THE REPRESENTATIONS OF CYCLOTOMIC BMW ALGEBRAS

HEBING RUI AND JIE XU

Abstract. In this paper, we prove that the cyclotomic BMW algebras $\mathcal{B}_{2p+1,n}$ are cellular in the sense of [16]. We also classify the irreducible $\mathcal{B}_{2p+1,n}$-modules over a field.

1. Introduction

Let $r, n$ be two positive integers. Haering–Oldenburg [17] introduced a class of finite dimensional algebras $\mathcal{B}_{r,n}$ called cyclotomic BMW algebras in order to study the link invariants. Such algebras are associative algebras over a commutative ring $R$, which are the cyclotomic quotients of affine BMW algebras in [17] and [13].

Motivated by Ariki, Mathas and Rui’s work on cyclotomic Nazarov–Wenzl algebras [3], we began to study $\mathcal{B}_{r,n}$ in August of 2004. By using the method given in section 3 in [3], we constructed all possible irreducible modules for $\mathcal{B}_{r,2}$. A by-product is the definition of $\mathfrak{u}$-admissible conditions given in Definition 3.15 for certain parameters in the ground ring $R$. We remark that there are two papers [14] and [24] in Arxiv which were posted at the end of 2006. In those papers, there are some results for $\mathfrak{u}$-admissible conditions.

In this paper, we will construct a class of rational functions, which are similar to those in [3] for cyclotomic Nazarov–Wenzl algebras. Such rational functions can be used to construct the seminormal representations for $\mathcal{B}_{r,n}$ under certain conditions given in Lemma 4.16 and Assumption 4.18. By Wedderburn–Artin Theorem on the semisimple finite dimensional algebras, we prove that the rank of $\mathcal{B}_{r,n}$ is no less than $r^n(2n-1)!$ if $\mathcal{B}_{r,n}$ is free. In order to prove that $\mathcal{B}_{r,n}$ is free over $R$ with rank $r^n(2n-1)!$, we have to find a subset $S$ of $\mathcal{B}_{r,n}$ whose cardinality is $r^n(2n-1)!$ such that $\mathcal{B}_{r,n}$ is generated by $S$ as $R$-module. If so, then $\mathcal{B}_{r,n}$ is free over $R$ with rank $r^n(2n-1)!$ as required.

In order to construct such a subset $S$, we construct a class of quotient modules of $\mathcal{B}_{r,n}$. Such modules can be used to construct a filtration of two–sided ideals of $\mathcal{B}_{r,n}$. Therefore, we can lift the set of generators for all such quotient modules to get a set of generators for $R$-module $\mathcal{B}_{r,n}$. Together with our previous results on the seminormal representations for $\mathcal{B}_{r,n}$, we prove that $\mathcal{B}_{r,n}$ is a cellular algebra in the sense of [16]. In particular, $\mathcal{B}_{r,n}$ is free over $R$ with rank $r^n(2n-1)!$ as required. For some technique reasons, we have to assume that $r$ is odd when we construct the quotient modules for $\mathcal{B}_{r,n}$. Using the results on the representation theory of cellular algebras given in [16], we find the relationship between the irreducible modules for Ariki–Koike algebras and $\mathcal{B}_{r,n}$. This will enable us to classify the irreducible $\mathcal{B}_{r,n}$-modules over a field.

We organize the paper as follows. In section 2, we recall the definition of $\mathcal{B}_{r,n}$ over a commutative ring. In section 3, we give all possible irreducible modules for $\mathcal{B}_{r,2}$. We also give the definition of $\mathfrak{u}$-admissible conditions. Under this assumption together with two conditions in Lemma 4.16 and Assumption 4.18, we construct the seminormal forms for $\mathcal{B}_{r,n}$ in section 4. In section 5, we construct the quotient modules for $\mathcal{B}_{r,n}$ for all odd positive integers $r$. At the end of section 5, we prove

\[\text{This is the main result of [26].}\]
that such a $B_{r,n}$ is cellular in the sense of [16]. Finally, we classify the irreducible $B_{r,n}$-modules for odd $r$ in section 6.

When we wrote the paper, we received Dr. Shona Yu’s Ph. D thesis entitled “The cyclotomic Birman-Murakami-Wenzl Algebras” [23]. In her thesis, Yu has proved that $B_{r,n}$ is cellular algebra for all $r$ by using different method. However, we could not understand why she had assumed that $\omega_0$, one of parameters appeared in the definition of $B_{r,n}$, is invertible when she constructed a subset of $B_{r,n}$ which generates $B_{r,n}$ as $R$-module.

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2. Cyclotomic BMW algebras

Throughout the paper, we fix two positive integers $r, n$ and a commutative ring $R$ with multiplicative identity $1_R$.

**Definition 2.1.** Suppose that $R$ is a commutative ring which contains $q^{\pm 1}, u_1^{\pm 1}, \ldots, u_r^{\pm 1}, q^{\pm 1}, q^{-1}$ with $\delta = q - q^{-1}$. Fix $\Omega = \{ \omega_a \mid a \in \mathbb{Z} \} \subseteq R$ such that 

$$\omega_0 = 1 - \delta^{-1}(q - q^{-1}).$$

The cyclotomic BMW algebras $B_{r,n}$ is the unital associative $R$–algebra generated by $\{ T_i, E_i, X_j^\pm \mid 1 \leq i < n \text{ and } 1 \leq j \leq n \}$ subject to the following relations:

1. $X_iX_i^{-1} = X_i^{-1}X_i = 1$ for $1 \leq i \leq n$.
2. (Kauffman skein relation) $1 = T_i^2 - \delta T_i + \delta qE_i$, for $1 \leq i < n$.
3. (braid relations)
   - (i) $T_iT_j = T_jT_i$ if $|i - j| > 1$,
   - (ii) $T_iT_{i+1}T_i = T_{i+1}T_iT_{i+1}$, for $1 \leq i < n - 1$,
   - (iii) $T_iX_j = X_jT_i$ if $j \neq i, i + 1$.
4. (Idempotent relations) $E_i^2 = \omega_0E_i$, for $1 \leq i < n$.
5. (Commutation relations) $X_iX_j = X_jX_i$, for $1 \leq i, j \leq n$.
6. (Skein relations)
   - (i) $T_iX_i - X_{i+1}T_i = \delta X_{i+1}(E_i - 1)$, for $1 \leq i < n$,
   - (ii) $X_iT_i - T_{i+1}X_i = \delta (E_i - 1)X_{i+1}$, for $1 \leq i < n$.
7. (Unwrapping relations) $E_iX_i^\pm E_i = \omega_aE_i$, for $a \in \mathbb{Z}$.
8. (Tangle relations)
   - (i) $E_iT_i = qE_i = T_iE_i$, for $1 \leq i \leq n - 1$,
   - (ii) $E_{i+1}E_i = E_{i+1}T_iT_{i+1} = T_iT_{i+1}E_i$, for $1 \leq i \leq n - 2$.
9. (Untwisting relations)
   - (i) $E_{i+1}E_iE_{i+1} = E_{i+1}$ for $1 \leq i \leq n - 2$,
   - (ii) $E_iE_{i+1}E_i = E_i$, for $1 \leq i \leq n - 2$.
10. (Anti–symmetry relations) $E_iX_iX_{i+1} = E_i = X_iX_{i+1}E_i$, for $1 \leq i < n$.
11. (Cyclotomic relation) $(X_1 - u_1)(X_1 - u_2) \cdots (X_1 - u_r) = 0$

By Definition 2.1(b)(f)(h), we have $X_i = T_{i-1}X_{i-1}T_{i-1}$ for $2 \leq i \leq n$. Therefore, $B_{r,n}$ can also be generated by $E_i, T_i, X_1$ for $1 \leq i \leq n - 1$. We will not need this fact. What we will need is $X_i = T_{i-1}X_{i-1}T_{i-1}$ later on.

$B_{r,n}$ was introduced by Haeber–Oldenburg in order to study the link invariants. It was re-defined by Goodman and Hauschild as the quotient algebra of affine BMW algebra in [13]. Further, Goodman and Hauschild [13] have proved that affine BMW algebras are free over $R$ by showing that they are isomorphic to the affine Kauffman tangle algebras. We do not recall such result since we do not need it when we discuss the $B_{r,n}$ later on.
The main purpose of this paper is to develop the representation theory of \( R_{r,n} \) by using the method in [3].

**Lemma 2.2.** There is a unique R-linear anti-isomorphism \( *: R_{r,n} \to R_{r,n} \) such that \( T_i^* = T_i, E_i^* = E_i \) and \( X_j^* = X_j \) for all positive integers \( i < n \) and \( j \leq n \).

**Proof.** By checking the defining relations for \( R_{r,n} \), it is easy to see that \( E_i T_i + T_i E_i = 1 \) for \( 1 \leq i \leq n - 2 \). In fact, By Definition 2.1(h)(ii), \( E_i T_i + T_i E_i = E_i + 1 \). So, \( E_i T_i = E_i + 1 \). \( T_i \) is an automorphism if \( \delta \) is an automorphism. \( \Box \)

**Lemma 2.3.** Given positive integers \( k \leq n - 1 \) and \( a \).

1. \( T_k X_k^a = X_{k+1}^a T_k + \sum_{i=1}^a \delta X_{k+1}^i (E_k - 1)X_k^{a-i} \).
2. \( T_k^{-1} X_k^a = X_{k+1}^a T_k^{-1} + \sum_{i=1}^a \delta X_{k+1}^i (E_k - 1)X_k^{a-i} \).
3. \( E_k X_k^a T_k = g E_k X_k^a + \delta \sum_{i=1}^a E_k X_k^{a-i} E_k X_k^{-i} - \delta \sum_{i=1}^a E_k X_k^{a-2i} \).
4. \( T_k X_k^a = X_{k+1}^a T_k - \sum_{i=1}^a \delta X_{k+1}^i (E_k - 1)X_k^{a-i} \).
5. \( T_k^{-1} X_k^a = X_{k+1}^a T_k^{-1} - \sum_{i=1}^a \delta X_{k+1}^i (E_k - 1)X_k^{a-i} \).
6. \( E_k X_k^a T_k = g E_k X_k^a - \delta \sum_{i=1}^a E_k X_k^{a-i} E_k X_k^{-i} + \delta \sum_{i=1}^a E_k X_k^{a-2i} \).

**Proof.** (1) can be proved by induction on \( a \). (2) follows from (1) and 2.1(b). Applying anti-automorphism * on (1), we get the formula for \( X_k^a T_k \). Multiplying \( E_k \) on such a formula, we get (3). (4) follows from (1). (5) follows from (4) and 2.1(b). (6) follows from (4). \( \Box \)

Acting \( E_k \) on both sides of \( X_k^a \prod_{i=1}^r (X_1 - u_i) = 0 \) for \( b \in \mathbb{Z} \), and Lemma 2.3(1) for \( k = 1 \), respectively, we have

\[
\sum_{s=0}^r (-1)^{r-s} \sigma_{r-s}(u) \omega_{s+b} E_1 = 0
\]

where \( \sigma_i(u) \) is the i-th basic symmetric polynomial in \( u_1, u_2, \ldots, u_r \).

If we assume that \( E_1 \) is not a torsion element, then

\[
\sum_{s=0}^r (-1)^{r-s} \sigma_{r-s}(u) \omega_{s+b} = 0, \forall b \in \mathbb{Z},
\]

where \( \omega_a = \omega_{a+b} + \sum_{i=1}^a q^{-1} \delta (\omega_{a-i} \omega_{a+i} - \omega_{a-2i}) \).

**Definition 2.6.** The parameters \( \omega_a \in R, a \in \mathbb{Z} \) are called admissible if they satisfy (2.5).

In [1], Ariki and Koike introduced \( H_{r,n}(u) := H_{r,n}, \) the cyclotomic Hecke algebra of type \( G(r, 1, n) \) or the Ariki-Koike algebra. By definition, it is the unital associative \( R \)-algebra generated by \( y_1, \ldots, y_n \) and \( g_1, g_2, \ldots, g_{n-1} \) subject to the following relations:

\[ a) \quad (g_i - q)(g_i + q^{-1}) = 0, \text{ if } 1 \leq i \leq n - 1, \]
\[ b) \quad g_i g_j = g_j g_i, \text{ if } |i - j| > 1, \]
\[ c) \quad g_i g_{i+1} g_i = g_{i+1} g_i g_{i+1}, \text{ for } 1 \leq i < n - 1, \]
\[ d) \quad g_i y_j = y_j g_i, \text{ if } j \neq i - 1, i + 1, \]
\[ e) \quad y_i y_j = y_j y_i, \text{ for } 1 \leq i, j \leq n, \]
\[ f) \quad y_{i+1} = y_i y_{i+1}, \text{ for } 1 \leq i < n - 1, \]
\[ g) \quad (y_1 - u_1)(y_1 - u_2) \ldots (y_1 - u_r) = 0. \]

In this paper, since we are assuming that \( u_1, \ldots, u_r \) are invertible in \( R, y_i \)'s are invertible in \( H_{r,n} \).
Let \( \langle E_1 \rangle \) be the two-sided ideal of \( R_{r,n} \) generated by \( E_1 \). It is not difficult to see that there is an epimorphism
\[
\eta : H_{r,n}(u) \longrightarrow R_{r,n}/\langle E_1 \rangle
\]
determined by: \( \eta(g_i) = T_i + \langle E_1 \rangle \) and \( \eta(g_j) = X_j + \langle E_1 \rangle \) for positive integers \( i < n \) and \( j \leq n \).

So, any \( R_{r,n} \)-module, which is annihilated by \( E_1 \), is an \( H_{r,n}(u) \)-module. We will use this fact frequently in the next section.

3. \( u \)-Admissible Conditions

In this section, unless otherwise stated, we always assume that \( R = \mathbb{Q}(u_1, u_2, \cdots, u_r, q) \), where \( q, u_1, u_2, \cdots, u_r \) are algebraically independent over \( \mathbb{Q} \).

Let \( q^\pm 1 \in R, \delta = q - q^{-1} \) and \( \Omega = \{ \omega_a \ | \ a \in \mathbb{Z} \} \subseteq R \) such that
\[
\omega_0 = 1 - \delta^{-1}(q - q^{-1}).
\]

The main purpose of this section is to construct all possible irreducible representations of \( R_{r,2} \) over \( R \). We find a set of conditions on the parameters \( q \) and \( \{ \omega_a \ | \ a \in \mathbb{Z} \} \), called \( u \)-admissible conditions, such that the dimension of the corresponding \( R_{r,2} \) is \( 3r^2 \). These conditions, which are similar to those for the cyclotomic Nazarov-Wenzl algebras in [3], are exactly what we need for general \( n \).

**Proposition 3.1.** Suppose that \( M \) is an irreducible \( R_{r,2} \)-module such that \( E_1 M = 0 \). Then one of the following results holds.

a) \( M = Rm \) is one dimensional and the action of \( R_{r,2} \) is determined by
\[
T_1 m = \varepsilon m, \quad E_1 m = 0, \quad X_1 m = u_i m, \quad \text{and} \quad X_2 m = \varepsilon^2 u_i m,
\]
where \( \varepsilon \in \{ q, -q^{-1} \} \) and \( 1 \leq i \leq r \). In particular, up to isomorphism, there are exact \( 2r \) such representations.

b) \( M \) is two dimensional and the action of \( R_{r,2} \) is given by
\[
T_1 \mapsto \frac{u_i}{u_j - u_i} \left( \begin{array}{cc} \delta & q - u_i q^{-1} u_j^{-1} \\ q^{-1} - q u_i u_j^{-1} & -\delta u_i u_j^{-1} \end{array} \right),
\]
\[
E_1 \mapsto \left( \begin{array}{cc} 0 & 0 \\ 0 & 0 \end{array} \right),
\]
\[
X_1 \mapsto \left( \begin{array}{cc} u_i & 0 \\ 0 & u_j \end{array} \right),
\]
\[
X_2 \mapsto \left( \begin{array}{cc} u_j & 0 \\ 0 & u_i \end{array} \right),
\]
where \( 1 \leq i, j \leq r \) with \( i \neq j \). In particular, up to isomorphism there are exact \( \binom{r}{2} \) such representations.

**Proof.** It follows from (2.7) that any irreducible \( R_{r,2} \)-module \( M \) has to be an irreducible \( H_{r,2} \)-module if \( E_1 M = 0 \). By the results for \( H_{r,2} \) in [1], \( M \) has to be one of the modules given in (a) and (b). By direct computation, both (a) and (b) do define the \( R_{r,2} \)-modules with trivial action of \( E_1 \) on them. \( \square \)

**Proposition 3.2.** Suppose \( \omega_0 \neq 0 \). There is a unique irreducible \( R_{r,2} \)-module \( M \) such that \( E_1 M \neq 0 \). Moreover, \( \dim_R M \leq r \). If \( d = \dim_R M \), then there exists a basis \( \{ m_1, m_2, \ldots, m_d \} \) of \( M \) and scalars \( \{ v_1, \ldots, v_d \} \subseteq \{ u_1, \ldots, u_r \} \), with \( v_i \neq v_j \) whenever \( i \neq j \), such that for \( 1 \leq i \leq d \) the following hold:

a) \( X_1 m_i = v_i m_i \) and \( X_2 m_i = v_i^{-1} m_i \),

b) \( E_1 m_i = \gamma_i(m_1 + \cdots + m_d) \)

\( ^3\)The results given in this section are the main results of [26]. Similar results can be found in [24]. We remark the method we are using here is almost the same as that used in [3].
c) $T_1 m_i = \frac{\delta(\gamma_i - 1)}{v_i^2 - 1} m_i + \sum_{j \neq i} \frac{\delta \gamma_i}{v_i v_j - 1} m_j,$

where $\omega_a = \sum_{j=1}^{d} v_j^a \gamma_j$, $a \in \mathbb{Z}$, and

(1) $\gamma_i = (\gamma_d(v_i) + \delta^{-1} q(v_i^2 - 1) \prod_{j \neq i} v_i - v_j) \delta v_i^{-1}$, where $\gamma_d(z) = 1$ for $2 \nmid d$ and $-z$, otherwise.

(2) $\varrho^{-1} = \alpha \prod_{i=1}^{d} v_i$ where $\alpha \in \{1, -1\}$ if $2 \nmid d$ and $\alpha \in \{q^{-1}, -q\}$, otherwise.

(3) $\omega_0 = \delta^{-1} q(\prod_{i=1}^{d} v_i^2 - 1) + 1 - \frac{1 - q^d + 1}{2} \prod_{i=1}^{d} v_i$.

Conversely, if $\gamma_j$, $\varrho^{-1}$, $\omega_a = \sum_{j=1}^{d} v_j^a \gamma_j$, $a \in \mathbb{Z}$, are defined as above, then (a)-(c) define an irreducible $\mathcal{B}_{r,2}$-module with $E_1 M \neq 0$.

**Proof.** The result can be proved by arguments similar to those for the cyclotomic Nazarov-Wenzl algebras in [3]. We include a proof here.

By direct computation, one can verify that the $R$-module $N$ generated by $\{X_1^\alpha X_2^\beta, X_1^\alpha X_2^\beta T_1, X_1^\alpha X_2^\beta E_1 X_1^\beta \mid 0 \leq \alpha, \beta \leq r - 1\}$ is a right $\mathcal{B}_{r,2}$-module. Since $1 \in N$, $N = \mathcal{B}_{r,2}$. In particular, $\mathcal{B}_{r,2}$ is of infinite dimension. So is any irreducible $\mathcal{B}_{r,2}$-module.

Suppose that $M$ is an irreducible $\mathcal{B}_{r,2}$-module such that $E_1 M \neq 0$ and $d = \dim_{R} M$. We first show that (a)-(c) hold.

Since $u_1, \ldots, u_r$ are pairwise distinct, we can fix a basis $\{m_1, \ldots, m_d\}$ of $M$ consisting of eigenvectors for $X_1$. Write $X_1 m_i = \omega_i m_i$, for some $\omega_i \in \{u_1, \ldots, u_r\}$. We remark that we will prove that $v_i \neq v_j$ whenever $i \neq j$.

Since we are assuming that $\omega_0 \neq 0$, we define $f = \frac{1}{\omega_0} E_1$. Then $f^2 = f$, $E_1 f = f E_1 = \omega_0 f$, and $f \neq 0$. Also, we have $f M \neq 0$ since $E_1 M \neq 0$.

Fix an element $0 \neq m \in f M$. Then $E_1 m = \omega_0 m$ and $T_1 m = \varrho m$ (since $T_1 E_1 = \varrho E_1$). As $X_1 X_2 E_1 = E_1$, $X_1 X_2 E_1 m = \omega_0 m$. Therefore, $X_1 X_2 m = m$.

We consider the $R$-module $M'$ generated by $\{m, X_1 m, \ldots, X_1^{d-1} m\}$. Since $E_1 X_1^k m = E_1 X_1^k \varrho m = \frac{1}{\omega_0} E_1 X_1^k E_1 m = \frac{\varrho}{\omega_0} E_1 m = \omega_k m$, $M'$ is stable under the action of $E_1$. By Lemma 2.3(3) for $k = 1$ and Lemma 2.2,

$$T_1 X_1^k m = \frac{1}{\omega_0} T_1 X_1^k E_1 m$$

$$= \frac{1}{\omega_0} (\varrho X_1^{-a} + \sum_{i=1}^{a} \delta(\omega_{a-i} X_1^{-i} - X_1^{-2i})) E_1 m$$

$$= \varrho X_1^{-a} m + \sum_{i=1}^{a} \delta(\omega_{a-i} X_1^{-i} m - X_1^{-2i} m)$$

Note that $(X_1 - v_1)(X_1 - v_2) \cdots (X_1 - v_d) = 0$ on $M$. Each $X_1^a$ for $a \in \mathbb{Z}$ can be written as a linear combination of $X_1^b$ with $0 \leq b \leq d - 1$. Therefore, $M'$ is closed under the action of $T_1$. Note that $X_2 = T_1 X_1 T_1$. So, $M'$ is closed under the action of $X_2$. We have $M' = M$ since $M$ is irreducible. In particular, by $\dim M = d$, $\{m, X_1 m, \ldots, X_1^{d-1} m\}$ is a basis of $M$. Since $E_1 X_1^k m = \omega_k m$, $E_1 M = Rm$.

Write $m = \sum_{i=1}^{d} r_i m_i$, for some $r_i \in R$. Suppose that $r_i = 0$ for some $i$. Then

$$\prod_{1 \leq j \leq d \atop j \neq i} (X_1 - v_j) \cdot m = 0.$$
Comparing the coefficients of $m_\delta$ where $X$ therefore, $X$ \( \delta \) is the Kronecker function. Therefore, \( m = m_1 + m_2 + \cdots + m_d \) in the following.

By the previous arguments, all of the eigenvalues $\{v_1, \ldots, v_d\}$ of $X_1$ must be distinct. It is easy to verify that $X_1X_2$ is central in $\mathcal{B}_{r,2}$. $X_1(X_1X_2)m_i = (X_1X_2)X_1m_i$. So $X_1X_2$ acts on $m_i$ as a scalar since $u_1$'s are algebraic independent. On the other hand, since $X_1X_2m = m$, by (3.3), $X_1X_2$ acts as 1 on $M$. Therefore, $X_2m_i = X_1^{-1}m_i = v_i^{-1}m_i$, for $i = 1, \ldots, d$, proving (a).

Since $u_1, \ldots, u_r$ are algebraically independent, $v_i^2 - 1$ and $v_iv_j - 1$, for $i \neq j$, are invertible in $R$. So the formula in part (c) makes sense.

As $E_1M = Rm$, we can define elements $\gamma_i \in R$ by

\[
E_1m_i = \gamma_i m = \gamma_i(m_1 + \cdots + m_d), \quad \text{for } i = 1, \ldots, d.
\]

Write $T_1m_i = \sum_{j=1}^d c_{ji}m_j$. By Definition 2.1(f) for $i = 1$, we have

\[
T_1X_2m_i - X_1T_1m_i = \delta X_2m_i - \delta E_1X_2m_i.
\]

So,

\[
\sum_{j=1}^d v_i^{-1}c_{ji}m_j - \sum_{j=1}^d c_{ji}v_jm_j = \delta v_i^{-1}m_i - \delta v_i^{-1}\gamma_i m.
\]

Comparing the coefficients of $m_j$ on both sides of the above equation, we have

\[
c_{ji} = \frac{\delta(\gamma_i - \delta_{ij})}{v_i^2 - 1},
\]

where $\delta_{ij}$ is the Kronecker function. Therefore,

\[
T_1m_i = \frac{\delta(\gamma_i - 1)}{v_i^2 - 1} m_i + \sum_{1 \leq j \leq d \atop j \neq i} \frac{\delta \gamma_i}{v_i v_j - 1} m_j.
\]

This proves (c).

Since $\varrho E_1 = T_1E_1$, we have, for any positive integer $i \leq d$:

\[
\varrho \gamma_i \sum_{j=1}^d m_j = \varrho E_1m_i = T_1E_1m_i = T_1(\gamma_i \sum_{j=1}^d m_j)
\]

\[
= \gamma_i \sum_{j=1}^d \left( \frac{\delta(\gamma_j - 1)}{v_j^2 - 1} m_j + \sum_{1 \leq k \leq d \atop k \neq j} \frac{\delta \gamma_j}{v_j v_k - 1} m_k \right)
\]

\[
= \gamma_i \left( \sum_{j=1}^d \frac{\delta(\gamma_j - 1)}{v_j^2 - 1} m_j + \sum_{j=1}^d \sum_{1 \leq k \leq d \atop k \neq j} \frac{\delta \gamma_k}{v_j v_k - 1} m_j \right)
\]

Since $E_1M \neq 0$, there is at least a non-zero $\gamma_i$. Therefore,

\[
\varrho \sum_{j=1}^d m_j = \sum_{j=1}^d \frac{\delta(\gamma_j - 1)}{v_j^2 - 1} m_j + \sum_{j=1}^d \sum_{1 \leq k \leq d \atop k \neq j} \frac{\delta \gamma_k}{v_j v_k - 1} m_j.
\]

Comparing the coefficients of $m_j$ on both sides of the equality, we have

\[
\sum_{k=1}^d \frac{\gamma_k}{v_j v_k - 1} = \delta^{-1} \varrho + \frac{1}{v_j^2 - 1}, \quad j = 1, 2, \ldots, d.
\]
The system of linear equations \((3.5)\) on \(\gamma_k\) has a unique solution since \(\det A_d \neq 0\) where \(A_d\) is the coefficient matrix such that the \((i, j)\)-th entry of \(A_d\) is \((v_i v_j - 1)^{-1}\). In fact, we have

\[
\det A_d = \prod_{1 \leq k < j \leq d} (v_k - v_j)^2 \prod_{1 \leq k, j \leq d} (v_k v_j - 1)^{-1}.
\]  

In order to verify \((3.6)\), we first observe that \(\prod_{1 \leq k, j \leq d} (v_k v_j - 1) \cdot \det A_d\) is a symmetric polynomial in \(v_1, \ldots, v_d\). So, it is divided by \(v_k - v_j, k \neq j\). Therefore, there is a \(f_d(v_1, \ldots, v_d) \in \mathbb{Q}[v_1, \ldots, v_d]\) such that

\[
\prod_{1 \leq k, j \leq d} (v_k v_j - 1) \cdot \det A_d = f_d(v_1, \ldots, v_d) \prod_{1 \leq k < j \leq d} (v_k - v_j)^2.
\]  

Comparing the coefficients of \(v_1\) on both sides of \((3.7)\), we know that the highest degree of \(v_1\) in \(f_d(v_1, v_2, \ldots, v_d)\) is zero. Since \(f_d(v_1, v_2, \ldots, v_d)\) is a symmetric polynomial in \(v_1, v_2, \ldots, v_d\), we set \(v_d = 0\) and get

\[
\det A_d|_{v_d=0} = -\det A_{d-1} \prod_{i=1}^{d-1} v_i^2.
\]  

Using \((3.7)\) to rewrite the above equation, we have \(c_d = c_{d-1}\). By induction assumption, \(c_d = c_j, 1 \leq j \leq d - 1\). An easy computation shows that \(c_1 = 1\). Thus \(c_d = 1\), proving \((3.6)\).

We have proved that the system of linear equations given in \((3.5)\) has a unique solution. We define

\[
f(z) = \frac{\gamma_d(z)}{(z^2 - 1)(v_j z - 1)} \prod_{l=1}^{d} \frac{v_l z - 1}{z - v_l} + \frac{\delta^{-1} g(z)}{z(v_j z - 1)} \prod_{l=1}^{d} \frac{v_l (v_j z - 1)}{z - v_l},
\]  

where \(\gamma_d(z)\) is defined in \((1)\) of Proposition 3.2. By residue theorem for complete non-singular curves for \(f(z)\), we have

\[
\sum_{k=1}^{d} \frac{1}{v_j v_k - 1} \gamma_d(v_k) + \delta^{-1} g(v_k^2 - 1) \prod_{l \neq k} v_l \prod_{l \neq k} \frac{v_k v_l - 1}{v_k - v_l} = \delta^{-1} g + \frac{1}{v_j - 1}.
\]  

The above equalities shows that \(\gamma_k\)’s given in \((1)\) in Proposition 3.2 satisfy \((3.5)\). We remark that the left (resp. right) side of the above equality can be interpreted as \(\sum_{k=1}^{d} \text{Res}_{z=v_k} f(z) dz\) (resp. \(-\sum_{v \in I} \text{Res}_{z=v} f(z) dz\) and \(I = \{\infty, \pm 1, 0\}\)). This completes the proof of \((b)\).

Now, we prove the formula about \(\omega_a, a \in \mathbb{Z}\). Since \(E_1 m = \omega_0 m\) and \(m = \sum_{i=1}^{d} m_i, \omega_0 = \sum_{i=1}^{d} \gamma_i\). Similarly, we have that \(\omega_a = \sum_{j=1}^{d} v_j^a \gamma_j\). In order to compute \(\omega_0\), we define

\[
g(z) = \frac{\gamma_d(z)}{z^2 - 1} \prod_{l=1}^{d} \frac{v_l z - 1}{z - v_l} + \frac{\delta^{-1} g(z)}{z} \prod_{l=1}^{d} \frac{v_l (v_j z - 1)}{z - v_l}.
\]  

Then \(\omega_0 = \sum_{i=1}^{d} \gamma_i = \sum_{i=1}^{d} \text{Res}_{z=v_i} g(z) dz\). By residue theorem for complete non-singular curves for \(g(z)\), \(\omega_0 = -\sum_{v \in I} \text{Res}_{z=v} f(z) dz\) and \(I = \{\infty, \pm 1, 0\}\). By direct computation,

\[
\omega_0 = \delta^{-1} g \left( \prod_{l=1}^{d} v_l^2 - 1 \right) + 1 - \frac{(-1)^d + 1}{2} \prod_{l=1}^{d} v_l.
\]  

By solving the equation \((\omega_0 - 1)\delta = \rho^{-1} - \rho\), we get \(\rho\) as required.
We next show that \( M \) is uniquely determined, up to isomorphism. Suppose that \( \mathcal{B}_{r,2} \) has another irreducible module of dimension \( d' \) upon which \( E_1 \) acts non-trivially. Then, by the previous arguments, 

\[
\omega_0 = \delta^{-1} g'(\prod_{i=1}^{d'} w_i^2 - 1) + 1 - \frac{(-1)^{d'} + 1}{2} \prod_{i=1}^{d'} w_i.
\]

for some \( \{ w_1, \ldots, w_{d'} \} \subseteq \{ u_1, \ldots, u_r \} \). Since we are assuming that \( u_1, \ldots, u_r, q \) are algebraically independent, \( d' = d, \rho = \rho' \) and \( w_i = u_{i(\sigma)} \), for some \( \sigma \in \mathfrak{S}_d \) and \( 1 \leq i \leq d \). By (a)–(c), \( M \cong M' \) as required.

Finally, it remains to verify that (a)–(e) do define a representation of \( \mathcal{B}_{r,2}(u) \) whenever \( \omega_0 = \sum_{i=1}^d v_i^2 \gamma_i \), for \( a \in \mathbb{Z} \) and \( \gamma_i, \rho \) as above. In other word, we need verify the defining relations for \( \mathcal{B}_{r,2} \). More explicitly, we need verify (a), (b), (d)–(h) and (j)-(k) in Definition 2.1.

In fact, it is easy to verify (a) and (j). Note that (d) and (g) can be verified easily by using the formula \( \omega_0 = \sum_{i=1}^d v_i^2 \gamma_i \). (f) follows from (3.4) and (h) follows from 

\[
\sum_{k=1}^{d} \frac{\gamma_k}{v_j v_k - 1} = \delta^{-1} \frac{1}{v_j^2 - 1}, \quad j = 1, \ldots, d.
\]

Finally, we verify (b) in Definition 2.1.

We have already proved that Definition 2.1(f) holds on \( M \). Thus, the following equalities hold in \( \text{End}_R(M) \):

\[
\begin{align*}
T_2^2 X_2 - T_2 X_1 T_1 & = \delta T_1 X_2 - \rho \delta E_1 X_2 \\
X_2 T_1^2 - T_1 X_2 X_1 & = \delta X_2 T_1 - \rho \delta E_2 X_1
\end{align*}
\]

Therefore, \((T_2^2 - T_1 + \rho \delta E_1)X_2 = X_2(T_1^2 - T_1 + \rho \delta E_1)\). Since \(v_i^{-1} \neq v_j^{-1} \) whenever \( i \neq j \), and \( X_2 m_i = v_i^{-1} m_i \), \( T_2^2 - T_1 + \rho \delta E_1 \) acts on \( \{ m_1, \ldots, m_d \} \) diagonally. So, for each positive integer \( i \leq d \), there is a \( c_i \in R \) such that

\[
(T_2^2 - T_1 + \rho \delta E_1)m_i = c_i m_i.
\]

So,

\[
(T_2^2 - T_1 + \rho \delta E_1)m_i = T_2^2 \sum_{1 \leq j \leq d} \frac{\delta \gamma_i - 1}{v_i - 1} m_i + \sum_{1 \leq j \leq d} \frac{\delta \gamma_i}{v_i v_j - 1} m_j
\]

\[
- \delta \frac{\delta \gamma_i - 1}{v_i - 1} m_i + \sum_{1 \leq j \leq d} \frac{\delta \gamma_i}{v_i v_j - 1} m_j + \rho \delta \gamma_i \sum_{j=1}^{d} m_j
\]

\[
= \frac{\delta \gamma_i - 1}{v_i - 1} \sum_{1 \leq j \leq d} \frac{\delta \gamma_i}{v_i v_j - 1} m_j + \sum_{1 \leq j \leq d} \frac{\delta \gamma_i}{v_i v_j - 1} m_j + \sum_{1 \leq j \leq d} \frac{\delta \gamma_i}{v_i v_j - 1} m_j
\]

\[
= \sum_{1 \leq j \leq d} \frac{\delta \gamma_j}{v_i v_j - 1} m_i + \sum_{1 \leq j \leq d} \frac{\delta \gamma_i}{v_i v_j - 1} m_j + \rho \delta \gamma_i \sum_{j=1}^{d} m_j
\]

Comparing the coefficient of \( m_i \) on both sides of the above equality, we have

\[
c_i = \delta^2 \gamma_i \sum_{j=1}^{d} \frac{\gamma_j}{(v_i v_j - 1)^2} + \delta \frac{v_i^2}{(v_i^2 - 1)^2} \gamma_i + \rho \delta \gamma_i.
\]
Suppose
\[ h(z) = \frac{\gamma_d(z)}{(z^2-1)(v_i z - 1)^2} \prod_{l=1}^{d} \frac{v_i z - 1}{z - v_l} + \frac{\delta^{-1}q}{z(v_i z - 1)^2} \prod_{l=1}^{d} \frac{v_i(v_i z - 1)}{z - v_l}. \]

We use the residue theorem for complete non-singular curves for \( h(z) \) and the equalities for \( \gamma_i, 1 \leq i \leq d \), in Proposition 3.2 to compute \( c_i \). We discuss the case \( 2 \nmid d \) and leave the other to the reader.

By direct computation, we have
\[ \text{Res}_{z=v_i} h(z) = \begin{cases} -\frac{1}{2}(v_i-1)^2, & \text{if } v = 1, \\ -\frac{1}{2}(v_i+1)^2, & \text{if } v = -1, \\ \delta^{-1}q, & \text{if } v = 0, \end{cases} \]

and
\[
\begin{aligned}
\text{Res}_{z=v_i^{-1}} h(z) &= \frac{v_i^2}{(1-v_i^2)^2} \prod_{l \neq i} \frac{v_i - v_l}{v_i v_l - 1} + \delta^{-1}q \frac{v_i^2}{1-v_i^2} \prod_{l \neq i} v_i \prod_{l \neq i} \frac{v_i - v_l}{v_i v_l - 1}, \\
\text{Res}_{z=v_i} h(z) &= \frac{1}{(1-v_i^2)^2} \prod_{l \neq i} \frac{v_i - v_l}{v_i v_l - 1} + \delta^{-1}q \frac{v_i^2 - 1}{(1-v_i^2)^2} \prod_{l \neq k} v_i \prod_{l \neq k} \frac{v_i^2 - 1}{v_i v_l - 1}.
\end{aligned}
\]

Using (1)–(2) of Proposition 3.2 to rewrite the above equalities yields
\[
\begin{aligned}
\text{Res}_{z=v_i^{-1}} h(z) &= \frac{v_i^2}{\gamma_i(1-v_i^2)^2} - \frac{1}{\gamma_i v_i^2}, \\
\text{Res}_{z=v_i} h(z) &= \frac{\gamma_i}{(1-v_i^2)^2}.
\end{aligned}
\]

By the residue theorem for complete non-singular curves for \( h(z) \), we have
\[ \text{Res}_{z \in I} h(z) + \sum_{k=1}^{d} \text{Res}_{z=v_k} h(z) = 0 \]

where \( I = \{ \pm 1,0,v_i^{-1} \} \). Rewriting the above equality yields
\[ \delta^{-1}q = \frac{v_i^2 + 1}{(v_i^2 - 1)^2} - \frac{v_i^2}{\gamma_i(1-v_i^2)^2} (1 - \frac{\delta^{-2}(v_i^2 - 1)^2}{v_i^2}) - \sum_{k=1}^{d} \frac{\gamma_k}{(1-v_i v_k)^2}. \]

So,
\[ c_i = \delta^{-1}q \gamma_i - \frac{v_i^2 + 1}{(v_i^2 - 1)^2} \delta^2 \gamma_i + \frac{v_i^2}{(1-v_i^2)^2} \delta^2 + \delta^2 \gamma_i \sum_{k=1}^{d} \frac{\gamma_k}{(1-v_i v_k)^2} = 1. \]

This shows that Definition 2.1(b) holds on \( M \). \( \square \)

**Theorem 3.8.** Suppose \( R = \mathbb{Q}(u_1,u_2,\ldots,u_r,q) \) where \( u_1, \ldots, u_r, q \) are algebraically independent over \( \mathbb{Q} \). If \( q \) and \( \omega_a, a \in \mathbb{Z} \), are given as in Proposition 3.2 for \( d = r \), then \( \mathcal{B}_{r,2} \) is semisimple over \( R \). Moreover, \( S = \{X_1^\alpha X_2^\beta, X_1^\alpha X_2^\beta T_1, X_1^\alpha E_1 X_2^\beta | 0 \leq \alpha, \beta \leq r - 1 \} \) is an \( R \)-basis of \( \mathcal{B}_{r,2} \).

**Proof.** By direct computation, one can verify that the \( R \)-submodule \( M \) of \( \mathcal{B}_{r,2} \) generated by \( S \) is a right \( \mathcal{B}_{r,2} \)-module. Since \( 1 \in S, M = \mathcal{B}_{r,2} \). In particular, \( \text{dim} \mathcal{B}_{r,2} \leq 3r^2 \).

Under the assumption, we have, by Proposition 3.2 that there is a unique irreducible \( \mathcal{B}_{r,2} \)-module \( M \) with \( E_1 M \neq 0 \). By Wedderburn-Artin Theorem for semisimple finite dimensional algebras together with Proposition 3.1 we have
\[ 3r^2 = 2r + \frac{r(r-1)}{2} + r^2 = \text{dim} \mathcal{B}_{r,2}/\text{Rad} \mathcal{B}_{r,2} \leq \text{dim} \mathcal{B}_{r,2} \leq 3r^2, \]

where \( \text{Rad} \mathcal{B}_{r,2} \) is the Jacobson radical of \( \mathcal{B}_{r,2} \). Thus all inequalities given above are equalities. In particular, \( \mathcal{B}_{r,2} \) is semisimple and \( S \) is an \( R \)-basis. \( \square \)
Definition 3.9. Suppose $x = (x_1, x_2, \cdots, x_r)$ are variables. For any non-negative integer $a$, Define $Q_a(x), Q'_a(x)$ such that

\begin{align*}
\prod_{i=1}^{r} \frac{y - x_i}{x_i^2 - 1} &= \sum_{a=0}^{\infty} Q_a(x)y^a, \\
\prod_{i=1}^{r} \frac{y - x_i}{x_i^2 - 1} &= \sum_{a=0}^{\infty} Q'_a(x)y^a.
\end{align*}

We set $Q_a(x) = Q'_a(x) = 0$ if $a < 0$. For each non-negative integer $a$, it is not difficult to verify that $Q_a(x)$ (resp. $Q'_a(x)$) is a symmetric polynomial in variables $x_1, x_2, \cdots, x_r$ (resp. $x_1^{-1}, \ldots, x_r^{-1}$). Further,

$$Q'_a(x) = Q_a(x_1^{-1}, \ldots, x_r^{-1}).$$

Lemma 3.11. Suppose $R$ is an integral domain which contains the identity $1_R$, and the units $u_1, u_2, \cdots, u_r, q, (q - q^{-1})$ such that $u_iu_j^{\pm 1} \neq 1$. Let $F$ be the quotient field of $R$. For $a \in \mathbb{Z}$, define

$$\omega_a = \sum_{i=1}^{r} (\gamma_{d}(u_i) + \delta^{-1} q(u_i^2 - 1) \prod_{j \neq i} u_j^a) \prod_{j \neq i} \frac{u_iu_j - 1}{u_i - u_j}$$

where $\gamma$ is defined in (2) of Proposition 3.2 for $d = r$. Suppose that $a \geq 0$. Then

\begin{align*}
\omega_a &= \begin{cases} 
A + \sum_{k=0}^{a-1} \frac{1 + (-1)^k}{2} Q_{a-1-k}(u) - \delta^{-1} q \delta_{a0}, & \text{if } 2 \mid r, \\
A - \sum_{k=0}^{a-1} \frac{1 + (-1)^k}{2} Q_{a-k}(u) - \delta^{-1} q \delta_{a0}, & \text{if } 2 \nmid r,
\end{cases}
\end{align*}

and for $a > 0$,

\begin{align*}
\omega_{-a} &= \begin{cases} 
B + \sum_{k=0}^{a-1} \frac{1 + (-1)^k}{2} Q'_{a-1-k}(u), & \text{if } 2 \mid r, \\
B - \sum_{k=0}^{a-1} \frac{1 + (-1)^k}{2} Q'_{a-k}(u), & \text{if } 2 \nmid r,
\end{cases}
\end{align*}

where

\begin{align*}
A &= \frac{1 + (-1)^a}{2} + \delta^{-1} q Q_a(u) \prod_{i=1}^{d} u_i, \\
B &= \frac{1 + (-1)^a}{2} - \delta^{-1} q Q'_a(u) \prod_{i=1}^{d} u_i.
\end{align*}

In particular, $\omega_a \in R$ for all $a \in \mathbb{Z}$.

Proof. The result for $a = 0$ follows from Proposition 3.2. Suppose $a$ is a positive integer. We define

$$f(z) = \frac{\gamma_{d}(z)z^a}{z^2 - 1} \prod_{l=1}^{r} \frac{u_lz - 1}{z - u_l} + \delta^{-1} q z^{a-1} \prod_{l=1}^{r} \frac{u_l(z - 1)}{z - u_l}.$$

Then $\omega_a = \sum_{i=1}^{r} \text{Res}_{z=u_i} f(z)dz$. By residue theorem for complete non-singular curves for $f(z)$,

\begin{align*}
\omega_a &= - \sum_{\nu \in \{\infty, \pm 1\}} \text{Res}_{z=\nu} f(z)dz.
\end{align*}

When $r$ is odd, $\text{Res}_{z=1} f(z)dz = -\frac{1}{2}$ and $\text{Res}_{z=-1} f(z)dz = -\frac{(-1)^a}{2}$. On the other hand, $-z^{-2} f(z^{-1}) = z^{-a} \phi_1(z) + z^{-a-1} \phi_2(z)$, where

\begin{align*}
\phi_1(z) &= (z^2 - 1)^{-1} \prod_{i=1}^{r} \frac{z - u_i}{u_i z - 1}, \\
\phi_2(z) &= -\delta^{-1} q \prod_{i=1}^{r} \frac{(z - u_i) u_i}{u_i z - 1}.
\end{align*}
Also, we have \((\varphi_2(z))^{(a)}|_{z=0} = -a!d^{-1}g\prod_{l=1}^{r}u_lQ_a(u)\) and
\[
(\varphi_1(z))^{(a-1)}|_{z=0} = \sum_{k=0}^{a-1} \left( \frac{1}{k!} \right) \frac{1}{z^k - 1} \left( \prod_{l=1}^{r} \frac{u_l}{z - u_l} \right) (a-1-k) \]
\[
= \sum_{k=0}^{a-1} \left( \frac{(-1)^k}{k!} \right) (1 + (-1)^k) \frac{1}{2} (a-1-k)!Q_a-1-k(u) \]
\[
= - (a-1)! \sum_{k=0}^{a-1} \frac{1 + (-1)^k}{2} Q_a-1-k(u). \]

Note that \(Res_{z=\infty} f(z)dz = Res_{z=0} f_1(z)dz + Res_{z=0} f_2(z)dz\) where \(f_1(z) = z^{-a}\varphi_1(z)\) and \(f_2(z) = z^{-a-1}\varphi_2(z)\). So,
\[
Res_{z=\infty} f(z)dz = -\sum_{k=0}^{a-1} \frac{1 + (-1)^k}{2} Q_a-1-k(u) - \delta^{-1} \prod_{l=1}^{r} u_l Q_a(u), \]
and \((3.12)\) follows immediately from \((3.13)\).

When \(r\) is even, \(\gamma_d(z) = -\bar{z}\). When we compute \(Res_{z=\infty} f(z)dz\), we need compute \((\varphi_1(z))^{(a)}|_{z=0}\). Therefore, we need use \(a\) instead of \(a-1\) in \((3.12)\) for odd \(r\). This implies the result for \((3.12)\) in the case \(2 | r\).

One can verify \((3.13)\) by similar arguments as above. Since \(Q_a(u)\) (resp. \(Q'_a(u)\)) are polynomials in variables \(u_1, u_2, \ldots, u_r\) (resp. \(u_1^{-1}, \ldots, u_r^{-1}\)) \(\omega_a \in R\) for all \(a \in \mathbb{Z}\).

**Definition 3.15.** Suppose \(R\) is a commutative ring which contains identity \(1_R\), and the units \(q, (q - q^{-1})\), \(u_i, 1 \leq i \leq r\). Let \(\Omega = \{\omega_a | a \in \mathbb{Z}\} \subseteq R\). Then \(\Omega \cup \{\omega\}\) is called \(u\)-admissible if \(\omega_a\) satisfy \((3.12), (3.13)\) for \(a \in \mathbb{Z}\) and \(g\) is given in Proposition \((3.2)\) for \(d = r\).

By Theorem \((3.8)\) \(\mathcal{B}_{r,2}\) is a free over \(R\) with dimension \(3r^2\) if \(R = \mathbb{Q}(u_1, u_2, \ldots, u_r, q)\), and \(\Omega \cup \{\omega\}\) is \(u\)-admissible. We will show that \(\mathcal{B}_{r,n}\) is free over a commutative ring with rank \(r^n(2n - 1)!\) if \(\Omega \cup \{\omega\}\) is \(u\)-admissible and \(2 \nmid r\).

Motivated by Nazarov’s work on Brauer algebras in \([19]\), we define two generating functions
\[
\begin{align*}
\tilde{w}_{1,+}(y) &= \sum_{a=0}^{\infty} \omega_a y^{-a}, \\
\tilde{w}_{1,-}(y) &= \sum_{a=1}^{\infty} \omega_{-a} y^{-a}
\end{align*}
\]

**Lemma 3.16.** Suppose \(y\) is an indeterminant. Then \(\Omega \cup \{\omega\}\) is \(u\)-admissible if and only if \(g\) is given in Proposition \((3.2)\) for \(d = r\) and
\[
\begin{align*}
\tilde{w}_{1,+}(y) &= \frac{y^2}{y^2} - \delta^{-1} q + (\delta^{-1} q \prod_{l=1}^{r} u_l + \frac{\gamma_d(y)}{\gamma_d(y-1)} \prod_{l=1}^{r} u_l \prod_{l=1}^{r} \frac{y - u_l}{y - u_l}), \\
\tilde{w}_{1,-}(y) &= \frac{1}{y^2} + \delta^{-1} q - \prod_{l=1}^{r} u_l (\delta^{-1} q \prod_{l=1}^{r} u_l - \frac{y}{\gamma_d(\gamma_d(y)^{-1})} \prod_{l=1}^{r} \frac{y - u_l}{y - u_l}).
\end{align*}
\]

**Proof.** When \(r\) is odd, we have \(\sum_{a=0}^{\infty} \frac{1 + (-1)^a}{2} y^{-a} = \frac{y^2}{y^2} - \). By \((3.10)\),
\[
\delta^{-1} q \prod_{l=1}^{r} u_l \sum_{a=0}^{\infty} \delta^{-1} Q_a(u) y^{-a} = \frac{y^2}{y^2} - \prod_{l=1}^{r} u_l \prod_{l=1}^{r} \frac{y - u_l}{y - u_l},
\]

and
\[ \sum_{a=0}^{\infty} \sum_{k=0}^{a-1} \frac{1 + (-1)^k}{2} Q_{a-1-k}(u)y^{-a} = (y^{-1} + y^{-3} + \cdots) \sum_{a=0}^{\infty} Q_a(u)y^{-a} \]
\[ = \frac{y}{y^2 - 1} \prod_{i=1}^{r} u_{i} \prod_{i=1}^{r} \frac{y - u_i^{-1}}{y - u_i} \]

Taking the sum of the above equalities yields the formula for \( \tilde{w}_{1,1}(y) \) in the case \( r \) is odd. Similarly, one can verify the formulae in other cases. \( \square \)

**Corollary 3.17.** \( \Omega \cup \{ \rho \} \) is \( u \)-admissible if and only if
\begin{enumerate} 
  \item \( \rho \) is defined by Proposition 3.12 for \( d = r \), 
  \item \( \omega_\rho \) for \( 0 \leq a \leq r - 1 \) is determined by \( \tilde{w}_{1,1}(y) \), 
  \item \( \Omega \) is admissible.
\end{enumerate}

**Proof.** \( \Longleftrightarrow \) Let \( c = \delta^{-1} \omega_1 \cdot \cdot \cdot \omega_r \). By Lemma 3.10 we have
\[(\tilde{w}_{1,1}(y) - \frac{y^2}{y^2 - 1} + \delta^{-1} \phi)(\tilde{w}_{1,1}(y) - \frac{1}{y^2 - 1} - \delta^{-1} \phi) = \frac{y^2}{1 - y^2} - c \left( \frac{y^2}{1 - y^2} - c \right). \]

For any positive integers \( r \) and all \( \rho \) in Proposition 3.12 for \( d = r \), we have
\[(\frac{y}{1 - y^2} - c) \left( \frac{y^2}{1 - y^2} - c \right) = \frac{y}{1 - y^2} - \delta^{-2}. \]

So,
\[(3.18) \quad (\tilde{w}_{1,1}(y) - \frac{y^2}{y^2 - 1} + \delta^{-1} \phi)(\tilde{w}_{1,1}(y) - \frac{1}{y^2 - 1} - \delta^{-1} \phi) = \frac{y^2}{1 - y^2} - \delta^{-2}. \]

Multiplying \( (1 - y^2)^2 \) on both sides of (4) yields an equality given by Laurent polynomials in indeterminate \( y \). Comparing the coefficients of \( y^1, i \leq 4 \) on both sides of such equality yields the second equality in (2.15). If \( \omega_\rho = \sum_{j=1}^{r} u_j^a \gamma_j \) where \( \gamma_j \)'s are given in Proposition 3.12 for \( d = r \), then
\[ \sum_{a=0}^{r} (-1)^{r-a} \sigma_{r-a}(u) \omega_{a+b} = \sum_{a=0}^{r} (-1)^{r-a} \sigma_{r-a}(u) \sum_{j=1}^{r} u_j^{a+b} \gamma_j \]
\[ = \sum_{j=1}^{r} \left( \sum_{a=0}^{r} (-1)^{r-a} \sigma_{r-a}(u) u_j^{a} \right) u_j^{b} \gamma_j \]
\[ = 0. \]

So, \( \Omega \) is admissible.

\( \Longleftarrow \) Conversely, if \( \Omega \) is admissible, then \( \omega_\rho \) for all \( a \in \mathbb{Z} \) are determined by \( \omega_0, \omega_1, \ldots, \omega_{r-1} \), uniquely. This implies the result. \( \square \)

4. The seminormal representations of \( B_{r,n}(u) \)

In this section, unless otherwise stated, we always keep the following assumptions:

**Assumption 4.1.** \( (1) \) \( R \) is a field which contains non zero elements \( q \), and \( u_i, 1 \leq i \leq r \) with \( o(q^2) > 2n \). \( (2) \) The parameters \( (u_1, u_2, \ldots, u_r) \) are generic in the sense of Definition 4.5. \( (3) \) The root conditions (4.18) hold. \( (4) \) \( \Omega \cup \{ \rho \} \) is \( u \)-admissible.

The main purpose of this section is to construct the seminormal representations for \( B_{r,n} \) over \( R \). We remark that the method we are using is the same as that for cyclotomic Nazarov-Wenzl algebras in [3]. We start by recalling some combinatorics.
A partition of \( m \) is a sequence of non-negative integers \( \lambda = (\lambda_1, \lambda_2, \ldots) \) such that \( \lambda_i \geq \lambda_{i+1} \) for all positive integers \( i \) and \( |\lambda| := \lambda_1 + \lambda_2 + \cdots = m \). Similarly, an \( r \)-partition of \( m \) is an ordered \( r \)-tuple \( \lambda = (\lambda^{(1)}, \ldots, \lambda^{(r)}) \) of partitions \( \lambda^{(s)} \) such that \( |\lambda| := |\lambda^{(1)}| + \cdots + |\lambda^{(r)}| = m \). Let \( \Lambda^+_{r}(m) \) be the set of all \( r \)-partitions of \( m \).

If \( \lambda \) and \( \mu \) are two \( r \)-partitions we say that \( \mu \) is obtained from \( \lambda \) by adding a box if there exists a pair \( (i, s) \) such that \( \mu_j^{(s)} = \lambda_j^{(s)} + 1 \) and \( \mu_{j+1}^{(s)} = \lambda_{j+1}^{(s)} \) for \( (j, t) \neq (i, s) \). In this situation we will also say that \( \lambda \) is obtained from \( \mu \) by removing a box and we write \( \lambda \subset \mu \) and \( \lambda = (s, i, \lambda_i^{(s)} + 1) \). We will also say that the triple \( (s, i, \lambda_i^{(s)} + 1) \) is an addable (resp. removable) node of \( \lambda \) (resp. \( \mu \)) which is in the \( i \)-th row, \( \lambda_n^{(s)} + 1 \)-column of \( s \)-th component of \( \lambda \) (resp. \( \mu \)).

Fix an integer \( m \) with \( 0 \leq m \leq \left\lfloor \frac{n}{2} \right\rfloor \). Let \( \lambda \in \Lambda^+_{r}(n-2m) \). It has been defined in [3] that an \( n \)-updown \( \lambda \)-tableau, or more simply an updown \( \lambda \)-tableau, is a sequence \( t = (t_0, t_1, t_2, \ldots, t_m) \) of \( r \)-partitions where \( t_0 = \lambda \) and the \( r \)-partition \( t_i \) is obtained from \( t_{i-1} \) by either adding or removing a box, for \( i = 1, \ldots, n \). When \( i = 0 \), we always assume that \( t_0 = \emptyset \). Let \( \mathcal{T}_{n}^{ud}(\lambda) \) be the set of \( n \)-updown \( \lambda \)-tableaux.

There is an equivalence relation \( \sim \) on \( \mathcal{T}_{n}^{ud}(\lambda) \), which has been defined in [3]. Suppose \( s, t \in \mathcal{T}_{n}^{ud}(\lambda) \). Then \( t \sim s \) if \( t_j = s_j \) whenever \( 1 \leq j \leq n \) and \( j \neq k \), for \( s, t \in \mathcal{T}_{n}^{ud}(\lambda) \). The following result has been proved in [3].

**Lemma 4.2.** Suppose \( s \in \mathcal{T}_{n}^{ud}(\lambda) \) with \( s_{k-1} = s_{k+1} \). Then there is a bijection between the set of all addable and removable nodes of \( s_{k-1} \) and the set of \( t \in \mathcal{T}_{n}^{ud}(\lambda) \) with \( t \sim s \).

Suppose that \( t \in \mathcal{T}_{n}^{ud}(\lambda) \) where \( \lambda \in \Lambda^+_{r}(n-2m) \) and \( 0 \leq f \leq \left\lfloor \frac{m}{2} \right\rfloor \). For each positive integer \( k \leq n \), either \( t_k \subset t_{k-1} \) or \( t_{k-1} \subset t_k \). We define

\[
\begin{align*}
  c_t(k) &= \begin{cases} 
    u_s q^{2(j-i)}, & \text{if } t_k \setminus t_{k-1} = (i, j, s), \\
    u_s^{-1} q^{-2(j-i)}, & \text{if } t_{k-1} \setminus t_k = (i, j, s).
  \end{cases}
\end{align*}
\]

We call \( c_t(k) \) the content of \( k \) in \( t \). Let \( \alpha = (i, j, s) \). We also define

\[
\begin{align*}
  c_\alpha(\alpha) &= \begin{cases} 
    u_s q^{2(j-i)}, & \text{if } \alpha \text{ is an addable node of } \lambda, \\
    u_s^{-1} q^{-2(j-i)}, & \text{if } \alpha \text{ is a removable node of } \lambda.
  \end{cases}
\end{align*}
\]

We write \( c(\alpha) \) instead of \( c_\alpha(\alpha) \) if there is no confusion.

The following condition, which is a counterpart of the generic condition for cyclotomic Nazarov-Wenzl algebras in [3], guarantees the existence of the seminormal representations for \( B_{r,n} \).

**Definition 4.5.** The parameters \( u = (u_1, \ldots, u_r) \) are generic for \( B_{r,n} \) if whenever there exists \( d \in \mathbb{Z} \) such that either \( u_i u_j^{\pm 1} = q^{2d} 1_R \) and \( i \neq j \), or \( u_i = \pm q^d \cdot 1_R \) then \( |d| \geq 2n \).

**Lemma 4.6.** Suppose that the parameters \( u \) are generic for \( B_{r,n} \) and that \( o(q^2) > 2n \). Let \( s, t \in \mathcal{T}_{n}^{ud}(\lambda) \) where \( \lambda \in \Lambda^+_{r}(n-2m) \) and \( 0 \leq f \leq \left\lfloor \frac{m}{2} \right\rfloor \). Then

a) \( s = t \) if and only if \( c_s(k) = c_t(k) \), for all positive integers \( k \leq n \).

b) \( c_s(k) \neq c_t(k+1) \), for all positive integers \( k < n \).

c) if \( s_{k-1} = s_{k+1} \) then \( c_s(k) \neq c_t(k)^{\pm 1} \), whenever \( t \sim s \) and \( t \neq s \),

d) \( c_s(k) \neq \pm q^k \), for all positive integers \( k \leq n \).

**Proof.** (a)-(c) can be proved by the arguments similar to those in [3]. The key point is that the assumptions imply that the contents of the addable and removable nodes in \( \lambda \) are distinct so an up-down \( \lambda \)-tableau \( s \) is uniquely determined by the sequence of contents \( c_s(k) \), for \( k = 1, \ldots, n \). (d) can be verified by direct computation. \( \square \)
Unless otherwise stated, we fix a \( \lambda \in \Lambda_0^+(n - 2f) \). Motivated by Ariki, Mathas and Rui’s work on cyclotomic Nazarov-Wenzl algebras in [3], we introduce the following rational functions in an indeterminate \( y \). Such functions will play a key role in the construction of seminormal representations of \( \mathcal{B}_{r,n} \).

**Definition 4.7.** Suppose that \( s \in \mathcal{F}_n^{wd}(\lambda) \). For \( 1 \leq k \leq n \), define rational functions \( W_k(y,s) \) by

\[
W_k(y,s) = \frac{y^2}{y^2 - 1} - \delta^{-1}q + (\delta^{-1}q) \prod\limits_{i=1}^{r} u_i + \frac{y^r(y)}{y^2 - 1} \prod\limits_{i=1}^{r} u_i \prod\limits_{\alpha} \frac{y - c^{-1}(\alpha)}{y - c(\alpha)},
\]

where \( \alpha \) runs over the addable and removable nodes of the \( r \)-partition \( s_{k-1} \).

**Lemma 4.8.** Suppose \( \lambda \) is an \( r \)-partition. Then

\[
\prod\limits_{\alpha} c(\alpha) = \prod\limits_{i=1}^{r} u_i
\]

where \( \alpha \) runs over all addable nodes and removable nodes of \( \lambda \).

**Proof.** It is known that the number of addable nodes of a partition, say \( \mu \), is equal to the number of the removable nodes of \( \mu \) plus 1. We arrange the removable nodes (resp. addable nodes) of \( \mu \) from top to bottom. Therefore, we assume that \((p, r_i, s_i), 1 \leq i \leq k \) (resp. \((p, a_i, b_i), 1 \leq i \leq k+1 \)) are all removable (resp. addable) nodes of \( \lambda^{(p)} \), the \( p \)-th component of \( \lambda \). We have \( a_1 = 1 \), \( b_{k+1} = 1 \), \( a_i = r_{i-1} + 1 \) and \( s_j = b_j - 1 \) for \( 2 \leq i \leq k+1 \) and \( 1 \leq j \leq k \). By (4.4),

\[
\prod\limits_{\alpha} c(\alpha) = u_p q^{2(\sum_{i=1}^{k+1}(b_i - a_i) + \sum_{i=1}^{k}(r_i - s_i))} = u_p.
\]

Multiplying the previous equality for all positive integers \( p \leq r \) yields (4.9). \( \square \)

**Lemma 4.10.** Suppose that \( u \) is generic and \( o(q^2) > 2n \). Let \( s \in \mathcal{F}_n^{wd}(\lambda) \) and \( 1 \leq k \leq n \). Then

\[
\frac{W_k(y,s)}{y} = \sum\limits_{\alpha} \left( \frac{\text{Res}_{y=c(\alpha)} W_k(y,s)}{y} \right) \cdot \frac{1}{y - c(\alpha)},
\]

where \( \alpha \) runs over the addable and removable nodes of \( s_{k-1} \).

**Proof.** Since \( u \) is generic and \( o(q^2) > 2n \), \( c(\alpha) \) are pairwise distinct for different addable and removable nodes \( \alpha \) of \( \lambda \). Further, we have \( c(\alpha) \not\in \{0, \pm 1\} \). Therefore, we can write

\[
\frac{W_k(y,s)}{y} = a + \frac{b}{y} + \frac{c}{y - 1} + \frac{d}{y + 1} + \sum\limits_{\alpha} \left( \frac{\text{Res}_{y=c(\alpha)} W_k(y,s)}{y} \right) \cdot \frac{1}{y - c(\alpha)},
\]

for some \( a, b, c, d \in R \), where \( \alpha \) runs over the addable and removable nodes of \( s_{k-1} \). In order to prove the result, we need verify \( a = b = c = d = 0 \). In fact, \( a = \lim_{y \to \infty} \frac{W_k(y,s)}{y} = 0 \). We have \( b = \text{Res}_{y=0} \frac{W_k(y,s)}{y} = -\delta^{-1}q + \delta^{-1} q \prod_{i=1}^{r} u_i c(\alpha)^{-2} \equiv 4.9 \equiv -\delta^{-1} q + \delta^{-1} q = 0 \). One can verify \( c = d = 0 \) similarly. \( \square \)

The following definition is the same as those for cyclotomic Nazarov-Wenzl algebras if we use our rational functions \( W_k(y,s) \) instead of those for cyclotomic Nazarov-Wenzl algebras in [3].
Definition 4.11. Let $\lambda \in \Lambda^+(n-2f)$ for some non-negative integer $f \leq \lfloor \frac{n}{2} \rfloor$. Assume that $k$ is a positive integer with $k \leq n$. If $s, t \in \mathcal{T}_n^{ad}(\lambda)$ with $s_{k-1} = s_{k+1}$, then we define the scalars $E_{st}(k) \in R$ by

$$E_{st}(k) = \begin{cases} \text{Res}_{y=c_k(k)} \frac{W_k(y,s)}{y}, & \text{if } s = t, \\ \sqrt{E_{ss}(k)} \sqrt{E_{tt}(k)}, & \text{if } s \neq t \text{ and } t \sim s, \\ 0, & \text{otherwise.} \end{cases}$$

We remark that we have to fix the choice of square roots $\sqrt{E_{ss}(k)}$, for $s \in \mathcal{T}_n^{ad}(\lambda)$ and $1 \leq k \leq n$, which we will illustrate later.

In [3], there is no definition for $E_{ss}(k)$ under the assumption $s_{k-1} \neq s_{k+1}$. In the current paper, we do not need such a definition either.

If $r$ is odd, then $\rho^{-1} \in \{u_1 \cdots u_r, -u_1 \cdots u_r\}$. It follows from Definition 4.17 that

$$E_{ss}(k) = \frac{1}{q\alpha c_k(k)} \left( \frac{c_k(k) - c_\alpha(k)}{c_k(k) - c_\alpha(k)} \right)^{-1} + \frac{1}{(q)^2 - c_k(k)} \prod_{\alpha} \frac{c_k(k) - c_\alpha(k)}{c_k(k) - c_\alpha(k)} \prod_{\alpha} c_k(k) - c_\alpha(k) \prod_{\alpha} c_k(k) - c_\alpha(k)$$

where $\alpha$ runs over all addable and removable nodes of $s_{k-1}$ with $\alpha \neq s_k \setminus s_{k-1}$.

If $r$ is even, then $\rho^{-1} \in \{q^{-1} u_1 \cdots u_r, -qu_1 \cdots u_r\}$. So,

$$E_{ss}(k) = \frac{1}{q\alpha c_k(k)} \left( \frac{c_k(k) - c_\alpha(k)}{c_k(k) - c_\alpha(k)} \right)^{-1} + \frac{1}{(q)^2 - c_k(k)} \prod_{\alpha} \frac{c_k(k) - c_\alpha(k)}{c_k(k) - c_\alpha(k)} \prod_{\alpha} c_k(k) - c_\alpha(k) \prod_{\alpha} c_k(k) - c_\alpha(k)$$

where $\alpha$ runs over all addable and removable nodes of $s_{k-1}$ with $\alpha \neq s_k \setminus s_{k-1}$.

It follows from Lemma 4.10 and (4.12) - (4.13) that

$$E_{st}(k) \neq 0, \text{ if } s \not\sim t.$$ 

Rewriting Lemma 4.10 yields the following equality:

$$\frac{W_k(y,s)}{y} = \sum_{t \sim s} E_{tt}(k)$$

Given two partitions $s$ and $t$ write $s \oplus t = \alpha$ if either $t \subset s$ and $s \setminus t = \alpha$, or $s \subset t$ and $t \setminus s = \alpha$.

Let $\mathfrak{S}_n$ be the symmetric group in $n$ letters. As an Coxeter group, $\mathfrak{S}_n$ is generated by $s_i := (i, i + 1)$ subject to the relations

$$s_i^2 = 1, \quad s_is_j = s_js_i \quad \text{if } |i-j| > 1$$

Let $s \in \mathcal{T}_n^{ad}(\lambda)$ with $s_{k-1} \neq s_{k+1}$, for some $k$, $1 \leq k < n$. Suppose that $s_k \cap s_{k-1}$ and $s_{k+1} \cap s_k$ are in different rows and in different columns. It is defined in [3] that

$$s_k s = (s_1, \ldots, s_{k-1}, t_k, s_{k+1}, \ldots, s_n) \in \mathcal{T}_n^{ad}(\lambda)$$

where $t_k$ is the $r$-partition which is uniquely determined by the conditions $t_k \ominus s_{k+1} = s_{k-1} \ominus s_k$ and $s_{k-1} \ominus t_k = s_k \ominus s_{k+1}$. If the nodes $s_k \ominus s_{k-1}$ and $s_{k+1} \ominus s_k$ are either in the same row, or in the same column, then $s_k s$ is not defined.

Definition 4.16. Let $s \in \mathcal{T}_n^{ad}(\lambda)$ with $s_{k-1} \neq s_{k+1}$, for some $k$, $1 \leq k < n$. We define

$$a_s(k) = \frac{\delta c_s(k + 1)}{c_s(k + 1) - c_s(k)} \quad \text{and} \quad b_s(k) = \sqrt{1 - a_s(k)^2 + \delta a_s(k)}.$$
We will fix the choice of square root for \( b_a(k) \) in Assumption 4.18. Since \( u \) is generic, by Lemma 4.10(b), \( c_a(k + 1) - c_a(k) \neq 0 \). So, the formula for \( a_a(k) \) makes sense.

As in [3], we do not define \( a_a(k) \) and \( b_a(k) \) when \( s_{k-1} = s_{k+1} \). The following result can be verified easily.

**Lemma 4.17.** Suppose that \( s \in \mathcal{F}_n^{ud}(\lambda) \) and \( 1 \leq k < n \). Then:

a) If \( s_k s \) is defined then \( a_a(k) = c_a(k+1) + c_a(k+1) = c_{s_k s}(k) \); consequently, \( a_{s_k s}(k) = \delta - a_a(k) \).

b) If \( s_k s \) is not defined then \( a_a(k) \in \{ q, q^{-1} \} \) and \( b_a(k) = 0 \).

Finally, if \( s_{k-1} = s_{k+1} \) and \( t \sim s \), where \( 1 \leq k < n \), we set

\[
T_{st}(k) = \frac{\delta}{c_a(k)c_t(k) - 1}.
\]

Note that \( c_a(k)c_t(k) \neq 1 \) by Lemma 4.16.

We will assume that we have chosen the square roots in the definitions of \( b_a(k) \) and \( E_{st}(k) \) so that the following equalities hold.

**Assumption 4.18 (Root conditions).** We assume that the ring \( R \) is large enough so that \( \sqrt{E_{ss}(k)} \in R \) and \( b_a(k) = \sqrt{1 - a_a(k)^2} + \delta a_a(k) \in R \), for all \( s, t \in \mathcal{F}_n^{ud}(\lambda) \) and \( 1 \leq k < n \), and that the following equalities hold:

a) If \( s_{k-1} \neq s_{k+1} \) and \( s_k s \) is defined then \( b_{s_k s}(k) = b_a(k) \).

b) If \( s_{k-1} \neq s_{k+1} \) and \( s \sim t \), where \( |k - l| > 1 \), then \( b_a(k) = b_a(k) \).

c) If \( s_{k-1} \neq s_{k+1} \), \( s_k \neq s_{k+2} \) and \( s_k s \) and \( s_{k+1} s \) are both defined then \( b_{s_{k+1} s}(k) = b_{s_k s}(k+1) \).

d) If \( s_{k-1} = s_{k+1} \) and \( s_k = s_{k+2} \) then \( \sqrt{E_{ss}(k)}\sqrt{E_{ss}(k+1)} = 1 \).

\( e) \) If \( s_{k-1} = s_{k+1} \), \( t_{k-1} = t_{k+1} \) and \( E_{st}(k) = E_{tt}(k) \) then \( \sqrt{E_{ss}(k)} = \sqrt{E_{tt}(k)} \).

f) If \( s_{k-1} = s_{k+1} \), \( s_k = s_{k+2} \) and \( t \sim s \), \( u \sim s \) with \( s_k t \) and \( s_{k+1} t \) both defined and \( s_k t = s_{k+1} u \) then \( b_t(k)\sqrt{E_{tt}(k+1)} = b_u(k+1)\sqrt{E_{uu}(k)} \).

The following is the main result of this section.

**Theorem 4.19.** Let \( \mathcal{B}_{r,n} \) be over a field \( R \) such that the Assumption 4.1 holds. Let \( \Delta(\lambda) \) be the \( R \)-vector space with basis \( \{ v_s \mid s \in \mathcal{F}_n^{ud}(\lambda) \} \). Then \( \Delta(\lambda) \) becomes a \( \mathcal{B}_{r,n} \)-module via

\[
T_k v_s = \begin{cases} \sum_{t \not\sim s} T_{st}(k)v_t, & \text{if } s_{k-1} = s_{k+1} \\ a_a(k)v_s + b_a(k)v_{s_k s}, & \text{if } s_{k-1} \neq s_{k+1} \end{cases}
\]

\[
E_k v_s = \begin{cases} \sum_{t \not\sim s} E_{st}(k)v_t, & \text{if } s_{k-1} = s_{k+1} \\ 0, & \text{if } s_{k-1} \neq s_{k+1} \end{cases}
\]

\[
X_i v_s = c_a(i)v_s,
\]

for \( 1 \leq k < n \) and \( 1 \leq i \leq n \). When \( s_k s \) is not defined, we set \( v_{s_k s} = 0 \).

**Definition 4.20.** We call \( \Delta(\lambda) \) the seminormal representation of \( \mathcal{B}_{r,n}(u) \) with respect to \( \lambda \) for \( \lambda \in \Lambda_+^+/(n - 2f) \), and \( 0 \leq f \leq \left\lfloor \frac{n}{2} \right\rfloor \).

In the remainder of this section, we will prove Theorem 4.19. The rational functions \( W_k(y, s) \) play the key role. As in [3], we will work with formal (infinite) linear combinations of elements of \( \Delta(\lambda) \) and \( \mathcal{B}_{r,n} \). Let \( Z(A) \) be the center of an algebra \( A \).
Lemma 4.21. Suppose that $R$ is a commutative ring which contains 1 and invertible elements $q, \delta, u_1, \ldots, u_r$ such that $\Omega \cup \{q\}$ is $u$-admissible. Given two integers $k \geq 1$ and $a \geq 0$. Then there is a $\omega_k^{(a)} \in Z(\mathcal{B}_{r,k-1}) \cap R[X_1^{\pm 1}, \ldots, X_{k-1}^{\pm 1}]$ such that

$$E_k X_k^a E_k = \omega_k^{(a)} E_k.$$ 

Moreover, the generating series $\tilde{W}_k(y) = \sum_{a=0}^{\infty} \omega_k^{(a)} y^{-a}$ satisfies

$$\frac{\tilde{W}_{k+1}(y) + \delta^{-1} q - \frac{y^2}{y^2 - 1}}{W_k(y) + \delta^{-1} q - \frac{y^2}{y^2 - 1}} = \frac{(y - X_k)^2}{(y - X_k^{-1})^2} \cdot \frac{y - q^{-2} X_k^{-1}}{y - q^{-2} X_k} \cdot \frac{y - q^2 X_k^{\pm 1}}{y - q^2 X_k}.$$ 

Proof. We prove the result by induction on $k$. When $k = 1$, the result follows from Definition 2.1(g). In this case, $W_1(y) = \tilde{w}_{1,+}(y)$. Suppose that we have already proved the result for all positive integers which are less than $k + 1$. In order to prove the result for $k + 1$, we start with the identity

$$T_k \frac{1}{y - X_k} = \frac{1}{y - X_{k+1}} T_k + \delta \frac{1}{y - X_k - 1} E_k \frac{1}{y - X_k} - \frac{\delta X_{k+1}}{(y - X_k)(y - X_{k+1})}.$$ 

We multiply $(y - X_{k+1})(yX_k - 1)$ (resp. $(y - X_k)$) on the left (resp. right) of (4.23). Then we use Definition 2.1(f),(j) to get the identity. Multiplying (4.23) on the left by $E_k$ and replacing $y$ by $y^{-1}$ yields

$$E_k \frac{1}{y - X_k} T_k = E_k \frac{\delta X_k}{yX_k - 1} - \frac{\delta}{(y - X_k)(yX_k - 1)} + \delta \frac{\tilde{W}_k(y)}{y} E_k \frac{1}{yX_k - 1}.$$ 

Multiplying $T_k$ on the right of (4.23) and using (4.23)–(4.24), we have

$$T_k \frac{1}{y - X_k} = \frac{1}{y - X_{k+1}} + \delta T_k \frac{1}{y - X_k} + \frac{\delta^2 X_{k+1}}{(y - X_k)(y - X_{k+1})}$$
$$\quad - \frac{\delta^2}{yX_k - 1} E_k \frac{1}{y - X_k - 1} E_k - \frac{\delta^2 X_{k+1}}{(y - X_k)^2 (y - X_{k+1})}$$
$$\quad + \frac{X_k(y - X_k)(yX_k - 1)}{(y - X_k)^2} \frac{1}{y - X_k} - \frac{\delta}{y - X_k} T_k \frac{X_k}{y - X_k}$$
$$\quad - \frac{\delta^2 X_{k+1}}{yX_k - 1} E_k \frac{1}{(y - X_k)(yX_k - 1)} + \frac{\delta^2 \tilde{W}_k(y)}{y (yX_k - 1)} E_k \frac{1}{yX_k - 1}.$$ 

Note that

$$\frac{\delta^2 X_{k+1}}{(y - X_k)(y - X_{k+1})} - \frac{\delta^2 X_{k+1}}{(y - X_k)^2 (y - X_{k+1})} = \frac{\delta^2 X_k}{(y - X_k)^2} - \frac{\delta^2 X_k}{y - X_{k+1}}.$$
By our induction assumption,

\[ E_{k+1}T_k \frac{1}{y-X_k} T_k E_{k+1} = \left( \frac{1}{y} - \frac{\delta^2 X_k}{(y-X_k)^2} \right) E_{k+1} \frac{y}{y-X_{k+1}} E_{k+1} + \frac{\delta^2 \omega_0 X_k}{(y-X_k)^2} - \frac{\delta^2}{(yX_k - 1)(y-X_k)} \]

\[ + \frac{\varphi^{-1} \delta}{y-X_k} + \frac{\delta^2 \omega_0 X_k}{(y-X_k)^2} - \frac{\delta^2}{(yX_k - 1)(y-X_k)} \]

\[- \frac{\varphi \delta X_k}{yX_k - 1} + \frac{X_k(y-X_k)^2 + \delta^2}{(yX_k - 1)^2} \]

\[= \left( \frac{1}{y} - \frac{\delta^2 X_k}{(y-X_k)^2} \right) E_{k+1} \frac{y}{y-X_{k+1}} E_{k+1} + \frac{\delta^2 \omega_0 X_k}{(y-X_k)^2} - \frac{\delta^2}{(yX_k - 1)(y-X_k)} \]

\[ + \frac{\varphi^{-1} \delta}{y-X_k} + \frac{\delta^2 \omega_0 X_k}{(y-X_k)^2} - \frac{\delta^2}{(yX_k - 1)(y-X_k)} \]

\[ + \frac{\delta^2 \tilde{W}_k(y)}{y(yX_k - 1)^2} E_{k+1} \]

On the other hand,

\[ E_{k+1}T_k \frac{1}{y-X_k} T_k E_{k+1} = E_{k+1} E_k \frac{1}{y-X_k} T_k E_{k+1} \]

\[= E_{k+1} E_k \frac{1}{y-X_k} E_k E_{k+1} - \delta E_{k+1} E_k \frac{1}{y-X_k} T_k E_{k+1} + \delta \varphi^{-1} E_{k+1} \frac{1}{y-X_k} \]

We use \([4.24]\) to compute the second term in the right hand side of the above equality. Thus,

\[ E_{k+1}T_k \frac{1}{y-X_k} T_k E_{k+1} = \left( \frac{\tilde{W}_k(y)}{y} + \frac{\varphi^{-1} \delta}{y-X_k} - \frac{\varphi \delta X_k}{yX_k - 1} \right) + \frac{\delta^2}{(yX_k - 1)^2} \]

Comparing the first and third expressions of \( E_{k+1}T_k \frac{1}{y-X_k} T_k E_{k+1} \) yields

\[ \left( \frac{1}{y} - \frac{\delta^2 X_k}{(y-X_k)^2} \right) E_{k+1} \frac{y}{y-X_{k+1}} E_{k+1} + \frac{\delta^2 \omega_0 X_k}{(y-X_k)^2} - \frac{\delta^2}{(yX_k - 1)(y-X_k)} \]

\[ + \frac{\varphi^{-1} \delta}{y-X_k} + \frac{\delta^2 \omega_0 X_k}{(y-X_k)^2} - \frac{\delta^2}{(yX_k - 1)(y-X_k)} \]

\[ = \left( \frac{\tilde{W}_k(y)}{y} + \frac{\varphi^{-1} \delta}{y-X_k} - \frac{\varphi \delta X_k}{yX_k - 1} + \frac{\delta^2}{(yX_k - 1)(y-X_k)} - \frac{\delta^2 \tilde{W}_k(y)}{y(yX_k - 1)} \right) E_{k+1} \]
Thus, 
\[
\left(1 - \frac{\delta^2X_k}{(y - X_k)^2}\right)E_{k+1} = \frac{y}{y - X_k}E_{k+1}
\]
\[
= \left(1 - \frac{\delta^2X_k}{y - X_k} + \frac{\delta^2\omega_kX_k}{(y - X_k)^2} + \frac{\delta\omega_kX_k}{y - X_k} - \frac{\delta^2}{(y - X_k)^2}\right)E_{k+1}
\]
\[
+ \frac{\delta^2}{(yX_k - 1)(y - X_k)} + \frac{\delta^2X_ky}{y - X_k} - \frac{\delta^2}{(yX_k - 1)^2}E_{k+1}
\]
\[
= \frac{1}{y} - \frac{\delta^2X_k}{(yX_k - 1)^2}W_k(y)E_{k+1} + \frac{\delta^2X_k}{y - X_k} - \frac{\delta^2}{(yX_k - 1)^2}E_{k+1}
\]

Multiplying \((yX_k - 1)^2(y - X_k)^2\) on both sides of above equation yields

\[
(yX_k - 1)^2 \left(\frac{(y - X_k)^2}{y} - \delta^2X_k\right)E_{k+1} = \frac{y}{y - X_k}E_{k+1}
\]
\[
= (y - X_k)^2 \left(\frac{(yX_k - 1)^2}{y} - \delta^2X_k\right)W_k(y)E_{k+1} + \frac{\delta^2X_k(y - X_k - 1)^2}{(yX_k - 1)^2}E_{k+1}
\]

So,

\[
(y - X_k)^2((y - X_k)^2 - \delta^2X_ky)E_{k+1} = \frac{y}{y - X_k}E_{k+1}
\]
\[
= (y - X_k)^2((y - X_k - 1)^2 - \delta^2X_k(y - X_k - 1)^2)W_k(y)E_{k+1} + \frac{\delta^2X_k(y - X_k - 1)^2}{(yX_k - 1)^2}E_{k+1}
\]

Therefore,

\[
(y - X_k)^2(y - q^{-2}X_k)(y - q^2X_k)E_{k+1} = \frac{y}{y - X_k}E_{k+1}
\]
\[
= (y - X_k)^2(y - q^{-2}X_k - 1)(y - q^2X_k)W_k(y)E_{k+1}
\]
\[
- \left(\delta^{-1} \frac{y^2}{y^2 - 1}\right)(y - X_k - 1)^2(y - q^{-2}X_k)(y - q^2X_k)
\]
\[
- (y - X_k)^2(y - q^{-2}X_k - 1)(y - q^2X_k)
\]

Thus,

\[
E_{k+1} = \frac{y}{y - X_k}E_{k+1} = \widetilde{W}_{k+1}(y)E_{k+1},
\]

where \(\widetilde{W}_{k+1}(y)\) satisfies the recursive relation (4.22).

Note that \(X_kT_j = T_jX_k\) and \(X_kE_j = E_jX_k\) for all positive integers \(j \leq k - 2\). By induction assumption and (4.22), \(\omega^{(a)}_{k+1}\) commutes with \(E_1, \ldots, E_{k-2}\) and \(T_1, \ldots, T_{k-2}\). In order to verify

(4.25) \(\omega^{(a)}_{k+1}Y = Y\omega^{(a)}_{k+1}\) for \(Y \in \{T_{k-1}, E_{k-1}\}\),
we use the following series

\[ \sum_{m \geq 0} a_m z^m = \frac{(1 + X_{k-1}z)(1 + X_kz)}{(1 + X_{k-1}^{-1}z)(1 + X_k^{-1}z)} \]

where \( z \in \{-y^{-1}, -q^{\pm 2}y\} \),

\[
\begin{align*}
a_0 &= 1, a_1 = X_{k-1} + X_k - X_{k-1}^{-1} - X_k^{-1}, \\
a_2 &= X_kX_kX_{k-1}^{-1}X_{k-1}^{-1} - (X_k^{-1} + X_{k-1}^{-1})(X_k + X_{k-1}^{-1} - X_k^{-1} - X_{k-1}^{-1}) \\
a_m &= -(X_{k-1}^{-1} + X_k^{-1})a_{m-1} - X_{k-1}^{-1}X_k^{-1}a_{m-2}, \text{ for } m \geq 3.
\end{align*}
\]

By induction, we have \( E_{k-1}a_j = a_j E_{k-1} = 0 \) for all non-negative integers \( j \).

We verify \( T_{k-1}a_m = a_m T_{k-1} \) for all \( m \geq 0 \) by induction on \( m \). There is nothing to be proved when \( m = 0 \) since \( a_0 = 1 \). By Definition 2.1(1),

\[
\begin{align*}
T_{k-1}(X_{k-1} + X_k) &= (X_{k-1} + X_k)T_{k-1} + \delta(X_k E_{k-1} - E_{k-1} X_k) \\
T_{k-1}(X_{k-1}^{-1} + X_k^{-1}) &= (X_{k-1}^{-1} + X_k^{-1})T_{k-1} + \delta(X_{k-1} E_{k-1} - E_{k-1} X_k^{-1})
\end{align*}
\]

So, \( T_{k-1}a_1 = a_1 T_{k-1} \). When \( m \geq 2 \), since \( T_{k-1} \) commute with \( X_kX_{k-1}^{-1} \) and \( X_k^{-1}X_{k-1}^{-1} \), we have, by induction assumption that

\[
\begin{align*}
T_{k-1}a_m - a_m T_{k-1} &= (X_{k-1}^{-1} + X_k^{-1})a_{m-1}T_{k-1} - T_{k-1}(X_{k-1}^{-1} + X_k^{-1})a_{m-1} \\
&= (X_{k-1}^{-1} + X_k^{-1})a_{m-1}T_{k-1} - (X_{k-1}^{-1} + X_k^{-1})T_{k-1}a_{m-1} \\
&\quad - \delta(X_{k-1}^{-1} E_{k-1} - E_{k-1} X_k^{-1})a_{m-1} = 0.
\end{align*}
\]

Using (4.22) twice, we have

\[
\begin{align*}
\tilde{W}_{k+1}(y) + \delta^{-1} y - \frac{y^2}{y - 1} &= (y - X_k)^2, \\
\tilde{W}_{k+1}(y) + \delta^{-1} y - \frac{y^2}{y - 1} &= \frac{(y - X_k)^2}{(y - X_k^{-1})^2}, \\
\tilde{W}_{l+1}(y) + \delta^{-1} y - \frac{y^2}{y - 1} &= \frac{(y - X_k^{-1})^2}{(y - X_k)^2}.
\end{align*}
\]

Thus, \( T_{k-1}, E_{k-1} \) commute with the right hand side of (4.26). By induction assumption, \( T_{k-1}, E_{k-1} \) commutes with \( \tilde{W}_{k-1}(y) \), it has to commute with \( \tilde{W}_{k+1}(y) \).

Thus, \( \omega_k^{(a)} \) commutes with \( T_{k-1} \) and \( E_{k-1} \).

Now, we prove \( \omega_k^{(a)} \in R[X_{k}^{\pm 1}, \ldots, X_{k-1}^{\pm 1}] \). Let \( g_k(X_k) = (y - X_k^{-1})^2(y - q^{-2}X_k)(y - q^2X_k) \). By (4.22),

\[ \tilde{W}_{k+1}(y)g_k(X_k) = \tilde{W}_{k}(y)g_k(X_k^{-1}) + (X_k - X_k^{-1})\delta^{-1}y((\delta^{-1} \rho - 1)y^2 - \delta^{-1} \rho). \]

Comparing the coefficients of \( y^j \) for \( j \leq 4 \) on both sides of the last equation, we have the following results:

\begin{itemize}
\item[a)] \( j = 4 \): \( \omega_{k+1}^{(0)} = \omega_k^{(0)} \). By induction assumption, \( \omega_k^{(0)} = \omega_0 \).
\item[b)] \( j = 3 \): \( \omega_{k+1}^{(1)} = \omega_k^{(1)} + \delta \rho^{-1}(X_k - X_k^{-1}) \).
\item[c)] \( j = 2 \):
\[ \omega_{k+1}^{(2)} = \omega_k^{(2)}(2X_k + (q^2 + q^{-2})X_k^{-1}) + \omega_k^{(1)}(2X_k^{-1} + (q^2 + q^{-2})X_k) \]
\item[d)] \( j = 1 \):
\[ \omega_{k+1}^{(3)} = \omega_k^{(3)} + (\omega_{k+1}^{(1)} - \omega_k^{(1)})(X_k^2 + X_k^{-2} + 2(q^2 + q^{-2})) \]
\[ = (\omega_{k+1}^{(0)} - \omega_{k+1}^{(0)})(2X_k^{-1} + (q^2 + q^{-2})X_k) - \delta \rho(X_k - X_k^{-1}) \]
\[ + (\omega_k^{(0)} - \omega_k^{(0)})(2X_k + (q^2 + q^{-2})X_k^{-1}) \).
\end{itemize}
Suppose that $\omega_k^{(j)} + \omega_k^{(-j)} + (\omega_k^{(j+2)} - \omega_k^{(-j+2)})(X_k^2 + X_k^{-2} + 2(q^2 + q^{-2}))$
\begin{align*}
&= \omega_k^{(j)} + \omega_k^{(-j)} + (\omega_k^{(j+1)} - \omega_k^{(-j+1)})(2X_k + (q^2 + q^{-2})X_k^{-1}) \\
&\quad + (\omega_k^{(j+3)} - \omega_k^{(-j+1)})(2X_k^{-1} + (q^2 + q^{-2})X_k).
\end{align*}

By induction assumptions on $k$ and $a$ together with the formulae in (a)-(e), we have that $\omega_k^{(a)} \in R[X_1^{\pm 1}, \ldots, X_k^{\pm 1}]$.

**Corollary 4.27.** Suppose that $R$ is a commutative ring which contains 1 and invertible elements $q, \delta, u_1, \ldots, u_r$ such that $\Omega \cup \{\varrho\}$ is $u$-admissible. Given a positive integer $k \leq n - 1$. If $a \in \mathbb{Z}$, then
a) $E_k X_k T_{k-1} E_k = \sum_{i=a}^a f_i X_{k-1} E_k$;
b) $E_k X_k T_{k+1} E_k = \sum_{i=a}^a g_i X_{k+2} E_k$,
where $f_i, g_i \in Z(\mathbb{R}_{r,K}) \cap R[X_1^{\pm 1}, \ldots, X_k^{\pm 1}]$ for $-a \leq i \leq a$.

**Proof.** Both (a) an (b) follow from Lemma 2.31(2)(4)(5) and Lemma 4.21 and Definition 2.1.

By Lemma 4.21, we have
\begin{equation}
(4.28) \quad \frac{\tilde{W}_k(y) + \delta^{-1}q - \frac{y^2}{y^2 - 1}}{W_1(y) + \delta^{-1}q - \frac{y^2}{y^2 - 1}} = \prod_{i=1}^{k-1} \frac{(y - X_i)^2}{(y - X_i^{-1})^2}, \quad \frac{y - q^2X_i^{-1}}{y - q^2X_i}, \quad \frac{y - q^2X_i}{y - q^2X_i^{-1}}.
\end{equation}

Since $\omega_k^{(a)} \in R[X_1^{\pm 1}, \ldots, X_k^{\pm 1}]$, we can define $\tilde{W}_k(y, s) \in R((y^{-1}))$ by
\[\tilde{W}_k(y) v_s = \tilde{W}_k(y, s) v_s.\]

**Proposition 4.29.** Given an $s \in \mathcal{F}_n(\lambda)$ and a positive integer $k \leq n$, we have $W_k(y, s) = \tilde{W}_k(y, s)$.

**Proof.** As $\Omega \cup \{\varrho\}$ is $u$-admissible, by Lemma 3.16 and 4.28, we have
\[
\tilde{W}_k(y) + \delta^{-1}q - \frac{y^2}{y^2 - 1} = A \cdot \prod_{\ell=1}^{r} \frac{(y - u_{\ell}^{(1)})}{(y - u_{\ell})} \times \prod_{i=1}^{k-1} \frac{(y - c_{\ell}(i))^{2}}{(y - c_{\ell}(i))^{2}}, \quad \frac{y - q^2c_{\ell}(i)^{-1}}{y - q^2c_{\ell}(i)}, \quad \frac{y - q^2c_{\ell}(i)}{y - q^2c_{\ell}(i)},
\]

where
\[A = (\varrho \delta^{-1} \prod_{\ell=1}^{r} u_{\ell} + \frac{y \gamma r(y)}{y^2 - 1}) \prod_{\ell=1}^{r} u_{\ell}.
\]

By arguments similar to those for [3, 4.17] we can verify that
\[
\tilde{W}_k(y) + \delta^{-1}q - \frac{y^2}{y^2 - 1} = A \prod_{\alpha} \frac{y - c(\alpha)^{-1}}{y - c(\alpha)},
\]
where $\alpha$ runs over the admissible and removable nodes of $s_{k-1}$. This proves $W_k(y, s) = \tilde{W}_k(y, s)$.

One can verify the following result by similar arguments to those for [3, 4.18].

**Corollary 4.30.** Suppose that $s \in \mathcal{F}_n(\lambda)$ and that $1 \leq k < n$ and $a \geq 0$. Then $E_k X_k E_k v_s = \omega_k^{(a)} E_k v_s$.

**Lemma 4.31.** Suppose that $s \in \mathcal{F}_n(\lambda)$ with $s_{k-1} = s_{k+1}$ and $s_k = s_{k+2}$. Then $E_{ss}(k) E_{ss}(k+1) = 1$. 

Proof. By (4.22) and Proposition 4.29,  
\[ W_{k+1}(y, s) + \delta^{-1} q - \frac{y^2}{y^2 - 1} = (W_k(y, s) + \delta^{-1} q - \frac{y^2}{y^2 - 1}) \times \frac{(y - c_s(k))^2}{(y - c_s(k) - 1)^2} \left( \frac{(y - q^{-2}c_s(k) - 1)}{(y - q^{-2}c_s(k))} \right), \]
where \( W_k(y, s) \) is given by Definition 4.7. Note that \( c_s(k)c_s(k + 1) = 1 \) and 
\[ E_{as}(k + 1) = \text{Res}_{y = c_s(k+1)} \frac{W_{k+1}(y, s)}{y} = \frac{\text{Res}_{y = c_s(k)} W_{k+1}(y, s)}{y}. \]
There are four cases we need to discuss:

**Case 1.** \( 2 \mid r \) and \( q^{-1} = u_1u_2 \ldots u_r \):

\[ E_{as}(k + 1) = (\delta^{-1} c_s(k) + \frac{c_s(k)^2}{1 - c_s(k)^2}) q^{-1} \prod_{t \neq s, t \neq s} \frac{c_s(k) - c_t(k)}{c_s(k) - c_t(k)^{-1}} \]
\[ \left( c_s(k) - c_s(k) \right) \frac{c_s(k) - q^{-2}c_s(k) - 1 \left( q^{-2}c_s(k) - 1 \right) - q^{-2}c_s(k) - 1 \left( q^{-2}c_s(k) - 1 \right)}{c_s(k) - 1 \left( q^{-2}c_s(k) - 1 \right) - q^{-2}c_s(k) - 1 \left( q^{-2}c_s(k) - 1 \right)} \]
\[ \frac{\rho \delta c_s(k)^2}{c_s(k)^2 - \delta c_s(k) - 1} \prod_{t \neq s, t \neq s} \frac{c_s(k) - c_t(k)}{c_s(k) - c_t(k)^{-1}} = \frac{1}{E_{as}(k)}. \]

**Case 2.** \( 2 \mid r \) and \( q^{-1} = -\prod_{i=1}^r u_i \):

\[ E_{as}(k + 1) = (\delta^{-1} c_s(k) + \frac{c_s(k)^2}{1 - c_s(k)^2}) \left( q^{-1} \right) \prod_{t \neq s, t \neq s} \frac{c_s(k) - c_t(k)}{c_s(k) - c_t(k)^{-1}} \]
\[ \left( c_s(k) - c_s(k) \right) \frac{c_s(k) - q^{-2}c_s(k) - 1 \left( q^{-2}c_s(k) - 1 \right) - q^{-2}c_s(k) - 1 \left( q^{-2}c_s(k) - 1 \right)}{c_s(k) - 1 \left( q^{-2}c_s(k) - 1 \right) - q^{-2}c_s(k) - 1 \left( q^{-2}c_s(k) - 1 \right)} \]
\[ \frac{\rho \delta c_s(k)^2}{c_s(k)^2 - \delta c_s(k) - 1} \prod_{t \neq s, t \neq s} \frac{c_s(k) - c_t(k)}{c_s(k) - c_t(k)^{-1}} = \frac{1}{E_{as}(k)}. \]

**Case 3.** \( 2 \mid r \) and \( q^{-1} = q^{-1} \prod_{i=1}^r u_i \):

\[ E_{as}(k + 1) = (q \delta^{-1} c_s(k) - \frac{c_s(k)^2}{1 - c_s(k)^2}) \left( q \delta^{-1} \right) \prod_{t \neq s, t \neq s} \frac{c_s(k) - c_t(k)}{c_s(k) - c_t(k)^{-1}} \]
\[ \left( c_s(k) - c_s(k) \right) \frac{c_s(k) - q^{-2}c_s(k) - 1 \left( q^{-2}c_s(k) - 1 \right) - q^{-2}c_s(k) - 1 \left( q^{-2}c_s(k) - 1 \right)}{c_s(k) - 1 \left( q^{-2}c_s(k) - 1 \right) - q^{-2}c_s(k) - 1 \left( q^{-2}c_s(k) - 1 \right)} \]
\[ \frac{\rho \delta c_s(k)^2}{c_s(k)^2 - q^2} \prod_{t \neq s, t \neq s} \frac{c_s(k) - c_t(k)}{c_s(k) - c_t(k)^{-1}} = \frac{1}{E_{as}(k)}. \]

**Case 4.** \( 2 \mid r \) and \( q^{-1} = -q \prod_{i=1}^r u_i \):
\[ E_{ss}(k+1) = \left( -q^{-1} \delta^{-1} c_s(k) - \frac{c_s(k)}{1 - c_s(k)^2} \right) \left( -q^{-1} \delta^{-1} \right) \prod_{t \neq \delta, t \neq s} \frac{c_s(k - 1) - c_i(k)}{c_s(k) - c_i(k)} \]

\[ \left( c_s(k - 1) - c_s(k) \right) \frac{c_s(k - 1) - q^{-2} c_s(k - 1)}{c_s(k) - q^{-2} c_s(k)} \frac{c_s(k - 1) - q^2 c_s(k - 1)}{c_s(k) - q^2 c_s(k)} \]

\[ = \frac{\varphi c_s(k)^2}{c_s(k)^2 - q^{-2}} \prod_{t \neq \delta, t \neq s} \frac{c_s(k) - c_i(k)}{c_s(k) - c_i(k)^{-1}} = \frac{1}{E_{ss}(k)}. \]

We remark that we use Lemma 4.2, 4.9 and (4.12)–(4.13) when we verify the equalities in cases 1–4. \( \square \)

The following result can be proved by similar arguments to those for \([3, 4.20]\).

**Lemma 4.32.** Fix an integer \( k \) with \( 1 \leq k < n - 1 \) and suppose that \( s, t, u \in S_n^{\text{ad}}(\lambda) \) such that \( s_{k-1} = s_{k+1}, s_k = s_{k+2}, t^{k+1} = s, u^{k} \sim s \) and that \( s_k t \) and \( s_{k+1} u \) are both defined with \( s_k t = s_{k+1} u \). Then

\[ b_t(k)^2 E_{tt}(k+1) = b_u(k+1)^2 E_{uu}(k). \]

The following combinatorial identities will be used in the proof of Theorem 4.19.

**Proposition 4.33.** Suppose that \( s, t' \in S_n^{\text{ad}}(\lambda) \) with \( s_{k-1} = s_{k+1}, s_k \neq s_{k+2}, t' \not\sim s \) and \( t' \neq s \), where \( 1 \leq k < n - 1 \). Then the following identities hold:

a) \( \sum_{t \neq s} \frac{E_{tt}(k)}{c_s(k) c_i(k) - 1} = \delta^{-1} q + \frac{1}{c_s(k)^2 - 1}, \)

b) \( \sum_{t \neq s} \frac{E_{tt}(k)}{(c_s(k) c_i(k) - 1)^2} = \frac{c_s(k)^2 + 1}{(c_s(k)^2 - 1)^2} - \delta^{-1} q + \frac{1}{c_s(k)^2 - 1} \frac{1}{E_{ss}(k)} \)

c) \( \sum_{t \neq s} \frac{E_{tt}(k)}{(c_s(k) c_i(k) - 1)(c_s(k) c_i(k) - 1)} = \frac{c_s(k) c_i(k) + 1}{(c_s(k)^2 - 1)(c_s(k)^2 - 1)} - \delta^{-1} q, \)

**Proof.** Evaluating both sides of (4.15) at \( y = c_s(k)^{-1} \) and using (4.20) gives (a). By Proposition 4.20 and Corollary 4.30 we have

\[ E_k \frac{1}{(v - X_k)(v - X_k)} E_k v_s = v - y \left( \frac{W_k(y, s)}{y} - \frac{W_k(v, s)}{v} \right) E_k v_s. \]

Comparing the coefficients of \( v_s \) on both sides of this equation yields

\[ \sum_{t \neq s} \frac{E_{tt}(k)}{(y - c_s(k))(v - c_s(k))} = \frac{1}{v - y} \left\{ \frac{W_k(y, s)}{y} - \frac{W_k(v, s)}{v} \right\}. \]

Let \( y = c_s(k)^{-1} \). We use (a) to rewrite the above equality and obtain the following equality:

\[ \sum_{t \neq s} \frac{E_{tt}(k)}{(1 - c_s(k) c_i(k))(v - c_i(k))} \]

\[ = \frac{W_k(v, s) + \delta^{-1} q - \frac{c_s(k)}{1 - c_s(k)}}{v(1 - c_i(k))} - \delta^{-1} q + \frac{v + c_s(k)}{(v^2 - 1)(1 - c_s(k)^2)} \]

Setting \( v = c_i(k)^{-1} \) gives (c). Now we set \( v = c_s(k)^{-1} \). There are four cases we need to discuss.
When $2 \nmid r$ and $\prod_{i=1}^{r} u_i = q^{-1}$, it follows from (4.34) that

$$\sum_{t \in \mathbb{Z}} \frac{E_{lt}(k)}{(1 - c_s(k)c_l(k))^2} \delta^{-1} q - \frac{1 + c_s(k)^2}{(1 - c_s(k))^2} = q^{-1} \left( \delta^{-1} c_s(k) + \frac{c_s(k)^2}{1 - c_s(k)^2} \right) \prod_{t \in \mathbb{Z}, t \neq s} \frac{c_l(k) - c_s(k)}{c_s(k) - c_l(k)} \frac{1 - c_s(k)}{c_s(k) - c_l(k)}$$

$$= \frac{c_s(k)^2 - \delta c_s(k) - 1}{\delta(c_s(k)^2 - 1)^2} \frac{1 - c_s(k)}{c_s(k) - c_l(k)} \prod_{t \in \mathbb{Z}, t \neq s} \frac{c_l(k) - c_s(k)}{c_s(k) - c_l(k)} \frac{1 - c_s(k)}{c_s(k) - c_l(k)}$$

$$= \frac{c_s(k)^2 - \delta c_s(k) - 1}{\delta(c_s(k)^2 - 1)^2} \frac{1 - c_s(k)}{c_s(k) - c_l(k)} \prod_{t \in \mathbb{Z}, t \neq s} \frac{c_l(k) - c_s(k)}{c_s(k) - c_l(k)} \frac{1 - c_s(k)}{c_s(k) - c_l(k)}$$

$$= (\delta^{-2} - \frac{c_s(k)^2}{(1 - c_s(k)^2)^2}) \frac{1}{E_{ss}(k)}$$

This proves (b) under the assumption $2 \nmid r$ and $\prod_{i=1}^{r} u_i = q^{-1}$. One can verify (b) in other cases.

We are going to check that the action of $\mathcal{B}_{r,n}(u)$ on $\Delta(\lambda)$ respects the defining relations of $\mathcal{B}_{r,n}(u)$.

**Lemma 4.35.** Suppose $s \in \mathcal{F}^{ud}(\lambda)$. Then

a) $E_{lt}^i v_s = \omega_0 E_{lt} v_s$, for $1 \leq i < n$.
b) $E_{lt}^i E_{kt} v_s = \omega_k E_{kt} v_s$, for $k \in \mathbb{Z}$.
c) $(X_i - u_1)(X_i - u_2) \cdots (X_i - u_t) v_s = 0$.
d) $X_i X_j v_s = X_j X_i v_s$ for $1 \leq i, j \leq n$.
e) $E_{lt}^i X_{i+1} v_s = X_{i+1} E_{lt}^i v_s = E_{lt}^i v_s$, $1 \leq i \leq n - 1$.
f) $(T_i X_i - X_i T_i + 1) v_s = \delta X_{i+1} (E_{i+1} - 1) v_s$, for $1 \leq i \leq n - 1$.
g) $(X_i T_i - (E_i - 1) X_{i+1} + 1)v_s = \delta (E_i - 1) X_{i+1} v_s$, for $1 \leq i \leq n - 1$.
h) $T_i X_i v_s = X_i T_{i+k} v_s$ if $i \nmid k, k + 1$.
i) $E_{lt}^i E_{kt} v_s = E_{kt} v_s$.
j) $E_{kt} T_i v_s = \delta E_{kt} v_s = T_i E_{kt} v_s$.
k) $T_i T_j v_s = T_j T_i v_s$ if $|i - j| > 1$.
l) $X_i X_i^{-1} = X_i^{-1} X_i = 1$ for $1 \leq i \leq n$.

**Proof.** We have already proved (a) and (b) for $k > 0$ in Corollary 4.30 (c)-(h) and (l) can be verified easily. By (c), we have (b) for all $k \in \mathbb{Z}$ with $k < 0$. (i)-(k) can be proved by arguments similar to those in [3, 4.234.2.25, 4.27a]. When we prove (j) we need to use Proposition 4.33 (a) instead of [3, 4.21a].

It remains to check the defining relations (b), (c)(ii), (h)(ii) in Definition 2.1

**Lemma 4.36.** Suppose that $s \in \mathcal{F}^{ud}(\lambda)$. Then $(T_i^2 - \delta T_i + \delta q E_k) v_s = v_s$.

**Proof.** We prove the result by computing the coefficient of $v_t$ in the expression of $(T_i^2 - \delta T_i + \delta q E_k) v_s$. There are two cases we have to discuss as follows.

**Case 1.** $s_{k-1} \neq s_{k+1}$.
Then $E_{lt} v_s = 0$. If $s_k s$ is not defined then $a_s(k) \in \{q, -q^{-1}\}$ and $b_s(k) = 0$ (see Lemma 4.17). So, $(T_i^2 - \delta T_i + \delta q E_k) v_s = v_s$. If $s_k s \in \mathcal{F}^{ud}(\lambda)$ then by the choice
of the square roots in (4.18) (a) we have

\[
(T_k^2 - \delta T_k + \delta \rho E_k)v_s = (T_k - \delta) \left( a_s(k)v_s + b_\delta(k)v_{s\delta} \right)
\]
\[
= a_s(k) \left( a_s(k)v_s + b_\delta(k)v_{s\delta} \right) - \delta \left( a_s(k)v_s + b_\delta(k)v_{s\delta} \right)
\]
\[
+ b_\delta(k) \left( a_{s\delta}(k)v_{s\delta} + b_{s\delta}(k)v_s \right)
\]
\[
v_s \quad \text{(by Lemma 4.37.)}
\]

**Case 2.** \(s_{k-1} = s_{k+1}^+\)

We have

\[
(T_k^2 - \delta T_k + \delta \rho E_k)v_s = \sum_{t \in \mathfrak{e}} \left( \sum_{v \in \mathfrak{e}} T_{sv}(k)T_{vt}(k) \right) v_t - \delta \sum_{t \in \mathfrak{e}} T_{st}(k) v_t + \rho \delta \sum_{t \in \mathfrak{e}} E_{st}(k) v_t.
\]

The coefficient of \(v_s\) in \((T_k^2 - \delta T_k + \delta \rho E_k)v_s\) is equal to 1 since

\[
\sum_{v \in \mathfrak{e}} T_{sv}(k)T_{sv}(k) - \delta T_{ss}(k) + \rho \delta E_{ss}(k)
\]
\[
= \sum_{v \in \mathfrak{e}} \frac{\delta^2 E_{ss}(k)E_{sv}(k)}{(c_s(k)c_v(k) - 1)^2} - \frac{\delta^2 E_{ss}(k)(1 - c_s(k)^2)(1 - c_s(k)^2)}{c_s(k)^2 - 1} + \rho \delta E_{ss}(k) + \frac{\delta^2(1 - 2E_{ss}(k))}{(c_s(k)^2 - 1)^2}
\]
\[
= \delta^2 E_{ss}(k) \left( \frac{1 + c_s(k)^2}{(1 - c_s(k)^2)^2} + \frac{1}{\delta^2 E_{ss}(k)} - \frac{c_s(k)^2}{(1 - c_s(k)^2)^2 E_{ss}(k)} \right)
\]
\[
+ \frac{\delta^2(1 - 2E_{ss}(k))}{(1 - c_s(k)^2)^2} + \frac{\delta^2(E_{ss}(k) - 1)}{1 - c_s(k)^2}
\]
\[
= 1. \quad \text{(by Proposition 4.33 (b))}
\]

If \(t \cong s\) and \(t \neq s\) then the coefficient of \(v_t\) in \((T_k^2 - \delta T_k + \delta \rho E_k)v_s\) is zero since

\[
\sum_{v \in \mathfrak{e}} T_{sv}(k)T_{st}(k) - \delta T_{st}(k) + \delta \rho E_{st}(k)
\]
\[
= \sum_{v \in \mathfrak{e}} \frac{\delta^2 E_{st}(k)E_{sv}(k)}{(c_s(k)c_v(k) - 1)(c_s(k)c_t(k) - 1)} + \frac{\delta(E_{ss}(k) - 1)}{c_s(k)^2 - 1} + \frac{\delta E_{st}(k)}{c_s(k)c_t(k) - 1}
\]
\[
+ \frac{\delta E_{st}(k)}{c_s(k)c_t(k) - 1} - \frac{\delta^2 E_{st}(k)}{c_s(k)c_t(k) - 1} + \rho \delta E_{st}(k)
\]
\[
= \delta^2 E_{st}(k) \left( \sum_{v \in \mathfrak{e}} \frac{E_{sv}(k)}{(c_s(k)c_v(k) - 1)(c_s(k)c_t(k) - 1)} - \frac{1}{(c_s(k)^2 - 1)(c_s(k)c_t(k) - 1)} \right)
\]
\[
= \delta^2 E_{st}(k) \left( \sum_{v \in \mathfrak{e}} \frac{E_{sv}(k)}{(c_s(k)c_v(k) - 1)(c_s(k)c_t(k) - 1)} - \frac{1}{(c_s(k)^2 - 1)(c_s(k)c_t(k) - 1)} + \rho \delta \right)
\]
\[
= 0. \quad \text{(by Proposition 4.33 (c)).}
\]

Therefore, \((T_k^2 - \delta T_k + \delta \rho E_k)v_s = v_s. \quad \square\)

**Proposition 4.37.** Suppose that \(s \in \mathfrak{T}^\text{ad}_{\nu}(\lambda)\). Then

a) \(E_{k+1}E_kv_s = T_kT_{k+1}E_kv_s.\)

b) \(E_{k+1}E_kv_s = E_{k+1}T_kT_{k+1}v_s.\)
Proof. (a) We assume that $s_{k-1} = s_{k+1}$ since otherwise $E_{k+1}v_a = T_k E_{k+1}v_a = 0$. Let $\tilde{s} \in \mathcal{F}^u_n(\lambda)$ be such that $s \sim \tilde{s}$ and $\tilde{s}_k = s_{k+2}$. Note that $E_{k}v_a = 0$ for any $s \in \mathcal{F}^u_n(\lambda)$ with $s_{k-1} \neq s_{k+1}$. So,

$$T_k E_{k+1}v_a = T_k E_{k+1}E_{k}v_a = \delta E_{k+1}E_{k}v_a + \delta E_{k}v_a$$

$$= (T_k - \delta) E_{k+1}E_{\tilde{s},\tilde{a}}(k)v_{\tilde{a}} + \delta \sum_{t \notin \tilde{s}} E_{st}(k)v_t$$

$$= (T_k - \delta) E_{\tilde{s},\tilde{a}}(k) \sum_{t \notin \tilde{s}} E_{\tilde{s},\tilde{a}}(k + 1)v_{t} + \delta \sum_{t \notin \tilde{s}} E_{st}(k)v_t$$

$$= E_{\tilde{s},\tilde{a}}(k)E_{\tilde{s},\tilde{a}}(k + 1) \sum_{t \notin \tilde{s}} T_{\tilde{s},t}(k)v_t + \sum_{t \notin \tilde{s}} E_{st}(k)E_{\tilde{s},\tilde{a}}(k + 1) (a_t(k + 1)v_t + b_t(k)v_{s_{k+1}})$$

$$- \delta E_{\tilde{s},\tilde{a}}(k) \sum_{t \notin \tilde{s}} E_{\tilde{s},\tilde{a}}(k + 1)v_{t} + \delta \sum_{t \notin \tilde{s}} E_{st}(k)v_t.$$

If $s_k t$ is defined, for $t$ in the second sum, then $(s_k t)_k \neq s_{k+2}$ and $u = s_{k+1}s_k t$ is also defined. Further, we have $u \sim \tilde{s}$ and $u \neq \tilde{s}$. Similarly,

$$T_k E_{k}v_a = E_{\tilde{s},\tilde{a}}(k) \sum_{t \notin \tilde{s}} T_{\tilde{s},t}(k)v_t + \sum_{t \notin \tilde{s}} E_{st}(k) (a_t(k + 1)v_t + b_t(k)v_{s_{k+1}})$$

We are going to compare the coefficients of $v_t$ in both $T_k E_{k+1}E_{k}v_a - \delta E_{k+1}E_{k}v_a + \delta E_{k}v_a$ and $T_{k+1}E_{k}v_a$. Note that $E_{\tilde{s},\tilde{a}}(k)E_{\tilde{s},\tilde{a}}(k + 1) = 1$ by Lemma 4.3.

**Case 1.** $t = \tilde{s}$:

Since $c_{\tilde{s}}(k)c_{\tilde{s}}(k + 1) = 1$, the definitions and the remarks above show that the coefficient of $v_t$ in $T_k E_{k+1}E_{k}v_a - \delta E_{k+1}E_{k}v_a + \delta E_{k}v_a$ is equal to

$$E_{\tilde{s},\tilde{a}}(k)E_{\tilde{s},\tilde{a}}(k + 1)(T_{\tilde{s},t}(k) - \delta) + \delta E_{\tilde{s},\tilde{a}}(k)$$

$$= E_{\tilde{s},\tilde{a}}(k)\left(\delta E_{\tilde{s},\tilde{a}}(k + 1)E_{\tilde{s},\tilde{a}}(k) - 1\right) c_{\tilde{a}}(k) c_{\tilde{a}}(k + 1) + \delta E_{\tilde{s},\tilde{a}}(k + 1) + \delta$$

$$= E_{\tilde{s},\tilde{a}}(k)\left(\frac{\delta E_{\tilde{s},\tilde{a}}(k + 1) - 1}{c_{\tilde{a}}(k + 1)}c_{\tilde{a}}(k) - 1\right) - \delta E_{\tilde{s},\tilde{a}}(k + 1) + \delta$$

$$= E_{\tilde{s},\tilde{a}}(k)T_{\tilde{s},\tilde{a}}(k + 1)$$

which is the coefficient of $v_t$ in $T_{k+1}E_{k}v_a$.

**Case 2.** $t \sim \tilde{s}$ and $t \neq \tilde{s}$:

Now, $c_{\tilde{s}}(k) = c_{\tilde{s}}(k + 2)$ and $c_{\tilde{s}}(k + 1) = c_{\tilde{s}}(k - 1)$, so the coefficient of $v_t$ in $T_k E_{k+1}E_{k}v_a - \delta E_{k+1}E_{k}v_a + \delta E_{k}v_a$ is

$$E_{\tilde{s},\tilde{a}}(k)E_{\tilde{s},\tilde{a}}(k + 1) T_{\tilde{s},t}(k) + \delta E_{st}(k)$$

$$= E_{\tilde{s},\tilde{a}}(k)E_{\tilde{s},\tilde{a}}(k + 1) \frac{\delta E_{st}(k)}{c_{\tilde{a}}(k)c_{\tilde{s}}(k - 1)} + \delta E_{st}(k)$$

$$= \delta E_{st}(k) \frac{1}{1 - c_{\tilde{a}}(k)c_{\tilde{s}}(k - 1)c_{\tilde{s}}(k - 1)}$$

$$= \delta E_{st}(k) \frac{c_{\tilde{s}}(k + 2)}{c_{\tilde{s}}(k + 2) - c_{\tilde{s}}(k + 1)}$$

$$= E_{st}(k) c_{\tilde{s}}(k + 1).$$

which is the coefficient of $v_t$ in $T_{k+1}E_{k}v_a$.

**Case 3.** $t \sim \tilde{s}$ and $t \neq \tilde{s}$:

Since $c_{\tilde{s}}(k)c_{\tilde{s}}(k + 1) = 1$, the coefficient of $v_t$ in $T_k E_{k+1}E_{k}v_a - \delta E_{k+1}E_{k}v_a + \delta E_{k}v_a$ is
is
\[(a_t(k) - \delta)E_{ss}(k + 1)E_{ss}(k)\]
\[= \left(\frac{\delta c_t(k + 1)}{c_t(k + 1) - c_t(k)} - \delta\right)E_{st}(k + 1)E_{ss}(k)\]
\[= E_{ss}(k)\frac{\delta E_{st}(k + 1)}{c_t(k + 1)c_s(k + 1) - 1}\]
\[= E_{ss}(k)E_{st}(k + 1)\]

which is the coefficient of \(v_t\) in \(T_{k+1}E_kv_s\).

Now suppose that \(s_k t\) is defined and let \(u = s_{k+1}s_k t\) be as above. Then the coefficient of \(v_{s_k t}\) in \(T_kE_{k+1}E_kv_s - \delta E_{k+1}E_kv_s + \delta E_kv_s\) is
\[E_{ss}(k)E_{st}(k + 1)b_t(k) = \sqrt{E_{ss}(k)}\sqrt{E_{ss}(k + 1)}b_t(k)\]
\[= \sqrt{E_{ss}(k)}\sqrt{E_{ss}(k + 1)}b_u(k + 1)\]
\[= E_{ss}(k)b_u(k + 1),\]

where the second equality comes from (4.18)(f). As \(s_k t = s_{k+1}u\) this is the coefficient of \(v_{s_k t}\) in \(T_{k+1}E_kv_s\).

In summary, we have proved that \(T_kE_{k+1}E_k - \delta E_{k+1}E_k + \delta E_kv_s = T_{k+1}E_kv_s\).

By Lemma (4.30) and Lemma (4.35)),
\[E_{k+1}E_kv_s = (T_k^2 - \delta T_k + \delta T_{E_k})E_{k+1}E_kv_s\]
\[= T_k(T_kE_{k+1}E_kv_s - \delta E_{k+1}E_kv_s + \delta E_kv_s)\]
\[= T_kT_{k+1}E_kv_s,\]

and (a) follows.

In order to prove (b), we need to consider four cases as follows.

**Case 1.** \(s_k = s_{k+2}\) and \(s_{k-1} = s_{k+1}\):

We have
\[E_{k+1}E_kT_{k+1}v_s - \delta E_{k+1}E_kv_s + \delta E_{k+1}v_s\]
\[= E_{k+1}E_kT_{ss}(k + 1)v_s - \delta E_{k+1}E_kv_s + \delta E_{k+1}v_s\]
\[= (T_{ss}(k + 1) - \delta)E_{k+1}E_{ss}(k)v_s + \delta E_{k+1}v_s\]
\[= ((T_{ss}(k + 1) - \delta)E_{ss}(k) + \delta)E_{k+1}v_s\]
\[= \frac{\delta c_s(k + 1)^2}{c_s(k + 1)^2 - 1}(1 - E_{ss}(k))E_{k+1}v_s\]
\[= \frac{\delta(E_{ss}(k) - 1)}{c_s(k + 1)^2 - 1}E_{k+1}v_s\]
\[= T_{ss}(k)E_{k+1}v_s = E_{k+1}T_kv_s.\]

**Case 2.** \(s_k \neq s_{k+2}\) and \(s_{k-1} = s_{k+1}\):

Define \(\tilde{s} \in \mathcal{T}_n^{ud}(\lambda)\) to be the unique updown tableau such that \(\tilde{s} \sim s\) and \(\tilde{s}_k = s_{k+2}\).
Then \( \tilde{s} \neq s \) and
\[
E_{k+1}E_kT_{k+1}v_s - \delta E_{k+1}E_kv_s + \delta E_{k+1}v_s
= E_{k+1}E_k(a_s(k+1)v_s + b_s(k+1)v_{s_{k+1}}) - \delta E_{k+1}E_kv_s
= (a_s(k+1) - \delta)E_{k+1}E_{\tilde{s}}(k)v_{\tilde{s}}
= \frac{\delta E_{\tilde{s}}(k)c_s(k+1)}{c_s(k)(c_s(k+1) - 1)}E_{k+1}v_{\tilde{s}}
= T_{\tilde{s}}(k)E_{k+1}v_{\tilde{s}} = E_{k+1}T_kv_{\tilde{s}}
\]

where the last second equality uses the facts that \( c_s(k+1)c_s(k) = 1 \), \( c_s(k+2) = c_s(k) \) and \( (s_{k+1}\tilde{s})_{k-1} \neq (s_{k+1}s)_{k+1} \).

**Case 3.** \( s_k = s_{k+2} \) and \( s_{k+1} \neq s_{k+1} \):
Let \( \tilde{s} \in \mathcal{G}_n^{ad}(\lambda) \) such that \( \tilde{s} \sim_{k+1} s \) and \( \tilde{s}_{k+1} = s_k \). Then
\[
E_{k+1}E_kT_{k+1}v_s - \delta E_{k+1}E_kv_s + \delta E_{k+1}v_s
= E_{k+1}E_kT_{\tilde{s}}(k+1)v_{\tilde{s}} + \delta E_{k+1}v_s
= T_{\tilde{s}}(k+1)E_{\tilde{s}}(k)E_{k+1}v_s + \delta E_{k+1}v_s
= T_{\tilde{s}}(k+1)E_{\tilde{s}}(k)\sum_{t \sim_{\tilde{s}} s} E_{\tilde{s}_t}(k+1)v_t + \delta \sum_{t \sim_{\tilde{s}} s} E_{\tilde{s}_t}(k+1)v_t
\]

and \( E_{k+1}T_kv_s = E_{k+1}(a_s(k)v_s + b_s(k)v_{s_{k+1}}) = a_s(k)\sum_{t \sim_{s} s} E_{s_t}(k+1)v_t \) since \( (s_k\tilde{s})_k \neq (s_k\tilde{s})_{k+2} \). However, since \( c_s(k+1) = c_s(k) \), we have
\[
T_{s\tilde{s}}(k+1)E_{s\tilde{s}}(k)E_{s\tilde{s}_t}(k+1) + \delta E_{s\tilde{s}_t}(k+1)
= \frac{\delta \sqrt{E_{s\tilde{s}}(k+1)\sqrt{E_{u\tilde{s}}(k+1)}}}{c_s(k+1)c_s(k) - 1} + \delta E_{s\tilde{s}_t}(k+1)
= \frac{\delta c_s(k+1)}{c_s(k+1) - c_s(k)}E_{s\tilde{s}_t}(k+1) = a_s(k)E_{s\tilde{s}_t}(k+1).
\]

So, \( E_{k+1}E_kT_{k+1}v_s - \delta E_{k+1}E_kv_s + \delta E_{k+1}v_s = E_{k+1}T_kv_s \).

**Case 4.** \( s_k \neq s_{k+2} \) and \( s_{k+1} \neq s_{k+1} \):
Under our assumptions, we have \( E_{k+1}E_kT_{k+1}v_s - \delta E_{k+1}E_kv_s + \delta E_{k+1}v_s = b_s(k+1)E_{k+1}E_{\tilde{s}}(k+1)v_{\tilde{s}} + b_s(k)E_{k+1}v_{s_{k+1}} \). If \( (s_k\tilde{s})_{k-1} \neq (s_k\tilde{s})_{k+1} \) then \( (s_k\tilde{s})_k \neq (s_k\tilde{s})_{k+2} \). So, \( E_{k+1}E_kT_{k+1}v_s - \delta E_{k+1}E_kv_s + \delta E_{k+1}v_s = 0 = E_{k+1}T_kv_s \).

Suppose now that \( (s_k\tilde{s})_{k-1} = (s_k\tilde{s})_{k+1} \) and let \( \tilde{s} \in \mathcal{G}_n^{ad}(\lambda) \) be the unique updown tableau such that \( \tilde{s} \sim_{k+1} s_k \) and \( \tilde{s}_k = s_{k+2} \). Set \( t = s_k\tilde{s} \) and \( u = s_{k+1}\tilde{s} \) and observe that the assumptions of (1.18) hold, so that \( b_t(k)\sqrt{E_{u\tilde{s}}(k+1)} = b_u(k+1)\sqrt{E_{s\tilde{s}}(k+1)} \). As \( b_t(k) = b_1(k) \) and \( b_u(k+1) = b_u(k+1) \). By (1.18)(d),

together with the fact that $t^k \sim s_k \bar{s}$, we have

$$E_{k+1}E_kv_{s_{k+1}v} - \delta E_{k+1}E_kv_v + \delta E_{k+1}v_s$$

$$= b_g(k + 1)E_{k+1} \sum_{\psi^{k+1}} E_{uv}(k)v_v$$

$$= b_g(k + 1)E_{uv}(k)v_v$$

$$= b_g(k + 1)E_{uv}(k)v_v$$

$$= b_g(k) \sum_{\psi^{k+1}} E_{uv}(k + 1)v_v$$

$$= b_g(k)E_{k+1}v_{sk} = E_{k+1}v_{sk}$$

In summary, we have proved that $E_{k+1}E_{k+1}v_{s_{k+1}v} - \delta E_{k+1}E_{k+1}v_v + \delta E_{k+1}v_s = E_{k+1}v_{sk}$ for any $s \in \mathcal{F}^u(\lambda)$. So, $E_{k+1}v_{sk} = (E_{k+1}E_{k+1} - \delta E_{k+1}E_{k+1} + \delta E_{k+1})(T_{k+1}v_s)$. Now, (b) follows from Lemma 4.36 and Lemma 4.35(i)(j), immediately.

**Lemma 4.38.** Suppose that $s \in \mathcal{F}^u(\lambda)$ with $s_{k-1} \neq s_{k+1}$ and $s_k \neq s_{k+2}$, where $1 \leq k < n - 1$. Then $T_kT_{k+1}v_s = T_{k+1}T_kT_{k+1}v_s$.

**Proof.** One can verify the result without difficulty if he uses the arguments in the proof of Lemma [3, 4.28]. We only give an example to illustrate it and leave the others to the reader.

Suppose that either $s_k$ is not defined, or $s_k$ is defined and $(s_k)_k \neq (s_k)_k$. In this case, the formulae for $T_kT_{k+1}v_s$ and $T_{k+1}T_kT_{k+1}v_s$ are exactly the same as those given in the proof of [3, 4.28] up to the definitions of $a_i(k), b_i(k)$ etc. We can verify $T_kT_{k+1}v_s = T_{k+1}T_kT_{k+1}v_s$ by comparing the coefficients of $v_s$ on both sides of the above equality. For example, we need to show

$$a_g(k)^2a_s(k + 1) + a_{sk}(k)(1 - a_s(k)^2 + \delta a_s(k))$$

$$= a_g(k)a_s(k + 1)^2 + a_{sk}(k)(1 - a_s(k + 1)^2 + \delta a_s(k + 1))$$

(4.39)

when we prove that the coefficients of $v_s$ in $T_kT_{k+1}v_s = T_{k+1}T_kT_{k+1}v_s$ are equal. The reader should compare the 4.39 with that given in line 4, in [3, p93]. In our case, $a_g(k) = \delta c(\beta)(c(\beta) - c(\alpha))^{-1}, a_s(k + 1) = \delta c(\gamma)(c(\gamma) - c(\beta))^{-1}, a_{sk}(k) = a_{sk}(k + 1) = \delta c(\gamma)(c(\gamma) - c(\alpha))^{-1}$ if we write $s_k \oplus s_{k+1} = \alpha, s_{k+1} \ominus s_k = \beta, s_{k+2} \ominus s_{k+1} = \gamma$. By direct computation, we can verify (4.39) easily.

**Lemma 4.40.** Suppose that $s \in \mathcal{F}^u(\lambda)$ and that either $s_{k-1} = s_{k+1}$ and $s_k \neq s_{k+2}$, or $s_{k-1} \neq s_{k+1}$ and $s_k = s_{k+2}$, for $1 \leq k < n - 1$. Then $T_kT_{k+1}v_s = T_{k+1}T_kT_{k+1}v_s$.

**Proof.** The result can be proved by arguments given in the proof of [3, 4.29]. Since it does not involve huge computation, we include a proof here.

**Case 1.** $s_{k+1} \bar{s}$ is defined:

Suppose first that $s_{k-1} = s_{k+1}$ and $s_k \neq s_{k+2}$. Then $t = s_{k+1} \bar{s} \in \mathcal{F}^u(\lambda)$ is well-defined. Furthermore, $t_k \neq t_{k+2}$ and $t_{k-1} \neq t_{k+1}$, so $T_kT_{k+1}v_{t_k} = T_{k+1}T_kT_{k+1}v_{t_k}$.
by Lemma 4.38. Now, \( T_{k+1}v_1 = a_t(k+1)v_1 + b_t(k+1)v_2 \) and \( b_t(k+1) \neq 0 \). Therefore
\[
T_k T_{k+1} T_{k} v_2 = \frac{1}{b_t(k+1)} T_k T_{k+1} T_{k} \left( T_{k+1}v_1 - a_t(k+1)v_1 \right) 
= \frac{1}{b_t(k+1)} \left( T_k (T_{k+1} T_{k} T_{k+1}) v_1 - a_t(k+1)(T_k T_{k+1} T_{k}) v_1 \right) 
= \frac{1}{b_t(k+1)} \left( T_k (T_{k+1} T_{k}) v_1 - a_t(k+1)(T_{k+1} T_{k+1}) v_1 \right) 
\]
by Lemma 4.38. Hence, using Lemma 4.36 twice,
\[
T_k T_{k+1} T_{k} v_2 = \frac{1}{b_t(k+1)} \left( 1 + \delta T_k - \delta E_k T_{k+1} T_k v_1 - a_t(k+1)(T_{k+1} T_{k+1}) v_1 \right) 
= \frac{1}{b_t(k+1)} \left( T_{k+1} T_k (1 + \delta T_k - \delta E_k + a_t(k+1)(T_{k+1} T_{k+1}) v_1) \right) 
= \frac{1}{b_t(k+1)} \left( T_{k+1} T_{k+1} T_k v_1 - a_t(k+1)v_1 \right) 
= (T_{k+1} T_{k+1} T_k) v_2 
\]
as required.

The case when \( s_{k-1} \neq s_{k+1} \) and \( s_k = s_{k+2} \) can be proved similarly.

**Case 2.** \( s_{k+1} \text{ is not defined:} \)

This is equivalent to saying that the two nodes \( s_{k+2} \ominus s_{k+1} \) and \( s_{k+1} \ominus s_k \) are either in the same row or in the same column. Therefore, either \( s_k \subset s_{k+1} \subset s_{k+2} \) or \( s_k \supset s_{k+1} \supset s_{k+2} \). Note that in either case \( s_{k-1} = s_{k+1} \), so we have
\[
E_k v_2 = \sum_{t \neq s} E_{st}(k) v_t + E_{sa}(k) v_s. 
\]
Using Lemma 4.37 and Lemma 4.35(j) twice, we have
\[
T_k T_{k+1} T_k E_k v_2 = \theta T_k T_{k+1} E_k v_2 = \theta E_{k+1} E_k v_2 = T_{k+1} E_{k+1} E_k v_2 = T_{k+1} E_{k+1} E_k v_2. 
\]

Suppose that \( t \sim s \) and \( t \neq s \). Since the two boxes \( s_{k+2} \ominus s_{k+1} \) and \( s_{k+1} \ominus t_k \) belong to different rows and columns, \( s_{k+1} \) is well-defined and \( t_k = t_{k+1} \). By Case 1, \( T_{k+1} T_k T_k v_1 = T_k T_{k+1} T_k v_1 \). Consequently, \( T_{k+1} T_k T_k + E_{ss}(k) v_2 = T_k T_{k+1} T_k E_{ss}(k) v_2 \). Canceling the non-zero factor \( E_{ss}(k) \) shows that \( T_k T_{k+1} T_k = T_{k+1} T_k T_k \).

**Proposition 4.41.** Suppose that \( 1 \leq k < n - 1 \) and \( s \in \mathcal{N}_n \). Then
\[
T_k T_{k+1} T_k = T_k T_{k+1} T_{k+1} v_2. 
\]

**Proof.** By Lemma 4.38 and Lemma 4.40 we need to consider the case when \( s_{k-1} = s_{k+1} \) and \( s_k = s_{k+2} \). By Proposition 4.37(a) and Proposition 4.35(j),
\[
T_{k+1} T_k T_k E_k v_2 = T_{k+1} E_{k+1} E_k v_2 = \theta E_{k+1} E_k v_2 = \theta T_k T_{k+1} E_k v_2 = T_k T_{k+1} T_k E_k v_2. 
\]
Therefore,
\[
(T_{k+1} T_k T_k - T_k T_{k+1} T_k) \left( E_{ss}(k) v_s + \sum_{t \neq s} E_{st}(k) v_t \right) = 0. 
\]
Now, if \( t \sim s \) and \( t \neq s \) then \( T_k T_{k+1} T_k v_1 = T_{k+1} T_k T_{k+1} v_1 \) by Lemma 4.40. Consequently, \( T_k T_{k+1} T_k v_2 = T_{k+1} T_k T_{k+1} v_2 \) since \( E_{ss}(k) \neq 0 \). This completes the proof.

**Proof of Theorem 4.49** We have already checked the defining relations for \( \mathcal{B}_{r,n} \) on \( \Delta(\lambda) \). So, \( \Delta(\lambda) \) is a \( \mathcal{B}_{r,n}(u) \)-module, as we wanted to show. □

The following result shows that we can chose \( u, q \in \mathbb{R} \) such that the root conditions can be satisfied in \( \mathbb{R} \).
Lemma 4.42. Suppose that $R = \mathbb{R}$. We choose $q, u_i \in \mathbb{R}^+$ in such a way that $|\log q^2 u_i| > \cdots > |\log q^2 u_r| \geq n$ and $|\log q^2 u_i| - |\log q^2 u_{i+1}| \geq 2n$, and one of the following conditions holds:

a) $q > 1$ if either $2 \nmid r$ and $q^{-1} = \prod_{i=1}^r u_i$ or $2 \mid r$ and $q^{-1} = q^{-1} \prod_{i=1}^r u_i$.

Furthermore, we assume that $log q^2 u_i < 0$ if $2 \mid i$ and $log q^2 u_i > 0$ if $2 \nmid i$.

b) $0 < q < 1$ if either $2 \nmid r$ and $q^{-1} = -\prod_{i=1}^r u_i$ or $2 \mid r$ and $q^{-1} = -q \prod_{i=1}^r u_i$.

Furthermore, we assume that $log q^2 u_i > 0$ if $2 \mid i$ and $log q^2 u_i < 0$ if $2 \nmid i$.

Suppose that $s \in \mathbb{F}_n^w(\lambda)$ and $1 \leq k < n$. Then $1 - a_s(k)^2 + \delta a_s(k) \geq 0$, if $s_{k-1} \neq s_{k+1}$, and $E_{ab}(k) > 0$, if $s_{k-1} = s_{k+1}$. In particular, the Root Condition (4.18) holds if we choose positive square roots $\sqrt{b_s(k)} \geq 0$ and $\sqrt{E_{ab}(k)} > 0$.

Proof. We start with the case $s_{k-1} \neq s_{k+1}$. Let $\alpha = s_k \oplus s_{k-1}$ and $\beta = s_{k+1} \ominus s_k$. Define $S = \{a \in \mathbb{R}^+ \mid |\log q^2 a| \geq 1\}$. By the definitions of $c(\alpha)$ and $c(\beta)$, $c(\beta)c(\alpha)^{-1} = u_i^\pm u_j^\pm q^{2(\pm k \pm l)}$ for some integers $i, j, k$ and $l$. We want to prove $c(\beta)c(\alpha)^{-1} \in S$. There are two cases we need to discuss:

Case 1: $u_i^\pm u_j^\pm = 1$.

In this case, $\alpha$ and $\beta$ are in the same component of $\lambda$. Also, both $\alpha$ and $\beta$ are either removable nodes or addable nodes of $\lambda$. By Lemma 4.10, $c(\beta)c(\alpha)^{-1} \neq 1$. Therefore, $c(\beta)c(\alpha)^{-1} = q^{2(\pm k \pm l)} \in S$.

Case 2: $u_i^\pm u_j^\pm \neq 1$.

We have

$$|\log q^2 (u_i^\pm u_j^\pm q^{2(\pm k \pm l)})| = |\log q^2 u_i \pm \log q^2 u_j \pm k \pm l| \geq |\log q^2 u_i \pm \log q^2 u_j| - |k \pm l| \geq 2n - |k \pm l| \geq 1.$$

Hence, $c(\beta)c(\alpha)^{-1} \in S$. So,

$$1 - a_s(k)^2 + \delta a_s(k) = \frac{(c(\beta) - q^{-2}c(\alpha))(c(\beta) - q^2c(\alpha))}{(c(\beta) - c(\alpha))^2} = \frac{(c(\beta) - q^{-2})(c(\beta) - q^2)}{(c(\beta) - c(\alpha))^2} > 0.$$

Now, we prove $E_{ab}(k) > 0$. Since we are assuming that $|\log q^2 u_i| - |\log q^2 u_{i+1}| \geq 2n$, $|\log q^2 u_i \pm \log q^2 u_j| \geq 2n$ if $t' \neq t$. Therefore, the signs of $\log q^2 u_i^\pm_{t'} q^{2(\pm k \pm d)}$ and $\pm \log q^2 u_i \pm \log q^2 u_{t'}$ are the same. In other words,

$$\frac{u_i^\pm u_j^\pm q^{2(\pm k \pm d)}}{u_i^\pm u_j^\pm - 1} > 0.$$

Similarly, we can verify

$$\frac{u_i^\pm q^{2(\pm k \pm l)}}{u_i^2 - 1} > 0.$$

Next we consider the case $s_{k-1} = s_{k+1}$. Let $\alpha = s_k \oplus s_{k-1}$ and $\lambda = s_{k-1}$. Write $\alpha = (i, j, t)$.

Let $u_i q^{2c_i}$, for $1 \leq i \leq l+1$, be the contents of the addable nodes of $\lambda^{(t)}$ and let $u_i^2 q^{-2d_i}$, for $1 \leq j \leq l$, be the contents of the removable nodes of $\lambda^{(t)}$. We may assume that

$$c_1 > d_1 > \cdots > c_l > d_l > c_{l+1}.$$

Let $\varepsilon_t$ be the sign of the product of $\frac{c(\alpha)c(\beta)}{c(\alpha) - c(\beta)}$, where $\beta$ runs over all of the addable and removable nodes of $\lambda^{(t)}$ such that $\beta \neq \alpha$. First we consider $\varepsilon_t$ where $t' \neq t$. 


By (4.43), \( \varepsilon_{t'} \) is equal to either the sign of
\[
\frac{u_{t}^{-1}u_{t'}^{-1} - 1}{u_{t}^{-1} - u_{t'}^{-1}} = \frac{u_{t'} - u_{t}}{1 - uu_{t'}}.
\]

or the sign of
\[
\frac{u_{t}u_{t'}^{-1} - 1}{u_{t} - u_{t'}^{-1}} = \frac{1 - uu_{t'}}{u_{t'} - u_{t}}.
\]

It is not difficult to see that the signs of the last two equations are the same. Suppose \( t' < t \) and \( q > 1 \). There are four cases we have to discuss.

- Both \( t' \) and \( t \) are odd. Then \( u_{t'} > u_{t} \) and \( uu_{t'} > 1 \).
- \( t' \) is odd and \( t \) is even. Then \( u_{t'} > u_{t} \) and \( uu_{t'} > 1 \).
- \( t' \) is even and \( t \) is odd. Then \( u_{t'} < u_{t} \) and \( uu_{t'} < 1 \).
- Both \( t' \) and \( t \) are even. Then \( u_{t'} < u_{t} \) and \( uu_{t'} < 1 \).

So, \( \varepsilon_{t'} < 0 \) if \( t' < t \). When \( t' > t \), we switch the role between \( t \) and \( t' \). So, \( \varepsilon_{t'} > 0 \). When \( 0 < q < 1 \), we use \( q^{-1} \) instead of the previous \( q \). Since \( |\log(q^{-1})u_{i}| = |\log_{q^{2}}u_{i}| \), we still have \( \varepsilon_{t'} < 0 \) if \( t' < t \) and \( \varepsilon_{t'} > 0 \) if \( t' > t \). Hence
\[
\prod_{t' \neq t} \varepsilon_{t'} = (-1)^{t'-1}.
\]

Suppose \( c(\alpha) = u_{t}q^{2c_{i}} \), for some \( i \). \( \varepsilon_{t} \) is equal to the sign of
\[
\prod_{k=1, k \neq i}^{l+1} q^{2(c_{i} - d_{k})} - 1 \prod_{k=1}^{l} q^{2(c_{i} - d_{k})} - 1.
\]

By (4.43), it is equal to the sign of
\[
\prod_{k=1, k \neq i}^{l+1} \frac{1}{q^{2c_{i} - d_{k}}} - 1 \prod_{k=1}^{l} \frac{1}{q^{2c_{i} - d_{k}}} - 1
\]
so \( \varepsilon_{t} = (-1)^{l'-1}(-1)^{l'-1} = 1 \) if \( q > 1 \) and \( \varepsilon_{t} = (-1)^{l-1+i}(-1)^{l-i+1} = 1 \) if \( 0 < q < 1 \).

If \( c(\alpha) = u_{t}^{-1}q^{-2d_{j}} \), for some \( j \), then \( \varepsilon_{t} \) is equal to the sign of
\[
\prod_{k=1, k \neq j}^{l} u_{t}^{-1}q^{-2d_{j} - d_{k}} - 1 \prod_{k=1}^{l+1} \frac{1}{u_{t}^{-1}q^{-2d_{j} - d_{k}}} - 1
\]
By (4.43), it is equal to the sign of
\[
\prod_{k=1, k \neq j}^{l} \frac{1}{q^{2(d_{j} - d_{k})}} - 1 \prod_{k=1}^{l+1} \frac{1}{q^{2(d_{j} - d_{k})}} - 1
\]
So \( \varepsilon_{t} = (-1)^{l-j}(-1)^{l+1-j} = -1 \) if \( q > 1 \) and \( \varepsilon_{t} = (-1)^{l-1}(-1)^{j} = -1 \) if \( 0 < q < 1 \).

In summary, we have proved
\[
\prod_{1 \leq t' \leq r} \varepsilon_{t'} = (-1)^{l'-1}, \quad \text{if } c(\alpha) = u_{t}q^{2c_{i}} \text{ for some } i,
\]
\[
\prod_{1 \leq t' \leq r} \varepsilon_{t'} = (-1)^{l}, \quad \text{if } c(\alpha) = u_{t}^{-1}q^{-2d_{j}} \text{ for some } j.
\]

We determine the sign of \( E_{sb}(k) \) as follows.
Case 1. $q > 1$, $r$ is odd and $q^{-1} = \prod_{i=1}^{r} u_i$:

$$E_{ss}(k) = \frac{1}{\phi(c)\delta}(c(\alpha) - c(\alpha)^{-1}) - 1 \prod_{\beta \neq \alpha} \frac{c(\alpha) - c(\beta)^{-1}}{c(\alpha) - c(\beta)}$$

$$= \frac{1}{\phi(c)\delta}(c(\alpha)^2 - q^{-1}(c(\alpha) + q) \prod_{\beta \neq \alpha} \frac{c(\alpha)c(\beta) - 1}{c(\alpha) - c(\beta)}$$

On the other hand, under our assumption, we have

- $c(\alpha) < q^{-1}$ if either $2 \mid t$ and $c(\alpha) = u_tq^{2r_i}$ or $2 \nmid t$ and $c(\alpha) = u_t^{-1}q^{-2d_i}$ for some $i$.
- $c(\alpha) > q^{-1}$ if either $2 \nmid t$ and $c(\alpha) = u_tq^{2r_i}$ or $2 \mid t$ and $c(\alpha) = u_t^{-1}q^{-2d_i}$ for some $i$.

Since we are assuming that $q > 1$, $\delta > 0$. By (4.45), $E_{ss}(k) > 0$ as required.

Case 2. $0 < q < 1$, $r$ is odd and $q^{-1} = -\prod_{i=1}^{r} u_i$:

$$E_{ss}(k) = \frac{1}{\phi(c)\delta}(c(\alpha) - c(\alpha)^{-1}) - 1 \prod_{\beta \neq \alpha} \frac{c(\alpha) - c(\beta)^{-1}}{c(\alpha) - c(\beta)}$$

$$= \frac{1}{\phi(c)\delta}(c(\alpha)^2 - q^{-1}(c(\alpha) + q) \prod_{\beta \neq \alpha} \frac{c(\alpha)c(\beta) - 1}{c(\alpha) - c(\beta)}$$

On the other hand, under our assumption, we have

- $c(\alpha) < q$ if either $2 \mid t$ and $c(\alpha) = u_tq^{2r_i}$ or $2 \nmid t$ and $c(\alpha) = u_t^{-1}q^{-2d_i}$ for some $i$.
- $c(\alpha) > q$ if either $2 \nmid t$ and $c(\alpha) = u_tq^{2r_i}$ or $2 \mid t$ and $c(\alpha) = u_t^{-1}q^{-2d_i}$ for some $i$.

Since we are assuming that $0 < q < 1$, $\delta < 0$. By (4.45), $E_{ss}(k) > 0$ as required.

Case 3. $q > 1$, $r$ is even and $q^{-1} = q^{-1}\prod_{i=1}^{r} u_i$:

$$E_{ss}(k) = \frac{1}{\phi(c)\delta}(1 - \frac{q^2}{c(\alpha)^2}) \prod_{\beta \neq \alpha} \frac{c(\alpha) - c(\beta)^{-1}}{c(\alpha) - c(\beta)}$$

$$= \frac{1}{\phi(c)\delta}(c(\alpha)^2 - q^2) \prod_{\beta \neq \alpha} \frac{c(\alpha)c(\beta) - 1}{c(\alpha) - c(\beta)}$$

On the other hand, under our assumption, we have

- $c(\alpha) < q$ if either $2 \mid t$ and $c(\alpha) = u_tq^{2r_i}$ or $2 \nmid t$ and $c(\alpha) = u_t^{-1}q^{-2d_i}$ for some $i$.
- $c(\alpha) > q$ if either $2 \nmid t$ and $c(\alpha) = u_tq^{2r_i}$ or $2 \mid t$ and $c(\alpha) = u_t^{-1}q^{-2d_i}$ for some $i$.

Since we are assuming that $q > 1$, $\delta > 0$. By (4.45), $E_{ss}(k) > 0$ as required.

Case 4. $0 < q < 1$, $r$ is even and $q^{-1} = -q\prod_{i=1}^{r} u_i$:
\[ E_{as}(k) = \frac{1}{\delta^a} \left( 1 - \frac{1}{q^2 c(\alpha)^2} \right) \prod_{\beta \neq \alpha} \frac{c(\alpha) - c(\beta)^{-1}}{c(\alpha) - c(\beta)} \]
\[ = \frac{1}{\delta^a c(\alpha)^2} \left( c(\alpha)^2 - q^{-2} \right) \prod_{i=1}^u \frac{c(\alpha) c(\beta) - 1}{c(\alpha) - c(\beta)} \]
\[ = \frac{-q}{\delta^a c(\alpha)} (c(\alpha) + q^{-1})(c(\alpha) - q^{-1}) \prod_{\beta \neq \alpha} \frac{c(\alpha) c(\beta) - 1}{c(\alpha) - c(\beta)} \]

On the other hand, under our assumption, we have
- \( c(\alpha) < q^{-1} \) if either \( 2 \mid t \) and \( c(\alpha) = u_i q^{2i} \) or \( 2 \nmid t \) and \( c(\alpha) = u_i^{-1} q^{-2d_i} \) for some \( i \),
- \( c(\alpha) > q^{-1} \) if either \( 2 \nmid t \) and \( c(\alpha) = u_i q^{2i} \) or \( 2 \mid t \) and \( c(\alpha) = u_i^{-1} q^{-2d_i} \) for some \( i \).

Since we are assuming that \( 0 < q < 1, \delta < 0 \). By (4.45), \( E_{as}(k) > 0 \) as required. \( \square \)

5. A Cellular Basis of \( \mathcal{B}_{r,n}(u) \) with Odd \( r \)

Throughout this section, unless otherwise stated, we always keep the following assumption:

**Assumption 5.1.** Let \( R \) be a commutative ring containing invertible elements \( q, q - q^{-1} \), and \( u_i, 1 \leq i \leq r \). We also assume that \( \Omega \cup \{ q \} \subseteq R \) is \( u \)-admissible.

The main purpose of this section is to construct a cellular basis for \( \mathcal{B}_{r,n} \). We remark that we assume that \( r \) is odd. In other words, \( r = 2n + 1 \) for some non-negative integer \( n \).

In [1], Ariki and Koike have proved that Ariki-Koike algebra \( \mathcal{H}_{r,n} \) is free over \( R \). Given a non-negative integer \( f \leq \frac{r}{2} \). Then \( \mathcal{H}_{r,n-2f} \) can be identified with the subalgebra of \( \mathcal{H}_{r,n} \). Since we have not proved that \( \mathcal{B}_{r,n} \) is free over \( R \), we could not say \( \mathcal{B}_{r,n} \) is a subalgebra of \( \mathcal{B}_{r,n} \) if \( r < n_2 \). However, there is an algebraic homomorphism from \( \mathcal{B}_{r,n} \) to \( \mathcal{B}_{r,n} \). So, we define \( \mathcal{B}'_{r,n} \) to be the image of \( \mathcal{B}_{r,n} \) in \( \mathcal{B}_{r,n} \).

**Proposition 5.2.** Given a positive integer \( n \geq 2 \). Let \( \mathcal{E}_n = \mathcal{B}_{r,n}E_1 \mathcal{B}_{r,n} \) be the twosided ideal of \( \mathcal{B}_{r,n} \) generated by \( E_1 \). Then there is a unique \( R \)-algebra isomorphism \( \varepsilon_n : \mathcal{H}_{r,n} \cong \mathcal{B}_{r,n}/\mathcal{E}_n \) such that

\[ \varepsilon_n(g_i) = T_i + \mathcal{E}_n \text{ and } \varepsilon_n(y_j) = X_j + \mathcal{E}_n, \]

for \( 1 \leq i < n \) and \( 1 \leq j \leq n \).

**Proof.** Let \( S = \{ X_j + \mathcal{E}_n, T_i + \mathcal{E}_n \mid 1 \leq i \leq n - 1, 1 \leq j \leq n \} \). By Definition 2.3, \( S \) generates \( \mathcal{B}_{r,n}/\mathcal{E}_n \). Therefore, \( \varepsilon_n \) is an algebraic epimorphism. We claim that \( \mathcal{B}_{r,n}/\mathcal{E}_n \) is free over \( R \) with rank \( r^n n! \). In fact, we consider \( \mathcal{B}_{r,n}/\mathcal{E}_n \) over \( R_0 := \mathbb{Z}[u^{\pm 1}, q^{\pm 1}, \delta^{\pm 1}] \) where \( \delta = q - q^{-1} \). Further, we assume that \( u, q \) are indeterminates. We have constructed the seminormal representations for \( \mathcal{B}_{r,n} \) with respect to all \( \lambda \in \Lambda^+_+(n - 2f), 0 \leq f \leq \frac{r}{2} \) under the conditions in Lemma 4.16 and 4.18. In particular, we have seminormal representations of \( \mathcal{B}_{r,n} \) over \( \mathbb{R} \). As \( \mathbb{R} \) is not finitely generated over \( \mathbb{Q} \), we can take \( r + 1 \) algebraically independent transcendental real numbers \( v_1 \in \mathbb{R} \) and \( q \). We define \( R_1 = \mathbb{Z}[v_1, v_2, \ldots, v_r, q^{\pm 1}, \delta^{\pm 1}] \). Also, we assume \( \Omega \cup \{ q \} \) is \( v \)-admissible. By Lemma 4.32, \( \Delta(\lambda) \) are \( \mathcal{B}_{r,n}/\mathcal{E}_n \)-modules for all \( \lambda \in \Lambda^+_+(1) \) over the field \( \mathbb{R} \). By Wedderburn-Artin theorem for semisimple finite dimensional algebra,

\[ \dim_{\mathbb{R}} \mathcal{B}_{r,n}/(E_1) \geq r^n n! \]
So, the image of an \( R \)-basis of \( \mathcal{H}_{r,n} \) has to be \( R \)-linear independent, and hence \( R_1 \)-linear independent. Therefore, \( \mathcal{B}_{r,n}/\mathcal{E}_n \) is free over \( R_1 \) with rank \( r^n n! \).

Note that \( R_1 \cong R_0 \) as rings. So, \( \mathcal{B}_{r,n}(u) \) over \( R_0 \) is isomorphic to \( \mathcal{B}_{r,n}(v) \) over \( R_1 \) as \( R_0 \)-modules. The corresponding isomorphism sends \( u_i \) (resp. \( q \)) to \( v_i \) (resp. \( q \)). So, \( \mathcal{B}_{r,n}/\mathcal{E}_n \) is free over \( R_0 \) with rank \( r^n n! \). By base change, it is free over an arbitrary commutative ring \( R \). So, \( \varepsilon_n \) is an isomorphism. \( \square \)

**Definition 5.3.** Given a non-negative integer \( f \leq \lfloor \frac{n}{2} \rfloor \). Let \( E^f = E_{n-1}E_{n-3} \cdots E_{n-2f+1} \) and let \( \mathcal{B}_{r,n} = \mathcal{B}_{r,n} E^f \mathcal{B}_{r,n} \). If \( f = \lfloor \frac{n}{2} \rfloor \in \mathbb{Z} \), then we set \( \mathcal{B}_{r,n}^{f+1} = 0 \).

By Definition 5.3 there is a filtration of two-sided ideals of \( \mathcal{B}_{r,n} \) as follows:

\[
\mathcal{B}_{r,n} = \mathcal{B}_{r,n}^0 \supset \mathcal{B}_{r,n}^1 \supset \cdots \supset \mathcal{B}_{r,n}^{\lfloor \frac{n}{2} \rfloor} \supset \mathcal{B}_{r,n}^{\lfloor \frac{n}{2} \rfloor+1} = 0.
\]

For \( 0 \leq f \leq \lfloor \frac{n}{2} \rfloor \) let \( \pi_{f,n} : \mathcal{B}_{r,n}^f \to \mathcal{B}_{r,n}^f/\mathcal{B}_{r,n}^{f+1} \) be the corresponding projection map of \( \mathcal{B}_{r,n} \)-bimodules.

Since we are assuming that \( r = 2p + 1 \) for some non-negative integer \( p \), we set \( \mathbb{N}_r = \{ -p, \ldots, -1, 0, 1, \ldots, p \} \) and define \( \mathbb{N}_r^f \) to be the set of \( n \)-tuples \( \kappa = (k_1, \ldots, k_n) \) such that \( k_i \in \mathbb{N}_r \) and \( k_i \neq 0 \) only for \( i = n - 2j + 1, 1 \leq j \leq f \). Thus, \( X^\kappa = X_{n-1}^{k_{n-1}} X_{n-3}^{k_{n-3}} \cdots X_{n-2f+1}^{k_{2f+1}} \).

**Lemma 5.4.** Suppose that \( \kappa \in \mathbb{N}_r^f \) with \( 0 \leq f \leq \lfloor \frac{n}{2} \rfloor \). Then \( E^f X^\kappa \mathcal{E}_{n-2f} \subset \mathcal{B}_{r,n}^{f+1} \) where \( \mathcal{E}_{n-2f} \) is the image of \( \mathcal{E}_{n-2f} \) in \( \mathcal{B}_{r,n} \).

**Proof.** The result follows since \( E^f X^\kappa \) commutes with \( \mathcal{E}_{n-2f} \). \( \square \)

By Proposition 5.2 and Lemma 5.4 there is a well-defined \( R \)-module homomorphism \( \sigma_f : \mathcal{H}_{r,n-2f} \to \mathcal{B}_{r,n}^f/\mathcal{B}_{r,n}^{f+1} \), for each non-negative integer \( f \leq \lfloor \frac{n}{2} \rfloor \), such that

\[
\sigma_f(h) = E^f \varepsilon_{n-2f}(h) + \mathcal{B}_{r,n}^{f+1}, \quad \text{for } h \in \mathcal{H}_{r,n-2f},
\]

where \( \varepsilon_{n-2f}(h) \) is the image of \( \varepsilon_{n-2f}(h) \) in \( \mathcal{B}_{r,n} \).

We recall the following definition in [3]. Suppose that \( f \) is a non-negative integer with \( f \leq \lfloor \frac{n}{2} \rfloor \). Let \( \mathcal{B}_f \) be the subgroup of \( \mathcal{S}_n \) generated by \( \{ s_{n-1} \} \cup \{ s_{n-2+2i}, s_{n-2+2i+1}, s_{n-2i+2i+2i-1}, s_{n-2i+2i-2+2i-1} \} \) \( 2 \leq i \leq f \). Let \( \tau = (n-2f, (2f)) \) and define

\[
\mathcal{D}_{f,n} = \left\{ d \in \mathcal{S}_n \mid \begin{array}{c} \text{\( \tau \)-cycle} \end{array} \right\}.
\]

For any positive integers \( i, j \), write

\[
s_{i,j} = \begin{cases} s_{i-1} s_{i-2} \cdots s_j, & \text{if } i > j, \\
1, & \text{if } i = j, \\
s_{i+1} s_{i+2} \cdots s_{j-1}, & \text{if } i < j.
\end{cases}
\]

**Lemma 5.5.** Suppose that \( 0 \leq f \leq \lfloor \frac{n}{2} \rfloor \). Then

\[
\mathcal{D}_{f,n} = \left\{ s_{n-2f+1}, s_{n-2f+2}, \ldots, s_{n-1}, s_n \mid \begin{array}{c} \text{\( 0 \leq i_f < \cdots < i_1 \leq n; \\
1 \leq i_k < \cdots < n-2k+2i_1 \leq f \leq n \)} \end{array} \right\}.
\]

**Proof.** It has been proved in [3] that \( \mathcal{D}_{f,n} \) is a complete set of right coset representatives for \( \mathcal{S}_{n-2f} \times \mathcal{B}_f \) in \( \mathcal{S}_n \). So,

\[
\# \mathcal{D}_{f,n} = \left| \mathcal{S}_n \right| / \left| \mathcal{S}_{n-2f} \right| = \frac{n!}{(n-2f)!f!2f}.
\]

Let \( \mathcal{D}_{f,n}^r \) be the set given in the right hand of (5.6). Then \( \# \mathcal{D}_{f,n}^r = \binom{n}{f} \). In general, since we are assuming that \( 1 \leq i_f < j_f \leq n-2f+2, i_f \in \{1, 2, \ldots, n-2f+1\} \), there are \( n + 2 - 2f - i_f \) choices for \( j_f \). In this case, by induction assumption,
there are \#D_{f−1,n−i} choices for the sequences \(i_1, j_1, \ldots, i_{f−1}, j_{f−1}\) which satisfy the inequalities in (5.9). So,

\[
#D'_{f,n} = (n−2f+1)#D'_{f−1,n−1} + \cdots + 1 \cdot #D'_{f−1,2f−1}.
\]

By induction assumption on \(#D'_{f−1,k}\) for \(2f−1 \leq k \leq n−1\),

\[
#D'_{f,n} = \sum_{k=2f}^{n} \frac{(k−2f+1)(k−1)!}{(k−2f+1)!(f−1)!2^{f−1}}.
\]

By induction on \(n\), we have \(#D'_{f,n} = \frac{n!}{(n−2f+1)!(2f−1)!} = #D_{f,n}\). Since \(D_{f,n} \supseteq D'_{f,n}\), \(D_{f,n} = D'_{f,n}\).

Given two standard \(\lambda\)-tableaux \(s, t\) for some \(r\)-partition \(\lambda\). It has been defined in [9] that

\[
m_{st} = g_{d(s)−1} \cdot \prod_{s=2}^{r} \prod_{i=1}^{a_{s−1}} (y_i - u_s) \sum_{w \in \mathfrak{S}_\lambda} g_w \cdot g_{d(t)},
\]

where \(a_{s−1} = |\lambda^{(1)}| + \cdots + |\lambda^{(s−1)}|\).

For each \(w \in \mathfrak{S}_\lambda\), let \(T_w = T_{i_1} \cdots T_{i_k}\) if \(s_1, s_2, \ldots, s_k\) is a reduced expression of \(w\) in \(\mathfrak{S}_n\). By Matsumoto's theorem (see e.g. [12, 1.2.2]), \(T_w \in \mathfrak{B}_{r,n}\) is independent of the reduced expression of \(w\).

**Definition 5.7.** Suppose that \(\lambda \in \Lambda^+_\big(n−2f\big)\), for some non-negative integer \(f \leq \lfloor \frac{n}{2} \rfloor\). For each pair \((s, t)\) of standard \(\lambda\)-tableaux define

\[
M_{st} = T_{d(s)−1} \cdot \prod_{s=2}^{r} \prod_{i=1}^{a_{s−1}} (X_i - u_s) \sum_{w \in \mathfrak{S}_\lambda} T_w \cdot T_{d(t)}.
\]

**Lemma 5.8.** Suppose that \(\lambda \in \Lambda^+_\big(n−2f\big)\), for some non-negative integer \(f \leq \lfloor \frac{n}{2} \rfloor\). For any \(s, t \in \mathfrak{T}^{std}(\lambda)\),

a) \(E_f M_{st} = M_{st} E_f \in \mathfrak{B}_{r,n}^f\).

b) If \(\kappa \in \mathbb{N}_f^{−}\) then \(M_{st} X^\kappa = X^\kappa M_{st}\).

c) If \(w\) is a permutation on \(\{n−2f+1, \ldots, n\}\) then \(M_{st} T_w = T_w M_{st}\).

d) \(\sigma_f(m_{st}) = \pi_f(m_{st}) E_f M_{st}\).

**Definition 5.9.** Given a \(\lambda \in \Lambda^+_\big(n−2f\big)\) and \(0 \leq f \leq \lfloor \frac{n}{2} \rfloor\). Define \(\mathfrak{B}_{r,n}^{\big(f, \lambda\big)}\) to be the two-sided ideal of \(\mathfrak{B}_{r,n}\) generated by \(\mathfrak{B}_{r,n}^{+1}\) and the elements

\[
\{ E_f M_{st} \mid s, t \in \mathfrak{T}^{std}(\mu) \text{ and } \mu \in \Lambda^+_\big(n−2f\big) \text{ with } \mu \supset \lambda \}.
\]

We also define \(\mathfrak{B}_{r,n}^{\big(f, \lambda\big)} = \sum_{\mu \supset \lambda} \mathfrak{B}_{r,n}^{\big(f, \mu\big)}\), where in the sum \(\mu \in \Lambda^+_\big(n−2f\big)\).

**Theorem 5.10.** Suppose that \(s \in \mathfrak{T}^{std}(\lambda)\). Let \(\Delta_s(f, \lambda)\) be the \(R\)-submodule of \(\mathfrak{B}_{r,n}^{\big(f, \lambda\big)} / \mathfrak{B}_{r,n}^{\big(f, \lambda\big)}\) spanned by \(\{ E_f M_{st} X^\kappa T_d + \mathfrak{B}_{r,n}^{\big(f, \lambda\big)} \mid (t, \kappa, d) \in \delta(f, \lambda) \}\), where \(\delta(f, \lambda) = \{ (t, \kappa, d) \mid t \in \mathfrak{T}^{std}(\lambda), \kappa \in \mathbb{N}_{f}^{−}\) and \(d \in D_{f,n}\}\). Then \(\Delta_s(f, \lambda)\) is a right \(\mathfrak{B}_{r,n}\)-module.

In order to prove Theorem 5.10, we need several Lemmas as follows.

**Lemma 5.11.** Let \(N_1\) be the \(R\)-submodule generated by \(\mathfrak{B}_{r,n} E_{n−1} X_{n−1}^\alpha T_d\) where \(d \in D_{1,n}\) and \(-p \leq \alpha \leq p\). Then \(N_1\) is a right \(\mathfrak{B}_{r,n}\)-module.

Proof. We need to verify

\[
E_{n−1} X_{n−1}^\alpha T_d h \in N_1
\]

for any \(h \in \{ T_i, E_{i}, X_{i}, 1 \leq i \leq n−1 \}\). Since \(X_{i} = T_{i−1}X_{i−1}T_{i−1}\), \(\{ T_i, E_{i}, X_{i}, 1 \leq i \leq n−1 \}\) are algebraic generators of \(\mathfrak{B}_{r,n}\). Since we are assuming that \(d \in D_{1,n}\), by Lemma 5.5 \(d = s_{n−1,i}s_{n,j}\) for some integers \(i, j \) with \(1 \leq i < j \leq n\).
First, we verify (5.12) for $h = E_k$, $1 \leq k \leq n - 1$. By Definition 2.1(b)(c),
$$E_{n-1}X^\alpha_{n-1}T_dE_k = E_kE_{n-1}X^\alpha_{n-1}T_d \in N_1$$
if $k \leq i - 2$. When $k = i$ and $j = i + 1$,
$$E_{n-1}X^\alpha_{n-1}T_{n-1,i}T_{n,j}E_k = E_{n-1}X^\alpha_{n-1}E_{n,i} = \omega^{(\alpha)}_{n-1}E_{n,i},$$
where $\omega^{(\alpha)}_{n-1} \in Z(\mathcal{B}_{r,n-2})$ is given in Lemma 4.22. For the simplification of notation, we use $\omega^{(\alpha)}_{n-1}$ to express its image in $\mathcal{B}_{r,n-2}$ in the previous equation. Since $E_{n,i} = E_{n-1}T_{n-1,i}T_{n,i+1}$, $\omega^{(\alpha)}_{n-1}E_{n,i} \in N_1$. If $j + 1 \leq k$,
$$E_{n-1}X^\alpha_{n-1}T_{n-1,i}T_{n,j}E_k$$
$$= E_{n-1}X^\alpha_{n-1}T_{n-1,i}T_{n,k-1}E_kT_{k-1,j}$$
$$= E_{n-1}X^\alpha_{n-1}T_{n-1,i}T_{n,k+1}E_kT_{k-1,j}$$
$$= E_{n-1}X^\alpha_{n-1}T_{n-1,k-2}E_kE_{k-1}T_{k-2,i}T_{n,k+1}E_kT_{k-1,j}$$
$$= E_{k-2}E_{n-1}X^\alpha_{n-1}T_{n-1,k}E_kE_{k-1}T_{k-2,i}T_{n,k+1}E_kT_{k-1,j}$$
$$= E_{k-2}T_{k-2,i}T_{n-1,k-1}E_{n-1}X^\alpha_{n-1}E_{n,k-1}E_kE_{k-1}E_kT_{k-1,j}$$
$$= E_{k-2}T_{k-2,i}T_{n-1,k-1}E_{n-1}X^\alpha_{n-1}T_{n-2,k-1}E_{n,k+1}E_kT_{k-1,j}$$
$$= E_{k-2}T_{k-2,i}T_{n-1,k-1}X^\alpha_{n-2}T_{n-2,k-1}E_{n,k+1}E_kT_{k-1,j}$$
where $\alpha > 1$.

It is not difficult to check (5.12) if one of the conditions holds: (1) $k = i - 1$,
(2) $k = i, j > i + 1$, (3) $i + 1 \leq k \leq j - 2$, (4) $k = j - 1$ and $j > i + 1$, (5) $k = j$. We leave the details to the reader. So, (5.12) holds for $h = E_k$ for all positive integers $k \leq n - 1$.

Now, suppose $h = T_k$. Similarly, there are eight cases we have to discuss. We only check three cases and leave the remainder cases to the reader. Note that the result for $k \leq i - 2$ follows immediately from Definition 2.1(b)(c).

If $k = i$ and $j > i + 1$, then
$$E_{n-1}X^\alpha_{n-1}T_{n-1,i}T_{n,j}E_k = E_{n-1}X^\alpha_{n-1}T_{n-1,i}T_{n,i+1}(1 + \delta T_i - \delta E_i)T_{n,j} \in N_1$$
since we have already proved that (5.12) holds for $h = E_i$. Similarly, when $k = j$,
$$E_{n-1}X^\alpha_{n-1}T_{n-1,i}T_{n,j}T_{n,j}^2 = E_{n-1}X^\alpha_{n-1}T_{n-1,i}T_{n,j+1}(1 + \delta T_j - \delta E_j) \in N_1.$$
Let $d \in \mathcal{D}_{1,n}$. By our induction assumption on Lemma 5.14, we can use $-p \leq k \leq p$ instead of $0 \leq k \leq p+1$ when $i = 1$.

Obviously, $E_{n,n-1}X_n^k \in N_1$. In general, $E_{n,i}X_k = E_{n,i+1}T_iX_kT_{i+1}$. By Lemma 2.3

$$E_{n,i}X_k = E_{n,i+1}(X_kT_i + \sum_{j=1}^{k} \delta X_{i+1}^j (E_i-1)X_k^{j-1})T_{i+1}.$$  

Since we have already proved that $N_1$ is stable under the actions of $E_j, T_j$, $1 \leq j \leq n-1$, by our induction assumption on $i$ and $k$, $E_{n,i}X_k \in N_1$.

When $i = 1$, we have to discuss the case when $-p \leq k \leq 0$. In fact, $E_{n-1}X_n^k \in N_1$ for any $-p \leq k \leq 0$. In general,

$$E_{n,i}X_k = E_{n,i+1}T_i^{-1}X_kT_i^{-1} \quad \text{(5.16)}$$

By our induction assumption on $i$ and $k$, we have $E_{n,i}X_k \in N_1$. Setting $i = 1$ yields the result as required. 

The following results can be proved by arguments in the proof of Lemma 5.14.

**Corollary 5.13.** Let $d \in \mathcal{D}_{1,n}$ and let $\alpha$ be integers with $-p \leq \alpha \leq p$.

1. $E_{n-1}X_n^{-1}E_{n-1}E_{n-2} \in \mathcal{B}_{r,n}^{-2}E_{n-1}E_{n-2}$.
2. $E_{n-1}X_n^{-1}T_dE_{n-1} \in \mathcal{B}_{r,n}^{-2}E_{n-1}$.

**Lemma 5.14.** For any $k \in \mathbb{Z}$, $E_{n-1}X_n^{-1}E_{n-3} \in N_3$ where $N_3$ is the $R$-submodule of $\mathcal{B}_{r,n}$ generated by $\mathcal{B}_{r,n}^{-4}E_{n-1}X_n^{-1}E_{n-3}$ and $\mathcal{B}_{r,n}^{-4}E_{n-3}X_n^{-3}E_{n-1}T_w$ where $w = d_1s_{n-1,n-3}s_{n-2}$ for some $1 \neq d_1 \in D_{1,n-2}$ and $\ell \in \mathbb{Z}$ with $|\ell| \leq p$.

**Proof.** Note that $E_{n-1}X_n^{-1}E_{n-3} = E_{n-3}X_n^{-3}E_{n-1}E_{n-2}E_{n-3}$. Applying Lemma 5.11 on $E_{n-3}X_n^{-3}$, we can write $E_{n-3}X_n^{-3}E_{n-1}E_{n-2}E_{n-3}$ as an $R$-linear combination of elements in $\mathcal{B}_{r,n}^{-4}E_{n-3}X_n^{-3}T_dE_{n-1}E_{n-2}E_{n-3}$ where $d_1 \in D_{1,n-2}$ and $\ell \in \mathbb{Z}$ with $|\ell| \leq p$. Such elements are in $N_3$ if $d_1 \neq 1$ since $E_{n-1}T_d = T_dE_{n-1}$ and $E_{n-1}E_{n-2}E_{n-3} = E_{n-1}T_{d_1}T_{d_2}E_{n-2}E_{n-3}$ when $d_1 = 1$. $E_{n-3}X_n^{-3}T_dE_{n-1}E_{n-2}E_{n-3} = E_{n-1}X_n^{-1}E_{n-3} \in N_3$.

**Lemma 5.15.** Given an integer $-p \leq k \leq p$, $E_{n-3}T_{r,n}^{-2}E_{n-1}E_{n-2}X_k^{-2} \in N_{4,k}$ where $N_{4,k}$ is the $R$-submodule of $\mathcal{B}_{r,n}$ generated by

$$\left\{ \mathcal{B}_{r,n}^{-4}E_{n-3}X_n^{-3}E_{n-1}X_{k_1}^{-1}T_dT_{d_1} d_2 \in \{1, s_{n-1,n-3}s_{n-2}, s_{n-2}s_{n-1}\}, d_1 \in D_{1,n-2}, |\ell_1| \leq p, |\ell_2| \leq |\ell| \right\}.$$

**Proof.** Applying Lemma 2.3(3)(6) (resp. Corollary 4.27) on $E_{n-1}X_n^{-1}T_{n-1}$ (resp. $E_{n-1}X_n^{-1}T_{n-2}E_{n-1}$), we see that both of them can be written as $R$-linear combinations of elements in $\mathcal{B}_{r,n}^{-2}E_{n-1}X_n^{n-1}$ with $|m| \leq |\ell|$. Now, one can verify $N_{4,k} \subseteq N_{4,k}$ by Definition 2.1 and Lemma 5.11 for $\mathcal{B}_{r,n}^{-2}$ without any difficulty.

Given a positive integer $k \leq p$ and an element $h \in \mathcal{B}_{r,n}^{-2}$. Then

$$E_{n-3}hE_{n-1}E_{n-2}X_k^{-2} = E_{n-3}hE_{n-1}T_{n-2}E_{n-2}X_k^{-2} \quad \text{(5.16)}$$

By our induction assumption on $i$ and $k$, we have $E_{n,i}X_k \in N_1$. Setting $i = 1$ yields the result as required. 

The following results can be proved by arguments in the proof of Lemma 5.14.

**Corollary 5.13.** Let $d \in \mathcal{D}_{1,n}$ and let $\alpha$ be integers with $-p \leq \alpha \leq p$.

1. $E_{n-1}X_n^{-1}T_dE_{n-2} \in \mathcal{B}_{r,n}^{-2}E_{n-1}E_{n-2}$.
2. $E_{n-1}X_n^{-1}T_dE_{n-1} \in \mathcal{B}_{r,n}^{-2}E_{n-1}$.

**Lemma 5.14.** For any $k \in \mathbb{Z}$, $E_{n-1}X_n^{-1}E_{n-3} \in N_3$ where $N_3$ is the $R$-submodule of $\mathcal{B}_{r,n}$ generated by $\mathcal{B}_{r,n}^{-4}E_{n-1}X_n^{-1}E_{n-3}$ and $\mathcal{B}_{r,n}^{-4}E_{n-3}X_n^{-3}E_{n-1}T_w$ where $w = d_1s_{n-1,n-3}s_{n-2}$ for some $1 \neq d_1 \in D_{1,n-2}$ and $\ell \in \mathbb{Z}$ with $|\ell| \leq p$.

**Proof.** Note that $E_{n-1}X_n^{-1}E_{n-3} = E_{n-3}X_n^{-3}E_{n-1}E_{n-2}E_{n-3}$. Applying Lemma 5.11 on $E_{n-3}X_n^{-3}$, we can write $E_{n-3}X_n^{-3}E_{n-1}E_{n-2}E_{n-3}$ as an $R$-linear combination of elements in $\mathcal{B}_{r,n}^{-4}E_{n-3}X_n^{-3}T_dE_{n-1}E_{n-2}E_{n-3}$ where $d_1 \in D_{1,n-2}$ and $\ell \in \mathbb{Z}$ with $|\ell| \leq p$. Such elements are in $N_3$ if $d_1 \neq 1$ since $E_{n-1}T_d = T_dE_{n-1}$ and $E_{n-1}E_{n-2}E_{n-3} = E_{n-1}T_{d_1}T_{d_2}E_{n-2}E_{n-3}$. When $d_1 = 1$, $E_{n-3}X_n^{-3}T_dE_{n-1}E_{n-2}E_{n-3} = E_{n-1}X_n^{-1}E_{n-3} \in N_3$.

**Lemma 5.15.** Given an integer $-p \leq k \leq p$, $E_{n-3}T_{r,n}^{-2}E_{n-1}E_{n-2}X_k^{-2} \in N_{4,k}$ where $N_{4,k}$ is the $R$-submodule of $\mathcal{B}_{r,n}$ generated by

$$\left\{ \mathcal{B}_{r,n}^{-4}E_{n-3}X_n^{-3}E_{n-1}X_{k_1}^{-1}T_dT_{d_1} d_2 \in \{1, s_{n-1,n-3}s_{n-2}, s_{n-2}s_{n-1}\}, d_1 \in D_{1,n-2}, |\ell_1| \leq p, |\ell_2| \leq |\ell| \right\}.$$

**Proof.** Applying Lemma 2.3(3)(6) (resp. Corollary 4.27) on $E_{n-1}X_n^{-1}T_{n-1}$ (resp. $E_{n-1}X_n^{-1}T_{n-2}E_{n-1}$), we see that both of them can be written as $R$-linear combinations of elements in $\mathcal{B}_{r,n}^{-2}E_{n-1}X_n^{n-1}$ with $|m| \leq |\ell|$. Now, one can verify $N_{4,k} \subseteq N_{4,k}$ by Definition 2.1 and Lemma 5.11 for $\mathcal{B}_{r,n}^{-2}$ without any difficulty.

Given a positive integer $k \leq p$ and an element $h \in \mathcal{B}_{r,n}^{-2}$. Then

$$E_{n-3}hE_{n-1}E_{n-2}X_k^{-2} = E_{n-3}hE_{n-1}T_{n-2}E_{n-2}X_k^{-2} \quad \text{(5.16)}$$
By Lemma 5.11 for $E_{n-3}h$, $E_{n-3}hE_{n-1}X_{n-1}^{k}T_{n-2}T_{n-1} \in N_{4,k}$. Also, we have

$$E_{n-3}hE_{n-1}X_{n-1}^{k}X_{n-2}^{i}T_{n-1} = E_{n-3}hX_{n-2}^{i}E_{n-1}X_{n-1}^{k}T_{n-1} \in N_{4,k}.$$  

One can verify the above inclusion by using $N_{4,k}T_{n-1} \subset N_{4,k}$ and Lemma 5.11 for $E_{n-3}hX_{n-2}^{i}$.

Finally, by induction assumption on $k$ (when $k = 0$, the result is trivial since $E_{n-1}E_{n-2} = E_{n-1}T_{n-2}T_{n-1}$), we have

$$h_1 := E_{n-3}hX_{n-2}^{i}E_{n-1}X_{n-2}^{k} \in N_{4,k}$$

for all positive integers $i \leq k$. Since $N_{4,k}T_{n-1} \subset N_{4,k}$, $h_1T_{n-1} \in N_{4,k}$.

So, $E_{n-3}hE_{n-1}E_{n-2}X_{n-2}^{k} \in N_{4,k}$.

The case when $-p \leq k \leq 0$ can be discussed similarly. The only difference is that we have to use Lemma 2.3(4) instead of Lemma 2.3(1). We leave the details to the reader.

\[ \square \]

**Lemma 5.17.** Let $N_{4}$ be the $R$-submodule of $\mathcal{R}_{r,n}$ generated by $\mathcal{R}_{r,n}E_{n-3}X_{n-3}^{k}E_{n-1}X_{n-2}^{i}T_{n-1}T_{n-2}$ where $d_2 \in \{ 1, s_{n-2}, s_{n-2}s_{n-1} \}$, $d_1 \in D_{1,n-2}$, $\ell, k \in \mathbb{Z}$ with $-p \leq k, \ell \leq p$. Then $N_{4} \subset h \subset h \in \{ T_{n-1}, T_{n-2}, E_{n-1}, E_{n-2} \}$.

**Proof.** By Corollary 5.13 and Lemma 5.11 we have $N_{4}E_{i} \subset N_{4}$ for $i \in \{ n-1, n-2 \}$. Using this result together with Lemma 2.3 and Definition 2.1(b), we have $N_{4}T_{i} \subset N_{4}$ for $i \in \{ n-1, n-2 \}$.

\[ \square \]

**Lemma 5.18.** Fix an integer $\ell$ with $-p \leq \ell \leq p$. Let $M_{\ell} \subset \mathcal{R}_{r,n}$ be the $R$-module generated by $\mathcal{R}_{r,n}E_{n-3}X_{n-3}^{k}E_{n-1}X_{n-2}^{i}T_{n-1}T_{n-2}$ where $d_2 \in \{ 1, s_{n-2}, s_{n-2}s_{n-1} \}$, $d_1 \in D_{1,n-2}$, $\ell, k \in \mathbb{Z}$ with $-p \leq k, \ell \leq p$. Let $N_{4}$ be the $R$-module defined in Lemma 5.17. If $E_{n-1}X_{n-3}^{a}E_{n-3}X_{n-2}^{i}T_{n-1}T_{n-2} \in N_{4}$ for all integers $|\ell| < |\ell|$ and $|m| \leq p$, then $M_{\ell} \subseteq N_{4}$.

**Proof.** First, we assume that $0 \leq i \leq |\ell|$. By Lemma 5.14 $M_{\ell} \subseteq N_{4}$ if

1. $E_{n-1}X_{n-3}^{a}E_{n-3}X_{n-2}^{i} \in N_{4}$ for all $a \in \mathbb{Z}$ and $|a| \leq p$,
2. $A \in N_{4}$ where $A = E_{n-3}X_{n-3}^{a}E_{n-1}T_{n-2}T_{n-1}T_{n-2}X_{n-3}^{i} = E_{n-3}X_{n-3}^{a}E_{n-1}T_{n-2}T_{n-1}T_{n-2}X_{n-3}^{i}$ for all $a \in \mathbb{Z}$, $|a| \leq p$ and $1 \neq d_1 \in D_{1,n-2}$.

By the definition of $N_{4}$, (1) holds. We prove (2) by induction on $i$.

Suppose $i = 0$. (2) holds since $d_1s_{n-2}s_{n-2}s_{n-1}s_{n-2} \in D_{2,n}$. When $i > 0$,

$$A = E_{n-3}X_{n-3}^{a}T_{d_1}E_{n-1}T_{n-2}X_{n-3}^{i}T_{n-1}T_{n-2}$$

(5.19)

$$+ E_{n-3}X_{n-3}^{a}T_{d_2}E_{n-1}T_{n-2} \sum_{j=1}^{i} \delta X_{n-2}^{j}(E_{n-3} - 1)X_{n-3}^{i-j}T_{n-1}T_{n-2}$$

We consider the second term (up to a scalar in $R$) of (5.19) as follows.

$$E_{n-3}X_{n-3}^{a}T_{d_1}E_{n-1}T_{n-2}X_{n-3}^{i-j}T_{n-1}T_{n-2}$$

$$= E_{n-3}X_{n-3}^{a}T_{d_1}X_{n-3}^{i-j}E_{n-1}E_{n-2}E_{n-3}X_{n-3}^{i-j}T_{n-2}$$

$$= E_{n-3}X_{n-3}^{a}T_{d_1}X_{n-3}^{i-j}E_{n-1}T_{n-2}T_{n-3}T_{n-1}T_{n-2}X_{n-3}^{i-j}T_{n-2}$$

$$(5.12) \sum_{-p \leq a' \leq p} \mathcal{R}_{r,n}E_{n-3}X_{n-3}^{a'}T_{d_1}E_{n-1}T_{n-2}T_{n-3}T_{n-1}T_{n-2}X_{n-3}^{i-j}T_{n-2}$$

If $d_1' \neq 1$, then $E_{n-3}X_{n-3}^{a'}E_{n-1}T_{n-2}T_{n-3}T_{n-1}T_{n-2}X_{n-3}^{i-j}T_{n-2} \in N_{4}$ by induction assumption on $i$ and Lemma 5.17.

If $d_1' = 1$, we still have $E_{n-3}X_{n-3}^{a'}E_{n-1}T_{n-2}T_{n-3}T_{n-1}T_{n-2}X_{n-3}^{i-j}T_{n-2} = E_{n-1}X_{n-3}^{a'}E_{n-3}X_{n-3}^{i-j}T_{n-2} \in N_{4}$, by Lemma 5.17.
We consider the third term on the right hand side of (5.19). We have
\[ E_n-3X_{n-3}^aT_dE_nT_n-2X_{n-3}^jE_nT_{n-3}T_n-2 \]
\[ = E_n-3X_{n-3}^aT_dE_nE_n-2X_{n-3}^jE_nT_n-2 \]
\[ = E_n-3X_{n-3}^aT_dX_{n-3}^jE_nE_n-2X_{n-2}^jT_n-2 \in N_4 \]
We remark that we use Lemma 5.13 for \( E_n-3X_{n-3}^aT_dX_{n-2}^jE_nE_n-2X_{n-2}^j \) and Lemma 5.17 to get the above inclusion.

We use Lemma 2.3 to express the first term on the right hand side of (5.19) as follows:
\[ E_n-3X_{n-3}^aT_dE_n-1(X_{n-1}^jT_n-2 + \sum_{j=1}^i \delta X_{n-1}^j(E_n-2 - 1)X_{n-2}^j)T_nE_nT_n-2. \]

Since we are assuming that \( d_1 \neq 1, d_1s_{n-1, n-3}s_{n-2} \in D_{2n} \). So, the first term on the right hand side of (5.20) is in \( N_4 \).

By Lemma 5.11 for \( E_n-3X_{n-3}^aT_dX_{n-2}^jT_n-3 \), and Lemma 5.17
\[ E_n-3X_{n-3}^aT_dX_{n-2}^jT_nE_n-1X_{n-1}^jT_nE_nT_n-2 \in N_4. \]

In other words, the third term on the right hand side of (5.20) is in \( N_4 \).

In order to show that the second term on the right hand side of (5.20) is in \( N_4 \), we need to show that
\[ B := E_n-3X_{n-3}^aT_dE_n-1X_{n-1}^jE_nE_n-2X_{n-2}^jT_nE_nT_n-2 \in N_4. \]

Note that
\[ E_n-3X_{n-3}^aT_dE_n-1X_{n-1}^jE_nE_n-2X_{n-2}^j = E_n-3X_{n-3}^aT_dX_{n-2}^jE_n-1E_nE_n-2X_{n-2}^j. \]

By Lemma 5.13, \( B \) can be written as linear combination of elements in \( D_{1, n-4} \) for \( E_n-3X_{n-3}^aT_dE_nX_{n-1}^uT_dE_n-1X_{n-1}^vT_dT_nE_n-1T_n \), where \( d, v \in \mathbb{Z} \) with \( |u| \leq p, |v| \leq i - j \leq i - 1 \), \( d' = 1 \). In order to finish the proof, we need to show that
\[ C := E_n-3X_{n-3}^aT_dE_nX_{n-1}^vT_dT_nE_nT_n-1 \in N_4. \]

There are four cases we have to discuss.

(1) \( d_2 = 1 \). By Lemma 5.11 and Lemma 5.17, \( C \in N_4 \) since
\[ E_n-3X_{n-3}^aT_dE_nX_{n-1}^vT_dT_nE_n-1X_{n-1}^v = E_n-3X_{n-3}^aT_dE_nX_{n-1}^vT_dT_nE_n-1X_{n-1}^v. \]

(2) \( d_2 = s_{n-2} \) and \( d' = 1 \). By Lemma 5.17 and our assumption \( C \in N_4 \) since
\[ C = E_n-3X_{n-3}^aE_nX_{n-1}^vT_dT_nE_nT_n-1 \] and \( |v| \leq i - 1 \leq \ell - 1 \).

(4) \( d_2 = s_{n-1} \). \( C \) follows from Lemma 5.17 and the result for \( d_2 = s_{n-2} \).

This completes the proof of the result under the assumption \( 0 \leq i \leq |\ell| \). When \(-|\ell| \leq i \leq 0 \), One can verify the result similarly by induction on \( i \). Note that the result holds when \( i = 0 \). We remark that we have to use Lemma 2.3 (5) instead of Lemma 2.3 (1). We also need \( N_1T_{n-1} \in N_4 \) for \( n = -2, n - 1 \) which follows from Lemma 5.17 and Definition 2.1 (b), immediately. We leave the details to the reader. □

**Lemma 5.22.** Fix an integer \( \ell \) with \(-p \leq \ell \leq p \). Let \( N_4 \) be the \( R \)-module defined in Lemma 5.17. If \( E_{n-1}X_{n-1}^uE_{n-3}X_{n-3}^mT_n-2E_{n-3}X_{n-3}^mT_n-3 \in N_4 \) for all integers \( |u| < |\ell| \) and \( |m| \leq p \), then \( E_{n-3}X_{n-3}^aE_{n-1E_n-2X_{n-2}^bT_n-3} \in N_4 \) with \( a \in \mathbb{Z} \) and \( |b| \leq |\ell| \).
Proof. First, we assume that 0 ≤ b ≤ |I|. Let \( h = E_{n-3} X_{n-3}^a E_{n-1} E_{n-2} X_{n-2}^b T_{n-3} \).

By Lemma 2.3(1), we have

\[
(5.23) \quad h = E_{n-3} X_{n-3}^a E_{n-1} E_{n-2} (T_{n-3} X_{n-3}^b - \sum_{i=1}^b \delta^i X_{n-2} (E_{n-3} - 1) X_{n-3}^{b-i})).
\]

The first term on the right hand side of (5.23) is equal to

\[
(5.24) \quad E_{n-3} X_{n-3}^a E_{n-1} E_{n-2} E_{n-3} X_{n-3}^b T_{n-2}^{-1} = E_{n-1} X_{n-1}^a E_{n-3} X_{n-3}^b T_{n-2}^{-1}
\]

which is in \( N_4 \) by Lemmas 5.17 and 5.18.

The second term on the right hand side of (5.23) (up to a scalar) is equal to \( E_{n-3} X_{n-3}^a E_{n-1} E_{n-2} E_{n-3} X_{n-3}^b \), which is in \( N_4 \) by (5.24). Finally, by Lemma 5.15 the last term is in \( N_4 \).

When \( b < 0 \), one can verify the result similarly. We remark that we have to use Lemma 2.3(4) instead of Lemma 2.3(1).

\[ \square \]

Proposition 5.25. \( E_{n-3} X_{n-3}^k E_{n-1} X_{n-1}^\ell T_{n-2} T_{n-3} \in N_4 \) for all integers \( k, \ell \) with \( -p \leq k, \ell \leq p \).

Proof. Let \( h = E_{n-3} X_{n-3}^k E_{n-1} X_{n-1}^\ell T_{n-2} T_{n-3} \). We prove \( h \in N_4 \) by induction on \(|\ell|\).

If \( \ell = 0 \), then

\[
h = E_{n-3} X_{n-3}^k E_{n-1} T_{n-2} T_{n-3} = E_{n-3} X_{n-3}^k E_{n-1} E_{n-2} T_{n-1} T_{n-3} = E_{n-1} X_{n-1}^k E_{n-3} T_{n-2} T_{n-1} T_{n-3}^{-1} \in N_4
\]

by Lemma 5.17. If \( \ell > 0 \), by Lemma 2.3(1), we have

\[
(5.26) \quad h = E_{n-3} X_{n-3}^k E_{n-1} (T_{n-2} X_{n-2}^\ell - \sum_{i=1}^\ell \delta^i X_{n-2} (E_{n-2} - 1) X_{n-2}^{\ell-i}) T_{n-3}
\]

The first term on the right hand side (5.26) is equal to

\[
E_{n-3} X_{n-3}^k E_{n-1} T_{n-2} X_{n-2}^\ell T_{n-3} = E_{n-3} X_{n-3}^k E_{n-1} E_{n-2} X_{n-2} T_{n-3} T_{n-1}^{-1}
\]

which is in \( N_4 \) by our induction assumption on for all integers \( \ell - 1 \), together with Lemma 5.22 and Lemma 5.17.

By induction and Lemma 5.22, \( E_{n-3} X_{n-3}^k E_{n-1} X_{n-1}^i E_{n-2} X_{n-2}^\ell T_{n-1} T_{n-3} = E_{n-3} X_{n-3}^k E_{n-1} E_{n-2} X_{n-2}^\ell T_{n-3} \in N_4 \).

So, The second term on the right hand side (5.20) is in \( N_4 \).

Finally, we consider the third term on the right hand side (5.20). However, this term is equal to \( E_{n-3} X_{n-3}^k X_{n-3}^\ell T_{n-3} E_{n-1} X_{n-1} \), which is in \( N_4 \) by Lemma 5.11.

When \( \ell < 0 \), one can verify the result similarly. We remark that we have to use Lemma 2.3(4) instead of Lemma 2.3(1).

\[ \square \]

Proposition 5.27. Let \( N_2 \) be the R-submodule of \( \mathcal{R}_{r,n} \) generated by \( \mathcal{R}_{r,n-4} E_{n-3} X_{n-3}^k E_{n-1} X_{n-1}^\ell T_d \) where \( -p \leq k, \ell \leq p \) and \( d \in D_{2,n} \). Then \( N_2 \) is a right \( \mathcal{R}_{r,n} \)-module.

Proof. Applying Lemma 5.11 twice, \( E_{n-3} X_{n-3}^k E_{n-1} X_{n-1}^\ell T_d h \), for \( h \in \mathcal{R}_{r,n} \), can be written as an R-linear combination of elements

\[
\mathcal{R}_{r,n} E_{n-3} X_{n-3}^k E_{n-3}^i T_{n-3,i} E_{n-1} X_{n-1}^\ell T_{n-1,i} T_{n,i}
\]

where \( i_1, i_2, i_3, i_4, k, \ell \in \mathbb{Z} \) with \( i_2 < i_1, i_4 < i_3, -p \leq k, \ell \leq p \). In order to show that \( E_{n-3} X_{n-3}^k E_{n-1} X_{n-1}^\ell T_d h \in N_2 \), we need to show

\[
(5.28) \quad A := E_{n-3} X_{n-3}^k E_{n-1} X_{n-1}^\ell T_d E_{n-1} X_{n-1}^\ell T_{n-1,i} T_{n,i} \in N_2.
\]

We are going to prove (5.28) by induction on \( i_2 \).
We assume that $i_4 \geq i_2$. Otherwise, $A \in N_2$ and follows. In particular, holds for $i_2 \in \{n-1, n-2\}$. By Lemma 5.11 and induction on $i_2$,

\[(5.29)\]

\[E_{n-3} R_{r,n-2} E_{n-1} X_{n-1}^{i} T_{n, i}, i_2 T_{n,i_2} \in N_2\]

for all integers $i, j, i_1, i_2$ with $i_2 < i_1$ and $i_2 > i_2$.

Since we are assuming that $i_4 \geq i_2$ and $i_2 < n-2$,

\[A = E_{n-3} X_{n-3}^{i_4} E_{n-1} X_{n-1}^{i} T_{n-1, i_2} T_{n-2, i_4+1} T_{n-1, i_1} T_{n,i_1}.\]

By Proposition 5.24, $A$ is in the $R$-submodule of $R_{r,n}$ generated by

\[R_{r,n-4} E_{n-3} X_{n-3}^{i_4} T_d E_{n-1} X_{n-1}^{i} T_{d_2} T_{n-2, i_2+1} T_{n,i_2} T_{n,i_1},\]

where $d_1 \in D_n, d_2 \in \{s_{n-2}, s_{n-2} s_{n-1}, 1\}$ and $-p \leq k_2, \ell_2 \leq p$. In order to prove \[(5.29),\] it suffices to prove \[(5.30)\]

\[B := E_{n-3} X_{n-3}^{i_4} T_{d_1} E_{n-1} X_{n-1}^{i} T_{d_2} T_{n-2, i_2+1} T_{n,i_2} T_{n,i_1} T_{n,i_1} \in N_2\]

There are three cases we have to discuss.

**Case 1.** $d_2 = 1$.

We have $B = E_{n-3} X_{n-3}^{i_4} T_{d_1} T_{n-2, i_2} T_{n-2, i_2+1} E_{n-1} X_{n-1}^{i} T_{n-1, i_1} T_{n,i_1}$, which is in $N_2$ by \[(5.29)\] if $i_3 + 1 < i_1$. When $i_3 + 1 \geq i_1$, $T_{n-1, i_1} T_{n,i_1} = T_{n,i_1} T_{n,i_1+1}$ and

\[(5.31)\]

\[B = E_{n-3} X_{n-3}^{i_4} T_{d_1} T_{n-2, i_2+1} E_{n-1} X_{n-1}^{i} T_{n,i_2} T_{n,i_2+2} + E_{n-1} X_{n-1}^{i} T_{n,i_1} T_{n,i_1+2}.\]

We use Lemma 2.8(3)(6) for $E_{n-1} X_{n-1}^{i} T_{n-1}$ to write $B$ as a linear combination of elements in $R_{r,n-2} E_{n-1} X_{n-1}^{i} T_{n,i_2+1}$ with $-p \leq j \leq p$. By \[(5.29),\] $B \in N_2$. This completes the proof for $d_2 = 1$.

**Case 2.** $d_2 = s_{n-2}$.

We have $B = E_{n-3} X_{n-3}^{i_4} T_{d_1} T_{n-2, i_2} T_{n-2, i_2+1} E_{n-1} X_{n-1}^{i} T_{n-1, i_1} T_{n,i_1}$, which is in $N_2$ by the result for $d_2 = 1$.

**Case 3.** $d_2 = s_{n-2} s_{n-1}$.

If $i_3 + 1 < i_1$, then

\[B = E_{n-3} X_{n-3}^{i_4} T_{d_1} T_{n-2, i_2} T_{n-2, i_2+1} E_{n-1} X_{n-1}^{i} T_{n-1, i_1} T_{n,i_1} T_{n,i_1+1} T_{n,i_1+2}\]

So, \[(5.30)\] follows from \[(5.29)\]. Finally, we assume that $i_3 + 1 \geq i_1$. Then

\[B = E_{n-3} X_{n-3}^{i_4} T_{d_1} T_{n-2, i_2} T_{n-2, i_2+1} E_{n-1} X_{n-1}^{i} T_{n-1, i_1} T_{n,i_1+2}.\]

By Definition 2.1(b),

\[(5.32)\]

\[B = E_{n-3} X_{n-3}^{i_4} T_{d_1} T_{n-2, i_2} T_{n-2, i_2+1} E_{n-1} X_{n-1}^{i} T_{n-1, i_1} T_{n,i_1+2} \times (1 + \delta E_{n-1} T_{n-1, i_1} T_{n,i_1+2} + \delta E_{n-1} T_{n-1, i_1} T_{n,i_1+2}.\]

The second term on the right hand side of \[(5.32)\] is equal to

\[\delta E_{n-3} X_{n-3}^{i_4} T_{d_1} T_{n-2, i_2} T_{n-2, i_2+1} E_{n-1} X_{n-1}^{i} T_{n-1, i_1} T_{n,i_1+2} T_{n,i_1+2}.\]

By our result for $d_2 = 1$, it is in $N_2$. Similarly, using Corollary 5.13, Lemma 5.11 \[(5.29)\], we see that the third term on the right hand side of \[(5.32)\] is in $N_2$. In order to prove $B \in N_2$, it remains to prove that

\[(5.33)\]

\[C := E_{n-3} X_{n-3}^{i_4} T_{d_1} T_{n-2, i_2} T_{n-2, i_2+1} E_{n-1} X_{n-1}^{i} T_{n-1, i_1} T_{n,i_1+2} \in M.\]

In fact, when $i_4 + 1 < i_1$,

\[C = E_{n-3} X_{n-3}^{i_4} T_{d_1} T_{n-2, i_2} T_{n-2, i_2+1} E_{n-1} X_{n-1}^{i} T_{n-1, i_1} T_{n,i_1+2} \in N_2.\]
Suppose that $i_4 + 1 \geq i_1$. By Definition 2.1(b),

$$C = E_{n-3} X_{n-3}^{k_2} T_{d_1} T_{n-3,i_2} E_{n-1} X_{n-1}^{\ell_2}$$

(5.34)

$$\times (1 + \delta T_{n-2} - \delta g E_{n-2}) T_{n-2,i_1} T_{n-1,i_4 + 2} T_{n,i_3 + 2}.$$  

Note that $T_{n-1,i_1} T_{n-1,i_4 + 2} T_{n,i_3 + 2} = T_{n-2,i_1} T_{n-1,i_4} T_{n,i_3 + 2}$. By (5.34), the first and the second terms on the right of (5.34) are in $N_2$. The third term on the right of (5.34) (up to a scalar) is equal to

$$E_{n-3} X_{n-3}^{k_2} T_{d_1} T_{n-3,i_2} E_{n-1} X_{n-1}^{\ell_2} E_{n-2} T_{n-2,i_1} T_{n-1,i_4 + 2} T_{n,i_3 + 2}$$

$$= E_{n-3} X_{n-3}^{k_2} T_{d_1} T_{n-3,i_2} X_{n-2}^{\ell_2} E_{n-1} E_{n-2} T_{n-2,i_1} T_{n-1,i_4 + 2} T_{n,i_3 + 2}$$

$$= E_{n-3} X_{n-3}^{k_2} T_{d_1} T_{n-3,i_2} X_{n-2}^{\ell_2} E_{n-1} T_{n-1,i_1} T_{n,i_4 + 2} T_{n,i_3 + 2}$$

$$= E_{n-3} X_{n-3}^{k_2} T_{d_1} T_{n-3,i_2} X_{n-2}^{\ell_2} T_{n-2,i_1} E_{n-1} T_{n-1,i_1} T_{n,i_4 + 2} T_{n,i_3 + 2} \text{ (5.29)}.$$

We have proved that (5.33) holds in any case. So, $B \in N_2$ and hence (5.28) holds.

**Proof of Theorem 5.10.** We claim that $M$ is a right $R_{r,n}$-module where $M$ is the $R$-module generated by $E'^f X'^s T_d$ with $\kappa \in N'_{f,n}$ and $d \in D_{f,n}$. If so, $E' f M_{s_1} X'^s T_d + \mathcal{B}_{r,n}'(f,\lambda)$ can be written as an $R$-linear combination of elements $M_{s_1} R_{r,n}' E'^s X'^s T_d + \mathcal{B}_{r,n}'(f,\lambda)$ for $\kappa' \in N'_{f,n}$ and $d' \in D_{f,n}$. Now, the result follows immediately from Lemma 5.11(d).

It remains to prove our claim. Let $d \in D_{f,n}$. By Lemma 7.5

$$d = s_{n-2} s_{n-2} s_{n-k} s_{n-k} \cdots s_{n-1} i_1 s_{n-j_1},$$

for some integers $i_1, \ldots, i_f, j_1, \ldots, j_f$ such that $1 \leq i_1 < \cdots < i_f \leq n$, $1 \leq i < j_k \leq n - 2k + 2$ for $1 \leq k \leq f$. Write $d = d_1 s_{n-1} \cdots s_{n-1, j_1}$. For any $h \in \mathcal{R}_{r,n}$, since

$$E' f X'^s T_d = \prod_{i=2}^{f} E_{n-2}^{s_{n-2} s_{n-k}} X_{n-2}^{s_{n-k}} T_{d_1} T_{n-1,i_1} T_{n,j_1} h,$$

by Lemma 5.11 $E' f X'^s T_d h \in N$ where $N$ is the $R$-submodule of $\mathcal{R}_{r,n}$ generated by

$$\prod_{i=2}^{f} E_{n-2}^{s_{n-2} s_{n-k}} X_{n-2}^{s_{n-k}} T_{d_1} T_{n-1,i_1} T_{n,j_1}$$

where $\kappa'_{i_1} k_1, \ell_1 \in \mathbb{Z}$ with $k_1 < \ell_1$ and $|\kappa'_{n-1}| \leq p$. By induction assumption on $n - 2$ for our claim, $E' f X'^s T_d h$ is in the $R$-submodule of $\mathcal{R}_{r,n}$ generated by

$$\mathcal{B}_{r,n}' f \prod_{i=1}^{f} E_{n-2}^{s_{n-2} s_{n-k}} X_{n-2}^{s_{n-k}} T_{d_1} T_{n-1,k_1} T_{n,\ell_1}$$

with $w_1 = s_{n-2} s_{n-2} \cdots s_{n-k} s_{n-k} w_2$ with $w_1 \in D_{f-1,n-2}$ and $|\kappa'_{n-1}| \leq p$ for $1 \leq i \leq f$. If $w_1 s_{n-1} s_{n-j_1} \in D_{f,n}$ then our claim follows. In particular, our claim follows if $k_1 \in \{n-2, n-1\}$. It remains to prove

$$A := \prod_{i=1}^{f} E_{n-2}^{s_{n-2} s_{n-k}} T_{d_1} T_{n-1,k_1} T_{n,\ell_1} \in M$$

for $k_2 \geq k_1$. We prove it by induction on $k_1$. In general, we have

$$A = \prod_{i=3}^{f} E_{n-2}^{s_{n-2} s_{n-k}} T_{d_2} B$$

where $B = E_{n-3} X_{n-3}^{k_2} E_{n-1} X_{n-1}^{k_1} T_{n-3,k_2} T_{n-2,\ell_2} T_{n-1,k_1} T_{n,\ell_1}$ and $d_2 = w_1 s_{n-3} s_{n-2} s_{n-2,\ell_2}$. 

$\square$
When \( k_1 \in \{n-1,n-2\} \), there is nothing to be proved since \( k_2 \leq n-3 < k_1 \). Suppose \( k_1 \leq n-3 \). By arguments in the proof of (5.28) for \( i_4 \geq i_2 \), we can write \( B \) as an \( R \)-linear combination of elements
\[
C := E_{n-3}B_{n-2}E_{n-1}X_{n-1}{^\ast}_T t_{n-1,k_1}t_{n,n},
\]
where \( k_1 < k'_1 < k'_2 \) and \( -p \leq k'_{n-1} \leq p \). By induction assumption on \( B_{r,n-2} \) for our claim, we can write \( E^fX^kT_dh \) as an \( R \)-linear combination of elements
\[
B_{r,n-2f} \prod_{i=1}^{f} E_{n-2i+1}X_{n-2i+1}{^\ast}_T t_{n-1,k'_1}t_{n,n'},
\]
where \( \alpha \in \mathbb{N}_{r,n} \), \( y_1 \in D_{r,n-2} \) and \( k'_1 > k_1 \). By our induction assumption on \( k_1 \), \( C \in M \). This completes the proof of our claim. \( \square \)

**Proposition 5.37.** Let \( * : \mathcal{B}_{r,n} \to \mathcal{B}_{r,n} \) be the \( R \)-linear anti–involution in Lemma 2.3. Suppose \( 0 \leq f \leq \lfloor \frac{n}{2} \rfloor \) and \( \lambda \in \Lambda^+_r(n-2f) \). Then \( \mathcal{B}_{r,n}^{(f,\lambda)} / \mathcal{B}_{r,n}^{(f,\lambda)} \) is spanned by the elements
\[
\{ T^*_fX^pE^fMstX^kT_d + \mathcal{B}_{r,n}^{(f,\lambda)} \mid (t, \kappa, d), (\sigma, \rho, \epsilon) \in \delta(f,\lambda) \}.
\]

**Proof.** Let \( W \) be the \( R \)-submodule of \( \mathcal{B}_{r,n}^{(f,\lambda)} / \mathcal{B}_{r,n}^{(f,\lambda)} \) spanned by the elements in (5.38). By Theorem 5.10 \( W \) is both left and right \( \mathcal{B}_{r,n} \)-submodule of \( \mathcal{B}_{r,n}^{(f,\lambda)} / \mathcal{B}_{r,n}^{(f,\lambda)} \). As the generators \( \{ E^fMst + \mathcal{B}_{r,n}^{(f,\lambda)} \} \) of \( \mathcal{B}_{r,n}^{(f,\lambda)} / \mathcal{B}_{r,n}^{(f,\lambda)} \) are contained in \( W \), \( W = \mathcal{B}_{r,n}^{(f,\lambda)} / \mathcal{B}_{r,n}^{(f,\lambda)} \). \( \square \)

**Definition 5.39.** Let \( \Lambda^+_r,n = \{ (f,\lambda) \mid 0 \leq f \leq \lfloor \frac{n}{2} \rfloor \) and \( \lambda \in \Lambda^+_r(n-2f) \} \). If \( (f,\lambda) \in \Lambda^+_r,n \) and \( (\sigma, \rho, \epsilon), (t, \kappa, d) \in \delta(f,\lambda) \) then we define
\[
C^{(f,\lambda)}_{(\sigma, \rho, \epsilon)(t, \kappa, d)} = T^*_fX^pE^fMstX^kT_d.
\]

We recall the definition of cellular algebra as follows.

**Definition 5.40.** [16] Let \( R \) be a commutative ring and \( A \) an \( R \)-algebra. Fix a partially ordered set \( \Lambda = (\Lambda, \triangleright) \) and for each \( \lambda \in \Lambda \) let \( T(\lambda) \) be a finite set. Finally, fix \( C^*_\lambda \in A \) for all \( \lambda \in \Lambda \) and \( s, t \in T(\lambda) \).

Then the triple \( (\Lambda, T, C) \) is a cell datum for \( A \) if:

a) \( \{ C^*_\lambda \mid \lambda \in \Lambda \text{ and } s, t \in T(\lambda) \} \) is an \( R \)-basis for \( A \);

b) the \( R \)-linear map \( * : A \to A \) determined by \( (C^*_\lambda)^* = C^*_\lambda \), for all \( \lambda \in \Lambda \) and all \( s, t \in T(\lambda) \) is an anti–isomorphism of \( A \);

c) for all \( \lambda \in \Lambda \), \( s \in T(\lambda) \) and \( a \in A \) there exist scalars \( r_{tu}(a) \in R \) such that
\[
C^*_\lambda a = \sum_{u \in T(\lambda)} r_{tu}(a)C^*_u \quad (\text{mod } A^{\triangleright \lambda}),
\]

where \( A^{\triangleright \lambda} = R \text–\text{span} \{ C^*_\mu \mid \mu \triangleright \lambda \text{ and } u, v \in T(\mu) \} \).

Furthermore, each scalar \( r_{tu}(a) \) is independent of \( s \). An algebra \( A \) is a cellular algebra if it has a cell datum and in this case we call \( \{ C^*_\lambda \mid s, t \in T(\lambda), \lambda \in \Lambda \} \) a cellular basis of \( A \).

We recall the representation theory of cellular algebras in [16]. Every irreducible \( A \)-module arises in a unique way as the simple head of some cell module. For each \( \lambda \in \Lambda \) fix \( s \in T(\lambda) \) and let \( C^*_t = C^*_s + A^{\triangleright \lambda} \). The cell modules of \( A \) are the modules \( \Delta(\lambda) \) which are the free \( R \)-modules with basis \( \{ C^*_t \mid t \in T(\lambda) \} \). The cell module \( \Delta(\lambda) \) comes equipped with a natural bilinear form \( \phi_\Lambda \) which is determined by the equation
\[
C^*_stC^*_t = \phi_\Lambda(C^*_t ; C^*_t) \cdot C^*_ss \quad (\text{mod } A^{\triangleright \lambda}).
\]
The form $\phi_\lambda$ is $A$–invariant in the sense that $\phi_\lambda(xa, y) = \phi_\lambda(x, ya^*)$, for $x, y \in \Delta(\lambda)$ and $a \in A$. Consequently,
\[ \operatorname{Rad} \Delta(\lambda) = \{ x \in \Delta(\lambda) \mid \phi_\lambda(x, y) = 0 \text{ for all } y \in \Delta(\lambda) \} \]
is an $A$–submodule of $\Delta(\lambda)$ and $D^\lambda = \Delta(\lambda) / \operatorname{Rad} \Delta(\lambda)$ is either zero or absolutely irreducible. Graham and Lehrer have proved that $\{D^\lambda \mid D^\lambda \neq 0\}$ consists of a complete set of pairwise non-isomorphic irreducible $A$-modules.

Now, we use the representation theory of a cellular algebra to prove Theorem 5.41 the main result of this section.

**Theorem 5.41.** Let $R$ be a commutative ring which contains the invertible elements $q, u_1, u_2, \ldots, u_r$ and $q - q^{-1}$. Suppose that $\Omega \cup \{\emptyset\}$ is $u$–admissible. Let $\mathcal{B}_{r,n}$ be the cyclotomic BMW algebras over $R$ with $2 \nmid r$. Then $\mathcal{B}_{r,n}$ is free over $R$ with
\[ \mathcal{C} = \{ C_{(s, \rho, e)}(t, \kappa, \delta) \mid (s, \rho, e), (t, \kappa, \delta) \in \delta(f, \lambda), \text{ where } (f, \lambda) \in \Lambda_{r,n}^+ \} \]
as its an $R$–basis. Further, $\mathcal{C}$ is a cellular basis of $\mathcal{B}_{r,n}(u)$.

**Proof.** By Proposition 5.37, $\mathcal{B}_{r,n}$ is an $R$-module spanned by $\mathcal{C}$. First, we assume $R = R_0$ where $R_0 = \mathbb{Z}[u, q^{\pm 1}, (q - q^{-1})^{-1}]$ and $u, q$ are indeterminates over $\mathbb{Z}$. we prove that $\mathcal{C}$ is $R_0$-linear independent. As $\mathbb{R}$ is not finitely generated over $\mathbb{Q}$, we can take $r + 1$ algebraically independent transcendental real numbers $v_i \in \mathbb{R}$ and $q$. We define $R_1 = \mathbb{Z}[v_1, v_2, \ldots, v_r, q^{\pm 1}, \delta]$. Then $R_1 \cong R_0$ as ring isomorphism. Therefore, $\mathcal{B}_{r,n}$ over $R_0$ is isomorphic to $\mathcal{B}_{r,n}$ over $R_1$ as $R_0$-algebra.

We have constructed the seminormal representations for $\mathcal{B}_{r,n}$ with respect to all $\lambda \in \Lambda_{r,n}^+(n - 2f)$, $0 \leq f \leq \lfloor \frac{n}{2} \rfloor$ under the conditions in Lemma 4.6 and (4.18). In particular, by Lemma 5.42 we have seminormal representations of $\mathcal{B}_{r,n}$ over $\mathbb{R}$. We remark that we are assuming that $\Omega \cup \emptyset$ is $v$–admissible. By arguments in the proof of Theorem 5.3 in [3], we have that $\Delta(\lambda)$ are irreducible $\mathcal{B}_{r,n}$-modules for all $\lambda \in \Lambda_{r,n}(n - 2f)$ and $0 \leq f \leq \lfloor \frac{n}{2} \rfloor$. Further, $\Delta(\lambda) \nmid \Delta(\mu)$ if $\lambda \neq \mu$. By Wedderburn–Artin theorem on semisimple finite dimension algebras,
\[ \dim_R \mathcal{B}_{r,n} \geq \dim_R \mathcal{B}_{r,n} / \operatorname{Rad} \mathcal{B}_{r,n} \geq \sum_{(f, \lambda) \in \Lambda_{r,n}^+} \# \mathcal{T}_n^{ud}(\lambda) = r^n (2n - 1)!! , \]
the last equality follows from classical branching rule for cyclotomic Brauer algebras, which was proved in Theorem 5.11 in [22]. It was also proved in [3, 5.2]. Therefore, $\dim \mathcal{B}_{r,n} = r^n (2n - 1)!!$ and $\mathcal{C}$ is $R_1$-linear independent. So is over $R_0$. This shows that $\mathcal{C}$ is an $R_0$ basis of $\mathcal{B}_{r,n}$. By base change, $\mathcal{C}$ is an $R$–basis for an arbitrary commutative ring. Further, by Proposition 5.37, $\mathcal{C}$ is a cellular basis of $\mathcal{B}_{r,n}$ as required.

In Theorem 5.41 we have assumed that $r$ is odd. We remark that the only place we need this assumption is that we use Proposition 5.37 to prove that $\mathcal{B}_{r,n}$ is an $R$-module spanned by $\mathcal{C}$.

When $r = 1$, $\mathcal{B}_{r,n}$ is the usual BMW algebra defined in [8]. It has been proved in [25] that BMW algebra is cellular. Late, Enyang gave another proof of this result in [11].

6. **Classification of the irreducible $\mathcal{B}_{r,n}(u)$ – modules**

In this section we assume that $F$ is a field which contains invertible elements $u_1, \ldots, u_r$, $q$ and $q - q^{-1}$. We also assume that $\Omega \cup \{\emptyset\}$ is $u$–admissible. By Theorem 5.41, $\mathcal{B}_{r,n}$ is a subalgebra of $\mathcal{B}_{r,n_1}$ if $n \leq n_1$. Therefore, we will identify $\mathcal{B}_{r,n}$ with $\mathcal{B}_{r,n}$ defined in the previous section.

We are going to classify the irreducible $\mathcal{B}_{r,n}$–modules over $F$. We remark that we assume that $r$ is odd.
All modules considered in this section are right modules.

**Lemma 6.1.** Given a positive integer \( f \leq \left\lceil \frac{n}{2} \right\rceil \). We have \( E_f \mathcal{B}_{r,n} E_f = \mathcal{B}_{r,n-2f} E_f \).

**Proof.** First, we assume that \( f = 1 \). By Theorem 5.41, the result follows if we prove \( E_{n-1} h E_{n-1} \in \mathcal{B}_{r,n-2} E_{n-1} \) for each cellular basis element \( h = T_{r,n} T_d \), where \((r, s, t, c, \kappa, d) \in \delta(f, \lambda)\).

By Lemma 5.11 \( E_{n-1} T_r X^p E_{n-1} M_{st} X^d = N_1 \) where \( N_1 \) is the \( R \)-submodule of \( \mathcal{B}_{r,n} \) generated by \( E_{n-1} E_{n-1} \in \mathcal{B}_{r,n-2} E_{n-1} \) where \( d \in D_{1,n} \) and \(-p \leq k \leq p\). Further, by Corollary 5.1, (b) for \( E_{n-1} X^k T_{d} E_{n-1} \), we have

\[
E_{n-1} T_r X^p E_{n-1} M_{st} X^d E_{n-1} = E_{n-1} E_{n-1} E_{n-1}.
\]

The inverse inclusion follows since \( \mathcal{B}_{r,n-2} E_{n-1} = E_{n-1} \mathcal{B}_{r,n-2} E_{n-1} \subset E_{n-1} \mathcal{B}_{r,n} E_{n-1} \). Using the result for \( f = 1 \) repeatedly, we have \( E_f \mathcal{B}_{r,n} E_f = \mathcal{B}_{r,n-2f} E_f \) for all positive integers \( f \leq \left\lceil \frac{n}{2} \right\rceil \). \( \square \)

It is proved in [9] that \( \bigcup_{\lambda \in \Lambda^+_r(n)} \{ m_{st} \mid s, t \in \mathcal{T}^{std}(\lambda) \} \) is a cellular basis for \( \mathcal{H}_{r,n} \). Let \( \Delta(\lambda) \) be the cell module of \( \mathcal{H}_{r,n} \) defined by this cellular basis. Let \( \phi_\lambda \) be the corresponding symmetric associative bilinear form. Let \( \phi_{f,\lambda} \) be the symmetric associative bilinear form on the cell module \( \Delta(f, \lambda) \) which is defined via the cellular basis of \( \mathcal{B}_{r,n} \) given in Theorem 5.41.

**Lemma 6.2.** Assume that \( (f, \lambda) \in \Lambda^+_r(n) \).

a) Let \( f \neq n/2 \). Then \( \phi_{f,\lambda} \neq 0 \) if and only if \( \phi_\lambda \neq 0 \).

b) Let \( f = n/2 \), and assume that \( \omega_\alpha \neq 0 \) for some non-negative integer \( \alpha \leq M - 1 \).

Then \( \phi_{f,\lambda} \neq 0 \).

c) If \( \omega_i = 0 \) for all non-negative integers \( i \leq r - 1 \), then \( \phi_{f,\lambda} = 0 \) for \( f = n/2 \).

**Proof.** (a) can be proved by arguments similar to those for [21, 3.1]. In order to prove (b), we assume that \( \ell \in \mathbb{Z} \) and \( k \in \mathbb{Z}^{\geq 0} \). We have

\[
\omega^{(\ell)}_{2k+1} E_{2k+1} E_{2k-1} \cdots E_1
= E_{2k+1} X^\ell_{2k+1} E_{2k+1} E_{2k-1} \cdots E_1
= E_{2k+1} X^\ell_{2k+1} E_{2k+1} E_{2k-1} E_{2k-1} E_{2k-1} \cdots E_1
= \vdots
= E_{2k+1} X^\ell_{2k+1} E_{2k+1} E_{2k-1} E_{2k-1} \cdots E_1.
\]

Since we are assuming that \( \Omega \cup\{q\} \) is \( u \)-admissible, by Theorem 5.41 \( C \) is an \( F \)-basis of \( \mathcal{B}_{r,n} \). Since \( E_{2k+1} E_{2k-1} E_{2k-1} \cdots E_1 \in C \), \( \omega^{(\ell)}_{2k+1} = \omega_\ell \). So,

\[
\phi_{\omega,0}(E^\omega, E^\omega X^\ell_{n-1} \cdots X^\omega_{X^\ell_1}) = (\omega_{\ell})^\omega \neq 0.
\]

This proves (b).

Suppose that \( \alpha, \beta \in \mathbb{N}^r \) for \( f = n/2 \). Using Lemma 6.1 repeatedly, we have, for any \( w \in \mathcal{S}_n \)

\[
E^\omega X^\alpha \cdot T_w \cdot X^\beta E^\omega = E^\omega h E_1
\]

for some \( h \in \mathcal{B}_{r,2} \). By direct computation, \( E_1 h E_1 = 0 \) for all \( h \in \mathcal{B}_{r,2} \), forcing \( E^\omega h E_1 = 0 \). Therefore, \( \phi_{\omega,0} = 0 \). This proves (c). \( \square \)
Lemma 6.2 sets up a relationship between the irreducible $\mathcal{B}_{r,n}$-modules and the irreducible $\mathcal{H}_{r,n-2f}$-modules for all non-negative integers $f \leq \lfloor \frac{n}{2} \rfloor$. Note that we can keep the assumption that $u_i = q^{k_i}, k_i \in \mathbb{Z}$ by using Dipper-James-Mathas’s Morita equivalent theorem for $\mathcal{H}_{r,n-2f}$. In “separate condition”, such a result was proved in [10]. By [2], [4] and [6], irreducible $\mathcal{H}_{r,n-2f}$-modules are indexed by $u$-Kleshchev $r$-multipartitions of $n - 2f$.

**Theorem 6.3.** Suppose $F$ is a field which contains non-zero elements $q, u_1, \ldots, u_r$ and $q - q^{-1}$. Assume that $\Omega \cup \{q\}$ is $u$-admissible. Let $\mathcal{B}_{r,n}, 2 \nmid r$ be the cyclotomic BMW algebra over $F$.

a) If $n$ is odd, then the set of all pair-wise non-isomorphic irreducible $\mathcal{B}_{r,n}$-modules are indexed by $(f, \lambda)$ where $0 \leq f \leq \lfloor \frac{n}{2} \rfloor$ and $\lambda$ are $u$-Kleshchev multipartitions of $n - 2f$.

b) Suppose that $n$ is an even number.

(i) If $\omega_i \neq 0$ for some non-negative integers $i \leq r - 1$, then the set of all pair-wise non-isomorphic irreducible $\mathcal{B}_{r,n}$-modules are indexed by $(f, \lambda)$ where $0 \leq f \leq \frac{n}{2}$ and $\lambda$ are $u$-Kleshchev multipartitions of $n - 2f$.

(ii) If $\omega_i = 0$ for all non-negative integers $i \leq r - 1$, then the set of all pair-wise non-isomorphic irreducible $\mathcal{B}_{r,n}$-modules are indexed by $(f, \lambda)$ where $0 \leq f < \frac{n}{2}$ and $\lambda$ are $u$-Kleshchev multipartitions of $n - 2f$.

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H. Rui, Department of Mathematics, East China Normal University, 200062 Shanghai, P.R. China.
E-mail address: hbrui@math.ecnu.edu.cn

J. Xu, Department of Mathematics, East China Normal University, 200062 Shanghai, P.R. China.
E-mail address: 52060601009@student.ecnu.edu.cn