ALGEBRAIC STRUCTURES DEFINED ON $m$-DYCK PATHS

DANIEL LÓPEZ N., LOUIS-FRANÇOIS PRÉVille-RATELLE, MARÍA RONCO

ABSTRACT. We introduce natural binary set-theoretical products on the set of all $m$-Dyck paths, which led us to define a non-symmetric algebraic operad $Dyck^m$. Our construction is closely related to the $m$-Tamari lattice, so the products defining $Dyck^m$ are given by intervals in this lattice. For $m = 1$, we recover the notion of dendriform algebra introduced by J.-L. Loday in [16], and there exists a natural operad morphism from the operad $Ass$ of associative algebras into the operad $Dyck^m$, consequently $Dyck^m$ is a Hopf operad. We give a description of the coproduct in terms of $m$-Dyck paths in the last section. As an additional result, for any composition of $m + 1 \geq 2$ in $r + 1$ parts, we get a functor from the category of $Dyck^m$ algebras into the category of $Dyck^r$ algebras.

INTRODUCTION

For $m \geq 1$, the $m$-Dyck paths are a particular family of lattice paths counted by Fuss-Catalan numbers, which are connected with the (bivariate) diagonal coinvariant spaces of the symmetric group. These representations are also called the Garsia-Haiman spaces, and they can be defined for an arbitrary number of sets of variables. Our work is motivated by the combinatorics of these spaces and by the Loday-Ronco Hopf algebra on binary trees.

The Garsia-Haiman spaces have influenced the work of many combinatorialists in the past 20 years (see for instance [13], [14], [15], [9]), and they are still a very active area of research today (see [5], [23], [24]) with many open problems. Note that the previous two lists of references are far from exhaustive. In particular we refer to the books of Bergeron (see [2]) and Haglund ([12]) for more explanations and references. Motivated by the combinatorics of the Garsia-Haiman spaces (see [13], [14], [15]) and by an enumerative formula of Chapoton counting intervals in the Tamari lattice (see [6]), F. Bergeron introduced the $m$-Tamari lattice, where the case $m = 1$ is the usual Tamari lattice. F. Bergeron and the second author

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(see [3]) showed that the trivariate diagonal coinvariant spaces are related to the intervals and the labelled intervals of the \( m \)-Tamari lattice. These labelled intervals are some generalizations of parking functions, where the latter is another family of combinatorial objects related with the (bivariate) Garsia-Haiman spaces. The \( m \)-Tamari lattice is the starting point of our work.

In [16], J.-L. Loday introduced the notion of dendriform algebra and proved that the algebraic operad of dendriform algebras is naturally described on the vector space \( \mathbb{K}[Y_\infty] \) spanned by planar rooted binary trees. Dendriform algebras are associative algebras whose product splits as the sum of two binary operations. In many associative algebras already known in literature, as the algebras defined by shuffles (see [7] or [20]) and the Rota-Baxter algebras (see [1]), the associative product comes from a dendriform structure. In [17], J.-L. Loday and the third author, proved that any free dendriform algebra has a natural structure of bialgebra, which is described in terms of admissible cuttings of trees.

The main goal of our work is to introduce a non-symmetric Hopf operad \( \text{Dyck}^m \) such that the space of \( n \)-ary operations of the theory is precisely the vector space \( \mathbb{K}[\text{Dyck}_n^m] \), spanned by all the \( m \)-Dyck paths of size \( n \), for any \( m \geq 1 \). When \( m = 1 \), we recover the operad of dendriform algebras.

Given an \( m \)-Dyck path of size \( n \), there is a unique way to color its down steps with elements of the set \( \{1, \ldots, n\} \) in such a way that F. Bergeron’s covering relation consists in increasing the level of a down step without changing its color. This condition characterizes the order and is the key ingredient of our construction. The operad \( \text{Dyck}^m \) is spanned by \( m + 1 \) binary operations \( \ast_0, \ldots, \ast_m \), which are given by intervals of F. Bergeron’s \( m \)-Tamari lattice. For readers interested in algebraic operads, let us point out that the operads \( \text{Dyck}^m \) are easily seen to be Koszul.

We also introduce the notion of \( \text{Dyck}^m \)-bialgebra and described the coproduct on the vector space \( \mathbb{K}[\text{Dyck}_n^m] \), spanned by the set of \( m \)-Dyck paths, in terms of admissible cuttings of the Dyck path, which seem to be a particular case of the cuttings of rooted trees introduced by R. Grossman and R. Larson in [10].

For \( m = 1 \), we know that the subspace of primitive elements of a dendriform bialgebra has a natural structure of brace algebra. For \( m > 1 \), the space of primitive elements of a \( \text{Dyck}^m \) algebra is a brace algebra equipped with some additional structure. In a forthcoming work we describe the operads associated to the primitive elements of \( \text{Dyck}^m \) bialgebras.

Before giving a more precise description of the contents of the manuscript, let us point out that in [22], J.-C. Novelli and J.-Y. Thibon introduced the notion of \( m \)-permutations and defined the Sylvester congruence in this new context. These construction led them to define \( m \)-trees as the classes of \( m \)-permutations modulo the generalized Sylvester congruence. In a second work, see [21], J.-C. Novelli introduced the notion of \( m \)-dendriform algebra
and showed that the vector space spanned by \( m \)-trees provide a natural description of this operad. Even if the dimension of the operad of \( m \)-dendriform algebras in degree \( n \) is the number of \( m \)-Dyck path of size \( n \) and both of them are generated by \( m + 1 \) products, J.-C. Novelli’s operad is different from \( \text{Dyck}^m \). In particular, our \( \text{Dyck}^m \) operad is defined by only two types of relations. A nice bijection between Dyck paths and \( m \)-trees still needs to be defined in order to compare both structures.

Contents

In the first section we recall some basic definitions and constructions of Dyck paths, needed in the sequel.

In Section 2 we introduce basic operations \( \times_j \) on the set of \( m \)-Dyck paths, and the notion of coloring of a Dyck path. The basic constructions of this section are used in Section 3 to define binary products \( \ast_0, \ldots, \ast_m \) on the space \( \mathbb{K}[\text{Dyck}^m] \), spanned by the set of \( m \)-Dyck paths, and to prove the relations between them.

In Section 4, we show that the \( \text{Dyck}^m \) algebra structure on the space spanned by Dyck paths is related to the \( m \)-Tamari lattice by the formulas:

\[
P \ast_i Q = \sum_{P \vdash_i Z \leq P \setminus_i Q} Z,
\]

for any pair of Dyck paths \( P \) and \( Q \) and any integer \( 0 \leq i \leq m \).

We introduce the formal definition of \( \text{Dyck}^m \) algebra in Section 5, and prove that the space \( \mathbb{K}[\text{Dyck}^m] \), equipped with the products \( \ast_i \) introduced in the previous section, is the free \( \text{Dyck}^m \) algebra spanned by one generator. As the operad of \( \text{Dyck}^m \) algebras is regular, the whole operad is described by the free object spanned by one generator, so the combinatorial properties of \( m \)-Dyck paths define completely the operad. We show that, given two non-negative integers \( h < m \), there is a natural way to define for any composition \( \underline{r} \) of \( m \) in \( h + 1 \) parts, an operad homomorphism \( F_{\underline{r}} \) from \( \text{Dyck}^h \) into \( \text{Dyck}^m \), which is compatible with the refinement of compositions. In particular, any \( \text{Dyck}^m \) algebra has an underlying associative structure, which describes the Hopf operad structure of \( \text{Dyck}^m \). To end Section 5 we prove that the image of a free \( \text{Dyck}^m \) algebra under the functor \( F_{\underline{r}} \) is a free \( \text{Dyck}^h \) algebra, for any composition \( \underline{r} \) of \( (m + 1) \) in \( (h + 1) \) parts.

The last section is devoted to define the coproduct on Dyck paths in terms of admissible cuttings.

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Preliminaries

All the vector spaces considered in the present work are over \( \mathbb{K} \), where \( \mathbb{K} \) is a field. For any set \( X \), we denote by \( \mathbb{K}[X] \) the vector space spanned by \( X \). For any \( \mathbb{K} \)-vector space \( V \), we denote by \( V^+ := \mathbb{K} \oplus V \) the augmented vector space. The set of non-negative integers is denoted \( \mathbb{Z}_+ \).

1. \( m \)-DYCK PATHS

In the present section we introduce basic notions of the combinatorial and algebraic structures we shall need in the rest of the work. For more detailed constructions and the proofs of the results we refer to [3], [4] and [5].

**Definition 1.1.** For \( m, n \geq 1 \), an \( m \)-Dyck path of size \( n \) is a path on the real plan \( \mathbb{R}^2 \), starting at \((0, 0)\) and ending at \((2mn, 0)\), consisting of up steps \((m, m)\) and down steps \((1, -1)\), which never goes below the \( x \)-axis. Note that the initial and terminal points of each step lean on \( \mathbb{Z}^2_+ \).

We denote by \( \text{Dyck}^m_n \) the set of all \( m \)-Dyck paths of size \( n \).

The number of elements of the set \( \text{Dyck}^m_n \) is \( d_{m,n} := \frac{1}{mn+1} \binom{(m+1)n}{n} \).

**Example 1.2.** For \( m = 2 \), we get that

1. the unique element of \( \text{Dyck}^2_1 \) is

   \[
   \begin{array}{c}
   \vdots \vdots \\
   \end{array}
   \]

2. the elements of \( \text{Dyck}^2_2 \) are

   \[
   \begin{array}{c}
   \vdots \vdots \vdots \vdots \\
   \end{array}
   \begin{array}{c}
   \vdots \vdots \vdots \vdots \\
   \end{array}
   \begin{array}{c}
   \vdots \vdots \vdots \vdots \\
   \end{array}
   \]

In order to define constructions on Dyck paths, we use a notation similar as the one employed by M. Bousquet-Mélo, E. Fusy and the second author in [4].

**Notation 1.3.** Let \( P \) be an \( m \)-Dyck path. We denote by \( \text{UP}(P) \) the set of up steps of \( P \) and by \( \text{DW}(P) \) the set of down steps of \( P \).

**Definition 1.4.** Let \( u \in \text{UP}(P) \) be an up step of an \( m \)-Dyck path \( P \), the rank of \( u \) is \( k \) if \( u \) is the \( k \)-th up step of \( P \), counting from left to right.

The shortest (translated) Dyck path which starts with \( u \) is called the excursion of \( u \) in \( P \), and is denoted \( P_u \). The down step \( w_u \in \text{DW}(P) \) matches \( u \) if it is the final step of the excursion of \( u \) in \( P \).

Finally, a down step \( d \) is at level \( k \) if the last up step \( u \) preceding \( d \) has rank \( k \).
Example 1.5. The up step $u$ in the path $P$ has rank 3 and the down step $d$ matches it.

Any up step $u$ in a Dyck path $P$ is determined by its rank, from now on we identify them, and denote the set of up steps of a Dyck path of size $n$ as $UP(P) = \{1, \ldots, n\}$.

2. Operations on Dyck paths

We want to describe basic operations on Dyck paths that we need in the sequel.

Notation 2.1. For a path $P \in Dyck^m_n$ and an integer $1 \leq k \leq n$, we denote by $DW_k(P)$ the set of down steps of level $k$ of $P$ and by $L_k(P)$ the number of elements of $DW_k(P)$. When no confusion is possible, we shall denote the last term of the sequence $L_n(P)$ simply by $L(P)$.

Note that $0 \leq \sum_{i=1}^j L_i(P) \leq mj$, for $1 \leq j \leq n$. A Dyck path $P$ is uniquely determined by the sequence $(L_1(P), \ldots, L_n(P))$.

Definition 2.2. Let $P$ and $Q$ be two $m$-Dyck paths of sizes $n_1$ and $n_2$, respectively. For $0 \leq i \leq L(P)$, define the $i^{th}$-concatenation of $P$ and $Q$, denoted $P \times_i Q$, as the Dyck path of size $n = n_1 + n_2$ obtained in the following way:

1. if $d_{1}^{P}, \ldots, d_{L(P)}^{P}$ denotes the ordered sequence of down steps of level $n_1$ of $P$, cut $P$ at the final vertex of $d_{L(P)}^{P}$,
2. glue the initial point of $Q$ (translated) at the final point of $d_{L(P)}^{P}$,
3. glue the down steps $d_{L(P)}^{P}$ at the end point of $Q$.

Example 2.3. Let $P$ and $Q$ be the 2-Dyck paths

the element $P \times_2 Q$ is the following 2-Dyck path:
Definition 2.4. An \( m \)-Dyck path \( P \) is called prime if there does not exist a pair of \( m \)-Dyck paths of smaller size \( Q \) and \( R \) such that \( P = Q \times R \).

Remark 2.5. For any \( m \)-Dyck path of size \( n \) there exist a unique composition \( (n_1, \ldots, n_r) \) of \( n \) (with \( n_i \geq 1 \) for each \( i \)) and a unique family of prime Dyck paths \( P_1 \in Dyck_{n_1}, \ldots, P_r \in Dyck_{n_r} \) such that \( P = P_1 \times \ldots \times P_r \).

The proof of the following Lemma is immediate.

Lemma 2.6. Let \( P \in Dyck_{n_1} \) be a prime Dyck path and let \( Q \in Dyck_{n_2} \) be another Dyck path. For any \( 1 \leq j \leq L(P) \), the Dyck path \( P \times_j Q \) is prime.

Define \( Dyck_0^m := \{ \bullet \} \), for \( m \geq 1 \). Any \( m \)-Dyck path \( P \) of size \( n \) may be written in two different ways, as:

- \( P = (((\rho_m \times_m P_0) \times_{m-1} P_1) \times_{m-2} \ldots) \times_0 P_m \), and
- \( P = P'_0 \times 0 (((\rho_m \times_m P'_1) \times_{m-1} P'_2) \times_{m-2} \ldots \times_{2m-1} P'_{m-1}) \times_1 P'_m \), for unique families of Dyck paths \( P_0, \ldots, P_m \) and \( P'_0, \ldots, P'_m \), with \( P_j \in Dyck_{n_j}^m \) and \( P'_j \in Dyck_{n'_j}^m \), \( 0 \leq n_j, n'_j \leq n - 1 \) and \( \sum_{j=0}^m n_j = \sum_{j=0}^m n'_j = n - 1 \).

For example, for the path \( P \) in the example above, we get

\[
\begin{align*}
\bullet \quad \bullet \quad \bullet \\
\quad \bullet \quad \bullet \quad \bullet \\
\quad \quad \bullet \quad \bullet \\
\end{align*}
\]

and

\[
\begin{align*}
\bullet \quad \bullet \quad \bullet \\
\quad \bullet \quad \bullet \quad \bullet \\
\quad \bullet \quad \bullet \\
\end{align*}
\]

Notation 2.7. For any Dyck path \( P = (((\rho_m \times_m P_0) \times_{m-1} P_1) \times_{m-2} \ldots) \times_0 P_m = P'_0 \times 0 (((\rho_m \times_m P'_1) \times_{m-1} \ldots \times_{2m-1} P'_{m-1}) \times_1 P'_m \)),

we denote it by \( P = \bigvee d(P_0, \ldots, P_m) = \bigvee u(P'_0, \ldots, P'_m) \).

Note that, the \( P_i \)'s and the \( P'_i \) may be just the point \( \bullet \).

Remark 2.8. Let \( d_m(x) \) be the generating series of \( K[Dyck]^m \), that is,

\[
d_m(x) := \sum_{n \geq 0} d_{m,n} x^n
\]

where \( d_{m,n} \) is the dimension of \( K[Dyck]^m \) and \( d_{m,0} = 1 \). The preceding discussion implies that the series \( d_m(x) \) satisfies the equation \( x \cdot d_m(x)^{m+1} = d_m(x) - 1 \).
Remark 2.9. A Dyck path $P$ is prime if, and only if, $P$ is of the form $P = \bigvee_d (P_0, \ldots, P_m) = \bigvee_u (P'_0, \ldots, P'_m)$, with $P'_0 = 0$.

Definition 2.10. Let $P$ be a $m$-Dyck path of size $n$. The standard coloring of $P$ is a map $\alpha_P$ from the set of down steps $DW(P)$ to the set $\{1, \ldots, n\}$, described recursively as follows:

1. For $P = \rho_n \in Dyck^n$, $\alpha_{\rho_n}$ is the constant function 1.
2. For $P = \bigvee_d (P_0, \ldots, P_m)$, with $P_j \in Dyck^n$, the set of down steps of $P$ is the disjoint union $\bigvee_d DW(P_j)$, where the first subset $\{1, \ldots, m\}$ corresponds to the down steps of $\rho_m$.

The map $\alpha_P$ is defined by:

$$\alpha_P(d) = \begin{cases} 1, & \text{for } d \in \{1, \ldots, m\}, \\ \alpha_{P_j}(d) + n_0 + \cdots + n_{j-1} + 1, & \text{for } d \in DW(P_j), \end{cases}$$

where $0 \leq j \leq m$.

In our last example, we get the following coloring for $P$:

Notation 2.11. For any path $P \in Dyck^n$ and any $1 \leq k \leq n$, we denote by $\omega_k(P)$ the word $\omega_k^P := \alpha_P(d_{k1}) \cdots \alpha_P(d_{kk}(P))$, which is the image under $\alpha_P$ of the sequence of level $k$ down steps of $P$ (from left to right).

Remark 2.12. Let $P$ be an $m$-Dyck path of size $n$.

1. For any down step $d \in DW(P)$, the color of $d$ coincides with the rank of the up step $u \in UP(P)$ which is the first intersection of the horizontal half-line beginning at the middle point of $d$ and going to the left side with the Dyck path $P$. In the example

2. We have that $|\alpha_P^{-1}(i)| = m$, for any $1 \leq i \leq n$.

3. For a fixed $1 \leq k \leq n$, the word $\omega_k^P = \alpha_P(d_{k1}^P) \cdots \alpha_P(d_{k(kP)}^P)$ is decreasing for the usual order of the natural numbers. Moreover, the first $m$ digits of $\omega_n^P$ are $n$'s.
(4) If $Q$ is another $m$-Dyck path, then $\mathcal{D}W(P \times_i Q) = \mathcal{D}W(P) \coprod \mathcal{D}W(Q)$, and $\alpha_{P \times_i Q}$ is described by:

$$
\alpha_{P \times_i Q}(d) = \begin{cases} 
\alpha_P(d), & \text{for any } d \text{ which belongs initially to } P, \\
\alpha_Q(d) + n, & \text{for any } d \text{ which belongs initially to } Q,
\end{cases}
$$

for any $0 \leq i \leq L(P)$.

3. Products on $m$-Dyck paths

**Definition 3.1.** For any positive integer $n$, a weak composition of $n$ with $r + 1$ parts is an ordered collection of non-negative integers $\lambda = (\lambda_0, \ldots, \lambda_r)$ such that $\sum_{i=0}^{r} \lambda_i = n$. We say that the length of $\lambda$ is $r + 1$.

**Notation 3.2.** Given an $m$-Dyck path $P$ of size $n$, the set of all weak compositions of $L(P)$ of length $r + 1$ is denoted $\Lambda_r(P)$.

Let $P \in \text{Dyck}_{n_1}^m$ and $Q = Q_1 \times_0 \cdots \times_0 Q_r \in \text{Dyck}_{n_2}^m$ be two Dyck paths, where $Q_j \in \text{Dyck}_{n_2}^m$ is prime, for $1 \leq j \leq r$.

Suppose that $\lambda = (\lambda_0, \ldots, \lambda_r)$ is a weak composition of $L(P)$. Define a Dyck path $P \star_{\lambda} Q$ of size $n_1 + n_2$ by the formula:

$$P \star_{\lambda} Q := (((P \times_{\lambda_1 + \cdots + \lambda_r} Q_1) \times_{\lambda_2 + \cdots + \lambda_r} Q_2) \times_{\lambda_3 + \cdots + \lambda_r} \cdots) \times_{\lambda_r} Q_r).$$

The product $\star_{\lambda}$ just divides the ordered set $\mathcal{D}W_{n_1}(P)$ of down steps of level $n_1$ of $P$ and glue, in order, the $i^{th}$ piece at the end of the path $Q_i$. If $\lambda_0 > 0$, the first $\lambda_0$ steps of $\mathcal{D}W_{n_1}(P)$ remain at the end of $P$.

**Example 3.3.** Let $P = (2, 3, 1, 6)$ be a path in $\text{Dyck}_2^3$ and let $Q = (1, 4, 4, 3, 2, 3, 4)$ be a 3-Dyck path of size 7, note that $Q = (1, 4, 4) \times_0 (3) \times_0 (2, 3, 4)$.

Consider the weak composition $\lambda_{\frac{P}{Q}} = (1, 2, 2, 1)$ of $L(P) = 6$ of length 4. The word on the top level of $P$ is $\omega_{\frac{P}{Q}} := 444331$. The path $P \times_{(1,2,2,1)} Q$ is:
The last point of Remark 2.12 implies that for any $P \in \text{Dyck}_m^n$, any $Q = Q_1 \times_0 \ldots \times_0 Q_r$ and any $\underline{\lambda} \in \Lambda_r(P)$, the set of down steps of $P * \underline{\lambda} Q$ is:

$$\mathcal{D}\mathcal{W}(P * \underline{\lambda} Q) = \mathcal{D}\mathcal{W}(P) \prod \mathcal{D}\mathcal{W}(Q),$$

and the standard coloring $\alpha_{P * \underline{\lambda} Q}$ is described by:

\[
\alpha_{P * \underline{\lambda} Q}(d) = \begin{cases} 
\alpha_P(d), & \text{for } d \in \mathcal{D}\mathcal{W}(P), \\
\alpha_Q(d) + n_1, & \text{for } d \in \mathcal{D}\mathcal{W}(Q).
\end{cases}
\]

**Notation 3.4.** Let $P$ be a Dyck path with $\mathcal{D}\mathcal{W}_n(P) = (d_1^P, \ldots, d_{L(P)}^P)$, and let $\underline{\lambda} = (\lambda_0, \ldots, \lambda_r)$ be a weak composition of $L(P)$. For $0 \leq i \leq m$, we denote by $\Lambda^i_r(P)$ the set of all weak compositions $\underline{\lambda}$ of length $r + 1$ such that the restriction $\alpha_P(d_{L(P) - \lambda_i + 1}^P), \ldots, \alpha_P(d_{L(P)}^P)$ of the word $\omega^P_n$ to its last $\lambda_r$ letters satisfies the following conditions:

1. any digit in the word $\alpha_P(d_{L(P) - \lambda_i + 1}^P), \ldots, \alpha_P(d_{L(P)}^P)$ appears at most $i$ times,
2. there exists at least one integer $1 \leq i_0 \leq n$ such that $i_0$ appears exactly $i$ times in $\alpha_P(d_{L(P) - \lambda_i + 1}^P), \ldots, \alpha_P(d_{L(P)}^P)$.

For example, for $P = (0, 2, 1, 3, 4) \in \text{Dyck}_3^2$,

we get that $\underline{\lambda}_1 = (1, 1, 2)$ belongs to $\Lambda_2^2(P)$, while $\underline{\lambda}_2 = (0, 3, 1)$ belongs to $\Lambda_2^1(P)$.

Observe that

$$\Lambda^i_r(P) = \{(\lambda_0, \ldots, \lambda_{r-1}, 0) \mid \sum_{i=0}^{r-1} \lambda_i = L(P) \text{ and } r \geq 1\}.$$

The set of all weak compositions of $L(P)$ is the disjoint union $\bigsqcup_{r \geq 0} \left( \prod_{i=0}^{m} \Lambda^i_r(P) \right)$, for any $m$-Dyck path $P$ of size $n$.

The following result is a straightforward consequence of Lemma 2.6 and the definition of $*_{\underline{\lambda}}$.

**Lemma 3.5.** Let $P = P_1 \times_0 \ldots \times_0 P_s$ in $\text{Dyck}_m^n$ and $Q = Q_1 \times_0 \ldots \times_0 Q_r$ in $\text{Dyck}_{n_2}^m$ be two Dyck paths, where $P_1, \ldots, P_s, Q_1, \ldots, Q_r$ are prime, and let $\underline{\lambda} \in \Lambda^i_r(P)$ be a weak composition. We have that:

1. if $i > 0$, then $P * \underline{\lambda} Q = P_1 \times_0 \ldots \times_0 P_{s-1} \times_0 (P_s * \underline{\lambda} Q)$, where $P_s * \underline{\lambda} Q$ is prime.
(2) if $i = 0$, then $\lambda = (\lambda_0, \ldots, \lambda_{r-1}, 0)$ and

$$P \ast \Delta Q = P_1 \times_0 \cdots \times_0 P_{s-1} \times_0 (P_s \ast \Delta (Q_1 \times_0 \cdots \times_0 Q_{j_0})) \times_0 Q_{j_0+1} \times_0 \cdots \times_0 Q_r,$$

where $j_0$ is the maximal element of $\{0, \ldots, r-1\}$ such that $\lambda_{j_0} \neq 0$.

The product on the graded vector space $\mathbb{K}[\text{Dyck}^m]$, spanned by the set of all $m$-Dyck paths, is defined as follows.

**Definition 3.6.** Let $P \in \text{Dyck}_{n_1}^m$ and $Q \in \text{Dyck}_{n_2}^m$ be two Dyck paths, such that $Q = Q_1 \times_0 \cdots \times_0 Q_r$ with $Q_i$ prime, $1 \leq i \leq r$. For any integer $0 \leq j \leq m$, define

$$P \ast_j Q = \sum_{\Delta \in \Lambda_j(P)} P \ast \Delta Q.$$

The product extends in a unique way to a linear map from $\mathbb{K}[\text{Dyck}^m] \otimes \mathbb{K}[\text{Dyck}^m]$ to $\mathbb{K}[\text{Dyck}^m]$.

**Example 3.7.** Let $P = (1, 3)$ be the 2-Dyck path

![2-Dyck path](image)

and let $Q = (0, 2, 4, 2) = (0, 2, 4) \times_0 (2)$ in $\text{Dyck}_4^2$.

![Example 3.7](image)

we get that $P \ast_0 Q = P \ast_{(3,0,0)} Q + P \ast_{(2,1,0)} Q + P \ast_{(1,2,0)} Q + P \ast_{(0,3,0)} Q = (1, 3, 0, 2, 4, 2) + (1, 2, 0, 2, 5, 2) + (1, 1, 0, 2, 6, 2) + (1, 0, 0, 2, 7, 2) =

![Example 3.7](image)

and

$$P \ast_1 Q = P \ast_{(2,0,1)} Q + P \ast_{(1,1,1)} Q + P \ast_{(1,0,2)} Q + P \ast_{(0,2,1)} Q + P \ast_{(0,1,2)} Q = (1, 2, 0, 2, 4, 3) + (1, 1, 0, 2, 5, 3) + (1, 1, 0, 2, 4, 4) + (1, 0, 0, 2, 6, 3) + (1, 0, 0, 2, 5, 4) =

![Example 3.7](image)
Proposition 3.8. Let \( P \in \text{Dyck}^m_{n_1} \) and \( Q = Q_1 \times_0 \ldots \times_0 Q_r \in \text{Dyck}^m_{n_2} \) be two Dyck paths, with \( Q_j \in \text{Dyck}^m_{n_2} \) prime for \( 1 \leq j \leq r \).

1. For nonnegative integers \( s \geq 1 \) and \( 0 \leq i < j \leq m \), the map
\[
\psi_{ij}(P, Q) : \{\lambda_i^j(P) \times \lambda_i^j(Q) \rightarrow \{(\lambda, \delta) | \lambda \in \Lambda_i^j(P) \text{ and } \delta \in \Lambda_i^j(P \ast \Lambda)\},
\]
which sends \( (\lambda, \tau) \mapsto (\lambda, \delta := (\tau_0, \tau_1, \tau_s + \lambda_r)) \) is bijective.

2. For any integer \( 0 \leq i \leq m \), the map \( \psi_i^j(P, Q)(\lambda, \tau) :=
\]
\[
((\lambda_0, \ldots, \lambda_r - 1, \lambda_r + \ldots + \lambda_{s + j_{\tau}}), (\tau_0, \ldots, \tau_{s + \lambda_r}) \ast \lambda_r, \lambda_{r + 1}, \ldots, \lambda_{r + s - j_{\tau}}))
\]
defines a bijection from the set \( \{(\lambda, \tau) \in \Lambda_s^0(Q) \text{ and } \lambda \in \Lambda_i^j(P \ast \Lambda)\} \)
to the set
\[
\{(\gamma, \delta) | \gamma \in \prod_{j=1}^m \Lambda_s^j(P) \text{ and } \delta \in \Lambda_s^j(P \ast \Lambda) \text{ such that } \delta_s \leq \gamma_r\},
\]
where \( j_{\tau} \) is the maximal integer \( 0 \leq j \leq s - 1 \) such that \( \tau_j > 0 \), and \( \prod \) denotes the disjoint union.

3. For any integer \( 0 \leq i \leq m \), the map
\[
\psi_i^j(P, Q)(\lambda, \tau) := (\lambda, \delta := (\tau_0, \ldots, \tau_{s - 1}, \tau_s + \lambda_r),
\]
from \( \Lambda_i^j(P) \times \prod_{j=1}^i \Lambda_s^j(Q) \) to the set
\[
\{(\gamma, \delta) | \gamma \in \Lambda_i^j(P) \text{ and } \delta \in \Lambda_i^j(P \ast \Lambda) \text{ such that } \gamma_r < \delta_s\},
\]
is bijective.

Proof. (1) For the first point, let \( \lambda \in \Lambda_i^j(P) \) and \( \tau \in \Lambda_i^j(Q) \) be two compositions.

If \( \mathcal{D}W_{n_1}(P) = (d_1^P, \ldots, d_{L(P)}^P) \) and \( \mathcal{D}W_{n_2}(Q) = (d_1^Q, \ldots, d_{L(Q)}^Q) \), then:
\[
\mathcal{D}W_{n_1 + n_2}(P \ast \Lambda) = (d_1^Q, \ldots, d_{L(Q)}^Q, d_{L(P) - \lambda_r + 1}^P, \ldots, d_{L(P)}^P).
\]
The map \( \psi_{ij} \) is defined by the formula:
\[
\psi_{ij}(\lambda, \tau) := (\lambda, \delta := (\tau_0, \ldots, \tau_{s - 1}, \tau_s + \lambda_r)).
\]
Clearly, \( \lambda \) belongs to \( \Lambda_i^j(P) \). On the other hand,
\[
\mathcal{D}W_{n_1 + n_2}(P \ast \Lambda) = (d_1^Q, \ldots, d_{L(Q)}^Q, d_{L(P) - \lambda_r + 1}^P, \ldots, d_{L(P)}^P),
\]
which implies that the subset of the last \( \tau_s + \lambda_r \) down steps of \( P \ast \Lambda \) is
\[
(d_{L(Q)}^Q, d_{L(P) - \lambda_r + 1}^P, \ldots, d_{L(P)})\).
\]
Note that:
\[
\alpha_{P \ast \Lambda}d_{L(P) - \lambda_r + 1}^P \ldots \alpha_{P \ast \Lambda}d_{L(P)}^P \text{ is a sequence of elements in the set } \{1, \ldots, n_1\} \text{ such that any digit appears at most } i \text{ times.}
(2) $\alpha_P(\omega_{\lambda}^Q(d_{L(Q)}^Q(\tau_{s+1})) \ldots \alpha_P(\omega_{\lambda}^Q(d_{L(Q)}^Q)))$ is a sequence of elements in the set $\{n_1 + 1, \ldots, n_1 + n_2\}$ where there exists at least one digit that appears $j$ times, and no digit appears more than $j$ times. So, $\delta$ belongs to $\Lambda_s^0(P \ast \underline{\lambda} Q)$.

For any pair of weak compositions $\underline{\lambda} \in \Lambda_s^i(P)$ and $\underline{\delta} \in \Lambda_s^i(P \ast \underline{\lambda} Q)$, we get:

$$\omega_{n_1 + n_2}^{P \ast \underline{\lambda} Q} = \alpha_P(\omega_{\lambda}^Q(d_{L(Q)}^Q(\tau_{s+1})) \ldots \alpha_P(\omega_{\lambda}^Q(d_{L(Q)}^Q))) + n_1, \alpha_P(\omega_{\lambda}^Q(d_{L(Q)}^Q(\tau_{s+1}))) \ldots, \alpha_P(\omega_{\lambda}^Q(d_{L(Q)}^Q))).$$

As the expression $\alpha_P(\omega_{\lambda}^Q(d_{L(Q)}^Q(\tau_{s+1})) \ldots \alpha_P(\omega_{\lambda}^Q(d_{L(Q)}^Q)))$ is a word in the alphabet $\{1, \ldots, n_1\}$ such that no digit appears more than $i$ times, and $i < j$, then $\tau := (0, \ldots, \delta_s - \lambda_r)$ must belong to $\Lambda_s^r(Q)$.

It is immediate to prove that the map $(\underline{\lambda}, \underline{\delta}) \mapsto (\underline{\lambda}, \tau)$ is inverse to $\psi_{ij}(P, Q)$, which ends the proof of (1).

(2) If $\underline{\lambda} \in \Lambda_s^{i-s-j}(P)$ and $\underline{\tau} \in \Lambda_s^0(Q)$, then it is immediate to verify that

(i) $\gamma = (\lambda_0, \ldots, \lambda_{r-1}, \lambda_r + \cdots + \lambda_{r+s-j})$ belongs to $\Lambda_s^i(P)$, for $i \leq j \leq m$,

(ii) $\delta = (\tau_0, \ldots, \tau_{s-j})$ belongs to $\Lambda_s^s(P \ast \underline{\lambda} Q)$,

(iii) $\delta_s = \lambda_r + \cdots + \lambda_{r+s-j} < \gamma_r = \lambda_r + \cdots + \lambda_{r+s-j}$.

Assume that we have two weak compositions $\underline{\gamma} = (\gamma_0, \ldots, \gamma_r) \in \prod_{j=i}^m \Lambda_s^i(P)$ and $\underline{\delta} = (\delta_0, \ldots, \delta_s) \in \Lambda_s^i(P \ast \underline{\lambda} Q)$ such that $\delta_s \leq \gamma_r$.

Let $j_0$ be the maximal integer $0 \leq j_0 \leq s-1$, such that $\delta_{j_0} + \cdots + \delta_s > \gamma_r$.

Define

(a) $\underline{\lambda} := (\gamma_0, \ldots, \gamma_{r-1}, \gamma_r - \delta_{j_0+1} - \cdots - \delta_s, \delta_{j_0+1}, \ldots, \delta_s)$,

(b) $\underline{\tau} := (\delta_0, \ldots, \delta_{j_0+1}, \delta_{j_0+1}, \ldots, \delta_s - \gamma_r, 0, \ldots, 0)$.

It is clear that $\underline{\lambda} \in \Lambda_s^{i-s-j_0}(P)$, $\underline{\tau} \in \Lambda_s^0(Q)$ and $\psi_1(P, Q)(\underline{\lambda}, \underline{\tau}) = (\gamma, \underline{\delta})$, which shows that $\psi_1$ is bijective, ending the proof of (2).

(3) For $\underline{\lambda} \in \Lambda_s^i(P)$ and $\underline{\tau} \in \Lambda_s^i(Q)$, for $1 \leq j \leq i$, we have that the weak composition $\psi_2^j(P, Q)(\underline{\lambda}, \underline{\tau}) = (\gamma, \underline{\delta})$ satisfies the following conditions:

(i) $\gamma = \underline{\lambda}$ belongs to $\Lambda_s^i(P)$,

(ii) the weak composition $\underline{\tau}$ belongs to $\Lambda_s^i(Q)$ for some $1 \leq j \leq i$. So, the sequence $\alpha_P(\omega_{\lambda}^Q(d_{L(Q)}^Q(\tau_{s+1})) \ldots \alpha_P(\omega_{\lambda}^Q(d_{L(Q)}^Q))) + n_1$ is a word in the digits of $\{n_1 + 1, \ldots, n_1 + n_2\}$ such that each sequence appears at most $j$ times.

On the other hand, the sequence $\alpha_P(\omega_{\lambda}^Q(d_{L(Q)}^Q(\tau_{s+1})) \ldots \alpha_P(\omega_{\lambda}^Q(d_{L(Q)}^Q)))$ is a word in $\{1, \ldots, n_1\}$ such that some digit appears exactly $i$ times in it and no digit appears more than $i$ times.
The sequence of level $n_1 + n_2$ of $P *_\omega Q$ is

$$P *_\omega Q = \omega^{n_1 + n_2} = \alpha_Q(d^Q_{L(Q) - \tau_0} + n_1 \alpha_P(d^P_{L(P) - \gamma_0}) + \alpha_P(d^P_{L(P) - \gamma_0}) \cdots \alpha_P(d^P_{L(P)}),$$

which shows that $\hat{\omega} = (\tau_0, \ldots, \tau_{s-1}, \tau_s + \lambda_r)$ belongs to $\Lambda^s_i(P *_\omega Q)$. (iii) As $\gamma = \Delta$ and $\hat{\omega} = (\tau_0, \ldots, \tau_{s-1}, \tau_s + \lambda_r)$, with $\tau_s > 0$, we get that $\gamma_r < \delta_s$.

The map $(\gamma, \hat{\omega}) \mapsto (\gamma, (\delta_0, \ldots, \delta_{s-1}, \delta_s - \gamma_r))$ is the inverse map of $\psi^2_i(P, Q)$.

Proof. Clearly, it suffices to prove the relations for any Dyck paths $P, Q$ and $Z$. Suppose that $P \in \text{Dyck}_{n_1}$, $Q = Q_1 \times_0 \cdots Q_r \in \text{Dyck}_{n_2}$ and $Z = Z_1 \times_0 \cdots \times_0 Z_s \in \text{Dyck}_{n_3}$, where $Q_1, \ldots, Q_r, Z_1, \ldots, Z_s$ are prime Dyck paths.

(1) For $0 \leq i < j \leq m$, applying a recursive argument on $s$ and Lemma 3.5 it is easy to see that, for any pair $(\Delta, \underline{\tau}) \in \Lambda^i(P) \times \Lambda^j(Q)$, we get:

$$P *_\Delta (Q *_\underline{\tau} Z) = ((P \times_{\lambda_1+\cdots+\lambda_r} Q_1) \times_{\lambda_2+\cdots+\lambda_r} \cdots) \times_{\lambda_r} (Q_r *_\underline{\tau} Z) = ((P \times_{\lambda_1+\cdots+\lambda_r} Q_1) \times_{\lambda_2+\cdots+\lambda_r} \cdots) \times_{\lambda_r} (Q_r) *_{\hat{\omega}} Z,$$

where $\hat{\omega} = (\tau_0, \ldots, \tau_{s-1}, \tau_s + \lambda_r)$.

Applying the same notation than in Proposition 3.8, we get that $P *_\Delta (Q *_\underline{\tau} Z) = (P *_{\hat{\omega}} Q) *_{\hat{\omega}} Z$ if, and only if, $\psi_{ij}(P, Q)(\Delta, \underline{\tau}) = (\Delta, \hat{\omega})$. The result follows applying point (1) of Proposition 3.8.

(2) We write $\sum_{j=0}^i P *_i (Q *_j Z) = P *_i (Q *_0 Z) + \sum_{j=1}^i P *_i (Q *_j Z)$ and we work the terms on the right hand side separately.

a) Suppose that $\underline{\tau} \in \Lambda^0_s(Q)$, by Lemma 3.5 we get that:

$$Q *_{\underline{\tau}} Z = Q_1 \times_0 \cdots \times_0 Q_{r-1} \times_0 (Q_r *_{\underline{\tau}'(Z_1 \times_0 \cdots \times_0 Z_j)}) \times_0 Z_{j+1} \times_0 \cdots \times_0 Z_s,$$

where $\underline{\tau}' = (\tau_0, \ldots, \tau_j)$ and $Q_r *_{\underline{\tau}'} (Z_1 \times_0 \cdots \times_0 Z_j)$ is prime.
Applying $P * \underline{\lambda}$, we obtain that:
\[
P (Q * Z) = (P * \underline{\lambda}^1 (Q \cdots \times Q \varrho \cdots Z)) * \underline{\lambda}^2 (Z) = (P * \underline{\lambda}^1 Q * \underline{\lambda}^2 (Z)) = (P * \underline{\lambda}^1 Q) * \underline{\lambda}^2 Z,
\]
for the weak compositions $\underline{\lambda}^1 = (\lambda_0, \ldots, \lambda_{r-1}, \lambda_r + \cdots + \lambda_{r+s-j_2})$, $\underline{\lambda}^2 = (\lambda_r, \ldots, \lambda_{r+s-j_2})$, $\underline{\delta} = (\tau_0, \ldots, \tau_{j_2-1}, \tau_{j_2} + \lambda_r + \cdots + \lambda_{r+s-j_2})$ and $\delta = (\tau_0, \ldots, \tau_{j_2-1}, \tau_{j_2} + \lambda_r, \lambda_{r+1}, \ldots, \lambda_{r+s-j_2})$.

The formula above implies that for any pair $(\underline{\lambda}, \underline{\tau}) \in \Lambda^i_{r+s-j_2}(P) \times \Lambda^0_s(Q)$, the elements $P * \underline{\lambda}^i (Q * Z)$ and $(P * \underline{\lambda}^i Q) * \underline{\delta} Z$ are equal whenever
\[
\psi_i^1 (P, Q) (\underline{\lambda}, \underline{\tau}) = (\gamma, \delta).
\]
So, we have proved that
\[
P * i (Q * Z) = \sum (P * \underline{\lambda}^i Q) * \underline{\delta} Z,
\]
where the sum is taken over all $\gamma \in \prod_{j=1}^m \Lambda^j_i (P)$ and $\delta \in \Lambda^j_i (P * \underline{\lambda}^i Q)$ such that $\delta_s \leq \gamma_r$.

b) Suppose now that $(\underline{\lambda}, \underline{\tau})$ belongs to $\Lambda^i_r (P) \times \prod_{j=1}^i \Lambda^j_s (Q)$. We have that:
\[
Q * Z = Q_1 \cdots Q_{r-1} \times Q_r * Z,
\]
with $Q_1, \ldots, Q_{r-1}, Q_r * Z$ prime. Let us compute
\[
P * \underline{\lambda} (Q * Z) = (P * \underline{\lambda}^1 (Q_1 \cdots Q_{r-1})) * \lambda_r (Q_r * Z) = (P * \underline{\lambda}^i Q) * \underline{\delta} Z,
\]
where $\underline{\lambda}^1 = (\lambda_0, \ldots, \lambda_{r-2}, \lambda_{r-1} + \lambda_r)$ and $\delta = (\tau_0, \ldots, \tau_{s-1}, \tau_s + \lambda_r)$.

Using the notation of Proposition 3.8, we have proved that
\[
P * \underline{\lambda} (Q * Z) = (P * \underline{\lambda}^i Q) * \underline{\delta} Z,
\]
whenever $\psi^2_i (P, Q) (\underline{\lambda}, \underline{\tau}) = (\gamma, \delta)$. So, we get:
\[
\sum_{j=1}^i P * i (Q * Z) = \sum (P * \underline{\lambda}^i Q) * \underline{\delta} Z,
\]
where the sum is taken over all $(\gamma, \delta) \in \Lambda^i_r (P) \times \Lambda^i_s (P * \underline{\lambda}^i Q)$ such that $\delta_s > \gamma_r$. 
Adding up a) and b), we get that:

\[ \sum_{j=0}^{i} P \ast_{j} (Q \ast_{j} Z) = \sum_{j=i}^{m} (P \ast_{j} Q) \ast_{j} Z, \]

which ends the proof. \( \square \)

4. Connection with the m-Tamari lattice

For \( n \geq 1 \), let \( Y_{n} \) denotes the set of planar rooted binary trees with \( n + 1 \) leaves.

**Notation 4.1.** Define binary operations \( \lor, / \) and \( \setminus \) on the set of trees as follows:

1. \( \lor \) is the map which sends an ordered pair of trees \( (t, w) \) to the tree obtained by joining the roots of \( t \) and \( w \) to a new root.
2. The element \( t/w \) is the tree obtained by joining the root of \( t \) to the first leaf of \( w \).
3. The element \( t \setminus w \) is the tree obtained by joining the root of \( w \) to the last leaf of \( t \).

for any \( t \) and \( w \) in \( Y_{\infty} := \bigcup_{n \geq 0} Y_{n} \).

The diagrams below show a more graphical description of the previous definitions,

\[ t \lor w = \begin{array}{c} w \\
\end{array} \quad t/w = \begin{array}{c} t \\
\end{array} \quad t \setminus w = \begin{array}{c} t \\
\end{array} \]

Note that, adding \( Y_{0} := \{\} \), for any \( t \in Y_{n} \) there exist unique trees \( t' \in Y_{n_{1}} \) and \( t'' \in Y_{n_{2}} \) such that \( t = t' \lor t'' \).

**Definition 4.2.** The Tamari order (see [8]) on \( Y_{n} \), \( n \geq 1 \), is the partial order transitively spanned by the following relations:

1. \( (t \lor w) \lor z < t \lor (w \lor z) \),
2. if \( t < w \), then \( t \lor z < w \lor z \),
3. if \( w < z \), then \( t \lor w < t \lor z \),

for \( t, w, z \in Y_{\infty} \).

It is well-known that the set \( Dyck_{n}^{1} \) of paths of size \( n \) has the same cardinal that the set of planar binary rooted trees \( Y_{n} \).

Consider the map \( \Gamma_{n} : Dyck_{n}^{1} \rightarrow Y_{n}, \ n \geq 0 \), defined by:

1. \( \Gamma_{0}(\bullet) := |, \) is the unique element of \( Y_{0} \),
2. \( \Gamma_{n}(P \times_{0} Q) := \Gamma_{n_{1}}(P)/\Gamma_{n_{2}}(Q), \)
3. \( \Gamma_{n+1}(P_{1} \times_{1} P) = | \lor \Gamma_{n_{1}}(P), \)

for any pair of Dyck paths \( P \in Dyck_{n_{1}}^{1} \) and \( Q \in Dyck_{n_{2}}^{1} \). The inverse application is defined recursively on \( n \) by:
(1) $\Gamma_0^{-1}(\emptyset) = \emptyset$.
(2) $\Gamma_n^{-1}(t^l \vee t^r) = \bigvee_u (\Gamma_{n_1}^{-1}(t^l), \Gamma_{n_2}^{-1}(t^r))$, for any $t^l \in \mathcal{Y}_{n_1}$ and $t^r \in \mathcal{Y}_{n_2}$.

So, the Tamari order is defined on $Dyck_n$, via the bijective map $\Gamma_n$, for $n \geq 1$.

F. Bergeron extended the Tamari order to the sets $Dyck_n$ of Dyck paths (see [3]) . Let us describe briefly the $m$-Tamari lattice $Dyck_n^m$.

Let $P$ be an $m$-Dyck path. For any down step $d_0 \in DW(P)$ which is followed by an up step $u \in UP(P)$, consider the excursion $P_u$ of $u$ in $P$ and its matching down step $w_u$ as described in Definition 1.4. Let $P_{(d_0)}$ be the Dyck path obtained by removing $d_0$ and gluing the initial vertex of $u$ to the end of the step preceding $d_0$, and attaching $d_0$ at the final point of $w_u$. For example

It is immediate to see that $\alpha_{P_{d_0}}(d) = \alpha_P(d)$, for any $d \in DW(P)$.

**Definition 4.3.** The $m$-Tamari order on $Dyck_n^m$ is the transitive relation spanned by the covering relation:

$$P \preceq P_{(d)},$$

for any $d \in DW(P)$ such that the final vertex of $d$ is the initial point of an up step $u \in UP(P)$. We use the symbol $\preceq$ for a covering relation.

The Hasse diagrams for $m = 2$ and $n = 1, 2$ are:

For $m = 1$, it is easy to see that the order defined on $Dyck_n^1$ in [3] is the order induced by the Tamari order on $\mathcal{Y}_n$ via the map $\Gamma_n^{-1}$. That is, $\Gamma_n$ is an isomorphism of partially ordered sets, for $n \geq 1$. 
The goal of the present section is to show that the binary operations $\ast_i : \mathbb{K}[\text{Dyck}_n^m] \otimes \mathbb{K}[\text{Dyck}_r^m] \to \mathbb{K}[\text{Dyck}_{n+r}^m]$ are described in terms of the $m$-Tamari order. Let us begin by describing the situation in the case $m = 1$.

**Definition 4.4.** (see [16]) A dendriform algebra over $\mathbb{K}$ is a vector space $A$ equipped with binary operations $\succ$ and $\prec$ satisfying the following conditions

1. $x \succ (y \succ z) = (x \succ y + x \prec y) \succ z$,
2. $x \succ (y \prec z) = (x \succ y) \prec z$,
3. $x \prec (y \succ z + y \prec z) = (x \prec y) \prec z$,

for $x, y, z \in A$.

In [17], J.-L. Loday and the third author showed that the vector space $\mathbb{K}[\mathcal{Y}_\infty]$, spanned by $\bigcup_{n \geq 1} \mathcal{Y}_n$, may be endowed with a natural dendriform structure, in such a way that $\mathbb{K}[\mathcal{Y}_\infty]$ is the free dendriform algebra on one generator.

The dendriform structure on $\mathbb{K}[\mathcal{Y}_\infty]$ is described in terms of the Tamari order and the binary operations $/$ and \ (see [18]) as follows:

1. $t \succ w = \sum_{t/w \leq z \leq (t/w)^{\vee} \wedge w} z$,
2. $t \prec w = \sum_{t^{\vee} \wedge (w^{\vee}) \leq z \leq t/w} z$.

It is not difficult to see that, for $m = 1$, we have:

1. $\Gamma_n(x) \succ \Gamma_r(y) = \Gamma_n(x) \ast_0 \Gamma_r(y)$,
2. $\Gamma_n(x) \prec \Gamma_r(y) = \Gamma_n(x) \ast_1 \Gamma_r(y)$,

for any pair of elements $x \in \text{Dyck}_n^1$ and $y \in \text{Dyck}_r^1$.

**Remark 4.5.** (1) Let $Q$ be a prime Dyck path, for any pair of $m$-Dyck path $P$, we get:

\[ P \times_0 Q < P \times_1 Q < \cdots < P \times_{L(P)} Q, \]

in the $m$-Tamari lattice.

(2) If $P < P'$ in $\text{Dyck}_{n_1}^m$ are such that $L(P) = L(P')$, and $Q < Q'$ in $\text{Dyck}_{n_2}^m$, then

(a) $P \times_k Q < P \times_k Q'$, for any $0 \leq k \leq L(P)$,
(b) $P \times_k Q < P' \times_k Q$, for any $0 \leq k \leq L(P)$.

For the rest of the section, the $m$-Dyck path $Q$ is supposed to be a product $Q = Q_0 \times_0 \cdots \times_0 Q_r$, where all the $Q_j$’s are prime Dyck paths.

**Lemma 4.6.** Let $P \in \text{Dyck}_{n_1}^m$ and $Q \in \text{Dyck}_{n_2}^m$ be two Dyck paths. Two weak compositions $\lambda$ and $\gamma$ in $\Lambda_r(P)$ satisfy that

\[ \lambda_j + \cdots + \lambda_r \leq \gamma_j + \cdots + \gamma_r, \]

for $1 \leq j \leq r$, if, and only if, $P \ast_\lambda Q \leq P \ast_\gamma Q$. 

Proof. If \( Q \) is prime, the result follows from point (1) of Remark 4.5. Suppose that \( Q = Q_1 \times_0 \cdots \times_0 Q_r \), for \( r > 1 \). A recursive argument shows that, for any pair of elements \( \lambda' \) and \( \gamma' \) in \( \Lambda_{r-1}(P) \), we have that

\[
P \ast \lambda' (Q_1 \times_0 \cdots \times_0 Q_{r-1}) \leq P \ast \gamma' (Q_1 \times_0 \cdots \times_0 Q_{r-1}),
\]

whenever \( \lambda'_1 + \cdots + \lambda'_{r-1} \leq \gamma'_1 + \cdots + \gamma'_{r-1} \), for \( 1 \leq j \leq r - 1 \).

We have

1. \( P \ast \lambda Q = (P \ast \lambda' (Q_1 \times_0 \cdots \times_0 Q_{r-1})) \times_{\lambda r} Q_r \),
2. \( P \ast \gamma Q = (P \ast \gamma' (Q_1 \times_0 \cdots \times_0 Q_{r-1})) \times_{\gamma r} Q_r \),

where \( \lambda' = (\lambda_0, \ldots, \lambda_{r-1}, \lambda_r) \) and \( \gamma' = (\gamma_0, \ldots, \gamma_{r-1}, \gamma_r) \). By the recursive hypothesis, we get that

\[
P \ast \lambda' (Q_1 \times_0 \cdots \times_0 Q_{r-1}) \leq P \ast \gamma' (Q_1 \times_0 \cdots \times_0 Q_{r-1}),
\]

and using that \( \lambda_r \leq \gamma_r \) we finally obtain \( P \ast \lambda Q \leq P \ast \gamma Q \).

Conversely, suppose that \( P \ast \lambda Q \leq P \ast \gamma Q \). Point (3) of Remark 4.5 implies that

\[\lambda_j + \cdots + \lambda_r \leq \gamma_j + \cdots + \gamma_r,\]

for \( 1 \leq j \leq r \), which ends the proof. \( \square \)

Notation 4.7. For any \( m \)-Dyck path \( P \) of size \( n \) and any \( 0 \leq i \leq m \), let

1. \( c_i(P) \) be the minimal number of elements such that the word

\[
\alpha P (d_{L(P)-c_i(P)+1}) \cdots \alpha P (d_{L(P)})
\]

contains \( i \) times an integer in \( \{1, \ldots, n\} \) and no integer more than \( i \) times,

2. \( c_i(P) \) be the maximal integer such that the word

\[
\alpha P (d_{L(P)-c_i(P)+1}) \cdots \alpha P (d_{L(P)})
\]

contains at least one integer repeated \( i \) times and no integer repeated \( i + 1 \) times.

Let \( P \in Dyck_{n_1}^m \) and \( Q \in Dyck_{n_2}^m \) be two Dyck paths. For any integer \( 0 \leq i \leq m \), let \( P/iQ \) and \( P\backslash iQ \) be the Dyck paths defined as follows:

1. \( P/iQ := P \times_{c_i(P)} Q \),
2. \( P\backslash iQ := (P \times_{L(P)} (Q_1 \times_0 \cdots \times_0 Q_{r-1})) \times_{c_i(P)} Q_r \).

Proposition 4.8. For any pair of Dyck paths \( P \in Dyck_{n_1}^m \) and \( Q \in Dyck_{n_2}^m \) and any integer \( 0 \leq i \leq m \), the product \( \ast_i \) is given in terms of the \( m \)-Tamari order by the following formula:

\[
P \ast_i Q = \sum_{P/iQ \leq Z \leq P\backslash iQ} Z.
\]
Lemma 4.6, it is easily seen that any down steps of \( R \) paths \( Z \) such that
\[ \lambda \in \Lambda^i_r(P). \]

The weak composition \( \lambda = (\lambda_0, \ldots, \lambda_r) \) satisfies that \( c_i(P) \leq \lambda_r \leq C_i(P) \)
and \( \sum_{j=0}^r \lambda_i = L(P). \)

As
\[ \begin{align*}
& \bullet \ P_i/\iota Q = P *_{(L(P) - c_i(P), 0, \ldots, 0, c_i(P))} Q, \text{ and} \\
& \bullet \ P_{\iota/\iota} Q = P *_{(0, \ldots, 0, L(P) - C_i(P), C_i(P))} Q,
\end{align*} \]
applying Lemma 4.6, it is easily seen that \( P / \iota Q \leq P * \lambda \leq P \setminus \iota Q. \)

Recall that, whenever \( R < S \) in the Tamari lattice, the set \( DW(R) \) of down steps of \( R \) is identified with the set \( DW(S) \). For any \( d \in DW(P) \) the levels of \( d \) in \( R \) and in \( S \) are different but \( \alpha_R(d) = \alpha_S(d). \)

Note that the unique down steps which have different levels in the Dyck paths \( P / \iota Q \) and \( P \setminus \iota Q \) are colored by the set of integers \( \{1, \ldots, n_1\} \). So, for any \( P / \iota Q \leq Z \leq P \setminus \iota Q \) and any \( 1 \leq l \leq r \), we get that
\[ (\ast) \ L_j(Z) = L_j(Q_l), \text{ for } n_1 + n_{21} + \cdots + n_{2(l-1)} < j < n_1 + n_{21} + \cdots + n_{2l}. \]

Define
\[ \lambda_j = \begin{cases} 
L_{n_1+n_2+\cdots+n_{2j}}(Z) - L(Q_j), & \text{for } 1 \leq j \leq r, \\
L_{n_1}(Z) - L(P), & \text{for } j = 0.
\end{cases} \]

The arguments above show that
\begin{align*}
(1) & \quad c_i \leq \lambda_r \leq C_i, \\
(2) & \quad c_i \leq \lambda_j + \cdots + \lambda_r \leq L(P), \text{ for } 1 \leq j \leq r - 1, \\
(3) & \quad 0 \leq L_{n_1}(Z) \leq L(P) - c_i.
\end{align*}

From \( \ast \), we get that \( Z = P * \lambda Q. \)

Lemma 4.6 and \( P / \iota Q \leq P * \lambda Q \leq P \setminus \iota Q \) imply that \( \lambda \in \Lambda^i_r(P). \)

Let us define the product \( * \) on \( \mathbb{K}[\text{Dyck}^m] \) as the sum \( * := \sum_{i=0}^m *_i. \) It is not difficult to see, using Proposition 4.8, that
\[ P * Q = \sum_{P / \iota Q \leq Z \leq P \setminus \iota Q} Z. \]

**Example 4.9.** Consider the Dyck paths \( P = (1, 3) \) and \( Q = (2, 2) \) in \( \text{Dyck}^2 \), the following diagram describes the Tamari interval \( I_{P * Q} \) of all \( Z \in \text{Dyck}^2 \)
such that \( P * Q = \sum_{Z \in I_{P * Q}} Z. \)

The Dyck paths in red are the terms of \( P *_0 Q \), the ones in green are the terms of \( P *_1 Q \), and the ones in blue are the terms appearing in \( P *_2 Q \).
5. Dyck\textsuperscript{m} algebras

We apply Theorem 3.9 to introduce the notion of Dyck\textsuperscript{m} algebra, for \( m \geq 1 \). When \( m = 1 \), we recover J.-L. Loday's dendriform algebras.

The present section contains two main results:

1. We prove that the vector space generated by all \( m \)-Dyck paths, with the products \( *_i \), \( 0 \leq i \leq m \), is the free Dyck\textsuperscript{m} algebra on one generator.

2. We define, for \( m \geq 2 \) and any composition \( r \) of \( m + 1 \) in \( l + 1 \) parts, a functor \( F_r \) from the category of Dyck\textsuperscript{m} algebras into the category of Dyck\textsuperscript{l} algebras, which sends free objects into free objects.

**Definition 5.1.** A Dyck\textsuperscript{m} algebra over \( \mathbb{K} \) is a vector space \( D \) equipped with \( m + 1 \) binary operations \( *_i : D \otimes D \rightarrow D \), for \( 0 \leq i \leq m \), satisfying the following relations:

1. \( x *_i (y *_j z) = (x *_i y) *_j z \), for \( 0 \leq i < j \leq m \),

2. \( \sum_{j=0}^{i} x *_i (y *_j z) = \sum_{k=i}^{m} (x *_k y) *_i z \),

for any elements \( x, y, z \) in \( D \).

**Remark 5.2.** Let \( D \) be a Dyck\textsuperscript{m} algebra. The relations of Definition 5.1 imply that,
(1) the underlying vector space \( D \), with the product \( \ast := \sum_{i=0}^{m} \ast_i \), is an associative algebra.

(2) for \( 1 \leq l \leq m \) and any composition \( \underline{r} = (r_0, \ldots, r_l) \) of \( m+1 \) of length \( l+1 \), the vector space \( D \) equipped with the binary operations

\[
x \ast_i y := \sum_{j=r_0+\cdots+r_{i-1}+1}^{r_0+\cdots+r_l} x \ast_j y,
\]

for \( 0 \leq i \leq l \), where \( r_{-1} = -1 \), is a Dyck\(^l\) algebra. So, they define a functor \( F_{\underline{r}} \) from the category of Dyck\(^m\) algebras into the category of Dyck\(^l\) algebras.

Note that, as particular cases of Remark 5.2 we get that for any Dyck\(^m\) algebra \( D \) and any \( 0 \leq k \leq m - 1 \), the vector space \( D \) equipped with the binary operations \( \succ^k := \sum_{i=0}^{k} \ast_i \) and \( \prec^k := \sum_{i=k+1}^{m} \ast_i \), is a dendriform algebra.

The following result is immediate to verify.

**Lemma 5.3.** For integers \( 0 \leq k < h < m \), let \( \underline{r} = (r_0, \ldots, r_h) \) be a composition of \( m+1 \) and \( \underline{s} = (s_0, \ldots, s_k) \) be a composition of \( h+1 \). Let \( \underline{s} \circ \underline{r} \) be the composition \( (r_0 + \cdots + r_{s_0}, r_{s_0+1} + \cdots + r_{s_0+s_1}, \ldots, r_{s_0+\cdots+s_{k-1}+1} + \cdots + r_h) \), we have that

\[
F_{\underline{s}} \circ F_{\underline{r}} = F_{\underline{s} \circ \underline{r}}.
\]

**Notation 5.4.** Theorem 3.9 asserts that the graded vector space \( \mathbb{K}[\text{Dyck}^m] \) spanned by the set of all \( m \)-Dyck paths, equipped with the operations \( \ast_i \) defined in Section 3, is a Dyck\(^m\) algebra, for all \( m \geq 1 \). From now on we denote this Dyck\(^m\) algebra by \( \mathcal{D}_m \).

As the relations of Definition 5.1 keep the order of the variables, the algebraic operad (see [19]) of Dyck\(^m\) algebras is regular, which means that the operad is described completely by the free object on one generator.

We now turn to prove that \( \mathcal{D}_m \) is in fact the free Dyck\(^m\) algebra on one generator. Before doing it, let us describe a simple way to describe the free Dyck\(^m\) algebra.

**Remark 5.5.** Let \( A \) be a vector space, equipped with a family \( \ast_0, \ldots, \ast_m \) of binary operations. Definition 5.1 states that \( (A, \ast_0, \ldots, \ast_m) \) is a Dyck\(^m\) algebra if, and only if, the operations \( \ast_i, 0 \leq i \leq m \), satisfy the following relations:

1. \( (x \ast_i y) \ast_j z = x \ast_i (y \ast_j z) \), for \( 0 \leq i < j \leq m \),
(2) \((x \ast_i y) \ast_i z = \sum_{j=0}^{i} x \ast_i (y \ast_j z) - \sum_{j=i+1}^{m} (x \ast_j y) \ast_i z\), for \(0 \leq i \leq m\),
for \(x, y, z\) in \(A\).

For \(n \geq 1\), let \(\mathcal{Y}_n^m\) be the set of all planar binary rooted trees with \(n + 1\) leaves (and \(n\) internal vertices), with the vertices colored by the elements of \(\{\ast_0, \ldots, \ast_m\}\). Given two colored trees, \(t\) and \(w\) and an integer \(0 \leq i \leq m\), we denote by \(t \lor_{i} w\) the colored tree obtained by connecting the roots of \(t\) and \(w\) to a new root colored by \(\ast_i\).

For any internal vertex \(v\) of a colored planar binary rooted tree \(t \in \mathcal{Y}_n^m\), we denote by \(t_v\) the colored subtree of \(t\) whose root is \(v\).

**Definition 5.6.** For \(n \geq 2\), define the set \(\mathcal{B}_n^m\) as the subset of all the elements \(t\) in \(\mathcal{Y}_{n-1}^m\) such that any subtree \(t_v\) satisfies the condition:

\(\text{(C)}\) if \(t_v = t_v^l \lor_{i} t_v^r\), then the color of the root of \(t_v^l\) is \(\ast_j\) for some \(j > i\).

For instance, the tree \(t = \begin{tikzpicture}[baseline=(current bounding box.center)]
  \begin{scope}[scale=0.4]
    \node (root) at (0,0) {$\ast_0$};
    \node (left) at (-2,-1) {$\ast_1$};
    \node (right) at (2,-1) {$\ast_2$};
    \node (left_left) at (-4,-2) {$\ast_3$};
    \node (left_right) at (-1,-2) {$\ast_4$};
    \node (right_left) at (1,-2) {$\ast_4$};
    \node (right_right) at (4,-2) {$\ast_3$};
    \draw (root) -- (left);
    \draw (root) -- (right);
    \draw (left) -- (left_left);
    \draw (left) -- (left_right);
    \draw (right) -- (right_left);
    \draw (right) -- (right_right);
  \end{scope}
\end{tikzpicture}\) does not belong to \(\mathcal{B}_6^3\), because in the subtree \(t_v = \begin{tikzpicture}[baseline=(current bounding box.center)]
  \begin{scope}[scale=0.4]
    \node (root) at (0,0) {$\ast_1$};
    \node (left) at (-1,-1) {$\ast_2$};
    \node (right) at (1,-1) {$\ast_2$};
    \draw (root) -- (left);
    \draw (root) -- (right);
  \end{scope}
\end{tikzpicture}\), the root of \(t_v^l\) is colored \(\ast_0\), while the root of \(t_v\) is colored with \(\ast_2\).

For \(n = 1\), \(\mathcal{B}_1^m\) is the set which has as unique element the tree with one leave and no vertex: \(|\cdot|\). Let \(\mathcal{B}^m = \bigcup_{n \geq 1} \mathcal{B}_n^m\).

Note that for any \(t = t^l \lor_{i} t^r \in \mathcal{B}^m\) the trees \(t^l\) and \(t^r\) belong to \(\mathcal{B}^m\).

For any set \(X\), let \(\mathcal{B}_n^m(X)\) denote the set of all trees in \(\mathcal{B}_n^m\) with leaves colored by the elements of \(X\). Let \(\text{Dyck}^m(X)\) be the graded vector space whose basis is the set \(\bigcup_{n \geq 1} \mathcal{B}_n^m(X)\).

For any pair of trees \(t \in \mathcal{B}_n^m(X)\) and \(w \in \mathcal{B}_n^m(X)\), with \(n, r \geq 1\), and any integer \(0 \leq i \leq m\), the product \(t \ast_i w \in \text{Dyck}^m(X)\) is defined recursively on \(n + r\) as follows,

1. for \(n = r = 1\), we have \(t \ast_i w := \begin{tikzpicture}[baseline=(current bounding box.center)]
  \begin{scope}[scale=0.4]
    \node (root) at (0,0) {$\ast_i$};
    \node (left) at (-1,-1) {$\ast_j$};
    \node (right) at (1,-1) {$\ast_j$};
    \draw (root) -- (left);
    \draw (root) -- (right);
  \end{scope}
\end{tikzpicture}\)
2. for \(t = t^l \lor_{i} t^r\) or \(n = 1\), with \(i < j \leq m\), the product \(\ast_i\) of \(t\) and \(w\) is \(t \ast_i w := t \lor_{i} w \in \mathcal{B}^m_{n+r}(X)\),
3. for \(t = t^l \lor_{i} t^r\), with \(0 \leq j \leq i\), we have that
   a. when \(j < i\), the recursive hypothesis states that \(t^r \lor_{i} w\) is defined, and we put \(t \ast_i w := t^l \lor_{i} (t^r \lor_{i} w)\),
   b. when \(j = i\), by Remark 5.5, we get
   \[t \ast_i w := (t^l \ast_i t^r) \ast_i w = \sum_{k=0}^{i} t^l \ast_k (t^r \ast_i w) - \sum_{k=i+1}^{m} (t^l \ast_k t^r) \ast_i w\]
For the second sum, for any $i < k \leq m$, by a recursive argument we suppose that $t^l \ast_k t^r = \sum_{\alpha} t^l_{\alpha}$. Moreover, Remark 5.5 implies that the root of any $t^l_{\alpha}$ is colored by an $*_{\alpha}$ with $i < h$.

So, $(t^l \ast_k t^r) \ast^*_i w := \sum_{\alpha} t^l_{\alpha} \lor_{*_{\alpha}} \ast^*_i w$.

For the first sum, as $t$ belongs to $B_m^n(X)$, we know that the root of $t^l$ is colored by a $*_{\alpha}$ with $h > i$, and therefore $h > k$ for all $0 \leq k \leq i$. On the other hand, the recursive hypothesis implies that $t^r \ast^*_i w = \sum_{\beta} t^r_{\beta} \lor_{*_{\beta}} \ast^*_i w$ is defined. So, $(t^l \ast_k t^r) \ast^*_i w := \sum_{\beta} t^l_\beta \lor_{*_{\beta}} \ast^*_i w$.

Finally, the formula for $t \ast^*_i w = (t^l \ast^*_i t^r) \ast^*_i w$ is

$$t \ast^*_i w := \sum_{k=0}^{i} \left( \sum_{\beta} t^l_\beta \lor_{*_{\beta}} \ast^*_i w \right) - \sum_{k=i+1}^{m} \left( \sum_{\alpha} t^l_{\alpha} \lor_{*_{\alpha}} \ast^*_i w \right),$$

where $t^l \ast_k t^r = \sum_{\alpha} t^l_{\alpha}$, for $i+1 \leq k \leq m$, and $t^r \ast^*_i w = \sum_{\beta} t^r_{\beta}$.

**Example 5.7.** Let

$$t = \ast^*_1 \ast^*_0 \ast^*_1$$

and

$$w = \ast^*_1$$

we get that

$$t \ast^*_2 w = \ast^*_1 \ast^*_0 \ast^*_1 \ast^*_2.$$

The result below follows immediately from Remark 5.5 and the construction above.

**Proposition 5.8.** For any set $X$, the graded vector space generated by the graded set $\bigcup_{n \geq 1} B_n^m(X)$ equipped with the binary products define above is the free Dyck$^m$ algebra on $X$.

**Notation 5.9.** We denote by Dyck$^m(X)$ the free Dyck$^m$ algebra generated by a set $X$.

In order to prove that the Dyck$^m$ algebra $D_m$ is the free Dyck$^m$ algebra on one element, we need the following Proposition.

**Proposition 5.10.** Any element of $P \in D_m$ is a linear combination of elements of the form $R_1 \ast_i R_2$, where $0 \leq i \leq m$ and the sizes of $R_1$ and $R_2$ are strictly smaller than the size of $P$. 
Proof. Let us point out that for $m = 1$, the result has been proved in [17].

For the general case, let

$$P = \bigvee_u (P_0, \ldots, P_m) = P_0 \times_0 ((\rho_m \times_m P_1) \times_{m-1} \ldots \times_1 P_m) \in \text{Dyck}^n_m.$$ 

It is immediate to see that $P' := ((\rho_m \times_m P_1) \times_{m-1} \ldots \times_1 P_m)$ is prime. So, if $P_0 \neq \bullet$, then

$$P = P_0 \times_0 P' = P_0 \ast_0 P',$$

and we are done.

Now suppose that $P_0 = \bullet$. The maximal element $P_{\text{max}(n)} = (0, \ldots, 0, nm)$ of the Tamari lattice $\text{Dyck}^n_m$ satisfies that $P_{\text{max}(n)} = \rho_m \ast_m P_{\text{max}(n-1)}$.

We may assume that the result is also true for elements $Q$ of size $n$ such that $P < Q \leq P_{\text{max}(n)}$ in the $m$-Tamari lattice. For $P = \bigvee_u (\bullet, P_1, \ldots, P_m)$, let $0 \leq i \leq m$ be the largest integer such that $P_i \neq \bullet$.

Let $P' := ((\rho_m \times_m P_1) \times_{m-1} \ldots \times_{m-i+2} P_{i-1}) \in \text{Dyck}^n_m$ be the Dyck path obtained from $P$ by collapsing $P_i$ to a point. We get that the sizes of both $P'$ and $P_i$ are smaller than $P$'s size, and that the last $m_i + 1$ letters of the word $\omega_{n-n_i}(P')$ are equal to 1.

So, we get that $P' \ast_{m-i+1} P_i = P + \sum_k Q_k$, with $P < Q_k$ for all $k$. As we have supposed that all Dyck paths $Q$ such that $P < Q < P_{\text{max}(n)}$ are linear combinations of elements of type $R_1 \ast_i R_2$, where $0 \leq i \leq m$ and the sizes of $R_1$ and $R_2$ are strictly smaller than the size of $Q$, the result also holds for $P$. \hfill \Box

The following theorem states that the graded vector space $\mathcal{D}_m$ also describes the algebraic operad $\text{Dyck}^m$.

**Theorem 5.11.** The free $\text{Dyck}^m$ algebra on one generator is isomorphic to $(\mathcal{D}_m, \ast_0, \ldots, \ast_m)$.

Proof. Let $\text{Dyck}^m(a)$ be the free $\text{Dyck}^m$ algebra on one generator $a$. As $\mathcal{D}_m$ is a $\text{Dyck}^m$ algebra, there exists a unique homomorphism $\phi : \text{Dyck}^m(a) \rightarrow \mathcal{D}_m$ such that $\phi(a)$ is $\rho_m$, the unique $m$-Dyck path of size 1. Proposition 5.10 implies that $\phi$ is surjective.

The subspace of homogeneous elements of degree $n$ of $\mathcal{D}_m$ is generated by the subset $\text{Dyck}^n_m$ of $m$-Dyck paths of size $n$. Let $\text{Dyck}^m(a)_n$ be the subspace of elements of degree $n$ of $\text{Dyck}^m(a)$.

As $\phi$ is surjective, to prove that $\phi$ is an isomorphism it suffices to show that the dimension of the vector space $\text{Dyck}^m(a)_n$ is the number of elements of the set $\text{Dyck}^m_n$, that is

$$\dim_K(\text{Dyck}^m(a)_n) = |\text{Dyck}^m_n| = d_{m,n}.$$
From Proposition 5.8, we know the underlying vector space of $\text{Dyck}^m(a)_n$ is generated by the set $B^m_n(a)$ of planar binary rooted trees with $n$ leaves colored by $a$ and the $(n-1)$ vertices colored by the elements of $\{\ast_0, \ldots, \ast_m\}$ satisfying condition 5.5. (1)

So, the dimension of $\text{Dyck}^m(a)_n$ over $\mathbb{K}$ is the number of elements of the set $B^m_n(a)$, which we denote by $b_{m,n}$, for $n \geq 1$.

The generating series of the set $\{b_{m,n}\}_{n \geq 1}$ is

$$f_m(x) = \sum_{n \geq 1} b_{m,n} x^n.$$  \hfill (1)

We need only to prove that $b_{m,n} = d_{m,n}$, the number of $m$-Dyck paths of size $n$, for $n \geq 1$.

From Remark 2.8, the generating series $d_m(x) = \sum_{n \geq 1} d_{m,n} x^n$ of the family of integers $\{d_{m,n}\}_{n \geq 1}$ satisfies

$$x \cdot (1 + d_m(x))^{m+1} = d_m(x).$$  \hfill (2)

Therefore, to end the proof, it suffices to show that the generating series of $\{b_{m,n}\}_{n \geq 1}$ satisfies the same recursion formula.

Note that $b_{m,1} = d_{m,1} = 1$.

For any colored tree $t \in B^m_n$, there exists a unique integer $r$, a unique collection of colored trees $w^1, \ldots, w^r$ in $B^m$ and a word $\ast_{i_1} \cdots \ast_{i_r}$ in the alphabet $\{\ast_0, \ldots, \ast_m\}$ such that $i_1 > \cdots > i_r$ and

$$t = (((\ast_{i_1} w^1) \vee \ast_{i_2} w^2) \cdots) \vee \ast_{i_r} w^r,$$

which implies that

$$f_m(x) = x \cdot (1 + f_m(x))^{m+1} \quad \square$$  \hfill (3)

**Corollary 5.12.** Let $V$ be a $\mathbb{K}$-vector space. The free $\text{Dyck}^m$ algebra on $V$ is the vector space

$$\text{Dyck}^m(V) := \bigoplus_{n \geq 1} D_{m,n} \otimes V^\otimes n,$$

equipped with the binary products given by:

$$P \otimes (v_1 \otimes \ldots \otimes v_m) \ast_i Q \otimes (w_1 \otimes \ldots \otimes w_n) := (P \ast_i Q) \otimes (v_1 \otimes \ldots \otimes v_i \otimes w_1 \otimes \ldots \otimes w_n),$$

for any integer $0 \leq i \leq m$, any $m$-Dyck paths $P \in \text{Dyck}^m_{n_1}$ and $Q \in \text{Dyck}^m_{n_2}$, and elements $v_1, \ldots, v_{n_1}, w_1, \ldots, w_{n_2} \in V$.

In Remark 5.2, we showed that, for $0 \leq h \leq m$ and any composition $\underline{r} = (r_0, \ldots, r_h)$ of $m$, there exists a functor $\mathbb{F}_{\underline{r}}$ from the category of $\text{Dyck}^h$-algebras into the category of $\text{Dyck}^m$ algebras (which is equivalent to an operad homomorphism from $\text{Dyck}^h$ to $\text{Dyck}^m$).
We want to show that the image under $\mathbb{F}_2$ of a free $Dyck^m$ algebra is free as a $Dyck^k$ algebra, too. From Corollary 5.12, we get that it suffices to prove that the image $\mathbb{F}_2(Dyck^m(a))$ of the free $Dyck^m$ algebra over one element, is free as a $Dyck^k$ algebra.

In order to do that, we need to introduce new basis of the underlying vector space of $Dyck(a)$, by modifying the basis $B^m$ described at Definition 5.6.

**Notation 5.13.** Given a family of colored trees $t_1, \ldots, t_p$ and a family of integers $0 \leq i_1, \ldots, i_p \leq m$, we denote by

1. $\Omega^L_{i_1, \ldots, i_p}(t_1, \ldots, t_p)$ the colored tree
   \[ \Omega^L_{i_1, \ldots, i_p}(t_1, \ldots, t_p) := (((\bigvee_{i_p} t_p) \bigvee_{i_{p-1}} t_{p-1}) \ldots) \bigvee_{i_1} t_1, \]

2. $\Omega^R_{i_1, \ldots, i_p}(t_1, \ldots, t_p)$ the colored tree
   \[ \Omega^R_{i_1, \ldots, i_p}(t_1, \ldots, t_p) := t_1 \bigvee_{i_1} (t_2 \bigvee_{i_2} (\ldots (t_{p-1} \bigvee_{i_{p-1}} (t_p \bigvee_{i_p} \ldots))). \]

That is

\[ \Omega^L_{i_1, \ldots, i_p}(t_1, \ldots, t_p) = \begin{cases} t_p & \text{if } i_p > i_1 \geq \cdots \geq i_2 > i_1, \\ t_1 & \text{if } i_1 > i_2 > \cdots > i_p, \end{cases} \quad \Omega^R_{i_1, \ldots, i_p}(t_1, \ldots, t_p) = \begin{cases} t_1 & \text{if } i_1 > i_2 > \cdots > i_p, \\ t_p & \text{if } i_p > i_1 \geq \cdots \geq i_2 > i_1. \end{cases} \]

Note first that for any tree $t \in \mathcal{Y}^m_{n-1}$ there exist unique non negative integers $p$ and $q$, such that:

\[ t = \Omega^L_{i_1, \ldots, i_p}(t_1, \ldots, t_p) = \Omega^R_{j_1, \ldots, j_q}(w_1, \ldots, w_q), \]

for a unique families of colored trees $t_1, \ldots, t_p$ and $w_1, \ldots, w_q$ and unique collections of integers $i_1, \ldots, i_p$ and $j_1, \ldots, j_q$ in $\{0, \ldots, m\}$ with $i_1 = j_1$. In particular, $t = t^l \bigvee_{k_0} t^r$, for

\[ t^l = \Omega^L_{i_2, \ldots, i_p}(t_2, \ldots, t_p) = w_1, \quad \text{and} \quad t^r = \Omega^R_{j_2, \ldots, j_q}(w_2, \ldots, w_q), \]

and $k_0 = i_1 = j_1$.

**Definition 5.14.** Given $0 \leq k \leq m$, define $B^m_{n,k}$ to be the set of planar binary rooted trees with $n$ leaves, with the vertices colored by the elements of $\{*_0, \ldots, *_m\}$ such that for any vertex $v$, the tree $t_v$ satisfies the following conditions:

1. if $t_v = \Omega^L_{i_1, \ldots, i_p}(t_1, \ldots, t_p)$, with the root colored by $*_i$ for $i_1 \neq k$, then either $i_2 = k$ or $i_2 > i_1$,
2. if $t_v = \Omega^L_{k, \ldots, i_p}(t_1, \ldots, t_p) = \Omega^R_{k, \ldots, j_q}(w_1, \ldots, w_q)$, then $i_s \in \{k+1, \ldots, m\}$ for $s \geq 2$, and $j_h \in \{0, \ldots, k\}$ for $h \geq 2$.

The basis $B^m$ coincides with the set $B^{m,m}$, under this notation.
Proposition 5.15. For any $0 \leq k \leq m$, the set $\mathcal{B}^{m,k} = \bigcup_{n \geq 1} \mathcal{B}^n_{m,k}$ is a basis of the underlying vector space of the free Dyck$^m$ algebra $\text{Dyck}^m(a)$.

Proof. We know that $\mathcal{B}^m$ is a linear basis of the $\mathbb{K}$-vector space $\text{Dyck}^m(a)$. We want to prove that there exists a bijective map $\varphi : \mathcal{B}^m \rightarrow \mathcal{B}^{m,k}$ satisfying that:

(i) $\varphi(t) = t$, for all $t \in \mathcal{B}^m \cap \mathcal{B}^{m,k}$,
(ii) if $t = t^l \vee s_i t^r$, with $i \neq k$, then $\varphi(t) = \varphi(t^l) \vee s_i \varphi(t^r)$, where $t^l, t^r \in \mathcal{B}^m$,
(iii) if $t = t^l \vee s_k t^r$, then the root of $\varphi(t)$, is $s_k$, for some $s \geq k$.
(iv) $t$ and $\varphi(t)$ represent the same element in $\text{Dyck}^m(a)$.

For a colored tree $t \in \mathcal{B}^m \cap \mathcal{B}^{m,k}$, we define $\varphi(t) := t$. Clearly $\mathcal{B}^m_n = \mathcal{B}^{m,k}_n$, for $n = 1, 2$.

If $t \notin \mathcal{B}^{m,k}$, we apply a recursive argument on $|t| > 2$.

(1) For $t = t^l \vee s_i t^r$, with $t^l$ and $t^r$ in $\mathcal{B}^m$ and $s \neq k$, define $\varphi(t) := \varphi(t^l) \vee s_i \varphi(t^r)$.

Note that, as $t \in \mathcal{B}^m$, we know that the root of $t^l$ is colored by $s_h$, for some $h > s$. The recursive hypothesis states that the colored planar rooted trees $\varphi(t^l)$ and $\varphi(t^r)$ belong to $\mathcal{B}^{m,k}$ and the color of the root of $\varphi(t^l)$ is $s_p$ for some $s < h \leq p$. So, $\varphi(t) \in \mathcal{B}^{m,k}$.

(2) If $t = t^l \vee s_k t^r$, with $t^l$ and $t^r$ in $\mathcal{B}^m$, then there exist unique pair of positive integers $p, q$ such that

(i) $t^l = \Omega^L_{j_2, \ldots, j_p}(t_2, \ldots, t_p)$, for a unique family of trees $t_2, \ldots, t_p$ in $\mathcal{B}^m$ and unique nonnegative integers $j_2 < \cdots < j_p$,
(ii) $t^r = \Omega^R_{j_2, \ldots, j_q}(w_2, \ldots, w_q)$, for a unique family of trees $w_2, \ldots, w_q$ in $\mathcal{B}^m$ and unique nonnegative integers $j_2, \ldots, j_q$.

(a) If $j_h \leq k$, for all $2 \leq h \leq q$, then a recursive argument on $q$ shows that

$$\varphi(t^r) = \Omega^R_{j_2, \ldots, j_q}(\varphi(w_2), \ldots, \varphi(w_q)),$$

where $w_h \in \mathcal{B}^m$, for $2 \leq h \leq q$. In this case, we define $\varphi(t) := \varphi(t^l) \vee s_k \varphi(t^r)$.

(b) If there exist at least one $2 \leq h \leq q$ such that $j_h > k$, let $s$ be the minimal integer such that $j_s > k$, $2 \leq s \leq q$. For $2 \leq h \leq s - 1$, we have that $j_h \leq k < j_s$.

Applying that $x \ast_i (y \ast_j z) = (x \ast_i y) \ast_j z$ in $\text{Dyck}^m(a)$, whenever $0 \leq i < j \leq m$, we get that the tree $t$ describes the same element than the tree:

$$(t^l \vee s_k (w_2 \vee_{s_{j_2}} (\ldots (w_{s-1} \vee_{s_{j_{s-1}}} w_s)))) \vee_{s_{j_s}} \Omega^R_{j_{s+1}, \ldots, j_q}(w_{s+1}, \ldots, w_q).$$

That is, we replace the tree...
if for any conditions satisfied by \(\varphi\),

For \(l \in B\), \(\varphi(l) = ((...((w_{2\leq l} \vee w_{s_{1\leq l}} (w_{s_{1\leq l}} \vee ...)))...))\) is colored by \(*_r\) with \(r > j_s\). So, we have to work a bit more to define \(\varphi(t)\).

Suppose that \(w_s = \Omega^L_{h_1,...,h_u}(w_{s,1},...,w_{s,u})\). As \(w_s \in B^m\), we get that \(j_s < h_1 < \cdots < h_u\).

The tree \(t_r \vee t_{j_s} (w_2 \vee ... (w_{s-1} \vee w_s))\) represents the same element than the tree:

\[
\tilde{t} := (((t_r \vee t_{j_s} (w_2 \vee ... (w_{s-1} \vee w_s))) \vee w_{h_u} w_{s,u})...)) \vee w_{s,1} = \Omega^R_{j_s+1,...,j_{s-1}} (w_{s,1}) \vee w_{s,u} w_{s,u}.
\]

Moreover, \(t_r \Omega^L_{i_2,...,i_p} (t_2,...,t_p)\) is such that \(k < i_r < \cdots < i_p\) and the root of \(\varphi(t_r)\) is colored by \(*_{i_2}\), which implies that

\[
\varphi(\tilde{t}) = (((\varphi(t_r)) \vee w_{s_k} \Omega^R_{j_2,...,j_{s-1}} (\varphi(w_2),...,\varphi(w_{s-1}))) \vee w_{s,u} \varphi(w_{s,u})...)) \vee w_{s,1} \varphi(w_{s,1}),
\]

and that the root of \(\varphi(t)\) is colored by \(*_{h_1}\).

Therefore the tree \(t\) describes the same element than \(\tilde{t} := \tilde{t} \vee w_{s,1} \Omega^R_{j_s+1,...,j_{s-1}} (w_{s,1},...,w_{q})\),

where the root of \(\varphi(t)\) is colored by \(*_{h_1}\), with \(h_1 > j_s > k\). We define

\[
\varphi(t) := \varphi(\tilde{t}) \vee w_{s,1} \varphi(\Omega^R_{j_s+1,...,j_{s-1}} (w_{s,1},...,w_{q})).
\]

To prove that \(\varphi\) is bijective, we give an explicit description of \(\varphi^{-1}\). Clearly, if \(t \in B^m \cap B^{m,k}\), then \(\varphi^{-1}(t) = t\).

For \(t \in B^{m,k} \setminus B^m\), we use a recursive argument on the degree \(|t|\). Suppose that for any \(r < n\), the map \(\varphi^{-1} : B^{m,k}_r \to B^m_r\) is defined. From the conditions satisfied by \(\varphi\), we get that its inverse satisfies that:

(iv) if the root of \(t\) is colored by \(*_{r}\), for some \(s \leq k\), then the root of \(\varphi^{-1}(t)\) is colored by \(*_{s}\),
(iiv) if the root of $t$ is colored by $*_s$, for some $s > k$, then the root of $\varphi^{-1}(t)$ is colored by $*_s$ or by $*_k$.

Let $t = t^1 \lor_{s_1} t^r \in B^{m,k}$. If $i_1 \leq k$, then we know that the root of $\varphi^{-1}(t^1)$ is colored by $*_s$, for some $s > i_1$. So, we get that:

$$\varphi^{-1}(t) = \varphi^{-1}(t^1) \lor_{s_1} \varphi^{-1}(t^r).$$

Suppose that $i_1 > k$ and that $t = \Omega_{i_1, \ldots, i_p}^{L}(t^r, t_1, \ldots, t_p) \notin B^m$. If $i_1 < i_2 < \cdots < i_p$, then it is immediate to see that

$$\varphi^{-1}(t) = \varphi^{-1}(t^1) \lor_{s_1} \varphi^{-1}(t^r) = \varphi^{-1}(\Omega_{i_1, \ldots, i_p}^{L}(\varphi^{-1}(t^r), \varphi^{-1}(t_2), \ldots, \varphi^{-1}(t_p))).$$

Otherwise, there exists a unique integer $1 \leq h \leq p$, such that

$$k < i_1 < \cdots < i_{h-1}, \quad \text{and} \quad i_h = k < i_{h+1} < \cdots < i_p.$$

Moreover, as $t \in B^{m,k}$, we have that $t_h = \Omega_{j_1, \ldots, j_q}^{R}(w_1, \ldots, w_q) \in B^{m,k}$, with $j_1 \leq k$.

From the definition of $\varphi$ and a recursive argument, we get that:

1. the tree $t$ represents the same element than the tree

$$\tilde{t} := \Omega_{i_{h+1}, \ldots, i_p}^{L}(t_{h+1}, \ldots, t_p) \lor_{s_k} (w_1 \lor_{s_{j_1}} (\ldots (w_q \lor_{s_{j_q}} \Omega_{i_1, \ldots, i_{h-1}}^{L}(t^r, t_2, \ldots, t_{h-1})))) =$$

2. $$\varphi^{-1}(t) = \varphi^{-1}(\Omega_{i_{h+1}, \ldots, i_p}^{L}(t_{h+1}, \ldots, t_p)) \lor_{s_k}$$

$$\left(\varphi^{-1}(w_1) \lor_{s_{j_1}} (\ldots (\varphi^{-1}(w_q) \lor_{s_{j_q}} \varphi^{-1}(\Omega_{i_1, \ldots, i_{h-1}}^{L}(t^r, t_2, \ldots, t_{h-1}))))\right).$$

We have that $\varphi^{-1}$ is well defined, a tedious but straightforward calculation shows that it is the inverse of $\varphi$. $\square$

**Lemma 5.16.** For any integer $0 \leq k < m$, let $r$ be the composition of $m+1$ in $m$ parts, given by $r_j = 1$ for $j \neq k$, and $r_k = 2$. For any set $X$, the image of $\text{Dyck}^m(X)$ under the functor $F_x$ is generated as $\text{Dyck}^{m-1}$ algebra by the graded set $A^{m,k}(X)$ of all colored trees $t$ in $\bigcup_{n \geq 1} B_{n,k}^m(X)$, such that $n = 1$, or $n > 1$ and the root of $t$ is colored by $*_k$.

**Proof.** Again, from the description of $\text{Dyck}^m(X)$, we have that it suffices to prove the result for the set with one element $X = \{a\}$. 
The $\text{Dyck}^{n-1}$ algebra structure of $\text{Dyck}^m(a)$ is given by the products

$$\overline{\tau}_j = \begin{cases} *_{j}, & \text{for } 0 \leq j < k, \\ *_{k} + *_{k+1}, & \text{for } j = k, \\ *_{j-1}, & \text{for } k < j < m. \end{cases}$$

The underlying vector space of $\mathbb{F}_a(\text{Dyck}^m(a))$ is equal to $\text{Dyck}^m(a)$. As the set $\bigcup_{n \geq 1} B_{n}^{m,k}$ is a basis of $\text{Dyck}^m(a)$ as a $\mathbb{K}$-vector space, it suffices to see that any element in $\bigcup_{n \geq 1} B_{n}^{m,k}$ belongs to the $\text{Dyck}^{n-1}$ algebra generated by the set $\mathcal{A}_{m,k}$, under the operations $\overline{\tau}_0, \ldots, \overline{\tau}_{m-1}$.

We proceed by induction on the degree $n$. For $n = 1, 2$, the result is immediate.

For $t = t^l \vee_\ast t^r \in B_{n}^{m,k}(X)$, the recursive hypothesis states that the trees $t^l$ and $t^r$ are obtained by applying the products $\overline{\tau}_0, \ldots, \overline{\tau}_{m-1}$ to elements of the set $\mathcal{A}_{m,k}$ of degree smaller than $n$.

We have to analyze three different cases:

1. For $i < k$, we have that $t = t^l \overline{\tau}_i t^r$, and as $t^l$ and $t^r$ are elements in the $\text{Dyck}^{n-1}$ algebra generated by $\mathcal{A}_{m,k}$, so is $t$.
2. For $i = k$, as $t \in B_{n}^{m,k}$, we get that $t \in \mathcal{A}_{m,k}$.
3. For $i = k + 1$, we have that $t = t^l \overline{\tau}_{k} t^r - t^l \ast_{k} t^r$ and the root of $t^l$ is colored by $*_{j}$, with $j > k + 1$ or $j = k$.

As $t^l$ and $t^r$ belong to $\text{Dyck}^{n-1}(\mathcal{A}_{m,k})$, the tree $t^l \overline{\tau}_{k} t^r$ is in $\text{Dyck}^{n-1}(\mathcal{A}_{m,k})$.

On the other hand, either $t^l \ast_{k} t^r \in \mathcal{A}_{m,k}$, or

$$t^l \ast_{k} t^r = (t^l \vee_* w^l) \vee_* w^r,$$

for some colored tree $w = w^l \vee_* w^r$ and $h > k$.

Applying a recursive argument to the degrees of the elements $t^l \vee_* w^l$ and $w^r$ the result follows.

4. For $i > k + 1$, we have that $t = t^l \vee_{i-1} t^r$, which belongs to $\text{Dyck}^{n-1}(\mathcal{A}_{m,k})$ by recursive hypothesis.

Lemma 5.16 states that $\mathbb{F}_a(\text{Dyck}^m(X))$ is a quotient of the free $\text{Dyck}^{n-1}$ algebra $\text{Dyck}^{n-1}(\mathcal{A}_{m,k}(X))$. For $X$ finite, the subspace of homogeneous elements of degree $n$ in $\mathbb{F}_a(\text{Dyck}^m(X))$ is $d_{m,n}|X|^n$.

So, to prove that $\mathbb{F}_a(\text{Dyck}^m(X))$ is isomorphic to $\text{Dyck}^{n-1}(\mathcal{A}_{m,k}(X))$, it suffices to show that the dimension of the subspace of homogeneous elements of degree $n$ in $\text{Dyck}^{n-1}(\mathcal{A}_{m,k})$ is $d_{m,n}$, where $\mathcal{A}_{m,k}$ is the set of trees in $B_n^m$ with the vertices colored by $*_{0}, \ldots, *_{m}$ and the root colored by $*_{k}$.

Recall that for any graded vector space $V = \bigoplus_{n \geq 1} V_n$ such that each $V_n$ is finite dimensional, the generating series of $V$ is $v(x) := \sum_{n \geq 1} \dim(V_n)x^n$. 

Lemma 5.17. Let $d_m(x)$ be the generating series of the free Dyck\(^m\) algebra Dyck\(^m\)(a). We have that:

$$d_m(x) = d_k(x \cdot (1 + d_m(x)^{m-k})),$$

for all $0 \leq k \leq m$.

Proof. Clearly, it is enough to prove this for $k = m - 1$. Let $g_m(x)$ be the inverse series of $d_m(x)$ (it exists because $d(0) = 0$).

Since $x \cdot (1 + d_m(x))^{m+1} = d_m(x)$, replacing $x$ by $g_m(x)$ we obtain:

$$g_m(x) = \frac{x}{1 + x^{m+1}},$$

which implies that $(1 + x) \cdot g_m(x) = g_{m-1}(x)$. So, replacing $x$ by $d_m(x)$ and applying $d_{m-1}(x)$ to both sides, we get the desired formula

$$d_{m-1}(x \cdot (1 + d_m(x))) = d_m(x).$$

\[\Box\]

Applying Lemmas 5.16 and 5.17, we get the following result.

Proposition 5.18. For a fixed $0 \leq k \leq m - 1$, let $r$ be the composition of $m + 1$ in $m$ parts, such that $r_i = 1$ for $i \neq k$ and $r_k = 2$. The Dyck\(^m\) algebra $\mathbb{F}_r(\text{Dyck}^m(X))$ is free.

Proof. Applying Lemmas 5.16 and 5.17, it suffices to prove that the number of elements in $A^m_k$ is $d_{m,n-1}$, for $0 \leq k \leq m$.

The number of elements of $B^m_k$ is $d_{m,n-1}$, to end the proof we define a bijective map $\theta_n$ from $B^m_k$ to $A^m_k$, for $n \geq 2$.

For $n = 2$, $\theta_1(\cdot)$ is the unique planar binary rooted tree with two leaves and the root colored by $\ast_k$.

Let $t = t^\ast \lor_s t^\ast$ be an element of $B^m_{n-1}$.

1. For $h > k$, let $t = \Omega^L_{h,i_2, \ldots, i_p}(t^r, t_2, \ldots, t_p)$.
   (a) If $i_1 > \cdots > i_2 > h > k$, then we define $\theta_n(t) := t \lor_s \cdot$.
   (b) If there exists one integer $1 \leq s \leq p$ such that $i_s = k$, then the $s$ is unique and $\theta_n(t)$ is defined by the formula:

   $$\theta_n(t) := \Omega^L_{h,i_2, \ldots, i_{s-1}}(t^r, t_2, \ldots, t_{s-1}) \lor_s \Omega^L_{k,i_{s+1}, \ldots, i_p}(t_s, \ldots, t_p).$$

2. For $h \leq k$, let $t = \Omega^R_{h,j_2, \ldots, j_q}(t^l, t_2, \ldots, t_q)$.
   (a) If $j_1 \leq k$ for any $2 \leq i \leq q$, then we define: $\theta_n(t) := | \lor_s t$.
   (b) Otherwise, let $2 \leq s \leq q$ be the minimal integer such that $j_s > k$. In this case, as $t \in B^m_k$, we know that $k \notin \{h, j_1, \ldots, j_s-1\}$.

   We define $\theta_n(t)$ to be the element:

   $$\theta_n(t) := \Omega^R_{j_2, \ldots, j_q}(w_s, \ldots, w_q) \lor_s \Omega^R_{h,j_2, \ldots, j_s-1}(t^l, t_2, \ldots, t_{s-1}).$$

   It is not difficult to verify that $\theta_n$ is bijective for all $n \geq 2$. So, the result is proved. \[\Box\]
Applying Lemma 5.3, as a straightforward consequence of Proposition 5.18, we get the following result.

**Theorem 5.19.** Let \(0 \leq i \leq m - 1\) be an integer and let \(r\) be a composition of \(m+1\) in \(s+1\) parts. The image of a free Dyck\(^m\) algebra \(\text{Dyck}^m(X)\) under the functor \(\mathbb{E}_\Sigma\) is a free Dyck\(^s\) algebra.

Note that, in particular we get that, for any free Dyck\(^m\) algebra, the associative algebra \((\text{Dyck}^m(X), \ast_0 + \cdots + \ast_m)\) is free.

### 6. A diagonal on \(m\)-Dyck Paths

As \(\text{Dyck}^m\) is a regular operad, given a \(\text{Dyck}^m\) algebra \((A, \{\ast_i\}_{0 \leq i \leq m})\) and an associative algebra \((B, \circ)\), the tensor product \(B \otimes A\) has a natural structure of \(\text{Dyck}^m\) algebra, where the products are given by the formula
\[
\ast_i^{B \otimes A} := \circ \otimes \ast_i,
\]
for \(0 \leq i \leq m\). In particular, when \((B, \circ) = (A, \ast := \sum_{i=0}^m \ast_i)\), the tensor product \(A \otimes A\) is a \(\text{Dyck}^m\) algebra. That is, the algebraic operad \(\text{Dyck}^m\) is a Hopf operad.

However, there does not exist a good notion of unit for \(\text{Dyck}^m\) algebras, when \(m \geq 1\).

In this section, we introduce the notion of \(\text{Dyck}^m\) bialgebra, and give an explicit description of the coproduct on the free algebra \(\mathcal{D}_m\), for \(m \geq 1\). For \(m = 1\) it coincides, via the linear map induced by the applications \(\Gamma_n : \text{Dyck}_m^1 \rightarrow \mathcal{Y}_n\), with the coproduct defined in [17] on the algebra \(\mathbb{K}[\mathcal{Y}_n]\) of planar binary rooted trees.

Given a vector space \(V\), recall that \(V^+\) is the vector space \(V^+ := \mathbb{K} \oplus V\) equipped with the usual augmentation map \(\epsilon : V^+ \rightarrow \mathbb{K}\). Let \(\overline{V^+} \otimes V^+\) denote the vector space \(\overline{V^+} \otimes V^+ := V^+ \otimes V \oplus V \otimes V^+\).

Let \((A, \{\ast_i\}_{0 \leq i \leq m})\) be a \(\text{Dyck}^m\) algebra. The products \(\ast_i\) are extended to linear maps \(\ast_i : A^+ \otimes A^+ \rightarrow A\), for \(0 \leq i \leq m\), by the formulas:

1. \(x \ast_0 1_\mathbb{K} = 0\) and \(1_\mathbb{K} \ast_0 x = x\),
2. \(x \ast_i 1_\mathbb{K} = 1_\mathbb{K} \ast_i x = 0\), for \(0 < i < m\),
3. \(x \ast_m 1_\mathbb{K} = x\) and \(1_\mathbb{K} \ast_m x = 0\),

for \(x \in A\).

Note that the element \(1_\mathbb{K} \ast_i 1_\mathbb{K}\) is not defined, for any \(0 \leq i \leq m\).

It is easily seen that the vector space \(\overline{A^+} \otimes A^+\), equipped with the operations \(\ast_i\) given by:

1. \((x_1 \otimes x_2) \ast_i (y_1 \otimes y_2) = (x_1 \ast y_1) \otimes (x_2 \ast_i y_2)\), for \(x_2 \in A\) or \(y_2 \in A\);
2. \((x_1 \otimes 1_\mathbb{K}) \ast_i (y_1 \otimes 1_\mathbb{K}) = (x_1 \ast_i y_1) \otimes 1_\mathbb{K}\),
for $x_1, x_2, y_1, y_2 \in A^+$, is a $Dyck^m$ algebra.

The previous construction motivates the following definition.

**Definition 6.1.** A $Dyck^m$ bialgebra over $\mathbb{K}$ is a $Dyck^m$ algebra $(A, \{\ast_i\}_{0 \leq i \leq m})$ equipped with a linear map $\Delta : A^+ \to A^+ \otimes A^+$ satisfying that:

1. the data $(A^+, \ast, \Delta, \iota, \epsilon)$ is a bialgebra in the usual sense, where
   a. the associative product $\ast$ is given by:
   
   $$x \ast y = \begin{cases} 
   \sum_{i=0}^{m} x \ast_i y, & \text{for } x, y \in A, \\
   x \cdot y, & \text{for } x \in \mathbb{K} \text{ or } y \in \mathbb{K},
   \end{cases}$$

   where $\cdot$ denotes indistinctly the product on $\mathbb{K}$ as well as the action of $\mathbb{K}$ on $A$, for $x, y \in A^+$.
   b. $\iota : \mathbb{K} \hookrightarrow A^+$ is the canonical inclusion of $\mathbb{K}$ into $A^+$, and $\epsilon : A^+ \to \mathbb{K}$ is the canonical projection.
2. the restriction of $\Delta$ from $A$ to the subspace $A^+ \otimes A^+$ is a homomorphism of $Dyck^m$ algebras.

A standard argument shows that for any free $Dyck^m$ algebra $Dyck^m(X)$, there exists a unique homomorphism $\Delta$ from $Dyck^m(X)^+$ into $Dyck^m(X)^+ \otimes Dyck^m(X)^+$ satisfying that:

1. $\Delta(1_\mathbb{K}) = 1_\mathbb{K} \otimes 1_\mathbb{K}$,
2. $\Delta(x) = x \otimes 1_\mathbb{K} + 1_\mathbb{K} \otimes x$, for $x \in X$,

giving $Dyck^m(X)$ a structure of $Dyck^m$ bialgebra.

Our aim is to give an explicit description, in terms of $m$-Dyck paths, of the coproduct $\Delta$ on the free $Dyck^m$ algebra $D_m$.

**Definition 6.2.** Let $P$ be a $m$-Dyck path. A **central step of $P$** is an up step of $P$ which is the initial step of $P$, or is preceded by another up step.

**Example 6.3.** Consider the following 2-Dyck path:

![2-Dyck path](image)

The central steps are marked in green.

**Notation 6.4.** Let $P$ be an $m$-Dyck path of size $n$. Given a pair of steps $(u, d) \in UP(P) \times DW(P)$, such that the starting vertex of $u$ and the final vertex of $d$ belong to the same horizontal line, we denote by $P_{u,d}$ the (translated) $m$-Dyck path obtained from $P$ which starts with $u$ and ends with $d$. 
Definition 6.5. A cut of $P$ is an $m$-Dyck path $P_{u,d}$ such that $u$ is a central step of $P$ and $P_{u,d} \neq P$. An admissible cutting of $P$ is a non-empty family of cuts $\mathcal{P} = \{P_{u_1,d_1}\}_{1 \leq i \leq s}$ of $P$ such that $P_{u_1,d_1}$ and $P_{u_3,d_3}$ are disjoint whenever $l \neq h$.

Remark 6.6. For any central step $u$ of $P$, the excursion $P_{u,u}$ of $u$ in $P$ is a cut of $P$.

Notation 6.7. Let $\mathcal{P} = \{P^1, \ldots, P^s\}$ be an admissible cutting of an $m$-Dyck path $P$, such that $P^l = P_{u_1,d_1}$, for $1 \leq l \leq s$.

Suppose that for any $1 \leq l \leq s$, the starting vertex of $u_l$ has coordinates $(a_l, b_l)$ and the final vertex of $d_l$ is $(c_l, d_l)$, we shall always assume that $\mathcal{P}$ is ordered in such a way that $a_1 < a_2 < \cdots < a_s$, which implies that:

$$a_1 < c_1 < a_2 < c_2 < \cdots < a_s < c_s.$$  

Example 6.8. Consider the Dyck path of the preceding example. The admissible cuts are the paths above the dotted red lines.

In the diagram, observe that the cuts $P_{u_1,d_1}$ and $P_{u_1,d_3} = P_{u_1,d_1} \times_0 P_{3,d_3}$ (where 3 denotes the third step of $P$) begin both with $u_1$, so there are two admissible cuts corresponding to the lowest red dotted line.

The admissible cuttings of $P$ are $\{P_{u_1,d_1}\}, \{P_{u_1,d_3}\}, \{P_{u_2,d_2}\}, \{P_{u_3,d_4}\}, \{P_{u_1,d_1}, P_{u_2,d_2}\}, \{P_{u_1,d_1}, P_{u_3,d_4}\}, \{P_{u_2,d_2}, P_{u_3,d_4}\}, \{P_{u_1,d_1}, P_{u_3,d_4}\}$ and $\{P_{u_1,d_1}, P_{u_2,d_2}, P_{u_3,d_4}\}$.

Let $\text{Ad}(P)$ denote the set of admissible cuttings of $P$.

Notation 6.9. Let $P$ be an element of $Dyck^m_n$. For any cut $P_{u,d}$ of $P$, denote by $P/P_{u,d}$ the Dyck path obtained from replacing the path $P_{u,d}$ by a point in $P$, that is, by taking off all the steps of $P_{u,d}$ and gluing the initial vertex of $u$ with the final vertex of $d$.

Remark 6.10. (1) Suppose that $P = P_1 \times_0 \cdots \times_0 P_r$, with $P_i$ prime for $1 \leq i \leq r$. The Dyck paths $P_1, P_1 \times_0 P_2, \ldots, P_1 \times_0 \cdots \times_0 P_{r-1}$ are cuts of $P$.

(2) Let $\mathcal{P} = \{P^1, \ldots, P^s\} \in \text{Ad}(P)$ be an admissible cutting of $P$. For any $1 \leq l \leq s$, the collection $\mathcal{P} \setminus \{P^l\}$ is an admissible cutting of $P/P^l$.

For any admissible cutting $\mathcal{P} = \{P^1, \ldots, P^s\}$ of a path $P \in Dyck^m_n$, the $m$-Dyck $P/\{P^1, \ldots, P^s\}$ is defined recursively by the formula:

$$P/\{P^1, \ldots, P^s\} := (P/\{P^1, \ldots, P^{s-1}\})/\{P^s\}.$$
Example 6.11. Let $P = (0,2,0,5,0,5) \in Dyck^2_6$ be the path of Example 6.8, and consider the admissible cutting $\{P_{u_1,d_3}\}$ of $P$. The 2-Dyck path $P/\{P_{u_1,d_3}\}$ is the path:

\[ \begin{array}{c}
| & | & | & | & | & | \\
\, & \, & \, & \, & \, & \, \\
\, & \, & \, & \, & \, & \, \\
\, & \, & \, & \, & \, & \, \\
\, & \, & \, & \, & \, & \, \\
\, & \, & \, & \, & \, & \, \\
\, & \, & \, & \, & \, & \, \\
\end{array} \]

\(u_3 \quad d_4 \quad P_{u_3,d_4} \quad d_1 \)

Definition 6.12. The (reduced) coproduct $\Delta : D_m \to D_m \otimes D_m$ on $D_m$ is defined by the following formula:

$$\Delta(P) = \sum_{P} P^1 \ast \cdots \ast P^s \otimes P/\{P^1, \ldots, P^s\},$$

for any $P \in Dyck^m_n$, where the sum ranges over all the admissible cuttings $P = \{P^1, \ldots, P^s\} \in Ad(P)$.

Remark 6.13. For any $m$-Dyck path $P \in Dyck^m_n$ and any admissible cutting $P = \{P^1, \ldots, P^s\}$ of $P$ such that $P^i \in Dyck^m_{n_i}$, we have that $P/P \in Dyck^m_N$ is a Dyck path, with $N = n - n_1 - \cdots - n_s$, so $L(P/P) \geq m$.

The reduced coproduct extends to a coproduct $\Delta : D_m \to \overline{D_m}^+ \otimes \overline{D_m}^+$ defining

$$\Delta(P) = P \otimes 1_K + \overline{\Delta}(P) + 1_K \otimes P.$$

Notation 6.14. Let $P$ be an $m$-Dyck path,

1. we use Sweedler’s notation for the coproduct, that is

$$\Delta(P) = \sum P_{(1)} \otimes P_{(2)},$$

for any $P \in D_m$, to denote the image of $P$ under the coproduct,
2. the image of $P$ under the reduced coproduct is denoted

$$\overline{\Delta}(P) = \sum \overline{P}_{(1)} \otimes \overline{P}_{(2)},$$

3. for any integer $0 \leq j \leq L(P)$, we denote by $\Delta_{\geq j}(P)$ (respectively, $\Delta_{= j}(P)$) the sum of the terms $P_{(1)} \otimes P_{(2)}$ appearing in $\Delta(P)$ such that $L(P_{(2)}) \geq j$ (respectively, $L(P_{(2)}) = j$).

We write $\sum P_{(1)}^{L \geq j} \otimes P_{(2)}^{L \geq j}$ for $\Delta_{\geq j}(P)$ (and similarly for $\Delta_{= j}(P)$).
4. for the reduced coproduct, we denote $\overline{\Delta}_{\geq j}(P) = \sum \overline{P}_{(1)}^{L \geq j} \otimes \overline{P}_{(2)}^{L \geq j}$ (respectively, $\overline{\Delta}_{= j}(P) = \sum \overline{P}_{(1)}^{L = j} \otimes \overline{P}_{(2)}^{L = j}$).

5. given an admissible cutting $P = \{P^1, \ldots, P^s\}$ of $P$, we use $P_{(1)}$ to denote the sum of elements $P^1 \ast \cdots \ast P^s$ and $P_{(2)}$ for the element $P/\{P\}$. 
From Remark 6.13 we get that \( \Delta_{L \geq m}(P) = \Delta(P) - P \otimes 1_k \), for any Dyck path \( P \in Dyck^m_1 \).

The main result of this section is the following Theorem.

**Theorem 6.15.** The coproduct \( \Delta \) defined on \( D_m \) satisfies the relation:

\[
\Delta(P \ast_i Q) = \Delta(P) \ast_i \Delta(Q) = \sum (P(1) \ast_i Q(1)) \otimes (P(2) \ast_i Q(2)),
\]

for any integer \( 0 \leq i \leq m \) and any pair of elements \( P, Q \in D_m \). In other words, the triple \( (D_m, \{ \ast_i \}_{0 \leq i \leq m}, \Delta) \) is a Dyck\(^m\) bialgebra.

The proof of Theorem 6.15 requires to prove some additional results first. Let us begin by extending the products \( \times_j \), defined in Section 2, to the \( D_m^+ \otimes D_m^+ \) in a trivial way.

**Definition 6.16.** For any pair of \( m \)-Dyck paths \( P \) and \( Q \), and any integer \( 0 \leq j \leq L(P) \), define:

\[
\begin{align*}
(1) & \quad P \times_j 1_k := \begin{cases} 0, & \text{for } 0 \leq j < L(P) \\ P, & \text{for } j = L(P). \end{cases} \\
(2) & \quad 1_k \times_j P := \begin{cases} 0, & \text{for } j > 0 \\ P, & \text{for } j = 0. \end{cases} \\
(3) & \quad (P \ast Q) \otimes (1_k \times_j 1_k) := (P \times_j Q) \otimes 1_k.
\end{align*}
\]

Extending by linearity, we get a well defined product \( \times_j \) on \( D_m^+ \otimes D_m^+ \), given by

\[
(P \otimes Q) \times_j (R \otimes S) = (P \ast R) \otimes Q \times_j S.
\]

**Lemma 6.17.** Let \( P \) be an \( m \)-Dyck path and \( P(2) \) the result of collapsing a set of admissible cuts of \( P \) to a point. For \( 0 < i < m \), we have \( C_i(P) = C_i(P(2)) \) and \( c_i(P) = c_i(P(2)) \). In particular, \( C_{m-1}(P) < L(P(2)) \).

**Proof.** Observe that the down steps of maximal level of \( P(2) \) are the last \( L(P(2)) \) down steps of \( P \) and the colors of both differ only by a renaming of colors. Therefore, for \( 0 < i < m \), we have that \( C_i(P) = C_i(P(2)) \) and \( c_i(P) = c_i(P(2)) \). Also, since \( P(2) \) is an \( m \)-Dyck path, it must have a color repeated \( m \) times, this implies that \( C_{m-1}(P) = C_{m-1}(P(2)) < L(P(2)) \). \( \square \)

**Proposition 6.18.** Let \( P \) be an \( m \)-Dyck path and \( Q \) a prime \( m \)-Dyck path. The coproduct \( \Delta \) satisfies that \( \Delta(P \times_0 Q) = \Delta(P) \times_0 \Delta(Q) \). Moreover, we have that \( \Delta(P \ast_0 Q) = \Delta(P) \ast_0 \Delta(Q) \).

**Proof.** Since \( Q \) is prime, a cut of \( P \times_0 Q \) is either a cut of \( P \), or \( P \) itself, or a cut of \( Q \). So, an admissible cutting \( R \) of \( P \times_0 Q \) satisfies one of the following conditions:
Lemma 6.19. Let \( P,Q \) be two \( m \)-Dyck paths, with \( P \in \text{Dyck}^n_1 \) and \( Q \) prime, and let \( j \) be an integer \( 0 < j \leq L(P) \). The coproduct \( \Delta \) on the elements \( P \times_j Q \) fulfills the following relation:

1. if \( 0 < j < L(P) \), then

\[
\Delta(P \times_j Q) = P \times_j Q \otimes 1_{X} + \Delta_{\leq j}(P) \times_j \Delta(Q) - \sum_{L(P(2)) \leq j} P^{L = 1} \otimes P^{L = j}
\]

where the sum is taken over all admissible cuttings \( \mathcal{P} = \{ P^1, \ldots, P^s \} \) of \( P \) such that \( L(P(2)) \leq j \).

2. if \( j = L(P) \), then

\[
\Delta(P \times_{L(P)} Q) = P \times_{L(P)} Q \otimes 1_{X} + \Delta_{L= L(P)}(P) \times_{L(P)} \Delta(Q)
\]

where the sum is taken over all admissible cuttings \( \mathcal{P} = \{ P^1, \ldots, P^s \} \) of \( P \) such that \( L(P(2)) < L(P) \).

Proof. For \( 0 < j \leq L(P) \), a cut \( R \) of \( P \times_j Q \) is of the form:

1. \( R \) is a cut of \( P \) such that \( L(P/R) \geq j \). Note that it means that either the level of the last step of \( R \) is smaller than \( n_1 \), or \( R = P_{u,d_k^e} \) with \( k \leq L(P) - j \) for some \( d = d_k^e \in \text{DW}_{n_1}(P) \).

2. \( R \) is a cut of \( Q \), for \( j < L(P) \). For \( j = L(P) \), \( R \) is a cut of \( Q \) or \( Q \).
(iii) \( R = P_{u,d}^{P} \times_{k-L(P)} j Q \), for some \( d_k^P \in DW_{n_1}(P) \) such that \( k \geq L(P) - j \).

So, any possible admissible cutting \( R \) of \( P \times j Q \) satisfies one of the following conditions:

(a) \( R \in \text{Ad}(P) \) is such that \( L(P/R) \geq j \), and \((P \times j Q)/R = (P/R) \times j Q\).

(b) \( R \in \text{Ad}(Q) \), for \( 1 \leq j \leq L(P) \), respectively \( R = \{Q\} \), for \( j = L(P) \).

In this case, \((P \times j Q)/R = P \times j Q/R\), respectively \((P \times j Q)/\{Q\} = P\).

(c) \( R \) is the disjoint union of \( P \in \text{Ad}(P) \), such that \( L(P/R) \geq j \), and \( Q \in \text{Ad}(Q) \), which does not contain \( Q \). So, \((P \times j Q)/R = P/P \times j Q/Q\).

(d) \( R = \{P^1, \ldots, P^{s-1}, P^s \times_{j-L(P_2)} Q\} \), where \( P = \{P^1, \ldots, P^s\} \in \text{Ad}(P) \) is such that \( \begin{cases} L(P_2) \leq j & \text{for } j < L(P), \\ L(P_2) < L(P) & \text{for } j = L(P). \end{cases} \)

For the previous two cases, we get that \((P \times_j Q)/R = P/P\).

(e) \( R = \{P^1, \ldots, P^{s-1}, P^s, Q\} \) for \( j = L(P) \), where \( P = \{P^1, \ldots, P^s\} \in \text{Ad}(P) \) is such that \( L(P_2) = L(P) \). Again, we get that \((P \times_j Q)/R = P/P\).

For any pair of Dyck paths \( R, S \), we have that:
\[
\begin{align*}
1_K \times_j R &= 0, & \text{for } 0 < j \leq L(R), \\
R \times_j Q &= 0, & \text{for } L(R) < j, \\
R \times_j 1_K &= 0, & \text{for } 0 < j < L(R).
\end{align*}
\]

An easy calculation shows that
\[
\Delta(P \times_j Q) = P \times_j Q \otimes 1_K + \sum_{L(P_2) \leq j} P^{L \geq j}_{(1)} \otimes P^{L \geq j}_{(2)} \times_j Q + \sum_{L(P_2) \leq j} P^{L \geq j}_{(1)} Q(1) \otimes P^{L \geq j}_{(2)} Q(2) + \sum_{L(P_2) \leq j} P^1 \times \ldots \times P^{s-1} \times (P^s \times_{j-L(P_2)} Q) \otimes P_{(2)} = \\
P \times_j Q \otimes 1_K + \Delta_{L \geq j}(P) \times_j \Delta(Q) - \sum_{L(P_2) \leq j} P^{L=j}_{(1)} Q \otimes P^{L=j}_{(2)} + \sum_{L(P_2) \leq j} P^1 \times \ldots \times P^{s-1} \times (P^s \times_{j-L(P_2)} Q) \otimes P_{(2)},
\]
for \( 1 \leq j < L(P) \), and
\[
\Delta(P \times_{L(P)} Q) = P \times_{L(P)} Q \otimes 1_K + \\
\sum_{L(P_2) \leq L(P)} P^{L=L(P)}_{(1)} \otimes P^{L=L(P)}_{(2)} \times_{L(P)} Q + \sum_{L(P_2) \leq L(P)} P^{L=L(P)}_{(1)} Q(1) \otimes P^{L=L(P)}_{(2)} \times_{L(P)} Q(2) + \\
\sum_{L(P_2) \leq L(P)} P^{L=L(P)}_{(1)} Q \otimes P^{L=L(P)}_{(2)} + \sum_{L(P_2) \leq L(P)} P^1 \times \ldots \times P^{s-1} \times (P^s \times_{L(P)-L(P_2)} Q) \otimes P_{(2)} = \\
P \times_{L(P)} Q \otimes 1_K + \Delta_{L=L(P)}(P) \times_{L(P)} \Delta(Q) + \sum_{L(P_2) \leq L(P)} P^1 \times \ldots \times P^{s-1} \times (P^s \times_{L(P)-L(P_2)} Q) \otimes P_{(2)},
\]
for $j = L(P)$, which ends the proof. □

**Proposition 6.20.** Let $Q$ be a prime $m$-Dyck path and $P$ any $m$-Dyck path. For any $1 \leq i \leq m$, the coproduct satisfies that:

$$
\Delta(P *_{i} Q) = \sum P_{(1)}^{*} Q_{(1)} \otimes P_{(2)} *_{i} Q_{(2)}.
$$

**Proof.** As $Q$ is prime, using the conventions of Notation 4.7, we have that

$$
P *_{i} Q = \sum_{j=c_{i}(P)} P \times_{j} Q.
$$

For $1 \leq i < m$, by Lemma 6.17, any $P_{(2)}$ coming from an admissible cutting of $P$ satisfies $L(P_{(2)}) \geq c_{m}(P) > j$, for $c_{i}(P) \leq j \leq C_{i}(P)$, which implies that $\sum P_{(1)}^{L=j} * Q \otimes P_{(2)}^{L=j} = 0$, and

$$
\sum_{L(P_{(2)}) \leq j} P_{1}^{1} \cdots P_{s-1}^{s-1} * (P_{s} \times_{j-L(P_{(2)})} Q) \otimes P_{(2)} = 0.
$$

So, $P_{(2)}$ satisfies $L(P_{(2)}) \geq j$ (with $c_{i}(P) \leq j \leq C_{i}(P)$) if, and only if, $P_{(2)} \neq 1_{K}$. Therefore, applying Lemma 6.19, we obtain

$$
\Delta(P \times_{j} Q) = \sum_{L(P_{(2)}) \geq j} P_{(1)}^{*} Q_{(1)} \otimes P_{(2)} \times_{j} Q_{(2)} + P \times_{j} Q \otimes 1_{K} =
$$

$$
= \sum_{P_{(2)} \neq 1_{K}} P_{(1)}^{*} Q_{(1)} \otimes P_{(2)} \times_{j} Q_{(2)} + P \times_{j} Q \otimes 1_{K}.
$$

Applying the formula above and Lemma 6.17, we get

$$
\Delta(P *_{i} Q) = \Delta \left( \sum_{j=c_{i}(P)}^{C_{i}(P)} \sum_{P_{(2)} \neq 1_{K}} P \times_{j} Q \right) =
$$

$$
\sum_{j=c_{i}(P)}^{C_{i}(P)} \left( \sum_{P_{(2)} \neq 1_{K}} P_{(1)}^{*} Q_{(1)} \otimes P_{(2)} \times_{j} Q_{(2)} \right) + P *_{i} Q \otimes 1_{K} =
$$

$$
= \sum_{P_{(2)} \neq 1_{K}} P_{(1)}^{*} Q_{(1)} \otimes \left( \sum_{j=c_{i}(P)}^{C_{i}(P)} P_{(2)} \times_{j} Q_{(2)} \right) + P *_{i} Q \otimes 1_{K} =
$$

$$
= \sum_{P_{(2)} \neq 1_{K}} P_{(1)}^{*} Q_{(1)} \otimes \left( \sum_{j=c_{i}(P_{(2)})}^{C_{i}(P_{(2)})} P_{(2)} \times_{j} Q_{(2)} \right) + P *_{i} Q \otimes 1_{K} =
$$

$$
= \sum_{P_{(2)} \neq 1_{K}} P_{(1)}^{*} Q_{(1)} \otimes P_{(2)} *_{i} Q_{(2)} + P *_{i} Q \otimes 1_{K} =
$$

$$
\sum_{P_{(2)} \neq 1_{K}} P_{(1)}^{*} Q_{(1)} \otimes P_{(2)} *_{i} Q_{(2)}.
$$
To prove the formula for $i = m$, we use that

$$\Delta(P \ast_m Q) = \sum_{j=e_m(P)}^{L(P)-1} \Delta(P \times_j Q) + \Delta(P \times_{L(P)} Q).$$

Applying Lemma 6.19, to both terms of the previous equality, we obtain

$$\sum_{j=e_m(P)}^{L(P)-1} \Delta(P \times_j Q) = \sum_{j=e_m(P)}^{L(P)-1} \left( P \times_j Q \otimes 1_{K} + \Delta_{L \geq j}(P) \times_j \Delta(Q) + \sum_{L(P(2)) \leq j} P^1 \ast \ldots \ast P^{s-1} \ast (P^s \times_{j - L(P(2))} Q) \otimes P(2) - \sum_{L(P(2)) \leq j} P^1 \ast \ldots \ast P^{L = j} \otimes P^{L = j}(2), \right),$$

and

$$\Delta(P \times_{L(P)} Q) = P \times_{L(P)} Q \otimes 1_{K} + \Delta_{L = L(P)}(P) \times_{L(P)} \Delta(Q) + \sum_{L(L(P(2)) \leq L(P))} P^1 \ast \ldots \ast P^{s-1} \ast (P^s \times_{L(P) - L(P(2))} Q) \otimes P(2).$$

We leave the proof of the following two equalities to the reader, from which the proof of the case $i = m$ is complete:

$$\Delta(P) \ast_m \Delta(Q) = \sum_{j=e_m(P)}^{L(P)} \left( P \times_j Q \otimes 1_{K} + \Delta_{L \geq j}(P) \times_j \Delta(Q) \right),$$

and

$$\sum_{j=e_m(P)}^{L(P)-1} \left( \sum_{L(P(2)) \leq j} P^{L = j} \otimes P^{L = j}(2) \right) = \sum_{j=e_m(P)}^{L(P)} \sum_{L(P(2)) \leq \min \{ j, L(P) - 1 \}} P^1 \ast \ldots \ast P^{s-1} \ast (P^s \times_{j - L(P(2))} Q) \otimes P(2).$$

We may prove now Theorem 6.15.

Proof. of Theorem 6.15 We prove the result applying a recursive argument on the number of prime factors of $Q$. Suppose $Q = Q_1 \times_0 \ldots \times_0 Q_r$, where the $Q_i$s are prime Dyck-paths.

For $r = 1$, the result is proved in Proposition 6.20.

Suppose that $r > 1$, and let $R := Q_1 \times_0 \ldots \times_0 Q_{r-1}$. Applying the relations satisfied by the products $\ast_i$’s, we get that:

$$P \ast_i Q = P \ast_i (R \ast_0 Q_r) = \sum_{j=1}^{m} (P \ast_j R) \ast_i Q_r - \sum_{j=1}^{i} (P \ast_j Q_r).$$
Since $Q_r$ is prime and $R$ is the product of $r-1$ prime factors, the recursive hypothesis states that
\[
\Delta((P *_j R) *_i Q_r) = (\Delta(P) *_j \Delta(R)) *_i \Delta(Q_r).
\]

By Lemma 3.5, if $j \geq 1$, then the element $R *_j Q_r$ has less than $r$ prime factors. So, we have that:
\[
\Delta(P *_i (R *_j Q_r)) = \Delta(P) *_i (\Delta(R) *_i \Delta(Q_r)).
\]

Since $D_m^+ \otimes D_m^+$ is a Dyck$m$ algebra, the substraction of these two terms gives exactly $\Delta(P) *_i (\Delta(R) *_0 \Delta(Q_r))$ which is equal to $\Delta(P) *_i \Delta(Q)$, which ends the proof of the theorem. \qed

**Corollary 6.21.** The coproduct $\Delta$ (hence also $\overline{\Delta}$) is coassociative.

**Proof.** We need to show that the composition
\[
(\Delta \otimes \text{Id} - \text{Id} \otimes \Delta) \circ \Delta : D_m \rightarrow D_m^+ \otimes D_m^+ \otimes D_m^+,
\]
is zero.

There is a Dyck$m$-algebra structure on $D_m^+ \otimes D_m^+ \otimes D_m^+$ given by:
\[
(x_1 \otimes x_2 \otimes x_3) *_i (y_1 \otimes y_2 \otimes y_3) = (x_1 * y_1) \otimes (x_2 * y_2) \otimes (x_3 * y_3),
\]
and we make similar considerations as in the case of $D_m^+ \otimes D_m^+$ when $x_3 = y_3 = 1$.

As $\Delta$ is a Dyck$m$ homomorphism, it is easy to see that both $\Delta \otimes 1, 1 \otimes \Delta$ are so.

By Theorem 5.11, coassociativity of $\Delta$ follows from the fact that
\[
(\Delta \otimes \text{Id} - \text{Id} \otimes \Delta)(\Delta \rho_m) = 0,
\]
on the generator $\rho_m$ of $D_m$. \qed

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