A characterization of a finite-dimensional commuting square producing a subfactor of finite depth

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Abstract

We give a characterization of a finite-dimensional commuting square of $C^*$-algebras with a normalized trace that produces a hyperfinite type $\text{II}_1$ subfactor of finite index and finite depth in terms of Morita equivalent unitary fusion categories. This type of commuting squares was studied by N. Sato, and we show that a slight generalization of his construction covers the fully general case of such commuting squares. We also give a characterization of such a commuting square that produces a given hyperfinite type $\text{II}_1$ subfactor of finite index and finite depth. These results also give a characterization of certain 4-tensors that appear in recent studies of matrix product operators in 2-dimensional topological order.

1 Introduction

Subfactor theory of Jones \cite{Jones} has revealed a wide range of interconnections among many different topics in mathematics and physics. In mathematical studies of sub-
factors, a finite dimensional commuting square, a combination of four finite dimensional C*-algebras with a trace satisfying a certain compatibility condition, has played important roles both in construction of examples and classification. (See the following section for a precise setting.) Repeated basic constructions of Jones applied to a finite dimensional commuting square produces a hyperfinite II$_1$ subfactor of finite index as the limit algebras. A type II$_1$ subfactor of finite index produces a sequence of finite dimensional commuting square arising as higher relative commutants through the Jones tower construction, called the canonical commuting squares. If the original subfactor is strongly amenable in the sense of [32], this sequence recovers the original subfactor completely, due to the celebrated classification theorem of Popa [32]. If the Bratteli diagrams of the higher relative commutants stabilize after a certain stage, the original subfactor is said to be of finite depth. A hyperfinite II$_1$ subfactor of finite index and finite depth is automatically strongly amenable. For this reason, a finite dimensional, but not necessarily canonical, commuting square producing a subfactor of finite index and finite depth has been important for more than 30 years.

The finite depth condition is also related to (2+1)-dimensional topological quantum field theory and 2-dimensional conformal field theory. A type II$_1$ subfactor of finite index and finite depth produces a fusion category of bimodules and the connections to quantum topology and mathematical physics work out with such categories. Its relation to 2-dimensional topological order has created recent interest in subfactors of finite index and finite depth. A finite dimensional commuting square produces a bi-unitary connection, a certain kind of 4-tensor, and a tensor network plays the key role in recent studies of 2-dimensional topological order [6], [22], [20]. (Also see [20], [24] for related topics.) For these reasons, we are interested in when a finite dimensional commuting square produces a hyperfinite II$_1$ subfactor of finite index and finite depth through repeated basic constructions. Our main theorem in this paper, Theorem 3.4, gives a characterization of such a commuting square. Sato’s results [33], [34], [35] play the key roles in this characterization. As a byproduct, we also give a characterization of such a commuting square that produces a given fixed subfactor of finite index and finite depth. It is easy to see that countably many different commuting squares produce the same subfactor of finite index and finite depth, so we never have uniqueness. We determine which commuting square produce a specific subfactor. We then give examples related to the A-D-E Dynkin diagrams and the Goodman-de la Harpe-Jones subfactors.

We list [10], [11] and [16] for general references on subfactor theory, [3], [19] for connections to conformal field theory and tensor categories, and [21] for relations to 2-dimensional topological order. This work was partially supported by JST CREST program JPMJCR18T6 and Grants-in-Aid for Scientific Research 19H00640 and 19K21832.
2 Finite dimensional commuting squares and construction of subfactors

We review a general construction of a hyperfinite type II$_1$ subfactor from repeated basic constructions of a commuting square of finite dimensional $C^*$-algebras after some basic definitions and results. They are standard materials as in books [10, Chapter 9], [11, Chapter 4], [16, Chapter 5], but there is a small subtlety about so-called non-degeneracy, so we explain the construction in detail and fix the notation.

Let $A \subset B$ be an inclusion of finite von Neumann algebras with a faithful normalized trace $\text{tr}$. (That is, we require $\text{tr}(1) = 1$.) Then the conditional expectation $E_A$ from $B$ to $A$ is uniquely determined by the property $\text{tr}(ab) = \text{tr}(aE_A(b))$ for $a \in A, b \in B$. We have the following definition as in [10, Definition 9.52], [11, Section 4.2], [16, Definition 5.1.7]. (This notion was introduced in [31, Lemma 1.2.2] originally.)

**Definition 2.1** Let

$B_{00} \subset B_{01} \cap B_{10} \subset B_{11}$

be inclusions of finite dimensional $C^*$-algebras with faithful normalized trace $\text{tr}$ on $B_{11}$. When one of the following, mutually equivalent conditions holds, we say that this is a commuting square.

1. We have $E_{B_{01}} = E_{B_{00}}$ on $B_{10}$.
2. We have $E_{B_{01}}(B_{10}) = B_{00}$.
3. We have $E_{B_{10}} = E_{B_{00}}$ on $B_{01}$.
4. We have $E_{B_{10}}(B_{01}) = B_{00}$.
5. We have $E_{B_{01}}E_{B_{10}} = E_{B_{00}}$.
6. We have $E_{B_{10}}E_{B_{01}} = E_{B_{00}}$.

We now assume that all the four Bratteli diagrams for $B_{00} \subset B_{01}$, $B_{00} \subset B_{10}$, $B_{10} \subset B_{11}$, and $B_{01} \subset B_{11}$ are connected. We then have the following definition.

**Definition 2.2** If we have span $B_{01}B_{10} = B_{11}$ for the above commuting square, we say that it is symmetric.

See [10, page 553], [16, Definition 5.3.6], [36, Definition 1.8] for other equivalent conditions of being symmetric. Also see [15 Corollary 5.3.4] for this equivalence. As in [16, Remark 5.3], a symmetric commuting square is also called a non-degenerate commuting square. We now consider a symmetric commuting square as in Definition 2.2.

We also recall the following definition of the basic construction [16, Definition 3.1.1], [10, Definition 9.21].
Definition 2.3 Let $A \subset B$ be an inclusion of finite von Neumann algebras with a normalized trace $\text{tr}$ on $B$. Let $L^2(B)$ be the completion of $B$ with respect to the inner product $\langle x, y \rangle = \text{tr}(y^*x)$ for $x, y \in B$ and identify $L^2(A)$ as the closure of $A$ within $L^2(B)$. Let $B$ act on $L^2(B)$ by the left multiplication. Let $e_A$ be the orthogonal projection from $L^2(B)$ onto $L^2(A)$. We call $e_A$ the Jones projection. We call the von Neumann algebra generated by $B$ and $e_A$ on $L^2(B)$ basic construction. This basic construction gives a finite von Neumann algebra with a natural trace.

We start with a commuting square as in Definition 2.2. We consider the basic construction $B_{02}$ for $B_{00} \subset B_{01}$ and the one $B_{12}$ for $B_{10} \subset B_{11}$. We can naturally identify the Jones projection $e_{B_{00}}$ and $e_{B_{10}}$ and regard $B_{02}$ as a subalgebra $B_{12}$. Then $B_{01} \subset B_{02}$ and $B_{11} \subset B_{12}$ is again a symmetric commuting square. We can repeat this procedure and obtain the following sequence of symmetric commuting squares.

$$B_{00} \subset B_{01} \subset B_{02} \subset B_{03} \subset \cdots$$

$$B_{10} \subset B_{11} \subset B_{12} \subset B_{13} \subset \cdots.$$

We can also apply the basic construction vertically to the original commuting square. The horizontal and vertical basic constructions are compatible and we have a double sequence $\{B_{kl}\}_{k,l}$ of finite dimensional $C^*$-algebras with trace $\text{tr}$. We label $e_k$ for the vertical Jones projection for $B_{k,l-1} \subset B_{k,l}$ and label $f_l$ for the horizontal Jones projection for $B_{0,l-1} \subset B_{0,l}$. Note that the vertical Jones projection $e_k$ is for the basic construction of $B_{k-1,l} \subset B_{k,l}$ for all $l$ and the horizontal Jones projection $f_l$ is for the basic construction of $B_{k,l-1} \subset B_{k,l}$ for all $k$. We now have the following relations.

$$B_{kl} = \begin{cases} B_{00}, & \text{if } k = 0 \text{ and } l = 0, \\ \langle B_{10}, e_1, e_2, \ldots, e_{k-1} \rangle, & \text{if } k > 0 \text{ and } l = 0, \\ \langle B_{01}, f_1, f_2, \ldots, f_{l-1} \rangle, & \text{if } k = 0 \text{ and } l > 0, \\ \langle B_{11}, e_1, e_2, \ldots, e_{k-1}, f_1, f_2, \ldots, f_{l-1} \rangle, & \text{if } k > 0 \text{ and } l > 0. \end{cases}$$

See [11, Proposition 4.1.2], [16, Corollary 5.5.5] for more details.

We next take a nonzero projection $p \in B_{00}$ and set $A_{kl} = pB_{kl}p$. We normalize the trace on $A_{kl}$. Then the following is a commuting square again by [11, Proposition 4.2.6].

$$A_{k,l} \subset A_{k,l+1} \subset A_{k,l+1} \subset A_{k+1,l+1}.$$

We then define $A_{k,\infty}$ to be the GNS-completion of $\bigcup_{l=0}^{\infty} A_{k,l}$ with respect to the trace $\text{tr}$ and $A_{\infty,l}$ to be the GNS-completion of $\bigcup_{k=0}^{\infty} A_{k,l}$ with respect to the trace.
tr. We also define \( A_{\infty, \infty} \) to be the GNS-completion of \( \bigcup_{k=0}^{\infty} A_{k, k} \). This is also the GNS-completions of \( \bigcup_{i=0}^{\infty} A_{\infty, i} \) and \( \bigcup_{k=0}^{\infty} A_{k, \infty} \). They are all hyperfinite II\(_1\) factors, because all the original Bratteli diagrams are finite and connected. The subfactors \( A_{k, \infty} \subseteq A_{k+1, \infty} \) and \( A_{\infty, j} \subseteq A_{\infty, j+1} \) have finite index. The isomorphism classes of these subfactors are independent of choice of \( p \) by the relative McDuff property \([2, \text{ Theorem 3.1}]\).

Jones asked in 1995 whether the subfactor \( A_{\infty, 0} \subseteq A_{\infty, 1} \) has a finite depth or not if so does \( A_{0, \infty} \subseteq A_{1, \infty} \), and Sato gave a positive answer to this question in \([33, \text{ Corollary 2.2}]\). That is, the subfactor \( A_{\infty, 0} \subseteq A_{\infty, 1} \) has a finite depth if and only if so does \( A_{0, \infty} \subseteq A_{1, \infty} \). He further studied this situation in \([34, 35]\).

We reformulate this construction in terms of string algebras and a bi-unitary connection as in \([10, \text{ Section 11.3}]\). Our symmetric commuting square

\[
\begin{array}{c}
B_{00} \subset B_{01} \\
\cap \quad \cap \\
B_{10} \subset B_{11}
\end{array}
\]

gives a bi-unitary connection on four connected graphs as in \([10, \text{ Theorem 11.2}], [36, \text{ Theorem 11.10}]\). (The notion of a bi-unitary connection was first introduced in \([28, 29]\) with an extra axiom of flatness. See \([10, \text{ Definition 11.3}]\) for the definition of a bi-unitary connection. It is also explained in \([22, \text{ Definition 2.2}]\).) Then our double sequence \( \{A_{kl}\}_{k,l} \) is described with the string algebra construction as in \([10, \text{ Section 11.3}]\). (If we have an increasing sequence of finite dimensional C*-algebras, then we have the Bratteli diagram. Conversely, if we have a Bratteli diagram given, we can construct the corresponding increasing sequence of finite dimensional C*-algebras canonically. A pair of paths on the Bratteli diagram starting at the top level and having the same ending vertex is called a \emph{string} and gives a canonical expression of a matrix unit in a finite dimensional C*-algebra. The Jones projection has a nice description in terms of string algebras as in \([10, \text{ Definition 11.5}]\). See \([10, \text{ Definition 11.1}]\) for more details.) However, there is a slight difference since we now have \( A_{00} \neq \mathbb{C} \) in general. This matter is handled as follows.

Let \( v_1, v_2, \ldots, v_m \) be the even vertices of \( G_0 \) \([10, \text{ Section 10.3}]\). (The set of these vertices is denoted by \( V_0 \) in \([22, \text{ Section 2}]\).) Let \( A_{00} = \bigoplus_{j=1}^{m} M_{n_j}(\mathbb{C}) \), where a direct summand \( M_{n_j}(\mathbb{C}) \) corresponds to the vertex \( v_j \). (Here we have \( n_j \geq 0 \) and at least one of \( n_j \) is strictly positive.) To deal with the matter of multiplicity, we set \( V \) be the set of \( v_j \), where \( 1 \leq j \leq m \) and \( 1 \leq i \leq n_j \). If \( v_j \) is connected to another vertex \( w \) on one of the four graphs, we interpret that \( v_j \) is connected to \( w \) below. Now our string is a pair \( (\xi, \xi') \) of paths on these graphs with \( r(\xi) = r(\xi') \) as usual as in \([10, \text{ Section 11.3}]\), but we now require \( s(\xi), s(\xi') \in V \) instead of \( s(\xi) = s(\xi') \). The multiplication and the \(*\)-operation are defined in the same way as in \([10, \text{ Definition 11.4}]\). (This is a slight generalization of an idea of double starting vertices in \([17, \text{ Section 5}]\).)

Ocneanu’s compactness argument \([29]\) says that we have \( A_{0, \infty} \cap A_{k, \infty} \subseteq A_{k, 0} \) for the double sequence \( \{A_{kl}\}_{k,l} \) as above. A proof of the compactness argument is given in \([10, \text{ Theorem 11.15}]\) only for the case \( A_{00} = \mathbb{C} \), but the same proof works for general \( A_{00} \) without any change.

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3  Sato’s construction and the main result

We recall Sato’s construction of a commuting square to realize two mutually opposite Morita equivalent subfactors in [35] in a slightly generalized form and prove our main result saying that this construction covers the fully general case.

We first recall some terminology related to (opposite) Morita equivalence of unitary fusion categories and subfactors. We refer readers to [8] for general theory of fusion categories and Morita equivalence. Also see [19] for a more brief description relevant to subfactor theory (and conformal field theory). A recent paper [7] also explains the framework in a concise manner.

Let \( N \subset M \) be a type II\(_1\) subfactor of finite index. The \( L^2 \)-completion \( L^2(M) \) of \( M \) with respect to the trace \( \text{tr} \) gives an \( N\)-\( M \) and \( M\)-\( N \) bimodules \( N M_M \) and \( M M_N \). (We simply write \( M \) instead of \( L^2(M) \).)

We look at all irreducible \( N\)-\( N \) bimodules arising from \( N M \otimes_N M \otimes_N \cdots \otimes_N M N \). If we have only finitely many isomorphism classes for such irreducible bimodules, then we say the subfactor has a finite depth. This is equivalent to the condition that we have only finitely many irreducible bimodules arising from irreducible decompositions of \( M M \otimes_N M \otimes_N \cdots \otimes_N M M \) up to isomorphism. In this case, we consider a fusion category of \( N\)-\( N \) bimodules that are finite direct sums of such irreducible \( N\)-\( N \) bimodules. We call it the fusion category of \( N\)-\( N \) bimodules arising from \( N \subset M \). We similarly define the fusion category of \( M\)-\( M \) bimodules arising from \( N \subset M \). These two fusion categories are Morita equivalent. We may realize any unitary fusion category in this form using the hyperfinite II\(_1\) factor and any Morita equivalence between two such fusion categories is realized in this way. For a fusion category \( \mathcal{C} \), we define its global index to be the square sum of the (Perron-Frobenius) dimensions of all irreducible objects up to isomorphism. This is an invariant for Morita equivalence. We define the global index of a subfactor \( N \subset M \) to be the global index of the fusion category of \( N\)-\( N \) bimodules arising from \( N \subset M \). (If a subfactor is of infinite depth, we define its global index to be infinite.)

For a subfactor \( N \subset M \), the opposite algebras, given by reversing the order of the multiplication, give the opposite subfactor \( N^{\text{opp}} \subset M^{\text{opp}} \). We say that the fusion category of the \( N\)-\( N \) bimodules arising from \( N \subset M \) and that of the \( N^{\text{opp}}\)-\( N^{\text{opp}} \) bimodules arising from \( N^{\text{opp}} \subset M^{\text{opp}} \) are opposite Morita equivalent. We also say that fusion categories \( \mathcal{C}_1 \) and \( \mathcal{C}_2 \) are opposite Morita equivalent, if \( \mathcal{C}_1 \) and \( \mathcal{C}_3 \) are Morita equivalent and \( \mathcal{C}_2 \) and \( \mathcal{C}_3 \) are opposite Morita equivalent for some fusion category \( \mathcal{C}_3 \). (See [35, Section 1] for its background.)

Remark 3.1 Sato simply said two fusion categories of bimodules are equivalent in [34], [35] when we say they are Morita equivalent today. His terminology is also inconsistent with standard terminology of equivalence of tensor categories, so we use “Morita equivalence” for this notion.

The following is essentially contained in [35, Section 3].
Theorem 3.2 Let $A \subset B$ and $C \subset D$ are Morita equivalent $II_1$ subfactors of finite index and finite depth and $C X_A$ be a bimodule giving the Morita equivalence. Define the finite dimensional $C^*$-algebras $A_{kl}$ as follows.

$$A_{kl} = \begin{cases} 
\text{End}(C D \otimes_C D \otimes \cdots \otimes_C D \otimes_C X \otimes_A B \otimes_A \cdots \otimes_A B_A), \\
(k/2 \text{ copies of } D \text{ and } l/2 \text{ copies of } B), \\
\text{if } k \text{ and } l \text{ are even}, \\
\text{End}(D D \otimes_C D \otimes \cdots \otimes_C D \otimes_C X \otimes_A B \otimes_A \cdots \otimes_A B_A), \\
((k+1)/2 \text{ copies of } D \text{ and } l/2 \text{ copies of } B), \\
\text{if } k \text{ is odd and } l \text{ is even}, \\
\text{End}(C D \otimes_C D \otimes \cdots \otimes_C D \otimes_C X_A B \otimes_A \cdots \otimes_A B_B), \\
(k/2 \text{ copies of } D \text{ and } (l+1)/2 \text{ copies of } B), \\
\text{if } k \text{ is even and } l \text{ is odd}, \\
\text{End}(D D \otimes_C D \otimes \cdots \otimes_C D \otimes_C X_A B \otimes_A \cdots \otimes_A B_B), \\
((k+1)/2 \text{ copies of } D \text{ and } (l+1)/2 \text{ copies of } B), \\
\text{if } k \text{ and } l \text{ are odd}. 
\end{cases}$$

Then the double sequence $\{A_{kl}\}_{k,l}$ of commuting squares gives a subfactor of finite index and finite depth.

The assumption on $C X_A$ means that all the irreducible bimodules appearing in $C X \otimes_A B \otimes_A \cdots \otimes_A B \otimes_A Z_C$ arise from the irreducible decompositions of $C D \otimes_C \cdots \otimes_C D C$. In this case, all the irreducible bimodules appearing in $C X \otimes_A B \otimes_A \cdots \otimes_A B_A$ arise from the irreducible decompositions of $A B \otimes_A \cdots \otimes_A B_A$ by the Frobenius reciprocity. Note that we do not require irreducibility of $C X_A$.

Another way to look at this $C X_A$ is as follows. We have a (possibly reducible) $Q$-system (in the sense of [25, Section 6]) within the fusion category of the $A$-$A$ bimodules arising from $A \subset B$ and it gives the same as the $C$-$C$ bimodules arising from $C \subset D$ as the dual fusion category. (Longo [25] uses type III factors and endomorphisms for $Q$-systems. See [27, Proposition 2.1] for the bimodule version of the $Q$-system.)

After [35, Remark 3.2], Sato used a more restrictive form of $C X_A$, but his construction and proof work in the above setting. Indeed, we need to check only connectedness of the four Brattelli diagrams. Suppose $C Y_A$ and $C Z_A$ are two irreducible bimodules arising in some irreducible decomposition in the above procedure. Since $A B A$ generates the fusion category of $A$-$A$ bimodules, we have $\dim \text{Hom}(C Y \otimes_A B \otimes_A \cdots \otimes_A B, C Z_A) > 0$ for sufficiently many copies of $A B A$. This means that the Brattelli diagram for $A_{2k,2l} \subset A_{2k,2l+1}$ is connected for all $k,l$. We can handle the other three Brattelli diagrams similarly.

Lemma 3.3 Let $A \subset B \subset C \subset D$ be inclusions of type $II_1$ factors. Suppose that $A \subset D$ is of finite index and finite depth and that the global indices of $A \subset D$, $A \subset B$ and $C \subset D$ are the same. Then the bimodule $p C_C$ gives Morita equivalence between the fusion categories of $B$-$B$ bimodules arising from $A \subset B$ and $C$-$C$ bimodules arising from $C \subset D$.

Proof. Let $C_A$ and $C_D$ be the fusion categories of $A$-$A$ bimodules and $D$-$D$ bimodules arising from $A \subset D$ respectively. Let $C_B$ be the fusion categories of $B$-$B$
bimodules arising from the relative tensor products of $B\otimes A$, $A\otimes A$ bimodules in $\mathcal{C}_A$ and $A\otimes B$. Similarly, let $\mathcal{C}_B$ be the fusion categories of $C\otimes C$ bimodules arising from the relative tensor products of $C\otimes D$, $D\otimes D$ bimodules in $\mathcal{C}_D$ and $D\otimes C$. Then it is easy to see that the bimodule $B\otimes C$ gives Morita equivalence between $\mathcal{C}_B$ and $\mathcal{C}_C$. Now by the assumption on the global indices, we see that $\mathcal{C}_B$ is given by $A \subset B$ and $\mathcal{C}_C$ is given by $C \subset D$. □

We now give our main result.

**Theorem 3.4** A double sequence $\{A_{kl}\}_{k,l}$ of commuting squares of finite dimensional $C^*$-algebras given in the form in Section 2 gives a subfactor of finite index and finite depth if and only if it is of the form of Theorem 3.2, where $A, B, C, D$ are hyperfinite.

In this case, the subfactor $A_{0,\infty} \subset A_{1,\infty}$ is anti-isomorphic to $B \subset B_1$, where $B_1$ is given by the basic construction of $A \subset B$.

**Proof.** The "if" part is already proved by Sato [35], so we give a proof for the converse.

Suppose that the subfactor $A_{0,\infty} \subset A_{1,\infty}$ has a finite depth. Let $N = A_{0,\infty}$, $M = A_{1,\infty}$, $P = A_{\infty,0}$, $Q = A_{\infty,1}$. We define $A_{k,-1} = A_{0,\infty}' \cap A_{k,\infty}$ for $k \geq 0$ and $A_{k,-2} = A_{1,\infty}' \cap A_{k,\infty}$ for $k \geq 1$. By the compactness argument, we have $A_{k,-2} \subset A_{k,-1} \subset A_{k,0}$. We define $R$ and $S$ to be the GNS-completions of $\bigcup_{k=0}^{\infty} A_{k,-1}$ and $\bigcup_{k=1}^{\infty} A_{k,-2}$ with respect to the trace $\text{tr}$, respectively. By the finite depth assumption on $N \subset M$, these are also hyperfinite type II$_1$ factors.

As noted in [33 page 371], the following is a commuting square for any $k \geq 0$.

$$
\begin{align*}
A_{k,-1} & \subset A_{k,0} \\
\cap & \cap \\
A_{k+1,-1} & \subset A_{k+1,0}.
\end{align*}
$$

For the same reason, the following also a commuting square for any $k \geq 1$.

$$
\begin{align*}
A_{k,-2} & \subset A_{k,-1} \\
\cap & \cap \\
A_{k+1,-2} & \subset A_{k+1,-1}.
\end{align*}
$$

We apply the compactness argument to the following commuting squares and the subfactor $R \subset A_{\infty,l}$ for some fixed $l$.

$$
\begin{align*}
A_{0,-1} & \subset A_{0,l} \\
\cap & \cap \\
A_{1,-1} & \subset A_{1,l} \\
\cap & \cap \\
A_{2,-1} & \subset A_{2,l} \\
\downarrow & \downarrow \\
R & \subset A_{\infty,l}.
\end{align*}
$$
Then we have $R' \cap A_{\infty,l} \subset A_{0,l}$. Since $A_{0,l} \subset N$, the converse inclusion is trivial, so we have $R' \cap A_{\infty,l} = A_{0,l}$. We similarly have $S' \cap A_{\infty,l} = A_{1,l}$. That is, the commuting square
\[
A_{k,l} \subset A_{k,l+1} \\
\cap \\
A_{k+1,l} \subset A_{k+1,l+1}
\]
is given as
\[
R' \cap A_{\infty,l} \subset R' \cap A_{\infty,l+1} \\
\cap \\
S' \cap A_{\infty,l} \subset S' \cap A_{\infty,l+1}.
\]
Using [10, Lemma 11.1], we identify the above commuting with the following commuting squares
\[
\text{End}(RQ \otimes P Q \otimes P \cdots \otimes P Q_P) \subset \text{End}(RQ \otimes P Q \otimes P \cdots \otimes P Q_P Q_Q) \\
\text{End}(sQ \otimes P Q \otimes P \cdots \otimes P Q_P) \subset \text{End}(sQ \otimes P Q \otimes P \cdots \otimes P Q_P Q_Q),
\]
where the number of copies of $Q$ is in the upper left corner is $l/2$ if $l$ is even and
\[
\text{End}(RQ \otimes P Q \otimes P \cdots \otimes P Q_P) \subset \text{End}(RQ \otimes P Q \otimes P \cdots \otimes P Q_P) \\
\text{End}(sQ \otimes P Q \otimes P \cdots \otimes P Q_Q) \subset \text{End}(sQ \otimes P Q \otimes P \cdots \otimes P Q_P Q_Q),
\]
where the number of copies of $Q$ is in the upper left corner is $(l + 1)/2$ if $l$ is odd. We further have $\text{End}(RQ \otimes P Q \otimes P \cdots \otimes P Q_P) = \text{End}(RQ \otimes P Q \otimes P Q \otimes P \cdots \otimes P Q_P)$ and similar identities for the other three End spaces. So our $A_{k,l}$ is of the form of Theorem 3.2, except for the matter of Morita equivalence.

Since the finite depth assumption implies strong amenability by [32, Theorem 1], the subfactor $S \subset R$ is anti-isomorphic to $N_1 \subset N$ given by the downward basic construction of $N \subset M$ by [32, Theorem 2]. This shows that the global indices of $N \subset M$ and $S \subset R$ are the same. Since the global indices of $P \subset Q$ and $N \subset M$ are the same by [33, Corollary 2.5], we know that the global indices of $P \subset Q$ and $S \subset R$ are the same.

By [34, Theorem 2.4], the fusion categories of $Q$-$Q$ bimodules arising from $P \subset Q$ and $S \subset Q$ are the same, so the global indices of $P \subset Q$ and $S \subset Q$ are the same. By Lemma 3.3, we now conclude that our $A_{k,l}$ is of the form of Theorem 3.2.

We have already seen in this case that $S \subset R$ is anti-isomorphic to $N_1 \subset N$.

\[\square\]

**Remark 3.5** In the above setting, we have $R = P$ if and only if the bi-unitary connection is flat. (See [10, Definition 11.16].) This is the case that was handled in [35, Lemma 3.1]. The canonical commuting square corresponds to the case where $S = P_1$, $R = P$ and $P_1$ is a downward basic construction of $P \subset Q$. 

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As a byproduct of the above characterization, we also characterize a commuting square producing a given hyperfinite $\text{II}_1$ subfactor of finite index and finite depth. Let $N \subset M$ be a hyperfinite type $\text{II}_1$ subfactor of finite index and finite depth. Let $P \subset Q$ be another hyperfinite $\text{II}_1$ subfactor of finite index and finite depth that is opposite Morita equivalent to $N \subset M$. Let $N \text{opp} = \text{opp}_Q X_P$ be a $N \text{opp}$-$P$ bimodule giving Morita equivalence between the fusion category of $N \text{opp}$-$N \text{opp}$ bimodules arising from $N \text{opp} \subset N \text{opp}$ and the one of $P$-$P$ bimodules arising from $P \subset Q$. Using $N \text{opp} N \text{opp} \otimes N \text{opp} \otimes Q$, as a generator of a paragroup (replacing $N M M$ in [10, Chapter 10]), we can realize the subfactor $N \text{opp} \subset Q$ and then we also realize $N \text{opp}$ and $P$ as intermediate algebras, so we have $N \text{opp} \subset N \text{opp} \subset P \subset Q$. We now see that the above construction of commuting squares $\{A_{0,1} \subset A_{1,1}\}$ gives a subfactor that is isomorphic to $N \subset M$. This gives the following.

**Corollary 3.6** Any finite dimensional commuting squares $\{A_{0,1} \subset A_{1,1}\}$ giving a hyperfinite $\text{II}_1$ subfactor $A_{0,\infty} \subset A_{1,\infty}$ that is isomorphic to $N \subset M$ is of the above form.

Note that for a given fusion category $\mathcal{C}$, we have only finitely many fusion categories that are Morita equivalent to $\mathcal{C}$. (This is because we have only finitely many irreducible $\mathcal{Q}$-systems [25, Section 6] for a given fusion category. A $\mathcal{Q}$-system is a unitary version of a Frobenius algebra.) But after fixing the fusion category of $P$-$P$ bimodules, we have countably many subfactors $P \subset Q$ and countably many bimodules $M \text{opp} X_P$. (Note that we do not require irreducibility of $P \subset Q$ or $M \text{opp} X_P$ even when $N \subset M$ is irreducible.) So the total number of such commuting squares for a fixed subfactor $N \subset M$ is countable. Note it is trivial that we have countably many commuting squares giving the same subfactor because if $\{A_{0,1} \subset A_{1,1}\}$ is one example, then $\{A_{0,\{m\}} \subset A_{1,\{m\}}\}$ is another example for any $m$.

As in [22], our bi-unitary connection is regarded as a 4-tensor related to 2-dimensional topological order. This theorem also characterizes which 4-tensor realizes a given subfactor $N \subset M$.

### 4 Examples

In this section, we work out some examples to illustrate our results in the previous section.

**Example 4.1** Let $N \subset M$ be the Jones subfactor with principal graph $A_n$. The fusion category of $N$-$N$ (or $M$-$M$) bimodules is isomorphic to the even part of the Wess-Zumino-Witten category $SU(2)_{n-1}$. All the fusion categories that are Morita equivalent to $SU(2)_{n-1}$ have been classified by Ocneanu [30]. We now look at only the even part of these fusion categories. (Also see [23, Section 2] for classification of $\mathcal{Q}$-systems.) If $n = 4m - 3$, then we have a fusion category arising from the $\alpha$-induction for a simple current extension of order 2. If $n = 11$ or $n = 29$, we also have fusion categories arising from the $\alpha$-induction for conformal embeddings $SU(2)_{10} \subset SO(5)_1$ or $SU(2)_{28} \subset (G_2)_1$. These exhaust all fusion categories that are
Morita equivalent to the one arising from a subfactor with principal graph $A_n$. (See [4] for a general theory of $\alpha$-induction and [5] Section 5] for this classification.) These give all possible commuting squares producing a subfactor with principal graph $A_n$ as in Corollary 3.6. They are related to the Goodman-de la Harpe-Jones subfactors [11, Section 4.5], [10, Section 11.6] as in [13].

The easiest nontrivial example among these is the Goodman-de la Harpe-Jones subfactor arising from $E_6$ having the Jones index $3+\sqrt{3}$. (See [11, Proposition 4.5.2], [10, Example 11.25].) This subfactor is given as $A_{0,\infty} \subset A_{1,\infty}$, and the subfactor $A_{\infty,0} \subset A_{\infty,1}$ has principal graph $A_{11}$. The graph $G_1$ in [10, Fig. 11.65] is not a principal graph of any subfactor, but it is understood as in Theorem 3.2.

**Example 4.2** If $n \neq 4m-3, 11$, then only fusion category that is Morita equivalent to the $N-N$ bimodules of the subfactor with principal graph $A_n$ is itself. In this case, the only commuting squares giving this subfactor are the canonical commuting square, its basic construction possibly cut with some projection in the higher relative commutants.

**Example 4.3** Let $A \subset B$ be a type $\Pi_1$ subfactor with principal graph $A_{4m-3}$, $C = B$ and $D$ be the crossed product of $B$ with the action of $\mathbb{Z}/2\mathbb{Z}$ arising from the two endpoints of the principal graph $A_{4m-3}$ as in $\alpha$ of [12, Corollary 3.6]. (We do not need hyperfiniteness of $B$ here, but if we do have hyperfiniteness, then this action is of $\mathbb{Z}/2\mathbb{Z}$ the “orbifold action” mentioned at the end of [18, Section 3] arising from [17].) If we apply the construction in Theorem 3.2 to this case, we get disconnected Bratteli diagrams for $A_{2k,2l} \subset A_{2k,2l+1}$ for sufficiently large $k$ and $l$, so the construction does not satisfy our requirement. This failure is due to the fact the global index of $C \subset D$ is 2, which is smaller than that of $A \subset B$.

In these examples so far, we have $P = R$ in Theorem 3.4. We now look at an example where we have $P \neq R$.

**Example 4.4** Consider a bi-unitary connection for the case all the four graphs are the Dynkin diagram $E_7$ as in [10, Fig. 11.32]. This bi-unitary connection is not flat and the principal graph of the resulting subfactor is $D_{10}$ as shown in [9, Section 3]. In this case, all the three subfactors $S \subset R$, $R \subset P$ and $P \subset Q$ have the principal graph $D_{10}$. (See [33, Example 2.7] for the subfactor $R \subset P$ in this example.)

**Example 4.5** Here is also another (rather trivial) example of $R \subsetneq P$. Let $\{B_{kl}\}_{k,l}$ be the canonical commuting square of a hyperfinite type $\Pi_1$ subfactor of finite index and finite depth. Take a nonzero projection $p \in B_{2m,2n}$ and set $A_{k,l} = pB_{2m+k,2n+l}p$, which gives commuting squares as above. Then we see that $S \subset R$ and $P \subset Q$ are isomorphic to the original subfactor and that $R \subset P$ is isomorphic to $qP \subset qQ_{2m-1}q$ where $P \subset Q \subset Q_1 \subset \cdots$ is the Jones tower and $q$ is some nonzero projection in $P' \cap Q_{2m-1}$. In this example, the bi-unitary connection for $\{A_{kl}\}_{k,l}$ is the same as the original one for $\{B_{kl}\}_{k,l}$, but the choice of the initial vertex $*$ for $A_{00}$ is different from the canonical one.

Also, fusion categories that are Morita equivalent to the $N-N$ bimodules of the Asaeda-Haagerup subfactor $N \subset M$ [11] are also classified in [14]. We can, in principle, list commuting squares producing the Asaeda-Haagerup subfactor.
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