DIAMETER ESTIMATION OF \((m, \rho)\)-QUASI EINSTEIN MANIFOLDS

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Abstract. This paper aims to study the \((m, \rho)\)-quasi Einstein manifold. This article shows that a complete and connected Riemannian manifold under certain conditions becomes compact. Also, we have determined an upper bound of the diameter for such a manifold. It is also exhibited that the potential function acquiesces to the Hodge-de Rham potential up to a real constant in an \((m, \rho)\)-quasi Einstein manifold. Later, some triviality and integral conditions are established for a non-compact complete \((m, \rho)\)-quasi Einstein manifold having finite volume. Finally, it is proved that with some certain constraints, a complete Riemannian manifold admits finite fundamental group. Furthermore, some conditions for compactness criteria have also been deduced.

1. Introduction and preliminaries

A non-gradient generalized \(m\)-quasi-Einstein manifold ([3]) is an \(n(> 2)\)-dimensional Riemannian manifold \((N, g)\) such that it preserves a smooth function \(\alpha \in C^\infty(N)\), with

\[
\alpha g + \frac{1}{m} W^g \otimes W^g = \frac{1}{2} L_W g + \text{Ric},
\]

where \(W \in \chi(N)\), it’s dual 1-form \(W^g\), \(L_W g\) indicates the Lie derivative of \(g\) along \(W\) and \(m\) is a scalar with \(0 < m \leq \infty\). In ([1]), if we replace \(\alpha\) by \(\lambda + \rho R\), for some real constants \(\lambda, \rho\) and the scalar curvature \(R\) of \(N\), then \((N, g)\) is called a non-gradient \((m, \rho)\)-quasi Einstein manifold ([16]), and in this case equation ([1]) reduces to

\[
\lambda g + \rho R g + \frac{1}{m} W^g \otimes W^g = \frac{1}{2} L_W g + \text{Ric}.
\]

If \(W\) is the gradient of a real valued smooth function \(f\) on \(N\), then the \((m, \rho)\)-quasi Einstein manifold is said to be a (gradient) \((m, \rho)\)-quasi Einstein manifold (for details see [7,10]), with

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potential function $f$. In this case, (2) reduces to

$$\lambda g + \rho R g + \frac{1}{m} df \otimes df = \nabla^2 f + Ric.$$  

If $m = \infty$ and $\rho = 0$ (resp., $m = \infty$), then (3) compresses to the Ricci soliton equation (see e.g. [6, 8, 9, 14, 18, 20]) (resp., $\rho$-Einstein soliton equation (see e.g. [17, 19])). In (3), if $m$ and $\lambda$ is taken from $\chi(N)$, then $(N, g)$ is known as the generalized $m$-quasi Einstein manifold [5]. Furthermore, tracing (3), we obtain

$$\lambda n + n \rho R + \frac{1}{m} |\nabla f|^2 = \Delta f + R.$$  

For a fixed $\phi \in C^\infty(N)$, the weighted Laplacian is defined by (see, [12])

$$\Delta_\phi f = \Delta f - \langle \nabla \phi, \nabla f \rangle.$$  

Catino [5] introduced the notion of a generalized quasi-Einstein manifold and characterized such type of manifolds with harmonic Weyl tensor. For a compact $m$-quasi Einstein metric having constant scalar curvature, a triviality result is proved by Case et al. [4]. They also determined that all compact 2-dimensional $m$-quasi Einstein manifolds are trivial. For a compact gradient $\rho$-Einstein soliton with some conditions, Shaikh et al. [19] have provided a lower bound of the diameter. Huang and Wei [10] have deduced some rigidity results on a compact $(m, \rho)$-quasi Einstein manifolds, in particular, by using some conditions on scalar curvature or on the constant $\rho$, they have showed that such a manifold is trivial. Demirbağ and Güler [7] have characterized an $(m, \rho)$-quasi Einstein manifold admitting closed conformal or parallel vector field. Also, they have established some rigidity results giving some new examples of the $(m, \rho)$-quasi Einstein manifold.

López and Rio [13] have proved a compactness theorem for a complete Riemannian manifold satisfying $\mathcal{L}_W g + Ric \geq \lambda g$, with potential vector field $W$ having bounded norm. Wylie [22] has proved that the fundamental group of a complete Riemannian manifold satisfying $\mathcal{L}_W g + Ric \geq \lambda g$, for $\lambda > 0$, is finite.

Hence inspiring by the above studies and the study of [2], in this article, we have showed the following:
In the first theorem we have generalized the work of Limoncu [11], and obtained an upper bound of the diameter.

**Theorem 1.1.** Let \((N, g)\) be a complete and connected Riemannian manifold satisfying \(\text{Ric} + \nabla^2 f - \frac{1}{m} df \otimes df \geq \lambda g + \rho Rg\). If \(|f| \leq K\) and \(\rho R \geq K_1\), for some real constants \(K, K_1\) choosing in such a way that \((\lambda + K_1) > 0\), then \(N\) is compact and the diameter satisfies

\[
diam(N) \leq \pi \sqrt{\frac{(n-1) + K \sqrt{2}}{\lambda + K_1}}.
\]

Next, we have generalized the work of Li and Wenyi [12], and obtained the following:

**Theorem 1.2.** Assume that \((N, g, e^{-f}dvol)\) is a smooth metric measure space satisfying \(\text{Ric} + \nabla^2 f - \frac{1}{m} df \otimes df \geq \lambda g + \rho Rg\), and \(u\) is a positive smooth \(f\)-harmonic function on \(N\), then

\[
\frac{1}{2} \Delta F \geq (\lambda + \rho R) F + \frac{1}{n} F^2 - \frac{2}{n} |\nabla f| F^2 - |\nabla F| F^2 + \frac{1}{m} \{df(\nabla \omega)\}^2 + \frac{1}{2} \langle \nabla f, \nabla F \rangle,
\]

where we denote \(F = |\nabla \log u|^2\).

**Theorem 1.3.** If \((N, g)\) is a compact and oriented \((m, \rho)\)-quasi Einstein manifold, then its potential function is given by \(f = \sigma + C_1\), where \(C_1\) is a constant and \(\sigma\) is the Hodge-de Rham potential.

Let \((N, g)\) be an oriented Riemannian manifold and \(A^k(N)\) be the collection of all \(k\)-th differential forms in \(N\). Now for given integer \(k \geq 0\), the global inner product in \(A^k(N)\) is defined by (see, [15])

\[
\langle \zeta, \omega \rangle = \int_N \zeta \wedge * \omega,
\]

for \(\zeta, \omega \in A^k(N)\). Here “*” denotes the Hodge star operator. The global norm of \(\zeta \in A^k(N)\) is given by \(\|\zeta\|^2 = \langle \zeta, \zeta \rangle\) with \(\|\zeta\|^2 \leq \infty\).

**Theorem 1.4.** Let \((N, g)\) be a complete \((m, \rho)\)-quasi Einstein manifold which is non-compact and of finite volume. If \(\rho > \frac{1}{n}\) and also \(W\) admits finite global norm, then the following holds:

(i) If \(R \geq \frac{\lambda_0}{1 - \rho m}\), then \(N\) is trivial,

(ii) If \((\lambda + \rho)R \geq 0\), then \(\frac{1}{m} \int_N |W|^2 dV \leq \int_N RdV\), and

(iii) \(\int_N RdV \leq \frac{\lambda_0}{1 - \rho m} \text{Vol}(N)\), where \(\text{Vol}(N)\) represents the volume of \(N\).
Corollary 1.4.1. Let \((N, g)\) be a complete \((m, \rho)\)-quasi Einstein manifold which is non-compact and of finite volume. If \(\rho < \frac{1}{n}\) and \(W\) is of finite global norm, then the following holds:

(i) If \(R \leq \frac{\lambda n}{1 - \rho n}\), then \(N\) is trivial,

(ii) If \((\lambda + \rho)R \leq 0\), then \(\int_N Rc \leq \frac{1}{m} \int_N |X|^2 dV\), and 

(iii) \(\int_N RdV \geq \frac{\lambda n}{1 - \rho n} Vol(N)\).

Next, we are interested in studying a complete Riemannian manifold \((N, g)\) with a vector field \(W\) such that,

\[
\lambda g + \rho Rg + \frac{1}{m} W^b \otimes W^b \leq \frac{1}{2} L_W g + Ric,
\]

In (7) the equality gives the fundamental equation of an \((m, \rho)\)-quasi Einstein manifold. Here the main aim is to prove the following results:

**Theorem 1.5.** If \((N, g)\) is a complete Riemannian manifold satisfying (7) along with \(\int_0^\rho R = K_3\), for some constant \(K_3\), then the fundamental group of \(N\) is finite.

**Theorem 1.6.** Let \((N, g)\) be a complete Riemannian manifold satisfying (7) with \(\|W\|\) bounded and \(\int_0^\rho R = K_3\), for some constant \(K_3\) with \((\lambda + K_3) > 0\). Then \(N\) becomes compact.

## 2. Proof of the results

**Proof of Theorem 1.1** Let \(p, q \in N\) and let \(\Gamma\) be a minimizing unit speed geodesic segment from \(p\) to \(q\) of length \(d\). Considering a parallel orthonormal frame \(\{\xi_1 = \Gamma', \xi_2, \ldots, \xi_n\}\) along \(\Gamma\) and a smooth function \(h \in C^\infty([0, d])\) such that \(h(0) = h(d) = 0\), we get (see, [11]),

\[
\sum_{i=2}^n \eta(h \xi_i, h \xi_i) = \int_0^d \{(n - 1)h^2 - h^2 Ric(\Gamma', \Gamma')\}dt,
\]

where \(\eta\) denotes the index form of \(\Gamma\).

With our assumption the relation (8), yields

\[
\sum_{i=2}^n \eta(h \xi_i, h \xi_i) \leq \int_0^d \left[ (n - 1)h^2 + h^2 \{\nabla^2 f - \frac{1}{m} df \otimes df - \lambda g - \rho Rg\}(\Gamma', \Gamma') \right] dt
\]

\[
= \int_0^d \left[ (n - 1)h^2 + h^2 \nabla^2 f(\Gamma', \Gamma') - \frac{h^2}{m} (df(\Gamma'))^2 - h^2 (\lambda + \rho R) \right] dt.
\]
Again, since $|f| \leq K$, we obtain (see, [11]),

\[
\int_0^d h^2 \nabla^2 f(\Gamma', \Gamma') dt = \int_0^d h^2 g(\nabla_{\Gamma'} \nabla f, \Gamma') dt = \int_0^d h^2 \Gamma'(g(\nabla f, \Gamma')) dt \\
\leq 2K \sqrt{d} \left( \int_0^d \left( \frac{d}{dt} (hh') \right)^2 dt \right)^{\frac{1}{2}}.
\]

(10)

The equations (9) and (10) together implies

\[
\sum_{i=2}^n \eta(h_\xi_i, h_\xi_i) \leq \int_0^d (n - 1) h^2 dt + 2K \sqrt{d} \left( \int_0^d \left( \frac{d}{dt} (hh') \right)^2 dt \right)^{\frac{1}{2}} \\
- \int_0^d \frac{h^2}{m} (df(\Gamma'))^2 dt - \int_0^d h^2(\lambda + \rho R) dt.
\]

(11)

If we take $h(t) = \sin(\frac{\pi}{d}t)$, then (11), yields

\[
\sum_{i=2}^n \eta(h_\xi_i, h_\xi_i) \leq (n - 1) \int_0^d \frac{\pi^2}{d^2} \cos^2 \left( \frac{\pi}{d} t \right) dt + 2K \sqrt{d} \left( \int_0^d \left( \frac{\pi^2}{d^2} \cos \left( \frac{2\pi}{d} t \right) \right)^2 dt \right)^{\frac{1}{2}} \\
- \lambda \int_0^d \sin^2 \left( \frac{\pi}{d} t \right) dt - \int_0^d \sin^2 \left( \frac{\pi}{d} t \right) \rho R dt \\
\leq (n - 1) \frac{\pi^2}{2d} + \frac{K\pi^2 \sqrt{2}}{d} - \frac{\lambda}{2} d - \frac{K_1}{2} d.
\]

Since $\Gamma$ is a minimizing geodesic, hence $\eta(h_\xi_i, h_\xi_i) \geq 0$. Consequently,

\[
0 \leq (n - 1) \frac{\pi^2}{2d} + \frac{K\pi^2 \sqrt{2}}{d} - \frac{\lambda}{2} d - \frac{K_1}{2} d.
\]

The above relation entails

\[
d \leq \pi \sqrt{2} \left( \frac{(n - 1) + K\sqrt{2}}{\lambda + K_1} \right).
\]

This completes the proof. \(\Box\)

**Proof of Theorem 1.2** Set $\omega = \log u$, then $F = |\nabla \omega|^2$ and $\omega$ satisfies the equation:

\[
\Delta_f \omega + |\nabla \omega|^2 = 0.
\]

Thus,

\[
\Delta_f \omega = -F.
\]
Therefore,
\[
|\nabla^2 \omega|^2 \geq \frac{1}{n} |\Delta \omega|^2 = \frac{1}{n} |F - \langle \nabla f, \nabla \omega \rangle|^2 \geq \frac{1}{n} F^2 - \frac{2}{n} \langle \nabla f, \nabla \omega \rangle F,
\]
and
\[
\langle \nabla \omega, \nabla \Delta f \omega \rangle = -\langle \nabla \omega, \nabla F \rangle.
\]
Now, the weighted Bochner formula:
\[
\frac{1}{2} \Delta_f |\nabla \omega|^2 = |\nabla^2 \omega|^2 + \langle \nabla \omega, \nabla \Delta f \omega \rangle + Ric_f (\nabla \omega, \nabla \omega),
\]
together with (12), (13) and our assumption, yields
\[
\frac{1}{2} \Delta_f F \geq \frac{1}{n} F^2 - \frac{2}{n} \langle \nabla f, \nabla \omega \rangle F - \langle \nabla \omega, \nabla F \rangle + (\lambda + \rho R) |\nabla \omega|^2 + \frac{1}{m} \{df \otimes df (\nabla \omega, \nabla \omega)\}.
\]
This implies
\[
\frac{1}{2} \Delta F - \langle \nabla f, \nabla F \rangle \geq \frac{1}{n} F^2 - \frac{2}{n} \langle \nabla f, \nabla \omega \rangle F - \langle \nabla \omega, \nabla F \rangle + (\lambda + \rho R) F + \frac{1}{m} \{df (\nabla \omega)\}^2.
\]
Applying the Cauchy-Schwarz inequality, we obtain
\[
\langle \nabla \omega, \nabla f \rangle \leq |\nabla \omega||\nabla f| = |\nabla f| F^{\frac{1}{2}}.
\]
Using the last inequality in (14), we get the estimation. \(\square\)

**Proof of Theorem 1.3** If \(W \in \chi(N)\), then as a consequence of Hodge-de Rham decomposition theorem, (see e.g. [21]), \(W\) can be written as
\[
W - \nabla \sigma = Y,
\]
where \(\sigma \in \chi(N)\) is the Hodge–de Rham potential and \(\text{div } Y = 0\). By considering an \((m, \rho)\)-quasi Einstein manifold \((N, g)\) so that the equation (2) holds, then we obtain
\[
R + \text{div} W = \lambda n + \frac{1}{m} |W|^2 + \rho R n.
\]
Therefore (15) concludes that \(\text{div} W = \Delta \sigma\) and hence (16) entails
\[
\lambda n + \frac{1}{m} |W|^2 + \rho R n = R + \Delta \sigma.
\]
Again (3) indicates that
\[(18) \quad \lambda n + \frac{1}{m} |\nabla f|^2 + \rho Rn = R + \Delta f.\]
From (17) and (18), we have
\[\Delta (f - \sigma) = 0,\]
which follows the result. \qed

**Proof of Theorem 1.4** We have, for \(l > 0\)
\[
\frac{1}{l} \int_{B(p,2l)} |W|dV \leq \left( \int_{B(p,2l)} \left( \frac{1}{l} \right)^2 dV \right)^{1/2} \left( \int_{B(p,2l)} \langle W, W \rangle dV \right)^{1/2}
\leq \frac{1}{l} \left( Vol(N) \right)^{1/2} \|W\|_{B(p,2l)},
\]
where \(B(p, l)\) is the open ball of radius \(l\) and center at \(p\). Thus
\[
\lim inf_{l \to \infty} \frac{1}{l} \int_{B(p,2l)} |W|dV = 0.
\]
Again, a Lipschitz continuous function \(\omega_l\) exists, (see [23]) such that for some real constant \(K_2 > 0\),
\[
0 \leq \omega_l(x) \leq 1 \quad \forall x \in N,
\text{supp } \omega_l \subset B(p, 2l),
|d\omega_l| \leq \frac{K_2}{l} \quad \text{almost everywhere on } N \quad \text{and}
\omega_l(x) = 1 \quad \forall x \in B(p, l).
\]
As \(\lim_{l \to \infty} \omega_l = 1\), using the function \(\omega_l\), we get
\[
\frac{C}{l} \int_{B(p,2l)} |W|dV \geq \left| \int_{B(p,2l)} \omega_l \operatorname{div} WdV \right|.
\]
From the defining relation of \((m, \rho)\)-quasi Einstein manifold, we obtain
\[(19) \quad \int_N \{\lambda n + \frac{1}{m} |W|^2 + (\rho n - 1)R\}dV = 0.
\]
By virtue of equation (19) and our assumption, we get the desired results. \qed

To prove Theorem 1.5 we need the following results:
Lemma 2.1 (22). Let \((N, g)\) be a complete Riemannian manifold and \(p, q \in N\) with \(r = d(p, q) > 1\). If \(\Gamma\) is a minimal geodesic joining from \(p\) to \(q\) and parametrized by the arc length \(s\), then

\[
2(n - 1) + \varepsilon_p + \varepsilon_q \geq \int_0^r \text{Ric}(\Gamma'(s), \Gamma'(s)) ds,
\]
where \(\varepsilon_q = \max\{0, \sup\{\text{Ric}_y(v, v) : y \in B(q, 1), ||v|| = 1\}\} \), for \(q \in N\).

Lemma 2.2. If \((N, g)\) is a complete Riemannian manifold satisfying (7), then for any \(p, q \in N\),

\[
\max\{1, \frac{1}{K_3 + \lambda} (2(n - 1) + \varepsilon_p + \varepsilon_q + ||W_p|| + ||W_q||)\} \geq d(p, q).
\]

Proof. Suppose that \(d(p, q) > 1\) and \(\Gamma\) is a minimal geodesic joining from \(p\) to \(q\) and parametrized by arc length \(s\). Then along \(\Gamma\), we can easily get

\[
\mathcal{L}_{W} g(\Gamma', \Gamma') = 2 \frac{d}{ds}[g(W, \Gamma')].
\]

Applying Lemma 2.1 we obtain

\[
2(n - 1) + \varepsilon_p + \varepsilon_q \geq \int_0^r \text{Ric}(\Gamma'(s), \Gamma'(s)) ds.
\]

Using the Cauchy-Schwarz inequality and (7), we have

\[
\int_0^r \text{Ric}(\Gamma'(s), \Gamma'(s)) ds \geq g_p(W, \Gamma'(0)) - g_q(W, \Gamma'(r)) + \int_0^r (W^p(\Gamma'(s)))^2 ds + \lambda d(p, q) + \int_0^r \rho R ds
\]
\[
\geq \lambda d(p, q) - ||W_p|| - ||W_q|| + K_3 d(p, q).
\]

In view of (22) and solving for \(d(p, q)\), the last inequality entails (21). \(\square\)

Proof of Theorem 1.5 We omit the proof of Theorem 1.5 as it is straightforward from the Lemma 2.2 and following the technique of [22, Theorem 1.1]. \(\square\)

Proof of Theorem 1.6 Let \(p \in N\) and \(\Gamma : [0, \infty) \to N\) be any geodesic starting from \(p\) and parametrized by the arc length \(s\). Hence (7) and the Cauchy-Schwarz inequality together imply that

\[
\int_0^r \text{Ric}(\Gamma'(s), \Gamma'(s)) ds \geq \int_0^r (W^p(\Gamma'(s)))^2 + \lambda r + \int_0^r \rho R g + 2g(W_p, \Gamma'(0)) - 2g(W_{\Gamma(r)}, \Gamma'(r)) ds
\]
\[
\geq 2g(W_p, \Gamma'(0)) - 2||W_{\Gamma(r)}|| + \lambda r + K_3 r.
\]
The boundedness of $\|W\|$ provides
\[
\int_{0}^{+\infty} \text{Ric}(\Gamma'(s), \Gamma'(s)) = +\infty,
\]
and hence Ambrose's compactness criteria \cite{1} concludes that $N$ is compact. \hfill \Box

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