The Structure of Integral Parabolic Subgroups of Orthogonal Groups

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Introduction

Arithmetic subgroups of algebraic groups, as discrete subgroups of finite co-volume appear in many branches of mathematics. One branch in which they play an important role is in algebraic geometry, since the Baily–Borel Theorem proves when $G$ is an algebraic group over $\mathbb{Q}$ such that the symmetric space $\mathcal{D}$ of $G(\mathbb{R})$ is Hermitian and $\Gamma \subseteq G(\mathbb{R})$ is arithmetic (with respect to the $\mathbb{Q}$-structure on $G$, say), the quotient $\Gamma \backslash \mathcal{D}$ consists of the complex points of a (typically non-compact) algebraic variety. This variety can be completed (in many different ways) to a complete variety, by adding appropriate boundary components which are associated with (orbits of) parabolic subgroups of $G(\mathbb{Q})$. The precise structure of these boundary components depends, especially in the toroidal compactifications of \cite{AMRT} and others, on the subtleties of the structure of the intersection of the corresponding parabolic group with $\Gamma$.

Over $\mathbb{Q}$ (and other fields), the parabolic subgroups of $G$ always have the same structure (up to a few cases in which some of the following components become trivial): It is a semi-direct product in which a Levi subgroup, which is the product of (usually) two or three simple algebraic groups operates on a unipotent group of Heisenberg type, which a central extension of two Abelian groups. However, the Levi group, as a specific subgroup, depends on the choice of additional data. While this data can be chosen almost arbitrarily over $\mathbb{Q}$, when working over $\mathbb{Z}$ one has to be much more careful. The same applies for the definition of the Heisenberg group, which typically involves the factor $\frac{1}{2}$, a fact that causes no problem over $\mathbb{Q}$ but does require care over $\mathbb{Z}$. Two incarnations of this fact are the definition of semi-characters (rather than characters) in Section 2.2 of \cite{BL}, and the condition about vanishing of Fourier coefficients of generalized Jacobi forms following Definition 2 of \cite{W} (note that the published version contains a mistake, which is corrected in the arXiv version).

In this paper we carry out the detailed analysis of the integral parabolic subgroups in the case where $G$ is the orthogonal group of a (non-degenerate) rational quadratic space $V$. Then the arithmetic subgroup is the discriminant kernel $\Gamma_L$ of an even lattice $L$ in $V$, and the parabolic subgroups are in correspondence with isotropic subspaces $U \subseteq V$, and hence denoted by $\mathcal{P}_U$ for such
When the lattice $I = U \cap L$ has rank 1, it is well-known that the intersection $\Gamma_{L,I}$ of $\Gamma_L$ with the connected component of $\mathcal{P}_U$ becomes just a semi-direct product in which the discriminant kernel $\Gamma_K$ of $K = (I^\perp \cap L)/I$ acts on $K$ (see Nak, Bo, Br, Z1, Z2, and others), since the unipotent radical is Abelian in this case. However, when $\dim U \geq 2$ the structure is more complicated: There are restrictions on the choices of complement $\tilde{U}$ for $U^\perp$ in $V$, and after taking such a choice the group is described in Theorem 3.1 and Proposition 3.2 below.

The symmetric space of the orthogonal group is Hermitian only when the signature is $(n,2)$ (or equivalently $(2,n)$), so that isotropic subspaces can be at most 2-dimensional. The toroidal compactifications of the resulting Shimura varieties are described in some detail in [F], but the more complicated structure of the integral parabolic groups associated with 2-dimensional subspaces seems to have been overlooked there. We remark that for 1-dimensional subspaces, where the group structure is simple, the toroidal boundary components depend on choices of fans and are hence not canonical. On the other hand, the components associated with 1-dimensional spaces are canonical, but their precise description as algebraic varieties depends, in general, on the subtleties of structure of the integral parabolic group. The result, which refines the statements from [F] and [BZ], is given in Theorem 4.5.

The paper is divided into 4 sections. Section 1 presents the well-known form of the parabolic subgroups of orthogonal groups over fields, mainly to introduce the necessary notation. Section 2 shows how lattices and their duals decompose in suitably chosen coordinates. Section 3 describes the structure of the parabolic subgroups over $\mathbb{Z}$, and Section 4 applies these results for determining the form of the canonical boundary components in the toroidal compactifications of the corresponding orthogonal Shimura varieties.

1 Parabolic Subgroups of Orthogonal Groups

Let $V$ be a non-degenerate finite-dimensional quadratic space over a field $F$ of characteristic different from 2. We shall denote the image of two vectors $u$ and $v$ in $V$ under the associated bilinear form by $(u,v)$, and we shorthand $(v,v)$ to $v^2$ for every $v \in V$. The quadratic form on $V$ therefore sends $v \in V$ to $v^2$, and for a subspace $U$ of $V$ we write $U^\perp = \{ v \in V | (u,v) = 0 \ \forall u \in U \}$ as usual. Given any vector space $X$ over $F$, we write

$$X^* := \text{Hom}_F(X,F)$$

for the dual space, and we have $V^* \cong V$ via the non-degenerate bilinear form. If $U$ is an isotropic subspace of $V$ (i.e., $(u,w) = 0$ for $u$ and $w \in U$, or equivalently $U \subseteq U^\perp$), then we set

$\mathcal{P}_U$ to be the stabilizer of $U$ in $O(V) := \text{Aut}_F(V,(\cdot,\cdot))$, as well as $W = U^\perp/U$.

They are the parabolic subgroup of $O(V)$ that is associated with $U$ and a non-degenerate non-degenerate quadratic space of dimension $\dim V - 2 \dim U$ respectively. The fact that elements of $\mathcal{P}_U$ must also preserve $U^\perp$ immediately yield the following first result.
**Lemma 1.1.** The parabolic group $P_U$ comes with a natural map into the product $GL(U) \times O(W)$.

Indeed, the element $M \in GL(U)$ that is associated with $A \in P_U$ is just the restriction $A|_U$, while the image $\gamma$ of $A$ in $O(W)$ is the action of $A$ on the quotient $W = U^\perp/U$. The remaining quotient $V/U^\perp$ is naturally identified with $U^*$ via the bilinear form, and the fact that for $v \in V$ such that $v + U^\perp$ corresponds to $v^* \in U^*$ and $u \in U$ we have

$$(v^*, u) = (v, u) = (Av, Au) = (Av, Mu) = ((Av)^*, Mu) = (M^*(Av)^*, u),$$

where $M^* \in GL(U^*)$ is the map that is dual to $M$, shows that the action of $A$ on $V/U^\perp \cong U^*$ must be like that of the inverse $M^{-1} = (M^*)^{-1}$ of $M^*$.

The kernel $W_U$ of the map from Lemma 1.1 is the unipotent radical of $P_U$. In order to analyze it, we first observe that $W^* \cong W$ (since $W$ is also non-degenerate), and recall that for every map $\psi : W \to U$ there is a dual map $\psi^* : U^* \to W$. In addition, a map $\eta : U^* \to U$ is called symmetric (resp. anti-symmetric) if the bilinear form

$$(u^*, v^*) \in U^* \times U^* \mapsto (u^*, \eta v^*) \in U^* \times U \mapsto \langle u^*, \eta v^* \rangle \in \mathbb{F}$$

is symmetric (resp. anti-symmetric). We denote the space of anti-symmetric linear maps from $U^*$ to $U$ by $\text{Hom}_F^{as}(U^*, U)$.

We also recall the following definition.

**Definition 1.2.** Let $X$ be a vector space over $\mathbb{F}$ supplied with an anti-symmetric bilinear map $B$ to another vector space $Y$ over $\mathbb{F}$. Then the associated Heisenberg group to be

$$H(X, Y) := X \times Y$$

with the product rule $(x, y) \cdot (\xi, \eta) = (x + \xi, y + \eta + B(x, \xi))$.

On the other hand, if a space $Z$ carries a $Y$-valued symmetric bilinear map $(z, \zeta) \mapsto \langle z, \zeta \rangle \in Y$, then on $Z \times Z$ we have the anti-symmetric bilinear map

$$(z, w) \mapsto \langle z, \omega \rangle - \langle w, \zeta \rangle,$$  

and we denote by $\tilde{H}(Z, Y)$ the associated Heisenberg group $H(Z \times Z, Y)$.

The group $H(X, Y)$ from Definition 1.2 lies in a short exact sequence

$$0 \to Y \to H(X, Y) \to X \to 0,$$

where $Y$ is contained in the center of $H(X, Y)$. When $B$ is non-degenerate, $Y$ is the full center of $H(X, Y)$. Since $Y$ is central and $X$ is commutative, the commutator of the pair $(x, y)$ and $(\xi, \eta)$ lies in $Y$ and depends only on $x$ and $\xi$—indeed, it equals $(0, 2B(x, \xi))$. In fact, this condition determines $H(X, Y)$ as an extension of $X$ by $Y$ (at least for finite-dimensional $X$):

**Proposition 1.3.** Let $H$ be a group mapping onto the finite-dimensional space $X$ with central kernel $Y$ as in Equation 1.2, and assume that when $h$ and $k$ are elements of $H$, with $X$-images $x$ and $\xi$ respectively, then the commutator $[h, k]$ is $2B(x, \xi) \in Y$. Then $H \cong H(X, Y)$ as extensions of $X$ by $Y$. 

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Proof. Choose a basis \((u_1, \ldots, u_{\dim X})\) for \(X\), and lift them arbitrarily to \(H\). Since elements of \(H\) mapping to multiples of the same vector in \(X\) commute (by the anti-symmetry of \(B\)), our lifts generate \(\dim X\) one-parameter subgroups of \(H\), mapping to the corresponding 1-dimensional vector spaces of \(X\). Denote the resulting lift of \(cu_i\) by \(h_{cu_i}\), and for an arbitrary element

\[
x = \sum_i c_i x_i \quad \text{we define the lift } \quad h_x = h_{c_1x_1} \cdots h_{c_{\dim X}x_{\dim X}} \bigg/ \sum_{i < j} B(c_i x_i, c_j x_j)
\]

(this is well-defined by the centrality of \(Y\)). By the equivalent of Equation (11), every element of \(H\) is of the form \(h_x y\) for some \(x \in X\) and \(y \in Y\), and we claim that the map sending this element to \((x, y) \in H(X, Y)\) yields the desired isomorphism. By this formula and the centrality of \(Y\), it suffices to verify that for \(x\) and above and another element \(\xi = \sum d_i u_i \in X\), say, we have

\[
h_x \cdot h_\xi = h_{x+\xi} \cdot B(x, \xi), \quad \text{with } B(x, \xi) \in Y
\]

(then our map will already be an isomorphism as extensions). But indeed, the left hand side is

\[
h_{c_1x_1} \cdots h_{c_{\dim X}x_{\dim X}} h_{d_1x_1} \cdots h_{d_{\dim X}x_{\dim X}} \bigg/ \sum_{i < j} B(c_i x_i, c_j x_j) B(d_i x_i, d_j x_j)
\]

by the centrality of \(Y\), which equals

\[
h_{(c_1+d_1)x_1} \cdots h_{(c_{\dim X}+d_{\dim X})x_{\dim X}} \sum_{i < j} 2B(c_j x_j, d_i x_i) - B(c_i x_i, c_j x_j) B(d_i x_i, d_j x_j)
\]

by the commutation relations, and the latter expression yields the desired right hand side by the definition of \(h_{x+\xi}\) and the anti-symmetry and bi-additivity of \(B\). This proves the proposition. \(\square\)

We remark that the anti-symmetry of \(B\) and the commutation relation also imply that the form of the lift of \(h_x\) in the proof of Proposition 1.3 does not depend on the ordering of the chosen basis \((u_1, \ldots, u_{\dim X})\) or of the product we take for defining \(h_x\), as long as the ordering of the indices coincides with the order in the product. In addition, one easily verifies that in any group \(H\) lying in a short exact sequence as in Equation (1) with \(Y\) central, the commutator map factors through a map from \(X \times X\) to \(Y\), which must be bi-additive and anti-symmetric. Hence every such extension over \(\mathbb{Q}\) is a Heisenberg group (by Proposition 1.3), and the same happens for continuous extensions over \(\mathbb{R}\) or \(\mathbb{Q}_p\), but in general such extensions may involve anti-symmetric bi-additive maps that are not bilinear (e.g., semi-bilinear maps involving Galois automorphisms). Proposition 1.3 may also be proved by general arguments relating group extensions to the cohomology of one Abelian group acting trivially on another Abelian group, under appropriate conditions (in particular, the action of 2 on the second group must be invertible).

The basic structure of the group \(W_X\) can now be described.
Proposition 1.4. The unipotent radical $W_U$ is a subgroup of the Heisenberg group $H(\text{Hom}_F(W,U), \text{Hom}_F^a(U^*,U))$, where the anti-symmetric map from $\text{Hom}_F(W,U) \to \text{Hom}_F^a(U^*,U)$ sends $\psi$ and $\varphi$ in the former space to $\frac{\psi^* - \varphi^*}{2}$ in the latter. The normal subgroup $\text{Hom}_F^a(U^*,U)$ of this Heisenberg group is contained in $W_U$.

Proof. Recall that an element $A \in W_U$ acts trivially on $U$, so that the map sending $v \in V$ to $v - Av$ is well-defined from $V/U$ to $V$. Moreover, since the restriction of $A$ to $U^\perp$ operates trivially on $W = U^\perp/U$, we deduce that if $z \in U^\perp$ then $z - Az \in U$. Combining these facts yields a linear map $\psi : W \to U$ (with $\psi(w) = z - Az$ for $z \in U^\perp$ with $z + U = w$), so that $A \mapsto \psi$ is a linear map from $W_U$ to $\text{Hom}_F(W,U)$.

Now, since $A$ is an isometry with $A|_U = \text{Id}_U$, it acts trivially also on $V/U^\perp$, and therefore $Av - v \in U^\perp$ for every $v \in V$. Combining all this with the isometry property of $A$, we obtain for $v \in V$ and $z \in U^\perp$ the equality

$$0 = (Av, Az) - (v, z) = (Av - v, z) + (v, Az - z) = (Av - v, z) - (v, \psi(z + U))$$

(since $Av - v \in U^\perp$ is perpendicular to $Az - z \in U$ and using the definition of $\psi$). The fact that $Av - v \in U^\perp$ allows us to write the latter equality as

$$(Av - v, w) = (v, \psi w) \quad \text{for} \quad v \in V \quad \text{and} \quad w \in W,$$

and since the previous paragraph shows that adding an element $z \in U^\perp$ to $v$ changes $Av - v$ only by $Az - z \in U$, which does not affect the pairing with $w \in W$, we can consider in the latter equality the image of $Av - v$ in $U^\perp/U = W$ as depending only on the image of $v$ in $V/U^\perp$. Recalling that $V/U^\perp \cong U^*$ canonically, we deduce that the map from $U^*$ to $W$ in which the element of $U^*$ which corresponds to $v + U^\perp \in V/U^\perp$ for $v \in V$ is sent to $Av - v$ coincides with the map $\psi^*$ dual to $\psi$.

Next, note that if $A \in \ker (W_U \to \text{Hom}_F(W,U))$ then $A|_U = \text{Id}|_U$. Moreover, the previous paragraph shows that in this case $v - Av$ lies in $U$ for every $v \in V$ (as its image modulo $U$ equals $\psi^*(u^*)$ where $u^* \in U^*$ corresponds to $v + U^\perp \in V/U^\perp \cong U^*$, and $\psi^* = 0$), and this vector depends only on the value of $v$ in $V/U^\perp$. Identify $V/U^\perp$ with $U^*$ once again, and denote the resulting map by $\eta : U^* \to U$. We write now for $v$ and $w$ in $V$, such that $v + U^\perp$ and $w + U^\perp$ correspond to $v^*$ and $w^*$ in $U^*$ respectively, the equality

$$0 = (Av, Aw) - (v, w) = (Av - v, w) + (v, Aw - w) = -(\eta v^*, w^*) - (v^*, \eta w^*)$$

(since $Av - v$ and $Aw - w$ are in $U$, hence they are perpendicular to one another and their pairing with element of $V$ reduces to the pairing with images in $V/U^\perp \cong U^*$, and by the definition of $\eta$), which shows that $\eta$ is indeed anti-symmetric. On the other hand, it is clear from the same calculation that given $\eta \in \text{Hom}_F^a(U^*,U)$, the map sending $v \in V$ to $v - \eta v^*$ is an element of $\mathcal{P}_U$ that lies in $W_U$ and maps to the trivial element of $\text{Hom}_F(W,U)$, so that $\ker (W_U \to \text{Hom}_F(W,U))$ is precisely $\text{Hom}_F^a(U^*,U)$. 

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Let us now evaluate the commutator of two elements \(A\) and \(B\) of \(W_U\), with respective images \(\psi\) and \(\varphi\) in \(\operatorname{Hom}_F(W,U)\). For this we write, for \(v \in V\) with \(v^* \in U^*\) corresponding to \(v + U^\perp \in V/U^\perp\), the equality

\[
v - ABv = (v - Av) + (v - Bv) + (z - Az) \quad \text{for} \quad z = Bv - v \in U^\perp \quad \text{with} \quad z + U = \varphi v^*,
\]

so that the summand \(z - Az\) is just \(\psi \varphi^* v^*\). A similar calculation, but now with \(BAv\), produces the equality

\[(BA - AB)v = (\psi \varphi^* - \varphi \psi^*) v^* \quad \text{for every} \quad v \in V \quad \text{with} \quad v^* \text{ as above.}\]

Now, as \(v^*\) depends only on \(v\) modulo \(U^\perp\), and the difference \(A^{-1} B^{-1} v - v\) lies in \(U^\perp\), we can replace \(v\) by \(A^{-1} B^{-1} v\) in the left hand side of the last equality, which produces \(v - ABA^{-1} B^{-1} v\). But by definition, this is the image of \(v^*\) under the element of \(\operatorname{Hom}_F(U^*, U)\) that is associated with the commutator \(ABA^{-1} B^{-1}\), and the right hand side is twice the asserted anti-symmetric map. An application of Proposition 1.3 now completes the proof of the proposition. \(\square\)

For determining the precise structure of \(\mathcal{P}_U\), as well as of groups over \(Z\) below, we shall need a complement \(\tilde{U}\) for \(U^\perp\) in \(V\), which is not necessarily isotropic (indeed, good isotropic complements over \(Z\) need not always exist). The bilinear form and the restriction of the projection from \(U^\perp\) onto \(W\) then give canonical isomorphisms

\[
\tilde{U} \cong V/U^\perp \cong U^* \quad \text{and} \quad \tilde{W} := (U + \tilde{U})^\perp \cong W \quad \text{respectively.}
\]

Composition with the latter isomorphisms gives identifications

\[
\operatorname{Hom}_F(W,U) \cong \operatorname{Hom}_F(W,W), \quad \operatorname{O}(W) \cong \operatorname{O}(W), \quad \operatorname{Hom}_F(U^*, U) \cong \operatorname{Hom}_F(U^*, U),
\]

\[
\operatorname{Hom}_F(U^*, W) \cong \operatorname{Hom}_F(U^*, W), \quad \text{and} \quad \operatorname{End}_F(U^*) \cong \operatorname{End}_F(U),
\]

which we denote by

\[
\psi \mapsto \tilde{\psi}, \quad \gamma \mapsto \tilde{\gamma}, \quad \eta \mapsto \tilde{\eta}, \quad \psi^* \mapsto \tilde{\psi}^*, \quad \text{and} \quad M^* \mapsto \tilde{M}^* \quad \text{respectively},
\]

for \(\psi \in \operatorname{Hom}_F(W,W)\), \(\gamma \in \operatorname{O}(W)\), \(\eta \in \operatorname{Hom}_F(U^*, U)\), and \(M \in \operatorname{End}_F(U)\), with \(\psi^* \in \operatorname{Hom}_F(U^*, W)\) and \(M^* \in \operatorname{End}_F(U^*)\).

It is clear that when \(\eta\) is symmetric or anti-symmetric then so is \(\tilde{\eta}\).

The fact that \(\tilde{U}\) may be non-isotropic produces the following canonical correction homomorphism.

**Lemma 1.5.** There exists a unique symmetric map \(\alpha : \tilde{U} \to U\) such that the space \(\{\tilde{u} - \alpha \tilde{u} | \tilde{u} \in \tilde{U}\}\) is isotropic.

**Proof.** Since \(U\) is isotropic, we find that

\[(\tilde{u} - \beta \tilde{u}, \tilde{v} - \beta \tilde{v}) = (\tilde{u}, \tilde{v}) - (\beta \tilde{u}, \tilde{v}) - (\tilde{u}, \beta \tilde{v}) \quad \text{for every} \quad \beta : \tilde{U} \to U.
\]
Hence we are interested in homomorphisms $\beta \in \text{Hom}_F(\tilde{U}, U)$ for which

the equality $(\beta \tilde{u}, \tilde{v}) + (\tilde{u}, \beta \tilde{v}) = (\tilde{u}, \tilde{v})$ holds for every $\tilde{u}$ and $\tilde{v}$ in $\tilde{U}$.

Now, the canonical isomorphism $U \cong \tilde{U}^*$ and the fact that $F$ is not of characteristic 2 produce a map

$$\alpha : \tilde{U} \to U$$ such that $(\tilde{u}, \tilde{v}) = 2(\alpha \tilde{u}, \tilde{v})$ for $\tilde{u}$ and $\tilde{v}$ in $\tilde{U}$,

and it is clear that $\alpha$ is symmetric and satisfies the desired equality. Finally, if $\beta : \tilde{U} \to U$ also satisfies this equality then we get

$$\beta = \alpha + (\beta - \alpha)$$ as well as $((\beta - \alpha) \tilde{u}, \tilde{v}) + (\tilde{u}, (\beta - \alpha) \tilde{v})$ for every $\tilde{u}$ and $\tilde{v}$, so that $\beta - \alpha$ is anti-symmetric. Hence $\beta$ is symmetric if and only if $\beta = \alpha$. This proves the lemma.

The stabilizer of $U \oplus \tilde{U}$ in $P_U$ (which stabilizes also $\tilde{W}$) contains a unique Levi subgroup of that parabolic group (namely the stabilizer of $U$ and of the isotropic subspace from Lemma [1.5]), and the choice of $\tilde{U}$ allows us to investigate $W_U$ as well in more detail. This establishes the following result.

**Proposition 1.6.** The group $W_U$ is the full Heisenberg group from Proposition [1.4] and the map from Lemma [1.2] is surjective and splits. In particular, $P_U$ is isomorphic to the semi-direct product in which $\text{GL}(U) \times \text{O}(W)$ acts on the Heisenberg group $W_U$ via $(M, \gamma) : (\psi, \eta) \mapsto (M \psi \gamma^{-1}, M \eta M^*)$.

**Proof.** Consider an element $A \in W_U$, lying over to the map $\psi \in \text{Hom}_F(W, U)$ as in Proposition [1.4] with the associated map $\tilde{\psi} \in \text{Hom}_F(\tilde{W}, U)$. The proof of that proposition shows that

$$A \big|_U = \text{Id}_U,$$ that for $w \in \tilde{W} \cong W$ we have $Aw = w - \tilde{\psi} w$,

and that there is a map $\xi \in \text{Hom}_F(U^*, U)$ such that

$$A\tilde{u} = \tilde{u} + \tilde{\psi}^* \tilde{u} - \xi \tilde{u}$$ for any $\tilde{u} \in \tilde{U}$, where $\tilde{\psi} \in \text{Hom}_F(\tilde{U}, U)$ is associated with $\xi$.

Since $\tilde{W}$ is perpendicular to $U$ and to $\tilde{U}$ and $U$ is isotropic, the fact that $A$ is an isometry implies that

$$0 = (\tilde{u}, \tilde{v}) - (A\tilde{u}, A\tilde{v}) = (\tilde{\xi} \tilde{u}, \tilde{v}) + (\tilde{u}, \tilde{\xi} \tilde{v}) - (\tilde{\psi}^* \tilde{u}, \tilde{\psi}^* \tilde{v})$$ for $\tilde{u}$ and $\tilde{v}$ in $\tilde{U}$,

so that after applying the isomorphism $\tilde{U} \cong U^*$ we get

$$(\xi u^*, v^*) + (u^*, \xi v^*) = (\psi^* u^*, \psi^* v^*) = (\psi \psi^* u^*, v^*)$$ for every $u^*$ and $v^*$ in $U^*$.

It follows that $\xi$ is the sum of the symmetric map $\frac{\psi \psi^*}{2}$ and an anti-symmetric map from $U^*$ to $U$. On the other hand, the same calculation shows that given any $\psi \in \text{Hom}_F(W, U)$, one can take $\xi$ to be $\frac{\psi \psi^*}{2}$ plus an arbitrary element of
\( \text{Hom}_U^a(U^*, U) \), and the map \( A \) defined by these formulae will be in \( O(V) \), hence in \( P_U \) and in \( W_U \), and will lie over \( \psi \), proving the first assertion.

For the second one, choose a pair of element \( M \in \text{GL}(U) \) and \( \gamma \in O(W) \), and we wish to construct an element \( A \in P_U \), that stabilizes \( U \oplus \tilde{U} \) and \( \tilde{W} \), and whose image in \( \text{GL}(U) \times O(W) \) is \( (M, \gamma) \). It is clear that we must take \( A|_U = M \) and \( A|_{\tilde{W}} = \gamma \), and using the map \( \alpha \) from Lemma 1.5 we set

\[
A\tilde{u} = \tilde{M}^{-1}\tilde{u} + M\alpha\tilde{u} - \alpha\tilde{M}^{-1}\tilde{u} \in U \oplus \tilde{U} \quad \text{for} \quad \tilde{u} \in \tilde{U}, \quad \text{where} \quad \tilde{M}^{-1} = (\tilde{M}^*)^{-1}
\]

(i.e., \( A \) takes the element \( \tilde{u} - \alpha\tilde{u} \) of the space from that lemma to its natural \( \tilde{M}^{-1} \)-image \( \tilde{M}^{-1}\tilde{u} \) in that space, to which we must add the image \( Ao\tilde{u} = M\alpha\tilde{u} \) of \( \alpha\tilde{u} \in U \) under \( A \). The fact that \( A \) thus defined lies in \( O(V) \) follows from the perpendicularity of \( \tilde{W} \) and \( U \oplus \tilde{U} \), the orthogonality of \( \gamma \), the isotropy of \( U \), and the fact that for \( \tilde{u} \) and \( \tilde{v} \) the pairing \( \langle A\tilde{u}, A\tilde{v} \rangle \) is the sum of

\[
(\tilde{M}^{-1}\tilde{u}, \tilde{M}^{-1}\tilde{v}), \quad (\tilde{M}^{-1}\tilde{u}, M\alpha\tilde{v}) = \frac{(\tilde{u}, \tilde{v})}{2}, \quad -(\tilde{M}^{-1}\tilde{u}, M\alpha\tilde{v}) = -\frac{(\tilde{M}^{-1}\tilde{u}, \tilde{M}^{-1}\tilde{v})}{2},
\]

\[
(M\alpha\tilde{u}, \tilde{M}^{-1}\tilde{v}) = \frac{(\tilde{u}, \tilde{v})}{2}, \quad \text{and} \quad -(\alpha\tilde{M}^{-1}\tilde{u}, \tilde{M}^{-1}\tilde{v}) = -\frac{(\tilde{M}^{-1}\tilde{u}, \tilde{M}^{-1}\tilde{v})}{2}
\]

by the proof of Lemma 1.5 and the isomorphism between \( \tilde{U} \) and \( U^* \). It is now immediate that \( A \) lies in \( P_U \), stabilizes \( U \oplus \tilde{U} \) and \( \tilde{W} \), and maps to \( (M, \gamma) \). It is fairly easy to verify that the map sending \( (M, \gamma) \) to \( A \) thus defined is an (injective) homomorphism of groups, whose image is the required Levi subgroup (this subgroup is, in fact, the stabilizer in \( P_U \) of the space from Lemma 1.5). This determines \( P_U \) as a semi-direct product, and a direct evaluation of the action by conjugation gives, using the fact that \( \gamma \in O(W) \), the asserted formula. This completes the proof of the proposition.

It follows from Proposition 1.6 that the choice of \( \tilde{U} \) defines a bijection of sets

\[
A \in P_U \leftrightarrow (M, \gamma, \psi, \eta) \in \text{GL}(U) \times O(W) \times \text{Hom}_F(W, U) \times \text{Hom}_U^a(U^*, U),
\]

where given \( u \in U \), \( w \in \tilde{W} \), and \( \tilde{u} \in \tilde{U} \), the element \( A \) on the left hand side satisfies

\[
Au = Mu \in U, \quad Aw = (\tilde{\gamma}w) - (\tilde{\psi}\tilde{\gamma}w) \in \tilde{W} \oplus U,
\]

and

\[
A\tilde{u} = (\tilde{M}^{-1}\tilde{u}) + (\psi*\tilde{M}^{-1}\tilde{u}) + (M\alpha\tilde{u} - \alpha\tilde{M}^{-1}\tilde{u} - \frac{\tilde{\psi}\tilde{m}^{-1}\tilde{u}}{2} - \eta\tilde{M}^{-1}\tilde{u}) \in U \oplus \tilde{W} \oplus U.
\]

Note that we have used the convention in which the element of the Levi subgroup from the proof of Proposition 1.6 operates first, and the one from \( W_U \) acts later. In the opposite order convention we have to replace \( \psi \) and \( \eta \) in Equation (2) by their images under the action of the Levi element, described in that proposition.

Considering Equation (2) in the natural quotients arising from \( U \) yields the following consequence.
Corollary 1.7. The action of \( P_U \) on the space \( V/U \perp \cong U^* \) is via (the dual of) the \( GL(U) \)-part of the quotient \( P_U/\mathcal{W}_U \) (or of the Levi subgroup). Its action on \( V/U \) is via the semi-direct product in which \( GL(U) \times O(W) \) acts on \( Hom_\mathbb{F}(W,U) \) via \((M,\gamma) : \psi \mapsto M\psi \gamma^{-1}\).

In particular, the kernel of the two actions of \( P_U \) appearing in Corollary 1.7 are the semi-direct product of \( O(W) \) on \( \mathcal{W}_U \) and the part \( Hom_\mathbb{F}^a(U^*,U) \) of \( \mathcal{W}_U \) respectively. We shall henceforth denote the finer quotient, namely \( \mathcal{P}_U/\mathcal{Hom}_\mathbb{F}^a(U^*,U) \), by \( \mathcal{P}_U \). The stabilizer of the direct sum \( U \oplus \bar{U} \) becomes the full group in the coarser quotient, and coincides with the Levi subgroup in the finer one.

It is standard to show that when \( \dim U \geq 2 \) the anti-symmetric map from Proposition 1.4 is non-degenerate. On the other hand, if \( \dim U = 1 \) then it is very degenerate, since \( Hom_\mathbb{F}^a(U^*,U) = 0 \) in this case (there are no anti-symmetric bilinear forms on a 1-dimensional space), and we then have \( \mathcal{P}_U = \mathcal{P}_U \).

Moreover, since in this case a choice of generator for \( U \) yields isomorphisms
\[
Hom_\mathbb{F}(W,U) \cong W^* \cong W, \quad \text{and} \quad GL(U) = \mathbb{F}^\times \quad \text{by definition when} \quad \dim U = 1,
\]
this special case of Proposition 1.6 yields the following result.

Corollary 1.8. The stabilizer of a 1-dimensional isotropic subspace \( U \) of \( V \) is the semi-direct product in which the product \( \mathbb{F}^\times \times O(W) \) operates on the additive group of \( W \) via \((e,\gamma) : w \mapsto e\gamma w \). It operates faithfully on the finer quotient \( V/U \) from Corollary 1.7 while the action on the 1-dimensional coarser quotient \( V/U \perp \) is only via the scalar part \( \mathbb{F}^\times \).

Note that replacing the choice of the generator of \( U \) that we used for identifying \( Hom_\mathbb{F}(W,U) \) with \( W \) in Corollary 1.8 corresponds to a scalar rescaling of \( W \), which does not affect the form of the semi-direct product there.

We also get a small simplification of Proposition 1.6 in case \( \dim U = 2 \), since the space \( Hom_\mathbb{F}^a(U^*,U) \), which is \( \bigwedge^2 U \) in the notation of [L], has dimension 1. Choosing a basis for \( U \), hence the dual basis for \( U^* \), yields a generator for \( Hom_\mathbb{F}^a(U^*,U) \), and we get isomorphisms
\[
GL(U) \cong GL_2(\mathbb{F}), \quad Hom_\mathbb{F}^a(U^*,U) \cong \mathbb{F}, \quad \text{and} \quad Hom_\mathbb{F}(W,U) \cong W^* \times W^* \cong W \times W.
\]

One verifies that the map from the latter space to \( Hom_\mathbb{F}^a(U^*,U) \cong \mathbb{F} \) is the anti-symmetrization of the pairing on \( W \), and as \( GL(U) \) has a natural action on \( Hom_\mathbb{F}^a(U^*,U) = \bigwedge^2 U = \det U \), the isomorphisms from Equation 3 expresses Proposition 1.6 as follows.

Corollary 1.9. When \( \dim U = 2 \) the Heisenberg group from Proposition 1.4 is isomorphic to the group \( \tilde{H}(W,F) \) from Definition 1.2. In \( \mathcal{P}_U \) it is acted upon by \( GL_2(F) \times O(W) \), where the former part acts on the part \( W \times W \) of \( \tilde{H}(W,F) \) as on (\( W \)-valued) length 2 vectors, while on \( F \) it is via the determinant.

There are two natural maps from \( O(V) \): One is the determinant to \( \{ \pm 1 \} \), and the other one is the spinor norm into \( \mathbb{F}^\times / (\mathbb{F}^\times)^2 \). Their restrictions to \( \mathcal{P}_U \) is evaluated as follows.
Proposition 1.10. Both the determinant and the spinor norm of elements of $\mathcal{W}_U$ are trivial, hence these maps factor through the quotient $\text{GL}(U) \times \text{O}(W)$. If an element of $\mathcal{P}_U$ maps to a pair $(M, \gamma)$ in that product, then its determinant is just $\det \gamma$, and its spinor norm is the product of the image of $\det M$ in $\mathbb{F}^\times / (\mathbb{F}^\times)^2$ with the spinor norm of $\gamma$.

Indeed, the triviality of both maps on $\mathcal{W}_U$ is easily verified (as a unipotent group), and if we identify the quotient with the Levi subgroup from Proposition 1.10 then up to the correction factor involving $\alpha$, an element of that Levi factor operates as $M$ on $U$, as $\gamma$ on $\tilde{W}$, and as $M^{-1}$ on $U^*$. The value of the determinant immediately follows, and for the spinor norm we just note that if $U = \mathbb{F}u$ is 1-dimensional and $\tilde{u} \in \tilde{U}$ satisfies $(\tilde{u}, u) = 1$ then the map acting on $U$ as the scalar $c$ and on the isotropic counterpart of $\tilde{U}$ from Lemma 1.5 as $\frac{1}{c}$ is the composition of the reflection in a vector $\tilde{u} - \alpha \tilde{u} - cu$ composed with the reflection in $\tilde{u} - \alpha \tilde{u}$, this gives the asserted spinor norm, and easily extends to $\tilde{U}$ of any dimension.

When $F$ is a subfield of $\mathbb{R}$ and we extend scalars to $\mathbb{R}$ if necessary, the spinor norm also becomes $\{\pm 1\}$-valued. Then for indefinite $V$, of some signature $(b_+, b_-)$, the group $\text{O}(V)$ has four connected components, the kernel $\text{SO}(V)$ of the determinant consists of two of them, and the identity component is denoted by $\text{SO}^+(V)$. When $V$ is definite we have $\text{SO}(V) = \text{SO}^-(V)$, and $\text{O}(V)$ has only two connected components. Recall that in this case the dimension of any isotropic subspace $U$ of $V$ lies between 0 and $\min\{b_+, b_-\}$, and when $\dim U = 0$ we have $\mathcal{P}_U = \text{O}(V)$. Since $\text{GL}(U)$ has two connected components wherever $U$ is non-trivial, the surjectivity of the map from Lemma 1.3 (proved in Proposition 1.10) combines with Proposition 1.10 and the connectivity of $\mathcal{W}_U$ to show that when $U$ is non-trivial and $W = U^\perp / U$, of signature $(b_+ - \dim U, b_- - \dim U)$, is indefinite (i.e., when $0 < \dim U < \min\{b_+, b_-\}$), the group $\mathcal{P}_U$ has eight connected components. For non-trivial $U$ of maximal dimension $\min\{b_+, b_-\}$ (hence definite $W$), this group has four connected components when $W$ is non-trivial (i.e., when $b_+ \neq b_-)$, and only two in case $\dim U = b_+ = b_- > 0$ and $W = \{0\}$. The intersection $\mathcal{P}_U \cap \text{SO}(V)$ always contain half of these connected components, and the connected components of $\mathcal{P}_U \cap \text{SO}^+(V)$ as well as of the intersection of $\mathcal{P}_U$ with the kernel $\text{O}^+(V)$ of the spinor norm on $\text{O}(V)$ alone can be easily determined (the latter will depend on the signature of $W$ when this space is definite and non-trivial).

We also note that Corollary 1.8 with $F = \mathbb{R}$ reproduces, when combined with Proposition 1.10 the result from [Nak] and [Z2] that the connected component of the stabilizer of the (oriented) line $\ell$, which is the stabilizer in $\text{SO}^+(V)$ of a ray in $\ell$, is the semi-direct product in which $\mathbb{R}^*_+ \times \text{SO}^+(W)$ operates on the additive group of the vector space $W = \ell^\perp / \ell$. If $W$ is indefinite then the stabilizer of $\ell$ in $\text{SO}^+(V)$ contains also elements inverting the orientation of $\ell$, provided that the corresponding element of $\text{O}(W)$ (or $\text{SO}(W)$) does not lie in $\text{SO}^+(W)$, in correspondence with Proposition 1.10 (examples for such elements are those denoted by $k_{\alpha, A}$ in [Nak] and in [Z2], with $a < 0$).
2 Lattices

From now on we consider the case \( F = \mathbb{Q} \), and we take \( L \) to be an even lattice in \( V \). This means that \( L \) is a finitely generated subgroup of full rank in \( V \), with \( (\lambda, \lambda) \in 2\mathbb{Z} \) when \( \lambda \in L \), hence \( (\lambda, \mu) \in \mathbb{Z} \) for \( \lambda \) and \( \mu \) in \( L \), and \( V = L_\mathbb{Q} \).

The notation for duals for lattices will mean the \( \mathbb{Z} \)-dual, and in particular \( L^* := \text{Hom}(L, \mathbb{Z}) \subseteq V^* \) is identified with \( \{ \nu \in V \mid (\nu, L) \subseteq \mathbb{Z} \} \subseteq V \).

We shall henceforth denote this subgroup of \( V \) by \( L^* \) as well, and we have \( L \subseteq L^* \), with the discriminant group \( \Delta_L := L^*/L \) of \( L \) being finite.

Since \( L \) is even, the quadratic form \( \lambda \mapsto \frac{\lambda^2}{2} \) yields a \( \mathbb{Q}/\mathbb{Z} \)-valued quadratic form, which we shall also denote by \( \mu \mapsto \frac{\mu^2}{2} \), in addition to the natural \( \mathbb{Q}/\mathbb{Z} \)-valued bilinear form, denoted by \( (\mu, \nu) \) as well. Since elements of \( O(V) \) (or of \( O(V_{\mathbb{R}}) \)) that preserve \( L \) must also preserve \( L^* \) hence act on \( \Delta_L \) (preserving the \( \mathbb{Q}/\mathbb{Z} \)-quadratic structure), we obtain a map \( \text{Aut}(L) \to \text{Aut}(\Delta_L) \), and we set \( \Gamma_L := \ker (\text{Aut}(L) \to \text{Aut}(\Delta_L)) \cap \text{SO}^+(V_{\mathbb{R}}) \).

The arithmetic subgroup \( \Gamma_L \) has better integral properties than \( \text{Aut}(L) \) itself (in particular it is more functorial and it is more adapted to theta lifts).

The isotropic subspaces of \( V \) are in one-to-one correspondence with primitive isotropic sublattices of \( L \), under the natural inverse maps \( (U \subseteq V \text{ isotropic}) \mapsto I = U \cap L \) and \( (I \subseteq L \text{ primitive isotropic}) \mapsto U = I_\mathbb{Q} \).

For \( I \) and \( U \) related in this way, we set \( P^0_{U_{\mathbb{R}}} \) to be the identity component of \( P_{U_{\mathbb{R}}} \), and \( \Gamma_{L,I} := P^0_{U_{\mathbb{R}}} \cap \Gamma_L \) (this is the same as \( P^0_U \cap \Gamma_L \) for \( P^0_U := P_U \cap P^0_{U_{\mathbb{R}}} \), and it typically has index 2 inside \( P_U \cap \Gamma_L \)). For analyzing it we shall require the following notion, in which we recall the difference in meaning between \( X^* \) for a vector space \( X \) and \( \Lambda^* \) for a lattice \( \Lambda \).

**Definition 2.1.** Let \( X \) and \( \tilde{X} \) be subspaces of \( V \) on which the restriction of the pairing yields a non-degenerate bilinear form on \( X \times \tilde{X} \), so that it identifies \( \tilde{X} \) with \( X^* \) (and equivalently \( X \) with \( \tilde{X}^* \)), and let \( \Lambda \subseteq X \) and \( \tilde{\Lambda} \subseteq \tilde{X} \) be lattices. We say that the pairing between \( \Lambda \) and \( \tilde{\Lambda} \) is *unimodular* if \( (\lambda, \tilde{\lambda}) \in \mathbb{Z} \) for every \( \lambda \in \Lambda \) and \( \tilde{\lambda} \in \tilde{\Lambda} \), and such that the resulting map from \( \tilde{\Lambda} \) to \( \Lambda^* \), or equivalently from \( \Lambda \) to \( \tilde{\Lambda}^* \), is an isomorphism of Abelian groups.

In particular, the pairing between \( L \) and \( L^* \subseteq V \) is unimodular.

Given isotropic \( U = I_\mathbb{Q} \subseteq V \) and \( I = U \cap L \subseteq \) as above, we denote by \( I_{L^*} := U \cap L^* \), \( I_{L^*}^\perp := I_{L^*}^\perp \cap L = U^\perp \cap L \), and \( I_{L^*}^\perp^* := I_{L^*}^\perp \cap L^* = U^\perp \cap L^* \).
where $I \subseteq I_{L^*}$ and $I_{L^*} \subseteq I_{L^*}$, with finite indices. The fact that $I$ and $I_{L^*}$ are primitive in $L$ and of $I_{L^*}$ and $I_{L^*}$, are primitive in $L^*$ (by definition) and the unimodularity of the pairing between $L$ and $L^*$ imply the natural identifications

$I^* = L^*/I_{L^*}^*$, \quad (I_{L^*})^* = L/I_{L^*}^*$, \quad (I_{L^*})^* = L^*/I_{L^*}^*$, and $(I_{L^*})^* = L/I^*$ over $\mathbb{Z}$.

It follows that

$\Lambda := I_{L^*}^*/I \subseteq W = U^*/U$ is an even lattice, and $\Lambda^* \subseteq W^* = W$ is given by

$\Lambda^* = \{ \xi : I_{L^*}^* \to \mathbb{Z}[\xi, I] = 0 \} = (I_{L^*}^*)^* \cap I = (L^*/I_{L^*}) \cap I = I_{L^*}/I_{L^*}$.

As expected we get

$\Lambda \subseteq \Lambda^*$ with finite index, and we set \quad $\Delta_\Lambda = \Lambda^* / \Lambda$ \quad and \quad $p : \Lambda^* \to \Delta_\Lambda$.

It will also be useful to consider

$L^*_I = \{ \mu \in L^* | \exists \nu \in L, \forall \lambda \in I, (\mu, \lambda) = (\nu, \lambda) \} = L + I_{L^*}^* \subseteq L^*$, \quad (4)

from which we deduce that

$L^*/L^*_I \cong L^*/I^*_{L^*} = L^*/I_{L^*}^* = (L + I_{L^*}^*)/I_{L^*}^* \cong L^*/I_{L^*}^* = L^*/I^*$,

and we conclude from

$\Delta_L = L^*/L$ \quad and \quad $H^\perp_I := I_{L^*}^*/I_{L^*}^* = L^*/L$ \quad that \quad $[\Delta_L : H^\perp_I] = [I_{L^*}^* : I]$.

Since one can also express $\Delta_\Lambda = \Lambda^* / \Lambda$ as

$I_{L^*}^*/I_{L^*}^* / I_{L^*}^*/I_{L^*}^* \cong I_{L^*}^*/I_{L^*}^* \cong I_{L^*}^*/I_{L^*}^* / I_{L^*}^*/I^*$,

we find that

$\Delta_\Lambda \cong H^\perp/I_H$ \quad with \quad $H_I = (L + I_{L^*})/L \subseteq \Delta_L$ isotropic with $H_I \cong I_{L^*}^*/I$,

and $H^\perp_I$ is the subgroup of $\Delta_L$ that is $\mathbb{Q}/\mathbb{Z}$-perpendicular to $H_I$ (whence the notation).

For giving good coordinates for $\mathcal{P}_U$ (as in, e.g., Equation (2)), we required a complementary subspace $\tilde{U}$ for $U^\perp$ in $V$, but here we shall need it to be complementary over $\mathbb{Z}$, as defined in the following lemma.

**Lemma 2.2.** Let $U$ be an isotropic subspace of $V$, and set $I = U \cap L$. Then there exists a sublattice $\tilde{I}$ of $L^*$ whose pairing with $I$ is unimodular in the sense of Definition 2.1. Such a sublattice is primitive in $L^*$.

**Proof.** The fact that $I$ is primitive in $L$ means that $L = I \oplus J$ for some subgroup $J$ of $L$ (not necessarily orthogonal to $I$), implying that $L^* = \text{Hom}(L, \mathbb{Z})$ is isomorphic to $I^* \oplus J^*$. Considering $L^*$ as a subgroup of $V$, the part corresponding to $I^*$ becomes the required sublattice $\tilde{I}$. Since $U^* = \text{Hom}(I, \mathbb{Q})$ is isomorphic to $\tilde{U} = \tilde{I}_0$, and the unimodularity from Definition 2.1 implies that elements of $U^*$ that have integral pairing with all of $I$ already lie in $I^*$, the primitivity immediately follows. This proves the lemma. \qed
We shall need only complements $\tilde{U}$ that are of the form $\tilde{I}_Q$ for a sublattice $\tilde{I}$ satisfying the condition from Lemma 2.2 (this is the reason why we do not take $\tilde{U}$ to be isotropic, since isotropic $I$ with this property need not always exist).

We shall also denote

$$\tilde{I}_L = \{ \mu \in \tilde{I} \mid \exists \nu \in L, \ \forall \lambda \in I, \ \langle \mu, \lambda \rangle = \langle \nu, \lambda \rangle \} = L^*_L \cap \tilde{I},$$  

(5)

which is a primitive sublattice of $L^*_L$. Note that $\tilde{I}_L$ is the subgroup of $\tilde{I}$ that pairs in a unimodular manner with $I_L^*$, so that $(\tilde{I}_L)^* \cong I_L^*$. However, we shall make use of the $\mathbb{Q}/\mathbb{Z}$-dual of $\tilde{I}_L$, which is

$$\text{Hom}(\tilde{I}_L, \mathbb{Q}/\mathbb{Z}) = \text{Hom}(\tilde{I}_L, \mathbb{Q})/\text{Hom}(\tilde{I}_L, \mathbb{Z}) = \tilde{U}^*/(\tilde{I}_L)^* \cong U/I_L.$$

because $(\tilde{I}_L)_Q = \tilde{I}_Q = \tilde{U}$ and $\tilde{U}^* \cong U$. Recall that the inverse of the natural projection yields an isomorphism $W \cong \tilde{W} = (U \oplus \tilde{U})^\perp$, and denote the images of $\Lambda \subseteq \Lambda^* \subseteq W$ by $\hat{\Lambda} \subseteq \hat{\Lambda}^* \subseteq \tilde{W}$. It follows that

$$\hat{\Lambda}^*/\hat{\Lambda} \cong \Lambda^*/\Lambda = \Delta_{\Lambda}, \quad \text{and we denote the projection } \hat{\Lambda}^* \rightarrow \Delta_{\Lambda} \text{ by } \hat{p}.$$

Analyzing $\Gamma_{L,I}$ will require the decompositions of $L$ and $L^*$ according to the splitting of $\bar{V}$ as $U \oplus \tilde{W} \oplus \tilde{U}$. The following lemma does this also for $L^*_L$.

**Proposition 2.3.** There exist a homomorphism

$$\iota : \tilde{I}_L \rightarrow \Delta_{\Lambda}, \quad \text{which satisfies } \frac{(\iota \tilde{u})^2}{2} = \frac{\tilde{u}^2}{2} + \mathbb{Z} \in \mathbb{Q}/\mathbb{Z} \text{ for every } \tilde{u} \in \tilde{I}_L,$$

such that for $u \in U$, $w \in \tilde{W}$, and $\tilde{u} \in \tilde{U}$ the sum $u + w + \tilde{u}$ is in $L$ if and only if

$$\tilde{u} \in \tilde{I}_L, \quad w \in \Lambda^* \text{ with } \tilde{p}w = w + \Lambda = \iota \tilde{u} \in \Delta_{\Lambda}, \quad \text{and } u \in -2\alpha \tilde{u} + I, \quad (6)$$

where $\alpha$ is the map from Lemma 2.2. On the other hand, given $u$, $w$, and $\tilde{u}$ as above, the $u + w + \tilde{u}$ lies in $L^*$ (resp. $L^*_L$) if and only if

$$w \in \Lambda^*, \quad u + I_{L^*} = -\iota^*(w + \Lambda) = -\iota^* \tilde{p}w, \quad \text{and } \tilde{u} \in \tilde{I} \text{ (resp. } \tilde{u} \in \tilde{I}_L), \quad (7)$$

where $\iota^* : \Delta_{\Lambda} \rightarrow U/I_{L^*}$ is the $\mathbb{Q}/\mathbb{Z}$-dual of $\iota$.

**Proof.** The unimodularity of the pairing of $\tilde{I}$ from Lemma 2.2 with $I$, the definition of $L^*_L$ and $\tilde{I}_L$ in Equations (1) and (5) respectively, and the fact that $I_{L^*} \subseteq L^*_L \subseteq L^*$, combine to show that

$$L^* = I_{L^*} \oplus \tilde{I} \quad \text{and} \quad L^*_L = I^*_L \oplus \tilde{I}_L.$$

This reduces the proof of Equation (7) to the determination of the decomposition of $I_{L^*}$ inside $U \oplus \tilde{W}$. On the other hand, we recall that

$$0 \rightarrow I \rightarrow I^*_L \rightarrow \Lambda \rightarrow 0 \text{ is exact, and that } (I_{L^*}^*, \tilde{I}) \subseteq \mathbb{Z} \quad \text{and } \tilde{I}^* \cong I,$$
from which it follows via the definition of $\tilde{W}$ that

$$L = I \oplus (L \cap \tilde{I}^\perp) \quad \text{and} \quad I_L^\perp = I \oplus (L \cap \tilde{W}), \quad \text{and therefore} \quad I_L^\perp = I \oplus \hat{\Lambda}. \quad (8)$$

Since the pairing of $I_L^\perp$ with $I_L^\perp$ is integral, and an element of $U \oplus \tilde{W}$ lies in $U \oplus \hat{\Lambda}^*$ if and only if it pairs integrally with $I \oplus \hat{\Lambda}$, we find that

$$I_L^\perp \subset U \oplus \hat{\Lambda}^*, \quad \text{and} \quad I_L^\perp \cap (U \oplus \hat{\Lambda}) = (I_L^\perp \cap U) \oplus I_L^\perp = I_L^\perp \cap \hat{\Lambda} = I_L^\perp \cap \hat{\Lambda}.$$  

Recalling that $I_L^\perp$ projects onto $\Lambda^*$, we deduce the existence of a homomorphism $\iota : \Delta_\Lambda \to U/I_L^\perp$ such that if $u \in U$ and $w \in \hat{\Lambda}^*$ then

$$u + w \in I_L^\perp \quad \iff \quad u + I_L^\perp = -i\hat{w} \in U/I_L^\perp, \quad \text{with} \quad \hat{w} = w + \hat{\Lambda} \in \Delta_\Lambda.$$  

The proof of Equation (7) thus reduces to finding the homomorphism $\iota$ for Equation (6), and showing that $\iota = \iota^*$.  

The definitions of $L_I^\perp$ and $I_L^\perp$ and our description of $I_L^\perp$ now imply that

$$L \subset L_I^\perp \subset U \oplus \Lambda^* \oplus \tilde{I}_L^\perp, \quad \text{with surjective projection onto } \tilde{I}_L,$$

and Equation (8) shows that $L \cap (U \oplus \Lambda^*) = I_L^\perp = I \oplus \hat{\Lambda}$. It follows that there are homomorphisms $\iota : \tilde{I}_L \to \Delta_\Lambda$ and $\hat{\iota} : \tilde{I}_L \to U/I$ such that for a triple $u \in U$, $w \in \Lambda^*$, and $\hat{w} \in \hat{U}$ we have $u + w + \hat{w} \in L$ if and only if

$$\hat{w} \in \tilde{I}_L, \quad w \in \Lambda^* \text{ with } \hat{w} = w + \hat{\Lambda} = \iota \hat{w} \in \Delta_\Lambda, \quad \text{and} \quad u + I = -\hat{\iota} \hat{w},$$

which is almost Equation (6). The inclusion $(L, I_L^\perp) \subset \mathbb{Z}$ and the description of the latter lattice using $\iota$ now easily imply (using the isotropy of $U$ and the perpendicularity of $\tilde{W}$ and $\tilde{U}$) the equality $\iota = \iota^*$, thus establishing Equation (7). For proving Equation (6) we only need to show that if $\hat{u} \in \tilde{I}_L$ and $\hat{\alpha}$ is the map from Lemma (5) then $\hat{\alpha} \hat{u}$ is the coset $2\alpha u + I$ in $U/I$. The decomposition of $L$ in Equation (8) and the fact that we require the image of $u$ modulo $I$ allows us to restrict attention to $u + w + \hat{u} \in (L \cap \tilde{I}^\perp)$, whose pairing with $\hat{v} \in \hat{U}$ vanishes. As $\hat{v} \in \hat{U}$ is perpendicular to $\tilde{W}$ by definition, we obtain from the proof of Lemma (5) that

$$(u + \hat{u}, \hat{v}) = 0 \quad \text{hence} \quad (-u, v) = (\hat{u}, \hat{v}) = 2(\alpha \hat{u}, \hat{v}) \quad \text{for} \quad \hat{v} \in \hat{U}, \quad \text{and thus} \quad u = -2\alpha \hat{u}$$

(since $u$ and $2\alpha \hat{u}$ are in $U$ and $U \cong \hat{U}^*$). This implies the desired relation $\hat{\alpha} \hat{u} = -u + I = 2\alpha u + I$ between $\hat{\alpha}$ and $\alpha$, and Equation (6) follows.

It only remains to prove the norm property of $\iota$. For this, take $u$, $w$, and $\hat{u}$ as in Equation (6) (so that in particular $u = -2\alpha \hat{u} + v$ for some $v \in I$), and recall that $L$ is an even lattice. It follows that

$$\iota(u + w + \hat{u})^2 = \frac{w^2 + \hat{u}^2}{2} + (\hat{u}, u) = \frac{w^2 + \hat{u}^2}{2} - 2(\hat{u}, \alpha \hat{u}) + (\hat{u}, v) = \frac{w^2 - \hat{u}^2}{2} + (\hat{u}, v) \in \mathbb{Z}$$

(using Lemma (5)), and since $(I, \tilde{I}_L) \subset \mathbb{Z}$, we may ignore the last summand. The fact that when $w \in \hat{\Lambda}^*$ the class of $\frac{w^2}{2}$ in $\mathbb{Q}/\mathbb{Z}$ depends only on $w + \hat{\Lambda} \in \Delta_\Lambda$, and this coset is $\hat{u} \hat{u}$ by Equation (6), thus proves the desired norm condition. This completes the proof of the proposition. \qed
Indeed, in this case they take the form

$$L^* = I_{L^*} \ast \Lambda^* \ast \tilde{I}, \quad L_I^* = I_{L^*} \ast \Lambda^* \ast \tilde{I}_L, \quad \text{and} \quad L \subseteq U \ast \Lambda \ast \tilde{I}_L \quad (9)$$

with only the \( \alpha \)-corrections in the \( U \)-coordinates of elements of \( L \) (which are therefore contained in \( I_{L^*} \)). In general, for a given choice of \( U \) (and \( I \)) there are many possible choices for the complement \( \tilde{U} \) (or \( \tilde{I} \)) satisfying the condition of Lemma 2.2. In order to compare the resulting maps from Proposition 2.3 we forget the norm condition, and consider \( \iota \) to be defined on \( (I_{L^*})^* \) (whose definition does not depend on \( \tilde{U} \)), and since this group is contained in \( I^* \) we have a restriction map

$$\text{Res}_{(I_{L^*})^*}^I : \text{Hom}(I^*, \Delta_{\Lambda}) \to \text{Hom}((I_{L^*})^*, \Delta_{\Lambda}).$$

The behavior of \( \iota \) under changing the choice of \( \tilde{U} \) is as follows.

**Corollary 2.4.** Replacing \( \tilde{U} \) and \( \tilde{I} \) by another complement as in Lemma 2.2 changes the map \( \iota : (I_{L^*})^* \to \Delta_{\Lambda} \) from Proposition 2.3 by the restriction of a homomorphism from \( I^* \) to \( \Delta_{\Lambda} \). In particular, the choice of \( U \) (and \( I \)) determines a class in \( \text{Hom}((I_{L^*})^*, \Delta_{\Lambda})/\text{Res}_{(I_{L^*})^*}^I(\text{Hom}(I^*, \Delta_{\Lambda})) \).

**Proof.** Denote the new complementary lattice, which satisfies the condition of Lemma 2.2 by definition, by \( \tilde{I} \), and set \( \hat{U} = \tilde{I}_Q \). As \( \tilde{I} \subseteq L^* \) pairs with \( I \) in a unimodular manner, decomposing \( L^* \) as in Equation (7) and projection onto \( \tilde{I} \) yields an isomorphism, whose inverse is a map \( \tilde{I} \to \hat{I} \) that we write as \( \tilde{u} \to \hat{u} \) (i.e., given \( \tilde{u} \in \tilde{I} \) we denote by \( \hat{u} \) the unique element of \( \tilde{I} \) whose \( \hat{U} \)-coordinate in Equation (7) is \( \tilde{u} \)). We therefore have \( \tilde{I} = \{ \hat{u} = (\beta \tilde{u}, \tilde{\varphi} \tilde{u})|\tilde{u} \in \tilde{U} \} \) in the coordinates of \( V \) as \( U \ast \hat{W} \ast \hat{U} \), for two homomorphisms

$$\varphi : \tilde{I} \to \Lambda^* \quad \text{and} \quad \beta : \tilde{I} \to U, \quad \text{with} \quad \beta \tilde{u} + I_{L^*} = -\iota^* \rho \varphi \tilde{u} \quad \text{for every} \quad \tilde{u} \in \tilde{I}, \quad (10)$$

where \( \tilde{\varphi} \tilde{u} \) is the image in \( \Lambda^* \) of \( \varphi \tilde{u} \in \Lambda^* \). We extend \( \varphi \) and \( \tilde{\varphi} \) to maps

$$\varphi : \hat{U} \to W \quad \text{and} \quad \tilde{\varphi} : \hat{U} \to \hat{W}, \quad \text{with duals} \quad \varphi^* : W \to U \quad \text{and} \quad \tilde{\varphi}^* : \hat{W} \to U,$$

and recall that when using \( \hat{I} \) and \( \hat{U} \), we consider

$$\Lambda \subseteq \Lambda^* \subseteq W \quad \text{via their images} \quad \Lambda \subseteq \Lambda^* \subseteq \hat{W} = (U + \hat{U})^\perp, \quad \text{with} \quad \hat{\rho} : \Lambda^* \to \Delta_{\Lambda}.$$

For determining the latter space, we take \( w \in \hat{W} \), and since

$$(w, \tilde{u}) = (w, \tilde{\varphi} \tilde{u}) = (\tilde{\varphi}^* w, \tilde{u}) = (\tilde{\varphi}^* w, \hat{u}), \quad \text{as} \quad \tilde{\varphi}^* w \in U \quad \text{and} \quad \hat{u} - \tilde{u} \in U^\perp,$$

we deduce that

$$\hat{W} = \{ \hat{w} = (-\tilde{\varphi}^* w, w, 0)|w \in \hat{W} \}, \quad \text{as well as} \quad \tilde{\varphi} \tilde{u} = (-\hat{\rho}^* \tilde{\varphi} \tilde{u}, \tilde{\varphi} \tilde{u}, 0) \quad \text{for} \quad \tilde{u} \in \tilde{U}$$
in the same coordinates. For the effect on $\iota$, consider a sum $u + w + \tilde{u} \in L$ as above, and write

$$\tilde{u} = \hat{v} - \frac{\hat{v}}{2} \beta \tilde{u},$$

as well as $w = \tilde{w} + \phi^*w$ and $\phi \tilde{u} = \tilde{\phi} \tilde{u}$. This changes the class $\iota \tilde{u} = \tilde{p} w = \tilde{p} \tilde{w}$ to $\tilde{p} (w - \tilde{\phi} \tilde{u})$, and therefore subtracts from $\iota$ the restriction to $\tilde{I}_L \cong (I_L)^*$ of the composition $\tilde{p} \phi$ on $\tilde{I} \cong I^*$. As for the condition on $\iota$ in Equation (6), we express it via Lemma 1.5 and the duality between $I$ and $\tilde{I}$ as the condition that $(u, \tilde{v}) \in - (\hat{u}, \tilde{v}) + \mathbb{Z}$ for every $\tilde{v} \in \tilde{I}$, and considering the modified value of $u$ we have to prove that

$$(u - \beta \tilde{u} + \phi^*w - \tilde{\phi}^* \phi \tilde{u}, \tilde{v}) \in - (\hat{u}, \tilde{v}) + \mathbb{Z} \text{ for every } \tilde{v} \in \tilde{I}.$$  

We can replace $\tilde{v}$ by $\hat{v}$ on the left hand side (since their difference is in $U^\perp$), and by duality, what we know on $(u, \hat{v})$, and the direct evaluation of $(\hat{u}, \hat{v})$, this side gives an element of

$$-(\hat{u}, \hat{v}) - (\beta \hat{u}, \hat{v}) + (w, \phi \hat{v}) - (\phi \hat{u}, \phi \hat{v}) + \mathbb{Z} = - (\hat{u}, \hat{v}) + (w, \varphi \hat{v}) + (\hat{u}, \beta \hat{v}) + \mathbb{Z}.$$  

But since $\varphi \hat{v} \in \Lambda^*$, the fact that $\hat{p} w = \hat{p} \hat{u}$ by Equation (6) combines with the expression for $\beta \hat{u} + I_L \varphi$ in Equation (10) and the containment $(u, I_L) \subseteq \mathbb{Z}$ (since $\hat{u} \in \tilde{I}_L$) to reduce the latter expression to $-(\hat{u}, \hat{v}) + \mathbb{Z}$ as desired. Similar considerations show that the $\mathbb{Q}/\mathbb{Z}$-image of $\frac{1}{2}$ coincides with $\frac{|(\iota - \varphi \hat{u})|^2}{2}$ when $\hat{u}$ is associated with $\hat{u} \in \tilde{I}_L$ as above, and as passing through the isomorphisms $I_L \cong (I_L)^* \cong I_L$ (with $\tilde{I}_L$ defined as in Equation (5) with $\tilde{I}$ replaced by $\tilde{I}$) allows us to write the latter expression as $\frac{|(\iota - \varphi \hat{u})|^2}{2}$, this verifies the norm condition from Proposition 2.4. Note from $\varphi (I) \subseteq \Lambda^*$ we obtain $\varphi^*(\Lambda) \subseteq I$ by dualizing and hence $\varphi^*(\Lambda) \subseteq I$, so that the form of $I_L^*$ as $I \oplus \Lambda$ as in Equation (5) is indeed preserved. This establishes the first assertion, from which the second one directly follows. This proves the corollary.

Recall that $\Lambda^*$ and $I_L^*$ are torsion-free, and the projection $p : \Lambda^* \to \Delta_\Lambda$ and the map from $I_L^*$ to $\Lambda^*$ are surjective. Hence homomorphisms from $\tilde{I}$ to $\Delta_\Lambda$ can always be lifted to maps $\varphi : \tilde{I} \to \Lambda^*$, and then to maps from $\tilde{I}$ to $I_L^*$ with an appropriate homomorphism $\beta$ as in Equation (10). Therefore no finer invariant can be associated to $U$ and $I$ themselves in Corollary 2.4. In particular, a necessary and sufficient condition for the existence of a convenient complement $\tilde{U}$ for which the associated map $\iota$ will vanish, so that Equations (6) and (7) will take the simpler form appearing in Equation (11), is that the class from Corollary 2.4 be trivial.

3 Intersection with Arithmetic Subgroups

We can now investigate the structure of the group $\Gamma_{L, I}$. First, the fact that we consider only elements in $\mathcal{P}_U$ means that in Equation (2) we have to consider only elements $\Lambda \in \mathcal{P}_U$ that are represented by parameters $M \in GL(U)$.
with positive determinant, \(\gamma \in O(W) \cap SO^+(W)\), \(\psi \in \text{Hom}_Q(W,U)\), and \(\eta \in \text{Hom}_G^+(U^*, U)\). In addition, we recall that if \(M \in \text{GL}(U)\) preserves \(I\) and \(\det M > 0\) then \(M\) is in the group \(\text{SL}(I)\). Moreover, for any lattice \(I \subseteq J \subseteq U\) (such as \(J = I_L^*\)), we define

\[
\text{SL}(J, I) := \{ M \in \text{SL}(I) | \forall u \in J, u - Mu \in I \} \subseteq \text{SL}(I) \cap \text{SL}(J).
\]

It follows that for every \(M \in \text{SL}(J, I)\), the map \(\text{Id}_U - M\) induces a well-defined map from \(U/J\) to \(U/I\). In addition, we have the group \(\Gamma_L \subseteq SO^+(W)\) defined in analogy to \(\Gamma \subseteq SO^+(V)\), and since our coordinates are already based on the choice of a complement \(U = I_Q\) as above, the map \(\iota : I_L \to \Delta_A\) from Proposition 2.3 as well as its dual \(\iota^* : \Delta_A \to U/I_L^*\), are given.

We can now give the characterization of \(\Gamma_{L,I}\) in these coordinates.

**Theorem 3.1.** The element \(A \in P^0_U\) that is associated with the parameters from above lies in \(\Gamma_{L,I}\) if and only if the following conditions are satisfied:

1. \(\gamma \in \Gamma_A\).
2. \(M \) is in \(\text{SL}(I_L^*, I)\). In particular, \(\text{Id}_U - M : U/I_L^* \to U/I\) is well-defined.
3. \(\psi(\Lambda) \subseteq I\), and the resulting map from \(\Delta_A\) to \(U/I\) equals the composition of \(\iota^* : \Delta_A \to U/I_L^*\) from Proposition 2.3 with \(\text{Id}_U - M : U/I_L^* \to U/I\) from part (ii).
4. \(\tilde{\eta} \in \text{Hom}_Q^+(\tilde{U}, U)\) is \((\text{Id}_U - M)\alpha (\text{Id}_U - M^*) + M\alpha - \alpha\tilde{M}^* - \tilde{\kappa} = 0\) for some \(\kappa \in \text{Hom}(I^*, I)\).

Moreover, \(\gamma\) in condition (i) can be arbitrary, for every \(M \in \text{SL}(I_L^*, I)\) as in condition (ii) there is some \(\psi \in \text{Hom}_Q(W, U)\) satisfying condition (iii), and for every such \(M\) and \(\psi\) there exists \(\eta \in \text{Hom}_G^+(U^*, U)\) for which condition (iv) is fulfilled (and then \(\eta\) is unique up to \(\text{Hom}_G^+(I^*, I)\)).

**Proof.** First, if \(AL = L\) and \(AU = U\) then \(M = A|_U\) preserves \(I = U \cap L\), and if \(\det M > 0\) then \(M \in \text{SL}(I)\). Now, the condition \(A \in \Gamma_L\) means that \(A\lambda - \lambda \in L\) for every \(\lambda \in L^*\), and we evaluate this difference for the two parts \(I\) and \(I_L^*\) of \(L^*\) from Proposition 2.3. Consider first an element \(\lambda \in I_L^*\), for which by Equations (7) and Equation (8) we get \(\lambda = u + w\) for

\[
w \in \tilde{\Lambda}^* \text{ and } u \in U \text{ with } u + I_L^* = -\iota^*\tilde{\kappa}w, \text{ and then } A\lambda - \lambda \in I_L^* = I \oplus \tilde{\Lambda}.
\]

Since this holds for every such \(u\) and \(w\), it follows that

\[
gw - w \in \tilde{\Lambda}^* \text{ for any } w \in \tilde{\Lambda}^*, \text{ and } Mu - u - \tilde{\psi}gw \in I \text{ when } u + I_L^* = -\iota^*\tilde{\kappa}w,
\]

from which condition (i) immediately follows, and by taking \(w = 0\) (hence \(u \in I_L^*\)) in Equation (11) we deduce condition (ii) as well. Now let \(w\) be any element of \(\tilde{\Lambda}\), which we write as \(\tilde{\gamma}^{-1}v\) for \(v \in \lambda\), and since \(\tilde{\kappa}w = 0\) we know that \(u \in I_L^*\) once again. Since \(Mu - u \in I\) by condition (ii), Equation (11)
yields \( \tilde{\psi}v \in I \) as well, proving the first part of condition \((iii)\). Considering 
\( \psi|_{\Lambda^*} : \Lambda^* \to U \), the condition \( \psi(\Lambda) \subseteq I \) produces a map \( \psi^\Delta : \Delta_\Lambda \to U/I \), and then condition \((i)\) and Equation \((11)\) imply that
\[
\psi^\Delta \tilde{\psi}w = \psi^\Delta (w + \tilde{\Delta}) = \psi^\Delta (\tilde{\gamma}w + \tilde{\Delta}) = Mu - u + I = (M - \text{Id}_U)u + I \subseteq U/I.
\]
As condition \((ii)\) shows that the right hand side depends only on the image of \( u \) in \( U/I_{\Lambda^*} \), which was seen to be \(-\nu^*\tilde{\psi}w\), we deduce that \( \psi^\Delta = (\text{Id}_U - M)\nu^* \), establishing condition \((iii)\).

Now, dualizing condition \((ii)\) and the isomorphisms \( \bar{I} \cong I^* \) and \( \bar{I}_L \cong (I_L)^* \) imply that \( \bar{M}^* \) and \( \bar{M}^{-*} \) preserve both \( \bar{I} \) and \( \bar{I}_L \), and that we have
\[
(\bar{M}^* - \text{Id}_{\bar{U}})(\bar{I}) \subseteq \bar{I}_L \text{ and } (\bar{M}^{-*} - \text{Id}_{\bar{U}})(\bar{I}) \subseteq \bar{I}_L.
\]
Hence when we take \( \lambda \in \bar{I} \) and subtract it from the expression for \( A\lambda \) in Equation \((2)\), the part \( \bar{u} = \bar{M}^{-*}\lambda - \lambda \) lies in \( \bar{I}_L \), as Equation \((9)\) requires. Dualizing condition \((iii)\) via these isomorphisms implies that
\[
\tilde{\psi}^* (\bar{I}) \subseteq \bar{\Lambda}^*, \text{ and } \tilde{\psi}\tilde{\psi}^* : \bar{I} \to \Delta_\Lambda \text{ equals } (\nu : \bar{I}_L \to \Delta_\Lambda) \circ (\text{Id}_{\bar{U}} - \bar{M}^* : \bar{I} \to \bar{I}_L).
\]
Since \( \mu = \bar{M}^{-*}\lambda \in \bar{I} \), the second part \( w = \tilde{\psi}^*\mu = \tilde{\psi}^*\bar{M}^{-*}\lambda \) from Equation \((2)\) lies in \( \bar{\Lambda}^* \), and we also obtain the equality
\[
w + \bar{\Lambda} = \tilde{\psi}w = \tilde{\psi}\tilde{\psi}^*\mu = \tilde{\psi}\tilde{\psi}^*\bar{M}^{-*}\lambda = \nu(\bar{M}^{-*}\lambda - \lambda) = \nu\bar{u},
\]
as Equation \((6)\) requires. Therefore, given conditions \((ii)\), and \((iii)\), the only additional requirement that remains in Equation \((9)\) is the last one, in which substituting the term for \( u \) in Equation \((2)\) yields that the sum of
\[
M\alpha\lambda - \alpha\bar{M}^{-*}\lambda - \tilde{\psi}\tilde{\psi}^*\bar{M}^{-*}\lambda - \bar{\eta}\bar{M}^{-*}\lambda \text{ and } 2\alpha\tilde{u} = 2\alpha(\bar{M}^{-*}\lambda - \lambda) \text{ is in } I.
\]
Writing this in terms of \( \mu = \bar{M}^{-*}\lambda \), and recalling that \( \bar{M}^{-*} \) takes \( \bar{I} \) onto itself, we deduce that
\[
\tilde{\kappa}\mu = \left[(M - \text{Id}_U)\alpha(\bar{M}^* - \text{Id}_{\bar{U}}) + M\alpha - \alpha\bar{M}^* - \tilde{\psi}\tilde{\psi}^* - \bar{\eta}\right]\mu \in I \text{ for every } \mu \in \bar{I}.
\]
Hence \( \eta \) must have the form required in condition \((iv)\), because \( \tilde{\kappa} \in \text{Hom}(\bar{I}, I) \) precisely when \( \kappa \in \text{Hom}(I^*, I) \). Since all of our arguments are invertible, we have proved that \( A \in \Gamma_L \) if and only if our four conditions are satisfied.

Now, if \( M \in \text{SL}(I_{\Lambda^*}, I) \) then the map \( \nu \circ (\text{Id}_U - M) : \Delta_\Lambda \to U/I \) has finite image \( J/I \) for some lattice \( I \subseteq J \subseteq U \). Since \( J \) is torsion-free, its composition with \( p \) can be lifted to a map from \( \Lambda^* \) to \( J \subseteq U \) (as in the remark following Corollary \((2.3)\), whose extension to an element \( \psi \in \text{Hom}_{Q}(W, U) \) clearly satisfies condition \((iii)\)).

Next, given both \( M \) and \( \psi \), showing the existence of \( \eta \) satisfying condition \((iv)\) amounts to proving that there exists an element \( \kappa \in \text{Hom}(I^*, I) \), or
equivalently \( \tilde{\kappa} \in \text{Hom}(\tilde{I}, I) \), such that the asserted formula for \( \tilde{\eta} \) there is anti-symmetric. As duality and Lemma 1.5 yield the equality
\[
\left((M \alpha - \alpha \tilde{M}^*) \tilde{u}, \tilde{v}\right) = (\alpha \tilde{u}, \tilde{M}^* \tilde{v}) - \frac{1}{2} (\tilde{u}, \tilde{M}^* \tilde{v}) - \frac{1}{2} (\tilde{M}^* \tilde{u}, \tilde{v})
\]
for \( \tilde{u} \) and \( \tilde{v} \) in \( \tilde{U} \), the part \( M \alpha - \alpha \tilde{M}^* \) from Equation (12) is always anti-symmetric. We now claim that \( (\text{Id}_U - M)\alpha(\text{Id}_{\tilde{U}} - \tilde{M}^*) - \frac{1}{2} \tilde{\psi} \tilde{\psi}^* \) defines a symmetric bilinear map \( B \) on \( \tilde{I} \) with values in \( \frac{1}{2} \mathbb{Z} \), such that \( B(\tilde{u}, \tilde{u}) \in \mathbb{Z} \) for every \( \tilde{u} \in \tilde{I} \). Indeed, duality and Lemma 1.5 express \( \tilde{B} \) on \( \tilde{U} \) as
\[
\left((\text{Id}_U - M)\alpha(\text{Id}_{\tilde{U}} - \tilde{M}^*) \tilde{u} - \frac{1}{2} \tilde{\psi} \tilde{\psi}^* \tilde{u}, \tilde{v}\right) = \frac{1}{2}\left((\text{Id}_{\tilde{U}} - \tilde{M}^*) \tilde{u}, (\text{Id}_{\tilde{U}} - \tilde{M}^*) \tilde{v}\right) - \frac{1}{2} \left(\tilde{\psi}^* \tilde{\psi} \tilde{u}, \tilde{v}\right),
\]
from which the symmetry is immediate. Next, the condition that \( B(\tilde{u}, \tilde{v}) \in \mathbb{Z} \) for every \( \tilde{u} \in \tilde{I} \), which by the usual relation between quadratic and bilinear forms implies the half-integrality of \( B(\tilde{u}, \tilde{v}) \) for every \( \tilde{u} \) and \( \tilde{v} \) there, is equivalent to the equality of \( \frac{1}{2} [\alpha(\text{Id}_{\tilde{U}} - \tilde{M}^*) \tilde{u}]^2 \) and \( \frac{1}{2} \tilde{\psi} \tilde{\psi}^* \tilde{u}^2 \) in \( \mathbb{Q}/\mathbb{Z} \). But as \( \tilde{\psi} \tilde{\psi}^* \tilde{u} \in \Lambda^* \) for \( \tilde{u} \in \tilde{I} \), condition (iii) on \( \psi \) and the norm condition on \( \iota \) in Proposition 2.3 imply that
\[
\frac{1}{2} [\tilde{\psi} \tilde{\psi}^* \tilde{u}]^2 + Z = \frac{1}{2} [\alpha(\text{Id}_{\tilde{U}} - \tilde{M}^*) \tilde{u}]^2 = \frac{1}{2} [\text{Id}_{\tilde{U}} - \tilde{M}^* \tilde{u}]^2 + Z
\]
in \( \mathbb{Q}/\mathbb{Z} \), as desired. The identification of \( B \) with a symmetric element of \( \text{Hom}(\tilde{I}, \frac{1}{2} I) \) via the isomorphism \( I \cong \tilde{I}^* \) thus implies the existence of an element \( \tilde{\kappa} \in \text{Hom}(\tilde{I}, I) \) such that the difference \( \tilde{\eta} \) will be anti-symmetric, as required for condition (iv) (to see this more explicitly, a basis for \( \tilde{I} \) and the dual basis for \( I \) will express the homomorphism \( (\text{Id}_U - M)\alpha(\text{Id}_{\tilde{U}} - \tilde{M}^*) - \frac{1}{2} \tilde{\psi} \tilde{\psi}^* \) and the bilinear form \( B \) by the same matrix, which is symmetric and half-integral with an integral diagonal, from which subtracting an appropriate integral matrix, corresponding to an element of \( \text{Hom}(\tilde{I}, I) \), yields an anti-symmetric matrix). As the difference between two possible \( \kappa \) must send \( \tilde{I} \) to \( I \) and be anti-symmetric, the choice \( \kappa \) is indeed unique up to \( \text{Hom}^{\text{as}}(\tilde{I}, I) \). This completes the proof of the theorem.

For a more succinct description of \( \Gamma_{L,I} \), we write the (finite) image of \( \iota^* \) in \( U/I_{L^*} \) as \( I_i/I_{L^*} \) for some lattice \( I_i \subseteq U \), and note that
\[
\text{the inclusion } I_{L^*} \subseteq I_i \text{ implies the inclusion } \text{SL}(I_i, I) \subseteq \text{SL}(I_{L^*}, I).
\]
Moreover, the intersection \( \Gamma_L \cap W_U = \Gamma_{L,I} \cap W_U \) will be a Heisenberg group over \( \mathbb{Z} \), which we shall denote by \( H(\text{Hom}(\Lambda^*, I), \text{Hom}^{\text{as}}(I^*, I)) \) (with the same anti-symmetric bilinear form from Proposition 1.4), but whose definition we need to make precise. For two elements \( \psi \) and \( \varphi \) in \( \text{Hom}(\Lambda^*, I) \), with dual maps \( \psi^* \) and \( \varphi^* \) in \( \text{Hom}(I^*, \Lambda) \), the combination \( \varphi \psi - \psi \varphi \) from Proposition 1.4 does not necessarily lie in \( \text{Hom}^{\text{as}}(I^*, I) \). Therefore the subset
\[
\text{Hom}(\Lambda^*, I) \times \text{Hom}^{\text{as}}(I^*, I) \subseteq \text{Hom}_q(W, U) \times \text{Hom}_q^{\text{as}}(U^*, U)
\]
is not closed under the law of multiplication from Definition 1.2 (with this anti-symmetric bilinear form). On the other hand, the proof of Theorem 3.1 shows that for any \( \psi \in \text{Hom}(\Lambda^*, I) \), the combination \( \frac{\psi \psi^*}{2} \), which also takes \( I^* \) to \( \frac{1}{2} I \), is such that the resulting half-integral symmetric bilinear form on \( I^* \) has integral diagonal, and that there is an element \( \kappa \in \text{Hom}(I^*, I) \), such that \( \frac{\psi \psi^* - \kappa}{2} \) is anti-symmetric. As the latter element is in \( \text{Hom}(I^*, \frac{1}{2} I) \) (since so are \( \frac{\psi \psi^*}{2} \) and \( \kappa \)), and more precisely in \( \text{Hom}^a(I^*, \frac{1}{2} I) \), the fact that \( \kappa \) is unique up to \( \text{Hom}^a(I^*, I) \) implies that \( \psi \) determines a coset

\[
c_{\psi} := \left\{ \eta \in \text{Hom}^a(I^*, \frac{1}{2} I) \mid \frac{\psi \psi^*}{2} - \eta \in \text{Hom}(I^*, I) \right\} \in \text{Hom}^a(I^*, \frac{1}{2} I) / \text{Hom}^a(I^*, I).
\]

Since for \( \psi \) and \( \varphi \) from \( \text{Hom}(\Lambda^*, I) \), with \( \eta \in c_{\psi} \) and \( \rho \in c_{\varphi} \) we have

\[
\frac{(\psi + \varphi)(\psi + \varphi)^*}{2} - (\eta + \rho + \frac{\psi \varphi^* - \varphi \psi^*}{2}) = \left( \frac{\psi \psi^*}{2} - \eta \right) + \left( \frac{\varphi \varphi^*}{2} - \rho \right) + \varphi \psi^* \in \text{Hom}(I^*, I)
\]

hence \( \eta + \rho + \frac{\psi \varphi^* - \varphi \psi^*}{2} \in c_{\psi + \varphi} \), we define \( H(\text{Hom}(\Lambda^*, I), \text{Hom}^a(I^*, I)) \) to be

\[
\left\{ (\psi, \eta) \in \text{Hom}(\Lambda^*, I) \times \text{Hom}^a(I^*, \frac{1}{2} I) \mid \eta + \text{Hom}^a(I^*, I) = c_{\psi} \right\},
\]

which is indeed a subgroup of \( \mathcal{W}_U = H(\text{Hom}_Q(W, U), \text{Hom}^a_Q(U^*, U)) \).

As an example, we consider the case where \( U \) over \( \mathbb{Q} \) that spans \( I \) over \( \mathbb{Q} \) restricts the isomorphisms from Equation (6) to

\[
\text{SL}(I) \cong \text{SL}_2(\mathbb{Z}), \quad \text{Hom}^a(I^*, I) \cong \mathbb{Z}, \quad \text{and} \quad \text{Hom}(\Lambda^*, I) \cong (\Lambda^*)^* \times (\Lambda^*)^* \cong \Lambda \times \Lambda,
\]

where the cyclic group in the middle is contained as an index 2 subgroup of \( \text{Hom}^a(I^*, \frac{1}{2} I) \cong \frac{1}{2} \mathbb{Z} \). The natural inclusion and Corollary 1.9 give

\[
H(\text{Hom}(\Lambda^*, I), \text{Hom}^a(I^*, I)) \subseteq H(\text{Hom}_Q(W, U), \text{Hom}^a_Q(U^*, U)) \cong \tilde{H}(W, \mathbb{Q}),
\]

and the image of the former subgroup inside the latter group will be naturally denoted by \( \tilde{H}(\Lambda, \mathbb{Z}) \). The choice of basis for \( I \) also yields we get isomorphisms

\[
\text{Hom}(I^*, I) \cong I \otimes I \cong M_2(\mathbb{Z}) \quad \text{inside} \quad \text{Hom}(I^*, U) \cong I \otimes U \cong M_2(\mathbb{Q}),
\]

and if \( \psi \in \text{Hom}(\Lambda^*, I) \) is associated with the \( (\lambda, \nu) \in \Lambda \times \Lambda \) then \( \frac{\psi \psi^*}{2} \) is taken to the matrix \( \frac{1}{2} \left( \begin{array}{cc} \langle \lambda, \lambda \rangle & \langle \lambda, \mu \rangle \\ \langle \mu, \lambda \rangle & \langle \mu, \mu \rangle \end{array} \right) \) \( \in M_2(\mathbb{Q}) \). It therefore follows that \( c_{\psi} \) is the trivial class in \( \text{Hom}^a(I^*, \frac{1}{2} I) / \text{Hom}^a(I^*, I) \cong \frac{1}{2} \mathbb{Z} / \mathbb{Z} \) when the pairing \( (\lambda, \nu) \) is even, but the non-trivial one in case it is odd, so that Equation (13) takes the form

\[
\tilde{H}(\Lambda, \mathbb{Z}) = \left\{ (\lambda, \nu, t) \in \Lambda \times \Lambda \times \frac{1}{2} \mathbb{Z} \mid t \in \frac{1}{2}(\lambda \mu) \right\}.
\]

In particular, since the map sending \( \lambda \) and \( \mu \) and \( \Lambda \) to \( (\lambda, \mu) \) modulo 2 is a homomorphism from \( \Lambda \times \Lambda \) to \( \mathbb{Z}/2\mathbb{Z} \), it has a kernel \( (\Lambda \times \Lambda)^0 \) and perhaps a non-trivial complement \( (\Lambda \times \Lambda)^1 \) in \( \Lambda \times \Lambda \), and Equation (14) becomes

\[
\tilde{H}(\Lambda, \mathbb{Z}) = \left( (\Lambda \times \Lambda)^0 \times \mathbb{Z} \right) \cup \left( (\Lambda \times \Lambda)^1 \times \frac{1}{2} \mathbb{Z} \right).
\]
which reduces to $\Lambda \times \Lambda \times \mathbb{Z}$ in case the homomorphism to $\mathbb{Z}/2\mathbb{Z}$ is trivial, but not otherwise.

The result of Theorem 3.1 can now be stated more neatly.

**Proposition 3.2.** The map from Lemma 1.1 restricts to a short exact sequence

$$1 \to \Gamma_{L, I} \cap \mathcal{W}_U = H(\text{Hom}(\Lambda^*, I), \text{Hom}^a(I^*, I)) \to \Gamma_{L, I} \to \text{SL}(I_{L^*}, I) \times \Gamma_{\Lambda} \to 1,$$

where the kernel is the group from Equation (13). Hence the image of $\Gamma_{L, I}$ in the coarser quotient $\text{GL}(U)$ from Corollary 1.3 is $\text{SL}(I_{L^*}, I)$. The image $\text{T}_{L, I} \cong \Gamma_{L, I}/\text{Hom}^a(I^*, I)$ of $\Gamma_{L, I}$ in $\mathcal{P}_U$ sits in a similar short exact sequence

$$1 \to \text{Hom}(\Lambda^*, I) \to \text{T}_{L, I} \to \text{SL}(I_{L^*}, I) \times \Gamma_{\Lambda} \to 1,$$

and after choosing a complement $\tilde{U} = \tilde{I}_Q$ as before, with the map $\iota$ from Proposition 2.3, the restriction of this short exact sequence to pre-images of $\text{SL}(I, I) \times \Gamma_{\Lambda}$ splits as a semi-direct product in the corresponding coordinates.

**Proof.** The fact that the image of $\Gamma_{L, I}$ in $\mathcal{P}_U/\mathcal{W}_U \cong \text{GL}(U) \times \text{O}(W)$ is the asserted one follows from conditions (i) and (ii) of Theorem 3.1 and the last assertions there. Conditions (iii) and (iv) there with $M = \text{Id}_U$ then determine the kernel, using the argument leading to Equation (13). The assertions about the image in $\text{GL}(U)$ and the second short exact sequence are now immediate, where for the splitting one notes that the map $\psi = 0$ satisfies condition (iii) of Theorem 3.1 if and only if $M \in \text{SL}(I, I)$ (by the definition of the latter group). This proves the proposition. \qed

When $\dim U = 2$, the previous considerations show that $\text{SL}(I_{L^*}, I)$ and $\text{SL}(I, I)$ become congruence subgroups of $\text{SL}_2(\mathbb{Z})$, which we denote by $\Gamma_{L^*}$ and $\Gamma_I$, respectively. For example, if $\alpha$ and $\beta$ are generators for $I$ such that $\frac{1}{D}\alpha$ and $\frac{1}{D}\beta$ generate $I_{L^*}$, with $D$ dividing $N$, then the group $\Gamma_{L^*}$ is the congruence subgroup $\Gamma^0(N, D) = \Gamma_1(N) \cap \Gamma^0(D)$. In correspondence with Corollary 1.9 Proposition 3.2 takes in this case the following form.

**Corollary 3.3.** If $U = I_Q$ is a 2-dimensional isotropic subspace of $V$ and we choose a basis for $I$ then we get the two short exact sequences

$$1 \to \tilde{H}(\Lambda, \mathbb{Z}) \to \Gamma_{L, I} \to \Gamma_{L^*} \times \Gamma_{\Lambda} \to 1$$

and

$$1 \to \Lambda \times \Lambda \to \tilde{T}_{L, I} \to \Gamma_{L^*} \times \Gamma_{\Lambda} \to 1,$$

where the kernel $\tilde{H}(\Lambda, \mathbb{Z})$ in the first sequence is defined in Equation (13), and a choice of a complement $\tilde{U} = \tilde{I}_Q$ yields a splitting of the second sequence over $\Gamma_{\Lambda}$, where $\iota$ is the map from Proposition 2.3.

Note that the pre-image of $\text{SL}(I, I) \times \Gamma_{\Lambda}$ in $\Gamma_{L, I}$ itself does not split the first short exact sequence from Proposition 3.2 and also not in Corollary 3.3. Indeed, the condition $M \in \text{SL}(I, I)$ implies that $\text{Id}_U - M^*(\tilde{I})$ is only contained in $\ker \iota \subseteq \tilde{I}_L$, on the pairing of which with $\tilde{I}$ we have no control, and while the proof of Theorem 3.1 shows that the symmetric part from Equation (12) (with
\(\psi = 0\) is in \(\text{Hom}(I^*, I/2I)\) with integral diagonal entries as before, the antisymmetric part \(M\alpha - \alpha M^*\) in that equation need not be in \(\text{Hom}^{ss}(I^*, I/2I)\). We remark that a direct verification shows that replacing \(\tilde{U}\) by another complement \(\tilde{U} = \tilde{I}_Q\) as in Corollary 2.4 preserves the description from Theorem 3.1 and Proposition 3.2.

Back to the group \(\Gamma_{L,I}\), condition (iii) of Theorem 3.1 shows that any element \(M \in \text{SL}(I_{L^*}, I)\) determines the class
\[
b_M := (\text{Id}_U - M) \circ \iota^* \in \text{Hom}(\Delta_{\Lambda, U/I}) = \text{Hom}(\Lambda, I)/\text{Hom}(\Lambda^*, I),
\]
(15)
such that \(\psi \in \text{Hom}_{\mathbb{Q}}(W, U)\) satisfies this condition if and only if \(\psi\) (is the extension of) an element of \(\text{Hom}(\Lambda, I)\) whose image modulo \(\text{Hom}(\Lambda^*, I)\) is in this class. As this class is trivial if and only if \(M\) is in the coordinates \(\text{GL}(U)\times\text{O}(W)\times\text{Hom}_{\mathbb{Q}}(W, U)\). Therefore the action of \((M, \gamma)\) on \(\text{Hom}(\Delta_{\Lambda, U/I})\) is by composition with \(M \in \text{SL}(I_{L^*}, I) \subseteq \text{SL}(I, I)\) on the right (which thus is well-defined) and with \(\gamma^{-1}\) on the left, the latter being trivial by the definition of \(\Gamma_{\Lambda}\). Hence \(b_{MN}\) is the coset containing the element \(\psi + M\varphi\gamma^{-1}\), which is indeed the asserted one, in correspondence with the equality
\[
b_{MN} = (\text{Id}_U - MN) \circ \iota^* = (\text{Id}_U - M) \circ \iota^* + M \circ (\text{Id}_U - N) \circ \iota^* = b_M + M(b_N).
\]
This proves the proposition. \(\square\)

Note that the map from Proposition 3.4 is not a group homomorphism from \(\text{SL}(I_{L^*}, I)\) to \(\text{Hom}(\Delta_{\Lambda, U/I})\) in general, and indeed, the set of \(M \in \text{SL}(I_{L^*}, I)\) with \(b_M\) as seen in Proposition 3.2 to be the subgroup \(\text{SL}(I, I)\), and it is not necessarily normal in \(\text{SL}(I_{L^*}, I)\). In the case where \(\dim U = 2\) considered in Corollary 3.3, the cocycle from Equation (15) can be seen as taking values in \((\Lambda^* \times \Lambda^*)/(\Lambda \times \Lambda)\) (or equivalently \(\Delta_{\Lambda} \times \Delta_{\Lambda}\)), but does not simplify more than in the general case.

Consider again the particular case where \(\iota = 0\), where \(L\) and \(L^*\) are described in the simpler Equation (9). Then \(I_i = I_{L^*}\) (hence \(\text{SL}(I_i, I) = \text{SL}(I_{L^*}, I)\)), the full group \(\Gamma_{L,I}\) is a semi-direct product (though \(\Gamma_{L,I}\) is not in general), and the cocycle \(M \mapsto b_M\) is trivial. On the other hand, recall the simpler form of \(\mathcal{P}_U\) appearing in Corollary 1.8 when \(\dim U = 1\). The structure of \(\Gamma_{L,I}\) in this case is also much simpler, regardless of \(\iota\).
Corollary 3.5. Let $I$ be the subgroup of $L$ that is generated by the primitive isotropic vector $z \in L$, and set $U = I_\mathbb{Q} = \mathbb{Q}z \subseteq V$. Then $\Gamma_{L,I}$ is isomorphic to the semi-direct product in which $\Gamma \Lambda$ operates on the group $(\Lambda, +)$.

Proof. We recall from Corollary 1.8 that $P_U = \overline{P_U}$ when $\dim U = 1$, so that we can describe $\Gamma_{L,I}$ itself by the second exact sequence from Proposition 3.2. Moreover, the group $SL(I)$ is trivial when $I$ is of rank 1 (in correspondence with $\mathbb{Q}_+ \times \mathbb{Q} \times +$ having no non-trivial integral points), so that the sequence already splits, and the choice of $z$ identifies the kernel in that sequence with $\Lambda$ as above. This proves the corollary.

In the 1-dimensional case $I = \mathbb{Z}z$ the lattice $\Lambda$ is typically denoted by $K$, so that Corollary 3.5 reproduces the semi-direct product of $\Gamma_K$ and $(K, +)$ appearing in, e.g., [Bo], [Br], [Z1], [Z2], among others.

4 Canonical Boundary Components of Toroidal Compactifications

As an application of our analysis, which was the original motivation for carrying it out, we determine the exact structure of the canonical boundary components of toroidal compactifications of orthogonal Shimura varieties. Let $L$ be an even lattice of signature $(n, 2)$ in the quadratic space $V = L_\mathbb{Q}$. Then the symmetric space $G(V_\mathbb{R}) = \{ v_- \subseteq V | v_- << 0, \dim v_- = 2 \}$, and for $v_- \in G(V_\mathbb{R})$ set $v_+ = v_-^\perp$ of $O(V_\mathbb{R})$ carries a natural (up to complex conjugation) structure of an $n$-dimensional complex manifold, as is described in Section 13 of [Bo], Section 3.2 of [Br], Section 1.2 of [Z1], Section 3.1 of [F] (all in the opposite signature), or Section 2 of [BZ]. For the theory of toroidal compactifications we refer to [AMRT] in general, to [Nam] for the symplectic case, and to Sections 3 and 5 of [F] or Sections 1 and 2 of [BZ] for our orthogonal case. The fiber of the toroidal compactification over a 0-dimensional cusp, which is related to the group $\Gamma_{L,I}$ for $I \subseteq L$ of rank 1 (which has simpler structure by Corollary 3.5), is not canonical in general, and depends on some choice of fan. On the other hand, over a 1-dimensional cusp lies a canonical toroidal boundary component, which is described grossly in Section 5 of [F], as well as precisely in Section 2 of [BZ] under some simplifying assumptions, as an open Kuga–Sato type variety. The goal of this section is to give the exact description of this boundary component without the simplifying assumption on $L$, $U$, and $I$ that appear in [BZ] (note that this is not yet the divisor on the toroidal compactification, and the form of the full divisor does depend on the choices of fans over 0-dimensional cusps).

For this let $I$ be a rank 2 isotropic lattice in $L$, set $U = I_\mathbb{Q}$, choose a basis $(z, w)$ for $I$ over $\mathbb{Z}$, and recall the coordinates from [K] (or [F], or [BZ]) that represent the complex manifold $G(V_\mathbb{R})$. Explicitly we have

$$G(V_\mathbb{R}) \cong \{ Z_V \in V_C | Z_V^2 = 0, (Z_V, Z_V) < 0, (Z_V, z) = 1, (\mathbb{R}Z_V, \mathbb{C}Z_V) \text{ oriented} \}$$
for some fixed orientation on each \( v_ - \in G(V_\mathfrak{R}) \) that is determined by \((z, w)\) hence depends continuously on \( v_-\). If \( \bar{I} \) is a complement for \( I^*_2 \) in \( L^* \) that satisfies the condition of Lemma 2.2 and \( \bar{U} = \bar{I}_Q \) then they are spanned over \( \mathbb{Z} \) (resp. \( \mathbb{Z} \)) by the basis \((\zeta, \omega)\) that is dual to \((z, w)\), and by setting \( \bar{W} = (U \oplus \bar{U}^\perp) \) as above the set of \( Z_\nu \in V_\mathcal{C} \) that represent \( G(V_\mathfrak{R}) \) becomes

\[
\left\{ Z_\nu = \zeta + \tau \omega + \bar{Z}_0 - \sigma w + \left( \tau \sigma - \frac{2\bar{z}^2 + (\zeta + \tau \omega)^2}{2} \right) z \middle| 3\tau > 0, \ 3\sigma > \frac{(3\tau)^2 \omega^2 + (3\bar{Z}_0)^2}{2\bar{z}^2}\right\}. \tag{16}
\]

In particular, the coordinate \( \tau \) from Equation (16) lies in the usual Poincaré upper half-plane \( \mathcal{H} = \{ \tau \in \mathbb{C} | 3\tau > 0 \} \), and \( \sigma \) lies in a translated copy of \( \mathcal{H} \). The projection modulo \( U_\mathcal{C} \) omits the coordinate \( \sigma \), and yields images in

\[
\bar{D}(F) := \bigcup_{\tau \in \mathcal{H}} W_\mathcal{C}^{1, \tau} \text{ with } \ W_\mathcal{C}^{1, \tau} := \left\{ \xi \in V_\mathcal{C} | (\xi, z) = 1, (\xi, w) = \tau \} / U_\mathcal{C}, \tag{17} \right.
\]

independently of the choice of \( \bar{I} \) and \( \bar{U} \), where \( F \) is the Baily–Borel cusp corresponding to \( U \). Recalling that the Baily–Borel cusp \( F \), or equivalently the base space \( D(F) \) from \([AMRT]\) and others, is isomorphic to \( \mathcal{H} \), this produces the following description of \( G(V_\mathfrak{R}) \).

**Proposition 4.1.** The symmetric space \( G(V_\mathfrak{R}) \) is an affine \( \mathcal{H} \)-bundle over the space \( \bar{D}(F) \) from Equations (17), which itself carries a structure of a holomorphic affine vector bundle over \( \mathcal{H} \).

Indeed, the affine vector bundle structure in Proposition (17) is obtained via the natural projection sending \( \xi \) in some \( W_\mathcal{C}^{1, \tau} \) to the corresponding \( \tau \in \mathcal{H} \).

Consider now the action of the group \( \Gamma_{L,1} \cap W_U = \bar{H}(\Lambda, \mathbb{Z}) \) and its subgroup \( \text{Hom}^a(I^*, I) \cong \Lambda^2 I \cong \mathbb{Z} \), the notation for which in the theory of toroidal compactifications is \( U_\mathcal{C}(F) \) and \( W_\mathcal{C}(F) \) respectively. Examining the latter yields the following consequence of Proposition (17).

**Corollary 4.2.** The quotient \( U_\mathcal{C}(F) \backslash G(V_\mathfrak{R}) \) is a punctured disc bundle over the space \( \bar{D}(F) \).

**Proof.** One easily verifies that elements of \( U_\mathcal{C}(F) \) operate by addition on the coordinate \( \sigma \), and dividing a translated upper half-plane by \( \mathbb{Z} \) gives a punctured disc. This proves the corollary. \( \square \)

The coordinate on the fibers of the map from Corollary (1.2) is the one denoted by \( q_2 \) in \([K]\). The toroidal boundary component in which we are interested is obtained by filling in the zero section of this disc bundle, and it is therefore isomorphic to the image of \( \bar{D}(F) \) under the action of \( \Gamma_{L,1} \). The first step of determining this image is dividing by the kernel \( \Lambda \times \Lambda \) of the associated short exact sequence from Corollary (3.3) (this is the group \( W_\mathcal{C}(F)/U_\mathcal{C}(F) \), denoted by \( V_\mathcal{C}(F) \) in \([AMRT]\) and others), and for describing the result we recall that \( \mathcal{E} \rightarrow \mathcal{H} \) is the universal elliptic curve, with fiber \( E_\tau := \mathbb{C}/(\mathbb{Z} \oplus \mathbb{Z} \tau) \) over \( \tau \in \mathcal{H} \).
Proposition 4.3. The quotient $\mathcal{V}_2(F) \setminus \bar{D}(F)$ is a principal homogenous space of the universal family $\mathcal{E} \otimes \Lambda$ over $\mathcal{H}$. It carries the punctured disc bundle $W_2(F) \setminus G(V_b)$, whose fiber coordinate is preserved by $(\Lambda \times \Lambda)^0$, but inverted under elements of $(\Lambda \times \Lambda)^1$.

Proof. It is easy to check that the action of $\mathcal{V}_2(F) \equiv \Lambda \times \Lambda$ from Corollary 3.3 becomes the additive action of the lattice $\Lambda \otimes \Lambda \tau$ on each fiber $W^{1,\tau}_C$ of the second map from Proposition 4.4. This proves the first assertion, since $W^{1,\tau}_C$ is an affine model, or a principal homogenous space, of the complex vector space $W_C = \Lambda_C$. The second one follows from the fact that an element of $W_2(F) = H(\Lambda, \mathbb{Z})$ with image in $(\Lambda \times \Lambda)^0$ adds an integer to $\sigma$ hence preserve $q_2$, but if the image is in $(\Lambda \times \Lambda)^1$ then our element adds a half-integer to $\sigma$ and therefore inverts the sign of $q_2$. This proves the proposition.

The preliminary version of the form of the toroidal boundary component that lies over $F$, which is the precise one if $\iota = 0$ and $\Gamma_L$ is trivial (the latter always happens when $\Gamma_L$ is assumed to be neat), and gives a finite cover of the precise one in general, is as follows. We denote by $W^{\Lambda}_L$ the open Kuga–Sato variety that lies over $\Gamma_L \setminus H$, in which the fiber over an element $\Gamma_L \tau$ is isomorphic to $E_\tau \otimes \Lambda$, and if $\tilde{I}$ and $\tilde{U}$ are chosen and $\iota$ is the map from Proposition 2.3 then $W^{\Lambda}_L$ is the similarly defined open Kuga–Sato variety over $\Gamma_L \setminus H$.

Proposition 4.4. Consider the pre-image of $\Gamma_i \times \{\text{Id}_\Lambda\} \subseteq \text{SL}(I) \times \Gamma_L$ in $\bar{D}_L, I$. The quotient of $\bar{D}(F)$ by this group is a principal homogenous space over $W^{\Lambda}_L$.

Proof. The statement follows from the structure of this pre-image as a semi-direct product over $\mathbb{Z}$, proved in Proposition 3.2 and Corollary 3.3. This proves the proposition.

Note that while the quotient of $G(V_b)$ by the pre-image in $\Gamma_L, I$ of the group from Proposition 4.4 is punctured disc bundle over the principal homogenous space there, the effect on the coordinate $q_2$ becomes more complicated. This happens for the same reason why this pre-image in not a semi-direct product over $\mathbb{Z}$ in general—the effect of the anti-symmetric term $M\alpha - \alpha M^*$ can multiply $q_2$ by some roots of unity, over which we have no control in general.

The structure of a principal homogenous space means that when we choose the complements $I$ and $U$, each affine space $W^{1,\tau}_C$ becomes naturally isomorphic to the vector space $\tilde{W}_C$, by subtracting $\zeta + \tau \omega$ from Equation (10). Hence after dividing the fiber over $\tau$ by $\Lambda \otimes \Lambda \tau$ we obtain in Proposition 4.4 a well-defined zero section for the projection from $W^{\Lambda}_L$ onto $\Gamma \setminus H$. As the general statement in Theorem 4.5 shows, such a zero section is not well-defined in general. Recall that each element $M \in \text{SL}(I_L, I)$ defines the class $b_M$ from Equation (15), and that when $\dim U = 2$ (hence $M \in \Gamma_{L^*}$), the class $b_M$ lies in $\Delta_\Lambda \times \Delta_\Lambda$. The cocycle condition from Proposition 3.4 allows us to define $W^{\Lambda, b}_L$ to be the quotient of the universal family $\mathcal{E} \otimes \Lambda$ from Proposition 4.4 under the action of $\Gamma_{L^*}$, in which an element $M$ in the latter group also acts on the fibers by translation by the image of $b_M$ in $\Delta_\Lambda \otimes \Delta_\Lambda \tau \subseteq E_\tau \otimes \Lambda$.  

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In addition, the group $\Gamma_\Lambda$ is a finite group (since $\Lambda$ is negative definite), and it acts on the space $W_\mathbb{C}$. The quotient $\Gamma_\Lambda \backslash W_\mathbb{C}$ is the affine algebraic variety $\text{Spec}[\langle \text{Sym}^* W \rangle]^{\Gamma_\Lambda}$ (under the identification $W \cong W^*$, and as $\Gamma_\Lambda$ operates on the fibers of the map $E \otimes \Lambda \to \mathcal{H}$, the structure sheaf of every fiber of the quotient is obtained by taking the $\Gamma_\Lambda$-invariant functions in the structure sheaf of $E_\tau \otimes \Lambda$. After dividing by $\Gamma$, we obtain a well-defined quotient $\Gamma_\Lambda \backslash W_\tau^\Lambda$ (with a similar structure sheaf), and since $\Gamma_\Lambda$ operates trivially on $\Delta_\Lambda$, the quotient $\Gamma_\Lambda \backslash W_\tau^\Lambda, b$ is also well-defined. Considering the actions of all the groups involved therefore yields the following description of our toroidal boundary component.

**Theorem 4.5.** The quotient $\Gamma_{L,I} \backslash \bar{D}(F)$ is isomorphic to $\Gamma_\Lambda \backslash W_{L^*}^{\Lambda,b}$.

Note that while $W_{L^*}^{\Lambda,b}$ does have a zero section that is defined up to a subgroup of $\Delta_\Lambda \times \Delta_\Lambda$, the more canonical definition of $\Gamma_{L,I} \backslash \bar{D}(F)$ in Theorem 4.5 does not have a well-defined zero section at all, since it is constructed from the affine vector bundle $\bar{D}(F)$ from Equation (17). As in Proposition 4.4, the bundle $\Gamma_{L,I} \backslash G(V_\mathbb{R})$ is again a punctured disc bundle over the variety from Theorem 4.5, but with the coordinate $q_2$ changing by roots of unity which may be wild.

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