OPERATOR AMENABILITY OF FOURIER–STIELTJES ALGEBRAS, II

VOLKER RUNDE AND NICO SPRONK

Abstract. We give an example of a non-compact, locally compact group $G$ such that its Fourier–Stieltjes algebra $B(G)$ is operator amenable. Furthermore, we characterize those $G$ for which $A^*(G)$—the spine of $B(G)$ as introduced by M. Ilie and the second named author—is operator amenable and show that $A^*(G)$ is operator weakly amenable for each $G$.

1. Introduction

1.1. History and context. Let $G$ be a locally compact group, and let $L^1(G)$ and $M(G)$ denote its group and measure algebra, respectively. In [12], P. Eymard introduced the Fourier algebra $A(G)$ and the Fourier–Stieltjes algebra $B(G)$ of $G$. In the framework of Kac algebras [11], $A(G)$ and $L^1(G)$ as well as $B(G)$ and $M(G)$ can be viewed as dual to one another: this duality generalizes the well known dual group construction for abelian groups.

It is a now classical theorem of B. E. Johnson [20] that $L^1(G)$ is amenable if and only if $G$ is an amenable group. On the other hand, $A(G)$ is an amenable Banach algebra only if $G$ has an abelian subgroup of finite index [15, 29]. In order to obtain the appropriate statement dual to Johnson’s theorem, we first need to recognize that $L^1(G)$—as the predual of the von Neumann algebra $L_\infty(G)$—is canonically equipped with an operator space structure. In [27], Z.-J. Ruan modified Johnson’s notion of Banach algebraic amenability from [20] by considering only completely bounded module actions and derivations and obtained the notion of operator amenability. Since $L_\infty(G)$ is abelian, the canonical operator space structure of $L^1(G)$ is max $L^1(G)$, so all bounded maps from $L^1(G)$ are automatically completely bounded; consequently, $L^1(G)$ is operator amenable if and only if it is amenable. As the predual of the group von Neumann algebra $VN(G)$, the Fourier algebra $A(G)$ also carries a natural operator space structure, and in [27], Ruan showed that $A(G)$ is operator amenable if and only if $G$ is amenable. We further note that Johnson [21] proved that $L^1(G)$ is always weakly amenable.

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(see [8] for a simpler proof) whereas $A(G)$ is known to be weakly amenable only if $G$ if the component of the identity is abelian [15]. On the other hand, $A(G)$ is always operator weakly amenable [33, 31, 32].

The questions for which $G$ the measure algebra $M(G)$ is amenable or weakly amenable, respectively, were eventually settled by H. G. Dales, F. Ghahramani and A. Ya. Helemskii [7]: $M(G)$ is amenable if and only if $G$ is discrete and amenable, and it is weakly amenable if and only if $G$ is discrete. In abelian group duality—and more generally in Kac algebra theory—, the property dual to discreteness is compactness. Thus, parallel to the $L^1(G)$-$A(G)$ situation, one is led to expect that $B(G)$ is operator (weakly) amenable if and only if $G$ is compact, as was conjectured in both [30] and [33] (see also [17, Problem 9]). In fact, it was shown in [30] that $B(G)$ is operator amenable with operator amenability constant $C < 5$ if and only if $G$ is compact. At the time [30] was written, the authors felt that the condition imposed on the amenability constant was unnecessary.

In the present article, we refute our previous conjecture by exhibiting examples of non-compact groups $G$ for which $B(G)$ is operator amenable. Moreover, we show that the operator amenability constant of $B(G)$ for those $G$ is precisely 5: this shows that the estimate for this constant from [30] cannot be improved. All of our examples, which are taken from work by L. Bagget [3] and G. Mauceri and M. A. Picardello [26], are separable Fell groups with countable dual spaces.

The groups $G$ for which $B(G)$ is amenable were characterized in [15]: they are precisely those compact $G$ with an abelian subgroup of finite index. In analogy with the corresponding result for $M(G)$, one might conjecture that $B(G)$ is weakly amenable exactly when $G$ is compact with an abelian connected component of the identity. Our examples show that that this natural conjecture is false too.

Furthermore, we establish some amenability results for the spine $A^*(G)$ of $B(G)$, which was introduced and studied by M. Ilie and the second-named author in [18].

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1.2. Definitions and notation. The Fourier and Fourier–Stieltjes algebra, $A(G)$ and $B(G)$, of a locally compact group $G$ were introduced in [12]. the Fourier–Stieltjes algebra $B(G)$ is the space of all coefficient functions of weakly operator continuous unitary representations on Hilbert spaces, i.e., $B(G) = \{ G \ni s \mapsto \langle \pi(s)\xi | \eta \rangle : (\pi, H) \in \Sigma_G, \xi, \eta \in H \}$, where $\Sigma_G$ denotes the class of such representations. Using direct sums and tensor products of representations, it can be verified that $B(G)$ is an algebra of functions. Moreover, $B(G)$ is the dual space of the enveloping $C^*$-algebra of $G$, and under this norm is, in fact, a Banach algebra. The left regular representation $(\lambda, L^2(G))$ is defined through left translation on the space of
square integrable functions with respect to left Haar measure. The Fourier algebra $A(G)$ is the space of all coefficients of $\lambda$, and is a closed ideal in $B(G)$. Moreover, $B(G)$ is the predual of the enveloping von Neumann algebra, $W^*(G)$, and $A(G)$ is the predual of the group von Neumann algebra, $VN(G) := \lambda(G)^\tau$.

In its capacity as the predual of $W^*(G)$, $B(G)$ is an operator space by [4]. Our standard reference for operator spaces and completely bounded maps is [10]. Taking the adjoint of the multiplication map $m_0 : B(G) \otimes B(G) \to B(G)$, one obtains a $^*$-homomorphism—and thus a complete contraction—$m_0^* : W^*(G) \to W^*(G) \overline{\otimes} W^*(G)$, where $\overline{\otimes}$ denotes the von Neumann tensor product. Consequently, $m_0$ extends to a complete contraction $m : B(G) \hat{\otimes} B(G) \to B(G)$, where $\hat{\otimes}$ stands for the operator space projective tensor product [5, 9]. All in all, $B(G)$ a completely contractive Banach algebra. It is clear that every closed subalgebra of $B(G)$—such as $A(G)$—is also a completely contractive Banach algebra. Note that that $B(G) \hat{\otimes} B(G)$ is canonically completely isometrically isomorphic to a closed subspace of $B(G \times G)$, and that $A(G) \hat{\otimes} A(G) \cong A(G \times G)$ holds completely isometrically isomorphically [9].

If $\mathcal{A}$ is a completely contractive Banach algebra, a completely bounded $\mathcal{A}$-bimodule is an operator space $\mathcal{V}$, which is a module for which the module maps $\mathcal{A} \times \mathcal{V} \ni (a, v) \mapsto a \cdot v$ and $\mathcal{V} \times \mathcal{A} \ni (v, a) \mapsto v \cdot a$ extend to completely bounded maps $\mathcal{A} \hat{\otimes} \mathcal{V} \to \mathcal{V}$ and $\mathcal{V} \hat{\otimes} \mathcal{A} \to \mathcal{A}$. Dual modules of completely bounded $\mathcal{A}$-bimodules with the dual space operator space structure and the dual action are also completely bounded $\mathcal{A}$-bimodules. A completely contractive Banach algebra $\mathcal{A}$ is said to be operator amenable if, for every completely bounded $\mathcal{A}$-bimodule $\mathcal{V}$, every completely bounded derivation $D : \mathcal{A} \to \mathcal{V}^*$ is inner.

The concept of a bounded approximate diagonal can be readily adapted from the Banach algebra context [19] to the operator space setting: a completely contractive Banach algebra is amenable if, and only if, it admits a completely bounded approximate diagonal, i.e., a bounded net $(d_\alpha)_\alpha$ in $\mathcal{A} \hat{\otimes} \mathcal{A}$ for which

$$a \cdot d_\alpha - d_\alpha \cdot a \xrightarrow{\alpha} 0$$

and

$$m(d_\alpha)a \xrightarrow{\alpha} a$$

for each $a \in \mathcal{A}$, where $a \cdot (b \otimes c) = (ab) \otimes c$ and $(b \otimes c) \cdot a = b \otimes (ca)$ for $b, c \in \mathcal{A}$, and $m : \mathcal{A} \hat{\otimes} \mathcal{A} \to \mathcal{A}$ is the multiplication map. We say that $\mathcal{A}$ has operator amenability constant $C$, or is $C$-operator amenable, if $C$ is the largest number for which $\limsup_\alpha \|d_\alpha\| \geq C$ for any completely bounded approximate diagonal. (This notion is again adapted from the corresponding notion for Banach algebras, which was developed in [22] specifically to address non-amenability of Fourier algebras for certain compact groups.)

Following [2], we say that a (completely contractive) commutative Banach algebra $\mathcal{A}$ is (operator) weakly amenable if every (completely) bounded derivation $D : \mathcal{A} \to \mathcal{V}$ into a (completely) bounded symmetric bimodule—i.e., satisfying $a \cdot v = v \cdot a$ for $a \in \mathcal{A}$ and $v \in \mathcal{V}$—is zero.
2. Some operator amenable Fourier-Stieltjes algebras

For any prime number \( p \), let \( \mathbb{Q}_p \) denote the field of \( p \)-adic numbers, which is a locally compact field. It is defined to be the completion of the rational numbers \( \mathbb{Q} \) by the \( p \)-adic valuation \(| \cdot |_p\), which is a multiplicative, non-Archimedean valuation, i.e., satisfies \(|rs|_p = |r|_p|s|_p\) and \(|r + s|_p \leq \max\{|r|_p, |s|_p\}\) for \( r, s \in \mathbb{Q}_p \). We then let \( \mathcal{O}_p := \{ r \in \mathbb{Q}_p : |r|_p \leq 1 \} \), the \( p \)-adic integers, which is a compact open subring of \( \mathbb{Q}_p \). The multiplicative group of \( \mathcal{O}_p \) is \( \mathcal{T}_p := \{ r \in \mathbb{Q}_p : |r|_p = 1 \} \). The family of sets \( \{ p^k \mathcal{O}_p \}_{p=0}^{\infty} \) forms a basis of neighborhoods of 0, so that \( \mathbb{Q}_p \) is totally disconnected.

Let \( GL(n, \mathcal{O}_p) \) denote the multiplicative group of \( n \times n \) matrices with entries in \( \mathcal{O}_p \) and determinant of valuation 1. This compact group acts on the vector space \( \mathbb{Q}_p^n \) by matrix multiplication, and we set \( G_{p,n} := GL(n, \mathcal{O}_p) \rtimes \mathbb{Q}_p^n \).

For \( n = 1 \), this is the group \( \mathcal{T}_p \rtimes \mathbb{Q}_p \) of \([3]\). It is reasonable to call \( G_{p,n} \) the “\( n \)th rigid \( p \)-adic motion group”.

In \([26]\), it was shown—using the “Mackey machine”—that the dual space of \( \hat{G}_{p,n} \)—the set of all (equivalence classes) of irreducible continuous unitary representations of \( G_{p,n} \)—is countable. In fact each group \( G_{p,n} \) is of the form \( G = K \rtimes A \) where

1. \( K \) is a compact group acting on an abelian group \( A \), with each of the groups separable, and
2. the dual space \( \hat{G} \) is countable and decomposes as \( \hat{K} \circ q \sqcup \{ \lambda_k \}_{k=1}^{\infty} \)
   where \( \hat{K} \) is the discrete dual space of \( K \), \( q: G \to K \) is the quotient map, and each \( \lambda_k \) is a subrepresentation of the left regular representation.

The following was proven for \( G_{p,1} \) independently in \([25]\) and \([35]\). A proof for general \( G_{p,n} \) can be obtained in a similar way. For the reader’s convenience, we give a proof for general groups of the form \( G = K \rtimes A \) satisfying (1) and (2).

**Proposition 2.1.** For \( G = K \rtimes A \) as above, \( B(G) = A(K)\circ q \oplus \ell_1 A(G) \) holds.

**Proof.** Let \( u \in B(G) \), so that \( u = \langle \pi(\cdot) \xi | \eta \rangle \) for some \((\pi, \mathcal{H}) \in \Sigma_G \) and \( \xi, \eta \in \mathcal{H} \). By \([34]\) Theorem 4.5] and (2) above, \( \pi \) is totally decomposable.

We may thus write we write

\[
\pi = \bigoplus_{\sigma \in \hat{K}} \alpha_{\sigma} \cdot \sigma \circ q \oplus \bigoplus_{k=1}^{\infty} \beta_k \cdot \lambda_k,
\]

where \( \alpha_{\sigma} \) and \( \beta_k \) are multiplicity constants. For \( \sigma \in \hat{K} \) and \( k \in \mathbb{N} \), let \( P_{\sigma} \) and \( P_k \) denote the orthogonal projection associated with \( \alpha_{\sigma} \cdot \sigma \circ q \) and \( \beta_k \cdot \lambda_k \),
respectively. We then obtain for $s \in G$ that

$$u(s) = \sum_{\sigma \in K} \langle \alpha_\sigma \cdot \sigma  \circ q(s) P_\sigma \xi | P_\sigma \eta \rangle + \sum_{k=1}^{\infty} \langle \beta_k \cdot \lambda_k(s) P_k \xi | P_k \eta \rangle.$$ 

By [14] (3.13) Corollaire, which uses standard von Neumann algebra techniques, this is an $\ell^1$-direct sum, i.e.,

$$\|u\| = \sum_{\sigma \in K} \|\langle \alpha_\sigma \cdot \sigma  \circ q(\cdot) P_\sigma \xi | P_\sigma \eta \rangle\| + \sum_{k=1}^{\infty} \|\langle \beta_k \cdot \lambda_k(\cdot) P_k \xi | P_k \eta \rangle\|.$$ 

Each $\langle \alpha_\sigma \cdot \sigma  \circ q(\cdot) P_\sigma \xi | P_\sigma \eta \rangle$ lies in $B(K) \circ q$, and each $\langle \beta_k \cdot \lambda_k(\cdot) P_k \xi | P_k \eta \rangle$ belongs to $A(G)$. All in all, we see that $u = u_1 + u_2$ with $\|u\| = \|u_1\| + \|u_2\|$, where $u_1 \in A(K) \circ q$ and $u_2 \in A(G)$.

Since $GL(n, \mathbb{O}_p)$ and $G_{p,n}$ are both disconnected, it follows from [14] Section 5], that $A(GL(n, \mathbb{O}_p)) \circ q \cong A(GL(n, \mathbb{O}_p))$ and $A(G_{p,n})$ are both generated by idempotents. Hence, it follows from Proposition 2.1 that $B(G_{p,n})$ is generated by idempotents as well.

From the remarks following the proof of [2] Theorem 1.4], we thus obtain:

**Corollary 2.2.** For each $n \in \mathbb{N}$, the Fourier–Stieltjes algebra $B(G_{p,n})$ is weakly amenable.

Since weak amenability implies operator weak amenability, Corollary 2.2 already shows that—contrary to what one might expect in view of [7]—there are non-compact, locally compact groups with an operator weakly amenable Fourier–Stieltjes algebra.

We shall now see that $B(G_{p,n})$ is even operator amenable. In fact, we will work again in the slightly more general setting of Proposition 2.1.

**Theorem 2.3.** For $G = K \ltimes A$ as in Proposition 2.1 $B(G) = A(K) \circ q \oplus \ell^1 A(G)$ is operator amenable with operator amenability constant 5.

**Proof.** This proof is adapted from [28] Theorem 3.1(i)]].

Since $K$ and $G = K \ltimes A$ are amenable groups, it follows from (an inspection of) [27] that $A(K) \circ q \cong A(K)$ and $A(G)$ are each 1-operator amenable. Let $(u_\alpha)_{\alpha \in A}$ be a norm 1 completely bounded approximate diagonal for $A(G)$ and let $(v_\beta)_{\beta \in B}$ be such for $A(K)$. Since $K$ is compact, we can arrange for $v_\beta(k, k) = 1$ for all $k \in K$. Again, $m : B(G) \otimes B(G) \to B(G)$ denotes the completely contractive multiplication map.

Let $\Gamma = A \times A^4 \times B$ be the product directed set [23] p. 69], let $e_\alpha = m(u_\alpha)$ for each $\alpha \in A$, and, for each $\gamma = (\alpha, (\alpha_{\alpha}'), \alpha_{\alpha}'' \in A, \beta)$ in $\Gamma$, set

$$w_\gamma = ((1 - e_\alpha) \otimes (1 - e_{\alpha'})) + u_{\alpha''} v_{\beta} \circ (q \times q)$$

$$= (1 \otimes 1 - e_\alpha \otimes 1 - 1 \otimes e_{\alpha'} + e_\alpha \otimes e_{\alpha'} + u_{\alpha''}) v_{\beta} \circ (q \times q).$$
Since
\[ B(G) \hat{\otimes} B(G) = A(K) \hat{\otimes} A(K) \]
\[ \oplus_{\ell^1} A(K) \otimes A(G) \oplus_{\ell^1} A(G) \otimes A(K) \]
by [30, Lemma 3.1], we see that
\[ \|w_\gamma\| \leq (1 + \|e_\alpha\| + \|e_{\alpha'\alpha}\| + \|u_{\alpha'\alpha}\|) \|v_\beta\| = 5 \]
for each \( \gamma \in \Gamma \).

We shall now verify that \((w_\gamma)_{\gamma \in \Gamma}\) is a completely bounded approximate diagonal for \(B(G)\).

We first check that \((w_\gamma)_{\gamma \in \Gamma}\) is asymptotically central for the right and left module actions. For \(u \in A(G)\), we have
\[ \|u \cdot w_\gamma - w_\gamma \cdot u\| \]
\[ \leq \|(u - u e_\alpha) \otimes (1 - e_{\alpha'\alpha}) - (1 - e_\alpha) \otimes (u - u e_{\alpha'\alpha}) + u u_{\alpha'\alpha} - u_{\alpha'\alpha} \cdot u\| \]
\[ \leq 2 \|u - u e_\alpha\| + 2 \|u - u e_{\alpha'\alpha}\| + \|u u_{\alpha'\alpha} - u_{\alpha'\alpha} \cdot u\|, \]
which can be made arbitrarily small for sufficiently large \(\alpha\) and \(\alpha'\). For \(v \in A(K)\), we have
\[ \|(v \cdot q) \cdot w_\gamma - w_\gamma \cdot (v \cdot q)\| \leq 5 \|v \cdot v_\beta - v_\beta \cdot v\| \]
which can be made arbitrarily small for sufficiently large choices of \(\beta\). For general \(w = v \cdot q + u \in B(G)\) with \(v \in A(K)\) and \(u \in A(G)\), it is thus clear that \(\|w \cdot w_\gamma - w_\gamma \cdot w\|\) can be made arbitrarily small for sufficiently large \(\gamma \in \Gamma\).

Next, we check that \((m(w_\gamma))_{\gamma \in \Gamma}\) is an approximate identity for \(B(G)\).

Note that, for \(\gamma = (\alpha, (\alpha'_{\alpha'\alpha})_{\alpha' \in A}, \beta) \in \Gamma\) we have
\[ m(w_\gamma) = 1 + (e_{\alpha'\alpha} - 1)e_\alpha. \]

For \(v \in A(K) \otimes q\), we then obtain
\[ \lim_{\gamma} m(w_\gamma)v = \lim_{\alpha} (e_{\alpha'\alpha} - 1)e_\alpha v = 0 \]
because \(\lim_{\alpha'} (e_{\alpha'\alpha} - 1)e_\alpha v = 0\) for each \(\alpha\) as \(e_{\alpha} v \in A(G)\). A similar calculation shows that \(\lim_{\gamma} m(w_\gamma)u = u\) for \(u \in A(G)\).

Consequently, \((w_\gamma)_{\gamma \in \Gamma}\) is indeed a completely bounded approximate diagonal for \(B(G)\), so that the operator amenability constant \(C\) of \(B(G)\) can at most be 5. Since \(G\) is not compact, \(C < 5\) cannot occur by [30, Theorem 3.2]. Hence, \(C = 5\) must hold. 

\[ \square \]

Remark 2.4. The operator amenability of \(B(G)\) in Theorem [28] can easily be obtained by observing that \(A(G)\) is an operator amenable ideal in \(B(G)\) with operator amenable quotient \(B(G)/A(G) \cong A(K)\) and then applying the operator space analog of [28, Theorem 2.3.7]. The disadvantage in doing this, however, is that it yields no information on the operator amenability constant of \(B(G)\).
Remark 2.5. In [30], we introduced, for an arbitrary locally compact group $G$, a decomposition $B(G) = A_{\mathcal{T}}(G) \oplus \ell_1 A_{\mathcal{T}_{\mathcal{LF}}}(G)$, which can be interpreted as dual to the decomposition of $M(G)$ into the discrete and the continuous measures. For non-discrete $G$, it is well known that there are continuous measures in $M(G) \setminus L^1(G)$, and we conjectured that, at least for amenable $G$, the inclusion $A(G) \subset A_{\mathcal{T}_{\mathcal{LF}}}(G)$ is proper unless $G$ is compact [30, p. 681, Remarks (2)]. Theorem 2.3 shows that this conjecture is false.

3. Operator amenability of the spine

3.1. The spine of $B(G)$. The main theorem of the previous section is actually a particular case of a more general result. The groups $G_{p,n}$ are all non-compact amenable groups for which $A^*(G_{p,n}) = B(G_{p,n})$ holds, where $A^*(G_{p,n})$ is the spine of $B(G_{p,n})$ as defined in [18]. In this section, we study amenability properties for spines.

We recall the definition of the spine of a Fourier–Stieltjes algebra $B(G)$ of a locally compact group $G$ below. Full details are presented in the article [18].

Let $\mathcal{T}_{\mathcal{aq}}(G)$ denote the family of all group topologies $\tau$ on $G$ with the following properties:

- The completion $G_{\tau}$ of $G$ with respect to the left uniformity generated by $\tau$ is a locally compact group. (This completion is unique up to homeomorphic isomorphism, and it is the same completion as gained from the right uniformity.)
- If $\tau_{ap}$ is the coarsest topology making the almost periodic compactification map $\eta : G \to G_{ap}$ continuous, then $\tau \supseteq \tau_{ap}$.

We call such topologies non-quotient locally precompact topologies. The family $\mathcal{T}_{\mathcal{aq}}(G)$ is a semilattice, i.e., a commutative, idempotent semigroup, under the operation $(\tau_1, \tau_2) \mapsto \tau_1 \lor \tau_2$, where $\tau_1 \lor \tau_2$ is the coarsest topology which is simultaneously finer than both $\tau_1$ and $\tau_2$. In particular, this semilattice is unital with unit $\tau_{ap}$.

For each $\tau \in \mathcal{T}_{\mathcal{aq}}(G)$ we let $\eta_{\tau} : G \to G_{\tau}$ be the natural map into the completion. Then the Fourier algebra $A(G_{\tau})$ is completely isometrically isomorphic to the subalgebra $A_{\tau}(G) = A(G_{\tau}) \circ \eta_{\tau}$ of $B(G)$. We have that $A_{\tau_1}(G) \cap A_{\tau_2}(G) = \{0\}$ if $\tau_1 \neq \tau_2$ and $\|u_{\tau_1} + u_{\tau_2}\| = \|u_{\tau_1}\| + \|u_{\tau_2}\|$ for $u_{\tau_j} \in A_{\tau_j}(G)$ ($j = 1, 2$), in this case. The spine is then the algebra

$$A^*(G) = \ell^1 - \bigoplus_{\tau \in \mathcal{T}_{\mathcal{aq}}(G)} A_{\tau}(G),$$

which is graded over $\mathcal{T}_{\mathcal{aq}}(G)$, i.e., $u_{\tau_1}u_{\tau_2} \in A_{\tau_1 \lor \tau_2}(G)$ for $u_{\tau_j} \in A_{\tau_j}(G)$ ($j = 1, 2$).

As in [18, Section 6.2], we can calculate that $\mathcal{T}_{\mathcal{aq}}(G_{p,n}) = \{\tau_{ap}, \tau_{p,n}\}$ for any of the groups $G_{p,n}$, where $\tau_{p,n}$ is the given topology on $G_{p,n}$. Moreover, the quotient map $q : G_{p,n} \to GL(n, \mathbb{C})$ is the almost periodic compactification map. Consequently, $B(G_{p,n}) = A^*(G_{p,n})$ holds.
3.2. Operator amenability of $A^*(G)$. For our discussion of the operator amenability of $A^*(G)$, we introduce some auxiliary notation. For any $F \subseteq T_{nq}(G)$, let $\langle F \rangle$ denote the sublattice of $T_{nq}(G)$ it generates, and define

$$A^*_F(G) = \ell^1 - \bigoplus_{\tau \in \langle F \rangle} A_\tau(G).$$

Note that, if $F \subseteq T_{nq}(G)$ is finite, then so is $\langle F \rangle$.

Since $T_{nq}(G)$ is finite for each of the groups $G_{p,n}$, the following lemma extends (the qualitative part of) Theorem 2.3.

**Proposition 3.1.** Let $G$ be an amenable, locally compact group, and let $F \subseteq T_{nq}(G)$ be finite. Then $A^*_F(G)$ is operator amenable.

**Proof.** We shall prove that $A^*_F(G)$ is amenable by using induction on $|F|$.

Suppose that $|F| = 1$, so that $F = \{\tau\}$ for some $\tau \in T_{nq}(G)$. Since $G$ is amenable, so is $G_\tau$ by [28, Proposition 1.2.1], which implies that $A^*_F(G) \cong A(G_\tau)$ is operator amenable by [27].

Now suppose that $|F| > 1$. Fix $\tau \in F$, let $F' := F \setminus \{\tau\}$, and set

$$I_{\tau,F} := A^*_{\tau \vee F'}(G) + A_\tau(G),$$

where $\tau \vee F' = \{\tau \vee \tau' : \tau' \in F'\}$. Then $I_{\tau,F}$ is an ideal in $A^*_F(G)$ containing $A^*_{\tau \vee F'}(G)$ as an ideal. Since $A^*_{\tau \vee F'}(G)$ is operator amenable by the induction hypothesis, and since $I_{\tau,F}/A^*_{\tau \vee F'}(G)$ is either $A_\tau(G)$ or $\{0\}$, we conclude from the completely bounded analogue of [28, Theorem 2.3.10] that $I_{\tau,F}$ is operator amenable. Since $A^*_F(G) = A^*_F(G) + I_{\tau,F}$ and $A^*_F(G)$ is operator amenable by induction hypothesis, a similar argument yields the operator amenability of $A^*_F(G)$. \(\square\)

It is immediate from Proposition 3.1 that $A^*(G)$ is operator amenable if $G$ is amenable and $T_{nq}(G)$ is finite. We shall see that these are the only conditions under which $A^*(G)$ can be operator amenable.

For the following lemma, note that, by linearity and continuity, the product of $T_{nq}(G)$ extends to $\ell^1(T_{nq}(G))$ turning it into a Banach algebra. Since the canonical operator space structure of $\ell^1(T_{nq}(G))$ is $\max \ell^1(T_{nq}(G))$, this Banach algebra is canonically completely contractive.

**Lemma 3.2.** Let $G$ be a locally compact group. Then the map

$$\Pi : A^*(G) \to \ell^1(T_{nq}(G)), \quad (u_\tau)_{\tau \in T_{nq}(G)} \mapsto \sum_{\tau \in T_{nq}(G)} u_\tau(e) \delta_\tau$$

is a complete quotient map and an algebra homomorphism.

**Proof.** Note that $VN^*(G) := \ell^\infty - \bigoplus_{\tau \in T_{nq}(G)} VN(G_\tau)$ is the dual space of $A^*(G)$. For each $\tau \in T_{nq}(G)$, let $p_\tau \in VN^*(G)$ be the central projection corresponding to the identity element of $VN(G_\tau)$. Then

$$\ell^\infty(T_{nq}(G)) \to VN^*(G), \quad (\lambda_\tau)_{\tau \in T_{nq}(G)} \mapsto (\lambda_\tau p_\tau)_{\tau \in T_{nq}(G)}$$
is a normal *-monomorphism and the adjoint of $\Pi$. This shows that $\Pi$ is indeed a complete quotient map.

To see that $\Pi$ is multiplicative, let $\tau_1, \tau_2 \in \mathcal{T}_{nq}(G)$, and let $u_{\tau_j} \in A_{\tau_j}(G)$ for $j = 1, 2$. It follows that $u_{\tau_1}u_{\tau_2} \in A_{\tau_1 \vee \tau_2}(G)$ and thus

$$\Pi(u_{\tau_1}u_{\tau_2}) = u_{\tau_1}(e)u_{\tau_2}(e)\delta_{\tau_1 \vee \tau_2} = u_{\tau_1}(e)\delta_{\tau_1} u_{\tau_2}(e)\delta_{\tau_2} = \Pi(u_{\tau_1})\Pi(u_{\tau_2}).$$

By linearity and continuity, this proves the multiplicativity of $\Pi$. \qed

**Theorem 3.3.** Let $G$ be a locally compact group. Then $A^*(G)$ is operator amenable if and only if $G$ is amenable and $\mathcal{T}_{nq}(G)$ is finite.

**Proof.** The “if” part is provided by Proposition 3.1.

For the “only if” part, suppose that $A^*(G)$ is operator amenable. The central projection $p_G \in VN^*(G)$ corresponding to the identity operator in $VN(G)$ forms a completely contractive projection onto $A(G)$ via the predual action, $A^*(G) \ni u \mapsto p_Gu$. Thus $A(G)$ is a completely complemented ideal in $A^*(G)$ and hence is operator amenable by the completely bounded analog of [28, Theorem 2.3.7]. Therefore, by [27], $G$ is amenable.

Since $A^*(G)$ is operator amenable, so is its quotient $\ell^1(\mathcal{T}_{nq}(G))$, and since the canonical operator space structure of $\ell^1(\mathcal{T}_{nq}(G))$ is max $\ell^1(\mathcal{T}_{nq}(G))$, it follows that $\ell^1(\mathcal{T}_{nq}(G))$ is amenable in the purely Banach algebraic sense. From [16, Theorem 2.7], we conclude that $\mathcal{T}_{nq}(G)$ is finite. \qed

**Example 3.4.** Using computations from [18, Section 6], we obtain that $A^*(G)$ is operator amenable for $G$ being any one of the following groups: the real numbers $\mathbb{R}$, the integers $\mathbb{Z}$, the Euclidean motion groups $M(n) = SO(n) \times \mathbb{R}^n$ for $n \in \mathbb{N}$, the $ax + b$ group, or $\mathbb{Q}_p$, where $p$ is any prime. On the other hand, the spine fails to be operator amenable for any of the groups $\mathbb{R}^n$ or $\mathbb{Z}^n$ with $n \geq 2$, for $\mathbb{Q}$ as a discrete group, and for any non-amenable group.

**Remark 3.5.** For any locally compact group $G$, let $B_0(G)$ denote the closed ideal of $B(G)$ consisting of functions vanishing at $\infty$. For the Euclidean motion groups, it is known (see, for example, the discussion on [6, p. 10]) that $B(M(n)) = A(SO(n))q \oplus_{eq} B_0(M(n))$. We suspect that $B_0(G)$ is never operator amenable when it is properly larger than $A(G)$ (see [13] for situations in which this is known to be the case). This would entail that $B(M(n))$ cannot be operator amenable.

**Remark 3.6.** In view of Theorems 2.3 and 3.3, we are prepared to make the conjecture that, for a locally compact group $G$, the Fourier–Stieltjes algebra $B(G)$ is operator amenable if and only if $B(G) = A^*(G)$, $G$ is amenable, and $\mathcal{T}_{nq}(G)$ is finite.

We note:

**Corollary 3.7.** Let $G$ be a locally compact group. Then $A^*(G)$ is amenable if and only if $G$ has an abelian subgroup group of finite index and $\mathcal{T}_{nq}(G)$ is finite.
Proof. If $A^*(G)$ is amenable, it is operator amenable, so that $T_{nq}(G)$ must be finite. Since $A(G)$ is a complemented ideal of $A^*(G)$, it must be amenable, too. Hence, $G$ has an abelian subgroup of finite index by [15, 29]. □

3.3. Operator weak amenability of $A^*(G)$. In contrast to Theorem 3.3, we have the following:

Proposition 3.8. Let $G$ be a locally compact group $G$. Then $A^*(G)$ is operator weakly amenable.

Proof. Let $V$ be an completely bounded symmetric $A^*(G)$-bimodule, and let $D : A^*(G) \to V$ be a completely bounded derivation. Then

$$D = \sum_{\tau \in T_{nq}(G)} D|_{A_{\tau}(G)}.$$ 

holds. Since $A_{\tau}(G) \cong A(G_{\tau})$ is operator weakly amenable by [33], it follows that $D|_{A_{\tau}(G)} = 0$ for each $\tau \in T_{nq}(G)$ and thus $D = 0$. □

Remark 3.9. It is not clear at all for which locally compact groups $G$, the spine $A^*(G)$ might be weakly amenable (in the original Banach algebraic sense). The spine is weakly amenable for any compact group with an abelian connected component of the identity by [15, Theorem 3.3] and also for any of the groups $G_{p,n}$ by Corollary 22. However, there are compact groups $K$ for which $A(K)$ is not weakly amenable [22]. If we let $G$ be any discrete group for which $G^{ap}$ admits such a group $K$ as a quotient, then $A^*(G)$ appears not to be weakly amenable. Indeed, $A_{G^{ap}}(G) \cong A(G^{ap})$ is a quotient of $A^*(G)$. Furthermore $A(G^{ap})$ contains an isometric copy of $A(K)$. Thus, it appears unlikely that $A(G^{ap})$ is weakly amenable, and we conjecture the same for $A^*(G)$.

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