Introduction to a Non-Commutative Version of the Standard Model

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The XIV International Hutsulian Workshop, Oct 28 - Nov 2, 2002, Chernivtsi, Ukraine

Abstract
This article provides a basic introduction to some concepts of non-commutative geometry. The importance of quantum groups and quantum spaces is stressed. Canonical non-commutativity is understood as an approximation to the quantum group case. Non-commutative gauge theory and the non-commutative Standard Model are formulated on a space-time satisfying canonical non-commutativity relations. We use *-formalism and Seiberg-Witten maps.
In these lectures I want to give a basic introduction to the Non-Commutative Standard Model advocated in [1]. The underlying mathematical ideas shall be introduced with some care. Although the ideas are in a way not necessary to understand the physics discussed in the late Chapters, they are nevertheless vital in gaining a better understanding and a sound picture. Why we discuss some important aspects of non-commutative geometry first. We will especially discuss the case of quantum groups and quantum spaces. Quantum spaces carry a (co-)representation of a quantum group. The quantum group describes the symmetry of that space. The notion of symmetry is a very important one in physics, and it can be generalised to some classes of non-commutative spaces. However, we will only discuss models on spaces obeying so-called canonical non-commutativity relation, which does not allow for a symmetry group. Classical Lorentz symmetry is broken, and there is no deformation of the symmetry present. In some sense, one can think of the canonical case as an approximation to the quantum group case. In both cases, the non-commutativity is characterised by a parameter $q$ or $\theta$, respectively. Of course, a very important feature is that in the limit of vanishing non-commutativity - $q \to 1, \theta \to 0$, respectively - we end up with the usual commutative theory. In Chapter 2, we will discuss a special approach to non-commutative geometry, namely $\ast$-products. One of the advantages of this approach is that the commutative limit is very transparent. In the third Chapter, we will discuss gauge theory on canonically non-commutative space-time [2]. The last two Chapters are devoted to the Standard Model and its generalisation to non-commutative space-time using techniques developed in Chapter 3.

1 Non-Commutative Geometry

Let me first try to give you some handwaving idea what picture we have in mind when we talk of non-commutative (nc) geometry. Some examples will show, where non-commutativity has already shown up in physics, and how these ideas might be useful. After all these motivations, I want to formulate some aspects of nc geometry mathematically. We will mainly be concerned with quantum groups and quantum spaces.

1.1 What is Non-Commutative Geometry?

As the word "non-commutative" says, the commutator of some elementary quantities is doomed not to vanish. Nc geometry is based on non-commutative coordinates

$$[\hat{x}^i, \hat{x}^j] \neq 0,$$

(1)

i.e., coordinates are non-commutative operators and we have to think in quantum mechanical terms. $\hat{x}^i$’s cannot all be diagonalised simultaneously. Space-time is the col-
lection of the eigenvalues (spectrum) of the operators $\hat{x}^i$. If the spectrum is discrete, space-time will be discrete. Commutative coordinates induce a continuous spectrum. Therefore, also space-time will be continuous. 

The theory of nc geometry is based on the simple idea of replacing ordinary coordinates with non-commuting operators. We will see how this idea can be formulated mathematically.

1.2 Physical Motivation for Non-Commuting Coordinates

But before we do so, let us consider some examples.

1.2.1 Divergencies in QFT

In quantum field theories, loop contributions to the transition amplitudes diverge. Consider a real scalar particle $\phi$ which is described by the action

$$\mathcal{L} = \frac{1}{2} \partial_\mu \phi \partial^\mu \phi - \frac{1}{2} m^2 \phi^2 + \frac{\lambda}{4!} \phi^4.$$ \hspace{1cm} (2)

The contribution of the diagram shown in Fig. 1 reads

$$\int \frac{d^4 q}{(2\pi)^4} \frac{1}{q^2 - m^2}.$$ \hspace{1cm} (3)

The result is divergent. The renormalisation procedure may remove some of the infinities. The theory is called renormalisable, if all divergencies can be removed with a finite number of counter terms. The theory defined by (2) is renormalisable in 4 dimensions. However, no renormalisable quantum field theory of gravity is known so far. Discretising space-time may introduce a momentum cutoff in a canonical way and render the theory finite or at least renormalisable. The hope is that this can be accomplished by making space-time non-commutative.

1.2.2 Quantum Gravity

All kind of models for and approaches to quantum gravity seem to lead to a fundamental length scale, i.e., to a lower bound to any position measurement. This seems to be a model independent feature. The uncertainty in space-time measurement can be explained by replacing coordinates by nc operators.
1.2.3 String Theory

In open string theory with a background B-field, the endpoints of the strings are confined to submanifolds (D-branes) and become non-commutative [4]. This is true even on an operator level,

\[ [X^i, X^j] = i\theta^{ij} \]

where \( \theta^{ij} = -\theta^{ji} \in \mathbb{R} \), and \( X^i \) are the coordinates of the 2-dimensional world sheet embedded in the target space (e.g., \( \mathbb{R}^{10} \)), i.e., operator valued bosonic fields. Therefore, we also have for the propagator

\[ \langle [X^i, X^j] \rangle = i\theta^{ij}. \]

1.2.4 Classical Non-Commuting Coordinates

Consider a particle with charge \( e \) moving in a homogeneous and constant magnetic field. The action is given by

\[ S = \int dt \left( \frac{1}{2} m \ddot{x}^\mu \dot{x}^\mu - \frac{e}{c} B_{\mu\nu} x^\mu \dot{x}^\nu \right), \]

where \( B_{\mu\nu} \) is an antisymmetric tensor defining the vector potential \( A^\mu, B_{\mu\nu} = -B_{\nu\mu} \) and \( A^\nu = B_{\mu\nu} x^\mu \). The classical commutation relations are

\[ \{ \pi^\mu, x^\nu \} = \delta^\nu_\mu, \]

where \( \{,\} \) is the classical Poisson structure. Writing it out explicitly, we get

\[ \{ \dot{x}^\mu, x^\nu \} + \frac{eB_{\mu\sigma}}{cm} \{ x^\sigma, x^\nu \} = \frac{1}{m} \delta^\mu_\nu, \]

where \( \pi^\mu = \frac{\partial L}{\partial \dot{x}^\mu} = m \dot{x}^\mu + \frac{e}{c} B_{\mu\nu} x^\nu \). Let us assume strong magnetic field \( B \) and small mass \( m \) - i.e., we restrict the particle to the lowest Landau level [5]. In this approximation, eqn. [8] simplifies, and we get [6]

\[ \{ x^\sigma, x^\nu \} = \frac{c(B^{-1})^{\sigma\nu}}{e}. \]

The coordinates perpendicular to the magnetic field do not commute, on a classical level.

1.3 Systematic Approach

Let us examine the classical situation depicted in Fig. [2]. We start with a smooth and compact manifold \( \mathcal{M} \). The topology of \( \mathcal{M} \) is uniquely determined by the algebra
of continuous complex (real) valued functions on $\mathcal{M}$, $C(\mathcal{M})$ with the usual involution (Urysohn’s Lemma [7]). The Gel’fand-Naimark theorem [8] relates the function algebra to an abelian $C^*$-algebra. The algebra of continuous functions over a compact manifold $\mathcal{M}$ is isomorphic to an abelian unital $C^*$-algebra. The algebra of continuous functions vanishing at infinity over a locally compact Hausdorff space $C^0(\mathcal{M})$ is isomorphic to an abelian $C^*$-algebra (not necessarily unital).

![Figure 2: Classical algebraic geometry](image)

Coordinates on the manifold are replaced by coordinate functions in $C(\mathcal{M})$, vector fields by derivations of the algebra. Points are replaced by maximal ideals, cf. Fig. 3.

![Figure 3: Algebraic geometry](image)

The trick in nc geometry is to replace the abelian $C^*$ algebra by a non-abelian one and to reformulate as much of the concepts of algebraic geometry as possible in terms of non-abelian $C^*$ algebras [9,10].

In the following, this nc algebra $\mathcal{A}$ will be given by the algebra of formal power series generated by the nc coordinate functions $\hat{x}^i$, divided by an ideal $\mathcal{I}$ of relations generated by the commutator of the coordinate functions,

$$\mathcal{A} = \frac{\mathbb{C}\langle\langle \hat{x}^1, \ldots, \hat{x}^n \rangle\rangle}{\mathcal{I}}, \quad (10)$$

where $[\hat{x}^i, \hat{x}^j] \neq 0$. Most commonly, the commutation relations are chosen to be either constant or linear or quadratic in the generators. In the canonical case the relations are
constant,
\[ [\hat{x}^i, \hat{x}^j] = i\theta^{ij}, \] (11)
where \( \theta^{ij} \in \mathbb{C} \) is an antisymmetric matrix, \( \theta^{ij} = -\theta^{ji} \). The linear or Lie algebra case
\[ [\hat{x}^i, \hat{x}^j] = i\lambda^{ij}_k \hat{x}^k, \] (12)
where \( \lambda^{ij}_k \in \mathbb{C} \) are the structure constants, basically has been discussed in two different approaches, fuzzy spheres [11] and \( \kappa \)-deformation [12,13,14]. Last but not least, we have quadratic commutation relations
\[ [\hat{x}^i, \hat{x}^j] = \frac{1}{q} \hat{R}^{ij}_{kl} \hat{x}^k \hat{x}^l, \] (13)
where \( \hat{R}^{ij}_{kl} \in \mathbb{C} \) is the so-called \( \hat{R} \)-matrix which will be discussed in some detail later, corresponding to quantum groups. For a reference, see e.g., [15,16].

Let us discuss one specific approach to nc geometry in some more detail, namely quantum groups. The main part of this talk will deal with the Standard Model on canonical space-time. We will consider canonical spaces as an approximation in some sense, to quantum spaces. The advantage of quantum spaces is that the concept of symmetry can be generalised to quantum groups. Whereas canonical space-time does not allow for a generalised Lorentz symmetry.

1.3.1 q-Deformed Case
Classically, symmetries are described by Lie algebras or Lie groups. Physical spaces are representation spaces of its symmetry algebra - or respectively co-representations of the function algebra over its symmetry group. Therefore, we will introduce nc spaces as co-representation spaces of some quantum group. The interpretation goes as follows: Space-time is a continuum in the low energy domain, at high energies - Planck energy or below - space-time undergoes a phase transition and becomes a "fuzzy" nc space. Therefore the symmetries are not broken, but deformed to a quantum group.

What is a quantum group? Let us start with the function algebra over a classical Lie algebra \( \mathcal{F}(\mathcal{G}) \). \( \mathcal{F}(\mathcal{G}) \) is a Hopf algebra which will be defined in a minute. Then there is a well defined transition from the classical function algebra to the respective quantum group, \( \mathcal{F}(\mathcal{G}) \rightarrow \mathcal{F}(\mathcal{G})_q \), introducing the non-commutativity parameter \( q \in \mathbb{C} \). In the classical limit, \( q \rightarrow 1 \), we have to regain the classical situation. This is the basic property of a deformation.

As we mentioned before, the enveloping algebra of a Lie algebra and the function algebra over a Lie group are in a natural way Hopf algebras. Most importantly, q-deformation does not lead out of the category of Hopf algebras. Therefore we will now define the concept of Hopf algebras and quantum groups.
**Hopf algebra.** A Hopf algebra $A$ (see, e.g., [17]) consists of an algebra and a co-algebra structure which are compatible with each other. Additionally, there is a map called antipode, which corresponds to the inverse of a group. $A$ is an algebra, i.e., there is a multiplication $m$ and a unit element $\eta$,

$$m : A \otimes A \to A, \quad \eta : \mathbb{C} \to A, \quad a \otimes b \mapsto ab, \quad c \mapsto c1_A,$$

such that the multiplication satisfies the associativity axiom (Fig. 4) and $\eta$ satisfies the

$$\begin{align*}
A \otimes A \otimes A & \xrightarrow{m \otimes id} A \otimes A \\
& \xrightarrow{id \otimes m} A \otimes A \otimes A \\
& \xrightarrow{id \otimes id} A \otimes A \otimes A \\
& \xrightarrow{m \otimes id} A \otimes A \\
& \xrightarrow{id \otimes m} A \otimes A \\
& \xrightarrow{m} A \\
\end{align*}$$

$$ab)(bc) = (ab)c = a(bc)$$

**Figure 4: Associativity**

Reversing all the arrows in Figs. 4 and 5 and replacing $m$ by the co-product $\Delta$ and $\eta$ by the co-unit $\epsilon$ gives us the axioms for the structure maps of a co-algebra. The co-product and the co-unit,

$$\begin{align*}
\Delta : A \to A \otimes A, \\
\epsilon : A \to \mathbb{C}
\end{align*}$$

**Figure 5: Unity axiom**
are the dual to $m$ and $\eta$, respectively. Compatibility between algebra and co-algebra structure means that the co-product $\Delta$ and the co-unit $\epsilon$ are algebra homomorphisms,

\begin{align}
\Delta(ab) &= \Delta(a)\Delta(b), \\
\epsilon(ab) &= \epsilon(a)\epsilon(b),
\end{align}

where $a, b \in A$. The antipode $S : A \rightarrow A$ satisfies the axiom shown in Fig. 6. It is an anti-algebra homomorphism.

\[
\begin{array}{cccc}
A \otimes A & \xrightarrow{\Delta} & A & \xrightarrow{\Delta} & A \otimes A \\
\downarrow{id \otimes S} & & \downarrow{\eta \circ \epsilon} & & \downarrow{id \otimes S} \\
A \otimes A & \xrightarrow{m} & A & \xrightarrow{m} & A \otimes A
\end{array}
\]

Figure 6: Antipode axiom

If $A$ is the algebra of functions over some matrix group, the antipode $S$ is the inverse,

\begin{equation}
S(t^i_j) = (t^{-1})^i_j,
\end{equation}

where $t^i_j$ are the coordinate functions and generate the algebra of functions.

Let me quote the structure maps for the function algebra and its dual. Let $G$ be an arbitrary, (for simplicity) finite group and $\mathcal{F}(G)$ the Hopf algebra of all complex-valued functions on $G$. Then the algebra structure of $\mathcal{F}(G)$ is given by

\begin{align}
m : \mathcal{F}(G) \otimes \mathcal{F}(G) &\rightarrow \mathcal{F}(G), \\
m(f_1 \otimes f_2)(g) &= f_1(g)f_2(g), \\
\eta : \mathbb{C} &\rightarrow \mathcal{F}(G), \\
\eta(k) &= k \mathbf{1}_{\mathcal{F}(G)}.
\end{align}

And we have the following co-algebra structure

\begin{align}
\Delta : \mathcal{F}(G) &\rightarrow \mathcal{F}(G) \otimes \mathcal{F}(G), \\
\Delta(f)(g_1 \otimes g_2) &= f(g_1g_2), \\
\epsilon : \mathcal{F}(G) &\rightarrow \mathbb{C}, \\
\epsilon(f) &= f(e),
\end{align}
where $e$ is the unit element of $G$. Eventually, the antipode is given by

$$ (S(f))(g) = f(g^{-1}). \quad (21) $$

$\mathcal{F}(G)$ is a commutative Hopf algebra.

Let us consider its dual. Let $\mathfrak{g}$ be a Lie algebra. The universal enveloping algebra is defined as

$$ U(\mathfrak{g}) = \frac{T(\mathfrak{g})}{x \otimes y - y \otimes x - [x, y]}, \quad (22) $$

where $T(\mathfrak{g})$ is the universal tensor algebra. Its algebra structure is given by the commutator and its unit element by the unit in $T(\mathfrak{g})$. The other structure maps are consistently defined as

$$ \Delta(x) = x \otimes 1 + 1 \otimes x, \quad (23) $$
$$ \epsilon(x) = 0, \quad (24) $$
$$ S(x) = -x. \quad (25) $$

$U(\mathfrak{g})$ is a co-commutative Hopf algebra, i.e., the co-product is symmetric.

Quantum Group. A quantum group is a Hopf algebra with one additional structure. Let us concentrate on the function algebra $\mathcal{F}(G)$ over some Lie group $G$ rather than on its dual, the universal enveloping algebra of the Lie algebra. The additional structure is the $R$-form,

$$ R : A \otimes A \to \mathbb{C}. $$

$\mathcal{F}(G)$ is a commutative algebra, the $R$-form describes the almost commutativity of the deformed product. Let us denote the quantum group $\mathcal{F}(G)_q$, since $R$ depends on the non-commutativity parameter $q$. Let $t^i_j$ be the coordinate functions generating $\mathcal{F}(G)_q$. The generators satisfy the so-called $RTT$-relations expressing the almost commutativity

$$ R^{ijk}_{kl} t^i_m t^j_n = t^j_l R^{kl}_{mn}, \quad (26) $$

where $R(t^i_k \otimes t^j_l) = R^{ij}_{kl}$. In the commutative limit, $R \to 1$, we have commutative generators,

$$ t^k_m t^l_n = t^l_n t^k_m. \quad (27) $$

$R$ is a solution of the Quantum-Yang-Baxter-Equation (QYBE)

$$ R_{12} R_{13} R_{23} = R_{23} R_{13} R_{12}, \quad (28) $$

where $(R_{13})^{ijk}_{lmn} = \delta^j_m t^{ik}_n$, $R_{12}$ and $R_{23}$ are defined accordingly.
Quantum spaces, \( \mathcal{M}_q \). A quantum space for a quantum group \( \mathcal{F}(\mathcal{G})_q \) has two basic properties. \( \mathcal{M}_q \) is a \( \mathcal{F}(\mathcal{G})_q \)-co-module algebra and in the commutative limit, \( q \to 1 \), \( \mathcal{M} \) is the proper \( \mathcal{F}(\mathcal{G}) \)-co-module space.

\[
\mathcal{M}_q \equiv \mathbb{C}\langle \langle \hat{x}^1, \ldots, \hat{x}^n \rangle \rangle / \mathcal{I}, \tag{29}
\]

where \( \mathcal{I} \) is the ideal generated by the commutation relations of the generators \( \hat{x}^i \). But how can the commutation relations be chosen consistently? The product in \( \mathcal{M}_q \) has to be compatible with the co-action of \( \mathcal{F}(\mathcal{G})_q \).

First let us introduce \( \hat{R} \equiv R \circ \tau \). In the classical limit, \( \hat{R} \) is just the permutation \( \tau \), \( \tau(a \otimes b) = b \otimes a \). \( \hat{R} \) can be decomposed into projectors,

\[
\hat{R} = \lambda_1 \hat{P}_S + \lambda_2 \hat{P}_A, \tag{30}
\]

where \( \hat{P}_A \) is the q-deformed generalisation of the antisymmetriser, \( \hat{P}_S \) of the symmetriser, respectively. The relations

\[
\hat{P}_A^{mn} \hat{x}^i \hat{x}^j = 0 \tag{31}
\]
on \( \mathcal{M}_q \) satisfy both requirements. In the commutative limit, (31) means that the commutator of two coordinates vanishes. It is also covariant under the co-action \( \rho \) of the quantum group

\[
\rho : \mathcal{M}_q \to \mathcal{F}(\mathcal{G})_q \otimes \mathcal{M}_q, \tag{32}
\]

\[
\rho(\hat{x}^i) = t^i_j \otimes \hat{x}^j, \tag{33}
\]

since

\[
\hat{P}_A^{mn} (t^i_k \otimes \hat{x}^k)(t^j_l \otimes \hat{x}^l) = t^m_i t^n_j \otimes \hat{P}_A^{ij} \hat{x}^k \hat{x}^l = 0. \tag{34}
\]

\( \hat{P}_A \) is a polynomial in \( \hat{R} \), and the \( \hat{R} TT \) relations (26) can be applied (\( R^{ij} = \hat{R}^{ij} \)).

**Differentials**, \( \hat{\partial}_A \). \( \hat{\partial}_A \) satisfy the same commutation relations as the coordinates [18],

\[
\hat{P}_A^{ij} \hat{\partial}_i \hat{\partial}_j = 0. \tag{35}
\]

This follows from the assumptions on the exterior derivative \( d \). The exterior derivative \( d = \xi^A \hat{\partial}_A \) shall have the same properties as in the classical case,

\[
d^2 = 0, \tag{36}
\]

\[
d \hat{x}^A = \xi^A + \hat{x}^A d, \tag{37}
\]

where the coordinate differentials \( \xi^A \) are supposed to anticommute, i.e.,

\[
\hat{P}_S^{AB} \xi^C \xi^D = 0. \tag{38}
\]
Consequently, the differentials satisfy a modified Leibniz rule
\[ \hat{\partial}_A(\hat{f}\hat{g}) = (\hat{\partial}_A\hat{f})\hat{g} + O_{A^B}(\hat{f})\hat{g}, \] (39)
where the operator \( O_{A^B} \) is a homomorphism \( O_{A^B}(\hat{f}\hat{g}) = O_{A^C}(\hat{f})O_{C^B}(\hat{g}) \).

I want to finish this Section on quantum groups and quantum spaces with a popular two dimensional example, the Manin plane.

**Example: Manin Plane**, see e.g., [19]. The Manin plane is generated by the two coordinates \( \hat{x}, \hat{y} \). The underlying symmetry is given by the quantum algebra \( \mathcal{U}_q(sl_2) \).

- The coordinates satisfy
  \[ \hat{x}\hat{y} = q\hat{y}\hat{x}. \] (40)
- The differentials fulfill the same relation, except for some scaling factor,
  \[ \hat{\partial}_x\hat{\partial}_y = \frac{1}{q}\hat{\partial}_{\hat{y}}\hat{\partial}_{\hat{x}}. \] (41)
- The crossrelations compatible with the above structures are given by
  \[ \hat{\partial}_x\hat{x} = 1 + q^2\hat{x}\hat{\partial}_x + q\lambda\hat{y}\hat{\partial}_y, \] (42)
  \[ \hat{\partial}_x\hat{y} = q\hat{y}\hat{\partial}_x, \] (43)
  \[ \hat{\partial}_y\hat{x} = q\hat{x}\hat{\partial}_y, \] (44)
  \[ \hat{\partial}_y\hat{y} = 1 + q^2\hat{y}\hat{\partial}_y, \] (45)
where \( \lambda = q - \frac{1}{q} \).
- The symmetry algebra \( \mathcal{U}_q(sl_2) \) is generated by \( T^+,T^-,T^3 \), which satisfy the following defining relations
  \[ \frac{1}{q}T^+T^- - qT^-T^+ = T^3, \]
  \[ q^2T^3T^+ - \frac{1}{q^2}T^+T^3 = (q + \frac{1}{q})T^+, \] (46)
  \[ q^2T^-T^3 - \frac{1}{q^2}T^3T^- = (q + \frac{1}{q})T^- . \]
The action of the generators on coordinates is given by

\[
\begin{align*}
T^3 \hat{x} &= q^2 \hat{x} T^3 - q \hat{x}, \\
T^3 \hat{y} &= \frac{1}{q^2} \hat{y} T^3 + \frac{1}{q} \hat{y}, \\
T^+ \hat{x} &= q \hat{x} T^+ + \frac{1}{q} \hat{y}, \\
T^+ \hat{y} &= \frac{1}{q} \hat{y} T^+, \\
T^- \hat{x} &= q \hat{x} T^-, \\
T^- \hat{y} &= \frac{1}{q} \hat{y} T^- + q \hat{x}.
\end{align*}
\]  

(47)

is defined via the ordinary integral introducing a weight function \(\omega\), \(\omega(x, y) = \frac{1}{xy}\). In the classical limit, the symmetry algebra fulfills the usual \(sl_2\) relations,

\[
\begin{align*}
[T^+, T^-] &= T^3, \\
[T^3, T^+] &= 2 T^+, \\
[T^-, T^3] &= 2 T^-.
\end{align*}
\]  

(48)

Eqns. (47) reduce to the usual action of the generators of angular momentum on a spin-1/2 state.

1.3.2 Canonical Case

For most parts of the rest of my lectures, we will concentrate on the canonical case. Thus let us summarize the most important properties. As discussed before, Minkowski space with canonical commutation relations does not allow for a Lorentz symmetry. Only a translational symmetry is present. Compared to the quantum group case or other more sophisticated examples, calculations can be done more easily and more interesting models can be studied. The commutator of two coordinates is a constant

\[
\begin{align*}
[\hat{x}^\mu, \hat{x}^\nu] &= i \theta^{\mu\nu},
\end{align*}
\]  

(49)

where \(\theta^{\mu\nu} = -\theta^{\nu\mu} \in \mathbb{C}\). The derivatives act on coordinates as in the classical case,

\[
\begin{align*}
[\hat{\partial}_\nu, \hat{x}^\mu] &= \delta^\mu_\nu.
\end{align*}
\]  

(50)

However, there are two consistent ways to define commutation relations of derivatives. By observing that

\[
\hat{x}^\mu - i \theta^{\mu\nu} \hat{\partial}_\nu
\]  

(51)
commutes with all coordinates $\hat{x}^\nu$ and all derivatives $\hat{\partial}_\nu$ one may assume that this expression equals some constant, 0 say. Thus, we can define a derivative in terms of the coordinates (for invertible $\theta$),

$$\hat{\partial}_\mu = -i\theta^{-1}_{\mu\nu} \hat{x}^\nu.$$  \hfill (52)

The commutator of derivatives is given by

$$[\hat{\partial}_\mu, \hat{\partial}_\nu] = i(\theta^{-1})_{\mu\nu}.$$  \hfill (53)

The other possibility is to define

$$\hat{\partial}_\mu \hat{f} = -i\theta^{-1}_{\mu\nu}[\hat{x}^\nu, \hat{f}]$$  \hfill (54)

which leads to

$$[\hat{\partial}_\mu, \hat{\partial}_\nu] = 0.$$  \hfill (55)

The integral is given by the usual four dimensional integral over commutative functions

$$\int \hat{f} \hat{g} := \int d^4x \, f \ast g(x) = \int d^4x \, f(x) \, g(x).$$  \hfill (56)

In the next Section we will discuss what we mean by the map $f \rightarrow \hat{f}$, mapping the function $f$ depending on commutative coordinates $x^\mu$ onto the nc function $\hat{f}$, and by the product $\ast$.

All the necessary prerequisites for field theory with action integral are met. But before we are going to turn to physics, to gauge theory, we will talk about the $\ast$-product approach. In this approach the commutative limit is very transparent.

## 2 Star Products

Let us consider the nc algebra of functions $\hat{A}$ on canonical Minkowski space

$$\hat{A} = \frac{\mathbb{C}\langle\langle \hat{x}^1, ..., \hat{x}^n\rangle\rangle}{\mathcal{I}},$$  \hfill (57)

where $\mathcal{I}$ is the ideal generated by the commutation relations of the coordinate functions, and the commutative algebra of functions

$$A = \frac{\mathbb{C}\langle\langle x^1, ..., x^n\rangle\rangle}{[x^i, x^j]} \equiv \mathbb{C}[x^1, ..., x^n],$$  \hfill (58)

i.e., $[x^i, x^j] = 0$. Our aim in this Section is to relate these algebras by an isomorphism. Let us first consider the vector space structure of the algebras, only. In order to construct
a vector space isomorphism, we have to choose a basis (ordering) in $\hat{\mathcal{A}}$ satisfying the Poincaré-Birkhoff-Witt property, e.g., the basis of symmetrically ordered monomials,

$$1, \hat{x}^i, \frac{1}{2}(\hat{x}^i\hat{x}^j + \hat{x}^j\hat{x}^i), \ldots$$

(59)

Now we map the basis monomials in $\mathcal{A}$ onto the according symmetrically ordered basis elements of $\hat{\mathcal{A}}$.

$$W : \mathcal{A} \rightarrow \hat{\mathcal{A}},$$

$$x^i \mapsto \hat{x}^i,$$

$$x^i x^j \mapsto \frac{1}{2}(\hat{x}^i\hat{x}^j + \hat{x}^j\hat{x}^i) \equiv : \hat{x}^i\hat{x}^j : .$$

The ordering is indicated by $:\cdot :$. $W$ is an isomorphism of vector spaces. In order to extend $W$ to an algebra isomorphism, we have to introduce a new non-commutative multiplication $\ast$ in $\mathcal{A}$. This $\ast$-product is defined by

$$W(f \ast g) := W(f) \cdot W(g) = \hat{f} \cdot \hat{g},$$

(61)

where $f, g \in \mathcal{A}, \hat{f}, \hat{g} \in \hat{\mathcal{A}}$.

$$\mathcal{A}, \ast \cong (\hat{\mathcal{A}}, \cdot),$$

(62)

i.e., $W$ is an algebra isomorphism. The information on the non-commutativity of $\hat{\mathcal{A}}$ is encoded in the $\ast$-product.

### 2.1 Construction of a $\ast$-Product of Functions

Let us choose symmetrically ordered monomials as basis in $\hat{\mathcal{A}}$. The commutation relations of the coordinates are

$$[\hat{x}^i, \hat{x}^j] = i\theta^{ij}(\hat{x}),$$

(63)

where $\theta(\hat{x})$ is an arbitrary expression in the coordinates $\hat{x}$, for now. In just a moment we will discuss the special cases (11 - 13). The Weyl quantisation procedure [20, 21] is given by the Fourier transformation,

$$\hat{f} = W(f) = \frac{1}{(2\pi)^n/2} \int d^n k e^{ik_j\hat{x}^j} \tilde{f}(k),$$

(64)

$$\tilde{f}(k) = \frac{1}{(2\pi)^n/2} \int d^n x e^{-ik_jx^j} f(x),$$

(65)

where we have replaced the commutative coordinates by nc ones ($\hat{x}^i$) in the inverse Fourier transformation (64). The exponential takes care of the symmetrical ordering. Using eqn. (61), we get

$$W(f \ast g) = \frac{1}{(2\pi)^n} \int d^n k d^n p e^{ik_j\hat{x}^j} e^{ip_j\hat{x}^j} \tilde{f}(k)\tilde{g}(p).$$

(66)
Because of the non-commutativity of the coordinates $\hat{x}^i$, we need the Campbell-Baker-Hausdorff (CBH) formula

$$e^A e^B = e^{A + B + \frac{1}{2}[A,B] + \frac{1}{12}([A,B], B) + \frac{1}{12}[[A,B], A] + \ldots}.$$  \hfill (67)

Clearly, we need to specify $\theta^{ij}(\hat{x})$ in order to evaluate the CBH formula.

**Canonical Case.** Due to the constant commutation relations, the CBH formula will terminate, terms with more than one commutator will vanish,

$$\exp(ik_i \hat{x}^i) \exp(ip_j \hat{x}^j) = \exp\left(i(k_i + p_i)\hat{x}^i - \frac{i}{2}k_i \theta^{ij} p_j\right).$$  \hfill (68)

Eqn. (66) now reads

$$f \ast g(x) = \int d^n k d^n p e^{i(k_i + p_i) x^i - \frac{i}{2}k_i \theta^{ij} p_j} \tilde{f}(k) \tilde{g}(p)$$  \hfill (69)

and we get for the $\ast$-product the Moyal-Weyl product \textsuperscript{[22]}

$$f \ast g(x) = \exp\left(\frac{i}{2} \frac{\partial}{\partial x^i} \theta^{ij} \frac{\partial}{\partial y^j}\right) f(x) g(y) \bigg|_{y \to x}. \hfill (70)$$

**Lie Algebra Case.** The coordinates build a Lie algebra

$$[\hat{x}^i, \hat{x}^j] = i\lambda_{kj}^i \hat{x}^k,$$  \hfill (71)

with structure constants $\lambda_{kj}^i$. In this case the CBH sum will not terminate and we get

$$\exp(ik_i \hat{x}^i) \exp(ip_j \hat{x}^j) = \exp\left(i(k_i + p_i)\hat{x}^i + \frac{i}{2}g_i(k, p)\hat{x}^i\right),$$  \hfill (72)

where all the terms containing more than one commutator are collected in $g_i(k, p)$. (66) becomes

$$f \ast g(x) = \int d^n k d^n p e^{i(k_i + p_i) x^i + \frac{i}{2}g_i(k, p) x^i} \tilde{f}(k) \tilde{g}(p).$$  \hfill (73)

The symmetrically ordered $\ast$-product takes the form

$$f \ast g(x) = e^{\frac{i}{2}z^i g_i(-i \frac{\partial}{\partial x^i}, \ldots, -i \frac{\partial}{\partial y^n})} f(x) g(y) \bigg|_{z \to x} \bigg|_{y \to x}. \hfill (74)$$

In general, it will not be possible to write down a closed expression for the $\ast$-product, since the CBH formula can be summed up only for very few examples.

**q-Deformed Case.** The CBH formula cannot be used explicitly, we have to use eqns. (60), instead. Let us first write functions as a power series in $x^i$,

$$f(x) = \sum_j c_j (x^1)^{j_1} \ldots (x^n)^{j_n},$$  \hfill (75)
where \( J = (j_1, \ldots, j_n) \) is a multi-index. In the same way, nc functions are given by power series in ordered monomials

\[
\hat{f}(\hat{x}) = \sum_{J} c_J : (\hat{x}^1)^{j_1} \cdots (\hat{x}^n)^{j_n} :.
\]  

(76)

In a next step, we have to express the product of two ordered monomials in the nc coordinates again in terms of ordered monomials, i.e., we have to find coefficients \( a_K \) such that

\[
: (\hat{x}^1)^{i_1} \cdots (\hat{x}^n)^{i_n} : : (\hat{x}^1)^{j_1} \cdots (\hat{x}^n)^{j_n} : = \sum_{K} a_K : (\hat{x}^1)^{k_1} \cdots (\hat{x}^n)^{k_n} :.
\]  

(77)

Knowing the \( a_K \), we know the \(*\)-product for monomials. It is simply given by

\[
(\hat{x}^1)^{i_1} \cdots (\hat{x}^n)^{i_n} * (\hat{x}^1)^{j_1} \cdots (\hat{x}^n)^{j_n} = \sum_{K} a_K (\hat{x}^1)^{k_1} \cdots (\hat{x}^n)^{k_n}
\]  

(78)

using the same coefficients \( a_K \) as in (77). The whole procedure makes use of the isomorphism \( W \) defined in eqns. (60) and (61). In a last step we generalise the above expression to functions \( f \) and \( g \), and express the \(*\)-product in terms of ordinary derivatives on the functions \( f \) and \( g \), respectively. This merely amounts to replacing \( q^{i_k} \) - where \( i_k \) refers to the power of the \( k^{th} \) coordinate in (78) - by the differential operator \( q^{x_k} \partial_k \), where no summation over \( i_k \) is implied. For the better illustration, let us consider some examples.

**Examples**

- The Manin plane is always an eligible candidate.

\[
\hat{x} \hat{y} = q \hat{y} \hat{x}.
\]

We consider normal ordering, i.e., a normal ordered monomial has the form

\[
:\hat{y}^3 \hat{x}^2 \hat{y} : = \hat{x}^2 \hat{y}^4.
\]  

(79)

Following the above prescription, we end up at the following \(*\)-product [2]

\[
\left. f * g (x, y) = q^{-\hat{x} \partial_x \hat{y} \partial_y} f(x, y) g(\hat{x}, \hat{y}) \right|_{\hat{x} \to x, \hat{y} \to y}.
\]

(80)

- As a further example we want to quote the \(*\)-products on \( q \)-deformed Euclidean space in 3 dimensions and of the \( q \)-Minkowski space [23, 24]. These products are more involved than the one on the Manin plane, but still the structures show many similarities.
In case of the q-deformed 3 dimensional Euclidean space, the algebra of functions is generated by the coordinates $\hat{x}^+, \hat{x}^3, \hat{x}^-$. Again, we consider normal ordering,

$$(\hat{x}^3)^i (\hat{x}^+)^j (\hat{x}^-)^l := (\hat{x}^+)^i (\hat{x}^3)^l (\hat{x}^-)^j.$$  

The $*$-product is quoted as

$$f * g = \sum_{i=0}^{\infty} \lambda^i \frac{x_i^{2\lambda}}{[i]_{q^3}} q^{|(\lambda_3^{\mu} + \lambda_3^{\nu}) - \lambda_3^{\nu} + i\lambda}} \left(D_{q^3}^i f(x) \cdot (D_{q^3}^i g(x'))\right)_{x' \rightarrow x}, \quad (81)$$

where $D_q A f(x) = \frac{f(x_A) - f(x_{A'})}{x_A - x_{A'}}$ is the discrete Jackson derivative and $q^{A_A} = q^{x_A/x_A}$, where no summation is implied. In [23] the $*$-product is also calculated in the four dimensional case. However, the q-deformed Minkowski space resembles a much more complicated structure than 4 dimensional Euclidean space. Therefore the $*$-product is much more involved, too. Again, we will only quote the result and will not bother about the details. The coordinates are $\hat{x}^0, \hat{x}^+, \hat{x}^3, \hat{x}^-$, normal ordering is defined by monomials of the form $\hat{x}^0, \hat{x}^+, \hat{x}^3, \hat{x}^-$. The $*$-product reads

$$f * g = \sum_{i=0}^{\infty} \left(\lambda^i \frac{x_i^{2\lambda}}{[i]_{q^3}} q^{|(\lambda_3^{\mu} + \lambda_3^{\nu}) - \lambda_3^{\nu} + i\lambda}} \times (82)$$

$$\times \left[D_{q^3}^i f(x_0, x_+, \hat{x}_3, q^{j-k} x_-) \cdot (D_{q^3}^i g(x_0, x_+, \hat{x}_3, q^{j-k} x_-))\right]_{x' \rightarrow x}, \quad (83)$$

where $R_{k,j}(x) = R_{k,j}(x_0, x_+, \hat{x}_3, q^{j-k} x_-)$ is some polynomial in the coordinates.

### 2.2 Mathematical Approach to $*$-Products

**Definition 1 (Poisson Bracket)**

Let $\mathcal{M}$ be a smooth manifold, a Poisson bracket is a bilinear map $\{,\} : \mathcal{C}^\infty(\mathcal{M}) \times \mathcal{C}^\infty(\mathcal{M}) \rightarrow \mathcal{C}^\infty(\mathcal{M})$ satisfying

- $\{f, g\} = -\{g, f\}$, antisymsmetry
- $\{\{f, g\}, h\} + \{\{g, h\}, f\} + \{\{h, f\}, g\} = 0$, Jacobi identity
- $\{f, gh\} = \{f, g\} h + g\{f, h\}$, Leibniz rule

Locally, we can always write the Poisson bracket with the help of a antisymmetric tensor

$$\{f, g\} = \theta^{ij} \partial_i f \partial_j g, \quad (84)$$

where $\theta^{ij} = -\theta^{ji}$. Because of the Jacobi identity $\theta^{ij}$ has to satisfy

$$\theta^{ij} \partial_j \theta^{kl} + \theta^{kj} \partial_j \theta^{il} = 0. \quad (85)$$
Definition 2 \((\ast\text{-Product})\)

Let \(f, g \in C^\infty(M)\) and \(C_i : C^\infty(M) \times C^\infty(M) \to C^\infty(M)\) be local bi-differential operators. Then we define the \(\ast\)-product \(\ast : C^\infty(M) \times C^\infty(M) \to C^\infty(M)[[h]],\) by

\[
\sum_{n=0}^{\infty} h^n C_n(f, g),
\]

(86)

such that the following axioms are satisfied:

- \(\ast\) is an associative product.
- \(C_0(f, g) = fg,\) classical limit.
- \(\frac{i}{h}[f^*g] = -\{f, g\},\) in the limit \(h \to 0,\) semiclassical limit.

The rhs. of definition (86) is an element of \(C^\infty(M)[[h]],\) the algebra of formal power series in the formal parameter \(h\) with coefficients in \(C^\infty(M).\) Therefore we can generalise the given definition of the \(\ast\)-product to a \(C[[h]]\)-linear product in \(B = C^\infty(M)[[h]]\) by

\[
(\sum_n f_nh^n) \ast (\sum_m g_mh^m) = \sum_{k,l} f_kg_lh^{k+l} + \sum_{k,l \geq 0, m \geq 1} C_m(f_k, g_l)h^{k+l+m},
\]

(87)

Theorem 3 \((\text{Theorem by M. Kontsevich }[25])\)

\(\ast\)-products exist for any Poisson structure on \(\mathbb{R}^n.\) He also gives an explicit construction for the functions \(C_n\) in (87).

Changing the ordering in the non-commutative algebra leads to gauge equivalent \(\ast\)-products. The gauge transformation of the \(\ast\)-product is given by

\[
\mathcal{D}f \ast \mathcal{D}g = \mathcal{D}(f \ast' g),
\]

(88)

where

\[
\mathcal{D}f = f + \sum_{n \geq 1} (ih)^n \mathcal{D}_n(f).
\]

(89)

\(\mathcal{D}_n\) is an differential operator of order \(n.\)

In the following we stick to the canonical case of commutation relations,

\[
[\hat{x}^i, \hat{x}^j] = i\theta^{ij}, \quad i, j = 1, ..., 4,
\]
with the Weyl-Moyal product of functions
\[ f \ast g(x) = \exp\left(\frac{i}{2} \partial_i \theta^{ij} \partial_j\right) f(x)g(y) \bigg|_{y \rightarrow x}. \]

Furtheron, we have classical derivatives
\[ [\partial_i, \partial_j] = 0, \]
and we use the ordinary integral for the integration
\[
\int f \ast g = \int d^4x (f \ast g)(x) = \int d^4x f(x)g(x),
\]
\[
\int f \ast g \ast h = \int g \ast h \ast f = \int (g \ast h) \cdot f.
\]

3 Gauge Theory on NC Spaces

We will now concentrate on physics. We want to discuss the Standard Model on a canonically deformed space-time. Before we can do so, we have to think about gauge theory on non-commutative spaces, in general. Let us first briefly recall classical gauge theory. We will discuss in some detail the features that are essential for the nc generalisation.

3.1 Gauge Theory on Classical Spaces

Internal symmetries are described by Lie groups or Lie algebras, respectively. The elements \( T^a \)
\[ [T^a, T^b] = f^{abc}_{\phantom{abc}d} T^d \]
are generators of the Lie algebra, where \( f^{abc}_{\phantom{abc}d} \) are its structure constants. Fields are given by \( n \)-dimensional vectors carrying a irreducible representation of the gauge group. Elements of the symmetry algebra are represented by \( n \times n \) matrices. The free action of the field \( \psi \) is given by
\[
S = \int d^4x \mathcal{L} = \int d^4x \partial_\mu \psi \partial^\mu \psi. \tag{91}
\]
Requiring the gauge invariance of the action \( S \), one has to introduce additional fields, gauge fields and to replace the usual derivatives by covariant derivatives \( D_\mu \).

Let us start with the field \( \psi \) building an irreducible representation of the gauge group, i.e.,
\[
\delta \psi(x) = i\alpha(x)\psi(x), \tag{92}
\]
where \( \alpha \) is Lie algebra valued,
\[
\alpha(x) = \alpha_a(x) T^a.
\]
Observe that the derivative of a field \( \psi \) does not transform covariantly, i.e.,

\[ \delta \partial_\mu \psi \neq i\alpha(x) \partial_\mu \psi(x). \]  

Replacing the usual derivatives \( \partial_\mu \) by covariant derivatives \( D_\mu \) and demanding that \( D_\mu \psi \) transforms covariantly, one has to introduce a gauge potential \( A_\mu(x) \),

\[
\begin{align*}
D_\mu &= \partial_\mu - igA_\mu(x), \\
A_\mu(x) &= A_\mu^a(x)T^a, \\
\delta A_\mu(x) &= \frac{1}{g} \partial_\mu \alpha(x) + [\alpha(x), A_\mu(x)].
\end{align*}
\]

As it is well known, the interaction fields are a consequence of the gauge invariance of the action. Interactions are gauge interactions. The modified action reads

\[ S = \int d^4x \, D_\mu \psi D^\mu \psi, \]  

including gauge Fields \( A_\mu \). Forgetting about mass terms, we still need a kinetic term for the gauge fields in our action. The only requirements are the gauge invariance of the kinetic term and renormalisabiliy of the theory. That fixes the kinetic term uniquely. This is a crucial point, and the situation will be different in the case of the NCSM. The action is given by

\[ S = \int d^4x \, (D_\mu \psi D^\mu \psi + Tr F_{\mu\nu} F^{\mu\nu}), \]  

where \( F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - ig[A_\mu, A_\nu] \) is the field strength. Considering abelian gauge symmetry, commutators in \( F_{\mu\nu} \) and in \( \delta A_\mu \) will vanish. Let us make one more important remark: there is a sharp distinction between internal and external symmetry transformations. As we will see, that is not true in the case of nc gauge theory.

### 3.2 Nc Gauge Theory

Non-commutative gauge theory, as presented in [2, 26], is based on essentially three principles,

- Covariant coordinates,
- Locality and classical limit,
- Gauge equivalence conditions.

Let us first briefly recall our starting point. We have non-commutative coordinates

\[
\begin{align*}
\{ \hat{x}^\mu, \hat{x}^\nu \} &= i\theta^{\mu\nu}, \\
\hat{A} &= \mathbb{C}\langle \hat{x}^1, \ldots, \hat{x}^n \rangle.
\end{align*}
\]

the product of function \( f, g \in \mathcal{A} \) is given by the Weyl-Moyal product.
3.2.1 Covariant Coordinates

Let $\psi$ be a non-commutative field, i.e., $\hat{\psi} \in \oplus_{i=1}^{n} \hat{A}$,

$$\hat{\delta}\hat{\psi}(\hat{x}) = i\hat{\alpha}\hat{\psi}(\hat{x})$$

(96)

or

$$\hat{\delta}\psi(x) = i\alpha \ast \psi(x),$$

(97)

in the $\ast$ formalism, where $W(\alpha) = \hat{\alpha}$. Now, a similar situation arises as in eqn. (93), only the derivatives are replaced by coordinates. The product of a field and a coordinate does not transform covariantly, since the $\ast$-product is not commutative,

$$\hat{\delta}(x \ast \psi(x)) = i x \ast \alpha(x) \ast \psi(x) \neq i \alpha(x) \ast x \ast \psi(x).$$

(98)

The arguments are the same as before, and we introduce covariant coordinates

$$X^\mu \equiv x^\mu + A^\mu,$$

(99)

such that

$$\hat{\delta}(X^\mu \ast \psi) = i\alpha \ast (X^\mu \ast \psi).$$

(100)

The covariant coordinates and the gauge potential transform under a nc gauge transformation in the following way

$$\hat{\delta}X^\mu = i[\alpha \ast X^\mu],$$

(101)

$$\hat{\delta}A^\mu = i[\alpha \ast x^\mu] + i[\alpha \ast A^\mu].$$

(102)

Other covariant objects can be constructed from covariant coordinates, such as a generalisation of the field strength,

$$F^{\mu\nu} = [X^\mu \ast X^\nu] - i\theta^{\mu\nu}, \quad \hat{\delta}F^{\mu\nu} = i[\alpha \ast F^{\mu\nu}].$$

(103)

For non degenerate $\theta$, we can define another gauge potential $V_\mu$

$$\hat{\delta}V_\mu = \partial_\mu \alpha + i[\alpha \ast V_\mu],$$

(104)

$$F_{\mu\nu} = \partial_\mu V_\nu - \partial_\nu V_\mu - i[V_\mu \ast V_\nu],$$

(105)

$$\hat{\delta}F_{\mu\nu} = i[\alpha \ast F_{\mu\nu}],$$

(106)

using

$$A^\mu = \theta^{\mu\nu}V_\nu, \quad F^{\mu\nu} = i\theta^{\mu\sigma}\theta^{\nu\tau}F_{\sigma\tau},$$

$$i\theta^{\mu\nu}\partial_\nu f = [x^\mu \ast f].$$

(107)
And we get for the covariant derivatives

\[ D_\mu \psi = (\partial_\mu - iV_\mu) \ast \psi, \]  
\[ \hat{\delta}(D_\mu \ast \psi) = i\alpha \ast D_\mu \psi. \]  

Even for abelian gauge groups, the \( \ast - \)commutators in eqns. (104) (105) do not vanish, and the theory has similarities to a non-abelian gauge theory on a commutative space-time.

Let us have a closer look at the gauge parameters and the gauge fields. In classical theory, the gauge parameter and the gauge field are Lie algebra valued, as we have mentioned before. Two subsequent nc gauge transformations are again a gauge transformation,

\[ \delta_\alpha \delta_\beta - \delta_\beta \delta_\alpha = \delta_{-i[\alpha, \beta]}, \]  

where \(-i[\alpha, \beta] = \alpha^a \beta_b f_{ab}^c T^c\). However, there is a remarkable difference to the non-commutative case. Let \( M^\alpha \) be some matrix basis of the enveloping algebra of the internal symmetry algebra. We can expand the gauge parameters in terms of this basis, \( \alpha = \alpha_a M^a, \beta = \beta_b M^b \). Two subsequent gauge transformations take again the form

\[ \hat{\delta}_\alpha \hat{\delta}_\beta - \hat{\delta}_\beta \hat{\delta}_\alpha = \hat{\delta}_{-i[\alpha, \beta]}, \]  

but the commutator of two gauge transformations involves the \( \ast - \)commutator of the gauge parameters, and

\[ [\alpha \ast, \beta] = \frac{1}{2} \{ \alpha_a \ast, \beta_b \} [M^a, M^b] + \frac{1}{2} [\alpha_a \ast, \beta_b] \{ M^a, M^b \}, \]  

where \( \{ M^a, M^b \} = M^a M^b + M^b M^a \) is the anti-commutator. The difference to (109) is the anti-commutator \( \{ M^a, M^b \} \), respectively the \( \ast - \)commutator of the gauge parameters, \([\alpha_a \ast, \beta_b]\). This term causes some problems. Let us assume that \( M^\alpha \) are the Lie algebra generators. Does the relation (111) close? Or does (111) rule out Lie algebra valued gauge parameters? Clearly, the only crucial term is the anti-commutator. The anti-commutator of two hermitian matrices is again hermitian. But the anti-commutator of traceless matrices is in general not traceless. Relation (111) will be satisfied for the generators of the fundamental representation of \( U(n) \). Therefore it has been argued [27] that \( U(n) \) - and with some difficulty \( SO(n) \) and \( Sp(n) \) [28] - is the only gauge group that can be generalised to nc spaces. But in fact arbitrary gauge groups can be tackled. But the gauge parameters \( \alpha, \beta \) and the gauge fields \( A_\mu \) have to be enveloping algebra valued [2][29], in general. Gauge fields and parameters now depend on infinitely many parameters, since the enveloping algebra is infinite dimensional. Luckily, the infinitely many degrees of freedom can be reduced to a finite number, namely the classical parameters, by the so-called Seiberg-Witten maps we will discuss in the next paragraph.
3.2.2 Locality and Classical Limit

The nc $\ast$-product can be written as an expansion in a formal parameter $h$,$$
 f \ast g = f \cdot g + \sum_{n=1}^{\infty} h^n C_n(f, g).
$$

In the commutative limit $h \to 0$, the $\ast$-product reduces to the pointwise product of functions. One may ask, if there is a similar commutative limit for the fields? The solution to this question was given for abelian gauge groups by [4],

\begin{align*}
\hat{A}_\mu[A] &= A_\mu + \frac{1}{2} \theta^{\sigma\tau} (A_\tau \partial_\sigma A_\mu + F_{\sigma \mu} A_\tau) + O(\theta^2), \\
\hat{\psi}[\psi, A] &= \psi + \frac{1}{2} \theta^{\mu \nu} A_\nu \partial_\mu \psi + O(\theta^2), \\
\hat{\alpha} &= \alpha + \frac{1}{2} \theta^{\mu \nu} A_\nu \partial_\mu \alpha + O(\theta^2).
\end{align*}

(112)

(113)

(114)

In this case, $\theta$ is the non-commutativity parameter. First of all, let me introduce an important convention to which we will stick from now on. Quantities with "hat" ($\hat{\psi}, \hat{A}, \hat{\alpha} \ldots \in (\mathcal{A}, \ast)$) refer to non-commutative fields and gauge parameters, respectively which can be expanded (cf. above) in terms of the ordinary commutative fields and gauge parameters, resp. ($\psi, A, \alpha$).

The Seiberg-Witten maps (112 - 114) reduce the infinitely many parameters of the enveloping algebra to the classical gauge parameters.

The origins of this map are in string theory. It is there that gauge invariance depends on the regularisation scheme applied [4]. Pauli-Villars regularisation provides us with classical gauge invariance

$$
\delta a_i = \partial_i \lambda,
$$

whence point-splitting regularisation comes up with nc gauge invariance

$$
\hat{\delta} \hat{A}_i = \partial_i \hat{\Lambda} + i[\hat{\Lambda} ; \hat{A}_i].
$$

(116)

Seiberg and Witten argued that consequently there must be a local map from ordinary gauge theory to non-commutative gauge theory

$$
\hat{A}[a], \hat{\Lambda}[\lambda, a]
$$

satisfying

$$
\hat{A}[a + \delta_\lambda a] = \hat{A}[a] + \hat{\delta}_\lambda \hat{A}[a].
$$

(118)

The Seiberg-Witten maps are solutions of (118). By locality we mean that in each order in the non-commutativity parameter $\theta$ there is only a finite number of derivatives.
3.2.3 Gauge Equivalence Conditions

Let us remember that we consider arbitrary gauge groups. Nc gauge fields \( \hat{A} \) and gauge parameters \( \hat{\Lambda} \) are enveloping algebra valued. Let us choose a symmetric basis in the algebra, \( T^a, \frac{1}{2}(T^aT^b + T^bT^a), \ldots \), such that

\[
\hat{\Lambda}(x) = \hat{\Lambda}_a(x)T^a + \hat{\Lambda}_{ab}(x) : T^aT^b : + \ldots ,
\]

(119)

\[
\hat{A}_\mu(x) = \hat{A}_{\mu a}(x)T^a + \hat{A}_{\mu ab}(x) : T^aT^b : + \ldots .
\]

(120)

Eqn. (118) defines the SW maps for the gauge field and the gauge parameter. However, it is more practical to find equations for the gauge parameter and the gauge field alone \[26\].

First we will concentrate on the gauge parameters \( \hat{\alpha} \). We already encountered the consistency condition

\[
\hat{\delta}_\alpha \hat{\delta}_\beta - \hat{\delta}_\beta \hat{\delta}_\alpha = \hat{\delta}_{\{\alpha,\beta\}}.
\]

More explicitly, it reads

\[
i\hat{\delta}_\alpha \hat{\beta}[A] - i\hat{\delta}_\beta \hat{\alpha}[A] + [\hat{\alpha}[A], \hat{\beta}[A]] = ([\alpha, \beta])[A].
\]

(121)

Keeping in mind the results from Section 3.2.2, we can expand \( \hat{\alpha} \) in terms of the non-commutativity \( \theta \),

\[
\hat{\alpha}[A] = \alpha + \alpha^1[A] + \alpha^2[A] + \ldots ,
\]

(122)

where \( \alpha^n \) is \( \mathcal{O}(\theta^n) \). The consistency relation (121) can be solved order by order in \( \theta \).

- 0th order: \( \alpha^0 = \alpha \),

(123)

- 1st order: \( \alpha^1 = \frac{1}{4} \theta^{\mu \nu} \{ \partial_\mu \alpha, A_\nu \}, \)

(124)

\[
= \frac{1}{2} \theta^{\mu \nu} \partial_\mu \alpha A_{\mu \nu} : T^aT^b : ,
\]

(125)

For fields \( \hat{\psi} \) the condition

\[
\delta_\alpha \hat{\psi}[A] = \hat{\delta}_\alpha \hat{\psi}[A] = i\hat{\alpha}[A] \ast \hat{\psi}[A]
\]

(126)

has to be satisfied, keeping in mind that \( \delta_\alpha \) denotes an ordinary gauge transformation and \( \hat{\delta}_\alpha \) a nc one. That means that the ordinary gauge transformation induces a nc gauge transformation. We expand the fields in terms of the non-commutativity

\[
\hat{\psi} = \psi^0 + \psi^1[A] + \psi^2[A] + \ldots
\]

(127)

and solve eqn. (126) order by order in \( \theta \). In first order, we have to find a solution to

\[
\delta_\alpha \psi^1[A] = i\alpha \psi^1 + i\hat{\alpha}\psi - \frac{1}{2} \theta^{\mu \nu} \partial_\mu \alpha \partial_\nu \psi.
\]

(128)
It is given by

\begin{align}
\text{0th order:} \quad & \psi^0 = \psi, \\
\text{1st order:} \quad & \psi^1 = -\frac{1}{2} \theta^{\mu\nu} A_\mu \partial_\nu \psi + \frac{i}{4} \theta^{\mu\nu} A_\mu A_\nu \psi. 
\end{align}

The gauge fields $\widehat{A}_\mu$ have to satisfy

\[ \delta_\alpha \widehat{A}_\mu[A] = \partial_\mu \widehat{\alpha}[A] + i[\widehat{\alpha}[A], \widehat{A}_\mu[A]]. \]

Using the expansion

\[ \widehat{A}_\mu[A] = A_\mu^0 + A_\mu^1[A] + A_\mu^2[A] + \ldots \]

and solving (131) order by order, we end up with

\begin{align}
\text{0th order:} \quad & A_\mu^0 = A_\mu, \\
\text{1st order:} \quad & A_\mu^1 = -\frac{1}{4} \theta^{\tau\nu} \{ A_\tau, \partial_\nu A_\mu + F_{\nu\mu} \},
\end{align}

where $F_{\nu\mu} = \partial_\nu A_\mu - \partial_\mu A_\nu - i[A_\nu, A_\mu]$. Similarly, we have for the field strength $\widehat{F}_{\mu\nu}$

\[ \delta_\alpha \widehat{F}_{\mu\nu} = i[\widehat{\alpha}, \widehat{F}_{\mu\nu}] \quad \text{and} \]

\[ \widehat{F}_{\mu\nu} = F_{\mu\nu} + \frac{1}{2} \theta^{\sigma\tau} \{ F_{\mu\sigma}, F_{\nu\tau} \} - \frac{1}{4} \theta^{\sigma\tau} \{ A_\sigma, (\partial_\tau + D_\tau) F_{\mu\nu} \}, \]

where $D_\mu F_{\tau\nu} = \partial_\mu F_{\tau\nu} - i[A_\mu, F_{\tau\nu}]$.

### 3.2.4 Remarks

Let us conclude this Section with some remarks and observations.

- SW maps provide solutions to the gauge equivalence relations.

- Gauge equivalence relations are not the only possible approach to SW maps. Another approach is via nc Wilson lines, see e.g., [30].

- However, a certain ambiguity in the SW map remains. They are unique modulo classical field redefinition and nc gauge transformation. We used these ambiguities in order to choose $\widehat{\lambda}, \widehat{A}_\mu$ hermitian. The freedom in SW map may also be essential for renormalisation issues. There, parameters characterising the freedom in the SW maps become running coupling constants [31]. Discussing tensor products of gauge groups, this freedom will also be of crucial importance in Section 5.
Gauge groups in non-commutative spaces contain space-time translations. Since
\[ \partial f = -i \theta^{-1}_{ij} [x^j, f], \]
we can express the translation of the field \( A_i \) as
\[ \delta A_i = v^j \partial_j A_i = i [\epsilon \gamma A_i], \]
where \( \epsilon = -v^j \theta^{-1}_{jk} x^k \). The gauge transformation of \( A_i \) with gauge parameter \( \epsilon \) gives
\[ \delta \epsilon A_i = i [\epsilon \gamma A_i] - v^j \theta^{-1}_{ji}. \]
This agrees with (3.2.4), ignoring the overall constant, which has no physical effect.

NC gauge theory allows the construction of realistic particle models on a nc spacetime with an arbitrary gauge group as internal symmetry group. Nc gauge parameters and gauge fields are enveloping algebra valued, in general (e.g., for \( SU(n) \)), but via SW maps the infinite number of degrees of freedom is reduced to the classical gauge parameters. Therefore these models will have the proper number of degrees of freedom.

4 Standard Model of Theoretical Particle Physics

The Standard Model of Particle Physics is a very successful and experimentally very well tested theory. It unifies strong, weak and electromagnetic interactions. The aim of this section is to give a very brief sketch. The Standard Model is a gauge theory with gauge group \( SU(3)_C \times SU(2)_L \times U(1)_Y \).

4.1 Particle Content

The particle content of the Standard Model is given in Table II. The action is given by
\[
S = \int d^4x \sum_{i=1,2,3} \bar{\Psi}_L^{(i)} i\gamma \Psi_L^{(i)} + \int d^4x \sum_{i=1,2,3} \bar{\Psi}_R^{(i)} i\gamma \Psi_R^{(i)}
- \int d^4x \frac{1}{4g} f_{\mu\nu} f^{\mu\nu}
- \int d^4x \frac{1}{2g} \text{tr} F_{\mu\nu}^L F_{\mu\nu}^L
- \int d^4x \frac{1}{2g_s} \text{tr} F_{\mu\nu}^S F_{\mu\nu}^S
+ \int d^4x \left( (D_\mu \Phi) \right)^i D^\mu \Phi - \mu^2 \Phi \Phi - \lambda \Phi \Phi \Phi \Phi
+ \int d^4x \left( -\sum_{i,j=1}^{3} (W_i^{ij} (L_i^L) \Phi e_R^{(i)} + W_i^{ij} e_R^{(i)} \Phi L_i^L) \right)
\] (139)
SU(3)_C  SU(2)_L  U(1)_Y  U(1)_Q
\begin{align*}
L_L &= \begin{pmatrix} \nu_L \\ e_L \end{pmatrix} \\
&= \begin{pmatrix} 1 \\ 1 \end{pmatrix} \\
u_R &= \begin{pmatrix} 3 \\ 1 \end{pmatrix} \\
d_R &= \begin{pmatrix} 3 \\ 1 \end{pmatrix} \\
Q_L &= \begin{pmatrix} u_L \\ d_L \end{pmatrix} \\
&= \begin{pmatrix} 1 \\ 2 \\ 1/6 \end{pmatrix} \\
\Phi &= \begin{pmatrix} \phi^+ \\ \phi^0 \end{pmatrix} \\
B^i &= \begin{pmatrix} 1 \\ 3 \\ 0 \end{pmatrix} \\
A &= \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \\
G^a &= \begin{pmatrix} 8 \\ 1 \\ 0 \end{pmatrix}
\end{align*}

Table 1: Particle content of the Standard Model

\begin{align*}
- \sum_{i,j=1}^3 \left( G_{ij}^{u_r} (\bar{Q}_L^{(i)} \Phi) u_R^{(j)} + G_{ij}^{d_r} (\bar{Q}_L^{(i)} \phi^+ \Phi) Q_L^{(j)} \right) \\
- \sum_{i,j=1}^3 \left( G_{ij}^{d_r} (\bar{Q}_L^{(i)} \Phi) d_R^{(j)} + G_{ij}^{q_r} (\bar{Q}_L^{(i)} \phi^0 \Phi) Q_L^{(j)} \right)
\end{align*}

where

\begin{align*}
\Psi_L^{(i)} &= \begin{pmatrix} L_L^{(i)} \\ Q_L^{(i)} \end{pmatrix}, & \Psi_R^{(i)} &= \begin{pmatrix} e_R^{(i)} \\ u_R^{(i)} \\ d_R^{(i)} \end{pmatrix}, & \Phi &= \begin{pmatrix} \phi^+ \\ \phi^0 \end{pmatrix}.
\end{align*}

Ψ_L denotes the left handed fermions, the leptons L and the quarks Q, Ψ_R denotes the right handed fermions. \( (i) \in \{1, 2, 3\} \) is the generation index, and \( \phi^+ \) and \( \phi^0 \) are the complex scalar fields of the scalar Higgs doublet. \( g' A(x) \) is the gauge field of the hypercharge symmetry, \( U(1)_Y \), \( B_{\mu}(x) = \frac{g}{2} B_{\mu a}(x) \sigma^a \) the field of the weak interaction, \( SU(2)_L \) and \( G_{\mu a}(x) = \frac{g}{2} G_{\mu a}(x) \lambda^a \) of the strong interaction, \( SU(3)_C \). Some exemplary covariant derivatives have the form

\begin{align*}
D_\mu \Phi &= \left( \partial_\mu - ig' A_\mu - ig' B_\mu \right) \Phi, \\
D_\mu Q_L^{(i)} &= \left( \partial_\mu - ig' A_\mu - \frac{g}{2} B_\mu \sigma^a - \frac{g}{2} G_{\mu b} \lambda^b \right) Q_L^{(i)}, \\
F_{\mu \nu}^L &= g \left( \partial_\mu B_\nu - \partial_\nu B_\mu - ig [B_\mu, B_\nu] \right).
\end{align*}

Fermions and bosons are massless, initially. Spontaneous breaking of the electroweak gauge symmetry \( SU(2)_L \times U(1)_Y \) is responsible for the masses of the gauge bosons, cf.
Section 4.2 The Yukawa coupling terms in (139) give masses to the fermions. \( W^{ij}, G^u_{ij} \) and \( G^d_{ij} \) in (139) are the Yukawa couplings.

4.2 Spontaneous Symmetry Breaking

Spontaneous symmetry breaking is characterised by two properties, the action is invariant under a symmetry, but the solutions of the equations of motion are not. A very popular example is the Mexican hat potential shown in Fig. 7. Let us consider the Higgs field. It has a \( SU(2) \times U(1) \) gauge symmetry. The action is given by

\[
S = \int d^4x \left( (D^\mu \phi)^\dagger D_\mu \phi - V(\phi) \right),
\]

where the potential has the form

\[
V(\phi) = \mu^2 \phi^\dagger \phi + \lambda (\phi^\dagger \phi)^2.
\]

Let us write the complex Higgs field as

\[
\phi(x) = \begin{pmatrix} \phi_1(x) + i\phi_2(x) \\ \phi_3(x) + i\phi_4(x) \end{pmatrix},
\]

where \( \phi_1, \phi_2, \phi_3 \) and \( \phi_4 \) are real fields. If the parameter \( \mu^2 > 0 \) in (145), (144) describes four real particles with mass \( \mu \). In case of \( \mu^2 < 0 \), the potential \( V \) resembles a Mexican hat potential.
hat potential and spontaneous symmetry breaking occurs. The minima of the potential lie on circle around $V(\phi = 0)$, e.g., the point

$$\phi_1 = \phi_2 = \phi_4 = 0,$$

$$\phi_3 = \sqrt{-\mu^2 / \lambda}$$

is an element of that circle. Each of these minima describes a possible vacuum state. Choosing one as the actual physical vacuum breaks the $SU(2)_L \times U(1)_Y$ gauge symmetry. Let us choose the point (147) as the vacuum, and let us expand around that vacuum, and let us fix the gauge freedom to the unitarity gauge,

$$\phi(x) = \frac{1}{\sqrt{2}} \begin{pmatrix} v \\ 0 \end{pmatrix} + h(x),$$

where $v = \sqrt{-\mu^2 / \lambda}$ is the vacuum expectation value. Expanding the action (144) will only leave us with the kinetic term, since

$$V(\phi) = V(\phi_0) + \Delta \phi \frac{\partial V}{\partial \phi} \bigg|_{\phi_0} + \mathcal{O}(\Delta \phi^2)$$

$$= 0 + \mathcal{O}(\Delta \phi^2),$$

where the chosen vacuum is denoted by $\phi_0$. The expansion of the kinetic term has the form

$$(D_\mu \phi)^\dagger (D_\mu \phi) = \frac{1}{2} \left( \frac{v g}{2} \right)^2 B_\mu^+ B^{-\mu} + \frac{1}{8} v^2 \left( B_\mu^3 A_\mu \right) \left( \begin{array}{cc} g^2 & -gg \\ -gg & g^2 \end{array} \right) \left( B_{3\mu}^3 A_\mu \right)$$

$$+ \partial_\mu h \partial^\mu h + \text{interaction terms},$$

where $B_\mu^\pm = \frac{1}{\sqrt{2}} \left( B_\mu^1 \pm i B_\mu^2 \right)$. We can identify $B^\pm$ with the $W^\pm$ bosons. As one can see from (150), they are massive. Their mass is given by

$$M_{W^\pm} = \frac{1}{2} v g.$$

By diagonalising the matrix in (150), we find the eigenvectors that correspond to the other two particles that are still missing, namely the $Z$ boson and the photon. Diagonalisation yields

$$Z_\mu = \frac{g B_\mu^3 - g A_\mu}{(g^2 + g^2)^{1/2}},$$

$$A_\mu = \frac{g B_\mu^3 + g A_\mu}{(g^2 + g^2)^{1/2}},$$

28
with masses $M_Z = \frac{g}{\sqrt{2}}(g^2 + g'^2)^{1/2}$ and $M_A = 0$, respectively. The mass of the electromagnetic photon is zero, that means that the $SU(2)_L \times U(1)_Y$ gauge symmetry is broken to $U(1)_{em}$.

The Yukawa coupling terms in the action of the Standard Model supply the fermions with masses. The resulting terms are of the form, e.g.,

$$- \frac{m_e}{2}(\bar{e}_R e_L + \bar{e}_L e_R).$$

(154)

### 5 Non-Commutative Standard Model

This Section is based on [1], a joint work with Xavier Calmet, Branislav Jurčo, Peter Schupp and Julius Wess. We are going to deal with the symmetry group $SU(3)_C \times SU(2)_L \times U(1)$. Let me stress that we will therefore not introduce any new degrees of freedom or any new parameters we would have to get rid off in the end. The aim is not to reduce the number of parameters in the Standard Model - the Higgs mass or parameters in the Higgs potential, the masses of the fermions or the Yukawa coupling constants, etc. - but to formulate a realistic gauge theoretical model on canonical space-time using the formalism introduced in Section 3. Especially interesting is whether the Higgs mechanism in the Standard Model can still be applied to provide the boson masses.

Naively spoken, the method we use is very simple. We just write down the Standard Model Lagrangian, replace $\cdot$ by $*$ and fields $\Psi, A$ by $\hat{\Psi}[\Psi, A], \hat{A}[A]$. Of course, it is not as easy as that. I have written down the non-commutative Lagrangian in (155). In the rest of my talk I am going to explain what all these terms mean. First of all, we have to address some restrictions on non-commutative gauge theories discussed in [27]. We have already discussed in Section 3.2 the problem of generalising other gauge groups than $U(n)$ to non-commutative spaces. Further it is argued that a non-commutative field may be charged under at most two gauge groups only. This problem will be solved in 5.1 combining all the gauge fields of the Standard Model into a single "master gauge field" and applying the SW map.

$$S_{NCSM} = \int d^4x \sum_{i=1}^{3} \bar{\Psi}_L^{(i)} * \hat{D}_\mu \hat{\Psi}_L^{(i)} + \int d^4x \sum_{i=1}^{3} \bar{\Psi}_R^{(i)} * \hat{D}_\mu \hat{\Psi}_R^{(i)}$$

$$- \int d^4x \frac{1}{2g_1} \text{tr} \hat{F}_{\mu\nu} * \hat{F}^{\mu\nu} - \int d^4x \frac{1}{2g_2} \text{tr} \hat{F}_{\mu\nu} * \hat{F}^{\mu\nu} - \int d^4x \frac{1}{2g_3} \text{tr} \hat{F}_{\mu\nu} * \hat{F}^{\mu\nu}$$

$$+ \int d^4x \left( \rho_0(\hat{D}_\mu \hat{\Phi}) \rho_0(\hat{D}^\mu \hat{\Phi}) - \mu^2 \rho_0(\hat{\Phi}) \rho_0(\hat{\Phi}) - \lambda \rho_0(\hat{\Phi}) \rho_0(\hat{\Phi}) \rho_0(\hat{\Phi}) \right)$$

(155)
\[
+ \int d^4x \left( - \sum_{i,j=1}^{3} \left( W_{ij} \bar{L}_i (\rho_L (\tilde{\Phi})) \star e_R (\rho_L (\tilde{\Phi})^\dagger \star \tilde{L}_L^{(j)}) + W_{ij}^\dagger \bar{e}_R (\rho_L (\tilde{\Phi})^\dagger \star \tilde{L}_L^{(j)}) \right) \\
- \sum_{i,j=1}^{3} \left( G_{uj}^{i} \bar{L}_i (\rho_Q (\tilde{\Phi})) \star u_R (\rho_Q (\tilde{\Phi})^\dagger \star \tilde{Q}_L^{(j)}) + G_{uj}^{i} \bar{u}_R (\rho_Q (\tilde{\Phi})^\dagger \star \tilde{Q}_L^{(j)}) \right) \\
- \sum_{i,j=1}^{3} \left( G_{ud}^{i} \bar{Q}_L (\rho_Q (\tilde{\Phi})) \star d_R (\rho_Q (\tilde{\Phi})^\dagger \star \tilde{Q}_L^{(j)}) + G_{ud}^{i} \bar{d}_R (\rho_Q (\tilde{\Phi})^\dagger \star \tilde{Q}_L^{(j)}) \right) \right)
\]

We have to discuss three serious problems, namely

the tensor product of gauge groups,

the so-called charge quantisation problem in nc QED,

the gauge invariance of the Yukawa couplings and

ambiguities in the choice of the kinetic terms for the gauge fields in the Lagrangian (155).

5.1 Tensor Product of Gauge Groups

There are several possibilities to deal with the tensor product of gauge groups which correspond to the freedom in the choice of SW maps. The most symmetric and natural choice is to take the classical tensor product

\[
V_\mu = g' A_\mu (x) Y + \frac{g}{2} \sum_{a=1}^{3} B_{\mu a} \sigma^a + \frac{g_S}{2} \sum_{a=1}^{8} G_{\mu a} \lambda^a, \tag{156}
\]

defining one overall, ”master” gauge field \( V_\mu \), with gauge paramter

\[
\Lambda = g' \alpha (x) Y + \frac{g}{2} \sum_{a=1}^{3} \alpha_{a}^\xi (x) \sigma^a + \frac{g_S}{2} \sum_{b=1}^{8} \alpha_{b}^S (x) \lambda^b. \tag{157}
\]

The nc gauge field \( \tilde{V}[V] \) and gauge parameter \( \tilde{\Lambda}[\Lambda, V] \) are given by

\[
\tilde{V}_\xi [V] = V_\xi + \frac{1}{4} \theta^{\mu \nu} \{ V_\nu, \partial_\mu V_\xi \} + \frac{1}{4} \theta^{\mu \nu} \{ F_{\mu \xi}, V_\nu \} + \mathcal{O}(\theta^2) \tag{158}
\]
\[
+ \mathcal{O}(\theta^2), \tag{159}
\]
\[
\tilde{\Lambda} = \Lambda + \frac{1}{4} \theta^{\mu \nu} \{ V_\nu, \partial_\mu \Lambda \} + \mathcal{O}(\theta^2). \tag{160}
\]

As a consequence, the gauge groups mix in higher order of \( \theta \) and cannot be viewed independently.
Let me also say a few words on the general tensor product of two gauge groups \[35\]. The most general solution of the gauge consistency condition \[121\] - for one gauge group - is given by

\[
\hat{\Lambda}[A] = \Lambda + \frac{1}{2} \theta^{\mu\nu} \{ A_\nu, \partial_\mu \Lambda \}_c + \mathcal{O}(\theta^2),
\]

(161)

where

\[
\{ A, B \}_c := \frac{1}{2} \{ A, B \} + (c - 1/2)[A, B],
\]

(162)

c is a complex function of space-time. We have \(\{ A, B \}_1/2 = 1/2 \{ A, B \}\). The according gauge field is of the form

\[
\hat{A}_\mu[A] = A_\mu + \frac{1}{2} \theta^{\nu\sigma} \{ A_\nu, \partial_\mu A_\sigma \}_c + \frac{1}{2} \theta^{\nu\sigma} \{ F_{\nu\mu}, A_\sigma \}_c + \mathcal{O}(\theta^2).
\]

(163)

The gauge parameter \(\hat{\Lambda}_{(\Lambda, \Lambda')}[A, A']\) of the tensor product of two gauge groups \(G\) and \(G'\) consists of two parts

\[
\hat{\Lambda}_{(\Lambda, \Lambda')}[A, A'] = \hat{\Lambda}_\Lambda[A, A'] + \hat{\Lambda}'_{\Lambda'}[A, A'],
\]

(164)

because of the linearity in the classical case. \(\hat{\Lambda}_{(\Lambda, \Lambda')}[A, A']\) has to fulfill consistency relation \[121\]. Therefore both, \(\hat{\Lambda}_\Lambda[A, A']\) and \(\hat{\Lambda}'_{\Lambda'}[A, A']\) have to satisfy \[121\] by their own, and there is an additional cross relation

\[
[\hat{\Lambda}_\Lambda * \hat{\Lambda}'_{\Lambda'}] + i\delta_{\Lambda} \hat{\Lambda}'_{\Lambda'} - i\delta_{\Lambda'} \hat{\Lambda}_\Lambda = 0.
\]

(165)

The solution is given by

\[
\hat{\Lambda}_{(\Lambda, \Lambda')}[A, A'] = \Lambda + \Lambda' + \frac{1}{2} \theta^{\mu\nu} \{ A_\nu, \partial_\mu \Lambda \}_c + \{ A'_\nu, \partial_\mu \Lambda' \}_d
\]

\[
+ (1 - \frac{\gamma(x)}{2}) \theta^{\mu\nu} A'_\nu \partial_\mu \Lambda + \frac{\gamma(x)}{2} \theta^{\mu\nu} A_\nu \partial_\mu \Lambda' + \mathcal{O}(\theta^2).
\]

(166)

\(\gamma(x)\) is a real function and \(c - 1/2\) and \(d - 1/2\) purely imaginary. Solving eqns. \[126\] and \[131\] using \[166\] will provide us with the SW map for matter fields and the gauge field. The symmetric choice in \[160\] is recovered by choosing \(\gamma = 1\) and \(c = d = 1/2\).

### 5.2 Charge Quantisation in NCQED

It seems that in ncQED only charges \(\pm q, 0\) can be accounted for, once \(q\) is fixed \[31\].

\[
\hat{D}_\mu \hat{\psi} = \partial_\mu \hat{\psi} - iq \hat{A}_\mu \* \hat{\Psi}
\]

(167)
the only other couplings of the field $\hat{A}_\mu$ to a matter field consistent with the nc gauge transformation $\delta_\alpha \hat{A}_\mu = \partial_\mu \hat{\alpha} + i[\hat{\alpha}^* ; \hat{A}_\mu]$ are

$$\hat{D}^- \hat{\psi}^- = \partial_\mu \hat{\psi}^- + iq\hat{\psi}^- \ast \hat{A}_\mu, \quad (168)$$

$$\hat{D}^0 \hat{\psi}^0 = \partial_\mu \hat{\psi}^0, \quad (169)$$

$$\hat{D}^0' \hat{\psi}^{0'} = \partial_\mu \hat{\psi}^{0'} - iq[\hat{A}_\mu ; \hat{\psi}^{0'}]. \quad (170)$$

Other charges $q^{(n)}$ cannot be absorbed into the respective field $\hat{A}_\mu$, because of the commutator in

$$\hat{F}_{\mu\nu} = \partial_\mu \hat{A}_\nu - \partial_\nu \hat{A}_\mu + ieq[\hat{A}_\mu ; \hat{A}_\nu], \quad (171)$$

$$\delta_\Lambda \hat{A}_\mu = \partial_\mu \hat{\Lambda} + i[\hat{\Lambda}^* ; \hat{A}_\mu]. \quad (172)$$

Classically, we can have two particles $\psi$ and $\psi'$ with charges $q$ and $q'$ coupling to the same gauge field. The gauge transformation of these fields has the form

$$\delta \psi = iq e\lambda \psi, \quad \delta \psi' = iq' e\lambda' \psi',$$

with covariant derivatives

$$\hat{D}_\mu \psi = \partial_\mu \psi - ieq a_\mu \psi,$$

$$\hat{D}_\mu \psi' = \partial_\mu \psi' - ieq' a'_\mu \psi',$$

where

$$\delta a_\mu = \partial_\mu \lambda, \quad \delta a'_\mu = \partial_\mu \lambda'.$$

Now, let us assume $\lambda' = \lambda$. We can consistently define

$$a_\mu = \frac{q'}{q} a'_\mu, \quad f_{\mu\nu} = \frac{q'}{q} f'_{\mu\nu}.$$

The $\ast$-commutators spoil this simple picture. The solution is again provided by SW maps. We have to introduce a different gauge field $\hat{a}_\mu^{(n)}$ for each distinct charge $q^{(n)}$ that appears in the theory. It seems that there are too many degrees of freedom, but the SW map for $a_\mu^{(n)}$ is an expansion in the commutative gauge field $a_\mu$ and $\theta$ only,

$$\hat{a}_\mu^{(n)} = a_\mu + \frac{eq^{(n)}}{4} \theta^{\sigma\tau} \{ \partial_\sigma a_\mu, a_\tau \} + \frac{eq^{(n)}}{4} \theta^{\sigma\tau} \{ f_{\sigma\mu}, a_\tau \} + O(\theta^2). \quad (173)$$

The degrees of freedom are reduced to the classical ones.
5.3 Yukawa Couplings

Let us now consider the Yukawa coupling terms in (155), and their behaviour under gauge transformation. They involve products of three fields, e.g.,

$$
-\sum_{i,j=1}^{3} \left( W_{ij}^{(i)}(\tilde{L}_{L} \ast \rho_{L}(\hat{\Phi})) \ast \tilde{e}_{R}^{(j)} + W_{ij}^{(i)}\bar{\tilde{e}}_{R}^{(j)} \ast (\rho_{L}(\hat{\Phi})^{\dagger} \ast \tilde{L}_{L}^{(j)}) \right)
$$

(174)

Only in the case of commutative space-time, $\Phi$ commutes with generators of $U(1)$ and $SU(3)$ groups. Therefore the Higgs field needs to transform from both sides, in order to “cancel charges” from fields on either side (e.g., $\tilde{L}_{L}^{(i)}$ and $\bar{\tilde{e}}_{R}^{(j)}$ in (174)). The expansion of $\hat{\Phi}$ transforming on the left and on the right under arbitrary gauge groups is called hybrid SW map,

$$
\hat{\Phi}[\Phi, A, A'] = \phi + \frac{1}{2}\theta^{\mu\nu}A_{\nu}\left(\partial_{\mu}\phi - \frac{i}{2}(A_{\mu}\phi + \phi A'_{\mu})\right)
$$

(175)

$$
-\frac{1}{2}\theta^{\mu\nu}\left(\partial_{\mu}\phi - \frac{i}{2}(A_{\mu}\phi + \phi A'_{\mu})\right)A'_{\nu} + O(\theta^{2}),
$$

(176)

with $\hat{\delta}\Phi = i\hat{\Lambda} \ast \hat{\Phi} - i\hat{\Phi} \ast \hat{\Lambda}'$. In the above Yukawa term (174), we have $\rho_{L}(\hat{\Phi}) = \hat{\Phi}[\phi, V, V']$, with

$$
V_{\mu} = -\frac{1}{2}g' A_{\mu} + g B_{\mu}^{a}T_{a},
$$

$$
V'_{\mu} = g' A_{\mu}.
$$

We further need a different representation for $\hat{\Phi}$ in each of the Yukawa couplings

$$
\rho_{Q}(\hat{\Phi}) = \hat{\Phi}[\phi, \frac{1}{6}g' A_{\mu} + g B_{\mu}^{a}T_{a} + g_{s}G_{\mu}^{a}T_{a}, \frac{1}{3}g' A_{\nu} - g_{s}G_{\nu}^{a}T_{a}],
$$

(177)

$$
\rho_{\bar{Q}}(\hat{\Phi}) = \hat{\Phi}[\phi, \frac{1}{6}g' A_{\mu} + g B_{\mu}^{a}T_{a} + g_{s}G_{\mu}^{a}T_{a}, -\frac{2}{3}g' A_{\nu} - g_{s}G_{\nu}^{a}T_{a}].
$$

(178)

The respective sum of the gauge fields on both sides gives the proper quantum numbers of the Higgs shown in Table 1.

5.4 Kinetic Terms for the Gauge Bosons

As we have mentioned earlier in Section 3.1, the kinetic terms for the gauge field in the classical theory are determined uniquely by the requirements of gauge invariance and renormalisability. In the non-commutative case, we do not have a principle like renormalisability at hand. Gauge invariance alone does not fix these terms in the Lagrangian. The non-commutative Standard Model as defined here, has rather to be considered as
an effective theory, where renormalisability is not applicable. Otherwise, the role of the non-commutativity \( \theta \) has to be considered very carefully. \( \theta \) may become a space-time field with a kinetic term of its own. Therefore, the representations used in the trace of the kinetic terms for the gauge bosons are not uniquely determined. We will take the simplest choice, since we are interested in a version of the Standard Model on non-commutative space-time with minimal modifications. This choice is discussed in Subsection 5.4.1. In Subsection 5.4.2 we will consider a maybe more physical and natural choice of representation. Considering a Standard Model originating from a \( SO(10) \) GUT theory \[35\], these terms have a unique non-commutative generalisation.

### 5.4.1 Minimal Non-Commutative Standard Model

The simplest choice for the gauge kinetic terms is named minimal Non-Commutative Standard Model. The gauge kinetic terms have the form displayed in eqn. \(155\),

\[
-\int d^4x \frac{1}{2g} \text{tr}_1 \widehat{F}_{\mu\nu} \ast \widehat{F}^{\mu\nu} - \int d^4x \frac{1}{2g} \text{tr}_2 \widehat{F}_{\mu\nu} \ast \widehat{F}^{\mu\nu} - \int d^4x \frac{1}{2g_S} \text{tr}_3 \widehat{F}_{\mu\nu} \ast \widehat{F}^{\mu\nu}.
\]

It is the sum of traces over the \( U(1)_Y, SU(2)_L \) and \( SU(3)_C \) sectors. \( \text{tr}_2 \) and \( \text{tr}_3 \) are the usual \( SU(2)_L \) and \( SU(3)_C \) traces, respectively. \( \text{tr}_1 \) is the trace over the \( U(1)_Y \) sector with

\[
Y = \frac{1}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}
\]

as representation of the charge generator.

### 5.4.2 Non-Minimal Non-Commutative Standard Model

A perhaps more physical version of the Non-Commutative Standard Model is obtained, if we consider a charge matrix \( Y \) containing all the fields of the Standard Model with covariant derivatives acting on them. For the simplicity of presentation we will only consider one family of fermions and quarks. Then the charge matrix has the form

\[
Y = \begin{pmatrix} -1 & -1/2 \\ -1/2 & 2/3 \\ 2/3 & 2/3 \\ 2/3 & \ldots \end{pmatrix}, \quad (179)
\]
according to Table I. It acts on fields given by column vectors containing all the particles of the theory,

$$\Psi = \begin{pmatrix}
e_R \\
L_L \\
u_R \\
d_R \\
Q_L \\
\phi
\end{pmatrix}. \quad (180)$$

The kinetic term for the gauge field is given by

$$S_{\text{gauge}} = - \int d^4x \text{Tr} \frac{1}{2G^2} \hat{F}_{\mu\nu} \hat{F}^{\mu\nu}, \quad (181)$$

where $\hat{F}_{\mu\nu} = \partial_\mu \hat{V}_\nu - \partial_\nu \hat{V}_\mu - i[\hat{V}_\mu, \hat{V}_\nu]$. The classical gauge field $V_\mu$ is the sum

$$V_\mu = g' A_\mu(x) Y + \frac{g}{2} \sum_{a=1}^{3} B_{\mu a} \sigma^a + \frac{g_2}{2} \sum_{a=1}^{8} G_{\mu a} \lambda^a,$$

where $Y$ is defined by (179), and $\hat{V}_\mu = \hat{V}_\mu[V]$. $G$ takes the role of the coupling constant for the "master" gauge field $\hat{V}_\mu$. It is an operator commuting with all the generators of $(Y, T_L^a, T_S^a)$. It can be expressed in terms of $Y$ and the six constants $g_1, \ldots, g_2$ referring to the six multiplets (cf. Table I),

$$Ge_R = g_1 e_R,$$
$$GL_L = g_2 L_L,$$
$$\vdots$$

In the classical limit only three combinations of these six constants are relevant. They correspond to the usual coupling constants, $g', g, g_S$,

$$\text{Tr} \frac{1}{G^2} Y^2 = \frac{1}{2g'^2}, \quad (182)$$
$$\text{Tr} \frac{1}{G^2} T_L^a T_L^b = \delta^{ab} \frac{1}{2g^2}, \quad (183)$$
$$\text{Tr} \frac{1}{G^2} T_S^a T_S^b = \delta^{ab} \frac{1}{2g_S^2}. \quad (184)$$

The above equations read, taking the charge matrix (179) into account

$$\frac{1}{g_1^2} + \frac{1}{2g_2^2} + \frac{4}{3g_3^2} + \frac{1}{3g_4^2} + \frac{1}{6g_5^2} + \frac{1}{2g_6^2} = \frac{1}{2g'^2}, \quad (185)$$
$$\frac{1}{g_2^2} + \frac{3}{g_3^2} + \frac{1}{g_4^2} = \frac{1}{g^2}, \quad (186)$$
$$\frac{1}{g_3^2} + \frac{1}{g_4^2} + \frac{2}{g_5^2} = \frac{1}{g_S^2}. \quad (187)$$
These three equations define for fixed $g'$, $g$, $g_S$ a three-dimensional simplex in the six-dimensional moduli space spanned by $1/g_1^2$, $\ldots$, $1/g_6^2$. The remaining three degrees of freedom become relevant at order $\theta$ in the expansion of the non-commutative action. Interesting are in particular the following traces corresponding to triple gauge boson vertices:

\[
\text{Tr} \frac{1}{G^2} Y^3 = -\frac{1}{g_1^2} - \frac{1}{4g_2^2} + \frac{8}{9g_3^2} - \frac{1}{9g_4^2} + \frac{1}{36g_5^2} + \frac{1}{4g_6^2},
\]

\[
\text{Tr} \frac{1}{G^2} Y^a T^b_L = \frac{1}{2} \delta^{ab} \left( -\frac{1}{2g_2^2} + \frac{1}{2g_5^2} + \frac{1}{2g_6^2} \right),
\]

\[
\text{Tr} \frac{1}{G^2} Y^c T^d_S = \frac{1}{2} \delta^{cd} \left( \frac{2}{3g_3^2} - \frac{1}{3g_4^2} + \frac{1}{3g_5^2} \right).
\]

We may choose, e.g., to maximise the traces over $Y^3$ and $Y^a T^b_L$. This will give $1/g_1^2 = 1/(2g^2) - 4/(3g_S^2) - 1/(2g^2)$, $1/g_3^2 = 1/g_S^2$, $1/g_6^2 = 1/g_2^2$, $1/g_2^2 = 1/g_4^2 = 1/g_5^2 = 0$ and

\[
\text{Tr} \frac{1}{G^2} Y^3 = -\frac{1}{2g^2} + \frac{3}{4g^2} + \frac{20}{9g_S^2},
\]

\[
\text{Tr} \frac{1}{G^2} Y^a T^b_L = \frac{1}{4g^2} \delta^{ab},
\]

\[
\text{Tr} \frac{1}{G^2} Y^c T^d_S = \frac{2}{6g_S^2} \delta^{cd}.
\]

In the scheme that we have presented in Section 5.4.1 all three traces are zero. One consequence is that while non-commutativity does not require a triple photon vertex, such a vertex is nevertheless consistent with non-commutativity. It is important to note that the values of all three traces are bounded for any choice of constants.

### 5.5 Higgs Mechanism

In the leading order of the expansion in $\theta$, we obtain

\[
S_{\text{Higgs}} = \int d^4x \left( (D^S_M \phi)\dagger D^SM^\mu \phi - \mu^2 \phi^3 \phi - \lambda(\phi^3 \phi)(\phi^3 \phi) \right) + \int d^4x \left( (D^S_M \phi)^\dagger \left( D^SM^\mu \rho_0(\phi^1) + \frac{1}{2} \theta^{\alpha\beta} \partial_\alpha V^\mu \partial_\beta \phi + \Gamma^\mu_\phi \right) \right)
\]

\[
+ \left( D^S_M \rho_0(\phi^1) + \frac{1}{2} \theta^{\alpha\beta} \partial_\alpha V^\mu_\beta \phi + \Gamma^\mu_\phi \right) \dagger D^SM^\mu \phi
\]

\[
+ \frac{1}{4} \mu^2 \theta^\mu_\nu \phi^3 (g' f^L_\mu + g F^L_\mu) - \lambda i \theta^{\alpha\beta} \phi^3 \phi (D^S_M \phi)^\dagger (D^S_M \phi) + O(\theta^2),
\]

where

\[
\Gamma^\mu_\phi = -i V^1_\mu = \frac{i}{4} \theta^{\alpha\beta} \left( g' A_\alpha + g B_\alpha, g' \partial_\beta A_\mu + g \partial_\beta B_\mu + g' f^L_\beta + g F^L_\beta \right).
\]
We have also used the representation \( \rho_0 \),

\[
\rho_0(\hat{\Phi}) = \phi + \rho_0(\phi^1) + \mathcal{O}(\theta^2),
\]

(196)

where

\[
\rho_0(\phi^1) = -\frac{1}{2} \theta^{\alpha\beta} (g' A_\alpha + g B_\alpha) \partial_\beta \phi + \frac{i}{8} \theta^{\alpha\beta} [g' A_\alpha + g B_\alpha, g' A_\beta + g B_\beta] \phi.
\]

(197)

For \( \mu^2 < 0 \) the \( SU(2)_L \times U(1)_Y \) gauge symmetry is spontaneously broken to \( U(1)_Q \), which is the gauge group describing the electromagnetic interaction. We have gauge invariance and may choose the so-called unitarity gauge,

\[
\phi = \left( \begin{array}{c} 0 \\ \eta + v \end{array} \right) \frac{1}{\sqrt{2}},
\]

(198)

where \( v \) is the vacuum expectation value. Since the zeroth order of the expansion of the non-commutative action corresponds to the Standard Model action, the Higgs mechanism generates the same masses for electroweak gauge bosons as in the commutative Standard Model,

\[
M_{W^\pm} = \frac{g v}{2} \quad \text{and} \quad M_Z = \sqrt{g^2 + g'^2} v,
\]

(199)

where the physical mass eigenstates \( W^\pm, Z \) and \( A \) are as usual defined by

\[
W^\mu = \frac{B^1_\mu \mp i B^2_\mu}{\sqrt{2}}, \quad Z_\mu = -\frac{g' A_\mu + g B^3_\mu}{\sqrt{g^2 + g'^2}} \quad \text{and} \quad A_\mu = \frac{g A_\mu + g' B^3_\mu}{\sqrt{g^2 + g'^2}}.
\]

(200)

There are no corrections to the masses of order \( \theta \) since these terms involve derivatives and therefore do not resemble mass terms. The Higgs mass is given by \( m_h^2 = -2\mu^2 \).

Rewriting the term \( \Gamma_\mu \) in terms of the mass eigenstates, using

\[
B^3_\mu = \frac{g Z_\mu + g' A_\mu}{\sqrt{g^2 + g'^2}} \quad \text{and} \quad A_\mu = \frac{g A_\mu - g' Z_\mu}{\sqrt{g^2 + g'^2}},
\]

(201)

one finds that besides the usual Standard Model couplings, numerous new couplings between the Higgs boson and the electroweak gauge bosons appear.

5.6 Conclusions and Remarks

- The action of the NCSM \(^{155}\) has been expanded up to first order in the non-commutativity \( \theta \). To 0th order, we get the commutative standard model.
• The different interactions cannot be considered separately in the NCSM, since we had to introduce the overall gauge field $V_\mu$ in (156). They mix due to the SW maps, where the nc gauge field is given by $\hat{V} = \hat{V}[V]$.

• Higgs mechanism and Yukawa sector can be implemented in NCSM. The masses of the gauge bosons and fermions equal those of the Standard Model on commutative space-time, at tree level.

• There is no coupling of the Higgs to the electromagnetic Photon, in the minimal version. That coupling might be expected due to eqn. (170).

• Furtheron, no self interaction of the $U(1)_Y$ bosons occur, as one can see from the expansion of the gauge kinetic terms. Vertices with five and six gauge bosons for $SU(3)_C$ and $SU(2)_L$ are present.

• UV/IR mixing [36] is not present due to $\theta$ expansion.

• New decay modes for hadrons are found, 2 Quarks-Gluon-Photon vertex occur.

• One has to search for experimental signature for non-commutativity. One suggestion is to search for $Z \rightarrow \gamma \gamma$ decays in the non-minimal NCSM [37] or other decays forbidden by Lorentz symmetry [38].

• Other suggestions come from astrophysics. A weak limit on the non-commutative scale is proposed in [39]. The coupling of neutrinos to photons implies new energy loss mechanisms in stars. This leads to an estimate for the non-commutativity scale. In the context of $\kappa$-deformation, it has been argued to search for observable effects in time delays of high-energy $\gamma$ rays or neutrinos from active galaxies, such as Markarian 142. These effects are due to modified dispersion relations, see e.g., [40].

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