Input to state estimates for 2D damped wave equations with localized and non-linear damping *

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Abstract

In this paper, we study input-to-state (ISS) issues for damped wave equations with Dirichlet boundary conditions on a bounded domain of dimension two. The damping term is assumed to be non-linear and localized to an open subset of the domain. In a first step, we handle the undisturbed case as an extension of [14], where stability results are given with a damping term active on the full domain. Then, we address the case with disturbances and provide input-to-state types of results.

1 Introduction

Consider the damped wave equation with localized damping, with Dirichlet boundary conditions given by

\begin{equation}
\begin{aligned}
\mathbf{P}_{\text{dis}}: \begin{cases} 
u_{tt} - \Delta u = -a(x)g(u_t + d) + e, & \text{in } \mathbb{R}_+ \times \Omega, \\
u = 0, & \text{on } \mathbb{R}_+ \times \partial \Omega, \\
u(0,.) = u^0, \ u_t(0,.) = u^1, & 
\end{cases}
\end{aligned}
\end{equation}

where $\Omega$ is a $C^2$ bounded domain of $\mathbb{R}^2$, $d$ and $e$ stand for a damping disturbance and a globally distributed disturbance for the wave dynamics respectively. The term $-a(x)g(u_t + d)$ stands for the (perturbed) damping term where $g : \mathbb{R} \rightarrow \mathbb{R}$ is a $C^1$ increasing function verifying $\xi g(\xi) > 0$ for $\xi \neq 0$ while $a : \overline{\Omega} \rightarrow \mathbb{R}$ is a continuous non negative function which is bounded below by a positive constant on some non-empty open subset $\omega$ of $\Omega$. Here, $\omega$ is the support region of the domain where the damping term is active. As for the initial condition $(u^0, u^1)$, it belongs to the standard Hilbert space $H^1_0(\Omega) \times L^2(\Omega)$.

In this paper, our aim is to obtain input-to-state (ISS) results for $\mathbf{P}_{\text{dis}}$, i.e., estimates of the norm of the state $u$ which, at once, show that trajectories tend to zero in the absence of disturbances and remain bounded by a function of the norms of the disturbances otherwise. One can refer to [18] for a thorough review of ISS results and techniques for finite dimension systems and to the recent survey [17] for infinite dimensional dynamical systems.

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In the case of the undisturbed dynamics, i.e., \((d, e) \equiv (0, 0)\) to which we will refer as the undisturbed problem \((P_0)\), there is a vast literature regarding the stability of the corresponding system with respect to the origin, which is the unique equilibrium state of the problem. This in turn amounts to have appropriate assumptions on \(a\) and \(g\), cf. [1] for extensive references. We will however point out the main ones that we need in order to provide the context of our work. To do so, we start by defining the energy of the system by

\[
E(t) = \frac{1}{2} \int_\Omega (|\nabla u|^2(t, \cdot) + u^2(t, \cdot)) \, dx,
\]

which defines a natural norm on the space \(H^1_0(\Omega) \times L^2(\Omega)\). Strong stabilization has been established in the early works [6] and [8], i.e., it is proved with an argument based on the Lasalle invariance principle that \(\lim_{t \to +\infty} E(t) = 0\) for every initial condition in \(H^1_0(\Omega) \times L^2(\Omega)\). However, no decay rate of convergence for \(E\) is established since it requires in particular extra assumptions on \(g\) and \(\omega\).

As a first working hypothesis, we will assume that \(g'(0) > 0\), classifying the present work in those that aimed at establishing results of exponential convergence for strong solutions. We refer to [1] for the line of work where \(g\) is assumed to be super-linear in a neighborhood of the origin (typically of polynomial type). Note that, in most of these works (except for the linear case) the rate of exponential decay of \(E\) depends on the initial conditions. That latter fact in turn relies on growth conditions of \(g\) at infinity. Regarding the assumptions on \(\omega\), they have been first put forward in the pioneering work [19] on semi-linear wave equations and its extension in [11], where the multiplier geometric conditions (MGC) have been characterized for \(\omega\) in order to achieve exponential stability. For linear equations, the sharpest geometrical results are obtained by microlocal techniques using the method of geometrical optics, cf [2] and [5].

In this paper, our objective is to obtain results for non-linear damping terms and one should think of the nonlinearity \(g\) not only as a mean to provide more general asymptotic behavior at infinity than a linear one but also as modeling an uncertainty of the shape of the damping term. Dealing with nonlinearities justifies why microlocal techniques are not suited here and we will be using the multiplier method as presented e.g. in [10]. Regarding non-linear damping terms, few general results are available under the condition \(g'(0) > 0\) and one can find a rather complete presentation of the available results in [14]. In particular, the proof of exponential stability along strong solutions has only been given for general nonlinearities \(g\), in dimension two and in the special case of a non-localized damping requiring only one multiplier coupled with a judicious use of Gagliardo-Nirenberg’s inequality. One of our results generalizes this finding (even though it has been mentioned in [14] with no proof that this is the case). It has also to be noted that similar results are provided in [13] in the localized case but the nonlinearity is lower bounded by a linear function for large values of its arguments. That simplifies remarkably some computations. Recall also that the purpose of [13] is instead to address issues when \(g'(0) = 0\) and to obtain accurate decay rates for \(E\).

Hence a possible interest of the present paper is the fact that it handles nonlinearities \(g\) so that \(g(v)/v\) tends to zero as \([v]\) tends to infinity with a linear behavior in a neighborhood of the origin.

As for ISS purposes, this paper can be seen as an extension to the infinite dimensional context of [12] where the nonlinearity is of the saturation type. Moreover, the present work extends to the dimension two the works [15] and [16], where this type of issues have been addressed by building appropriate Lyapunov functions and by providing results in dimension one. Here, we are not able to construct Lyapunov functions and we rely instead on energy estimates based on the multiplier method, first in the undisturbed case, and next showing how these estimates are modified change
when adding the two disturbances $d$ and $e$. To develop that strategy, we must impose additional assumptions on $g'$, still handling saturation functions. As a final remark, we must recall that [14] contains other stability results in two directions. On one hand, $g'$ can simply admit a (possibly) negative lower bound and on the other hand, the space dimension $N$ can be larger than 2, at the price of more restrictive assumptions on $g$, in particular, by assuming quasi-linear lower bounds for its asymptotic behavior at infinity. One can readily extend the results of the present paper in both directions by eventually adding growth conditions on $g$.

2 Statement of the problem and main results

We next provide assumptions on the data needed to precisely define (1) and, from now on, we will refer to it as the disturbed problem ($P_{\text{dis}}$) while ($P_0$) will be used to denote the undisturbed one, i.e., (1) with $(d,e) \equiv (0,0)$,

\[
(P_0) \begin{cases} 
    u_{tt} - \Delta u = -a(x)g(u_t), & \text{in } \mathbb{R}^+ \times \Omega, \\
    u = 0, & \text{on } \mathbb{R}^+ \times \partial \Omega, \\
    u(0,\cdot) = u^0, \quad u_t(0,\cdot) = u^1.
\end{cases}
\]

Throughout the paper, the domain $\Omega$ is a bounded open subset of $\mathbb{R}^2$ of class $C^2$, the assumptions on $g$ for ($P_0$) are the following.

- $H_1$: The function $g : \mathbb{R} \rightarrow \mathbb{R}$ is a $C^1$ non-decreasing function, such that
  \[
  g(0) = 0, \quad g'(0) > 0, \quad g(x) > 0 \quad \text{for } x \neq 0,
  \]
  \[
  \exists \, C > 0, \quad \exists \, q > 1, \quad \forall \, |x| \geq 1, \quad |g(x)| \leq C|x|^q. \tag{4}
  \]
  As for ($P_{\text{dis}}$), they are more restrictive since we replace (5) by

- $H'_1$: The function $g$ is as ($H_1$) except that (5) is replaced by
  \[
  \exists \, 0 < m < 4, \quad \exists \, C > 0, \quad \forall \, |x| > 1, \quad |g'(x)| \leq C|x|^m. \tag{6}
  \]

Remark 2.1 The hypothesis given by (6) imposes that $q$ in (5) is less than 5.

- $H_2$: The localization function $a : \overline{\Omega} \rightarrow \mathbb{R}$ is a continuous function such that
  \[
  a \geq 0 \quad \text{on } \Omega \quad \text{and} \quad \exists \, a_0 > 0, \quad a \geq a_0 \quad \text{on } \omega. \tag{7}
  \]

In order to prove the stability of solutions, we impose a multiplier geometrical condition (MGC) on $\omega$.

- $H_3$: There exists an observation point $x_0 \in \mathbb{R}^2$ for which $\omega$ contains the intersection of $\Omega$ with an $\epsilon$-neighborhood of
  \[
  \Gamma(x_0) = \{ x \in \partial \Omega, \quad (x - x_0).\nu(x) \geq 0 \}, \tag{8}
  \]
  where $\nu$ is the unit outward normal vector for $\partial \Omega$ and an $\epsilon$-neighborhood of $\Gamma(x_0)$ is defined by
  \[
  \mathcal{N}_\epsilon(\Gamma(x_0)) = \{ x \in \mathbb{R}^2 : \text{dist}(x, \Gamma(x_0)) \leq \epsilon \}. \tag{9}
  \]
Remark 2.2 The results and techniques of this work extend readily to a weaker and more general MGC than (H₃) introduced in [14], called piecewise MGC in [11].

Regarding the disturbances $d$ and $e$, we make the following hypotheses:

(H₄): the disturbance function $d: \mathbb{R}^+ \times \Omega \rightarrow \mathbb{R}$ belongs to $L^1(\mathbb{R}^+, L^2(\Omega))$ and

$$d(0, \cdot) \in H_0^1(\Omega) \cap L^{2q}(\Omega), \quad \int_0^\infty \Delta d(s, \cdot) ds - d_t(\cdot, \cdot) \in Lip(\mathbb{R}^+, H_0^1(\Omega)), \quad (10)$$

where $\text{Lip}$ denotes the space of Lipschitz continuous functions. We also impose that

$$C_1(d) = \int_0^\infty \int_\Omega (|d| + |d|^{2q}) \, dx \, dt, \quad C_2(d) = \int_0^\infty \int_\Omega |d|^m (d_t)^2 \, dx \, dt,$$

$$C_3(d) = \int_0^\infty \int_\Omega (d_t)^2 \, dx \, dt, \quad C_4(d) = \int_0^\infty \left( \int_\Omega |d_t|^{2\left(\frac{q}{p+1}\right)} \, dx \right)^{\frac{p+1}{p}} dt < \infty, \quad (11)$$

where $p$ is a fixed real number so that, if $0 < m \leq 2$, then $p > \frac{2}{m}$ and if $2 < m < 4$, then $p \in (1, \frac{m}{m-2})$.

(H₅): the disturbance function $e: \mathbb{R}^+ \times \Omega \rightarrow \mathbb{R}$ belongs to $W^{1,1}(\mathbb{R}^+, L^2(\Omega))$ and

$$e \in Lip(\mathbb{R}^+, H_0^1(\Omega)), \quad e(0, \cdot) \in L^2(\Omega), \quad C_5(e) = \int_0^\infty \int_\Omega e^2 \, dx \, dt < \infty. \quad (12)$$

Remark 2.3 For the sake of simplicity, the symbol $C$ will denote positive constants independent of initial conditions and disturbances, i.e., only depending on the domains $\Omega, \omega$ and the functions $a$ and $g$. The symbol $C_u$ ($C_{e,d}$ resp.) will denote positive constants that depend furthermore on initial conditions (on disturbances resp.) but not on disturbances (on initial conditions resp.). Note also that it will always be the case that both the $C_u$’s and the $C_{e,d}$’s are $\mathcal{K}$-functions of the respective norms, in particular those defined in (12) and (11). Here $\mathcal{K}$ denotes the set of continuous increasing functions $\gamma: \mathbb{R}^+ \rightarrow \mathbb{R}_+$ with $\gamma(0) = 0$, cf. [17].

We gather our findings in the following theorems regarding both the undisturbed problem ($P_0$) and the disturbed one ($P_{\text{dis}}$).

Theorem 2.1 Suppose that Hypotheses (H₁), (H₂) and (H₃) are satisfied. Then, given $(u^0, u^1) \in (H^2(\Omega) \cap H_0^1(\Omega)) \times H_0^1(\Omega)$, Problem ($P_0$) admits a unique strong solution $u$ such that

$$u \in C^1(\mathbb{R}^+, H_0^1(\Omega)) \cap C(\mathbb{R}^+, H^2(\Omega) \cap H_0^1(\Omega)),$$

$$\forall t \in \mathbb{R}^+, \quad (u(t), u_t(t)) \in (H^2(\Omega) \cap H_0^1(\Omega)) \times H_0^1(\Omega).$$

Furthermore, the energy of the solution decays exponentially, i.e., there exists an explicit constant $C_u > 0$ depending on $u^0, u^1$ such that

$$\forall t \geq 0, \quad E(t) \leq E(0)e^{1-\frac{t}{C_u}}. \quad (13)$$
Theorem 2.2 Suppose that Hypotheses (H')_1, (H_2) to (H_5) are satisfied. Then, given (u^0, u^1) \in (H^2(\Omega) \cap H^1_0(\Omega)) \times H^1_0(\Omega), Problem (P_{dis}) has a unique strong solution u such that

\[ u \in C^1(\mathbb{R}_+, L^2(\Omega)) \cap C(\mathbb{R}_+, H^1_0(\Omega)), \forall t \in \mathbb{R}_+, (u(t), u_t(t)) \in (H^2(\Omega) \cap H^1_0(\Omega)) \times H^1_0(\Omega). \]

Furthermore, the following energy estimate holds:

\[ E(t) \leq E(0)e^{-\frac{\sigma}{C_u}} + C_{d,e}(C_u' + 1), \quad (14) \]

where the positive constants \( C_u, C_u' \) depend only on the initial conditions and the positive constant \( C_{d,e} \) depends only on the disturbances \( d \) and \( e \).

Remark 2.4 Theorem 2.2 still holds true if the Lipschitz assumptions in (10) and (12) are replaced by bounded variation ones.

Remark 2.5 Note that (14) is an ISS-type estimate but it fails to be a strict one (let say in the sense of Definition 1.6 in [17]) for two facts. First of all, the estimated quantity \( E \) is the norm of a trajectory in the space \( H^1_0(\Omega) \times L^2(\Omega) \) while the constants \( C_u, C_u' \) depend on the initial condition by its norm in the smaller space \( (H^2(\Omega) \cap H^1_0(\Omega)) \times H^1_0(\Omega) \). This difference seems unavoidable since in the undisturbed case exponential decay can be proved only for strong solutions as soon as the nonlinearity \( g \) is not assumed to be bounded below at infinity by a linear function. As a matter of fact, it would be interesting to prove that strong stability is the best convergence result one could get for weak solutions, let say with damping functions \( g \) of saturation type functions and in dimension at least two.

The second difference lies in the second term in (14), namely it is not just a \( K \)-function of the norms of the disturbances. We can get such a result if we have an extra assumption on \( g \), typically \( g \) of growth at most linear at infinity (i.e., \( q = 1 \)) with bounded derivative (i.e., \( m = 0 \)). In particular, this covers the case of regular saturation functions (increasing bounded functions \( g \) with bounded derivatives).

We give next the proof of the well-posedness parts of both Theorem 2.1 and Theorem 2.2. Note that it suffices to prove the well-posedness of (P_{dis}) and we obtain that of (P_0) as a sub-case. The argument is standard and starts by defining \( D(t, x) = \int_0^t d(s, x)ds \). After setting \( U = u + D \), it is immediate to see that solving (P_{dis}) for \( u \) is equivalent to solving for the function \( U = u + D \) the following system

\[
\begin{cases}
U_{tt} - \Delta U + a(x)g(U_t) = \tilde{d} & \text{in } \mathbb{R}_+ \times \Omega, \\
U = 0 & \text{on } \mathbb{R}_+ \times \partial \Omega, \\
U(0, \cdot) = U^0, & U_t(0, \cdot) = U^1,
\end{cases}
\quad (15)
\]

where \( \tilde{d} = D_{tt} - \Delta D - e, U^0 = u^0 \) and \( U^1 = u^1 + d(0, \cdot) \).

Define the unbounded operator

\[ A : H^1_0(\Omega) \times L^2(\Omega) \longrightarrow H^1_0(\Omega) \times L^2(\Omega), \]

\[ (x_1, x_2) \longmapsto (x_2, -\Delta x_1 + ag(x_2)), \]

with domain

\[ D(A) = (H^2(\Omega) \cap H^1_0(\Omega)) \times H^1_0(\Omega). \quad (16) \]
Remark 2.6 In [14], the domain of the operator has been chosen as
\[ Z = \{(u, v) \in H^1_0(\Omega) \times H^1_0(\Omega) : -\Delta u + g(v) \in L^2(\Omega)\}, \]
but since we are working in dimension two, taking the domain of \( \Delta \) (domain in this case is nothing else but \( V \) and \( \Delta \)) and \( G \) where \((d, e) \equiv (0, 0)\) and using the hypothesis on \( g \) given by \( \tilde{d} \), we deduce that \( g(v) \in L^2(\Omega) \). Then, by using Lemma 3.8 (also with \((d, e) \equiv (0, 0)\)), we have \(-\Delta u + ag(v) \in L^2(\Omega)\), which means that \( \Delta u \in L^2(\Omega) \), but since \( \|\Delta u\|_{L^2(\Omega)} \) is an equivalent norm to the norm of \( H^2(\Omega) \) when \( u \in H^1_0(\Omega) \) and \( \Delta u \in L^2(\Omega) \) the proof is a direct result of Theorem 4 of Section 6.3 in [7]. As a result, the domain in this case is nothing else but \((H^2(\Omega) \cap H^1_0(\Omega)) \times H^1_0(\Omega)\).

We set \( V = \begin{pmatrix} U_0 \\ U_1 \end{pmatrix} \). Then, Problem (15) can be written as
\[ V_t = AV + G(t), \quad V(0) = V_0 = \begin{pmatrix} U_0^0 \\ U_1^0 \end{pmatrix}, \]
where \( G \in \text{Lip}(\mathbb{R}_+, L^2(\Omega) \times H^1_0(\Omega)). \) is defined \( \forall t \in \mathbb{R}_+ \) as \( G(t) = \begin{pmatrix} 0 \\ \tilde{d} \end{pmatrix} \).

We apply Theorem 3.4 combined with Propositions 3.2 and 3.3 in [3], but first, we have to prove that \(-A\) is a maximal monotone operator on \( H^1_0(\Omega) \times L^2(\Omega) \), which has already been proved in [9].

3 Proofs of the energy estimates

3.1 Proof of (13)

We start by providing an argument for the undisturbed case \((P_0)\) and this amounts to carefully combining the computations in [13] and [14].

3.1.1 Reduction to Proposition 3.1

The argument requires some results which we state below in Lemma 3.1. For their proofs, we will refer to [13] for the first two results, even though they treat the non-localized case, the proofs remain the same with the existence of \( a(x) \), i.e., in the localized case, as for the third result, it is a result of Theorem A.3 in dimension two.

Lemma 3.1 [14] Lemma 1, Lemma 2] Under the hypotheses of Theorem (2.1), we have for strong solutions \( u \) of \((P_0)\)
\[ E(T) - E(S) = -\int_S^T \int_\Omega a(\cdot) u_t g(u_t) \, dx \, dt \leq 0, \quad 0 \leq S \leq T. \tag{17} \]

We also have the existence of a positive constant \( C_u \) such that
\[ \forall t \geq 0, \quad \| -\Delta u(t, \cdot) + a(\cdot) g(u_t(t, \cdot)) \|^2_{L^2(\Omega)} + \| u_t(t, \cdot) \|^2_{H^1_0(\Omega)} \leq C_u. \tag{18} \]

Moreover, for all \( q > 2 \) and strong solutions \( u \) of \((P_0)\), we have
\[ \| u_t(t, \cdot) \|^q_{L^q(\Omega)} \leq C_u E(t), \tag{19} \]
We finish the proof of (13) assuming that the following proposition holds true:

**Proposition 3.1** Under the hypotheses of Theorem (2.1), we have the following energy estimate:

\[ \forall S \leq T \in \mathbb{R}_+, \quad \int_S^T E \, dt \leq C_u E(S), \quad (20) \]

where \( C_u \) is a positive constant that depends on initial conditions.

One concludes at once by applying Gronwall’s inequality given in Theorem A.1 to obtain the desired exponential decay of the energy.

### 3.1.2 Proof of Proposition 3.1

We now embark on an argument for Proposition 3.1. It is based on the use of several multipliers, which will be applied to the partial differential equation of (3). For that purpose, we need to define several functions associated with \( \Omega \).

Let \((u^0, u^1) \in (H^2(\Omega) \cap H^1_0(\Omega)) \times H^1_0(\Omega), S \leq T\) two non negative times and \(x_0 \in \mathbb{R}^2\) an observation point and \(\epsilon_0 < \epsilon_1 < \epsilon_2 < \epsilon\) where \(\epsilon\) is the same defined in 9 and let us define \(Q_i\) for \(i = 0, 1, 2\).

Since \((\Omega \setminus Q_1) \cap \overline{Q_0} = \emptyset\) we can define a function \(\psi \in C^\infty_0(\mathbb{R}^2)\) such that

\[
\begin{cases} 
0 \leq \psi \leq 1, \\
\psi = 1 \text{ on } \overline{\Omega \setminus Q_1}, \\
\psi = 0 \text{ on } Q_0.
\end{cases}
\]

We define the \(C^1\) vector field \(h\) on \(\Omega\) and the function \(\rho : \mathbb{R}_+ \times \Omega\) as follows:

\[
h(x) := \psi(x)(x - x_0), \quad \rho(t, x) = a(x)g(u_t), \quad (t, x) \in \mathbb{R}_+ \times \Omega.
\]

When the context is clear, we will omit the arguments of \(h\) and \(\rho\).

We use the multiplier \(M := h\nabla u + \frac{u}{2}\) to deduce the following:

**Lemma 3.2** [13, (5.13)] Under the hypotheses of Proposition 3.1, we have the following inequality

\[
\int_S^T E \, dt \leq - \left[ \int_S^T u_t M \, dx \right]_S^{T} + C \int_S^T \int_{\Omega \cap Q_1} \nabla u|^2 \, dx \, dt - \int_S^T \int_{\Omega} \rho M(u) \, dx \, dt
\]

\[ + C \int_S^T \int_{\Omega \cap Q_1} u_t^2 \, dx \, dt. \quad (22) \]

Note that the last term of (22) is upper bounded by the integral term \(\int_S^T \int_{\omega} u_t^2 \, dx \, dt\) since \(\Omega \cap Q_1 \subset \omega\). We next estimate the other integral terms on the right hand side of (22). As for \(T_1\), we have the following:

**Lemma 3.3** [13, (5.14)] There exists a positive constant \(C\) such that

\[
|T_1| \leq CE(S). \quad (23)
\]
Handling $T_2$ is less obvious and is not fully contained in [13]. Its estimate is summarized in the following lemma.

**Lemma 3.4** Under the hypotheses of Proposition 3.1, we have for every $\eta > 0$,

$$ T_2 \leq C \left( \frac{1}{\eta} + 1 \right) \int_S^T \int_\omega |u|^2 \, dx \, dt + C_\omega (\eta + \eta^{q+1}) \int_S^T E \, dt + C \left( \frac{1}{\eta^{q+1} \eta + 1} \right) E(S). \quad (24) $$

**Proof of Lemma 3.4.** The argument requires a new multiplier, namely $\xi u$, where the function $\xi \in C_0^\infty (\mathbb{R}^2)$ is defined by

$$
\begin{cases}
0 \leq \xi \leq 1, \\
\xi = 1 \text{ on } Q_1, \\
\xi = 0 \text{ on } \mathbb{R}^2 \setminus Q_2.
\end{cases}
$$

Such a function $\xi$ exists since $\mathbb{R}^2 \setminus Q_2 \cap Q_1 = \emptyset$. Using the multiplier $\xi u$ yields the following identity, which is obtained as in the proof of Lemma 9 in [13],

$$
\int_S^T \int_\Omega |n u|^2 \, dx \, dt = \int_S^T \int_\Omega |u|^2 \, dx \, dt + \frac{1}{2} \int_S^T \int_\Omega \Delta \xi u^2 \, dx \, dt \left[ \int_\Omega \xi u_t \, dx \right]_S^T - \int_S^T \int_\Omega \xi u_p \, dx \, dt. \quad (26)
$$

Since $\xi \in C_0^\infty (\mathbb{R}^2)$, $\Delta \xi$ is bounded and, by using the definition of $\xi$, we can rewrite (26) as

$$
T_2 \leq \int_S^T \int_{\Omega \cap Q_2} |u|^2 \, dx \, dt + C \left[ \int_{\Omega \cap Q_2} u_t \, dx \right]_S^T \left[ \int_{\Omega \cap Q_2} u^2 \, dx \right]_S^T + C \left[ \int_{\Omega \cap Q_2} |(\mathring{u} - g(u))| \, dx \right]_S^T. \quad (27)
$$

First note that the first term of (27) is upper bounded by $\int_S^T \int_\omega |u|^2 \, dx \, dt$ since $\Omega \cap Q_2 \subset \omega$. We next estimate the other terms in the right-hand side of (27). We start by treating $S_1$ and we easily get by Young’s and Poincaré inequalities that

$$
\int_{\Omega \cap Q_2} |u u_t| \, dx \leq \frac{1}{2} \int_{\Omega \cap Q_2} |u|^2 \, dx + \frac{1}{2} \int_{\Omega \cap Q_2} |u_t|^2 \, dx \leq CE. \quad (28)
$$

Using (17), we obtain from (28) that

$$
S_1 \leq CE(S). \quad (29)
$$

Then, we estimate $S_2$. Since $(\Omega \setminus \omega) \cap (Q_2 \cap \Omega) = \emptyset$, there exists a function $\beta \in C_0^\infty (\mathbb{R}^2)$ such that

$$
\begin{cases}
0 \leq \beta \leq 1, \\
\beta = 1 \text{ on } Q_2 \cap \Omega, \\
\beta = 0 \text{ on } \Omega \setminus \omega.
\end{cases}
$$

For every $t \geq 0$, let $z$ be the solution of the following elliptic problem:

$$
\begin{cases}
\Delta z = \beta u \text{ in } \Omega, \\
z = 0 \text{ on } \partial \Omega.
\end{cases}
$$

One can prove the following:
Lemma 3.5 \[\text{[13, (5.22) to (5.26)]}\] Under the hypotheses of Proposition 3.1 with \( z \) as defined in (31), it holds that

\[
\|z\|_{L^2(\Omega)} \leq C' \|u\|_{L^2(\Omega)}, \quad \|z_t\|_{L^2(\Omega)}^2 \leq C'' \int_{\Omega} \beta |u|^2 \, dx, \quad \|\nabla z\|_{L^2(\Omega)} \leq C \|\nabla u\|_{L^2(\Omega)},
\] (32)

\[
\forall S \leq T \in \mathbb{R}_+, \quad \int_S^T \int_{\Omega} \beta u^2 \, dx \, dt = \left[ \int_{\Omega} z u_t \, dx \right]_S^T + \int_S^T \int_{\Omega} (-z_t u_t + z \rho) \, dx \, dt.
\] (33)

Equation (32) gathers standard elliptic estimates from the definition of \( z \) as a solution of (31) while (33) is obtained by using \( z \) as a multiplier for \((P_0)\). Since the non-negative \( \beta \) is equal to 1 on \( Q_2 \) and 0 on \( \mathbb{R}^2 \setminus \omega \), we deduce from (33) that

\[
S_2 \leq \left[ \int_{\Omega} z u_t \, dx \right]_S^T \underbrace{- \int_S^T \int_{\Omega} z_t u_t \, dx \, dt}_U + \int_S^T \int_{\Omega} z \rho \, dx \, dt.
\] (34)

We start by handling \( U_1 \). One has

\[
\left| \int_{\Omega} z u_t \, dx \right| \leq \|z\|_{L^2(\Omega)} \|u_t\|_{L^2(\Omega)} \leq C \|\nabla u\|_{L^2(\Omega)} \|u_t\|_{L^2(\Omega)} \leq CE(t),
\] (35)

after using Cauchy-Schwarz inequality, then (32) and Poincaré inequality, and finally the definition of \( E(t) \).

Using (35) and the fact that \( E \) is non-increasing, it is then immediate to derive that

\[
|U_1| = \left| \left( \int_{\Omega} z u_t \, dx \right) (T) - \left( \int_{\Omega} z u_t \, dx \right) (S) \right| \leq C(E(T) + E(S)) \leq CE(S).
\] (36)

As for \( U_2 \), one starts by Young inequality to get

\[
|U_2| \leq \int_S^T \int_{\Omega} \frac{1}{2 \eta} |z_t|^2 \, dx \, dt + \int_S^T \int_{\Omega} \frac{\eta}{2} |u_t|^2 \, dx \, dt,
\]

and then, from (32) and since \( 0 \leq \beta \leq 1 \), we deduce

\[
|U_2| \leq \frac{C}{\eta} \int_S^T \int_{\omega} u_t^2 \, dx \, dt + C \eta \int_S^T E \, dx \, dt.
\] (37)

Finally, for \( U_3 \), we first have that \( |U_3| \leq V_1 + V_2 \) where, for all \( S \leq T \in \mathbb{R}_+ \),

\[
V_1 := \int_S^T \int_{|u_t| \leq 1} |ag(u_t)z| \, dx \, dt, \quad V_2 := \int_S^T \int_{|u_t| > 1} |ag(u_t)z| \, dx \, dt.
\]

Using (32) as well as Young and Poincaré inequalities, we have

\[
V_1 \leq \int_S^T \int_{|u_t| \leq 1} \left( \frac{\eta}{2} |z|^2 + \frac{1}{2 \eta} |ag(u_t)|^2 \right) \, dx \, dt
\]

\[
\leq C \eta \int_S^T \int_{\Omega} |u|^2 \, dx \, dt + \frac{C}{\eta} \int_S^T \int_{|u_t| \leq 1} |ag(u_t)|^2 \, dx \, dt
\]

\[
\leq C \eta \int_S^T E \, dt + \frac{C}{\eta} \int_S^T \int_{|u_t| \leq 1} |ag(u_t)|^2 \, dx \, dt,
\] (38)
Then, $g(0) = 0$ implies that there exists a constant $C > 0$ such that $|g(x)| \leq C|x|$ for $|x| \leq 1$. We can then obtain

$$
\int_T \int_{|u_t| \leq 1} |ag(u_t)|^2 \, dx \, dt \leq \int_T \int_{|u_t| \leq 1} au_t g(u_t) \, dx \, dt \leq C \int_T (-E') \, dt.
$$

(39)

Combining (38) and (39) yields

$$
V_1 \leq C \eta \int_T E \, dt + \frac{C}{\eta} E(S).
$$

(40)

To estimate $V_2$, we start by the following remark: by Rellich-Kondrachov’s theorem in dimension two (cf. [4]) and using the fact that $z \in H^1_0(\Omega)$, we deduce from (32) that

$$
\|z\|_{L^{q+1}(\Omega)} \leq C \sqrt{E}.
$$

(41)

Now using Holder’s inequality, we get

$$
V_2 \leq \left( \int_{|u_t| > 1} |z|^{q+1} \, dx \right)^{\frac{1}{q+1}} \left( \int_{|u_t| > 1} |ag(u_t)|^{\frac{q+1}{\eta}} \, dx \right)^{\frac{\eta}{q+1}}.
$$

(42)

Using (32) with (11) and the fact that $a$ is bounded, we obtain

$$
V_2 \leq C \int_T E^{\frac{1}{\eta}} \left( \int_{|u_t| > 1} |ag(u_t)||g(u_t)|^{\frac{1}{\eta}} \, dx \right)^{\frac{q}{q+1}} \, dt.
$$

(43)

By [3], (17) and Young’s inequality, we rewrite (43)

$$
V_2 \leq C \int_T E^{\frac{1}{\eta}} \left( \int_{|u_t| > 1} ag(u_t)u_t \, dx \right)^{\frac{q}{q+1}} \, dt \leq C \int_T E^{\frac{1}{\eta}} \left( \int_{\Omega} ag(u_t)u_t \, dx \right)^{\frac{q}{q+1}} \, dt
$$

$$
\leq C \int_T E^{\frac{1}{\eta}} (-E')^{\frac{q}{q+1}} \, dt \leq \int_T E^{\frac{1}{\eta}} (-E')^{\frac{q}{q+1}} \, dt
$$

$$
\leq C \eta^{q+1} \int_T E^{\frac{q+1}{\eta}} \, dt + \frac{C}{\eta^{q+1}} \int_T (-E')^{\frac{q}{q+1}} \, dt
$$

$$
\leq C \eta^{q+1} \int_T E \, dt + \frac{C}{\eta^{q+1}} \int_T E \, dt + \frac{C}{\eta} E(S).
$$

(44)

By first combining (40) and (44) to get an estimate of $|U_3|$ and then gathering the resulting inequality with (37), (36) and (34), we arrive at the following estimate

$$
S_2 \leq (\eta + \eta^{q+1}) C_u \int_T E \, dt + C \left( \frac{1}{\eta^{q+1}} + \frac{1}{\eta} + 1 \right) E(S) + \frac{C}{\eta} \int_T \int_{\omega} u_t^2 \, dx \, dt.
$$

(45)

It remains to estimate $S_3$. We start by the obvious inequality

$$
|S_3| \leq \int_T \int_{\Omega \cap Q_2} |ag(u_t)| \, dx \, dt = \int_T \int_{|u_t| \leq 1} |ag(u_t)| \, dx \, dt + \int_T \int_{|u_t| > 1} |ag(u_t)| \, dx \, dt.
$$

Using Young’s inequality, one has, for every $\eta > 0$,

$$
W_1 \leq \int_T \int_{|u_t| \leq 1} \left( \frac{\eta}{2} |u_t|^2 + \frac{1}{2\eta} |ag(u_t)|^2 \right) \, dx \, dt \leq \frac{\eta}{2} \int_T E \, dt + \frac{1}{2\eta} \int_T \int_{|u_t| \leq 1} |ag(u_t)|^2 \, dx \, dt.
$$

$$
W_2 \leq \int_T \int_{|u_t| > 1} \left( \frac{\eta}{2} |u_t|^2 + \frac{1}{2\eta} |ag(u_t)|^2 \right) \, dx \, dt \leq \frac{\eta}{2} \int_T E \, dt + \frac{1}{2\eta} \int_T \int_{|u_t| > 1} |ag(u_t)|^2 \, dx \, dt.
$$

$$
W_3 \leq \int_T \int_{|u_t| > 1} \left( \frac{\eta}{2} |u_t|^2 + \frac{1}{2\eta} |ag(u_t)|^2 \right) \, dx \, dt \leq \frac{\eta}{2} \int_T E \, dt + \frac{1}{2\eta} \int_T \int_{|u_t| > 1} |ag(u_t)|^2 \, dx \, dt.
$$

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which yields with (39) that, for every \( \eta > 0 \),
\[
W_1 \leq C \eta \int_S^T E \, dt + \frac{C}{\eta} E(S). \tag{46}
\]

As for \( W_2 \), we first notice that \( u \) satisfies the estimate (41) satisfied by \( z \). Hence, by following the same steps followed to obtain (44), we similarly get that
\[
W_2 \leq C_u \eta^{q+1} \int_S^T E \, dt + \frac{C}{\eta^{q+1}} E(S). \tag{47}
\]

Combining (46) and (47), we get that, for every \( \eta > 0 \),
\[
|S_3| \leq (\eta + \eta^{q+1}) C_u \int_S^T E \, dt + C \left( \frac{1}{\eta^{q+1}} + \frac{1}{\eta} \right) E(S). \tag{48}
\]

Gathering (48), (45), (29) and (27), we have now the upper bound for \( T_2 \), which completes the proof of Lemma 3.4.

The final step to prove Proposition 3.1 consists of estimating \( T_3 \). This is the purpose of the following lemma:

**Lemma 3.6** Under the hypotheses of Proposition 3.1, we have the following estimation, for every \( \eta > 0 \),
\[
|T_3| \leq C_u (\eta + \eta^{q+1} + \eta^{\frac{2(q+1)}{q+2}}) \int_S^T E dt + C_u \left( \frac{1}{\eta^{q+1}} + \frac{1}{\eta} \right) E(S). \tag{49}
\]

**Proof of Lemma 49.** We have \( |T_3| \leq \frac{1}{2} S_3 + X_1 + X_2 \) where
\[
X_1 = \int_S^T \int_{|u_t| \leq 1} |ag(u_t) \nabla u.h| \, dx \, dt, \quad X_2 = \int_S^T \int_{|u_t| > 1} |ag(u_t) \nabla u.h| \, dx \, dt.
\]

Since we have (48), it remains to estimate \( X_1 \) and \( X_2 \). Using Young and Poincaré’s inequalities and the fact that \( |h| \) is bounded, we get
\[
X_1 \leq C \int_S^T \int_{|u_t| \leq 1} \left( \frac{\eta}{2} |u|^2 + \frac{1}{2 \eta} |ag(u_t)|^2 \right) dx \, dt \leq C \int_S^T \left( C \eta E + \frac{1}{2 \eta} \int_{|u_t| \leq 1} |ag(u_t)|^2 dx \right) dt.
\]

The above combined with (39) yields, for every \( \eta > 0 \),
\[
X_1 \leq C \eta \int_S^T E \, dt + \frac{C}{\eta} E(S). \tag{50}
\]

Using Cauchy-Schwarz’s inequality, one has
\[
X_2 \leq C \int_S^T \left( \int_{|u_t| > 1} |\nabla u|^2 \, dx \right)^\frac{1}{2} \left( \int_{|u_t| > 1} |ag(u_t)|^2 \, dx \right)^\frac{1}{2} \, dt \leq C \int_S^T E^{\frac{1}{2}} \left( \int_{|u_t| > 1} |ag(u_t)|^2 \, dx \right)^\frac{1}{2} \, dt. \tag{51}
\]
On another hand, using Holder’s inequality, \((5)\) and \((7)\), we have
\[
\int_{|u_t|>1} |ag(u_t)|^2\,dx \leq \left(\int_{|u_t|>1} |ag(u_t)|^{q+1}\,dx\right)^{\frac{1}{q+1}} \left(\int_{|u_t|>1} |ag(u_t)|^{\frac{q}{q+1}}\,dx\right)^{\frac{q}{q+1}} \\
\leq C \left(\int_{|u_t|>1} |u_t|^{q(q+1)}\,dx\right)^{\frac{1}{q+1}} \left(\int_{|u_t|>1} ag(u_t)u_t\,dx\right)^{\frac{q}{q+1}} \\
\leq C \left(\int_{|u_t|>1} |u_t|^{q(q+1)}\,dx\right)^{\frac{1}{q+1}} (-E')^{\frac{q}{q+1}}. \tag{52}
\]

From \((19)\) with \(p = q(q + 1)\), we have
\[
\|u_t\|_{L^q(q+1)(\Omega)} \leq C_u E(t),
\]
which gives
\[
\left(\int_{|u_t|>1} |u_t|^{q(q+1)}\,dx\right)^{\frac{1}{q+1}} = \|u_t\|_{L^q(q+1)(\Omega)} \leq C_u E^{\frac{1}{q+1}}. \tag{53}
\]

Combining \((52)\) and \((53)\), we obtain
\[
\int_{|u_t|>1} |ag(u_t)|^2\,dx \leq C_u E(t)^{\frac{1}{q+1}} (-E')^{\frac{q}{q+1}}. \tag{54}
\]

Now combining \((51)\) and \((54)\) and then using Young inequality and \((17)\), we have for every \(\eta > 0\)
\[
X_2 \leq \int_S^T C_u E^{\frac{1}{q+1}} E(t)^{\frac{1}{q+1}} (-E')^{\frac{q}{2(q+1)}} dt \leq C_u \int_S^T E(t)^{\frac{q+2}{2(q+1)}} (-E')^{\frac{q}{2(q+1)}} dt \\
\leq C_u \eta^{\frac{2(q+1)}{q+2}} \int_S^T E\,dt + \frac{C_u}{\eta^{\frac{q}{q+1}}} E(S). \tag{55}
\]

Combining \((48)\), \((50)\) and \((55)\), we have the required estimate for \(|T_3|\) and that concludes the proof of Lemma 3.6.

We start by noticing that
\[
T_4 \leq C \int_S^T \int_\Omega a|u_t|^2\,dx\,dt.
\]
By following Section 4 in [14], we get that
\[
T_4 \leq \frac{C_u}{R} \int_S^T E(t)\,dt + \left(\frac{1}{\alpha_R} + 1\right) E(S), \tag{56}
\]
for any \(R > 0\), where \(\alpha_R\) is any positive real number such that \(|g(v)| \geq \alpha_R|v|\) for \(|v| \leq R\). Such an \(\alpha_R\) exists because the function defined by \(\frac{g(v)}{v}\) for \(v \neq 0\) and \(g'(0)\) for \(v = 0\) is continuous on \(\mathbb{R}\) and takes values in \(\mathbb{R}^*_+\) (since \(g'(0) > 0\)).

By choosing \(R\) big enough in \((56)\), and \(\eta\) small enough in \((19)\) and \((24)\), we can finally complete the proof of Proposition 3.1 by combining \((22)\) with the resulting estimations of \(T_1, T_2, T_3\) and \(T_4\), which proves \((20)\).
3.2 Proof of (14)

We follow the lines of the proof of (13) and establish the several results corresponding to \((P_{\text{dis}})\). The main difference is that the energy \(E\) is no longer non-increasing. This requires extra work.

3.2.1 Reduction to Proposition 3.2

We start with the following lemma stating that the energy \(E\) is bounded along trajectories of \((P_{\text{dis}})\).

**Lemma 3.7** Under the hypotheses of Theorem (2.2), the energy of a solution of Problem \((P_{\text{dis}})\), satisfies

\[
E'(t) = -\int_{\Omega} au_t g(u_t + d) \, dx - \int_{\Omega} u_t e(e) \, dx, \quad \forall t \geq 0.
\]

Furthermore, there exist positive constants \(C\) and \(C_{d,e}\) such that

\[
E(T) \leq CE(S) + C_{d,e}, \quad \forall 0 \leq S \leq T.
\]

**Proof of Lemma 3.7.** Equation (57) follows after multiplying the first equation of (1) by \(u_t\) and standard computations. Next, notice that we do not have the dissipation of \(E\) since the sign of \(E'\) is not necessarily constant, to get (58), we first write

\[
-\int_{\Omega} au_t g(u_t + d) \, dx = -\int_{|u_t| \leq |d|} au_t g(u_t + d) \, dx - \int_{|u_t| > |d|} au_t g(u_t + d) \, dx.
\]

From (4) and the fact that \((u_t + d)\) and \(u_t\) have the same sign if \(|u_t| > |d|\), we deduce that

\[
-\int_{|u_t| > |d|} au_t g(u_t + d) \, dx \leq 0.
\]

On another hand, since \(g\) is non-decreasing, has linear growth in a neighborhood of zero by (4), and satisfies (5), we have

\[
-\int_{|u_t| \leq |d|} au_t g(u_t + d) \, dx \leq C\int_{|u_t| \leq |d|} |d| g(|2d|) \, dx \leq C\int_{\Omega} |d| g(|2d|) \, dx \leq C\int_{\Omega} (|d|^2 + |d|^{q+1}) \, dx.
\]

Combining (59), (60), (61) and (57), we obtain

\[
E' \leq C\int_{\Omega} (|d|^2 + |d|^{q+1}) \, dx - \int_{\Omega} u_t e(e) \, dx dt.
\]

Using Cauchy-Schwarz inequality

\[
E' \leq C\int_{\Omega} |d|^{q+1} \, dx + \left(\int_{\Omega} |e|^{2} \, dx dt\right)^{\frac{1}{2}} \left(\int_{\Omega} |u_t|^{2} \, dx\right)^{\frac{1}{2}} \leq C\int_{\Omega} |d|^{q+1} \, dx + C\|e\|_{L^2(\Omega)} \sqrt{E},
\]

which gives when integrating between two arbitrary non negative times \(S \leq T\)

\[
E(T) \leq E(S) + C\int_{S}^{T} |d|^{q+1} \, dx dt + C\int_{S}^{T} \|e\|_{L^2(\Omega)} \sqrt{E} dt \leq E(S) + CC_1(d) + C\int_{S}^{T} \|e\|_{L^2(\Omega)} \sqrt{E} dt,
\]
which allows us to apply Theorem [A.2], we obtain
\[ E(T) \leq 2E(S) + 2CC_1(d) + \left( C \int_S^T \|c\|_{L^2(\Omega)} dt \right)^2 = CE(S) + C_{d,e}, \]
which proves (58). Hence, the proof of Lemma 3.7.

We provide now an extension of (18) to (P\text{dis}).

**Lemma 3.8** Under the hypotheses of Theorem 2.2, for every solution of Problem (P\text{dis}) with initial conditions \((u^0, u^1) \in (H^2(\Omega) \cap H^1_0(\Omega)) \times H^1_0(\Omega)), there exist explicit positive constants \(C_u\) and \(C_{d,e}\) such that
\[ \forall t \geq 0, \quad \| - \Delta u(t, \cdot) + a(\cdot)g(u(t, \cdot) + d(t, \cdot)) + e(t, \cdot) \|_{L^2(\Omega)}^2 + \| u_t(t, \cdot) \|_{H^1_0(\Omega)}^2 \leq C_u + C_{d,e}. \quad (63) \]

**Proof of Lemma 3.8.** If \(w\) denotes \(u_t\), then the first part of Theorem 2.2 says that \(u_t(t) \in H^1_0(\Omega)\) for every \(t \geq 0\), which allows us to differentiate the three equations of (11) with respect to \(t\), obtaining (in the distributional sense)
\[
\begin{cases}
  w_{tt} - \Delta w + ag'(w + d)(w_t + d_t) + e_t = 0 & \text{in } \Omega \times \mathbb{R}_+, \\
  w(0) = u^1, \quad w_t(0) = \Delta u^0 - g(u^1 + d(0)) - e(0), & \text{on } \partial \Omega \times \mathbb{R}_+, \quad (64)
\end{cases}
\]
Set \(E_w(t)\) for the energy of \(w\) and \(t \geq 0\), i.e.,
\[ E_w(t) = \frac{1}{2} \int_\Omega \left( w^2(t, x) + |\nabla w(t, x)|^2 \right) dx. \]
Using \(w_t\) as a test function in the distributional version of (64), we deduce, after integrating over \(\Omega \times [0, t], t \in \mathbb{R}_+\) and standard computations, that
\[ E_w(t) - E_w(0) = - \int_0^t \int_\Omega \left( ag'(w + d)(d_t + w_t)w_t + e_tw_t \right) dxd\tau. \quad (65) \]
Let \(I := \int_0^t \int_\Omega a(\cdot)g'(w + d)(d_t + w_t)w_t \, dxd\tau \). We cut it in two parts according to whether \(|d_t| \leq |w_t|\) or not. Clearly the part corresponding to \(|d_t| \leq |w_t|\) is non negative since \(g' \geq 0, a \geq 0\) and \((d_t + w_t)\) and \(w_t\) have the same sign. From (6), one has the immediate estimate
\[ g'(a + b) \leq C(1 + |a + b|^m) \leq C(1 + |a|^m + |b|^m), \quad \forall a, b \in \mathbb{R}. \]
Using the above, we can hence rewrite (65) as,
\[
\begin{align*}
E_w(t) - E_w(0) & \leq \int_0^t \int_{|d_t| > |w_t|} ag'(w + d)(d_t + w_t)w_t \, dxd\tau + \int_0^t \int_\Omega |e_t||w_t| \, dxd\tau \\
& \leq C \int_0^t \int_\Omega g'(w + d)d_t^2 \, dxd\tau + \int_0^t |e_t|_{L^2(\Omega)} \sqrt{E_w} d\tau \\
& \leq C \int_0^t \int_\Omega (1 + |w|^m + |d|^m)d_t^2 \, dxd\tau + \int_0^t \|e_t\|_{L^2(\Omega)} \sqrt{E_w} d\tau. \quad (66)
\end{align*}
\]
Using Holder’s inequality, we have
\[
\int_0^t \int_\Omega |w|^p dx dg \leq \left( \int_0^t \int_\Omega |w|^{pm} dx \right)^{\frac{1}{p}} \left( \int_0^t \int_\Omega |d|^{2p'} dx \right)^{\frac{1}{p'}} dt,
\]
with \( p \) defined in (11) and \( p' > 1 \) is its conjugate exponent given by \( \frac{1}{p} + \frac{1}{p'} = 1 \). Thanks to the assumptions on \( p \), one can use Gagliardo-Nirenberg’s inequality for \( w \) to get
\[
\left( \int_\Omega |w(t, x)|^{pm} dx \right)^{\frac{1}{p}} \leq CE_w(t) E(t) \frac{1-(\theta)m}{2}, \quad t \geq 0,
\]
where \( \theta = 1 - \frac{2}{mp} \). Combining (68), (67) and (66)
\[
E_w(t) - E_w(0) \leq C \int_0^t E^{\frac{m \theta}{2}} E^{\frac{1-(\theta)m}{2}} \int_\Omega (|d|^{2p'} dx)^{\frac{1}{p'}} dt
+ \int_0^t \left( 1 + |d|^m \right) dt \int_\Omega E \leq \int_0^t \|e\|_{L^2(\Omega)} \sqrt{E(w)} dt.
\]
Note that \( \frac{m \theta}{2} < 1 \). Setting \( h_1(t) = \|e\|_{L^2(\Omega)} \), and \( h_2(t) = \int_\Omega (|d|^{2p'} dx)^{\frac{1}{p'}} \), (69) becomes
\[
E_w(t) \leq E_w(0) + C_2(d) + C_3(d) + C(C_u + C_{d,e}) \int_0^t E^{\frac{m \theta}{2}} h_1(s) ds + \int_0^t h_2(s) \sqrt{E_w ds}
\]
We can now apply Theorem A.2 on (70) with
\[
S = 0, \quad T = t, \quad \alpha_1 = \frac{m \theta}{2}, \quad \alpha_1 = \frac{1}{2}, \quad F = E_w, \quad C = C_2(d) + C_3(d), \quad C_1 = 1, \quad C_2 = C_u + C_{d,e}.
\]
We get the conclusion after using Young’s inequality.

We next provide the estimate corresponding to (19) for (P_dis).

**Lemma 3.9** For all \( q > 2 \), we have
\[
\|u_t(t, \cdot)\|_{L^q(\Omega)}^q \leq (C_u + C_{d,e}) E(t), \quad t \geq 0.
\]

**Proof of Lemma 3.9** From (63) we have \( \|u_t\|_{H^3_0(\Omega)} \leq C_u + C_{d,e} \). On another hand from Gagliardo-Nirenberg’s theorem, we have
\[
\|u_t\|_{L^q(\Omega)}^q \leq C \|u_t\|_{H^3_0(\Omega)}^{q-2} \|u_t\|^2_{L^2(\Omega)} \leq (C_u + C_{d,e}) E.
\]

We next show how to derive the second part of Theorem 2.2 as a consequence of the next proposition.

**Proposition 3.2** Suppose that the hypotheses of Theorem (2.2) are satisfied, then the energy \( E \) of the solution \( u \) of (P_dis) with \( (u^0, u^1) \in (H^2(\Omega) \cap H^1_0(\Omega) \times H^1_0(\Omega)) \), satisfies the following estimate:
\[
\int_0^T E dt \leq C_u E(S) + (1 + C_{d,e}) C_{d,e}, \quad (74)
\]
where the positive constants \( C_u, C_{d,e} \) depend only on the initial condition and the positive constant \( C_{d,e} \) depends only on the disturbances \( d \) and \( e \) respectively.
Assuming that the previous proposition holds true, we get at once from Theorem A.1 that (112) holds true with \( T = C_0 \) and \( C_0 = (1 + C_u)C_{d,e} \). Using (58) for \( t \geq 1 \) with \( T = t \) and \( S \in [t - 1, t] \) and integrating it over \([t - 1, t]\), one gets that

\[
E(t) \leq C \int_{t-1}^{t} E(s) \, ds + C \int_{t-1}^{\infty} E(s) \, ds + C_{d,e}.
\]

Combining the above with (112) yields (14) for \( t \geq 1 \). In turn, (58) with \( T \in [0, 1] \) and \( S = 0 \) provides (14) for \( t \leq 1 \).

### 3.2.2 Proof of Proposition 3.2

Consider a strong solution \( u \) of (P\text{dis}) associated with an initial condition \((u^0, u^1) \in (H^2(\Omega) \cap H^1_0(\Omega)) \times H^1_0(\Omega)\), two non-negative times \( S \leq T \) and let \( x_0 \in \mathbb{R}^2 \) be an observation point and \( Q_i \) for \( i = 0, 1, 2 \) as defined in the proof of Proposition 3.1. We have the following first estimate:

**Lemma 3.10** Under the hypotheses of Proposition 3.2, we have the following inequality:

\[
\int_{S}^{T} E \, dt \leq \left[ \int_{\Omega} u_i M(u) \, dx \right]_{S}^{T} + C \int_{S}^{T} \int_{\Omega \cap Q_1} \nabla u_i^2 \, dx \, dt + \int_{S}^{T} \int_{\Omega \cap Q_2} a g(u_i + d) M(u) \, dx \, dt \nonumber
\]

\[
+ \int_{S}^{T} \int_{\Omega \cap Q_3} e M(u) \, dx \, dt + C \int_{S}^{T} \int_{\Omega \cap Q_4} u_i^2 \, dx \, dt,
\]

where \( h \) is defined in (21) and \( M(u) \) is the multiplier \( h \cdot \nabla u + \frac{u}{2} \).

**Proof of Lemma 3.10** The proof is essentially the same as the proof of Lemma 3.2 by simply taking \( \rho \) equal to \( a(x)g(u_i + d) + e \) instead of \( a(x)g(u_i) \).

We now embark on estimating the terms \( T_1 \) to \( T_5 \).

Exactly as in Lemma 3.3 except that we use (58) in the very last step, we deduce that

\[
T_1' \leq C E(S) + C_{d,e}.
\]

The estimate of \( T_2 \) requires more work and it is given in the following lemma:

**Lemma 3.11** Under the hypotheses of Proposition 3.2, we have that

\[
T_2' \leq \sigma_2 \int_{S}^{T} E \, dt + C \int_{S}^{T} \int_{\Omega} u_i^2 \, dx \, dt + (C_u + C_{d,e}) E(S) + C_{d,e} C_u + C_{d,e},
\]

where \( \sigma_2 < 1 \) is an arbitrary positive number to be chosen later.

**Proof of Lemma 3.11** Similarly to (26), one gets

\[
\int_{S}^{T} \int_{\Omega} \xi |\nabla u|^2 \, dx \, dt = \int_{S}^{T} \int_{\Omega} \xi |u_i|^2 \, dx \, dt + \frac{1}{2} \int_{S}^{T} \int_{\Omega} \Delta \xi u^2 \, dx \, dt \left[ \int_{\Omega} \xi u_i \, dx \right]_{S}^{T} - \int_{S}^{T} \int_{\Omega} \xi u(a(.)) g(u_i + d) + e \, dx \, dt,
\]

\[
(78)
\]

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where \( \xi \) is defined in (25). One deduces from the above

\[
T_2' \leq \int_{S}^{T} \int_{\Omega \cap Q_2} |u_t|^2 \, dx \, dt + \left[ \int_{\Omega \cap Q_2} uu_t \, dx \right]_{S}^{T} + C \int_{S}^{T} \int_{\Omega \cap Q_2} u^2 \, dx \, dt \\
+ \int_{S}^{T} \int_{\Omega \cap Q_2} |uag(u_t + d)| \, dx \, dt + \int_{S}^{T} \int_{\Omega} |ue| \, dx \, dt,
\]

which corresponds to the estimate (27) of \( T_2 \).

As in the estimation of \( S_1 \) in (29), we have from (28) that \( \int_{\Omega \cap Q_2} |uu_t| \, dx \leq \max(C(\Omega)^2, 1)E(t) \), which gives that

\[
S_1' \leq CE(S) + C_{d,e}.
\]

The first estimate of \( S_2' \) is in correspondence with Lemma 3.5 and reads as follows.

\[
\int_{S}^{T} \int_{\Omega} \beta u^2 \, dx \, dt = \left[ \int_{\Omega} z u_t \, dx \right]_{S}^{T} + \int_{S}^{T} \int_{\Omega} (-z_t u_t + z(a g(u_t + d) + e)) \, dx \, dt,
\]

with \( z \) as defined in (31) and \( \beta \) as defined in (30). We then deduce for \( S_2' \) the estimate (34) with the corresponding quantities \( U_1' \) to \( U_3' \) where the only change is \( \rho = a(x)g(u_t + d) + e \) instead of \( a(x)g(u_t) \).

As for \( U_1 \), (35) holds true for \( U_1' \), which yields the estimate corresponding to (36) namely

\[
|U_1'| \leq CE(S) + C_{d,e}.
\]

Following the argument yielding (37), we have

\[
|U_2'| \leq C \int_{S}^{T} \int_{\omega} u_t^2 \, dx \, dt + \eta_0 \int_{S}^{T} E \, dt.
\]

**Remark 3.1** Note that the constant \( C \) before \( \int_{S}^{T} \int_{\omega} u_t^2 \, dx \, dt \) depends on the choice of \( \eta_0 \) but since \( \eta_0 \) is a real number that is going to be chosen independently of initial conditions and disturbances, the constant \( C \) is going to be a constant real number as well, for the simplicity of notations, we denote it by \( C \) despite its dependence on \( \eta_0 \). We will do the same with many other constants in the rest of the paper.

Similarly to \( U_3 \), we now write \( U_3' \) as the sum \( V_1' + V_2' + V_3' \) where the first two correspond to whether \( |u_t + d| \leq 1 \) or not and \( V_3' \) is equal to \( \int_{S}^{T} \int_{\Omega} z e \, dx \, dt \).

Then, as \( V_1 \), the quantity \( V_1' \) satisfies (38) where one simply replaces \( u_t \) by \( u_t + d \), which means that that for every \( \eta > 0 \)

\[
V_1' \leq \eta \int_{S}^{T} E \, dt + \frac{1}{\eta} \int_{S}^{T} \int_{|u_t| \leq 1} |ag(u_t + d)|^2 \, dx \, dt,
\]

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we fix \( \eta = \eta_1 \) small enough to be chosen later, we obtain

\[
V_1' \leq \eta_1 \int_S^T E dt + C \int_S^T \int_{|u_t| \leq 1} |ag(u_t + d)|^2 dx dt, \tag{83}
\]

Using the same arguments as for \( (39) \) plus the handling of the disturbances \( d, e \), one gets that

\[
\int_S^T \int_{|u_t + d| \leq 1} |ag(u_t + d)|^2 dx dt \leq \int_S^T \int_{|u_t + d| \leq 1} \alpha(\cdot)(u_t + d)g(u_t + d) dx
\]

\[
\leq \int_S^T \int_{|u_t + d| \leq 1} a u_t g(u_t + d) dx + \int_S^T \int_{\Omega} u_t e dx dt - \int_S^T \int_{\Omega} u_t e dx dt + \int_S^T \int_{|u_t + d| \leq 1} adg(u_t + d) dx
\]

\[
\leq C \int_S^T (-E') dt + C \int_S^T \int_{|u_t + d| \leq 1} |d| dx dt + \int_S^T \int_{\Omega} |u_t| |e| dx dt
\]

\[
\leq CE(S) + C \int_S^T \int_{\Omega} |d| dx dt + \eta \int_S^T E dt + \frac{C}{\eta} \int_S^T \int_{\Omega} |e|^2 dx dt.
\]

We fix as \( \eta = \eta_2 \) with \( \eta_2 < 1 \) is a small arbitrary number to be chosen later, we obtain

\[
\int_S^T \int_{|u_t + d| \leq 1} |ag(u_t + d)|^2 dx dt \leq CE(S) + \eta_2 \int_S^T E dt + C_{d,e}. \tag{84}
\]

Combining \( (83) \) and \( (84) \), we derive

\[
V_1 \leq C (\eta_1 + \eta_2) \int_S^T E dt + CE(S) + CC_1(d) + CC_5(e)
\]

\[
\leq C (\eta_1 + \eta_2) \int_S^T E dt + CE(S) + C_{d,e}. \tag{85}
\]

Since \( z \) satisfies \( (41) \), we have

\[
V_2' \leq \int_S^T \left( \int_{|u_t + d| > 1} (a|g(u_t + d)|)^{\frac{q+1}{q}} dx \right)^{\frac{q}{q+1}} \left( \int_{|u_t + d| > 1} |z|^{\frac{q+1}{q}} dx \right)^{\frac{q}{q+1}} dt
\]

\[
\leq C \int_S^T \left( \int_{|u_t + d| > 1} a|u_t + d||g(u_t + d)| dx \right)^{\frac{q}{q+1}} \left( \int_{|u_t + d| > 1} |z|^{\frac{q+1}{q}} dx \right)^{\frac{q}{q+1}} dt
\]

\[
\leq C \int_S^T \left( \frac{1}{\eta^{\frac{q+1}{q}}} \int_{|u_t + d| > 1} a(x)(u_t + d)g(u_t + d) dx + \eta^{q+1} \int_{\Omega} |z|^{\frac{q+1}{q}} dx \right) dt
\]

\[
\leq C \int_S^T \left( \frac{1}{\eta^{\frac{q+1}{q}}} \int_{|u_t + d| > 1} au_t g(u_t + d) dx + \frac{1}{\eta^{q+1}} \int_{|u_t + d| > 1} adg(u_t + d) dx + \eta^{q+1} \int_{\Omega} |z|^{\frac{q+1}{q}} dx \right) dt
\]

\[
\leq C \int_S^T \left( \frac{1}{\eta^{\frac{q+1}{q}}} (-E') - \frac{1}{\eta^{q+1}} \int_{\Omega} u_t e dx + \frac{1}{\eta^{q+1}} \left( \int_{|u_t + d| > 1} dg(u_t + d) dx \right) + \eta^{q+1} E^{\frac{q+1}{q}} dx \right) dt
\]

\[
\leq \frac{C}{\eta^{\frac{q+1}{q}}} E(S) + C \eta^{q+1}(C_u + C_{d,e}) \int_S^T E dt + \frac{C}{\eta^{q+1}} \int_S^T \left( - \int_{\Omega} u_t e dx + \int_{|u_t + d| > 1} dg(u_t + d) dx \right) dt. \tag{86}
\]
We fix \( \eta = \left( \frac{\eta_3}{\epsilon(C_u + C_{d,e})} \right)^{\frac{1}{q+1}} \), where \( \eta_3 < 1 \) to be chosen later. We obtain
\[
C \eta^{q+1} (C_u + C_{d,e}) = \eta_3,
\]
\[
\frac{C}{\eta^{q+1}} = C \left( \frac{C_u + C_{d,e}}{\eta_3} \right)^q \leq \frac{C}{\eta_3} \left( C_u^q + C_{d,e}^q \right) = C_u + C_{d,e},
\]
which yields
\[
V_2' \leq (C_u + C_{d,e}) E(S) + \eta_3 \int_S^T E dt + (C_u + C_{d,e}) \int_S^T \left( - \int_{\Omega} u_t e dx + \int_{|u_t + d| > 1} dg(u_t + d) \right) dt.
\]

To estimate \( V_2' \), we still have to handle the last two integral terms in the above.
\[
(C_u + C_{d,e}) \int_S^T \int_{|u_t + d| > 1} dg(u_t + d) \ dx \ dt \leq (C_u + C_{d,e}) \int_S^T \int_{|u_t + d| > 1} |d|(|u_t|^q + |d|^q) \ dx \ dt
\]
\[
\leq (C_u + C_{d,e}) \int_S^T \int_{\Omega} |d||u_t|^q \ dx \ dt + (C_u + C_{d,e}) C_1(d)
\]
\[
\leq (C_u + C_{d,e}) \frac{C}{\eta} \int_S^T \int_{\Omega} |d|^2 \ dx \ dt + C(C_u + C_{d,e}) \eta \int_S^T \int_{\Omega} |u_t|^{2q} \ dx \ dt + (C_u + C_{d,e}) C_{d,e}
\]
\[
\leq \left( \frac{1}{\eta} + 1 \right) (C_u + C_{d,e}) C_{d,e} + \eta(C_u + C_{d,e})^2 \int_S^T E \ dt \ dx \ dt
\]
\[
\leq C \eta(C_u + C_{d,e}) \int_S^T E \ dt + \left( \frac{1}{\eta} + 1 \right) (C_{d,e} C_u + C_{d,e}).
\]

We fix \( \eta = \frac{\eta_3}{\epsilon(C_u + C_{d,e})} \), we obtain
\[
C \eta(C_u + C_{d,e}) = \eta_4,
\]
\[
\left( \frac{1}{\eta} + 1 \right) (C_{d,e} C_u + C_{d,e}) = \left( \frac{C}{\eta_4} (C_u + C_{d,e}) + 1 \right) (C_{d,e} C_u + C_{d,e}) = (C_{d,e} C_u + C_{d,e}),
\]
which yields
\[
(C_u + C_{d,e}) \int_S^T \int_{|u_t + d| > 1} dg(u_t + d) \ dx \ dt \leq \eta_4 \int_S^T E \ dt + C_{d,e} C_u + C_{d,e}.
\]

On another hand, we have
\[
(C_u + C_{d,e}) \int_S^T \int_{\Omega} u_t e \ dx \ dt \leq \eta(C_u + C_{d,e}) \int_S^T E \ dt + \frac{C}{\eta} (C_{d,e} C_u + C_{d,e}),
\]
using the same concept as before we fix \( \eta = \frac{\eta_3}{\epsilon(C_u + C_{d,e})} \), we obtain
\[
(C_u + C_{d,e}) \int_S^T \int_{\Omega} u_t e \ dx \ dt \leq \eta_5 \int_S^T E \ dt + C_{d,e} C_u + C_{d,e},
\]

Combining (87), (88) and (89)
\[
V_2' \leq (C_u + C_{d,e}) E(S) + C (\eta_3 + \eta_4 + \eta_5) \int_S^T E dt + C_{d,e} C_u + C_{d,e}.
\]
On another hand we have

\[ V_3' \leq \eta_6 \int_S^T E \, dt + \frac{C}{\eta_6} C_5(e) \leq C \eta_6 \int_S^T E \, dt + C_{d,e}. \quad (91) \]

Now combining (81), (82), (85), (90) and (91), we obtain an estimation of \( S_2' \)

\[ S_2' \leq C \left( \sum_{i=0}^{6} \eta_i \right) \int_S^T E \, dt + C \int_\omega u_t^2 \, dx \, dt + (C_u + C_{d,e}) E(S) + C_{d,e} C_u + C_{d,e}. \quad (92) \]

Regarding \( S_3' \), we follow the same steps as to get \( V_1' + V_2' \) since \( u \) satisfies the same result (81). Hence, we obtain by following the same concept we used to estimate the other terms \( S_1, S_2, S_3 \)

\[ S_3' \leq C \eta_7 \int_S^T E \, dt + (C_u + C_{d,e}) E(S) + C_{d,e} C_u + C_{d,e}. \quad (93) \]

Finally, we simply have

\[ S_4' \leq \eta_8 \int_S^T E \, dt + \frac{C}{\eta_8} \int_S^T \int_\Omega |e|^2 \, dx \, dt \leq \eta_8 \int_S^T E \, dt + C_{d,e}, \quad (94) \]

which completes the estimate of \( T_2' \) in estimating the right side of (79) by taking \( \sigma_2 = C \sum_{i=0}^{8} \eta_i \), which is also an arbitrary constant that can be chosen as small as we want it to be, and hence the proof of Lemma 3.11.

The estimation of \( T_3' \) is provided in the next lemma. As for the estimate of \( T_2' \), we have to introduce a new parameter \( R_1 > 0 \) with respect to the estimate of \( T_3 \) and, this will play a crucial role to estimate this term.

**Lemma 3.12** Under the hypotheses of Proposition 3.2, we have the following estimation:

\[ T_3' \leq \sigma_3 \int_S^T E \, dt + (C_u + C_{d,e}) E(S) + C_{d,e} C_u + C_{d,e}, \quad (95) \]

where \( \sigma_3 < 1 \) is a positive arbitrary real numbers to be chosen later.

**Proof of Lemma 3.12** We have:

\[ T_3' = \frac{1}{2} S_3' + \int_S^T \int_\Omega a g(u_t + d) \nabla u . h \, dx \, dt . \quad (96) \]

We have already estimated \( S_3' \) in (93) and it remains to deal with \( X' \). We have

\[
X' \leq \frac{C}{\eta} \int_\Omega (a |g(u_t + d)|)^2 \, dx \, dt + C\eta \int_\Omega |\nabla u|^2 \, dx \, dt \\
\leq C \int_\Omega a |g(u_t + d)| \, dx \, dt + C\eta_9 \int_S^T E \, dt.
\] (97)
Now, set $R_1 > 1$ to be chosen later, We have
\[
\int_S \int_\Omega a|g(u_t + d)|^2 \, dx \, dt = \int_S \int_{|u_t + d| \leq R_1} a|g(u_t + d)|^2 \, dx \, dt + \int_S \int_{|u_t + d| > R_1} a|g(u_t + d)|^2 \, dx \, dt.
\]
Since $g(0) = 0$ we have $|g(x)| \leq C_{R_1} |x|$ for $|x| < R_1$, which gives
\[
\mathcal{Y}_1' \leq C_{R_1} \int_S \int_{|u_t + d| \leq R_1} |a(g(u_t + d)||u_t + d| \, dx \, dt
\]
\[
\leq C_{R_1} \int_S \int_{|u_t + d| \leq R_1} a(g(u_t + d)u_t \, dx \, dt + C \int_S \int_{|u_t + d| \leq R_1} a(g(u_t + d)d \, dx \, dt
\]
\[
\leq C_{R_1} \int_S \int_{|u_t + d| \leq R_1} (-E') \, dx \, dt + C_{R_1} \int_S \int_{|u_t + d| \leq R_1} |d| \, dx \, dt - C \int_S \int_\Omega u_t e \, dx \, dt
\]
\[
\leq C_{R_1} E(S) + C_{R_1} \int_S \int_\Omega |d| \, dx \, dt + C_{R_1} \eta \int_S \int_\Omega E \, dt + \frac{C_{R_1}}{\eta} C_5(e)
\]
\[
\leq C_{R_1} E(S) + C_{R_1} \eta \int_S \int_\Omega E \, dt + C_{R_1} C_{d,e}, \quad (98)
\]
with $\eta$ was chosen to be equal to $\frac{\eta_0}{C_{R_1}}$, where $0 < \eta_0 < 1$ is an arbitrary constant to be chosen later. On the other hand, we have
\[
\mathcal{Y}_2' \leq C \int_S \int_{|u_t + d| > R_1} |u_t + d|^{2q} \, dx \, dt
\]
\[
\leq C \int_S \int_{|u_t + d| > R_1} |u_t|^{2q} \, dx \, dt + \int_S \int_{|u_t + d| > R_1} |d|^{2q} \, dx \, dt
\]
\[
\leq C \int_S \int_{|u_t + d| > R_1} \frac{|u_t + d|}{R_1} |u_t|^{2q} \, dx \, dt + \int_S \int_\Omega |d|^{2q} \, dx \, dt
\]
\[
\leq C \int_S \int_\Omega \frac{|u_t|}{R_1} |u_t|^{2q} \, dx \, dt + C \int_S \int_\Omega |d|^{2q} \, dx \, dt + C_1(d)
\]
\[
\leq \frac{C}{R_1} \int_S \int_\Omega |u_t|^{2q+1} \, dx \, dt + \frac{C}{R_1^2} \int_S \int_\Omega |u_t|^{4q} \, dx \, dt + C_{d,e}.
\]
By using Lemma 3.9 as well as the fact that $R_1 > 1$, it follows that
\[
\mathcal{Y}_2' \leq \frac{C}{R_1} (C_u + C_{d,e}) \int_S \int_\Omega E \, dt + C_{d,e}.
\]
we fix $R_1 = \frac{C(C_u + C_{d,e})}{\eta_1}$, where $\eta_1$ is an arbitrary number to be chosen small enough later; we obtain
\[
\mathcal{Y}_2' \leq \eta_1 \int_S \int_\Omega E \, dt + C_{d,e}. \quad (99)
\]
**Remark 3.2** For such a choice of $R_1$, and based on how $C_{R_1}$ is defined, we can assume that $C_{R_1}$ is also a constant of the type $C_u + C_{d,e}$.

By combining (97), (98) and (99)

$$X' \leq C(\eta_9 + \eta_{10} + \eta_{11})\int_t^T E\ dt + (C_{d,e} + C_u)E(S) + C_{d,e}C_u + C_{d,e}.$$  \hspace{1cm} (100)

Finally, we combine (95) and (100) with the estimation of $S_3^r$ and we set $\sigma_3 = C(\eta_7 + \eta_9 + \eta_{10} + \eta_{11})$, we obtain (95).

We next seek an upper bound for $T'_4$. We have

$$T'_4 = \frac{1}{2} \int_t \int \int e\ dx\ dt + \int_t \int e\nabla u.h\ dx\ dt,$$  \hspace{1cm} (101)

where $0 < \eta_{12} < 1$ is an arbitrary number to be chosen later. On one hand, we have

$$\int_t \int e\ dx\ dt \leq \eta_{12} \int_t^T E\ dt + C \int_t^T \int |e|^2\ dx\ dt.$$  \hspace{1cm} (102)

On another hand, we have

$$\int_t \int e\nabla u.h\ dx\ dt \leq \eta_{13} \int_t^T E\ dt + C \int_t^T \int |e|^2\ dx\ dt.$$  \hspace{1cm} (103)

Taking $\sigma_4 = \eta_{12} + \eta_{13}$ and combining (101), (102) and (103), we obtain

$$T'_4 \leq \sigma_4 \int_t^T E\ dt + C_{d,e}.$$  \hspace{1cm} (104)

It remains to handle the last term $T'_5$. For every $R_2 > 1$, we have

$$T'_5 \leq \frac{1}{a_0} \int_t \int a(x)u_t^2\ dx\ dt \leq \int_t \int a(x)(u_t + d)^2\ dx\ dt + \int_t \int a(x)\ dx\ dt$$

$$\leq C \int_t \int [a(x)(u_t + d)^2 + C \int_t \int a(x)\ dx\ dt + C_{d,e}.$$  \hspace{1cm} (105)

On one hand, recall that there exists $\alpha_{R_2} > 0$ such that $|g(v)| \geq \alpha_{R_2}|v|$ for $|v| \leq R_2$, which gives

$$Z'_1 \leq \int_t^T \int [a(x)(u_t + d)g(u_t + d) \frac{(u_t + d)}{g(u_t + d)}\ dx\ dt$$

$$\leq \frac{R_2}{\alpha_{R_2}} \int_t \int a(x)(u_t + d)g(u_t + d)\ dx\ dt$$

$$\leq \frac{R_2}{\alpha_{R_2}} \int_t \int a(x)u_tg(u_t + d)\ dx\ dt + \frac{R_2}{\alpha_{R_2}} \int_t \int |d|\ dx\ dt$$

$$\leq \frac{R_2}{\alpha_{R_2}}\ E(S) + \frac{R_2}{\alpha_{R_2}} \eta \int_t^T E\ dt + \frac{CR_2}{\alpha_{R_2}} \eta \int_t^T \int |e|^2\ dx\ dt + \frac{R_2}{\alpha_{R_2}} \int_t^T \int |d|\ dx\ dt.$$
We choose \( \eta = \frac{\alpha R_2 \eta_4}{R_2} \), we obtain
\[
Z'_1 \leq C \frac{R_2}{\alpha R_2} E(S) + \eta_4 \int_S^T E \, dt + \frac{R_2^3}{\alpha R_2} C_{d,e} + \frac{R_2}{\alpha R_2} C_{d,e}.
\]

On another hand, we have
\[
Z'_2 \leq C \int_S^T \int_{|u_t| > R_2} |u_t|^2 \, dx \, dt + C \int_S^T \int_{|u_t| > R_2} |d|^2 \, dx \, dt
\]
\[
\leq C \int_S^T \int_{|u_t| > R_2} \frac{|u_t + d|}{R_2} |u_t|^2 \, dx \, dt + CC_1(d)
\]
\[
\leq C \int_S^T \int_{|u_t| > R_2} \frac{|u_t|^3}{R_2} \, dx \, dt + C \int_S^T \int_{|u_t| > R_2} \frac{|u_t|^2 |d|}{R_2} \, dx \, dt + CC_1(d)
\]
\[
\leq \frac{C}{R_2} \int_S^T \int_{|u_t| > R_2} |u_t|^3 \, dx \, dt + \frac{C}{R_2^2} \int_S^T \int_{|u_t| > R_2} |u_t|^4 \, dx \, dt + CC_1(d). \tag{106}
\]

By using Lemma 3.9 and the fact that \( R_2 > 1 \), one derives
\[
\frac{C}{R_2} \int_S^T \int_{|u_t| > R_2} |u_t|^3 \, dx \, dt + \frac{C}{R_2^2} \int_S^T \int_{|u_t| > R_2} |u_t|^4 \, dx \, dt \leq C \left( \frac{C_u + C_{d,e}}{R_2} \right) \int_S^T E \, dt. \tag{107}
\]

We choose \( R_2 = \frac{C(C_u + C_{d,e})}{\eta_5} \) and we combine (106) and (107) we have
\[
Z'_2 \leq \eta_5 \int_S^T E \, dt + C_{d,e}. \tag{108}
\]

**Remark 3.3** For such a choice of \( R_2 \), and based on how \( \alpha R_2 \) is defined, we can assume that \( \frac{1}{\alpha R_2} \) is also a constant of the type \( C_u + C_{d,e} \). As a result we have
\[
Z'_1 \leq (C_u + C_{d,e}) E(S) + \eta_4 \int_S^T E \, dt + C_{d,e} C_u + C_{d,e}. \tag{109}
\]

Combining (105), (108) and (109) and using (11) and (12), we obtain :
\[
T'_5 \leq (C_u + C_{d,e}) E(S) + (\eta_4 + \eta_5) \int_S^T E \, dt + C_{d,e} C_u + C_{d,e},
\]
which gives when taking \( \sigma_5 = \eta_14 + \eta_5 \)
\[
T'_5 \leq \sigma_5 \int_S^T E \, dt + (C_u + C_{d,e}) E(S) + C_{d,e} C_u + C_{d,e}. \tag{110}
\]

We can finish the proof of Proposition 3.2, we combine the estimations of \( T'_i \), \( i = 1, 2, 3, 4, 5 \), which are given by (76), (77), (85), (104) and (110) with (75), we obtain
\[
\int_S^T E \, dt \leq (\sigma_2 + \sigma_3 + \sigma_4 + \sigma_5) \int_S^T E \, dt + C_u E(S) + C_{d,e} E(S) + C_{d,e} C_u + C_{d,e}. \tag{111}
\]

We choose \( \sigma_i = \frac{1}{8}, \ i = 2, 3, 4 \) we obtain that \( \sigma_2 + \sigma_3 + \sigma_4 + \sigma_5 = \frac{1}{2} \), then we use the fact that \( C_{d,e} E(S) \leq C_{d,e} (E(0) + C_{d,e}) = C_{d,e} C_u + C_{d,e} \), we obtain (74).
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A Appendix

We next list technical results used in the core of the paper.

**Theorem A.1 Gronwall integral lemma**

Let \( E : \mathbb{R}_+ \to \mathbb{R}_+ \) satisfy, for some \( C_0, T > 0 \):

\[
\int_t^{+\infty} E(s)ds \leq TE(t) + C_0, \quad \forall \ t \geq 0.
\]

Then, the following estimates hold true

\[
\int_t^{+\infty} E(s)ds \leq TE(0)e^{-\frac{T}{t}} + 2C_0, \quad \forall \ t \geq 0.
\]

and, if in addition, \( t \mapsto E(t) \) is non increasing, one has

\[
E(t) \leq E(0)e^{1-\frac{T}{t}} + \frac{C_0}{T}, \quad \forall \ t \geq 0.
\]

The proof is classical, cf. for instance [1].

**Theorem A.2 Generalized Gronwall lemma**

Let \( F, h_1 \) and \( h_2 \) non negative functions defined on \( \mathbb{R}_+ \) satisfying

\[
\|h_1\|_1 := \int_0^{\infty} h_1(t)dt < \infty, \quad \|h_2\|_1 := \int_0^{\infty} h_2(t)dt < \infty,
\]

and

\[
F(T) \leq F(S) + C + C_1 \int_S^{T} h_1(s)F^{\alpha_1}(s)ds + C_2 \int_S^{T} h_2(s)F^{\alpha_2}(s)ds, \quad \forall \ S \leq T,
\]

where \( C, C_1, C_2 \) are positive constants and \( 0 \leq \alpha_1, \alpha_2 < 1 \). Then, \( F \) satisfies the following bound

\[
\sup_{t \in [S,T]} F(t) \leq \max \left( 2(F(S) + C), (2\bar{C})^{\frac{1}{\alpha}} \right), \quad \text{with} \quad \bar{C} := C_1\|h_1\|_1 + C_2\|h_2\|_1,
\]

where \( \alpha := \max(\alpha_1, \alpha_2) \) if \( 2\bar{C} \geq 1 \) or \( \alpha := \min(\alpha_1, \alpha_2) \) if \( 2\bar{C} < 1 \).

**Proof of Theorem A.2.** Fix \( T \geq S \geq 0 \). For \( t \in [S,T] \) set \( Y(t) \) for the right-hand side of (115) applied at the pair of times \( S \leq t \). It defines a non decreasing absolutely continuous function. Since \( F(t) \leq Y(t) \leq Y(T) \) for \( t \in [S,T] \), one deduces that \( F_T := \sup_{t \in [S,T]} F(t) \) is finite for every \( t \in [S,T] \). One gets from (115)

\[
F_T \leq F(S) + C + \bar{C} \max(F^{\alpha_1}_T, F^{\alpha_2}_T),
\]

with the notations of (116). The latter follows at once by considering whether \( F(S) + C > \bar{C} \max(F^{\alpha_1}_T, F^{\alpha_2}_T) \) or not.

We recall the following useful result, cf. for instance [14].
Theorem A.3 Gagliardo–Nirenberg interpolation inequality
Let $\Omega$ be a bounded Lipschitz domain, $1 \leq r < p \leq \infty, 1 \leq q \leq p$ and $m \geq 0$. Then the inequality

$$\|v\|_p \leq C \|v\|_{m,q}^{\theta} \|v\|_r^{1-\theta} \quad \text{for} \quad v \in W^{m,q}(\Omega) \cap L^r(\Omega).$$

(117)

holds for some constant $C > 0$ and

$$\theta = \left(\frac{1}{r} - \frac{1}{p}\right) \left(\frac{m}{N} + \frac{1}{r} - \frac{1}{q}\right)^{-1},$$

(118)

where $0 < \theta \leq 1$ (0 $< \theta < 1$ if $p = \infty$ and $mq = N$) and $\|\cdot\|_p$ denotes the usual $L^p(\Omega)$ norm and $\|\cdot\|_{m,q}$ the norm in $W^{m,q}(\Omega)$. 

