Research Article

Some Fixed Point Results in Function Weighted Metric Spaces

Awais Asif,1 Nawab Hussain,2 Hamed Al-Sulami,2 and Muahammad Arshad1

1Department of Math & Stats, International Islamic University Islamabad, Islamabad, Pakistan
2Department of Mathematics, King Abdulaziz University, P.O. Box 80203, Jeddah 21589, Saudi Arabia

Correspondence should be addressed to Nawab Hussain; nhusain@kau.edu.sa

Received 11 December 2020; Revised 4 January 2021; Accepted 10 April 2021; Published 26 April 2021

Academic Editor: Efthymios G. Tsionas

Copyright © 2021 Awais Asif et al. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

After the establishment of the Banach contraction principle, the notion of metric space has been expanded to more concise and applicable versions. One of them is the conception of \( F \)-metric, presented by Jleli and Samet. Following the work of Jleli and Samet, in this article, we establish common fixed points results of Reich-type contraction in the setting of \( F \)-metric spaces. Also, it is proved that a unique common fixed point can be obtained if the contractive condition is restricted only to a subset closed ball of the whole \( F \)-metric space. Furthermore, some important corollaries are extracted from the main results that describe fixed point results for a single mapping. The corollaries also discuss the iteration of fixed point for Kannan-type contraction in the closed ball as well as in the whole \( F \)-metric space. To show the usability of our results, we present two examples in the paper. At last, we render application of our results.

1. Introduction and Preliminaries

In recent years, along with \( F \)-metric presented by Jleli et al. [1], many authors presented interesting generalizations of metric spaces [2–9]. Jleli and Samet introduced generalized metric spaces, known as \( F \)-metric spaces, and proved their generality to metric spaces with the help of concrete examples. The idea of \( F \)-metric spaces was compared with \( b \)-metric and \( s \)-relaxed metric spaces, and hence, the Banach contraction principle was established in the frame of \( F \)-metric spaces.

Banach contraction principle states that any contraction on a complete metric space has a unique fixed point. This principle guarantees the existence and uniqueness of the solution of considerable problems arising in mathematics. Because of its importance for mathematical theory, the Banach contraction principle has been extended and generalized in many directions [10, 11]. The fixed point theory of multivalued contraction mappings using the Hausdorff metric was initiated by Nadler [12], who extended the Banach contraction principle to multivalued mappings. Since then, many authors have studied various fixed point results for multivalued mappings. Nazam et al. [13] proved fixed point theorems for Kannan-type contractions on closed balls in complete partial metric spaces. The above-mentioned results and its generalizations are recently investigated for fixed point in the setting of \( F \)-metric space (see [14–16]).

In this article, we prove fixed point and common fixed points results of Reich-type contractions for single-valued mappings in \( F \)-metric spaces.

This article is organized into three sections. Section 2 contains a short history of the previous literature that becomes a motivation for this article. There are some basic definitions which help readers to understand our results easily. In Section 3, we established theorems of fixed points and common fixed points of single-valued Reich contractions in \( F \)-metric spaces. An example is provided to explain our results. Section 4 deals with fixed point theorems of contractions with respect to closed balls in \( F \)-metric spaces along with an example.

2. Basic Relevant Notions

Definition 1 (see [1]). A self-mapping \( g \) on a nonempty set \( A \) is said to be Kannan contraction if there exists a number \( k \), \( 0 < k < (1/2) \), such that, for each \( a, b \in A \), we have...


\[
d(g(a), g(b)) \leq k[d(a, g(a)) + d(b, g(b))].
\]  

(1)

Let \( f: (0, \infty) \to \mathbb{R} \) with the following characteristics:

(F1) \( f \) is strictly increasing

(F2) For any sequence \( \{t_n\} \subset (0, \infty) \), we have

\[
\lim_{n \to \infty} t_n = 0 \Leftrightarrow \lim_{n \to \infty} f(t_n) = -\infty.
\]  

(2)

The collection of all such functions satisfying (F1) and (F2) is denoted by \( \mathcal{F} \). The concept of \( \mathcal{F} \)-metric is generalized as follows:

**Definition 2** (see [1]). Suppose \( A \) is a nonempty set and \( (f, a) \in \mathcal{F} \times [0, \infty) \). Let the function \( d: A \times A \to [0, \infty) \) be such that

1. (d1) For all \( (a, b) \in A \times A \), \( d(a, b) = 0 \Leftrightarrow a = b \)
2. (d2) For all \( (a, b) \in A \times A \), \( d(a, b) = d(b, a) \)
3. (d3) \( \{t_n\} \in \mathcal{F} \times [0, \infty) \) for every \( (a, b) \in A \times A \), for each \( N' \in \mathbb{N}, N' \geq 2 \) and for every with \( (t_1, t_2, \ldots, t_{N'}) = (a, b) \), we have

\[
d(a, b) > 0 \Rightarrow f\left(\sum_{i=1}^{N'} d(t_i, t_{i+1})\right) + a.
\]  

(3)

Then, \( d \) is known as an \( \mathcal{F} \)-metric on \( A \), and the pair \( (A, d) \) is called an \( \mathcal{F} \)-metric space.

**Example 1** (see [1]). Let \( A = \mathbb{N} \) (set of natural numbers) and \( d: A \times A \to (0, \infty) \) be defined by

\[
d(a, b) = \begin{cases} 0, & \text{if } a = b \\ |a - b|, & \text{if } a \neq b \end{cases}
\]  

(4)

for all \( (a, b) \in A \times A \). It can easily be seen that \( d \) is an \( \mathcal{F} \)-metric with \( f(x) = \ln(x) \).

**Example 2** (see [1]). Let \( A = \mathbb{N} \) and \( d: A \times A \to (0, \infty) \) is defined as

\[
d(a, b) = \begin{cases} 0, & \text{if } a = b \\ e^{\frac{|a - b|}{1 + |a - b|}}, & \text{if } a \neq b \\ e^{\frac{|a - b|}{1 + |a - b|}}, & \text{if } a \neq b \\ e^{\frac{|a - b|}{1 + |a - b|}}, & \text{if } a \neq b \end{cases}
\]  

(5)

for all \( (a, b) \in A \times A \). Then, \( d \) is \( \mathcal{F} \)-metric on \( A \).

**Definition 3** (see [1]). Suppose \( \{a_n\} \) is a sequence in \( A \). Then,

1. (i) \( \{a_n\} \) is \( \mathcal{F} \)-convergent to a point \( a \in A \) if \( \lim_{n \to \infty} d(a_n, a) = 0 \)
2. (ii) \( \{a_n\} \) is an \( \mathcal{F} \)-Cauchy sequence if \( \lim_{m,n \to \infty} d(a_m, a_n) = 0 \)
3. (iii) The space \( (A, d) \) is \( \mathcal{F} \)-complete if every \( \mathcal{F} \)-Cauchy sequence \( \{a_n\} \subset A \) is \( \mathcal{F} \)-convergent to a point \( a \in A \)

**Definition 4** (see [1]). Let \( (A, d) \) be an \( \mathcal{F} \)-metric space. A subset \( O \) of \( A \) is said to be \( \mathcal{F} \)-open if, for every \( a \in O \), there is some \( r > 0 \) such that \( B(a, r) \subset O \), where

\[
B(a, r) = \{b \in A : d(a, b) < r\}.
\]  

(6)

We say that a subset \( C \) of \( A \) is \( \mathcal{F} \)-closed if \( A \setminus C \) is \( \mathcal{F} \)-open.

**Definition 5** (see [1]). Let \( (A, d) \) be an \( \mathcal{F} \)-metric space and \( B \) be a nonempty subset of \( A \). Then, the following statements are equivalent:

1. (i) \( B \) is \( \mathcal{F} \)-closed.
2. (ii) For any sequence \( \{a_n\} \subset B \), we have

\[
\lim_{n \to \infty} d(a_n, a) = 0, \quad a \in A \implies a \in B.
\]  

(7)

**Theorem 1** (see [1]). Suppose \( (f, a) \in \mathcal{F} \times [0, \infty) \) and \( (A, d) \) is an \( \mathcal{F} \)-complete \( \mathcal{F} \)-metric space. Let \( g: A \to A \) be a given mapping. Suppose that there exists \( k \in (0, 1) \) such that

\[
d(g(a), g(b)) \leq k d(a, b), \quad (a, b) \in A \times A.
\]  

(8)

Then, \( g \) has a unique fixed point \( a^* \in A \). Moreover, for any \( a_0 \in A \), the sequence \( \{a_n\} \subset A \) defined by \( a_{n+1} = g(a_n), n \in \mathbb{N} \) is \( F \)-convergent to \( a^* \).

**Theorem 2** (see [17]). Suppose \( A \) is a complete metric space with metric \( d \), and let \( g: A \to A \) be a function such that

\[
d(g(a), g(b)) \leq \alpha d(a, b) + \beta d(a, g(a)) + \gamma d(b, g(b)),
\]  

(9)

for all \( a, b \in A \), where \( \alpha, \beta, \gamma \) are nonnegative integers and satisfy \( \alpha + \beta + \gamma < 1 \). Then, \( g \) has a unique fixed point.

**Lemma 1** (see [18]). In Banach space \( (B(W), \| \cdot \|) \) along with the metric \( d \) defined by

\[
d(g, h) = \| g - h \| = \max_{w \in W} \| g(w) - h(w) \|,
\]  

(10)

is an \( \mathcal{F} \)-metric space.

### 3. Fixed Points of Reich-Type Contractions in \( \mathcal{F} \)–Metric Spaces

In this section, we construct fixed point and common fixed points results for single-valued Reich-type and Kannan-type contractions in the setting of \( \mathcal{F} \)-metric space.

**Theorem 3.** Suppose \( (f, a) \in \mathcal{F} \times [0, \infty) \) and \( (X, d) \) is an \( \mathcal{F} \)-complete \( \mathcal{F} \)-metric space. Let \( S, T: X \to X \) be self-mappings such that
Proof. Suppose $x_0$ is an arbitrary point and define a sequence $(x_n)$ by

$$d(x_{2j+1}, x_{2j+2}) = d(Sx_{2j}, x_{2j+1}) = a d(x_{2j}, x_{2j+1}) + b d(Sx_{2j}),$$

which yields

$$d(x_{2j+1}, x_{2j+2}) = a d(x_{2j}, x_{2j+1}) + b d(Sx_{2j}) + c d(x_{2j+1}, Tx_{2j+1}).$$

Using (20), we write

$$f \left( \frac{\lambda}{1 - \lambda} d(x_0, x_1) \right) < f(\varepsilon) - \alpha, \quad m > n \geq n'.$$

By (d3) and above equation, we obtain

$$d(x_m, x_n) < 0, \quad m > n \geq n' \implies f(d(x_m, x_n)) < f(\varepsilon).$$

This shows that

$$d(x_n, x_m) < \varepsilon, \quad m > n \geq n'.$$

Hence, we showed that $(x_n)$ is an $F$-Cauchy sequence in $X$. Since $(X, d)$ is $F$-complete, there exists $z^* \in X$ such that $(x_n)$ is $F$-convergent to $z^*$, i.e.,

$$\lim_{n \to \infty} d(x_n, z^*) = 0.$$

To prove that $z^*$ is the fixed point of $S$, assume $d(Sz^*, z^*) > 0$. Then,

$$d(Sz^*, x_{2j+1}) = d(Sz^*, Tx_{2j+1})$$

which implies $(1 - b)d(Sz^*, z^*) < 0$, which is a contradiction. Hence, $d(Sz^*, z^*) = 0$, i.e., $Sz^* = z^*$. Similarly, suppose $d(z^*, Tz^*) > 0$:
\[(1 - c)d(z^*, Tz^*) < 0,\]  
which is contradiction to the assumption. Therefore, we get

\[Tz^* = z^*.\]  
Hence, \[Tz^* = Sz^* = z^*\].

**Uniqueness.** Assume that \[z^{**}\] is also a common fixed point of \(S\) and \(T\) and \(z^* \neq z^{**}\). Then,

\[d(z^*, z^{**}) = d(Sz^*, Tz^{**}) \leq a d(z^*, z^{**}) + b d(z^*, Sz^{**}) + c d(z^{**}, Tz^{**})\]
\[= a d(z^*, z^{**}) + b d(z^*, z^{**}) + c d(z^{**}, z^{**}).\]  

(30)

We get \((1-a)d(z^*, z^{**}) < 0\), which is a contradiction. Hence, \(z^* = z^{**}\) \(\blacksquare\)

**Example 3.** Suppose

\[Y = \{Y_j = \frac{6j + 1}{2}, j \in \mathbb{N}\},\]  
\[d(x, y) = \begin{cases} 0, & \text{if } x = y \\ e^{\|x - y\|}, & \text{if } x \neq y \end{cases}\]  

(31)

Let \(f(x) = \ln x\) and \(S, T : X \to X\) are defined by

\[T(Y_j) = \begin{cases} Y_1, & \text{if } j = 1, 2, \\ Y_{j-1}, & \text{if } j > 2, \end{cases}\]

\[S(Y_j) = \begin{cases} 1, & \text{if } j = 1, \\ Y_2, & \text{if } j = 2, \\ Y_{j-2}, & \text{if } j > 2. \end{cases}\]  

(32)

It can be easily verified that \(d\) is an \(\mathcal{F}\)-metric and \(f\) satisfies \((F1)-(F2)\). Fix \(b = c = 0\) and \((x, y) \in X \times X\).

Suppose \(i \neq j\), then

\[d(SY_j, TY_i) = d(Y_{j-1}, Y_{i-1}) = e^{\|Y_j - Y_{i-1}\|} = e^{\|j - (i-1)\|} = e^{\|j-i-1\|} = e^{\min(|j-i|-1)} < e^{-2} < e^{\|j-i\|} = d(Y_j, Y_i)\]
\[= a d(Y_j, Y_i) + b d(Y_j, SY_j) + c d(Y_i, TY_i),\]  

(33)

where \(a = e^{-2}\). The inequality (11) holds true. Moreover, it is clear that \(Y_1\) is the only common fixed point of \(S\) and \(T\).

Taking \(a = 0\) in Theorem 1, we get the following result of Kannan contractions.

Replacing \(S\) in Theorem 3, we get the following corollary.

**Corollary 1.** Suppose \((f, a) \in \mathcal{F} \times [0, \infty)\) and \((X, d)\) is an \(\mathcal{F}\)-complete \(\mathcal{F}\)-metric space. Let \(T : X \to X\) is a self-mapping such that

\[d(Tx, Ty) \leq a d(x, y) + b d(x, Tx) + c d(y, Ty),\]  

(34)

for \(a, b, c \in [0, \infty)\) such \(a + b + c < 1\) for all \((x, y) \in X \times X\). Then, \(T\) has at most one fixed point in \(X\).

Taking \(b = c = 0\) in Corollary 1, we get the following result.

**Corollary 2.** Suppose \((f, a) \in \mathcal{F} \times [0, \infty)\) and \((X, d)\) is an \(\mathcal{F}\)-complete \(\mathcal{F}\)-metric space. Let \(T : X \to X\) is a self-mapping such that

\[d(Tx, Ty) \leq a d(x, y),\]  

(35)

for \(a \in (0, \infty)\) and \((x, y) \in X \times X\). Then, \(T\) has at least one fixed point in \(X\).

Besides the above important results, Theorem 3 also led us to the following fixed point result of Kannan-type contraction.

**Corollary 3.** Suppose \((f, a) \in \mathcal{F} \times [0, \infty)\) and \((X, d)\) is an \(\mathcal{F}\)-complete \(\mathcal{F}\)-metric space. Let \(S, T : X \to X\) be self-mappings. Suppose that, for \(k \in [0, 1]\) such that

\[d(Sx, Ty) \leq \frac{k}{2}(d(x, Sx) + d(y, Ty)),\]  

(36)

for all \((x, y) \in X \times X\), then \(S\) and \(T\) have at most one common fixed point in \(X\).

**Proof.** Suppose \(x_0\) is an arbitrary point and define a sequence \((x_n)\) by \(Sx_2 = x_{2j+1}\) and \(Tx_{2j+1} = x_{2j+2}\) ; \(j = 0, 1, 2, \ldots\).

Using the contraction and the iteration given above, we can write

\[d(x_{2j+1}, x_{2j+2}) = d(Sx_{2j}, Tx_{2j+1})\]
\[= \frac{k}{2} \left[ d(x_{2j}, Sx_{2j}) + d(x_{2j+1}, Tx_{2j+1}) \right]\]
\[= \frac{k}{2} \left[ d(x_{2j}, Sx_{2j}) + d(x_{2j+1}, x_{2j+2}) \right].\]  

(37)

This implies

\[\left(1 - \frac{k}{2}\right) d(x_{2j+1}, x_{2j+2}) \leq \frac{k}{2} d(x_{2j}, Sx_{2j}).\]  

(38)

or

\[d(x_{2j+1}, x_{2j+2}) \leq \frac{k}{2 - k} d(x_{2j}, Sx_{2j})\]  

(39)

\[d(x_{2j+1}, x_{2j+2}) \leq \lambda d(x_{2j+1}, x_{2j+1}).\]  

(40)

where \((k/(2 - k)) = \lambda\). Similarly,

\[d(x_{2j}, x_{2j+1}) \leq \frac{k}{2 - k} d(x_{2j+1}, x_{2j+1}) = \lambda d(x_{2j+1}, x_{2j+1}).\]  

(41)

Continuing the same way as in Theorem 3, we get the common fixed point of \(S\) and \(T\).

Replacing \(S\) with \(T\), we get the following result of single mapping. \(\blacksquare\)
Corollary 4. Suppose \((f, a) \in \mathcal{F} \times [0, \infty)\) and \((X, d)\) is an \(\mathcal{F}\)-complete \(\mathcal{F}\)-metric space. Let \(T : X \to X\) be a self-mapping. Suppose that, for each \(k \in [0, 1]\) such that
\[
d(Tx, Ty) \leq \frac{k}{2} (d(x, Tx) + d(y, Ty)),
\]
for all \((x, y) \in X \times X\), then \(T\) has at most one fixed point in \(X\).

4. Fixed Points of Reich-Type Contractions on \(\mathcal{F}\)-Closed Balls

This portion of the paper deals with the fixed points theorems of Reich-type contractions that hold true only on the closed balls rather than on the whole space \(X\).

Definition 6. Let \((X, d)\) be an \(\mathcal{F}\)-complete \(\mathcal{F}\)-metric space and \(S, T : X \to X\) be self-mappings. Suppose that \(a + b + c < 1\) for all \(a, b, c \in (0, \infty)\). Then, the mappings \(S\) and \(T\) are called Reich-type contractions on \(B(x_0, r) \subseteq X\) such that
\[
d(Sx, Ty) \leq a d(x, y) + b d(Sx, x) + c d(y, Ty), \quad \forall x, y \in B(x_0, r).
\]

Theorem 4. Suppose \((f, a) \in \mathcal{F} \times [0, \infty)\) and \((X, d)\) is an \(\mathcal{F}\)-complete \(\mathcal{F}\)-metric space. Let \(S\) and \(T\) be the Reich-type \(F\)-contractions on \(B(x_0, r)\). Suppose that for \(x_0 \in X\) and \(r > 0\), the following conditions are satisfied:
\begin{enumerate}[(a)]
\item \(B(x_0, r)\) is \(\mathcal{F}\)-closed;
\item \(d(x_0, x_1) \leq (1 - \lambda)r\), for \(x_1 \in X\) and \(\lambda = ((a + b)/(1 - c))\);
\item There exist \(0 < e < r\) such that \(f((1 - \lambda^{k+1})r) \leq f(e) - \alpha\), where \(k \in \mathbb{N}\).
\end{enumerate}

Then, \(S\) and \(T\) have at most one common fixed point in \(B(x_0, r)\).

Proof. Suppose \(x_0\) is an arbitrary point and define a sequence \((x_n)\) by \(T(x_n) = x_{n+1}\) and \(S(x_{n+1}) = x_{n+2}\) for \(j = 0, 1, 2, \ldots\).

We need to show that \(x_n\) is in \(B(x_0, r)\) for all \(n \in \mathbb{N}\). We show it by mathematical induction. By (b), we write
\[
d(x_0, x_1) < r. \tag{43}
\]
Therefore, \(x_1 \in B(x_0, r)\). We know by previous theorems that
\[
d(x_1, x_2) \leq \lambda d(x_0, x_1). \tag{44}
\]

Now,
\[
f(d(x_0, x_2)) \leq f\left(\sum_{i=1}^{k+1} d(x_{i-1}, x_i)\right) + \alpha
\leq f\left(\sum_{i=1}^{k+1} \lambda d(x_0, x_1)\right) + \alpha
= f\left(\lambda(1 + \lambda)\right) + \alpha
\leq f\left(\lambda(1 + \lambda)\right) + \alpha
\leq f\left(\frac{1 - \lambda^{k+1}}{1 - \lambda}\right). \tag{55}
\]

Using (b), we write
\[
f(d(x_0, x_k)) \leq f\left(\frac{1 - \lambda^{k+1}}{1 - \lambda}\right) + \alpha
= f\left((1 - \lambda^{k+1})\right) + \alpha. \tag{56}
\]

Using (c), we deduce that
\[
d(x_0, x_2) < r, \tag{46}
\]
i.e., \(x_2 \in B(x_0, r)\). Suppose \(x_3, \ldots, x_k \in B(x_0, r)\) for some \(k \in \mathbb{N}\). Now, if \(x_{k+1} \leq x_{k+1}\), then by (42), we can write
\[
d(x_{k+1}, x_{k+2}) = d(Sx_{k+2}), Tx_{k+2}) \leq a \cdot d(x_{k+1}, x_{k+2})
+ b \cdot d(x_{k+1}, Sx_{k+2}) + c \cdot d(x_{k+1}, Tx_{k+2})
= a \cdot d(x_{k+1}, x_{k+2}) + b \cdot d(x_{k+1}, x_{k+2}) + c \cdot d(x_{k+1}, x_{k+2}). \tag{47}
\]

This implies
\[
(1 - c)d(x_{k+1}, x_{k+2}) \leq (a + b)d(x_{k+1}, x_{k+2}), \tag{48}
\]
or
\[
d(x_{k+1}, x_{k+2}) \leq \frac{a + b}{1 - c} \cdot d(x_{k+1}, x_{k+2}). \tag{49}
\]

Let \((a + b)/(1 - c) = \lambda\), we get
\[
d(x_{k+1}, x_{k+2}) \leq \lambda d(x_{k+1}, x_{k+2}). \tag{50}
\]

Similarly, if \(x_1 \leq x_2\), then
\[
d(x_1, x_2) \leq \frac{a + b}{1 - c} \cdot d(x_1, x_2), \tag{51}
\]

Therefore, from inequality (50) and (51), we write
\[
d(x_{k+1}, x_{k+2}) \leq \lambda d(x_{k+1}, x_{k+2}), \tag{52}
\]

and
\[
d(x_{k+1}, x_{k+2}) \leq \lambda^2 d(x_{k+1}, x_{k+2}), \tag{53}
\]

From (52) and (53), we write
\[
d(x_k, x_{k+1}) \leq \lambda^k d(x_0, x_1), \quad \text{for some } k \in \mathbb{N}. \tag{54}
\]

Now, using (54), we have
\[
f(d(x_0, x_k)) \leq f\left(\sum_{i=1}^{k+1} d(x_{i-1}, x_i)\right) + \alpha
= f\left(\sum_{i=1}^{k+1} \lambda d(x_0, x_1)\right) + \alpha
\leq f\left(\frac{1 - \lambda^{k+1}}{1 - \lambda}\right) + \alpha
\leq f\left(\lambda^k d(x_0, x_1)\right) + \alpha. \tag{55}
\]

Using (b), we write
\[
f(d(x_0, x_k)) \leq f\left(\frac{1 - \lambda^{k+1}}{1 - \lambda}\right) + \alpha
= f\left((1 - \lambda^{k+1})\right) + \alpha. \tag{56}
\]

Using (c), we deduce that
\[ f(d(x_0, x_{k+1})) \leq f(\varepsilon) < f(r). \] (57)

Hence, by (F1), we notice that
\[ d(x_0, x_{k+1}) \leq r. \] (58)

This implies that \( x_{k+1} \in B(x_0, r) \). Therefore, \( x_n \in B(x_0, r) \) for all \( n \in \mathbb{N} \). Now, we have by (42)
\[ d(x_{2n+1}, x_{2n+2}) = d(Sx_{2n}, Tx_{2n+1}) \leq a d(x_{2n}, x_{2n+1}) + b d(x_{2n}, x_{2n+1}) + c d(x_{2n+1}, x_{2n+2}) = a d(x_{2n}, x_{2n+1}) + b d(x_{2n+1}, x_{2n+2}) + c d(x_{2n+1}, x_{2n+2}). \]

(59)

Following the same steps of proof of Theorem 3 and using (a), we obtain that the sequence \( (x_n) \) is \( \mathcal{F} \)-convergent to some \( z^* \) in \( B(x_0, r) \cdot z^* \) can be proved as common fixed point of \( S \) and \( T \) in the same way as in Theorem 3.

Taking \( S = T \) in Theorem 4, we get the following result of single mappings.

**Corollary 5.** Suppose \((f, a) \in \mathcal{F} \times [0, \infty), (X, d) \) is an \( \mathcal{F} \)-complete \( \mathcal{F} \)-metric space and \( T : X \longrightarrow X \) is a self-mapping. Suppose that \( a + b + c < 1, \) fora, b, c \( \in [0, \infty) \). Suppose that for \( x_0 \in X \) and \( r > 0 \), the following conditions are satisfied:

(a) \( B(x_0, r) \subseteq X \) is \( \mathcal{F} \)-closed

(b) \( d(Tx, Ty) \leq d(x, y) + b d(x, Tx) + c d(y, Ty) \), for all \( x, y \in B(x_0, r) \)

(c) \( d(x_0, x_1) \leq (1 - \lambda)r \), for \( x_1 \in X \) and \( \lambda = ((a + b)/ (1 - c)) \)

(d) There exists \( 0 < \varepsilon < r \) such as \( f((1 - \lambda^{k+1})r) \leq f(\varepsilon) - \alpha \), where \( k \in \mathbb{N} \)

Then, There is at most one fixed point in \( B(x_0, r) \).

**Example 4.** Let \( X = [0, \infty) \) and \( f(x) = \ln x \). Define \( T : X \longrightarrow X \) by
\[ Tx = \begin{cases} x & \text{if } x \in [0, 1], \\ \frac{x}{3} & \text{if } x \in (1, \infty), \end{cases} \]
and define \( d \) by
\[ d(x, y) = \begin{cases} (x - y)^2, & \text{if } (x, y) \in [0, 1] \times [0, 1], \\ |x - y|, & \text{if } (x, y) \notin [0, 1] \times [0, 1]. \end{cases} \]

(60)

(61)

\[ d(x_0, x_1) = d(x_0, Tx_0) = \left(1 - \frac{1}{12}\right)^2 = \frac{1}{36} \] (62)

\[ \left(1 - \frac{3}{4}\right)^2 = (1 - \lambda)r. \]

This shows that condition (b) is fulfilled. Furthermore, suppose \( k = 1 \), then
\[ f\left(\left(1 - \lambda^{k+1}\right)r\right) = \ln\left(\left(1 - \frac{3}{4}\right)^2\right)^{\frac{1}{2}} = \ln\left(\frac{7}{64}\right) \]
\[ \ln\left(\frac{8}{64}\right) - \ln\left(\frac{8}{7}\right) = f(\varepsilon) - \alpha, \]
i.e.,
\[ f\left(\left(1 - \lambda^{k+1}\right)r\right) = f(\varepsilon) - \alpha. \]

(63)

(64)

Hence, condition (d) is satisfied for \( \varepsilon = (8/64) \leq (1/4) = r \) and \( \alpha = \ln(8/7) \). Similarly, for all values of \( k \in \mathbb{N} \), we can find some \( 0 < \varepsilon < r \) and \( \alpha \) such that condition (d) is fulfilled.

Now, checking for condition (b), we have two cases:

(i) If \((x, y) \in B(x_0, r) \times B(x_0, r) \), then
\[ d(Tx, Ty) = \left(\frac{x^2}{3} \right)^2 = \frac{1}{9}(x - y)^2 < \left(\frac{3}{4}(x - y)^2\right) \]
\[ = a d(x, y) + 0. d(x, Tx) + 0. d(y, Ty) \]
\[ = a d(x, y) + b d(x, Tx) + c d(y, Ty), \]

(65)

as \( b = c = 0 \).

Therefore, for all \((x, y) \in B(x_0, r) \times B(x_0, r) \), condition (d) is also satisfied.

(ii) If \((x, y) \notin B(x_0, r) \times B(x_0, r) \), e.g., \( x = 2 \) and \( y = 3 \), then
\[ d(Tx, Ty) = |2^3 - 3^3| > \left(\frac{3}{4}(2 - 3)\right) \]
\[ = a d(x, y) + a d(y, Ty) \]
\[ = a d(x, y) + b d(x, Tx) + c d(y, Ty), \]

(66)

Hence, condition (d) holds only for \( B(x_0, r) \) and not on \( X \times X \). Moreover, \( 0 \in B(x_0, r) \) is the fixed point of \( T \).

**Corollary 6.** Suppose \((f, a) \in \mathcal{F} \times [0, \infty) \) and \((X, d) \) is an \( \mathcal{F} \)-complete \( \mathcal{F} \)-metric space. Let \( S, T : X \longrightarrow X \) are self-mappings and \( k \in [0, 1) \), assume that, for \( x_0 \in X \) and \( r > 0 \), the following conditions are satisfied:

(a) \( B(x_0, r) \subseteq X \) is \( \mathcal{F} \)-closed

(b) \( d(Sx, Ty) \leq (k/2)(d(x, Sx) + d(y, Ty)) \), for all \( x, y \in B(x_0, r) \)

(c) \( d(x_0, x_1) \leq (1 - \lambda)r \), for \( x_1 \in X \) and \( \lambda = (k/(2 - k)) \)

(d) there exist \( 0 < \varepsilon < r \) such as \( f\left((1 - \lambda^{k+1})r\right) \leq f(\varepsilon) - \alpha \), where \( k \in \mathbb{N} \)
Then, $S$ and $T \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$ have at most one common fixed point in $B(x_0, r)$. 

**Corollary 7.** Suppose $(f, \alpha) \in F \times [0, \infty)$ and $(X, d)$ is an $F$-complete $\mathcal{F}$-metric space. Let $S, T: X \to X$ are self-mappings and $k \in [0, 1)$, assume that, for $x_0 \in X$ and $r > 0$, the following conditions are satisfied:

(a) $B(x_0, r) \subseteq X$ is $\mathcal{F}$-closed
(b) $d(Sx, Ty) \leq k \ d(x, y)$ for all $x, y \in B(x_0, r)$
(c) $d(x_0, x_1) \leq (1 - \lambda) r$, for $x_1 \in X$ and $\lambda = (k/(2-k))$
(d) there exist $0 < \epsilon < r$ such as $f((1 - \lambda^{k+1}) \ r) \leq f(\epsilon) - \alpha$, where $k \in N$

Then, $S$ and $T$ have at most one common fixed point in $B(x_0, r)$.

An example can be proved in a similar way as that to the previous examples.

5. **Application**

This section is concerned with the application of the main result proved in Section 2, in finding a unique common solution of the functional equations that are used in dynamic programming.

The two main components of dynamic programming are decision space (DS) and a state space (SS). The SS includes different states such as transitional states, initial, and action states, while the DS is composed of the steps that are taken for locating the possible solution point of the problem. Optimization and computer programming are based on this system. In particular, a problem of dynamic programming is converted to functional equations as

$$p(u) = \max_{v \in V} \{ F(u, v) + f_1(u, v, p(\eta(u, v))) \}, \quad \text{for } u \in U,$$

$$x(u) = \max_{v \in V} \{ F(u, v) + f_2(u, v, p(\eta(u, v))) \}, \quad \text{for } u \in U,$$

where $Y$ and $Z$ are Banach spaces such as $U \subseteq Y$ and $V \subseteq Z$ and

$$\eta: U \times V \to U$$

$$F: U \times V \to R,$$

$$f_1, f_2: U \times V \times R \to R.$$ 

Suppose $U$ and $V$ are the DS and SS, respectively. We aim to locate a single common solution point for equations (67) and (68). We denote the set of all bounded real-valued mappings on $U$ by $W(U)$. Let $j$ be arbitrary member of $W(U)$ and say $\|j\| = \max_{u \in U} |j(u)|$. Then, the duplet $(W(U), \| \cdot \|)$ is a Banach space with $d$ defined by

$$d(j, k) = \max_{u \in U} |j(u) - k(u)|.$$

Let the following conditions holds true:

(C1) $F, f_1, f_2$ are bounded.

(C2) For $u \in U$ and $j \in W(U)$, define $S, T: W(U) \to W(U)$ by

$$Sj(u) = \max_{v \in V} \{ F(u, v) + f_1(u, v, j(\eta(u, v))) \}, \quad \text{for } u \in U,$$

$$Tj(u) = \max_{v \in V} \{ F(u, v) + f_2(u, v, j(\eta(u, v))) \}, \quad \text{for } u \in U.$$

Observe that, $S$ and $T$ are well-defined whenever the functions $F, f_1$ and $f_2$ are bounded.

(C3) For $(u, v) \in U \times V$, $j, k \in W(U)$ and $l \in U$, we write

$$\begin{align*}
[ f_1(u, v, j(l)) - f_1(u, v, k(l)) ] & \leq M(j, k),
\end{align*}$$

where

$$M(j, k) = \alpha d(j, k) + \beta d(j, Sj) + \gamma d(k, Tj),$$

for $\alpha, \beta, \gamma \in [0, \infty)$ and $\alpha + 2\beta + 2\gamma < 1$.

Now, we develop the following theorem.

**Theorem 5.** Suppose conditions (C1)–(C3) hold true, then there exists a single bounded common solution of equations (67) and (68).

**Proof.** From Lemma 1.10, we have $(W(U), d)$ is an $F$-complete $F$–MS. $d$ is defined by (70), and from (C1), we deduce that $S$ and $T$ are self-mappings on $W(U)$. Let $\omega$ be an arbitrary positive number and $j_1, j_2 \in W(U)$. Take $u \in U$ and $v_1, v_2 \in V$ such as

$$Sj_1(u, v_1) + f_1(u, v_1, j_1(\eta(u, v_1))) + \omega,$$

$$Tj_1(u, v_1) + f_2(u, v_1, j_1(\eta(u, v_1))) + \omega,$$

and

$$Sj_1 \geq F(u, v_1) + f_1(u, v_1, j_1(\eta(u, v_1))),$$

$$Tj_1 \geq F(u, v_1) + f_1(u, v_1, j_1(\eta(u, v_1))).$$

Then, using (74) and (77), we obtain

$$Sj_1(u) - Tj_2(u) \leq f_1(u, v_1, j_1(\eta(u, v_1))) - f_1(u, v_1, j_2(\eta(u, v_1))) + \omega \leq M(j_1(u), j_2(u)) + \omega.$$
Also, from (75) and (76), we get
\[ T_{j_2}(u) - S_{j_1}(u) < M \left( j_1(u), j_2(u) \right) + \omega. \] (79)
Merging the above two inequalities, we write
\[ |S_{j_1}(u) - T_{j_2}(u)| < M \left( j_1(u), j_2(u) \right) + \omega, \] (80)
for all \( \omega > 0 \). Thus,
\[ d(S_{j_1}(u), T_{j_2}(u)) \leq M \left( j_1(u), j_2(u) \right), \] (81)
i.e.,
\[ d(S_{j_1}, T_{j_2}) \leq M \left( j_1, j_2 \right), \] (82)
for every \( \epsilon \in U \). All the requirements of Theorem 3 are fulfilled. Therefore, by using Theorem 3, \( S \) and \( T \) have a unique bounded and common solution for equations (67) and (68).

6. Conclusion
This article has furthered the idea of \( F \)-metric space and fixed point and common fixed point results are elaborated in the setting of \( F \)-metric space. It is obtained that the fixed point and common fixed point of a contraction mapping can be available even if the contractive condition is restricted to only a subset closed ball of the whole \( F \)-metric space. Examples have been provided for both locally and globally contractions and a comparison between them is made for better understanding. Some important corollaries have been developed from the proved results. At last, application of the main result in finding a unique solution of the functional equation is given. In future, we opt to explore similar results in the frame of fuzzy cone metric space. Fixed point of Reich-type contractions will be investigated in picture fuzzy metric space, fuzzy soft sets, and other applicable abstract spaces. The proposed research will be primarily based upon some existing literature on the topics ([19–22]).

Data Availability
No data were used to support this study.

Conflicts of Interest
The authors declare that they have no conflicts of interest.

Authors’ Contributions
All the authors contributed equally to the research.

Acknowledgments
This project was funded by the Deanship of Scientific Research (DSR), King Abdulaziz University, Jeddah, Saudi Arabia. The authors, therefore, acknowledge with thanks DSR technical and financial support.

References
[1] M. Jleli and B. Samet, “On a new generalization of metric spaces,” Journal of Fixed Point Theory and Applications, vol. 20, no. 3, p. 128, 2018.
[2] A. Branciari, “A fixed point theorem of Banach-Caccioppoli type on a class of generalized metric spaces,” Publicationes Mathematicae Debrecen, vol. 57, pp. 31–37, 2000.
[3] S. Czerwik, “Contraction mappings in \( b \)-metric spaces,” Acta Mathematica et Informatica Universitatis Ostraviensis, vol. 1, no. 1, pp. 5–11, 1993.
[4] R. Fagin, R. Kumar, and D. Sivakumar, “Comparing top \( k \) lists,” SIAM Journal on Discrete Mathematics, vol. 17, no. 1, pp. 134–160, 2003.
[5] S. Gähler, “\( 2 \)-metrische Räume und ihre topologische Struktur,” Mathematische Nachrichten, vol. 26, no. 1-4, pp. 115–148, 1964.
[6] M. Jleli and B. Samet, “A generalized metric space and related fixed point theorems,” Fixed Point Theory and Algorithms for Sciences and Engineering, vol. 2015, no. 1, p. 14, 2015.
[7] M. A. Khamsi and N. Hussain, “KKM mappings in metric type spaces,” Nonlinear Analysis: Theory, Methods & Applications, vol. 73, no. 9, pp. 3123–3129, 2010.
[8] S. G. Matthews, “Partial metric topology,” Annals of the New York Academy of Sciences, vol. 728, pp. 183–197, 1994.
[9] Z. Mustafa and B. Sims, “A new approach to generalized metric spaces,” Journal Of Nonlinear And Convex Analysis, vol. 7, no. 2, pp. 289–297, 2006.
[10] S. Reich, “Fixed points of contractive functions,” Bolletino dell Unione Matematica Italiana, vol. 5, pp. 26–42, 1972.
[11] D. Wardowski, “Fixed points of a new type of contractive mappings in complete metric spaces,” Fixed Point Theory and Algorithms for Sciences and Engineering, vol. 1, p. 94, 2012.
[12] S. Nadler, “Multi-valued contraction mappings,” Pacific Journal of Mathematics, vol. 30, no. 2, pp. 475–488, 1969.
[13] M. Nazam, C. Park, A. Hussain, M. Arshad, and J. R. Lee, “Fixed point theorems for \( F \)-contractions on closed ball in partial metric spaces,” Journal of Computational Analysis and Applications, vol. 26, no. 1, pp. 759–769, 2019.
[14] A. Hussain, H. Al-Sulami, H. Hussain, and H. Farooq, “Newly fixed disc results using advanced contractions on \( F \)-metric space,” Journal of Applied Analysis & Computation, vol. 10, no. 6, pp. 2313–2322, 2020.
[15] H. Isik, N. Hussain, and A. R. Khan, “\( F \)-metric spaces with an application,” International Journal of Nonlinear Analysis and Applications, vol. 11, no. 2, pp. 351–361, 2020.
[16] Z. D. Mitrović, H. Aydi, N. Hussain, and A. Mukheimer, “Reich, Jungck, and berinde common fixed point results on \( F \)-metric spaces and an application,” Mathematics, vol. 7, p. 387, 2019.
[17] S. Reich, “Some remarks concerning contraction mappings,” Canadian Mathematical Bulletin, vol. 14, no. 1, pp. 121–124, 1971.
[18] A. Hussain and T. Kanwal, “Existence and uniqueness for a neutral differential problem with unbounded delay via fixed point results,” Transactions of A. Razmadze Mathematical Institute, vol. 172, no. 3, pp. 481–490, 2018.
[19] A. Asif, S. U. Khan, T. Abdeljawad, M. Arshad, and A. Ali, “3D dynamic programming approach to functional equations with applications,” Journal of Function Spaces, vol. 2020, Article ID 9485620, 9 pages, 2020.
[20] A. Asif, S. U. Khan, S. Ullah Khan, T. Abdeljawad, M. Arshad, and E. Savas, “3D analysis of modified \( F \)-contractions in convex \( b \)-metric spaces with application to Fredholm integral
[21] A. Asif, M. Nazam, M. Arshad, and S. O. Kim, “F-metric, F-contraction and common fixed-point theorems with applications,” Mathematics, vol. 7, no. 7, pp. 586–599, 2019.
[22] A. Asif, M. Alansari, N. Hussain, M. Arshad, and A. Ali, “Iterating fixed point via generalized mann’s iteration in convex b-metric spaces with application,” Complexity, vol. 2012, p. 12, 2012.