An optimal investment strategy aimed at maximizing the expected utility across all intermediate capital levels

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Abstract

This study investigates an optimal investment problem for an insurance company operating under the Cramer-Lundberg risk model, where investments are made in both a risky asset and a risk-free asset. In contrast to other literature that focuses on optimal investment and/or reinsurance strategies to maximize the expected utility of terminal wealth within a given time horizon, this work considers the expected value of utility accumulation across all intermediate capital levels of the insurer. By employing the Dynamic Programming Principle, we prove a verification theorem, in order to show that any solution to the Hamilton-Jacobi-Bellman (HJB) equation solves our optimization problem. Subject to some regularity conditions on the solution of the HJB equation, we establish the existence of the optimal investment strategy. Finally, to illustrate the applicability of the theoretical findings, we present numerical examples.

Keywords. Stochastic control · dynamic programming principle · Hamilton-Jacobi-Bellman equation · optimal investment · expected utility · risk process
1 Introduction

The optimal investment problem has become an attractive research area in actuarial science, financial mathematics, and quantitative finance. For insurance companies, this problem involves determining the best way to allocate wealth among different asset classes, such as equity, debt and real estate, over a given period. The goal is to achieve this allocation but also reducing risk, minimizing exposure to potential losses, and ensuring sufficient funds to meet future cash flow obligations.

After the classical collective risk model introduced by [Lundberg (1903)], the ruin probability of such a portfolio became a principal focus in this field, with various approaches. Nowadays, there are several additional methods for studying the wealth of an insurance portfolio and reducing or controlling the ruin probability. In stochastic control theory applied to insurance models, reinsurance or a combination of reinsurance with dividends and optimal investment are other possibilities for controlling certain risk measures of the insurance portfolio (see for instance [Hipp and Vogt (2003)], [Asmusen and Albrecher (2010)], [Schmidli (2008)], [Azcue and Muler (2014)], [Diko et al. (2011)], [Hipp (2004)], [Korn and Wiese (2008)], and references therein).

In this paper, we adopt the approach of maximizing the expected utility of an insurance portfolio with investment. In practical terms, most insurance contracts are inherently tied to financial markets, whether through mortality rates, interest rates, financial products, or direct connections to stocks or indices (see, for instance, [Dhaene at al. (2013)], [Artzner at al. (2023)], and [Robben et al. (2022)]). Therefore, it is essential to continually assess the risk and wealth of the insurance portfolio at all times $t$ in order to minimize exposure or risk. Additionally, numerous insurance companies calibrate their portfolios based on the severity of claims per policy or in response to an increase in the portfolio’s risk, aiming to reduce fluctuations over the long term. To address these issues, unlike other literature that primarily focuses on optimal investment strategies to maximize the expected utility of terminal wealth within a specified time horizon (see, for instance, [Cadenillas and Zou (2014)], [Badaoui at al. (2018)], [Merton R. (1969)]), our proposal specifically considers the expected value of utility accumulation across all intermediate capital levels of the insurer.

Following the ideas of [Cadenillas and Zou (2014)] and [Badaoui at al. (2018)], our model, described in Section 2, proposes a new stochastic optimization problem. This problem differs from existing ones in the literature (see [Schmidli (2008)]) for a summary of stochastic control problems applied to insurance) by considering the expected value of utility accumulation across all intermediate capital levels of the insurer, because most insurance contracts are inherently tied to stocks or financial instruments, exposing the insurance portfolio to greater market risks [Dhaene at al. (2013)], [Artzner at al. (2023)].

[Ferguson (1965)] was the first to apply stochastic control theory to solve the problem of the expected utility of wealth for the investor in the discrete case. Subsequently,
in his seminal paper, [Merton R. (1969)] introduced the fundamental classical optimal investment-consumption model, laying the foundation for future continuous-time stochastic optimization problems. Inspired by [Merton R. (1969)], [Browne (1995)] considered a risk process modeled by a Brownian motion with drift, incorporating the possibility of investment in a risky asset that follows a geometric Brownian motion, but without a risk-free interest rate. In that work, Browne verified the conjecture announced by Ferguson and, for the first time, established a relationship between minimizing the ruin probability and maximizing the exponential utility of terminal wealth. This connection provides a clear link between insurance and finance. Our work also establishes this relationship (see Lemmas 4 and 3). Other related works that we can mention, where the problem of maximizing the expected utility of terminal wealth was studied, include [Zariphopoulou (2001)], [Badaoui and Fernández (2013)]. In several of these works, general properties were proven, along with an existence and uniqueness theorem, leading to a closed form of the optimal strategy.

To solve our optimization problem, we explored the properties of the value function to derive the dynamic programming principle (DPP). Subsequently, we obtained the Hamilton-Jacobi-Bellman (HJB) equation. To find solutions to the HJB equation, we imposed additional conditions on the utility function and applied an appropriate boundary condition to the value function. Finally, we also demonstrate that in some cases, the Merton ratio serves as the optimal strategy, aligning with findings reported in [Merton R. (1969)] and [Browne (1995)], among others.

The rest of the paper is organized as follows. In Section 2, basic definitions are described and the optimization problem is formulated. Section 3 focuses on studying the properties of the value function. In Section 4, the Dynamic Programming Principle for the problem is validated. In Section 5, it is demonstrated that the value function is a solution to the HJB equation. In Section 6, it is presented numerical results in order to compare the behavior of the ruin probability with and without optimal investment, using exponential, Pareto, and Weibull claim size distributions.

2 Preliminaries and problem formulation

The well-known Cramér-Lundberg risk model, with application to insurance, is driven by equation

\[ X_t = x + ct - Q_t = x + ct - \sum_{i=1}^{N_t} U_i, \quad t \in [0, T] \tag{2.1} \]

where \( X_0 = x \) is the initial surplus or the surplus known at a giving or starting instant, \( X_t \) represents the dynamics of surplus of an insurance company up to time \( t \), \( c \) is the premium income per unit time, assumed deterministic and fixed, and \( Q_t \) es the total claim amount process. \( \{U_i\}_{i \geq 1} \) is a sequence of i.i.d. random variables with common cumulative distribution \( F \), with \( F(0) = 0 \). We assume the existence of \( \mu = \mathbb{E}(U_1) \). \( N_t = \max\{k \geq 1 : T_k \leq t\} \) denote the number of claims occurring before or at a given time \( t \), where the random variable \( T_i \) denote the arrival times. We
assume that \( \{N_t\}_{t \geq 0} \) is a Poisson process with Poisson intensity \( \lambda \), independent of the sequence \( \{U_i\}_{i \geq 1} \). The process \( Q_t = \sum_{i=1}^{N_t} U_i \) is then a compound Poisson process. We will assume particular distributions in some sections of this manuscript and we state it appropriate and clearly. An important condition for the model is the so called *income condition* or *net profit condition*, in the case positive loading condition: \( c \mathbb{E}(T_1) > \mathbb{E}(U_1) \). It brings an economical sense to the model: it is expected that the income until the next claim is greater than the size of the next claim. If we denote \( Y_1 = T_1 \) and \( Y_i = T_i - T_{i-1}, i \geq 2 \), then the net income between the \((i-1)\)-th and the \(i\)-th claims is \( cY_i - U_i \)

Let the time to ruin of the Cramé-Lundberg process \( X_t \) be denoted by \( \tau = \inf\{t > 0 : X_t < 0\} \). The ruin probability is defined as \( \psi(x) = \mathbb{P}(\tau < \infty | X_0 = x) \) and the corresponding survival probability is \( \varphi(x) = 1 - \psi(x) \).

For \( t \geq 0 \) define \( \mathcal{F}_t = \sigma\{X_s : 0 \leq s \leq t\} \), the smallest \( \sigma \)-algebra making the family \( \{X_t : 0 \leq s \leq t\} \) measurable. \( \{\mathcal{F}_t : t \geq 0\} \) is the filtration generated by \( \{X_t\} \). Cramer-Lundberg risk process \( X_t \) has an independent increment property, i.e., for any \( 0 \leq s \leq t \), the sub \( \sigma \)-algebra \( \mathcal{F}_s \) and the random variable \( X_t - X_s \) are independent.

We shall assume that the insurance company invests its surplus in a financial market described by the standard Black - Scholes model, i.e., on the market there is a riskless bond and risky assets satisfying the following SDEs

\[
\begin{align*}
    dS_t^0 &= rS_t^0 dt, & dS_t &= \mu S_t dt + \sigma S_t dB_t, & t \in [0, T]
\end{align*}
\]

respectively, where \( r \) is the risk-free rate and \( \mu, \sigma \) are the expected return and volatility of the stock market, and \( dB_t \) is the increment of a standard Brownian motion. We assume that \( \mu > r \). At time \( t \) the insurer must choose what fraction of the surplus invest in a stock portfolio, \( \pi_t \) (the remaining fraction \( 1 - \pi_t \) being invested in the riskless bond), i.e., \( \pi_t X_t \) is the amount invested into the risky asset and \( (1 - \pi_t) X_t \) is the amount invested into the riskless bond. We assume that the processes \( S_t \) and \( Q_t \) are independent, this means that the sub \( \sigma \)-algebras \( \sigma\{S_t : t \geq 0\} \) and \( \sigma\{Q_t : t \geq 0\} \) are independent. Denote by \( X_t^{\pi} \) the process with investment strategy \( \pi = \{\pi_t\} \) where \( \pi_t \in [0,1] \). Given an investment strategy \( \pi = \{\pi_t\} \) it is easy to prove that the controlled risk process \( X_t^{\pi} \) can be written as

\[
    dX_t^{\pi} = [c + \mu \pi_t X_t^{\pi} + r(1 - \pi_t)X_t^{\pi}] dt + \sigma \pi_t X_t^{\pi} dW_t - dQ_t
\]

The controlled risk process \( X_t^{\pi} \) can be viewed as the wealth of an risk averse economic agent (insurer) at time \( t \) with cash injection to the fund due to the income per unit time, assumed deterministic and constant. Note that, in this case, the wealth of the agent have an stochastic dynamic with initial condition \( X_t^{\pi}_0 = x \).

We denote by \( \prod_x^{ad} \) the set of all the admissible investment strategies with initial value \( x \). An investment strategy is *admissible* if the process \( \{\pi_t\} \) is predictable with respect to the filtration \( \mathcal{F}_t = \sigma\{X_t, S_t\} \). We allow all adapted cadlag control processes \( \pi = \{\pi_t\} \in \prod_x^{ad} \) such that there is a unique solution \( \{X_t^{\pi}\} \) to the stochastic differential equation (2.3).

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We define a stationary investment strategy as the one where the investment decision depends only on the current surplus, i.e., \( \pi_t = \pi(X_t^\pi) \) is the fraction of the surplus invest in a stock portfolio when the current surplus is \( X_t^\pi \). Thus the controlled investment process \( X_t^\pi \) should satisfy

\[
\begin{align*}
    dX_t^\pi &= [c + \mu(X_t^\pi) + r(X_t^\pi - \pi(X_t^\pi))]dt + \sigma\pi(X_t^\pi)dW_t - dQ_t \\
    (2.4)
\end{align*}
\]

If \( \pi : [0, \infty) \to \mathbb{R} \) is a Lipschitz continuous function, using general results of diffusion processes is possible to prove that there is a unique strong solution to the SDE (2.4) (see [Karatzas and Shreve, 1991], [Ikeda and Watanabe, 1981]). We further restrict to strategies such that \( X_t^\pi \geq 0 \), i.e., the agent is not allowed to have debts. This means that if there is no money left the agent can no longer invest. We will further have to assume that our probability space \((\Omega, \mathcal{F}, \mathbb{P})\) is chosen in such a way that for the optimal strategy found below a unique solution \( \{X_t^\pi\} \) exists. Let us define the time to ruin \( \tau^\pi = \inf \{t : X_t^\pi < 0\} \) and the ruin probability \( \psi^\pi(x) = \mathbb{P}(\tau^\pi < \infty | X_0^\pi = x) \).

According to the utility theory, in a financial market where investors are facing uncertainty, an investor is not concerned with wealth maximization per se but with utility maximization. It is therefore possible to introduce an increasing and concave utility function \( \phi(x, t) \) representing the expected utility of a risk averse investor.

We now describe our optimization problem. Given a strategy \( \pi = \{\pi_t\} \in \Pi^{ad} \), we define the value of the strategy \( \pi \) as follow

\[
    V^\pi(x) = \mathbb{E} \left[ \int_0^{T \wedge \tau^\pi} \phi(X_s^\pi, s) ds \mid X_0^\pi = x \right].
\]

Now, the goal of the problem is not anymore to maximize the expected portfolio value or minimize the ruin probability or maximizing the expectation of the present value of all dividends paid to the shareholders up to the ruin but to maximize the expected utility stemming from the wealth during the contract \([0, T]\), where \( T \) is the maturity date of the contract. If the initial portfolio value is \( X_0^\pi = x \), then our objective is to find the optimal strategy \( \pi^* \in \Pi^{ad}_x \) such that

\[
    V(x) = \sup_{\pi \in \Pi^{ad}_x} V^\pi(x) = V^\pi^*(x) \quad (2.6)
\]

For simplicity of notation, we will omit \( \pi \) of the stopping time \( \tau^\pi \). We just make the convention that \( X_t^\pi = 0 \) for \( t \geq \tau \).

Note that this system have a two random components, the total claim amount \( Q_t \) and the unit price \( S_t \) of the risky asset, but the unique component that the insurer can be to control is the fraction of wealth that is invested in the risky asset.

Because the agent prefer value growth of the surplus \( X_t^\pi \) (intuitively that property reduces ruin probability) the utility function \( \phi(x, t) \) is assumed to be strictly increasing. Because the agent is risk-averse, the utility \( \phi(x, t) \) is assumed to be strictly concave. In other words, a monetary unit means less to the agent if \( x \) is large than if
$x$ is small. Finally, because the agent’s preferences do not change rapidly, we assume that $\phi(x, t)$ is continuous in $t$. Note that as a concave function $\phi(x, t)$ is continuous in $x$. For simplicity of the notation we norm the utility functions such that $\phi(0, t) = 0$. To avoid some technical problems we suppose that $\phi(x, t)$ is continuously differentiable with respect to $x$ and that $\lim_{x \to \infty} \phi_x(x, t) = 0$, where $\phi_x(x, t)$ denotes the derivative with respect to $x$.

3 Basic properties of the value function

In this section, we present some results that characterize the regularity of the value function $V$ of our optimization problem.

Lemma 1. The function $V$ is strictly increasing and concave with boundary value $V(0) = 0$, and hence continuous in the interior of the domain.

Proof. Note that if $x = 0$, any strategy with $\pi_t \neq 0$ would immediately lead to ruin. Without wealth the economic agent don’t obtain any utility. Thus, $V(0) = \int_0^T \phi(0, s)ds = 0$. Let $\pi$ be a strategy for initial capital $x$ and let $y > x$. We denote the surplus process starting in $x$ by $X^\pi_x$ and the surplus process starting in $y$ by $Y^\pi_y$. Note that $X^\pi_t < Y^\pi_t$, for all $t \geq 0$. As $\phi$ is strictly increasing, then $\phi(X^\pi_t, t) < \phi(Y^\pi_t, t)$. Thus, we obtain $V(y) > V(x)$.

Let $z = \alpha x + (1 - \alpha)y$ for $\alpha \in [0, 1]$. Let $\pi^1$ the strategy for initial value $x$ and $\pi^2$ the strategy for initial value $y$. Consider the strategy $\pi_t = \alpha \pi^1_t + (1 - \alpha)\pi^2_t$. Then $X^\pi_t = \alpha X^\pi^1_t + (1 - \alpha)X^\pi^2_t$. The value of this new strategy now becomes

$$V(z) \geq V^\pi(z) = E \left[ \int_0^{T \wedge \tau^\pi} \phi \left( \alpha X^\pi^1_s + (1 - \alpha)X^\pi^2_s, s \right) ds \right]$$

$$\geq E \left[ \int_0^{T \wedge \tau^\pi} \alpha \phi \left( X^\pi^1_s, s \right) + (1 - \alpha)\phi \left( X^\pi^2_s, s \right) ds \right]$$

$$= \alpha V^{\pi^1}(x) + (1 - \alpha)V^{\pi^2}(y)$$

Taking the supremum on the right-hand side we obtain $V(z) \geq \alpha V(x) + (1 - \alpha)V(y)$. 

The following lemma establishes that for any strategy $\pi \in \Pi^ad_x$, such that $\tau^\pi$ occurs between claim times, can never be optimal.

Lemma 2. Suppose that $\pi \in \Pi^ad_x$ is such that $P(T_i \wedge T < \tau^\pi < T_{i+1} \wedge T) > 0$, for some $i \in N$, where $T_i$’s are the jump times of the Poisson process $\{N_t : t \geq 0\}$, then exist $\bar{x} \in \Pi^ad_x$ such that $P(\tau^\bar{x} = T_i$ for some $i) = 1$, and $V(\bar{x}) > V(x)$.

Proof. First note that on the set $\{T_i \wedge T < \tau^\pi < T_{i+1} \wedge T\}$, one must have that $X^\pi_{t \wedge T} = X^\pi_{\tau^\pi} = 0$. Now, define the strategy $\bar{\pi}_t = \pi_t 1_{\{t < \tau^\pi\}}$ and denote $\bar{X} = X^\bar{\pi}$.
Then, it is clearly that \( \tilde{X}_t = X_t^\pi \) for all \( t \in [0, \tau^\pi] \), \( \mathbb{P} \)-a.s. Consequently, \( \tilde{X} \) satisfies the following stochastic differential equation

\[
\begin{cases}
  d\tilde{X}_t = dX_t^\pi, & t \in [0, \tau^\pi] \\
  dX_t = (c + rX_t)dt - dQ_t, & t > \tau^\pi 
\end{cases}
\] (3.1)

Note that the process \( dX_t = (c + rX_t)dt - dQ_t \) in the absence of claims (or between the jumps of \( N_t \)) is increasing, and the ruin can occur only at some time \( T_i \). Therefore the \( \tau^\pi = T_k \) for some \( k > i \). Thus

\[
V^\pi(x) = \mathbb{E} \left[ \int_0^{T \wedge \tau^\pi} \phi(X_s^\pi, s)ds|X_0^\pi = x \right] 
\]

\[
= \mathbb{E} \left[ \int_0^{T \wedge \tau^\pi} \phi(X_s^\pi, s)ds|X_0^\pi = x \right] + \mathbb{E} \left[ \int_{T \wedge \tau^\pi}^{\tau^\pi} \phi(X_s^\pi, s)ds|X_0^\pi = x \right] 
\]

\[
= V^\pi(x) + \mathbb{E} \left[ \int_{\tau^\pi}^{T \wedge \tau^\pi} \phi(X_s^\pi, s)ds|X_0^\pi = x \right] 
\]

\[
\geq V^\pi(x) + \mathbb{E} \left[ \int_{\tau^\pi}^{T \wedge T \wedge T} \phi(X_s^\pi, s)ds|X_0^\pi = x \right] 
\]

\[
> V^\pi(x) 
\]

since \( \mathbb{P}(T_i \wedge T < \tau^\pi < T_{i+1} \wedge T) > 0 \). This prove the lemma.

Using the before lemma we obtain the following result about the ruin probability.

**Lemma 3.** Suppose that \( \pi \in \prod_{i=1}^{\mathbb{N}} \) is such that \( \mathbb{P}(T_i \wedge T < \tau^\pi < T_{i+1} \wedge T) > 0 \), for some \( i \in \mathbb{N} \), where \( T_i \)'s are the jump times of the Poisson process \( \{ N_t : t \geq 0 \} \). Then, exists \( \tilde{\pi} \in \prod_{i=1}^{\mathbb{N}} \) such that \( \mathbb{P}_x \left( \tilde{\tau}^\pi = T_i \right) \) for some \( i \) = 1, and

\[
\psi^\pi(x) < \psi^\pi(x) 
\]

**Proof.** On the event \( \{ T_i \wedge T < \tau^\pi < T_{i+1} \wedge T \} \), with the notation as before lemma, we have that \( X_{\tau^\pi} = X_0^\pi = 0 \). Now, define the strategy \( \pi_t = \pi_i 1_{\{ t < \tau^\pi \}} \) and define by \( \tilde{X} \) the risk process with investment, i.e., \( \tilde{X} := X^\pi \). Then, it is clearly that \( \tilde{X}_t = X_t^\pi \) for all \( t \in [0, \tau^\pi] \), \( \mathbb{P} \)-a.s. Consequently, \( \tilde{X} \) satisfies the following stochastic differential equation

\[
\begin{cases}
  d\tilde{X}_t = dX_t^\pi, & t \in [0, \tau^\pi] \\
  dX_t = (c + rX_t)dt - dQ_t, & t > \tau^\pi 
\end{cases}
\] (3.2)

Note that the process \( dX_t = (c + rX_t)dt - dQ_t \) in the absence of claims is increasing, and the ruin can occur only at some time \( T_i \). Therefore the \( \tau^\pi = T_k \) for some \( k > i \). Thus,

\[
\tau^\pi = \tau^\pi + \theta 
\]
where \( \theta \) is the ruin time of the process \( \{ Y_t = \hat{X}_{t+\tau} : t \geq 0 \} \), with \( Y_0 = 0 \). Since \( \mathbb{P}(T_1 \wedge T < \tau^\pi < T_{1+1} \wedge T) > 0 \), we have that \( \mathbb{P}(\theta > 0) > 0 \). Thus,

\[
\psi(x) = \mathbb{P}[\tau^\pi < \infty | X_0^\pi = x] = \mathbb{P}[\tau^\pi < \infty \land \theta < \infty | X_0^\pi = x]
\]

\[
= \mathbb{P}[\tau^\pi < \infty | X_0^\pi = x] \mathbb{P}[\theta < \infty | \tau^\pi < \infty]
\]

Remember that the initial surplus of the process \( \{ Y_t : t \geq 0 \} \) is \( Y_0 = 0 \), and the fraction of the surplus invested in the risky asset and the riskless bond is zero and one, respectively, i.e., \( \pi_t = 0 \) for all \( t \geq 0 \). Denote by \( \psi^J(x) \) the ruin probability of the process \( Y_t \). It is clear that \( \psi^J(x) < \psi(x) \), where \( \psi(x) \) is the classical Cramér-Lundberg risk process with the same arrival times, starting at \( \tau^\pi \), and the same claim size. Since we assume the net profit condition to the process without investment, we obtain \( \psi(x) < 1 \). Therefore, we have \( \mathbb{P}_x[\theta < \infty | \tau^\pi < \infty] < 1 \), proving the lemma.

**Lemma 4.** For any \( \varepsilon > 0 \), there exist \( \delta > 0 \) such that for any strategy \( \pi \in \prod_x^{ad} \) and \( h > 0 \) with \( 0 < h < \delta \), we can find \( \hat{\pi}^h = \prod_x^{ad} \) such that

\[
V^\pi(x) - V^{\hat{\pi}^h}(x-h) < \varepsilon \tag{3.3}
\]

**Proof.** In order to solve the problem, we introduce the value function

\[
V^\pi(t, x) = \mathbb{E}\left[ \int_t^{\tau^\pi \wedge T} \phi(X_s^\pi, s) ds \mid X_t^\pi = x \right]. \tag{3.4}
\]

The value function becomes \( V(x, t) = \sup_{\pi} V^\pi(x, t) \). The function we are looking for is \( V(x) = V(x, 0) \). Now, we consider the following modified strategy

\[
\hat{\pi}_t = 1_{\{ T_1 > h \}} 1_{\{ h, \tau^\pi \wedge T \}(t)} \pi_t \tag{3.5}
\]

where \( T_1 \) is the first interarrival time. If \( t < h \) and on the set \( \{ T_1 > h \} \) we have that \( \hat{\pi}_t \equiv 0 \) and the solution of \( dX^\pi_t = (c + r X^\pi_t) dt \) will be continuous and increasing for \( t \in [0, h] \), so that

\[
X^\pi_t = xe^{rt} + \frac{c}{r}(e^{rt} - 1), t \in [0, h] \quad \mathbb{P}\text{-a.s. on } \{ T_1 > h \}
\]

For fixed \( \varepsilon > 0 \), we have

\[
V^\pi(x) - V^{\hat{\pi}^h}(x-h) = \left( V^\pi(x) - V^{\hat{\pi}^h}(x) \right) + \left( V^{\hat{\pi}^h}(x) - V^{\hat{\pi}^h}(x-h) \right) = J_1 + J_2
\]

We shall estimate \( J_i \)'s separately. First, since \( V(t, x) \) is increasing, concave and continuous in \([0, \infty)\) with \( \lim_{x \to \infty} V(t, x) < \infty \), we have that \( V(t, x) \) is uniformly continuous on the variables \( (t, x) \), for all strategy \( \pi \in \prod_x^{ad} \). Thus, there exists \( \delta_1 \) such that, for \( h < \delta_1 \), then \( J_2 < \varepsilon/2 \).
Now, to estimate $J_1$, we use the following inequality

$$V(\hat{x}, 0) = \mathbb{E} \left[ \int_0^{\tau_{\hat{x}} \land T} \phi(X_s, s) ds \mid X_0 = x \right]$$

$$\geq \mathbb{E}_x \left[ \int_0^{\tau_{\hat{x}} \land T} \phi(X_s, s) ds \mid T_1 > h \right] \mathbb{P}(T_1 > h)$$

$$\geq e^{-\lambda h} \mathbb{E}_x \left[ \int_0^{\tau_{\hat{x}} \land T} \phi(X_s, s) ds \mid T_1 > h \right]$$

Thus, using the previous argument and since $V$ is increasing, concave and continuous in $[0, \infty)$ with $\lim_{x \to \infty} V(x) < \infty$, we have that

$$V(\hat{x}, 0) - V^\pi(x) \leq \left( 1 - e^{-\lambda h} \right) V^\pi(h, x) \leq C_1 h$$

(3.6)

In addition, it is clear that $V(t, x)$ is uniformly continuous on the variables $(t, x)$ with $\lim_{h \downarrow 0} [V^\pi(h, x) - V^\pi(0, x)] = 0$, for all strategy $\pi \in \Pi^{ad}$. Therefore, there exists $\delta_2$ such that $J_1 < \varepsilon/2$ for $h < \delta_2$. Taking $\delta = \min\{\delta_1, \delta_2\}$ we prove (3.3), whence the lemma.

4 Dynamic programming principle

In this section, we show the Dynamic Programming Principle for our optimization problem.

**Theorem 4.1.** For any initial capital $x$ and stopping time $\tau \in [0, T]$, the value function satisfies

$$V(x) = \sup_{\pi \in \Pi^{ad}} \mathbb{E}_x \left[ \int_0^{\tau \land \tau^\pi} \phi(X_s, s) ds + V(X_{\tau \land \tau^\pi}) \right]$$

(4.1)

**Proof.** We shall first argue the theorem for deterministic time $h \in [0, T]$. Define the function

$$A(x, h) = \sup_{\pi \in \Pi^{ad}} \mathbb{E}_x \left[ \int_0^{h \land \tau^\pi} \phi(X_s, s) ds + V(X_{h \land \tau^\pi}) \right]$$

Now, we show that $V(x) = A(x, h)$. Let $\pi \in \Pi^{ad}$, and write

$$V^\pi(x) = \mathbb{E}_x \left[ \int_0^{h \land \tau^\pi} \phi(X_s, s) ds \right] + \mathbb{E}_x \left[ \int_h^{\tau^\pi} \mathbb{1}_{\{\tau^\pi > h\}} \phi(X_s, s) ds \right]$$

(4.2)
and
\[
\mathbb{E}_x \left[ \int_h^{\tau} 1_{\{\tau > h\}} \phi(X_s^\pi, s) \, ds \right] \\
= \mathbb{E}_x \left[ 1_{\{\tau > h\}} \mathbb{E} \left[ \int_h^{\tau} \phi(X_s^\pi, s) \, ds \mid \mathcal{F}_h \right] \right] \\
\leq \mathbb{E}_x \left[ 1_{\{\tau > h\}} V(X_h^\pi) \right] \\
\leq \mathbb{E}_x [V(X_{h \wedge \tau}^\pi)]
\]
Then
\[
V^{\pi}(x) \leq \mathbb{E}_x \left[ \int_0^{h \wedge \tau} \phi(X_s^\pi, s) \, ds \right] + \mathbb{E}_x [V(X_{h \wedge \tau}^\pi)]
\]
Taking supremum over strategies $\pi$, we obtain $V(x) \leq A(x, h)$.

The reverse inequality is a bit more complicated. Let $\varepsilon > 0$ be a constant. Since $V$ is increasing, concave and continuous in $[0, \infty)$ with $\lim_{x \to \infty} V(x) < \infty$, we can find an increasing sequence $\{x_n : n = 0, 1, 2, \ldots\}$ with $x_0 = 0$ and $\lim_{n \to \infty} x_n = \infty$, such that, if $x \in [x_n, x_{n+1})$ then
\[
V(x) - V(x_n) < \frac{\varepsilon}{3} \quad (4.3)
\]
Now, by definition of function $V$, for each $x_i$ there exists an admissible strategy $\pi^{(i)} \in \Pi^{ad}$ such that
\[
V(x_i) - V^{\pi^{(i)}}(x_i) < \frac{\varepsilon}{3} \quad (4.4)
\]
If $x \in [x_i, x_{i+1})$ then there exists an admissible strategy $\bar{\pi}^{(i)} \in \Pi^{ad}$ such that
\[
V^{\pi^{(i)}}(x_i) - V^{\bar{\pi}^{(i)}}(x_i) < \frac{\varepsilon}{3} \quad (4.5)
\]
Then, we obtain the following inequalities
\[
V^{\bar{\pi}^{(i)}}(x) > V^{\bar{\pi}^{(i)}}(x_i) - \frac{\varepsilon}{3} \\
> V(x_i) - \frac{2\varepsilon}{3} \\
> V(x) - \varepsilon \quad (4.6)
\]
Therefore, we get that $V(x) - V^{\bar{\pi}^{(i)}}(x) < \varepsilon$. Now, for any $\pi \in \Pi^{ad}$ we will define a new strategy $\pi^*$ as follow:
\[
\pi^*_t = \pi_t 1_{[0, h]}(t) + \sum_{i=0}^{\infty} 1_{[h, \tau]}(t) 1_{A_i}(X_h^\pi) \bar{\pi}^{(i)}(t)
\]
\[
\pi^*_t = \pi_t 1_{[0, h]}(t) + \sum_{i=0}^{\infty} 1_{[h, \tau]}(t) 1_{A_i}(X_h^\pi) \bar{\pi}^{(i)}(t)
\]
where $A_i = [x_i, x_{i+1})$ and $h > 0$. Notice that, for $t \leq h$ we have that $\pi^*_t = \pi_t$. In addition, in the case $h < \tau^\pi$, we take the index $i$ such that $X_h^\pi = X_{h \wedge \tau}^\pi \in A_i$,
and follow the strategy $\pi^*_t = \pi^{(i)}_t$, for $t \in [h, \tau^\pi]$. Figure 1 shows the graphical representation of the strategy $\pi^*$. In addition, notice that if $h < \tau^\pi$, then $X^\pi_t > 0$, for all $t \in [0, h)$, $\pi^*_t = \pi_t$ for $t \in [0, h)$ and $\pi^*_t = \pi^{(i)}_t$ for $t \in [h, \tau^\pi]$. Therefore, we have that \{\tau^\pi ≤ h\} = \{\tau^\pi ≤ h\}. Furthermore, on \{\tau^\pi > h\}, by Equation (4.6) we obtain

$$V^\pi(X^\pi_h) ≥ V(X^\pi_h) - \varepsilon, \text{ P - a.s. on } \{\tau^\pi > h\} \quad (4.8)$$

Consequently, similar to Equation (4.2) we have that

$$V(x) ≥ V^\pi(x) = \mathbb{E}_x \left[ \int_0^{h \land \tau^\pi} \phi(X^\pi_s, s)ds \right] + \mathbb{E}_x \left[ \int_{h}^{\tau^\pi} 1_{\{\tau^\pi > h\}} \phi(X^\pi_s, s)ds \right]$$

$$= \mathbb{E}_x \left[ \int_0^{h \land \tau^\pi} \phi(X^\pi_s, s)ds \right] + \mathbb{E}_x \left[ 1_{\{\tau^\pi > h\}} \int_{h}^{\tau^\pi} \phi(X^\pi_s, s)ds \right]$$

$$= \mathbb{E}_x \left[ \int_0^{h \land \tau^\pi} \phi(X^\pi_s, s)ds \right] + \mathbb{E} \left[ 1_{\{\tau^\pi > h\}} \int_{h}^{\tau^\pi} \phi(X^\pi_s, s)ds \mid X^\pi = x \right]$$

$$= \mathbb{E}_x \left[ \int_0^{h \land \tau^\pi} \phi(X^\pi_s, s)ds \right] + \mathbb{E} \left[ 1_{\{\tau^\pi > h\}} \mathbb{E} \left[ \int_{h}^{\tau^\pi} \phi(X^\pi_s, s)ds \mid X^\pi = x \right] \mid X^\pi = x \right]$$

$$= \mathbb{E}_x \left[ \int_0^{h \land \tau^\pi} \phi(X^\pi_s, s)ds \right] + \mathbb{E}_x \left[ 1_{\{\tau^\pi > h\}} V^\pi(X^\pi_h \land \tau^\pi) \right]$$

Now, we use the fact that

$$1_{\{\tau^\pi ≤ h\}} V^\pi(X^\pi_h \land \tau^\pi) = 1_{\{\tau^\pi ≤ h\}} V^\pi(X^\pi_\tau) = 1_{\{\tau^\pi ≤ h\}} V^\pi(0) = 0$$
Therefore, using Equation (4.8), we get
\[
V(x) \geq V^\pi(x) = \mathbb{E}_x \left[ \int_0^{h \wedge \tau^*} \phi(X^\pi_s, s) ds + V^\pi(X^\pi_{h \wedge \tau^*}) \right]
\geq \mathbb{E}_x \left[ \int_0^{h \wedge \tau^*} \phi(X^\pi_s, s) ds + V^\pi(X^\pi_{h \wedge \tau^*}) \right] - \varepsilon
\] (4.9)

Since \( \varepsilon > 0 \) is arbitrary, we obtain that \( V(x) \geq A(x, h) \), proving the theorem for \( \tau = h \).

We now consider the general case, when \( \tau \in [0, T] \). Let \( t_0 = 0, t_1, \ldots, t_n = T \) be a partition of \([0, T] \), where \( t_k = \frac{k}{n}T, k = 0, 1, \ldots, n \). Define
\[
\tau_n = \sum_{k=0}^{n-1} t_k 1_{[t_k, t_{k+1})}(\tau) = \begin{cases} 
  t_0 & \text{if } \tau \in [t_0, t_1) \\
  t_1 & \text{if } \tau \in [t_1, t_2) \\
  \vdots & \\
  t_{n-1} & \text{if } \tau \in [t_{n-1}, t_n) 
\end{cases}
\]

Note that \( \{\tau_n\} \) are simple functions such that \( \tau_n \to \tau \), \( \mathbb{P} \)-a.s. Further, \( \tau_n \) take only a finite number of values. Further, \( \tau_n \equiv 0 \) if \( \tau \in [0, t_1) \) and \( \tau_n \geq t_1 \) if \( \tau \in [t_1, T) \), for all \( n \geq 1 \). Using the same argument as (4.2), it is easy to show that \( V(x) \leq A(x, \tau_n) \) for all \( n \geq 1 \).

The reverse inequality shall prove utilizing induction on \( n \), in order to
\[
V(x) \geq A(x, \tau_n) \quad \text{for all } n \geq 1
\] (4.10)

For \( n = 1 \) we have that \( \tau_1 \equiv 0 \) on \([0, T] \), so there is nothing to prove. Suppose that (4.10) holds for \( \tau_n \). We shall argue that (4.10) holds for \( \tau_{n+1} \) as well. For any \( \pi \in \Pi^{ad} \), on \( \{\tau_{n+1} < t_1\} \) we have that \( \tau_{n+1} \equiv 0 \) and
\[
\mathbb{E}_x \left[ 1_{\{\tau_{n+1} < t_1\}} \left( \int_0^{\tau_{n+1} \wedge \tau^*} \phi(X^\pi_s, s) ds + V(X^\pi_{\tau_{n+1} \wedge \tau^*}) \right) \right] = \mathbb{E}_x \left[ V(X^\pi_0) \right]
\]

Then, we have \( V(x, \tau_{n+1}) = V(x) \) \( \mathbb{P} \)-a.s. on \( \{\tau_{n+1} < t_1\} \) for all \( n \). Now, on \( \{\tau_{n+1} \geq t_1\} \) we have
\[
\mathbb{E}_x \left[ \int_0^{\tau_{n+1} \wedge \tau^*} \phi(X^\pi_s, s) ds + V(X^\pi_{\tau_{n+1} \wedge \tau^*}) \right]
= \mathbb{E}_x \left[ \int_0^{\tau^*} \phi(X^\pi_s, s) ds \right]
+ \mathbb{E}_x \left[ \int_0^{\tau_{n+1} \wedge \tau^*} \phi(X^\pi_s, s) ds + V(X^\pi_{\tau_{n+1} \wedge \tau^*}) \right] 1_{\{\tau_{n+1} > t_1\}} 1_{\{\tau \geq t_1\}}
+ \left[ \int_0^{t_1} \phi(X^\pi_s, s) ds + V(X^\pi_{t_1}) \right] 1_{\{\tau_{n+1} = t_1\}} 1_{\{\tau \geq t_1\}}
\] (4.11)
\( \mathbb{P} \)-a.s. on \( \{ \tau_{n+1} \geq t_1 \} \). Note that on the set \( \{ \tau_{n+1} > t_1 \} \), \( \tau_{n+1} \) takes only \( n \) values, then by inductional hypothesis and the Markov property of the process \( \{ X_t^\pi \} \), we have that

\[
\mathbb{E}_x \left( \int_{t_1}^{\tau_{n+1} \wedge \tau^\pi} \phi(X_s^\pi, s) ds + V(X_{\tau_{n+1} \wedge \tau^\pi}^\pi) \right) 1_{\{\tau_{n+1} > t_1\}} 1_{\{\tau^\pi \geq t_1\}} \leq \mathbb{E}_x \left( V(X_{t_1}^\pi) 1_{\{\tau_{n+1} > t_1\}} 1_{\{\tau^\pi \geq t_1\}} \right)
\]

(4.12)

Utilizing this inequality into (4.11) we obtain

\[
\mathbb{E}_x \left[ \int_0^{\tau_{n+1} \wedge \tau^\pi} \phi(X_s^\pi, s) ds + V(X_{\tau_{n+1} \wedge \tau^\pi}^\pi) \right] \leq \mathbb{E}_x \left[ 1_{\{\tau^\pi < t_1\}} \int_0^{\tau^\pi} \phi(X_s^\pi, s) ds \right] + \mathbb{E}_x \left[ \int_0^{t_1} \phi(X_s^\pi, s) ds + V(X_{t_1}^\pi) \right] 1_{\{\tau_{n+1} = t_1\}} 1_{\{\tau^\pi \geq t_1\}} \]

+ \mathbb{E}_x \left[ \int_0^{t_1} \phi(X_s^\pi, s) ds + V(X_{t_1}^\pi) \right] 1_{\{\tau_{n+1} > t_1\}} 1_{\{\tau^\pi \geq t_1\}} \]

(4.13)

\[
\leq \mathbb{E}_x \left[ 1_{\{\tau^\pi < t_1\}} \int_0^{\tau^\pi} \phi(X_s^\pi, s) ds \right] + \mathbb{E}_x \left[ 1_{\{\tau^\pi \geq t_1\}} \left( V(X_{t_1}^\pi) + \int_0^{t_1} \phi(X_s^\pi, s) ds \right) \right]
\]

\[
= \mathbb{E}_x \left[ \int_0^{t_1 \wedge \tau^\pi} \phi(X_s^\pi, s) ds + V(X_{t_1}^\pi) \right] \leq V(x)
\]

\( \mathbb{P} \)-a.s. on \( \{ \tau_{n+1} \geq t_1 \} \). The last inequality is due to (4.1) for fixed time \( t_1 \). Consequently, we obtain \( A(x, \tau_n) \leq V(x) \) for all \( n \), whence \( A(x, \tau_n) = V(x) \) for all \( n \). Finally, utilizing dominated convergence theorem, together with the continuity of the value function, we proof the general identity of (4.1). This complete the proof. \( \square \)

5 The Hamilton-Jacobi-Bellman equation

We are now ready to calculate the Hamilton-Jacobi-Bellman (HJB) equation associated to our optimization problem (2.6). Given \( \varepsilon > 0 \) be a constant there exist a strategy \( \pi \in \prod_{ad}^\pi \) and \( h > 0 \) such that \( V(h, x) < V^\pi(h, x) + \varepsilon \). Using Itô’s Lemma for a function of two variables and a similar argument as in Section 2.2 of Azcue and Muler [Azcue and Muler (2014)], we find

\[
\lim_{t \downarrow 0} \frac{1}{t} \mathbb{E}_x \left[ f(\tau^\pi, t, X_{\tau_{n+1} \wedge t}^\pi) - f(0, x) \right] = \frac{1}{2} \left[ \sigma^\pi(x) \right]^2 f_{xx}(0, x) + \mu^\pi(x) f_x(0, x) + f_t(0, x) + \lambda \int_{(0, \infty)} \left[ f(0, x - z) - f(0, x) \right] dF_U(z)
\]

(5.1)
where \( x \in \mathbb{R} \), \( \mu^\pi(x) = c + \mu \pi(x) + r(x - \pi(x)) \), \( \sigma^\pi(x) = \sigma \pi(x) \) and \( F_U \) is the distribution function of claim size. By Theorem 4.1 we have that

\[
V(0, x) = \sup_{\pi} V^\pi(0, x) \\
\geq \mathbb{E}_x \left[ \int_0^{\tau \wedge h} \phi(s, X^\pi_s) ds + V^\pi(\tau \wedge h, X^\pi_{\tau \wedge h}) \right] \\
> \mathbb{E}_x \left[ \int_0^{\tau \wedge h} \phi(s, X^\pi_s) ds + V(\tau \wedge h, X^\pi_{\tau \wedge h}) \right] - \varepsilon
\]

(5.2)

In the before equation we can let \( \varepsilon \) tend to zero. Assuming that \( V(t, x) \) is twice continuously differentiable in \( x \) and continuously differentiable in \( t \), and dividing by \( h \) and letting \( h \downarrow 0 \) we get

\[
\phi(0, x) + \frac{1}{2} [\sigma^\pi(x)]^2 f_{xx}(0, x) + \mu^\pi(x) f_x(0, x) + f_t(0, x) + \lambda \mathcal{I}[V](0, x) \leq 0
\]

where \( \mathcal{I}[V](0, x) = \int_{(0, \infty)} [V(0, x - z) - V(0, x)] dF_U(z) \). This inequality has to hold for all \( \pi \). Therefore, this motives the Hamilton-Jacobi-Bellman equation by our optimization problem

\[
\phi(t, x) + V_t(t, x) + \mathcal{L}[V](t, x) = 0, \quad (t, x) \in [0, T) \times [0, \infty)
\]

(5.3)

with the boundary conditions \( V(t, 0) = V(T, x) = 0 \), and where \( \mathcal{L} \) is the following second order partial integro-differential operator

\[
\mathcal{L}[V](t, x) = \sup_{\pi} \left\{ \frac{1}{2} [\sigma^\pi(x)]^2 V_{xx}(t, x) + \mu^\pi(x) V_x(t, x) + \lambda \mathcal{I}[V](t, x) \right\}
\]

for \( V \in C^1_0([0, T] \times [0, \infty)) \), the set of all functions twice continuously differentiable in \( x \) and continuously differentiable in \( t \).

In addition, by Lemma 1, \( V : [0, T] \times \mathbb{R}_+ \to \mathbb{R}_+ \) is strictly increasing and concave in \( x \) with the boundary condition \( V(t, 0) = V(T, x) = 0 \). Because the left hand side of (5.3) is quadratic in \( \pi \), we find that

\[
\pi^* = -\frac{(\mu - r)V_x}{\sigma^2 V_{xx}}
\]

(5.4)

Strategy \( \pi^* \) will be our candidate for the optimal strategy. As \( \pi(X^\pi_t) \) is the amount invested in the risky asset, we have that \( \pi(X^\pi_t) = \theta(t, X^\pi_t) X^\pi_t \), where \( \theta \) represent the proportion of the surplus invested in the risky asset at time \( t \) (hence \( \theta(t, X^\pi_t) \in [0, 1] \) for all \( t \in [0, T] \)). Then, using (5.4) we find that

\[
\theta^*(t, x) = -\frac{(\mu - r)V_x}{\sigma^2 x V_{xx}}
\]

(5.5)
Note that, if the expected return of the stock market $\mu$ is greater than the risk-free rate $r$, then $\theta^*(t, x) \geq 0$, but that does not guarantee that $\theta^*(t, x)$ is less than 1. Thus, if $\mu > r$ the right strategy would be $\min\{\theta^*(t, x), 1\}$. In the other hand, if $\mu \leq r$ the right strategy will be $\theta^*(t, x) \equiv 0$.

**Theorem 5.1.** Suppose that there exists a solution $f(t, x)$ of (5.3) that is a function twice continuously differentiable in $x$ and continuously differentiable in $t$ with boundary conditions $f(T, x) = 0$. Then $V(t, x) \leq f(t, x)$. If

$$\theta^*(t, x) = \frac{\mu - r}{\sigma^2 x^2}$$

is bounded and $f(t, 0) = 0$ for all $t$, then $V(t, x) = f(t, x)$ and an optimal strategy is given by $\{\pi^*(t, X^\pi_t) : t \in [0, \tau_{\pi} \wedge T]\}$.

**Proof.** Let $\pi \in \Pi_{ad}$ and let $0 < T_1 < \cdots < T_i < \cdots < t$ be the interarrival times in $[0, t)$. The controlled investment process $X^\pi_t$ has finite jumps on each finite time interval $[0, t)$. The jump size of the process $X^\pi_t$ at time $t$ is denoted by $\Delta X^\pi_t = X^\pi_{T_i} - X^\pi_{T_i-} = -U_i$ where $U_i$ is the claim size at time $T_i$. Consider the process $f(t, X^\pi_t)$ conditioned on $X_s = x$. Using Itô’s Lemma for a jump process (see for details [Cont and Tankov (2004)]), we have that

$$f(t, X^\pi_t) - f(s, x) = \int_s^t \left[ \frac{\partial f}{\partial t}(s, X^\pi_s) + \mu^\pi \frac{\partial f}{\partial x}(s, X^\pi_s) + \frac{(\sigma^\pi)^2}{2} \frac{\partial^2 f}{\partial x^2}(s, X^\pi_s) \right] ds$$

$$+ \int_s^t \sigma^\pi \frac{\partial f}{\partial x}(s, X^\pi_{s-}) dW_s + \sum_{i \geq 1: s \leq T_i < t} \left[ f(T_i, X^\pi_{T_i}) - f(T_i, X^\pi_{T_i-}) + \Delta X^\pi_{T_i} \right]$$

It is known that the stochastic integral is a martingale. Thus

$$\mathbb{E}_x \left[ \int_s^t \sigma^\pi \frac{\partial f}{\partial x}(s, X^\pi_{s-}) dW_s \right] = 0.$$

Then, by the Hamilton-Jacobi-Bellman equation (5.3) we have that

$$\mathbb{E}_x \left[ f(r^\pi \wedge T, X^\pi_t) + \int_s^{r^\pi \wedge T} \phi(r, X^\pi_r) dr \right] \leq f(s, x)$$

Because $f(r^\pi \wedge T, X^\pi_t) \geq 0$, we obtain the following inequality

$$V^\pi(s, x) = \mathbb{E}_x \left[ \int_s^{r^\pi \wedge T} \phi(r, X^\pi_r) dr \right] \leq f(s, x)$$

Now, if we take supremum over all strategies $\pi \in \Pi_{ad}$, we have that $V(s, x) \leq f(s, x)$. 

Suppose now that \( \theta^* \) is bounded by a value \( \bar{\theta} = \sup \theta(t, x) \). Note that by the Hamilton-Jacobi-Bellman equation (5.3), \( \theta^* > 0 \) whenever \( X_t^* > 0 \), where \( \{X_t^*\} \) is the process with optimal investment strategy \( \pi^* \). Choose \( \varepsilon < x \) and define \( \tau_{\varepsilon} = \inf \{t > s : X_t^* < \varepsilon \} \). Note that \( \tau_{\varepsilon} \to \tau^* \) converges monotonically as \( \varepsilon \to 0 \), where \( \tau^* \) is the time to ruin of the process \( X_t^* \).

As \( f \) is a concave function, we have that \( f_{xx} < 0 \). Thus, \( f_x \) is increasing on \( x \) and \( f_x(t, x) \leq f_x(t, \varepsilon) \) for all \( t \). Then, \( f_x \) is bounded on \([0, T] \times [\varepsilon, \infty)\). Now, by the Hamilton-Jacobi-Bellman equation (5.3) and the Itô formula we obtain

\[
\mathbb{E}_x \left[ f(t, X_t^*) + \int_s^t \phi(r, X_r^*)dr \right] = f(s, x) + \mathbb{E}_x \left[ \int_s^T \sigma \pi^* \frac{\partial f}{\partial x}(r, X_r^*)dW_r \right] \tag{5.7}
\]

In addition, as \( \theta^* \) is bounded, then the second moment of \( X_t^* \) is bounded for all \( t \), and \( \int_0^T \mathbb{E}[X_t^*]dt < \infty \). Thus, we have that \( \int_s^T \sigma \pi^* \frac{\partial f}{\partial x}(r, X_r^*)dW_r \) is a martingale (for more details see [Karatzas and Shreve, 1991], [Ikeda and Watanabe, 1981]). Therefore, we obtain the following equation,

\[
\mathbb{E}_x \left[ f(\tau_{\varepsilon} \wedge T, X_{\tau_{\varepsilon} \wedge T}^*) + \int_s^{\tau_{\varepsilon} \wedge T} \phi(r, X_r^*)dr \right] = f(s, x)
\]

Notice that, as \( \varepsilon \to 0 \) we have that

\[
\int_s^{\tau_{\varepsilon} \wedge T} \phi(r, X_r^*)dr \to \int_s^{\tau^* \wedge T} \phi(r, X_r^*)dr
\]

monotonically. The first term can be written as follow

\[
\mathbb{E}_x \left[ f(\tau_{\varepsilon}, \varepsilon : \tau_{\varepsilon} \leq T) \right] + \mathbb{E}_x \left[ f(T, X_T^*) : \tau_{\varepsilon} > T \right]
\]

The term \( \mathbb{E}_x \left[ f(\tau_{\varepsilon}, \varepsilon : \tau_{\varepsilon} \leq T) \right] \) is bounded by \( f(0, \varepsilon)\mathbb{P}[\tau_{\varepsilon} \leq T] \), which is uniformly bounded. Thus, by the boundary condition, it converges to \( \mathbb{E}_x \left[ f(\tau^*, 0) : \tau^* \leq T \right] = 0 \). Again, by boundary condition, the second term converges monotonically to

\[
\mathbb{E}_x \left[ f(T, X_T^*) : \tau^* > T \right] = 0
\]

This completes the proof.
Finally, suppose that \( f(t, x) \) is a solution of (5.3) twice continuously differentiable in \( x \) and continuously differentiable in \( t \) with boundary conditions \( f(T, x) = 0 \). Assume that there exists some constant \( M < \infty \) such that
\[
\theta^*(t, x) = -\frac{(\mu - r)f_x}{\sigma^2 x f_{xx}} \leq M
\]
Remember that \( f_x > 0 \) and \( f_{xx} < 0 \). Then
\[
\frac{\partial}{\partial x} \ln (f_x(t, x)) \leq -\frac{(\mu - r) M}{x}
\]
Therefore, for some constants \( \alpha \in (0, 1) \) and \( K > 0 \) we have that
\[
f(t, x) \leq K x^{1-\alpha}
\]
This bound for the solution motivate trying a solution of the form \( V(t, x) = f^\alpha(t)x^{1-\alpha} \).

6 Numerical example

We apply the method to the Cobb-Douglas type utility function \( \phi(t, x) = x^{1-\alpha}e^{-\kappa \alpha t} \), where \( \alpha \in (0, 1), \kappa \in \mathbb{R} \), in order to solve the optimization problem (2.6) trying a solution of the form \( V(t, x) = f^\alpha(t)x^{1-\alpha} \) for a continuously differentiable function \( f : [0, T] \to \mathbb{R}_+ \) with boundary condition \( f(T) = 0 \). By (5.4) we have that the optimal strategy for our problem is given by
\[
\pi^*(x) = \frac{(\mu - r)}{\sigma^2 \alpha} x
\]
and the optimal proportion of the surplus invested in the risky asset at time \( t \) is given by
\[
\theta^* = \frac{(\mu - r)}{\sigma^2 \alpha}
\]
Notice that in this case solving our optimization problem we obtain that the optimal proportion to be invested in the risky asset is the Merton ratio, introduced in [Merton R., (1969)]. Next, by the HJB equation (5.3) we obtain the following equation
\[
\phi(x, t) + \alpha f^{\alpha-1}(t)f'(t)x^{1-\alpha} + [c + rx + (\mu - r)\theta^* x] (1 - \alpha) f^\alpha(t)x^{-\alpha} - \frac{1}{2} \sigma^2 (\theta^*)^2 x^2 \alpha (1 - \alpha) f^\alpha(t)x^{-\alpha-1} + \lambda f^\alpha(t)E \left((x - U)^{1-\alpha} - x^{1-\alpha}\right) = 0
\]
Thus, the function \( f(t) \) satisfies the following relation
\[
\frac{f'(t)}{f(t)} + e^{-\kappa \alpha t} - \frac{1}{2} \alpha (1 - \alpha) \sigma^2 (\theta^*)^2 = K(x)
\]
with boundary condition \( f(T) = 0 \), where
\[
K(x) = \frac{\lambda}{x^{1-\alpha}} E \left(x^{1-\alpha} - (x - U)^{1-\alpha}\right) - \frac{c(1 - \alpha)}{x} - [r + (\mu - r)\theta^*] (1 - \alpha)
\]
for all $t$ and $x$. Thus, we have that $f$ satisfies the differential equation
\[
\alpha f'(t) + f^{1-\alpha}(t)e^{-\kappa\alpha t} = \left(K + \frac{1}{2}\alpha(1-\alpha)\sigma^2(\theta^*)^2\right)f(t)
\]
In order to solve these we are now going to use the substitution $z(t) = f^{\alpha}(t)$ obtaining the differential equation $z'(t) - Rz(t) = e^{-\kappa\alpha t}$, where $R = K + \frac{1}{2}\alpha(1-\alpha)\sigma^2(\theta^*)^2$. We thus find that the solution is given by
\[
f(t) = \frac{e^{-\kappa t}}{(R + \kappa\alpha)^{1/\alpha}} \left(e^{-(R/\kappa\alpha)(T-t)} - 1 \right)
\] (6.3)

**Exponential claim sizes:** Assume that the claim sizes $U_i$ has $\text{Exp}(\theta)$ distribution, i.e., $U_i \sim \text{Exp}(\theta)$. Then, we obtain the following identity
\[
\mathbb{E}(x - U)^{1-\alpha} = B(2 - \alpha, 1)x^{2-\alpha}F_1(1, 3-\alpha, -\theta x) \leq B(2 - \alpha, 1)x^{2-\alpha}e^{-\theta x}
\]
Thus, we have that $\mathbb{E}(x - U)^\alpha \to 0$ as $x \to \infty$. Using this result is straightforward prove the following asymptotic result for the function $K$.

**Proposition 1.** Let $\{U_i\}$ be a sequence of independent, identically distributed (i.i.d). random variables having $\text{Exp}(\theta)$ distribution. Then the following hold.

1. If $x \to \infty$, then
   \[
   K(x) \to \lambda - \alpha(r + \theta^*(\mu - r))
   \]
   where $\theta^*$ is the optimal proportion of the surplus invested in the risky asset.

2. If $x \to 0^+$, then $K(x) \to -\infty$.

### 6.1 Simulation of the ruin probability with and without optimal investment

In this section, we conduct Monte Carlo simulation in order to compute and compare the behavior of the ruin probability with and without optimal investment. For the purpose of comparison, ruin probability with and without optimal investment are calculated to Pareto and Weibull distribution as claim sizes. The ruin probability $\psi(\pi, x, T)$ for the risk process with investment $X_\pi t$ can be simulated by randomly drawing sample paths according to the process $X_\pi t$ and counting the trajectories that lead to ruin and dividing this number by the total number $N$ of simulated trajectories. Thus, we get an unbiased estimator of the ruin probability
\[
\psi(\pi, x, T) = \mathbb{P}_x[\tau^\pi < T] \approx \hat{\psi}(\pi, x, T) = \frac{1}{N} \sum_{i=1}^{N} 1_A(w_i)
\] (6.4)
where $A$ is the set of all trajectories $w_i$ that lead to ruin up to time $T$. Notice that to estimate the ruin probability it is necessary to simulate a sample trajectories of the jump-diffusion process

$$dX^\pi_t = [c + \mu \pi_t X^\pi_t + r(1 - \pi_t)X^\pi_t] \, dt + \sigma \pi_t X^\pi_t \, dW_t - dQ_t$$

where $Q_t$ is a compound Poisson process.

Now, we wish to simulate paths of the process $\{X^\pi_t\}$ without knowing its distribution or an explicit solution to the equation SDE in order to know if each simulated trajectory goes to ruin or not. We can simulate a discretized version of the jump-diffusion process. In particular, we simulate a discretized trajectories, $\{\hat{X}_0, \hat{X}_h, \hat{X}_{2h}, ..., \hat{X}_{nh}\}$ where $n$ is the number of time steps, $h$ is a constant and $h = T/n$. The smaller the value of $h$, the closer our discretized path will be to the continuous-time path of $X^\pi_t$ that we wish to simulate (for more details see example [Karatzas and Shreve, 1991], [Oksendal (2000)] and [Ikeda and Watanabe, 1981]).

In the literature there are several discretization schemes available, the simplest approach is the Euler scheme. The Euler method is intuitive and easy to implement, and in our case we get the following discretize version

$$\hat{X}^\pi_{kh} = \hat{X}^\pi_{(k-1)h} + \left[c + \mu \pi \hat{X}^\pi_{(k-1)h} + r(1 - \pi_t)\hat{X}^\pi_{(k-1)h}\right]h + \sigma \pi \hat{X}^\pi_{(k-1)h}Z_k - \Delta Q_{kh}$$

where the $Z_k$ are i.i.d. $N(0, h)$.

Note that the risk process with investment $X^\pi_t$ has finite jumps on any finite interval $[0, T)$, and in absence of claims (or between the jumps of $N_t$) is continuous and satisfies the discretized version of the process

$$\hat{X}^\pi_{kh} = \hat{X}^\pi_{(k-1)h} + \left[c + \mu \pi \hat{X}^\pi_{(k-1)h} + r(1 - \pi_t)\hat{X}^\pi_{(k-1)h}\right]h + \sigma \pi \hat{X}^\pi_{(k-1)h}Z_k$$

The jump size of the process $X^\pi_t$ at time $t$ is denoted by $\Delta X^\pi_t = X^\pi_t - X^\pi_{t-}$. The notation $X^-_t$ refers to $\lim_{s \to t^-} X_s$. Thus, if the $n^{th}$ jump in the compound Poisson process occurs at time $t$ we have

$$X^\pi_t - X^\pi_{t^-} = -U_n$$

where $U_n$ is the claim size at time $t = T_n$.

Taking in account previous remarks, an approach to stimulating a discretized version of $X^\pi_t$ on the interval $[0, T]$ is given by

1. First simulate the arrival times in the compound Poisson process up to time $T$.
2. Use a pure diffusion discretization between the jump times.
3. At the $n^{th}$ arrival time $T_n$, simulate the $n^{th}$ claim size $U_n$ conditional on the value of the discretized process, $\hat{X}^\pi_{T_n}$, immediately before $T_n$.  

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We consider the case of a Weibull and Pareto distribution for the claim size (severity) \( U \) with \( U \sim \text{Weibull}(1, 50) \) and \( U \sim \text{Pareto}(25, 2) \), frequency parameter \( \lambda = 1 \), premium rate \( c = 65 \) and safety loading factor \( \rho = 30\% \). Parameters to the optimal investment are: risk-free rate \( r = 8.4 \times 10^{-4} \), expected return \( \mu = 10^{-3} \), volatility \( \sigma^2 = 10^{-3} \) and risk aversion \( \alpha = 0.2 \). In our simulation we have around of 0.48\% monthly excess return and Merton ratio \( \pi = 0.8 \). The choice of the parameters is purely academic and \( T = 1 \) year.

Figure 3: Ruin probability with claim size having Weibull and Pareto distribution: without investment (solid line) and with optimal investment (dashed line).

Using the discretized version of the process 6.5, we simulate \( n = 10000 \) trajectories for each case. Numerical results, in Figure 3, show that the optimal strategy in one year reduce the ruin probability of the portfolio for different claim size distributions. Solid line is the ruin probability without investment and dashed line is the ruin probability with investment. The difference in ruin probabilities seems to be small, but notice that, with Exponential claim size for capital risk \( x = 100 \) we have a ruin probability \( \hat{\psi}^\pi(x, T) = 0.4306 \) with optimal investment, while the ruin probability is 0.4406 without investment \( \hat{\psi}(x, T) \). In Table 1, we summarize our results for Exponential, Pareto and Weibull distributions for the claim size, and different values of risk capital \( x \).

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Table 1: Ruin probability with claim size having Exponential, Pareto and Weibull distribution: without investment $\psi(x, T)$ and with optimal investment $\psi^\pi(x, T)$.

| x  | Exponential $\psi(x, T)$ | Pareto $\psi^\pi(x, T)$ | Weibull $\psi(x, T)$ | Weibull $\psi^\pi(x, T)$ |
|----|--------------------------|--------------------------|----------------------|--------------------------|
| 100| 0.4406                   | 0.4372                   | 0.377                | 0.3728                   |
| 200| 0.274                    | 0.2676                   | 0.2588               | 0.247                    |
| 400| 0.1014                   | 0.097                    | 0.1464               | 0.1418                   |

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