semigroup on $L^2_\beta$ and

   $$||e^{t\tilde{B}(y)}||_{L(L^2_\beta)} \leq e^{t||K||_{L(L^2_\beta)}}.$$  

3. Consider $L^2$ to be the whole space. Let $\lambda \in \sigma_p(\tilde{B}(y))$, then $\text{Re} \lambda \leq 0$ by (3.1). Suppose $\text{Re} \lambda = 0$ and let $f \neq 0, f \in L^2$ be an eigenfunction of $\tilde{B}(y)$ corresponding to $\lambda$, then

   $$\text{Re}(\tilde{B}(y)f, f)_{L^2} = (Lf, f)_{L^2} = 0.$$  

Then $f \in \text{Ker} L, Lf = 0$, and so $2\pi i y \cdot \xi f = 0$. But $y \neq 0, f \neq 0$. This is a contradiction.

Let $\lambda \in \sigma_p(\tilde{B}(y) - P)$ and $f \neq 0$ to be a eigenfunction of $\tilde{B}(y) - P$ corresponding to $\lambda$ in $L^2$, then

$$\text{Re}((\tilde{B}(y) - P)f, f)_{L^2} = (Lf, f)_{L^2} - ||Pf||_{L^2}^2 = 0.$$  

Noticing $(Lf, f)_{L^2} \leq 0$, we have $(Lf, f)_{L^2} = ||Pf||_{L^2} = 0$. Then $f \in \text{Ker} L \cap \text{Ker} L^\perp$, and thus $f = 0$. This is a contradiction.

$\square$
Theorem 3.3. Assume $\gamma \in [0, d)$, $Re \lambda \geq 0$, $p \in (1, \infty)$, $R > 0$, $y \in \mathbb{R}^d$, $y_1 > 0$. Then the followings are valid.

(1) $\lambda \in \rho(\hat{A}(y))$ and $(\lambda I - \hat{A}(y))^{-1} : L^p_{\beta+\gamma} \to L^p_{\beta}$ is linear continuous with

\[
\|(\lambda I - \hat{A}(y))^{-1} u\|_{L^p_{\beta+\gamma}} \leq \frac{1}{\nu_0} \|u\|_{L^p_{\beta}}, \quad (3.6)
\]

\[
\|(\lambda I - \hat{A}(y))^{-1} \chi_{|\xi| \geq R} u\|_{L^p_{\beta}} \leq C\nu\|u\|_{L^p_{\beta+\gamma}(|\xi| \geq R)}. \quad (3.7)
\]

(2) If $|y| \leq y_1$, $|\lambda| \geq 4\pi y_1 R$, then

\[
\|(\lambda I - \hat{A}(y))^{-1} \chi_{|\xi| \leq R} u\|_{L^p_{\beta}} \leq C\|u\|_{L^p_{\beta}} \frac{1}{|\lambda|}. \quad (3.8)
\]

Consequently,

\[
\|(\lambda I - \hat{A}(y))^{-1} u\|_{L^p_{\beta}} \leq C\nu\|u\|_{L^p_{\beta+\gamma}(|\xi| \geq R)} + C\|u\|_{L^p_{\beta}} \frac{1}{|\lambda|}, \quad (3.9)
\]

\[
\|(\lambda I - \hat{A}(y))^{-1} K u\|_{L^p_{\beta}} \leq C\nu\|u\|_{L^p_{\beta-1} (1 + R)^{-1}} + C\|u\|_{L^p_{\beta-\gamma} (|\xi| \leq R)} \frac{1}{|\lambda|}. \quad (3.10)
\]

(3) If $|y| \geq y_1$, then

\[
\|(\lambda I - \hat{A}(y))^{-1} \chi_{|\xi| \leq R} u\|_{L^p_{\beta}} \leq C\nu\|u\|_{L^p_{\beta+\gamma}(|\xi| \leq R)} \frac{1}{y_1} + C\nu\|u\|_{L^p_{\beta}} \frac{1}{\nu_0} \frac{1}{y_1}. \quad (3.11)
\]

Consequently,

\[
\|(\lambda I - \hat{A}(y))^{-1} K u\|_{L^p_{\beta}} \leq C\nu,_{d, \gamma}\|u\|_{L^p_{\beta-\gamma}(|\xi| \leq R)} + C\nu,_{d, \gamma}\|u\|_{L^p_{\beta-1}} \left(\frac{R(\pi y, \gamma)}{y_1^{\gamma+1}} + \frac{R^{\frac{1}{2}} y_1}{\nu_0}\right), \quad (3.12)
\]

where $\delta = \frac{2}{p_0 - 2} \in (0, 1)$ and $p_0 := \frac{d}{d - \gamma} + \frac{d}{2}$.

Proof. 1. For $Re \lambda \geq 0$, $p \in (1, \infty)$, we have for $u \in L^p_{\beta+\gamma}$,

\[
(\lambda I - \hat{A}(y))^{-1} = \frac{1}{\lambda + \nu(\xi) + 2\pi iy \cdot \xi},
\]

\[
\|(\lambda I - \hat{A}(y))^{-1} u\|_{L^p_{\beta}} \leq \int_{\mathbb{R}^d} \frac{|(1 + |\xi|)^3 u(\xi)|^p}{|\nu(\xi)|^p} \, d\xi \leq \frac{1}{\nu_0^p} \|u\|_{L^p_{\beta+\gamma}}^p.
\]

Thus $\lambda \in \rho(\hat{A}(y))$ and $(\lambda I - \hat{A}(y))^{-1} : L^2_{\beta+\gamma} \to L^2_{\beta}$ is linear continuous. For $R > 0$, similarly, we have

\[
\|(\lambda I - \hat{A}(y))^{-1} \chi_{|\xi| \geq R} u\|_{L^p_{\beta}} \leq C\nu\|u\|_{L^p_{\beta+\gamma}(|\xi| \geq R)}. \quad (3.6)
\]

2. Fix $y_1 > 0$. If $|y| \leq y_1$, $|\xi| \leq R$, then $|2\pi y \cdot \xi| \leq 2\pi y_1 R$. Thus for $|\lambda| \geq 4\pi y_1 R$,

\[
\|(\lambda I - \hat{A}(y))^{-1} \chi_{|\xi| \leq R} u\|_{L^p_{\beta}} \leq \|u\|_{L^p_{\beta}} \sup_{|\xi| \leq R} \frac{1}{\lambda + \nu(\xi) + 2\pi iy \cdot \xi}
\]

\[
\leq \sqrt{2} \|u\|_{L^p_{\beta}} \sup_{|\xi| \leq R} \frac{1}{Re \lambda + |Im \lambda| - 2\pi iy_1 R}
\]

\[
\leq C\|u\|_{L^p_{\beta}} \frac{1}{|\lambda|}.
\]
Thus for \( \alpha > \beta + \gamma \), using (3.7), we have
\[
\left\| (\lambda - \tilde{A}(y))^{-1}u \right\|_{L^p_{\beta}} \leq C_{\nu} \left\| u \right\|_{L^p_{\beta+\gamma}, (\left\| \xi \right\| \geq R)} + C \left\| u \right\|_{L^p_{\beta}} \frac{1}{|\lambda|},
\]
\[
\leq C_{\nu} \left\| u \right\|_{L^p_{\beta}} (1 + R)^{\beta+\gamma-\alpha} + C \left\| u \right\|_{L^p_{\beta}} \frac{1}{|\lambda|}.
\]
Pick \( \alpha = \beta + \gamma + 1 \), then
\[
\left\| (\lambda - \tilde{A}(y))^{-1}Ku \right\|_{L^p_{\beta}} \leq C_{\nu} \left\| Ju \right\|_{L^p_{\beta+\gamma+1}} (1 + R)^{-1} + C \left\| Ku \right\|_{L^p_{\beta}} \frac{1}{|\lambda|}
\]
\[
\leq C_{\nu} \left\| u \right\|_{L^p_{\beta-1}} (1 + R)^{-1} + C \left\| u \right\|_{L^p_{\beta-\gamma-2}} \frac{1}{|\lambda|}.
\]

3. If \( |y| \geq y_1 \), we will use another method. For \( R > 0 \),
\[
\left\| (\lambda - \tilde{A}(y))^{-1}Ku \right\|_{L^p_{\beta}} \leq C_{\nu} \int_{R^d} (1 + |\xi|)^{\beta+\gamma} |u(\xi)|^p \left( \frac{|\nu(\xi)|^2}{|Re(\lambda + \nu|^2 + |Im(\lambda + 2\pi y \cdot \xi)|^2} \right)^{p/2} d\xi.
\]
We divide this integral into two parts:
\[
\int_{|\xi| \leq R} + \int_{|\xi| \leq R}.
\]
Notice if \( \xi \in \{ \xi : |Re(\lambda + \nu|^2 + |Im(\lambda + 2\pi y \cdot \xi)|^2 \leq 1 \} \), and if \( \xi \in \{ \xi : Re(\lambda + \nu|^2 + |Im(\lambda + 2\pi y \cdot \xi)|^2 \geq \frac{|\nu|}{\sqrt{y_1}} \}, \)
\(|y| \geq y_1 \), we have \( \frac{|\nu(\xi)|^2}{|Re(\lambda + \nu|^2 + |Im(\lambda + 2\pi y \cdot \xi)|^2} \leq \frac{\nu_1}{|\nu|} \leq \frac{\nu_1}{\sqrt{y_1}} \). Therefore if \( |y| \geq y_1 \),
\[
\left\| (\lambda - \tilde{A}(y))^{-1}Ku \right\|_{L^p_{\beta}} \leq C_{\nu} \left\| u \right\|_{L^p_{\beta+\gamma}, (|\xi| \leq R, |Re(\lambda + \nu|^2 + |Im(\lambda + 2\pi y \cdot \xi)|^2 \leq \frac{|\nu|}{\sqrt{y_1}})} + C_{\nu} \left\| u \right\|_{L^p_{\beta+\gamma}, (|\xi| \leq R)} \frac{1}{y_1}
\]

(3.13)

4. Suppose \( |y| \geq y_1 \). To prove the last assertion, we need to use the properties of \( K \) from 2.3. Pick \( p_0 = \frac{d}{d-\gamma} + \frac{d}{2} \), \( \theta = \frac{p_0}{2(p_0-1)} \) and let \( \delta = \frac{1}{4} - \delta > 0 \). Then \( p_0 = 2, q_0 = 2 + \delta \) satisfy (2.9) and
\[
\left\| Kf \right\|_{L^p_{\beta+\gamma+1}} \leq C_{\gamma,d,q,p,d} \left\| f \right\|_{L^p_{\beta}}.
\]
(3.14)

Pick \( p = 2 \) in (3.13), then using the fact \( \left\{ |\xi| \leq R, |Re(\lambda + \nu|^2 + |Im(\lambda + 2\pi y \cdot \xi)|^2 \leq \frac{|\nu|}{\sqrt{y_1}} \right\} \leq \frac{C_{\nu} R^{d-1}}{\sqrt{y_1}} \) and Hölder’s inequality, we have
\[
\left\| (\lambda - \tilde{A}(y))^{-1}Ku \right\|_{L^p_{\beta}} \leq C_{\nu} \left\| u \right\|_{L^p_{\beta+\gamma}, (|\xi| \leq R)} \left( \int_{|\xi| \leq R} \left\| Ku \right\|_{L^p_{\beta+\gamma}, (|\xi| \leq R)} \right) \left( \int_{|\xi| \leq R} \left\| Ku \right\|_{L^p_{\beta+\gamma}, (|\xi| \leq R)} \right) \frac{1}{y_1}
\]
\[
= C_{\nu,d,\gamma} \left\| u \right\|_{L^p_{\beta+\gamma}, (|\xi| \leq R)} \left( \frac{R^{d+1}}{2^{d+1} y_1^{1-\gamma}} + \frac{R^{d+1}}{2^{d+1} y_1^{1-\gamma}} \right).
\]
Thus
\[
\left\| (\lambda - \tilde{A}(y))^{-1}Ku \right\|_{L^p_{\beta}} \leq C_{\nu,d,\gamma} \left\| u \right\|_{L^p_{\beta-1}} \left( \frac{R^{d+1}}{2^{d+1} y_1^{1-\gamma}} + \frac{R^{d+1}}{2^{d+1} y_1^{1-\gamma}} \right).
\]
With (3.7), we can get (3). 

The following theorem gives the existence of inverse \((I - (\lambda - \tilde{A}(y))^{-1}K)^{-1}\) and \((I - (\lambda - \tilde{A}(y))^{-1}K_0)^{-1}\) on \( L^p_{\beta} \).
Theorem 3.4. Fixed $\beta \in \mathbb{R}$, we consider $L^2_\beta$ to be the whole space.

1. For $y \in \mathbb{R}^d$, $\text{Re} \lambda \geq 0$, if $y \neq 0$ or $\lambda \neq 0$, then

$$ 1 \in \rho((\lambda I - \hat{A}(y))^{-1}K). $$

(3.15)

2. For $y \in \mathbb{R}^d$, $\text{Re} \lambda \geq 0$, we have

$$ 1 \in \rho((\lambda I - \hat{A}(y))^{-1}K_0). $$

(3.16)

Proof. 1. If not, we suppose $1 \notin \sigma((\lambda I - \hat{A}(y))^{-1}K)$. Since $(\lambda I - \hat{A}(y))^{-1} : L^2_{\beta+\gamma} \to L^2_{\beta}$ is linear continuous and $K : L^2_{\beta} \to L^2_{\beta+\gamma+1}$ is compact, we have $(\lambda I - \hat{A}(y))^{-1}K : L^2_{\beta+\gamma} \to L^2_{\beta+\gamma}$ is compact, for $\beta \in \mathbb{R}$. Thus by Fredholm alternative, on $L^2_{\beta+\gamma}$, we have

$$ 1 \in \sigma_p((\lambda I - \hat{A}(y))^{-1}K). $$

Thus for some $0 \neq u \in L^2_{\beta+\gamma}$,

$$ u = (\lambda I - \hat{A}(y))^{-1}Ku, $$

(3.17)

and hence $u \in L^2_{\beta+\gamma+2}$. By using (3.17) inductively, we have $u \in \cap_{\beta \in \mathbb{R}} L^2_{\beta}$. Let $v = (\lambda I - \hat{A}(y))^{-1}u$, we have $v \in \cap_{\beta \in \mathbb{R}} L^2_{\beta} \subset D_0(\hat{B}(y))$ and

$$ 0 = (\lambda I - \hat{B}(y))v. $$

(3.18)

Thus $\lambda$ is an eigenvalue of $\hat{B}(y)$ and $\text{Re} \lambda = 0$ by theorem 3.2 (1). But $(y, \lambda) \neq 0$, so equation (3.18) contradicts to theorem 3.2 (3).

2. The proof of the second assertion is very similar, but in this case we need to use the fact that $\sigma_p(\hat{B}_0(y) - P) \subset \{\text{Re} \lambda < 0\}$. \hfill $\square$

Remark 3.5. $v \in D_0(\hat{B}(y))$ is necessary. But if $\beta \geq 0$, then we don't have to use (3.17) inductively.

The following lemma is useful for proving the uniformly boundedness of $(\lambda I - \hat{B}(y))^{-1}$ and $(\lambda I - \hat{B}_0(y))^{-1}$. Let $X$ to be a metric space and $Y$ to be a normed space and define $C(X; Y)$ and $BC(X; Y)$ as the following.

$$ C(X; Y) = \{f : X \to Y \mid f \text{ is continuous from } X \text{ to } Y\}, $$

$$ BC(X; Y) = \{f \in C(X; Y) \mid \sup_{x \in X} ||f||_Y < \infty\}. $$

Lemma 3.6. Assume $\gamma \in [0, d)$.

1. For $\text{Re} \geq 0$, $\alpha > \beta + \gamma$, we have

$$ (\lambda I - \hat{A}(y))^{-1} \in C(\overline{C_+} \times \mathbb{R}^d; L(L^2_{\alpha}, L^2_{\beta})), $$

$$ (\lambda I - \hat{A}(y))^{-1}K \in C(\overline{C_+} \times \mathbb{R}^d; L(L^2_{\alpha}, L^2_{\beta+\gamma+2})). $$

(3.19)

2. The inverse $(I - (\lambda I - \hat{A}(y))^{-1}K)^{-1}$ exists on $L^2_{\beta}$ for $(\lambda, y) \in (\overline{C_+} \times \mathbb{R}^d) \setminus \{(0, 0)\})$. Also For $r > 0$,

$$ (I - (\lambda I - \hat{A}(y))^{-1}K)^{-1} \in BC(\overline{C_+} \times (\mathbb{R}^d \setminus B_r); L(L^2_{\beta}, L^2_{\beta})) $$

$$ \cap BC((\overline{C_+} \setminus B_r) \times \mathbb{R}^d; L(L^2_{\beta}, L^2_{\beta})), $$

(3.20)

where $B_r \in \mathbb{R}^d$ is the closed ball in $\mathbb{R}^d$ with center $0$ and radius $r$.

3. $(I - (\lambda I - \hat{A}(y))^{-1}K_0)^{-1} : \overline{C_+} \times \mathbb{R}^d \to L(L^2_{\beta}, L^2_{\beta})$ is linear continuous and

$$ (I - (\lambda I - \hat{A}(y))^{-1}K_0)^{-1} \in BC(\overline{C_+} \times \mathbb{R}^d; L(L^2_{\beta}, L^2_{\beta})). $$

(3.21)
Proof. 1. Let $\alpha > \beta + \gamma$. For $(\lambda_1, y_1), (\lambda_2, y_2) \in \mathbb{C}_+ \times \mathbb{R}^d$, we have

$$(\lambda_1 I - \hat{A}(y_1))^{-1} - (\lambda_2 I - \hat{A}(y_2))^{-1} = \frac{1}{\lambda_1 + \nu(\xi) + 2\pi iy_1 \cdot \xi} - \frac{1}{\lambda_2 + \nu(\xi) + 2\pi iy_2 \cdot \xi}$$

On one hand, by using (3.6) in 3.3, we have

$$\left\| [ (\lambda_1 I - \hat{A}(y_1))^{-1} - (\lambda_2 I - \hat{A}(y_2))^{-1} ] \chi_{|\xi| \leq R} \right\|_{L(L^2_{\gamma+\gamma}, \lambda I)} \leq C_\nu.$$

(3.22)

On the other hand,

$$\left| \frac{1}{\lambda_1 + \nu(\xi) + 2\pi iy_1 \cdot \xi} - \frac{1}{\lambda_2 + \nu(\xi) + 2\pi iy_2 \cdot \xi} \right| \leq C_\nu(1 + |\xi|)^{2\gamma}(|\lambda_2 - \lambda_1| + 2|y_2 - y_1| |\xi|).$$

Thus

$$\left\| [ (\lambda_1 I - \hat{A}(y_1))^{-1} - (\lambda_2 I - \hat{A}(y_2))^{-1} ] \chi_{|\xi| \leq R} \right\|_{L(L^2_{\gamma+\gamma}, \lambda I)} \leq C_\nu(1 + R)^{2\gamma}(|\lambda_2 - \lambda_1| + R|y_2 - y_1|).$$

(3.23)

Combining equation (3.22) and (3.23), we have for $\alpha > \beta + \gamma$,

$$\| (\lambda_1 I - \hat{A}(y_1))^{-1} - (\lambda_2 I - \hat{A}(y_2))^{-1} \|_{L(L^2_{\gamma+\gamma}, \lambda I)} \leq C_\nu(1 + R)^{\beta + \gamma - \alpha} + C_\nu(1 + R)^{2\gamma}(|\lambda_2 - \lambda_1| + R|y_2 - y_1|).$$

Thus

$$\| (\lambda_1 I - \hat{A}(y_1))^{-1} - (\lambda_2 I - \hat{A}(y_2))^{-1} \|_{L(L^2_{\gamma+\gamma}, \lambda I)} \to 0,$$

as $|(\lambda_1, y_1) - (\lambda_2, y_2)| \to 0$ and $(\lambda_1, y_1), (\lambda_2, y_2) \in \mathbb{C}_+ \times \mathbb{R}^d$. Therefore,

$$(\lambda I - \hat{A}(y))^{-1} \in C (\mathbb{C}_+ \times \mathbb{R}^d; L(L^2_{\alpha}, L^2_{\beta})), \quad (\lambda I - \hat{A}(y))^{-1} K \in C (\mathbb{C}_+ \times \mathbb{R}^d; L(L^2_{\alpha}, L^2_{\beta+\gamma+2})).$$

(3.24)

2. For $\beta \in \mathbb{R}$, pick $\alpha = \beta + \gamma + 2$ in (3.24), then we have

$$(\lambda I - \hat{A}(y))^{-1} K \in C (\mathbb{C}_+ \times \mathbb{R}^d; L(L^2_{\alpha}, L^2_{\beta})).$$

(3.25)

Also theorem 3.4 shows that $(I - (\lambda I - \hat{A}(y))^{-1} K)^{-1}$ exists on $L^2_{\beta}$ for $(\lambda, y) \neq (0, 0)$. Firstly we state as basic resolvent theorem. For any bounded linear operators $T_1, T_2$ defined on the same Banach space, if $\eta \in \rho(T_1) \cap \rho(T_2)$, then whenever $\|T_1 - T_2\| \leq \frac{1}{\eta^2(\eta^2 - 1)^{\frac{1}{2}}}$,

$$(\eta I - T_1)^{-1} - (\eta I - T_2)^{-1} = (\eta I - T_1)^{-1}(T_1 - T_2)(\eta I - T_2)^{-1} = \sum_{n=1}^{\infty} ((\eta I - T_2)^{-1}(T_1 - T_2))^n(\eta I - T_2)^{-1},$$

$$\| (\eta I - T_1)^{-1} - (\eta I - T_2)^{-1} \| \leq \frac{1}{2} \| T_1 - T_2 \| \| (\eta I - T_2)^{-1} \|^2.$$ 

And so the resolvent $(\lambda I - T)^{-1}$ of $T$ is continuous with respect to $T$. Now $(I - (\lambda I - \hat{A}(y))^{-1} K)^{-1}$ exists on $L^2_{\beta}$ for $(\lambda, y) \neq (0, 0)$, thus using (3.25), we have

$$(I - (\lambda I - \hat{A}(y))^{-1} K)^{-1} \in C (\mathbb{C}_+ \times (\mathbb{R}^d \setminus \{0\}); L(L^2_{\alpha}, L^2_{\beta})), \quad \cap C((\mathbb{C}_+ \setminus \{0\}) \times \mathbb{R}^d; L(L^2_{\alpha}, L^2_{\beta})).$$
Similarly, since \((I - (\lambda I - \hat{A}(y))^{-1}K_0)^{-1}\) exists on \(L^2_{\beta}\) for \((\lambda, y) \in \overline{C_+} \times \mathbb{R}^d\) and

\[
(I - (\lambda I - \hat{A}(y))^{-1}K_0)^{-1} \in C(\overline{C_+} \times \mathbb{R}^d; L(L^2_{\beta}, L^2_{\beta})).
\]

These prove the continuity.

3. Fix \(y_1 > 0, R > 0\), we shall use theorem 3.3 to prove the boundedness. If \(|y| \leq y_1\), \(\Re \lambda > 0\) with \(|\lambda| \geq 4\pi y_1 R\), then

\[
\|(\lambda I - \hat{A}(y))^{-1}K\|_{L^2_{\beta} \rightarrow L^2_{\beta}} \leq C_{\nu,d}[(1 + R)^{-1} + \frac{1}{|\lambda|}]^3.
\]  

(3.26)

If \(|y| \geq y_1\),

\[
\|(\lambda I - \hat{A}(y))^{-1}K\|_{L^2_{\beta} \rightarrow L^2_{\beta}} \leq C_{\nu,d,q,\gamma}\left( (1 + R)^{-1} + R\frac{\beta y_1^2}{\nu^2 + y_1^2} + R\frac{\beta y_1^2}{\nu^2 + y_1^2} y_1^{-\delta} \right).
\]  

(3.27)

Write \(C_0 = \max(C_{\nu,d}, C_{\nu,d,q,\gamma})\). Firstly we pick a sufficiently large \(R_1 = R_1(\nu, d, q, \gamma)\) s.t.

\[
C_0(1 + R_1)^{-1} < 1/4.
\]

Then with this \(R_1\), we pick \(y_1 = y_1(R_1)\) so large that

\[
C_0\left(R\frac{\beta y_1^2}{\nu^2 + y_1^2} + R\frac{\beta y_1^2}{\nu^2 + y_1^2} y_1^{-\delta}\right) < 1/4.
\]

Let \(r_1 = \max(4\pi y_1 R_1, 1 + R_1)\) and

\[
B_0(r, r_1, y_1) := \{ (\lambda, y) : \Re \lambda \geq 0, |\lambda| \leq r_1, |y| \leq y_1, |y| \geq r \},
\]

\[
B_1(r, r_1, y_1) := \{ (\lambda, y) : \Re \lambda \geq 0, |y| \geq r \} \setminus B_0(r, r_1, y_1).
\]

Since \((I - (\lambda I - \hat{A}(y))^{-1}K)^{-1} \in C(\overline{C_+} \times \mathbb{R}^d \setminus \{0\}; L(L^2_{\beta}, L^2_{\beta}))\), and \(B_0(r, r_1, y_1)\) is a compact set, we have

\[
\sup_{(\lambda, y) \in B_0(r, r_1, y_1)} \|(I - (\lambda I - \hat{A}(y))^{-1}K)^{-1}\|_{L(L^2_{\beta})} < \infty.
\]  

(3.28)

For \((\lambda, y) \in B_1(r, r_1, y_1)\), we have \(|\lambda| > r_1\) or \(|y| > y_1\). If \(|y| > y_1\), then by (3.27), we have

\[
\|(\lambda I - \hat{A}(y))^{-1}K\|_{L(L^2_{\beta})} \leq 1/2.
\]

If \(|y| \leq y_1\), then \(|\lambda| > r_1\) and then by (3.26),

\[
\|(\lambda I - \hat{A}(y))^{-1}K\|_{L(L^2_{\beta})} \leq 1/2.
\]

Thus for \((\lambda, y) \in B_1(r, r_1, y_1)\), we have \((I - (\lambda I - \hat{A}(y))^{-1}K)^{-1}\) exists on \(L^2_{\beta}\) and

\[
\|(I - (\lambda I - \hat{A}(y))^{-1}K)^{-1}\|_{L(L^2_{\beta})} \leq 2.
\]  

(3.29)

Combining (3.28) and (3.29), we have for \(r > 0\).

\[
(I - (\lambda I - \hat{A}(y))^{-1}K)^{-1} \in BC(\overline{C_+} \times \mathbb{R}^d \setminus B_r; L(L^2_{\beta})).
\]

4. By digging out a ball near \(\lambda = 0\) instead of \(y = 0\), we can get the second boundedness for \((I - (\lambda I - \hat{A}(y))^{-1}K)^{-1}\).

5. The proof of the last assertion is similar to step 3, but in this case we don’t need to dig out the ball \(B_r\), since \(\hat{B}(y) - P\) has no zero eigenvalue at \(y = 0\).
Noticing that

\[(\lambda I - \hat{B}(y))^{-1} = (I - (\lambda I - \hat{A}(y))^{-1}K)^{-1}(\lambda I - \hat{A}(y))^{-1},\]  

we have the following corollary.

**Corollary 3.7.** For any \(r > 0\), we have

\[(\lambda I - \hat{B}(y))^{-1} \in BC(\overline{C_+ \times (R^d \setminus B_r)}; L(L^2_{\beta+\gamma}, L^2_{\beta})) \cap BC((\overline{C_+ \setminus B_r}) \times R^d; L(L^2_{\beta+\gamma}, L^2_{\beta}))\]  

Consequently,

\[
\left(\sup_{\text{Re}\lambda \geq 0, y \in R^d, |y| \geq r} + \sup_{\text{Re}\lambda \geq 0, |\lambda| \geq r, y \in R^d}\right)\|\lambda I - \hat{B}(y)\|_{L(L^2_{\beta+\gamma}, L^2_{\beta})} < \infty.
\]  

Furthermore, we need the following invertibilities.

**Lemma 3.8.** Let \(\text{Re}\lambda \geq 0, y \in R^d, \beta \in R\). Then the inverse \((\lambda I - \hat{B}_0(y))^{-1}\) exists on \(L^2_{\beta+\gamma}\) and

\[
\sup_{\text{Re}\lambda \geq 0, y \in R^d}\|\lambda I - \hat{B}_0(y)\|_{L^2_{\beta+\gamma}} < \infty.
\]  

If \(y \neq 0\), then the inverse \((I - (\lambda I - \hat{B}_0(y))^{-1}P)^{-1}\) exists on \(L^2_{\beta}\) and \((I - P(\lambda I - \hat{B}_0(y))^{-1}P)^{-1}\) exists on \(\text{Ker}L \subset L^2_{\beta}\).

**Proof.** The idea is similar to the proof of inverse of \((I - (\lambda I - \hat{A}(y))^{-1}K)^{-1}\).

1. Fix any \(\beta \in R\). By theorem 3.3, \((\lambda I - \hat{A}(y))^{-1}\) exists on \(L^2_{\beta+\gamma}\) and is bounded from \(L^2_{\beta+\gamma}\) to \(L^2_{\beta}\). By lemma 3.4 that \((I - (\lambda I - \hat{A}(y))^{-1}K_0)^{-1}\) exists on \(L^2_{\beta+\gamma}\). Also since \(\hat{B}_0(y) = \hat{A}(y) + K_0\), we have

\[
\lambda I - \hat{B}_0(y) = (\lambda I - \hat{A}(y))(I - (\lambda I - \hat{A}(y))^{-1}K_0).
\]

Thus for \(\text{Re}\lambda \geq 0\), \((\lambda I - \hat{B}_0(y))^{-1}\) exists on \(L^2_{\beta+\gamma}\) and

\[
(\lambda I - \hat{B}_0(y))^{-1} = (I - (\lambda I - \hat{A}(y))^{-1}K_0)^{-1}(\lambda I - \hat{A}(y))^{-1}.
\]  

Then by theorem 3.3 and 3.6,

\[
\sup_{\text{Re}\lambda \geq 0, y \in R^d}\|\lambda I - \hat{B}_0(y)\|_{L^2_{\beta+\gamma}} < \infty.
\]

2. If not, we suppose \(1 \in \sigma((\lambda I - \hat{B}_0(y))^{-1}P)\). Since \((\lambda I - \hat{B}_0(y))^{-1} : L^2_{\beta+\gamma} \rightarrow L^2_{\beta}\) is linear continuous and \(P : L^2_{\beta} \rightarrow L^2_{\beta+\gamma}\) is compact, we have \((\lambda I - \hat{B}_0(y))^{-1}P : L^2_{\beta+\gamma} \rightarrow L^2_{\beta+\gamma}\) is compact, for \(\beta \in R\). Thus by Fredholm alternative, on \(L^2_{\beta+\gamma}\), we have \(1 \in \sigma_p((\lambda I - \hat{B}_0(y))^{-1}P)\). Thus for some \(0 \neq u \in L^2_{\beta+\gamma}\),

\[
u = (\lambda I - \hat{B}_0(y))^{-1}Pu,
\]

and hence \(u \in \cap_{\beta \in R}L^2_{\beta}\) and

\[
(\lambda I - \hat{B}(y))u = 0.
\]  

Thus \(\lambda\) is an eigenvalue of \(\hat{B}(y)\) and \(\text{Re}\lambda = 0\) by theorem 3.2 (1). But \(y \neq 0\), so equation (3.35) contradicts to theorem 3.2 (3).

3. The proof existence of \((I - P(\lambda I - \hat{B}_0(y))^{-1}P)^{-1}\) is similar to step 2. \(\square\)
4 Eigenvalue Structure near \( y = 0 \)

In this section, we will give the proof of the existence of eigenvalues to operator \( P(\lambda - \hat{B}_0(y))^{-1}P \) as well as the asymptotic behavior of the singular points of \( (I - P(\lambda - \hat{B}_0(y))^{-1}P)^{-1} \) as \( y \to 0 \). These theorems are necessary for the estimate on semigroup \( e^{t\hat{B}(y)} \). Also we can prove that the singular points of \( (I - P(\lambda - \hat{B}_0(y))^{-1}P)^{-1} \) is the same as the resolvent \( (\lambda - \hat{B}(y))^{-1} \).

Assume \( \text{Re}\lambda \geq 0, y \neq 0, \beta \in \mathbb{R} \). From theorem 3.8, \( (\lambda - \hat{B}_0(y))^{-1} \) is a linear bounded operator form \( L^2_{\beta + \gamma} \) to \( L^2_{\beta} \) with estimate

\[
\sup_{\text{Re}\lambda \geq 0, y \in \mathbb{R}^d} \|(\lambda - \hat{B}_0(y))^{-1}\|_{L(L^2_{\beta + \gamma}, L^2_{\beta})} < \infty. \quad (4.1)
\]

Applying resolvent identities:

\[
(\lambda I - \hat{B}(y))^{-1} = (\lambda I - \hat{B}_0(y))^{-1}(I - P(\lambda I - \hat{B}_0(y))^{-1})^{-1} = (\lambda I - \hat{B}_0(y))^{-1} - (\lambda I - \hat{B}(y))^{-1}P(\lambda I - \hat{B}_0(y))^{-1},
\]

we have

\[
(\lambda I - \hat{B}_0(y))^{-1} = (\lambda I - \hat{B}_0(y))^{-1} - (\lambda I - \hat{B}_0(y))^{-1}(I - P(\lambda I - \hat{B}_0(y))^{-1})^{-1}P(\lambda I - \hat{B}_0(y))^{-1}. \quad (4.2)
\]

To simplify this form further, we need some basic lemmas.

**Lemma 4.1.** Let \( A \) be a linear continuous operator from \( L^2_{\beta + \gamma} \) to \( L^2_\beta \), for any \( \beta \in \mathbb{R} \). If the inverse in the following statement exists, then they are valid.

1. For \( f \in L^2_\beta \), we have \( (I - PA)^{-1}Pf \in \text{Ker}L \). Consequently,

\[
(I - PA)^{-1}Pf = P(I - PA)^{-1}Pf. \quad (4.3)
\]

2. On \( L^2_\beta \), for any \( \beta \in \mathbb{R} \), we have

\[
(I - PA)^{-1}P = (I - PAP)^{-1}P. \quad (4.4)
\]

**Proof.**

1. For \( f \in L^2_\beta \), then \( Pf \in \cap_{\beta \in \mathbb{R}} L^2_\beta \). Let \( g = (I - PA)^{-1}Pf \). Then

\[
(I - PA)g = Pf,
\]

\[
g = PAg + Pf \in \text{Ker}L.
\]

2. Let \( f \in L^2_\beta \), then by (4.3),

\[
Pf = (I - PA)(I - PA)^{-1}Pf = (I - PA)(I - PA)^{-1}Pf,
\]

\[
(I - PAP)^{-1}Pf = (I - PA)^{-1}Pf.
\]

**Lemma 4.2.** For \( \text{Re}\lambda \geq 0, \beta \in \mathbb{R} \), for \( f \in L^2_\beta \),

\[
P(\lambda I - L + P)^{-1}f = (\lambda I - L + P)^{-1}Pf = \frac{Pf}{\lambda + 1}. \quad (4.5)
\]

**Proof.** Let \( f \in L^2_\beta \) and \( g := (\lambda I - L + P)^{-1}f \), then \( (\lambda I - L + P)g = f \), and \( Pg = \frac{Pf}{\lambda + 1} \). For the second equality, it suffices to show that \( (\lambda I - L + P)^{-1}Pf \in \text{Ker}L \). Let \( h := (\lambda I - L + P)^{-1}Pf \), then \( (\lambda I - L + P)h = Pf \). Taking inner product with \( P^2h \), we have

\[
\lambda\|P^2h\|^2_{L^2} + \langle -Lh, P^2h \rangle = 0
\]

But \( L \) is a non-positive operator, thus \( \langle -Lh, h \rangle = 0 \) and \( h \in \text{Ker}L \). \( \Box \)
Applying lemma 4.1 to (4.2) with $A = (\lambda I - \widehat{B}_0(y))^{-1}$, we have

$$(\lambda I - \widehat{B}(y))^{-1} = (\lambda I - \widehat{B}_0(y))^{-1} + (\lambda I - \widehat{B}_0(y))^{-1}P(I - P(\lambda I - \widehat{B}_0(y))^{-1}P)^{-1}(\lambda I - \widehat{B}_0(y))^{-1}.$$  

Here $\|P\|_{L^2_{\beta} \to L^2_{\beta+\gamma}} < \infty$. Thus

$$\sup_{\Re \lambda \geq 0, y \in \mathbb{R}^d} \|(\lambda I - \widehat{B}_0(y))^{-1}P\|_{L^2_{\beta} \to L^2_{\beta}} < \infty,$$

$$\sup_{\Re \lambda \geq 0, y \in \mathbb{R}^d} \|P(\lambda I - \widehat{B}_0(y))^{-1}\|_{L^2_{\beta} \to L^2_{\beta}} < \infty.$$

Then $(\lambda I - \widehat{B}_0(y))^{-1}P$ and $P(\lambda I - \widehat{B}_0(y))^{-1}$ are bounded on $L^2_{\beta}$, so the singularity of resolvent $(\lambda I - \widehat{B}(y))^{-1}$ near $y = 0$ comes from $(1 - P(\lambda I - \widehat{B}_0(y))^{-1}P)^{-1}: \text{Ker} L \to \text{Ker} L$. So next we study the behavior of this operator.

Formally by the second resolvent identity, on $\text{Ker} L$, we have

$$(\lambda I - \widehat{B}_0(y))^{-1} = (\lambda I + L + P)^{-1} + (\lambda I - \widehat{B}_0(y))^{-1}(-2\pi iy \cdot \xi)(\lambda I - L + P)^{-1}. \tag{4.6}$$

It is valid only on $\text{Ker} L$, so here we check this identity carefully. Indeed for $f \in \text{Ker} L$, by using 4.2, we have

$$(\lambda I - \widehat{B}_0(y))f = (\lambda I + 2\pi iy \cdot \xi - L + P)f = (\lambda + 1 + 2\pi iy \cdot \xi)f,$$

$$f = (\lambda I - \widehat{B}_0(y))^{-1}(\lambda + 1 + 2\pi iy \cdot \xi)f,$$

$$(\lambda I - \widehat{B}_0(y))^{-1}f = \frac{1}{\lambda + 1}(I - (\lambda I - \widehat{B}_0(y))^{-1}(2\pi iy \cdot \xi))f.$$  

Write $y = r\omega$, with $\omega \in S^{d-1}$, $r = |y|$ and write $\lambda = \sigma + i\tau$. Define

$$D(\sigma, \tau, r, \omega) = P((\sigma + i\tau + 2\pi ir\omega \cdot \xi)I - L + P)^{-1}(\omega \cdot \xi)P. \tag{4.7}$$

Then on $L^2_{\beta}$, we have

$$P(\lambda I - \widehat{B}_0(y))^{-1}P = \frac{1}{\lambda + 1}(P - 2\pi irD(\sigma, \tau, r, \omega)). \tag{4.8}$$

Here we can assume $r \in \mathbb{R}$ instead of $r > 0$.

**Remark 4.3.** When considering operator $D(\sigma, \tau, r, \omega)$, we can assume $r \in \mathbb{R}$, but when we go back to $(I - P(\lambda I - \widehat{B}_0(y))^{-1}P)^{-1}$, we should assume $y \neq 0$.

### 4.1 Eigenvalues of $D(\sigma, \tau, r, \omega)$

Define an orthonormal basis $\{\psi_j\}_{j=0}^{d+1}$ of $\text{Ker} L$ in $L^2(\mathbb{R}^d)$ as following,

$$\psi_0 = M^{1/2},$$

$$\psi_j = \xi_j M^{1/2}, \quad \text{if } j = 1, \ldots, d$$

$$\psi_{d+1} = \frac{1}{\sqrt{2d}}(\|\xi\|^2 - d) M^{1/2}. \tag{4.9}$$

Fix $\omega \in S^{d-1}$. Define rotation $R \in O(d)$ on $\mathbb{R}^d$ s.t. $R\omega = e_1$, where $e_1 = (1, 0, \ldots, 0)$. Now we investigate the eigenvalues of

$$D(\sigma, \tau, r, \omega) = P((\sigma + i\tau + 2\pi ir\omega \cdot \xi)I - L + P)^{-1}(\omega \cdot \xi)P,$$

where $\sigma \geq 0$, $\tau \in \mathbb{R}$, $r \in \mathbb{R}$, $\omega \in S^{d-1}$. Notice $D$ maps $\text{Ker} L$ into $\text{Ker} L$, so under the orthonormal basis $\{R^T\psi_j\}$, we can obtain its matrix representation.
Definition 4.4. For $j, k = 0, \ldots, d + 1$, define
\[
D_{jk}(\sigma, \tau, r) := (D(\sigma, \tau, r, e_1)\psi_j, \psi_k)_{L^2}.
\]

Lemma 4.5. For $\sigma \geq 0$, $\tau \in \mathbb{R}$, $r \in \mathbb{R}$, $\omega \in S^{d-1}$, $j, k = 0, \ldots, d + 1$, we have
\[
(D(\sigma, \tau, r, \omega)R^T \psi_j, R^T \psi_k)_{L^2} = (D(\sigma, \tau, r, e_1)\psi_j, \psi_k)_{L^2},
\]
and
\[
(D(\sigma, \tau, r, e_1)\psi_j, \psi_k)_{L^2} \in C^\infty(\mathbb{R}_+ \times \mathbb{R} \times \mathbb{R}; \mathbb{C}).
\]  

Proof. 1. For any rotation $R \in O(d)$ acting on variable $\omega$, we know that $R$ commutes with $P$, $I$, $L$, thus for $\sigma \geq 0$,
\[
RD(\sigma, \tau, r, \omega) = R((\sigma + i\tau + 2\pi i r \omega)I - L + P)^{-1}(\omega \cdot \xi)P
\]
\[
= P((\sigma + i\tau + 2\pi i r \omega \cdot R^T \xi)I - L + P)^{-1}(\omega \cdot R^T \xi)PR
\]
\[
= D(\sigma, \tau, r, R^T \omega)R.
\]
Then (4.10) follows from $R^T \omega = e_1$.

2. Recall theorem 3.8 that $((\sigma + i\tau + 2\pi i r \xi)I - L + P)^{-1}$ is a linear bounded operator form $L^2_{\beta+\gamma}$ to $L^2_\beta$ and is continuous with respect to $\sigma \geq 0$, $\tau \in \mathbb{R}$, and $r \in \mathbb{R}$. By the second resolvent identity, for any $\sigma_1, \sigma_2 \geq 0$,
\[
d_{\sigma}D_{jk}(\sigma_1, \tau, r) - D_{jk}(\sigma_2, \tau, r)
\]
\[
= \left[ ((\sigma_1 + i\tau + 2\pi i r \xi_1)I - L + P)^{-1} - ((\sigma_2 + i\tau + 2\pi i r \xi_1)I - L + P)^{-1} \right] \xi_1 \psi_j, \psi_k
\]
\[
= ((\sigma_2 + i\tau + 2\pi i r \xi_1)I - L + P)^{-1}(\sigma_2 - \sigma_1)((\sigma_1 + i\tau + 2\pi i r \xi_1)I - L + P)^{-1} \xi_1 \psi_j, \psi_k,
\]
and so whenever $\sigma > 0$,
\[
\partial_\sigma D_{jk}(\sigma, \tau, r) = -((\sigma + i\tau + 2\pi i r \xi_1)I - L + P)^{-2} \xi_1 \psi_j, \psi_k)_{L^2}.
\]
Inductively,
\[
\partial^n_\sigma D_{jk}(\sigma, \tau, r) = (-1)^n n!((\sigma + i\tau + 2\pi i r \xi_1)I - L + P)^{-n-1} \xi_1 \psi_j, \psi_k)_{L^2},
\]
Similarly,
\[
\partial^n_\sigma D_{jk}(\sigma, \tau, r) = (-1)^n n!((\sigma + i\tau + 2\pi i r \xi_1)I - L + P)^{-n-1} \xi_1 \psi_j, \psi_k)_{L^2}.
\]

For the derivative with respect to $r$, we need to be more careful. For $r_1, r_2 \in \mathbb{R}$,
\[
D_{jk}(\sigma, \tau, r_1) - D_{jk}(\sigma, \tau, r_2)
\]
\[
= \left[ ((\sigma + i\tau + 2\pi i r_2 \xi_1)I - L + P)^{-1} - ((\sigma + i\tau + 2\pi i r_1 \xi_1)I - L + P)^{-1} \right] \xi_1 \psi_j, \psi_k
\]
Use the uniformly boundedness of $((\sigma + i\tau + 2\pi i r \xi_1)I - L + P)^{-1}$ from $L^2_{\beta+\gamma}$ to $L^2_\beta$ and notice $\psi_j \in \cap_{\beta \in \mathbb{R}} L^2_{\beta}$, we have
\[
\partial_r D_{jk}(\sigma, \tau, r) = -2\pi i ((\sigma + i\tau + 2\pi i r \xi_1)I - L + P)^{-1} \xi_1 \psi_j, \psi_k.
\]
Inductively,
\[
\partial^n_r D_{jk}(\sigma, \tau, r) = (-2\pi i)^n n!((\sigma + i\tau + 2\pi i r \xi_1)I - L + P)^{-n-1} \xi_1 \psi_j, \psi_k).
\]
All these derivatives are right-continuous at $\sigma = 0$ and so $D_{jk}(\sigma, \tau, r) \in C^\infty(\mathbb{R}_+ \times \mathbb{R} \times \mathbb{R})$. $\square$
Here we need a $C^\infty$ extension theorem from [8].

**Theorem 4.6.** (Seeley). Suppose $f(x, \sigma) \in C^\infty(\mathbb{R}^d \times \{\sigma \geq 0\})$. Let $\phi \in C^\infty(\mathbb{R}^d)$ such that $\phi = 1$ on $0 \leq |t| \leq 1$ and $0$ if $|t| \geq 2$. There exists $\{a_k\}, \{b_k\}$ such that (i). $b_k < 0$; (ii). $\sum |a_k||b_k|^n < \infty$, for $n = 0, 1, \ldots$; (iii). $\sum a_k(b_k)^n = 1$, for $n = 0, 1, \ldots$; (iv). $b_k \to -\infty$. Define for $\sigma < 0$,

$$f(x, \sigma) := \sum_{k=0}^{\infty} a_k \phi(b_k \cdot \sigma) f(x, b_k \sigma).$$

Then $f(x, \sigma) \in C^\infty(\mathbb{R}^d \times \mathbb{R})$.

Applying this $C^\infty$ extension theorem, we can extend $D_{jk}$ to all $\sigma \in \mathbb{R}$ such that

$$D_{jk}(\sigma, \tau, r) \in C^\infty(\mathbb{R} \times \mathbb{R} \times \mathbb{R}),$$

and for $\sigma < 0$,

$$D_{jk}(\sigma, \tau, r) = \sum_{k=0}^{\infty} a_k \phi(b_k \cdot \sigma) D_{jk}(b_k \sigma, \tau, r).$$

**4.1.1 The Eigenvalues Equation.**

For $\sigma \geq 0$,

$$D(\sigma, \tau, r, e_1) = P((\sigma + i\tau + 2\pi i r \xi_1)I - L + P)^{-1}(\xi_1) P. \quad (4.14)$$

Thus for $j = 2, \ldots, d$, the reflection $r_j : \xi \to (\xi_1, \ldots, -\xi_j, \ldots, \xi_d)$ commutes with $D(\sigma, \tau, r, e_1)$. Also for $j = 2, \ldots, d$, $

\psi_j$ is odd with respect to $\xi_j$, and for $k \neq j$, $\psi_k$ is even with respect to $\xi_j$. Thus

$$D_{jk}(\sigma, \tau, r) = 0, \quad \text{if } 2 \leq j \leq d, 0 \leq k \leq d + 1, k \neq j$$

or $2 \leq k \leq d, 0 \leq j \leq d + 1, k \neq j$.

So the eigenvalues equation of operator $\mathcal{D}(\sigma, \tau, r, \omega)$ under basis $\{R^T \psi_j\}_{j=0}^{d+1}$ is

$$\eta I_{(d+2) \times (d+2)} = (D_{jk})_{j,k=0}^{d+1}. \quad (4.15)$$

That is

$$\eta I_{(d+2) \times (d+2)} - \begin{pmatrix}
D_{00} & D_{01} & 0 & \ldots & 0 & D_{0,d+1} \\
D_{10} & D_{11} & 0 & \ldots & 0 & D_{1,d+1} \\
0 & 0 & D_{22} & \ldots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \ldots & D_{dd} & 0 \\
D_{d+1,0} & D_{d+1,1} & 0 & \ldots & 0 & D_{d+1,d+1}
\end{pmatrix} = 0,$$

where the matrix $(D_{jk})_{j,k=0,\ldots,d+1}$ is smooth in $(\sigma, \tau, r) \in \mathbb{R}^3$.

**4.1.2 Eigenvalues of $(D_{jk})_{j,k=0}^{d+1}$.**

Firstly, we can easily get $(d - 1)$ eigenvalues. That is, for $j = 2, \ldots, d$,

$$\eta_j(\sigma, \tau, r) = D_{jj}(\sigma, \tau, r) \in C^\infty(\mathbb{R} \times \mathbb{R} \times \mathbb{R}). \quad (4.16)$$

Also, one can pick the eigenvector corresponding to $\eta_j(\sigma, \tau, r)$ to be the unit vector $e_j \in \mathbb{R}^{d+2}$.

The remaining part is

$$\eta I_{3 \times 3} - \begin{pmatrix}
D_{00} & D_{01} & D_{0,d+1} \\
D_{10} & D_{11} & D_{1,d+1} \\
D_{d+1,0} & D_{d+1,1} & D_{d+1,d+1}
\end{pmatrix} = 0. \quad (4.17)$$

We want to find its eigenvalues and the corresponding eigenvectors. Here we shall use the method in [5].

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\textbf{Theorem 4.7.} There exists $r_1 > 0$ such that the eigenvalues $\eta_j(\sigma, \tau, r)$ and the corresponding right eigenvectors $\mathbf{z}_j(\sigma, \tau, r)$ of $(D_{jk})_{j,k=0,1,d+1}$ exist and are smooth in $B(0, r_1) \subset \mathbb{R}^3$. Furthermore, for $j = 0, 1, d+1$,

$$\eta_j(0, 0, 0) = \eta_{0,j},$$

where

$$\eta_{0,0} = \sqrt{\alpha_1^2 + \alpha_2^2}, \quad \eta_{0,1} = 0, \quad \eta_{0,d+1} = -\sqrt{\alpha_1^2 + \alpha_2^2}.$$ 

\textbf{Proof.} 1. Denote matrix $F(\sigma, \tau, r) := (D_{jk})_{j,k=0,1,d+1}$ and define

$$f(\sigma, \tau, r, z, \eta) := ((F - \eta I)z, |z|^2) \in C^\infty(\mathbb{R}^7),$$ (4.18)

with $z \in \mathbb{R}^3$, $\eta \in \mathbb{R}$. We intend to use implicit function theorem near $f(\sigma, \tau, r, z, \eta) = (0, 1) \in \mathbb{R}^4$.

2. If $\sigma = \tau = r = 0$, we have

$$(D_{jk})_{j,k=0,1,d+1}|_{r=\tau=\sigma=0} = \begin{pmatrix} 0 & \alpha_1 & 0 \\ \alpha_1 & 0 & \alpha_2 \\ 0 & \alpha_2 & 0 \end{pmatrix},$$

where $\alpha_1 = (\xi_1^2 \mathbf{M}^{1/2} \mathbf{M}^{1/2})_{L^2}$, $\alpha_2 = (\xi_1^2 \mathbf{M}^{1/2}, \psi_{d+1})_{L^2}$. Thus we can obtain three distinct real eigenvalues of $F(0,0,0)$ and their corresponding eigenvectors:

$$\eta_{0,0} = \sqrt{\alpha_1^2 + \alpha_2^2}, \quad z_{0,0} = (-\alpha_2, 0, \alpha_1)^T / \sqrt{\alpha_1^2 + \alpha_2^2},$$

$$\eta_{0,1} = 0, \quad z_{0,1} = (\alpha_1, -\sqrt{\alpha_1^2 + \alpha_2^2}, \alpha_2) / \sqrt{2\alpha_1^2 + 2\alpha_2^2},$$

$$\eta_{0,d+1} = -\sqrt{\alpha_1^2 + \alpha_2^2}, \quad z_{0,d+1} = (\alpha_1, \sqrt{\alpha_1^2 + \alpha_2^2}, \alpha_2) / \sqrt{2\alpha_1^2 + 2\alpha_2^2}.$$ Then for $j = 0, 1, d+1$,

$$f(0, 0, 0, z_{0,j}, \eta_{0,j}) = (0, 1).$$

3. In order to use implicit function theorem, we need to verify that

$$\det \nabla_{z,\eta} f(0, 0, 0, z_{0,j}, \eta_{0,j}) \neq 0.$$

Here

$$\nabla_{z,\eta} f(\sigma, \tau, r, z, \eta) = \begin{pmatrix} F - \eta I & -z \\ 2z^T & 0 \end{pmatrix}_{4 \times 4}.$$ Let $F_{\varepsilon} = F(0, 0, 0, z_{0,j}, \eta_{0,j}) - \eta_{0,j} I - \varepsilon I$. Then $F_{\varepsilon}z_{0,j} = -\varepsilon z_{0,j}$ and so

$$\begin{pmatrix} F_{\varepsilon} \\ 2z_{0,j}^T \end{pmatrix} - \begin{pmatrix} z_{0,j} \\ 0 \end{pmatrix} \times \begin{pmatrix} I_{3 \times 3} & -z_{0,j} \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} F_{\varepsilon} \\ 2z_{0,j}^T \end{pmatrix} - \begin{pmatrix} 0 \\ -\varepsilon \end{pmatrix}.$$ Taking the determinant, we have

$$\left| \begin{pmatrix} F_{\varepsilon} \\ 2z_{0,j}^T \end{pmatrix} - \begin{pmatrix} z_{0,j} \\ 0 \end{pmatrix} \right| = \det(F(0, 0, 0, z_{0,j}, \eta_{0,j}) - \eta_{0,j} I - \varepsilon I)(-2) \varepsilon)$$

$$= 2 \prod_{k=0,1,d+1,k \neq j} (\eta_{0,k} - \eta_{0,j} - \varepsilon).$$

Letting $\varepsilon \to 0$, we have

$$\det \nabla_{z,\eta} f(0, 0, 0, z_{0,j}, \eta_{0,j}) = 2\prod_{k=0,1,d+1,k \neq j} (\eta_{0,k} - \eta_{0,j} - \varepsilon) \neq 0.$$ Then we can apply the implicit function theorem to get the smooth eigenvalues and eigenvectors near $(\sigma, \tau, r) = (0, 0, 0)$.

Therefore, for $j = 0, 1, d+1$, we can get the eigenvalues $\eta_j(\sigma, \tau, r) \in C^\infty(B(0, r_1); \mathbb{R})$ and eigenvectors $\mathbf{z}_j(\sigma, \tau, r) \in C^\infty(B(0, r_1); \mathbb{R}^{d+2})$ to $(D_{jk})_{j,k=0}^{d+1}$, while the eigenvectors is still denoted by $\mathbf{z}_j$ by keeping the $0^{th}, 1^{th}, (d+1)^{th}$ component the same and supplementing the $2^{nd}$ to $d^{th}$ to be 0. \hfill \square
4.1.3 Asymptotic Behavior of $\eta_j$.

Here we will investigate the derivatives of $\eta_j$ with respect to $\tau$ and $r$ at $(\sigma, \tau, r) = (0, 0, 0)$.

For $j = 2, \ldots, d$, we know

$$\eta_j(\sigma, \tau, r) = D_{jj}(\sigma, \tau, r).$$

Thus from (4.12), (4.13) and recall lemma 4.2, we have

$$\eta_j(0, 0, 0) = D_{jj}(0, 0, 0) = (\xi_1 \psi_j, \psi_j)_{L^2} = 0,$$

$$\partial_r \eta_j(0, 0, 0) = -i((-L + P)^{-1} \xi_1 \psi_j, \psi_j)_{L^2}$$

$$= -i(\xi_1 \xi_j \mathbf{M}^{1/2}, \xi_1 \xi_j \mathbf{M}^{1/2})_{L^2} = 0,$$

$$\partial_r \eta_j(0, 0, 0) = -2\pi i((-L + P)^{-1} \xi_1 (L + P)^{-1} \xi_1 \psi_j, \psi_j)_{L^2}$$

$$= -2\pi i((-L + P)^{-1} \xi_1 \psi_j, \xi_1 \psi_j)_{L^2}.$$

For the inner product $((-L + P)^{-1} \xi_1 \psi_j, \xi_1 \psi_j)_{L^2}$, we shall use the following lemma to deal with it.

**Lemma 4.8.** Let $f \in \cap_{y \in \mathbb{R}} L^2$, then $(-L + P)^{-1} P^f \in (\text{Ker} L)^\perp$ and so

$$((-L + P)^{-1} f, f)_{L^2} = (P f, P f)_{L^2} + (-L^{-1} P^f, P^f)_{L^2},$$

where $(-L^{-1} P^f, P^f)_{L^2} > 0$ whenever $P^f \neq 0$.

**Proof.** Let $g = (-L + P)^{-1} P^f$, then $(-L + P) g = P f$. Taking inner product with any $\psi \in \text{Ker} L$, we have $(g, \psi) = 0$. Thus $g \in (\text{Ker} L)^\perp$ and so $-L g = P f$, $g = -L^{-1} P f$. Thus

$$((-L + P)^{-1} f, f)_{L^2} = ((-L + P)^{-1} P f, f)_{L^2} + ((-L + P)^{-1} P^f, f)_{L^2} = (P f, P f)_{L^2} + (-L^{-1} P^f, P^f)_{L^2}.$$

Also if $P^f \neq 0$, then $(-L^{-1} P^f, P^f)_{L^2} = (h, -L h)_{L^2} > 0$, where $h = L^{-1} P f$. 

With this lemma and noticing $P(\xi_1 \psi_j) = 0$, we have

$$\partial_r \eta_j(0, 0, 0) = 2\pi i(L^{-1} \xi_1 \xi_j \mathbf{M}^{1/2}, \xi_1 \xi_j \mathbf{M}^{1/2})_{L^2}.$$

For $j = 0, 1, d + 1$, in order to obtain the asymptotic behavior of $\eta_j$, we shall use the determinant. That is to let

$$f(\sigma, \tau, r, \eta) := \det(\eta I_{3 \times 3} - (D_{jk})_{j,k=0,1,d+1}).$$

Then $f(\sigma, \tau, r, \eta_j(\sigma, \tau, r)) = 0$, for $(\sigma, \tau, r) \in B(0, r_1)$, since $\eta_j$ is the eigenvalue of $(D_{jk})_{j,k=0,1,d+1}$. Taking the derivatives with respect to $\tau$, for $|(\sigma, \tau, r)| < r_1$, we have

$$\partial_\tau f + \partial_\tau f \cdot \partial_\tau \eta_j = 0,$$

$$\partial_\tau f + \partial_\tau \eta_j \cdot \partial_\tau \eta_j = 0. \tag{4.20}$$

$$\partial_\tau f + \partial_\tau \eta_j \cdot \partial_\tau \eta_j = 0. \tag{4.21}$$

Since $f(\sigma, \tau, r) = \det(\eta I - (D_{jk})_{j,k=0,1,d+1})$, we shall use the Jacobi's formula to calculate the derivative to the determinant of a matrix. Recall that $\alpha_1 = (\xi_1 \mathbf{M}^{1/2}, \mathbf{M}^{1/2})_{L^2}$, $\alpha_2 = (\xi_1 \mathbf{M}^{1/2}, \psi_{d+1})_{L^2}$ and applying (4.12), we have

$$\partial_\tau f + \partial_\tau \eta_j \cdot \partial_\tau \eta_j = 2i(\alpha_1^2 + \alpha_2^2) \eta_{0,j}. \tag{4.22}$$
If we define $\alpha_3 = ((-L + P)^{-1}\xi_1\psi_1,\xi_1\psi_1)_{L^2}$, $\alpha_4 = ((-L + P)^{-1}\xi_1\psi_{d+1},\xi_1\psi_{d+1})_{L^2}$, then similar to (4.22) and applying (4.13), we have

$$\partial_r f(0,0,0,\eta_{0,j}) = \text{tr} \left( \begin{pmatrix} \eta_{0,j} - \alpha_1^2 & \alpha_1\eta_{0,j} & \alpha_1\eta_{0,j} & \alpha_2\eta_{0,j} \\ \alpha_1\eta_{0,j} & \eta_{0,j} - \alpha_1^2 & \alpha_2\eta_{0,j} & \alpha_1\eta_{0,j} \\ \alpha_2\eta_{0,j} & \alpha_1\eta_{0,j} & \eta_{0,j} - \alpha_1^2 & \alpha_2\eta_{0,j} \\ \alpha_2\eta_{0,j} & \alpha_1\eta_{0,j} & \alpha_2\eta_{0,j} & \eta_{0,j} - \alpha_1^2 \end{pmatrix} \times \begin{pmatrix} 2\pi i\alpha_1 & 0 & 2\pi i\alpha_2 \\ 0 & 2\pi i\alpha_3 & 0 \\ 2\pi i\alpha_2 & 0 & 2\pi i\alpha_4 \end{pmatrix} \right)$$

$$= 2\pi i(\alpha_1\eta_{0,j}^2 + \alpha_3\eta_{0,j}^2 + \alpha_4\eta_{0,j}^2 + \alpha_1\alpha_2 - \alpha_1^2\alpha_4). \quad (4.23)$$

Also one can easily get

$$\partial_\eta f(0,0,0,\eta_{0,j}) = 3\eta_{0,j}^2 - \alpha_1^2 - \alpha_2^2. \quad (4.24)$$

Thus from (4.20), (4.21) and use (4.22), (4.23), (4.24), we can summarize:

**Theorem 4.9.** For cases $j = 2,\ldots,d$, we have

$$\eta_j(0,0,0) = \partial_r \eta_j(0,0,0) = 0,$$

$$\partial_r \eta_j(0,0,0) = 2\pi i(L^{-1}\xi_1\xi_j M^{1/2},\xi_1\xi_j M^{1/2})_{L^2},$$

where $(L^{-1}\xi_1\xi_j M^{1/2},\xi_1\xi_j M^{1/2})_{L^2} < 0$. For cases $j = 0,1,d+1$, we have

$$\eta_{0,0} = \sqrt{\alpha_1^2 + \alpha_2^2}, \quad \eta_{0,1} = 0, \quad \eta_{0,d+1} = -\sqrt{\alpha_1^2 + \alpha_2^2},$$

and

$$\partial_r \eta_j(0,0,0) = \begin{cases} 0, & \text{if } j = 1, \\ -i\eta_{0,j}, & \text{if } j = 0,d+1, \\ \frac{2\pi i(\alpha_1\alpha_2^2 - \alpha_1^2\alpha_4)}{\alpha_1^2 + \alpha_2^2}, & \text{if } j = 1, \\ -\pi i((\alpha_1 + \alpha_3 + \alpha_4)(\alpha_1^2 + \alpha_2^2) + \alpha_1\alpha_2 - \alpha_1^2\alpha_4)}{\alpha_1^2 + \alpha_2^2}, & \text{if } j = 0,d+1. \end{cases}$$

### 4.1.4 Eigen-projection of $D(\sigma,\tau,\rho,\omega)$.

In the last section, we obtained $d+2$ smooth eigenvalues $\eta_j(\sigma,\tau,\rho,\omega)$ and $d+2$ smooth right eigenvectors $z_j(\sigma,\tau,\rho,\omega) \in \mathbb{R}^{d+2}$ to $(D_{jk})_{j,k=0}^{d+1}$, when $(\sigma,\tau,\rho,\omega) \in B(0,r_1)$. Notice that the dimension of $\text{Ker}L$ is $d+2$ and $(D_{jk})_{j,k=0,\ldots,d+1}$ is the matrix representation of $D(\sigma,\tau,\rho,\omega)$ under the basis $\{R^T \psi_j\}$ of $\text{Ker}L$:

$$D(\sigma,\tau,\rho,\omega)(R^T \psi_0,\ldots,R^T \psi_{d+1}) = (R^T \psi_0,\ldots,R^T \psi_{d+1})(D_{jk})^{d+1}_{j,k=0}. \quad (4.25)$$

So we know that $\{\eta_j(\sigma,\tau,\rho,\omega)\}_{j=0,\ldots,d+1}$ are the eigenvalues of $D(\sigma,\tau,\rho,\omega)$ and the eigenspace of $D(\sigma,\tau,\rho,\omega)$ is exactly $\text{Ker}L$. Thus $D(\sigma,\tau,\rho,\omega)$ has eigenvectors:

$$\phi_j(\sigma,\tau,\rho,\omega) = \sum_{k=0}^{d+1} z_j^{(k)}(\sigma,\tau,\rho,\omega) R^T \psi_k \in C^\infty(B(0,r_1);L^2),$$

where $(z_j^{(0)},\ldots,z_j^{(d+1)}) = z_j$. Define the smooth eigen-projections of $D(\sigma,\tau,\rho,\omega)$ on $L^2$ by

$$P_j(\sigma,\tau,\rho,\omega)f := (f,\phi_j)_{L^2} \phi_j \in C^\infty(B(0,r_1);L^2_{\beta}), \quad (4.26)$$

for any $f \in L^2_{\beta}$, $\beta \in \mathbb{R}$. Then $\sum_{j=0}^{d+1} P_j(\sigma,\tau,\rho,\omega) = P$.  

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4.2 Eigenvalues of $P(\lambda I - \hat{B}_0(y))^{-1}P$.

Recall (4.8) that

$$P(\lambda I - \hat{B}_0(y))^{-1}P = \frac{1}{\lambda + 1} (P - 2\pi irD(\sigma, \tau, r, \omega)).$$

We regard $P(\lambda I - \hat{B}_0(y))^{-1}P$ as an operator on Ker$L$, then we know its $j^{th}$ ($j = 0, \ldots, d + 1$) eigenvalue is

$$\mu_j(\sigma, \tau, r) := \frac{1}{\lambda + 1} (1 - 2\pi ir \eta_j(\sigma, \tau, r)) \in C^\infty(B(0, r_1)).$$

(4.25)

**Theorem 4.10.** There exists $0 < r_2 \leq r_1$ and $\sigma_j(r), \tau_j(r) \in C^\infty([-r_2, r_2])$, s.t. for $r \leq r_2$,

$$\mu_j(\sigma_j(r), \tau_j(r), r) = 1.$$  

(4.26)

Moreover,

$$\sigma_j(r) = \sigma_j^{(2)} r^2 + O(r^3),$$  

(4.27)

$$\tau_j(r) = \tau_j^{(1)} r + O(r^3),$$  

(4.28)

as $r \to 0$, where $\sigma_j^{(2)} < 0$, $\tau_j^{(1)} \in \mathbb{R}$ with explicit expression

$$\tau_j^{(1)} = \begin{cases} 0, & \text{if } j = 1, \ldots, d, \\ - 2\pi \sqrt{1 + 2/d}, & \text{if } j = 0, \\ 2\pi \sqrt{1 + 2/d}, & \text{if } j = d + 1, \end{cases}$$

$$\sigma_j^{(2)} = \begin{cases} 8\pi^2 (L^{-1} \xi_j \psi_j, \xi_j \psi_j)_{L^2}, & \text{if } j = 2, \ldots, d, \\ \frac{8\pi^2}{1 + 2/d} (L^{-1} P^\perp (\xi_j \psi_{d+1}), P^\perp (\xi_j \psi_{d+1}))_{L^2}, & \text{if } j = 1, \\ \frac{4\pi^2}{1 + 2/d} (L^{-1} P^\perp (\xi_j \psi_1), P^\perp (\xi_j \psi_1))_{L^2} + \frac{8\pi^2}{d + 2} (L^{-1} P^\perp (\xi_j \psi_{d+1}), P^\perp (\xi_j \psi_{d+1}))_{L^2}, & \text{if } j = 0, d + 1. \end{cases}$$

**Proof.** 1. Define $f = (\text{Re} \mu_j, \text{Im} \mu_j) : B(0, r_1) \subset \mathbb{R}^3 \to \mathbb{R}^2$ to be a smooth function. Notice

$$\mu_j(0, 0, 0) = 1, \ \partial_\sigma \mu_j(0, 0, 0) = -1, \ \partial_\tau \mu_j(0, 0, 0) = -i.$$  

(4.29)

Thus $f(0, 0, 0) = (1, 0)$, det $\nabla_{\sigma, \tau} f(0, 0, 0) = 1$. By implicit function theorem, there exists $r_2 \in (0, r_1]$ and functions $\sigma_j(r), \tau_j(r) \in C^\infty(|r| \leq r_2)$ such that for $|r| \leq r_2$,

$$\sigma(0) = \tau(0) = 0, \ \mu_j(\sigma_j(r), \tau_j(r), r) = 1.$$  

2. For $|r| \leq r_2$, by (4.25),

$$1 = \mu_j(\sigma_j(r), \tau_j(r), r) = \frac{1}{\sigma_j(r) + i \tau_j(r) + 1} (1 - 2\pi i r \eta_j(\sigma_j(r), \tau_j(r), r)), $$  

$$\sigma_j(r) + i \tau_j(r) = -2\pi i r \eta_j(\sigma_j(r), \tau_j(r), r).$$  

(4.30)

Using (4.30) and applying the behavior of $\eta_j$ from 4.9, we have

$$\sigma_j'(0) + i \tau_j'(0) = -2\pi i \eta_j(0, 0, 0) = \begin{cases} 0, & \text{if } j = 1, \ldots, d, \\ - 2\pi i \sqrt{\alpha_1^2 + \alpha_2^2}, & \text{if } j = 0, \\ 2\pi i \sqrt{\alpha_1^2 + \alpha_2^2}, & \text{if } j = d + 1. \end{cases}$$
So $\sigma_j'(0) = 0$ and $\tau_j'(0) = -2\pi\eta_j(0, 0, 0)$, then
\[
\sigma_j''(0) + i\tau_j''(0) = -4\pi i (\partial_\sigma\eta_j \cdot \sigma_j' + \partial_\tau\eta_j \cdot \tau_j' + \partial_\omega \eta_j)|_{\sigma=\tau=r=0} = 8\pi^2 i \partial_\tau \cdot \eta_j(0, 0, 0)\eta_j(0, 0, 0) - 4\pi i \partial_\eta_j(0, 0, 0).
\]

If $j = 2, \ldots, d$, then
\[
\sigma_j''(0) + i\tau_j''(0) = 8\pi^2(L^{-1}\xi_j M^{1/2}, \xi_j M^{1/2})_{L^2} < 0.
\]

If $j = 0, 1, d + 1$, then
\[
\sigma_j''(0) + i\tau_j''(0) = 4\pi^2(2(\alpha_1^2 + \alpha_2^2) - \frac{(\alpha_1 + \alpha_3 + \alpha_4)(\alpha_1^2 + \alpha_2^2) + \alpha_1\alpha_2 - \alpha_2^2\alpha_4}{\alpha_1^2 + \alpha_2^2}),
\]
\[
\quad \text{if } j = 0, d + 1.
\]

Using Gamma function, we can calculate that
\[
\alpha_1 = (\xi_1 M^{1/2}, \xi_1 M^{1/2})_{L^2} = 1, \quad \alpha_2 = (\xi_1 M^{1/2}, \xi_1 \psi_{d+1})_{L^2} = \sqrt{\frac{2}{d}}.
\]

Also by lemma 4.8, we have
\[
\alpha_3 = ((-L + P)^{-1}\xi_1 \psi_1, \xi_1 \psi_1)_{L^2} = \| P(\xi_1 \psi_1) \|^2_{L^2} + (-L^{-1}P^\perp(\xi_1 \psi_1), P^\perp(\xi_1 \psi_1))_{L^2},
\]
\[
\alpha_4 = ((-L + P)^{-1}\xi_1 \psi_{d+1}, \xi_1 \psi_{d+1})_{L^2} = \| P(\xi_1 \psi_{d+1}) \|^2_{L^2} + (-L^{-1}P^\perp(\xi_1 \psi_{d+1}), P^\perp(\xi_1 \psi_{d+1}))_{L^2},
\]
and here
\[
\| P(\xi_1 \psi_1) \|^2_{L^2} = \sum_{j=0}^{d+1} |(\xi_1^2 M^{1/2}, \psi_j)_{L^2}|^2 = |(\xi_1^2 M^{1/2}, M^{1/2})_{L^2}|^2 + |(\xi_1^2 M^{1/2}, \psi_{d+1})_{L^2}|^2 = 1 + \frac{2}{d},
\]
\[
\| P(\xi_1 \psi_{d+1}) \|^2_{L^2} = \sum_{j=0}^{d+1} |(\xi_1 \psi_{d+1}, \psi_j)_{L^2}|^2 = |(\xi_1 \psi_{d+1}, \xi_1 M^{1/2})_{L^2}|^2 = \frac{2}{d}.
\]

Substitute these values into (4.31) and we will get the explicit expression of $\tau_j''(0)$ and $\sigma_j''(0)$.

Denote
\[
H(\sigma, \tau, r, \omega) = \frac{1}{\lambda + 1}(P - 2\pi i r D(\sigma, \tau, r, \omega)).
\]

As an operator defined on finite dimensional space Ker$L$, for $|r| \leq r_2$, we have
\[
I - H(\sigma, \tau, r, \omega) = \sum_{j=0}^{d+1} (1 - \mu_j(\sigma, \tau, r)) P_j(\sigma, \tau, r, \omega).
\]

Then we claim that on Ker$L$,
\[
(I - H(\sigma, \tau, r, \omega))^{-1} = \sum_{j=0}^{d+1} (1 - \mu_j(\sigma, \tau, r))^{-1} P_j(\sigma, \tau, r, \omega),
\]
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here the inverse is taken in $\text{Ker} L$. Indeed, on $\text{ker} L$,

\[
(I - H(\sigma, \tau, r, \omega)) \sum_{j=0}^{d+1} (1 - \mu_j(\sigma, \tau, r))^{-1} P_j(\sigma, \tau, r, \omega)
\]

\[
= \sum_{j=0}^{d+1} (1 - \mu_j(\sigma, \tau, r)) P_j(\sigma, \tau, r, \omega) \times \sum_{k=0}^{d+1} (1 - \mu_k(\sigma, \tau, y))^{-1} P_k(\sigma, \tau, y)
\]

\[
= \sum_{j=0}^{d+1} P_j(\sigma, \tau, r, \omega) = P.
\]

Therefore for $\text{Re}\lambda \geq 0$, $y \in \mathbb{R}^d \setminus \{0\}$,

\[
(\lambda I - \hat{B}(y))^{-1} = (\lambda I - \hat{B}_0(y))^{-1} + \sum_{j=0}^{d+1} (1 - \mu_j(\sigma, \tau, r))^{-1} P_j(\sigma, \tau, r, \omega) (\lambda I - \hat{B}_0(y))^{-1}.
\]

(4.33)

Denote $U_j(\sigma, \tau, y) = (\lambda I - \hat{B}_0(y))^{-1} P_j(\sigma, \tau, r, \omega) (\lambda I - \hat{B}_0(y))^{-1}$. Differentiate (4.33) with respect to $\tau$, we have

\[
(\lambda I - \hat{B}(y))^{-n-1} = (\lambda I - \hat{B}_0(y))^{-n-1} + \sum_{j=0}^{d+1} \sum_{k=0}^{n} (1 - \mu_j(\sigma, \tau, r))^{-k-1} U_{j,k}^{(n)}(\sigma, \tau, y),
\]

(4.34)

where $U_{j,k}^{(n)}(\sigma, \tau, y) \in C^\infty([\{\sigma, \tau, y\} : |\sigma, \tau, y| \leq r_2]; L(L^2_\sigma))$ is given as a linear combination of products of $\mu_j, U_j$ and their derivatives with respect to $\tau$, and

\[
\sup_{(\sigma, \tau, \tau) \leq \mathbb{R}(0, r_2)} \|U_{j,k}^{(n)}(\sigma, \tau, y)\|_{L(L^2_\sigma)} < \infty.
\]

5 Estimate on $e^{t\hat{B}(y)}$

In this section, we will give the proof of boundedness of semigroup $e^{t\hat{B}(y)}$ and its asymptotic behavior when $t \to \infty$.

Fistly we need the resolvent identities and the inversion semigroup formula.

\[
(\lambda I - \hat{B}(y))^{-1} = (I - (\lambda I - \hat{A}(y))^{-1} K)^{-1} (\lambda I - \hat{A}(y))^{-1},
\]

(5.1)

and for $\sigma > 0$,

\[
e^{t\hat{B}(y)} = e^{s\text{lim}_{a \to \infty} \frac{1}{2\pi i} \int_{\sigma - ia}^{\sigma + ia} e^{\lambda t} (\lambda I - \hat{B}(y))^{-1} u d\lambda},
\]

(5.2)

for $u \in D_0(\hat{B}(y))$, where the limit in taken with respect to $L^2(\mathbb{R}^d_\sigma)$ norm.

Now we investigate the right hand side of (5.2). Consider $L^2(\mathbb{R}^d)$ to be the whole space. For $y \in \mathbb{R}^d \setminus \{0\}$, we have $\{\text{Re}\lambda \geq 0\} \subset \rho(\hat{B}(y))$ and thus $(\lambda I - \hat{B}(y))^{-1} : \{\text{Re}\lambda \geq 0\} \to L(L^2)$ is a holomorphic operator-valued function with respect to $\lambda$ in $\{\text{Re}\lambda \geq 0\}$. Applying Cauchy theorom, for $u \in L^2$, $\sigma > 0$,

\[
\frac{1}{2\pi i} \int_{\sigma - ia}^{\sigma + ia} e^{\lambda t} (\lambda I - \hat{B}(y))^{-1} u d\lambda = \frac{1}{2\pi i} \left( \int_{-ia}^{ia} + \int_{\sigma - ia}^{\sigma + ia} + \int_{ia}^{\sigma - ia} \right) e^{\lambda t} (\lambda I - \hat{B}(y))^{-1} u d\lambda
\]

\[
= I_1 u + I_2 u + I_3 u.
\]
Remark 5.1. The whole space really matters, since only in $L^2$ we can use $\{\mathfrak{R}\lambda \geq 0\} \subset \rho(\hat{B}(y))$. But later we will assume $u \in L^2_{\beta} \subset L^2$, with $\beta \geq 0$.

Firstly we consider the part $I_2$ and $I_3$. If furthermore $u \in L^2_{\beta+\gamma}$, then

$$I_2u = \frac{1}{2\pi i} \int_{\mathbb{R}} e^{\lambda t} (\lambda I - \hat{B}(y))^{-1} u \, d\lambda = \frac{1}{2\pi i} \int_{\mathbb{R}} e^{\lambda t} (I - (\lambda I - \hat{A}(y))^{-1} K)^{-1} (\lambda I - \hat{A}(y))^{-1} u \, d\lambda.$$  

Notice here $y$ is fixed, then by theorem 3.6, $\sup_{\lambda \in C_{+\gamma}} \| (I - (\lambda I - \hat{A}(y))^{-1} K)^{-1} \|_{L(L^2_{\beta})} \leq C_{y,\beta} < \infty$. Thus by theorem 3.3,

$$\| I_2u \|_{L^2_{\beta}} \leq \frac{C_u}{2\pi} \int_{\mathbb{R}} e^{\sigma t} \| (\lambda I - \hat{A}(y))^{-1} u \|_{L^2_{\beta}} \, d\lambda \leq C_{y,\nu} e^{\sigma t} \int_{\mathbb{R}} \lambda \left( \| u \|_{L^2_{\beta+\gamma}(|\xi| \geq R)} + C_{\nu} \| u \|_{L^2_{\beta+\gamma} |\alpha|} \right) \, d\lambda \rightarrow 0,$$

as $R \rightarrow \infty$, and hence $a \geq 2\pi |y|R \rightarrow \infty$. The part $I_3$ is similar. Thus $I_2u, I_3u \rightarrow 0$ in $L^2_{\beta}$ norm as $a \rightarrow \infty$ if $u \in L^2_{\beta+\gamma}$.

To deal with the part $I_1$, we need the following lemma.

Lemma 5.2. For any $y_1 > 0$, $\sigma \geq 0$, we have

$$\int_{\mathbb{R}} \| ((\sigma + i\tau)I - \hat{A}(y))^{-1} u \|_{L^2_{\beta}}^2 \, d\tau \leq C_{\nu} \| u \|_{L^2_{\beta+\gamma/2}}^2.$$  

Thus for $|y| \geq y_1$,

$$\int_{\mathbb{R}} \| (\sigma + i\tau)I - \hat{B}(y))^{-1} u \|_{L^2_{\beta}}^2 \, d\tau \leq C_{\nu, y_1, \beta} \| u \|_{L^2_{\beta+\gamma/2}}^2.$$  

Proof. For $\sigma \geq 0$,

$$\int_{\mathbb{R}} \| ((\sigma + i\tau)I - \hat{A}(y))^{-1} u \|_{L^2_{\beta}}^2 \, d\tau = \frac{1}{(1 + |\xi|)^{2\beta}} |u(\xi)|^2 \int_{\mathbb{R}} \frac{1}{\sigma + \nu(\xi)^2 + |\tau + 2\pi y \cdot \xi|^2} \, d\sigma d\xi \leq \frac{1}{(1 + |\xi|)^{2\beta}} |u(\xi)|^2 \frac{1}{|\nu(\xi)|} \int_{\mathbb{R}} \frac{1}{1 + \tau^2} \, d\tau d\xi \leq C_{\nu} \| u \|_{L^2_{\beta+\gamma/2}}^2.$$  

Then using the fact that $\sup_{\lambda \in C_{+\gamma}} \| (I - (\lambda I - \hat{A}(y))^{-1} K)^{-1} \|_{L(L^2_{\beta})} \leq C_{y,\beta} < \infty$, and the resolvent identity (5.1), we get the second assertion. \hfill $\square$

Now for the part $I_1$,

$$I_1u = \frac{1}{2\pi i} \int_{-\alpha}^{+\alpha} e^{i\tau t} (i\tau I - \hat{B}(y))^{-1} u \, d\tau.$$ \hspace{1cm} (5.3)

Notice for $\tau \in \mathbb{R}$, $y \in \mathbb{R}^{d} \setminus \{0\}$, we have $i\tau \in \rho(\hat{B}(y))$, and

$$\frac{d^k}{d\tau^k} ((i\tau I - \hat{B}(y))^{-1} u(\xi)) = i^k k! (-i)^k (i\tau I - \hat{B}(y))^{-k-1} u(\xi).$$ \hspace{1cm} (5.4)
Thus using integration by parts,
\[
\int_{-a}^{a} e^{i\tau t} (i\tau I - \hat{B}(y))^{-1} u(\xi) \, d\tau
= e^{i\tau t} (i\tau I - \hat{B}(y))^{-1} u(\xi)\bigg|_{\tau = a}^{\tau = -a} - \int_{-a}^{a} \frac{d}{d\tau} ((i\tau I - \hat{B}(y))^{-1} u(\xi)) e^{i\tau t} \, d\tau
= \ldots
= \sum_{k=1}^{n} \frac{(-1)^{k-1} e^{i\tau t}}{(it)^k} \frac{d}{d\tau} ((i\tau I - \hat{B}(y))^{-1} u(\xi))\bigg|_{\tau = a}^{\tau = -a}
+ (-1)^n \int_{-a}^{a} \frac{d^n}{d\tau^n} ((i\tau I - \hat{B}(y))^{-1} u(\xi)) e^{i\tau t} \, d\tau
= \sum_{k=1}^{n} \frac{e^{i\tau t}}{it^k} (k-1)! (i\tau I - \hat{B}(y))^{-k} u(\xi)\bigg|_{\tau = a}^{\tau = -a} + n! \int_{-a}^{a} (i\tau I - \hat{B}(y))^{-n-1} u(\xi) e^{i\tau t} \, d\tau.
\]

Recall theorem 3.3 that for $R > 0$, $\text{Re} \lambda > 0$ with $|\lambda| \geq 4\pi |y|R$,
\[
\| (\lambda I - \hat{A}(y))^{-1} u \|_{L^2_{\beta}} \leq C \| u \|_{L^2_{\beta+\gamma,\{\xi|\geq R\}}} + C \| u \|_{L^2_{\beta+\gamma,\{|\xi|< R\}}} \frac{1}{|\lambda|}.
\]
Without loss of generality, we can let $a \geq 1$. Notice that here $y \neq 0$ is fixed, then applying 3.7 and 3.6, we have
\[
\| (ia I - \hat{B}(y))^{-k} u(\xi) \|_{L^2_{\beta}} \leq C_{k-1} \| (ia I - \hat{B}(y))^{-1} u(\xi) \|_{L^2_{\beta+2+\gamma}}
\leq C_k \left( \| u \|_{L^2_{\beta+k,\{\xi|\geq R\}}} + \| u \|_{L^2_{\beta+k+\gamma,\{|\xi|< R\}}} \right) \to 0,
\]
as $R \to \infty$ and $a \to \infty$, if $u \in L^2_{\beta+2+\gamma}$. Thus it suffices to deal with following term when $a \to \infty$.
\[
n! \int_{-a}^{a} (i\tau I - \hat{B}(y))^{-n-1} u(\xi) e^{i\tau t} \, d\tau.
\]

### 5.1 Away from origin

Notice from 3.7, we have for $\beta \in \mathbb{R}$, $b > 0$ that
\[
C_b := \sup_{\text{Re} \lambda \geq 0, |y| \geq b} \| (\lambda I - \hat{B}(y))^{-1} \|_{L(L^2_{\beta+\gamma},L^2_{\beta})} < \infty.
\]
So if $|y| \geq b$, for $u \in L^2_{\beta+n,\gamma}$, $v \in L^2_{\beta}$, we have
\[
\left| (n! \int_{-a}^{a} (i\tau I - \hat{B}(y))^{-n-1} u(\xi) \frac{e^{i\tau t}}{t^n} \, d\tau, v) \right|_{L^2_{\beta}}
\leq \frac{n!}{t^n} \int_{-a}^{a} \| (i\tau I - \hat{B}(y))^{-n} u(\xi) \|_{L^2_{\beta+n/2,\{\xi|\geq R\}}} \| (-i\tau I - \hat{B}(y)^*)^{-1} v(\xi) \|_{L^2_{\beta-n/2}} \, d\tau
\leq \frac{n!}{t^n} \left( \int_{-a}^{a} \| (i\tau I - \hat{B}(y))^{-n} u(\xi) \|_{L^2_{\beta+n/2}}^2 \, d\tau \right)^{1/2} \left( \int_{-a}^{a} \| (-i\tau I - \hat{B}(y)^*)^{-1} v(\xi) \|_{L^2_{\beta-n/2}}^2 \, d\tau \right)^{1/2}
\leq \frac{C_{\gamma,n} n!}{t^n} \| u \|_{L^2_{\beta+n,\gamma}} \| v \|_{L^2_{\beta}}.
\]
Notice here $\hat{B}(y)^* = 2\pi iy \cdot \xi + L$ has the same boundedness as $\hat{B}(y)$. Thus if $u \in L^2_{\beta+n,\gamma}$, \(n! \int_{-a}^{a} (i\tau I - \hat{B}(y))^{-n-1} u(\xi) \frac{e^{i\tau t}}{t^n} \, d\tau \) is a bounded sequence in $\left( L^2_{\beta}\right)^*$, hence has a weakly * convergent subsequence (denoted
by \( \{ a_k \} \) and its weak limit \( Iu \) is controlled by \( \frac{n!}{|u|_{L^2_{\lambda + n, \gamma}}} \). That is

\[
Iu := \operatorname{weak-lim}_{a_k \to \infty} I_1 u = \operatorname{weak-lim}_{a_k \to \infty} \frac{n!}{2\pi i t^n} \int_{a_k}^{a} e^{\tau t} (i\tau I - \hat{B}(y))^{-n-1} u(\xi) \, d\xi,
\]

and

\[
\|Iu\|_{L^2_{\lambda}} = \|\operatorname{weak-lim}_{a_k \to \infty} I_1 u\|_{L^2_{\lambda}} \leq \frac{C_{n, n!}}{t^n} \|u\|_{L^2_{\lambda + n, \gamma}}.
\]

**Remark 5.3.** Here the integral region can be replaced by \((b, a)\), for any \( b > 0 \).

### 5.2 Near origin

For \( y \) near the origin but \( y \neq 0 \), we can also use the method of integration by parts to obtain

\[
\operatorname{s-lim}_{a \to \infty} \frac{1}{2\pi i} \int_{a-i}^{a+i} e^{\lambda t} (\frac{\lambda}{\sqrt{2\pi}})^{-n-1} u \, d\lambda = \operatorname{s-lim}_{a \to \infty} \frac{n!}{2\pi i t^n} \int_{-a}^{a} e^{\tau t} (i\tau I - \hat{B}(y))^{-n-1} u(\xi) \, d\xi,
\]

where the limit is taken in \( L^2_{\lambda} \). Then for \( b \in (0, a) \), we divide the integral region to be

\[
\int_{-a}^{a} = \int_{b}^{a} + \int_{-b}^{b} + \int_{-a}^{a}.
\]

The first and the last term has a weak limit by same argument as in section 5.1, where we essentially need the fact from 3.7 that

\[
C_b := \sup_{\lambda \geq 0, |\text{Im}\lambda| \geq b, y \in \mathbb{R}^d} \|\lambda - \hat{B}(y)\|^{-n} \|u\|_{L^2_{\lambda, \lambda^2}} < \infty.
\]

So it remains to deal with the integral

\[
\int_{-b}^{b} e^{\tau t} (i\tau I - \hat{B}(y))^{-n-1} u(\xi) \, d\xi.
\]

We will use the identity (4.34) from section 4. The following lemma is used for controlling the term \((1 - \mu_j(\sigma, \tau, r))^{-1}\) in (4.34).

**Lemma 5.4.** Let \( f(x, y, z) \in C^1((x, y, z) \in \mathbb{R}^3 : |(x, y, z)| \leq r) \). Suppose \( \nabla_{(x, y, z)} f(0, 0, 0) = -(1, i, a) \), with some constant \( a \in \mathbb{C} \). Then for \( |(x_1, y_1)| \leq r_1, |(x_2, y_2)| \leq r_1 \),

\[
\frac{1}{2} |(x_1 - x_2, y_1 - y_2)| \leq |f(x_1, y_1, z) - f(x_2, y_2, z)| \leq \frac{3}{2} |(x_1 - x_2, y_1 - y_2)|.
\]

**Proof.** By mean value theorem,

\[
f(x_1, y_1, z) - f(x_2, y_2, z) = \nabla_{(x, y, z)} f(x_\theta, y_\theta, 0) \cdot (x_1 - x_2, y_1 - y_2, 0),
\]

where \( x_\theta = x_1 + (1 - \theta)x_2, y_\theta = y_1 + (1 - \theta)y_2, \) with \( \theta \in (0, 1) \). Take \( r_1 \in (0, r) \) so small that for \( |(x_\theta, y_\theta)| \leq r_1 \),

\[
|\nabla_{(x, y, z)} f(x_\theta, y_\theta, 0) - \nabla_{(x, y, z)} f(0, 0, 0)| < \frac{1}{2}
\]

Then

\[
f(x_1, y_1, z) - f(x_2, y_2, z) = \nabla_{(x, y, z)} f(0, 0, 0) \cdot (x_1 - x_2, y_1 - y_2, 0)
\]

\[
+ (\nabla_{(x, y, z)} f(x_\theta, y_\theta, 0) - \nabla_{(x, y, z)} f(0, 0, 0)) \cdot (x_1 - x_2, y_1 - y_2, 0),
\]

\[
|f(x_1, y_1, z) - f(x_2, y_2, z)| \geq |(1, i, a) \cdot (x_1 - x_2, y_1 - y_2, 0)| - \frac{1}{2} |(x_1 - x_2, y_1 - y_2, 0)|
\]

\[
= \frac{1}{2} |(x_1 - x_2, y_1 - y_2)|,
\]

and the second inequality is similar. \( \square \)
Write \( y = r\omega \), with \( r \in \mathbb{R}_+ \), \( \omega \in \mathbb{S}^{d-1} \). Applying theorem 4.10, for \( r \leq r_2 \),
\[(1 - \mu_j(\sigma, \tau, r))^{-1} = (\mu_j(\sigma_j(r), \tau_j(r), r) - \mu_j(\sigma, \tau, r))^{-1}.
\]
Thus by lemma 5.4,
\[
\frac{2}{3} |(\sigma_j(r) - \sigma_j(r) - \tau)| \leq |1 - \mu_j(\sigma, \tau, r)|^{-1} \leq 2|(\sigma_j(r) - \sigma_j(r) - \tau)|.
\]
By asymptotic behavior of \( \sigma(r) \) and \( \tau(r) \) in 4.10, there exists \( \eta_0 > 0 \), and \( r_3 \in (0, r_2) \) such that for \( r \leq r_3 \),
\[\sigma_j(r) \leq -2\eta_0 r^2, \quad |\tau_j(r) - \tau_j^{(1)} r| \leq \eta_0 r^2.\]
Thus for \( r \leq r_3 \), the equation (5.7) becomes
\[
\int_{-b}^{b} e^{i\tau t} (i\tau I - \tilde{B}(y))^{-n-1} u(\xi) d\tau
= \int_{-b}^{b} e^{i\tau t} \left( (\lambda I - \tilde{B}_0(y))^{-n-1} + \sum_{j=0}^{d+1} \sum_{k=0}^{n} (1 - \mu_j(\sigma, \tau, r))^{-k-1} U_{j,k}^{(n)}(\sigma, \tau, y) \right) u(\xi) d\tau
= \int_{-b}^{b} e^{i\tau t} \left( (\lambda I - \tilde{B}_0(y))^{-n-1} + \sum_{j=0}^{d+1} \sum_{k=0}^{n} \int_{-b}^{b} e^{i\tau t} (1 - \mu_j(\sigma, \tau, r))^{-k-1} U_{j,k}^{(n)}(\sigma, \tau, y) u(\xi) d\tau
= I_0 u + \sum_{j=0}^{d+1} \sum_{k=0}^{n} I_{j,k}^{(n)} u.
\]
Noticing that \( C_\beta := \sup_{\lambda \geq 0, y \in \mathbb{R}^d} ||(\lambda I - \tilde{B}_0(y))^{-1}||_{L^2_\beta \rightarrow L^2_\beta} < \infty \), we have
\[
||I_0 u||_{L^2_\beta} \leq C_\beta \sup_{\lambda \geq 0, y \in \mathbb{R}^d} ||(\lambda I - \tilde{B}_0(y))^{-1}||_{L^2_\beta \rightarrow L^2_\beta} ||u||_{L^2_\beta} \leq C_{b,\beta} ||u||_{L^2_\beta}.
\]
On the other hand, since \( U_{j,k}^{(n)}(0, \tau, y) \) is smooth in \( \{(\tau, y) \in [-r_3, r_3] \times \mathbb{R}^d \} \), we have
\[
\sup_{|\tau| \leq r_3, y \in \mathbb{R}^d} ||U_{j,k}^{(n)}(0, \tau, y)||_{L^2_\beta} = C_\beta < \infty.
\]
and then if \( b \in (0, r_3] \)
\[
||I_{j,k}^{(n)} u||_{L^2_\beta} \leq \left\| \int_{-b}^{b} e^{i\tau t} (1 - \mu_j(0, \tau, r))^{-k-1} U_{j,k}^{(n)}(0, \tau, y) u(\xi) d\tau \right\|_{L^2_\beta}
\leq C_{\beta, n} ||u(\xi)||_{L^2_\beta} \int_{-b}^{b} \left| 1 - \mu_j(0, \tau, r) \right|^{-k-1} d\tau
\leq C_{\beta, n} ||u(\xi)||_{L^2_\beta} \int_{-b}^{b} \frac{1}{\left| (\sigma_j(r) + |\tau - \tau_j(r)|) \right|^{k+1}} d\tau.
\]
Now for \( r \leq r_3 \), we have \( \sigma_j(r) \leq -2\eta_0 r^2, |\tau_j(r) - \tau_j^{(1)} r| \leq \eta_0 r^2 \). Thus for \( k \in \mathbb{N} \),
\[
\int_{-b}^{b} \frac{1}{\left| (\sigma_j(r) + |\tau - \tau_j(r)|) \right|^{k+1}} d\tau \leq \int_{-b}^{b} \frac{1}{(\eta_0 r^2 + |\tau - \tau_j^{(1)} r|)^{k+1}} d\tau
\leq \int_{0}^{b+|\tau_j^{(1)} r|} \frac{2}{(\eta_0 r^2 + \tau)^{k+1}} d\tau.
\]
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If \( k = 0 \), then
\[
\int_{-b}^{b} \frac{1}{(|\sigma_j(r)| + |r - \tau_j(r)|)^{k+1}} d\tau \leq 2 \log \left( \frac{b + |\tau_j^{(1)} r + \eta_0 r^2|}{\eta_0 r^2} \right) \leq C \log(r^{-1} + e).
\]

If \( k > 0 \), then
\[
\int_{-b}^{b} \frac{1}{(|\sigma_j(r)| + |r - \tau_j(r)|)^{k+1}} d\tau \leq \frac{2}{k} \left[ (\eta_0 r^2 + b + |\tau_j^{(1)} r|)^{-k} - (\eta_0 r^2)^{-k} \right] \leq \frac{C_k}{r^{2k}}.
\]

Denote
\[
\rho_n(y) := \begin{cases} 
\log(|y|^{-1} + e), & \text{if } n = 0, \\
\frac{1}{|y|^{2n}}, & \text{if } n \geq 1.
\end{cases}
\]

Then for \( k = \mathbb{N}, r \leq r_3 < 1, \)
\[
\| f^{(n)}_{j,k} u \|_{L_\beta^3} \leq C_{\beta, \rho_k(y)} \| u(\xi) \|_{L_\beta^3}, \\
\| \sum_{j=0}^{d+n} \sum_{k=0}^{n} f^{(n)}_{j,k} u \|_{L_\beta^3} \leq C_{\beta, n, \rho_n(y)} \| u(\xi) \|_{L_\beta^3}, \tag{5.9}
\]

For \( |y|, b \in (0, r_3) \), combining (5.8) and (5.9), we have
\[
\| \int_{-b}^{b} e^{i\tau t} (i\tau I - \hat{B}(y))^{-n-1} u(\xi) d\tau \|_{L_\beta^3} \leq C_{r_3, b, \beta} \| u \|_{L_{\beta+n\gamma}^3} + C_{\beta, n, \rho_n(y)} \| u(\xi) \|_{L_\beta^3}.
\]

Thus for \( u \in D_{\beta}(\hat{B}(y)), \beta \geq 0, \)
\[
\| e^{i\beta}(i\tau I - \hat{B}(y))^{-n-1} u(\xi) d\tau \|_{L_\beta^3} \leq \| \sum_{j=0}^{d+n} \sum_{k=0}^{n} f^{(n)}_{j,k} u \|_{L_\beta^3} \leq C_{n, \beta, \rho_n(y)} \| u(\xi) \|_{L_\beta^3}. \tag{5.10}
\]

### 5.3 Result

Now for \( |y| \geq r_3 \), we have (5.6). Together with the estimate (5.10) for \( |y| \leq r_3 \), we have
\[
\| e^{i\beta}(y) u \|_{L_\beta^3} \leq \frac{C_{r_3, n, \beta, \rho_n(y)}}{t^n} \| u \|_{L_{\beta+n\gamma}^3} + \rho_n(y) \chi_{|y| \leq r_3} \| u \|_{L_\beta^3}.
\]

But also the semigroup estimate on \( e^{i\beta}(y) \) gives \( \| e^{i\beta}(y) u \|_{L_\beta^3} \leq e^{tK\|L_\beta^3\|} \| u \|_{L_\beta^3} \). Thus for \( n \in \mathbb{N} \),
\[
\| e^{i\beta}(y) u \|_{L_\beta^3} \leq \frac{C_{r_3, n, \beta, \rho_n(y)}}{(1 + t)^n} \| u \|_{L_{\beta+n\gamma}^3} + \rho_n(y) \chi_{|y| \leq r_3} \| u \|_{L_\beta^3}.
\]

But here in order to use interpolation theorem, we can only use a weaker result:
\[
\| e^{i\beta}(y) u \|_{L_\beta^3} \leq \frac{C_{r_3, \beta, d}(1 + \rho_n(y)) \chi_{|y| \leq r_3}}{(1 + t)^n} \| u \|_{L_{\beta+n\gamma}^3}.
\]

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Notice for $n \in \mathbb{N}$, $\theta \in (0,1)$, we have

\[
(1 + \rho_n(y)\chi_{|y| \leq r_3})^\theta (1 + \rho_{n+1}(y)\chi_{|y| \leq r_3})^{1-\theta} \\
\leq C_\theta (1 + \rho_{n}^\alpha(y)\rho_{n+1}^\alpha(y)\chi_{|y| \leq r_3}) \\
\leq C_\theta + C_\theta \chi_{|y| \leq r_3} \begin{cases} 
|y|^{-2(\alpha + (1-\theta)\alpha_{n+1})}, & \text{if } n \geq 1, \\
C_\theta |y|^{-2(1-\theta)} \log^\theta(|y|^{-1} + e), & \text{if } n = 0.
\end{cases}
\]

Define

\[
\rho_{\alpha}(y) := \begin{cases} 
\frac{1}{|y|^{2\alpha}} \log(|y|^{-1} + e) & \text{if } \alpha \in (0,1), \\
\frac{1}{|y|^{2\alpha}} & \text{if } \alpha \geq 1.
\end{cases}
\]

Then by an interpolation theorem in appendix 7.4, we have the final result:

**Theorem 5.5.** Assume $\gamma \in [0,d)$, $\alpha \in [0,\infty)$, $\beta \geq 0$, then there exists $r_3 \in (0,1)$ such that

\[
\|e^{t\hat{B}(y)}u\|_{L_\beta^2} \leq \frac{C_{\alpha,d,r_3,\beta}(1 + \rho_{\alpha}(y)\chi_{|y| \leq r_3})}{(1+t)^{\alpha}}
\]

(5.11)

### 6 Estimate on $e^{tB}$ and Global Existence

In this section, we will give the proof on estimate on $e^{tB}$ and the proof of our main existence theorem 1.1.

Assume $d \geq 3$, $\gamma \in [0,d)$. By semigroup theory, for $u \in D_\beta(B)$, we have

\[
e^{tB}u = \mathcal{F}^{-1} e^{t\hat{B}(y)} \mathcal{F}u,
\]

where the equality "=" means the almost everywhere equality on $\mathbb{R}^d_x \times \mathbb{R}^d_y$.

**Theorem 6.1.** Let $\gamma \in [0,d)$, $l \geq 0$, $p \in [1,2]$, $\alpha \in [0,\frac{d}{2}(\frac{2}{p} - 1))$, $\beta \geq 0$. Suppose $u \in D_\beta(B) \cap L_{\beta+\alpha,\gamma}(H') \cap L_{\beta+\alpha,\gamma}(L^p)$, then

\[
\|e^{tB}u\|_{L_\infty^\infty(L_\beta^2(H'))} \leq C_{\alpha,r_3,\beta,l,p} \left(\|u\|_{L_{\beta+\alpha,\gamma}(H')} + \|u\|_{L_{\beta+\alpha,\gamma}(L^p)}\right).
\]

(6.1)

**Proof.** For $l \geq 0$,

\[
\|e^{tB}u\|_{L_{\beta+\alpha,\gamma}(H')} = \left(\int_{\mathbb{R}^d} \int_{\mathbb{R}^d} (1 + |\xi|)^{2\beta}(1 + |y|)^{2l}e^{t\hat{B}(y)}\hat{u}(y,\xi)^2 \, dy \, d\xi\right)^{1/2}.
\]

We split the integral on $y$ into two parts: $|y| \geq r_3$ and $|y| \leq r_3$. Then on one hand,

\[
\left(\int_{|y| \geq r_3} \int_{\mathbb{R}^d} (1 + |\xi|)^{2\beta}(1 + |y|)^{2l}e^{t\hat{B}(y)}\hat{u}(y,\xi)^2 \, dy \, d\xi\right)^{1/2} \\
\leq C_{r_3,\beta} \delta \left(\int_{|y| \geq r_3} (1 + |y|)^{2l} (1+t)^{-2\alpha} \|\hat{u}(y,\xi)\|_{L_{\beta+\alpha,\gamma}}^2 \, d\xi\right)^{1/2} \\
\leq C_{r_3,\beta} (1+t)^{-\alpha} \|u(x,\xi)\|_{L_{\beta+\alpha,\gamma}(H')}.
\]
On the other hand, let \((2q') = p \in [1, \infty]\), then by Hölder’s inequality and Hausdorff-Young inequality, we have

\[
\left( \int_{|y| \leq r_3} (1 + |\xi|)^{2\beta}(1 + |y|)^{2(|e^{t\hat{A}}(y)\hat{u}(y, \xi)|^2 d\xi dy) \right)^{1/2}
\leq \frac{C_{r_3, \beta, d}}{(1 + t)^d} \left( \int_{|y| \leq r_3} (1 + |y|)^{2\beta}\left( \|\hat{u}\|_{L^2_{\beta + \alpha \gamma}(H^1)} + |y|^{-2\alpha} \log(|y|^{-1} + e)\|\hat{u}\|_{L^2_{\beta + \alpha \gamma}} \right)^2 d\xi dy \right)^{1/2}
\leq \frac{C_{r_3, \beta, d}}{(1 + t)^d} \|u\|_{L^2_{\beta + \alpha \gamma}(H^1)} + C_{r_3, \beta, d, \alpha, \omega, \gamma} |u(x, \xi)|_{L^2_{\beta + \alpha \gamma}(L^p)},
\]

provided \(4\alpha q' \in [0, d)\). That is \(\alpha \in [0, \frac{d}{4q'}] = [0, \frac{d}{4p} - 1)\). Therefore,

\[
\|e^{tA}u\|_{L^p_{\beta + \alpha \gamma}(H^1)} \leq C_{\alpha, r_3, d, \beta, \gamma} (1 + t)^{-\alpha} \|u\|_{L^2_{\beta + \alpha \gamma}(H^1)} + C_{\alpha, r_3, d, \beta, \gamma} (1 + t)^{-\alpha} \|u\|_{L^2_{\beta + \alpha \gamma}(L^p)}.
\]

Futhermore, we need an estimate on the semigroup generated by \(A\) on \(L^p_{\beta}\).

**Lemma 6.2.** For \(\alpha \geq 0, \beta \in \mathbb{R}, l \in \mathbb{R}\),

\[
\|e^{tA}u\|_{L^p_{\beta}(H^1)} \leq C_{\nu_0} (1 + t)^{-\alpha} \|u\|_{L^p_{\beta + \alpha \gamma}(H^1)}.
\]

**Proof.** The semigroup generated by \(\hat{A}(y)\) is \(e^{t\hat{A}}(y)u = e^{t\hat{A}}(y)u\). For \(u \in D(A) \subset L^p_{\beta}(H^1)\), we have \(u(y, \cdot) \in D_{\beta}(\hat{A}(y))\), for a.e. \(y \in \mathbb{R}^d\) and

\[
(1 + |\xi|)^{-\alpha \gamma} |e^{t\hat{A}}(y)u(y, \cdot)| \leq \sup_{\xi \in \mathbb{R}^d} e^{-\nu_0(1 + |\xi|)^{-\gamma} (1 + |\xi|)^{-\alpha \gamma} |u(y, \xi)|} \leq C_{\nu_0} (1 + t)^{-\alpha} |u(y, \xi)|,
\]

since for \(t, \alpha \geq 0\), we have \(\sup_{t \geq 0} (xt)^\alpha e^{-\nu_0 tx} \leq C_{\nu_0}\). Then

\[
\|e^{tA}u\|_{L^p_{\beta}(H^1)} = \| \mathcal{F}^{-1} e^{t\hat{A}}(y) \mathcal{F} u\|_{L^p_{\beta}(H^1)}
\leq \|\left( \int_{\mathbb{R}^d} (1 + |y|)^{2\beta} |e^{t\hat{A}}(y)u(y, \cdot)|^2 d\xi dy \right)^{1/2} \|_{L^p_{\beta}}
\leq C_{\nu_0} (1 + t)^{-\alpha} \|u\|_{L^p_{\beta + \alpha \gamma}(H^1)}.
\]

**Remark 6.3.** This lemma show that the weighted normed space \(L^p_{\beta + \alpha \gamma}\) is essential for our analysis.

**Lemma 6.4.** For \(0 \leq \alpha < 1 < \alpha_0\),

\[
\int_0^t \frac{1}{(1 + t - s)^\alpha(1 + s)^{\alpha_0}} ds \leq C_{\alpha, \alpha_0} \frac{1}{(1 + t)^\alpha}.
\]

**Proof.**

\[
\int_0^t \frac{1}{(1 + t - s)^\alpha(1 + s)^{\alpha_0}} ds
\leq \frac{1}{(1 + t/2)^\alpha} \int_0^{t/2} \frac{1}{(1 + s)^{\alpha_0}} ds + \frac{1}{(1 + t/2)^\alpha} \int_{t/2}^t \frac{1}{(1 + t - s)^\alpha} ds
\leq C_{\alpha, \alpha_0} \left( \frac{1}{(1 + t)^\alpha} + \frac{1}{(1 + t)^{\alpha_0 + \alpha - 1}} \right) \leq C_{\alpha, \alpha_0} \frac{1}{(1 + t)^\alpha}.
\]
Recall that \( e^{tB} \) can be viewed as a semigroup on \( L_{\alpha}^p(H^l) \), for \( p \in [1, 2] \). Then by the Duhamel principle, we can have another boundedness of semigroup \( e^{tB} \).

**Theorem 6.5.** Let \( \gamma \in [0, d] \), \( l \geq 0 \), \( p \in [1, 2] \), \( \alpha \in \left[ 0, \frac{d}{\min \{2, \gamma \}} \right) \). Suppose \( u \in L_{\beta+\alpha\gamma}(H^l) \cap L_{\beta+\alpha\gamma}^2(L^p) \), then

\[
\|e^{tB}u\|_{L_{\gamma}^p(H^l)} \leq C_{\nu, \gamma, \alpha, \gamma, \beta, p} \left( \|u\|_{L_{\beta+\alpha\gamma}^\infty(H^l)} + \|u\|_{L_{\beta+\alpha\gamma}^2(L^p)} \right).
\]  

(6.3)

**Proof.** Let \( u \in L_{\beta+\alpha\gamma}(H^l) \cap L_{\beta+\alpha\gamma}^2(L^p) \), \( v = e^{tB}u \in D_\beta(B) \). Then by Duhamel principle, we have for \( t \geq 0 \),

\[
v(t) = e^{tA}u + \int_0^t e^{(t-s)A}K e^{sB}u \, ds.
\]  

(6.4)

Thus for \( p \in [1, \infty] \), \( \alpha_0 > 1 \),

\[
\|v(t)\|_{L_{\gamma}^p(H^l)} \leq \|e^{tA}u\|_{L_{\gamma}^p(H^l)} + \int_0^t \|e^{(t-s)A}K e^{sB}u\|_{L_{\gamma}^p(H^l)} \, ds
\]

\[
\leq C_{\nu_0} (1 + t)^{-\alpha} \|u\|_{L_{\beta+\alpha\gamma}^p(H^l)} + \int_0^t (1 + t - s)^{-\alpha_0} \|K v(s)\|_{L_{\beta+\alpha\gamma}^p(H^l)} \, ds
\]

\[
\leq C_{\nu_0} (1 + t)^{-\alpha} \|u\|_{L_{\beta+\alpha\gamma}^p(H^l)} + C_{\alpha, \alpha_0} (1 + t)^{-\alpha} \|K v\|_{L_{\beta+\alpha\gamma}^p(H^l)}.
\]

Thus when \( p = \infty \),

\[
\|v\|_{L_{\gamma}^\infty(H^l)} \leq C_{\nu_0, \gamma, \alpha, \alpha_0} \left( \|u\|_{L_{\beta+\alpha\gamma}^\infty(H^l)} + \|v\|_{L_{\beta+\alpha\gamma}^\infty(H^l)} \right).
\]  

(6.5)

Pick \( \alpha_0 \in (1, \frac{3d}{2\gamma}) \), then \( \alpha_0 \gamma - \gamma - 2 < 0 \). Use equation (6.5) inductively, we have

\[
\|v\|_{L_{\gamma}^\infty(H^l)} \leq C_{\nu_0, \gamma, \alpha, \alpha_0} \left( \|u\|_{L_{\beta+\alpha\gamma}^\infty(H^l)} + \|v\|_{L_{\beta+\alpha\gamma}^\infty(H^l)} \right).
\]

Recall the important property of \( K \) from 2.3 that for \( p > \max\left(\frac{d}{\alpha - \gamma}, \frac{d}{\beta} \right) \), \( \theta \in (0, 1) \),

\[
\|K f\|_{L_{\beta+\alpha\gamma}^\infty(H^l)} \leq C_{\gamma, \alpha, \gamma, \beta} \|f\|_{L_{\gamma}^p(H^l)},
\]

(6.6)

\[
\|K f\|_{L_{\beta+\alpha\gamma}^p(H^l)} \leq C_{\gamma, \alpha, \gamma, \beta} \|f\|_{L_{\gamma}^p(H^l)}.
\]  

(6.7)

with \( \frac{1}{q_0} = \frac{\theta}{\infty} + \frac{1 - \theta}{p_0} \), \( \frac{1}{p_0} = \frac{\theta}{\beta} + \frac{1 - \theta}{\alpha} \), where \( p_0 = \frac{d}{\alpha - \gamma} + \frac{d}{\beta} \). Pick a sequence \( \{p_j\}_j \in (1, \infty) \) by letting

\[
p_1 = 2, \quad p_{j+1} = p_j + \frac{p_j (p_j - 1)}{p_0 - p_j}, \quad (j = 1, 2, \ldots).
\]  

(6.8)

Then \( \{p_j\} \) satisfies

\[
\frac{1}{p_{j+1}} = \frac{1 - \theta}{1}, \quad \frac{1}{p_j} - \frac{\theta}{p_0} = \frac{1 - \theta}{1}, \quad (j = 1, 2, \ldots)
\]

(6.9)

where \( \theta = \frac{1 - \frac{1}{p_0}}{1 - \frac{1}{p_0}} \). One can observe from (6.8) that \( p_{j+1} - p_j \geq \frac{2}{p_0 - 2} \). Thus there exists a finite \( n \in \mathbb{N} \) such that \( p_{n-1} \leq \max\left(\frac{d}{\alpha - \gamma}, \frac{d}{2} \right) \). Then we can apply (6.6) to \( p_n \). (Be careful that we can’t use \( p_{n+1} \), since \( p_n \) may be larger than \( p_0 \) and we won’t have (6.9) with \( \theta > 0 \).) Thus using Duhamel’s formula (6.4) and the boundedness of \( K \) (6.6) inductively,

\[
\|v\|_{L_{\gamma}^\infty(H^l)} \leq C_{\nu_0, \gamma, \alpha, \alpha_0, n} \left( \|u\|_{L_{\beta+\alpha\gamma}^\infty(H^l)} + \sum_{j=0}^{n} \|u\|_{L_{\beta+\alpha\gamma}^p(H^l)} + \|v\|_{L_{\beta+\alpha\gamma}^\infty(H^l)} \right).
\]
Let $\beta > 1$ to get that $\|u\|_{L^p_{\alpha + \gamma}(H^l)} \leq \|u\|_{L^\infty_{\alpha + \gamma}(H^l)}$, for $j = 0, 1, \ldots, n$. Then by theorem 6.1, we have

$$\|v\|_{L^\infty_{\beta}(L^\infty(H^l))} \leq C_{\nu, \gamma, \alpha, \alpha_0, n, r_1, d, \beta, \rho} \left( \|u\|_{L^\infty_{\beta + \alpha}(H^l)} + \|u\|_{L^2_{\beta}(L^p)} \right).$$

The following lemma is well-studied in [9] and I will put the proof in appendix. Notice this lemma is still valid for $\gamma \in [0, d)$.

**Lemma 6.6.** Assume $\gamma \in [0, d)$, $\alpha \geq 0$.

1. For $l > \frac{4}{\beta}$, $\beta \in \mathbb{R}$,

$$\|\Gamma(f, g)\|_{L^\infty_{\beta + \alpha}(H^l)} \leq C_{\nu} \|f\|_{L^\infty_{\beta}(H^l)} \|g\|_{L^\infty_{\beta}(H^l)}. \quad (6.10)$$

2. For $\beta > \frac{4}{\beta} - \gamma + \alpha \gamma$, $l \geq 0$,

$$\|\Gamma(f, g)\|_{L^2_{\beta}(L^1)} \leq C_{\nu, \beta_0} \|f\|_{L^\infty_{\beta + \alpha}(H^l)} \|g\|_{L^\infty_{\beta + \alpha}(H^l)}. \quad (6.11)$$

**Theorem 6.7.** (Existence). Assume the cross-section $q$ satisfies the angular cut-off assumption (1.5). Assume $d \geq 3$, $\gamma \in [0, d)$, $\alpha \in \left(\frac{d}{2}, \min\left(\frac{d}{2}, 1\right)\right)$. Let $l > \frac{1}{2}$, $\beta > \frac{4}{\beta} - \gamma + \alpha \gamma$. There exists constants $A_0, A_1$ such that if the initial data $f_0 \in L^\infty_{\beta + \alpha}(H^l) \cap L^2_{\alpha}(L^1)$ satisfies

$$\|f_0\|_{L^\infty_{\beta + \alpha}(H^l)} + \|f_0\|_{L^2_{\alpha}(L^1)} \leq A_0. \quad (6.12)$$

Let $X = \{f \in L^\infty_{\alpha}(L^\infty_{\beta}(H^l)) : \|f\|_{L^\infty_{\alpha}(L^\infty_{\beta}(H^l))} \leq A_1\}$. Then the Cauchy problem to Boltzmann equation

$$\begin{cases}
  f_t + \xi \cdot \nabla_x f = Q(f, f), \\
  g|_{t=0} = f_0.
\end{cases} \quad (6.13)$$

posses a unique solution $f = f(t) \in X \cap C^0([0, \infty); L^\infty_{\beta}(H^l))$ and

$$\|f\|_{L^\infty_{\alpha}(L^\infty_{\beta}(H^l))} \leq C_{\nu, \gamma, \alpha, \beta, d} \left( \|f_0\|_{L^\infty_{\beta + \alpha}(H^l)} + \|f_0\|_{L^2_{\alpha}(L^1)} \right). \quad (6.14)$$

The uniqueness is taken in the sense that $f \in X$.

**Proof.** By semigroup theory, it suffices to find the fixed point of

$$\Phi[f] := e^{tB} f_0 + \int_0^t e^{(t-s)B} \Gamma(f(s), f(s)) \, ds. \quad (6.15)$$

Now pick $\alpha \in \left(0, \frac{d}{4}\right) \cap (0, 1)$, $\beta > \frac{4}{\beta} - \gamma + \alpha \gamma$, then

$$\|\Phi[f]\|_{L^\infty_{\beta}(H^l)} \leq \|e^{tB} f_0\|_{L^\infty_{\beta}(H^l)} + \int_0^t \|e^{(t-s)B} \Gamma(f(s), f(s))\|_{L^\infty_{\beta}(H^l)} \, ds. \quad (6.16)$$

For the first term,

$$\|e^{tB} f_0\|_{L^\infty_{\beta}(H^l)} C_{\nu, \gamma, \alpha, \beta, d} \frac{1}{(1 + t)^\alpha} \left( \|f_0\|_{L^\infty_{\beta + \alpha}(H^l)} + \|f_0\|_{L^2_{\alpha}(L^1)} \right).$$

Pick $\alpha \in \left(0, \frac{d}{4}\right) \cap (\frac{1}{4}, 1)$, which is non-empty since $d \geq 3$, then the second term in (6.16) becomes

$$\int_0^t \|e^{(t-s)B} \Gamma(f(s), g(s))\|_{L^\infty_{\beta}(H^l)} \, ds$$

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also

Then

and

by letting

\[ \| \mathbf{e} \|_2 \Phi \leq 2 \| \mathbf{e} \|_{C\ell} \int_{0}^{t} \frac{1}{(1 + t - s)^{2\alpha}} ds \]

\[ \leq C_{\nu, \gamma, \alpha, \beta, d} \int_{0}^{t} \frac{1}{(1 + t - s)^{\alpha}} \| f(s) \|_{L_{\beta}^{\infty}(H^1)} \| g(s) \|_{L_{\beta}^{\infty}(H^1)} ds \]

\[ \leq C_{\nu, \gamma, \alpha, \beta, d} \int_{0}^{t} \frac{1}{(1 + t - s)^{\alpha}} \| f(s) \|_{L_{\beta}^{\infty}(H^1)} \| g(s) \|_{L_{\beta}^{\infty}(H^1)} ds \]

Thus

\[ \| \Phi[f] \|_{L_{\beta}^{\infty}(H^1)} \leq C_{\nu, \gamma, \alpha, \beta, d} \left( \| f_0 \|_{L_{\beta}^{\infty}(H^1)} + \| f_0 \|_{L_{\alpha}^{\infty}(L^1)} + \| f \|_{L_{\beta}^{\infty}(L_{\beta}^{\infty}(H^1))} \right) \]

\[ = C_1 \left( \| f_0 \|_{L_{\beta}^{\infty}(H^1)} + \| f_0 \|_{L_{\alpha}^{\infty}(L^1)} \right) + C_2 \| f \|_{L_{\beta}^{\infty}(L_{\beta}^{\infty}(H^1))}. \]

On the other hand, noticing that \( \Gamma(f, f) - \Gamma(g, g) = \Gamma(f + g, f - g) \), we have

\[ \| \Phi[f] - \Phi[g] \|_{L_{\beta}^{\infty}(H^1)} \leq \int_{0}^{t} \| e^{(t-s)B} \Gamma(f + g)(s), (f - g)(s) \|_{L_{\beta}^{\infty}(H^1)} ds \]

\[ \leq C_{\nu, \gamma, \alpha, \beta, d} \int_{0}^{t} \frac{1}{(1 + t - s)^{\alpha}} \| f + g \|_{L_{\beta}^{\infty}(H^1)} \| f - g \|_{L_{\beta}^{\infty}(H^1)}, \]

and

\[ \| \Phi[f] - \Phi[g] \|_{L_{\alpha}^{\infty}(L_{\beta}^{\infty}(H^1))} \leq C_2 \| f + g \|_{L_{\beta}^{\infty}(L_{\beta}^{\infty}(H^1))} \| f - g \|_{L_{\beta}^{\infty}(L_{\beta}^{\infty}(H^1))}, \]

by letting \( C_2 \) large enough. Define \( C_3 := \frac{1}{4C_2C_2} \), then if \( \left( \| f_0 \|_{L_{\beta}^{\infty}(H^1)} + \| f_0 \|_{L_{\alpha}^{\infty}(L^1)} \right) < C_3 \), we can let

\[ C_4 := 1 - 4C_1C_2 \left( \| f_0 \|_{L_{\beta}^{\infty}(H^1)} + \| f_0 \|_{L_{\alpha}^{\infty}(L^1)} \right) > 0. \]

Choose \( C_5 := \frac{1}{4C_2} (1 - \sqrt{C_4}) \), then \( C_2C_5^2 - C_5 + C_1 \left( \| f_0 \|_{L_{\beta}^{\infty}(H^1)} + \| f_0 \|_{L_{\alpha}^{\infty}(L^1)} \right) = 0. \)

Finally we pick a normed space

\[ X := \{ f \in L_{\alpha}^{\infty}(L_{\beta}^{\infty}(H^1)) : \| f \|_{L_{\beta}^{\infty}(L_{\beta}^{\infty}(H^1))} \leq C_5 \}. \]

Then \( X \) is a complete with norm \( \| \cdot \|_{L_{\beta}^{\infty}(L_{\beta}^{\infty}(H^1))} \), and for \( f \in X \),

\[ \| \Phi[f] \|_{L_{\beta}^{\infty}(L_{\beta}^{\infty}(H^1))} \leq C_1 \left( \| f_0 \|_{L_{\beta}^{\infty}(H^1)} + \| f_0 \|_{L_{\alpha}^{\infty}(L^1)} \right) + C_2C_5^2 = C_5, \]

\[ \| \Phi[f] - \Phi[g] \|_{L_{\beta}^{\infty}(L_{\beta}^{\infty}(H^1))} \leq 2C_2C_5 \| f - g \|_{L_{\beta}^{\infty}(L_{\beta}^{\infty}(H^1))} = (1 - \sqrt{C_5}) \| f - g \|_{L_{\beta}^{\infty}(L_{\beta}^{\infty}(H^1))}. \]

This proves that \( \Phi \) is a contraction map on \( X \). Thus there exists a unique fixed point \( f \in X \) to \( \Phi \), which is the solution of

\[ f := e^{tB} f_0 + \int_{0}^{t} e^{(t-s)B} \Gamma(f(s), f(s)) ds. \]

Also

\[ \| f \|_{L_{\beta}^{\infty}(L_{\beta}^{\infty}(H^1))} \leq C_5 = \frac{1}{2C_2} (1 - \sqrt{C_4}) = \frac{2C_1}{1 + \sqrt{C_4}} \left( \| f_0 \|_{L_{\beta}^{\infty}(H^1)} + \| f_0 \|_{L_{\alpha}^{\infty}(L^1)} \right). \]
7 Appendix

7.1 Hilbert-Schmidt Operator

Let $H_1$ be a separable Hilbert space, $H_2$ be a Hilbert space.

**Definition 7.1.** $T \in L(H_1, H_2)$ is called a Hilbert-Schmidt operator if there exists an orthonormal basis $\{e_n\}_{n=1}^{\infty}$ of $H_1$ such that

$$\sum_{n=1}^{\infty} \|Te_n\|_{H_2}^2 < \infty. \quad (7.1)$$

**Theorem 7.2.** If $T \in L(H_1, H_2)$ is a Hilbert-Schmidt operator, then $T$ is compact.

**Proof.** Let $\{e_n\}_{n=0}^{\infty}$ be an orthonormal basis of $H_1$ such that $\sum_{n=1}^{\infty} \|Te_n\|_{H_2}^2 < \infty$. Then for $x \in H_1$,

$$\|Tx\|_{H_2}^2 = \sum_{n=0}^{\infty} (x, e_n) Te_n,$$

since the right hand side is absolutely convergent. Define $T_n : H_1 \to H_2$ by

$$T_n(x) := \sum_{k=1}^{n} (x, e_k) Te_k.$$

Then $T_n$ has finite rank and hence is compact.

On the other hand, for $x \in H_1$, $\|x\|_{H_1} \leq 1$, we have

$$\|Tx - T_n x\|_{H_2}^2 \leq \sum_{k=n+1}^{\infty} (x, e_k) Te_k \|_{H_2}^2 \leq \|x\|_{H_1}^2 \cdot \sum_{k=n+1}^{\infty} \|Te_k\|_{H_2}^2,$$

$$\|T - T_n\| \leq \left( \sum_{k=n+1}^{\infty} \|Te_k\|_{H_2}^2 \right)^{\frac{1}{2}} \to 0,$$

as $n \to \infty$. Thus $T$ is compact since it’s the limit of a sequence of compact operators. □

**Theorem 7.3.** Suppose $L^2(X, \mu)$ and $L^2(Y, \lambda)$ are two separable Hilbert space. If $k \in L^2(X \times Y, \mu \otimes \lambda)$, then

$$Kf := \int_Y k(x, y) f(y) \, d\lambda(y) : L^2(Y, \lambda) \to L^2(X, \mu) \quad (7.2)$$

is a Hilbert-Schmidt operator.

**Proof.** Let $\{e_n\}_{n=0}^{\infty}$ be an orthonormal basis of $L^2(Y, \lambda)$, then so is $\{\overline{e_n}\}_{n=0}^{\infty}$. Since $k \in L^2(X \times Y, \mu \otimes \lambda)$, we have $k(x, \cdot) \in L^2(Y, \lambda)$, for almost all $x \in X$. Thus $Ke_n$ is well-defined for almost all $x \in X$ and

$$Ke_n(x) = \int_Y k(x, y) f(y) \, d\lambda(y),$$

$$\sum_{n=0}^{\infty} \|Ke_n\|_{L^2(X, \mu)}^2 = \sum_{n=0}^{\infty} \int_X \left( \int_Y k(x, y) f(y) \, d\lambda(y) \right)^2 \, d\mu(x)$$

$$= \int_X \sum_{n=0}^{\infty} |(k(x, \cdot), \overline{e_n})_{L^2(Y)}|^2 \, d\mu(x)$$

$$= \|k\|_{L^2(X \times Y)} < \infty.$$ □
7.2 Interpolation

Define \( \|f\|_{L^p_\beta} = \|(1 + |\xi|)^\beta f\|_{L^p(\mathbb{R}^d)}, \ p \in [1, \infty) \).

**Theorem 7.4.** Let \( \beta \in \mathbb{R}, \ \gamma \in \mathbb{R} \setminus \{0\}, \ p \in [1, \infty] \). Suppose \( T \) is a linear operator defined on \( L^p_{\beta + \alpha \gamma} \) such that

\[
\|Tf\|_{L^p_\beta} \leq A_n \|f\|_{L^p_{\beta + \alpha \gamma}}
\]

for some constants \( A_n > 0 \), for all \( n \in \{0, 1, 2, \ldots\} \). Let \( \theta \in (0, 1) \), \( n, m \in \{0, 1, 2, \ldots\} \), pick \( \alpha = n \cdot \theta + m \cdot (1 - \theta) \). Then \( T \) is bounded linear operator from \( L^p_{\beta + \alpha \gamma} \) to \( L^p_{\beta} \), with

\[
\|Tf\|_{L^p_\beta} \leq A_n A_m^{1 - \theta} \|f\|_{L^p_{\beta + \alpha \gamma}}.
\]

To prove this theorem, we will need the Hadamard’s three lines theorem.

**Theorem 7.5.** Let \( F \) be an analytic function in the open strip \( S = \{z \in \mathbb{C} : 0 < \text{Re} \lambda < 1\} \). Suppose \( F \) is continuous and bounded on \( S \) with

\[
|F(z)| \leq \begin{cases} 
A_0, & \text{if Re} \lambda = 0, \\
A_1, & \text{if Re} \lambda = 1,
\end{cases}
\]

for some positive constants \( A_0, A_1 \). Then for any \( \theta \in [0, 1] \), if \( \text{Re} z = \theta \), we have

\[
|F(z)| \leq A_0^{1 - \theta} A_1^\theta.
\]

**Proof.**

1. With loss of generality, assume \( m > n \). Fix \( \theta \in (0, 1) \) and Let \( \alpha = n \cdot \theta + m \cdot (1 - \theta) \).

Notice \( T \) is well-defined on \( L^p_{\beta + \alpha \gamma} \), since \( L^p_{\beta + \alpha \gamma} \subset L^p_{\beta + m \gamma} \).

2. Let \( f(\xi) = \sum_{k=1}^K a_k e^{i\alpha_k} \chi_{A_k}(\xi) \) be any simple complex function on \( \mathbb{R}^d \), with \( a_k > 0 \), \( \alpha_k \in \mathbb{R} \), \( \{A_k\} \) are pairwise disjoint bounded measurable subsets of \( \mathbb{R}^d \). We would like to control

\[
\|Tf\|_{L^p_\beta} = \sup_{\|g\|_{L^p'} \leq 1, \ g \text{ is simple}} \left| \int_{\mathbb{R}^d} Tg(\xi) |1 + |\xi||^\beta d\xi \right|.
\]

Write \( g = \sum_{j=1}^J b_j e^{i\beta_j} \chi_{B_j}(\xi) \), with \( b_j > 0 \), \( \beta_j \in \mathbb{R} \), \( \{B_j\} \) are pairwise disjoint bounded measurable subsets of \( \mathbb{R}^d \). Suppose \( z \in \{z \in \mathbb{C} : 0 \leq \text{Re} z \leq 1\} \). Define

\[
f_z(\xi) := \sum_{k=1}^K a_k e^{i\alpha_k} \chi_{A_k}(\xi) \cdot (1 + |\xi|)^{(\alpha - m - (1 - \beta))\gamma}.
\]

Then \( f_0(\xi) = f(\xi), f_0(\xi) = f(\xi)(1 + |\xi|)^{(\alpha - m)\gamma}, f_1(\xi) = f(\xi)(1 + |\xi|)^{(\alpha - n)\gamma} \). Define

\[
F(z) := \int_{\mathbb{R}^d} T(f_z)(1 + |\xi|)^\beta d\xi
\]

\[
= \sum_{k=1}^K \sum_{j=1}^J a_k b_j e^{i\alpha_k} e^{i\beta_j} \int_{\mathbb{R}^d} T(\chi_{A_k}(\xi)(1 + |\xi|)^{(\alpha - m - (1 - \beta))\gamma}) \chi_{B_j}(\xi)(1 + |\xi|)^\beta d\xi.
\]

Now we need to check \( F \) satisfies the assumptions in three-lines theorem 7.5.

(i). Claim: \( F(z) \) is continuous and bounded in \( \{0 \leq \text{Re} z \leq 1\} \).

Indeed,

\[
|F(z)| \leq \sum_{k=1}^K \sum_{j=1}^J a_k b_j \|T(\chi_{A_k}(1 + |\xi|)^{(\alpha - m - (1 - \beta))\gamma})\|_{L^p_\beta} \|\chi_{B_j}(\xi)\|_{L^{p'}}.
\]
\[ \leq C \sum_{k=1}^{K} \| \chi A_k \|_{L^p_{\beta + (a - n z + m z) \gamma}} < \infty. \]

Notice \( A_k \) is bounded, there exists an open ball \( B(0, R) \) with radius \( R > 0 \) such that \( A_k \subseteq B(0, R) \), for all \( 1 \leq k \leq K \). Then for \( z_1, z_2 \in \{ 0 \leq \text{Re} z \leq 1 \} \), by Hölder’s inequality and the boundedness of \( T \), we have

\[
|F(z_1) - F(z_2)| \\
\leq \sum_{k=1}^{K} \sum_{j=1}^{J} a_k b_j \left\| \chi A_k (1 + |\xi|)^{\beta + \alpha \gamma} \left[ (1 + |\xi|)^{(m-n)z_1 \gamma} - (1 + |\xi|)^{(m-n)z_2 \gamma} \right] \right\|_{L^p_{\gamma}} \left\| \chi B_j (\xi) \right\|_{L^{p'}} \\
\leq C_{\gamma, m, n, g} \sum_{k=1}^{K} \left\| \chi A_k (1 + |\xi|)^{\beta + \alpha \gamma} \min \{ 1, (1 + |\xi|)^{(m-n) \gamma} \} \ln (1 + |\xi|) \right\|_{L^p_{\gamma}} |z_1 - z_2| \\
\leq C_{\gamma, m, n, K} |z_1 - z_2| \to 0,
\]
as \( |z_1 - z_2| \to 0 \). This proves the claim.

(ii). Claim: \( F(z) \) is analytic in \( \{ 0 < \text{Re} z < 1 \} \).

Indeed, similarly, for \( z, z_0 \in \{ 0 < \text{Re} z < 1 \} \), by Hölder’s inequality and the boundedness of \( T \) and noticing that \( \{ A_k \}_{k=1}^{K} \) is uniformly bounded, we have

\[
\left| \frac{F(z) - F(z_0)}{z - z_0} \right| \\
= \sum_{k=1}^{K} \sum_{j=1}^{J} a_k b_j e^{i\alpha_k} e^{i\beta_j} \\
\int_{\mathbb{R}^d} T(x) \frac{\chi A_k \partial_z \left[ (1 + |\xi|)^{(a - n z - m (1 - z)) \gamma} \right] \chi B_j (\xi) (1 + |\xi|)^{\beta}}{z - z_0} \\
= C_{\alpha, \beta, \gamma, K, J} \sum_{k=1}^{K} \left\| \chi A_k \chi B_j (\xi) (1 + |\xi|)^{\beta} \right\|_{L^p_{\gamma}} (z - z_0) \\
\leq C_{\alpha, \beta, \gamma, K, J} \sum_{k=1}^{K} |m - n| \left\| \chi A_k \chi B_j (\xi) (1 + |\xi|)^{\beta} \right\|_{L^p_{\gamma}} (z - z_0),
\]
for some \( s, t \in (0, 1) \). Notice \( z \to z_0 \) implies \( t, s \to 0 \) and \( A_k \) are uniformly bounded, thus the limit \( \lim_{z \to z_0} \frac{F(z) - F(z_0)}{z - z_0} \)
exists and hence \( F(z) \) is analytic.

(iii). If \( \text{Re} z = 0 \), by using Hölder’s inequality and the boundedness of \( T \), we have

\[
|F(z)| \leq \| T f_z \|_{L^p_{\beta}} \| g \|_{L^{p'}} \leq A_n \| f_z \|_{L^p_{\beta + \alpha \gamma}} \| g \|_{L^{p'}} = A_n \| f \|_{L^p_{\beta + \alpha \gamma}} \| g \|_{L^{p'}}.
\]
If \( \text{Re} z = 1 \), similarly we have

\[
|F(z)| \leq \| T f_z \|_{L^p_{\beta}} \| g \|_{L^{p'}} \leq A_m \| f_z \|_{L^p_{\beta + \alpha \gamma}} \| g \|_{L^{p'}} = A_m \| f \|_{L^p_{\beta + \alpha \gamma}} \| g \|_{L^{p'}}.
\]

Therefore we can apply the Hadamard’s three-lines theorem. When \( \text{Re} z = 0 \),

\[
|F(z)| \leq A_n^0 A_{m}^{1-\theta} \| f \|_{L^p_{\beta + \alpha \gamma}} \| g \|_{L^{p'}}.
\]

Thus for any simple function \( f \) in \( L^p_{\beta + \alpha \gamma} \),

\[
\| T f \|_{L^p_{\gamma}} = \sup_{\| g \|_{L^{p'}} \leq 1, \text{g is simple}} |F(\theta)| \leq A_n^0 A_{m}^{1-\theta} \| f \|_{L^p_{\beta + \alpha \gamma}}.
\]

Since simple functions are dense in \( L^p_{\beta + \alpha \gamma} \), we proved the theorem.
Lemma 7.6. For $A_1, A_2 > 0$, $d \geq 3$. Denote
\[
    b = \left( \frac{\xi + \xi_2}{2}, \frac{\xi_2 - \xi}{|\xi|} \right) \frac{\xi_2 - \xi}{|\xi|}.
\]

Then
(1). For $\alpha \in (-\infty, d)$,
\[
    \int_{\mathbb{R}^d} \frac{1}{|\xi|\alpha} e^{-A_1|\xi_2| - A_2|\xi|} d\xi \leq C_{\alpha, A_1, A_2, d} \frac{1}{1 + |\xi|}. \tag{7.6}
\]

(2). For $\alpha \in (-\infty, d)$,
\[
    \int_{\mathbb{R}^d} \frac{1}{|\xi|\alpha} e^{-A_1|\xi|} d\xi \leq \int_{\mathbb{R}^d} \frac{1}{|\xi|\alpha} e^{-A_1|\xi|} d\xi \leq C_{\alpha, A_1, A_2, d} \frac{1}{(1 + |\xi|)^d}. \tag{7.7}
\]

(3). For $\alpha \in [0, d)$, $\beta \in \mathbb{R}$,
\[
    \int_{\mathbb{R}^d} \frac{1}{|\xi - \xi_2|\alpha} e^{-A_1|\xi| - A_2|\xi_2|} d\xi \leq C_{\alpha, A_1, A_2, d} \frac{1}{(1 + |\xi|)^d+1}. \tag{7.8}
\]

Proof. 1. If $\alpha < d$,
\[
    \int_{\mathbb{R}^d} \frac{1}{|\xi|\alpha} \exp(-A_1|\xi_2| - A_2|\xi|) d\xi
    = \int_{\mathbb{R}^d} \frac{1}{|\xi|\alpha} \exp(-A_1|\xi|^2 - A_2(2\xi_2 \cdot \xi)^2) d\xi
    = \int_0^\infty \frac{1}{\alpha - d + 1} \exp(-A_1 r^2) \int_{S_{d-1}} \exp(-A_2 r^2 + 2\xi_2 \cdot \xi) d\sigma(\xi_2) dr.
\]

Here
\[
    \int_{S_{d-1}} \exp(-A_2 r^2 + 2\xi_2 \cdot \xi) d\sigma(\xi_2) dr = \frac{2\pi^{d-1}}{\Gamma\left(\frac{d-1}{2}\right)} \int_{-1}^1 \exp(-A_2 (r - 2s|\xi|)^2) (1 - s^2)^{\frac{d-2}{2}} ds
    \leq C_d \int_{-1}^1 \exp(-A_2 (r - 2s|\xi|)^2) ds.
\]

If $|\xi| \geq 1$, we have
\[
    \int_{-1}^1 \exp\left(\frac{-A_2 (r - 2s|\xi|)^2}{4}\right) ds = \frac{1}{2|\xi|} \int_{r - 2|\xi|}^{r + 2|\xi|} \exp\left(\frac{-A_2 s^2}{4}\right) ds \leq C_{d, A_2} \frac{1}{1 + |\xi|},
\]

and if $|\xi| \leq 1$, we have
\[
    \int_{-1}^1 \exp\left(\frac{-A_2 (r - 2s|\xi|)^2}{4}\right) ds \leq \frac{C}{1 + |\xi|}.
\]

Thus
\[
    \int_{\mathbb{R}^d} \frac{1}{|\xi|\alpha} \exp(-A_1|\xi_2| - A_2|\xi|) d\xi \leq \int_0^\infty \frac{C_{d, A_2}}{r^\alpha - d + 1(1 + |\xi|)} \exp(-A_1 r^2) dr
    \leq C_{\alpha, d, A_1, A_2} \frac{1}{1 + |\xi|}.
\]
This is (7.6).
2. Fix $\alpha < d$. If $|\xi| \leq 1$,
\[
\int_{\mathbb{R}^d} \frac{1}{|\xi - \xi'|^\alpha} \exp(-A_1|\xi'|^2) \, d\xi,
\]
\[
= \int_{|\xi - \xi'| > \frac{|\xi|}{2}} \frac{1}{|\xi - \xi'|^\alpha} \exp(-A_1|\xi'|^2) \, d\xi + \int_{|\xi - \xi'| \leq \frac{|\xi|}{2}} \frac{1}{|\xi - \xi'|^\alpha} \exp(-A_1|\xi'|^2) \, d\xi,
\]
\[
\leq \int_{|\xi - \xi'| > \frac{|\xi|}{2}} \frac{1}{\alpha} \exp(-A_1|\xi'|^2) \, d\xi + \int_{|\xi - \xi'| \leq \frac{|\xi|}{2}} \frac{1}{|\xi - \xi'|^\alpha} \exp(-A_1|\xi'|^2) \, d\xi,
\]
\[
\leq C_{A_1,d,\alpha}.
\]
If $|\xi| \geq 1$, notice $|\xi - \xi| \leq \frac{|\xi|}{2}$ implies $|\xi| \geq |\xi|/2 \geq |\xi' - \xi|$, we have
\[
\int_{\mathbb{R}^d} \frac{1}{|\xi - \xi'|^\alpha} \exp(-A_1|\xi'|^2) \, d\xi,
\]
\[
\leq \int_{|\xi - \xi'| > \frac{|\xi|}{2}} \frac{1}{|\xi - \xi'|^\alpha} \exp(-A_1|\xi'|^2) \, d\xi + \int_{|\xi - \xi'| \leq \frac{|\xi|}{2}} \frac{1}{|\xi - \xi'|^\alpha} \exp(-A_1|\xi'|^2) \, d\xi,
\]
\[
\leq \int_{|\xi - \xi'| > \frac{|\xi|}{2}} \frac{1}{|\xi - \xi'|^\alpha} \exp(-A_1|\xi'|^2) \, d\xi + \int_{|\xi - \xi'| \leq \frac{|\xi|}{2}} \frac{1}{|\xi - \xi'|^\alpha} \exp(-A_1|\xi'|^2) \, d\xi,
\]
\[
\leq C_{A_1,d,\alpha} \left( \frac{1}{|\xi|^\alpha} + \exp\left( -\frac{A_1|\xi|^2}{8} \right) \right)
\]
\[
\leq \frac{1}{(1 + |\xi|)^\alpha}.
\]
This proves (7.7).
3. Notice that $|\xi| \leq |\xi|/2$ implies $|\xi' - \xi| \geq |\xi|/2$. Thus by (7.6),
\[
\int_{\mathbb{R}^d} \frac{1}{|\xi - \xi'|^\alpha (1 + |\xi'|^2)} \exp(-A_1|\xi' - \xi'|^2 - A_2|b|^2) \, d\xi',
\]
\[
\leq \left( \int_{|\xi'| > \frac{|\xi|}{2}} + \int_{|\xi'| \leq \frac{|\xi|}{2}} \right) \frac{1}{|\xi - \xi'|^\alpha (1 + |\xi'|^2)} \exp(-A_1|\xi' - \xi'|^2 - A_2|b|^2) \, d\xi',
\]
\[
\leq \frac{1}{(1 + |\xi|/2)^\alpha} \int_{|\xi'| > \frac{|\xi|}{2}} \frac{1}{|\xi - \xi'|^\alpha} \exp(-A_1|\xi' - \xi'|^2 - A_2|b|^2) \, d\xi',
\]
\[
+ \int_{|\xi'| \leq \frac{|\xi|}{2}} \frac{1}{|\xi - \xi'|^\alpha} \exp(-A_1|\xi' - \xi'|^2 - A_2|b|^2) \, d\xi' \exp\left( -\frac{A_1|\xi|^2}{8} \right)
\]
\[
\leq C_{\beta,\alpha, A_1, A_2} \left( \frac{1}{1 + |\xi|} + \exp\left( -\frac{A_1|\xi|^2}{8} \right) \right).
\]
This gives (7.8).

Proof of lemma 6.6. 1. Notice
\[
1 + |\xi| \leq (1 + |\xi|)(1 + |\xi'|),
\]
\[
1 + |\xi| \leq 1 + (|\xi'|^2 + |\xi'|^2)^{1/2} \leq 1 + |\xi'| + |\xi'| \leq (1 + |\xi'|)(1 + |\xi'|).
\]
Then for $f, g \in L^\infty_b(\mathbb{R}^d)$,
\[
|\Gamma(f, g)| = M^{-1/2} \int_{\mathbb{R}^d} \int_{S^{d-1}} |M^{1/2} f' \left( M^{1/2} \right)' g' + \left( M^{1/2} \right)' f' M^{1/2} g'|
\]
\[
\leq C_{\beta,\alpha} \left( \frac{1}{1 + |\xi|} + \exp\left( -\frac{A_1|\xi|^2}{8} \right) \right).
\]
\[-M^{1/2}fM^{1/2}g - M^{1/2}f_\ast M^{1/2}gq(\xi_\ast, \theta)\, d\omega d\xi_\ast\]

\[= \int_{\mathbb{R}^d} \int_{S^{d-1}} M^{1/2}(|f'g'| + |f_g'| + |f_g| + |f_\ast g|)q(\xi - \xi_\ast, \theta)\, d\omega d\xi_\ast\]

Thus
\[
\|\Gamma(f, g)\|_{L^\infty_{\beta_0}(\mathbb{R}^d)} \\
\quad \leq \sup_{\xi \in \mathbb{R}^d} \int_{\mathbb{R}^d} \int_{S^{d-1}} M^{1/2}(|f'g'| + |f_g'| + |f_g| + |f_\ast g|)(1 + |\xi|)^{2\gamma} q(\xi - \xi_\ast, \theta)\, d\omega d\xi_\ast \\
\quad \leq \sup_{\xi \in \mathbb{R}^d} (1 + |\xi|)^\gamma \int_{\mathbb{R}^d} \int_{S^{d-1}} M^{1/2}\left( |(1 + |\xi'|)^3 f'(1 + |\xi'|)^3 g'| + |(1 + |\xi'|)^3 f_\ast'(1 + |\xi'|)^3 g| \right) q(\xi - \xi_\ast, \theta)\, d\omega d\xi_\ast \\
\quad \leq 4\|f\|_{L^\infty_{\beta_0}} \|g\|_{L^\infty_{\beta_0}} \sup_{\xi \in \mathbb{R}^d} (1 + |\xi|)^\gamma \int_{\mathbb{R}^d} \int_{S^{d-1}} M^{1/2}q(\xi - \xi_\ast, \theta)\, d\omega d\xi_\ast \\
\quad \leq 4\nu_1 \|f\|_{L^\infty_{\beta_0}} \|g\|_{L^\infty_{\beta_0}}.
\]

On the other hand, \(l > \frac{d}{2}\) implies that the Sobolev space \(H^l\) is a Banach algebra. Thus for \(f, g \in L^\infty_{\beta_0}(H^l)\), \(\|\Gamma_j(f, g)\|_{H^l} \leq \Gamma_j(\|f\|_{H^l}, \|g\|_{H^l}).\) This proves assertion (1).

2. For \(\beta_0 > \frac{d}{2}\), we have
\[
\|f\|_{L^2_{\beta_0, \gamma}} \leq \|f\|_{L^\infty_{\beta_0 + \alpha \gamma}} (1 + |\xi|)^{-2\beta_0} \|L^2 \leq C_{\beta_0} \|f\|_{L^\infty_{\beta_0 + \alpha \gamma}}.
\]

For \(f, g \in L^2_{\alpha \gamma}(L^1), \beta_0 > \frac{d}{2}\),
\[
\|\Gamma(f, g)\|_{L^2_{\beta_0, \gamma}(L^1)} \\
\quad \leq \int_{\mathbb{R}^d} \int_{S^{d-1}} M^{1/2}(|f'g'| + |f_g'| + |f_g| + |f_\ast g|)q(\xi - \xi_\ast, \theta)\, d\omega d\xi_\ast \|\Gamma_{\beta_0}(f, g)\|_{L^2_{\beta_0, \gamma}(L^1)} \\
\quad \leq C_{\beta_0} \|\Gamma(f)\|_{L^2_{\beta_0, \gamma}(L^2)} \|g\|_{L^2_{\beta_0, \gamma}} \|f\|_{L^\infty_{\beta_0 + \alpha \gamma}} \\
\quad \leq C_{\beta_0, \nu_1} \|f\|_{L^\infty_{\beta_0 - \gamma + \alpha \gamma}(L^2)} \|g\|_{L^\infty_{\beta_0 - \gamma + \alpha \gamma}(L^2)}.
\]

If \(\beta > \frac{d}{2} - \gamma + \alpha \gamma\), then we can find \(\beta_0 \in (\frac{d}{2}, \beta + \gamma - \alpha \gamma)\) to make the above inequality valid. Then
\[
\|\Gamma(f, g)\|_{L^2_{\beta_0}(L^1)} \leq C_{\beta_0, \nu_1} \|f\|_{L^\infty_{\beta_0}(L^2)} \|g\|_{L^\infty_{\beta_0}(L^2)} \\
\quad \leq C_{\beta_0, \nu_1} \|f\|_{L^\infty_{\beta_0}(H^l)} \|g\|_{L^\infty_{\beta_0}(H^l)}.
\]

\[\square\]

7.4 Properties of \(L\)

Proof of Theorem 2.1. 1. Firstly I will give a derivation of expression to \(L\). Recall that \(Lf := M^{-\frac{\star}{2}}Q(M^\ast f, M) + M^{-\star}Q(M, M^\ast f)\). Using the fact that \(MM_\ast = M'M'^\ast\), we have
\[
Lf = \int_{\mathbb{R}^d} \int_{S^{d-1}} M^\ast \left( f'_\ast \left(M^\ast\right)' + f' \left(M^\ast\right)' \right) q(\xi - \xi_\ast, \theta)\, d\omega d\xi_\ast.
\]

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- \mathbf{M}^2 \int_{\mathbb{R}^d} \int_{S^{d-1}} \mathbf{M}^2 \mathbf{q}(\xi - \xi_*, \theta) \, d\omega d\xi_* - \mathbf{f} \int_{\mathbb{R}^d} \int_{S^{d-1}} \mathbf{M}_* \mathbf{q}(\xi - \xi_*, \theta) \, d\omega d\xi_*.

It suffices to deal with the first term \( I \). Consider the unit vector \( m \in \text{Span}\{\xi - \xi_*, \omega\} \) such that \( m \perp \omega \). Then performing a changing variable from \( \omega \) to \( m \), (where one need to use polar coordinate on \( S^{d-1} \) to do the changing variable.) we have

\[
\int_{\mathbb{R}^d} \int_{S^{d-1}} \mathbf{M}^2 \left( \mathbf{M}^2 \right)' f'(\xi - \xi_*, \theta) \, d\omega d\xi_* = \int_{\mathbb{R}^d} \int_{S^{d-1}} \mathbf{M}^2 \left( \mathbf{M}^2 \right)' f'(\xi - \xi_*, \theta) \, d\omega d\xi_*,
\]

Thus by Fubini’s theorem,

\[
I = 2 \int_{S^{d-1}} \int_{\mathbb{R}^d} \mathbf{M}^2 \left( \mathbf{M}^2 \right)' f'(\xi - \xi_*, \theta) \, d\omega d\xi_*
\]

\[
= 2 \int_{S^{d-1}} \int_{\mathbb{R}^d} \mathbf{M}^2(\xi_*) \mathbf{M}^2(\xi + \xi - (\xi_* \cdot \omega)\omega) f(\xi + (\xi_* \cdot \omega)\omega) q(\xi_* \theta) \, d\xi_* d\omega
\]

\[
= 4 \int_{S^{d-1}} \int_{\mathbb{R}^d} \mathbf{M}^2(x + r\omega + \xi) \mathbf{M}^2(x + \xi) f(x + r\omega) q(x + r\omega, \theta) \, dx d\omega(\xi_* \omega),
\]

where \( P_\omega \) is the hyperplane in \( \mathbb{R}^d \) that orthogonal to \( \omega \) containing the origin. Here we remark that one actually needs the boundedness on \( K \) to make sure that Fubini’s theorem can be applied. Then

\[
I = 4 \int_{\mathbb{R}^d} \int_{S^{d-1}} \mathbf{M}^2 \left( \mathbf{M}^2 \right)' f'(\xi - \xi_*, \theta) \, d\omega d\xi_*
\]

\[
= 4 \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \left( \frac{1}{(2\pi)^{-\frac{d}{2}}} \right) \exp \left( -\frac{|x + a|^2}{2} - \frac{|b|^2}{2} - \frac{|\xi_* - \xi|^2}{8} \right) \frac{f(\xi_*) q(x + \xi_*, \xi)}{|\xi_*|^{d-1}} \, dx d\xi_*
\]

\[
= 4 \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \left( \frac{1}{(2\pi)^{-\frac{d}{2}}} \right) \exp \left( -\frac{|x + a|^2}{2} - \frac{|b|^2}{2} - \frac{|\xi_* - \xi|^2}{8} \right) \frac{f(\xi_*) q(x + \xi_*, \xi)}{|\xi_*|^{d-1}} \, dx d\xi_*
\]

where \( \theta \) is angle between \( x - a + \xi_* - \xi \) and \( \xi_* - \xi \). This gives the expression of \( k_1 \).

2. \( \nu \) has expression:

\[
\nu(\xi) = \int_{\mathbb{R}^d} \int_{S^{d-1}} \mathbf{M}^2 \mathbf{q}(\xi - \xi_*, \theta) \, d\omega d\xi_*.
\]

Recall the cut-off assumption (1.5), then we have

\[
\nu(\xi) = q_0(2\pi)^{d/4} \int_{\mathbb{R}^d} \exp(-\frac{|\xi_*|^2}{4}) |\xi - \xi_*|^{-\gamma} d\xi_*.
\]

On one hand, by (7.7) in lemma 7.6, we have

\[
\nu(\xi) = q_0(2\pi)^{d/4} \int_{\mathbb{R}^d} \exp(-\frac{|\xi_*|^2}{4}) |\xi - \xi_*|^{-\gamma} d\xi_* \leq C_{d,\gamma, q_0} (1 + |\xi|)^{-\gamma}.
\]

On the other hand, notice \( |\xi_* - \xi| \leq 1 + |\xi_*| \leq (1 + |\xi_*|)(1 + |\xi|) \), we have

\[
\nu(\xi) \geq q_0(2\pi)^{d/4} \int_{\mathbb{R}^d} \exp(-\frac{|\xi_*|^2}{4})(1 + |\xi_*|)^{-\gamma} d\xi_* (1 + |\xi|)^{-\gamma}
\]

\[
\geq C_{d,\gamma, q_0} (1 + |\xi|)^{-\gamma}.
\]

This proves statement (i).
3. For the part $k_1$, we firstly estimate $J := \int_{P_{\xi, \xi}} e^{-|x|^2} q(x - a + \xi, \theta) \, dx$. Notice $a \in P_{\xi, \xi}$ and integral is taken in $P_{\xi, \xi}$, we have

$$J \leq \int_{P_{\xi, \xi}} e^{-|x|^2} |x - a + \xi|^{-\gamma} |\cos \theta| \, dx$$

$$\leq \int_{P_{\xi, \xi}} e^{-|x|^2} |x - a + \xi|^{-\gamma} |(x - a + \xi - \xi) \cdot (\xi - \xi)| \, dx$$

$$= \int_{P_{\xi, \xi}} \frac{e^{-|x|^2}}{(|x - a|^2 + |\xi - \xi|^2)^{(\gamma + 1)/2}} \, dx.$$  

If $|\xi - \xi| \leq 1$, by (7.7) in lemma 7.6,

$$\frac{J}{|\xi - \xi|} \leq \int_{P_{\xi, \xi}} e^{-|x|^2} \frac{1}{(|x - a|^2 + |\xi - \xi|^2)^{(\gamma + 1)/2}} \, dx$$

$$\leq C_{d, \gamma} \left( \frac{1}{(1 + |a|)^{\gamma + 1}} \right).$$

If $|\xi - \xi| > 1$, notice $|x - a| \leq \frac{|a|}{2}$ implies $|x| > \frac{|a|}{2}$, then

$$\frac{J}{|\xi - \xi|} \leq \left( \int_{P_{\xi, \xi}} + \int_{|x - a| > \frac{|a|}{2}} \right) e^{-|x|^2} \frac{1}{(|x - a|^2 + |\xi - \xi|^2)^{(\gamma + 1)/2}} \, dx$$

$$\leq \int_{P_{\xi, \xi}} \frac{e^{-|x|^2}}{|\xi - \xi|^{\gamma + 1}} \, dx + \int_{|x - a| > \frac{|a|}{2}} \frac{1}{(|x - a|^2 + |\xi - \xi|^2)^{(\gamma + 1)/2}} \, dx$$

$$\leq C_{d, \gamma} \left( \frac{1}{|\xi - \xi|^{\gamma + 1}} \right).$$

by using the fact that

$$\sup_{a \in \mathbb{R}^d, \xi - \xi \geq 1} e^{-|x|^2} \frac{|a + \xi - \xi|^{\gamma + 1}}{|\xi - \xi|^{\gamma + 1}} \leq \begin{cases} e^{-|x|^2} (2|a|)^{\gamma + 1}, & \text{if } |a| > |\xi - \xi|, \\ \frac{1}{2^{\gamma + 1}}, & \text{if } |a| \leq |\xi - \xi|. \end{cases}$$

Thus for any $\varepsilon \in (0, 1)$,

$$|k_1(\xi, \psi)| \leq C_{d, \gamma} \left| \frac{1}{|\xi - \xi|^{d-2}} (1 + |a| + |\xi - \xi|)^{\gamma + 1} \exp\left(-\frac{|b|^2}{2} - \frac{|\xi - \xi|^2}{8}\right), \right.$$
\[
(f, -\nu g) + \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} k(\xi, \xi_*) g(\xi) \, d\xi f(\xi_*) \, d\xi_* = (f, -\nu g) + (f, Kg) = (f, Lg)_{L^2}.
\]

2. For the non-positiveness, we can use a well-known fact that
\[
(Q(f, g), \psi)_{L^2} = \frac{1}{4} \int_{\mathbb{R}^d} \int_{\mathbb{S}^{d-1}} (f'_* g' + f' g'_* - f_* g - fg'_*) q(\xi - \xi_*, \theta)(\psi + \psi - \psi'_*) \, d\omega d\xi_* d\xi,
\]
which is valid whenever the integral is absolutely convergent. Then for \( f \in L^2(\mathbb{R}^d) \), since \( K \) is bounded on \( L^2 \) and \( \nu \in L^\infty \), we can apply this identity to get
\[
(Lf, f)_{L^2} = -\frac{1}{4} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \int_{\mathbb{S}^{d-1}} \left| f'_* (M^\frac{1}{2})' + f' (M^\frac{1}{2})'_* - f_* M^\frac{1}{2} - f M^\frac{1}{2} \right|^2 q(\xi - \xi_*, \theta) \, d\omega d\xi_* d\xi \leq 0. \tag{7.9}
\]

3. Let \( f \in L^2 \) such that \( Lf = 0 \). Then \( (Lf, f)_{L^2} = 0 \). Using (7.9) and the fact \( MM_* = M'M'_* \), we have for a.e. \( \xi, \xi_* \in \mathbb{R}^d, \omega \in \mathbb{S}^{d-1} \) that
\[
f'_* (M^{-\frac{1}{2}})' + f' (M^{-\frac{1}{2}})'_* = f_* M^{-\frac{1}{2}} + f M^{-\frac{1}{2}}.
\]
Thus by the theory of collision invariant, \( f \in \text{Span}\{M^\frac{1}{2}, \xi_1 M^\frac{1}{2}, \ldots, \xi_d M^\frac{1}{2}, |\xi|^2 M^\frac{1}{2}\} \). \( \square \)

References

[1] Russel E. Caflisch. The Boltzmann equation with a soft potential. *Communications in Mathematical Physics*, 74(2):97–109, jun 1980.

[2] Carlo Cercignani, Reinhard Illner, and Mario Pulvirenti. *The Mathematical Theory of Dilute Gases*, volume 106 of *Applied Mathematical Sciences*. Springer Science+Business Media New York, 1994.

[3] Richard S. Ellis and Mark A. Pinsky. The First and Second Fluid Approximations to the Linearized Boltzmann Equation. *Journal de Mathématiques Pures et Appliquées*, 54:125–156, 1975.

[4] Klaus-Jochen Engel, Rainer Nagel, Rainer Nagel, M. Campiti, and T. Hahn. *One-Parameter Semigroups for Linear Evolution Equations*. Springer New York, 1999.

[5] Lawrence C. Evans. *Partial Differential Equations: Second Edition (Graduate Studies in Mathematics)*. American Mathematical Society, 2010.

[6] Israel Gohberg, Seymour Goldberg, and Marinus Kaashoek. *Classes of Linear Operators Vol. I (Operator Theory: Advances and Applications) (v. 1)*. Birkhuser, 1990.

[7] Tai-Ping Liu and Shih-Hsien Yu. Solving Boltzmann equation I, Green function. *Bulletin of the Institute of Mathematics Academia Sinica (New Series)*, 6(2):115–243, 2011.

[8] R. T. Seeley. Extension of \( C^\infty \) functions defined in a half space. *Proceedings of the American Mathematical Society*, 15(4):625–625, apr 1964.

[9] Seiji Ukai and Kiyoshi Asano. On the Cauchy problem of the Boltzmann equation with a soft potential. *Publications of the Research Institute for Mathematical Sciences*, 18(2):477–519, 1982.

[10] Tong Yang and Hongjun Yu. Spectrum Analysis of Some Kinetic Equations. *Archive for Rational Mechanics and Analysis*, 222(2):731–768, may 2016.

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