Properties of low-energy excitations in one-dimensional superconductors and density-wave systems are examined by the bosonization technique. In addition to the usual spin and charge quantum numbers, a new, independently measurable attribute is introduced to describe elementary, low-energy excitations. It can be defined as a number $w$ which determines, in multiple of $\pi$, how many times the phase of the order parameter winds as an excitation is transposed from one to another. The winding number is zero for electrons and holes with conventional quantum numbers, but it acquires a nontrivial value $w = 1$ for neutral spin-$\frac{1}{2}$ excitations and for spinless excitations with a unit electron charge. It may even be irrational, if the charge is irrational. Thus, these excitations are topological, and they can be viewed as composite particles made of spin or charge degrees of freedom and dressed by kinks in the order parameter.

In spite of the success of the basic concept, it cannot be justified rigorously because of the break-down of the renormalizing procedure. Indeed, there are many interesting instances where the physical picture of weakly interacting elementary excitations with quantum numbers equal to those of bare ones becomes invalid. Examples of such cases are the quantum Hall effect and quasi-one-dimensional conductors, where low-energy excitations have unusual fermion quantum numbers in spin- and charge-density-wave systems which have continuous symmetry and whose ground states develop quasi-long-range order in one dimension and true long-range order in higher dimensions. In addition to superconductors, kinks must then be introduced in spin- and charge-density-wave systems. Our conclusion complements the earlier observation that soliton excitations have unusual fermion quantum numbers in charge-density-wave systems where massless fermions are coupled to a boson field. Here we demonstrate that also the converse is true: elementary charge and spin excitations always carry kinks in one-dimensional systems with quasi-long-range order.

Specifically, consider the one-dimensional Hamiltonian

$$
H = -t \sum_{n \sigma} (c_{n+1}^\dagger c_n + h.c.) + \frac{1}{2} \sum_{n,m} V_{n-m} \rho_n \rho_m, \quad (1)
$$

where $c_{n\sigma}$ is the fermion operator for an electron of spin $\sigma$ at site $n$, $\rho_n = \sum_\sigma c_{n\sigma}^\dagger c_{n\sigma}$ is the electron number density, $V_{n-m}$ is the electron-electron interaction, and $t$ is the nearest-neighbor hopping matrix element. To address the low-energy and long-wavelength excitations, the energy spectrum is linearized at the Fermi energy and the fermionic degrees of freedom are expressed by slowly varying fields $\psi_{\sigma \pm}(x)$:

$$
c_{n\sigma} / \sqrt{a} \sim \psi_{\sigma +}(x_n) e^{-ik_F x_n} + \psi_{\sigma -}(x_n) e^{ik_F x_n}, \quad (2)
$$
where $k_F$ is the Fermi wavevector, $a$ is the lattice spacing, and $x_n = na$. Subsequently, the free part of the Hamiltonian, $H_0 = H_0 + H_1$, becomes

$$H_0 = v_F \int dx \left[ \hat{p}_\sigma(x_-) \hat{\psi}_\sigma(x_-) - \hat{p}^\dagger_\sigma(x_+) \hat{\psi}_\sigma(x_+) \right],$$

where $\hat{p} = -i\hbar \partial_x$ is the momentum operator and $v_F = (2\pi/\hbar) k_F a$ is the Fermi velocity (hereafter a sum over repeated indices is implied unless otherwise noted). The left (+) and right (−) moving electrons with spin $\sigma$ are labeled according to their arguments, denoting both the space and time coordinate, $x_\pm \equiv (x,t)\pm$; namely, $\psi_\sigma(x_\pm) = \psi_{\sigma\pm}(x,t)$. In the case of free fermions, the Heisenberg equations of motion, $i\hbar \partial_t \psi_\sigma(x_\pm) = [\psi_\sigma(x_\pm), H]$, lead to the field operators which are functions of variables $x \pm v_F t$, where ± refer to the left- and right-moving electrons, respectively: $\psi_\sigma(x_\pm) = \psi_\sigma(x \pm v_F t)$.

It is convenient to define left- and right-moving currents as $J_{\sigma\sigma'}(x_\pm) = \psi^\dagger_\sigma(x_\pm) \psi_{\sigma'}(x_\pm)$. The colons denote normal ordering with respect to the filled Fermi sea of the noninteracting system. In terms of these currents, the Hamiltonian describing the electron-electron interactions may be rewritten as

$$H_1 = \int dx \left( -V_1 J_{\sigma\mu}(x_+):J_{\mu\sigma}(x_-) + V_2 J_{\sigma\sigma'}(x_+):J_{\mu\mu}(x_-) \right) + \frac{1}{2} V_3 \left[ J_{\sigma\sigma}(x_+):J_{\mu\mu}(x_-) + J_{\sigma\sigma}(x_-):J_{\mu\mu}(x_+) \right].$$

As usual, $V_1$ is the backward-scattering constant, $V_2$ is the forward-scattering constant, $V_3$ is the Umklapp-scattering constant, and $V_4$ is another forward scattering constant. Because $V_3$ corresponds to the scattering processes which violate momentum conservation by $4k_F$, it is important only if $4k_F$ is equal to the reciprocal lattice constant (the second-order commensurability). Note that $V_{1,3} = V(2k_F)$ and $V_{2,4} = V(0)$, where $V(k) = \sum_n a V_n e^{i k x_n}$. For simplicity, we consider a limit where the backward and Umklapp terms are either zero or scale to zero, so that the model is exactly solvable by bosonization. For example, far away from the second-order commensurability, the Umklapp processes are effectively turned off and remain so under the renormalization-group flow. As long as the backward and Umklapp scattering terms remain irrelevant in the sense of scaling, they can be neglected. Finally, the coupling constants $V_k$ are allowed to be independent, as implied by many important interactions that are not depicted by the original Hamiltonian, Eq. (3).

In Abelian bosonization, the left- and right-moving fermion operators are expressed in terms of boson fields, $\psi_\sigma(x_\pm) \sim K_\sigma \exp[\mp i \Phi_\sigma(x_\pm)]$; (no sum over repeated indices is implied in this paragraph), where $K_\sigma$ is the Klein phase-operator which establishes the correct anticommutation relations for different fields [5]. For instance, with this definition, the current operator becomes $J_{\sigma\sigma}(x_\pm) = (1/2\pi) \partial_x \Phi_\sigma(x_\pm)$. For $V_1 = V_3 = 0$, the Hamiltonian is quadratic in the boson fields $\Phi_\sigma$:

$$H_0 = \sum_\sigma \hbar v_F \int \frac{dx}{4\pi} \left( [\partial_x \Phi_\sigma(x_\pm)]^2 + [\partial_x \Phi_\sigma(x_-)]^2 \right),$$

$$H_1 = \sum_{\sigma\mu} \frac{v_F}{4\pi} \left( V_2 [\partial_x \Phi_\sigma(x_+)] [\partial_x \Phi_\mu(x_-)] + \frac{1}{2} V_3 [\partial_x \Phi_\sigma(x_+)]^2 + [\partial_x \Phi_\mu(x_-)]^2 \right).$$

In the absence of interactions, $\Phi_\sigma(x_\pm) = \Phi_\sigma(x \pm v_F t)$. $H$ is readily diagonalized by a set of unitary transformations. First, define $\Phi_\sigma(x_\pm) = [\Phi_\sigma(x_\pm) - \Phi_\pm(x_\pm)]/\sqrt{2}$ and $\Phi_\pm(x_\pm) = [\Phi_\pm(x_\pm) + \Phi_\pm(x_\pm)]/\sqrt{2}$. Second, define $\varphi_\pm(x_\pm) = \Phi_\pm(x_\pm)$ and $\varphi_\pm(x_\pm) = \Phi_\pm(x_\pm) \cos \phi + i \Phi_\pm(x_\pm) \sin \phi$, with $g = V_2/(\hbar v_F + V_4)$ and $\tan 2\phi = g$. The Hamiltonian is transformed into

$$H = \sum_{\nu=\pm,c} \hbar v_{\nu} \int \frac{dx}{4\pi} \left( [\partial_x \varphi_\nu(x_\pm)]^2 + [\partial_x \varphi_\nu(x_-)]^2 \right),$$

where $v_s = v_F$ and $v_c = v_F/cosh 2\theta$ are the spin and charge velocities. In the Heisenberg picture, $\varphi_\nu(x_\pm) = \varphi_\nu(x \pm v_s t)$ and $\varphi_\nu(x_-) = \varphi_\nu(x \pm v_c t)$. A useful relation in computing correlation functions of interacting fields is the two-point correlation, $\langle \varphi_\nu(x_\pm) \varphi_\nu(y_\pm) \rangle = -\log(1 \mp i x_{\nu \pm})$, where $x_{\nu \pm} = x \pm v_s t \ (\nu = s, c)$. The ultraviolet cutoff, the Fermi length, is formally associated with the lattice spacing $a$.

For $g < 0$, the superconducting (SC) instability is the most dominant one at zero temperature. Defining the singlet pairing field as

$$\Delta(x) = [\varphi_\nu(x_+)]^\dagger \varphi_\nu(x_-) - [\varphi_\nu(x_+)] \varphi_\nu(x_-)]/\sqrt{2},$$

the correlation function

$$G_{SC}(x - y) = \langle 0 | \Delta(x) \Delta(y) | 0 \rangle$$

probes the ordering fluctuations of interest in the ground state $| 0 \rangle$. Below, we will always consider cases where $x$ and $y$ are measured at equal times $t$: $x = (x,t)$ and $y = (y,t)$. Using the identity,

$$e^{i\alpha \varphi(x)} : e^{i\beta \varphi(y)} : = e^{i(\alpha \varphi(x) + i\beta \varphi(x))} e^{-\alpha \beta \varphi(x) \varphi(y)},$$

one can show that

$$
\Delta(x) \Delta(y) = G_{SC}(x - y) e^{i \alpha A_c} \cos A_c,
$$

where the prefactor is

$$G_{SC}(r) = \left( \frac{1}{a} \right)^{\mu} \left( \frac{a^2 + r^2}{a^2} \right)^{\mu_{SC}/2}.$$

The operators, $A_\nu = [\varphi_\nu(x) - \varphi_\nu(y)]/\sqrt{2} \ (\nu = s, c)$, are expressed in terms of new fields, $\varphi_\nu(x) = \varphi_\nu(x_+) + \varphi_\nu(x_-)$.
\( \phi_\ast(x_\pm) \) and \( \phi_\ast(x) = \phi_\ast(x_+) - \phi_\ast(x_-) \). The exponents are defined as \( \mu_{SC} = 1 + \eta^2 \), \( \mu = 1 + 1/\sqrt{1-g^2} \), and \( \eta = [(1+g)/(1-g)]^{1/4} \). In the ground state, \( G_{SC}^{(0)}(x) = G_{SC}(x) \) — thus, the asymptotic forms of the correlation function are \( G_{SC}^{(0)}(x) \propto x^{-\mu_{SC}} \) and \( G_{SC}^{(0)}(q) = 0, \omega \propto \omega^{-\alpha_{SC}} \), where \( \alpha_{SC} = 2 - \mu_{SC} \). The superconducting correlation function is singular \( \alpha_{SC} > 0 \) at low energies and small momenta, for \( g < 0 \).

The effect of excitations on pairing fluctuations can be examined by computing the correlation function in the state \( |\Psi_\uparrow\rangle = \psi_\uparrow(0)\{0\} \), where \( \psi_\uparrow(0) = [\psi_\uparrow(0_+) + \psi_\uparrow(0_-)]/\sqrt{2} \) adds an electron into the system at the origin and time \( t = 0 \). First, let the electron be far away from the points \( x \) and \( y \) where the pairing fields are measured, \( |r| \ll |R| \); here, \( r = x - y \) and \( R = (x+y)/2 \). Then, writing \( A_\nu = (r/\sqrt{2})\partial_x \phi_\nu(R) + O(r^3) \), an operator-product expansion can be developed for Eq. \( [14] \). The correlation function,

\[ G_{SC}^{(1)}(x,y) = \langle \Psi_\uparrow | \Delta_\uparrow(x) \Delta_\downarrow(y) | \Psi_\uparrow \rangle, \]

simply becomes for \( t = 0 \)

\[ G_{SC}^{(1)}(x,y) = \left( \frac{1}{a} \right)^{3\mu/2} \left( \frac{a^2}{a^2 + r^2} \right)^{\mu_{SC}/2} \cos^2 \frac{ar/2}{a^2 + R^2}. \]

(13)

For given \( r \), the influence of the electron on the correlation function decays as \( R^{-4} \). This behavior is universal as the scaling exponent is independent of the strength of interactions. The result is consistent with the natural observation that a single electron has no effect on the bulk properties of a superconductor.

Second, let \( x \) and \( y \) be arbitrary. The correlation function can be written in the form

\[ G_{SC}^{(1)}(x,y) = G_{SC}(r) F(x,y), \]

where \( F(x,y) = [F_\uparrow(x,y) + F_-(-x,y)]/2 \) and

\[ F_\pm(x,y) = \cos[\theta_a(x_\pm - \theta_a(y_\pm))] \times e^{\pm iw_1[\theta_\uparrow(x_{1\pm}) - \theta_\downarrow(y_{1\pm})]} e^{\pm iw_2[\theta_\uparrow(x_{2\pm}) - \theta_\downarrow(y_{2\pm})]}, \]

(15)

We have defined \( \theta_a(x) = \arctan(x/a) \), \( w_1 = \eta \cos \phi \), and \( w_2 = -\eta \sinh \phi \) \( (w_1 > w_2 > 0, \text{for } g < 0) \). Note that \( \theta_\uparrow(x) \rightarrow \pm \pi \text{sgn}(x) \), as \( |x|/a \rightarrow \infty \). Initially \((t = 0)\), the function \( F \) reduces to \( F(x,y) = \cos^2[\theta_\uparrow(x) - \theta_\downarrow(y)] \).

For \( |r| \ll |R| \), we recover Eq. \( [13] \). In the scaling limit \( r, R \rightarrow \infty \) with \( z \equiv r/R < 1 \) fixed, the asymptotic behavior of \( G_{SC}^{(1)} \)

\[ G_{SC}^{(1)}(x,y) \propto \left( \frac{1}{R} \right)^{\mu_{SC}} \left[ 1 - \left( \frac{\lambda}{R} \right)^2 \right], \]

(16)

where \( \lambda/a = [2z/(1-z^2)]^{1/2} \). Again, the scaling exponent describing the suppression of superconducting correlations near the electron is universal.

In general, the order parameter \( \Delta \) is obtained from \( \langle \Delta_\uparrow(x) \Delta_\downarrow(y) \rangle \rightarrow |\Delta|^2 \), as \( r \rightarrow \infty \). Likewise, the influence of an excitation on the superconductivity can be studied by letting \( x \) to probe the order parameter in the neighborhood of the excitation and \( y \rightarrow \infty \), even though strictly speaking \( \Delta \) vanishes in one dimension. For \( t = 0 \), the solution is

\[ F(x,\infty) = 1 - \frac{a^2}{a^2 + x^2}, \]

(17)

which shows that the electron destroys the order parameter in its vicinity while preserving \( F \geq 0 \). The order-parameter suppression dies out as \( x^{-2} \). However, in the interacting system, the electron decays instantaneously into elementary excitations involving either spin or charge degrees of freedom. The kink-like nature of these excitations is revealed, when the system is allowed to evolve in time. For illustrative purposes, consider the ground state with one right-moving electron initially added at the origin, \( |\Psi_\uparrow\rangle = \psi_\uparrow(0)\{0\} \), so that \( F = F_- \). The spin and the charge excitations carry kinks in the order parameter, and they both separately contribute to the total phase shift an equal amount of \( \pi \), because \( w_1 + w_2 = 1 \). Initially, the kinks overlap but then split as the excitations move apart. If only the spin excitation is located between the points \( x \) and \( y \) at a later time \( t \), the correlation function is negative: \( G_{SC}^{(1)} \) changes sign if either \( x \) or \( y \) crosses the position of the spin. This clearly shows that the state \( |\Psi_\uparrow\rangle \) has an ordinary kink, or an antiphase domain wall, in the order parameter precisely at the position of the spin \( [14] \). The winding number of a spin-1/2 particle is 1. In contrast, the splitting of the charge into right and left moving charge components, \( q_1^- \) and \( q_2^- \), leads to two irrational kinks whose winding numbers are \( w_1 \) and \( w_2 \). The winding number of the kink is uniquely related to its charge — in units where the electron charge is one, the relation is particularly simple: \( w_{1,2} = q_{1,2}^- \). The total winding number, as well as the total charge, is conserved. Because the spin and charge velocities differ, the spin and charge components move independently, and the kink splits into two.
charge kinks also propagate with the different velocities. This suggests a new and potentially attractive way to detect spin-charge separation in a superconductor by using the Josephson effect. It would also serve as evidence of irrational charge. Figure 1 describes schematically the effect of spin and charge on the superconducting order parameter.

The above result is generalized in a straightforward manner for $N$ electrons initially injected into the system at positions $R_i = (R_i, 0) \ (i = 1, \ldots, N)$:

$$G_{SC}^{(N)}(x, y) = G_{SC}(r) \prod_{i=1}^{N} F(x - R_i, y - R_i).$$ (18)

If an infinite number of electrons is randomly distributed with a mean distance $\ell$, the correlation function decays exponentially:

$$G_{SC}^{(\infty)}(x, y) \propto \left( \frac{1}{r} \right)^{\mu_{SC}} e^{-4r/t}\ell,$$ (19)

with $r(t) = \min [\|r\|, (v_x - v_y)|t|]$. Thus, for any nonzero concentration of injected electrons, $n_{ex} = a/\ell \neq 0$, superconductivity is destroyed.

One may also ask whether other quantities than the order parameter show any signatures of kinks. Indirect evidence of kinks could be looked for in the density of states and the conductivity, for example. They are probed in specific heat and optical absorption measurements. (i) That there are new states at low energies (“midgap” states) associated with the kinks is evident from the density of states $N(\omega)$. In the absence of excitations, at zero temperature, $N(\omega) \sim \omega^\alpha$, where $\alpha = \frac{1}{2}(1/\sqrt{1 - \eta^2} - 1)$. Enhanced superconducting fluctuations will always lead to a pseudo-gap in the density of states, because $\alpha > 0$, for $g < 0$. If there are excitations present in the system, the density of states must correspondingly be modified at low energies such that

$$N(\omega) \propto (\omega^2 + \gamma_1^2 \omega^{1/2}) + (\omega^2 + \gamma_2^2 \omega^{1/2}),$$ (20)

where $\gamma_1 = (2v_c/\ell)[\cos^2 \phi - \sin(\frac{\pi}{2} \cos 2\phi)]$ and $\gamma_2 = (2v_c/\ell)[1 - \cos(\frac{\pi}{2} \sin 2\phi)]$. Thus, nonzero $N(\omega \to 0)$ implies that kinks give rise to new states at low energies. (ii) The conductivity, however, does not exhibit any unusual behavior due to kinks, because it is sensitive to the concentration of charges and the interactions between them — injected electrons behave undistinguishably from the existing particles, and the charge dynamics remains the same. If kinks and excitations were pinned, different kind of behavior would be expected.

In general, a nonzero backward scattering term which scatters electrons of opposite spin across the Fermi surface in opposite directions must be included. As a result, left- and right-moving spin degrees of freedom are coupled so that the spin will also split into left- and right-moving components. While it appears that the spin by itself cannot have other values than integers and half integers, irrational values are allowed for the average spin and the concomitant kink.

The concept of a winding number applies equally to charge-density-wave and spin-density-wave instabilities (at $2k_F$) that are described by operators

$$\rho(x) = \psi_+^\dagger(x_+) \psi_-(x_-) e^{i2kFx} + h.c.$$ (21a) (charge-density-wave) and

$$S_\sigma(x) = \psi_+^\dagger(x_+) \tau_{\sigma\sigma'}^i \psi_-(x_-) e^{i2kFx} + h.c.$$ (21b) (spin-density-wave); $\tau^a$ are the three Pauli matrices ($a = 1, 2, 3$). The only difference is that the functional forms of the irrational winding numbers $w$ depend in a unique fashion on the nature of quasi-long-range order.

In conclusion, elementary excitations in interacting one-dimensional conductors always carry kinks in the order parameter. The winding number characterizing a kink cannot be arbitrary but is determined by the spin and the charge of the excitation and by quasi-long-range order of the ground state — in other words, by the interactions. This is not an entirely unexpected result, because electrons are interpreted in bosonization as solitons [8]. The novelty of our formulation is that the kink structure of the excitations becomes observable, if the ground state develops quasi-long-range order. This implies a completely new conception of probing unusual quantum numbers of elementary excitations, which manifest themselves through structural and dynamical modulations in the order parameter. For example, in a superconductor, the Josephson effect could provide a method to measure winding numbers associated with injected elementary excitations. Using Hartree-Fock theory, we have already pointed out that spin excitations in superconductors tend to form antiphase domain walls (kinks) in the order parameter [3]. An analogous observation concerning impurity spinons has been made in the context of one-dimensional Kondo lattices [1]. Thus, these results corroborate the conclusion that kinks are generic excitations of superconductors.

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