Partial Differential Equations/Complex Analysis

A flower structure of backward flow invariant domains for semigroups

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Abstract

In this Note, we study conditions which ensure the existence of backward flow invariant domains for semigroups of holomorphic self-mappings of a simply connected domain $D$. More precisely, the problem is the following. Given a one-parameter semigroup $S$ on $D$, find a simply connected subset $\Omega \subset D$ such that each element of $S$ is an automorphism of $\Omega$, in other words, such that $S$ forms a one-parameter group on $\Omega$.

Résumé

Une structure en rosace de domaines invariants par flot rétrograde de semi-groupes. Dans cette Note nous établissons des conditions qui assument l’existence de domaines invariants par flot rétrograde de semi-groupes d’applications holomorphes d’un domaine $D$, simplement connexe, dans lui-même. De manière plus précise, étant donné un semi-groupe $S$ à un paramètre sur $D$, trouver un sous-ensemble connexe $\Omega \subset D$ tel que chaque élément de $S$ soit un automorphisme de $\Omega$, en d’autres termes tel que $S$ soit un groupe à un paramètre sur $\Omega$. Pour citer cet article: M. Elin et al., C. R. Acad. Sci. Paris, Ser. I 346 (2008).

Let $D$ be a simply connected domain in the complex plane $\mathbb{C}$. By $\text{Hol}(D, \Omega)$ we denote the set of all holomorphic functions on $D$ with values in a domain $\Omega$ in $\mathbb{C}$. We write $\text{Hol}(D)$ for $\text{Hol}(D, D)$, the set of holomorphic self-mappings of $D$. This set is a topological semigroup with respect to composition. We denote by $\text{Aut}(D)$ the group of all automorphisms of $D$; thus $F \in \text{Aut}(D)$ if and only if $F$ is univalent on $D$ and $F(D) = D$.

Definition 1. A family $S = \{F_t\}_{t \geq 0} \subset \text{Hol}(D)$ is said to be a one-parameter continuous semigroup (semiflow) on $D$ if:

(i) $F_t(F_s(z)) = F_{t+s}(z)$ for all $t, s \geq 0$;
(ii) $\lim_{t \to 0^+} F_t(z) = z$ for all $z \in D$. 

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If, in addition, condition (i) holds for all $t, s \in \mathbb{R}$, then $(F_t)^{-1} = F_{-t}$ for each $t \in \mathbb{R}$; and $S$ is called a one-parameter continuous group (flow) on $D$. In this case, $S \subset \text{Aut}(D)$.

In this Note, we study the following problem: **Given a one-parameter semigroup $S \subset \text{Hol}(D)$, find a simply connected domain $\Omega \subset D$ (if it exists) such that $S \subset \text{Aut}(\Omega)$.**

It follows by a result of E. Berkson and H. Porta [4] that each continuous semigroup is differentiable in $t \in \mathbb{R}^+ = [0, \infty)$, (see also [1] and [13]). So, for each continuous semigroup (semiflow) $S = \{F_t\}_{t \geq 0} \subset \text{Hol}(D)$, the limit,

$$\lim_{t \to 0^+} \frac{z - F_t(z)}{t} = f(z), \quad z \in D,$$

exists and defines a holomorphic mapping $f \in \text{Hol}(D, \mathbb{C})$. This mapping $f$ is called the (infinitesimal) generator of $S = \{F_t\}_{t \geq 0}$.

Let now $D = \Delta$ be the open unit disk in $\mathbb{C}$.

Observe that if a semigroup $S = \{F_t\}_{t \geq 0}$ does not contain an elliptic automorphism of $\Delta$, then there is a unique point $\tau \in \Delta$ which is the attractive point for the semigroup $S$, i.e., for all $z \in \Delta$,

$$\lim_{t \to \infty} F_t(z) = \tau. \quad (1)$$

This point is usually referred as the **Denjoy–Wolff point** of $S$. In addition,

- if $\tau \in \Delta$, then $\tau = F_t(\tau)$ is a unique fixed point of $S$ in $\Delta$;
- if $\tau \in \partial \Delta$, then $\tau = \lim_{t \to 1^-} F_t(\tau)$ is a common boundary fixed point of $S$ in $\Delta$, and no element $F_t$ ($t > 0$) has an interior fixed point in $\Delta$.

Also, we note that if $\tau$ in (1) belongs to $\partial \Delta$, then if follows by a result in [10] that the angular limits,

$$f(\tau) := \zeta \lim_{z \to \tau} f(z) = 0 \quad \text{and} \quad f'(\tau) := \zeta \lim_{z \to \tau} f'(z) = \beta$$

exist and that $\beta$ is a nonnegative real number (see also [6]). Moreover, if for some point $\zeta \in \partial \Delta$ there are limits,

$$\zeta \lim_{z \to \zeta} f(z) = 0 \quad \text{and} \quad \zeta \lim_{z \to \zeta} f'(z) = \gamma,$$

with $\gamma \geq 0$, then $\gamma = \beta$ and $\zeta = \tau$ (see [10] and [15]).

In the case where $\beta > 0$, the semigroup $S = \{F_t\}_{t \geq 0}$ consists of mappings $F_t \in \text{Hol}(\Delta)$ of **hyperbolic type**, $\zeta \lim_{z \to \tau} \frac{\partial F_t(z)}{\partial z} = e^{-i \beta} < 1$. Otherwise ($\beta = 0$), it consists of mappings of **parabolic type**, $\zeta \lim_{z \to \tau} \frac{\partial F_t(z)}{\partial z} = 1$ for all $t \geq 0$.

**Definition 2.** A point $\eta \in \partial \Delta$, is said to be a boundary regular null point of $f \in \text{Hol}(D, \mathbb{C})$ if $f(\eta) := \zeta \lim_{z \to \eta} f(z) = 0 \quad \text{and} \quad \gamma = \zeta \lim_{z \to \eta} f'(z)$ exists finitely.

It follows by a result in [15] (see also [6]) that if $f \in \text{Hol}(D, \mathbb{C})$ is the generator of a semigroup $S = \{F_t\}_{t \geq 0}$ having a boundary regular null point $\eta \in \partial \Delta$ with $\gamma = \zeta \lim_{z \to \eta} f'(z)$, then $\gamma$ is a real number. Moreover, $\gamma \geq 0$ if and only if $\eta \in \partial \Delta$ is the Denjoy–Wolff point of $S$; otherwise ($\gamma < 0$), $\eta$ is a repelling fixed point for $S$.

It turns out that if a semigroup $S$ generated by $f \in \text{Hol}(D, \mathbb{C})$ contains neither elliptic automorphisms of $\Delta$ nor a parabolic type self-mapping of $\Delta$, then the solvability of our problem mentioned above is equivalent to the existence of a boundary regular null point of the generator $f$ different from the Denjoy–Wolff point of $S$. Actually, more is true.

**Definition 3.** Let $S = \{F_t\}_{t \geq 0}$ be a semiflow on $\Delta$. A domain $\Omega \subset \Delta$ is called a **backward flow-invariant domain** (shortly, BFID) for $S$ if $S \subset \text{Aut}(\Omega)$.

**Theorem 1.** Let $S = \{F_t\}_{t \geq 0}$ be a nontrivial semiflow on $\Delta$ generated by $f \in \text{Hol}(D, \mathbb{C})$ which does not contain an elliptic automorphism of $\Delta$. The following assertions are equivalent:
(i) \( f \) has a boundary regular null point \( \eta \in \partial \Delta \) different from the Denjoy–Wolff point of \( S \), i.e.,

\[ \gamma = \lim_{z \to \eta} f'(z) < 0; \]

(ii) for some \( \alpha > 0 \), the differential equation,

\[ \alpha \varphi'(z)(z^2 - 1) = 2f(\varphi(z)), \]

has a locally univalent solution \( \varphi \) with \( |\varphi(z)| < 1 \) when \( z \in \Delta \).

Moreover, in this case, \( \alpha \geq -\gamma \), \( \varphi \) is univalent and is a Riemann conformal mapping of \( \Delta \) onto a backward flow invariant domain \( \Omega \subset \Delta \), so \( S \subset \text{Aut}(\Omega) \).

The following result contains a partial converse:

**Theorem 2.** Let \( S = \{F_t\}_{t \geq 0} \) be a semiflow on \( \Delta \) generated by \( f \), and let \( \tau \in \bar{\Delta} \) be its Denjoy–Wolff point with \( f(\tau) = 0 \) and \( f'(\tau) = \beta, \Re \beta > 0 \). If \( \Omega \subset \Delta \) is a nonempty backward flow invariant domain for \( S \), then it is a Jordan domain such that \( \tau \in \partial \Omega \), and there is a point \( \eta \in \partial \Omega \cap \partial \Delta \) such that \( \lim_{\eta \to \eta} F_t(z) = \eta \) whenever \( z \in \Omega \). \( \lim_{\eta \to \eta} f(z) = 0 \) and \( \lim_{\eta \to \eta} f'(z) = -\gamma \) exists with \( \gamma < 0 \). In addition, there is a conformal mapping \( \varphi \) of \( \Delta \) onto \( \Omega \) which satisfies Eq. (2) with some \( \alpha \geq -\gamma \).

**Definition 4.** A backward flow invariant domain (BFID) \( \Omega \subset \Delta \) for \( S \) is said to be maximal if there is no \( \Omega_1 \supset \Omega \), \( \Omega_1 \neq \Omega \), such that \( S \subset \text{Aut}(\Omega_1) \).

**Theorem 3.** Let \( S = \{F_t\}_{t \geq 0} \) be a semiflow on \( \Delta \) generated by \( f \), and let \( \eta \in \partial \Delta \) be a boundary regular null point of \( f \) with \( \gamma = \lim_{z \to \eta} f'(z) < 0 \). Let \( \varphi \) be a (univalent) solution of (2) with \( \alpha \geq -\gamma \) normalized by \( \varphi(1) = \tau \) and \( \varphi(-1) = \eta \). The following assertions are equivalent:

(i) \( \Omega = \varphi(\Delta) \) is a maximal BFID;
(ii) \( \alpha = -\gamma \);
(iii) \( \varphi \) is isogonal at the boundary point \( z = -1 \).

In general, a maximal BFID for \( S \) need not be unique. Moreover, if a semigroup \( S = \{F_t\}_{t \geq 0} \) contains neither elliptic automorphisms of \( \Delta \) nor a self-mapping of parabolic type, then there is a one-to-one correspondence between maximal flow invariant domains for \( S \) and repelling fixed points. This fact determines a flower structure of the collection of BFID’s around the Denjoy–Wolff point (see Fig. 1).

**Theorem 4.** Let \( S = \{F_t\}_{t \geq 0} \) be a semiflow on \( \Delta \) generated by \( f \). Assume that there is a sequence \( \{\eta_k\} \in \partial \Delta \) of boundary regular null points of \( f \), i.e., \( f(\eta_k) = 0 \) and \( \gamma_k = f'(\eta_k) > -\infty \). Then the following assertions hold.
(i) There is $\delta > 0$ such that $\gamma_k < -\delta < 0$ for all $k = 1, 2, \ldots$.

(ii) For each $a < -\delta < 0$ there is at most a finite number of the points $\eta_k$ such that $a \leq \gamma_k < -\delta$. Consequently, Eq. (2) has a (univalent) solution $\varphi \in \text{Hol}(\Delta)$ for each $\alpha \geq -\max\{\gamma_k\} > -\delta$.

(iii) If $\varphi_k$ is a solution of (2) normalized by $\varphi_k(1) = \tau$, $\varphi_k(-1) = \eta_k$ with $\alpha = \gamma_k$ and $\Omega_k = \varphi_k(\Delta)$ (i.e., $\Omega_k$ are maximal), then for each pair $\Omega_{k_1}$ and $\Omega_{k_2}$ such that $\eta_{k_1} \neq \eta_{k_2}$ either $\overline{\Omega_{k_1}} \cap \overline{\Omega_{k_2}} = \{\tau\}$ or $\overline{\Omega_{k_1}} \cap \overline{\Omega_{k_2}} = l$, where $l$ is a continuous curve joining $\tau$ with a point on $\partial \Delta$.

The proofs of our theorems are based on linearization models for semigroups constructed by solutions of Schröder’s and Abel’s functional equations (see, for example, [12,3,7,8] and [5]). The main tools in the study of geometric properties of these solutions are recent developments in the theory of starlike and spirallike functions with respect to a boundary point (see [14,11,16,9] and [2]). On the way to solving these problems, we prove a new angle distortion theorem for starlike and spiral-like functions with respect to interior and boundary points.

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